1. Introduction

Toric geometry has always been closely intertwined with combinatorics. Toric geometry has also figured prominently in mirror symmetry, particularly in the derivation of Hori and Vafa, who define the mirrors of toric varieties and hypersurfaces therein. (Indeed, toric geometry is responsible for the vast majority of examples
of mirror symmetry.) In this survey, we give an exposition without proofs of the results of [38, 37, 16], which together link up these various relations.

Briefly, mirror symmetry relates coherent sheaves on a toric variety to a Fukaya-type category of Lagrangian submanifolds of an affine space \((\mathbb{C}^*)^n \cong T^*((S^1)^n))\). An equivariant version of this duality relates equivariant coherent sheaves to a Fukaya category of \(T^*B\), where \(B\) is the universal cover of \(S^1\). The theorems of [38, 37] equate the Fukaya category of a cotangent \(T^*B\) to the category of constructible sheaves on the base \(B\). In our application \(B \cong \mathbb{R}^n\) is a vector space and the constructible category is generated by constant sheaves on polytopes. It is thus a combinatorial category which completes a triangle of linkages between equivariant coherent sheaves on a toric variety, a Fukaya category on a cotangent space of a vector space, and a constructible sheaf category generated by polytopes on that vector space. The resulting map from coherent to constructible sheaves is a categorification of Morelli’s combinatorial description of the equivariant K-theory of a toric variety [36].

1.1. Outline. In Section 2 we discuss the categories arising in homological mirror symmetry and give a list of some results to date. In Section 3 we review the toric geometry of the B-side and the different geometries of the A-side. We then describe how, beginning with an equivariant ample line bundle on a toric variety \(X_\Sigma\) constructed from a fan \(\Sigma \subset N_\mathbb{R}\), we construct a Lagrangian submanifold of the T-dual mirror geometry. This defines a functor from coherent sheaves on a toric variety to a Fukaya category of a cotangent bundle. Section 4 is a largely self-contained summary of the equivalence constructed in [38, 37], and can therefore be read (or skipped) independently. This section relates the Fukaya category of a cotangent bundle to constructible sheaves on the base manifold. In Section 5 we review the direct map from equivariant coherent sheaves on \(X_\Sigma\) to constructible sheaves on a vector space \(M_\mathbb{R} = N_\mathbb{R}\), and discuss the relation to Morelli’s theorem. Finally, we try to make these results as accessible as possible by providing a selection of examples in Section 6.

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2. Mirror Symmetry for Toric Manifolds

Mathematical treatments of mirror symmetry can take different forms. We will focus on the one most suited to our purpose, the homological mirror symmetry between a projective toric variety and its Hori-Vafa Landau-Ginzburg mirror.

\footnote{Several possible versions of this Fukaya category arise in the literature – see Section 2.2.3, 6.3 and References [11, 2, 23] – though we mainly work with the one defined in [38].}
2.1. **Hori-Vafa mirror.** In [26], Hori and Vafa derived from physics that the mirror geometry of a projective toric Fano manifold \(X\) is a Landau-Ginzburg model \(((\mathbb{C}^*)^n, W)\), where \(n = \dim_{\mathbb{C}} X\), and \(W : (\mathbb{C}^*)^n \to \mathbb{C}\) is a holomorphic function known as the superpotential [26]. Their method involves T-duality for the real torus acting on \(X\). On \(X\) one can consider the A-model \(A(X, \omega)\), which only depends on the symplectic structure \(\omega\) of \(X\), and the B-model \(B(X, J)\) which only depends on the complex structure \(J\) of \(X\). One is then interested in the mathematical structures that arise in these models. For example, quantum cohomology and Lagrangian submanifolds arise in the A-model, while singularity theory and coherent sheaves arise in the B-model. Schematically, mirror symmetry postulates the following equivalences:

\[
\begin{align*}
A(X, \omega) &\simeq B((\mathbb{C}^*)^n, W), \\
B(X, J) &\simeq A((\mathbb{C}^*)^n, W).
\end{align*}
\]

In [4], Auroux described a construction of the mirror of a non-toric Fano manifold, or more generally, a Kähler manifold with a nonzero, effective anti-canonical divisor.

2.2. **Categories in mirror symmetry.** Kontsevich’s homological mirror symmetry conjecture [30] relates categories of D-branes in mirror-dual physical theories. We now introduce the categories that arise in \(A(X, \omega)\), \(B(X, J)\), \(A((\mathbb{C}^*)^n, W)\), and \(B((\mathbb{C}^*)^n, W)\).

2.2.1. **The Fukaya category \(\text{Fuk}(X)\).** The category that arises in \(A(X, \omega)\) is the Fukaya category \(\text{Fuk}(X)\). The Fukaya category is a rather vast subject in general. We give a superficial treatment here, and a somewhat expanded discussion in Section 4 – see [20, 41] for foundational material.

An object in \(\text{Fuk}(X)\) is a closed Lagrangian submanifold of \((X, \omega)\). The hom space of two Lagrangian submanifolds \(L, L'\) is the Floer complex:

\[
\text{hom}_{\text{Fuk}(X)}(L, L') = CF^*(L, L') = \bigoplus_{p \in L \cap \phi(L')} \Lambda_{\text{nov}} p
\]

where \(\phi\) is a Hamiltonian symplectic automorphism of \((X, \omega)\) such that \(L\) and \(\phi(L')\) intersect transversally, and

\[
\Lambda_{\text{nov}} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{n_i} \mid a_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}_{\geq 0}, n_i \in \mathbb{Z}, \lim_{i \to \infty} \lambda_i = \infty \right\}
\]

is the Novikov ring (we will see later that the sequence \(\lambda_i\) consists of possible symplectic areas of certain holomorphic polygons in \(X\)). \(\text{Fuk}(X)\) is an \(A^\infty\)-category: there are maps

\[
m_k : \text{hom}_{\text{Fuk}(X)}(L_0, L_1) \otimes \cdots \otimes \text{hom}_{\text{Fuk}(X)}(L_{k-1}, L_k) \to \text{hom}_{\text{Fuk}(X)}(L_0, L_k)[k-2]
\]

satisfying certain bilinear relations. In particular,

\[
d := m_1 : \text{hom}_{\text{Fuk}(X)}(L_0, L_1) \to \text{hom}_{\text{Fuk}(X)}(L_0, L_1)[-1]
\]

\[\footnote{The conjecture as originally posed in [30] involved only mirror symmetry of Calabi-Yau manifolds, but Kontsevich later generalized the conjecture to the toric Fano case.} \]
is the differential of the Floer complex\[^3\] and $\circ := m_2 : \text{hom}_{Fuk}(X)(L_0, L_1) \otimes \text{hom}_{Fuk}(X)(L_1, L_2) \to \text{hom}_{Fuk}(X)(L_0, L_2)$ is a (not necessarily associative) compatible product: $d(p \circ q) = (dp) \circ q \pm p \circ (dq)$.

The composition maps $m_k$ are defined by summing over holomorphic maps $u$ satisfying boundary conditions as in Figure 1 (assuming $L_i$’s intersect transversally), weighted by $T$ symplectic area of $u$ e Maslow index of $u \in \Lambda_{nov}$.

![Figure 1. Holomorphic polygons](image)

More precisely, note that the symplectic area and Maslov index of $u$ is determined by its relative homotopy class $\beta \in \pi_2 := \pi_2(X, L_0 \cup \cdots \cup L_k)$.

Let $\int_\beta \omega$ and $\mu(\beta)$ denote the symplectic area and the Maslov index of the class $\beta$, and let $n_\beta(p_1, \ldots, p_k, q)$ be the number of holomorphic maps in Figure 1 in class $\beta$ (sometimes this is a “virtual” number which is a rational number instead of an integer). Then

$$m_k(p_1, \ldots, p_k) = \sum_q \sum_{\beta \in \pi_2} n_\beta(p_1, \ldots, p_k, q) T \int_\beta \omega e^{\mu(\beta)} \cdot q.$$  

See [20] for details.

**Remark 2.1.** In this paper, we are more interested in exact Fukaya categories for a noncompact manifold $X$ equipped with an exact symplectic form $\omega = d\theta$, where $\theta$ is a 1-form on $X$. A Lagrangian submanifold of $(X, \omega)$ is exact if $\theta|_L = df$ for some function $f : L \to \mathbb{R}$. We build the category from exact, possibly noncompact Lagrangian submanifolds (satisfying certain regularity properties near infinity). In this case, we may use $\mathbb{Q}$ or $\mathbb{C}$ instead of the Novikov ring $\Lambda_{nov}$ in (3), and define

$$m_k(p_1, \ldots, p_k) = \sum_q n(p_1, \ldots, p_k, q) \cdot q$$

where $n(p_1, \ldots, p_k, q)$ is the number of holomorphic maps satisfying boundary conditions as in Figure 1. See [41] for details.

**Remark 2.2.** In general Lagrangians are equipped with flat complex line bundles. On the right hand side of (3), $\Lambda_{nov}$ is replaced by $\Lambda_{nov} \otimes \text{Hom}_\mathbb{C}(V_p, V'_p) \cong \Lambda_{nov} \otimes \mathbb{Q} \mathbb{C}$, where $V$ and $V'$ are flat complex line bundles on $L$ and $L'$, respectively. The right hand side of (4) contains an additional factor which is the holonomy of the flat line

$^3$There may be a “curved” obstruction, meaning $d^2 \neq 0$. 
bundle along $\partial \Omega$. These extra data can be ignored in the exact case described in
the above Remark 2.1.

The bounded derived category $DFuk(X)$ is a triangulated category obtained
by taking cohomology $H^0$ of the triangulated envelope $TrFuk(X)$ of $Fuk(X)$ (see
Section 4.1).

2.2.2. The dg category of coherent sheaves $Coh(X)$. The category that arises in
$B(X,J)$ is the derived category (or rather an appropriate dg enhancement) of
coherent sheaves on $X$. The shortest way to define $DCoh(X)$ is as the dg category
whose objects are complexes of injective quasicoherent sheaves $I^\bullet$ satisfying

The cohomology sheaves of $I^\bullet$ are coherent, and vanish in all but finitely
many degrees.

Then $\text{hom}(I^\bullet, J^\bullet)$ is the usual chain complex of homomorphisms.

2.2.3. The Fukaya-Seidel category $FS((\mathbb{C}^*)^n, W)$. The category of D-branes that
arises in $A((\mathbb{C}^*)^n, W)$ is the Fukaya-Seidel category $FS((\mathbb{C}^*)^n, W)$.

Assume that the superpotential $W$ has isolated nondegenerate critical points
$x_1, \ldots, x_v$, so that $W : (\mathbb{C}^*)^n \to \mathbb{C}$ is a Lefschetz fibration. Let $b \in \mathbb{C}$ be a regular
value, so that the fiber $W^{-1}(b)$ is a smooth complex hypersurface in $(\mathbb{C}^*)^n$. An
object in $FS((\mathbb{C}^*)^n, W)$ is a vanishing cycle which is a Lagrangian sphere in $W^{-1}(b)$
associated to a path from $b$ to a critical point. Alternatively, it is a Lagrangian
thimble which is a Lagrangian submanifold of $(\mathbb{C}^*)^n$ with boundary in $W^{-1}(b)$; its
image under $W$ is a path from a critical value to $b$. As in Section 2.2.1, the hom
space of two vanishing cycles is a Floer complex, and the comosition maps are
defined by counting holomorphic polygons in $W^{-1}(b)$ with boundaries in vanishing
cycles.

We write $DFS((\mathbb{C}^*)^n, W)$ for the bounded derived category of the Fukaya-Seidel
category $FS((\mathbb{C}^*)^n, W)$.

2.2.4. $DSing((\mathbb{C}^*)^n, W)$. The derived category that arises in the Landau-Ginzburg
B-model $B((\mathbb{C}^*)^n, W)$ is $DSing((\mathbb{C}^*)^n, W)$.

We briefly describe the definition, following Orlov [39]. Assume that the su-
perpotential $W$ has isolated nondegenerate critical points $x_1, \ldots, x_v$. Let $W_i$
be the closed subscheme of $(\mathbb{C}^*)^n$ defined by $W(z) - x_i = 0$. Define the triangulated
category $DSing(W_i)$ as the quotient of the bounded derived category of coher-
ent sheaves on $W_i$, $DCoh(W_i)$, by the full triangulated subcategory generated by
perfect complexes, $\text{Perf}(W_i)$. Then

$$DSing((\mathbb{C}^*)^n, W) = \prod_{i=1}^v DSing(W_i).$$

Orlov, using the theorem of Eisenbud (Section 5 of [14]), relates this category to the
category of matrix factorizations. We will not use this description or say anything
more about it, however.

