Kinematics of flows on curved, deformable media

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Abstract

Kinematics of geodesic flows on specific, two dimensional, curved surfaces (the sphere, hyperbolic space and the torus) are investigated by explicitly solving the evolution (Raychaudhuri) equations for the expansion, shear and rotation, for a variety of initial conditions. For flows on the sphere and on hyperbolic space, we show the existence of singular (within a finite value of the time parameter) as well as non-singular solutions. We illustrate our results through a phase diagram which demonstrates under which initial conditions (or combinations thereof) we end up with a singularity in the congruence and when, if at all, we can obtain non–singular solutions for the kinematic variables. Our analysis portrays the differences which arise due to positive or negative curvature and also explores the role of rotation in controlling singular behaviour. Subsequently, we move on to geodesic flows on two dimensional spaces with varying curvature. As an example, we discuss flows on a torus. Characteristic oscillatory features, dependent on the ratio of the two radii of the torus, emerge in the solutions for the expansion, shear and rotation. Singular (within a finite time) and non–singular behaviour of the solutions are also discussed. Finally, we conclude with a generalisation to three dimensional spaces of constant curvature, a summary of some of the generic features obtained and a comparison of our results with those for flows in flat space.

Keywords: Kinematics, deformable media, Raychaudhuri equations, curved spacetime and geodesic congruence.

PACS numbers: 83.10.Bb, 04.40.-b, 4.20.Jb
I. INTRODUCTION

Geodesic flows in curved spaces have been a topic of interest among mathematicians for many years. However, as far as we are aware, a detailed study of the kinematics of such flows, does not exist in the literature. The kinematics of flows can either be linked with the evolution of particle distribution on a manifold, or with the evolution of deformations of a medium itself. This aspect may be understood as follows.

![Particle distributions at two time instants](image)

**FIG. 1**: Particle distributions at two time instants $t_1$ and $t_2$ evolving under (geodesic) motion on a spherical surface.

Imagine a spherical surface on which we have particles sprinkled randomly as shown in Fig. 1. Given some initial motion, these particles will move along trajectories governed by the equations of motion (geodesics, say, as a special case). In an active viewpoint, we might look at a region on the sphere containing a certain number of particles at an initial time and ask—do the particles accrete, spread out or redistribute themselves in an area-preserving way, as time evolves? The same question may also be posed for understanding deformations of a medium as well, where the particles now imply the points of the medium. Such investigations in a real four dimensional scenario, are of fundamental importance in studying accretion near black hole horizons, structure formation and evolution of the universe at different scales, and motion of fluid flows to name a few.

In [1], we restricted ourselves to flows in flat space where the evolution of each point on the medium was along a curve which solved the Newton’s second law with damping and elastic forces added. What happens if we have flows on surfaces of nontrivial curvature? To this end, here, we look at geodesic flows on a sphere, a saddle surface as well as a surface
of varying curvature—the torus, and figure out the kinematics of flows on them subject to specific initial conditions. Curvature creates non-trivial effects on the flows, which need to be quantified. We are fortunate that in some cases, the set of equations governing the behaviour of flows can indeed be solved analytically. Therefore, we are able to see explicitly how the kinematical quantities evolve. Where analytical solutions are impossible, we resort to numerical methods in order to arrive at our conclusions.

The kinematics of flows on deformable media is characterised by the kinematical quantities—the isotropic expansion, shear and rotation or vorticity [2] (henceforth referred as ESR). The deformation of the medium (or a local region in the medium) is related to the behaviour, with time, of the congruence of geodesics and can be characterised in terms of the time evolution of some deformation vector [1, 3]. The analysis of this evolution ultimately leads to a local, first order, nonlinear and coupled set of equations for the ESR variables, known in the literature, as the Raychaudhuri equations [4].

The Raychaudhuri equations are well-known in the study of spacetime singularities in gravitation and cosmology [5]-[9]. An important consequence which emerges from the analysis of the equation for the expansion is the focusing of geodesics [10]. Moreover, the use of Raychaudhuri equations in the proofs of singularity theorems [11, 12] (through the notion of focusing of geodesics) is, today, a well-established fact [6, 7].

An important assumption about our forthcoming analysis may be noted here. In [1], we assumed that the medium (more precisely, the particles that make the medium) does not develop curvature due to the deformation (i.e the flat space remains flat). Here too, we assume that the deformations do not change the curvature of the medium.

Our plan in this paper is as follows. We first derive the Raychaudhuri equations in two dimensional curved spaces of constant curvature. We then recall the exact solutions of the geodesic equations in the spherical and hyperbolic geometry, which are required as inputs while solving the Raychaudhuri equations. The appearance of a finite time singularity depending on the initial conditions on the expansion, shear and rotation variables is investigated analytically by including the effects of stiffness and viscous damping. The generic features of the solutions for different values of the parameters appearing in the Raychaudhuri equations and the initial conditions are clearly brought out and summarized. Some specific cases are also solved numerically and discussed in detail. The equations presented in this two dimensional case can easily be extended to curved higher dimensional spacetimes with
space-like sections of constant curvature (maximally symmetric spacetimes). This aspect of our work here is one of our motivating factors because, in future, we intend to carry out similar analysis in spacetimes which represent specific gravitational fields and arise as solutions in General Relativity.

Following the analysis of flows on constant curvature spaces, we move on to spaces of varying curvature. We discuss the specific example of flows on a torus for various classes of geodesics on it and comment on particular features in the solutions for the expansion, shear and rotation.

Finally, in the last section, we offer our comments and conclusions as well as our views on possible future work.

