Trotter product formulas and global regular upper bounds of the Navier Stokes equation solution

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Abstract

Global upper bounds with respect to regular norms of the incompressible Navier Stokes equation solution with regular data are constructed by an infinite scheme, where we work in bounded ZFC with bounded quantifiers and explicit infinitesimals. Trotter product formulas with infinitesimal error are obtained, which simplify for calculi with explicit infinitesimal and make the spatial effects needed in order to obtain global schemes more transparent.

1 A global upper bound theorem

Global regular existence for the incompressible Navier Stokes equation

\[
\frac{\partial v_i}{\partial t} - \nu \sum_{j=1}^{n} \frac{\partial^2 v_i}{\partial x_j^2} + \sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j} = \\
\sum_{j,m=1}^{n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^{n} \left( \frac{\partial v_m}{\partial x_j} \frac{\partial v_j}{\partial x_m} \right) (t,y) dy,
\]

(1)

v(0,.) = h,

to be solved for \( v = (v_1, \ldots, v_n)^T \) on the domain \([0, \infty) \times \mathbb{T}^n\) follows from the existence of global regular upper bounds. Here, \( \mathbb{T}^n \) is a torus of dimension \( n \) (equivalent to periodic boundary condition), the equation in \( \Box \) is written in Leray projection form, where the constant \( \nu > 0 \) is the viscosity and \( K_n \) is the Laplacian kernel of dimension \( n \geq 3 \), and \( h = (h_1, \ldots, h_n)^T \) are the initial data. We shall give a non-standard proof of the following non-standard theorem

**Theorem 1.1.** Let \( m \geq n + 2 \), and \( H^m = H^m(\mathbb{T}^n) \) the standard Sobolev space of order \( m \), and let \( C^m \) denote the space of continuous functions with continuous spatial derivatives up to order \( m \) as usual. We assume \( \nu > 0 \) and that for all \( 1 \leq i \leq n \) we have \( h_i \in H^m \) for some \( m \geq n + 2 \), where \( H^m \) denotes the standard Sobolev space of order \( m \). Then we claim that there is a global regular solution \( v_i \), \( 1 \leq i \leq n \) of \( \Box \) with

\[
v_i \in C^1((0, \infty), H^m \cap C^m),
\]

(2)

where for each \( T > 0 \) there exists a constant \( C > 0 \) such that for all \( t \in [0,T] \) we have

\[
|v_i(t,.)|_{H^m} \leq C
\]

(3)

1
2 Proof of the theorem 1.1

In the following we describe a proof of this standard theorem in a framework of analysis with explicit infinitesimals (such as the functional analytic framework by Connes or the framework nonstandard analysis). We refer to time intervals \([t_0, t_e]\) for finite times \(t_0, t_e \geq 0\) in the usual fashion where we assume that real finite numbers plus multiple infinitesimals are included (in nonstandard terminology we may use hyperfinite discretizations of finite time intervals). The advantage of calculi with explicit infinitesimals is that we do not need a local iteration and that the Trotter product formula has a simpler expression. This way we can observe the damping via the viscosity term in each infinitesimal time step more directly. Readers not intimate with calculus with explicit infinitesimals may read the following on an intuitive level (as Leibniz or Diderot would have read it), and then it should be possible for the reader to translate the following into a proof without explicit infinitesimals.

Writing the velocity component \(v_i = v_i(t, x)\) for fixed \(t \geq 0\) in the analytic basis \(\{\exp\left(\frac{2\pi i\alpha x}{l}\right), \alpha \in \mathbb{Z}^n\}\)

\[
v_i(t, x) := \sum_{\alpha \in \mathbb{Z}^n} v_{i\alpha}(t) \exp\left(\frac{2\pi i\alpha x}{l}\right), \quad (4)
\]

the initial value problem in (1) is equivalent an infinite ODE initial value problem for the infinite time dependent vector function of velocity modes \(v_{i\alpha}, \alpha \in \mathbb{Z}^n, 1 \leq i \leq n\), where

\[
\begin{align*}
\frac{dv_{i\alpha}}{dt} &= \sum_{j=1}^{n} \nu_j \left(\frac{-4\pi^2\alpha_j^2}{l^2}\right) v_{i\alpha} - \sum_{j=1}^{n} \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i\gamma_j}{l} v_{j(\alpha-\gamma)} v_{i\gamma} \\
&\quad + 2\pi i\alpha_i 1_{\{\alpha \neq 0\}} \sum_{k=1}^{n} \sum_{\gamma \in \mathbb{Z}^n} \frac{4\pi^2\gamma_j (\alpha_k - \gamma_k) v_{i\alpha} v_{k(\alpha-\gamma)}}{\sum_{i=1}^{n} 4\pi^2\alpha_i^2},
\end{align*} \quad (5)
\]

for all \(1 \leq i \leq n\) and where for all \(\alpha \in \mathbb{Z}^n\) we have \(v_{i\alpha}(0) = h_{i\alpha}\). We denote \(v^F = (v^F_1, \cdots, v^F_n)^T\) with \(n\) infinite vectors \(v^F_i = (v_{i\alpha})_{\alpha \in \mathbb{Z}^n}\). In order to have some terminology (abbreviations) available we use some nonstandard analytic terms in the following. The framework with its definitions can be found in the next section. Note that little of this is actually needed, and we show in section 4 that there is a simple functional analytic proof in ZFC which uses a simple elementary construction of infinitesimals and a classical transfinite induction principle (which is most basic set theory) only. For arbitrary \(t_e > 0\), a hyperfinite number \(N\) and an infinitesimal \(\delta t\) with \(N\delta t = t_e\) we define the nonstandard scheme where for time steps

\[
m\delta t \in \{0, \delta t, 2\delta t, \cdots, (N-1)\delta t, N\delta t = t_e\}.
\]

We may assume that the letter set is an internal set such that we may apply the internal induction principle and the internal set definition principle. The following argument may be rephrased in other calculi with infinitesimal numbers but it seems that the nonstandard calculus is the most convenient in order to
define a transfinite scheme. We have the infinitesimal Euler scheme

\[ v_{\alpha}(m + 1)\delta t = v_{\alpha}(m\delta t) + \sum_{j=1}^{n} \nu \left( -\frac{4\pi^2\alpha_j^2}{\nu} \right) v_{\alpha}(m\delta t)\delta t \]

\[ -\sum_{j=1}^{n} \sum_{\gamma \in \mathbb{Z}^{n}} \frac{2\pi i\alpha_j}{\nu} v_{\gamma}(m\delta t) v_{\gamma}(m\delta t)\delta t \]

\[ + 2\pi i\alpha_1 \delta t \sum_{\gamma \in \mathbb{Z}^{n}} 4\pi^2 \frac{\gamma_{\gamma}(\alpha_{\gamma} - \gamma_{\gamma}) v_{\gamma}(m\delta t) v_{\gamma}(m\delta t)\delta t}{\sum_{i=1}^{n} 4\pi\alpha_i^2}. \]

The last two terms on the right side of (6) correspond to the spatial part of the incompressible Euler equation, where for the sake of simplicity of notation we abbreviate (after same renaming with respect to the Burgers term)

\[ e_{ij\alpha\gamma}(m\delta t) = \frac{2\pi i(\alpha_{\gamma} - \gamma_{\gamma})}{\nu} v_{ij}(m\delta t) \]

\[ + 2\pi i\alpha_1 \delta t \sum_{\gamma \in \mathbb{Z}^{n}} 4\pi^2 \frac{\gamma_{\gamma}(\alpha_{\gamma} - \gamma_{\gamma}) v_{\gamma}(m\delta t) v_{\gamma}(m\delta t)\delta t}{\sum_{i=1}^{n} 4\pi\alpha_i^2}. \]

