A PRIORI BOUNDS AND EXISTENCE RESULT OF POSITIVE SOLUTIONS FOR FRACTIONAL LAPLACIAN SYSTEMS

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ABSTRACT. In this paper, we consider the fractional Laplacian system
\[
\begin{cases}
(\Delta)_{\alpha}^{2} u + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + C(x)u = f(x, v), & x \in \Omega, \\
(\Delta)_{\beta}^{2} v + \sum_{i=1}^{N} c_i(x) \frac{\partial v}{\partial x_i} + D(x)v = g(x, u), & x \in \Omega, \\
u > 0, v > 0, & x \in \Omega, \\
u = 0, v = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \), \( \alpha \in (1, 2) \), \( \beta \in (1, 2) \), \( N > \max\{\alpha, \beta\} \). Under some suitable conditions on potential functions and nonlinear terms, we use scaling method to obtain a priori bounds of positive solutions for the fractional Laplacian system with distinct fractional Laplacians.

1. Introduction. We consider the following fractional Laplacian system
\[
\begin{cases}
(\Delta)_{\alpha}^{2} u + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + C(x)u = f(x, v), & x \in \Omega, \\
(\Delta)_{\beta}^{2} v + \sum_{i=1}^{N} c_i(x) \frac{\partial v}{\partial x_i} + D(x)v = g(x, u), & x \in \Omega, \\
u > 0, v > 0, & x \in \Omega, \\
u = 0, v = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \), \( \alpha \in (1, 2) \), \( \beta \in (1, 2) \), \( N > \max\{\alpha, \beta\} \). Under some suitable conditions on potential functions and nonlinear terms, we want to obtain a priori bounds of positive solutions for the fractional Laplacian system.

The fractional Laplacian \((\Delta)_{\alpha}^{2} u(x)\) is defined by
\[
(\Delta)_{\alpha}^{2} u(x) = C_{N, \alpha} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+\alpha}} dy,
\]
where \( P.V. \) denotes the Cauchy principal value of the integral (see [15]). The fractional Laplacian appears in many areas, for example, physics, probability, mathematical finances and so on, see [2, 6, 14] and reference therein.

The fractional Laplacian has been extensively studied during the last decade in mathematics. When we consider the scalar fractional Laplacian equations, a large number of articles have used variational methods to obtain existence and multiplicity of solutions for the fractional Laplacian equations, see [1, 3, 4, 12, 13, 16, 17, 19, 20, 26, 31, 32, 33, 34, 35] and reference therein. Recently, Chen

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et al. in [9, 10] developed a direct method of moving planes for the fractional Laplacian. There is also a vast literature concerning fractional Laplacian systems. Some articles considered Liouville type theorems for the equations and systems involving fractional Laplacian, see [24, 28, 29, 36] and reference therein.

Leite and Montenegro in [22] used the mountain pass theorem to prove existence of positive viscosity solutions for the following system

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u = v^q, & x \in \Omega, \\
(-\Delta)^{\frac{\alpha}{2}} v = u^p, & x \in \Omega, \\
u > 0, v > 0, & x \in \Omega, \\
u = 0, v = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(2)

when \(pq \neq 1, p, q > 0\) and

\[
\frac{1}{p+1} + \frac{1}{q+1} < \frac{N - \alpha}{N}.
\]

By a monotonicity result, Quaas and Xia in [28] obtained the non-existence of positive viscosity solutions for the following system

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u = v^q, & x \in \mathbb{R}^N, \\
(-\Delta)^{\frac{\alpha}{2}} v = u^p, & x \in \mathbb{R}^N, \\
u = 0, v = 0, & x \notin \mathbb{R}^N,
\end{cases}
\]

(3)

where \(1 < p, q < \frac{N+\alpha}{N-\alpha}\).

Recently, Li and Ma in [25] used the iteration method and a direct method of moving planes to consider the following system

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u(x) = f(v(x)), & x \in \mathbb{R}^N, \\
(-\Delta)^{\frac{\alpha}{2}} v(x) = g(u(x)), & x \in \mathbb{R}^N, \\
u, v \geq 0, & x \in \mathbb{R}^N,
\end{cases}
\]

(4)

where \(f \geq 0\) and \(g \geq 0\) are non-decreasing continuous and they satisfy some more conditions.

When the fractional Laplacian equations and systems are not of variational type, we can not use variational methods to solve the problems. As we know, the related results are considered in few articles.

For the fractional Laplacian equation

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} K u = u^p + g(x, u), & x \in \Omega, \\
u = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(5)

and

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} K u = u^p + h(x, u, \nabla u), & x \in \Omega, \\
u = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(6)

by the classical scaling method of Gidas and Spruck and topological degree, Barrios et al. in [5] obtained a priori bounds and existence of positive solutions for equation (5) and equation (6) under the condition \(1 < p < \frac{N+\alpha}{N-\alpha}\).

Recently, Leite and Montenegro in [23] considered the following strongly coupled fractional Laplacian system in non-variational form

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u = v^q, & x \in \Omega, \\
(-\Delta)^{\frac{\alpha}{2}} v = u^p, & x \in \Omega, \\
u > 0, v > 0, & x \in \Omega, \\
u = 0, v = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(7)
They established a priori bounds of positive solutions for subcritical and superlinear nonlinearities by means of blow-up method. They also derived the existence of positive solutions through topological method.

It is worth to point out that our paper is inspired by [30]. For the fractional Laplacian system

$$
\begin{align*}
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u &= a(x)u^{\beta_1} + b(x)v^{\beta_2} + f(x, u, v, \nabla u, \nabla v), & x \in \Omega, \\
(-\Delta)^{\frac{\alpha}{2}} v &= c(x)u^{\beta_3} + p(x)v^{\beta_4} + g(x, u, v, \nabla u, \nabla v), & x \in \Omega,
\end{cases}
\end{align*}
$$

(8)

where $1 < \alpha < 2$, $\beta_1 > 1$, $\beta_2 > 1$, and some suitable conditions hold, Quaas and Xia in [30] obtained a priori bounds and existence of positive solutions for system (8). It is a natural question: do we have a priori bounds if the system contains linear terms $u, v, \nabla u$ and $\nabla v$? Furthermore, as far as we know, there are few works about the fractional system with distinct fractional operators. In our paper, we solve the problem when the fractional operators are distinct.