II. GEODESIC FLOWS ON TWO DIMENSIONAL SPACES WITH CONSTANT CURVATURE

A. Expansion, rotation and shear

The deformations in a two dimensional deformable medium can be characterised in terms of the time evolution of the deformation vector \( \xi^i \) (where \( i = 1, 2 \)) which connects two infinitesimally separated points of the medium at any time instant \([1, 3]\). The points at a later time instant move an infinitesimal distance along the respective instantaneous velocity directions \([1]\). For the purpose, the time evolution of the deformation vector for short time intervals can be given as,

\[
\frac{d\xi^i}{dt} = B^i_j(t) \xi^j + \mathcal{O}(\Delta t^2),
\]

(2.1)

where the time dependent second rank tensor \( B^i_j(t) \) governs the dynamics of the deformable medium. Further, using the equation (2.1), the second order derivative of the deformation vector \( \xi^i \) with respect to time can be calculated as follows,

\[
\left( \frac{dB^i_j}{dt} + B^i_k B^k_j \right) \xi^j = \ddot{\xi}^i.
\]

(2.2)

In order to describe the congruence’s behaviour, the evolution tensor \( B^i_j \) in equations (2.1) and (2.2) can be decomposed as the linear combination of expansion, shear and rotation in the following form,

\[
B^i_j = \frac{1}{2} \theta \delta^i_j + \sigma^i_j + \omega^i_j,
\]

(2.3)
where $\theta$, $\sigma_i^j$ and $\omega_i^j$ represent, respectively, the expansion, shear and rotation. The kinematics of deformations can now be quantified in terms of expansion, shear and rotation (ESR) of the medium. The equation (2.3) can be written in the following $2 \times 2$ matrix form,

$$B_{ij}^i = \begin{pmatrix} \frac{1}{2} \theta & 0 \\ 0 & \frac{1}{2} \theta \end{pmatrix} + \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix} + \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix},$$

(2.4)

where $\sigma_+$ and $\sigma_\times$ represent the shear parameters while $\omega$ is the rotation parameter which along with the expansion scalar $\theta$, characterise the deformations of the medium [1, 3].

**B. The evolution equations**

The evolution equations for the congruence of geodesics consist of the Raychaudhuri equations for the ESR variables and the geodesic equations (corresponding to the metric of interest) which are presented below.

1. *Raychaudhuri equations*

In order to have the evolution (Raychaudhuri) equations for ESR variable, we need to consider a general expression for $\ddot{\xi}^i$ is as follows,

$$\ddot{\xi}^i = -R_t^i_{ljm}u^l u^m \xi^j - K_{ij} \xi^j - \beta \dot{\xi}^i,$$

(2.5)

where $K_{ij}$ and $\beta$ are the stiffness and viscous damping parameters in the medium, respectively. It may be noted that the first term on the right hand side of equation (2.5) is clearly because of the curvature contribution and is absent while dealing with the case of flat space [1]. The $K_{ij}$ in equation (2.5) has the form given as,

$$K_{ij} = \begin{pmatrix} k + k_+ & k_\times \\ k_\times & k - k_+ \end{pmatrix}.$$  

(2.6)

Using equations (2.2) and (2.5) lead to,

$$\dot{B}_{ij}^i + B_{ik}^i B_{kj}^k + K_{ij} + \beta B_{ij}^i = -R_t^i_{ljm}u^l u^m.$$  

(2.7)

In order to derive the evolution equations for ESR variables, let us consider a two dimensional metric in the following form,

$$ds^2 = \frac{dr^2}{1 - \kappa r^2} + r^2 d\varphi^2,$$

(2.8)
where $\kappa = 0, +1, -1$ denotes the curvature corresponding to the flat, spherical and hyperbolic geometries, respectively. Finally, using the decomposition of evolution tensor (2.3) in the equation (2.7), one obtains the Raychaudhuri equations for the ESR variables as follows,

\begin{align*}
\dot{\theta} + \frac{1}{2} \theta^2 + \beta \theta + 2 (\sigma^2 + \sigma_x^2 - \omega^2) + \kappa + 2k &= 0, \\
\dot{\sigma}_+ + (\beta + \theta) \sigma_+ - \frac{\kappa}{2} + k_+ &= 0, \\
\dot{\sigma}_x + (\beta + \theta) \sigma_x + k_x &= 0, \\
\dot{\omega} + (\beta + \theta) \omega &= 0.
\end{align*}

For $\kappa = 0$, these equations exactly reduce to the case of flat space time [1]. It may be recalled that $R_{ij} = g_{ij} R/2$ in two dimensions. The analytical solutions of the equations (2.9)-(2.12) can be found by solving the geodesics equations corresponding to the metric (2.8).

2. Geodesic equations

The geodesic equations derived from the metric (2.8) are as follows,

\begin{align*}
\ddot{r} + \frac{\kappa r}{(1 - \kappa r^2)} \dot{r}^2 - r (1 - \kappa r^2) \dot{\phi}^2 &= 0, \\
r \ddot{\phi} + 2 \dot{r} \dot{\phi} &= 0,
\end{align*}

where the dot denotes the derivative with respect to an affine parameter $\lambda$ which we shall identify, without any loss of generality, with the parameter $t$ quoted earlier. The equation (2.14) leads to $\dot{\phi} = c/r^2$ where $c$ is an integration constant. These equations can be solved separately for $\kappa = \pm 1$ which are discussed below.

(i) For $\kappa = 1$:

Let us consider $r = \sin \phi$, which transforms the equations (2.13) and (2.14) as follows,

\begin{align*}
\ddot{\phi} - \sin \phi \cos \phi \dot{\phi}^2 &= 0, \\
\ddot{\phi} + 2 \cot \phi \dot{\phi}^2 &= 0.
\end{align*}

The equations (2.15) and (2.16) are satisfied with $\phi = \pi/2$ and $\dot{\phi} = d$ (where $d$ is a constant) which represents an equatorial great circle.
(ii) For $\kappa = -1$:

With $r = \sinh \phi$, the equations (2.13) and (2.14) transform in the following form,

$$\ddot{\phi} - \sinh \phi \cosh \phi \dot{\varphi}^2 = 0, \quad (2.17)$$

$$\ddot{\varphi} + 2 \coth \phi \dot{\phi} \dot{\varphi} = 0. \quad (2.18)$$

The equations (2.17) and (2.18) have a solution with $\phi = \lambda$ and $\varphi = f$, where $f$ is a constant. In order to evaluate the constants involved in the solutions of the geodesic equations, the normalisation condition $g_{\alpha\beta}u^\alpha u^\beta = 1$ leads to,

$$\dot{r}^2 + r^2 (1 - \kappa r^2) \dot{\varphi}^2 = (1 - \kappa r^2). \quad (2.19)$$

In view of the solution for geodesic equations (2.15) and (2.16) with (2.19), the constant $c = d = \pm 1$ for the case $\kappa = 1$. On the other hand $c = 0$ for $\kappa = -1$ for both the solutions considered for the geodesic equations (2.17) and (2.18).

