Abstract
We study various properties of composition operators $C_\psi$ acting between generalized Fock spaces $\mathcal{F}_p^\varphi$ and $\mathcal{F}_q^\varphi$ with weight functions $\varphi$ growing faster than the classical Gaussian function $\frac{1}{2}|z|^2$ and satisfy some mild smoothness conditions. We show that if $p \neq q$, then $C_\psi : \mathcal{F}_p^\varphi \rightarrow \mathcal{F}_q^\varphi$ is bounded if and only if it is compact. This result shows a significance difference from the analogous result for the case when $C_\psi$ acts between the classical Fock spaces or generalized Fock spaces where the weight functions grow slower than the Gaussian function. We further described the Schatten $S_p(\mathcal{F}_2^2)$ class, hyponormal, unitary, cyclic and supercyclic composition operators on the spaces. As an application, we also characterized the compact differences, the isolated and essentially isolated points, and connected components of the space of the operators under the operator norm topology.

Keywords Generalized Fock spaces · Bounded · Compact · Composition · Normal · Unitary · Cyclic · Supercyclic · Connected · Isolated

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1 Introduction

For a given holomorphic mappings $\psi$ and $f$ on the complex plane $\mathbb{C}$, we define the composition operator induced by $\psi$ as $C_\psi f = f(\psi)$. Composition operators have been extensively studied on various spaces of holomorphic functions over several settings in the past many years. It is rather difficult to give a comprehensive list of related works on the subject now. For an overview in the framework of Fock spaces, which we are interested in, one may consult the materials for example in [4,5,9,11,14] and the references therein. On the other hand, over the unit disc of the complex plane or the unit ball in $\mathbb{C}^n$, the monographs in [7,10,18] provide a comprehensive expositions specially on the early developments of the area. The study of composition operators has continued to attract interest partly because it finds itself at the interface of both operator and function theories.

On the classical Fock spaces setting, the boundedness and compactness properties of the operators were studied for example in [4,5,15,16]. On the other hand, when the weight function generating the generalized Fock spaces grow slower than the classical Gaussian weight function, a thorough look into the proof of Proposition 2.1 of [14] shows that the forms of the symbol $\psi$ inducing bounded and compact $C_\psi$ are just like that of the classical case. A similar result can be also read in [5,9,13] where the form of the operator is discussed on Fock–Sobolev spaces which are typical examples of generalized Fock spaces with weight functions growing slower than the Gaussian function. A natural question is what happens to these properties when the weight functions grow faster than the Gaussian function. The aim of this work is taking further the study of the operators on such spaces, and answer these and other related basic topological and dynamical questions. It turns out that while the dynamical structures of the operators behave like that of the classical setting, the faster growth of the weight functions result in a poorer structure in the forms of the symbols inducing bounded composition operators acting between two generalized Fock spaces.

We begin by setting the growth and smoothness conditions for the weight function. Let $\varphi : [0, \infty) \to [0, \infty)$ be a twice continuously differentiable function. We extend $\varphi$ to the whole complex plane by setting $\varphi(z) = \varphi(|z|)$. We further assume that its Laplacian, $\Delta \varphi$, is positive and set $\tau(z) \simeq (\Delta \varphi(z))^{-1/2}$ when $|z| \geq 1$, and $\tau(z) \simeq 1$ whenever $0 \leq |z| < 1$, where $\tau$ is a radial differentiable function satisfying the admissibility conditions

$$\lim_{r \to \infty} \tau(r) = 0 \quad \text{and} \quad \lim_{r \to \infty} \tau'(r) = 0,$$

and there exists a constant $C > 0$ such that $\tau(r)r^C$ increases for large $r$ or

$$\lim_{r \to \infty} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

By the notation $U(z) \lessapprox V(z)$ (or equivalently $V(z) \simeq U(z)$) means that there is a constant $C$ such that $U(z) \leq CV(z)$ holds for all $z$ in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lessapprox V(z)$ and $V(z) \lessapprox U(z)$. 


We may note that there are many concrete examples of weight functions \( \varphi \) that satisfy the above smoothness and admissibility conditions. The power functions \( \varphi_\alpha(r) = r^\alpha, \ \alpha > 2 \), the exponential functions such as \( \varphi_\beta(r) = e^{\beta r}, \ \beta > 0 \), and the supper exponential functions \( \varphi(r) = e^{\epsilon r} \) are all typical examples of such weight functions.

Having set forth the conditions on \( \varphi \), we may now define the associated generalized Fock spaces \( \mathcal{F}_\varphi^p \) as spaces consisting of all entire functions \( f \) for which

\[
\|f\|_p = \int_{\mathbb{C}} |f(z)|^p e^{-p\varphi(z)} \, dA(z) < \infty,
\]

where \( dA \) denotes the usual Lebesgue area measure on \( \mathbb{C} \). These spaces have been studied in various contexts in the past years for instance in [2,6,12].

It has been known that the Laplacian \( \Delta \varphi \) of the weight function \( \varphi \) plays a significant role in the study of various operators on generalized Fock spaces. Now it is found that the structure of the symbol \( \psi \) inducing a bounded map \( C_\psi \) is totally determined based on the growth of \( \Delta \varphi \). More specifically, if \( C_\psi \) acts between two different generalized Fock spaces and \( \Delta \varphi(z) \to \infty \) as \( |z| \to \infty \), then one of our main result shows that \( C_\psi \) experiences a poorer boundedness structure than the case when the Laplacian is uniformly bounded. As a consequence of the growth of the Laplacian, we will in addition see that the space of all bounded composition operators \( C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q, \ p \neq q \) is connected under the operator norm topology. On the other hand, as will be seen in Sect. 2.3, the cyclicity, hypercyclicity and supercyclicity dynamical structures show no dependency on the growth of the Laplacian.

### 2 Main Results

In this section, we state the main results and defer their proofs for Sect. 4. Our first main result describes the bounded and compact composition operators acting between the generalized Fock spaces.

**Theorem 2.1** Let \( 0 < p, q < \infty \) and \( \psi \) be a nonconstant holomorphic map on the complex plane \( \mathbb{C} \). If

(i) \( p \neq q \), then the following statements are equivalent.

(a) \( C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q \) is bounded;
(b) \( C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q \) is compact;
(c) \( \psi(z) = az + b \) for some complex numbers \( a \) and \( b \) such that \( |a| < 1 \).

(ii) \( p = q \), then \( C_\psi : \mathcal{F}_\varphi^p \to \mathcal{F}_\varphi^q \) is

(a) bounded if and only if \( \psi(z) = az + b \) for some complex numbers \( a \) and \( b \) such that \( |a| \leq 1 \), and \( b = 0 \) whenever \( |a| = 1 \).
(b) compact if and only if \( \psi(z) = az + b \) for some complex numbers \( a \) and \( b \) such that \( |a| < 1 \).

Part (i) of the result shows a significance difference with the corresponding result for the case when \( C_\psi \) acts between the classical Fock spaces or generalized Fock
spaces where the weight function $\phi$ grows slower than the Gaussian weight function \cite{4,5,9,13–15}. On the other hand, part (ii) of the result simply extends the classical results. It is interesting to observe the sharp contrast between the cases when $p = q$ and $p \neq q$. Unlike the classical setting where these two cases have brought no different conditions, the boundedness structure gets poorer when the weight grows faster than the Gaussian case. Furthermore, apart from the fact that $p \neq q$, all of the results are independent of the size of the exponents $p$ and $q$ in the range $0 < p, q < \infty$.

### 2.1 Essential Norms and Schatten Class Membership

In this section, we turn our attention to the study of essential norms and Schatten class membership of composition operators on the spaces $\mathcal{F}_\psi^p$. For a bounded linear operator $T$ on a Banach space $\mathcal{H}$, we recall that the essential norm $\|T\|_e$ of $T$ is the norm of its equivalence classes in the Calkin algebra. In other words,

$$\|T\|_e = \inf \{\|T - K\|; K \text{ is a compact operator}\}.$$  

It follows that $\|T\|_e \leq \|T\|$ and $\|T\|_e = 0$ whenever $T$ is a compact operator. Computing the values of the norms and essential norms of composition operators is not an easy task and hence, not much is known on these values over various settings. In this section, we will estimate the values for $C_\psi$ on the spaces $\mathcal{F}_\psi^p$ for $p \geq 1$. For a noncompact operator $C_\psi$, we will indeed show that its essential norm $\|C_\psi\|_e$ is comparable to its operator norm $\|C_\psi\|$. In the Hilbert space setting $\mathcal{F}_\psi^2$, we have rather obtained the precise values of the norms, namely that $\|C_\psi\|_e = \|C_\psi\| = 1$.

If $C_\psi$ is a compact operator on $\mathcal{F}_\psi^2$, then it admits a Schmidt decomposition, that is there exist orthonormal bases $(e_n)_{n\in\mathbb{N}}$ and $(\sigma_n)_{n\in\mathbb{N}}$, and a sequence of nonnegative numbers $(\lambda_{n,\psi})_{n\in\mathbb{N}}$ with $\lambda_{n,\psi} \to 0$ as $n \to \infty$ such that for all $f$:

$$C_\psi f = \sum_{n=1}^{\infty} \lambda_{n,\psi}(f, e_n)\sigma_n.$$  

The operator $C_\psi$ with such a decomposition belongs to the Schatten $S_p(\mathcal{F}_\psi^2)$ class if and only if

$$\|C_\psi\|_{S_p}^p = \sum_{n=1}^{\infty} |\lambda_{n,\psi}|^p < \infty.$$  

We refer to \cite{19,20} for a more detailed account of the theory of Schatten classes. It turns out that all compact composition operators on $\mathcal{F}_\psi^2$ belong to the Schatten $S_p$ class for all $0 < p < \infty$. We may now summarize all the above observations and record them in the following theorem.

