Numerical Study of Inhomogeneous Pre-Big-Bang Inflationary Cosmology

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Abstract

We study numerically the inhomogeneous pre-big-bang inflation in a spherically symmetric space-time. We find that a large initial inhomogeneity suppresses the onset of the pre-big-bang inflation. We also find that even if the pre-big-bang inflationary stage is realized, the initial inhomogeneities are not homogenized. Namely, during the pre-big-bang inflation “hairs” (irregularities) do not fall, in sharp contrast to the usual (potential energy dominated) inflation where initial inhomogeneity and anisotropy are damped and thus the resulting universe is less sensitive to initial conditions.

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I. INTRODUCTION

The pre-big-bang(PBB) cosmological scenario inspired by string theory has been suggested as a possible working mechanism of inflation within the framework of the low energy effective action of string theory [1]. In this scenario, the inflation occurs by the kinetic energy of a massless dilaton, and the curvature and the string coupling grow until the effective action breaks down. After that epoch stringy nonperturbative effects become important and hopefully enable the universe to make a smooth transition to a standard Friedmann-Robertson-Walker(FRW) epoch of decelerating expansion (for recent attempts toward graceful exit, see [2]).

One of the aims for the introduction of the inflationary universe scenario [3] was to solve the homogeneity problem. However, most of works on the PBB inflation is done within the context of a homogeneous space-time (however, see [4]). So, questions arise as to a homogenization process due to the PBB inflation: Does the PBB inflation really homogenize initial inhomogeneities? Can initial inhomogeneities prevent the onset of the PBB inflation? The purpose of this paper is to answer these questions in a spherically symmetric space-time.

The peculiarity of the PBB inflation is that even classically the weak energy condition [5] is violated during that epoch when the equation are written in the form of the Einstein equations [6], while the strong energy condition is violated but the weak energy condition is respected classically in the case of the usual (potential energy dominated) inflation. This suggests that the behaviour of inhomogeneities in the PBB inflation is quite different from that in the usual inflation.

According to the linear perturbation analysis, the density perturbation $\delta$ of $p = w\rho$ matter, with $w$ being a constant, behaves as [7] (in the long wavelength limit) $\delta \propto t^{2(1+3w)/(1+6w)}$. Similar behaviour is found by the approximation method to describe the super-horizon scale inhomogeneity (the gradient expansion method) [8]. Therefore, for the strong energy violating (but the weak energy condition respecting) matter $(-1 < w < -1/3)$, the density perturbation decays with time. An inflaton generically satisfies this property. This behavior is related to the so-called cosmic no hair conjecture, which states that all expanding universe models with positive cosmological constant asymptotically approach the de Sitter space-time [9]. On the other hand, however, for the weak energy condition violating matter ($w < -1$), the density perturbation grows with time.

Of course, the dilaton does not have a constant $w$ and so the above consideration is not directly related to the PBB inflationary cosmology, but it strongly suggests that the behavior of the density perturbation in the PBB inflation is quite different from that in the usual inflation. In order to study how initial inhomogeneity affects the onset and the duration of the PBB inflation, we need to consider large inhomogeneities, and hence we have to resort to fully numerical treatment. In this paper we investigate the inhomogeneous pre-big-bang inflation in a spherically symmetric space-time by solving the field equations fully numerically. A related numerical work based on a spectral method was appeared [10], however, the analysis is limited to weakly nonlinear perturbations. There also appeared a related analytical work [11] where a criterion for PBB inflation is proposed, which is consistent with our numerical results.

This paper is organized as follows. In Sec.2, basic equations based on the ADM formalism are given. In Sec.3, after the details of the initial condition and the numerical method is
described, the numerical results are given. Sec.4 is devoted to summary.

