Scalar mode analysis of the warped Salam-Sezgin model

Antonios Papazoglou
École Polytechnique Fédérale de Lausanne,
Institute of Theoretical Physics,
SB ITP LLPF BSP 720, CH 1015, Lausanne, Switzerland
E-mail: antonios.papazoglou@epfl.ch

Abstract. We study the scalar perturbations (which mix with the dilaton) of the general axisymmetric warped Salam-Sezgin model with codimension-2 branes. We show that the scalar fluctuation analysis can be reduced to studying two scalar modes of constant wavefunction, plus modes of non-constant wavefunction which obey a single Schrödinger equation. From the obtained explicit solution of the scalar modes, we point out the importance of the non-constant modes in describing the four dimensional effective theory. Furthermore, we show that due to these modes, the warped solutions can be unstable for a certain region of the parameter space.

1. Introduction
Among six dimensional supergravities, the Salam-Sezgin model [1, 2] (the supersymmetric analogue of [3]) has received particular attention for the past decades. A particular characteristic of this model, that was noticed recently [4], is that all the non-singular (i.e., with no singularities worse than conical) maximally symmetric vacua are of the type \((\text{Minkowski})^4 \times X_2\), with \(X_2\) a two dimensional manifold. The generic vacuum solutions of this type have been found to get a warping in front of the four dimensional line element [4,5]. This warping leads always to the appearance of conical singularities which have to be supported by codimension-2 branes. The unique vacuum that preserves \(N = 1\) supersymmetry is the one that has no warping (and with no branes present) in the four dimensional world-volume and that has its gauge field flux embedded in the gauged \(U(1)\) direction. The appearance of the warping breaks the remaining \(N = 1\) supersymmetry.

The importance of such kind of compactifications has been increased the last years with the consideration of models which try to ameliorate the cosmological constant problem, the so called selftuning models [6]. In these models, use was made of a special property of the codimension-2 branes, that they do not curve their world-volume, but instead induce a deficit angle in the bulk. Thus, the vacuum energy of fields living on these branes does not gravitate in four dimensions. This would give the hope to solve the puzzle of the smallness of the cosmological
constant. However, from the available models, the ones with flux compactifications have a hidden fine-tuning related to the flux quantization or conservation condition [4, 5, 7].

In the present talk, we will discuss the linearized scalar fluctuations of the Salam-Sezgin model for the general warped background of the form given in Refs. [4, 5]. A more detailed presentation of the contents of the present talk can be found at the original paper [8]. Although we focus on the warped solution with axial symmetry of extra dimensions, our fluctuation analysis is also applied to the general warped background without axial symmetry in the local patch coordinate for each brane. We will study only the fluctuations which are coupled with the dilaton perturbations. We see that they are divided to fluctuations with constant profile along the extra dimensions and to fluctuations with non-trivial wavefunctions. The lowest massive non-constant mode mixes however with the massive constant mode. This mixing is always present irrespective of the presence of warping and has been neglected in the literature when discussing the effective four-dimensional physics of the model [9]. In particular, in the four-dimensional supergravity description of the unwarped solution without branes, the new non-constant mode has to be included as a new massive chiral multiplet relevant for low energy physics. Moreover, in the warped case, we have found the interesting result that the mixing with the new mode plays a crucial role in determining the instability of the solution, from the wrong sign of the kinetic term of one mode for some region of the parameter space.

2. Salam-Sezgin model: general axisymmetric vacua

We will first review the general vacuum solutions of Salam-Sezgin model [1] with axial symmetry. The bosonic sector of the system consists of the metric $g_{MN}$, a Kalb-Ramond field $B_{MN}$, a dilaton $\phi$ and a gauge field $A_M$. For the purpose of this work, we will set the Kalb-Ramond field to its zero background value, so we will not include it in the action. Then, the bosonic bulk action of the system is given by

$$S = \int d^6 x \sqrt{-g} \left[ R - \frac{1}{4} e^{2\phi} F_{MN}^2 - \frac{1}{4} (\partial_M \phi)^2 - 8 g^2 e^{-\frac{1}{2} \phi} \right] - \int d^4 x \sqrt{-\hat{g}} V_s ,$$

