A general formalism for the stability of Kerr

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Abstract

The goal of this paper is to provide a geometric framework for analyzing the uniform decay properties of solutions to the Teukolsky equation in the fully nonlinear setting of perturbations of Kerr. It contains the first nonlinear version of the Chandrasekhar transformation introduced in the linearized setting in [7] and [14] with the intent to use it in our ongoing project to prove the full nonlinear stability of slowly rotating Kerr as solution to the Einstein vacuum equations.

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1 Introduction

The goal of this paper is to provide the geometric framework for analyzing the uniform decay properties of solutions to the Teukolsky equation in the fully nonlinear setting of perturbations of Kerr, as solutions to the Einstein vacuum equations

$$\text{Ric}(g) = 0.$$  (1)

In the linearized setting, i.e. the Teukolsky equation on a fixed Kerr background with small angular momentum, this was achieved independently in [7] and [14] by a suitable version of the so-called Chandrasekhar transformation. We recall that this transformation was first introduced by Chandrasekhar to pass from the Teukolsky equation to a Regge-Wheeler equation in the context of mode stability for the linearized gravity system around Schwarzschild, see [3]. A first physical version of that transformation appeared in [6] and was used by the authors to derive robust uniform decay estimates for the Teukolsky equation in Schwarzschild. Its nonlinear version appears in [11] where it plays an important role in the proof of the full nonlinear stability of Schwarzschild for axially symmetric polarized perturbations. In this paper we derive the first nonlinear version of the transformations introduced in [7] and [14] with the intent to use it in our ongoing project to prove the full nonlinear stability of slowly rotating Kerr as solution to
the Einstein vacuum equations. Note also that the result of [14] was used in [2] to provide a first proof of linearized stability of Kerr. A different proof, based on generalized wave coordinates, was also given in [9].

1.1 General comments on perturbations of Kerr

A proof of the nonlinear stability\(^1\) of Kerr, at least for small angular momentum, requires the following major ingredients.

1. A formalism to derive the Teukolsky and Regge-Wheeler type equations in the nonlinear setting.
2. An analytic mechanism to derive estimates for these.
3. A dynamical mechanism to identify the final mass and final angular momentum.
4. A dynamical mechanism for finding the right gauge conditions in which the very notion of convergence to the final state can be formulated.
5. A precisely formulated continuity argument based on a grand bootstrap scheme which assigns to all geometric quantities involved in the process specific decay rates, which can be dynamically recovered from the initial conditions by a long series of estimates, and thus ensure convergence to a final Kerr state.

In [12] and [13], the last two authors have provided a framework for dealing with the issue 4, by constructing generalized notions of generally covariant modulated (GCM) spheres\(^2\) in the asymptotic region of a general perturbation of Kerr. In connection to the issue 3, it is expected, as in [11], that the Hawking mass associated to the 2-spheres of a well adapted spacetime foliation will converge to the final mass. In [13], a definition of angular momentum for GCM spheres was also given with the expectation that it will play a similar role in detecting the final angular momentum of a general perturbation of Kerr.

In this paper we deal with the first issue, i.e. we provide a geometric framework in which the Teukolsky and Regge-Wheeler equations can be derived for general, nonlinear, realistic perturbations of Kerr. We emphasize the word realistic by which we mean that the perturbations have decay properties consistent with the bootstrap assumptions mentioned in 5. We make these assumptions here by adapting those made in [11].

\(^1\)We refer here to a geometric proof similar in spirit to that of the recent proof of the nonlinear stability of Schwarzschild under polarized perturbations [11]. Other approaches are also being pursued, most notably those based on a generalized notion of wave coordinates.

\(^2\)Generalizing those used in the nonlinear stability of Schwarzschild in the polarized case, see [11].
1.2 Teukolsky equation in linearized theory

It has been known since the work of Teukolsky [16] [17] that, for linearized perturbations of Kerr, the extreme components of the curvature tensor verify decoupled wave equations in a fixed Kerr background. As mentioned above, a first robust derivation of decay estimates for solutions of the Teukolsky equation in Schwarzschild was given in [6], based on a physical space variant of the Chandrasekhar transformation\(^3\). The method was recently extended to the Teukolsky equation in Kerr spacetimes \(Kerr(a, m)\) with \(|a| \ll m\) independently in [14] and [7].

Both [7] and [14] start from the well known second order Teukolsky equation \(L(\alpha^{\pm 2}) = 0\) in a Kerr spacetime \(Kerr(a, m)\). The complex scalar \(\alpha^{\pm 2}\) (for spin +2 or spin −2) is defined, in the context of the Newman-Penrose (NP) formalism, to be one of the extreme curvature components, relative to a principal null frame\(^4\). The Teukolsky equation for \(\alpha^{\pm 2}\) is a decoupled wave equation which is obtained as consequence of the linearized Einstein vacuum equations. Both [7] and [14] transform \(\alpha^{\pm 2}\), by an appropriate generalized Chandrasekhar transformation, into a new complex scalar \(\Psi = Q(\alpha^{\pm 2})\), with \(Q\) a second order operator (in the direction of the ingoing null direction for spin +2, outgoing for spin −2), expressed relative to the standard Kerr coordinates. The complex scalar \(\Psi\) verifies a generalized Regge-Wheeler equation of the schematic form

\[
\Box_{a,m} \Psi + ia c(r, \theta) \partial_t \Psi + V(r, \theta) \Psi = a L_Q(\alpha^{\pm 2})
\]  

(2)

where \(\Box_{a,m}\) is the D’Alembertian operator associated to the Kerr metric \(Kerr(a, m)\), \(c(r, \theta)\) is a function of \(r\) and \(\theta\), \(V(r, \theta)\) is a favorable potential and \(L_Q(\alpha^{\pm 2})\) are lower order terms in the derivatives of \(\alpha^{\pm 2}\). Though the complex scalar \(\alpha^{\pm 2}\) is defined using a principal null tetrad via the NP formalism, all calculations which take the Teukolsky equation to (2) are done in coordinates. This approach leads to serious difficulties when one tries to extend the calculations in the nonlinear setting where the precise structure of the nonlinear terms is essential.

In the setting of polarized perturbations of Schwarzschild [11], this calculation was performed using null frames, which are both adapted to an integrable foliation and almost diagonalize the curvature tensor\(^5\). One could thus rely on the geometric formalism developed in the context of the proof of the nonlinear stability of Minkowski space [5]. This latter cannot be straightforwardly applied from Minkowski to perturbations of Kerr. To capture the almost diagonalizable properties of principal null frames one has to give up on integrability and thus, seemingly, forced to work with a general Newman-Penrose (NP) formalism, as introduced in [15]. The NP formalism, however, presents its own set of difficulties. Indeed complex calculations, such those needed to derive the nonlinear analogue of (2), depend on higher derivatives of all connection

\(^3\)Which takes solutions of the Teukolsky equation into solutions of a Regge-Wheeler type equation, that can then be analyzed using a variant of the vectorfield method based on Morawetz and \(r^p\)-weighted estimates.

\(^4\)We recall that principal null frames have the very important property of diagonalizing the curvature tensor.

\(^5\)i.e. such frames diagonalize the curvature tensor up to terms which are small with respect to the perturbation. These frames are thus small perturbations of standard null frames in Schwarzschild which are both principal and integrable.
coefficients of the NP frame rather than only those which are geometrically significant. This could seriously affect the structure of non-linear corrections.

In our work we rely instead on a tensorial approach, based on horizontal structures which closely mimics the calculations done in integrable settings while maintaining the important diagonalizable properties of the principal directions. This allows us to maintain, with minimal changes, the geometric formalism of [5] widely used today in mathematical GR. We give a full account of it in this paper\(^6\) and use it to derive the proper nonlinear corrections to (2). We will show in fact that these corrections admit a structure very similar to the one found in [11], see Theorem 2.4.7 in that paper.

1.3 Short summary of the formalism

We give here a short summary of the formalism, which is extensively described in section 2.

1.3.1 Basic definitions

Let \((\mathcal{M}, g)\) a Lorentzian space-time. Consider a fixed null pair of vectorfields \((e_3, e_4)\), i.e.

\[
g(e_3, e_3) = g(e_4, e_4) = 0, \quad g(e_3, e_4) = -2,
\]

and denote by \(\mathbf{O}(\mathcal{M})\) the vector space of horizontal vectorfields \(X\) on \(\mathcal{M}\), i.e. \(g(e_3, X) = g(e_4, X) = 0\). A null frame \((e_3, e_4, e_1, e_2)\) on \(\mathcal{M}\) consists, in addition to the null pair \((e_3, e_4)\), of a choice of horizontal vectorfields \((e_1, e_2)\), such that

\[
g(e_a, e_b) = \delta_{ab} \quad a, b = 1, 2.
\]

The commutator \([X, Y]\) of two horizontal vectorfields may fail however to be horizontal. We say that the pair \((e_3, e_4)\) is integrable if \(\mathbf{O}(\mathcal{M})\) forms an integrable distribution, i.e. \(X, Y \in \mathbf{O}(\mathcal{M})\) implies that \([X, Y] \in \mathbf{O}(\mathcal{M})\). As it is well-known, the principal null pair in Kerr fails to be integrable, see also Remark 1.1. Given an arbitrary vectorfield \(X\) we denote by \((^hX)\) its horizontal projection,

\[
(^hX) = X + \frac{1}{2}g(X, e_3)e_4 + \frac{1}{2}g(X, e_4)e_3.
\]

For any \(X, Y \in \mathbf{O}(\mathcal{M})\) we define \(\gamma(X, Y) = g(X, Y)\) and\(^7\)

\[
\chi(X, Y) = g(D_Xe_4, Y), \quad \chi(X, Y) = g(D_Xe_3, Y).
\]

\(^6\)Elements of the horizontal structure framework were also shortly discussed in the appendix to [10] as an alternative to the NP formalism. Relations between the two formalisms are addressed there as well.

\(^7\)In the particular case where the horizontal structure is integrable, \(\gamma\) is the induced metric, and \(\chi\) and \(\chi\) are the null second fundamental forms.
Observe that $\chi$ and $\chi$ are symmetric if and only if the horizontal structure is integrable. Indeed this follows easily from the formulas\footnote{Note that we can view $\chi$ and $\chi$ as horizontal 2-covariant tensor-fields by extending their definition to arbitrary $X, Y$, by $\chi(X, Y) = \chi^{(b)}(X^{(h)} Y)$, $\chi(X, Y) = \chi^{(b)}(X^{(h)} Y)$.},

$$
\chi(X, Y) - \chi(Y, X) = g(D_\chi e_4, Y) - g(D_\chi e_4, X) = -g(e_4, [X, Y]), \\
\chi(X, Y) - \chi(Y, X) = g(D_\chi e_3, Y) - g(D_\chi e_3, X) = -g(e_3, [X, Y]).
$$

We define their trace $\text{tr} \chi$, $\text{tr} \chi$, and anti-trace $\text{tr} \chi$, $\text{tr} \chi$ as follows

$$
\text{tr} \chi := \delta^{ab} \chi_{ab}, \quad \text{tr} \chi := \delta^{ab} \chi_{ab}, \quad (a)\text{tr} \chi := \varepsilon^{ab} \chi_{ab}, \quad (a)\text{tr} \chi := \varepsilon^{ab} \chi_{ab}.
$$

Accordingly, we decompose $\chi, \chi$ as follows

$$
\chi_{ab} = \dot{\chi}_{ab} + \frac{1}{2} \delta_{ab} \text{tr} \chi + \frac{1}{2} \varepsilon_{ab} (a)\text{tr} \chi, \\
\chi_{ab} = \dot{\chi}_{ab} + \frac{1}{2} \delta_{ab} \text{tr} \chi + \frac{1}{2} \varepsilon_{ab} (a)\text{tr} \chi.
$$

**Remark 1.1.** The non integrability of $(e_3, e_4)$ corresponds to non vanishing $(a)\text{tr} \chi$ and $(a)\text{tr} \chi$. A celebrated example of a non integrable null frame is the principal null frame of Kerr for which $(a)\text{tr} \chi$ and $(a)\text{tr} \chi$ are indeed non trivial.

We define the horizontal covariant operator $\nabla$ as follows:

$$
\nabla_X Y := (h)(D_X Y) = D_X Y - \frac{1}{2} \chi(X, Y) e_4 - \frac{1}{2} \chi(X, Y) e_3, \quad X, Y \in \mathcal{O}(M).
$$

Note that,

$$
\nabla_X Y - \nabla_Y X = [X, Y] - \frac{1}{2} \left((a)\text{tr} \chi e_4 + (a)\text{tr} \chi e_3 \right) \in (X, Y).
$$

Note that $\nabla$ acts like a Levi-Civita connection i.e., for all $X, Y, Z \in \mathcal{O}(M),$

$$
Z \gamma(X, Y) = \gamma(\nabla_Z X, Y) + \gamma(X, \nabla_Z Y).
$$

We can then define connection and curvature coefficients similar to those in the integrable case, as in [5],

$$
\chi_{ab} = g(D_a e_3, e_b), \quad \chi_{ab} = g(D_a e_4, e_b), \quad \xi_a = \frac{1}{2} g(D_3 e_3, e_a), \quad \xi_a = \frac{1}{2} g(D_4 e_4, e_a), \\
\omega = \frac{1}{4} g(D_3 e_3, e_4), \quad \omega = \frac{1}{4} g(D_4 e_4, e_3), \quad \eta_a = \frac{1}{2} g(D_3 e_4, e_a), \quad \eta_a = \frac{1}{2} g(D_4 e_4, e_a), \\
\zeta_a = \frac{1}{2} g(D_a e_4, e_3),
$$

$$
\alpha_{ab} = R_{a4b4}, \quad \beta_a = \frac{1}{2} R_{a434}, \quad \beta_a = \frac{1}{2} R_{a334}, \quad \rho_{ab} = R_{a3b3}, \quad \rho = \frac{1}{4} R_{3434}, \quad \rho = \frac{1}{4} R_{3434},
$$

and derive the corresponding null structure and null Bianchi equations, see Propositions 2.29 and 2.32.
Remark 1.2. The main advantage of the formalism presented above is that, with the exception of the presence of the integrable defect scalars \((a)\tr \chi, (a)\tr \chi', \tr \chi', (a)\tr \chi, \tr \chi'_a = \lambda^{a-1} \chi\), the null structure and null Bianchi equations look very similar to the familiar equations in [5].

### 1.3.2 Conformally invariant derivative operators

The Ricci and curvature coefficients depend, of course, on the particular null frame we choose. Of particular importance in our work here are the conformal frame transformations

\[
e'_{3} = \lambda e_{3}, e'_{4} = \lambda e_{4}, e'_{a} = e_{a}.
\]

Note, in particular, that under such a frame transformation we have

\[
\tr \chi' = \lambda^{-1} \tr \chi, \quad (a)\tr \chi' = \lambda^{a-1} (a)\tr \chi, \quad \tr \chi' = \lambda \tr \chi, \quad (a)\tr \chi' = \lambda^{a} (a)\tr \chi,
\]

\[
\xi' = \lambda \xi, \quad \eta' = \eta, \quad \eta' = \eta, \quad \xi' = \lambda^{-1} \xi,
\]

\[
\alpha' = \lambda^{2} \alpha, \quad \beta' = \lambda \beta, \quad \rho' = \rho, \quad \omega' = \omega, \quad \zeta' = \zeta, \quad (a)\tr \chi' = \lambda^{a} (a)\tr \chi,
\]

and

\[
\omega' = \lambda^{-1} \left( \omega + \frac{1}{2} e_{3} (\log \lambda) \right), \quad \omega' = \lambda \left( \omega - \frac{1}{2} e_{4} (\log \lambda) \right), \quad \zeta' = \zeta - \nabla (\log \lambda).
\]

We say that a horizontal tensor \(f\) is \(s\)-conformal invariant if, under the conformal frame transformation above it changes as \(f' = \lambda^{s} f\). According to this definition \(\chi\) is 1-conformal, \(\alpha\) is 2-conformal, \ldots, while \(\omega, \omega, \zeta\) fail to be conformal.

Remark 1.3. If \(f\) verifies \(f' = \lambda^{s} f\), then the derivatives \(\nabla_{3} f, \nabla_{4} f, \nabla_{a} f\) are not conformal invariant. Note however that

- \(\nabla_{3} f := \nabla_{3} f - 2 s \omega f\) is \((s - 1)\)-conformally invariant.
- \(\nabla_{4} f := \nabla_{4} f + 2 s \omega f\) is \((s + 1)\)-conformally invariant.
- \(\nabla_{A} f := \nabla_{A} f + s \zeta_{A} f\) is \(s\)-conformally invariant.

The conformal invariant operator \(\nabla_{3}\) played an important role in the derivation of the Regge-Wheeler equation in [11]. Here they play an equally important role.

### 1.3.3 Complexification and values in Kerr

The null structure and null Bianchi equations simplify considerably by introducing complex notations such as

\[
A := \alpha + i \* \alpha, \quad B := \beta + i \* \beta, \quad P := \rho + i \* \rho, \quad B := \beta + i \* \beta, \quad A := \alpha + i \* \alpha,
\]

\[
X := \chi + i \* \chi, \quad X := \chi + i \* \chi, \quad H := \eta + i \* \eta, \quad H := \eta + i \* \eta, \quad Z := \zeta + i \* \zeta.
\]

\(\text{Note that } s \text{ is precisely what is called in [5] the signature of the tensor.}\)
In particular, note that $\mathrm{tr} X = \mathrm{tr} \chi - i^{(a)} \mathrm{tr} \chi$, $\mathrm{tr} \overline{X} = \mathrm{tr} \chi - i^{(a)} \mathrm{tr} \chi$.

These complexified tensors take a particularly simple form in Kerr, relative to a principal null frame\(^{10}\), see section 5.1,

$$\tilde{X} = \check{X} = \Xi = \Xi = \omega = 0, \quad A = B = \overline{A} = 0,$$

and

$$\mathrm{tr} X = \frac{2}{q}, \quad \mathrm{tr} \overline{X} = -\frac{2\Delta q}{|q|^2}, \quad P = -\frac{2m}{q^3},$$

where $q = r + i a \cos \theta$ and $\Delta = r^2 + a^2 - 2mr$ relative to the Boyer-Lindquist coordinates $(r, \theta)$.

We note also that $q$ verifies the equations, see (67),

$$\nabla_4 q = \frac{1}{2} \mathrm{tr} X q, \quad \nabla_3 q = -\frac{1}{2} \overline{\mathrm{tr} X} q, \quad \mathcal{D} q = q \mathcal{H}, \quad \overline{\mathcal{D}} q = q \overline{\mathcal{H}}.$$

(4)

### 1.4 General perturbations of Kerr

To state our main theorem we need to have a sufficiently general candidate for spacetime structures which are small perturbations of Kerr.

**Definition 1.4.** We say that a spacetime $\mathcal{M}$ is an $O(\epsilon)$ perturbation of Kerr$(a, m)$, $|a| < m$, if $\mathcal{M}$ comes equipped with two coordinate functions $r : \mathcal{M} \rightarrow (0, \infty)$, $\theta : \mathcal{M} \rightarrow [0, \pi]$ and a null frame $(e_3, e_4, e_1, e_2)$ such that all Ricci and curvature coefficients which vanish in Kerr are $O(\epsilon)$ and all other quantities differ by $O(\epsilon)$ from their corresponding values in Kerr, expressed with respect to $(r, \theta)$.

We introduce a schematic notation, similar to the one used in [11], see Definition 2.3.8 in that paper, to keep track of the specific structure of error terms. We divide the connection coefficient terms into

$$\Gamma^{(0)}_g = \left\{ r\Xi, \quad \tilde{X}, \quad \check{Z}, \quad \mathcal{H}, \quad \frac{1}{r} \nabla_4 q - \frac{1}{2r} \mathrm{tr} X q, \quad \frac{1}{r} (\mathcal{D} q - q \mathcal{H}), \quad \frac{1}{r} (\overline{\mathcal{D}} q - q \overline{\mathcal{H}}) \right\},$$

$$\Gamma^{(0)}_b = \left\{ \tilde{\mathcal{H}}, \quad \tilde{X}, \quad \Xi, \quad \frac{1}{r} \nabla_3 q - \frac{1}{2r} \overline{\mathrm{tr} X} q \right\},$$

where, given a quantity $Q$, we have denoted by $\check{Q}$ the renormalized quantity $\check{Q} = Q - Q_{Kerr}$, with $Q_{Kerr}$ the corresponding value of $Q$ in Kerr, expressed in terms of the variables $r$ and $\theta$.

Thus, for example,

$$\check{P} = P + \frac{2m}{q^3}, \quad q = r + i a \cos \theta.$$

\(^{10}\)There is an indeterminacy in the principal null frame as one may replace the null pair $(e_3, e_4)$ with $(\lambda^{-1} e_3, \lambda e_4)$ for any $\lambda > 0$. The formulas provided here correspond to the arbitrary choice of $\lambda > 0$ ensuring $\nabla_4 e_4 = 0$ and thus $\omega = 0$. Note that our main result, stated in Theorem 1.8, is independent of this particular choice.
For higher derivatives, we denote
\[ \Gamma^{(s)}_g = \mathcal{D} \leq^s \Gamma_g, \quad \Gamma^{(s)}_b = \mathcal{D} \leq^s \Gamma_b, \]
where
\[ \mathcal{D} = \{ \nabla_3, r\nabla_4, r\mathcal{D} \}. \]

**Remark 1.5.** The notations above embody the expected decay properties of the Ricci coefficients, that are better for \( \Gamma_g \) than for \( \Gamma_b \). The notation \( \mathcal{D} \) corresponds to the fact that the decay properties of derivatives behave differently in different directions.

### 1.5 Statement of the main theorems

#### 1.5.1 Teukolsky equation

Making use of the null Bianchi equations for \( A \) and \( B \), expressed using our conformal derivatives, we derive the following version of the Teukolsky equation.

**Proposition 1.6** (Teukolsky equation). The complex tensor \( A \) satisfies the following equation:

\[
\mathcal{L}(A) = \text{Err}[\mathcal{L}(A)]
\]

where\(^{11}\)

\[
\mathcal{L}(A) = - \mathfrak{c}\nabla_4 \mathfrak{c}\nabla_3 A + \frac{1}{2} \mathfrak{c}\mathcal{D} \hat{\otimes} (\mathfrak{c}\mathcal{D} \cdot A) + \left( -\frac{1}{2} \text{tr} X - 2\text{tr} X \right) \mathfrak{c}\nabla_3 A - \frac{1}{2} \text{tr} A \mathfrak{c}\nabla_4 A
\]
\[ + (4\mathcal{H} + \hat{ \mathcal{H} } + \mathcal{H} ) \cdot \mathfrak{c} \nabla A + (-\text{tr} X \text{tr} X + 2\overline{\mathcal{P}}) A + 2 \mathcal{H} \hat{\otimes} (\overline{\mathcal{H}} \cdot A) \]

with error term expressed schematically\(^{12}\)

\[
\text{Err}[\mathcal{L}(A)] = r^{-1} \mathcal{D} \leq^1 (\Gamma_g B) + \Xi \nabla_3 B + \text{t.o.t.}
\]

A comparison between this version of the Teukolsky equation and the traditional version\(^{13}\) derived in the NP formalism is done in Appendix A.

\(^{11}\)The operators \((c)\nabla_4, (c)\nabla_3, (c)\nabla, (c)\mathcal{D}\) are extensions of the conformal operators, introduced in Remark 1.3, for complex tensors.

\(^{12}\)The error terms denoted t.o.t. are quadratic in the perturbation and enjoy better decay properties, or are higher order and decay at least as good.

\(^{13}\)Note that \( A \) here is a complex horizontal 2-tensor, while the standard Teukolsky equation is expressed relative to the complex scalar \( \alpha^{(\pm)} \).
1.5.2 Generalized Regge-Wheeler equation

We look for a quantity which generalizes the quantity $q$ used in [11]. In analogy with [11], it is natural to look for a renormalized version of the conformally invariant tensor

$$Q(A) = \langle c \rangle \nabla_3 \langle c \rangle \nabla_3 A + C \langle c \rangle \nabla_3 A + D A$$

(9)

for some well chosen complex scalar functions $C, D$. Note that any such expression is $O(\epsilon^2)$ invariant and vanishes in Kerr.

**Definition 1.7.** Given a general null frame $(e_3, e_4, e_1, e_2)$ and given scalar functions $r$ and $\theta$ satisfying the assumptions in section 5.3, we define our main quantity $q$ as

$$q = qT^3 Q(A) = qT^3 \left( \langle c \rangle \nabla_3 \langle c \rangle \nabla_3 A + C \langle c \rangle \nabla_3 A + D A \right)$$

(10)

where $q = r + ia \cos \theta$, and where the complex scalar functions $C, D$ are to be suitably chosen.

We are now ready to state the main result of the paper, concerning the wave equation satisfied by $q$.

**Theorem 1.8** (The generalized Regge-Wheeler equation). There exist choices of complex scalar functions $C, D$ such that $q$ defined above verifies the equation

$$\square_2 q - \frac{4ia \cos \theta}{|q|^2} T(q) - Vq = a L_q[A] + Err[\square_2 q]$$

(11)

where

- $T$ is a vectorfield defined by (71) in the nonlinear case, which reduces to $\partial_t$ in Kerr,
- the potential $V$ is a complex scalar function whose real part coincides with the potential of the Regge-Wheeler equation in [11], i.e. $V = -tr \chi tr \chi + O \left( \frac{|a|}{r^3} \right)$,
- $L_q[A]$ is a linear second order operator in $A$, given by

$$L_q[A] = c_1 \langle c \rangle \nabla_2 \langle c \rangle \nabla_3 A + c_2 \langle c \rangle \nabla_3 A + c_3 \langle c \rangle \nabla(A) + c_4 A$$

with $c_1, \ldots, c_4$ smooth functions of $(r, \theta)$ having the following fall-off in $r$

$$c_1 = O(r), \quad c_2 = O(1), \quad c_3 = O(1), \quad c_4 = O \left( \frac{1}{r} \right),$$

- $Err[\square_2 q]$ is the nonlinear correction term, which is given schematically by the expression

$$Err[\square_2 q] = r^2 \delta^{c2} (\nabla g \cdot (A, B)) + \nabla_3 (r^2 \delta^{c2} (\nabla g \cdot (A, B)))$$

$$+ \delta^{c1} (\nabla g \cdot q) + r^2 \tilde{H} \nabla^{c1} \nabla_3^{c1} A + l.o.t.$$

\[14\]The error terms denoted l.o.t. are quadratic in the perturbation and enjoy better decay properties, or are higher order and decay at least as good.
Remark 1.9. Note the similarity between the structure of the nonlinear terms here with that of Theorem 2.4.7 in [11].

Remark 1.10. Observe that the generalized Regge-Wheeler equation (11) for the complexified symmetric traceless 2-tensor \( q \) reduces to the equation (2) obtained in [14] and [7] for complex scalar functions once the equation is projected to the component \( q(e_1, e_1) \), as shown in section 8.2. A general discussion of such projections can be found in section 6.3.

1.6 Structure of the paper

The structure of the paper is as follows.

- In section 2, we introduce our general formalism.
- In section 3, we introduce our complex notations and rewrite the equations of section 2 that considerably simplify in this setting.
- In section 4, we express the covariant wave operator for complexified symmetric traceless 2-tensors using our complex notations.
- In section 5, we specify our general formalism in the particular case of Kerr.
- In section 6, we derive the Teukolsky equation in the nonlinear setting.
- In section 7, we derive the generalized Regge-Wheeler equation in the nonlinear setting.
- In section 8, we derive useful identities involving the complexified symmetric traceless 2-tensor \( q \).

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2 A general formalism

2.1 Null pairs and horizontal structures

Let \((M, g)\) a Lorentzian space-time. Consider an arbitrary null pair \(e_3 = L, e_4 = L\), i.e.
\[ g(e_3, e_3) = g(e_4, e_4) = 0, \quad g(e_3, e_4) = -2. \]

**Definition 2.1.** We say that a vectorfield \(X\) is \((L, L)\)-horizontal, or simply horizontal, if
\[ g(L, X) = g(L, X) = 0. \]
We denote by \(O(M)\) the set of horizontal vectorfield on \(M\). Given a fixed orientation on \(M\), with corresponding volume form \(\in\), we define the induced volume form on \(O(M)\) by,
\[ \in(X, Y) := \frac{1}{2} \in(X, Y, L, L). \]

Clearly, any linear combination of horizontal vectorfields is again horizontal. The commutator \([X, Y]\) of two horizontal vectorfields may fail however to be horizontal. We say that the pair \((L, L)\) is integrable if the set of horizontal vectorfields forms an integrable distribution, i.e. \(X, Y \in O(M)\) implies that \([X, Y] \in O(M)\). In this work we will work with general, not necessarily integrable null pairs.

Given an arbitrary vectorfield \(X\) we denote by \(^{(h)}X\) its horizontal projection,
\[ ^{(h)}X = X + \frac{1}{2}g(X, L)L + \frac{1}{2}g(X, L)L. \]

**Definition 2.2.** A \(k\)-covariant tensor-field \(U\) is said to be horizontal, \(U \in O_k(M)\), if for any \(X_1, \ldots X_k\) we have,
\[ U(X_1, \ldots X_k) = U^{(h)}X_1, \ldots ^{(h)}X_k. \]

We can define the projection operator,
\[ \Pi^{\mu\nu} = g^{\mu\nu} + \frac{1}{2}(L^\mu L^\nu + L^\mu L^\nu). \]

Clearly \(\Pi^\mu_\mu \Pi^\alpha_\nu = \Pi^\alpha_\nu\). An arbitrary tensor \(U_{\alpha_1 \cdots \alpha_m}\) is horizontal, if
\[ \Pi^{\beta_1}_\alpha \cdots \Pi^{\beta_m}_\alpha U_{\beta_1 \cdots \beta_m} = U_{\alpha_1 \cdots \alpha_m}. \]

**Definition 2.3.** For any horizontal \(X, Y\) we define\(^{15}\)
\[ \gamma(X, Y) = g(X, Y) \]
\[ \chi_1 \chi_2 \]
\[ \chi_3 \chi_4 \]
\[ \chi_5 \chi_6 \]
\[ \chi_7 \chi_8 \]

\(^{15}\)In the particular case where the horizontal structure is integrable, \(\gamma\) is the induced metric, and \(\chi\) and \(\chi\) are the null second fundamental forms.
and
\[
\begin{align*}
\chi(X,Y) &= g(D_X L, Y), \\
\bar{\chi}(X,Y) &= g(D_X L, Y).
\end{align*}
\] (14)

Observe that \(\chi\) and \(\bar{\chi}\) are symmetric if and only if the horizontal structure is integrable. Indeed this follows easily from the formulas,
\[
\begin{align*}
\chi(X,Y) - \chi(Y,X) &= g(D_X L, Y) - g(D_Y L, X) = -g(L, [X,Y]), \\
\bar{\chi}(X,Y) - \chi(Y,X) &= g(D_X L, Y) - g(D_Y L, X) = -g(L, [X,Y]).
\end{align*}
\]

We can view \(\gamma, \chi\) and \(\bar{\chi}\) as horizontal 2-covariant tensor-fields by extending their definition to arbitrary vectorfields \(X, Y\) according to,
\[
\gamma(X,Y) = \gamma(hX, hY), \quad \chi(X,Y) = \chi(hX, hY), \quad \bar{\chi}(X,Y) = \bar{\chi}(hX, hY).
\]

Given a general 2- covariant horizontal tensor \(U\) we decompose it in its symmetric and antisymmetric part as follows,
\[
\begin{align*}
{}^\text{s}U(X,Y) &= \frac{1}{2}(U(X,Y) + U(Y,X)), \\
{}^\text{a}U(X,Y) &= \frac{1}{2}(U(X,Y) - U(Y,X)).
\end{align*}
\] (15)

Given an horizontal structure defined by \(e_3 = L, e_4 = L\) we associate a null frame by choosing orthonormal horizontal vectorfields \(e_1, e_2\) such that \(\gamma(e_a, e_b) = \delta_{ab}\). By convention, we say that \((e_1, e_2)\) is positively oriented on \(O(\mathcal{M})\) if,
\[
\in(e_1, e_2) = \in(e_1, e_2, e_3, e_4) = 1.
\] (16)

Given a covariant, horizontal, 2-tensor \(U\) and an arbitrary orthonormal horizontal frame \((e_a)_{a=1,2}\) we have,
\[
\begin{align*}
{}^\text{s}U_{ab} &= \frac{1}{2}(U_{ab} + U_{ba}), \\
{}^\text{a}U_{ab} &= \frac{1}{2}(U_{ab} - U_{ba}).
\end{align*}
\]

We define the trace and anti-trace according to,

**Definition 2.4.** The trace of a horizontal 2-tensor \(U\) is defined by \(\text{tr}(U) := \delta^{ab}U_{ab} = \delta^{ab}(s)U_{ab}\). (17)

We define the anti-trace of \(U\) by,
\[
(\text{a}tr)(U) := \epsilon^{ab}U_{ab} = \epsilon^{ab}(a)U_{ab}.
\] (18)

Observe that the first trace is independent of the particular choice of the frame \(e_1, e_2\). On the other hand, for fixed \(e_3, e_4\), \((\text{a}tr)\) depends on the orientation of \(e_1, e_2\). Also, by interchanging \(e_3, e_4\), \((\text{a}tr)\) changes sign.
A general horizontal, 2-tensor $U$ can be decomposed according to,

$$ U_{ab} = (s)U_{ab} + (a)U_{ab} = \hat{U}_{ab} + \frac{1}{2} \delta_{ab} \text{tr}(U) + \frac{1}{2} \epsilon_{ab} (a)\text{tr}(U). \quad (19) $$

In what follows we fix a null pair $e_3, e_4$ and an orientation on $O(M)$.

**Definition 2.5.** We define the left and right duals of horizontal 1-forms $\omega$ and 2- covariant tensor-fields $U$,

$$ \ast \omega_a = \epsilon_{ab} \omega_b, \quad \omega^* a = \omega_b \in ba, \quad (20) $$

$$ (\ast U)_{ab} = \epsilon_{ac} U_{cb}, \quad (U^*)_{ab} = U_{ac} \in cb. \quad (21) $$

**Lemma 2.6.** Given a horizontal 1-form $\omega$, we have

$$ \ast(\ast \omega) = -\omega, \quad \ast \omega = -\omega^*. $$

**Lemma 2.7.** Given an arbitrary covariant, horizontal 2-tensor $U$, we have

1. $\ast (\ast U) = -U$.
2. If $U$ is symmetric, then $\ast U = -U^*$.
3. If $U = \hat{U}$ is symmetric, traceless, then, $\ast \hat{U} = -\hat{U}^*$ is also symmetric traceless.
4. In general,

$$ \text{tr}(\ast U) = \text{tr}(U^*) = -(a)\text{tr}(U), $$

$$ (a)\text{tr}(\ast U) = (a)\text{tr}(U^*) = \text{tr}(U), $$

$$ \hat{\ast U} = \ast \hat{U}. $$

Given a general horizontal 2-form $U$ we have, according to (19)

$$ U_{ab} = \hat{U}_{ab} + \frac{1}{2} \delta_{ab} \text{tr}(U) + \frac{1}{2} \epsilon_{ab} (a)\text{tr}(U). $$

Therefore,

$$ U^*_{ab} = \hat{U}^*_{ab} + \frac{1}{2} \epsilon_{ab} \text{tr}(U) - \frac{1}{2} \delta_{ab} (a)\text{tr}(U), $$

$$ \ast U_{ab} = \ast \hat{U}_{ab} + \frac{1}{2} \epsilon_{ab} \text{tr}(U) - \frac{1}{2} \delta_{ab} (a)\text{tr}(U). $$

Hence,

$$ U^*_a b = -U^*_a b + \epsilon_{ab} \text{tr}(U) - \delta_{ab} (a)\text{tr}(U). \quad (22) $$

We note the following lemma,
Lemma 2.8. Given two 1-forms $\xi, \eta$ we have,

$$^*\xi \cdot \eta = -^*\eta \cdot \xi = \xi \cdot \eta^*.$$  

Given a 1-form $\xi$ and 2-tensor $U$ we have,

$$\xi_a U^*_{ab} = ^*\xi_a U_{ab} - ^*\xi_b (\text{tr}U) - \xi_b (^a \text{tr}U),$$  

$$^*\xi_a U^*_{ab} = -\xi_a U_{ab} + \xi_b (\text{tr}U) - ^*\xi_b (^a \text{tr}U).$$  

Thus,

$$^*\xi_a U^*_{ab} + \xi_a U_{ab} = \xi_b (\text{tr}U) - ^*\xi_b (^a \text{tr}U).$$

Also,

$$^*\xi_a U^*_{ab} - \xi_a U_{ab} = -2\xi_a \hat{U}_{ab}.$$  

Proof. The last formula follows from,

$$^*\xi_a U^*_{ab} = ^*\xi_a \left( \hat{U}_{ab} + \frac{1}{2} \varepsilon_{ab} \text{tr}(U) - \frac{1}{2} \delta_{ab} (^a \text{tr}(U)) \right)$$

$$= -\xi_a \hat{U}_{ab} + \frac{1}{2} \xi_b \text{tr}(U) - ^*\xi_b (^a \text{tr}(U)),$$

$$\xi_a U_{ab} = \xi_a \left( \hat{U}_{ab} + \frac{1}{2} \delta_{ab} \text{tr}(U) + \frac{1}{2} \varepsilon_{ab} (^a \text{tr}(U)) \right)$$

$$= \xi_a \hat{U}_{ab} + \frac{1}{2} \xi_b \text{tr}(U) - \xi_b (^a \text{tr}(U)).$$

\[\Box\]

Definition 2.9. We denote by $S_0 = S_0(M)$ the set of pairs of scalar functions on $M$, $S_1 = S_1(M)$ the set of horizontal 1-forms on $M$, and by $S_2 = S_2(M)$ the set of symmetric traceless horizontal 2-forms on $M$.

Definition 2.10. Given $\xi, \eta \in S_1$ we denote

$$\xi \cdot \eta := \delta^{ab} \xi_a \eta_b,$$

$$\xi \wedge \eta := \varepsilon^{ab} \xi_a \eta_b = \xi \cdot \eta^*,$$

$$(\xi \hat{\otimes} \eta)_{ab} := \frac{1}{2} (\xi_a \eta_b + \xi_b \eta_a - \delta_{ab} \xi \cdot \eta).$$

Given $\xi \in S_1$, $U \in S_2$ we denote

$$(\xi \cdot U)_a := \delta^{bc} \xi_b U_{ac}.$$  

Given $U, V \in S_2$ we denote

$$(U \wedge V)_{ab} := \varepsilon^{ab} U_{ac} V_{cb}.$$
Remark 2.11. Notice that the definition of $\hat{\xi} \hat{\otimes} \eta$ differ by $\frac{1}{2}$ from the one given in [5].

The following two lemmas are immediate.

**Lemma 2.12.** Given a horizontal vector $\xi$ and a symmetric 2-form $U$, we have

$$U_{ab} \xi^b = \left( \hat{U} \cdot \xi + \frac{1}{2} \tr(U) \xi + \frac{1}{2} ({}^a \tr(U) {}^a \tr(\xi)) \right)_a.$$

**Lemma 2.13.** Given two symmetric, traceless, horizontal 2-tensors $\hat{U}, \hat{V}$ we have,

$$\hat{U}_{ac} \hat{V}_{cb} + \hat{V}_{ac} \hat{U}_{cb} = \delta_{ab} \hat{U} \cdot \hat{V}$$

where,

$$\hat{U} \cdot \hat{V} = \delta^{ac} \delta^{bd} \hat{U}_{ab} \hat{V}_{cd}.$$

In particular,

$$\hat{V}_{ac} \hat{V}_{cb} = \frac{1}{2} \delta_{ab} |\hat{V}|^2$$

with, $|\hat{V}|^2 = \hat{V} \cdot \hat{V}$.

