Abstract

It is shown that several Hamiltonian systems possessing dynamical or hidden symmetries can be realized within the framework of Nambu’s generalized mechanics. Among such systems are the SU(n)-isotropic harmonic oscillator and the SO(4)-Kepler problem. As required by the formulation of Nambu dynamics, the integrals of motion for these systems necessarily become the so-called generalized Hamiltonians. Furthermore, in most of these problems, the definition of these generalized Hamiltonians is not unique.
1 Introduction

More than two decades ago, Nambu [1] proposed a generalization to classical Hamiltonian mechanics. In his formalism, he replaced the usual pair of canonical variables found in Hamiltonian mechanics with a triplet of coordinates in an odd dimensional phase space. Furthermore, he formulated his dynamics via a ternary operation, the Nambu bracket, as opposed to the usual binary Poisson bracket. Yet, the fundamental principles of a canonical form of Nambu’s generalized mechanics, similar to the invariant geometrical form of Hamiltonian mechanics, has only recently been given [2]. The re-emergence of this little known theory is possibly due to its relevance to the recent mathematical structures having their basis in Hamiltonian mechanics such as the Poisson-Lie group, quantum groups, and the Yang-Baxter equation. Since the basic idea of Nambu mechanics is to extend the usual binary operation on phase space to multiple operations of higher order, this theory may also give some insights into the theory of higher order algebraic structures and their possible physical significance.

The fact that the development of Nambu mechanics is still at the preliminary stages can be seen by the relatively few known examples of dynamical systems which admit a Nambu-type formulation. Nambu himself came up with only one example; the Euler equations for the angular momentum of a rigid body in three dimensions [1]. Another (somewhat exotic) example is that of Nahm’s system of equations in the theory of static SU(2)-monopoles [3], [4]. A third example was found in [2]. Here, we present four new examples of systems taking on the Nambu form. These systems share the common property of possessing dynamical or hidden symmetries resulting in extra integrals of motion beyond those needed for complete integrability. It is felt that these new examples may help in further understanding the elements of Nambu’s theory such as its algebraic structure and its (possible) quantization. We begin by stating the basic facts of Nambu mechanics leaving all details to the recent comprehensive study by Takhtajan [2].

2 Nambu Mechanics

For comparisons sake, let us first review certain fundamental definitions and results of Hamiltonian mechanics (see [5]). Let $M$ denote a smooth manifold of finite dimension and $C^\infty(M)$ the algebra of smooth real-valued functions on $M$. A Poisson Bracket on $M$ is a $\mathbb{R}$-bilinear map

$$\{ \ , \ \} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M),$$

such that $\forall f_1, f_2, \text{ and } f_3 \in C^\infty(M)$,

$$\{f_1, f_2\} = -\{f_2, f_1\} \quad \text{(skew symmetry)},$$

$$\{f_1 f_2, f_3\} = f_1 \{f_2, f_3\} + \{f_1, f_3\} f_2 \quad \text{(Leibniz rule)},$$

$$\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0 \quad \text{(Jacobi identity)}.$$
By property (3), it is clear that $\forall \, h \in C^\infty(M)$, the map $f \to \{f, h\}$ is a derivation of $C^\infty(M)$. Thus, one may define a vector field $X_h$ such that $X_h(f) = \{f, h\}$, $\forall \, f \in C^\infty(M)$. In Hamilton’s formulation of dynamics, one defines a very special vector field $X_H$ on the phase space $M$ via a Hamiltonian function $H \in C^\infty(M)$ such that

$$X_H(f) = \{f, H\} = \frac{df}{dt}, \quad \forall \, f \in C^\infty(M).$$

(5)

This, of course, produces Hamiltons’ equations of motion.

The Poisson bracket given above is explicitly defined by a $C^\infty$ tensor field $\omega \in \wedge^2 TM$, an element of the square of the tangent bundle $TM$ of the phase space $M$, such that

$$\{f, g\} = \omega(df, dg) = \omega^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$

(6)

where $(x^i)$ are some local coordinates on $M$ and we use the Einstein summation convention. For the purposes of regular Hamiltonian dynamics, $\omega^{ij}$ is a constant antisymmetric two-tensor. Outside of Hamiltonian mechanics, there exists linear Poisson structures such that if $(x_1, x_2, ..., x_n)$ is a basis for a Lie algebra, the Poisson bracket is given by

$$\{f, g\} = [x_i, x_j]_{\text{Lie}} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$

(7)

$[\ , \ ]_{\text{Lie}}$ denoting the Lie algebra bracket. A more modern example is that of a Poisson-Lie group where the phase space $M$ is some Lie group $G$ such that the group multiplication of $G$ is compatible, in some sense, with the Poisson-structure on $G$ (for a precise definition, see [6]). A complete description of such structures (Poisson-Lie Manifolds) may be found in the monograph [7].

