ON THE ARCHIMEDEAN CHARACTERIZATION OF PARABOLAS

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Abstract. Archimedes knew that the area between a parabola and any chord $AB$ on the parabola is four thirds of the area of triangle $\Delta ABP$ where $P$ is the point on the parabola at which the tangent is parallel to $AB$. We consider whether this property (and similar ones) characterizes parabolas. We present five conditions which are necessary and sufficient for a strictly convex curve in the plane to be a parabola.

1. Introduction

A parabola is the set of points in the plane which are equidistant from a point $F$ called the focus and a line $l$ called the directrix. Archimedes found some interesting area properties of parabolas.

Consider the region bounded by a parabola and a chord $AB$. Let $P$ be the point on the parabola where the tangent is parallel to the chord $AB$. The line through $P$ parallel to the axis of the parabola meets chord $AB$ at a point $V$. Then, he showed that the area of the parabolic region is $a|PV|^{3/2}$ for some constant $a$, which depends only on the parabola.

Furthermore, he proved that the area of the parabolic region is $4/3$ times the area of triangle $\Delta ABP$ whose base is the chord and whose third vertex is $P$. For the proofs of Archimedes, see Chapter 7 of [8].

In this paper, we consider whether this property (and similar ones) characterizes parabolas. As a result, we present five conditions which are necessary and sufficient for a strictly convex curve in the plane to be a parabola.

Usually, a curve $X$ in the plane $\mathbb{R}^2$ is called convex if it bounds a convex domain in the plane $\mathbb{R}^2$.

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Hereafter, we will say that a convex curve $X$ in the plane $\mathbb{R}^2$ is strictly convex if the curve is smooth (that is, of class $C^3$) and is of positive curvature $\kappa$ with respect to the unit normal $N$ pointing to the convex side. Hence, in this case we have $\kappa(s) = \langle X''(s), N(X(s)) \rangle > 0$, where $X(s)$ is an arclength parametrization of $X$.

For a smooth function $f : I \to \mathbb{R}$ defined on an open interval, we will also say that $f$ is strictly convex if the graph of $f$ has positive curvature $\kappa$ with respect to the upward unit normal $N$. This condition is equivalent to the positivity of $f''(x)$ on $I$.

First of all, we prove the following characterization of parabolas:

**Theorem 1.** Let $X$ be the graph of a strictly convex function $f : I \to \mathbb{R}$ in the plane $\mathbb{R}^2$. Then $f$ is a quadratic polynomial if and only if $X$ satisfies Condition:

(A): For a point $P$ on $X$ and a chord $AB$ of $X$ parallel to the tangent of $X$ at $P$, let $V$ denote the point where the line through $P$ parallel to the $y$-axis meets $AB$. Then the area of the region bounded by the curve and $AB$ is $a|PV|^3/2$, where $a$ is a positive constant which depends only on the curve $X$.

Second, we prove

**Theorem 2.** Let $X$ be the graph of a strictly convex function $f : I \to \mathbb{R}$ in the plane $\mathbb{R}^2$. Then $f$ is a quadratic polynomial if and only if $X$ satisfies Condition:

(B): For a sufficiently small $k > 0$, let $X_k$ denote the graph of $y = f(x) + k$. For any point $V$ on $X_k$, let the tangent at $V$ meet the curve $X$ at $A$ and $B$. Then the region $S$ bounded by $X$ and the chord $AB$ has constant area (say, $\phi(k)$) independent of the choice of $V$.

Since $|PV| = k$, Theorem 1 is a special case of Theorem 2 for $\phi(k) = ak^{3/2}$, where $a$ is a constant.

Now, for an arbitrary strictly convex curve $X$ in the plane $\mathbb{R}^2$ which is not necessarily the graph of a function, we consider the following condition:

(C): For a point $P$ on $X$ and a chord $AB$ of $X$ parallel to the tangent of $X$ at $P$, the area of the region bounded by the curve and $AB$ is $4/3$ times the area of triangle $\triangle ABP$.

Then, we prove the following characterization of parabolas, which is the main theorem of this article.

