Shear Viscosity of Turbulent Chiral Plasma

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Abstract

It is well known that the difference between the chemical potentials of left-handed and right-handed particles in a parity violating (chiral) plasma can lead to an instability. We show that the chiral instability may drive turbulent transport. Further we estimate the anomalous viscosity of chiral plasma arising from the enhanced collisionality due to turbulence.

Keywords: Chiral Imbalance, Berry curvature, anomalous viscosity

1. Introduction

The suggestion that the strongly interacting matter created in the relativistic heavy-ion collision experiments can have local P and CP violations has created a lot of excitement. According to Refs. \cite{1,2,3,4} the proposed P and CP violations in QCD can be due to finite nonzero topological charges present at high-temperature and density. In presence of a very strong magnetic field (which can be created during the heavy-ion collision) the nonzero topological charge can induce a net chiral imbalance. As a result particles with positive and negative charges will traverse in opposite directions along the magnetic field and thus a net charge separation can occur. This phenomenon is called ‘chiral magnetic effect’ (CME)\cite{3,5,6}. Recently an experiment with the STAR detector at RHIC has been performed to observe the CME by measuring the three particle azimuthal correlators sensitive to the charge separation. It has been found that in case of Au-Au and Cu-Cu collisions at $\sqrt{s} = 200$ GeV correlation of opposite charges separates out \cite{10,11} which can be an indication of CME or P and CP violation. These developments have created a lot of interests in this field. Theoretical models that study these aspects of strongly interacting matter consider a plasma of massless fermions which interact with each other in chiral invariant way. There exists both hydrodynamical and kinetic theory based models describing such a plasma in which the quantum mechanical nature of the chiral anomaly can have a macroscopic consequences. In this paper we shall focus on the kinetic theory approach. Recently it was shown that the CME and other CP violating effects can be incorporated within a kinetic theory framework \cite{12,13,14,15} by using the Berry curvature \cite{16} corrections. The kinetic theory approach is more general in comparison with a hydrodynamical framework and can be applied to study various equilibrium and nonequilibrium situations.

It should be noted here that the effect of parity violation because of weak-interaction considered to be important in the context of core collapsing supernova and the formation of neutron stars\cite{17,18} e.g. the peculiar velocity of pulsar \cite{19} or in the generation of magnetic field during the core collapsing neutron star \cite{20,21,22}. However, the role of parity violating effects due to the strong sector in a quark star is not fully explored. In the present work we consider the chiral-plasma instability (CPI) which may arise either in core collapsing supernova due to weak process\cite{23} or in a quark matter in the interior of a neutron star due to strong process. Such instabilities have
been studied in the context of electromagnetic and quark-gluon plasma at finite temperature using the Berry-curvature modified kinetic equation[24, 25]. A similar kind of instability can exist in case of a electroweak plasma and early universe [21]. In Ref. [24] it was argued that the chiral-imbalance instability can lead to the growth of Chern-Simons number (or magnetic-helicity in plasma physics context) at expense of the chiral imbalance. Subsequently in Refs.[21, 26] it was shown that the generation of magnetic helicity in presence of chiral instability may lead to the growth of Chern-Simons number or magnetic-helicity transfer in a medium is usually governed by collision. But in case of turbulence interaction between particles and field can enhance the decorrelation frequency and the effective viscosity can be written as,

\[ \eta \sim \frac{\text{Stress}}{v_{\text{collision}} + v_{\text{decorrelation}}}, \]  

where, \( v_{\text{collision}} \) and \( v_{\text{decorrelation}} \) respectively denote the collision and decorrelation frequencies. In the case of a neutron star collision frequency can become very small as temperature \( T \) become small [33] and thus the decorrelation frequency can have dominant contribution in determining \( \eta \).

2. Linear Response Analysis and Chiral Instability

We start with the Berry-curvature modified collisionless kinetic (Vlasov) equation at the leading order in \( \mathcal{N} \) [15] given as:

\[ (\partial_t + \mathbf{v} \cdot \nabla) n_p + \epsilon_p \mathbf{E} + e \mathbf{v} \times \mathbf{B} - \partial_x \epsilon_p \cdot \partial_p n_p = 0 \]  

(2)

where \( \mathbf{v} = \frac{p}{\epsilon} \), \( \epsilon_p = p(1 - e \mathbf{B} \cdot \mathbf{\Omega}_p) \) and \( \mathbf{\Omega}_p = \pm p/2p^2 \) is the Berry curvature. ± sign corresponds to right and left-handed fermions respectively. Note that when \( \Omega_p=0 \), energy of a chiral fermion \( \epsilon_p \) is independent of \( x \), Eq.(2) reduces to the standard Vlasov equation.

