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A hybrid fractional COVID-19 model with general population mask use: Numerical treatments

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Abstract In this work, a novel mathematical model of Coronavirus (2019-nCov) with general population mask use with modified parameters. The proposed model consists of fourteen fractional-order nonlinear differential equations. Grünwald-Letnikov approximation is used to approximate the new hybrid fractional operator. Compact finite difference method of six order with a new hybrid fractional operator is developed to study the proposed model. Stability analysis of the used methods are given. Comparative studies with generalized fourth order Runge–Kutta method are given. It is found that, the proposed model can be described well the real data of daily confirmed cases in Egypt.

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1. Introduction

In present time the whole globe is facing a threatful outbreak called COVID-19 which originated from Wuhan; a large city in China. Since December 2019 to the 18th of July 2020, nearly 590,000 people died due to the mentioned disease and about 13.81 millions were all over the world. Also 7.718606 millions infected people have recovered from the disease, this data was recorded by WHO [1].

A novel coronavirus is a serious global issue and has a negative impact on the economy of Egypt. According to the publicly reported data, the first case of the novel corona virus in Egypt was reported on 14 February 2020. Total of 96753 cases were recorded in Egypt from the beginning of the pandemic until the eighteenth of August, where 96,581 individuals were Egyptians and 172 were foreigners. The total number of people recovered reached 61,562, or 63.6%, and the infection rate is 967 infectious per million of the population. The total number of deaths recorded 5,184 deaths, of which 5,165 were Egyptians, 19 were foreigners, and the death rate reached 5.3% of the infected.
To understand the spread different infectious diseases including COVID-19 among plants, human or any other animals, mathematicians implement the concept of differentiation and integration to model the dynamics of that disease in the form of ODEs or PDEs that could be used to predict the transmission of that disease within a specific population. For this task, they divide the total population into different compartments. The solution of these systems may deliver an indication to health and other sectors that how severe the spread can be and what parameter is needed to control the spread of virus or disease under consideration.

Researchers around the world focus on the study the coronavirus infection in different scenarios. Some study the statistical tools to understand the infected cases and to find some relationships that can be used further to help the minimize the spread of the infection. Some studying the virus biologically and to think of some vaccine developments. Besides this, others develop mathematical models in terms to predict the infection eradication stage. Mathematical models are considered fast for many infectious diseases and its outbreak. Like corona infection, the authors used the theory of differential equations to formulate mathematical models with different characteristics and provided useful results for its eliminations. Several mathematical models are considered in literature in order to investigate and analyze the complex transmission pattern of the novel ongoing COVID-19 pandemic, see ([3–5, 23, 6–24, 30, 34]) and the references therein.

The fractional mathematical models that are generalized model and considered useful for modeling purposes in epidemiology. Various benefits can be obtained from a fractional order system in the sense of best data fitting, information about its memory and to identify the best possible value of the fractional order that can best describe the model for the real cases. In addition, the hereditary properties make the models constructed in fractional derivatives stronger and more useful for describing the real phenomenon [31]. It is useful than the ordinary order and with the benefit of crossover behaviors. Different mathematical models with interesting results are proposed in ([30] [13, 17–20, 21, 22, 23]) and the references therein. Moreover, it can be analyzed complex phenomena including disease models, see for example ([9, 11, 14–17, 20, 32, 33]). One of the most effective and reliable operators is the hybrid fractional operator, it can be expressed as a linear combination of the Riemann–Liouville fractional integral and the Caputo fractional derivative. As in [5] this operator is general than Caputo fractional operator.

The analytic solution of such fractional order differential equations is difficult to be found. Hence, it is very important to develop numerical techniques to approximate the solutions of these models. High-order compact finite difference schemes are becoming increasingly common because of their high precision and the advantages associated with compact cells for more details see ([11, 25]).

The aim of this work is to extend the model of COVID-19 with general population mask use which given as integer order in [3] to hybrid fractional order model and modified parameters. The boundedness, existence, uniqueness and the basic reproduction number of the present model will be discussed. Two high accuracy numerical methods are given to study the proposed model numerically. These methods are the compact finite difference of six order method with the Caputo proportional constant hybrid operator (CPC-CFD6M) and Generalized fourth order Runge–kutta method (GRK4M). Stability analysis of the used methods are given. Comparison between two methods will be given. We show that our COVID-19 model describes well the real data of daily confirmed cases in Egypt from 9 March to 13 June, 2020 [2].

