ON C*-ALGEBRAS AND K-THEORY FOR INFINITE-DIMENSIONAL FREDHOLM MANIFOLDS

DORIN DUMITRĂȘCU AND JODY TROUT†

Abstract. Let $M$ be a smooth Fredholm manifold modeled on a separable infinite-dimensional Euclidean space $E$ with Riemannian metric $g$. Given an augmented Fredholm filtration $\mathcal{F}$ of $M$ by finite-dimensional submanifolds $\{M_n\}_{n=k}^\infty$, we associate to the triple $(M, g, \mathcal{F})$ a non-commutative direct limit $C^*$-algebra

$$A(M, g, \mathcal{F}) = \lim_{\to} A(M_n)$$

that can play the role of the algebra of functions vanishing at infinity on the non-locally compact space $M$. The $C^*$-algebra $A(\mathcal{F})$, as constructed by Higson-Kasparov-Trout for their Bott periodicity theorem, is isomorphic to our construction when $M = E$. If $M$ has an oriented Spin$q$-structure ($1 \leq q \leq \infty$), then the $K$-theory of this $C^*$-algebra is the same (with dimension shift) as the topological $K$-theory of $M$ defined by Mukherjea. Furthermore, there is a Poincaré duality isomorphism of this $K$-theory of $M$ with the compactly supported $K$-homology of $M$, just as in the finite-dimensional spin setting.

1. Introduction

Infinite-dimensional Hilbert manifolds have been studied since the 1960’s, with main applications in infinite-dimensional differential topology, global analysis, non-linear PDEs, and other areas. This paper is concerned with constructing $C^*$-algebras and computing the $K$-theory for a particular class of infinite-dimensional Hilbert manifolds, namely Fredholm manifolds \cite{18,20,21}. This is part of a research program to introduce concepts and techniques from Alain Connes’ noncommutative geometry \cite{11}, e.g., $C^*$-algebras, $K$-theory, cyclic (co)homology, and spectral triples, into the study of Fredholm manifolds.

But first, let us review the finite-dimensional case. Given $M$ a finite-dimensional Riemannian manifold, let $C_0(M)$ be the commutative $C^*$-algebra of all continuous complex-valued functions which vanish at infinity on $M$. This $C^*$-algebra categorically encodes the topological properties of $M$ \cite{46} and, by the Serre-Swan theorem, plays a dual role in the $K$-theory of $M$:

$$K^j(M) \cong K_j(C_0(M)), \quad j = 0, 1,$$

where $K^j(M)$ is the (reduced) topological $K$-theory of $M$ \cite{48}. Furthermore, if $M$ has a spin (or spin°) structure \cite{33}, there is a Poincaré duality isomorphism \cite{26,33}:

$$K^{n-j}(M) \cong K_j^c(M), \quad j = 0, 1,$$

1991 Mathematics Subject Classification. 19, 46, 47, 55, 57, 58.

Key words and phrases. $C^*$-algebra, Fredholm manifold, direct limit, $K$-theory, $K$-homology, Poincaré duality.

† The second author was partially supported by NSF Grant DMS-0071120.
where $K^j_*(M)$ denotes the dual (compactly supported) $K$-homology of $M$ and $n$ is the dimension of $M$.

The other $C^*$-algebra for a finite-dimensional $M$ is non-commutative and constructed using the Riemannian metric $g$. For each $x \in M$, the tangent space $T_x M$ of $M$ is a finite-dimensional Euclidean space with inner product $g_x$. Thus, we can form the complex Clifford algebra $\text{Cliff}(T_x M, g_x)$ (see Section 2). It has a canonical structure as a finite-dimensional $\mathbb{Z}_2$-graded $C^*$-algebra. The family of $C^*$-algebras $\{\text{Cliff}(T_x M, g_x)\}_{x \in M}$ naturally forms a $\mathbb{Z}_2$-graded, $C^*$-algebra vector bundle $\text{Cliff}(TM) \to M$, called the Clifford algebra bundle of $M$. We then can define

$$C(M) = C_0(M, \text{Cliff}(TM))$$

to be the $C^*$-algebra of continuous sections of the Clifford algebra bundle of $M$ vanishing at infinity. This $C^*$-algebra was used by Kasparov in studying the Novikov Conjecture, where he used the notation $C_\tau(M)$. If $M$ is even-dimensional and has a spin structure (or, more generally, a spin-structure) then this $C^*$-algebra is Morita equivalent to $C_0(M)$. (In general, $C(M)$ is Morita equivalent to $C_0(TM)$.)

By the Morita invariance of $K$-theory, it follows that

$$K_j(C(M)) \cong K_j(C_0(M)) \cong K^j(M), \quad j = 0, 1.$$ 

For $M$ odd-dimensional and spin, this is more complicated. (See Proposition 5.14.)

If $M$ is an infinite-dimensional Hilbert manifold modeled on a separable infinite-dimensional Euclidean (i.e., real Hilbert) space $\mathcal{E}$, then these two constructions do not work. Both fail since compact subsets of $M = \mathcal{E}$ are “thin”, i.e., contained in finite-dimensional subspaces. Thus, $C_0(\mathcal{E}) = \{0\}$ since there are no compactly supported continuous functions on $\mathcal{E}$ which are non-zero. However, the Clifford $C^*$-algebra has been generalized by Higson-Kasparov-Trout to the case $M = \mathcal{E}$, by a direct limit construction that exploits an important property of Clifford algebras with respect to orthogonal sums (see equation (2)). The component $C^*$-algebras in the direct limit are given by

$$\mathcal{A}(E^n) = C_0(\mathbb{R}) \hat{\otimes} C(\mathcal{E}) \cong C_0(\mathbb{R}) \hat{\otimes} C_0(E^n, \text{Cliff}(E^n))$$

where $\hat{\otimes}$ denote the $\mathbb{Z}_2$-graded tensor product and $C_0(\mathbb{R})$ is graded by even and odd functions. Since the map $E^n \to \mathcal{A}(E^n)$ is functorial with respect to inclusions of finite-dimensional subspaces, one can construct a non-commutative direct limit $C^*$-algebra (in the better notation of [24]):

$$\mathcal{A}(\mathcal{E}) = \lim_{\overset{\longrightarrow}{E^n}} \mathcal{A}(E^n)$$

where the direct limit is taken over all finite-dimensional subspaces $E^n \subset \mathcal{E}$. (See Example 4.13 for more on this construction and how it fits into our theory.) This $C^*$-algebra was used to prove an equivariant Bott periodicity theorem for infinite-dimensional Euclidean spaces [26] and has had applications to proving cases of the Novikov Conjecture and, more generally, the Baum-Connes Conjecture [24, 49].

Now, suppose the Hilbert manifold $M$ is fibered as the total space of a smooth infinite rank Euclidean vector bundle $p : F \to X$, with fiber $\mathcal{E}$ and compatible affine connection $\nabla$, over a finite-dimensional Riemannian manifold $X$. Let $p_u : F^n \to X$ be a finite rank subbundle of $F$. Using the connection $\nabla$ and the metrics on $F$ and $X$, we can give the total space $F_u$ a canonical structure of a Riemannian manifold
and define the component $C^*$-algebra
\[ A(F^a) = C_0(\mathbb{R}) \hat{\otimes} \mathcal{C}(F^a) \cong C_0(\mathbb{R}) \hat{\otimes} C_0(F^a, \text{Cliff}(TF^a)). \]
Since the map $F^a \mapsto A(F^a)$ is functorial with respect to inclusions of finite-dimensional subbundles \[45\], we can then construct a direct limit $C^*$-algebra:
\[ A(F, \nabla) = \lim_{\rightarrow} A(F^a) \]
where the direct limit is taken over all finite rank subbundles $p_a : F^a \to X$ of $F$.
Trout \[45\] used this $C^*$-algebra to prove an equivariant Thom isomorphism theorem for infinite rank Euclidean bundles, which reduces to the Higson-Kasparov-Trout Bott periodicity theorem when the base manifold $X$ is a point.

For a more general curved Hilbert manifold $M$, with Riemannian metric $g$, there does not seem to be a natural generalization of the previous constructions. Based on the above, one would be tempted to construct a direct limit $C^*$-algebra
\[ “A(M) = \lim_{M_n \subset M} A(M_n)” \]
where the component $C^*$-algebras should be given by
\[ A(M_n) = C_0(\mathbb{R}) \hat{\otimes} C(M_n) \]
and the direct limit is taken over all finite-dimensional submanifolds $M_n \subset M$. The problem is that, even though the component $C^*$-algebras have many functoriality properties (as discussed in Section 2), if we are given smooth (isometric) inclusions $M_n \subset M_b \subset M_c$ of finite-dimensional submanifolds of $M$, there is no obvious way to define a commuting diagram (as there is in the Bott periodicity and Thom isomorphism cases)
\[ \begin{array}{ccc}
A(M_b) & \rightarrow & A(M_c) \\
\downarrow & & \downarrow \\
A(M_n) & \rightarrow & A(M_n+1)
\end{array} \]
needed to construct the corresponding direct limit.

However, if the Hilbert manifold $M$ has a Fredholm structure, then we can construct a direct limit $C^*$-algebra by choosing an appropriate countable sequence \{${M_n}$\}_{n=1}^{\infty} of expanding, topologically closed, finite-dimensional submanifolds of $\text{dim}(M_n) = n$. The sequence \{${M_n}$\}_{n=1}^{\infty} is called a Fredholm filtration of $M$. (See Section 3 for the geometric definitions and details.) The countability of this sequence of submanifolds clearly simplifies the direct limit construction since only each “Gysin” map $A(M_n) \to A(M_{n+1})$ needs to be constructed, which will require some non-trivial geometry (i.e., connections and normal bundles.)

Equip the Riemannian Fredholm manifold $(M, g)$ with an augmented Fredholm filtration $\mathcal{F} = (M_n, U_n)_{n=1}^{\infty}$ (as in Definition \[63\]) where $U_n$ is a total open tubular neighborhood of $M_n \hookrightarrow M_{n+1}$. Section 4 contains the construction of a noncommutative direct limit $C^*$-algebra for the triple $(M, g, \mathcal{F})$:
\[ A(M, g, \mathcal{F}) = \lim_{\rightarrow} A(M_n) \]
that can play the role of the algebra of functions vanishing at infinity on $M$. 
Using ideas of Mukherjea \cite{34,35} to associate cohomology functors to Fredholm manifolds via Fredholm filtrations, the topological K-theory groups of \((M,F)\) are defined as the direct limit:

\[
K^{\infty-j}(M,F) = \lim_{\rightarrow} K^{n-j}(M_n), \quad j = 0, 1,
\]

where the connecting map \(K^{n-j}(M_n) \to K^{(n+1)-j}(M_{n+1})\) is the Gysin (or shriek) map (Definition \ref{def:shriek}) of the embedding \(M_n \hookrightarrow M_{n+1}\), and the inspiration for our connecting map \(\mathcal{A}(M_n) \to \mathcal{A}(M_{n+1})\). Note that this definition does, in general, depend on the choice of Fredholm filtration, since the sequence \(\{M_n\}_{n=k}^{\infty}\) may not be K-orientable \cite{17,14}.

But, using appropriate notions of Spin\(_q\)-structures (see Section 5.2) for Riemannian Fredholm manifolds, originally investigated by Anastasiei \cite{2} and de la Harpe \cite{14}, the following Serre-Swan and Poincaré duality isomorphism theorem (combining Theorems \ref{thm:serre} and \ref{thm:poincare}) is obtained:

**Theorem 1.1.** Let \((M,g)\) be a smooth Fredholm manifold with oriented Riemannian \(q\)-structure \(1 \leq q \leq \infty\). If \(M\) has a Spin\(_q\)-structure then there are isomorphisms

\[
K^{\infty-j}(M,F) \cong K_{j+1}(\mathcal{A}(M,g,F)) \cong K^j(M), \quad j = 0, 1,
\]

where \(\mathcal{F} = (M_n,U_n)_{n=k}^{\infty}\) is any augmented Fredholm filtration of \(M\).

Thus, the K-theory groups of \((M,F)\) and of the \(C^*\)-algebra \(\mathcal{A}(M,g,F)\) do not depend on the choice of the Riemannian metric \(g\) or the (augmented) Fredholm filtration \(\mathcal{F}\). The dimension shift and the relation with Poincaré duality for finite-dimensional spin manifolds then justifies our interpretation of \(\mathcal{A}(M,g,F)\) as an appropriate non-commutative (suspension of the) “algebra of functions vanishing at infinity” on \(M\).

