Strongly Maximal Submodules with A Study of Their Influence on Types of Modules

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Abstract

Let S be a commutative ring with identity, and A is an S-module. This paper introduced an important concept, namely strongly maximal submodule. Some properties and many results were proved as well as the behavior of that concept with its localization was studied and shown.

Keywords: Maximal submodule, regular module, regular ring, semi-simple module, prime module.

1. Introduction

Along with this paper, S is a commutative ring with identity, and A is an S-module. A proper submodule N of an S-module A is named maximal if there exists a submodule D of A such that N ⊆ D ⊆ A, then D=A[1][11]. Equivalently, N is the maximal submodule in A if and only if A/N is a simple S-module [1][12]. Maximal submodules may not exist; for instance, the Z-module ℂ has no maximal submodules. The main goal of this paper is to introduce a new concept called strongly maximal submodule (for short, SM-submodule) where a proper non-zero submodule B of an S-module A is said to be strongly maximal submodule if and only if, for every non-zero ideal E of S implies A/E ⊇ B is a regular module. Field house is defined in [9], a pure submodule of the form: A submodule D of an S-module A is pure if IA∩D = ID for every ideal I of S and Sahera introduced in [2], the definition of F-regular module, where module A is said to be F-regular if and only if every submodule of A is pure. Every strongly maximal submodule is a maximal submodule, but the opposite does not true. This paper is divided into three sections. We reviewed some basic definitions and properties needed in our next work. Section three introduced the definition of strongly maximal submodule. Lots of properties and examples of this concept were shown. In section four, the
behavior of strongly maximal submodules under localization was some characterized and results were proved.

2. Basic concepts and Results

This part includes some well-known definitions, concepts, and results that are useful for us in our study of the next section.

Proposition (2.1)

Every submodule of the regular module is regular [2].

Proposition (2.2)

An S-module A is cyclic if and only if it is isomorphic to a factor module of S [4].

Proposition (2.3)

An S-module A is simple if and only if A \cong S/E for some maximal ideal E of S [4] [1].

Definition (2.4)

A submodule B of an S-module A is called prime if B \neq A, and whenever tx \in B for t \in S and x \in A we have either t \in [B: A] or x \in B [5].

Definition (2.5)

A submodule B of an S-module A is called a semimaximal submodule if and only if A/B is a semi-simple S-module [3, definition (2.1.1), p32].

Definition (2.6)

Let B be a submodule of an S-module A, the closure of B is denoted by \text{CL}(B) = \{ x \in A : [B:(x)] essential in S \}. It is clear that \text{CL}(B) is a submodule of A containing B. That is B \subseteq \text{CL}(B) [7].

S3: Strongly Maximal Submodules with its advantages

In this section, the concept of strongly maximal submodule (for short, SM-submodule) was introduced, which was a generalization of the concept the strongly maximal ideal in a ring S. Several examples and properties were proved also a lot of characterizations, and different results were presented.

Let us start with our basic definition.

Definition (3.1)

Let A be an S-module, and B be a non-zero proper submodule of A. Then B is named strongly maximal submodule (for short SM-submodule) if and only if, for every non-zero ideal E of S implies A/E^2B is a regular module.

Examples and Remarks (3.2)
1. All the following modules have no SM-submodules.

(i) $Z$ as a $Z$-module.

(ii) $Z_p$ as a $Z$-module, $p$ is a prime number.

(iii) $Z_p$ as a $Z_p$-module, $p$ is a prime number.

2. Every simple $S$-module has no SM-submodule. But the opposite is not true and the following example shows that: The module $A = Z_4 \oplus Z$ as a $Z$-module. Since $A$ has no SM-submodules, $A$ is not a simple module. Also notice examples (ii) and (iii) in no.(1).

3. It is important to note that it is not necessary that all modules contain SM-submodules; for example $Z_{p^\infty}$ as a $Z$-module. Since, all the submodules of $Z_{p^\infty}$ are of the form $<1/p^i + Z>$, where $p$ is a prime number and $i = 0, 1, 2…$

Now, we write $N = <1/p^i + Z>$ and let $E$ be an ideal of $Z$. If we take $E = <1>$, then $Z_{p^\infty}/E^2B = Z_{p^\infty}/<1>^2B = Z_{p^\infty}/B \cong Z_{p^\infty}$ is not a regular module and hence $B = <1/p^i + Z>$ is not SM-submodule of $Z_{p^\infty}$. That is, $Z_{p^\infty}$ has no SM-submodules.