To state our main result with respect to system (1), let us introduce the notations (cf. Chapter 6 in [18])

$$
\|u\|_{0, \gamma} = \sup_{\Omega} (d(x)^\gamma |u(x)|)
$$

and

$$
\|u\|_{1, \gamma} = \sup_{\Omega} (d(x)^\gamma |u(x)| + d(x)^{\gamma+1} |\nabla u(x)|),
$$

where $\gamma \in \mathbb{R}$, $u \in C^1(\Omega)$ and $d(x) := \text{dist}(x, \partial \Omega)$.

We now formulate the assumptions.

(V1) $b_i(x), c_i(x), C_i, D(x) : \bar{\Omega} \to [0, \infty)$ are continuous functions, $i = 1, 2, \ldots, N$.

(F1) $f(x, t), g(x, t) : \bar{\Omega} \times \mathbb{R} \to [0, \infty)$ are continuous functions.

(F2) $\lim_{t \to \infty} \frac{f(x, t)}{t^q} = K(x)$, $\lim_{t \to \infty} \frac{g(x, t)}{t^q} = h(x)$, here $K(x), h(x) : \bar{\Omega} \to (0, \infty)$ are continuous functions, $1 < p < \frac{n+\theta}{n-\alpha}$ and $1 < q < \frac{n+\theta}{n-\beta}$.

(F3) $\lim_{t \to 0} \frac{f(x, t)}{t^q} = K_1(x)$, $\lim_{t \to 0} \frac{g(x, t)}{t^q} = h_1(x)$, here $K_1(x), h_1(x) : \bar{\Omega} \to (0, \infty)$ are continuous functions, $p_1 > 1$, and $q_1 > 1$.

We take $\theta$ satisfying

$$
\max\left\{\frac{\alpha}{2} - 1, \frac{\beta}{2} - 1\right\} < \theta < 0. \tag{9}
$$

Then we have the following main result.

**Theorem 1.1.** Suppose (V1) and (F1) – (F2) hold. If $1 < \alpha < 2, 1 < \beta < 2$ and $(u, v)$ is a positive viscosity solution of system (1), then there exists a positive number $C$, such that $\|u\|_{1, \theta} \leq C$, $\|v\|_{1, \theta} \leq C$.

When the potential functions $b_i(x) = c_i(x) = C(x) = D(x) = 0, \forall x \in \Omega$, then system (1) turns into the following system

$$
\begin{align*}
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u &= f(x, v), & x \in \Omega, \\
(-\Delta)^{\frac{\alpha}{2}} v &= g(x, u), & x \in \Omega,
\end{cases}
\end{align*}
$$

(10)

Similar to the proof of Theorem 3.1 in [30], we can have a priori bounds for system (10).
Theorem 1.2. Suppose \((F_1) - (F_2)\) hold. If \(0 < \alpha < 1, 0 < \beta < 1\) and \((u,v)\) is a positive viscosity solution of system \((10)\), then there exists a positive number \(C\), such that \(\|u\|_{L^\infty(\Omega)} \leq C, \|v\|_{L^\infty(\Omega)} \leq C\).

Furthermore, using topological degree and the priori bounds in Theorem 1.2, we can prove the existence of positive solutions for system \((10)\).

Theorem 1.3. Suppose \((F_1) - (F_3)\) hold and \(\alpha \in (0,1), \beta \in (0,1)\). Then there exists at least one positive viscosity solution of system \((10)\).

This paper is organized as follows. In Section 2, preliminary results are revisited. We prove Theorem 1.1 in Section 3. In Section 4, we obtain some results and prove Theorem 1.3.

2. Preliminaries. To begin with, we introduce some notations. For \(x \in \Omega\), \(d(x) = \text{dist}(x, \partial\Omega)\). For \(r > 0\), \(B_r := \{x \in \mathbb{R}^N : |x| < r\}\) represents the ball with radius \(r\). We shall use \(C\) and \(C_i\) to represent positive constants which may be distinct even in the same line.

Theorem 2.1. ([28], Theorem 2.5) Let \(g\) bounded in \(\mathbb{R}^N \setminus \Omega\) and \(f \in C^\gamma_{\text{loc}}(\Omega)\). Suppose \(u\) is a viscosity solution of
\[
(\Delta)u^f = f \quad \text{in } \Omega, \quad u = g \quad \text{in } \mathbb{R}^N \setminus \Omega.
\]
Then there exists a \(\gamma > 0\) such that \(u \in C^{\alpha+\gamma}_{\text{loc}}(\Omega)\).

Theorem 2.2. ([21], Theorem 1.2) Assume that \(\alpha \in (1,2)\). Suppose \(u\) is a viscosity solution of
\[
(\Delta)u^f = f \quad \text{in } \Omega,
\]
where \(f \in L^\infty_{\text{loc}}(\Omega)\). Then there exists \(\gamma = \gamma(N,\alpha) \in (0,1)\) such that \(u \in C^1_{\text{loc}}(\Omega)\). Moreover, for every ball \(B_R \subset \subset \Omega\), there exists a positive constant \(C = C(N,\alpha,R)\) such that
\[
\|u\|_{C^1(\overline{B}_R)} \leq C(\|f\|_{L^\infty(B_R)} + \|u\|_{L^\infty(\mathbb{R}^N)}).
\]

Theorem 2.3. ([7], Lemma 5) Let \(\{u_k\}, k \in \mathbb{N}\) be a sequence of functions that are bounded in \(\mathbb{R}^N\) and continuous in \(\Omega\), \(f_k\) and \(f\) are continuous in \(\Omega\) such that

1. \((\Delta)\mathbf{\overline{u}} u_k \leq f_k\) in \(\Omega\) in viscosity sense.
2. \(u_k \to u\) locally uniformly in \(\Omega\).
3. \(u_k \to u\) a. e. in \(\mathbb{R}^N\).
4. \(f_k \to f\) locally uniformly in \(\Omega\).