It is worth noting that for the corresponding case in $2+1$ dimensions with $-dt^2$ added in the line element given in equation (2.8), one will obtain an additional geodesic equation corresponding to $t$ as $\ddot{t} = 0$. Thus, $t = a\lambda + b$ and with the choices $a = 1$ and $b = 0$, we get back the above set of the geodesic equations. So, the generic features of geodesics, will remain same as they were in two dimensions. One may also note in the passing that in the case of cylindrical geometry with constant radius $\rho$ represented by the metric $ds^2 = dz^2 + \rho^2 d\vartheta^2$, the geodesic equations are obtained as $\ddot{z} = 0$ and $\ddot{\vartheta} = 0$. In such a case, the Raychaudhuri equations essentially reduce to those obtained for the case of flat (Euclidean) space (see [1]).

C. Analytical solutions of Raychaudhuri equations

For the above mentioned choice of solutions of geodesic equations, the Raychaudhuri equations (2.9 - 2.12) for ESR variables reduce to the following simplified form,

$$\dot{\theta} + \frac{1}{2} \theta^2 + \beta \theta + 2(\sigma_+^2 + \sigma_-^2 - \omega^2) + \kappa + 2k = 0, \quad (2.20)$$

$$\dot{\sigma}_+ + (\beta + \theta) \sigma_+ - p = 0, \quad (2.21)$$

$$\dot{\sigma}_- + (\beta + \theta) \sigma_- - q = 0, \quad (2.22)$$

$$\dot{\omega} + (\beta + \theta) \omega = 0, \quad (2.23)$$
where \( p = \frac{\kappa}{2} - k_+ \) and \( q = -k_\times \). The equations (2.20)-(2.23) represent the dynamics of the ESR variables. It may be mentioned here that for flat (Euclidean) geometry, the evolution equations for the ESR variables (see [1]) are recovered by putting \( \kappa = 0 \) and redefining \( p = -k_+ \).

Using the substitution \( \theta = \ddot{Z}/\dot{Z} - \beta \) in (2.20)-(2.23) leads to,

\[
\frac{1}{2} \dddot{Z} \dot{Z} - \frac{1}{4} \ddot{Z}^2 + \dddot{Z}^2 (\sigma_+^2 + \sigma_\times^2 - \omega^2) + \left( \kappa + 2k - \frac{\beta^2}{2} \right) \frac{\ddot{Z}^2}{2} = 0, \tag{2.24}
\]

\[
\dot{Z}\dot{\sigma}_+ + \ddot{Z}\sigma_+ - p\dot{Z} = 0, \tag{2.25}
\]

\[
\dot{Z}\dot{\sigma}_\times + \ddot{Z}\sigma_\times - q\dot{Z} = 0, \tag{2.26}
\]

\[
\dot{Z}\dot{\omega} + \ddot{Z}\omega = 0. \tag{2.27}
\]

Solving for \( \sigma_+ \), \( \sigma_\times \) and \( \omega \), respectively, from (2.25)-(2.27), and substituting in (2.24) followed by one further time differentiation yields,

\[
\dddot{Z} + 2\alpha_2 \ddot{Z} + \alpha_1 Z = D, \tag{2.28}
\]

where \( \alpha_1 = 4(p^2 + q^2) \), \( \alpha_2 = \kappa + 2k - \beta^2/2 \), and \( D = -4(pE + qF) \) with \( E \) and \( F \) as constants of integration. Now, one can write the general solution of the ESR variables as,

\[
\theta = \frac{\ddot{Z}}{Z} - \beta, \quad \sigma_+ = \frac{pZ + E}{Z}, \quad \sigma_\times = \frac{qZ + F}{Z}, \quad \omega = \frac{G}{Z}, \tag{2.29}
\]

where \( G \) is also a constant of integration, and \( Z(t) \) is the solution of the differential equation.

This scheme of solution of course leads to more number of constants of integration than conditions available from the initial values of the ESR variables. Therefore, one has to also ensure the satisfaction of the individual equations in (2.20)-(2.23), which yields appropriate number of constraint equations required for the solution of all the constants of integration involved.

It is evident from (2.28) that the solution of \( Z(t) \) is dependent on the values of \( \alpha_1 \) and \( \alpha_2 \), which are in turn dependent on the curvature, stiffness and damping parameter values. The solutions for all possible cases are straightforward, though may be tedious. Therefore, for the purpose of illustration, we consider only some special cases in detail in the rest of this section. The nature of solutions for all possible cases are discussed and summarized separately later.
Case I: \( k = k_+ = k_x = 0 \) and \( \beta = 0 \)

We start with the case without stiffness and damping. Here, we have some simplifications in (2.28) with \( \alpha_1 = 1 \) and \( \alpha_2 = \kappa \). The solutions for the spherical and hyperbolic geometries are presented below.

