Claw-freeness, 3-homogeneous subsets of a graph and a reconstruction problem

Maurice Pouzet, Hamza Si Kaddour, Nicolas Trotignon

To cite this version:

Maurice Pouzet, Hamza Si Kaddour, Nicolas Trotignon. Claw-freeness, 3-homogeneous subsets of a graph and a reconstruction problem. Contributions to Discrete Mathematics, 2011, 6, pp.92-103. hal-00839758

HAL Id: hal-00839758
https://hal.science/hal-00839758
Submitted on 29 Jun 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Abstract. We describe \(\text{Forb}(K_{1,3}, \overline{K_{1,3}})\), the class of graphs \(G\) such that \(G\) and its complement \(\overline{G}\) are claw-free. With few exceptions, it is made of graphs whose connected components consist of cycles of length at least 4, paths, and of the complements of these graphs. Considering the hypergraph \(\mathcal{H}(3)(G)\) made of the 3-element subsets of the vertex set of a graph \(G\) on which \(G\) induces a clique or an independent subset, we deduce from above a description of the Boolean sum \(G + G'\) of two graphs \(G\) and \(G'\) giving the same hypergraph. We indicate the role of this latter description in a reconstruction problem of graphs up to complementation.

1. Results and motivation

Our notations and terminology mostly follow [3]. The graphs we consider in this paper are undirected, simple and have no loop. That is a graph is a pair \(G := (V, E)\), where \(E\) is a subset of \([V]^2\), the set of 2-element subsets of \(V\). Elements of \(V\) are the vertices of \(G\) and elements of \(E\) its edges. We denote by \(V(G)\) the vertex set of \(G\) and by \(E(G)\) its edge set. We look at members of \([V]^2\) as unordered pairs of distinct vertices. If \(A\) is a subset of \(V\), the pair \(G_{|A} := (A, E \cap [A]^2)\) is the graph induced by \(G\) on \(A\). The complement of \(G\) is the simple graph \(\overline{G}\) whose vertex set is \(V\) and whose edges are the unordered pairs of nonadjacent and distinct vertices of \(G\), that is \(\overline{G} = (V, \overline{E})\), where \(\overline{E} = [V]^2 \setminus E\). We denote by \(K_3\) the complete graph on 3 vertices and by \(K_{1,3}\) the graph made of a vertex linked to a \(\overline{K}_3\). The graph \(K_{1,3}\) is called a claw, the graph \(K_{1,3}\) a co-claw.

In [4], Brandstädt and Mahfud give a structural characterization of graphs with no claw and no co-claw; they deduce several algorithmic consequences (relying on bounded clique width). We will give a more precise characterization of such graphs. We denote by \(A_6\) the graph on 6 vertices made of a \(K_3\) bounded by three \(K_3\) (cf. Figure 1) and by \(C_n\) the \(n\)-element cycle, \(n \geq 4\). We denote by \(P_9\) the Paley graph on 9 vertices (cf. Figure 1). Note that \(P_9\) is isomorphic to its complement \(\overline{P}_9\), to the line-graph of \(K_{3,3}\) and also to \(K_3 \square K_3\), the cartesian product of \(K_3\) by itself (see [3] page 30 if needed for a definition of the cartesian product of graphs, and see [15] page 176 and [3] page 28 for a definition and basic properties of Paley graphs).
Given a set $\mathcal{F}$ of graphs, we denote by $\text{Forb}(\mathcal{F})$ the class of graphs $G$ such that no member of $\mathcal{F}$ is isomorphic to an induced subgraph of $G$. Members of $\text{Forb}(K_3)$, resp. $\text{Forb}(K_{1,3})$ are called triangle-free, resp. claw-free graphs.

The main result of this note asserts:

**Theorem 1.1.** The class $\text{Forb}(K_{1,3}, \overline{K_{1,3}})$ consists of the induced subgraphs of $P_9$, of graphs whose connected components consist of cycles of length at least 4, paths, and of the complements of these graphs.

As an immediate consequence of Theorem 1.1, note that the graphs $A_6$ and $\overline{A_6}$ are the only members of $\text{Forb}(K_{1,3}, \overline{K_{1,3}})$ which contain a $K_3$ and a $\overline{K_3}$ with no vertex in common. Note also that $A_6$ and $\overline{A_6}$ are very important graphs for the study of how maximal cliques and stable sets overlap in general graphs. See the main theorem of [7], see also [8]. Also, in [10], page 31, a list of all self-complementary line-graphs is given. Apart from $C_5$, there are all induced subgraphs of $P_9$.