Then \((\Delta)\mathbf{\overline{u}} u \leq f\) in \(\Omega\) in viscosity sense.

Let \(\xi \in \partial\Omega\) and \(\lambda > 0\). We denote
\[
\Omega_\lambda = \{x \in \mathbb{R}^N : \lambda x + \xi \in \Omega\}
\]
and
\[
d_\lambda(x) := \text{dist}(x, \partial\Omega_\lambda).
\]

Lemma 2.4. ([5], Lemma 6) Suppose \(0 < \alpha < 2\). For every \(\tau \in \left(\frac{\alpha}{2}, \alpha\right)\), if \(u\) satisfies
\[
(\Delta)\mathbf{\overline{u}} u \leq C_1 d^\tau_\lambda(x), \quad x \in \Omega_\lambda,
\]
for some \(C_1 > 0\) with \(u = 0\) in \(\mathbb{R}^N \setminus \Omega\), then
\[
u(x) \leq C_2 (C_1 + \|u\|_{L^\infty(\Omega_\lambda)})d^{-\tau}_\lambda(x), \quad x \in (\Omega_\lambda)_\delta,
\]
for some \(C_2 > 0\) depending on \(\alpha, \delta, \tau\).
Theorem 2.5. ([29], Theorem 1.1) Let $p, q > 0$ and $pq > 1$. Suppose
\[
\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1} \in \left(\frac{N-\alpha}{2}, N-\alpha\right)
\]
and
\[
\left(\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\right) \neq \left(\frac{N-\alpha}{2}, N-\alpha\right).
\]
Then, for some $\sigma > 0$, there exists no positive $L^\frac{2}{p}(\mathbb{R}^N) \cap C^{\alpha+\sigma}(\mathbb{R}^N)$ if $0 < \alpha < 1$ or in $L^\frac{2}{p}(\mathbb{R}^N) \cap C^{1,\alpha+\sigma-1}(\mathbb{R}^N)$ if $\alpha > 1$ type solution to system
\[
\begin{cases}
(\triangle)^{\frac{2}{p}} u = v^q, & x \in \mathbb{R}^N, \\
(\triangle)^{\frac{2}{p}} v = u^p, & x \in \mathbb{R}^N.
\end{cases}
\tag{11}
\]

The following result is important to obtain a priori bounds of positive viscosity solutions for system (1), which is inspired by [30].

Lemma 2.6. Under the assumptions in Theorem 1.1, if $u, v \in C^1(\Omega) \cap L^\infty(\mathbb{R}^N)$ and $(u, v)$ is a positive viscosity solution of the following system
\[
\begin{cases}
(\triangle)^{\frac{2}{p}} u + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + C(x) u = f(x, u), & x \in \Omega, \\
(\triangle)^{\frac{2}{p}} v + \sum_{i=1}^{N} c_i(x) \frac{\partial v}{\partial x_i} + D(x) v = g(x, u), & x \in \Omega, \\
u > 0, v > 0, & x \in \Omega, \\
u = 0, v = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\tag{12}
\]
then there exists a positive constant $C$ such that
\[
u(x) \leq C(1 + d^{-\beta_1}(x)), \quad |\nabla u(x)| \leq C(1 + d^{-\beta_1}(x)),
\]
\[
u(x) \leq C(1 + d^{-\beta_2}(x)), \quad |\nabla v(x)| \leq C(1 + d^{-\beta_2}(x)),
\]
where $\beta_1 = \frac{\alpha + 2\beta}{pq-1}, \beta_2 = \frac{\alpha + p\beta}{pq-1}$.