**Spherical geometry \((\kappa = 1)\)**

The exact solutions for the ESR variables in this case are obtained as follows. The solution for the equation for \( Z(t) \) is obtained as:

\[
Z(t) = (A_1 + B_2 + B_1 t) \sin t + (B_1 - A_2 - B_2 t) \cos t - 2E. \tag{2.30}
\]

From this expression, using the above-mentioned definitions for the ESR variables we obtain:

\[
\theta = \frac{(A_2 + B_1 + B_2 t) \cos t + (B_2 - A_1 - B_1 t) \sin t}{(A_1 + B_1 t) \cos t + (A_2 + B_2 t) \sin t}, \tag{2.31}
\]

\[
\sigma_+ = \frac{2E - \{(A_1 + B_2 + B_1 t) \sin t + (B_1 - A_2 - B_2 t) \cos t\}}{2[(A_1 + B_1 t) \cos t + (A_2 + B_2 t) \sin t]}, \tag{2.32}
\]

\[
\sigma_\times = \frac{F}{[(A_1 + B_1 t) \cos t + (A_2 + B_2 t) \sin t]}, \tag{2.33}
\]

\[
\omega = \frac{G}{[(A_1 + B_1 t) \cos t + (A_2 + B_2 t) \sin t]}, \tag{2.34}
\]

where, \( A_1, B_1, A_2, B_2, E, F \) and \( G \) are the integration constants and will also be used throughout in the text for the analytical solutions of other cases as well. These can be defined in terms of the initial conditions on the ESR variables at \( t = 0 \) which are represented as \( \theta_0, \sigma_{+0}, \sigma_{\times0} \) and \( \omega_0 \). The integration constants are calculated by putting back the solutions (2.31)-(2.34) in equation (2.9) and using the initial conditions on the ESR variables as,

\[
A_1 = \frac{4}{(\theta_0^2 + 4\sigma_{+0}^2) - 4I_0}, \quad A_2 = \frac{A_1(\theta_0 + 2\sigma_{+0})}{2}, \quad B_1 = \frac{A_1(\theta_0 - 2\sigma_{+0})}{2}, \]

\[
B_2 = 1, \quad E = 0, \quad F = A_1 \sigma_{\times0}, \quad G = A_1 \omega_0, \tag{2.35}
\]

where \( I_0 = \sigma_{\times0}^2 - \omega_0^2 \). The constant \( B_2 \) is arbitrarily chosen as unity.

**Hyperbolic geometry \((\kappa = -1)\)**

In this case, the exact solutions of the ESR variables can be expressed as,

\[
\theta = \frac{(A_2 + B_1 + B_2 t) \cosh t + (A_1 + B_2 + B_1 t) \sinh t}{(A_1 + B_1 t) \cosh t + (A_2 + B_2 t) \sinh t}, \tag{2.36}
\]
\[ \sigma_+ = \frac{2E - \{(A_1 - B_2 + B_1 t) \sinh t + (A_2 - B_1 + B_2 t) \cosh t\}}{2\{(A_1 + B_1 t) \cosh t + (A_2 + B_2 t) \sinh t\}}, \] (2.37)

\[ \sigma_\times = \frac{F}{\{(A_1 + B_1 t) \cosh t + (A_2 + B_1 t) \sinh t\}}, \] (2.38)

\[ \omega = \frac{G}{\{(A_1 + B_1 t) \cosh t + (A_2 + B_1 t) \sinh t\}}. \] (2.39)

The integration constants for this case remain same as given by (2.35). It may be noted that, in the absence of stiffness and damping, the solutions of the ESR variables in spherical geometry are in terms of harmonic functions, while the solutions are composed of hyperbolic functions in hyperbolic geometry. Therefore, while in spherical geometry a finite time singularity is inevitable (irrespective of initial conditions), in hyperbolic geometry a solution singularity may be absent for specific initial conditions. This fact can be noted by looking at the values of \( t \) for which the denominator can go to zero in each case. For the spherical geometry the denominator goes to zero if:

\[ \tan t = \frac{A_1 + B_1 t}{A_2 + B_2 t}. \] (2.40)

The right hand side has value \( \frac{A_1}{A_2} \) at \( t = 0 \), and as \( t \to \infty \), we get \( -\frac{B_1}{B_2} \). Since the left hand side is not a bounded function there will always be solutions of the above equation at finite values of \( t \). On the other hand, for a hyperbolic geometry, we have,

\[ \tanh t = -\frac{A_1 + B_1 t}{A_2 + B_2 t}. \] (2.41)

The left hand side is now a bounded function (takes on values between \(-1\) and \(+1\)). Thus, if \( A_1 > A_2 \) and \( B_1 > B_2 \), we find that there are no solutions which reach a finite time singularity.

**Case II**: \( \alpha_2 = 0 \) and \( \kappa = \pm 1 \)

This case involves both stiffness and damping. The choice \( \alpha_2 = 0 \) implies \( 2k - \beta^2/2 = \mp 1 \) (for \( \kappa = \pm 1 \)) which signifies a relation between the damping parameter \( \beta \) and the parameter \( k \) related to the elastic force. Hence, one might be tempted to note a similarity between this case and that of standard critical damping (say for a harmonic oscillator).

The solution of equation (2.28) then, consequently leads to the following ESR variables,

\[ \theta = \frac{2\gamma [e^{\gamma t}\{B_1 \cos \gamma t - A_1 \sin \gamma t\} + e^{-\gamma t}\{A_2 \sin \gamma t - B_2 \cos \gamma t\}]}{e^{\gamma t}\{D_1 \sin \gamma t + C_1 \cos \gamma t\} + e^{-\gamma t}\{D_2 \cos \gamma t - C_2 \sin \gamma t\}} - \beta, \] (2.42)
\[
\sigma_+ = \frac{E - (\frac{1}{2} + k_\gamma)[\tilde{G} + e^{\gamma t}\{A_1 \cos \gamma t + B_1 \sin \gamma t\} + e^{-\gamma t}\{A_2 \cos \gamma t + B_2 \sin \gamma t\}]}{\gamma[e^{\gamma t}\{D_1 \sin \gamma t + C_1 \cos \gamma t\} + e^{-\gamma t}\{D_2 \cos \gamma t - C_2 \sin \gamma t\}]}, \quad (2.43)
\]
\[
\sigma_\times = \frac{F - k_x[\tilde{G} + e^{\gamma t}\{A_1 \cos \gamma t + B_1 \sin \gamma t\} + e^{-\gamma t}\{A_2 \cos \gamma t + B_2 \sin \gamma t\}]}{\gamma[e^{\gamma t}\{D_1 \sin \gamma t + C_1 \cos \gamma t\} + e^{-\gamma t}\{D_2 \cos \gamma t - C_2 \sin \gamma t\}]}, \quad (2.44)
\]
\[
\omega = \frac{G}{\gamma[e^{\gamma t}\{D_1 \sin \gamma t + C_1 \cos \gamma t\} + e^{-\gamma t}\{D_2 \cos \gamma t - C_2 \sin \gamma t\}]}, \quad (2.45)
\]

