DYNAMICAL BEHAVIOR OF A ROTAVIRUS DISEASE MODEL WITH TWO STRAINS AND HOMOTYPIC PROTECTION

KUN LU*
Department of Mathematics, Shaanxi University of Science and Technology
Xi’an 710021, China

WENDI WANG
School of Mathematics and Statistics, Southwest University
Chongqing 400715, China

JIANQUAN LI
Department of Mathematics, Shaanxi University of Science and Technology
Xi’an 710021, China

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Abstract. A two-strain rotavirus model with vaccination and homotypic protection is proposed to study the survival of the two strains of rotavirus within the host. Corresponding to the different efficacy of monovalent vaccine against different strains, the vaccination reproduction numbers of the two strains and the reproduction numbers of their mutual invasion are found. Based on the existence and local stability of equilibria, our results suggest that the obtained reproduction numbers determine together the dynamics of the model, and that the two-strain rotavirus dies out as both the numbers is less than unity. The coexistence of two strains, one of which is dominant, is also related to the two reproduction numbers.

1. Introduction. Rotavirus (RV) is one of the leading causes of severe diarrhoea in children under five [4], severe cases of which can lead to fatal gastroenteritis and even death from dehydration [10]. Rotavirus infection causes 138.5 million cases of gastroenteritis and 453,000 deaths in children under 5 years of age worldwide every year, and is a serious global public health problem [14, 15]. There are seven rotavirus groups. For a few groups, some different strains can be produced by recombination of their different genes[3]. This results in the broad diversity within the strains of rotavirus[8, 23]. There is evidence that, for children infected by one strain, the secondary infection is more likely from another strain due to homotypic protection [18], but the corresponding incidence could decrease due to the partial immunity obtained from the infection of previous strain[6]. Immune response is also an important factor influencing infectious disease models[21]. For

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* Corresponding author: Kun Lu.
rotavirus infection, the immunity of children can also be obtained in two other ways. One part is the immunity inherited from the maternal antibody for infant\cite{2,7}, another acquires immunity by vaccination\cite{1}.

Vaccination programs are considered to be the most effective public health strategies for reducing the incidence rate of rotavirus infection\cite{22,17,11,19,14,13}. Omondi, et al. \cite{9} and Shim, et al. \cite{12} established respectively an ordinary differential equation model and a partial differential equation model to investigate the transmission of single strain rotavirus under vaccination. So far, models including multiple strains of rotavirus infection are rare. White et al. \cite{20} established and investigated dynamics of a series of two-strain rotavirus transmission models according to the heterogeneity of the multi-strain rotavirus for immunity. Young et al.\cite{22} considered a more complex case that the model involves vaccination and vaccine attenuation, and examined the effects of vaccine on the qualitative behaviors of infection levels in a population. But only local dynamics of the models were obtained in \cite{20,22} because of the complexity of the models.

Motivated by the character of rotavirus transmission mentioned above and the corresponding modelling ideas, we will propose a new dynamic model of rotavirus infection with two strains, vaccination, homotypic protection and the partial immunity. The paper is organized as follows. In next section, we present the model based on some assumptions and discuss its boundedness. In Section 3, the existence of equilibria is determined, and the corresponding regions are shown in the plane determined by the related reproduction numbers. The stability of the boundary equilibria are analyzed in Section 4. At last, the conclusion and discussion are stated.

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2. Formulation and boundedness of the model. According to the introduction in the previous section, we make the following assumptions for the transmission of two strain rotavirus:

1. The infant is protected by maternal antibodies. That is, the new born has the immunity against infection of rotavirus.
2. There are two strains of rotavirus spreading in the population. Strain 2 is an evolution of strain 1, so that the children recovered from strain 2 infection could not be reinfected by strain 1. The children recovered from strain 1 infection could be reinfected by strain 2, but the incidence decreases.
3. There is vaccination in infancy. If an infant is not vaccinated, he/she will become susceptible for any strain rotavirus.
4. The vaccine here is mainly for strain 1, then the vaccinated children have complete immunity against infection of strain 1, and partial immunity for strain 2.

Therefore, we divide the total population into seven subpopulations: the breast-fed infants ($X$), the susceptible subpopulation ($S$), the vaccinated subpopulation ($V$), the infectious subpopulation with only strain $i$ ($I_i$) ($i = 1,2$), the recovered subpopulation from infection with strain $i$ ($R_i$) ($i = 1,2$). The transmission mechanism of the two strains of rotavirus is given by the diagram (Figure 1), where $X = X(t)$, $S = S(t), V = V(t), I_i = I_i(t)$ and $R_i = R_i(t)$ ($i = 1,2$) represent the numbers
of individuals in the corresponding compartments at time \( t \), respectively. \( A \) is the birth rate of the population, \( \mu \) is the natural death rate of the population, \( \tau \) is the rate at which maternal protection against rotavirus infection wears off, \( \phi \) is the fraction of vaccination coverage for the individuals leaving the breast-fed infant subpopulation, \( \beta_i \) \( (i = 1, 2) \) is the transmission coefficients of strain \( i \) \( (i = 1, 2) \), \( k \) is the rate at which the immunity from the vaccine weakens, \( \xi \) reflects the efficiency of the vaccine against strain 2, \( r_i \) \( (i = 1, 2) \) is the rate of recovery from infection by strain \( i \) \( (i = 1, 2) \), \( \sigma \) reflects the homotypic protection and represents the reduction fraction of infection force of strain 2 for the individuals recovered from the infection of strain 1 relative to the susceptible individuals. Here \( 0 < \sigma, \xi, \phi < 1 \) and other coefficients are all positive constants.

\[
\begin{align*}
\frac{dX}{dt} &= A - (\tau + \mu)X, \\
\frac{dS}{dt} &= (1 - \phi)\tau X - \beta_1SI_1 - \beta_2SI_2 - \mu S + kV, \\
\frac{dV}{dt} &= \phi\tau X - (1 - \xi)\beta_2I_2V - (k + \mu)V, \\
\frac{dI_1}{dt} &= \beta_1SI_1 - (r_1 + \mu)I_1, \\
\frac{dI_2}{dt} &= \beta_2SI_2 + (1 - \xi)\beta_2I_2V + \sigma\beta_2I_2R_1 - (r_2 + \mu)I_2, \\
\frac{dR_1}{dt} &= r_1I_1 - \mu R_1 - \sigma\beta_2I_2R_1, \\
\frac{dR_2}{dt} &= r_2I_2 - \mu R_2.
\end{align*}
\]

(1)

It is easy to know from the first equation of (1) that \( \lim_{t \to \infty} X(t) = \frac{A}{\tau + \mu} \). And the variable \( R_2 \) does not appears in other equations of (1). Then the system consisting
of the middle five equations of (1) has the following limit system,

\[\begin{align*}
\frac{dS}{dt} &= (1-\phi)M - \beta_1 SI_1 - \beta_2 SI_2 - \mu S + kV, \\
\frac{dV}{dt} &= \phi M - (1-\xi)\beta_2 I_2 V - (k+\mu)V, \\
\frac{dI_1}{dt} &= \beta_1 SI_1 - (r_1+\mu)I_1, \\
\frac{dI_2}{dt} &= \beta_2 SI_2 + (1-\xi)\beta_2 I_2 V + \sigma \beta_2 I_2 R_1 - (r_2+\mu)I_2, \\
\frac{dR_1}{dt} &= r_1 I_1 - \mu R_1 - \sigma \beta_2 I_2 R_1,
\end{align*}\]