**Theorem 3.** Let $X$ be a strictly convex curve in the plane $\mathbb{R}^2$. Then $X$ is a parabola if and only if it satisfies Condition (C).
In order to prove Theorems 1, 2 and 3, first of all, in Section 2 we establish a new geometric meaning of curvature $\kappa$ of a plane convex curve $X$ at a point $P \in M$ with $\kappa(P) > 0$ (Lemma 6). For the curvature function $\kappa$ of a plane curve, we refer to [3].

As applications of Theorem 3, we may prove some generalizations of Theorems 1 and 3 as follows.

**Corollary 4.** Let $X$ be a strictly convex curve in the plane $\mathbb{R}^2$. Then $X$ is a parabola if and only if it satisfies Condition:

(D): For a point $P$ on $X$ and a chord $AB$ of $X$ parallel to the tangent of $X$ at $P$, the area of the region bounded by the curve and $AB$ is $a(P)|\triangle ABP|^b(P)$, where $a(P)$ and $b(P)$ are some functions of $P$ and $|\triangle ABP|$ denotes the area of the triangle $\triangle ABP$.

Finally, for the graph $X$ of a strictly convex function $f : I \to \mathbb{R}$ in the plane $\mathbb{R}^2$, we consider the following Condition:

(E): For a point $P$ on $X$ and a chord $AB$ of $X$ parallel to the tangent of $X$ at $P$, let $V$ denote the point where the line through $P$ parallel to the $y$-axis meets $AB$. Then the area of the region bounded by the curve and $AB$ is $a(P)|PV|^b(P)$, where $a(P)$ and $b(P)$ are some functions of $P$.

Then we prove

**Corollary 5.** Let $X$ be the graph of a strictly convex function $f : I \to \mathbb{R}$ in the plane $\mathbb{R}^2$. Then $X$ satisfies Condition (E) if and only if $X$ is a parabola, which is given by either a quadratic polynomial $f$ or a function $f$ in (3.26) according as the function $a(P)$ is constant or not.

It follows from Corollary 5 that Theorem 1 is a corollary of Theorem 3.

To prove Corollaries 4 and 5, first of all, we show that $b(P)$ must be 1 in Corollary 4 (respectively, $3/2$ in Corollary 5). Then we can show that $X$ satisfies Condition (C). Hence, it follows from Theorem 3 that Corollaries 4 and 5 hold.

Among the graphs of functions, Á. Bényi et al. proved some characterizations of parabolas ([1,2]) and B. Richmond and T. Richmond established a dozen characterizations of parabolas using elementary techniques ([7]). In their papers, parabola means the graph of a quadratic polynomial in one variable.

For an example, consider a function $f(x) = b\{(1 - cx) - \sqrt{1 - 2cx}\}$ in (3.26) with $b, c > 0$ defined on $I = (−\infty, \frac{1}{2c})$. Then, the function $f$ is strictly convex and its graph $X$ satisfies Condition (C) (but neither (A) nor (B)). Note that $X$ is not the graph of a quadratic polynomial, but an open part of the parabola given in (3.27).

Throughout this article, all curves are of class $C^3$ and connected, unless otherwise mentioned.
Suppose that $X$ is a strictly convex curve in the plane $\mathbb{R}^2$ with the unit normal $N$ pointing to the convex side. For a fixed point $P \in X$, and for a sufficiently small $h > 0$, consider the line $l$ passing through $P + hN(P)$ which is parallel to the tangent of $X$ at $P$. Let’s denote by $A$ and $B$ the points where the line $l$ intersects the curve $X$.

We denote by $S_P(h)$ (respectively, $R_P(h)$) the area of the region bounded by the curve $X$ and chord $AB$ (respectively, of the rectangle with a side $AB$ and another one on the tangent of $X$ at $P$ with height $h > 0$). We also denote by $L_P(h)$ the length $|AB|$ of the chord $AB$. Then we have $R_P(h) = hL_P(h) = 2|\triangle ABP|$, where $|\triangle ABP|$ denotes the area of the triangle $\triangle ABP$.