**Theorem 2.2** Let $0 < p < \infty$ and $\psi$ be a nonconstant holomorphic map on $\mathbb{C}$ that induces a bounded operator $C_\psi$ on $\mathcal{F}_\psi^p$. Then if $C_\psi$ is
(i) not compact on $\mathcal{F}_\varphi^p$ for $p \geq 1$, then its essential norm is comparable with its operator norm and

$$1 \geq \|C_\psi\|_e \simeq \|C_\psi\|. \quad (2.1)$$

On the Hilbert space $\mathcal{F}_\varphi^2$ case, we have equality and

$$\|C_\psi\|_e = \|C_\psi\| = 1. \quad (2.2)$$

(ii) compact on $\mathcal{F}_\varphi^2$, then it belongs to the Schatten $S_p(\mathcal{F}_\varphi^2)$ class for all $0 < p < \infty$. In the classical setting for $p = 2$, it has been proved [4] that the norm and essential norm of $C_\psi$ are equal and

$$\|C_\psi\|_e = \|C_\psi\| = 1, \quad (2.3)$$

where $C_\psi$ is noncompact bounded operator. The proof of (2.3) uses Hilbert spaces techniques based on an explicit expression of the reproducing kernel. In the current setting, such kind of an expression for the kernel function is still an open problem. Yet, we managed in circumventing this difficulty by using an asymptotic estimate of $\|K_z\|_2$ as $|z| \to \infty$ and arrive at (2.2). For $p \neq 2$, we will instead use another sequence of test functions where we only know the estimated values of the functions. It remains an interesting open problem to compute the precise values of the estimates in (2.1). On the other hand, results in [15] show that every compact composition operator acting on the classical Fock space belongs to the Schatten $S_p$ class for all positive $p$. In spirit of this, part (ii) of our theorem shows that the result remains valid in generalized Fock spaces generated by fast growing weight functions.

### 2.2 Normal and Unitary Composition Operators

In this section we characterize mappings $\psi$ which induce hyponormal and unitary composition operators $C_\psi$ on the spaces $\mathcal{F}_\varphi^2$. Recall that a bounded linear operator $T$ on a complex Hilbert space $\mathcal{H}$ is said to be hyponormal if $T^*T \geq TT^*$ where $T^*$ is the adjoint of $T$. The operator is normal if $TT^* = T^*T$, and unitary whenever $TT^* = T^*T = I$, where $I$ is the identity operator on $\mathcal{H}$. Note that while a hyponormal operator is a generalization of a normal operator, not all normal operators are unitary. Our next main result shows that only non compact composition operators on $\mathcal{F}_\varphi^2$ are unitary.

**Theorem 2.3** Let $\psi(z) = az + b$ induces a bounded composition operator $C_\psi$ on $\mathcal{F}_\varphi^2$. Then $C_\psi$ is

(i) unitary if and only if $|a| = 1$.
(ii) hyponormal if and only if it is normal.

On the classical Fock space, this equivalency was proved recently in [11]. Our result now shows that these properties are independent of the fast growth of the inducing weight function $\varphi$. 
An interesting related property is the notion of essentially normal. Recall that a bounded $C_\psi$ is essentially normal if the commutator $[C_\psi^*, C_\psi] = C_\psi^* C_\psi - C_\psi C_\psi^*$ is compact. Then, the following is an immediate consequence of Theorems 2.1 and 2.3.

**Corollary 2.4** Let $\psi(z) = az + b$ induces a bounded composition operator $C_\psi$ on $F_2^2$. Then $C_\psi$ is essentially normal.

By Theorem 2.1 either $|a| = 1$ in which case by Theorem 2.3, the operator is normal or $|a| < 1$ and the operator becomes compact. Since normal and compact operators are essentially normal, the corollary trivially holds.

### 2.3 Dynamics of the Composition Operators on $F^p_\varphi$

A bounded linear operator $T$ on a Banach space $\mathcal{H}$ is said to be cyclic if there exists a vector $x$ in $\mathcal{H}$ such that the linear span of its orbit under $T$,

$$\text{Orb}(T, x) := \{T^n x : n = 0, 1, 2, \ldots\},$$

is dense in $\mathcal{H}$. Such a vector $x$ is called cyclic for the operator $T$. The operator is hypercyclic if the orbit itself is dense in $\mathcal{H}$, and supercyclic if there exists a vector $x$ in $\mathcal{H}$ such that the projective orbit,

$$\text{Projorb}(T, x) := \{\lambda T^n x : \lambda \in \mathbb{C}, n = 0, 1, 2, \ldots\},$$

is dense in $\mathcal{H}$. Clearly any hypercyclic operator is cyclic, but the cyclic operators form a much larger class while supercyclicity is an intermediate property between the two. It is worth mentioning that if an operator $T$ has a hypercyclic vector, then each element in the orbit of such vector is also hypercyclic which implies that a hypercyclic operator has a dense set of hypercyclic vectors. For more information about hypercyclicity and supercyclicity, one may consult the monographs by Bayart and Matheron [1], and by Grosse–Erdmann and Peris Manguillot [8].

There have been much interest for a long time in studying these properties partly because of their relations to the famous invariant subspaces open problem which conjectures that every bounded linear operator on a Banach space has a non-trivial closed invariant subspace. On the other hand, the operator $C^n_\psi$ is itself a composition operator induced by the $n^{th}$ iterate of $\psi$,

$$C^n_\psi = C_{\psi^n}, \quad \psi^n = \underbrace{\psi \circ \psi \circ \psi \circ \cdots \circ \psi}_{n\text{-times}},$$

(2.4)

which obviously makes the study of the dynamical properties a natural subject. Furthermore, the relation in (2.4) indicates that the dynamical behavior of a composition operator is heavily dependent on the dynamical properties of its inducing map $\psi$.

In this section, we study the dynamical properties of the composition operator on $F_p$. We may first make the following simple observation, namely that no bounded
composition operator on $\mathcal{F}_p^p$ can be hypercyclic. To notice this, set $\psi(z) = az + b$ and observe that if $|a| < 1$, then the operator $C_\psi$ is compact and hence by Corollary 1.22 of [1], it can not be hypercyclic. On the other hand, if $|a| = 1$, we may deny the assertion and assume that the operator is hypercyclic with hypercyclic vector $f$. By extracting a subsequence $\psi_{n_k}$ such that $\psi_{n_k}z \to az$ as $k \to \infty$, we observe that applying (2.4) for any univalent function $g$ in the orbit of $f$

$$g(z) = \lim_{k \to \infty} C_{n_k} f(z) = \lim_{k \to \infty} C_{n_k} f(z) = f(az).$$

It follows that $f$ itself is a univalent function and hence its orbit contains only univalent functions which is a contradiction. Our next main result shows that the operator can not be supercyclic either.

**Theorem 2.5** Let $1 \leq p < \infty$ and $\psi(z) = az + b$ be a nonconstant map on $\mathbb{C}$ that induces a bounded composition operator $C_\psi$ on $\mathcal{F}_p^p$. Then $C_\psi$

(i) is cyclic on $\mathcal{F}_p^p$ if and only if $a^n \neq a$ for all $n > 1$. Furthermore, a function $h \in \mathcal{F}_p^p$ with Taylor series expansion

$$h(z) = \sum_{n=0}^{\infty} a_n \left( \frac{z - b}{1 - a} \right)^n$$

is cyclic vector for $C_\psi$ if and only if $a_n \neq 0$ for all $n \in \mathbb{Z}_+ := \{0, 1, 2, 3, \ldots\}$.

(ii) can not be supercyclic on $\mathcal{F}_p^p$.

The cyclicity and supercyclicity problems have not been solved in the classical Fock spaces settings either except for the Hilbert space case which were studied respectively in [9] and [11]. As can be seen in Sect. 4, our approach, which neither uses Hilbert spaces techniques nor the fast growth property of the weight function $\varphi$, shows that the same result holds for all $p$ on the classical spaces as well.

### 2.4 Connected Components and Isolated Points of the Space of Composition Operators

In the present section we consider some topological structures of bounded compositions operators $C_\psi : \mathcal{F}_p^p \to \mathcal{F}_q^q$ for all $0 < p, q < \infty$. We denote by $C(\mathcal{F}_p^p, \mathcal{F}_q^q)$ the space of such operators equipped with the operator norm topology. The first natural question to pose in this direction is when the difference of two elements from $C(\mathcal{F}_p^p, \mathcal{F}_q^q)$ becomes compact. It turns out that the difference is compact if and only if both of the operators are compact. Another natural point of interest is to identify the isolated and connected components of the space $C(\mathcal{F}_p^p, \mathcal{F}_q^q)$ which we give a complete identification below.

**Theorem 2.6** (i) Let $0 < p, q < \infty$ and $C_\psi$ be in $C(\mathcal{F}_p^p, \mathcal{F}_q^q)$. If

(a) $p \neq q$, then the space $C(\mathcal{F}_p^p, \mathcal{F}_q^q)$ is connected.
(b) \( p = q \), then \( C_\psi \) is isolated if and only if it is not compact. In this case, the set of all compact composition operators on \( F_p^\psi \) is a connected component of \( C(F_p^\psi, F_p^\psi) \).

(ii) Let \( 0 < p < \infty \) and \( C_{\psi_1}, C_{\psi_2} \in C(F_p^\psi, F_p^\psi) \) where \( \psi_1 \neq \psi_2 \). Then

(a) \( C_{\psi_1} - C_{\psi_2} \) is compact on \( F_p^\psi \) if and only if both \( C_{\psi_1} \) and \( C_{\psi_2} \) are compact.