(2)

where \( M = \frac{\tau A}{\tau + \mu} \). It is easy to know that solutions of (2) with the nonnegative initial conditions keep nonnegative. Then, from (2) it follows that

\[\frac{d}{dt}(S+V+I_1+I_2+R_1) = M - \mu(S+V+I_1+I_2+R_1) - r_2 I_2 \leq M - \mu(S+V+I_1+I_2+R_1).\]

It implies that

\[\limsup_{t \to \infty}(S+V+I_1+I_2+R_1) \leq \frac{M}{\mu}.\]

From the second equation of (2), we have

\[\frac{dV}{dt} \leq \phi M - (k+\mu)V.\]

So

\[\limsup_{t \to \infty} V(t) \leq \frac{\phi M}{k+\mu} := V^{(0)}.\]

(3)

Hence, for an arbitrary number \( \varepsilon > 0 \), there is a \( T \) large enough such that \( V < V^{(0)} + \varepsilon \) for \( t > T \). Thus, for \( t > T \), from the first equation of (2) we have

\[\frac{dS}{dt} \leq (1-\phi)M + k(V^{(0)} + \varepsilon) - \mu S.\]

It follows that

\[\limsup_{t \to \infty} S(t) \leq \frac{1}{\mu}[(1-\phi)M + k(V^{(0)} + \varepsilon)].\]

Due to the arbitrariness of \( \varepsilon \), we can get

\[\limsup_{t \to \infty} S(t) \leq \frac{1}{\mu}[(1-\phi)M + kV^{(0)}] := S^{(0)}.\]

(4)

Summarizing the inferences above, we know that the set

\[\Omega = \left\{(S,V,I_1,I_2,R_1) \in R_+^5 | S+V+I_1+I_2+R_1 \leq \frac{M}{\mu}, S \leq S^{(0)}, V \leq V^{(0)}\right\}\]

is positively invariant and attractive for system (2). Therefore, hereafter we consider the dynamic behavior of (2) on \( \Omega \).
3. Existence of equilibria. Set

\[ R_{10} = \frac{\beta_1 S^{(0)}}{r_1 + \mu}, \quad R_{20} = \frac{\beta_2 [S^{(0)} + (1 - \xi)V^{(0)}]}{r_2 + \mu}. \]

For the boundary equilibria of system (2), we have the following statements.

**Theorem 3.1.** System (2) always has the disease-free equilibrium \( E_0 (S^{(0)}, V^{(0)}, 0, 0, 0) \). When \( R_{10} > 1 \), there is a strain 1 endemic equilibrium \( E_1 (S^{(1)}, V^{(1)}, I_1^{(1)}, 0, R_1^{(1)}) \); when \( R_{20} > 1 \), there is a strain 2 endemic equilibrium \( E_2 (S^{(2)}, V^{(2)}, 0, I_2^{(2)}, 0) \). Here

\[ S^{(1)} = \frac{r_1 + \mu}{\beta_1} V^{(1)} = \frac{\phi M}{k + \mu}, I_1^{(1)} = \frac{\mu}{\beta_1} (R_{10} - 1), R_1^{(1)} = \frac{r_1}{\beta_1} (R_{10} - 1). \]

\[ V^{(2)} = \frac{\phi M}{(1 - \xi) \beta_2 I_2^{(2)} + (k + \mu)}, S^{(2)} = \frac{(1 - \phi) M + kV^{(2)}}{\beta_2 I_2^{(2)} + \mu}, \]

and \( I_2^{(2)} \) is the positive zero of function \( f(I_2) = 0, f(I_2) \) is defined in (10).

**Proof.** By direct calculation, we get the coordinates of \( E_0 \) and \( E_1 \). For the equilibrium \( E_2 (S^{(2)}, V^{(2)}, 0, I_2^{(2)}, 0) \) (\( f(I_2) > 0 \) of system (2), its coordinates are determined by the equations

\[ (1 - \phi) M - \beta_2 S I_2 - \mu S + kV = 0, \]

\[ \phi M - (1 - \xi) \beta_2 I_2 V - (k + \mu) V = 0, \]

\[ \beta_2 S + (1 - \xi) \beta_2 V - (r_2 + \mu) = 0. \]

From the second equation of system (6) it follows that

\[ V = \frac{\phi M}{(1 - \xi) \beta_2 I_2 + (k + \mu)}, \]

which is less than \( V^{(0)} \) for \( I_2 > 0 \). Substituting (7) into the first equation of (6), we have

\[ S = \frac{1}{\beta_2 I_2 + \mu} \left[ (1 - \phi) M + \frac{k \phi M}{(1 - \xi) \beta_2 I_2 + (k + \mu)} \right], \]

which is less than \( S^{(0)} \) for \( I_2 > 0 \). Further, substituting equations (7) and (8) into the last equation of (6) yields,

\[ \frac{1}{\beta_2 I_2 + \mu} \left[ (1 - \phi) M + \frac{k \phi M}{(1 - \xi) \beta_2 I_2 + (k + \mu)} \right] + \frac{(1 - \xi) \phi M}{(1 - \xi) \beta_2 I_2 + (k + \mu)} - \frac{r_2 + \mu}{\beta_2} = 0. \]

Substituting \( \phi M = (k + \mu)V^{(0)} \) and \( (1 - \phi) M = \mu S^{(0)} - kV^{(0)} \) into equation (9) yields \( f(I_2) = 0 \), where

\[ f(I_2) = \frac{1}{\beta_2 I_2 + \mu} \left[ \mu S^{(0)} - kV^{(0)} + \frac{k(k + \mu)V^{(0)}}{(1 - \xi) \beta_2 I_2 + (k + \mu)} \right] + \frac{(1 - \xi)(k + \mu)V^{(0)}}{(1 - \xi) \beta_2 I_2 + (k + \mu)} - \frac{r_2 + \mu}{\beta_2}. \]

It is obvious that function \( f(I_2) \) is decreasing, and that \( \lim_{I_2 \to \infty} f(I_2) = -\frac{\mu S^{(0)}}{\beta_2} < 0 \). Then, equation \( f(I_2) = 0 \) has a unique positive root if and only if \( f(0) > 0 \), i.e., \( R_{20} > 1 \). Correspondingly, system (2) has a unique strain 2 endemic equilibrium \( E_2 \) if and only if \( R_{20} > 1 \). \( \square \)
Next, we discuss the existence of positive equilibrium (i.e. equilibrium with two strains) \( E^*(S^*, V^*, I_1^*, I_2^*, R_1^*, R_2^*) \) of system (2). For \( E^* \), the coordinates satisfy the following equations

\[
\begin{align*}
(1 - \phi)M - \beta_1 SI_1 - \beta_2 SI_2 - \mu S + kV &= 0, \\
\phi M - (1 - \xi)\beta_2 I_2 V - (k + \mu) V &= 0, \\
\beta_1 S - (r_1 + \mu) &= 0, \\
\beta_2 S + (1 - \xi)\beta_2 V + \sigma \beta_2 R_1 - (r_2 + \mu) &= 0, \\
r_1 I_1 - \mu R_1 - \sigma \beta_2 I_2 R_1 &= 0.
\end{align*}
\]

(11)

For simplicity, we define

\[
\theta_1 = \frac{r_1 + \mu}{\beta_1}, \quad \theta_2 = \frac{r_2 + \mu}{\beta_2}.
\]

(12)

From the second and the third equations of (11), we have

\[
S = \frac{r_1 + \mu}{\beta_1} = \theta_1, \quad V = \frac{\phi M}{(1 - \xi)\beta_2 I_2 + (k + \mu)}.
\]

(13)