II. FIELD EQUATIONS

The low-energy effective action is given by
\[ S = \int d^4x \sqrt{-g} \frac{e^{-\phi}}{l_s^2} \left( (4) R + g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi \right), \]
(2.1)
where \( l_s \) is the string scale which is of order the Planck scale, \( \phi \) is the dilaton, and \( (4) R \) is the four dimensional Ricci scalar. We have omitted other matter degrees of freedom for simplicity. Varying the action by \( g^{\mu \nu} \) and \( \phi \) yield
\[ G_{\mu \nu} = -\nabla_\mu \nabla_\nu \phi + \frac{1}{2} g_{\mu \nu} (\nabla \phi)^2 \equiv T_{\mu \nu}, \]
(2.2)
\[ \Box \phi = (\nabla \phi)^2, \]
(2.3)
where \( \nabla_\mu \) and \( \Box \) are a covariant derivative and the d’Alembertian of \( g_{\mu \nu} \), respectively.

Although after the conformal transformation such that \( g_{E \mu \nu} = e^{-\phi} g_{\mu \nu} \) the action is reduced to that of a massless scalar field coupled minimally to the Einstein gravity, we shall work consistently in the string frame. This is because the gauge condition in the string frame metric has a direct geometrical meaning and because the interpretation of the numerical results is direct and clear as suggested in [10].

A. Basic Equations in (3+1) Form

In the ADM formalism, the line element generally takes of the form
\[ ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(\beta^i dt + dx^i)(\beta^j dt + dx^j), \]
(2.4)
where \( \alpha, \beta^i \) is the lapse function, the shift vector, respectively. \( \gamma_{ij} \) is the three-metric of the spacelike hypersurface. A timelike vector \( n_\mu \) normal to the hypersurface is given as
\[ n_\mu = (-\alpha, 0, 0, 0). \]
(2.5)
The extrinsic curvature \( K_{ij} \) is then defined by
\[ K_{ij} = -\frac{1}{2} \mathcal{L}_n \gamma_{ij}, \]
(2.6)
where \( \mathcal{L}_n \) denotes the Lie derivative along the integral curve of \( n^\mu \). The field equations Eq.(2.2) then break up into evolution equations
\[ \gamma_{ij,t} - \mathcal{L}_{\beta} \gamma_{ij} = -2\alpha K_{ij}, \]
(2.7)
\[ K_{ij,t} - \mathcal{L}_{\beta} K_{ij} = -D_i D_j \alpha + \alpha \left[ R_{ij} + KK_{ij} - 2K_{it} K^t_j - \left( S_{ij} + \frac{1}{2} \gamma_{ij} (\rho_H - S^t_t) \right) \right] \]
(2.8)
and constraint equations
Here $D_i$ is the covariant derivative operator with respect to $\gamma_{ij}$, $R_{ij}$ is the three-dimensional Ricci tensor, $K$ is the trace of the extrinsic curvature, and $\gamma_{ij,t}$ denotes the partial derivative of $\gamma_{ij}$ with respect to $t$. $\rho_H$, $J_i$ and $S_{ij}$ are defined by the right-hand-side of Eq. (2.2) (denoted as $T_{\mu\nu}$) as

$$\rho_H = T_{\mu\nu}n^\mu n^\nu, \quad J_i = -T_{\mu\nu}n^\mu h_i^\nu, \quad S_{ij} = T_{\mu\nu}h_i^\mu h_j^\nu,$$

where

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

is the projection operator on these hypersurfaces. We also need to write the equation of motion of dilaton Eq. (2.3) in the first-order form. To do so, we introduce the following variable likewise $K_{ij}$

$$\Pi = -\mathcal{L}_n \phi = -\frac{1}{\alpha} \left( \phi, t - \beta^i \phi, i \right).$$

Then the dilaton equation of motion can be written in the first-order form as

$$\phi, t - \beta^i \phi, i = -\alpha \Pi, \quad \Pi, t - \beta^i \Pi, i = \alpha \left( K \Pi - \Pi^2 + (D\phi)^2 - D^2 \phi \right) - \gamma^{ij} \alpha, i \phi, j.$$

