Nodes, Monopoles and Confinement
in 2 + 1-Dimensional Gauge Theories

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Abstract

In the presence of Chern-Simons interactions the wave functionals of physical states in 2 + 1-dimensional gauge theories vanish at a number of nodal points. We show that those nodes are located at some classical configurations which carry a non-trivial magnetic charge. In abelian gauge theories this fact explains why magnetic monopoles are suppressed by Chern-Simons interactions. In non-abelian theories it suggests a relevant role for nodal gauge field configurations in the confinement mechanism of Yang-Mills theories. We show that the vacuum nodes correspond to the chiral gauge orbits of reducible gauge fields with non-trivial magnetic monopole components.
The role of magnetic monopoles in the confinement mechanism still remains elusive in QCD$_{3+1}$ in spite of the appealing conjectures based on the dual superconductor picture [1]. However in 2+1 dimensions where they play the role of instantons their contribution seems to be crucial for confinement. In compact lattice QED$_{2+1}$ it has been shown that the logarithmic perturbative Coulomb potential becomes linear by means of Debye screening of electric charges in a monopole gas [2] in a similar manner as vortices drive the Kosterlitz-Thouless phase transition in the XY model [3].

On the other hand, in 2+1 dimensional gauge theories there exists the possibility of having massive gluons while keeping gauge invariance. This is possible because in 2+1 dimensional space-time massless and massive vector-like particles have the same number of degrees of freedom. The generation of mass for gauge fields can be explicitly achieved in local terms thanks to the peculiar properties of the Chern-Simons term.

In such a case quarks are deconfined because no condensation of pseudoparticles can dramatically modify the exponential decay of gauge propagators. In fact, it has been shown that in compact QED$_{2+1}$ the confinement of electric charges is traded by that of magnetic monopoles [4] (see also [5]), and the magnetic superconductivity picture of confinement is traded by a standard electric superconducting scenario. In this sense a topological mass perturbation realizes an electromagnetic duality transformation.

In this note we elaborate on the role of magnetic monopoles in a topologically massive Yang-Mills theory (YM$_{TM}$). The Hamiltonian of the theory in Schrödinger representation reads

$$ H = -\frac{\Lambda}{2} \int \frac{d^2x}{\sqrt{h}} \text{Tr} \left\{ \frac{\delta}{\delta A_\mu(x)} + \frac{ik}{4\pi} \epsilon^{\mu\nu} A_\nu(x) \right\}^2 $$

$$ + \frac{1}{4\Lambda} \int d^2x \sqrt{h} \text{Tr} F_{\mu\nu} F^{\mu\nu}(x) $$

(1)

where $h$ is the metric of the 2D space. Physical states are constrained by the Gauss law condition

$$ D_\mu \frac{\delta}{\delta A_\mu(x)} \psi(A) = \frac{i k}{4\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu(x) \psi(A) $$

(2)
where $D_\mu$ stands for the covariant derivative.

If $k$ is not an integer there are no solutions of Gauss law (2). This is the way the quantization of Chern-Simons coupling constant $k$ appears in the canonical formalism [6]. When $k$ is an integer the solutions of the constraint equation (2) can be geometrically characterized as sections of a non-trivial line bundle defined over the space of gauge orbits $\mathcal{M} = \mathcal{A}/\mathcal{G}$, i.e. the space of gauge fields $\mathcal{A}$ moded out by the group of gauge transformations $\mathcal{G}$ [6]. The Chern class of this bundle where the physical states live $\mathcal{E}(\mathcal{M}, \mathbb{C})$ is $c_1(\mathcal{E}) = k$.

Now, because the bundle is non-trivial any section (i.e. any physical state) must vanish at some gauge field configurations [7]. Such a behavior is in contrast with Feynman’s claim on the absence of nodes in the ground state of pure Yang-Mills theory in 2+1 dimensions. He used that property, suggested by a formal maximum principle, to argue that quarks are confined in 2+1 dimensions [8].

If there is any relationship between the absence of confinement and the existence of nodes in the vacuum state of the theory with a Chern-Simons term it is possible to establish a connection between the configurations where the vacuum vanishes and permanent confinement.

In quantum mechanics the existence of nodes in the ground state is usually related to its degeneracy. In such a case, the position of nodal points is not relevant because they change from one state to another. However, in topologically massive gauge theories it has been claimed that the vacuum is not degenerated [8]. In such a case, the nodal points are also unique and the corresponding gauge configurations would play a role in the confinement mechanism.

