ON THE RIGIDITY OF MODULI OF WEIGHTED POINTED STABLE CURVES

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Abstract. Let $\overline{M}_{g,A[n]}$ be the Hassett moduli stack of weighted stable curves, and let $\overline{M}_{g,A[n]}$ be its coarse moduli space. These are compactifications of $\mathcal{M}_{g,n}$ and $\mathcal{M}_{g,n}$ respectively, obtained by assigning rational weights $A = (a_1, \ldots, a_n)$, $0 < a_i \leq 1$ to the markings; they are defined over $\mathbb{Z}$, and therefore over any field. We study the first order infinitesimal deformations of $\overline{M}_{g,A[n]}$ and $\overline{M}_{g,A[n]}$. In particular, we show that $\overline{M}_{0,A[n]}$ is rigid over any field, if $g \geq 1$ then $\overline{M}_{g,A[n]}$ is rigid over any field of characteristic zero, and if $g + n > 4$ then the coarse moduli space $\overline{M}_{g,A[n]}$ is rigid over an algebraically closed field of characteristic zero. Finally, we take into account a degeneration of Hassett spaces parametrizing rational curves obtained by allowing the weights to have sum equal to two. In particular, we consider such a Hassett 3-fold which is isomorphic to the Segre cubic hypersurface in $\mathbb{P}^4$, and we prove that its family of first order infinitesimal deformations is non-singular of dimension ten, and the general deformation is smooth.

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Introduction

In [Has03] B. Hassett introduced new compactifications $\overline{M}_{g,A[n]}$ of the moduli stack $\mathcal{M}_{g,n}$ parametrizing smooth genus $g$ curves with $n$ marked points, where the notion of stability is defined in terms of a fixed vector of rational weights $A[n] = (a_1, \ldots, a_n)$, on the markings. The classical Deligne-Mumford compactification corresponds to the weights $a_1 = \ldots = a_n = 1$; Hassett construction requires that $0 < a_i \leq 1$ for every $i$ and that $\sum a_i > 2 - 2g$.

As the stack $\overline{M}_{g,n}$, the stacks $\overline{M}_{g,A[n]}$ are smooth and proper over $\mathbb{Z}$, and therefore $\overline{M}_{g,A[n]}^R$ is defined over any commutative ring $R$ via base change. By [KM97] the formation of the coarse moduli space is compatible with flat base change; we write $\overline{M}_{g,n}^R$ for the coarse moduli scheme of $\overline{M}_{g,n}^R$, and refer to it as a Hassett moduli space. Again in analogy with the Deligne-Mumford case, Hassett stacks for $g = 0$ are already schemes, hence coincide with the corresponding Hassett spaces.

Hassett spaces are central objects in the study of the birational geometry of $\overline{M}_{g,n}$. Indeed, in genus zero some of these spaces appear as intermediate steps of the blow-up construction of $\overline{M}_{0,n}$ developed by M. Kapranov in [Ka93] and some of them turn out to be Mori Dream Spaces [AM16 Section 6], while in higher genus they may be related to the LMMP on $\overline{M}_{g,n}$ [Mo13].

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In this paper we push forward the techniques developed in [FM16] to study the infinitesimal deformations of Hassett moduli stacks and spaces over an arbitrary field. The results in Theorems 2.6 and 2.7 can be summarized as follows.

**Theorem 1.** Let \( K \) be an arbitrary field, and \( n \geq 3 \) an integer. Then the genus zero Hassett moduli space \( \overline{M}_{0,A[n]}^K \) is rigid for any vector of weights \( A[n] \).

Let \( g \geq 1 \) and assume \( K \) is a field of characteristic zero. Then Hassett stack \( \overline{M}_{g,A[n]}^K \) is rigid for any vector of weights \( A[n] \).

For a field \( K \) of characteristic zero we then apply the deformation theory of varieties with transversal \( A_1 \) and \( 1/3(1,1) \) singularities developed in [FM16] Sections 5.4, 5.5] to the study of infinitesimal deformations of the coarse moduli spaces \( \overline{M}_{g,A[n]}^K \). In the following statement we summarize the results on deformations of \( \overline{M}_{g,A[n]}^K \) in Proposition 2.6 and Theorem 2.7.

**Theorem 2.** Let \( K \) be a field of characteristic zero. If \( g + n \geq 4 \) then the coarse moduli space \( \overline{M}_{g,A[n]} \) does not have locally trivial first order infinitesimal deformations for any vector of weights \( A[n] \).

If \( K \) is an algebraically closed field of characteristic zero and \( g + n > 4 \) then \( \overline{M}_{g,A[n]}^K \) is rigid for any vector of weights \( A[n] \).

In Section 3 we consider a natural variation \( \overline{M}_{0,\bar{A}[0]} \) on the moduli problem of weighted pointed rational curves, introduced by B. Hassett in [Has03 Section 2.1.2] by allowing the weights to have sum equal to two.

In particular, we consider Hassett space \( \overline{M}_{0,\bar{A}[0]} \) with weights \( a_1 = \ldots = a_6 = 1/3 \). This space is isomorphic to the Segre cubic, a 3-fold of degree three in \( \mathbb{P}^4 \) with ten nodes which carries a very rich projective geometry [Do15]. In Section 3 we study the infinitesimal deformations of \( \overline{M}_{0,\bar{A}[0]} \), that is of the Segre cubic.

In Theorem 3.2 we prove that \( \overline{M}_{0,\bar{A}[0]} \) does not have locally trivial deformations, while its family of first order infinitesimal deformations is non-singular of dimension ten and the general deformation is smooth.

Finally, in Section 2.1 we apply the rigidity results in Section 2 and the techniques developed in [FM16] Section 1 to lift automorphisms from zero to positive characteristic, in order to extend the main results on the automorphism groups of Hassett spaces in [MM14], [MM16], [BM13], [Ma14] and [Ma16] over an arbitrary field.

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## 1. Preliminaries on Hassett moduli spaces

Let \( S \) be a noetherian scheme and \( g,n \) two non-negative integers. A family of nodal curves of genus \( g \) with \( n \) marked points over \( S \) consists of a flat proper morphism \( \pi : C \to S \) whose geometric fibers are nodal connected curves of arithmetic genus \( g \), and sections \( s_1, \ldots, s_n \) of \( \pi \).

A collection of input data \( (g,A[n]) := (g,a_1,\ldots,a_n) \) consists of an integer \( g \geq 0 \) and the weight data: an element \( (a_1,\ldots,a_n) \in \mathbb{Q}^n \) such that \( 0 < a_i \leq 1 \) for \( i = 1,\ldots,n \), and

\[
2g - 2 + \sum_{i=1}^{n} a_i > 0.
\]

The vector \( A[n] \) in the input data \( (g,A[n]) \) is called an admissible weight data.

**Definition 1.1.** A family of nodal curves with marked points \( \pi : (C,s_1,\ldots,s_n) \to S \) is stable of type \( (g,A[n]) \) if

- the sections \( s_1,\ldots,s_n \) lie in the smooth locus of \( \pi \), and for any subset \( \{s_{i_1},\ldots,s_{i_r}\} \) with non-empty intersection we have \( a_{i_1} + \ldots + a_{i_r} \leq 1 \),
isomorphism and

Remark 1.2.

We denote by

\[ D \]

the divisor parametrizing curves with two smooth components, of genus

\[ g \]

component respectively .

\[ g,A \]

may appear boundary divisors parametrizing smooth curves. For instance, as soon as there exist two indices \[ i,j \] such that \[ a_i + a_j \leq 1 \] we get a boundary divisor whose general point represents a smooth curve where the marked points labeled by \[ i \] and \[ j \] collide.