Current density \( \mathbf{j} \) is defined as:

\[ \mathbf{j} = -e \int \frac{d^3p}{(2\pi)^3} \left[ \epsilon_p \partial_p n_p + e \left( \mathbf{\Omega}_p \cdot \partial_p \mathbf{n}_p \right) \right] + e\mathbf{E} \times \sigma, \]  

(3)

where, \( \partial_p = \frac{\partial}{\partial p} \) and \( \partial_x = \frac{\partial}{\partial x} \). The \( \mathbf{eE} \times \sigma \) of the above equation represents the anomalous Hall current. Where \( \sigma \) is as follows:

\[ \sigma = e \int \frac{d^3p}{(2\pi)^3} \Omega_p n_p. \]  

(4)

Let us first consider right handed fermions with chemical potential \( \mu \). In this case we can take equilibrium distribution function of the form \( n_p^0 = 1/(e^{\epsilon_p/\mu - i\Omega_p/T} + 1) \).

Now for a linear response analysis we express Eq.(2) and \( \Omega_p \) by a linear-order deviation in the gauge field. We consider the power counting scheme [15] for gauge field as small and independent parameters. In this scheme one can write the distribution function in Eq.(2) as follows,

\[ n_p = n_p^0 + e(n_p^{(e)} + n_p^{(\delta e)}), \]  

(5)

where, \( n_p^0 = n_p^{(0)} + e n_p^{(0)\delta e} \) with \( n_p^{(0)} = \frac{1}{e^{\epsilon_p/\mu - i\Omega_p/T} + 1} \) and \( n_p^{(\delta e)} = \frac{\mathbf{v} \times \mathbf{\Omega}}{e} \left( e^{\epsilon_p/\mu - i\Omega_p/T} - 1 \right) \).

Now from Eq.(3), the anomalous Hall-current term \( \mathbf{eE} \times \sigma \), can be of order \( O(\epsilon \delta) \) or higher. Here we are interested in finding deviations in current up to order \( O(\epsilon \delta) \) therefore, only \( n_p^{(0)} \) should contribute to \( \sigma \) in the anomalous Hall term. Hence \( \sigma \) from Eq.(4) will be

\[ \sigma = \frac{e}{2} \int d\Omega d\mathbf{p} \frac{\mathbf{v}}{(1 + e^{\epsilon_p/\mu - i\Omega_p/T})} = 0. \]  

(6)

Thus the anomalous Hall current term will not contribute.

Now the kinetic equation (2) at \( O(\epsilon) \) and \( O(\epsilon \delta) \) scales of distribution function can be written as,

\[ (\partial_t + \mathbf{v} \cdot \partial_x) n_p^{(\epsilon)} = -(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \partial_p n_p^{(0)} \]  

(7)

\[ (\partial_t + \mathbf{v} \cdot \partial_x) n_p^{(\delta \epsilon)} + n_p^{(\delta \epsilon)} = -\frac{1}{e} \partial_x \epsilon \cdot \partial_p n_p^{(0)} \]  

(8)
Similarly equation for the current defined in Eq. 3 at $O(\varepsilon)$ and $O(\varepsilon^2)$ can be written as,

$$J^{\mu(\varepsilon)} = e^2 \int \frac{d^3 p}{(2\pi)^3} \nu' \nu'^{(\varepsilon)}$$

(9)

$$J^{\mu(\varepsilon^2)} = e^2 \int \frac{d^3 p}{(2\pi)^3} \left[ \nu' \nu'^{(\varepsilon^2)} - \frac{\nu'^{(0)}}{2} \partial_p \frac{\partial}{\partial p} \right] B' - e^2 \partial_p \frac{\partial}{\partial p} \delta'$$

(10)

Using Eqs. 7, 8, 9, and the expression $j^{\mu}_{\text{ind}} = \Pi^\mu(\varepsilon)A_{\varepsilon}(K)$ one can obtain the expression for the spatial part of self energy, $\Pi^\mu = \Pi^\mu_{\text{r}} + \Pi^\mu_{\text{l}}$ for right handed particles. If we have contribution from all type of species i.e. right/left fermions with charge $e$ and chemical potential $\mu_R/\mu_L$ as well as right/left handed antifermions with charge $-e$ and chemical potential $-\mu_R/\mu_L$, then, $\Pi^\mu_{\text{r}}$ (parity even part of polarization tensor) and $\Pi^\mu_{\text{l}}$ (parity-odd part) can be written as,

$$\Pi^\mu_{\text{r}}(K) = m_D^2 \int \frac{d\Omega}{4\pi} v^i v^j \left( \delta^\mu_{\text{r}} + \nu_\text{r} v^k \right)$$

(11)

$$\Pi^\mu_{\text{r}}(K) = C_E \int \frac{d\Omega}{4\pi} v^i v^j \left( \delta^\mu_{\text{r}} + \nu_\text{r} v^k \right)$$