We may adopt a coordinate system $(x, y)$ of $\mathbb{R}^2$ in such a way that $P$ is taken to be the origin $(0, 0)$ and the $x$-axis is the tangent line of $X$ at $P$. Furthermore, we may assume that $X$ is locally the graph of a non-negative strictly convex function $f : \mathbb{R} \to \mathbb{R}$.

For a sufficiently small $h > 0$, we have

$$S_P(h) = \int_{f(x)<h} \{h - f(x)\}dx,$$
$$R_P(h) = hL_P(h) = h \int_{f(x)<h} 1dx.$$

The integration is taken on the interval $I_P(h) = \{x \in \mathbb{R} | f(x) < h\}$.

On the other hand, we also have

$$S_P(h) = \int_{y=0}^{h} L_P(y)dy.$$

This shows that

$$S'_P(h) = L_P(h), \quad \text{and thus} \quad R_P(h) = hS'_P(h).$$

First of all, we prove the following lemma, which acts a key role in this article.

**Lemma 6.** Suppose that $X$ is a strictly convex curve in the plane $\mathbb{R}^2$ with the unit normal $N$ pointing to the convex side. Then we have

$$\lim_{h \to 0} \frac{1}{\sqrt{h}}L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}},$$

where $\kappa(P)$ is the curvature of $X$ at $P$ with respect to the unit normal $N$.

**Proof.** As above, we may adopt a coordinate system $(x, y)$ of $\mathbb{R}^2$ in such a way that $P$ is taken to be the origin $(0, 0)$ and $X$ is locally the graph of a non-negative strictly convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = f'(0) = 0$. Then $N$ is the upward unit normal.
Since the curve $X$ is of class $C^3$, the Taylor’s formula of $f(x)$ is given by

$$f(x) = ax^2 + f_3(x)$$

(2.4)

where $a = f''(0)/2$, and $f_3(x)$ is an $O(|x|^3)$ function. From $\kappa(P) = f''(0) > 0$, we see that $a$ is positive.

Now, we let $x = \sqrt{h}\xi$. Then, together with (2.1), (2.4) gives

$$\frac{1}{\sqrt{h}}L_P(h) = \frac{1}{\sqrt{h}} \int_{f(x)<h} 1dx$$

(2.5)

$$= \int_{a\xi^2 + g_3(\sqrt{h}\xi) < 1} 1d\xi,$$

where $g_3(\sqrt{h}\xi) = f_3(\sqrt{h}\xi)/h$. Since $f_3$ is an $O(|x|^3)$ function, we have

$$|g_3(\sqrt{h}\xi)| \leq C\sqrt{h}|\xi|^3,$$

(2.6)

where $C$ is a constant. As $h \to 0$, it follows from (2.5) and (2.6) that

$$\lim_{h \to 0} \frac{1}{\sqrt{h}}L_P(h) = \int_{\alpha\xi^2 < 1} 1d\xi$$

(2.7)

$$= \frac{2}{\sqrt{a}}.$$

Since $\kappa(P) = 2a$, this completes the proof of Lemma 6. \qed

**Remark.** From Lemma 6, we get a new geometric meaning of curvature $\kappa(P)$ of a plane convex curve $X$ at a point $P \in M$ with $\kappa(P) > 0$. That is, we obtain

$$\kappa(P) = \lim_{h \to 0} \frac{8h}{L_P(h)^2}.$$

Now, we give a proof of Theorem 1.

Let $X$ be the graph of a strictly convex function $f : I \to \mathbb{R}$, where $I$ is an open interval. Then $N$ is given by the upward unit normal. For a fixed point $P = (x, f(x))$ on $X$ and a small number $h > 0$, consider the line $l$ passing through the point $P + hN(P)$ which is parallel to the tangent to $X$ at $P$.

