The Banks-Zaks Expansion and
"Freezing" in Perturbative QCD

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Abstract:

The recent calculation of the four-loop $\beta$-function in QCD provides further evidence that the Banks-Zaks expansion in $16\frac{1}{2} - n_f$ is sufficiently well behaved to be useful even for $n_f \approx 2$ light flavours. This expansion inherently predicts "freezing" of the QCD couplant at low energies as a perturbative effect. We consider the $e^+e^-$ and Bjorken-sum-rule cases as examples.
1 Introduction

The old idea that the QCD running coupling “freezes” at low energies has been phenomenologically successful in a wide variety of contexts. (See Refs. [1, 2] and references therein.) Theoretical evidence that “freezing” does occur, and for purely perturbative reasons, comes from the third-order calculation of $R_{e^+e^-}$ which, when “optimized” with respect to renormalization scheme, yields such behaviour [3]. Padé approximant methods indicate a similar conclusion [4]. Another approach is the Banks-Zaks (BZ) expansion [5, 6, 7, 8], within which “freezing” is natural and ubiquitous. The relevance of the BZ expansion to low-energy QCD phenomenology hinges on an extrapolation in the number of (massless) flavours, $n_f$, from 16$\frac{1}{2}$ down to about 2. Our point in this paper is that the credibility of this extrapolation has been significantly enhanced by the recent calculation of the QCD $\beta$ function to 4 loops [9].

The BZ expansion [5]-[8] is an expansion in $n_f^{\text{crit}} - n_f$, where $n_f^{\text{crit}}$ is the critical number of flavours at which asymptotic freedom is lost; i.e., where the first $\beta$-function coefficient changes sign. In QCD $n_f^{\text{crit}}$ is 16$\frac{1}{2}$. For $n_f$ slightly less than 16$\frac{1}{2}$ the $\beta$ function starts out negative, but quickly turns positive, and there is thus an infrared fixed point close to the origin [10, 5]. Provided the couplant $a(\mu)$ initially lies between 0 and $a^*$ (necessary for asymptotic freedom), the couplant will “freeze” to $a^*$ as the energy scale $\mu$ tends to zero. Thus, in the BZ expansion, “freezing” is natural and universal and will occur for all $n_f < 16\frac{1}{2}$, unless or until the BZ expansion breaks down.

Ref. [8] has suggested that the BZ expansion is qualitatively relevant to the real world where only 2 or 3 quark flavours are light compared to the QCD $\Lambda$ scale. The crucial issue is whether the expansion shows reasonable numerical convergence when $n_f \approx 2$. Assuming good behaviour, Ref [8] made a prediction for a certain coefficient. As we discuss, this prediction is borne out by the recent calculation of the QCD $\beta$ function to four-loops [9].

The plan of the paper is as follows: Section 2 presents the notation and extracts the relevant coefficients. Sections 3 and 4 discuss the BZ expansion for quantities at $Q = 0$ and at general $Q$, respectively. An important role is played by $\gamma^*$, the slope of the $\beta$ function at the fixed point, and we discuss its coefficients (the “universal invariants” of Grunberg [7]) to fourth order. Sect. 5 contains concluding remarks. Appendix A summarizes the results for a general $SU(N)$ colour group. Appendix B discusses the issue of renormalization-scheme (RS) invariance and the notion of ‘regular’ and ‘irregular’ schemes. The Higgs-decay case, which we argue is not a good guide to infrared behaviour, is discussed in Appendix C.
2 Notation

We write the $\beta$ function in the form:

$$\beta(a) \equiv \mu \frac{da}{d\mu} = -ba^2 \left(1 + ca + c_2a^2 + c_3a^3 + \ldots\right),$$

(1)

where $a \equiv \alpha_s/\pi$. The coefficients, in the $\overline{\text{MS}}$ scheme, are \([10, 11, 9]\):

$$\beta_0 = 2b = 11 - \frac{2}{3}n_f,$$

$$\beta_1 = 8bc = 102 - \frac{38}{3}n_f,$$

$$\beta_2 = 32bc_2 = \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2,$$

(2)

$$\beta_3 = 128bc_3 = \left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n_f$$

$$+ \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n_f^2 + \frac{1093}{729}n_f^3.$$

Here $\zeta_s$ is the Riemann zeta-function ($\zeta_3 = 1.202056903\ldots$, $\zeta_5 = 1.036927755\ldots$).

For $n_f$ just below $16\frac{1}{2}$, the $\beta$ function has a zero at $a^* \sim -\frac{1}{c}$, and $a^*$ is asymptotically proportional to $(16\frac{1}{2} - n_f)$. Its limiting form:

$$a_0 \equiv \frac{8}{321}(16\frac{1}{2} - n_f)$$

(3)

serves as the expansion parameter for the BZ expansion \([8]\). Because the constant of proportionality is so small, $a_0$ remains small ($\leq 0.4$) even with $n_f = 0$. To proceed, one re-writes all perturbative coefficients, eliminating $n_f$ in favour of $a_0$. The first two $\beta$-function coefficients, which are RS invariant, become:

$$b = \frac{107}{8}a_0,$$

(4)

$$c = -\frac{1}{a_0} + \frac{19}{4}.$$  (5)