The relevance of nodes was anticipated in ref. [7], however, nodal configurations of the vacuum functional where unknown for a long time. In the present note we provide the solution for this longstanding problem.

In the abelian case, $G = U(1)$, there is no topological reason for physical states to have nodes. If the space is compactified to become a 2D sphere $S^2$ the first and second homotopy groups of the orbit space vanish, $\pi_1(\mathcal{M}) = \pi_2(\mathcal{M}) = 0$ and, thus, any line bundle
over $\mathcal{M}$ is trivial. Physical states are sections on a trivial bundle and, thus, they can be non-null for any gauge field configuration.

The space of orbits splits into several disjoint pieces ($\pi_0(\mathcal{M}) = \mathbb{Z}$) each one containing abelian gauge fields carrying the same magnetic charge. Since the magnetic charge is quantized by Dirac condition, the different connected components of $\mathcal{M}$ are parametrized by an integer number $n$, $\mathcal{M} = \bigcup_{n=-\infty}^{\infty} \mathcal{M}_n$. From a topological viewpoint all the connected components $\mathcal{M}_n$ of $\mathcal{M}$ are equal. Actually, they are diffeomorphic to the component without magnetic charge $\mathcal{M}_0$.

If all the sections of the bundle $\mathcal{M} \times \mathbb{C}$ were physical states, the Hilbert space would be a sum $\mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n$ of Hilbert spaces, each one corresponding to different monopole backgrounds. The energies from each $n$-monopole sector would be shifted by $n^2/2\Lambda$ by the effect of the potential term of the Hamiltonian.

However this is not the case because Gauss law imposes a very restrictive condition on physical states. In fact, if we integrate both sides of the Gauss law (2) we get

$$\int d^2x \partial_\mu \frac{\delta}{\delta A_\mu(x)} \psi(A) = \frac{i k}{4\pi} \int d^2x B(A) \psi(A),$$

where $B(A) = F_{12}(A)$ is the magnetic field strength. The left hand side vanishes because under the integral we have a pure differential whereas the right hand side reduces to $(kn/2)\psi(A)$, $n$ being the magnetic charge carried out by the gauge field $A$. Consequently, for gauge fields with non-trivial magnetic charge ($n \neq 0$) the wave functional $\psi(A)$ must vanish. This means that when $k \neq 0$ only the sections over the $\mathcal{M}_0$ sector do correspond to physical states, i.e. $\mathcal{H}_{\text{phys}} \equiv \mathcal{H}_0$. The theory is exactly solvable and the vacuum state reads

$$\Psi_0(A) = \exp \left\{ \int d^2x \text{Tr} \left[ \frac{i k}{4\pi} \partial^\mu A_\mu \Delta^{-1} B(A) \right. \right.$$

$$\left. \left. \left. - \frac{1}{2\Lambda \sqrt{\hbar}} B(A) \Delta^{-1} (m^2 + \Delta) \right] \right. \right.$$

for gauge fields without magnetic charge $\Phi(A) = \int d^2x B(A) = 0$ and vanishes $\Psi_0(A) = 0$ for magnetic monopole configurations ($\Phi(A) \neq 0$). The spectrum corresponds to a free massive photon with mass $m = k\Lambda/2\pi$. The basic property involved in the above argument is
that constant gauge transformations do not transform abelian gauge fields which implies the
vanishing of the left hand side of the Gauss law whereas space constant temporal component
of gauge fields ($A_0=\text{cte}$) does couple to the other components by means of both, Yang-Mills
and Chern-Simons, terms of the action, which leads to the equality of both sides of the Gauss
law equation. In physical terms what happens is that the Chern-Simons term generates a
transmutation of magnetic charge into electric charge which is reflected in the anomalous
terms of Gauss law.

The same physical argument applies for higher genus ($g > 0$) compactifications $\Sigma$ of the
physical space. In such a case the topology of the space of gauge orbits without magnetic
charge, $\mathcal{M}_0$, becomes more sophisticate and in fact for $k \neq 0$ the bundle of physical states
$\mathcal{E}(\mathcal{M}, \mathbb{C})$ is non-trivial. The quantization of the theory implies the existence of additional
nodes also on the orbit space $\mathcal{M}_0$. However in these cases the vacuum is degenerated
and the configurations with vanishing amplitudes $\psi(A) = 0$ are not physically relevant. From the above results, some of them anticipated in Ref. [9], we conclude that in abelian
2+1 dimensional gauge theories Chern-Simons interactions are absolutely incompatible with
monopoles.