where \( \gamma = \alpha^{1/4} \) and \( C_1 = A_1 + B_1, \ D_1 = B_1 - A_1, \ C_2 = A_2 + B_2, \ D_2 = B_2 - A_2, \ E_1, \ E_2 \) and \( G_1 \) are the integration constants. These constants are calculated by putting back the solutions in the corresponding evolution equations and using the initial conditions of the ESR variables as follows,

\[
A_1 = \frac{1}{4}(2P - Q + 1), \quad B_1 = \frac{1}{4}(2P + Q - 1),
\]
\[
A_2 = \frac{1}{4\gamma}(\theta_0 + \gamma Q + \gamma), \quad B_2 = \frac{1}{4\gamma}(-\theta_0 + \gamma Q + \gamma),
\]
\[
E = \gamma \sigma_{+0}, \quad F = \gamma \sigma_{\times0}, \quad G = \gamma \omega_0, \quad (2.46)
\]

where \( P \) and \( Q \) are given as follows,

\[
P = -\frac{1}{2\gamma^3}[(1 + 2k_\gamma)\sigma_{+0} - 2k_x\sigma_{\times0}], \quad Q = \frac{1}{2\gamma^2}[\theta_0^2 - 2(\sigma_{+0}^2 + \sigma_{\times0}^2 - \omega_0^2)]. \quad (2.47)
\]

In calculating the integration constants \( E \) and \( F \), we have further used \( Z(0) = 0 \) without any loss of generality. It may be noted that the solution of \( Z(t) \) consists of one exponentially growing and the other exponentially decaying harmonic mode. Thus, there is a finite time at which \( \dot{Z}(t) = 0 \) irrespective of the initial conditions. This happens when the denominator in the above equations goes to zero implying,

\[
\tan \gamma t = \frac{C_1 e^{\gamma t} + D_2 e^{-\gamma t}}{C_2 e^{-\gamma t} - D_1 e^{\gamma t}}. \quad (2.48)
\]

The fact that the above condition can be satisfied for some values of \( t \) confirms that the ESR variables can have finite time singularities. It may also be noted that a difference in the above-mentioned forms of the ESR variables for the \( \kappa = 1 \) and \( \kappa = -1 \) cases arise only through the relation between the allowed values of \( k \) and \( \beta \) (i.e. \( 2k - \beta^2/2 = \mp 1 \) for \( \kappa = \pm 1 \)). Only the value of the additive constant piece \( \beta \) changes in the functional form of \( \theta \), when we change \( \kappa \) from +1 to -1. The various generic features based on the analytical solutions for all possible cases are presented in the next section.
D. Nature of solutions

The generic features of the solutions of (2.28) for different values of $\alpha_1$ and $\alpha_2$ are straightforward to obtain from the original fourth order differential equation. To see this, substitute $Z(t) = Y(t) + D/\alpha_1$ to get rid of the constant inhomogeneous term $D$. The equation for $Y(t)$ can now be easily solved by the substitution $Y(t) = e^{irt}$. We obtain the relation $r^4 - 2\alpha_2 r^2 + \alpha_1 = 0$ from which, by solving, we get:

$$r = \pm \left( \alpha_2 \pm \sqrt{\alpha_2^2 - \alpha_1} \right)^{1/2}. \quad (2.49)$$

Thus, the nature of the solutions crucially depends on the value of $\alpha_2^2 - \alpha_1$, the sign of $\alpha_2$ and also on whether $\alpha_1 > 0$ or $\alpha_1 = 0$.

The nature of possible solutions is schematically presented in the $\alpha_1$-$\alpha_2$ parameter plane in Fig. 2. Based on this figure, one can comment on the nature of solutions of the ESR.
\[
\alpha_2 > \alpha_1
\]

| Conditions | Nature of Solutions |
|------------|---------------------|
| \(\alpha_2 > \alpha_1\) | Finite time singularity irrespective of initial conditions |
| (i) \(\alpha > 0\): Region I | |
| \(\alpha_1 > 0\) | Initial condition dependent oscillatory solution, or finite time singularity |
| \(\alpha_1 = 0\) | |
| (ii) \(\alpha < 0\): Region III | |
| \(\alpha_1 > 0\) | Hyperbolic solutions: initial condition dependent stable solution, or finite time singularity |
| \(\alpha_1 = 0\) | Initial condition dependent stable solution, or finite time singularity |

| \(\alpha_2 = \alpha_1\) | Finite time singularity irrespective of initial conditions |
| \(\alpha_2 < \alpha_1\): Region II | Oscillatory solutions with one growing and one decaying mode |

| \(\alpha_2 > \alpha_1\) | Finite time singularity irrespective of initial conditions |
| \(\alpha > 0\): Region I | |
| \(\alpha_1 > 0\) | Initial condition dependent oscillatory solution, or finite time singularity |
| \(\alpha_1 = 0\) | |
| (ii) \(\alpha < 0\): Region III | |
| \(\alpha_1 > 0\) | Hyperbolic solutions: initial condition dependent stable solution, or finite time singularity |
| \(\alpha_1 = 0\) | Initial condition dependent stable solution, or finite time singularity |

\[
\alpha_2 = \alpha_1
\]

| \(\alpha_2 < \alpha_1\): Region II | Oscillatory solutions with one growing and one decaying mode |

TABLE I: Different possible nature of solutions of the ESR variables corresponding to different regions as marked in Fig. 2.

variables, as summarised in Table I.