2.2.5. Kontsevich’s conjecture. The homological mirror conjecture for toric Fano
manifolds postulates the following quasi-equivalences of triangulated categories:

(a) $DFuk(X) \cong DSing((\mathbb{C}^*)^n, W)$.
(b) $DCoh(X) \cong DFuk((\mathbb{C}^*)^n, W)$. 

where items (a) and (b) correspond to Equations (1) and (2) in Section 2.1 respectively.

2.3. Results to date. Here is a rundown of some results in establishing Kontsevich’s conjecture in the special case of a non-Calabi-Yau toric variety. Many other results in mirror symmetry are omitted, with apologies.

The equivalence (a) has been studied by Cho [7, 8], Cho-Oh [9]; Fukaya-Oh-Ohta-Ono studied (a) for any projective toric manifolds, including non-Fano ones [21, 22].

The equivalence (b) has been studied in the following works:
(i) Hori, Iqbal, and Vafa define a correspondence of branes for statement (b) in the case of projective spaces and toric del Pezzo surfaces in [27].
(ii) In [5], Auroux-Katzarkov-Orlov prove (b) when \( \dim C = 2 \). They allow orbifold singularities, and in particular, weighted projective planes. They also study Hirzebruch surfaces \( \mathbb{F}_m = \mathbb{P}(\mathbb{O}_1 \oplus \mathbb{O}_1 (m)) \) \( (m \geq 0) \), which are not Fano when \( m \geq 2 \).
(iii) Bondal-Ruan announced a proof of (b) for weighted projective spaces of any dimension.
(iv) In [2], Abouzaid studied the equivalence (b) for projective toric manifolds, including non-Fano ones. Abouzaid established a quasi-equivalence between \( \text{DCoh}(X) \) and a full subcategory of \( D^n \text{Fuk}((\mathbb{C}^*)^n, W^{-1}(0)) \), where \( \text{Fuk}((\mathbb{C}^*)^n, W^{-1}(0)) \) is certain relative Fukaya category, and the superscript \( \pi \) stands for the split closure. An object in \( \text{Fuk}((\mathbb{C}^*)^n, W^{-1}(0)) \) is a compact Lagrangian submanifold in \((\mathbb{C}^*)^n \) with boundary in the complex hypersurface \( W^{-1}(0) \).
(v) In [15], the first author proved a version of (b) for projective spaces \( \mathbb{P}^n \) using T-duality. On the right hand side of (b), he uses a Fukaya category of the cotangent bundle \( T^*(S^1)^n \cong (\mathbb{C}^*)^n \) constructed by Nadler and the fourth author in [38] (see Section 4). The quasi-equivalence is established via a category of constructible sheaves on the torus \( (S^1)^n \).

We will give an exposition of the recent work by the authors [16] generalizing (v) to all projective toric manifolds, including non-Fano ones.

3. T-duality

In [46], mirror symmetry for Calabi-Yau manifolds was described by dualizing the fibers of a conjectural special Lagrangian torus fibration with singularities. This “T-duality” provides a transformation of branes. This transformation was investigated in [3, 34], and can be applied to the open orbit \( Y \cong (\mathbb{C}^*)^n \) inside \( X \). This is the method we will employ to construct Lagrangian objects from holomorphic ones.

To be more explicit, we need to introduce some notation.

3.1. Moment Polytope. Here we recall the moment map construction, since it figures prominently in the sequel. We ignore many other constructions of toric geometry. Readers may consult [24] for foundations on toric geometry, or [16] Section 2 for material immediately related to present purposes.

Let \( X \) be a projective toric manifold, and let \( T \cong (\mathbb{C}^*)^n \) act on \( X \), where \( n \) is the complex dimension of \( X \). Let \( N = \text{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^n \) be the lattice of 1-parameter subgroups of \( T \), and let \( M = \text{Hom}(T, \mathbb{C}^*) \) be the group of irreducible characters of
Note that $M$ is the dual lattice of $N$. Let $T_R \cong U(1)^n$ be the maximal compact subgroup of $T$. Define

$$N_R = N \otimes \mathbb{Z} \mathbb{R}, \quad M_R = M \otimes \mathbb{Z} \mathbb{R}.$$  

Then $N_R$ is canonically identified with the Lie algebra $t_R$ of $T_R$, while $M_R$ is canonically identified with $t_R^\vee$, the dual real vector space of $t_R$. We have

$$N_R/N \cong T_R, \quad M_R/M \cong T_R^\vee$$

where $T_R^\vee$ is the dual torus.

Let $\omega$ be a $T_R$-invariant symplectic form on $X$ such that the $T_R$-action on $X$ is Hamiltonian. Let $\mu_{T_R} : X \rightarrow M_R$ be the moment map of the $T_R$-action. It is determined by $\omega$ up to addition of a constant vector in $M_R$. The image of the moment map is a convex polytope $\Delta \subset M_R$, known as the moment polytope. The fibers of the moment maps are $T_R$-orbits. We have a homeomorphism $X/T_R \cong \Delta$, and the natural projection $X \rightarrow \Delta$ restricts to a $T_R$-fibration from the open orbit $Y \cong (\mathbb{C}^*)^n$ to the interior $\Delta^o$ of the convex polytope $\Delta$.

**Example 3.1.** $X = \mathbb{P}^2$, and $\omega$ is the Fubini-Study Kähler form, so that $\int_{\mathbb{P}^2} \omega^2 = 4\pi^2$. The moment map $\mu_{T_R} : \mathbb{P}^2 \rightarrow M_R \cong \mathbb{R}^2$ is given by

$$\mu_{T_R}([X_0, X_1, X_2]) = \frac{(|X_1|^2, |X_2|^2)}{|X_0|^2 + |X_1|^2 + |X_2|^2}.$$  

The moment polytope is the triangle:

```
(0, 1)    
|  
(0, 0)----(1, 0)
```

**Figure 2.** Moment polytope of $\mathbb{P}^2$

### 3.2. Geometry of the open orbit.
Here we discuss the various geometries arising from the open orbit of the complex torus under T-duality.

Let $Y \subset X$ be the open orbit, equipped with the Kähler structure inherited from $X$. Let $\hat{Y} \rightarrow Y$ be the universal cover, equipped with the pull back Kähler structure. The complex and symplectic structures on $\hat{Y}$ and $Y$ are summarized in the following diagram:

---

4We use the convention that the dual of a circle of radius $R$ is a circle of radius $1/R$. An alternative convention is that the dual of a circle of circumference $\ell$ is a circle of circumference $1/\ell$.  

\( T_{\mathbb{R}} N_{\mathbb{R}} \cong \mathbb{C}^n \) (complex)
\( \cong N_{\mathbb{C}} \quad \text{anti-symplectic} \)
\( \exp_T \quad \text{complex} \quad \text{anti-symplectic} \)
\( T \cong Y \cong T_{\mathbb{R}} \times \Delta^o \)
\( \log_T \quad \text{complex} \quad \text{anti-symplectic} \)
\( \mu_{\mathbb{R}} \)
\( N_{\mathbb{R}} \leftarrow \pi \quad Y \rightarrow p \quad \Delta^o \quad \subset M_{\mathbb{R}} \)

In the above diagram,
\[
\exp_T : N_{\mathbb{C}} \cong \mathbb{C}^n \rightarrow T \cong (\mathbb{C}^*)^n, \quad (w_1, \ldots, w_n) \mapsto (e^{w_1}, \ldots, e^{w_n}),
\]
is the exponential map from the Lie algebra of \( T \) to the Lie group \( T \), and
\[
\log_T : T \cong (\mathbb{C}^*)^n \rightarrow N_{\mathbb{R}} \cong \mathbb{R}^n, \quad (t_1, \ldots, t_n) \mapsto (\log |t_1|, \ldots, \log |t_n|),
\]
is the logarithm map. \( T^* \Delta^o \cong N_{\mathbb{R}} \times \Delta^o \) is equipped with the canonical symplectic form
\[
diag(\sum_{i=1}^n \theta_i dx_i) = \sum_{i=1}^n \theta_i dx_i.
\]
This descends to a symplectic form on \( T_{\mathbb{R}} \times \Delta^o = (N_{\mathbb{R}}/N) \times \Delta^o \). The map \( T^* \Delta^o \cong N_{\mathbb{R}} \times \Delta^o \rightarrow T_{\mathbb{R}} \times \Delta^o \) is given by \( (\theta, x) \mapsto (\exp_{T_{\mathbb{R}}}(\theta), x) \). Here
\[
\exp_{T_{\mathbb{R}}} : N_{\mathbb{R}} \cong \mathbb{R}^n \rightarrow T_{\mathbb{R}} \cong U(1)^n, \quad (\theta_1, \ldots, \theta_n) \mapsto (e^{\sqrt{-1} \theta_1}, \ldots, e^{\sqrt{-1} \theta_n})
\]
is the exponential map from the Lie algebra \( t_{\mathbb{R}} \) to the Lie group \( T_{\mathbb{R}} \); the kernel \( N_{\mathbb{R}} \subset N_{\mathbb{R}} \) of \( \exp_{T_{\mathbb{R}}} \) is given by \( \theta_i \in 2\pi \mathbb{Z} \). The map \( T_{\mathbb{R}} \times \Delta^o \rightarrow \Delta^o \) is the projection to the second factor; it also is the moment map of the \( T_{\mathbb{R}} \)-action on \( T_{\mathbb{R}} \times \Delta^o \) by multiplication on the first factor. Note that [25, Theorem 6.4]
\[
\int_X \frac{\omega^n}{n!} = \int_Y \frac{\sum_{i=1}^n (\theta_i dx_i)^n}{n!} = (2\pi)^n \int_{\Delta^o} dx_1 \cdots dx_n = (2\pi)^n \text{volume}(\Delta).
\]
Following [31], one may apply \( T \)-duality to the \( T_{\mathbb{R}} \)-fibrations \( \pi : Y \rightarrow N_{\mathbb{R}} \) and \( p : Y \rightarrow \Delta^o \) to obtain the \( T \)-dual \( Y^\dual \) together with a Kähler structure. This construction is described explicitly by Auroux [4, Section 4] and Chan-Leung ([12, Section 3] and [13, Section 2.1]). In particular, they found that as a complex manifold, \( Y^\dual \) is an open subset of the Hori-Vafa mirror \( T^\dual \cong (\mathbb{C}^*)^n \).

Let \( \hat{Y}^\dual \rightarrow Y^\dual \) be the universal cover, equipped with the pull back Kähler structure. The complex and symplectic structures on \( Y^\dual \) and \( Y^\dual \) are summarized in the following diagram:
\[
\begin{array}{cccccc}
T^* N_{\mathbb{R}} & \cong & Y^\dual & \cong & M_{\mathbb{R}} \times \Delta^o \subset M_{\mathbb{C}} & = T M_{\mathbb{R}} \\
\downarrow & & \downarrow & & \downarrow & \\
T^\dual \times N_{\mathbb{R}} & \cong & Y^\dual & \cong & \log^{-1}(\Delta^o) \subset T^\dual & \xrightarrow{\text{holomorphic}} \mathbb{C} \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \\
N_{\mathbb{R}} & \xleftarrow{\pi^\dual} & Y^\dual & \xrightarrow{p^\dual} & \Delta^o & \subset M_{\mathbb{R}}
\end{array}
\]
In the above diagram, \( \exp_{T^\vee} : M_C \cong \mathbb{C}^n \to T^\vee \cong (\mathbb{C}^*)^n \) is the exponential map from the Lie algebra of \( T^\vee \) to the Lie group \( T^\vee \), and \( \log_{T^\vee} : T \cong (\mathbb{C}^*)^n \to N_R \cong \mathbb{R}^n \) is the logarithm map. \( T^*N_R \cong M_R \times N_R \) is equipped with the canonical symplectic form
\[
d(\sum_{i=1}^n \gamma_i dy_i) = \sum_{i=1}^n d\gamma_i \wedge dy_i.
\]
This descends to a symplectic form on \( T^\vee_R \times N_R = (M_R/M) \times N_R \). The map \( M_R \times N_R \to T^\vee_R \times N_R \) is given by \((\gamma, y) \mapsto (\exp_{T_R^\vee}(\gamma), y)\). Here \( \exp_{T_R^\vee} : M_R \to T^\vee_R \) is the exponential map from the Lie algebra \( t_R^\vee \) to the Lie group \( T^\vee_R \); the kernel \( M \subset M_R \) is given by \( \gamma_i \in 2\pi \mathbb{Z} \). The map \( T^\vee_R \times N_R \to N_R \) is the projection to the second factor; it is also the moment map of the \( T^\vee \)-action on \( T^\vee_R \times N_R \) by multiplication on the first factor. Finally, we have
\[
\begin{array}{ccc}
T^*M^\text{symplectic}_R & \cong & \tilde{Y}^\vee \\
\downarrow & & \downarrow \\
T^*T^\vee_R & \cong & Y^\vee
\end{array}
\]
where \( T^*M_R \) and \( T^*T^\vee_R \) are equipped with the canonical symplectic form of the cotangent bundle:
\[
d(\sum_{i=1}^n y_i d\gamma_i) = \sum_{i=1}^n dy_i \wedge d\gamma_i.
\]

Example 3.2. \( X = \mathbb{P}^1 \).