(b) if \( C_{\psi_1} - C_{\psi_2} \) is compact on \( F_q^\psi \), then it belongs to the Schatten \( S_p(F_q^\psi) \) class for all \( p \).

Observe that the validity of the result in part (a) of (i) is dependent on the fast growth of the weight function \( \varphi \) while part (b) does not. Part (a) of (ii) shows that the cancellation property of the inducing maps plays no roll for compactness of the difference. On the contrary, it is worth mentioning that compactness of the differences of two composition operators on the weighted Bergman spaces over the unit disc has been characterized by some suitable cancellation property of the inducing symbols at each boundary points \([17]\). Such property makes it possible for each composition operator in the difference not necessarily to be compact.

A natural question following Theorem 2.6 is whether every isolated composition operator in \( C(F_p^\psi, F_p^\psi) \) is still isolated under the essential norm topology which is weaker than the topology induced by the operator norm. Our next main result shows this is in deed the case.

**Theorem 2.7** Let \( 1 \leq p < \infty \). Then a composition operator \( C_\psi \) in \( C(F_p^\psi, F_p^\psi) \) is essentially isolated if and only if it is isolated.

The isolated and essentially isolated points of the space of the operators on the classical Fock spaces have not been also identified as far as we know. As will be seen in Sect. 4.6, the method we use to prove Theorem 2.7 can be easily adopted to the classical setting. In stead of using the sequence of the functions \( f_w^{\ast}, R \), one can use the sequence of the normalized reproducing kernels to conclude the analogous results.

### 3 Preliminaries and Auxiliary Results

Here we collect background materials and present some auxiliary results which will be used to prove our main results in the sequel. Our first lemma shows that every symbol \( \psi \) that induces a bounded operator on generalized Fock spaces fixes a point in the complex plane.

**Lemma 3.1** Let \( 0 < p, q < \infty \) and \( \psi = az + b \) induces a bounded composition operator \( \psi : F_p^\psi \rightarrow F_q^\psi \). Then \( C_\psi \) has a fixed point.

**Proof** By Theorem 2.1, boundedness implies that \( |a| \leq 1 \) and \( b = 0 \) whenever \( |a| = 1 \). Thus, in the case when \( a = 1 \), \( \psi \) fixes the origin. On the other hand, if \( a \neq 1 \), then \( \psi \) fixes the point \( b/(1-a) \).

In the next section we will see that the fixed point behaviour of the map \( \psi \) plays an important role in proving our supercyclicity result.
Another important ingredient in our subsequent consideration is the following. By Proposition A and Corollary 8 of [6] where the original idea comes from [2], for a sufficiently large positive number $R$, there exists a number $\eta(R)$ such that for any $w \in \mathbb{C}$ with $|w| > \eta(R)$, there exists an entire function $f(w, R)$ such that

$$|f(w, R)(z)|e^{-\varphi(z)} \leq C \min \left\{ 1, \left( \frac{\min\{\tau(w), \tau(z)\}}{|z - w|} \right)^{\frac{R^2}{2}} \right\}$$

(3.1)

for all $z$ in $\mathbb{C}$, and for some constant $C$ that depends on $\psi$ and $R$. In particular, when $z$ belongs to $D(w, R \tau(w))$, the estimate becomes

$$|f(w, R)(z)|e^{-\varphi(z)} \simeq 1,$$

(3.2)

where $D(a, r)$ denotes the Euclidean disk centered at $a$ and radius $r > 0$. Furthermore, the functions $f(w, R)$ belong to $\mathcal{F}_\varphi^p$ for all $p$ with norms estimated by

$$\|f(w, R)\|_p^p \simeq \tau(w)^2, \quad \eta(R) \leq |w|.$$  

(3.3)

For $p = 2$, the space $\mathcal{F}_\varphi^2$ is known to be a reproducing kernel Hilbert space. An explicit expression for the kernel function is still an interesting open problem. However, an asymptotic estimation of the norm

$$\|K_w\|_2^2 \simeq \tau(w)^{-2}e^{2\varphi(w)}.$$  

(3.4)

holds for all $w \in \mathbb{C}$. Furthermore, for subharmonic functions $\varphi$ and $f$, it also holds a local pointwise estimate

$$|f(z)|^p e^{-\beta \varphi(z)} \lesssim \frac{1}{\sigma^2 \tau(z)^2} \int_{D(z, \sigma \tau(z))} |f(w)|^p e^{-\beta \varphi(w)} dA(w)$$

(3.5)

for all finite exponent $p$, any real number $\beta$, and a small positive number $\sigma$: see Lemma 7 of [6] for more details.

### 3.1 Density of the Complex Polynomials in $\mathcal{F}_\varphi^p$

The various dynamical property of operators are closely related to the density of polynomials in various functional spaces; see for example [3,9,18]. This fact will be an important tool in proving part (i) of Theorem 2.5 in our current setting too. Thus, we shall first present the following density result.

**Lemma 3.2** Suppose $0 < p < \infty$ and $f \in \mathcal{F}_\varphi^p$. Then there is a sequence of polynomials $(P_n)$ such that $\|P_n - f\|_p \to 0$ as $n \to \infty$.

This lemma was first proved in [6, Theorem 28]. We provide here another proof using the notions of inclusion and dilation techniques.
Proof For $f \in \mathcal{F}_p^\psi$ and $0 < r < 1$, define a sequence of dilation functions $f_r$ by $f_r(z) = f(rz)$. Then it suffices to show that $\|f_r - f\|_p \to 0$ as $r \to 1^-$ and $\|f_r - P_n\|_p \to 0$ as $n \to \infty$ where $P_n$ is a sequence of complex polynomials. To show the first, we may compute

$$\|f_r\|_p^p = \int_\mathbb{C} |f(rz)|^p e^{-p\psi(z)} dA(z) = \frac{1}{r^2} \int_\mathbb{C} |f(w)|^p e^{-p\psi(r^{-1}w)} dA(w)$$

$$= \frac{1}{r^2} \int_\mathbb{C} |f(w)|^p e^{-p\psi(w)} e^{-p(\psi(r^{-1}w) + \psi(w))} dA(w).$$

Since $\psi$ is an increasing radial function and $0 < r < 1$ we have $e^{-p(\psi(r^{-1}w) - \psi(w))} \leq 1$ for all $w \in \mathbb{C}$. Applying Lebesgue dominated convergence theorem,

$$\lim_{r \to 1^-} \|f_r\|_p^p = \lim_{r \to 1^-} \frac{1}{r^2} \int_\mathbb{C} |f(w)|^p e^{-p\psi(w)} \left(e^{-p(\psi(r^{-1}w) - \psi(w))}\right) dA(w) = \|f\|_p^p,$$

showing that $\|f_r\|_p \to \|f\|_p$ and hence $f_r(z) \to f(z)$ as $r \to 1^-$. Therefore,

$$\lim_{r \to 1^-} \|f_r - f\|_p = 0. \quad (3.6)$$

Next, we fix some $r \in (0, 1), \alpha \in (r^2, \frac{1}{2})$ and proceed to show that $f_r \in \mathcal{F}_{(\psi, \alpha)}^2$ where $\mathcal{F}_{(\psi, \alpha)}^2 \subset \mathcal{F}_p^\psi$ and

$$\mathcal{F}_{(\psi, \alpha)}^2 := \left\{ f \text{ entire : } \|f\|_{(2, \alpha)}^2 = \int_\mathbb{C} |f(z)|^2 e^{-2\alpha\psi(z)} dA(z) < \infty \right\}.$$

To prove the first, we may apply (3.5) and estimate

$$\|f_r\|_{(2, \alpha)}^2 = \int_\mathbb{C} |f(rw)|^2 e^{-2\alpha\psi(w)} dA(w) \leq \|f\|_p^2 \int_\mathbb{C} \frac{1}{\tau(rw)^{\frac{4}{p}}} e^{2\varphi(wr) - 2\alpha\psi(w)} dA(w).$$

By definition of $\tau$ and $\varphi$, we also observe that

$$\frac{1}{\tau(rw)^{\frac{4}{p}}} \lesssim e^{\varphi(wr)} \text{ as } |w| \to \infty.$$

Taking this into account and the fact that $\alpha > r^2$ we further estimate

$$\int_\mathbb{C} \frac{1}{\tau(rw)^{\frac{4}{p}}} e^{2\varphi(wr) - 2\alpha\psi(w)} dA(w) \lesssim \int_\mathbb{C} e^{4\varphi(wr) - 2\alpha\psi(w)} dA(w)$$

$$\lesssim \int_\mathbb{C} e^{2r^2\varphi(w) - 2\alpha\psi(w)} dA(w) < \infty,$$
here we used the fact that \( \varphi \) grows faster than the classical function \( |z|^2/2 \) and hence \( \varphi(rw) \lesssim \frac{r^2}{2} \varphi(w) \) whenever \( |w| \to \infty \).

For the inclusion property, we consider \( h \in \mathcal{F}^2_{(\varphi, \alpha)} \) and applying (3.5) again and proceed to estimate

\[
\int_{\mathbb{C}} |h(z)|^p e^{-p\varphi(z)} dA(z) \lesssim \|h\|^p_{(2, \alpha)} \int_{\mathbb{C}} e^{p\alpha \varphi(z) - p\varphi(z)} \tau(z)^p dA(z)
\leq \|h\|^p_{(2, \alpha)} \int_{\mathbb{C}} e^{2p\alpha \varphi(z) - p\varphi(z)} dA(z) \lesssim \|h\|^p_{(2, \alpha)}.
\]

Now, since the set of all holomorphic complex polynomials is dense in the Hilbert space \( \mathcal{F}^2_{\varphi} \), taking \( P_n \) be the \( n^{th} \) Taylor polynomial of \( f_r \), we deduce from the inclusion property that

\[
\|f_r - P_n\|_p \leq C\|f_r - P_n\|_{(2, \alpha)} \to 0
\]
as \( n \to \infty \). From this and (4.5), the result follows. \( \square \)

The next lemma will find application in proving the equality of the norm and essential norm of the composition operator when it acts on the generalized Hilbert space \( \mathcal{F}^2_{\varphi} \).

