Group Analysis of Non-autonomous Linear Hamiltonians through Differential Galois Theory.

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Abstract

In this paper we introduce a notion of integrability in the non autonomous sense. For the cases of $1 + \frac{1}{2}$ degrees of freedom and quadratic homogeneous Hamiltonians of $2 + \frac{1}{2}$ degrees of freedom we prove that this notion is equivalent to the classical complete integrability of the system in the extended phase space. For the case of quadratic homogeneous Hamiltonians of $2 + \frac{1}{2}$ degrees of freedom we also give a reciprocal of the Morales-Ramis result. We classify those systems by terms of symplectic change of frames involving algebraic functions of time, and give their canonical forms.

Keywords: Hamiltonian Systems, Integrability, Differential Galois Theory.

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1 Introduction and Main Results

Differential Galois theory has been fruitful applied to the study of integrability of Hamiltonian systems, see for instance [10, 11, 12, 15, 13, 14, 4] and many others. In a recent paper [1], this Morales-Ramis approach is applied to the study of non-autonomous Hamiltonian systems. In order to do that, we extend a non-autonomous Hamiltonian system of $n + \frac{1}{2}$ degrees of freedom to an autonomous system of $n + 1$ degrees of freedom, and we apply Morales-Ramis techniques to this last one. However, there is a number of open questions here. How do the integrability of the original and extended systems relate? The variational equation itself is a linear non-autonomous Hamiltonian system, but its extension is not linear anymore. The Morales-Ramis approach provide us necessary condition for the integrability. However, in the linear context. Do the Morales-Ramis approach give us sufficient conditions for the integrability? We arrive to the following result.
Theorem 4.1. Let \( H \in \overline{M(\Gamma)}[x_1, x_2, y_1, y_2]_2 \) be a quadratic homogeneous non-autonomous Hamiltonian of \( 2 + \frac{1}{2} \) degrees of freedom, with coefficients meromorphic in \( \Gamma \) a ramified covering of \( \Gamma \). The following are equivalent:

1. The associated extended autonomous system \( \hat{H} = H + h \) is completely integrable by meromorphic functions in \( \hat{\Gamma} \times V \times \mathbb{C}_h \) for some ramified covering \( \hat{\Gamma} \) of \( \Gamma \).

2. \( H \) is integrable in the non-autonomous sense by meromorphic functions in \( \hat{\Gamma} \times V \) for some ramified covering \( \hat{\Gamma} \) of \( \Gamma \).

3. \( H \) is integrable in the non-autonomous sense by quadratic first integrals \( F_1, F_2 \in \overline{M(\Gamma)}[x_1, x_2, y_1, y_2]_2 \).

4. The connected component of the Galois group of \( \hat{X}_H \) is a abelian.

In general, the classification of integrable systems is, in general an interesting and difficult problem. See for instance [16], for related results. In this work, we arrive to a classification of all integrable quadratic (and non-linear in \( t \)) \( 2 + \frac{1}{2} \) degrees of freedom Hamiltonians. The classification we give here is related with to Williamson’s canonical forms for autonomous quadratic Hamiltonians ([17], also in [2], appendix 6), but it is not an extension of it. Let us recall that Williamson classification is a classification under real independent of time changes of frame.

In the list below, canonical forms are listed up to a factor of proportionality in \( \overline{M(\Gamma)} \) which does not alter the integrability class. However, we should notice that, as can be easily seen in the list, for certain specific values of \( f(t) \in \overline{M(\Gamma)} \) in the cases (3), (4) and (5), the Galois group collapses and we fall into the cases (1) or (2).

Theorem 6.1. Let \( H(t, x_1, x_2, y_1, y_2) \in \overline{M(\Gamma)}[V]_2 \) be an integrable quadratic homogeneous Hamiltonian of \( 2 + \frac{1}{2} \) degrees of freedom. Then, there exist a symplectic change of frame,

\[
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\eta_1 \\
\eta_2
\end{pmatrix} = B(t) \begin{pmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2
\end{pmatrix}
\]

with \( B(t) \in \text{Sp}(4, \overline{M(\Gamma)}) \) such that, for the transformed Hamiltonian \( \tilde{H}(\xi_1, \xi_2, \eta_1, \eta_2) \),

\[
\tilde{H} = H - (\xi_1, \xi_2, \eta_1, \eta_2)J\hat{B}B^{-1}J = \begin{pmatrix}
\xi_1 \\
\xi_2 \\
\eta_1 \\
\eta_2
\end{pmatrix}, \quad J = \begin{pmatrix}
-I \\
I
\end{pmatrix}
\]

belongs to one of the following categories:
| Normal Form | Galois | Quadratic Invariants | Parameters |
|-------------|--------|----------------------|------------|
| $f(t)(\xi_1\eta_1 + \frac{p}{q}\xi_2\eta_2)$ | $\mathbb{C}^*$ | $\xi_1\eta_1, \xi_2\eta_2$ | $f(t), \frac{p}{q}$ |
| $f(t)(\xi_1\eta_1 + \xi_2\eta_2)$ | $\mathbb{C}^*$ | $\xi_1\eta_1, \xi_2\eta_2, \xi_1\eta_2 - \xi_2\eta_1$ | $f(t)$ |
| $f(t)\xi_1\eta_1$ | $\mathbb{C}^*$ | $\xi_1\eta_1, \xi_2^2, \eta_2^2, \xi_2\eta_2$ | $f(t)$ |
| $f(t)\eta_1^2 + \eta_2^2$ | $\mathbb{C}$ | $\eta_1^2, \xi_2^2, \xi_2\eta_2, \eta_2^2$ | $f(t)$ |
| $f(t)(\xi_2\eta_1 + \lambda\eta_1^2 + \frac{\lambda}{q}\eta_2^2)$ | $\mathbb{C}$ | $2\xi_2\eta_1 + \eta_2^2, \eta_1^2$ | $f(t), \lambda$ |
| $f(t)\xi_1\eta_1 + g(t)\xi_2\eta_2$ | $(\mathbb{C}^*)^2$ | $\xi_1\eta_1, \xi_2\eta_2$ | $f(t), g(t)$ |
| $f(t)\frac{\eta_1^2 + g(t)\eta_2^2}{2}$ | $\mathbb{C} \times \mathbb{C}^*$ | $\eta_1^2, \xi_2\eta_2$ | $f(t), g(t)$ |
| $f(t)\frac{\eta_1^2 + g(t)\eta_2^2}{2}$ | $\mathbb{C}^2$ | $\eta_1^2, \eta_2^2$ | $f(t), g(t)$ |
| $f(t)\eta_1\xi_2 + g(t)\eta_1 + \frac{\eta_2^2}{2}$ | $\mathbb{C}^2$ | $2\eta_1\xi_2 + \eta_2^2, \eta_1^2$ | $f(t), g(t)$ |

Where $f(t)$ and $g(t)$ are arbitrary meromorphic functions, $\lambda$ is an arbitrary constant, and $p, q$ are coprime integers.

