EXTRAPOLATION AND FACTORIZATION OF MATRIX WEIGHTS

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ABSTRACT. In this paper we prove the Jones factorization theorem and the Rubio de Francia extrapolation theorem for matrix \( A_p \) weights. These results answer longstanding open questions in the study of matrix weights. The proof requires the development of the theory of convex-set valued functions and measurable seminorm functions. In particular, we define a convex-set valued version of the Hardy Littlewood maximal operator and construct an appropriate generalization of the Rubio de Francia iteration algorithm, which is central to the proof of both results in the scalar case.

1. INTRODUCTION

The purpose of this paper is to extend the theory of matrix \( A_p \) weights by proving the Jones factorization theorem [43] and the Rubio de Francia extrapolation theorem [57] in this setting. Our work answers a longstanding open question first raised (we believe) by Nazarov and Treil in 1996 [50, Section 11.5.4]. To provide some context for our results, we briefly recall some earlier work. For further details, we refer the reader to [25, 30]. The now classical \( A_p \) weights were introduced by Muckenhoupt and others in the 1970s. A weight (i.e., a non-negative, measurable function \( w \) that satisfies \( 0 < w(x) < \infty \) a.e.) is said to satisfy \( w \in A_p, 1 < p < \infty \), if

\[
[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{p-1} < \infty,
\]

where the supremum is taken over all cubes in \( \mathbb{R}^n \) with sides parallel to the coordinate axes. A weight \( w \) is in \( A_1 \) if

\[
[w]_{A_1} = \sup_Q \sup_{x \in Q} w(x)^{-1} \int_Q w(y) \, dy < \infty.
\]

It was shown that (1.1) is a sufficient condition, when \( 1 < p < \infty \), for norm inequalities of the form

\[
\int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx,
\]

and (1.2) is sufficient for the corresponding weak type estimate when \( p = 1 \), where \( T \) is the Hardy-Littlewood maximal operator, a Calderón-Zygmund singular integral, a square function, and other classical operators of harmonic analysis.

Two fundamental and closely related results in the study of weighted norm inequalities are the Jones factorization theorem and the Rubio de Francia extrapolation theorem.

**Theorem 1.1 (Jones Factorization Theorem).** Given a weight \( w \) and \( 1 < p < \infty \), \( w \in A_p \) if and only if there exist weights \( w_0, w_1 \in A_1 \) such that \( w = w_0 w_1^{1-p} \).
Theorem 1.2 (Rubio de Francia Extrapolation). Given $1 \leq p_0 < \infty$, suppose that an operator $T$ is such that for every $w_0 \in A_{p_0}$ and $f \in L^{p_0}(w_0)$,
\[
\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w_0(x) \, dx \leq C_0([w_0]_{A_{p_0}}) \int_{\mathbb{R}^n} |f(x)|^{p_0} w_0(x) \, dx.
\]
Then for every $p$, $1 < p < \infty$, every $w \in A_p$, and every $f \in L^p(w)$,
\[
\int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq C([w]_{A_p}) \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx.
\]

The proofs of both of these results are very closely related: each depends on the properties of the Rubio de Francia iteration algorithm
\[
\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|_{L^p(w)}^k},
\]
where $M$ is the Hardy-Littlewood maximal operator. (See the above references and also [14, 18].)

Rubio de Francia extrapolation has had many important applications in harmonic analysis and PDEs: see, for instance, [2, 27, 28]. In particular, it was was central to the original proofs of the so-called “$A_2$ conjecture”: that is, the sharp constant estimate
\[
\|Tf\|_{L^p(w)} \leq C(n, p, T)[w]_{A_{p}^\text{max} \{1, p' - 1\}} \|f\|_{L^p(w)},
\]
where $T$ is a Calderón-Zygmund singular integral. By using a sharp, quantitative version of extrapolation, the proof is reduced to showing this inequality holds for $p = 2$. See Hytönen [37, 38] and Lerner [44].

We now turn from the theory of scalar weights to matrix weights. Given a Calderón-Zygmund singular integral operator $T$, it extends to an operator on vector-valued functions $f = (f_1, \ldots, f_d)$ by applying it to each coordinate: $Tf = (Tf_1, \ldots, Tf_d)$. In a series of papers in the 1990s [50, 61–64], Nazarov, Treil and Volberg considered the question of whether there existed a corresponding “matrix” $A_p$ condition on positive semidefinite, symmetric (i.e., real self-adjoint) matrix functions $W$ such that
\[
\int_{\mathbb{R}^n} |W^{1/p}(x)Tf(x)|^p \, dx \leq C \int_{\mathbb{R}^n} |W^{1/p}(x)f(x)|^p \, dx,
\]
This problem was motivated by applications to stationary processes and to Toeplitz operators acting on vector-valued functions. It was first solved on the real line when $T$ is the Hilbert transform and $p = 2$ by Treil and Volberg [63]. They showed that a sufficient condition on the matrix $W$ is a matrix analog of the $A_2$ condition:
\[
[W]_{A_2} = \sup_Q \left( \frac{1}{Q} \int_Q W(x) \, dx \right)^{\frac{1}{2}} \left( \frac{1}{Q} \int_Q W^{-1}(x) \, dx \right)^{\frac{1}{2}} < \infty.
\]
This condition, however, does not extend to the case $p \neq 2$. An equivalent, but more technical definition of matrix $A_p$ in terms of norm functions was conjectured by Treil [61] and used by Nazarov and Treil [50] and separately by Volberg [64] to prove matrix weighted norm inequalities for the Hilbert transform. These authors noted two significant technical obstructions. The first was the lack of a “vector-valued” version of the Hardy-Littlewood maximal operator that could bound vector-valued operators but not lose the geometric information imbedded in the vector structure. The second was that proofs were much easier in the case $p = 2$, but that there was no version of the Rubio de Francia extrapolation theorem to extend these results to $p \neq 2$.

These results were extended to general Calderón-Zygmund singular integrals in $\mathbb{R}^n$ by Christ and Goldberg [11, 32]. A key component of their proofs is to define for each $p$ a scalar-valued, matrix
weighted maximal operator:

\[ M_W f(x) = \sup_Q \int_Q |W^{1/p}(x)W^{-1/p}(y)f(y)| \ dy \cdot \chi_Q(x). \]

While sufficient for their approach, here we note one drawback of this operator: while \( f \) and \( T f \) are vector-valued operators, \( M_W f \) is scalar-valued, and so cannot be iterated.

Finally, we note that Roudenko [56] gave an equivalent definition of matrix \( A_p \) that looked more like the definition in the scalar case: \( W \in A_p \) if and only if

\[ [W]_{A_p} = \sup_Q \left( \int_Q |W^{1/p}(x)W^{-1/p}(y)|_{p'} \ dy \right)^{\frac{1}{p'}} dx < \infty. \]

All of the estimates for singular integrals were qualitative: like the early proofs in the scalar case they did not give good estimates on the dependence of the constant on the value of \([W]_{A_p}\). After the sharp result in the scalar case was proved by Hytönen, it was natural to conjecture that the same result holds in the matrix case: more precisely, that

\[ \left( \int_{\mathbb{R}^n} |W(x)^{1/p} T f(x)|^p dx \right)^{1/p} \leq C [W]_{A_p}^{\max\{1,p'-1\}} \left( \int_{\mathbb{R}^n} |W(x)^{1/p} f(x)|^p dx \right)^{1/p}. \]

This problem is referred to as the matrix \( A_2 \) conjecture; it was first considered by Bickel, Petermichl and Wick [6] and by Pott and Stoica [52] when \( p = 2 \). More recently, Nazarov, Petermichl, Treil and Volberg [49] showed that when \( p = 2 \), they can get a constant of the form \( C(n,d,T)[W]_{A_2}^{3/2} \). (Also see [21].) A great deal of attention has been focused on the case \( p = 2 \) since this case is frequently easier. The first quantitative results for \( p \neq 2 \) were proved by the second author, Isralowitz and Moen [16], who got a constant of the form

\[ C(n,d,p,T)[W]_{A_p}^{1+\frac{1}{p-1} - \frac{1}{p}}. \]

A very important tool in the more recent proofs of the \( A_2 \) conjecture in the scalar case is the domination of singular integrals by sparse operators introduced by Lerner [44], Nazarov, Petermichl, Treil and Volberg [49] extended this result to vector-valued singular integrals by interpreting the vector \( T f \) as a point in a convex set. More precisely, they showed that there exists a sparse collection of dyadic cubes \( S \), depending on \( T \) and \( f \), such that

\[ T f(x) \in C \sum_{Q \in S} \langle \langle f \rangle \rangle_Q \chi_Q(x), \]

where \( \langle \langle f \rangle \rangle_Q \) is the convex set

\[ \langle \langle f \rangle \rangle_Q = \left\{ \int_Q k(y)f(y) \ dy : k \in L^\infty(Q), \|k\|_\infty \leq 1 \right\}, \]

and the sum is the (infinite) Minkowski sum of convex sets. However, instead of working directly with convex-set valued functions, they reduced the problem to estimating vector-valued sparse operators of the form

\[ T^S f(x) = \sum_{Q \in S} \int_Q \varphi_Q(x,y) f(y) \ dy, \]

where for each \( Q \), \( \varphi_Q \) is a real-valued function supported on \( Q \times Q \) such that, for each \( x \), \( \|\varphi_Q(x, \cdot)\|_\infty \leq 1 \).

Given this background, we can now describe our main results. To do so we must first introduce a change in notation. For a number of reasons connected to our proofs, we have chosen to write matrix
weighted norms in the form
\[
\left( \int_{\mathbb{R}^n} |W(x)f(x)|^p \, dx \right)^{\frac{1}{p}}.
\]
This is equivalent to replacing the matrix weight \( W \) by \( W^p \). In doing this we replace the class \( A_p \) with the equivalent class \( A_p \):

\[
[W]_{A_p} = \sup_Q \left( \int_Q \left( \int_Q |W(x)W^{-1}(y)|_L^p \, dy \right)^{\frac{1}{p}} \, dx \right)^{\frac{1}{p}} < \infty.
\]
We also define the classes \( A_1 \) and \( A_\infty \) by

\[
[W]_{A_1} = \sup \mathrm{ess sup} \int_Q |W^{-1}(x)W(y)|_{op} \, dy < \infty,
\]
and

\[
[W]_{A_\infty} = \sup \mathrm{ess sup} \int_Q |W(x)W^{-1}(y)|_{op} \, dy < \infty.
\]

The class \( A_1 \) was first introduced by Frazier and Roudenko [29]; the class \( A_\infty \) is new, though it was implicit in the literature in the scalar case. Note that in the scalar case we can write the definition of \( A_p \) as

\[
[w]_{A_p} = \sup_Q |Q|^{-1} \|w\chi_Q\|_{L^p} \|w^{-1}\chi_Q\|_{L^{p'}} < \infty,
\]
and this makes sense even when \( p = 1 \) or \( p = \infty \). This definition of \( A_\infty \) was implicit in Muckenhoupt [47] but mostly overlooked. It has been used to define a uniform \( A_p \) condition, \( 1 \leq p \leq \infty \); see Nieraeth [51]. We also remark that this approach to scalar weighted norm inequalities is used for off-diagonal inequalities and norm inequalities on Banach function spaces: see, for instance, [13, 15, 48]).

With this notation, our main results are the following.

**Theorem 1.3.** Fix \( 1 < p < \infty \). Given a matrix weight \( W \), we have \( W \in A_p \) if and only if

\[
W = W_0^{1/p}W_1^{1/p'},
\]
for some commuting matrix weights \( W_0 \in A_1 \) and \( W_1 \in A_\infty \).

**Theorem 1.4.** Let \( T \) be a sublinear operator. Suppose that for some \( p_0 \), \( 1 \leq p_0 \leq \infty \), there exists an increasing function \( K_{p_0} \) such that for every \( W_0 \in A_{p_0} \),

\[
\|Tf\|_{L^{p_0}(\mathbb{R}^n,W_0)} \leq K_{p_0}([W_0]_{A_{p_0}})\|f\|_{L^{p_0}(\mathbb{R}^n,W_0)}.
\]

Then for all \( p \), \( 1 < p < \infty \), and for all \( W \in A_p \),

\[
\|Tf\|_{L^p(\mathbb{R}^n,W)} \leq K_p(p,p_0,n,d,[W]_{A_p})\|f\|_{L^p(\mathbb{R}^n,W)},
\]

where

\[
K_p(p,p_0,n,d,[W]_{A_p}) = C(p,p_0)K_{p_0}\left(C(n,d,p,p_0)\max\left\{\frac{1}{p_0},\frac{1}{p'}\right\}\right).
\]

**Remark 1.5.** For simplicity and ease of comparison to the scalar case, we state Theorem 1.3 assuming that the matrices \( W_0 \) and \( W_1 \) commute. We can remove this hypothesis, but to do so we must replace the product \( W_0^{1/p}W_1^{1/p'} \) with the geometric mean of the two matrices. See Proposition 8.7. One interesting feature of our proof is that in constructing the matrices \( W_0 \) and \( W_1 \), we show that they can be realized as scalar multiples of \( W \).
Remark 1.6. We actually prove a more general version of Theorem 1.4, replacing the operator \( T \) by a family of pairs of functions \((f, g)\). This more abstract approach to extrapolation was first suggested in [20] and systematically developed in [18].

Remark 1.7. In Theorem 1.4 the function \( K_p \) depending on \( K_{p_0} \) has exactly the same form as the function gotten in the sharp constant extrapolation theorem of Dragičević, et al. [23]. Note also that we are able to begin the extrapolation from \( p_0 = \infty \); this gives a quantitative version of a result proved in the scalar case by Harboure, et al. [33]; this quantitative version was recently proved by Nieraeth [51, Corollary 4.14].

Remark 1.8. As has been noted in the literature (e.g., in [22, 40]), many problems in matrix weighted inequalities are significantly easier to prove when \( p = 2 \) than for all \( p \) (see, for instance, [5, 6, 11, 49, 52, 63]), and the lack of an extrapolation theorem requires more difficult proof techniques to be introduced. We therefore expect that Theorem 1.4 will prove useful in extending results from the case \( p = 2 \) to the full range of \( p \). We note, in particular, that because we have a sharp constant version of extrapolation, to prove the full matrix \( A_2 \) conjecture it will be enough to prove it in the case \( p = 2 \). Further, we note that using the results in [42], the sharp constant estimates for commutators proved by Chung et al. [12] in the scalar case can be immediately extended to the matrix weighted setting once the matrix \( A_2 \) conjecture is proved for \( p = 2 \).

To prove the Jones factorization theorem and Rubio de Francia extrapolation for matrix weights, we considerably expand upon the ideas underlying the convex-set sparse domination theorem described above. To do so, we draw upon an extensive literature on convex-set analysis (see, for instance, [1, 10, 54]) which does not seem to have been previously applied to problems in harmonic analysis. We define measurable functions \( F : \mathbb{R}^n \to \mathcal{K} \), where \( \mathcal{K} \) is (a subset of) the collection of convex sets in \( \mathbb{R}^d \), and develop the connection between norm functions and convex-set valued functions. There is a one-to-one correspondence between measurable norm functions and measurable convex-set valued functions. As noted above, the matrix \( A_p \) condition was originally defined in terms of norm functions, but the trend, at least since the work of Roudenko [56] and Goldberg [32], has been to interpret it only in terms of matrices. We go back to this definition in terms of norm functions; this proved to be essential at several points in our proofs as it provides the necessary link between matrices and convex-set valued functions.

We define a convex-set valued version of the Hardy-Littlewood maximal operator by using the so-called Aumann integral of convex-set valued functions (see [1]) to define the maximal operator

\[
MF(x) = \text{cvar} \left( \bigcup_Q \int_Q F(y) \, dy \cdot \chi_Q(x) \right).
\]

With this definition we get analogs of all the properties of the scalar maximal operator: in particular, it dominates \( F \) via inclusion, \( F(x) \subset MF(x) \); and is bounded on \( L^p(W) \) when \( W \in A_p \). Most importantly, it maps convex-set valued functions to convex-set valued functions, and therefore can be iterated. This allows us to define the Rubio de Francia iteration algorithm for convex-set valued functions:

\[
\mathcal{R}H(x) = \sum_{k=0}^{\infty} \frac{M^k H(x)}{2^k \|M\|_{L^p(W)}^k}.
\]

This operator has properties analogous to the scalar operator: \( H(x) \subset \mathcal{R}H(x); \|\mathcal{R}H\|_{L^p(W)} \leq 2 \|H\|_{L^p(W)} \); and \( \mathcal{R}H \) satisfies a convex-set valued \( A_1 \) condition; \( M(\mathcal{R}H)(x) \subset \mathcal{CR}H(x) \). This property is closely related to the \( A_1 \) condition for norm functions (and so for matrix weights). With this version of the iteration algorithm, we are able to extend the scalar proofs of factorization and extrapolation to the matrix case. The overall outline of the proofs is similar to those of the scalar results (see [13, 18], but there are a significant number of technical obstacles which must be addressed. Here we note the two most difficult: first, matrix functions do not, in general, commute. Second, while it is possible to define powers of
matrices (and so of ellipsoids), it is not possible to define powers of arbitrary convex sets. (See Milman and Rotem [45, 46].)

Remark 1.9. The fact that we must specialize to consider ellipsoid valued functions might suggest that we could simplify our approach to extrapolation by restricting to these kinds of functions rather than working with the more general convex-set valued functions. However, even for vector-valued functions, the Aumann averages and the convex-set valued maximal operator will yield convex sets that are not ellipsoids. We give an example in Section 5.

Remark 1.10. We believe that convex-set valued functions will have additional applications in the study of matrix weighted norm inequalities. For instance, see the very recent paper by Vuorinen [65], who uses them to prove estimates for a “strong” version of the Christ-Goldberg maximal operator defined using the basis of rectangles.

The remainder of this paper is organized as follows. To prove factorization and extrapolation we need to establish a large number of preliminary results. This is done in Sections 2–4. In Section 2 we present a number of results about convex sets and seminorms. Most of these results are known and we gather them here for ease of reference and to establish consistent notation. However, some results are new (or rather, we could not find them in the literature). In particular, we prove some basic results about the geometric mean of two norms that are essential to the proof of factorization.

In Section 3 we define measurable, convex-set valued functions and establish the properties of the Aumann integral necessary to define the maximal operator on convex-set valued functions. We have gathered together, with consistent hypotheses and notation, a number of theorems from across the literature and proved some results specific to our needs, such as a version of Minkowski’s inequality for the Aumann integral (Proposition 3.21). Since much of this material appears unfamiliar to most harmonic analysts, and since there are a number of delicate issues related to measurability of convex-set valued functions, we have included most details and we give extensive references to the literature.

In Section 4 we define seminorm functions, explore their connection with measurable convex-set valued functions, and define the norm-weighted \( L^p \) spaces of convex-set valued functions. These are not Banach spaces, but have most of the same properties, which allows us to rigorously define the Rubio de Francia iteration algorithm. Finally, we make explicit the connection between measurable seminorm functions and matrix weights.

In the remaining sections we develop our new results. In Section 5 we define averaging operators and the maximal operator on convex-set valued functions. We prove that this maximal operator has properties that are the exact analogs of those of the scalar maximal operator, and we prove unweighted norm inequalities by adapting the scalar proof (using dyadic cubes) to the setting of convex-set valued functions (Theorem 5.10).

In Section 6 we turn to the definition of matrix \( A_p \) in terms of norm functions. Many of these results are already in the literature, but, because we have chosen to take a different approach than what has been done previously, we believed it was important to carefully restate these results to incorporate the endpoint results when \( p = 1 \) and \( p = \infty \). The main result of this section is that the convex-set valued maximal operator is bounded on \( L^p(W) \), \( 1 < p \leq \infty \), when \( W \in A_p \). The proof uses a measurable version of the John ellipsoid theorem (Theorem 3.7) to reduce to norm inequalities for the Christ-Goldberg matrix weighted maximal operator.

In Section 7 we define convex-set valued \( A^F_k \) weights and show that there is a one-to-one correspondence between them and norm functions in \( A_1 \) (Theorem 7.3); this gives us a connection between convex-set \( A^F_k \) and matrix \( A_1 \) weights that is needed for the proof of extrapolation. We then define a generalized Rubio de Francia iteration algorithm which includes the version given above and which covers the various forms of the operator used in the proofs of factorization and extrapolation (Theorem 7.6).
In Section 8 we state and prove our version of the Jones factorization theorem (restated there as Theorem 8.1). The proof is based on that of the scalar version given in [14]. In the scalar case, the difficult direction is to prove that an $A_p$ weight can be factored as the product of $A_1$ weights; the other direction, sometimes referred to as “reverse factorization”, is an immediate consequence of the definition of $A_p$ weights. In the matrix case, however, both directions are difficult. The proof of factorization is based on the Rubio de Francia iteration algorithm and follows the scalar proof given in [14]. The proof of reverse factorization is more delicate: it is here that we were required to work with the definition of $A_p$ in terms of norm functions. Our final proof is, implicitly, based on an interpolation argument between finite dimensional spaces.

Finally, in Section 9 we state and prove a sharp constant version of Rubio de Francia extrapolation for matrix weights (Theorem 9.1). The proof is based on the approach to extrapolation developed in [18], and so reverse factorization is a central part of the proof. We adopt the perspective of working with families of extrapolation pairs $(f, g)$, which completely avoids any mention of operators. Using our definition of $A_p$ weights, we are also able to give a uniform proof that includes the endpoint results when $p = \infty$. This yields a quantitative version of a result proved in the scalar case by Harboure, Macías and Segovia [33].

Throughout this paper we will use the following notation. We will develop some things in the setting of abstract measure spaces; in this setting $(\Omega, A, \mu)$ will denote a $\sigma$-finite, complete measure space endowed with a positive measure $\mu$. In Euclidean space the constant $n$ will denote the dimension of $\mathbb{R}^n$, which will be the domain of our functions. The value $d$ will denote the dimension of vector and set-valued functions. In $\mathbb{R}^d$, $B$ will denote the $\sigma$-algebra of Borel sets, and $m_d$ will denote the Lebesgue measure. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^n)$ will denote the Lebesgue space of scalar functions, and $L^p(\mathbb{R}^n, \mathbb{R}^d)$ will denote the Lebesgue space of vector-valued functions.

Given $v = (v_1, \ldots, v_d)^t \in \mathbb{R}^d$, the Euclidean norm of $v$ will be denoted by $|v|$. The standard orthonormal basis in $\mathbb{R}^d$ will be denoted by $\{e_i\}_{i=1}^d$. The open unit ball in $\{v \in \mathbb{R}^d : |v| < 1\}$ will be denoted by $B$ and its closure by $\overline{B}$. Matrices will be $d \times d$ matrices with real-valued entries unless otherwise specified. The set of all such matrices will be denoted by $\mathcal{M}_d$. The set of all $d \times d$, symmetric (i.e., self-adjoint), positive semidefinite matrices will be denoted by $\mathcal{S}_d$. We will denote the transpose of a matrix $A$ by $A^\ast$. Given two quantities $A$ and $B$, we will write $A \lesssim B$, or $B \gtrsim A$ if there is a constant $c > 0$ such that $A \leq cB$. If $A \lesssim B$ and $B \gtrsim A$, we will write $A \approx B$.

2. CONVEX SETS AND SEMINORMS

In this section we develop the connections between convex sets in $\mathbb{R}^d$ and seminorms defined on $\mathbb{R}^d$. We begin with some basic definitions and notation. Given a set $E \subset \mathbb{R}^d$, let $\overline{E}$ denote the closure of $E$. Given two sets $E$, $F \subset \mathbb{R}^d$, define their Minkowski sum to be the set

$$E + F = \{x + y : x \in E, y \in F\}.$$ 

For $\lambda \in \mathbb{R}$, define $\lambda E = \{\lambda x : x \in E\}$. A set $E$ is symmetric if $-E = E$. A set $E$ is absorbing if for every $v \in \mathbb{R}^d$, $v \in tE$ for some $t > 0$.

A set $K \subset \mathbb{R}^d$ is convex if for all $x, y \in K$ and $0 < \lambda < 1$, $\lambda x + (1 - \lambda)y \in K$. For the basic properties of convex sets, see [54, 59]. Given a set $E$, let $\text{conv}(E)$ denote the convex hull of $E$: the smallest convex set that contains $E$. Equivalently, $\text{conv}(E)$ consists of all finite convex combinations elements in $E$:

$$\text{conv}(E) = \left\{ \sum_{i=1}^k \alpha_i x_i : x_i \in E, \alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1 \right\}.$$

The convex hull is additive: given two sets $E$, $F$, $\text{conv}(E) + \text{conv}(F) = \text{conv}(E + F)$. We will denote the closure of the convex hull of $E$ by $\overline{\text{conv}}(E)$.
Let $\mathcal{K}(\mathbb{R}^d)$ be the collection of all closed, nonempty subsets of $\mathbb{R}^d$. The subscripts $a, b, c, s$ appended to $\mathcal{K}$ will denote absorbing, bounded, convex, and symmetric sets, respectively. Since $\mathbb{R}^d$ is finite dimensional, a convex set $K$ is absorbing if and only if $0 \in \text{int}(K)$. We are particularly interested in the following two subsets of $\mathcal{K}$:

- $\mathcal{K}_{acs}(\mathbb{R}^d)$: absorbing, convex, symmetric, and closed subsets of $\mathbb{R}^d$;
- $\mathcal{K}_{abcs}(\mathbb{R}^d)$: bounded, convex, symmetric, and closed subsets of $\mathbb{R}^d$.

We generalize the norm on $\mathbb{R}^d$ by introducing the concept of a seminorm.

**Definition 2.1.** A seminorm is a function $p : \mathbb{R}^d \to [0, \infty)$ that satisfies the following properties: for all $u, v \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$,

\begin{align*}
(2.1) & \quad p(u + v) \leq p(u) + p(v), \\
(2.2) & \quad p(\alpha v) = |\alpha| p(v).
\end{align*}

A seminorm is a norm if $p(v) = 0$ if and only if $v = 0$.

**Definition 2.2.** Given $K \in \mathcal{K}_{acs}(\mathbb{R}^d)$ define the corresponding Minkowski functional $p_K : \mathbb{R}^d \to [0, \infty)$ by

$$p_K(v) = \inf\{ r > 0 : v/r \in K\}.$$ 

**Definition 2.3.** Given a seminorm $p$, define the unit ball of $p$ to be the set

$$K(p) = \{ v \in \mathbb{R}^d : p(v) \leq 1 \}.$$ 

By properties (2.1) and (2.2), the unit ball $K(p)$ is a convex, absorbing, symmetric set. In fact, using the Minkowski functional, there is a one-to-one correspondence between sets $K \in \mathcal{K}_{acs}(\mathbb{R}^d)$ and seminorms $p$. For a proof of the following result, see [31, p. 210] or [58, Theorems 1.34 and 1.35].

**Theorem 2.4.** Given any $K \in \mathcal{K}_{acs}(\mathbb{R}^d)$, the Minkowski functional $p_K$ satisfies seminorm properties (2.1) and (2.2). Conversely, given any seminorm $p$, the unit ball $K(p) \in \mathcal{K}_{acs}(\mathbb{R}^d)$. This correspondence between sets in $\mathcal{K}_{acs}(\mathbb{R}^d)$ and seminorms is one-to-one.

Since $\mathbb{R}^d$ is finite dimensional, all norms on it are equivalent. Therefore, given a norm $p$, $K(p)$ is bounded. Conversely, if $K \in \mathcal{K}_{abcs}(\mathbb{R}^d)$, then $p_K$ is a norm [31, p. 210]. This gives the following corollary to Theorem 2.4.

**Corollary 2.5.** There is a one-to-one correspondence between norms on $\mathbb{R}^d$ and the set $\mathcal{K}_{abcs}(\mathbb{R}^d)$, given by the map $K \mapsto p_K$.

There is another correspondence between convex sets and seminorms, one which will be more useful for our purposes below. To state it, we need to introduce the concept of the dual seminorm and the polar of a convex set. The proof of the following result follows at once from the properties of a seminorm.

**Lemma 2.6.** Given a seminorm $p$, define $p^* : \mathbb{R}^d \to [0, \infty)$ by

$$p^*(v) = \sup_{w \in \mathbb{R}^d, p(w) \leq 1} |\langle v, w \rangle|.$$ 

Then $p^*$ is a seminorm. If $p$ is a norm, the definition may be written as

\begin{equation}
(2.3) \quad p^*(v) = \sup_{w \in \mathbb{R}^d, w \neq 0} \frac{|\langle v, w \rangle|}{|p(w)|}.
\end{equation}

**Definition 2.7.** Given $K \in \mathcal{K}_{cs}(\mathbb{R}^d)$, define its polar set by

$$K^\circ = \{ v \in \mathbb{R}^d : |\langle v, w \rangle| \leq 1 \text{ for all } w \in K \}.$$
A polar set can be thought of as the “dual” of a convex set. This is made precise by the following result; for a proof, see [54, Theorem 14.5].