Note that with this abbreviation (6) becomes

\[ v_{\alpha}(m + 1)\delta t = v_{\alpha}(m\delta t) + \sum_{j=1}^{n} \nu \left( -\frac{4\pi^2\alpha_j^2}{\nu} \right) v_{\alpha}(m\delta t)\delta t \]

\[ + \sum_{j=1}^{n} \sum_{\gamma \in \mathbb{Z}^{n}} e_{ij\alpha\gamma}(m\delta t) v_{ij}(m\delta t)\delta t. \]

Similar schemes can be derived for real mode schemes with sin, cos-basis of course, but it is easy to check that for real data \( h_i \) the above scheme leads to real solutions, i.e.

\[ \forall 1 \leq i \leq n \ \forall m \geq 1 \ \forall x : v_i((m\delta t, x) \in \mathbb{R}. \]

Here we may assume that the velocity components \( v_i \) have their values in the field of standard real numbers where standard parts are taken tacitly if internal counterparts of value functions are considered. For the sake of simplicity (and without loss of generality) we consider the case \( l = 1 \) in the following. In this form the damping effect of the unbounded Laplacian is not obvious. Therefore, we derive the Trotter product formula stating that for all \( t_\nu = N_0\delta t \) (where \( N_0 \) may be a natural number or a hyperfinite number) we have

\[ V^F(t_\nu) = \Pi_{m=0}^{N_0-1} \left( \delta_{ij\alpha\beta} \exp \left( -\nu4\pi^2 \sum_{i=1}^{n} \alpha_i^2\delta t \right) \right) \left( \exp \left( \left( e_{ij\alpha\beta}(m\delta t) \right)\delta t \right) \right)^F, \]

and where \( \hat{\cdot} \) means that the identity holds up to an infinitesimal error. Furthermore, the entries in \( \delta_{ij\alpha\beta} \) are Kronecker-\( \delta \)s which describe the unit \( n\mathbb{Z}^{n} \times n\mathbb{Z}^{n} \) matrix. The formula in (10) is easily verified by showing that at each time step \( m \)

\[ \left( \delta_{ij\alpha\beta} \exp \left( -\nu4\pi^2 \sum_{i=1}^{n} \alpha_i^2\delta t \right) \right) \left( \exp \left( \left( e_{ij\alpha\beta}(m\delta t) \right)\delta t \right) \right)^F(m\delta t) \]

(as a representation for \( V^F(m\delta t) \)) solves the equation (6) with an error of order \( \delta t^2 \). The use of explicit infinitesimals allow us to have an effective use of first
order equality \( \doteq \) for arbitrary finite time where on an infinitesimal time level we have simplifications of the formula in \( (10) \) in the sense that

\[
    w^F(t_e) = \Pi_{m=0}^{N_0} \left( \delta_{ij\alpha\beta} \left( 1 - \nu \pi^2 \sum_{i=1}^{n} \alpha_i^2 \delta t \right) \right) \left( 1 + \left( e_{ij\alpha\beta} \delta t \right) \right) H^F,
\]

(12)
is also valid up to order \( O(\delta t^2) \).

For the logician we note that in a nonstandard framework of enlarged universes we would write down an internal function counterpart of \( (10) \) and then show that the formula in \( (10) \) holds. Now, since \( h_i \in H^{n+2}, 1 \leq i \leq n \) we surely have

\[
    \forall \alpha \in \mathbb{Z}^n : |h_{i\alpha}| \leq \frac{C}{1 + |\alpha|^{n+2}}.
\]

(13)

We may use that for some constant \( c = c(n) \) depending only on the dimension \( n \) we have

\[
    \sum_{\beta \in \mathbb{Z}^n} \frac{C}{1 + |\alpha - \beta|^{n+2}} \frac{C}{1 + |\beta|^{n+2}} \leq \frac{cC^2}{1 + |\alpha|^{n+2}}.
\]

(14)

It is well-known that \( \nu \) can be made large by transformation. We can also consider the function \( w_i, 1 \leq i \leq n \) with

\[
    v_i = \lambda w_1, \quad \lambda w_i(0, \cdot) = h_i,
\]

(15)

where we have

\[
    \left\{ \begin{array}{l}
        \frac{\partial w_i}{\partial t} - \nu \sum_{j=1}^{n} \frac{\partial^2 w_i}{\partial x_j^2} + \lambda \sum_{j=1}^{n} w_j \frac{\partial w_i}{\partial x_j} = \\
        \lambda \sum_{j,m=1}^{n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_i} K_n(x - y) \right) \sum_{j,m=1}^{n} \left( \frac{\partial w_m}{\partial x_j} \frac{\partial w_i}{\partial x_m} \right) (t, y) dy,
    \end{array} \right.
\]

(16)

This means that we get a small parameter with the nonlinear terms. The Trotter product for \( w = (w_1, \ldots, w_n)^T \) then becomes

\[
    w^F(t_e) = \Pi_{m=0}^{N_0} \left( \delta_{ij\alpha\beta} \exp \left( -4\pi^2 \nu \sum_{i=1}^{n} \alpha_i^2 \delta t \right) \right) \left( \exp \left( \left( e_{ij\alpha\beta} \delta t \right) \right) \right) \lambda^{-1} H^F,
\]

(17)

where

\[
    e_{ij\alpha\beta} (m \delta t) = -\lambda \frac{2\pi i(\alpha - \gamma)}{4\pi^2} w_i(\alpha - \gamma)(m \delta t)
\]

(18)

As \( \lambda > 0 \) is small enough the Trotter product formula in \( (17) \) allows us to show that the damping via the viscosity term is dominant for non-zero modes at each time step. Note that for the zero modes we have neither damping nor contribution of the Leray projection term. The zero mode can be controlled by another damping. It is convenient to give \( v_i \) an auto-control, i.e. to compare the value function \( v_i, 1 \leq i \leq n \) for time step interval \([t_0, t_0 + 1)\) smaller than \( 1 \) with a time dilated function \( \lambda(t) : [0,1] \times \mathbb{T}^n \rightarrow \mathbb{R} \), \( 1 \leq i \leq n \) along with

\[
    \lambda(1 + \rho t) \lambda(t_0^0(s, \cdot)) = v_i(t, \cdot), \quad s = \frac{t - t_0}{\sqrt{1 - (t - t_0)^2}}.
\]

(19)
for \( s \in [0, 1) \), and where it is clearly sufficient to prove the preservation of a upper bound for a time step scheme for the function \( u_i \), \( 1 \leq i \leq n \). Here the parameters \( \lambda, \rho > 0 \) are a small parameters, where \( \lambda > 0 \) ensures that the nonlinear terms in the equation for \( u_i \) get this additional small parameter, while \( \rho > 0 \) is used in order to ensure that the upper bound is preserved on the global time level (cf. below). In general we shall have \( 0 < \lambda \leq \rho \) in order to have a comparatively strong damping term.