### B. Basic Equations in Spherical Symmetry

Now we specialize the above general considerations to a spherically symmetric space-time. The most general line element of a spherically symmetric space-time is written in the form

$$ds^2 = -(\alpha^2 - A^2 \beta^2)dt^2 + 2A^2 \beta dt dr + A^2 dr^2 + B^2 r^2 d\Omega^2,$$

where $\beta$ is the radial component of the shift vector. Because of the spherical symmetry, only two components of $K_{ij}$ are regarded as independent variables. We introduce $K_1 \equiv K^r_r$. Because there does not appear “bare” $\phi$ in Eqs. (2.2) and (2.3), it is convenient to introduce the radial derivative of it as another independent variable:

$$\Phi = \frac{\phi, r}{r}.$$

Then the evolution equations in (3+1) form are written as
\[ A_t - \beta A_x = -\alpha AK_1 + \beta_r A, \quad (2.20) \]
\[ B_t - \beta B_x = -\frac{1}{2} \alpha B(K - K_1) + \frac{\beta}{r} B, \quad (2.21) \]
\[ K_t - \beta K_x = -\frac{1}{A^2} \left( 4x\alpha_{,xx} + 6\alpha_{,x} + 4x\alpha_{,x} \left( \frac{2B_x}{B} - \frac{A_x}{A} \right) \right) \]
\[ + \alpha \left[ \frac{1}{2} K^2 - KK_1 + \frac{3}{2} K^2 + K\Pi - \Pi^2 + \frac{x\Phi^2}{A^2} \right. \]
\[ \left. - \frac{1}{A^2} \left( \left( \frac{4B_x}{B} - 2\frac{A_x}{A} + 3 \right) \Phi + 2x\Phi_{,x} \right) \right], \quad (2.22) \]
\[ K_{1,t} - \beta K_{1,x} = -\frac{1}{A^2} \left( 4x\alpha_{,xx} + 2\alpha_{,x} - 4x\frac{A_x}{A} \alpha_{,x} \right) \]
\[ + \alpha \left[ \frac{4}{A^2} \left( -2x\frac{B_{xx}}{B} - 3\frac{B_x}{B} + \frac{A_x}{A} + 2\frac{A_x B_x}{AB} \right) \right. \]
\[ + KK_1 - K_1\Pi + \frac{1}{A^2} \left( 2x\Phi_{,x} + \Phi - 2x\frac{A_x}{A} \Phi \right) \right], \quad (2.23) \]
\[ \Pi_t - \beta \Pi_x = \alpha K\Pi - \alpha \left( \Pi^2 - \frac{x\Phi^2}{A^2} \right) \]
\[ - \frac{\alpha}{A^2} \left( \frac{2x\alpha_x}{A} - 2\frac{x A_x}{A} + 4\frac{x B_x}{B} + 3 \right) \Phi - 2\frac{x\alpha}{A^2} \Phi_{,x}, \quad (2.24) \]
\[ \Phi_{,t} - \beta \Phi_{,x} = -2\alpha_x \Pi - 2\alpha \Pi_{,x} + \left( \beta_r + \frac{\beta}{r} \right) \Phi, \quad (2.25) \]

where we have set \( x = r^2 \) to include the regularity condition at the origin. There are also constraint equations
\[ \frac{2}{A^2} \left[ \frac{4A_x}{A} - 8x\frac{B_{xx}}{B} - 16\frac{B_x}{B} - 4\frac{B^2}{B^2} + 8\frac{x A_x B_x}{AB} + \frac{A^2}{x B^2} - \frac{1}{x} \right] + 1 \frac{K^2}{2} + KK_1 - \frac{3}{2} K^2 \]
\[ = 2K\Pi - \Pi^2 + x\frac{\Phi^2}{A^2} - \frac{2}{A^2} \left( \left( \frac{4B_x}{B} - 2\frac{A_x}{A} + 3 \right) \Phi + 2x\Phi_{,x} \right) \]
\[ \quad \left( 2K_1 - K \right) = -\Pi_{,x} + \frac{1}{2} K_1 \Phi. \quad (2.26) \]