To the best of our knowledge, the numerical studies using the high accuracy of compact finite difference of six order method for a hybrid fractional Coronavirus (2019-nCov) model with general population mask use and modified parameters have never been explored.

The rest of this paper is structured as follows: Preliminaries and notations are introduced in Section 2. The hybrid fractional order models are given and some properties of the proposed model such as the boundedness, existence, uniqueness and the basic reproduction number of the proposed model are proved in Section 3. Section 4, Numerical methods for solving the proposed model are presented. In Section 5, we discuss the numerical simulations. The conclusions are given in Section 6.

2. Background materials

In the following, some mathematical tools used in this work are introduced:

- We can define Caputo fractional order derivative as follows [4]:

\[
C_0^\alpha D_t^\alpha y(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t y(t)(t - s)^{-\alpha}ds, \quad 0 < \alpha < 1,
\]

where \(\Gamma\) is the Euler gamma function.

- The Riemann–Liouville integral can be defined as follows [4]:

\[
RL_0^\alpha I_t^\alpha y(t) = \left[\int_0^t y(s)(t - s)^{\alpha - 1}ds\right] \frac{1}{\Gamma(\alpha)}.
\]

where, \(0 < \alpha < 1\) and \(y(t)\) is an integrable function.

- The hybrid fractional operator is defined as follows [5]:

\[
CP_0^\alpha D_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(y(s)K_1(x, s) + y(t)K_0(x, s)\right)(t - s)^{-\alpha}ds.
\]

where, \(K_0(x, t) = x(t^{\alpha - 1}), \ K_1(x, t) = (1 - x)t^\alpha\), \(0 < \alpha < 1\).

Definition 2.1. [5] The Caputo proportional hybrid operator (CP) given in (3) is defined either as general way:

\[
CP_0^\alpha D_t^\alpha y(t) = \left(\int_0^t (t - s)^{-\alpha}(y(s)K_1(x, s) + y(t)K_0(x, s))ds\right)\frac{1}{\Gamma(1 - \alpha)},
\]

\[
= K_1(x, t)y(t) + K_0(x, t)y(t)\left(\frac{t^\alpha}{\Gamma(1 - \alpha)}\right).
\]

Or as the Caputo proportional constant hybrid operator (CPC) [5]:

\[
CPC_0^\alpha D_t^\alpha y(t) = \left(\int_0^t (t - s)^{-\alpha}(y(s)K_1(x) + y(t)K_0(x))ds\right)\frac{1}{\Gamma(1 - \alpha)},
\]

\[
= K_1(x)RL_0^\alpha I_t^\alpha y(t) + K_0(x)C_0^\alpha D_t^\alpha y(t),
\]

where, \(K_0(x) = xQ^{1-\alpha}\), \(K_1(x) = (1 - x)Q^\alpha\), \(Q\) is a constant.
Definition 2.2. The inverse operators to the fractional CPC derivatives is given by [5]:

\[
\text{CPC}_a D_t^\gamma y(t) = \left( \int_0^t \exp \left[ \frac{K_i(z)}{K_0(z)} (t-s) \right] K_t^\gamma(s) ds \right) \frac{1}{K_0(z)} . \tag{6}
\]

3. Mathematical models of a hybrid fractional order

In the following, two Coronavirus spreading models with and without mask use which presented in [3] will be developed. We extended these models using a new hybrid fractional derivatives [5], moreover, the parameters dimensions will be adapted in [20,23]). These models given as follows:

3.1. A hybrid fractional order COVID-19 model without mask use

The hybrid fractional COVID-19 model without any mask use is given:

\[
\text{CPC}_a D_t^\gamma S = -\beta^I(I + \eta A) \frac{S}{N} , \tag{7}
\]

\[
\text{CPC}_a D_t^\gamma E = \beta^I(I + \eta A) \frac{S}{N} - \sigma^E E ,
\]

\[
\text{CPC}_a D_t^\gamma I = \gamma^I I - \gamma^R I ,
\]

\[
\text{CPC}_a D_t^\gamma A = (1 - \xi) \sigma^A E - \gamma^A A ,
\]

\[
\text{CPC}_a D_t^\gamma H = \phi^I I - \delta^H H - \gamma^H H ,
\]

\[
\text{CPC}_a D_t^\gamma R = \gamma^I I + \gamma^R A + \gamma^H H ,
\]

\[
\text{CPC}_a D_t^\gamma D = \delta^H H .
\]

\[
R + E + I + S + A = N ,
\]

where, \((E)\) denotes the exposed class, \((I)\) denotes the symptomatic infectious class, \((S)\) denotes the class of susceptible, \((H)\) denotes the hospitalized class, \((A)\) denotes the hospitalized asymptomatic infectious class, \((R)\) denotes the recovered class and \((D)\) denotes the cumulative deaths. Moreover, we assume that some fraction of symptomatic infectious individuals progress to the hospitalized class \(H\) are unable to pass the disease to the general public [3].

3.2. A hybrid fractional COVID-19 model with general population mask use

In the following, we extended the COVID-19 model with general population mask use which given in [3] to a hybrid fractional order model with modified parameters. We consider the fractions of the general population wears mask outward efficiency and uniform inward. Let us consider \(U\) and \(M\) represent all population variables that typically do and do not wear masks, respectively. Both the variables and parameters of proposed models are given in Tables 1, 2 respectively. The hybrid COVID-19 multi-group fractional order model given as follows:

\[
\text{CPC}_a D_t^\gamma S_U = -\beta^I(I + \eta A) \frac{S}{N} - \beta^I(I_U(1 - \eta U) + (1 - \eta U)\eta A) \frac{S}{N} ,
\]

\[
\text{CPC}_a D_t^\gamma E_U = \beta^I(I_U + \eta A) \frac{S}{N} + \beta^I(I_U(1 - \eta U) + (1 - \eta U)\eta A) \frac{S}{N} - \sigma^E E ,
\]

\[
\text{CPC}_a D_t^\gamma I_U = \gamma^I I_U - \gamma^R I_U ,
\]

\[
\text{CPC}_a D_t^\gamma A_U = (1 - \xi) \sigma^A E_U - \gamma^A A_U ,
\]

\[
\text{CPC}_a D_t^\gamma H_U = \phi^I I_U - \delta^H H_U - \gamma^H H_U ,
\]

\[
\text{CPC}_a D_t^\gamma R_U = \gamma^I I_U + \gamma^R A_U + \gamma^H H_U ,
\]

\[
\text{CPC}_a D_t^\gamma D_U = \delta^H H_U .
\]

\[
E_U + I_U + S_U + R_U + A_U + R_M + E_M + S_M + A_M + I_M = N .
\]

Table 1 The variables of system (8) and their definitions [3].

| The abbreviation | Definition |
|------------------|------------|
| \( R_U \) | Recovery class wears mask with uniform inward efficiency. |
| \( H_U \) | Hospitalized class wears mask with uniform inward efficiency. |
| \( E_U \) | Exposed class wears mask with uniform inward efficiency. |
| \( I_U \) | Infectious and symptomatic class wears mask with uniform inward efficiency. |
| \( S_U \) | Susceptible class wears mask with uniform inward efficiency. |
| \( D_U \) | Fatality class wears mask with uniform inward efficiency. |
| \( A_U \) | Infectious but asymptomatic class wears mask with uniform inward efficiency. |
| \( R_M \) | Recovery class wear general face mask. |
| \( H_M \) | Hospitalized class wear general face mask. |
| \( E_M \) | Exposed class wear general face mask. |
| \( I_M \) | Symptomatic and infectious class wear general face mask. |
| \( S_M \) | Susceptible class wear general face mask. |
| \( D_M \) | Fatality class wear general face mask. |
| \( A_M \) | Infectious but asymptomatic class wear general face mask. |

Table 2 All parameters of systems (7) and (8) and their definitions.