where $V_s$ is the brane tension and $\hat{g}_{\mu\nu}$ is the metric pulled back to the brane worldvolume. The gauge coupling $g$ corresponds to the gauged U(1)$_R$ of the model and in principle different from the the gauge coupling $\tilde{g}$ of the gauge field $A_M$. Assuming the axial symmetry in the internal space, there will be in general, two 3-branes sitting in the antipodal points of the axis of symmetry. The general warped solution in this case can be analytically found in the following gauge [4, 5]

$$ds^2 = W^2(r) \eta_{\mu\nu} dx^\mu dx^\nu + \gamma^2(r) [dr^2 + \lambda^2 \alpha^2(r) d\psi^2] ,$$

$$\phi = 4 \ln W ,$$

with the various functions given by

$$\gamma(r) = \frac{W}{f_0} , \quad \alpha(r) = r \frac{f_0}{f_1} ,$$

$$F_{mn} = \epsilon_{mn} \frac{\lambda q \gamma^2 \alpha}{W^6} ,$$

$$W^4 = \frac{f_1}{f_0} , \quad f_0 = 1 + \frac{r^2}{r_0^2} , \quad f_1 = 1 + \frac{r^2}{r_1^2} ,$$

with $r_0$ and $r_1$ being the positions of the two 3-branes.
where flux $q$ is a constant and the two radii are given by

$$r_0^2 = \frac{1}{2g^2}, \quad r_1^2 = \frac{8}{q^2}.$$  

(7)

The quantization condition of the gauge field flux is given by the relation

$$\frac{4\lambda \tilde{g}}{q} = n, \quad n = \text{integer}.$$  

(8)

In this general solution, the metric has two conical singularities, one at $r = 0$ and the other at $r = \infty$\textsuperscript{1}, with deficit angles $\delta_s$ (supported by tensions $V_s = 2\delta_s$) given by

$$\frac{\delta_0}{2\pi} = 1 - \lambda,$$
$$\frac{\delta_\infty}{2\pi} = 1 - \lambda \frac{r_1^2}{r_0^2} = 1 - \frac{n^2}{\lambda} \left( \frac{g}{\tilde{g}} \right)^2.$$  

(9)\quad (10)

For $r_0 = r_1$, i.e., for $q = 4g$, we have the unwarped model. In the case when $\lambda = 1$ and $\tilde{g} = g$, the unwarped case is possible only if $n = 1$, i.e., with no branes present. In all cases where 3-branes are present in the vacuum, supersymmetry is completely broken.

Finally, let us go to a coordinate system which is Gaussian-normal with respect to the two branes. In this new radial coordinate the perturbation equations in the next section will be expressed in a more convenient way. Thus, if we define

$$d\rho = \gamma dr, \quad a = \gamma a,$$

the metric is expressed as

$$ds_6^2 = W^2 \eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + \lambda^2 a^2 d\theta^2.$$  

(12)

3. Linearized scalar perturbations

We would like in this section to perturb the above general vacuum solution and in particular the spin-0 sector which mixes with the dilaton. For this purpose we consider the following ansatz for the perturbed metric:

$$ds_6^2 = e^{-\psi} W^2 \eta_{\mu\nu} dx^\mu dx^\nu + e^\xi (d\rho^2 + e^{2(\psi - \xi)} \lambda^2 a^2 d\theta^2).$$  

(13)

The perturbation in front of $d\theta^2$ is the right one to avoid mixing with the graviton - or in other words it comes from the $(\mu\nu)$ Einstein equations with $\mu \neq \nu$. The gauge field perturbation is considered as following

$$F_{\mu\theta} = \nabla_\mu a_\theta, \quad F_{\rho\theta} = \frac{\lambda q a}{W^6} + a'_\theta,$$

with all the other components vanishing. In addition, the scalar field perturbation is

$$\phi = 4 \ln W + f.$$  

(15)

In all the above perturbations $\psi, \xi, f, a_\theta$ are functions of the 4d coordinates ($x$) and the radial one ($\rho$). We will not consider the $\theta$-dependence which will provide the angular excitations of the resulting modes, i.e., we will restrict ourselves to the $s$-mode of the excitations.