It is worth noticing from the expression of \(\alpha_2 = \kappa + 2k - \beta^2/2\) that a large value of \(k\) in hyperbolic geometry can make the solutions similar to that in spherical geometry. This will occur for values of \(k\) such that \(\alpha_2 > \alpha_1 > 0\). On the other hand, a large value of \(\beta\) such that \(\alpha_2 > \alpha_1\) and \(\alpha_2 < 0\), in a spherical geometry the flow characteristics will resemble those in a hyperbolic geometry. Further, with suitable values of stiffness and damping, the evolution of the ESR variables in a curved space (spherical or hyperbolic) can exhibit features of evolution in a flat (Euclidean) space without stiffness and damping. This will occur whenever \(\alpha_2 = 0\), \(k_+ = -1/2\) and \(k_- = 0\).

The nature of solutions depending on initial conditions can also be easily determined. As a specific interesting example, consider the case with \(\alpha_1 = 0\) and \(\alpha_2 > 0\). In this case (2.28) reduces to,

\[
\ddot{Z} + 2\alpha_2 \dot{Z} = 0,
\]

which has the general solution,

\[
Z(t) = A_1 \cos \sqrt{2\alpha_2} t + B_1 \cos \sqrt{2\alpha_2} t + A_2 t + B_2,
\]

where \(A_1\), \(B_1\), \(A_2\) and \(B_2\) are constants of integration, which are determined from the
initial conditions. Now, as is evident from (2.29), the ESR variables will exhibit finite time singularity whenever \( \dot{Z} = 0 \). This will occur whenever \(|A_2| < \sqrt{2\alpha_2(A_1^2 + B_1^2)}\), which can now be explicitly written in terms of the initial conditions on the ESR variables. It may be noted that if there is no singularity, the ESR solution in this case will exhibit stable oscillations.

E. Numerical examples

In this section, we numerically investigate the nature of solutions for different initial conditions and parameter values (\(\alpha_1\) and \(\alpha_2\)) in different regions as indicated in Fig. 2.

![Graph A](image)

**FIG. 3:** Evolution of ESR variables with parameters \(k = 1.0, k_+ = k_- = 0, \beta = 2.0\), for (a) \(\kappa = 1\), and (b) \(\kappa = -1\).

1. \(\alpha_1 > 0\) and \(\alpha_2^2 = \alpha_1\)

   These parameter values lie exactly on the parabola in Fig. 2. When \(\alpha_2 > 0\), the solutions of \(Z(t)\) are composed of harmonic functions with secular terms. This makes the solution of the ESR variables singular in finite time irrespective of the initial
conditions, as shown in Fig. 3(a). On the other hand, when $\alpha_2 < 0$, the solutions of $Z(t)$ have hyperbolic functions with secular terms. In this case, it is possible to find initial conditions on the ESR variables for which the solutions are non-singular. One such case is shown in Fig. 3(b).

2. $\alpha_1 = 0$ and $\alpha_2^2 > 0$

In this case, the parameter values lie on the positive $y$-axis in Fig. 2. As discussed before, the ESR solutions can show finite time singularity or pure oscillatory modes, depending on the initial conditions. Both these cases are presented in Fig. 4(a) and (b). It is interesting to note that the difference in initial conditions in the two sub-plots is only in the rotation initial condition (i.e., $\omega_0$). The stabilizing effects of rotation (also observed in [1]) is evident once again.

![Graph showing evolution of ESR variables](image)

**FIG. 4:** Evolution of ESR variables with parameters $\kappa = 1$, $k = 1.0$, $k_+ = -0.5$, $k_\times = 0$, $\beta = 2.0$ for two different initial conditions.
FIG. 5: Evolution of ESR variables with parameters $k = 1.0$, $k_+ = 1.0$, $k_\times = 0.5$, $\beta = 2.0$ for (a) $\kappa = 1$ and (b) $\kappa = -1$, both of which lie in Region II in Fig. 2.

3. $\alpha_1 > 0$ and $\alpha_2 < \alpha_1$

These parameter values lie in Region II in Fig. 2. Hence, they will all show finite time singularity as discussed before.

Two such cases have been presented in Figs. 5(a) and (b) for spherical and hyperbolic geometries, respectively.

4. $\alpha_1 > 0$ and $\alpha_2 > \alpha_1$
We consider here parameter values in Region III in Fig. 2. The solutions of $Z(t)$ here are composed of hyperbolic functions. The nature of solutions of the ESR variables are now dependent on the initial conditions. One can, therefore, have finite time singularity, as shown in Fig. 6(a), or stable solutions, as observed in Fig. 6(b).

III. GEODESIC FLOWS ON A TORUS

In this section, we consider kinematics of deformations on a torus which represents a curved medium with varying curvature. The line element for a ring torus is given as,

$$ds^2 = (1 + a \cos \phi)^2 d\psi^2 + a^2 d\phi^2,$$

where $\psi, \phi \to [0, 2\pi]$ and $a = r_1/r_2$ with $r_1$ and $r_2$ as the minor and major radius of the torus, respectively.
1. Raychaudhuri equations

Using (2.7) and (3.1), the Raychaudhuri equations for ESR variables can now be obtained as follows,

\[
\dot{\theta} + \frac{1}{2} \theta^2 + \beta \theta + 2(\sigma_+^2 + \sigma_-^2 - \omega^2) + \frac{\cos \phi}{a(1 + a \cos \phi)} + 2k = 0, \tag{3.2}
\]

\[
\dot{\sigma}_+ + (\beta + \theta) \sigma_+ - \frac{\cos \phi}{2a(1 + a \cos \phi)} + k_+ = 0, \tag{3.3}
\]

\[
\dot{\sigma}_- + (\beta + \theta) \sigma_- + k_- = 0, \tag{3.4}
\]

\[
\dot{\omega} + (\beta + \theta) \omega = 0. \tag{3.5}
\]

One can try to solve these equations by simultaneously solving the geodesic equations corresponding to the metric (3.1).