In the case $k = 0$ the theory reduces to a pure Maxwell continuum ("non compact")
theory. External charges are confined by a logarithmic potential. Monopoles are not con-
fined and in fact they have a finite mass $M = 1/(2\Lambda)$. When photons become massive by
the effect of the Chern-Simons interaction electric charges are deconfined whereas magnetic
monopoles decouple from the physical degrees of freedom, i.e. their mass becomes infinite
and their correlators vanish. In this theory the dual superconductor picture, i.e. confinement/condensation of magnetic/electric charges, is explicitly realized and the insertion of
the Chern-Simons term makes the transition from one regime to another.

The same analysis holds in the non-abelian theory for gauge field configurations with
non-zero total magnetic charge. However we will see that even in the sector of zero net
charge there are some configurations where the states vanish.

Let us restrict ourselves, for simplicity, to the $SU(N)$ case, although our analysis can
be easily generalized for arbitrary gauge groups. Gauge fields are defined on a trivial bundle \( P = \Sigma \times SU(N) \). Reducible gauge fields \( A \) can, actually, be defined on subbundles \( P_r(\Sigma, U(N_1) \times \cdots \times U(N_r)) \) of \( P \) with structure group \( U(N_1) \times \cdots \times U(N_r) \). They are decomposed into a sum \( A = A_1 + A_2 + \cdots + A_r \) of elementary gauge fields \( A_i \) with values in \( u(N_i) \). The gauge field elementary components \( A_i \) of \( A \) are defined on principal bundles \( P_i(\Sigma, U(N_i)) \) whose first chern classes \( c_1(P_i) \) represent the magnetic charges of the different components of \( A \). Since \( A \) is a connection with gauge group \( SU(N) \) the total magnetic charge \( \sum_{i=1}^{r} c_1(P_i) = 0 \) vanishes.

Reducible gauge fields are invariant under the following group of gauge transformations

\[
\Phi_t = \begin{pmatrix}
    e^{i\mu_1 t}I_1 & 0 & \cdots & 0 \\
    0 & e^{i\mu_2 t}I_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & e^{i\mu_r t}I_r
\end{pmatrix},
\]

with \( \mu_i = c_1(P_i)/N_i \) and where \( I_i \) denotes the identity matrix of \( U(N_i) \). Thus, the infinitesimal generator of the group \( K_t \) of gauge transformations \( \phi(A) = \Phi_t|_{t=0} \) satisfies that \( d_A\phi(A) = 0 \), and

\[
\int d^2 x \text{Tr} \phi(A) D^\mu \frac{\delta}{\delta A^\mu} \psi(A) = 0,
\]

for any functional \( \psi(A) \). In particular, for physical states Gauss’ law (2) implies that \( \int \text{Tr} \phi(A) dA \psi(A) \) must vanish. Now,

\[
\int \text{Tr} \phi(A) dA = \int \text{Tr} \phi(A)(dA + dA^A) = 2 \int \text{Tr} \phi(A) F(A) = 4\pi \sum_{i=1}^{r} \frac{c_1(P_i)^2}{N_i}.
\]

and, therefore, if one of the magnetic charges \( c_1(P_i) \) of the components of \( A \) is non null every physical state must vanish at that gauge field configuration, i.e. \( \psi(A) = 0 \).

However, reducible configurations do not exhaust all nodal configurations. The reason is that the topological arguments leading to the existence of nodes discussed at the beginning also apply to the orbit space of irreducible connections. Therefore, there should exist other
genuine non-abelian configurations where physical states vanish. To find those configurations a more elaborate dynamical argument is required. In general, nodal configurations will not be the same for all physical states, only those with net magnetic charge satisfy this property. Non-abelian, irreducible nodes depend on the physical state we consider. In the following, we will analyse the nodes of the vacuum state.

To study the vacuum state we need to minimize expectation value the quantum hamiltonian (1). For that it will be useful to introduce the chiral components of the connection. Having fixed a 2-dimensional metric $h$ our 2D space $\Sigma$ acquires a complex structure and this induces a chiral decomposition $A = A_z dz + A_\bar{z} d\bar{z}$ of the gauge field. The component $A_\bar{z}$ defines an holomorphic structure on the vector bundle $E(\Sigma, \mathbb{C}^N)$ associated to $P$. Conversely, once one fix an hermitian structure on $E$ any holomorphic structure on $E(\Sigma, \mathbb{C}^N)$ defines a unique unitary connection $A$ on $P$. This correspondence induces an isomorphism between the space of gauge fields $A$ and the space $A_\bar{z}$ of holomorphic structures on $E(\Sigma, \mathbb{C}^N)$.