Finally, the dynamical picture resulting from a solution to (5) produces a phase space (time) flow

$$x \mapsto T_t(x), \quad x \in M$$

(8)

and an evolution operator

$$U_t : C^\infty(M) \to C^\infty(M),$$

$$U_t(f)(x) = f(T_t(x)),$$

(9)

where $x \in M$, $f \in C^\infty(M)$. For consistency, $U_t$ must preserve the two algebraic operations defined on $C^\infty(M)$, the usual multiplication of functions and the Poisson bracket. This requirement of $U_t$ to be an algebra isomorphism $(U_t(f_1 f_2) = U_t(f_1) U_t(f_2)$ and $U_t(\{f_1, f_2\}) = \{U_t(f_1), U_t(f_2)\}$) is equivalent to the properties (3) and (4) of the Poisson bracket. A similar requirement will lead to a certain fundamental identity in Nambu mechanics.

Nambu’s generalization of mechanics is based upon a higher order ($n > 2$) algebraic structure defined on a (possibly odd dimensional) phase space $M$. A Nambu Bracket of order $n$ on a manifold $M$ is an $\mathbb{R}$-multilinear map

$$\{ \ , \ , \ , \} : [C^\infty(M)]^n \to C^\infty(M)$$

(10)

2
such that $\forall f_1, f_2, \ldots, f_{2n-1} \in C^\infty(M)$,

$$
\{f_1, \ldots, f_n\} = (-1)^{P(\sigma)}\{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\},
$$

$$
\{f_1f_2, f_3, \ldots, f_{n+1}\} = f_1\{f_2, f_3, \ldots, f_{n+1}\} + \{f_1, f_3, \ldots, f_{n+1}\}f_2,
$$

and

$$
\{\{f_1, \ldots, f_{n-1}, f_n\}, f_{n+1}, \ldots, f_{2n-1}\} + \{f_n, \{f_1, \ldots, f_{n-1}, f_{n+1}\}, f_{n+2}, \ldots, f_{2n-1}\}
$$

$$
+ \ldots + \{f_n, \ldots, f_{2n-2}, \{f_1, \ldots, f_{n-1}, f_{2n-1}\}\} = \{f_1, \ldots, f_{n-1}, f_n, \ldots, f_{2n-1}\},
$$

where $\sigma \in S_n$ (the permutation group) and $P$ is the parity of the permutation $\sigma$. Conditions (11) and (12) are the familiar skew-symmetric and derivation properties found for the Poisson Bracket. On the other hand, (13) is a generalized Jacobi identity called the Fundamental Identity (FI). The FI was first introduced by Takhtajan [2] and by M. Flato and C. Fronsdal. It was also independently found by Sahoo and Valsakumar [8], [9].

Now, analogous to (5), Nambu dynamics is determined by a special Nambu-Hamiltonian vector field $X_{NH}$ given by

$$
X_{NH}(f) = \{f, H_1, \ldots, H_{n-1}\} = \frac{df}{dt},
$$

$\forall f \in C^\infty(M)$ where $H_1, \ldots, H_{n-1}$ are the generalized Hamiltonians of the system. A solution to the above equations of motion produces an evolution operator $U_t$ as described earlier by (8) and (9). As is the case in Hamiltonian mechanics, Nambu dynamics is consistent if and only if $U_t$ is an isomorphism of the above defined algebraic structure on $C^\infty(M)$. It can be shown that the corresponding isomorphism requirement of

$$
U_t(\{f_1, \ldots, f_n\}) = \{U_t(f_1), \ldots, U_t(f_n)\}
$$

is equivalent to the FI (13) for the Nambu bracket. Furthermore, the FI provides a very important dynamical result. First of all, a function $I \in C^\infty(M)$ is an integral of motion if $\{I, H_1, \ldots, H_{n-1}\} = 0$. Then, using the FI, one can prove that the Nambu bracket of $n$ integrals of motion is also an integral of motion (analogous to Poisson’s Theorem in Hamiltonian mechanics). Consequently, one can define the concept of integrability in Nambu mechanics.