Then the hypothesis shows that $S_P(h) = a|PV|^{3/2}$ for small $h > 0$, where $a$ is a constant depending only on $X$. Note that $|PV| = h \sec \theta$, where $f'(x) = \tan \theta$ is the slope of the tangent line at $P$. Hence we have

$$S_P(h) = a(\sec \theta)^{3/2}h^{3/2}$$

(2.8)

$$= aW(x)^{3/2}h^{3/2},$$
where \( W(x) = \sqrt{1 + f'(x)^2} \). Thus (2.2) yields

\[
L_P(h) = \frac{3}{2}aW(x)^{3/2}h^{1/2}.
\]

(2.9)

Therefore it follows from Lemma 6 that

\[
\kappa(P) = \frac{32}{9a^2W(x)^3}.
\]

(2.10)

Since the curvature \( \kappa(P) \) of \( X \) at \( P = (x, f(x)) \) is given by

\[
\kappa(P) = \frac{f''(x)}{W(x)^3},
\]

(2.11)

we see that \( f''(x) \) is a constant. Hence \( f(x) \) is a quadratic polynomial. This completes the proof of the if part of Theorem 1.

By a straightforward calculation, it is trivial to prove the only if part of Theorem 1. This completes the proof of Theorem 1.

Second, we give a proof of Theorem 2.

Let \( X \) be the graph of a strictly convex function \( f : I \to \mathbb{R} \), where \( I \) is an open interval. Then \( N \) is given by the upward unit normal. We fix a point \( P(x, f(x)) \) on \( X \). For a sufficiently small \( h > 0 \), consider the line \( l \) passing through \( P + hN(P) \) which is parallel to the tangent of \( X \) at \( P \). Let’s denote by \( A \) and \( B \) the points where the line \( l \) intersects the curve \( X \).

Then the chord \( AB \) is tangent to \( X_k \) at \( V(x, f(x) + k) \), where \( k = hW \) and \( W(x) = \sqrt{1 + f'(x)^2} \). The hypothesis shows that \( S_P(h) = \phi(k) \). It follows from (2.2) that

\[
L_P(h) = S'_P(h) = W(x)\phi'(hW),
\]

\[
R_P(h) = hL_P(h) = hW(x)\phi'(hW).
\]

(2.12)

Hence we have

\[
\frac{L_P(h)}{\sqrt{h}} = \frac{\phi'(k)}{\sqrt{k}}W(x)^{3/2}.
\]

(2.13)

For a fixed point \( P(x, f(x)) \) on \( X \), it follows from \( k = hW(x) \) that \( h \to 0 \) is equivalent to \( k \to 0 \). Thus, Lemma 6 implies that

\[
\lim_{k \to 0} \frac{\phi'(k)}{\sqrt{k}} = W(x)^{-3/2} \lim_{h \to 0} \frac{1}{\sqrt{h}}L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}W(x)^{-3/2}.
\]

(2.14)

If we denote by \( \alpha \) the limit of the left hand side of (2.14), which is independent of \( P \), then we have

\[
\kappa(P) = \frac{8}{\alpha^2W(x)^3}.
\]

(2.15)
Similarly to the proof of Theorem 1, we see that $f(x)$ is a quadratic polynomial. This completes the proof of the if part of Theorem 2.

For a proof of the only if part of Theorem 2, see Example 1.2 in [6, p.6]. This completes the proof of Theorem 2.

3. Main Theorem

In this section, we prove Theorem 3, which is the main theorem of this article.

Let $X$ denote a strictly convex curve in the plane $\mathbb{R}^2$ with the unit normal $N$ pointing to the convex side. Suppose that $X$ satisfies Condition (C). Then, for $P \in X$ and a sufficiently small $h > 0$ we have

$$S_P(h) = \frac{2}{3} R_P(h). \quad (3.1)$$

By differentiating (3.1) with respect to $h$, it follows from (2.2) that

$$L_P(h) = 2h L'_P(h). \quad (3.2)$$

Therefore, we get

$$L_P(h) = c(P) \sqrt{h}, \quad (3.3)$$

where $c = c(P)$ is a constant depending on $P$. Furthermore, Lemma 6 implies that

$$c(P) = \frac{2 \sqrt{2}}{\sqrt{\kappa(P)}}. \quad (3.4)$$

In order to prove Theorem 3, first, we fix an arbitrary point $A$ on $X$.

As before, we take a coordinate system $(x, y)$ of $\mathbb{R}^2$: $A$ is taken to be the origin $(0, 0)$ and $x$-axis is the tangent line of $X$ at $A$. Furthermore, we may regard $X$ to be locally the graph of a non-negative strictly convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = f'(0) = 0$ and $f''(0) > 0$.