(12)

where,

$$m_D^2 = -\frac{e^2}{2\pi} \int_0^\infty dp \left[ \frac{\partial \rho_{\mu\nu}(p-\mu) + \partial \rho_{\mu\nu}(p + \mu)}{\partial p} + \frac{\partial \rho_{\mu\nu}(p-\mu) + \partial \rho_{\mu\nu}(p + \mu)}{\partial p} \right]$$

$$C_E = -\frac{e^2}{4\pi} \int_0^\infty dp \left[ \frac{\partial \rho_{\mu\nu}(p-\mu) + \partial \rho_{\mu\nu}(p + \mu)}{\partial p} - \frac{\partial \rho_{\mu\nu}(p-\mu) + \partial \rho_{\mu\nu}(p + \mu)}{\partial p} \right].$$

(13)

Note that while deriving these expression we have chosen the temporal gauge i.e. $A_0 = 0$. It is easy to perform above integrations and get $m_D^2 = e^2 \left( \frac{\mu_R^2 + \mu_L^2}{2\pi} \right)$ and $C_E = e^2 \left( \frac{\mu_R^2 + \mu_L^2}{2\pi} \right)$, where $\mu_S = \mu_R - \mu_L$. From here it is clear that that when there is no chiral imbalance $C_E = 0$ whereas $m_D^2 \neq 0$. Introduction of chemical potential $\mu_S$ for chiral fermions requires some clarification. Physically it can be interpreted as the imbalance between the right handed and left handed fermion and arises because of the topological charge [5, 27].

Now Maxwell’s equation is

$$\partial_i F^{\nu\mu} = j^{\mu}_{\text{ind}} + j^{\mu}_{\text{ext}}$$

(14)

Taking the fourier transform and using the expression of the induced current $j^{\mu}_{\text{ind}} = \Pi^{\nu\mu}(K)A_{\varepsilon}(K)$ and choosing temporal gauge $A_0 = 0$ as one can get,

$$(k^2 - \omega^2)\delta^{ij} - k^j k^i + \Pi^{ij}(K)E^j = i\omega j^{\varepsilon\mu}_{\text{ext}}(k).$$

(15)

One can define inverse of the propagator as,

$$[\Delta^{-1}(K)]^{ij} = (k^2 - \omega^2)\delta^{ij} - k^j k^i + \Pi^{ij}(K).$$

(16)

Dispersion relation can be obtained by finding the poles of $[\Delta(K)]^{ij}$. In order to find the poles of the propagator $\Delta^{ij}$ we write $\Pi^{ij}$ in a tensor decomposition. For the current problem we need three independent projectors, transverse $P^T_{\mu\nu} = \delta^{ij} - k^i k^j/k^2$, longitudinal $P^L_{\mu\nu} = k^i k^j/k^2$ and a parity odd tensor projector $P^A_{\mu\nu} = i\epsilon^{ijk}k_k$. Thus we write $\Pi^{ij}$ as:

$$\Pi^{ij} = P_T^{ij}_{\mu\nu} + P_L^{ij}_{\mu\nu} + P_A^{ij}_{\mu\nu}$$

(17)

where, $P_T, P_L$ and $P_A$ are some scalar functions of $k$ and $\omega$ and need to be determined.

Following the decomposition of $\Pi^{ij}$, one can also decompose $[\Delta^{-1}(K)]^{ij}$ appearing in Eq. 16 as

$$[\Delta^{-1}(K)]^{ij} = C_T P^T_{\mu\nu} + C_L P^L_{\mu\nu} + C_A P^A_{\mu\nu}.$$ 

(18)

where coefficient $C$s are related to the scalar functions defined in Eq. 17 by following equation:

$$C_T = k^2 - \omega^2 + P_T, C_L = -\omega^2 + P_L, C_A = A.$$ 

Now using Eq. 17 one can write $P_T = \frac{1}{2} P_T^T P_T^V$, $P_L = P_L^V P_T^V$ and $P_A = -\frac{1}{2} P_A^V P_T^V$ and then using the Eqs. 11-12 for $\Pi^{ij}$ one can obtain,

$$\Pi_T = m_D^2 \frac{\omega^2}{2k^2} \left[ 1 + \frac{k^2 - \omega^2}{2\omega k} \ln \frac{\omega + k}{\omega - k} \right],$$

$$\Pi_L = m_D^2 \frac{\omega^2}{k^2} \left[ -\frac{\omega}{2k} \ln \frac{\omega + k}{\omega - k} \right],$$

$$\Pi_A = k C_E \left[ 1 - \frac{\omega^2}{k^2} \right] \left[ 1 - \frac{\omega}{2k} \ln \frac{\omega + k}{\omega - k} \right].$$

(19)