In terms of the chiral components of the gauge field the physical states that minimize the “kinetic” term of the Hamiltonian (1) are of the form

$$\psi(A) = \exp\left\{\frac{ki}{8\pi} \int dz d\bar{z} \text{Tr} A_\bar{z} A_z \right\} \xi(A_\bar{z}), \tag{7}$$

$\xi(A_\bar{z})$ being any holomorphic functional of $A_\bar{z}$. The restriction of the Gauss law to those states reads

$$D_\bar{z} \frac{\delta}{\delta A_\bar{z}} \xi(A_\bar{z}) = \frac{k}{\pi} \partial_\bar{z} A_\bar{z} \xi(A_\bar{z}), \tag{8}$$

and it is analogue to the Gauss law of pure Chern-Simons topological field theory. In fact, one can identify these states with those of the Chern-Simons theory in holomorphic quantization [13], [9].

In $A_\bar{z}$ there is an action of a larger group of symmetries, the group of chiral or complex gauge transformations $G^C$. The action of $h \in G^C$ on $A_\bar{z}$ is given by

$$^h A_\bar{z} = h A_\bar{z} h^{-1} + i h \partial_\bar{z} h^{-1}, \tag{9}$$
and the isomorphism between $\mathcal{A}$ and $\mathcal{A}_z$ induces an action of $\mathcal{G}^C$ in $\mathcal{A}$ that extends the ordinary or unitary gauge transformations in $\mathcal{G}$.

The relevance of these transformations comes from the fact that integration of the Gauss law (8) determines how the states with minimal kinetic energy change under chiral gauge transformations of the gauge field. In [13] it is shown that the states are multiplied by a non-null factor depending on $A$ and $h \in \mathcal{G}^C$. Now, we have shown that physical states must vanish for reducible gauge fields with magnetic monopole components, therefore the states with minimal kinetic energy also vanish along their complex orbits. Generically, the gauge fields in those orbits are non-reducible and thus we obtain this way a larger set of nodes for these states.

One could argue that most of the fields belong to orbits of gauge fields without monopole components which define a dense set of $\mathcal{A}$. From a quantum point of view it means that they are the most relevant configurations for the dynamical behavior of the theory. However, it turns out that they are the other orbits which are relevant for the discussion of the structure of vacuum states. In some sense, one can think of the space of gauge fields expanded by complex gauge transformations from configurations with non-trivial monopole components as a boundary of the space of all gauge field configurations. In this picture, the topological effects would arise as boundary conditions to be satisfied by the quantum states at those special configurations.

Atiyah and Bott [10] have studied in detail the action of the complex gauge group and they have shown that the chiral gauge orbits can be organized in strata of $\mathcal{A}$. The main stratum, $\mathcal{A}_0$, is made up of the gauge fields such that all subbundles of the associated holomorphic bundle $E$ have non-positive first Chern class; It is an open dense submanifold of $\mathcal{A}$. All flat connections belong to this stratum and the union of their complex gauge orbits is dense in $\mathcal{A}_0$. Then the physical states with minimal kinetic energy (7) are completely determined, like Chern-Simons states, by their values at flat connections [14].

In the case of the sphere $\Sigma = S^2$ the complex gauge orbit of the trivial connection $A = 0$ expands the main stratum $\mathcal{A}_0$ of $\mathcal{A}$. Then there is a unique minimal state as in Chern-
Simons theory. Its wave functional $\psi_0$ is completely determined by its value at $A = 0$ and it vanishes nowhere in $A_0$. All other orbits have a gauge configuration decomposable into a sum of monopole/anti-monopole gauge field components. Therefore, the physical states with minimal kinetic energy must vanish along all complex gauge orbits unless $A_0$. This result is compatible with the fact that for any value of $\psi_0(0)$ the functional $\psi_0(A)$ converges to 0 as $A$ approaches an orbit of a field configuration with monopole components. A result that can be obtained following the techniques of Ref. [13].

For higher genus Riemann surfaces the structure of the moduli space of flat connections is non-trivial and the states with minimal kinetic energy are not uniquely determined from the Gauss law (see [14] for the toroidal topology). There is a finite-dimensional space of physical states and they can have additional nodes for some particular flat connections. This nodes, however, are not physically relevant as they change from one state to another.