Substituting them into the first equation of (11) yields

\[
I_1 = \frac{1}{r_1 + \mu} \left[ (1 - \phi)M - (\mu + \beta_2 I_2)\theta_1 + \frac{k\phi M}{(1 - \xi)\beta_2 I_2 + (k + \mu)} \right].
\]

Since \((1 - \phi)M = \mu S^{(0)} - kV^{(0)}\) and \(\phi M = (k + \mu)V^{(0)}\), we get

\[
I_1 = \frac{1}{r_1 + \mu} \left[ (\mu S^{(0)} - kV^{(0)}) - (\mu + \beta_2 I_2)\theta_1 + \frac{k(\mu + \mu)V^{(0)}}{(1 - \xi)\beta_2 I_2 + (k + \mu)} \right] := f_1(I_2).
\]

It is easy to see that function \(f_1(I_2)\) is decreasing, and \(\lim_{I_2 \to \infty} f_1(I_2) = -\infty\),

\[
f_1(0) = \frac{\mu}{r_1 + \mu} \frac{S^{(0)} - \theta_1}{S^{(0)} - \theta_1} + \frac{S^{(0)} - \theta_1}{S^{(0)} - \theta_1} + \frac{S^{(0)} - \theta_1}{S^{(0)} - \theta_1}.
\]

then \(f_1(I_2) < 0\) for \(I_2 > 0\) if \(f_1(0) \leq 0\), i.e., \(R_{10} \leq 1\). It implies that \(R_{10} > 1\) is necessary for (11) to have a positive solution.

On the other hand, substituting (13) into the fourth equation of (11), it follows that

\[
R_1 = \frac{1}{\sigma} \left[ \theta_2 - \theta_1 - \frac{(1 - \xi)(k + \mu)V^{(0)}}{(1 - \xi)\beta_2 I_2 + (k + \mu)} \right].
\]

Again, substituting it into the last equation of (11), we have

\[
I_1 = \frac{\mu + \sigma \beta_2 I_2}{\sigma r_1} \left[ \theta_2 - \theta_1 - \frac{(1 - \xi)(k + \mu)V^{(0)}}{(1 - \xi)\beta_2 I_2 + (k + \mu)} \right] := f_2(I_2).
\]

Obviously, that \(\theta_2 > \theta_1\) is necessary for \(f_2(I_2) > 0\). Therefore, in the following, we consider two cases:

(i) If \(\theta_2 - \theta_1 \geq (1 - \xi)V^{(0)}\), then \(f_2(0) \geq 0\) function \(f_2(I_2)\) is increasing and keeps positive for \(I_2 > 0\), and \(\lim_{I_2 \to \infty} f_2(I_2) = +\infty\). Hence, under the condition that \(R_{10} > 1\) and \(\theta_2 - \theta_1 \geq (1 - \xi)V^{(0)}\), equation \(f_1(I_2) = f_2(I_2)\) has a unique positive solution if \(f_1(0) > f_2(0)\), that is,

\[
\frac{(1 - \sigma)r_1 + \mu}{r_1 + \mu} (S^{(0)} - \theta_1) < S^{(0)} + (1 - \xi)V^{(0)} - \theta_2.
\]
which is determined by equation \( S(0) + (1 - \xi)V(0) - \theta_2 \leq S(0) - \theta_1 \), equation \( f_1(I_2) = f_2(I_2) \) has a unique positive solution when \( R_{10} > 1 \) and

\[
\frac{(1 - \sigma)r_1 + \mu}{r_1 + \mu} (S(0) - \theta_1) < S(0) + (1 - \xi)V(0) - \theta_2 \leq S(0) - \theta_1. \tag{14}
\]

From the definition of \( R_{10} \) and \( R_{20} \) it follows that

\[
\theta_1 = \frac{S(0)}{R_{10}}, \quad \text{and} \quad \theta_2 = \frac{S(0) + (1 - \xi)V(0)}{R_{20}}. \tag{15}
\]

Then substituting them into (14), we have

\[
\frac{(r_1 + \mu)[S(0) + (1 - \xi)V(0)]R_{10}}{[(1 - \sigma)r_1 + \mu]S(0) + \sigma r_1 S(0) + (r_1 + \mu)(1 - \xi)V(0)]R_{10} < R_{20} \leq \frac{[S(0) + (1 - \xi)V(0)]R_{10}}{S(0) + (1 - \xi)V(0)R_{10}}. \tag{16}
\]

(ii) When \( 0 < \theta_2 - \theta_1 < (1 - \xi)V(0) \), which implies that \( f_2(I_2) < 0 \), from \( f_2(I_2) = 0 \) we define

\[
\hat{I}_2 := \frac{k + \mu}{1 - \xi} \beta_2 \left[ \frac{(1 - \xi)V(0)}{\theta_2 - \theta_1} - 1 \right]. \tag{17}
\]

Then \( f_2(I_2) < 0 \) for \( 0 < \hat{I}_2 < I_2 \), \( f_2(I_2) > 0 \) for \( I_2 > \hat{I}_2 \), and \( f_2(I_2) \) is increasing for \( I_2 > \hat{I}_2 \). Thus, for the case that \( R_{10} > 1 \) and that \( 0 < \theta_2 - \theta_1 < (1 - \xi)V(0) \), equation \( f_1(I_2) = f_2(I_2) \) has a unique positive solution greater than \( \hat{I}_2 \) if and only if \( f_1(\hat{I}_2) > 0 \), that is,

\[
\mu S(0) - kV(0) - \frac{(k + \mu)\theta_1 V(0)}{\theta_2 - \theta_1} + \frac{\mu \xi \theta_1 + k \theta_2}{1 - \xi} > 0. \tag{18}
\]

It is equivalent to \( H(\theta_1, \theta_2) > 0 \), where

\[
H(\theta_1, \theta_2) = (\theta_2 - \theta_1)^2 - \frac{(1 - \xi)(k + \mu)V(0)\theta_1}{k} + m \frac{(\theta_2 - \theta_1)}{k},
\]

and

\[
m = (1 - \xi) \left( \mu S(0) - kV(0) \right) + (\mu \xi + k)\theta_1. \tag{19}
\]

Since

\[
H(\theta_1, \theta_1) = - \frac{(1 - \xi)(k + \mu)V(0)\theta_1}{k} < 0
\]

and

\[
H(\theta_1, \theta_1 + (1 - \xi)V(0)) = \frac{\mu(1 - \xi)^2 V(0)}{k} \left( S(0) - \theta_1 \right)
\]

\[
= \frac{\mu(1 - \xi)^2 V(0)}{k} \left( 1 - \frac{1}{R_{10}} \right) > 0
\]

for \( R_{10} > 1 \), then there is a function \( \theta_2 = \Theta(\theta_1) \in (\theta_1, \theta_1 + (1 - \xi)V(0)) \), where

\[
\Theta(\theta_1) = \theta_1 + \frac{1}{2} \left[ \frac{m^2}{k^2} + \frac{4(1 - \xi)(k + \mu)V(0)\theta_1}{k} - m \right], \tag{20}
\]

which is determined by equation \( H(\theta_1, \theta_2) = 0 \). Further, we know that \( H(\theta_1, \theta_2) < 0 \) for \( \theta_1 < \theta_2 < \Theta(\theta_1) \), and \( H(\theta_1, \theta_2) > 0 \) for \( \Theta(\theta_1) < \theta_2 < \theta_1 + (1 - \xi)V(0) \). Thus, when \( R_{10} > 1 \), the condition that \( 0 < \theta_2 - \theta_1 < (1 - \xi)V(0) \) and \( H(\theta_1, \theta_2) > 0 \) can be expressed with \( \Theta(\theta_1) < \theta_2 < \theta_1 + (1 - \xi)V(0) \).