Within the class of so-called ‘regular’ schemes \([5, 8]\), which includes $\overline{\text{MS}}$, perturbative coefficients have a polynomial dependence on $n_f$, and we may write

$$c_i = \frac{1}{a_0} \left(c_{i,-1} + c_{i,0}a_0 + c_{i,1}a_0^2 + \ldots\right).$$

(6)

The coefficients, in $\overline{\text{MS}}$, are collected in the table below.
\[
\begin{align*}
c_{1,0} &= \frac{19}{4} = 4.75 \\
c_{2,-1} &= -\left(\frac{8}{107}\right) \left(\frac{37117}{708}\right) = -3.61 \\
c_{2,0} &= \frac{243}{32} = 7.59 \\
c_{2,1} &= \left(\frac{107}{8}\right) \left(\frac{325}{192}\right) = 22.6 \\
c_{3,-1} &= \left(\frac{8}{107}\right) \left(\frac{53981}{1152} + \frac{5335}{32} \zeta_3\right) = 18.5 \\
c_{3,0} &= -\frac{1544327}{13824} - \frac{16171}{288} \zeta_3 = -179 \\
c_{3,1} &= \left(\frac{107}{8}\right) \left(\frac{2587}{96} + \frac{809}{144} \zeta_3\right) = 451 \\
c_{3,2} &= -\left(\frac{107}{8}\right)^2 \left(\frac{1093}{336}\right) = -56.6
\end{align*}
\]

The BZ expansion can be applied to any perturbatively calculable physical quantity of the form:

\[
\mathcal{R} = a \left(1 + r_1 a + r_2 a^2 + r_3 a^3 + \ldots\right).
\]  

(7)

In a ‘regular’ scheme the coefficients \( r_i \) are polynomials in \( n_f \), and hence in \( a_0 \):

\[
r_i = r_{i,0} + r_{i,1} a_0 + r_{i,2} a_0^2 + \ldots
\]  

(8)

Note that a term \( r_{i,j} a_0^p \) or \( c_{i,j} a_0^p \) can be assigned a degree \( i + j - p \), and all terms in any formula must have matching degree.

The prototypical example is the \( e^+ e^- \) ratio:

\[
R_{e^+ e^-} (Q) \equiv \frac{\sigma_{\text{tot}} (e^+ e^- \to \text{hadrons})}{\sigma (e^+ e^- \to \mu^+ \mu^-)}.
\]  

(9)

where, neglecting quark masses, we can write \( R_{e^+ e^-} (Q) = 3 \Sigma q_i^2 (1 + R_{e^+ e^-}) \), where \( R_{e^+ e^-} \) has the form (8). [Actually, for \( R_{e^+ e^-} \) there is a problem in that \( r_2 \) involves a term \( 1.2395 (\Sigma q_i^2)^2 / (3 \Sigma q_i^2) \) whose \( n_f \) dependence is ambiguous because it depends on the electric charges we assign to the additional, ficticious quarks. This arises because \( R_{e^+ e^-} \) involves not just QCD, but its coupling to electromagnetism. Fortunately, this term seems to make only a small numerical contribution. We shall ignore it henceforth.] The coefficients, in \( \text{MS} \) with the renormalization scale \( \mu \) equated with \( Q \), are collected in the table below [3]:

| Coefficients in \( R_{e^+ e^-} \). [MS(\( \mu = Q \))] | 
|--------------------------------------------------------|---|
| \( r_{1,0} = \frac{1}{12} \) | = 0.0833 |
| \( r_{1,1} = \left(\frac{107}{8}\right) \left(\frac{11}{4} - 2 \zeta_3\right) \) | = 4.63 |
| \( r_{2,0} = -\frac{12521}{288} + 13 \zeta_3 \) | = -27.85 |
| \( r_{2,1} = \left(\frac{107}{8}\right) \left(\frac{401}{24} - \frac{53}{3} \zeta_3 + \frac{25}{3} \zeta_5\right) \) | = 55.0 |
| \( r_{2,2} = \left(\frac{107}{8}\right)^2 \left(\frac{151}{12} - \frac{19}{4} \zeta_3 - \frac{e^2}{12}\right) \) | = -8.34 |
Another example is the Bjorken sum rule:

$$\int_0^1 dx g_1^{ep-en}(x, Q^2) = \frac{1}{3} \left| \frac{g_A}{g_V} \right| (1 - R_{\text{Bj}}).$$

(The same QCD corrections, apart from a $(\Sigma q_i)^2/(3\Sigma q_i^2)$ term, appear in the Gross Llewellyn-Smith sum rule.) The coefficients, from Ref. [12] are listed below.

| Coefficients in $R_{\text{Bj}} \, [\overline{\text{MS}}(\mu = Q)]$ |
|-----------------|-----------------|
| $r_{1,0}$       | $-\frac{11}{12}$ |
| $r_{1,1}$       | $\frac{107}{8}$ |
| $r_{2,0}$       | $-\frac{1385}{72} - \frac{55}{4} \zeta_3$ |
| $r_{2,1}$       | $\left(\frac{107}{8}\right) \left(\frac{2749}{432} + \frac{61}{18} \zeta_3 - 5\zeta_5\right)$ |
| $r_{2,2}$       | $\left(\frac{107}{8}\right)^2 \left(\frac{115}{72}\right)$ |

We mention that the same decomposition of coefficients is needed in the “large-$b$” approximation [13], which employs the opposite limit ($b \to \infty$, rather than $b = (107/8)a_0 \to 0$ as here).

