Slowed yet explosive global cascades driven by response heterogeneity in multiplex networks

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Emergent social phenomena arise from multiple types of influence. We study threshold cascades on a multiplex network of nodes following one of two rules: some nodes activate when, in some layer, enough neighbors are active; the other nodes activate when, in all layers, enough neighbors are active. Varying the fractions of nodes following either rule facilitates or inhibits cascades. Moreover, near the inhibition regime, global cascades appear discontinuously, suggesting a way in which civil movements, product adoption, or financial knock-on contagion might appear abruptly.

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Introduction.—Multiple channels of interaction in a network (or network layers) can have nontrivial consequences in the system’s dynamics and function \[1\]–\[10\]. Effects of introducing new network layers include catastrophic cascades of failure [1], facilitated cascades [5], and super-diffusive spreading [7]. Most such studies on multiplex networks have assumed identical dynamics for every node. However, real-world complex systems such as human society and the financial system consist of heterogeneous individuals who respond differently to their multiplex environment. Changes in the way people and institutions respond to multiple channels of influence can have dramatic results. For instance, new social media such as Twitter and YouTube have increasingly been changing the way people communicate, which have reportedly contributed to unexpected, explosive democratic movements in Arabic countries [11]. Similarly, banks continue to find new ways to lend, which has allegedly played a role in banking crises over the centuries [12] and has become the subject of reform by new regulations (e.g., the Volcker Rule [13]). However, the consequences of the heterogeneous response have yet to be fully characterized and understood.

The threshold cascade model provides a theoretical tool for understanding the spread of behavior in a social network [14]–[17] and for studying the cascades of “knock-on” default among financial institutions [18]–[21]. In this stylized model, nodes exist in one of two states, active or inactive (e.g., a person changed behavior or not, a bank has defaulted or not). Initially, each node draws a threshold from a distribution \(Q(r)\). An inactive node with degree \(k\) and with \(m\) active neighbors activates when its fraction of active neighbors, \(m/k\), exceeds its threshold \(r\). The dynamics are iterated starting from a small fraction \(p_0\) of initially active “seed” nodes, and then the cascade size \(\rho\), the fraction of active nodes in the steady state, is observed. Previous studies showed that, for a wide range of network densities and threshold distributions, even an extremely small seed fraction \(p_0\) can activate a large number of nodes, an event called a global cascade \([16],[17]\).

Model.—In this paper, we explore the effects of introducing new network layers and of heterogeneous responses of nodes to those layers in threshold cascade dynamics. As a starting point, we consider a mixture of two specific response rules by which a node activates, defined as follows.

Some nodes activate as soon as, in at least one layer, a sufficiently large fraction of their neighbors in that layer are active. In social systems, this rule would mean that just one social sphere can convince someone to change behavior. In banking systems, it would mean that a bank engaging in multiple kinds of lending defaults if, for at least one type of lending, sufficiently many of its borrowers of that type defaulted and cannot repay the bank. If all nodes respond this way, then the existence of multiple layers facilitates global cascades \([1]\) (i.e., behavior change and bank default spread widely with ease).

The remaining nodes are more stubborn: they activate as soon as, in each and every layer, a sufficiently large fraction of their neighbors in that layer are active. In social systems, this rule would mean that a person waits to change behavior until receiving enough influence from all social spheres. In banking systems, this rule would mean that a bank defaults once enough of its borrowers of every type have defaulted. If more nodes follow this rule, then global cascades become rare or even impossible. As the system approaches this extreme of inhibited cascades, new behavior emerges: global cascades appear discontinuously as the network densifies. We show that this phenomenon is associated with a cusp catastrophe and that it suggests ways to promote or inhibit cascading phenomena in social, financial, and other multiplex networks.

Formulation.—Let us formulate the model of threshold dynamics on multiplex networks with heterogeneous response rules. A node in a multiplex network with \(\ell\) layers (an “\(\ell\)-plex network”) has \(k_\alpha\) neighbors in each layer \(\alpha \in \{1, \ldots, \ell\}\). At a certain point in time, this node sees that \(m_\alpha\) out of its \(k_\alpha\) neighbors in layer \(\alpha\) are active, and the node responds according to one of the two response rules, \(F_{\text{OR}}\) or \(F_{\text{AND}}\), defined as follows. \(F_{\text{OR}}\) denotes the “OR rule”, for which an inactive node activates when, in at least one layer \(\alpha\) in which it has neighbors (i.e., \(k_\alpha > 0\)), the fraction of active neighbors, \(m_\alpha/k_\alpha\), exceeds its threshold \(r_\alpha\); it can be written as

\[
F_{\text{OR}}(m, k, r) = \max \left\{ 1 - \frac{m_\alpha}{k_\alpha} : 1 \leq \alpha \leq \ell, k_\alpha > 0 \right\},
\]

where \(m \equiv (m_1, \ldots, m_\ell)\) and \(k \equiv (k_1, \ldots, k_\ell)\) denote the numbers of active neighbors and the numbers of neighbors,
respectively, of the node, and \( I_A \) is the indicator function (\( I_A = 1 \) if \( A \) is true, else \( I_A = 0 \)). Likewise, the response function of the “AND rule”, for which nodes activate when enough neighbors in every layer are active, can be written as

\[
F_{\text{AND}}(m, k, r) = \min \left\{ \frac{m_{\alpha}}{r_{\alpha}} : 1 \leq \alpha \leq \ell, k_\alpha > 0 \right\}. \tag{2}
\]