Substituting (15) into \( \Theta(\theta_1) < \theta_2 < \theta_1 + (1 - \xi)V(0) \), it follows that

\[
\frac{[S(0) + (1 - \xi)V(0)] R_{10}}{S(0) + (1 - \xi)V(0)R_{10}} < R_{20} < \frac{S(0) + (1 - \xi)V(0)}{\Theta(S(0)/R_{10})}. \tag{21}
\]
Note that the inequalities (16) and (21) can be merged into the following inequalities:

\[
\frac{(r_1 + \mu) \left[ S^{(0)} + (1 - \xi)V^{(0)} \right]}{[(1 - \sigma)r_1 + \mu]S^{(0)} + [\sigma r_1 S^{(0)} + (r_1 + \mu)(1 - \xi)V^{(0)}]} R_{10} < R_{20} < \frac{S^{(0)} + (1 - \xi)V^{(0)}}{\Theta(S^{(0)}/R_{10})}.
\]

(22)

For simplicity, we denote

\[
R_{20}^{(1)} = \frac{(r_1 + \mu) \left[ S^{(0)} + (1 - \xi)V^{(0)} \right]}{[(1 - \sigma)r_1 + \mu]S^{(0)} + [\sigma r_1 S^{(0)} + (r_1 + \mu)(1 - \xi)V^{(0)}]} R_{10},
\]

\[
R_{20}^{(2)} = \frac{S^{(0)} + (1 - \xi)V^{(0)}}{\Theta(S^{0}/R_{10})}.
\]

Then, from the discussions on the cases (i) and (ii), we know that, when \( R_{10} > 1 \), equation \( f_1(I_2) = f_2(I_2) \) has a unique positive root if the inequalities \( R_{20}^{(1)} < R_{20} < R_{20}^{(2)} \) hold. Therefore, we have the following results with respect to the existence of positive equilibrium of system (2).

**Theorem 3.2.** System (2) has a unique positive equilibrium \( E^*(S^*, V^*, R_1^*, I_2^*, R_1^*) \) if \( R_{10} > 1 \) and \( R_{20}^{(1)} < R_{20} < R_{20}^{(2)} \).

Note that \( R_{20}^{(1)} > 1 \) as \( R_{10} > 1 \). Then \( R_{20}^{(1)} < R_{20} \) implies that \( R_{20} > 1 \) is a necessary condition of Theorem 3.2.

For function \( \theta_2 = \Theta(\theta_1) \) determined by equation \( H(\theta_1, \theta_2) = 0 \), the straightforward calculation shows

\[
\frac{d\Theta(\theta_1)}{d\theta_1} = -\frac{\partial H}{\partial \theta_1} \left/ \frac{\partial H}{\partial \theta_2} \right. = 1 + \frac{(1 - \xi)(k + \mu)V^{(0)} - (\mu \xi + k)(\theta_2 - \theta_1)}{2(\theta_2 - \theta_1)k + m}.
\]

(23)

Then \( \theta_2 > \theta_1 \) implies that the denominator in (23) is positive. And from \( \theta_2 < \theta_1 + (1 - \xi)V^{(0)} \) it follows that

\[
(1 - \xi)(k + \mu)V^{(0)} - (\mu \xi + k)(\theta_2 - \theta_1) > (1 - \xi)^2 \mu V^{(0)} > 0.
\]

Therefore, the inequality that \( \frac{d\Theta}{d\theta_1} > 1 \) holds. Further, \( \Theta(S^{(0)}) = S^{(0)} + (1 - \xi)V^{(0)} \)

implies that \( R_{20}^{(2)} < 1 \) as \( R_{10} < 1 \), \( R_{20}^{(2)} = 1 \) as \( R_{10} = 1 \), and \( R_{20}^{(2)} > 1 \) as \( R_{10} > 1 \). Meantime, it is also easy to know that \( R_{20}^{(1)} < 1 \) as \( R_{10} < 1 \), \( R_{20}^{(1)} = 1 \) as \( R_{10} = 1 \), and \( R_{20}^{(1)} > 1 \) as \( R_{10} > 1 \).

Therefore, the two curves in the \( R_{10} - R_{20} \) plane, \( R_{20} = R_{20}^{(1)} \) and \( R_{20} = R_{20}^{(2)} \), are both in the region \( \{(R_{10}, R_{20}) : R_{10} > 1, R_{20} > 1 \} \). It is shown in Figure 2.

On the other hand, we define

\[
R_{21} = \frac{\beta_2}{r_2 + \mu} \left[ S^{(1)} + (1 - \xi)V^{(1)} + \sigma R_1^{(1)} \right]
\]

(24)
as \( R_{10} > 1 \), and

\[
R_{12} = \frac{\beta_1 S^{(2)}}{r_1 + \mu}
\]

(25)
as \( R_{20} > 1 \), where \( S^{(1)}, S^{(2)}, V^{(1)} \) and \( R_1^{(1)} \) are defined in (5). Then, with respect to \( R_{21} \) and \( R_{12} \), we have the following statements.

**Proposition 1.** (i) When \( R_{10} > 1 \), \( R_{21} > 1 \) is equivalent to \( R_{20} > R_{20}^{(1)} \).

(ii) When \( R_{20} > 1 \), \( R_{12} > 1 \) is equivalent to \( R_{20} < R_{20}^{(2)} \).
Figure 2. The existence of equilibria of system (2).

Proof. (i) When \( R_{10} > 1 \), from (5) and the definition of \( R_{10} \) it follows that

\[
R_{21} = \frac{\beta_2}{r_2 + \mu} \left[ \frac{S^{(0)}(r_1 + \mu) + \sigma \tau_1 (R_{10} - 1) S^{(0)}}{(r_1 + \mu) R_{10}} + (1 - \xi) V^{(0)} \right].
\]

Further, applying the expressions of \( R_{20} \) and \( R_{20}^{(1)} \) yields

\[
R_{21} = R_{20} R_{20}^{(1)}.
\]

Then \( R_{21} > 1 \) is equivalent to \( R_{20} > R_{20}^{(1)} \) when \( R_{10} > 1 \).

(ii) Note that \( S^{(2)} \) and \( V^{(2)} \) satisfy \( \beta_2 S + (1 - \xi) \beta_2 V = r_2 + \mu \) as \( R_{20} > 1 \). Then applying the definition of \( V^{(0)} \) gives

\[
R_{12} = \frac{\theta_2 - (1 - \xi) V^{(2)}}{\theta_1} = \frac{1}{\theta_1} \left[ \theta_2 - \frac{(1 - \xi)(k + \mu)V^{(0)}}{(1 - \xi)\beta_2 I_2^{(2)} + k + \mu} \right].
\]

So \( R_{12} > 1 \) is equivalent to \( \frac{(1 - \xi)(k + \mu)V^{(0)}}{(1 - \xi)\beta_2 I_2^{(2)} + k + \mu} < \theta_2 - \theta_1 \), that is,

\[
I_2^{(2)} > \frac{k + \mu}{(1 - \xi)\beta_2} \left[ \frac{(1 - \xi)V^{(0)}}{\theta_2 - \theta_1} - 1 \right] = \hat{I}_2.
\]

According to the definition of \( I_2^{(2)} \) and the monotonicity of function \( f(I_2) \) defined by (10), \( I_2^{(2)} > \hat{I}_2 \) is equivalent to \( f(\hat{I}_2) > 0 \), that is,

\[
\mu S^{(0)} - kV^{(0)} - \frac{(k + \mu)\theta_1 V^{(0)}}{\theta_2 - \theta_1} + \frac{\mu \xi \theta_1 + k \theta_2}{1 - \xi} > 0,
\]

which is the same as the inequality (18). From the proof of the existence of positive equilibrium of (2) it follows that the inequality (26) holds if and only if \( R_{20} < R_{20}^{(2)} \).