### 3 BZ Expansion: $Q = 0$

The fixed-point condition $\beta(a^*) = 0$ always has a solution as a power series in $a_0$:

$$a^* = a_0 \left[1 + v_1a_0 + v_2a_0^2 + v_3a_0^3 + \ldots\right].$$

A straightforward calculation yields:

$$v_1 = c_{1,0} + c_{2,-1},$$

$$v_2 = (c_{1,0} + 2c_{2,-1})(c_{1,0} + c_{2,-1}) + c_{2,0} + c_{3,-1},$$

$$v_3 = c_{1,0}^3 + 6c_{1,0}^2c_{2,-1} + c_{1,0}(3c_{2,0} + 4c_{3,-1} + 10c_{2,-1})$$

$$+ c_{2,-1}(4c_{2,0} + 5c_{3,-1}) + 5c_{2,-1}^3 + c_{2,1} + c_{3,0} + c_{4,-1}.$$  

(Numerically, $v_1 = 1.1366$, $v_2 = 23.27$, $v_3 = c_{4,-1} - 138.6$, in the $\overline{\text{MS}}$ scheme. The poor apparent convergence of the $a^*$ series need not bother us, since $a^*$ is RS dependent.)

A physical quantity $R$ will also have an infrared limit given by a power series in $a_0$. One simply takes the perturbative expansion of $R$, Eq. (7); substitutes $a = a^*$, given by (12); and re-expands in powers of $a_0$. This yields:

$$R^* = a_0 \left[1 + w_1a_0 + w_2a_0^2 + w_3a_0^3 + O(a_0^4)\right],$$

(13)
where

\begin{align}
    w_1 &= v_1 + r_{1,0}, \\
    w_2 &= v_2 + 2r_{1,0}v_1 + r_{2,0} + r_{1,1}, \\
    w_3 &= v_3 + (2v_2 + v_1^2)r_{1,0} + v_1(2r_{1,1} + 3r_{2,0}) + r_{2,1} + r_{3,0}.
\end{align}  

(14)

These coefficients are RS-scheme independent (see Appendix B) and so their numerical values are significant. They should be order-1 numbers, if all is to be well.

For the $e^+e^-$ case, Ref. [8] obtained the value of the first coefficient, $w_1 = 1.22$, but $w_2$ could only be obtained as $-18.25 + c_{3,-1}$, since $c_{3,1}$ was then unknown. To quote Ref. [8]: “For the expansion to be credible one needs $c_{3,-1}(\text{MS})$ to be in the range, say, $+13$ to $+21$.” This prediction is confirmed by the new $\beta$-function result [9], which yields $c_{3,-1} = 18.5$. Therefore, $w_2$ is quite small, 0.23, giving a respectable series:

\[
    \mathcal{R}^{*}_{e^+e^-} = a_0 \left[ 1 + 1.22a_0 + 0.23a_0^2 + \ldots \right].
\]

(15)

The next coefficient,

\[
    w_3(e^+e^-) = c_{4,-1} + r_{3,0}(e^+e^-) - 164.0,
\]

(16)

would require calculation of both $\beta$ and $\mathcal{R}^{*}_{e^+e^-}$ to one more order.

In the Bj-sum-rule case the corresponding result is

\[
    \mathcal{R}^{*}_{\text{Bj}} = a_0 \left[ 1 + 0.22a_0 - 1.21a_0^2 + \ldots \right],
\]

(17)

where the coefficients are also of order unity. It is interesting that the $\mathcal{R}^{*}$ in this case is even smaller than in the $e^+e^-$ case. The next coefficient is

\[
    w_3(\text{Bj}) = c_{4,-1} + r_{3,0(\text{Bj})} - 203.7.
\]

(18)

4 \hspace{1em} BZ Expansion: Nonzero $Q$

A formulation of the BZ expansion for quantities at a general $Q$ was derived in Ref. [8]. We briefly review the main ingredients. First, we need a suitable form of the boundary condition for the $\beta$-function equation. Setting $\hat{\beta}(x) \equiv \beta(x)/b$, we write

\[
    b \ln \left( \frac{\mu}{\Lambda} \right) = \lim_{\delta \to 0} \left[ \int_{\delta}^{a} \frac{dx}{\hat{\beta}(x)} + C(\delta) \right].
\]

(19)
The constant of integration $C(\delta)$ needs to be suitably singular as $\delta \to 0$ and we choose \[14, 8\]:

$$C(\delta) = \text{P.V.} \int_{\delta}^{\infty} \frac{dx}{x^2(1+cx)}$$

$$= \frac{1}{\delta} + c \ln \delta + c \ln |c| + O(\delta). \quad (20)$$

Note that Cauchy’s principal value (P.V.) is introduced to deal with the pole at $x = -1/c$ when $c < 0$. This choice amounts to a definition of $\tilde{\Lambda}$, within a given RS. [We use a tilde to distinguish it from the older, but still widely used, definition of the $\Lambda$ parameter \[15\].