Finally we recall the notion of \( \psi \)-classes on \( \overline{M}_{g,A[n]} \). Let \( (\pi: C \to \overline{M}_{g,A[n]}, (s_1, ..., s_n)) \) be the universal family on \( \overline{M}_{g,A[n]} \). The \( \psi \)-classes on \( \overline{M}_{g,A[n]} \) are defined as

\[ \psi_i = \pi_*(-s_i^2) = c_1(\pi^*\omega) \]

for \( i = 1, ..., n \).

1.1. Kapranov’s blow-up construction. In [Ka93] M. Kapranov works, for sake of simplicity, on an algebraically closed field of characteristic zero. On the other hand Kapranov’s arguments are purely algebraic and his description works over \( \mathbb{Z} \).

In [Ka93] Kapranov proved that \( \overline{M}_{0,n} \) can be constructed as an iterated blow-up \( f_n: \overline{M}_{0,n} \to \mathbb{P}^{n-3} \) induced by \( \psi_n \) which is big and globally generated.

Construction 1.3. [Ka93] More precisely, fix \( (n-1) \)-points \( p_1, ..., p_{n-1} \in \mathbb{P}^{n-3} \) in linear general position.

(1) Blow-up the points \( p_1, ..., p_{n-2} \), the strict transforms of the lines spanned by two of these \( n-2 \) points,..., the strict transforms of the linear spaces spanned by the subsets of cardinality \( n-4 \) of \( \{p_1, ..., p_{n-2}\} \).

(2) Blow-up \( p_{n-1} \), the strict transforms of the lines spanned by pairs of points including \( p_{n-1} \) but not \( p_{n-2} \),...,..., the strict transforms of the linear spaces spanned by the subsets of cardinality \( n-4 \) of \( \{p_1, ..., p_{n-1}\} \) containing \( p_{n-1} \) but not \( p_{n-2} \).

\( \vdots \)

(\( r \)) Blow-up the strict transforms of the linear spaces spanned by subsets of the form

\[ \{p_{n-1}, p_{n-2}, ..., p_{n-r+1}\} \]
so that the order of the blow-ups in compatible by the partial order on the subsets given by inclusion.

\( (n - 3) \) Blow-up the strict transforms of the codimension two linear space spanned by the subset 
\{ \( p_{n-1}, p_{n-2}, ..., p_4 \) \}.

The composition of these blow-ups is the morphism \( f_n : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3} \) induced by the psi-class \( \psi_n \).

We denote by \( W_{r,s}[n] \), where \( s = 1, ..., n - r - 2 \), the variety obtained at the \( r \)-th step once we finish blowing-up the subspaces spanned by subsets \( S \) with \( |S| \leq s + r - 2 \), and by \( W_r[n] \) the variety produced at the \( r \)-th step. In particular, \( W_{1,1}[n] = \mathbb{P}^{n-3} \) and \( W_{n-3}[n] = \overline{M}_{0,n} \).

In [Has03, Section 6.1], Hassett interprets the intermediate steps of Construction 1.3 as moduli spaces of weighted rational curves. Consider the weight data

\[ A_{r,s}[n] := \begin{pmatrix} 1/(n-r-1), & \ldots, & 1/(n-r-1), & s/(n-r-1), & 1, & \ldots, & 1 \end{pmatrix}_{(n-r-1) \text{ times}} \]

for \( r = 1, \ldots, n-3 \) and \( s = 1, \ldots, n - r - 2 \). Then \( W_{r,s}[n] \cong \overline{M}_{0,A_{r,s}[n]} \), and the Kapranov’s map \( f_n : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3} \) factorizes as a composition of reduction morphisms

\[ \rho_{A_{r-1,s},A_{r,s}} : \overline{M}_{0,A_{r-1,s}[n]} \rightarrow \overline{M}_{0,A_{r,s}[n]}, \quad s = 2, \ldots, n - r - 2, \]

\[ \rho_{A_{r,n-r-2},A_{r+1,1}[n]} : \overline{M}_{0,A_{r,n-r-2}[n]} \rightarrow \overline{M}_{0,A_{r+1,1}[n]} \].

**Remark 1.4.** Hassett space \( \overline{M}_{0,A_{1,n-3}[n]} \) that is \( \mathbb{P}^{n-3} \) blown-up at all the linear spaces of codimension at least two spanned by subsets of \( n-2 \) points in linear general position, is the Losev-Manin’s moduli space \( \overline{L}_{n-2} \) introduced by A. Losev and Y. Manin in [LM00], see [Has03, Section 6.4]. The space \( \overline{L}_{n-2} \) parametrizes \( (n-2) \)-pointed chains of projective lines \( (C, x_0, x_\infty, x_1, \ldots, x_{n-2}) \) where:

- \( C \) is a chain of smooth rational curves with two fixed points \( x_0, x_\infty \) on the extremal components,
- \( x_1, \ldots, x_{n-2} \) are smooth marked points different from \( x_0, x_\infty \) but non necessarily distinct,
- there is at least one marked point on each component.

By [LM00, Theorem 2.2] there exists a smooth, separated, irreducible, proper scheme representing this moduli problem. Note that after the choice of two marked points in \( \overline{M}_{0,n} \) playing the role of \( x_0, x_\infty \) we get a birational morphism \( \overline{M}_{0,n} \rightarrow \overline{L}_{n-2} \) which is nothing but a reduction morphism.

For example, \( \overline{L}_1 \) is a point parametrizing a \( \mathbb{P}^1 \) with two fixed points and a free point, \( \overline{L}_2 \cong \mathbb{P}^1 \), and \( \overline{L}_3 \) is \( \mathbb{P}^2 \) blown-up at three points in general position, that is a del Pezzo surface of degree six, see [Has03, Section 6.4] for further generalizations.

### 1.2. Some notions of deformation theory

Let us recall some basic notions of deformation theory to which we will constantly refer along the paper.

Let \( X \) be a scheme over a field \( K \). An Artinian \( K \)-algebra with residue field \( K \). A deformation \( X_A \) of \( X \) over Spec(\( A \)) is called **trivial** if it is isomorphic to \( X \times_K \text{Spec}(A) \); it is **locally trivial** if there is an open cover of \( X \) by open affines \( U \) such that the induced deformation \( U_A \) is trivial.

We recall some well-known facts about infinitesimal deformations of normal varieties. By [Hil71] the tangent and obstruction spaces to deformations of \( X \) are given by Ext\(^1\)(\( L_X, \mathcal{O}_X \)) and Ext\(^2\)(\( L_X, \mathcal{O}_X \)) where \( L_X \) is the cotangent complex; when \( X \) is a normal variety, these spaces are actually Ext\(^1\)(\( \Omega_X, \mathcal{O}_X \)) and Ext\(^2\)(\( \Omega_X, \mathcal{O}_X \)) respectively. Locally trivial infinitesimal deformations have as tangent and obstruction spaces \( H^1(X, T_X) \) and \( H^2(X, T_X) \), respectively.

**Definition 1.5.** Let \( X \) be a scheme over a field. We will say it is **rigid** if it has no non-trivial infinitesimal deformations. If \( X \) is smooth, this is equivalent to \( H^1(X, T_X) = 0 \), and if \( X \) is generically reduced this is equivalent to Ext\(^1\)(\( \Omega_X, \mathcal{O}_X \)) = 0.
By the exact sequence

\[ 0 \rightarrow H^1(X, T_X) \rightarrow \text{Ext}^1(\Omega_X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \rightarrow H^2(X, T_X) \rightarrow \text{Ext}^2(\Omega_X, \mathcal{O}_X) \]

induced by the local-to-global spectral sequence for Ext, if \( H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = 0 \) then all deformations are locally trivial, while if \( H^1(X, T_X) = 0 \) then all locally trivial deformations are trivial.