Remark 2.1. An alternative localized version of the scheme uses the transformation
\[
\lambda(1 + \rho(t - t_0))u_i^{0,n}(s,.) = v_i(t,.) \quad s = \frac{t - t_0}{\sqrt{1 - (t - t_0)^2}}. \tag{20}
\]
The scheme for \( u_i^{t_0} \), \( 1 \leq i \leq n \) becomes
\[
u_i^{t_0}((m + 1)\delta t) = \nu_i^{t_0}((m\delta t) + \mu \sum_{j=1}^{m} \left(-\frac{4\pi^2\alpha_j^2}{\mu}\right) u_{i,\alpha}(m\delta t)\delta t
+ \sum_{j=1}^{m} \sum_{\gamma \in \mathbb{Z}^n} \nu^{\mu,\gamma}_{ij}(m\delta t) u_{i,\gamma}(m\delta t)\delta t,
\]
where \( \mu = \sqrt{1 - (t - t_0)^2} \) is evaluated at \( t_0 + m\delta t \), and
\[
\nu_{ij}^{\mu,\gamma}(m\delta t) = -\lambda \mu^1,\nu_{ij} \left(2\pi i(\alpha_j - \beta_j)\right) u_{i,\alpha}(m\delta t)
+ \lambda \mu^1,\nu_{ij} \left(2\pi i(\alpha_j - \beta_j)\right) \sum_{\gamma \in \mathbb{Z}^n} \nu^{\mu,\gamma}_{ij}(m\delta t) - \rho^{0,n}(m\delta t)\delta_{ij}\gamma
\]
along with \( \mu^1,\nu(t) := (1 + \rho(t)) \sqrt{1 - (t - t_0)^2} \) and \( \rho^{0,n}(t) := \frac{\rho}{\sqrt{1 - (t - t_0)^2}} \). The last term in (22) is related to the damping term of the equation for the function \( u_i^{t_0} \), \( 1 \leq i \leq n \). Still it may not be obvious that the damping term can offset the growth caused by the nonlinear terms, where we use [13] for an estimate of the latter. Note that a small parameter \( \lambda > 0 \) ensures that the nonlinear terms become small and this ensures that even the damping via the viscosity dominates the growth caused by the nonlinear terms. Consider the infinitesimal Trotter product formula for this variation. Defining \( t_i^0 = N_0\delta t + l_i^0 \) and for some times \( 0 \leq t^0_i \) and \( t^0_i - t^0_i < 1 \) along with \( t^0_i - t^0_i = N_0\delta t \) for some number \( N_0 \) (which may be a natural number or a hyperfinite number) we get
\[
u^F,\nu_i(t_i^0) = \Pi_{m=0}^N \left(\delta_{ij,\alpha} \exp \left(-\nu \sum_{i=1}^{n} \alpha_j^2 \delta t\right)\right) \times
\]
\[
\times \left(\exp \left(\left(\nu_{ij}^{\mu,\gamma}(m\delta t)\delta_{ij}\gamma\right)\delta t\right)\right) \nu^F,\nu_i(t_i^0). \tag{23}
\]

Remark 2.2. You may expect that we separate the damping term \(-\rho^{0,n}(m\delta t)\delta_{ij}\gamma\) in (22) in order to have another exponential factor of the damping term in (23). Indeed this leads to another Trotter product formula, but as we have formulas based on explicit infinitesimals and maintain equality up to infinitesimal error, we may use the simple form in (23).

The main theorem then follows from the inductive application of the following lemma written with the norm \( \|\cdot\|_H^m \) which is the dual of the norm \( \|\cdot\|_{H^{-m}} \).
Lemma 2.3. For \( m \geq n + 2 \) we have for \( C' > 0 \)
\[
|u^F(0)|_{h^m}^n \leq C' \Rightarrow |u^F(s)|_{h^m}^n \leq C'
\]
for \( s \in \left[ 0, \frac{1}{\sqrt{3}} \right] \) corresponding to \( t \in \left[ 0, \frac{1}{2} \right] \), where for \( s \geq 0 \) we define
\[
|u^F(s)|_{h^m} := \max_{1 \leq i \leq n} |u^F_i(s)|_{h^m},
\]
and \( |.|_{h^m} \) is the dual Sobolev norm corresponding to the standard Sobolev norm \( |.|_{H^m} \). Here we recall that time intervals \( \left[ 0, \frac{1}{\sqrt{3}} \right] \) may be hyperfinite discretizations (if we work in the theoretical context of hyperfinite numbers).

Remark 2.4. Note that \( u_i \in h^m \equiv h^m(\mathbb{Z}^n) \iff \sum_{\alpha \in \mathbb{Z}^n} |u_i^\alpha|^2m_1 + |\alpha|^2m < \infty \) and \( m \geq n + 2 \) implies that for some constant \( c > 0 \) we have
\[
|u_i^\alpha| \leq \frac{c}{1 + |\alpha|^{n+2}} < \infty.
\]

We shall prove Lemma 2.3 in section 4. This lemma may then be applied for all \( k \in \frac{1}{4} \mathbb{N} = \left\{ \frac{k}{2^i} | k \in \mathbb{N} \ldots \right\} \) with \( t_0 = k - 0.5 \) such that we get an upper bound for \( u_i^{k-0.5} \) at each time step \( k \) for the time interval \( [k - 0.5, k] \). Note that we use step size 0.5 in order to have effective damping terms. In section 5 then we shall show that we can sharpen the lemma a bit such that it leads to a global upper bound for \( v_i^\alpha \), \( \alpha \in \mathbb{Z}^n, 1 \leq i \leq n \) first. In order to see what is going on at each induction step consider for \( t_0 \in \frac{1}{4} \mathbb{N} \) the transformation
\[
v_i^{t_0} = \lambda (1 + \rho t) u_i^{t_0}, \text{ where } |v_i^{t_0}(t_0,.)|_{H^m} \leq C.
\]
This means that we have
\[
|u_i^{t_0}(0,.)|_{H^m} \leq \frac{C}{\lambda}.
\]
Now if we apply the Lemma 2.3 directly, then we get
\[
|u_i^{t_0}(s,.)|_{H^m} \leq \frac{C}{\lambda}
\]
for \( s \in \left[ 0, \frac{1}{\sqrt{3}} \right] \), and this implies for \( t = \frac{1}{2} \) (corresponding to \( s \in \frac{1}{\sqrt{3}} \)) that we have
\[
|v_i^{t_0} \left( t_0 + \rho \frac{1}{2} \right) |_{H^m} \leq (1 + \rho 0.5) C.
\]
In order to prove lemma 2.3 we may use transfinite induction (with respect to time) or hyperfinite induction in a framework of an enlarged universe. The enlarged universe is constructed in order to have infinitesimal entities and rigorous definitions of terms like hyperfinite numbers etc.. We note that this allows us to apply genuine tools of nonstandard analysis such as the internal set definition principle and internal induction (applied to internal version of the Trotter product formula), where this internal induction principle has ordinary transfinite induction counterparts which shows that our proof may be rephrased in Connes’ functional analytic setting or in other calculi with explicit infinitesimals.
Remark 2.5. We note that the Trotter product formula applied to $\lambda w_i = v_i$, $1 \leq i \leq n$ for $\lambda > 0$ small implies that we get the upper bound

$$|w_{\gamma}(m\delta t)| \leq \frac{C}{1 + |\gamma|^{n+2}}.$$  \hspace{1cm} (32)

for some constant $C > 0$ and all $1 \leq i \leq n$. A closer analysis shows that we do not even need the damping terms as we can derive an upper bound for the $v_{i\gamma}$-scheme of the form

$$|v_{i\gamma}(m\delta t)| \leq \frac{\sum_{k=0}^{2^m} a_k C^{2^k} \delta t^{2^k-1}}{1 + |\gamma|^{n+2}}.$$ \hspace{1cm} (33)

with an appropriate growth of coefficients $a_k$. Here the coefficients $a_k$ are polynomials in $\delta t$ themselves, where it suffices to estimate them up to first order.

