On Serre dimension of monoid algebras and Segre extensions

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Abstract
Let $R$ be a commutative noetherian ring of dimension $d$ and $M$ be a commutative, cancellative, torsion-free monoid of rank $r$. Then $S\text{-}dim(R[M]) \leq \max\{1, \text{dim}(R[M]) - 1\} = \max\{1, d + r - 1\}$. Further, we define a class of monoids $\mathcal{M}_n$ such that if $M \in \mathcal{M}_n$ is seminormal, then $S\text{-}dim(R[M]) \leq \text{dim}(R[M]) - n = d + r - n$, where $1 \leq n \leq r$. As an application, we prove that for the Segre extension $S_mn(R)$ over $R$, $S\text{-}dim(S_mn(R)) \leq \text{dim}(S_mn(R)) - \left[\frac{m+n-1}{\text{gcd}(m,n)}\right] = d + m + n - 1 - \left[\frac{m+n-1}{\text{gcd}(m,n)}\right]$.

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1. Introduction
In the search for an answer to his conjecture, Serre [20] gave a splitting theorem which states that for a commutative noetherian ring $R$ of dimension $d$, if the rank of an $R$-projective module $P$ exceeds $d$, then $P$ admits a decomposition with a free direct summand. This shrinks the class of projective $R$-modules one needs to study, to projective modules of rank $\leq d$. Such a decomposition of $P$ is possible if there exists a $p \in P$ and a $\phi \in \text{Hom}_R(P, R)$ such that $\phi(p) = 1$. These elements are called unimodular elements of $P$ and $U_m(P)$ denotes the set of such elements. The Serre dimension of $R$, written as $S\text{-}dim(R)$, is defined to be the smallest integer $s$, such that if rank($P$) $\geq s + 1$, then $U_m(P) \neq \emptyset$.

Serre’s splitting theorem thus gives $S\text{-}dim(R) \leq \dim(R)$. Plumstead [16] proved $S\text{-}dim(R[X]) \leq \dim(R)$. Bhatwadekar-Roy [3] generalized the said splitting theorem to polynomial rings $R[X_1, \ldots, X_m]$ and in [2], Bhatwadekar, et. al. extended this result further to Laurent polynomial rings $R[X_1, \ldots, X_m, Y_1^\pm 1, \ldots, Y_n^\pm 1]$. Lindel [13] gave an independent proof of the same by employing semi-linear maps on the graded structure of such rings.

In another direction, Weimers [24] proved the result for discrete Hodge algebras. When $R$ is a PID, the corresponding result was proved by Gubeladze [5] for monoid algebras $R[M]$, where $M$ is a commutative, cancellative, torsion-free and seminormal monoid. He further conjectured in [7], the existence of unimodular elements in a general setup $R[M]$. Swan in [23], proved such an existence for any Dedekind domain $R$, when rank($P$) $\geq 2$ and $M$ is a commutative, cancellative and torsion-free monoid. Keshari-Sarwar in [10], gave an affirmative answer to the same for a certain class of monoids $C(\phi)$, which covered the case for positive rank 2 normal monoids. We prove the splitting theorem in the top rank case (see Theorem 3.4):

**Theorem 1.1.** Let $R$ be a ring of dimension $d$ and $M$ be a monoid of rank $r \geq 1$. Then $S\text{-}dim(R[M]) \leq \max\{1, \text{dim}(R[M]) - 1\} = \max\{1, d + r - 1\}$.

In Lemma 3.3, we show the existence of a positive submonoid $V$ of $M$ such that $M = U(M)V$ and study $S\text{-}dim(R[M])$. As a consequence, for rank 2 normal monoids (not necessarily positive) we obtain

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$S$-$dim(R[M]) \leq dim(R)$. To tackle the case when $\text{rank}(P) < dim(R[M])$, we define a descending chain of class of monoids $\{\mathfrak{M}_n\}_{n \geq 1}$, and prove the following (see Theorem 4.1):

**Theorem 1.2.** Let $R$ be a ring of dimension $d$ and $A = R[M]$, where $M \in \mathfrak{M}_n$ is a seminormal monoid of rank $r \geq 1$. Assume $P$ to be a projective $A$-module of rank $\geq dim(A) - n = d + r - n$. Then

1. the map $Um(P) \to Um(P/A_{i,j} P)$ is surjective and
2. $S$-$dim(A) \leq dim(A) - n = d + r - n$.

In particular, if $M \in \mathfrak{M}_r$, then $S$-$dim(A) \leq d$.

Utilizing the techniques developed, we discuss examples of monoids in $\mathfrak{M}_n$. In Section 5, we show the existence of unimodular elements of projective modules over Serre extension $S_{mn}(R)$ of $R$ and its application, investigate the Serre dimension of Rees algebras $R[T]$. For $m,n \in \mathbb{Z}_{\geq 0}$, define $Y$ to be the $m \times n$ matrix of indeterminates $y_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Let $I$ be the ideal of the polynomial algebra $R[y_{ij} | 1 \leq i \leq m, 1 \leq j \leq n] = R[F]$ generated by the binomial relations obtained from the $2 \times 2$ minor of $Y$. Then the Serre extension $S_{mn}(R)$ of $R$ over $mn$ variables is defined as $S_{mn}(R) = R[F]/I$. From ([23], Lemma 12.11), $S_{22}(R)$ is a monoid algebra over $R$. If $R$ is a field, then Lindel ([13], Example 1.9) proved that projective $S_{22}(R)$-modules are free. This was later extended by Swan ([23], Theorem 1.5) for any Dedekind domain $R$. Krishna-Sarwar ([11], Theorem 1.3) proved that if $d = 1$ and $Q \subset R$, then $S$-$dim(S_{22}(R)) \leq 1$. In a general $m \times n$ setup, we prove that the monoid corresponding to $S_{mn}(R)$ is a member of $\mathfrak{M}_{k(m,n)}$, where $k(m,n) = \left\lceil \frac{m+n-1}{\min(m,n)} \right\rceil$ (see Theorem 5.1):

**Theorem 1.3.** Let $R$ be a ring of dimension $d$ and $A = S_{mn}(R)$ be the Serre extension of $R$ over $mn$ variables. Let $k(m,n) = \left\lceil \frac{m+n-1}{\min(m,n)} \right\rceil$. Then there exists a monoid $M \in \mathfrak{M}_{k(m,n)}$ such that $A \simeq R[M]$. As a consequence, $S$-$dim(A) \leq dim(A) - k(m,n) = d + m + n - 1 - k(m,n)$. In addition, if $N \in PS(M)$ is a seminormal monoid, then $N \in \mathfrak{M}_{\max(m,n)} \subset \mathfrak{M}_{k(m,n)}$.

The central theme in the theorems above is to check the invariance of the bound of Serre dimension under monoid extensions of the ring $R$. We next attempt to improve upon this result for a certain class of rings $R$. In ([2], Theorem 5.2), for a normal $d$-dimensional ring $R$ and a projective $R[T]$-module of rank $\geq d$, the authors proved the map $Um(P) \to Um(P/TP)$ to be surjective, if $Um(P_f) \neq \emptyset$ for some $f \in R[T]$ monic in $T$. Taking cue from them, we prove the following (see Theorem 5.5):

**Theorem 1.4.** Let $R$ be a normal ring of dimension $d$, $M \in \mathfrak{M}_n$ a normal $\phi$-simplicial monoid of rank $r > 0$ and $A = R[M]$. Let $P$ be a projective $A$-module of rank $\dim(A) - n$ and $J = J(R,P)$ be the Quillen ideal of $P$. Assume

1. $Um(P_f) \neq \emptyset$ for some $f \in R[M]$ monic in $t_1$;
2. When $n > 1$, $M \in \mathfrak{M}_n$ is such that the automorphism $\tilde{\eta}$ obtained has the form $\tilde{\eta}(t_i) \in t_i + M_1$ for $i > 1$.

Then the map $Um(P) \to Um(P/A_{i,j} P)$ is surjective.

As a corollary to the above, we show that if $R$ is a normal ring, $B$ a birational overring of $R[X]$ and $M$ a positive normal rank 2 monoid, then $S$-$dim(B[M]) \leq dim(R)$ (Corollary 5.8).

2. Preliminaries

Throughout, all rings are commutative noetherian with unity and projective modules are finitely generated of constant rank. All monoids considered are commutative, cancellative and torsion-free.
For a projective $R$-module $P$ and $p \in P$, we denote the order ideal of $p$ as $\mathcal{O}_p(p) = \{ \phi(p) \mid \phi \in \text{Hom}_R(P, R) \}$. If $\mathcal{O}_p(p) = R$, then $p$ is called a unimodular element of $P$. The set of such elements is denoted by $Um(P)$ and is our main object of study in this paper. The existence of a unimodular element $p \in P$ decomposes it $P \simeq Rp \oplus Q$, where $Q$ is a projective $R$-module. This is often referred to as a splitting of $P$.

