A majority of elliptic curves over $\mathbb{Q}$ satisfy the 
Birch and Swinnerton-Dyer conjecture

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Abstract

We prove that a majority (in fact, > 66%) of all elliptic curves over $\mathbb{Q}$, when ordered by height, satisfy the Birch and Swinnerton-Dyer rank conjecture.

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1 Introduction

Any elliptic curve $E$ over $\mathbb{Q}$ is isomorphic to a unique curve of the form $E_{A,B} : y^2 = x^3 + Ax + B$, where $A, B \in \mathbb{Z}$ and for all primes $p$: $p^6 \mid B$ whenever $p^4 \mid A$. The (naive) height $H(E_{A,B})$ of the elliptic curve $E = E_{A,B}$ is then defined by

$$H(E_{A,B}) := \max\{4|A^3|, 27B^2\}.$$

The purpose of this article is to prove the following theorem:

**Theorem 1** A majority of elliptic curves over $\mathbb{Q}$, when ordered by height, satisfy the Birch and Swinnerton-Dyer rank conjecture.
As a consequence of our methods, we also obtain:

**Theorem 2**  A majority of elliptic curves over \(\mathbb{Q}\), when ordered by height, have finite Tate–Shafarevich group.

In fact, we will prove that the words “A majority” in Theorems 1 and 2 can be replaced with “At least 66.48%”.

More precisely, for any elliptic curve \(E\) over \(\mathbb{Q}\), let us denote by \(\text{rk}(E)\) the algebraic rank of \(E\), i.e., the rank of the Mordell–Weil group \(E(\mathbb{Q})\). Let \(\text{rk}_{\text{an}}(E)\) denote the analytic rank of \(E\), i.e., the order of vanishing at the central critical point of the \(L\)-function of \(E\) (an entire function by the modularity theorem [28, 25, 7]). Let \(\text{III}(E)\) denote the Tate–Shafarevich group of \(E\). Then we prove

\[
\liminf_{X \to \infty} \frac{\# \{ E/\mathbb{Q} : \text{rk}(E) = \text{rk}_{\text{an}}(E), \text{III}(E) \text{ is finite}, \text{ and } H(E) < X \}}{\# \{ E/\mathbb{Q} : H(E) < X \}} > 66.48%,
\]

although this percentage can likely be improved with a more careful analysis (see, e.g., the last paragraph of this introduction).

We also obtain new lower bounds for the proportion of elliptic curves having both algebraic and analytic rank zero, and for the proportion having both algebraic and analytic rank one:

**Theorem 3** When ordered by height, at least 16.50% of elliptic curves over \(\mathbb{Q}\) have algebraic and analytic rank zero, and at least 20.68% of elliptic curves have algebraic and analytic rank one.

As a consequence, we also obtain a new lower bound on the (lim inf of the) average rank of all elliptic curves, when ordered by height:

**Corollary 4** The average (algebraic or analytic) rank of elliptic curves over \(\mathbb{Q}\), when ordered by height, is at least 0.2068.

The observant reader will have noticed that the sum of the percentages in Theorem 3 is less than the 66.48% stated above for the majority in Theorems 1 and 2. This is because our method, described below, enables us to prove better lower bounds on the proportion of curves that have analytic rank 0 or 1 than for the sum of individual lower bounds on the proportions of curves having analytic rank 0 and those having analytic rank 1!

The proofs of Theorems 1, 2, and 3 make combined use of results from [3], [4], [10], [21], [22], [24], and [30]. The works [21] and [22] allow one to deduce sufficient \(p\)-adic conditions for an elliptic curve to have algebraic and analytic rank zero, assuming \(p \geq 3\). The works [24] and [30] (which also rely on [21] and [22], as well as on the Gross–Zagier formula [13], its extension to Shimura curves by Shouwu Zhang [29], Kolyvagin’s theory of Euler systems of Heegner points [14], and the variant described by Bertolini–Darmon [1]) give sufficient \(p\)-adic conditions for an elliptic curve to have algebraic and analytic rank one, assuming \(p \geq 5\). In particular, one can deduce certain sufficient conditions for an elliptic curve to have algebraic and analytic rank zero or one, in accordance with whether its \(p\)-Selmer group has rank zero or one. We deduce two such theorems—namely, Theorems 5 and 9—in §2. On the other hand, the works [3] and [4] allow one to determine the average sizes of \(p\)-Selmer groups, for \(p = 3\) and \(5\), respectively, in any large family of elliptic curves defined by congruence conditions (see [2, 3] for the definition of a large family).

We combine these results as follows. Aside from requiring the \(p\)-Selmer groups to have rank zero or one, the conditions required of the elliptic curves in Theorems 5 and 9 are either congruence conditions (e.g., having good ordinary or multiplicative reduction at \(p\)) or conditions that hold for 100% of elliptic curves (e.g., \(E[p]\) being an irreducible Galois representation). In some cases,
indefinitely many congruence conditions are required, but nevertheless the families are large in the sense of [2,3]. We then estimate the density of the elliptic curves in the large families defined by the congruence conditions in these theorems. In both cases, the density is shown to be close to 80%. Combined with the results from [3] and [4] on average sizes of $p$-Selmer groups in large families, this is then already enough to deduce that a sizable proportion of curves have algebraic and analytic rank zero or algebraic and analytic rank one.

However, to deduce that there are many curves of analytic rank zero and many curves of analytic rank one, we need to know that there are a large number of curves with root number +1 and root number −1. In fact, using the results of [4], we show that each of the large families of elliptic curves that we consider contains a large subfamily of density at least 55% for which the root numbers are equidistributed, when the curves are ordered by height. Using this and the theorem of Dokchitser and Dokchitser [10] on the correspondence between root numbers and $p$-Selmer parity, and then optimizing the combinatorics, enables us to deduce that many elliptic curves have algebraic and analytic rank zero, many have algebraic and analytic rank one, and most have algebraic and analytic rank zero or algebraic and analytic rank one, yielding Theorems 1 and 3 and Corollary 4. Since we show that the majority of elliptic curves over $\mathbb{Q}$ have analytic rank at most one, we then obtain Theorem 2 through the work of Kolyvagin et al. proving the finiteness of the Tate-Shafarevich group in these cases.

In the earlier papers [3] and [5], it was shown that a positive proportion of elliptic curves, when ordered by height, have algebraic and analytic rank zero, and a positive proportion have algebraic and analytic rank one, respectively. In the rank one case, the proof relied on a different $p$-adic criterion, proved in [20], for an elliptic curve to have rank one, together with an equidistribution result on the local images of Selmer elements. We expect that forthcoming work that combines some of the techniques of [20] and [30] and extends the results to cases of supersingular reduction will substantially improve the percentages given in this paper (see §4). It is also worth noting that if the Selmer average results of [2], [3], and [4] could be extended to infinitely many primes, then the words “A majority” in Theorems 1 and 2 could be replaced with “100%” (see §4, Theorem 27).

2 Preliminaries

In this section, we collect the results that we need as ingredients for the proofs of Theorems 1–3.