**Theorem 2.8.** Let \( K \in \mathcal{K}_{\text{cs}}(\mathbb{R}^d) \). The following statements hold:

(a) If \( K \in \mathcal{K}_{\text{acs}}(\mathbb{R}^d) \), then \( K^\circ \in \mathcal{K}_{\text{cs}}(\mathbb{R}^d) \).

(b) If \( K \in \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \), then \( K^\circ \in \mathcal{K}_{\text{acs}}(\mathbb{R}^d) \).

(c) \( (K^\circ)^\circ = K \).

(d) If \( K \) is bounded and absorbing, then \( p_K \) is a norm and

\[ p_K^\circ = (p_K)^* . \]

As a corollary to Theorem 2.8 and Corollary 2.5 we have the following.

**Corollary 2.9.** Given a norm \( p \), then the dual seminorm \( p^* \) is a norm. Moreover, \( p^{**} = (p^*)_* = p \).

We can now state another characterization of seminorms in terms of convex sets. The proof is an immediate consequence of Theorems 2.4 and 2.8.

**Theorem 2.10.** The mapping \( K \mapsto p_K^\circ \) defines a one-to-one correspondence between bounded, convex, symmetric sets and seminorms on \( \mathbb{R}^d \). More precisely, if \( K \in \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \), then \( p_K^\circ \) is a seminorm; conversely, if \( p \) is a seminorm, then \( K(p)^\circ \in \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \).

The next result gives some important properties of seminorms induced by polar sets.

**Theorem 2.11.** The following are true:

(a) Let \( K_1, K_2 \in \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \). Then \( K_1 + K_2 \in \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \) and

\[ p(K_1 + K_2)^\circ = pK_1^\circ + pK_2^\circ . \]

(b) Let \( K \in \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \) and \( \alpha \in \mathbb{R} \). Then \( \alpha K = |\alpha|K \in \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \) and

\[ p(\alpha K)^\circ = |\alpha|pK^\circ . \]

(c) Let \( \{K_i\}_{i \in \mathbb{N}} \subset \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \). If \( K = \text{conv}(\bigcup_{i \in \mathbb{N}} K_i) \) is bounded, then

\[ pK^\circ = \sup_{i \in \mathbb{N}} p(K_i)^\circ . \]

(d) Let \( \{K_i\}_{i \in \mathbb{N}} \subset \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \) be a family of nested sets, with \( K_{i+1} \subset K_i \) for all \( i \). If \( K = \bigcap_{i \in \mathbb{N}} K_i \), then

\[ pK^\circ = \inf_{i \in \mathbb{N}} p(K_i)^\circ . \]

To prove Theorem 2.11 we introduce the concept of a support function.

**Definition 2.12.** Given \( K \in \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \), its support function \( h_K : \mathbb{R}^d \to [0, \infty) \) is defined to be

\[ h_K(v) = \sup_{w \in K} \langle v, w \rangle . \]

Note that since \( K \) is symmetric, we can write \( |\langle v, w \rangle| \) in the definition of the support function.

**Lemma 2.13.** Given \( K \in \mathcal{K}_{\text{bcsc}}(\mathbb{R}^d) \), \( h_K = p_K^\circ \).

**Proof.** By Definitions 2.2, 2.7, and 2.12, for any \( v \in \mathbb{R}^d \),

\[ p_K^\circ(v) = \inf \{ r > 0 : v/r \in K^\circ \} = \inf \{ r > 0 : |\langle v/r, w \rangle| \leq 1 \text{ for all } w \in K \} \]

\[ = \inf \{ r > 0 : |\langle v, w \rangle| \leq r \text{ for all } w \in K \} = \sup \{|\langle v, w \rangle| : w \in K \} = h_K(v) . \]
Proof of Theorem 2.11. To prove (a), first note that by Lemma 2.13 applied twice,
\[ p_{(K_1 + K_2)}(v) = \sup_{w \in K_1 + K_2} \langle v, w \rangle = \sup_{w_1 \in K_1} \langle v, w_1 \rangle + \sup_{w_2 \in K_2} \langle v, w_2 \rangle = p_{K_1}(v) + p_{K_2}(v). \]
Part (b) is proved similarly. To prove (c) we use Lemma 2.13 and the definition of the convex hull:
\[ p_{K^*}(v) = \sup_{w \in K} \langle v, w \rangle = \sup \left\{ \sum_{i=1}^{k} \alpha_i \langle v, w_i \rangle : w_i \in K_i, k > 0, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1 \right\} \]
\[ = \sup \left\{ \sum_{i=1}^{k} \alpha_i p_{K_i^*}(v) : k > 0, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1 \right\} = \sup_{i} p_{K_i^*}(v). \]
Finally, to prove (d), first note that for each \( i, K \subset K_i \), so we have
\[ p_{K^*}(v) = \sup_{w \in K} \langle v, w \rangle \leq \sup_{w_i \in K_i} \langle v, w_i \rangle = p_{K_i^*}(v). \]
Hence, \( p_{K^*}(v) \leq \inf_{i} p_{K_i^*}(v). \)
To prove that equality holds, suppose to the contrary that this is a strict inequality. Let
\[ \epsilon = \inf_{i} p_{K_i^*}(v) - p_{K^*}(v) > 0. \]
Hence, for each \( i \), \( p_{K_i^*}(v) - p_{K^*}(v) \geq \epsilon \). By Lemma 2.13, for each \( i \) there exists \( w_i \in K_i \) such that
\[ \langle v, w_i \rangle + \frac{\epsilon}{2} > p_{K_i^*}(v). \]
Hence,
\[ \langle v, w_i \rangle - p_{K^*}(v) > \frac{\epsilon}{2}. \]
Since the \( K_i \) are nested, by passing to a subsequence we may assume that the \( w_i \) converge to a point \( w \in K \) as \( i \to \infty \). Therefore, by continuity,
\[ \langle v, w \rangle - p_{K^*}(v) \geq \frac{\epsilon}{2}, \]
which contradicts the fact that \( \langle v, w \rangle \leq h_k(v) = p_{K^*}(v) \). Hence, equality holds. \( \square \)

We now consider the weighted geometric mean of two norms. These results will be very important in the proof of reverse factorization in Section 8. Let \( p_0, p_1 \) be two norms. For \( 0 < t < 1 \), define for all \( v \in \mathbb{R}^d \),
\[ p_t(v) = p_0(v)^{1-t} p_1(v)^t. \]
The function \( p_t \) need not be a norm, though it is homogenous: \( p_t(\alpha v) = |\alpha| p_t(v) \). However, we can still use the definition in Lemma 2.6 to define \( p_t^* \), which will be norm.

**Lemma 2.14.** Given norms \( p_0, p_1 \) and \( 0 < t < 1 \), define \( p_t \) by (2.4). If we define \( p_t^* \) by equation (2.3), then \( p_t^* \) is a norm.

*Proof.* First note that since \( p_0, p_1 \) are norms, if \( w \neq 0 \), \( p_t(w) \neq 0 \), so \( p_t^* \) is well defined. That it is a norm then follows immediately from the properties of the Euclidean inner product. \( \square \)

By Corollary 2.9 we also have that \( p_t^{**} \) is a norm; though it is not equal to \( p_t \) we will be able to use it in place of \( p_t \).

**Lemma 2.15.** Given norms \( p_0, p_1 \) and \( 0 < t < 1 \), define \( p_t \) by (2.4). Then for all \( v \in \mathbb{R}^d \), \( p_t^{**}(v) \leq p_t(v) \).
There exists an invertible matrix

4.1

4

4.11

there exist positive definite matrices

(see Theorem 2.6) and

\text{Proposition 2.7.}

Remark D. In particular, we can choose such that

\text{Lemma 2.17.}

By the definition of \text{Lemma 1.6.1}. For completeness we include the short proof.

\text{Lemma 2.18.}

The following hold for all \( v \in \mathbb{R}^d \):

1. Given a norm \( p \), there exists a positive definite matrix \( A \) such that \( p(v) \approx |Av| \), where the implicit constants depend only on \( d \).

2. Given any invertible matrix \( A \in \mathcal{M}_d \), \( p(v) = |Av| \) is a norm and \( p^*(v) = |(A^*)^{-1}v| \).

The following result shows two positive definite matrices are simultaneously congruent to diagonal matrices, see [4, Ex. 1.6.1]. For completeness we include the short proof.

\text{Lemma 2.19.}

Given norms \( p_0 \) and \( p_1 \) by (2.4). Then for all \( v \in \mathbb{R}^d \),

\text{Proposition 2.19.}

Given norms \( p_0 \) and \( p_1 \) by (2.4). Then for all \( v \in \mathbb{R}^d \),

\text{Theorem 3.13}. Since we prove this result in detail for measurable norm functions in Section 4.1 (see Theorem 4.11 and Proposition 4.12) we omit the simpler proof here.

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The following result shows two positive definite matrices are simultaneously congruent to diagonal matrices, see [4, Ex. 1.6.1]. For completeness we include the short proof.

\text{Lemma 2.18.}

Let \( A, B \in S_d \). Then, there exists an invertible matrix \( S \), and diagonal matrices \( D_A \) and \( D_B \) such that

\text{Lemma 2.19.}

Given norms \( p_0 \), \( p_1 \) and \( 0 < t < 1 \), define \( p_t \) by (2.4). Then for all \( v \in \mathbb{R}^d \),

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Given norms \( p_0 \) and \( p_1 \) by (2.4). Then for all \( v \in \mathbb{R}^d \),

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Given norms \( p_0 \), \( p_1 \) and \( 0 < t < 1 \), define \( p_t \) by (2.4). Then for all \( v \in \mathbb{R}^d \),

\text{Theorem 3.13}. Since we prove this result in detail for measurable norm functions in Section 4.1 (see Theorem 4.11 and Proposition 4.12) we omit the simpler proof here.
Since \( \{S*e_i\}_{i=1}^d \) is a basis, we can write \( w \in \mathbb{R}^d \) as
\[
w = \sum_{i=1}^d b_i S*e_i.
\]

If we let \( v = S^{-1}e_i \), then \( p_t(v) = \lambda_i^{1-t} \mu_i^t \). Hence,
\[
(2.7) \quad p_t^* (w) = \sup_{v \in \mathbb{R}^d} \frac{\langle w, v \rangle}{p_t(v)} \geq \max_{1 \leq i \leq d} \frac{\langle w, S^{-1}e_i \rangle}{p_t(S^{-1}e_i)} = \max_{1 \leq i \leq d} \frac{|b_i|}{\lambda_i^{1-t} \mu_i^t}.
\]

Similarly, since \( \{S^{-1}e_i\} \) is a basis, we can write any \( v \in \mathbb{R}^d \) as
\[
v = \sum_{i=1}^d c_i S^{-1}e_i.
\]

Since
\[
p_t(v) \approx \left( \sum_{i=1}^d |c_i|^2 \lambda_i^2 \right)^{(1-t)/2} \left( \sum_{i=1}^d |c_i|^2 \mu_i^2 \right)^{t/2}
\]
we have \( p_t(v) \gtrsim |c_i| \lambda_i^{1-t} \mu_i^t \) for any \( v \in \mathbb{R}^d \) such that \( |\langle v, S*e_i \rangle| = |c_i| \). Thus,
\[
(2.8) \quad p_t^* (w) \leq \sum_{i=1}^d |b_i| |p_t^* (S*e_i) \leq \sum_{i=1}^d \frac{|b_i|}{\lambda_i^{1-t} \mu_i^t}.
\]

Combining (2.7) and (2.8) yields
\[
(2.9) \quad p_t^* (w) \approx \left( \sum_{i=1}^d \left( \frac{|b_i|}{\lambda_i^{1-t} \mu_i^t} \right)^2 \right)^{1/2} = |D_A^{-(1-t)/2} D_B^{-t/2} (S^{*})^{-1} w|.
\]

Define \( q(w) = p_0^*(w)^{1-t} p_t^*(w)^t \). By Lemma 2.17,
\[
p_0^*(w) \approx |D_A^{-1/2} (S^{*})^{-1} w| \quad \text{and} \quad p_t^*(w) \approx |D_A^{-1/2} (S^{*})^{-1} w|.
\]

Hence, applying (2.9) for \( p_0^* \) and \( p_t^* \) yields its dual analogue
\[
(2.10) \quad q^* (v) \approx \left( \sum_{i=1}^d (|c_i| \lambda_i^{1-t} \mu_i^t)^2 \right)^{1/2} = |D_A^{(1-t)/2} D_B^{t/2} S v|.
\]

Combining (2.9) and (2.10) with Lemma 2.17 yields for any \( v \in \mathbb{R}^d \),
\[
(2.11) \quad p_t^{**} (v) \approx |D_A^{(1-t)/2} D_B^{t/2} S v| \approx q^* (v).
\]

Finally we connect the concept of a weighted geometric mean of norms with that of matrices [4].

**Definition 2.20.** Let \( A \) and \( B \) be two symmetric positive definite matrices. For \( 0 < t < 1 \) define the weighted geometric mean of \( A \) and \( B \) by
\[
A \#_t B = A^{1/2} (A^{-1/2} BA^{-1/2})^t A^{1/2}.
\]

Lemma 2.21 gives an equivalent definition of the weighted geometric mean.

**Lemma 2.21.** Let \( A, B \in S_d \). Suppose that for some invertible matrix \( S \), and diagonal matrices \( D_A \) and \( D_B \) we have
\[
A = S*D_A S, \quad B = S*D_B S.
\]

Then, the weighted geometric mean of \( A \) and \( B \) satisfies
\[
(2.12) \quad A \#_t B = S^*(D_A)^{1-t} (D_B)^t S.
\]
Proof. We need to recall some useful facts about the set $S_d$ of symmetric positive $d \times d$ matrices from [4, Chapter 6]. The set $S_d$ is an open subset of the space of all $d \times d$ symmetric matrices, which is equipped with the inner product $\langle A, B \rangle = \text{tr} A^* B$. Hence, $S_d$ is a differentiable manifold equipped with a natural Riemannian metric. By [4, Theorem 6.1.6], there exists a unique geodesic path joining any two points $A, B \in S_d$, which has a parametrization $A\#_t B$, $0 \leq t \leq 1$. For each $d \times d$ invertible matrix $X$, define the congruence transformation

$$\Gamma_X : S_d \to S_d, \quad \Gamma_X(A) = X^* A X, \ A \in S_d.$$ 

By [4, Lemma 6.1.1], $\Gamma_X$ preserves lengths of differentiable paths in $S_d$. Hence, if the $\gamma : [0, 1] \to S_d$ is a geodesic path in $S_d$, so is $\Gamma_X \circ \gamma$.

Let $\gamma(t) = A\#_t B$, $0 \leq t \leq 1$, be a geodesic path between $A$ and $B$. Then,

$$\Gamma_{S^{-1}}(\gamma(t)) = (S^{-1})^* A\#_t B S^{-1}$$

is a geodesic path between the diagonal matrices $D_A$ and $D_B$. Since the matrices $D_A$ and $D_B$ commute, by [4, Proposition 6.1.5 et seq.], their geodesic path is given by $t \mapsto (D_A)^{1-t} (D_B)^t$. Hence, since geodesic paths are unique, we have

$$(S^{-1})^* A\#_t B S^{-1} = (D_A)^{1-t} (D_B)^t, \quad 0 \leq t \leq 1,$$

which yields (2.12). \qed

As a consequence Proposition 2.19 and Lemma 2.21 we have the following corollary.

**Corollary 2.22.** Suppose that $A, B \in S_d$ and the norms $p_0$ and $p_1$ are given by

$$p_0(v) = |A^{1/2} v| \quad \text{and} \quad p_1(v) = |B^{1/2} v| \quad \text{for} \ v \in \mathbb{R}^d.$$ 

Then the double dual of the weighted geometric mean $p_1$ (2.4) satisfies

$$(p_1^*)^*(v) \approx |(A\# B)^{1/2} v| \quad \text{for} \ v \in \mathbb{R}^d.$$ 

**Proof.** This is an immediate consequence of (2.11) and (2.12) since

$$|(A\# B)^{1/2} v|^2 = \langle (A\# B) v, v \rangle = \langle (D_A)^{1-t} (D_B)^t S v, S v \rangle = |D_A^{(1-t)/2} D_B^{t/2} S v|^2. \quad \Box$$

### 3. Convex-set valued functions

**Measurable convex-set valued functions.** In this section we develop the properties of measurable convex-set valued functions. Recall our standing assumption that $(\Omega, \mathcal{A}, \mu)$ is a positive, $\sigma$-finite, and complete measure space. We start with a definition of measurability of functions taking values in closed sets $K(\mathbb{R}^d)$.

**Definition 3.1.** Given a function $F : \Omega \to K(\mathbb{R}^d)$, we say that $F$ is measurable if for every open set $U \subset \mathbb{R}^d$, $F^{-1}(U) = \{ x \in \Omega : F(x) \cap U \neq \emptyset \} \in \mathcal{A}$.

We shall employ the following characterization of closed-set valued measurable functions. For a proof, see [1, Theorems 8.1.4, 8.3.1]. The equivalence of (i) and (iv) is known as the Castaing representation theorem.

**Theorem 3.2.** Given $F : \Omega \to K(\mathbb{R}^d)$, the following are equivalent:

(i) $F$ is measurable;

(ii) the graph of $F$, given by

$$\text{Graph}(F) = \{(x, v) \in \Omega \times \mathbb{R}^d : v \in F(x)\},$$

belongs to the product $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}$, where $\mathcal{B}$ is the Borel $\sigma$-algebra of $\mathbb{R}^d$;

(iii) for any $v \in \mathbb{R}^d$, the distance map $x \mapsto d(v, F(x))$ is measurable;
(iv) there exists a sequence of measurable selection functions $f_k : \Omega \to \mathbb{R}^d$, $k \in \mathbb{N}$, of $F$ such that for all $x \in \Omega$,

\begin{equation}
F(x) = \{f_k(x) : k \in \mathbb{N}\}.
\end{equation}

We will denote the set of all measurable selection functions for a convex-set valued function $F$, that is all measurable functions $f$ such that $f(x) \in F(x)$ a.e., by $\mathcal{S}^0(\Omega, F)$. Note that by (iv), this set is non-empty.

Measurability is preserved by taking the intersection or the convex hull of the union of a sequence of measurable closed-set valued functions. For a proof, see [1, Theorems 8.2.2, 8.2.4].

**Theorem 3.3.** Given a family $F_k : \Omega \to \mathcal{K}(\mathbb{R}^d)$, $k \in \mathbb{N}$, of measurable maps, the convex hull union map $G : \Omega \to \mathcal{K}(\mathbb{R}^d)$ defined by

\[ G(x) = \overline{\text{conv}} \left( \bigcup_{k \in \mathbb{N}} F_k(x) \right) \]

is measurable. Likewise, the intersection map $H : \Omega \to \mathcal{K}(\mathbb{R}^d)$ defined by

\[ H(x) = \bigcap_{k \in \mathbb{N}} F_k(x) \]

is measurable.

As a consequence of Theorem 3.3 we can prove that the polar of a measurable map with values in convex symmetric sets $\mathcal{K}_{cs}(\mathbb{R}^d)$ is again measurable.

**Theorem 3.4.** Given a measurable map $F : \Omega \to \mathcal{K}_{cs}(\mathbb{R}^d)$, the polar map $F^\circ : \Omega \to \mathcal{K}_{cs}(\mathbb{R}^d)$, defined by $F^\circ(x) = F(x)^\circ$, $x \in \Omega$, is also measurable.

**Proof.** By Theorem 3.2(iv), there exists measurable selection functions $f_k \in \mathcal{S}^0(\Omega, F)$, $k \in \mathbb{N}$, such that (3.1) holds. For each $k \geq 1$, define $F_k : \Omega \to \mathcal{K}_{cs}(\mathbb{R}^d)$ by

\[ F_k(x) = \{v \in \mathbb{R}^d : |\langle v, f_k(x) \rangle| \leq 1\}. \]

Since the mapping from $\Omega \times \mathbb{R}^d$ to $\mathbb{R}$ given by $(x, v) \mapsto \langle v, f_k(x) \rangle$ is measurable on the product $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}$, the graph of $F_k$ is measurable, so by Theorem 3.2(ii) $F_k$ is a measurable convex-set valued function. But by Definition 2.7,

\[ F(x)^\circ = \bigcap_{k \in \mathbb{N}} F_k(x), \]

so by Theorem 3.3, $F^\circ$ is measurable as well. \hfill \Box

When a convex-set valued map $F : \Omega \to \mathcal{K}_b(\mathbb{R}^d)$ takes values in compact sets, we have yet another equivalent definition of measurability. Recall that the Hausdorff distance between two nonempty compact sets $K_1, K_2 \subset \mathbb{R}^d$ is defined by

\begin{equation}
\begin{aligned}
d(K_1, K_2) &= \max \left\{ \sup_{v \in K_1} \inf_{w \in K_2} |v - w|, \sup_{v \in K_2} \inf_{w \in K_1} |v - w| \right\}.
\end{aligned}
\end{equation}

It is well-known that the collection of nonempty compact sets $\mathcal{K}_b(\mathbb{R}^d)$ equipped with the Hausdorff distance is a complete and separable metric space; see, for instance, [10, Theorem II.8]. Both $\mathcal{K}_{bc}(\mathbb{R}^d)$ and $\mathcal{K}_{bs}(\mathbb{R}^d)$ are closed subsets of $\mathcal{K}_b(\mathbb{R}^d)$: see [59, Theorem 1.8.5]. We can characterize the measurability of compact-set valued mappings in terms of this topology; this result is due to Castaing and Valadier [10, Theorem III.2].

**Theorem 3.5.** Given $F : \Omega \to \mathcal{K}_b(\mathbb{R}^d)$, $F$ is measurable in the sense of Definition 3.1 if and only if $F$ is measurable as a function into $\mathcal{K}_b(\mathbb{R}^d)$ with the Hausdorff topology. That is, if $U \subset \mathcal{K}_b(\mathbb{R}^d)$ is open in the Hausdorff topology, then $F^{-1}(U)$ is measurable.
We also need a characterization of the measurability of functions taking values in the set of subspaces of $\mathbb{R}^d$.

**Theorem 3.6.** Let $F : \Omega \to \mathcal{K}(\mathbb{R}^d)$ be such that $F(x)$ is a (linear) subspace of $\mathbb{R}^d$ for all $x \in \Omega$. For each $x \in \Omega$, let $P(x) \in \mathcal{M}_d$ be the matrix of the orthogonal projection of $\mathbb{R}^d$ onto $F(x)$. Then $F$ is measurable in the sense of Definition 3.1 if and only if the matrix-valued mapping $P : \Omega \to \mathcal{M}_d$ is measurable.

Theorem 3.6 is actually a special case of the theory of range functions, which were introduced and studied by Helson in the context of shift-invariant subspaces [34]. In general, a range function takes values in the set of closed subspaces of a separable Hilbert space. A range function is defined to be measurable precisely if the projection map is measurable. Theorem 3.6 can be proved using results about multiplication-invariant spaces in [8] although the assumption that $L^2(\Omega, \mu)$ is a separable Hilbert space is needed. To avoid this extra assumption, we give a short direct proof.

**Proof.** Suppose first that $F$ is measurable in the sense of Definition 3.1. Then there exists a sequence of measurable selection functions $\{f_k\}_{k \in \mathbb{N}}$ such that (3.1) holds. For each $x \in \Omega$, apply Gram-Schmidt orthogonalization to the vectors $\{f_k(x)\}_{k \in \mathbb{N}}$ to obtain a collection of orthogonal vectors $\{g_k(x)\}_{k \in \mathbb{N}}$ that span $F(x)$ and whose norms are either 0 or 1. Since each $g_k$ is a finite linear combination of the functions $f_k$, we have that each $g_k : \Omega \to \mathbb{R}^d$ is measurable. The orthogonal projection $P$ is given by $P(x)v = \sum_{k \in \mathbb{N}} \langle v, g_k(x) \rangle g_k(x)$ for $v \in \mathbb{R}^d$, and so $P$ is a measurable matrix-valued function.

Conversely, suppose the function $P : \Omega \to \mathcal{M}_d$ is measurable and takes its values in the set of orthogonal projections. Define a countable collection of measurable selection functions $f_q(x) = P(x)q$ which are indexed by $q \in \mathbb{Q}^d$. Since

$$F(x) = P(x)(\mathbb{R}^d) = \{P(x)q : q \in \mathbb{Q}^d\},$$

by Theorem 3.2 the mapping $F$ is measurable in the sense of Definition 3.1. \qed

It is well known that given a set $K \in \mathcal{K}_{bcs}(\mathbb{R}^d)$, there exists a unique ellipsoid $E$ of maximal volume such that $E \subset K \subset \sqrt{d}E$. This is referred to as the John ellipsoid [60, Theorem 3.13]. Given a convex-set valued function $F$ taking values in $\mathcal{K}_{bcs}(\mathbb{R}^d)$, we can define an associated function $G$ such that $G(x)$ is the John ellipsoid of $F(x)$. It turns out that this mapping is measurable in the sense of Definition 3.1: see Lemma 3.8 below. For our purposes, we state this result in a slightly different form. We note in passing that the measurability of the John ellipsoid has been implicitly assumed in the literature; see, for instance, [32, Proposition 1.2].

**Theorem 3.7.** Suppose that $F : \Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d)$ is measurable in the sense of Definition 3.1. Then there exists a measurable matrix-valued mapping $W : \Omega \to \mathcal{M}_d$ such that:

(i) the columns of the matrix $W(x)$ are mutually orthogonal;

(ii) for all $x \in \Omega$,

$$W(x)B \subset F(x) \subset \sqrt{d}W(x)B.$$

The proof of Theorem 3.7 requires three lemmas. First, let $\mathcal{E}$ be the set of all ellipsoids in $\mathbb{R}^d$ (possibly lower dimensional):

$$\mathcal{E} = \{P\mathbb{B} : P \in \mathcal{M}_d\} \subset \mathcal{K}_{bcs}(\mathbb{R}^d).$$

We note that by a compactness argument (see the proof of Lemma 3.8 below) we have that $\mathcal{E}$ is a closed subset of $\mathcal{K}_b(\mathbb{R}^d)$ with respect to the Hausdorff distance (3.2).

**Lemma 3.8.** Given a measurable convex-set valued function $F : \Omega \to \mathcal{K}_{abc}(\mathbb{R}^d)$, there exists a measurable mapping $G : \Omega \to \mathcal{K}_{abc}(\mathbb{R}^d)$ such that $G(x) \in \mathcal{E}$ and for all $x \in \Omega$,

$$G(x) \subset F(x) \subset \sqrt{d}G(x).$$

(3.3)
Proof. For each \( x \in \Omega \), define \( G(x) \) to be the John ellipsoid; as noted above, this is the unique ellipsoid of maximal volume that satisfies (3.3). To complete the proof we only have to show that \( G : \Omega \to K_{abc}(\mathbb{R}^d) \) is measurable in the sense of Definition 3.1.

Let \( P_1, P_2, \ldots \) be a dense collection of invertible matrices in \( \mathcal{M}_d \). In particular, for any ellipsoid \( E = P_0 \overline{B} \in \mathcal{E} \) of positive volume and any \( \epsilon > 0 \) there exists \( i \in \mathbb{N} \) such that

\[
P_i \overline{B} \subset E \subset (1 + \epsilon)P_i \overline{B}.
\]

We now define a sequence of measurable functions \( G_i : \Omega \to \mathcal{E}, i \in \mathbb{N} \), by induction. Let

\[
G_1(x) = \begin{cases} 
P_i \overline{B} & \text{if } P_i \overline{B} \subset F(x), \\ \{0\} & \text{otherwise.} \end{cases}
\]

For any invertible \( P \in \mathcal{M}_d \) we have that

\[
\{ x \in \Omega : P \overline{B} \nsubseteq F(x) \} = \{ x \in \Omega : F(x) \cap P(\mathbb{R}^d \setminus \overline{B}) \neq \emptyset \}
\]

is measurable by Definition 3.1. Therefore, \( G_1 \) is a measurable function. Suppose for some \( i \geq 1 \), we have defined measurable functions \( G_1, \ldots, G_i \). Define

\[
G_{i+1}(x) = \begin{cases} 
P_{i+1} \overline{B} & \text{if } P_{i+1} \overline{B} \subset F(x) \text{ and } m_d(P_{i+1} \overline{B}) > m_d(G_i(x)), \\ G_i(x) & \text{otherwise.} \end{cases}
\]

(Recall that \( m_d \) denotes Lebesgue measure on \( \mathbb{R}^d \).) To show that \( G_{i+1} \) is measurable, first note that the volume functional \( K \mapsto m_d(K) \) is a continuous mapping of \( K_{bc}(\mathbb{R}^d) \) to \([0, \infty)\) \([59, \text{Theorem 1.8.16}]\). Hence, since \( G_i : \Omega \to K_{bc}(\mathbb{R}^d) \) is measurable, so is \( m_d(G_i) : \Omega \to [0, \infty) \) by Theorem 3.5. But then for any open set \( U \),

\[
\{ x \in \Omega : G_{i+1}(x) \cap U \neq \emptyset \} = \big( \{ x \in \Omega : m_d(G_{i+1}(x)) \geq m_d(P_{i+1} \overline{B}) \} \cap \{ x \in \Omega : G_i(x) \cap U \neq \emptyset \} \big)
\]

\[
\cup \big( \{ x \in \Omega : m_d(G_{i+1}(x)) < m_d(P_{i+1} \overline{B}) \} \cap \{ x \in \Omega : P_{i+1} \overline{B} \subset F(x) \text{ and } P_{i+1} \overline{B} \cap U \neq \emptyset \} \big).
\]

Since each set on the right-hand side is measurable, we conclude that \( G_{i+1} \) is a measurable function.