Finally, it should be noted that, given a Fredholm filtration \(\{M_n\}_{n=k}^{\infty}\) of \(M\), we can also naturally associate an inverse limit algebra, called by Phillips \cite{37} a \(\sigma\)-\(C^*\)-algebra,

\[
C_0^{\text{inv}}(M) = \lim_{\leftarrow} C_0(M_n)
\]

where the connecting map \(C_0(M_{n+1}) \to C_0(M_n)\) is the pullback under the inclusion \(M_n \hookrightarrow M_{n+1}\). However, this algebra does not have the structure of a \(C^*\)-algebra, in general. Moreover, if we try to define the “topological K-theory” of \(M\) as the inverse limit (using contravariance of topological K-theory)

\[
K_{\text{inv}}^j(M) = \lim_{\leftarrow} K^j(M_n), \quad j = 0, 1,
\]

then we do not get a well-behaved functor. Indeed, as Buhstaber and Mishchenko have shown, the resulting K-theory sequence of a pair \((M,N)\) is not exact, in general \cite{30,10} even for CW-complexes. Also, K-theory does not behave well with respect to inverse limits since there is a Milnor \(\lim^1\)-sequence (Theorem 3.2 \cite{38}):

\[
0 \to \lim^1 K^{j+1}(M_n) \to RK_j(C_0^{\text{inv}}(M)) \to K_{\text{inv}}^j(M) \to 0
\]

where \(RK_j\) is the representable K-theory for \(\sigma\)-\(C^*\)-algebras developed by Phillips \cite{38} and Weidner \cite{17}. Hence, there would be no corresponding Serre-Swan duality theorem as in the finite-dimensional category.

The authors would like to thank John Roe, Carolyn Gordon, David Webb, Dana Williams, Gregory Leibon, and the referee for interesting discussions and helpful suggestions.
2. Clifford $C^*$-algebras and the Thom $\ast$-Homomorphism

In this section we assemble the constructions and results for finite-dimensional manifolds that are needed to carry out the direct limit construction of the $C^*$-algebra of an infinite-dimensional Fredholm manifold. All of the manifolds in this section are assumed to be smooth, Hausdorff, paracompact, and finite-dimensional. For a detailed discussion of most of the results in this section, including more proofs, see Section 2 of Trout [45].

Let $V$ be a finite-dimensional Euclidean vector space with inner product $\langle \cdot, \cdot \rangle$. The complex Clifford algebra of $V$, denoted $\text{Cliff}(V)$, is the universal complex $C^*$-algebra (with unit) generated by the elements of $V$ such that $v^* = v$ and $v \cdot w + w \cdot v = 2 \langle v, w \rangle 1$ for all $v, w \in V$. It has a natural $\mathbb{Z}_2$-grading by declaring that all elements of $V$ have odd degree. The universal property [33, 23] of $\text{Cliff}(V)$ is that if $f : V \to A$ is a real linear map of $V$ into a unital complex $C^*$-algebra $A$ such that $f(v)^2 = \langle v, v \rangle 1_A$ for all $v \in V$ then there is an induced $C^*$-algebra homomorphism $\tilde{f} : \text{Cliff}(V) \to A$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Cliff}(V) & \xrightarrow{f} & A \\
C \uparrow & & \downarrow \\
V & \xrightarrow{f} & A
\end{array}
\]

where we denote by $C : V \hookrightarrow \text{Cliff}(V)$ the canonical inclusion. However, we will usually identify $v = C(v) \in \text{Cliff}(V)$ for all $v \in V$. An important property of these $\mathbb{Z}_2$-graded $C^*$-algebras is their behavior with respect to orthogonal sums:

(2) $\text{Cliff}(V \oplus W) \cong \text{Cliff}(V) \hat{\otimes} \text{Cliff}(W)$

where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product. (See the books [33, 23] for a review of Clifford algebras and Blackadar [6] for a review of graded $C^*$-algebras.)

Let $M_n$ be a finite-dimensional smooth Riemannian manifold of dimension $n$ with Riemannian metric $g$. Let $TM_n \to M_n$ denote the tangent bundle of $M_n$. Let $\text{Cliff}(TM_n) \to M_n$ denote the Clifford bundle [4, 5] of $TM_n$, i.e., the bundle of Clifford algebras over $M_n$ whose fiber at $x \in M_n$ is the complex Clifford algebra $\text{Cliff}(T_x M_n)$ of the Euclidean tangent space $T_x M_n$. It has an induced $\mathbb{Z}_2$-graded $C^*$-algebra bundle structure.

**Definition 2.1.** [29] Denote by $\mathcal{C}(M_n)$ the $C^*$-algebra

\[
\mathcal{C}(M_n) = \mathcal{C}_0(M_n, \text{Cliff}(TM_n))
\]

of continuous sections of $\text{Cliff}(TM_n)$ which vanish at infinity on $M_n$, with induced $\mathbb{Z}_2$-grading from $\text{Cliff}(TM_n)$. (Kasparov [29] used the notation $C_r(M_n)$.)

For example, if $M_n = V$ is a finite dimensional Euclidean vector space, then $TM_n \cong V \times V$ and so $\mathcal{C}(M_n) \cong C_0(V, \text{Cliff}(V))$ as in Definition 2.2 [25]. A priori, this $C^*$-algebra depends on the Riemannian metric $g$ of $M_n$. However, the universal property of Clifford algebras shows that the $C^*$-algebra structure on $\mathcal{C}(M_n)$ depends only on the manifold $M_n$ and not the chosen metric $g$. Indeed, if $h$ is another
Riemannian metric on $M_n$, then $\alpha = h^{-1} \circ \hat{g} : TM_n \to TM_n$ is an automorphism of the tangent bundle $TM_n$, where $\hat{g} : TM_n \to T^*M_n$ is the (co)tangent bundle isomorphism induced by any metric $g$. It satisfies
\[
h(\alpha(X), X) = g(X, X) \geq 0
\]
for any vector field $X$. Thus, $\alpha$ is positive definite with respect to the metric $h$ and so has a positive square root, i.e., a bundle automorphism $\beta : TM_n \to TM_n$ such that
\[
h(\beta(X), \beta(X)) = h(\alpha(X), X) = g(X, X).
\]
If $\text{Cliff}(TM_n, h)$ denotes the Clifford bundle of $M_n$ with respect to the metric $h$, then
\[
\beta(X)^2 = g(X, X)1
\]
in $\text{Cliff}(TM_n, h)$. By the universal property above (applied to each fiber) $\beta$ extends to an isomorphism $\hat{\beta} : \text{Cliff}(TM_n, g) \to \text{Cliff}(TM_n, h)$ of Clifford bundles. (See also Section 9.1 [23].) By taking sections, there is a canonically induced isomorphism
\[
\hat{\beta} : \mathcal{C}(M, g) \to \mathcal{C}(M, h)
\]
of $\mathbb{Z}_2$-graded $C^*$-algebras.
Let $C_0(M_n)$ denote the commutative $C^*$-algebra of continuous complex-valued functions on $M_n$ vanishing at infinity. We always consider $C_0(M_n)$ to be trivially graded. If a $\mathbb{Z}_2$-graded $C^*$-algebra $A$ is equipped with a (fixed) $*$-homomorphism $\Theta : C_0(M_n) \to Z(M(A))$ that is nondegenerate and has grading degree zero, where $Z(M(A))$ denotes the center of the multiplier algebra of $A$, then we say that $A$ has a $\mathbb{Z}_2$-graded $C_0(M_n)$-algebra structure [45]. We denote $\Theta(f) a = f \cdot a$ for all $f \in C_0(M_n)$ and $a \in A$. Note that pointwise multiplication $(fs)(x) = f(x)s(x), \forall x \in M_n$, where $f \in C_0(M_n)$ and $s \in C(M_n)$, determines a nondegenerate $*$-homomorphism $C_0(M_n) \to ZM(C(M_n))$ into the center of the multiplier algebra of $C(M_n)$ of grading degree zero. Thus, we have the following.

**Corollary 2.2.** The $C^*$-algebra $\mathcal{C}(M_n)$ has a canonical $\mathbb{Z}_2$-graded $C_0(M_n)$-algebra structure, and up to $\mathbb{Z}_2$-graded isomorphism, is independent of the Riemannian metric on $M_n$.

**Definition 2.3.** Let $\mathcal{S}$ denote the $C^*$-algebra $C_0(\mathbb{R})$ of continuous complex-valued functions on the real line which vanish at infinity, with $\mathbb{Z}_2$-grading by even and odd functions. If $A$ is any $\mathbb{Z}_2$-graded $C^*$-algebra then we let $\mathcal{S}A$ be the graded (max) tensor product $\mathcal{S} \otimes A$. In particular, let
\[
A(M_n) = \text{det} \mathcal{S}\mathcal{C}(M_n) = \mathcal{S} \otimes C_0(M_n, \text{Cliff}(TM_n))
\]
which can be viewed as a non-commutative topological suspension of $M_n$.

The following functoriality result will be used when we identify the total space of the normal bundle of an embedding with an open tubular neighborhood.

**Lemma 2.4.** [45] Let $\phi : M_n \to N_n$ be a diffeomorphism of Riemannian manifolds. There is an induced $\mathbb{Z}_2$-graded $C^*$-algebra isomorphism
\[
\phi_* : \mathcal{A}(M_n) \to \mathcal{A}(N_n).
\]

\(^1\text{Recall that the suspension of a } C^*\text{-algebra } A \text{ is the } C^*\text{-algebra } SA = C_0(\mathbb{R}) \otimes A. \text{ In particular, } SC_0(M_n) \cong C_0(\mathbb{R} \times M_n) \text{ where } \mathbb{R} \times M_n \text{ is the (reduced) topological suspension of } M_n.\)
Proof. Let $g$ denote the metric on $M_n$ and $h$ denote the metric on $N_n$. If
\[ \phi : M_n \to N_n \]
is a diffeomorphism, then, using the pullback metric $\phi^*(h)$, we have that
\[ \phi : (M_n, \phi^*(h)) \to (N_n, h) \]
is an isometry of Riemannian manifolds, which clearly induces a canonical isomorphism
\[ \hat{\phi} : \mathcal{C}(M_n, \phi^*(h)) \to \mathcal{C}(N_n, h) \]
of $Z_2$-graded $C^*$-algebras. By the argument above, we have a canonical isomorphism
\[ \hat{\beta} : \mathcal{C}(M_n, g) \to \mathcal{C}(M_n, \phi^*(h)). \]
Taking the composition and tensoring with the identity of $\mathcal{S}$ gives the required canonical isomorphism
\[ \phi_* = id_{\mathcal{S}} \otimes (\hat{\beta} \circ \hat{\phi}) : \mathcal{A}(M_n) = \mathcal{S} \otimes \mathcal{C}(M_n) \to \mathcal{S} \otimes \mathcal{C}(N_n) = \mathcal{A}(N_n) \]
of $Z_2$-graded $C^*$-algebras. The following is an easy functoriality property for open inclusions.

Lemma 2.5. \[ \square \]
Let $U_n$ be an open subset of the Riemannian manifold $M_n$. The inclusion $i : U_n \to M_n$ induces a short exact sequence
\[ 0 \to \mathcal{A}(U_n) \xrightarrow{1 \otimes i_*} \mathcal{A}(M_n) \to \mathcal{A}(M_n \setminus U_n) \to 0 \]
of $C^*$-algebras. Thus, $\mathcal{A}(U_n) \triangleleft \mathcal{A}(M_n)$ as a (two-sided) $C^*$-ideal.

Let $p : E \to M_n$ be a smooth finite rank Euclidean vector bundle. We will show that there is a natural “Thom” $*$-homomorphism
\[ \Psi_p : \mathcal{A}(M_n) \to \mathcal{A}(E), \]
where we consider $E$ as a finite-dimensional manifold with Riemannian structure to be constructed as follows. The main example we have in mind is where $E = \nu M_n$ is the (total space of the) normal bundle of an isometric embedding $M_n \to M_{n+1}$.

Given $p : E \to M_n$, there is a short exact sequence\[ \square \] of real vector bundles
\[ 0 \to VE \to TE \xrightarrow{T_p} p^*TM_n \to 0 \]
where the vertical subbundle $VE = \ker(T^*p)$ is isomorphic to $p^*E$. This sequence does not have a canonical splitting, in general, but choosing a compatible connection $\nabla$ on $E$ determines an associated vector bundle splitting. Recall that a connection $\nabla : C^\infty(M_n, E) \to C^\infty(M_n, T^*M_n \otimes E)$ on $E$ is compatible\[ \square \] with the bundle metric $(\cdot, \cdot)$ on $E$ if
\[ d(s_1, s_2) = (\nabla s_1, s_2) + (s_1, \nabla s_2) \]
for all smooth sections $s_1, s_2 \in C^\infty(M_n, E)$. If $p : E \to M_n$ is equipped with a compatible connection $\nabla$, then we call $E$ an affine Euclidean bundle.