At this stage, it may be instructive to recover the homogeneous and isotropic solutions. This can be done by taking \( \alpha = 1, \beta = 0, A = B \) and neglecting the spatial dependence in the variables. Then we arrive at the following equations:
\[ \frac{A_t}{A} = -\frac{1}{3} K \quad (2.28) \]
\[ K_t = \frac{1}{3} K^2 + K\Pi - \Pi^2 \quad (2.29) \]
\[ \Pi_t = K\Pi - \Pi^2 \quad (2.30) \]
\[ \frac{2}{3} K^2 = 2K\Pi - \Pi^2, \quad (2.31) \]

with \( \Pi = -\phi_t \). By rewriting as \( H \equiv -K/3 \), Eq.(2.28) is nothing but the definition of the Hubble parameter. Eq.(2.31) is the Friedmann equation. Eq.(2.29) and Eq.(2.30) are the evolution equations. Either of them is redundant. Eq.(2.31) admits the following solution:
\[ K = -3H = \frac{3 \pm \sqrt{3}}{2} \Pi. \]  

(2.32)

It is easily found that minus sign corresponds to the PBB branch solution, while positive sign corresponds to the post-big-bang branch solution.

### III. NUMERICAL RESULTS

#### A. Initial Condition

Initial data must satisfy the constraint equations Eq.\((2.26)\) and Eq.\((2.27)\). Because of spherical symmetry, we can consider, without loss of generality, the conformally flat three-metric:

\[ dl^2 = \gamma_{ij} dx^i dx^j = \psi(r)^4 (dr^2 + r^2 d\Omega^2). \]  

(3.1)

As a choice of the initial time slice (or choice of the extrinsic curvature), we adopt the following extrinsic curvature considering the similarity with the homogeneous universe

\[ K_{ij} = \frac{1}{3} \mathcal{K} \gamma_{ij} + \sigma_{ij} = \frac{1}{2} \left(1 - \frac{1}{\sqrt{3}}\right) \Pi \gamma_{ij}, \]  

(3.2)

where \(\sigma_{ij}\) is the traceless part of the extrinsic curvature. That is, we take \(\sigma_{ij} = 0\) and \(K = (3 - \sqrt{3})\Pi/2\). The former condition is for simplicity while the latter condition is chosen so that it corresponds to the Friedmann equation in the PBB phase in the homogeneous universe (see Eq.\((2.32)\)).

Then the constraint equations become

\[ 4x \psi_{xx} + 6\psi_x = \frac{3}{4} \psi \Phi + \frac{x}{2} \psi \Phi_x - \frac{x}{8} \psi \Phi^2, \]  

(3.3)

\[ \Pi_x = \frac{\sqrt{3} - 1}{4} \Phi \Pi. \]  

(3.4)

Note that because of the choice Eq.\((3.2)\) the Hamiltonian constraint Eq.\((3.3)\) is now independent of \(K_{ij}\) and \(\Pi\). Given \(\Phi\), we solve Eq.\((3.3)\) iteratively to get \(\psi\) by the cyclic reduction method for a tridiagonal matrix. However, the right-hand-side source term does not have an appropriate \(\psi\)-dependence for numerical treatment: as \(\psi\) increases, the source term also increases, and therefore the equation is numerically unstable. So, we “redefine” \(\Phi\) by \(\Phi_I \equiv \psi \Phi\), and rewrite the right-hand-side of Eq.\((3.3)\) in terms of \(\Phi_I\). That is,

\[ 4x \psi_{xx} + 6\psi_x = \frac{x \psi_x}{2 \psi} \Phi_I + \frac{3}{4} \Phi_I + \frac{x}{2} \Phi_{I,x} - \frac{x}{8} \psi \Phi_I^2. \]  

(3.5)

Physically, fixing \(\Phi_I\) corresponds (roughly) to fixing the proper kinetic energy density of the dilaton field. The boundary conditions are

\[ \psi_x = 0 \]  

(3.6)

at the origin for regularity and
\[(r\psi - 1)_r = 0 \quad (3.7)\]

at the numerical outer boundary to guarantee the asymptotically flat Friedmann property. Eq. (3.7) corresponds to setting the asymptotic scale factor to be unity. We then solve Eq. (3.4) to get \( \Pi \).