To complete the proof, we need to show that \( G_i(x) \) converges to \( G(x) \) in the Hausdorff distance (3.2). Because then \( G \) is a measurable function with respect to the Hausdorff topology, and so by Theorem 3.5 is measurable in the sense of Definition 3.1. We will prove this by contradiction. Fix \( x \in \Omega \) and suppose to the contrary that \( G_i(x) \) does not converge to \( G(x) \). Since \( G(x) \) is the maximal ellipsoid contained in \( F(x) \), by (3.4) and the definition of the \( G_i \)'s we have that as \( i \to \infty \),

\[
m_d(G_i(x)) \to m_d(G(x)).
\]

By the Blaschke selection theorem \([59, \text{Theorem 1.8.6}]\), \( \{ E \in \mathcal{E} : E \subset F(x) \} \) is a compact subset of \( K_{bc}(\mathbb{R}^d) \). Hence, some subsequence \( G_{i_j}(x) \) converges as \( j \to \infty \) to an ellipsoid \( E' \in \mathcal{E} \), and by assumption \( E' \neq G(x) \). But we have \( m_d(E') = m_d(G(x)) \), and this contradicts the fact that the John ellipsoid \( G(x) \) is unique. Thus \( G_i(x) \to G(x) \) and our proof is complete.

Lemma 3.9. Let \( F : \Omega \to K_{bc}(\mathbb{R}^d) \) be a measurable mapping. Then there exists a measurable mapping \( v : \Omega \to \mathbb{R}^d \) such that for all \( x \in \Omega \)

\[
v(x) \in F(x) \quad \text{and} \quad |v(x)| = \sup\{|v| : v \in F(x)\}.
\]

Proof. Let \( \{ f_k \}_{k \in \mathbb{N}} \) be a sequence of measurable selection functions such that (3.1) holds. Define \( g_0 : \Omega \to [0, \infty) \) by

\[
g_0(x) = \sup\{|v| : v \in F(x)\} = \sup\{|f_k(x)| : k \in \mathbb{N}\}.
\]
Then \( g_0 \) is measurable. Now define \( F_0 : \Omega \to \mathcal{K}_b(\mathbb{R}^d) \) by
\[
F_0(x) = \{ v \in F(x) : |v| = g_0(x) \} = F(x) \cap g_0(x)S,
\]
where \( S = \{ u \in \mathbb{R}^d : |u| = 1 \} \). Then by Theorem 3.3, \( F_0 \) is measurable since \( F \) and \( g_0S \) are.

We now show that we can choose \( v(x) \) from \( F_0(x) \) in such a way that \( v(x) \) is a measurable function. We do this iteratively by choosing the vectors \( v \) that are maximal in each coordinate. For \( v = (v_1, \ldots, v_d) \in \mathbb{R}^d \), let \( P_1(v) = v_1 \) be the projection onto the first coordinate. Since \( P_1 \) is continuous, if we define \( g_1 : \Omega \to [0, \infty) \) by
\[
g_1(x) = \sup\{ v_1 : (v_1, \ldots, v_d) \in F(x) \} = \sup_{k} P_1(f_k(x)),
\]
then \( g_1 \) is measurable. Define \( F_1 : \Omega \to \mathcal{K}_b(\mathbb{R}^d) \) by
\[
F_1(x) = \{(v_1, \ldots, v_d) \in F_0(x) : v_1 = g_1(x) \} = F(x) \cap (\{g_1(x)\} \times \mathbb{R}^{d-1}).
\]

By Theorem 3.3, \( F_1 \) is measurable.

We repeat this argument: by induction, for each \( i \geq 1 \) we define \( F_{i+1} : \Omega \to \mathcal{K}_b(\mathbb{R}^d) \) such that \( F_{i+1}(x) \) consists of all points in \( F_i(x) \) that have a maximal \( i + 1 \) coordinate. Here we mean maximal in norm: in \( g_i \) we fixed \( v_1 \) to be positive in order to be explicit, but in the subsequent steps the maximal coordinate could be negative. After \( d \) steps this yields a measurable function \( F_d : \Omega \to \mathcal{K}_b(\mathbb{R}^d) \). By the maximality of each coordinate we must have that \( F_d(x) \) is a singleton: i.e., \( F_d(x) = \{v(x)\} \) for some measurable function \( v : \Omega \to \mathbb{R}^d \).

**Lemma 3.10.** Let \( G : \Omega \to \mathcal{K}_bcs(\mathbb{R}^d) \) be a measurable mapping such that \( G(x) \in \mathcal{E} \) for all \( x \in \Omega \). Then there exists a measurable mapping \( W : \Omega \to \mathcal{M}_d \) such that:

(i) the columns of the matrix \( W(x) \) are mutually orthogonal;

(ii) \( G(x) = W(x)B \) for all \( x \in \Omega \).

**Proof.** We will construct the columns \( v_1, \ldots, v_d : \Omega \to \mathbb{R}^d \) of \( W \) inductively. Let \( v_1 \) be the vector-valued function given by Lemma 3.9 corresponding to \( G \). Define the mapping \( J_1 : \Omega \to \mathcal{K}(\mathbb{R}^d) \) by \( J_1(x) = \text{span}\{v_1(x)\} \). By Theorem 3.2 it is measurable, since the collection of linear multiples of \( v_1(x) \) by rational numbers forms a countable collection of measurable selection functions. Then by Theorem 3.6, \( J_1 \) is a measurable range function: i.e., the associated projection matrix is a measurable function. Hence, so is orthogonal projection, and thus the range function \( J_1^\perp : \Omega \to \mathcal{K}(\mathbb{R}^d) \), defined as the orthogonal complement \( J_1^\perp(x) = (J_1(x))^\perp \), is measurable.

We now proceed by induction. If for some \( i \geq 1 \) we have defined measurable, vector-valued functions \( v_1, \ldots, v_i \), then we can define a mapping
\[
J_i : \Omega \to \mathcal{K}(\mathbb{R}^d), \quad J_i(x) = \text{span}\{v_1(x), \ldots, v_i(x)\}.
\]
The map \( J_i \) is measurable by Theorem 3.2 since linear combinations of the vectors \( v_1, \ldots, v_i \) with rational coefficients form a countable family of measurable selection functions. Define the measurable mapping \( G_i : \Omega \to \mathcal{K}_bcs(\mathbb{R}^d) \) by
\[
G_i(x) = G(x) \cap J_i(x)^\perp.
\]
We can now apply Lemma 3.9 to get a vector-valued function \( v_{i+1} \) which is orthogonal to \( v_1, \ldots, v_i \). For every \( x \in \Omega \) the vectors \( v_1(x), \ldots, v_d(x) \) define semi-axes of an ellipsoid; since at every step we chose \( v_i \) to be maximal, they are given in decreasing order and the ellipsoid must equal \( G(x) \). Hence, if \( W(x) \) is the \( d \times d \) matrix with columns \( v_i \), then we have \( G(x) = W(x)B \). □

**Proof of Theorem 3.7.** If the function \( F \) in our hypothesis is absorbing for all \( x \), then the desired conclusion follows from Lemmas 3.8 and 3.10. Since this need not be the case, we need to consider the
“dimension” of \( F \) at each point. More precisely, we argue as follows. Given an arbitrary measurable mapping \( F : \Omega \to \mathcal{K}_{bc}(\mathbb{R}^d) \), define the new function
\[
J(x) = \text{span } F(x) = \bigcup_{r > 0} r F(x) = \text{conv} \left( \bigcup_{r > 0, r \in \mathcal{Q}} r F(x) \right);
\]
then by Theorem 3.3, \( J \) is measurable. For \( k = 0, \ldots, d \) define the sets
\[
\Omega_k = \{ x \in \Omega : \dim J(x) = k \}.
\]
Equivalently, if we let \( P \) be the measurable projection matrix in Theorem 3.6, then \( \Omega_k \) is the set where \( P(x) \) has rank \( k \). Since the rank can be computed by taking the determinant of all the \( k \times k \) minors, it is a measurable mapping, and so \( \Omega_k \) is a measurable set. Therefore, to complete the proof it will suffice to show the conclusion for each restriction \( F|_{\Omega_k} \), \( k = 1, \ldots, d \).

Fix \( k \). By [35, Theorem 2 in Section 1.3] we can find measurable functions \( w_1, \ldots, w_k : \Omega_k \to \mathbb{R}^d \) such that \( w_1(x), \ldots, w_k(x) \) form an orthonormal basis of \( J(x) \) for \( x \in \Omega_k \). This follows from the Gram-Schmidt process as in the proof of Theorem 3.6. Denote the collection of \( s \times t \) matrices by \( M_{s \times t} \). Let \( M_k(x) \in M_{t \times k} \) be the matrix whose columns are the vectors \( w_1(x), \ldots, w_t(x) \). Then \( M_k(x) \) is an isometry of \( \mathbb{R}^k \) onto \( J(x) \) and the transpose \( M^*_k(x) \in M_{k \times d} \) is its inverse. Consequently, \( F_k : \Omega_k \to \mathcal{K}_{bc}(\mathbb{R}^k) \), defined by \( F_k(x) = M^*_k(x) F(x), x \in \Omega_k \), is a measurable convex-set valued mapping such that \( F_k(x) \) is absorbing. Therefore, we can apply Lemmas 3.8 and 3.10 to get a measurable mapping \( W_k : \Omega_k \to \mathcal{M}_d \) such that the columns of \( W_k(x) \) are mutually orthogonal and \( F_k(x) = W_k(x) \overline{B}_k \), where \( \overline{B}_k \) is the closed unit ball in \( \mathbb{R}^k \).

Finally, define \( W(x) = M_k(x) \circ W_k(x) \circ P_k \in \mathcal{M}_d \) for \( x \in \Omega_k \), where \( P_k \in \mathcal{M}_d \) is the coordinate projection of \( \mathbb{R}^d \) onto \( \mathbb{R}^k \). Then the columns of \( W(x) \) are orthogonal and
\[
W(x) \overline{B} = (M_k(x) \circ W_k(x)) \overline{B}_k = M_k(x) F_k(x) = F(x)
\]
for \( x \in \Omega_k \). This defines the required mapping \( W : \Omega_k \to \mathcal{M}_d \); combining these functions we get the desired mapping on \( \Omega \).

**Integrals of convex-set valued maps.** In this section we define the integral of convex-set valued functions using the Aumann integral. We follow the treatment given in [1, Section 8.6]. As before, the underlying measure space is \((\Omega, \mathcal{A}, \mu)\).

**Definition 3.11.** Suppose \( F : \Omega \to \mathcal{K}(\mathbb{R}^d) \) is a measurable map. Define the set of all integrable selection functions of \( F \) by
\[
S^1(\Omega, F) = \{ f \in L^1(\Omega, \mathbb{R}^d) : f \in S^0(\Omega, F) \}.
\]
The Aumann integral of \( F \) is the set of integrals of integrable selection functions of \( F \), i.e.,
\[
\int_\Omega F \, d\mu = \left\{ \int_\Omega f \, d\mu : f \in S^1(\Omega, F) \right\}.
\]

A priori a measurable map \( F \) may not have any integrable selection functions. We therefore introduce a class of maps for which this set is non-empty.

**Definition 3.12.** A measurable closed-set valued function \( F : \Omega \to \mathcal{K}(\mathbb{R}^d) \) is integrably bounded if there exists a non-negative function \( k \in L^1(\Omega, \mathbb{R}) \) such that
\[
F(x) \subset k(x) \overline{B} \quad \text{for a.e. } x \in \Omega.
\]
If \( \Omega \) is a metric space (in particular if \( \Omega = \mathbb{R}^n \)) we say \( F \) is locally integrably bounded if this holds for \( k \in L^1_{\text{loc}}(\Omega, \mathbb{R}) \).

Below, we will want to treat the integral of a vector-valued function as the integral of a convex-set valued function. We will be able to do this using the following lemma.
Lemma 3.13. Let \( f \in L^1(\Omega, \mathbb{R}^d) \). Then, the convex-set valued map
\[
F(x) = \text{conv} \{ f(x), -f(x) \}, \quad x \in \Omega,
\]
is measurable and integrably bounded. Moreover, its Aumann integral satisfies
\[
\int_{\Omega} F \, d\mu = \left\{ \int_{\Omega} kf \, d\mu : k \in L^\infty(\Omega; \mathbb{R}), ||k||_\infty \leq 1 \right\}.
\]

Proof. Let \( \{\alpha_i\}_{i \in \mathbb{N}} \) be a dense subset of the interval \([-1, 1]\). Then,
\[
F(x) = \{\alpha_i f(x) : i \in \mathbb{N}\}.
\]
Hence, by Theorem 3.2 (see also [1, Theorem 8.2.2]) \( F \) is measurable as a convex-set valued mapping. Moreover, it is clear that if \( f \in S^1(\Omega, F) \), then \( g(x) = k(x) f(x) \), where \( |k(x)| \leq 1 \). Hence, (3.7) follows from the definition of the Aumann integral. \( \square \)

When the measure \( \mu \) is non-atomic, the integral of any closed-set valued map \( F \) is convex, even when the values of \( F \) are not necessarily convex. For proof of this highly non-trivial result, see [1, Theorem 8.6.3].

Theorem 3.14. Suppose that the measure \( \mu \) is nonatomic. Given a measurable mapping \( F : \Omega \to \mathcal{K}(\mathbb{R}^d) \), let \( K = \int_{\Omega} F \, d\mu \) be the Aumann integral of \( F \). Then \( K \) is a convex, though not necessarily closed, subset of \( \mathbb{R}^d \). In addition, if \( F \) is integrably bounded, then \( K \subseteq \mathcal{K}_{bc} \).

In this paper we are primarily interested in convex-set valued mappings. In this case, the assumption that \( \mu \) is nonatomic can be dropped and we have the following result.

Theorem 3.15. Given a measurable mapping \( F : \Omega \to \mathcal{K}_{bc}(\mathbb{R}^d) \), let \( K = \int_{\Omega} F \, d\mu \) be the Aumann integral of \( F \). Then \( K \) is a convex, symmetric set in \( \mathbb{R}^d \), and so \( K \subseteq \mathcal{K}_{cs}(\mathbb{R}^d) \). In addition, if \( F \) is integrably bounded, then \( K = \overline{K} \subseteq \mathcal{K}_{bc}(\mathbb{R}^d) \).

Proof. This result follows from the corresponding properties of the integrable selection functions. Since \( F(x) \in \mathcal{K}_{bc}(\mathbb{R}^d) \) for all \( x \in \Omega \), if \( f, g \in S^1(\Omega, F) \), then \(-f \in S^1(\Omega, F) \) and for \( 0 < \lambda < 1 \), \( \lambda f + (1 - \lambda)g \in S^1(\Omega, F) \). Hence, it follows from Definition 3.11 that \( K \) is a convex, symmetric set in \( \mathbb{R}^d \).

Now suppose that \( F \) is integrably bounded by \( k \in L^1(\Omega) \). If \( f \in S^1(\Omega, F) \), then \( f(x) \in F(x) \), so \( |f(x)| \leq k(x) \) a.e. In particular,
\[
\left| \int_{\Omega} f(x) \, d\mu \right| \leq \int_{\Omega} k(x) \, dx,
\]
and so \( K \) is bounded.

Finally, to show that \( K \) is closed, first note that \( S^1(\Omega, F) \) is closed in \( L^1(\Omega) \). For if \( f \in L^1(\Omega) \) is a limit point, there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) that converges to \( f \) in \( L^1 \) and (by passing to a subsequence) pointwise almost everywhere. Since \( f_n(x) \in F(x) \) and \( F(x) \) is closed, \( f(x) \in F(x) \), so \( f \in S^1(\Omega, F) \).

Second, given any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( E \subseteq \Omega \) satisfies \( |E| < \delta \), then \( \int_E k \, d\mu < \epsilon \). Therefore, if we replace \( \Omega \) by \( E \) in the above argument, then we have that that \( S^1(\Omega, F) \) is equi-integrable. Hence, by the Dunford-Pettis theorem (see [24, Theorem IV.8.9, p. 292] or [66, Theorem III.1.12]), \( S^1(\Omega, F) \) is weakly compact in \( L^1(\Omega) \). Let \( v \) be a limit point of \( K \). Then there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( S^1(\Omega, F) \) such that \( \int_{\Omega} f_n \, d\mu \to v \) as \( n \to \infty \). By weak compactness, if we pass to a subsequence, there exists \( f \in L^1(\Omega) \) such that \( f_n \to f \) weakly; in particular, \( \int_{\Omega} f \, d\mu = v \). Moreover, by Mazur’s lemma [9, Corollary 3.8], there exists a sequence \( \{g_k\}_{k \in \mathbb{N}} \), where each \( g_k \) is a convex combination of the functions \( f_n \), that converges to \( f \) in \( L^1 \) norm. However, as we noted above, \( S^1(\Omega, F) \) is convex, and so each \( g_k \) is contained in it. Therefore, since \( S^1(\Omega, F) \) is closed, \( f \in S^1(\Omega, F) \), and so \( v \in K \). Thus \( K \) is closed, which completes our proof. \( \square \)
We now show that integrable selection functions are additive. Our proof is adapted from [36, Theorem 1.4].

**Theorem 3.16.** Suppose that $F_i : \Omega \rightarrow \mathcal{K}_{bcs}(\mathbb{R}^d)$, $i = 1, 2$, are measurable and integrably bounded. Then

$$S^1(\Omega, F_1 + F_2) = S^1(\Omega, F_1) + S^1(\Omega, F_2).$$

**Proof.** One direction is immediate: if $f_i \in S^1(\Omega, F_i)$, $i = 1, 2$, then $f_1 + f_2 \in S^1(\Omega, F_1 + F_2)$. To prove the converse, note that by Theorem 3.2, we have sequences of selection functions $\{f_k\}_{k \in \mathbb{N}} \subset S^1(\Omega, F_1)$, $\{g_j\}_{j \in \mathbb{N}} \subset S^1(\Omega, F_2)$ such that for each $x \in \Omega$,

$$F_1(x) + F_2(x) = \{f_k(x) + g_j(x) : j, k \in \mathbb{N}\}.$$

Therefore, if we fix $f \in S^1(\Omega, F_1 + F_2)$, there exist sequences of measurable functions $\{h_n\}_{n \in \mathbb{N}}$ and $\{k_n\}_{n \in \mathbb{N}}$ such that for almost every $x \in \Omega$ and $n \in \mathbb{N}$, $h_n(x) \in \{f_1(x), \ldots, f_n(x)\}$, $k_n(x) \in \{g_1(x), \ldots, g_n(x)\}$, and $h_n(x) + k_n(x) \to f(x)$ as $n \to \infty$. The construction of these functions follows the argument in [36, Lemma 1.3] with $p = 1$, which yields a sequence that converges in norm; by passing to a subsequence we get a sequence that converges pointwise almost everywhere.

We now argue as in the proof of Theorem 3.15. There we showed that $S^1(\Omega, F_i)$, $i = 1, 2$, is closed and weakly compact subset of $L^1(\Omega)$. Hence, by passing to a subsequence, there exists $h, k \in L^1(\Omega)$ such that $h_n \to h$ and $k_n \to k$ weakly in $L^1(\Omega)$ as $n \to \infty$. By Mazur’s lemma, we can replace the functions $h_n$ and $k_n$ by convex combinations of them to get sequences that converge in $L^1(\Omega)$ norm.

Since the sets $S^1(\Omega, F_i)$, $i = 1, 2$, are convex and closed, we have $h \in S^1(\Omega, F_1)$ and $k \in S^1(\Omega, F_2)$. Therefore, $f = h + k \in S^1(\Omega, F_1) + S^1(\Omega, F_2)$ and this completes our proof. □

As a consequence of Theorem 3.16 we can prove that the Aumann integral is linear and monotonic.

**Theorem 3.17.** Suppose that $F_i : \Omega \rightarrow \mathcal{K}_{bcs}(\mathbb{R}^d)$, $i = 1, 2$, are measurable and integrably bounded. Then, for any $\alpha_i \in \mathbb{R}$, $i = 1, 2$, we have

$$\int_\Omega (\alpha_1 F_1 + \alpha_2 F_2) \, d\mu = \alpha_1 \int_\Omega F_1 \, d\mu + \alpha_2 \int_\Omega F_2 \, d\mu.$$

Moreover, if $F_1(x) \subset F_2(x)$ for all $x \in \Omega$, then

$$\int_\Omega F_1 \, d\mu \subset \int_\Omega F_2 \, d\mu.$$

**Proof.** The monotonicity of the Aumann integral follows at once from Definition 3.11 and from the fact that if $F_1 \subset F_2$, then $S^1(\Omega, F_1) \subset S^1(\Omega, F_2)$.

To show that it is linear, note first that it is immediate from Definition 3.11 that

$$\int_\Omega \alpha_i F_i \, d\mu = \alpha_i \int_\Omega F_i \, d\mu.$$

Finally, by Theorem 3.16 we have that

$$\int_\Omega F_1 + F_2 \, d\mu = \left\{ \int_\Omega f_1 + f_2 \, d\mu : f_i \in S^1(\Omega, F_i), i = 1, 2 \right\} = \int_\Omega F_1 \, d\mu + \int_\Omega F_2 \, d\mu.$$ □

**Corollary 3.18.** Given a locally integrably bounded function $F : \mathbb{R}^n \to \mathcal{K}_{bcs}(\mathbb{R}^d)$ and bounded sets $A \subset B$,

$$\int_A F(x) \, dx \subset \int_B F(x) \, dx.$$

**Proof.** Since $F(x)\chi_A(x) \subset F(x)\chi_B(x)$, this follows at once from Theorem 3.17. □
Finally, we prove versions of Hölder’s inequality and Minkowski’s inequality for the Aumann integral. The proof requires one lemma.

**Lemma 3.19.** Given $F : \Omega \to K_{bc\text{s}}(\mathbb{R}^d)$ measurable,

\[(3.8) \quad \int_{\Omega} F(x) \, d\mu = \{0\}\]

if and only if $F(x) = \{0\}$ a.e.

**Proof.** Suppose that (3.8) holds. Let $f \in L^1(\Omega, F)$. Take any $v \in \mathbb{R}^d$. Then

\[ f^+ = f\chi_{\{x \in \Omega : \langle f(x), v \rangle \geq 0\}} \quad \text{and} \quad f^- = f\chi_{\{x \in \Omega : \langle f(x), v \rangle < 0\}} \]

are also selection functions since $0 \in F(x)$. By Definition 3.11 we deduce that

\[ \int_{\Omega} (f^+(x), v) \, d\mu = \int_{\Omega} (f^-(x), v) \, d\mu = 0. \]

Hence, $(f^+(x), v) = (f^-(x), v) = 0$ for a.e. $x \in \Omega$, and hence $(f(x), v) = 0$ for a.e. $x \in \Omega$. Since $v \in \mathbb{R}^d$ is arbitrary we have that all the integrable selection functions of $F$ are trivial. If $i$ is an arbitrary selection of $F$, then there exists a strictly positive function $k$ on $\Omega$ such that $kf \in L^1(\Omega, F)$. By the previous argument $k(x)f(x) = 0$ a.e. Hence, Theorem 3.2 yields that $F(x) = \{0\}$ a.e. The converse implication is trivial. \hfill \Box

**Proposition 3.20.** Let $\rho$ be a norm on $\mathbb{R}^d$ and fix $1 < p < \infty$. Suppose $H : \Omega \to K_{bc\text{s}}(\mathbb{R}^d)$ and $f, g : \Omega \to [0, \infty)$ are measurable, and $f^pH$ and $g^pH$ are integrably bounded. Then

\[ \rho \left( \int_{\Omega} f(x)g(x)H(x) \, d\mu \right) \leq \rho \left( \int_{\Omega} f(x)^pH(x) \, d\mu \right)^\frac{1}{p} \rho \left( \int_{\Omega} g(x)^pH(x) \, d\mu \right)^\frac{1}{p}. \]

**Proof.** The proof is an adaptation of the standard proof of Hölder’s inequality for scalar functions. Since $f^pH$ and $g^pH$ are integrably bounded, the integrals on the righthand side are finite. If either is equal to 0, then by Lemma 3.19, either $fH = \{0\}$ or $gH = \{0\}$ a.e., so the lefthand side is 0 as well. Therefore, since the desired inequality is homogeneous, we may assume without loss of generality that

\[ \rho \left( \int_{\Omega} f(x)^pH(x) \, d\mu \right) = \rho \left( \int_{\Omega} g(x)^pH(x) \, d\mu \right) = 1. \]

Then by Young’s inequality and Theorem 3.17,

\[ \rho \left( \int_{\Omega} f(x)g(x)H(x) \, d\mu \right) \leq \rho \left( \int_{\Omega} \frac{1}{p} f(x)^pH(x) \, d\mu + \int_{\Omega} \frac{1}{p} g(x)^pH(x) \, d\mu \right) \leq \frac{1}{p} \rho \left( \int_{\Omega} f(x)^pH(x) \, d\mu \right) + \frac{1}{p} \rho \left( \int_{\Omega} g(x)^pH(x) \, d\mu \right) = 1. \] \hfill \Box

**Proposition 3.21.** Let $\rho$ be a norm on $\mathbb{R}^d$ and fix $1 < p < \infty$. Suppose $H : \Omega \to K_{bc\text{s}}(\mathbb{R}^d)$ and $f, g : \Omega \to [0, \infty)$ are measurable, and $f^pH$ and $g^pH$ are integrably bounded. Then

\[ \rho \left( \int_{\Omega} (f(x) + g(x))^pH(x) \, d\mu \right)^\frac{1}{p} \leq \rho \left( \int_{\Omega} f(x)^pH(x) \, d\mu \right)^\frac{1}{p} + \rho \left( \int_{\Omega} g(x)^pH(x) \, d\mu \right)^\frac{1}{p}. \]

**Proof.** The proof is again an adaptation of the standard proof of Minkowski’s inequality for scalar functions. First note that since

\[ [f + g]^pH \subset 2^{p-1}(f^p + g^p)H = 2^{p-1}f^pH + 2^{p-1}g^pH, \]
\[ (f + g)^p H \] is integrably bounded, so the left hand side of the inequality is finite. We may also assume without generality that it is positive since otherwise there is nothing to prove. But then, by Proposition 3.17,

\[
\rho \left( \int_\Omega [f(x) + g(x)]^p H(x) \, d\mu \right) \\
= \rho \left( \int_\Omega f(x)[f(x) + g(x)]^{p-1} H(x) \, d\mu + \int_\Omega g(x)[f(x) + g(x)]^{p-1} H(x) \, d\mu \right) \\
\leq \rho \left( \int_\Omega f(x)[f(x) + g(x)]^{p-1} H(x) \, d\mu \right) + \rho \left( \int_\Omega g(x)[f(x) + g(x)]^{p-1} H(x) \, d\mu \right) \\
\leq \rho \left( \int_\Omega f(x)^p H(x) \, d\mu \right)^{\frac{1}{p'}} \rho \left( \int_\Omega [f(x) + g(x)]^p H(x) \, d\mu \right)^{\frac{1}{p'}} + \rho \left( \int_\Omega g(x)^p H(x) \, d\mu \right)^{\frac{1}{p'}} \rho \left( \int_\Omega [f(x) + g(x)]^p H(x) \, d\mu \right)^{\frac{1}{p'}}. 
\]

The last step follows from Proposition 3.20 since \((p - 1)p' = p\). The desired inequality now follows immediately. \(\square\)

4. Seminorm functions

In this section we introduce seminorm functions, which we will use below to define \(L^p\) spaces of convex-set valued functions. We will show an equivalence between the Aumann integral of such a function and the seminorm associated with the convex bodies. We begin with a definition.

**Definition 4.1.** A seminorm function \(\rho\) on \(\Omega\) is a mapping \(\rho : \Omega \times \mathbb{R}^d \to [0, \infty)\) such that:

(i) \(x \mapsto \rho_x(v) = \rho(x, v)\) is a measurable function for any \(v \in \mathbb{R}^d\),

(ii) for all \(x \in \Omega\), \(\rho_x(\cdot)\) is a seminorm on \(\mathbb{R}^d\).

Our first result shows that there is a one-to-one correspondence between seminorm functions and measurable convex-set valued maps \(\Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d)\). For a variant of Theorem 4.2 for bounded convex-set valued mappings into separable Banach space, see [1, Theorem 8.2.14], and a version for compact convex-set valued mappings into a locally convex, metrizable, separable space, see [10, Theorem III.15].

