**p-nomial Coefficients and p-nomial Theorem**

Aziz ATTA*

Mathematics and Structural Analysis, Atta Engineering Design Office, El Jadida, Morocco

*Corresponding author: azizatta20@gmail.com

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**Abstract** Binomial coefficients have long been studied by several mathematicians for several centuries [1] and they are currently grouped in what is called Pascal's triangle [2]. These coefficients are very useful not only in combinatorics, but also, they intervene in many fields such as enumeration, development of the binomial in algebra, development in series, and in probability distributions and statistics [3]. In addition, Pascal's formula, generative of these coefficients, leads us to think of a generalization of these coefficients and of the other strongly related mathematical tools. In this article, we will try to generalize the binomial coefficients, the Newton’s binomial as well as the Fibonacci sequence and establish their expressions. We will call these generalizations *p-*nomial coefficients, Atta’s *p-*nomial and *p-*bonacci sequence.

**Keywords:** *p-*nomial coefficients, *p-*nomial identity, staircase of *p-*nomials, *p-*bonacci sequence, *p-*Bernstein polynomial

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1. **Introduction**

Binomial coefficients \( \binom{n}{k} \), Newton's binomial and the Fibonacci sequence have always fascinated not only mathematicians, but also lovers of nature and the fine arts. Countless works and research have been devoted to these themes, which have contributed to the study of other related subjects not only in mathematics but also in other fields such as physics, chemistry, computer science, drawing and painting as well as the construction fields [4].

The history of these three notions of mathematics is very long and dates back several centuries. Indeed, these tools have been studied by several mathematicians in India, Persia, China (Zhu Shijie and Yang Hui), Maghreb, Germany and Italy [1].

We also know that there are strong links between the Fibonacci sequence, the binomial coefficients and the Newton’s binomial that can be established using analysis techniques such as recurrent sequences, polynomials and descending and ascending diagonals from Pascal's triangle. But we can also use the techniques of matrix calculation [4]. Indeed, the binomial coefficients can be obtained by calculating the powers of a strictly triangular matrix having just the non-zero diagonal and sub-diagonal whose coefficients are all equal to 1.

In the same way, we will try to generalize the binomial coefficients under the name of *p-*nominal coefficients using the technique of matrix calculation; which will allow us to introduce a generalization of Pascal’s triangle [3] called staircase of *p-*nomial coefficients. Note that these coefficients and multinomial coefficients should not be mixed. However, there is an obvious relation between these two notions.

At this stage, we will hope to introduce a concept similar to Newton's binomial called *Atta’s *p-*nomial or *p-*nomial identity which should not be mixed with the multinomial (linked to the multinomial coefficients). Thereafter, the generalization of the Fibonacci sequence known as the *p-*bonacci sequence will be accessible and the establishment of its expression will be done using the staircase of the *p-*nomial coefficients by the same method followed to establish the expression of the Fibonacci sequence from Pascal's Triangle.

We will finally present some applications and establish some useful formulas using these tools such as the introduction of a new probability distribution (generalization of Binomial Distribution) call *p-*nomial distribution and new polynomial similar to Bernstein's polynomial [5] called *p-*Bernstein polynomial (Writing in italic) without forgetting that they will have other applications in other fields.

2. **Definition of *p-*nomial Coefficients**

Let \( p \) be a non-zero natural integer. We define *p-*nomial coefficient and we note \( \pi^k_{[p-1]} \) the \( k \)th *p-*nomial coefficient among \( (p-1)n \) by the following recurrent relation:

\[
\forall n \geq 0 \; \forall k \in \left[0, (p-1)(n+1)\right]
\]
we have:

\[
\begin{align*}
    \pi^0_{n|p-1} &= \pi^0_{n|p-1} = \cdots = \pi^0_{n|p-1} = 1 \\
    \pi^k_{(n+1)|p-1} &= \pi^k_{n|p-1} + \pi^{k+p-2}_{n|p-1} + \pi^{k-p+1}_{n|p-1}
\end{align*}
\]

Such that \( \pi^q_{n|p} = 0 \) if \( q < 0 \) or \( q > np \) \([6]\).

**Example.** For \( n = 1 \) and \( p > 1 \)

\[
\forall k \in [0, p-1] \left[ \begin{array}{c}
    \pi^k_{n|p-1} = \pi^k_{n|p-1} + \pi^{k+p}_{n|p-1} = k + 1 \\
    \pi^{k-p}_{n|p-1} = \pi^k_{n|p-1} + \pi^{k-p}_{n|p-1} = p - k
\end{array} \right]
\]

### 3. Properties of \( p \)-nomial Coefficients

**Property 1**

The \((p+1)\)-nomial coefficients verify the following property:

\[
\forall k \in [0, np] \quad \pi^k_{n|p} = \pi^{np-k}_{n|p}
\]

**Proof.** Using recurrence on \( n \). For \( n = 0 \), it’s trivial. For \( n = 1 \), \( \pi^k_{1|p} = \pi^{p-k}_1 = 1 \) for all \( k \in [0, p] \) (by definition).