**Remark 2.14.** The previous lemma implies in particular $\hat{U} \cdot \hat{V} = 0$.

We generalize the lemma as follows,

**Lemma 2.15.** Given $U, V$ arbitrary 2-covariant horizontal tensor-fields, we have,

$$\delta^{ab} U_{ac} V_{cb} = \hat{U} \cdot \hat{V} + \frac{1}{2} (\tr(U) \tr(V) - {}^a \tr(U) {}^a \tr(V)),$$

$$\varepsilon^{ab} U_{ac} V_{cb} = \hat{U} \land \hat{V} + \frac{1}{2} ({}^a \tr(U) \tr(V) + \tr(U) {}^a \tr(V)),$$

$$\hat{U}_{ac} \hat{V}_{cb} = \frac{1}{2} (\hat{U}_{ab} \tr(V) + \hat{V}_{ab} \tr(U)) + \frac{1}{2} \left( - {}^a \hat{U}_{ab} \tr(V) + {}^a \hat{V}_{ab} \tr(U) \right),$$

where,

$$\hat{U} \cdot \hat{V} = \delta^{ac} \delta^{bd} \hat{U}_{ab} \hat{V}_{cd},$$

$$\hat{U} \land \hat{V} = \hat{U} \cdot \ast \hat{V} = \varepsilon^{ab} \hat{U}_{ac} \hat{V}_{cb}.$$ 

**Proof.** In view of the decomposition (19), we have

$$U_{ac} V_{cb} = \hat{U}_{ac} \hat{V}_{cb} + \frac{1}{2} (\tr(V) \hat{U}_{ab} + \tr(U) \hat{V}_{ab}) + \frac{1}{2} ({}^a \tr(U) {}^a \tr(V) \hat{V}_{ab} + \hat{V}_{ab} \tr(V))$$

$$+ \frac{1}{4} (\tr(U) \tr(V) - {}^a \tr(U) {}^a \tr(V)) \delta_{ab} + \frac{1}{4} \tr(U) \tr(U) \tr(V) + {}^a \tr(U) \tr(V)) \varepsilon_{ab}$$

and the proof easily follows, using also the fact that $\hat{U} \cdot \hat{V} = 0$ according to Remark 2.14. \(\square\)
The following is an immediate consequence of the lemma.

**Corollary 2.16.** In the particular case when \( U = V \) the lemma becomes,

\[
\begin{align*}
\delta^{ab}U_{ac}U_{cb} &= |\hat{U}|^2 + \frac{1}{2}\left((tr(U))^2 - (\langle u \rangle tr(U))^2\right), \\
\varepsilon^{ab} U_{ac}U_{cb} &= tr(U)\langle u \rangle tr(U), \\
\hat{U}_{ac}U_{cb} &= tr(U)\hat{U}_{ab}.
\end{align*}
\]

As another corollary to Lemma 2.15 we have

**Lemma 2.17.** Let \( u \) be an arbitrary 2-horizontal tensor and \( v \in S_2 \). Then

\[
uacv_{cb} + ubcv_{ca} = \delta_{ab}u \cdot v + \left( tr(u)v_{ab} + \frac{1}{2}\left( uac - uca \right)v_{cb} + \left( ubc - ucb \right)v_{ca}\right) \]

Proof. We give below a direct proof based on Lemma 2.13 according to which, given \( u, v \in S_2 \), we have

\[
uacv_{cb} + ubcv_{ca} = \delta_{ab}u \cdot v.
\]

If \( u \) is only symmetric and \( v \in S_2 \) we can write,

\[
uacv_{cb} + ubcv_{ca} = \left( \hat{u}_{ac} + \frac{1}{2}\delta_{ac}tr(u)\right)v_{cb} + \left( \hat{u}_{bc} + \frac{1}{2}\delta_{bc}tr(u)\right)v_{ca}
\]

= \delta_{ab}\hat{u} \cdot v + \left( tr(u)v_{ab} \right).

If \( u \) is an arbitrary 2-tensor and \( v \in S_2 \),

\[
uacv_{cb} + ubcv_{ca} = \frac{1}{2}\left( uac + uca + \left( uac - uca \right) \right)v_{ca} + \frac{1}{2}\left( ubc + ubc + \left( ubc - ucb \right) \right)v_{ca}
\]

= \frac{1}{2}\delta_{ab}\left( uac + uca \right)v_{ac} + \frac{1}{2}\left( uac - uca \right)v_{cb} + \left( ubc - ucb \right)v_{ca}
\]

= \delta_{ab}u \cdot v + \left( tr(u)v_{ab} + \frac{1}{2}\left( uac - uca \right)v_{cb} + \left( ubc - ucb \right)v_{ca}\right).

\]

**Lemma 2.18.** The following hold true

- Given two 1-forms \( \xi, \eta \) we have,

\[
*\xi \cdot \eta = -\xi \cdot *\eta, \\
*\xi \cdot *\eta = \xi \cdot \eta, \\
*\xi \wedge *\eta = -\xi \wedge *\eta, \\
*\xi \wedge *\eta = \xi \wedge *\eta, \\
*\xi \hat{\wedge} *\eta = \xi \hat{\wedge} *\eta, \\
*\langle (\hat{\wedge} \eta) \rangle = *\xi \hat{\wedge} *\eta, \\
*\xi \hat{\wedge} *\eta = -\xi \hat{\wedge} *\eta.
\]
• Given a horizontal vector $\xi$ and a symmetric traceless 2-form $U$, we have
  
  $\ast(\xi \cdot U) = \xi \cdot \ast U,$
  
  $\ast \xi \cdot U = -\xi \cdot \ast U,$
  
  $\ast \xi \cdot \ast U = \xi \cdot U.$

• Given 2 symmetric traceless 2-forms $U, V$, we have,
  
  $\ast U \cdot V = -U \cdot \ast V,$
  
  $\ast U \cdot \ast V = U \cdot V,$
  
  $\ast U \wedge V = -U \wedge \ast V,$
  
  $\ast U \wedge \ast V = U \wedge V.$

Proof. The statements follow from the above results except the ones involving $\hat{\otimes}$. To check those, we write

  $\ast(\xi \hat{\otimes} \eta)_{11} = (\xi \hat{\otimes} \eta)_{21} = \xi_2 \eta_1 + \xi_1 \eta_2,$
  
  $(\xi \hat{\otimes} \ast \eta)_{11} = \xi_1 (\ast \eta)_1 - \xi_2 (\ast \eta)_2 = \xi_1 \eta_2 + \xi_2 \eta_1,$
  
  $(\ast \xi \hat{\otimes} \eta)_{11} = (\ast \xi)_1 \eta_1 - (\ast \xi)_2 \eta_2 = \xi_2 \eta_1 + \xi_1 \eta_2.$

Also,

  $2 \ast(\xi \hat{\otimes} \eta)_{12} = \xi_2 \eta_2 - \xi_1 \eta_1,$
  
  $2(\xi \hat{\otimes} \ast \eta)_{12} = \xi_1 \ast \eta_2 + \xi_2 \ast \eta_1 = -\xi_1 \eta_1 + \xi_2 \eta_2,$
  
  $2(\ast \xi \hat{\otimes} \eta)_{12} = \ast \xi_1 \eta_2 - \ast \xi_2 \eta_1 = \xi_2 \eta_1 - \xi_1 \eta_2.$

Hence,

  $\ast(\xi \hat{\otimes} \eta) = \ast \xi \hat{\otimes} \eta = \xi \hat{\otimes} \ast \eta.$

\[\square\]

**Lemma 2.19.** Given $\xi, \eta \in S_1$, $u \in S_2$ we have

  $\xi \hat{\otimes} (\eta \cdot u) + \eta \hat{\otimes} (\xi \cdot u) = (\xi \cdot \eta)u.$

Proof. We check first the identities

  $\left(\xi \hat{\otimes} (\eta \cdot u)\right)_{ab} = \frac{1}{2} (\xi_a \eta_b u_{bc} + \xi_b \eta_a u_{ac} - \delta_{ab} (\xi \hat{\otimes} \eta) \cdot u),$
  
  $\left(\eta \hat{\otimes} (\xi \cdot u)\right)_{ab} = \frac{1}{2} (\eta_a \xi_b u_{bc} + \eta_b \xi_a u_{ac} - \delta_{ab} (\eta \hat{\otimes} \xi) \cdot u),$

which follows easily from the definition of $\hat{\otimes}$ and

  $(\xi \hat{\otimes} \eta) \cdot u = \xi \cdot (\eta \cdot u) = \eta (\xi \cdot u).$

20
Note also that,

\[
(\xi \hat{\otimes} \eta)_{ac}u_{cb} + (\xi \hat{\otimes} \eta)_{bc}u_{ca} = \frac{1}{2} (\xi_a \eta_c + \xi_c \eta_a - \delta_{ac} (\xi \cdot \eta))u_{cb} + \frac{1}{2} (\xi_b \eta_c + \xi_c \eta_b - \delta_{bc} (\xi \cdot \eta))u_{ca}
\]

\[
= (\xi \hat{\otimes} (\eta \cdot u))_{ab} + \frac{1}{2} \delta_{ab} (\xi \hat{\otimes} \eta) \cdot u + (\eta \hat{\otimes} (\xi \cdot u))_{ab} + \frac{1}{2} \delta_{ab} (\xi \hat{\otimes} \eta) \cdot u
\]

\[
- (\xi \cdot \eta)u_{ab}
\]

\[
= (\xi \hat{\otimes} (\eta \cdot u))_{ab} + (\eta \hat{\otimes} (\xi \cdot u))_{ab} - (\xi \cdot \eta)u_{ab} + \delta_{ab} (\xi \hat{\otimes} \eta) \cdot u.
\]

We now apply Lemma 2.13 to the $S_2$ tensors $\xi \hat{\otimes} \eta$ and $u$ to deduce,

\[
(\xi \hat{\otimes} \eta)_{ac}u_{cb} + (\xi \hat{\otimes} \eta)_{bc}u_{ca} = (\xi \hat{\otimes} \eta) \cdot u \delta_{ab}.
\]

Hence,

\[
(\xi \hat{\otimes} \eta) \cdot u \delta_{ab} = (\xi \hat{\otimes} \eta)_{ac}u_{cb} + (\xi \hat{\otimes} \eta)_{bc}u_{ca}
\]

\[
= (\xi \hat{\otimes} (\eta \cdot u))_{ab} + (\eta \hat{\otimes} (\xi \cdot u))_{ab} - (\xi \cdot \eta)u_{ab} + \delta_{ab} (\xi \hat{\otimes} \eta) \cdot u
\]

i.e.,

\[
(\xi \hat{\otimes} (\eta \cdot u))_{ab} + (\eta \hat{\otimes} (\xi \cdot u))_{ab} - (\xi \cdot \eta)u_{ab} = 0
\]

as desired. \qed

### 2.2 Horizontal covariant derivative

We are now ready to define the horizontal covariant operator $\nabla$ as follows: Given $X, Y \in O(M)$ the covariant derivative $D_X Y$ fails in general to be horizontal. We thus define,

\[
\nabla_X Y := (h)(D_X Y) = D_X Y - \frac{1}{2} \chi'(X, Y) L - \frac{1}{2} \chi(X, Y) L. \quad (26)
\]

**Proposition 2.20.** For all $X, Y \in O(M)$,

\[
\nabla_X Y - \nabla_Y X = [X, Y] - ^{(a)} \chi(X, Y) L - ^{(a)} \chi(X, Y) L
\]

\[
= [X, Y] - \frac{1}{2} \left( ^{(a)} \text{tr} L + ^{(a)} \text{tr} L \right) \in (X, Y).
\]

In particular,

\[
[X, Y]^{\perp} = \frac{1}{2} \left( ^{(a)} \text{tr} L + ^{(a)} \text{tr} L \right) \in (X, Y). \quad (27)
\]

For all $X, Y, Z \in O(M)$,

\[
Z \gamma(X, Y) = \gamma(\nabla_Z X, Y) + \gamma(X, \nabla_Z Y).
\]
Remark 2.21. In the integrable case, $\nabla$ coincides with the Levi-Civita connection of the metric induced on the integral surfaces of $O(M)$.

Given a general covariant, horizontal tensor-field $U$ we define its horizontal covariant derivative according to the formula,

$$
\nabla_Z U(X_1, \ldots, X_k) = Z(U(X_1, \ldots, X_k)) - U(\nabla_Z X_1, \ldots, X_k) - \ldots - U(X_1, \ldots, \nabla_Z X_k).
$$

(28)

2.3 Ricci coefficients

Definition 2.22. We define the horizontal 1-forms,

$$
\eta(X) := \frac{1}{2} g(X, D_L L), \quad \eta(X) := \frac{1}{2} g(X, D_L L),
$$

(29)

$$
\xi(X) := \frac{1}{2} g(X, D_L L), \quad \xi(X) := \frac{1}{2} g(X, D_L L).
$$

(30)

We can extend the operators $\nabla_L$ and $\nabla_L$ to arbitrary $k$-covariant, horizontal tensor-fields $U$ as follows,

$$
\nabla_L U(X_1, \ldots, X_k) = L(U(X_1, \ldots, X_k)) - U(\nabla_L X_1, \ldots, X_k) - \ldots - U(X_1, \ldots, \nabla_L X_k),
$$

$$
\nabla_L U(X_1, \ldots, X_k) = L(U(X_1, \ldots, X_k)) - U(\nabla_L X_1, \ldots, X_k) - \ldots - U(X_1, \ldots, \nabla_L X_k).
$$

(31)

The following proposition follows easily from the definition.

Proposition 2.23. The operators $\nabla$, $\nabla_L$ and $\nabla_L$ take horizontal tensor-fields into horizontal tensor-fields. We have,

$$
\nabla \gamma = \nabla_L \gamma = \nabla_L \gamma = 0.
$$

(31)
In addition to the horizontal tensor-fields $\chi, \chi, \eta, \eta, \xi, \xi$ introduced above we also define the scalars,

$$\omega := \frac{1}{4} g(D_L L, L), \quad \omega := \frac{1}{4} g(D_L L, L),$$

and the horizontal 1-form,

$$\zeta(X) = \frac{1}{2} g(D_X L, L).$$

We summarize below the definition of the the horizontal 1-forms $\xi, \xi, \eta, \eta, \zeta \in O_1$:

$$\begin{cases}
\xi(X) = \frac{1}{2} g(D_L L, X), & \xi(X) = \frac{1}{2} g(D_L L, X), \\
\eta(X) = \frac{1}{2} g(D_L L, X), & \eta(X) = \frac{1}{2} g(D_L L, X), \\
\zeta(X) = \frac{1}{2} g(D_X L, L),
\end{cases}$$

and the real scalars

$$\omega = \frac{1}{4} g(D_L L, L), \quad \omega = \frac{1}{4} g(D_L L, L).$$

**Definition 2.24.** The horizontal tensor-fields $\chi, \chi, \eta, \eta, \xi, \xi, \omega, \omega$ are called the connection coefficients of the null pair $(L, L)$. Given an arbitrary basis of horizontal vectorfields $e_1, e_2$, we write using the short hand notation $D_a = D_{e_a}, a = 1, 2,$

$$\begin{align*}
\chi_{ab} &= g(D_a L, e_b), & \chi_{ab} &= g(D_a L, e_b), \\
\xi_a &= \frac{1}{2} g(D_L L, e_a), & \xi_a &= \frac{1}{2} g(D_L L, e_a), \\
\omega &= \frac{1}{4} g(D_L L, L), & \omega &= \frac{1}{4} g(D_L L, L), \\
\eta_a &= \frac{1}{2} g(D_L L, e_a), & \eta_a &= \frac{1}{2} g(D_L L, e_a), \\
\zeta_a &= \frac{1}{2} g(D_e L, L).
\end{align*}$$

We easily derive the Ricci formulae,

$$\begin{align*}
D_a e_b &= \nabla a e_b + \frac{1}{2} \chi_{ab} e_3 + \frac{1}{2} \chi_{ab} e_4, \\
D_a e_4 &= \chi_{ab} e_b - \zeta_a e_4, \\
D_a e_3 &= \chi_{ab} e_b + \zeta_a e_3, \\
D_3 e_a &= \nabla_3 e_a + \eta_a e_3 + \xi_a e_4, \\
D_3 e_3 &= -2 \omega e_3 + 2 \xi_a e_b, \\
D_3 e_4 &= 2 \xi_a e_4 + 2 \eta_a e_b, \\
D_4 e_a &= \nabla_4 e_a + \eta_a e_4 + \xi_a e_3, \\
D_4 e_4 &= -2 \omega e_4 + 2 \xi_b e_b, \\
D_4 e_3 &= 2 \omega e_3 + 2 \eta_b e_b.
\end{align*}$$
Definition 2.25. We introduce the notation
\[ t_r \chi := \text{tr}(\chi), \quad (a) t_r \chi := (a) \text{tr}(\chi), \quad tr \chi := \text{tr}(\chi), \quad (a) tr \chi := (a) \text{tr}(\chi). \] (37)
\( \hat{\chi}, t_r \chi \) and \( \hat{\chi}, tr \chi \) are called, respectively, the shear and expansion of the horizontal distribution \( O(M) \). The scalars \( (a) tr \chi \) and \( (a) tr \chi \) measure the integrability defects of the distribution.

Definition 2.26. For a given horizontal 1-form \( \omega \), we define the frame dependent operators,
\[ \text{div} \omega = \delta^{ab} \nabla_b \omega_a, \quad \text{curl} \omega = \varepsilon^{ab} \nabla_a \omega_b, \]
\[ (\nabla \otimes \omega)_{ba} = \frac{1}{2} (\nabla_b \omega_a + \nabla_a \omega_b - \delta_{ab} (\text{div} \omega)). \]

2.4 Curvature and Weyl fields

Assume that \( W \in T^0_4(M) \) is a Weyl field, i.e.
\[ \begin{cases} W_{\alpha \beta \mu \nu} = -W_{\beta \alpha \mu \nu} = -W_{\alpha \beta \nu \mu} = W_{\mu \nu \alpha \beta}, \\ W_{\alpha \beta \mu \nu} + W_{\alpha \mu \nu \beta} + W_{\alpha \nu \beta \mu} = 0, \\ g^{\beta \mu} W_{\alpha \beta \mu \nu} = 0. \end{cases} \] (38)

We define the null components of the Weyl field \( W, \alpha(W), \beta(W), \varrho(W) \in O_2(M) \) and \( \beta(W), \beta(W) \in O_1(M) \) by the formulas
\[ \begin{align*}
\alpha(W)(X,Y) &= W(L,X,L,Y), \\
\beta(W)(X) &= \frac{1}{4} W(X, L, L, L), \\
\varrho(W)(X,Y) &= W(X, L, L, Y).
\end{align*} \] (39)

Recall that if \( W \) is a Weyl field its Hodge dual \( *W \), defined by \( *W_{\alpha \beta \mu \nu} = \frac{1}{4} \varepsilon_{\mu \nu \rho \sigma} W_{\alpha \beta \rho \sigma} \), is also a Weyl field. We easily check the formulas,
\[ \begin{cases} \alpha( *W) = *\alpha(W), \quad \alpha( *W) = - *\alpha(W), \\ \beta( *W) = *\beta(W), \quad \beta( *W) = - *\beta(W), \\ \varrho( *W) = *\varrho(W). \end{cases} \] (40)

It is easy to check that \( \alpha, \alpha \) are symmetric traceless horizontal tensor-fields. On the other hand the horizontal 2-tensorfield \( \varrho \) is neither symmetric nor traceless. It is convenient to express it in terms of the following two scalar quantities,
\[ \rho(W) = \frac{1}{4} W(L, L, L, L), \quad *\rho(W) = \frac{1}{4} *W(L, L, L, L). \] (41)

Observe also that,
\[ \rho( *W) = *\rho(W), \quad *\rho( *W) = - \rho. \]
Thus,

\[ \varrho(X, Y) = (\rho \gamma(X, Y) + \ast \rho \in (X, Y)) \], \quad \forall X, Y \in \mathcal{O}(\mathcal{M}). \quad (42) \]

We have

\[
\begin{align*}
W_{a3b4} &= \varrho_{ab} = (\rho \delta_{ab} + \ast \rho \in_{ab}), \\
W_{ab34} &= 2 \in_{ab} \ast \rho, \\
W_{abcd} &= \in_{ab} \ast \beta_{cd}, \\
W_{abc3} &= \in_{ab} \ast \beta_{c}, \\
W_{abc4} &= -\in_{ab} \ast \beta_{c}.
\end{align*}
\]

**Remark 2.27.** In addition to the Hodge duality we will need to take into account the duality with respect to the interchange of \( L, L \), which we call a pairing transformation. Clearly, under this transformation, \( \alpha \leftrightarrow \overline{\alpha} \), \( \beta \leftrightarrow -\overline{\beta} \), \( \rho \leftrightarrow \rho \), \( \ast \rho \leftrightarrow -\ast \rho \), \( \overline{\varrho}_{ab} := \varrho_{ba} \). One has to be careful however when combining the Hodge dual and pairing transformations. In that case we have, \( \ast \alpha \leftrightarrow -\ast \alpha \), \( \ast \beta \leftrightarrow \ast \beta \). This is due to the fact that under the pairing transformation \( \in_{ab} \rightarrow -\in_{ab} \) (since \( \in_{ab} = \in_{ab}^{34} \)). Indeed, for example,

\[
\begin{align*}
\ast \alpha_{ab} &= \alpha(\ast W)_{ab} = \ast W_{a3b4} = -\in_{a3c4} W_{c3b4} = \in_{ac} \varrho_{cb}, \\
\ast \alpha_{ab} &= \alpha(\ast W)_{ab} = \ast W_{a4b4} = -\in_{a4c3} W_{c4b4} = -\in_{cb} \varrho_{ac}.
\end{align*}
\]

The decomposition above applies in particular to the Riemann curvature tensor \( \mathbf{R} \) of a vacuum spacetime.

**Definition 2.28** (Horizontal curvature tensor). We define the curvature tensor \( R_{cdab} \) of the horizontal structure by the usual formula,

\[
\nabla_a \nabla_b X_c - \nabla_b \nabla_a X_c = R_{cdab} X^d.
\]

### 2.5 Connection to the Newman-Penrose formalism

In the Newman-Penrose NP formalism, one choses a specific orthonormal basis of horizontal vectors \((e_1, e_2)\) and defines all connection coefficients relative to the complexified frame \((e_3, e_4, m, \overline{m})\) where \( m = e_1 + ie_2, \overline{m} = e_1 - ie_2 \). Thus, all quantities of interest are complex scalars instead of our horizontal tensors such as \( S_1, S_2 \). The NP formalism works well for deriving the basic equations, but has the disadvant of substantially increasing the number of variables. Moreover, the calculations become far more cumbersome when deriving equations involving higher derivatives of the main quantities, in perturbations of Kerr. Another advantage of the formalism used here is that all important equations look similar to the ones in [5].

We refer to [15] for the original form of the NP formalism and to the appendix in [10] for a more detailed comparison between the NP and our formalism.
2.6 Null structure equations

We state below the null structure equation in the general setting discussed above. We assume given a vacuum spacetime endowed with a general null frame \((e_3, e_4, e_1, e_2)\) relative to which we define our connection and curvature coefficients.

**Proposition 2.29** (Null structure equations). The connection coefficients verify the following equations

\[
\nabla_3 tr \chi = -|\hat{\chi}|^2 - \frac{1}{2} (tr \chi^2 - (a) tr \chi^2) + 2 \text{div} \xi - 2 \omega tr \chi + 2 \hat{\xi} \cdot (\eta + \eta - 2 \zeta),
\]

\[
\nabla_3 (a) tr \chi = -tr \chi (a) tr \chi + 2 \text{curl} \xi - 2 \omega (a) tr \chi + 2 \hat{\xi} \wedge (-\eta + \eta + 2 \zeta),
\]

\[
\nabla 3 \hat{\chi} = -tr \hat{\chi} + 2 \nabla \hat{\xi} - 2 \omega \hat{\chi} + 2 \hat{\xi} \hat{\eta} (\eta + \eta - 2 \zeta) - \alpha,
\]

\[
\nabla_4 tr \chi = -|\hat{\chi}|^2 - \frac{1}{2} (tr \chi^2 - (a) tr \chi^2) + 2 \text{div} \xi - 2 \omega tr \chi + 2 \hat{\xi} \cdot (\eta + \eta + 2 \zeta),
\]

\[
\nabla_4 (a) tr \chi = -tr \chi (a) tr \chi + 2 \text{curl} \xi - 2 \omega (a) tr \chi + 2 \hat{\xi} \wedge (-\eta + \eta - 2 \zeta),
\]

\[
\nabla_4 \hat{\chi} = -tr \hat{\chi} + 2 \nabla \hat{\xi} - 2 \omega \hat{\chi} + 2 \hat{\xi} \hat{\eta} (\eta + \eta + 2 \zeta) - \alpha.
\]
Also,
\[
\nabla_3 \zeta + 2\nabla \omega = -\hat{\chi} \cdot (\zeta + \eta) - \frac{1}{2} \text{tr} \chi (\zeta + \eta) - \frac{1}{2} (^{(a)} \text{tr} \chi (^{*} \zeta + ^{*} \eta) + 2\omega (\zeta - \eta) \\
+ \hat{\chi} \cdot \xi + \frac{1}{2} \text{tr} \chi \xi + \frac{1}{2} (^{(a)} \text{tr} \chi ^{*} \xi + 2\omega \xi - \beta),
\]
\[
\nabla_4 \zeta - 2\nabla \omega = \hat{\chi} \cdot (-\zeta + \eta) + \frac{1}{2} \text{tr} \chi (-\zeta + \eta) + \frac{1}{2} (^{(a)} \text{tr} \chi (-^{*} \zeta + ^{*} \eta) + 2\omega (\zeta + \eta) \\
- \hat{\chi} \cdot \xi - \frac{1}{2} \text{tr} \chi \xi - \frac{1}{2} (^{(a)} \text{tr} \chi ^{*} \xi - 2\omega \xi - \beta),
\]
\[
\nabla_3 \eta - \nabla_4 \xi = -\hat{\chi} \cdot (\eta - \eta) - \frac{1}{2} \text{tr} \chi (\eta - \eta) + \frac{1}{2} (^{(a)} \text{tr} \chi (^{*} \eta - ^{*} \eta) - 4\omega \xi + \beta),
\]
\[
\nabla_4 \eta - \nabla_3 \xi = -\hat{\chi} \cdot (\eta - \eta) - \frac{1}{2} \text{tr} \chi (\eta - \eta) + \frac{1}{2} (^{(a)} \text{tr} \chi (^{*} \eta - ^{*} \eta) - 4\omega \xi - \beta,
\]

and
\[
\nabla_3 \omega + \nabla_4 \omega - 4\omega = \xi \cdot \xi - (\eta - \eta) \cdot \zeta + \eta \cdot \eta = \rho.
\]

Also,
\[
\text{div} \hat{\chi} + \zeta \cdot \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi + \frac{1}{2} \text{tr} \chi \zeta - \frac{1}{2} \nabla (^{(a)} \text{tr} \chi - \frac{1}{2} (^{(a)} \text{tr} \chi ^{*} \zeta - (^{(a)} \text{tr} \chi ^{*} \eta - (^{(a)} \text{tr} \chi ^{*} \xi - \beta,
\]
\[
\text{div} \hat{\chi} - \zeta \cdot \hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi - \frac{1}{2} \text{tr} \chi \zeta - \frac{1}{2} \nabla (^{(a)} \text{tr} \chi + \frac{1}{2} (^{(a)} \text{tr} \chi ^{*} \zeta - (^{(a)} \text{tr} \chi ^{*} \eta - (^{(a)} \text{tr} \chi ^{*} \xi + \beta,
\]

and\(^{16}\)
\[
\text{curl} \zeta = -\frac{1}{2} \hat{\chi} \wedge \hat{\chi} + \frac{1}{4} (\text{tr} \chi (^{(a)} \text{tr} \chi - \text{tr} \chi (^{(a)} \text{tr} \chi) + \omega (^{(a)} \text{tr} \chi - \omega (^{(a)} \text{tr} \chi + ^{*} \rho.
\]

\textbf{Proof.} Except for the fact that the order of indices in \(\chi, \bar{\chi}\) is important, since they are no longer symmetric, the derivation is exactly as in [5]. \qed

Note that we are missing the traditional Gauss equation which, in the integrable case, connects the Gauss curvature of a sphere to a Riemann curvature component. In what follows we state a result which is its non-integrable analogue.

**Proposition 2.30.** The following identity holds true.
\[
\nabla_a (\nabla_b X_c) - \nabla_b (\nabla_a X_c) = R_{cdab} X^d + \frac{1}{2} e_{ab} \left( (^{(a)} \text{tr} \chi \nabla_3 + (^{(a)} \text{tr} \chi \nabla_4 \right) X_c \\
- \frac{1}{2} \left( \chi_{ac} \chi_{bd} + \chi_{ac} \chi_{bd} - \chi_{bc} \chi_{ad} - \chi_{bc} \chi_{ad} \right) X^d.
\]

\(^{16}\)Note that this equation follows from expanding \(R_{34ab} \).
Proof. Given $X \in S_1$ we have,

$$D_bX_c = \nabla_bX_c, \quad D_3X_c = \nabla_3X_c, \quad D_4X_c = \nabla_4X_c, \quad D_bX_3 = -\chi_{bd}X_d, \quad D_bX_4 = -\chi_{ba}X_d.$$  

Also,

$$D_aD_bX_c = \nabla_a(\nabla_bX_c) - \frac{1}{2} \chi_{ab} D_3X_c - \frac{1}{2} \chi_{ab} D_4X_c - \frac{1}{2} \chi_{ac} D_bX_3 - \frac{1}{2} \chi_{ac} D_bX_4 \quad \nabla_a(\nabla_bX_c) - \frac{1}{2} \chi_{ab} D_3X_c - \frac{1}{2} \chi_{ab} D_4X_c + \frac{1}{2} \chi_{ac} \chi_{bd} X_d + \frac{1}{2} \chi_{ac} \chi_{bd} X_d.$$  

Hence,

$$D_aD_bX_c = \nabla_a(\nabla_bX_c) - \frac{1}{2} \chi_{ab} D_3X_c - \frac{1}{2} \chi_{ab} D_4X_c - \frac{1}{2} \chi_{ac} \chi_{bd} X_d + \frac{1}{2} \chi_{ba} \chi_{ad} X_d.$$  

Subtracting we derive

$$R_{cdab}X^d = D_aD_bX_c - D_bD_aX_c = \nabla_a(\nabla_bX_c) - \nabla_b(\nabla_aX_c) - \frac{1}{2} (\chi_{ab} - \chi_{ba}) \nabla_3X_c - \frac{1}{2} (\chi_{ab} - \chi_{ba}) \nabla_4X_c + \frac{1}{2} \chi_{ac} \chi_{bd} X_d + \frac{1}{2} \chi_{ac} \chi_{bd} X_d.$$  

Thus,

$$\nabla_a(\nabla_bX_c) - \nabla_b(\nabla_aX_c) = \frac{1}{2} (\chi_{ab} - \chi_{ba}) \nabla_3X_c + \frac{1}{2} (\chi_{ab} - \chi_{ba}) \nabla_4X_c - \frac{1}{2} \chi_{ac} \chi_{bd} X_d + \frac{1}{2} \chi_{ac} \chi_{bd} X_d.$$  

To end the proof we set

$$\chi_{ab} - \chi_{ba} = \varepsilon_{ab} (a) \text{tr} \chi, \quad \chi_{ab} - \chi_{ba} = \varepsilon_{ab} (a) \text{tr} \chi.$$  

\[ \square \]

Remark 2.31. Recall that

$$\nabla_a(\nabla_bX_c) - \nabla_b(\nabla_aX_c) = R_{cdab}X^d.$$  

Thus, taking $X_d = e_d$, we can rewrite our Gauss formula in the form

$$R_{cdab} = \frac{1}{2} (a) \text{tr} \chi \varepsilon_{ab} g(\nabla_3e_d, e_c) + \frac{1}{2} (a) \text{tr} \chi \varepsilon_{ab} g(\nabla_4e_d, e_c) - \frac{1}{2} \chi_{ac} \chi_{bd} - \chi_{ac} \chi_{bd} - \chi_{bd} \chi_{ad} + \chi_{bd} \chi_{ad} + R_{cdab}.$$  

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2.7 Null Bianchi identities

We state below the equations verified by the null curvature components of an Einstein vacuum manifold.

Proposition 2.32 (Null Bianchi). We have,

\[
\nabla_3 \alpha - 2\nabla \hat{\otimes} \beta = -\frac{1}{2} \left( \text{tr} \chi \alpha + (a) \text{tr} \chi^* \alpha \right) + 4 \omega \alpha + 2(\zeta + 4\eta) \hat{\otimes} \beta - 3(\rho \hat{\chi} + *\rho *\hat{\chi}),
\]

\[
\nabla_4 \beta - \text{div} \alpha = -2(\text{tr} \chi \beta - (a) \text{tr} \chi^* \beta) - 2\omega \beta + \alpha \cdot (2\xi + \eta) + 3(\xi \rho + \xi *\rho),
\]

\[
\nabla_3 \beta + \text{div} \varrho = -(\text{tr} \chi \beta + (a) \text{tr} \chi^* \beta) + 2\omega \beta + 2\zeta \cdot \hat{\chi} + 3(\rho \eta + \rho *\eta + \alpha \cdot \xi),
\]

\[
\nabla_4 \rho - \text{div} \beta = -\frac{3}{2} \left( \text{tr} \chi \rho + (a) \text{tr} \chi^* \rho \right) + (2\eta + \zeta) \cdot \beta - 2\xi \cdot \beta - \frac{1}{2} \hat{\chi} \cdot \alpha,
\]

\[
\nabla_4 \star \rho + \text{curl} \beta = -\frac{3}{2} (2(\text{tr} \chi \star \rho) - (2\eta + \zeta) \cdot \star \beta - 2\xi \cdot \star \beta + \frac{1}{2} \hat{\chi} \cdot \star \alpha,
\]

\[
\nabla_3 \star \rho + \text{curl} \beta = -\frac{3}{2} (2(\text{tr} \chi \star \rho) - (2\eta - \zeta) \cdot \star \beta + 2\xi \cdot \star \beta - \frac{1}{2} \hat{\chi} \cdot \star \alpha.
\]

\[
\nabla_4 \chi + 2\nabla \hat{\otimes} \beta = -\frac{1}{2} \left( \text{tr} \chi \alpha - (a) \text{tr} \chi^* \alpha \right) + 4 \omega \alpha + 2(\zeta - 4\eta) \hat{\otimes} \beta - 3(\rho \hat{\chi} - *\rho \hat{\chi}).
\]

Here,

\[
div \varrho = -(\nabla \rho + *\nabla *\rho),
\]

\[
div \hat{\varrho} = -(\nabla \rho - *\nabla *\rho).
\]

Proof. The proof follows line by line from the derivation in [5] except, once more, for keeping track of the lack of symmetry for \(\chi, \hat{\chi}\). Note also that \(\hat{\varrho}_{ab} = \varrho_{ba}\) and that \((\text{div} \varrho)_b = \nabla^a \hat{\varrho}_{ab}\). \(\square\)

We also recall, see [5], the basic Hodge operators

- \(P_1\) takes \(S_1\) into \(^{17}S_0\)

\[
P_1 \xi = (\text{div} \xi, \text{curl} \xi),
\]

- \(P_1\) takes \(S_2\) into \(S_1\)

\[
(P_2 \xi)_a = \nabla^b \xi_{ab},
\]

\(^{17}\text{Recall that } S_0 \text{ refers to pairs of scalar functions } (a, b).\)
\[ D^*_1 \text{ takes } S_0 \text{ into } S_1 \]
\[ D^*_1(f, f^*) = -\nabla_a f + \varepsilon_{ab} \nabla_b f^* , \]

\[ D^*_2 \text{ takes } S_1 \text{ into } S_2 \]
\[ D^*_2 \xi = -\nabla \otimes \xi . \]

### 2.8 Commutation formulas

**Lemma 2.33.** Let \( U_A = U_{a_1...a_k} \) be a general \( k \)-horizontal tensorfield.

1. We have
\[
[\nabla_3, \nabla_b] U_A = -\chi_{bc} \nabla_c U_A + (\eta_b - \zeta_b) \nabla_3 U_A + \sum_{i=1}^{k} (\chi_{a_i b} \eta_c - \chi_{bc} \eta_{a_i}) U_{a_1...c...a_k} \\
+ \text{Err}_{3bA}[U],
\]
\[
\text{Err}_{3bA}[U] = \sum_{i=1}^{k} \left( \chi_{a_i c} \xi_c - \chi_{bc} \xi_{a_i} - \varepsilon_{a_i c} \right) U_{a_1...c...a_k} + \xi_b \nabla_3 U_A .
\]

2. We have
\[
[\nabla_4, \nabla_b] U_A = -\chi_{bc} \nabla_c U_A + (\eta_b + \zeta_b) \nabla_4 U_A + \sum_{i=1}^{k} (\chi_{a_i b} \eta_c - \chi_{bc} \eta_{a_i}) U_{a_1...c...a_k} \\
+ \text{Err}_{4bA}[U],
\]
\[
\text{Err}_{4bA}[U] = \sum_{i=1}^{k} \left( \chi_{a_i c} \xi_c - \chi_{bc} \xi_{a_i} + \varepsilon_{a_i c} \right) U_{a_1...c...a_k} + \xi_b \nabla_3 U_A .
\]

3. We have,
\[
[\nabla_4, \nabla_3] U_A = 2(\eta_b - \eta_b) \nabla_b U_A + 2 \sum_{i=1}^{k} (\eta_{a_i} \eta_b - \eta_{a_i} \eta_b - \varepsilon_{a_i b} \xi_b) U_{a_1...b...a_k} \\
+ 2\omega \nabla_3 U_A - 2\omega \nabla_4 U_A + \text{Err}_{43A},
\]
\[
\text{Err}_{43A} = 2 \sum_{i=1}^{k} (\xi_{a_i} \xi_b - \xi_{a_i} \xi_b) U_{a_1...b...a_k} .
\]

**Proof.** It suffices to consider the case \( k = 1 \). We write,
\[
D^2_{3b} U_a = \nabla_3 \nabla_b U_a + (\eta_a \chi_{bc} + \xi_a \chi_{bc}) U_{c} - \eta_b \nabla_3 U_a - \xi_b \nabla_4 U_a ,
\]
\[
D^2_{3b} U_a = \nabla_3 \nabla_3 U_a + (\chi_{ab} \xi_{c} + \chi_{ac} \eta_{c}) U_{c} - \chi_{bc} \nabla c U_a - \xi_b \nabla_3 U_a ,
\]
\[
D^2_{3b} U_a - D^2_{5a} U_a = R_{ac3b} U_c = -\varepsilon_{ac} \beta_{a} U_c .
\]
Hence,

\[ [\nabla_3, \nabla_b] U_a = \nabla_3 \nabla_b U_a - \nabla_b \nabla_3 U_a \]

\[ = -(\eta_a \chi_{bc} + \xi_a \chi_{bc}) U_c + \eta_b \nabla_3 U_a + 2 \xi_4 U_a + D^2_{ab} U_a \]

\[ + (\chi_{ab} \xi_c + \chi_{ab} \xi_c) U_c - \chi_{bc} \nabla_c U_a - \eta_b \nabla_3 U_a - D^2_{bc} U_a \]

\[ = -\chi_{bc} \nabla_c U_a + (\chi_{ab} \eta_c - \chi_{bc} \eta_a) U_c + (\eta_b - \zeta_b) \nabla_3 U_a \]

\[ + (\chi_{ab} \xi_c - \chi_{bc} \xi_a) + \xi \nabla_4 U_a - \varepsilon_{ac} \beta_a U_c \]

i.e.,

\[ [\nabla_3, \nabla_b] U_a = -\chi_{bc} \nabla_c U_a + (\chi_{ab} \eta_c - \chi_{bc} \eta_a) U_c + (\eta_b - \zeta_b) \nabla_3 U_a \]

\[ + (\chi_{ab} \xi_c - \chi_{bc} \xi_a) + \xi \nabla_4 U_a - \varepsilon_{ac} \beta_a U_c \]

as stated. The commutator formula for \([\nabla_4, \nabla_b] U_a\) is derived easily by symmetry. Also,

\[ D^2_{ab} U_a = \nabla_4 \nabla_3 U_a - 2 \eta \nabla_3 U_a - 2 \xi_4 \eta U_a + 2 \eta_4 U_a \xi U_c, \]

\[ D^2_{ab} U_a = \nabla_3 \nabla_4 U_a - 2 \omega \nabla_4 U_a - 2 \xi_4 \eta U_a + 2 \eta_4 \xi U_a \xi U_c, \]

\[ D^2_{ab} U_a - D^2_{ab} U_a = R_{ab34} U^b = -2 \rho_{ab} U^b \].

