Large deviations for the height in 1D Kardar-Parisi-Zhang growth at late times

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Abstract – We study the atypically large deviations of the height $H \sim O(t)$ at the origin at late times in $(1+1)$-dimensional growth models belonging to the Kardar-Parisi-Zhang (KPZ) universality class. We present exact results for the rate functions for the discrete single-step growth model, as well as for the continuum KPZ equation in a droplet geometry. Based on our exact calculation of the rate functions we argue that models in the KPZ class undergo a third-order phase transition from a strong-coupling to a weak-coupling phase, at late times.

The celebrated Tracy-Widom (TW) distribution was discovered originally in random matrix theory (RMT) \cite{1,2}. In RMT, it describes the probability distribution of the typical fluctuations of the largest eigenvalue of a Gaussian random matrix. Since then, this distribution has emerged in a variety of problems \cite{3,4} (unrelated \emph{a priori} to RMT), ranging from random permutations \cite{5} all the way up to the Yang-Mills gauge field theory \cite{6}. Why is the TW distribution so ubiquitous? It was recently shown that in several systems there is indeed a third-order phase transition in these systems \cite{20–22}. The appearance of the TW distribution in these models then raises a natural question: is there a third-order phase transition between a strong- and a weak-coupling phase in such growth models? If so, how can one describe these two phases? In this letter, we show that there is indeed a third-order phase transition in these growth models by studying the probability distribution $P(H,t)$ of the height $H$ at the origin (suitably centered) renormalized at late times $t \gg 1$. Specifically, we find that $P(H,t)$, for $t \gg 1$, has three different behaviors, see fig. 1:

\begin{equation}
P(H,t) \sim \begin{cases} 
\frac{1}{\lambda^{1/3}} f \left( \frac{H}{\lambda^{1/3}} \right), & H \sim O(t^{1/3}) \text{ II}, \\
\exp^{-t \Phi_{+}(H/t)}, & H \sim O(t^{1/3}) \text{ III}. 
\end{cases}
\end{equation}

The regime II is well known and it describes the typical height fluctuations ($H \sim O(t^{1/3})$) and the scaling function $f(s)$ is given by the TW distribution. The scaling function depends on the initial conditions: for the flat geometry it corresponds to $f_1(s)$ (i.e., TW for the Gaussian orthogonal ensemble, GOE), while for the curved (or droplet)
Thus as to the left (red line) and to the right (green line) are described universally, $\Phi$ will be explained later. Note that on the scale the central part of width $H$ the large positive fluctuations ($s \to -\infty$) and $s \to +\infty$, respectively, by the left and right large deviation functions in eq. (1).

The new results in this letter concern the atypical large height fluctuations in regimes I and III in eq. (1). The regime I corresponds to the large negative fluctuations ($H \sim O(t) < 0$) and is characterized by the left large deviation function $\Phi_-(z)$. Similarly, the regime III describes the large positive fluctuations ($H \sim O(t) > 0$) and is characterized by the right large deviation function $\Phi_+(z)$. These two rate functions $\Phi_-(z)$ are the characteristics of the two phases: $\Phi_-(z)$ corresponds to the strong-coupling phase, while $\Phi_+(z)$ describes the weak-coupling phase (as will be explained later). Note that on the scale $H \sim O(t)$, the central part of width $O(t^{1/3})$ is effectively reduced to a point $z = 0$ as $t \to \infty$. Indeed, it follows from eq. (1) that

$$\lim_{t \to \infty} \frac{-1}{t^2} \ln P(H = z, t) = \begin{cases} \Phi_-(z), & z \leq 0, \\ \Phi_+(z), & z > 0. \end{cases}$$

Thus as $t \to \infty$, $z = 0$ becomes a critical point and $\Phi_-(z)$ can be interpreted as the “free energy” of the strong-coupling phase. We further show that it vanishes universally, $\Phi_-(z) \propto |z|^3$, as $z \to 0^-$, thus indicating a third-order phase transition. Therefore, in order to probe this third-order transition it is important to compute the large deviation functions. In this letter, we compute $\Phi_\pm(z)$ explicitly for the droplet geometry in i) a discrete single-step growth model belonging to the KPZ class and ii) the continuum KPZ equation. In general, $\Phi_\pm(z)$ are non-universal and depend on the model. However, their small arguments behaviors are universal: $\Phi_-(z) \propto |z|^3$ as $z \to 0^-$ and $\Phi_+(z) \propto z^{3/2}$ as $z \to 0^+$. Indeed, as the critical point $z = 0$ is approached from either side, the large deviation behaviors smoothly match with the asymptotic tails of the TW distribution (2).