**Theorem 4.2.** Suppose that \(\rho : \Omega \times \mathbb{R}^d \to [0, \infty)\) is a seminorm function. Then the convex-set valued mapping \(F : \Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d)\) defined for each \(x \in \Omega\) by

\[
F(x) = \{ v \in \mathbb{R}^d : \rho_x(v) \leq 1 \}
\]

is measurable. Conversely, given a measurable mapping \(F : \Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d)\), define a function \(\rho : \Omega \times \mathbb{R}^d \to [0, \infty)\) by

\[
\rho_x(v) = p_{F(x)\circ}(v) \quad (x, v) \in \Omega \times \mathbb{R}^d.
\]

Then \(\rho\) is a seminorm function. Moreover, the correspondence between seminorm functions \(\rho\) and convex-set valued mappings \(F\) is one-to-one.

**Proof.** Suppose first that \(\rho : \Omega \times \mathbb{R}^d \to [0, \infty)\) is a seminorm function. To show that \(F\) is measurable, we will first prove that \(\rho\) satisfies a stronger version of (i) in Definition 4.1:

(i-a) \(\rho : \Omega \times \mathbb{R}^d \to [0, \infty)\) is measurable with respect to the product \(\sigma\)-algebra \(\mathcal{A} \otimes \mathcal{B}\), where \(\mathcal{B}\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}^d\).
We could derive this from the fact that $\rho$ is a Carathéodory map, using [1, Lemma 8.2.6]. Instead, however, we will give a direct proof of (i-a). Let $\mathcal{D} = \{v_i\}_{i \in \mathbb{N}}$ be a countable dense subset of $\mathbb{R}^d$. For any functional $l \in (\mathbb{R}^d)^*$, the set

$$A_l = \{x \in \Omega : |l(v)| \leq \rho_x(v) \text{ for all } v \in \mathbb{R}^d\}$$

is a measurable subset of $\Omega$. To see this, note that $A_l$ can be written as a countable intersection of measurable sets in $\mathcal{A}$:

$$A_l = \bigcap_{i=1}^{\infty} \{x \in \Omega : |l(v_i)| \leq \rho_x(v_i)\}.$$

Now let $\mathcal{D}' = \{l_i\}_{i \in \mathbb{N}}$ be a countable dense subset of functionals on $\mathbb{R}^d$ given by $l_i(v) = \langle v, v_i \rangle$, $v \in \mathbb{R}^d$. For any norm $p$ on $\mathbb{R}^d$ we claim that

$$p(v) = \sup\{\|l_i(v)\| : i \in \mathbb{N}\} \text{ is such that } |l_i(w)| \leq p(w) \text{ for all } w \in \mathbb{R}^d.$$

To see this, fix $v \in \mathbb{R}^d$ and let $E = \{\alpha v : \alpha \in \mathbb{R}\}$ be the subspace generated by $v$. Define the linear functional $\lambda$ on $E$ by $\lambda(\alpha v) = \alpha p(v)$. Then by the Hahn-Banach theorem, $\lambda$ extends to an element of $(\mathbb{R}^d)^*$ such that $|\lambda(w)| \leq p(w)$ for all $w \in \mathbb{R}^d$. The identity (4.4) now follows from the density of $\mathcal{D}'$.

To prove (i-a), assume for the moment that for all $x \in \Omega$, $\rho_x(\cdot)$ is a norm on $\mathbb{R}^d$. Then, if we combine (4.3) and (4.4) we get that

$$\rho_x(v) = \sup_{i \in \mathbb{N}} |l_i(v)\chi_{A_l}(x) \quad \text{for all } (x, v) \in \Omega \times \mathbb{R}^d.$$

Hence, $\rho$ is measurable with respect to the $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}$. Finally, if $\rho$ is an arbitrary seminorm function, define the sequence of norm functions

$$\rho^t(x, v) = \rho_x(v) + \frac{1}{t}|v| \quad (x, v) \in \Omega \times \mathbb{R}^d.$$

Each $\rho^t$ is measurable with respect to the $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}$, and so their limit $\rho$ is measurable as well. This proves (i-a).

We can now prove that $F : \Omega \rightarrow \mathcal{K}_{bcs}(\mathbb{R}^d)$ defined by (4.1) is measurable. By (i-a)

$$\text{Graph}(F^o) = \rho^{-1}([0, 1]) = \{(x, v) \in \Omega \times \mathbb{R}^d : \rho_x(v) \leq 1\}$$

is a measurable set in $\mathcal{A} \otimes \mathcal{B}$. By Theorem 3.2, $F^o$ is measurable. Consequently, $F = (F^o)^o$ is a measurable convex-set valued mapping by Theorem 2.8.

The converse is much easier to prove. Suppose $F : \Omega \rightarrow \mathcal{K}_{bcs}(\mathbb{R}^d)$ is measurable and define $\rho$ by (4.2). By Theorem 3.4, $F^o : \Omega \rightarrow \mathcal{K}_{cs}(\mathbb{R}^d)$ is measurable. Hence, by (4.6), $\rho^{-1}([0, t])$ is a measurable subset in $\mathcal{A} \otimes \mathcal{B}$ for $t = 1$. By scaling, the same is true for any $t > 0$. Since the $\sigma$-algebra of open sets in $[0, \infty)$ is generated by sets of the form $[0, t]$, by a standard measure theory argument $\rho^{-1}(U)$ is measurable for any open set $U \subset [0, \infty)$. Thus, $\rho$ is measurable in the sense of (i-a) and so $\rho$ is a seminorm function. Finally, the one-to-one correspondence is a consequence of Theorems 2.10 and 3.4.

As a corollary of Theorem 3.4 and Theorem 4.2 we have the following.

**Corollary 4.3.** If $\rho : \Omega \times \mathbb{R}^d \rightarrow [0, \infty)$ is a norm function, then $\rho^* : \Omega \times \mathbb{R}^d \rightarrow [0, \infty)$, defined by $\rho^*_x(v) = (\rho_x)^*(v)$, is a measurable norm function.

The following lemma, whose proof makes use of seminorm functions, will be used below.

**Lemma 4.4.** Every measurable mapping $F : \Omega \rightarrow \mathcal{K}_{bcs}(\mathbb{R}^d)$ is the pointwise limit of simple measurable mappings with respect to the Hausdorff distance on $\mathcal{K}_{bcs}(\mathbb{R}^d)$.
Proof. Suppose first that $F : \Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d)$. Then the corresponding seminorm function $\rho : \Omega \times \mathbb{R}^d \to [0, \infty)$ from Theorem 4.2 is actually a norm function. With the same notation as in the proof of Theorem 4.2, for each $n \in \mathbb{N}$ we define the seminorm function

$$
\rho^n(x, v) = \sup_{1 \leq i \leq n} |l_i(v)|\chi_{A_i}(x) \quad \text{for all } (x, v) \in \Omega \times \mathbb{R}^d.
$$

Let $F_n : \Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d)$ be the corresponding convex-valued function. Clearly, $F_n$ is a simple measurable function. Since $\rho^n(x, v) \not\to \rho(x)$ as $n \to \infty$ for all $(x, v) \in \Omega \times \mathbb{R}^d$, we have that

$$
F_1(x) \subset F_2(x) \subset \cdots \quad \text{and} \quad F(x) = \bigcup_{n \in \mathbb{N}} F_n(x).
$$

For any $x \in \Omega$, $F_n(x) \in \mathcal{K}_{bcs}(\mathbb{R}^d)$ for sufficiently large $n$. By the characterization of the convergence of convex bodies in [59, Theorem 1.8.7], we have that $F_n(x) \to F(x)$ as $n \to \infty$ with respect to the Hausdorff distance in $\mathcal{K}_{bcs}(\mathbb{R}^d)$.

Finally, let $F : \Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d)$ be any measurable function. Define the measurable functions $G_n : \Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d)$, $n \in \mathbb{N}$, by $G_n(x) = F_n(x) + \frac{1}{n}\mathbf{B}$. Since each $G_n$ is a pointwise limit of simple measurable mappings, a Cantor diagonalization argument shows that so is $F$. \hfill $\Box$

The next result extends the correspondence between convex-set valued mappings and seminorm functions in Theorem 4.2 to their respective integrals.

**Theorem 4.5.** Let $F : \Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d)$ be a measurable mapping, and let $\rho$ be the corresponding seminorm function given by (4.2). Then $F$ is integrably bounded if and only if for all $v \in \mathbb{R}^d$,

$$
(4.7) \quad p(v) := \int_{\Omega} \rho_x(v) \, d\mu(x) < \infty.
$$

In this case $p$ is a seminorm which coincides with the Minkowski functional of the polar set of $\int_{\Omega} F \, d\mu$. In other words,

$$
(4.8) \quad \int_{\Omega} F \, d\mu = \left\{ v \in \mathbb{R}^d : \int_{\Omega} \rho_x(v) \, d\mu(x) \leq 1 \right\}^\circ.
$$

Proof. We first consider the special case when $F : \Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d)$ is a simple mapping: i.e., $F$ takes only finitely many values $K_1, \ldots, K_m \in \mathcal{K}_{bcs}(\mathbb{R}^d)$. Hence, it can be written in the form

$$
F(x) = \sum_{i=1}^m \chi_{A_i}(x)K_i \quad x \in \Omega,
$$

where $A_1, \ldots, A_m$ are disjoint measurable sets such that $\bigcup_{i=1}^m A_i = \Omega$. The corresponding seminorm function $\rho$ also takes on finitely many values and satisfies

$$
\rho_x(v) = \sum_{i=1}^m \chi_{A_i}(x)p_{(K_i)^\circ}(v), \quad (x, v) \in \Omega \times \mathbb{R}^d.
$$

The mapping $F$ is integrably bounded if and only if $\mu(A_i) < \infty$ for any $i$ such that $K_i \neq \{0\}$. In this case the Aumann integral of $F$ equals

$$
K = \int_{\Omega} F \, d\mu = \sum_{i=1}^m \mu(A_i)K_i,
$$

where we use the convention that $\mu(A_i)K_i = \{0\}$ if $\mu(A_i) = \infty$ and $K_i = \{0\}$. Hence, by Theorem 2.11(a)(b) the seminorm $p$ given by (4.7) satisfies

$$
p = \sum_{i=1}^m \mu(A_i)p_{(K_i)^\circ} = p_{K^\circ}.
$$
Thus, (4.7) and (4.8) hold exactly when $F$ is integrably bounded. This proves Theorem 4.5 for simple mappings $F : \Omega \to K_{bcs}(\mathbb{R}^d)$.

Now fix a general measurable mapping $F : \Omega \to K_{bcs}(\mathbb{R}^d)$ and let $\rho$ be the associated seminorm function. Suppose first that $F$ is integrably bounded. We need to show that for all $v \in \mathbb{R}^d$, $x \mapsto \rho_x(v)$ is in $L^1(\Omega)$. Since $F$ is integrably bounded, there exists $k \in L^1$ such that for all $x$, $F(x) \subset k(x)B$. But then by Lemma 2.13,

$$\rho_x(v) = p_{F(x)}(v) = h_{F(x)}(v) = \sup_{w \in F(x)} \langle v, w \rangle \leq \sup_{w \in k(x)B} \langle v, w \rangle = k(x)|v|.$$  \hfill (4.9)

It is immediate that $x \mapsto \rho_x(v)$ is in $L^1$.

Conversely, suppose that (4.7) holds for all $v \in \mathbb{R}^d$. Define $k : \Omega \to [0, \infty)$ by

$$k(x) := \sup_{v \in \mathbb{R}^d} \rho_x(v), \quad x \in \Omega.$$  

Then, since $v$ is a convex combination of the standard basis vectors $\pm e_i$, $i = 1, \ldots, d$, by the triangle inequality

$$\int_\Omega k(x)d\mu(x) \leq \int_\Omega \sum_{i=1}^d \rho_x(e_i)d\mu(x) = \sum_{i=1}^d p(e_i) < \infty.$$  

Thus, $k \in L^1(\Omega)$. Furthermore, if we let $\rho_k$ be the seminorm function defined by $x \mapsto k(x)|\cdot|$, then $\rho_x(v) \leq \rho_k(v)$, and arguing as we did in (4.9) we conclude that $F$ is integrably bounded.

We now prove (4.8). We will first prove the special case where $F(x) \subset \mathbb{R}^d$ is absorbing for all $x \in \mathbb{R}^d$. If we argue as we did in the proof of Lemma 4.4, then we have that there exists a sequence of simple, convex-set value mappings $F_n : \Omega \to K_{bcs}(\mathbb{R}^d)$ such that $F_n(x) \subset F_{n+1}(x)$ for all $n \in \mathbb{N}$ and

$$F(x) = \bigcup_{n \in \mathbb{N}} F_n(x).$$

We now apply the Lebesgue dominated convergence theorem for convex-set valued mappings [1, Theorem 8.6.7] to the sequence $\{F_n\}_{n \in \mathbb{N}}$ to get that

$$K = \int_\Omega Fd\mu = \lim_{n \to \infty} \int_\Omega F_n d\mu = \bigcup_{n \in \mathbb{N}} \int_\Omega F_n d\mu = \text{conv} \left( \bigcup_{n \in \mathbb{N}} K_n \right),$$

where $K_n = \int_\Omega F_n d\mu$. Here we interpret the limit in the middle term as the Kuratowski limit of closed sets [1, Section 1.1]. Similarly, if we let $\rho^n$ be the seminorm associated with $F_n$ (by Theorem 4.2), and if we apply the monotone convergence theorem to the sequence $\{\rho^n\}_{n \in \mathbb{N}}$, we get that for all $v \in \mathbb{R}^d$,

$$p(v) = \int_\Omega \rho_x(v) d\mu(x) = \lim_{n \to \infty} \int_\Omega \rho^n_x(v) d\mu(x) = \sup_{n \in \mathbb{N}} p_n(v),$$

where $p_n(v) = \int_\Omega \rho^n_x(v)$. As we proved above for simple functions, for each $n \in \mathbb{N}$,

$$K_n = \left\{ v \in \mathbb{R}^d : p_n(v) \leq 1 \right\}^\circ.$$  \hfill (4.12)

Therefore, by Theorems 2.10 and 2.11(c),

$$p(v) = \sup_{n \in \mathbb{N}} p_n(v) = \sup_{n \in \mathbb{N}} p_{K_n^\circ}(v) = p_{K^\circ}(v).$$

It follows at once that (4.8) holds.

Finally, we consider the general case where $F : \Omega \to K_{bcs}(\mathbb{R}^d)$ is an arbitrary integrably bounded mapping. For each $j \in \mathbb{N}$, define a new convex-set valued mapping $F_j = F + \frac{k}{j}B$ and its corresponding
semimorm function $\rho^j$. Then $F_j$ is integrably bounded and absorbing, and so $\rho^j_x(\cdot)$ is a norm for all $x \in \Omega$. Therefore, by the previous case,

$$K_j = \int_\Omega F_j \, d\mu = \left\{ v \in \mathbb{R}^d : \int_\Omega \rho^j_x(v) \, d\mu(x) \leq 1 \right\}^\circ.$$  

The convex sets $K_j$ form a nested, decreasing sequence, so again by the Lebesgue dominated convergence theorem for convex-set valued mappings \cite[Theorem 8.6.7]{1} we have that

$$K = \int_\Omega F \, d\mu = \lim_{j \to \infty} \int_\Omega F_j \, d\mu = \bigcap_{j \in \mathbb{N}} \int_\Omega F_j \, d\mu = \bigcap_{j \in \mathbb{N}} K_j,$$

where the limit is the Kuratowski limit. By the dominated convergence theorem applied to the sequence $\rho^j$, we have that for all $v \in \mathbb{R}^d$,

$$p(v) = \lim_{j \to \infty} \int_\Omega \rho^j_x(v) \, d\mu(x) = \inf_{j \in \mathbb{N}} \int_\Omega \rho^j_x(v) \, d\mu(x) = \inf_{j \in \mathbb{N}} p(K_j)^c(v).$$

Therefore, by Theorems \ref{thm:2.10} and \ref{thm:2.11}(d), the identity \eqref{eq:4.8} follows at once.

**L_p spaces of convex-set valued functions.** In this section we define a natural generalization of the space $L^p(\Omega, \rho)$ of vector-valued functions $f : \Omega \to \mathbb{R}^d$ equipped with the norm

$$\|f\|_{L^p(\Omega, \rho)} = \left( \int_\Omega \rho_x(f(x))^p \, d\mu(x) \right)^{\frac{1}{p}} < \infty.$$  

Recall our standing assumption that $(\Omega, \mathcal{A}, \mu)$ is a positive, $\sigma$-finite, and complete measure space. Given a fixed seminorm function $\rho : \Omega \times \mathbb{R}^d \to [0, \infty)$ we will define the space $L^p_K(\Omega, \rho)$ of convex-set valued mappings $F : \Omega \to K_{bcs}(\mathbb{R}^d)$. To do so, we first prove a basic measurability lemma. For any seminorm $p$ on $\mathbb{R}^d$ and $K \in K_{bcs}(\mathbb{R}^d)$, define $p(K) = \sup\{p(v) : v \in K\}$.

**Lemma 4.6.** Let $\rho : \Omega \times \mathbb{R}^d \to [0, \infty)$ be a seminorm function and let $F : \Omega \to K_{bcs}(\mathbb{R}^d)$ be a measurable convex-set valued mapping. Then

$$x \mapsto \rho_x(F(x)) = \sup\{\rho_x(v) : v \in F(x)\}$$

is a measurable function from $\Omega$ to $[0, \infty)$.

**Proof.** If $f = \sum v_i \chi_{A_i}$ is a simple, vector-valued function, then the map $x \mapsto \rho_x(f(x)) = \sum_i \rho_x(v_i) \chi_{A_i}(x)$, is measurable by the definition of seminorm functions. By Theorem \ref{thm:3.2} there exists a sequence of measurable selection functions $\{f_k\}_{k \in \mathbb{N}}$ of $F$ such that \eqref{eq:3.1} holds. Since for each $k \geq 1$, the function $f_k : \Omega \to \mathbb{R}^d$ is a pointwise limit of simple measurable functions, so by the above observation we have that

$$x \mapsto \rho_x(F(x)) = \sup_{k \in \mathbb{N}} \rho_x(f_k(x))$$

is measurable. \hfill \square

**Definition 4.7.** Suppose that $\rho$ is a seminorm function on $\Omega$. For each $p$, $0 < p < \infty$, define the Lebesgue space of convex-set valued mappings $L^p_K(\Omega, \rho)$ to be the set of measurable mappings $F : \Omega \to K_{bcs}(\mathbb{R}^d)$ such that

$$\|F\|_{L^p_K(\Omega, \rho)} = \|F\|_p = \left( \int_\Omega \rho_x(F(x))^p \, d\mu(x) \right)^{\frac{1}{p}} < \infty.$$  

When $p = \infty$, define $L^\infty(\Omega, \rho)$ to be the set of all such $F$ that satisfy

$$\|F\|_{L^\infty_K(\Omega, \rho)} = \|F\|_\infty = \text{ess sup}_{x \in \Omega} \rho_x(F(x)) < \infty.$$  

A straightforward argument shows that $\|\cdot\|_p$, $1 \leq p \leq \infty$, satisfies the usual properties of a seminorm:
We claim that
\begin{equation}
(4.14)
\end{equation}
However, unlike its classical vector-valued analog, \(L^K_p(\Omega, \rho)\) is not a vector space because \(K_{becs}(\mathbb{R}^d)\) equipped with the Minkowski addition is only a semigroup: the additive inverse does not exist. Nevertheless, we have that \(L^K_p(\Omega, \rho)\) is a complete metric space.

Recall that as we noted above, the set of nonempty, compact, convex sets equipped with the Hausdorff distance \(d\) is a complete metric space. Given a norm \(\rho_x\) on \(\mathbb{R}^d\) and two compact sets \(K_1, K_2 \subset \mathbb{R}^d\), define the corresponding Hausdorff distance function
\begin{equation}
(4.15)
\end{equation}
If in \(4.14\) the sets \(K_1\) and \(K_2\) are replaced by countable dense subsets, this function is measurable. By Lemma 4.4, any measurable convex-set valued mapping \(F : \Omega \to K_{becs}(\mathbb{R}^d)\) is a pointwise limit of simple measurable mappings with respect to the Hausdorff topology on \(K_{becs}(\mathbb{R}^d)\). Hence, given any \(F, G \in L^K_p(\Omega, \rho)\), we have that \(x \mapsto d_{H,x}(F(x), G(x))\) is measurable, so we can define the distance function
\begin{equation}
(4.16)
\end{equation}
Moreover, we have that \(d_p\) is a metric and we have the following analogue of the classical result for vector-valued \(L^K_p(\Omega, \rho)\) spaces.

\section*{Theorem 4.8} Given a norm function \(\rho : \Omega \times [0, \infty) \to \mathbb{R}^d\), the space \(L^K_p(\Omega, \rho)\), \(1 \leq p < \infty\), equipped with \(d_p\) is a complete metric space (after identifying functions that are equal to \(\{0\}\) a.e.). In addition, this metric is invariant and homogeneous: that is, for \(F, G, H \in L^K_p(\Omega, \rho)\) and \(\alpha \in \mathbb{R}\),
\begin{equation}
(4.17)
\end{equation}
Proof. The proof that \(d_p\) is a metric is straightforward: it follows from the triangle inequality for the Hausdorff distance \(d_{H,x}\) and Minkowski’s inequality on the scalar-valued spaces \(L^p(\Omega)\). Properties \(4.16\) and \(4.17\) then follow immediately from the analogous properties for the Hausdorff distance \(d_{H,x}\). It remains to prove that \(L^K_p(\Omega, \rho)\) is complete. We will do this in the case \(1 \leq p < \infty\) by adapting the proof of the classical Riesz-Fischer theorem. The case \(p = \infty\) is much easier: the proof is similar to that of the completeness of \(L^\infty(\Omega)\) and we leave the details to the reader.

Let \(\{F_n\}_{n \in \mathbb{N}}\) be a Cauchy-Fischer sequence in \(L^K_p(\Omega, \rho)\). Then there exists a strictly increasing sequence \(\{n_i\}_{i \in \mathbb{N}}\) such that for \(i \geq 1\), \(d_p(F_{n_{i+1}}, F_{n_i}) < 2^{-i}\). Let \(B_x = \{v \in \mathbb{R}^d : \rho_x(v) \leq 1\}\). For each \(k \in \mathbb{N}\), define \(g_k : \Omega \to [0, \infty)\) by
\begin{equation}
(4.18)
\end{equation}
and define \(g : \Omega \to [0, \infty]\) by
\begin{equation}
(4.19)
\end{equation}
We claim that \(g(x) < \infty\) for a.e. \(x \in \Omega\). To see this, note that by Minkowski’s inequality,
\[ \|g_k\|_p = \left( \int_{\Omega} \left( \sum_{i=1}^{k} d_{H,x}(F_{n_{i+1}}(x), F_{n_i}(x)) \right)^p d\mu(x) \right)^{\frac{1}{p}} \]

\[ \leq \sum_{i=1}^{k} d_p(F_{n_{i+1}}, F_{n_i}) \leq \sum_{i=1}^{k} 2^{-i} < 1. \]

Then by Fatou’s lemma we have that
\[ \|g\|_p^p \leq \liminf_{k \to \infty} \|g_k\|_p^p \leq 1. \]

For each \( k \in \mathbb{N} \), define \( h_k : \Omega \to [0, \infty] \) by
\[ h_k(x) = \sum_{i=k}^{\infty} d_{H,x}(F_{n_{i+1}}(x), F_{n_i}(x)), \quad x \in \Omega. \]

For any \( i, j \geq k \), the triangle inequality implies that
\[ d_{H,x}(F_{n_i}(x), F_{n_j}(x)) \leq h_k(x) \leq g(x). \]

Since \( g(x) < \infty \) for a.e. \( x \in \Omega \), we have \( h_k(x) \to 0 \) as \( k \to \infty \). Hence, for a.e. \( x \in \Omega \), the sequence \( \{F_{n_i}(x)\}_{i \in \mathbb{N}} \) is Cauchy in \( \mathcal{K}_{bcs}(\mathbb{R}^d) \) with respect to the Hausdorff distance \( d_{H,x} \) given by (4.14). Since \( \mathcal{K}_{bcs}(\mathbb{R}^d) \) is a closed subset of \( \mathcal{K}_{b}(\mathbb{R}^d) \) in the Hausdorff topology, the sequence \( \{F_{n_i}(x)\}_{i \in \mathbb{N}} \) converges to some set \( F(x) \in \mathcal{K}_{bcs}(\mathbb{R}^d) \) for a.e. \( x \in \Omega \). Since \( F : \Omega \to \mathcal{K}_{bcs}(\mathbb{R}^d) \) is the pointwise a.e. limit of measurable functions \( F_n, i \in \mathbb{N}, F \) is measurable as well. Finally, we have that \( F \) is the limit of \( \{F_n\}_{n \in \mathbb{N}} \) in \( L^p_K \). Since \( F_{n_{k}}(x) \to F(x) \) in \( d_{H,x} \) distance, by the triangle inequality,
\[ d_{H,x}(F(x), F_{n_k}(x)) \leq \lim_{i \to \infty} d_{H,x}(F_{n_i}(x), F_{n_k}(x)) \leq h_k(x). \]

Therefore, as \( k \to \infty \), by the Lebesgue dominated convergence theorem we have that
\[ \lim_{k \to \infty} d_p(F, F_{n_k})^p = \lim_{k \to \infty} \int_{\Omega} d_{H,x}(F(x), F_{n_k}(x))^p d\mu(x) \]
\[ \leq \lim_{k \to \infty} \int_{\Omega} \left( \sum_{i=k}^{\infty} d_{H,x}(F_{n_{i+1}}(x), F_{n_i}(x)) \right)^p d\mu(x) = 0. \]

Finally, since \( \{F_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( L^p_K(\Omega, \rho) \), by a standard argument we have that \( d_p(F, F_n) \to 0 \) as \( n \to \infty \). This completes the proof. \( \square \)

**Remark 4.9.** While \( L^p_K(\Omega, \rho) \) is not a Banach space, one can show that it is a convex cone of some Banach space. By the Rådström embedding theorem [53, Theorem 1], the collection of all nonempty, compact convex subsets of a normed, real vector space (endowed with the Hausdorff distance) can be isometrically embedded as a convex cone in a normed real vector-space. Thus, for a.e. \( x \), the Hausdorff distance \( d_{H,x} \) comes from a certain norm \( \|\cdot\|_x \) on a vector space \( V \), which consists of equivalence classes of pairs of compact convex sets under the relation:
\[ (K_1, K_2) \sim (K_3, K_4) \quad \text{if and only if} \quad K_1 + K_4 = K_2 + K_3. \]

The space \( V \) with norm \( \|\cdot\|_x \) is a complete separable normed space; all the norms \( \|\cdot\|_x \) are mutually equivalent since they come from equivalent norms \( \rho_x \) on a finite dimensional space \( \mathbb{R}^d \). Thus, \( L^p_K(\Omega, \rho) \) can be identified with a weighted vector-valued space \( L^p(\Omega, \|\cdot\|_x) \) consisting of all measurable functions \( f : \Omega \to V \) such that
\[ \|f\|_p = \left( \int_{\Omega} \|f(x)\|_x^p d\mu(x) \right)^{\frac{1}{p}} < \infty \].

Since we will not use this fact elsewhere, we leave the details to the interested reader.
Matrix weights and seminorms. Let $A : \Omega \to \mathcal{M}_d$ be a measurable matrix mapping. Then we can define a seminorm function $\rho_A$ by

$$\rho_A(x, v) = |A(x)v|, \quad x \in \Omega, \ v \in \mathbb{R}^d.$$ 

Clearly, for each $x$, $\rho_A(x, \cdot)$ is a seminorm, and since $A$ is measurable, the map $x \mapsto \rho_A(x, v)$ is measurable for all $v$. Moreover, in defining seminorms it suffices to restrict ourselves to measurable, positive semidefinite matrix mappings $W : \Omega \to \mathcal{S}_d$.

**Theorem 4.10.** Given a measurable matrix mapping $A : \Omega \to \mathcal{M}_d$, there exists a measurable matrix mapping $W : \Omega \to \mathcal{S}_d$ such that for all $x \in \Omega$ and $v \in \mathbb{R}^d$, $\rho_A(x, v) = \rho_W(x, v)$. If $A$ is invertible for a.e. $x \in \Omega$, then $W$ is positive definite almost everywhere.

**Proof.** Given a matrix $A \in \mathcal{M}_d$, it is well-known that if we form the polar decomposition of $A$ we can write $A = UW$, where $U$ is orthogonal and $W \in \mathcal{S}_d$. Further, if $A$ is invertible, then $W$ is positive definite. But then, for any $v \in \mathbb{R}^d$,

$$|Av| = |UWv| = |Wv|.$$ 

Therefore, it suffices to show that we can take $W$ to be a measurable function. We can define $W$ by $W = (A^t A)^{1/2}$, so we need to show that we can measurably define the square root of a positive semidefinite matrix.