**Lemma 3.3** The normalized reproducing kernel \( K_z/\|K_z\|_2 \) converges weakly to \( 0 \) in \( \mathcal{F}^2_{\varphi} \) when \( |z| \to \infty \).

**Proof** The sequence \( e_n(z) = z^n/\|z^n\|_2, \ n \geq 0 \) represents the standard orthonormal basis for \( \mathcal{F}^2_{\varphi} \). It means that holomorphic polynomials are dense in \( \mathcal{F}^2_{\varphi} \), and hence suffices to show that for any nonnegative integer \( m \)

\[
\left\langle \left\{ w^m, \frac{K_z}{\|K_z\|_2} \right\} \right\rangle = \frac{|z|^m}{\|K_z\|_2} \to 0, \ |z| \to \infty.
\]

But this holds trivially as

\[
\|K_z\|_2^2 = \sum_{n=0}^{\infty} |e_n(z)|^2 = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\|z^n\|_2^2},
\]

which is a power series on \( |z|^2 \) with positive coefficients. \( \square \)

We close this section with a lemma that will be used to prove Theorems 2.5 and 2.6 in the next section

**Lemma 3.4** Let \( 0 < p, q < \infty \), \( \psi_n(z) = a_n z + b_n \), and \( \psi(z) = az + b \) where \( (a_n) \) and \( (b_n) \) are sequences of complex numbers such that \( 0 \leq |a_n| \leq 1 \) for all \( n \), \( a_n \to a \) and \( b_n \to b \) as \( n \to \infty \). Then for any \( f \in \mathcal{F}^p_{\varphi} \) and \( C\psi, C\psi_n \in C(\mathcal{F}^p_{\varphi}, \mathcal{F}^q_{\varphi}) \)

\[
\lim_{n \to \infty} \|C\psi_n f - C\psi f\|_q = 0.
\] (3.7)
Proof If \( a_n = 0 \) for all \( n \), then the lemma trivially follows. Thus, assuming \( 0 < |a| \leq 1 \), we compute

\[
\|C_{\psi_n} f\|_q^q = \int_{\mathbb{C}} |f(a_n z + b_n)|^q e^{-q \varphi(z)} dA(z)
\]

\[
= \int_{\mathbb{C}} |f(a_n z + b)|^q e^{-q \varphi(a_n z + b_n)} \left( e^{q \varphi(a_n z + b_n)} - q \varphi(z) \right) dA(z)
\]

\[
= \int_{\mathbb{C}} |f(w)|^q e^{-q \varphi(w)} \left( |a_n|^{-2} e^{q \varphi(w)} - q \varphi((w - b_n)/a_n) \right) dA(w).
\]

Since \( |a_n| \leq 1 \), the quantity \( e^{q \varphi(w)} - q \varphi((w - b_n)/a_n) \) is uniformly bounded on \( \mathbb{C} \). Applying Lebesgue dominated convergence theorem and smoothness of the weight function \( \varphi \), we obtain

\[
\lim_{n \to \infty} \|C_{\psi_n} f\|_q^q = \lim_{n \to \infty} \int_{\mathbb{C}} |f(w)|^q e^{-q \varphi(w)} \left( |a_n|^{-2} e^{q \varphi(w)} - q \varphi((w - b_n)/a_n) \right) dA(w)
\]

\[
= \int_{\mathbb{C}} |f(w)|^q e^{-q \varphi(w)} \left( |a|^{-2} e^{q \varphi(w)} - q \varphi((w - b)/a) \right) dA(w)
\]

\[
= \int_{\mathbb{C}} |f(az + b)|^q e^{-q \varphi(z)} dA(z) = \|C_{\psi} f\|_q^q
\]

from which (3.7) follows. \( \square \)

4 Proof of the Main Results

We now turn to the proofs of the main results.

4.1 Proof of Theorem 2.1

We may first assume that \( 0 < p, q < \infty \) and reformulate the boundedness and compactness properties in terms of embedding properties between \( \mathcal{F}_p^p \) and \( \mathcal{F}_q^q \). We set a pullback measure \( \mu(\psi, q) \) on \( \mathbb{C} \) as

\[
\mu(\psi, q)(E) = \int_{\psi^{-1}(E)} e^{-q \varphi(w)} dA(w)
\]

for every Borel subset \( E \) of \( \mathbb{C} \). Then we observe

\[
\|C_{\psi} f\|_q^q = \int_{\mathbb{C}} |f(\psi(z))|^q e^{-q \varphi(z)} dA(z) = \int_{\mathbb{C}} |f(z)|^q d\mu(\psi, q)(z).
\]

From this, it follows that \( C_{\psi} : \mathcal{F}_p^p \to \mathcal{F}_q^q \) is bounded if and only if the embedding map \( i_d : \mathcal{F}_p^p \to L^q(\mu(\psi, q)) \) is bounded. To study this reformulation further, we may consider first part (i) of the theorem along the following two cases:
Case 1: Assume $0 < p < q < \infty$. By Theorem 1 of [6], the map $i_d : \mathcal{F}_\varphi^p \to L^q(\mu(\psi,q))$ is bounded if and only if for some $\delta > 0$,

$$
\sup_{w \in \mathbb{C}} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta \tau(w))} \epsilon^{\varphi(z)} d\mu(\psi,q)(z) < \infty.
$$

Using (4.1), we may rewrite this condition again as

$$
I := \sup_{w \in \mathbb{C}} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta \tau(w))} \epsilon^{\varphi(z)} d\mu(\psi,q)(z)
= \sup_{w \in \mathbb{C}} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta \tau(w))} \epsilon^{\varphi(z) - \varphi(\psi^{-1}(z))} dA(\psi^{-1}(z)) < \infty. \tag{4.3}
$$

Having singled out this equivalent reformulation, the next task is to examine condition (4.3) and arrive at the assertion of the theorem. Let us first assume that (4.3) holds and show that

$$
\psi(z) = az + b
$$

for some $|a| < 1$. Applying (3.5) and estimating further on the right-hand side of (4.3) gives

$$
I \geq \tau(\psi(w))^{\frac{2(p-q)}{p}} \epsilon^{\varphi(\psi(w)) - \varphi(w)}. \tag{4.4}
$$

for all $w$ in $\mathbb{C}$ which implies

$$
\tau(\psi(w))^{\frac{2(q-p)}{p}} \geq \epsilon^{\varphi(\psi(w)) - \varphi(w)}. \tag{4.5}
$$

We claim that

$$
\limsup_{|w| \to \infty} (\varphi(\psi(w)) - \varphi(w)) < 0.
$$

If not, then there exists a sequence $w_j \in \mathbb{C}$ such that $|w_j| \to \infty$ as $j \to \infty$ and

$$
\limsup_{j \to \infty} \varphi(\psi(w_j)) - \varphi(w_j) \geq 0.
$$

This along with (4.5) and applying the admissibility assumptions on (1.1), and the fact that $\psi$ is a nonconstant entire function, we get

$$
0 = \limsup_{j \to \infty} \tau(\psi(w_j))^{\frac{2(q-p)}{p}} \geq \limsup_{j \to \infty} \epsilon^{\varphi(\psi(w_j)) - \varphi(w_j)}.
\epsilon^{\limsup_{j \to \infty} \varphi(\psi(w_j)) - \varphi(w_j)) \geq 1,
$$

which is a contradiction. By the growth assumption on $\psi$ and (4.5) we see that $\psi(z) = az + b$ for some $a, b$ in $\mathbb{C}$ and $|a| < 1$.\]
Next, we assume that $\psi$ has the above linear form with $|a| < 1$, and proceed to show that $C_\psi$ is a compact map. Using the preceding embedding formulation and Theorem 1 of [6], $C_\psi : \mathcal{F}_\psi^p \rightarrow \mathcal{F}_\psi^q$ is compact if and only if
\[
\lim_{|w| \to \infty} \frac{1}{\tau(w)^{2q/p}} \int_{D(w, \delta \tau(w))} e^{q \psi(z) - q \psi(\psi^{-1}(z))} dA(\psi^{-1}(z)) = 0. \tag{4.6}
\]
Since $|a| < 1$, the integrand above is a decaying function. Thus,
\[
\frac{1}{\tau(w)^{2q/p}} \int_{D(w, \delta \tau(w))} e^{q \psi(z) - q \psi(\psi^{-1}(z))} dA(\psi^{-1}(z)) \lesssim \frac{\tau(w)^2}{\tau(w)^{2q/p}} e^{q \psi(w) - q \psi(\psi^{-1}(w))} = \tau(a z + b)^{\frac{2p-2q}{p}} e^{q \psi(a z + b) - q \psi(z)}. \tag{4.7}
\]
By definition of $\varphi$ and $\tau$, we notice that the last quantity in (4.7) tends to zero as $|w| \to \infty$ and hence (4.6) holds. Since compactness obviously implies boundedness, we are finished with the proof for the case $p < q$.