2.1 $p$-adic conditions for an elliptic curve to have algebraic and analytic rank 0

We begin with the following criterion, deduced from results in [22] and [21], which gives a sufficient condition for an elliptic curve over $\mathbb{Q}$ to have algebraic and analytic rank zero:

**Theorem 5** Let $p$ be an odd prime. Let $E$ be an elliptic curve over $\mathbb{Q}$ with conductor $N$ such that:

(a) $E$ has good ordinary or multiplicative reduction at $p$;

(b) $E[p]$ is an irreducible $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module;

(c) there is at least one prime $\ell \neq p$ such that $\ell \nmid N$ and $E[p]$ is ramified at $\ell$;

(d) the $p$-Selmer group $S_p(E)$ of $E$ is trivial.

Then the rank and analytic rank of $E$ are both equal to 0.
Proof: If the $p$-Selmer group $S_p(E)$ of $E$ is trivial, then so is the $p^\infty$-Selmer group (denoted $\text{Sel}_{p^\infty}(E/\mathbb{Q})$ and $\text{Sel}_{p^\infty}(E)$ in [22] and [21], respectively), and hence the Mordell–Weil group $E(\mathbb{Q})$ is finite. It also follows from [22] Thm. 2(b)] (for the case of good reduction at $p$) and [21] Thm. B] (for the case of multiplicative reduction at $p$) that under the hypotheses on $E$ stated in the theorem, we have $L(E,1) \neq 0$. \square

Remark 6 Work in progress [23] should also allow $E$ to have good supersingular reduction, remove the condition (b), and replace (c) with just the existence of one prime $\ell \neq p$ at which $E$ has potentially multiplicative or supercuspidal reduction.

Remark 7 Suppose $\ell \parallel N$ and $\ell \neq p$. The condition that $E[p]$ be ramified at $\ell$ is equivalent to $p \nmid \text{ord}_\ell(\Delta_E)$ for a minimal discriminant $\Delta_E$ of $E$ at $\ell$ [19] Prop. V.6.1 & Ex. V.5.13. In particular, for $\ell \geq 5$, a Weierstrass equation $E_{A,B} : y^2 = x^3 + Ax + B$ that satisfies $\ell^6 \nmid B$ if $\ell^4 \nmid A$ is minimal at $\ell$, and so if $\ell \parallel N$ then $E_{A,B}[p]$ is ramified at $\ell$ if and only if $p \nmid \text{ord}_\ell(\Delta(E_{A,B}))$. This will be used later to interpret hypothesis (c) of Theorem 5 (and hypotheses (c) and (d) of Theorem 9 below) as congruence conditions at $\ell$.

Remark 8 Both [22] Thm. 2(b)] and [21] Thm. B], on which the proof of Theorem 3 relies, identify the power of $p$ dividing the product of the order of the $p^\infty$-Selmer group $\text{Sel}_{p^\infty}(E)$ of $E$ and the Tamagawa factors of $E$ as the power of $p$ dividing $L(E,1)/\Omega_E$, where $\Omega_E$ is the real period of $E$. As explained in [21], this relation is a consequence of the Iwasawa–Greenberg main conjecture for $E$ (which is the main result of [22] and [21]). One consequence of this equality of powers of $p$ is that if $\text{Sel}_{p^\infty}(E)$ is finite, then $L(E,1) \neq 0$. To conclude further that the Tate–Shafarevich group $\text{III}(E)$ of $E$ is finite then requires an appeal to the work of Gross–Zagier [13] and Kolyvagin [14] Thm. A and Cor. B].

2.2 $p$-adic conditions for an elliptic curve to have algebraic and analytic rank 1

Next, for $p \geq 5$ we have sufficient $p$-adic conditions, deduced from [30] and [24], for an elliptic curve over $\mathbb{Q}$ to have algebraic and analytic rank one:

**Theorem 9** Let $p \geq 5$ be a prime. Let $E$ be an elliptic curve over $\mathbb{Q}$ with conductor $N$ such that:

(a) $E$ has good ordinary or multiplicative reduction at $p$;

(b) $E[p]$ is an irreducible $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module;

(c) for all primes $\ell \parallel N$ such that $\ell \equiv \pm 1 \pmod{p}$, $E[p]$ is ramified at $\ell$;

(d) if $N$ is not squarefree, then there exist at least two prime factors $\ell \parallel N$ with $\ell \neq p$ and where $E[p]$ is ramified;

(e) if $E$ has multiplicative reduction at $p$ then $E[p]$ is not finite at $p$, and if $E$ has split multiplicative reduction at $p$ then the $p$-adic Mazur–Tate–Teitelbaum $\mathcal{L}$-invariant $\mathcal{L}(E)$ of $E$ satisfies $\text{ord}_p(\mathcal{L}(E)) = 1$;

(f) the $p$-Selmer group $S_p(E)$ of $E$ has order $p$.

Then the rank and analytic rank of $E$ are both equal to 1.
the rank of $E$ that once it is known that $E$ rank one and finite Tate–Shafarevich group follows from the main results of [29]. We note in passing in the setting of [13] are also non-torsion, but they may have indices divisible by $p$.

We recall what is meant by a “large family” of elliptic curves. For each prime $p$, let $\Sigma = (\Sigma_\ell)_{\ell}$, where $\Sigma_\ell$ be a closed subset of $(\mathbb{Z}/\ell\mathbb{Z})^2 : \Delta(A,B) := -4A^3 - 27B^2 \neq 0$ with boundary of measure 0. To such a collection $\Sigma = (\Sigma_\ell)_{\ell}$, we associate the set $F_\Sigma$ of elliptic curves over $\mathbb{Q}$, where $E_{A,B} \in F_\Sigma$ if and only

Proof: By a theorem of Cassels, the $p^\infty$-Selmer group of $E$ is isomorphic as a $\mathbb{Z}_p$-module to $(\mathbb{Q}_p/\mathbb{Z}_p)^r \oplus F \oplus F$ for some integer $r$ (expected to be the rank of $E(\mathbb{Q})$) and a finite group $F$. Since $E(\mathbb{Q})[p]$ is trivial under (b), the $p$-Selmer group $S_p(E)$ is the $p$-torsion of the $p^\infty$-Selmer group. Hence if $S_p(E)$ has order $p$, then $F = 0$ and $r = 1$. It follows from this observation and [30, Thm. 1.3] (for the case of good reduction) and [24, Thm. 1.1] (for the case of multiplicative reduction) that under the hypotheses stated for $E$ in the theorem, we have $\text{ord}_{s=1} L(E,s) = 1$ and the rank of $E(\mathbb{Q})$ is 1. □

Remark 10 We expect that combining the methods used to prove Theorem 9 with the approach in [20] should allow $E$ to have supersingular reduction at $p$. Furthermore, it should be possible to extend this result to $p = 3$ under the additional restrictions of good reduction at $p$ and that the image of the local restriction $S_p(E) \to E(\mathbb{Q}_p)/pE(\mathbb{Q}_p)$ not lie in the image of $E(\mathbb{Q}_p)[p]$. These extensions require corresponding extensions of the main results in [27].