2 Main Concepts

We are interested in non-autonomous Hamiltonian systems. By technical reasons, we are going to consider the coefficients of those Hamiltonians to be multivalued meromorphic functions of time, for which we will allow finite ramification points. Therefore we will consider a Riemann surface $\Gamma$ endowed with a meromorphic derivation. In most cases $\Gamma$ will be an open subset of the complex projective line $\mathbb{C}$. The field $\mathcal{M}(\Gamma)$ of meromorphic functions on $\Gamma$ is then a differential field (see Section 3) and so is its algebraic closure $\overline{\mathcal{M}(\Gamma)}$. By abuse of notation, we will denote this field by $\mathcal{M}(\Gamma)$ and its dual 1-form as $dt$, even in the case in which there is no $t \in \mathcal{M}(\Gamma)$ (for instance, its happens when $\Gamma$ is a complex torus).

The construction that we are going to give here is, in fact, suitable for for coefficients in any differential field of characteristic zero with algebraically closed field of constant.

Let $V$ be a symplectic vector space of complex dimension $2n$, endowed with a system of symplectic coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the symplectic form on $V$ is written,

$$\Omega_2 = \sum_{i=1}^{n} dx_i \wedge dy_i. \quad (2.1)$$

The ring $\mathcal{M}(V)$ of meromorphic functions on $V$ in then endowed with a Poisson Bracket,

$$\{F, G\} = \sum_{i=1}^{n} (F_{x_i}G_{y_i} - F_{y_i}G_{x_i}). \quad (2.2)$$
2.1 Non-autonomous Hamiltonians

We want to consider Hamiltonian systems depending on time through algebraic functions. Therefore, we consider $\Gamma$ to be any ramified covering of $\Gamma$ and the field $\mathbb{M}(\Gamma \times V)$ of meromorphic functions on $\Gamma \times V$. This manifold, $\Gamma \times V$, is not a symplectic manifold. However, we can consider the 2-form $\Omega_2$ of equation (2.1), which becomes now a degenerated 2-form, and the Poisson bracket induced, that we denote using a double bracket:

$$\{\{F, G\}\} = \sum_{i=1}^{n} \left( F_{x_i} G_{y_i} - F_{y_i} G_{x_i} \right).$$

For a given $H \in \mathbb{M}(\Gamma \times V)$, there exist a unique meromorphic vector field $\vec{X}_H$ in $\Gamma \times V$ such that,

$$i_{\vec{X}_H} \Omega_2 = H dt - dH, \quad \langle \vec{X}_H, dt \rangle = 1. \quad (2.3)$$

We call $\vec{X}_H$ the Hamiltonian vector field associated with the Hamiltonian function $H$. The equation for integral curves of $\vec{X}_H$ are the usual Hamilton-Jacobi equations of motion:

$$\frac{dx_i}{dt} = H_{y_i}, \quad \frac{dy_i}{dt} = -H_{x_i}.$$

It is important to remark that the definition of $\vec{X}_H$ is sensible to changes of frame in the bundle $\Gamma \times V$. As it is written in formula (2.3), the vector field $\vec{X}_H$ depends on the derivative of $H$ with respect to $t$. So that, we understand that the vector field $\frac{\partial}{\partial t}$ is given as a data of the problem. In general, we can write the motion equation:

$$\frac{d\xi}{dt} = \{\xi, H\} + \xi_t, \quad \xi \in \mathbb{M}(\Gamma \times V).$$

We are specifically interested in non-autonomous quadratic homogeneous Hamiltonians $H \in \mathbb{M}(\Gamma)[V]_2$. They are written as,

$$H = \sum_{i=1, j=1}^{n} a_{ij} \frac{x_i x_j}{2} + b_{ij} \frac{y_i y_j}{2} + c_{ij} x_i y_j,$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are symmetric and $C = (c_{ij})$ is square matrix with coefficients in $\mathbb{M}(\Gamma)$.

For such Hamiltonians, the vector field $\vec{X}_H$ is a meromorphic vector field in $\Gamma \times V$ for a suitable ramified covering $\Gamma$ of $\Gamma$ and the equations of motion

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C^t & B \\ -A & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (2.4)$$

form a system of $2n$ linear differential equations that can be investigated from the standpoint of differential Galois theory.
2.2 Non-autonomous Integrability

We present a definition of complete integrability for non-autonomous Hamiltonian systems. Let us consider $H \in M(\Gamma \times V)$. For the understanding of our definition is essential to note that the ramified covering $\tilde{\Gamma}$ of $\Gamma$ we consider is just geometric tool we need in order to deal with finitely-many valued functions of $t$. Whenever new algebraic functions appear we just consider a new ramified covering $\hat{\Gamma}$ of $\tilde{\Gamma}$ and lift up all the structure. We recall that a ramified covering induces a natural algebraic field extension, $M(\Gamma \times V) \subset M(\hat{\Gamma} \times V)$ compatible with all the differential calculus we are using here.

Hence, in the following definition we should say integrability by functions which are meromorphic on $V$ and algebraic (finitely-many valued) on $t$. However, in order to make the exposition clear we will write just integrable in the non-autonomous sense.

**Definition 2.1.** Let $H \in M(\Gamma \times V)$ a Hamiltonian function. We say that $H$ is integrable in the non-autonomous sense if there exist a ramified covering $\hat{\Gamma}$ of $\Gamma$ and functions $F_1, \ldots, F_n \in M(\hat{\Gamma} \times V)$ such that:

1. $\{\{F_i, F_j\}\} = 0$
2. $\vec{X}_H F_i = 0$
3. $F_1, \ldots, F_n$, and $t$ are functionally independent.