Now using the fact that a vector and its inverse exists in same space, we can expand $[\Delta(K)]^{ij}$ in the tensor projector basis as:

$$[\Delta(K)]^{ij} = a P^T_{\mu\nu} + b P^L_{\mu\nu} + c P^A_{\mu\nu}.$$ 

(20)
Now using the relation $[\Delta^{-1}(K)]^{ij} \Delta(K) = \delta^{ij}$ one can obtain the coefficients $a, b, c$ in terms of the coefficients $C$’s appearing in Eq.\((18)\). Poles of the $[\Delta^{-1}(K)]^{ij}$ can be obtained by equating denominators of the expressions for $a, b, c$ with zero. In the present case we have same denominator for $b$ and $c$ while it is different for $a$ therefore the dispersion relation:

$$C_A^2 - C_I^2 = 0,$$  \hspace{1cm} (21)  

$$C_L = 0.$$  \hspace{1cm} (22)  

Here we would like to note that the dispersion relation given by Eq.\((22)\) gives only oscillations and do not have instability therefore, it is not of our interest. Dispersion relation given by Eq.\((21)\) can be written as:

$$\omega^2 = k^2 + \Pi_T \mp \Pi_A$$  \hspace{1cm} (23)  

In the quasi-static limit i.e. $|a| << k$, one can write $\Pi_T \Pi_L$ and $\Pi_A$ as:

$$\Pi_T(k,\omega) = \left( \mp \frac{\sqrt{\alpha} \omega}{4 \pi} \right) m^2_D,$$

$$\Pi_L(k,\omega) = O(\omega^2/k^2) + ~....$$

$$\Pi_A(k,\omega) = \mp \frac{\mu \omega^2}{4\pi} \left( \mp \frac{\sqrt{\alpha} \omega}{2} - 1 \right).$$  \hspace{1cm} (24)  

In this limit Eq.\((23)\) with the minus sign will give the dispersion relation $\omega = ip(k)$ where $\rho(k)$ is given by:

$$\rho(k) = \left( 4\frac{\alpha \mu}{n^2 m_D^2} \right) k^2 \left[ 1 - \frac{\pi k}{\mu \omega} \right]$$  \hspace{1cm} (25)  

Here we have used and defined $\alpha = \frac{e^2}{4\pi}$ as the electromagnetic coupling. It is clear from Eq.\((25)\) that $\omega$ is purely an imaginary number and its real-part is zero i.e. $Re(\omega) = 0$. Positive $\rho(k) > 0$ implies an instability as $e^{-i\rho(k)t} \sim e^{\mu t/\omega}$ due net chiral chemical potential $\mu$. Thus plasma has exponential instability that can drive turbulence. Instability will be maximum at $k_{\text{max}} = \frac{2\mu \omega}{\sqrt{\alpha}}$. For simplicity, in the next section we consider the case of right handed particles only.

3. Diffusion via nonlinear particle-wave interaction, decorrelation time

We shall use Resonance Broadening theory \[34, 35, 36, 37\] \[38, 39\]. First we consider the case of high density and low temperature, it can shown $\epsilon_p = p - e \left( \frac{B_{\omega k} \cdot v}{2\mu_R} \right) + O(\frac{1}{\rho^2})$\[15\]. Now consider the distribution function,

$$n_p = n_p^{(0)} + \epsilon_p n_{p,\omega k}^4,$$  \hspace{1cm} (26)  

where $\langle n_p \rangle = \langle n_p^{(0)} \rangle$, $\langle \rangle$ represents the spatial averaging. $n_{p,\omega k}^4$ is the coherent response to field fluctuations. Taking the spatial averaging Berry curvature modified kinetic Eq.\((2)\) can be written as,

$$\partial_t \langle n_p \rangle = -e^2 \left( \text{E}_{\omega k} + v \times \text{B}_{\omega k} + ik \left( \frac{B_{\omega k} \cdot v}{2\mu_R} \right) \right) \cdot \partial_p n_{p,\omega k}^4$$  \hspace{1cm} (27)  

In the quasilinear theory trajectories of the particles are assumed to be unperturbed irrespective of the presence of fluctuating fields. As a result coherent response $n_{p,\omega k}^4$ has a peak $1/(\omega - k \cdot v)$. In the resonance broadening theory one considers the perturbed trajectories of the particles due to effects of random fields and calculate the approximate coherent response function $n_{p,\omega k}^4$ as an average over a statistical ensemble or perturbed trajectories. As a results the peak in the coherent response gets broadened \[34, 37\]. In the case of resonance broadening theory, response function can be written as \[34, 37\],

$$n_{p,\omega k}^4 = \int_0^\infty d\omega \text{e}^{-R(k,\omega) \cdot t} \left( \text{E}_{\omega k} + v \times \text{B}_{\omega k} + ik \left( \frac{B_{\omega k} \cdot v}{2\mu_R} \right) \right) \partial_p \langle n_p \rangle$$  \hspace{1cm} (28)  

We take Gaussian probability distribution as,

$$pdf(\delta \rho) = \frac{1}{\sqrt{2\pi} \sigma} \text{e}^{-\frac{(\delta \rho)^2}{2\sigma^2}}.$$  \hspace{1cm} (29)  

With the above probability distribution one can get,

$$\langle \text{e}^{-R(k,\omega) \cdot t} \rangle_{pdf} \approx \text{e}^{-\frac{t^2}{2\sigma^2}}.$$  \hspace{1cm} (30)  

Here, $t_s$ is given by following equation,

$$t_s^2 = \frac{4E_p^2}{k^2 \mu}.$$  \hspace{1cm} (31)  

where, $E_p^2 = \int \rho_p E_p / \rho_p$ \[10\] \[15\].