The monoid algebra $R[M]$ is generated as a free $R$-module with basis as elements of the monoid $M$ and coefficients in $R$. For a finite set $T$ we denote by $\mathbb{Z}_+[T]$, the monoid generated by finite multiplicative $\mathbb{Z}_+$-combinations of elements in $T$. A monoid $M$ is defined to be

- **affine**, if it is finitely generated, i.e., there exists $m_1, \ldots, m_k \in M$ such that $M = \mathbb{Z}_+[m_1, \ldots, m_k]$;

- **positive**, if its group of units $U(M)$ is trivial;

- **cancellative**, if $zx = zy$ implies $x = y$ for $x, y, z \in M$;

- **torsion-free**, if for $x, y \in M$ and $n > 0$, $x^n = y^n$ implies $x = y$;

- **normal**, if $x \in gp(M)$ and $x^n \in M$ for some $n > 0$, then $x \in M$;

- **seminormal**, if $x \in gp(M)$ and $x^2, x^3 \in M$, then $x \in M$.

Since $M$ is cancellative, the torsion-freeness of $M$ is equivalent to that of its group completion $gp(M)$. The rank of $M$ is the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes gp(M)$.

Gubeladze in ([9], Corollary 3.2), proved the following:

**Theorem 2.1.** Let $k$ be a field and $A = k[M]$ be the monoid algebra. Then $M$ is seminormal if and only if projective $A$-modules are free.

For a submonoid $L \subset M$, one can define the localization of $M$ at $L$, as the submonoid of $gp(M)$ given by $L^{-1}M = \{ m-l \mid m \in M, l \in L \}$. An affine monoid $M$ of rank $r$ is $\phi$-simplicial if there exists an embedding $M \hookrightarrow \mathbb{Z}^r_+$, which is integral. Geometrically speaking, it means we can find a hyperplane $H \subset \mathbb{R}^r$, such that $H$ intersects $\mathbb{R}_+ M$ (the cone generated by $M$) in a simplex. The equivalent definition that we will use during the course of our discussion is that $M \subset \mathbb{Z}_+[t_1, \ldots, t_r]$ and for all $i$ there exists $p_i \in \mathbb{Z}_{>0}$ such that $t_i^{p_i} \in M$.

As is usually the case if $M \subset \mathbb{Z}_+[t_1, \ldots, t_r]$ is not $\phi$-simplicial, we can define a new class of monoids containing $M$, denoted by $PS(M)$. We say $N \subset PS(M)$, if $M \subset N \subset \mathbb{Z}_+[t_1, \ldots, t_r]$ and if for an $i$, $t_i^{p_i} \not\in M$ for all $p_i \in \mathbb{Z}_{>0}$, then there exists a $s_i \in \mathbb{Z}_{>0}$ such that $t_i^{s_i} \not\in N$. In simpler terms we can identify elements of $PS(M)$ as a monoid obtained by adjoining elements to $M$ to make it $\phi$-simplicial.

If $M$ is a positive affine monoid of rank $r$, then we know that $M$ can be embedded in $\mathbb{Z}^r_+$ (i.e., $\mathbb{Z}_+[t_1, \ldots, t_r]$).

For $T \subset \{ 1, \ldots, r \}$, define $\widetilde{M}_T = \mathbb{Z}_+[t_i \mid i \not\in T] \cap M$. Let $M_1 = M \cap \mathbb{Z}_+ \left[ \prod_{i=1}^r t_i^{p_i} \mid p_i > 0, p_i \in \mathbb{Z}_+ \right]$. Given $1 \leq j \leq r$, one can assign a positive grading to $A = R[M]$ via $t_j$, by defining $A = \bigoplus_{i \geq 0} A_i^j = A_0^j \oplus A_+^j$ where

1. $M_1^j := \mathbb{Z}_+[t_1, \ldots, \hat{t}_j, \ldots, t_r] \cap M$;

2. $A_i^j$ is the $R[M_0^j]$-module generated by $M_1^j$;

3. $A_+^j = \bigoplus_{i \geq 1} A_i^j$.

Observe $A_0^j = R[M_0^j]$ is a monoid algebra where $M_0^j = \widetilde{M}_j$ is a positive monoid (and seminormal, if $M$ is so) of rank $r - 1$.

Unless specified, $R[M]$ is assumed to have the $\mathbb{Z}_+^r$-grading corresponding to the lexicographic order $t_1 < t_2 < \ldots < t_r$. One may note that in this case the zeroth homogeneous component $A_0 = R$, and the irrelevant ideal $A_+$ is generated by the generators of the monoid. Let $A = R[t_1, \ldots, t_r]$ and $f \in A$. 

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Then corresponding to this order we define $H(f)$ to be the highest degree component of $f$ and by $L(f)$, the coefficient of $H(f)$ in $R$. We say $f \in A$ is a quasi-monic if $L(f) \in R^\times = U(R)$.

The following result of Lindel ([13], Proposition 1.8) is vital to our discussion:

**Proposition 2.2.** Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded algebra and $P$ be a projective $A$-module. Let $P_*$ be free for some $s \in A_0$. If there exists $p \in P$ such that the extension $A_0/O_P(p) \cap A_0 \rightarrow A/O_P(p)$ is integral and $O_P(p) + sA_* = A$, then there exists $q \in Um(P)$ such that $q \equiv p$ modulo $sA_+P$.

We tweak Lemma 3.1 of [1] and write

**Lemma 2.3.** Let $R \subset S$ be a finite extension of reduced rings. Assume $(R/C)_{red} = (S/C)_{red}$, where $C = Ann_R(S/R)$. Assume $P$ is a projective $R$-module of rank $\geq 2$. If $Um(P \otimes_R S) \neq \emptyset$, then $Um(P) \neq \emptyset$.

Let $sn(R)$ and $sn(M)$ denote the seminormalization of the respective entities. We invoke the following Theorem ([4], Corollary 4.76):

**Theorem 2.4.** Let $R$ be a reduced ring and $M$ a monoid. Then $sn(R[M]) \simeq sn(R)[sn(R[M])]$.

Let $R$ be a reduced ring and $P$ a projective $R[M]$-module of rank $\geq 2$. Then using the above two results one can show that if $Um(P \otimes sn(R)[sn(M)]) \neq \emptyset$, then $Um(P) \neq \emptyset$. This is of immense utility in investigating unimodular elements of projective $R[M]$-module when $M$ is not seminormal.

3. The top rank case

This section is dedicated to proving the top rank case for monoid algebras $R[M]$. The following lemma is a crucial step towards the proof of Theorem 1.1.

**Lemma 3.1.** Let $R$ be a ring of dimension $d$ and $A = R[X_1, \ldots, X_m][M]$, where $M = Z^+_{x}[W_1, \ldots, W_I]$ is a positive monoid of rank $r \geq s$. Let $I \subset A$ be an ideal of height $d + r - s$. If $S = \{W_{j_1}, \ldots, W_{j_l}\} \subset \{W_1, \ldots, W_I\}$ is an algebraically independent subset of $A$, then $I \cap R[S, X_1, \ldots, X_m]$ contains a quasi-monic.

**Proof.** Let $W' = \{Z_1, \ldots, Z_l, X_1, \ldots, X_m\}$ be a set of variables over $R$. Consider the following composition of maps:

$$R[Z_1, \ldots, Z_l, X_1, \ldots, X_m] \xrightarrow{\beta} R[W'] \xrightarrow{\beta} A,$$

where $\beta|_R = Id_R$ and $\beta(Z_j) = W_j$ for all $1 \leq j \leq l$ and $i$ is the natural inclusion. Then $ht(\beta^{-1}(I)) > (d + r - s) + ht(ker(\beta)) \geq d + l - s$. This in turn implies $ht(\beta^{-1}(I) \cap R[Z_{j_1}, \ldots, Z_{j_l}, X_1, \ldots, X_m]) > d$. The result follows from ([12], Ch. III, Lemma 3.2), which implies the existence of $f \in I \cap R[S][X_1, \ldots, X_m]$ with $L(f) \in R^\times = U(R)$.

**Proposition 3.2.** Let $R$ be a ring of dimension $d > 0$, $M \subset Z^+_r$ a positive monoid of rank $r$ and $A = R[M]$. Let $P$ be a projective $A$-module of rank $> d$. Consider the following patching diagram:

$$\begin{array}{ccc}
B & \xrightarrow{\beta} & B_1 \\
\downarrow{\pi_1} & & \Downarrow{\pi_2} \\
B_2 & \xrightarrow{\pi_2} & B'.
\end{array}$$

1. Let $rank(P) > 2$ and $s \in R$ be such that $B = A, B_1 = A_s, B_2 = A_{1+sR}, B' = A_s(1+sR)$ or

2. Let $s \in R$ be a non-zerodivisor such that $B = A/sA^+_s, B_1 = A/sA, B_2 = A/A^+_s$ and $B' = A/(s, A^+_s)$.

If $Um(P \otimes_A B_j) \neq \emptyset$ for $j = 1, 2$, then $Um(P \otimes_A B) \neq \emptyset$. 