We start by analyzing a directed polymer model belonging to the KPZ universality class, introduced by Rost [23] and studied extensively by Johansson [10]. This model can be translated to a discrete space-time $(x, t)$ growth model in a “droplet” geometry. The growth takes place on the substrate $-t \leq x \leq t$ (see fig. 2), starting from the seed at the origin $x = 0$ at $t = 0$. The interface height $h(x, t)$, at site $x$ and at time $t$, evolves in the bulk $-t < x < t$ as

$$h(x, t) = \max[h(x-1, t-1), h(x+1, t-1)] + \eta(x, t),$$

where $\eta(x, t)$ are independent and identically distributed (i.i.d.) non-negative random variables each drawn from an exponential distribution: $p(\eta) = e^{-\eta}$ for $\eta \geq 0$. This model (4) also maps onto the usual totally asymmetric exclusion process (TASEP) on the infinite lattice, $x \in \mathbb{Z}$, with step initial condition (i.e., initially filled for $x \leq 0$ and empty for $x > 0$): $h(0, t)$ then corresponds to the total integrated current up to time $t$ at $x = 0$. Johansson showed that at late times, the average height $h(x, t) = v(x/t) t$ with $v(z) = 1 + \sqrt{1-z^2}$ exhibiting a semi-circular droplet shape (see fig. 2). Moreover, the height at the origin at late times behaves as $h(0, t) \approx 2 t + 2 t^{1/3} \chi_2$, where $\chi_2$ is a $t$-independent random variable distributed via the TW distribution for the GUE, $f_2(s)$ [10]. By exploiting an exact mapping to the largest eigenvalue of complex Wishart matrices [10], and using the results for the large deviations of the latter [24,25], we establish the result in eq. (1) (with $H = h(0, t) - t$). In regime I, we get [26] (see also [31] for related results for

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The heights at the two edge points $x = \pm t$, $h_{\pm}(t) = h(x = \pm t, t)$, evolve differently: $h_{\pm}(t) = h_{\pm}(t-1) + \eta(\pm t, t)$. 

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the model in (4) with geometric disorder:
\[ \Phi_-(z) = \frac{1}{8} \left(2z - z^2 - 2\ln(1 + z)\right), \quad -1 < z < 0, \]
where \( z > -1 \) since the height \( h(0,t) > 0 \). As \( z \to 0^- \), one gets \( \Phi_-(z) \sim |z|^3/12 \) as announced at the beginning of the paper. In regime III, we find
\[ \Phi_+(z) = 2\sqrt{2(z+1)} + \ln \left(2z + 1 - 2\sqrt{2(z+1)}\right), \quad z \geq 0, \]
which behaves as \( \Phi_+(z) \sim (4/3)|z|^{3/2} \) as \( z \to 0^+ \). Note that in regime II, if we make \( H \sim \mathcal{O}(t) \) and use the asymptotic behaviors of the TW distribution in eq. (2) with \( \beta = 2 \), it can be checked that it matches smoothly with the large deviation regimes on both sides. Interestingly, in this height model (4) there is a clear physical explanation as to why the left tail (regime I in (1)) scales like \( \sim e^{-t^2} \) while the right tail (regime III in (1)) behaves like \( \sim e^{-t} \). Indeed, in order to realize a configuration of \( H \) much smaller than its typical value (regime I), the noise variables \( \eta(x,t) \) at all \( x \) within the \( (1 + 1) \)-dimensional wedge (cf. fig. 2) should be small. Indeed, if any of the \( \eta(x,t) \) within this wedge is big, the dynamics in eq. (4) would force the neighboring sites at the next time step to be big. The probability of this event, where \( \text{collectively all the noise variables } \eta(x,t) \text{ inside the wedge } (|x| < t), \text{ of area } \propto t^2, \text{ are all small, is proportional to } e^{-t^2} \) (the noise variables being i.i.d.). In contrast, a configuration where \( H \) is much bigger than its typical value (regime III) can be realized by adding large positive noise variables at the origin \( \eta(x = 0, \tau) \) at all times \( \tau \) between 0 and \( t \). The probability of this event is simply \( \sim e^{-t^2} \) as the noises at different times are i.i.d. Hence this event is not a collective one, unlike the left large deviation. Thus, the left large deviation (regime I in eq. (1)) is the analogue of the “strong-coupling phase” and the right large deviation (regime II in eq. (1)) corresponds to the “weak-coupling” phase. The transition between the two phases is a third-order phase transition, as \( \Phi_-(z) \propto |z|^{3/2} \) as \( z \to 0^- \), as mentioned above. This picture is similar to other third-order phase transitions observed before in RMT and reviewed recently in ref. [7].