**Case 2:** $0 < q < p < \infty$. Invoking the reformulation in (4.2) again, $C_\psi : \mathcal{F}_\psi^p \rightarrow \mathcal{F}_\psi^q$ is bounded (compact) if and only if the embedding map $i_d : \mathcal{F}_\psi^p \rightarrow L^q(\mu(\psi, q))$ is bounded (compact). By Theorem 1 of [6], boundedness or compactness of $i_d$ holds if and only if for some $\delta > 0$,
\[
T(z) := \frac{1}{\tau(z)^2} \int_{D(z, \delta \tau(z))} e^{q \psi(w)} d\mu(\psi, q)(w)
\]
belongs to $L^{\frac{p}{p-q}}(\mathbb{C}, dA)$. We plan to show that this holds if and only if $\psi(w) = az + b$ with $|a| < 1$. Assuming the latter and applying Hölder’s inequality
\[
\int_{\mathbb{C}} |T(z)|^{\frac{p}{p-q}} dA(z) = \int_{\mathbb{C}} \left( \frac{1}{\tau(z)^2} \int_{D(z, \delta \tau(z))} e^{q \psi(w)} dA(\psi^{-1}(w)) \right)^{\frac{p}{p-q}} dA(z)
\]
\[
\lesssim \int_{\mathbb{C}} \tau(z)^{-2} \int_{D(z, \delta \tau(z))} e^{\frac{qp}{p-q} \psi(w)} dA(\psi^{-1}(w)) dA(z) =: T_1
\]
Since $w \in D(z, \delta \tau(z))$, by Lemma 5 of [6] there exists a positive constant $C$ with
\[
\frac{1}{C} \tau(w) \leq \tau(z) \leq C \tau(w).
\]
Then, for any $\zeta \in D(z, \delta \tau(z))$
\[
|\zeta - w| \leq |\zeta - z| + |z - w| \leq 2\delta \tau(z) \leq 2\delta C \tau(w) = \beta \tau(w), \quad \beta := 2\delta C.
\]
This shows that $D(z, \delta \tau(z)) \subset D(w, \beta \tau(w))$ which together with Fubini’s theorem and Lemma 5 of [6] again imply

$$T_1 = \int_{\mathbb{C}} \tau(z)^{-2} \int_{\mathbb{C}} \chi_{D(z, \delta \tau(z))}(w) \frac{\frac{d\mu}{\rho - q} \phi(w)}{e^{\rho \phi(\psi^{-1}(w))}} dA(\psi^{-1}(w)) dA(z)$$

$$\leq \int_{\mathbb{C}} \frac{\frac{d\mu}{\rho - q} \phi(w)}{e^{\rho \phi(\psi^{-1}(w))}} \left( \int_{\mathbb{C}} \chi_{D(w, \beta \tau(w))}(z) \tau(z)^{-2} dA(z) \right) dA(\psi^{-1}(w))$$

$$= \int_{\mathbb{C}} \frac{\frac{d\mu}{\rho - q} \phi(w)}{e^{\rho \phi(\psi^{-1}(w))}} \left( \int_{D(w, \beta \tau(w))} \tau(z)^{-2} dA(z) \right) dA(\psi^{-1}(w))$$

$$\gtrsim \int_{\mathbb{C}} \frac{\frac{d\mu}{\rho - q} \phi(w)}{e^{\rho \phi(\psi^{-1}(w))}} dA(\psi^{-1}(w)) < \infty.$$ 

On the other hand, if $T$ is $L^\frac{\mu}{\rho - q}$ integrable over $\mathbb{C}$, then $C_\psi : \mathcal{F}_p^\mu \to \mathcal{F}_q^\mu$ is bounded, and applying $C_\psi$ to the sequence of test functions $f_{(w, R)}$ and using a weaker version of the point estimate in (3.5)

$$\|f_{(w, R)}\|_p \gtrsim \|C_\psi f_{(w, R)}\|_q \gtrsim |f_{(w, R)}(\psi(z))| \tau(z)^{\frac{\mu}{\rho - q}} e^{-\phi(z)}$$

for all points $w, z \in \mathbb{C}$. Setting, in particular, $w = \psi(z)$ and invoking the estimates in (3.2) and (3.3) gives

$$\tau(\psi(z))^{\frac{\mu}{\rho - q}} \gtrsim \tau(z)^{\frac{\mu}{\rho - q}} e^{\phi(\psi(z)) - \phi(z)}.$$ 

(4.8)

Since $\psi$ is a nonconstant entire function, the left-hand side of (4.8) tends to zero as $|z| \to \infty$. So does the right-hand side and that happens only if

$$\sup_{z \in \mathbb{C}} e^{\phi(\psi(z)) - \phi(z)} < \infty.$$ 

(4.9)

Arguing as in the proof of the corresponding part in case 1, we observe that (4.9) holds only if $\psi$ has a linear form $\psi(z) = az + b$ with $|a| \leq 1$ and $b = 0$ whenever $|a| = 1$. We further claim that $|a| < 1$. If not, using again the $L^\frac{\mu}{\rho - q}$ integrability of $T$

$$\int_{\mathbb{C}} |T(z)|^{\frac{\mu}{\rho - q}} dA(z) = \int_{\mathbb{C}} \tau(z)^{-2} \left( \int_{D(z, \delta \tau(z))} \frac{e^{\rho \phi(w)}}{e^{\rho \phi(\psi^{-1}(w))}} dA(\psi^{-1}(w)) \right)^{\frac{\mu}{\rho - q}} dA(z)$$

$$\gtrsim \int_{\mathbb{C}} \tau(z)^{-2} \left( \int_{D(z/\alpha, \delta \tau(z))} \frac{e^{\rho \phi(aw)}}{e^{\rho \phi(\psi(w))}} dA(w) \right)^{\frac{\mu}{\rho - q}} dA(z)$$

$$= \int_{\mathbb{C}} \tau(z)^{-2} \tau(z/\alpha)^{\frac{2\mu}{\rho - q}} dA(z) = \infty$$

which is a contradiction as $\tau(z/\alpha) = \tau(z)$ whenever $|a| = 1$. 

(ii) The corresponding proofs for this part follows rather easily by simply setting $p = q$ in the arguments made in part (i). Thus, we skip it.

### 4.2 Proof of Theorem 2.2

(i) If $C_\psi$ is bounded but not compact, then by Theorem 2.1, $\psi(z) = az$ where $|a| = 1$. Consequently, $\varphi(\psi(z)) = \varphi(az) = \varphi(|az|) = \varphi(z)$. With this, we find an upper bound for the norm of the operator

$$
\|C_\psi f\|_p^p = \int_C \left| \frac{f(\psi(z))}{e^{p\varphi(z)}} \right|^p dA(z) \leq \sup_{z \in C} \left( e^{p\varphi(\psi(z)) - p\varphi(z)} \right) \int_C \left| \frac{f(\psi(z))}{e^{p\varphi(\psi(z))}} \right|^p dA(z)
$$

$$
= \sup_{z \in C} e^{p\varphi(\psi(z)) - p\varphi(z)} \|f\|_p^p = \|f\|_p^p.
$$

Therefore,

$$
1 \geq \|C_\psi\| \geq \|C_\psi\|_e. \quad (4.10)
$$

A common way to prove lower bounds for essential norms is to find a suitable weakly null sequence of functions $f_n$ and use the fact that

$$
\|C_\psi\|_e \geq \limsup_{n \to \infty} \|C_\psi f_n\|_p. \quad (4.11)
$$

On classical Fock spaces, the sequence of the reproducing kernels does this job. Since no explicit expression is known for the kernel function in our current setting, we will instead use the sequence of functions

$$
f^*_n(w, R) = f(w, R) / \|f(w, R)\|_p \quad (4.12)
$$

as described by the properties in (3.1), (3.2), and (3.3). Obviously, the sequence $f^*_n$ is uniformly bounded, and due to the relation in (3.1), $f^*_n \to 0$ uniformly on compact subset of $\mathbb{C}$ as $|w| \to \infty$. Thus, $f^*_n \to 0$ weakly as $|w| \to \infty$. With this, we proceed to make further estimates on the right-hand side of the norm in (4.11).

Making use of (3.5) for some small positive number $\delta$

$$
\|C_\psi\|_e \geq \limsup_{|w| \to \infty} \|C_\psi f^*_n((\psi(w), R))\|_p
$$

$$
\simeq \limsup_{|w| \to \infty} \frac{1}{\tau(w)} \left( \int_C |f((\psi(w), R)(\psi(z)))|^{p} e^{-p\varphi(z)} dA(z) \right)^{\frac{1}{p}}
$$

$$
\geq \limsup_{|w| \to \infty} \frac{1}{\tau(w)} \left( \int_{D(\psi(w), \delta \tau(\psi(w)))} |f((\psi(w), R)(\psi(z)))|^{p} e^{-p\varphi(\psi(z))} dA(z) \right)^{\frac{1}{p}}
$$
\[
\begin{align*}
\vartriangleleft \limsup_{|w| \to \infty} & \frac{\tau(\psi(w))^2}{\tau(w)^2} \left| f_{\left(\psi(w), R\psi(w)\right)} \right| P e^{-p\varphi(\psi(w))} \\
\simeq \limsup_{|w| \to \infty} & \frac{\tau(\psi(w))^2}{\tau(w)^2} = \limsup_{|w| \to \infty} \frac{\tau(w)^2}{\tau(w)^2} = 1
\end{align*}
\]

which completes the proof of the lower estimate.

For the Hilbert space case, applying Lemma 3.3, we have

\[
\|C_{\psi}\|_e \geq \limsup_{|w| \to \infty} \left\| K_w \right\|^{-1} C_{\psi} K_w
\]

\[
= \limsup_{|w| \to \infty} \left\| K_w \right\|^{-1/2} \left( \int_{\mathbb{C}} |K_w(\psi(z))|^2 e^{-2\varphi(z)} dA(z) \right)^{1/2}
\]

\[
= \limsup_{|w| \to \infty} \left\| K_w \right\|^{-1/2} \left( \int_{\mathbb{C}} |K_w(az)|^2 e^{-2\varphi(az)} dA(z) \right)^{1/2} = 1,
\]

from which and (4.10) we arrive at the asserted equality.