For any point $B(x, f(x))$ with $x \neq 0$, we denote by $P$ the point on $X$ such that the chord $AB$ is parallel to the tangent of $X$ at $P$. Then we have $P = (g(x), f(g(x)))$, for a function $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ which satisfies $|g(x)| < |x|$ and

$$xf'(g(x)) = f(x). \quad (3.5)$$

Since $g(x)$ tends to 0 as $x \to 0$, we may assume that $g(0) = 0$.

We prove the following lemma, which plays a crucial role in the proof of Theorem 3.

**Lemma 7.** $f(x)$ and $g(x)$ satisfy

$$x^3 f''(g(x)) = 8 \{f(x)g(x) - xf(g(x))\}. \quad (3.6)$$
\[ xf(x) = \frac{4}{3} \{ f(x)g(x) - xf(g(x)) \} + 2 \int_0^x f(t)dt. \] (3.7)

**Proof.** Consider the triangle \( \triangle ABC \), where \( C \) denotes the point \((x, 0)\). Then we have \( |AC|^2 + |BC|^2 = |AB|^2 \). Note that by definition, \( |AB|^2 = L_P(h)^2 \), where \( h \) denotes the distance from \( P \) to the chord \( AB \). This shows that

\[ x^2 + f(x)^2 = L_P(h)^2. \] (3.8)

The distance \( h \) from \( P \) to the chord \( AB \) is given by

\[ h = \frac{\epsilon \{ f(x)g(x) - xf(g(x)) \}}{\sqrt{x^2 + f(x)^2}}, \] (3.9)

where \( \epsilon = 1 \) for \( x > 0 \) and \( \epsilon = -1 \) for \( x < 0 \).

Since the curvature \( \kappa(P) \) of \( X \) at \( P \) is given by

\[ \kappa(P) = \frac{f''(g(x))}{(\sqrt{1 + f'(g(x))^2})^3}, \] (3.10)

it follows from (3.3), (3.4) and (3.5) that

\[ L_P(h)^2 = \frac{8h}{\kappa(P)} = \frac{8(x^2 + f(x)^2)}{f''(g(x))x^3} \{ f(x)g(x) - xf(g(x)) \}. \] (3.11)

Together with (3.8), this implies that (3.6) holds.

In order to prove (3.7), we consider the area of triangle \( \triangle ABC \). Then we have

\[ \frac{\epsilon}{2} xf(x) = S_P(h) + \epsilon \int_0^x f(t)dt, \] (3.12)

where \( \epsilon = 1 \) for \( x > 0 \) and \( \epsilon = -1 \) for \( x < 0 \). By assumption, we have \( S_P(h) = (4/3) | \triangle ABP | \). Hence we get

\[ S_P(h) = \frac{2\epsilon}{3} \{ f(x)g(x) - xf(g(x)) \}. \] (3.13)

Together with (3.12), this implies that (3.7) holds. \( \square \)

Next, with the help of Lemma 7, we show that in a neighborhood of an arbitrary point \( A \in X \), the curve \( X \) is a parabola.

By differentiating (3.7) with respect to \( x \), it follows from (3.5) that

\[ f(g(x)) = g(x)f'(x) - \frac{3}{4} \{ xf'(x) - f(x) \}. \] (3.14)
Differentiating (3.5) with respect to $x$, and using again (3.5), we get

$$f''(g(x)) = \frac{xf'(x) - f(x)}{x^2g'(x)}.$$  \hspace{1cm} (3.15)

On the other hand, together with (3.14), (3.6) shows that

$$f''(g(x)) = \frac{xf'(x) - f(x)}{x^3}\{6x - 8g(x)\}.$$  \hspace{1cm} (3.16)

It follows from (3.15) and (3.16) that

$$\{xf'(x) - f(x)\}\{8g(x)g'(x) - 6xg'(x) + x\} = 0.$$  \hspace{1cm} (3.17)

Since $f(x)$ is strictly convex, we obtain

$$8g(x)g'(x) - 6xg'(x) + x = 0.$$  \hspace{1cm} (3.18)

If we let $y = g(x)$, then (3.18) becomes $xdx + (8y - 6x)dy = 0$. By putting $y = vx$, we get a separable differential equation, and hence we can solve (3.18). Since $g(0) = 0$, we see that $g(x) = x/2, x/4$ or

$$g(x) = \frac{1}{4c}(cx + 1 - \sqrt{1-2cx}),$$  \hspace{1cm} (3.19)

where $c$ is a nonzero constant.

