On the weighted \( q \)-Bernoulli numbers and polynomials

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Abstract In this paper we discuss new concept of the \( q \)-extension of Bernoulli numbers and polynomials with weight \( \alpha \). From these \( q \)-Bernoulli numbers with weight \( \alpha \), we establish some interesting identities and relations.

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1. Introduction

Let \( p \) be a fixed odd prime. Throughout this paper \( \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C} \) and \( \mathbb{C}_p \) will, respectively, denote the ring of \( p \)-adic rational integers, the field of \( p \)-adic rational numbers, the complex number field and the completion of algebraic closure of \( \mathbb{Q}_p \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = p^{-1} \). When one talks of \( q \)-extension, \( q \) is variously considered as an indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \), one normally assume \( |q| < 1 \). If \( q \in \mathbb{C}_p \), then we assume \( |q - 1|_p < p^{-\frac{1}{p-1}} \), so that \( q^x = \exp(x \log q) \) for \( |x|_p < 1 \). In this paper we use the following notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad \text{(see \([1-16]\))}.
\]

Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), let us start with the expressions

\[
\frac{1}{[p^N]_q} \sum_{0 \leq x < p^N} q^x f(x) = \sum_{0 \leq x < p^N} f(x) \mu_q(x + p^N \mathbb{Z}_p),
\]

representing \( p \)-adic \( q \)-analogue of Riemann sums for \( f \) (see \([3-7]\) ).
The integral of $f$ on $\mathbb{Z}_p$ will be defined as the limit ($N \to \infty$) of these sums, when it exists. The $p$-adic $q$-integral of function $f(\in UD(\mathbb{Z}_p))$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \quad (see \ [7]). \quad (1)$$

From (1), we note that

$$\left| \int_{\mathbb{Z}_p} f(x)d\mu_q(x) \right| \leq p\|f\|_1, \quad (see \ [3-7]),$$

where

$$\|f\|_1 = \sup \left\{ \left| f(0) \right|_p, \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right|_p \right\}.$$

In [2], Carlitz defined a set of numbers $\zeta_k = \zeta_k(q)$ inductively by

$$\zeta_0,q = 1, \quad (q\zeta + 1)^k - \zeta_k,q = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \quad (2)$$

with the usual convention of replacing $\zeta^k$ by $\zeta_{k,q}$. These numbers are the $q$-extension of ordinary Bernoulli numbers. But they do not remain finite when $q = 1$. So, Carlitz modified Eq.(2) as follows:

$$\beta_0,q = 1, \quad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing $\beta^k$ by $\beta_{k,q}(see \ [2])$. These numbers $\beta_{k,q}$ are called the $n$-th Carlitz’s $q$-Bernoulli numbers.

In [1], Carlitz also considered the extension of Carlitz’s $q$-Bernoulli numbers as follows:

$$\beta^h_{0,q} = \frac{h}{[h]_q}, \quad q^h(q\beta^h + 1)^k - \beta^h_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing $(\beta^h)^k$ by $\beta^h_{k,q}$.

In this paper we construct $q$-Bernoulli numbers with weight $\alpha$ which are different the extension of Carlitz’s $q$-Bernoulli numbers. By using $p$-adic $q$-integral equations on $\mathbb{Z}_p$, we investigate some interesting identities and relations on the $q$-Bernoulli numbers with weight $\alpha$.

2. On the weighted $q$-Bernoulli numbers and polynomials

Let $f_n(x) = f(x + n)$. By (1), we see that
\[ qI_q(f_{1}) = q \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x+1)q^x, \]

\[ = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x + \lim_{N \to \infty} \frac{f(p^N)q^{p^N} - f(0)}{[p^N]_q} \]

\[ = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) + (q-1)f(0) + \frac{q-1}{\log q} f'(0), \]  

and

\[ q^2I_q(f_2) = q^2 \int_{\mathbb{Z}_p} f(x+2)d\mu_q(x) \]

\[ = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x+1)q^{x+1} + \lim_{N \to \infty} \frac{f(1+p^N)q^{p^N+1} - f(1)q}{[p^N]_q} \]

\[ = q \int_{\mathbb{Z}_p} f(x+1)d\mu_q(x) + q(q-1)f(1) + \frac{q-1}{\log q} f'(1) \]

\[ = I_q(f) + (q-1)f(0) + q(q-1)f(1) + \frac{q-1}{\log q} f'(0) + \frac{q(q-1)}{\log q} f'(1). \]

Thus, we have

\[ q^2I_q(f_2) - I_q(f) = \sum_{l=0}^{1} q^l(q-1)f(l) + \frac{q-1}{\log q} \sum_{l=0}^{1} q^l f'(l). \]

Continuing this process, we obtain the following theorem.

