RULED MINIMAL SURFACES IN PRODUCT SPACES

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ABSTRACT. It is well known that the helicoids are the only ruled minimal surfaces in \( \mathbb{R}^3 \). The similar characterization for ruled minimal surfaces can be given in many other 3-dimensional homogeneous spaces. In this note we consider the product space \( M \times \mathbb{R} \) for a 2-dimensional manifold \( M \) and prove that \( M \times \mathbb{R} \) has a nontrivial minimal surface ruled by horizontal geodesics only when \( M \) has a Clairaut parametrization. Moreover such minimal surface is the trace of the longitude rotating in \( M \) while translating vertically in constant speed in the direction of \( \mathbb{R} \).

1. Introduction

In Euclidean 3-space the only ruled minimal surfaces are the planes and the helicoids which are the surfaces obtained by rotating a geodesic in \( \mathbb{R}^2 \) while translating vertically in constant speed. The similar result can be derived in \( S^2 \times \mathbb{R} \) and \( H^2 \times \mathbb{R} \) (cf. [1, 2]). For other homogeneous space such as \( S^3, H^3 \), Nil\(^3\), Berger sphere and \( SL(2, \mathbb{R}) \), we also have complete characterization for the ruled minimal surfaces (cf. [3, 5, 6, 7]). In this note we will consider ruled minimal surfaces in the product space \( M \times \mathbb{R} \) where \( M \) is a 2-dimensional Riemannian manifold.

When \( M \) is a surface of revolution we can construct a “helicoid” in \( M \times \mathbb{R} \) by rotating the longitude in \( M \) while translating in the direction of \( \mathbb{R} \) in a constant speed. In Sec. 2, we show that such helicoids are ruled minimal surfaces in \( M \times \mathbb{R} \). In fact these are the only nontrivial minimal surfaces ruled by horizontal geodesics. More generally, when \( M \) has a parametrization \( \varphi(x, y) \) with the metric of the form

\[
(g_{ij}) = \begin{bmatrix}
1 & 0 \\
0 & \beta(x)^2
\end{bmatrix},
\]

the surfaces given by \( X(s, t) = (\varphi(t, s), as) \) in \( M \times \mathbb{R} \) are the only nontrivial minimal surfaces ruled by horizontal geodesics. Here, we call these surfaces as ‘helicoids’ for convenience. In general, a parametrization is called a Clairaut...
parametrization when its metric coefficients satisfy $g_{12} = 0$ and both $g_{11}$ and $g_{22}$ are functions of $x$ only. (cf. [4, p. 340]) Our $\varphi(x,y)$ above is a Clairaut parametrization.

In Section 3, we will give the main theorem which states that this is the only possible case of nontrivial horizontally ruled minimal surfaces in product space $M \times \mathbb{R}$. More precisely, if there exist a nontrivial horizontally ruled minimal surface in general product space $M \times \mathbb{R}$, then $M$ must have a Clairaut parametrization and the minimal surface should be one of the helicoids.

2. Helicoids in $M \times \mathbb{R}$ for surface $M$ with Clairaut parametrization

Let $M$ be a 2-dimensional manifold given by Clairaut parametrization $\varphi(x,y)$ with Riemannian metric

$$(g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta(x)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

From elementary computation, we note that the $x$-parameter curves of $\varphi$ are geodesics which we will call longitudes. Any surface of revolution in $\mathbb{R}^3$ has such parametrization with longitude as $x$-parameter curves. More generally, such manifolds can be characterized as having a non-zero Killing field on $M$.

In the product space $M \times \mathbb{R}$, the natural generalization of the helicoids is the surfaces given by the parametrization $X(s,t) = (\varphi(t,s), as)$ which we will call as helicoids in $M \times \mathbb{R}$. The next lemma states that the helicoids are in fact a ruled minimal surface in $M \times \mathbb{R}$. Even though the proof of the lemma is a straightforward computation, we give a proof for the sake of completeness.

Lemma 2.1. Let $M$ be a 2-dimensional manifold given by Clairaut parametrization $\varphi(x,y)$ as above. In the product space $M \times \mathbb{R}$, the parametrization $X(s,t) = (\varphi(t,s), as)$ gives a ruled minimal surface for any $a \in \mathbb{R}$.