3 Background of nonstandard analysis, bounded Zermelo and enlargements of universes

There has been some criticism of nonstandard analysis, especially by A. Connes, and some criticism of this criticism (cf. [4]). Our point of view is consistent with the view expressed in [4]. From a theoretical (logical) point of view nonstandard analysis is equiconsistent with ZFC. Infinitesimals can never be exhibited in space-time, neither the infinitesimals of nonstandard analysis nor the compact operators considered by Connes. Connes’ interpretations of infinitesimals as compact operators (cf [2]) leads to noncommuting infinitesimals. However, we do not need this if we consider schemes for classical equations. So nonstandard analysis is a legitimate possible framework for the scheme considered here, but we emphasize that it can be rephrased in Connes’ theory. An elementary introduction into nonstandard analysis may be found in [5]. In order to have all terms defined we need to say what an infinitesimal number and a hyperfinite number is in an enlarged universe. We remark that the proof can be rephrased in an argument which works only with basic set theory such as the transfinite induction principle (defined there), and where we define infinitesimal numbers directly in ZFC using non-principal ultrafilters bases on the class of co-finite subsets of the set of natural numbers. No specific features of this construction are needed, and interpretations of infinitesimals as compact operators (as is done by Connes) provide an alternative valid framework for the argument of this paper. First we define the latter term starting with a recapture of bounded Zermelo. Then we define the former terms in the framework of enlarged universes. Bounded Zermelo is ZFC with quantification restricted to existing sets. The language of bounded Zermelo is a normal set theoretic language, with the exception of the restricted quantification rule, i.e. for any formula $\phi(x)$ with the free variable $x$ and sets $a, b$

$$\exists x \in a \phi(x), \forall x \in b \phi(x)$$ \hspace{1cm} (34)

are formulas. Denote the usual set theoretic language with bounded quantifiers by $\mathcal{L}_R$. We list the axioms of restricted ZFC (RZFC in symbols).

(E) (Extensionality) $y = x$ iff, for all $z, z \in x$ iff $z \in y$.  

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(RC) (Restricted Comprehension) If $\phi(x)$ is an $L_R$-formula (with quantifiers restricted) and free variable $x$ and $a$ is a set, then there exists a set $b$ with $x \in b$ iff $x \in a$ and $\phi(x)$.

(NS) (Null Set) There exists a set $\emptyset$ such that for all $x \notin \emptyset$.

(P) (Pair) For all $x$ and $y$ there exists $z$ with $u \in z$ iff $u = x$ or $u = y$.

(U) (Union) For all $x$ there exists $y$ with $z \in y$ iff there exists $w$ with $z \in w \in y$.

(PS) (Power Set) for all $x$ there exists $y$ with $z \in y$ iff $z \subseteq x$. The set $y$ will be denoted by $P(x)$

(F) (Foundation) For all $x \neq \emptyset$ there exists a set $y \in x$ with $y \cap x = \emptyset$.

(I) (Axiom of Infinity) There exists a set $\mathbb{N}$ such that $\emptyset \in \mathbb{N}$ and $x \in \mathbb{N}$ implies $x \cup \{x\} \in \mathbb{N}$

(AC) (Axiom of Choice) if $I \neq \emptyset$ is an index set and for all $i \in I \ X_i \neq \emptyset$, then $\Pi_{i \in I} X_i \neq \emptyset$.

Bounded Zermelo is well known to be equiconsistent to a model of the first order theory of well-pointed topoi. However, as explained before, we are not dealing with the topos-theoretic view here.

Next we consider enlarged universes. Several types of extensions of universes may be considered (systems of sets with certain closure properties). A standard extension is called enlargement of universes is an axiomatic enlargement of an universe $U$ and is defined as an embedding

$$U \overset{\gamma}{\rightarrow} U'$$

(35)

which satisfies some axioms. Another one is by an ultrafilter construction with an index set $I$. So we seek for some object of the form

$$"U^* := U^I / \mathcal{F},$$

(36)

where $\mathcal{F}$ is a nonprincipal ultrafilter. Let us sketch this.

The axiomatic system RZFC tells us what sets are. In set theory sometimes you take entities for granted (as Kronecker expresses that the natural numbers are given by god, and the rest is constructed by mankind). Similar in nonstandard theory we tend to consider some set as the set of urelements. In ZFC people realised that urelements are superfluous. However, when we talk about universes over ..., we include urelements as convenient. We denote sets by capital letters and elements which are either sets or urelements by small letters. E.g. in $A \in U \ A$ is a set, while in $a \in U$ a may be a set or an urelement. A universe is a set with certain properties. First if $A \in U$ is a set, then we want all elements of $a$ to be present in $U$, i.e.

$$a \in A \in U \Rightarrow a \in U.$$ (37)

Any set $U$ which satisfies 37 is called transitive. Furthermore, in a universe we want to have with a set $A \in U$ its transitive closure $Tr(A) \in U$. Here $Tr(A)$ is the
smallest transitive set that contains \( A \). If \( A \) is transitive itself, then \( A = \text{Tr}(A) \), of course. We require:

\[
\text{if } A \in \mathcal{U}, \text{ then there ex. a transitive set } B \in \mathcal{U} \text{ with } A \subseteq B \subseteq \mathcal{U}. \quad (38)
\]

Finally we require (and are allowed to by RZFC) that

\[
\text{if } a, b \in \mathcal{U}, \text{ then } \{a, b\} \in \mathcal{U}
\]

\[
\text{if } A, B \in \mathcal{U} \text{ are sets, then } A \cup B \in \mathcal{U} \quad (39)
\]

\[
\text{if } A \in \mathcal{U} \text{ is a set, then } P(A) \in \mathcal{U}
\]

If the universe contains a set \( S \) such that the members of \( S \) are individuals in the sense that

\[
\forall x \in S [x \neq \emptyset \land (\forall y \in \mathcal{U} (y \notin x))]. \quad (40)
\]

The next step of course is to show that universes exist. They are realized by superstructures. Let \( S \) be a set. We define a series cumulative power set by

\[
\mathcal{U}_0(S) = S,
\]

\[
\mathcal{U}_{n+1}(S) = \mathcal{U}_n(S) \cup P(\mathcal{U}_n(S)).
\]

Then it is easy to check that

\[
\mathcal{U}(S) := \bigcup_{n \in \mathbb{N}} \mathcal{U}_n(S) \quad (41)
\]

is a universe. If the set \( S \) is known from the context, then we . The language \( \mathcal{L}_R \) with quantification restricted to the sets of the universe \( \mathcal{U} \) is denoted \( \mathcal{L}^\mathcal{U}_R \). Next, a nonstandard framework for a set \( S \) comprises a universe \( \mathcal{U} \) over \( S \) and a map

\[
\mathcal{U}^* \to \mathcal{U}', \quad (43)
\]

which satisfies

\[
a^* = a \text{ for } a \in S,
\]

\[
\emptyset^* = \emptyset, \quad (44)
\]

the \( \mathcal{L}^\mathcal{U}_R \)-sentence \( \phi \) is true iff \( \phi^* \) is true.

Such a nonstandard framework is called an enlargement if the following condition is satisfied:

if \( A \in \mathcal{U} \) is a collection of sets with the finite intersection property, then there exists an element \( z \in \mathcal{U}' \) such that

\[
z \in \bigcap \{Z^* | Z \in A\}. \quad (45)
\]

**Example 3.1.** If \( \mathcal{U} \) is a universe on \( \mathbb{R} \) and \( A \) is the set of intervals \( \{(0, r) | r > 0 \} \), then the latter set satisfies the finite intersection property. Then the enlargement principle tells us that there exists a positive infinitesimals, i.e. there exists

\[
b \in \bigcap \{(0, r)^* | r > 0 \} = \text{’set of positive infinitesimals’}. \quad (46)
\]
Example 3.2. Consider a universe \( U \) on \( \mathbb{N} \) and an enlargement \( U \to U' \). The set
\[
A = \{ \mathbb{N}_{\geq n} | n \in \mathbb{N} \}
\]
with \( \mathbb{N}_{\geq n} := \{ m \in \mathbb{N} | m \geq n \} \) satisfies the finite intersection property. Then the enlargement principle tells us that
\[
\exists b : b \in \cap \{ \mathbb{N}_{\geq n} | n \in \mathbb{N} \} = \mathbb{N}^* \setminus \mathbb{N},
\]
i.e. there is a set of unlimited numbers.