We suppose by recurrence that:

\[
\forall k \in [0, np] \quad \pi^k_{n|p} = \pi^{np-k}_{n|p}
\]

For \( n+1 \), we have:

\[
\begin{align*}
    \pi^k_{(n+1)|p} &= \pi^k_{n|p} + \pi^{k+p}_{n|p} + \pi^{k-p}_{n|p} \\
    &= \pi^k_{n|p} + \pi^{k+p}_{n|p} + \pi^{k-p}_{n|p} \\
    &= \pi^k_{n|p} + \pi^{k+p}_{n|p} + \pi^{k-p}_{n|p} \\
    &= \pi^k_{n|p} + \pi^{k+p}_{n|p} + \pi^{k-p}_{n|p}
\end{align*}
\]

This completes demonstration by recurrence.

This property allow us to build a staircase each line of which consists of \((p-1)n+1\) elements numbered from 0 to \((p-1)n\) that we call the staircase of \( p \)-nomials. For \( p = 2 \), it’s exactly Pascal’s triangle.

The number \( \mu = \left\lfloor \frac{p}{2} \right\rfloor \) is called staircase step.

**Property 2**

The \((p+1)\)-nomial coefficients verify the following property:

\[
\forall n \in \mathbb{N} \quad \sum_{k=0}^{np} \pi^k_{n|p} = (p+1)^n
\]

**Proof.** Using recurrence on \( n \).

For \( n = 0 \), it’s trivial. For \( n = 1 \):

\[
\forall n \in \mathbb{N} \quad \sum_{k=0}^{np} \pi^k_{n|p} = \sum_{k=0}^{p} 1 = p + 1 \text{ (by definition)}.
\]

Suppose by recurrence that, for \( n \in \mathbb{N} : \]

\[
\sum_{k=0}^{np} \pi^k_{n|p} = (p+1)^n.
\]

For \( n+1 \), we have:

\[
\begin{align*}
    \sum_{k=0}^{p(n+1)} \pi^k_{n+1|p} &= \sum_{k=0}^{p(n+1)} \pi^k_{n|p} + \pi^{p(n+1)}_{n|p} \\
    &= \sum_{k=0}^{p} \pi^k_{n|p} + \pi^{p(n+1)}_{n|p} \\
    &= (p+1)^n + (p+1)^{n+1}
\end{align*}
\]

This completes demonstration by recurrence.

**Property 3**

\[
\forall n \in \mathbb{N}^*, \quad \text{we have:}
\begin{align*}
    \forall p \geq 2 \quad \pi^1_{n|p-1} &= n \\
    \forall p \geq 3 \quad \pi^2_{n|p-1} &= \frac{n(n+1)}{2}
\end{align*}
\]

**Proof.** Using recurrence:

For \( n = 1 \), \( \pi^1_{1|p-1} = 1 \) and \( \pi^2_{1|p-1} = 1 = \frac{1 \times 2}{2} \) (by definition because \( p-1 \geq 2 \)). Now, we suppose by recurrence that, for \( n \in \mathbb{N}^* : \]

\[
\begin{align*}
    \forall p \geq 2 \quad \pi^1_{n|p-1} &= n \\
    \forall p \geq 3 \quad \pi^2_{n|p-1} &= \frac{n(n+1)}{2}
\end{align*}
\]

For \( n+1 \), we have:

\[
\begin{align*}
    \forall p \geq 2 \quad \pi^1_{n+1|p-1} &= \pi^1_{n|p-1} + \pi^0_{n|p-1} = n + 1 \\
    \forall p \geq 3 \quad \pi^2_{n+1|p-1} &= \pi^2_{n|p-1} + \pi^0_{n|p-1} + \pi^0_{n|p-1}
\end{align*}
\]

This completes demonstration by recurrence.

**1-nominal coefficients (uninomial)**

For the case \( p = 1 \), the 1-nomial coefficients are given by:

\[
\forall n \geq 0 \quad \pi^0_{n|p} = 1
\]
2-nomial coefficients (binomial)

For the case $p = 2$, the 2-nomial coefficients are given by:

\[
\binom{0}{q}_p = \binom{0}{q} = 1 \\
\forall n \geq 0 \quad \forall k \in [0, n+1] \quad \binom{k}{n}_p = \binom{k}{n} + \binom{k-1}{n}_p
\]

We easily notice that the sequence $\left(\binom{k}{n}_p\right)_{n \in \mathbb{N}}$ verify relation of the binomial coefficients [2]:

\[
\binom{0}{q}_k = \binom{0}{q} = 1 \\
\forall k \in [0, n+1] \quad \binom{k}{n}_p = \binom{k}{n} + \binom{k-1}{n}_p
\]

We thus deduce that:

\[
\forall n \geq 0 \quad \forall k \in [0, n] \quad \binom{k}{n}_p = \binom{k}{n}
\]

Therefore, 2-nomial coefficients are exactly the binomial coefficients.

4. Fundamental Theorem

Introduction

$[x]$ denote the integer part of the real number $x$.