The moment map \( \mu_{\mathbb{P}^1} : \mathbb{P}^1 \to M_R \cong \mathbb{R} \) is given by
\[
\mu_{\mathbb{P}^1}(X_0, X_1) = \frac{a|X_1|^2}{|X_0|^2 + |X_1|^2}, \quad a > 0.
\]
The image is \([0, a] \subset \mathbb{R}\).

The complex coordinates on \( \tilde{Y} \cong \mathbb{C} \) and \( Y \cong \mathbb{C}^* \) are \( y + \sqrt{-1} \theta \) and \( e^{y+\sqrt{-1} \theta} = X_1/X_0 \), respectively. The restrictions of the symplectic form \( \omega \) and the Riemannian metric \( g \) on \( X = \mathbb{P}^1 \) to \( Y = \mathbb{C}^* \) are given by
\[
\omega = dx \wedge d\theta = \frac{2ae^{2y} dy \wedge d\theta}{(1 + e^{2y})^2}, \quad y \in N_R
\]
and
\[
g = \frac{a}{2x(a-x)} dx^2 + \frac{2x(a-x)}{a} d\theta^2 = \frac{2ae^{2y}}{(1 + e^{2y})^2} (dy^2 + d\theta^2), \quad y \in \mathbb{R}
\]
where
\[
x \in (0, a) \subset M_R \cong \mathbb{R}, \quad y \in N_R \cong \mathbb{R}, \quad \theta \in N_R/N \cong \mathbb{R}/2\pi \mathbb{Z}.
\]
The symplectic form \( \omega^\vee \) and the Riemannian metric \( g^\vee \) on \( Y^\vee \) are given by
\[
\omega^\vee = \frac{adx \wedge d\gamma}{2x(a-x)} = dy \wedge d\gamma,
\]
\[
g^\vee = \frac{a}{2x(a-x)} (dx^2 + d\gamma^2) = \frac{2ae^{2y}}{(1 + e^{2y})^2} dy^2 + \frac{1 + e^{2y}}{2ae^{2y}} d\gamma^2.
\]
where
\[
x \in (0, a) \subset M_R \cong \mathbb{R}, \quad y \in N_R \cong \mathbb{R}, \quad \gamma \in M_R/M \cong \mathbb{R}/2\pi \mathbb{Z}.
\]
Recall that under the T-duality, a circle of radius $R$ is dual to a circle of radius $1/R$. In this example, the fiber $p^{-1}(x)$ of $p : Y \to \Delta^o = (0, a)$ is a circle of radius $\sqrt{\frac{2x(a-x)}{a}}$, while the fiber $(p^\vee)^{-1}(x)$ of $p^\vee : Y^\vee \to \Delta^o = (0, a)$ is a circle of radius $\sqrt{\frac{a}{2x(a-x)}}$.

3.3. Statement of symplectic results. We now describe the symplectic results from [10]. The description of the T-duality transformation of objects is postponed until Section 3.4

A smooth projective toric variety is defined by a fan $\Sigma \subset \mathbb{N}_\mathbb{R}$. Fans of some toric surfaces are shown in Figure 3, where $F_m$ is the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m))$ ($m \geq 0$), and $B_k$ is the toric blowup of $\mathbb{P}^2$ at $k$ points.

![Figure 3. Fans of some toric surfaces](image)

The first row in Figure 3 consists of all projective smooth toric Fano surfaces. Note that the anti-canonical divisor $-K_{F_m}$ is ample for $m = 0, 1$, numerically effective for $m = 2$, and not numerically effective for $m > 2$. In general, the fan of a smooth projective toric surface is determined by its 1-dimensional cones.

Let $\langle , \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ denote the natural pairing. Given a $d$-dimensional cone $\sigma \subset N_\mathbb{R}$, define an $(n-d)$-dimensional subspace

$$\sigma^\bot = \{ m \in M_\mathbb{R} \mid \langle m, v \rangle = 0 \ \forall v \in \sigma \} \subset M_\mathbb{R}.$$

Let $X_\Sigma$ denote the toric variety defined by a fan $\Sigma$. Given a fan, define a conical Lagrangian submanifold of $M_\mathbb{R} \times N_\mathbb{R} \cong T^* M_\mathbb{R}$:

$$\Theta_\Sigma = \bigcup_{\sigma \in \Sigma} \bigcup_{m \in M} (\sigma^\bot + M) \times (-\sigma)$$

where $\sigma^\bot + M = \{ x + m \mid x \in \sigma^\bot, \ m \in M \}$ is the union of all translations of $\sigma^\bot$ by a point in the lattice $M$. Note that $\Theta_\Sigma$ descends to a conical Lagrangian

$$\Theta_\Sigma/M \subset (M_\mathbb{R}/M) \times N_\mathbb{R} \cong T^* T^\vee_\mathbb{R}.$$

For example, when $\dim \mathbb{C} X_\Sigma = 2$ is a projective surface, $\Theta_\Sigma$ is determined by the 1-dimensional cones $\rho_i$, $i = 1, \ldots, r$. Each 1-dimensional cone $\rho_i$ determines a circle

$$\rho_i^\bot/M \subset M_\mathbb{R}/M \cong (S^1)^2.$$
The fiber of $\Lambda^\Sigma/M \to M_R/M$ over a (generic) point in $\rho^+_i/M$ is $-\rho_i \subset N_R$. We use an inner product to identify $N_R$ with $M_R$ and view $-\rho_i$ as a ray normal to $\rho^+_i/M$. Therefore $\Lambda^\Sigma$ can be drawn on a square:

![Figure 4. Conical Lagrangians of some toric surfaces](image)

Let $\iota : T^* M_R \to D^* M_R$ be the diffeomorphism from the cotangent bundle to its disk bundle. Given a Lagrangian submanifold or Lagrangian subvariety $L \subset T^* M_R$, defined the conic limit of $L$ to be

$$L^\infty = \iota(L) \cap S^* M_R \subset S^* M_R,$$

where $S^* M_R$ is the sphere bundle of the cotangent bundle $T^* M_R$. The conic limit of a Lagrangian submanifold of $T^* T_R$ is defined similarly.

In [16], the authors derived the following equivariant version of homological mirror symmetry (“(b)”) for any projective toric manifold:

$$DCoh_T(X^\Sigma) \cong DFuk(T^* M_R; \Lambda^\Sigma).$$

Here $Coh_T(X^\Sigma)$ is the category of $T$-equivariant coherent sheaves. $DCoh_T(X^\Sigma)$ is generated by $T$-equivariant ample line bundles. An object in $Fuk(T^* M_R; \Lambda^\Sigma)$ is an exact Lagrangian submanifold $L$ of $T^* M_R$ with compact horizontal support such that $L^\infty \subset \Lambda^\Sigma^\infty$. The hom spaces are defined as in Section 2.2.1.

3.4. T-dual of an equivariant line bundle. The T-duality functor expressing the equivalence above is constructed on a generating set of holomorphic objects, equivariant ample line bundles. We describe this here.

Let $D_i$, $i = 1, \ldots, r$ be the codimension one orbit closures of $X$. Then any $T$-invariant divisor is of the form

$$D = \sum_{i=1}^r c_i D_i,$$

and any $T$-equivariant line bundle is of the form $O_X(D)$. There is a $T$-invariant meromorphic section $s$ of $O_X(D)$, unique up to multiplication by $C^*$, such that $\text{div}(s) = D$. $s$ is holomorphic and nonvanishing on $Y$, so it is a holomorphic frame of $O_X(D)|_Y$. Let $\nabla$ be the $U(1)$-connection determined by a $T_R$-invariant hermitian metric $h$ on $O_X(D)$, and let $\alpha$ be the connection 1-form with respect to the unitary frame $s/\|s\|_h$. Then $\alpha$ is purely imaginary, and the restriction of $\sqrt{-1}\alpha$
to a fiber $Y_x := p^{-1}(x)$ of the $T_R$-fibration $p : Y \to \triangle^o$ is a real harmonic 1-form on $Y_x$, which can be identified with an element in $H^1(Y_x; \mathbb{R})$, the universal cover of $Y_x^\vee \cong H^1(Y_x; \mathbb{R})/H^1(Y_x; \mathbb{Z})$. Letting $x$ vary yields a section of $\hat{Y}^\vee \cong M_R \times N_R \to N_R$, which is the same as a map $\Psi_h : N_R \to M_R$. The map $\Psi_h$ has the following interpretation. Let $F_h = \frac{\partial^2}{\partial t^2}$ be a closed $T$-invariant 2-form, then $\omega_h = \sqrt{-1}F_h$ is a presymplectic form in the sense of Karshon-Tolman [33]. Note that 
\[
[\omega_h] = 2\pi c_1(\mathcal{O}_X(D_{c})) \in H^2(X; \mathbb{R}).
\]
$\omega_h$ defines a moment map $\Phi_h : X \to M_R$ up to a constant vector in $M_R$; the constant is determined by the equivariant structure on $\mathcal{O}_X(D_{c})$. Let $j_0 : N_R \to X$ be the composition of $\exp : N_R \to Y$ and the open embedding $Y \to X$. Then $\Psi_h = \Phi_h \circ j_0$. The T-dual Lagrangian of the the $T$-equivariant line bundle $\mathcal{O}_X(D_{c})$ equipped with a $T_R$-invariant hermitian metric $h$ is given by 
\[
L_{c,h} = \{(\Phi_h \circ j_0(y), y) \mid y \in N_R\} \subset M_R \times N_R = T^*M_R.
\]

**Example 3.3.** For $\mathbb{P}^1$, the 1-cones in the fan $\Sigma$ are generated by $v_1 = 1$, $v_2 = -1$. Let $D_1$ and $D_2$ be two $T$-invariant divisors corresponding to these 1-cones, and $z$ be the complex coordinate such that $z|_{D_1} = 0$ and $z|_{D_2} = \infty$. The exponential map $j_0 : N_R \to \mathbb{P}^1$ is simply given by $y_1 \mapsto e^{y_1}$. The $T$-equivariant line bundle $\mathcal{O}(c_1D_1 + c_2D_2)$ has a canonical meromorphic section $s$. Endow this line bundle with a hermitian metric $h$
\[
\|s\|_h^2 = \frac{|z|^{2c_1}}{(1 + |z|^2)^{c_1 + c_2}}.
\]
The moment map $\Phi_h$ associated to the presymplectic form $\omega_h = \sqrt{-1}F_h$ is given by 
\[
z \mapsto \frac{(c_1 + c_2)|z|^2}{1 + |z|^2} - c_1.
\]
Thus the map $\Psi_h : N_R \to M_R$ is given by the formula 
\[
\frac{\gamma_1}{2\pi} = \frac{(c_1 + c_2)e^{2y_1}}{1 + e^{2y_1}} - c_1.
\]
This equation characterizes the T-dual Lagrangian $L_{c,h}$ where $c = c_1v_1 + c_2v_2$.

$D\text{Coh}_T(X)$ is generated by equivariant ample line bundles, so it suffices to define the functor $D\text{Coh}_T(X) \to DFuk(T^*M_R; \Lambda_\Sigma)$ on equivariant ample line bundles. When $\mathcal{O}_X(D_{c})$ is an ample, there exists a $T_R$-invariant hermitian metric $h$ on $\mathcal{O}_X(D_{c})$ such that $\omega_h$ is a symplectic form. Then the image of the moment map $\Phi_h$ is a convex polytope 
\[
\triangle_{c} = \{m \in M_R \mid \langle m, v_i \rangle \geq -c_i \mid i = 1, \ldots, r\},
\]
where $v_i \in N$ is primitive, and $\mathbb{R}_{\geq 0}v_i$ is the 1-dimensional cone associated to $D_i$. The map $\Psi_h = \Phi_h \circ j_0 : N_R \to \triangle^c$ is a diffeomorphism from $N_R$ to the interior of $\triangle_{c}$. Let $F_{\sigma}$ be the face of $\triangle_{c}$ associated to the cone $\sigma \in \Sigma$, and define a conical Lagrangian 
\[
\Lambda_{c} = \bigcup_{\sigma \in \Sigma} F_{\sigma} \times (-\sigma) \subset \Lambda_\Sigma.
\]
Then $L_{c,h}^\infty = \Lambda_{c}^\infty \subset \Lambda_\Sigma^\infty$. It is indeed an object in $Fuk(T^*M_R; \Lambda_\Sigma)$. Different metrics define equivalent objects.
4. Microlocalization

We are proving an equivalence between coherent sheaves, Lagrangians, and constructible sheaves. The first equivalence is described in the preceding section. The last equivalence – between the Fukaya category $\text{Fuk}(T^*B)$ of a cotangent bundle and constructible sheaves $\text{Sh}_c(B)$ on the base – is provided by the papers [38, 37], whose results and ideas we describe in this section. The strategy in relating the Fukaya category of a cotangent to constructible sheaves will be to exploit how both are computed by Morse theory. The relation to Morse theory is found following the lead of Kontsevich-Soibelman [32] and Fukaya-Oh [19], who study a similar geometry.