Substituting Eq.\((30)\) in Eq.\((28)\) one gets,

$$n_{p,\omega k}^4 = \int_0^\infty d\omega \text{e}^{-R(k,\omega) \cdot t} \left( \text{E}_{\omega k} + v \times \text{B}_{\omega k} + ik \left( \frac{B_{\omega k} \cdot v}{2\mu_R} \right) \right) \partial_p \langle n_p \rangle$$  \hspace{1cm} (32)  

Now,

$$\int_0^\infty d\omega \text{e}^{-R(k,\omega) \cdot t} \approx -\frac{i}{\omega - k \cdot v + it_s}.$$  \hspace{1cm} (33)  

Using Eq.\((32)\) one can write the following Diffusion equation,

$$\left( \partial_t - \partial_p \cdot D(p) \cdot \partial_p \langle n_p \rangle \right) = 0,$$  \hspace{1cm} (34)  

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where,
\[
D(p) = -\int d\omega dk \left( F_{\omega, k} \frac{\delta}{\omega - k \cdot v + i/t_c} F_{\omega, k} \right)
\]  
(35)
and
\[
F_{\omega, k} = e^{-i \left( E_{\omega, k} + \mathbf{v} \times \mathbf{B}_{\omega, k} + \frac{1}{2} \frac{e \mathbf{B}_{\omega, k} \cdot \mathbf{v}}{m^2} \right)}
\]  
(36)
In this problem we are interested in the studying diffusion only due to color magnetic excitations. In this case the Diffusion coefficient can be written as,
\[
D(\omega) = \int d\omega dk \left( \mathbf{v} \times \mathbf{B}_{\omega, k} - i k \frac{e \mathbf{B}_{\omega, k} \cdot \mathbf{v}}{m^2} \right) \left( \mathbf{v} \times \mathbf{B}_{\omega, k} + i k \frac{e \mathbf{B}_{\omega, k} \cdot \mathbf{v}}{m^2} \right).
\]  
(37)
Now, choosing \( k = \lambda \mathbf{z}, \mathbf{B}_{\omega, k} = \partial \mathbf{B}_{\omega, k} \mathbf{z}. \) and considering \( \omega = i \gamma. \) Then the diffusion coefficient,
\[
D = e^2 \sum_{\omega, k} \left( \frac{\partial^2 B_{\omega, k}^2 \mathbf{z} \mathbf{z} + v_i \partial B_{\omega, k}^2 \mathbf{z} \mathbf{z} + \partial B_{\omega, k}^2 \mathbf{z} \mathbf{z} + \frac{\partial B_{\omega, k}^2 \frac{e \mathbf{B}_{\omega, k} \cdot \mathbf{v}}{2m^2}}{2m^2}}{(\gamma + 1/t_c + ikv_i)} \right).
\]  
(38)
For strong turbulence we can use approximation \((1/t_c)^2 \gg (kv_i)^2\). In this case, at saturation \((\gamma = 0)\) the diffusion coefficient can be written as,
\[
D = e^2 \sum_{\omega, k} \left( \frac{\partial^2 B_{\omega, k}^2 \mathbf{z} \mathbf{z} + v_i \partial B_{\omega, k}^2 \mathbf{z} \mathbf{z} + \partial B_{\omega, k}^2 \mathbf{z} \mathbf{z} + \frac{\partial B_{\omega, k}^2 \frac{e \mathbf{B}_{\omega, k} \cdot \mathbf{v}}{2m^2}}{2m^2}}{(1/t_c)} \right).
\]  
(39)
Now, taking thermal average of velocities and using Eqs.\(^{(37)}^{(39)}\) one can get the decorrelation time as,
\[
\left( \frac{1}{t_c} \right) = \frac{e^2 k^2}{24 E^2} \sum_{\omega, k} \left( \frac{\partial^2 B_{\omega, k}^2 \mathbf{z} \mathbf{z} + v_i \partial B_{\omega, k}^2 \mathbf{z} \mathbf{z} + \partial B_{\omega, k}^2 \mathbf{z} \mathbf{z} + \frac{\partial B_{\omega, k}^2 \frac{e \mathbf{B}_{\omega, k} \cdot \mathbf{v}}{2m^2}}{2m^2}}{2m^2} \right).
\]  
(40)
where, \( v_i^2 = \frac{\int v_i v_i f(v)}{\int v^2 f(v)} \). This is the relation between \( t_c \) and the intensity of color magnetic excitations. Now we calculate the decorrelation time by incorporating the non-linear corrections in the self energy due to resonance broadening.