Let $V : \Omega \to \mathcal{S}_d$ be a measurable mapping, then by [55, Lemma 2.3.5] there exists a measurable matrix mapping $U$ such that $U(x)$ is orthogonal and $U^t(x)V(x)U(x)$ is diagonal. Denote this matrix by $D = \text{diag}(\lambda_1, \ldots, \lambda_d)$, and define its square root to be the diagonal matrix $D^{1/2} = \text{diag}(\lambda_1^{1/2}, \ldots, \lambda_d^{1/2})$. If we now define $V^{1/2} = UD^{1/2}U^t$, then $V^{1/2}$ is measurable and $V^{1/2}V^{1/2} = V$. \(\square\)

Conversely, given a norm function $\rho_x$, we can associate to it a matrix norm $\rho_W$. This result was proved in [32, Proposition 1.2]. For completeness, and to emphasize the role of measurability, we include the short proof.

**Theorem 4.11.** Let $\rho$ be a norm function. Then there exists a measurable matrix mapping $W : \Omega \to \mathcal{S}_d$ such that for a.e. $x \in \Omega$, $W$ is positive definite, and for every $v \in \mathbb{R}^d$,

$$\rho_W(x, v) \leq \rho(x, v) \leq \sqrt{d}\rho_W(x, v).$$

**Proof.** Let

$$K(x) = \{v \in \mathbb{R}^d : \rho_x(v) \leq 1\}$$

be the unit ball of $\rho_x$, $x \in \Omega$. Then by Corollary 2.5, $K(x) \in \mathcal{K}_{abcd}(\mathbb{R}^d)$. Further, $K : \Omega \to \mathcal{K}_{abcd}(\mathbb{R}^d)$ is a measurable mapping. To see this, note that by Theorem 4.2, $K^o$ is measurable, so by Theorem 3.4, $K$ is measurable. Therefore, by Theorem 3.7 there exists a measurable matrix mapping $A : \Omega \to \mathcal{M}_d$ such that

$$A(x)\overline{B} \subset K(x) \subset \sqrt{d}A(x)\overline{B}. \tag{4.18}$$

We therefore have that for $x \in \Omega$ and $v \in \mathbb{R}^d$,

$$p_{A(x)\overline{B}}(v) \leq \rho(x, v) \leq \sqrt{d}p_{A(x)\overline{B}}(v),$$

where $p_{A(x)\overline{B}}$ is the Minkowski functional of $A(x)\overline{B}$. (See Definition 2.2.) It follows from (4.18) that $A$ is invertible. Thus,

$$p_{A(x)\overline{B}}(v) = \inf\{r > 0 : \frac{v}{r} \in A(x)\overline{B}\} = \inf\{r > 0 : A^{-1}(x)v \in r\overline{B}\} = |A^{-1}(x)v| = \rho_{A^{-1}}(x, v).$$

Finally, by Theorem 4.10, there exists a measurable, positive definite matrix mapping $W : \Omega \to \mathcal{S}_d$ such that $\rho_W(x, v) = \rho_{A^{-1}}(x, v)$. This completes the proof. \(\square\)
Proposition 4.12. If \( W : \Omega \to \mathcal{M}_d \) is invertible a.e., then for a.e. \( x \in \Omega \) and every \( v \in \mathbb{R}^d \), \( \rho_W(x, v) = \rho_{(W^*)^{-1}}(x, v) \). In particular, if \( W \) is symmetric a.e., then \( \rho_W^* = \rho_W^{-1} \).

Proof. Arguing as in the proof of Theorem 4.11, we have that the unit ball of \( \rho_W \) is \( K(x) = W^{-1}(x)\overline{B} \), and so by Theorem 2.8, the unit ball of \( \rho_W^* \) is

\[
K(x)^o = \{ v \in \mathbb{R}^d : |\langle v, W^{-1}(x)y \rangle| \leq 1, y \in \overline{B} \} = \{ v \in \mathbb{R}^d : |\langle (W^*)^{-1}(x)v, y \rangle| \leq 1, y \in \overline{B} \} = W^*(x)\overline{B}.
\]

As above, \( W^*(x)\overline{B} \) is the unit ball of \( \rho_{(W^*)^{-1}} \). By Corollary 2.5, if two norms have the same unit ball, they are the same norm, so \( \rho_W^*(x, v) = \rho_{(W^*)^{-1}}(x, v) \). \( \square \)

We refer to the matrix \( W \) in Theorem 4.11 as the matrix weight associated with the norm function \( \rho \). Then we have that a function \( F \in L^p_K(\Omega, \rho) \) if and only if \( F \in L^p_K(\Omega, \rho_W) \), and

\[
\| F \|_{L^p_K(\Omega, \rho_W)} \leq \| F \|_{L^p_K(\Omega, \rho)} \leq \sqrt{d} \| F \|_{L^p_K(\Omega, \rho_W)}.
\]

We will thus be able to pass between these spaces depending on which is most convenient. The spaces \( L^p_K(\Omega, \rho_W) \) are referred to as matrix weighted spaces; for simplicity we will often denote them by \( L^p_K(\Omega, W) \). Closely connected to these spaces are the matrix-weighted spaces of vector-valued functions, which we will denote \( L^p(\Omega, W) \). This space can be identified with a subset of \( L^p_K(\Omega, W) \) using the mapping defined in Lemma 3.13.

5. The Maximal Operator on Convex-set valued Functions

In this section we generalize the Hardy-Littlewood maximal operator to the setting of convex-set valued functions. Throughout this section, we will take our underlying measure space to be \( \mathbb{R}^n \) equipped with Lebesgue measure. We will use the standard Euclidean norm on \( \mathbb{R}^d \), and given a set \( K \subset \mathbb{R}^d \), we define the norm of a set by

\[
|K| = \sup \{ |v| : v \in K \}.
\]

Hereafter, by a cube \( Q \) we will always mean a cube whose sides are parallel to the coordinate axes. Unless we indicate otherwise, all integrals are taken with respect to the Lebesgue measure \( m_n \) of \( \mathbb{R}^n \). The volume of the cube \( Q \) is denoted by \( m_n(Q) \) rather than the customary \( |Q| \) to avoid ambiguity with the norm of a set given by (5.1).

Averaging operators. We first consider the simpler case of averaging operators. Given a function \( F : \mathbb{R}^n \to \mathcal{K}_{bcs}(\mathbb{R}^d) \) that is locally integrably bounded, if we fix a cube \( Q \), then we define the averaging operator \( A_Q \) by

\[
A_Q F(x) = \int_Q F(y)\,dy \cdot \chi_Q(x) = \frac{1}{m_n(Q)} \int_Q F(y)\,dy \cdot \chi_Q(x).
\]

Therefore, \( A_Q F(x) \) is the “average” of \( F \) on \( Q \) if \( x \in Q \), and is the set \( \{0\} \) otherwise. However, the associated convex set can be quite different from \( F \), even if it is the convex-set valued function associated to a vector-valued function (as in Lemma 3.13). For example, let \( f : \mathbb{R} \to \mathbb{R}^2 \) be defined by

\[
f(x) = \begin{cases} (1, 1)^t, & x \geq 0; \\
(-1, 1)^t, & x < 0.
\end{cases}
\]

and let \( F(x) = \operatorname{conv}\{f(x), -f(x)\} \). Let \( Q = [-1, 1] \). Then for \( x \in Q \)

\[
A_Q F(x) = \left\{ \int_Q k(y)f(y)\,dy : k \in L^\infty(Q), \|k\|_\infty \leq 1 \right\}.
\]
Fix any \( k \in L^\infty(Q) \); then
\[
\int_Q k(y)f(y) \, dy = \frac{1}{2} \int_{-1}^{0} k(y) \, dy \left( \frac{-1}{1} \right) + \frac{1}{2} \int_{0}^{1} k(y) \, dy \left( \frac{1}{1} \right).
\]
The integrals are constants with values in \([-1, 1]\) so without loss of generality we may assume that \( k \) is constant on \([-1, 0) \) and \([0, 1]\); denote these values by \( a \) and \( b \). Hence,
\[
A_Q F(x) = \left\{ \frac{a}{2} \left( \frac{-1}{1} \right) + \frac{b}{2} \left( \frac{1}{1} \right) : |a|, |b| \leq 1 \right\}.
\]
It follows immediately that \( A_Q F(x) \) is equal to the square with vertices \((\pm 1, 0), (0, \pm 1)\).

By Theorem 3.15, \( A_Q : \mathbb{R}^n \to \mathcal{K}_{bc}(\mathbb{R}^d) \) and is a measurable mapping. The averaging operators are linear operators in the sense of Lemma 5.1 below. We use this terminology, even though \( \mathcal{K}_{bc}(\mathbb{R}^d) \) is not a vector space, because of the compelling form of the identities (5.3) and (5.4). Lemma 5.1 is an immediate consequence of the linearity of the Aumann integral, Theorem 3.17.

**Lemma 5.1.** Given any cube \( Q \), the averaging operator \( A_Q \) is linear: if \( F, G : \mathbb{R}^n \to \mathcal{K}_{bc}(\mathbb{R}^d) \) are locally integrably bounded mappings, and \( \alpha \in \mathbb{R} \), then
\[
(5.3) \quad A_Q(F + G)(x) = A_Q F(x) + A_Q G(x),
\]
\[
(5.4) \quad A_Q(\alpha F)(x) = \alpha A_Q F(x).
\]

For \( 1 \leq p < \infty \), if \( F \in L_p^m(\mathbb{R}^d, |\cdot|) \), then \( F \) is locally integrably bounded: if we define \( k(x) = |F(x)| \), then by definition, \( k \in L_p(\mathbb{R}^n) \), and so \( k \in L_p^1(\mathbb{R}^n) \). Since \( F(x) \subseteq k(x)B \), \( F \) is locally integrably bounded. In particular, averaging operators are well-defined on \( L_p^m(\mathbb{R}^d, |\cdot|) \).

**Proposition 5.2.** Given a cube \( Q \), for \( 1 \leq p \leq \infty \), \( A_Q : L_p^m(\mathbb{R}^n, |\cdot|) \to L_p^m(\mathbb{R}^n, |\cdot|) \), and \( \| A_Q F \|_p \leq \| F \|_p \).

**Proof.** Fix \( p, 1 \leq p < \infty \). By the definition of the norm in \( L_p^m(\mathbb{R}^n, |\cdot|) \),
\[
\| A_Q F \|_{L_p^m(\mathbb{R}^n, |\cdot|)} = \left( \int_{\mathbb{R}^n} \left( \frac{1}{Q} \int_{Q} F(y) \, dy \cdot \chi_Q(x) \right)^p \, dx \right)^{\frac{1}{p}} = \left( \frac{1}{Q} \int_{Q} F(y) \, dy \right)^{\frac{1}{p}} m_n(Q)^{\frac{1}{p}}.
\]
Since \( F \) is locally integrably bounded, by Lemma 3.9 there exists a selection function \( v_F \in S^1(Q, F) \) such that \(|v_F(x)| = |F(x)|\) for all \( x \in Q \). In particular, given any selection function \( f \in S^1(Q, F) \), \(|f(x)| \leq |v_F(x)|\). Therefore,
\[
\left| \frac{1}{Q} \int_{Q} F(y) \, dy \right| = \sup \left\{ \left| \frac{1}{Q} \int_{Q} f(y) \, dy \right| : f \in S^1(Q, F) \right\}
\leq \frac{1}{Q} \int_{Q} |v_F(y)| \, dy \leq \left( \frac{1}{Q} \int_{Q} |v_F(y)|^p \, dy \right)^{\frac{1}{p}} = \| F \|_{L_p(\mathbb{R}^n, |\cdot|)} m_n(Q)^{-\frac{1}{p}}.
\]
If we combine these estimates we get the desired inequality.

When \( p = \infty \), the proof is similar but simpler. By the definition of the \( \| \cdot \|_\infty \) norm, for a.e. \( x \), \(|v_F(x)| = |F(x)| \leq \| F \|_{L_\infty(\mathbb{R}^n, |\cdot|)} \). Hence, arguing as above,
\[
\| A_Q F \|_{L_\infty(\mathbb{R}^n, |\cdot|)} \leq \frac{1}{Q} \int_{Q} |v_F(y)| \, dy \leq \| F \|_{L_\infty(\mathbb{R}^n, |\cdot|)}.
\]

\[ \square \]

**Remark 5.3.** We can also define the averaging operator by taking averages over balls \( B \) instead of cubes. Every result above remains true for these averaging operators.
The convex-set valued maximal operator. We now extend the definition of the Hardy-Littlewood maximal operator to convex-set valued functions.

**Definition 5.4.** Given a locally integrably bounded function $F : \mathbb{R}^n \to \mathcal{K}_{bcs}(\mathbb{R}^d)$, define the maximal operator acting on $F$ by

$$MF(x) = \overline{\text{conv}} \left( \bigcup_Q A_Q F(x) \right),$$

where the union is taken over all cubes $Q$ whose sides are parallel to the coordinate axes.

It is immediate from the definition that since $F(x) \in \mathcal{K}_{bcs}(\mathbb{R}^d)$, $MF(x) \in \mathcal{K}_{cs}$. The set $MF(x)$ can be a considerably larger set than $F(x)$. For example, if we let $f$ be the vector-valued function (5.2) and define $F$ as before, then for $x > 0$, arguing as we did above, we can show that

$$MF(x) = \overline{\text{conv}} \left\{ \begin{pmatrix} -as/(t-s) & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} bt \\ t-s \end{pmatrix} : |a|, |b| \leq 1; t, s \in \mathbb{R}, s < 0 < x < t \right\}.$$

(The case $0 \leq s < x < t$ should also be included, but it is easy to check that it does not add anything to the set.) If we reparameterize by setting $s = -rt, 0 < r < \infty$, and then making the change of variables $v = \frac{1}{1+r^2}$, we get that

$$MF(x) = \overline{\text{conv}} \left\{ \left( \begin{array}{c} -a \\ a \end{array} \right) + v \left( \begin{array}{c} a+b \\ -a+b \end{array} \right) : |a|, |b| \leq 1; 0 < v < 1 \right\}.$$

By varying the parameters, it is straightforward to see that we get all points in the square with vertices $(\pm 1, \pm 1)$.

**Lemma 5.5.** The maximal operator is sublinear: for any locally integrably bounded mappings $F, G : \mathbb{R}^n \to \mathcal{K}_{bcs}(\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$,

$$M(F + G)(x) \subset MF(x) + MG(x), \quad M(\alpha F)(x) = \alpha MF(x).$$

Further, the maximal operator is monotone: if $F(x) \subset G(x)$ for all $x$, then $MF(x) \subset MG(x)$.

**Proof.** Sublinearity follows from Lemma 5.1 and the linearity of the convex hull with respect to Minkowski sum:

$$M(F + G)(x) = \overline{\text{conv}} \left( \bigcup_Q \left[ A_Q F(x) + A_Q G(x) \right] \right) \subset \overline{\text{conv}} \left( \bigcup_Q A_Q F(x) + \bigcup_Q A_Q G(x) \right) = MF(x) + MG(x).$$

Similarly,

$$M(\alpha F)(x) = \overline{\text{conv}} \left( \bigcup_Q \alpha A_Q F(x) \right) = \alpha MF(x).$$

Monotonicity follows from the definition of the maximal operator and Theorem 3.17. □

Below we will also need a version of sublinearity that generalizes the fact that in the scalar case, for $1 < p < \infty$, the operator $M_p f(x) = M(|f|^p)(x)^{\frac{1}{p}}$ is sublinear.

**Lemma 5.6.** Given $1 < p < \infty$, a locally integrably bounded mapping $H : \mathbb{R}^n \to \mathcal{K}_{bcs}(\mathbb{R}^d)$, non-negative functions $f, g \in L^p_{\text{loc}}(\mathbb{R}^n)$, and a norm $\rho$,

$$\rho(M((f + g)^p H)(x))^\frac{1}{p} \leq \rho(M(f^p H)(x))^\frac{1}{p} + \rho(M(g^p H)(x))^\frac{1}{p}.$$

Proof. We introduce an auxiliary operator that simplifies our argument. Given a locally integrably bounded, convex-set valued function $F$, define 

$$
\hat{M}F(x) = \bigcup_Q A_Q F(x).
$$

Note that in contrast to the maximal operator $M$, the operator $\hat{M}$ may not be convex-set valued since the convex hull is not present in the definition of $\hat{M}$. This is not a problem since the operators $M$ and $\hat{M}$ share the same boundedness characteristics. Indeed, we claim that $\rho(MF(x)) = \rho(\hat{M}F(x))$. Clearly, $\rho(MF(x)) \geq \rho(\hat{M}F(x))$. To see the reverse inequality, fix $v$ in 

$$
\text{conv} \left( \bigcup_Q A_Q F(x) \right).
$$

Then we can write $v$ as the finite sum $v = \sum \alpha_i v_i$, where $v_i \in A_Q F(x)$, $Q_i \in \mathcal{D}$, $\alpha_i \geq 0$, and $\sum \alpha_i = 1$. But then it is immediate that 

$$
\rho(v) \leq \sum \alpha_i \rho(v_i) \leq \rho(\hat{M}F(x)),
$$

and the desired inequality follows at once.

Given this equality we can argue as follows: by Proposition 3.21,

$$
\rho(\hat{M}(f + g)^p H)(x)^{\frac{1}{p}} = \rho \left( \bigcup_{x \in Q} A_Q ((f + g)^p H)(x) \right)^{\frac{1}{p}}
$$

$$
= \sup_Q \rho \left( A_Q ((f + g)^p H)(x) \right)^{\frac{1}{p}}
$$

$$
\leq \sup_Q \rho \left( A_Q (f^p H)(x) \right)^{\frac{1}{p}} + \sup_Q \rho \left( A_Q (g^p H)(x) \right)^{\frac{1}{p}}
$$

$$
= \rho(\hat{M}(f^p H)(x))^{\frac{1}{p}} + \rho(\hat{M}(g^p H)(x))^{\frac{1}{p}}. \quad \square
$$

We claim that $MF$ is a measurable function. To show this, let $Q$ denote the countable set of all cubes with edges parallel to the coordinate axes, all of whose vertices have rational coordinates.

**Proposition 5.7.** Given a locally integrably bounded function $F : \mathbb{R}^n \to \mathcal{K}_{bcs}(\mathbb{R}^d)$,

$$
MF(x) = \text{conv} \left( \bigcup_{P \in Q} A_P F(x) \right).
$$

Consequently, $MF : \mathbb{R}^n \to \mathcal{K}_{cs}(\mathbb{R}^d)$ is a measurable function.

Proof. Fix $x \in \mathbb{R}^n$; then it is immediate that 

$$
\text{conv} \left( \bigcup_{P \in Q} A_P F(x) \right) \subset MF(x).
$$

To prove the reverse inclusion, fix a cube $Q$ containing $x$. Then for any $\epsilon > 0$, there exists a cube $P \in Q$ containing $Q$ such that $m_n(P) \leq (1 + \epsilon)m_n(Q)$. Hence, by Corollary 3.18,

$$
\int_Q F(y) \, dy \subset \frac{m_n(P)}{m_n(Q)} \int_P F(y) \, dy \subset (1 + \epsilon) \int_P F(y) \, dy.
$$

Therefore,

$$
MF(x) \subset (1 + \epsilon) \text{conv} \left( \bigcup_{P \in Q} A_P F(x) \right).
$$

Since $\epsilon > 0$ is arbitrary, we get that equality holds.
Finally, since each averaging operator \( A_P, P \in Q \), is measurable, by Theorem 3.3, \( MF \) is a measurable function.

\[ \square \]

**Lemma 5.8.** Given a locally integrably bounded function \( F : \mathbb{R}^n \to K_{bcs}(\mathbb{R}^d) \), for almost every \( x \in \mathbb{R}^n \), \( F(x) \subset MF(x) \).

**Proof.** By Theorem 3.2, we can write

\[
F(x) = \{ f_k(x) : k \in \mathbb{N} \},
\]

where \( f_k \in S^0(\mathbb{R}^n, F) \). Since \( F \) is locally integrably bounded, \( f_k \in L^1_{loc}(\mathbb{R}^n) \), so the restriction \( f_k|_Q \in S^1(Q, F) \) for any cube \( Q \). Since the collection \( \{ f_k \}_{k \in \mathbb{N}} \) is countable, by the Lebesgue differentiation theorem, for almost every \( x \in \mathbb{R}^n \),

\[
f_k(x) = \lim_{m \to 0} \int_Q f_k(y) \, dy.
\]

By the definition of the Aumann integral,

\[
\int_Q f_k(y) \, dy \in \int_Q F(y) \, dy,
\]

and therefore \( f_k(x) \) is a limit point of \( \bigcup_Q A_QF(x) \).

Since \( MF \) has values in closed sets, \( f_k(x) \in MF(x) \) and the desired inclusion follows. \( \square \)

There are alternative definitions of the maximal operator that are analogous to the ones from the classical theory. Let \( Q(x, r) \) be the cube centered at \( x \) with side length \( r \). Then we can define the centered maximal operator

\[
M^cF(x) = \text{conv} \left( \bigcup_{r>0} A_{Q(x,r)}F(x) \right).
\]

We can also define a maximal operator where the averages are over balls containing \( x \) instead of cubes,

\[
\overline{MF}(x) = \text{conv} \left( \bigcup_B A_BF(x) \right),
\]

where

\[
A_BF(x) = \int_B F(y) \, dy \cdot \chi_B(x)
\]

is the averaging operator defined with respect to balls. Similarly we can restrict to balls centered at \( x \),

\[
\overline{M^cF}(x) = \text{conv} \left( \bigcup_{r>0} A_{B(x,r)}F(x) \right).
\]

All of these maximal operators are equivalent to the maximal operator \( M \) as originally defined. This follows from Corollary 3.18, using the fact that given a point \( x \) and cube \( Q \), then \( Q \subset Q(x, 2\ell(Q)) \), and the fact that given a ball \( B(x, r) \),

\[
Q(x, n^{-n/2}r) \subset B(x, r) \subset Q(x, 2r).
\]

Since we will not use this result, we leave the details to the interested reader.

More important is a dyadic version of the convex-set valued maximal operator. Given the collection of dyadic cubes

\[
\mathcal{D} = \{ 2^k([0,1]^n + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^n \},
\]
we can define the dyadic maximal operator

\[ M^d F(x) = \text{conv} \left( \bigcup_{Q \in D} A_Q F(x) \right). \]

It is immediate that \( M^d \) has all the same properties as the maximal operator \( M \). Moreover, from the definition we have that for any locally integrably bounded convex-set valued function \( F \), \( M^d F(x) \subset MF(x) \).

The converse inclusion is not true, but if we define a larger family of dyadic operators, a closely related inclusion is true. For \( \tau \in \{0, \pm 1/3\}^n \), define the translated dyadic grid

\[ D^\tau = \{2^k((0, 1)^n + m + (-1)^k \tau) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}. \]

Then \( D^0 = D \); moreover, all of the dyadic grids \( D^\tau \) have the same essential properties as \( D \). (See [13, 39].) We define the generalized dyadic maximal operator

\[ M^\tau F(x) = \text{conv} \left( \bigcup_{Q \in D^\tau} A_Q F(x) \right). \]

**Lemma 5.9.** Given a locally integrably bounded, convex-set valued function \( F : \mathbb{R}^n \to \mathcal{K}_{bcs}(\mathbb{R}^d) \),

\[ MF(x) \subset C \sum_{\tau \in \{0, \pm 1/3\}^n} M^\tau F(x), \]

where the constant \( C \) depends only on the dimension \( n \).

**Proof.** Fix \( x \in \mathbb{R}^n \) and a cube \( Q \) containing \( x \). Then there exists \( \tau \in \{0, \pm 1/3\}^n \) and a cube \( P \subset D^\tau \) such that \( Q \subset P \) and \( \ell(P) \leq 3\ell(Q) \) [13, Theorem 3.1]. Therefore,

\[ \int_Q F(y) \, dy \subset 3^n \int_P F(y) \, dy. \]

Since \( 0 \in F(y) \), \( 0 \in \int_P F(y) \, dy \), and so

\[ \bigcup_{Q} A_Q F(x) \subset 3^n \bigcup_{\tau \in \{0, \pm 1/3\}^n} \bigcup_{P \in D^\tau} A_P F(x). \]

By the linearity of the convex hull,

\[ MF(x) \subset C \sum_{\tau \in \{0, \pm 1/3\}^n} M^\tau F(x). \]

**\( L^p \) norm inequalities for the convex-set valued maximal operator.** In this section we prove strong and weak-type norm inequalities for the convex-set valued maximal operator.

**Theorem 5.10.** For \( 1 < p \leq \infty \), \( M : L^p_K(\mathbb{R}^n, |\cdot|) \to L^p_K(\mathbb{R}^n, |\cdot|) \) is bounded. When \( p = 1 \), \( M : L^1_K(\mathbb{R}^n, |\cdot|) \to L^{1,\infty}_K(\mathbb{R}^n, |\cdot|) \) is bounded. That is, for all \( \lambda > 0 \) and \( F \in L^1_K(\mathbb{R}^n, |\cdot|) \),

\[ m_n(\{x \in \mathbb{R}^n : |MF(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |F(x)| \, dx. \]

**Proof.** Our proof adapts the classic proof of the boundedness of the dyadic maximal operator, which uses the Calderón-Zygmund cubes, to the convex-set valued maximal operator. For the theory of the scalar maximal operator, which extends to vector-valued functions without change, see [25, 30]. We begin with several reductions. First, by Lemma 5.9,

\[ |MF(x)| \leq C \sum_{\tau \in \{0, \pm 1/3\}^n} |M^\tau F(x)|, \]
and so it will suffice to prove the strong and weak-type inequalities for \(M^*\). In fact, given that all of the dyadic grids \(\mathcal{D}^*\) have the same properties as the standard dyadic grid \(\mathcal{D}\), it will suffice to prove them for the dyadic convex-set valued maximal operator, \(\hat{M}^d\). Moreover, arguing as we did in the proof of Lemma 5.6, it will suffice to prove our estimates for the auxiliary operator \(\hat{M}^d\) with omitted convex hull, defined like \(\hat{M}\) in Lemma 5.6, but only using dyadic cubes.

First note that by Proposition 5.2, for a.e. \(x \in \mathbb{R}^n\), \(|A_Q F(x)| \leq \|F\|_{L^\infty_k(\mathbb{R}^n, |\cdot|)}\), and so we have that 
\[
\|\hat{M}^d F\|_{L^\infty_k(\mathbb{R}^n, |\cdot|)} \leq \|F\|_{L^\infty_k(\mathbb{R}^n, |\cdot|)}.
\]

We will now prove the weak \((1, 1)\) inequality by adapting the Calderón-Zygmund decomposition to convex-set valued functions. Fix \(\lambda > 0\) and define
\[
\Omega^d_\lambda = \{x \in \mathbb{R}^n : |\hat{M}^d F(x)| > \lambda\}.
\]
If \(\Omega^d_\lambda\) is empty, there is nothing to prove. Otherwise, given \(x \in \Omega^d_\lambda\), there must exist a cube \(Q \in \mathcal{D}\) such that \(x \in Q\) and
\[
\left|\int_Q F(y) \, dy\right| > \lambda.
\]
We claim that among all the dyadic cubes containing \(x\), there must be a largest one with this property. Arguing as we did above, we have that
\[
\left|\int_Q F(y) \, dy\right| \leq \int_Q |F(y)| \, dy \leq m_n(Q)^{-1} \|F\|_{L^1_k(\mathbb{R}^n, |\cdot|)}.
\]
Since the right-hand side goes to 0 as \(m_n(Q) \to \infty\), we see that such a maximal cube must exist. Denote this cube by \(Q_x\). Since the set of dyadic cubes is countable, we can enumerate the set \(\{Q_x : x \in \Omega^d_\lambda\}\) by \(\{Q_j\}_{j \in \mathbb{N}}\). The cubes \(Q_j\) must be disjoint, since if one was contained in the other, it would contradict the maximality. By our choice of these cubes, \(\Omega^d_\lambda \subset \bigcup_j Q_j\). Hence, we have that
\[
m_n(\Omega^d_\lambda) \leq \sum_j m_n(Q_j) \leq \frac{1}{\lambda} \sum_j m_n(Q_j) \left|\int_{Q_j} F(y) \, dy\right| \leq \frac{1}{\lambda} \sum_j \int_{Q_j} |F(y)| \, dy \leq \lambda^{-1} \|F\|_{L^1_k(\mathbb{R}^n, |\cdot|)}.
\]

To complete the proof, fix \(1 < p < \infty\). For each \(\lambda > 0\) we can decompose \(F = F^\lambda_1 + F^\lambda_2\), where
\[
F^\lambda_1(x) = F(x)\chi_{\{x \in \mathbb{R}^n : |F(x)| > \lambda/2\}}, \quad F^\lambda_2(x) = F(x)\chi_{\{x \in \mathbb{R}^n : |F(x)| \leq \lambda/2\}}.
\]
Since the operator \(\hat{M}^d\) is bounded on \(L^\infty_k(\mathbb{R}^n, |\cdot|)\), by Lemma 5.5,
\[
|\hat{M}^d F(x)| \leq |\hat{M}^d F^\lambda_1(x)| + |\hat{M}^d F^\lambda_2(x)| \leq |\hat{M}^d F^\lambda_1(x)| + \lambda/2.
\]
Therefore, by the weak \((1, 1)\) inequality and Fubini’s theorem,
\[
\|\hat{M}^d F(x)\|_{L^p_k(\mathbb{R}^n, |\cdot|)}^p \leq \int_0^\infty \lambda^{p-1} m_n(\{x \in \mathbb{R}^n : |\hat{M}^d F^\lambda_1(x)| > \lambda/2\}) \, d\lambda \\
\leq \int_0^\infty \lambda^{p-2} \int_{\{x \in \mathbb{R}^n : |F(x)| > \lambda/2\}} |F(x)| \, dx \, d\lambda \\
= \int_{\mathbb{R}^n} |F(x)| \int_0^2 |F(x)| \lambda^{p-2} \, d\lambda \, dx \\
= 2^{p-1} \|F\|_{L^p_k(\mathbb{R}^n, |\cdot|)}^p,
\]
where \(1/p + 1/p' = 1\). \(\square\)
Even though \( L^p_K(\mathbb{R}^n, |\cdot|) \) is not a normed vector space, the maximal operator is still continuous.