The question now is whether enlargements really exist. This can be shown with the ultrafilter construction. Two types of sets are of special interest for us: the first type are the internal sets:
\[
a \in U \text{ is internal if } a \in A^* \text{ for some } A \in U;
\]
the second type are the hyperfinite sets: let
\[
P_F(A) = \{ B \subseteq A | B \text{ is finite} \}.
\]
Then the hyperfinite sets are the members of \( P_F(A)^* \in U' \).

Let \( I \) be an infinite set and let \( \mathcal{F} \) be a nonprincipal ultrafilter on \( I \). Let \( S \) be a set and let \( U \) be a universe over \( S \). To \( a \in U \) assign \( a_{\infty} \in U_{\infty} \), the function with constant value \( a \). This way we embed \( U \) in a larger universe. Similarly as in the ultrafilter construction on \( \mathbb{R} \) we have to consider equivalence of elements with respect to the ultrafilter, this time of functions \( f,g \in U_{\infty} \). We say
\[
f \sim g \text{ iff } \{ i | f(i) = g(i) \} \in \mathcal{F}
\]
\[
f \in g \sim f' \in g' \text{ iff } \{ i | f(i) \in g(i) \& f'(i) \in g'(i) \} \in \mathcal{F}
\]
We denote equivalence classes \( [\cdot] \) as before and define
\[
W_n := \{ f \in U^n | \{ i | f(i) \in U_n \} \in \mathcal{F} \}.
\]
\[
W := \cup_{n \in \mathbb{N}} W
\]
Now for \( f \in W_0 \) let \( [f] := \{ [h] | h \in W_0 \} \), and let
\[
\mathcal{Y} = \{ [f] | f \in W_0 \}
\]
This defines
\[
U_0 (\mathcal{Y}) = \mathcal{Y}
\]
Inductively, having defined \( U_n (\mathcal{Y}) \), for \( f \in W_{n+1} \setminus W_n \) define
\[
[f] = \{ [h] | h \in W_n \text{ and } \{ i | h(i) \in f(i) \} \in \mathcal{F} \}
\]
Then
\[
U_{n+1} (\mathcal{Y}) = U_n (\mathcal{Y}) \cup \{ [f] | f \in W_{n+1} \setminus W_n \},
\]
and
\[
U (\mathcal{Y}) = \cup_{n \in \mathbb{N}} U_n (\mathcal{Y}).
\]
Now, $U(Y)$ is the ultrafilter-enlargement we had looked for. For each $f, g \in W$ it is easy to see that
\[ [f] \in [g] \iff \{ i | f(i) \in g(i) \} \in \mathcal{F}, \quad [f] = [g] \iff \{ i | f(i) = g(i) \} \in \mathcal{F} \quad (59) \]
The map \[ * : U(X) \to U(Y), \quad a \to a^* = [a_1] \quad (60) \]
is an embedding of the universe $U(X)$ in the universe $U(Y)$, and we observe that
\[ \emptyset^* = \emptyset, \quad \text{and} \quad X^* = Y. \quad (61) \]
Since $\mathcal{F}$ is an ultrafilter, we know that the enlargement $U(Y)$ has nonstandard members. Let $L_{U(X)}$ and $L_{U(Y)}$ be the formal languages of the respective universes. Denote the model of the ultrafilter-enlargement by $U_Y = (U(Y), \in)$ the model of the original universe by $U_X = (U(X), \in)$. Then we get the following version of the theorem of Loos.

**Theorem 3.3.** For any $L_{U(X)}$-formula $\phi(x_1, \cdots, x_m)$ and $f_1, \cdots, f_m \in W$
\[ U_Y \models \phi \left( e_{[f_1, \cdots, f_m]}^{x_1, \cdots, x_m} \right) \iff \{ i | U_X \models \phi \left( e_{f_1(i), \cdots, f_m(i)}^{x_1, \cdots, x_m} \right) \} \in \mathcal{F} \quad (62) \]

Let $U$ be a universe (which contains the real numbers as individuals) and let $U \overset{\phi}{\to} U'$ be an enlargement. For $A \in U$ and let
\[ P_F(A) = \{ b \subseteq A | B \text{ is finite } \}. \quad (63) \]
$P_F(A)^*$ are called hyperfinite subsets of $A$. As an example, consider $P_F(\mathbb{N})$ and the $L_U$-sentence
\[ \forall n \in \mathbb{N} \exists A \in P_F(\mathbb{N}) \forall m \in \mathbb{N} \ [m \in A \leftrightarrow m \leq n], \quad (64) \]
i.e. the sentence which has the meaning that for each natural number $n \in \mathbb{N}$ there is a set $A = \{ 1, \cdots, n \}$ in $P_F(\mathbb{N})$. The transfer sentence is
\[ \forall n \in \mathbb{N^*} \exists A \in P_F(\mathbb{N})^* \forall m \in \mathbb{N^*} \ [m \in A \leftrightarrow m \leq n]. \quad (65) \]
Hence, for all $n \in \mathbb{N}^*$
\[ A = \{ 1, \cdots, n \} \in P_F(\mathbb{N})^*. \quad (66) \]
Note that $n$ can be infinite, i.e. $n \in \mathbb{N}^* \setminus \mathbb{N}$. We prove

**Theorem 3.4.** $A$ is hyperfinite iff there exists $n \in \mathbb{N}^*$ and an internal bijection
\[ f : \{ 1, \cdots, n \} \to A. \quad (67) \]
Here a function $f : A \to B$ is called internal if the set $\text{graph}(f) \subseteq A \times B$ is internal.

Proof. Consider a $L_U$-formula
\[ \phi(X,Y,n,f) \quad (68) \]
which expresses that $f : X \to Y$ with $X = \{ m \in \mathbb{N} | m \leq n \}$ is a bijection. Then the $L_U$-sentence
\[ \psi \equiv \forall Y \in P_F(B) \exists n \in \mathbb{N} \exists f \in P(\mathbb{N} \times B) \exists X \in P(\mathbb{N}) \phi(X,Y,n,f) \quad (69) \]
asserts that for all $Y \in P_F(B)$ there is a number $n$ and a bijection between $X = \{1, \ldots, n\}$ and $Y$, a sentence which is true. The sentence $\psi^*$ is true by transfer. So if $B \in U$ and $A \in P_F(B)^*$ then the claim follows from the truth of $\psi^*$. For the converse suppose that there is an internal bijection $f : X = \{1, \ldots n\} \to A$ for some $n \in \mathbb{N}^*$. Then $A$ is internal, because it is the range of an internal function. We want to show that $A$ is hyperfinite. First we observe that

$$\exists X \in P(\mathbb{N})^* \phi(X, A, n, f)^* \quad (70)$$

is true. Hence the claim that $A$ is hyperfinite follows from transfer of the true $L_U$-sentence

$$\forall Y \in B \exists n \in \mathbb{N} \exists f \in P(\mathbb{N} \times A) \exists X \in P(\mathbb{N}) \left( \phi(X, A, n, f) \to Y \in P_F(A) \right) . \quad (71)$$

Having defined enlargements and hyperfinite numbers we now can easily define all other terms which are used in a nonstandard form of the argument. These are the internal induction principle and the internal set definition principle. For a hyperfinite number the discretization

$$\left\{ \frac{k}{N} \mid k \text{ hypernatural } \& k \leq N \right\} = \left\{ 0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N - 1}{N}, 1 \right\} \quad (72)$$

is an internal set. Internal sets have the advantage that they obey an internal induction principle based on the fact that each internal subset of the set of hypernaturals has a least member. We have

**Theorem 3.5.** An internal subset $S$ of the set of hypernatural numbers $\mathbb{N}^*$ which contains 1, i.e., $1 \in S$, and is closed under the successor operation $n \to n + 1$, i.e., $n \in S \to n + 1 \in S$ equals the whole set of hypernatural numbers, i.e., $S = \mathbb{N}^*$.