\textsuperscript{1} Note that this point is at a finite proper distance from $r = 0$. 

3
3.1. Linearized equations of motion

Writing down the equations of motion, we can see that the perturbation $a_\theta$ can be solved as a function of the other three perturbations

$$a_\theta = \frac{\lambda a W^4}{q} \left[ \psi' + \xi' + 2 \frac{W'}{W} (\psi + \xi) - 2 \frac{a'}{a} (\psi - \xi) - 2 \frac{W'}{W} f \right].$$  \hfill (16)

The other three perturbations $\psi, \xi, f$ satisfy a coupled system of three equations which can be written as

$$\square \psi + \frac{3}{2} \psi'' + \frac{1}{2} \xi'' + 9 \frac{W'}{W} \psi' + 5 \frac{W'}{W} \xi' + \frac{1}{2} \frac{a'}{a} \psi' + \frac{3}{2} \frac{a'}{a} \xi' - \frac{W'}{W} f' - \frac{4g^2}{W^2} (2 \xi - f) = 0,$$  \hfill (17)

$$\square \xi - \frac{1}{2} \psi'' + \frac{1}{2} \xi'' + \frac{W'}{W} \psi' + 5 \frac{W'}{W} \xi' + \frac{5}{2} \frac{a'}{a} \psi' + \frac{3}{2} \frac{a'}{a} \xi' + \frac{W'}{W} f' - \frac{4g^2}{W^2} (2 \xi - f) = 0,$$  \hfill (18)

$$\square f + f'' - \psi'' - \xi'' + 6 \frac{W'}{W} f' - 10 \frac{W'}{W} (\psi' + \xi') + \frac{a'}{a} f' + \frac{a'}{a} \psi' - 3 \frac{a'}{a} \xi' + \frac{8g^2}{W^2} (2 \xi - f) = 0.$$  \hfill (19)

The boundary conditions which the above perturbations satisfy at the singular points $\rho_s$ are found by matching the singular terms at the original equations of motion and read

$$\psi(\rho_s) = \xi(\rho_s),$$  \hfill (20)

$$f'(\rho_s) = \psi'(\rho_s) = \xi'(\rho_s) = 0.$$  \hfill (21)

We have found that the above system has solutions where the wavefunctions are related as

$$\xi(x, y) = A \psi(x, y) \quad \text{and} \quad f(x, y) = B \psi(x, y),$$  \hfill (22)

and it collapses to a single differential equation. The solutions fall into two classes:

Conformal wavefunctions

The first case is the one where the wavefunctions are constant, i.e., $\psi = \psi(x)$. Then, it is easy to see from the above system, that there are two possible solutions for $(A, B)$

$$(A, B) = (1, 2) \quad \text{and} \quad (A, B) = (1, -2).$$  \hfill (23)

Thus, we have $\xi = \psi$ and in addition two possibilities for $f = \pm 2\psi$. The first mode corresponds to the massless mode with $\square \psi = 0$ and the second one to a massive mode with $\square \psi = 16g^2 \psi$. For following use, we will call the first mode $\psi_0$ and the second one $\psi_1$. The above shows that the two modes which one finds in the unwarped case [9], maintain their form even when we introduce a warping.

Let us note here that the massless mode, which corresponds to the breathing mode of the internal space, has the same relative wavefunction as the graviton zero mode. By this we mean that in the four dimensional part of the metric we have

$$ds_4^2 = \{ W^2(\rho)\eta_{\mu\nu} + W^2(\rho)[h_{\mu\nu}(x) - \psi(x)\eta_{\mu\nu}] \} dx^\mu dx^{\nu}.$$  \hfill (24)

This is in contrast with the five dimensional case of e.g., the Randall-Sundrum model, where the relative wavefunctions of the radion and the zero mode graviton were different [10].
Figure 1. The wavefunctions $\chi(\rho)$ for first three non-constant modes and for three different ratios of $4g/q$. The first state wavefunctions are plotted with thick lines, the second with thin lines and the third with dashed lines.