**Theorem 1.** For \( n \in \mathbb{N} \), we have

\[ q^nI_q(f_n) - I_q(f) = (q-1) \sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} q^l f'(l). \]

In particular, \( n = 1 \),

\[ qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0). \]

For \( \alpha \in \mathbb{N} \), we evaluate the following bosonic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \):

\[ \bar{I}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} [x]_{q^n}d\mu_q(x) = \frac{1-q}{(1-q^\alpha)n} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) (-1)^l \frac{\alpha l + 1}{1 - q^{\alpha l+1}}. \]  

From (4), we note that

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\[ \frac{1 - q}{1 - q^\alpha} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{\alpha l + 1}{1 - q^{\alpha l + 1}} \]

\[ = \frac{1 - q}{(1 - q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{\alpha l}{1 - q^{\alpha l + 1}} + \frac{1 - q}{(1 - q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1 - q^{\alpha l + 1}} \]

\[ = \frac{n\alpha (1 - q)}{(1 - q^\alpha)^n} \sum_{l=1}^n \left( \frac{n - 1}{l - 1} \right) \frac{(-1)^l}{1 - q^{\alpha l + 1}} + \frac{1 - q}{(1 - q^\alpha)^n} \sum_{m=0}^\infty q^m (1 - q^\alpha)^n \]

\[ = -\frac{n\alpha}{(1 - q)^n} \sum_{m=0}^\infty q^m (1 - q^\alpha)^n \]

\[ = -\frac{n\alpha}{[\alpha]_q} \sum_{m=0}^\infty q^m [m]_q^{n-1} + (1 - q) \sum_{m=0}^\infty q^m [m]_q^n. \]

Therefore, by (4) and (5), we obtain the following theorem.

**Theorem 2.** Let \( n, \alpha \in \mathbb{N} \). Then we have

\[ -\frac{\beta_n^{(\alpha)}}{n} = \frac{\alpha}{[\alpha]_q} \sum_{m=0}^\infty q^m [m]_q^{n-1} - \frac{(1 - q)}{n} \sum_{m=0}^\infty q^m [m]_q^n. \]

For \( \alpha = 1 \), we note that \( \beta_n^{(1)} \) are same the Carlitz \( q \)-Bernoulli numbers. In this paper, \( \beta_n^{(\alpha)} \) are called the \( n \)-th \( q \)-Bernoulli numbers with weight \( \alpha \).

By (4) and (5), we easily get

\[ \int_{\mathbb{Z}_p} e^{[x]_q \alpha t} d\mu_q(x) = -t \frac{\alpha}{[\alpha]_q} \sum_{m=0}^\infty q^m [m]_q^{n-1} + (1 - q) \sum_{m=0}^\infty q^m [m]_q^n. \]

From (6), we have

\[ \sum_{n=0}^\infty \frac{\beta_n^{(\alpha)} t^n}{n!} = -t \frac{\alpha}{[\alpha]_q} \sum_{m=0}^\infty q^m [m]_q^{n-1} + (1 - q) \sum_{m=0}^\infty q^m [m]_q^n. \]

Therefore, we obtain the following corollary.