| Name | Definition | Value (per day^{-1}) | Reference |
|------|------------|----------------------|-----------|
| \( \beta^I \) | Infectious contact rate | 2.5^a | Fitting [3] |
| \( \sigma^E \) | Transition exposed to infectious | (1/1.5)^a | dimensionless |
| \( \eta \) | Coefficient transmission due to super-spreaders | 1.9 | Fitting [3] |
| \( \gamma^I \) | A part of infections that will be symptomatic | 0.5 | dimensionless |
| \( \phi \) | Rate of hospitalization | 0.0025^a | dimensionless |
| \( \gamma^R \) | Recovery rate | (1/7)^a | dimensionless |
| \( \gamma^H \) | Recovery rate, Asymptomatic | (1/7)^a | dimensionless |
| \( \delta^H \) | Rate of disease induced death of infected class | 0.015^a | dimensionless |
We can define the feasible region for model (8) as follows:

$$\Omega = \{ (E_U, S_U, A_U, R_U, I_U, R_M, E_M, S_M, A_M, I_M) \in \mathbb{R}^{10} : E_U + I_U + S_U + R_U + A_U + R_M + E_M + S_M + A_M + I_M = N, \ H_U, H_M, D_U, D_M, \psi \in \mathbb{R}^+ \}.$$  

Boundedness of the proposed model solution can be verified by adding all equations of system (8) as follows:

$$D^{\alpha}_{t_0} N_g(t) = 0, \quad N_g(0) = A \geq 0,$$  

where, $A$ is a constant, $N_g$ is total summation of population in system (8). The solution of (9) is given as follows [5]:

$$N_g(t) \geq Ae^{\frac{K(t - t_0)}{K(t_0)}}.$$  

It can be clearly seen from (10) at $t \to \infty$, then $N_g(t) \geq 0$. Hence the solutions of the system (8) are bounded.

3.3. Existence and Uniqueness

In the following, we will use the fixed point theory. Let us rewrite (8) as follows:

$$c^{\alpha}_{C} \frac{D^\alpha}{D^\alpha t} y(t) = g(y(t), t), \quad y(0) = y_0 \geq 0.$$  

The vector $y(t) = (E_U, H_U, D_U, S_U, A_U, R_U, I_U, R_M, E_M, H_M, D_M, S_M, A_M, I_M)^T$, represents the state variables and $b$ is a continuous vector function such that:

$$b(t) = \begin{pmatrix}
-\beta(I_U + \eta A_U) \frac{S_U}{N} - \beta(I_U(1 - \epsilon_1) + (1 - \epsilon_1)\eta A_U) \frac{S_U}{N}, \\
\beta(I_U + \eta A_U) \frac{S_U}{N} + \beta(I_U(1 - \epsilon_0) + (1 - \epsilon_0)\eta A_U) \frac{S_U}{N} - \Delta^s E_U,
\beta(I_U + \eta A_U) \frac{S_U}{N} + \beta(I_U(1 - \epsilon_0) + (1 - \epsilon_0)\eta A_U) \frac{S_U}{N} - \Delta^s E_U,
\gamma_1^a I_U + \gamma_1^a A_U + \gamma_1^\alpha H_U, \\
\Delta^s H_U, \\
-\beta(1 - \epsilon_0)(I_U + \eta A_U) \frac{S_U}{N} - \beta(1 - \epsilon_0)(I_M(1 - \epsilon_0) + (1 - \epsilon_0)\eta A_M) \frac{S_M}{N}, \\
\beta(1 - \epsilon_0)(I_U + \eta A_U) \frac{S_U}{N} + \beta(1 - \epsilon_0)(I_M(1 - \epsilon_0) + (1 - \epsilon_0)\eta A_M) \frac{S_M}{N} - \Delta^s E_M,
\gamma_1^a I_M + \gamma_1^a A_M + \gamma_1^\alpha H_M,
\Delta^s H_M
\end{pmatrix}.$$  

with initial condition $y_0$. Moreover, the Lipschitz condition is satisfied by $g$, where $g$ is a quadratic vector function i.e. there exists $M^g \in \mathbb{R}$, such that [26]:

$$\|g(y_1(t), t) - g(y_2(t), t)\| = M^g \|y_1(t) - y_2(t)\|.$$  

**Theorem 3.1.** The fractional proposed model (8) has unique solution if below condition holds:

$$\frac{M^g T_\max^{\alpha} \gamma^{\alpha}_{\max}}{\Gamma(\alpha - 1) K_0(z)} \leq 1.$$  