**Corollary 5.11.** For \( 1 < p \leq \infty \), the maximal operator is continuous on \( L^p_K(\mathbb{R}^n, |\cdot|) \) with respect to the metric (4.15).

**Proof.** Let \( d_H \) denote the Hausdorff distance defined by (4.14) with respect to the Euclidean metric. Given compact sets \( F, G \in \mathbb{R}^d \), if \( d_H(F, G) \leq r \), then it follows at once from the definition that \( F \subset G + rB \) and \( G \subset F + rB \).

Fix a sequence \( \{F_n\}_{n \in \mathbb{N}} \) that converges to \( F \) in \( L^p_K(\mathbb{R}^n, |\cdot|) \). For each \( n \in \mathbb{N} \) define
\[
H_n(x) = d_H(F_n(x), F(x))B.
\]
Then \( \|H_n\|_{L^p_K(\mathbb{R}^n, |\cdot|)} \to 0 \) as \( n \to \infty \), and
\[
F(x) \subset F_n(x) + H_n(x), \quad F_n(x) \subset F(x) + H_n(x).
\]
by Proposition 5.5 the maximal operator is sublinear, so we have that
\[
MF(x) \subset MF_n(x) + MH_n(x), \quad MF_n(x) \subset MF(x) + MH_n(x).
\]
Therefore, by Theorem 5.10,
\[
d_p(MF, MF_n) \leq d_p(MF_n + MH_n, MF_n) = \|MH_n\|_{L^p_K(\mathbb{R}^n, |\cdot|)} \leq C\|H_n\|_{L^p_K(\mathbb{R}^n, |\cdot|)}.
\]
The desired conclusion follows at once. □

**Remark 5.12.** The proof of Corollary 5.11 is not specific to the maximal operator: in fact, we have that any linear or sublinear operator that is bounded on \( L^p_K(\mathbb{R}^n, |\cdot|) \) is continuous.

6. **Matrix \( A_p \) Weights and Weighted Norm Inequalities**

In this section we extend Theorem 5.10 to the spaces \( L^p_K(\mathbb{R}^n, \rho) \), where the norm function \( \rho \) satisfies a generalized Muckenhoupt \( A_p \) condition. To prove our results, we first need to develop the theory of \( A_p \) norms. Throughout this section, let \( \rho : \mathbb{R}^n \times \mathbb{R}^d \to [0, \infty) \) be a norm function, such that if \( \rho_x(v) = \rho_{F(x)}(v) \) as in (4.2), then \( F \) is locally integrably bounded. Hence, by Theorem 4.5, given any cube \( Q \) and \( v \in \mathbb{R}^d \),
\[
\int_Q \rho_x(v) \, dx < \infty.
\]

**\( A_p \) norms and matrix \( A_p \) weights.** The classical Muckenhoupt \( A_p \) condition (1.1) is defined in terms of averages of scalar weights. Here we will first define the corresponding “average” of a norm. Fix \( 1 \leq p < \infty \) and suppose \( \rho(\cdot, v) \in L^p_{\infty}(\mathbb{R}^d) \) for all \( v \in \mathbb{R}^d \). We define \( \rho_{p,Q} : \mathbb{R}^d \to [0, \infty) \) by
\[
\langle \rho \rangle_{p,Q}(v) = \|\rho(\cdot, v)\|_{p,Q} = \left( \int_Q \rho_x(v)^p \, dx \right)^{\frac{1}{p}}.
\]
Similarly, if \( \rho(\cdot, v) \in L^\infty(\mathbb{R}^d) \) for all \( v \in \mathbb{R}^d \), we define
\[
\langle \rho \rangle_{\infty,Q}(v) = \|\rho(\cdot, v)\|_{\infty,Q} = \text{ess sup}_{x \in Q} \rho_x(v).
\]
Since \( \| \cdot \|_{p,Q}, 1 \leq p \leq \infty \), is a norm, it follows that \( \langle \rho \rangle_{p,Q} \) is a norm. Let \( \rho_x^* \) be the dual norm function and let \( \langle \rho^* \rangle_{p,Q} \) be the average of the dual norm (see Corollary 2.9). These are related by the following inequality. When \( 1 < p < \infty \), this was proved in [32, Proposition 1.1]; for completeness we include the short proof which immediately extends to \( p = 1 \) and \( p = \infty \).

**Lemma 6.1.** Given a norm function \( \rho : \mathbb{R}^n \times \mathbb{R}^d \to [0, \infty) \) and \( 1 \leq p \leq \infty \), then for every cube \( Q \) and \( v \in \mathbb{R}^d \),
\[
\langle \rho \rangle_{p,Q}^* \leq \langle \rho^* \rangle_{p',Q}(v).
\]
Proof. Fix $1 < p < \infty$. By Hölder’s inequality, given two vectors $v$, $w \in \mathbb{R}^d$,

$$\langle v, w \rangle \leq \int_Q \rho_x^*(v) \rho_x(w) \, dx \leq \left( \int_Q \rho_x^*(v)^{p'} \, dx \right)^{\frac{1}{p'}} \left( \int_Q \rho_x(w)^p \, dx \right)^{\frac{1}{p}} = \langle \rho^* \rangle_{p',Q(v)} \langle \rho \rangle_{p,Q(w)}.$$ 

The desired inequality now follows by the definition of the dual norm. When $p = 1$ or $p = \infty$, we repeat this argument but use the $L^\infty$ norm in place of the $L^{p'}$ or $L^p$ norm. \hfill \square

An $A_p$ norm is one for which the reverse of inequality (6.1) holds. The following definition and lemma first appeared in the work of Nazarov, Treil, and Volberg \cite{NazarovTreilVolberg1997, NazarovVolberg2001} when $1 < p < \infty$. Note that our definition of an $A_\infty$ norm is different from the one that is given there.

**Definition 6.2.** Given a norm function $\rho : \mathbb{R}^n \times \mathbb{R}^d \to [0, \infty)$, then for $1 \leq p \leq \infty$ we say that $\rho \in A_p$ if for every cube $Q$ and $v \in \mathbb{R}^d$,

$$\langle \rho^* \rangle_{p',Q(v)} \lesssim \langle \rho \rangle_{p,Q(v)}.$$ 

The infimum of the constants which make this inequality true is denoted by $[\rho]_{A_p}$.

**Lemma 6.3.** Given $1 \leq p \leq \infty$ and norm function $\rho : \mathbb{R}^n \times \mathbb{R}^d \to [0, \infty)$, if $\rho \in A_p$, then $\rho^* \in A_{p'}$ and $\langle \rho^* \rangle_{A_{p'}} = [\rho]_{A_p}$.

Proof. It is immediate from the definition of the dual norm that if $p_1$ and $p_2$ are two norms, and $p_1(v) \leq p_2(v)$ for all $v \in \mathbb{R}^d$, then $p_2^*(v) \leq p_1^*(v)$. But then from (6.2) we have that $\langle \rho \rangle_{p,Q(v)} \leq [\rho]_{A_p} \langle \rho^* \rangle_{p',Q}^*(v)$, and since $\rho^{**} = \rho$, it follows that $\rho^* \in A_{p'}$ and $\langle \rho^* \rangle_{A_{p'}} = [\rho]_{A_p}$. \hfill \square

We can also characterize $A_p$ norms in terms of their associated matrices; in doing so, we also give our definition of matrix $A_p$. As we noted in the Introduction, our definition is different from, but equivalent to, the definition used previously when $1 < p < \infty$, and corresponds to replacing the matrix $W$ by $W^p$ in that definition; see \cite{DDP}. We give two characterizations. To do so, we first define the notion of a reducing operator. These were first introduced in \cite{NazarovVolberg2001} for norms; here we will follow the definition in \cite{2012arXiv1202.6406B} in terms of matrices. Given a norm function $\rho$, by Theorem 4.11 there exists a positive definite matrix mapping $W : \mathbb{R}^n \to S_d$ such that $\rho_x(v) \approx |W(x)v|$. By Proposition 4.12 we have that $\rho^*_x(v) \approx |W^{-1}(x)v|$. In both cases the implicit constants depend only on $d$. Given a cube $Q$ and $1 \leq p \leq \infty$, $\langle \rho \rangle_{p,Q}$ is also a norm and by the John ellipsoid theorem there exists a matrix $W_Q^p$ such that for all $v \in \mathbb{R}^d$,

$$\langle \rho \rangle_{p,Q}(v) \approx \|W(\cdot)v\|_{p,Q} \approx |W_Q^p v|.$$ 

The matrix $W_Q^p$ is referred to as the reducing operator associated to $\rho$ on $Q$. For the reducing operators associated to the dual norm we will use the notation $\overline{W}_Q^p$, i.e.,

$$\langle \rho^* \rangle_{p,Q}(v) \approx \|W^{-1}(\cdot)v\|_{p,Q} \approx |\overline{W}_Q^p v|.$$ 

**Proposition 6.4.** Given a norm function $\rho : \mathbb{R}^n \times \mathbb{R}^d \to [0, \infty)$ with associated matrix mapping $W$, and given $1 \leq p \leq \infty$, $\rho \in A_p$ if and only if

$$\langle W \rangle_{A_p}^R = \sup_Q |\overline{W}_Q^p W_Q^p|_{op} < \infty.$$ 

Moreover, $|W|_{A_p}^R \approx [\rho]_{A_p}$ with implicit constants that depend only on $d$. 

Proof. Suppose first that \((6.3)\) holds. Then given any cube \(Q\) and vector \(v \in \mathbb{R}^d\),
\[
\langle \rho^* \rangle_{p,Q}(v) \approx |W_Q^p v| = |W_Q^{p'} |W_Q^{p'} (W_Q^{p'})^{-1} v| \leq |W_Q^{p'} W_Q^{p} | \langle \rho \rangle_{p,Q} \langle \rho \rangle_{p,Q}(v).
\]
Hence, \(\rho \in A_p\).

Conversely, if \(\rho \in A_p\), then given any vector \(v \in \mathbb{R}^d\),
\[
|W_Q^{p'} W_Q^{p} v| \approx \langle \rho^* \rangle_{p',Q} (W_Q^{p} v) \leq [\rho]_{A_p} \langle \rho \rangle_{p,Q} (W_Q^{p} v) \approx |W_Q^{p} v| = |v|.
\]
It follows at once that \((6.3)\) holds and the constants are comparable.

If a matrix mapping \(W\) is such that \(\rho_W\) is an \(A_p\) norm, we say that \(W\) is in matrix \(A_p\), and write \(W \in A_p\). Note that it follows immediately from Proposition 6.4, analogous to Lemma 6.1, that \(W \in A_p\) if and only if \(W^{-1} \in A_p\).

We can give another characterization of matrix \(A_p\) using integral averages that strongly resembles the Muckenhoupt \(A_p\) condition for scalar weights. When \(1 < p < \infty\), this condition is due to Roudenko [56]; when \(p = 1\) it was used as the definition of \(A_1\) by Frazier and Roudenko [29]. Here we give the proof when \(p = 1\) (equivalently, when \(p = \infty\)) and refer the reader to [56] for the case \(1 < p < \infty\).

Proposition 6.5. Given a norm function \(\rho : \mathbb{R}^n \times \mathbb{R}^d \to (0, \infty)\) with associated matrix mapping \(W\), and given \(1 < p < \infty\), \(\rho \in A_p\) if and only if
\[
[W]_{A_p} = \sup_{Q} \left( \frac{\int_{Q} \left( \int_{Q} |W(x)W^{-1}(y)|_{\text{op}}^p dy \right)^{\frac{1}{p}} dx}{\int_{Q} |W^{-1}(x)W(y)|_{\text{op}}^p dy} \right) < \infty.
\]

When \(p = 1\), \(\rho \in A_1\) if and only if
\[
[W]_{A_1} = \sup_{Q} \sup_{x \in Q} \int_{Q} |W^{-1}(x)W(y)|_{\text{op}} dy < \infty.
\]

When \(p = \infty\), \(\rho \in A_{\infty}\) if and only if
\[
[W]_{A_{\infty}} = \sup_{Q} \sup_{x \in Q} \int_{Q} |W(x)W^{-1}(y)|_{\text{op}} dy < \infty.
\]

For all \(p\), \([W]_{A_p} \approx [W]_{A_p}^{R} \approx [\rho]_{A_p}\) with constants that depend only on \(d\).

Proof. Recall that if \(A\) and \(B\) are two matrices in \(S_d\), then
\[
|AB|_{\text{op}} = |(AB)^t|_{\text{op}} = |B^tA^t|_{\text{op}} = |BA|_{\text{op}}.
\]

Suppose first that \(\rho \in A_1\). Let \(\{e_i\}_{i=1}^d\) be the standard basis in \(\mathbb{R}^d\). Fix a cube \(Q\); then for almost every \(x \in Q\),
\[
\int_{Q} |W^{-1}(x)W(y)|_{\text{op}} dy = \int_{Q} |W(y)W^{-1}(x)|_{\text{op}} dy \approx \sum_{i=1}^d \int_{Q} |W(y)W^{-1}(x)e_i| dy \approx \sum_{i=1}^d |W_Q^{1}W_Q^{1}(x)e_i|_{\text{op}} \approx \sum_{i=1}^d |W_Q^{1}W_Q^{1}(x)e_i| \leq \sum_{i=1}^d |W_Q^{\infty}W_Q^{\infty}(x)e_i| \approx |W_Q^{\infty}W_Q^{\infty}|_{\text{op}} < \infty;
\]
the last inequality follows from Proposition 6.4. This gives us inequality \((6.4)\).

Conversely, suppose \((6.4)\) holds. If we fix a cube \(Q\), then there exists a vector \(v \in \mathbb{R}^d\), \(|v| = 1\), and \(x \in Q\) such that
\[ |W_Q^\infty W_Q^1|_{op} \lesssim |W_Q^\infty W_Q^1 v| \lesssim |W^{-1}(x)W_Q^1 v| \leq |W^{-1}(x)W_Q^1|_{op} = |W_Q^1 W^{-1}(x)|_{op} \]
\[ \lesssim \sum_{i=1}^d |W_Q^1 W^{-1}(x)e_i| \approx \sum_{i=1}^d \int_Q |W(y)W^{-1}(x)e_i| \, dy \]
\[ \lesssim \int_Q |W(y)W^{-1}(x)|_{op} \, dy \lesssim \int_Q |W^{-1}(x)W(y)|_{op} \, dy < \infty. \]

So again by Proposition 6.4, \( \rho \in \mathcal{A}_1 \) and the constants are comparable. \( \square \)

Weighted norm inequalities for averaging and maximal operators. In this section we generalize Proposition 5.2 and Theorem 5.10 to the case of matrix weights.

**Proposition 6.6.** Given \( 1 \leq p \leq \infty \) and a matrix weight \( W \), the following are equivalent:

1. \( W \in \mathcal{A}_p \).
2. Given any cube \( Q \), \( A_Q : L^p_{\mathcal{K}}(\mathbb{R}^n, W) \to L^p_{\mathcal{K}}(\mathbb{R}^n, W) \), and \( \|A_Q\|_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)} \leq K \).

Moreover, we have that \( [W]_{\mathcal{A}_p} \approx \sup_Q \|A_Q\|_{L^p_{\mathcal{K}}(\mathbb{R}^n, W)}. \)

**Proof.** We first assume \( W \in \mathcal{A}_p \). When \( 1 \leq p < \infty \), this result was originally proved for vector-valued functions in [19, Proposition 4.7], but the proof readily extends to convex-set valued functions. Here we prove the case when \( p = \infty \). Fix \( W \in \mathcal{A}_\infty \) and \( F \in L^\infty_{\mathcal{K}}(\mathbb{R}^n, W) \). Given a cube \( Q \), for almost every \( x \in Q \),

\[ |W(x)A_Q F(x)| = \sup \left\{ \left|W(x)\int_Q f(y) \, dy\right| : f \in S^1(Q, F) \right\} \]
\[ \leq \sup \left\{ \int_Q |W(x)W^{-1}(y)W(y)f(y)| \, dy : f \in S^1(Q, F) \right\} \]
\[ \lesssim \sup \left\{ [W]_{\mathcal{A}_\infty} \|W f\|_\infty : f \in S^1(Q, F) \right\} \]
\[ = [W]_{\mathcal{A}_\infty} \|F\|_{L^\infty_{\mathcal{K}}(\mathbb{R}^n, W)}. \]

To prove necessity, first note that it follows at once from the mapping from vector-valued to convex-set valued functions given in Lemma 3.13, that to prove necessity it suffices to prove it for vector-valued functions. This was proved when \( 1 \leq p < \infty \) in [16, Theorem 1.18]. The proof for \( 1 < p < \infty \) immediately extends to the case \( p = \infty \), using the fact that the dual of \( L^1 \) is \( L^\infty \). \( \square \)

As a corollary to Proposition 6.6 we deduce that if \( W \in \mathcal{A}_p \), then the operator norm \( |W|_{op} \) is a scalar weight in \( \mathcal{A}_p \). This was proved by Goldberg [32, Corollary 2.3] for \( p < \infty \) and the same proof holds for \( p = \infty \). We omit the details.

**Corollary 6.7.** For \( 1 \leq p \leq \infty \), if \( W \in \mathcal{A}_p \) and \( w = |W(\cdot)|_{op} \) is an operator norm of \( W \), then \( w \in \mathcal{A}_p \) with \( [w]_{\mathcal{A}_p} \lesssim [W]_{\mathcal{A}_p} \).

To prove norm inequalities for the convex-set valued maximal operator, we need an auxiliary weighted maximal operator first introduced by Christ and Goldberg [11, 32]. Given a matrix weight \( W \), for any function \( f \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^d) \) define

\[ M_W f(x) = \sup_Q \int_Q |W(x)W^{-1}(y)f(y)| \, dy \cdot \chi_Q(x). \]

**Proposition 6.8.** Fix \( 1 < p \leq \infty \). Given a matrix weight \( W \in \mathcal{A}_p \), \( M_W : L^p(\mathbb{R}^n, \mathbb{R}^d) \to L^p(\mathbb{R}^n) \).

Moreover,

\[ \|M_W f\|_{L^p(\mathbb{R}^n)} \leq C(n, d, p)[W]_{\mathcal{A}_p}^p \|f\|_{L^p(\mathbb{R}^n, \mathbb{R}^d)}. \]
Proof. For $1 < p < \infty$, this inequality, without a quantitative estimate of the constant, was proved in \cite{11, 32}. The given estimate was proved by Isralowitz and Moen \cite[Theorem 1.3]{41}. We will prove the case when $p = \infty$.

Give a vector function $f \in L^\infty(\mathbb{R}^n, \mathbb{R}^d)$, then for any cube $Q$ and a.e. $x \in Q$, we have by Proposition \ref{prop:6.5} that

$$
\int_Q |W(x)W^{-1}(y)f(y)| \, dy \leq \int_Q |W(x)W^{-1}(y)|_{op}|f(y)| \, dy \lesssim [W]_{A_\infty} \|f\|_\infty.
$$

If we now fix $x$ and take the supremum over all cubes containing $x$, we get the desired estimate. \hfill \Box

**Theorem 6.9.** Given $1 < p \leq \infty$ and a matrix weight $W \in \mathcal{A}_p$, then the convex-set valued maximal operator satisfies $M : L^p_K(\mathbb{R}^n, W) \to L^p_K(\mathbb{R}^n, W)$. Moreover,

$$
\|MF\|_{L^p_K(\mathbb{R}^n, W)} \leq C(n, d, p)\|W\|_{A_p} \|F\|_{L^p_K(\mathbb{R}^n, W)}.
$$

**Proof.** We will prove this by reducing to the corresponding inequalities for $M_W$. First note that by replacing $F$ by $W^{-1}F$, we have that $M : L^p_K(\mathbb{R}^n, W) \to L^p_K(\mathbb{R}^n, W)$ is bounded if and only if

$$
\|WM(W^{-1}F)\|_{L^p_K(\mathbb{R}^n, |\cdot|)} \lesssim \|F\|_{L^p_K(\mathbb{R}^n, |\cdot|)}.
$$

Given $F \in L^p_K(\mathbb{R}^n, |\cdot|)$, by Theorem \ref{thm:3.7} there exists a measurable matrix $A : \mathbb{R}^n \to \mathcal{M}_d$ such that

$$
A(x)B \subset F(x) \subset \sqrt{d}A(x)B,
$$

Let $a_i(x), 1 \leq i \leq d$, be the columns of $A(x)$. Then $a_i(x) \in F(x)$, and conversely, if $v \in F(x),

$$
v = \sum_{i=1}^d \lambda_i a_i(x), \quad \text{where} \quad \sum_{i=1}^d |\lambda_i| \leq \sqrt{d} \left( \sum_{i=1}^d |\lambda_i|^2 \right)^{1/2} \leq d.
$$

Since $F \in L^p_K(\mathbb{R}^n, |\cdot|)$, $F$ is locally integrably bounded, and so $a_i \in L^1_{\text{loc}}(\mathbb{R}^d)$. Define

$$
F_i(x) = \text{conv}\{a_i(x), -a_i(x)\}.
$$

Then by Lemma \ref{lem:3.13}, $F_i$ is measurable and locally integrably bounded, and by the above estimate,

$$
F(x) \subset C(d) \sum_{i=1}^d F_i(x).
$$

Hence, by Lemma \ref{lem:5.5},

$$
W(x)M(W^{-1}F)(x) \subset C(d) \sum_{i=1}^d W(x)M(W^{-1}F_i)(x).
$$

Again by Lemma \ref{lem:3.13}, for $1 \leq i \leq d$ and any cube $Q$ containing $x$,

$$
|W(x)A_Q(W^{-1}F_i)(x)| = \sup \left\{ \left| W(x) \int_Q k(y)W^{-1}(y)a_i(y) \, dy \right| : \|k\|_\infty \leq 1 \right\}
\leq \int_Q |W(x)W^{-1}(y)a_i(y)| \, dy \leq M_Wa_i(x).
$$

Therefore, we have that $|W(x)M(W^{-1}F_i)(x)| \leq M_Wa_i(x)$. But then by Proposition \ref{prop:6.8},

$$
\|WM(W^{-1}F)\|_{L^p_K(\mathbb{R}^n, |\cdot|)} \leq C(d) \sum_{i=1}^d \|WM(W^{-1}F_i)\|_{L^p_K(\mathbb{R}^n, |\cdot|)}
$$

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\[ \leq C(d) \sum_{i=1}^{d} \| MW a_i \|_{L^p(\mathbb{R}^n, \mathbb{R})} \]
\[ \leq C(n, d, p) \sum_{i=1}^{d} \| a_i \|_{L^p(\mathbb{R}^n, \mathbb{R}^d)} \]
\[ \leq C(n, d, p) \sum_{i=1}^{d} \| F \|_{L^p(\mathbb{R}^n, |\cdot|)} \].

**Remark 6.10.** Recently, it was shown in [17] that for \( p = 1 \), if \( W \in A_1 \), then \( MW : L^1(\mathbb{R}^n, \mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^n) \) with a constant proportional to \( [W]^2_{A_1} \). The above proof can be modified to show that

\[ m_n(\{ x \in \mathbb{R}^n : |W(x)M(W^{-1}F)(x)| > \lambda \}) \leq \frac{C}{\lambda} [W]^2_{A_1} \int_{\mathbb{R}^n} |F(x)| \, dx. \] (6.9)

7. **Convex-set valued \( A_1 \) and the Rubio de Francia iteration algorithm**

In this section we give a new characterization of the matrix \( A_1 \) condition that is a close analog of the classical Muckenhoupt \( A_1 \) condition. We then use this to define a convex-set valued version of the Rubio de Francia iteration algorithm.

**Convex-set valued \( A_1 \).** Given a locally integrably bounded function \( F : \mathbb{R}^n \to \mathcal{K}_{\text{bcs}}(\mathbb{R}^d) \), we showed in Lemma 5.8 that \( F(x) \subseteq MF(x) \) almost everywhere. This motivates the following definition which adapts the definition of \( A_1 \) in the scalar case.

**Definition 7.1.** Given a locally integrably bounded function \( F : \mathbb{R}^n \to \mathcal{K}_{\text{bcs}}(\mathbb{R}^d) \), we say that \( F \) is in convex-set valued \( A_1^K \), if there exists a constant \( C \) such that for almost every \( x \),

\[ MF(x) \subseteq CF(x). \]

Denote the infimum of all such constants by \([F]_{A_1^K} \).

There is an alternative characterization of convex-set valued \( A_1 \) in terms of averaging operators.

**Lemma 7.2.** Given a locally integrably bounded function \( F : \mathbb{R}^n \to \mathcal{K}_{\text{bcs}}(\mathbb{R}^d) \), \( F \in A_1^K \) if and only if there exists a constant \( C \) such that for every cube \( Q \) and almost every \( x \in Q \),

\[ \int_Q F(y) \, dy \subseteq CF(x). \] (7.1)

The infimum of all such constants equals \([F]_{A_1^K} \).

**Proof.** One direction is immediate: if \( F \in A_1^K \), then for every cube \( Q \) and almost every \( x \in Q \),

\[ \int_Q F(y) \, dy \subseteq MF(x) \subseteq [F]_{A_1^K} F(x). \]

Conversely, suppose that (7.1) holds. Recall that \( Q \) is the countable collection of cubes whose vertices have rational coordinates. For each \( P \in Q \), let \( E_P \) be the set of \( x \in P \) such that (7.1) does not hold. If we define

\[ E = \bigcup_{P \in Q} E_P, \]

then \( m_n(E) = 0 \). Fix \( x \notin E \). Then

\[ \bigcup_{P \in Q} \int_{x \in P} F(y) \, dy \subseteq CF(x). \]
Since $F \in \mathcal{K}_{\text{bcs}}$, the closed convex hull of the left-hand side is also contained in $CF(x)$. Therefore, by Proposition 5.7, $MF(x) \subseteq CF(x)$. This, together with the above estimate, shows that the infimum of all such constant $C$ must be $[F]_{\mathcal{A}_1^K}$.

There is a one-to-one correspondence between $\mathcal{A}_1^K$ and matrix $A_1$ weights. To prove this we use the characterization of (locally) integrably bounded convex-set valued mappings from Theorem 4.5.

**Theorem 7.3.** Given a convex-set valued function $F : \mathbb{R}^n \to \mathcal{K}_{\text{bcs}}(\mathbb{R}^d)$ that is locally integrably bounded, $F \in \mathcal{A}_1^K$ if and only if the norm function $\rho(x,v) = p_{F(x)}(v)$ satisfies $\rho \in A_1$. Moreover, $[F]_{\mathcal{A}_1^K} \approx [\rho]_{A_1}$ with implicit constants that depend only on $d$.

