HARMONIC MAPS AND SELF-DUAL EQUATIONS FOR
IMMERSED SURFACES

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Abstract

The immersion of the string world sheet, regarded as a Riemann surface, in $\mathbb{R}^3$ and $\mathbb{R}^4$ is described by the generalized Gauss map. When the Gauss map is harmonic or equivalently for surfaces of constant mean curvature, we obtain Hitchin’s self-dual equations, by using $SO(3)$ and $SO(4)$ gauge fields constructed in our earlier studies. This complements our earlier result that $h\sqrt{g} = 1$ surfaces exhibit Virasaro symmetry. The self-dual system so obtained is compared with self-dual Chern-Simons system and a generalized Liouville equation involving extrinsic geometry is obtained.

The immersion in $\mathbb{R}^n$, $n > 4$ is described by the generalized Gauss map. It is shown that when the Gauss map is harmonic, the mean curvature of the immersed surface is constant. $SO(n)$ gauge fields are constructed from the geometry of the surface and expressed in terms of the Gauss map. It is found Hitchin’s self-duality relations for the gauge group $SO(2) \times SO(n - 2)$. 
I. INTRODUCTION

The study of Yang-Mills connections on Riemann surfaces is of importance in string theory. The space of self-dual connections provides a model for Teichmüller space [1]. Clearly, the world sheet of a string is a 2-dimensional surface immersed in $R^n$ (For convenience we consider both the world sheet and the target space as Euclidean). The immersion induces a metric on the world sheet. The second fundamental form of the surface determines its extrinsic geometry. We [2] have developed a formalism to study the dynamics of the world sheet conformally immersed (by conformal immersion it is meant that the induced metric is in the conformal gauge) in $R^3$ and $R^4$ using the generalized Gauss map [3]. The extrinsic curvature action can be written as a constrained Grassmannian $\sigma$-model action and the theory is asymptotically free.

Subsequently [4] we found a hidden Virasaro symmetry for surfaces of constant scalar mean curvature density ($h\sqrt{g} = 1$). An action exhibiting this symmetry has recently been constructed [5] and it is a WZNW action. The quantum theory of this action has also been studied in [5]. It would be of interest to know if surfaces of constant scalar mean curvature ($h=$constant) exhibit novel properties, so that we get a better understanding of the many facets of the string world sheet. For such surfaces, it is known that the Gauss map from the world sheet $M$ (regarded as a Riemann surface) into the Grassmannian $G_{2,n} \cong SO(n)/(SO(2) \times SO(n-2))$ is harmonic for $n = 3,4$. It is the purpose of this paper to show that there exists a non-Abelian self-dual system for such surfaces.
It is instructive to compare our results with those of Hitchin [1] and Donaldson [6]. Hitchin [1] has obtained two dimensional self-dual Yang-Mills-Higgs system by dimensional reduction of the Euclidean 4-dimensional self-dual $SU(2)$ Yang-Mills fields. By dimensional reduction here, one means that the fields are independent of $x_3$ and $x_4$. The complex Higgs field is identified with $\mu = 3$ and 4 component of the Yang-Mills field. Donaldson considers harmonic map from a Riemann surface $M$ into $H^3$, an hyperbolic 3-space of constant negative curvature. The harmonic map extremizes the ‘energy functional’. By showing an existence of a flat $PSL(2, C)$ connection and using the harmonic map equation (the Euler-Lagrange equation), Hitchin’s equations were obtained.

In this paper, we first consider the Gauss map of a 2-dimensional surface into the Grassmannian $G_{2,n} \simeq SO(n)/(SO(2) \times SO(n - 2))$ for $n = 3, 4$. We [4] have previously constructed $SO(n)$ connections on the string world sheet. We project these onto $SO(2) \times SO(n - 2)$ and the coset. The projection onto $SO(2) \times SO(n - 2)$ is identified as the gauge field (see sec.III) and that on the coset as the Higgs field. Using the Euler-Lagrange equations (harmonic map equations) for the surface, we show that this system is self-dual. In this analysis, the harmonic map has a geometrical interpretation. For immersion in $R^3$ and $R^4$, the Gauss map is harmonic if the mean curvature scalar $h$ is constant [7]. We next take up immersion in $R^n$ for $n > 4$ and show explicitly that when the Gauss map is harmonic, the mean scalar curvature of the surface is constant. The $SO(n)$ gauge fields are constructed from the geometry of the surface. Using their projections onto the subgroup $SO(2) \times SO(n - 2)$ and its complement in $G_{2,n}$, the self-dual system of Hitchin is obtained when and
only when the Gauss map is harmonic. Thus our main result is:

**Theorem.1**

Let $M$ be an oriented surface immersed in $R^n$ and let $M_0$ be the Riemann surface obtained by the induced conformal structure on $M$. Let $\mathcal{G} : M_0 \to G_{2,n}$ be the Gauss map. Let $A$ be the flat $SO(n)$ connection induced on $M_0$ defined by the adapted frame of the tangents and (n-2) normals to $M$. The projection of $A$ expressed in terms of the coordinates of the quadric $Q_{n-2}$ taken as the model for $G_{2,n}$, onto $SO(n-2) \times SO(2)$ and its complement in $G_{2,n}$ satisfy Hitchin’s self dual system of equations when the Gauss map is harmonic.

Recently, Dunne, Jackiw, Pi and Trugenberger [8] made a systematic analysis of the Yang-Mills non-linear Schrödinger equation and demonstrated self-dual Chern-Simons equations for static configurations. Here the matter density is in the adjoint representation. By choosing the Chern-Simons gauge field in the commuting set of the Cartan subalgebra and $\Psi (\rho = -i[\Psi, \Psi^\dagger])$ in terms of the ladder operators with positive roots, they [8] and Dunne [9] obtain Toda equations. We obtain similar results for the Gauss map.

In this way, we find that the (extrinsic) geometry of the string world sheet is closely related to the self-dual system of Hitchin [1], Donaldson [6] and to the static configuration of (2+1) self-dual non-Abelian Chern-Simons theory at the classical level.