(ii) Since Schatten class membership has the nested property in the sense that \( S_p \subseteq S_q \) for \( p \leq q \), it suffices to verify the theorem only for the case when \( p \) is in the range \( 0 < p < 2 \). Recall that a compact operator \( T \) belongs to the Schatten \( S_p \) class if and only if the positive operator \( (T^* T)^{p/2} \) belongs to the trace class \( S_1 \). Furthermore, \( T \in S_p \) if and only if \( T^* \in S_p \), and \( \|T\|_{S_p} = \|T^*\|_{S_p} \). Thus, we may estimate the trace of \( (C_{\psi} C_{\psi}^*)^{p/2} \) by

\[
\text{tr}\left((C_{\psi} C_{\psi}^*)^{p/2}\right) = \int_{\mathbb{C}} \left| C_{\psi} C_{\psi}^* k_z \right|^p dA(z) \leq \int_{\mathbb{C}} \left| C_{\psi} C_{\psi}^* k_z \right|^p dA(z)
\]

\[
= \int_{\mathbb{C}} \|C_{\psi}^* k_z\|^p dA(z), \quad (4.13)
\]

where the inequality holds since \( 0 < p \leq 2 \), \( C_{\psi} C_{\psi}^* \) is a positive operator, and \( k_z = K_z/\|K_z\|_2 \) is a unit norm vector, see [20, Proposition 1.31]. On the other hand, by the reproducing property of the kernel function, we have the adjoint property

\[
C_{\psi}^* K_w(z) = \left\langle C_{\psi}^* K_w, K_z \right\rangle = \left\langle K_w, C_{\psi} K_z \right\rangle = \left\langle C_{\psi} K_z, K_w \right\rangle = K_{\psi(w)}(z).
\]

From this estimate and (3.4), we have that

\[
\|C_{\psi}^* k_w\|_2 \simeq \frac{\tau(w)}{\tau(\psi(w))} e^{\varphi(\psi(w)) - \varphi(w)}.
\]
This along with (4.13) and compactness of $C_\psi$ implies

\[
\text{tr}\left( (C_\psi C_\psi^*)^2 \right) \leq \int_C \left( \frac{\tau(w)}{\tau(\psi(w))} \right)^p e^{p(\psi(\psi(w)) - \psi(w))} dA(z) \\
= \int_C \left( \frac{\tau(w)}{\tau(aw + b)} \right)^p e^{p(\psi(\psi(w)) - \psi(w))} dA(z) \\
\lesssim \int_C e^{p(\psi(\psi(w)) - \psi(w))} dA(z) < \infty,
\]

from which and condition (4.13), we conclude that $\text{tr}\left( (C_\psi C_\psi^*)^2 \right)$ is finite.

### 4.3 Proof of Theorem 2.3

(i) Suppose $C_\psi$ is unitary. Applying the adjoint property for each $z \in C$

\[
||K_z||_2^2 = ||C_\psi K_z||_2^2 = ||C_\psi^* K_z||_2^2 = ||K_\psi(z)||_2^2.
\]

Considering the asymptotic relation in (3.4), we further have

\[
e^{2\phi(z)} \tau(z)^2 \sim e^{2\phi(az+b)} \tau(az+b)^2
\]

By definition of $\tau$ and the admissibility condition on the weight function $\phi$, the above estimate holds for $|z| \to \infty$ only if $b = 0$ and $|a| = 1$.

Conversely, if $C_\psi = az$, with $|a| = 1$, then we need to show that $C_\psi$ is surjective and preserves the inner product on $F_\psi^2$. Thus, for each $f, g$ in $F_\psi^2$:

\[
\langle C_\psi f, C_\psi g \rangle = \int_C f(az) \overline{g(az)} e^{-2\phi(z)} dA(z) \\
= \frac{1}{|a|^2} \int_C f(w) \overline{g(w)} e^{-2\phi(w)} dA(w) = \langle f, g \rangle.
\]

which shows that the operator preserves the inner product. It remains to show that the operator is also surjective. But this follows easily since $C_\psi^{-1} = C_\psi^{-1}$ exists in this case. Thus, $C_\psi$ is unitary.

(ii) Obviously a normal operator is hyponormal. Thus, assume that $C_\psi$ is hyponormal.

If $C_\psi$ is compact in addition, then it is normal. On the other hand, if $C_\psi$ is not compact, then $\psi(z) = az$, $|a| = 1$. By (i), it follows that $C_\psi$ is unitary in this case hence normal again.

### 4.4 Proof of Theorem 2.5

**Part (i).** As pointed earlier, this part of the theorem was proved for the special case $p = 2$ and $\phi(z) = z^s$, $s \leq 1$ in [Theorem 4.2] [9]. The proof in [9] is based on Hilbert
Let us first assume that $C_\psi$ is cyclic and prove the necessity of the condition. Arguing on the contrary, if $a^k = a$ for some $k \geq 2$, then $|a| = 1$ and hence $\psi(z) = az$. For any cyclic vector $f_0$ in $F_\psi$, it follows that $C_\psi^k f_0(z) = f_0(a^k z) = f_0(az) = C_\psi f_0(z)$ which implies

$$\{C_\psi^n f_0, n \in \mathbb{Z}_+\} = \{C_\psi^n f_0 : n = 0, 1, 2, 3, \ldots k\}.$$ 

This shows that the closed linear span of the orbit is finite dimensional, and hence $C_\psi$ can not be cyclic.

Conversely, suppose $\psi(z) = az + b$ and $a^n \neq a$ for every $n \geq 2$ which obviously implies that $a \neq 1$. Then we proceed to show that there exists a cyclic vector $h \in F_\psi$ with Taylor series expansion at $z = \frac{b}{1-a}$

$$h(z) = \sum_{n=0}^{\infty} a_n \left(z - \frac{b}{1-a}\right)^n.$$ 

Let us first make a short argument verifying the necessity that for $h$ to be a cyclic vector, $a_n \neq 0$ for all $n \in \mathbb{Z}_+$. If $a_n = 0$ for some $n = m$, it follows from the fact that

$$C^k_\psi h(z) = \sum_{n=0}^{\infty} a_n a^k n (z - \frac{b}{1-a})^n,$$

all functions $f$ in the closed linear span of $\{C^k_\psi h : k \in \mathbb{Z}_+\}$ satisfy $\frac{d^m}{dz^m} f \bigg|_{z = \frac{b}{1-a}} = 0$ which contradicts the cyclic behaviour of $h$.

We may now consider the case when $|a| = 1$ and hence $b = 0$. This together with the assumption $a^n \neq a$ for every $n \geq 2$ imply

$$\{a^k, k \in \mathbb{Z}_+\} = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$ 

Thus, for each $w \in \mathbb{T}$ there exists a sequence $\{k_j\}_j$ in $\mathbb{Z}_+$ such that $a^{k_j} \to w$ as $j \to \infty$. Let $\psi_w(z) = wz$. Then we claim

$$\lim_{j \to \infty} \|C^{k_j}_\psi h - C_{\psi_w} h\|_p = 0. \quad (4.14)$$

Using the radial property $\varphi(a^{k_j} z) = \varphi(z)$ and change of variables, we compute

$$\lim_{j \to \infty} \|C^{k_j}_\psi h\|_p = \lim_{j \to \infty} \int_{\mathbb{C}} |h(a^{k_j} z)|^p e^{-p\varphi(a^{k_j} z)} dA(z)$$
\[
\lim_{j \to \infty} \int_C |h(z)|^p e^{-p\varphi(z)} dA(z) = \frac{1}{|w|^2} \int_C |h(z)|^p e^{-p\varphi(z)} dA(z)
\]

\[
= \int_C |h(wz)|^p e^{-p\varphi(z)} dA(z) = \|C\psi_w h\|_p
\]

from which (4.14) follows. This verifies that \(C\psi_w h\) belongs to the closed linear span of \(\{C^k\psi h : k \in \mathbb{Z}_+\}\).

The mapping \(G : \mathbb{T} \to \mathcal{F}_\psi^p\) defined by \(G(w) = C\psi_w h\) is continuous, which can be extended to analytic function \(\tilde{G}\) in \(\mathbb{D}\) with \(\tilde{G}(w) = G(w)\) on the boundary of \(\mathbb{D}\). Then, by Cauchy Integral Formula (using \(C\psi_w (z) = G(w)(z) = \tilde{G}(w)(z)\))

\[
a_n z^n = \frac{1}{2\pi i} \int_{|w|=1} \frac{C\psi_w h(z)}{w^{n+1}} d\omega.
\]

Hence the set of polynomials \(a_n z^n, \ n \in \mathbb{Z}_+\) belongs to the closed linear span of \(\{C^k\psi h : k \in \mathbb{Z}_+\}\). From this, the fact that \(a_n \neq 0\) for all \(n \in \mathbb{Z}_+,\) and Lemma 3.2, the conclusion of the theorem follows for this case.

It remains to show the case when \(0 < |a| < 1\). For each \(m \in \mathbb{Z}_+\), we decompose the function \(h\) as \(h = h_m + g_m\) where

\[
h_m(z) = \sum_{n=0}^{m} a_n \left(z - \frac{b}{1-a}\right)^n \quad \text{and} \quad g_m(z) = \sum_{n=m+1}^{\infty} a_n \left(z - \frac{b}{1-a}\right)^n.
\]

(4.15)

Using induction we plan to prove that for every \(m \in \mathbb{Z}_+\)

\(h_m \in \text{span} \{C^k\psi h : k \in \mathbb{Z}_+\}\).