Also, we can give another example $Z_8$ as a $Z_8$-module that has no SM-submodules. Since $<2>$ and $<4>$ are not SM-submodules.

4. The submodule $<3>$ of a $Z_6$-module $Z_6$ is an SM-submodule. To clarify, let $E$

be an ideal of a ring $Z_6$.

if $E = <1>$, then $Z_6/<1>^2 <3> = Z_6/<3> \cong Z_3$ is a regular module.

if $E = <2>$, then $Z_6/<2>^2 <3> = Z_6/<0> \cong Z_6$ is a regular module.

if $E = <3>$, then $Z_6/<3>^2 <3> = Z_6/<3> \cong Z_3$ is a regular module.

Therefore $<3>$ is an SM-submodule of $Z_6$.

In general, all submodules of $Z_6$ as a $Z_6$-module are SM-submodules.

5. In $Z_{10}$ as a $Z_{20}$-module, the submodule $B = <\overline{5}>$ is an SM-submodule. Since if we take $E = <\overline{1}>$, $<\overline{2}>$, $<\overline{10}>$, $<\overline{4}>$, $<\overline{5}>$, where $E$ is ideal of $Z_{20}$, then $Z_{20}/E^2<\overline{5}>$ is a regular module for all ideal $E$ of $Z_{20}$. This ends the proof of example.

6. Consider $Z_4$ as a $Z$-module. The submodule $<\overline{2}>$ is not SM-submodule of $Z_4$. Since $Z_4/E^2<\overline{2}>$ is not regular, $E$ is an ideal of $Z$. To prove that, take $E = <\overline{4}>$. Then, $Z_4/E^2<\overline{2}> \cong Z_4$ is not regular module.

7. Let $Z_4$ as a $Z_4$-module. Then the submodule $B = <\overline{2}>$ is not an SM-submodule. Notice if $E = <\overline{2}>$ then $Z_4/E^2<\overline{2}>$ is not regular module.
8. Let $B = \langle \overline{2} \rangle$ be a submodule of a $Z_{20}$-module $Z_{10}$. Since $Z_{10}/E^2B = Z_{10}/\langle \overline{2} \rangle^2\langle \overline{2} \rangle \cong Z_8$ is not a regular module, where $E = \langle \overline{2} \rangle$ is an ideal of $Z_{20}$.

9. Let $A = Z_6 \oplus Z_3$ be an $Z_{12}$-module and $B = \langle \overline{2} \rangle \oplus \langle \overline{0} \rangle$ be a submodule of $A$. Then $A/E^2B = Z_6 \oplus Z_3/E^2(\langle \overline{2} \rangle \oplus \langle \overline{0} \rangle) \cong Z_4 \oplus \langle \overline{6} \rangle$ is a regular module where $E = \langle \overline{1} \rangle$, $\langle \overline{2} \rangle$, $\langle \overline{3} \rangle$, $\langle \overline{4} \rangle$ and $\langle \overline{6} \rangle$ be ideals of $Z_{12}$. Therefore $B = \langle \overline{2} \rangle \oplus \langle \overline{0} \rangle$ is an SM-Submodule of $A = Z_6 \oplus Z_3$.

10. Consider $A = Z_6 \oplus Z$ as a $Z$-module. Then the submodule $B = \langle \overline{3} \rangle \oplus \langle \overline{2} \rangle$ of $A$ is not SM-submodule. Since $A/E^2B = Z_6 \oplus Z/<\overline{2}>(\langle \overline{3} \rangle \oplus \langle \overline{2} \rangle)$ is not a regular module, where $E = \langle \overline{1} \rangle$ is an ideal of $Z$.

11. Every SM-submodule is maximal but the opposite is not true and the following example shows that: The submodule $\langle \overline{2} \rangle$ of a $Z$-module $Z_4$ is a maximal submodule in $Z_4$ but it is not a SM-submodule, see no.(6).