**Proof.** We apply the definition (5) on (11), we get:

$$y(t) = y(t_0) + \frac{1}{K_0(z)} \int_{t_0}^{t} \exp(-\frac{K_1(z)}{K_0(z)}(t - s))^{R_L} D^{\alpha}_{s} = g(y(s), s) ds.$$  

Let $K = (0, T)$ and the operator $B : C(K, \mathbb{R}^{14}) \to C(K, \mathbb{R}^{14})$ such that:

$$B[y(t)] = y(t_0) + \frac{1}{K_0(z)} \int_{t_0}^{t} \exp(-\frac{K_1(z)}{K_0(z)}(t - s))^{R_L} D^{\alpha}_{s} = g(y(s), s) ds.$$  

It gives:

$$B[y(t)] = y(t).$$

Let $\|\cdot\|_K$ denotes the supremum norm on $K$. Thus

$$\|y(t)\|_K = \sup_{t \in K} \|y(t)\|, \quad y(t) \in C(K, \mathbb{R}^{14}).$$

So, $C(K, \mathbb{R}^{14})$ with $\|\cdot\|_K$ is a Banach space. Moreover, the following relation holds:

$$\| \int_{t_0}^{t} \varphi(s, t) y(s) ds\| \leq \Lambda \|\varphi(s, t)\|_K \|y(s)\|_K,$$

with $y(t) \in C(K, \mathbb{R}^{14}), \varphi(s, t) \in C(K^2, \mathbb{R}^{14})$ such that:

$$\|\varphi(s, t)\|_K = \sup_{t \in K} \|\varphi(s, t)\|.$$  

Thus (15) can be written as:

Finally, we obtain:

$$\|B[y_1(t)] - B[y_2(t)]\|_K \leq L \|y_1(t) - y_2(t)\|_K,$$  

Finally, we obtain:

$$\|B[y_1(t)] - B[y_2(t)]\|_K \leq L \|y_1(t) - y_2(t)\|_K.$$  

(17)
where
\[ L = \frac{M^\alpha \Gamma_\alpha - \lambda_\alpha}{\Gamma_\alpha - \lambda_\alpha} \]
If \( L \leq 1 \), then the operator \( B \) is called a contraction. Hence, the fractional system \((8)\) possesses a unique solution.

### 3.4. Basic reproduction number

We use the next generation method \([6]\) to find the basic reproduction number. Consider the following matrices \( F \) and \( V \), where \( F \) represents the new infection terms, \( V \) represents the remaining transfer terms \([6]\):

\[
F = \begin{pmatrix}
0 & \beta_0 \frac{S(t)}{N(t)} & \beta_0 \frac{S(t)}{N(t)} & 0 & \beta_0 (1 - \epsilon_0) \frac{S(t)}{N(t)} & \beta_0 \eta_1 (1 - \epsilon_0) \frac{S(t)}{N(t)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
V = \begin{pmatrix}
\sigma^\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sigma^\alpha (1 - \delta_1) & 0 & \gamma_* & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Then,
\[
R_0 = \rho(FV^{-1})
= \beta_0 \left[ \frac{S_\in T(0)}{N(0)} + \frac{S_\in T(0)(1 - \epsilon_0)(1 - \delta_1)}{N(0)} \right] \left( \frac{\sigma_1}{\phi^\alpha + \gamma_*} + \frac{\eta(1 - \epsilon_0)}{\gamma_*^\alpha} \right),
\]
where, \( R_0 \) is the basic reproduction number of the model, and \( \rho \) indicates the spectral radius of \( FV^{-1} \).