By differentiating (3.14) with respect to $x$, it follows from (3.5) that

$$\{xg(x) - \frac{3}{4}x^2\}f''(x) + xg'(x)f'(x) - g'(x)f(x) = 0.$$  \hspace{1cm} (3.20)

If $g(x) = x/2$, then (3.20) shows that

$$x^2f''(x) - 2xf'(x) + 2f(x) = 0,$$  \hspace{1cm} (3.21)

of which general solutions are given by $ax^2 + bx$ for some $a, b \in \mathbb{R}$. Since $f(0) = f'(0) = 0$, it follows from (3.21) that $f(x) = ax^2$ for some positive constant $a$. Thus, in a neighborhood of $A$, the curve $X$ is a parabola.

If $g(x) = x/4$, then (3.20) yields that

$$2x^2f''(x) - xf'(x) + f(x) = 0.$$  \hspace{1cm} (3.22)

For some $a, b \in \mathbb{R}$, the general solutions of (3.22) are given by

$$f(x) = ax + b\sqrt{|x|}.$$  \hspace{1cm} (3.23)

This contradicts to $f'(0) = 0$.

If $g(x) = \frac{1}{4c}(cx + 1 - \sqrt{1-2cx})$, it follows from (3.20) that

$$(1-2cx)(\sqrt{1-2cx}-(1-cx))f''(x) + c^2xf'(x) - c^2f(x) = 0.$$  \hspace{1cm} (3.24)
The general solutions of (3.24) are given by
\[ f(x) = ax + b(1 - \sqrt{1 - 2cx}), \]  
(3.25)
where \( a, b \in \mathbb{R} \). Since \( f(x) \) satisfies \( f(0) = f'(0) = 0 \) and \( f''(0) > 0 \), (3.25) shows that
\[ f(x) = b\{(1 - cx) - \sqrt{1 - 2cx}\}, \]  
(3.26)
where \( b \) is a positive constant. Hence, in a neighborhood of \( A \), the curve \( X \) is given by
\[ b^2c^2x^2 + 2bcxy + y^2 - 2by = 0. \]  
(3.27)

It follows from the classification theorem of quadratic polynomials in \( x \) and \( y \) that the curve defined by (3.27) is a parabola.

Summarizing the above discussions, we see that the curve \( X \) is locally a parabola.

Finally, we show that the curve \( X \) is a parabola as follows.

First, consider two parabolas \( \Phi_1 \) and \( \Phi_2 \) in the plane \( \mathbb{R}^2 \). For each \( i = 1, 2 \), let’s denote by \( \phi_i \) a connected open arc of the parabola \( \Phi_i \).

Suppose that the two arcs \( \phi_1 \) and \( \phi_2 \) share a common subarc \( \phi \). We fix a point \( A \) on the subarc \( \phi \). As before, we take a coordinate system \( (x, y) \) of \( \mathbb{R}^2 \): \( A \) is taken to be the origin \((0, 0)\), \( x \)-axis is the tangent line of \( \phi \) at \( A \) and \( \phi \) lies in the upper half plane. Then for each \( i = 1, 2 \), the parabolic arc \( \phi_i \) is locally the graph of \( f_i \) which is either of the form \( f_i(x) = a_i x^2 \) with \( a_i > 0 \) or of the form in (3.26) with \( b = b_i > 0, c = c_i \neq 0 \). That is, the parabola \( \Phi_i \) is of the form \( y = a_i x^2 \) with \( a_i > 0 \) or of the form in (3.27) with \( b = b_i > 0, c = c_i \neq 0 \).