Let $\nabla^* : C^\infty(E, p^*E) \to C^\infty(E, T^*E \otimes p^*E)$ denote the pullback of $\nabla$ on the bundle $p^*E \to E$, which is defined by the formula:
\[ \nabla^*(fp^*s) = df \otimes p^*s + fp^*(\nabla s) \]
for $f \in C^\infty(M_n)$ and $s \in C^\infty(M_n, E)$. The tautological section $\tau \in C^\infty(E, p^*E)$ is the smooth section of $p^*E \to E$ defined by the formula $\tau(e) = (e, e)$ for all $e \in E$. The derivative of $\tau$ will be denoted by
\[ \omega = \nabla^* \tau \in C^\infty(E, T^*E \otimes p^*E) = \Omega^1(E, p^*E) \cong \Omega^1(E, VE) \]

Theorem 2.8. Let $E$ be a finite rank affine Euclidean bundle on the Riemannian manifold $M$. There is an induced orthogonal splitting of the exact sequence

$$0 \to p^*E \to TE \to p^*TM_n \to 0$$

and so there is a canonical isomorphism of Euclidean vector bundles

$$TE \cong p^*E \oplus p^*TM_n$$

where $p^*E$ and $p^*TM_n$ have the pullback metrics. Thus, the manifold $E$ has a canonical Riemannian metric.

Hence, given a compatible connection $\nabla$ on the Euclidean bundle $E$, we can define the $C^*$-algebra $\mathcal{C}(E)$ as above using the induced Riemannian structure on the manifold $E$. However, we also have the $C^*$-algebra $C_0(E, \text{Cliff}(p^*E))$ associated to the pullback bundle $p^*E \to E$.\footnote{Note: Although $C_0(E, \text{Cliff}(p^*E)) \cong p^*C_0(M_n, \text{Cliff}(E))$, we will not need this isomorphism.}

Definition 2.7. Let $A$ and $B$ be $\mathbb{Z}_2$-graded $C_0(M_n)$-algebras. The balanced tensor product over $M_n$, denoted $A \hat{\otimes}_{M_n} B$, is the quotient of the maximal graded tensor product $A \otimes B$ by the ideal $J$ generated by

$$\left\{(f \cdot a) \hat{\otimes} b - a \hat{\otimes} (f \cdot b) : a \in A, b \in B, f \in C_0(M_n)\right\}.$$ 

For example $C_0(M_n) \hat{\otimes}_{M_n} A \cong A$ via the map induced by $f \hat{\otimes} a \mapsto f \cdot a$.

The following is an important result that relates these two $C^*$-algebras to the $C^*$-algebra $\mathcal{C}(M_n)$ of the base manifold $M_n$.

Theorem 2.8. Let $p : E \to M_n$ be a finite rank affine Euclidean bundle on the Riemannian manifold $M_n$. There is a natural isomorphism of graded $C^*$-algebras

$$\mathcal{C}(E) \cong C_0(E, \text{Cliff}(p^*E)) \hat{\otimes}_{M_n} \mathcal{C}(M_n).$$

Proof. By the previous lemma, there is an induced orthogonal splitting

$$TE = p^*E \oplus p^*TM_n.$$

Thus, we have an induced isomorphism of $\mathbb{Z}_2$-graded Clifford algebra bundles

$$\text{Cliff}(TE) \cong \text{Cliff}(p^*E \oplus p^*TM_n) = \text{Cliff}(p^*E) \hat{\otimes} p^*\text{Cliff}(TM_n).$$
Therefore, by taking sections, we have canonical balanced tensor product isomorphisms (see Proposition A.7 [15])

\[ \mathcal{C}(E) \overset{\text{def}}{=} C_0(E, \text{Cliff}(TE)) \cong C_0(E, \text{Cliff}(p^*E) \hat{\otimes} \text{Cliff}(TM_n)) \cong C_0(E, \text{Cliff}(p^*E) \hat{\otimes}_E C_0(E, \text{Cliff}(p^*TM_n))). \]

But, we have, using pullbacks along \( p : E \rightarrow M_n \), that there are canonical pullback isomorphisms (see Proposition A.9 [15])

\[ C_0(E, \text{Cliff}(p^*TM_n)) \cong p^*C_0(M_n, \text{Cliff}(TM_n)) = p^* \mathcal{C}(M_n) = \text{def} C_0(E) \hat{\otimes}_M \mathcal{C}(M_n). \]

Hence, it follows that

\[ C(E) \cong C_0(E, \text{Cliff}(p^*E)) \hat{\otimes}_E C_0(E, \text{Cliff}(p^*TM_n)) \cong C_0(E, \text{Cliff}(p^*E)) \hat{\otimes}_E C_0(E) \hat{\otimes}_M \mathcal{C}(M_n) \]

using the canonical isomorphism \( A \hat{\otimes}_E C_0(E) \cong A \) for graded \( C_0(E) \)-algebras. \( \square \)

We now wish to define a certain “Thom operator” for the “vertical” algebra \( C_0(E, \text{Cliff}(p^*E)). \) Associate to the Euclidean bundle \( E \) an unbounded section

\[ C_E : E \rightarrow \text{Cliff}(p^*E) : e \mapsto C_{p(e)}(e) \]

where \( C_{p(e)} \) is the Clifford operator on the Euclidean space \( E_{p(e)} \) from Definition 2.4 of [25]. It is given globally by the composition

\[ E \xrightarrow{\tau} p^*E \xrightarrow{C} \text{Cliff}(p^*E) \]

where \( \tau \in C^\infty(E, p^*E) \) is the tautological section (see above) and \( C : p^*E \hookrightarrow \text{Cliff}(p^*E) \) is the canonical inclusion \( C(e_1, e_2) = C_{p(e_1)}(e_2) \). The following is then easy to prove.

**Theorem 2.9.** Let \( E \) be a finite rank Euclidean bundle on \( M_n \). Multiplication by the section \( C_E : E \rightarrow \text{Cliff}(p^*E) \) determines a degree one, essentially self-adjoint, unbounded multiplier (see Definition A.1 [15]) of the \( C^\ast \)-algebra \( C_0(E, \text{Cliff}(p^*E)) \) with domain \( C_c(E, \text{Cliff}(p^*E)) \).

We will call \( C_E \) the **Thom operator** of \( E \rightarrow M_n \). Thus, we have a functional calculus homomorphism

\[ \mathcal{S} \rightarrow M(C_0(E, \text{Cliff}(p^*E))) : f \rightarrow f(C_E) \]

from \( \mathcal{S} \) to the multiplier algebra of \( C_0(E, \text{Cliff}(p^*E)) \). Note that \( f(C_E) \) goes to zero in the “fiber” directions on \( E \) (since \( p(e) \) is constant), but is only bounded in the “manifold” directions on \( E \). Indeed, for the generators \( f(x) = \exp(-x^2) \) and \( g(x) = x \exp(-x^2) \) of \( \mathcal{S} \), we have that \( f(C_E) \) and \( g(C_E) \) are, respectively, multiplication by the following functions on \( E \):

\[ f(C_E)(e) = \exp(-\|e\|^2) \quad \text{and} \quad g(C_E)(e) = e \cdot \exp(-\|e\|^2), \quad \forall e \in E. \]

**Definition 2.10.** Let \( X \) denote the degree one, essentially self-adjoint, unbounded multiplier of \( \mathcal{S} \), with domain the compactly supported functions, given by multiplication by \( x \), i.e., \( Xf(x) = xf(x) \) for all \( f \in C_c(\mathbb{R}) \) and \( x \in \mathbb{R} \).
By Lemma A.3, the operator $X \hat{\otimes} 1 + 1 \hat{\otimes} C_E$ determines a degree one, essentially self-adjoint, unbounded multiplier of the tensor product

$$\mathcal{S} \hat{\otimes} C_0(E, \text{Cliff}(p^*E)) = SC_0(E, \text{Cliff}(p^*E))$$

with domain $C_c(\mathbb{R}) \hat{\otimes} C_c(E, \text{Cliff}(p^*E))$. We obtain a functional calculus homomorphism

$$\beta_E : \mathcal{S} \rightarrow M(SC_0(E, \text{Cliff}(p^*E))) : f \mapsto f(\hat{\otimes} 1 + 1 \hat{\otimes} C_E)$$

from $\mathcal{S}$ into the multiplier algebra of $SC_0(E, \text{Cliff}(p^*E))$. Now we can define our “Thom $*$-homomorphism” for a finite rank affine Euclidean bundle. This will provide part of the connecting map in Section 4 when we define the direct limit $C^*$-algebra for an infinite-dimensional Riemannian Fredholm manifold.

**Theorem 2.11.** Let $p : E \rightarrow M_n$ be a finite rank affine Euclidean bundle on the Riemannian manifold $M_n$. With respect to the isomorphism

$$\mathcal{A}(E) \cong \mathcal{S} \hat{\otimes} C_0(E, \text{Cliff}(p^*E)) \hat{\otimes} M_n \mathcal{C}(M_n)$$

from Theorem 2.8 there is a graded $*$-homomorphism

$$\Psi_p = \beta_E \hat{\otimes} M_n \text{id}_{M_n} : \mathcal{A}(M_n) \rightarrow \mathcal{A}(E)$$

which on elementary tensors $f \hat{\otimes} s \in \mathcal{S} \hat{\otimes} \mathcal{C}(M_n) = \mathcal{A}(M_n)$ is given by

$$f \hat{\otimes} s \mapsto f(\hat{\otimes} 1 + 1 \hat{\otimes} C_E) \hat{\otimes} M_n s.$$

**Proof.** From the discussion above, we have that $\beta_E \hat{\otimes} M_n \text{id}_{M_n}$ is the composition

$$\mathcal{A}(M_n) \xrightarrow{\beta_E \hat{\otimes} \text{id}} M(SC_0(E, \text{Cliff}(p^*E))) \hat{\otimes} \mathcal{C}(M_n) \rightarrow M(SC_0(E, \text{Cliff}(p^*E))) \hat{\otimes} M_n \mathcal{C}(M_n)$$

Checking on the generator $f(x) = \exp(-x^2)$ of $\mathcal{S}$, we compute that

$$f(\hat{\otimes} 1 + 1 \hat{\otimes} C_E) \hat{\otimes} s = \exp(-x^2) \hat{\otimes} \exp(-\|e\|^2) \hat{\otimes} M_n s \in \mathcal{A}(E)$$

Similarly for $g(x) = x \exp(-x^2)$, we find that

$$g(\hat{\otimes} 1 + 1 \hat{\otimes} C_E) \hat{\otimes} M_n s = x \exp(-x^2) \hat{\otimes} \exp(-\|e\|^2) \hat{\otimes} M_n s$$

$$+ \exp(-x^2) \hat{\otimes} e \cdot \exp(-\|e\|^2) \hat{\otimes} M_n s \in \mathcal{A}(E).$$

It follows that the range of $\Psi_p = \beta_E \hat{\otimes} M_n \text{id}_{M_n}$ is in $\mathcal{A}(E)$ as desired. □

Since the space of compatible connections $\nabla$ on $E \rightarrow M_n$ is convex, we have the following result.

**Proposition 2.12.** Let $p : E \rightarrow M_n$ be a smooth finite rank affine Euclidean bundle on the Riemannian manifold $M_n$. The homotopy class of the $*$-homomorphism $\Psi_p : \mathcal{A}(M_n) \rightarrow \mathcal{A}(E)$ is independent of the choice of compatible connection on $E$.

**Proposition 2.13.** If $p : E = M_n \times V \rightarrow M_n$ is a trivial finite rank affine Euclidean bundle (with trivial connection $\nabla_0 = d$) then we have a $\mathbb{Z}_2$-graded isomorphism

$$\mathcal{C}(E) \cong \mathcal{C}(V) \hat{\otimes} \mathcal{C}(M_n)$$

such that the Thom map has the form

$$\Psi_p \cong \beta_V \hat{\otimes} \text{id}_{\mathcal{C}(M_n)} : \mathcal{A}(M_n) = \mathcal{S} \hat{\otimes} \mathcal{C}(M_n) \rightarrow \mathcal{A}(V) \hat{\otimes} \mathcal{C}(M_n) \cong \mathcal{A}(E)$$

where $\beta_V : \mathcal{S} \rightarrow \mathcal{A}(V) : f \mapsto f(\hat{\otimes} 1 + 1 \hat{\otimes} C_V)$ is the Thom map for $V \rightarrow \{0\}$. 
Proof. The trivial connection $\nabla_0 = d$ gives the manifold $E = M_n \times V$ the Riemannian metric induced by the isomorphism

$$TE = TM_n \times TV \rightarrow M_n \times V = E.$$  

The pullback vector bundle $p^* E \rightarrow E$ has the form

$$p^* E = (M_n \times V) \times V \rightarrow M_n \times V = E$$

and so the Clifford bundle $\text{Cliff}(p^* E) = (M_n \times V) \times \text{Cliff}(V)$, which gives:

$$C_0(E, \text{Cliff}(p^* E)) = C_0(M_n \times V, (M_n \times V) \times \text{Cliff}(V)) \cong C_0(V, \text{Cliff}(V)) \hat{\otimes} C_0(M_n).$$

By Theorem 2.8 it follows that

$$C(E) \cong C_0(E, \text{Cliff}(p^* E)) \hat{\otimes} M_n C(M_n)$$

$$\cong C_0(V, \text{Cliff}(V)) \hat{\otimes} C_0(M_n) \hat{\otimes} M_n C(M_n)$$

$$\cong C(V) \hat{\otimes} C(M_n).$$

where we used the isomorphism $C_0(M_n) \hat{\otimes} M_n C(M_n) \cong C(M_n)$. The result now easily follows. \qed

For example, if $p : E_b \rightarrow E_a$ is the orthogonal projection of a finite dimensional Euclidean vector space $E_b$ onto a linear subspace $E_a$ then $\Psi_p = \beta_\delta$ is the “Bott homomorphism” from Definition 3.1 of Higson-Kasparov-Trout [24].