We say that $F_1, \ldots, F_n$ form a complete system of first integrals for $H$. It is well know that, by algebraic reasons, we can never have more that $n$ independent first integrals in involution. However, is some cases it is possible to find different complete systems of first integrals for a given Hamiltonians. Non-autonomous Hamiltonian systems with this properties are usually called superintegrable. For an algebraic treatment of the superintegrability of autonomous Hamiltonian systems see [9]

2.3 Extended autonomous System

Let us consider $H \in M(\Gamma \times V)$. There is a classical way to extend the phase space in such a way that we obtain a new autonomous Hamiltonian system in $n + 1$ degrees of freedom which is, in many ways, equivalent to our original non-autonomous system. When the integrability of a non-autonomous Hamiltonian system is studied, usually it done in this way. It is the extended system the one which is investigated. See, for instance [1].

We consider a new variable $h$ called dissipation that can take any arbitrary complex value. Then, we are considering a new phase space, $\Gamma \times V \times \mathbb{C}_h$. This phase space is a symplectic manifold endowed with the symplectic form,

$$\hat{\Omega}_2 = \Omega_2 + dt \wedge dh.$$
We consider the extended Hamiltonian $\tilde{H}$ defined:

$$\tilde{H} = H + h.$$ 

The autonomous Hamiltonian vector field $\vec{X}_H$, gives us the following equation of motion,

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad \dot{h} = -\frac{\partial H}{\partial t}, \quad i = 1. \quad (2.5)$$

Let us consider the natural projection $\pi: \tilde{T} \times V \times \mathbb{C}_h \to \tilde{T} \times V$. It is clear, from the equations (2.5) that the vector field $\vec{X}_H$ is projectable by $\pi$ and it is projected onto $\vec{X}_H$.

On the other hand, let $\lambda$ be any complex constant value. We can consider the hypersurface $H_\lambda = \{(x, y, t, h) \in \tilde{T} \times V \times \mathbb{C}_h : H(x, y, t) + h = \lambda\}$, which is a constant energy hypersurface. It is also clear that $H_\lambda$ is isomorphic by $\pi$ with $\tilde{V} \times \mathbb{T}_h$, and that this isomorphism conjugates $\vec{X}_H|_{H_\lambda}$ with $\vec{X}_H$.

As a first test for our definition, we expect the extended autonomous system associated to an integrable non-autonomous system to be a completely integrable Hamiltonian system. This fact is easy to verify.

**Proposition 2.1.** Assume that $H$ is integrable in the non-autonomous sense, and let $\tilde{\Gamma}$ be a ramified covering of $\Gamma$ such that there is a complete systems of first integrals of $\vec{X}_H$ in $M(\tilde{\Gamma} \times V)$. Then, $\tilde{H}$ is completely integrable by meromorphic functions in $\tilde{\Gamma} \times V \times \mathbb{C}_h$.

**Proof.** Let us consider $F_1, \ldots, F_n$ a complete system of first integrals in involution of $\vec{X}_H$. Then, it is clear that $\tilde{H}, F_1, \ldots, F_n$ are in involution, and they are meromorphic functionally independent functions in $\tilde{\Gamma} \times V \times \mathbb{C}_h$. 

However, the reciprocal does not hold in general. Let us assume that $\vec{X}_H$ is completely integrable by meromorphic functions. Then, there exist a complete system of first integrals including $\tilde{H}$, take $F_1, \ldots, F_n, \tilde{H}$. We take the energy level hypersurface $H_\lambda$. We can construct first integrals of $\vec{X}_H$ easily by restricting the first integrals we had to $H_\lambda$ which is isomorphic to $\tilde{\Gamma} \times V$. We get the first integrals, $F_i(x, y, t) = F_i(x, y, t, \lambda - H(x, y, t))$.

Then we have,

$$\{\{\tilde{F}_i, \tilde{F}_j\} = \{\{F_i, F_j\} + \frac{\partial F_i}{\partial h}\{F_j, H\} + \frac{\partial F_j}{\partial h}\{F_i, H\},$$

and therefore the $\tilde{F}_i$ are not in general in involution. In spite of it, we can can state the reciprocal for the case of Hamiltonians of $1 + \frac{1}{2}$ degrees of freedom, in which the hypothesis of involution is not necessary.
Theorem 2.1. Let be $H \in \mathcal{M}(\Gamma \times \mathbb{C}_{x,y}^2)$ a non-autonomous Hamiltonian of $1 + \frac{1}{2}$ degrees of freedom. Then $H$ is integrable in the non-autonomous sense by meromorphic functions in $\tilde{\Gamma} \times \mathbb{C}_{x,y}^2$ if and only if its associated autonomous extended Hamiltonian $\tilde{H} = H + h$ is completely integrable by meromorphic functions in $\tilde{\Gamma} \times \mathbb{C}_{x,y}^2 \times \mathbb{C}_h$.

3 Differential Galois Theory

The differential Galois theory deals with the integrability by quadratures of systems of linear differential equations. In this section we will develop only the part of the theory we need for our purposes, and we will give no proofs of the facts we expose. The interested reader may consult more complete references like [7, 8, 10, 3].

Let $K$ be a field of characteristic zero. A derivation in $K$ is an additive map $\partial : K \rightarrow K$ which satisfies the Leibniz rule

$$\partial(ab) = b\partial(a) + a\partial(b).$$

A differential field is a pair $(K, \partial_K)$ consisting on a field and a derivation on it. By abuse of notation we will write $K$ instead of the pair $(K, \partial_K)$ whenever it does not lead to confusion.

Given a differential field $K$, we denote by $C(K)$ the field of constants of $K$ which consists of the elements $a \in K$ such that $\partial_K a$ vanish. From now on we will consider always a differential field $K$ whose field of constants is an algebraically closed field $\mathbb{C}$ of characteristic zero. In most examples we are going to consider, this field of constants is the field of the complex numbers.

Example 3.1. Let us consider a non-autonomous Hamiltonian $H \in \mathcal{M}(V \times T)$, as in subsection 2.1. Then, the field $\mathcal{M}(V \times T)$ endowed with the derivation,

$$\partial F = -\{H, F\} = \vec{X}_H F$$

is a differential field. Its field of constants consist of the meromorphic first integrals of the Hamiltonian vector field $\vec{X}_H$.

3.1 Picard-Vessiot Extensions

An extension of differential fields $K \subset L$ is an inclusion of $K$ in $L$ which is an extension of fields and $\partial_L|_K = \partial_K$. In what follows all extensions of fields we consider are differential extensions.

