A Pedagogically Sound yet Efficient Deletion algorithm for Red-Black Trees: The Parity-Seeking Delete Algorithm

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Abstract

Red-black (RB) trees are one of the most efficient variants of balanced binary search trees. However, they have always been blamed for being too complicated, hard to explain, and not suitable for pedagogical purposes. Sedgewick (2008) proposed left-leaning red-black (LLRB) trees in which red links are restricted to left children, and proposed recursive concise insert and delete algorithms. However, the top-down deletion algorithm of LLRB is still very complicated and highly inefficient. In this paper, we first consider 2-3 red-black trees in which both children cannot be red. We propose a parity-seeking delete algorithm with the basic idea of making the deficient subtree on a par with its sibling: either by fixing the deficient subtree or by making the sibling deficient, as well, ascending deficiency to the parent node. This is the first pedagogically sound algorithm for the delete operation in red-black trees. Then, we amend our algorithm and propose a parity-seeking delete algorithm for classical RB trees. Our experiments show that, despite having more rotations, 2-3 RB trees are almost as efficient as RB trees and twice faster than LLRB trees. Besides, RB trees with the proposed parity-seeking delete algorithm have the same number of rotations and almost identical running time as the classic delete algorithm. While being extremely efficient, the proposed parity-seeking delete algorithm is easily understandable and suitable for pedagogical purposes.

Keywords: red-black trees, 2-3 red-black trees, parity-seeking, delete, pedagogical, efficient

1. Introduction

Bayer & McCreight (1970, 1972) invented B-trees which are balanced tree data structures appropriate for organization and maintenance of large ordered indices, especially on disks. Since each node of a B-tree should allocate room for a predetermined maximum number of keys, B-trees are memory-inefficient. By linking keys of a B-tree node by left arcs, Bayer (1971) introduced a binary tree representation of B-trees which avoided their storage overhead. Bayer (1972) introduced symmetric binary trees, which were binary tree representations of 2-3-4 trees and allowed the keys within a B-tree to be either linked by left arcs or right arcs. Symmetric binary trees were named red-black (RB) trees thereafter when Guibas & Sedgewick (1978) proposed a dichromatic framework for balanced trees. Since then, many improvements to RB trees have been proposed. Some authors (Andersson et al., 1990; Roura, 2013) tried to decrease the maximum height of RB trees, which is 2 log(n) in the worst case. Others tried to uncouple updating from rebalancing, allowing a greater degree of concurrency and postponed processing (Boyar & Larsen, 1994; Park & Park, 2001; Larsen, 2002; Besa & Eterovic, 2013; Howard & Walpole, 2014). While being extremely useful in applications, RB trees have always been criticized for being baffling and inappropriate for pedagogical purposes. To simplify RB trees, Andersson (1993) proposed right-leaning red-black trees in which only the right nodes could be red. In another attempt to simplify RB trees, Okasaki (1999) proposed an algorithm for insertion into RB...
trees using functional programming in Haskell. By temporarily introducing a third "double-black" color, Germane & Might (2014) proposed a functional delete algorithm for RB trees. Attempting to simplify RB trees for pedagogical purposes, Sedgewick (2008) proposed left-leaning red-black (LLRB) trees. Although the insert algorithm of LLRB trees is simple, the delete algorithm is even more incomprehensible than classical RB trees. In fact, the real problem with classical RB trees is the delete algorithm which its rationale is unclear (Germane & Might, 2014; Sen et al., 2016).

In this paper, we initially consider 2-3 RB trees, in which children cannot both be red, and propose an insertion algorithm and an intuitive parity-seeking delete algorithm that is highly suitable for educational purposes. We then show that, with a simple amendment, the proposed parity-seeking delete algorithm can be used in ordinary 2-3-4 RB trees, yielding the first pedagogically sound algorithm for RB trees. Besides, our experiments on 2-3 and 2-3-4 RB trees show that the proposed parity-seeking delete algorithm is extremely efficient. The rest of the paper proceeds as follows. In Section 2, we review the classical algorithm of RB trees as was explained in (Cormen et al., 2009). In Section 3, we review the more recent LLRB trees (Sedgewick, 2008), and show that, despite the claims of the author, the deletion algorithm is extremely inefficient and unintuitive. In Section 4, we consider 2-3 RB trees and propose an insertion algorithm along with a novel parity-seeking delete algorithm that is much simpler than the delete algorithm of classical RB trees. In Section 6, we experimentally evaluate the performance of the standard RB trees, as described by Cormen et al. (2009), LLRB, and the proposed 2-3 and 2-3-4 RB trees. We conclude the paper in Section 7.