The $p$-nomial coefficients $\binom{q}{k}_{p-1}$ are obtained by calculation of the matrix $\left[PT\right]_n\left(\frac{1...1}{p \text{ times}}\right)^k$, where:

\[
\left[PT\right]_n\left(\frac{1...1}{p \text{ times}}\right) = \\
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 1 & \cdots & 0 \\
0 & \cdots & \cdots & 1 & 1 \\
1 & \cdots & \cdots & \cdots & 1 \\
0 & \cdots & 0 & \cdots & 1 \\
0 & \cdots & \cdots & \cdots & 1
\end{pmatrix}
\]

This matrix is called $p$-unit matrix of size $n$.

Indeed, the non-zero coefficients of the columns of this matrix $\left[PT\right]_n\left(\frac{1...1}{p \text{ times}}\right)^k$ are identical and therefore we will be interested in the first column of this matrix. The $q^{th}$ ($q \in [0,(p-1)k]$) nonzero coefficient in this column represents the $\binom{q}{k}_{p-1}$. To find a relation between $\binom{q}{k}_{p-1}$ and $\binom{q}{k-1}_{p-1}$, proceed as follows. Let’s write the $(p+1)$-unit matrix of size $n$ as:

\[
\left[PT\right]_n\left(\frac{1...1}{p+1 \text{ times}}\right) = \left[N_T\right]_n\left(\frac{1...1}{p \text{ times}}\right) + I_n
\]

Where:

\[
\left[N_T\right]_n\left(\frac{1...1}{p \text{ times}}\right) = \\
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 1 \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 1 & \cdots & 0
\end{pmatrix}
\]

So, we will have:

\[
\left[PT\right]_n\left(\frac{1...1}{p \text{ times}}\right)^k = \sum_{q=0}^{k} \binom{q}{k} \left[N_T\right]_n\left(\frac{1...1}{p \text{ times}}\right)^q.
\]

Well, the first column of the strictly triangular matrix $\left[N_T\right]_n\left(\frac{1...1}{p \text{ times}}\right)$ contains $p$ non-zero elements, which means that $\left[N_T\right]_n\left(\frac{1...1}{p \text{ times}}\right)^q$ contains the $p$-nomial coefficients $\binom{q}{k}_{p-1}$. So, we can conclude that the $\binom{m}{k}_{p}$ are combinations of $\binom{q}{k}_{p-1}$ ($q \in [0,k]$) with $C_k^q$ as coefficients of sum. The $m^{th}$ line $L_m$ ($m \in [0, pk]$) of matrix $\left[PT\right]_n\left(\frac{1...1}{p \text{ times}}\right)^k$ contains the $p$-nomial coefficients:

\[
\binom{m}{k}_{p}, \binom{m-1}{k}_{p}, \binom{m-2}{k}_{p}, \cdots, \binom{m-min(m,k)}{k}_{p-1}.
\]

Identity of complements

We have the following identity:

\[
\forall m \in \mathbb{N} \quad \forall p \in \mathbb{N}^* \quad \frac{m+p-1}{p} + \frac{(p-1)}{p} = m
\]

which means that the sum is independent of $p$.

Proof. If $m = 0 \left(\begin{array}{c}
p \\
1
\end{array}\right)$ then:

\[
\frac{m+p-1}{p} + \frac{(p-1)}{p} = \frac{m+1-1}{p} + \frac{m-m}{p} = \frac{m}{p} + \frac{m-m}{p} = m.
\]

If $m = r \left(\begin{array}{c}
p \\
r
\end{array}\right)$ (1 $\leq r < p$) then $m = qr+r$ and we have:

\[
\frac{m+p-1}{p} + \frac{(p-1)}{p} = \frac{q+1+r-1}{p} + \frac{m-q-r}{p} = q + m - q - r = m.
\]
Hence the validity of the identity. This will help us to write the expression of the recurrent relation linking \((p + 1)\)-nomial and \(p\)-nomial coefficients.

**Lemma 1**

Let \(m\) a natural number such as its Euclidean division by \(p\) is written \(m = sp + r\). We have:

\[
\pi_s^{m-s-r} + \ldots + \pi_s^{m-s-p} = \pi_s^{m-s+1}\]

**Proof.** Let \(j\) a non-zero natural integer such that \(j < r\). Because of \(r < 0\), we get:

\[
\pi_s^{m-s-j} = \pi_s^{s(p-1)-m+s+j} = \pi_s^{s(p-1)\cdot s^p + s^p + m + s + j} = \pi_s^{j-r} = 0
\]

So, we can write:

\[
\pi_s^{m-s-r} + \ldots + \pi_s^{m-s-p} = \pi_s^{m-s+1} + \pi_s^{s(p-1)} + \pi_s^{m-s-r} + \ldots + \pi_s^{m-s-p} = \pi_s^{m-s+1}.
\]

**Lemma 2**

Let \(m, q, p\) and \(s\) four natural numbers such as \(p \geq 1, s \geq 1\) and \(0 \leq m - q \leq p - 1\). We have the following equality:

\[
\pi_s^{m-q} + \ldots + \pi_s^0 = \pi_s^{m-q} + \ldots + \pi_s^{p+q-p-1}
\]

**Proof.** Indeed, according to the definition formula of \(p\)-nomials coefficients:

\[
\pi_s^{m-q} = \pi_s^{m-q} + \ldots + \pi_s^{p+q-p-1}
\]

But, \(m - q - p + 1 \leq 0\) and \(0 \leq m - q \leq p - 1 \leq s\) (\(p - 1\)), so:

\[
\pi_s^{m-q} = \pi_s^{m-q} + \ldots + \pi_s^0
\]

**Fundamental theorem**

Let \(p \in \mathbb{N}^+\), \(k \in \mathbb{N}\) and \(m \in \{0, kp\}\).