Since \( f_1 \) is equal to \( f_2 \) around \( x = 0 \), \( f_1 \) and \( f_2 \) have the same derivatives at the origin. Hence, we immediately see that \( \Phi_1 = \Phi_2 \) because \( f''_i(0) = 2a_i, f''''_i(0) = 0 \) or \( f''_i(0) = b_i^2c^2, f''''_i(0) = 3b_i^2c^3 \) in each case for \( i = 1, 2 \).

Next, let’s fix a point \( A \) on the curve \( X \). Then an open arc of \( X \) containing \( A \) is a parabolic arc \( \phi_0 \) of a parabola \( \Phi_0 \). For an arbitrary point \( B \) on the curve \( X \), the compactness of the closed arc \( AB \) of \( X \) shows that there exist consecutive points \( A = P_0, P_1, \ldots, P_n = B \) on \( X \) and open arcs \( \phi_0, \phi_1, \ldots, \phi_n \) of \( X \) such that 1) for each \( i = 0, 1, \ldots, n \), \( P_i \) lies on \( \phi_i \), 2) each \( \phi_i \) is a parabolic arc, 3) \( \{\phi_i\} \) covers the closed arc \( AB \) of \( X \).

Since \( \phi_i \) and \( \phi_{i+1} \) share a common subarc for each \( i = 0, 1, \ldots, n-1 \), a successive use of the above argument shows that every \( \phi_i \) is an arc of the parabola \( \Phi_0 \), and hence \( B \in \Phi_0 \). Therefore we see that \( X \) is the parabola \( \Phi_0 \).

This completes the proof of the if part of Theorem 3.

For a proof of the only if part of Theorem 3, see Chapter 7 of [8], which is originally due to Archimedes. This completes the proof of Theorem 3.

4. Corollaries and Remarks

In this section, first of all, we prove Corollaries 4 and 5.
First, suppose that a strictly convex curve \( X \) in the plane \( \mathbb{R}^2 \) satisfies Condition (D) with \( b(P) = 1 \). Then we have

\[
S_P(h) = \frac{a(P)}{2} h L_P(h). \tag{4.1}
\]

By differentiating (4.1) with respect to \( h \), it follows from (2.2) that

\[
(2 - a(P)) L_P(h) = a(P) h L'_P(h). \tag{4.2}
\]

Solving (4.2), we get

\[
L_P(h) = c(P) h^{d(P)}, \tag{4.3}
\]

where \( c = c(P) \) is a constant depending on \( P \) and \( d(P) = (2 - a(P))/a(P) \).

It follows from (4.3) and Lemma 6 that \( d(P) = 1/2 \), and hence, \( a(P) = 4/3 \). Thus, the curve \( X \) satisfies Condition (C).

Now, suppose that \( X \) satisfies Condition (D) with \( b(P) \neq 1 \). Then we have

\[
S_P(h) = a(P) 2^{-b(P)} \{ h L_P(h) \}^{b(P)}, \tag{4.4}
\]

which shows that \( b(P) > 0 \). By differentiating (4.4) with respect to \( h \), it follows from (2.2) that

\[
L'_P(h) + h^{-1} L_P(h) = c(P) h^{-b(P)} L_P(h)^{2-b(P)}, \tag{4.5}
\]

where \( c(P) = 2^{b(P)} a(P)^{-1} b(P)^{-1} \). Solving the Bernoulli equation (4.5), we get

\[
\{ h L_P(h) \}^{b(P)-1} = c(P) (b(P) - 1) \ln h + d(P), \tag{4.6}
\]

where \( d(P) \) is a constant depending on \( P \).

In case \( b(P) > 1 \), by letting \( h \to 0 \), (4.6) leads to a contradiction. In case \( b(P) \in (0,1) \), multiplying the both sides of (4.6) by \( h^{\alpha(P)} \) with \( \alpha(P) = (1 - b(P))/2 > 0 \), and then by letting \( h \to 0 \), we get a contradiction. This shows that \( b(P) \) must be 1.

Together with the above discussion on the case \( b(P) = 1 \), Theorem 3 completes the proof of Corollary 4.

Next, we prove Corollary 5.