Example 3.2. Let $\overline{K}$ be the algebraic closure of $K$. Then, the derivation extends to $\overline{K}$ in a unique way and $K \subset \overline{K}$ is a differential field extension. Furthermore, $C(\overline{K}) = \mathbb{C}$ since $\mathbb{C}$ is algebraically closed.
Let $K \subset L$ be a differential field extension. A differential automorphism of $L$ over $K$ is a field automorphism $\sigma$ of $L$ which commutes with the derivation and fix $K$ point-wise. The set of all differential automorphisms of $L$ over $K$ is clearly a group that we denote by $\text{Aut}_K(L)$.

Let us consider a system of linear homogeneous differential equations with coefficients in $K$,

\[ y' = Ay, \quad A \in \mathfrak{gl}(n, K) \tag{3.6} \]

and an extension $K \subset E$. The set of solution of (3.6) in $E^n$ form a subset $S_E \subset E^n$ which is a vector space over $C(E)$ of dimension leq or equal than $n$. We denote by $K(S_E)$ to the smallest differential field containing both $K$ and the coordinates of elements of $S_E$.

A differential field extension $K \subset L$ is called a Picard-Vessiot extension for (3.6) if the following conditions hold:

1. There is a fundamental matrix of solutions of (3.6) in $\text{GL}(n, L)$.
2. $L$ is generated over $K$ by the solutions of (3.6), id est, $L = K(S_L)$.
3. There is no new constants in $L$, $C(L) = C$.

Any system of linear homogeneous differential equations as (3.6) admits a Picard-Vessiot extension. Those extensions are unique up to an isomorphism that fixes $K$ point-wise. Therefore, we will speak on the Picard-Vessiot extension associated with (3.6).

In general, a differential field extension $K \subset L$ is called a Picard-Vessiot extension if it is a Picard-Vessiot for certain system of linear homogeneous equations with coefficients in $K$. If $K \subset L$ is a Picard-Vessiot extension then for any intermediate extension $K \subset L_1 \subset L$ we have that $L_1 \subset L$ is also a Picard-Vessiot extension. A remarkable property of Picard-Vessiot extensions is the normality, for any $a \in L$ not in $K$ there exist an automorphism $\sigma \in \text{Aut}_K(L)$ such that $\sigma(a) \neq a$.

### 3.2 Differential Galois Group

Let $K \subset L$ a Picard-Vessiot extension with common field of constants $C$ for the system (3.6). Let us consider $S_L \subset L^n$ the set of solutions of such system. It is a $C$-vector space. Any differential automorphism $\sigma$ fix point-wise $K$ and then let the equations (3.6) invariant. Therefore, it induces a $C$-linear map $\phi_\sigma: V \to V$. This gives us a faithful representation,

\[ \text{Aut}_K(L) \to \text{GL}(S_L, C). \]

The image of this map is called the differential Galois group of the system (3.6). By abuse of notation it will be denoted by $\text{Gal}_L/K$. However, let us recall that this group, is associated to the equation, not to the differential field extension.
The most remarkable property of $\text{Gal}_{L/K}$ is that it is an linear algebraic subgroup of $\text{GL}(S_L, \mathbb{C})$. A linear algebraic subgroup is just a matrix group defined by polynomial equations in the matrix elements. In linear algebraic groups is natural to consider the Zariski topology, for what closed subsets are defined by polynomial equations. With this topology any algebraic group has a finite number of connected components and the connected component which contains the identity is the biggest normal algebraic subgroup of finite index.

Let $G \subset \text{Gal}_{L/K}$ be an algebraic subgroup. We can assign to it the intermediate extension $K \subset L^G \subset L$, being $L^G$ the field of elements that are fixed by $G$. Reciprocally, for any intermediate extension $K \subset F \subset L$ the group $G_F$ of automorphisms of $L$ that fix $F$ point-wise is an algebraic subgroup $G \subset \text{Gal}_{L/K}$.

**Proposition 3.1.** The assignation $G \sim L^G$ gives a one to one correspondence between the set of all algebraic subgroups of $\text{Gal}_{L/K}$ and intermediate extensions $K \subset L$. A subgroup $G$ is normal in $\text{Gal}_{L/K}$ if and only if $K \subset L^G$ is a Picard-Vessiot extension. In such case its Galois group is $\text{Gal}_{L/K}/G$.

Note that a Picard-Vessiot extension is algebraic if only if its Galois group is finite and purely transcendental if and only if its Galois group is connected. Let us denote $\text{Gal}^0_{L/K}$ to the connected component of the identity of the Galois group of $L$ over $K$. Using the Proposition 3.1 we can split out the extension in two Picard-Vessiot extensions, $K \subset K^0 \subset L$ such that $K \subset K^0$ is an algebraic extension with finite Galois group $\text{Gal}_{L/K}/\text{Gal}^0_{L/K}$ and $K^0 \subset L$ is purely transcendental with connected Galois group $\text{Gal}^0_{L/K}$.

In particular, if $K$ is algebraically closed, any Picard-Vessiot extension has connected Galois group.

For an algebraic subgroup $G \subset \text{GL}(n, \mathbb{C})$ with Lie algebra $g \subset \text{gl}(n, \mathbb{C})$ and any field extension $C \subset F$ we will denote by $G(F)$ the algebraic subgroup of $\text{GL}(n, F)$ defined by the same equations and by $g(F)$ its Lie algebra. The Galois group of a system (3.6) is bounded by the matrix of coefficients. If this matrix takes values in a fixed Lie subalgebra of $g(n, \mathbb{C})$ the Galois group can not growth beyond. The following result is well known and can be found in [3].

**Proposition 3.2.** Let $G \subset \text{GL}(n, \mathbb{C})$ be an algebraic subgroup of $\text{GL}(n, \mathbb{C})$ and let $g$ its Lie algebra. Let us consider a system of equations,

$$y' = Ay, \quad A \in g(K) \quad (3.7)$$

such that the matrix of coefficients $A$ relies in the Lie algebra of $G$, and $K \subset L$ its Picard-Vessiot extension. We can take a basis of $S_L$, the space of solutions of (3.7) in $L^n$, in such way that, $\text{Gal}_{L/K} \subset G$.
3.3 Lie-Kolchin Reduction

Let us consider the system (3.6), and $\mathbb{K} \subset \mathbb{K}_1$ a field extension with no new constants. We can take a change of variables, $z = By$, with $B \in \text{GL}(n, \mathbb{K}_1)$ obtaining a new equation for $z$,

$$z' = (B'B^{-1} + BAB^{-1})z \quad (3.8)$$

where the new matrix of coefficients $(B'B^{-1} + BAB^{-1})$ is now in $\mathfrak{gl}(n, \mathbb{K}_1)$.