From Theorem 1.1 we obtain a characterization of the Boolean sum of two graphs having the same 3-homogeneous subsets. For that, we say that a subset of vertices of a graph $G$ is homogeneous if it is a clique or an independent set (note that the word homogeneous is used with this meaning in Ramsey theory; in other areas of graph theory it has other meanings, several in fact). Let $H^{(3)}(G)$ be the hypergraph having the same vertices as $G$ and whose hyperedges are the 3-element homogeneous subsets of $G$. Given two graphs $G$ and $G'$ on the same vertex set $V$, we recall that the **Boolean sum** $G\oplus G'$ of $G$ and $G'$ is the graph on $V$ whose edges are unordered pairs $e$ of distinct vertices such that $e \in E(G)$ if and only if $e \notin E(G')$. Note that $E(G\oplus G')$ is the symmetric difference $E(G)\Delta E(G')$ of $E(G)$ and $E(G')$. The graph $G\oplus G'$ is also called the **symmetric difference** of $G$ and $G'$ and denoted by $G\Delta G'$ in [3]. Given a graph $U$ with vertex set $V$, the **edge-graph** of $U$ is the graph $S(U)$ whose vertices are the edges $u$ of $U$ and whose edges are unordered pairs $uv$ such that $u = xy$, $v = xz$ for three distinct elements $x, y, z \in V$ such that $yz$ is not an edge of $U$. Note that the edge-graph $S(U)$ is a spanning subgraph of $L(U)$, the **line-graph** of $U$, not to be confused with it.

Claw-free graphs and triangle-free graphs are related by means of the edge-graph construction. Indeed, as it is immediate to see, for every graph $U$, we have:

$$U \in \text{Forb}(K_{1,3}) \iff S(U) \in \text{Forb}(K_3) \quad (*)$$

Our characterization is this:

**Theorem 1.2.** Let $U$ be a graph. The following properties are equivalent:
There are two graphs $G$ and $G'$ having the same 3-element homogeneous subsets such that $U := G \oplus G'$; 

(2) $S(U)$ and $S(\overline{U})$ are bipartite; 

(3) Either (i) $U$ is an induced subgraph of $P_9$, or (ii) the connected components of $U$, or of its complement $\overline{U}$, are cycles of even length or paths.

As a consequence, if the graph $U$ satisfying Property (1) is disconnected, then $U$ contains no 3-element cycle, moreover, if $U$ contains no 3-element cycle then each connected component of $U$ is a cycle of even length, or a path, in particular $U$ is bipartite.

The implication (2) $\Rightarrow$ (3) in Theorem 1.2 follows immediately from Theorem 1.1. Indeed, suppose that Property (2) holds, that is $S(U)$ and $S(\overline{U})$ are bipartite, then from Formula (⋆) and from the fact that $S(A_6)$ and $S(C_{2n})$, $n \geq 4$, are respectively isomorphic to $C_9$ and to $C_{2n}$, we have:

$$U \in \text{Forb}\{K_{1,3}, K_{1,3}, A_6, C_{2n+1}, C_{2n+1} : n \geq 2\}.$$ 

From Theorem 1.1, Property (3) holds. The other implications, obtained by more straightforward arguments, are given in Subsection 2.3.

This leaves open the following:

**Problem 1.3.** Which pairs of graphs $G$ and $G'$ with the same 3-element homogeneous subsets have a given Boolean sum $U := G \oplus G'$?

A partial answer, motivated by the reconstruction problem discussed below, is given in [6]. We mention that two graphs $G$ and $G'$ as above are determined by the graphs induced on the connected components of $U := G \oplus G'$ and on a system of distinct representatives of these connected components (Proposition 10 [6]).

A quite natural problem, related to the study of Ramsey numbers for triples, is this:

**Problem 1.4.** Which hypergraphs are of the form $H^{(3)}(G)$?

An asymptotic lower bound of the size of $H^{(3)}(G)$ in terms of $|V(G)|$ was established by A.W. Goodman [9].