Suppose that the graph \( X \) of a strictly convex function \( f : I \to \mathbb{R} \) in the plane \( \mathbb{R}^2 \) satisfies Condition (E). Then for a fixed point \( P(x, f(x)) \) on \( X \) and for \( h > 0 \), we have

\[
S_P(h) = a(P) \vert PV \vert^{b(P)}. \tag{4.7}
\]

Since \( \vert PV \vert = h W(x) \) with \( W(x) = \sqrt{1 + f'(x)^2} \), by differentiating (4.7) with respect to \( h \), we get

\[
L_P(h) = a(P) b(P) W(x)^{b(P)} h^{b(P)-1}. \tag{4.8}
\]

Hence, it follows from Lemma 6 that \( b(P) = 3/2 \). This shows that

\[
S_P(h) = a(P) W(x)^{3/2} h^{3/2} \text{ and } L_P(h) = \frac{3}{2} a(P) W(x)^{3/2} \sqrt{h}. \tag{4.9}
\]
Thus we get
\[ \Delta ABP = \frac{1}{2} hL_P(h) = \frac{3}{4} a(P)W(x^{3/2}h^{3/2}) = \frac{3}{4} S_P(h), \tag{4.10} \]
which shows that \( X \) satisfies Condition (C). Therefore, it follows from the proof of Theorem 3 that \( X \) is a parabola, which is given by either a quadratic polynomial \( f \) or a function \( f \) in (3.26).

Conversely, if \( f \) is a quadratic polynomial, Theorem 1 shows that the graph \( X \) of \( f \) satisfies Condition (E) with a constant \( a(P) \) and \( b(P) = 3/2 \). If \( f \) is a function in (3.26), it is straightforward to show that the graph \( X \) of \( f \) satisfies Condition (E) with a nonconstant function \( a(P) \) and \( b(P) = 3/2 \).

This completes the proof of Corollary 5.

Together with (3.1)-(3.4) and Theorem 3, the same argument as in the proof of Corollary 5 shows

**Corollary 8.** Let \( X \) denote a strictly convex curve in the plane \( \mathbb{R}^2 \). Then, the following are equivalent.

1) \( X \) satisfies Condition (C).
2) \( S_P(h) = a(P)h^{3/2} \), where \( a(P) \) is a function of \( P \in X \).
3) \( S_P(h) = a(P)h^{b(P)} \), where \( a(P) \) and \( b(P) \) are some functions of \( P \in X \).
4) \( X \) is a parabola.

**Remark 9.** It follows from our proofs that Theorem 3 holds even if a strictly convex (hence, \( C^{(3)} \)) curve \( X \) satisfies Condition (C) for sufficiently small \( h > 0 \) at every point \( P \in X \).

Finally, we give an example of a convex curve which satisfies Condition (C) for sufficiently small \( h > 0 \) at every point \( P \in X \), but it is not a parabola. Note that the example is not of class \( C^{(3)} \), and hence it is not strictly convex either.

**Example 10.** Consider the graph \( X \) of a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) which is given by
\[ f(x) = \begin{cases} 9x^2, & \text{if } x < 0, \\ \frac{9}{4}x^2, & \text{if } x \geq 0. \end{cases} \tag{4.11} \]

Then, the function \( f \) is not of class \( C^{(3)} \) at the origin, and hence the curve \( X \) is not strictly convex. It is straightforward to show that if \( P \) is the origin, then for all \( h > 0 \) we have
\[ L_P(h) = \sqrt{h}, \quad \text{and} \quad S_P(h) = \frac{2}{3} R_P(h). \tag{4.12} \]

Hence \( X \) satisfies Condition (C) at the origin for all \( h > 0 \). If \( P \in X \) is not the origin, then there exists a positive number \( \varepsilon(P) \) such that for every positive number \( h \) with \( h < \varepsilon(P) \), \( X \) satisfies Condition (C).
Thus, $X$ satisfies Condition (C) for sufficiently small $h > 0$ at every point $P \in X$. But it is not a parabola.

**Remark 11.** In [4] and [5], the authors proved the higher dimensional versions of Theorems 1 and 2, respectively.

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