The connected component of the identity of the Galois group represent the smallest subgroup to which our differential equation can be reduced by means of a change of variables involving algebraic functions. The following result if due to Kolchin and Kovacic [8], and is very close to a method of reduction of differential equations due to Lie. For a modern presentation, and comparison between those two see [3].

**Theorem 3.1.** Let us consider the system (3.7) and its associated Picard-Vessiot extension $\mathbb{K} \subset \mathbb{L}$. Let us fix any basis of $S_\mathbb{L}$, the space of solutions of (3.6) in $\mathbb{L}$ so that we get a representation of $\text{Gal}_{\mathbb{L}/\mathbb{K}}$ in $G$. Let $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$. Then, there exist $B \in G(\overline{\mathbb{K}})$ such that $\overline{A} = B'B^{-1} + BAB^{-1}$ is in the Lie algebra of the Galois group $\text{gal}_{\mathbb{L}/\mathbb{K}}(\overline{\mathbb{K}})$:

$$z = By, \quad z' = \overline{A}z, \quad \overline{A} \in \text{gal}_{\mathbb{L}/\mathbb{K}}(\overline{\mathbb{K}}).$$

3.4 Morales-Ramis Integrability Condition

Let $V$ be a complex symplectic manifold of dimension $2n$, with symplectic form $\Omega_2$ that we write in local coordinates,

$$\Omega_2 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

The field of meromorphic functions $\mathbb{M}(V)$ is then endowed with a Poisson bracket as in (2.2) and to any function $H \in \mathbb{M}(V)$ its correspond a Hamiltonian vector field $\vec{X}_H$.

Let us recall that two functions $F, G$ are said to be in *involution* if $\{F, G\}$ vanish, and a Hamiltonian $H \in \mathbb{M}(V)$ is called completely integrable by meromorphic functions in $V$ if there exist $n$ first integrals of $\vec{X}_H$, $F_1, \ldots, F_n$, functionally independent and in involution.

Let $\Gamma$ be an integral curve of $\vec{X}_H$. Let us consider $\mathbb{K}$ to be the field of meromorphic functions on $\Gamma$. The vector field $\vec{X}_H$ gives to $\mathbb{K}$ the structure of a differential field. We consider the first variational equation of $\vec{X}_H$ along $\Gamma$ as a system of linear differential equations with coefficients in $\mathbb{K}$. This equation can be written as follows:

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial y_j} |_{\Gamma} & \frac{\partial H}{\partial y_i} |_{\Gamma} \\ -\frac{\partial H}{\partial x_j} |_{\Gamma} & -\frac{\partial H}{\partial x_i} |_{\Gamma} \end{pmatrix} \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \quad (3.9)$$
This equation has always a known solution given by the Hamiltonian vector field, \( \left( \frac{\partial H}{\partial y_i}|\Gamma, -\frac{\partial H}{\partial y_i}|\Gamma \right) \) and a know invariant \( \frac{\partial H}{\partial x_i} \xi_i + \frac{\partial H}{\partial y_i} \eta_i \) given by \( dH|\Gamma \). They allow us to reduce the system (3.9) to a system of dimension \( 2(n-1) \) called the normal variational equation.

Let us recall that an algebraic group is called virtually abelian if an only if its Lie algebra is abelian, if and only if its connected component of the identity is abelian. The following result [11, 10] gives us an algebraic criterium for the complete integrability of Hamiltonian systems.

**Theorem 3.2.** Assume that \( H \) is completely integrable by terms of meromorphic first integrals. Let \( \Gamma \) be an integral curve of \( \vec{X}_H \). Then the Galois groups of the variational equation and the normal variational equation of \( \vec{X}_H \) along \( \Gamma \) are virtually abelian.

### 4 Linear non-autonomous Hamiltonian Systems

Let us consider \( V \) a \( 2n \)-dimensional symplectic vector space over \( \mathbb{C} \) with symplectic form \( \Omega_2 = \sum_{i=1}^{n} dx_i \wedge dy_i \) and \( \Gamma \) a Riemann surface endowed with a meromorphic derivation. We have seen that for any quadratic homogeneous non-autonomous Hamiltonian \( H \in \overline{M(\Gamma)}[V]_2 \) the equations of motion take the form given by formula (2.4).

Let us consider the matrix,

\[
J = \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix},
\]

where \( I \) denotes the identity matrix of rank \( n \). We define the symplectic group \( \text{Sp}(2n, \mathbb{C}) \) to the the group of all non-degenerate matrices \( \sigma \) such that \( \sigma^t J \sigma = J \). It is an algebraic subgroup of \( \text{GL}(2n, \mathbb{C}) \). Its lie algebra \( \text{sp}(2n, \mathbb{C}) \) consist in all matrices \( A \) such that \( A^t J + JA = 0 \). It is clear that the matrix of coefficients of (2.4) is in \( \text{sp}(2n, \mathbb{C}(t)) \), and then the Galois group of such equation is a subgroup of \( \text{Sp}(2n, \mathbb{C}) \).

#### 4.1 Lie Algebra Structure

Let us consider \( \mathbb{C}[V]_2 \) the space of quadratic homogeneous polynomials on the linear coordinates of \( V \) with complex coefficients. Then, the assignation:

\[
\mathbb{C}[V]_2 \xrightarrow{\sim} \text{sp}(2n, \overline{M(\Gamma)})
\]

\[
\sum_{i=1}^{n} \left( \frac{a_{ij}}{2} x_i x_j + \frac{b_{ij}}{2} y_i y_j + c_{ij} x_i y_j \right) \rightarrow \begin{pmatrix} C^t & B \\ -A & -C \end{pmatrix}
\]

brings us an isomorphism between the Poisson structure of \( \mathbb{C}[V]_2 \) and the Lie algebra structure of \( \text{sp}(2n, \overline{M(\Gamma)}) \). This fact is well known (see for instance [10], page 64).
4.2 Changes of Frame

Quadratic homogeneous Hamiltonian behave nicely with respect to linear changes of frame. Let us take a new system of coordinates,

\[
\begin{pmatrix}
    \xi \\
    \eta
\end{pmatrix} = B(t) \begin{pmatrix} x \\ y \end{pmatrix}
\]

with \( B(t) \in \text{Sp}(4, \mathbb{M}(\Gamma)) \). Then, by using (3.8) and (4.10) we can write down a transformed Hamiltonian,

\[
\bar{H} = H - (\xi, \eta)J\dot{B}^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad J = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},
\]

which gives the equations of the movement in the new system of coordinates,

\[
\dot{\xi} = \{\xi, \bar{H}\}, \quad \dot{\eta} = \{\eta, \bar{H}\}.
\]

4.3 Integrability

**Lemma 4.1.** Let \( H \in \mathbb{M}(\Gamma)[V]_2 \) be a quadratic homogeneous non-autonomous Hamiltonian. Let us consider \( \hat{X}_H^\Gamma \) the associated extended autonomous vector field. Then, the normal variational equation of \( \hat{X}_H^\Gamma \) along any integral curve \( \Gamma \) coincides with \( \hat{X}_H^\Gamma \) itself.