The motivation for Theorem 1.2 (and thus Theorem 1.1) originates in a reconstruction problem on graphs that we present now. Considering two graphs $G$ and $G'$ on the same set $V$ of vertices, we say that $G$ and $G'$ are isomorphic up to complementation if $G'$ is isomorphic to $G$ or to the complement $\overline{G}$ of $G$. Let $k$ be a non-negative integer, we say that $G$ and $G'$ are $k$-hypomorphic up to complementation if for every $k$-element subset $K$ of $V$, the graphs $G|_K$ and $G'|_K$ induced by $G$ and $G'$ on $K$ are isomorphic up to complementation. Finally, we say that $G$ is $k$-reconstructible up to complementation if every graph $G'$ which is $k$-hypomorphic to $G$ up to complementation is in fact isomorphic to $G$ up to complementation.

The following problem emerged from a question of P.Ille [12]:

**Problem 1.5.** For which pairs $(k, v)$ of integers, $k < v$, every graph $G$ on $v$ vertices is $k$-reconstructible up to complementation?

It is immediate to see that if the conclusion of the problem above is positive for some $k, v$, then $v$ is distinct from 3 and 4 and, with a little bit of thought, that if $v \geq 5$ then $k \geq 4$ (see Proposition 4.1 of [5]). With J. Dammak, G. Lopez [5] and [6] we proved that the conclusion is positive if:
equal to $G(U)$ (Harary and Holzmann [13]) A graph $G$ is the line-graph of a triangle-free graph if and only if $G$ contains no claw and no diamond.

Proof. Since [13] is very difficult to find, we include a short proof. Checking that a line-graph of a triangle-free graph contains no claw and no diamond is a routine matter. Conversely, let $G$ be graph with no claw and no diamond. A theorem of Beineke [1] states that there exists a list $L$ of nine graphs such any graph that does not contain a graph from $L$ is a line-graph. One of the nine graphs is the claw and the eight remaining ones all contain a diamond. So, $G = L(R)$ for some graph $R$. Let $R'$ be the graph obtained from $R$ by replacing each connected component of $R$ isomorphic to a triangle by a claw. So, $L(R) = L(R') = G$. We claim that $R'$ is triangle-free. Else let $T$ be a triangle of $R'$. From the construction of $R'$, there is a vertex $v \notin T$ in the connected component of $R'$ that contains $T$. So we may choose $v$ with a neighbor in $T$. Now the edges of $T$ and one edge from $v$ to $T$ induce a diamond of $G$, a contradiction.

Let $G$ be in the class $\text{Forb}\{K_{1,3}, \overline{K_{1,3}}\}. $
We may assume that $G$ and $\overline{G}$ are connected.

Else, up to symmetry, $G$ is disconnected. If $G$ contains a vertex $v$ of degree at least 3, then $N_G(v)$ contains an edge (for otherwise there is a claw), so $G$ contains a triangle. This is a contradiction since by picking a vertex in another component we obtain a co-claw. So all vertices of $G$ are of degree at most 2. It follows that the components of $G$ are cycles (of length at least 4, or there is a co-claw) or paths, an outcome of the theorem. This proves (1).

We may assume that $G$ contains no induced path $P_6$.

Else $G$ has an induced subgraph $H$ that is either a path on at least 6 vertices or a cycle on at least 7 vertices. Suppose $H$ maximal with respect to this property. If $G = H$, an outcome of the theorem is satisfied. Else, by (1), we pick a vertex $v$ in $G \setminus H$ with at least a neighbor in $H$. From the maximality of $H$, $v$ has a neighbor $p_i$ in the interior of some $P_6 = p_1p_2p_3p_4p_5p_6$ of $H$. Up to symmetry we assume that $v$ has a neighbor $p_i$ where $i \in \{2, 3\}$. So $N_G(v) \cap \{p_1, p_2, p_3, p_4\}$ contains an edge $e$ for otherwise $\{p_i, p_{i-1}, p_{i+1}, v\}$ induces a claw. If $e = p_1p_2$ then $v$ must be adjacent to $p_4, p_5, p_6$ for otherwise there is a co-claw; so $\{v, p_1, p_4, p_6\}$ induces a claw. If $e = p_2p_3$ then $v$ must be adjacent to $p_5, p_6$ for otherwise there is a co-claw, so from the symmetry between $\{p_1, p_2\}$ and $\{p_5, p_6\}$ we may rely on the previous case. If $e = p_4p_5$ then $v$ must be adjacent to $p_1, p_6$ for otherwise there is a co-claw; so $\{v, p_1, p_3, p_6\}$ induces a claw. In all cases there is a contradiction. This proves (2).