Proof. Assume on the contrary that there exists a sequence $\{(u_k, v_k)\}$ of positive solutions of system (12) and $y_k \in \Omega$ such that $W_k(y_k) > 2k(1 + d^{-1}(y_k))$, here $W_k(x) := (u_k(x))^{\frac{1}{q}} + |\nabla u_k(x)|^{\frac{1}{pq-1}} + (v_k(x))^{\frac{1}{q}} + |\nabla v_k(x)|^{\frac{1}{pq-1}}$. By Lemma 5.1 in [27], there exists $x_k \in \Omega$ such that
\[
W_k(x_k) \geq W_k(y_k), \quad W_k(x_k) > 2kd^{-1}(x_k),
\]
\[
W_k(z) \leq 2W_k(x_k), \forall z \in B(x_k, kW_k^{-1}(x_k)).
\]
We know that $W_k(x_k) \to \infty, k \to \infty$. Let $\lambda_k := \frac{1}{W_k(x_k)} \to 0, k \to \infty$. Define
\[
\tilde{u}_k(x) = \lambda_k^{\beta_1} u_k(\lambda_k x + x_k)
\]
and
\[
\tilde{v}_k(x) = \lambda_k^{\beta_2} v_k(\lambda_k x + x_k)
\]
in $B_k := \{x \in \mathbb{R}^N : |x| < k\}$. By direct calculation, we know that $(\tilde{u}_k, \tilde{v}_k)$ is a positive solution of the following system
\[
\begin{cases}
(\triangle)^{\frac{2}{p}} \tilde{u}_k(x) + \lambda_k^{\beta_1-1} \sum_{i=1}^{N} b_i(\lambda_k x + x_k) \frac{\partial \tilde{u}_k(x)}{\partial x_i} + \lambda_k^{\beta_1} C(\lambda_k x + x_k) \tilde{u}_k(x) = \lambda_k^{\beta_1+\beta_1} f(\lambda_k x + x_k, \lambda_k^{-\beta_1} \tilde{u}_k(x)), & x \in B_k, \\
(\triangle)^{\frac{2}{p}} \tilde{v}_k(x) + \lambda_k^{\beta_2-1} \sum_{i=1}^{N} c_i(\lambda_k x + x_k) \frac{\partial \tilde{v}_k(x)}{\partial x_i} + \lambda_k^{\beta_2} D(\lambda_k x + x_k) \tilde{v}_k(x) = \lambda_k^{\beta_2+\beta_2} g(\lambda_k x + x_k, \lambda_k^{-\beta_2} \tilde{u}_k(x)), & x \in B_k.
\end{cases}
\tag{13}
For \( k \) large enough, we obtain that
\[
(\bar{u}_k(x))^{\frac{1}{q}} + |\nabla \bar{u}_k(x)|^{\frac{1}{q-1}} + (\bar{v}_k(x))^{\frac{1}{p}} + |\nabla \bar{v}_k(x)|^{\frac{1}{p-1}} \\
= W_k(\lambda_k x + x_k) \\
\leq 2
\] (14)
and
\[
(\bar{u}_k(0))^{\frac{1}{q}} + |\nabla \bar{u}_k(0)|^{\frac{1}{q-1}} + (\bar{v}_k(0))^{\frac{1}{p}} + |\nabla \bar{v}_k(0)|^{\frac{1}{p-1}} = 1.
\] (15)
So \( \bar{u}_k, \nabla \bar{u}_k, \bar{v}_k, \nabla \bar{v}_k \) are uniformly bounded in \( B_k \). By Theorem 2.2, we get that there exist \( \beta' \in (0, 1), \beta'' \in (0, 1) \), such that \( \bar{u}_k \in C^{0, \beta'}(\Omega), \bar{v}_k \in C^{0, \beta''}(\Omega) \). For any \( B_R \subset \subset \Omega \), there exists a constant \( C = C(R) > 0 \), such that
\[
\| \bar{u}_k \|_{C^{0, \beta'}(\Omega)} \leq C, \| \bar{u}_k \|_{C^{0, \beta''}(\Omega)} \leq C.
\] (16)
By Ascoli-Arzelà’s theorem and the diagonal argument, up to a subsequence, we have that \( \bar{u}_k \to u, \bar{v}_k \to v \) in \( C^{0, \beta'}_\text{loc}(\mathbb{R}^N) \) as \( k \to \infty \). Taking limit in (15), we have
\[
u(0)^{\frac{1}{q}} + |\nabla \nu(0)|^{\frac{1}{q-1}} + \nu(0)^{\frac{1}{p}} + |\nabla \nu(0)|^{\frac{1}{p-1}} = 1.
\]
So \( (u, v) \neq (0, 0) \).

**Case (I).** If up to a subsequence, \( v_k(\lambda_k x + x_k) \to \infty, k \to \infty \), then \( \lambda_k^{-\beta_2} \bar{v}_k(0) = v_k(\lambda_k x + x_k) \to \infty, k \to \infty \). By (F2) and (14), for \( k \) large enough, we have
\[
\lambda_k^{\alpha + \beta_1} f(\lambda_k x + x_k, \lambda_k^{-\beta_2} \bar{v}_k(x)) \\
= \lambda_k^{\alpha + \beta_1} \frac{f(\lambda_k x + x_k, \lambda_k^{-\beta_2} \bar{v}_k(0))}{(\lambda_k^{-\beta_2} \bar{v}_k(x))^{\eta}} \\
= \lambda_k^{\alpha + \beta_1 - \eta \beta_2} (K(\lambda_k x + x_k) + o(1))(\bar{v}_k(x))^{\eta} \\
= K(\lambda_k x + x_k)(\bar{v}_k(x))^{\eta} + o(1).
\] (17)

**Case (II).** If up to a subsequence, \( v_k(\lambda_k x + x_k) \to c, k \to \infty \), \( \lambda_k^{-\beta_2} \bar{v}_k(0) = v_k(\lambda_k x + x_k) \to c, k \to \infty \). By the fact that \( \alpha + \beta_1 > 0 \) and (F1), we have
\[
\lambda_k^{\alpha + \beta_1} f(\lambda_k x + x_k, \lambda_k^{-\beta_2} \bar{v}_k(x)) \\
= \lambda_k^{\alpha + \beta_1} f(\lambda_k x + x_k, v_k(\lambda_k x + x_k)) \\
\to 0, \text{ as } k \to \infty.
\] (18)

**Case (III).** If up to a subsequence, \( u_k(\lambda_k x + x_k) \to \infty, k \to \infty \), then \( \lambda_k^{-\beta_1} \bar{u}_k(0) = u_k(\lambda_k x + x_k) \to \infty, k \to \infty \). By (F2) and (14), for \( k \) large enough, we have
\[
\lambda_k^{\beta + \beta_2} g(\lambda_k x + x_k, \lambda_k^{-\beta_1} \bar{u}_k(x)) \\
= \lambda_k^{\beta + \beta_2} \frac{g(\lambda_k x + x_k, \lambda_k^{-\beta_1} \bar{u}_k(0))}{(\lambda_k^{-\beta_1} \bar{u}_k(x))^p} \\
= \lambda_k^{\beta + \beta_2 - p \beta_1} (h(\lambda_k x + x_k) + o(1))(\bar{u}_k(x))^p \\
= h(\lambda_k x + x_k)(\bar{u}_k(x))^p + o(1).
\] (19)
Case (IV). If up to a subsequence, \( u_k(\lambda_k x + x_k) \to c, k \to \infty \), then \( \lambda_k^{-\beta_1} u_k(x) = u_k(\lambda_k x + x_k) \to c, k \to \infty \). By the fact that \( \beta + \beta_2 > 0 \) and (F₁), we have
\[
\lambda_k^{\beta + \beta_2} g(\lambda_k x + x_k, \lambda_k^{-\beta_1} u_k(x)) \\
= \lambda_k^{\beta + \beta_2} g(\lambda_k x + x_k, u_k(\lambda_k x + x_k)) \\
\to 0, \text{ as } k \to \infty.
\] (20)

If the case (I+ III) holds, then we may assume \( x_k \to x_0 \). Taking the limit in system (13) by Theorem 2.3, we have
\[
\begin{cases}
(-\Delta) \frac{\partial}{\partial x} u = K(x_0) v^p, & x \in \mathbb{R}^N, \\
(-\Delta) \frac{\partial}{\partial x} v = h(x_0) u^q, & x \in \mathbb{R}^N, \\
u \ge 0, v \ge 0, & x \in \mathbb{R}^N.
\end{cases}
\] (21)

But by Theorem 1.2 in [23] we know that system (21) has no positive solution. So We get the contradiction.