Next we recall the internal set definition principle.

**Theorem 3.6.** Let $U'$ be an enlargement of an universe and let $\psi(x)$ be an internal $L_{U'}$ formula, where $x$ is the only free variable. Then for any internal set $S \in U'$ the subset

$$R := \{ x \in S \mid \phi(x) \} \quad (73)$$

is internal. Here 'is an internal formulas' means that $\psi(x) \equiv \psi(x, a_1, \ldots, a_m)$ for some internal constants $a_i \in U'$.

### 4 Proof(s) of Lemma 2.3

The theoretical minimum for the proof of Lemma 2.3 is a calculus with infinitesimal entities in order to have an exact meaning of the scheme described above together with the principle of classical transfinite induction. However, the internal set definition principle and internal induction are a convenient tool. Clearly, both principles have their counterpart in ZFC, or, more naturally in NBG, where they can be rephrased with the transfinite induction principle. Therefore, the following argument can be rephrased in a functional analytic setting with explicit infinitesimals such as Connes’ theory. In order to define the theoretical minimum for the argument, recall the transfinite induction principle. First, a nonempty linear ordered set $S$ is called well-ordered if any nonempty subset of $S$ as a least element. Next recall
Definition 4.1. An ordinal number is a transitive set which is well-ordered by the relation $\in$. The class of all ordinals is denoted by $\text{On}$. Furthermore, for $\alpha \in \text{On}$ we define $\alpha + 1 := \alpha \cup \{\alpha\}$ to be the successor ordinal.

The well-known transfinite induction principle (proofs can be found in standard textbooks of set theory) then can be stated as follows.

Theorem 4.2. Let $R \subset \text{On}$ be a class of ordinals where $\text{On}$ is the class of all ordinals, and let $\phi$ be a property. Assume that

i) $\emptyset \in R$ and $\phi(\emptyset)$ is valid;

ii) if $\alpha \in R$, then $\alpha + 1 \in R$, and if $\phi(\alpha)$ holds, then $\phi(\alpha + 1)$ holds;

iii) if $\alpha \neq \emptyset$ is a nonzero limit ordinal, where $\beta \in R$ for all $\beta \in \alpha$ and such that $\phi(\beta)$ holds for all $\beta \in \alpha$, then $\alpha \in R$ and $\phi(\alpha)$ holds.

Then $R$ is the class of all ordinals and $\phi(\alpha)$ holds for all $\alpha \in \text{On}$. Moreover, the transfinite induction principle can be extended to every transitive class $T$ where we replace items [i] − [iii]) by just two items $a)$ and $b)$.

a) $\emptyset \in T$ and $\phi(\emptyset)$ is valid;

b) if $\alpha \in T$ and $\phi(\beta)$ holds for all $\beta \in \alpha$, then $\phi(\alpha)$ holds.

Then for every $\alpha \in T \phi(\alpha)$ holds.

The latter theorem may be applied to the transitive class of ordinal numbers $T \subset \text{On}$ which is itself linearly ordered by $\in$. The transitive class used then corresponds to a hyperfinite set, where for a given time step $k \in \mathbb{N}$ we consider a hyperfinite function $u_{\text{lo}}^t = u_{k-1}^t$, $1 \leq i \leq n$ on a hyperfinite time interval

$$I_N := \left\{0, \frac{1}{2N}, \frac{2}{2N}, \ldots, \frac{N-1}{2N}, \frac{1}{2} \right\},$$

(74)

where $N$ is a hyperfinite number (note that hyperfinite numbers are also denoted by $\mathbb{N}^*$. As a property to preserve for transfinite induction we consider for $l \in I_N$ and some $C' > 0$

$$\phi(l) \equiv \|u^F(l\delta t)\|_{h^m}^n \leq C'.$$

(75)

The Trotter product formulas are defined with classical external sets but they can trivially extended such that the formulas in the extended universe contain only internal constants (such as $\mathbb{Z}^*$ instead of $\mathbb{Z}$ etc.). Hence, it is clear that $\phi(l)$ may be assumed to be defined by an internal formula with internal constants, where such that

$$\phi(l) \equiv \{l \in I_N|\phi(l) \text{ is true} \}$$

(76)

In order to apply the internal induction principle we may define the extension $\psi(l)$ for all hyperfinite numbers, where

$$\psi(l) \equiv \begin{cases} 
\phi(l) \text{ if } l \in I_N \\
\text{true if } l \in \mathbb{N}^* \setminus I_N.
\end{cases}$$

(77)

We show

$$\forall l \in I_N : \phi(l),$$

(78)
a statement, which corresponds to
\[ \forall l \in \mathbb{N}^+ : \psi(l). \]  

(79)

Inductively, we assume that for \( k \in \mathbb{N} \) we have for \( u^{F,t_0}(t_i^0) \) with \( t_0 = k - 1 \) and \( t_i = 0 \) that
\[ |u^{F,k-1}(t_i^0)|_{h^m} \leq C'. \]  

(80)

This assumption holds for \( C' = C/\lambda \) clearly for \( k = 1 \) where \( u^{F,0}(t_i^0) = \frac{1}{\lambda} h^F \) such that we can assume that (80) holds. In the following we always tacitly assume that the formulas, especially Trotter product formulas are internal formulas which can always be achieved by trivial extension). In the following we use \( C > 0 \) generically for the simplicity of notation, i.e. \( C' \) may be \( C' \). Using the internal induction principle (or transfinite induction in a related classical argument) we show that for some constant \( C > 0 \) we have
\[ \forall l \in I_N : |u^{F,k-1}((l - 1)\delta t)|_{h^m} \leq C \Rightarrow |u^{F,k-1}(l\delta t)|_{h^m} \leq C \]  

(81)

where we may use that for \( l \in I_N \) the Trotter product formula
\[ u^{F,k-1}(l\delta t) \equiv \Pi_{n=0}^{l-1} (\delta_{i\alpha j\beta} \exp \left( -\nu 4\pi^2 \sum_{i=1}^{m} \alpha_i^2 \delta t \right)) \times \left( \exp \left( \left( \left( \epsilon_{i\alpha j\beta} \right)_{i\alpha j\beta} (m\delta t) \right) \delta t \right) \right) u^{F,k-1}(0). \]  

(82)

holds. Assuming inductively that
\[ |u^{F,k-1}((l - 1)\delta t)|_{h^m} \leq C \]  

(83)

from (82) we have for each substep \( l \) (at the main step \( k \))
\[ u^{F,k-1}(l\delta t) \equiv (\delta_{i\alpha j\beta} \exp \left( -\nu 4\pi^2 \sum_{i=1}^{m} \alpha_i^2 \delta t \right)) \times \left( \exp \left( \left( \left( \epsilon_{i\alpha j\beta} \right)_{i\alpha j\beta} ((l - 1)\delta t) \right) \delta t \right) \right) u^{F,k-1}((l - 1)\delta t). \]  

(84)