**Proof.** Let \( \hat{H} = \frac{a}{2}x_i x_j + \frac{b}{2}y_i y_j + c_i x_i y_j + h \). Then, the variational equation of \( \hat{X}_H^\Gamma \) around an integral curve \( \Gamma \) is:

\[
\begin{pmatrix}
    \xi' \\
    \eta' \\
    \tau' \\
    \chi'
\end{pmatrix} = \begin{pmatrix}
    C^\Gamma & B & H_{yi,\xi}|_{\Gamma} & 0 \\
    -A & -C & -H_{yi,\tau}|_{\Gamma} & 0 \\
    0 & 0 & 0 & 0 \\
    -H_{tx,j}|_{\Gamma} & -H_{ty,j}|_{\Gamma} & -H_{tt}|_{\Gamma} & 0
\end{pmatrix} \begin{pmatrix}
    \xi \\
    \tau \\
    \eta \\
    \chi
\end{pmatrix}
\]

The Normal Variational Equation is obtained by using the know solution \( \tau = 1, \xi = \chi = \eta = 0 \) and restriction to the hyperplane \( \chi = 0 \). We easily get,

\[
\begin{pmatrix}
    \xi' \\
    \eta'
\end{pmatrix} = \begin{pmatrix} C^\Gamma & B \\ -A & -C \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix},
\]

which gives us the normal variational equation of the statement.

**Theorem 4.1.** Let \( H \in \mathbb{M}(\Gamma)[V]_2 \) be a quadratic homogeneous non-autonomous Hamiltonian of \( 2 + \frac{1}{2} \) degrees of freedom, with coefficients meromorphic in \( \tilde{\Gamma} \) a ramified covering of \( \Gamma \). The following are equivalent:

1. The associated extended autonomous system \( \hat{H} \) is completely integrable by meromorphic functions in \( \tilde{\Gamma} \times V \times \mathbb{C}_h \) for some ramified covering \( \tilde{\Gamma} \) of \( \Gamma \).
(2) $H$ is integrable in the non-autonomous sense by meromorphic functions in $\hat{\Gamma} \times V$ for some ramified covering $\hat{\Gamma}$ of $\Gamma$.

(3) $H$ is integrable in the non-autonomous sense by quadratic first integrals $F_1, F_2 \in \overline{M(\Gamma)}[V]_2$.

(4) The connected component of the Galois group of $\hat{X}_H$ is an abelian.

**Proof:**

(4) $\Rightarrow$ (3) This part of the proof relies on the classification of connected abelian subgroups of the symplectic which is made in section 5. By Lie-Kolchin reduction, Theorem 3.1, there exist a symplectic change of frame $B \in Sp(4, \overline{M(\Gamma)})$ such that,

$$
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\eta_1 \\
\eta_2
\end{pmatrix} =
B
\begin{pmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2
\end{pmatrix}
$$

(4.11)

and

$$
\begin{pmatrix}
\xi'_1 \\
\xi'_2 \\
\eta'_1 \\
\eta'_2
\end{pmatrix} =
A
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\eta_1 \\
\eta_2
\end{pmatrix},
A \in g(\overline{M(\Gamma)}).
$$

(4.12)

Where $g$ is an abelian subalgebra of $\mathfrak{sp}(2, \mathbb{C})$. By Corollary 5.1, $g$ is contained in an abelian subalgebra, spanned by two linear Hamiltonian vector fields $\tilde{Y}_1$ and $\tilde{Y}_2$ with constants coefficients. By the dictionary between $\mathfrak{sp}(2, \mathbb{C})$ and $\mathbb{C}[V]_2$ there are two quadratic polynomials $F_1, F_2$ such that $\tilde{Y}_1 = \tilde{X}_{F_1}$ and $\tilde{Y}_2 = \tilde{X}_{F_2}$. Then, its is clear that $F_1(\xi_1, \xi_2, \eta_1, \eta_2)$ and $F_2(\xi_1, \xi_2, \eta_1, \eta_2)$ are first integrals of (4.12) in involutions. Substituting these variables using (4.11) we get two first integrals of $\tilde{X}_H$ in involution in $\overline{M(\Gamma)}[V]_2$.

(3) $\Rightarrow$ (2) In particular the first integrals in $\overline{M(\Gamma)}[V]_2$ are meromorphic functions in $\hat{\Gamma} \times V$ for some covering $\hat{\Gamma}$ of $\Gamma$.

(2) $\Rightarrow$ (1) It is Proposition 2.1.

(1) $\Rightarrow$ (4) By Lemma 4.1, we have that $\tilde{X}_H$ coincides with the normal variational equation of its associated extended autonomous system $\hat{\tilde{X}}_H$ along any particular solution. We take $\Gamma$ the particular solution corresponding to $x_i = 0$, $y_i = 0$, $h = 0$, $t = t$, therefore the field of coefficients is here $\overline{M(\Gamma)}$ which is an algebraic extension of $M(\Gamma)$. By Theorem 3.2, we have that the Galois group of this equation is virtually abelian. It finish the proof. $\square$

**Remark 4.1.** The proof of the above systems relies on Corollary 5.1 which is proven in the next section. It is well known that any abelian subalgebra of $\mathfrak{sp}(2n, \mathbb{C})$ has dimension at most $n$, and there are maximal abelian subalgebras of dimension $n$. However, it is beyond the knowledge of the authors if any maximal abelian subalgebra realizes dimension $n$. If this holds, then Theorem 4.1 will hold for any number of degrees of freedom.
5 Classification of Connected Abelian Subgroups of Sp(4, \mathbb{C})

In this section we classify the connected abelian subgroups of Sp(4, \mathbb{C}). In the paper [5] there are shown the canonical form of 2-Ziglin subgroups that lead to certain obstructions to integrability. In a similar way, the classification of connected abelian connected subgroups completes the proof of Theorem 4.1 and allow us to compute canonical forms for integrable system.

