A survey on star edge-coloring of graphs

Hui Lei\textsuperscript{1}, Yongtang Shi\textsuperscript{2}
\textsuperscript{1} School of Statistics and Data Science, LPMC and KLMDASR
Nankai University, Tianjin 300071, China
\textsuperscript{2} Center for Combinatorics and LPMC
Nankai University, Tianjin 300071, China
hlei@nankai.edu.cn; shi@nankai.edu.cn

Abstract

The star chromatic index of a multigraph $G$, denoted $\chi'_{st}(G)$, is the minimum number of colors needed to properly color the edges of $G$ such that no path or cycle of length four is bicolored. We survey the results of determining the star chromatic index, present the interesting proofs and techniques, and collect many open problems and conjectures.

Keywords: star edge-coloring; subcubic multigraphs; bipartite graphs; planar graphs; maximum average degree

AMS subject classification 2010: 05C15

1 Introduction

All multigraphs in this paper are finite and loopless; and all graphs are finite and without loops or multiple edges. Given a multigraph $G$, let $c : E(G) \rightarrow [k]$ be a proper edge-coloring of $G$, where $k \geq 1$ is an integer and $[k] := \{1, 2, \ldots, k\}$. We say that $c$ is a star $k$-edge-coloring of $G$ if no path or cycle of length four in $G$ is bicolored under the coloring $c$; and $G$ is star $k$-edge-colorable if $G$ admits a star $k$-edge-coloring. The star chromatic index of $G$, denoted $\chi'_{st}(G)$, is the smallest integer $k$ such that $G$ is star $k$-edge-colorable. The chromatic index and chromatic number of $G$ are denoted by $\chi'(G)$ and $\chi(G)$. As pointed out in [9], the definition of star edge-coloring of a graph $G$ is equivalent to the star vertex-coloring of its line graph $L(G)$. Star edge-coloring of a graph was initiated by Liu and Deng [29], motivated by the vertex version (see [1, 4, 5, 8, 24, 35]). Given a multigraph $G$, we use $|G|$ to denote the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, and $\Delta(G)$ the maximum degree of $G$, respectively. We use $K_n$, $P_n$ and $C_n$ to denote the complete graph, the path and the cycle on $n$ vertices, respectively. The maximum average degree of a
multigraph \( G \), denoted \( \text{mad}(G) \), is defined as the maximum of \( 2e(H)/|H| \) taken over all the non-empty subgraphs \( H \) of \( G \). The girth of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth. The following first upper bound is a result of Liu and Deng [29].

**Theorem 1.1 ([29])** Every graph \( G \) with \( \Delta(G) \geq 7 \) satisfies \( \chi'_st(G) \leq \lceil 16(\Delta(G) - 1)^{3/2} \rceil \).

Theorem 1.2 below is a result of Dvořák, Mohar, and Šámal [9], which give an upper and a lower bounds for complete graphs.

**Theorem 1.2 ([9])** The star chromatic index of the complete graph \( K_n \) satisfies

\[
2n(1 + o(1)) \leq \chi'_st(K_n) \leq n \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.
\]

In particular, for every \( \epsilon > 0 \), there exists a constant \( c \) such that \( \chi'_st(K_n) \leq cn^{1+\epsilon} \) for every integer \( n \geq 1 \).

They proved the upper bound using a nontrivial result about sets without arithmetic progressions, and up till now, it is still the best known. For the lower bound, they used an elegant double counting approach. The authors of [2] observed a improvement in their proof and obtained the bound \( \chi'_st(K_n) \geq 3n(n - 1)/(n + 4) \) (see [33] for a proof).

Currently the best upper bound in terms of the maximum degree for general graphs is also derived in [9] using Theorem 1.2.

**Theorem 1.3 ([9])** For any graph \( G \) with maximum degree \( \Delta \), \( \chi'_st(G) \leq \Delta \cdot 2^{O(1)\sqrt{\log \Delta}} \).

The true order of magnitude of \( \chi'_st(K_n) \) is still unknown. Dvořák, Mohar, and Šámal [9] made the following question.

**Question 1.4 ([9])** What is the true order of magnitude of \( \chi'_st(K_n) \)? Is \( \chi'_st(K_n) = O(n) \)?

Dvořák, Mohar, and Šámal [9] also mentioned that, if convenient, one may start with other special graphs in place of \( K_n \), in particular, from Observation 1.5 it follows that if \( \chi'_st(K_{n,n}) \) is \( O(n) \) or \( n(\log n)^{O(1)} \), \( n^{1+o(1)} \), respectively), then \( \chi'_st(K_n) \) is \( O(n \log n) \) or \( n(\log n)^{O(1)} \), \( n^{1+o(1)} \), respectively).

**Observation 1.5 ([9])** \( \chi'_st(K_n) \leq \sum_{i=1}^{\lceil \log_2 n \rceil} 2^{i-1} \chi'_st(K_{[n/2^i],[n/2^i]}) \).

The paper is organized as follows. The first Section 2 contains the results on algorithmic aspects of the star edge-coloring. In Sections 3, 4, 5 we present the results of star chromatic
index on subcubic graphs, bipartite graphs, and planar graphs, respectively. Star edge-coloring is naturally generalized to the list version and it was pointed out in [9]: It would be interesting to understand the list version of star edge-coloring. The results of list star edge-coloring are presented in Section 6. Finally in Section 7 we summarize the results of the star chromatic index of another family of graphs such as Halin graphs, generalized Petersen graphs, products of graphs and so on.

2 Complexity

It is well-known [39] that the chromatic index of a graph with maximum degree \( \Delta \) is either \( \Delta \) or \( \Delta + 1 \). However, it is NP-complete [18] to determine whether the chromatic index of an arbitrary graph with maximum degree \( \Delta \) is \( \Delta \) or \( \Delta + 1 \). The problem remains NP-complete even for cubic graphs (the degree of all vertices is 3). Lei, Shi, and Song [27] proved the following result.

**Theorem 2.1 ([27])** It is NP-complete to determine whether \( \chi'_{st}(G) \leq 3 \) for an arbitrary graph \( G \).

**Proof.** First let us denote by SEC the problem stated in Theorem 2.1, and we denote by 3EC the following well-known NP-complete problem of Holyer [18]:

Given a cubic graph \( G \), is \( G \) 3-edge-colorable?

Clearly, SEC is in the class NP. We shall reduce 3EC to SEC.

Let \( H \) be an instance of 3EC. We construct a graph \( G \) from \( H \) by replacing each edge \( e = uv \in E(H) \) with a copy of graph \( H_{ab} \), identifying \( u \) with \( a \) and \( w \) with \( b \), where \( H_{ab} \) is depicted in Figure 1. The size of \( G \) is clearly polynomial in the size of \( H \), and \( \Delta(G) = 3 \).

![Figure 1: Graph \( H_{ab} \).](image)
It suffices to show that $\chi'(H) \leq 3$ if and only if $\chi'_{st}(G) \leq 3$. Assume that $\chi'(H) \leq 3$. Let $c : E(H) \to \{1, 2, 3\}$ be a proper 3-edge-coloring of $H$. Let $c^*$ be an edge coloring of $G$ obtained from $c$ as follows: for each edge $e = uw \in E(H)$, let $c^*(av_1^3) = c^*(v_3^3v_7^3) = c^*(v_7^6b) = c(uw)$, $c^*(v_1^3v_2^3) = c^*(v_3^3v_7^3) = c^*(v_5^3v_6^3) = c^*(v_5^6v_6^6) = c(uw) + 1$, and $c^*(v_2^3v_3^3) = c^*(v_5^3v_6^3) = c(uw) + 2$, where all colors here and henceforth are done modulo 3. Notice that $c^*$ is a proper 3-edge-coloring of $G$. Furthermore, it can be easily checked that $G$ has no bicolored path or cycle of length four under the coloring $c^*$. Thus $c^*$ is a star 3-edge-coloring of $G$ and so $\chi'_{st}(G) \leq 3$.