Remark 11 We recall that $E[p]$ is “finite at $p$” if as a $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation, $E[p]$ is isomorphic to the representation on the $\mathbb{Q}_p$-points of a finite flat group scheme over $\mathbb{Z}_p$ (this is always the case if $E$ has good reduction at $p$). If $E$ has multiplicative reduction at $p$, then $E[p]$ is finite at $p$ if and only if $p \nmid \text{ord}_p(\Delta)$ for a minimal discriminant $\Delta$ at $p$ [21, §2.9, Prop. 5]. In particular, for $p \geq 5$, a Weierstrass equation $E_{A,B} : y^2 = x^3 + Ax + B$ that satisfies $p^6 \nmid B$ if $p^4 \nmid A$ is minimal at $p$, and so if $p \nmid N$ then $E_{A,B}[p]$ is finite at $p$ if and only if $p \nmid \text{ord}_p(\Delta(E_{A,B}))$. This will be used later to interpret hypothesis (e) of Theorem 9 as congruence conditions at $p$.

Remark 12 The theorems [30, Thm. 1.3] and [24, Thm. 1.1], on which the proof of Theorem 9 relies, are consequences of the main results of [30] and [24], respectively. These results show that for an elliptic curve $E$ as in Theorem 9 the indices of certain Heegner points on $E$ over appropriate imaginary quadratic fields are not divisible by $p$ (in particular these Heegner points are non-torsion!). The proof of this indivisibility further relies on comparing [22, Thm. 2(b)] and [21, Thm. B] with a special value formula of Gross [26 (7-8)], and owes much to ideas of Bertolini and Darmon [11]. These Heegner points do not necessarily come from uniformizations by modular curves as in the work of Gross and Zagier [13], but possibly only from uniformizations by Shimura curves, which is treated in the work of Shouwu Zhang [29]. That $E$ then has both rank and analytic rank one and finite Tate–Shafarevich group follows from the main results of [29]. We note in passing that once it is known that $E$ has analytic rank one, then it follows that the Heegner points on $E$ in the setting of [13] are also non-torsion, but they may have indices divisible by $p$.

2.3 Average sizes of p-Selmer groups of elliptic curves in large families

For $p \in \{3, 5\}$, it is easy to see that, when ordered by height, a large proportion of elliptic curves already satisfy conditions (a)–(c) of Theorem 5 or conditions (a)–(e) of Theorem 9. To show that a positive proportion of elliptic curves also satisfy the hypotheses (d) and (f) of Theorems 5 and 9, respectively, we require the following results on average orders of $p$-Selmer groups.

Theorem 13 ([24, 38, 41]) Let $p \leq 5$ be a prime. When elliptic curves over $\mathbb{Q}$ in any large family are ordered by height, the average size of the $p$-Selmer group is $p + 1$.

We recall what is meant by a “large family” of elliptic curves. For each prime $\ell$, let $\Sigma_\ell$ be a closed subset of $(\mathbb{Z}/\ell\mathbb{Z})^2 : \Delta(A,B) := -4A^3 - 27B^2 \neq 0$ with boundary of measure 0. To such a collection $\Sigma = (\Sigma_\ell)_\ell$, we associate the set $F_\Sigma$ of elliptic curves over $\mathbb{Q}$, where $E_{A,B} \in F_\Sigma$ if and only
(A, B) ∈ Σ_ℓ for all ℓ. We then say that F_Σ is a family of elliptic curves over \( \mathbb{Q} \) that is *defined by congruence conditions*. (Although we shall not do so in this article, we can also impose “congruence conditions at infinity” on \( F_Σ \), by insisting that an elliptic curve \( E_{A,B} \) belongs to \( F_Σ \) if and only if \( (A, B) \) belongs to \( Σ_∞ \), where \( Σ_∞ \) consists of all \( (A, B) \) with \( \Delta(A, B) \) positive, or negative, or either.)

A family \( F = F_Σ \) of elliptic curves defined by congruence conditions is said to be *large* if, for all sufficiently large primes \( ℓ \), the set \( Σ_ℓ \) contains all \( E_{A,B} \) with \( (A, B) \) ∈ \( \mathbb{Z}_2^ℓ \) such that \( ℓ^2 \nmid \Delta(A, B) \).

For example, the family of all elliptic curves is large, as is any family of elliptic curves \( E_{A,B} \) defined by finitely many congruence conditions on \( A \) and \( B \). The family of all semistable elliptic curves is also large. Any large family makes up a positive proportion of all elliptic curves [2, Thm. 3.17].

**Remark 14** Theorem 13 is proven in [2, 3, 4] by applying techniques from the geometry-of-numbers and sieve methods to count integral models of \( p \)-Selmer group elements—as binary quartic forms, ternary cubic forms, and quintuples of quinary alternating 2-forms—for \( p = 2, 3, \) and 5, respectively. The theory of such integral models was developed in works of Birch–Swinnerton-Dyer [6], Cremona–Fisher–Stoll [8], and Fisher [12], respectively. We note that a refinement of Theorem 13 establishing the equidistribution of the local images in \( E(\mathbb{Q}_ℓ)/pE(\mathbb{Q}_ℓ) \) of \( p \)-Selmer group elements of elliptic curves \( E \) lying in sufficiently small \( ℓ \)-adic discs, was given in [5]. Such a refinement may be useful in future improvements to Theorem 1, particularly when using \( p = 3 \) (see Remark 10).

### 2.4 Root numbers and Selmer parity

Theorem 13 alone is not sufficient to guarantee the existence of curves satisfying either Condition (d) of Theorem 5 or Condition (f) of Theorem 9. In order to deduce positive proportion statements for the individual ranks 0 and 1, we will make use of information regarding the distribution of the *parity* of the ranks—or of the \( p \)-Selmer ranks—of these curves. In this direction, in [4], a union \( F \) of large families was constructed in which half of the members of \( F \) have root number +1 and half −1. The following theorem of Dokchitser–Dokchitser [10] (see also Nekovář [15]) then allows us to relate these root numbers with \( p \)-Selmer ranks.

**Theorem 15 (Dokchitser–Dokchitser)** Let \( E \) be an elliptic curve over \( \mathbb{Q} \) and let \( p \) be any prime. Let \( s_p(E) \) and \( t_p(E) \) denote the rank of the \( p \)-Selmer group of \( E \) and the rank of \( E(\mathbb{Q})[p] \), respectively. Then the quantity \( r_p(E) := s_p(E) - t_p(E) \) is even if and only if the root number of \( E \) is +1.

The theorem of Dokchitser and Dokchitser, together with the following theorem which follows from [4, §5], will allow us to deduce that many elliptic curves have even (resp. odd) \( p \)-Selmer parity.

**Theorem 16** Let \( F \) be any large family of elliptic curves over \( \mathbb{Q} \) defined by congruence conditions modulo powers of primes \( p \) such that \( p \equiv 1 \pmod{4} \). Then there exists a finite union \( F' \) of large subfamilies of \( F \) such that, when elliptic curves in \( F \) and \( F' \) are ordered by height, the root numbers in \( F' \) are equidistributed and \( F' \) contains a density of greater than 55.01% of the elliptic curves in \( F \).