2. Red-Black (RB) Trees

RB trees can be defined both for general binary trees that preserve the inorder iteration of elements (Sahni, 1998, 2005) and more specifically for binary search trees (Cormen et al., 2009). In this paper, for simplicity, we define RB trees as binary search trees. The generalization of the proposed method to general binary trees is straightforward.

Definition 1 (RB Trees). An RB tree is a binary search tree with one additional attribute in each node: its color, which can be either red or black. RB trees have the following properties:

1. The root node is black.
2. If a node is red, then its parent is black.
3. The number of visited black nodes from the root to all external nodes is the same.

Sometimes it is useful to refer to the color of a link. The color of the link between a child node and its parent, is the color of the child node.

2.1. Relation between RB trees and B-trees of order 4 (2-3-4 trees)

Considering an RB tree, if we draw the red links horizontally and the black links vertically, then a representation is obtained in which, due to the 3rd property in definition 1, all leaves are drawn at the same level. Furthermore, if we place the horizontally connected nodes in one compound node, then the 2-3-4 tree equivalent of the very RB tree is obtained. Figure 1 shows an RB tree along with its other equivalent representations. In illustrations of this paper, we depict black nodes and links by solid lines, the red nodes and links by solid double lines, and those with either red or black colors by dotted lines.

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1 We assume that the null pointers of the leaf and degree-1 nodes are replaced by pointers to some imaginary nodes called external nodes. In fact, since we use the nil trick (Cormen et al., 2009), in our implementation, all external nodes are represented by the nil node.
2.2. Basic operations in RB Trees

After inserting/deleting a node into/from an RB tree, the properties of definition 1 may be violated. While modifying the tree in order to comply with definition 1, it is important that the order of the nodes in the inorder traversal of the tree does not change, so that the resulting tree would remain a valid binary search tree. In this section, we introduce the basic operations that preserve the properties of binary search trees. These operations are left rotation and right rotation, which are shown in Figure 2. Furthermore, changing the color of nodes is another operation that preserves the properties of binary search trees. To understand the color of nodes after rotation, it is easier to assume that the links are rotated and infer the color of nodes from the color of their links to their parents.

2.3. Insertion algorithm of RB trees

The \textit{insert} algorithm of RB trees works in two steps. Initially, the new data is inserted according to the rules of binary search trees in a new red node. Then, if any property of definition 1 is violated, the tree is fixed with appropriate fix-up operations. The 3rd property of definition 1 could not be violated as the newly inserted node is colored red. If the insertion is applied to an empty tree, then the 1st property of definition 1 is violated, which is simply fixed by changing the color of the root node to black. The only potential problem is the violation of the 2nd property of definition 1, i.e. the occurrence of two consecutive red nodes. Assuming that a child node and its parent are both red, and that the parent node is a left child, the tree is fixed using the following rules:

1. If the sibling of the parent node is red, then the parent node and its sibling are turned black and the grandparent node is turned red. Checking for two consecutive red-nodes is continued from the grandparent node (Figure 3a and Figure 3b).
2. If the sibling of the parent node is black, and the current node is a right child, then a left rotation is performed on its parent node (Figure 3c). The situation becomes ready for applying the next rule.
3. If the sibling of the parent node is black, and the current node is a left child, then a right rotation is performed on the grandparent node (Figure 3d).

The rules for the case that the parent node is a right child, are obtained by exchanging "left" and "right" in the above statements.
2.4. Deletion algorithm of RB trees

The delete operation may happen at the root node, an internal node, or a leaf node. Firstly, if the to-be-deleted node is of degree 2, its value is replaced by the greatest value in the left subtree or the smallest value in the right subtree, transferring the deletion to a degree-1 node or a leaf node. Then, the actual deletion is performed according to the following rules:

1. **Deleting a degree-1 node:** Since degree-1 nodes do not possess a child on one side, the existence of a black node further down their subtree is precluded. Also, since a node and its child cannot both be red, it is only possible for a degree-1 node to be a black node with a single red child. In this case, the value of the red child node is copied to the degree-1 node, and the red child node is deleted.