We may assume that $G$ contains no $A_6$ and no $\overline{A_6}$.

Else up to a complementation, let $aa', bb', cc'$ be three disjoint edges of $G$ such that the only edges between them are $ab, bc, ca$. If $V(G) = \{a, a', b, b', c, c'\}$, an outcome of the theorem is satisfied, so let $v$ be a seventh vertex of $G$. We may assume that $av \in E(G)$ (else there is a co-claw). If $a'v \in E(G)$ then $vb', vc' \in E(G)$ (else there is a co-claw) so $\{v, a', b', c'\}$ is a claw. Hence $a'v \notin E(G)$. We have $vb \in E(G)$ (or $\{a, a', v, b\}$ is a claw) and similarly $vc \in E(G)$. So $\{a', v, b, c\}$ is a co-claw. This proves (3).

We may assume that $G$ contains no diamond.

We prefer to think about this in the complement, so suppose for a contradiction that $G$ contains a co-diamond, that is four vertices $a, b, c, d$ that induce only one edge, say $ab$. By (1), there is a path $P$ from $\{c, d\}$ to some vertex $w$ that has a neighbor in $\{a, b\}$. We choose such a path $P$ minimal and we assume up to symmetry that the path is from $c$.

If $w$ is adjacent to both $a, b$ then $\{a, b, w, d\}$ induces a co-claw unless $w$ is adjacent to $d$, similarly $w$ is adjacent to $c$, so $\{w, a, c, d\}$ induces a claw. Hence $w$ is adjacent to exactly one of $a, b$, say to $a$. So, $P' = cPwab$ is an induced path and for convenience we rename its vertices $p_1, \ldots, p_k$. If $d$ has a neighbor in $P'$ then, from the minimality of $P'$, this neighbor must be $p_2$. So, $\{p_2, p_1, p_3, d\}$ induces a claw. Hence, $d$ has no neighbor in $P'$.

By (1), there is a path $Q$ from $d$ to some vertex $v$ that has a neighbor in $P'$. We choose $Q$ minimal with respect to this property. From the paragraph above, $v \neq d$. Let $p_i$ (resp. $p_j$) be the neighbor of $v$ in $P'$ with minimum (resp. maximum) index. If $i = j = 1$ then $dQvp_1Pwp_{k-1}p_k$ is a path on at least 6 vertices a contradiction to (2). So, if $i = j$ then $i \neq 1$ and symmetrically, $i \neq k$, so $\{p_{i-1}, p_i, p_{i+1}, v\}$ is a claw.
Hence $i \neq j$. If $j > i + 1$ then $\{v, v', p_i, p_j\}$, where $v'$ is the neighbor of $v$ along $Q$, is a claw. So, $j = i + 1$. So $ep_jp_i$ is a triangle. Hence $P' = p_ip_jp_kp_q$, $Q = dv$ and $i = 2$, for otherwise there is a co-claw. Hence, $P' \cup Q$ form an induced $\overline{K_6}$ of $G$, a contradiction to (3). This proves (4).

Now $G$ is connected and contains no claw and no diamond. So, by Theorem 2.1, $G$ is the line-graph of some connected triangle-free graph $R$. Symmetrically, $\overline{G}$ is also a line-graph. These graphs are studied in [2].

If $R$ contains a vertex $v$ of degree at least 4 then all edges of $R$ must be incident to $v$, for else an edge $e$ non-incident to $v$ together with three edges of $R$ incident to $v$ and non-incident to $e$ form a co-claw in $G$. So all vertices of $R$ have degree at most 3 since otherwise, $G$ is a clique, a contradiction to (1). We may assume that $R$ has a vertex $a$ of degree 3 for otherwise $G$ is a path or a cycle. Let $b, b', b''$ be the neighbors of $a$. Since $a$ has degree 3, all edges of $R$ must be incident to $b, b'$ or $b''$ for otherwise $G$ contains a co-claw.