Abbreviating \( E^{u^{k-1},\lambda,l-1} = E^{u^{k-1},\lambda,(l - 1)\delta t} = \left( \epsilon_{i\alpha j\beta} \right)_{i\alpha j\beta} ((l - 1)\delta t) \) we may write
\[ u^{F,k-1}(l\delta t) \equiv (\delta_{i\alpha j\beta} \exp \left( -\nu 4\pi^2 \sum_{i=1}^{m} \alpha_i^2 \delta t \right)) \times \left( \exp \left( E^{u^{k-1},\lambda,l-1}\delta t \right) \right) u^{F,k-1}((l - 1)\delta t). \]  

(85)

Note that the formula
\[ \left( \exp \left( E^{u^{k-1},\lambda,l-1}\delta t \right) \right) u^{F,k-1}((l - 1)\delta t) - 1 \left( E^{u^{k-1},\lambda,l-1}\delta t \right) u^{F,k-1}((l - 1)\delta t) \]  

\[ = \sum_{j=2}^{\infty} \frac{E^{u^{k-1},\lambda,l-1}\delta t}{j!} u^{F,k-1}((l - 1)\delta t) \]  

(86)

can be used in order to determine the error in (85). The precise expression is not of interest for us but we not the there is a vector \( r^F = (r_i^F, \ldots, r_n^F) \) with \( r_i^F \in \mathbb{Z}^m (\mathbb{Z}^n) \) and such that for each entry \( r_{i\alpha} \) in \( r^F = (r_{i\alpha})_{\alpha \in \mathbb{Z}^n} \) we have
\[ r_{i\alpha} \in O \left( \delta t^2 \right), \]  

(87)
and, hence, for each substep \( l \) we have

\[
\mathbf{u}^{E,k-1}(l\delta t) = (\delta_i \alpha_{j\beta} \exp(-\nu 4\pi^2 \sum_{i=1}^n \alpha_i^2 \delta t)) \times \left( \exp\left(E_{u^{k-1},\lambda}(l-1)\delta t\right) \right) \mathbf{u}^{F,k-1}((l-1)\delta t) + \mathbf{r}^F. \tag{88}
\]

Using the definition of the Euler terms (torus size one)

\[
e_{ij\alpha\gamma}^{u_{\alpha\lambda},k-1}((l-1)\delta t) = -\lambda \mu^{1,\lambda} \sum_{j=1}^n (\alpha_j - \gamma_j) u_{i(\alpha-j)}^{k-1}((l-1)\delta t)
\]

\[
+ \lambda \mu^{1,\lambda} 2\pi\sum_{j=1}^n (\alpha_j - \gamma_j) u_{i(\alpha-j)}^{k-1}((l-1)\delta t) - \mu^{0,k-1}((l-1)\delta t) \delta_i \alpha_{j\alpha}\tag{89}
\]

we get for all \( 1 \leq i \leq n \) and all \( \alpha \in \mathbb{Z}^n \) (summmands in \( \mathbb{Z}^n \setminus \mathbb{Z} \) are zero in a trivial extension such that we may assume that the involved formulas are internal), and up to order \( O(\delta t^2) \) we have

\[
u^{k-1}((l-1)\delta t) = \exp(-\nu 4\pi^2 \sum_{i=1}^n \alpha_i^2 \delta t) \times \sum_{j=1}^n (\alpha_j - \gamma_j) u_{i(\alpha-j)}^{k-1}((l-1)\delta t)
\]

\[
+ \lambda \mu^{1,\lambda} 2\pi\sum_{j=1}^n (\alpha_j - \gamma_j) u_{i(\alpha-j)}^{k-1}((l-1)\delta t) - \mu^{0,k-1}((l-1)\delta t) \delta_i \alpha_{j\alpha}. \tag{90}
\]

Note that for \( \alpha = 0 \) the Leray projection term vanishes and we have

\[
e_{ij\alpha\gamma}^{u_{\alpha\lambda},k-1}((l-1)\delta t) = -\lambda \mu^{1,\lambda} \sum_{j=1}^n (\alpha_j - \gamma_j) u_{i(\alpha-j)}^{k-1}((l-1)\delta t) - \mu^{0,k-1}((l-1)\delta t) \delta_i \alpha_{j\alpha}. \tag{91}
\]

where the \( \gamma \)-modes contribute only for \( \gamma \neq 0 \) because of the presence of one spatial derivative in the nonlinear Burgers term. The coefficients of the equation for the localized comparison functions in (81) are globally bounded. For an arbitrary given finite time horizon \( T > 0 \) this holds also for the scheme considered in this paper, such that we do not need to consider localized schemes in this paper. Especially, for given \( T > 0 \) we have some parameter \( \rho > 0 \) such that we surely have the upper bounds \( |\mu^{0,k-1}((l-1)\delta t)| \leq 2 \) and \( |\mu^{1,k-1}((l-1)\delta t)| \leq 2 \). Hence, we get the upper bounds

\[
|\lambda \mu^{1,\lambda} 2\pi\sum_{j=1}^n (\alpha_j - \gamma_j) u_{i(\alpha-j)}^{k-1}((l-1)\delta t) | \leq \frac{\lambda M^\pi C^2}{1+|\alpha|^n+2} \tag{92}
\]

for the Burgers term and

\[
|\lambda \mu^{1,\lambda} 2\pi\sum_{j=1}^n (\alpha_j - \gamma_j) u_{i(\alpha-j)}^{k-1}((l-1)\delta t) | \leq \frac{\lambda M^\pi C^2}{1+|\alpha|^n+2} \tag{93}
\]
for the Leray projection term. Furthermore, for ‘global time step size’ \( \frac{1}{2} \) we have \( |\mu^{0,k-1}(l-1)\delta t| \geq \frac{\nu_{\rho}}{1+\rho T} \). Hence we get

\[
|u_{\alpha}^{k-1}(l\delta t)| \leq |u_{\alpha}^{k-1}((l-1)\delta t)| \left( 1 - \nu_4^2 \sum_{j=1}^{n} \alpha_j^2 \delta t - \frac{\nu_{\rho}}{1+\rho T} \delta t \right) + \frac{\lambda_{\alpha} c C^2}{(1 + |\alpha|^{n+2}) \delta t} + c \delta t^2
\]

for some finite number \( c \). Here the first term in brackets on the right side \( 1 - \nu \sum_{j=1}^{n} \alpha_j^2 \delta t \) is a first order approximation of \( \exp \left( -\nu \sum_{j=1}^{n} \alpha_j^2 \delta t \right) \), such that

\[
|u_{\alpha}^{k-1}(l\delta t)| \leq |u_{\alpha}^{k-1}((l-1)\delta t)| \left( 1 - \nu_4^2 \sum_{j=1}^{n} \alpha_j^2 \delta t - \frac{\nu_{\rho}}{1+\rho T} \delta t \right) + \frac{\lambda_{\alpha} c C^2}{(1 + |\alpha|^{n+2}) \delta t} + c \delta t^2
\]

(94)
is an alternative upper bound at time step \( l \). A useful choice between (94) and (95) depends on the infinitesimal stepsite of course, where ‘size’ means size relative to the spatial decay. We shall discuss this matter of time step size in the next section. Assuming \( 0 < \nu < 1 \) without loss of generality, we are free to choose \( \lambda \) once and for all, i.e., independently of the time step number. Given arbitrary \( T > 0 \) and having chosen \( \rho > 0 \) as above which choose

\[
0 < \lambda \leq \frac{\nu_{\rho}}{1+\rho T} \frac{c}{16\pi^2 c C^2}.
\]