Therefore, when \( R_{20} > 1 \), \( R_{12} > 1 \) is equivalent to \( R_{20} < R_{20}^{(2)} \).

The proof of Proposition 1 is complete.

Thus, according to Proposition 1 Theorem 3.2 can be restated as follows.
Theorem 3.3. System (2) has a unique positive equilibrium $E^*$ if $R_{10} > 1$, $R_{20} > 1$, $R_{12} > 1$, and $R_{21} > 1$, i.e., $\min\{R_{10}, R_{20}, R_{12}, R_{21}\} > 1$.

4. Stability of the boundary equilibria of system (2). In this section, we first investigate the local stability of the boundary equilibria of system (2), and then discuss their global stability. The necessary and sufficient conditions for the local stability of all boundary equilibria and the global stability of disease-free equilibrium are obtained. But for the boundary equilibria $E_1$ and $E_2$ we have only the sufficient conditions for global stability.

4.1. Local stability of the boundary equilibria of system (2). With respect to the local stability of the boundary equilibria of system (2), we have the following results.

Theorem 4.1. (i) The disease-free equilibrium $E_0$ of system (2) is locally asymptotically stable if and only if $R_{10} < 1$ and $R_{20} < 1$, unstable if either $R_{10} > 1$ or $R_{20} > 1$.

(ii) Given $R_{10} > 1$, the strain 1 endemic equilibrium $E_1$ of system (2) is locally asymptotically stable if $R_{21} < 1$ and unstable if $R_{21} > 1$.

(iii) When $R_{20} > 1$, the strain 2 endemic equilibrium $E_2$ of system (2) is locally asymptotically stable if $R_{12} < 1$ and unstable if $R_{12} > 1$.

Proof. (i) Direct calculation shows that the eigenvalues of the Jacobian matrix of system (2) at $E_0$ are
\[
\lambda_1 = \lambda_2 = -\mu < 0, \lambda_3 = -k - \mu < 0, \lambda_4 = \beta_1 S(0) - r_1 - \mu = (r_1 + \mu)(R_{10} - 1),
\]
and
\[
\lambda_5 = \beta_2 \left[ S(0) + (1 - \xi)V(0) \right] - (r_2 + \mu) = (r_2 + \mu)(R_{20} - 1).
\]
When $R_{10} < 1$ and $R_{20} < 1$, all the eigenvalues are negative, $E_0$ is locally asymptotically stable. If $R_{10} > 1$ or $R_{20} > 1$, the disease-free equilibrium $E_0$ is unstable.

(ii) The Jacobian matrix of system (2) at $E_1$ is given by
\[
J(E_1) = \begin{pmatrix}
-\beta_1 I_1^{(1)} - \mu & k & -\beta_1 S^{(1)} & -\beta_2 S^{(1)} & 0 \\
0 & -k - \mu & 0 & -(1 - \xi)\beta_2 V^{(1)} & 0 \\
\beta_1 I_1^{(1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & K & 0 \\
0 & 0 & r_1 & -\sigma \beta_2 R_1^{(1)} & -\mu
\end{pmatrix},
\]
where $K = \beta_2 S^{(1)} + (1 - \xi)\beta_2 V^{(1)} + \sigma \beta_2 r_1^{(1)} - (r_2 + \mu)$. It is easy to see that $\lambda_1 = -\mu < 0, \lambda_2 = -k - \mu < 0$, and $\lambda_3 = K = (r_2 + \mu)(R_{21} - 1)$ are three eigenvalues of matrix $J(E_1)$, and that the other two eigenvalues are determined by
\[
\bar{\lambda}_1(E_1) = \begin{pmatrix}
-\beta_1 r_1^{(1)} - \mu & -\beta_1 S^{(1)} \\
\beta_1 I_1^{(1)} & 0
\end{pmatrix}.
\]
Obviously, the eigenvalues of matrix $\bar{\lambda}_1(E_1)$ are all with negative real parts. Then, the local stability of $E_1$ is determined by $\lambda_3 = (r_2 + \mu)(R_{21} - 1)$. Therefore, Theorem 4.1(ii) holds.

(iii) For the Jacobian matrix of system (2) at $E_2$, $J(E_2)$, it is easy to know that
\[
\lambda_1 = -\sigma \beta_2 I_2^{(2)} - \mu < 0 \quad \text{and} \quad \lambda_2 = \beta_1 S^{(2)} - r_1 - \mu = (r_1 + \mu)(R_{12} - 1)
\]
are its two eigenvalues, and the other three eigenvalues are determined by the matrix

\[
\bar{J}(E_2) = \begin{pmatrix}
-\beta_2 I_2^{(2)} - \mu & k & -\beta_2 S^{(2)} \\
0 & -(1-\xi)\beta_2 I_2^{(2)} - k - \mu & -(1-\xi)\beta_2 V^{(2)} \\
\beta_2 I_2^{(2)} & (1-\xi)\beta_2 I_2^{(2)} & 0
\end{pmatrix}.
\]

We claim that all the eigenvalues of \(\bar{J}(E_2)\) are with negative real parts. In fact, direct calculation shows that the characteristic equation of \(\bar{J}(E_2)\) is \(\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0\), where

\[
c_1 = (\beta_2 I_2^{(2)} + \mu) + [(1-\xi)\beta_2 I_2^{(2)} + k + \mu] > 0,
\]

\[
c_2 = \beta_2 I_2^{(2)} + \mu \left[(1-\xi)\beta_2 I_2^{(2)} + k + \mu \right] + \beta_2 I_2^{(2)} [S^{(2)} + (1-\xi)^2 V^{(2)}] > 0,
\]

\[
c_3 = \beta_2^2 S^{(2)} I_2^{(2)} \left[(1-\xi)\beta_2 I_2^{(2)} + k + \mu \right] + (1-\xi)^2 \beta_2^2 V^{(2)} I_2^{(2)} (\beta_2 I_2^{(2)} + \mu) + k(1-\xi)\beta_2^2 V^{(2)} I_2^{(2)} > 0.
\]

Then

\[
c_1c_2 - c_3 = \left\{ (\beta_2 I_2^{(2)} + \mu) + [(1-\xi)\beta_2 I_2^{(2)} + k + \mu] \right\} (\beta_2 I_2^{(2)} + \mu) \times \left\{ (1-\xi)\beta_2 I_2^{(2)} + k + \mu \right\} - k(1-\xi)\beta_2^2 V^{(2)} I_2^{(2)}
\]

\[
+ \beta_2^2 I_2^{(2)} \left\{ S^{(2)} \left( \beta_2 I_2^{(2)} + \mu \right) + (1-\xi)^2 V^{(2)} \left[(1-\xi)\beta_2 I_2^{(2)} + k + \mu \right] \right\},
\]

applying

\[
S^{(2)} \left( \beta_2 I_2^{(2)} + \mu \right) = (1-\phi)M + kV^{(2)}
\]

to the last expression yields, we have

\[
c_1c_2 - c_3 = \left\{ (\beta_2 I_2^{(2)} + \mu) + [(1-\xi)\beta_2 I_2^{(2)} + k + \mu] \right\} (\beta_2 I_2^{(2)} + \mu) \times \left\{ (1-\xi)\beta_2 I_2^{(2)} + k + \mu \right\} - k(1-\xi)\beta_2^2 V^{(2)} I_2^{(2)}
\]

\[
+ \beta_2^2 I_2^{(2)} \left\{ (1-\phi)M + kV^{(2)} + (1-\xi)^2 V^{(2)} \left[(1-\xi)\beta_2 I_2^{(2)} + k + \mu \right] \right\}
\]

\[
= \left\{ (\beta_2 I_2^{(2)} + \mu) + [(1-\xi)\beta_2 I_2^{(2)} + k + \mu] \right\} (\beta_2 I_2^{(2)} + \mu) \times \left\{ (1-\xi)\beta_2 I_2^{(2)} + k + \mu \right\} + k\xi \beta_2^2 V^{(2)} I_2^{(2)}
\]

\[
+ \beta_2^2 I_2^{(2)} \left\{ (1-\phi)M + (1-\xi)^2 V^{(2)} \left[(1-\xi)\beta_2 I_2^{(2)} + k + \mu \right] \right\} > 0.
\]