We consider the following Gaussian form for \( \Phi_I \):

\[
\Phi_I = -\frac{\phi_0}{\Delta^2} \exp \left( -\frac{x}{\Delta^2} \right). \quad (3.8)
\]

We examine whether changing the scale of inhomogeneity (i.e. \( \Delta \)) can affect the onset or the duration of the PBB inflation. We use the unit where \( H_i(r \to \infty) = -K_i(r \to \infty)/3 = 1 \) initially, which determines the time scale \( t_0 \) to the big bang singularity (in FRW universe) such that \( t_0 = H_i^{-1}/\sqrt{3} \). The normalization factor \( \phi_0 \) is arbitrary and we set it to one.

B. Numerical Details

We describe the numerical details for solving the field equations by the finite difference method. Time steps are labeled by the index \( n \) and spatial grid points are labeled by \( i \). We define the time derivative operator

\[
(Z_t)_i^n \equiv \frac{\Delta t_{n-1}}{\Delta t_n + \Delta t_{n-1}} \frac{Z_i^{n+1} - Z_i^n}{\Delta t_n} + \frac{\Delta t_n}{\Delta t_n + \Delta t_{n-1}} \frac{Z_i^n - Z_i^{n-1}}{\Delta t_{n-1}}, \quad (3.9)
\]

where \( \Delta t_n \equiv t_{n+1} - t_n \) and a operator for a derivative with respect to \( x = r^2 \)

\[
(Z_x)_i^n \equiv \frac{x_i - x_{i-1}}{x_{i+1} - x_i} (Z_i^{n+1} - Z_i^n) + \frac{x_{i+1} - x_i}{x_i - x_{i-1}} (Z_i^n - Z_{i-1}^n). \quad (3.10)
\]

Then for a grid uniform in \( r \), the evolution equations are second-order accurate in both space and time. We take a nonuniform grid because we need to perform numerical calculations with various initial conditions and therefore it is desirable to use small grid points. Typically we use 512 spatial grid points in this work. The time step is constrained by the Courant condition. For a nonuniform grid, we use

\[
\Delta t = \varepsilon_1 \min_i \left( \frac{r_{i+1} - r_i}{\beta_i \pm 2\alpha_i/A_i} \right), \quad (3.11)
\]

and we take \( \varepsilon_1 = 0.2 \) for accuracy. The Courant condition is sufficient for stability, however, not for accuracy. Therefore we impose the additional requirement that \( A, B, \Phi \) do not change too much (near origin) in one time step:

\[
dt = \varepsilon_2 \min \left( \frac{A}{A_t}, \frac{B}{B_t}, \frac{\Phi}{\Phi_t} \right), \quad (3.12)
\]

and choice of the value of \( \varepsilon_2 \) depends on the initial condition. For example, we choose \( \varepsilon_2 = 10^{-3} \) for \( \Delta/H_i^{-1} = 1.0 \).

Regarding the boundary conditions, Eq. (2.27) requires a single boundary condition. We set \( K_1 = K/3 \) at the origin for regularity. Evolution equations Eq. (2.22)-Eq. (2.23) require
an outer boundary condition as well as the regularity condition at the origin of kind Eq. 3.6. To determine the value of a variable at $i_{\text{max}}$, we extrapolate linearly in $x$ using the values of the variable at $i_{\text{max}} - 1$ and $i_{\text{max}} - 2$. This is because it is not obvious that the conventional outgoing-wave boundary condition should be imposed in the present context. Since we locate the numerical outer boundary at the point much further than the scale of inhomogeneity ($> 10 \times \Delta$) where the universe is approximately FRW, such a choice is sufficient in practice.