The relation is $\ln(\Lambda/\tilde{\Lambda}) = (c/b) \ln(2|c|/b)$. While the two definitions are not dissimilar for small $n_f$, they become infinitely different as $n_f \to 16\frac{1}{2}$. In the BZ-expansion context the use of $\tilde{\Lambda}$ is much more convenient.]

As explained in Ref. \[8\], it is convenient to put the $\beta$ function into the form

$$\frac{1}{\beta(x)} = -\frac{1}{x^2} + \frac{c}{x} - \frac{1}{\hat{\gamma}^* (a^* - x)} + H(x). \quad (21)$$

where $\hat{\gamma}^*$ is $\gamma^*/b$, with $\gamma^*$ being the slope of the $\beta$ function at the fixed point:

$$\gamma^* \equiv \frac{\beta(x)}{dx} \bigg|_{x=a^*} = -ba^* \left(1 + 2ca^* + 3c_2a^{*2} + 4c_3a^{*3} + \ldots \right). \quad (22)$$

As discussed below, $\hat{\gamma}^*$ can be obtained as a series in $a_0$. The remainder function $H(x)$ can be expanded as a power series, $H_0 + H_1x + \ldots$, whose coefficients are of order $a_0$.

One now inserts (21) into (19) and performs the integration. One can then eliminate $a$ and $a^*$ in favour of $R$ and $R^*$. In fact, since the result must be RS invariant, one can — without loss of generality — short-cut this step by utilizing the “effective-charge” RS in which $a \equiv R$. This leads to the formula \[8\]:

$$\rho_1 = \frac{1}{R} + \frac{1}{\hat{\gamma}^*(n)} \ln \left(1 - \frac{R}{R^*} \right) + c \ln (|c| R) + \sum_{i=0}^{n-4} \frac{H_i^{(ec)} R^{i+1}}{i+1}. \quad (23)$$

The last term, involving the $H_i^{(ec)}$ coefficients (of the effective-charge scheme), is only relevant in fourth order and beyond. Thus, for the first three orders the equation takes the same form, just with the parameters $\hat{\gamma}^*$ and $R^*$ approximated to the appropriate order.

On the left-hand side, $\rho_1$ is the RS invariant \[14\]

$$\rho_1 \equiv b \ln \left(\frac{\mu}{\Lambda} \right) - r_1 \equiv b \ln \left(\frac{Q}{\Lambda_{eff}} \right), \quad (24)$$

where $\Lambda_{eff}$ is a characteristic scale specific to the particular physical quantity $R$. It is related to the $\tilde{\Lambda}$ parameter of some reference scheme (eg. $\overline{\text{MS}}$) by an exactly calculable
factor \( \exp(r_1/b) \) involving the \( r_1 \) coefficient in that scheme, evaluated at \( \mu = Q \). (We caution that \( r_1 \) cannot be split into \( \mathcal{O}(1) \) and \( \mathcal{O}(a_0) \) pieces in a RS-invariant way.)

Numerically inverting Eq. (23) provides \( R \) as a function of \( Q \). The resulting \( R(Q) \) naturally agrees with ordinary perturbation theory to the corresponding order at large \( Q \), but freezes to the value \( R^*(n) \) as \( Q \to 0 \). The BZ series expansion for \( R^* \) was discussed in the previous section. The BZ expansion for \( \hat{\gamma}^* \) is obtained straightforwardly by substituting the expansion of \( a^* \) (Eqs. (11) and (12)) into (22). This gives:

\[
\hat{\gamma}^* = a_0 \left[ 1 + g_1 a_0 + g_2 a_0^2 + g_3 a_0^3 + \ldots \right]
\]

where,

\[
\begin{align*}
g_1 &= c_{1,0}, \\
g_2 &= c_{1,0}^2 - c_{2,-1}^2 - c_{3,-1}, \\
g_3 &= c_{1,0}^3 - 4c_{2,-1}^3 - 5c_{1,0}c_{2,-1}^2 - 4c_{1,0}c_{3,-1} \\
 &\quad - 2c_{2,-1}c_{2,0} - 6c_{2,-1}c_{3,-1} - c_{3,0} - 2c_{4,-1}.
\end{align*}
\]

It is noteworthy that certain terms of degree \( n \) are absent in \( g_n \): \( g_1 \) does not contain \( c_{2,-1} \); \( g_2 \) does not contain \( c_{2,0} \) or \( c_{2,-1}c_{1,0} \); and \( g_3 \) does not contain \( c_{2,1} \) or \( c_{2,0}c_{1,0} \) or \( c_{2,-1}c_{1,0}^2 \).

The significance of \( \gamma^* = b\hat{\gamma}^* \) is that it is the ‘critical exponent’ governing how \( R \) approaches \( R^* \) as \( Q \to 0 \); asymptotically, \( R - R^* \propto Q^{\gamma^*} \). As pointed out by Grunberg, the \( g_n \) coefficients are RS invariants, and are universal, in the sense that they are not specific to some particular physical quantity \( R \). (See Appendix B for discussion of some subtleties.)