So the claim holds by the Routh-Hurwitz Criterion. Hence, the local stability of \(E_2\) is determined by \(\lambda_2 = (r_1 + \mu)(R_{12} - 1)\). Therefore, Theorem 4.1(iii) holds. \(\Box\)

According to the results on the existence of equilibria of system (2) in Section 3, we show the local stability of its boundary equilibria in Figure 3, which corresponds to Figure 2.

4.2. Global stability of the boundary equilibria of system (2). In the following, we investigate the global stability of the boundary equilibria of system (2) by applying the theory of limit systems and constructing the appropriate Lyapunov functions. In order to make the proof of the global stabilities of equilibria \(E_0, E_1\) and \(E_2\) simple, we first give the following statements about the change of the variables \(I_1, R_1\) and \(I_2\).

**Lemma 4.2.** For system (2), \(\lim_{t \to \infty} I_1(t) = \lim_{t \to \infty} R_1(t) = 0\) if \(R_{10} < 1\).
Figure 3. The local stability of the boundary equilibria of system (2), where LAS denotes locally asymptotically stable, \( R_{20} = R_{20}^{(2)} \) is equivalent to \( R_{12} = 1 \), \( R_{20} = R_{20}^{(1)} \) is equivalent to \( R_{21} = 1 \).

Proof. For \((S, V, I_1, I_2, R_1) \in \Omega\), from the third equation of (2) it follows that

\[
\frac{dI_1}{dt} \leq [\beta_1 S^{(0)} - (r_1 + \mu)] I_1 = (r_1 + \mu)(R_{10} - 1) I_1,
\]

which implies that \( I_1(t) \) is bounded for \( I_2 \geq 0 \). Then we have \( \lim_{t \to \infty} I_1(t) = 0 \) for \( R_{10} < 1 \). Thus, when \( R_{10} < 1 \), the last equation of (2) has the limit system

\[
\frac{dR_1}{dt} = -\mu R_1 - \frac{r_2 + \mu}{\beta_2} R_1.
\]

Then, from \( I_2 \geq 0 \) and \( R_1' \leq -\mu R_1 \) it follows that \( \lim_{t \to \infty} R_1(t) = 0 \). The proof of Lemma 4.2 is complete.

Lemma 4.3. For system (2), \( \lim_{t \to \infty} I_2(t) = 0 \) if one of the following two sets of conditions holds, \( 1 - \xi - \sigma \geq 0 \) and \( R_{20} < 1 \); (d2) \( 1 - \xi - \sigma < 0 \) and \( R_{20} \leq \min \{\rho_1, \rho_2\} \), where

\[
\rho_1 = \frac{S^{(0)} + (1 - \xi)V^{(0)}}{S^{(0)} + (1 - \xi + \sigma)V^{(0)}}, \quad \rho_2 = \frac{S^{(0)} + (1 - \xi)V^{(0)}}{S^{(0)} + V^{(0)}}.
\]

Proof. Letting \( N = S + V + I_1 + I_2 + R_1 \), the fourth equation of (2) can be rewritten as

\[
\frac{dI_2}{dt} = \beta_2 I_2 \left[ S + (1 - \xi) V + \sigma(N - S - V - I_1 - I_2) - \frac{r_2 + \mu}{\beta_2} \right] = \beta_2 I_2 \left[ \sigma N + (1 - \sigma) S + (1 - \xi - \sigma) V - \sigma(I_1 + I_2) - \frac{r_2 + \mu}{\beta_2} \right].
\]

From (2) it follows that

\[
N' = M - \mu N - r_2 I_2 \leq M - \mu N
\]

for \( I_2 \geq 0 \). It implies that \( \limsup_{t \to \infty} N(t) \leq \frac{M}{\mu} \).
(d1) When \( 1 - \xi - \sigma \geq 0 \), substituting \( N \leq \frac{M}{\mu}, S \leq S^{(0)}, V \leq V^{(0)} \) and \( M = \mu (S^{(0)} + V^{(0)}) \) to (28), we have
\[
\frac{dI_2}{dt} \leq \beta_2 I_2 \left[ S^{(0)} + (1 - \xi)V^{(0)} - \frac{r_2 + \mu}{\beta_2} \right] = (r_2 + \mu)(R_{20} - 1) I_2.
\]
Then \( \lim_{t \to \infty} I_2(t) = 0 \) when \( 1 - \xi - \sigma \geq 0 \) and \( R_{20} < 1 \).

(d2) Note that \( 1 - \xi \geq 0 \) and \( 1 - \sigma \geq 0 \). Then, from (28) we can get
\[
\frac{dI_2}{dt} \leq \beta_2 I_2 \left[ \sigma N + (1 - \sigma)S + (1 - \xi)V - \frac{r_2 + \mu}{\beta_2} \right],
\]
and
\[
\frac{dI_2}{dt} \leq \beta_2 I_2 \left[ \sigma N + (1 - \sigma)S + (1 - \sigma)V - \frac{r_2 + \mu}{\beta_2} \right].
\]

Further, applying \( N \leq \frac{M}{\mu}, S \leq S^{(0)}, V \leq V^{(0)} \) and \( M = \mu (S^{(0)} + V^{(0)}) \) to (29)
and (30), we have
\[
\frac{dI_2}{dt} \leq \beta_2 I_2 \left[ S^{(0)} + (1 - \xi + \sigma)V^{(0)} - \frac{r_2 + \mu}{\beta_2} \right] = (r_2 + \mu)I_2 \left( \frac{R_{20}}{\rho_2} - 1 \right),
\]
and
\[
\frac{dI_2}{dt} \leq \beta_2 I_2 \left[ S^{(0)} + V^{(0)} - \frac{r_2 + \mu}{\beta_2} \right] = (r_2 + \mu)I_2 \left( \frac{R_{20}}{\rho_2} - 1 \right).
\]
Then, when \( R_{20} \leq \min\{\rho_1, \rho_2\} \), from (31) and (32) it follows that \( \lim_{t \to \infty} I_2(t) = 0 \).
Note that \( \rho_1 \leq 1 \) and \( \rho_2 \leq 1 \). Therefore, Lemma 4.3 is true.

With respect to the global stability of the boundary equilibria, \( E_0, E_1 \) and \( E_2 \) of system (2), we have the following results.

**Theorem 4.4.** (i) For system (2), if \( R_{10} < 1 \) and \( R_{20} < 1 \), the disease-free equilibrium \( E_0 \) is globally asymptotically stable on \( \Omega \).

(ii) When \( R_{10} > 1 \), the strain 1 endemic equilibrium \( E_1 \) is globally asymptotically stable in \( \Omega \) if one of the two sets of conditions in Lemma 4.3 holds.