Let us recall that any connected abelian subgroups of Sp(4, \mathbb{C}) must be of dimension one or two, and that any connected abelian linear group is direct product of multiplicative \mathbb{C}^* and addictive \mathbb{C} groups. The following technical lemma can be proved by direct computation.

All computations of this section are easy to reproduce, so that we will give just sketches of the proofs.

Lemma 5.1. Let $A$ be a nihilpotent matrix in $\mathfrak{sp}(4, \mathbb{C})$ then one of the following conditions holds:

(1) $A^2 = 0$, ker$(A)$ is of dimension 3, and $A$ is conjugated to a matrix of the form:

$$
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

(2) $A^2 = 0$, ker$(A)$ is of dimension 2, and $A$ is conjugated to a matrix of the form:

$$
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

(3) $A^3 \neq 0$, ker$(A)$ is of dimension 1, and $A$ is conjugated to a matrix of the form:

$$
\begin{pmatrix}
0 & 1 & \lambda & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}
$$

with $\lambda \in \mathbb{C}$.

Proof. Let $A$ be a nihilpotent matrix in $\mathfrak{sp}(4, \mathbb{C})$. It is well know that there
exist a non-degenerate matrix $B$ such that,

$$\bar{A} = B^{-1}AB = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with $a$, $b$ and $c$ equal to 0 or 1. In this base, the matrix of the symplectic metric is a non-degenerate skew-symmetric matrix,

$$\bar{J} = B^tJB = \begin{pmatrix} 0 & j_{12} & j_{13} & j_{14} \\ -j_{12} & 0 & j_{23} & j_{24} \\ -j_{13} & -j_{23} & 0 & j_{34} \\ -j_{14} & -j_{24} & -j_{34} & 0 \end{pmatrix}$$

Applying that $\bar{A}^t\bar{J} + \bar{J}\bar{A} = 0$ we see that the cases $a = b = 1$, $c = 0$ and $a = 0$, $b = c = 1$ lead $|\bar{J}| = 0$. Then, it is clear that these cases can not happen for a symplectic matrix $A$. The cases $a = 1, b = c = 0$; $a = c = 0, b = 1$ and $a = b = 0, c = 1$ are clearly equivalent and lead to the case (1) of the statement. The cases $a = c = 1, b = 0$ lead to the case (2) of the statement. And the case $a = b = c = 1$ leads to the case (3) of the statement, just by applying $\bar{A}^t\bar{J} + \bar{J}\bar{A} = 0$ and looking for the matrices that conjugate $\bar{J}$ to canonical form $J$. □

**Proposition 5.1.** Let $G$ be a 1-dimensional connected abelian subgroup of $\text{Sp}(4, \mathbb{C})$, then $G$ is conjugated to one of the following list:

1. Case $G$ isomorphic to the multiplicative group $\mathbb{C}^*$.

1.a $$G \equiv \left\{ \begin{pmatrix} \lambda^p & 0 & 0 & 0 \\ 0 & \lambda^q & 0 & 0 \\ 0 & 0 & \lambda^{-p} & 0 \\ 0 & 0 & 0 & \lambda^{-q} \end{pmatrix} : \lambda \in \mathbb{C}^* \right\}$$

with $(p, q)$ relative primes.

1.b $$G \equiv \left\{ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{C}^* \right\}$$

1.c $$G \equiv \left\{ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C}^* \right\}$$
(2) Case $G$ isomorphic to the additive group $\mathbb{C}$.

(2.a)

\[
G \equiv \left\{ \begin{pmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C} \right\}
\]

(2.b)

\[
G \equiv \left\{ \begin{pmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & \lambda \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C} \right\}
\]

(2.c)

\[
G \equiv \left\{ \begin{pmatrix} 1 & \lambda & -\frac{\lambda^2}{6} + k\lambda & \frac{\lambda^2}{2} \\ 0 & 1 & -\frac{\lambda^2}{2} & \lambda \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 \end{pmatrix} : \lambda \in \mathbb{C} \right\}
\]

with $k \in \mathbb{C}$.

Proof. Multiplicative groups are always contained into a maximal torus and then they are classified. See for instance [6]. Then, we have just to classify additive group. A one dimensional subgroup of $\text{GL}(n, \mathbb{C})$ is isomorphic to the additive group if and only if its Lie algebra is generated by a nilpotent matrix. Conjugacy classes of nilpotent matrix are given by Lemma 5.1. Cases 2. a, b, and c are given just by the exponential of these nilpotent matrices. □

Theorem 5.1. Let $G$ be a maximal connected abelian subgroup of $\text{Sp}(4, \mathbb{C})$, then $G$ is conjugated to one of the following list:

(3) Case $G$ isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.

\[
G \equiv \left\{ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \mu^{-1} \end{pmatrix} : \lambda, \mu \in \mathbb{C}^* \right\}
\]

(4) Case $G$ isomorphic to $\mathbb{C} \times \mathbb{C}^*$.

\[
G \equiv \left\{ \begin{pmatrix} 1 & 0 & \lambda & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu^{-1} \end{pmatrix} : \lambda \in \mathbb{C}, \mu \in \mathbb{C}^* \right\}
\]

(5) Case $G$ isomorphic to $\mathbb{C} \times \mathbb{C}$. 16
Proof. First, let us see that any one dimensional subgroup of $\text{Sp}(4, \mathbb{C})$ is contained in a two dimensional abelian subgroup. If our group is multiplicative, this result is well known. If our group is additive just note that cases 2.a and 2.b are included into case 5.a here and case 2.c is included into case 5.b.

Second, let us see that any 2-dimensional abelian subgroup of $\text{Sp}(4, \mathbb{C})$ falls in one of the cases we list above. Let $G$ such a group. Then, it is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$, $\mathbb{C} \times \mathbb{C}^*$ or to $\mathbb{C} \times \mathbb{C}$. In the first case, it is a maximal torus and it is well known that it falls into case 3.