If the case (I+ IV) holds, then we may assume \( x_k \to x_0 \). Taking the limit in system (13) by Theorem 2.3, we have
\[
\begin{cases}
(-\Delta) \frac{\partial}{\partial x} u = K(x_0) v^p, & x \in \mathbb{R}^N, \\
(-\Delta) \frac{\partial}{\partial x} v = 0, & x \in \mathbb{R}^N, \\
u \ge 0, v \ge 0, & x \in \mathbb{R}^N.
\end{cases}
\] (22)

Using Liouville theorem to the second equation in system (22), we obtain \( v \equiv C \ge 0 \). But by the case (IV), We can know \( \tilde{u}_k(x) \to 0 \). Therefore we get \( u \equiv 0 \). By the first equation in system (22), we have \( u \equiv 0 \). So we get the contradiction.

In the same way, we can also get the contradiction in case (II+ III) and in case (II+ IV). Thus we get the result of Lemma 2.6. \( \square \)

3. Result about a priori bounds. We prove Theorem 1.1 in this section.

Proof of Theorem 1.1. Assume on the contrary that there exists a sequence \( \{(u_k, v_k)\} \) of positive solutions of system (1) such that
\[
\|u_k\|_{1,\theta} \to \infty \text{ or } \|v_k\|_{1,\theta} \to \infty, \text{ as } k \to \infty.
\]
We may assume that
\[
(\|u_k\|_{1,\theta})^{\beta_2 - \theta} \ge (\|v_k\|_{1,\theta})^{\beta_1 - \theta},
\] (23)
here \( \beta_1, \beta_2 \) are positive numbers which are the same as the numbers in Lemma 2.6.

Denote
\[
\lambda_k = (\|u_k\|_{1,\theta})^{-\frac{1}{\beta_1}},
\]
\[
M_k(x) = d^\theta(x) u_k(x) + d^{\theta + 1}(x) |\nabla u_k(x)|,
\]
\[
N_k(x) = d^\theta(x) v_k(x) + d^{\theta + 1}(x) |\nabla v_k(x)|.
\]
There exists \( x_k \in \Omega \), such that \( M_k(x_k) \ge \sup_{\Omega} M_k - \frac{1}{k}, M_k(x_k) \to +\infty \). Let \( \xi_k \) be a projection of \( x_k \) on \( \partial \Omega \). Denote
\[
\bar{u}_k(x) = \lambda_k^{\beta_1} u_k(\lambda_k x + \xi_k),
\]
\[
\bar{v}_k(x) = \lambda_k^{\beta_2} v_k(\lambda_k x + \xi_k),
\]
\[
\Omega_k = \{ x \in \mathbb{R}^N : \lambda_k x + \xi_k \in \Omega \}.
\]
By direct calculation, we can deduce that \((\overline{u}_k, \overline{v}_k)\) is a positive solution of the following system

\[
\begin{cases}
(\Delta)^2 \overline{u}_k + \lambda_k^{\alpha - 1} \sum_{i=1}^{N} b_i (\lambda_k x + \xi_k) \frac{\partial \overline{u}_k(x)}{\partial x_i} + \lambda_k^\alpha C(\lambda_k x + \xi_k) \overline{u}_k(x) \\
= \lambda_k^{\alpha + \beta} f(\lambda_k x + \xi_k, \lambda_k^{-\beta} \overline{u}_k(x)), \quad x \in \Omega_k,
\end{cases}
\]

\[
(\Delta)^2 \overline{v}_k(x) + \lambda_k^{\beta - 1} \sum_{i=1}^{N} c_i (\lambda_k x + \xi_k) \frac{\partial \overline{v}_k(x)}{\partial x_i} + \lambda_k^\beta D(\lambda_k x + \xi_k) \overline{v}_k(x)
\]

\[
= \lambda_k^{\beta + \beta_k} g(\lambda_k x + \xi_k, \lambda_k^{-\beta} \overline{v}_k(x)), \quad x \in \Omega_k,
\]

\[
\overline{u}_k(x) > 0, \overline{v}_k(x) > 0, \quad x \in \Omega_k,
\]

\[
\overline{u}_k(x) = 0, \overline{v}_k(x) = 0, \quad x \in \mathbb{R}^N \setminus \Omega_k.
\]

Now we want to estimate \(\|\overline{u}_k\|_{1,\theta} \).

Denote

\[
d_k(x) := \text{dist}(x, \partial \Omega_k).
\]

It is easy to see that

\[
d_k(x) = \frac{\text{dist}(\lambda_k x + \xi_k, \partial \Omega)}{\lambda_k}.
\]

For \(x \in \Omega_k\), by direct calculation and (23), we have

\[
d_k^0(\lambda_k x) \overline{u}_k(x) + d_k^{0+1}(\lambda_k x) \| \nabla \overline{u}_k(x) \|
\]

\[
= \lambda_k^{\beta - \theta} M_k(\lambda_k x + \xi_k)
\]

\[
= \frac{M_k(\lambda_k x + \xi_k)}{\|u_k\|_{1,\theta}}
\]

\[
\leq 1 \quad (25)
\]

and

\[
d_k^0(\lambda_k x) \overline{v}_k(x) + d_k^{0+1}(\lambda_k x) \| \nabla \overline{v}_k(x) \|
\]

\[
= \lambda_k^{\beta - \theta} N_k(\lambda_k x + \xi_k)
\]

\[
= \frac{N_k(\lambda_k x + \xi_k)}{\|v_k\|_{1,\theta}}
\]

\[
\leq 1 \quad (26)
\]

Choosing the point \(z_k = \frac{x_k - \xi_k}{\lambda_k}\), by the definition of \(M_k\) and choice of \(x_k\), we know that

\[
d_k^0(z_k) \overline{u}_k(z_k) + d_k^{0+1}(z_k) \| \nabla \overline{u}_k(z_k) \| \to 1, \quad k \to \infty. \quad (27)
\]

There exists a subsequence of \(\{x_k\}\) which is also denoted by \(\{x_k\}\) such that \(x_k \to x^0 \in \Omega\).