Conversely, assume that $\chi'_{st}(G) \leq 3$. Let $c^* : E(G) \to \{1, 2, 3\}$ be a star 3-edge-coloring of $G$. Let $c$ be an edge-coloring of $H$ obtained from $c^*$ by letting $c(e) = c^*(av_1^3)$ for any $e = uw \in E(H)$. Clearly, $c$ is a proper 3-edge-coloring of $H$ if for any edge $e = uw$ in $G$, $c^*(av_1^3) = c^*(v_7^6b)$. We prove this next. Let $e = uw$ be an edge of $H$. We consider the following two cases.

**Case 1:** $c^*(v_3^3v_7^3) = c^*(v_5^3v_6^3)$.

In this case, let $c^*(v_3^3v_7^3) = c^*(v_5^3v_6^3) = \alpha$, where $\alpha \in \{1, 2, 3\}$. We may further assume that $c^*(v_3^3v_7^3) = \beta$ and $c^*(v_3^3v_7^3) = c^*(v_5^3v_6^3) = \gamma$, where $\{\beta, \gamma\} = \{1, 2, 3\} \setminus \alpha$. This is possible because $d_G(v_3^3) = d_G(v_5^3) = 3$ and $c^*$ is a proper 3-edge-coloring of $G$. Since $c^*$ is a star edge-coloring of $G$, we see that $c^*(v_1^3v_2^3) = c^*(v_5^3v_6^3) = \alpha$ and so $c^*(av_1^3) = c^*(v_6^6b) = \beta$.

**Case 2:** $c^*(v_3^3v_7^3) \neq c^*(v_5^3v_6^3)$.

In this case, let $c^*(v_3^3v_7^3) = \alpha$, $c^*(v_3^3v_7^3) = \beta$, $c^*(v_3^3v_7^3) = \gamma$, where $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$. This is possible because $\alpha \neq \beta$ by assumption. Since $c^*$ is a proper edge-coloring of $G$, we see that $c^*(v_3^3v_7^3) = \beta$ and $c^*(v_3^3v_7^3) = \alpha$. One can easily check now that $c^*(v_1^3v_2^3) = \alpha$ and $c^*(v_5^3v_6^3) = \beta$, and so $c^*(av_1^3) = c^*(v_6^6b) = \gamma$, because $c^*$ is a star edge-coloring of $G$.

In both cases we see that $c^*(av_1^3) = c^*(v_6^6b)$. Therefore $c$ is a proper 3-edge-coloring of $H$ and so $\chi'(H) \leq 3$. This completes the proof of Theorem 2.1.

There are few results for the computational complexity of the star edge-coloring problem. In [36], Oomomi, Roshanbin, and Dastjerdi presented a polynomial time algorithm that finds an optimum star edge-coloring for every tree.

**Theorem 2.2** ([36]) There is a polynomial time algorithm for computing the star chromatic index of every tree and presenting an optimum star edge-coloring of it.
3 Subcubic graphs

A multigraph $G$ is subcubic if all its vertices have degree less than or equal to three. Dvořák, Mohar, and Šámal [9] considered the star chromatic index of subcubic multigraphs. To state their result, we need to introduce one notation. A graph $G$ covers a graph $H$ if there is a mapping $f : V(G) \to V(H)$ such that for any $uv \in E(G)$, $f(u)f(v) \in E(H)$, and for any $u \in V(G)$, $f$ is a bijection between $N_G(u)$ and $N_H(f(u))$. They proved the following.

**Theorem 3.1 ([9])** Let $G$ be a multigraph.

(a) If $G$ is subcubic, then $\chi'_{st}(G) \leq 7$.

(b) If $G$ is cubic and has no multiple edges, then $\chi'_{st}(G) \geq 4$ and the equality holds if and only if $G$ covers the graph of 3-cube.

As observed in [9], $K_{3,3}$ and the Heawood graph are star 6-edge-colorable. No subcubic multigraphs with star chromatic index seven are known. Dvořák, Mohar, and Šámal [9] proposed the following conjecture.

**Conjecture 3.2 ([9])** Let $G$ be a subcubic multigraph. Then $\chi'_{st}(G) \leq 6$.

It was shown that every subcubic outerplanar graph is star 5-edge-colorable in [2] and every cubic Halin graph is star 6-edge-colorable in [6, 19]. Lei, Shi, and Song [27] proved that

**Theorem 3.3 ([27])** Let $G$ be a subcubic multigraph.

(a) If $\text{mad}(G) < 2$, then $\chi'_{s}(G) \leq 4$ and the bound is tight.

(b) If $\text{mad}(G) < 24/11$, then $\chi'_{s}(G) \leq 5$.

(c) If $\text{mad}(G) < 5/2$, then $\chi'_{s}(G) \leq 6$.

Later, Lei, Shi, Song, and Wang [28] improved Theorem 3.3(b) by showing the following result.

**Theorem 3.4 ([28])** Let $G$ be a subcubic multigraph with $\text{mad}(G) < 12/5$. Then $\chi'_{st}(G) \leq 5$.

We don’t know if the bound $12/5$ in Theorem 3.4 is best possible. The graph depicted in Figure 2 has maximum average degree $14/5$ but is not star 5-edge-colorable.

In [43], Wang, Wang, and Wang focused on the star edge-coloring of graphs with maximum degree four and proved the following result.

**Theorem 3.5 ([43])** Every graph $G$ with $\Delta(G) = 4$ satisfies $\chi'_{st}(G) \leq 14$. 


4 Bipartite graphs

In this section we consider the star edge-coloring of bipartite graphs. We first consider complete bipartite graphs. Let $K_{m,n}$ denote the complete bipartite graph in which the orders of its bipartition sets are $m$ and $n$, where $m \leq n$. Trivially $\chi'_{st}(K_{1,n}) = n$.

Dvorak, Mohar, and Šámal [9] obtained Observation 4.1, thus proved an asymptotically bound on $\chi'_{st}(K_{n,n})$: it follows from Theorem 1.2 that for every $\varepsilon > 0$, there is a constant $C > 0$, such that for every $n \geq 1$, $\chi'_{st}(K_{n,n}) \leq Cn^{1+\varepsilon}$.

**Observation 4.1 ([9])** $\chi'_{st}(K_{n,n}) \leq \chi'_{st}(K_{n}) + n$.

Wang, Wang, and Wang [43] proved that $\chi'_{st}(K_{3,4}) = 7$, and it is known that $\chi'_{st}(K_{3,3}) = 6$ in [9]. Casselgren, Granholm, and Raspaud [6] obtained the following results.

**Theorem 4.2 ([6])** Let $m, n$ be positive integers. The following holds.

(a) $\chi'_{st}(K_{2,n}) = 2n - \lfloor \frac{n}{2} \rfloor$.

(b) $\chi'_{st}(K_{3,n}) = 3\lceil \frac{n}{2} \rceil$ for $n \neq 4$.

(c) $\frac{5n}{3} \leq \chi'_{st}(K_{4,n}) \leq 20\lceil \frac{n}{12} \rceil$.