Non-constant wavefunctions

The second case is the one where the wavefunctions $\psi(x, y)$ have non-trivial profiles. Then, one can see that the only way that all three equations (17), (18), (19) collapse to the same second order equation for $\psi$ is when 

$$(A, B) = (-1, -2),$$

in other words, when $\xi = -\psi$ and $f = -2\psi$. In this case we obtain the differential equation for the fluctuation $\psi$

$$\frac{\Box \psi}{W^2} + \psi'' + \left(6 \frac{W'}{W} - \frac{a'}{a}\right) \psi' = 0.$$ 

(26)

Summarizing, the spectrum of the scalar excitations of the model consists of a zero mode, a first excited state with constant wavefunction and a tower of additional excited states with non-constant wavefunctions. In the unwarped case, the gauge field perturbation is zero for the zero mode and the massive constant wavefunction state, but non-trivial for all the other modes. On the other hand, in the warped case, the gauge field perturbation is nontrivial for all the massive states.

3.2. Solutions for the non-constant modes

To solve Eqn. (26) for the non-constant modes we will first separate variables as $\psi(x, \rho) = \tilde{\psi}(x) \chi(\rho)$ with $\Box \tilde{\psi} = m^2 \tilde{\psi}$. Then Eqn. (26) becomes

$$\chi'' + \left(6 \frac{W'}{W} - \frac{a'}{a}\right) \chi' + \frac{m^2}{W^2} \chi = 0.$$ 

(27)

It can be easily shown that the solutions to the above equation are

$$\chi_n = \sqrt{\frac{2n - 1}{2n(n - 1)}} \cdot \frac{2r/r_0}{1 + r^2/r_0^2} \cdot P_{n-1}^1 \left(\frac{1 - r^2/r_0^2}{1 + r^2/r_0^2}\right)$$

(28)

and the spectrum is given by

$$m_n^2 = \frac{4}{r_0^2} n(n - 1) \quad \text{with} \quad l = 2, 3, 4, \ldots$$

(29)
and is independent of the warping but depends only on \( r_0 \) (i.e., on \( g \)).

The scalar modes satisfy the following orthogonality condition

\[
\int d\rho \frac{W^A}{a} \chi_m \chi_n = \delta_{mn} .
\] (30)

The wavefunctions of the modes with \( 4g/q > 1 \) are localised closer to the brane sitting at \( r = \infty \), while the modes with \( 4g/q < 1 \) are localised closer to the brane sitting at \( r = 0 \). Examples of these modes are given in Fig.1.

4. Effective action for scalar modes

We would like now to calculate the quadratic four dimensional effective action of the perturbations that we considered in the previous section. In order to do that, we have to expand the action (1) up to quadratic orders of scalar perturbations, substitute the modes that we have already found and integrate the extra two dimensions. This leads to the following quadratic effective action for the scalar modes

\[
\mathcal{L}_{\text{eff}} = 2M_P^2 \left[ \psi_0 \Box \psi_0 + A \psi_1 (\Box - 16g^2) \psi_1 
+ B(\psi_1 \Box \tilde{\psi}_2 + \tilde{\psi}_2 \Box \psi_1 - 32g^2 \psi_1 \tilde{\psi}_2) + C \sum_{l \geq 2} \tilde{\psi}_l (\Box - m_l^2) \tilde{\psi}_l \right] 
+ \frac{1}{2} W^2(\rho_s) (\psi_0 + \psi_1 + \sum_{l \geq 2} \tilde{\psi}_l \chi_l(\rho_s)) \eta^{\mu \nu} T^\text{brane}_{\mu \nu} ,
\] (31)

where \( M_P^2 = \lambda \pi r_0^2 \) is the effective four dimensional Planck mass. The constants appearing in the action depend on the ratio \( 4g/q \) and read

\[
A = \frac{5}{6} + \frac{1}{12} \left( \frac{4g}{q} \right)^2 + \frac{1}{12} \left( \frac{q}{4g} \right)^2 ,
\] (32)

\[
B = \frac{1}{2\sqrt{3}} \left[ 1 + \left( \frac{4g}{q} \right)^2 \right] ,
\] (33)

\[
C = \left( \frac{4g}{q} \right)^2 .
\] (34)

In the above action, we also added the coupling of the scalar modes to the brane matter by the brane energy-momentum tensor \( T^\text{brane}_{\mu \nu} \).