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Proof. Since \( \text{nil}(R) \subseteq \text{nil}(A) \), we can assume \( R \) is a reduced ring. As \( \text{rank}(P) \geq 2 \), we may further assume \( R[M] \) to be seminormal using Lemma 2.3. By Theorem 2.4 we have \( M \) is seminormal. Let \( P \) be a projective \( A \)-module satisfying the conditions of the hypothesis. Choose \( u \in Um(P \otimes_A B_1) \) and \( v \in Um(P \otimes_A B_2) \). Consider the decomposition arising from \( \pi_1(u) \in Um(P \otimes_A B') \) as \( P \otimes_A B' = B' \oplus Q \), where \( Q \) is projective \( B' \)-module of rank \( > d - 1 \). If \( \text{rank}(Q) \geq \max\{2, d\} \), then by ([15], Theorem 3.4) there exists a \( \tilde{\sigma} \in E(P \otimes B') \) such that \( \tilde{\sigma}(\pi_2(v)) = \pi_1(u) \). The corresponding diagrams will be

\[
\begin{array}{cc}
A & \xrightarrow{\pi_1} & A/sA_i \\
A_{1+sR} \cong R_{1+sR}[M] & \xrightarrow{\pi_2} & A_{s(1+sR)} \\
A/sA_i & \xrightarrow{\pi_2} & A/(s, A_i).
\end{array}
\]

(1) By ([17], Lemma 1 and Theorem 1), there exists a decomposition \( \tilde{\sigma} = \alpha_{1+sR} \circ \beta_s \), where \( \alpha \in \text{Aut}(P_s) \) and \( \beta \in \text{Aut}(P_{1+sR}) \). As \( \beta_s(v_s) = \alpha_{1+sR}(u_1+sR) \), patch \( \beta(v) \) and \( \alpha_{-1}(u) \) to get \( p \in Um(P) \).

(2) Lift \( \tilde{\sigma} \) to \( \sigma \in E(P/A_i, P') \), and patch \( \sigma(v) \) and \( u \), to get \( \tilde{p} \in Um(P/sA_i, P) \). If \( d = 1 \) and \( \text{rank}(Q) = 1 \), then follow the approach as in Case 2 of ([10], Theorem 3.4) to get \( Um(P/sA_i, P) \neq \emptyset \). □

The following lemma restructures the generators of \( M \):

**Lemma 3.3.** Let \( M \subset \mathbb{Z}[t_1, \ldots, t_r] \) be an affine monoid of rank \( r \). Then there exists generators \( \{W_1, \ldots, W_l\} \subset M \) such that

1. \( \{W_1, \ldots, W_{2r'}\} = \text{gen}(U(M)) \), where \( r' = \text{rank}(U(M)) \)
2. \( W_i = W_{i+1}^{-1}, \) for \( 1 \leq i \leq r' \)
3. \( V = Z_+\{W_{2r'+1}, \ldots, W_l\} \) is a positive monoid such that \( V \subset M \setminus U(M) \).

**Proof.** Let \( W' = \{W_1', \ldots, W_k\} \subset \mathbb{Z}[t_1, \ldots, t_r] \) be a minimal set of generators of \( M \). Then for some \( k \leq l' \), \( \{W_1', \ldots, W_k\} \) acts as a generating set of \( U(M) \). As \( U(M) \) is a finitely generated torsion-free group, it is free and generated by \( r' \) elements such that \( r \geq r' = \text{rank}(U(M)) \). Then \( U(M) \cong \mathbb{Z}_+[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] \) and \( \theta^{-1}x_i = W_i, \) where \( W_i \in Z_+[W_{i+1}, \ldots, W_k] \). Then \( Z_+[W_1^{-1}, \ldots, W_{r'}^{-1}] = Z_+[W_1, \ldots, W_k] = U(M) \). Then \( W = \{W_1^{\pm 1}, \ldots, W_k^{\pm 1}, W_{k+1}, \ldots, W_l\} \) is a generating set of \( M \) with the desired properties. □

The advantage of such a decomposition is that it gives a useful way to cleave the units and gives sufficient data about its components. We henceforth refer to \( V \) as a positive component, \( p(M) \) of \( M \). A nontrivial consequence of the above lemma would be that if \( M \) is not free, then \( \text{rank}(U(M)) < \text{rank}(M) \). Now we are in a position to prove Theorem 1.1.

**Theorem 3.4.** Let \( R \) be a ring of dimension \( d \) and \( M \) be a monoid of rank \( r \geq 1 \). Then \( S\text{-dim}(R[M]) \leq \max\{1, \text{dim}(R[M]) - 1\} = \max\{1, d + r - 1\} \).

**Proof.** We may assume \( R \) to be a reduced ring. Let \( M \) be affine and \( A = R[M] \). Assume \( P \) to be a projective \( A \)-module of rank \( > \max\{1, \text{dim}(A) - 1\} = \max\{1, d + r - 1\} \). As \( \text{rank}(P) \geq 2 \), we may further assume \( R[M] \) to be seminormal using Lemma 2.3. By Theorem 2.4, \( M \) is seminormal. We will induct on the dimension of the base ring \( R \). If \( d = 0 \), then \( R \) is a finite product of fields and by Theorem 2.1, \( P \) is free. If \( r = 1 \), then as \( M \) is seminormal, we have \( M = \mathbb{Z}_+ \) or \( M = \mathbb{Z} \) and we are done by [2].

Let \( d > 0 \) and \( r > 1 \). Then \( \text{rank}(P) > \max\{1, d + r - 1\} > 2 \). Denote by \( S \) the set of non-zerodivisors of \( R \). Then by the \( d = 0 \) case on \( S^{-1}A \), there exists a non-zerodivisor \( s \in R \) such that \( P_s \) is free. By induction on \( d, Um(P/sP) \neq \emptyset \). Thus there exists a \( p \in P \) such that \( (p, s) \in Um(P \otimes A) \), the image of \( p \) is unimodular in \( P/sP \) and \( O_P(p) + sA = A \). By ([13], Corollary 1.3) we may assume \( ht(O_P(p)) = \text{rank}(P) \geq d + r \).
By Lemma 3.3 we may assume existence of \( W = \{ W_{t_1}^{\pm 1}, \ldots, W_{t_r}^{\pm 1}, \ldots, W_{2t_r+1} \} \subset M \subset \mathbb{Z}[t_1, \ldots, t_r] \) such that \( M = \mathbb{Z}_+[W] \). Employ the following composition of maps:

\[
R[X_i^{\pm 1}] \xrightarrow{\gamma} R[X_1^{\pm 1}, \ldots, X_i^{\pm 1}, X_{2t'+1}, \ldots, X_{2t'+1}] \xrightarrow{\beta} A,
\]

where \( 1 \leq i \leq r' \), \( 2t' + 1 < j \leq 2t' + l \), \( \beta|_R = Id_R \) and \( \beta(X_k) = W_k \) for all \( k \). As height of both the ideals \( O_P(p) \cap R[X_i^{\pm 1}] \) and \( O_P(p) \cap R[X_j] \) exceed \( d \), by Lemma 3.1 and ([14], Lemma 2.3), there exists monics \( f_j \in O_P(p) \cap R[X_j] \) and special monics \( f_i \in O_P(p) \cap R[X_i] \), with coefficients in \( R \). This implies the extension \( R/O_P(p) \cap R \hookrightarrow A/O_P(p) \) is integral. As \( O_P(p) + sA = A \), hence \( (O_P(p) \cap R) + sR = R \). This in turn means \( U_{M}(P_{1+sR}) \neq \emptyset \) and by Proposition 3.2 we can conclude the proof.

If \( M \) is not affine, then \( M \) can be written as a filtered union of affine monoids \( M_i \), where \( i \in \mathcal{I} \). As seminormalization of affine monoids is again affine, we can write \( M \) as the filtered union of affine seminormal monoids. Thus a projective \( R[M] \)-module is extended from a projective \( R[M_i] \)-module for some \( i \in \mathcal{I} \) and the rest follows.

If \( M \) is a positive seminormal monoid of rank \( r \), then by [2], \( S-dim(R[M \oplus \mathbb{Z}^n]) \leq dim(R[M]) = d + r. \)

One way of looking at this would be that the units of monoids play no role in the Serre dimension in the above case. A natural question in a general setup would be whether \( S-dim(R[M]) \leq d + r - rank(U(M)) \).

The corollary below gives a partial answer to this query:

**Corollary 3.5.** Let \( R \) be a ring of dimension \( d \) and \( M \) be a normal monoid of rank \( r \). Then \( S-dim(R[M]) \leq d + r - rank(U(M)) \). In particular, if

1. \( rank(U(M)) = r - 1 \), then \( S-dim(R[M]) \leq d; \)
2. \( r = 2 \), then \( S-dim(R[M]) \leq d. \)

**Proof.** Let \( M \) be affine and \( rank(U(M)) = r' \). Then by ([4], Proposition 2.26) we get that \( M \simeq U(M) \oplus M' \) and by [2], \( S-dim(R[M]) = S-dim(R[M' \oplus U(M)]) \leq d + r - r' \). For (1) observe that \( M' \) will be a free positive monoid of rank 1 and use [2] to get the indicated. If \( r = 2 \) and \( M \) is not positive \( (r' > 0) \) then the assertion follows from (1). When \( M \) is positive, invoke ([10], Corollary 3.6) to prove the required. If \( M \) is not affine, then by ([4], Proposition 2.22) we may write \( M = \lim M_i \), where \( M_i \)'s are affine normal monoids. Then a projective \( R[M] \)-module is extended from a projective \( R[M_i] \)-module and thus the assertions follow.