Thus due to non-linear wave particle interactions self energy calculated in Eqs.\(^{(12)}^{(15)}\) acquires a corrections as \( \omega \to \omega + i/t_c. \)
\[
\Pi(\omega) = m^2 D\int \frac{d\Omega}{4\pi} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3l}{(2\pi)^3} \left( \frac{1}{2} \frac{e \mathbf{B}_{\omega, k} \cdot \mathbf{v}}{m^2} \right)
\]  
(41)
\[
\Pi^\omega(\omega) = C_E \left( \frac{4m^2}{2\pi^2} \right) \int \frac{d\Omega}{4\pi} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3l}{(2\pi)^3} \left( \frac{1}{2} \frac{e \mathbf{B}_{\omega, k} \cdot \mathbf{v}}{m^2} \right)
\]  
(42)
It is important to note that here we have considered only right handed particles so in the Eq.\(^{(13)}^{(15)}\) \( m^2 \) and \( C_E \) will have contribution from right handed particles only. Now, using the similar decomposition of self energy as in case of linear stability analysis one can calculate \( \Pi_T, \) and \( \Pi_A \) to be of the form,
\[
\Pi_T = -\frac{m^2}{4k} \left( \frac{1 - \frac{2}{\omega} + i\pi}{1 + \frac{2}{\omega} + i\pi} \right) \right) \left[ \ln \left( 1 + \frac{1}{\omega} + i\pi \right) \right]
\]  
(43)
Now at saturation \( \omega = 0 \), therefore
\[
\Pi_T = -\frac{m^2}{4k} \left( \frac{1 - \frac{2}{\omega} + i\pi}{1 + \frac{2}{\omega} + i\pi} \right) \right) \left[ \ln \left( 1 + \frac{1}{\omega} + i\pi \right) \right]
\]  
(43)
We determine the decorrelation time from the dispersion relation given in Eq.\(^{(23)}\) with \( \omega = 0. \)
\[
k^2 - \frac{m^2}{2k} \left( \frac{1}{t_c} - \frac{2k}{T} \right) \right) \left[ \ln \left( 1 + \frac{1}{\omega} + i\pi \right) \right]
\]  
(44)
This is transcendental equation decorrelation can be obtained by solving this equation.
We consider the case \( \mu_B \gg T \), in this case \( m^2 \sim \left( \frac{2m}{\pi} \right)^2 \). Further we consider \( k = k_{max} = \frac{2m}{\pi} \) which correspond to maximum growth rates of chiral instability. In this case decorrelation time will be dependent on \( \alpha \) and \( \mu_B \). For \( \alpha = 1/\lambda_\gamma \) the solution for \( t_c \) of above equation in terms of \( \mu_B \) is shown in the following figure.
Note that the strong turbulence require that the condition $\frac{1}{\omega_{\text{max}}} \gg v_t$ is satisfied in $\mu_R \gg T$ regime. Now if we take $t_c \sim \frac{1}{k}$, $k = k_{\text{max}} = \frac{2\omega_{\text{max}}}{3\pi}$ and $E_p - \mu_R$ we can determine the saturation level of color magnetic excitations using Eq.(40) as,

$$\delta B_{\text{sat}} \sim \frac{\mu_R^2}{\sqrt{\alpha}}. \quad (45)$$

4. Calculation of anomalous viscosity

We follow Ref.[41][42] to calculate the anomalous viscosity. For simplicity we make $v_\alpha$ depend on $x$ as,

$$v_\alpha \rightarrow v_\alpha - u(x) \quad (46)$$

where, $u(x)$ is the mean flow variable.

Now using Eq.(49) one can write the diffusion equation (Eq.(54)) as,

$$(\partial_t + v \cdot \partial_x)(\eta_p) \simeq \frac{c^2}{1/t_c} \left( \left< v_\alpha^2 \delta B_{\text{sat}}^2 \right> T'_{\text{sat}}(\eta_p) + \left( \frac{v_\alpha^2 k^2 |\delta B_{\text{sat}}^2|}{4\mu_R^4} \right) \delta \eta_p(\eta_p) \right). \quad (47)$$

Second term can be written as;

$$v_\alpha\left( \frac{1}{l_c} \right) \left( \frac{1}{d'p} \right) \frac{d(\eta_p)}{dp} \partial_x \eta(x). \quad (48)$$