In the second case, let us consider $G$ isomorphic to $\mathbb{C} \times \mathbb{C}^*$. Let $g$ be its Lie algebra. It is clear that there is a unique line in $g$ spanned by a nilpotent matrix, since there is only one algebraic morphism from $\mathbb{C}$ into $G$. Let $A$ be such a matrix, then it falls in one of the three cases of Lemma 5.1. Assume that $A$ falls in case (2) of (3) of Lemma 5.1, we can compute explicitly the commutator of such matrices, but we find that all matrices that commute with $A$ are also nilpotent. But, by hypothesis, there is a non nilpotent matrix in $g$ which commutes with $A$. Therefore, $A$ must fall in case (1) of Lemma 5.1. The space of matrices in that commute with $A$ is then easily computed and leads us to case 5.

In the third case, let us consider $G$ isomorphic to $\mathbb{C} \times \mathbb{C}$. Then, its Lie algebra $g$ is spanned by two nilpotent matrix. We have to split in two cases. If any matrix in $g$ falls in the case (3) of Lemma 5.1, we can arrive easily to canonical form (5.b). If there is a matrix $A \in g$ that falls into case (3) of Lemma 5.1, then $g$ is completely determined by $A$. Any other matrix in $g$ is in the commutator of $A$ which is a Lie algebra of dimension 2. The exponential of this Lie algebra lead us to canonical form (5.b).

\textbf{Corollary 5.1.} Any maximal connected abelian subgroup of $\text{Sp}(4, \mathbb{C})$ is of dimension 2.

6 Canonical Forms of Integrable Systems

Let $H(t, x_1, x_2, y_1, y_2) \in M(\Gamma)[V]_2$ be an integrable quadratic homogeneous Hamiltonian of $2 + \frac{1}{2}$ degrees of freedom. By theorem 4.1 we know that the notion of
complete integrability and integrability in the non-autonomous sense are equivalent, so that we will just speak of an integrable non-autonomous Hamiltonian.

We also know, by Theorem 4.1 that its differential Galois group is a connected abelian subgroup of $\text{Sp}(4, \mathbb{C})$. Proposition 5.1 and Theorem 5.1 give us a complete list of all conjugacy classes of abelian subgroups of $\text{Sp}(4, \mathbb{C})$. We can then apply Lie-Kolchin reduction (Theorem 3.1) to $\vec{X}_H$ obtaining a canonical form for the Hamiltonian.

At this point we should remark that, when applying a time dependent change of frame to a non-autonomous Hamiltonian system, the Hamiltonian function is then modified in the following form. If $B(t) \in \text{Sp}(4, \mathcal{M}(\Gamma))$ is a time dependent symplectic matrix that give us a change of frame,

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta_1 \\ \eta_2 \end{pmatrix} = B(t) \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$$

then, the new Hamiltonian function for our Hamiltonian systems is,

$$\tilde{H} = H - (\xi_1, \xi_2, \eta_1, \eta_2)J\dot{B}(t)B^{-1}(t) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta_1 \\ \eta_2 \end{pmatrix}, \quad J = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

as it follows from the change of frame formula for linear systems 3.8. We can state directly the following result.

**Theorem 6.1.** Let $H(t, x_1, x_2, y_1, y_2) \in \mathcal{M}(\Gamma)[V]_2$ be an integrable quadratic homogeneous Hamiltonian of $2 + \frac{1}{2}$ degrees of freedom. Then, there exist a symplectic change of frame,

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta_1 \\ \eta_2 \end{pmatrix} = B(t) \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$$

with $B(t) \in \text{Sp}(4, \mathcal{M}(\Gamma))$ such that, for the transformed Hamiltonian $\tilde{H}(\xi_1, \xi_2, \eta_1, \eta_2)$,

$$\tilde{H} = H - (\xi_1, \xi_2, \eta_1, \eta_2)J\dot{B}B^{-1}(t) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta_1 \\ \eta_2 \end{pmatrix}, \quad J = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

belongs to one of the following categories:
Normal Form | Galois | Quadratic Invariants | Parameters
--- | --- | --- | ---
0 | \{1\} | All | \(f(t), \frac{p}{q}\)
\(f(t)\left(\xi_1 \eta_1 + \frac{p}{q} \xi_2 \eta_2\right)\) | \(\mathbb{C}^*\) | \(\xi_1 \eta_1, \xi_2 \eta_2\) | \(f(t)\)
\(f(t)(\xi_1 \eta_1 + \xi_2 \eta_2)\) | \(\mathbb{C}^*\) | \(\xi_1 \eta_1, \xi_2 \eta_2, \xi_1 \eta_2 - \xi_2 \eta_1\) | \(f(t)\)
\(f(t)\xi_1 \eta_1\) | \(\mathbb{C}^*\) | \(\xi_1 \eta_1, \xi_2^2, \eta_2^2, \xi_2 \eta_2\) | \(f(t)\)
\(f(t)\eta_1^2 + \eta_2^2\) | \(\mathbb{C}\) | \(\eta_1^2, \xi_2^2, \xi_2 \eta_2, \eta_2^2\) | \(f(t)\)
\(f(t)\left(\xi_2 \eta_1 + \lambda \eta_2^2 + \frac{g(t)}{q}\right)\) | \(\mathbb{C}\) | \(2\xi_2 \eta_1 + \eta_2^2, \eta_1^2\) | \(f(t), \lambda\)
\(f(t)\xi_1 \eta_1 + g(t)\xi_2 \eta_2\) | \((\mathbb{C}^*)^2\) | \(\xi_1 \eta_1, \xi_2 \eta_2\) | \(f(t), g(t)\)
\(f(t)\eta_1^2 + g(t)\xi_2 \eta_2\) | \(\mathbb{C} \times \mathbb{C}^*\) | \(\eta_1^2, \xi_2 \eta_2\) | \(f(t), g(t)\)
\(f(t)\eta_1^2 + g(t)\eta_2^2\) | \(\mathbb{C}^2\) | \(\eta_1^2, \eta_2^2\) | \(f(t), g(t)\)
\(f(t)\eta_1(\xi_2 + g(t)\eta_1 + \frac{g(t)}{2})\) | \(\mathbb{C}^2\) | \(2\eta_1 \xi_2 + \eta_2^2, \eta_1^2\) | \(f(t), g(t)\)

Where \(f(t)\) and \(g(t)\) are arbitrary meromorphic functions, and \(\lambda\) is an arbitrary constant, and \(p, q\) are coprime integers.

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