### 4. Numerical Methods

#### 4.1. CPC-CFD6M

Let us consider the following fractional order differential equation:
\[
cpc y(t) = f(t, y(t)), \quad 0 \leq x \leq 1, \quad y(0) = y_0.
\]

The first derivative approximation using the compact difference of six order method (CFD6M) is given as follows \([10]\):
\[
y_1'(t) = \frac{-1}{60} y_{i-3} + \frac{9}{60} y_{i-2} - \frac{45}{60} y_{i-1} + \frac{45}{60} y_{i+1} - \frac{9}{60} y_{i+2} + \frac{1}{60} y_{i+3}
+ O(\tau^8),
\]
(20)

Now, we can discretize \((19)\) using \((20)\) and definition \((5)\) with Grünwald-Letnikov approximation as follows:
\[
\frac{K(x)}{\tau^\alpha} \left( y_{i+1} + \sum_{j=1}^{i+1} \alpha_j y_{i+1-j} + \sum_{j=0}^{i+1} \alpha_j y_{i+j} - \frac{25}{60} y_{i-1} + \frac{25}{60} y_{i+1} - \frac{1}{60} y_{i+3} - q_i y(0) \right) = f(t_i, y_i),
\]
where, \( f' = f \), \( \tau = \frac{\tau}{\tau} \), \( N_i \in \mathbb{N} \), \( \mu_i = \frac{1}{\tau} \), \( \mu_\epsilon = \alpha \), \( q_i = \frac{\mu_\epsilon}{\tau^\alpha} \) and \( i = 1, 2, \ldots, n + 1 \). \( \alpha_0 = 1, \alpha_k = (1 - \epsilon_i) \alpha_{k-1} \).

Additionally, consider \([23]\):

0 < \mu_{i+1} < \mu_i < \ldots < \mu_1 = \alpha < 1,

0 < q_{i+1} < q_i < \ldots < q_1 = \frac{1}{\Gamma(1 - \alpha)},

Put \( C = \frac{K(x)}{\tau^\alpha} \), \( B = \frac{K(x)}{\tau^\alpha} \) in \((21)\). Then, we have:
\[
y_{i+1} = \frac{1}{\tau} \left[ f(t_i, y_i) + \frac{25}{60} y_{i+1} - \frac{1}{60} y_{i+3} + \frac{25}{60} y_{i+1} + \frac{1}{60} y_{i+3} + \frac{25}{60} y_{i+1} + \frac{1}{60} y_{i+3} \right]
+ q_i y(0) - C \sum_{j=1}^{i+1} \alpha_j y_{i+1-j}.
\]

#### Remark 1

If \( K(x) = 0 \) and \( K_0(x) = 1 \) in \((21)\), then we have the discretization of CPC-CFD6M with Caputo operator (CFD6M).

#### 4.2. Stability of CPC-CFD6M

In order to investigate the stability of the proposed method \(22\) consider the test problem of linear fractional differential equation \([24]\):
\[
cpc y(t) = v(t), \quad t \in [0, T], \quad x \in (0, 1), \quad v < 0,
\]
\[
y(0) = y_0.
\]

Using the approximation of CPC-CFD6M \((22)\) we can discretize \((23)\) as follows:
Fig. 1  Real data verses model (7) fitting at different \( \alpha \).
\[ y_{j+1} = \frac{1}{(\alpha + 1)C} \left( (\alpha y(t) + \frac{9}{45} y_{j-1} - \frac{2}{45} y_{j-2} + \frac{2}{3} y_{j-1} + \frac{1}{3} y_{j+2} - \frac{2}{3} y_{j+1}) \right) \]

\[ -B \sum_{i=1}^{n} \mu_i \left( (\alpha y(t) + \frac{9}{45} y_{j-2} - \frac{2}{45} y_{j-1} + \frac{2}{3} y_{j+1} + \frac{1}{3} y_{j+2} - \frac{2}{3} y_{j+1}) \right) + q_j y(0) - C \sum_{i=1}^{n} \alpha_i y_{n+1-i}. \]