The Nambu bracket is explicitly generated by a Nambu tensor field $\eta \in \wedge^n TM$ such that

$$
\{f_1, \ldots, f_n\} = \eta(df_1, \ldots, df_n) = \eta_{i_1\ldots i_n}(x)\frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_n}},
$$

where $(x_1, x_2, \ldots)$ are some local coordinates on $M$ and repeated indices are summed. The FI imposes serious constraints on the Nambu tensor field $\eta$. Yet, for our purposes, we only need to define the simplest type of Nambu bracket. Let $M = \mathbb{R}^n$ be our phase space with coordinates $x_1, \ldots, x_n$. Then the so-called ‘canonical’ Nambu bracket is

$$
\{f_1, \ldots, f_n\} = C\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}
$$

where the right hand side is the Jacobian of the mapping $(f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ and $C$ is a constant factor. This bracket will be used throughout the rest of the paper.
3 The SU(n)-Isotropic Harmonic Oscillator

It is well known that the Hamiltonian for the n-dimensional simple harmonic oscillator where all the frequencies have been set to one,

$$H = \sum_{i=1}^{n} (p_i^2 + q_i^2)$$  \hspace{1cm} (18)

is invariant under the symmetry group SU(n) and has the following integrals of motion,

$$L_{ij} = q_ip_j - q_jp_i,$$ \hspace{1cm} (19)

and

$$A_{ij} = p_ip_j + q_iq_j,$$ \hspace{1cm} (20)

where \(i, j = 1, \ldots, n\). Obviously, the \(L_{ij}\)'s are the angular momenta of the system whereas the diagonal components of \(A_{ij}\) are the individual energies associated with the separate one-dimensional oscillations. It is the off-diagonal components of \(A_{ij}\) which provide the hidden integrals of motion thus forcing all the trajectories in phase space to lie on curves. That is, the \(L_{ij}\)'s and the \(A_{ij}\)'s provide \((2^n - 1)\) independent integrals of motion within the \(2n\) dimensional phase space, and therefore all orbits lie on one-dimensional manifolds. By Bertrand’s theorem (see [10]), these orbits are closed implying that the extra dynamical integrals of motion are simple functions of the phase space coordinates.

Beginning with the two-dimensional case, if one defines the following functions

$$S_1 = (A_{12} + A_{21})/2,$$
$$S_2 = (A_{22} - A_{11})/2,$$
$$S_3 = L_{12}/2,$$ \hspace{1cm} (21)

it is easy to verify that

$$\{S_i, S_j\} = \epsilon_{ijk}S_k$$ \hspace{1cm} (22)

which are simply the commutation relations for the Lie algebra su(2). Furthermore, the Casimir function is related to the Hamiltonian since

$$S_1^2 + S_2^2 + S_3^2 = S^2 = \frac{H^2}{4}.$$ \hspace{1cm} (23)

The integrals of motion for this system are,

$$I_1 = p_1^2 + q_1^2 = C_1,$$
$$I_2 = p_2^2 + q_2^2 = C_2,$$
$$I_3 = q_1p_2 - q_2p_1 = C_3,$$
$$I_4 = p_1p_2 + q_1q_2 = C_4,$$ \hspace{1cm} (24)
where the $C_i$’s are the constant values taken by the $I_i$’s. Using these $I_i$’s as the generalized Hamiltonians, we can describe the corresponding Nambu dynamics as follows. Consider the following Nambu bracket,

$$\{ f, I_1, I_2, I_3 \} = \left( \frac{-1}{2C_4} \right) \frac{\partial(f, I_1, I_2, I_3)}{\partial(p_1, p_2, q_1, q_2)}, \quad (25)$$

where, as before, the right hand side symbolizes the Jacobian operation and $f$ is some function of the phase space coordinates. Straightforward calculations will show that this bracket in fact produces all the correct equations of motion, i.e. $dp_1/dt = \{p_1, I_1, I_2, I_3 \} = -2q_1$, etc. Note that since only three generalized Hamiltonians were needed, $I_4$ was not used at all. Another definition, using $I_4$, and giving the correct equations of motion is

$$\{ f, I_1, I_2, I_4 \} = \left( \frac{1}{2C_3} \right) \frac{\partial(f, I_1, I_2, I_4)}{\partial(p_1, p_2, q_1, q_2)}. \quad (26)$$