C. Numerical Test

As a test of the numerical code, we have used homogeneous initial conditions and compared the numerical solutions with the exact homogeneous solutions. The results are shown in Fig. 1. There the exact solutions are shown as solid curves, while the numerical solutions at the center ($r = 0$) are plotted as crossed points. We find excellent agreement. In fact, the maximum relative error between the numerical solution and the exact solution was less than $10^{-4}$ and the maximum relative error in the Hamiltonian constraint was less than $10^{-8}$ because the variables appear only quadratically in the constraint.

Barrow and Kunze derived an exact solution of the collapse of a homogeneous spherical region of the stiff ($p = \rho$) fluid [14]. The comparison with their solution is difficult in the present work because the spatial distribution of the dilaton is step-function-like and hence $\Phi$ becomes singular near the surface (delta-function-like). Furthermore, Eq. (2.24) contains the spatial derivative of $\Phi$, which develops a further ill-behaved singularity.

D. Results of Numerical Calculations

We choose the geodesic slicing condition, $\alpha = 1$ with zero shift, $\beta = 0$. We use Eqs. (2.20-2.22) and (2.24-2.25) to solve $A, B, K, \Pi, \Phi$ and use the momentum constraint Eq. (2.27) instead of Eq. (2.23) to solve $K_1$. The Hamiltonian constraint equation Eq. (2.26) is used as a check of the numerical accuracy. As a geometrically invariant diagnostics for the inhomogeneity, we calculate the Bel-Robinson tensor [15]. The Bel-Robinson tensor is defined in terms of the Weyl tensor as

$$T_{\mu\nu\rho\sigma} \equiv C_{\mu\rho\delta\epsilon} C_{\nu\sigma}^{\delta\epsilon} + *C_{\mu\rho\delta\epsilon} *C_{\nu\sigma}^{\delta\epsilon},$$

(3.13)

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and $*C_{\mu\nu\rho\sigma} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} C^{\alpha\beta\rho\sigma} \rho_{\alpha\beta}$ is its dual. The “superenergy” density on the three-dimensional spacelike hypersurface with $n^\nu$ being the unit normal vector is defined by

$$W \equiv T_{\mu\nu\rho\sigma} n^\mu n^\nu n^\rho n^\sigma = E_{\mu\nu} E^{\mu\nu} + B_{\mu\nu} B^{\mu\nu},$$

(3.14)

where

$$E_{\mu\nu} \equiv C_{\mu\nu\beta\epsilon} n^\alpha n^\beta, \quad B_{\mu\nu} \equiv *C_{\mu\nu\beta\epsilon} n^\alpha n^\beta.$$  

(3.15)

They can be written in the present case as
\[ E_{ij} = R_{ij} + KK_{ij} - K_{i}^l K_{lj} - \frac{1}{2} (S_{ij} - \frac{1}{3} \gamma_{ij} S_l^l) - \frac{2}{3} \gamma_{ij} \rho_H, \]
\[ B_{ij} = \epsilon_{lm(i} D^j K^{m}_{j)}. \]

We have numerically solved the field equations with various choice of \( \Delta \). Numerical results are shown in Figs.2,3,4 for \( \Delta/H_i^{-1} = 1, 5, 0.2 \).