4.1. Algebraic preliminaries. Here we review some of the algebraic structures which appear in the argument. We will work with dg and $A_\infty$ categories. Recall that a dg algebra is a complex with differential $d$ and a degree-zero product $\circ$ obeying the Leibnitz rule. The path from dg algebras to dg categories is clear: the hom spaces in dg categories have the structure of a differential complex, with compositions morphisms of complexes. $A_\infty$ algebras have degree-one differentials $m_1 := d$, Leibnitz degree-zero products $m_2 = \circ$, and higher compositions $m_k$ of degree $2 - k$ (so a dg algebra is an $A_\infty$ algebra with $m_{k\geq 3} = 0$). Recall here the central example of the $A_\infty$ algebra of chains on a loop space: the concatenation product is not associative, but is associative at the level of homology. Further, the different homotopies from $((a \ast b) \ast c) \ast d$ to $a \ast (b \ast (c \ast d))$ are themselves homotopic, and so on, so there are higher relations as well. (N.B.: The suspension from a space to its loops helps to explain the unusual grading conventions one often encounters.) There is a clear path from $A_\infty$ algebras to $A_\infty$ categories.

Given an $A_\infty$ category $\mathcal{C}$ we can consider the dg category $\mathcal{C}\text{-mod}$ of (contravariant) $A_\infty$ functors from $\mathcal{C}$ to the category of chain complexes. The Yoneda embedding is a functor from $\mathcal{C}$ to $\mathcal{C}\text{-mod}$ sending an object $a$ to $\text{hom}_\mathcal{C}(\_ , a)$.

If $\mathcal{C}$ is a dg or $A_\infty$ category (the former a special case of the latter), then taking cohomology of the hom spaces yields an ordinary (associative) $\mathbb{Z}$-graded category $H(\mathcal{C})$. $H(\mathcal{C})$ has a triangulated structure: a triangle is distinguished if its image under Yoneda is (isomorphic to) an exact triangle. $A_\infty$ categories with shift functors satisfying some conditions are said to be triangulated. The triangulated envelope of $\mathcal{C}$, denoted $\text{Tr}\mathcal{C}$, is, informally, the smallest $A_\infty$ category generated by $\mathcal{C}$. It is unique up to quasi-equivalence, and can be constructed explicitly from twisted complexes or as the envelope within $\mathcal{C}\text{-mod}$ of the image under the Yoneda embedding.

4.2. The Cast of Categories. Some intermediate categories between $\text{Fuk}(T^*B)$ and $\text{Sh}_c(B)$ figure prominently in the proof of equivalence. After recalling the definition of $\text{Sh}_c(B)$ below, we will describe these intermediate categories and the quasi-equivalences between them.

$\text{Sh}_c(B)$. We recall what we mean by the category of constructible sheaves, $\text{Sh}_c(B)$. Recall that a constructible sheaf on a topological space $B$ is a sheaf (of $\mathbb{C}_B$ modules) that is locally constant on the strata of a Whitney stratification. We use the term

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5 We refer the reader to [29] for rigorous definitions and an excellent discussion.

6 We recall that a stratification is a decomposition of a manifold into strata of pairwise disjoint manifolds such that one intersects the closure of another if and only if it is contained therein.
"constructible sheaf" (sometimes just "sheaf") to describe a complex of sheaves with bounded, constructible cohomology. Acyclic complexes form a null system within this dg category, and we write \( Sh_c(B) \) for the associated quotient. Despite the quotient, \( Sh_c(B) \) retains the structure of a \( dg \) category. Taking cohomology in degree zero produces \( DSh_c(B) \), the bounded derived category of constructible sheaves on \( B \). As an example of an object, to an open submanifold \( U \supset \hookrightarrow B \) (compatible with a Whitney stratification), we can associate the sheaf \( i_* \mathcal{C}_U \). We call this a "standard open" sheaf. In this section, we will assume \( B \) is compact. When \( B \) is noncompact we write \( Sh_{cc}(B) \) for the subcategory with compactly supported cohomology\(^7\) and \( DSh_{cc}(B) \) for its derived category.

4.2.1. The Morse Category. As discovered by Fukaya\(^8\) (and Cohen-Jones-Segal\(^9\)), Morse theory contains an \( A_\infty \) category structure. Define \( Mor(B) \) to be the category with objects consisting of open sets with defining functions for their boundaries, i.e. pairs \((U, m)\) with \( U \subset B \) an open submanifold and \( m : B \to \mathbb{R} \) a function obeying \( m|_U > 0 \), \( m|_{\partial U} = 0 \). We put \( f = \log m|_U : U \to \mathbb{R} \). Hom spaces are made from the critical points graded by Morse index, that is \( \text{hom}_{Mor(B)}((U, m), (U', m')) = \bigoplus_{p \in \text{Crit}(f' - f)} \mathbb{C}_p[-\text{deg}(p)] \). The differential \( d \) is defined by counting (with sign) points in the space of gradient flow trajectories \( \{ \dot{y} = \nabla(f' - f) \} \) modulo time translation. Higher compositions, such as the product, are defined by counting gradient flow trees. For example, if we want to compute a product \( p \circ q \) and find the coefficient of a critical point \( r \), we count (with sign) trajectories of gradient flow lines for \( f' - f \) emanating from \( p \) that meet flows for \( f'' - f' \) from \( q \) (at some point) and continue flowing by \( f'' - f \) to \( r \). The "tree" here is shaped like the letter \( Y \).

The conditions on \( m \) ensure that gradient flows of \( f = \log m \) do not leave the open sets, and this local version of Morse theory makes sense.

4.2.2. The De Rham Category. In fact, this \( A_\infty \) structure is a consequence of the \( dg \) structure of a de Rham category we now describe. The objects of \( DR(B) \) are also pairs \((U, m)\) and the hom spaces are relative de Rham complexes \( \text{hom}_{DR(B)}(U, U') = (\Omega^*(\overline{U} \cap U'), \partial U \cap U', d) \) (if one likes, one can omit the inconsequential data of \( m \) in the objects). For example, when \( U = U' = B \) we get \( \Omega^*(B) \) with de Rham differential.

The point is that a \( dg \) algebra \( A \) induces an \( A_\infty \) structure on a subvector space \( B \) which appears as a strong deformation retract, i.e. the image of a projector. That is, we have \( p : A \to A, B = \text{Image}(p), i : B \to A \) with \( p \circ i = \text{id}_B \) and a homotopy \( h : A \to B \) of degree \(-1\) such that \( dh + hd = i \circ p - \text{id}_A \). A simple example is when \( A \) is the de Rham complex of a Riemannian manifold and \( B \) is the vector space of harmonic differential forms. Let \( \Delta \) be the Laplacian and \( G \) the Green operator. Then \( i \) is inclusion, \( p \) is projection \( 1 - G\Delta \) and \( h = Gd^1 \).

The induced compositions \( m_k(b_1, \ldots, b_k) \) are defined as follows. First define \( a_j = i(b_j) \) from the inclusion, then form a sum over all trivalent trees connecting vertices \( a_j \) to a terminal root, where each tree counts by forming the \( dg \) product \( \circ \) from \( A \) at

\[^7\] Whitney condition avoids some kinds of pathological behavior on the local behavior of different strata.

\[^8\] So the first subscript "c" is for "constructible," the second for "compact."

\[^9\] In case it was not clear from the context, \( p \in \text{hom}((U, m), (U', m')) \), \( q \in \text{hom}((U', m'), (U'', m'')) \) and \( r \in \text{hom}((U, m), (U'', m'')) \).
each internal vertex and the homotopy \( h \) is applied at each internal edge. Finally, the projection \( p \) is applied at the terminal root so that the result lies in \( B \). For example, the contribution to \( m_i(b_1, b_2, b_3, b_4) \) from the tree \( \mathcal{T} \) is \( p(h(i(b_1) \circ i(b_2)) \circ h(i(b_3) \circ i(b_4))) \). Then \( i \) and \( p \) are quasi-equivalences of \( A_\infty \) algebras. The same construction holds for categories.

Harvey and Lawson found the homotopy relating DR(\( B \)) to Mor(\( B \)) using the language of currents. As submanifold of \( B \times B \) or “kernel” defines, through pull-back and push-forward (integration), an operator on currents (which we will describe as differential forms for simplicity). Given a function \( f \), the kernel corresponding to gradient flow \( \varphi_t \) for time \( t \) is graph \( \Gamma_t := \{ x, \varphi_t(x) \} \). Letting \( t \) range from 0 to a fixed point at \( t = \infty \), we get a homotopy between the identity \( \varphi_0 \) and a the space of currents supported at critical points (thought of as the Morse complex vector space), given by the union: \( h = \cup_{0 \leq t \leq \infty} \Gamma_t \).

4.2.3. The Category Open(\( B \)). The category DR(\( B \)) looks a lot like the dg subcategory Open(\( B \)) \( \subset \text{Sh}_c(\( B \)) \) whose objects are again pairs \( (U, m) \), thought of as representing the “standard open” sheaves \( i_* \mathbb{C}_U \) (again, \( m \) is inconsequential). In fact, one can easily show

\[
\text{hom}_{\text{DR}(\( B \))}(U, U') = (\Omega^*(\overline{U} \cap U'), \partial U \cap U', d) \cong \text{hom}_{\text{Sh}_c(\( B \))}(i_* \mathbb{C}_U, i_* \mathbb{C}_{U'}). 
\]

Further, the compositions coincide. As a result, these categories have the same triangulated envelope. But the triangulated envelope TrOpen(\( B \)) is all of Sh(\( B \))!

Informally, “standard opens generate.”\(^9\)

Taken together, these results show the following \( A_\infty \) quasi-equivalences:

\[
\text{TrMor}(\( B \)) \cong \text{TrDR}(\( B \)) \cong \text{Sh}_c(\( B \)).
\]

What remains is to show that Mor(\( B \)) is equivalent to the Fukaya category Fuk(T\(^*\)B). To do so, first, we find a quasi-embedding, then we construct an inverse.

4.3. Fukaya-Oh Theorem. Central to the construction of an equivalence is the theorem of Fukaya and Oh\(^{10}\) relating the subcategory of Mor(\( B \)) defined by global functions, i.e. objects \( (U = B, m = \exp(f)) \), to the subcategory of Fuk(T\(^*\)B) defined by global graphs \( \Gamma_{\text{gf}} \). To explain the idea, let us recall that Fukaya theory is like Morse theory on the space of paths \( c \) between pairs of Lagrangian submanifolds (for the Morse action functional \( a(c) = \int_c \theta \), where \( \theta \) is the canonical one-form on \( T^*B \))\(^{11}\). Floer\(^{12}\) related this Morse theory for pairs of Lagrangian graphs in a cotangent bundle to ordinary Morse theory on the base. Specifically, he showed that pseudoholomorphic strips between such Lagrangian graphs are in correspondence with gradient flow lines for the difference of the Morse functions. Fukaya and Oh extended this idea to more general pseudoholomorphic disks\(^{13}\).

\(^9\)This follows from two facts. First, any complex of sheaves can be constructed from cones and shifts of its cohomology sheaves, so it remains to prove that constructible sheaves are generated by standard opens. This follows by triangulating the space and writing a non-open simplex \( T \) as the “difference” between the open “star” \( \text{Star}(T) \) of all simplices containing \( T \) in their closure and the complement \( \text{Star}(T) \setminus T \), with \( T \) removed.

\(^{10}\)Let \( \pi : T^*B \to B \) be the projection. Then \( \theta_{(x, \xi)}(v) = \xi(\pi_*v) \). On a general symplectic manifold we put \( a(c) = \int_D \omega \), where \( D \) is a homotopy between \( c \) and some fiducial path \( c_0 \).