**Proof.** Let $\rho : \mathbb{R}^n \times \mathbb{R}^d \to [0,\infty)$ be a norm function; then by Theorem 4.2, $\rho(x,v) = p_{F(x)}$, where $F : \mathbb{R}^n \to \mathcal{K}_{\text{bcs}}(\mathbb{R}^d)$ is the measurable mapping

$$F(x) = \{x \in \mathbb{R}^n : \rho_x(v) \leq 1\}.$$ 

Conversely, given $F$ we can define the norm function $\rho$ in this way. By Theorem 4.5, $F$ is locally integrably bounded if and only if for every cube $Q$ and $v \in \mathbb{R}^d$, the norm

$$\langle \rho \rangle_{1,Q}(v) = \int_Q \rho_x(v) \, dx < \infty.$$

By Theorem 4.11, there exists a measurable matrix $W : \mathbb{R}^n \to S_d$, positive definite almost everywhere, such that $\rho_x(v) \approx |W(x)v|$, with constants depending only on $d$. By Proposition 6.5, $\rho$ is an $A_1$ norm if and only if $W$ is in matrix $A_1$, and satisfies (6.4), which is equivalent to the existence of a constant $C_0$ such that

$$\int_Q \frac{|W(y)W^{-1}(x)v|}{|v|} \, dy \leq C_0 < \infty$$

for any cube $Q$, almost every $x \in Q$, and every $v \in \mathbb{R}^d \setminus \{0\}$. By the change of variables $v \mapsto W(x)v$, this is equivalent to

$$\int_Q |W(y)v| \, dy \leq C_0|W(x)v|$$

for all $v \in \mathbb{R}^d$, which in turn is equivalent to saying that for every cube $Q$, almost every $x \in Q$, and every $v \in \mathbb{R}^d$,

$$\langle \rho \rangle_{1,Q}(v) \leq C_1 \rho_x(v),$$

where $C_1 = c(d)C_0$.

We will now show that (7.3) is equivalent to the $\mathcal{A}_1^K$ condition for $F$. By Theorem 4.5, for every cube $Q$, we have that $\langle \rho \rangle_{1,Q}(v) = p_{K_Q}(v)$, where

$$K_Q = \left(\int_Q F(y) \, dy\right)^\circ.$$

On the other hand, we have that

$$K_Q = \left\{ v \in \mathbb{R}^d : \int_Q \rho_y(v) \, dy \leq 1 \right\},$$

and so (7.3) is equivalent to the inclusion

$$\left(\int_Q F(y) \, dy\right)^\circ = K_Q \supset \left\{ v \in \mathbb{R}^d : C_1 \rho_x(v) \leq 1 \right\} = C_1^{-1} F(x)^\circ = (C_1 F(x))^\circ.$$
If we take the polar of the sets we reverse the inclusion, so this is equivalent to
\[ \int_Q F(y) dy \subset C_1 F(x), \]
and by Lemma 7.2 this is equivalent to \( F \in A^K_\infty \). Therefore, we have that \( \rho \) is an \( A_1 \) norm if and only if \( F \) is locally integrably bounded and in \( A^K_1 \). By taking the infima of the respective constants we see that \( [F]_{A_1^K} \approx [\rho]_{A_1} \).

**Corollary 7.4.** Given a matrix weight \( W, W \in A_1 \) if and only if \( W \overline{B} \in A^K_1 \).

**Proof.** Let \( F = WB \). By Theorem 7.3, \( F \in A^K_1 \) if and only if \( \rho(x, v) = p_{F(x)\circ}(v) \). We compute \( \rho \) explicitly: if we argue as in the proof of Proposition 4.12, \( F(x) = W^{-1}(x) \overline{B} \), and so \( p_{F(x)\circ}(v) = \rho_W(x) \).

**The Rubio de Francia iteration algorithm.** Our goal now is to show that the Rubio de Francia iteration algorithm [18, Chapter 2] can be extended to the convex-set valued maximal operator. Given \( 1 < p \leq \infty \), suppose that \( \rho \in A_p \). Let \( \|M\|_\rho = \|M\|_{L^p_\infty(\mathbb{R}^n, \rho)} \) denote the norm of the convex-set valued maximal operator on \( L^p_\infty(\mathbb{R}^n, \rho) \): that is, the infimum of all constants \( C \) such that \( \|MF\|_{L^p_\infty(\mathbb{R}^n, \rho)} \leq C\|F\|_{L^p_\infty(\mathbb{R}^n, \rho)} \).

Given \( G \in L^p_\infty(\mathbb{R}^n, \rho) \) we formally define the Rubio de Francia iteration algorithm to be the sum
\[
RG(x) = \sum_{k=0}^\infty 2^{-k} \|M\|^{-k} M^k G(x),
\]
where \( M^k G = M \circ M \circ \cdots \circ MG \) for \( k \geq 1 \) and \( M^0 G(x) = G(x) \). We can show that this series converges to a convex-set valued function that has exactly the same properties as in the scalar setting.

**Theorem 7.5.** Suppose that \( \rho \) is an \( A_p \) norm for some \( 1 < p \leq \infty \). Fix \( G \in L^p_\infty(\mathbb{R}^n, \rho) \) and define \( RG \) by (7.4); then this series converges in \( L^p_\infty(\mathbb{R}^n, \rho) \) and \( RG : \mathbb{R}^n \to K_{bcs}(\mathbb{R}^d) \) is a measurable mapping. Moreover, it has the following properties:

1. \( G(x) \subset RG(x) \);
2. \( \|RG\|_{L^p_\infty(\mathbb{R}^n, \rho)} \leq 2\|G\|_{L^p_\infty(\mathbb{R}^n, \rho)} \);
3. \( RG \in A^K_1 \) and \( M(RG)(x) \subset 2\|M\|_\rho RG(x) \).

Since variants of the iteration algorithm will play a central role in subsequent sections, we are instead going to prove a more general result which has Theorem 7.5 as an immediate corollary using Lemma 5.5 and Theorem 6.9.

**Theorem 7.6.** Fix \( 1 \leq p \leq \infty \) and a norm function \( \rho \). Suppose \( T \) is a convex-set valued operator with the following properties:

1. \( T : L^p_\infty(\mathbb{R}^n, \rho) \to L^p_\infty(\mathbb{R}^n, \rho) \) with norm \( \|T\|_\rho \).
2. \( T \) is sublinear and monotone in the sense of Lemma 5.5.

Given \( G \in L^p_\infty(\mathbb{R}^n, \rho) \), define
\[
SG(x) = \sum_{k=0}^\infty 2^{-k} \|T\|^{-k} T^k G(x),
\]
where \( T^k G = T \circ T \circ \cdots \circ TG \) for \( k \geq 1 \) and \( T^0 G(x) = G(x) \). Then this series converges in \( L^p_\infty(\mathbb{R}^n, \rho) \) and \( SG : \mathbb{R}^n \to K_{bcs}(\mathbb{R}^d) \) is a measurable mapping. Moreover, it has the following properties:

1. \( G(x) \subset SG(x) \);
2. \( \|SG\|_{L^p_\infty(\mathbb{R}^n, \rho)} \leq 2\|G\|_{L^p_\infty(\mathbb{R}^n, \rho)} \);
3. \( T(SG)(x) \subset 2\|T\|_\rho SG(x) \).
Proof. For brevity, in this proof we will denote \( \| \cdot \|_{L^p_{\infty}(\mathbb{R}^n, \rho)} \) simply as \( \| \cdot \|_p \). To prove that the series (7.5) converges in norm, we apply Theorem 4.8. Let

\[
S_n G(x) = \sum_{k=0}^{n} 2^{-k} \| T \|_p^{-k} T^k G(x)
\]
denote the partial sums of the series \( SG \). We claim that this sequence is Cauchy with respect to the metric \( d_\rho \) defined in (4.15). Indeed, if \( n > m \), then by (4.16) and the boundedness of \( T \),

\[
(7.6) \quad d_\rho(S_n G, S_m G) = d_\rho\left( \sum_{k=m+1}^{n} 2^{-k} \| T \|_p^{-k} T^k G, \{0\} \right)
\]

\[
= \left\| \sum_{k=m+1}^{n} 2^{-k} \| T \|_p^{-k} T^k G \right\|_p \leq \sum_{k=m+1}^{n} 2^{-k} \| T \|_p^{-k} \| T^k G \|_p \leq 2^{-m} \| G \|_p.
\]

By Theorem 4.8, \( L^p_{\infty}(\mathbb{R}^n, \rho) \) is complete with respect to the metric \( d_\rho \), so the sequence \( \{S_n G\}_{n \in \mathbb{N}} \) converges. Let \( SG \) denote the limit.

We now prove the desired properties. To prove the first, since \( d_\rho(S_n G, SG) \to 0 \), there exists a subsequence such that for almost every \( x \in \mathbb{R}^n \),

\[
\lim_{j \to \infty} d_{H,x}(S_{n_j} G(x), SG(x)) = 0.
\]

But for all \( n \in \mathbb{N} \), \( S_{n-1} G(x) \subset S_n G(x) \). Therefore, we have that \( S_n G(x) \to SG(x) \) in the Hausdorff metric. It then follows from [59, Theorem 1.8.7] that

\[
(7.7) \quad SG(x) = \bigcup_{n \in \mathbb{N}} S_n G(x).
\]

Property (1) follows immediately.

To prove the second property, note that for every \( n \geq 1 \),

\[
\| SG \|_p = d_\rho(SG, \{0\}) \leq d(SG, S_n G) + d(S_n G, \{0\}) = d(SG, S_n G) + \| S_n G \|_p.
\]

But then, if we take the limit as \( n \to \infty \) and estimate the second norm as we did above in (7.6), we have that

\[
\| SG \|_p \leq \limsup_{n \to 0} [d(SG, S_n G) + \| S_n G \|_p] \leq 2 \| G \|_p.
\]

Finally, we show the third property. For each \( n \geq 1 \), we can write

\[
SG(x) = S_n G(x) + E_n(x), \quad \text{where } E_n(x) = \sum_{k=n+1}^{\infty} 2^{-k} \| T \|_p^{-k} T^k G(x);
\]

by assumption \( T \) is sublinear, so

\[
T(SG)(x) \subset T(S_n G)(x) + TE_n(x).
\]

We estimate each term on the right separately. To estimate the first, we argue as above. Since by (4.16),

\[
d_\rho(SG, S_n G) = d_\rho(E_n, \{0\}) = \| E_n \|_p,
\]

we have that \( \| E_n \|_p \to 0 \) as \( n \to \infty \). Since \( T \) is bounded, \( \| TE_n \|_p \to 0 \) as \( n \to \infty \). Therefore, there exists a subsequence \( \{E_{n_j}\} \) such that

\[
\rho_\rho(TE_{n_j}(x)) \to 0
\]

almost everywhere as \( j \to \infty \). However, the sets \( E_n \) are nested, \( E_{n+1}(x) \subset E_n(x) \); since by assumption \( T \) is monotone, \( TE_{n+1}(x) \subset TE_n(x) \). Therefore, we have that for a.e. \( x \)

\[
\rho_\rho(TE_n(x)) \to 0
\]
as \( n \to \infty \). Hence, given any \( \epsilon > 0 \), for all \( n \) sufficiently large, \( TE_n(x) \subset B(\epsilon, 0) \).

On the other hand, again since \( T \) is sublinear,

\[
T(S_nG(x)) \subset \sum_{k=0}^{n} 2^{-k} ||T||^{p_k} T^{k+1} G(x) \subset 2||T||_{p} S_{n+1} G(x) \subset 2||T||_{p} S G(x).
\]

The last inclusion follows from (7.7). Combining these two estimates, we see that for every \( \epsilon > 0 \),

\[
T(SG)(x) \subset 2||T||_{p} SG(x) + B(\epsilon, 0).
\]

Since \( \epsilon > 0 \) is arbitrary and since \( SG(x) \) is closed, it follows that \( T(SG)(x) \subset 2||T||_{p} SG(x). \)

\[\square\]

8. Factorization of Matrix Weights

In this section we prove the Jones factorization theorem [14] for matrix weights, Theorem 1.3 in the Introduction. We restate it here.

**Theorem 8.1.** Fix \( 1 < p < \infty \). Given a matrix weight \( W \), we have \( W \in A_p \) if and only if

\[
W = W_0^{1/p} W_1^{1/p'},
\]

for some commuting matrix weights \( W_0 \in A_1 \) and \( W_1 \in A_\infty \).

To make clear the connection with the classical factorization theorem for scalar \( A_p \) weights, recall that a scalar weight \( w \in A_p \) if and only if \( w^p \in A_p \). Thus, we can restate the Jones factorization theorem as \( w \in A_p \) if and only if \( w = w_0^{1/p} w_1^{1/p'} \), where \( w_0, w_1 \in A_1 \) and so \( w_1^{-1} \in A_\infty \). Any two scalar weights commute, hence the assumption of commutativity is moot. In higher dimensions the situation is more complicated. For non-commuting matrix weights \( W_0 \) and \( W_1 \), it is necessary to replace the product \( W_0^{1/p} W_1^{1/p'} \) by their weighted geometric mean \( ((W_0)^2 \#_{1/p'} (W_1)^2)^{1/2} \). For that reason Theorem 8.1 splits into two more precise statements generalizing the scalar theorem.

8.1. Factorization. We divide the proof of Theorem 8.1 into two propositions. In the first we prove factorization proper, which in the scalar case is the more difficult half of the proof. The proof is a modification of the proof in the scalar case [14, Theorem 4.2] using the Rubio de Francia iteration algorithm, which yields matrix weights \( W_0 \) and \( W_1 \), which are not only commuting, but also scalar multiples of one another.

**Proposition 8.2.** Fix \( 1 < p < \infty \). Given a matrix weight \( W \in A_p \), there exist matrix weights \( W_0 \) and \( W_1 \) such that:

- \( W_0 \in A_1 \) with \( [W_0]_{A_1} \lesssim [W]^{p}_{A_p} \).
- \( W_1 \in A_\infty \) with \( [W_1]_{A_\infty} \lesssim [W]^{p}_{A_p} \).
- \( W_0 = rW, W_1 = sW \) for some measurable scalar functions \( r, s \), and

\[
W = W_0^{1/p} W_1^{1/p'}.
\]

The proof requires several lemmas which will also be used in the proof of extrapolation in Section 9. The key technical idea is that we replace the convex-set valued maximal operator with a slightly larger, ellipsoid-valued maximal operator configured to the matrices.

**Definition 8.3.** Let \( W \) be an invertible matrix weight. Given a measurable function \( H : \mathbb{R}^n \to \mathcal{K}_{bcs}(\mathbb{R}^d) \), define the exhausting operator \( N_W \) with respect to \( W \), which acts on \( H \) by

\[
N_W H(x) = |W(x)H(x)||W(x)^{-1}\mathbf{1}|.
\]

The following lemma shows that the exhausting operator is sublinear, monotone, and an isometry on \( L^p_{K}(\mathbb{R}^n, W) \).
Lemma 8.4. Given $1 < p < \infty$, a matrix $W$, and $H \in L^p_K(\mathbb{R}^n, W)$, the operator $N_W$ satisfies the following:

1. $H(x) \subset N_W H(x)$.
2. $N_W$ is an isometry: $\|N_W H\|_{L^p_K(\mathbb{R}^n, W)} = \|H\|_{L^p_K(\mathbb{R}^n, W)}$.
3. $N_W$ is sublinear and monotone in the sense of Lemma 5.5.

Proof. To prove the inclusion, note that if $v \in H(x)$, then $W(x)v \in W(x)H(x)$, and so $|W(x)v| \leq |W(x)H(x)|$. Hence $v \in |W(x)H(x)|^{-1}(x)\overline{B} = N_W H(x)$.

To prove $N_W$ is an isometry, it is enough to observe that for almost every $x \in \mathbb{R}^n$,

$$|W(x)N_W H(x)| = |W(x)H(x)|\overline{B} = |W(x)H(x)|.$$ 

Finally, to prove that $N_W$ is sublinear, fix $G, H \in L^p_K(\mathbb{R}^n, W)$. If $w(x) \in W(x)G(x) + W(x)H(x)$, then $|w(x)| \leq |W(x)G(x)| + |W(x)H(x)|$. Hence, $|W(x)(G + H)(x)| \leq |W(x)G(x)| + |W(x)H(x)|$, and so $N_W(G + H)(x) \subset N_WG(x) + N_WH(x)$. Similarly, if $\alpha \in \mathbb{R}$, then $N_W(\alpha H)(x) = \alpha N_W H(x) = \alpha N_W H(x)$.

In the proof of factorization and extrapolation in the next section, we consider two special classes of convex-set valued functions: a function $G$ is ball-valued if there exists a non-negative scalar function $r$ such that $G(x) = r(x)\overline{B}$. Similarly, given a matrix $W$, $G$ is said to be ellipsoid-valued with respect to $W$ if $G(x) = r(x)W(x)\overline{B}$.

Lemma 8.5. Fix $1 \leq p \leq \infty$ and a norm function $\rho$. Let $T$ be a convex-set valued operator that satisfies the hypotheses of Theorem 7.6, and suppose that if $G \in L^p_K(\mathbb{R}^n, \rho)$ is a ball-valued function, then $TG$ is as well. If $S$ is the associated iteration operator, then $SG$ is ball-valued. Moreover, if whenever $G$ is an ellipsoid-valued function with respect to a matrix $W$, then $TG$ is, we have that $SG$ is also ellipsoid-valued function with respect to $W$.

Proof. Let $G = r_0 W \overline{B}$ for some scalar function $r_0$ and matrix $W$. Then by induction, we have that for all $k \geq 0$, $T^kG$ is ellipsoid-valued, so we have that $T^kG = r_k W \overline{B}$. Since the Minkowski sum of two ellipsoids of the form $r W \overline{B}$ and $s W \overline{B}$ is again an ellipsoid of this form, we have, in the notation of Theorem 7.6, that for all $n \in \mathbb{N}$, $S^nG$ is an ellipsoid-valued function with respect to $W$. But then it follows at once from (7.7) that $SG$ is ellipsoid-valued with respect to $W$.

We now define the powers of a ball-valued function. If $G(x) = r(x)\overline{B}$ is a ball-valued function, for all $t > 0$ let $G^t = r^t \overline{B}$. The following lemma is an immediate consequence of this definition.

Lemma 8.6. If $G$ is a ball-valued function, then for all $t > 0$, the mapping $G \mapsto G^t$ is monotone.

Proof of Proposition 8.2. To apply the Rubio de Francia iteration algorithm, we define two auxiliary operators, $T_1$ and $T_2$. Let $P_W = N_W M$, where $M$ is the convex-set valued maximal operator. Let $q = pp'$, $t > 1$. For $G \in L^p_K(\mathbb{R}^n, |\cdot|)$ define

$$T_1 G(x) = \left[W(x)P_W(W^{-1}(N_I G))^{p'}(x)\right]^{1/p'}.$$ 

Here, $N_I$ is the exhausting operator with respect to the identity matrix $I$. By the definition of $N_W$ and $N_I$, the two exponents appear on ball-valued functions and so are well defined. We claim that $T_1$ satisfies the hypotheses of Theorem 7.6. First, by Lemma 8.4 and Theorem 6.9,

$$\int_{\mathbb{R}^n} |T_1 G(x)|^q dx = \int_{\mathbb{R}^n} |W(x)P_W(W^{-1}(N_I G))^{p'}(x)|^p dx \leq C[W]_A q_{pp'} \int_{\mathbb{R}^n} |N_I G(x)|^{pp'} dx = C[W]_A q \int_{\mathbb{R}^n} |G(x)|^q dx.$$ 

We now prove that $T_1$ is monotone. By Lemmas 5.5, 8.4, and 8.6, all its component functions are monotone, so $T_1$, their composition, is as well.
To prove that it is sublinear, first note that since \( T_1 F \) is a ball-valued function, sublinearity is equivalent to showing that for \( G, H \in L^q_k(\mathbb{R}^n, |\cdot|) \) and \( x \in \mathbb{R}^n \),
\[
|T_1(G + H)(x)| \leq |T_1 G(x)| + |T_2 H(x)|.
\]
To prove this, fix \( x \) and define the norm \( \rho(v) = |W(x)v| \). For any locally integrably bounded function \( F \), by the definition of \( N_I \) and by (the proof of) Lemma 8.4,
\[
|(W(x)N_W(MF)(x))^{1/p'}| = |(W(x)MF(x))^{1/p'}| \leq |W(x)MF(x)|^{1/p'} = \rho(N_W(MF)(x))^{1/p'} = \rho(MF(x))^{1/p'}.
\]
Note that \( N_I \) produces ball-valued functions and is sublinear by Lemma 8.4; hence, \(|(G + H)(y)| \leq |G(y)| + |H(y)| \). Therefore, if we combine these two observations, by Lemma 5.6,
\[
|T_1(G + H)(x)| \leq \rho(M(|G(x) + |H(x)|)^{p'}W^{-1}(x)\B)|^{1/p'}
\]
\[
\leq \rho(M(|G(x)|^{p'}W^{-1}(x)\B))^{1/p'} + \rho(M(|H(x)|^{p'}W^{-1}(x)\B))^{1/p'} = |T_1 G(x)| + |T_2 H(x)|.
\]
We define \( T_2 \) similarly:
\[
T_2 G(x) = \left[ W^{-1}(x)P_{W^{-1}}(W(N_I G)^p)(x) \right]^{1/p}.
\]
Since \( W^{-1} \in \mathcal{A}_{p'} \), the same argument as above shows that
\[
||T_2 G||_{L^q_k(\mathbb{R}^n, |\cdot|)} \leq C[W^{-1}]_{\mathcal{A}_{p'}} ||G||_{L^q_k(\mathbb{R}^n, |\cdot|)},
\]
and that \( T_2 \) is sublinear and monotone.

Define the operator \( T = T_1 + T_2 \). Then \( T \) satisfies the hypotheses of Theorem 7.6 with operator norm \( ||T||_{L^q_k(\mathbb{R}^n, |\cdot|)} \leq \left[ W \right]_{\mathcal{A}_p} \) in light of (8.1) and (8.2). Hence, if we define
\[
SG(x) = \sum_{k=0}^{\infty} 2^{-k} ||T||_{L^q_k(\mathbb{R}^n, |\cdot|)}^{-k} T^k G(x),
\]
then \( ||SG||_{L^q_k(\mathbb{R}^n, |\cdot|)} \leq 2 ||G||_{L^q_k(\mathbb{R}^n, |\cdot|)} \) and
\[
T(SG)(x) \subset 2 ||T||_{L^q_k(\mathbb{R}^n, |\cdot|)} SG(x).
\]
Now fix a ball-valued function \( G = r \B \in L^q_k(\mathbb{R}^n, |\cdot|) \). Then by Lemma 8.5, \( SG = r \B \). It follows from (8.3) that \( T_1(SG)(x) \subset 2 ||T||_{L^q_k(\mathbb{R}^n, |\cdot|)} SG(x) \); equivalently,
\[
W(x)P_{W^{-1}(SG)}(x) \subset C_1 SG(x)^{p'},
\]
where \( C_1 = 2^{p'} ||T||_{L^q_k(\mathbb{R}^n, |\cdot|)}^{p'} \). Define the matrix \( W_1(x) = r(x)^{-p'} W(x) \). Then
\[
M(W_1^{-1}\B)(x) \subset P_{W^{-1}(SG)}(x) \subset C_1 W(x)^{-1} SG(x)^{p'} = C_1 W_1(x)^{-1} \B.
\]
Therefore, \( W_1^{-1} \B \in \mathcal{A}_{p'}^1 \); by Corollary 7.4, \( W_1^{-1} \in \mathcal{A}_1 \), and so \( W_1 \in \mathcal{A}_\infty \). Moreover, by (8.1)
\[
[W_1]_{\mathcal{A}_\infty} = [W_1^{-1}]_{\mathcal{A}_1} \lesssim C_1 \lesssim [W]_{\mathcal{A}_p}^{p'}.
\]
We can repeat the above argument, replacing \( T_1 \) by \( T_2 \); if we define \( W_0(x) = r(x)^{p'} W(x) \), then we get that \( W_0 \B \in \mathcal{A}_{p'}^1 \), and so \( W_0 \in \mathcal{A}_1 \). Moreover, by (8.2)
\[
[W_0]_{\mathcal{A}_1} \lesssim ||T||_{L^q_k(\mathbb{R}^n, |\cdot|)} \lesssim [W]_{\mathcal{A}_p}^{p'}
\]
Finally, we have that
\[ W_0^{1/p}(x)W_1^{1/p'}(x) = \tilde{r}(x)W_1^{1/p}(x)\tilde{r}(x)^{-1}W_1^{1/p'}(x) = W(x). \quad \square \]

8.2. Reverse factorization. We now prove the so-called “reverse factorization” property, that the product of suitable powers of \( A_1 \) and \( A_\infty \) weights is an \( A_p \) weight. In the scalar case this is an immediate consequence of the definitions. However, in the matrix case it is much more difficult since the statement involves a weighted geometric mean of two matrices, while the proof requires working with norms rather than matrix weights. To state our result, recall Definition 2.20: given two symmetric, positive definite matrices, for \( 0 < t < 1 \), let \( A\#_tB = A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2} \).

**Proposition 8.7.** Suppose that \( W_0 \in A_1, W_1 \in A_\infty \), and \( 1 < p < \infty \). Then,
\[ \tilde{W} = (((W_0)^2\#_t(W_1)^2)^{1/2} \in A_p. \]

In particular, if \( W_0 \) and \( W_1 \) commute, then \( W_0^{1/p}W_1^{1/p'} \in A_p. \)

The second half of Theorem 8.1 follows immediately from Proposition 8.7. We will in fact prove a much more general result.

**Proposition 8.8.** Given \( 1 \leq q_0, q_1 \leq \infty \), suppose that \( W_0 \in A_{q_0} \) and \( W_1 \in A_{q_1} \). Fix \( 0 < t < 1 \) and define \( \tilde{W} = (((W_0)^2\#_t(W_1)^2)^{1/2} \in A_p \), where \( \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1} \). Moreover,
\[ [\tilde{W}]_{A_q} \leq c(d) [W_0]^{1-t}_{A_{q_0}} [W_1]_A^{t q_1}. \]

Proposition 8.7 follows at once from this if we take \( q_0 = 1, q_1 = \infty \), and \( t = 1/p' \). Beyond its intrinsic interest, we prove Proposition 8.8 because the following corollary, which again follows at once by the correct choice of \( q_0 \) and \( q_1 \), plays an important role in the proof of extrapolation in Section 9. In the scalar case this result is used to prove sharp constant extrapolation and is due to Duoandikoetxea [26, Lemma 2.1] (see also [18, Theorem 3.22]).