3. Fredholm Manifolds and Filtrations

Fredholm manifolds are a particular case of Hilbert manifolds, i.e., manifolds modeled on a separable infinite-dimensional real Hilbert space. Most of the standard constructions from the differential geometry of finite-dimensional manifolds carry on in the infinite dimensional situation (as reference see Lang’s book [32]). All the Hilbert manifolds that we consider in this paper are assumed to be connected, separable, paracompact, Hausdorff, and infinitely smooth.

Let $\mathcal{E}$ be a separable infinite-dimensional Euclidean space, i.e., a real Hilbert space of countably infinite dimension. We will use the following notation: $\mathcal{L}(\mathcal{E})$ denotes the real $C^*$-algebra of bounded linear operators on $\mathcal{E}$; $\mathcal{F} = \mathcal{F}(\mathcal{E})$ denotes the finite rank operators; $\mathcal{K} = \mathcal{K}(\mathcal{E})$ denotes the closed ideal of compact operators; $\Phi = \Phi(\mathcal{E})$ denotes the Fredholm operators; and $\mathcal{GL}(\mathcal{E})$ denotes the Banach-Lie group of units of $\mathcal{L}(\mathcal{E})$, with identity $I$.

**Definition 3.1.** A perturbation class $P$ of $\mathcal{E}$ is a subspace $P = P(\mathcal{E})$ of $\mathcal{L}(\mathcal{E})$ such that: (1) $\mathcal{F}(\mathcal{E}) \subseteq P(\mathcal{E})$, (2) $P(\mathcal{E})$ is an ideal in $\mathcal{L}(\mathcal{E})$, and (3) $\Phi(\mathcal{E}) + P(\mathcal{E}) = \Phi(\mathcal{E})$.

As examples of perturbation classes we have: the finite rank operators $\mathcal{F}(\mathcal{E})$, the compact operators $\mathcal{K}(\mathcal{E})$, or indeed any proper two-sided ideal included in $\mathcal{K}$. For $1 \leq q < \infty$, let $P_q$ be the perturbation class defined as the closure of $\mathcal{F}(\mathcal{E})$ under the norm

$$\|T\|_q = \left(\text{Trace}(T^*T)^{q/2}\right)^{1/q}.$$  

If $q = 1$ one obtains the trace-class operators, and if $q = 2$ the Hilbert-Schmidt operators. If $q = \infty$, then we set $P_{\infty} = \mathcal{K}(\mathcal{E})$ with norm $\|T\|_{\infty} = \|T\|$.  

Given a perturbation class $P$ of $\mathcal{E}$, we let

$$\mathcal{GL}_P(\mathcal{E}) = \mathcal{GL}(\mathcal{E}) \cap (I + P(\mathcal{E}))$$

$$= \{ T = I + K \mid T \in \mathcal{GL}(\mathcal{E}), K \in P(\mathcal{E}) \}.$$
For $1 \leq q < \infty$, we abbreviate $GL_p(E) = GL_q(E)$. For $p = \infty$, we abbreviate $GL_{K(\mathcal{E})}(\mathcal{E}) = GL_{K}(\mathcal{E})$. We topologize $GL_q(\mathcal{E})$ by requiring that the map $GL_q(\mathcal{E}) \to O \subset P_q$, $K \to I + K$, be a homeomorphism, where $O$ is the set of all $K$ with $I + K$ invertible \cite{30}. In general, $GL_p(\mathcal{E})$ is a normal subgroup of $GL(\mathcal{E})$, but, where $GL(\mathcal{E})$ is contractible (by Kuiper’s theorem \cite{31}), $GL_p(\mathcal{E})$ may not be contractible.

For example, by Theorem B of Palais \cite{36}, we have that $\pi_0(GL_q(\mathcal{E})) = \mathbb{Z}/2\mathbb{Z}$ for all $1 \leq q \leq \infty$. However, $GL_p(\mathcal{E})$ is not a closed subgroup of $GL(\mathcal{E})$ unless $P = P_\infty = K(\mathcal{E})$.

**Definition 3.2.** Let $M$ be a Hilbert manifold modeled on $\mathcal{E}$. A Fredholm structure on $M$ is an integrable reduction of the principal $GL(\mathcal{E})$-bundle of $M$ to $GL_K(\mathcal{E})$. Equivalently, it is a maximal atlas of $M$ such that the differential of the change of coordinates maps is an element of $GL_K(\mathcal{E})$ at every point. A Fredholm manifold is a Hilbert manifold with a specified Fredholm structure.

Since there is a natural inclusion $GL_p(\mathcal{E}) \hookrightarrow GL_K(\mathcal{E})$ induced by the inclusion $P(\mathcal{E}) \hookrightarrow K(\mathcal{E})$, if a Hilbert manifold $M$ is equipped with a reduction of it’s structure group from $GL(\mathcal{E})$ to $GL_p(\mathcal{E})$ then we can give $M$ a canonical Fredholm structure in the sense of the previous definition. We will make use of this fact when discussing spin structures for Fredholm manifolds in Section 5.

**Note.** A $C^\infty$-map $f : M \to N$ between Hilbert manifolds is called a Fredholm map if, for every $x \in M$, $Df(x) : T_xM \to T_{f(x)}N$ is a Fredholm operator. Fredholm manifolds are exactly the manifolds on which Fredholm maps can be constructed. Results of Elworthy and Tromba \cite{21} show that for a Fredholm manifold $M$ there is an index zero (even bounded and proper) Fredholm map $f : M \to \mathcal{E}$.

The following decomposition theorem is crucial in the study of Fredholm manifolds (\cite{31} Thm 2.2):

**Theorem 3.3.** Let $M$ be a Fredholm manifold. There exists a sequence $\{M_n\}_{n=k}^\infty$ of finite dimensional closed submanifolds such that:

(i) $\dim M_n = n$; $M_n \subset M_{n+1}$;

(ii) the inclusions $M_n \hookrightarrow M_{n+1}$ and $M_n \hookrightarrow M$ have trivial normal bundles;

(iii) $M_\infty = \bigcup_{n \geq k} M_n$ is dense in $M$; and

(iv) the natural inclusion map $M_\infty \hookrightarrow M$ is a homotopy equivalence, if $M_\infty$ is given the direct limit topology.

A sequence $\{M_n\}_{n=k}^\infty$ as in the theorem above is called a Fredholm filtration of $M$.

We will now give some examples (and a non-example) of Fredholm manifolds and filtrations.

**Examples 3.4.** (i) The Euclidean space $M = \mathcal{E}$ has an obvious Fredholm structure, determined by a single chart $I : \mathcal{E} \to \mathcal{E}$. It is the only possible structure. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of $\mathcal{E}$, and $E_n$ be the linear span of $\{e_1, e_2, \ldots, e_n\}$. The sequence $\{E_n\}_n$ is known as a flag of $\mathcal{E}$, and it forms a Fredholm filtration.

(ii) The unit sphere of $\mathcal{E}$, $S_\mathcal{E} = \{x \in \mathcal{E} | \|x\| = 1\}$, gets by restriction from $\mathcal{E}$ a Fredholm structure. As a Fredholm filtration we have

\[ S^1 \subset S^2 \subset \cdots \subset S^n \subset \cdots \subset S_\mathcal{E}. \]

(iii) The following is a non-example. The sequence of real projective spaces

\[ \mathbb{R}P^1 \subset \mathbb{R}P^2 \subset \cdots \subset \mathbb{R}P^n \subset \cdots \subset \mathbb{R}P_\mathcal{E} \]
is not a Fredholm filtration of the infinite dimensional real projective space $\mathbb{R}P_\infty$ of $\mathcal{E}$, for any choice of Fredholm structure, because the inclusions $\mathbb{R}P^n \subset \mathbb{R}P^{n+1}$ do not have trivial normal bundles.

To get an idea how Fredholm filtrations are constructed in general, we briefly outline the procedure as follows. Let $M$ be a Fredholm manifold modeled on $E$. Let $\{E_n\}$ be a flag for $E$ as in Example 3.4 (i) above. Choose an index zero Fredholm map $f : M \to E$ which is transversal to the $E_n$’s, and define $M_n = f^{-1}(E_n)$. Each $M_n$ (when nonempty) is a finite-dimensional submanifold of $M$ of dimension $n$ and $M_n \subset M_{n+1}$. The normal bundle $\nu M_n \to M_n$ of the inclusion $M_n \subset M_{n+1}$ is the pullback $\nu M_n = f^*(\nu E_n)$ of the (trivial) normal bundle $\nu E_n = E_n^\perp \cap E_{n+1}$ and, hence, is trivial. The sequence $\{M_n\}_{n=k}^\infty$, where $M_k \neq \emptyset$ is the first nonempty submanifold, forms a Fredholm filtration of $M$. Note that since there is always a bounded, proper index zero Fredholm map $f : M \to E$, the $M_n$’s can be chosen to be compact. See the Addendum to Theorem 2C in Eells and Elworthy [19].

One can actually say more about the Fredholm filtrations of a Fredholm manifold, but we need to recall first some facts about the differential geometry of infinite dimensional manifolds.

**Definition 3.5.** Let $N$ be a submanifold of $M$. A tubular neighborhood of $N$ in $M$ consist of the following data: a vector bundle $\pi : B \to N$ over $N$, an open neighborhood $V$ of the zero section $\zeta(N)$ in $B$, an open set $U$ in $M$ containing $N$, and a diffeomorphism $f : V \to U$ which commutes with the zero section $\zeta : N \to V$:

\[
\begin{array}{ccc}
V & \xleftarrow{\zeta} & \pi^{-1}(N) \\
\downarrow & & \downarrow \\
N & \xrightarrow{\zeta} & U
\end{array}
\]

$U$ is called the tube of the tubular neighborhood. The tubular neighborhood is called total if $V = B$ the total space of the bundle.

Using the notion of spray [32 IV.3], its associated exponential map, and restriction to the normal bundle of the inclusion $i : N \to M$, one can prove the existence and uniqueness of tubular neighborhoods, if $M$ is a Hilbert manifold ([32, Theorems IV.5.1 and IV.6.2]). On a Riemannian manifold one can always choose tubular neighborhood to be total.

**Definition 3.6.** A Riemannian manifold is a pair $(M, g)$, where $M$ is a Hilbert manifold, and $g$ is a metric on $M$, i.e., $g_x$ is a (smoothly varying) positive-definite non-singular symmetric bilinear form on $T_xM$, for every $x \in M$.

According with [32 Cor.II.3.8], every paracompact $C^\infty$-manifold modeled on a separable Hilbert space admits partitions of unity of class $C^\infty$. It follows that Hilbert manifolds admit Riemannian metrics:

**Proposition 3.7.** [32 Prop.VII.1.1] Let $M$ be a manifold admitting partitions of unity, and let $\pi : B \to M$ be a vector bundle whose fibers are Hilbertable vector spaces. Then $\pi$ admits a Riemannian metric.

Granted all of this, the next statement is a combination of [34 Thm 2.3] and remarks from [35] and [19].
Theorem 3.8. Let $M$ be a Fredholm manifold with Riemannian metric $g$ compatible with the topology of $M$. There exists a Fredholm filtration $\{M_n\}_{n=1}^{\infty}$ of $M$ for which geodesically defined exponential neighborhoods $Z_n$ of $M_n$ in $M$ can be constructed satisfying:

$$Z_n \subset Z_{n+1} \text{ and } \bigcup_{n \geq k} Z_n = M.$$ 

Moreover $U_n = Z_n \cap M_{n+1}$ is a tubular neighborhood of $M_n$ in $M_{n+1}$, for each $n \geq k$.

Definition 3.9. We call a Fredholm filtration $\{M_n\}_{n=1}^{\infty}$ together with a collection $\{U_n\}_{n=1}^{\infty}$, where $U_n$ is a total tubular neighborhood of $M_n \subset M_{n+1}$, an augmented Fredholm filtration and we shall denote this by $\mathcal{F} = (M_n, U_n)_{n=1}^{\infty}$. Note that we assume that each $U_n$ is equipped with a fixed diffeomorphism $\phi_n : \nu M_n \to U_n$.

Fredholm manifolds often arise as spaces of paths and we end this section with one more example. In Section 5, Example 5.10, we will discuss examples of Fredholm manifolds arising from loop groups $\Omega \mathcal{G}$ of certain compact Lie groups $\mathcal{G}$ (and their associated spin structures.)