If one of $b, b', b''$, say $b$, is of degree 3, then $N_R(b) = \{a, a', a''\}$ and all edges of $R$ are incident to one of $a, a', a''$ (or there is a co-claw). So $R$ is a subgraph of $K_{3,3}$. So, since $P_3 = L(K_{3,3})$, $G = L(R)$ is an induced subgraph of $P_3$, an outcome of the theorem. Hence we assume that $b, b', b''$ are of degree at most 2. If $|N_R(\{b, b', b''\}) \setminus \{a\}| \geq 3$, then $R$ contains the pairwise non-incident edge $bc, b'c', b''c''$ say, and the edges $ab, ab', ab''$, $bc, b'c', b''c''$ are vertices of $G$ that induce an $\overline{K_6}$, a contradiction to (3). So, $|N_R(\{b, b', b''\}) \setminus \{a\}| \leq 2$ which means again that $R$ is a subgraph of $K_{3,3}$. □

2.2. Ingredients for the proof of Theorem 1.2. The proof of the equivalence between Properties (1) and (2) of Theorem 1.2 relies on the following lemma.

Lemma 2.2. Let $G$ and $G'$ be two graphs on the same vertex set $V$ and let $U := G + G'$. Then, the following properties are equivalent:

(a) $G$ and $G'$ have the same 3-element homogeneous subsets;
(b) $U(\{x, y\}) = U(\{x, z\}) \neq U(\{y, z\}) \implies G(\{x, y\}) \neq G(\{x, z\})$ for all distinct elements $x, y, z$ of $V$.
(c) The sets $A_1 := E(U) \cap E(G)$ and $A_2 := E(U) \setminus E(G)$ divide $V(S(U))$ into two independent sets and also the sets $B_1 := E(\overline{U}) \cap E(G)$ and $B_2 := E(\overline{U}) \setminus E(G)$ divide $V(S(\overline{U}))$ into two independent sets.

Proof. Observe first that Property (b) is equivalent to the conjunction of the following properties:

(b$_U$): If $uv$ is an edge of $S(U)$ then $u \in E(G)$ iff $v \notin E(G)$.

and

(b$_\overline{U}$): If $uv$ is an edge of $S(\overline{U})$ then $u \in E(G)$ iff $v \notin E(G)$.

(a) $\implies$ (b$_U$). Let us show (a) $\implies$ (b$_U$).

Let $uv \in E(S(U))$, then $u, v \in E(U)$. By contradiction, we may suppose that $u, v \in E(G)$ (the other case implies $u, v \in E(G')$ thus is similar). Since $u$ and $v$ are edges of $U = G + G'$ then $u, v \notin E(G')$. Let $w := yz$ such that $u = xy, v = xz$. Then $w \notin E(U)$ and thus $w \in E(G)$ iff $w \in E(G')$.

If $w \in E(G)$, $\{x, y, z\}$ is an homogeneous subset of $G$. Since $G$ and $G'$ have the same 3-element homogeneous subsets, $\{x, y, z\}$ is a homogeneous subset of $G'$. Hence, since $u, v \notin E(G')$, $w = yz \notin E(G')$, thus $w \notin E(G)$, a contradiction.

If $w \notin E(G)$, then $w \notin E(G')$; since $u, v \notin E(G')$ it follows that $\{x, y, z\}$ is an homogeneous subset of $G'$. Consequently $\{x, y, z\}$ is an homogeneous subset of $G$. Consequently $\{x, y, z\}$ is an homogeneous subset of $G$. Consequently
Since \( u, v \in E(G) \), then \( w \in E(G) \), a contradiction.

The implication \( (a) \implies (b) \) is similar.

\( (b) \implies (a) \). Let \( T \) be a \( K_3 \) of \( G \). Suppose that \( T \) is not an homogeneous subset of \( G' \) then we may suppose \( T = \{ u, v, w \} \) with \( u, v \in E(G') \) or \( u, v \in E(G') \) and \( w \notin E(G') \). In the first case \( uv \in E(S(U)) \), which contradicts Property \((b)\), in the second case \( uv \notin E(S(U)) \), which contradicts Property \((b)\).

\( (b) \implies (c) \). First \( V(S(U)) = E(U) = A_1 \cup A_2 \) and \( V(S(U)) = E(U) = B_1 \cup B_2 \). Let \( u, v \) be two distinct elements of \( A_1 \) (respectively \( A_2 \)). Then \( u, v \in E(G) \) (respectively \( u, v \notin E(G) \)). From \((b)\) we have \( uv \notin E(S(U)) \). Then \( A_1 \) and \( A_2 \) are independent sets of \( V(S(U)) \). The proof that \( B_1 \) and \( B_2 \) are independent sets of \( V(S(U)) \) is similar.