(iii) For system (2), if \( R_{10} < 1 < R_{20} \), the strain 2 endemic equilibrium \( E_2 \) is globally asymptotically stable.

**Proof.** (i) When \( R_{10} < 1 \), by Lemma 4.2 we have that \( \lim_{t \to \infty} I_1(t) = \lim_{t \to \infty} R_1(t) = 0 \). Then, for this case, the fourth equation of (2) has the limit equation
\[
\frac{dI_2}{dt} = \beta_2 S + (1 - \xi)\beta_2 V - (r_2 + \mu)I_2.
\]
For \( S \leq S^{(0)} \) and \( V \leq V^{(0)} \), from (33) we have
\[
\frac{dI_2}{dt} \leq \beta_2[S^{(0)} + (1 - \xi)V^{(0)}] - (r_2 + \mu) = (r_2 + \mu)(R_{20} - 1)I_2.
\]
So \( \lim_{t \to \infty} I_2(t) = 0 \) for \( R_{20} < 1 \). Thus, the equation
\[
\frac{dV}{dt} = \phi M - (k + \mu)V = (k + \mu) \left( V^{(0)} - V \right)
\]
is the limit equation of the second equation of (2). Then \( \lim_{t \to \infty} V = V^{(0)} \).

Lastly, from \( \lim_{t \to \infty} I_1(t) = \lim_{t \to \infty} I_2(t) = 0 \) and \( \lim_{t \to \infty} V(t) = V^{(0)} \) obtained above for \( R_{10} < 1 \) and \( R_{20} < 1 \), the first equation of (2) has the limit system
\[
\frac{dS}{dt} = (1 - \phi)M - \mu S + kV^{(0)}.
\]
It implies that \( \lim_{t \to \infty} S(t) = S^{(0)} \). The above inferences show that, as \( R_{10} < 1 \) and \( R_{20} < 1 \), equilibrium \( E_0 \) is globally asymptotically attractive. Therefore, the local stability implies that \( E_0 \) is globally asymptotically stable on \( \Omega \).

(ii) Under the condition (d1) \( 1 - \xi - \sigma \geq 0 \) and \( R_{20} < 1 \) or (d2) \( 1 - \xi - \sigma < 0 \) and \( R_{20} \leq \min \{ \rho_1, \rho_2 \} \), Lemma 4.3 implies that the system, consisting of the first three equations of (2), has the following limit system

\[
\begin{align*}
\frac{dS}{dt} &= (1 - \phi)M - \beta_1 SI_1 - \mu S + kV, \\
\frac{dV}{dt} &= \phi M - (k + \mu)V, \\
\frac{dI_1}{dt} &= \beta_1 SI_1 - (r_1 + \mu)I_1.
\end{align*}
\]

By the expressions of coordinates of equilibrium \( E_1 \), (37) can be rewritten as

\[
\begin{align*}
\frac{dS}{dt} &= S \left( (1 - \phi)M \left( \frac{1}{S} - \frac{1}{S^{(1)}} \right) - \beta_1 (I_1 - I_1^{(1)}) + k \left( V - V^{(1)} \right) \right), \\
\frac{dV}{dt} &= \phi MV \left( \frac{1}{V} - \frac{1}{V^{(1)}} \right), \\
\frac{dI_1}{dt} &= \beta_1 I_1 (S - S^{(1)}).
\end{align*}
\]

Define a Lyapunov function \( L_1 \) by

\[
L_1 = \left( S - S^{(1)} - S^{(1)} \ln \frac{S}{S^{(1)}} \right) + \left( V - V^{(1)} - V^{(1)} \ln \frac{V}{V^{(1)}} \right) + \left( I_1 - I_1^{(1)} - I_1^{(1)} \ln \frac{I_1}{I_1^{(1)}} \right).
\]

Then the derivative of \( L_1 \) with respect to \( t \) along solutions of (38) is given by

\[
\begin{align*}
\frac{dL_1}{dt} &= (1 - \phi)M (S - S^{(1)}) \left( \frac{1}{S} - \frac{1}{S^{(1)}} \right) + k(S - S^{(1)}) \left( V - V^{(1)} \right) \\
&\quad + \phi M (V - V^{(1)}) \left( \frac{1}{V} - \frac{1}{V^{(1)}} \right) \\
&\quad + (1 - \phi)M \left( 2 - \frac{S}{S^{(1)}} - \frac{S^{(1)}}{S} \right) + \mu V^{(1)} \left( 2 - \frac{V}{V^{(1)}} - \frac{V^{(1)}}{V} \right) \\
&\quad + kV^{(1)} \left( 3 - \frac{S}{S^{(1)}} - \frac{S^{(1)}}{S} \right) \left( V - V^{(1)} \right) + \frac{dS}{dt} \left( \frac{V}{V^{(1)}} - \frac{V^{(1)}}{V} \right)
\end{align*}
\]

The property that the arithmetic mean is greater than or equal to the geometric mean implies that \( \frac{dL_1}{dt} \leq 0 \), and that the equality holds if and only if \( S = S^{(1)} \) and \( V = V^{(1)} \). From the first equation of (38) we know easily that singleton \( \{ \hat{E}_1 \left( S^{(1)}, V^{(1)}, I_1^{(1)} \right) \} \) is the largest invariant set of system (38) (i.e. (37)) on the set satisfying \( \frac{dL_1}{dt} = 0 \). By LaSalle Invariance Principle [5] Theorem 4.4 (ii) holds.

(iii) By Lemma 4.2, \( R_{10} < 1 \) implies that \( \lim_{t \to \infty} I_1(t) = \lim_{t \to \infty} R_1(t) = 0 \). Then, when \( R_{10} < 1 \), the system, consisting of the first three equations of (2), has the limit system