\(^{11}\)In a recent preprint of Iacovino\(^{23}\), a cleverly chosen inhomogeneous term is added to the pseudoholomorphic curve equation equation to show that the image of such a pseudoholomorphic disk is a gradient flow tree.
Consider a pair of functions \((f, f')\) and a pair of exact Lagrangian graphs \(\Gamma_{df}, \Gamma_{df'}\). Fukaya and Oh prove not only that for small enough \(\epsilon\) the Morse moduli space of gradient flow lines for \(f' - f\) is diffeomorphic to the moduli space of holomorphic strips bounding the two Lagrangians. They also show that the same is true for collections \(\bar{f} = (f_0, \ldots, f_k)\) and \(\Gamma_{\epsilon df\bar{f}} = (\Gamma_{\epsilon df_0}, \ldots, \Gamma_{\epsilon df_k})\) when we consider the Morse moduli space of gradient flow trees and the Fukaya moduli space of disks. The case of three objects with a gradient flow tree a letter \(Y\) is illustrative: the corresponding disk looks like a thickening. Fukaya and Oh construct from the \(Y\) tree an approximate holomorphic disk, then prove that an actual holomorphic disk exists nearby. The “nearby” aspect is important, since it is clear from the discussion in Section 4.2 that we will need a local version of the Fukaya-Oh result.

4.4. Building the Equivalence. This discussion, plus our notation, suggests that to construct an embedding from \(\text{Mor}(B)\) to \(\text{Fuk}(T^*B)\) we should map \((U, m)\) to \(\Gamma_{df}\), a Lagrangian graph over \(U\). In fact, as discussed in Section 4.2, the category \(\text{Sh}_c(B)\) is generated by “standard open” sheaves \(i_*\mathbb{C}_U\), so we can define a functor by defining it on the full subcategory \(\text{Open}(B)\) consisting of these objects and extending as a triangulated functor to the triangulated envelope, which is \(\text{Sh}_c(B)\). This will be our strategy for proving \(\text{Sh}_c(B) \cong \text{Fuk}(T^*B)\).

Two simple examples should help develop some intuition for why the embedding is constructed this way.

**Example 4.1.** Consider the constant sheaf \(\mathbb{C}_B\). Then \(\text{hom}_{\text{Sh}_c(B)}(\mathbb{C}_B, \mathbb{C}_B)\) can be taken to be any model for sheaf cohomology. \(\mathbb{C}^*H^*(B, \mathbb{C}_B) \cong H^*(B)\) such as Čech or Morse complexes (or de Rham, if we take coefficients in \(\mathbb{R}\) or \(\mathbb{C}\)). The Morse complex is obtained by first choosing a Morse function (which we think of as a section of \(\mathbb{C}_B \otimes_{\mathbb{R}} C^\infty(B)\)) and then analyzing its gradient flow trajectories.

We now recall that the Piunkinin-Salamon-Schwartz (PSS) isomorphism in Floer theory relates the Floer cohomology of a Lagrangian \(L \subset M\) (which we think of as \(H^*\text{hom}_{\text{Fuk}(M)}(L, L)\)) to the ordinary (e.g., singular) cohomology \(H^*(L)\). We take this as the first suggestion of a functor relating \(\mathbb{C}_B\) to the zero section \(B \subset T^*B\). (The zero section is also the characteristic cycle of \(\mathbb{C}_B\) — see Section 4.4.)

In fact, the Morse function \(f\) on \(L\) can be thought of as perturbation data for the Lagrangian, since \(\omega^{-1}(df, -)\) is a normal vector field to \(L\). As discussed in Section 4.3 above, Fukaya and Oh showed that for small enough \(\epsilon\) the Morse complex of \(\epsilon f\) is equal to the Floer complex of \(L\) and its the perturbation. They further proved that every moduli space used in computing compositions for the \(A_\infty\) Morse category of functions (gradient flow trees) \(f : B \to \mathbb{R}\) is oriented-diffeomorphic to the Fukaya moduli spaces (holomorphic disks) for the category of graphs \(\Gamma_{df}\) in the cotangent bundle \(T^*B\).

**Example 4.2.** Now consider a hom from \(\mathbb{C}_B\) to a standard \(i_*\mathbb{C}_U\). Again, the hom in \(\text{Sh}_c(B)\) is any model for the sheaf cohomology of \(i_*\mathbb{C}_U\), i.e. \(H^*(U)\), such as the de Rham complex. Morse theory on \(U\) would work, with a proper Morse function \(f\). This can be achieved by taking \(f = \log m\), where \(m > 0\) on \(U\) and \(m = 0\) on \(\partial U\).

12Recall that in \(\text{Sh}_c(B)\) we have quotiented by homotopies.
are diffeomorphic to the spaces of gradient flow lines of \( f \). As shown in [38], this turns out to be the case. Even more: the same is also true for the moduli space of a composition among \( \Gamma_{df_1}, \ldots, \Gamma_{df_k} \) and the corresponding moduli space for Morse functions \( f_1, \ldots, f_k \).

With these motivations, we define a functor \( \mu_0 : \text{Open}(B) \to \text{Fuk}(T^*B) \) by \( \mu_0((U, m)) = \Gamma_{df} \), where \( f = \log m \) as usual. Since \( \text{TrOpen}(B) \cong \text{Sh}_c(B) \), \( \mu_0 \) extends to a microlocalization functor \( \mu : \text{Sh}_c(B) \to \text{Fuk}(T^*B) \) sending \( i_*\mathcal{C}_U \) to \( \Gamma_{df} \). By Equation (7) and the local Fukaya-Oh theorem, \( \mu \) is a quasi-embedding (an isomorphism on the cohomology of the hom complexes). The Verdier dual of direct image \( i_* \) is the proper direct image \( i_! \). It turns out that Verdier duality in \( \text{Fuk}(T^*B) \) is multiplication by \((-1)^n\) in the fibers. We write \( L_{U*} = \mu(i_*\mathcal{C}_U) \) for the “standard Lagrangian brane” or simply “standard Lagrangian” and \( L_{U!} = \mu(i_!\mathcal{C}_U) \) for the “costandard brane” or “costandard Lagrangian.” One can extend the definition of standard branes to non-open submanifolds, as well (we use this below).

### 4.5. Equivalence and the Inverse Functor

The microlocalization functor \( \mu : \text{Sh}_c(B) \to \text{Fuk}(T^*B) \) is a quasi-embedding, and we can define an obvious candidate for its inverse. We want to associate a sheaf to a Lagrangian. Recall that T-duality associates a bundle to a section, and a higher rank bundle to a multi-section. Here we have an analogous story, and the key is to note that the stalk of the sheaf at a point where it is locally constant is generated by the points of the Lagrangian over that point. There may be disks relating the points, however, and a more invariant definition would involve the whole Fukaya hom complex between a fiber and a Lagrangian (thought of as the fiber of a complex of vector bundles). We will therefore define for a Lagrangian a sheaf of complexes. The natural guess (taking contravariance into account) is that a Lagrangian brane \( P \) is mapped to a sheaf of complexes \( F_P \) defined by

\[
F_P(U) = \text{hom}_{\text{Fuk}(T^*B)}(L_{U!}, P),
\]

where \( L_{U!} \) is the costandard brane on \( U \), as defined above\(^{13}\).

In [37], it is proven that the microlocalization functor \( \mu \) is a quasi-equivalence and that the functor \( v \) mapping \( P \) to \( F_P \) is a quasi-inverse to \( \mu \). The latter assertion follows readily from the former; proving the essential surjectivity of \( \mu \) is the difficult part. To do this, Nadler considers the toy problem of showing that a collection of vectors \( v_t \) spans a vector space \( V \). It is enough to express the identity map in the form \( I = \sum v_t \otimes w^t \) for some (co-)vectors \( w^t \). Then applying \( I \) to \( v \) expresses \( v \) in terms of the \( v_t \). Analogously, Beilinson’s resolution of the diagonal \( \Delta_B \subset \mathbb{P}^n \times \mathbb{P}^n \) (thought of as the kernel for the identity functor) leads to a resolution of a coherent sheaf in terms of a vector bundles. A similar trick works here.

First, it suffices to prove surjectivity for a class of Fukaya objects that intersect infinity of \( T^*B \) inside the boundary-at-infinity of some fixed (but arbitrary) conical Lagrangian \( \Lambda \) at infinity in \( T^*B \). Then find a triangulation \( \{ T_\alpha \} \) of \( B \) whose associated conical Lagrangian \( \cup_\alpha T^*_\alpha \) contains \( \Lambda \). Of course the diagonal \( \Delta_B \subset B \times B \) is not expressible in terms of products \( T_\alpha \times T_\beta \) – otherwise, we’d nearly be finished. The trick in the proof is that the piece of the diagonal \( \Delta_{T_\alpha} \subset T_\alpha \times T_\alpha \) over \( T_\alpha \) is homotopic to \( \{ t_\times \} \times T_\alpha \) for any \( t_\times \in T_\alpha \), and Nadler shows that corresponding

---

\(^{13}\)In fact, this is actually an \( A_\infty \) sheaf since the composition of restrictions will not be associative on the nose – see [37] for details.
to this homotopy is an isomorphism of functors between a piece of the diagonal \( Y(L_{\triangle \tau^{*}}) \) and an external product \( Y(L_{\tau^{*}}) \otimes Y(L_{(1,1)}) \), where \( Y \) is the Yoneda embedding. Applying the identity operator in this form shows that the (Yoneda modules corresponding to) \( L_{\tau^{*}} \) generate the category.

4.6. Singular support and characteristic cycles. One can describe the results relating constructible sheaves to the Fukaya category as a categorification of the characteristic cycle construction of Kashiwara-Schapira (see [31] for foundational material). The path is straightforward, with some hindsight and a selective look at some facts involving of characteristic cycles. We review this interpretation here.

First recall that given a constructible complex of sheaves \( F \) on \( B \), its singular support, \( SS(F) \), is a conical Lagrangian subvariety in \( T^{*}B \) which encodes Morse-theoretic obstructions to extending local sections of \( F \). That is, if a covector is not in \( SS(F) \), it means that there is no obstruction to propagating local sections of \( F \) in directions which are positive on the covector.

There is a finer invariant of \( F \) called its characteristic cycle, or \( CC(F) \), which is a linear combination of Lagrangian components of \( SS(F) \). The multiplicity of \( CC(F) \) at a given covector is the Euler characteristic of the local Morse groups of the complex with respect to the covector (roughly, the restriction map that the sheaf associates to an open neighborhood and to the smaller open set of points evaluating negatively on the covector).

So, for example, the characteristic cycle of a flat vector bundle on \( B \) is the zero section in \( T^{*}B \) with multiplicity equal to the rank of the vector bundle. More generally, the characteristic cycle of a flat vector bundle on a submanifold of \( B \) is its conormal bundle with multiplicity equal to the rank of the vector bundle.

**Remark 4.3.** The singular support is additive on exact triangles. That is, we have \( SS(\text{Cone}(F \to G)) \subset SS(F) \cup SS(G) \) for any morphism of complexes \( F \to G \). It follows that if we fix a conical Lagrangian \( \Lambda \subset T^{*}B \), the full subcategory \( Sh_{c}(B; \Lambda) \subset Sh_{c}(B) \) of sheaves with singular support contained in \( \Lambda \) is triangulated. These subcategories appear repeatedly in applications—most commonly, \( \Lambda \) is the conormal variety to a Whitney stratification of \( B \), and then \( Sh_{c}(B; \Lambda) \) is the same as the category of complexes constructible with respect to the stratification. The more general notion, when \( \Lambda \) is not conormal to a stratification, occurs in our work on toric varieties below.

The calculation of Schmid-Vilonen [40] is very suggestive. These authors compute the characteristic cycle of the standard open \( i_{*}\mathbb{C}_{U} \) to be the limit \( \lim_{\epsilon \to 0} \Gamma_{\epsilon \log m} \). Note that \( \epsilon \log m = \log m^{\epsilon} \), and as \( \epsilon \) becomes small \( m^{\epsilon} \) approaches the indicator function of \( U \), which can be thought of informally as a section of \( i_{*}\mathbb{C}_{U} \). The Morse theory of \( \log(m^{\epsilon}) \) is independent of \( \epsilon > 0 \). On the other hand, the limit \( \epsilon \to 0 \) seems to relate Morse theory (functions) to constructible sheaves (constant functions). In an entirely nonrigorous sense, the family \( m^{\epsilon} \) is a kind of isotopy between the Morse functions of the Morse category and the indicator functions of constructible sheaves.