While the right tail rate function \( \Phi_+(z) \) (z) has been studied numerically [32] and, more recently, analytically [33] in discrete growth models, the left tail \( \Phi_-(z) \) is much harder to compute, and there are very few exact results, an exception being the longest increasing subsequence in random permutations (for both tails) [34]. We now show that these rate functions can also be calculated for the continuum KPZ equation itself, where the height field \( h(x,t) \) evolves as
\[ \partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \sqrt{D} \xi(x,t), \]
where \( \nu > 0 \) is the coefficient of diffusive relaxation, \( \lambda_0 > 0 \) is the strength of the non-linearity and \( \xi(x,t) \) is a Gaussian white noise with zero mean and \( \langle (\xi(x,t)\xi(x',t')) \rangle = \delta(x-x')\delta(t-t') \). We use everywhere the natural units of space \( x' = (2\nu)^3/(D\lambda_0^2) \), time \( t' = 2(2\nu)^{3/2}/(D\lambda_0^2) \) and height \( h' = \frac{\delta}{D\lambda_0^2} \).

Here, for definiteness we focus on the narrow wedge initial condition, \( h(x,0) = -|x|/\delta \); with \( \delta < 1 \), which gives rise to a curved (“droplet”) mean profile as time evolves [15–19]. We focus on the shifted height at the origin \( x = 0 \), \( H(t) = h(0,t) + \frac{\nu}{2\lambda_0} \), which fluctuates typically on a scale \( t^{1/3} \) around its mean at large time, as described by the regime II in eq. (1) with \( f(s) = f_2(s) \), the TW distribution for the GUE.

We show below that for the continuum KPZ equation, in a droplet geometry, the generic result in eq. (1) holds in regimes I and III as well. Interestingly the rate functions turn out to be rather simple in this case,
\[ \Phi_-(z) = \frac{1}{12} |z|^3, \quad z \leq 0, \]
\[ \Phi_+(z) = \frac{4}{3} |z|^{3/2}, \quad z \geq 0. \]

Thus, the continuum KPZ equation also exhibits a third-order phase transition at the critical point \( z = 0 \).

To derive the rate functions for the continuum KPZ case, we start from an exact formula [15–18], valid at all times \( t \) in the droplet geometry. It relates the following generating function to a Fredholm determinant (FD):
\[ g_t(s) := \langle \exp(-\int_{-t}^{t/3} H) \rangle = \det(I - P_s K_t P_s), \]
where the finite time kernel is
\[ K_t(r, r') = \int_{-\infty}^{\infty} du \frac{A_i(r + u)A_i(r' + u)}{1 + e^{r/r' + u}}, \]
and \( P_s \) is the projector on the interval \([s, +\infty)\) (see footnote 2). In eq. (11), \( A_i(x) \) denotes the Airy function.

Let us recall that to obtain the typical fluctuations regime (II) in formula (1), where \( H(t) \sim t^{1/3} \), one needs to take the limit \( t \to +\infty \) at fixed \( s \) in (10). In that limit \( K_t(r, r') \) converges to the standard Airy kernel, \( K_{Ai}(r, r') = \int_{-\infty}^{\infty} du A_i(r + u)A_i(r' + u) \) and the right-hand side (r.h.s.) converges to the GUE-TW distribution. The left-hand side (l.h.s.) of (10) converges to \( (\theta(s-t^{-1/3}H)(y)) \) (where \( \theta(x) \) is the Heaviside step function), and one obtains
\[ \lim_{t \to +\infty} \text{Prob}(\chi_{t,s} < s) = \det(I - P_s K_{Ai} P_s) = f_2(s), \]
where \( f_2(s) = \int_s^{\infty} f_2(s')ds' \) is the cumulative distribution function (CDF) of the GUE-TW distribution. To compute the rate functions \( \Phi_{\pm}(z) \) we now consider the

\[ \text{Footnote 2: We recall that, for a trace-class operator } K(x,y) \text{ such that } \text{Tr}K = \int dx K(x,x) \text{ is well defined, } \text{det}(I - K) = \exp[-\sum_{n} \text{Tr}K^n/n], \text{ where } \text{Tr}K^n = \int dx_1 \cdots \int dx_n K(x_1, x_2)K(x_2, x_3) \cdots K(x_n, x_1). \]

The effect of the projector \( P_s \) in (10) is simply to restrict the integrals over \( x_i \)’s to the interval \([s, +\infty)\).
formula (10) in the limit when $s$ and $t$ are both large, keeping the ratio $y = s/t^{3/2}$ fixed.