By virtue of Lemma 2.6 and the boundedness of \(d(x_k)\), we obtain

\[
M_k(x_k) = \frac{d^0(x_k) u_k(x_k) + d^{0+1}(x_k) |\nabla u_k(x_k)|}{\|u_k\|_{1,\theta}}
\]

\[
\leq C(d^0(x_k)(1 + d^{-\beta_1}(x_k)) + d^{0+1}(x_k)(1 + d^{-\beta_1^{-1}}(x_k)))
\]

\[
= C d^0(x_k)(1 + 2d^{-\beta_1}(x_k) + d(x_k))
\]

\[
\leq C d^0(x_k)(1 + d^{-\beta_1}(x_k)). \quad (28)
\]
By (28), we have
\[ \| u_k \|_{1, \theta} = \sup_{\Omega} M_k \leq M_k(x_k) + \frac{1}{k} \leq Cd^\theta(x_k)(1 + d^{-\beta_1}(x_k)). \]  
(29)

Combining the definition of \( \lambda_k \) with (29), we obtain that
\[ \frac{d(x_k)}{\lambda_k} = \frac{d(x_k)}{\| u_k \|_{1, \theta}^{1-\theta}} \leq Cd(x_k)(d^\theta(x_k)(1 + d^{-\beta_1}(x_k)))^{\frac{1}{1-\theta}} \]
\[ \leq Cd(x_k)d^{\frac{1}{1-\theta}}(x_k)(1 + d^{-\beta_1}(x_k)) \]
\[ = Cd^{\frac{\beta_1}{1-\theta}}(x_k)(1 + d^{-\beta_1}(x_k)) \]
\[ = C(1 + d^{\frac{\beta_1}{1-\theta}}(x_k)). \]  
(30)

Here we require \( pq > 1, \theta < \frac{\alpha+\beta_1}{pq-1}, \) so \( \frac{\beta_1}{\beta_1 - \theta} > 0. \) Up to a subsequence of \( \{ x_k \}, \) we can get that
\[ \lim_{k \to \infty} \frac{d(x_k)}{\lambda_k} = d \geq 0. \]  
(31)

If there exists a subsequence of \( \{ \overline{v}_k \} \) (we still denote it by \( \{ \overline{v}_k \} \)) such that \( \lambda_k^{-\beta_2}\overline{v}_k(x) \to \infty, k \to \infty, \) then by (F2), (26) and the fact \( \alpha + \beta_1 - 2\beta q = 0, \) as \( k \) large enough, we have
\[ \lambda_k^{\alpha+\beta_1} f(\lambda_k x + \xi_k, \lambda_k^{-\beta_2}\overline{v}_k(x)) \]
\[ = \lambda_k^{\alpha+\beta_1} f(\lambda_k x + \xi_k, \lambda_k^{-\beta_2}\overline{v}_k(x)) \]
\[ = (\lambda_k^{-\beta_2}\overline{v}_k(x))^q \]
\[ \leq C d^\theta(x). \]
(32)

We can use (25) to obtain
\[ |\lambda_k^{\alpha-1} \sum_{i=1}^{N} b_i(\lambda_k x + \xi_k) \frac{\partial \overline{v}_k(x)}{\partial x_i} | \leq C \lambda_k^{\alpha-1} d_k^{\theta-1}(x) \]
(33)

and
\[ \lambda_k^\alpha C(\lambda_k x + \xi_k) \overline{v}_k(x) \leq C \lambda_k^\alpha d_k^{-\theta}(x). \]
(34)

Combining (24), (32), (33) with (34), we deduce that
\[ (-\Delta)^{-\frac{1}{2}} \overline{v}_k(x) \leq C d_k^{\theta-1}(x), \text{ when } d_k(x) < \delta. \]
(35)