The new \( \beta \)-function result [9] enables us to determine the numerical value of the second invariant; \( g_2 = -8.99 \). (The exact expression, for general \( N \), is given in Appendix A.) Hence the \( \hat{\gamma}^* \) series is:

\[
\hat{\gamma}^* = a_0 \left[ 1 + 4.75a_0 - 8.99a_0^2 + \ldots \right]
\]

Clearly, the \( \hat{\gamma}^* \) series is not as well behaved as the \( R^* \) series that we saw earlier. In Fig. 1(a) we show \( \gamma^* (= b\hat{\gamma}^*) \) as a function of \( n_f \). (Fig. 1(b) shows the same quantity normalized by \( 1/(ba_0) \).) The lower and upper solid curves are the first- and second-order results, respectively, while the middle solid curve is the third-order result. The dashed curve represents the third-order result re-cast as a Padé approximant:

\[
\hat{\gamma}^* \approx a_0 \frac{(1 + 6.64a_0)}{(1 + 1.89a_0)}.
\]
For comparison we also give, as the dotted curve, the prediction arising from an optimized-perturbation-theory analysis of the $e^+e^-$ case [1] (see comments in Appendix B). The reasonable agreement between the last three curves gives us some confidence that the extrapolation to low $n_f$ is qualitatively valid, even if the quantitative precision is not good.

The next coefficient is

$$g_3 = 269.44 - 2c_{4,-1}.$$  \hspace{1cm} (29)

We therefore predict that $c_{4,-1}(\overline{\text{MS}})$ will turn out to be somewhere around $135 \pm 10$. Assuming that the $u_3$ coefficients in (16) and (18) are modest, we can expect that $r_{3,0}(e^+e^-) \approx 29 \pm 10$ and $r_{3,0}(\text{Bj}) \approx 68 \pm 10$.

Knowing $\hat{\gamma}^*$ and $R^*$, we can numerically solve Eq. (23) to obtain $R(Q)$ as a function of $Q/\tilde{\Lambda}_{\text{eff}}$. The result, for the $e^+e^-$ case in third order, is shown in Fig. 2 for $n_f = 14, 10, 6$ and 2. (It may be directly compared with similar figures for first and second order in Ref. [8].) Corresponding results in the Bjorken-sum-rule case are shown in Fig. 3. Note that the $\tilde{\Lambda}_{\text{eff}}$’s defining the units of $Q$ in Figs. 2 and 3 are not the same: however, they are easily converted to a common $\tilde{\Lambda}$ using Eq. (24).

5 Concluding Remarks

The result of van Ritbergen et al.’s [9] calculation of the $\beta$ function to four-loops sheds much light on the BZ expansion. It supports the idea that the expansion is relevant to the phenomenologically interesting case of only two light quark flavors. That, in turn, implies perturbatively explicable “freezing” of the QCD running coupling constant.

We mention again the very interesting work on the “large-$b$” approximation (see [13] and references therein) which draws on large-$n_f$ results. This is the opposite approximation to ours. It extrapolates upwards from $n_f = -\infty$ (minus infinity if the theory is to be asymptotically free) towards $n_f \approx 2$, whereas we are extrapolating down from $n_f = 16\frac{1}{2}$. We think both approximations are useful; neither is very precise, but both seem to be qualitatively valid, and offer a great deal of insight into QCD. Our preliminary studies indicate that the large-$b$ approximation also predicts “freezing,” and we hope to report on this shortly.

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Appendix A: SU($N$) generalization

The critical number of flavours is:

\[ n_{f}^{\text{crit}} = \frac{11}{2} N. \]  \hfill (30)

The $\beta$-function coefficients in $\overline{\text{MS}}$ are

\[ \beta_0 = 2b = \frac{1}{3} (11N - 2n_f) \]
\[ \beta_1 = 8bc = \frac{1}{3N} (34N^3 - n_f(13N^2 - 3)) \]
\[ \beta_2 = 32bc^2_{\overline{\text{MS}}} = \frac{1}{108N^2} \left( 5714N^5 + n_f(-3418N^4 + 561N^2 + 27) + n_f^2(224N^3 - 66N) \right) \]
\[ \beta_3 = 128bc^3_{\overline{\text{MS}}} = \frac{1}{1944N^3} \left( 601892N^7 - 25920N^5 + \zeta_3(9504N^7 + 684288N^5) \right) 
+ n_f \left[ -485513N^6 + 58583N^4 - 21069N^2 - 5589 \right] 
+ \zeta_3(-4320N^6 - 118368N^4 + 9504N^2) \right] 
+ n_f^2 \left[ 69232N^5 - 19816N^3 - 22428N + 18144N^5 - 13824N^3 + 52704N \right] 
+ n_f^3 \left[ 1040N^4 - 616N^2 \right] \]  \hfill (31)

The BZ expansion parameter becomes

\[ a_0 = \frac{16}{3(25N^2 - 11)} (n_{f}^{\text{crit}} - n_f) \]  \hfill (32)

The invariant coefficients are

\[ b = \frac{(25N^2 - 11)}{16} a_0, \]  \hfill (33)
\[ c = -\frac{1}{a_0} + \frac{(13N^2 - 3)}{8N}. \]  \hfill (34)