[i.e., replace \( \max \) with \( \min \) in (1)]. In our model, a random fraction \( \mathcal{E} \) (respectively, \( 1 - \mathcal{E} \)) of nodes in the network follow the OR (AND) rule, hereafter called the OR (AND) nodes. Such an uncorrelated population, with the additional assumption that nodes independently draw their thresholds \( r \in [0, 1]^\ell \) from a distribution \( Q(r) \), can be treated mean-field-theoretically using the mean response function given by

\[
\bar{F}(m, k) \equiv \int Q(r) F(m, k, r) \, dr, \tag{3}
\]

with \( F \) taken to be the additive mixture of two response functions parametrized by \( \mathcal{E} \in [0, 1] \),

\[
F(m, k, r) = \mathcal{E} F_{\text{OR}}(m, k, r) + (1 - \mathcal{E}) F_{\text{AND}}(m, k, r). \tag{4}
\]

To investigate these dynamics, we extend the analysis due to [17] to obtain the cascade size \( \rho \) for random initial seeds of fraction \( \rho_0 \) on locally treelike networks as

\[
\rho = \rho_0 + (1 - \rho_0) \sum_{k=0}^{\infty} P(k) \prod_{m=0}^{\ell} B_m(q_\nu) \bar{F}(m, k). \tag{5}
\]

Here, we approximate the locally treelike graph as a tree, and \( q_\nu^{(\nu)} \) is the (limiting) probability that a node is activated by its children given that its parent in layer \( \nu \) is inactive; the joint degree distribution of the \( \ell \)-plex network is denoted by \( P(k) \); the sum \( \sum_{k=0}^{\infty} \) runs over all length-\( \ell \) vectors \( k \) with entries \( m_\nu \in \{0, \ldots, k_\nu\} \) for each layer index \( \nu \in \{1, \ldots, \ell\} \); and \( B_m(q) \) is shorthand notation for the binomial probability distribution, \( \binom{k}{m} q^m (1 - q)^{k-m} \). The probability \( q_\nu^{(\nu)} \) is obtained by iteratively solving for the fixed point of the coupled recursion relations, written vectorially as

\[
q_{n+1} = g(q_n), \tag{6}
\]

with the \( \alpha \)-th component of (6) given by

\[
q_{n+1}^{(\alpha)} = g^{(\alpha)}(q_n) = \rho_0 + (1 - \rho_0) \sum_{k=0}^{\infty} \frac{k_\alpha P(k)}{z_\alpha} \times \sum_{m_\alpha=0}^{k_\alpha} \sum_{m_\nu=0, \nu \neq \alpha}^\ell B_{m_\alpha}^{k_\alpha - 1}(q_n^{(\alpha)}) \prod_{\nu \neq \alpha} B_{m_\nu}^{k_\nu}(q_n^{(\nu)}) \bar{F}(m, k). \tag{7}
\]

which updates the chance that a node \( n \)-hop from the leaves of the tree are activated by its children given that its parent in layer \( \alpha \) is inactive, starting from \( q_0^{(\alpha)} = \rho_0 \).

**Results.**—We investigate a simple yet rich illustrative case: an uncorrelated, two-layer (duplex) Erdős-Rényi network [22] with identical mean degree \( z \) and identical uniform threshold \( R \) in each layer. Thus, \( P(k) = P(k_1)P(k_2) \) and \( Q(r) = \delta(r_1 - R)\delta(r_2 - R) \). Extending the formalism to more than two layers is straightforward and presented in part in [22]. (The degree distribution of correlated multiplex networks was introduced [4], and their robustness was studied recently [23].)

First we illustrate how changing the fraction of OR nodes, \( \mathcal{E} \), either facilitates or inhibits global cascades and can change the nature of the appearance of global cascades. We calculate in Fig. 1(a) the regions of mean degree \( z \) and threshold \( R \)
for which global cascades are likely and unlikely. To obtain the boundary separating these parameter regions, we find local maxima in the number of iterations (NOI) of the recursion relation (3), a procedure analogous to examining the divergence of susceptibility at a phase transition in critical phenomena (25) and recently applied to cascading failures in interdependent networks (24). This method more accurately locates the boundaries than the first-order cascade condition used in previous studies (5) (16) (17) (26) because it allows for nodes to be activated by more than one neighbor.

If , then all nodes follow the OR rule (1), which maximally facilitates global cascades compared to the single-layer case [compare the red boundary with the gray region in Fig. 1(a)] (5). As decreases, more nodes follow the AND rule (2), which inhibits cascades and hence shrinks the cascade region [see the orange, green, and blue boundaries in Fig. 1(a)]. When , the cascade region becomes smaller than the single-layer case, showing that multiplexity can also impede cascades. If all nodes follow the AND rule (i.e., ), then global cascades are nearly impossible [see the dot-dashed boundary in Fig. 1(a)].