Because of (35), we use Lemma 2.4 to deduce that
\[ \overline{v}_k(x) \leq C_0 d_k^{\theta-1+\alpha}(x), \text{ when } d_k(x) < \delta. \]
(36)
By (25), we have
\[ d_k^\alpha(x) = d_k^{1-\alpha}(x) d_k^{\alpha-1}(x) \leq \delta^{1-\alpha} d_k^{\alpha-1}(x), \text{ when } d_k(x) \geq \delta. \]
Therefore
\[ \mathfrak{u}_k(x) \leq \delta^{1-\alpha} d_k^{\theta-1+\alpha}(x), \text{ when } d_k(x) \geq \delta. \]
We take \( \delta > C_0^{-\epsilon} \). Thus we know that \( \mathfrak{u}_k(x) \leq C_0 d_k^{\theta-1+\alpha}(x) \). By Lemma 5 in [5], we get that |\nabla \mathfrak{u}_k| \leq Cd_k^{\theta-2+\alpha}(x). So we can get that
\[ d_k^\theta(z_k) \mathfrak{u}_k(z_k) + d_k^{\theta+1}(z_k)|\nabla \mathfrak{u}_k(z_k)| \leq Cd_k^{\theta-1}(z_k). \]
We can use (27) together with (37) to deduce that \( d_k(z_k) \) is bounded away from 0 as \( k \) large enough. Using the fact \( d_k(z_k) = \frac{d(z_k)}{\lambda_k} \), we know \( d > 0 \). By the fact 0 \( \in \partial \Omega_k \), we can obtain that \( z_k \) is also bounded away from 0. Thus up to a subsequence, \( z_k \to z_0 \). We denote \( \mathbb{R}_+^N = \{ x \in \mathbb{R}^N | x_N \geq -d \} \). Using Ascoli-Arzelà’s theorem, we have
\[ \mathfrak{u}_k \to \mathfrak{u}_0 \text{ in } C^1_{loc}(\mathbb{R}_+^N), \]
\[ \mathfrak{v}_k \to \mathfrak{v}_0 \text{ in } C^1_{loc}(\mathbb{R}_+^N), \]
and
\[ \text{dist}^\theta(0, \mathbb{R}_+^N) \mathfrak{u}_0(0) + \text{dist}^{\theta+1}(0, \mathbb{R}_+^N)|\nabla \mathfrak{u}_0(0)| = 1. \]
As a result, we get that \((\mathfrak{u}_0, \mathfrak{v}_0) \neq (0, 0)\).
\[ \mathfrak{u}_0(x) \leq C\text{dist}^{-\theta+1+\alpha}(x, \mathbb{R}_+^N), \text{ if dist}(x, \mathbb{R}_+^N) < \delta, \]
\[ \mathfrak{v}_0(x) \leq C\text{dist}^{-\theta+1+\alpha}(x, \mathbb{R}_+^N), \text{ if dist}(x, \mathbb{R}_+^N) < \delta. \]
Therefore \( \mathfrak{u}_0(x) = \mathfrak{v}_0(x) = 0 \) in \( \mathbb{R}^N \setminus \mathbb{R}_+^N \).
Furthermore, \((\mathfrak{u}_0, \mathfrak{v}_0)\) is a positive solution of the following system
\begin{align*}
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} \mathfrak{u}_0(x) = K(\mathfrak{u}_0)^p(\mathfrak{u}_0), & \text{in } \mathbb{R}_+^N, \\
(-\Delta)^{\frac{\alpha}{2}} \mathfrak{v}_0(x) = h(\mathfrak{v}_0)^q(\mathfrak{u}_0), & \text{in } \mathbb{R}_+^N, \\
\mathfrak{u}_0(x) = 0, \mathfrak{v}_0(x) = 0, & \text{in } \mathbb{R}^N \setminus \mathbb{R}_+^N.
\end{cases}
\end{align*}
By (F2) and Theorem 1.3 in [23], system (38) has no positive solution. Thus we deduce the contradiction. Therefore we get the result of Theorem 1.1. \( \Box \)

4. Proof of Theorem 1.3. In this section, we use the following important topological degree theory to consider the existence of positive solutions for system (10).

Theorem 4.1. ([8], Theorem 3.6.3) Suppose that \( E \) is an ordered Banach space with positive cone \( P \), and \( U \subset P \) is an open bounded set containing \( 0 \). Let \( \rho > 0 \) be such that \( B_{\rho}(0) \cap P \subset U \). Assume \( T : U \to P \) is compact and satisfies
\begin{enumerate}
\item[(a)] for every \( \mu \in [0, 1] \), we have \( u \neq \mu T(u) \) for every \( u \in P \) with \( \|u\| = \rho \);
\item[(b)] there exists \( \psi \in P \setminus \{0\} \) such that \( u - T(u) \neq t\psi \), for every \( u \in \partial U \), for every \( t \geq 0 \).
\end{enumerate}
Then \( T \) has a fixed point in \( U \setminus B_{\rho}(0) \).

Given \((u, v) \in C(\Omega) \times C(\Omega)\), we consider the following system
\begin{align*}
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} \bar{u} = f(x, v), & x \in \Omega, \\
(-\Delta)^{\frac{\alpha}{2}} \bar{v} = g(x, u), & x \in \Omega, \\
\bar{u} > 0, \bar{v} > 0, & x \in \Omega, \\
\bar{u} = 0, \bar{v} = 0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
\end{align*}
(39)
We choose \( \gamma \) from Lemma 12.3.5 in [11], we can obtain that there exists \( C \) bounded in \( \mathbb{R}^N \). It is easy to see that (\( \bar{u} \)) subsequence of \( \{u_k\} \)

Similarly, we also have that up to a subsequence of \( \{v_k\} \), there exists \( \bar{v}_0 \) such that

where \( G_1(x, y) \) is the Green’s function for \((-\Delta)^\frac{3}{2} \) in \( \mathbb{R}^N \).

In the same way, we have

We know that \((-\Delta)^\frac{3}{2} \bar{u}_k = f(x, v_k) \) is bounded. According to Lemma 12.3.5 in [11], we can obtain that there exists \( \gamma_0 > 0 \) such that \( \bar{u}_k \) is bounded in \( C^{\gamma_0}(\Omega) \). The embedding \( C^{\gamma_0}(\Omega) \hookrightarrow C_0(\Omega) \) is compact. Then up to a subsequence of \( \{\bar{u}_k\} \), there exists \( u_0 \) such that

Similarly, we also have that up to a subsequence of \( \{\bar{v}_k\} \), there exists \( \bar{v}_0 \) such that

It is easy to see that \((\bar{u}_0, \bar{v}_0) \in P \). Therefore we finish the proof.

**Proof of Theorem 1.3.** We choose \( \rho > 0 \) small enough and define

\[
B_\rho(0) = \{(u, v) \in E \mid \| (u, v) \|_E < \rho \},
\]

\[
\partial B_\rho(0) = \{(u, v) \in E \mid \| (u, v) \|_E = \rho \}.
\]

For \((u, v) \in \partial B_\rho(0) \cap P \), assume that \((u, v) \) satisfies \((u, v) = \mu T(u, v) \) for some \( \mu \in [0, 1) \). We know that \((u, v) \) is a solution of the following system

By \((F_3)\), we get that

\[
\mu f(x, v) \leq f(x, v) \leq C_1 v^{\alpha_1} \leq C_1 \rho^{\alpha_1} \quad (42)
\]

and

\[
\mu g(x, u) \leq g(x, u) \leq C_1 u^{\beta_1} \leq C_1 \rho^{\beta_1}. \quad (43)
\]

Using (42) and (43), we have

\[
\mu T_1(u, v) = \mu \int_\Omega G_1(x, y) f(y, v) dy \leq C_1 \rho^{\alpha_1} \int_\Omega G_1(x, y) dy \leq C_1 \rho^{\alpha_1} \quad (44)
\]
and

\[ \mu T_2(u, v) = \mu \int_{\Omega} G_2(x, y)f(y, u)dy \leq C_1 \rho^{p_1} \int_{\Omega} G_2(x, y)dy \leq C \rho^{p_1}, \]  

(45)

where \( G_2(x, y) \) is the Green’s function for \((-\Delta)^{\frac{\alpha}{2}}\) in \(\mathbb{R}^N\). Thus we deduce that

\[ \|(u, v)\|_E \leq C \rho^{\min\{p_1, q_1\}}. \]  

(46)

We take \( \rho > 0 \) small enough. Then it is impossible for (46) if \((u, v) \in \partial B_\rho(0) \cap P\). Therefore for every \( \mu \in [0, 1) \), we have \((I - \mu T)(u, v) \neq 0\) for every \((u, v) \in P\) with \(\|(u, v)\|_E = \rho\).