12. A submodule of an SM-submodule is not necessary to be an SM-submodule, for example: A submodule $\langle \overline{2} \rangle$ in a $Z_6$-module $Z_6$ is SM-submodule. See no.(4), While $\langle \overline{0} \rangle$ is a submodule of $\langle \overline{2} \rangle$ and it is not SM-submodule.

13. The intersection of two SM-submodules is not condition to be SM-submodule, for example : The two submodules $\langle \overline{2} \rangle$, $\langle \overline{3} \rangle$ in $Z_6$-module $Z_6$ are SM-Submodules but $\langle \overline{2} \rangle \cap \langle \overline{3} \rangle = \langle \overline{0} \rangle$ is not an SM-submodule.

14. More generally, let $\{B_j\}_{j=1}^n$ be a finite collection of SM-submodules of an $S$-module $A$. Then $\cap_{j=1}^n B_i$ is not always SM-submodule.

15. The direct sum of two SM-submodules of an $S$-module $A$ is not necessary to be an SM-submodule, for example: Let $\langle \overline{2} \rangle$, $\langle \overline{3} \rangle$ be two SM-submodules of a $Z_6$-module $Z_6$, but $\langle \overline{2} \rangle \oplus \langle \overline{3} \rangle = Z_6$ is not an SM-submodule.

16. From the fact that every maximal submodule is a semimaximal by [3, Remarks and Examples (2.1.2), p32], and the fact no.(11), we obtain that every SM-submodule of an $S$-module $A$ is a semimaximal while the converse is not true in general and the following shows that: Let $6Z$ be a submodule of a $Z$-module $Z$. Then, $6Z$ is a semimaximal submodule of $Z$. Since $6Z = 2Z \cap 5Z$ where $2Z, 5Z$ are maximal submodules of $Z$. But $6Z$ is not an SM-submodule of $Z$. Since $Z/(2Z)^2(6Z) = Z/24Z \cong Z_{24}$ is not a regular module.

**Proposition (3.3)**

Let $B, D$ be two submodules of an $S$-module $A$ with $B \subseteq D$. Then $B$ is SM-submodule in $D$ if and only if $B$ is SM-submodule in $A$.

**Proof:**

Suppose $B$ is SM-submodule in $A$, then $A/E^2B$ is regular module for every non-zero ideal $E$ of $S$. Since $D/E^2B$ is a submodule of $A/E^2B$ (Notice, $E^2B \subseteq B \subseteq D$), and hence by proposition (2.1), $D/E^2B$ is a regular submodule of $A/E^2B$. Thus $B$ is SM-submodule in $D$. 

Next, we will give an application to a proposition (3.3)

**Corollary (3.4)**

Let $A$ be an $S$-module and $B$ be a proper submodule of $A$. If $B$ is an SM-submodule of $[B_A : A]$ and $[B_A : A]$ is an SM-submodule in $A$, then, $B$ is an SM-submodule in $A$.

**Proof:**

It clear that $B \subseteq [B_A : A] \subseteq A$. Therefore, by using proposition (3.3), we conclude that $B$ is SM-submodule in $A$.

Now, we will give the sufficient condition for a submodule to not be SM-submodule.

**Proposition (3.5)**

Let $A$ be an $S$-module. If $A$ is cyclic module (if $A=Sx$ for some $x \in A$) and $\text{ann}_s(x)$ is maximal ideal of $S$, then $A$ has no SM-submodule.

**Proof:**

Since $A= Sx$ for some $x \in A$, then $A$ is isomorphic to a factor module of $S$ by proposition (2.2). We can define $f: S \to A$ such that $f(r) = rx$. It is easily to show that $f$ is well-defined and epimorphism, by the first fundamental theorem of isomorphism $S/\text{Ker} f \cong A$. Next, $\text{Ker} f = \{r \in S : f(r) = 0_A\} = \{r \in S : rx=0_A\} = \text{ann}_s(x)$ That is $S/\text{ann}_s(x) \cong A$. Also, we have $\text{ann}_s(x)$ is maximal ideal of $S$ which implies $S/\text{ann}_s(x)$ is simple and hence $A$ is simple. Therefore, by examples and remarks ((3.2) No. (2)), we get the result.