We deduce,

\[ [\nabla_4, \nabla_3] U_a = 2 \omega \nabla_3 U_a + 2 \eta \nabla_3 U_a - 2 \xi_4 \eta U_a - 2 \xi_4 \xi U_c \]

\[ - 2 \omega \nabla_4 U_a - 2 \xi_4 \nabla_b U_a - 2 \eta_4 \eta_4 U_a - 2 \xi_4 \xi U_c - 2 \rho_{ab} U^b \]

\[ = 2 (\eta_b - \eta_b) \nabla_b U_a + 2 \omega \nabla_3 U_a - 2 \omega \nabla_4 U_a + 2 (\eta_a \eta_b - \eta_a \eta_b) U^b - 2 \rho_{ab} U^b \]

\[ + 2 (\xi_a \xi_c - \xi_a \xi_c) U^c \].

Thus,

\[ [\nabla_4, \nabla_3] U_a = 2 (\eta_b - \eta_b) \nabla_b U_a + 2 \omega \nabla_3 U_a - 2 \omega \nabla_4 U_a + 2 (\eta_a \eta_b - \eta_a \eta_b - \rho_{ab}) U^b \]

\[ + 2 (\xi_a \xi_c - \xi_a \xi_c) U^c \]

as stated. \qed

\textbf{Remark 2.34.} The formulas of Lemma 2.33 remain true if we substitute in the main terms

\[ \chi_{ab} \rightarrow \frac{1}{2} (\text{tr} \chi_{ab} + (a) \text{tr} \chi \in ab ) \]

\[ \chi_{ab} \rightarrow \frac{1}{2} (\text{tr} \chi_{ab} + (a) \text{tr} \chi \in ab ) \]

and add in the error terms the contributions due to \( \hat{\chi}, \hat{\chi} \).

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Thus the main terms in the commutator formula $[\nabla_3, \nabla_b]U_A$ take the form

$$[\nabla_3, \nabla_b]U_A = -\chi_{bc} \nabla_c U_A + (\eta_b - \zeta_b) \nabla_3 U_a + \sum_{i=1}^{k} \left( \chi_{a,b} \eta_c - \chi_{bc} \eta_a \right) U_{a_1...a_k}$$

$$= -\frac{1}{2} \left( \text{tr} \chi \nabla_b U_A + (\text{tr} \chi) \nabla_b U_A \right) + (\eta_b - \zeta_b) \nabla_3 U_A$$

$$+ \frac{1}{2} \sum_{i=1}^{k} \left( \delta_{a,b} \text{tr} \chi + \epsilon_{a,b} (\text{tr} \chi) \eta_c U_{a_1...a_k} \right)$$

$$- \frac{1}{2} \sum_{i=1}^{k} \eta_a \left( \text{tr} \chi U_{a_1...b...a_k} + (\text{tr} \chi) \nabla U_{a_1...b...a_k} \right).$$

If in addition $U$ is symmetric traceless in all indices, i.e. $U \in S_k$ we deduce,

$$[\nabla_3, \nabla_b]U_A = -\frac{1}{2} \left( \text{tr} \chi \nabla_b U_A + (\text{tr} \chi) \nabla_b U_A \right) + (\eta_b - \zeta_b) \nabla_3 U_A$$

$$+ \frac{1}{2} \sum_{i=1}^{k} \left( \delta_{a,b} \text{tr} \chi + \epsilon_{a,b} (\text{tr} \chi) \eta_c U_{a_1...a_k} \right)$$

$$- \frac{1}{2} \sum_{i=1}^{k} \eta_a \left( \text{tr} \chi U_{a_1...b...a_k} + (\text{tr} \chi) \nabla U_{a_1...b...a_k} \right).$$

In the following Lemma we specialize to the case of $S_0, S_1$ and $S_2$.

**Lemma 2.35.** The following commutation formulas hold true.

1. Given $f \in S_0$ we have

$$[\nabla_3, \nabla_a]f = -\frac{1}{2} \left( \text{tr} \chi \nabla_a f + (\text{tr} \chi) \nabla_f \nabla_a \right) + (\eta - \zeta_a) \nabla_3 f - \chi_{ab} \nabla_b f + \xi_a \nabla_4 f,$$

$$[\nabla_4, \nabla_a]f = -\frac{1}{2} \left( \text{tr} \chi \nabla_a f + (\text{tr} \chi) \nabla_f \nabla_a \right) + (\eta + \zeta_a) \nabla_4 f - \chi_{ab} \nabla_b f + \xi_a \nabla_3 f,$$

$$[\nabla_4, \nabla_3]f = 2(\eta - \eta) \nabla f + 2\omega \nabla_3 f - 2\omega \nabla_4 f.$$

2. Given $u \in S_1$ we have

$$[\nabla_3, \nabla_a]u_b = -\frac{1}{2} \left( \text{tr} \chi \nabla_a u_b + \eta_b u_a - \delta_{ab} \eta \cdot u \right) - \frac{1}{2} \left( \text{tr} \chi \nabla_a u_b + \eta_b u_a - \epsilon_{ab} \eta \cdot u \right)$$

$$+ (\eta - \zeta_a) \nabla_3 u_b + \text{Err}_{3ab}[u].$$

$$\text{Err}_{3ab}[u] = -\beta_a u_b + \xi_a \nabla_4 u_b - \xi_b \chi_{ac} u_c + \chi_{ab} \xi \cdot u - \chi_{ac} \nabla_c u_b - \eta_b \hat{\chi}_{ac} u_c + \hat{\chi}_{ab} \eta \cdot u,$$

$$[\nabla_4, \nabla_a]u_b = -\frac{1}{2} \left( \text{tr} \chi \nabla_a u_b + \eta_b u_a - \delta_{ab} \eta \cdot u \right) - \frac{1}{2} \left( \text{tr} \chi \nabla_a u_b + \eta_b u_a - \epsilon_{ab} \eta \cdot u \right)$$

$$+ (\eta + \zeta_a) \nabla_4 u_b + \text{Err}_{4ab}[u],$$

$$\text{Err}_{4ab}[u] = -\beta_a u_b + \xi_a \nabla_3 u_b - \xi_b \chi_{ac} u_c + \chi_{ab} \xi \cdot u - \hat{\chi}_{ac} \nabla_c u_b - \eta_b \hat{\chi}_{ac} u_c + \hat{\chi}_{ab} \eta \cdot u,$$

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We deduce the following commutation formulas.

1. Given \( u^3 \in S_3 \) we have

\[
\begin{align*}
[\nabla_4, \nabla_3]u_a &= 2\omega \nabla_3u_a - 2\omega \nabla_4u_a + 2(\eta_a - \eta_3)\nabla_bu_a + 2(\eta \cdot u)\eta_a - 2(\eta \cdot u)\eta_a - 2\rho ^* u_a \\
+ & \ Err_{3\alpha}[u], \\
Err_{3\alpha}[u] &= 2(\xi_a\xi_b - \xi_a\xi_b)u^b.
\end{align*}
\]  

(49)

3. Given \( u \in S_2 \) we have

\[
\begin{align*}
[\nabla_3, \nabla_a]u_{bc} &= -\frac{1}{2} tr \chi (\nabla_a u_{bc} + \eta_b u_{ac} + \eta_c u_{ab} - \delta_{ab}(\eta \cdot u)c - \delta_{ac}(\eta \cdot u)b) \\
&- \frac{1}{2} \beta_a u_{bc} + \xi_a \nabla^4 u_{bc} - \xi_b \nabla^4 u_{ac} - \xi_c \nabla^4 u_{ab} + \chi_{ab} \xi_d u_{dc} + \chi_{ac} \xi_d u_{bd} \\
&- \nabla_a \nabla_d u_{bc} - \eta_b \nabla_a u_{dc} - \eta_c \nabla_a u_{bd} + \nabla_a \eta_d u_{bc} + \nabla_a \eta_d u_{bd}, \\
+ & \ Err_{a\beta}[u], \\
Err_{a\beta}[u] &= 2(\beta_a u_{bc} + \xi_a \nabla^4 u_{bc} - \xi_b \nabla^4 u_{ac} - \xi_c \nabla^4 u_{ab} + \chi_{ab} \xi_d u_{dc} + \chi_{ac} \xi_d u_{bd} \\
&- \nabla_a \nabla_d u_{bc} - \eta_b \nabla_a u_{dc} - \eta_c \nabla_a u_{bd} + \nabla_a \eta_d u_{bc} + \nabla_a \eta_d u_{bd}, \\
+ & \ Err_{3\beta}[u], \\
Err_{3\beta}[u] &= 2(\xi_a \xi_b - \xi_a \xi_b)u^c b + 2(\xi_a \xi_b - \xi_a \xi_b)u_a c.
\end{align*}
\]  

(50)

We deduce the following commutation formulas.

**Corollary 2.36.** The following commutation formulas hold true.

1. Given \( u \in S_1 \) we have,

\[
\begin{align*}
[\nabla_4, \nabla_3]u_a &= 2\omega \nabla_3u_a - 2\omega \nabla_4u_a + 2(\eta_a - \eta_3)\nabla_bu_a + 4\eta \nabla(\eta \cdot u) - 4\eta \nabla(\eta \cdot u) - 4\rho ^* u_a \\
+ & \ Err_{3\alpha}[u], \\
Err_{3\alpha}[u] &= 2(\xi_a \xi_b - \xi_a \xi_b)u^b.
\end{align*}
\]  

(52)

\[
\begin{align*}
[\nabla_3, \nabla_a]u_{ab} &= 2\omega \nabla_3u_{ab} - 2\omega \nabla_4u_{ab} + 2(\eta_a - \eta_3)\nabla_cu_{ab} + 4\eta \nabla(\eta \cdot u) - 4\eta \nabla(\eta \cdot u) - 4\rho ^* u_{ab} \\
+ & \ Err_{3\beta}[u], \\
Err_{3\beta}[u] &= 2(\xi_a \xi_b - \xi_a \xi_b)u_a c.
\end{align*}
\]  

(53)
Also,

\begin{align}
[\nabla_3, \nabla \otimes] u &= -\frac{1}{2} \text{tr} \chi (\nabla \otimes u + \eta \otimes u) - \frac{1}{2} (a) \text{tr} \chi \left( (\nabla \otimes u + \eta \otimes u) + (\eta - \zeta) \otimes \nabla_3 u + \text{Err}_{3\otimes} [u],
\right.

\text{Err}_{3\otimes} [u] &= - \ast \beta \otimes \ast u + \xi \otimes \nabla_4 u - \xi \otimes (\chi \cdot u) + \tilde{\chi} (\xi \cdot u) - \tilde{\chi} \cdot \nabla u - \eta \otimes (\tilde{\chi} \cdot u) + \tilde{\chi} (\eta \cdot u),
\end{align}

(54)

\begin{align}
[\nabla_4, \nabla \otimes] u &= -\frac{1}{2} \text{tr} \chi (\nabla \otimes u + \eta \otimes u) - \frac{1}{2} (a) \text{tr} \chi \left( (\nabla \otimes u + \eta \otimes u) + (\eta + \zeta) \otimes \nabla_4 u + \text{Err}_{4\otimes} [u],
\right.

\text{Err}_{4\otimes} [u] &= \ast \beta \otimes \ast u + \xi \otimes \nabla_3 u - \xi \otimes (\chi \cdot u) + \tilde{\chi} (\xi \cdot u) - \tilde{\chi} \cdot \nabla u - \eta \otimes (\tilde{\chi} \cdot u) + \tilde{\chi} (\eta \cdot u).
\end{align}

2. Given \( u \in S_2 \) we have

\begin{align}
[\nabla_3, \text{div}] u &= -\frac{1}{2} \text{tr} \chi \left( (\text{div} u - 2\eta \cdot u) + \frac{1}{2} (a) \text{tr} \chi \left( (\text{div} * u - 2\eta \cdot * u) + (\eta - \zeta) \cdot \nabla_3 u + \text{Err}_{3\text{div}} [u],
\right.

\text{Err}_{3\text{div}} [u] &= -2 \ast \beta \otimes * u + \xi \cdot \nabla_4 u - \xi \cdot \chi \cdot u - (\chi \cdot u) \xi + \xi \cdot u \cdot \chi - \tilde{\chi} \cdot \nabla u

- \eta \cdot \tilde{\chi} \cdot u - (\tilde{\chi} \cdot u) \eta + \eta \cdot u \cdot \tilde{\chi}.
\end{align}

(55)

\begin{align}
[\nabla_4, \text{div}] u &= -\frac{1}{2} \text{tr} \chi \left( (\text{div} u - 2\eta \cdot u) + \frac{1}{2} (a) \text{tr} \chi \left( (\text{div} * u - 2\eta \cdot * u) + (\eta + \zeta) \cdot \nabla_4 u + \text{Err}_{4\text{div}} [u],
\right.

\text{Err}_{4\text{div}} [u] &= 2 \ast \beta \otimes * u + \xi \cdot \nabla_3 u - \xi \cdot \chi \cdot u - (\chi \cdot u) \xi + \xi \cdot u \cdot \chi - \tilde{\chi} \cdot \nabla u

- \eta \cdot \tilde{\chi} \cdot u - (\tilde{\chi} \cdot u) \eta + \eta \cdot u \cdot \tilde{\chi}.
\end{align}

Proof. We check (54). From (48) we have

\begin{align}
2[\nabla_4, \nabla \otimes] u_{ab} &= [\nabla_4, \nabla_a] u_b + [\nabla_4, \nabla_b] u_a - \delta_{ab} [\nabla_4, \text{div}] u

&= -\text{tr} \chi (\nabla \otimes u + \eta \otimes u) + (\eta + \zeta) \nabla_4 u_b - \frac{1}{2} (a) \text{tr} \chi H_{ab}

+ \text{Err}_{4ab} [u] + \text{Err}_{4ba} [u] - \delta_{ab} \text{Err}_{4\text{div}} [u]
\end{align}

where

\begin{align}
H_{ab} &= (\ast \nabla a u_b + \eta_b \ast u_a - \epsilon_{ab} \eta \cdot u) + (\ast \nabla b u_a + \eta_a \ast u_b - \epsilon_{ba} \eta \cdot u) - \delta_{ab} (\ast \nabla \cdot u + \eta \cdot * u)

&= 2 (\ast \nabla \otimes u)_{ab} + 2 (\eta \otimes * u)_{ab}.
\end{align}

Recalling that \( \ast \xi \otimes \eta = \xi \otimes \ast \eta = \ast (\xi \otimes \eta) \) we infer that \( H = 2 \ast (\nabla \otimes u + \eta \otimes u) \). This proves the desired result.

We check the last statement in item 2. From (52)

\begin{align}
[\nabla_4, \nabla_a] u_{bc} &= -\frac{1}{2} \text{tr} \chi (\nabla_a u_{bc} + \eta_b \ast u_{ac} + \eta_c \ast u_{ab} - \delta_{ab} (\eta \cdot u)_c - \delta_{ac} (\eta \cdot u)_b)

- \frac{1}{2} (a) \text{tr} \chi (\ast \nabla a u_{bc} + \eta_b \ast u_{ac} + \eta_c \ast u_{ab} - \epsilon_{ab} (\eta \cdot u)_c - \epsilon_{ac} (\eta \cdot u)_b)

+ (\eta_a + \zeta a) \nabla_4 u_{bc}
\end{align}

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we deduce, recalling that $\delta_{ab}u_{ab} = 0$,

$$\begin{align*}
[\nabla_4, \text{div}]u_c &= \delta_{ab}[\nabla_4, \nabla_a]u_{bc} \\
&= -\frac{1}{2} \text{tr} \chi (\text{div} u_c + (\eta \cdot u)_c - 2(\eta \cdot u)_c) \\
&\quad - \frac{1}{2} (a) \text{tr} \chi (-\text{div} * u_c + (\eta \cdot * u)_c + *(\eta \cdot u)_c) + ((\eta + \zeta) \cdot \nabla_4 u)_c
\end{align*}$$

which proves the desired result.

2.9 Null structure and Bianchi using conformally invariant derivatives

Consider frame transformations of the form

$$e'_3 = \lambda^{-1} e_3, \quad e'_4 = \lambda e_4, \quad e'_a = e_a.$$

Note that under the above mentioned frame transformation we have

$$\begin{align*}
\text{tr} \chi' &= \lambda^{-1} \text{tr} \chi, \quad (a) \text{tr} \chi' = \lambda^{-1} (a) \text{tr} \chi, \\
\xi' &= \lambda \xi, \quad \eta' = \eta, \quad \eta' = \eta, \quad \zeta' = \lambda^{-1} \zeta, \\
\alpha' &= \lambda^2 \alpha, \quad \beta' = \lambda \beta, \quad \rho' = \rho, \quad \rho' = \rho, \quad \beta' = \lambda^{-1} \beta, \quad \alpha' = \lambda^{-2} \alpha.
\end{align*}$$

and

$$\omega' = \lambda^{-1} \left( \omega + \frac{1}{2} e_3 (\log \lambda) \right), \quad \omega' = \lambda \left( \omega - \frac{1}{2} e_4 (\log \lambda) \right), \quad \zeta' = \zeta - \nabla (\log \lambda).$$

Remark 2.37. If $f$ verifies $f' = \lambda^s f$, then $\nabla_3 f, \nabla_4 f, \nabla_a f$ are not conformal invariant.

We correct the lacking of being conformal invariant by making the following definition.

Lemma 2.38. If $f$ verifies $f' = \lambda^s f$, then

1. $(c) \nabla_3 f := \nabla_3 f - 2s \omega f$ is $(s - 1)$-conformally invariant.
2. $(c) \nabla_4 f := \nabla_4 f + 2s \omega f$ is $(s + 1)$-conformally invariant.
3. $(c) \nabla_A f := \nabla_A f + s \zeta_A f$ is $s$-conformally invariant.

Proof. Immediate verification.

Remark 2.39. Note that $s$ is precisely what in [5] is called the signature of the tensor.

Using these definitions we rewrite the main equations as follows
Proposition 2.40.

\[
\begin{align*}
(c) \nabla_3 tr \chi &= -|\hat{\chi}|^2 - \frac{1}{2} (tr \chi^2 - (a) tr \chi^2) + 2(c) \text{div} \xi + 2 \xi \cdot (\eta + \eta), \\
(c) \nabla_3 (a) tr \chi &= - tr \chi (a) tr \chi + 2 (c) \text{curl} \xi + 2 \xi \wedge (-\eta + \eta), \\
(c) \nabla_3 \hat{\chi} &= - tr \chi \hat{\chi} + 2 \nabla \hat{\xi} + 2 \xi \hat{(\eta + \eta)} - \alpha,
\end{align*}
\]

\[
\begin{align*}
(c) \nabla_3 tr \chi &= - \hat{\chi} \cdot \hat{\chi} - \frac{1}{2} tr \chi tr \chi + (a) tr \chi + \frac{1}{2} (a) tr \chi + 2(c) \text{div} \eta + 2(\xi \cdot \xi + |\eta|^2) + 2\rho, \\
(c) \nabla_3 (a) tr \chi &= - \hat{\chi} \wedge \hat{\chi} - \frac{1}{2} (a) tr \chi tr \chi + tr \chi (a) tr \chi + 2 (c) \text{curl} \eta + 2 \xi \wedge \xi - 2 * \rho, \\
(c) \nabla_3 \hat{\chi} &= - \frac{1}{2} (tr \chi \hat{\chi} + tr \chi \hat{\chi}) - \frac{1}{2} ( - * \hat{\chi} (a) tr \chi + * \hat{\chi} (a) tr \chi) + 2 (c) \nabla \hat{\eta} + 2 \xi \hat{\xi} + 2 \eta \hat{\eta},
\end{align*}
\]

\[
\begin{align*}
(c) \nabla_4 tr \chi &= - |\hat{\chi}|^2 - \frac{1}{2} (tr \chi^2 - (a) tr \chi^2) + 2 (c) \text{div} \xi + 2 \xi \cdot (\eta + \eta), \\
(c) \nabla_4 (a) tr \chi &= - tr \chi (a) tr \chi + 2 (c) \text{curl} \xi + 2 \xi \wedge (-\eta + \eta), \\
(c) \nabla_4 \hat{\chi} &= - tr \chi \hat{\chi} + 2 (c) \nabla \hat{\xi} + 2 \xi \hat{(\eta + \eta)} - \alpha,
\end{align*}
\]

\[
\begin{align*}
(c) \nabla_3 \eta - (c) \nabla_4 \xi &= - \hat{\chi} \cdot (\eta - \eta) - \frac{1}{2} tr \chi (\eta - \eta) + \frac{1}{2} (a) tr \chi (\eta - \eta) - 4 \omega \xi + \beta, \\
(c) \nabla_4 \eta - (c) \nabla_3 \xi &= - \hat{\chi} \cdot (\eta - \eta) - \frac{1}{2} tr \chi (\eta - \eta) + \frac{1}{2} (a) tr \chi (\eta - \eta) - 4 \omega \xi - \beta.
\end{align*}
\]

Also,

\[
\begin{align*}
(c) \text{div} \hat{\chi} &= \frac{1}{2} (c) \nabla (tr \chi) - \frac{1}{2} (c) \nabla (a) tr \chi - (a) tr \chi * \eta - (a) tr \chi * \xi - \beta, \\
(c) \text{div} \hat{\xi} &= \frac{1}{2} (c) \nabla (tr \chi) - \frac{1}{2} (c) \nabla (a) tr \chi - (a) tr \chi * \eta - (a) tr \chi * \xi + \beta.
\end{align*}
\]

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We define the complexified version of horizontal tensors on type, i.e.

By Definition 2.5, the duals of real horizontal tensors are real horizontal tensors of the same type.

Recall Definition 2.9 of the set of real horizontal -tensors and symmetric traceless -tensors already introduced, we define their complexified version which components with the objective of simplifying the main equations.

From the real scalars, 1-tensors and symmetric traceless 2-tensors already introduced, we define their complexified version which results in anti-self dual tensors.

### 3 Main equations in complex notations

In this section we introduce complex notations for the Ricci coefficients and the curvature components with the objective of simplifying the main equations. From the real scalars, 1-tensors and symmetric traceless 2-tensors already introduced, we define their complexified version which results in anti-self dual tensors.

### 3.1 Complex notations

Recall Definition 2.9 of the set of real horizontal -tensors \( S_k = S_k(\mathcal{M}, \mathbb{R}) \) on \( \mathcal{M} \). For instance,

- \( a \in S_0 \) is a real scalar function on \( \mathcal{M} \),
- \( f \in S_1 \) is a real horizontal 1-tensor on \( \mathcal{M} \),
- \( u \in S_2 \) is a real horizontal symmetric traceless 2-tensor on \( \mathcal{M} \).

By Definition 2.5, the duals of real horizontal tensors are real horizontal tensors of the same type, i.e. \( *f \in S_1 \) and \( *u \in S_2 \).

We define the complexified version of horizontal tensors on \( \mathcal{M} \).

### Proposition 2.41

We have,

\[
\begin{align*}
\text{(c)} \nabla_3 \alpha - 2 \text{(c)} \nabla \beta &= -\frac{1}{2} (\text{tr} \chi \alpha + \text{(a)} \text{tr} \chi \star \alpha) + 2 \cdot 4\eta \hat{\otimes} \beta - 3(\rho \hat{\chi} + \star \rho \star \chi), \\
\text{(c)} \nabla_4 \beta - \text{(c)} \text{div} \alpha &= -2(\text{tr} \chi \beta - \text{(a)} \text{tr} \chi \star \beta) + \alpha \cdot \eta + 3(\xi \rho + \star \xi \star \rho), \\
\text{(c)} \nabla_3 \beta + \text{(c)} \text{div} \varrho &= - (\text{tr} \chi \beta + \text{(a)} \text{tr} \chi \star \beta) + 2\xi \cdot \chi + 3(\rho \eta + \star \rho \star \eta) + \alpha \cdot \xi, \\
\text{(c)} \nabla_4 \rho - \text{(c)} \text{div} \beta &= -\frac{3}{2} (\text{tr} \chi \rho + \text{(a)} \text{tr} \chi \star \rho) + 2\eta \cdot \beta - 2\xi \cdot \beta - \frac{1}{2} \hat{\chi} \cdot \alpha, \\
\text{(c)} \nabla_4 \ast \rho + \text{(c)} \text{curl} \beta &= -\frac{3}{2} (\text{tr} \chi \ast \rho - \text{(a)} \text{tr} \chi \rho) - 2\eta \cdot \ast \beta - 2\xi \cdot \ast \beta + \frac{1}{2} \hat{\chi} \cdot \ast \alpha, \\
\text{(c)} \nabla_3 \ast \rho + \text{(c)} \text{curl} \beta &= -\frac{3}{2} (\text{tr} \chi \ast \rho - \text{(a)} \text{tr} \chi \rho) - 2\eta \cdot \ast \beta + 2\xi \cdot \beta - \frac{1}{2} \hat{\chi} \cdot \ast \alpha, \\
\text{(c)} \nabla_4 \ast \beta - \text{(c)} \text{div} \tilde{\varrho} &= - (\text{tr} \chi \ast \beta + \text{(a)} \text{tr} \chi \star \beta) + 2\beta \cdot \tilde{\chi} - 3(\rho \tilde{\eta} + \ast \rho \star \eta) - \alpha \cdot \xi, \\
\text{(c)} \nabla_3 \ast \beta + \text{(c)} \text{div} \alpha &= -2(\text{tr} \chi \ast \beta - \text{(a)} \text{tr} \chi \star \beta) - \alpha \cdot \eta - 3(\xi \rho - \star \xi \star \rho), \\
\text{(c)} \nabla_4 \hat{\alpha} + 2 \text{(c)} \nabla \hat{\beta} &= -\frac{1}{2} (\text{tr} \chi \hat{\alpha} - \text{(a)} \text{tr} \chi \star \hat{\alpha}) - 2 \cdot 4\eta \hat{\otimes} \hat{\beta} - 3(\rho \hat{\chi} - \star \rho \star \hat{\chi}).
\end{align*}
\]
Definition 3.1. We denote by $S_k(\mathbb{C}) = S_k(M, \mathbb{C})$ the set of complex anti-self dual $k$-tensors on $M$. More precisely,

- $a + ib \in S_0(\mathbb{C})$ is a complex scalar function on $M$ if $(a, b) \in S_0$,
- $F = f + i^* f \in S_1(\mathbb{C})$ is a complex anti-self dual 1-tensor on $M$ if $f \in S_1$,
- $U = u + i^* u \in S_2(\mathbb{C})$ is a complex anti-self dual symmetric traceless 2-tensor on $M$ if $u \in S_2$.

Observe that $F \in S_1(\mathbb{C})$ and $U \in S_2(\mathbb{C})$ are indeed anti-self dual tensors, i.e.:

$^*F = -iF, \quad ^*U = -iU.$

More precisely

$$U_{12} = U_{21} = i^* U_{12} = i U_{12}, \quad U_{11} = i U_{11}.$$ 

Recall that the derivatives $\nabla_3, \nabla_4$ and $\nabla_a$ are real derivatives. We can use the dual operators to define the complexified version of the $\nabla_a$ derivative, which allows to simplify the notations in the main equations.

Definition 3.2. We define the complexified version of the horizontal derivative as

$$\mathcal{D} = \nabla + i^* \nabla, \quad \overline{\mathcal{D}} = \nabla - i^* \nabla$$ 

More precisely, we have

- for $a + ib \in S_0(\mathbb{C})$,
  $$\mathcal{D}(a + ib) := (\nabla + i^* \nabla)(a + ib), \quad \overline{\mathcal{D}}(a + ib) := (\nabla - i^* \nabla)(a + ib).$$ 

- For $f + i^* f \in S_1(\mathbb{C})$,
  $$\mathcal{D} \cdot (f + i^* f) := (\nabla + i^* \nabla) \cdot (f + i^* f) = 0,$$
  $$\overline{\mathcal{D}} \cdot (f + i^* f) := (\nabla - i^* \nabla) \cdot (f + i^* f),$$
  $$\mathcal{D} \hat{\otimes} (f + i^* f) := (\nabla + i^* \nabla) \hat{\otimes} (f + i^* f).$$ 

- For $u + i^* u \in S_2(\mathbb{C})$,
  $$\mathcal{D}(u + i^* u) := (\nabla + i^* \nabla)(u + i^* u) = 0,$$
  $$\overline{\mathcal{D}}(u + i^* u) := (\nabla - i^* \nabla)(u + i^* u).$$
Note that

\[ ^*D = -iD. \]

**Remark 3.3.** For \( F = f + i^*f \in S_1(\mathbb{C}) \) the operator \(-\frac{1}{2}D^\mathbb{C}\) is formally adjoint to the operator \( \overline{D} \cdot U \) applied to \( U \in S_2(\mathbb{C}) \). For \( h = a + ib \in S_0(\mathbb{C}) \) the operator \(-Dh\) is formally adjoint to the operator \( \overline{D} \cdot F \) applied to \( F \in S_1(\mathbb{C}) \). These notions makes sense literally only if the horizontal structure is integrable.

**Lemma 3.4.** The following holds.

- If \( \xi, \eta \in S_1 \)
  
  \[
  \xi \cdot \eta + i^* \xi \cdot i^* \eta = \frac{1}{2} \left( (\xi + i^* \xi) \cdot (\eta + i^* \eta) \right), 
  \]
  
  \[
  \xi \otimes \eta + i^* (\xi \otimes i^* \eta) = \frac{1}{2} \left( (\xi + i^* \xi) \otimes (\eta + i^* \eta) \right). 
  \]

- If \( \eta \in S_1, u \in S_2 \)
  
  \[
  u \cdot \eta + i^* u \cdot \eta = \frac{1}{2} \left( (u + i^* u) \cdot (\eta + i^* \eta) \right), 
  \]
  
  \[
  u \cdot \eta + i^* (u \cdot \eta) = \frac{1}{2} \left( (u + i^* u) \cdot (\eta + i^* \eta) \right). 
  \]

- If \( u, v \in S_2 \)
  
  \[
  u \cdot v + i^* u \cdot v = \frac{1}{2} \left( (u + i^* u) \cdot (v + i^* v) \right). 
  \]

- If \( a, b \in S_0 \)
  
  \[
  \nabla a - i^* \nabla b + i (\nabla a + \nabla b) = D(a + ib). 
  \]

- If \( \xi \in S_1 \)
  
  \[
  \text{div} \xi + i \text{curl} \xi = \frac{1}{2} D \cdot (\xi + i^* \xi), 
  \]
  
  \[
  \nabla \otimes \xi + i^* (\nabla \otimes \xi) = \frac{1}{2} D^\mathbb{C} \otimes (\xi + i^* \xi). 
  \]

- If \( u \in S_2 \)
  
  \[
  \text{div} u + i^* (\text{div} u) = \frac{1}{2} D \cdot (u + i^* u). 
  \]
Proof. The first identities rely on Lemma 2.18. The other rely on the following identities, for \( \xi \in S_1, \ u \in S_2 \):

\[
\nabla \cdot * \xi = \text{curl} \xi, \quad * \nabla \cdot \xi = -\text{curl} \xi, \quad * \nabla \cdot * \xi = \nabla \xi, \quad *(\text{div} \ u) = \nabla \cdot * \ u, \quad * \nabla \cdot \ u = -*(\text{div} \ u), \quad * \nabla \cdot * \ u = \nabla \cdot \ u.
\]

For example, we check that \(* (\nabla \hat{\otimes} \xi) = \nabla \hat{\otimes} * \xi\). Let \( C = \nabla \hat{\otimes} \xi \) and \( D = \nabla \hat{\otimes} * \xi \). We have,

\[
C_{11} = \nabla_1 \xi_1 - \nabla_2 \xi_2 = -C_{22}, \quad C_{12} = \nabla_1 \xi_2 + \nabla_2 \xi_1 = C_{21}.
\]

Hence,

\[
(*C)_{11} = C_{21} = \nabla_2 \xi_1 + \nabla_1 \xi_2, \quad (*C)_{12} = C_{22} = -C_{11} = -(\nabla_1 \xi_1 - \nabla_2 \xi_2),
\]

\[
(*C)_{21} = -C_{11} = -(\nabla_1 \xi_1 - \nabla_2 \xi_2), \quad (*C)_{22} = -C_{12} = -(\nabla_1 \xi_2 + \nabla_2 \xi_1).
\]

On the other hand,

\[
D_{11} = \nabla_1 * \xi_1 - \nabla_2 * \xi_2 = \nabla_1 \xi_2 + \nabla_2 \xi_1, \quad D_{12} = \nabla_1 * \xi_2 + \nabla_2 * \xi_1 = -\nabla_1 \xi_1 + \nabla_2 \xi_2.
\]

Hence,

\[
D_{11} = (*C)_{11}, \quad D_{12} = (*C)_{12},
\]

i.e. \(*C = D\) as desired. \(\square\)

Lemma 3.5. Let \( E = \xi + i * \xi \in S_1(\mathbb{C}) \) and \( F = \eta + i * \eta \in S_1(\mathbb{C}) \) and \( U = u + i * u \in S_2(\mathbb{C}) \). Then

\[
E \hat{\otimes} (F \cdot U) + F \hat{\otimes} (E \cdot U) = (E \cdot F + E \cdot F) \cdot U. \quad (56)
\]

Proof. Recall, see Lemma 2.19,

\[
\xi \hat{\otimes} (\eta \cdot u) + \eta \hat{\otimes} (\xi \cdot u) = (\xi \cdot \eta)u.
\]

Now

\[
E \hat{\otimes} (F \cdot U) = (\xi + i * \xi) \hat{\otimes} ((\eta + i * \eta) \cdot (u + i * u)) = 2(\xi + i * \xi) \hat{\otimes} ((u \cdot \eta) + i * (u \cdot \eta)) = 4 \left( \xi \hat{\otimes} (u \cdot \eta) + i * (\xi \hat{\otimes} (u \cdot \eta)) \right),
\]

\[
F \hat{\otimes} (E \cdot U) = 4 \left( \eta \hat{\otimes} (u \cdot \xi) + i * (\eta \hat{\otimes} (u \cdot \xi)) \right).
\]
Therefore
\[ E\hat{\otimes}(F \cdot U) + F\hat{\otimes}(E \cdot U) = 4(\xi \hat{\otimes}(u \cdot \eta) + i^*(\xi \hat{\otimes}(u \cdot \eta))) + 4(\eta \hat{\otimes}(u \cdot \xi) + i^*(\eta \hat{\otimes}(u \cdot \xi))) \]
\[ = 4((\xi \cdot \eta) u + 4i^*((\xi \cdot \eta) u)) \]
\[ = 4((\xi \cdot \eta) u + 4i^*((\xi \cdot \eta) u)) \]
while
\[ E \cdot F + E \cdot F = 2(\xi \cdot \eta + i^*(\xi \cdot \eta)) + 2(\eta \cdot \xi + i^*(\eta \cdot \xi)) = 4(\xi \cdot \eta). \]
Hence,
\[ E\hat{\otimes}(F \cdot U) + F\hat{\otimes}(E \cdot U) = (E \cdot F + E \cdot F)U \]
as stated.

3.2 Leibniz formulas

We collect here Leibniz formulas involving the derivative operators defined above.

**Lemma 3.6.** Let \( h \in S_0(\mathbb{C}), F = f + i^*f \in S_1(\mathbb{C}), U = u + i^*u \in S_2(\mathbb{C}). \) Then
\[ \overline{\mathcal{D}} \cdot (hF) = h\overline{\mathcal{D}} \cdot F + \overline{\mathcal{D}}(h) \cdot F, \]
\[ D\hat{\otimes}(hF) = hD\hat{\otimes}F + D(h)\hat{\otimes}F, \]
\[ \overline{\mathcal{D}} \cdot (hU) = \overline{\mathcal{D}}(h) \cdot U + h(\overline{\mathcal{D}} \cdot U), \]
\[ D\hat{\otimes}(F \cdot U) = (D \cdot F)U + (F \cdot D)U. \]

**Proof.** Note that
\[ \text{curl} (hf) = e^{ab}\nabla_a(hf)_b = e^{ab}\nabla_a(h) f_b + e^{ab} h \nabla_a(f)_b - e^{ba} \nabla_a(h) f_b + h\text{curl}f \]
\[ = -*\nabla h \cdot f + h\text{curl}f, \]
\[ \text{div} (hu) = \nabla^a(hu)_{ab} = \nabla^a(h)u_{ab} + h\nabla^a u_{ab} = \nabla h \cdot u + h\text{div}u, \]
and
\[ (\nabla\hat{\otimes}(hf))_{AB} = \frac{1}{2}((\nabla_A(hf)_B + \nabla_B(hf)_A) - \delta_{AB}(\text{div}(hf))) \]
\[ = \frac{1}{2}(\nabla_A(h) f_B + h\nabla_A f_B + \nabla_B h f_A + h\nabla_B f_A - \delta_{AB}(\nabla h \cdot f + h\text{div}(f))) \]
\[ = h(\nabla\hat{\otimes}f)_{AB} + (\nabla h \otimes f)_{AB}. \]
Now,

\[
\mathcal{D} \cdot (hF) = \mathcal{D} \cdot (hf + i \ast (hf)) = 2\text{div} (hf) + 2i\text{curl} (hf) \\
= 2(\nabla h \cdot f + h\text{div} f) + 2i(-\ast \nabla h \cdot f + h\text{curl} f) \\
= h\mathcal{D} \cdot F + 2\nabla h \cdot f + 2i(\nabla h \cdot \ast f) \\
= h\mathcal{D} \cdot F + F \cdot (\nabla h - i \ast \nabla h) \\
= h\mathcal{D} \cdot F + \mathcal{D}(h) \cdot F.
\]

We have

\[
\mathcal{D}\hat{\otimes}(hF) = 2\nabla \hat{\otimes}(hf) + 2i \ast (\nabla \hat{\otimes}(hf)) \\
= 2(h(\nabla \hat{\otimes}f) + (\nabla h \hat{\otimes}f)) + 2i \ast (h(\nabla \hat{\otimes}f) + (\nabla h \hat{\otimes}f)) \\
= 2h(\nabla \hat{\otimes}f) + 2ih \ast (\nabla \hat{\otimes}f) + 2(\nabla h \hat{\otimes}f) + 2i \ast (\nabla h \hat{\otimes}f) \\
= h\mathcal{D}\hat{\otimes}F + 2(\nabla h \hat{\otimes}f) + 2i \ast (\nabla h \hat{\otimes}f) \\
= h\mathcal{D}\hat{\otimes}F + (\nabla h + i \ast \nabla h) \hat{\otimes}(f + i \ast f) \\
= h\mathcal{D}\hat{\otimes}F + \mathcal{D}(h) \hat{\otimes}F.
\]

We have

\[
\mathcal{D} \cdot (hu) = \mathcal{D} \cdot (hu + i \ast (hu)) = 2\text{div} (hu) + 2i \ast (\text{div} (hu)) \\
= 2(\nabla h \cdot u + h\text{div} u) + 2i \ast (\nabla h \cdot u + h\text{div} u) \\
= 2\nabla h \cdot u + 2i \ast (\nabla h \cdot u) + h(\mathcal{D} \cdot U) \\
= \nabla h \cdot u + \ast \nabla h \cdot \ast u + i \nabla h \cdot \ast u - i \ast \nabla h \cdot u + h(\mathcal{D} \cdot U) \\
= \nabla h \cdot u - i \ast \nabla h \cdot u + i(\nabla h \cdot \ast u - i \ast \nabla h \cdot \ast u) + h(\mathcal{D} \cdot U) \\
= (\nabla h - i \ast \nabla h) \cdot (u + i \ast u) + h(\mathcal{D} \cdot U) \\
= (\mathcal{D}(h) \cdot U + h(\mathcal{D} \cdot U)
\]
as desired.