**Right tail:** We start with the right large deviation function, therefore we consider formula (10) in the regime of large $s > 0$. Consider first the l.h.s. of eq. (10). It is convenient to introduce a random variable $y$ (independent of $H$) distributed via the Gumbel distribution, of CDF given by

$$\langle \theta(b - y) \rangle_y = e^{-e^{-b}}. \tag{13}$$

Substituting $b = st^{1/3} - H$ in (13) allows us to rewrite the l.h.s of (10) as

$$1 - \langle \exp(-e^{H(t) - t^{1/3}s}) \rangle = \langle \text{Prob}(H > st^{1/3} - y) \rangle_y. \tag{14}$$

Now consider the r.h.s of eq. (10) for $s \gg 1$. Expanding the FD in powers of $K_t$ and keeping only the first two terms one obtains

$$\det(I - P_sK_sP_s) \approx 1 - \int_s^{+\infty} dr K_t(r, r). \tag{15}$$

Equating eqs. (14) and (15) and taking a derivative with respect to $s$, for $s \gg 1$,

$$t^{1/3}(P(H = st^{1/3} - y, t)) \approx K_t(s, s). \tag{16}$$

We first study the asymptotics of $K_t(s, s)$ for large $s \sim t^{2/3}$. Performing a change of variable $u = -t^{1/3}v$, eq. (11) becomes

$$K_t(yt^{2/3}, yt^{2/3}) = t^{2/3} \int_{-\infty}^{+\infty} dv \frac{A_t^2(t^{2/3}(y - v))}{1 + e^{tv}} \tag{17}$$

with $y = \mathcal{O}(1)$. This integral can be analyzed for large $t$ [26] and we obtain

$$K_t(yt^{2/3}, yt^{2/3}) \sim e^{-t(1)}(y). \tag{18}$$

where the pre-exponential factors are given in [26]. Having obtained the r.h.s of (16) we now consider its l.h.s. We anticipate (and verify a posteriori) that in this right tail the PDF has the form (setting $z = H/t$)

$$\ln P(H, t) = -t^{4} / 3z^{3/2} - a \ln t - \chi_{\text{droplet}}(z) + o(1), \tag{19}$$

where the constant $a$ and the function $\chi_{\text{droplet}}(z)$ are yet to be determined. Inserting this form on the l.h.s. of (16), analyzing the resulting integral [26] and comparing it to the r.h.s. in (18), we find that indeed the ansatz in (19) is correct with $a = 1$ and an explicit form for $\chi_{\text{droplet}}(z)$.

3In eq. (18) the change of behavior at $y = 1/4$ can be simply understood as follows. The expansion of the generating function in the left-hand side of (10) gives $\approx 1 - e^{-t^{1/3}y}e^{Ht} = 1 - e^{-t^{1/3}y(1-1/12)}$ where we used that $e^{Ht} = e^{t^{1/3}}$ and $y = st^{1/3}$. Taking a derivative with respect to $s$ then gives the second line of (18).

given in eq. (53) of the Supplementary Material in [26]. Finally, keeping only the leading behavior of (19) gives us the exact right rate function,

$$\Phi_+(z) = \frac{4}{3} z^{3/2}, \quad z \geq 0, \tag{20}$$

as announced in eq. (9). For the pre-exponential factor in the flat case we find $a = 1/2$ and $\chi_{\text{flat}}(z)$ given in eq. (60) of the Supplementary Material in [26].