**II.PRELIMINARIES**
Consider a 2-dimensional (Euclidean) string world sheet $M$ immersed in $R^n$. Let $M_0$ be the Riemann surface obtained by the induced conformal structure on $M$. The induced metric is $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\mu$, with $X^\mu(\xi_1, \xi_2)$ as immersion coordinates ($\mu = 1, 2, \ldots, n$) and $\xi_1, \xi_2$ as local isothermal coordinates on the surface. The Gauss-Codazzi equations introduce the second fundamental form $H^i_{\alpha\beta}$, $i = 1, 2, \ldots, (n-2)$. Locally on the surface, we have two tangents and $(n-2)$ normals. The Gauss map is,

$$G : M_0 \rightarrow G_{2,n} \simeq SO(n)/(SO(2) \times SO(n-2)).$$

$G_{2,n}$ admits a complex structure. It is convenient to regard $G_{2,n}$ as a quadric in $CP^{n-1}$ defined by $\sum_{i=1}^n Z_i^2 = 0$, where $Z_i$ are the homogeneous coordinates on $CP^{n-1}$. A local tangent 2-plane to $M$ is an element of $G_{2,n}$, or equivalently a point in $Q_{n-2}$. Then [3] we have,

$$\partial_z X^\mu = \psi \Phi^\mu,$$

where $z = \xi_1 + i\xi_2, \bar{z} = \xi_1 - i\xi_2$, $\Phi^\mu \in Q_{n-2}, \Phi^\mu \Phi^\mu = 0$ and $\psi$ is a complex function of $z$ and $\bar{z}$ which can be determined in terms of the geometrical properties of the surface. As not every element of $G_{2,n}$ is a tangent plane to $M$, the Gauss map (2) has to satisfy $(n-2)$ conditions of integrability [3]. These were explicitly obtained in Ref.3 for immersion in $R^3$ and $R^4$ and by us [11] for $R^n (n > 4)$. We first consider immersion in $R^3$ and $R^4$.

For immersion in $R^3$, $\Phi^\mu$ is parametrized as,

$$\Phi^\mu = [1 - f^2, i(1 + f^2), 2f],$$

where $f$ is a function of immersion coordinates.
where $f$ is complex. The Gauss map integrability condition is,

\[ \text{Im} \left[ \frac{f_{z\bar{z}}}{f_z} - \frac{2\bar{f}f_z}{1 + |f|^2} \right]_{\bar{z}} = 0. \quad (4) \]

The mean curvature $h(= N^\mu H^\mu_\alpha)$ is given by,

\[ (\ell \text{nh})_z = \frac{f_{z\bar{z}}}{f_z} - \frac{2\bar{f}f_z}{1 + |f|^2}, \quad (5) \]

which is known as the Kenmotsu equation \[10\]. The normal $N^\mu$ to the surface can be expressed in terms of $f$ as,

\[ N^\mu = \frac{1}{1 + |f|^2} \left[ f + \bar{f}, -i(f - \bar{f}), |f|^2 - 1 \right]. \quad (6) \]

The energy integral of the surface is,

\[ S = \int \left| \frac{f_{\bar{z}}}{1 + |f|^2} \right|^2 dz d\bar{z}, \quad (7) \]

which is also the extrinsic curvature action $\int \sqrt{g} |H|^2$. The Euler-Lagrange equations are,

\[ L(f) = f_{z\bar{z}} - \frac{2\bar{f}f_zf_{\bar{z}}}{1 + |f|^2} = 0. \quad (8) \]

The Gauss map is said to be harmonic if $f$ satisfies the Euler-Lagrange equations and it then follows from (5) that $h$ is constant \[7\].

For immersion in $R^4$, we have $G_{2,4} \simeq SO(4)/(SO(2) \times SO(2)) \simeq CP^1 \times CP^1$ and so $\Phi^\mu$ is parametrized in terms of the two $CP^1$ fields, $f_1$ and $f_2$ as,

\[ \Phi^\mu = \left[ 1 + f_1 f_2, i(1 - f_1 f_2), f_1 - f_2, -i(f_1 + f_2) \right]. \quad (9) \]

The Gauss map integrability conditions are,

\[ \text{Im} \left[ \sum_{i=1}^2 \frac{f_{iz\bar{z}}}{f_{iz}} - \frac{2\bar{f}_i f_{iz}}{1 + |f_i|^2} \right]_{\bar{z}} = 0, \]

\[ |F_1| = |F_2|, \quad (10) \]
where \( F_i = \frac{f_i}{1 + |f_i|^2} \). There are two normals \( N_1^\mu, N_2^\mu \) to the surface which can be written in terms of \( f_1 \) and \( f_2 \) as,

\[
\begin{align*}
D &= \left((1 + |f_1|^2)(1 + |f_2|^2)\right)^{\frac{1}{4}}, \\
A^\mu &= \left[f_2 - \tilde{f}_1, -i(f_2 + \tilde{f}_1), 1 + \tilde{f}_1 f_2, -i(1 - \tilde{f}_1 f_2)\right], \\
N_1^\mu &= \frac{1}{2}(A^\mu + \bar{A}^\mu)/D, \\
N_2^\mu &= \frac{1}{2i}(A^\mu - \bar{A}^\mu)/D.
\end{align*}
\]

Then the components of \( H_\alpha^\mu \) along \( N_1^\mu \) and \( N_2^\mu \) are given by,

\[
\begin{align*}
h_1 &= \frac{F_1 - F_2}{2\psi D}, \\
h_2 &= \frac{i(F_1 + F_2)}{2\psi D},
\end{align*}
\]

and the mean curvature \( h^2 = (h_1^2 + h_2^2) \) satisfies the equation [2],

\[
2(\ell n h)_z = \sum_{i=1}^{2} \left[ \frac{f_1 z \bar{f}_i}{\bar{f}_i} - \frac{2\tilde{f}_i f_1 z}{1 + |f_1|^2} \right].
\]

The energy integral of the surface is,

\[
S = \int \sum_{i=1}^{2} |F_i|^2 + |\dot{F}_i|^2,
\]

where \( \dot{F}_i = \frac{f_1 z}{1 + |f_1|^2} \). The Euler-Lagrange equations are obtained as,

\[
L(f_1) = 0 ; \quad L(f_2) = 0.
\]