The \((p + 1)\)-nomial coefficients and the \(p\)-nomial coefficients are linked by the following recurrent relation called \(p\)-nominal formula:

\[
\pi_{kp}^{m} = \sum_{q=\min(k,m)}^{\min(k,m)+1} C_k^q \pi_{kp}^{m-q} = \sum_{q=\min(k,m)+1}^{\min(k,m)+p-1} C_k^q \pi_{kp}^{m-q}
\]

**Proof.** The proof of the fundamental theorem of \(p\)-nomials is based on a proof by recurrence. We will thus use recurrence many times. To initiate the recurrence, let us demonstrate this formula for the first value of \(p\). For \(p = 1\):

\[
\forall (k, m) \in \mathbb{N}^2 \quad (m \leq k),
\]

So, it’s true for \(p = 1\). Now suppose that for \(p \geq 1\), we have:

\[
\pi_s^{m-p} = \sum_{q=\min(k,m)+1}^{\min(k,m)+p-1} C_k^q \pi_{kp}^{m-q}
\]

Now, let’s prove the recurrence for \(k + 1\). Using definition formula and recurrence hypothesis, we can write that:

\[
\pi_s^{m-(k+1)} = \sum_{j=0}^{\min(k,m)-j} C_{k+1}^j \pi_{k+1}^{m-j-q} = \sum_{j=0}^{\min(k,m)-j} \frac{C_k^j \pi_{kp}^{m-j-q}}{p}
\]

The first proposition of demonstration consists in treating the possible cases according to the Euclidean division of \(m\) by \(p\) and for each case, treat different possible cases by comparing the numbers \(m - p, k\) and \(m - p < k < m\) or \(k = m\).

We need also to use a lot lemmas 1 and 2. We note the Euclidean division of \(m\) by \(p\) as follows:

\[
m = sp + r \quad (0 \leq r < p)
\]

but this demonstration is very long since it discusses several cases. We can use a more concise demonstration which consists in noticing that in formula (1), we can sum from the minimum of the index \(q\) which is \(\lfloor \frac{m-1}{p} \rfloor\) to the maximum which is \(\min(k,m)\) since the terms which will be added are null. We should also notice that:

\[
\min(m, k) + 1 = \min(m, k + 1) = k + 1
\]

\[
m \leq k \Rightarrow \begin{cases} \min(m, k) = \min(m, k + 1) = m \\ \pi_{kp}^{m-\min(k,m)+1} \pi_{kp}^{m-\min(k,m)+1} = 0 \end{cases}
\]

For that, we’ll discuss just two cases:

**First case:** If \(r = 0 \Rightarrow m = \frac{m+p-1}{p}\) (**) and:

\[
m - q \leq q(p - 1) = q = \frac{m+p-1}{p}
\]

Then, we get:

\[
\pi_{kp}^{m} = \sum_{j=0}^{\min(k,m)} \sum_{q=\min(k,m)}^{\min(k,m)-j} C_k^j \pi_{kp}^{m-q} = \sum_{q=\min(k,m)-j}^{\min(k,m)-j} \sum_{j=0}^{\min(k,m)-j} C_k^j \pi_{kp}^{m-q-j}
\]
Using (**) and (***) we conclude that:

$$\pi^m_{kk\|p} = \sum_{q=m-p+1}^{\min(k,m)+1} C_k^q \pi^{m-q}_{q\|p-1}$$

**Second case:** If $1 \leq r < p$:

$$\begin{align*}
\left\lceil \frac{m+p}{p} \right\rceil &= \left\lceil \frac{s+1+r}{p} \right\rceil = \left\lceil \frac{m+p}{p} \right\rceil = \left\lceil \frac{m+p+1}{p} \right\rceil \\
\left\lceil \frac{m+p-1}{p} \right\rceil &= \left\lceil \frac{s+1+r-1}{p} \right\rceil = \left\lceil \frac{m+p-1}{p} \right\rceil
\end{align*}$$

and: $m-q \leq (p-1) \Rightarrow q \geq m \Rightarrow q \geq \left\lceil \frac{m+p+1}{p} \right\rceil$.

