Generating matrix of the bi-periodic Lucas numbers

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Abstract

In this paper, firstly, we introduce the $Q_l$-Generating matrix for the bi-periodic Lucas numbers. Then, by taking into account this matrix representation, we obtain some properties for the bi-periodic Fibonacci and Lucas numbers.

Keywords: bi-periodic Fibonacci sequence, bi-periodic Lucas sequence matrix method.

1 Introduction

Fibonacci and Lucas numbers have attracted the attention of mathematicians because of their intrinsic theory and applications [11, 15]. Though closely related in definition, Lucas and Fibonacci numbers exhibit distinct properties. After Fibonacci numbers firstly defined by Leonardo da Pisa at the beginning of the thirteenth century, many authors have generalized this sequence by using different methods [4, 6, 9, 16, 18].

Edson and Yayanie, in [4], defined bi-periodic Fibonacci sequence $\{q_n\}_{n \in \mathbb{N}}$ as

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & n \text{ even} \\ bq_{n-1} + q_{n-2}, & n \text{ odd} \end{cases}$$

where $q_0 = 0$, $q_1 = 1$ and $a, b$ are nonzero real numbers.

Also, the Binet formula of generalized Fibonacci sequence is given by

$$q_n = \frac{1}{a^{\lfloor \frac{n-1}{2} \rfloor} b^{\lfloor \frac{n}{2} \rfloor}} \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$$

where $\alpha = \frac{a+b+\sqrt{a^2b^2+4ab}}{2}$, $\beta = \frac{a+b-\sqrt{a^2b^2+4ab}}{2}$ and $\varepsilon(n) = n - 2 \left[ \frac{n}{2} \right]$. 

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Cassini identities for bi-periodic Fibonacci sequence is defined by the form
\[ a^{1-\varepsilon(n)}b^{\varepsilon(n)}q_{n-1}q_{n+1} - a^{\varepsilon(n)}b^{1-\varepsilon(n)}q_n^2 = a(-1)^n. \] (3)

On the other hand, the Lucas numbers are an integer sequence named after the mathematician François Édouard Anatole Lucas (1842–91), who studied both that sequence and the closely related Fibonacci numbers. Bilgici, in [2], defined a generalization of Lucas sequence by the recurrence relation
\[ l_n = \begin{cases} bl_{n-1} + l_{n-2}, & n \text{ even} \\ al_{n-1} + l_{n-2}, & n \text{ odd} \end{cases} \tag{4} \]
where \( l_0 = 2, \ l_1 = a \) and \( a, b \) are nonzero real numbers. The Binet formula of this sequence is
\[ l_n = \frac{1}{a^{\lfloor n/2 \rfloor}b^{\lfloor n+1/2 \rfloor}} (\alpha^n + \beta^n). \tag{5} \]

And Cassini identities for bi-periodic Lucas sequence is given by
\[ \left( \frac{b}{a} \right)^{\varepsilon(n+1)} l_{n-1}l_{n+1} - \left( \frac{b}{a} \right)^{\varepsilon(n)} l_n^2 = (-1)^{n+1} (ab + 4). \tag{6} \]

Additionally, some authors studied matrix representation of special number sequences [1, 3, 5, 7, 8, 10, 12, 14, 17]. In one of these studies, Sylvester [12] gave Fibonacci \( Q \)-matrix as
\[ Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \]

Then, for a positive integer \( n \), the author found that \( Q^n \) has the form
\[ Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}. \]

Also, by using Fibonacci \( Q \)-matrix, Cassini’s Fibonacci formula can be found as follows:
\[ F_{n+1}F_{n-1} - F_n^2 = (-1)^n. \]

Similarly, in [10], Koken and Bozkurt defined the Lucas \( Q_L \)-matrix as
\[ Q_L = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \]

Also, they found some well-known equalities and a Binet-like formula for the Lucas numbers.