A formula of Kashiwara-Dubson further suggests a relationship to the Fukaya category. The \( K \)-theory of constructible sheaves is isomorphic to the abelian group of constructible functions, with the local Euler characteristic providing the isomorphism. The characteristic cycle maps this isomorphically to the group of closed, conical Lagrangian cycles \( L_{\text{con}}(T^{*}B) \). Dubson-Kashiwara prove that for constructible
sheaves $F_1$, $F_2$ (thought of as elements of the K-theory),
\[
\chi(F_1, F_2) = CC(DF_1) \cdot CC(F_2),
\]
where $D$ is the Verdier duality functor. ($CC(DF) = a(CC(F))$, where $a$ is (-1) in the fibers of $T^*B$.) On the left, the expression is the descendent in K-theory of the chain complex $\text{hom}_{Shc}(B)(F_1, F_2)$ computed in the dg category $Shc(B)$. On the right, the objects were not taken to lie in any category, but now we understand the $CC(F)$ as the K-theory descendent of the brane $\mu(F) \in \text{Ob}(TrFuk(T^*B))$. Therefore, on the right side we can begin with the complex $\text{hom}_{TrFuk}(T^*B)(\mu(F_1), \mu(F_2))$ and compute its Euler characteristic to get the required result. The explicit map from the Fukaya category to its K-theory (conical Lagrangian cycles) is simply the limit of dilation toward the zero section (this leaves infinity intact), which preserves the intersection pairing.

We arrive at the interpretation that the microlocalization quasi-equivalence is a categorification of the characteristic cycle.

4.7. Comments on technicalities. The reader might be dissatisfied with our informal discussion, and will want to convince her/himself of the details. We must refer such a reader to the papers [37, 38]; here, as a preview, we make a few remarks on the technical issues arising in the proofs, and how they are dealt with. (Even still, we omit many.) Most readers will gladly skip this section.

(1) $\text{Mor}(B)$. To define this category, one must ensure that the moduli spaces of gradient flows have compactifications, no matter the behavior of the open sets. To do so, one uses stratification theory to conclude that the set $\{m > \epsilon\}$ is a good approximation to $U$ with a smooth boundary, and similarly for $U'$ and another $\epsilon'$, and that these two intersect transversely. One also needs compatibility with the (possibly perturbed) metric so that the inward/outward behavior of $\nabla \log f$ and $-\nabla \log f'$ remain along the “new” open sets. These conditions ensure that sequences of gradient flows do not “wander off” the intersection of the open sets, so that the standard compactifications by broken trajectories can be constructed.

(2) Brane structures in $Fuk(T^*B)$. As is beautifully described in [41], the Fukaya category requires that $\text{hom}$ spaces are graded and that the moduli spaces defining compositions are oriented (compatibly with topological field theory gluings). Also, Lagrangian objects can carry unitary local systems, which we have not yet discussed. The existence of a grading follows from a “grading” of the Lagrangian submanifolds, i.e. a lift of the Maslov phase from $U(1)$ to its universal cover $\mathbb{R}$ (in particular, the Lagrangians must be Maslov-trivial). Orientability requires that the Lagrangians are relatively pin with respect to a background class in $w_2(T^*B)$ Happily, Lagrangian graphs over $U$ are homotopic to the zero section along $U$ (which is given the zero grading), and therefore are canonically graded and relatively pin with respect to the class $\pi^*w_2(B)$. By assigning the trivial local system, we find canonical brane structures on the Lagrangians corresponding to standard opens. We call the resulting objects “standard branes.”

(3) Regularity. One requires the moduli spaces to be manifolds, which depends on the vanishing of obstructions to infinitesimal deformations of each pseudoholomorphic map in the moduli space. The situation here is no different
than in the compact Fukaya category (see the excellent [41]) or even symplectic Gromov-Witten theory. Regularity is achieved by perturbing the equations of pseudoholomorphicity.

(4) **Perturbations and Infinity.** Nontransversal intersections of Lagrangians are treated through perturbation data (as in the case of the compact Fukaya category). The novelty in the noncompact case arises from intersections “at infinity.” These are treated by perturbing with geodesic flow for small time, which is the Hamiltonian perturbation associated to the distance (from the zero section) function. For multi-compositions involving several Lagrangians intersecting at infinity, the composition order determines the ordering of times of geodesic flow: perturbations propagate forward in time: for example in $T^*\mathbb{R} = \mathbb{R}^2$, $\text{hom}([x = 0], [y = 1/x]) = 0$ while $\text{hom}([y = 1/x], [x = 0]) \neq 0$.

For these perturbations to separate points at infinity (and bring Lagrangian intersections into finite space), we must avoid pathologies such as intersection points accumulating at infinity (the helix $\theta = \xi$ in $T^*S^1$ is disallowed). This leads us to require that the Lagrangians are “good” subsets of the compactification $\overline{T^*B}$, such as subanalytic sets.

(5) **Compactifying moduli spaces.** Compositions in $\text{Fuk}(T^*B)$ are defined by intersections in moduli spaces of pseudoholomorphic disks, which therefore must be compactified. The novelty in the noncompact case is that sequences of pseudoholomorphic disks may wander off to infinity. A tameness condition on the Lagrangians prevents this from happening [45]. In fact, we need a slightly weaker notion to accommodate standard branes, which are not necessarily tame. We require simply a tame perturbation – essentially, a family of Lagrangians $L_t$ indexed by small $t \geq 0$ which is tame for all $t > 0$. Also, an almost complex structure is chosen on the cotangent bundle which is asymptotically conical (in particularly, not Sasakian near infinity). The resulting theory is well-defined.

(6) **Increasing unions of finite calculations** The arguments here apply to any finite calculation among a finite collection of branes The issue of compatibility for increasing unions of finite calculations is discussed in the appendix of [37].

(7) **Equivalence and the inverse functor.** The main analytical issue to deal with here is in expressing the identify functor in terms of standard branes. The diagonal in $B \times B$ is nowhere a Cartesian product. However, inside the cell $T \times T$, the diagonal $\Delta_T$ is homotopic to $T \times \{t_\ast\}$ where $t_\ast$ is any point in $T$. Correspondingly, one can construct a family of Lagrangian branes interpolating between $L_\Delta(T^*)$ and $L_{T \times \{t_\ast\}}$. The key is that the interpolating family only moves through infinity without intersecting the conical set $\Lambda$, and that this “noncharacteristic isotopy” of branes induces isomorphic functors. The analytical argument proves that the moving moduli spaces of holomorphic disks which realize isomorphisms of branes (defined as you would for Hamiltonian isotopies of compact branes) make sense for small motions. The trick is a kind of “shimmy,” breaking up the motion of a brane into two parts: first, the motion near infinity is done far away from the zero section so as not to affect any holomorphic disks (which are controlled by area bounds); then, the motion in finite space is performed and
the moving moduli spaces are compactified as usual. Finally, any compact motion is divided up into a finite sequence of small motions, all of which lead to isomorphic objects.

The preceding (1–5) ensure well-definedness of the Fukaya category. We need more to apply the Fukaya-Oh theorem relating to the Morse category. The issue is that the theorem as stated only applies to global graphs $\Gamma_d$ with $f : B \to \mathbb{R}$, whereas standard branes are graphs only over open sets $U$. The main thing to prove is that the bounds ensuring compact moduli spaces do not break down under the dilation from $\Gamma_d$ to $\Gamma_{\epsilon d}$.

4.8. Statement of results. The arguments of this section are the ideas behind the theorem, stated formally here:

**Theorem 4.4.** [35 37] Let $B$ be a real analytic manifold. There is a quasi-equivalence of $A_\infty$ categories

$$\text{Sh}_c(B) \cong \text{TrFuk}(T^*B)$$

which sends $i_*\mathcal{C}_U$ to $\Gamma_{d\log m}$, where $m$ is a defining function for $\partial U$. Taking cohomology $H^0$ leads to a quasi-equivalence of bounded, derived categories

$$D\text{Sh}_c(B) \cong D\text{Fuk}(T^*B).$$

The theorem makes sense for noncompact $B$ if we restrict to objects with compact support. The theorem also makes sense if we fix a conical Lagrangian inside $T^*B$ and consider the subcategory of constructible sheaves with singular support contained in it. Applied to the conifold Lagrangian $\Lambda_\Sigma \subset T^*M_{\mathbb{R}}$ defined in (5), we have

$$D\text{Sh}_{c,\epsilon}(M_{\mathbb{R}}; \Lambda_\Sigma) \cong D\text{Fuk}(T^*M_{\mathbb{R}}; \Lambda_\Sigma).$$

5. Coherent-Constructible Correspondence

The results of Sections 3 and 4 produce a functor between coherent and constructible sheaves that goes through the Fukaya category. In this section we cut out the middleman and construct the coherent-constructible correspondence directly. In so doing, we discover that we have stumbled upon a categorification of Morelli’s description of the equivariant K-theory of a toric variety [36], which we now describe.

We use the notation in Section 3.3 and Section 3.4. Let $X_\Sigma$ be a smooth projective toric variety defined by a complete fan $\Sigma \subset N_{\mathbb{R}}$. Let $D_\epsilon$ be an ample $T$-divisor, so that $\mathcal{O}_{X_\Sigma}(D_\epsilon)$ is an equivariant ample line bundle. Let $1_{\Delta_\epsilon} : M_{\mathbb{R}} \to \mathbb{R}$ denote the indicator function on the moment polytope $\Delta_\epsilon$ of $\mathcal{O}_{X_\Sigma}(D_\epsilon)$. Then $\mathcal{O}_{X_\Sigma}(D_\epsilon) \to 1_{\Delta_\epsilon}$ defines a homomorphism

$$I_T : K_T(X_\Sigma) \to L_M(M_{\mathbb{R}})$$

between abelian groups, where $L_M(M_{\mathbb{R}})$ is the group of functions on $M_{\mathbb{R}}$ generated over $\mathbb{Z}$ by the indicator functions of convex lattice polyhedra. Morelli [36] has the following characterization of the image of $I_T$. Define the group of polyhedra germs $S_M(M_{\mathbb{R}})$ to be the abelian group generated by rational convex cones in $M_{\mathbb{R}}$, with relations

$$[\sigma \cup \tau] = [\sigma] + [\tau] - [\sigma \cap \tau]$$

where $\sigma, \tau, \sigma \cup \tau$ are rational convex cones. Then $S_M(M_{\mathbb{R}})$ is the group of germs of functions in $L_M(M_{\mathbb{R}})$ at the origin (or at any point in $M$). Let $S_{M}(M_{\mathbb{R}})$ be the
subgroup of $S_M(MR)$ generated by $\{\sigma^\vee \mid \sigma \in \Sigma\}$. Then the image $L_\Sigma(MR)$ of $I_T$ is the subgroup of $L_M(MR)$ consisting of functions whose germ at any point $m \in M$ lies in $S_\Sigma(MR)$, and $I_T$ defines an isomorphism

$$K_T(X_\Sigma) \cong L_\Sigma(MR)$$

of abelian groups.

The coherent-constructible correspondence

$$DCoh_T(X_\Sigma) \cong DSh_{cc}(MR; \Lambda_\Sigma)$$

can be viewed as categorification of Morelli’s results \cite{morelli94}. In \cite{morelli94}, the authors defined a functor $\kappa : DCoh_T(X_\Sigma) \to Sh_{cc}(MR; \Lambda_\Sigma)$ and proved that it is a quasi-equivalence of dg categories. Taking the cohomology $H^0$ gives \cite{morelli94}. We outline the argument in the remainder of this section.

Recall that $DCoh_T(X_\Sigma)$ is generated by equivariant ample line bundles, so $\kappa$ is determined by its restriction to equivariant ample line bundles. Given an equivariant ample line bundle $\mathcal{O}_X(\mathcal{E})$, which is an object in $DCoh_T(X_\Sigma)$, let $i : \Delta^2 \hookrightarrow MR$ be the inclusion map. Then $i_! \mathcal{C}_{\Delta^2}[n]$ is a constructible sheaf on $MR$ with compact support, where $n = \dim_c X_\Sigma$, and $SS(i_! \mathcal{C}_{\Delta^2}) \subset \Lambda_\Sigma$, so it is an object of $Sh_{cc}(MR; \Lambda_\Sigma)$. We want to show that $\mathcal{O}_X(\mathcal{E}) \mapsto i_! \mathcal{C}_{\Delta^2}[n]$ defines a functor $\kappa : DCoh_T(X_\Sigma) \to Sh_{cc}(MR; \Lambda_\Sigma)$ between dg categories. To verify this, it is more convenient to express $\kappa$ in terms of different generators (which do not actually belong to $DCoh_T(X_\Sigma)$).

For any pair $(\chi, \sigma) \in M \times \Sigma$, we will define a constructible sheaf $\Theta(\chi, \sigma)$ on $MR$ and an equivariant quasi-coherent sheaf $\Theta'(\chi, \sigma)$ on $X_\Sigma$.