Next is the application of proposition (3.5)

**Corollary (3.6)**

If a non-zero prime and semi-simple $S$-module $A$, then $A$ has no SM-submodule.

**Proof:**

Suppose that $A$ is a prime and semi-simple module, then, we obtain $A$ is simple module. To prove this, assume that $A$ is not simple which implies $A$ is a direct sum of simple $S$-modules. Then, there exists a simple module $M_1$ and $M_2$ which are a direct summand of $A$ with $M_1 \neq M_2$. $M_1 \cong S/E$, $M_2 \cong S/D$ where $E$ and $D$ are maximal ideals of $S$, by proposition (2.3). But $A$ is prime module, then $\text{ann}_s(M_1) = E = \text{ann}_s(A)$ and $\text{ann}_s(M_2) = D = \text{ann}_s(A)$. Thus $E = D$ which implies that $M_1 = M_2$, and this is a contradiction. Hence, $A$ is a simple module and by using proposition (3.5), we have $A$ has no SM-submodule.

As a direct of corollary (3.6), we have the following.

**Corollary (3.7)**
Let $P$ be a prime and semimaximal submodule of an $S$-module $A$. Then the quotient module by $P$ has no SM-submodule.

**Proof:**

Since $P$ is a semimaximal submodule of $A$, then by definition (2.5), we have $A/P$ is a semi-simple $S$-module. On the other hand, $P$ is a prime submodule of $A$, then, by [3, proposition (1.1.51)] we obtain that $A/P$ is a prime $S$-module, and hence by corollary (3.6), then $A/P$ is a simple $S$-module and hence $A/P$ has no SM-submodule.

**S4: The behavior of SM-submodules under localization.**

Let $K$ be a subset of a ring $S$, $W$ is multiplication closed if the two condition hold:

1. $I \in W$.
2. $xy \in W$ for every $x, y \in W$.

We know that every proper ideal $E$ in $S$ is prime if and only if $S$-E is multiplicatively closed, see [8].

If $A$ is an $S$-module and $W$ be a multiplicatively closed on $S$ such that $W\neq<0>$, then $S_w$ be the set for all fractional $r/w$ where $r \in S$ and $w \in W$ and $A_w$ be the set of all fractional $m/w$ where $m \in A$ and $w \in W$. For $m_1, m_2 \in A$ and $w_1, w_2 \in W$, $m_1/w_1=m_2/w_2$ if and only if $\exists t \in W$ such that $t(w_1m_1-w_2m_2)=0$. Also, we can make $A_w$ in to $S_w$-module by setting $m_1/w_1+m_2/w_2=(w_2m_1+w_1m_2)/w_1w_2$ and $(r/w_1) (m_1/w_2) = rm_1/w_1w_2$ for every $m_1, m_2 \in A$ and every $r \in S$, $w_1, w_2 \in W$. If $W =S$-E where $E$ is a prime ideal, we used $A_E$ instead of $A_w$ and $S_E$ instead of $S_w$. If a ring has only one maximal ideal, then it is called a local ring. Hence $S_E$ is often called the localization of $S$ at $E$, similar $A_E$ is the $r/1$, $r \in S$ and $\Phi : A \rightarrow A_w$ such that $\Phi(m)=m/1 \forall m \in A$. Furthermore, if $B$ is a submodule of an $S$-module $A$ and $W$ be a multiplicatively closed in $S$, then $B_w = \{n/w: n \in B, w \in W\}$ be a submodule on $S_w$-module, see [8].

In this section we study the behavior of an SM-submodule under localization and several of results have been proved.

The following lemma is needed in our next result.

**Lemma (4.1) [10]**

Let $A$ be an $S$-module and $B, L$ are two submodules of $A$. Then, $B=L$ if and only if $B_P=L_P$ for every maximal ideal $P$ of $S$.

The following proposition study the relationship between a module $A$ and its locally and prove that they are equivalent.

**Proposition (4.2):**

Let $A$ be an $S$-module and $B$ is nonzero proper submodule of $A$. Then, $B_P$ is SM-submodule of an $S_P$-submodule $A_P$ if and only if $B$ is SM-submodule of an $S$-module $A$.