In the light of the above studies, in here, we define the \( Q_l \)-Generating matrix for bi-periodic Lucas numbers. This matrix is generalization form of well-known Lucas \( Q_L \)-matrix. After, by using this matrix representation, we have found some equalities for the bi-periodic Fibonacci and Lucas numbers.
2 The properties of $Q_l$-Generating matrix

In this section we firstly introduce a generalized matrix representation of the bi-periodic Lucas numbers. Then, we investigate the $n$th power, determinant and inverse of $Q_l$-Generating matrix.

**Definition 1** Bi-periodic Lucas $Q_l$-Generating matrix define by

$$Q_l = \begin{bmatrix} a^2 + 2 \frac{a}{b} & \frac{a^2}{2} \\ a & \frac{2a}{b} \end{bmatrix}. \quad (7)$$

**Theorem 2** Let $Q_l$-Generating matrix be as in (7). Then, for every $n \in \mathbb{Z}^+$, we have

$$Q_l^n = \begin{cases} \left( \frac{a}{b} \right)^n (ab + 4)^n \begin{bmatrix} q_{n+1} & q_n \\ \frac{b}{a}q_n & q_{n-1} \end{bmatrix}, & n \text{ even} \\ \left( \frac{a}{b} \right)^n (ab + 4)^{\frac{n-1}{2}} \begin{bmatrix} l_{n+1} & l_n \\ \frac{b}{a}l_n & l_{n-1} \end{bmatrix}, & n \text{ odd} \end{cases}, \quad (8)$$

where $q_n, l_n$ are the $n$th bi-periodic Fibonacci and Lucas numbers, respectively.

**Proof.** We use mathematical induction on $n$. Since $l_0 = 2, l_1 = a, l_2 = ab + 2, q_1 = 1, q_2 = a$ and $q_3 = ab + 1$ we write

$$Q_l = \begin{bmatrix} a^2 + 2 \frac{a}{b} & \frac{a^2}{2} \\ a & \frac{2a}{b} \end{bmatrix} = \left( \frac{a}{b} \right) (ab + 4)^0 \begin{bmatrix} l_2 & l_1 \\ \frac{b}{a}l_1 & l_0 \end{bmatrix}$$

$$Q_l^2 = \begin{bmatrix} a^4 + 5 \frac{a^3}{b} + 4 \frac{a^2}{b^2} & a^4 + 4 \frac{a^3}{b} \\ a^3 + 4 \frac{a^2}{b} & a^3 + 4 \frac{a^2}{b^2} \end{bmatrix} = \left( \frac{a}{b} \right)^2 (ab + 4) \begin{bmatrix} q_3 & q_2 \\ \frac{b}{a}q_2 & q_1 \end{bmatrix}$$

which show that the equation (8) is true for $n = 1$ and $n = 2$. Now we suppose that it is true for $n = k$, that is,

$$Q_l^k = \begin{cases} \left( \frac{a}{b} \right)^k (ab + 4)^{\frac{k}{2}} \begin{bmatrix} q_{k+1} & q_k \\ \frac{b}{a}q_k & q_{k-1} \end{bmatrix}, & k \text{ even} \\ \left( \frac{a}{b} \right)^k (ab + 4)^{\frac{k-1}{2}} \begin{bmatrix} l_{k+1} & l_k \\ \frac{b}{a}l_k & l_{k-1} \end{bmatrix}, & k \text{ odd} \end{cases}.$$

If we supposed that $k$ is even, by using properties of the bi-periodic Fibonacci numbers, we obtain

$$Q_l^{k+2} = Q_l^k Q_l^2 = \left( \frac{a}{b} \right)^{k+2} (ab + 4)^{\frac{k+1}{2}} \begin{bmatrix} q_{k+3} & q_{k+2} \\ \frac{b}{a}q_{k+2} & q_{k+1} \end{bmatrix}.$$
Similarly, for \( k \) is odd, we can write

\[
Q_{k+2} = Q_k Q_{l+2} = \left( \frac{a}{b} \right)^{k+2} (ab + 4)^{\frac{k+1}{2}} \begin{bmatrix}
    l_{k+3} & l_{k+2} \\
    \frac{b}{a} l_{k+2} & l_{k+1}
\end{bmatrix}
\]