**4. Serre dimension of monoid algebras corresponding to \( \mathfrak{M}_n \) - Lower rank case**

Define \( \mathfrak{M}_1 \) to be the class of affine positive monoids. Let \( M \in \mathfrak{M}_1 \) be a submonoid of \( \mathbb{Z}_+[t_1, \ldots, t_r] \) of rank \( r \). Fix \( W = \{ W_1, \ldots, W_l \} \subset M \) such that \( M = \mathbb{Z}_+[W] \), where \( W_i \)'s are monomials in \( \mathbb{Z}_+[t_1, \ldots, t_r] \).

Define \( gen(M)_1 = W \cap (M \setminus \tilde{M}_1) = \{ U_1, \ldots, U_{g_1} \} \), which is a subset of generators of \( M \) having some positive power of \( t_1 \).

For \( n \geq 2 \), define class of monoids \( \mathfrak{M}_n \subset \mathfrak{M}_1 \) as \( M \in \mathfrak{M}_n \) if

1. \( n \leq rank(M); \)
2. For each \( U_i \in gen(M)_1 \), there exists algebraically independent set \( S_i = \{ W_{i1}, \ldots, W_{ig_i} \} \subset W \) such that if \( f_i \in R[S_i] \) is quasi-monic, then there exists \( \eta \in Aut(R[M]) \), which is the restriction of \( \tilde{\eta} \in Aut(R[t_1, \ldots, t_r]) \), such that \( \eta(f_i) \) is monic in \( U_i \) with coefficients in \( A_0^1 \) for each \( 1 \leq i \leq g_1; \)
3. \( \tilde{M}_1 \in \mathfrak{M}_{n-1}. \)
For the sake of convenience if $M \in \mathfrak{M}_n$, we will denote by $(U_j, S_j)_{1 \leq j \leq n}$ its relevant information at the $n$'th level. The second property guarantees the existence of monic polynomials in all of the elements of $\text{gen}(M)_1$ in ideals of large enough height. The third ensures a smooth inductive process. These two steps, in combination with a Milnor patching diagram, lead to the required result. One may note that the classes of monoids $\{\mathfrak{M}^n\}_{n>1}$ defined above, form a descending chain since $\mathfrak{M}_j \subseteq \mathfrak{M}_i$ if $j \geq i$.

Let $M$ be $\phi$-simplicial and therefore for all $i$, $t^p_i \in M$ for some $p_i \geq 1$. Then $t^p_i \in \text{gen}(M)_1$, irrespective of one’s choice for generators for $M$. To decrease the Serre dimension when $M$ is $\phi$-simplicial monoid, we don’t need to find monics in all the elements of $\text{gen}(M)_1$ but just $t^p_i \in \text{gen}(M)_1$. Thus if $M$ is $\phi$-simplicial then condition (2) can be relaxed to

$$(2') \text{ If } f \in R[t_1, \ldots, t_n] \cap R[M], \text{ then exists quasi-monic, then there exists an } \tilde{g} \in \text{Aut}(R[t_1, t_2, \ldots]), \text{ such that its restriction } \eta \in \text{Aut}(R[M]) \text{ and } \eta(f) \text{ is monic in } t_1 \text{ with coefficients in } A^i_0.$$ 

**Remark 4.1.** If $M$ is a $\phi$-simplicial normal monoid, then by ([6], Lemma 6.3) we can relax the definition of $\mathfrak{M}_n$ and demand that under the automorphism we get a monic with coefficients in $R[M]$ instead of $R[M^i_0]$.

If $M$ is $\phi$-simplicial, then $M \in \mathfrak{M}_n$ means $M$ satisfies conditions (1), (2') and (3). The index chosen in the above definitions hold no relevance as we can permute the variables. For projective modules $P$ of rank $d < \text{rank} P < d + r$, we will now prove Theorem 1.2.

**Theorem 4.1.** Let $R$ be a ring of dimension $d$ and $A = R[M]$, where $M \in \mathfrak{M}_n$ is a seminormal monoid of rank $r \geq 1$. Assume $P$ to be a projective $A$-module of rank $\dim(A) - n = d + r - n$. Then

1. the map $U^{m}(P) \to U^{m}(P/A^i_1 P)$ is surjective and
2. $S\text{-dim}(A) \leq \dim(A) - n = d + r - n$.

In particular, if $M \in \mathfrak{M}_r$, then $S\text{-dim}(A) \leq d$.

**Proof.** One can assume that $R$ is reduced. We will proceed by induction on $d$. If $d = 0$, then $R$ is a product of fields and by Theorem 2.1, both $P$ and $U^{m}(P/A^i_1 P)$ are free. As $R[M] \to R[M^i_0]$ is a retraction, the surjection follows. Assume $d > 0$. Let $S$ be the set of non-zerodivisors of $R$. Then $\text{dim}(S^{-1}R) = 0$, and by $d = 0$ case we can find an $s \in S$ such that $P_s$ is free.

Let $p_i \in U^{m}(P/A^i_1 P)$. By the inductive process, $U^{m}(P/sP) \neq \emptyset$ and thus Proposition 3.2 gives $\tilde{p} \in U^{m}(P/sA^i_1 P)$ such that $p \equiv p_i$ modulo $A^i_1 P$, where ‘‘$\equiv$’’ denotes reduction modulo $sA^i_1 P$. As $O_P(p) + sA^i_1 = A$, choose $a \in A^i_1$ such that $1 + sa \in O_P(p)$. By ([13], Lemma 1.2 and Corollary 1.3) there exists $q \in P$ such that $ht(O_P(p + saq)) = \text{rank}(P) \geq d + r - n + 1$. As $p$ and $p + saq$ have the same image modulo $sA^i_1 P$, we can assume $ht(O_P(p)) \geq d + r - n + 1$.

Let $(U_j, S_j)_j$ be the relevant information of $M$ at the $n$'th level. As per definition, corresponding to each $U_j \in \text{gen}(M)$ there exist algebraically independent subset $S_j = \{W_{j_1}, \ldots, W_{j_n}\}$ of $M$. Consider the composition of maps

$$R[X_{j_1}, \ldots, X_{j_n}] \overset{\eta_j}{\longrightarrow} R[X_{1}, \ldots, X_{j}] \overset{\beta}{\longrightarrow} R[M].$$

By Lemma 3.1 there exists quasi-monic $f_j \in O_P(p) \cap R[S_j]_j$ for all $j$. By definition of $\mathfrak{M}_n$, there exists a common $\tilde{g} \in U^{m}_j(R[t_1, \ldots, t_r])$ such that its restriction $\eta \in U^{m}(A)$ and $\eta(f_j) \in O_P(p)$ is monic in $U_j$ with coefficients in $A^i_1$ for all $j$. As $\tilde{g}$ fixes $t_1$, we have $\eta(O_P(p)) + s\tilde{g}(A^i_1) = A$. Replacing $A$ by $\eta(A)\text{ we can assume } O_P(p) \text{ contains monics in } U_j \text{ with coefficients in } A^i_1 \text{ for all } j \text{ and } O_P(p) + sA^i_1 = A.$ Hence the extension $A^i_1/O_P(p) \cap A^i_1 \to A/O_P(p)$ is integral. Using Proposition 2.2 we obtain a $p_0 \in U^{m}(P)$ such that $p_0 \equiv \tilde{p}$. Note that $p_0 \equiv p \equiv p_i$ modulo $A^i_1 P$. Thus the map $U^{m}(P) \to U^{m}(P/A^i_1 P)$ is surjective. The second conclusion follows using process similar to above with an added induction on $n$, where $n = 1$ case follows through by Theorem 3.4.

Given a projective $A$-module $P$, we denote by $\mu(P)$ the number of minimal generators of $P$.  

\[ \]
Corollary 4.2. Let $R$ be a ring of dimension $d$ and $A = R[M]$, where $M$ is a seminormal monoid of rank $r$. Let $P$ be a projective $A$-module. Then $\mu(P) \leq \text{rank}(P) + d + r - 1$. If $M \in \mathfrak{M}_n$, then $\mu(P) \leq \text{rank}(P) + d + r - n$.

Proof. Let if possible $\mu(P) = m > \text{rank}(P) + d + r - n$. Consider the natural surjection $\phi : A^m \to P$, where $Q = \ker(\phi)$. By Theorem 4.1, there exists $q \in Um(Q)$. As $q \in Q = \ker(\phi)$, $\phi$ restricts to a surjection $\phi : A^m/qA \to P$. By ([15], Theorem 3.4) $A^m/qA \simeq A^{m-1}$, which implies $P$ is generated by $m-1$ elements, a contradiction.