In $\overline{\text{MS}}$

\[ c_{2,-1} = \frac{(-1402N^4 + 242N^2 + 33)}{48N(25N^2 - 11)}, \]  \hfill (35)
\[ c_{2,0} = \frac{(318N^4 + 55N^2 - 9)}{384N^2}, \]  \hfill (36)
\[ c_{2,1} = \frac{(25N^2 - 11)(112N^2 - 33)}{3072N}, \]  \hfill (37)
\[ c_{3,-1} = \frac{(14731N^6 - 30047N^4 - 58839N^2 - 2277)}{1152N^2(25N^2 - 11)} 
+ \frac{11}{8} \frac{(25N^4 - 18N^2 + 77)}{(25N^2 - 11)} \zeta_3. \]  \hfill (38)
The Grunberg invariants, the coefficients in the expansion of $\hat{\gamma}^*$ are:

$$g_1 = \frac{(13N^2 - 3)}{8N}, \quad (39)$$

$$g_2 = \frac{(366782N^8 - 865400N^6 + 1599316N^4 - 571516N^2 - 3993)}{768N^2(25N^2 - 11)^2}
- \frac{11(25N^4 - 18N^2 + 77)}{8(25N^2 - 11)} \zeta_3. \quad (40)$$

**Appendix B: Regular and Irregular Schemes**

In this appendix we discuss renormalization-scheme invariance of the BZ expansion results. Since the expansion parameter $a_0 = \frac{8}{321}(16\frac{1}{2} - n_f)$ is an RS-invariant pure number, one expects the coefficients in the BZ expansion of a physical quantity $R$ to be RS-invariant, but it is important to have confirmation. We first consider the fixed point results at $Q = 0$ (which are independent of the $\tilde{\Lambda}$ parameter).

In the text, as in Ref [8], we limited the discussion to so-called ‘regular schemes’ in which the coefficients in $R$ and $\beta(a)$ have a polynomial dependence on $n_f$. Such schemes are natural in diagrammatic terms, since each fermion loop gives an $n_f$ factor. They are convenient for our purposes, since the BZ-expansion coefficients are easily extracted from calculations made in those schemes. However, we emphasize that “irregular” schemes are not necessarily “bad.” They will also lead to the same BZ-expansion results, but the extraction of the BZ coefficients from calculations in those schemes will be less straightforward. An analogous situation arises, for example, with gauge invariance; a certain class of gauges may be convenient for some purposes, but inconvenient for others.

It is easy to show the invariance of the BZ coefficients within the class of ‘regular’ schemes. This was done in Ref. [8] by considering the RS-invariants $\rho_2, \rho_3, \ldots$ [14] that are invariant combinations of $R$ and $\beta(a)$ coefficients, and expanding them in powers of $a_0$. A ‘low-brow’ version of the proof is also instructive. Consider a change of RS, $a \mapsto a'$, where

$$a' = a \left(1 + u_1a + u_2a^2 + \ldots \right). \quad (41)$$

The coefficients in the expansion of $R$ change to

$$r'_1 = r_1 - u_1,$$
$$r'_2 = r_2 - u_2 - 2u_1r_1 + 2u_1^2,$$

etc.. The $\beta$ function transforms to $\beta'(a') = (\partial a'/\partial a)\beta(a)$, whose coefficients are:

$$c' = c$$

10
\begin{align}
    c'_2 &= c_2 + u_2 - u_1^2 - u_1 c, \\
    c'_3 &= c_3 + 2u_3 + cu_1^2 - 2c_2u_1 - 6u_1u_2 + 4u_1^3, \quad (43)
\end{align}

etc.. If both the primed and unprimed scheme are regular, then the \( c_i, c'_i \) and \( r_i, r'_i \) coefficients are expandable as in Eqs. \([4],[5]\), and the \( u_i \) coefficients in the scheme transformation can be expanded as:

\begin{align}
    u_1 &= u_{1,0} + u_{1,1}a_0, \\
    u_2 &= u_{2,0} + u_{2,1}a_0 + u_{2,2}a_0^2, \quad (44)
\end{align}

etc.. It is then straightforward to prove the invariance of the combinations appearing in the BZ expansion of \( \mathcal{R} \) and \( \gamma^* \). For instance, from the relations

\begin{align}
    r'_{1,0} &= r_{1,0} - u_{1,0}, \\
    c'_{2,-1} &= c_{2,-1} + u_{1,0}, \\
    c'_{3,-1} &= c_{3,-1} - u_{1,0}^2 - 2c_{2,-1}u_{1,0}, \quad (45)
\end{align}

one sees that \( r'_{1,0} + c'_{2,-1} = r_{1,0} + c_{2,-1} \), and \( c'_{3,-1} + (c'_{2,-1})^2 = c_{3,-1} + (c_{2,-1})^2 \), showing that these combinations are RS invariant. Extending this procedure one can prove the invariance of the higher-order \( w_i \) and \( g_i \) coefficients.