2. ON THE RIGIDITY OF \( \overline{\mathcal{M}}_{g,A[n]} \) AND \( \overline{\mathcal{M}}_{g,A[n]} \)

Let \( \rho : \overline{\mathcal{M}}_{g,A[n]} \rightarrow \overline{\mathcal{M}}_{g,B[n]} \) be a reduction morphism between Hassett moduli stacks. By [Has03, Proposition 4.5] the morphism \( \rho \) contract the boundary divisors \( D_{I,J} = \overline{\mathcal{M}}_{g,A_I} \times \overline{\mathcal{M}}_{g,A_J} \) with \( A_I = (a_{i_1}, ..., a_{i_r}, 1) \), \( A_J = (a_{j_1}, ..., a_{j_{n-r}}, 1) \) and \( c = b_1 + ... + b_r \leq 1 \) for \( 2 < r \leq n \).

By [Has03] Remark 4.6 the morphism \( \rho \) can be factored as a composition of reduction morphisms \( \rho = \rho_1 \circ ... \circ \rho_s \) where \( \rho_i : \overline{\mathcal{M}}_{g,A'[n]} \rightarrow \overline{\mathcal{M}}_{g,B'[n]} \) is the blow-up of \( \overline{\mathcal{M}}_{g,B'[n]} \) along the image of a single divisor of type \( D_{I,J} \).

We will need the following commutative algebra result.

**Lemma 2.1.** Let \( R \) be a commutative ring. Given the following commutative diagram of \( R \)-modules

\[
\begin{array}{ccccccc}
0 & \rightarrow & A & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & B & \rightarrow & 0 \\
\pi_1 & | & \gamma & | & \pi_2 & | & & & | & & \\
0 & \rightarrow & A & \rightarrow & B_1 & \rightarrow & B_2 & \rightarrow & B & \rightarrow & 0 \\
\pi_1 & | & \gamma & | & \pi_2 & | & & & | & & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

there exists an isomorphism \( \delta : C_1 \rightarrow C_2 \).

**Proof.** For any \( c_1 \in C_1 \) there exists \( a_1 \in A_1 \) such that \( c_1 = \pi_1(a_1) \). We define \( \delta(c_1) = \pi_2(\gamma(a_1)) \).

It is straightforward to check that \( \delta \) is well defined and that it is an isomorphism. \( \square \)

Now, we are ready to explicit the normal bundle of \( \overline{\mathcal{M}}_{g,C[1]} \subset \overline{\mathcal{M}}_{g,B[n]} \) in terms of the first psi-class.

**Proposition 2.2.** Let \( \rho : \overline{\mathcal{M}}_{g,A[n]} \rightarrow \overline{\mathcal{M}}_{g,B[n]} \) be a reduction morphism contracting a single boundary divisor \( D_{I,J} \) as above, and let \( \rho(D_{I,J}) = \overline{\mathcal{M}}_{g,C[s+1]} \) be its image, with \( s = n - r \). Then

\[ N_{\overline{\mathcal{M}}_{g,C[s+1]}/\overline{\mathcal{M}}_{g,B[n]}} = (\psi_1^s)^{(r-1)} \]

**Proof.** The reduction morphism \( \rho : \overline{\mathcal{M}}_{g,A[n]} \rightarrow \overline{\mathcal{M}}_{g,B[n]} \) is the blow-up of \( \overline{\mathcal{M}}_{g,B[n]} \) along \( \rho(D_{I,J}) = \overline{\mathcal{M}}_{g,C[s+1]} \) with \( C = (c, b_{j_1}, ..., b_{j_{n-r}}) \). We identify \( \overline{\mathcal{M}}_{g,C[s+1]} \) with the image of the embedding

\[
\begin{array}{c}
[\mathcal{C}, x_1, ..., x_{s+1}] \rightarrow \overline{\mathcal{M}}_{g,B[n]} \\
\end{array}
\]

Let \( [C, x_1, ..., x_{s+1}] \in \overline{\mathcal{M}}_{g,C[s+1]} \subset \overline{\mathcal{M}}_{g,B[n]} \) be a point. On the curve \( C \) we have the exact sequence
Now, since $\text{Hom}(\Omega_C(\sum_{i=1}^{n} x_i), \mathcal{O}_C) = T_{Id} \text{Aut}(C, (x_1, ..., x_n))$ and $(C, (x_1, ..., x_n))$ is stable we have that
\[
\text{Hom}(\Omega_C(x_1 + ... + x_n), \mathcal{O}_C) = 0
\]

Therefore, applying the functor $\text{Hom}(-, \mathcal{O}_C)$ and taking stalks at the point $x_1 \in C$ we get the following exact sequence
\[
0 \rightarrow \text{Hom}(\Omega_C, \mathcal{O}_C) \rightarrow \bigoplus_{i=1}^{n} \text{Ext}^1(\mathcal{O}_{C,x_i}, \mathcal{O}_C) \rightarrow \text{Ext}^1(\Omega_C(\sum_{i=1}^{n} x_i), \mathcal{O}_C)) \rightarrow \text{Ext}^1(\Omega_C, \mathcal{O}_C) \rightarrow 0
\]

On the other hand we have the same exact sequence on $[C, x_1, ..., x_1, ..., x_{s+1}]$ seen as a point in $\overline{\mathcal{M}}_{g,C[s+1]}$. Therefore we may consider the following diagram
\[
\begin{array}{ccc}
(T_{x_1}C)^{\otimes r}/T_{x_1}C & \longrightarrow & N_{\overline{\mathcal{M}}_{0,C[s+1]}/\overline{\mathcal{M}}_{0,B[n]}|[C,x_1]} \\
\text{Hom}(\Omega_C, \mathcal{O}_C) & \longrightarrow & \bigoplus_{i=1}^{n} \text{Ext}^1(\mathcal{O}_{C,x_i}, \mathcal{O}_C) \\
\downarrow & & \uparrow \alpha \\
\text{Hom}(\Omega_C, \mathcal{O}_C) & \longrightarrow & \bigoplus_{i=1}^{s+1} \text{Ext}^1(\mathcal{O}_{C,x_i}, \mathcal{O}_C)
\end{array}
\]

where the vertical maps are defined as
\[
\alpha : \bigoplus_{i=1}^{s+1} \text{Ext}^1(\mathcal{O}_{C,x_i}, \mathcal{O}_C) \longrightarrow \bigoplus_{i=1}^{n} \text{Ext}^1(\mathcal{O}_{C,x_i}, \mathcal{O}_C)
\]

and
\[
\beta : T^1\text{Def}(C, x_1, ..., x_{s+1}) \longrightarrow T^1\text{Def}(C, x_1, ..., x_1, ..., x_{s+1})
\]

with the identifications $T^1\text{Def}(C, x_1, ..., x_{s+1}) = \text{Ext}^1(\Omega_C(\sum_{i=1}^{s+1} x_i), \mathcal{O}_C)) = T_{[C,x_1]}\overline{\mathcal{M}}_{g,C[s+1]}$ and $T^1\text{Def}(C, x_1, ..., x_1, ..., x_{s+1}) = \text{Ext}^1(\Omega_C(\sum_{i=1}^{s+1} x_i), \mathcal{O}_C)) = T_{[C,x_1]}\overline{\mathcal{M}}_{g,B[n]}$. Furthermore, we have that $\text{Ext}^1(\mathcal{O}_{C,x_i}, \mathcal{O}_C) = T_{x_1}C$. Hence
\[
\bigoplus_{i=1}^{n} \text{Ext}^1(\mathcal{O}_{C,x_i}, \mathcal{O}_C)/\bigoplus_{i=1}^{s+1} \text{Ext}^1(\mathcal{O}_{C,x_i}, \mathcal{O}_C) = (T_{x_1}C)^{\otimes r}/T_{x_1}C = (T_{x_1}C)^{(r-1)}
\]

By Lemma 2.1 we get $N_{\overline{\mathcal{M}}_{g,C[s+1]}/\overline{\mathcal{M}}_{g,B[n]}|[C,x_1]} \cong (T_{x_1}C)^{(r-1)}$, and hence
\[
N_{\overline{\mathcal{M}}_{g,C[s+1]}/\overline{\mathcal{M}}_{g,B[n]}} = (\psi^\vee)^{(r-1)}
\]

Note that $\text{codim}_{\overline{\mathcal{M}}_{g,B[n]}}(\overline{\mathcal{M}}_{g,C[s+1]}) = n - 3 - (n - r - 2) = r - 1$.