The Gauss map is harmonic if \( f_1 \) and \( f_2 \) satisfy (15) and by (13) harmonicity of the Gauss map from \( M \) into \( G_{2,4} \) implies that the immersed surface has constant \( h \) for immersion in \( \mathbb{R}^4 \). It is to be noted that \( f_1 \) and \( f_2 \) should also satisfy the second requirement in (10) for them to describe the Gauss map.
In Ref. 4, we have considered tangents to $M$ as $\hat{e}_1 = \frac{1}{\sqrt{2|\Phi|}}(\Phi^\mu + \bar{\Phi}^\mu)$ and $\hat{e}_2 = \frac{1}{\sqrt{2|\Phi|}}(\Phi^\mu - \bar{\Phi}^\mu)$ along with the (n-2) normals. Then, the local orthonormal frame $(\hat{e}_1, \hat{e}_2, N_i^\mu)$ satisfies,

$$\partial_z \hat{e}_i = (A_z)_{ij} \hat{e}_j; \quad i, j = 1 \text{ to } n,$$

(16)

where $\hat{e}_i = N_i^\mu$, for $i = 3$ to $n$ and $(A_z)_{ij}$ is an antisymmetric $n \times n$ matrix. A similar equation for the $\bar{z}$ derivative defines $(A_{\bar{z}})_{ij}$. $A_z$ and $A_{\bar{z}}$ are easily seen to transform as $SO(n, C)$ gauge fields under local $SO(n)$ transformations of $(\hat{e}_1, \hat{e}_2, N_i^\mu)$ which follows from (16) [4]. These non-Abelian gauge fields are constructed from the geometrical properties of the surface alone and so they are characteristics of the world sheet. Using equations (3) and (6), it is easily verified that $A_z$ for immersion in $R^3$ is given by,

$$A_z = \frac{1}{1 + |f|^2} \begin{bmatrix} 0 & -i(f \bar{f} - \bar{f}f) & -(f + \bar{f}) \\ i(f \bar{f} - \bar{f}f) & 0 & i(f - \bar{f}) \\ f + \bar{f} & -i(f - \bar{f}) & 0 \end{bmatrix}$$

(17)

Similarly, using equations (9) and (10), and denoting $d_i = 1 + |f_i|^2$, $m_i = f_i \bar{f}_{iz} - \bar{f}_i f_{iz}$, $p_i = f_{iz} + \bar{f}_{iz}$, and $q_i = f_{iz} - \bar{f}_{iz}$, $A_z$ for immersion in $R^4$ is,

$$A_z = \frac{1}{2} \begin{bmatrix} 0 & -i\left(\frac{m_1}{d_1} + \frac{m_2}{d_2}\right) & \frac{p_1}{d_1} - \frac{p_2}{d_2} & i\left(\frac{q_1}{d_1} + \frac{q_2}{d_2}\right) \\ i\left(\frac{m_1}{d_1} + \frac{m_2}{d_2}\right) & 0 & -i\left(\frac{q_1}{d_1} - \frac{q_2}{d_2}\right) & \frac{p_1}{d_1} + \frac{p_2}{d_2} \\ -\left(\frac{p_1}{d_1} - \frac{p_2}{d_2}\right) & i\left(\frac{q_1}{d_1} - \frac{q_2}{d_2}\right) & 0 & i\left(\frac{m_1}{d_1} - \frac{m_2}{d_2}\right) \\ -i\left(\frac{q_1}{d_1} + \frac{q_2}{d_2}\right) & -\left(\frac{p_1}{d_1} + \frac{p_2}{d_2}\right) & -i\left(\frac{m_1}{d_1} - \frac{m_2}{d_2}\right) & 0 \end{bmatrix}$$

(18)

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The gauge field $A_{\bar{z}}$ can be obtained by replacing $z$ derivatives by $\bar{z}$ derivatives and it is seen that $(A_{\bar{z}})^\dagger = -A_{\bar{z}}$. Further, from (16) it is easily verified that the gauge fields satisfy,

$$\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}] = 0.$$  \hfill (19)

**III. HARMONIC MAP AND SELF-DUAL SYSTEM**

We now project the gauge fields constructed in the previous section onto $SO(2) \times SO(n - 2)$ and its orthogonal complement in $G_{2,n}$ for $n = 3$ and 4. The general procedure is briefly outlined here. (For details see Ref.13). Consider a $G/H$ sigma model on a two dimensional Riemann surface. Denote the generators of the Lie algebra $L_G$ of $G$ by $L(\bar{\sigma}), \bar{\sigma} = 1, 2, \ldots [G]$ and those of $L_H$ of $H$ by $L(\bar{\sigma}); \bar{\sigma} = 1, 2, \ldots [H]; [H] < [G]$. The remaining generators of $L_G$ will be denoted by $L(\sigma)$. Consider a local gauge group associated with $G$. We have,

$$M \ni (z, \bar{z}) \rightarrow g(z, \bar{z}) \in G,$$  \hfill (20)

and introduce,

$$\omega_\alpha(g) = g^\dagger \partial_\alpha g.$$  \hfill (21)

The field strength associated with $\omega_\alpha(g)$ is zero. In fact, $\omega_\alpha(g)$ is same as $-A_z$ and $-A_{\bar{z}}$ and is equivalent to (16) with $g(z, \bar{z})$ as the $n \times n$ matrix formed by the two tangent vectors $\hat{e}_1$ and $\hat{e}_2$ and the $(n - 2)$ normals $N^\mu_i$. Under a local gauge transformation generated by $u(z, \bar{z}) \in H$, we have,

$$g(z, \bar{z}) \rightarrow g(z, \bar{z})u(z, \bar{z}),$$

$$\omega_\alpha(g) \rightarrow \omega_\alpha(gu) = u^\dagger \omega_\alpha(g)u + u^\dagger \partial_\alpha u.$$  \hfill (22)
Thus $-A_z$ and $-A_{\bar{z}}$ transform as gauge fields under $SO(2) \times SO(n-2)$ gauge transformation. The projection of $\omega_\alpha(g)$ onto $L_H$ and its orthogonal complement are,

\[
\begin{align*}
    a_\alpha(g) &= L(\bar{\sigma})tr(L(\bar{\sigma})\omega_\alpha(g)), \\
    b_\alpha(g) &= L(\sigma)tr(L(\sigma)\omega_\alpha(g)),
\end{align*}
\]

and it is straightforward to verify that under (22),

\[
\begin{align*}
    a_\alpha(g) &\rightarrow a_\alpha(gu) = u^\dagger a_\alpha u + u^\dagger \partial_\alpha u, \\
    b_\alpha(g) &\rightarrow b_\alpha(gu) = u^\dagger \partial_\alpha u.
\end{align*}
\]

So, $a_\alpha(g)$ transforms as a gauge field under local gauge transformations belonging to $H$ and $b_\alpha(g)$ transforms homogeneously.