**Proof:**
Suppose that B is nonzero proper submodule of A. We must prove that A/E^2B is a regular S-module for every nonzero ideal E of S; that is, every submodule of A/E^2B is pure. Let L/E^2B be a submodule of A/E^2B. It is clear that I(L/E^2B) ⊇ I(A/E^2B)∩(L/E^2B) where I is an ideal of S. Now, to prove I(L/E^2B) ⊆ I(A/E^2B)∩(L/E^2B). Let x ∈ I(L/E^2B). Then x = Σ^n_{i=1} a_i(l+E^2B). Therefore x/s = (Σ^n_{i=1} a_i(l+E^2B))/s ∈ I_p(L_p/E_p^2B_p) but L_p is SM-submodule in A_p, then I_p(L_p/E_p^2B_p) = (I_p(A_p/E_p^2B_p))∩(L_p/E_p^2B_p) which implies x/s ∈ I_p(A_p/E_p^2B_p)∩(L_p/E_p^2B_p).

Proof:

Proposition (4.3):

Let L, B be two finitely generated submodules of an S-module A. If L_p, B_p are SM-submodules of A_p, then L∩B is an SM-submodule of A.

Proposition (4.4):

Let L, B be two finitely generated submodules of an S-module A. Then, L+B is an SM-submodules of A if L_p+B_p are SM-submodules of an S_p-module A_p.
Proof:

Let $L$, $B$ be two finitely generated submodules of $A$. Then by \cite{10, p24}, we have 
\[ [L_P:B_P]+[B_P:L_P] = S_P \]
for every maximal ideal $P$ of $S$. Let $y_1 \in [L_P:B_P]$ and $y_2 \in [B_P:L_P]$ such that $y_1+y_2 = 1 = \text{unity of } S_P$. Then, either $y_1$ is a unit element or $y_2$ is a unit element (Since $S_P$ is local ring). Therefore 
\[ [L_P:B_P] = S_P \text{ or } [B_P:L_P] = S_P \]
and hence either $L_P \subseteq B_P$ or $B_P \subseteq L_P$ which implies $L_P + B_P = L_P$ or $L_P + B_P = B_P$, but $L_P$, $B_P$ are SM-submodules of $A_P$. Thus $L_P + B_P$ is an SM-submodule and $(L+B)_P$ is an SM-submodule and by proposition(4.2), $L+B$ is an SM-submodule of $A$.

5. Conclusion

The conclusion of this work is to study an important concept, namely strongly maximal submodule. Some properties and many results were proved and the behavior of that concept with its localization were studied and shown.

References

1. Burton, D. M. 1970. A first course in rings and ideals. Addison-Wesley.
2. Sahera Mahmod Yasin About regular module type-$F$, Master letter, Science College, Univ. of Baghdad, 1993.
3. Khalaf, H.Y. Semimaximal Submodules Ph.D. Thesis, Univ. of Baghdad, 2007.
4. Frank W., Anderson K. and Fuller R., Rings and Categories of Modules, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
5. Lu.C.P., M-Radicals of Submodules in Modules, Math. Japon., 34 1989, 211-219.
6. Kazem, A.D. Some Types of Visible Submodules and Fully Visible Modules M.sc.Thesis, Univ. of Baghdad, 2020.
7. Goldie, K.R. Torsion Free Modules and Rings, J. Algebra, 1, 268-287, 1964.
8. Larson M.D., Mc Carthy P.J., 1971, Multiplication theory of Ideals, Academic press, New York and London.
9. Fieldhouse, D.J. Pure Theories, Math. Ann., 184, 1-18, 1969.
10. Mijbass A.S., On Cancelation modules, M.Sc. Thesis, Baghdad university, 1992.
11. Faris, H. I., Jasim, R. H., & Mohammed, N. J. 2021, March. Pseudo maximal submodules. In Journal of Physics: Conference Series (Vol. 1818, No. 1, p. 012055). IOP Publishing.
12. Mohammed, A. S., & Sallman, M. D. 2017. 2-Maximal Submodules and Related Concepts. Journal of university of Anbar for Pure science, 11(3).
13. Jud H.M., Some types of fully cancelation modules, M.sc.Thesis, Baghdad University, 2016.