We obtain thus:

$$\pi^m_{kk\|p} = \sum_{j=0}^{p} \min(k,m-j) \sum_{q=m-j-1}^{\min(k,m)} C_k^q \pi^{m-q-j}_{q\|p-1}$$

$$= \sum_{j=0}^{p} \sum_{q=m-j-1}^{\min(k,m)} C_k^q \pi^{m-q-j}_{q\|p-1} = \sum_{j=0}^{p} \sum_{q=m-j-1}^{\min(k,m)} C_k^{q+j} \pi^{m-q-j}_{q\|p-1}$$

$$= \sum_{q=m-1}^{\min(k,m)} C_k^q \left( \pi^{m-q}_{q\|p-1} + \sum_{j=0}^{p-1} C_k^q \pi^{m-q-j}_{q\|p-1} - \pi^{m-q-1}_{q\|p-1} \right)$$

$$= \sum_{q=m-1}^{\min(k,m)} C_k^q \pi^{m-q}_{q\|p-1} + \sum_{q=m-1}^{\min(k,m)} C_k^q \pi^{m-q-1}_{q\|p-1}$$

$$= \sum_{q=m-1}^{\min(k,m)} C_k^q \pi^{m-q}_{q\|p-1} + \sum_{q=m-1}^{\min(k,m)} C_k^{q+j} \pi^{m-q-j}_{q\|p-1}$$

We thus treated all cases; which completes the proof of the fundamental theorem of $p$-nomials by recurrence.

**Expressions of $p$-nomial coefficients**

Using the fundamental theorem, we can establish an expression for the $p$-nomial coefficients. We give below the expression of $p$-nomials:

$$\pi^q_{q\|p} = \sum_{j=0}^{p} \min(q-p+j, q-p+1) \sum_{i=0}^{q-p-j} C_p^i \pi^{q-p-j}_{q-p-j-i\|p-j-1}$$

To simplify the writing, we introduce the symbol $\Pi$ to group the summations and order them by an index that varies in a given interval. For $p \geq 1$:

$$\pi^q_{q\|p} = \prod_{j=0}^{p-1} \sum_{q-j+q-p-j+1}^{p-1} C_p^i \pi^{q-p-j}_{q-p-j-i\|p-j-1}$$

We can also notice that:
Let $x$ and $y$ two complex numbers and $p$ a non-zero natural integer. We define Atta’s $(p+1)$-nomial by:

$$A_{n+1[p]}(x,y) = (x^p + x^{p-1}y + \ldots + xy^{p-1} + y^p)^n$$

**$p$-nomial theorem**

We have the following equality:

$$A_{n+1[p]}(x_1,x_2) = \sum_{k=0}^{np} \xi^k_{n[p]} x_1^k x_2^{np-k}$$

**Proof.** Using recurrence. For $n = 0$, we get: $1 = x_0^0$.

For $n = 1$, we get:

$$\sum_{k=0}^{p} \xi^k_{p} x_1^k x_2^{p-k} = \sum_{k=0}^{p} \xi^k_{p} x_1^k x_2^{p-k}$$

Let suppose that for $n \in \mathbb{N}$:

$$A_{n+1[p]}(x_1,x_2) = \sum_{k=0}^{np} \xi^k_{n[p]} x_1^k x_2^{np-k}.$$

For $n + 1$, we have:

$$A_{n+2[p]}(x_1,x_2) = \left(x_1^p + x_1^{p-1}x_2 + \ldots + x_1 x_2^{p-1} + x_2^p\right) A_{n+1[p]}(x_1,x_2)$$

which completes the proof of $p$-nomial theorem by recurrence.

**6. $p$-bonacci Sequence**

**Definition**

We define a generalization of the Fibonacci sequence (which is the 2-bonacci) [1,6,8]. $p$-bonacci is defined by:

$$F_{k[p]} = \begin{cases} 0 & k = 0; F_{1[p]} = 1; F_{q[p]} = 2^{q-2} & q \in \left[2, p - 1\right] \\ F_{k+p[p]} = F_{k[p]} + \ldots F_{k+q[p]} & k \geq 0 \end{cases}$$

**Expression**

Let $p \geq 2$ be a natural integer. The $n$th number in the sequence of $p$-bonacci is given by the formula:

$$\forall n \geq 1 \quad F_{[p]}(n) = \sum_{k=n+1}^{n-1} \xi^k_{n[p]} x_1^{k-n+1}$$

**Proof.** The demonstration is done by recurrence on $n$ using the Euclidean division of $n$ by $p$ and by discussing according to the rest of the division $r$, the three cases $r = 0$, $r = 1$ et $2 \leq r < p$. Furthermore, it is assumed by recurrence that:

$$\forall k \leq n \quad F_{k[p]} = \sum_{q=0}^{k-1} \xi^q_{n-k[p]} x_1^{q+1}$$

For example, Tetra-Bonacci sequence is given by using the staircase of tetra-nomials as follow (with $F_0[3] = 0$):
Golden numbers sequence

We define the golden ratio of order \( p \) as the limit of:

\[
\lim_{n \to +\infty} \frac{F_{np}}{F_n} \quad \forall n \in \mathbb{N}
\]

The sequence \((\phi_n)_{n \in \mathbb{N}}\) is called golden numbers sequence.