Now we consider immersion in $\mathbb{R}^3$. The $SO(3)$ gauge fields $A_z$ and $A_{\bar{z}}$ in (17) are projected onto $SO(2)$ and its orthogonal complement in $G_{2,3}$. Denoting the anti-Hermitian generators of $SO(3)$ as $T_1, T_2, T_3, [T_1, T_2] = T_3$, etc., we have,

\[
\begin{align*}
    a_z &= \frac{1}{2} T_3 tr(T_3 A_z), \\
    b_z &= \frac{1}{2} T_1 tr(T_1 A_z) + \frac{1}{2} T_2 tr(T_2 A_z).
\end{align*}
\]

The gauge group in (22) is $U(1)$. Similar projections for $A_{\bar{z}}$ are made. It can be verified that $a_z + b_z = -A_z = g^\dagger \partial_z g$. Once we have a flat connection which can be decomposed as $a_z$ and $b_z$ (and similarly for $A_{\bar{z}}$), we have,

\[
\begin{align*}
    \partial_z a_z - \partial_{\bar{z}} a_z + [a_z, a_{\bar{z}}] + [b_z, b_{\bar{z}}] &= 0, \\
    \partial_{\bar{z}} b_z + [a_z, b_{\bar{z}}] &= \partial_z b_{\bar{z}} + [a_z, b_z],
\end{align*}
\]

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where we have made use of the group structure underlying (25), namely; the first equation in (26) is in the Cartan subalgebra while the second in $T_1$ and $T_2$ directions: hence both must separately vanish. The second equation in (26) gives the self-dual property if each side vanishes. This we shall prove by using the equations of motion (15), namely for harmonic Gauss map. Explicitly, from (17), we find,

$$a_z = \frac{1}{1+|f|^2} \begin{bmatrix} 0 & i(f \bar{f} - \bar{f} f) & 0 \\ -i(f \bar{f} - \bar{f} f) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (27)$$

and,

$$b_z = \frac{1}{1+|f|^2} \begin{bmatrix} 0 & 0 & f_z + \bar{f}_z \\ 0 & 0 & -i(f_z - \bar{f}_z) \\ -(f_z + \bar{f}_z) & i(f_z - \bar{f}_z) & 0 \end{bmatrix}. \quad (28)$$

$a\bar{z} = -a_z^\dagger$; $b\bar{z} = -b_z^\dagger$. Then we find that,

$$\partial_z b_z + [a_z, b_z] = 0, \quad (29)$$

if and only if $L(f) = 0$. Thus for surfaces of constant $h$, i-e for harmonic Gauss maps, we find that,

$$\partial_z a_z - \partial_z a_z + [a_z, a_z] + [b_z, b_z] = 0,$$

$$\partial_z b_z + [a_z, b_z] = 0,$$

$$\partial_z b_z + [a_z, b_z] = 0. \quad (30)$$

Further $a_z$ transforms as an $SO(2)$ gauge field while $b_z$ transforms homogeneously.

We have thus obtained Hitchin’s self-dual Yang-Mills-Higgs system for harmonic
maps in $G_{2,3}$. Equations (30) are also equivalent to static self-dual Chern-Simons system if we set the matter density $\rho = -i [b_z, b^\dagger_z]$ which lies in the Cartan subalgebra of $SO(3)$. Stated differently, we have thus shown that harmonic Gauss maps of immersed surfaces in $R^3$ represented by $f(z)$ in Eqn.(3) satisfies the self-dual equations of Hitchin.

Next consider surfaces in $R^4$. Here the Gauss map maps $M_0$ into $G_{2,4} \simeq SO(4)/(SO(2) \times SO(2))$. We choose $T_1$ to $T_6$ as generators of $SO(4)$ such that $[T_1, T_2] = T_3$, et.cyc; $[T_4, T_5] = T_6$, et.cyc; and $[T_i, T_j] = 0$ for $i = 1, 2, 3; j = 4, 5, 6$. The explicit form of $A_z$ has been given in Eqn.(18) and the projection of $A_z$ and $A_z^\dagger$ onto $SO(2) \times SO(2)$ and its complement in $G_{2,4}$ are,

\[
a_z = T_3 \text{tr}(T_3 A_z) + T_6 \text{tr}(T_6 A_z),
\]

\[
b_z = T_1 \text{tr}(T_1 A_z) + T_2 \text{tr}(T_2 A_z) + T_4 \text{tr}(T_4 A_z) + T_5 \text{tr}(T_5 A_z).
\]

Equations similar to (26) readily follow from (18). The explicit forms of $a_z$ and $b_z$ are not displayed as the procedure is straightforward. The self-dual property can be verified for harmonic maps by computing each side of the second equation in (26) for this case, immersion in $R^4$. Introducing,

\[
\mathcal{L}(f_i) = \frac{(L(f_i) + \bar{L}(f_i))}{1 + |f_i|^2},
\]

\[
\mathcal{L}'(f_i) = \frac{(L(f_i) - \bar{L}(f_i))}{1 + |f_i|^2},
\]

\[
S = \mathcal{L}(f_1) + \mathcal{L}(f_2),
\]

\[
D = \mathcal{L}(f_1) - \mathcal{L}(f_2),
\]

\[
S' = \mathcal{L}'(f_1) + \mathcal{L}'(f_2),
\]

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\[ D' = L'(f_1) - L'(f_2), \]  
\[(32)\]

for \( i = 1, 2 \) and where \( L(f) \) is defined in (8), we find,

\[
\partial_z b_z + [a_z, b_z] = \frac{1}{2} \times \begin{bmatrix}
0 & 0 & -D & -iS' \\
0 & 0 & iD' & -S \\
D & -iD' & 0 & 0 \\
iS' & S & 0 & 0
\end{bmatrix}. \tag{33}
\]