Our next aim is to prove a vanishing result for the higher cohomology groups of $\psi$-classes on Hassett space.

**Proposition 2.3.** Let $\rho : \overline{\mathcal{M}}_{g,A[n]} \rightarrow \overline{\mathcal{M}}_{g,B[n]}$ be a reduction morphism contracting a single boundary divisor $D_{I,J}$ as above, and let $\rho(D_{I,J}) = \overline{\mathcal{M}}_{g,C[s+1]}$ be its image, with $s = n - r$. Assume $H^i(\overline{\mathcal{M}}_{g,C[m]}, \psi^\vee_j) = 0$ for any $i = 1, ..., m$, $j > 0$, for any Hassett stack $\overline{\mathcal{M}}_{g,C[m]}$ with $m < n$. Then
\[
H^j(\overline{\mathcal{M}}_{g,A[n]}, \psi^\vee_j) = 0 \implies H^j(\overline{\mathcal{M}}_{g,B[n]}, \psi^\vee_j) = 0
\]

for any $j > 0$.

**Proof.** Let us write the exceptional divisor as $D_{I,J} = \overline{\mathcal{M}}_{0,C_1}[r+1] \times \overline{\mathcal{M}}_{g,C_2}[s+1]$. We distinguish two cases: $i > r$ and $i \leq r$.

If $i > r$ then $\psi^\vee_j \cong \rho^* \psi^\vee_j$. Since $R^j \rho_* \psi^\vee_j = 0$ we have that
\[
H^j(\overline{\mathcal{M}}_{g,B[n]}, \psi^\vee_j) = H^j(\overline{\mathcal{M}}_{g,A[n]}, \rho^* \psi^\vee_j) = H^j(\overline{\mathcal{M}}_{g,A[n]}, \psi^\vee_j) = 0
\]
Lemma 2.4. Let \( X \) be a smooth stack and \( Z \subseteq X \) be a smooth substack. Then
\[
T^1\operatorname{Def}_f(X, Z) = H^1(X, T_X(-\log Z)) = H^1(\operatorname{Bl}_Z X, T_{\operatorname{Bl}_Z X}) = T^1\operatorname{Def}_{\operatorname{Bl}_Z X}.
\]
Furthermore, the following diagram
\[
\begin{array}{ccc}
0 & \rightarrow & T_X \\
\downarrow & & \downarrow \\
0 & \rightarrow & \epsilon^* T_X | \mathcal{E} \\
\end{array}
\]
induces an isomorphism \( H^1(Z, N_{Z/X}) \cong H^1(E, Q) \) for any \( i \geq 1 \), where \( L \) is implicitly defined by requiring the second row to be exact.

Proof. The argument is the same as for smooth varieties. Let \( \tilde{X} = \operatorname{Bl}_Z X \) be the blow-up and \( \epsilon : \tilde{X} \rightarrow X \) be the blow-up morphism with exceptional divisor \( E = \mathbb{P}(N_{Z/X}) \).

By hypothesis we have \( H^1(\overline{M}_{g,A|n}, \psi_i^\vee) = 0 \). Furthermore, \( R^ip^*_2 \rho_1^* \psi_i^\vee = 0 \) implies
\[
H^1(D_{I,J}, \rho_1^* \psi_i^\vee) = H^1(D_{I,J}, p^*_2 \psi_i^\vee) = H^1(\overline{M}_{g,C_2[s+1]}, \psi_i^\vee) = 0
\]
by induction hypothesis on \( \dim \overline{M}_{g,C_2[s+1]} < \dim \overline{M}_{g,B|n} \). Since \( R^1 \rho_1^* \psi_i^\vee = 0 \) taking the long exact sequence in cohomology we conclude that \( H^1(\overline{M}_{g,B|n}, \psi_i^\vee) = H^1(\overline{M}_{g,B|n}, \rho_1^* \psi_i^\vee) = 0 \).

We will need the following lemma relating the first order infinitesimal deformations of a stack to the deformations of its blow-up along a smooth substack.

Proof. Let \( \tilde{X} = \operatorname{Bl}_Z X \) be the blow-up and \( \epsilon : \tilde{X} \rightarrow X \) be the blow-up morphism with exceptional divisor \( E = \mathbb{P}(N_{Z/X}) \).

Now \( \epsilon_! T_{\tilde{X}} | \mathcal{E} = 0 \) for any \( i \geq 0 \) implies \( \epsilon_! T_{\tilde{X}} = 0 \) for any \( i \geq 0 \) and \( \epsilon_! T_{\tilde{X}} \cong 0 \) for any \( i \geq 0 \).

Furthermore, \( H^i(\epsilon_! T_{\tilde{X}}) = 0 \) for any \( i \geq 0 \) and \( \epsilon_! T_{\tilde{X}} \cong 0 \) for any \( i \geq 0 \).

By Equation (3.1) in the proof of [FM16, Theorem 3.1] we get the case in the statement with \( g = 0 \) over any field. Let us prove the same for \( g \geq 1 \) over any field of characteristic zero.

By [Kn83, Theorem 4] the line bundle \( \psi_n \) on \( \overline{M}_{g,n} \) is identified with the pull-back of the line bundle \( \omega_\pi (Z) \) via the isomorphism \( c : \overline{M}_{g,n} \rightarrow \mathcal{U}_{g,n-1} \), where \( \pi : \mathcal{U}_{g,n-1} \rightarrow \overline{M}_{g,n-1} \) is the...
universal curve over \(\overline{\mathcal{M}}_{g,n-1}\). Furthermore, by [Ke99, Theorem 0.4] the \(\mathbb{Q}\)-line bundle \(p_\ast \omega_\pi(\Sigma)\) is nef and big, where \(p : U_{g,n-1} \to U_{g,n-1}\) is the map on the coarse moduli space.

Since we are over a field of characteristic zero we can apply Kodaira vanishing [Hac08, Theorem A.1] to the line bundle \(p_\ast \omega_\pi(\Sigma)\). In particular, we get

\[
H^1(\overline{\mathcal{M}}_{g,n}, \psi_\pi' \otimes \omega_\pi(\Sigma)) = 0
\]

for \(j > 0\).

Now, let us consider the second statement. Since by [FM16, Theorem 3.1] we know that

\[
H^1(\overline{\mathcal{M}}_{0,n}, T_{\overline{\mathcal{M}}_{g,n}}) = 0
\]

for \(g \geq 1\) over any field of characteristic zero we may proceed by induction on \(k\) and prove the second statement for a single morphism \(\rho : \overline{\mathcal{M}}_{g,A'}[n] \to \overline{\mathcal{M}}_{g,B'}[n]\).

Let \(E\) be the exceptional locus of the morphism \(\rho\), and let \(Z = \rho(E)\). We denote by \(T^1\text{Def}(\overline{\mathcal{M}}_{g,B'}[n], Z)\) the space of first order infinitesimal deformation of the couple \((\overline{\mathcal{M}}_{g,B'}[n], Z)\).