In particular, Theorems 15 and 16 together imply that, for any prime \( p \), a density of greater than 55.01% of all elliptic curves over \( \mathbb{Q} \) have equidistributed parity of \( p \)-Selmer rank. By construction, the elliptic curves \( E \in F' \) in Theorem 16 have the property that \( E \) and its −1-twist \( E_{-1} \) both lie in \( F' \) and, moreover, have root numbers of opposite sign.
3 Proofs of Theorems 1 and 2

3.1 Setup and notation

For a large family $F$ of elliptic curves $E_{A,B}$, let $\mu(F)$ denote the density of all elliptic curves over $\mathbb{Q}$, when ordered by height, lying in $F$. When $F = F_\Sigma$ is a large family defined by congruence conditions, by [2 Thm. 3.17] the density $\mu(F)$ equals the product of the local densities $\mu_\ell(F)$ over all primes $\ell$, where $\mu_\ell(F) = \mu_\ell(\Sigma_\ell)$ equals the measure of $\Sigma_\ell \subset \mathbb{Z}_\ell^2$ divided by $1 - \ell^{-10}$; here $1 - \ell^{-10}$ is the measure of the set of $(A, B) \in \mathbb{Z}_\ell^2$ such that $\ell^6 \nmid B$ whenever $\ell^4 \mid A$.

We begin this section by estimating the densities of three large families, $S_0(5)$, $S'_1(5)$, and $S_1(5)$, that satisfy many of the conditions of Theorems 5 or 9 for the prime $p = 5$. These families are defined as follows. For any prime $p \geq 5$, let $S_0(p)$ be the set of elliptic curves $E_{A,B}$ over $\mathbb{Q}$ such that:

- $E_{A,B}$ has good ordinary or multiplicative reduction at $p$;

let $S'_1(p)$ be the subset of curves $E_{A,B} \in S_0(p)$ also satisfying:

- if $E_{A,B}$ has multiplicative reduction at $p$, then $p \nmid \text{ord}_p(\Delta(A, B))$,

- if $E_{A,B}$ has split multiplicative reduction at $p$, then $\text{ord}_p(\Xi(E_{A,B})) = 1$;

and let $S_1(p)$ be the subset of curves $E_{A,B} \in S'_1(p)$ also satisfying:

- $p \nmid \text{ord}_p(\Delta(A, B))$ for all primes $\ell \equiv \pm 1 \pmod{p}$ such that $\text{ord}_\ell(\Delta(A, B)) > 0$.

Then $S_0(p)$ is the set of curves satisfying (a) of Theorem 5. $S'_1(p)$ is contained in the set of curves satisfying (a) and (e) of Theorem 9 (see Remark 11); and $S_1(p)$ is contained in the set of curves satisfying (a), (c), and (e) of Theorem 9 (see Remark 7). All three of these sets $S_0(p) \supset S'_1(p) \supset S_1(p)$ are large families; this will be demonstrated during the course of the proofs of Lemmas 17, 18, and 19.

For any large family $F$, let $\mu_{\text{equi}}(F)$ denote the supremum of $\mu(F')$ over all finite unions $F'$ of large subfamilies of $F$ for which the root numbers of elliptic curves in $F'$ are equidistributed. We conjecture that for any large family $F$, we have $\mu_{\text{equi}}(F) = \mu(F)$. Although this seems difficult to prove at the moment, we are nevertheless able to show that for both of the sets $F = S_0(5)$ or $S'_1(5)$, we have $\mu_{\text{equi}}(F) = \kappa \cdot \mu(F)$ with $\kappa \geq 0.5501$; this will follow from Theorem 16 together with the fact (demonstrated in §3.2) that $S_0(5)$ and $S'_1(5)$ are large families defined by congruence conditions modulo powers of 5.

3.2 The densities of $S_0(5)$, $S'_1(5)$, and $S_1(5)$

The next three lemmas give the densities of $S_0(5)$, $S'_1(5)$, and $S_1(5)$ among all elliptic curves $E$ over $\mathbb{Q}$.

Lemma 17 We have $\mu(S_0(5)) = \frac{4 \cdot 5^{10}}{5(5^{10} - 1)} > .8$.

Proof: Let $p \geq 5$ be a prime. It follows from [18 Rem. VII.1.1] that $E_{A,B}$ is a minimal Weierstrass equation at $p$. Therefore, $E_{A,B}$ has good reduction at $p$ if and only if $p \nmid \Delta(A, B)$ [18 Prop. VII.5.1(a)]. In this case, the condition that $E_{A,B}$ has ordinary reduction is that $p$ not divide the coefficient of $x^{p-1}$ in $(x^3 + Ax + B)^{(p-1)/2}$ [18 Thm. V.4.1(a)]. Similarly, $E_{A,B}$ has multiplicative
reduction at $p$ if and only if $p \mid \Delta(A, B)$ but $p \nmid A$ [18 Prop. VII.5.1(b)]. These are just congruence conditions modulo $p$, and so $S_0(p)$ is a large family. If $p = 5$, then these conditions together amount to

$$5 \nmid A.$$  

The measure of the set of $(A, B) \in \mathbb{Z}_p^2$ satisfying this condition is just $\frac{4}{5}$, while the measure of the set of $(A, B) \in \mathbb{Z}_p^2$ satisfying $5^0 \nmid B$ whenever $5^4 \mid A$ is $1 - \frac{1}{5^5}$. It follows that

$$\mu(S_0(5)) = \frac{4}{5} \cdot \left(1 - \frac{1}{5^{10}}\right) = \frac{4 \cdot 5^{10}}{5^{5 \cdot 10} - 1}.$$  

\[ \square \]

**Lemma 18** We have $\mu(S_1(5)) = \left(\frac{99}{125} - \frac{19}{125} \cdot \frac{4}{5^5 - 1}\right) \left(1 - \frac{1}{5^{10}}\right)^{-1} = 0.7918054 \ldots.$

**Proof:** Let $p \geq 5$ be a prime. As noted in the proof of Lemma 17, the Weierstrass equation $E_{A, B}$ is minimal at $p$, and $E_{A, B}$ has multiplicative reduction at $p$ if and only if $p \mid \Delta(A, B)$ but $p \nmid A$. We claim that if $E_{A, B}$ has multiplicative reduction at $p$, then it has split multiplicative reduction if and only if $\omega^4 \equiv -3A \pmod{p}$ has a solution. To see this, we work over $\mathbb{F}_p$. Let $x_0 \in \mathbb{F}_p$ be the double root of $x^2 + Ax + B$, so $3x_0^2 = -A$. The equations for the tangent lines at the nodal point $(x_0, 0)$ on the singular curve $E_{A, B}/\mathbb{F}_p$ are $y - \alpha(x - x_0) = 0$ and $y - \beta(x - x_0) = 0$ for some $\alpha, \beta \in \mathbb{F}_p^\times$, and satisfy

$$y^2 - x^3 - Ax - B = (y - \alpha(x - x_0)) \cdot (y - \beta(x - x_0)) - (x - x_0)^3.$$  

Equating the respective coefficients of $x$ and $x^2$ on each side, we find that $\alpha + \beta = 0$ and $2\alpha \beta x_0 + 3x_0^2 = A$. From this, together with the equality $3x_0^2 = -A$, we conclude that $-\alpha^2 x_0 = A$. Squaring and again using that $3x_0^2 = -A$, we see that $\alpha^4 = -3A$. As $E_{A, B}$ has split multiplicative reduction if and only if $\alpha, \beta \in \mathbb{F}_p^\times$, the claim follows.