And if we compose this partial function, we conclude

\[
Q_{k+2} = \begin{cases}
    \left( \frac{a}{b} \right)^{k+2} (ab + 4)^{\frac{k+1}{2}} \begin{bmatrix}
        q_{k+3} & q_{k+1} \\
        b a q_{k+2} & q_{k+2}
    \end{bmatrix}, & k + 2 \text{ even} \\
    \left( \frac{a}{b} \right)^{k+2} (ab + 4)^{\frac{k+1}{2}} \begin{bmatrix}
        l_{k+3} & l_{k+2} \\
        \frac{b}{a} l_{k+2} & l_{k+1}
    \end{bmatrix}, & k + 2 \text{ odd}
\end{cases}
\]

which is desired. Hence the proof is completed.

By using above theorem, we can give the following corollary.

**Corollary 3** Let \( Q_{n} \) be as in (8). Then the following equality is valid for all positive integers:

\[
\det(Q_{n}) = \left( \frac{a^2}{b^2} (ab + 4) \right)^{n}.
\]

**Proof.** If we prove using iteration, then we obtain

\[
\det(Q_{1}) = \begin{vmatrix}
    a + 2 \frac{a}{b} \\
    a^2 + 4 \frac{a^2}{b^2}
\end{vmatrix} = \frac{a^3}{b} + 4 \frac{a^2}{b^2} = \frac{a^2}{b^2} (ab + 4),
\]

\[
\det(Q_{l}) = \begin{vmatrix}
    a^2 + 3 \frac{a}{b} + 4 \frac{a^2}{b^2} \\
    a^3 + 4 \frac{a^3}{b^2}
\end{vmatrix} = \left( \frac{a^2}{b^2} (ab + 4) \right)^{2},
\]

\[
\vdots
\]

\[
\det(Q_{n}) = \begin{cases}
    \left( \frac{a}{b} \right)^{2n} (ab + 4)^{n} \left( q_{n+1} q_{n-1} - \frac{b}{a} q_{n}^2 \right), & n \text{ even} \\
    \left( \frac{a}{b} \right)^{2n} (ab + 4)^{n-1} \left( l_{n+1} l_{n-1} - \frac{b}{a} l_{n}^2 \right), & n \text{ odd}
\end{cases}
\]

\[
= \left( \frac{a}{b} \right)^{2n} (ab + 4)^{n}.
\]

Cassini and Binet formulas for bi-periodic Fibonacci and Lucas sequences are given in [2, 4]. In here, with a new perspective, we rewrite these properties using \( Q_l \)-Generating matrix.

**Theorem 4** The following equalities are valid for all positive integers:

1. \( a^{1-n} b^{2n} q_{n+1} q_{n-1} - a^{1-n} b^{2n} q_n^2 = a (-1)^n \),
2. \( \left( \frac{a}{b} \right)^{n+1} l_{n-1} l_{n+1} - \left( \frac{a}{b} \right)^n l_n^2 = (ab + 4) (-1)^{n+1} \).
Proof. By using Theorem 2 and Corollary 3, we obtain for even and odd $n$

$$q_{n+1}q_{n-1} - \frac{b}{a}q_n^2 = 1,$$

$$l_{n-1}l_{n+1} - \frac{b}{a}l_n^2 = (ab + 4),$$

respectively. That is, for $n \in \mathbb{Z}^+$,

$$aq_{2n+1}q_{2n-1} - bq_{2n} = a,$$

$$al_{2n}l_{2n+2} - bl_{2n+1}^2 = a(ab + 4).$$

Then, by using $q_{2n-1} = bq_{2n} - q_{2n+1}$ and $l_{2n+2} = bl_{2n+1} + l_{2n}$, we obtain