1. Given $(\chi, \sigma) \in M \times \Sigma$, let $(\chi + \sigma^\vee)^\circ$ be the interior of the translated dual cone $\chi + \sigma^\vee \subset MR$. Define

$$\Theta(\chi, \sigma) = i_! \mathcal{C}_{(\chi + \sigma^\vee)^\circ}.$$ 

Then $\Theta(\chi, \sigma)$ is a constructible sheaf on $MR$ whose microlocal support is contained in $\Lambda_\Sigma$, but its support is not compact. Therefore $\Theta(\chi, \sigma)$ is an object of $Sh_{cc}(MR; \Lambda_\Sigma)$. Let $\langle \Theta \rangle$ denote the full triangulated dg subcategory of $Sh_{cc}(MR; \Lambda_\Sigma)$ generated by

$$\{\Theta(\chi, \sigma) \mid (\chi, \sigma) \in M \times \Sigma\}.$$ 

The hom space between any two of these generators is simple: the computation can be reduced to computing relative cohomology groups of (contractible set, point) or (contractible set, empty set), which is either 0, or $C$ (at degree zero).

2. Given $\sigma \in \Sigma$, $X_\sigma = \text{Spec} \mathbb{C}[\sigma^\vee \cap M]$ is an affine open subvariety of $X$. Let $\mathcal{O}_{\sigma}(\chi)$ be the equivariant quasicoherent sheaf on $X_{\sigma}$ corresponding to the $M$-graded $\mathbb{C}[\sigma^\vee \cap M]$-module freely generated by $\chi \in M$. Define

$$\Theta'(\chi, \sigma) = j_{\sigma, *} \mathcal{O}_{\sigma}(\chi)$$

where $j_{\sigma, *} : X_{\sigma} \hookrightarrow X$ is the open embedding. Then $\Theta'(\chi, \sigma)$ is an equivariant quasi-coherent sheaf on $X_{\Sigma}$. Let $\langle \Theta' \rangle$ denote the full triangulated dg subcategory of $Q_T(X_\Sigma)$ (localization of the dg category of equivariant quasi-coherent sheaves on $X_\Sigma$ with respect to acyclic complexes) generated by

$$\{\Theta'(\chi, \sigma) \mid (\chi, \sigma) \in M \times \Sigma\}.$$
The hom space between any two of these generators is simple: the computation can be reduced to computing \( \text{Ext}^* \) between two equivariant line bundles on the affine toric variety \( X_\sigma \), which is either 0, or \( \mathbb{C} \) (at degree zero). By \v{C}ech resolution we see that any equivariant line bundle is in the subcategory \( \langle \Theta' \rangle \), so

\[
DCoh_T(X_\Sigma) \subset \langle \Theta' \rangle \subset Q_T(X_\Sigma).
\]

(3) Comparing the calculations in (1) and (2), we conclude that \( \Theta'(\chi, \sigma) \mapsto \Theta(\chi, \sigma)[n] \) defines a quasi-equivalence \( \langle \Theta' \rangle \cong \langle \Theta \rangle \). Moreover, under this quasi-equivalence an ample line bundle \( O_{X_\Sigma}(\bar{D}) \) is mapped to \( i_! C_{\Delta} \circ \bar{D} \). Therefore we have a full embedding

\[
\kappa : DCoh_T(X_\Sigma) \rightarrow Sh_{cc}(M_\mathbb{R}; \Lambda_\Sigma).
\]

(4) In [16], it is shown that \( \kappa \) is essentially surjective, i.e. that every sheaf \( F \in Sh_{cc}(M_\mathbb{R}; \Lambda_\Sigma) \) is quasi-isomorphic to one of the form \( \kappa(G) \). This requires a careful induction on the “height” of \( F \), which is a measure of the complexity of its singular support.

(5) The argument can be extended to any complete toric variety, including singular and non-projective varieties. The category \( DCoh_T(X_\Sigma) \) is replaced by \( \mathcal{P}erf_T(X_\Sigma) \), the dg category of “perfect complexes,” which are by definition bounded complexes of equivariant vector bundles.

(6) The techniques of [16] produce a full embedding

\[
\bar{\kappa} : DCoh(X_\Sigma) \rightarrow Sh_{cc}(T_\mathbb{R}^\vee, \Lambda_\Sigma/M)
\]

which makes the following square commute:

\[
\begin{array}{ccc}
DCoh_T(X_\Sigma) & \xrightarrow{\kappa} & Sh_{cc}(M_\mathbb{R}; \Lambda_\Sigma) \\
f \downarrow & & p \downarrow \\
DCoh(X_\Sigma) & \xrightarrow{\bar{\kappa}} & Sh_{cc}(T_\mathbb{R}^\vee, \Lambda_\Sigma/M).
\end{array}
\]

where \( f \) is obtained by forgetting the \( T \)-equivariant structure, and \( p \) is induced by the natural projection \( p : M_\mathbb{R} \rightarrow T_\mathbb{R}^\vee = M_\mathbb{R}/M \). Presumably \( \bar{\kappa} \) is always an equivalence, though as of this writing we do not have a proof of this fact. (For example, \( \bar{\kappa} \) is an equivalence in the examples in Section 6.1 and Section 6.2 below.)

6. Examples

6.1. Taking the mapping cone. On \( B = S^1 = \mathbb{R}/\mathbb{Z} \), let \( p = 0 \) and \( U = (0, 1) \). We have \( S^1 = \{ p \} \cup U \). There is an exact triangle in the derived category

\[
\mathbb{C}_{S^1} \rightarrow i_U^* \mathcal{C}_U \rightarrow \mathcal{O}_p \rightarrow
\]

where \( i_U : U \rightarrow S^1 \) is the embedding map and \( \mathcal{O}_p \) is the skyscraper sheaf at \( p \).

Let \( F_p \) be the fiber of \( T^*S^1 \rightarrow S^1 \) at \( p \), \( L_{U^*} \) be the standard Lagrangian over the interval \( U \). The microlocalization functor

\[
\mu : Sh_{cc}(S^1) \rightarrow TrFuk(T^*S^1)
\]

takes the above exact cone to

\[
B \rightarrow L_{U^*} \rightarrow F_p \rightarrow .
\]
The brane $F_p$ and $L_{U*}$ are equipped with the trivial gradings, and $B$ is simply the Lagrangian brane supported at the zero section $S^1$ (with the trivial grading).

Under this functor, $K$-theory classes $K(i_{U*}C_{U}) = K(C_{S^1} \oplus \mathcal{O}_p)$ map to $K(L_{U*}) = K(B \oplus F_p)$, which reflects the dilation limit

$$\lim_{\epsilon \to 0} \epsilon L_{U*} = F_p \cup S^1.$$ 

Taking the mapping cone of the morphism $B \rightarrow L_{U*}$ is geometrically interpreted as taking symplectic surgery, as depicted in the Figure 5.

Figure 5. The mapping cone over the morphism $B \rightarrow L_{U*}$ in the Fukaya category is obtained by taking the symplectic surgery at the intersection point (left). The resulting brane $\text{Cone}(B \rightarrow L_{U*})$ (right) is isotopic to the fiber brane $F_p$. Notice in both pictures the cotangent bundle $T^*S^1$ is cut along $F_p$ for illustration.

**Remark 6.1.** When $X_\Sigma = \mathbb{P}^1$, $M_\mathbb{R} \cong S^1$ and $\Lambda_{\Sigma}/M = F_p \cup B$. We have the (non-equivariant) mirror symmetry

$$DFuk(T^*(M_\mathbb{R}/M); \Lambda_{\Sigma}/M) \cong DCoh(\mathbb{P}^1).$$

Let $D_0$ be one of the $T$-invariant divisors. The above exact triangle comes from

$$\mathcal{O}_{D_0}[-1] \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow .$$

**6.2. Toric Fano surfaces.** Let $X_\Sigma$ be a smooth projective toric Fano surface, so that $\Sigma$ is one of the five fans in the first row of Figure 3. We have

$$DCoh_T(X_\Sigma) \cong DSh_{c}c(M_\mathbb{R}; \Lambda_{\Sigma}), \quad DCoh(X_\Sigma) \cong DSh_{c}(T^*_\mathbb{R}; \Lambda_{\Sigma}/M).$$

Moreover, in this case there is a constructible (but possibly non-Whitney) stratification of $T^*_\mathbb{R}$ (see Figure 6), such that each stratum is contractible, and any object in $Sh_{c}(T^*_\mathbb{R}; \Lambda_{\Sigma}/M)$ is constant on each stratum. This stratification coincides with the one defined by Bondal [6].

\[\text{Cone = Surgery}\] is the mantra found from the lessons of [20 43 44].
6.3. Hirzebruch surfaces. The Hirzebruch surface \( \mathbb{F}_m \) \( (m \geq 0) \) is defined by a fan \( \Sigma_m \) spanned by four 1-cones \( \rho_i = \mathbb{R}_{\geq 0} v_i \), where

\[
v_1 = (1, 0), \ v_2 = (0, 1), \ v_3 = (-1, -m), \ v_4 = (0, -1).
\]

See Figure 3 for the fans \( \Sigma_m, m = 0, 1, 2, 3 \). Let \( D_i \) be the \( T \)-divisor corresponding to \( v_i \). Then \( D_2, D_4 \) are sections of the projection \( \mathbb{F}_m \to \mathbb{P}^1 \), and \( D_1, D_3 \) are fibers. We have \( D_2, D_4 = -D_2 \cdot D_4 \). The fan data determines a conical Lagrangian \( \Lambda_m := \Lambda_{\Sigma_m} \) in \( T^* M_\mathbb{R} \). See Figure 4 for the conical Lagrangians \( \Lambda_m, m = 0, 1, 2, 3 \).

Let us discuss the various categories for this family of examples.

6.3.1. Category of equivariant coherent sheaves \( \text{Coh}_T(\mathbb{F}_m) \). This category is generated by line bundles

\[
\mathcal{O}_{\mathbb{F}_m}(D_{\mathcal{E}}),
\]

where \( D_{\mathcal{E}} = \sum_{i=1}^4 c_i D_i \). This category is also generated by ample line bundles.

6.3.2. Equivariant category of constructible sheaves \( \text{Sh}_{cc}(M_\mathbb{R}; \Lambda_m) \). This category consists of compactly-supported constructible sheaves whose singular supports lie in \( \Lambda_m \). It is generated by costandard sheaves \( i_{\Delta_{\mathcal{E}}^\circ} \mathcal{C}_{\Delta_{\mathcal{E}}^\circ}[2] \) for all ample divisors \( D_{\mathcal{E}} \). When \( D_{\mathcal{E}} \) is ample, i.e.

\[
c_2 + c_4 > 0, \ c_1 + c_3 > mc_4,
\]

the polytope \( \Delta_{\mathcal{E}} \) is a polytope on the \( M_\mathbb{R} \) plane with four vertices

\[
(-c_1, -c_2), \ (c_3 + mc_2, -c_2), \ (c_3 - mc_4, c_4), \ (-c_1, c_4).
\]

The map \( i_{\Delta_{\mathcal{E}}^\circ} \) is the inclusion \( \Delta_{\mathcal{E}}^\circ \to M_\mathbb{R} \) of the interior.

In particular, the anti-canonical divisor of \( \mathbb{F}_m \) is given by

\[
-K_{\mathbb{F}_m} = D_1 + D_2 + D_3 + D_4.
\]

It is ample when \( m = 0, 1 \), and is numerically effective when \( m = 2 \). It is not numerically effective when \( m > 2 \).

6.3.3. Equivariant Fukaya category \( \text{Fuk}(T^* M_\mathbb{R}; \Lambda_m) \) associated to the universal cover of the mirror \( Y^\vee \cong T^* M_\mathbb{R} \) (symplectically). The category \( \text{Fuk}(T^* M_\mathbb{R}; \Lambda_m) \) consists of Lagrangian branes whose conical limits are subsets of \( \Lambda_m \). It is generated by costandard Lagrangian branes \( L_{\mathcal{E}, h} \) over \( \Delta_{\mathcal{E}} \) for all ample divisors \( D_{\mathcal{E}} \).