We write

\[
2\mathcal{D}\hat{\otimes}(F \cdot U)_{ab} = \mathcal{D}_a(F \cdot U)_b + \mathcal{D}_b(F \cdot U)_a - \delta_{ab}D^e(F \cdot U)_c \\
= \mathcal{D}_a(F^e U_{cb}) + \mathcal{D}_b(F^e U_{ca}) - \delta_{ab}D^d(F^e U_{cd}) \\
= \mathcal{D}_a F^e U_{cb} + \mathcal{D}_b F^e U_{ca} - \delta_{ab}D^d F^e U_{cd} + \mathcal{F}^e (\mathcal{D}_a U_{cb} + \mathcal{D}_b U_{ca} - \delta_{ab}D^d U_{cd}).
\]

Now, in view of Lemma 2.17,

\[
\mathcal{D}_a F^e U_{cb} + \mathcal{D}_b F^e U_{ca} = \delta_{ab}(\mathcal{D}^d F^e) U_{cd} + (D \cdot F) U_{ab} + \frac{1}{2} \left( (\mathcal{D}_a F^e - \mathcal{D}_c F^e) U_{cb} + (\mathcal{D}_b F^e - \mathcal{D}_c F^e) U_{ca} \right).
\]
Hence
\[ \mathcal{D}_a \mathcal{F}^c U_{cb} + \mathcal{D}_b \mathcal{F}^c U_{ca} - \delta_{ab} \mathcal{D}^d \mathcal{F}^c U_{cd} = (\mathcal{D} \cdot \mathcal{F}) U_{ab} + \frac{1}{2} \left( (\mathcal{D}_a \mathcal{F}_c - \mathcal{D}_c \mathcal{F}_a) U_{cb} + (\mathcal{D}_b \mathcal{F}_c - \mathcal{D}_c \mathcal{F}_b) U_{ca} \right). \]

Recall that \(* F = -iF, \quad *U = -iU, \quad *D = -iD. We deduce,
\[ \mathcal{D}_a \mathcal{F}_b - \mathcal{D}_b \mathcal{F}_a = i \varepsilon_{ab} (\mathcal{D} \cdot \mathcal{F}). \] (59)

To check note that
\[ \mathcal{D}_1 \mathcal{F}_2 - \mathcal{D}_2 \mathcal{F}_1 = 2 \left[ (\nabla_1 f_2 - \nabla_2 f_1) + i(\nabla_1 f_1 + \nabla_2 f_2) \right], \]
\[ (\mathcal{D} \cdot \mathcal{F}) = 2 \left[ (\nabla_1 f_1 + \nabla_2 f_2) - i(\nabla_1 f_2 - \nabla_2 f_1) \right]. \]

We deduce,
\[ \mathcal{D}_a \mathcal{F}^c U_{cb} + \mathcal{D}_b \mathcal{F}^c U_{ca} - \delta_{ab} \mathcal{D}^d \mathcal{F}^c U_{cd} = (\mathcal{D} \cdot \mathcal{F}) U_{ab} + \frac{1}{2} i(\mathcal{D} \cdot \mathcal{F}) (\varepsilon_{ac} U_{cb} + \varepsilon_{bc} U_{ca}) \]
\[ = (\mathcal{D} \cdot \mathcal{F}) U_{ab} + \frac{1}{2} i(\mathcal{D} \cdot \mathcal{F})(-2iU_{ab}) \]
\[ = 2(\mathcal{D} \cdot \mathcal{F}) U_{ab}. \]

Therefore,
\[ 2 \mathcal{D} \otimes (\mathcal{F} \cdot U)_{ab} = 2(\mathcal{D} \cdot \mathcal{F}) U_{ab} + \mathcal{F}^c (\mathcal{D}_a U_{cb} + \mathcal{D}_b U_{ca} - \delta_{ab} \mathcal{D}^d U_{cd}). \]

It remains to re-express the tensor
\[ \mathcal{D}_a U_{cb} + \mathcal{D}_b U_{ca} - \delta_{ab} \mathcal{D}^d U_{cd}. \]

Note also that \( \mathcal{D}^d U_{cd} = 0. \) We claim
\[ \mathcal{D}_a U_{cb} + \mathcal{D}_b U_{ca} = 2 \mathcal{D}_c U_{ab}. \]

Indeed, for \( a = b = 1, \ c = 2, \)
\[ 2 \mathcal{D}_1 U_{21} = 2(\nabla_1 + i \nabla_1) U_{21} = 2\nabla_1 U_{21} + i(i \nabla_1) U_{21} = -2i(\nabla_1 + i \nabla_1) U_{12} = 2(\nabla_2 - i \nabla_1) U_{11}, \]
\[ 2 \mathcal{D}_2 U_{11} = 2(\nabla_2 + i \nabla_2) U_{11} = 2\nabla_2 U_{11} + i(i \nabla_1) U_{11}. \]

For \( a = c = 1, \ b = 2, \)
\[ \mathcal{D}_1 U_{12} + \mathcal{D}_2 U_{11} = (\nabla_1 + i \nabla_2) U_{12} + (\nabla_2 - i \nabla_1) U_{11} = (\nabla_1 + i \nabla_2) U_{12} + i(\nabla_2 - i \nabla_1) U_{12} \]
\[ = 2(\nabla_1 + i \nabla_2) U_{12} = 2 \mathcal{D}_1 U_{12}. \]

We deduce,
\[ 2 \mathcal{D} \otimes (\mathcal{F} \cdot U)_{ab} = 2(\mathcal{D} \cdot \mathcal{F}) U_{ab} + 2 \mathcal{F}^c \mathcal{D}_c U_{ab}. \]

This implies the lemma. \( \square \)
Lemma 3.7. Let $F = f + i \ast f \in S_1(\mathbb{C})$, $U = u + i \ast u \in S_2(\mathbb{C})$. Then

\[ F \otimes (\overline{D} \cdot U) = (F \cdot \overline{D})U, \]
\[ (F \cdot \overline{D})U + (\overline{F} \cdot D)U = 4f \cdot \nabla U, \]
\[ (F \cdot \overline{D})U = 2F \cdot \nabla U. \]

As a consequence,

\[ 2f \cdot \nabla U = (F + \overline{F}) \cdot \nabla U. \quad (60) \]

Proof. We have

\[ (\overline{D} \cdot U)_1 = \overline{D}^a U_{1a} = (\nabla - i \ast \nabla)^a U_{1a} \]
\[ = (\nabla_1 - i \ast \nabla_1)U_{11} + (\nabla_2 - i \ast \nabla_2)U_{12} \]
\[ = (\nabla_1 - i \nabla_2)U_{11} + (\nabla_2 + i \nabla_1)(-iU_{11}) \]
\[ = 2(\nabla_1 - i \nabla_2)U_{11}, \]

and

\[ (\overline{D} \cdot U)_2 = \overline{D}^a U_{2a} = (\nabla - i \ast \nabla)^a U_{2a} \]
\[ = (\nabla_1 - i \ast \nabla_1)U_{21} + (\nabla_2 - i \ast \nabla_2)U_{22} \]
\[ = (\nabla_1 - i \nabla_2)(-iU_{11}) + (\nabla_2 + i \nabla_1)(-U_{11}) \]
\[ = -2i(\nabla_1 - i \nabla_2)U_{11}. \]

Therefore

\[ 2(F \otimes (\overline{D} \cdot U))_{11} = 2F_1(\overline{D} \cdot U)_1 - \delta_{11} F \cdot (\overline{D} \cdot U) \]
\[ = F_1(\overline{D} \cdot U)_1 - F_2(\overline{D} \cdot U)_2 \]
\[ = (f_1 + i \ast f_1)2(\nabla_1 - i \nabla_2)U_{11} - (f_2 + i \ast f_2)(-2i(\nabla_1 - i \nabla_2)U_{11}) \]
\[ = 4f_1 \nabla_1 U_{11} - 4i f_1 \nabla_2 U_{11} + 4i f_2 \nabla_1 U_{11} + 4f_2 \nabla_2 U_{11}. \]

On the other hand,

\[ 2(F \otimes (\overline{D} \cdot U))_{12} = F_1(\overline{D} \cdot U)_2 + F_2(\overline{D} \cdot U)_1 \]
\[ = (f_1 + i \ast f_1)(2(\nabla_1 - i \nabla_2)U_{12}) + (f_2 + i \ast f_2)(2(\nabla_1 - i \nabla_2)iU_{12}) \]
\[ = (f_1 + if_1)(2(\nabla_1 - i \nabla_2)U_{12}) + (f_2 - if_1)(2(\nabla_1 - i \nabla_2)iU_{12}) \]
\[ = 4f_1 \nabla_1 U_{12} - 4if_1 \nabla_2 U_{12} + 4if_2 \nabla_1 U_{12} + 4f_2 \nabla_2 U_{12} \]

which therefore gives

\[ (F \otimes (\overline{D} \cdot U))_{ab} = 2f_1 \nabla_1 U_{ab} - 2if_1 \nabla_2 U_{ab} + 2if_2 \nabla_1 U_{ab} + 2f_2 \nabla_2 U_{ab}. \]
On the other hand,

\[(F \cdot \overline{D}U)_{ab} = F^c \overline{D}_c U_{ab} = F_1 \overline{D}_1 U_{ab} + F_2 \overline{D}_2 U_{ab}\]

\[= (f_1 + i^* f_1)(\nabla_1 - i^* \nabla_1)U_{ab} + (f_2 + i^* f_2)(\nabla_2 - i^* \nabla_2)U_{ab}\]

\[= (f_1 + i f_2)(\nabla_1 - i \nabla_2)U_{ab} + (f_2 - i f_1)(\nabla_2 + i \nabla_1)U_{ab}\]

\[= 2f_1 \nabla_1 U_{ab} - 2if_1 \nabla_2 U_{ab} + 2if_2 \nabla_1 U_{ab} + 2f_2 \nabla_2 U_{ab}.\]

From the above we also have

\[(F \cdot D U)_{ab} = 2f_1 \nabla_1 U_{ab} + 2if_1 \nabla_2 U_{ab} - 2if_2 \nabla_1 U_{ab} + 2f_2 \nabla_2 U_{ab}\]

which implies

\[(F \cdot \overline{D}U)_{ab} + (F \cdot D U)_{ab} = 4f_1 \nabla_1 U_{ab} + 4f_2 \nabla_2 U_{ab} = 4f \cdot \nabla U\]

as stated. Finally,

\[F^c \overline{D}_c U = f^c \overline{D}_c U + i(\star_f)\overline{D}_c U = f^c \overline{D}_c U - i f^c(\star \overline{D}_c U) = 2f^c \overline{D}_c U = 2F^c \nabla_c U.\]

3.3 Complex notations for the Ricci coefficients and curvature components

We now extend the definitions for the Ricci coefficients and curvature components given in Section 2 to the complex case by using the anti-self dual tensors defined above.

**Definition 3.8.** Let \((\mathcal{M}, g)\) be a manifold satisfying the Einstein vacuum equation. Then we define the following complex anti-self dual tensors:

- \(A := \alpha + i^* \alpha, \quad B := \beta + i^* \beta, \quad P := \rho + i^* \rho, \quad B := \beta + i^* \beta, \quad A := \alpha + i^* \alpha,\)
- \(X := \chi + i^* \chi, \quad X := \chi + i^* \chi, \quad H := \eta + i^* \eta, \quad H := \eta + i^* \eta, \quad Z := \zeta + i^* \zeta, \quad \Xi := \xi + i^* \xi, \quad \Xi := \xi + i^* \xi.\)

In particular, note that

\[trX = tr \chi - i^{(a)} tr \chi, \quad \hat{X} = \hat{\chi} + i^* \hat{\chi}, \quad tr \chi = tr \chi - i^{(a)} tr \chi, \quad \hat{X} = \hat{\chi} + i^* \hat{\chi}.\]

3.4 Main equations in complex form

The complex notations allow us to rewrite the Ricci equations in a more compact form.
Proposition 3.9.

\[ \nabla_3 \text{tr} X + \frac{1}{2} (\text{tr} X)^2 + 2 \omega \text{tr} X = D \cdot \Xi + \Xi \cdot H + \Xi \cdot (H - 2Z) - \frac{1}{2} \hat{\Xi} \cdot \hat{X}, \]

\[ \nabla_3 \hat{X} + \mathfrak{R}(\text{tr} X) \hat{X} + 2 \omega \hat{X} = D \otimes \Xi + \Xi \otimes (H + H - 2Z) - \Delta, \]

\[ \nabla_3 \text{tr} X + \frac{1}{2} \text{tr} X \text{tr} X - 2 \omega \text{tr} X = D \cdot \mathcal{H} + H \cdot \mathcal{H} + 2 \mathfrak{P} + \Xi \cdot \Xi - \frac{1}{2} \hat{\Xi} \cdot \hat{X}, \]

\[ \nabla_3 \hat{X} + \frac{1}{2} \text{tr} X \hat{X} - 2 \omega \hat{X} = D \otimes H + H \otimes H - \frac{1}{2} \text{tr} X \hat{X} + \frac{1}{2} \Xi \otimes \Xi, \]

\[ \nabla_4 \text{tr} X + \frac{1}{2} (\text{tr} X)^2 + 2 \omega \text{tr} X = D \cdot \Xi + \Xi \cdot \mathcal{H} + \Xi \cdot (H + 2Z) - \frac{1}{2} \hat{\Xi} \cdot \hat{X}, \]

\[ \nabla_4 \hat{X} + \mathfrak{R}(\text{tr} X) \hat{X} + 2 \omega \hat{X} = D \otimes \Xi + \Xi \otimes (H + H + 2Z) - A. \]

Also,

\[ \nabla_3 \text{tr} X + \frac{1}{2} \text{tr} X (Z + H) - 2 \omega (Z - H) = -2 D \omega - \frac{1}{2} \hat{\Xi} \cdot (\Xi + \mathcal{H}) \]

\[ + \frac{1}{2} \text{tr} X \Xi + 2 \omega \Xi - B + \frac{1}{2} \Xi \cdot \hat{X}, \]

\[ \nabla_4 \text{tr} X (Z - H) - 2 \omega (Z + H) = 2 D \omega + \frac{1}{2} \hat{\Xi} \cdot (Z + \mathcal{H}) \]

\[ - \frac{1}{2} \text{tr} X \Xi - 2 \omega \Xi - B - \frac{1}{2} \Xi \cdot \hat{X}, \]

\[ \nabla_3 H - \nabla_4 \Xi = -\frac{1}{2} \text{tr} X (H - H) - \frac{1}{2} \hat{\Xi} \cdot (\mathcal{H} - \mathcal{H}) - 4 \omega \Xi + B, \]

\[ \nabla_4 H - \nabla_3 \Xi = -\frac{1}{2} \text{tr} X (H - H) - \frac{1}{2} \hat{\Xi} \cdot (\mathcal{H} - \mathcal{H}) - 4 \omega \Xi - B, \]

and

\[ \nabla_3 \omega + \nabla_4 \omega - 4 \omega \omega - \xi \cdot \xi - (\eta - \tilde{\eta}) \cdot \zeta + \eta \cdot \eta = \rho. \]

Also,

\[ \frac{1}{2} \mathcal{D} \cdot \hat{X} + \frac{1}{2} \hat{X} \cdot Z = \frac{1}{2} D \text{tr} X + \frac{1}{2} \text{tr} X Z - i \mathfrak{S}(\text{tr} X) H - i \mathfrak{S}(\text{tr} X) \Xi - B, \]

\[ \frac{1}{2} \mathcal{D} \cdot \hat{X} - \frac{1}{2} \hat{X} \cdot Z = \frac{1}{2} D \text{tr} X - \frac{1}{2} \text{tr} X Z - i \mathfrak{S}(\text{tr} X) H - i \mathfrak{S}(\text{tr} X) \Xi + B, \]

and,

\[ \text{curl} \zeta = -\frac{1}{2} \chi \wedge \hat{X} + \frac{1}{4} (\text{tr} \chi^{(a)} \text{tr} X - \text{tr} \chi^{(a)} \text{tr} X) + \omega^{(a)} \text{tr} X - \omega^{(a)} \text{tr} X + \ast \rho. \]
The complex notations allow us to rewrite the Bianchi identities as follows.

**Proposition 3.10.** We have,

\[
\begin{align*}
\nabla_3 A - \mathcal{D} \hat{\otimes} B & = -\frac{1}{2} \text{tr} X A + 4\omega A + (Z + 4H) \hat{\otimes} B - 3\mathcal{P} \hat{X}, \\
\nabla_4 B - \frac{1}{2} \mathcal{T} \cdot A & = -2 \text{tr} X B - 2\omega B + \frac{1}{2} A \cdot (2Z + \hat{H}) + 3\mathcal{P} \Xi, \\
\nabla_3 B - \mathcal{D} \hat{\mathcal{P}} & = -\text{tr} X B + 2\omega B + \mathcal{B} \cdot \hat{X} + 3\mathcal{P} H + \frac{1}{2} A \cdot \Xi, \\
\nabla_4 P - \frac{1}{2} \mathcal{T} \cdot B & = -\frac{3}{2} \text{tr} X P + \frac{1}{2} (2H + Z) \cdot \mathcal{B} - \Xi \cdot B - \frac{1}{4} \hat{X} \cdot \mathcal{A}, \\
\nabla_3 P + \frac{1}{2} \mathcal{T} \cdot B & = -\frac{3}{2} \text{tr} X P - \frac{1}{2} (2H - Z) \cdot B + \Xi \cdot B - \frac{1}{4} \hat{X} \cdot \mathcal{A}, \\
\n\nabla_4 B + \mathcal{D} P & = -\text{tr} X B + 2\omega B + \mathcal{B} \cdot \hat{X} - 3\mathcal{P} H - \frac{1}{2} A \cdot \Xi, \\
\n\nabla_3 B + \frac{1}{2} \mathcal{T} \cdot A & = -2 \text{tr} X B - 2\omega B - \frac{1}{2} A \cdot (-2Z + \hat{H}) - 3\mathcal{P} \Xi, \\
\n\nabla_4 A + \frac{1}{2} \mathcal{T} \hat{\otimes} B & = -\frac{1}{2} \text{tr} X A + 4\omega A + \frac{1}{2} (Z - 4H) \hat{\otimes} B - 3\mathcal{P} \hat{X}.
\end{align*}
\]

**Proof.** We derive the equation for $A$ and $B$. Observe that from the Bianchi identity

\[
\nabla_3 \alpha = 2\nabla \hat{\otimes} \beta - \frac{1}{2} \left(\text{tr} X \alpha + (^a)\text{tr} X \alpha \right) + 4\omega \alpha + 2(\zeta + 4\eta) \hat{\otimes} \beta - 3(\rho \hat{\chi} + \ast \rho \ast \hat{\chi})
\]

we obtain

\[
\ast \nabla_3 \alpha = 2 \ast (\nabla \hat{\otimes} \beta) - \frac{1}{2} \left(\text{tr} X \ast \alpha - (^a)\text{tr} X \alpha \right) + 4\omega \ast \alpha + 2 \ast ((\zeta + 4\eta) \hat{\otimes} \beta) - 3(\rho \ast \hat{\chi} - \ast \rho \hat{\chi}).
\]

This implies

\[
\begin{align*}
\nabla_3 A = & \quad \nabla_3 (\alpha + i \ast \alpha) \\
= & \quad 2\nabla \hat{\otimes} \beta + 2i \ast (\nabla \hat{\otimes} \beta) - \frac{1}{2} \left(\text{tr} X \alpha + (^a)\text{tr} X \alpha \right) - \frac{1}{2} i \left(\text{tr} X \ast \alpha - (^a)\text{tr} X \alpha \right) + 4\omega (\alpha + i \ast \alpha) \\
& + 2(\zeta + 4\eta) \hat{\otimes} \beta + 2i \ast ((\zeta + 4\eta) \hat{\otimes} \beta) - 3(\rho \ast \hat{\chi} - \ast \rho \hat{\chi}) \\
= & \quad \mathcal{D} \hat{\otimes} (\beta + i \ast \beta) - \frac{1}{2} \left(\text{tr} X - i (^a)\text{tr} X \right) \alpha - \frac{1}{2} \left(\text{tr} X - i (^a)\text{tr} X \right) \ast \alpha + 4\omega (\alpha + i \ast \alpha) \\
& + ((\zeta + 4\eta + i \ast (\zeta + 4\eta) \hat{\otimes} (\beta + i \ast \beta)) - 3(\rho - i \ast \rho) \hat{\chi} - 3(\rho - i \ast \rho) i \ast \hat{\chi}
\end{align*}
\]

which finally gives

\[
\begin{align*}
\nabla_3 A = & \quad \mathcal{D} \hat{\otimes} B - \frac{1}{2} \text{tr} X A + 4\omega A + (Z + 4H) \hat{\otimes} B - 3\mathcal{P} \hat{X}.
\end{align*}
\]

From the equation

\[
\nabla_4 \beta = \text{div} \alpha - 2(\text{tr} X \beta - (^a)\text{tr} X \beta) - 2\omega \beta + \alpha \cdot (2\zeta + \eta) + 3(\xi \rho + \ast \xi \ast \rho)
\]

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we obtain
\[ *(\nabla^4 \beta) = *\text{div} \alpha - 2(\text{tr} \chi \cdot *\beta + (a) \text{tr} \chi *\beta) - 2\omega *\beta + *(\alpha \cdot (2\zeta + \eta)) + 3(*\xi \rho - \xi *\rho). \]

This implies
\[ \nabla^4 B = \nabla^4 (\beta + i *\beta) \]
\[ = \text{div} \alpha + i *\text{div} \alpha - 2(\text{tr} \chi \beta - (a) \text{tr} \chi *\beta) - 2i(\text{tr} \chi *\beta + (a) \text{tr} \chi \beta) - 2\omega (\beta + i *\beta) \]
\[ + \alpha \cdot (2\zeta + \eta) + i *(\alpha \cdot (2\zeta + \eta)) + 3(\xi \rho + *\xi *\rho) + 3i(*\xi \rho - \xi *\rho) \]
\[ = \frac{1}{2} \nabla \cdot (\alpha + i *\alpha) - 2(\text{tr} \chi + i (a) \text{tr} \chi) \beta - 2(\text{tr} \chi + i (a) \text{tr} \chi) i *\beta - 2\omega (\beta + i *\beta) \]
\[ + \frac{1}{2}(\alpha + i *\alpha) \cdot (2\zeta + \eta - i *(2\zeta + \eta)) + 3(\rho - i *\rho) \xi + 3(\rho - i *\rho)i *\xi \]

which finally gives
\[ \nabla^4 B = \frac{1}{2} \nabla \cdot A - 2\text{tr} XB - 2\omega B + \frac{1}{2} A \cdot (2\overline{Z} + \overline{H}) + 3\overline{P} \Xi \]
as stated. \(\square\)

### 3.5 Main equations in complex form using conformal operators

**Definition 3.11.** We define the following conformal angular derivatives in the complex notation:

- For \(a + ib \in \mathcal{S}_0(\mathbb{C})\) we define
  \[ (c)D(a + ib) := (\nabla^{(c)} + i *(\nabla^{(c)}) (a + ib). \]

- For \(f + i *f \in \mathcal{S}_1(\mathbb{C})\) we define
  \[ (c)D(f + i *f) := (\nabla^{(c)} + i *(\nabla^{(c)}) \cdot (f + i *f), \]
  \[ (c)\tilde{D}(f + i *f) := (\nabla^{(c)} + i *(\nabla^{(c)}) \tilde{\cdot} (f + i *f). \]

- For \(u + i *u \in \mathcal{S}_2(\mathbb{C})\) we define
  \[ (c)\cdot (u + i *u) := (\nabla^{(c)} + i *(\nabla^{(c)}) \cdot (u + i *u). \]

- In all the above cases we set
  \[ (c)\overline{D} := (\nabla^{(c)} - \overline{i(c)} \nabla. \]

These complex notations allow us to rewrite the null structure equations as follows.
Proposition 3.12.

\[(c)\nabla_3 \text{tr} X + \frac{1}{2} \text{tr} X^2 = (c) D \cdot \Xi + \Xi \cdot \mathcal{H} + \Xi \cdot H - \frac{1}{2} \mathcal{X} \cdot \mathcal{X},\]

\[(c) \nabla_3 \hat{X} + \Re(\text{tr} X) \hat{X} = (c) D \hat{\otimes} \Xi + \Xi \hat{\otimes} (H + H) - A,\]

\[(c) \nabla_4 \text{tr} X + \frac{1}{2} \text{tr} X \text{tr} X = (c) D \cdot \mathcal{H} + H \cdot \mathcal{H} + 2P + \Xi \cdot \Xi - \frac{1}{2} \mathcal{X} \cdot \mathcal{X},\]

\[(c) \nabla_4 \hat{X} + \frac{1}{2} \text{tr} X \hat{X} = (c) D \hat{\otimes} H + H \hat{\otimes} H - \frac{1}{2} \text{tr} X \hat{X} + \Xi \hat{\otimes} \Xi,\]

\[(c) \nabla_4 \text{tr} X + \frac{1}{2} \text{tr} X^2 = (c) D \cdot \Xi + \Xi \cdot \mathcal{H} + \Xi \cdot H - \frac{1}{2} \mathcal{X} \cdot \mathcal{X},\]

\[(c) \nabla_4 \hat{X} + \Re(\text{tr} X) \hat{X} = (c) D \hat{\otimes} \Xi + \Xi \hat{\otimes} (H + H) - A,\]

\[(c) \nabla_3 H - (c) \nabla_4 \Xi = -\frac{1}{2} \text{tr} X (H - H) - \frac{1}{2} \mathcal{X} \cdot (\mathcal{H} - \mathcal{H}) + B,\]

\[(c) \nabla_4 H - (c) \nabla_3 \Xi = -\frac{1}{2} \text{tr} X (H - H) - \frac{1}{2} \mathcal{X} \cdot (\mathcal{H} - \mathcal{H}) - B.\]

Also,

\[\frac{1}{2} (c) D \cdot \hat{X} = \frac{1}{2} (c) D \text{tr} X - i\Im(\text{tr} X) H - i\Im(\text{tr} X) \Xi - B,\]

\[\frac{1}{2} (c) \overline{D} \cdot \hat{X} = \frac{1}{2} (c) D \overline{\text{tr} X} - i\Im(\text{tr} X) H - i\Im(\text{tr} X) \Xi + B.\]

The complex notations allow us to rewrite the Bianchi identities as follows.
Proposition 3.13. We have,

\[
\begin{align*}
(c)\nabla_3 A - (c)\mathcal{D} \otimes B &= -\frac{1}{2} \text{tr} X A + 4H \otimes B - 3\mathcal{P} \hat{X}, \\
(c)\nabla_4 B - \frac{1}{2} (c)\mathcal{D} \cdot A &= -2\text{tr} X B + \frac{1}{2} A \cdot \overline{B} + 3\mathcal{P} \Xi, \\
(c)\nabla_3 B - (c)\mathcal{D} \mathcal{P} &= -\text{tr} X B + \overline{B} \cdot \hat{X} + 3\mathcal{P} \mathcal{H} + \frac{1}{2} A \cdot \Xi, \\
(c)\nabla_4 P - \frac{1}{2} (c)\mathcal{D} \cdot \overline{B} &= -\frac{3}{2} \text{tr} X P + \mathcal{H} \cdot \overline{B} - \Xi \cdot B - \frac{1}{4} \hat{X} \cdot \overline{A}, \\
(c)\nabla_3 P + \frac{1}{2} (c)\mathcal{D} \cdot B &= -\frac{3}{2} \text{tr} X P - \mathcal{H} \cdot B + \Xi \cdot B - \frac{1}{4} \hat{X} \cdot \overline{A}, \\
(c)\nabla_4 B + (c)\mathcal{D} P &= -\text{tr} X B + \overline{B} \cdot \hat{X} - 3\mathcal{P} \mathcal{H} - \frac{1}{2} A \cdot \Xi, \\
(c)\nabla_3 B + \frac{1}{2} (c)\mathcal{D} \cdot A &= -2\text{tr} X B - \frac{1}{2} A \cdot \overline{B} - 3\mathcal{P} \Xi, \\
(c)\nabla_4 A + \frac{1}{2} (c)\mathcal{D} \otimes B &= -\frac{1}{2} \text{tr} X A - 2 \mathcal{H} \otimes B - 3\mathcal{P} \hat{X}.
\end{align*}
\]

4 Invariant wave operators

Recall that given a horizontal structure \( \mathcal{O}(M) \) and \( X, Y \in \mathcal{O}(M) \) we have defined \( \nabla_X Y \) to be the horizontal projection of \( D_X Y \). We extend this definition to \( X \in T(M) \) and \( Y \in \mathcal{O}(M) \) as follows

**Definition 4.1.** Given \( X \in T(M) \) and \( Y \in \mathcal{O}(M) \) we define,

\[
\hat{D}_X Y := (\mathcal{D}_X Y)^\perp.
\]

Given an orthonormal frame \( e_1, e_2 \in \mathcal{O}(M) \) we write

\[
\hat{D}_\mu e_a = \sum_{b=1,2} (\Lambda_\mu)_{ab} e_b, \quad (\Lambda_\mu)_{\alpha\beta} := g(\mathcal{D}_\mu e_\beta, e_\alpha).
\]

**Definition 4.2.** Given a general, covariant, \( S \)-horizontal tensor-field \( U \) we define its horizontal covariant derivative according to the formula,

\[
\hat{D}_X U(Y_1, …, Y_k) = X(U(Y_1, …, Y_k)) - U(\hat{D}_X Y_1, …, Y_k) - … - U(Y_1, …, \hat{D}_X Y_k) \quad (61)
\]

where \( X \in T_M \) and \( Y_1, …, Y_k \in T_S M \).

**Proposition 4.3.** For all \( X \in T_M \) and \( Y_1, Y_2 \in T_S M \),

\[
X h(Y_1, Y_2) = h(\hat{D}_X Y_1, Y_2) + h(Y_1, \hat{D}_X Y_2).
\]
\textbf{Proof.} Indeed,

\begin{align*}
X h(Y_1, Y_2) &= X g(Y_1, Y_2) = g(D_X Y_1, Y_2) + g(Y_1, D_X Y_2) = h(D_X Y_1, Y_2) + h(Y_1, D_X Y_2).
\end{align*}

\section*{4.1 Mixed tensors}

We consider tensors $T^k M \otimes O M$, i.e. tensors of the form $U_{\mu_1 \ldots \mu_k, A_1 \ldots A_L}$ for which we define,

\begin{align*}
\dot{D}_\mu U_{\nu_1 \ldots \nu_k, A_1 \ldots A_L} &= e_\mu U_{\nu_1 \ldots \nu_k, A_1 \ldots A_l \ldots} - U_{\nu_1 \ldots \nu_k, \nu_i A_1 \ldots A_l} - \ldots - U_{\nu_1 \ldots \nu_k, A_1 \ldots A_l \ldots \nu_1}.
\end{align*}

We are now ready to prove the following,

\textbf{Proposition 4.4.} We have the curvature formula

\begin{align*}
(\dot{D}_\mu \dot{D}_\nu - \dot{D}_\nu \dot{D}_\mu) \Psi_A &= R_{A B} \Psi_B.
\end{align*}

More generally for a mixed tensor $\Psi_{\lambda A}$

\begin{align*}
(\dot{D}_\mu \dot{D}_\nu - \dot{D}_\nu \dot{D}_\mu) \Psi_{\lambda A} &= R_{A B} \Psi_{\lambda B} + R_{A B} \Psi_{\mu \nu}.
\end{align*}

\textbf{Proof.} Straightforward verification. \hfill \Box

Observe that all the definitions above hold true for both real and complex tensors.

\section*{4.2 Invariant lagrangian}

Recall from Definition 2.3 that $\gamma$ is a horizontal 2-tensor with $\gamma_{AB} = g(e_A, e_B)$. We introduce for a complex tensor $\Psi \in S_k(C)$

\begin{align*}
\mathcal{L} &= g^{\mu \nu} \gamma_{AB} \dot{D}_\mu \Psi^A \dot{D}_B \Psi^B + V \gamma_{AB} \Psi^A \Psi^B.
\end{align*}

for $V$ real.

\textbf{Proposition 4.5.} The Euler Lagrange equations are given by:

\begin{align*}
\Box \Psi^A &= V \Psi^A
\end{align*}

where $\Box \Psi^A := g^{\mu \nu} \dot{D}_\mu \dot{D}_\nu \Psi^A$. 

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Proof. The variation of the action is given by,

\[
\begin{align*}
0 &= \int_M \gamma_{AB} \left( g^{\mu \nu} D^A \Psi^B + g^{\mu \nu} D^A (\delta \Psi)^B + V \Psi^A (\delta \Psi)^B + V \Psi^A (\delta \Psi)^B \right) \, dv_g \\
&= \int_M D^\nu \left( g^{\mu \nu} \gamma_{AB} D^A (\delta \Psi)^B \right) \, dv_g + \int_M \gamma_{AB} \left( g^{\mu \nu} \gamma_{AB} D^A (\delta \Psi)^B \right) \, dv_g \\
&- \int_M \gamma_{AB} \left( g^{\mu \nu} D^A (\delta \Psi)^B + V \Psi^A (\delta \Psi)^B + g^{\mu \nu} D^A (\delta \Psi)^B + V \Psi^A (\delta \Psi)^B \right) \, dv_g \\
&= - \int_M \gamma_{AB} \left( g^{\mu \nu} D^A (\delta \Psi)^B + V \Psi^A (\delta \Psi)^B \right) \, dv_g \\
&- \int_M \gamma_{AB} \left( g^{\mu \nu} D^A (\delta \Psi)^B + V \Psi^A (\delta \Psi)^B \right) \, dv_g
\end{align*}
\]

from which the proposition follows. \( \square \)

**Definition 4.6.** We introduce the energy-momentum tensor

\[
Q_{\mu \nu} := \frac{1}{2} \dot{D}_\mu \Psi \cdot \dot{D}_\nu \Psi + \frac{1}{2} \dot{D}_\mu \Psi \cdot \dot{D}_\nu \Psi - \frac{1}{2} g_{\mu \nu} \left( \dot{D}_\lambda \Psi \cdot \dot{D}_\lambda \Psi + V \Psi \cdot \Psi \right)
\]

where the dot product here denotes full contraction with respect to the horizontal indices, i.e. \( D_\mu \Psi \cdot D_\nu \Psi = h^{ac} h^{bd} D_\mu \Psi_{ab} \cdot D_\nu \Psi_{cd} \).