\( (c) \implies (b) \). This implication is trivial. \( \square \)

2.3. Proof of Theorem 1.2. Implication \( (1) \implies (2) \) follows directly from implication \( (a) \implies (c) \) of Lemma 2.2. Indeed, Property \((c)\) implies trivially that \( S(U) \) and \( S(U) \) are bipartite.

\( (2) \implies (1) \). Suppose that \( S(U) \) and \( S(U) \) are bipartite. Let \( \{ A_1, A_2 \} \) and \( \{ B_1, B_2 \} \) be respectively a partition of \( V(S(U)) = E(U) \) and \( V(S(U)) = E(U) \) into independent sets. Note that \( A_1 \cap B_j = \emptyset \), for \( i, j \in \{ 1, 2 \} \). Let \( G, G' \) be two graphs with the same vertex set as \( U \) such that \( E(G) = A_1 \cup B_1 \) and \( E(G') = A_2 \cup B_1 \).

Clearly \( E(G) \cup E(G') = A_1 \cup A_2 = E(U) \). Thus \( U = G \cup G' \). To conclude that Property \((1)\) holds, it suffices to show that \( G \) and \( G' \) have the same 3-element homogeneous subsets, that is Property \((a)\) of Lemma 2.2 holds. For that, note that \( A_1 = E(U) \setminus E(G) \), \( A_2 = E(U) \setminus E(G) \), \( B_1 = E(U) \setminus E(G) \) and \( B_2 = E(U) \setminus E(G) \) and thus Property \((c)\) of Lemma 2.2 holds. It follows that Property \((a)\) of this lemma holds.

The proof of implication \( (2) \implies (3) \) was given in Section 1. For the converse implication, let \( U \) be a graph satisfying Property \((3)\). It is clear from Figure 1 that \( S(P_9) \) is bipartite (vertical edges and horizontal edges form a partition). Since \( P_9 \) is isomorphic to \( P_9 \), \( S(P_9) \) is bipartite too. Thus, if \( U \) is isomorphic to an induced subgraph of \( P_9 \), Property \((2)\) holds. If not, we may suppose that the connected components of \( U \) are cycles of even length, paths (otherwise, replace \( U \) by \( \overline{U} \)).

In this case, \( S(U) \) is trivially bipartite. In order to prove that Property 2 holds, it suffices to prove that \( S(U) \) is bipartite too. This is a direct consequence of the following claim:

Claim 2.3. If \( U \) is a bipartite graph, then \( S(U) \) is bipartite too.

Proof. If \( c : V(U) \to \mathbb{Z}/2\mathbb{Z} \) is a colouring of \( U \), set \( c' : V(S(U)) \to \mathbb{Z}/2\mathbb{Z} \) defined by \( c'(\{x, y\}) := c(x) + c(y) \).

With this, the proof of Theorem 1.2 is complete.

2.4. A direct proof for \((3) \implies (1)\) of Theorem 1.2. In [6] we gave all possible decompositions of \( U \). When \( U = P_9 \), a decomposition \( U = G \cup G' \) can be given by a picture (see figure 2).
Let $n \geq 2$. Let $X_n$ be an $n$-element set, $x_0, \ldots, x_{n-1}$ be an enumeration of $X_n$, $X_n^0 := \{x_i \in X_n : i \equiv 0 \pmod{2}\}$ and $X_n^1 := X_n \setminus X_n^0$. Set $R_n := [X_n^0]^2 \cup [X_n^1]^2$, $S_n := \{(x_{2i}, x_{2i+1}) : 2i < n\}$, $S'_n := \{(x_{2i+1}, x_{2i+2}) : 2i < n - 1\}$. Let $M_n$ and $M'_n$ be the graphs with vertex set $X_n$ and edge sets $E(M_n) := R_n \cup S_n$ and $E(M'_n) := R_n \cup S'_n$ respectively. Let $M''_n := (X_n, R_n \cup S'_n \cup \{(x_0, x_{n-1})\})$ for $n$ even, $n \geq 4$. For $n \in \{6, 7\}$ we give a picture (see figure 3). For convenience, we set $M_1 = M'_1$ the graph with one vertex and we put $V(M_1) := X_1^0 := \{x_0\}$. When $G$ is a graph of the form $M_n$, $M'_n$, or $M''_n$, with $n \geq 1$, we put $V^0(G) := X_n^0$ and $V^1(G) := X_n^1$.