(96)
such that from (96) we have for some \( c > 0 \)

\[
|u_{\alpha}^{k-1}(l\delta t)| \leq |u_{\alpha}^{k-1}((l-1)\delta t)| \left( 1 - \nu_4^2 \sum_{j=1}^{n} \alpha_j^2 \delta t - \frac{\nu_{\rho}}{1+\rho T} \delta t \right) + \frac{\lambda_{\alpha} c C^2}{(1 + |\alpha|^{n+2}) \delta t} + c \delta t^2.
\]

(97)
From the latter representation it is easy to prove the preservation of upper bounds. Having

\[
|u_{\alpha}^{k-1}(l\delta t)| \leq \frac{C}{1 + |\alpha|^{n+2}},
\]

(98)
for some \( C > \nu > 0 \) consider the set \( M \) of all \( \alpha \) with

\[
|u_{\alpha}^{k-1}(l\delta t)| \in \left[ \frac{C}{1 + |\alpha|^{n+2}} \right]^{\gamma} \left[ \frac{1 + |\alpha|^{n+2}}{1 + |\alpha|^{n+2}} \right].
\]

(99)
If no such mode \( \alpha \) exists, i.e., if \( M = \emptyset \) then the upper bound in (95) is clearly preserved at the next time step. Moreover, as \( \rho > 0 \) is small, for all \( \alpha \neq 0 \) where (97) holds we clearly have inductively that

\[
|u_{\alpha}^{k-1}(l\delta t)| \leq |u_{\alpha}^{k-1}((l-1)\delta t)|.
\]

(100)
Furthermore, for \( \alpha = 0 \) the Leray projection term cancels and the formula in (97) simplifies to

\[
u_{\alpha}^{k-1}(l\delta t) \doteq \left( u_{\alpha}^{k-1}(l\delta t) - \lambda_{\alpha} \mu_{\alpha}^{k-1} 2\pi l \sum_{j=1}^{n} \sum_{\gamma \in \mathbb{Z}^n} (-\gamma_j) u_{\alpha}^{k-1}((l-1)\delta t) u_{j\gamma}(l\delta t) \delta t \right. \\
u_{\alpha}^{k-1}(l\delta t) u_{\alpha}^{k-1}((l-1)\delta t) \delta t.
\]

(101)
Hence if \(0 = (0, \cdots, 0) \in M\), then we have

\[
|w^{k-1}_0(l \delta t)| \leq |w^{k-1}_0((l - 1) \delta t)|
\]

(102)

by the choice of \(\lambda\). Summing in (101) with weight \(\frac{1}{1+|\alpha|+\pi}\) over \(\alpha\) we observe that the upper bound \(\max_{1 \leq i \leq n} |w^F_i(l \delta t)|_{\mathcal{H}^m} \leq C\) is preserved.

5 Global transfinite schemes

We summarize the scheme for the computation of the velocity components \(v_i, 1 \leq i \leq n\) up to arbitrary given \(T > 0\) with \(T \in \frac{1}{\mathbb{N}}\), and show how the upper bounds are preserved on a global time level. We use a time discretization \(t_k = \frac{k}{r}, k \geq 0\). In order to gain one small parameter \(\lambda\) for the nonlinear terms we consider the function

\[
\lambda w^F = v^F.
\]

(103)

For a certain hyperfinite \(N_0\) we have at the first time step \(k = 1\)

\[
\begin{aligned}
    w^F(N_0 \delta t) &= \Pi_{m=0}^{N_0} \left( \delta_{ij,\alpha,\beta} \exp \left( -\nu 4\pi \sum_{i=1}^n \alpha_i^2 \delta t \right) \right) \\
    &\times \left( \exp \left( \left( \frac{e^{\lambda}}{ij,\alpha,\beta} \right) \delta t \right) \right) \lambda^{-1} h^F ,
\end{aligned}
\]

(104)

where \(N_0 \in \mathbb{N} \cup \{\infty\}\) and \(N_0 \delta t = \frac{1}{r}\). Here the Euler matrix \(n\mathbb{Z}^n \times n\mathbb{Z}^n\)-matrix has the entries

\[
e_{ij,\alpha,\beta}(mdt) = -\lambda 2\pi i \nu (\alpha - \gamma) \nu (\alpha - \gamma) (mdt)
\]

\[
+ \lambda 2\pi i \nu (\alpha - \gamma) 4\pi \sum_{i=1}^n \gamma_i (\alpha - \gamma) \nu (\alpha - \gamma) (mdt) \sum_{i=1}^n 4\pi \nu (\alpha - \gamma)
\]

(105)

Having computed \(w^F(\frac{(k-1)}{2} \delta t)\) after time step \(k - 1\) we get

\[
\begin{aligned}
    w^F \left( N_0 \delta t + \frac{(k-1)}{2} \right) &= \Pi_{m=0}^{N_0} \left( \delta_{ij,\alpha,\beta} \exp \left( -\nu 4\pi \sum_{i=1}^n \alpha_i^2 \delta t \right) \right) \\
    &\times \left( \exp \left( \left( \frac{e^{\lambda}}{ij,\alpha,\beta} \right) \delta t \right) \right) \lambda^{-1} w^F \left( \frac{(k-1)}{2} \right) ,
\end{aligned}
\]

(106)

where \(\lim_{N_0 \to \infty} N_0 \delta t = \frac{1}{r}\). At each time step \(k \geq 1\) we consider a subscheme for the function \(u^{k-1} = (u_i^{k-1}, \cdots, u_n^{k-1})^T\) with

\[
(1 + \rho t) u_i^{k-1}(s,.) = w_i 
\]

(107)

for \(t \in \left[ \frac{k-1}{2}, \frac{k}{r} \right]\) and corresponding \(s \in \left[ 0, \frac{1}{\sqrt{3}} \right]\). Having computed \(v_i \left( \frac{k-1}{2},.. \right), 1 \leq i \leq n\) with

\[
|v_i \left( \frac{k-1}{2},.. \right)|_{\mathcal{H}^m} \leq C \left( 1 + \rho \frac{k-1}{2} \right),
\]

(108)

we have

\[
|u_i^{k-1}(0,.)|_{\mathcal{H}^m} \leq C
\]

(109)
Hence by Lemma 2.3 we have

$$|u_i^{k-1}(s,.)|_{H^m} \leq \frac{C}{\lambda},$$

(110)

for $s \in \left[0, \frac{1}{\sqrt{3}} \right]$. Hence, for corresponding $t \in \left[\frac{k-1}{2}, \frac{k}{2} \right]$ we get

$$|v_i^{k-1}(t,.)|_{H^m} = \lambda (1 + \rho t) \left| u_i^{k-1}(s,.)\right|_{H^m} \leq \frac{C}{\lambda}(1 + \rho t),$$

(111)

which shows that the transfinite subscheme of time step $k$ is well-defined, and that the scheme is global. As we have a transfinite scheme we are rather flexible with respect to the choice of the time step size $\delta t$. However, depending on the spatial regularity (degree of polynomial decay) we have to be a little careful depending on the first order approximation which we choose, i.e., if we use the first order approximation $1 - \nu \pi^2 \sum_{i=1}^{n} \alpha_i^2$ for $\exp(-\nu \pi^2 \sum_{i} \alpha_i^2)$ as a factor in our scheme, it is better to consider smaller time step size (otherwise the only preservation with polynomial decay of order $n + 1$ can be proved. Finally, we mention that auto-controlled schemes do not provide uniqueness, but, to the contrary, variations of such schemes may be used in order to construct singular solutions such as for the Euler equation in [3]. Furthermore, such ideas may be used in order to extend results for certain classes of hyperbolic equation as considered in [1] via inviscid limits. Explicit forms and classical interpretations of the scheme proposed will be considered elsewhere.

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