To this end, consider a function \(g\) in \(\mathcal{F}_\psi^p\) and observe that

\(C^k\psi g(z) = g \left(a^k z + \frac{b(1-a^k)}{1-a}\right)\).

Since \(|a| < 1\), we also have \(a^k z + (1-a)^{-1} b(1-a^k) \to (1-a)^{-1} b\) and by Lemma 3.4

\[
\lim_{k \to \infty} \|C^k\psi g - C_{\frac{b}{1-a}} g\|_p = 0.
\]

It follows from this and (4.15) that

\[
\lim_{k \to \infty} \|C^k\psi g_0 - C_{\frac{b}{1-a}} g_0\|_p = \lim_{k \to \infty} \|C^k\psi g_0\|_p = 0
\]

from which we further deduce

\[
\|C^k\psi h - a_0\|_p = \|C^k\psi (a_0 + g_0) - a_0\|_p \leq \|C^k\psi (a_0) - a_0\|_p + \|C^k\psi (g_0)\|_p \to 0
\]
as \( k \to \infty \). Therefore,
\[
h_0 \in \text{span} \{ C^k_\psi h : k \in \mathbb{Z}_+ \}.
\]

Suppose now that \( h_0, h_1, \ldots, h_{N-1} \in \text{span} \{ C^k_\psi h : k \in \mathbb{Z}_+ \} \). Then by the decomposition in (4.15) it holds that \( g_{N-1} \in \text{span} \{ C^k_\psi h : k \in \mathbb{Z}_+ \} \), and hence
\[
C^j_\psi g_{N-1} \in \text{span} \{ C^k_\psi h : k \in \mathbb{Z}_+ \}
\tag{4.16}
\]
for every \( j \in \mathbb{Z}_+ \). We next compute
\[
C^j_\psi g_{N-1}(z) = C^j_\psi \sum_{n=N}^\infty a_n \left( z - \frac{b}{1-a} \right)^n = \sum_{n=N}^\infty a_n a^n \left( z - \frac{b}{1-a} \right)^n
\]
\[
= a^N \left( z - \frac{b}{1-a} \right)^N \sum_{n=N}^\infty a_n a^{j(n-N)} \left( z - \frac{b}{1-a} \right)^{n-N}
\]
\[
= a^N \left( z - \frac{b}{1-a} \right)^N C^j_\psi \sum_{n=N}^\infty a_n \left( z - \frac{b}{1-a} \right)^{n-N}
\]
\[
= a^N \left( z - \frac{b}{1-a} \right)^N C^j_\psi f_{N-1}(z)
\tag{4.17}
\]
where \( \psi^j(z) = a^j z + \frac{b(1-a^j)}{1-a} \) and
\[
f_{N-1}(z) = a_N + \sum_{n=N+1}^\infty a_n \left( z - \frac{b}{1-a} \right)^{n-N}.
\]

From (4.16) and (4.17) we also obtain
\[
\left( z - \frac{b}{1-a} \right)^N C^j_\psi f_{N-1} \in \text{span} \{ C^k_\psi h : k \in \mathbb{Z}_+ \}.
\tag{4.18}
\]

By Lemma 3.4 we have that
\[
\lim_{j \to \infty} \| C^j_\psi f_{N-1} - a_N \|_p = \lim_{j \to \infty} \| C^j_\psi f_{N-1} - C^j_\psi \frac{b}{1-a} f_{N-1} \|_p = 0.
\tag{4.19}
\]

To this end, we further claim that
\[
\Gamma_j(z) := \left( z - \frac{b}{1-a} \right)^N C^j_\psi f_{N-1} - a_N \left( z - \frac{b}{1-a} \right)^N =: \Gamma(z)
\tag{4.20}
\]
in $\mathcal{F}_\psi^p$ as $j \to \infty$ as well. We may compute

$$
\|\Gamma_j\|_p^p = \int_C \left| \left( z - \frac{b}{1-a} \right)^N C_j f_{N-1}(z) \right|^p e^{-p\psi(z)} dA(z) = \int_C f_{N-1}\left( a^j z + \frac{b(1-a^j)}{1-a} \right) e^{-p\psi(a^j z + \frac{b(1-a^j)}{1-a})} U_j(z) dA(z)
$$

where

$$
U_j(z) = \left| z - \frac{b}{1-a} \right|^p e^{p\psi(a^j z + \frac{b(1-a^j)}{1-a} - \psi(z))}
$$

We also observe that since $\psi$ is an increasing weight function, and $|a^j| < 1$, the sequence of functions $U_j$ are uniformly bounded over $\mathbb{C}$. Furthermore, since norm convergence in $\mathcal{F}_\psi^p$ implies pointwise convergence, by (4.19) for each $z \in \mathbb{C}$

$$
C_{\psi,j} f_{N-1}(z) \to C_{\frac{b}{1-a}} f_{N-1}(z)
$$

as $j \to \infty$. With this, an application of Lebesques convergence theorem implies

$$
\lim_{j \to \infty} \|\Gamma_j\|_p^p = \lim_{j \to \infty} \int_C f_{N-1}\left( a^j z + \frac{b(1-a^j)}{1-a} \right) e^{-p\psi(a^j z + \frac{b(1-a^j)}{1-a})} U_j(z) dA(z) = \int_C C_{\frac{b}{1-a}} f_{N-1}(z) \left| z - \frac{b}{1-a} \right|^p e^{-p\psi(z)} dA(z) = \|\Gamma\|_p^p.
$$

Thus, the claim in (4.20) follows which along with (4.18) give

$$
a_N \left( z - \frac{b}{1-a} \right)^N \in \text{span} \left\{ C^k_{\psi} h : k \in \mathbb{Z}_+ \right\}, \quad h_N \in \text{span} \left\{ C^k_{\psi} h : k \in \mathbb{Z}_+ \right\},
$$

Therefore,

$$
h_m \in \text{span} \left\{ C^k_{\psi} h : k \in \mathbb{Z}_+ \right\},
$$

for every $m \in \mathbb{Z}_+$ which in turn results in

$$
a_n \left( z - \frac{b}{1-a} \right)^n \in \text{span} \left\{ C^k_{\psi} h : k \in \mathbb{Z}_+ \right\},
$$

for every $n \in \mathbb{Z}_+$. Then, since $a_n \neq 0$ for all $n \in \mathbb{Z}_+$, by Lemma 3.2 the assertion of the theorem follows.

**Part (ii)** We now proceed to show that $C_{\psi}$ can not be supercyclic. We set $\psi(z) = az + b$ and argue in the direction of contradiction, and assume that $C_{\psi}$ has a supercyclic vector $f \in \mathcal{F}_\psi^p$. If $0 < |a| < 1$, then by Lemma 3.1, $\psi$ fixes the point $b/(1-a)$.
It follows that \( f(b/(1-a)) \neq 0 \). If not, the projective orbit contains only functions which vanishes at \( b/(1-a) \). Now for each function \( g \) in the projective orbit of \( f \), there exists a sequence \((\lambda_{nk})\) such that

\[
\lim_{k \to \infty} \| \lambda_{nk} C_{\psi}^{nk} f - g \|_p = 0.
\]

Then we compute

\[
g\left( \frac{b}{1-a} \right) = \lim_{k \to \infty} \lambda_{nk} C_{\psi}^{nk} f\left( \frac{b}{1-a} \right) = \lim_{k \to \infty} \lambda_{nk} C_{\psi}^{nk} f\left( \frac{b}{1-a} \right) = f\left( \frac{b}{1-a} \right) \lim_{k \to \infty} \lambda_{nk},
\]

where we used here the fact that norm convergence implies pointwise convergence. Thus, for all \( z \in \mathbb{C} \), applying the fact that \( a^{nk} \to 0 \) as \( k \to \infty \) and (2.4)

\[
g(z) = \lim_{k \to \infty} \lambda_{nk} C_{\psi}^{nk} f(z) = \lim_{k \to \infty} \lambda_{nk} f\left( a^{nk} z + \frac{b(1-a^{nk})}{1-a} \right)
\]

\[
= \left[ f\left( \frac{b}{1-a} \right) \right]^{-1} g\left( \frac{b}{1-a} \right) \lim_{k \to \infty} f\left( a^{nk} z + \frac{b(1-a)}{1-a} \right)
\]

\[
= \left[ f\left( \frac{b}{1-a} \right) \right]^{-1} g\left( \frac{b}{1-a} \right) f\left( \frac{b}{1-a} \right) = g\left( \frac{b}{1-a} \right),
\]

showing that only constant functions are in the projective orbit of \( f \) resulting a contradiction.

If \( \psi(z) = az \) with \( |a| = 1 \), then it fixes the origin. We may choose a univalent function \( g \in F_p^\psi \) such that \( g(0) \neq 0 \), and pick a subsequence \( \psi^{nk} \) such that \( \psi^{nk}(z) \to az \) as \( k \to \infty \). Then

\[
g(z) = \lim_{k \to \infty} \lambda_{nk} C_{\psi}^{nk} f(z) = \lim_{k \to \infty} \lambda_{nk} f\left( a^{nk} z \right) = g(0) f(az).
\]

It follows that

\[
f(z) = \frac{f(0)}{g(0)} g\left( \frac{z}{a} \right)
\]

is univalent as \( f(0) \neq 0 \). Consequently, the projective orbits of \( f \) contains only univalent functions which is again a contradiction.