2. **Deleting a red leaf node:** In this case, the node is simply removed and the resulting tree is a legitimate RB tree.

3. **Deleting a black leaf node:** After deleting a black leaf node, the number of black nodes from the root node to the leaves of the left and right subtrees of its parent would be different, and the 3rd property of definition 1 would be violated. In this case, until at least one of the rules of Figure 4 is applicable, the fix-up operations are continued.

The main problem with the rules of Figure 4 is not their number, but their unclear rationale. For example, the rule of Figure 4e states that if the root of the deficient subtree is black, its sibling is black, and the right child of the sibling is red, then make the right child of the sibling black, and perform a left rotation on the sibling. From an educational point of view, the problem with this rule is that one has no idea what the rationale behind it is.

3. Left-Leaning Red-Black (LLRB) Trees

For pedagogical purposes, Sedgewick (2008) proposed LLRB trees to lessen the complexity of classical red-black trees. An LLRB tree is a red-black tree in which all red nodes are left children of their parents.
Figure 4: Rules for fixing RB trees after deleting a black leaf node. The rules are applied recursively. The root of the deficient subtree, which has one fewer black node, is shown with a square and is called $x$. Each rule, except (a), has a dual rule which is obtained by symmetry and is not shown. Rule (a) applies when the root of the deficient subtree is red. Cormen et al. (2009) did not explicitly mention (a) as a rule, however, it is necessary if the rules are to be complete. Rule (b) prepares a black sibling for $x$. Rule (c) applies when the sibling and both of its children are black. It elevates deficiency to the parent node. Rule (d) applies when the sibling and its right child are black, but the left child of the sibling node is red. It makes the right child of the sibling red and prepares the situation for the next rule. Rule (e) applies when the sibling is black and its right child is red. After applying this rule, the deficiency is removed altogether and the algorithm terminates.

LLRB trees have a one-to-one correspondence with 2-3 trees. Figure 5 shows an example of this one-to-one correspondence. Sedgewick (2008) proposed a neat insertion algorithm and taught it in his MOOC algorithms course on Coursera (Wayne & Sedgewick, 2012). However, as we will show, the deletion algorithm of LLRB is neither efficient nor suitable for educational purposes.

3.1. Insertion algorithm of LLRB tree

As in classical RB trees, the insert algorithm of LLRB trees starts by inserting a new leaf node into a binary search tree with the color red. In addition to the possibility of having double red links, which is a violation of the 2nd property of RB trees in definition 1, the inserted node could be a right child, violating the sole new constraint of LLRB trees. Sedgewick (2008) proposed the three operations of left rotation, right rotation, and color flip to transform the resulting tree into a correct LLRB tree (Figure 6). Note that in contrast to classical RB trees where there were 3 other symmetric cases, since LLRB trees do not permit red right children, here all cases are the three ones shown in Figure 6. One of the important weaknesses of the insert algorithm of LLRB is that these rules should be applied until reaching the root node, even though it is possible to infer that the tree has been fixed up long before reaching the root. The reason for
this inefficiency is that the insert algorithm is implemented recursively and there is no way to empty the call stack except throwing an exception. In fact, our attempt to modify the code of LLRB to terminate the fix-up operation by throwing an exception led to the severe slow down of the algorithm.

Figure 6: Basic operations of LLRB trees to fix up a tree after an insertion. Note that the rules for applying the left and right rotations are not completely symmetric. The right rotation is applied when two consecutive left children are red, while the left rotation is applied on a right red child.

3.2. Deletion algorithm of LLRB tree

Sedgewick (2008) proposed a recursive top-down algorithm for deletion in LLRB trees. To delete a node, the algorithm starts from the root node and moves left/right towards the to-be-deleted node. The algorithm prepares the scene to apply the actual deletion to a red node and, therefore, as it descends the tree it ensures that either the current node or its left child is red. If it is not the case, the algorithm enforces this property by two methods named "moveRedLeft" and "moveRedRight". As the deletion algorithm descends the tree, it modifies the tree extensively and causes immense changes. This is awful since it is possible that the query node does not exist, or it is already red and, therefore, can be simply deleted. Figure 7 shows an example of a tree in which the deletion operation is as simple as solely deleting the node with the given key, while the delete algorithm of LLRB engages in immense modifications to the tree.