Even products of $I_i$’s can be used as generalized Hamiltonians. For instance, the bracket

$$\{ f, I_1, I_2, I_3 I_4 \} = \frac{1}{2(C_3^2 - C_4^2)} \frac{\partial(f, I_1, I_2, I_3 I_4)}{\partial(p_1, p_2, q_1, q_2)} \quad (27)$$

also works. By the derivation property (12) for Nambu brackets, it can be checked that the alternate definitions (25)-(27) are consistent with each other. For example, let $f = p_1$ in (27) and add to the brackets in (25), (26), and (27) the extra subscripts $I_3$, $I_4$, and $I_{34}$ respectively to distinguish them. Then

$$\{ p_1, I_1, I_2, I_3 I_4 \}_{I_{34}} = I_3\{ p_1, I_1, I_2, I_4 \}_{I_{34}} + \{ p_1, I_1, I_2, I_3 \}_{I_{34}} I_4$$

$$= \frac{I_3}{2(C_3^2 - C_4^2)} \frac{\partial(p_1, I_1, I_2, I_4)}{\partial(p_1, p_2, q_1, q_2)} + \frac{I_4}{2(C_3^2 - C_4^2)} \frac{\partial(p_1, I_1, I_2, I_3)}{\partial(p_1, p_2, q_1, q_2)}$$

$$= \frac{2C_3^2}{2(C_3^2 - C_4^2)} \{ p_1, I_1, I_2, I_4 \}_{I_3} - \frac{2C_4^2}{2(C_3^2 - C_4^2)} \{ p_1, I_1, I_2, I_3 \}_{I_3}$$

$$= \frac{2C_3^2}{2(C_3^2 - C_4^2)} (-2q_1) - \frac{2C_4^2}{2(C_3^2 - C_4^2)} (-2q_1) = -2q_1$$

as needed. It is a straightforward exercise to verify that the above calculation also holds for the remaining phase space variables.

By the above arguments and the fact that

$$\frac{\partial(I_1, I_2, I_3, I_4)}{\partial(p_1, p_2, q_1, q_2)} = 0,$$

it is clear that one may choose any three of the four integrals of motion (24) as the generalized Hamiltonians. One must simply find the correct factors in front of the common Jacobian term. Once these basic brackets have been established, linear combinations such as
\{f, I_1, I_2 + I_3, I_4\} and those of the type similar to that of (27) can easily be found using the linearity and derivation properties of the Nambu bracket. This extra flexibility not found in Hamiltonian mechanics is obviously due to the multiple Hamiltonian structure of Nambu mechanics. Finally, it is relatively straightforward to extend the above arguments to higher dimensional systems. One simply uses the integrals of motion given by (19) and (20) as the generalized Hamiltonians and finds the correct constants related to these integrals to multiply the Jacobian term in the definition of the Nambu bracket. Thus, the SU(n)-isotropic harmonic oscillator is realizable as an \(n^{th}\) order Nambu mechanical system.

4 The SO(4)-Kepler Problem

The well known Kepler Hamiltonian is

\[ H = \frac{\vec{p}^2}{2} - \frac{1}{r}, \]  

(28)

where \(r = \sqrt{x^2 + y^2 + z^2}\). Because \(H\) possesses rotational symmetry, the orbital angular momentum \(\vec{L} = \vec{r} \times \vec{p}\) is an integral of motion. This rotational symmetry implies that the orbit lies in some two dimensional plane, though it is not enough to ensure that the orbit is closed. An extra dynamical symmetry must exist for a closed orbit since the integrals of motion \(H\) and \(\vec{L}\) only reduce the phase space to a two dimensional (as opposed to a one dimensional) manifold. Such an integral was first discovered by Laplace (but is called the Runge-Lenz vector in classical mechanics or the Lenz-Pauli vector in quantum mechanics) and is given by

\[ \vec{A} = \vec{p} \times \vec{L} - \frac{\vec{r}}{r}. \]  

(29)

One can easily check that

\[ \{A_i, L_j\} = \epsilon_{ijk}A_k \]

and

\[ \{A_i, A_j\} = -\epsilon_{ijk} \left( \frac{\vec{p}^2}{2} - \frac{2}{r} \right) L_k = -2HL_k = -2EL_k, \]

where \(E\) is the constant value taken by \(H\). For bound state problems \((E < 0)\), one can define a new conserved vector \(\vec{D}\) as

\[ \vec{D} = \frac{\vec{A}}{\sqrt{-2E}}. \]