**Corollary 3.** Let \( F_q^{(\alpha)}(t) = \sum_{n=0}^\infty \beta_n^{(\alpha)} \frac{t^n}{n!} \). Then we have

\[ F_q^{(\alpha)}(t) = -t \frac{\alpha}{[\alpha]_q} \sum_{m=0}^\infty q^m [m]_q^{n-1} + (1 - q) \sum_{m=0}^\infty q^m [m]_q^n. \]

Now, we consider the \( q \)-Bernoulli polynomials with weight \( \alpha \) as follows:
\[ \tilde{\beta}_{n,q}(x) = \int_{\mathbb{Z}_p} [x+y]^n d\mu_q(y), \text{ for } n \in \mathbb{Z}_+ \text{ and } \alpha \in \mathbb{N}. \] (7)

By (7), we see that
\[
\tilde{\beta}_{n,q}(x) = \frac{1 - q}{(1 - q^n)} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l} x^{l} \frac{\alpha l + 1}{1 - q^{\alpha l+1}} = -n \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{m \alpha + m}[m + x]^{n-1} + (1 - q) \sum_{m=0}^{\infty} q^m [m + x]^n. \] (8)

Let \( F^{(\alpha)}_q(t, x) = \sum_{n=0}^{\infty} \tilde{\beta}_{n,q}(x) \frac{t^n}{n!} \). Then we have
\[
F^{(\alpha)}_q(t, x) = -t \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{m \alpha + m}[m + x]^{n-1} t + (1 - q) \sum_{m=0}^{\infty} q^m [m + x]^n. \] (9)

By simple calculation, we easily get
\[
\tilde{\beta}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]^{n-l} q^{\alpha l} x^l \int_{\mathbb{Z}_p} [y]^{l} d\mu_q(y) = \sum_{l=0}^{n} \binom{n}{l} [x]^{n-l} q^{\alpha l} \tilde{\beta}_{l,q}. \] (10)

Therefore, by (8), (9) and (10), we obtain the following theorem.

**Theorem 4.** For \( n \in \mathbb{Z}_+ \) and \( \alpha \in \mathbb{N} \), we have
\[
\tilde{\beta}_{n,q}(x) = \frac{(1 - q)}{(1 - q^n)} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l} x^l \frac{\alpha l + 1}{1 - q^{\alpha l+1}} = -n \frac{\alpha}{[\alpha]_q} \sum_{m=0}^{\infty} q^{m \alpha + m}[m + x]^{n-1} + (1 - q) \sum_{m=0}^{\infty} q^m [m + x]^n. \]

Moreover,
\[
\tilde{\beta}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]^{n-l} q^{\alpha l} \tilde{\beta}_{l,q}. \]

By Theorem 1, we see that
\[
q^n \tilde{\beta}_{m,q}(n) - \tilde{\beta}_{m,q} = (q - 1) \sum_{l=0}^{n-1} q^l [y]^{m} + m \frac{\alpha}{[\alpha]_q} \sum_{l=0}^{n-1} q^{\alpha l+t}[y]^{m-1}. \]
Therefore, we obtain the following theorem.

**Theorem 5.** For $m \in \mathbb{Z}_+$ and $\alpha, n \in \mathbb{N}$, we have

$$q^n \beta_{m,q}^{(\alpha)}(n) - \beta_{m,q}^{(\alpha)} = (q - 1) \sum_{l=0}^{n-1} q^l [l]_q^m + \frac{m \alpha}{[\alpha]_q} \sum_{l=0}^{n-1} q^{\alpha l + 1} [l]_q^{m-1}.$$

In (3), it is known that

$$q I_q (f_1) - I_q (f) = (q - 1) f(0) + \frac{q - 1}{\log q} f'(0).$$

If we take $f(x) = e^{[x]_{\alpha} t}$, then we have

$$(q - 1) + \frac{\alpha}{[\alpha]_q} t = q \int_{\mathbb{Z}_p} e^{[x+1]_{\alpha} t} d\mu_q(x) - \int_{\mathbb{Z}_p} e^{[x]_{\alpha} t} d\mu_q(x)
= \sum_{n=0}^{\infty} \left(q \beta_{n,q}^{(\alpha)}(1) - \beta_{n,q}^{(\alpha)}\right) \frac{t^n}{n!}.$$  (11)

Therefore, by (11), we obtain the following theorem.

**Theorem 6.** For $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we have

$$\beta_{0,q}^{(\alpha)} = 1, \quad q \beta_{n,q}^{(\alpha)}(1) - \beta_{n,q}^{(\alpha)} = \left\{ \begin{array}{ll}
\frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\
0, & \text{if } n > 1.
\end{array} \right.$$  (10)

By (10) and Theorem 6, we obtain the following corollary.