6.3.4. Tri-Equivalence. The three categories above are equivalent, by [16]:

\[
\text{Coh}_T(\mathbb{F}_m) \xrightarrow{\cong} \text{Sh}_{cc}(M_\mathbb{R}; \Lambda_m) \xrightarrow{\cong} \text{Fuk}(T^* M_\mathbb{R}; \Lambda_m).
\]

The correspondence between a generating class of objects is

\[
\mathcal{O}_{\mathbb{F}_m}(D_{\mathcal{E}}) \mapsto L_{\mathcal{E}, h} \mapsto i_{\Delta_{\mathcal{E}}^\circ} \mathcal{C}_{\Delta_{\mathcal{E}}^\circ}[2].
\]
Figure 7. The polytope $\Delta_{(0,0,2,1)}$ for $F_1$. The constructible sheaf corresponding to the ample bundle $O_{F_1}(2D_3 + D_4)$ is the standard constructible sheaf over this polytope (i.e. $i_{\Delta_{(0,0,2,1)}(0,0,2,1)}[2]$).

6.3.5. Relative Fukaya category $\text{Fuk}((\mathbb{C}^*)^2, W_{-1}^m(0))$. Let $\vec{t} = (t_1, t_2, t_3, t_4)$ be chosen such that $D_{\vec{t}} = \sum_{i=1}^4 t_i D_i$ is an ample divisor. Then the Poincaré dual of this divisor class is a Kähler class in $H^2(F_m; \mathbb{R})$. The superpotential $W_m : (\mathbb{C}^*)^2 \to \mathbb{C}$ of the Hori-Vafa mirror of $F_m$ is given by

$$W_m = e^{-t_1} z_1 + e^{-t_2} z_2 + e^{-t_3} z_1^{-1} z_2^{-1} + e^{-t_4} z_2^{-1}.$$  

An object in $\text{Fuk}((\mathbb{C}^*)^2, W_{-1}^m(0))$ is an exact compact Lagrangian submanifold with boundary in the complex hypersurface $W_{-1}^m(0)$. Abouzaid defined a subcategory $\text{TFuk}((\mathbb{C}^*)^2, W_{-1}^m(0))$ of tropical Lagrangian sections of $\text{Fuk}((\mathbb{C}^*)^2, W_{-1}^m(0))$, and showed that $\text{TFuk}((\mathbb{C}^*)^2, W_{-1}^m(0))$ is equivalent to the category of line bundles on $F_m$. Since line bundles generate $\text{DCoh}(X)$, he concluded that

$$\text{DCoh}(X) \cong D^* \text{TFuk}((\mathbb{C}^*)^2, W_{-1}^m(0)) \xrightarrow{i_T} D^* \text{Fuk}((\mathbb{C}^*)^2, W_{-1}^m(0))$$

where $i_T$ is embedding of a full subcategory. (Indeed, Abouzaid proved a statement like (13) for any smooth projective toric variety.)

To describe a tropical Lagrangian section, we consider the tropical limit $M_\infty$ of $W_{-1}^m(0)$ defined by the ample divisor $D_{\vec{t}}$ [35]. The image of $M_\infty$ under the logarithm map $\text{log} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$ is a graph known as the tropical amoeba (see Figure 8). The tropical amoeba divides $\mathbb{R}^2$ into connected components. When $m = 0, 1, 2$, there is a unique bounded component which can be identified with the moment polytope of the anti-ample divisor $D_{-\vec{t}}$. When $m > 2$, there are more than one bounded
components, one of which can be identified with the moment polytope of \( D^- \). In the first row of Figure 8, we take \( D^- \) to be the anti-canonical divisor, which is ample; in the second row, we take \( D^- = D_1 + D_2 \). An object in \( TFuk((\mathbb{C}^*)^2, M_\infty) \) is a section of \( \log^{-1}(\Delta_{-\mathbf{t}}) \to \Delta_{-\mathbf{t}} \) with boundary in \( M_\infty \). Let \( e_i \) be the face of \( \Delta_{-\mathbf{t}} \) which corresponds to \( D_i \). Then \( e_i \) is a closed interval parallel to the line \( \rho_i^+ \subset M_\mathbb{R} \). Let \( e_i^- \) be the interior of \( e_i \). Then

\[
\log^{-1}(e_i^-) \cap M_\infty = T e_i^-/(M \cap \rho_i^+)
\]

Given an object

\[
L_{e_i,h} = \{(\Phi_h \circ j_0(y), y) \mid y \in N_\mathbb{R} \} \subset M_\mathbb{R} \times N_\mathbb{R}
\]

in \( Fuk(T^*M_\mathbb{R}; \Lambda_m) \), we associate a tropical Lagrangian section. Define

\[
L_{\mathbf{t},e_i,h} = \{(\Phi_h \circ j_0(y), -\Phi_{\mathbf{t}^i} \circ j_0(y)) \mid y \in N_\mathbb{R} \} \subset M_\mathbb{R} \times M_\mathbb{R}
\]

where \( \Phi_{\mathbf{t}^i} : \mathbb{F}_m \to \mathbb{R}^2 \) is a moment map of the ample divisor \( D_{\mathbf{t}^i} \). Then \( L_{\mathbf{t},e_i,h} \) is a section of \( M_\mathbb{R} \times \Delta_{-\mathbf{t}} \to \Delta_{-\mathbf{t}} \); its closure is

\[
\mathcal{T}_{\mathbf{t},e_i,h} = \{(\Phi_h(x), -\Phi_{\mathbf{t}^i}(x)) \mid x \in \mathbb{F}_m \}
\]

which is a section of \( M_\mathbb{R} \times \Delta_{-\mathbf{t}} \to \Delta_{-\mathbf{t}} \). Moreover, the boundary of \( \mathcal{T}_{\mathbf{t},e_i,h} \) is contained in

\[
\bigcup_{i=1}^{4}(M + \rho_i^+) \times F_i,
\]

so \( \mathcal{T}_{\mathbf{t},e_i,h} \) descends to a section \( T_{\mathbf{t},e_i,h} \) of \( \log^{-1}(\Delta_{-\mathbf{t}}) \to \Delta_{-\mathbf{t}} \) with boundary in \( M_\infty \).

Abouzaid showed that one can smooth \( T_{\mathbf{t},e_i,h} \) to obtain a tropical section \( L'_{\mathbf{t},e_i,h} \) which is compact Lagrangian with boundary in \( W_m^{-1}(0) \), a smooth hypersurface very close to \( M_\infty \) (up to scaling). Then \( L'_{\mathbf{t},e_i,h} \) is an object in \( TFuk((\mathbb{C}^*)^2, W_m^{-1}(0)) \).

It is expected that the embedding \( i_\mathbf{t} \) in \([13]\) is a quasi-equivalence when \( m = 0, 1 \). (More generally, one expects such a quasi-equivalence for any smooth projective toric Fano variety.) When \( m > 2 \), there are other bounded regions giving rise to Lagrangian sections which do not correspond to objects in \( D\text{Coh}(\mathbb{F}_m) \), so \( i_\mathbf{t} \) cannot be a quasi-equivalence.

6.3.6. Fukaya-Seidel category \( FS((\mathbb{C}^*)^2, W_m) \). Let the superpotential \( W_m \) be defined by \([12]\) as above. When \( m = 0, 1 \), \( \mathbb{F}_m \) is Fano, we may take \( D^- \) to be the anti-canonical class, i.e., \( t_1 = t_2 = t_3 = t_4 = 1 \). So

\[
W_1 = e^{-1}(z_1 + z_2 + z_1^{-1}z_2^{-1} + z_2^{-1}).
\]

There are four critical values of \( W_1 \), namely, \( \lambda_1 > 0, \lambda_3 < 0 \) on the real line, and two mutually conjugate \( \lambda_0 \) and \( \lambda_2 \) with \( \text{Im}\lambda_0 > 0 \). To each critical value, there is a unique critical point in \( (\mathbb{C}^*)^2 \). Choose a generic fiber \( W_1^{-1}(\lambda) \) and choose four paths from \( \lambda \) to the critical values. The vanishing cycles associated to these paths are the objects of the Fukaya-Seidel category \( FS((\mathbb{C}^*)^2, W_1) \). For example, Figure 9 describes a particular layout of paths and thus defines a Fukaya-Seidel category. (DifferentDenote the vanishing cycle associated to the critical point \( \lambda_i \) by \( L_i \).

Auroux, Katzarkov and Orlov [2] prove the quasi-equivalence \( DC\text{Coh}(\mathbb{F}_1) \cong DFS((\mathbb{C}^*)^2, W_1) \). Let \( \pi \) be the blow-up map \( \mathbb{F}_1 \to \mathbb{P}^2 \) and let \( E = D_4 \) be the exceptional curve in \( F_1 \). Object by object, they show the mirror correspondence

\[
L_0 \sim \mathcal{O}_{F_1}, \quad L_1 \sim \pi^*(T_{\mathbb{P}^2}(-1)), \quad L_2 \sim \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)), \quad L_3 \sim \mathcal{O}_E.
\]
Figure 9. A choice of paths defining the Fukaya-Seidel category for \(((\mathbb{C}^*)^2, W_1)\). The category consists of four vanishing cycles in \(W_1^{-1}(\lambda)\). A different homotopy type of these paths gives rise to the mutation of objects, as described in [42].

The full strong exceptional collection \(\{\mathcal{O}_{\mathbb{F}_1}, \pi^*(T_{\mathbb{P}^2}(-1)), \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)), \mathcal{O}_E\}\) implies \(DFS((\mathbb{C}^*)^2, W_1) \cong DCoh(\mathbb{F}_1)\).

In general, we may take \(D_\ell = D_1 + D_2\), i.e., \(t_1 = t_2 = 1, t_3 = t_4 = 0\). Then
\[
W_m = e^{-1}z_1 + e^{-1}z_2 + z_1^{-1}z_2^{-m} + z_2^{-1}.
\]

There are \(m + 2\) critical points \(\lambda_0, \lambda_1, \ldots, \lambda_{m+1}\) of \(W_m\). Then \(DFS((\mathbb{C}^*)^2, W_m)\) is generated by \(m + 2\) vanishing cycles \(L_0, L_1, \ldots, L_{m+1}\). Auroux, Katzarkov and Orlov showed that, in the limit
\[
W_{m,t_3} = e^{-1}z_1 + e^{-1}z_2 + e^{-t_3}z_1^{-1}z_2^{-m} + z_2^{-1}, \quad t_3 \to \infty,
\]
four of the critical points stay in a bounded region (we may assume they are \(\lambda_0, \ldots, \lambda_3\)), while the other \(m - 2\) critical points go to infinity. They showed that
\[
DCoh(\mathbb{F}_m) \cong DBFS((\mathbb{C}^*)^2, W_m) \xrightarrow{i} DFS((\mathbb{C}^*)^2, W_m)
\]
where \(BFS((\mathbb{C}^*)^2, W_m) \subset FS((\mathbb{C}^*)^2, W_m)\) is the subcategory generated by \(L_0, L_1, L_2, L_3\).

The inclusion \(i\) in (14) is not an equivalence when \(m > 2\).

We may take \(\lambda = 0\). Then the tropical Lagrangian section \(L_{\ell,h}^\prime\) is a circle in \(W_m^{-1}(0)\). In particular, the zero section can be identified with \(L_0\). \(L_{\ell,h}^\prime\) is contained in a bounded region. It is expected that \(L_{\ell,h}^\prime\) is contained in the subcategory \(DBFS((\mathbb{C}^*)^2, W_m)\), and that the Lagrangian sections \(L_{\ell,h}^\prime\)'s also generate \(DBFS((\mathbb{C}^*)^2, W_m)\).

**Remark 6.2.** Here the superpotential \(W_m\) corresponds to \(\mathfrak{W}_0\), the leading order of the full potential \(\mathfrak{W}\), in Fukaya-Oh-Ohta-Ono [21]. The full potential
\[
\mathfrak{W} = \mathfrak{W}_0 + \text{higher order terms}
\]
counts holomorphic disks of Maslov index 2 in \(\mathbb{F}_m\) with boundary in a Lagrangian torus fiber of the moment map. The higher order terms in (15) come from disks with sphere bubbles; such index 2 disks do not exist in the Fano case \((m = 0, 1)\), but exist and contribute to \(\mathfrak{W}\) when \(m \geq 3\). Fukaya-Oh-Ohta-Ono showed that \(\mathfrak{W}\) has four critical points in \(Y^\vee \subset T^\vee = \text{Spec} \mathbb{C}[z_1, z_1^{-1}, z_2, z_2^{-1}]\) (or rather \(\text{Spec}(\Lambda_{\text{nov}} \otimes \mathbb{C})[z_1, z_1^{-1}, z_2, z_2^{-1}]\)) [21, Example 7.2]. In a forthcoming sequel of [21, 22], Fukaya-Oh-Ohta-Ono prove that the number of critical points of \(\mathfrak{W}\) in \(Y^\vee\) is equal to \(\dim \mathbb{Q} H^*(X, \mathbb{Q})\) for any smooth projective toric variety \(X\).

\footnote{When there are degenerate critical points, one counts with multiplicities.}
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