Figure 7: An example showing the inefficiency and complexity of the delete operation in LLRB. Deletion of 25 leads to two right rotations on 18 and 24 in the top-down pass. Then the algorithm deletes 25 and continues with two left rotations on 24 and 9. This is while, in this example, 25 is a red node that could be simply deleted without any fix-ups.

4. The considered framework: 2-3 RB Trees

We define a 2-3 RB tree as a red-black tree in which both children of a node can not be red. Note that, like (Bayer, 1972) and in contrast to (Bayer, 1971; Andersson, 1993; Sedgewick, 2008), 2-3 RB trees treat the left and right children symmetrically. While LLRB trees are in one-to-one correspondence with 2-3 trees, there might be multiple equivalent 2-3 RB trees for a given 2-3 tree. Figure 8 illustrates a 2-3 tree and two of its equivalent 2-3 RB trees.

4.1. Proposed Insertion algorithm for 2-3 RB trees

To insert a value in a 2-3 RB tree, we initially insert it with the color red in the position determined by the rules of binary search trees. Then, if necessary, we perform fix-up operations until we obtain a legitimate
2-3 RB tree. For two reasons the resulting tree, after the initial insertion, might not be a legitimate 2-3 RB tree: (I) the parent of the just-inserted node is red, or (II) its sibling is red. Let us denote the node of the tree which has one of these problems by $x$. Our proposed rules for case I, in which the node $x$ and its parent are both red, are shown in Figure 9a and Figure 9b. In case II, in which the just-inserted node and its sibling are red, we propose a color-flip operation as shown in Figure 9c. We terminate the fix-up operations as soon as the color of $x$ becomes black. We make root the child of a dummy node with the color black, to ensure that a black node is eventually visited, and the procedure terminates. Finally, we reset the color of the root to black.

**Proposition 1.** The fix-up operations of the insert algorithm of 2-3 RB trees terminate.

**Proof.** As is clear from Figure 9 at each step, the node marked with $x$ becomes one level closer to the root node. Therefore, the maximum possible number of fix-up operations is the height of the tree.

4.2. The proposed parity-seeking delete algorithm for 2-3 RB trees

In this section, we describe our proposed parity-seeking delete algorithm in the context of 2-3 RB trees. First, according to the deletion rules of binary search trees, the initial delete operation is transferred to
a leaf or a degree-1 node. Now, if the degree of the to-be-deleted node is one, then, from property 3 of definition [1] it follows that its whole subtree is a single red child. Therefore, to delete a degree-1 node, it suffices to delete its red child and put its value in its parent. Now, consider the case of deleting a leaf node. If the leaf node is red, then it can be simply deleted and the resulting tree is a valid 2-3 RB tree (Figure 10). The hard case is deleting a black leaf node. First, let us define deficient subtrees.

**Definition 2** (Deficient subtree). A subtree rooted at a node \( x \) is deficient if (1) neglecting the color of \( x \), it is a 2-3 RB tree, and (2) the number of visited black nodes from \( x \) to the leaves is one less than that of \( x \)’s sibling.

Assume that we want to delete a black leaf node named \( z \). After deleting \( z \), we replace it with \( nil \) and set the parent of \( nil \) to the parent of \( z \). Therefore, initially, \( nil \) is the root of the deficient subtree. Inductively, assume that \( x \) is the root of the deficient subtree, and \( y \) is its sibling. Our parity-seeking delete algorithm works as follows: it either fixes the deficiency of the node \( x \) or also makes its sibling \( y \) deficient, elevating the deficiency to the parent node. There are three possibilities:

I. \( x \) is red.

II. \( x \) and \( y \) are both black.

III. \( x \) is black and \( y \) is red.

Case I is simply handled by changing the color of \( x \) to black, which resolves the deficiency of \( x \). In the following subsections, we explain our algorithm for the other two cases.