Example 3.10. See [20]. Let $X$ be a complete finite-dimensional Riemannian manifold, and $a \in X$. Let $M = P_a(X)$ be the space of paths $\gamma : [0,1] \to X$, with $\gamma(0) = a$ and $\gamma$ absolutely continuous with square integrable derivative. Then $M$ is a separable smooth Hilbert manifold. Moreover a complete Riemannian structure on $M$ is given by

$$g_\gamma(u,v) = \langle u,v \rangle_\gamma = \int_0^1 \langle D_\gamma u, D_\gamma v \rangle_\gamma,$$

for $u, v \in T_\gamma M$, where $D_\gamma$ denotes the covariant derivative along $\gamma$. There is natural diffeomorphism

$$\delta : P_a(X) \to P_0(T_aX), \delta(\gamma)(t) = \int_0^t \tau_0^s \gamma'(s) \, ds,$$

where $\tau_0^s$ denotes parallel transport along $\gamma$ from $T_{\gamma(s)}X$ to $T_aX$. This map $\delta$, called E. Cartan’s development map, gives a diffeomorphism of $M = P_a(X)$ with the Hilbert space $P_0(T_aX)$ and, hence, a unique Fredholm structure on the contractible space $M$.

4. The $C^*$-algebra of a Fredholm Manifold

Let $M$ be a smooth, separable, connected, paracompact Hilbert manifold modeled on the separable, infinite-dimensional Euclidean space $\mathcal{E}$. We assume that $M$ is equipped with a Riemannian Fredholm structure, i.e., a reduction of the structure group of $M$ from $GL(\mathcal{E})$ to $GL_K(\mathcal{E})$ and a Riemannian metric $g$ that is compatible with the topology of $M$. This is equivalent to a reduction of the structure group from $GL(\mathcal{E})$ to $O_K(\mathcal{E}) = GL_K(\mathcal{E}) \cap O(\mathcal{E})$. (See section 5.2).

Let $\mathcal{F} = (M_n, U_n)_{n=1}^{\infty}$ be an augmented Fredholm filtration of $M$ by closed $n$-dimensional submanifolds $M_n$ with total tubular neighborhoods $M_n \subset U_n \subset M_{n+1}$, as in Definition 3.10. Let $p_n : \nu M_n \to M_n$ denote the normal bundle of the embedding $j_n : M_n \hookrightarrow M_{n+1}$. That is, we have a short exact sequence

$$0 \to TM_n \to TM_{n+1}|_{M_n} \to \nu M_n \to 0$$

of finite rank vector bundles.
These geometric considerations lead us to the following topological diagram of bundles and spaces:

\[ \begin{array}{cccccc}
\nu M_n & \xrightarrow{\phi_n \text{ diffeo}} & U_n & \xrightarrow{k_n \text{ open}} & M_{n+1} \\
\downarrow p_n & & \downarrow & & \\
M_n & & & & \\
\end{array} \]

where the tubular neighborhood \( U_n \) is identified with the total space of the normal bundle \( \nu M_n \) via a fixed diffeomorphism \( \phi_n : \nu M_n \rightarrow U_n \) and \( k_n : U_n \hookrightarrow M_{n+1} \) denotes the (open) inclusion.

For each \( n \), let \( M_n \) have the induced Riemannian metric \( g_n = i_n^*(g) \) where \( i_n : M_n \rightarrow M \) denotes the inclusion. Thus, for each \( n \geq k \), we have the associated \( C^* \)-algebra

\[ A(M_n) = SC(M_n) = S \hat{\otimes} C_0(M_n, \text{Cliff}(T M_n)) \]

as in Definition 2.3. Recall that \( S \) denotes the \( C^* \)-algebra \( C_0(\mathbb{R}) \) graded by even and odd functions.

The restricted bundle \( TM_{n+1}|_{M_n} \) is the pullback bundle \( j_n^*(TM_{n+1}) \) under the inclusion \( j_n : M_n \hookrightarrow M_{n+1} \). Thus, there is an induced pullback metric \( j_n^*(g) \) and pullback connection \( j_n^*(\nabla^{n+1}) \) on \( TM_{n+1}|_{M_n} \), where \( \nabla^{n+1} \) is the Levi-Civita connection of \( M_{n+1} \) [5]. Using this pullback metric we have an orthogonal splitting

\[ TM_{n+1}|_{M_n} \cong TM_n \oplus \nu M_n \]

of vector bundles on \( M_n \). Give \( \nu M_n \) the induced bundle metric and projected connection \( \nabla^{\nu M_n} \). Thus, \( p_n : \nu M_n \rightarrow M_n \) has a canonical structure as an affine Euclidean bundle. By Theorem 2.11 there is an induced \( C^* \)-algebra homomorphism

\[ \Psi_{p_n} : A(M_n) \rightarrow A(\nu M_n) \]

where \( \nu M_n \) is given the Riemannian metric from Lemma 2.6.

Give the open set \( U_n \subset M_{n+1} \) the induced Riemannian metric \( k_n^*(g_{n+1}) \) from \( M_{n+1} \). By Lemma 2.4 we have an inclusion of \( C^* \)-algebras

\[ (k_n)_* : A(U_n) \hookrightarrow A(M_{n+1}) \]

induced by the inclusion \( k_n : U_n \hookrightarrow M_{n+1} \). Finally, we have by Lemma 2.4 a canonical \( C^* \)-algebra isomorphism

\[ (\phi_n)_* : A(\nu M_n) \cong A(U_n) \]

induced by the diffeomorphism \( \phi_n : \nu M_n \rightarrow U_n \) of the tubular neighborhood \( U_n \) with the total space \( \nu M_n \) of the normal bundle.

Thus, we have the following diagram of \( C^* \)-algebras and \( * \)-homomorphisms, which can be considered as the non-commutative version of diagram (5) above:

\[ \begin{array}{cccccc}
A(\nu M_n) & \xrightarrow{(\phi_n)_* \cong} & A(U_n) & \xrightarrow{(k_n)_*} & A(M_{n+1}) \\
\downarrow \Psi_{p_n} & & \downarrow \cong & & \\
A(M_n) & & & & \\
\end{array} \]
The dotted arrow, which is by definition the composition of the other three, gives the connecting map \( \alpha_n : \mathcal{A}(M_n) \rightarrow \mathcal{A}(M_{n+1}) \) in the definition of our \( C^* \)-algebra \( \mathcal{A}(M, g, \mathcal{F}) \).

**Definition 4.1.** Let \( M \) be a smooth Fredholm manifold\(^3\), modeled on the separable infinite-dimensional Euclidean space \( \mathcal{E} \), equipped with a Riemannian metric \( g \) compatible with the topology of \( M \), and an augmented Fredholm filtration \( \mathcal{F} = (M_n, U_n)_{n=k}^{\infty} \). The \( C^* \)-algebra of the triple \( (M, g, \mathcal{F}) \) is the direct limit \( C^* \)-algebra

\[
\mathcal{A}(M, g, \mathcal{F}) = \lim_{\rightarrow} \mathcal{A}(M_n)
\]

where the direct limit is taken over the directed system \( \{\mathcal{A}(M_n), \alpha_n\}_{n=k}^{\infty} \) and the connecting maps \( \alpha_n \) are given by diagram (6).

It easily follows that \( \mathcal{A}(M, g, \mathcal{F}) \) has the structure of a \( \mathbb{Z}_2 \)-graded, separable, nuclear \( C^* \)-algebra. One can also show (using Lemma 2.4 and the construction in Lemma 2.6) that \( \mathcal{A}(M, g, \mathcal{F}) \) does not depend, up to isomorphism of \( \mathbb{Z}_2 \)-graded \( C^* \)-algebras, on the choice of the Riemannian metric \( g \) of \( M \). Indeed, we have:

**Lemma 4.2.** Let \( M \) be a smooth Fredholm manifold with augmented Fredholm filtration \( \mathcal{F} = (M_n, U_n)_{n=k}^{\infty} \). If \( g \) and \( h \) are Riemannian metrics on \( M \) compatible with the topology, there is a canonical map

\[
\Phi : \mathcal{A}(M, g, \mathcal{F}) \rightarrow \mathcal{A}(M, h, \mathcal{F})
\]

which is an isomorphism of \( \mathbb{Z}_2 \)-graded \( C^* \)-algebras.

**Proof.** The identity map \( \text{id}_M : (M, g) \rightarrow (M, h) \) is a diffeomorphism of Riemannian Fredholm manifolds and induces for each \( n \geq k \) a commuting diagram

\[
\begin{array}{ccccccccc}
\mathcal{A}(M_n, g_n) & \longrightarrow & \mathcal{A}(\nu M_n, g'_n) & \longrightarrow & \mathcal{A}(U_n, k^*_n(g_{n+1})) & \longrightarrow & \mathcal{A}(M_{n+1}, g_{n+1}) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\mathcal{A}(M_n, h_n) & \longrightarrow & \mathcal{A}(\nu M_n, h'_n) & \longrightarrow & \mathcal{A}(U_n, k^*_n(h_{n+1})) & \longrightarrow & \mathcal{A}(M_{n+1}, h_{n+1})
\end{array}
\]

where \( g'_n \) and \( h'_n \) are the Riemannian metrics induced on the total space \( \nu M_n \) by Lemma 2.5 and the vertical maps are the \( \mathbb{Z}_2 \)-graded \( C^* \)-algebra isomorphisms induced by \( \text{id}_{M_n} : (M_n, g_n) \rightarrow (M_n, h_n) \) from Lemma 2.4. The result now easily follows by the universal property for direct limits [46] since the composition of the top and bottom rows are the connecting maps in the direct limits \( \mathcal{A}(M, g, \mathcal{F}) \) and \( \mathcal{A}(M, h, \mathcal{F}) \), respectively. \( \square \)

The \( C^* \)-algebra \( \mathcal{A}(M, g, \mathcal{F}) \) does indeed depend on the choice of the augmented Fredholm filtration \( \mathcal{F} = (M_n, U_n)_{n=k}^{\infty} \). However, we will see in the next section that the \( K \)-theory groups of \( \mathcal{A}(M, g, \mathcal{F}) \) do not depend on the choice of the tubular neighborhoods \( \{U_n\}_{n=k}^{\infty} \) and, moreover, if \( M \) has an appropriate spin structure then the \( K \)-theory groups do not depend on the choice of filtrating manifolds \( \{M_n\}_{n=k}^{\infty} \).

We will now consider two examples from the literature that are directly related to this construction.

**Example 4.3.** Consider \( M = \mathcal{E} \), with metric \( g \) induced by the inner product \( \langle \cdot, \cdot \rangle \), and Fredholm filtration given by a flag \( \{E_n\}_n \) of \( \mathcal{E} \) as in Example 3.4 (i).
Setting $\nu E_n = U_n = E_{n+1}$, we obtain an augmented Fredholm filtration $F = (E_n, E_{n+1})_{n=1}^\infty$ of $E$. We thus have the $C^*$-algebra
\[ \mathcal{A}(E, g, F) = \lim_{\rightarrow} \mathcal{A}(E_n) \]
as constructed above. Since $TE_n \cong E_n \times E_n$ is trivial, we have that
\[ \mathcal{A}(E_n) \cong \mathcal{S} \otimes C_0(E_n, \text{Cliff}(E_n)) = \mathcal{SC}(E_n) \]
as in Definition 3.1 of Higson-Kasparov-Trout [24]. Also, since $\nu E_n = U_n = E_{n+1}$, it follows that the connecting map $\alpha_n : \mathcal{A}(E_n) \to \mathcal{A}(E_{n+1})$ can be canonically identified with the Bott periodicity map
\[ \beta_{(n+1)n} = \alpha_n : \mathcal{A}(E_n) \to \mathcal{A}(E_{n+1}) \]
of Definition 3.1 in [25]. Using an approximation argument to deal with the dense subalgebra of compactly supported functions, it follows that the $C^*$-algebra $\mathcal{A}(E, g, F)$ is isomorphic to the $C^*$-algebra
\[ \mathcal{A}(E) = \lim_{E_n \subset E} \mathcal{A}(E_n) \]
where the direct limit is taken over the directed system of all finite dimensional subspaces $E_n \subset E$. See also Lemma 2.6 and the discussion after Definition 4.6 of Higson-Connes and Novikov Conjectures [24]. This $C^*$-algebra has important applications to the Baum-Connes and Novikov Conjectures [24, 25, 49].

**Example 4.4.** Another example, which generalizes the previous one, comes from the Thom isomorphism theorem for infinite rank Euclidean vector bundles [44]. Suppose $M$ is the total space of a smooth (locally trivial) vector bundle $p : M \to X$, with fiber $E$ and structure group $\text{GL}(E)$, over a smooth, finite-dimensional Riemannian manifold $X$ of dimension $k$. Since the fiber $E$ is infinite-dimensional, we may assume [10] that $M = X \times E$ is trivial. The inner product $\langle \cdot, \cdot \rangle$ on $E$ then canonically induces a Euclidean metric structure on the bundle $M$. Using the isomorphism
\[ TM \cong TX \times TE = TX \times (E \times E) \]
we canonically endow the total space $M$ with the structure of a Riemannian Hilbert manifold. Also, since $TM$ is trivial, it follows that $M$ has a canonical structure as a Fredholm manifold.