Corollary 4.3. Let $M \in \mathfrak{M}_n$ be a $\phi$-simplicial monoid of rank $r$. Then $M \oplus \mathbb{Z}^n_+ \in \mathfrak{M}_{n+m}$.

Proof. Let $M' = M \oplus \mathbb{Z}^n_+$ and $m > 0$. Let $Z_i$ represent the variables in the $\mathbb{Z}^n_+$ direct summand of $M'$. Let $f \in R[t_1, \ldots, t_n, Z_1, \ldots, Z_m] \cap R[M']$ be a quasi-monic. Choose $c \in \mathbb{Z}_+$ large enough so that the $R[M']$-automorphism of $R[M']$ given by

$$Z_i \mapsto Z_i + ct_n,$$

for all $i$, is such that the coefficient of highest degree component of $\theta(f)$ is given by $u \prod_{1 \leq i \leq n} t_i^{e_i}$ and $u \in R^\times$.

Then $\theta(f)$ is a quasi-monic in $R'[t_1, \ldots, t_n] \cap R'[M']$, where $R' = R[\mathbb{Z}^n_+]$. By definition of $\mathfrak{M}_n$, there exists $\tilde{\eta} \in \text{Aut}_{R[t_1]}(R'[t_1, \ldots, t_n])$ such that its restriction $\eta \in \text{Aut}(R'[M]) = \text{Aut}(R[M'])$ and $\eta(\theta(f))$ is monic in $t_1$ with coefficients in $(R[M')]_0 = R[\mathbb{Z}^n_+\{m\}]$. As $\tilde{\eta} \circ \theta \in \text{Aut}_{R[t_1]}(R[t_1, \ldots, t_n])$, we have our result.

If $M$ is $\phi$-simplicial, we can assume the much simpler condition (2') instead of (2). The above approach can be replicated subject to minor changes and one can arrive at the same conclusion.

Theorem 4.4. Let $R$ be a ring of dimension $d$ and $A = R[M]$, where $M \in \mathfrak{M}_n$ is a $\phi$-simplicial seminormal monoid of rank $r \geq 1$. Assume $P$ to be a projective $A$-module of rank $\geq \dim(A) - n = d + r - n$. Then

1. the map $Um(P) \to Um(P/A_1^d P)$ is surjective and
2. $S\text{-dim}(A) \leq \dim(A) - n = d + r - n$.

In particular, if $M \in \mathfrak{M}_n$, then $S\text{-dim}(A) \leq d$.

By $M_*$ we mean the monoid $(\text{int}(\mathbb{R}_+(M)) \cap M) \cup \{0\}$. For the definitions of quasi-normal and quasi-truncated monoids refer to ([6], Section 5 and 6).

Theorem 4.5. Let $M$ be a quasi-truncated quasi-normal monoid of rank $\geq 2$. Then $M \in \mathfrak{M}_2$. As a consequence, if $M$ is $\phi$-simplicial seminormal of rank $\geq 2$, then $S\text{-dim}(R[M_*]) \leq d + \text{rank}(M) - 2$.

Proof. Let $M$ be a quasi-truncated quasi-normal monoid of rank $\geq 2$ and $f \in R[t_1, t_2] \cap R[M]$, be a quasi-monic. Then by ([6], Lemma 6.7) there exists an $R[t_1]$-automorphism $\eta$ of $R[M]$ given by $\eta(t_2) = t_2 + ct^s$ where $c \in \mathbb{Z}_{\geq 0}$ and thus $\eta(f)$ is monic in $t_1$. As $M$ is quasi-normal, by Remark 4.1 $M \in \mathfrak{M}_2$. For the second part, as $M$ is seminormal by ([4], Proposition 2.40) we have $M_* = (n(M))_*$, where $n(M)$ denotes normalization of $M$. Invoke ([6], Theorem 3.1) to obtain quasi-truncated normal monoids $Q_i$'s such that $n(M)_*$ is the filtered union of $Q_i$'s. This concludes the proof.

In [8], Gubeladze introduced the concept of tilted $R$-subalgebras of $R[t_1, \ldots, t_r]$. We repurpose this concept from the point of view of the variables and define $t_i$ to be a strongly tilted variable of the monoid $M \subset \mathbb{Z}_+[t_1, \ldots, t_r] = F$, if there exists $c \in \mathbb{Z}_+$ such that $t_i^s F \subset M$ for all $s \geq c$. If $M$ is affine, then we say $t_i$ is a tilted variable of $M$ if for all $j \neq i$, there exists a $c_j \in \mathbb{Z}_+$ such that $t_i^c t_j \in M$ and $t_j \in M$. It follows from definition that if $t_i$ is a tilted variable of $M \subset F$, then any monoid $N$ such that $M \subset N \subset F$ has $t_i$ as its tilted variable. As a straightforward example of such monoid, consider $M = \mathbb{Z}_+[t_1, t_1t_2] \subset \mathbb{Z}_+[t_1, t_2]$, where $t_2$ is a tilted variable of $M$, often rewritten as $R[M]$ is $t_1$-tilted. The polynomial algebra $R[t_1, \ldots, t_r]$ is tilted in all it’s variables. The existence of a tilted variable leads to our required type of automorphism:
Lemma 4.6. Let \( R \) be a ring and \( A = R[M] \), where \( M \) is an affine positive monoid of rank \( r \). Let \( t_1 \) be a tilted variable of \( A \). If \( f \in A \) is a quasi-monic, then there exists an \( \eta \in \text{Aut}(A) \) such that \( \eta(f) \) is monic in \( t_1 \).

Proof. Let \( F = \mathbb{Z}_+[t_1, \ldots, t_r] \) and \( f = f_0 + f_1 + \ldots + f_l \in A \subset R[F] \) be a quasi-monic of degree \( l \geq 1 \). Let \( f_i = u \sum_{k=1}^r t_k^p k \) for some \( u \in R^x \) and \( p_k \in \mathbb{Z}_+ \) for \( 1 \leq k \leq r \). As \( A \) is \( t_1 \)-tilted, there exists a \( c_j \in \mathbb{Z}_{>0} \) such that \( t_1^{c_j} t_j \in M \) for all \( j \neq 1 \) and \( t_1 \in M \).

We wish to define an \( R[t_1] \)-automorphism of \( A \) of the form \( t_k \to t_k + t_1^l \) for \( 2 \leq k \leq r \), which satisfies the required. As \( M \) is affine we have \( A = R[m_1, \ldots, m_k] \). Let \( c > \max \{\text{tot-deg}(m_i)\} \). Choose \( q > \max \{cc, l\} \). Then \( \eta(f) \) is monic in \( t_1 \) and \( \eta(m_i) \in A \) for all \( i \), and hence \( \eta \in \text{Aut}(A) \).

The above lemma holds for any (not necessarily affine) \( R \)-algebra \( A = R[t_1, t_2, \ldots, t_r] \), which has a strongly tilted variable and proves that \( \eta(f) - f \in A \) for all \( f \in R[t_1, \ldots, t_r] \). As we are dealing with affine monoid algebras, we refrain from using the said version in the interest of simplicity.

Example 4.7. Using techniques developed in the previous discussion, we improve the Serre dimension of some monoid algebras:

1. If \( M = \mathbb{Z}_+ / (\) free positive monoid of rank \( r \), then \( M \in \mathfrak{M}_r \).

2. The results corresponding to the special class of monoid \( C(\phi) \) introduced in [10], can be subsumed into that of \( \mathfrak{M}_r \). To be exact, if \( M \in C(\phi) \) is of rank \( r \), then \( M \in \mathfrak{M}_r \). Hence all normal rank 2 monoids belong to \( \mathfrak{M}_2 \) by ([10], Corollary 3.6).

3. Let \( M \subset \mathbb{Z}_+[t_1, \ldots, t_r] \) be a \( t_1 \)-tilted \( \phi \)-simplicial quasi-normal monoid of rank \( r \). Then the enveloping normal monoid corresponding to \( M \) is the free monoid \( \mathbb{Z}_+^r \). Then \( M \) is of rank \( r \).

4. Let \( M = \mathbb{Z}_+[x_1, x_2, x_1x_4, x_2^2x_4, x_3x_4] \). Then \( M \) is a non \( \phi \)-simplicial seminormal monoid of rank 4.

5. Applications

5.1. Segre Extension

In the following theorem, we identify the monoid corresponding to the monoid algebra \( S_{mn}(R) \) and maneuver it by means of a monoid isomorphism. This identification in combination with Theorem 4.1, yields a proof of Theorem 1.3.

Theorem 5.1. Let \( R \) be a ring of dimension \( d \) and \( A = S_{mn}(R) \) be the Segre extension of \( R \) over \( mn \) variables. Let \( k(m, n) = \left[ \frac{m+n-1}{\min(m, n)} \right] \). Then there exists a monoid \( M \in \mathfrak{M}_{k(m, n)} \) such that \( A \simeq R[M] \). As a consequence, \( S\text{-dim}(A) \leq \dim(A) - k(m, n) = d + m + n - 1 - k(m, n) \). In addition, if \( N \in PS(M) \) is a seminormal monoid, then \( N \in \mathfrak{M}_{\max(m, n)} \subset \mathfrak{M}_{k(m, n)} \).