\subsection*{4.5 Proof of Theorem 2.6}

We consider first (a) of part (i) and assume that \( p \neq q \). We plan to show that \( C(F_p^\psi, F_q^\psi) \) is connected. Aiming to argue in the direction of contradiction, suppose there exists an isolated point \( C_{\psi} \in C(F_p^\psi, F_q^\psi) \). Since \( p \neq q \), by Theorem 2.1, \( C_{\psi} \) is a compact operator and hence \( \psi(z) = az + b, \ |a| < 1 \). Then, choose two sequences of numbers \((a_n)\) with \( |a_n| < 1 \) and \( a_n \neq 0 \) for all \( n \) and \( b_n \) such that \( a_n \to a \) and \( b_n \to b \) as
\( n \to \infty \). It follows that \( \psi_n(z) = a_n z + b_n \to a z + b = \psi(z) \). Then for any \( f \in \mathcal{F}_\psi^p \), by Lemma 3.4
\[
\| C_{\psi_n} f - C_{\psi} f \|_q \to 0 \text{ as } n \to \infty.
\]

Using this we find
\[
\lim_{n \to \infty} \| C_{\psi_n} f - C_{\psi} f \| \leq \| C_{\psi_n} f - C_{\psi} f \|_q = 0
\]
contradicting our assumption.

(b) Let \( p = q \) and assume that \( C_{\psi} \in C(\mathcal{F}_\psi^p, \mathcal{F}_\psi^p) \) is not compact. Then by Theorem 2.1, \( \psi(z) = a z, \ |a| = 1 \). We proceed to show that \( C_{\psi} \) is isolated. That is there exists a positive number c such that
\[
\| C_{\psi} - C_{\psi_1} \| \geq c \tag{4.22}
\]
for all \( C_{\psi_1} \in C(\mathcal{F}_\psi^p, \mathcal{F}_\psi^p) \) for which \( \psi_1 \neq \psi \). We may first consider the forms \( \psi_1(z) = a_1 z, \ |a_1| = 1 \) and \( a_1 \neq a \). Since the polynomials are contained in \( \mathcal{F}_\psi^p \),
\[
\| C_{\psi} - C_{\psi_1} \| \geq \sup_{n \geq 0} \| z^n \|_p^{-1} \| (C_{\psi} - C_{\psi_1}) z^n \|_p
= \sup_{n \geq 0} \| z^n \|_p^{-1} |a^n - a_1^n| z^n \|_p = \sup_{n \geq 0} |a^n - a_1^n| \geq 2. \tag{4.23}
\]

On the other hand, if \( C_{\psi_1} \) is compact, then \( \psi_1 = a_1 z + b, \ |a_1| < 1 \) and using the unit norm sequence of functions \( f_{(w,R)}^* \) in (4.12)
\[
\| C_{\psi} - C_{\psi_1} \| \geq \sup_{w \in \mathbb{C}} \| (C_{\psi} - C_{\psi_1}) f_{(w,R)}^* \|_p \geq \sup_{w \in \mathbb{C}} \left( \| C_{\psi} f_{(w,R)}^* \|_p - \| C_{\psi_1} f_{(w,R)}^* \|_p \right)
\geq \sup_{w \in \mathbb{C}} \left( 1 - \| C_{\psi_1} f_{(w,R)}^* \|_p \right). \tag{4.24}
\]

Now, \( f_{(w,R)}^* \to 0 \) weakly as \( |w| \to \infty \), and as \( C_{\psi_1} \) is compact, we have
\[
\| C_{\psi_1} f_{(w,R)}^* \|_p \to 0
\]
as \( |w| \to \infty \). This together with (4.24) for sufficiently big \( |w| \) gives
\[
\| C_{\psi} - C_{\psi_1} \| \gtrsim 1. \tag{4.25}
\]

From (4.25) and (4.23), the claim in (4.22) follows.
(ii) If both operators are compact, obviously the difference is also compact. Thus, we shall prove the other implication, i.e., assuming the difference is compact, we need to verify that both composition operators are compact. We plan to argue in the direction of contradiction again, and assume that one of them \( C_\psi \) is not compact. It follows that \( C_\psi \) is not compact either since for any \( f \in F_\psi \)

\[
|C_\psi f(z)|^p \lesssim |(C_\psi - C_\psi^*) f(z)|^p + |C_\psi^* f(z)|^p.
\]

Thus, we may set \( \psi_1(z) = a_1 z \) and \( \psi_2(z) = a_2 z \) where \( a_1 \neq a_2 \) and \( |a_j| = 1, j = 1, 2 \). Since the unit norm sequence \( f^*_w \) is weakly convergent, compactness of the difference operator implies

\[
\|(C_\psi - C_\psi^*) f^*_w \|_p \to 0 \text{ as } |w| \to \infty. \tag{4.26}
\]

On the other hand, we have a lower estimate

\[
\|(C_\psi - C_\psi^*) f^*_w \|_p^p = \int_C |C_\psi f^*_w (z) - C_\psi^* f^*_w (z)|^p e^{-p\psi(z)} dA(z) \\
\geq \int_{D(z_0, \tau(z_0))} |C_\psi f^*_w (z) - C_\psi^* f^*_w (z)|^p e^{-p\psi(z)} dA(z).
\]

From this and applying (3.5) and (3.3) we estimate

\[
\|C_\psi - C_\psi^* f^*_w \|_p \geq \frac{\tau(z_0)^p}{\tau(w)^p} |C_\psi f^*_w (z_0) - C_\psi^* f^*_w (z_0)| e^{-p\psi(z)} dA(z) \\
\approx \frac{\tau(z_0)^p}{\tau(w)^p} |f^*_w (\psi_1(z_0)) - C_\psi^* f^*_w (\psi_2(z_0))| e^{-p\psi(z)}.
\]

Setting \( w = \psi_1(z_0) \) on the right-hand side above, applying (3.1) and (3.2) and observing that \( \tau(\psi_1(z_0)) = \tau(\psi_2(z_0)) = \tau(z_0) \) leads to

\[
\|(C_\psi - C_\psi^*) f^*_w \|_p \geq \frac{\tau(z_0)^p}{\tau(\psi_1(z_0))^p} |f(\psi_1(z_0)) - f(\psi_2(z_0))| e^{-p\psi(z)} \\
\geq \left( e^{\psi(z)} - e^{\psi(z_0)} \right) \left( \frac{\tau(z_0)}{|z_0||a_1 - a_2|} \right) e^{-p\psi(z)} \\
\approx \left( e^{\psi(z)} - e^{\psi(z_0)} \right) \left( \frac{\tau(z_0)}{|z_0||a_1 - a_2|} \right)^2 e^{-p\psi(z)} = 1 - \left( \frac{\tau(z_0)}{|z_0||a_1 - a_2|} \right)^2 = 1
\]

when \( |z_0| \to \infty \) which contradicts the fact in (4.26).

The statement in part (b) is an immediate consequence of part (a) and part (ii) of Theorem 2.2.
4.6 Proof of Theorem 2.7

Since the essential norm topology is weaker than the operator norm topology, each essentially isolated point is isolated. Thus, we consider an operator \( C_{\psi_1} \in C(\mathcal{F}_p, \mathcal{F}_p) \), and assume that it is isolated in the operator norm topology. Then we plan to show that it is also essentially isolated. We may let \( \psi_1(z) = a_1 z \) with \( |a_1| = 1 \). It suffices to show that for all bounded composition operators \( C_{\psi_2} \in C(\mathcal{F}_p, \mathcal{F}_p) \), the estimate

\[
\|C_{\psi_1} - C_{\psi_2}\|_e \gtrsim 1
\]

holds. If \( \psi_2 \) is not compact either, then we may set \( \psi_2(z) = a_2 z \) where \( a_1 \neq a_2 \) and \( |a_2| = 1 \). Then for any compact operator \( Q \) on \( \mathcal{F}_p \) we have

\[
\| (C_{\psi_1} - C_{\psi_2}) - Q \| \gtrsim \lim_{|w| \to \infty} \| (C_{\psi_1} - C_{\psi_2}) f^*_R(w, R) \|_p
\]

\[
\geq \lim_{|w| \to \infty} \| (C_{\psi_1} - C_{\psi_2}) f^*_R \|_p - \| Q f^*_R \|_p
\]

\[
= \lim_{|w| \to \infty} \| (C_{\psi_1} - C_{\psi_2}) f^*_R \|_p.
\]

Arguing as in the preceding proof and setting \( w = \psi_1(z_0) \) we find,

\[
\|C_{\psi_1} - C_{\psi_2}\|_e \gtrsim \lim_{|z_0| \to \infty} \left( |f_{R}(\psi_1(z_0), R)(\psi_1(z_0))| - |f_{R}(\psi_1(z_0), R)(\psi_2(z_0))| \right) e^{-\varphi(z_0)}
\]

\[
\gtrsim \lim_{|z_0| \to \infty} \left( 1 - \left( \frac{\tau(z_0)}{|z_0| |a_1 - a_2|} \right) \right) = 1.
\]

On the other hand, if \( C_{\psi_2} \) is compact, we set \( \psi_2(z) = a_2 z + b \) with \( |a_2| < 1 \), and repeating the preceding arguments

\[
\|C_{\psi_1} - C_{\psi_2}\|_e \gtrsim \lim_{|z_0| \to \infty} \left( |f_{R}(\psi_1(z_0), R)(\psi_1(z_0))| - |f_{R}(\psi_1(z_0), R)(\psi_2(z_0))| \right) e^{-\varphi(z_0)}
\]

\[
\gtrsim \lim_{|z_0| \to \infty} \left( 1 - \left( \frac{\min\{\tau(z_0), \tau(a_2 z_0 + b_2)\}}{|z_0(a_1 - a_2) + b_2|} \right) \right) = 1,
\]

and completes the proof.

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