Then we have the following exact sequence

\[
H^0(Z, N_Z/\overline{\mathcal{M}}_{g,B'}[n]) \to T^1\text{Def}(\overline{\mathcal{M}}_{g,B'}[n], Z) \to T^1\text{Def}(\overline{\mathcal{M}}_{g,B'}[n]) = H^1(Z, N_Z/\overline{\mathcal{M}}_{g,B'}[n])
\]

By Proposition 2.2 we have \(N_Z/\overline{\mathcal{M}}_{g,B'}[n] \cong (\psi_\pi')^{(r-1)}\) and by Proposition 2.3 we have

\[
H^1(\overline{\mathcal{M}}_{g,B'}[n], (\psi_\pi')^{(r-1)}) = 0
\]

Therefore \(H^1(Z, N_Z/\overline{\mathcal{M}}_{g,B'}[n]) = 0\). By Lemma 2.4 we get

\[
T^1\text{Def}(\overline{\mathcal{M}}_{g,B'}[n], Z) = H^1(\overline{\mathcal{M}}_{g,B'}[n], T_{\overline{\mathcal{M}}_{g,B'}}(-\log Z)) = H^1(\overline{\mathcal{M}}_{g,A'}[n], T_{\overline{\mathcal{M}}_{g,A'}})
\]

Furthermore, we have \(H^1(\overline{\mathcal{M}}_{g,A'}[n], T_{\overline{\mathcal{M}}_{g,A'}}) = 0\) by induction hypothesis. Therefore

\[
T^1\text{Def}(\overline{\mathcal{M}}_{g,B'}[n]) = H^1(\overline{\mathcal{M}}_{g,B'}[n], T_{\overline{\mathcal{M}}_{g,B'}}) = 0
\]

Finally, by induction we conclude that \(H^1(\overline{\mathcal{M}}_{g,A'}[n], T_{\overline{\mathcal{M}}_{g,A'}}) = 0\), that is \(\overline{\mathcal{M}}_{g,A'}[n]\) is rigid.

Now, let us consider Hassett moduli spaces \(\overline{\mathcal{M}}_{g,A}[n]\) such that \(g+n \geq 4\). We begin by studying locally trivial deformations.

**Proposition 2.6.** If \(g+n \geq 4\), over a field of characteristic zero, the coarse moduli space \(\overline{\mathcal{M}}_{g,A}[n]\) does not have locally trivial first order infinitesimal deformations for any vector of weights \(A[n]\).

**Proof.** Without loss of generality we can assume that there exists a reduction morphism \(\rho : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,A}[n]\) contracting a single boundary divisor \(D := D_{I,J}\). Let \(\rho(D) = \overline{\mathcal{M}}_{g,C}[s+1]\) be the image of the exceptional divisor. We have the following diagram

\[
\begin{array}{ccc}
0 & \to & \overline{T_{\mathcal{M}}_{g,n}} \\
0 & \to & \overline{L}
\end{array}
\]

\[
\begin{array}{ccc}
\overline{T_{\mathcal{M}}_{g,A}[n]} & \to & Q \\
\overline{T_{\mathcal{M}}_{g,A}[n]} & \to & Q
\end{array}
\]

\[
\begin{array}{ccc}
\rho^\ast \overline{T_{\mathcal{M}}_{g,A}[n]} & \to & \overline{Q} \\
\rho^\ast \overline{T_{\mathcal{M}}_{g,A}[n]} & \to & \overline{Q}
\end{array}
\]

where \(\overline{N_{\mathcal{M}_{g,C}[s+1]}/\mathcal{M}_{g,A}[n]} = \pi_\ast \overline{N_{\mathcal{M}_{g,C}[s+1]}/\mathcal{M}_{g,A}[n]}\) and \(\pi : \overline{\mathcal{M}}_{g,C}[s+1] \to \overline{\mathcal{M}}_{g,C}[s+1]\) is the coarse moduli map. By Lemma 2.4 we have

\[
H^1(D, Q) \cong H^1(\overline{\mathcal{M}}_{g,C}[s+1], \overline{N_{\mathcal{M}_{g,C}[s+1]/\mathcal{M}_{g,A}[n]}})
\]

and by Lemma 2.2 \(N_{\overline{\mathcal{M}}_{g,C}[s+1]/\mathcal{M}_{g,A}[n]} = (\psi_\pi')^{(r-1)}\). Furthermore by Theorem 2.8 we get

\[
H^1(\overline{\mathcal{M}}_{g,C}[s+1], (\psi_\pi')^{(r-1)}) = 0
\]

and

\[
H^1(\overline{\mathcal{M}}_{g,C}[s+1], N_{\overline{\mathcal{M}}_{g,C}[s+1]/\mathcal{M}_{g,A}[n]}) = 0
\]

Therefore

\[
H^1(\overline{\mathcal{M}}_{g,C}[s+1], \pi_\ast \overline{T_{\mathcal{M}}_{g,A}[n]}) = 0
\]

and

\[
H^1(\overline{\mathcal{M}}_{g,C}[s+1], \pi_\ast \overline{T_{\mathcal{M}}_{g,A}[n]}) = 0
\]
as well. Let us consider the exact sequence in cohomology
\[ \cdots \to H^1(\mathcal{M}_{g,n}, T_{\mathcal{M}_{g,n}}) \to H^1(\mathcal{M}_{g,n}, \rho^* T_{\mathcal{M}_{g,A[n]}}) \to H^1(D, Q) \to \cdots \]

Since \( H^1(D, Q) = 0 \) we get \( H^1(\mathcal{M}_{g,n}, \rho^* T_{\mathcal{M}_{g,A[n]}}) \cong H^1(\mathcal{M}_{g,n}, T_{\mathcal{M}_{g,n}}) \). Furthermore \( \rho \) is birational and \( \rho_* \mathcal{O}_{\mathcal{M}_{g,n}} = \mathcal{O}_{\mathcal{M}_{g,A[n]}} \). By the projection formula we have \( \rho_* \rho^* T_{\mathcal{M}_{g,A[n]}} = T_{\mathcal{M}_{g,A[n]}} \).

Since \( R^i \rho_* \rho^* T_{\mathcal{M}_{g,A[n]}} = 0 \) for \( i > 0 \) we conclude
\[ H^1(\mathcal{M}_{g,A[n]}, T_{\mathcal{M}_{g,A[n]}}) \cong H^1(\mathcal{M}_{g,A[n]}, \rho_* \rho^* T_{\mathcal{M}_{g,A[n]}}) \cong H^1(\mathcal{M}_{g,n}, \rho^* T_{\mathcal{M}_{g,A[n]}}) \]

On the other hand, if \( g + n \geq 4 \) then
\[ H^1(\mathcal{M}_{g,A[n]}, \rho^* T_{\mathcal{M}_{g,A[n]}}) \cong H^1(\mathcal{M}_{g,n}, T_{\mathcal{M}_{g,n}}) = 0 \]

by [Hac08] Theorem 2.3. We conclude that, if \( g + n \geq 4 \) then \( H^1(\mathcal{M}_{g,A[n]}, T_{\mathcal{M}_{g,A[n]}}) = 0 \), that is \( \mathcal{M}_{g,A[n]} \) does not have locally trivial first order infinitesimal deformations. \( \square \)

Finally, we get the following rigidity result for the coarse moduli spaces \( \mathcal{M}_{g,A[n]} \).

**Theorem 2.7.** If \( g + n > 4 \), over an algebraically closed field of characteristic zero, the coarse moduli space \( \mathcal{M}_{g,A[n]} \) is rigid for any vector of weights \( A[n] \).

**Proof.** Let \( \rho : \mathcal{M}_{g,n} \to \mathcal{M}_{g,A[n]} \) be the reduction morphism. Let \([C, (x_1, ..., x_n)] \in \mathcal{M}_{g,A[n]} \) be a point with \( x_{i_1} = ... = x_{i_r} \), and \([\Gamma, (y_1, ..., y_n)] \in \rho^{-1}([C, (x_1, ..., x_n)]) \subset \mathcal{M}_{g,n} \). Then \( \Gamma = \Gamma_1 \cup \cdots \cup \Gamma_k \) where \( \Gamma_1, ..., \Gamma_k \) are rational components contracted to the point \( x_{i_1} = ... = x_{i_r} \in C \), and \( \Gamma' \) is isomorphic to \( C \). Therefore, we have that \( Aut(C, (x_1, ..., x_n)) \cong Aut(\Gamma, (y_1, ..., y_n)) \).