Then one finds that the commutation relations reduce to

\[ \{L_i, L_j\} = \epsilon_{ijk}L_k, \]

\[ \{D_i, L_j\} = \epsilon_{ijk}D_k, \]

\[ \{D_i, D_j\} = \epsilon_{ijk}L_k, \]
which is the Lie algebra \( \text{so}(4) \). (Note that for scattering problems where \( E > 0 \), one instead finds the Lie algebra \( \text{so}(3,1) \)).

Explicitly, we have

\[
H = \frac{p_1^2 + p_2^2 + p_3^2}{2} - \frac{1}{(q_1^2 + q_2^2 + q_3^2)^{1/2}},
\]

(30)

and,

\[
\begin{align*}
I_1 &= p_2(q_1p_2 - q_2p_1) - p_3(q_3p_1 - q_1p_3) - \frac{q_1}{(q_1^2 + q_2^2 + q_3^2)^{1/2}} = C_1, \\
I_2 &= p_3(q_2p_3 - q_3p_2) - p_1(q_1p_2 - q_2p_1) - \frac{q_2}{(q_1^2 + q_2^2 + q_3^2)^{1/2}} = C_2, \\
I_3 &= p_1(q_3p_1 - q_1p_3) - p_2(q_2p_3 - q_3p_2) - \frac{q_3}{(q_1^2 + q_2^2 + q_3^2)^{1/2}} = C_3,
\end{align*}
\]

(31)

\[
\begin{align*}
I_4 &= (q_2p_3 - q_3p_2) = C_4, \\
I_5 &= (q_3p_1 - q_1p_3) = C_5, \\
I_6 &= (q_1p_2 - q_2p_1) = C_6,
\end{align*}
\]

with the relations

\[
\begin{align*}
I_1^2 + I_2^2 + I_3^2 &= 1 + 2H(I_4^2 + I_5^2 + I_6^2), \\
I_1I_4 + I_2I_5 + I_3I_6 &= 0,
\end{align*}
\]

(32)

where, as before, the \( C_i \)'s are the constant values taken by the \( I_i \)'s. Therefore, there exists only five independent constants of motion as expected. As was the case for the harmonic oscillator, one may choose any five of the above six \( I_i \)'s (or products thereof) as the generalized Hamiltonians. For example, one may define the Nambu bracket for this system as

\[
\{f, I_2, I_3, I_4, I_5, I_6\} = \left(\frac{1}{C_4C_5^2 + C_6^2} \right) \frac{\partial(f, I_2, I_3, I_4, I_5, I_6)}{\partial(p_1, p_2, p_3, q_1, q_2, q_3)},
\]

or,

\[
\{f, I_1, I_3, I_4, I_5, I_6\} = \left(\frac{-1}{C_5C_4^2 + C_6^2} \right) \frac{\partial(f, I_1, I_3, I_4, I_5, I_6)}{\partial(p_1, p_2, p_3, q_1, q_2, q_3)},
\]

(33)

or,

\[
\{f, I_1, I_2, I_3, I_4, I_5\} = \left(\frac{-1}{C_3C_4^2 + C_5^2} \right) \frac{\partial(f, I_1, I_2, I_3, I_4, I_5)}{\partial(p_1, p_2, p_3, q_1, q_2, q_3)},
\]

etc. An interesting fact to note here is that we were unable to (directly) incorporate the original Hamiltonian \( H \) into a form of the Nambu bracket.

\section{Two More Examples}

First of all, let us analyse a Hamiltonian related to the motion of two vortices in an ideal incompressible fluid. A physical description of this system may be found in the monograph [10]. The Hamiltonian and the corresponding integrals of motion are

\[
H = \ln\left[(q_1 - q_2)^2 + (p_1 - p_2)^2\right] = E,
\]

(34)
and,
\[
I_1 = q_1 + q_2 = C_1, \\
I_2 = p_1 + p_2 = C_2, \\
I_3 = p_1^2 + p_2^2 + q_1^2 + q_2^2 = C_3. \\
\]

(35)

Since we have three independent integrals of motion and a four dimensional phase space, (35) incorporates a dynamical symmetry reducing the flow in phase space to a (not necessarily closed) curve. This system, like the previous examples, has several different Nambu brackets all producing the correct equations of motion. Below, we simply list the basic choices.