Now we want to check condition (b) in Theorem 4.1, namely there exist \( U \subset P \), \((\phi, \psi) \in P\), such that if \( R > 0 \) large enough, then

\[ (u, v) - T(u, v) \neq t(\phi, \psi), \forall t \geq 0, \forall (u, v) \in \partial B_R(0). \]  

(47)

Taking \((\phi, \psi) \in P\), \(\phi\) and \(\psi\) are the corresponding unique solution for the following equations

\[ \begin{cases} 
(-\Delta)^{\frac{\alpha}{2}} \phi = 1, & x \in \Omega, \\
\phi > 0, & x \in \Omega, \\
\phi = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases} \]  

(48)

and

\[ \begin{cases} 
(-\Delta)^{\frac{\alpha}{2}} \psi = 1, & x \in \Omega, \\
\psi > 0, & x \in \Omega, \\
\psi = 0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases} \]  

(49)

We want to prove (47), which is equivalent to prove that the following system has no positive solution,

\[ \begin{cases} 
(-\Delta)^{\frac{\alpha}{2}} u = f(x, v) + t, & x \in \Omega, \\
(-\Delta)^{\frac{\alpha}{2}} v = g(x, u) + t, & x \in \Omega, \\
u > 0, v > 0, & x \in \Omega, \\
u = 0, v = 0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases} \]  

(50)

Let

\[ \lambda_1 = \inf\{ \int_{\Omega} \left| (-\Delta)^{\frac{\alpha}{2}} u \right|^2 + \left| (-\Delta)^{\frac{\alpha}{2}} v \right|^2 dx \mid (u, v) \in H_0^{\frac{\alpha}{2}}(\Omega) \times H_0^{\frac{\alpha}{2}}(\Omega), \int_{\Omega} u^+ v^+ dx = 1 \}, \]

where \( u^+ = \max\{u, 0\}, v^+ = \max\{v, 0\} \). We can know that \( \lambda_1 > 0 \) and it is achieved by \((\omega_1, \omega_2) \in H_0^{\frac{\alpha}{2}}(\Omega) \times H_0^{\frac{\alpha}{2}}(\Omega)\). By the weak maximum principle, \( \omega_1 \neq 0, \omega_2 \geq 0 \) in \(\Omega\) and \( \omega_1 \neq 0, \omega_2 \neq 0 \). \((\omega_1, \omega_2)\) satisfies

\[ \begin{cases} 
(-\Delta)^{\frac{\alpha}{2}} \omega_1(x) = \mu_1 \omega_2(x), & x \in \Omega, \\
(-\Delta)^{\frac{\alpha}{2}} \omega_2(x) = \mu_2 \omega_1(x), & x \in \Omega, \\
\omega_1(x) = 0, \omega_2(x) = 0, & x \in \mathbb{R}^N \setminus \Omega,
\end{cases} \]  

(51)

where \( \mu_1 + \mu_2 = \lambda_1 \).

Choosing \( \lambda > \lambda_1 \), there exists \( C_0 > 0 \) such that

\[ f(x, v) - \lambda v \geq -C_0 \]  

(52)

and

\[ g(x, u) - \lambda u \geq -C_0. \]  

(53)
Combining (50), (52) with (53), we have
\[ (-\Delta)^{\frac{\alpha}{2}} u \geq \lambda v - C_0 + t \] \hfill (54)
and
\[ (-\Delta)^{\frac{\beta}{2}} v \geq \lambda u - C_0 + t. \] \hfill (55)

When \( t > C_0 \), we have
\[ (-\Delta)^{\frac{\alpha}{2}} u \geq \lambda v \] \hfill (56)
and
\[ (-\Delta)^{\frac{\beta}{2}} v \geq \lambda u. \] \hfill (57)

Combining (51), (56) with (57), we have
\[ \mu_1 \int_{\Omega} u(x) \omega_2(x) dx \geq \lambda \int_{\Omega} v(x) \omega_1(x) dx \] \hfill (58)
and
\[ \mu_2 \int_{\Omega} v(x) \omega_1(x) dx \geq \lambda \int_{\Omega} u(x) \omega_2(x) dx. \] \hfill (59)

From (58) and (59), we have \( \mu_1 \mu_2 \geq \lambda^2 > \lambda_1^2 \). Hence at least one of \( \mu_1 \) and \( \mu_2 \) is greater than \( \lambda_1 \), which is obviously not true.

When \( t \leq C_0 \), we have
\[ (-\Delta)^{\frac{\alpha}{2}} u \leq f(x,v) + C_0 \leq k(x) |v|^q + C_0 \] \hfill (60)
and
\[ (-\Delta)^{\frac{\beta}{2}} v \leq g(x,u) + C_0 \leq h(x) |u|^p + C_0. \] \hfill (61)

Similarly, by means of scaling method, we can get a priori bounds of the solutions for (60) and (61). So there exists \( M > 0 \), such that for any solution \((u,v)\) of (60) and (61), \( \|(u,v)\|_E < M \). Taking \( R = M \), when \((u,v) \in \partial B_M(0)\), then \( \|(u,v)\|_E = M \). Thus we get the contradiction. Therefore system (50) has no positive solution. As a result, we prove Theorem 1.3.

\[ \square \]

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