$$bq_{2n+2}q_{2n} - aq_{2n+1}^2 = a(-1),$$

$$bl_{2n+1}l_{2n+2} - al_{2n+1}^2 = a(ab + 4)(-1).$$

If we compose these equations, we conclude

$$a^{1-\varepsilon(n)}b^{\varepsilon(n)}q_{n+1}q_{n-1} - a^{1-\varepsilon(n)}b^{\varepsilon(n)}q_n^2 = a(-1)^n,$$

$$\left(\frac{b}{a}\right)^{\varepsilon(n+1)}l_{n-1}l_{n+1} - \left(\frac{b}{a}\right)^{\varepsilon(n)}l_n^2 = (ab + 4)(-1)^{n+1}.$$

\[\blacksquare\]

Theorem 5 Let $n$ be any integer. The Binet formulas of bi-periodic Fibonacci and Lucas numbers are

$$q_n = \left(\frac{a^{1-\varepsilon(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}}\right)\frac{\alpha^n - \beta^n}{\alpha - \beta},$$

$$l_n = \frac{1}{a\left\lceil \frac{n}{2} \right\rceil b\left\lfloor \frac{n+1}{2} \right\rfloor}(\alpha^n + \beta^n),$$

where $\alpha$ and $\beta$ are roots of $X^2 - abX - ab = 0$ equation.

Proof. Let the matrix $Q_l$ be as in (7). Characteristic equation of $Q_l$-Generating matrix is

$$\lambda^2 - \left(a^2 + 4\frac{a}{b}\right)\lambda + \frac{a^3}{b} + 4\frac{a^2}{b^2} = 0.$$

Then, eigenvalues and eigenvectors of the matrix $Q_l$ are

$$\lambda_1 = \left(\frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}\right)(ab + 4)^{\frac{1}{2}}\alpha, \lambda_2 = \left(\frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}\right)(ab + 4)^{\frac{1}{2}}(-\beta)$$
and
\[ u_1 = \left( \frac{a^2}{b}, -\frac{a}{b} \beta \right), u_2 = \left( \frac{a^2}{b}, -\frac{a}{b} \alpha \right) \]
where \( \alpha = \frac{ab + \sqrt{ab^2 + 4ab}}{2} \) and \( \beta = \frac{ab - \sqrt{ab^2 + 4ab}}{2} \). Q Generating matrix can be diagonalized by using
\[ V = U^{-1} Q U, \]
which
\[ U = \begin{pmatrix} u_1^T & u_2^T \end{pmatrix} = \begin{bmatrix} \frac{a}{b} & \frac{a^2}{b^2} \\ -\frac{a}{b} \beta & -\frac{a}{b} \alpha \end{bmatrix} \]
and
\[ V = \text{diag}(\lambda_1, \lambda_2) \]
\[ = \begin{bmatrix} \left( \frac{a}{b} \right)^\frac{1}{2} & (ab + 4)^\frac{1}{2} \alpha \\ 0 & \left( \frac{a}{b} \right)^\frac{1}{2} (ab + 4)^\frac{1}{2} (-\beta) \end{bmatrix}. \]
From properties of similar matrices, for \( n \) is any integer, we obtain
\[ Q_l^n = U V^n U^{-1}. \]
Thus, we get
\[ Q_l^n = \left( \frac{a}{b} \right)^\frac{1}{2} \frac{(ab + 4)^\frac{1}{2}}{\alpha - \beta} \begin{bmatrix} \alpha^{n+1} - \beta (-\beta)^n & a\alpha^n - a (-\beta)^n \\ b\alpha^n - b (-\beta)^n & -\beta\alpha^n + a (-\beta)^n \end{bmatrix} \]
\[ = \left( \frac{a}{b} \right)^\frac{1}{2} \frac{(ab + 4)^\frac{1}{2}}{\alpha - \beta} \begin{bmatrix} a^{n+1} - \beta (-\beta)^n & a\alpha^n - a (-\beta)^n \\ b\alpha^n - b (-\beta)^n & -\beta\alpha^n + a (-\beta)^n \end{bmatrix}. \]
Taking into account the Theorem 2, for the case \( n \) is even and odd, we can write
\[ q_n = \left( \frac{a}{(ab)^\frac{1}{2}} \right) \frac{\alpha^n - \beta^n}{\alpha - \beta}, \]
\[ l_n = \frac{1}{a^{n+1} b^{n+1}} (\alpha^n + \beta^n). \]
respectively. That is,
\[ q_{2n} = \left( \frac{a}{(ab)^n} \right) \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}, \]
\[ l_{2n+1} = \frac{1}{a^{2n+1} b^{2n+1}} (\alpha^{2n+1} + \beta^{2n+1}). \]
And also, since \( q_{2n+2} = aq_{2n+1} + q_{2n} \) and \( l_{2n+1} = al_{2n} + l_{2n-1} \), we have
\[ q_{2n+1} = \left( \frac{1}{(ab)^n} \right) \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta}. \]
Comparing the entries (1,2) and (2,1) for the matrices (9) and (10), we find