(d) $\chi'_{st}(K_{m,n}) \leq 15\lceil \frac{n}{8} \rceil \lceil \frac{m}{8} \rceil$.

Some exact values of $\chi'_{st}(K_{m,n})$ are given by Casselgren, Granholm, and Raspaud [6] in the following Table 1.

Moreover, using the idea in the proof of Theorem 4.2(c), one can prove lower bounds on the star chromatic index for further families of complete bipartite graphs. Let us here just list a few cases corresponding to the values in the Table 1(see [6]).
\[ \chi'_st(K_5, n) \geq \frac{5n}{3}. \]
\[ \chi'_st(K_6, n) \geq \frac{7n}{4}. \]
\[ \chi'_st(K_7, n) \geq \frac{7n}{4}. \]
\[ \chi'_st(K_8, n) \geq \frac{9n}{5}. \]

Next, we turn to general bipartite graphs with restrictions on the vertex degrees. In the following we use the notation \( G = (X, Y; E) \) for a bipartite graph \( G \) with parts \( X \) and \( Y \) and edge set \( E = E(G) \). We denote by \( \Delta(X) \) and \( \Delta(Y) \) the maximum degrees of the vertices in the parts \( X \) and \( Y \), respectively. A bipartite graph \( G = (X, Y; E) \) where all vertices in \( X \) have degree \( r \) and all vertices in \( Y \) have degree \( d \) is called \((r, d)\)-biregular.

If the vertices in one part of a bipartite graph \( G \) have maximum degree 1, then trivially \( \chi'_st(G) = \Delta(G) \). For the case when the vertices in one of the parts have maximum degree two, Casselgren, Granholm, and Raspaud \([6]\) proved the following.

**Theorem 4.3 \([6]\)** Let \( G = (X, Y; E) \) be a bipartite graph with \( \Delta(X) = 2 \) and \( k \geq 1 \). Then

- \( \chi'_st(G) \leq 3k \) if \( \Delta(Y) = 2k \).
- \( \chi'_st(G) \leq 3k + 2 \) if \( \Delta(Y) = 2k + 1 \).

Note that the upper bounds in Theorem 4.3 are sharp. This can be obtained from Theorem 4.2(a).

By Theorem 4.3, one can get the following corollary for \((2, d)\)-biregular graphs.

**Corollary 4.4 \([6]\)** If \( G \) is a \((2, d)\)-biregular graph with \( d \geq 3 \), then \( \chi'_st(G) \leq \frac{3d + 1}{2} \).

Melinder \([32]\) gave a general upper bound on the star chromatic index of biregular graphs.
Theorem 4.5 ([32]) Let $G$ be an $(r,d)$-biregular graph, where $r \geq d > 1$. Then $\chi'_{st}(G) \leq r^2 - 2r + d + 1$.

Now we consider the star chromatic index of bipartite graphs in terms of the maximum degree. By Theorem 3.1, we have $\chi'_{st}(G) \leq 7$ if $G$ is a bipartite graph with $\Delta(G) = 3$. Wang, Wang, and Wang [43] considered bipartite graphs with $\Delta(G) = 4$ and proved the following.

Theorem 4.6 ([43]) Let $G$ be a bipartite graph with $\Delta(G) = 4$. Then $\chi'_{st}(G) \leq 13$.

Since $K_{4,4}$ requires 7 colours, and no graph requiring 8 colours has been found, the smallest upper bound is at least 7 [6].

Recently, Melinder [32] gave a general upper bound for bipartite graphs with maximum degree.

Theorem 4.7 ([32]) If $G$ is a bipartite graph with maximum degree $\Delta$, then

$$\chi'_{st}(G) \leq \Delta^2 - \Delta + 1.$$ 

Melinder [32] also proved an upper bound for a special case of multipartite graphs.

Theorem 4.8 ([32]) Let $K_{1,\ldots,1,n}$ be a complete multipartite graph with $m$ parts of size one. Then

$$\chi'_{st}(K_{1,\ldots,1,n}) \leq \begin{cases} 
n & \text{if } m = 1, \\
2n - \lfloor \frac{n}{2} \rfloor + 1 & \text{if } m = 2, \\
3\lceil \frac{n}{2} \rceil + 3 & \text{if } m = 3, \\
20\lceil \frac{n}{12} \rceil + 6 & \text{if } m = 4, \\
15\lceil \frac{n}{8} \rceil \lceil \frac{n}{8} \rceil + \frac{m(m-1)}{2} & \text{if } m \geq 5.
\end{cases}$$

5 Planar graphs

Deng, Liu, and Tian [11] and Bezegová, Lužar, Mockovčiaková, Soták, and Škrekovski [2] independently proved that for each tree $T$ with maximum degree $\Delta$, its star chromatic index $\chi'_{st}(T) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor$. Moreover, the bound is tight. Bezegová, Lužar, Mockovčiaková, Soták, and Škrekovski [2] investigated the star edge-coloring of outerplanar graphs by showing the following results.

Theorem 5.1 ([2]) Let $G$ be an outerplanar graph. Then

(a) $\chi'_{st}(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 12$ if $\Delta(G) \geq 4$.

(b) $\chi'_{st}(G) \leq 5$ if $\Delta(G) \leq 3$. 

8
In [2], the authors pointed out that the constant 12 in Theorem 5.1(a) can be decreased to 9 by using more involved analysis. Moreover, they put forward the following conjecture.

**Conjecture 5.2 ([2])** Every outerplanar graph $G$ has $\chi'_{st}(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 1$ if $\Delta(G) \geq 4$.

Wang, Wang, and Wang [42] improved Theorem 5.1(a) by edge partition method.

**Theorem 5.3 ([42])** If $G$ is an outerplanar graph, then $\chi'_{st}(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 5$.

A cactus is a graph in which every edge belongs to at most one cycle. Since these graphs are outerplanar, in order to prove Conjecture 5.2, it is worth to study the star edge coloring of cactus graphs. Omoomi, Dastjerdi, and Yektacian [37] proved Conjecture 5.2 for cactus graphs.

**Theorem 5.4 ([37])** If $G$ is a cactus, then $\chi'_{st}(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 1$.

An outerplanar graph $G$ is maximal if $G + uv$ is not outerplanar for any two non-adjacent vertices $u$ and $v$ of $G$. Deng and Tian [14] determined the exact values of the star chromatic index for all maximal outerplanar graphs with order $3 \leq n \leq 8$ and proved the following results.

**Theorem 5.5 ([14])** Let $G$ be a maximal outerplanar graph with order $n$ and maximum degree $\Delta$. Then

- $\chi'_{st}(G) = 6$ if $\Delta = 4$.
- $6 \leq \chi'_{st}(G) \leq n - 1$ if $n \geq 8$. The bounds are tight.

Recently, Deng, Yao, Zhang, and Cui [10] studied the star chromatic index of 2-connected outerplanar graphs and proved that if $G$ is a 2-connected outerplanar graph with diameter $d$ and maximum degree $\Delta$, then $\chi'_{st}(G) \leq \Delta + 6$ if $d = 2$ or 3; $\chi'_{st}(G) \leq 9$ if $\Delta = 5$.

Wang, Wang, and Wang [42] investigated the star edge-coloring of planar graphs by using an edge-partition technique and a useful relation between star chromatic index and strong chromatic index.