When $U$ is a cycle of even size $2n$, a decomposition $U = G + G'$ can be given by $G = M_{2n}$ and $G' = M'_{2n}$. When $U$ is a path of size $n$, a decomposition $U = G + G'$ can be given by $G = M_n$ and $G' = M'_n$.

When the connected components of $U$ are cycles of even length or paths, we define $G$ and $G'$ satisfying $U = G + G'$ as follows: For each connected component $C$ of $U$, $(G_C, G'_C)$ is given by the previous step. For distinct connected components $C$ and $C'$ of $U$, $x \in C$, $x' \in C'$, $xx' \in E(G)$ (and $xx' \in E(G')$) if and only if $x \in V^0(G_C)$ and $x' \in V^0(G'_C)$, or $x \in V^1(G_C)$ and $x' \in V^1(G'_C)$.

When the connected components of $U$ are cycles of even length or paths, from $U = G + G'$, the previous step gives a pair $(G, G')$, then a pair $(G, G')$.

**Acknowledgements**

We thank S. Thomassé for his helpful comments.
References

[1] L.W. Beineke, Characterizations of derived graphs, J. Combinatorial Theory 9 (1970) 129-135.
[2] L.W. Beineke, Derived graphs with derived complements, In Lecture Notes in Math. (Proc. Conf., New York, 1970) pages 15-24. Springer (1971).
[3] J.A. Bondy, U.S.R. Murty, Graph Theory, Basic Graph Theory, Graduate Texts in Mathematics, vol 244, Springer, 2008, 654 pp.
[4] A. Brandstädt and S. Mahfud, Maximum weight stable set on graphs without claw and co-claw (and similar graph classes) can be solved in linear time, Information Processing Letters 84 (2002) 251-259.
[5] J. Dammak, G. Lopez, M. Pouzet, H. Si Kaddour, Hypomorphy up to complementation, JCTB, Series B 99 (2009) 84-96.
[6] J. Dammak, G. Lopez, M. Pouzet, H. Si Kaddour, Reconstruction of graphs up to complementation, in Proceedings of the First International Conference on Relations, Orders and Graphs: Interaction with Computer Science, ROGIC08, May 12-15 (2008), Mahdia, Tunisia, pp. 195-203.
[7] X. Deng, G. Li, W. Zang, Proof of Chvátal’s conjecture on maximal stable sets and maximal cliques in graphs, JCTB, Series B 91 (2004) 301-325.
[8] X. Deng, G. Li, W. Zang, Corrigendum to proof of Chvátal’s conjecture on maximal stable sets and maximal cliques in graphs, JCTB, Series B 94 (2005) 352-353.
[9] A.W. Goodman, On sets of acquaintances and strangers at any party, Amer. Math. Monthly 66 (1959) 778-783.
[10] A. Farrugia, Self-complementary graphs and generalisations: a comprehensive reference manual, Master’s thesis, University of Malta (1999).
[11] D.H. Gottlieb, A class of incidence matrices, Proc. Amer. Math. Soc. 17 (1966) 1233-1237.
[12] P. Ille, personal communication, September 2000.
[13] F. Harary and C. Holzmann, Line graphs of bipartite graphs, Rev. Soc. Mat. Chile 1 (1974) 19-22.
[14] W. Kantor, On incidence matrices of finite projective and affine spaces, Math. Zeitschrift 124 (1972) 315-318.
[15] J.H. Van Lint, R.M. Wilson, A course in Combinatorics, Cambridge University Press (1992).
[16] R.M. Wilson, A Diagonal Form for the Incidence Matrices of t-Subsets vs. k-Subsets, Europ J. Combinatorics 11 (1990) 609-615.

ICJ, Université de Lyon, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France
E-mail address: pouzet@univ-lyon1.fr

ICJ, Université de Lyon, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France
E-mail address: sikaddour@univ-lyon1.fr

CNRS, LIAFA, Université Paris Diderot, Paris 7, Case 7014, 75205 Paris Cedex 13, France
E-mail address: nicolas.trotignon@liafa.jussieu.fr