It can be seen that when the Euler-Lagrange equations of motion are satisfied, \( L(f_i) = 0 \) for \( i = 1, 2 \) it follows that \( \partial_z b_z = [a_z, b_z] = 0 \), which is the self-dual equation. It is pertinent to note that \( b_z \) transforms homogeneously under the local \( SO(2) \times SO(2) \) gauge transformation. The field \( a_z \) which is in the Cartan subalgebra \( SO(2) \times SO(2) \) transforms as a gauge field. It is important to reiterate that \( a_z \) and \( b_z \) are embedded in \( SO(4) \). In this way we realize Hitchin’s equations for two dimensional gauge fields (constructed from the surface itself) on the world sheet immersed in \( R^4 \), when the Gauss map is harmonic. Conversely, explicit solutions to the self-dual equations for the gauge group studied here are given by (27) and (28) for \( R^3 \) and (18) and (31) for \( R^4 \), where the complex functions \( f_1 \) and \( f_2 \) satisfy the equations, \( L(f_1) = 0 \) and \( L(f_2) = 0 \). Since we have seen that harmonicity of the Gauss map implies that the surfaces have constant mean curvature, it can be concluded that such surfaces possess the properties of a self dual 2-dimensional field theory.

The basic equations (30) are very similar to the self-dual Chern-Simons system considered in Ref.8. Fujii [14] examined the relationship between Toda systems and
the Grassmannian $\sigma$-models. We now extend our considerations for $R^3$. In analogy with [8], the matter density $\rho = \rho_3 T_3$ is,

$$\rho_3 = \frac{2(f_z \bar{f}_z - f_{\bar{z}} \bar{f}_{\bar{z}})}{(1 + |f|^2)^2}. \quad (34)$$

We recall the following relations for Gauss map in $R^3$. With $\sqrt{g} = \exp(\phi)$, $\hat{F} = \frac{f}{1 + |f|^2}$, we have,

$$|F|^2 = \frac{h}{2} H_{zz} = \frac{h^2}{2} \exp(\phi),$$

$$|\hat{F}|^2 = \frac{h}{2} \frac{H_{zz} H_{\bar{z}\bar{z}}}{H_{z\bar{z}}} = \frac{1}{2} |H_{zz}|^2 \exp(-\phi),$$

$$H_{zz} = h\sqrt{g} = h \exp(\phi),$$

$$\rho_3 = 2(|\hat{F}|^2 - |F|^2), \quad (35)$$

where $H_{\alpha\beta}$ is the second fundamental form. It can be verified by using the equation of motion (8) and the Gauss map relation, $(\ell n \psi)_z = -\frac{2ff_z}{(1 + |f|^2)}$, that $\partial_z \partial_{\bar{z}} \ell n H_{zz} |^2 = 0$. The Gauss curvature $R = -\exp(-\phi) \partial_z \partial_{\bar{z}} \phi$. Writing the Gauss curvature in terms of $H_{\alpha\beta}$, we have a modified Liouville equation for extrinsic curvature as,

$$\partial_z \partial_{\bar{z}} \phi = -2h^2 \exp(\phi) + 2 \exp(-\phi) \exp(\phi_E), \quad (36)$$

where $|H_{zz}|^2 = \exp(\phi_E)$ and $\partial_z \partial_{\bar{z}} \phi_E = 0$. When we consider $f$ to be anti-holomorphic, then $\hat{F} = 0$ and (39) reduces to the Liouville equation for $h = 1$,

$$\partial_z \partial_{\bar{z}} \phi = -2 \exp(\phi), \quad (37)$$

which is also the Toda equation for $SO(3)$. Hence Eqn.(36) may be viewed as the generalized Liouville equation for immersed surfaces whose induced Riemann surface has genus $g > 1$. (See below)
For immersion in $R^4$, when we consider $f_1$ and $f_2$ both anti-holomorphic, we obtain,

$$
\rho = \frac{f_{1\bar{z}\bar{f}_{1z}}}{(1 + |f_1|^2)^2} T_6 + \frac{f_{2\bar{z}\bar{f}_{2z}}}{(1 + |f_2|^2)^2} T_3. \quad (38)
$$

The Cartan matrix for $SO(4)$ is $K_{\alpha\beta} = -2\delta_{\alpha\beta}$. Then we obtain,

$$
\partial_z \partial_{\bar{z}} \ell n \rho_6 = -2\rho_6,
\partial_z \partial_{\bar{z}} \ell n \rho_3 = -2\rho_3, \quad (39)
$$

where $\rho_6$ and $\rho_3$ are the coefficients of $T_6$ and $T_3$ in (42). When $f$ satisfies the Euler-Lagrange equation $L(f) = 0$, the immersed surface has $h$ constant. If $f$ is anti-holomorphic, the Gauss map is a map from $S^2$ into $G_{2,3}$. This can be seen by noting that,

$$
2\pi(1 - g) = \int \frac{|f_{\bar{z}}|^2 - |f_z|^2}{(1 + |f|^2)^2} dz d\bar{z},
= \int \frac{|f_{\bar{z}}|^2}{(1 + |f|^2)^2} dz d\bar{z}, \quad (40)
$$

where $g$ is the genus of the surface. Since the right hand side is positive definite, it follows that $g = 0$ unless $f = 0$.

**IV. HARMONIC MAPS IN $R^n(n > 4)$**

We now consider immersion of 2-dimensional surfaces in $R^n$, $n > 4$. There are two reasons for this consideration. First of all, the gauge field $a_z$ in the two cases considered ($n=3$ and 4) is Abelian embedding in $SO(3)$ and $SO(4)$. This is similar
to the choice in Ref. 8 and 9. We would like to realize the non-Abelian property of $a_z$ which occurs when $n > 4$. Secondly, the result that harmonic Gauss map implies constant mean curvature scalar, has been proved for immersion in $R^3$ by Ruh and Vilms [7] and can be proved from our [2] results for $h$ and the Euler-Lagrange equations for immersion in $R^4$. For immersion in $R^n$, $n > 4$, such a result has not yet been explicitly obtained to the best of our knowledge. In this paper we prove this and use it to obtain the self-duality equations for harmonic Gauss maps.