Now, let us consider the following codimension two, that is of maximal dimension, irreducible components of the singular locus of \( \mathcal{M}_{g,A[n]} \):

- \( Z_i \) for \( i = 4, 6 \), is the codimension two loci parametrizing curves with an elliptic tail having four and six automorphisms respectively;
- \( Y \) is the locus parametrizing reducible curves \( E \cup C \) where \( E \) is an elliptic curve with a marked point which is fixed by the elliptic involution, and \( C \) is a curve of genus \( g - 1 \) with \( n - 1 \) marked points;
- \( W \) is the locus parametrizing reducible curves \( C_1 \cup C_2 \) where \( C_1 \) and \( C_2 \) are of genus two and \( g - 2 \) respectively, the marked points are on \( C_2 \), and \( C_1 \cap C_2 \) is a fixed point of the hyperelliptic involution on \( C_1 \).

By the observation on the automorphism groups of the curves in the first part of the proof and [FM16 Proposition 5.7], we have that when \( g + n > 4 \) the only codimension two irreducible components of \( \text{Sing}(\mathcal{M}_{g,A[n]}) \) are \( Z_4, Z_6, Y \) and \( W \). Furthermore, each component contains dense open subsets, denoted by a superscript zero, with complement of codimension at least two such that \( \mathcal{M}_{g,A[n]} \) has transversal \( A_1 \) singularities along \( Z_4^0, Y^0 \) and \( W^0 \), and transversal \( \frac{1}{2}(1, 1) \) singularities along \( Z_6^0 \).

By Proposition 2.6 we know that \( \mathcal{M}_{g,A[n]} \) does not have locally trivial deformations. Therefore, it is enough to prove that \( H^0(\mathcal{M}_{g,A[n]}, \mathcal{E}xt^1(\mathcal{O}_{\mathcal{M}_{g,A[n]}}, \mathcal{O}_{\mathcal{M}_{g,A[n]}})) = 0 \).

Note that \( \mathcal{E}xt^1(\mathcal{O}_{\mathcal{M}_{g,A[n]}}, \mathcal{O}_{\mathcal{M}_{g,A[n]}}) \) is a coherent sheaf supported on \( \text{Sing}(\mathcal{M}_{g,A[n]}) \). By [Fa95 Lemmas 2.4, 2.5] there are no sections of \( \mathcal{E}xt^1(\mathcal{O}_{\mathcal{M}_{g,A[n]}}, \mathcal{O}_{\mathcal{M}_{g,A[n]}}) \) supported on the components of \( \text{Sing}(\mathcal{M}_{g,A[n]}) \) of codimension greater than two.

Now, to conclude it is enough to argue as in [FM16 Theorem 5.13] by using the deformation theory of varieties with transversal \( A_1 \) and \( \frac{1}{2}(1, 1) \) singularities developed in [FM16 Sections 5.4, 5.5]. \( \square \)

**Remark 2.8.** Note that by Remark 1.2 we have that \( \mathcal{M}_{1,A[2]} \cong \mathcal{M}_{1.2} \) for any weight data. Therefore, by [FM16 Theorem 4.8] \( \mathcal{M}_{1,A[2]} \) does not have locally trivial deformations, while its family of first order infinitesimal deformations is non-singular of dimension six and the general deformation is smooth.
2.1. Automorphisms of Hassett spaces in arbitrary characteristic. In this section we apply the rigidity results in Section 2 to extend the main results on the automorphism groups of Hassett spaces in [MM16] over an arbitrary field. In order to lift automorphisms from zero to positive characteristic we will use the techniques developed in [FM16 Section 1] considering the ring \( W(K) \) of Witt vectors over \( K \), see [Wi36] for details.

For our purposes it is enough to keep in mind that \( W(K) \) is a discrete valuation ring with a closed point \( x \in \text{Spec}(W(K)) \) with residue field \( K \), and a generic point \( \xi \in \text{Spec}(W(K)) \) with residue field of characteristic zero.

Note that not all permutations of the markings define an automorphism of the space \( \overline{M}_{g,A[n]} \). Indeed in order to define an automorphism, permutations have to preserve the weight data in a suitable sense.

For instance, consider Hassett space \( \overline{M}_{1,A[4]} \) with weights \((1,1/3,1/3,1/3)\) and the divisor parametrizing reducible curves \( C_1 \cup C_2 \), where \( C_1 \) has genus zero and markings \((1,1/3,1/3)\), and \( C_2 \) has genus one and marking 1/3. After the transposition \( 1 \leftrightarrow 4 \) the genus zero component has markings \((1/3,1/3,1/3)\), so it is contracted. This means that the transposition \( 1 \leftrightarrow 4 \) induces just a birational automorphism of \( \overline{M}_{1,A[4]} \) contracting a divisor on a codimension two subscheme. This example leads us to the following definition.

**Definition 2.9.** A transposition \( i \leftrightarrow j \) of two marked points is admissible if and only if for any \( h_1,\ldots,h_r \in \{1,\ldots,n\} \setminus \{i,j\} \), with \( r \geq 2 \),

\[
a_i + \sum_{k=1}^r a_{h_k} \leq 1 \iff a_j + \sum_{k=1}^r a_{h_k} \leq 1.
\]

We denote by \( S_{A[n]} \subseteq S_n \) the subgroup of permutations generated by admissible transpositions.

We begin by taking into account Hassett spaces appearing in Construction 1.3.

**Theorem 2.10.** Let \( K \) be any field. For Hassett spaces appearing in Construction 1.3 we have that if \( 2 \leq r \leq n-4 \) then:

- \( \text{Aut}(\overline{M}^K_{0,A,[1]}[n]) \cong S_{n-r} \times S_r, \)
- \( \text{Aut}(\overline{M}^K_{0,A,[r]}[n]) \cong S_{n-r-1} \times S_r, \) if \( 1 < s < n-r-2, \)
- \( \text{Aut}(\overline{M}^K_{0,A,-r-2}[n]) \cong S_{n-r-1} \times S_{r+1}, \)

and if \( r = n-3 \) then \( s = 1, \overline{M}^K_{0,A,-3,1}[n] \cong \overline{M}^K_{0,n}, \) and \( \text{Aut}(\overline{M}^K_{0,A,-n-3,1}[n]) \cong S_n \) for any \( n \geq 5. \)

Finally, if \( \text{char}(K) = 0 \) for the Losev-Manin moduli space we have:

\[
\text{Aut}(\overline{M}^K_{0,A,-n-3,1}[n]) \cong (K^*)^{n-3} \times (S_2 \times S_{n-2})
\]

**Proof.** Let \( K \) be a field of characteristic zero, and let \( \overline{K} \) be its algebraic closure. By [FM16 Proposition 1] there exists an injective morphism of groups

\[
\chi : \text{Aut}(\overline{M}^K_{0,A,[n]}) \to \text{Aut}(\overline{M}^\overline{K}_{0,A,[n]})
\]

for any weight data \( A[n] \). To conclude that, for Hassett spaces appearing in the statement, \( \chi \) is surjective it is enough to apply [MM16 Theorem 1].