**Definition 5.6** A proper $k$-edge-coloring of $G$ is called a **strong $k$-edge-coloring** if any two edges of distance at most two get distinct colors. That is, each color class is an induced matching in the graph $G$. The **strong chromatic index**, denoted $\chi'_s(G)$, of $G$ is the smallest integer $k$ such that $G$ has a strong $k$-edge-coloring.

From the definitions, it is straightforward to see that $\chi'_{st}(G) \leq \chi'_s(G)$.
**Definition 5.7** Suppose that $H$ is a subgraph of a graph $G$. A *restricted-strong* $k$-edge-coloring of $H$ on $G$ is a $k$-edge-coloring such that any two edges having a distance at most two in $G$ get distinct colors. The *restricted strong chromatic index* of $H$ on $G$, denoted $\chi'_s(H|G)$, is the smallest integer $k$ such that $H$ has a restricted strong $k$-edge-coloring on $G$.

**Definition 5.8** An *edge-partition* of $G$ is a decomposition of $G$ into subgraphs $G_1, G_2, \ldots, G_m$ such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots E(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$.

**Theorem 5.9** ([42]) If a graph $G$ can be edge-partitioned into two graphs $F$ and $H$, then

$$\chi'_st(G) \leq \chi'_st(F) + \chi'_s(H|G).$$

**Theorem 5.10** ([41, 42]) Every planar graph $G$ with maximum degree $\Delta$ can be edge-partitioned into two forests $F_1$, $F_2$ and a subgraph $K$ such that $\Delta(K) \leq 10$ and $\Delta(F_i) \leq \lceil(\Delta - 9)/2\rceil$ for $i = 1, 2$.

Using Theorems 5.9 and 5.10, currently the best bound for planar graphs can be obtained.

**Theorem 5.11** ([42]) Let $G$ be a planar graph with maximum degree $\Delta$. Then

$$\chi'_st(G) \leq 2.75\Delta + 18.$$  

Similarly, Wang, Wang, and Wang [42] proved more specific results (together with the result for outerplanar graphs from Theorem 5.3).

**Theorem 5.12** ([42]) Let $G$ be a planar graph with maximum degree $\Delta$ and girth $g$. Then

(a) $\chi'_st(G) \leq 2.25\Delta + 6$ if $G$ is $K_4$-minor free.

(b) $\chi'_st(G) \leq \lceil1.5\Delta\rceil + 18$ if $G$ has no 4-cycles.

(c) $\chi'_st(G) \leq \lceil1.5\Delta\rceil + 13$ if $g \geq 5$.

(d) $\chi'_st(G) \leq \lceil1.5\Delta\rceil + 3$ if $g \geq 8$.

6 **List star edge-coloring of graphs**

A natural generalization of star edge-coloring is the list star edge-coloring. Given a list assignment $L$ which assigns to each edge $e$ a finite set $L(e)$, a graph is said to be *L-star edge-colorable* if $G$ has a star edge-coloring $c$ such that $c(e) \in L(e)$ for each edge $e$. The *list-star chromatic index*, $ch'_st(G)$ of a graph $G$ is the minimum $k$ such that for every edge
list $L$ for $G$ with $|L(e)| = k$ for every $e \in E(G)$, $G$ is $L$-star edge-colorable. For any graph $G$, it is obvious that $\chi'_{st}(G) \leq ch'_{st}(G)$.

Moser and Tardos [34] designed an algorithmic version of Lovász Local Lemma by means of the so-called Entropy Compression Method. In [7], by employing Entropy Compression Method, Cai, Yang, and Yu gave an upper bound for the list-star chromatic index of a graph.

**Theorem 6.1 ([7])** For any graph $G$ with maximum degree $\Delta$,

$$ch'_{st}(G) \leq \left\lceil 2\Delta^{3/2}(\frac{1}{\Delta} + 2)^{3/2} + 2\Delta \right\rceil.$$  

Dvořák, Mohar, and Šámal [9] asked the following question.

**Question 6.2 ([9])** Is it true that $ch'_{st}(G) \leq 7$ for every subcubic graph $G$? (perhaps even $\leq 6$?)

Kerdjoudj, Kostochka, and Raspaud [20] gave a partial answer to this question. They proved that $ch'_{st}(G) \leq 8$ for every subcubic graph. Subsequently, Lužar, Mockovčiaková, and Soták [31] answered Question 6.2 in affirmative.

**Theorem 6.3 ([31])** Let $G$ be a subcubic graph. Then $ch'_{st}(G) \leq 7$.

Kerdjoudj, Kostochka, and Raspaud [20], Kerdjoudj and Kostochka [21] and Kerdjoudj, Pradeep, and Raspaud [22] studied the list star edge-coloring of graphs with small maximum average degree.

**Theorem 6.4 ([20, 21, 22])** Let $G$ be a graph with maximum degree $\Delta$.

(a) If mad$(G) < 7/3$, then $ch'_{st}(G) \leq 2\Delta - 1$.

(b) If mad$(G) < 5/2$, then $ch'_{st}(G) \leq 2\Delta$.

(c) If mad$(G) < 8/3$, then $ch'_{st}(G) \leq 2\Delta + 1$.

(d) If mad$(G) < 14/5$, then $ch'_{st}(G) \leq 2\Delta + 2$.

(e) If mad$(G) < 3$, then $ch'_{st}(G) \leq 2\Delta + 3$.

In [30], Li, Horacek, Luo, and Miao presented, in contrast, a best possible linear upper bound for the list-star chromatic index for some graph classes. Specifically they showed the following.
**Theorem 6.5 ([30])** Let $\varepsilon > 0$ be a real number and $d = 2^{\left\lfloor \frac{8-3\varepsilon}{9\varepsilon} \right\rfloor}$. Let $G$ be a graph with maximum degree $\Delta$ and $\text{mad}(G) < \frac{8}{3} - \varepsilon$. Then

$$
\text{ch}'_{st}(G) \leq \max\{\frac{3}{2}\Delta + \frac{d}{2} + 2, \Delta + 2d + 1\}.
$$

To overcome some difficulties in the proof, they developed a new coloring extension method by requiring certain edges to be colored with different size of sets of colors.

It was observed in [3] that every planar graph with girth $g$ satisfies $\text{mad}(G) < \frac{2g}{g-2}$. This implies the following by Theorems 3.4, 6.4, and 6.5.

**Corollary 6.6 ([20, 21, 22, 28, 30])** Let $G$ be a planar graph with maximum degree $\Delta$ and girth $g$.

(a) If $\Delta = 3$ and $g \geq 12$, then $\chi'_{st}(G) \leq 5$.

(b) If $g \geq 14$, then $\text{ch}'_{st}(G) \leq 2\Delta - 1$.

(c) If $g \geq 10$, then $\text{ch}'_{st}(G) \leq 2\Delta$.

(d) If $g \geq 8$, then $\text{ch}'_{st}(G) \leq 2\Delta + 1$.

(e) If $g \geq 7$, then $\text{ch}'_{st}(G) \leq 2\Delta + 2$.

(f) If $g \geq 6$, then $\text{ch}'_{st}(G) \leq 2\Delta + 3$.

(g) If $g \geq 9$, then $\text{ch}'_{st}(G) \leq \max\{\frac{3\Delta}{2} + 11, \Delta + 37\}$.

(h) If $g \geq 16$, then $\text{ch}'_{st}(G) \leq \frac{3\Delta}{2} + 4$.

For planar graphs with girth 4, Li, Horacek, Luo, and Miao [30] obtained an infinite family of such graphs whose list-star chromatic index can not be bounded above by $\frac{3}{2}\Delta + c$.