\[
\begin{align*}
\frac{dS}{dt} &= (1 - \phi)M - \beta_2 SI_2 - \mu S + kV, \\
\frac{dV}{dt} &= \phi M - (1 - \xi)\beta_2 I_2 V - (k + \mu)V, \\
\frac{dI_2}{dt} &= \beta_2 SI_2 + (1 - \xi)\beta_2 I_2 V - (r_2 + \mu)I_2.
\end{align*}
\]
Since $S^{(2)}, V^{(2)}$ and $I_2^{(2)}$ satisfy the equations
\[
\begin{align*}
\mu &= \frac{(1-\phi)M}{S} - \beta_2 I_2 + k \frac{V}{S}, \\
k + \mu &= \frac{\phi M}{V} - (1 - \xi) \beta_2 I_2,
\end{align*}
\]
then system (39) can be rewritten as
\[
\begin{align*}
\frac{dS}{dt} &= S \left[ (1-\phi)M \left( \frac{1}{S} - \frac{1}{S^{(2)}} \right) - \beta_2 (I_2 - I_2^{(2)}) + k \left( \frac{V}{S} - \frac{V^{(2)}}{S^{(2)}} \right) \right], \\
\frac{dV}{dt} &= V \left[ \phi M \left( \frac{1}{V} - \frac{1}{V^{(2)}} \right) - (1 - \xi) \beta_2 (I_2 - I_2^{(2)}) \right], \\
\frac{dI_2}{dt} &= I_2 \beta_2 (S - S^{(2)}) + (1 - \xi)(V - V^{(2)}).
\end{align*}
\]
Define a Lyapunov function $L_2$ by
\[
L_2 = \left( S - S^{(2)} - S^{(2)} \ln \frac{S}{S^{(2)}} \right) + \left( V - V^{(2)} - V^{(2)} \ln \frac{V}{V^{(2)}} \right) + \left( I_2 - I_2^{(2)} - I_2^{(2)} \ln \frac{I_2}{I_2^{(2)}} \right).
\]
Then the derivative of $L_2$ with respect to $t$ along solutions of (40) (i.e. (39)) is
\[
\frac{dL_2}{dt} = \left( (1-\phi)M (S - S^{(2)}) \left( \frac{1}{S} - \frac{1}{S^{(2)}} \right) + k (S - S^{(2)}) \left( \frac{V}{S} - \frac{V^{(2)}}{S^{(2)}} \right) \right) \\
+ \phi M (V - V^{(2)}) \left( \frac{1}{V} - \frac{1}{V^{(2)}} \right) \\
= (1 - \phi)M \left( 2 - \frac{S}{S^{(2)}} - \frac{S^{(2)}}{S} \right) + kV^{(2)} \left( 3 - \frac{S}{S^{(2)}} - \frac{V^{(2)}}{V^{(2)}} - \frac{V^{(2)}}{S^{(2)}} \right) \\
+ [(1 - \xi) \beta_2 I_2^{(2)} V^{(2)} + \mu V^{(2)}] \left( 2 - \frac{V^{(2)}}{V^{(2)}} - \frac{V^{(2)}}{V^{(2)}} \right).
\]
By the inequality of arithmetic and geometric means (AM-GM inequality) we know that $\frac{dL_2}{dt} \leq 0$ and that $\frac{dL_2}{dt} = 0$ if and only if $S = S^{(2)}$ and $V = V^{(2)}$. Further, it is easy to see that the largest invariant set of (40) (i.e. (39)) in the set satisfying $\frac{dL_2}{dt} = 0$ is the singleton $\{ \bar{E}_2 \left( S^{(2)}, V^{(2)}, I_2^{(2)} \right) \}$. Hence, according to the local stability of equilibrium $E_2$ of system (2), LaSalle Invariance Principle [5] implies that equilibrium $\bar{E}_2$ of (40) (i.e. (39)) is globally asymptotically stable. Therefore, from the theory of limit system [16] it follows that equilibrium $E_2$ of system (2) is globally asymptotically stable in $\Omega$. 

According to Theorems 4.1 and 4.4, the local stability of $E_0$ implies its global stability. Then the analysis for the stability of $E_0$ is complete. When $R_{10} < 1 < R_{20}$, the local stability of $E_2$ also implies its global stability. For the case that $R_{10} \geq 1$ and $R_{20} > R_{20}^{(2)}$, the numerical simulation shows that $E_2$ is also globally stable (Figure 4) although the global stability of $E_2$ is not proved.

Clearly, Theorem 4.4 provides only sufficient conditions for the global stability of $E_1$ and no condition for $E^*$. However, the numerical simulations show that the local stability of $E_1$ can imply its global stability (Figures 5 and 6), and that $E^*$ is globally stable only if it exists (Figure 7).

With respect to numerical simulations here, except for $\beta_1$ and $\beta_2$, we choose the fixed values for parameters, $M = 1 \text{year}^{-1}, \phi = 0.8, \mu = 0.1 \text{year}^{-1}, k = 0.6 \text{year}^{-1}, \xi = 0.8, r_1 = r_2 = 0.3 \text{year}^{-1}$ and $\sigma = 0.2$. That is, we simulate various cases by changing the values of $\beta_1$ and $\beta_2$. Figure 4 is for the case that
$R_{10} > 1$ and $R_{20} > R_{20}^{(2)}$; Figure 5 is for $R_{10} > 1$ and $R_{20} < 1$; Figure 6 is for $R_{10} > 1$ and $1 < R_{20} < R_{20}^{(1)}$; Figure 7 is for the case that $E^*$ exists.

Figure 4. The trajectories of $I_1$ and $I_2$ for the case that $R_{10} \geq 1$ and $R_{20} > R_{20}^{(2)}$. Here, $\beta_1 = 0.075$ and $\beta_2 = 0.2$. Correspondingly, $R_{10} = 1.66$, $R_{20} = 4.54$, $R_{20}^{(2)} = 1.64$, $E_2$ is globally stable.

Figure 5. The trajectories of $I_1$ and $I_2$ for the case that $R_{10} > 1$ and $R_{20} < 1$. Here, $\beta_1 = 0.08$ and $\beta_2 = 0.04$. Correspondingly, $R_{10} = 1.77$, $R_{20} = 0.68$, $E_1$ is globally stable.

5. Conclusion and discussion. For model (2) proposed here, we obtained almost complete results on the existence and local stability of equilibria of the model except for some critical cases (such as $R_{10} = 1$ or $R_{20} = 1$). With respect to the corresponding conditions, $R_{10}$ and $R_{20}$ are two important quantities. Note that, in the presence of vaccination which is fully effective for the strain 1 and partially effective for the strain 2, $S_0^0$ and $S_0^0 + (1 - \xi)V_0^0$ are the numbers of susceptible individuals for the two strains without infection, respectively. And $\frac{1}{r_i + \mu}$ ($i = 1, 2$) is the average infectious period of the strain $i$. Therefore,

$$R_{10} = \beta_1 \cdot S_0^0 \cdot \frac{1}{r_1 + \mu}$$

is the vaccination reproduction number of the strain 1 rotavirus for model (2), and

$$R_{20} = \beta_2 \cdot [S_0^0 + (1 - \xi)V_0^0] \cdot \frac{1}{r_2 + \mu}$$

is the vaccination reproduction number of the strain 2 rotavirus for model (2).
In addition, we also give two quantities $R_{12}$ and $R_{21}$, which are defined in (24) and (25), respectively. As shown in Theorems 3.3 and 4.4, they determine the existence of the positive equilibrium $E^*$ and the stability of equilibria $E_1$ and $E_2$.

The condition that $R_{21}$ is defined is that $R_{10} > 1$. For this case, $E_1$ exists. According to assumption that the children recovered from strain 1 infection could be reinfected by strain 2, then when population stay in equilibrium $E_1$, the individuals susceptible for strain 2 consist of $S, V$ and $R_1$. Therefore, by the biological meanings of the parameters $\xi$ and $\sigma$ we know easily that $R_{21}$ is the invasion reproduction number of strain 2 rotavirus under the case that the infection of strain 1 is at the steady state.

The condition that $R_{12}$ is defined is that $R_{20} > 1$, which assures that $E_2$ exists. According to assumption that the vaccinated children have complete immunity against infection of strain 1 and that the children recovered from strain 2 infection could not be reinfected by strain 1, then when population stay in equilibrium $E_2$, the individuals susceptible for strain 2 include only $S$. Therefore, $R_{12}$ is the invasion reproduction number of strain 1 rotavirus under the case that the infection of strain 2 is at the steady state.

Thus, Theorem 3.3 suggests that the coexistence of the two strains implies the success of their mutual invasion.

With respect to the global stability of the boundary equilibria of the model, the almost necessary and sufficient conditions are found for the infection-free equilibrium $E_0$ by applying the theory of limit systems and the comparison theorem,
but only the sufficient conditions are obtained for the single strain equilibrium by applying the theory of limit systems and constructing the appropriate Lyapunov functions. Many numerical simulation showed that the local stability of the strain $i$ ($i = 1, 2$) equilibrium implies its global stability.

For the positive equilibrium of the model (coexisting equilibrium of the two strains), $E^*$, only the existence is determined, the condition on its local and global stability are not obtained. However, our numerical simulation shows that $E^*$ should be globally stable if it exists. For these results shown by numerical simulation but not obtained by theoretical analysis, we will continue to explore the analytical methods in subsequent studies.

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E-mail address: luk@sust.edu.cn
E-mail address: wendi@swu.edu.cn
E-mail address: jianqil@263.net