Let $\{E_n\}_{n=1}^\infty$ be a flag for $E$. For each $n \geq k + 1$, let
\[ M_n = X \times E_{n-k} \to X \]
denote the trivial vector subbundle of rank $n - k$. One can then check that the collection of submanifolds $\{M_n\}_{n=k+1}^\infty$ determines a Fredholm filtration of $M$ such that we can canonically identify the total space $M_n$ of the normal bundle of $M_n \leftrightarrow M_{n+1}$ as $M_{n+1}$. We then have that $F = (M_n, M_{n+1})_{n=k+1}^\infty$ is an augmented Fredholm filtration for $M$. Since $M_n = X \times E_{n-k}$ we have
\[ \mathcal{A}(M_n) \cong \mathcal{A}(E_{n-k}) \otimes \mathcal{C}(X) \cong \mathcal{S} \otimes \mathcal{C}(E_{n-k}) \otimes \mathcal{C}(X). \]
It follows from Proposition 2.13, the results in [44], and a similar approximation argument that
\[ \mathcal{A}(M, g, F) \cong \mathcal{A}(E) \otimes \mathcal{C}(X) \cong \mathcal{A}(M, \nabla_0, X) \]
where $\mathcal{A}(M, \nabla_0, X)$ is the $C^*$-algebra of the affine Euclidean bundle $p : M \to X$, equipped with the trivial connection $\nabla_0 = d$, as in Definition 3.11 of [44].
5. K-theory, Spin Structures and Poincaré Duality

In this section we discuss the relationship between the topological K-theory groups, the (compactly supported) K-homology groups of a Fredholm manifold $M$ and the $K$-theory groups of the $C^*$-algebra $A(M, g, F)$ we constructed in the last section. When an oriented Riemannian Fredholm manifold $M$ has been equipped with an appropriate infinite-dimensional spin structure, we will see that all of these groups coincide, as in the finite-dimensional spin manifold setting.

5.1. The topological $K$-theory of a Fredholm manifold. Mukherjea [35 Sec.2], in the context of generalized cohomologies obtained from a spectrum on the category of compact spaces, defined the corresponding cohomology groups for Fredholm categories. Based on his work, we are led to make the following definition.

Definition 5.1. Let $M$ be smooth Fredholm manifold with augmented Fredholm filtration $\mathcal{F} = (M_n, U_n)_{n=0}^\infty$. The $j$th topological $K$-theory group of $(M, \mathcal{F})$, denoted $K^{\infty-j}(M, \mathcal{F})$, is defined to be the direct limit

$$K^{\infty-j}(M, \mathcal{F}) = \lim_{\rightarrow} K^{n-j}(M_n),$$

for $j = 0, 1$, where the connecting maps are the Gysin (or shriek) maps $27$

$$(j_n): K^{n-j}(M_n) \to K^{n+1-j}(M_{n+1})$$

associated to the inclusions $j_n : M_n \hookrightarrow M_{n+1}$. These may be obtained from diagram $5$, via the functoriality properties of topological $K$-theory, as the composition of Gysin maps

$$\begin{array}{c}
K^{n+1-j}(\nu M_n) \\
\cong \\
K^{n-j}(M_n)
\end{array} \xrightarrow{(\phi_n)_!} \xrightarrow{(k_n)_!} \xrightarrow{(j_n)_!} K^{n+1-j}(M_{n+1})$$

where the map $s_1$ is the Gysin map associated to the zero section $s : M_n \to \nu M_n$, and which induces the Thom isomorphism. (Compare this with diagram $6$.)

Clearly, the definition of the topological $K$-theory of $M$ does not depend on the choice of tubular neighborhoods $\{U_n\}_n$ (or any Riemannian metric $g$) but does, a priori, depend on the choice of Fredholm filtration $\{M_n\}_n$, as does the definition of $A(M, g, \mathcal{F})$. However, if $M$ has a certain infinite-dimensional spin structure, then these topological $K$-theory groups $K^{\infty-j}(M, \mathcal{F})$ do not depend on the choice of $\mathcal{F} = (M_n, U_n)_n$.

5.2. Fredholm Spin$_{\nu}$-structures. Recall the notation introduced at the beginning of Section 3. Let $\mathcal{E}$ be a separable infinite-dimensional Euclidean space. For $1 \leq q \leq \infty$, let $GL_q(\mathcal{E}) = GL(\mathcal{E}) \cap (I + P_q)$, where $P_q$ is the $q$-th Schatten-von Neumann perturbation class. Let $\partial(\mathcal{E})$ denote the orthogonal operators on $\mathcal{E}$. We let $\bar{\partial}(\mathcal{E}) = \partial(\mathcal{E}) \cap GL_q(\mathcal{E})$ and let $\mathcal{S}\bar{\partial}_q(\mathcal{E})$ denote the connected component of $I$ in $\bar{\partial}_q(\mathcal{E})$. All of these groups are infinite-dimensional Banach-Lie groups $13$, with manifold topology given by the restriction of the norm $\| \cdot \|_q$. Note that since $P_q \subset K(\mathcal{E})$, it follows that $GL_q(\mathcal{E}) \subset GL_K(\mathcal{E})$ and so any Hilbert manifold with $GL_q(\mathcal{E})$ as structure group has a canonical Fredholm structure as in Definition $7$.

Let $M$ be a smooth, paracompact, connected Hilbert manifold, without boundary, modeled on $\mathcal{E}$. Let $\xi : E \to M$ be a smooth (locally trivial) vector bundle over
M, with fiber E, endowed with a reduction of the structure group from GL(E) to GL_q(E). A Riemannian q-structure [2 Def 2.1] on ξ is a reduction of the structure group from GL_q(E) to O_q(E). Since M is paracompact, this may be accomplished by using a partition of unity to define a smooth bundle metric g_x on the fibers E_x of ξ. If ξ is the tangent bundle π : TM → M, with Fredholm structure group GL_q(E), then we say that M has a Riemannian q-structure.

Definition 5.2. [2 Def 2.2] A Riemannian q-structure on ξ : E → M is orientable if ξ admits a further reduction of its structure group to SØ_q(E). A given reduction will be called an orientation and ξ will be said to have an oriented Riemannian q-structure.

A proof of the following can be found in [30 Prop 6.2] or [2 Thm 2.1].

Theorem 5.3. A Riemannian q-structure on ξ : E → M is orientable if and only if the first Stieffel-Whitney class w_1(ξ) ∈ H^1(M, Z_2) vanishes. In particular, if M has a Riemannian q-structure, then M is orientable if and only if w_1(M) = w_1(TM) = 0.

For the theory of Stieffel-Whitney classes associated to Hilbert bundles over Hilbert manifolds that we are considering, see Koschorke [30]. Note that, contrary to the finite-dimensional case, these characteristic classes are not diffeomorphism invariants, in general. (See [30 Example 6.2] for details.)

Since SO_q(E) is of index 2 in Ø_q(E), it follows that the universal covering Spin_q(E) is a Banach-Lie group and the covering map is 2-sheeted. We thus have an exact sequence of (paracompact) topological groups

\[ 1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}_q(E) \rightarrow SO_q(E) \rightarrow 1 \]

Concrete realizations of these infinite-dimensional spin groups were constructed for q = 1 by P. de la Harpe [13] and for q = 2 by Plymen and Streater [40]. However, we will not need explicit constructions of these spin groups, only the fact that they are 2-sheeted covering groups of the associated special orthogonal groups, as in the finite-dimensional case. In the following, we may abbreviate Spin_q and SO_q for Spin_q(E) and SO_q(E), respectively.

Definition 5.4. [2 Def 2.4]) Suppose ξ : E → M has an SO_q-structure, i.e., an oriented Riemannian q-structure. A Spin_q-structure on ξ is a principal bundle extension associated to the covering map

\[ \rho : \text{Spin}_q \rightarrow SO_q \]

of the principal SO_q-bundle of linear frames of ξ. If M is a Fredholm manifold with oriented Riemannian q-structure, then a Spin_q-structure on M is a Spin_q-structure on π : TM → M. We will then call M a Fredholm Spin_q-manifold.

That is, if p : L → M is the principal SO_q-bundle of oriented orthonormal frames of ξ : E → M, then a Spin_q-structure for ξ is a principal Spin_q-bundle q : Σ → M such that Σ is a 2-fold covering of L, the restriction of the covering map \( \tilde{\rho} : \Sigma \rightarrow L \) to the fibers are 2-sheeted coverings and

\[ \tilde{\rho}(s \cdot g) = \tilde{\rho}(s) \rho(g) \quad \text{and} \quad q(s) = p(\tilde{\rho}(s)) \]
for all \( s \in \Sigma \) and \( g \in \text{Spin}_q \). Thus, the following diagram commutes:

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\tilde{\rho}} & L \\
\downarrow q & & \downarrow q \\
M & \xrightarrow{\text{Id}_M} & M
\end{array}
\]

For \( q = 1 \) de la Harpe has shown that the existence of a \( \text{Spin}_q \)-structure on a Fredholm manifold \( M \) with oriented Riemannian \( q \)-structure is equivalent to the vanishing \( w_2(M) = 0 \) of the second Stieffel-Whitney class in \( H^2(M, \mathbb{Z}_2) \). We wish to extend his result to all values \( 1 \leq q \leq \infty \) and all \( SO_q \)-vector bundles. Although his argument for \( q = 1 \) almost certainly holds in the general case, we will provide a more direct proof using an argument of Lawson and Michelsohn [33] from the finite-dimensional spin case. In order to do that, we need the following cohomology computation, which follows from some results in the literature [13, 15], but we provide a proof for completeness.

Lemma 5.5. For \( 1 \leq q \leq \infty \), \( H^1(\text{SO}_q(E), \mathbb{Z}_2) \cong \mathbb{Z}_2 \).

Proof. Choose a flag \( \{ E_n \} \) for \( E \) as in Example 3.4 (i). This induces an inclusion of topological groups

\[
\text{SO}(\infty) = \lim_{\rightarrow} \text{SO}(n) \hookrightarrow \text{SO}_q(E)
\]

which, by Proposition 3 in [13], is a homotopy equivalence. Hence, using the identity as basepoint, we have by Bott periodicity [7]:

\[
\pi_1(\text{SO}_q(E)) \cong \pi_1(\text{SO}(\infty)) \cong \lim_{\rightarrow} \pi_1(\text{SO}(n)) \cong \mathbb{Z}_2.
\]

Since \( \text{SO}_q(E) \) is connected with abelian fundamental group, it follows that

\[
H_1(\text{SO}_q(E)) \cong \pi_1(\text{SO}_q(E)) \cong \mathbb{Z}_2.
\]

The result now follows from the Universal Coefficient Theorem in cohomology:

\[
H^1(\text{SO}_q(E), \mathbb{Z}_2) \cong \text{Hom}(H_1(\text{SO}_q(E), \mathbb{Z}), \mathbb{Z}_2) \cong \text{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2
\]

and we are done. \( \Box \)

Theorem 5.6. Let \( \xi : E \to M \) be a Hilbert bundle with oriented Riemannian \( q \)-structure. Then \( \xi \) has a \( \text{Spin}_q \)-structure if and only if the second Stieffel-Whitney class \( w_2(\xi) \in H^2(M, \mathbb{Z}_2) \) vanishes. In particular, if \( M \) is a Fredholm manifold with oriented Riemannian \( q \)-structure, then there exists a \( \text{Spin}_q \)-structure on \( M \) if and only if \( w_2(M) = 0 \).

For the following, recall that in principal bundle theory, if \( M \) is a paracompact space and \( G \) is a topological group, then \( H^1(M, G) \) is isomorphic to the set of isomorphism classes of principal \( G \)-bundles on \( M \), where we are using Čech cohomology. (See Appendix A of Lawson and Michelsohn [33].)

Proof. Let \( p : L \to M \) be the principal \( \text{SO}_q \)-bundle of oriented orthonormal frames of \( \xi \). We then have a fibration

\[
\begin{array}{ccc}
\text{SO}_q(E) & \xrightarrow{i} & L \\
\downarrow p^* & & \downarrow p \\
H^1(M, \mathbb{Z}_2) & \xrightarrow{i^*} & H^1(L, \mathbb{Z}_2) \\
\downarrow \delta & & \downarrow \delta \\
H^2(M, \mathbb{Z}_2)
\end{array}
\]

which induces an exact sequence

\[
H^1(M, \mathbb{Z}_2) \xrightarrow{p^*} H^1(L, \mathbb{Z}_2) \xrightarrow{i^*} H^1(\text{SO}_q(E), \mathbb{Z}_2) \xrightarrow{\delta} H^2(M, \mathbb{Z}_2)
\]
in Čech cohomology. It follows by the above discussion (see also [2 Thm 2.3]) that \( \xi \) has a Spin\(_q\)-structure if and only if there is a cohomology class \( \alpha = \alpha(\xi) \in H^1(L, \mathbb{Z}_2) \) such that \( i^*(\alpha) \neq 0 \) since a Spin\(_q\)-structure on \( \xi \) determines a nontrivial 2-sheeted covering of \( L \). Let \( g_2 \) be the generator of \( H^1(SO_q(\mathcal{E}), \mathbb{Z}_2) \cong \mathbb{Z}_2 \). It follows that \( \xi \) has a Spin\(_q\)-structure if and only if there is a cohomology class \( \alpha = \alpha(\xi) \in H^1(L, \mathbb{Z}_2) \) such that \( i^*(\alpha) = g_2 \). Consequently, by exactness of the sequence above, we have that this holds if and only if

\[
  w_2(\xi) = \delta(\xi(g_2)) = \delta(i^*(\alpha)) = 0 \in H^2(M, \mathbb{Z}_2).
\]

The fact that the second Stieffel-Whitney class of \( \xi \) is given by

\[
  w_2(\xi) = \delta(\xi(g_2)) \in H^2(M, \mathbb{Z}_2)
\]

follows from the universal properties of these classes [30 Proposition 6.3].