### 4.2.1. Case II: both the root of the deficient subtree, and its sibling are black

Assume that both the root of the deficient subtree, i.e. \( x \), and its sibling, i.e \( y \), are black. We attempt to move the deficiency one level higher by turning \( y \) red. If one of \( y \)’s children is red, a vertical double-red link situation arises. Our handling for cases that one of \( y \)’s children is red is shown in Figures 11c and 11d. Please note that at the moment we are fixing the subtree rooted at the common parent of \( x \) and \( y \), and a potential vertical double-red link between \( y \) and its parent will be resolved when deficiency reaches \( y \)’s grandparent. If none of \( y \)’s children are red, the deficiency is transferred to the parent of \( x \) and \( y \) (Figure 11b). Please note that there is no special handling for the case that the whole tree becomes deficient as it is automatically handled by cases I and II.

### 4.2.2. Case III: the root of the deficient subtree is black and its sibling is red

In this case \( x \) is black and \( y \) is red. Therefore, children of \( y \) are black. In this case, we can neither fix the deficiency of \( x \) as \( x \) is black, nor can make the sibling deficient as \( y \) is red. We perform a rotation on the common parent of \( x \) and \( y \) so that the new sibling of \( x \) becomes one of the children of \( y \). Since the new sibling of \( x \) is black, the algorithm returns to case II. Figure 11e illustrates this situation. In contrast to the insert algorithm in which the considered node was steadily moving up the tree, in the delete algorithm the deficient subtree can both move up or down the tree. In the following proposition, we prove that, despite this, the delete algorithm of 2-3 RB trees terminates.

**Proposition 2.** The proposed parity-seeking algorithm for deletion in 2-3 RB trees terminates and generates a legitimate 2-3 RB tree.

**Proof.** We need to prove that, in all the three cases of the delete algorithm, the problem of deficiency is resolved. We have:

- In case I, where \( x \) was red, the deficiency problem was completely resolved by making \( x \) black (Figure 11a). In this case the algorithm clearly terminates.

- In case II, where both \( x \) and \( y \) were black, the deficiency problem was either completely resolved (Figures 11c and 11d), or the deficiency moved one level closer to the root node (Figure 11b).
Figure 10: If the target of deletion is red, it is simply removed, and a valid 2-3 RB tree is obtained.

- In case III, where $x$ is black and $y$ is red, the algorithm eventually moves to case II. Considering Figure 11e if at least one of Cs children are red, the deficiency problem is resolved immediately as was shown in Figures 11c and 11d. On the other hand, if both children of $C$ were black, then, after applying rules of case II, $C$ becomes red and the deficiency problem transfers to the red node $B$. The deficiency of the red node $B$ is then immediately resolved by changing its color to black by case I.

5. A Parity-Seeking delete algorithm for classical RB trees

After preparing this manuscript, we noticed the high similarity between the proposed parity-seeking delete algorithm of 2-3 RB and the delete algorithm of classical RB trees. Rules (a), (b), and (c) in Figure 4 for deletion in RB trees are identical to rules (a), (e), and (b) in Figure 11 for deletion in 2-3 RB trees. The series of operations performed in rules (c) and (d) in Figures 11 for 2-3 RB trees have the same effect as rules (c) and (d) in Figure 4 for RB trees. The only difference is that, in 2-3 RB trees, the case where $y$ has two red children is impossible, while this situation is subsumed in case (e) of Figure 4a for classical RB trees. By substituting rule (d) in Figure 11d with the new rule shown in Figure 12, we obtain an intuitive parity-seeking delete algorithm for RB trees. It must be mentioned that, in our implementation, we follow all intermediate steps shown in Figures 11 and 12. To distinguish it from classical RB trees, we call a red-black tree with the new parity-seeking delete algorithm, a 2-3-4 RB tree.