Recall that the group $\mu_{p - 1}$ of $(p - 1)$-st roots of unity is a subgroup of $\mathbb{Z}_p^\times$. For a positive integer $k$ we define a subset $S_k \subset \mathbb{Z}_p^\times$ of size $p - 1$ by

$$S_k = \begin{cases} \{\omega - 24p \omega^2 : \omega \in \mu_{p - 1}\} & k = 1 \\ \mu_{p - 1} & k > 1. \end{cases}$$  

Suppose $E_{A, B}$ has split multiplicative reduction at $p$. Let $k = \text{ord}_p(\Delta(A, B)) > 0$. We claim that if $p \nmid k$, then

$$\frac{\Delta(A, B)}{p^k} \pmod{p^2} \notin S_k \pmod{p^2} \iff \mathfrak{L}(E_{A, B}) \in p\mathbb{Z}_p^\times.$$  

To see this, recall that since $E_{A, B}$ has split multiplicative reduction at $p$, it has a Tate parameterization and a corresponding Tate period $q \in p\mathbb{Z}_p$, and $\mathfrak{L}(E_{A, B}) = \frac{\log_p q}{\text{ord}_p(q)}$. The discriminant $\Delta(E_{A, B}) \in p\mathbb{Z}_p$ is the value of a convergent series

$$\Delta(E_{A, B}) = \Delta(q) = q - 24q^2 + \cdots$$  

with $\mathbb{Z}$-coefficients. From this we see that $k = \text{ord}_p(q)$ and that if $p \nmid k$, then $\mathfrak{L}(E_{A, B}) \in p\mathbb{Z}_p^\times$ if and only if $\log_p q \in p\mathbb{Z}_p^\times$. (Here $\log_p$ is the Iwasawa $p$-adic logarithm; in particular, $\log_p p = 0$ and $\log_p 1 = 0$ for $\omega \in \mu_{p - 1}$.) We can uniquely write $q = p^k \omega u$ for some $\omega \in \mu_{p - 1}$ and $u \in 1 + p\mathbb{Z}_p =$
From this it easily follows that, given $A$ such that $\text{ord}_p A \equiv k (\text{mod } p)$, we see that if
\[
B \equiv \omega u_0 + p(12a_0 a_1^2 + 27b_0^2) \text{ (mod } p^2),
\]
we see that if $k > 1$, then $u \equiv 1 (\text{mod } p^2)$ if only if $\frac{\Delta(A)}{p^k} \equiv \omega (\text{mod } p^2)$. Similarly, if $k = 1$, then $u \equiv 1 (\text{mod } p^2)$ if and only if $\frac{\Delta(A)}{p} \equiv \omega - 24p\omega^2 (\text{mod } p^2).

Suppose $p \nmid a_0$ and $k = \text{ord}_p(\Delta(a_0, b_0)) > 0$ (so $p \nmid b_0$). Writing $A = a_0 + p^k a_1$ and $B = b_0 + p^k b_1$, and letting $\Delta_0 = \Delta(a_0, b_0)$, we find that
\[
\frac{\Delta(A, B)}{p^k} \equiv \Delta_0 - 16 \left\{ \begin{array}{l}
12a_0^2a_1 + 54b_0b_1 + p(12a_0a_1^2 + 27b_0^2) \quad k = 1 \\
12a_0^2a_1 + 54b_0b_1 \quad k > 1
\end{array} \right. (\text{mod } p^2).
\]

From this it easily follows that, given $A$ modulo $p^{k+2}$ belonging to any one of the $\frac{1}{2}(p - 1)p^{k+1}$ primitive residue classes modulo $p^{k+2}$ such that $27B^2 \equiv -4A^3 (\text{mod } p)$ is solvable, there exist exactly $(2p - 1)(p - 1)$ residue classes for $B$ modulo $p^{k+2}$ (depending on $A$ modulo $p^{k+2}$) such that $\text{ord}_p(\Delta(A, B)) = k$ and $\frac{\Delta(A, B)}{p^k} \not\in S_k (\text{mod } p^2)$. In particular, the measure of the set of $(A, B) \in \mathbb{Z}_p^2$ such that $p \nmid A$, ord$_p(\Delta(A, B)) = k$ for a given $k > 0$ with $p \nmid k$, $E_{A,B}$ has split multiplicative reduction at $p$, and $\frac{\Delta(A, B)}{p^k} \not\in S_k (\text{mod } p^2)$ is equal to $\frac{(p - 1)^2(2p - 1)}{5p^3}$ if $p \equiv 1 (\text{mod } 4)$ and to $\frac{(p - 1)^2(2p - 1)}{5p^3}$ if $p \equiv 3 (\text{mod } 4)$.

Since $S_1'(5)$ is defined only by congruence conditions modulo powers of 5, we have
\[
\mu(S_1'(5)) = \mu_5(S_1'(5)).
\]

We define three sets $\Sigma_5^{\text{good}}$, $\Sigma_5^{\text{nonsplit}}$, and $\Sigma_5^{\text{split}} \subset \mathbb{Z}_5^2$:

- $\Sigma_5^{\text{good}}$ is the set of those $(A, B) \in \mathbb{Z}_5^2$ with $5 \nmid A$ and $5 \nmid \Delta(A, B)$ (this is the set of $(A, B)$ such that $E_{A,B}$ has good ordinary reduction at 5);
- $\Sigma_5^{\text{nonsplit}}$ is the set of those $(A, B) \in \mathbb{Z}_5^2$ with $A \equiv 2 \text{ (mod } 5)$ and $B \equiv \pm 2 \text{ (mod } 5)$ (so $5 \nmid \Delta(A, B)$ and $5 \nmid \text{ord}_5(\Delta(A, B))$ (this is the set of $(A, B)$ such that $E_{A,B}$ has non-split multiplicative reduction at 5 and $5 \nmid \text{ord}_5(\Delta(A, B))$);
- $\Sigma_5^{\text{split}}$ is the set of those $(A, B) \in \mathbb{Z}_5^2$ with $A \equiv 3 \text{ (mod } 5)$ and $B \equiv \pm 1 \text{ (mod } 5)$ (so $5 \mid \Delta(A, B)$, $5 \nmid k = \text{ord}_5(\Delta(A, B))$, and $\frac{\Delta(A, B)}{p^k} \not\in S_k (\text{mod } p^2)$ (this is the set of $(A, B)$ with $E_{A,B}$ having split multiplicative reduction at 5, $5 \mid \Delta(A, B)$, and $\text{ord}_5(\text{L}(E_{A,B})) = 1$).

Then $\mu_5(S_1'(5)) = \mu_5(\Sigma_5^{\text{good}}) + \mu_5(\Sigma_5^{\text{nonsplit}}) + \mu_5(\Sigma_5^{\text{split}})$.