Consequently, if \( \xi : E \to M \) admits a Spin\(_q\)-structure determined by \( \alpha(\xi) \in H^1(L, \mathbb{Z}_2) \) then the most general Spin\(_q\)-structure on \( \xi \) is of the form \( \alpha(\xi) + p^*(\beta) \) where \( \beta \in H^1(M, \mathbb{Z}_2) \). Thus, there is a bijection between the set of (isomorphism classes of) Spin\(_q\)-structures on \( \xi \) and \( H^1(M, \mathbb{Z}_2) \). It follows that a Spin\(_q\)-structure on \( \xi \) (or \( M \)) is unique if \( H^1(M, \mathbb{Z}_2) = 0 \).

The next two results are immediate corollaries (see Theorems 2.5 and 2.6 of [2]).

**Proposition 5.7.** Given Spin\(_q\)-structures on two out of the three vector bundles \( \xi_1, \xi_2, \) and \( \xi_1 \oplus \xi_2 \) on \( M \), there is a uniquely determined Spin\(_q\)-structure on the third.

**Proposition 5.8.** If \( \xi : E \to M \) admits a Spin\(_q\)-structure and \( f : N \to M \) is smooth, then the pull-back vector bundle \( f^*\xi : f^*E \to N \) admits a Spin\(_q\)-structure.

In the context of Fredholm manifolds, the above give:

**Corollary 5.9.** Let \( M \) be a Fredholm Spin\(_q\)-manifold. If \( \{M_n\}_n \) is any associated Fredholm filtration of \( M \) then each \( M_n \) has a canonical (finite-dimensional) spin structure.

Indeed, associated to the inclusion \( i_n : M_n \to M \) we have a split short exact sequence

\[
0 \to TM_n \to TM|_{M_n} \to \mu M_n \to 0.
\]

The normal bundle \( \mu M_n \) has a Spin\(_q\)-structure being trivial, and \( TM|_{M_n} = i_n^*(TM) \) has one because of Proposition 5.8. Thus, we have

\[
w_2(\mu M_n) = w_2(TM|_{M_n}) = 0
\]

and finally Proposition 5.7 gives the result since \( w_2(M_n) = 0 \).

We end this subsection about spin structures with an example coming from certain based loop groups.

**Example 5.10.** Consider a compact, connected, simply connected, simple Lie group \( G \). Let \( \Omega_s G = H^s_0(S^1, G) \) be the group of based loops on \( G \), i.e., maps from the circle to \( G \) in the \( s \)th Sobolev space \( H^s \) which take a fixed point on \( S^1 \) into the identity element of \( G \), where \( s \geq 1/2 \). \( \Omega_s G \) is a (real) Hilbert Lie group.

D. Freed constructed in [22, Sec.5] a particular Fredholm 1-structure, coming from a classifying map

\[
\Omega_s G \to BGL(\infty; \mathbb{C}) \sim \Phi_0,
\]
where $\Phi_0$ denotes the Fredholm operators of index zero. The resulting frame bundle was called the geometric frame bundle. He concluded that the realization of this geometric frame bundle is trivial and that the Stiefel-Whitney classes of $\Omega_sG$ vanish ([22 Thm 5.30]). Our Theorem 5.16 now shows that this is the unique $\text{Spin}_1$-structure on $\Omega_sG$. Indeed, the hypothesis on $G$ implies that $\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$, and $\pi_3(G) = \mathbb{Z}$. Consequently $H_1(\Omega_sG, \mathbb{Z}) = 0$ and $H_2(\Omega_sG, \mathbb{Z}) \cong \pi_2(\Omega_sG) \cong \pi_3(G) = \mathbb{Z}$. These imply that $H^1(\Omega_sG, \mathbb{Z}_2) = 0$ and $H^2(\Omega_sG, \mathbb{Z}_2) = \mathbb{Z}/2$. As $w_2(\Omega_sG) = 0$ by Freed’s Corollary 5.31, and as $\text{Spin}_1$-structures on $\Omega G$ are parametrized by $H^1(\Omega_sG, \mathbb{Z}_2) = 0$, we obtain the claimed uniqueness of the $\text{Spin}_1$-structure on $\Omega_sG$. Moreover, Freed’s Fredholm structure is actually the unique $\text{Spin}_1$-structure, for all $1 \leq q \leq \infty$.

5.3. $K$-homology and Poincaré duality. Recall that if $X$ is a compact space then the $j$-th $K$-homology group of $X$ is the abelian group $K_j(X) = KK^j(C(X), \mathbb{C})$ which is dual to the $j$-th $K$-theory group $K^j(X) = KK^j(\mathbb{C}, C(X))$. The map $X \mapsto K_j(X)$ defines a generalized homology theory on the category of compact spaces and continuous maps [8, 28, 26].

Definition 5.11. Let $M$ be a paracompact space. The $j^{th}$ compactly supported $K$-homology group of $M$ is

$$K^j_c(M) = \lim_{\longrightarrow} K_j(X),$$

where the direct limit is over all the compact subsets $X \subset M$, and $j = 0, 1$.

In order to prove our Poincaré duality result, we need the following result, whose proof requires the $KK$-theory for pro-$C^*$-algebras developed by Weidner [47] and Phillips [38]. A heuristic proof would be that since $M \sim M_{\infty} = \lim_{\longrightarrow} M_n$, we have in compactly supported $K$-homology that $K^j_c(M) \cong K^j_c(M_{\infty}) \cong \lim_{\longrightarrow} K^j_c(M_n)$.

Proposition 5.12. Let $M$ be a smooth Fredholm manifold. If $\{M_n\}_{n=k}^{\infty}$ is any Fredholm filtration of $M$ then there is an isomorphism of abelian groups

$$K^j_c(M) \cong \lim_{\longrightarrow} K^j_c(M_n), \quad j = 0, 1,$$

where the connecting map $K^j_c(M_n) \to K^j_c(M_{n+1})$ in the direct limit is induced by the inclusion $M_n \hookrightarrow M_{n+1}$.

Proof. Let $g$ be a Riemannian metric on $M$ compatible with the topology (which exists via paracompactness). Thus, $(M, g)$ is a metric space. Since metric spaces are compactly generated [38 I.4.3], it follows that the algebra $C(M)$ of all continuous complex-valued functions on $M$, with the topology of uniform convergence on compact subsets, is a pro-$C^*$-algebra with involution given by pointwise complex conjugation [37 Ex 1.3.3]. Let $X \subset M$ denote the collection of all compact subsets $X$ of $M$ ordered by inclusion. Since $M$ is regular, it is completely Hausdorff [37 Def 2.2], and so by Corollary 2.9 of [37], it follows that there is an isomorphism

$$C(M) \cong \lim_{\longrightarrow} C(X),$$

of pro-$C^*$-algebras. Similarly, for each $n$, we have an isomorphism

$$C(M_n) \cong \lim_{\longrightarrow} C(K_n)$$
of pro-$\mathcal{C}^*$-algebras where $\mathcal{C}_{M_n}$ denotes the set of all compact subsets $K_n$ of $M_n$ ordered by inclusion. Let $M_\infty = \bigcup_n M_n = \lim M_n$ with the direct limit topology. Since $M_\infty$ is countably compactly generated in the direct limit topology, we then have an isomorphism

\begin{equation}
C(M_\infty) \cong \lim_{\to} C(M_n)
\end{equation}

of pro-$\mathcal{C}^*$-algebras. By Theorem 3.3 the inclusion $M_\infty \hookrightarrow M$ is a homotopy equivalence, hence the pro-$\mathcal{C}^*$-algebras $C(M)$ and $C(M_\infty)$ have the same homotopy type.

Using the fact that Weidner’s $KK$-groups $KK^j_W(A, B)$ for pro-$\mathcal{C}^*$-algebras [47, 38] extend Kasparov’s $KK$-groups for $\mathcal{C}^*$-algebras [28], are homotopy-invariant, and convert inverse limits to direct limits in the $K$-homology variable, we compute as follows:

\[ K^j_\infty(M) = \lim_{\to} K^j(C(X), \mathbb{C}) \quad \text{(Definition 5.11)} \]
\[ \cong KK^j_W(\lim_{\to} C(X), \mathbb{C}) \quad \text{(By [47, Thm 5.1])} \]
\[ \cong KK^j_W(C(M), \mathbb{C}) \quad \text{(By Eqn (9))} \]
\[ \cong KK^j_W(C(M_\infty), \mathbb{C}) \quad \text{(homotopy invariance)} \]
\[ \cong KK^j_W(\lim_{\to} C(M_n), \mathbb{C}) \quad \text{(By Eqn (11))} \]
\[ \cong \lim_{\to} KK^j_W(C(M_n), \mathbb{C}) \quad \text{(By [47, Thm 5.1])} \]
\[ \cong \lim_{\to} KK^j_W(\lim_{\to} C(K_n), \mathbb{C}) \quad \text{(By Eqn (10))} \]
\[ \cong \lim_{\to} \lim_{\to} KK^j_W(C(K_n), \mathbb{C}) \quad \text{(By [47, Thm 5.1])} \]
\[ \cong \lim_{\to} K^j_\infty(M_n) \quad \text{(Definition 5.11)} \]

□

Compare the following result for Fredholm Spin$_q$-manifolds with [35, Thm 2.1].

**Theorem 5.13 (Poincaré duality).** If $M$ is a smooth Fredholm Spin$_q$-manifold with augmented Fredholm filtration $\mathcal{F}$, there is an isomorphism

\[ K^{\infty-j}(M, \mathcal{F}) \cong K^j_\infty(M) \]

**Proof.** Let $\mathcal{F} = (M_n, U_n)_{n=k}^\infty$ be the augmented Fredholm filtration. Since $M$ is a Fredholm Spin$_q$-manifold, each $M_n$ has a canonical spin structure by Corollary [31, Cor 31] (or [26, Exercise 11.8.11]) we have a natural Poincaré duality isomorphism

\[ P_n : K^{n-j}(M_n) \xrightarrow{\cong} K^j_\infty(M_n) \]

\[ \text{Note that there is a typo in the statement of [47, Thm 5.1]} \]
given by the cap product with the fundamental class \([M_n]\). Naturality is the assertion that the Poincaré duality diagram

\[
\begin{array}{ccc}
K^{n-j}(M_n) & \xrightarrow{j_n!} & K^{n+1-j}(M_{n+1}) \\
\cong & \cong & \cong \\
K^j_\ast(M_n) & \xrightarrow{j_n} & K^j_\ast(M_{n+1})
\end{array}
\]

commutes, where \(j_n : M_n \to M_{n+1}\). It now follows that:

\[
\begin{align*}
K^j_\ast(M) & \cong \lim K^j_\ast(M_n) \quad \text{(Proposition 5.12)} \\
& \cong \lim K^{n-j}(M_n) \quad \text{(classical Poincaré duality)} \\
& = K^{\infty-j}(M, \mathcal{F}) \quad \text{(Definition 5.1)}
\end{align*}
\]

as desired.

5.4. **K-theory of the \(C^\ast\)-algebra** \(\mathcal{A}(M, g, \mathcal{F})\). First we discuss the finite dimensional results we will need. Let \(M_n\) be an oriented Riemannian \(n\)-manifold. An important relationship between the non-commutative \(C^\ast\)-algebra \(\mathcal{C}(M_n) = C_0(M_n, \text{Cliff}(TM_n))\) and the commutative \(C^\ast\)-algebra \(C_0(M_n)\) is given by \(\text{spin}^c\)-structures [33]. Let \(\mathcal{C}_1 = \text{Cliff}(\mathbb{R})\) denote the first complex Clifford algebra. The following is adapted from Theorem 2.11 of Plymen [39] and Proposition II.A.9 of Connes [11].

**Proposition 5.14.** If \(n = 2k\) is even, there is a bijective correspondence between \(\text{spin}^c\)-structures on \(M_n\) and Morita equivalences (in the sense of Rieffel [42, 44]) between the \(C^\ast\)-algebras \(C_0(M_n)\) and \(\mathcal{C}(M_n)\). Thus, \(\mathcal{A}(M_n)\) is Morita equivalent to \(C_0(\mathbb{R} \times M_n)\). If \(n = 2k + 1\) is odd, then \(\text{spin}^c\)-structures on \(M\) are in bijective correspondence with Morita equivalences \(C_0(M_n) \sim \mathcal{C}(M_n) \otimes \mathcal{C}_1\).