6. Experiments

In this section, we experimentally compare our proposed 2-3 and 2-3-4 RB trees with classical RB trees and LLRB trees in inserting and removing random sequences of numbers. For LLRB trees, we started from the implementation of Sedgewick (2008) in java and translated it to C++ for fair comparison. We were forced to modify the code slightly and handle some null references since even the original java implementation crashed in our extensive tests. We implemented RB trees based on (Cormen et al., 2009) with a nil node, trying to make it similar to the elegantly concise implementation of LLRB. Then, we implemented our 2-3 and 2-3-4 RB trees with as few modifications as possible to the implementation of RB trees. Our goal of having a common basis for the implementation of RB, 2-3 RB, and 2-3-4 RB trees was to ensure that any difference in performance is solely due to algorithmic issues and all codes have been optimized to the same level. For fair comparison, we added the nil node to the implementation of LLRB, which helped in removing some conditional statements. All experiments have been performed on a UX310UQ notebook PC with an Intel(R) Core(TM) i7-6500U CPU @ 2.5GHz and 12 GB memory on a 64-bit windows 10 operating system. We report both the average number of rotations and the average execution time. Table 1 shows the number of rotations for each algorithm, normalized by $n \log n$ and multiplied by 1000 for better readability. As it can be seen, the average number of rotations in LLRB is almost 2 times of RB and 2-3-4 RB in the insert and almost 20 times in the delete algorithm, showing extreme inefficiency of LLRB. Comparing RB and 2-3 RB, we observe that the number of rotations in the insert algorithm of 2-3 RB trees is almost 1.5 times of that of RB trees. The number of rotations in the delete algorithm of RB and 2-3 RB trees are almost equal. In fact, the number of rotations of the delete algorithms of RB and 2-3 RB are identical and the observed difference is solely due to the different initial trees obtained by different insertion algorithms. As expected, the number of rotations of RB and 2-3-4 RB trees are identical.
Figure 11: Rules for the case of deleting a leaf black node from 2-3 RB trees. After deleting a leaf black node, its subtree becomes deficient. The root of the deficient subtree is denoted by $x$ and its sibling is shown by $y$. These rules are applied recursively until the deficiency is fixed. Each rule has a dual rule which is obtained by symmetry. The rationale behind these rules is very simple: either fix the deficiency, or make the sibling deficient, lifting up the deficiency to the parent node. (a) Case I in which $x$ is red. The deficiency is simply solved by making $x$ black. (b,c,d) Case II in which both $x$ and $y$ are black. The solution is to make $y$ red and lift deficiency to the parent of $x$. However, if one of $y$’s children is red, a vertical double-red link is constructed, which is fixed by rotations. Finally, a horizontal double-red appears which helps to solve the deficiency altogether. (e) Case III in which $x$ is black and $y$ is red. The solution is to perform a rotation so that the new sibling of $x$ becomes black. Since $y$ is red, $y$’s children are certainly black and, therefore, the new sibling of $x$ would be black. The algorithm returns to case II in which the sibling of $x$ is black.

Figure 12: A new rule that, when substituted with rule (d) of Figure 11, collectively provide an intuitive delete algorithm for classical RB trees. The new rule handles the case when $y$ has two red children.
Table 2 reports the running time of RB, LLRB, and 2-3 RB, and 2-3-4 RB trees, normalized by \( n \log n \). As it can be seen, the running time of RB, 2-3 RB, and 2-3-4 RB trees are almost equal, while the running time of LLRB trees is almost twice of them. This shows that the number of rotations is not an appropriate unit for measuring the running time of red-black trees as it does not reflect the actual running time. Although, our motivation for introducing the parity-seeking delete algorithm was pedagogical, we observe that the resulting algorithm is also very efficient.

### Table 1: Average number of rotations, over \( \# \text{rep} \) random samples, of each variant of red-black trees, normalized by \( n \log n \). For better visibility, the values are multiplied by 1000. The column \( n \) shows the number of random integers inserted and removed from the red-black trees.