Any scheme related to \( \overline{\text{MS}}(\mu = Q) \) by a transformation \([41]\) with \( u_i \)'s expandable as in \( [44] \) is a ‘regular’ scheme. [It is noteworthy that in a general, regular scheme the coefficients \( c_{2,2}, c_{3,3}, \ldots \) are non-zero, while in a more restrictive class of ‘strictly regular’ schemes, of which \( \overline{\text{MS}}(\mu = Q) \) is an example, these coefficients vanish. This distinction is unimportant for the BZ expansion, but matters for the large-\( b \) approximation \([13] \).] The ‘effective charge’ (or ‘FAC’) scheme, in which \( a = \mathcal{R} \) for some specific physical quantity, is a ‘regular’ (but not ‘strictly regular’) scheme. However, it is easy to construct RS’s that are ‘irregular’ simply by considering a transformation in which the \( u_i \) depend on \( n_f \) in a non-polynomial fashion, so that the coefficients \( r'_i \) are no longer expandable in positive powers of \( a_0 \). There is nothing intrinsically ‘bad’ about such schemes. They can arise quite naturally. For example, the ’t Hooft scheme, in which \( c_2 = c_3 = \ldots = 0 \), is ‘irregular’ \([13]\). [Recall that, in ‘regular’ schemes, \( c_{3,-1} + (c_{2,-1})^2 \) is invariant and does not vanish.] The principle-of-minimal-sensitivity (PMS) scheme for any given physical quantity is also ‘irregular.’ In both these cases the \( r_i \) coefficients have \( 1/a_0 \) pieces.

In a ‘regular’ scheme, obtaining the BZ expansion to \( n \)th order requires terms of order \( n + 1 \) in the \( \beta \) function, and of order \( n \) in the physical quantity \( \mathcal{R} \). In an ‘irregular’ scheme
the same information is distributed among higher-order coefficients as well. Starting from an ‘irregular’ scheme, one would require some knowledge of perturbative coefficients to higher orders; maybe to all orders. However, provided one carefully kept all terms that could contribute to a given order in $a_0$, one would obtain the same BZ expansion.

Finally, we discuss the finite-$Q$ case. Following Ref. [8], we have formulated the result as Eq. (23), which is to be solved numerically to obtain $\mathcal{R}$ as a function of $Q$. This formula involves the RS-invariant quantities $\rho_1$, $\hat{\gamma}^*$, $\mathcal{R}^*$, $c$, and $H_i^{(ec)}$. The $\tilde{\Lambda}$ parameter and $Q$ appear only in $\rho_1$. The $H_i^{(ec)}$ coefficients (relevant only in 4th order and beyond) are directly related to the $\tilde{\rho}_2, \tilde{\rho}_3, \ldots$ invariants of Ref. [14] which can be conveniently redefined (hence the tilde) to coincide with the $\beta$-function coefficients of the ‘effective charge’ scheme [16].

The RS invariance of the BZ expansion of $\hat{\gamma}^*$ [7] is verifiable by the procedure discussed above. It is expected because of the well-known result [17] that the slope of the $\beta$ function at a fixed point is an invariant. However, there is an important caveat to the last statement [18], which necessitates some further discussion. The quoted result follows by differentiating the $\beta$-function transformation, $\beta'(a') = (\partial a'/\partial a)\beta(a)$, to give

$$\frac{\partial \beta'}{\partial a'} = \frac{\partial a}{\partial a'} \frac{\partial^2 a'}{\partial a^2} \beta(a) + \frac{\partial \beta}{\partial a}. \quad (46)$$

The first term vanishes at the fixed point — provided that neither $\partial a/\partial a'$ nor $\partial^2 a'/\partial a^2$ is singular there [17]. Chýla [18] has pointed out that, in general, it can be quite natural for those factors to be singular (and arbitrary scheme transformation can of course make fixed points appear and disappear!). In general, then, the critical exponent $\gamma^*$ in $\mathcal{R} - \mathcal{R}^* \propto Q^{\gamma^*}$ as $Q \to 0$ is not the same as $\partial \beta/\partial a|_{a=a^*}$. However, these two quantities will coincide in a large class of schemes. It also seems safe to assume that they coincide in the context of the BZ expansion, where there is necessarily a fixed point at $a^* = a_0 + \ldots$.

[We find that the 3rd-order PMS scheme, though ‘irregular,’ also yields $\gamma^* = \partial \beta/\partial a|_{a=a^*}$. This requires a detailed analysis of the optimization equations [14, 1] as $Q \to 0$. The PMS results (in the $e^+e^-$ case) for $\gamma^*$ are shown as the dotted curve in Fig. 1.]

**Appendix C: Higgs decay**

In this appendix we discuss the case of Higgs-boson decay into hadrons. This seems to be a much more problematic than the cases discussed earlier. It could be viewed as conflicting with the general picture we have presented. We shall argue, though, that the crucial role
of quark masses in this quantity makes it unsuitable as a guide to the infrared behaviour of perturbative QCD. First, let us discuss the numbers.