Although \(\mathcal{C}(M_n)\) and \(\mathcal{A}(M_n)\) carry natural \(\mathbb{Z}_2\)-gradings, when we consider their \(C^\ast\)-algebra \(K\)-theory, we will ignore these gradings. That is, if \(A\) is any \(C^\ast\)-algebra — graded or not — then \(K_j(A)\) \((j = 0, 1)\) will denote the \(K\)-theory group of the underlying \(C^\ast\)-algebra, without the grading. Since \(C^\ast\)-algebra \(K\)-theory is Morita invariant, we have the following.

**Corollary 5.15.** If \(M_{2k}\) is an even-dimensional oriented Riemannian manifold with \(\text{spin}^c\)-structure, there is a canonical \(K\)-theory isomorphism\(^5\)

\[
K_j(\mathcal{A}(M_{2k})) \cong K^{j+1}(M_{2k}).
\]

The next result is proved by Trout [15, Thm 2.14]:

**Thom Isomorphism Theorem 5.16.** If \(E \to M\) is a smooth finite-rank affine Euclidean bundle, then the \(*\)-homomorphism \(\Psi_p : \mathcal{A}(M_n) \to \mathcal{A}(E)\) from Theorem 5.17 induces an isomorphism of abelian groups:

\[\Psi_* : K_j(\mathcal{A}(M_n)) \to K_j(\mathcal{A}(E)), \text{ for } j = 0, 1.\]

In fact, it is the \(C^\ast\)-algebraic formulation of the classical Thom isomorphism \(\Phi : K^j(M) \to K^j(E)\) from topological \(K\)-theory.

\(^5\)It is also true that \(K_j(\mathcal{C}(M_{2k})) \cong K^j(M_{2k})\), but we shall not use this here.
Corollary 5.17. [45 Cor 2.20] If \( E \) is a finite even-rank oriented Euclidean spin\(^c\)-bundle (with spin connection \( \nabla \)) on an even-dimensional oriented Riemannian spin\(^c\)-manifold \( M \), then \( \Psi_p : A(M) \to A(E) \) induces the topological Thom isomorphism \( \Phi \), as depicted in the following commutative diagram:

\[
\begin{array}{ccc}
K_J(A(M)) & \xrightarrow{\Psi_*} & K_J(A(E)) \\
\downarrow & & \downarrow \\
K_{J+1}(M) & \xrightarrow{\Phi} & K_{J+1}(E)
\end{array}
\]

Although the connecting maps \( \alpha_n : A(M) \to A(M_{n+1}) \) are not functorial at the \( C^* \)-algebra level (as in diagram (1)), they are at the level of \( K \)-theory.

Lemma 5.18. The following diagram of abelian groups

\[
\begin{array}{ccc}
K_J(A(M_{n+1})) & \xrightarrow{(\alpha_n)_*} & K_J(A(M_{n+1})) \\
\downarrow & & \downarrow \\
K_J(A(M_n)) & \xrightarrow{(\alpha_{n+1})_*} & K_J(A(M_{n+2}))
\end{array}
\]

commutes for all \( n \geq k \) and \( j = 0, 1 \), where \( \alpha_{n+2}^n : A(M_n) \to A(M_{n+2}) \) is any Gysin map induced by the inclusion \( M_n \hookrightarrow M_{n+2} \) (as in Diagram (6)).

Proof. The functor \( M_n \mapsto K_j(A(M_n)) \) from the category of finite-dimensional smooth (Riemannian) manifolds is homotopy-invariant, has Gysin maps (independent of the choice of tubular neighborhood) and, most importantly, a transitive Thom homomorphism [45 Lem 3.10]. The result now follows from the corresponding proof in Karoubi [27 Props 5.22 and 5.24] for topological \( K \)-theory. \( \square \)

We now come to the main result of our paper.

Theorem 5.19. Let \( M \) be a smooth Fredholm Spin\(_q\)-manifold with Riemannian metric \( g \) and augmented Fredholm filtration \( \mathcal{F} = (M_n, U_n)_{n=k}^\infty \). With a dimension shift, the \( K \)-theory of \( A(M, g, \mathcal{F}) \) coincides with the topological \( K \)-theory of \( (M, \mathcal{F}) \) and the (compactly supported) \( K \)-homology of \( M \):

\[
K_{j+1}(A(M, g, \mathcal{F})) \cong K^{\infty-j}(M, \mathcal{F}) \cong K^c_j(M).
\]

Proof. Indeed, using the fact that \( 2\mathbb{Z} \) is cofinal in \( \mathbb{Z} \), we can restrict to the even-dimensional subsequences in the directed limits under consideration:

\[
K^{\infty-j}(M, \mathcal{F}) = \lim_{\rightarrow} K^{n-j}(M_n) \quad \text{(Definition [51])}
\]

\[
\cong \lim_{\rightarrow} K^{2n-j}(M_{2n}) \quad \text{(cofinal property of direct limits)}
\]

\[
\cong \lim_{\rightarrow} K^{j+2}(M_{2n}) \quad \text{(Bott periodicity)}
\]

\[
\cong \lim_{\rightarrow} K_{j+1}(A(M_{2n})) \quad \text{(Corollary [51])}
\]

\[
\cong \lim_{\rightarrow} K_{j+1}(A(M_n)) \quad \text{(cofinal property of direct limits)}
\]

\[
\cong K_{j+1}(\lim_{\rightarrow} A(M_n)) \quad \text{(continuity of \( K \)-theory)}
\]

\[
= K_{j+1}(A(M, g, \mathcal{F})) \quad \text{(Definition [51])}
\]

\( \square \)
As the compactly supported $K$-homology of $M$ does not depend on the metric and on the choice of augmented filtration, we get in particular the following independence on the metric and the filtration (compare again with [35, Thm 2.1]):

**Corollary 5.20.** If $M$ is a smooth Fredholm Spin$_q$-manifold, as above, then its topological $K$-theory $K^\infty_{-j}(M, F)$ and the $K$-theory of $\mathcal{A}(M, g, F)$ do not depend on the choices of the metric $g$ and augmented Fredholm filtration $F$.

Another easy consequence is:

**Corollary 5.21.** If $E$ is a separable infinite-dimensional Euclidean space, then

$$K_j(A(E)) \cong K_j(A(S_E)) \cong \begin{cases} 0, & \text{if } j = 0, \\ \mathbb{Z}, & \text{if } j = 1 \end{cases}$$

where $S_E$ denotes the unit sphere in $E$.

**References**

1. R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, 2nd ed., Applied Mathematical Sciences, vol. 75, Springer-Verlag, New York, 1988.
2. M. Anastasiei, *Spin structures on Hilbert manifolds*, An. Ştiinţ. Univ. “Al. I. Cuza” Iaşi Secţ. Ia Mat. (N.S.) 24 (1978), no. 2, 367–373.
3. M. F. Atiyah, *K-theory*, W. A. Benjamin, New York, 1967.
4. M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology 3 (1964), no. Suppl. 1, 3–38.
5. N. Berline, E. Getzler, and M. Vergner, *Heat Kernels and Dirac Operators*, Grundlehren der mathematischen Wissenschaften, vol. 298, Springer-Verlag, Berlin, 1986.
6. B. Blackadar, *K-theory for Operator Algebras*, 2nd ed., MSRI Publication Series 5, Springer-Verlag, New York, 1998.
7. R. Bott, *The stable homotopy of the classical groups*, Ann. Math. 70 (1959), 313–337.
8. L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Extensions of C*-algebras and K-homology*, Ann. of Math. (2) 105 (1977), no. 2, 265–324.
9. V. M. Bukhshtaber and A. S. Miščenko, *Elements of infinite filtration in K-theory*, Dokl. Akad. Nauk SSSR 178 (1968), 1234–1237.
10. A. Connes, *A K-theory on the category of infinite cell complexes*, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968), 560–604.
11. A. Connes, *Noncommutative geometry*, Academic Press, 1994.
12. A. Connes and G. Skandalis, *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci. 20 (1984), no. 6, 1139–1183.
13. P. de la Harpe, *Classical Banach-Lie algebras and Banach-Lie groups of operators in Hilbert space*, Springer-Verlag, Berlin, 1972, Lecture Notes in Mathematics, Vol. 285.
14. E. Dyer, *Cohomology theories*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York-Amsterdam, 1969.
15. J. Dixmier, *Champs continus d’espaces hilbertiens et de C*-algèbres*, J. Math. Pure Appl. 42 (1993), 1–20.
16. J. Dixmier, *The Clifford algebra and the spinor group of a Hilbert space*, Compositio Math. 25 (1972), 245–261.
17. J. Dixmier, *Some Properties of Infinite-Dimensional Orthogonal Groups*, Global analysis and its applications., Internat. Atomic Energy Agency, Vienna, 1974, pp. 71–77.
18. E. Dyer, *Nonlinear functional analysis* (Proc. Sympos. Pure Math., Vol. XVIII, Part 1, Chicago, Ill., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 62–85.
19. J. Elsken and K. D. Elworthy, *Open embeddings of certain Banach manifolds*, Ann. of Math. (2) 91 (1970), 465–485.
20. K. D. Elworthy and A. J. Tromba, *Differential structures and Fredholm maps on Banach manifolds*, Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 45–94.
22. D. S. Freed, *The geometry of loop groups*, J. Differential Geom. 28 (1988), no. 2, 223–276.
23. J. M. Gracia-Bondía, J. C. Varilly, and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, Birkhäuser, Boston, Basel, Berlin, 2001.
24. N. Higson and G. Kasparov, *E-theory and K-theory for groups which act properly and isometrically on Hilbert space*, Invent. Math. 144 (2001), 23–74.
25. N. Higson, G. Kasparov, and J. Trout, *A Bott Periodicity Theorem for Infinite Dimensional Euclidean Space*, Advances in Mathematics 135 (1998), no. 1, 1–40.
26. N. Higson and J. Roe, *Analytic K-Homology*, Oxford Mathematical Monographs, Oxford University Press., Oxford, England., 2000.
27. M. Karoubi, *K-theory, An Introduction*, Grundlehren der Mathematischen Wissenschaften, vol. 226, Springer-Verlag, New York, 1978.
28. G. G. Kasparov, *The operator K-functor and extensions of C*-algebras*, Math. USSR Izvestija 16 (1981), 513–572.
29. , *Equivariant KK-theory and the Novikov Conjecture*, Invent. Math. 91 (1988), 147–201.
30. U. Koschorke, *Infinite dimensional K-theory and characteristic classes of Fredholm bundle maps*, Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 95–133.
31. N. H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, Topology 3 (1965), 19–30.
32. S. Lang, *Differential and Riemannian manifolds*, third ed., Graduate Texts in Mathematics, vol. 160, Springer-Verlag, New York, 1995.
33. H. B. Lawson and M-L Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, 1989.
34. K. K. Mukherjea, *The homotopy type of Fredholm manifolds*, Trans. Amer. Math. Soc. 149 (1970), 655–663.
35. , *The algebraic topology of Fredholm manifolds*, Analyse globale, (Sém. Math. Supérieures, No. 42, Univ. Montréal, Montreal, Que., 1969), Presses Univ. Montréal, Montreal, Que., 1971, pp. 163–177.
36. R. S. Palais, *On the homotopy type of certain groups of operators*, Topology 3 (1965), 271–279.
37. N. Christopher Phillips, *Inverse limits of C*-algebras*, J. Operator Theory 19 (1988), no. 1, 159–195.
38. , *Representable K-theory for σ-C*-algebras*, K-Theory 3 (1989), no. 5, 441–478.
39. R. Plymen, *Strong Morita Equivalence, Spinors and Symplectic Spinors*, J. Operator Theory 16 (1986), 305–324.
40. R. J. Plymen and R. F. Streater, *A model of the universal covering group of SO(H)2*, Bull. London Math. Soc. 7 (1975), no. 3, 283–288.
41. I. Raeburn and D. P. Williams, *Pull-backs of C*-algebras and crossed products by certain diagonal actions*, Trans. Amer. Math. Soc. 287 (1985), 755–777.
42. , *Morita Equivalence and Continuous-Trace C*-Algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, 1998.
43. A. Rennie, *Smoothness and locality for nonunital spectral triples*, K-Theory 28 (2003), no. 2, 127–165.
44. M. A. Rieffel, *Morita equivalence for operator algebras*, Proc. Symp. Pure Math 38 (1982), 285–298.
45. J. Trout, *A Thom Isomorphism for Infinite Rank Euclidean Bundles*, Homology, Homotopy and Applications 5 (2003), 121–159.
46. N. E. Wegge-Olsen, *K-theory and C*-algebras*, Oxford University Press, New York, 1993.
47. J. Weidner, *KK-groups for generalized operator algebras. I, II*, K-Theory 3 (1989), no. 1, 57–77, 79–98.
48. G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York, 1978.
49. G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. 139 (2000), no. 1, 201–240.