Clearly,
\[
\mu_5(\Sigma_5^{\text{good}}) = \frac{16}{25} \cdot \left( 1 - \frac{1}{5^{10}} \right)^{-1}.
\]

As was explained above, for any $A \equiv 2, 3 \text{ (mod } 5)$, there are $2(5 - 1)$ residue classes modulo $5^{k+2}$ (depending on $A$ modulo $5^{k+2}$) such that: $\text{ord}_5\Delta(A, B) = k$ if and only if $B$ belongs to one of these residue classes. Hence
\[
\mu_5(\Sigma_5^{\text{nonsplit}}) = \left( 1 - \frac{1}{5^{10}} \right)^{-1} \cdot \sum_{k=1}^{\infty} \frac{2(5 - 1)}{5^{k+2}} = \frac{2}{25} \left( 1 - \frac{4}{5^{5} - 1} \right) \left( 1 - \frac{1}{5^{10}} \right)^{-1},
\]

9
and

$$
\mu_5(\Sigma_{5}^{\text{split}}) = \left(1 - \frac{1}{5^{10}}\right)^{-1} \cdot \sum_{\substack{k=1 \\ 5 \mid k}}^{\infty} \frac{(5 - 1)(2 \cdot 5 - 1)}{5^{k+3}} = \frac{9}{125} \left(1 - \frac{4}{5^5 - 1}\right) \left(1 - \frac{1}{5^{10}}\right)^{-1}.
$$

Therefore,

$$
\mu_5(S_1'(5)) = \left(\frac{99}{125} - \frac{19}{125} \cdot \frac{4}{5^5 - 1}\right) \left(1 - \frac{1}{5^{10}}\right)^{-1} = .7918054 \ldots.
$$

\hfill \Box

**Lemma 19** We have $\mu(S_1(5)) > .7917957$.

**Proof:** The density of $S_1(p)$ equals the density of $S_1'(p)$ times the product over all primes $\ell \equiv \pm 1 \pmod{p}$ of the local densities $\mu_\ell(\Sigma_\ell)$ where $\Sigma_\ell$ is the set of $(A, B) \in \mathbb{Z}_\ell^2$ such that $p \nmid \text{ord}_\ell(\Delta(A, B))$ whenever $\ell \mid \Delta(A, B)$.

We note that for a prime $\ell \geq 5$, the measure of the set of $(A, B) \in \mathbb{Z}_\ell^2$ that satisfy $\ell \nmid A$ and $\text{ord}_\ell(\Delta(A, B)) = k$ for a given integer $k > 0$ is $\frac{(\ell-1)^2}{\ell^{k+2}}$. For given an $A$ belonging to one of the $\frac{\ell-1}{2}$ primitive residue classes modulo $\ell$ for which $27B^2 \equiv -4A^3 \pmod{\ell}$ is solvable in $B$, we have $\text{ord}_\ell(\Delta(A, B)) = k$ if and only if $B$ belongs to one of $2(\ell-1)$ residue classed modulo $\ell^{k+1}$ (which may depend on $A$ modulo $\ell^{k+1}$). It follows that for any prime $\ell \equiv \pm 1 \pmod{p}$, $\mu_\ell(\Sigma_\ell)$ is at least

$$
\left(1 - \frac{1}{\ell^{10}}\right)^{-1} \left(1 - \sum_{n=1}^{\infty} \frac{(\ell-1)^2}{\ell^{6n+2}} - \frac{1}{\ell^5}\right) = \left(1 - \frac{1}{\ell^{10}}\right)^{-1} \left(1 - \frac{(\ell-1)^2}{\ell^2(\ell^5 - 1)} - \frac{1}{\ell^5}\right).
$$

The first term being subtracted in the second factor on the left-hand side is the measure of those $(A, B) \in \mathbb{Z}_\ell^2$ such that $\ell \nmid A$, $\text{ord}_\ell(\Delta(A, B)) > 0$, and $5 \mid \text{ord}_\ell(\Delta(A, B))$. The second term being subtracted is the measure of the set of $(A, B)$ such that $\ell^2 \mid A$ and $\ell^3 \mid B$, which contains the set of the $(A, B)$ with $\ell^4 \mid A$ and $\ell^6 \mid B$ and those $(A, B)$ such that $\ell \mid A$, $\text{ord}_\ell(\Delta(A, B)) > 0$, and $5 \mid \text{ord}_\ell(\Delta(A, B))$.

Therefore,

$$
\mu(S_1(5)) \geq \mu(S_1'(5)) \cdot \prod_{\ell \equiv \pm 1 \pmod{5}} \left(1 - \frac{1}{\ell^{10}}\right)^{-1} \left(1 - \frac{(\ell-1)^2}{\ell^2(\ell^5 - 1)} - \frac{1}{\ell^5}\right) = .7917957 \ldots.
$$

\hfill \Box

In particular, we note that

$$
\mu(S_1'(5)) - \mu(S_1(5)) \leq .00001. \quad (2)
$$

Finally, we end with a lemma which explains why we only used Condition (a) of Theorem 5 to define $S_0(p)$, and only used Conditions (a), (c), and (e) of Theorem 9 to define $S_1(p)$: namely, 100% of elliptic curves over $\mathbb{Q}$ satisfy Conditions (b) and (c) of Theorem 5 and Conditions (b) and (d) of Theorem 9.

**Lemma 20** Let $p$ be any prime. Then, when ordered by height, a density of 100% of elliptic curves $E$ over $\mathbb{Q}$ possess the following two properties:

- $E[p]$ is an irreducible $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module;

10
There exist at least two prime factors \( \ell \mid N(E), \ell \neq p \), such that \( E[p] \) is ramified at \( \ell \).

**Proof:** That 100% of elliptic curves satisfy the first property follows easily, e.g., by Hilbert irreducibility. (See also the work of Duke [11], who shows that in fact 100% of all elliptic curves \( E \) over \( \mathbb{Q} \), when ordered by height, have the property that the action of \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \) on \( E[p] \) is irreducible for all primes \( p \).

To see that 100% of elliptic curves over \( \mathbb{Q} \) also satisfy the second property, we will show that the density of elliptic curves not satisfying this property is zero. For a large integer \( L \) and for each prime \( 5 \leq \ell \leq L, \ell \neq p \), let \( S(L, \ell) \) be the set of elliptic curves \( E_{A,B} \) such that \( \text{ord}_{\ell}(\Delta(A, B)) \leq 1 \) and for all primes \( 5 \leq q \leq L \) with \( q \notin \{\ell, p\} \), we have \( \text{ord}_q(\Delta(A, B)) \neq 1 \). Let \( S(L) = \cup_{5 \leq \ell \leq L, \ell \neq p} S(L, \ell) \).

Any curve in the complement of \( S(L) \) satisfies the second property of the lemma with two primes \( 5 \leq \ell_1, \ell_2 \leq L \). So it suffices to show that \( \mu(S(L)) \) tends to 0 as \( L \) gets large.

The set \( S(L) \) is a disjoint union of the large sets \( S(L, \ell) \), and the density \( \mu(S(L, \ell)) \) of \( S(L, \ell) \) is a product of local densities. Calculating these local densities as in the proof of Lemma [19] we find

\[
\mu(S(L)) = \sum_{5 \leq \ell \leq L, \ell \neq p} (1 - \ell^{-10})^{-1} \left( 1 - \frac{1}{\ell^2} - \frac{(\ell - 1)}{\ell^3} \right) \prod_{5 \leq q \leq L, q \neq \ell, p} (1 - q^{-10})^{-1} \left( 1 - \frac{(q - 1)^2}{q^3} \right),
\]

which tends to 0 as \( L \) tends to \( \infty \). \( \square \)

### 3.3 A lower bound on the proportion of curves having algebraic and analytic rank 0

We first treat the case of rank 0.