The hadronic decay width of the Higgs has the form \( \Gamma_H = \frac{3G_F}{4\sqrt{2}\pi} M_H \sum_q m_q^2 \Gamma(a) \) with \( \Gamma(a) = 1 + \Gamma_1 a + \ldots \), and where \( m_q \) is the running quark mass evaluated at some scale. For \( \Gamma_H \) there is a factorization-scheme ambiguity (how much of the radiative corrections should be absorbed into \( m_2^q \), and much should be left in the explicit series \( \Gamma(a) \)?). However, one can define the quantity \( \mathcal{R}_{\text{Higgs}} = -\frac{1}{2} \frac{d \ln (\Gamma_H/M_H)}{d \ln M_H^2} \) (47) which is free of this factorization-scheme ambiguity, and is a physical quantity of the same form \( \mathcal{R} = a(1 + r_1 a + \ldots) \) considered earlier. The coefficients, from Ref. [19] are collected in the table below.

| Coefficients in \( \mathcal{R}_{\text{Higgs}} \). [\( \overline{\text{MS}}(\mu = Q) \)] |  |
|---|---|
| \( r_{1,0} \) | \( \frac{23}{12} \left( \frac{11}{6} \right) \) | = 1.917 |
| \( r_{1,1} \) | \( \frac{107}{8} \) | = 24.52 |
| \( r_{2,0} \) | \( \frac{503}{18} - \frac{55}{2} \zeta_3 \) | = -44.47 |
| \( r_{2,1} \) | \( \frac{107}{8} \left( \frac{295}{144} - \frac{7}{4} \zeta_3 - \frac{\pi^2}{3} \right) \) | = 200.5 |
| \( r_{2,2} \) | \( \frac{107}{8} \left( \frac{275}{72} - \zeta_3 - \frac{\pi^2}{12} \right) \) | = 321.1 |
| \( r_{3,0} \) | \( -\frac{22631621}{82944} - \frac{13939}{432} \zeta_3 + \frac{4675}{48} \zeta_5 + \frac{107}{24} \frac{\pi^2}{12} \) | = -166.6 |
| \( r_{3,1} \) | \( \frac{107}{8} \left( \frac{208411}{3456} - \frac{20755}{288} \zeta_3 - \frac{355}{48} \zeta_5 - \frac{137}{102} \frac{\pi^2}{12} \right) \) | = -2162 |
| \( r_{3,2} \) | \( \frac{107}{8} \left( \frac{694303}{6912} - \frac{119}{4} \zeta_3 + \frac{25}{4} \zeta_5 - \frac{317}{96} \frac{\pi^2}{12} \right) \) | = 6901 |
| \( r_{3,3} \) | \( \frac{107}{8} \left( \frac{985}{192} - \frac{5}{2} \zeta_3 - \frac{11}{24} \frac{\pi^2}{12} \right) \) | = 3808 |

In Ref. [20] it was observed that, at third order in the effective-charge (or ‘FAC’) scheme, there is a fixed point with \( \mathcal{R}_{\text{Higgs}}^* \sim a^* \approx 0.15 \). The authors viewed this as probably spurious. Indeed, it is odd that it is only about half the size of the frozen couplant found in the \( e^+e^- \) case, and so is far from the leading-order BZ expectation that \( a^* \approx a_0 \). At 4th order Ref. [19] finds that this fixed point is no longer present. We have checked that the situation is much the same in optimized perturbation theory [14, 21].

The \( Q = 0 \) BZ series in this case is:

\[
\mathcal{R}_{\text{Higgs}} = a_0(1 + 3.05 a_0 + 7.67 a_0^2 + \ldots)
\] (48)

whose coefficients are considerably larger than in the \( e^+e^- \) or Bjorken-sum-rule cases (Eqs. (15), (17)). For a low number of flavours the “corrections” are as big as the leading term,
and both the same sign. (The next coefficient in the expansion is $c_{4,-1} - 109$ and is still unknown; our estimate (see Eq. (29) suggests that it is around $26 \pm 10$.) One could conclude that this is perhaps a case where the BZ expansion breaks down before $n_f = 2$, so that maybe there is no freezing in this case.

We believe, however, the problem is that this quantity is just not a useful indicator of massless QCD’s infrared behaviour, because of the way that quark masses are involved. For exactly massless quarks the hadronic Higgs-decay rate is zero, because the Higgs-quark coupling is proportional to quark mass. (The calculations neglect quark masses in the radiative corrections but not, of course, in the overall coupling factor.) If we keep the quark masses finite when we consider, theoretically, the limit $M_H \to 0$, we will trivially get zero as soon as the decay becomes kinematically forbidden. To avoid this we would need to consider a limit in which $m_q$ tends to zero at least as fast as $M_H$; say $m_q \propto (M_H)^\kappa$ with $\kappa \geq 1$. But then $R_{\text{Higgs}}$ is not of the form $a(1 + r_1 a + \ldots)$, and depends on $\kappa$, making it ill-defined. These issues do not arise for $R_{e^+e^-}$ or $R_{\text{Bj}}$, which are meaningful for massless quarks.
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Fig. 1. (a) The critical exponent $\gamma^*$ in first, second, and third orders of the BZ expansion (lower, upper, and middle solid curves). A Padé approximant form of the third-order result is shown as the dashed curve. The dots represent the result of an optimized-perturbation-theory analysis [1]. (b) The same, normalized by $1/(ba_0)$; i.e., $\hat{\gamma}^*/a_0$. 
Fig. 2. $R_{e^+e^-}$ as a function of $Q/\bar{\Lambda}_{\text{eff}}$ to third order in the BZ expansion for $n_f = 14, 10, 6, 2$. 
Fig. 3. $R_{\text{Bj}}$ as a function of $Q/\tilde{\Lambda}_{\text{eff}}$ to third order in the BZ expansion for $n_f = 14, 10, 6, 2$. 