**Corollary 8.9.** Given \( 1 < p < \infty \) and \( W \in A_p \), suppose there exists a scalar function \( s \) such that \( W_1 = sW \in A_\infty \). Then for \( p < p_0 < \infty \), \( \tilde{W} = W_0^{p/p_0}W_1^{1-p/p_0} \in A_{p_0} \); moreover,
\[ [\tilde{W}]_{A_{p_0}} \leq c(d) [W_0]^{p/p_0} [W_1]_{A_\infty}^{1-p/p_0}. \]

Similarly, if there exists a scalar function \( s \) such that \( W_0 = pW \in A_1 \), then for \( 1 < p_0 < p \), \( \tilde{W} = W_0^{1-p/p_0}W_1^{p'/p_0} \in A_{p_0} \); moreover,
\[ [\tilde{W}]_{A_{p_0}} \leq c(d) [W_0]_A^{1-p/p_0} [W_1]^{p'/p_0}_{A_p}. \]

**Proof of Proposition 8.8.** We define three norm functions:
\[ \rho_0(x, v) = |W_0(x)v|, \]
\[ \rho_1(x, v) = |W_1(x)v|, \]
\[ \rho(x, v) = |\tilde{W}(x)v|, \quad \text{where } \tilde{W} = (((W_0)^2\#_t(W_1)^2)^{1/2}. \]

For fixed point \( x \), define \( p_t(v) = \rho_0(x, v)^{1-t}\rho_1(x, v)^t \), \( v \in \mathbb{R}^d \). By Corollary 2.22 followed by Lemma 2.15 we have
\[ \rho(x, v) \approx p_t^*(v) \leq p_t(v) = \rho_0(x, v)^{1-t}\rho_1(x, v)^t. \]

Since
\[ A^{-1} \#_t B^{-1} = (A\#_t B)^{-1}, \quad \text{for } A, B \in S_d, \ 0 < t < 1, \]
by Proposition 4.12 we have a similar inequality for dual norms
\[ \rho^*(x, v) \lesssim \rho_0^*(x, v)^{1-t}\rho_1^*(x, v)^t. \]
Fix a cube $Q$. Since $1 = \frac{q(1-t)}{q_0} + \frac{qt}{q_1}$, by (8.4) and Hölder’s inequality (if $q_0, q_1 < \infty$),

\begin{equation}
\langle \rho \rangle_{q,Q}(v) \lesssim \left( \int_Q \rho_0(x,v)^{q(1-t)} \rho_1(x,v)^{qt} \, dx \right)^{\frac{1}{t}} \\
\leq \left( \frac{1}{q_0} \right)^{\frac{q_0}{q_0-t}} \left( \frac{1}{q_1} \right)^{\frac{1}{q_1-t}} \langle \rho_0 \rangle_{q_0,Q}(v)^{1-t} \langle \rho_1 \rangle_{q_1,Q}(v)^{t}.
\end{equation}

A simple modification of this argument shows that this inequality holds if $q_0$ or $q_1 = \infty$. Since we also have that $1 = \frac{q(1-t)}{q_0} + \frac{qt}{q_1}$, we can repeat this argument using (8.5) to get that

\begin{equation}
\langle \rho^* \rangle_{q',Q}(v) \lesssim \langle \rho^*_0 \rangle_{q'_0,Q}(v)^{1-t} \langle \rho^*_1 \rangle_{q'_1,Q}(v)^t.
\end{equation}

Since $W_0 \in A_{q_0}$ and $W_1 \in A_{q_1}$, by Definition 6.2 we have that

\begin{equation}
\langle \rho^*_0 \rangle_{q'_0,Q}(v)^{1-t} \langle \rho^*_1 \rangle_{q'_1,Q}(v)^t \leq [\rho_0]_{A_{q_0},Q}^t [\rho_1]_{A_{q_1},Q}^t \langle \rho^*_0 \rangle_{q_0,Q}(v)^{1-t} \langle \rho^*_1 \rangle_{q_1,Q}(v)^t.
\end{equation}

To simplify notation, we define several norms:

\begin{align*}
\sigma_0 &= \langle \rho^*_0 \rangle_{q'_0,Q}, & \tau_0 &= \langle \rho_0 \rangle_{q_0,Q}, \\
\sigma_1 &= \langle \rho^*_1 \rangle_{q'_1,Q}, & \tau_1 &= \langle \rho_1 \rangle_{q_1,Q}, \\
\sigma &= \langle \rho^* \rangle_{q,Q}, & \tau &= \langle \rho \rangle_{q,Q},
\end{align*}

and the two geometric means

\begin{align*}
\sigma_t(v) &= \sigma_0(v)^{1-t} \sigma_1(v)^t, & \tau_t(v) &= \tau_0(v)^{1-t} \tau_1(v)^t.
\end{align*}

By inequality (8.7), $\sigma(v) \lesssim \sigma_0(v)^{1-t} \sigma_1(v)^t$. By the definition of the dual norm (2.3) it follows that $\sigma^*(v) \gtrsim \sigma_t^*(v)$. By Corollary 2.9 and Lemma 2.14, these are both norms and if we dualize again, we get

\begin{equation}
\sigma(v) = \sigma^{**}(v) \lesssim \sigma_t^{**}(v).
\end{equation}

Similarly, by inequality (8.6), $\tau(v) \lesssim \tau_t(v)$, so we can repeat the above argument to get that $\tau(v) \lesssim \tau_t^{**}(v)$. If we dualize yet again, we get

\begin{equation}
\tau_t^{**}(v) = \tau_t^{***}(v) \lesssim \tau^*(v).
\end{equation}

Finally, by inequality (8.8) we have that

\[ \sigma_t(v) \leq [\rho_0]_{A_{q_0},Q}^{1-t} [\rho_1]_{A_{q_1},Q}^t \tau_0(v)^{1-t} \tau_1^*(v)^t. \]

If we dualize twice, and then apply the dual of equivalence (2.6) in Proposition 2.19, we get that

\begin{equation}
\sigma_t^{**}(v) \leq [\rho_0]_{A_{q_0},Q}^{1-t} [\rho_1]_{A_{q_1},Q}^t \tau_0^*(v)^{1-t} \tau_1^*(v)^t \lesssim [\rho_0]_{A_{q_0},Q}^{1-t} [\rho_1]_{A_{q_1},Q}^t \tau^*(v) = [\rho_0]_{A_{q_0},Q}^{1-t} [\rho_1]_{A_{q_1},Q}^t \tau_t^*(v).
\end{equation}

If we now combine inequalities (8.9), (8.10), and (8.11), we have

\[ \langle \rho^* \rangle_{q',Q}(v) = \sigma(v) \lesssim \sigma_t^{**}(v) \lesssim [\rho_0]_{A_{q_0},Q}^{1-t} [\rho_1]_{A_{q_1},Q}^t \tau_0^*(v)^{1-t} \tau_1^*(v)^t \lesssim [\rho_0]_{A_{q_0},Q}^{1-t} [\rho_1]_{A_{q_1},Q}^t \tau^*(v) = [\rho_0]_{A_{q_0},Q}^{1-t} [\rho_1]_{A_{q_1},Q}^t \langle \rho \rangle_{q,Q}(v). \]

Since the cube $Q$ is arbitrary, we get the desired result. \qed
Remark 8.10. Proposition 8.8 can be also shown using the complex interpolation method. Here we give only a brief sketch of the argument. Applying the exactness of the complex interpolation functor of exponent $t$ [3, Theorem 4.1.2] to the identity operator on $\mathbb{R}^d$ equipped with norms appearing in Definition 6.2, we deduce that the complex interpolation norms satisfy

$$\langle \rho_0^* \rangle_q, Q : \langle \rho_1^* \rangle_{q_1}, Q \rangle_t \leq [\rho_0]_{A_{q_0}}^{1-t} [\rho_1]_{A_{q_1}}^t \langle \rho_0 \rangle_{q_0}, Q : \langle \rho_1 \rangle_{q_1}, Q \rangle_t \quad (8.12)$$

The last identity is a consequence of the duality theorem [3, Corollary 4.5.2]. Then, we use the fact that for any two norms $p_0$ and $p_1$ on $\mathbb{R}^d$, the complex interpolation norm satisfies

$$[p_0, p_1]_t \approx (p_0^{1-t}, p_1^t)^{*}.$$  

This can be shown using Corollary 2.22 and the complex interpolation of weighted $L^p$ spaces [3, Theorem 5.5.3]. Hence, applying the double dual to (8.7) followed by (8.12) and then by the triple dual of (8.6) yields

$$\langle \rho^* \rangle_q, Q \lesssim [\langle \rho_0^* \rangle_q, Q : \langle \rho_1^* \rangle_{q_1}, Q \rangle_t \lesssim [\rho_0]_{A_{q_0}}^{1-t} [\rho_1]_{A_{q_1}}^t \langle \rho_0 \rangle_{q_0}, Q : \langle \rho_1 \rangle_{q_1}, Q \rangle_t \lesssim [\rho_0]_{A_{q_0}}^{1-t} [\rho_1]_{A_{q_1}}^t \langle \rho \rangle_{q_0}, Q.$$  

This shows that $\rho$ belongs to $A_q$ with appropriate bound on $[\rho]_{A_q}$.

9. Extrapolation of matrix weights

In this section we state and prove the Rubio de Francia extrapolation theorem for matrix $A_p$ weights, originally stated as Theorem 1.4 in the Introduction. As we noted there, we prove a version of sharp constant extrapolation; this proof requires multiple cases. A simpler proof, with only one case but which does not give the best possible constant or include the endpoint result $p_0 = \infty$, is possible, following the proof given in [18, Theorem 3.9]. We leave the details to the interested reader.

To state our result, we introduce the convention of extrapolation pairs. This approach to extrapolation was developed in [18]. Hereafter, $\mathcal{F}$ will denote a family of pairs $(f, g)$ of measurable, vector-valued functions such that neither $f$ nor $g$ is equal to 0 almost everywhere. If we write an inequality of the form

$$\|f\|_{L^p(\mathbb{R}^n, W)} \leq C\|g\|_{L^p(\mathbb{R}^n, W)}, \quad (f, g) \in \mathcal{F},$$

we mean that this inequality holds for all pairs $(f, g) \in \mathcal{F}$ such that the lefthand side of this inequality is finite. The constant, whether given explicitly or implicitly, is assumed to be independent of the pair $(f, g)$ and to depend only on $[W]_{A_p}$ and not on the particular weight $W$. We want to stress that $\|f\|_{L^p(\mathbb{R}^n, W)} < \infty$ is a crucial technical assumption in our proof, and to apply extrapolation an appropriate family $\mathcal{F}$ must be constructed. In the scalar case this can easily be done via a truncation argument and approximation: see [14, Section 6]. In the case of matrix weights a similar argument can be applied: see below.

Theorem 9.1. Suppose that for some $p_0$, $1 \leq p_0 \leq \infty$, there exists an increasing function $K_{p_0}$ such that for every $W_0 \in A_{p_0}$,

$$\|f\|_{L^p(\mathbb{R}^n, W_0)} \leq K_{p_0}([W_0]_{A_{p_0}})\|g\|_{L^p(\mathbb{R}^n, W_0)}, \quad (f, g) \in \mathcal{F}, \quad (9.1)$$

Then for all $p$, $1 < p < \infty$, and for all $W \in A_p$,

$$\|f\|_{L^p(\mathbb{R}^n, W)} \leq K_p(p, p_0, n, d, [W]_{A_p})\|g\|_{L^p(\mathbb{R}^n, W)}, \quad (f, g) \in \mathcal{F}, \quad (9.2)$$

where

$$K_p(p, p_0, n, d, [W]_{A_p}) = C(p, p_0)K_{p_0} \left( C(n, d, p, p_0)[W]_{A_p} \max \left\{ \frac{p}{p_0}, \frac{p_0}{p} \right\} \right).$$
Before giving the proof of Theorem 9.1, we want to sketch how to use it to prove weighted norm inequalities. We will consider the case of proving inequalities of the form \( \| T f \|_{L^p} \leq C \| f \|_{L^p} \), where \( T \) is a sublinear operator that is bounded on the scalar weighted spaces \( L^p(w) \), where \( w \in A_p \). This is the case, for example, if \( T \) is a Calderón-Zygmund singular integral operator.

We would like to apply extrapolation to pairs of the form \((Tf, f)\); but to do so, we need to construct a family \( \mathcal{F} \) such that given any \( p, 1 < p < \infty \), and \( W \in A_p \), then for any pair \((Tf, f) \in \mathcal{F}, \| T f \|_{L^p(W)} < \infty \). Given a function \( f \in L^\infty_c(\mathbb{R}^n, \mathbb{R}^d) \), we have that \( f \in L^p(\mathbb{R}^n, w) \) for any scalar weight \( w \in A_p \). Moreover, by Corollary 6.7, if matrix \( W \in A_p \), then \( |W|_{op} \) is a scalar \( A_p \) weight. Hence, since \( T \) is bounded on the scalar weighted spaces,

\[
\int_{\mathbb{R}^n} |W(x)T f(x)|^p \, dx \leq \int_{\mathbb{R}^n} (|W(x)|_{op}|T f(x)|)^p \, dx \leq \int_{\mathbb{R}^n} (|W(x)|_{op}|f(x)|)^p \, dx < \infty.
\]

Therefore, if we form the family of extrapolation pairs

\[ \mathcal{F} = \{(Tf, f) : f \in L^\infty_c(\mathbb{R}^n, \mathbb{R}^d)\}, \]

for all \( 1 < p < \infty \) we can apply the conclusion of Theorem 9.1 to every pair in \( \mathcal{F} \). This is sufficient to establish the desired result for every \( f \in L^p(\mathbb{R}^n, W) \). From [19, Propositions 3.6, 3.7] we have that \( L^\infty_c(\mathbb{R}^n, \mathbb{R}^d) \) and \( C^\infty_c(\mathbb{R}^n, \mathbb{R}^d) \) are dense in \( L^p(\mathbb{R}^n, W) \) for any matrix weight \( W \) and \( 1 \leq p < \infty \). Therefore, by a standard approximation argument we get the desired inequality.

**Proof of Theorem 9.1.** The proof has four cases and is modeled on the proof of sharp-constant extrapolation in [18, Theorem 3.22].

Fix \( 1 < p < \infty \) and \( W \in A_p \). We begin by defining two iteration operators. To define the first, let \( P_W = N_WM \), where \( M \) is the convex-set valued maximal operator and \( N_W \) is from Definition 8.3. By Lemma 8.4 and Theorem 6.9,

\[
\|P_W\|_{L^p(K, W)} = \|M\|_{L^p(K, W)} \leq C(n, d, p) |W|_{op}^p.
\]

Moreover, by Lemmas 5.5 and 8.4, \( P_W \) is sublinear and monotone. Therefore, by Theorem 7.6 we can define

\[
\mathcal{R}_W H(x) = \sum_{k=0}^\infty 2^{-k} \|P_W\|^k_{L^p(K, W)} \mathcal{P}_W H(x),
\]

and we have that

(A) \( H(x) \subset \mathcal{R}_W H(x) \),

(B) \( \|\mathcal{R}_W H\|_{L^p(K, W)} \leq 2\|H\|_{L^p(K, W)} \),

(C) \( \mathcal{R}_W H \in A^K_1 \) and \( M(\mathcal{R}_W H)(x) \subset P_W(\mathcal{R}_W H)(x) \subset 2C(n, d, p) |W|_{op}^p \mathcal{R}_W H(x) \); the first inclusion in (C) follows from Lemma 8.4. Further, by the definition of \( N_W \), we have that if \( H = rW^{-1}M \), then \( P_W H \) is also an ellipsoid-valued function with respect to \( W^{-1} \). Hence, by Lemma 8.5, \( \mathcal{R}_W H \) is also of this form.

We now define the second iteration operator. Since \( W \in A_p \), \( W^{-1} \in A_{p'} \), so by Theorem 6.9, \( M \) is bounded on \( L^p(K, W^{-1}) \) and

\[
\|M\|_{L^p(K, W^{-1})} \leq C(n, d, p) \|W^{-1}\|_{A_{p'}} = C(n, d, p) |W|_{op}^p.
\]

Define \( M' H(x) = W^{-1}(x)M(WH)(x) \). Then

\[
\|M' H\|_{L^p(K, W^{-1})} = \|M(WH)\|_{L^p(K, W^{-1})} \leq C(n, d, p) |W|_{op}^p \|H\|_{L^p(K, W^{-1})},
\]

and we have that

(A) \( H(x) \subset \mathcal{R}_W H(x) \),

(B) \( \|\mathcal{R}_W H\|_{L^p(K, W^{-1})} \leq 2\|H\|_{L^p(K, W^{-1})} \),

(C) \( \mathcal{R}_W H \in A^K_1 \) and \( M(\mathcal{R}_W H)(x) \subset P_{W'}(\mathcal{R}_W H)(x) \subset 2C(n, d, p) |W|_{op}^p \mathcal{R}_W H(x) \); the first inclusion in (C) follows from Lemma 8.4.
Now let \( P'_I = N_I M' \); again by Lemmas 5.5 and 8.4, \( P'_I \) is sublinear and monotone, so by Theorem 7.6 we can define
\[
\mathcal{R}'_I H(x) = \sum_{k=0}^{\infty} 2^{-k} \| P'_I \|^{k} L_{K}(\mathbb{R}^n,|\cdot|) (P'_I)^k H(x),
\]
and we have that
\[
\begin{align*}
(\text{A'}) & \quad H(x) \subset \mathcal{R}'_I H(x), \\
(\text{B'}) & \quad \| \mathcal{R}'_I H \|_{L_{K}(\mathbb{R}^n,|\cdot|)} \leq 2 \| H \|_{L_{K}(\mathbb{R}^n,|\cdot|)}, \quad \\
(\text{C'}) & \quad W \mathcal{R}'_I H \in A^C_I \text{ and } M(W \mathcal{R}'_I H)(x) \subset 2C(n,d,p)[W]^p \mathcal{R}'_I H(x).
\end{align*}
\]
To see why (C') holds, note that by Theorem 7.6 and Lemma 8.4 we have that
\[
W^{-1}(x)M(W \mathcal{R}'_I H)(x) \subset W^{-1}(x)N_I M(W \mathcal{R}'_I H)(x) \subset 2\| P'_I \|_{L_{K}(\mathbb{R}^n,|\cdot|)} \mathcal{R}'_I H(x).
\]
Finally, by Lemma 8.5, if \( H \) is a ball-valued function, then so is \( \mathcal{R}'_I H \).

To prove extrapolation we consider four cases, depending on the relative sizes of \( p \) and \( p_0 \).

**Case I:** \( 1 < p < p_0 < \infty \). Fix \( (f,g) \in \mathcal{F} \). To prove inequality (9.2), we may suppose, by our assumptions on the family \( \mathcal{F} \), that \( 0 < \| f \|_{L_p(\mathbb{R}^n,W)} < \infty \). Similarly, we may assume that \( 0 < \| g \|_{L_p(\mathbb{R}^n,W)} < \infty \); we may assume the second inequality since otherwise (9.2) is trivially true. Define the functions
\[
\begin{align*}
F(x) &= \text{conv}\{-f(x), f(x)\}, \quad N_W F(x) = |W(x)F(x)|W^{-1}(x)\mathbb{B}, \\
G(x) &= \text{conv}\{-g(x), g(x)\}, \quad N_W G(x) = |W(x)G(x)|W^{-1}(x)\mathbb{B},
\end{align*}
\]
where we have that
\[
|W(x)F(x)| = |W(x)f(x)|, \quad |W(x)G(x)| = |W(x)g(x)|.
\]
Now define the ellipsoid-valued function
\[
\tilde{H}(x) = \left( \frac{|W(x)f(x)|}{\|f\|_{L_p(\mathbb{R}^n,W)}} + \frac{|W(x)g(x)|}{\|g\|_{L_p(\mathbb{R}^n,W)}} \right) W^{-1}(x)\mathbb{B} = \tilde{h}(x)W^{-1}(x)\mathbb{B}.
\]
Then we have that \( \| \tilde{H} \|_{L_p(\mathbb{R}^n,W)} \leq 2 \). The function \( \mathcal{R}_W \tilde{H} \) is also ellipsoid-valued with respect to \( W^{-1} \).

Hence, there exists a scalar function, which we denote by \( \mathcal{R}_W \tilde{h} \), such that
\[
\mathcal{R}_W \tilde{H}(x) = \mathcal{R}_W \tilde{h}(x)W^{-1}(x)\mathbb{B}.
\]
By property (A), \( \tilde{H}(x) \subset \mathcal{R}_W \tilde{H}(x) \), which implies that \( \tilde{h}(x) \leq \mathcal{R}_W \tilde{h}(x) \).

By Hölder’s inequality with exponents \( p_0/p \) and \( (p_0/p)' = \frac{p_0}{p_0-p} \),
\[
\left( \int_{\mathbb{R}^n} |W(x)f(x)|^p \, dx \right)^{\frac{1}{p}} \leq \left( \int_{\mathbb{R}^n} |\mathcal{R}_W \tilde{h}(x)|^{\frac{p_0-p}{p_0}} W(x)f(x)|^{p_0} \, dx \right)^{\frac{1}{p_0}} \mathcal{R}_W \tilde{h}(x)^{\frac{p}{p_0}} \left( \int_{\mathbb{R}^n} \mathcal{R}_W \tilde{h}(x)^p \, dx \right)^{\frac{p_0-p}{p_0}} \leq I_1 \left( \int_{\mathbb{R}^n} |\mathcal{R}_W \tilde{h}(x)|^{\frac{p_0-p}{p_0}} W(x)f(x)|^{p_0} \, dx \right)^{\frac{1}{p_0}} \mathcal{R}_W \tilde{h}(x)^{\frac{p}{p_0}} \left( \int_{\mathbb{R}^n} \mathcal{R}_W \tilde{h}(x)^p \, dx \right)^{\frac{p_0-p}{p_0}}.
\]

We estimate \( I_1 \) and \( I_2 \) separately. To estimate the latter: by the definition of \( \mathcal{R}_W \tilde{h} \), by property (B), and by Lemma 8.4,
\[
I_2 = \int_{\mathbb{R}^n} \mathcal{R}_W \tilde{h}(x)^p \, dx = \int_{\mathbb{R}^n} |W(x)\mathcal{R}_W \tilde{H}(x)|^p \, dx \leq 2^p \int_{\mathbb{R}^n} |W(x)\tilde{H}(x)|^p \, dx \leq 4^p.
\]
To estimate $I_1$ note first that by property (C),
\[(R_W \tilde{h}) W^{-1} \bar{R} = R_W \tilde{h} \in A^p_1.\]
By Corollary 7.4, $(R_W \tilde{h})^{-1} W^{-1} \in A_1$, and so $(R_W \tilde{h})^{-1} W \in A_\infty$. Thus, by Corollary 8.9,
\[W_0 = (R_W \tilde{h}) \frac{W_0 - p}{R_0} W = [(R_W \tilde{h})^{-1} W] \frac{W_0 - p}{R_0} W \frac{R}{R_0} \in A_{p_0},\]
and
\[(9.3) \quad [W_0]_{A_{p_0}} \leq C(n, d, p, p_0) \left[ W_{p_0}^\frac{p}{p_0} [W]_{A_p} \right]_{A_{p_0}} = C(n, d, p, p_0) \left[ W_{p_0}^\frac{p}{p_0} \right]_{A_p}.\]
Second, since $R_W \tilde{h}(x) \geq \tilde{h}(x) \geq |W(x)f(x)|/\|f\|_{L^p(\mathbb{R}^n, W)}$, we have
\[
I_1 = \|f\|_{L^p(W_0, \mathbb{R}^d)}^{p_0} = \int_{\mathbb{R}^n} \left| R_W \tilde{h}(x) \frac{W_0 - p}{R_0} W(x)f(x) \right|^p \, dx \\
\leq \int_{\mathbb{R}^n} \left| R_W \tilde{h}(x) \frac{W_0 - p}{R_0} W(x)F(x) \right|^p \, dx \\
\leq \|f\|_{L^p(W_0, \mathbb{R}^n, W)}^{p_0} \int_{\mathbb{R}^n} |W(x)F(x)|^p \, dx = \|f\|_{L^p(W_0, \mathbb{R}^n, W)}^{p_0} < \infty.
\]
Likewise, using the fact that $R_W \tilde{h}(x) \geq \tilde{h}(x) \geq |W(x)G(x)|/\|g\|_{L^p(\mathbb{R}^n, W)}$, we have
\[
\|g\|_{L^p(W_0, \mathbb{R}^d)}^{p_0} = \int_{\mathbb{R}^n} \left| R_W \tilde{h}(x) \frac{W_0 - p}{R_0} W(x)g(x) \right|^p \, dx \leq \|g\|_{L^p(W_0, \mathbb{R}^n, W)}^{p_0} < \infty.
\]
Taken together, these estimates imply that we can apply our hypothesis (9.1) to the pair $(f, g)$ with the weight $W_0$. Therefore, by (9.3) we have
\[
I_1^{\frac{1}{p_0}} = \|f\|_{L^p(W_0, \mathbb{R}^d)}^{p_0} \leq K_{p_0}([W_0]_{A_{p_0}}) \|g\|_{L^p(W_0, \mathbb{R}^d)}^{p_0} \leq K_{p_0} \left( C(n, d, p, p_0) \left[ W_{p_0}^\frac{p}{p_0} \right]_{A_p} \right) \|g\|_{L^p(\mathbb{R}^n, W)}.\]
Combining this inequality with the estimate for $I_2$ yields (9.2).

**Case II: $p_0 = \infty$.** As in the previous case we have $W_0 = (R_W \tilde{h})^{-1} W \in A_\infty$. Moreover, for almost every $x$ we have that
\[
|W(x)F(x)|/R_W \tilde{h}(x)^{-1} \leq |W(x)F(x)|/\tilde{h}(x)^{-1} \leq \|f\|_{L^p(\mathbb{R}^n, W)}.
\]
Thus, $\|f\|_{L^\infty(\mathbb{R}^n, W_0)} \leq \|f\|_{L^p(\mathbb{R}^n, W)} < \infty$. The same argument also shows that
\[
\|g\|_{L^\infty(\mathbb{R}^n, W_0)} \leq \|g\|_{L^p(\mathbb{R}^n, W)} < \infty.
\]
Therefore, we can apply (9.1) to the pair $(f, g) \in \mathcal{F}$ and argue as in Case I to get
\[
\|f\|_{L^p(\mathbb{R}^n, W)}^p = \int_{\mathbb{R}^n} |W(x)f(x)|^p \, dx \\
= \int_{\mathbb{R}^n} \left| R_W \tilde{h}(x)^{-1} W(x)f(x) \right|^p \, dx \\
\leq \|f\|_{L^\infty(\mathbb{R}^n, W_0)}^{p_0} \int_{\mathbb{R}^n} R_W \tilde{h}(x)^p \, dx \\
\leq 4^p K_\infty([W_0]_{A_{p_0}})^p \|g\|_{L^\infty(\mathbb{R}^n, W_0)}^p \\
\leq 4^p K_\infty(C(n, d, p) [W]_{A_p}^{p_0}) \|g\|_{L^p(\mathbb{R}^n, W)}^p.\]
Case III: $1 < p_0 < p$. Fix $(f, g) \in \mathcal{F}$. To prove inequality (9.2), we may again assume that $0 < \|f\|_{L^p(\mathbb{R}^n, W)}$, $\|g\|_{L^p(\mathbb{R}^n, W)} < \infty$. Since the dual of the scalar function space $L^p(\mathbb{R}^n)$ is $L^{p'}(\mathbb{R}^n)$, there exists $h \in L^{p'}(\mathbb{R}^n)$, $\|h\|_{L^{p'}(\mathbb{R}^n)} = 1$, such that

$$
\|f\|_{L^p(\mathbb{R}^n, W)} = \int_{\mathbb{R}^n} |W(x)f(x)|h(x)\,dx.
$$

Define the ball-valued function $H(x) = h(x)B$; since $H \in L^{p'}(\mathbb{R}^n)$, $R_h^tH$ is defined and is ball-valued function; set $R_h^tH(x) = R_h^t h(x)B$. As before, by (A'), we have that $h(x) \leq R_h^t h(x)$. Therefore, by Hölder’s inequality with exponent $p_0$, we have that

$$
\int_{\mathbb{R}^n} |W(x)f(x)|h(x)\,dx \leq \int_{\mathbb{R}^n} |R_h^t h(x)|^{1-p'/p_0} W(x)f(x)|h(x)|^{p'/p_0} \,dx
$$

\[\leq \left( \int_{\mathbb{R}^n} |R_h^t h(x)|^{1-p'/p_0} W(x)f(x)|h(x)|^{p_0} \,dx \right)^{\frac{1}{p_0}} \left( \int_{\mathbb{R}^n} h(x)^{p_0} \,dx \right)^{\frac{1}{p_0}}.
\]

To complete the estimate, first note that by (C'), $(R_h^t h)W\overline{B} \in A_{\infty}$. Hence, by Corollary 7.4, $(R_h^t h)W \in A_1$. Therefore, by Corollary 8.9, $W_0 = R_h^t h(x)^{1-p'/p_0} W(x) \in A_{p_0}$ and

$$
[W_0]_{A_{p_0}} \leq C(n, d, p, p_0)[W]_{A_p}^{p(1-p'/p_0)} [W]_{A_p}^{p'/p_0} = C(n, d, p, p_0)[W]_{A_p}^{p/p_0}.
$$

Moreover, the above estimates yield

$$
\|f\|_{L^p(\mathbb{R}^n, W)} \leq \|f\|_{L^{p_0}(\mathbb{R}^n, W_0)}.
$$

On the other hand, by Hölder’s inequality with exponent $p/p_0$ and property (B'),

$$
\|f\|_{L^{p_0}(\mathbb{R}^n, W_0)} = \int_{\mathbb{R}^n} |R_h^t h(x)|^{1-p'/p_0} W(x)f(x)|^{p_0} \,dx
$$

\[\leq \left( \int_{\mathbb{R}^n} |W(x)f(x)|^p \,dx \right)^{p_0/p} \left( \int_{\mathbb{R}^n} |R_h^t h(x)|^{p_0} \,dx \right)^{{1/p_0}'} \leq 2^{1/(p/p_0)'} \|f\|_{L^p(\mathbb{R}^n, W)} < \infty.
\]

Likewise, we have

$$
\|g\|_{L^{p_0}(\mathbb{R}^n, W_0)} \leq 2^{1/(p/p_0)'} \|g\|_{L^p(\mathbb{R}^n, W)} < \infty.
$$

Therefore, we can apply our hypothesis (9.1) to the pair $(f, g)$ with the weight $W_0$

$$
\|f\|_{L^p(\mathbb{R}^n, W)} \leq \|f\|_{L^{p_0}(\mathbb{R}^n, W_0)} \leq K_{p_0}([W_0]_{A_{p_0}}) \|g\|_{L^{p_0}(\mathbb{R}^n, W_0)}
$$

\[\leq C(p, p_0)K_{p_0} (C(n, d, p, p_0)[W]_{A_p}^{p/p_0}) \|g\|_{L^p(\mathbb{R}^n, W)}.
\]

Case IV: $p_0 = 1$. We make the same assumptions and use the same notation as in the previous case. Then we have that $W_0 = (R_h^t h)W \in A_1$, and the above argument shows that

$$
\|f\|_{L^1(\mathbb{R}^n, W_0)} \leq 2^{1/p'} \|f\|_{L^p(\mathbb{R}^n, W)} < \infty.
$$

The same inequality holds for $g$. Therefore, we can apply inequality (9.1) to the pair $(f, g)$ to get

$$
\|f\|_{L^p(\mathbb{R}^n, W)} = \int_{\mathbb{R}^n} |W(x)f(x)|h(x)\,dx \leq \int_{\mathbb{R}^n} |R_h^t h(x)W(x)f(x)| \,dx = \|f\|_{L^1(\mathbb{R}^n, W_0)}
$$

\[\leq K_1([W_0]_{A_1}) \|g\|_{L^1(\mathbb{R}^n, W_0)} \leq 2K_1(C(n, d, p)[W]_{A_p}) \|g\|_{L^p(\mathbb{R}^n, W)}. \quad \square
\]
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