**Proposition 6.7 ([30])** For each integer $\Delta \geq 3$, there exists a planar graph $G$ of girth 4 with maximum degree $\Delta$ such that

$$
\text{ch}'_{st}(G) \geq \chi'_{st}(G) \geq \frac{13}{8}\Delta - \frac{3}{4}.
$$

In view of Theorem 5.12(c), Corollary 6.6(g) and Proposition 6.7, Li, Horacek, Luo, and Miao [30] conjectured that

**Conjecture 6.8 ([30])** There exists a constant $c > 0$ such that for any planar graph $G$ of girth at least 5 with maximum degree $\Delta$, we have

$$
\text{ch}'_{st}(G) \leq \frac{3}{2}\Delta + c.
$$
Let \( k \in \mathbb{N} \). A graph \( G \) is \( k \)-degenerate if \( \delta(H) \leq k \) for every subgraph \( H \) of \( G \). Kerdjoudj and Raspaud [23] and Han, Li, Luo, and Miao [16] proved:

**Theorem 6.9 ([23])** Every \( k \)-degenerate graph \( G \) of maximum degree \( \Delta \) and \( k \geq 2 \) satisfies

\[
ch'_st(G) \leq (3k - 2)\Delta - k^2 + 2.
\]

**Theorem 6.10 ([16])** Let \( k \geq 2 \) be an integer. For every \( k \)-degenerate graph \( G \) with maximum degree \( \Delta \), we have the following two upper bounds:

1. \( ch'_st(G) \leq \frac{5k-1}{2}\Delta - \frac{k(k+3)}{2} \). The bound is tight for \( C_5 \) as \( ch'_st(C_5) = 4 \)
2. \( ch'_st(G) \leq 2k\Delta + k^2 - 4k + 2 \).

Kerdjoudj and Raspaud [23] also considered the class of \( K_4 \)-minor free graphs. Since the \( K_4 \)-minor free graphs are 2-degenerate[15], by applying Theorem 6.9 we have \( ch'_st(G) \leq 4\Delta - 2 \) for any graph \( G \) which is \( K_4 \)-minor free and contains at least one edge. Kerdjoudj and Raspaud [23] improved this bound by proving the following.

**Theorem 6.11 ([23])** Let \( G \) be a \( K_4 \)-minor free graph with maximum degree \( \Delta \). Then

\[
ch'_st(G) \leq \begin{cases} 
4 & \text{if } \Delta = 2, \\
3\Delta - 3 & \text{if } \Delta \geq 3.
\end{cases}
\]

Notice that Theorem 6.11 implies that Conjecture 3.2 is true for every \( K_4 \)-minor free subcubic graph.

Han, Li, Luo, and Miao [16] extended this result of the star chromatic index of trees to list-star chromatic index.

**Theorem 6.12 ([16])** For every tree \( T \) with maximum degree \( \Delta \), \( ch'_st(T) \leq \lfloor \frac{3\Delta}{2} \rfloor \). This bound is tight.

7 Another family of graphs

7.1 Paths, Cycles, Fan graphs, Wheel graphs

For any integer \( n \geq 4 \), the wheel graph \( W_n \) (resp. fan graph \( F_n \)) is the \( n \)-vertex graph obtained by joining a vertex to each of the \( n - 1 \) vertices of the cycle graph \( C_{n-1} \) (resp. the path graph \( P_{n-1} \)). The following results were gave by Deng [13] and Wang, Wang, and Wang [43].

**Theorem 7.1 ([13, 43])** For \( P_n \) (\( n \geq 2 \)) and \( C_n \) (\( n \geq 3 \)),

\[
\]
Theorem 7.2 ([13]) Let $F_n$ be a fan graph on order $n \geq 3$ and $W_n$ be a wheel graph on order $n \geq 4$. Then

(a) $\chi_{st}'(F_n) = \chi_{st}'(P_n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } 3 \leq n \leq 4, \\ 3 & \text{if } n \geq 5. \end{cases}$

(b) $\chi_{st}'(C_n) = \chi_{st}'(P_n) = \begin{cases} 3 & \text{if } n \neq 5, \\ 4 & \text{if } n = 5. \end{cases}$

7.2 Halin graphs

A Halin graph is a planar graph that consists of a plane embedding of a tree $T$ and a cycle $C$, in which the cycle $C$ connects the leaves of the tree such that it is the boundary of the exterior face and the degree of each interior vertex of $T$ is at least three. A complete Halin graph is a graph that all leaves of the characteristic tree are at the same distance from the root vertex.

A caterpillar is a tree whose removal of leaves results in a path called the spine of the caterpillar. The necklace graph denoted by $N_n$ is a cubic Halin graph obtained by joining a cycle with all vertices of degree 1 of a caterpillar having $n$ vertices of degree 3 and $n + 2$ vertices of degree 1.

Hou, Li, and Wang [19] studied the star chromatic index of Halin graphs. Theorem 7.3 was also proved by Casselgren, Granholm, and Raspaud [6]

Theorem 7.3 ([19, 6]) If $G$ is a cubic Halin graph, then

$$\chi_{st}'(G) \leq 6.$$ 

Furthermore, the bound is tight.

Theorem 7.4 ([19]) Let $N_n$ be a necklace such that $n \geq 1$ is odd. Then

$$\chi_{st}'(N_n) \leq 5.$$
Theorem 7.5 ([19]) Let $G$ be a complete Halin graph with maximum degree $\Delta(G) \geq 6$. Then
\[ \chi'_st(G) \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1. \]

7.3 $k$-power graphs
The $k$-power of a graph $G$ is the graph $G^k$ whose vertex set is $V(G)$, two distinct vertices being adjacent in $G^k$ if and only if their distance in $G$ is at most $k$.

Hou, Li, and Wang [19] studied the star chromatic index of path square graphs and cyclic square graphs.

Theorem 7.6 ([19]) For the graph $P^2_n$ with $n \geq 3$,
\[ \chi'_st(P^2_n) = \begin{cases} 3 & \text{if } n = 3, \\ 4 & \text{if } n = 4, \\ 6 & \text{if } n \geq 5. \end{cases} \]

Theorem 7.7 ([19]) For the graph $C^2_n$ with $n \geq 3$,
\[ \chi'_st(C^2_n) \leq \begin{cases} 8 & \text{if } n \text{ is odd (}n \neq 5 \text{ or } 9), \\ 9 & \text{if } n \text{ is even}. \end{cases} \]

7.4 Generalized Petersen graphs
Let $n$ and $k$ be positive integers, $n \geq 2k + 1$ and $n \geq 3$. The generalized Petersen graph $P(n,k)$, which was introduced in [40], is a cubic graph with $2n$ vertices, denoted by $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$, and all edges are of the form $u_iu_{i+1}, u_iv_i, v_iv_{i+k}$ for $0 \leq i \leq n - 1$. In the absence of a special claim, all subscripts of vertices of $P(n,k)$ are taken modulo $n$ in the following.

By Theorem 3.1(b), we have $\chi'_st(P(n,k)) \geq 4$. Zhu and Shao [45] gave a necessary and sufficient condition of $\chi'_st(P(n,k)) = 4$ and showed that “almost all” generalized Petersen graphs have a star 5-edge-coloring. Hou, Li, and Wang [19] proved that $\chi'_st(P(3n,n)) = 5$ for $n \geq 2$, which is a special case of Theorems 7.8 and 7.9.

Theorem 7.8 ([45]) $\chi'_st(P(n,k)) = 4$ if and only if $n \equiv 0 \pmod{4}$ and $k \equiv 1 \pmod{2}$.