Proof. We take $\Psi(x,y,z) = (\varphi(x,y), z)$ as a coordinate of $M \times \mathbb{R}$. Then, the coefficients for the first fundamental form are

$$(g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta(x)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the parametrization of the surface is $X(s,t) = \Psi(t,s,as)$. For the coordinate frame $\{\partial_x, \partial_y, \partial_z\}$ the Riemannian connection becomes

$$\nabla_{\partial_x} \partial_x = 0, \quad \nabla_{\partial_y} \partial_y = -\beta(x)\beta'(x)\partial_x, \quad \nabla_{\partial_z} \partial_z = 0$$

$$\nabla_{\partial_x} \partial_y = \nabla_{\partial_y} \partial_x = \frac{\beta'(x)}{\beta(x)}\partial_y, \quad \nabla_{\partial_y} \partial_z = \nabla_{\partial_z} \partial_y = 0, \quad \nabla_{\partial_x} \partial_z = \nabla_{\partial_z} \partial_x = 0.$$

Now for the surface $X(s,t) = \Psi(t,s,as)$, we have

$$X_s = \partial_y + a\partial_z, \quad X_t = \partial_z.$$
and
\[ X_{ss} = \nabla_{X_s} X_s = \nabla_{\partial_s} \partial_y + 2a \nabla_{\partial_s} \partial_z + a^2 \nabla_{\partial_t} \partial_z = -\beta(t)\beta'(t) \partial_x, \]
\[ X_{st} = \nabla_{X_s} X_t = \nabla_{\partial_s} \partial_y + a \nabla_{\partial_t} \partial_z = \frac{\beta'(t)}{\beta(t)} \partial_y = X_{ts}, \]
\[ X_{tt} = \nabla_{X_t} X_t = \nabla_{\partial_t} \partial_x = 0. \]

Taking the unit normal vector field \( n \) to the surface as
\[ n = \frac{1}{\sqrt{a^2 + \beta^2(t)}} \left( \frac{a}{\beta(t)} \partial_y - \beta(t) \partial_z \right), \]
we have
\[ E = \langle X_s, X_s \rangle = a^2 + \beta^2(t), \]
\[ F = \langle X_s, X_t \rangle = 0, \]
\[ G = \langle X_t, X_t \rangle = 1 \]
and
\[ l = \langle X_{ss}, n \rangle = 0, \]
\[ m = \langle X_{st}, n \rangle = \frac{a\beta'(t)}{\sqrt{a^2 + \beta^2(t)}}, \]
\[ n = \langle X_{tt}, n \rangle = 0. \]

Therefore the mean curvature \( H \) of the surface is
\[ H = \frac{1}{2} \frac{Gl - 2Fm + En}{EG - F^2} = 0 \]
and the surface is minimal in \( M \times \mathbb{R} \). Noting that \( X_{tt} = 0 \) and \( \langle X_t, \partial_z \rangle = 0 \), the \( t \)-parameter curves of \( X \) are horizontal geodesics.

As mentioned before, the 2-dimensional manifold given by Clairaut parametrization can be characterized as a Riemannian 2-manifold with a non-zero Killing field. And from the exactly same computation, the above lemma can be stated as the following.

When \( M \) is a 2-manifold with a nonzero Killing field \( K \), let \( F_s \) be the flow of \( K \) on \( M \). Then, \( \tilde{F}_s(p, z) = (F_s(p), z) \) and \( T_s(p, z) = (p, z + s) \) are flows of Killing fields in \( M \times \mathbb{R} \). Moreover for each \( z \in \mathbb{R} \) the orthogonal trajectories of orbits \( \{ \tilde{F}_s(p, z) \} \subset M \times \{ z \} \), \( p \in M \) are all geodesics in \( M \times \mathbb{R} \) which is horizontal in the sense that it is perpendicular to \( \frac{\partial}{\partial z} \) everywhere. Let \( \gamma(t) \) be one of such geodesics, then the surface in \( M \times \mathbb{R} \) given by the parametrization
\[ X(s, t) = T_{as}(\tilde{F}_s(\gamma(t))) \]
is a minimal surface ruled by horizontal geodesics which we will call as horizontally ruled minimal surfaces. In fact, these are the only possible cases of nontrivial horizontally ruled minimal surfaces in a product space \( M \times \mathbb{R} \) which we will prove in the next section.
There are examples of the minimal surfaces ruled by non-horizontal geodesics in product space. If we consider the Euclidean 3-space $\mathbb{R}^2 \times \mathbb{R}$ with a distinguished vertical direction, then a usual helicoid with oblique (and therefore non vertical) axis serves as one.