Then from boundness theorem [23] we have:

\[ |y_{j+1}| = \frac{1}{(\alpha + 1)C} \left| (\alpha y(t) + \frac{9}{45} y_{j-1} - \frac{2}{45} y_{j-2} + \frac{2}{3} y_{j-1} + \frac{1}{3} y_{j+2} - \frac{2}{3} y_{j+1}) \right| \]

\[ -B \sum_{i=1}^{n} \mu_i \left( (\alpha y(t) + \frac{9}{45} y_{j-2} - \frac{2}{45} y_{j-1} + \frac{2}{3} y_{j+1} + \frac{1}{3} y_{j+2} - \frac{2}{3} y_{j+1}) \right) + q_j y(0) - C \sum_{i=1}^{n} \alpha_i y_{n+1-i}. \]

Since, \( (\alpha + 1)C > 1 \), then we have: \( |y_{j}| < |y_{0}| \) and \( |y_{0}| \geq |y_{1}| \geq \ldots \geq |y_{n-1}| \geq |y_{n}| \geq |y_{n+1}| \). This means that, the proposed scheme is stable.

4.3. GRK4M

Consider the following FODE:

\[ c_0 D_0^\alpha y(t) = f(t, y(t)), \quad 0 < \alpha \leq 1, \quad 0 < t \leq T, \]

\[ y(0) = y_0. \]

Then the approximate solution of (26) using GRK4M [7] is given as follows:

\[ y_{n+1} = y_n + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4), \]

\[ K_1 = k f(t_n, y_n), \]

\[ K_2 = k f(t_n + \frac{1}{2} \xi, y_n + \frac{1}{2} K_1), \]

\[ K_3 = k f(t_n + \frac{1}{2} \xi, y_n + \frac{1}{2} K_2), \]

\[ K_4 = k f(t_n + \xi, y_n + K_3), \]

where \( \xi = \frac{\eta}{(\alpha + 1)} \).

Fig. 2 Simulations of model (7) using at different values of \( \alpha \) and \( \beta = 2.5, \eta = 1.9 \) using CPC-CFD6M.
4.4. Stability of GRK4M

In order to study the stability of GRK4M. Consider for simplicity the following test problem:

\[ \frac{\partial}{\partial t} y(t) = t y(t), \quad 0 < t \leq T, \quad 0 < x \leq 1, \quad v < 0, \]  
\[ y(0) = y_0. \]

Using GRK4M [7], Eq. (28) can be written as follows:

\[ y(t_{j+1}) = y(t_j) + \frac{1}{6} \tau t_j y(t_j), \quad j = 0, 1, \ldots, n - 1. \]

The stability analysis of GRK4M is similar to the GEM method [8], when the terms are regrouped, the following equation is achieved:

\[ y(t_{j+1}) = (1 + \frac{1}{6} \tau t_j y(t_j)) y_0, \quad j = 0, 1, \ldots, n - 1. \]  

Then the stability condition [8] is given as follows:

\[ -1 < (1 + \frac{1}{6} \tau t_j y(t_j)) < 1. \]  

![Fig. 3](image1.png) Numerical simulations of $E_U, I_U, A_U$ and $H_U$ with and without mask use at $x = 1$, $\beta = 0.5$, $\eta = 0.5$ using ode45 and CPC-CFD6M.

![Fig. 4](image2.png) Numerical simulations of $S$ and $D$ with and without mask use at $x = 0.85$ using CPC-CFD6M.
5. Numerical experiments

In this section, CPC-NSFDM and GRK4M are constructed to obtain the numerical results of hybrid fractional models (7) and (8). For data fitting of model (7) we have taken some parameters values from the literature and the remaining values are fitted for the data collected for Egypt. We have fitted our model solutions with the real data collected from WHO for Egypt from 9th March 2020 to 13 June 2020 [2]. According to the publically reported data, the total population of Egypt for the year 2020 is 100500159. For the initial values we have considered \( S(0) = \left( \frac{100500159}{21793} \right) - 10, I(0) = 2, E(0) = 0, A(0) = 5, H(0) = 3, R(0) = 0 \) and \( D(0) = 0 \). We get different parameter values as shown in Table 2. This section provides graphical results of the given study. Fig. 1, shows the comparison between real data from WHO verses present considered model using CPC-CFD6M. It can be noted from this figure that the proposed model shows a strong agreement with real data collected by WHO. Fig. 2, depicts the influence of different values of parameter \( \alpha \) on the dynamics and transmission of COVID-19 for different subclasses of the total population. In Figs. 3–10, simulations the model (8) at different values of the fractional order derivative, \( \alpha \), in the interval \((0,1]\). The parameter values used in the simulations can be found in Table 2 and we put \( \beta = 0.5, \eta = 0.5 \) as given in [3]. The initial conditions are