**Theorem 21** Suppose \( F \) is a finite union of large families of elliptic curves such that exactly 50% of the curves in \( F \), when ordered by height, have root number +1. Then at least 3/8 of the curves in \( F \), when ordered by height, have 5-Selmer rank 0.

**Proof:** Let \( p = 5 \). Then by Theorem [13] the average size of the \( p \)-Selmer group of curves in \( F \) is \( p + 1 \). On the other hand, by Theorem [15] we know that that 1/2 of the curves in \( F \) have odd \( p \)-Selmer rank and thus have at least \( p \) elements in the \( p \)-Selmer group. Hence the lim sup of the average size of the \( p \)-Selmer group among the half of elliptic curves in \( F \) having even \( p \)-Selmer rank is at most \( p + 2 \). Now if the \( p \)-Selmer group of an elliptic curve has even rank, then it must have order 1, \( p^2 \), or more than \( p^2 \). Since the lim sup of the average of such orders is at most \( p + 2 \), a lower density of at least \( (p^2 - p - 2)/(p^2 - 1) = (p - 2)/(p - 1) \) of these orders must be equal to 1. Thus among these 1/2 of curves in \( F \) with even \( p \)-Selmer rank, a lower density of at least \( (p - 2)/(p - 1) \) have trivial \( p \)-Selmer group; i.e., a lower density of at least \( 1/2 \cdot (p - 2)/(p - 1) = (p - 2)/(2p - 2) \) of curves in \( F \) have \( p \)-Selmer rank 0. For \( p = 5 \), this yields a lower proportion of 3/8 of curves in \( F \) having 5-Selmer rank 0. \( \square \)

**Corollary 22** When ordered by height, at least 16.50% of elliptic curves over \( \mathbb{Q} \) have both algebraic and analytic rank 0.

**Proof:** By definition, a proportion of \( \mu(S_0(5)) \) of all elliptic curves over \( \mathbb{Q} \) satisfy condition (a) of Theorem [5]. By Theorem [16] inside the family \( S_0(5) \) there exists a finite union \( F \) of large subfamilies
of density \( \mu(F) = \kappa \cdot \mu(S_0(5)) \), with \( \kappa \geq .5501 \), and having equidistributed root number. By Theorem 21 a proportion of \( 3/8 \) of elements of \( F \) have 5-Selmer rank 0. Thus, by Theorem 5 and Lemma 20, a proportion of at least \( 3/8 \cdot \kappa \cdot \mu(S_0(5)) \) of all elliptic curves over \( \mathbb{Q} \) have both algebraic and analytic rank 0.

By Theorem 16 and Lemma 17, we thus obtain a lower density of greater than

\[
\frac{3}{8} \cdot .5501 \times .8 = .16503
\]

of all curves have both algebraic and analytic rank 0, as stated in the corollary. \( \square \)

In other words, at least 16.50% of all elliptic curves have rank 0 and satisfy the Birch and Swinnerton-Dyer conjecture.

### 3.4 A lower bound on the proportion of curves having algebraic and analytic rank 1

We are now ready to treat the case of rank 1.

**Theorem 23** Suppose \( F \) is a finite union of large families of elliptic curves such that exactly 50% of the curves in \( F \), when ordered by height, have root number \(+1\). Then at least \( 19/40 \) of the curves in \( F \), when ordered by height, have 5-Selmer rank 1.

**Proof:** Let \( p = 5 \). Again, by Theorem 13, the average size of the \( p \)-Selmer group of curves in \( F \) is \( p + 1 \); by Theorem 15, we know that that 1/2 of the curves in \( F \) have even \( p \)-Selmer rank. Since every \( p \)-Selmer group has at least one element (namely, the identity element), the limsup of the average order of the \( p \)-Selmer groups among the half of elliptic curves in \( F \) that have odd \( p \)-Selmer rank is at most \( 2p + 1 \). Now if the \( p \)-Selmer group of an elliptic curve has odd rank, then it must have order \( p \), \( p^3 \), or more than \( p^3 \). Since the limsup of the average of such orders is at most \( 2p + 1 \), a lower density of at least \( (p^3 - 2p - 1)/(p^3 - p) = (p^2 - p - 1)/(p^2 - p) \) of these orders must be equal to \( p \). Thus among these 1/2 of curves in \( F \) with odd \( p \)-Selmer rank, at least \( (p^2 - p - 1)/(p^2 - p) \) have \( p \)-Selmer rank 1; i.e., a lower density of at least \( 1/2 \cdot (p^2 - p - 1)/(p^2 - p) = (p^2 - p - 1)/(2p^2 - 2p) \) of curves in \( F \) have \( p \)-Selmer rank 1. For \( p = 5 \), this yields a lower proportion of 19/40 of curves in \( F \) having 5-Selmer rank 1. \( \square \)

**Corollary 24** When ordered by height, at least 20.68% of elliptic curves over \( \mathbb{Q} \) have both algebraic and analytic rank 1.

**Proof:** By definition, a proportion of \( \mu(S'_1(5)) \) of all elliptic curves over \( \mathbb{Q} \) satisfy Conditions (a) and (e) of Theorem 9. By Theorem 16 there exists a finite union \( F \) of large subfamilies in \( S'_1(5) \) with density \( \kappa \cdot \mu(S'_1(5)) \), with \( \kappa \geq .5501 \), and having equidistributed root numbers. By Theorem 23 a proportion of at least 19/40 of the elliptic curves \( E \) in \( F \) have the property that \#\( S_5(E) = 5 \). We conclude by Theorem 9, Lemma 20, and (2) that a proportion of at least

\[
\frac{19}{40} \times .5501 \times .7918054 - .00001 > .20688
\]

of all elliptic curves lie in \( S_1(5) \) and have both algebraic and analytic rank 1. \( \square \)

In other words, at least 20.68% of all elliptic curves have rank 1 and satisfy the Birch and Swinnerton-Dyer conjecture.
3.5 A lower bound on the proportion of curves having algebraic and analytic rank 0 or 1

Corollaries 22 and 24 together already show that a proportion of at least 16.50 + 20.68 = 37.18% of all elliptic curves satisfy the Birch and Swinnerton-Dyer conjecture. However, we can do much better if we do not consider the individual ranks 0 and 1 separately. Specifically, we can prove that a large proportion of curves have algebraic and analytic rank 0 or 1 even in large families where the root number is not equidistributed:

**Theorem 25** Let $F \subseteq S_0(5) \cap S_1(5)$ be any finite union of large families of elliptic curves. Then at least $19/24$ of the elliptic curves in $F$ have algebraic and analytic rank 0 or 1. If, furthermore, root numbers are equidistributed in $F$, then at least $7/8$ of the elliptic curves in $F$ have algebraic and analytic rank 0 or 1.

**Proof:** Let $p = 5$. By Theorem 13 the average size of the $p$-Selmer group of curves in $F$ is $p + 1$. Let $x_{0 \text{ or } 1}$ be the lower density of elliptic curves in $F$ having 5-Selmer rank 0 or 1. Then

$$x_{0 \text{ or } 1} + p^2(1 - x_{0 \text{ or } 1}) \leq p + 1,$$

and hence $x_{0 \text{ or } 1} \geq (p^2 - p - 1)/(p^2 - 1)$. The bound is achieved when a proportion of $(p^2 - p - 1)/(p^2 - 1)$ of elliptic curves in $F$ have $p$-Selmer rank 0, and a proportion of $p/(p^2 - 1)$ have $p$-Selmer rank 2. Setting $p = 5$, and then applying Theorem 9 now proves the first part of Theorem 25.