3. Horizontally ruled minimal surfaces in $M \times \mathbb{R}$

In this section we consider a ruled minimal surfaces in product space $M \times \mathbb{R}$ for general 2-dimensional Riemannian manifold $M$. Of course the horizontal section $M \times \{z_0\}$ and the vertical cylinder $\{\gamma(t)\} \times \mathbb{R}$ over a geodesic $\gamma$ of $M$ are ruled minimal surfaces in $M \times \mathbb{R}$ ruled by horizontal geodesics. These surfaces are referred as the trivial ruled minimal surfaces. Note that these surfaces are totally geodesic in $M \times \mathbb{R}$. For the existence of nontrivial horizontally ruled minimal surfaces in $M \times \mathbb{R}$, the next theorem states that $M$ must have a Clairaut parametrization and the ruled minimal surfaces must be the helicoid considered in Sec. 2 at least locally.

**Theorem 3.1.** If there is a ruled minimal surface $\Sigma$ in $M \times \mathbb{R}$ through $P = (p_0, z_0)$ ruled by horizontal geodesics and $T_P \Sigma$ is neither parallel nor perpendicular to the vertical direction of $\mathbb{R}$, then $p_0 \in M$ has a neighborhood $U$ with Clairaut parametrization and the surface $\Sigma$ is a part of a helicoid in $U \times \mathbb{R}$ near $(p_0, z_0)$.

**Proof.** Since $T_P \Sigma$ is transversal to the vertical direction, there exist a neighborhood of $P$ in $\Sigma$ on which the projection map $\pi : M \times \mathbb{R} \to M$ is a diffeomorphism. Noting that the ruling geodesics are projected to the geodesics in $M$, we can take a ruled parametrization of $\Sigma$ on a neighborhood of $P$ such that

$$
\begin{cases}
X(s, t) = (\varphi(s, t), h(s)) \subset M \times \mathbb{R}, \\
\varphi(s, t) = \exp_{\alpha(s)}(tv(s))
\end{cases}
$$

for some functions $h(s)$ where $\alpha(s)$ is a unit speed curve with $\alpha(0) = p_0$ in $M$ and $v(s)$ is a tangent vector field to $M$ along $\alpha$ of unit length with $\langle v(s), \alpha'(s) \rangle \equiv 0$. Noting that $\varphi(x, y)$ is a geodesic coordinate in some neighborhood $U$ of $p_0$ in $M$, we can take a coordinate patch $\Psi(x, y) = (\varphi(x, y), z)$ on $U \times \mathbb{R}$. For this coordinate, the coefficients for the first fundamental form are

$$
(g_{ij}) = \begin{bmatrix}
 f^2(x, y) & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
\end{bmatrix}
$$

for some function $f(x, y) > 0$ with $f(x, 0) = 1$ and the Riemannian connection becomes

$$
\nabla_{\partial_x} \partial_x = \frac{1}{f} \frac{\partial f}{\partial x} \partial_x - \frac{\partial f}{\partial y} \partial_y, \quad \nabla_{\partial_y} \partial_y = 0, \quad \nabla_{\partial_z} \partial_z = 0,
$$