\[ l_{2n} = \frac{1}{(ab)^n} (\alpha^{2n} + \beta^{2n}) . \]

If we compose the obtained results, the desired result is obtained. □

In following theorem, we offer relationships between bi-periodic Fibonacci sequence and bi-periodic Lucas sequence.

**Theorem 6** For \( m, n \in \mathbb{Z} \), the following statements are true:

i) \( (ab+4)q_{2(m+n+1)} = l_{2m+1}l_{2(n+1)} + l_{2m}l_{2n+1} \),

ii) \( q_{2(m+n)} = q_{2m}q_{2n+1} + q_{2m-1}q_{2n} \),

iii) \( l_{2(m+n)+1} = l_{2m+1}q_{2n+1} + l_{2m}q_{2n} \),

iv) \( (ab+4)q_{2(m-n)} = l_{2m+1}l_{2(n+1)} - l_{2(m+1)}l_{2n+1} \),

v) \( q_{2(m-n)} = q_{2m}q_{2n+1} - q_{2m-1}q_{2n} \),

vi) \( l_{2(m-n)+1} = q_{2m+1}l_{2n+1} - q_{2(m+1)}l_{2n} \).

**Proof.** Using (8), \( Q_i^{m+n} \) can be written as

\[
Q_i^{m+n} = \begin{cases} 
\left( \frac{a}{b} \right)^{m+n} (ab+4) \frac{m+n}{2} & \begin{bmatrix} q_{m+n+1} & q_{m+n} \\
\frac{b}{a} q_{m+n} & q_{m+n-1} \end{bmatrix} , \ m+n \text{ even} \\
\left( \frac{b}{a} \right)^{m+n} (ab+4) \frac{m+n-1}{2} & \begin{bmatrix} l_{m+n+1} & l_{m+n} \\
\frac{b}{a} l_{m+n} & l_{m+n-1} \end{bmatrix} , \ m+n \text{ odd} 
\end{cases} . \tag{9}
\]

For the case of odd \( m \) and \( n \), we can write

\[
Q_i^m Q_i^n = \left( \frac{a}{b} \right)^{m+n} (ab+4) \frac{m+n-1}{2} \begin{bmatrix} l_{m+1}l_{n+1} + \frac{b}{a} l_{m}l_{n} & l_{m+1}l_{n} + l_{m}l_{n-1} \\
\frac{b}{a} l_{m+1}l_{n+1} + l_{m}l_{n} & \frac{b}{a} l_{m+1}l_{n} + l_{m-1}l_{n-1} \end{bmatrix} . \tag{10}
\]

If we compare the 1st row and 2nd column entries of the matrices (9) and (10), we get

\[ (ab+4) q_{m+n} = l_{m+1}l_{n} + l_{m}l_{n-1} . \]