Indeed, we can prove that the golden ratio \( \phi \) exists using recurrence relation \((b)\). In addition, we have \([9]\):

\[
\phi_1 = 1; \quad \phi_2 = \frac{\sqrt{5} + 1}{2}; \quad \phi_3 = \frac{3\sqrt{19 + 3\sqrt{33}} + 3\sqrt{19 - 3\sqrt{33}} + 1}{3}
\]

And for \( p \geq 2 \), the golden number is the unique positive solution different from 1 of the equation:

\[ x^{p+1} - 2x^p + 1 = 0. \quad (eq) \]

We can easily demonstrate using the intermediate value theorem that:

\[ \forall p \geq 1 \quad \frac{2p}{p^2 + 1} \leq \phi_p < 2. \]

So, we can deduce that:

\[ \lim_{p \to +\infty} \phi_p = 2. \]

In addition, one can establish an asymptotic expression with the golden ratio \( \phi_p \) as follows. We note this expression:

\[ \phi_p \sim 2 - \varepsilon_p \quad \left( \lim_{p \to +\infty} \varepsilon_p = 0 \right) \]

Since we have \((eq)\):

\[ \phi_p^{p+1} - 2\phi_p^p = -1 \]

By replacing by the expression of \( \phi_p \):

\[ (2 - \varepsilon_p)^{p+1} - 2(2 - \varepsilon_p)^p = -1 \]

By proceeding with a limited development:

\[ 2^{p+1} \left( 1 - \frac{(p+1)}{2} \right)^p - 2^{p+1} \left( 1 - \frac{p}{2} \right)^p \sim -1 \]

We thus obtain \( \varepsilon_p \sim \frac{1}{2^p} \) and:

\[ \phi_p \sim 2 - \frac{1}{2^p}. \]

7. Trinomial Coefficients

Definition

The \( k^{th} \) trinomial coefficient among \( 2n \) noted \( \pi_n^k \) verify following recurrence relation:

\[
\forall n \geq 0 \quad \forall k \in \left[ 0, 2(n+1) \right] \quad \begin{cases} 
\pi_n^0 = 1 \\
\pi_n^{k+1} = \pi_n^k + \pi_n^{k-1} + \pi_n^{k-2} 
\end{cases}
\]

Such that \( \pi_n^q = 0 \) if \( q < 0 \) or \( q > 2n \).

Trinomial staircase (triangle)

It is easy to construct triangle of trinomials. Indeed, we follow the same approach of construction of the Pascal’s triangle:

"We sum the three successive elements of line \( n \) and put it below the central element of the three elements summed for line number \( n + 1 \):"

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 2 & 1 & 1 & 1 \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\
1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1 \\
\end{array}
\]

The triangle above is the triangle of trinomials up to \( n = 5 \).

Atta’s trinomial

For all complex numbers \( x_1 \) and \( x_2 \):

\[
\forall n \in \mathbb{N} \quad \left( x_1^2 + x_1x_2 + x_2^2 \right)^n = \sum_{k=0}^{2n} \pi_n^k x_1^{2n-k} \]

Expression of trinomial coefficients.

Let \( k \in \mathbb{N} \) and \( m \in \left[ 0, 2k \right] \).

The trinomial coefficients and the binomial coefficients are linked by the following recurrent relation:

\[
\pi_n^m = \sum_{q=m+1}^{\min(k,m)} \binom{2q}{q} C_q^{m-q} 
\]

Properties 1

\[ \forall k \in \left[ 0, 2n \right] \quad \pi_n^k \pi_n^{k-1} = \pi_n^{2n-k} \quad \text{and} \quad \sum_{k=0}^{2n} \pi_n^k = 3^n. \]

Proof. By using Atta’s trinomial for \( x_1 = x_2 = 1 \).

Property 2

Let \( p \) a prime number. We have following properties:
Proof. By using the expression of trinomial coefficients:
\[
\pi^k_{\mathcal{e}(p)} = \min(p, k) - 1 \leq q \leq k \min(p, k),
\]
which gives: \(k - q < \frac{k + 1}{2}\) and \(2q - k \leq 2\min(p, k) - k\).

If \(k - q = p\), then \(p < \frac{k + 1}{2}\) or more \(2p - 1 < k\).

Because of \(k \in [1, 2p - 1] \setminus \{p\}\), we deduce that \(p\) doesn't divide \((p - q)!\). If \(2q - k = p\) then \(p + k \leq 2\min(p, k)\) and this is not possible because \(k \neq p\), so we deduce that \(p\) doesn't divide \((2q - k)!\). Thus, \(p\) doesn't divide \((p - q)!\) \((k - q)!\) \((2q - k)!\) and we deduce that:
\[
\forall k \in [1, 2p - 1] \setminus \{p\}, \quad p / \pi^k_{\mathcal{e}(p)}.
\]