Here, we bring back the term $v_\alpha \cdot \partial_x$. Now, if we consider, $k = k_{\text{max}}$, in this case the summation on $\omega$ and $k$ can be lifted out and we can write the diffusion equation in terms of mean flow variable as,

$$\partial_t \delta \eta_p(\eta_p) - v_\alpha^2 p \left( \frac{d(\eta_p)}{dp} \right) \partial_x \eta(x) \simeq \frac{c^2}{1/t_c} \left( \left< v_\alpha^2 \delta B_{\text{sat}}^2 \right> T'_{\text{sat}}(\eta_p) + \left( \frac{v_\alpha^2 k^2 |\delta B_{\text{sat}}^2|}{4\mu_R^4} \right) \delta \eta_p(\eta_p) \right). \quad (49)$$

Note that in the above equation $\omega$ and $k$ respectively corresponds to $\omega_{\text{max}}$ and $k_{\text{max}}$. Now taking moment $J_1 \equiv \int (v_\alpha p \delta B_{\text{sat}}^2) d\eta_p$, the left hand side of above equation will become,

$$LHS = \partial_t \left( T^{\alpha \alpha} - T^{\gamma \gamma} - T^{\text{zz}} \right) - \int d\Omega (2v_\alpha^2 - v_\gamma^2 - v_\text{zz}^2) \times \left( \frac{e^2 B_{\text{sat}} \cdot v^2}{4\mu_R^2} \right) \left| \int_0^\infty \int_0^d \frac{d\eta_p}{dp} \frac{d\eta_p}{dp} \partial_x \eta(x) \right| \quad (50)$$

Note that in the above expression we have used the definition of energy momentum tensor as,

$$T^{\mu \nu} = \int d^3p \left( 1 + \frac{e^2 B_{\text{sat}} \cdot \eta_p}{\epsilon_p} \right) p^\mu p^\nu \eta_p(n_p). \quad (51)$$

Simplifying above Eq.(50) we can write,

$$LHS = \partial_t \left( T^{\alpha \alpha} - T^{\gamma \gamma} - T^{\text{zz}} \right) + \int d\Omega (2v_\alpha^2 - v_\gamma^2 - v_\text{zz}^2) \times \left( \frac{e^2 B_{\text{sat}} \cdot v^2}{4\mu_R^2} \right) \left| \int_0^\infty \int_0^d \frac{d\eta_p}{dp} \frac{d\eta_p}{dp} \partial_x \eta(x) \right| \quad (52)$$

Now,

$$RHS = \frac{c^2 v_\alpha^2 |\delta B_{\text{sat}}^2|}{1/t_c} \left( \frac{1}{(2\pi)^2} \right) \left( \int d\Omega \left( 2v_\alpha^2 - v_\gamma^2 - v_\text{zz}^2 \right) \left( \frac{e^2 B_{\text{sat}} \cdot v^2}{4\mu_R^2} \right) \times \left( \int_0^\infty \int_0^d \frac{d\eta_p}{dp} \frac{d\eta_p}{dp} \partial_x \eta(x) \right) \right) \quad \left( \frac{1 + \frac{k^2}{4\mu_R^2}}{1 + \frac{v_\alpha^2}{p^2}} \right) \left( \int_0^\infty \int_0^d \frac{d\eta_p}{dp} \frac{d\eta_p}{dp} \partial_x \eta(x) \right). \quad (53)$$

Now using,

$$\delta \eta_p(\eta_p) = \frac{p^2}{p} \frac{d\eta_p}{dp} + \frac{1}{p} \frac{d\eta_p}{dp} + \frac{1}{p} \frac{d\eta_p}{dp} \quad (54)$$

and writing $\partial_x \eta(x)$ in a similar fashion. One can get;

$$RHS = \frac{c^2 v_\alpha^2 |\delta B_{\text{sat}}^2|}{1/t_c} \left( \frac{1}{(2\pi)^2} \right) \left( \int d\Omega \left( 2v_\alpha^2 - v_\gamma^2 - v_\text{zz}^2 \right) \left( \frac{e^2 B_{\text{sat}} \cdot v^2}{4\mu_R^2} \right) \times \left( \int_0^\infty \int_0^d \frac{d\eta_p}{dp} \frac{d\eta_p}{dp} \partial_x \eta(x) \right) \right) \quad \left( \frac{1 + \frac{k^2}{4\mu_R^2}}{1 + \frac{v_\alpha^2}{p^2}} \right) \left( \int_0^\infty \int_0^d \frac{d\eta_p}{dp} \frac{d\eta_p}{dp} \partial_x \eta(x) \right). \quad (55)$$

With further simplification we can write above equation as,

$$RHS = \frac{c^2 v_\alpha^2 |\delta B_{\text{sat}}^2|}{1/t_c} \left( \frac{1}{(2\pi)^2} \right) \left( \int d\Omega \left( 2v_\alpha^2 - v_\gamma^2 - v_\text{zz}^2 \right) \left( \frac{e^2 B_{\text{sat}} \cdot v^2}{4\mu_R^2} \right) \times \left( \int_0^\infty \int_0^d \frac{d\eta_p}{dp} \frac{d\eta_p}{dp} \partial_x \eta(x) \right) \right) \quad \left( \frac{1 + \frac{k^2}{4\mu_R^2}}{1 + \frac{v_\alpha^2}{p^2}} \right) \left( \int_0^\infty \int_0^d \frac{d\eta_p}{dp} \frac{d\eta_p}{dp} \partial_x \eta(x) \right). \quad (56)$$