| n    | \#rep | RB   | LLRB | 2-3 RB | 2-3-4 RB | RB   | LLRB | 2-3 RB | 2-3-4 RB |
|------|-------|------|------|--------|----------|------|------|--------|----------|
| 1K   | 1000  | 193 ± 9 | 569 ± 10 | 275 ± 9 | 193 ± 9 | 126 ± 8 | 2539 ± 95 | 136 ± 8 | 126 ± 8  |
| 10K  | 100   | 146 ± 2 | 430 ± 2  | 208 ± 2 | 146 ± 2 | 95 ± 2  | 2940 ± 70 | 103 ± 2 | 95 ± 2   |
| 100K | 100   | 117 ± 1 | 345 ± 1  | 167 ± 1 | 117 ± 1 | 76 ± 0  | 3178 ± 70 | 82 ± 1  | 76 ± 0   |
| 1M   | 10    | 97 ± 0  | 287 ± 0  | 139 ± 0 | 97 ± 0  | 63 ± 0  | 3320 ± 50 | 68 ± 0  | 63 ± 0   |
| 10M  | 10    | 83 ± 0  | 246 ± 0  | 119 ± 0 | 83 ± 0  | 54 ± 0  | 3377 ± 39 | 59 ± 0  | 54 ± 0   |

Table 2: Average running time (in nanoseconds), over \( \# \text{rep} \) random samples, of each variant of red-black trees, normalized by \( n \log n \). The column \( n \) shows the number of random integers inserted and removed from the red-black trees.

| n    | \#rep | RB   | LLRB | 2-3 RB | 2-3-4 RB | RB   | LLRB | 2-3 RB | 2-3-4 RB |
|------|-------|------|------|--------|----------|------|------|--------|----------|
| 1K   | 1000  | 50 ± 4 | 75 ± 12 | 49 ± 4  | 48 ± 4   | 40 ± 3 | 107 ± 19 | 42 ± 4  | 40 ± 5   |
| 10K  | 100   | 49 ± 4 | 66 ± 7  | 51 ± 10 | 51 ± 9   | 44 ± 4 | 102 ± 12 | 48 ± 10 | 46 ± 11  |
| 100K | 100   | 60 ± 8 | 98 ± 9  | 59 ± 5  | 56 ± 6   | 58 ± 11| 153 ± 10 | 59 ± 6  | 53 ± 6   |
| 1M   | 10    | 102 ± 1 | 168 ± 2 | 109 ± 0 | 101 ± 0  | 113 ± 1 | 248 ± 4  | 129 ± 2 | 112 ± 1  |
| 10M  | 10    | 157 ± 1 | 210 ± 8 | 176 ± 1 | 156 ± 1  | 181 ± 1 | 316 ± 24 | 206 ± 1 | 181 ± 1  |

### 7. Conclusions

In this paper, we introduced the parity-seeking delete algorithm for 2-3 and classic RB trees. Our goal was to introduce a pedagogically sound and easily understandable algorithm for deletion in red-black trees. The proposed parity-seeking delete algorithm is very natural and easily understandable. Specifically, the rationale behind the parity-seeking delete algorithm is to balance the deficient subtree and its sibling by either fixing the deficient subtree or making the sibling also deficient, elevating the deficiency one level higher. In our experiments, we found that the performance of 2-3 RB trees is very close to classical RB trees both in the insert and delete operations. Besides, we also introduced a parity-seeking delete algorithm for classical RB trees which its performance is almost identical to the classic delete algorithm of RB trees. The goal of devising a simple yet efficient algorithm for the delete operation in red-black trees is finally achieved.

### Author Contributions

The parity-seeking delete algorithm came to the mind of Kamaledin Ghiasi-Shirazi when he taught LLRB trees in his data structure course. He invited his former students, Taraneh Ghandi, Ali Taghizadeh, and Ali Rahimi-Baigi, to participate in the preparation of this paper. All authors validated the idea in common sessions, and Ali Taghizadeh, Ali Rahimi-Baigi, and Taraneh Ghandi implemented 2-3 RB along with the
competing methods of RB and LLRB. Ali Taghizadeh and Ali Rahimi-Baigi carefully studied RB and LLRB trees and explained it to other members of the team. The paper was initially written on the blackboard of a classroom in Persian, with all authors participating and discussing. The paper was then translated to English by Taraneh Ghandi and Kamaledin Ghiassi-Shirazi. All graphics have been produced by Taraneh Ghandi. Considering the extreme importance of the topic, Kamaledin Ghiassi-Shirazi re-implemented RB, 2-3 RB, and 2-3-4 RB trees in a unified framework for a fair comparison. Kamaledin Ghiassi-Shirazi revised the manuscript and prepared the final manuscript. All authors carefully read and commented on the final manuscript.

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