Proof. It is sufficient to provide a proof for the case \( m \leq n \). We will induct on \( n \). If \( n = 1 \), then \( A = S_{mn}(R) \) is a polynomial algebra and we are done by [3]. Assume \( n \geq 2 \). From ([23], Lemma 12.11) \( A \) is isomorphic to the monoid ring \( R[M'] \) presented by generators \( \{y_{ij} \}_{1 \leq i \leq m, 1 \leq j \leq n} \) where \( y_{ij}y_{kl} = y_{il}y_{kj} \) for \( i \neq k \) and \( j \neq l \). Define \( M \) as

\[
M = \mathbb{Z}_+[x_1, \ldots, x_n, x_i x_j \mid 1 \leq i \leq n \text{ and } n + 1 \leq j \leq m + n - 1].
\]
Using $gp(M) = \mathbb{Z} \cdot [x_1^{+1}, \ldots, x_n^{+1}, x_{n+1}^{+1}, \ldots, x_{n+m-1}^{+1}]$, we have $\text{rank}(M) = m + n - 1$. Consider the monoid homomorphism $\theta : M' \rightarrow M$, where the generators of $M'$ are mapped to generators of $M$ in the given order:

$$\theta(y_{ij}) = \begin{cases} x_j & i = 1, \\ x_ix_{n+i-1} & i > 1. \end{cases}$$

Note that $\theta$ preserves the relations of $M'$, i.e., $\theta(y_{ij}y_{kl}) = \theta(y_{ld}y_{kj})$ for $i \neq k$ and $j \neq l$. We claim that $\theta$ is an isomorphism. Surjectivity of $\theta$ is straightforward. For injectivity, consider the group homomorphism $\phi : gp(M) \rightarrow gp(M')$ defined by

$$\phi(x_j) = \begin{cases} y_{ij} & 1 \leq j \leq n, \\ y_{i-1,j+1-n} & n < j \leq m + n - 1. \end{cases}$$

For $1 \leq i \leq n$ and $1 \leq j \leq m - 1$, we can deduce $\phi(x_{i,n+i}) = y_{ij}y_{i,j+1}^{-1}y_{i+1,j+1} = y_{i+1,j+1}$, using the relation between $y_{ij}$'s. Restricting $\phi$ to $M$, we get $\phi(M) = M'$, $\theta \circ \phi |_{M} = 1_{M'}$ and $\phi |_{M} \circ \theta = 1_{M'}$. Thus $\theta$ is an isomorphism and $\dim(A) = \dim(R) + \text{rank}(M) = d + m + n - 1$.

**Claim:** $M \in \mathfrak{M}_k$, where $k = k(m,n) = \left[ \frac{m+n-1}{m} \right]$. Observe $\text{gen}(M)_1 = \{x_1, x_1x_{n+1}, \ldots, x_1x_{n+m-1}\}$ and identify it's generators as $U_1 = x_1$ and $U_j = x_1x_{n+j-1}$ for $2 \leq j \leq m$. Choose $S_1 = \{x_1, \ldots, x_k\}$ and $S_j = \{x_{n+j-1}x_1, x_{n+j-1}x_{j-1}, \ldots, x_{n+j-1}x_k\}$ for all $j \geq 2$. Then given quasi-monomics $f_j \in R[S_j]$, consider the following map:

$$\tilde{\eta}(x_i) = \begin{cases} x_i + x_1^{d_i} & 2 \leq i \leq k, \\ x_i + x_1^{d_i-1}x_{n+i-1} & jk - (k + j - 2) < i \leq jk - (j - 1) \text{ and } j \geq 2, \\ x_i & \text{else.} \end{cases}$$

where $d_j > \text{tot-deg}(f_j)$. Here $\tilde{\eta}$ is an $R[x_1]$-automorphism of $R[x_1, x_n, x_{n+1}, x_{n+2}, \ldots, x_{m+n-1}]$. On restricting to $R[M]$, one may see that $\eta = \tilde{\eta}|_{R[M]} \in \text{Aut}_{R[x_1]}(R[M])$ and $\theta(f_j)$ is monic in $U_j$ with coefficients in $R[M_1^\theta]$. As

$$M_1 = M_1^\theta = \mathbb{Z} \cdot [x_2, \ldots, x_n, x_i, x_j | 2 \leq i \leq n + 1 \leq j \leq m + n - 1] \cong \{y_{ij} | 1 \leq i \leq m, 1 < j \leq n\},$$

by induction $R[M_1] \cong S_{m(n-1)}(R)$. Note that if $k > 1$, then $m < n$. By the inductive process we have $\tilde{M}_1 \in \mathfrak{M}_k$, where $k - 1 \leq k' = \left[ \frac{m+n-2}{m} \right] \leq k$. Hence $M \in \mathfrak{M}_k$. As $M$ is seminormal, the conclusion holds by Theorem 4.1.

We now prove the second part of the proof again by induction on $n$. Let $N \in PS(M)$ be a seminormal monoid. If $n = 1$, then $N = \mathbb{Z}_+ \in \mathfrak{M}_1$, by [3]. Assume $n > 1$. Then as $N$ is a $x_1$-tilted $\phi$-simplicial monoid, we only need to find a monic in $x_1$. Let $f \in R[x_1, \ldots, x_n]$ be a quasi-monic. Then there exists large enough $p$ such that the $R[x_1, x_{n+1}, \ldots, x_{m+n-1}]$-automorphism of $R[x_1, \ldots, x_{m+n-1}]$ given by $x_i \mapsto x_i + x_1^p$ for $1 \leq i \leq n$, restricts to an $R[x_1]$-automorphism $\eta$ of $R[N]$. By Lemma 4.6, $\eta(f)$ is monic in $x_1$ with coefficients in $R[x_2, \ldots, x_n] \subset R[N_1^\theta]$. As $N_1 = N_1^\theta \in PS(M_1)$ is seminormal and $R[N_1^\theta] \cong S_{m(n-1)}(R)$, by induction $N_1 \in \mathfrak{M}_n$. Thus $N \in \mathfrak{M}_n$. Since we have assumed $m \leq n$, we get $N \in \mathfrak{M}_{\max\{m,n\}} \subset \mathfrak{M}_{k(m,n)}$. \hspace{1cm} \Box

### 5.2. Rees Algebra

Let $R$ be a ring and $\mathbb{I} = \{I_n\}$ a filtration of $R$, where $I_j \subset I_{j-1}$ and $I_0 = R$. Denote by $R[I,t] = \bigoplus_{n \geq 0} I_n t^n$ and $R[\mathbb{I},t^{-1}] = \bigoplus_{n \in \mathbb{Z}} I_n t^n$, the Rees algebra and extended Rees algebra of $R$ w.r.t. $\mathbb{I}$, respectively. If $\mathbb{I}_n$ is the $I$-adic filtration of $R$, then by $R[I]$ and $R[I,t^{-1}]$ we denote the corresponding Rees algebras. These rings are also referred to as blowup algebras as $\text{Proj}(R[I])$ is the blowup of $\text{Spec}(R)$ along the subscheme defined by $I$ and have their application in the study of desingularization. From an algebraic viewpoint it was studied by Rees in [19].

In [18], Rao-Sarwar proved that $S\cdot \dim(R[I]) \leq \dim(R)$, when $R$ is a domain. The following generalizes this result when $R$ and $\mathbb{Z}_n$ have a certain form and follows as a straightforward corollary to Theorem 1.1.
Proposition 5.2. Let \( R \) be a ring of dimension \( d \) and \( B = R[X_1, \ldots, X_m] \). Let \( \mathcal{I} = \{ I_n \} \) be a filtration of \( B \), where \( I_n \subset B \) are ideals generated by non-constant monomials for all \( n \), and \( A = B[\mathcal{I}] \) or \( B[\mathcal{I}, t \mapsto 1] \). Then \( S\text{-dim}(A) \leq \max\{1, d\} \).

Proof. We may assume \( A \) is reduced. Since \( I_n \) consists of monomials in \( R[X_1, \ldots, X_m] \), \( A \) is a monoid algebra, say \( R[M] \), where \( M \) is a positive monoid. Let \( P \) be a projective \( A \)-module of rank \( > \max\{1, d\} \). As \( \text{rank}(P) \geq 2 \), we may further assume \( R[M] \) to be seminormal using Lemma 2.3. By Theorem 5.2, \( M \) is seminormal. The conclusion thus follows from Theorem 3.4.

What is of interest is, if and when we can prove that the corresponding monoid \( M \in \mathfrak{M}_n \) for \( n > 1 \). We discuss one such example:

Corollary 5.3. Let \( B = R[X_1, \ldots, X_m] \) and \( I \) = \( (X_1, \ldots, X_m) \). If \( A = R[I]\), then

\[
S\text{-dim}(A) \leq \begin{cases} 
\dim(R) + (m + 1)/2 & \text{m odd;} \\
\dim(R) + (m/2) + 1 & \text{else.}
\end{cases}
\]

Proof. Observe that \( A \cong S_{2m}(R) \). By Theorem 5.1, we have \( A = R[M] \), where \( M \in \mathfrak{M}_n \) and \( n = \left\lceil \frac{m+2-1}{2} \right\rceil \). As \( \dim(A) = \dim(R) + m + 1 \), the conclusion follows.