Theorem 7.9 ([45]) Let $\ell$ be the greatest common divisor of $n$ and $k$. Then $P(n,k)$ has a star 5-edge-coloring when
- $\ell \geq 3$, with the exception of $\ell = 3, k \neq \ell$, and $\frac{n}{3} \equiv 1 \pmod{3}$,
• \( \ell = 1, n \equiv 0 \pmod{2} \) and \( k \equiv 1 \pmod{2} \),
• \( k = 1 \) and \( n \geq 5 \),
• \( k = 2 \) and \( n \equiv 0 \pmod{6} \).

Furthermore, Zhu and Shao [45] found that the generalized Petersen graph \( P(n, k) \) with \( n = 3, k = 1 \) is the only graph with a star chromatic index of 6 among the investigated graphs. Based on these results, they conjectured that \( P(3, 1) \) is the unique generalized Petersen graph that admits no star 5-edge-coloring.

7.5 Cartesian product of graphs

The Cartesian product of two graphs \( G \) and \( H \), denoted by \( G \square H \), is a graph with vertex set \( V(G) \times V(H) \), and \((a, x)(b, y) \in E(G \square H)\) if either \( ab \in E(G) \) and \( x = y \), or \( xy \in E(G) \) and \( a = b \). A \( d \)-dimensional grid \( G_{\ell_1,\ell_2,\ldots,\ell_d} = P_{\ell_1} \square P_{\ell_2} \square \cdots \square P_{\ell_d} \) is the Cartesian product of \( d \) paths. A \( d \)-dimensional hypercube \( Q_d \) is the Cartesian product of \( P_2 \) by itself \( d \) times. A \( d \)-dimensional toroidal grid \( T_{\ell_1,\ell_2,\ldots,\ell_d} = C_{\ell_1} \square C_{\ell_2} \square \cdots \square C_{\ell_d} \) is the Cartesian product of \( d \) cycles.

Omoomi and Dastjerdi [38] first gave an upper bound on the star chromatic index of the Cartesian product of two graphs in terms of their star chromatic indices and their chromatic numbers.

**Theorem 7.10 ([38])** For any two graphs \( G \) and \( H \),

\[
\chi'_{st}(G \square H) = \chi'_{st}(H \square G) \leq \min\{\chi'_{st}(G)\chi(H) + \chi'_{st}(H)\chi(G) + \chi'_{st}(G)\}.
\]

Moreover, this bound is tight.

**Theorem 7.11 ([38])** For every graph \( G \) and a positive integer \( n \), the following hold.

- If \( n \geq 2 \), then \( \chi'_{st}(G \square P_n) \leq \chi'_{st}(G \square C_{2n}) \leq \chi'_{st}(G) + 2\chi(G) \).
- If \( n \geq 2\chi(G) + 1 \) is odd, then \( \chi'_{st}(G \square C_n) \leq \chi'_{st}(G) + 2\chi(G) + 1 \).
- If \( n \geq 3 \) is odd, then \( \chi'_{st}(G \square C_n) \leq \chi'_{st}(G) + 2\chi(G) + \left\lceil \frac{2\chi(G)}{n-1} \right\rceil \leq \chi'_{st}(G) + 2\chi(G) + 3 \).

Then they determined the exact value of the star chromatic index of 2-dimensional grids and extended this result to get an upper bound on the star chromatic index of \( d \)-dimensional grids, where \( d \geq 3 \). The following theorem 7.12, corollary 7.13 and corollary 7.15 also were proved by Deng, Liu, and Tian [12].
Theorem 7.12 ([12, 38]) For all integers $2 \leq m \leq n$,

$$\chi'_st(P_m \Box P_n) = \begin{cases} 
3 & \text{if } m = n = 2, \\
4 & \text{if } m = 2, n \geq 3, \\
5 & \text{if } m = 3, n \in \{3, 4\}, \\
6 & \text{otherwise.}
\end{cases}$$

Corollary 7.13 ([12, 38]) If $G_{\ell_1, \ell_2, \ldots, \ell_d}$ is a $d$-dimensional grid, where $d \geq 2$, then

$$\left\lceil \frac{8}{3}(d - \sum_{i=1}^{d} \frac{1}{\ell_i}) \right\rceil \leq \chi'_st(G_{\ell_1, \ell_2, \ldots, \ell_d}) \leq 4d - 2.$$

Moreover, for $d = 2$ and $\ell_1, \ell_2 \geq 4$, the bound is tight.

In the following theorem, Omoomi and Dastjerdi [38] considered the star chromatic index of the Cartesian product of paths and cycles.

Theorem 7.14 ([38]) For all integers $m \geq 2$ and $n \geq 3$,

$$\chi'_st(P_m \Box C_n) \leq \begin{cases} 
7 & \text{if } n \text{ is even,} \\
8 & \text{if } n \text{ is odd.}
\end{cases}$$

Now, by Theorems 7.10 and 7.14, Omoomi and Dastjerdi [38] obtained an upper bound on the star chromatic index of hypercube $Q_d$, where $d \geq 3$, as follows.

Corollary 7.15 ([12, 38]) If $Q_d$ is a $d$-dimensional hypercube, where $d \geq 3$, then

$$\left\lceil \frac{4}{3}d \right\rceil \leq \chi'_st(Q_d) \leq 2d - 2.$$ 

Moreover the bound is tight.

As a corollary of Theorem 7.12, Holub, Lužar, Mihalíková, Mockovčiaková, and Soták [17] established the lower bound of 6 colors for the Cartesian products, where one factor is a cycle and the other is a path of length at least 2. Then they also obtained the exact values of the star chromatic index for specific lengths of paths and cycles.

Corollary 7.16 ([17]) For all integers $m, n \geq 3$,

$$\chi'_st(P_m \Box C_n) \geq 6.$$
Theorem 7.17 ([17]) For all integers $m \geq 2$ and $n \geq 3$,

$$\chi'_{st}(P_m \Box C_n) = \begin{cases} 
4 & \text{if } m = 2 \text{ and } n \equiv 0 \pmod{4}, \\
5 & \text{if } m = 2 \text{ and } n \equiv 3, n \not\equiv 0 \pmod{4}, \\
6 & \text{if } m = 2, n = 3 \text{ or } m \in \{3, 4, 5, 6\} \text{ or } m \geq 7, n \equiv 0 \pmod{2, 3}, \\
7 & \text{if } m \geq 7 \text{ and } n \in \{5, 7\}.
\end{cases}$$

It remains to determine $\chi'_{st}(P_m \Box C_n)$ for $m \geq 7$ and $n \geq 11$, $n \not\equiv 0 \pmod{2, 3}$. Holub, Lužar, Mihalíková, Mockovčiaková, and Soták [17] proposed the following conjecture.

Conjecture 7.18 ([17]) There exist constants $c_1$ and $c_2$ such that for all integers $m \geq c_1$ and $n \geq c_2$, we have

$$\chi'_{st}(P_m \Box C_n) = 6.$$ 

Finally, Omoomi and Dastjerdi [38] gave some upper bounds on the star chromatic index of the Cartesian product of two cycles and $d$-dimensional toroidal grids as follows.