Applications

Formulas

Using the \(p\)-nomial, we can have several formulas among which we cite:

\[
\forall n \in \mathbb{N}, \quad \forall x \in \mathbb{C}, \quad \left(1 + x + x^2 + \ldots + x^p\right)^n = \sum_{k=0}^{n p} \pi^k_{\mathcal{e}(p)} x^k
\]

\[
\forall n \in \mathbb{N}, \quad \forall p \in \mathbb{N}, \quad \sum_{k=0}^{n p} \pi^k_{\mathcal{e}(p)} = (p + 1)^n
\]

\[
\forall n \in \mathbb{N}, \quad \forall p \in \mathbb{N}, \quad \sum_{k=0}^{n p} (-1)^k \pi^k_{\mathcal{e}(p)} = \left(\frac{1 + (-1)^p}{2}\right)^n
\]

\[
\forall (n, m) \in \mathbb{N}^2, \quad \forall p \in \mathbb{N}, \quad \sum_{k=0}^{n p} \pi^k_{\mathcal{e}(p)} = \sum_{j=0}^{n} \left(\pi^j_{\mathcal{e}(p)} \right)^{n-j}
\]

\[
\forall n \in \mathbb{N}, \quad \forall p \in \mathbb{N}, \quad \pi^k_{\mathcal{e}(p)} = \sum_{j=0}^{n} \left(\pi^j_{\mathcal{e}(p)} \right)^{n-j}
\]

\[
\forall n \in \mathbb{N}, \quad \forall x \in \mathbb{C}, \quad \sum_{k=1}^{n p} k x^k \pi^k_{\mathcal{e}(p)} = \frac{np}{2} (p + 1)^n
\]

\[
\forall \theta \in \mathbb{C} \setminus \{2\pi k\}, \quad \sum_{k=0}^{n p} \pi^k_{\mathcal{e}(p)} \cos(k\theta) = \left(\frac{\sin\left(p + \frac{1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}\right)^n \cos\left(\frac{np\theta}{2}\right)
\]

\[
\forall \theta \in \mathbb{C} \setminus \{2\pi k\}, \quad \sum_{k=0}^{n p} \pi^k_{\mathcal{e}(p)} \sin(k\theta) = \left(\frac{\sin\left(p + \frac{1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}\right)^n \sin\left(\frac{np\theta}{2}\right)
\]

\[
\forall n \in \mathbb{N}, \quad \forall x \in \mathbb{R}, \quad \sum_{k=0}^{n p} \pi^k_{\mathcal{e}(p)} \cos(kx) = \left(\frac{\sin\left(p + \frac{1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}\right)^n \cosh\left(\frac{np x}{2}\right)
\]

\[
\forall n \in \mathbb{N}, \quad \forall x \in \mathbb{R}, \quad \sum_{k=0}^{n p} \pi^k_{\mathcal{e}(p)} \sin(kx) = \left(\frac{\sin\left(p + \frac{1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}\right)^n \sinh\left(\frac{np x}{2}\right)
\]

\[
\forall n \in \mathbb{N}, \quad \forall x \in \mathbb{R}, \quad \sum_{k=0}^{n p} \pi^k_{\mathcal{e}(p)} (1 - q)^{p-k} = n \left(\sum_{k=0}^{p} q^k (1 - q)^{p-k}\right)^{n-1} \left(\sum_{k=0}^{p} k q^k (1 - q)^{p-k}\right)
\]

\(p\)-Bernstein polynomial

For a degree \(m \geq 0\), there are \(mp + 1\) \(p\)-Bernstein polynomials, defined over the interval \([0,1]\), by:
\[
B^q_{m[p]}(x) = \pi^q_{m[p]} x^q (1 - x)^{mp-q}.
\]

These polynomials form the basis of the vector space of polynomials of degrees less than or equal to \(mp\).

These polynomials verify the following properties:
\[
B^q_{m[p]} = \frac{\pi^q_{m[p]} B^{q}_{mp}}{C_{mp}^q}
\]

where \(B^q_{m[p]}\) Bernstein polynomial [5].
\[
B^q_{m[p]}(u) = B^q_{m[p]}(1 - u)
\]

\[
\sum_{q=0}^{mp} B^q_{m[p]}(x) = \left(\sum_{q=0}^{p} x^q (1 - x)^{mp-q}\right)^m
\]

\[
B^q_{m[p]} = \pi^q_{m[p]} \sum_{k=0}^{mp-q} (-1)^{k} C_{mp-q}^{k} x^q (1 - x)^{k}
\]

\[
\forall n \in \mathbb{N}, \quad \forall p \in \mathbb{N}, \quad \sum_{k=0}^{np} \pi^k_{\mathcal{e}(p)} B^k_{mp-k}(x) = \sum_{k=0}^{mp-q} C_{mp-q}^{k} B^k_{m[p]}(1 - x)
\]

\(p\)-nomial distribution

Definition

Let \(n\) and \(p\) two non-zero natural integers. A random variable \(X\) follows a \((p + 1)\)-nomial distribution \(\pi^n_{p}(nq)\) (generalization of binomial distribution) [10], where \(0 < q < 1\) if:
\[
\forall k \in \mathbb{N}, \quad P(X = k) = \pi^k_{n[p]} \frac{q^k (1-q)^{n-k}}{\sum_{j=0}^{n} q^j (1-q)^{n-j}}
\]

Indeed, this law is well defined:
\[
\sum_{j=0}^{p} q^j (1-q)^{p-j} = \begin{cases} 
\frac{p+1}{2^p} & \text{if } q = \frac{1}{2} \\
\frac{q^{p+1}-(1-q)^{p+1}}{2q-1} & \text{if } q \neq \frac{1}{2} 
\end{cases}
\]

Which shows that: \( \forall q \in [0,1], \sum_{j=0}^{p} q^j (1-q)^{p-j} \neq 0. \)

Moreover:
\[
\forall k \in \mathbb{N}, \quad \sum_{k=0}^{np} q_k^{p} = \sum_{k=0}^{np} q_k^{p} (1-q)^{np-k} = 1,
\]

**Esperance**

Note \( Q_p = q^p + q^{p-1} (1-q) + \ldots + q (1-q)^{p-1} + (1-q)^p. \)

The Esperance of the p-nominal law is given by:
\[
E_p(X) = n q \left( p + (1-q) \frac{1}{Q_p} \frac{dQ_p}{dq} \right).
\]

For \( p = 1 \), we have \( Q_1 = q + 1 - q = 1 \). We find \( E_1(X) = n q \) : this is Esperance of binomial law.