![Graphical results](image-url)

Fig. 5  Numerical simulations of \( S_t \) and \( S_M \) with mask use at different \( \alpha \) using CPC-CFD6M in (a) and by using GRK4M in (b).
set to $(3000, 0, 600, 100, 300, 0, 0, 3000, 0, 600, 100, 300, 0, 0)$.

Fig. 3 shows the numerical solution of the model (8) using the Matlab solver ode45 and CPC-CFD6M at $z = 1$, where the time levels are chosen in days. Fig. 4 illustrates the behavior of $D$ and $S$ with and without mask use at $z = 0.96$ using CPC-CFD6M. We noted that, the number of susceptible individuals who use a mask is more than the number of susceptible individuals who do not use a mask. Also, the number of $D$ who use a mask is less than the number of $D$ who do not use a mask. Fig. 5 shows the numerical simulation of susceptible individuals who use a mask using CPC-CFD6M and GRK4M at different values of $z$.

Fig. 6 shows the variables behavior of the model (8) at different value of $z$ using GRK4M. Variables behavior of (8) are shown in Fig. 7, it changes with different values of $z$ using CPC-CFD6M. Moreover, $I_U, R_U$ and $I_M, R_M$ behaviors are shown in Fig. 8, it changes when we use CPC-CFD6M and GRK4M. We noted that the number of $I_U, I_M$ when we use CPC-CFD6M are less than the number of $I_U$ when we use GRK4M. Also the number of $R_U, R_M$ when we use CPC-
CFD6M are more than the number of $R_U, R_M$ when we use GRK4M. We can conclude that, CPC operator is preferable to simulate the biological models than the Caputo operator. We can obtain the Caputo operator as a spatial case from the new operator CPC when we put $K_0(x) = 1$ and $K_1(x) = 0$. Fig. 9, shows the numerical simulations of $I_U$ and $R_U$ at $x = 0.9$ using C-CFD6M and GRK4M. We noted that results which obtain by C-CFD6M are better than the results which obtained by GRK4M. Also Fig. 10 shows how the behavior of $I_U, A_U, H_U$ and $I_M, A_M, H_M$ are changed with different values of $x$ when we use C-CFD6M. Table 3 shows the CPU time for CPC-CFD6M and GRK4M. We noted that at $x = 1$, the Matlab solver ode 45 is the fastest one, and for $0.8 < x < 1$, GRK4M is faster than CPC-CFD6M.

Fig. 7 Simulations of model (8) using CPC-CFD6M at different values of $x$. 
Fig. 8  Numerical simulations of $I_M, A_M, H_M$ and $R_M$ at $z = 0.9$ using CPC-CFD6M and GRK4M.

Fig. 9  Numerical simulations of $I_U$ and $R_U$ at $z = 0.9$ using C-CFD6M and GRK4M.
6. Conclusions

In this article a novel hybrid fractional COVID-19 model with general population mask use and modified parameters is presented. This fractional model can be described the phenomena with memory than the integer order model. Also, the model in [3] can be obtained as a special case from the proposed model when $\alpha = 1$. We can conclude from the obtained results in this work that, the proposed COVID-19 model describes well the real data collected by WHO [1] of daily confirmed cases in Egypt. Existence, uniqueness and boundedness of the solutions are proved, moreover, the basic reproduction number is discussed. Two highly accurate numerical methods are used to study the presented model. These methods are GRK4M and CPC-CFD6M. Mathematical analysis for GRK4M and CPC-CFD6M are introduced. Comparative studies are done and we can conclude from Table 3 and the graphical results that the CPC-CFD6M is prefable to describe the biological results than the GRK4M, but GRK4M is faster than the CPC-CFD6M. Moreover, Caputo operator can be obtained as a special case from the new hybrid operator CPC.

### Declaration of Competing Interest

None.

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