Theorem 7.19 ([38]) For all integers $m, n \geq 3$,

$$\chi'_{st}(C_m \Box C_n) \leq \begin{cases} 
7 & \text{if } m \text{ and } n \text{ are even}, \\
8 & \text{if } m \not\equiv 3 \pmod{2} \text{ is odd and } n \text{ is even}, \\
9 & \text{if } m = 3 \text{ and } n \text{ is even}, \\
10 & \text{if } m \text{ and } n \text{ are odd}.
\end{cases}$$

Corollary 7.20 ([38]) For all integers $d \geq 2$ and $\ell_1, \ell_2, \ldots, \ell_d \geq 3$,

$$\chi'_{st}(T_{2\ell_1, 2\ell_2, \ldots, 2\ell_d}) \leq 4d - 1 \text{ and } \chi'_{st}(T_{\ell_1, \ell_2, \ldots, \ell_d}) \leq 7d - 4.$$ 

Holub, Lužar, Mihalíková, Mockovčiaková, and Soták [17] obtained the exact values of the star chromatic index for specific lengths of cycles and showed the star chromatic index of the Cartesian product of two cycles is at most 7. Note that Corollary 7.16 implies that the Cartesian product of any two cycles will need at least 6 colors for a star edge-coloring.

Theorem 7.21 ([17]) For all integers $m, n \geq 3$,

$$6 \leq \chi'_{st}(C_m \Box C_n) \leq 7.$$ 

Theorem 7.22 ([17]) For all integers $m, n \geq 3$,

$$\chi'_{st}(C_m \Box C_n) = \begin{cases} 
6 & \text{if } m, n \equiv 0 \pmod{3} \text{ or } m \equiv 0 \pmod{4}, n \equiv 0 \pmod{2} \\
or m = 6, n \equiv 0 \pmod{3, 4}, \\
7 & \text{if } m = 3, n \not\equiv 0 \pmod{3} \text{ or } m = 4, n \in \{5, 7, 9, 11\} \\
or m \in \{5, 7\} \text{ or } m = 6, n \not\equiv 0 \pmod{3, 4} \text{ or } m = 8, n = 9.
\end{cases}$$
Note that Conjecture 7.18 is equivalent to the following.

**Conjecture 7.23 ([17])** There exists a constant c such that for every integer \( m \geq c \), there exists an integer \( n \) such that
\[
\chi'_{st}(C_m \Box C_n) = 6.
\]

In fact, Holub, Lužar, Mihalíková, Mockovčiaková, and Soták [17] believed (with a bit lower confidence) that the following stronger version of Conjecture 7.23 can also be confirmed.

**Conjecture 7.24 ([17])** There exists a constant c such that for all integers \( m, n \geq c \),
\[
\chi'_{st}(C_m \Box C_n) = 6.
\]

### 7.6 Corona product of graphs

The **corona product** of two graphs \( G \) and \( H \) is the graph \( G \circ H \) formed from one copy of \( G \) and \( |V(G)| \) copies of \( H \) where the \( i \)th vertex of \( G \) is adjacent to every vertex in the \( i \)th copy of \( H \). We first state some special graph classes as follows.

- The **\( n \)-sunlet** graph on \( 2n \) vertices is obtained by attaching \( n \) pendant edges to the cycle \( C_n \) and is denoted by \( C'_n \).

- **Double star** \( K_{1,n,n} \) is a tree obtained from the star \( K_{1,n} \) by adding a new pendant edge of the existing \( n \) pendant vertices. It has \( 2n + 1 \) vertices and \( 2n \) edges.

- The **helm graph** \( H_n \) is the graph obtained from the wheel graph \( W_{n+1} \) by adjoining a pendant edge at each vertices of the outer cycle.

- The **gear graph** \( G_n \), also known as a bipartite wheel graph, is the wheel graph \( W_{n+1} \) with a vertex added between each pair of adjacent vertices of the outer cycle.

In [25] and [26], the authors obtained the star edge chromatic number of the corona product of paths with paths, paths with cycles, paths with wheels, paths with helms, paths with gear graphs, paths with \( n \)-sunlet graphs, paths with Double star graphs, and paths with complete bipartite graphs, respectively.

**Theorem 7.25 ([25])** For all positive integers \( m \) and \( n \),
\[
\chi'_{st}(P_m \circ P_n) = \begin{cases} 
  n & \text{if } m = 1, \\
  n + 1 & \text{if } m = 2, \\
  n + 2 & \text{if } m \geq 3.
\end{cases}
\]
Theorem 7.26 ([26]) Let $G \in \{P_m \circ C_n, P_m \circ W_n, P_m \circ H_n, P_m \circ G_n\}$, where $m \geq 1$ and $n > 4$. Then

$$
\chi'_st(G) = \Delta(G).
$$

Theorem 7.27 ([25]) For all positive integers $m$ and $n \geq 3$, then

$$
\chi'_st(P_m \circ C'_n) = \begin{cases} 
2n & \text{if } m = 1, \\
2n + 1 & \text{if } m = 2, \\
2n + 2 & \text{if } m \geq 3.
\end{cases}
$$

Theorem 7.28 ([25]) For all positive integers $m$ and $n$, then

$$
\chi'_st(P_m \circ K_{1,n,n}) = \begin{cases} 
2n + 1 & \text{if } m = 1, \\
2n + 2 & \text{if } m = 2, \\
2n + 3 & \text{if } m \geq 3.
\end{cases}
$$

Theorem 7.29 ([25]) For all positive integers $\ell \geq 3$, $m \geq 3$ and $n \geq 3$, then

$$
\chi'_st(P_\ell \circ K_{m,n}) = m + n + 2.
$$

7.7 Graph join of two graphs

Let $G$ and $H$ be two graphs with $V(G) \cap V(H) = \emptyset$. The graph join $G \vee H$ of $G$ and $H$ has $V(G \vee H) = V(G) \cup V(H)$ and two different vertices $u$ and $v$ of $G \vee H$ are adjacent if $u \in V(G)$ and $v \in V(H)$, or $uv \in E(G)$ or $uv \in E(H)$. Yang, Liu, and Chen [44] considered the star chromatic index of the graph join of paths and paths and proved the following results.

Theorem 7.30 ([44]) For any two graphs $G$ and $H$,

$$
\chi'_st(G \vee H) \leq \max\{\chi'_st(G), \chi'_st(H)\} + |G||H|.
$$

Theorem 7.31 ([44]) For any integer $n \geq 2$,

$$
\chi'_st(P_n \vee P_2) = \begin{cases} 
5 & \text{if } n = 2, \\
\frac{3n-1}{2} + 4 & \text{if } n \geq 3 \text{ and } n \equiv 1 \pmod{2}, \\
\frac{3n}{2} + 3 & \text{if } n \geq 4 \text{ and } n \equiv 0 \pmod{2}.
\end{cases}
$$

Theorem 7.32 ([44]) $\chi'_st(P_3 \vee P_3) = 10$ and $\chi'_st(P_n \vee P_n) < n^2 - 3n + 9$ when $n \geq 4$.

Corollary 7.33 ([44]) $m + 2 \leq \chi'_st(P_m \vee P_n) < mn - 3n + 9$ when $m > n \geq 4$. 

20
Acknowledgments. Hui Lei was partially supported by the National Natural Science Foundation of China and the Fundamental Research Funds for the Central Universities, Nankai University. Yongtang Shi was partially supported by the National Natural Science Foundation of China (No. 11922112), the Natural Science Foundation of Tianjin and the Fundamental Research Funds for the Central Universities, Nankai University.

References

[1] M.O. Albertson, G.G. Chappell, H.A. Kierstead, A. Kündgen, and R. Ramamurthi, Coloring with no 2-colored $P_4$’s, Electron. J. Combin. 11(2004), #R26.