We recall the essential details of the Gauss map of surfaces in $R^n$ from our earlier paper [11] and from Hoffman and Osserman [10]. $\Phi^\mu$ of $Q_{n-2}$ in (2) is parametrized in the following manner. Let $(z_1, z_2, \ldots, z_n)$ be the homogeneous coordinates of $CP^{n-1}$. The quadric $Q_{n-2} \subset CP^{n-1}$ is defined by,

$$\sum_{k=1}^n z_k^2 = 0.$$  \hfill (41)

Let $H$ be the hyperplane in $CP^{n-1}$ defined by $H : (z_1 - iz_2) = 0$. Then $Q_{n-2}^* = Q_{n-2} \setminus \{H\}$ is biholomorphic to $C^{n-2}$ under the correspondence [10],

$$(z_1, \ldots, z_n) = \frac{z_1 - iz_2}{2} [1 - \zeta_k^2, i(1 + \zeta_k^2), 2\zeta_1, \ldots, 2\zeta_{n-2}],$$  \hfill (42)

where,

$$\zeta_j = \frac{z_{j+2}}{z_1 - iz_2},$$  \hfill (43)

for $j = 1, 2, \ldots, n-2$. In (42) and in what follows we use the summation convention that repeated indices are summed from 1 to $n-2$, unless otherwise stated.

Conversely, given any $(\zeta_1, \ldots, \zeta_{n-2}) \in C^{n-2}$, the point,

$$\Phi^\mu = [1 - \zeta_k^2, i(1 + \zeta_k^2), 2\zeta_1, \ldots, 2\zeta_{n-2}],$$  \hfill (44)
satisfies (41) and hence defines a point in the complex quadric $Q_{n-2}$. The Fubini-Study metric on $CP^{n-1}$ induces a metric on $Q_{n-2}$ [11] which is computed as,

$$g_{ij} = \frac{4}{|\Phi|^2} \delta_{ij} + \frac{16}{|\Phi|^2} \left[ \zeta_i \bar{\zeta}_j - \zeta_j \bar{\zeta}_i + 2 \zeta_i \bar{\zeta}_j |\zeta_k|^2 - \zeta_i \bar{\zeta}_j \zeta_k - \bar{\zeta}_i \zeta_j \zeta_k - \bar{\zeta}_i \zeta_j \zeta_k \right],$$  

(45)

where,

$$|\Phi|^2 = 2 + 4 \zeta_k \bar{\zeta}_k + 2 \zeta_k^2 \zeta_m^2.$$  

(46)

We [11] found it convenient to introduce an n-vector,

$$A^\mu_k = -[\bar{\zeta}_k + \zeta_k \zeta_m] \Phi^\mu + \frac{|\Phi|^2}{2} v^\mu_k,$$  

(47)

for $k = 1, 2, ..., (n-2)$ and,

$$v^\mu_k = (-\zeta_k, i\zeta_k, 0, 0, 0, 0, 0, 0, 0, 0),$$  

(48)

where $1_k$ stands for 1 in the $(k+2)$th position. The algebraic properties of $A^\mu_k$ and $a^\mu_k$ have been established in [11]. The $(n-2)$ real normals to the surface have been obtained as,

$$N_i^\mu = \frac{4}{|\Phi|^2} (O^T)_{ij} A^\mu_j,$$  

(49)

where the $(n-2) \times (n-2)$ matrix $O$ has been defined in [11].

The $(n-2)$ complex functions $\zeta_i(z, \bar{z})$ where $z, \bar{z}$ are the isothermal coordinates on $M$ have been shown to satisfy $(n-2)$ conditions so that they can represent the Gauss map [11]. The mean curvature scalar $h$ of the surface has been shown to be related to Gauss map by,

$$(lh)_z = \frac{\sum_{j=1}^{n-2} \zeta_{jz} \bar{\zeta}_{j\bar{z}}}{\sum_{j=1}^{n-2} (\zeta_{jz})^2} = \frac{4}{|\Phi|^2} \sum_{j=1}^{n-2} \zeta_{jz} \left[ \bar{\zeta}_j + \zeta_j \sum_{k=1}^{n-2} \bar{\zeta}_k \right],$$  

(50)
which is the generalization of the Kenmotsu equation to immersion in $R^n$.

**Theorem 2**

Let $M$ and $M_0$ be as defined in Theorem 1. Then, if the Gauss map $\mathcal{G} : M \to G_{2,n}$ is harmonic, the mean curvature scalar $h$ of $M$ is constant.

**Proof**

The Gauss map is said to be harmonic if the $(n-2)$ complex functions $\zeta_i$ satisfy the Euler-Lagrange equations of the ‘energy integral’[11],

$$\mathcal{E} = \int g_{ij} \zeta_i \bar{\zeta}_j.$$

(51)

The above ‘energy integral’ is also the action for the extrinsic curvature of the surface $M$, namely, $\int \sqrt{g} |H|^2$. The Euler-Lagrange equations that follow from the extremum of $\mathcal{E}$ are given by,

$$\zeta_{k\bar{z}} = -\frac{4}{|\Phi|^2} \sum_{i=1}^{n-2} \zeta_{i\bar{z}} \zeta_{iz} (\bar{\zeta}_k + \zeta_k \sum_{m=1}^{n-2} \bar{\zeta}_m)$$

$$+ \frac{4}{|\Phi|^2} \sum_{i=1}^{n-2} (\bar{\zeta}_i + \zeta_i \sum_{m=1}^{n-2} \zeta^2_m) [\zeta_{k\bar{z}} \zeta_{iz} + \zeta_{k\bar{z}} \zeta_{iz}].$$

(52)

Upon using the above Euler-Lagrange equations in the expression for $(\ell n h)_z$ in (51) it follows that the mean curvature scalar $h$ of the surface $M$ is constant. Q.E.D

We now construct $SO(n)$ gauge fields on the surface $M$. The defining equation for them is (16) with $\Phi^\mu$ given by (44) and the $(n-2)$ normals by (49). The various
components of $A_z$ are given below.

\[
A_z = \begin{bmatrix}
0 & (A_z)_{12} & (A_z)_{1i} & (A_z)_{2i} & (A_z)_{ij} & \cdots & (A_z)_{ij} \\
(A_z)_{21} & 0 & (A_z)_{2i} & 0 & 0 & \cdots & 0 \\
(A_z)_{31} & (A_z)_{32} & \cdots & \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(A_z)_{n1} & (A_z)_{n2} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]  