2. Geodesic equations

The geodesic equations corresponding to the above metric are given by,

\[
\ddot{\psi} - \frac{2a \sin \phi}{(1 + a \cos \phi)} \dot{\psi} \dot{\phi} = 0, \tag{3.6}
\]

\[
\ddot{\phi} + \frac{(1 + a \cos \phi) \sin \phi}{a} \dot{\psi}^2 = 0. \tag{3.7}
\]

The first integrals of these geodesic equations are obtained as,

\[
\dot{\psi} = \frac{l_1}{(1 + a \cos \phi)^2}, \tag{3.8}
\]

\[
\dot{\phi} = \sqrt{-\frac{l_2}{a^2(1 + a \cos \phi)^2}} + l_2, \tag{3.9}
\]

where \(l_1\) and \(l_2\) are integration constants and their different choices lead to five different families of geodesics [13]. The normalisation \(g_{\alpha\beta} u^\alpha u^\beta = 1\) for this case yields \(l_2 = 1/a^2\), and the equation (3.7) is therefore only dependent on \(l_1\). We now consider some special cases in order to perform a detailed analysis of the deformations as obtained by integrating the Raychaudhuri equations (3.2)-(3.5).
3. Kinematics of ESR for certain geodesic families

We comment here on the nature of solutions of Raychaudhuri equations for certain geodesics families.

**Case I:** $\dot{\phi} = 0$ and $\dot{\psi} \neq 0$

This case corresponds to the inner equator geodesic of a torus. From equations (3.8) and (3.9), we have $\phi = \text{constant}$ and $\dot{\psi} = 1/l_1$.

![Geodesics on a torus](image)

**FIG. 7:** Geodesics on a torus considered for the numerical integration of the Raychaudhuri equations for (a) $a = 0.2$, (b) $a = 0.5$, and (c) $a = 0.7$.

The Raychaudhuri equations are given by (2.20)-(2.23) with the substitutions $\kappa = \gamma = \cos \phi/a(1 + a \cos \phi)$ (which is constant in this case), $p = \gamma/2 - k_\perp$ and $q = -k_\times$. The value of $\gamma$ can be positive, negative or zero. For positive (negative) value of $\gamma$, the solution can be obtained from the spherical (hyperbolic) solution as presented in the previous section. On the other hand, for $\gamma = 0$ (corresponding to $\phi = \pi/2, 3\pi/2$), the solution follows from the flat space solutions as obtained in [1]. Therefore, the kinematics of deformations are similar to the cases discussed previously.

**Case II:** $\dot{\phi} \neq 0$ and $\dot{\psi} = 0$

This case corresponds to the meridians of a torus. From equations (3.8) and (3.9), we now have $\dot{\phi} = 1/a$ and $\psi = \text{constant}$. The Raychaudhuri equations have the same structure as in Case I above with $\gamma = \gamma(\lambda)$. It is difficult to integrate these equations analytically because...
of the variation of $\gamma$ with $\lambda$. However, we can integrate them numerically.

We now present numerical solutions of the Raychaudhuri equations corresponding to different values of the dimensionless parameter $a$. The corresponding geodesics are shown in Fig. 7 which belong to two geodesic families on a torus [13]. Excluding the geodesics corresponding to the above mentioned Case I, the evolution of the ESR variables is observed to be oscillatory. These oscillations are basically due to periodic variations of $\gamma = R/2$ (where $R$ is Ricci scalar) with $\phi$. The frequency and amplitude of oscillations for small values of $a$ is high and reduces with the increase in $a$. The evolution of the ESR variables is presented in Fig. 8 for three different values of $a$. Solutions may or may not exhibit a finite time singularity, depending on the initial values of $\theta_0$ and $\omega_0$. The Raychaudhuri equations for other known families of geodesics on the torus are, similarly, difficult to solve analytically and can be studied numerically. We do not indulge in discussing them further in order to avoid repeating qualitatively similar conclusions.

![Graph](image)

**FIG. 8:** Evolution of ESR variables for different values of $a$ corresponding to the three geodesics shown in Fig. 7 ($l_1 = 0.3$, $l_2 = 1/a^2$).
IV. COMMENT ON EVOLUTION EQUATIONS IN THREE DIMENSIONAL MEDIA

The general expression for the evolution tensor $B_{ij}$ in in three dimensional media [1] is expressed as:

$$B_{ij} = \frac{1}{3} \theta \delta_{ij} + \sigma_{ij} + \omega_{ij},$$  \hspace{1cm} (4.1)

where the expansion ($\theta$), shear ($\sigma_{ij}$) and rotation ($\omega_{ij}$) in the equation (4.1) have the following $3 \times 3$ matrix form:

$$B^i_j = \begin{pmatrix} \frac{1}{3} \theta & 0 & 0 \\ 0 & \frac{1}{3} \theta & 0 \\ 0 & 0 & \frac{1}{3} \theta \end{pmatrix} + \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & -(\sigma_{11} + \sigma_{22}) \end{pmatrix} + \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.2)

It may be noted that, for simplicity, we have used the definition $\sigma^i_j = \sigma_{ij}$ and $\omega^i_1 = \omega_1$, $\omega^i_2 = \omega_2$ and $\omega^i_3 = -\omega_3$. In order to derive the evolution equations in three dimensions let us consider, the following three dimensional line element,

$$ds^2 = \frac{dr^2}{1 - \kappa r^2} + r^2(d\varphi^2 + \sin^2 \varphi d\chi^2).$$  \hspace{1cm} (4.3)

This line element represents three dimensional maximally symmetric spaces of constant curvature which are topologically $S^3 (\kappa = 1)$, $H^3 (\kappa = -1)$ or $R^3 (\kappa = 0)$.