[2] L. Bezegová, B. Lužar, M. Mockovčiaková, R. Soták, and R. Škrekovski, Star edge coloring of some classes of graphs, J. Graph Theory 81(1)(2016), 73–82.

[3] O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud, and E. Sopena, On the maximum average degree and the oriented chromatic number of a graph, Discrete Math. 206(1999), 77–89.

[4] Y. Bu, D.W. Cranston, M. Montassier, A. Raspaud, and W. Wang, Star coloring of sparse graphs, J. Graph Theory 62(2009), 201–219.

[5] M. Chen, A. Raspaud, and W. Wang, 6-star-coloring of subcubic graphs, J. Graph Theory 72(2013), 128–145.

[6] C.J. Casselgren, J.B. Granholm, and A. Raspaud, On star edge colorings of bipartite and subcubic graphs, arXiv:1912.02467, 2019.

[7] J. Cai, C. Yang, and j. Yu, An upper bound for the choice number of star edge coloring of graphs, Appl. Math. Comput. 348(2019), 588–593.

[8] J. Cai, J. Wang, and X. Zhang, Restricted Colorings of Graphs, Science Press, Beijing, 2019.

[9] Z. Dvořák, B. Mohar, and R. Šámal, Star chromatic index, J. Graph Theory 72(2013), 313–326.

[10] X. Deng, Q. Yao, Y. Zhang, and X. Cui, A note on a conjecture of star chromatic index for outerplanar graphs, Appl. Math. Comput. 384(2020), 125353.

[11] K. Deng, X.S, Liu, and S.L. Tian, Star edge coloring of trees (in Chinese), J. Shandong Univ. Nat. Sci. 46(8)(2011), 84–88
[12] K. Deng, X.S. Liu, and S.L. Tian, Star edge coloring of $d$-dimensional grids, *J. East China Norm. Univ. Nat. Sci. Ed.* **3**(2012), 13–16.

[13] K. Deng, Star edge-coloring of graphs, M.D. Thesis, Northwest Normal University, 2007.

[14] K. Deng and S.L. Tian, Star edge coloring of maximal outer plane graphs, *Appl. Math. A J. Chin. Univ.* **26**(4)(2011), 489–494.

[15] R.J. Duffin, Topology of series-parallel networks, *J. Math. Anal. Appl.* **10**(2)(1965), 303–318.

[16] M. Han, J. Li, R. Luo, and Z.K. Miao, List star edge coloring of $k$-degenerate graphs, *Discrete Math.* **342**(6)(2019), 1838–1848.

[17] P. Holub, B. Lužar, E. Mihalíková, M. Mockovčiaková, and R. Soták, Star edge-coloring of square grids, arXiv:2005.0286, 2020.

[18] I. Holyer, The NP-completeness of edge-coloring, *SIAM J. Comput.* **10**(1981), 718–720.

[19] X.L. Hou, L.X. Li, and T. Wang, Star edge coloring of some special graphs, preprint.

[20] S. Kerdjoudj, A. V. Kostochka, and A. Raspaud, List star edge coloring of subcubic graphs, *Discuss. Math. Graph Theory* **38**(4)(2018), 1037–1054.

[21] S. Kerdjoudj and A. Raspaud, List star edge coloring of sparse graphs, *Discrete Appl. Math.* **238**(2018), 115–125.

[22] S. Kerdjoudj, K. Pradepp, and A. Raspaud, List star chromatic index of sparse graphs, *Discrete Math.* **341**(2018), 1835–1849.

[23] S. Kerdjoudj and A. Raspaud, List star edge-coloring of $k$-degenerate graphs and $K_4$-minor free graphs, *Discrete Appl. Math.* **261**(2019), 268–275.

[24] H.A. Kierstead, A. Kündgen, and C. Timmons, Star coloring bipartite planar graphs, *J. Graph Theory* **60**(2009), 1–10.

[25] K. Kaliraj, R. Sivakami, and V.J. Vivin, Star edge coloring of corona product of path with some graphs, *International J.Math. Combin.* **3**(2016), 115–122.

[26] K. Kaliraj, R. Sivakami, and V.J. Vernold, Star edge coloring of corona product of path and wheel graph families, *Proyecciones (Antofagasta)* **37**(4)(2018), 593–601.

[27] H. Lei, Y. Shi, and Z-X. Song, Star chromatic index of subcubic multigraphs, *J. Graph Theory* **88**(4)(2018), 566–576.
[28] H. Lei, Y. Shi, Z-X. Song, and T. Wang, Star 5-edge-colorings of subcubic multigraphs, *Discrete Math.* 341(4)(2018), 950–956.

[29] X.-S. Liu and K. Deng, An upper bound on the star chromatic index of graphs with $\Delta \geq 7$, *J. Lanzhou Univ. Nat. Sci.* 44(2008), 98–100.

[30] J.A. Li, K. Horacek, R. Luo, and Z.K. Miao, Upper bounds on list star chromatic index of sparse graphs, *Acta. Math. Sin.-English Ser.*, 36(1)(2020) 1–12.

[31] B. Lužar, M. Mockovčiaková, and R. Soták, Note on list star edge-coloring of subcubic graphs, *J. Graph Theory* 90(3)(2019), 304–310.

[32] V. Melinder, Upper bounds on the star chromatic index for bipartite graphs, PhD thesis, Linköping University, 2020.

[33] M. Mockovčiaková, Distance constrained edge colorings of graphs, PhD thesis, P. J. Šafárik University, Faculty of Science, Košice, 2013.

[34] R.A. Moser and G. Tardos, A constructive proof of the general Lovász local lemma, *J. ACM* 57(2)(2010), 1–15.

[35] J. Nešetřil and P. Ossona de Mendez, Colorings and homomorphisms of minor closed classes, Algorithms and Combinatorics, Vol. 25, Springer, Berlin, 2003, pp. 651–664.

[36] B. Omoomi, E. Roshanbin, and M.V. Dastjerdi, A polynomial time algorithm to find the star chromatic index of trees, arXiv:1805.09586, 2018.

[37] B. Omoomi, M.V. Dastjerdi, and Y. Yektaeian, Star edge coloring of cactus graphs, *Iran. J. Sci. Technol. Trans. Sci.* (2020), 1–7.

[38] B. Omoomi and M.V. Dastjerdi, Star edge coloring of the Cartesian product of graphs, arXiv:1802.01300, 2018.

[39] V. G. Vizing, On an estimate of the chromatic class of a p-graph, *Diskret Analiz* 3(1964), 25–30 (in Russian).

[40] M.E. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs, *J. Combin. Theory* 6(1969), 152–164.

[41] Y. Wang, X. Hu, and W. Wang, A note on the linear 2-arboricity of planar graphs, *Discrete Math.* 340(7)(2017), 1449–1455.
[42] Y. Wang, W. Wang, and Y. Wang, Edge-partition and star chromatic index, *Appl. Math. Comput.* **333**(2018), 480–489.

[43] Y. Wang, Y. Wang, and W. Wang, Star edge-coloring of graphs with maximum degree four, *Appl. Math. Comput.* **340**(2019), 268–275.

[44] Y. Yang, X. Liu, and X. Chen, The star-edge chromatic index on join-graph $P_m \vee P_n$, *J. Northwest Norm. Univ. Nat. Sci.* **44**(6)(2008), 26–28.

[45] E. Zhu and Z. Shao, On the star chromatic index of generalized Petersen graphs, *Discuss. Math. Graph Theory* (2019), doi:10.7151/dmgt.2195.