**Corollary 7.** For $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we have

$$\beta_{0,q}^{(\alpha)} = 1, \quad q(q^{\alpha} \beta_{n,q}^{(\alpha)} + 1)^n - \beta_{n,q}^{(\alpha)} = \left\{ \begin{array}{ll}
\frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\
0, & \text{if } n > 1,
\end{array} \right.$$  (11)

with the usual convention of replacing $(\beta_q^{(\alpha)})^n$ by $\beta_{n,q}^{(\alpha)}$.

From (7), we can easily derive the following equation (12).

$$\int_{\mathbb{Z}_p} [x + y]_{\alpha,q}^n d\mu_q(y) = \frac{[d]_{\alpha,q}^n}{[d]^n} \sum_{a=0}^{d-1} q^a \int_{\mathbb{Z}_p} \left[ \frac{x + a}{d} + y \right]^n \frac{d\mu_q(y)}{q^{\alpha d}}
= \frac{[d]_{\alpha,q}^n}{[d]^n} \sum_{a=0}^{d-1} q^a \beta_{n,q}^{(\alpha)} \left( \frac{x + a}{d} \right).$$  (12)
Therefore, by (12), we obtain the following theorem.

**Theorem 8.** For \( n \in \mathbb{Z}_+ \) and \( \alpha, d \in \mathbb{N} \), we have

\[
\tilde{\beta}_{n,q}^{(\alpha)}(x) = \frac{[d]_q^n}{[d]_q} \sum_{a=0}^{d-1} q^\alpha \tilde{\beta}_{n,q}^{(\alpha)} \left( \frac{x + a}{d} \right).
\]

From (7), we note that

\[
\tilde{\beta}_{n,q-1}^{(\alpha)}(1 - x) = \int_{\mathbb{Z}_p} [1 - x + x_1]_{q-\alpha}^n d\mu_{q-1}(x_1)
\]

\[
= (-1)^n q^{\alpha n} \int_{\mathbb{Z}_p} [x + x_1]_{q}^n d\mu_q(x_1)
\]

\[
= (-1)^n q^{\alpha n} \tilde{\beta}_{n,q}^{(\alpha)}(x).
\]

Therefore, by (13), we obtain the following theorem.

**Theorem 9.** For \( n \in \mathbb{Z}_+ \) and \( \alpha \in \mathbb{N} \), we have

\[
\tilde{\beta}_{n,q}^{(\alpha)}(1 - x) = (-1)^n q^{\alpha n} \tilde{\beta}_{n,q}^{(\alpha)}(x).
\]

It is easy to show that

\[
\int_{\mathbb{Z}_p} [1 - x]_{q-\alpha}^n d\mu_q(x) = \int_{\mathbb{Z}_p} (1 - [x]_q^n) d\mu_q(x)
\]

\[
= (-1)^n q^{\alpha n} \int_{\mathbb{Z}_p} [x - 1]_{q}^n d\mu_q(x).
\]

By (13) and (14), we obtain the following corollary.

**Corollary 10.** For \( \alpha \in \mathbb{N} \) and \( n \in \mathbb{Z}_+ \), we have

\[
\int_{\mathbb{Z}_p} [1 - x]_{q-\alpha}^n d\mu_q(x) = \sum_{l=0}^{n} \binom{n}{l} (-1)^l \tilde{\beta}_{l,q}^{(\alpha)}
\]

\[
= (-1)^n q^{\alpha n} \tilde{\beta}_{n,q}^{(\alpha)}(-1)
\]

\[
= \tilde{\beta}_{n,q}^{(\alpha)}(2).
\]

From Theorem 4, we have

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Therefore, we obtain the following theorem.

**Theorem 11.** For \( n \in \mathbb{Z}_+ \) with \( n > 1 \), we have

\[
q^2 \bar{\beta}_{n,q}(2) = nq^{1+\alpha} \frac{\alpha}{[\alpha]_q} - q^2 + q
\]

Therefore, we obtain the following theorem.

**Theorem 11.** For \( n \in \mathbb{Z}_+ \) with \( n > 1 \), we have

\[
q^2 \bar{\beta}_{n,q}(2) = nq^{1+\alpha} \frac{\alpha}{[\alpha]_q} - q^2 + q
\]

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