$$
\nabla_{\partial_x} \partial_y = \nabla_{\partial_y} \partial_x = \frac{1}{f} \frac{\partial f}{\partial y} \partial_x, \quad \nabla_{\partial_y} \partial_z = \nabla_{\partial_z} \partial_y = 0, \quad \nabla_{\partial_x} \partial_z = \nabla_{\partial_z} \partial_x = 0.
$$
Now for the ruled parametrization \( X(s, t) = \Psi(s, t, h(s)) \) of the surface \( \Sigma \), we have
\[
X_s = \partial_x + h'(s)\partial_z, \quad X_t = \partial_y
\]
and
\[
X_{ss} = \nabla X_s X_s = h''(s)\partial_z + \nabla \partial_x \partial_x + 2h'(s)\nabla \partial_x \partial_z + (h')^2(s)\nabla \partial_z \partial_z
= \frac{1}{f(s, t)} \frac{\partial f(s, t)}{\partial s} \partial_x - f(s, t) \frac{\partial f(s, t)}{\partial t} \partial_t + h''(s)\partial_z.
\]
\[
X_{st} = \nabla X_s X_t = \nabla \partial_x \partial_x + h'(s)\nabla \partial_y \partial_z = \frac{1}{f(s, t)} \frac{\partial f(s, t)}{\partial t} \partial_x = X_{ts},
\]
\[
X_{tt} = \nabla X_t X_t = \nabla \partial_y \partial_y = 0.
\]
Taking the unit normal vector field \( n \) to the surface as
\[
n = \frac{1}{\sqrt{f^2(s, t) + (h')^2(s)}} \left( h'(s) \frac{\partial f(s, t)}{\partial s} f(s, t) \right),
\]
we have
\[
E = \langle X_s, X_s \rangle = f^2(s, t) + (h')^2(s),
\]
\[
F = \langle X_s, X_t \rangle = 0,
\]
\[
G = \langle X_t, X_t \rangle = 1
\]
and
\[
l = \langle X_{ss}, n \rangle = \frac{1}{\sqrt{f^2(s, t) + (h')^2(s)}} \left( h'(s) \frac{\partial f(s, t)}{\partial s} + h''(s)f(s, t) \right),
\]
\[
m = \langle X_{st}, n \rangle = \frac{h'(s)}{\sqrt{f^2(s, t) + (h')^2(s)}} \frac{\partial f(s, t)}{\partial s},
\]
\[
n = \langle X_{tt}, n \rangle = 0.
\]
Therefore the mean curvature \( H \) of the surface is
\[
H = \frac{1}{2} \left( \frac{Gl - 2Fm + En}{f^2(s, t) + (h')^2(s)} \right)
= \frac{1}{\sqrt{f^2(s, t) + (h')^2(s)}} \left( h'(s) \frac{\partial f(s, t)}{\partial s} + h''(s)f(s, t) \right).
\]
Since the mean curvature \( H = 0 \),
\[
h'(s) \frac{\partial f(s, t)}{\partial s} + h''(s)f(s, t) = \frac{\partial}{\partial s} \left( h'(s)f(s, t) \right) = 0.
\]
This implies \( h'(s)f(s, t) = \zeta(t) \) for some function \( \zeta(t) \) and since \( f(s, 0) = 1 \), \( h'(s) = \zeta(0) \) is a constant. But from the fact that \( T_P \Sigma \) is not horizontal, \( h'(0) \neq 0 \) and \( h(s) = c_1s + c_0 \) for some constants \( c_0 \) and \( c_1 = \zeta(0) \neq 0 \). Therefore \( f(s, t) = \frac{1}{c_1} \zeta(t) \) is independent of \( s \) and the coordinate \( \varphi \) of \( M \) is a Clairaut parametrization and clearly \( X(s, t) \) gives a helicoid in \( U \times \mathbb{R} \). □
For the ruled parametrization $X(s, t) = (\varphi(s, t), h(s))$ in above proof, $h'(s) = c_1$ is a constant and the angle between $T_{X(s,t)}\Sigma$ and $\partial_z$ is given by
\[
\arccos \left( \frac{h'(s)}{\sqrt{f^2(s, t) + (h'(s))^2}} \right) = \arccos \left( \frac{c_1}{\sqrt{\xi^2(t) + c_1^2}} \right)
\]
which is independent to $s$. From this we can conclude that for any horizontally ruled minimal surface $\Sigma \subset M \times \mathbb{R}$, the angle between the tangent space of $\Sigma$ and the vertical direction is constant along any orthogonal trajectories of the ruling geodesics in $\Sigma$.

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