(53)

where,

\[
(A_z)_{12} = \frac{2}{i |\Phi|^2} [\bar{\zeta}_j \zeta_i^2 \bar{\zeta}_j - (\bar{\zeta}_j + \zeta_j \bar{\zeta}_i^2) \zeta_j] \\
(A_z)_{1i} = \frac{2\sqrt{2}}{i |\Phi|^2} (OT)_{ij} |\Phi|^2 \bar{\zeta}_j - 4(\bar{\zeta}_j + \zeta_j \bar{\zeta}_m^2)(\zeta_k + \bar{\zeta}_k \zeta_q^2) \bar{\zeta}_k \\
&+ |\Phi|^2 (2\bar{\zeta}_j \bar{\zeta}_k \bar{\zeta}_m + \bar{\zeta}_j) \\
(A_z)_{2i} = \frac{2\sqrt{2}}{i |\Phi|^2} (OT)_{ij} |\Phi|^2 \bar{\zeta}_j + 4(\bar{\zeta}_j + \zeta_j \bar{\zeta}_m^2)(\zeta_k + \bar{\zeta}_k \zeta_q^2) \bar{\zeta}_k \\
&- |\Phi|^2 (2\bar{\zeta}_j \bar{\zeta}_k \bar{\zeta}_m + \bar{\zeta}_j) \\
(A_z)_{ij} = -\frac{1}{|\Phi|^2} \partial_z (|\Phi|^2) \delta_{ij} + \frac{4}{|\Phi|^4} (OT)_{jk} \partial_z O_{ki} \\
&+ \frac{16}{|\Phi|^4} (\bar{\zeta}_k + \zeta_k \bar{\zeta}_q^2) \zeta_m [(OT)_{jk}(OT)_{im} - (OT)_{jm}(OT)_{ik}],
\]  

(54)

$A_z$ can be obtained by replacing the $z$-derivatives by $\bar{z}$ derivatives. They define the $SO(n)$ gauge fields on the surface $M$. We now project them on to $SO(2) \times SO(n-2)$ and its orthogonal complement in $G_{2,n} \simeq SO(n)/(SO(2) \times SO(n-2))$.

Denoting these projections by $a_z$ and $b_z$ respectively (and similarly for $a_{\bar{z}}$ and $b_{\bar{z}}$),
we have,

\[
\begin{align*}
a_z &= - \begin{bmatrix} 0 & (A_z)_{12} & 0(13 \text{ to } 1n) \\ (A_z)_{21} & 0 & 0(23 \text{ to } 2n) \\ 0 & 0 & \vdots \vdots \vdots \vdots & (A_z)_{ij} \\ 0 & 0 & \vdots \vdots \vdots \vdots \end{bmatrix} 
\end{align*}
\]

and,

\[
\begin{align*}
b_z &= - \begin{bmatrix} 0 & 0 & (A_z)_{1i}(i = 3 \text{ to } n) \\ (A_z)_{31} & (A_z)_{32} & (A_z)_{2i}(i = 3 \text{ to } n) \\ \vdots & \vdots & 0 \\ (A_z)_{n1} & (A_z)_{n2} \end{bmatrix}.
\end{align*}
\]

From the general considerations described in (24), it follows that \(a_z\) transforms as a gauge field under local \(SO(2) \times SO(n - 2)\) gauge transformation while \(b_z\) transforms homogeneously. It can be verified that \(a_z\) in (55) is indeed a non-Abelian gauge field when \(n > 4\). Further the construction and the gauge group structure of \(a_z\) are quite different from those of Donaldson [6]. The gauge connection \(a_z\) contains contributions from both the tangent space and the normal frame to \(M\) reflecting \(SO(2) \times SO(n - 2)\) group structure. The orthogonal complement \(b_z\) on the other hand receives from interaction of tangents with the normals. As \(b_z\) transforms homogeneously under the local \(SO(2) \times SO(n - 2)\) gauge transformations, it can be identified with the Higg's field of Hitchin. Realizing that \(A_z = -(a_z + b_z)\) and using (19) we immediately obtain (26) exploiting the group structure underlying
and (56). A similar feature has been used by Donaldson [6]. In order to prove that second equation in (26) gives the self duality, namely the vanishing of both the sides, we make use of the Euler-Lagrange equation (52) or harmonic Gauss map. To prove the self-duality equation, namely, \( \partial_z b_z + [a_z, b_z] = 0 \), when the Gauss map is harmonic, that is, when (52) is satisfied, we proceed as below. We consider \((1i)\) component of the self-duality equation for \( i \geq 3 \) which follows from (55) and (56).

\[
(\partial_z b_z + [a_z, b_z])_{1i} = \partial_z (b_z)_{1i} + (a_z)_{12} (b_z)_{2i} - (b_z)_{1i} (a_z)_{ji}, \quad j \geq 3,
\]

\[
= -\partial_z (A_z)_{1i} + (A_z)_{12} (A_z)_{2i} - (A_z)_{1j} (A_z)_{ji}, \quad (57)
\]

where the structure of (55) and (56) have been used. Using the definition that \((A_z)_{1i} = N_i^\mu (\partial_z \hat{e}_1)\) and (16), we find,

\[
(\partial_z b_z + [a_z, b_z])_{1i} = (A_z)_{12} (A_z)_{2i} + (A_z)_{12} (A_z)_{2i} - N_i^\mu (\partial_z \partial_z \hat{e}_1). \quad (58)
\]

The expressions for \((A_z)_{12} , (A_z)_{2i} , (A_z)_{12} \) and \((A_z)_{2i} \) have been given in (54).