5.3. Monic Inversion

A ring \( R \) is said to be normal if for every prime ideal \( p \subset R \), \( R_p \) is a normal domain. Let \( A = \bigoplus_{i \geq 0} A_i \) be a positively graded ring and \( P \) be a projective \( A \)-module. The Quillen ideal of \( P \), denoted by \( J(A_0, P) \), is defined to be the set of elements \( a \in A_0 \) such that \( P_a \) is extended from \( (A_0)_a \). When \( \dim(A_0) \geq 1 \), it can be deduced from Theorem 2.1 that \( \text{ht}(J(A_0, P)) \geq 1 \). Let \( R \) be a \( d \)-dimensional normal ring and \( P \) be a projective \( R[T] \)-module of rank \( d \). Then by ([2], Theorem 5.2) the authors proved that the map \( \text{Um}(P) \to \text{Um}(P/T \cdot P) \) is surjective, if \( \text{Um}(P_f) \neq \emptyset \) for some \( f \in R[T] \) monic in \( T \). Further, using the techniques of ([1], Lemma 3.2), the assumption of normality on \( R \) can be relaxed. In ([9], Corollary 3.1) this was generalized to the ring \( R[T_1, \ldots, T_n] \) to show surjection of \( \text{Um}(P) \to \text{Um}(P/A_{+}^n \cdot P) \) subject to the existence of \( f \in R[T_1, \ldots, T_n] \) monic in \( T_1 \) with \( \text{Um}(P_f) \neq \emptyset \). We further generalize it to monoid algebras and give a proof for Theorem 1.4 using the theorem below ([2], Criterion 1):

Theorem 5.4. Let \( A = \bigoplus_{i \geq 0} A_i = A_0 \oplus A_+ \) be a positively graded ring. Let \( P \) be a projective \( A \)-module and \( J = J(A_0, P) \). If \( q \in P \) such that \( q_{1+A_+} \in \text{Um}(A_{1+A_+}) \) and \( q_{1+J} \in \text{Um}(A_{1+J}) \), then there exists a \( p \in \text{Um}(P) \) such that \( p \equiv q \) modulo \( A_{+}^n \).

Theorem 5.5. Let \( R \) be a normal ring of dimension \( d \), \( M \in \mathfrak{M}_n \) a normal \( \phi \)-simplicial monoid of rank \( r > 0 \) and \( A = R[M] \). Let \( P \) be a projective \( A \)-module of rank \( \dim(A) - n \) and \( J = J(R, P) \) be the Quillen ideal of \( P \). Assume

1. \( \text{Um}(P_f) \neq \emptyset \) for some \( f \in R[M] \) monic in \( t_1 \);
2. When \( n > 1 \), \( M \in \mathfrak{M}_n \) is such that the automorphism \( \tilde{\eta} \) obtained has the form \( \tilde{\eta}(t_i) \in t_i + M_1 \) for \( i > 1 \).

Then the map \( \text{Um}(P) \to \text{Um}(P/A_{+}^n \cdot P) \) is surjective.

Proof. We may assume \( R \) to have a connected spectrum. Let \( J = J(R, P) \subset R \). If \( d = 0 \), then \( R \) is a field and by Theorem 2.1, both \( P \) and \( P/A_{+}^n \cdot P \) are free. As \( P \) is extended from \( R[M_0] \) and \( R[M] \to R[M_0] \) is a retraction, we have the required surjection. Let \( d \geq 1 \). Let \( \sim \) denote reduction modulo \( JA \) and \( \sim_+ \) denote reduction modulo \( (J, A_{+}^n) \).

Case 1: If \( d = 1 \), then \( R \) is a regular ring. Let \( p \in \text{Spec}(R) \). As \( M \) is positive, by ([23], Theorem 1.2), \( P_p \) is extended from \( R_p \). By the graded version of Quillen’s local-global principle ([4], Theorem 8.11), \( P \) is extended
from \(A/A_+\) and hence from \(A/A_+^1\). Since \(R[M] \to R[M_0]\) is a retraction, the map \(Um(P) \to Um(P/A_+^1 P)\) is surjective.

Case 2: Let \(d > 1\). Next we want to prove \(ht(J) \geq 2\). Let if possible \(p \in Spec(R)\) be a height 1 minimal prime of \(J\). Then \(R_p\) is a PID by and Theorem 2.1, \(P_p\) is free. This would imply the existence of \(s \in J \cap (R \setminus p)\), a contradiction. Therefore \(ht(J) \geq 2\). Let \(p_2 \in Um(P/A_+^1 P)\). As \(M_0^1 \in \mathfrak{M}_{n-1}\), \(dim(R/J) \leq d-2\) and \(rank(\tilde{P}) = d + r - n > (d - 2) + r - n\), by Theorem 4.4, the map \(Um(\tilde{P}) \to Um(\tilde{P})\) is onto. Choose \(p_1 \in Um(\tilde{P})\) such that \(\pi_1(p_1) = \pi_2(p_2)\). Consider the following patching diagram:

\[
\begin{array}{ccc}
A/\mathcal{J}A_+^1 A & \xrightarrow{\theta_1} & A/\mathcal{J}A \cong (R/JR)[M] \\
\downarrow{\pi_2} & & \\
A/\mathcal{J}A_+^1 \cong R[M_0^1] & \xrightarrow{\pi_2} & A/(J, A_+^1)A \cong (R/JR)[M_0^1].
\end{array}
\]

Then there exist \(p' \in Um(P/A(\mathcal{J}A_+^1 P)\) such that \(\theta_1(p') = p_1 \in Um(\tilde{P})\) and \(\theta_2(p') = p_2 \in Um(P/A_+^1 P)\). Using \(p_1 \in Um(\tilde{P})\) we may decompose \(\tilde{P} = A\mathcal{P}_1 \oplus Q\). Let \(q \in P\) be such that \(J \subset \theta_1(q)\). Using the decomposition above write \(q = (\tilde{a}_1, q')\). As a consequence of Eisenbud-Evans in [16] there exists a transvection \(\tau \in Aut(P)\) such that \(\tau(q) = (\tilde{a}_1, q')\) and \(ht(\tilde{A}_0(\tilde{O}_Q(q'))) \geq rank(P) - 1 = d - 1 + r - n\). From ([3], Proposition 4.1) we may lift \(\tau\) to \(\tau \in Aut(P)\). On replacing \(P\) by \(\tau(P)\), we may assume \(ht(\tilde{A}_0(\tilde{O}_Q(q'))) \geq rank(P) - 1 = d - 1 + r - n\).

Claim: \(\tilde{O}_Q(q')\) contains a monic in \(t_1\).

Let \(\{p_1, \ldots, p_\ell\}\) be minimal primes of \(\tilde{O}_Q(q')\) not containing \(\tilde{a}\). Then \(ht(\cap p_i) \geq ht(\tilde{A}_0(\tilde{O}_Q(q'))) \geq rank(P) - 1 = d - 1 + r - n\). Let \(M = \mathbb{Z}[W_1, \ldots, W_\ell] \subset \mathbb{Z}[t_1, \ldots, t_\ell]\). As \(M\) is \(\phi\)-simplicial, the first \(r\) elements can be chosen to be \(t_{i_1}\) for some \(s_i \in \mathbb{Z}\). Consider the composition of maps

\[
(R/JR)[X_1, \ldots, X_\ell] \xrightarrow{\beta_j} (R/JR)[X_1, \ldots, X_\ell] \xrightarrow{\beta} (R/JR)[M] \cong \tilde{A},
\]

where \(\beta(X_i) = W_i\) for all \(i\). If \(n = 1\), then \(\cap p_i\) contains monic in \(t_1\) with coefficients in \(R/J\). Therefore all minimal primes of \(O_Q(q')\) and hence \(O_Q(q')\) contains a monic in \(t_1\) with coefficients in \(R/J\).

Let \(n > 1\). As \(dim(R/JR) \leq d - 2\), by Lemma 3.1 there exists quasi-monic \(g \in \cap p_i \cap (R/JR)[t_1^{p_1}, \ldots, t_\ell^{p_\ell}]\). Since \(M \in \mathfrak{M}_n\), there exists an \((R/JR)[t_i]\)-automorphism \(\tilde{g}\) such that \(\tilde{g}(g) \in \cap p_i\) is monic in \(t_i\) with coefficients in \((R/JR)[M_0]\). By (2), we may lift \(\tilde{g}\) to \(g \in Aut(R[t_i][R[M]])\) and replace \(A\) by \(A\). If \(\mathfrak{p}\) a minimal prime ideal of \(O_Q(q')\) containing \(\tilde{a}\), then \(Op(\mathfrak{p}) \subset \mathfrak{p}\). As \(\mathfrak{p}\) preserves monic in \(t_1\), \(\mathfrak{p}(f)\) is again monic in \(t_1\). Therefore all minimal primes of \(O_Q(q')\) and hence \(O_Q(q')\) contains a monic in \(t_1\) with coefficients in \((R/J)[M_0]\).