The evolution equations (i.e. Raychaudhuri equations) for the expansion, shear and rotation can be derived following standard methods. There are nine coupled, first order, nonlinear equations involving $\theta$ (expansion), $\sigma_{11}, \sigma_{12}, \sigma_{22}, \sigma_{13}, \sigma_{23}$ (shear) and $\omega_1, \omega_2, \omega_3$ (rotation) and their first derivatives.

The geodesic equations in the above mentioned line element are as follows,

$$\ddot{r} + \frac{\kappa r}{(1 - \kappa r^2)} \dot{r}^2 - r (1 - \kappa r^2) \dot{\varphi}^2 - r (1 - \kappa r^2) \sin^2 \varphi \dot{\chi}^2 = 0,$$  \hspace{1cm} (4.4)

$$r \ddot{\varphi} + 2 \dot{r} \dot{\varphi} - r \sin \varphi \cos \varphi \dot{\chi}^2 = 0,$$  \hspace{1cm} (4.5)

$$r \ddot{\chi} + 2 \dot{r} \dot{\chi} + 2 \cot \varphi \dot{\varphi} \dot{\chi} = 0.$$  \hspace{1cm} (4.6)

For the three dimensional case, it may be noted that the equations (2.13) and (2.14) are modified with additional terms and we also have a third equation (4.6). These equations reduce to the geodesic equations for the two dimensional case for a constant value of $\chi$.  \hspace{1cm} (4.3)
We make a particular choice of geodesics for which \( \chi = \text{constant}, \dot{\phi} = d, r = 1 \) and with \( \kappa = 1 \) (i.e. the case of spherical geometry). Further, \( K_{ij} = k\delta_{ij} \). It is also possible to consistently choose \( \sigma_{13} = \sigma_{23} = 0, \omega_1 = \omega_2 = 0 \). With all of these choices, the evolution equations for the five remaining variables turn out to be,

\[
\dot{\theta} + \frac{1}{3} \theta^2 + \beta \theta + 2(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{12}^2 + \sigma_{11}\sigma_{22} - \omega_3^2) + 3k + 6 = 0, \tag{4.7}
\]

\[
\dot{\sigma}_{11} + (\beta + \frac{2}{3} \theta) \sigma_{11} + \frac{1}{3}(\sigma_{11}^2 + \sigma_{12}^2 - \omega_3^2) - \frac{2}{3}(\sigma_{22}^2 + \sigma_{11}\sigma_{22}) - 2 = 0, \tag{4.8}
\]

\[
\dot{\sigma}_{12} + (\beta + \frac{2}{3} \theta + \sigma_{11} + \sigma_{22}) \sigma_{12} = 0, \tag{4.9}
\]

\[
\dot{\sigma}_{22} + (\beta + \frac{2}{3} \theta) \sigma_{22} + \frac{1}{3}(\sigma_{21}^2 + \sigma_{22}^2 - \omega_3^2) - \frac{2}{3}(\sigma_{11}^2 + \sigma_{11}\sigma_{22}) - 2 = 0, \tag{4.10}
\]

\[
\dot{\omega}_3 + (\beta + \frac{2}{3} \theta + \sigma_{11} + \sigma_{22}) \omega_3 = 0. \tag{4.11}
\]

These equations, as well as more general cases can indeed be solved numerically. We have worked through some examples (which we do not quote here). Generic features are not drastically different.

V. SUMMARY AND SCOPE

In this article, we have investigated the kinematics of flows on curved, deformable media by solving the evolution equations for the expansion, shear and rotation of geodesic congruences in two dimensional spaces of constant as well as varying curvature. Let us briefly point out the novelties that arise while considering flows on curved spaces.

- In [1], where we looked at flows on flat space, geodesics were straight lines and the need for solving the geodesic equations did not arise. Here, however, this is a necessity and we have looked at some simple solutions of the geodesic equations which are required as inputs in the evolution equations for ESR.

- In a flat space setting, the evolution equations do not have a contribution from geometric terms involving Ricci and Weyl. Here, though any contribution from the Weyl term is absent (two dimensions !), the Ricci term does indeed contribute. Thus, curvature effects on the nature of the flows do appear through the non-zero value of \( \kappa \).
• The analytic solutions obtained for the special cases illustrate the possibilities of the existence of finite time singularities in the geodesic congruence under study. We are able to point out, in a fairly general setting, the conditions on the parameters which imply the existence of finite time singularities/nonsingular solutions. The representation of regions in parameter space where singular/non-singular solutions exist is also obtained and explained.

• A major difference between flows in positive and negative curvature spaces is the fact, that in the latter, we do have the possibility of exclusively non-singular solutions. This is illustrated through the analytic solutions as well as the numerical results where we explicitly show the differences that arise between results in spaces of positive and negative curvature.

• The stabilizing effect of rotation, as observed in the flat space analysis in [1] persists. We have illustrated this in our numerical solutions and plots thereof.

• The role of stiffness, damping and the initial conditions are pointed out in each case with representative plots.

• In the case of flows on the torus (varying curvature) the oscillatory nature of the inhomogeneous term in the Raychaudhuri equations leads to oscillations in the ESR variables. The nature of the oscillations (i.e. amplitude and frequency) seem to depend on the parameter $a$. Further, the solutions exhibit finite time singularities and the role of rotation is expected to be similar, i.e., it can inhibit the formation of finite time singularities.

Finally, it is customary to ask in conclusion—what next? We have mentioned briefly the equations in three dimensional spaces of constant curvature and pointed out that the results will be similar to those in two dimensions. It will surely be of greater value if a complete D-dimensional analysis is carried out. In two dimensions, it will be useful to look at further examples of flows in spaces with varying curvature—in particular we can look at higher genus tori and flows on them. Furthermore, a case of physical interest could be the kinematics of geodesic flows in black holes spacetimes, where the spaces are not usually of constant curvature and also involve a Lorentzian signature line element.

We hope to report on some of the above questions in the near future.
Acknowledgments

The authors thank the Department of Science and Technology (DST), Government of India for financial support through a sponsored project (grant number: SR/S2/HEP-10/2005).

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