Furthermore, tuning the fraction of OR nodes can cause cascades to appear discontinuously. Previous work has shown that, for fixed threshold , the mean cascade size grows continuously and then drops discontinuously with increasing mean degree (17). What is novel about our multiplex model with a mixture of response rules is that the small- transition for global cascade changes from continuous to discontinuous when becomes sufficiently small [Fig. 1(b)]. We find that the NOI for these small- transitions exhibits a peak at (Fig. 1(c), inset), and later we show that this is where the continuous transition becomes discontinuous.

The bifurcation diagram (Fig. 2 inset) confirms this discontinuous transition. The fixed points of recursion (6) are roots of with solid and dotted curves denoting the stable and unstable solutions, respectively. The red curve denotes the physical solutions for cascades starting from the small seed . Global cascades appear discontinuously at .

we observe only the smallest root (plotted as red in the inset of Fig. 2) because we consider small seed-sizes . Bifurcation analysis of reveals that the system undergoes a fold catastrophe: as increases from 0, two new roots appear in a saddle-node bifurcation, and one of those roots (the unstable one) annihilates the small, stable root at another saddle-node bifurcation at the small- transition point, leaving only a large stable root (Fig. 2). Thus, as the network densifies, global cascades appear discontinuously. Increasing the fraction of OR nodes beyond eliminates the fold catastrophe, thereby restoring the familiar continuous transition (16) (17).

In short, the model undergoes a cusp catastrophe (22), illustrated in Fig. 3. The cusp point (, ) marks the parameters at which the line of continuous transitions and the line of discontinuous transitions join. This point can be associated with tricritical behavior when (28–30), which are tangential and normal, respectively, to the continuous transition line. In the new coordinate system centered at the estimated cusp point (, ) = (0.28, 1.36), the two scaling fields obey a power-law relation near the origin as

with the crossover exponent (36) (Fig. 3 inset). Other choices for (, ) were not compatible with the scaling.

Response heterogeneity affects not only whether cascades appear discontinuously; it also affects who activates when and how slowly the cascade progresses, which are questions of interest to marketers, financial regulators and others trying to
measure, facilitate, or inhibit cascades. As depicted in Fig. 4, a typical global cascade near the cusp point can be qualitatively divided into four stages [labeled I–IV in Fig. 4(e)]. Initially, activation grows exponentially, and the more easily agitated OR nodes activate in greater numbers than the AND nodes, even though OR nodes are less numerous ($E' = 0.2$ in Fig. 4). In stage II, the rates of activation slow for both types of nodes. What is particularly interesting about stage II is that the presence of AND nodes significantly delays the global cascade. If the goal is to prevent large cascades (as in bank regulation), then stage II provides a crucial window of opportunity for intervention. After sufficiently many nodes have activated, the AND nodes activate explosively and eventually overtake the number of active OR nodes (stage III) until the activations saturate for a finite system (stage IV).

**Summary and Discussion.**—Introducing new network layers, such as adding new social media or creating novel ways of lending, can facilitate or impede threshold-driven cascades, depending on how nodes respond to their multiplex surroundings (Fig. 1). For networks in which most nodes can be activated through any one of the layers (i.e., for large $E$), global cascades become likely—even for networks that would have too dense to allow global cascades if there were just one channel of influence. By contrast, if most nodes wait to activate until their thresholds are met in each and every layer (i.e., if $E$ is small), then global cascades occur rarely, if at all. However, when global cascades do occur in this small-$E$ regime, they occur discontinuously at a larger network density as the network densifies (Figs. 1 and 3). This delayed yet explosive emergence of global cascades resembles the abrupt emergence of connectivity in percolation with the product rule [31], with the additional feature of being slowed (Fig. 4).

Our model results suggest several hypotheses for applications. In marketing and product adoption, the value of $E'$ may depend on the type of product, possibly inversely with the price [32]; for example, people often seek input from multiple social spheres when buying a car (small $E'$). By contrast, influence from just one social sphere often suffices to convince someone to buy a particular smartphone application, which can more easily spread in a viral manner (large $E'$).

In financial systems with many types of lending, regulators currently aim to prevent stress in one lending layer from affecting an institution’s activity in another layer [13]. A financial system in which most banks’ subsidiaries are “bankruptcy-remote” (i.e., the bankruptcy of the subsidiary does not significantly distress the parent institution) resembles the case of small $E$, for which global cascades are rare. During crises, however, subsidiaries may not be bankruptcy-remote after all [33], so $E$ may become large, which makes the system more prone to global cascades. Regulation that reduces $E$ (e.g., by ensuring bankruptcy remoteness) may reduce the chance of global cascades of default, but as the network density changes, global cascades may appear abruptly (Fig. 3). Furthermore, when global cascades occur with small $E$, they take a relatively long amount of time (Fig. 4), providing the illusion that the cascade is waning but also an opportunity to prevent the cascade from becoming global.

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