**Lemma 4.7.** We have,

\[
\begin{align*}
D^\nu Q_{\mu \nu} &= \frac{1}{2} \dot{D}_\mu \Psi \cdot (\Box \Psi - V \Psi) + \frac{1}{2} \dot{D}_\mu \Psi \cdot (\Box \Psi - V \Psi) \\
&+ \frac{1}{2} D^\nu \Psi A R_{A B \nu \mu} \Psi^B + \frac{1}{2} D^\nu \Psi A R_{A B \nu \mu} \Psi^B - \frac{1}{2} D_\mu \Psi V \Psi \cdot \Psi
\end{align*}
\]

Proof. We have,

\[
\begin{align*}
D^\nu Q_{\mu \nu} &= \frac{1}{2} \dot{D}^\nu \dot{D}_\nu \Psi \cdot \dot{D}_\mu \Psi + \frac{1}{2} \dot{D}^\nu \dot{D}_\nu \Psi \cdot \dot{D}_\mu \Psi \\
&+ \frac{1}{2} D^\nu \Psi \left( \dot{D}_\mu D_\mu - D_\mu D_\mu \right) \Psi + \frac{1}{2} D^\nu \Psi \left( \dot{D}_\mu D_\mu - D_\mu D_\mu \right) \Psi \\
&- \frac{1}{2} V D_\mu \Psi \cdot \Psi - \frac{1}{2} V D_\mu \Psi \cdot \Psi - \frac{1}{2} D_\mu \Psi \cdot \Psi \\
&= \frac{1}{2} \dot{D}_\mu \Psi (\Box \Psi - V \Psi) + \frac{1}{2} \dot{D}_\mu \Psi (\Box \Psi - V \Psi) + \frac{1}{2} D^\nu \Psi A R_{A B \nu \mu} \Psi^B + \frac{1}{2} D^\nu \Psi A R_{A B \nu \mu} \Psi^B \\
&- \frac{1}{2} D_\mu \Psi V \Psi \cdot \Psi
\end{align*}
\]

as desired. \( \square \)
4.3 Standard calculation

**Proposition 4.8.** Consider $\Psi \in S_2(\mathbb{C})$ and $X$ a vectorfield on $M$.

1. The 1-form $P_\mu = Q_{\mu\nu} X^\nu$ verifies,

$$D_\mu P_\mu = \frac{1}{2} X^\mu \dot{D}_\mu \Psi \cdot (\dot{\square} \Psi - V \Psi) + \frac{1}{2} X^\mu \dot{D}_\mu \Psi \cdot (\dot{\square} \Psi - V \Psi) - X(V) \Psi \cdot \Psi.$$ 

2. Let $X$ as above, $w$ a scalar and $M$ a one form. Define,

$$P_\mu = P_\mu[X,w,M] = Q_{\mu\nu} X^\nu + \frac{1}{4} w \Psi \cdot \dot{D}_\mu \Psi + \frac{1}{4} w \Psi \cdot \dot{D}_\mu \Psi - \frac{1}{4} |\Psi|^2 \partial_\mu w + \frac{1}{4} |\Psi|^2 M_\mu.$$ 

with $|\Psi|^2 := \Psi \cdot \Psi$. Then,

$$D_\mu P_\mu[X,w,M] = \frac{1}{2} Q \cdot (X) \pi - \frac{1}{2} X(V) \Psi \cdot \Psi + \frac{1}{2} w \mathcal{L}[\Psi] - \frac{1}{4} |\Psi|^2 \partial_\mu w$$

$$+ \frac{1}{4} |\Psi|^2 \text{Div} M + \frac{1}{4} \Psi \cdot \dot{D}_\mu \Psi M^\mu + \frac{1}{4} \Psi \cdot \dot{D}_\mu \Psi M^\mu$$

$$+ \frac{1}{2} \left(X(\Psi) + \frac{1}{2} w \Psi\right) \cdot (\dot{\square} \Psi - V \Psi) + \frac{1}{2} \left(X(\Psi) + \frac{1}{2} w \Psi\right) \cdot (\dot{\square} \Psi - V \Psi)$$

$$+ \frac{1}{2} X^\mu \dot{D}^\nu \Psi^A R_{AB\nu\mu} \Psi^B + \frac{1}{2} X^\mu \dot{D}^\nu \Psi^A R_{AB\nu\mu} \Psi^B.$$ 

**Proof.** See Chapter 10 in [11].

4.4 Canonical form of the wave equation

Consider the wave equation for $S_2$-tensors

$$\dot{\square} \Psi_{ab} := g^{\mu\nu} \dot{D}_\mu \dot{D}_\nu \Psi_{ab}.$$ 

**Lemma 4.9.** We have, for $\Psi \in S_2(\mathbb{C})$

1. We have

$$\square_2 \Psi = -\frac{1}{2} (\nabla_3 \nabla_4 \Psi + \nabla_4 \nabla_3 \Psi) + \Delta_2 \Psi + \left(\omega - \frac{1}{2} \text{tr} \chi\right) \nabla_4 \Psi + \left(\omega - \frac{1}{2} \text{tr} \chi\right) \nabla_3 \Psi$$

$$+ (\eta + \eta) \cdot \nabla \Psi. \quad (62)$$

2. We have

$$[\nabla_3, \nabla_4] \Psi = 4 \ast_\rho \ast \Psi - 2 \omega \nabla_3 \Psi + 2 \omega \nabla_4 \Psi - 2(\eta - \eta) \cdot \nabla \Psi. \quad (63)$$

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3. We also have

\[\square_2 \Psi = -\nabla_4 \nabla_3 \Psi + \Delta_2 \Psi + \left(2\omega - \frac{1}{2} \text{tr} \chi\right) \nabla_3 \Psi - \frac{1}{2} \text{tr} \chi \nabla_4 \Psi + 2\eta \cdot \nabla \Psi = -\rho \Psi. \ (64)\]

**Proof.** By definition

\[\Box_2 \Psi_{ab} = g^{34} \hat{D}_3 \hat{D}_4 \Psi_{ab} + g^{23} \hat{D}_4 \hat{D}_3 \Psi_{ab} + g^{ca} \hat{D}_c \hat{D}_d \Psi_{ab}.\]

We write

\[
\begin{align*}
\hat{D}_4 \Psi_{ab} & = \nabla_4 \Psi_{ab}, \\
\hat{D}_3 \hat{D}_4 \Psi_{ab} & = \nabla_3 \nabla_4 \Psi_{ab} - 2\omega \nabla_4 \Psi_{ab} - 2\eta \cdot \nabla \Psi_{ab}, \\
\hat{D}_4 \hat{D}_3 \Psi_{ab} & = \nabla_4 \nabla_3 \Psi_{ab} - 2\omega \nabla_3 \Psi_{ab} - 2\eta \cdot \nabla \Psi_{ab}, \\
\hat{D}_d \Psi_{ab} & = \nabla_d \Psi_{ab}, \\
\hat{D}_c \hat{D}_d \Psi_{ab} & = \nabla_c \nabla_d \Psi_{ab} - \frac{1}{2} \chi_{cd} \nabla_3 \Psi_{ab} - \frac{1}{2} \chi_{cd} \nabla_4 \Psi_{ab}. 
\end{align*}
\]

Hence

\[
\Box_2 \Psi_{ab} = -\frac{1}{2} \hat{D}_3 \hat{D}_4 \Psi_{ab} - \frac{1}{2} \hat{D}_4 \hat{D}_3 \Psi_{ab} + g^{cd} \hat{D}_c \hat{D}_d \Psi_{ab}
\]

\[= -\frac{1}{2} \left(\nabla_3 \nabla_4 \Psi_{ab} + \nabla_4 \nabla_3 \Psi_{ab}\right) + g^{cd} \left(\nabla_c \nabla_d \Psi_{ab} - \frac{1}{2} \chi_{cd} \nabla_3 \Psi_{ab} - \frac{1}{2} \chi_{cd} \nabla_4 \Psi_{ab}\right)
\]

\[+ \cdot \nabla \Psi_{ab} + \frac{1}{2} \chi_{cd} \nabla_3 \Psi_{ab} + \frac{1}{2} \chi_{cd} \nabla_4 \Psi_{ab}
\]

\[= -\frac{1}{2} \left(\nabla_3 \nabla_4 \Psi_{ab} + \nabla_4 \nabla_3 \Psi_{ab}\right) + \Delta_2 \Psi_{ab} - \frac{1}{2} \text{tr} \chi \nabla_3 \Psi_{ab} - \frac{1}{2} \text{tr} \chi \nabla_4 \Psi_{ab}
\]

\[+ \cdot \nabla \Psi_{ab} + \frac{1}{2} \chi_{cd} \nabla_3 \Psi_{ab} + \frac{1}{2} \chi_{cd} \nabla_4 \Psi_{ab}.
\]

Hence

\[\Box_2 \Psi = -\frac{1}{2} \left(\nabla_3 \nabla_4 \Psi + \nabla_4 \nabla_3 \Psi\right) + \Delta_2 \Psi + \left(\omega - \frac{1}{2} \text{tr} \chi\right) \nabla_3 \Psi + \left(\omega - \frac{1}{2} \text{tr} \chi\right) \nabla_4 \Psi
\]

\[+(\eta + \gamma) \cdot \nabla \Psi.
\]

To prove the second statement we recall

\[(\hat{D}_\mu \hat{D}_\nu - \hat{D}_\nu \hat{D}_\mu) \Psi_{ab} = R_a \psi c_{\mu \nu} \Psi_{cb} + R_b \psi c_{\mu \nu} \Psi_{ac}.
\]

Hence,

\[(\hat{D}_4 \hat{D}_3 - \hat{D}_3 \hat{D}_4) \Psi_{ab} = R_a \psi c_{43} \Psi_{cb} + R_b \psi c_{43} \Psi_{ac} = -2 \in_{ac} \rho \Psi_{cb} - 2 \in_{bc} \rho \Psi_{ac}
\]

\[= -4 \rho \psi ab.
\]

We deduce,

\[-4 \rho \psi ab = (\hat{D}_4 \hat{D}_4 - \hat{D}_3 \hat{D}_3) \Psi_{ab}
\]

\[= (\nabla_4 \nabla_4 - \nabla_3 \nabla_4) \Psi_{ab} + 2\omega \nabla_4 \Psi_{ab} + 2\eta \cdot \nabla \Psi_{ab} - 2\omega \nabla_3 \Psi_{ab} - 2\eta \cdot \nabla \Psi_{ab}
\]
and thus
\[ [\nabla_4, \nabla_3] \Psi = -4 \ast \rho \ast \Psi + 2 \omega \nabla_3 \Psi - 2 \overline{\omega} \nabla_4 \Psi + 2 (\eta - \eta) \cdot \nabla \Psi \]
as stated.

\section{Perturbations of Kerr}

Before discussing perturbations of Kerr, we first provide basic facts concerning Kerr in sections 5.1 and 5.2.

\subsection{The Kerr metric}

We consider the Kerr metric in standard Boyer-Lindquist coordinates
\[ g = -\frac{|q|^2 \Delta (dt)^2}{\Sigma^2} + \frac{\Sigma^2 (sin \theta)^2}{|q|^2} \left( d\varphi - \frac{2amr}{\Delta} dt \right)^2 + \frac{|q|^2}{\Delta} (dr)^2 + |q|^2 (d\theta)^2, \]
where
\[ q = r + ia \cos \theta \] (65)
and
\[ \begin{cases} 
\Delta &= r^2 - 2mr + a^2, \\
|q|^2 &= r^2 + a^2 (cos \theta)^2, \\
\Sigma^2 &= (r^2 + a^2)|q|^2 + 2mra^2 (sin \theta)^2 = (r^2 + a^2)^2 - a^2 (sin \theta)^2 \Delta.
\end{cases} \]
The null frame is given by
\[ e_4 = \frac{r^2 + a^2}{\Delta} \partial_t + \partial_r + \frac{a}{\Delta} \partial_\varphi, \quad e_3 = \frac{r^2 + a^2}{|q|^2} \partial_t - \frac{\Delta}{|q|^2} \partial_r + \frac{a}{|q|^2} \partial_\varphi, \]
\[ e_1 = \frac{1}{|q|} \partial_\theta, \quad e_2 = \frac{a sin \theta}{|q|} \partial_t + \frac{1}{|q| sin \theta} \partial_\varphi. \] (66)

\textbf{Remark 5.1.} There is an indeterminacy in the principal null frame as one may replace the null pair \((e_3, e_4)\) with \((\lambda^{-1} e_3, \lambda e_4)\) for any \(\lambda > 0\). In this section, the formulas correspond to the arbitrary choice of \(\lambda > 0\) ensuring \(\nabla_4 e_4 = 0\) and thus \(\omega = 0\). Note that our main result, on generalized Regge-Wheeler equation, is independent of this choice.

The complex Ricci coefficients are given by
\[ \hat{X} = \hat{X} = \Xi = \overline{\Xi} = \omega = 0, \quad H = -Z. \]
\[\text{tr}X = \frac{2}{q}, \quad \text{tr}X = -\frac{2\Delta q}{|q|^4},\]

\[H_1 = \frac{ai \sin \theta q}{|q|^3}, \quad H_2 = \frac{a \sin \theta q}{|q|^3},\]

\[Z_1 = \frac{ai \sin \theta \overline{q}}{|q|^3}, \quad Z_2 = \frac{a \sin \theta \overline{q}}{|q|^3},\]

\[\omega = \frac{a^2 \cos^2 \theta (r - m) + mr^2 - a^2 r}{|q|^4}.\]

**Remark 5.2.** Note the identities

\[H_1 = -Z_1, \quad H_2 = Z_2\]

and also

\[H_1 = \overline{H}_1, \quad H_2 = -\overline{H}_2.\]

The complex curvature components are given by

\[A = B = B = A = 0, \quad P = -\frac{2m}{q^r}.\]

### 5.2 Equations for \(q\) in Kerr

Recall the definition (65) of \(q\), \(q = r + ia \cos \theta\). We have the following equations.

**Lemma 5.3.** The scalar \(q\) satisfies

\[\nabla_4 q = \frac{1}{2} \text{tr}X q, \quad \nabla_3 q = \frac{1}{2} \text{tr}\overline{X} q, \quad \mathcal{D}q = q H, \quad \overline{\mathcal{D}}q = q \overline{H}.\] (67)

Also

\[q H = -\overline{q} H.\] (68)

In particular \(|H|^2 = |\overline{H}|^2\).

**Proof.** From the value of \(\text{tr}X = \frac{2}{q}\), and the reduced equation \(\nabla_4 \text{tr}X + \frac{1}{2}(\text{tr}X)^2 = 0\) we deduce the equation for \(\nabla_4 q\). From the value of \(P = -\frac{2m}{q^r}\) and the reduced Bianchi identity \(\nabla_3 P = -\frac{3}{2} \text{tr}X P\) we deduce the equation for \(\nabla_3 q\). Similarly, the equation \(\mathcal{D}\overline{P} = -3\overline{P} H\) becomes \(\mathcal{D}\overline{q} = H \overline{q}\) or \(\overline{\mathcal{D}}q = q \overline{H}\). The last equation in (67) follows in the same manner from \(\mathcal{D}P = -3P \overline{H}\).

To derive (68) we write, recalling that \(q = r + ia \cos \theta\) and using \(e_a(r) = 0\), i.e. \(\mathcal{D}r = 0\),

\[q H = \mathcal{D}q = \mathcal{D}(r + ia \cos \theta) = \nabla (r + ia \cos \theta) + i \ast \nabla (r + ia \cos \theta)
= ia \nabla \cos \theta - a \ast \nabla \cos \theta,\]

\[q \overline{H} = \overline{\mathcal{D}}q = \overline{\mathcal{D}}(r + ia \cos \theta) = \nabla (r + ia \cos \theta) - i \ast \nabla (r + ia \cos \theta)
= ia \nabla \cos \theta + a \ast \nabla \cos \theta.\]
We deduce $qH = -qH$ i.e. $qH = -\bar{q}H$ as stated.

**Lemma 5.4.** In Kerr we have

$$\nabla(tr \chi) = -\frac{3}{2} tr(\eta + \eta) - \frac{1}{2} (a tr(\eta - \eta)),$$

$$\nabla ((a)tr \chi) = -\frac{3}{2} (a tr(\eta + \eta)) + \frac{1}{2} tr(\alpha tr(\eta - \eta)).$$

**Proof.** Using that $\nabla q = \frac{1}{2} q(H + \bar{H})$, and since $trX = -\frac{2\Delta}{q^2}$ we have

$$\nabla(trX) = \frac{2\Delta}{q^2} \nabla q + \frac{4\Delta}{qq^2} \nabla q = \frac{2\Delta}{q^2} \left( \frac{1}{2} (H + \bar{H}) + \frac{2\Delta}{q^2} (\bar{H} + H) \right) = \frac{2\Delta}{qq^2} \left( \frac{1}{2} H + \frac{1}{2} \bar{H} + H \right) = -trX \left( \frac{1}{2} H + \frac{1}{2} \bar{H} + H \right).$$

We compute

$$-trX \left( \frac{1}{2} H + \frac{1}{2} \bar{H} + H \right) = -(trX - i(a)trX) \left( \frac{1}{2} \eta + i \eta + \frac{1}{2} (\eta - i \eta) + \eta - i \eta + \eta + i \eta \right) = -(trX - i(a)trX) \left( \frac{3}{2} \eta + \frac{3}{2} \eta + i \frac{1}{2} \eta - \frac{1}{2} \eta \right).$$

Since $\nabla trX = \nabla tr X - i\nabla (a)trX$, and each term is real, we obtain the lemma.

**Lemma 5.5.** In Kerr we have

$$\frac{1}{2} trX(\eta + \eta) - \frac{1}{2} (a tr(\eta - \eta)) = 0.$$  \hspace{1cm} (69)

**Proof.** Recall that

$$trX = -\frac{2\Delta}{|q|^4}, \quad (a)trX = \frac{2a\Delta \cos \theta}{|q|^4},$$

and

$$H_1 = -\frac{a^2 \sin \theta \cos \theta}{|q|^3} + i \frac{a \sin \theta r}{|q|^3}, \quad H_2 = \frac{a \sin \theta r}{|q|^3} + i \frac{a^2 \sin \theta \cos \theta}{|q|^3},$$

$$H_1 = -\frac{a^2 \sin \theta \cos \theta}{|q|^3} - i \frac{a \sin \theta(r)}{|q|^3}, \quad H_2 = -\frac{a \sin \theta r}{|q|^3} + i \frac{a^2 \sin \theta \cos \theta}{|q|^3}.$$
Therefore

\[\eta_1 = -\frac{a^2 \sin \theta \cos \theta}{|q|^3}, \quad \eta_2 = \frac{a \sin \theta r}{|q|^3},\]

\[\ast \eta_1 = \frac{a \sin \theta r}{|q|^3}, \quad \ast \eta_2 = \frac{a^2 \sin \theta \cos \theta}{|q|^3},\]

\[\eta_1 = -\frac{a^2 \sin \theta \cos \theta}{|q|^3}, \quad \eta_2 = \frac{a \sin \theta r}{|q|^3},\]

\[\ast \eta_1 = -\frac{a \sin \theta (r)}{|q|^3}, \quad \ast \eta_2 = \frac{a^2 \sin \theta \cos \theta}{|q|^3}.\]

We evaluate the above expression in 1:

\[-\frac{1}{2} \mathrm{tr} \chi \eta_1 - \frac{1}{2} \mathrm{tr} \chi \eta_1 + \frac{1}{2} (a) \mathrm{tr} \chi (\ast \eta_1) - \frac{1}{2} (a) \mathrm{tr} \chi \ast \eta_1\]

\[= -\frac{1}{2} \frac{2r \Delta a^2 \sin \theta \cos \theta}{|q|^4} - \frac{1}{2} \frac{2r \Delta a^2 \sin \theta \cos \theta}{|q|^4} - \frac{1}{2} \frac{2a \Delta \cos \theta (a \sin \theta r)}{|q|^4} + \frac{1}{2} \frac{2a \Delta \cos \theta \ar \sin \theta}{|q|^4} = 0.\]

We evaluate the expression in 2:

\[-\frac{1}{2} \mathrm{tr} \chi \eta_2 - \frac{1}{2} \mathrm{tr} \chi \eta_2 + \frac{1}{2} (a) \mathrm{tr} \chi (\ast \eta_2) - \frac{1}{2} (a) \mathrm{tr} \chi \ast \eta_2\]

\[= -\frac{1}{2} \frac{2r \Delta a \sin \theta r}{|q|^4} - \frac{1}{2} \frac{2r \Delta a \sin \theta r}{|q|^4} + \frac{1}{2} \frac{2a \Delta \cos \theta (a^2 \sin \theta \cos \theta)}{|q|^4} - \frac{1}{2} \frac{2a \Delta \cos \theta a^2 \sin \theta \cos \theta}{|q|^4} = 0.\]

This proves the lemma. \[\square\]

5.3 Perturbations of Kerr

Recall that in Kerr the Ricci coefficients \(\hat{X}, \hat{X}, \Xi, \Xi\) and the curvature components \(A, B, B, A\) vanish identically. We therefore expect that in perturbations of Kerr these quantities stay small, i.e. of order \(O(\epsilon)\) for a sufficiently small \(\epsilon\).

**Definition 5.6.** We say that a spacetime \(\mathcal{M}\) is an \(O(\epsilon)\) perturbation of \(\text{Kerr}(a, m)\) if \(\mathcal{M}\) comes equipped with two functions \(r : \mathcal{M} \rightarrow (0, \infty), \, \theta : \mathcal{M} \rightarrow [0, \pi]\) and a null frame \((e_3, e_4, e_1, e_2)\) such that the following conditions are verified.

1. The connection and curvature coefficients verify

\[
\Xi, \quad \Xi, \quad \hat{X}, \quad \hat{X}, \quad A, \quad B, \quad B, \quad A = O(\epsilon),
\]

\[
\tilde{H}, \quad \overline{H}, \quad \tilde{Z}, \quad \overline{Z}, \quad \overline{\mathrm{tr} X}, \quad \overline{\mathrm{tr} X}, \quad \tilde{\omega}, \quad \tilde{\omega}, \quad \overline{P} = O(\epsilon),
\]
where, given a quantity $Q$, we denoted by $\bar{Q}$ the renormalized quantity $\bar{Q} = Q - Q_{\text{Kerr}}$ where $Q_{\text{Kerr}}$ denotes the value of $Q$ expressed in terms of the variables $r$ and $\theta$.

2. The complex scalar $q = r + ia \cos \theta$ verifies

\[
\nabla_4 q - \frac{1}{2} \text{tr} X q = O(\epsilon), \quad \nabla_3 q - \frac{1}{2} \text{tr} X q = O(\epsilon), \quad Dq - q \bar{H} = O(\epsilon), \quad \overline{Dq} - q \bar{H} = O(\epsilon).
\]

We introduce a schematic notation, similar to the one used in [11], see Definition 2.3.8, to keep track of the error terms. We divide the connection coefficient terms into

\[
\Gamma^{(0)}_g = \left\{ r \Xi, \quad \tilde{X}, \quad \tilde{Z}, \quad \tilde{H}, \quad \frac{1}{r} \nabla_4 q - \frac{1}{2r} \text{tr} X q, \quad \frac{1}{r}(Dq - q \bar{H}), \quad \frac{1}{r}(\overline{Dq} - q \bar{H}) \right\},
\]

\[
\Gamma^{(0)}_b = \left\{ \tilde{H}, \quad \tilde{X}, \quad \Xi, \quad \frac{1}{r} \nabla_3 q - \frac{1}{2r} \text{tr} X q \right\}.
\]

(70)

For higher derivatives we denote

\[
\Gamma^{(s)}_g = \mathcal{D}^{\leq s} \Gamma_g, \quad \Gamma^{(s)}_b = \mathcal{D}^{\leq s} \Gamma_b,
\]

where

\[
\mathcal{D} = \{ \nabla_3, r \nabla_4, rD \}.
\]

For an $O(\epsilon)$-perturbation of Kerr we can also define the following two vectorfields:

\[
T = \frac{1}{2} e_3 + \frac{\Delta}{2|q|^2} e_4 - \frac{a \sin \theta}{|q|} e_2, \quad (71)
\]

\[
Z = \left( |q| \sin \theta + \frac{a^2 \sin^3 \theta}{|q|} \right) e_2 - \frac{1}{2} a \sin^2 \theta e_3 - \frac{a \sin^2 \theta \Delta}{2|q|^2} e_4. \quad (72)
\]

In Kerr spacetime we have $T = \partial_t$ and $Z = \partial_\varphi$.

5.4 Commutation formulas for perturbations of Kerr

In this section, we collect commutation formulas where the error terms are written schematically by making use of the notation introduced in (70).

5.4.1 Commutation formulas in complex form

Lemma 5.7. The following commutation formulas hold true.
1. Let \( F = f + i \cdot f \in S_1(\mathbb{C}) \). Then

\[
\begin{align*}
[N_4, D\hat{\otimes}]F &= -\frac{1}{2} \text{tr} X (D\hat{\otimes}F + H\hat{\otimes}F) + (H + Z)\hat{\otimes}N_4F + \text{Err}_4 D\hat{\otimes}[F], \\
[N_3, D\hat{\otimes}]F &= -\frac{1}{2} \text{tr} X (D\hat{\otimes}F + H\hat{\otimes}F) + (H - Z)\hat{\otimes}N_3F + \text{Err}_3 D\hat{\otimes}[F],
\end{align*}
\]

where

\[
\begin{align*}
\text{Err}_4 D\hat{\otimes}[F] &= r^{-1} \Gamma_g \cdot \otimes_1 F + \text{ l.o.t.}, & \text{Err}_3 D\hat{\otimes}[F] &= \Gamma_g \cdot \otimes_1 F + \text{ l.o.t.}
\end{align*}
\]

2. Let \( U = u + i \cdot u \in S_2(\mathbb{C}) \). Then

\[
\begin{align*}
[N_3, N_4]U &= -2 \omega N_3U + 2 \omega N_4U + 2(\eta - \eta) \cdot N_4U - 4 \eta \otimes (\eta \cdot U) + 4 \eta \otimes (\eta \cdot U) + 4i \cdot \rho U + \text{Err}_{34}[U],
\end{align*}
\]

where

\[
\text{Err}_{34}[U] = (\Gamma_g \cdot \Gamma_g) U = \text{ l.o.t.}
\]

3. For \( U \in S_2(\mathbb{C}) \)

\[
\begin{align*}
[N_4, \mathcal{B}]U &= -\frac{1}{2} \text{tr} X (\mathcal{B} \cdot U - 2 \mathcal{H} \cdot U) + (H + Z) \cdot N_4U + \text{Err}_4 \mathcal{B}[U], \\
[N_3, \mathcal{B}]U &= -\frac{1}{2} \text{tr} X (\mathcal{B} \cdot U - 2 \mathcal{H} \cdot U) + (H - Z) \cdot N_3U + \text{Err}_3 \mathcal{B}[U],
\end{align*}
\]

where

\[
\begin{align*}
\text{Err}_4 \mathcal{B}[U] &= r^{-1} \Gamma_g \cdot \otimes_1 U + \text{ l.o.t.}, & \text{Err}_3 \mathcal{B}[U] &= \Gamma_g \cdot \otimes_1 U + \text{ l.o.t.}
\end{align*}
\]

Also,

\[
\begin{align*}
[N_3, N_a]U_{bc} &= -\frac{1}{2} \text{tr} X \left( N_a U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab} (\eta \cdot U)_c - \delta_{ac} (\eta \cdot U)_b \right) \\
&- \frac{1}{2} \text{tr} X \left( (\eta_a - \zeta_a) N_3 U_{bc} + \text{Err}_{3abc}[U] \right)
\end{align*}
\]

where

\[
\text{Err}_{3abc}[U] = \Gamma_g \cdot \otimes_1 U + \text{ l.o.t.}
\]

where l.o.t. denotes error terms which are quadratic in the perturbation and enjoy better decay properties, or are higher order and decay at least as good.
Similarly, to prove the last part of the Lemma we recall from Corollary 2.36, from (54) we have

\[ [\nabla_4, \nabla \otimes] f = -\frac{1}{2} \text{tr} \chi (\nabla \otimes f + \nabla \otimes f) - \frac{1}{2} \langle a \rangle \text{tr} \chi ^* (\nabla \otimes f + \nabla \otimes f) + (\eta + \zeta) \nabla \otimes f + \text{Err} \nabla \otimes [f]. \]

Hence for \( F = f + i * f \)

\[ [\nabla_4, \nabla \otimes] F = -\frac{1}{2} \text{tr} \chi (\nabla \otimes F + \nabla \otimes F) - \frac{1}{2} \langle a \rangle \text{tr} \chi ^* (\nabla \otimes F + \nabla \otimes F) + (\eta + \zeta) \nabla \otimes F + \text{Err} \nabla \otimes [F]. \]

Recalling that \( *F = -iF \) and \( *(\nabla \otimes f) = \nabla \otimes * f \) we deduce,

\[ [\nabla_4, \nabla \otimes] F = -\frac{1}{2} \text{tr} \chi (\nabla \otimes F + \nabla \otimes F) + i \frac{1}{2} \langle a \rangle \text{tr} \chi (\nabla \otimes F + \nabla \otimes F) + (\eta + \zeta) \nabla \otimes F + \text{Err} \nabla \otimes [F] \]

\[ = -\frac{1}{2} (\text{tr} \chi - i \langle a \rangle \text{tr} \chi) (\nabla \otimes F + \nabla \otimes F) + (\eta + \zeta) \nabla \otimes F + \text{Err} \nabla \otimes [F] \]

\[ = -\frac{1}{2} \text{tr} X (\nabla \otimes F + \nabla \otimes F) + (\eta + \zeta) \nabla \otimes F + \text{Err} \nabla \otimes [F]. \]

Taking the dual

\[ [\nabla_4, * \nabla \otimes] F = -\frac{1}{2} \text{tr} X ( * \nabla \otimes F + * \nabla \otimes F) + *(\eta + \zeta) \nabla \otimes F + * \text{Err} \nabla \otimes [F]. \]

Finally adding we derive

\[ [\nabla_4, D \otimes] F = -\frac{1}{2} \text{tr} X (D \otimes F + H \otimes F) + (H + Z) \nabla \otimes F + \text{Err} \nabla \otimes D \otimes [F] \]

as stated.

Recall from (52),

\[ [\nabla_3, \nabla_4] u = -2\omega \nabla_3 u + 2\omega \nabla_4 u + 2(\eta c - \eta c) \nabla c u - 4\eta \nabla (\eta \cdot u) + 4H \nabla (\eta \cdot u) + 4 \rho \cdot u + \text{Err} \nabla_3 [u]. \]

Adding the corresponding formula for the dual we derive for \( U = u + i * u \),

\[ [\nabla_3, \nabla_4] U = -2\omega \nabla_3 U + 2\omega \nabla_4 U + 2(\eta c - \eta c) \nabla c U - 4\eta \nabla (\eta \cdot U) + 4H \nabla (\eta \cdot U) + 4i \rho \cdot U + \text{Err} \nabla_3 [U]. \]

To prove the last part of the Lemma we recall from Corollary 2.36

\[ [\nabla_4, \text{div}] u = -\frac{1}{2} \text{tr} \chi (\text{div} u - 2\eta \cdot u) + \frac{1}{2} \langle a \rangle \text{tr} \chi (\text{div} u - 2\eta \cdot u) + (\eta + \zeta) \cdot \nabla u + \text{Err} \nabla \text{div} [u]. \]

Similarly

\[ [\nabla_4, \text{div}] * u = -\frac{1}{2} \text{tr} \chi (\text{div} * u - 2\eta \cdot * u) - \frac{1}{2} \langle a \rangle \text{tr} \chi (\text{div} * u - 2\eta \cdot * u) + (\eta + \zeta) \cdot \nabla * u + \text{Err} \nabla \text{div} ^* [u]. \]

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Thus, for $U = u + i \ast u$,

$$[\nabla_4, \text{div}]U = -\frac{1}{2} \text{tr} \chi (\text{div} U - 2 \eta \cdot U) - \frac{1}{2} (\eta + \zeta) \cdot \nabla_4 U + \text{Err}_{4D}[U]$$

Hence, since $\ast U = -iU$,

$$[\nabla_4, \bar{D}] \cdot U = [\nabla_4, \nabla] \cdot U - i [\nabla_4, \ast \nabla] \cdot U = [\nabla_4, \nabla] \cdot U + i [\nabla_4, \nabla] \cdot \ast U = 2[\nabla_4, \nabla] \cdot U$$

Also,

$$\bar{D} U = \nabla \cdot U - i \ast \nabla \cdot U = \nabla \cdot U + i \nabla \cdot \ast U = 2\nabla \cdot U.$$

Hence,

$$[\nabla_4, \bar{D}] \cdot U = 2[\nabla_4, \text{div}]U = -\frac{1}{2} \text{tr} \chi (\bar{D} U - 2 \eta \cdot U) + 2(\eta + \zeta) \cdot \nabla_4 U + \text{Err}_{4D}[U].$$

Note also that since $U = i \ast U$ and $\eta \cdot \ast U = -\ast \eta \cdot U$

$$2\eta \cdot U = \eta \cdot U + \eta \cdot (i \ast U) = \eta \cdot U - i \ast \eta \cdot U = (\eta - i \ast \eta) \cdot U = \bar{H} \cdot U.$$

Similarly

$$2(\eta + \zeta) \cdot \nabla_4 U = (\bar{H} + \bar{Z}) \cdot \nabla_4 U.$$

Hence,

$$[\nabla_4, \bar{D}] \cdot U = -\frac{1}{2} \text{tr} \chi (\bar{D} U - 2 \bar{H} \cdot U) + (\bar{H} + \bar{Z}) \cdot \nabla_4 U + \text{Err}_{4D}[U]$$

as stated.

Starting with (52)

$$[\nabla_3, \nabla_a]u_{bc} = -\frac{1}{2} \text{tr} \chi \left( \nabla_a u_{bc} + \eta_b u_{ac} + \eta_c u_{ab} - \delta_{ab}(\eta \cdot u)_c - \delta_{ac}(\eta \cdot u)_b \right)$$

$$- \frac{1}{2} (\eta + \zeta) \left( \ast \nabla_a u_{bc} + \eta_b \ast u_{ac} + \eta_c \ast u_{ab} - \varepsilon_{ab}(\eta \cdot u)_c - \varepsilon_{ac}(\eta \cdot u)_b \right)$$

and adding the same expression for $u$ replaced with $i \ast u$ we derive

$$[\nabla_3, \nabla_a]U_{bc} = -\frac{1}{2} \text{tr} \chi \left( \nabla_a U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta \cdot U)_c - \delta_{ac}(\eta \cdot U)_b \right)$$

$$- \frac{1}{2} \text{tr} \chi \left( \ast \nabla_a U_{bc} + \eta_b \ast U_{ac} + \eta_c \ast U_{ab} - \varepsilon_{ab}(\eta \cdot U)_c - \varepsilon_{ac}(\eta \cdot U)_b \right)$$

$$+ (\eta_a - \zeta_a) \nabla_3 U_{bc} + \text{Err}_{3abc}[U]$$

as stated.
5.4.2 Commutation formulas for conformally invariant derivatives

**Lemma 5.8.** The following commutation formulas hold true.

1. Let $F = f + i \ast f \in S_1(\mathbb{C})$. Then

$$\begin{align*}
\left[ c \nabla_4, (c)D \otimes \right] F &= -\frac{1}{2} \text{tr} X \left( (c)D \otimes F + (1 - s) H \otimes F \right) + H \otimes (c) \nabla_4 F + \text{Err}_{4D \otimes} [F], \\
\left[ c \nabla_3, (c)D \otimes \right] F &= -\frac{1}{2} \text{tr} X \left( (c)D \otimes F + (1 + s) H \otimes F \right) + H \otimes (c) \nabla_3 F + \text{Err}_{3D \otimes} [F],
\end{align*}$$

(77)

where

$$\begin{align*}
\text{Err}_{4D \otimes} [F] &= r^{-1} \Gamma_g \cdot \partial \leq 1 F + \text{ l.o.t.}, \\
\text{Err}_{3D \otimes} [F] &= \Gamma_g \cdot \partial \leq 1 F + \text{ l.o.t.}
\end{align*}$$

2. Let $U = u + i \ast u \in S_2(\mathbb{C})$. Then

$$\begin{align*}
\left[ c \nabla_3, (c) \nabla_4 \right] U &= 2((\eta - \eta) \cdot (c) \nabla) U + ((s - 2) P + (s + 2) \overline{P} - 2s \eta \cdot \eta) U \\
&- 4 \eta \otimes (\eta \cdot U) + 4 \eta \otimes (\eta \cdot U) + \text{Err}_{34} [U]
\end{align*}$$

(78)

where

$$\begin{align*}
\text{Err}_{34} [U] &= (\Gamma_g \cdot \Gamma_g) U = \text{ l.o.t.}
\end{align*}$$

3. For $U \in S_2(\mathbb{C})$

$$\begin{align*}
\left[ c \nabla_3, (c) \overline{D} \cdot \right] U &= -\frac{1}{2} \text{tr} \chi \left( (c) \overline{D} \cdot U + (s - 2) \overline{H} \cdot U \right) + \overline{H} \cdot \nabla_3 U + \text{Err}_{3 \overline{D}} [U]
\end{align*}$$

(79)

where

$$\begin{align*}
\text{Err}_{3 \overline{D}} [U] &= \Gamma_g \cdot \partial \leq 1 U + \text{ l.o.t.}
\end{align*}$$

Also,

$$\begin{align*}
\left[ c \nabla_3, (c) \nabla_a \right] U_{bc} &= \eta_a (c) \nabla_3 U_{bc} \\
&- \frac{1}{2} \text{tr} \chi \left( (c) \nabla_a U_{bc} + s(\eta_a)U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta \cdot U)_c - \delta_{ac}(\eta \cdot U)_b \right) \\
&- \frac{1}{2} (a) \text{tr} \chi \left( * (c) \nabla_a U_{bc} + s(\ast \eta_a)U_{bc} + \eta_b \ast U_{ac} + \eta_c \ast U_{ab} \right. \\
&\left. - \epsilon_{ab} (\eta \cdot U)_c - \epsilon_{ac} (\eta \cdot U)_b \right) + \text{Err}_{3abc} [U]
\end{align*}$$

where

$$\begin{align*}
\text{Err}_{3abc} [U] &= \Gamma_g \cdot \partial \leq 1 U + \text{ l.o.t.}
\end{align*}$$
Recall that \((c)\) we finally obtain
\[
\nabla_3(D \otimes F) = \nabla_3(D \otimes F + sZ \otimes F) - 2s_\omega(D \otimes F + sZ \otimes F)
\]
\[
= \nabla_3(D \otimes F) + s\nabla_3(Z \otimes F) + sZ \nabla_3 F - 2s_\omega(D \otimes F + sZ \otimes F).
\]

Using (73) and the null structure equation
\[
\nabla_3 Z = -\frac{1}{2} \text{tr} X(Z + H) + 2\omega(Z - H) - 2D_\omega
\]
\[
- \frac{1}{2} \hat{X} \cdot (Z + H) + \frac{1}{2} \text{tr} X \Xi + 2\omega \Xi - B + \frac{1}{2} \Xi \cdot \hat{X}
\]
we obtain
\[
(c) \nabla_3(D \otimes F) = D \otimes (\nabla_3 F) - \frac{1}{2} \text{tr} X(D \otimes F + H \otimes F) + (H - Z) \otimes \nabla_3 F
\]
\[
+ s\left( -\frac{1}{2} \text{tr} X(Z + H) + 2\omega(Z - H) - 2D_\omega \right) \otimes F
\]
\[
+ sZ \otimes \nabla_3 F - 2s_\omega(D \otimes F + sZ \otimes F) + \text{Err}_3(c,D \otimes \omega)[F]
\]
where
\[
\text{Err}_3(c,D \otimes \omega)[F] = \text{Err}_3D \otimes \omega[F] + s\left( -\frac{1}{2} \hat{X} \cdot (Z + H) + \frac{1}{2} \text{tr} X \Xi + 2\omega \Xi - B + \frac{1}{2} \Xi \cdot \hat{X} \right) \otimes F.
\]

We can rewrite the above as
\[
(c) \nabla_3(D \otimes F) = D \otimes (\nabla_3 F) + sZ \otimes \nabla_3 F + (H - Z) \otimes \nabla_3 F + 2s_\omega(Z - H) \otimes F
\]
\[
- \frac{1}{2} \text{tr} X(D \otimes F + H \otimes F + s(Z + H) \otimes F)
\]
\[
- 2s_\omega D \otimes F - 2s_\omega Z \otimes F - 2sD_\omega \otimes F + \text{Err}_3(c,D \otimes \omega)[F]
\]
which gives
\[
(c) \nabla_3(D \otimes F) = D \otimes (\nabla_3 F) + sZ \otimes (c) \nabla_3 F + (H - Z) \otimes (c) \nabla_3 F
\]
\[
- \frac{1}{2} \text{tr} X(D \otimes F + H \otimes F + s(Z + H) \otimes F) + \text{Err}_3(c,D \otimes \omega)[F].
\]

Recall that \((c)\) \nabla_3 F is conformal of type \(s - 1\), therefore
\[
(c)D \otimes (c) \nabla_3 F = D \otimes (c) \nabla_3 F + (s - 1)Z \otimes (c) \nabla_3 F.
\]

We finally obtain
\[
(c) \nabla_3(D \otimes F) = (c)D \otimes (c) \nabla_3 F - \frac{1}{2} \text{tr} X \left( (c)D \otimes F + (s + 1)H \otimes F \right) + H \otimes (c) \nabla_3 F + \text{Err}_3(c,D \otimes \omega)[F]
\]

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as stated.

Similarly,

\[
\nabla_4(c\mathcal{D}\mathcal{F}) = \nabla_4(D\mathcal{F} + sZ\mathcal{F})
\]

Using (73) and the null structure equation for \(\nabla Z\), we obtain

\[
\nabla_4(c\mathcal{D}\mathcal{F}) = \mathcal{D}\mathcal{F} + sZ\nabla_4 F + 2s\omega(Z + H)\mathcal{F}
\]

which gives

\[
\nabla_4(c\mathcal{D}\mathcal{F}) = \mathcal{D}\mathcal{F} + sZ\nabla_4 F + 2s\omega(Z + H)\mathcal{F}.
\]

Recall that \(\nabla_4 F\) is conformal of type \(s + 1\), therefore

\[
\nabla_4(c\mathcal{D}\mathcal{F}) = \mathcal{D}\mathcal{F} + 2s\omega(Z + H)\mathcal{F}.
\]

We finally obtain

\[
\nabla_4(c\mathcal{D}\mathcal{F}) = \mathcal{D}\mathcal{F} + 2s\omega(Z + H)\mathcal{F}.
\]

as stated.

We have

\[
\left\{\nabla_3, \nabla_4\right\} U = \nabla_3(c\mathcal{D}\mathcal{F}) - \nabla_4(c\mathcal{D}\mathcal{F})
\]

\[
= \mathcal{D}\mathcal{F} - 2s\omega(Z + H)\mathcal{F}.
\]
Using (74) and
\[
\nabla_3 \omega + \nabla_4 \omega - 4 \omega_{\omega} = \rho + (\eta - \eta) \cdot \zeta - \eta \cdot \eta + \xi \cdot \xi
\]
we obtain
\[
\begin{align*}
\left[ \nabla_3, \nabla_4 \right] U &= -2 \omega \nabla_3 U + 2 \omega \nabla_4 U + 2(\eta c - \eta) \nabla_\epsilon U - 4 \eta \tilde{\omega}(\eta \cdot U) + 4 \eta \tilde{\omega}(\eta \cdot U) + \text{Err}_{34}[U] \\
&= -2(\epsilon - \epsilon) U - 2 \omega \nabla_4 U + 2 \omega \nabla_3 U + 2s(\rho + (\eta - \eta) \cdot \zeta - \eta \cdot \eta) U \\
&= 2(\eta - \eta)(\nabla_\epsilon U + ((s - 2) \epsilon + (s + 2) \tilde{\omega} - 2 \rho \cdot \eta) U \\
&= -4 \eta \tilde{\omega}(\eta \cdot U) + 4 \eta \tilde{\omega}(\eta \cdot U) + \text{Err}_{34}[U]
\end{align*}
\]
as stated.