For \( p = 2 \), \( Q_2 = q^2 - q + 1 \). We find \( E_2(X) = \frac{n q (q + 1)}{q^2 - q + 1} \):
this is Esperance of trinomial distribution.

**Inverse of a triangular matrix**

We consider the following triangular matrix:
\[
T_{n+1} = \begin{pmatrix}
1 & 0 & 0 & \cdots & \cdots & 0 \\
1 & 1 & 0 & \cdots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 1 & 0 & \vdots \\
1 & \cdots & 1 & 0 & 0 & \vdots \\
1 & \cdots & 1 & 1 & 0 & \vdots \\
1 & \cdots & 1 & 1 & 1 & 1
\end{pmatrix}
\]

The inverse of this matrix is given by:
\[
T_{n+1}^{-1} = \begin{pmatrix}
t_0 & 0 & 0 & \cdots & \cdots & 0 \\
t_1 & t_0 & 0 & \cdots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
t_{q+1} & t_q & \cdots & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
t_n & \cdots & t_q & t_1 & t_0 & 0 \\
\end{pmatrix}
\]

Where: \( t_q = \sum_{k=0}^{np+1} \min(q,k) \frac{q_k^{p}}{n} C_{n+1}^{q+1} C_k \frac{q^m}{n-m+1}. \)

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Table 1A. Staircase of trinomial coefficients \( (n \text{ from 0 to 12}) \)

| \( n \) | 0    | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   | 11   | 12   |
|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 0     | 1    | 1    | 1    |      |      |      |      |      |      |      |      |      |      |
| 1     |      |      |      |      | 1    | 3    | 6    | 7    | 6    | 3    |      |      |      |
| 2     | 1    | 4    | 10   | 16   | 19   | 16   | 10   | 4    | 1    |      |      |      |      |
| 3     | 1    | 5    | 15   | 30   | 45   | 51   | 45   | 30   | 15   | 5    |      |      |      |
| 4     | 1    | 6    | 21   | 50   | 90   | 126  | 141  | 126  | 90   | 50   | 21   | 6    |      |
| 5     | 1    | 7    | 28   | 77   | 161  | 266  | 357  | 393  | 357  | 266  | 161  | 77   | 28   |
| 6     | 1    | 8    | 36   | 112  | 266  | 504  | 784  | 1016 | 1107 | 1016 | 784  | 504  | 266  |
| 7     | 1    | 9    | 45   | 156  | 414  | 882  | 1554 | 2304 | 2907 | 3139 | 2907 | 2304 | 1554 |
| 8     | 1    | 10   | 55   | 210  | 615  | 1452 | 2850 | 4740 | 6765 | 8530 | 8953 | 8350 | 6765 |
| 9     | 1    | 11   | 66   | 275  | 880  | 2277 | 4917 | 9642 | 14355 | 19855 | 24068 | 25653 | 24068 |
| 10    | 1    | 12   | 78   | 352  | 1221 | 3432 | 8074 | 16236 | 28314 | 43252 | 58278 | 69576 | 58278 |
| 11    | 1    |      |      |      |      |      |      |      |      |      |      |      |      |
| 12    | 1    |      |      |      |      |      |      |      |      |      |      |      |      |

Table 1B. Staircase of trinomial coefficients \( (n \text{ from 0 to 12}) \) organized to find tribonacci sequence as shown by the arrows

Table 1C. First values of tribonacci sequence \( (n \text{ from 0 to 13}) \)

| \( n \) | \( T_0 \) | \( T_1 \) | \( T_2 \) | \( T_3 \) | \( T_4 \) | \( T_5 \) | \( T_6 \) | \( T_7 \) | \( T_8 \) | \( T_9 \) | \( T_{10} \) | \( T_{11} \) | \( T_{12} \) | \( T_{13} \) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0     | 0     | 1     | 1     | 2     | 4     | 7     | 13    | 24    | 44    | 81    | 149   | 274   | 504   | 927   |
| \( T_0/T_{n+1} \) |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| \( T_n/T_{n+1} \) |      |      |      |      |      |      |      |      |      |      |      |      |      |      |

We observe that \( \frac{T_{n+1}}{T_n} \) tends to \( \phi^2 = \frac{\sqrt{19} + 3\sqrt{33} + \sqrt{3} \cdot (\sqrt{19} - 3\sqrt{33}) + 1}{3} \approx 1.8392867 \).

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