The quantity \(N_i^\mu \partial_z \partial_z \hat{e}_1\) is calculated using (44) to be,

\[
N_i^\mu \partial_z \partial_z \hat{e}_1 = \frac{4}{|\Phi|^4} (O_T)_{ik} [\partial_z (\frac{1}{\sqrt{2} |\Phi|}) (A_k^\mu \partial_z \Phi^\mu + A_k^\mu \partial_z \Phi^\mu)]
\]

\[
+ \partial_z (\frac{1}{\sqrt{2} |\Phi|}) (A_k^\mu \partial_z \Phi^\mu + A_k^\mu \partial_z \Phi^\mu)
\]

\[
+ (\frac{1}{\sqrt{2} |\Phi|}) (A_k^\mu \partial_z \partial_z \Phi^\mu + A_k^\mu \partial_z \partial_z \Phi^\mu)]. \quad (59)
\]

Using the expressions for \(A_k^\mu\) in (47) and \(\Phi^\mu\) in (44), the above quantity has been evaluated using,

\[
A_k^\mu \partial_z \Phi^\mu = |\Phi|^2 \zeta_{kz},
\]

\[
A_k^\mu \partial_z \Phi^\mu = |\Phi|^2 \zeta_{kz},
\]

22
\begin{align*}
A_k^\mu \partial_z \bar{\Phi}^\mu &= -4(\bar{\zeta}_k + \zeta_k \bar{\zeta}_q) (\zeta_j + \bar{\zeta}_j \zeta_m^2) \bar{\zeta}_j z + |\Phi|^2 (\bar{\zeta}_k z + 2\zeta_k \bar{\zeta}_m \zeta_{mz}) \\
A_k^\mu \partial_{\bar{z}} \bar{\Phi}^\mu &= -4(\bar{\zeta}_k + \zeta_k \bar{\zeta}_q) (\zeta_j + \bar{\zeta}_j \zeta_m^2) \bar{\zeta}_j \bar{z} + |\Phi|^2 (\bar{\zeta}_k z + 2\zeta_k \bar{\zeta}_m \zeta_{m\bar{z}}) \\
A_k^\mu \partial_z \partial_{\bar{z}} \Phi^\mu &= 4(\bar{\zeta}_k + \zeta_k \bar{\zeta}_m) \zeta_{qz} \zeta_{q\bar{z}} + |\Phi|^2 \zeta_{kz \bar{z}} \\
A_k^\mu \partial_{\bar{z}} \partial_z \bar{\Phi}^\mu &= -4(\bar{\zeta}_k + \zeta_k \bar{\zeta}_m) [(\zeta_j + \bar{\zeta}_j \zeta_q^2) \bar{\zeta}_j \bar{z} + \zeta_q \zeta_j \bar{\zeta}_j \bar{z}] \\
&\quad + |\Phi|^2 [2\zeta_k \bar{\zeta}_j \bar{\zeta}_j \bar{z} + 2\zeta_k \bar{\zeta}_j \bar{\zeta}_j \bar{z} + \bar{\zeta}_k \bar{\zeta}_m], \quad (60)
\end{align*}

and the Euler-Lagrange equations of motion (52) (harmonic map requirement) for the $z\bar{z}$-derivatives of $\zeta$'s. We then find,

\begin{align*}
(\partial_z b_z + [a_z, b_z])_{1i} &= 0, \quad (61)
\end{align*}

which is the required self-duality equation. Similarly the other components have been verified. This proves our main result that for immersed surfaces in $R^n$ for $n > 4$, the surface $M$ admits Hitchin’s self-dual system when the Gauss map is harmonic.

Explicit solutions to the self-dual equations for the gauge group $SO(2) \times SO(n-2)$ are given by (55) and (56), where the complex functions $\zeta$'s satisfy the equation of motion (52).

\section{Conclusions}

We have considered certain properties of oriented string world sheet immersed in $R^n$. The immersion is described by the Gauss map from the Riemann surface induced by the conformal structure on the immersed surface into $G_{2,n}$. For $n = 3$ and $n = 4$, when the Euler-Lagrange equations following from the extrinsic curvature action expressed in terms of the coordinates of the Grassmanian are satisfied, the
Gauss map is harmonic and the mean curvature scalar of the immersed surface is constant. For such a class of surfaces, we have made use of the $SO(3)$ and $SO(4)$ two dimensional gauge fields constructed by us [4] and projected them onto the subgroup and its orthogonal complement in the Grassmannian $G_{2,3}$ and $G_{2,4}$. The projection onto the subgroup transforms as a gauge field belonging to the subgroup while the complement transforms homogeneously. By identifying the complement with the complex Higg’s field, we are able to prove the existence of solutions to Hitchin’s self-dual equation for constant $h$ immersions in $\mathbb{R}^3$ and $\mathbb{R}^4$. This study complements our earlier result that $h\sqrt{g} = 1$, surfaces exhibit Virasaro symmetry. The quantum theory of these surfaces have recently been studied by us [5].

The self-dual system so obtained for harmonic maps is compared with the self-dual Chern-Simons system. A generalized Liouville equation involving extrinsic geometry is obtained. As a particular case, when the map is anti-holomorphic the familiar Toda equations are obtained.

We have generalized the results to conformal immersion of 2-dimensional surfaces in $\mathbb{R}^n$, for arbitrary $n$, using the results of the generalized Gauss map. We prove that the surface has constant mean curvature when the Gauss map is harmonic. This harmonicity condition or Euler-Lagrange equation is used to show that for such surfaces, there exists Hitchin’s self-dual system.

The general action for the string theory will be a sum of the Nambu-Goto action and the action involving extrinsic geometry. When the theory is described in terms of the Gauss map, we have earlier [2] noted that both the actions can be expressed
as Grassmannian sigma model. Explicitly,

\[
S = S_{NG} + S_{Extrinsic}
\]

\[
= \mu \int \sqrt{g} dz \wedge d\bar{z} + \sigma \int \sqrt{|H|} dz \wedge d\bar{z},
\]

\[
= \mu \int \frac{1}{h^2} g_{ij} \zeta_i \zeta_j d\bar{z} \wedge d\bar{z} + \sigma \int g_{ij} \bar{\zeta}_i \bar{\zeta}_j dz \wedge d\bar{z}.
\]

(62)

In general the mean curvature scalar \( h \) will be a (real) function of \( \zeta \) and so the study of the action \( S \) will be complicated since we have a space-dependent coupling for \( S_{NG} \) in this framework. When surfaces of constant \( h \) are considered or when the Gauss map is harmonic, it is easy to see that the total action is just a Grassmannian sigma model with one effective coupling constant. Since the classical equations of motion for this action are identical to (52), it is possible to study the quantization in the background field method.

**Acknowledgements**

We are thankful to Prof.I.Volovich and Prof.I.Y.Arefeva for useful discussions. One of us (R.P) wishes to thank Prof.C.S.Seshadri for valuable discussions. This work has been supported by an operating grant (K.S.V) from the Natural Sciences and Engineering Council of Canada. R.P thanks the Department of Physics, Simon Fraser University, for the hospitality during the summer 1993.

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