We choose stationary limit in this case $\partial_t (2T^{\alpha \alpha} - T^{\gamma \gamma} - T^{\text{zz}}) = 0$, therefore from the diffusion equation (L.H.S=R.H.S) we can get,

$$\delta \eta(x) = \frac{c^2 v_\alpha^2 |\delta B_{\text{sat}}^2|}{1/t_c} \left( \frac{8I_1 J_1 - 4I_2 J_1 + 8 \left( 1 + \frac{k^2}{4\mu_R^2} \right) I_1 J_1}{5I_1 J_2} \right) \quad (54)$$
where,
\[ I_1 = \frac{1}{(2\pi)^3} \int d\Omega (2v_1^2 - v_1^2) \frac{(e\sigma_{B_{v, k}} \cdot v)^2}{4\mu_k^4} = \frac{e^2 \sigma_{B_{v, k}}^2}{105\pi^2 \mu_k^6} \]
\[ I_2 = \frac{1}{(2\pi)^3} \int d\Omega (2v_1^2 - v_1^2) \frac{(e\sigma_{B_{v, k}} \cdot v)^2}{4\mu_k^4} = \frac{e^2 \sigma_{B_{v, k}}^2}{15\pi^2 \mu_k^6} \]
\[ I_3 = \frac{1}{(2\pi)^3} \int d\Omega (2v_1^2 - v_1^2) \frac{(e\sigma_{B_{v, k}} \cdot v)^2}{4\mu_k^4} = \frac{e^2 \sigma_{B_{v, k}}^2}{210\pi^2 \mu_k^6} \]
\[ J_1 = \int_0^\infty dp (n_p), \]
\[ J_2 = \int_0^\infty dp \langle n_p \rangle. \]

Now, using Eq. (55), one can write,
\[ (2T^\alpha - T^\alpha - T^\alpha) = \int \frac{d\Omega}{(2\pi)^3} (e\sigma_{B_{v, k}} \cdot v)(2v_1^2 - v_1^2) \int_0^\infty dp \langle n_p \rangle. \]

The definition of shear viscosity is,
\[ \eta_{\alpha} = \frac{(2T^\alpha - T^\alpha - T^\alpha)}{-4\theta_{v, a}(x)} \]

Taking the distribution function of the form, \( \langle n_p \rangle = \frac{1}{e^{\epsilon_{B_{v, k}} - \epsilon_{\beta} - \varepsilon}} \), considering \( \mu_B \gg T \) and using Eqs. (54,55) and \( k = k_{max} = \frac{2\pi^2}{\omega_{max}} \), one can estimate anomalous shear viscosity,
\[ \eta_{\alpha} \sim \frac{\mu_B}{T} \left( 1 + \frac{11\pi^2 T^2}{3\mu_B^2} \right) \]

One can notice from Fig.1 that for the case \( \mu_B \gg T \) and \( k = k_{max} \), \( 1/t_c \) depends on \( \mu_B \) in an approximately linear way i.e. \( 1/t_c \propto \mu_B \).

Slope of the curve can be found by a linear fit. For \( \alpha = 1/137 \) the slope is \( \sim 5.09 \times 10^{-7} \). The slope increases by increasing \( \alpha \). Thus \( \eta \) scales like \( \mu_B^3 \).

5. Conclusion

We have calculated the coefficient of shear viscosity based on the strong turbulence argument. For the case when \( \mu_B/T > 1 \), the collision rates becomes insignificant at low temperature, in this regime the decorrelation frequency \( 1/t_c \) can have a significant contribution in determining \( \eta \). In this low temperature limit the entropy density \( s \) scales as \( \mu_B^3 T \) and the ratio \( \eta/s \propto \mu_B/T \) and it can be a large number. In deriving the above expression of \( \eta \) we have ignored non-linear wave-wave interaction which can play a role in case of non-Abelian plasmas. However to address this question one require to numerically simulate the chiral plasma instability with the full nonlinearity.

Note that dimensional argument suggests that for the case when \( \mu_B \ll T \), stress (energy density) \( \sim \mu_B^2 T^3 \), decorrelation frequency \( (1/t_c \sim \omega_{max}) \) of CPI \( \sim \mu_B [25] \) and \( \eta \) scales as \( \mu_B T^2 \). Therefore, \( \eta/s \propto \mu_B/T \), which could be a small number. We hope that this analytic study will help in understanding the viscosity due to turbulent transport in parity violating plasma and can be useful in it numerical simulations.

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