This result is also consistent with the known exact large time behavior of the moments, $e^{nH} \sim \mathcal{O}(1)$ calculated using the Bethe ansatz [35]. Indeed a saddle point calculation using $P(H, t) \sim e^{-\frac{4}{3}(\frac{n}{tn^{3/2}})^{3/2}}$ reads

$$\int dH e^{-\frac{4}{3}(\frac{n}{tn^{3/2}})^{3/2}} \sim e^{-\frac{4}{3}n^{3/2}}, \tag{21}$$

where the saddle point, at $H_n = n^2t/4$ for fixed integer $n$, is precisely in the right large deviation regime. Note that the dependence on the initial condition appears only in the (subdominant) pre-exponential factor of the moments, as discussed in [26] where we establish that $\Phi_+(z) \sim \frac{4}{3} z^{3/2}$ for droplet and flat initial conditions.

**Left tail:** We now focus on the left tail where we set $H/t \sim \mathcal{O}(1) < 0$. In this case, one can show [26] that the l.h.s. of (10) scales as $\sim e^{-t^{1/3}(y-s)/t^{1/3}}$ for $y = \mathcal{O}(1)$. The r.h.s of (10), $Q_t(s) := \det(I - P_sK_sP_s)$, is not easy to analyze in the regime of large negative $s$. Fortunately in ref. [18] the authors proved an exact differential equation satisfied by $Q_t(s)$:

$$\partial_s^2 \ln Q_t(s) = - \int_{-\infty}^{+\infty} dv \sigma_t(v)[Q_t(s, v)]^2, \tag{22}$$

where

$$\sigma_t(v) = \frac{1}{1 + e^{-t^{1/3}v}}, \tag{23}$$

and $\sigma_t(v) = \sigma_0(\sigma_t(v))$. The function $q_t(s, v)$ satisfies a non-linear integro-differential equation in the $s$ variable,

$$\partial_s^2 q_t(s, v) = (s + v + 2) \int_{-\infty}^{+\infty} dv \sigma_t(v)[q_t(s, v)]^2 q_t(s, v) \tag{24}$$

with the boundary condition $q_t(s, v) \approx \mathcal{O}(1)$ for $v$. In the long limit $t \to +\infty$, $\sigma_t(v) \to \delta(v)$ and hence $q_t(s, 0)$ satisfies the standard Painlevé II equation [1].

For large but finite $t$, we substitute the anticipated scaling form $Q_t(s) \sim e^{-t^{1/3}(y-s)/t^{1/3}}$ in (22). The consistency then suggests that $q_t(s, v)$ takes the scaling form

$$q_t(s, v) \sim t^{1/3} q(s/t^{2/3}, vt^{1/3}), \quad \text{for } t \to +\infty, \tag{25}$$

and the scaling function $\tilde{q}(y, v)$ satisfies

$$\int_{-\infty}^{+\infty} dv \tilde{q}(y, v)^2e^{-v} = \Phi_-(y) \tag{26}$$

4We have tried other different forms but none of them were found consistent with eq. (24).
Substituting further the scaling form (25) in the differential equation (24) we obtain as $t \to \infty$

$$y + 2 \int_{-\infty}^{\infty} dv \frac{g(y,v)^2 e^{-v}}{(1 + e^{-v})^2} = 0.$$  \tag{27}

Comparing with (26) immediately gives for all $z \leq 0$, $\Phi^\nu_-(z) = -\frac{z}{2}$. Solving with the boundary condition $\Phi_-(z) \approx z_0 z/12$, coming from matching with the left tail of the TW GUE distribution as $z \to 0^-$, implies

$$\Phi_-(z) = \frac{1}{12} |z|^3, \quad z \leq 0,$$  \tag{28}

as announced in eq. (8).

In summary, our results on large deviations for the height at late times for growth models in the KPZ class suggest a third-order phase transition between a strong- and a weak-coupling phase. Generically the associated rate functions are non-universal but their small argument behaviors are universal, as they match the TW tails. In the case of the continuum KPZ equation these functions are universal, as they match the TW tails. In the left tail in \eqref{eq:late-time-continuum-KPZ}, manifestly different from the late-time behavior \cite{38}, showing that the TW universality extends all the way to the large-deviation regime. A natural question is how this late-time behavior is approached as time increases. Weak noise expansion and instanton calculations in the tails (for the flat geometry) indicate a different behavior $P(H,t) \sim e^{-(|H|^{5/2}t)^{1/2}}$ in the left tail in the early-time regime $t \ll 1$ \cite{36,37}. In fact we have computed exactly the short time height distribution in the droplet geometry which exhibits a similar $|H|^{5/2}$ left tail behavior \cite{38}, manifestly different from the late-time behavior $|H|^3$ obtained here at late-times. In contrast, the right tail $H^{3/2}$ is already attained at early time.

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