For \( \Delta/H_i^{-1} = 1 \) (Fig.2), which can be regarded as a fiducial case, the evolution was terminated at \( t \approx 0.49 \) beyond which numerical calculations became inaccurate near the origin. The inhomogeneities in \( \Phi \) and \( \Pi \) always grows during the evolution. The superenergy of the Bel-Robinson tensor shows that after the decay of the initial irregularities, new irregularities emerge and grow. This phenomena can be understood as follows. Initially inhomogeneous region expands being dragged by the cosmic expansion. However, its self gravitational energy dominates over the background energy density soon, and then that lump decouples from the cosmic expansion and begin to collapse. We note that although irregularities are increasing, the universe nonetheless expands rapidly since the expansion rate is increasing (See Fig.2(a)(b)) The situation is to be contrasted with the usual potential energy dominated inflation (for example, see Fig.14 in [16] and Fig.3 in [17]).

For \( \Delta/H_i^{-1} = 5 \) (Fig.3), the evolution was terminated at \( t \approx 0.46 \). Further evolution made the numerical calculations inaccurate near the origin. We find that the behavior similar to \( \Delta/H_i^{-1} = 1 \) case appears again. For \( \Delta/H_i^{-1} = 0.2 \) (Fig.4), the evolution was terminated at \( t \approx 0.12 \). Even if we perform numerical evolution further, the evolution was forced to stop soon. Presumably the singularity is reached at the origin first. Note that \( K \) and \( K_1 \) are growing positively near origin. That means the expansion is delayed and eventually turns into the contraction to the singularity. Consequently, the metric (or scale factor) does not grow much near the origin. We do not see the growth of inhomogeneity in the Bel-Robinson tensor probably because there is no time for them to overcome the cosmic expansion. If we could perform the numerical evolution any further more accurate, we could see the same structure as in Fig.2 and Fig.3.

We also calculate the e-folding number \( N \) of the expansion at the origin

\[ N = \ln \left\{ \left( A(r = 0) B(r = 0)^2 \right)^{1/3} / \psi(r = 0)^2 \right\}, \]

and the results are shown in Fig.5. For comparison we also plotted the e-folding number for the FRW universe. We find that for \( \Delta \lesssim 0.3 \times H_i^{-1} \) the universe does not inflate sufficiently and/or recollapses. Note that the e-folding number for \( \Delta \gtrsim 0.4H_i^{-1} \), where the onset of PBB-inflation is not prevented (in agreement with the rough estimate in [11]), is larger than that of the homogeneous solution. This is because initially the inhomogeneous dilaton is superposed on the background homogeneous solution and that inhomogeneous energy contributes in turn to the superinflationary expansion. In [11], a criterion for PBB inflation is proposed that is translated roughly as, \( \Delta \gtrsim H_i^{-1} \), which is consistent with our numerical results.

Our results may be understood intuitively in the Einstein frame. In the Einstein frame, the PBB phase corresponds to a recollapsing universe with a massless scalar field rather than an expanding universe. So, inhomogeneities should grow.

To conclude, we find that a large inhomogeneity, \( \Delta \lesssim 0.3H_i^{-1} \), reduces the amount of the PBB inflation and sometimes even does not allow the development of the PBB inflation as
in the case of the usual inflation \[18\]. However, even if the universe enters the PBB inflation stage, the initial inhomogeneity grows and is not smoothed out globally at all.

\section*{E. Fine-Tuning Problem}

We consider the meaning of the condition for the onset of the PBB inflation($\Delta \gtrsim 0.4H_i^{-1}$) from the viewpoint of the resolution of the horizon problem \[19\].

Let the time of the beginning of the PBB inflation be $t_i$ and the final time be $t_f$ and the time of the final singularity be $t_s$. $t_f$ is supposed to be the time when the stringy nonperturbative effects become dominant and the low energy effective action Eq.\ref{eq:2.1} is no longer valid. The decrease of the comoving Hubble length, $(H a)^{-1}$, measures the amount of inflation. Using the PBB solution in FRW background \[1,19\], we have

$$Z \equiv H_f a_f = \frac{(t_s - t_i)}{(t_s - t_f)}^{1+1/\sqrt{3}} \approx \frac{(t_s - t_i)}{l_s}^{1+1/\sqrt{3}}, \quad (3.19)$$