On the other hand, comparing the entries 2nd row and 1st column, we obtain

\[ (ab+4) q_{m+n} = l_{m}l_{n+1} + l_{m-1}l_{n} . \]

For the case of even \( m \) and \( n \),

\[
Q_i^m Q_i^n = \left( \frac{a}{b} \right)^{m+n} (ab+4) \frac{m+n}{2} \begin{bmatrix} q_{m+1}q_{n+1} + \frac{b}{a} q_{m}q_{n} & q_{m+1}q_{n} + q_{m}q_{n-1} \\
\frac{b}{a} q_{m+1}q_{n+1} + q_{m}q_{n} & \frac{b}{a} q_{m+1}q_{n} + q_{m-1}q_{n-1} \end{bmatrix} . \tag{11}
\]

Comparing the entries (1,2) and (2,1) for the matrices (9) and (11), we find

\[ q_{m+n} = q_{m+1}q_{n} + q_{m}q_{n-1} , \]

\[ q_{m+n} = q_{m}q_{n+1} + q_{m-1}q_{n} . \]
For the case of odd \(m\) and even \(n\) (or case of even \(m\) and odd \(n\)),

\[
Q^m_l Q^n_l = \left(\frac{a}{b}\right)^{m+n} \binom{m+n}{ab+4} \left[ \frac{l_{m+1} q_{n+1} + \frac{b}{a} l_m q_n}{\frac{b}{a} (m q_{n+1} + l_{m-1} q_n)} \frac{l_{m+1} q_n + l_m q_{n-1}}{\frac{b}{a} l_m q_n + l_{m-1} q_n} \right].
\]

And if we compare the entries (1,2) and (2,1) for the matrices \(9\) and \(12\), we acquire

\[
l_{m+n} = l_{m+1} q_n + l_m q_{n-1},
\]

\[
l_{m+n} = l_m q_{n+1} + l_{m-1} q_n.
\]

Similarly, by calculating inverse of the matrix \(Q^n_l\) in \(8\), we conclude

\[
Q^{-n}_l = \begin{cases} \left(\frac{a}{b}\right)^{-n} (ab+4)^{-\frac{n}{2}} \left[ \frac{q_{n-1}}{q_n} - q_n \begin{array}{cc} q_{n-1} & -q_n \\ -q_n & q_{n+1} \end{array} \right], & n \text{ even} \\ \left(\frac{a}{b}\right)^{-n} (ab+4)^{-\frac{n+1}{2}} \left[ \frac{l_{n-1}}{l_n} - l_n \begin{array}{cc} l_{n-1} & -l_n \\ -l_n & l_{n+1} \end{array} \right], & n \text{ odd} \end{cases}.
\]

Benefiting from the equality \(Q^{m-n}_l = Q^m_l Q^{-n}_l\) and by comparing the entries (1,2) and (2,1) of these matrices, the desired result can be obtained. That is, for the case of odd \(m\) and \(n\), we get

\[
(ab+4) q_{m-n} = l_m l_{n+1} - l_{m+1} l_n.
\]

For the case of even \(m\) and \(n\), we obtain

\[
q_{m-n} = q_m q_{n+1} - q_{m+1} q_n.
\]

Finally, for the case of even \(m\) and odd \(n\) (or case of odd \(m\) and even \(n\)), we acquire

\[
l_{m-n} = q_m l_{n+1} - q_{m+1} l_n,
\]

Thus, we have the desired expressions. ■

3 Conclusion

This paper presents some properties of bi-periodic Fibonacci and Lucas numbers and relationships between these sequences by using the \(Q_l\)-Generating matrix. Also, some well-known matrices are special cases of this generating matrix. For example, if we choose \(a = b = 1\) and \(a = b = k\), we get the Lucas \(Q_L\)-matrix and \(k\)-Lucas Companion matrix, respectively.

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