We have
\[
\nabla_3 \left[ \nabla_3, \nabla_4 \right] U = \nabla_3 \left( [D + sZ] \cdot U \right)
\]
\[
= \nabla_3([D + sZ] \cdot U) - 2s \omega([D + sZ] \cdot U)
\]
\[
= \nabla_3[D \cdot U] + s \nabla_3 Z \cdot U + sZ \cdot \nabla_3 U - 2s \omega([D + sZ] \cdot U).
\]

Using the equation for \( \nabla_3 Z \) and (79), we have
\[
\nabla_3 \left[ \nabla_3, \nabla_4 \right] U = D \cdot \nabla_3 U - \frac{1}{2} \text{tr} \chi(D \cdot U - 2 \tilde{H} \cdot U) + (H - Z) \cdot \nabla_3 U + \text{Err}_{34}[U]
\]
\[
+ s \left( -\frac{1}{2} \text{tr} \chi(Z + \tilde{H}) + 2 \omega(Z - \tilde{H}) - 2 \tilde{\omega} \right) \cdot U + sZ \cdot \nabla_3 U - 2s \omega([D + sZ] \cdot U).
\]

We can rewrite the above as
\[
\nabla_3 \left[ \nabla_3, \nabla_4 \right] U = \left( \nabla_3 \cdot \left( \nabla_3 U \right) - \frac{1}{2} \text{tr} \chi \left( \nabla_3 \cdot U + (s - 2) \tilde{H} \cdot U \right) + \tilde{H} \cdot \nabla_3 U + \text{Err}_{34}[U] \right)
\]

which gives the desired formula.

We have
\[
\nabla_3(\nabla_a U_{bc}) = \nabla_3(\nabla_a U_{bc} + s \zeta_a U_{bc})
\]
\[
= \nabla_3 \nabla_a U_{bc} + s \nabla_3 \zeta_a U_{bc} + s \zeta_a \nabla_3 U_{bc} - 2s \omega(\nabla_a U_{bc} + s \zeta_a U_{bc}).
\]

Using (76) and the null structure equation
\[
\nabla_3 \zeta + 2 \omega = -\frac{1}{2} \text{tr} \chi(\zeta + \eta) - \frac{1}{2} \text{tr} \chi(\zeta + \eta) + 2 \omega(\zeta - \eta)
\]
\[
+ \tilde{\chi} \cdot \xi + \frac{1}{2} \text{tr} \chi \xi + \frac{1}{2} \text{tr} \chi \xi + 2 \omega \xi - \tilde{\chi} \cdot (\zeta + \eta) - \beta
\]

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we obtain

\[
(\nabla \nabla) \nabla_a U_b - \frac{1}{2} \text{tr} \chi \left( \nabla_a U_b + \eta_b U_{ac} + \eta_c U_{ab} - \epsilon_{ab} (\eta \cdot U)_c - \delta_{ac} (\eta \cdot U)_b \right) - \frac{1}{2} \text{tr} \chi \left( \nabla_a U_b + \eta_b * U_{ac} + \eta_c * U_{ab} - \epsilon_{ab} (\eta \cdot U)_c - \delta_{ac} (\eta \cdot U)_b \right) + (\eta_a - \zeta_a) \nabla_3 U_{bc} + s(\nabla_a \omega - \frac{1}{2} \text{tr} \chi (\zeta_a + \eta_a) - \frac{1}{2} \text{tr} \chi (\zeta_a + \eta_a)) + 2 \omega (\zeta_a - \eta_a) U_{bc} + s \zeta_a \nabla_3 U_{bc} - 2 s \omega (\nabla_a U_{bc} + s \zeta_a U_{bc}) + \text{Err}_{3abc}[U].
\]

We can rewrite the above as

\[
(\nabla \nabla) \nabla_a U_b = \nabla_a (\nabla \nabla) U_{bc} + s \zeta_a (\nabla \nabla) U_{bc} + (\eta_a - \zeta_a) (\nabla \nabla) U_{bc} + 2 s \omega (\zeta_a - \eta_a) U_{bc} - \frac{1}{2} \text{tr} \chi \left( \nabla_a U_b + s (\zeta_a + \eta_a) U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab} (\eta \cdot U)_c - \delta_{ac} (\eta \cdot U)_b \right) - \frac{1}{2} \text{tr} \chi \left( \nabla_a U_b + s (\zeta_a + \eta_a) U_{bc} + \eta_b * U_{ac} + \eta_c * U_{ab} - \epsilon_{ab} (\eta \cdot U)_c - \epsilon_{ac} (\eta \cdot U)_b \right) + 2 s \omega (\nabla_a U_{bc} + s \zeta_a U_{bc}) + \text{Err}_{3abc}[U]
\]

which gives

\[
(\nabla \nabla) \nabla_a U_b = \nabla_a (\nabla \nabla) U_{bc} + s \zeta_a (\nabla \nabla) U_{bc} + (\eta_a - \zeta_a) (\nabla \nabla) U_{bc} - \frac{1}{2} \text{tr} \chi \left( \nabla_a U_b + s (\eta_a) U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab} (\eta \cdot U)_c - \delta_{ac} (\eta \cdot U)_b \right) - \frac{1}{2} \text{tr} \chi \left( \nabla_a U_b + s (\eta_a) U_{bc} + \eta_b * U_{ac} + \eta_c * U_{ab} - \epsilon_{ab} (\eta \cdot U)_c - \epsilon_{ac} (\eta \cdot U)_b \right) + \text{Err}_{3abc}[U].
\]

Recall that \( (\nabla \nabla) U \) is of conformal type \( s - 1 \), therefore

\[
(\nabla \nabla) (\nabla \nabla) U = \nabla_a (\nabla \nabla) U_{bc} + (s - 1) \zeta_a (\nabla \nabla) U_{bc}.
\]

We finally obtain

\[
(\nabla \nabla) (\nabla \nabla) U = (\nabla \nabla) (\nabla \nabla) U_{bc} + (\eta_a - \zeta_a) (\nabla \nabla) U_{bc} - \frac{1}{2} \text{tr} \chi \left( \nabla_a U_b + s (\eta_a) U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab} (\eta \cdot U)_c - \delta_{ac} (\eta \cdot U)_b \right) - \frac{1}{2} \text{tr} \chi \left( \nabla_a U_b + s (\eta_a) U_{bc} + \eta_b * U_{ac} + \eta_c * U_{ab} - \epsilon_{ab} (\eta \cdot U)_c - \epsilon_{ac} (\eta \cdot U)_b \right) + \text{Err}_{3abc}[U]
\]

as stated.
6 The wave operator for perturbations of Kerr

In this section we derive the expression for the wave operator for anti-self dual tensors in perturbations of Kerr.

6.1 The Gauss equation

To derive the wave operator, we need to specialize the Gauss equation (43) to a tensor \( \Psi \in \mathcal{S}_2(\mathbb{C}) \) in a perturbation of Kerr. We will use this formula in Proposition 6.2.

**Proposition 6.1.** We have for \( \Psi \in \mathcal{S}_2(\mathbb{C}) \):

\[
(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \Psi = \frac{1}{2} ([ (a) tr \chi \nabla_3 + (a) tr \chi \nabla_4] \Psi + 2i \left( \frac{1}{4} tr \chi tr \chi + \frac{1}{4} (a) tr \chi (a) tr \chi + \rho \right) \Psi) + \text{Err}_{12}[\Psi]
\]

or also

\[
(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \Psi = \frac{1}{2} ([ (a) tr \chi \nabla_3 + (a) tr \chi \nabla_4] \Psi + i \left( \frac{1}{4} tr \chi tr \chi + \frac{1}{4} tr \chi tr \chi + P + \overline{P} \right) \Psi) + \text{Err}_{12}[\Psi]
\]

where

\[
\text{Err}_{12}[\Psi] = r^{-1} \Gamma_g \cdot \Psi.
\]

**Proof.** From (43), we have

\[
(\nabla_a \nabla_b - \nabla_b \nabla_a) \Psi_{st} = \frac{1}{2} \epsilon_{ab} ([ (a) tr \chi \nabla_3 + (a) tr \chi \nabla_4] \Psi_{st} - \frac{1}{2} E_{sadb} \Psi_{dt} - \frac{1}{2} E_{tdab} \Psi_{sd} + (- \epsilon_{sd} \epsilon_{ab} \rho) \Psi_{dt} + (- \epsilon_{td} \epsilon_{ab} \rho) \Psi_{sd}
\]

where

\[
E_{cdab} = \chi_{ac} X_{bd} + \chi_{ac} X_{bd} - \chi_{bc} X_{ad} - \chi_{bc} X_{ad}, \quad (82)
\]

We now compute the \( E \) as given by (82). We have:

\[
\begin{align*}
\chi_{ac} X_{bd} &= \frac{1}{4} (tr \chi \delta_{ac} + (a) tr \chi \epsilon_{ac}) (tr \chi \delta_{bd} + (a) tr \chi \epsilon_{bd}) + r^{-1} \Gamma_g \\
&= \frac{1}{4} (tr \chi tr \chi \delta_{ac} \delta_{bd} + (a) tr \chi (a) tr \chi \epsilon_{ac} \epsilon_{bd} + \delta_{ac} \epsilon_{bd} tr \chi (a) tr \chi + \epsilon_{ac} \delta_{bd} (a) tr \chi tr \chi) + r^{-1} \Gamma_g,
\end{align*}
\]

\[
\begin{align*}
\chi_{ac} \chi_{bd} &= \frac{1}{4} (tr \chi tr \chi \delta_{ac} \delta_{bd} + (a) tr \chi (a) tr \chi \epsilon_{ac} \epsilon_{bd} + \delta_{ac} \epsilon_{bd} tr \chi (a) tr \chi + \epsilon_{ac} \delta_{bd} (a) tr \chi tr \chi) + r^{-1} \Gamma_g,
\end{align*}
\]

\[
\begin{align*}
\chi_{bc} X_{ad} &= \frac{1}{4} (tr \chi tr \chi \delta_{bc} \delta_{ad} + (a) tr \chi (a) tr \chi \epsilon_{be} \epsilon_{ad} + \delta_{bc} \epsilon_{ad} tr \chi (a) tr \chi + \epsilon_{bc} \delta_{ad} (a) tr \chi tr \chi) + r^{-1} \Gamma_g,
\end{align*}
\]

\[
\begin{align*}
\chi_{bc} \chi_{ad} &= \frac{1}{4} (tr \chi tr \chi \delta_{bc} \delta_{ad} + (a) tr \chi (a) tr \chi \epsilon_{be} \epsilon_{ad} + \delta_{bc} \epsilon_{ad} tr \chi (a) tr \chi + \epsilon_{bc} \delta_{ad} (a) tr \chi tr \chi) + r^{-1} \Gamma_g.
\end{align*}
\]
Thus,
\[
E_{cdab} = \chi_{ac}\chi_{bd} + \chi_{ad}\chi_{bc} - \chi_{bd}\chi_{ac} - \chi_{bc}\chi_{ad} \\
= \frac{1}{2} \text{tr} \chi \text{tr} \chi (\delta_{ac} \delta_{bd} - \delta_{bc} \delta_{ad}) + \frac{1}{2} (a) \text{tr} \chi (a) \text{tr} \chi (\epsilon_{ac} \epsilon_{bd} - \epsilon_{bc} \epsilon_{ad}) \\
+ \frac{1}{4} (\delta_{ac} \epsilon_{bd} + \epsilon_{ac} \delta_{bd} - \epsilon_{bc} \delta_{bd} - \epsilon_{bc} \delta_{bd}) \text{tr} \chi (a) \text{tr} \chi \\
+ \frac{1}{4} (\epsilon_{ac} \delta_{bd} + \delta_{ac} \epsilon_{bd} - \epsilon_{bc} \delta_{ad} - \epsilon_{bc} \delta_{ad}) (a) \text{tr} \chi + r^{-1} \Gamma_g.
\]

Observe that
\[
E_{1112} = \frac{1}{4} (\delta_{11} \in_{21} - \in_{21} \delta_{11}) \text{tr} \chi (a) \text{tr} \chi + \frac{1}{4} (\delta_{11} \in_{21} - \in_{21} \delta_{11}) (a) \text{tr} \chi \text{tr} \chi + r^{-1} \Gamma_g = r^{-1} \Gamma_g, \\
E_{2212} = \frac{1}{4} (\in_{12} \delta_{22} - \delta_{22} \in_{12}) \text{tr} \chi (a) \text{tr} \chi + \frac{1}{4} (\in_{12} \delta_{22} - \delta_{22} \in_{12}) (a) \text{tr} \chi \text{tr} \chi + r^{-1} \Gamma_g = r^{-1} \Gamma_g,
\]
and
\[
E_{1212} = \frac{1}{4} \text{tr} \chi \text{tr} \chi (\delta_{11} \delta_{22} - \frac{1}{2} (a) \text{tr} \chi (a) \text{tr} \chi (- \in_{21} \in_{12}) + r^{-1} \Gamma_g = \frac{1}{2} \text{tr} \chi \text{tr} \chi + \frac{1}{2} (a) \text{tr} \chi (a) \text{tr} \chi + r^{-1} \Gamma_g, \\
E_{2112} = \frac{1}{2} \text{tr} \chi \text{tr} \chi (\in_{12} \in_{21} + \frac{1}{2} (a) \text{tr} \chi (a) \text{tr} \chi (\in_{12} \in_{21}) + r^{-1} \Gamma_g = -\frac{1}{2} \text{tr} \chi \text{tr} \chi - \frac{1}{2} (a) \text{tr} \chi (a) \text{tr} \chi + r^{-1} \Gamma_g.
\]

Define \( Y_{sd} = -\frac{1}{2} E_{sd12} \Psi_{dt} - \frac{1}{2} E_{td12} \Psi_{sd} \). Since \( \Psi \in S_2 \) we have that \( \Psi_{12} = -i \Psi_{11} \) and \( \Psi_{11} = i \Psi_{12} \) and therefore evaluating \( Y \) in coordinates:
\[
Y_{11} = -\frac{1}{2} E_{1112} \Psi_{12} - \frac{1}{2} E_{1121} \Psi_{1d} = -\frac{1}{2} E_{1112} \Psi_{12} - \frac{1}{2} E_{1121} \Psi_{21} - \frac{1}{2} E_{1212} \Psi_{11} - \frac{1}{2} E_{1112} \Psi_{12} \\
= (E_{1112} + i E_{1212}) \Psi_{11}, \\
Y_{12} = -\frac{1}{2} E_{1112} \Psi_{12} - \frac{1}{2} E_{1121} \Psi_{22} - \frac{1}{2} E_{2112} \Psi_{11} - \frac{1}{2} E_{2212} \Psi_{12} = (E_{1112} + i E_{1212}) \Psi_{12}.
\]

This implies
\[
-\frac{1}{2} E_{sd12} \Psi_{dt} - \frac{1}{2} E_{td12} \Psi_{sd} = \frac{1}{2} i \left( \text{tr} \chi \text{tr} \chi + (a) \text{tr} \chi (a) \text{tr} \chi \right) \Psi_{st} + r^{-1} \Gamma_g \Psi.
\]

Define \( W_{st} = R_{sd12} \Psi_{dt} + R_{td12} \Psi_{sd} \). Evaluating \( W \) in coordinates we have
\[
W_{11} = R_{1d12} \Psi_{d1} + R_{1d12} \Psi_{1d} = 2 R_{1212} \Psi_{12} = -2i R_{1212} \Psi_{11} = 2i \rho \Psi_{11}.
\]

We obtain
\[
R_{sd12} \Psi_{dt} + R_{td12} \Psi_{sd} = 2i \rho \Psi_{st}.
\]

Putting the above together we obtain the final formula. \( \square \)
6.2 The wave equation using complex operators

We now express the Laplacian appearing in the canonical form of the wave equation in terms of the complex derivative. We summarize the result in the following.

Proposition 6.2. Given $\Psi \in S_2(\mathbb{C})$ we have

$$D\hat{\otimes}(D \cdot \Psi) = 2\Delta_2 \Psi - i (a) trX \nabla_3 + (a) tr\nabla_4) \Psi + \left(\frac{1}{2} trX tr\nabla + \frac{1}{2} tr\nabla trX + 2P + 2\bar{P}\right) \Psi + Err_{D\hat{\otimes}D}[\Psi]$$

where

$$Err_{D\hat{\otimes}D}[\Psi] = r^{-1} \Gamma_g \cdot \Psi.$$

Proof. We define $Y_{ab} := 2(D\hat{\otimes}(D \cdot \Psi))_{ab}$, and express $Y$ in coordinates. We have

$$Y_{ab} = D_a D^c \Psi_{cb} + D_b D^c \Psi_{ca} - \delta_{ab} D^d D^c \Psi_{dc}.$$

By construction, $Y$ is symmetric and traceless. We calculate first $Y_{11} = -Y_{22}$. For $a = b = 1$ we derive

$$Y_{11} = D_1 D^c \Psi_{c1} + D_1 D^c \Psi_{c1} - \delta_{11} D^d D^c \Psi_{dc}$$

$$= 2D_1 D^c \Psi_{c1} - D^d D^c \Psi_{dc}$$

$$= 2D_1 D_1 \Psi_{11} + 2D_1 D_2 \Psi_{21} - D^d D^1 \Psi_{d1} - D^d D^2 \Psi_{d2}$$

$$= 2D_1 D_1 \Psi_{11} + 2D_1 D_2 \Psi_{21} - D_1 D_1 \Psi_{11} - D_2 D_1 \Psi_{21} - D_1 D_2 \Psi_{12} - D_2 D_2 \Psi_{22}$$

$$= D_1 D_1 \Psi_{11} + D_1 D_2 \Psi_{21} - D_2 D_1 \Psi_{21} - D_2 D_2 \Psi_{22}.$$

Writing $\Psi_{22} = -\Psi_{11}$, we have

$$Y_{11} = (D_1 D_1 + D_2 D_2) \Psi_{11} + (D_1 D_2 - D_2 D_1) \Psi_{12}.$$

We compute

$$Y_{11} = ((\nabla_1 + i * \nabla_1)(\nabla_1 - i * \nabla_1) + (\nabla_2 + i * \nabla_2)(\nabla_2 - i * \nabla_2)) \Psi_{11}$$

$$+ ((\nabla_1 + i * \nabla_1)(\nabla_1 - i * \nabla_2) - (\nabla_2 + i * \nabla_2)(\nabla_1 - i * \nabla_1)) \Psi_{12}$$

$$= (\nabla_1 + i \nabla_2)(\nabla_1 - i \nabla_2) + (\nabla_2 - i \nabla_1)(\nabla_2 + i \nabla_1) \Psi_{11}$$

$$+ ((\nabla_1 + i \nabla_2)(\nabla_2 + i \nabla_1) - (\nabla_2 - i \nabla_1)(\nabla_1 - i \nabla_2)) \Psi_{12}$$

$$= 2\Delta \Psi_{11} - 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \Psi_{11} + 2(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \Psi_{12} + 2i \Delta \Psi_{12}.$$

Using that $\Psi_{12} = -i \Psi_{11}$, we obtain

$$Y_{11} = 2\Delta \Psi_{11} - 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \Psi_{11} - 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \Psi_{11} + 2\Delta \Psi_{11}$$

$$= 4\Delta \Psi_{11} - 4i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) \Psi_{11}.$$
We now compute $Y_{12} = Y_{21}$. For $a = 1, b = 2$ we derive

\[ Y_{12} = D_1 D_{\Psi_{12}} + D_2 D_{\Psi_{22}} - \delta_{12} D D^* \Psi_{dc} \]

\[ = D_1 D_{\Psi_{12}} + D_1 D_{\Psi_{22}} + D_2 D_{\Psi_{11}} + D_2 D_{\Psi_{21}} \]

\[ = (D_1 D_1 + D_2 D_2)\Psi_{12} + (D_2 D_1 - D_1 D_2)\Psi_{11}. \]

We therefore obtain

\[ Y_{12} = 2\Delta \Psi_{12} - 2i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1)\Psi_{12} - 2(\nabla_1 \nabla_2 - \nabla_2 \nabla_1)\Psi_{11} - 2i \Delta \Psi_{11}. \]

Using that $\Psi_{11} = i \Psi_{12}$, we obtain

\[ Y_{12} = 4\Delta \Psi_{12} - 4i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1)\Psi_{12}. \]

Putting the above together we have

\[ Y_{ab} = 4\Delta \Psi_{ab} - 4i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1)\Psi_{ab}. \]

Using the Gauss formula (81), we prove the desired formula.

By putting together the canonical expression for the wave equation and the above expression for the Laplacian, we obtain the following.

**Corollary 6.3.** The wave equation for $\Psi \in S_2(\mathbb{C})$ can be written as

\[ \Box_2 \Psi = -\nabla_1 \nabla_2 \Psi + \frac{1}{2} D \nabla_1 D \Psi + \left(2\omega - \frac{1}{2} tr X\right) \nabla_2 \Psi - \frac{1}{2} tr X \nabla_4 \Psi \]

\[ + \frac{1}{2}(H \cdot D)\Psi + \frac{1}{2}(H \cdot D)\Psi + \left(-\frac{1}{4} tr X tr X - \frac{1}{4} tr X tr X - 2P\right) \Psi + \text{Err}_{\Box_2} \Psi \]

where

\[ \text{Err}_{\Box_2} \Psi = r^{-1} \Gamma_g \cdot \Psi. \]

**Proof.** Consider (64): \[ \Box_2 \Psi = -\nabla_1 \nabla_2 \Psi + \Delta_2 \Psi + \left(2\omega - \frac{1}{2} tr X\right) \nabla_3 \Psi - \frac{1}{2} tr X \nabla_4 \Psi + 2\eta \cdot \nabla \Psi + (P - \overline{P}) \Psi \]

where $\Delta_2$ is the Laplacian on the horizontal structure for a 2-tensor. Using (83) to write

\[ \Delta_2 \Psi = \frac{1}{2} D \nabla_1 D \Psi + \frac{1}{2} (a) tr X \nabla_3 \Psi + \left(a \nabla_4 \Psi - \frac{1}{4} tr X tr X + \frac{1}{4} tr Y tr X + P + \overline{P}\right) \Psi + \Gamma_g \Psi \]

we obtain

\[ \Box_2 \Psi = -\nabla_1 \nabla_2 \Psi + \frac{1}{2} D \nabla_1 D \Psi + \left(2\omega - \frac{1}{2} tr X\right) \nabla_3 \Psi - \frac{1}{2} tr X \nabla_4 \Psi + \frac{1}{2}(H \cdot D)\Psi + \frac{1}{2}(H \cdot D)\Psi \]

\[ + \left(-\frac{1}{4} tr X tr X - \frac{1}{4} tr X tr X - 2P\right) \Psi + \Gamma_g \Psi \]

where we used $(H \cdot D)\Psi + (H \cdot D)\Psi = 4\eta \cdot \nabla \Psi$. \qed
For $\Phi \in \mathcal{S}_2(\mathbb{C})$ of conformal type 0 we can write

$$\Box_2 \Phi = - \frac{(c)}{2} \nabla_4 (c) \nabla_3 \Phi + \frac{1}{2} (c) D \otimes ((c) D \cdot \Phi) + \left( - \frac{1}{4} \text{tr} X \bar{\text{tr}} X - \frac{1}{4} \text{tr} X \bar{\text{tr}} X - 2P \right) \Phi$$

(85)

which is the invariant conformal wave operator for $\Phi \in \mathcal{S}_2$ of conformal type 0.

### 6.3 The projection of the wave operator

In what follows, we consider the wave equation satisfied by symmetric traceless complex 2-tensors, $A, Q(A), q \in \mathcal{S}_2(\mathbb{C})$. On the other hand, the equations governing linearized gravity around Kerr which are known in the literature, for instance the Teukolsky equation, are equations for complex scalars of spin $\pm 2$. Such equations for complex scalars can be obtained by projecting the corresponding 2-tensor to its first components.

In the following Lemma, we make the explicit connection between the wave operator applied to a 2-tensor and the wave equation verified by its projection, which is a scalar function of spin $\pm 2$.

**Proposition 6.4.** Let $\Phi \in \mathcal{S}_2(\mathbb{C})$ and let $\psi$ be its projection to its first components, i.e. $\psi = \Phi(e_1, e_1) = \Phi_{11}$. Then

$$\Box_2 \Phi_{11} = \Box_g \psi + i \frac{4}{|q|^2} \cos \theta \mathcal{Z}(\psi) - \left( \frac{4}{|q|^2} \cot^2 \theta + a \bar{V} \right) \psi + r^{-1} \Gamma_g \psi$$

(86)

where $\Box_2$ and $\Box_g$ are the wave operators for 2-tensors and scalars respectively, for perturbations of Kerr.

**Proof.** Using Corollary 6.3 and Lemma A.1, the projection of the wave operator $\Box_2$ for a perturbation of Kerr applied to a 2-tensor $\Phi$ can be computed:

$$\Box_2 \Phi_{11} = - (\nabla_4 \nabla_3 \Phi)_{11} + \frac{1}{2} D \otimes (D \cdot \Phi)_{11} + \left( 2 \omega - \frac{1}{2} \text{tr} X \right) (\nabla_3 \Phi)_{11} - \frac{1}{2} \text{tr} X (\nabla_4 \Phi)_{11} + (2 \eta \cdot \nabla \Phi)_{11}$$

$$+ \left( - \frac{1}{4} \text{tr} X \bar{\text{tr}} X - \frac{1}{4} \text{tr} X \bar{\text{tr}} X - 2P \right) \Phi_{11} + r^{-1} \Gamma_g \Phi_{11}$$

$$= e_4 e_3 \psi - i (a) \text{tr} \chi e_4 (\psi) - i (a) \text{tr} \chi e_3 (\psi)$$

$$+ \Delta \psi - \frac{1}{2} (a) \text{tr} \chi e_3 (\psi) - \frac{1}{2} (a) \text{tr} \chi e_4 (\psi) + 4 i \Lambda e_2 (\psi) + \left( -4 \Lambda^2 + \frac{1}{2} \text{tr} \chi \bar{\text{tr}} X + 2 \rho \right) \psi$$

$$- \frac{1}{2} (a) \text{tr} \chi e_3 (\psi) - \frac{1}{2} (a) \text{tr} \chi e_4 (\psi) - i (a) \text{tr} \chi e_3 (\psi)$$

$$+ \Delta \psi + 2 \eta_1 e_1 (\psi) + 2 \eta_2 e_2 (\psi) + \left( - \frac{1}{2} \text{tr} \chi \bar{\text{tr}} X - 2 \rho \right) \psi + a \bar{V} \psi + r^{-1} \Gamma_g \psi$$

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where $a\tilde{V}\psi$ denotes zero-th order term in $\psi$. This gives, using (125):

\[
(\Box_2\Psi)_{11} = -e_4e_3\psi - \frac{1}{2} tr \chi e_3(\psi) - \frac{1}{2} tr \chi e_4(\psi) + \Delta \psi + 2\eta_4 e_1(\psi) + 2\eta_4 e_2(\psi) - i (^{(a)}tr\chi e_4(\psi)) - i (^{(a)}tr\chi e_3(\psi)) + 4i\Lambda e_2(\psi) - 4\Lambda^2\psi + a\tilde{V}\psi
\]

Recall that $\Lambda = \frac{r^2 + a^2}{|q|^2} \cot \theta$. Using (71) and (72) we have

\[
^{(a)}tr\chi e_4(\psi) + ^{(a)}tr\chi e_3(\psi) - 4\Lambda e_2 = 2a\Delta \cos \theta \frac{|q|^4}{|q|^2} e_4 - \frac{2}{|q|^3} \sin \theta e_2
\]

which therefore gives (86).

\begin{proof}

Remark 6.5. More generally, for a $k$-tensor $\Psi \in S_k(\mathbb{C})$, the projection of the wave equation for $\Psi$ to its first component $\psi = \Psi_{1...1}$ satisfies

\[
(\Box_k\Psi)_{11} = \Box_g\psi + \frac{1}{2} k \cos \theta \frac{|q|^2}{|q|^2} Z(\psi) - \left( \frac{k^2}{|q|^2} \cot^2 \theta + a\tilde{V} \right) \psi + r^{-1}\Gamma_g\psi.
\]

In the physics literature, this is referred to as the $k$-spin weighted wave equation, see for example [2].

\end{proof}

7 The Teukolsky equation

The curvature components $A$ and $\underline{A}$ are the only quantities which are invariant up to quadratic and higher order error terms to null frame transformations and vanish in Kerr. In the language of the previous section they are $O(\epsilon^2)$-invariant.

It is known that these curvature components satisfy wave equations which decouple from all other components at the linear level; the celebrated Teukolsky equations. In this section we derive, using our formalism, the corresponding Teukolsky equation for $A$ while keeping track of the error terms generated by the perturbation from Kerr.
Proposition 7.1. The complex tensor $A$ satisfies the following equation:

$$
\mathcal{L}(A) = \text{Err}[\mathcal{L}(A)]
$$

where

$$
\mathcal{L}(A) = -(c)\nabla_4 (c) \nabla_3 A + \frac{1}{2} (c) D \tilde{\otimes} (c) \nabla \cdot A + \left( -\frac{1}{2} \tr X - 2 \tr X \right) (c) \nabla_3 A - \frac{1}{2} \tr X (c) \nabla_4 A
$$

+ $(4H + H + \mathcal{H}) \cdot (c) \nabla A + (-\tr X \tr X + 2 \mathcal{P}) A + 2H \tilde{\otimes} (\mathcal{H} \cdot A)$

with error term expressed schematically\(^{18}\)

$$
\text{Err}[\mathcal{L}(A)] = r^{-1} d \leq 1 (\Gamma_\beta B) + \Xi \nabla_3 B + \text{ l.o.t.}
$$

Proof. According to Proposition 3.13, we have the following Bianchi identity for $A$:

$$
(c) \nabla_3 A - (c) D \tilde{\otimes} B = -\frac{1}{2} \tr X A + 4H \tilde{\otimes} B - 3\mathcal{P} \nabla. \nabla.
$$

(90)

Apply $(c) \nabla_4$ to (90):

$$(c) \nabla_4 (c) \nabla_3 A = (c) \nabla_4 (c) D \tilde{\otimes} B - \frac{1}{2} (c) \nabla_4 \tr X A - \frac{1}{2} \tr X (c) \nabla_4 A
$$

+ $4(c) \nabla_4 (H) \tilde{\otimes} B + 4H \tilde{\otimes} (c) \nabla_4 (B) - 3 (c) \nabla_4 \mathcal{P} \nabla - 3 \mathcal{P} (c) \nabla_4 \nabla.

We compute the right hand side of the above equation.

According to Lemma 5.8, we apply the commutation formula (77) to $B$, which is a one form of conformal type 1. Using the definition of $\Gamma_\beta$ and $\Gamma_\eta$ as in (70), we have

$$
[ (c) \nabla_4, (c) D \tilde{\otimes} ] B = -\frac{1}{2} \tr X (c) D \tilde{\otimes} B + H \tilde{\otimes} (c) \nabla_4 B + \text{Err}_4 (c) D \tilde{\otimes} [B]
$$

where

$$
\text{Err}_4 (c) D \tilde{\otimes} [B] = \frac{1}{r} \Gamma_\beta (\nabla_3 + rD) B + \left( B + \frac{a}{r^2} \Gamma_\eta \right) B.
$$

We therefore obtain

$$
(c) \nabla_4 (c) D \tilde{\otimes} B + 4H \tilde{\otimes} (c) \nabla_4 (B)
$$

$$
= (c) D \tilde{\otimes} (c) \nabla_4 B + 4H \tilde{\otimes} (c) \nabla_4 (B) - \frac{1}{2} \tr X (c) D \tilde{\otimes} B + H \tilde{\otimes} (c) \nabla_4 B + \text{Err}_4 (c) D \tilde{\otimes} [B]
$$

$$
= (c) D \tilde{\otimes} (c) \nabla_4 B + (4H + H) \tilde{\otimes} (c) \nabla_4 B - \frac{1}{2} \tr X (c) D \tilde{\otimes} B + \text{Err}_4 (c) D \tilde{\otimes} [B].
$$

\(^{18}\)The error terms denoted l.o.t. are quadratic in the perturbation and enjoy better decay properties, or are higher order and decay at least as good.
According to Proposition 3.13, we have the following Bianchi identity for $B$:

$$(c)\nabla_4 B - \frac{1}{2}(c)D \cdot A = -2\text{tr}XB + \frac{1}{2}A \cdot \overline{H} + 3\overline{P} \Xi.$$ 

We therefore obtain

$$(c)\nabla_4((c)D \otimes B) + 4H \hat{\otimes} (c)\nabla_4(B)$$

$$= (c)D \hat{\otimes} \left(\frac{1}{2}(c)D \cdot A - 2\text{tr}XB + \frac{1}{2}A \cdot \overline{H} + 3\overline{P} \Xi\right)$$

$$+ \left(4H + \overline{H}\right) \hat{\otimes} \left(\frac{1}{2}(c)D \cdot A - 2\text{tr}XB + \frac{1}{2}A \cdot \overline{H} + 3\overline{P} \Xi\right) - \frac{1}{2}\text{tr}X (c)D \otimes B + \text{Err}_4(c)D \otimes [B]$$

$$= \frac{1}{2}(c)D \hat{\otimes} \left(\frac{1}{2}(c)D \cdot A + \overline{H} \cdot A\right) + \left(\frac{1}{2}\text{tr}X - 2\text{tr}X\right) (c)D \hat{\otimes} B$$

$$+ \left(2H + \frac{1}{2}H\right) \hat{\otimes} \left(\frac{1}{2}(c)D \cdot A + \overline{H} \cdot A\right) + 2(- (c)D\text{tr}X - \text{tr}X (4H + H)) \hat{\otimes} B$$

$$+ 3\overline{P} \left((c)D \hat{\otimes} \Xi + (4H + H) \hat{\otimes} \Xi\right) + \text{Err}_4(c)D \hat{\otimes} [B] + 3 (c)D \overline{P} \hat{\otimes} \Xi.$$ 

Using the null structure equation

$$(c)\nabla_4\text{tr}X + \frac{1}{2}\text{tr}X\text{tr}X = (c)D \cdot \overline{H} + H \cdot \overline{H} + 2\overline{P} + \Gamma_A \Gamma_g$$

we obtain

$$-\frac{1}{2}(c)\nabla_4\text{tr}X A - \frac{1}{2}\text{tr}X(c)\nabla_4 A$$

$$= \left(\frac{1}{2}\text{tr}X \text{tr}X - \frac{1}{2}(c)D \cdot \overline{H} + H \cdot \overline{H}\right) A - \frac{1}{2}\text{tr}X (c)\nabla_4 A + \text{cubic terms}.$$ 

Using the null structure equation

$$(c)\nabla_4 H - (c)\nabla_3 \Xi = \frac{1}{2}\text{tr}X(H - H) - \frac{1}{2} \hat{\nabla} \cdot (\overline{H} - \overline{H}) - B$$

we obtain

$$4 (c)\nabla_4(H) \hat{\otimes} B = -2\text{tr}X(H - H) \hat{\otimes} B + \left((c)\nabla_3 \Xi + \frac{a}{r^2} \Gamma_g + B\right) B.$$ 

Finally using

$$(c)\nabla_4 \hat{X} + \mathfrak{R}(\text{tr}X) \hat{X} = (c)D \hat{\otimes} \Xi + \Xi \hat{\otimes} (H + H) - A,$$

$$(c)\nabla_4 P - \frac{1}{2}(c)D \cdot \overline{B} = -\frac{3}{2}\text{tr}XP + H \cdot \overline{B} + (\Xi) \overline{B} + \Gamma_g A,$$

we obtain

$$-3 (c)\nabla_4 P \hat{X} - 3 \overline{P} (c)\nabla_4 \hat{X}$$

$$= -3\left(-\frac{3}{2}\text{tr}X \overline{P}\right) \hat{X} - 3\overline{P}\left(-\frac{1}{2}(\text{tr}X + \text{tr}X) \hat{X} - A + (c)D \hat{\otimes} \Xi + \Xi \hat{\otimes} (H + H)\right) + \Gamma_g (c)D \overline{B} + \frac{a}{r^2} \Gamma_g B.$$ 

$$= \left(\frac{3}{2}\text{tr}X + 6\text{tr}X\right) \overline{P} \hat{X} + 3\overline{P}A - 3\overline{P}\left((c)D \hat{\otimes} \Xi + \Xi \hat{\otimes} (H + H)\right) + \Gamma_g (c)D \overline{B} + \frac{a}{r^2} \Gamma_g B.$$
Summing (91), (92), (93) and (94), we obtain

\[
\nabla^4_{(c)} \nabla_3 A = \left( -\frac{1}{2} \text{tr} X - 2\text{tr} \hat{X} \right) \nabla_3 A - \frac{1}{2} \text{tr} X \nabla_4 A \\
+ 2 \left( -\nabla^4_{(c)} X - (\text{tr} X - \text{tr} \hat{X}) H \right) \hat{B} \\
+ \left( -\text{tr} X \text{tr} X - \frac{1}{2} (\nabla^4_{(c)} \nabla \hat{H} + \hat{H} \cdot \hat{H}) + 2\hat{P} \right) A \\
+ \frac{1}{2} \nabla^4_{(c)} \left( (\nabla_3 A + \hat{H} \cdot A) + \left( 2H + \frac{1}{2} \hat{H} \right) \hat{X} \right) \\
+ \frac{1}{r} \Gamma_1 (\nabla_3 + rD) B + \left( (\nabla_3 \Xi + \frac{a}{r^2} \Gamma_1 + B) B. \right)
\]

Using again (90) we have

\[
\left( -\frac{1}{2} \text{tr} X - 2\text{tr} \hat{X} \right) \nabla_3 A - \frac{1}{2} \text{tr} X \nabla_4 A \\
= \left( -\frac{1}{2} \text{tr} X - 2\text{tr} \hat{X} \right) \left( \nabla_3 A - \frac{1}{2} \text{tr} X \nabla_4 A \right) \\
= \left( -\frac{1}{2} \text{tr} X - 2\text{tr} \hat{X} \right) \left( \nabla_3 A + \frac{1}{2} \text{tr} X A - 4\hat{H} \hat{B} \right).
\]

This gives

\[
\nabla^4_{(c)} \nabla_3 A = \left( -\frac{1}{2} \text{tr} X - 2\text{tr} \hat{X} \right) \nabla_3 A - \frac{1}{2} \text{tr} X \nabla_4 A \\
+ 2 \left( -\nabla^4_{(c)} X - (\text{tr} X - \text{tr} \hat{X}) H \right) \hat{B} \\
+ \left( -\text{tr} X \text{tr} X - \frac{1}{2} (\nabla^4_{(c)} \nabla \hat{H} + \hat{H} \cdot \hat{H}) + 2\hat{P} \right) A \\
+ \frac{1}{2} \nabla^4_{(c)} \left( (\nabla_3 A + \hat{H} \cdot A) + \left( 2H + \frac{1}{2} \hat{H} \right) \hat{X} \right) \\
+ \frac{1}{r} \Gamma_1 (\nabla_3 + rD) B + \left( (\nabla_3 \Xi + \frac{a}{r^2} \Gamma_1 + B) B. \right)
\]

By Codazzi equation

\[
\frac{1}{2} \nabla^4_{(c)} \hat{X} = \frac{1}{2} \nabla^4_{(c)} \text{tr} X - i3(\text{tr} X)(H + \Xi) - B
\]

therefore the second line is absorbed by the quadratic terms in the last line. We therefore have

\[
\nabla^4_{(c)} \nabla_3 A = \left( -\frac{1}{2} \text{tr} X - 2\text{tr} \hat{X} \right) \nabla_3 A - \frac{1}{2} \text{tr} X \nabla_4 A \\
+ \left( -\text{tr} X \text{tr} X - \frac{1}{2} (\nabla^4_{(c)} \nabla \hat{H} + \hat{H} \cdot \hat{H}) + 2\hat{P} \right) A \\
+ \frac{1}{2} \nabla^4_{(c)} \left( (\nabla_3 A + \hat{H} \cdot A) + \left( 2H + \frac{1}{2} \hat{H} \right) \hat{X} \right) \\
+ \frac{1}{r} \Gamma_1 (\nabla_3 + rD) B + \left( (\nabla_3 \Xi + \frac{a}{r^2} \Gamma_1 + B) B. \right)
\]
where
\[
\text{Err}[\mathcal{L}(A)] = \frac{1}{r} \Gamma_g (\nabla_3 + r \mathcal{D}) B + \left( ^{(c)} \nabla_3 \Xi + \frac{a}{r^2} \Gamma_g + B \right) B.
\]

The last line of the above equation involving the angular derivatives of \( A \) can be written as
\[
\frac{1}{2} (^{(c)} \mathcal{D} \hat{\otimes} (^{(c)} \nabla_3 A) + H \cdot A) + \left( 2H + \frac{1}{2} H \right) \hat{\otimes} (^{(c)} \mathcal{D} \cdot A + H \cdot A)
\]
\[
= \frac{1}{2} (^{(c)} \mathcal{D} \hat{\otimes} (^{(c)} \nabla_3 A) + 2H \hat{\otimes} (^{(c)} \mathcal{D} \cdot A) + \left( 2H + \frac{1}{2} H \right) \hat{\otimes} (H \cdot A).
\]

Applying (56) and (58), we write
\[
(^{(c)} \mathcal{D} \hat{\otimes} (H \cdot A) = (^{(c)} \mathcal{D} \cdot H) A + (H \cdot H) A,
\]

which implies
\[
^{(c)} \nabla_4 ^{(c)} \nabla_3 A = \frac{1}{2} (^{(c)} \mathcal{D} \hat{\otimes} (^{(c)} \nabla_3 A) + \left( -\frac{1}{2} \text{tr} X - 2 \text{tr} X \right) (^{(c)} \nabla_3 A - \frac{1}{2} \text{tr} X (^{(c)} \nabla_4 A
\]
\[
+ \left( -\text{tr} X \text{tr} X - \frac{1}{2} (^{(c)} \mathcal{D} \cdot H + H \cdot H) + 2 \mathcal{P} \right) A
\]
\[
+ \frac{1}{2} (^{(c)} \mathcal{D} \cdot H) A + (H \cdot H) A
\]
\[
+ \left( 2H + \frac{1}{2} H \right) \hat{\otimes} (^{(c)} \mathcal{D} \cdot A) + (\text{tr} X \text{tr} X + 2 \mathcal{P}) A
\]
\[
= \frac{1}{2} (^{(c)} \mathcal{D} \hat{\otimes} (^{(c)} \nabla_3 A) + \left( -\frac{1}{2} \text{tr} X - 2 \text{tr} X \right) (^{(c)} \nabla_3 A - \frac{1}{2} \text{tr} X (^{(c)} \nabla_4 A
\]
\[
+ 2H \hat{\otimes} (H \cdot A) + \text{Err}[\mathcal{L}(A)]
\]

Using Lemma 3.7, we further simplify the angular part writing
\[
\frac{1}{2} (H \cdot (^{(c)} \mathcal{D}) A + \left( 2H + \frac{1}{2} H \right) \hat{\otimes} (^{(c)} \nabla_3 A) = \frac{1}{2} (H \cdot (^{(c)} \mathcal{D}) A + \left( 2H + \frac{1}{2} H \right) \hat{\otimes} (^{(c)} \mathcal{D}) A
\]
\[
= \frac{1}{2} (H \cdot (^{(c)} \mathcal{D} + H \cdot (^{(c)} \mathcal{D}) A + (2H \cdot (^{c)} \mathcal{D}) A
\]
\[
= (2\eta + 4H) \cdot (^{(c)} \nabla A
\]
\[
= (4H + H + H) \cdot (^{(c)} \nabla A.
\]

This proves the proposition. \(\square\)
Remark 7.2. The Teukolsky equation (87) is a tensorial equation for $A \in \mathcal{S}_2(\mathbb{C})$, as defined in our formalism. The standard derivation of the equation, in linear theory, is done instead with respect to the Newmann-Penrose formalism, see section 2.5. To relate the Teukolsky equation in our formalism to the classical one in NP formalism we have to project it with respect to the horizontal frame $e_1, e_2$. One can check in fact that the standard Teukolsky variable, which we denote by $\alpha^{[+2]}$, is related to $A$ via the formula

$$\alpha^{[+2]} := -\frac{q}{q} A_{11}$$

where $A_{11} = A(e_1, e_1)$. One can check, see Appendix A, that in the particular case of the Kerr metric we have

$$|q|^2 \Box_{m,a} \alpha^{[+2]} = -4 (r - m) \partial_r \alpha^{[+2]} - 4 \left( \frac{m r^2 - 2 a^2}{\Delta} - r - i a \cos \theta \right) \partial_t \alpha^{[+2]}$$

$$-4 \left( \frac{a (r - m)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right) \partial_\varphi \alpha^{[+2]} + (4 \cot^2 \theta - 2) \alpha^{[+2]}$$

where $\Box_{m,a}$ is the D’Alembertian relative to the Kerr metric. This is the standard form of the Teukolsky equation in Boyer-Lindquist coordinates, see [16].