To prove the second part of Theorem 25 let $x_{0 \text{ or } 1}$ again denote the lower density of elliptic curves with 5-Selmer rank 0 or 1. Also, let $x_0$ (resp. $x_1$) denote the lower density of elliptic curves with 5-Selmer rank 0 (resp. 1). Then, by Theorem 13 and 15 we have

$$x_0 + p^2(1/2 - x_0) + p(x_1 + p^2(1/2 - x_1)) \leq p + 1.$$

Thus, we obtain

$$(p^2 - 1)x_0 + (p^3 - p)x_1 \geq (p^3 + p^2)/2 - p - 1.$$

In conjunction with the constraint $x_1 \leq 1/2$, it follows that

$$x_{0 \text{ or } 1} \geq x_0 + x_1 \geq [(p^2 + p)/2 - p - 1]/(p^2 - 1) + 1/2 = (2p - 3)/(2p - 2).$$

Again, this bound is achieved when a proportion of $(p - 2)/(2p - 2)$ of elliptic curves over $\mathbb{Q}$ have $p$-Selmer rank 0, a proportion of $1/2$ of elliptic curves have 5-Selmer rank 1, and a proportion of $1/(2p - 2)$ of elliptic curves have 5-Selmer rank 2. Setting $p = 5$, and applying Theorem 9 now yields the second part of Theorem 25. $\square$

**Corollary 26** When ordered by height, at least 66.48% of elliptic curves over $\mathbb{Q}$ have algebraic and analytic rank 0 or 1.

**Proof:** By Theorem 10 there is a finite union $F'$ of large subfamilies in $S_0(5) \cap S_1(5)$ of density $\kappa \mu(S_0(5) \cap S_1(5))$ with $\kappa \geq .5501$ and such that for all $E \in F'$ the root number of $E$ and its $-1$-twist have opposite signs. Let $F = F' \cap S_1(5)$. Then $F$ is also a finite union of large subfamilies and, since $S_1(5)$ is stable under $-1$-twist, $F$ also has the property that for all $E \in F$ the root number of $E$ and its $-1$-twist are both in $F$ and have opposite signs. In particular, the root numbers of the curves in $F$ are equidistributed. By 2, the density of $F$ satisfies

$$\mu(F) \geq \kappa \mu(S_0(5) \cap S_1(5)) - .00001.$$
By the second part of Theorem 25, a proportion of at least $7/8$ of the curves in $F$ have algebraic and analytic rank 0 or 1.

Next we consider the set $F''$ of curves in $S_0(5) \cap S_1(5)$ on which the above arguments have not been applied. This set contains the complement of $F$ in $S_0(5) \cap S_1(5)$, which is a (possibly infinite) disjoint union of large subfamilies. So for any $\epsilon > 0$, $F''$ contains a finite union $F''_{\epsilon}$ of large subfamilies such that

$$\mu(F''_{\epsilon}) \geq \mu(S_0(5) \cap S_1(5)) - \mu(F) - \epsilon \geq (1 - \kappa) \cdot \mu(S_0(5) \cap S'_1(5)) - .00001 - \epsilon.$$  

By Theorem 25, a proportion of at least $19/24$ of the curves in $F''_{\epsilon}$ have algebraic and analytic rank 0 or 1.

For the set of elliptic curves in $S_0(5)$ on which the above arguments have not been applied, which has density at least $0.8 - 0.79179 \ldots = 0.00820 \ldots$, we can apply the arguments of Corollary 22. This gives an additional set of curves of density at least $3/8 \times 0.5501 \times 0.00820 = 0.00169 \ldots$ that have algebraic and analytic rank 0.

It follows that a total proportion of at least

$$\left(\frac{7}{8} \kappa + \frac{19}{24} (1 - \kappa)\right) \times \mu(S_0(5) \cap S_1(5)' \cap S_0(5)) - \left(\frac{7}{8} + \frac{19}{24}\right) \times 0.00001 + 0.00169$$

of elliptic curves have algebraic and analytic rank 0 or 1. Since $\kappa \geq 0.5501$, we conclude by Lemma 18 that this proportion is at least

$$\left(\frac{7}{8} \times 0.5501 + \frac{19}{24} \times 0.4499\right) \times 0.7918054 \ldots - \left(\frac{7}{8} + \frac{19}{24}\right) \times 0.00001 + 0.00169 = 0.664816 \ldots.$$  

Thus at least 66.48% of all elliptic curves over $\mathbb{Q}$ satisfy the Birch and Swinnerton-Dyer conjecture, yielding Theorem 1.

### 4 Conclusion and future work

The proportions in Corollaries 22, 24, and 26 can be improved by strengthening Theorems 5, 9, 13, or 16. In particular, if Theorem 5 is improved as indicated in Remark 6, then the lower bound on the proportion of elliptic curves having algebraic and analytic rank 0 increases to 19.8%, and the lower bound on the proportion of elliptic curves having algebraic and analytic rank 0 or algebraic and analytic rank 1 increases to 69.6%. If Theorem 9 can be improved as indicated in Remark 10, then the lower bound on the proportion of elliptic curves having algebraic and analytic rank 1 increases to 24.8%, and the lower bound on the proportion of elliptic curves satisfying the Birch and Swinnerton-Dyer rank conjecture increases to 79.7% (working also with the prime 3 would push this lower bound over 80%).

Finally, there have been a number of recent heuristics (cf. Delaunay, Poonen–Rains, and a conjecture) that independently suggest that, for all $p$, the average size of the $p$-Selmer group of all elliptic curves, when ordered by height, should be $p+1$. Indeed, Theorem 13 gives excellent evidence for this conjecture, confirming the conjecture for the primes $p = 2, 3$, and 5.

Tracing through the methods of the previous sections, with a general value of $p$ in place of $p = 5$, immediately allows us to deduce:
Theorem 27 Consider the family of all elliptic curves over $\mathbb{Q}$ ordered by height. Suppose that for all primes $p$, the average size of the $p$-Selmer group of elliptic curves over $\mathbb{Q}$ is $p + 1$. Then the Birch and Swinnerton-Dyer rank conjecture is true for 100% of elliptic curves over $\mathbb{Q}$.

Proof: Let $p$ be a prime. A lower bound on the density $\mu(S_0(p) \cap S_1(p))$ is

$$(1 - 1/p)^2 \cdot \prod_{\ell \equiv \pm 1 \pmod{p}} (1 - 1/\ell^{10})^{-1}(1 - 1/\ell^5)^2.$$ (3)

Of these, by the proof of Corollary 26, a proportion of at least

$$\frac{p^2 - p - 1}{p^2 - 1}$$ (4)

have algebraic and analytic rank 0 or 1. As $p$ tends to infinity, the product of (3) and (4) tends to 1. $\square$

Indeed, in this paper, we have used the $p$-Selmer average for the prime $p = 5$ to prove unconditionally that at least 66.48% of elliptic curves over $\mathbb{Q}$ satisfy the Birch and Swinnerton-Dyer rank conjecture.

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