So far we have considered only the kinetic term of the hamiltonian (1) but the vacuum state should minimize the whole hamiltonian, which also includes a potential term. To understand how this term could affect the vacuum structure we need a more elaborate argument. The crucial remark is that, as we have seen, the vacuum as any other physical state has to vanish at reducible gauge field configurations with monopole components, but among those configurations there are the absolute minima of the potential term when we restrict to the corresponding strata [10]. This can be understood from the following remarks. The flow defined by the gradient field of the potential term is tangent to the complex gauge orbits and such a flow has critical points at reducible configurations that are solutions of Yang-Mills equations. These critical gauge configurations can be found in any strata of $A$, and it can be shown that the negative modes of the second variation of the potential at these critical points are orthogonal to their strata, which implies that they are local minima of the potential restricted to the those strata. The fact that they are global minima follows from the results of Atiyah and Bott [10].

Vanishing of the wave functional $\Psi_0(A)$ along the chiral gauge orbits containing solutions of Yang-Mills equation with magnetic monopole components is, then, necessary to minimize
the expectation value of $H$. It is not only required for the minimization of the kinetic term, but also for that of the Yang-Mills potential term, and it is a consequence of Ritz variational principle. Both terms of the Hamiltonian, the kinetic and potential terms conspire to make the vacuum to vanish on the orbits of gauge field configurations with monopole components. It is obvious that we can not extend this argument to higher energy states.

This result explains why in the limit of infinite topological mass $\Lambda \rightarrow \infty$ we recover the Chern-Simons states which by the same argument also vanish for the same configurations, and are completely determined by their values at flat gauge field configurations \cite{I3}. A similar result holds for an arbitrary gauge group $G$.

In summary, magnetic monopoles in YM$_{TM}$ are suppressed in any physical state by kinematical constraints, but the gauge field configurations on their complex gauge orbits are also suppressed in the vacuum state. They only give non-trivial contributions to excited states. Since the YM$_{TM}$ is not confining it is natural to speculate about the connection between the existence of nodes and the absence of confinement induced by the Chern-Simons interaction. If this connection exists an important role will be played by those configurations in the confinement mechanism for pure gauge theories. So far, most of the confinement scenarios gave a leading role to magnetic monopoles. From the above analysis it might be inferred that the gauge fields which are chiral gauge equivalent to those monopoles also play a relevant role. This opens a new possibility for understanding the mechanism of permanent confinement in 2+1 dimensional gauge theories.

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REFERENCES

[1] G. ’t Hooft, in High Energy Physics, Ed. A. Zichichi, Ed. Compositori, Bologna (1976), Nucl. Phys. B 190 [FS3] (1981) 455;
S. Mandelstam, Phys. Rep. 23 (1976) 245, Phys. Rep. 67 (1980) 109.

[2] A. Polyakov, Phys. Lett. B 59 (1975) 82;
Nucl. Phys. 120 (1977) 429.

[3] V.L. Berezinskii, Zh. Eksp. Theor. Fiz. 59 (1970) 907.

[4] I. Affleck, J. Harvey, L. Palla, G. W. Semenoff, Nucl. Phys. B328 (1989) 575-58.

[5] M.C. Diamantini, P. Sodano, C.A. Trugenberger, Phys. Rev. Lett. 71 (1993) 1969.

[6] M. Asorey, P.K. Mitter, Phys. Lett. B 153 (1985) 147.

[7] M. Asorey, in Geometry and Fields ed. A. Jadczyk, World Sci, Singapore (1986).

[8] R. Feynman, Nucl. Phys. B 188 (1981) 479.

[9] M. Asorey, J. Geom. Phys. 11 (1993) 63;
M. Asorey, S. Carlip, F. Falceto, Phys. Lett. B 312 (1993) 477.

[10] M. Atiyah, R. Bott, Philos. Trans. R. Soc. London A 308 (1982) 523.

[11] M. Asorey, P.K Mitter, Ann. Inst. H. Poincaré 45 (1986) 61.

[12] S. Deser, R. Jackiw, S. Templeton, Phys. Rev. Lett. 48 (1982) 975; Ann. Phys. (NY) 140 (1982) 372.

[13] K. Gawędzki, A. Kupiainen, Commun. Math. Phys. 135 (1991) 531.

[14] F. Falceto, K. Gawędzki, Commun. Math. Phys. 159 (1994) 549.