8 The Regge-Wheeler-type equation

Just like in Schwarzschild, there are other $O(\epsilon^2)$ invariant quantities to be considered in Kerr.

Lemma 8.1. The expression

$$Q(A) = (c) \nabla_3 (c) \nabla_3 A + C (c) \nabla_3 A + D \ A \in \mathcal{S}_2$$

(95)

for any scalar functions $C$ and $D$ is $O(\epsilon^2)$ invariant. Moreover, it is conformally invariant of type 0 provided $C$ is conformally invariant of type $-1$ and $D$ is conformally invariant of type $-2$.

Proof. Clearly the quantity vanishes in Kerr and is an $O(\epsilon^2)$ invariant. By construction, it is also conformally invariant under the above conditions on $C$ and $D$. \qed

We will also need the following rescaled version of $Q$.

Definition 8.2. Given a general null frame $(e_3, e_4, e_1, e_2)$ and given scalar functions $r$ and $\theta$ satisfying the assumptions in Section 5.3, we define our main quantity $q \in \mathcal{S}_2$ as

$$q = q^3 Q(A) = q^3 \left( (c) \nabla_3 (c) \nabla_3 A + C (c) \nabla_3 A + D \ A \right)$$

(96)

where $q = r + i a \cos \theta$. 78
The quantity $q$ can be seen as a second order differential operator in $A$. In particular, this is a physical space version of the Chandrasekhar transformation, which transforms the Teukolsky equation satisfied by $A$, as in Proposition 7.1, into a Regge-Wheeler-type equation. We now state the main result of the paper concerning the wave equation satisfied by $q$.

**Theorem 8.3.** There exists choices of complex scalar functions $C$, $D$, see (104) (105), such that the invariant symmetric traceless 2-tensor $q \in S_2$ verifies the equation

$$\Box^2 q - \frac{4ia \cos \theta}{|q|^2} T(q) - V q = a L_q[A] + Err[\Box^2 q]$$

(97)

where

- the potential $V$ is a complex scalar function, see also Remark 8.5 below,
- $L_q[A]$ is a linear second order operator in $A$, given by
  $$L_q[A] = c_1(e) \nabla_2 (e) \nabla_3 A + c_2(e) \nabla_3 A + c_3(e) \nabla(A) + c_4 A$$
  with $c_1, \ldots, c_4$ smooth functions of $(r, \theta)$ which have the following fall-off in $r$
    $$c_1 = O(r), \quad c_2 = O(1), \quad c_3 = O(1), \quad c_4 = O \left( \frac{1}{r} \right),$$

- The error term is given schematically by
  $$Err[\Box^2 q] = r^2 \partial^{\leq 2} (\Gamma_g \cdot (A, B)) + \nabla_3 (r^3 \partial^{\leq 2} (\Gamma_g \cdot (A, B)))$$
  $$+ \partial^{\leq 1} (\Gamma_g q) + r^2 \mathcal{H} \nabla^{\leq 1} \nabla_3 A + l.o.t.$$ 

**Remark 8.4.** The Regge-Wheeler type equation in Kerr was obtained by Ma [14] and Dafermos, Holzegel and Rodnianski [7] starting with the NP complex scalar curvature component $\Psi_0$, corresponding to $A_{11} = \alpha_{11} + i \ast \alpha_{11}$. Our equation (97) is instead a tensorial equation in curved background with precise reference to the error terms, see also the similar equation derived in [11]. We note that, in the particular case of Kerr, equations derived in [14], [7] can be obtained from (97) by projection to the 1-1 component. Such a projection modifies the equations by the appearance of Christoffel symbols of the horizontal distributions, see Section 8.2 for the projection of the Regge-Wheeler type equation to its first component.

**Remark 8.5.** The potential term $V$ coincides with the potential $-tr \chi tr \chi$, appearing in [11] in the context of perturbations of Schwarzschild, plus terms multiplied by the angular momentum $a$. More precisely,

$$\Re(V) - V_0 = O \left( \frac{|a|}{r^4} \right), \quad \Im(V) = O \left( \frac{|a|}{r^8} \right), \quad V_0 := -tr \chi tr \chi.$$ 

In particular, for small angular momentum, $\Re(V) - V_0$ and $\Im(V)$ can be treated as lower order terms, and absorbed by the left hand side.

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19The error terms denoted l.o.t. are quadratic in the perturbation and enjoy better decay properties, or are higher order and decay at least as good.
8.1 Proof of Theorem 8.3

Recall the Teukolsky equation as in Proposition 7.1, i.e.
\[ \mathcal{L}(A) = \text{Err}[\mathcal{L}(A)]. \] (98)

We want to apply the Chandrasekhar transformation, i.e. the operator \( Q \), to the above. We compute the commutator \([Q, \mathcal{L}]A\) between \( \mathcal{L} \) and the second order differential operator \( Q \), as defined in (95), for any scalar functions \( C \) and \( D \). In order to cancel out the highest order terms in the equation for \( Q(A) \) we need to impose differential equations on the functions \( C \) and \( D \). We obtain the following.

**Proposition 8.6.** Let \( Q(A) = (c)\nabla_3 (c)\nabla_3 A + C (c)\nabla_3 A + D A \) such that \( C \) and \( D \) satisfy
\[
(c)\nabla_3 C + \frac{C}{2} (\text{tr} X + \text{tr} \bar{X}) - \text{tr} X \text{tr} \bar{X} = \Gamma_g, \tag{99}
\]
\[
(c)\nabla_3 D + D (\text{tr} X + \text{tr} \bar{X}) - \frac{C}{4} \text{tr} X \text{tr} \bar{X} = r^{-1} \Gamma_g. \tag{100}
\]
Then the commutator between \( Q \) and \( \mathcal{L} \) is given by
\[
[Q, \mathcal{L}](A) = 4\eta \cdot (c)\nabla Q(A) - (\text{tr} X + \text{tr} \bar{X}) (c)\nabla_4 Q(A) + C_0(Q(A))
+ a L_Q(A) + \text{Err}[[Q, \mathcal{L}]A] \tag{101}
\]
where \( C_0(Q(A)) \) are zero-th order terms in \( Q(A) \), \( L_Q(A) \) are linear lower order terms in \( Q(A) \) of the schematic form
\[ L_Q(A) = d_1 (c)\nabla (c)\nabla_3 A + d_2 (c)\nabla_3 A + d_3 (c)\nabla (A) + d_4 A, \]
with \( d_1, \ldots, d_4 \) smooth functions of \((r, \theta)\) which have the following fall-off in \( r \)
\[ d_1 = O \left( \frac{1}{r^3} \right), \quad d_2 = O \left( \frac{1}{r^4} \right), \quad d_3 = O \left( \frac{1}{r^4} \right), \quad d_4 = O \left( \frac{1}{r^5} \right). \]
The error terms \( \text{Err}[[Q, \mathcal{L}]A] \) are given by
\[ \text{Err}[[Q, \mathcal{L}]A] = (c)\nabla_3 \left( \frac{1}{r} \Gamma_g \cdot \delta \leq 2 A \right) + \text{l.o.t.} \]

**Proof.** See Appendix B. \( \square \)

Observe that the transport equation given by (99) only imposes conditions on the real part of the coefficient \( C \). Indeed, any function \( C \) of the form \( C = 2\text{tr} \bar{X} + i\epsilon^{(a)} \text{tr} \bar{X} \) for any constant \( \epsilon \).
satisfies (99). Indeed
\[
^{(c)}\nabla_3 C + \frac{C}{2} (\text{tr} X + \text{tr} X) - \text{tr} X \text{tr} X
\]
\[
= ^{(c)}\nabla_3 (2\text{tr} \chi + i\epsilon (a)\text{tr}\chi) + (2\text{tr} \chi + i\epsilon (a)\text{tr}\chi)\text{tr} \chi - (\text{tr} \chi^2 + (a)\text{tr}\chi^2)
\]
\[
= 2 \left( -\frac{1}{2} (\text{tr} \chi^2 - (a)\text{tr}\chi^2) + \Gamma_g \right) + i\epsilon (\text{tr} \chi (a)\text{tr}\chi + \Gamma_g)
\]
\[
+ (2\text{tr} \chi + i(\epsilon (a)\text{tr}\chi))\text{tr} \chi - (\text{tr} \chi^2 + (a)\text{tr}\chi^2) = \Gamma_g.
\]
Similarly, let \( D \) be a function of the form \( D = \frac{1}{2} \text{tr} \chi^2 + f (a)\text{tr}\chi^2 + i\epsilon \text{tr} \chi (a)\text{tr}\chi \). Then we have
\[
^{(c)}\nabla_3 D + D (\text{tr} \chi + \text{tr} X) - \frac{C}{4} \text{tr} X (\text{tr} X)
\]
\[
= ^{(c)}\nabla_3 \left( \frac{1}{2} \text{tr} \chi^2 + f (a)\text{tr}\chi^2 + i\epsilon \text{tr} \chi (a)\text{tr}\chi \right) + \left( \frac{1}{2} \text{tr} \chi^2 + f (a)\text{tr}\chi^2 + i\epsilon \text{tr} \chi (a)\text{tr}\chi \right) 2\text{tr} \chi
\]
\[
- \frac{1}{4} (2\text{tr} \chi + i\epsilon (a)\text{tr}\chi)(\text{tr} \chi^2 + (a)\text{tr}\chi^2)
\]
\[
= \text{tr} \chi \left( -\frac{1}{2} (\text{tr} \chi^2 - (a)\text{tr}\chi^2) + \Gamma_g \right) + 2f (a)\text{tr}\chi (-\text{tr} \chi (a)\text{tr}\chi + \Gamma_g)
\]
\[
+ i\epsilon \left( -\frac{1}{2} (\text{tr} \chi^2 - (a)\text{tr}\chi^2) + \Gamma_g \right) \text{tr} \chi + i\epsilon \text{tr} \chi (\text{tr} \chi^2 + (a)\text{tr}\chi^2)
\]
\[
+ \left( \frac{1}{2} \text{tr} \chi^2 + f (a)\text{tr}\chi^2 + i\epsilon \text{tr} \chi (a)\text{tr}\chi \right) 2\text{tr} \chi - \frac{1}{4} (2\text{tr} \chi + i\epsilon (a)\text{tr}\chi)(\text{tr} \chi^2 + (a)\text{tr}\chi^2)
\]
\[
= i (a)\text{tr} \chi \left( \text{tr} \chi^2 + (a)\text{tr}\chi^2 \right) \left( \frac{1}{2} \epsilon - \frac{1}{4} \epsilon \right) + r^{-1} \Gamma_g.
\]
In particular, for any constants \( f \) and \( \epsilon \)
\[
C = 2\text{tr} \chi + i\epsilon (a)\text{tr}\chi, \quad \text{and} \quad D = \frac{1}{2} \text{tr} \chi^2 + f (a)\text{tr}\chi^2 + i\epsilon \text{tr} \chi (a)\text{tr}\chi.
\]
satisfy (99) and (100).

We now select a choice for the imaginary part of the function \( C \) so that the linear lower order terms can be simplified. We have the following lemma.

\textbf{Lemma 8.7.} Let \( C \) be given by
\[
C = 2\text{tr} \chi - 4i (a)\text{tr}\chi.
\]
Then \( C \) satisfies the assumptions of Proposition 8.6 and the linear lower order terms have the following structure:
\[
L_Q (A) = d_1^{(c)}\nabla_2^{(c)}\nabla_3 A + d_2^{(c)} \nabla_3 A + d_3^{(c)} \nabla (A) + d_4 A,
\]
with the fall-off in \( r \) given by Proposition 8.6.
Proof. See Appendix B.2.

Remark 8.8. The choice (104) corresponds to fixing \( \epsilon = -4 \) which yields for \( D \)

\[
D = \frac{1}{2} \text{tr} \chi^2 + f^{(a)} \text{tr} \chi^2 - 2i \text{tr} \chi^{(a)} \text{tr} \chi
\]  

(105)

for any constant \( f \). Note that with such choices, \( C \) is conformally invariant of type \(-1\) and \( D \) is conformally invariant of type \(-2\), so that \( Q(A) \) is conformally invariant of type 0 according to Lemma 8.1.

We therefore consider \( Q(A) \) defined using these functions \( C \) and \( D \), i.e.

\[
Q(A) = (c) \nabla_3 (c) \nabla_3 A + \left( 2 \text{tr} \chi - 4i \text{tr} \chi \right) \quad (c) \nabla_3 A + \left( \frac{1}{2} \text{tr} \chi^2 - 2 \text{tr} \chi^{(a)} \text{tr} \chi \right) A.
\]  

(106)

Applying the operator \( Q \) to (98), we obtain

\[
\mathcal{L}(Q(A)) + [Q, \mathcal{L}](A) = Q(\text{Err}[\mathcal{L}(A)]).
\]  

(107)

Recall that

\[
\mathcal{L}(A) = - (c) \nabla_4 (c) \nabla_3 A + \frac{1}{2} (c) \nabla_3 (c) \nabla_3 A + \left( -\frac{1}{2} \text{tr} X - 2 \text{tr} \chi \right) \quad (c) \nabla_3 A - \frac{1}{2} \text{tr} X (c) \nabla_4 A
\]  

\[+ (4H + H + \bar{H}) \cdot (c) \nabla A + \left( -\text{tr} \chi \text{tr} \chi + 2 \bar{P} \right) A + 2H \bar{\otimes} (\bar{H} \cdot A).\]

which gives\(^{20}\) for \( Q = Q(A) \)

\[
\mathcal{L}(Q) = - \nabla_4 \nabla_3 Q + \frac{1}{2} \bar{D} \bar{\otimes} (\bar{D} \cdot Q) + \left( -\frac{1}{2} \text{tr} X - 2 \text{tr} \chi \right) \nabla_3 Q - \frac{1}{2} \text{tr} X \nabla_4 Q
\]  

\[+ (4H + 2H + 2 \bar{H}) \cdot \nabla Q + \left( -\text{tr} \chi \text{tr} \chi + 2 \bar{P} \right) Q + 2H \bar{\otimes} (\bar{H} \cdot Q).\]

Using Proposition 8.6, equation (107) gives

\[
- \nabla_4 \nabla_3 Q + \frac{1}{2} \bar{D} \bar{\otimes} (\bar{D} \cdot Q) + \left( -\frac{1}{2} \text{tr} X - 2 \text{tr} \chi \right) \nabla_3 Q - \left( \frac{3}{2} \text{tr} X + \text{tr} \chi \right) \nabla_4 Q + 2 \eta \cdot \nabla Q
\]  

\[+ (4H + 2H + 2 \bar{H}) \cdot \nabla Q + \bar{C}_0(Q)
\]

\[= a L_Q(A) + Q(\text{Err}[\mathcal{L}(A)]) + \text{Err}[[Q, \mathcal{L}]A]\]

where we wrote \( 4 \eta \cdot \nabla Q = 2 \eta \cdot \nabla Q + (H + \bar{H}) \cdot \nabla Q \) using (60).

Recall (85) applied to \( Q \):

\[
\Box Q = - (c) \nabla_4 (c) \nabla_3 Q + \frac{1}{2} \bar{D} \bar{\otimes} (\bar{D} \cdot Q) - \frac{1}{2} \text{tr} X (c) \nabla_3 Q - \frac{1}{2} \text{tr} X^{(c)} \nabla_4 Q + 2 \eta \cdot \nabla Q
\]

\[+ \left( -\frac{1}{4} \text{tr} X \text{tr} \chi - \frac{1}{4} \text{tr} X \text{tr} \chi - 2 \bar{P} \right) Q + r^{-1} \Gamma_3 Q.
\]

\(^{20}\)Recall that \( Q(A) \) is of conformal type 0, therefore all conformal derivatives coincide with the non-conformal ones.
We can therefore write the equation for $Q$ as

$$
\Box Q = 2\text{tr} X \nabla_3 Q + (\text{tr} X + \text{tr} X) \nabla_4 Q - (4H + 2H + 2\overline{H}) \cdot \nabla Q + \tilde{V} Q
+ a \cdot L_Q(A) + \text{Err}[\Box Q] \tag{108}
$$

where $\tilde{V} Q$ collects the linear zero-th order terms in $Q$, and the error terms are given by $\text{Err}[\Box Q] = Q(\text{Err}[L(A)]) + \text{Err}[Q, L] A$.

We now want to rescale $Q$ in order to absorb the first order terms in (108) into the wave operator. Observe that for a scalar function $f$, we have

$$
\Box (fQ) = \Box (f) Q + f \Box (Q) - \nabla_3 f \nabla_4 Q - \nabla_4 f \nabla_3 Q + 2 \nabla f \cdot \nabla Q. \tag{109}
$$

Let $f$ be given by

$$
f = q^{3/2}.
$$

Recalling that

$$
\nabla_3 q = \frac{1}{2} \text{tr} X q + r \Gamma_b,
$$

$$
\nabla_4 q = \frac{1}{2} \text{tr} X q + r \Gamma_g,
$$

we deduce

$$
\nabla_3 (f) = \left( \frac{1}{2} \text{tr} X + \frac{3}{2} \overline{\text{tr} X} \right) f + r^4 \Gamma_b,
$$

$$
\nabla_4 (f) = \left( \frac{1}{2} \text{tr} X + \frac{3}{2} \overline{\text{tr} X} \right) f + r^4 \Gamma_g.
$$

Recalling that

$$
\overline{D} q = q H + r \Gamma_g,
$$

$$
D q = q H + r \Gamma_g,
$$

we deduce

$$
2 \nabla q = (\nabla + i \ast \nabla) q + (\nabla - i \ast \nabla) q = D q + \overline{D} q = q (H + \overline{H}) + r \Gamma_g,
$$

$$
2 \nabla q = q (H + \overline{H}) + r \Gamma_g,
$$

and therefore

$$
\nabla f = \frac{1}{2} (H + \overline{H}) f + \frac{3}{2} (\overline{H} + H) f + r^4 \Gamma_g,
$$

$$
= \left( \frac{3}{2} H + \frac{1}{2} \overline{H} + \frac{1}{2} H + \frac{3}{2} \overline{H} \right) f + r^4 \Gamma_g.
$$
Defining
\[ q = fQ = q\overline{q}^3Q \]
we combine (108) and (109) and obtain
\[
\Box_2 q = 2\text{tr}X \nabla_3 q + (\text{tr}X + \text{tr}X) \nabla_4 q - (4H + 2H + 2\overline{H}) \cdot \nabla q + Vq \\
- \left( \frac{1}{2}\text{tr}X + \frac{3}{2}\text{tr}X \right) \nabla_4 q - \left( \frac{1}{2}\text{tr}X + \frac{3}{2}\text{tr}X \right) \nabla_3 q + 2 \left( \frac{3}{2}H + \frac{1}{2}\overline{H} + \frac{1}{2}H + \frac{3}{2}\overline{H} \right) \cdot \nabla q \\
+ fa L_Q(A) + f\text{Err}[\Box_2 Q] \\
= \frac{1}{2} \left( \text{tr}X - \text{tr}X \right) \nabla_4 q + \frac{1}{2} \left( \text{tr}X - \text{tr}X \right) \nabla_3 q + (-H + \overline{H} - H + \overline{H}) \cdot \nabla q + Vq \\
+ a L_q(A) + \text{Err}[\Box_2 q]
\]
where
\[ L_q(A) = f (L_Q(A)) = c_1 (c^{(c)} \nabla (c^{(c)} \nabla_3 A) + c_2 (c^{(c)} \nabla_3 A) + c_3 (c^{(c)} \nabla (A) + c_4 A. \]
with \( c_1, \ldots, c_4 \) smooth functions of \((r, \theta)\) which have the following fall-off in \( r \)
\[ c_1 = O(r), \quad c_2 = O(1), \quad c_3 = O(1), \quad c_4 = O \left( \frac{1}{r} \right). \]
and \( Vq \) collects the zero-th order terms in \( q \).

We are left to analyze the first order terms in the above equation for \( q \). We have
\[
\frac{1}{2} \left( \text{tr}X - \text{tr}X \right) \nabla_4 q + \frac{1}{2} \left( \text{tr}X - \text{tr}X \right) \nabla_3 q = i \left( (a)\text{tr}X \nabla_3 + (a)\text{tr}X \nabla_4 \right).
\]
Observe that, in view of our definition of \( T \) and \( Z \), see (71) (72),
\[
\frac{|q|^2}{2} \left( e_3 + \frac{\Delta}{|q|^2} e_4 \right) = (r^2 + a^2)T + aZ
\]
so that
\[
(a)\text{tr}X \nabla_3 + (a)\text{tr}X \nabla_4 = \frac{2a \cos \theta}{|q|^2} e_3 + \frac{2a \Delta \cos \theta}{|q|^4} e_4 + \Gamma g \mathfrak{d} \\
= \frac{4a \cos \theta (r^2 + a^2)}{|q|^4} T + \frac{4a^2 \cos \theta}{|q|^4} Z + \Gamma g \mathfrak{d}.
\]
This gives
\[
\frac{1}{2} \left( \text{tr}X - \text{tr}X \right) \nabla_4 q + \frac{1}{2} \left( \text{tr}X - \text{tr}X \right) \nabla_3 q = \frac{4ia \cos \theta (r^2 + a^2)}{|q|^4} T(q) + \frac{4ia^2 \cos \theta}{|q|^4} Z(q) + \Gamma g \mathfrak{d}.
\]
In view of our definition of \( T \) and \( Z \), see (71) (72), we have
\[
e_2 = \frac{a \sin \theta}{|q|} T + \frac{1}{|q| \sin \theta} Z
\]
(111)
Also, observe that in Kerr $H_1 = \overline{H}_1$, $H_2 = -\overline{H}_2$. Therefore

$$( -H + \overline{H} - H + \overline{H} ) \cdot \nabla = ( -H_1 + \overline{H}_1 - H_1 + \overline{H}_1 ) \nabla_1 + ( -H_2 + \overline{H}_2 - H_2 + \overline{H}_2 ) \nabla_2$$

$$= 2( \overline{H}_2 - H_2 ) \nabla_2$$

$$= 2 \left( -\frac{a \sin \theta (r + ia \cos \theta)}{|q|^3} + \frac{a \sin \theta (r - ia \cos \theta)}{|q|^3} \right) \nabla_2$$

$$= -4i \frac{a^2 \sin \theta \cos \theta}{|q|^3} \left( \frac{a \sin \theta}{|q|} \mathbf{T} + \frac{1}{|q| \sin \theta} \mathbf{Z} \right)$$

$$= -4i \frac{a^3 \sin^2 \theta \cos \theta}{|q|^4} \mathbf{T} - 4i \frac{a^2 \cos \theta}{|q|^4} \mathbf{Z}.$$ 

This gives

$$( -H + \overline{H} - H + \overline{H} ) \cdot \nabla q = -4i \frac{a^3 \sin^2 \theta \cos \theta}{|q|^4} \mathbf{T}(q) - 4i \frac{a^2 \cos \theta}{|q|^4} \mathbf{Z}(q) + \Gamma_\theta.$$

We finally obtain

$$\Box_2 q = \frac{4ia \cos \theta (r^2 + a^2)}{|q|^4} \mathbf{T}(q) + \frac{4i a^2 \cos \theta}{|q|^4} \mathbf{Z}(q) - 4i \frac{a^3 \sin^2 \theta \cos \theta}{|q|^4} \mathbf{T}(q) - 4i \frac{a^2 \cos \theta}{|q|^4} \mathbf{Z}(q) + V q$$

$$+ a L_q(A) + \text{Err}[\Box_2 q]$$

$$= \frac{4i a \cos \theta (r^2 + a^2 - a^2 \sin^2 \theta)}{|q|^4} \mathbf{T}(q) + V q + a L_q(A) + \text{Err}[\Box_2 q]$$

which proves Theorem 8.3.

### 8.2 The projection of the Regge-Wheeler type equation

Using formula (86) for the projection of the wave operator for a 2-tensor to its first components, and neglecting error terms, we obtain

$$(\Box_2 q)_{11} = \Box g(q_{11}) + \frac{4}{|q|^2} \cos \theta \mathbf{Z} q_{11} - 4 \frac{2m}{|q|^2} \cot^2 \theta q_{11} + a \tilde{V} q_{11}.$$ 

By projecting (97) to the first component we then obtain for $\psi = q_{11}$:

$$\Box g \psi + \frac{4i}{|q|^2} \left( \frac{\cos \theta}{\sin^2 \theta} \mathbf{Z} - a \cos \theta \mathbf{T} \right) \psi - \frac{4}{|q|^2} \left( \cot^2 \theta + 1 - \frac{2m}{r} \right) \psi = F_{11} + a \tilde{V} \psi$$

where $F_{11}$ is the projection of the right hand side of (97). In the particular case of Kerr, the above equation can be written as

$$\Box_{g_{\text{Kerr}}} \psi + \frac{4i}{|q|^2} \left( \frac{\cos \theta}{\sin^2 \theta} \partial_r - a \cos \theta \partial_t \right) \psi - \frac{4}{|q|^2} \left( \cot^2 \theta + 1 - \frac{2m}{r} \right) \psi = \tilde{F}. \quad (112)$$
Remark 8.9. Observe that equation (112) is exactly the one obtained by Ma in [14], see equation (1.27b) for \( s = 2 \) in that paper. The estimates in [14] are derived through energy estimates, Morawetz estimates obtained by decomposing in modes as in [7], and transport estimates for the lower order terms. Nevertheless, the estimates obtained in the particular case of Kerr cannot be easily generalized to the general case of perturbations of Kerr, since the difference between the wave operators gives rise to error terms which are not of the acceptable form in the sense of \( \text{Err}[\Box_{2\mathbf{q}}] \) in Theorem 8.3. A physical space analysis which makes use of approximated hidden symmetries for the perturbed metric as in [1] may be a better way to approach the analysis of the above equation.

9 Additional useful identities involving \( Q(A) \)

We collect here some relations involving \( Q(A) \) and its derivative. We summarize them in the following propositions.

Proposition 9.1. The symmetric traceless 2 tensor \( Q(A) \) with \( C = 2\text{tr}X \) and \( D = \frac{1}{2}(\text{tr}X)^2 \) satisfies

\[
Q(A) = (-)^D\otimes(-)^D\mathcal{P} + \frac{3}{2}\mathcal{P}\left(\frac{1}{2}\text{tr}X\hat{X} + \overline{\text{tr}X\hat{X}}\right) + \left(-(-)^D\text{tr}X + 4(-)^D\nabla_3H - \frac{1}{2}\text{tr}XH\right)\otimes B + \left(8(-)^D\mathcal{P} + 12\mathcal{P}H\right)\otimes H + \text{Err}[Q(A)]
\]

where

\[
\text{Err}[Q(A)] = \text{Err}_3(-)^D\otimes B + (-)^D(\Gamma_gB + \Gamma_bA) + H(\Gamma_gB + \Gamma_bA) + r^{-2}\Gamma_g\Gamma_g.
\]

In particular, we can write

\[
Q(A) = (-)^D\otimes(-)^D\mathcal{P} + \frac{3}{2}\mathcal{P}\left(\frac{1}{2}\text{tr}X\hat{X} + \overline{\text{tr}X\hat{X}}\right) + O_r + \text{Err}[Q(A)]
\]

where \( O_r \) are terms overshooting in \( r \) with respect to the other ones, according to the expected bootstrap assumptions.

Proof. Recall that by Proposition 3.13, we have,

\[
(-)^D\nabla_3A + \frac{1}{2}\text{tr}XA = (-)^D\otimes B + 4H\otimes B - 3\mathcal{P}\hat{X}.
\]

We infer

\[
(-)^D\nabla_3\left((-)^D\nabla_3A + \frac{1}{2}\text{tr}XA\right) = (-)^D\otimes(-)^D\nabla_3B + [(-)^D\nabla_3, (-)^D\otimes]B + 4H\otimes(-)^D\nabla_3B + 4(-)^D\nabla_3H\otimes B - 3\mathcal{P}(-)^D\nabla_3\hat{X} - 3(-)^D\nabla_3\mathcal{P}\hat{X}.
\]
By Lemma 5.8, we have
\[
[(c)\nabla_3, (c)\mathcal{D}\otimes]B = -\frac{1}{2} \text{tr} X \left((c)\mathcal{D}\otimes B + 3H\otimes B\right) + H\otimes (c)\nabla_3 B + \text{Err}_3 (c)\mathcal{D}\otimes [B]
\]
and hence
\[
(c)\nabla_3 \left((c)\nabla_3 A + \frac{1}{2} \text{tr} X A\right) = (c)\mathcal{D}\otimes (c)\nabla_3 B + 5H\otimes (c)\nabla_3 B - \frac{1}{2} \text{tr} X \left((c)\mathcal{D}\otimes B + 3H\otimes B\right)
\]
\[+ 4(c)\nabla_3 H\otimes B - 3\overline{\mathcal{P}}(c)\nabla_3 \hat{X} - 3(c)\nabla_3 \overline{\mathcal{P}} \hat{X} + \text{Err}_3 (c)\mathcal{D}\otimes [B].\]

Next, using Propositions 3.12 and 3.13,
\[
(c)\nabla_3 B - (c)\mathcal{D}\overline{\mathcal{P}} = -\text{tr} X B + 3\overline{\mathcal{P}} H + \Gamma_g B + \Gamma_b A,
\]
\[
(c)\nabla_3 P + \frac{1}{2} (c)\mathcal{D} \cdot B = -\frac{3}{2} \text{tr} X P - \overline{\mathcal{H}} \cdot B + \Gamma_b B + \Gamma_g A,
\]
\[
(c)\nabla_3 \hat{X} + \frac{1}{2} \text{tr} X \hat{X} = (c)\mathcal{D}\otimes H + H\otimes H - \frac{1}{2} \text{tr} X H + \Gamma_b \Gamma_g,
\]
we deduce
\[
(c)\nabla_3 \left((c)\nabla_3 A + \frac{1}{2} \text{tr} X A\right) = (c)\mathcal{D}\otimes \left((c)\mathcal{D}\overline{\mathcal{P}} - \text{tr} X B + 3\overline{\mathcal{P}} H\right)
\]
\[+ 5H\otimes \left((c)\mathcal{D}\overline{\mathcal{P}} - \text{tr} X B + 3\overline{\mathcal{P}} H\right) - \frac{1}{2} \text{tr} X \left((c)\mathcal{D}\otimes B + 3H\otimes B\right)
\]
\[+ 4(c)\nabla_3 H\otimes B - 3\overline{\mathcal{P}} \left(\frac{1}{2} \text{tr} X \hat{X} + (c)\mathcal{D}\otimes H + H\otimes H - \frac{1}{2} \text{tr} X H + \Gamma_b \Gamma_g\right)
\]
\[- 3 \left(-\frac{3}{2} \text{tr} X \overline{\mathcal{P}}\right) \hat{X} + \text{Err}
\]
where
\[
\text{Err} = \text{Err}_3 (c)\mathcal{D}\otimes [B] + (c)\mathcal{D}(\Gamma_g B + \Gamma_b A) + H(\Gamma_g B + \Gamma_b A) + r^{-2} \Gamma_g \Gamma_g.
\]

Hence, using (57), we have
\[
(c)\nabla_3 \left((c)\nabla_3 A + \frac{1}{2} \text{tr} X A\right) = (c)\mathcal{D}\otimes \left((c)\mathcal{D}\overline{\mathcal{P}} + \frac{3}{2} \overline{\mathcal{P}} \left(\text{tr} X \hat{X} + \overline{\text{tr}} X \hat{X}\right)\right)
\]
\[+ \left(-\frac{3}{2} \text{tr} X \left((c)\mathcal{D}\otimes B - 3\overline{\mathcal{P}} \hat{X}\right)\right)\otimes B
\]
\[+ \left(8(c)\mathcal{D}\overline{\mathcal{P}} + 12\overline{\mathcal{P}} H\right) \otimes H + \text{Err}.
\]

Since
\[
(c)\mathcal{D}\otimes B - 3\overline{\mathcal{P}} \hat{X} = (c)\nabla_3 A + \frac{1}{2} \text{tr} X A - 4H\otimes B
\]

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this infer

\[ \nabla_3 \left( \nabla_3 A + \frac{1}{2} \text{tr} X A \right) + \frac{3}{2} \nabla X \left( \nabla_3 A + \frac{1}{2} \text{tr} X A \right) \]

\[ = \nabla_3 \left( \nabla_3 A + \frac{1}{2} \text{tr} X A \right) \]

Using

\[ \nabla_3 \text{tr} X + \frac{1}{2} (\text{tr} X)^2 = \delta^{\leq 1} \Gamma_g \]

the left hand side of the above is given by

\[ \nabla_3 \left( \nabla_3 A + \frac{1}{2} \text{tr} X A \right) + \frac{1}{2} \nabla_3 \text{tr} X A + \frac{3}{2} \nabla X \left( \nabla_3 A + \frac{1}{2} \text{tr} X A \right) \]

\[ = \nabla_3 \left( \nabla_3 A + \frac{1}{2} \text{tr} X A + \frac{1}{2} (\text{tr} X)^2 A + \delta^{\leq 1} \Gamma_g A \right) \]

which coincides with \( Q(A) \). This concludes the proof.

\[ \Box \]

**Proposition 9.2.** The symmetric traceless 2-tensor \( Q(A) \) with \( C = 2 \text{tr} X \) and \( D = \frac{1}{2} (\text{tr} X)^2 \) satisfies

\[ \nabla_3 \left( \nabla_3 \left( \nabla_3 Q(A) \right) \right) = \nabla \left\{ \frac{1}{2} \nabla_3 \left( \nabla_3 \left( \nabla_3 Q(A) \right) \right) \right\} \]

where \( \hat{O}_r \) is overshooting in powers of \( r \) and

\[ \text{Err} \left( \nabla_3 \left( \nabla_3 Q(A) \right) \right) = r^2 \delta^{\leq 2} \left( \Gamma_g \Gamma_b \right) + \frac{5}{2} r \text{tr} X \nabla \left( \text{Err} \left( \nabla_3 Q(A) \right) \right) + r^5 \Gamma_b Q(A) + \nabla \left( \nabla_3 \text{Err} \left( \nabla_3 Q(A) \right) \right). \]

**Proof.** We start by formula (113):

\[ \nabla \left( \nabla_3 \left( \nabla_3 Q(A) \right) \right) = \nabla \left\{ \frac{1}{2} \nabla_3 \left( \nabla_3 \left( \nabla_3 Q(A) \right) \right) \right\}. \]

Taking \( \nabla_3 \) derivative we deduce

\[ \nabla_3 \quad \left( \nabla \left( \nabla_3 Q(A) \right) \right) = \nabla_3 \left\{ \frac{1}{2} \nabla_3 \left( \nabla_3 \left( \nabla_3 Q(A) \right) \right) \right\} \]

\[ \quad L = \nabla_3 \left\{ \frac{1}{2} \nabla_3 \left( \nabla_3 \left( \nabla_3 Q(A) \right) \right) \right\}. \]
We calculate \( L \) as follows

\[
L = \langle (c) D \hat{\otimes} (c) D \rangle \nabla^3 \mathcal{P} + \langle (c) \nabla_3, (c) D \hat{\otimes} (c) D \rangle \mathcal{P} + \frac{3}{2} \langle (c) \nabla_3 (\mathcal{P} \nabla_X \hat{X}) \rangle + \frac{3}{2} \langle (c) \nabla_3 (\mathcal{P} \nabla_X \hat{X}) \rangle.
\]

Ignoring cubic and higher order terms, we have

\[
(c) \nabla_3 (\mathcal{P} \nabla_X \hat{X}) = \mathcal{P} \nabla_X (c) \nabla_3 (\hat{X}) + \mathcal{P} (c) \nabla_3 (\nabla_X \hat{X}) + (c) \nabla_3 (\mathcal{P} \nabla_X \hat{X})
\]

\[
= \mathcal{P} \nabla_X \left( - \frac{1}{2} \mathcal{P} \nabla_X \hat{X} + (c) D \otimes H + H \otimes H - \frac{1}{2} \mathcal{P} \nabla_X \hat{X} \right)
\]

\[
+ \mathcal{P} \left( - \frac{1}{2} (\nabla_X \hat{X})^2 \right) \hat{X} + \left( - \frac{3}{2} \mathcal{P} \nabla_X \hat{X} \right) \mathcal{P} \nabla_X \hat{X} + r^{-3} \mathcal{P} \leq 1 (\Gamma_g \Gamma_b)
\]

where

\[
O_1 = \mathcal{P} \nabla_X \hat{X} \otimes H
\]

is overshooting in powers of \( r \). Also,

\[
(c) \nabla_3 (\mathcal{P} \nabla_X \hat{X}) = \mathcal{P} \nabla_X (c) \nabla_3 (\hat{X}) + \mathcal{P} (c) \nabla_3 (\nabla_X \hat{X}) + (c) \nabla_3 (\mathcal{P} \nabla_X \hat{X})
\]

\[
= \mathcal{P} \nabla_X \left( - \mathcal{R} (\nabla_X \hat{X}) \hat{X} + (c) D \otimes \Xi + \Xi \otimes (H + H) - A \right)
\]

\[
+ \mathcal{P} \left( - \frac{1}{2} \mathcal{P} \nabla_X \hat{X} + (c) D \cdot H + H \cdot H + 2 \mathcal{P} \right) \hat{X}
\]

\[
+ \left( - \frac{3}{2} \mathcal{P} \nabla_X \hat{X} \right) \mathcal{P} \nabla_X \hat{X} + r^{-3} \mathcal{P} \leq 1 (\Gamma_g \Gamma_b)
\]

where

\[
O_2 = \mathcal{P} \nabla_X \left( \Xi \otimes (H + H) \right) + \mathcal{P} (H \cdot H) \hat{X}
\]

is overshooting in powers of \( r \). Also we have

\[
(c) D \otimes (c) D \otimes (c) \nabla_3 \mathcal{P} = \langle (c) D \otimes (c) D \rangle \left( - \frac{3}{2} \mathcal{P} \nabla_X \mathcal{P} - \frac{1}{2} \langle (c) D \cdot \mathcal{B} \rangle - H \cdot \mathcal{B} \right)
\]

\[
= \langle (c) D \otimes \rangle \left( - \frac{3}{2} \langle (c) D \nabla_X \mathcal{P} \rangle - \frac{3}{2} \mathcal{P} \langle (c) D \nabla_X \mathcal{P} \rangle - \frac{1}{2} \langle (c) D (c) D \cdot \mathcal{B} \rangle - (c) D (H \cdot \mathcal{B}) \right)
\]

\[
= - \frac{1}{2} \langle (c) D \otimes (c) D \rangle (c) D \cdot \mathcal{B} \rangle - \frac{3}{2} \mathcal{P} \langle (c) D \otimes (c) D \rangle (c) D \mathcal{P} \rangle - \frac{3}{2} \mathcal{P} \langle (c) D \otimes (c) D \rangle (c) D \mathcal{P} \rangle + O_3
\]

where

\[
O_3 = -3 \langle (c) D \nabla_X (c) D \mathcal{P} \rangle - \langle (c) D \otimes (c) D \rangle (H \cdot \mathcal{B})
\]
is overshooting in powers of $r$. Now in view of Lemma 5.8 applied to $F = \Gamma P \Gamma$ which is of conformal type 0,

\[
[(c)\nabla_3, (c)D\hat{\otimes} (c)\nabla_3] (c)\nabla_3 = - \frac{1}{2} \text{tr} X ( (c)D\hat{\otimes} (c)\nabla_3 + H\hat{\otimes} (c)\nabla_3) + H\hat{\otimes} \nabla_3 (c)\nabla_3 + \text{Err}_{3(c)D\hat{\otimes}} [ (c)\nabla_3 P]
\]

where

\[
O_4 = - \frac{1}{2} \text{tr} X \left( H\hat{\otimes} (c)\nabla_3 P \right) + H\hat{\otimes} \nabla_3 [ (c)\nabla_3 P]
\]

is overshooting in powers of $r$.