Since \( \psi_1 \) and \( \tilde{\psi}_2 \) are degenerate in mass, i.e., \( m_2^2 = 16g^2 \), we can choose a basis such that there is no mixing term between these two states in the quadratic action. This is given by

\[
\phi_\pm = \frac{1}{\sqrt{2}} \left( \pm d + \sqrt{1 + d^2} \psi_1 \pm \tilde{\psi}_2 \right) \quad \text{with} \quad d \equiv \frac{C - A}{2B} .
\] (35)

Thus, we obtain the effective action in a diagonal form as

\[
\mathcal{L}_{\text{eff}} = 2M_P^2 \left[ \psi_0 \Box \psi_0 + K_+ \phi_+ (\Box - 16g^2) \phi_+ + K_- \phi_- (\Box - 16g^2) \phi_- 
+ \sum_{l \geq 3} \tilde{\psi}_l (\Box - m_l^2) \tilde{\psi}_l \right] 
+ \frac{1}{2} W^2(\rho_s) \left[ \psi_0 + \frac{1}{\sqrt{2}(1 + d^2)} (\phi_+ + \phi_-) \right] \eta^{\mu \nu} T^\text{brane}_{\mu \nu} ,
\] (36)
Figure 2. The coefficients of the kinetic terms of the two canonical modes, $K_+$ for $\phi_+$ and $K_-$ for $\phi_-$, as a function of $4g/q$. The mode $\phi_-$ is ghost-like for $4g/q \lesssim 0.82$.

with

$$K_\pm = \pm B + C(\mp d + \sqrt{1 + d^2}) \over \sqrt{1 + d^2}.$$  

(37)

Here, we note that there is no coupling of higher KK modes to the brane matter. So, from the brane perspective, we can think of the four dimensional effective theory, only consisting of the fields ($\psi_0, \phi_+, \phi_-$). The two massive modes $\phi_+$ and $\phi_-$, although they are degenerate in mass, they have different couplings to brane fields, since when canonically normalised their couplings will get a contribution from the different $K_+$ and $K_-$.

In particular, for the unwarped solution, we have $A = C = 1$ and $B = \frac{1}{\sqrt{3}}$ and the kinetic terms for the scalar modes are positive definite. On the other hand, the warped solution can be unstable in some of parameter space. The values of $K_+$ and $K_-$ as a function of $4g/q$ are depicted in Fig.2. The mode $\phi_-$ can in principle be a ghost when $K_- < 0$. This happens when

$$\frac{4g}{q} < \sqrt{\frac{3 + 2\sqrt{5}}{11}} \approx .82.$$  

(38)

Consequently, we observe that the mixing with the additional degenerate massive mode, which has been ignored in the literature, plays a crucial role in determining the instability of the warped solution. For the embedding of $\tilde{g} = g$ and $\lambda = 1$, the mode $\phi_-$ has always positive norm (since then $4g/q = n \geq 1$ from Eqn.(8)), but otherwise it depends on the values of $\lambda$ and $g/\tilde{g}$.

5. Conclusions

We have analyzed the linearized scalar perturbations (which couple to the dilaton) of the general axisymmetric vacuum of the Salam-Sezgin model. We have found that the mass eigenstates consist of a zero mode with constant wavefunction, a degenerate pair of first excited states, one with constant and the other with non-constant wavefunctions, and a tower of heavier states with non-constant wavefunctions. The degenerate pair has quadratic mixing in the effective action, even in the case with no warping. The orthogonal combinations couple to the branes at the two poles with different strengths. An important observation is that for a certain region of the parameter space, one of the canonical fields acquires a ghost-like kinetic term, which signals the onset of a dynamical instability.
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