Now, let \( K_p \) be a field of characteristic \( p > 0 \), and \( W(K_p) \) the ring of Witt vectors of \( K_p \) with residue field \( K \) of characteristic zero. By Theorem 2.5 we have that \( H^1(\overline{M}^K_{0,A,[n]}, T_{\overline{M}^K_{0,A,[n]}}) = 0. \)

Furthermore, if \( r \neq 1 \) and \( s \neq n-3 \) by the first part of the proof we know that \( \text{Aut}(\overline{M}^K_{0,A,r,[n]}) \) is finite. Since \( \text{char}(K) = 0 \) this yields \( H^0(\overline{M}^K_{0,A,[n]}, T_{\overline{M}^K_{0,A,[n]}}) = 0. \) Now, by [FM16 Theorem 1.6] we get an injective morphism of groups

\[
\chi_p : \text{Aut}(\overline{M}^K_{0,A,r,[n]}) \to \text{Aut}(\overline{M}^K_{0,A,r,[n]})
\]

which by the first part of proof is surjective as well. \( \Box \)

Now, let us move to the case \( g \geq 1. \)
ON THE RIGIDITY OF MODULI OF WEIGHTED POINTED STABLE CURVES

Theorem 2.11. Let $K$ be a field with $\text{char}(K) \neq 2$. If $g \geq 1$ and $2g - 2 + n \geq 3$ then

$$\text{Aut}(\overline{M}^K_{g,A[n]}) \cong \text{Aut}(\overline{M}^K_{g,A[n]}) \cong S_{A[n]}.$$  

Furthermore, if $K$ is algebraically closed with $\text{char}(K) \neq 2, 3$ then we have

- $\text{Aut}(\overline{M}^K_{1,A[2]}) \cong (K^*)^2$ while $\text{Aut}(\overline{M}^K_{1,A[2]})$ is trivial,
- $\text{Aut}(\overline{M}^K_{1,A[1]}) \cong \text{PGL}(2, K)$ while $\text{Aut}(\overline{M}^K_{1,A[1]}) \cong K^*.$

Proof. First of all, note that by Remark 1.2 we have

$$\text{Aut}(\overline{M}^K_{1,A[2]}) \cong (K^*)^2 \text{ while } \text{Aut}(\overline{M}^K_{1,A[2]}) \text{ is trivial},$$

$$\text{Aut}(\overline{M}^K_{1,A[1]}) \cong \text{PGL}(2, K) \text{ while } \text{Aut}(\overline{M}^K_{1,A[1]}) \cong K^*.$$

By [Has03, Section 2.1.2] we may construct an explicit family of such weighted curves

$$\overline{\mathcal{C}}(\tilde{A}) \to \overline{M}_{0,n} \text{ over } \overline{M}_{0,n} \text{ as an explicit blow-down of the universal curve over } \overline{M}_{0,n}.$$  

Furthermore, if $a_i < 1$ for any $i = 1, ..., n$ we may interpret the geometric invariant theory quotient $(\mathbb{P}^1)^n/\text{SL}_2$ with respect to the linearization $\mathcal{O}(a_1, ..., a_n)$ as the moduli space of such weighted curves

$$\overline{M}_{0,\tilde{A}[n]}$$

associated to the family $\overline{\mathcal{C}}(\tilde{A}).$

In this section we will show that Hassett spaces with weights summing to two can have non-trivial first order infinitesimal deformations by considering a specific example.

Let us consider the weight data

$$A[6] = (1, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3), \quad \tilde{A}[6] = (1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3)$$

and the reduction morphism

$$\rho : \overline{M}_{0,A[6]} \to \overline{M}_{0,\tilde{A}[6]}.$$  

By [Has03, Section 6.2] the moduli space $\overline{M}_{0,A[6]}$ is the blow-up of $\mathbb{P}^3$ at five points $p_i \in \mathbb{P}^3$ in linear general position.
Let \( \{f_p \} \subset |O_{P^3}(2)| \) be the linear system of quadrics in \( P^3 \) through the \( p_i \)'s. Note that \( |f_p \) induces a rational map \( \phi : P^3 \dashrightarrow P^4 \) whose image is a hypersurface \( S \subset P^4 \) of degree \( \deg(S) = 2^3 - 5 = 3 \).

Since the base locus of \( \phi \) consists exactly of the \( p_i \)'s we get a morphism \( \tilde{\phi} : M_{0, [6]} \to P^4 \) fitting in the following commutative diagram

\[
\begin{array}{ccc}
M_{0, [6]} & \xrightarrow{\tilde{\phi}} & P^4 \\
\pi_{p_i} \downarrow & & \downarrow \phi \\
\mathbb{P}^3 & \xrightarrow{\phi} & S \subset P^4
\end{array}
\]

where \( \pi_{p_i} : M_{0, [6]} \to \mathbb{P}^3 \) is the blow-up of the \( p_i \)'s. Note that the only curves contracted by \( \tilde{\phi} \) are the strict transforms \( \tilde{L}_{i,j} \) of the then lines \( L_{i,j} = \langle p_i, p_j \rangle \). Therefore \( \tilde{\phi} \) is a small contraction, and since

\[
N_{L_{i,j}/M_{0, [6]}} \cong O_{L_{i,j}}(-1) \oplus O_{L_{i,j}}(-1)
\]

we conclude that \( S = \tilde{\phi}(M_{0, [6]}) \subset P^4 \) is a cubic surface singular at the ten points \( q_{i,j} = \tilde{\phi}(\tilde{L}_{i,j}) \), and these ten points are nodes of \( S \), that is ordinary singularities with \( \mult q_{i,j}(S) = 2 \). Note that in dimension greater than two nodes are not finite quotient singularities, therefore they may contribute to the infinitesimal deformations of \( S \).

A cubic hypersurface in \( P^4 \) whose singular locus consists of ten nodes is, up to a change of coordinates, a Segre cubic \( \text{[Do15, Proposition 2.1]} \), that is the hypersurface defined by the equations

\[
\begin{align*}
0 & = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 \\
0 & = x_0 + x_1 + x_2 + x_3 + x_4 + x_5.
\end{align*}
\]

where \( [x_0 : \ldots : x_5] \) are homogeneous coordinates on \( P^5 \). The ten nodes are located at the points conjugate to \( [1 : 1 : 1 : -1 : -1 : -1] \) under the action of \( S_6 \) permuting the coordinates.

The Segre cubic is a very interesting and peculiar variety in classical algebraic geometry, indeed its Hessian is the Barth-Nieto quintic, its intersection with a hyperplane of the form \( x_i = 0 \) is the Clebsch cubic surface, while its intersection with a hyperplane of the type \( x_i = x_j \) is the Cayley’s nodal cubic surface, and finally it is dual to the Igusa quartic 3-fold in \( P^4 \) \( \text{[Do15]} \).

The discussion above shows that we may interpret the morphism \( \tilde{\phi} \) as the reduction morphism \( \rho : M_{0, [6]} \to M_{0, \widehat{[6]}} \), and Hassett space \( \overline{M}_{0, [6]} \) as the Segre cubic \( S \subset P^4 \).

**Proposition 3.1.** Let \( S \subset P^4 \) be the Segre cubic. Then \( \text{Aut}(S) \cong S_6 \) is the permutation group on six elements.

**Proof.** Let us consider the weights \( A[6] \) and \( \widehat{A}[6] \) in (3.1), and the small resolution \( \rho : M_{0, [6]} \to M_{0, \widehat{[6]}} \) in (3.2). Since \( \text{Sing}(S) \) consists of ten points which are ordinary double points we may resolve the singularities of \( S \) just by blowing-up these ten points. Let \( f : X \to S \) be the blow-up.

Now, by Construction (1.3) the blow-up of \( M_{0, [6]} \) along the strict transforms of the ten lines through two of the five points is isomorphic to \( M_{0, 6} \), and we have a reduction morphism \( \overline{\rho} : M_{0, 6} \to M_{0, [6]} \).