Let \(p \in P\) be a lift of \(p' \in P/A(\mathcal{J}A_+^1 P)\). In the final step we shift \(p\) to \(p_0 = p + t_1^N g g q\) for a well chosen \(N\) and show that \((p_0)_{1 + A_+^1} \in Um(P_1 + A_+^1)\) and \((p_0)_{1 + J(R[M_0], P)} \in Um(P_1 + J(R[M_0], P))\). We arrive at our desired conclusion by invoking Theorem 5.4.

We can choose \(N\) large enough so that \(Op(p_0)\) contains a monic in \(t_1\), say \(h\). As \(\tilde{g} = p_1\) we get \(\tilde{g} = (\tilde{a} \tilde{q}', q')\). Thus \(\tilde{g}_0 = ((1 + t_1^{N - \tilde{g}} \tilde{g}) \tilde{p}', t_1^{N} g q')\). Since \(O_Q(q')\) contains \(\tilde{g}\), we have \(Op(p_0) \subset Op(\tilde{g}_0)\). Which in turn implies \(\tilde{p}_0 \in Um(\tilde{P})\) and thus \((p_0)_{1 + A_+} \in Um(P_1 + A_+^1)\). By ([6], Lemma 6.3) the extension \(R[M_0] \to R[M_0]\) is integral and thus \((R/J)[M_0]/(Op(p_0) \cap R[M_0]) \to (R/J)[M]/Op(p_0)\) is integral. As \(Op(p_0)\) and \(J\) are comaximal in \(A\), we have \(Op(p_0) \cap R[M_0] \cap JR[M_0]\) are comaximal in \(R[M_0]\). Therefore \(p_0 \in Um(P_{1 + J(R[M_0])}) \subset Um(P_1 + J(R[M_0], P))\) (Note that \(JR[M_0] \subset J(R[M_0], P))\).

As \((p_0)_{1 + A_+^1} \in Um(P_1 + A_+^1)\), by Theorem 5.4, there exists \(p_3 \in Um(P)\) such that \(p_3 \equiv p_0\) modulo \(A_+^1 P\). As \(p_0 \equiv p_2\) modulo \(A_+^1 P\), the surjection follows.

Note that in all of the examples discussed before, the automorphisms occurring out of \(M\) being in \(\mathfrak{M}_n\) satisfy the condition (2) above. The above theorem results in a slew of corollaries.
Corollary 5.6. Let $R$ be a normal ring of dimension $d$, $M \in C(\phi)$ a normal monoid of rank $r > 0$ and $A = R[M]$. Assume $P$ to be a projective $A$-module of rank $d$. If $Um(P_f) \neq \emptyset$ for some $f \in R[M]$ monic in $t_1$, then the map $Um(P) \to Um(P/A_1^1 P)$ is surjective.

Proof. Since $M \in C(\phi) \subset \mathfrak{M}_r$, by definition, for $2 \leq i \leq r$, $\exists c_i \in \mathbb{N}$ and $\tilde{\eta} \in Aut_{R[M]}(R[t_1, \ldots, t_i])$ given by $\tilde{\eta}(t_i) \mapsto t_i + t_i^c_i$.

$M$ clearly satisfies the second condition of the hypothesis of Theorem 5.5 and thus the result follows.

Corollary 5.7. Let $R$ be a normal ring of dimension $d$, $M \in C(\phi)$ a normal monoid of rank $r > 0$ and $A = R[M]$. Assume $P$ to be a projective $A$-module of rank $d$. If for each $i$, there exists $f_i \in R[M]$ monic in $t_i$ such that $Um(P_{f_i}) \neq \emptyset$, then the map $Um(P) \to Um(P/A_1^1 P)$ is surjective.

Proof. This follows from the above corollary by inducting on the rank of the monoid. If $r = 0$, then $M = 0$ and we are done. Let $r > 0$ and $-\vdash$ denote reduction modulo $A_1^1$. By induction on $R[M_0^1]$, we get $Um(P) \to Um(P/(R[M_1^1])_+) \simeq Um(P/A_1^1 P)$.

is surjective. The previous corollary gives $Um(P) \to Um(P)$ is surjective. This proves the required.

The following was proved in ([2], Theorem 5.1) when $M$ is free:

Corollary 5.8. Let $R$ be a normal ring of dimension $d$. Let $B$ be a birational overring of $R[X]$ and $M$ be a normal monoid of rank $2$. Then $S\text{-dim}(B[M]) \leq d$.

Proof. Let $P$ be a projective $B[M]$-module of rank $d + 1$. As $M$ is $\phi$-simplicial, there exists $p_i \in \mathbb{Z}_{>0}$, such that $t_i^{p_i} \in M$, for $i = 1, 2$. Choose $f_i = t_i^{p_i}$ for all $i$. Since $C[f_i^{-1}M] \simeq C[M]$, $M$ is normal, from ([4], Theorem 4.40) we may infer $f_i^{-1}M$ is a normal monoid for all $i$. From ([4], Proposition 2.26) we can further deduce that $f_i^{-1}M \simeq \mathbb{Z} \oplus \mathbb{Z}_i$. By ([2], Theorem 5.1), $Um(P_{f_i}) \neq \emptyset$, for $i = 1, 2$. Thus using Corollary 5.7 we can conclude $Um(P) \neq \emptyset$.

Corollary 5.9. Let $R$ be a ring of dimension $d > 1$, $M \in \mathfrak{M}_n$ a normal $\phi$-simplicial monoid of rank $r > 0$ and $A = R[M]$. Let $P$ be a projective $A$-module of rank $\dim(A) - n$, and $J = J(R, P)$ be the Quillen ideal of $P$ of height$(J) > 1$. Assume

1. $Um(P_f) \neq \emptyset$ for some $f \in R[M]$ monic in $t_1$;
2. When $n > 1$, $M \in \mathfrak{M}_n$ is such that the automorphism $\tilde{\eta}$ obtained has the form $\tilde{\eta}(t_i) \in t_i + M_1$ for $i > 1$.

Then the map $Um(P) \to Um(P/A_1^1 P)$ is surjective.

Proof. This is a restatement of Theorem 5.5, except here the restriction of $R$ normal is removed and additional conditions of $ht(J) > 1$ and $d > 1$ is added. In proof of Theorem 5.5, the normality of $R$ is used in two places, first in the case $d = 1$, and then in case $d \geq 2$ to show $ht(J) \geq 2$. The rest is identical to the proof of Theorem 5.5.

The following (also proved in ([9], Corollary 3.1)) is a consequence of the above corollary:

Corollary 5.10. Let $R$ be a ring of dimension $d$ and $A = R[T_1, \ldots, T_n]$. If $P$ is a projective $A$-module of rank $d$ such that $Um(P_f) \neq \emptyset$ for some $f \in A$ monic in $T_1$. If $Um(P/A_1^1 P) \neq \emptyset$, then $Um(P)$.

Proof. We may assume $R$ is a reduced ring. If $d = 1$, then $P_f$ is free. Applying ([17], Theorem 3) and ([21], Theorem 1) to the ring $(R[T_2, \ldots, T_n])[T_1]$, we get $P$ is free. Let $d = 2$. As $\text{rank}(P) = 2$, by ([1], Proposition 3.3), $Um(P) \neq \emptyset$. 

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Let \( d \geq 3 \) and \( A' = sn(R)[T_1, \ldots, T_n] \). Denote by \( D \) the determinant of \( P, P' = P \otimes A' \) and by 
\( J' = J(sn(R), P') \) the Quillen ideal of \( P' \). Choose \( p \in Spec(sn(R)) \) of height 1 and the multiplicative subset 
\( S = sn(R) \setminus p \). Further, \( dim(sn(R)_p) = 1 \) and \( S^{-1}A' = sn(R)_p[T_1, \ldots, T_n] \). Since \( rank(P) \geq 3 \), 
\[ S^{-1}(P') \simeq S^{-1}(D \otimes A') \oplus (S^{-1}A')^{d-1}, \]
by [3]. By [22], \( D \otimes A' \) is extended from \( sn(R) \). This in turn implies \( S^{-1}(D \otimes A') \) is extended from the local 
ring \( sn(R)_p \) and therefore \( S^{-1}(D \otimes A') = S^{-1}A' \). Therefore \( ht(J') \geq 2 \). Assume \( Um(P/A'_1 P) \neq \emptyset \). As 
\[ P/A_1^2 P \simeq P \otimes R[T_2, \ldots, T_n] \]
\[ P'/A'_1 P' \simeq P \otimes sn(R)[T_2, \ldots, T_n], \]
we have \( Um(P'/A'_1 P') \neq \emptyset \). Apply hypothesis of Corollary 5.9 to the ring \( sn(R) \) and 
\( M = \mathbb{Z}_+[T_1, \ldots, T_n] \). Since conditions (1) and (2) hold, we get \( Um(P') \neq \emptyset \). By Lemma 2.3 and Theorem 2.4, \( Um(P') \neq \emptyset \). \( \square \)

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