where $a$ is the scale factor and $H_f^{-1} \approx t_s - t_f \approx l_s$ is assumed. $Z > e^{60}$ is required to solve the horizon problem \[20\]. On the other hand, we find that the lump of the initial size $\Delta \gtrsim 0.4H_i^{-1}$ inflates. Using Eq.\ref{eq:3.19} this inequality can be rewritten as

$$\Delta \gtrsim 0.4H_i^{-1} \approx 0.4(t_s - t_i) > e^{37}l_s, \quad (3.20)$$

which shows that initially extremely large inhomogeneous region (in the unit of the string scale) is required in order to get the PBB inflation and to solve the horizon problem. This large number may be regarded as fine-tuning of the initial condition because $l_s$ is a natural length scale in string theory.

\section*{IV. SUMMARY}

We have studied numerically the effect of the initial inhomogeneities on the onset and duration of the PBB inflation. We found that a large initial inhomogeneity does suppress the onset of the PBB inflation as in the case of the usual inflation \[18\]. Further, even if the PBB inflation is realized, the initial inhomogeneity grows contrary to the usual inflation. The initial scale of inhomogeneity is required to be extremely large (in the unit of the string scale) to realize the PBB inflation and to solve the horizon problem.

The fine-tuning problem of the PBB inflation was recently addressed in \[19,21\] for homogeneous cosmologies in the context of the horizon problem and the flatness problem: the pre-big-bang scenario is quite sensitive to spatial curvature and anisotropy. Combining these results with our results, it may be said that if the present universe would be resulted from the PBB inflation, the universe would have to be initially extremely homogeneous and isotropic and flat.

Of course, our study does not exhaust all the possibility; (i) we did not include other massless degrees of freedom (for example, the axion), (ii) the asymptotic condition was limited to the flat FRW universe, (iii) the space-time dimensionality was fixed to four. It would be interesting to relax the above situations.
Hawking and Moss once said in [9] that the asymptotic approach to the de Sitter state (in the usual inflation) is “very similar to the way that a gravitational collapse rapidly approaches a stationary black-hole state (outside of the black-hole horizon) which depends only on the mass and angular momentum but which is otherwise independent of the nature of the collapsing body”. By the same black-hole analogy, the PBB inflation may be similar to the “interior” of a black-hole which is dependent on the nature of the collapsing body (initial conditions).

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FIGURE CAPTION

Fig.1. Solutions with homogeneous initial conditions. Solid curves correspond to the exact homogeneous solutions, while crossed points correspond to the numerical solutions. We find excellent agreement.

Fig.2. Evolutions from inhomogeneous initial conditions up to $t \simeq 0.49 \times H_i^{-1}$ for $\Delta = 1.0 \times H_i^{-1}$. All variables are normalized by $H_i$. Thick curves are initial values. Time evolution is in the direction from top to bottom for $K, K_1, \Phi, \Pi$ and the relative error, while bottom to top for $A$ and $B$. For the superenergy of the Bel-Robinson tensor, first the initial lump decays, and then new lump appears near the origin.

Fig.3. Evolutions from inhomogeneous initial conditions up to $t \simeq 0.46 \times H_i^{-1}$ for $\Delta = 5.0 \times H_i^{-1}$. Thick curves are initial values.

Fig.4. Evolutions from inhomogeneous initial conditions up to $t \simeq 0.12 \times H_i^{-1}$ for $\Delta = 0.20 \times H_i^{-1}$. Thick curves are initial values.

Fig.5. E-folding number at the center as a function of time. $\Delta/H_i^{-1} = 5.0, 1.0, 0.5, 0.4, 0.3, 0.2, 0.1$ from top to bottom. Dashed line corresponds to the homogeneous solution. We find that for $\Delta \lesssim 0.3 \times H_i^{-1}$ the universe does not inflate sufficiently.
Fig. 2(d)
Fig. 4(c)