Note that the morphism \( \rho \circ \overline{\rho} : M_{0, 6} \to S \) maps the exceptional divisor \( E_{i,j} \) over the strict transform \( \tilde{L}_{i,j} \) to the singular point \( q_{i,j} \in S \). Therefore, by the universal property of the blow-up \( \text{[Har77, Proposition 7.4]} \) there exists a unique morphism \( \xi : M_{0, 6} \to X \) such that the following diagram

\[
\begin{array}{ccc}
M_{0, 6} & \xrightarrow{\xi} & X \\
\overline{\rho} \downarrow & & \downarrow f \\
M_{0, [6]} & \xrightarrow{\rho} & S
\end{array}
\]

commutes. Now, note that since \( X \) is smooth \( \xi \) can not be a small contraction. On the other hand, since \( \rho \) is small and \( \xi \) must map the exceptional divisor \( E_{i,j} \) onto the exceptional divisor
ep is the Tyurina number of the node $p_i$ by applying $H(3.5)$. Hence Equation (3.4) yields
\[
\chi \leq \chi(3.4)
\]
and since $\chi$ is an isomorphism, $X \cong M_{0,6}$.

Now, let $\phi \in \operatorname{Aut}(S)$ be an automorphism. Then $\phi$ must preserve the set $\operatorname{Sing}(S)$ of the ten singular points. Therefore, by [Har77, Corollary 7.15] $\phi$ lifts to an automorphism $\tilde{\phi}$ of $X \cong M_{0,6}$, and we get an injective morphism of groups
\[
\chi : \operatorname{Aut}(S) \to \operatorname{Aut}(M_{0,6})
\]
Now, to conclude it is enough to recall that $S \cong M_{0,6}$ is the geometric invariant theory quotient $[\mathbb{P}^1]^{\geq 3}/\operatorname{SL}_2$ with respect to the symmetric linearization $\mathcal{O}(1/3,\ldots,1/3)$, hence $S_6$ acts on $S \cong M_{0,6}$ by permuting the marked points, and that $\operatorname{Aut}(M_{0,6}) \cong S_6$ [BM13 Theorem 3], [Ma14 Theorem 3.10], [FMI0 Theorem 1.1]. Hence $\chi$ is surjective as well. $\Box$

Now, we are ready to study the infinitesimal deformations of the Segre cubic.

**Theorem 3.2.** Hassett moduli space $M_{0,6}$ with weights $a_1 = \ldots = a_6 = \frac{1}{3}$ does not have locally trivial deformations, while its family of first order infinitesimal deformations is non-singular of dimension ten and the general deformation is smooth.

**Proof.** All along the proof we will identify $M_{0,6}$ with the Segre cubic $S \subset \mathbb{P}^4$. The first order infinitesimal deformations of $S$ are parametrized by the group $\operatorname{Ext}^1(\Omega_S, \mathcal{O}_S)$. The sheaf $\operatorname{Ext}^1(\Omega_S, \mathcal{O}_S)$ is supported on the singularities of $S$, and since $S$ has isolated singularities $\operatorname{Ext}^1(\Omega_S, \mathcal{O}_S)$ can be computed separately for each singular point.

Recall that $S$ is singular at ten nodes, let $p \in S$ be one of these nodes. Then, étale locally, in a neighborhood of $p$ the Segre cubic $S$ is isomorphic to an étale neighborhood of the singularity $S = \{f(x, y, z, w) = x^2w + xy - zw = 0\} \subset \mathbb{A}^4$.

Indeed, note that the partial derivatives of $f$ vanish simultaneously just at $(0, 0, 0, 0) \in \mathbb{A}^4$, and $\partial f / \partial x \partial y = 1$. Furthermore, the projective tangent cone of $S$ at $(0, 0, 0, 0)$ is a smooth quadric surface in $\mathbb{P}^3$. This means that $S$ has an ordinary singularity of multiplicity two at the origin.

Let $R = K[x, y, z, w] / (x^2 w + xy - zw)$, and let us consider the free resolution
\[
0 \to R \to R^{\oplus 4} \to \Omega_R \to 0
\]
of $\Omega_R$, where $\psi_j$ is the matrix of the partial derivatives of $f = x^2w + xy - zw$. Therefore, we get
\[
\operatorname{Ext}^1(\Omega_R, R) \cong R / \operatorname{Im}(\psi_j) \cong K[x, y, z, w] / (x^2 w + xy - zw, 2xw + y, x, -w, x^2 - z) \cong K,
\]
\[
\operatorname{Ext}^2(\Omega_R, R) = 0.
\]
Now, let us consider the exact sequence
\[
0 \to \mathcal{O}_S(-3) \to \mathcal{O}_{\mathbb{P}^4}[S] \to \mathcal{O}^1_{\mathbb{P}^4} \to 0
\]
by applying $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{O}_S, \mathcal{O}_S)$ we get
\[
(3.3) \quad 0 \to T_S \to T_{\mathbb{P}^4} \to \mathcal{O}_S(3) \to \operatorname{Ext}^1(\Omega_S, \mathcal{O}_S) \to 0
\]
Therefore, $H^i(S, T_S) = 0$ for $i \geq 2$, and by taking Euler-Poincaré characteristics we have
\[
\chi(T_S) + \chi(\mathcal{O}_S(3)) = \chi(T_{\mathbb{P}^4}) + \chi(\mathcal{O}_S)
\]
and since $\chi(T_{\mathbb{P}^4}) = 24$, and $\chi(\mathcal{O}_S(3)) = \binom{4+3}{3} - 1 = 34$ we get
\[
(3.4) \quad \chi(T_S) = \chi(\mathcal{O}_S) - 10
\]
Note that we may interpret $\chi(\mathcal{O}_S) = h^0(\mathcal{O}_S) = h^0(S, \mathcal{O}_S) = \sum_{i=1}^{10} \tau_{p_i}$, where $\tau_{p_i}$ is the Tyurina number of the node $p_i \in S$, that is the rank at $p_i$ of the skyscraper sheaf $\mathcal{O}_S$. Note that since $\mathcal{O}_S = 0$ we have $\tau_{p_i} = 1$ for any $i = 1, \ldots, 10$, and hence Equation (3.4) yields
\[
(3.5) \quad h^0(S, T_S) - h^1(S, T_S) = 0
\]
Now, by Proposition 3.1 we have that $\text{Aut}(\mathcal{S}) \cong S_6$. Therefore, $\mathcal{S}$ does not have infinitesimal automorphisms and $H^0(\mathcal{S}, T_\mathcal{S}) = 0$. This last fact together with Equation (3.3) forces $H^1(\mathcal{S}, T_\mathcal{S}) = 0$ as well.

So the sequence

$$H^1(\mathcal{S}, T_\mathcal{S}) \to \text{Ext}^1(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S}) \to H^0(\mathcal{S}, \text{Ext}^1(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S})) \to H^2(\mathcal{S}, T_\mathcal{S})$$

yields

$$\dim_K(\text{Ext}^1(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S})) = h^0(\mathcal{S}, \text{Ext}^1(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S})) = 10$$

Finally, to compute the dimension of the obstruction space $\text{Ext}^2(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S})$ we use the local-to-global Ext spectral sequence

$$H^i(\mathcal{S}, \text{Ext}^j(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S})) \Rightarrow \text{Ext}^{i+j}(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S})$$

Note that $H^1(\mathcal{S}, \text{Ext}^1(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S})) = 0$ because $\text{Ext}^1(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S})$ is supported on a zero dimensional scheme. Moreover, by (3.3) we have $H^2(\mathcal{S}, \text{Ext}^0(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S})) = H^2(\mathcal{S}, T_\mathcal{S}) = 0$. Finally, $\text{Ext}^2(\Omega_R, R) = 0$ yields $H^0(\mathcal{S}, \text{Ext}^2(\Omega_\mathcal{S}, \mathcal{O}_\mathcal{S})) = 0$ as well.

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