\[
\{ f, I_1, I_2, I_3 \} = \left( -\frac{1}{\exp(E)} \right) \frac{\partial(f, I_1, I_2, I_3)}{\partial(p_1, p_2, q_1, q_2)} \\
\{ f, H, I_1, I_2 \} = \left( \frac{-1}{2} \right) \frac{\partial(f, H, I_1, I_2)}{\partial(p_1, p_2, q_1, q_2)} \\
\{ f, H, I_1, I_3 \} = \left( \frac{-1}{2C_2} \right) \frac{\partial(f, H, I_1, I_3)}{\partial(p_1, p_2, q_1, q_2)} \\
\{ f, H, I_2, I_3 \} = \left( \frac{1}{2C_1} \right) \frac{\partial(f, H, I_2, I_3)}{\partial(p_1, p_2, q_1, q_2)}. \\
\]

(36)

(37)

(38)

(39)

Note that the constant factor in the above examples is usually related to the integral of motion not chosen to be one of the Hamiltonians (as was the case in the oscillator example of Section 3). One can use the derivation property of the Nambu bracket to find other brackets from the four listed above. For instance, one can show that (37) and (38) are consistent with

\[
\{ f, H, I_1, I_2, I_3 \} = \left( \frac{-1}{2(C_3 + C_2^2)} \right) \frac{\partial(f, H, I_1, I_2, I_3)}{\partial(p_1, p_2, q_1, q_2)}, \\
\]

(40)

and so on.

Finally, consider the (unphysical) Hamiltonian

\[
H = q_1(p_1 - q_1) - q_2(p_2 - q_2), \\
\]

(41)

and its integrals of motion

\[
I_1 = q_1(p_1 - q_1) = C_1, \\
I_2 = q_2(p_2 - q_2) = C_2, \\
I_3 = q_1q_2 = C_3. \\
\]

(42)

Since \( H = I_1 - I_2 \), there exists only the following basic Nambu bracket,

\[
\{ f, I_1, I_2, I_3 \} = \left( \frac{1}{C_3} \right) \frac{\partial(f, I_1, I_2, I_3)}{\partial(p_1, p_2, q_1, q_2)}. \\
\]

(43)

This two-dimensional system can be extended to higher dimensions by simply adding the appropriate terms to the Hamiltonian (41) (such as \( q_3(p_3 - q_3) \)) and finding the extra integrals of motion (i.e. \( q_1q_3 \) or \( q_2q_3 \)).
6 Conclusion

We have demonstrated that several Hamiltonian systems possessing dynamical symmetries can be realized in the Nambu formalism of generalized mechanics. For all but one of these systems, an extra freedom was found in the choice of the generalized Hamiltonians needed for their Nambu construction. Finally, one may speculate that since the harmonic oscillator is a very important example in quantum mechanics, its Nambu formulation may lead to a better understanding of the yet unsolved problem of the quantization of Nambu mechanics.

Acknowledgements
The author is grateful to L.A. Takhtajan for suggesting this problem and for many helpful discussions. Thanks also go to M. Flato for suggesting improvements in Section 4 of this paper.

7 References

1. Nambu, Y., Phys. Rev. D. 7, 2405 (1973).
2. Takhtajan, L.A., Comm. Math. Phys. 160, 295 (1994).
3. Chakravarty, S., and Clarkson, P., Phys. Rev. Lett. 65, 1085 (1990).
4. Takhtajan, L.A., preprint PAM no.121, University of Colorado (1991).
5. Arnold, V.I., Mathematical Methods Of Classical Mechanics, Springer-Verlag, Berlin (1978).
6. Takhtajan, L.A., in Mo-Lin Ge and Bao-Heng Hao (eds), Introduction To Quantum Groups and Integrable Massive Models of Quantum Field Theory, World Scientific, Singapore, (1990).
7. Vaisman, I., Lectures on the Geometry of Poisson Manifolds, Birkhauser, Berlin, (1994).
8. Sahoo, D., and Valsakumar, M.C., Mod. Phys. Lett. A. 9, 2727, (1994).
9. Sahoo, D., and Valsakumar, M.C., Phys. Rev. A. 46, 4410 (1992).
10. Arnold, V.I., Dynamical Systems 3, Springer-Verlag, Berlin, (1988).