From Lemma 2.35, we deduce

\[
[\nabla_3, \nabla_a]P = - \frac{1}{2} \left( \text{tr} X \nabla_a P + \eta_a \nabla_3 P - \hat{\nabla}_a \nabla_b P + \xi_a \nabla_4 P \right)
\]

Taking the dual

\[
[\nabla_3, \ast \nabla_a]P = - \frac{1}{2} \left( \text{tr} X \ast \nabla_a P - \eta_a \nabla_3 P - \hat{\nabla}_a \nabla_b P + \xi_a \nabla_4 P \right)
\]

Summing them we derive

\[
[\nabla_3, D]P = - \frac{1}{2} \text{tr} X \ast D P + (H - Z) \nabla_3 P - \hat{X} \cdot \nabla P + \Xi \nabla_4 P
\]

and therefore

\[
[(c)\nabla_3, (c)D] P = - \frac{1}{2} \text{tr} X ( (c)D\nabla_3 + H (c)\nabla_3 P - \frac{1}{2} \hat{X} \cdot (c)D\nabla_3) + \Xi (c)\nabla_4 P
\]

\[
= - \frac{1}{2} \text{tr} X (c)D\nabla_3 + H \left( - \frac{1}{2} \text{tr} X \nabla_3 P - \frac{3}{2} \text{tr} X H - \frac{1}{4} \hat{X} \cdot \hat{X} \right)
\]

\[
= - \frac{1}{2} \text{tr} X (c)D\nabla_3 - \frac{3}{2} \text{tr} X \left( H + \text{tr} X \Xi \right) + O_5 + r^{-3}\delta \leq (\Gamma_g \Gamma_b)
\]

where

\[
O_5 = H \left( - \frac{1}{2} \text{tr} X \nabla_3 P - \frac{3}{2} \text{tr} X H + \text{tr} X \Xi \right) + O_5 + r^{-3}\delta \leq (\Gamma_g \Gamma_b)
\]

is overshooting in powers of $r$. We deduce

\[
(c)D\hat{\otimes} [ (c)\nabla_3, (c)D] P = (c)D\hat{\otimes} \left( - \frac{1}{2} \text{tr} X (c)D\nabla_3 P + \frac{3}{2} \text{tr} X H + \text{tr} X \Xi \right) + O_5 + r^{-3}\delta \leq (\Gamma_g \Gamma_b)
\]

\[
= - \frac{1}{2} \text{tr} X (c)D\hat{\otimes} (c)D\nabla_3 + H (c)\nabla_3 (c)D\hat{\otimes} P + \Xi (c)\nabla_4 P
\]

\[
= - \frac{1}{2} \text{tr} X (c)D\hat{\otimes} (c)D\nabla_3 - \frac{3}{2} \text{tr} X H + \text{tr} X \Xi
\]

\[
+ (c)D\hat{\otimes} O_5 + r^{-3}\delta \leq (\Gamma_g \Gamma_b)
\]

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which gives

\[
(\overset{\circ}{c}D \overset{\circ}{\otimes} (\overset{\circ}{c}\nabla_3, \overset{\circ}{c}D)\bar{P}) = -\frac{1}{2} \text{tr} X (\overset{\circ}{c}D \overset{\circ}{\otimes} (\overset{\circ}{c}D)\bar{P}) - \frac{3}{2} \bar{P} \left( \text{tr} X (\overset{\circ}{c}D \overset{\circ}{\otimes} H + \text{tr} X (\overset{\circ}{c}D \overset{\circ}{\otimes} \Xi) + O_6 + r^{-3} \delta \leq 2 (\Gamma_g \Gamma_b) \right)
\]

where

\[
O_6 = -\frac{1}{2} \overset{\circ}{c}D \text{tr} X (\overset{\circ}{c}D)\bar{P} - \frac{3}{2} \overset{\circ}{c}D \bar{P} \left( \text{tr} X H + \text{tr} X \Xi \right)
\]

is overshooting in powers of \( r \).

Hence, since \([\overset{\circ}{c}\nabla_3, \overset{\circ}{c}D \overset{\circ}{\otimes} (\overset{\circ}{c}D)\bar{P}] = [\overset{\circ}{c}\nabla_3, \overset{\circ}{c}D\overset{\circ}{\otimes}] (\overset{\circ}{c}D)\bar{P} + \overset{\circ}{c}D \overset{\circ}{\otimes} [\overset{\circ}{c}\nabla_3, \overset{\circ}{c}D] \bar{P} \), we obtain

\[
[\overset{\circ}{c}\nabla_3, \overset{\circ}{c}D \overset{\circ}{\otimes} (\overset{\circ}{c}D)\bar{P}] = -\text{tr} X (\overset{\circ}{c}D \overset{\circ}{\otimes} (\overset{\circ}{c}D)\bar{P}) - \frac{3}{2} \bar{P} \left( \text{tr} X (\overset{\circ}{c}D \overset{\circ}{\otimes} H + \text{tr} X (\overset{\circ}{c}D \overset{\circ}{\otimes} \Xi) \right) + O_7 + r^{-3} \delta \leq 2 (\Gamma_g \Gamma_b)
\]

with \( O_7 = O_4 + O_6 \). We deduce

\[
L = (\overset{\circ}{c}D \overset{\circ}{\otimes} (\overset{\circ}{c}D) \overset{\circ}{\nabla_3} \bar{P} + [\overset{\circ}{c}\nabla_3, \overset{\circ}{c}D \overset{\circ}{\otimes} (\overset{\circ}{c}D)\bar{P} + \frac{3}{2} \overset{\circ}{c}D \text{tr} X (\overset{\circ}{c}D)\bar{P} + \frac{3}{2} \overset{\circ}{c}D \overset{\circ}{\nabla_3} (\overset{\circ}{c}D \text{tr} X \overset{\circ}{X} \overset{\circ}{\hat{X}}) \right.
\]

where \( O_L = O_3 + O_7 + \frac{3}{2} O_1 + \frac{3}{2} O_2 \). On the other hand, writing \( \overset{\circ}{c}\nabla_3 \bar{q} = \frac{1}{2} \text{tr} X \bar{q} + r \Gamma_b \), we have

\[
5 \bar{q}^4 (\overset{\circ}{c}\nabla_3 (\bar{q}) Q(A) = \frac{5}{2} \text{tr} X \bar{q}^5 Q(A) + r \Gamma_b Q(A)
\]

Hence, we finally obtain

\[
(\overset{\circ}{c}\nabla_3 (\bar{q}^5 Q(A)) = \bar{q}^5 \left\{ -\frac{1}{2} (\overset{\circ}{c}D \overset{\circ}{\otimes} (\overset{\circ}{c}D) \overset{\circ}{\nabla_3} \bar{P}) - \frac{3}{2} \bar{P} (\overset{\circ}{c}D \overset{\circ}{\otimes} (\overset{\circ}{c}D) \text{tr} X \right.
\]

where

\[
\bar{\hat{O}}_r = \bar{q}^5 O_L + \frac{5}{2} \text{tr} X \bar{q}^5 O_r + \bar{q}^5 (\overset{\circ}{c}\nabla_3 O_r
\]

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is overshooting in powers of \( r \), and the error term is given by

\[
\text{Err}[\nabla_3(q^5 Q(A))] = r^2 \delta^{22}(\Gamma_g \Gamma_b) + \frac{5}{2} \text{tr} q^5 \{\text{Err}[Q(A)]\} + r^5 \Gamma_b Q(A) + q^5 (\nabla_3 \text{Err}[Q(A)]
\]
as desired.

\section{Relation with the Teukolsky equation in the literature}

The goal of this appendix is to relate the Teukolsky equation obtained here to the Teukolsky equation as appears in the literature for the scalar curvature component \( \alpha^{[2]} \) in Newman-Penrose formalism:

\[
|q|^2 \Box_g \alpha^{[2]} = -4 (r - m) \partial_r \alpha^{[2]} - 4 \left( \frac{m(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right) \partial_t \alpha^{[2]} - 4 \left( \frac{a(r - m)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right) \partial_r \alpha^{[2]} + (4 \cot^2 \theta - 2) \alpha^{[2]}. \tag{114}
\]

We will show how the above Teukolsky equation (114) and the Teukolsky equation (87) are consistent one with another.

\subsection{Relation between \( \alpha^{[2]} \) and projected \( A \)}

In the literature, the curvature component \( \alpha^{[2]} \) is defined in Newman-Penrose formalism as

\[
\alpha^{[2]} = -W(l, m, l, m)
\]

where

\[
l = e_4, \quad m = \frac{|q|}{\sqrt{2q}} (e_1 + ie_2),
\]

where the vectors \( e_4, e_3, e_1 \) and \( e_2 \) are given in Boyer-Lindquist coordinates by (66). We therefore deduce

\[
\alpha^{[2]} = \frac{|q|^2}{2q^2} W(e_4, (e_1 + ie_2), e_4, (e_1 + ie_2)) = -\frac{q}{2q} (W_{4141} + iW_{4142} + iW_{4241} - W_{4242}) = -\frac{q}{2q} (W_{4141} - W_{3242} + 2iW_{4142})
\]

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which gives, using that \( W_{4242} = - W_{4141} \),

\[
\alpha^{[+2]} = -\frac{q}{q} (W_{4141} + iW_{4142}).
\]

Observe that

\[
A_{11} = W_{4141} + iW_{4142}
\]

and therefore denoting by \( a \) the projection of \( A \) to the 11 component:

\[
a = A_{11}
\]

we obtain the following relation:

\[
\alpha^{[+2]} = -\frac{q}{q} a.
\] (115)

In particular \( \alpha^{[+2]} \) and \( a \) differ by a scalar factor given by 

\[
-\frac{q}{q} = -\frac{r - ia \cos \theta}{r + ia \cos \theta},
\]

which is constant (equal to -1) in the case of Schwarzschild.

A.2 The equation for \( a \) from the Teukolsky in the literature

We deduce the equation for \( a \) from the standard Teukolsky equation in the literature (114) and the relation between \( \alpha^{[+2]} \) and \( a \) given by (115). We have

\[
|q|^2 \Box_g a = -|q|^2 \Box_g \left( \frac{q}{q} \alpha^{[+2]} \right)
\]

\[
= -|q|^2 \Box_g \left( \frac{q}{q} \right) \alpha^{[+2]} - \frac{q}{q} |q|^2 \Box_g (\alpha^{[+2]}) - 2\Delta \partial_r \left( \frac{q}{q} \right) \partial_r \alpha^{[+2]} - 2\partial_q \left( \frac{q}{q} \right) \partial_q \alpha^{[+2]}.
\]

Using that \( \partial_r q = 1 \) and \( \partial_q q = -ia \sin \theta \), we compute

\[
\partial_r \left( \frac{q}{q} \right) = \frac{\eta - q}{q^2} = \frac{-2ia \cos \theta}{q^2},
\]

\[
\partial_q \left( \frac{q}{q} \right) = -ia \sin \theta \eta - qia \sin \theta = \frac{-2ira \sin \theta}{q^2}.
\]

Using (114), we obtain

\[
|q|^2 \Box_g a = \left( -|q|^2 \Box_g \left( \frac{q}{q} \right) \right) \alpha^{[+2]} + \frac{q}{q} 4(r - m) \partial_r \alpha^{[+2]} + 4 \frac{q}{q} \left( \frac{m(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right) \partial_r \alpha^{[+2]}
\]

\[
+ 4 \frac{q}{q} \left( \frac{a(r - m)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right) \partial_\phi \alpha^{[+2]} - \frac{q}{q} (4 \cot^2 \theta - 2) \alpha^{[+2]}
\]

\[
- 2\Delta - 2ia \cos \theta \frac{q}{q^2} \partial_r \alpha^{[+2]} - 2 \frac{2ira \sin \theta}{q^2} \partial_q \alpha^{[+2]}
\]
which gives
\[|q|^2 \Box_g a = -4 \left( r - m + \Delta \frac{ia \cos \theta}{|q|^2} \right) \left( \frac{-q}{q} \partial_t \alpha^{[+2]} \right) - 4 \left( \frac{m(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right) \left( -\frac{q}{q} \theta_t \alpha^{[+2]} \right) - 4 \left( \frac{a(r - m)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right) \left( -\frac{q}{q} \theta_\varphi \alpha^{[+2]} \right) - 4 \frac{ira \sin \theta}{|q|^2} \left( -\frac{q}{q} \theta_\theta \alpha^{[+2]} \right) \left( 4 \cot^2 \theta - 2 + q^2 \Box_g \left( \frac{q}{q} \right) \right)\].

Since
\[\partial_t a = \partial_t \left( -\frac{q}{q} \alpha^{[+2]} \right) = -2ia \cos \theta a - \frac{q}{q} \partial_t \alpha^{[+2]},\]
\[\partial_\theta a = \partial_\theta \left( -\frac{q}{q} \alpha^{[+2]} \right) = -2ira \sin \theta a - \frac{q}{q} \partial_\theta \alpha^{[+2]},\]

we can write
\[-\frac{q}{q} \partial_t \alpha^{[+2]} = \partial_t a + \frac{2ia \cos \theta}{|q|^2} a, \quad -\frac{q}{q} \partial_\theta \alpha^{[+2]} = \partial_\theta a + \frac{2ira \sin \theta}{|q|^2} a,\]

We therefore obtain
\[|q|^2 \Box_g a = -4 \left( r - m + \Delta \frac{ia \cos \theta}{|q|^2} \right) \left( \partial_t a + \frac{2ia \cos \theta}{|q|^2} a \right) - 4 \left( \frac{m(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right) \partial_t a - 4 \left( \frac{a(r - m)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right) \partial_\varphi a - 4 \frac{ira \sin \theta}{|q|^2} \left( \partial_\theta a + \frac{2ira \sin \theta}{|q|^2} a \right) \left( 4 \cot^2 \theta - 2 + q^2 \Box_g \left( \frac{q}{q} \right) \right) a\]

which gives the following wave equation for the scalar component \(a\):

\[|q|^2 \Box_g a = -4 \left( r - m + \Delta \frac{ia \cos \theta}{|q|^2} \right) \partial_t a - 4 \left( \frac{m(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right) \partial_t a - 4 \left( \frac{a(r - m)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right) \partial_\varphi a - 4 \frac{ira \sin \theta}{|q|^2} \partial_\theta a - 4 \left( \frac{4 \cot^2 \theta - 2 + q^2 \Box_g \left( \frac{q}{q} \right)}{q^2} \right) a.\]

We will now make use of the null frames to rewrite the above equation in a more convenient form.
We substitute the first order terms given as partial derivatives \( \partial_r, \partial_t, \partial_\theta, \partial_\varphi \) with the derivatives expressed in terms of null frames. From (66), we can write

\[
\partial_r = -\frac{|q|^2}{2\Delta} e_3 + \frac{1}{2} e_4,
\]
\[
\partial_t = \frac{1}{2} e_3 + \frac{\Delta}{2|q|^2} e_4 - \frac{a \sin \theta}{|q|} e_2,
\]
\[
\partial_\theta = |q| e_1,
\]
\[
\partial_\varphi = \left( |q| \sin \theta + \frac{a^2 \sin^3 \theta}{|q|} \right) e_2 - \frac{1}{2} a \sin^2 \theta e_3 - \frac{a \sin^2 \theta \Delta}{2|q|^2} e_4.
\]

This implies from (116):

\[
|q|^2 \Box_g a = -4 \left( r - m + \frac{\Delta}{|q|^2} \right) \left( \frac{|q|^2}{2\Delta} e_3 a + \frac{1}{2} e_4 a \right)
\]
\[
-4 \left( \frac{m(r^2 - a^2)}{\Delta} - r - i a \cos \theta \right) \left( \frac{1}{2} e_3 a + \frac{\Delta}{2|q|^2} e_4 a - \frac{a \sin \theta}{|q|} e_2 a \right)
\]
\[
-4 \left( \frac{a(r - m)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right) \left( \left( |q| \sin \theta + \frac{a^2 \sin^3 \theta}{|q|} \right) e_2 a - \frac{1}{2} a \sin^2 \theta e_3 a - \frac{a \sin^2 \theta \Delta}{2|q|^2} e_4 a \right)
\]
\[
-4 \frac{ira \sin \theta}{|q|^2} |q| e_1 a + \bar{W} a
\]

where the potential is given by

\[
\bar{W} = 4 \cot^2 \theta - 2 + q^2 \Box_g \left( \frac{q}{\bar{q}} \right) - 4 \left( r - m + \frac{\Delta}{|q|^2} \right) \frac{2i a \cos \theta}{|q|^2} + \frac{8r^2 a^2 \sin^2 \theta}{|q|^4}. \quad (117)
\]

The above equation simplifies to

\[
|q|^2 \Box_g a = 2 \left( \frac{|q|^2}{\Delta} \left( r - m \right) - \frac{m(r^2 - a^2)}{\Delta} + a \sin^2 \theta \frac{a(r - m)}{\Delta} + r + 3 i a \cos \theta \right) e_3 a
\]
\[
+ 2 \left( -r - m - \frac{m(r^2 - a^2)}{|q|^2} + \frac{\Delta}{|q|^2} \right) a \sin^2 \theta \frac{a(r - m)}{\Delta} + \frac{i \Delta}{|q|^2} e_4 a
\]
\[
+ 4 \frac{a \sin \theta}{|q|} \left( \frac{m(r^2 - a^2)}{|q|^2} - r - i a \cos \theta \right) - 4 \left( \frac{a(r - m)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right) \left( \left( |q| \sin \theta + \frac{a^2 \sin^3 \theta}{|q|} \right) \right) e_2 a
\]
\[
-4 \frac{ira \sin \theta}{|q|} e_1 (a) + \bar{W} a.
\]

We now use the following values in Kerr to rewrite the coefficients of the first order terms in the equation. Recall:

\[
\text{tr} \chi = \frac{2r}{|q|^2}, \quad (a)\text{tr} \chi = \frac{2a \cos \theta}{|q|^2}, \quad \text{tr} \chi = -2r \frac{\Delta}{|q|^4}, \quad (a)\text{tr} \chi = \frac{2a \Delta \cos \theta}{|q|^4},
\]
\[
\omega = \frac{-a^2 \cos^2 \theta (r - m) - mr^2 + a^2 r}{|q|^4},
\]
\[
\eta_1 = -\frac{a^2 \sin \theta \cos \theta}{|q|^3}, \quad \eta_2 = -\frac{a \sin \theta r}{|q|^3}.
\]

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From $P = -\frac{2m}{q^3}$, we deduce
\[ P = -\frac{2m}{|q|^6}(r - ia \cos \theta)^3 = \frac{2m}{|q|^6}(-r^3 + 3ra^2 \cos^2 \theta + ia \cos \theta(3r^2 - a^2 \cos^2 \theta)) \]
and therefore
\[ \rho = \frac{2m}{|q|^6}(-r^3 + 3ra^2 \cos^2 \theta), \quad \ast \rho = \frac{2ma \cos \theta}{|q|^6}(3r^2 - a^2 \cos^2 \theta). \]
Define also
\[ \Lambda = \frac{r^2 + a^2}{|q|^3} \cot \theta. \]

We rewrite the coefficients in the following way.

- The real part of the coefficient of $e_3a$ is given by
  \[ \frac{|q|^2}{\Delta} (r - m) - \frac{m(r^2 - a^2)}{\Delta} + a \sin^2 \theta \frac{a(r - m)}{\Delta} + r \]
  \[ = \frac{1}{\Delta} \left( (r^2 + a^2 \cos^2 \theta)(r - m) - m(r^2 - a^2) + a^2(1 - \cos^2 \theta)(r - m) + r(r^2 - 2mr + a^2) \right) \]
  \[ = \frac{1}{\Delta} \left( r^3 - mr^2 - mr^2 + a^2 r + r(r^2 - 2mr + a^2) \right) \]
  \[ = \frac{1}{\Delta} (2r \Delta) = 2r. \]
  Therefore the coefficient of $e_3a$ is given by
  \[ 4r + 6ia \cos \theta = |q|^2 \left( 2tr \chi + 3i^{(a)}tr \chi \right). \]

- The real part of the coefficient of $e_4a$ is given by
  \[ -r + m - \frac{m(r^2 - a^2)}{|q|^2} + \frac{\Delta}{|q|^2} r + \frac{a \sin^2 \theta}{|q|^2} a(r - m) \]
  \[ = \frac{1}{|q|^2} \left( (r^2 + a^2 \cos^2 \theta)(-r + m) - m(r^2 - a^2) + \Delta r + a^2(1 - \cos^2 \theta)(r - m) \right) \]
  \[ = \frac{1}{|q|^2} \left( -r^3 - 2a^2 \cos^2 \theta r + (2a^2 \cos^2 \theta)(m) + (r^2 - 2mr + a^2)r + a^2(r) \right) \]
  \[ = \frac{1}{|q|^2} \left( -2a^2 \cos^2 \theta(r - m) - 2mr^2 + 2a^2 r \right). \]
  Therefore the coefficient of $e_4a$ is given by
  \[ \frac{2}{|q|^2} \left( -2a^2 \cos^2 \theta(r - m) - 2mr^2 + 2a^2 r \right) + 2i \frac{a \Delta}{|q|^2} \cos \theta = |q|^2 \left( -4\omega + i^{(a)}tr \chi \right). \]
The coefficient of $e_2 a$ is given by

\[
-4 a \sin \theta \left( \frac{m(r^2 - a^2) - r - ia \cos \theta}{|q|} \right) - 4 \left( \frac{a(r - m)}{|q|} \right) \left( |q| \sin \theta + \frac{a^2 \sin^3 \theta}{|q|} \right)
\]

Its real part simplifies to

\[
-4 a \sin \theta \left( \frac{m(r^2 - a^2) - r}{|q|} \right) - 4 \left( \frac{a(r - m)}{|q|} \right) \left( |q| \sin \theta + \frac{a^2 \sin^3 \theta}{|q|} \right)
\]

\[
= -4 \frac{a \sin \theta}{|q|} \left( m(r^2 - a^2) - r(2r - 2mr + a^2) - (r - m)(|q|^2 + a^2 \sin^2 \theta) \right)
\]

\[
= -4 \frac{a \sin \theta}{|q| \Delta} \left( mr^2 - a^2 m - r^3 + 2mr^2 - a^2 r - (r - m)(r^2 + a^2) \right)
\]

\[
= -4 \frac{a \sin \theta}{|q| \Delta} (-2r^3 + 4mr^2 - 2a^2 r) = 4 \frac{a \sin \theta}{|q| \Delta} (-2r \Delta) = -\frac{8ra \sin \theta}{|q|}.
\]

Therefore the coefficient of $e_2 a$ is given by

\[
-\frac{8ra \sin \theta}{|q|} - 4i \left( \frac{2a^2 \sin \theta \cos \theta}{|q|} + \frac{\cos \theta}{|q|} \right) = |q|^2 \left( 8\eta_2 - 4i(\Lambda - \eta_1) \right).
\]

Indeed,

\[
\Lambda - \eta_1 = \frac{r^2 + a^2}{|q|^3} \cot \theta + \frac{a^2 \sin \theta \cos \theta}{|q|^3} = \left( \frac{r^2 + a^2 + a^2 \sin^2 \theta}{|q|^3 \sin \theta} \right) \cos \theta
\]

\[
= \left( \frac{r^2 + a^2 \cos^2 \theta + 2a^2 \sin^2 \theta}{|q|^3 \sin \theta} \right) \cos \theta = \left( \frac{1}{|q|^3 \sin \theta} + \frac{2a^2 \sin \theta}{|q|^3} \right) \cos \theta.
\]

The coefficient of $e_1 a$ is given by

\[
-4 \frac{ira \sin \theta}{|q|} = |q|^2 \left( i4\eta_2 \right)
\]

We can therefore write

\[
\Box_{gKer} a = \left( 2tr \chi + 3i^{(a)} tr \chi \right) e_3 a + \left( -4\omega + i^{(a)} tr \chi \right) e_4 a
\]

\[
+ i4\eta_2 e_1 (a) + \left( 8\eta_2 - 4i(\Lambda - \eta_1) \right) e_2 a + Wa
\]

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where the potential is given by

$$W = \frac{1}{|q|^2} \left( 4 \cot^2 \theta - 2 + q^2 \Box_g \left( \frac{q}{q} \right) - 4 \left( r - m + \Delta \frac{i a \cos \theta}{|q|^2} \right) \frac{2 i a \cos \theta}{|q|^2} + \frac{8 r^2 a^2 \sin^2 \theta}{|q|^4} \right).$$

This gives

$$W = \frac{1}{|q|^2} \left( 4 \cot^2 \theta - 2 + 8 ( -2 m r + a^2 ) \frac{a^2 \cos^2 \theta}{|q|^4} + \frac{8 r^2 a^2}{|q|^4} - 8 ( r - m ) \frac{i a \cos \theta}{|q|^2} + q^2 \Box_g \left( \frac{q}{q} \right) \right).$$

Putting at common denominator we obtain

$$W = \frac{1}{|q|^2} \left( \frac{4 \cot^2 \theta - 2}{|q|^4} ( r^4 + 2 r^2 a^2 \cos^2 \theta + a^4 \cos^4 \theta ) + 8 ( -2 m r + a^2 ) \frac{a^2 \cos^2 \theta}{|q|^4} + \frac{8 r^2 a^2}{|q|^4} - 8 ( r - m ) \frac{i a \cos \theta}{|q|^2} ( r^2 + a^2 \cos^2 \theta ) + q^2 \Box_g \left( \frac{q}{q} \right) \right)$$

which gives

$$W = \frac{1}{|q|^2} \left( ( 4 \cot^2 \theta - 2 ) r^4 + ( 8 \cot^2 \theta a^2 \cos^2 \theta - 4 a^2 \cos^2 \theta + 8 a^2 ) r^2 - 16 m a^2 \cos^2 \theta r \\
+ 4 a^4 \cos^4 \theta \cot^2 \theta - 2 a^4 \cos^4 \theta + 8 a^4 \cos^2 \theta \\
- 8 i a \cos \theta ( r^3 - m r^2 + a^2 \cos^2 \theta r - m a^2 \cos^2 \theta ) + |q|^4 \Box_g \left( \frac{q}{q} \right) \right).$$

We now compute the term $|q|^4 q^2 \Box_g \left( \frac{q}{q} \right)$. Recall that in Kerr:

$$|q|^2 \Box_g = \Delta \partial_r^2 + \left( -\frac{(r^2 + a^2)^2}{\Delta} + a^2 \sin^2 \theta \right) \partial_t^2 + \partial_\theta^2 + \left( \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) \partial_\phi^2 \\
- \frac{4 m a r}{\Delta} \partial_r \partial_\phi + 2 ( r - m ) \partial_r + \frac{\cos \theta}{\sin \theta} \partial_\theta.$$

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We therefore have

\[
|q|^2 \partial_q \left( \frac{q}{q} \right) = \Delta \partial_q^2 \left( \frac{q}{q} \right) + \partial_{q_r}^2 \left( \frac{q}{q} \right) + 2(r-m) \partial_r \left( \frac{q}{q} \right) + \frac{\cos \theta}{\sin \theta} \partial_q \left( \frac{q}{q} \right)
\]

\[
= \Delta \partial_r \left( \frac{-2ia \cos \theta}{q^2} \right) + \partial_q \left( \frac{-2ira \sin \theta}{q^2} \right) + 2(r-m) \frac{-2ia \cos \theta}{q^2} + \frac{\cos \theta}{\sin \theta} \left( \frac{-2ira \sin \theta}{q^2} \right)
\]

\[
= \left( r^2 - 2mr + a^2 \right) - \frac{4ia \cos \theta}{q^3} - 2r(a \cos \theta + 2a \sin^2 \theta)
\]

\[
+ 2(r-m) \frac{-2ia \cos \theta}{q^2} + \cos \theta \left( \frac{-2ira \sin \theta}{q^2} \right)
\]

\[
= \frac{4iar^2 \cos \theta}{q^3} - \frac{8m i a \cos \theta}{q^3} + \frac{4ia^2 \cos \theta}{q^3} - \frac{4ra^2 \sin^2 \theta}{q^3} + \frac{4mia \cos \theta}{q^3} - \frac{-8ira \cos \theta}{q^3}
\]

\[
+ \frac{-8ira \cos \theta(r - ia \cos \theta)}{q^3}
\]

\[
= \frac{1}{q^3} \left( -4ra^2 \cos^2 \theta - 4ra^2 + 4ma^2 \cos^2 \theta + 4ia \cos \theta(-r^2 - mr + a^2) \right).
\]

Therefore

\[
|q|^4 \partial_q^2 \left( \frac{q}{q} \right) = q \left( -4ra^2 \cos^2 \theta - 4ra^2 + 4ma^2 \cos^2 \theta + 4ia \cos \theta(-r^2 - mr + a^2) \right)
\]

\[
= (r + ia \cos \theta) \left( -4ra^2 \cos^2 \theta - 4ra^2 + 4ma^2 \cos^2 \theta + 4ia \cos \theta(-r^2 - mr + a^2) \right)
\]

\[
= -4r^2 a^2 \cos^2 \theta - 4r^2 a^2 + 4ma^2 \cos^2 \theta r + 4ia \cos \theta(-r^2 - mr^2 + a^2 r)
\]

\[
+ (ia \cos \theta) \left( -4ra^2 \cos^2 \theta - 4ra^2 + 4ma^2 \cos^2 \theta \right) - \left( 4a^2 \cos^2 \theta (-r^2 - mr^2 + a^2) \right)
\]

\[
= (-4a^2)r^2 + 8ma^2 \cos^2 \theta r - 4a^4 \cos^2 \theta + 4ia \cos \theta(-r^3 - mr^2 - ra^2 \cos^2 \theta + ma^2 \cos^2 \theta).
\]

We therefore obtain

\[
W = \frac{1}{|q|^6} \left( (4 \cot^2 \theta - 2) r^4 + (8 \cot^2 \theta a^2 \cos^2 \theta - 4a^2 \cos^2 \theta + 8a^2) r^2 - 16ma^2 \cos^2 \theta r
\]

\[
+ 4a^4 \cos^4 \theta \cot^2 \theta - 2a^4 \cos^4 \theta + 8a^4 \cos^2 \theta
\]

\[
- 8ia \cos \theta(r^3 - mr^2 + a^2 \cos^2 \theta - ma^2 \cos^2 \theta)
\]

\[
+ (-4a^2)r^2 + 8ma^2 \cos^2 \theta r - 4a^4 \cos^2 \theta + 4ia \cos \theta(-r^3 - mr^2 - ra^2 \cos^2 \theta + ma^2 \cos^2 \theta)
\]

which gives

\[
|q|^6 W = (4 \cot^2 \theta - 2)r^4 + (8 \cot^2 \theta a^2 \cos^2 \theta - 4a^2 \cos^2 \theta + 4a^2) r^2 - 8ma^2 \cos^2 \theta r
\]

\[
+ 4a^4 \cos^4 \theta \cot^2 \theta - 2a^4 \cos^4 \theta + 4a^4 \cos^2 \theta
\]

\[
+ 4ia \cos \theta(-3r^3 + mr^2 - 3a^2 \cos^2 \theta r + 3ma^2 \cos^2 \theta).
\]

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In particular writing

\[ \cot^2 \theta \cos^2 \theta = \cot^2 \theta (1 - \sin^2 \theta) = \cot^2 \theta - \cos^2 \theta \]

we obtain

\[ |q|^6 \Re(W) = (4 \cot^2 \theta - 2) r^4 + (8 \cot^2 \theta + 4 - 12 \cos^2 \theta) a^2 r^2 - 8 m a^2 \cos^2 \theta + 4a^4 \cot^2 \theta - 6a^4 \cos^4 \theta, \]

\[ |q|^6 \Im(W) = a \cos \theta (-12 r^3 + 4 m r^2 - 12 a^2 \cos^2 \theta + 12 m a^2 \cos^2 \theta). \]

From considering the case of Schwarzschild, we know that the real part of the potential should contain the following terms:

\[ 4\Lambda^2 + \frac{1}{2} \tr \chi \tr \chi - 10 \omega \tr \chi - 8 \rho \]

\[ = 4 \left( \frac{r^2 + a^2)^2}{|q|^6} \right) \cot^2 \theta - \frac{1}{2} \frac{2r e^\Delta}{|q|^2} + 10 \frac{2r}{|q|^2} \frac{-a^2 \cos^2 \theta (r - m) - m r^2 + a^2 r}{|q|^4} - \frac{8m}{|q|^6} \left( -3 r^3 + 3 a^2 \cos^2 \theta \right) \]

\[ = \frac{1}{|q|^6} \left( 4 (r^4 + 2r^2 a^2 + a^4) \cot^2 \theta - 2r^2 (r^2 - 2m r + a^2) + 20r (-m r^2 + a^2 r - a^2 \cos^2 \theta r + a^2 \cos^2 \theta m) - 16m (-r^3 + 3 a^2 \cos^2 \theta) \right) \]

It could also contain terms like

\[ (a) \tr \chi \quad (a), \quad \eta_1, \quad \eta_2, \quad \eta_1, \quad \Lambda \eta_1, \]

which are given by

\[ (a) \tr \chi (a) \tr \chi = \frac{2a \cos \theta \ 2a \Delta \cos \theta}{|q|^2} \frac{4a^2 \cos^2 \theta (r^2 - 2m r + a^2)}{|q|^2} \]

\[ = \frac{1}{|q|^6} \left( 4 \cos^2 \theta a^2 r^2 - 8 m a^2 \cos^2 \theta + 4 a^4 \cos^2 \theta \right), \]

\[ \eta_1 \eta_1 = \frac{a^4 \sin^2 \theta \cos^2 \theta}{|q|^6} = \frac{1}{|q|^6} a^4 \cos^2 \theta (1 - \cos^2 \theta), \]

\[ \eta_2 \eta_2 = \frac{a^2 \sin^2 \theta r^2}{|q|^6} = \frac{1}{|q|^6} (1 - \cos^2 \theta) a^2 r^2, \]

\[ e_1(\eta_1) = -a^2 \frac{1}{|q|} \partial_\theta \left( \frac{\sin \theta \cos \theta}{|q|^3} \right) = \frac{1}{|q|^6} ((-2 \cos^2 \theta + 1) a^2 r^2 + a^4 \cos^2 \theta (-2 + \cos^2 \theta)), \]

\[ \Lambda \eta_1 = \frac{-r^2 + a^2}{|q|^3} \cot \theta \frac{a^2 \sin \theta \cos \theta}{|q|^3} = \frac{1}{|q|^6} ((-\cos^2 \theta) a^2 r^2 - a^4 \cos^2 \theta). \]
From the above it is clear that we need to have a term \(-\frac{5}{2} (a)\text{tr}\chi(a)\text{tr}\chi\). We have

\[
4\Lambda^2 + \frac{1}{2} \text{tr} \chi \text{tr} \chi - 10\omega \text{tr} \chi - 8\rho - \frac{5}{2} (a)\text{tr}\chi(a)\text{tr}\chi
\]

\[
= \frac{1}{|q|^6}\left((4\cot^2 \theta - 2)r^4 + (8\cot^2 \theta + 18 - 20\cos^2 \theta)a^2 r^2 - 28mra^2\cos^2 \theta + 4a^4\cot^2 \theta\right)
\]

\[
+ \frac{1}{|q|^6}\left(-(10\cos^2 \theta)a^2 r^2 + 20mra^2\cos^2 \theta - 10a^4\cos^2 \theta\right)
\]

\[
= \frac{1}{|q|^6}\left((4\cot^2 \theta - 2)r^4 + (8\cot^2 \theta + 18 - 30\cos^2 \theta)a^2 r^2 - 8mra^2\cos^2 \theta + 4a^4\cot^2 \theta - 10a^4\cos^2 \theta\right).
\]

Observe that

\[
|q|^6\left(\Re(W) - (4\Lambda^2 + \frac{1}{2} \text{tr} \chi \text{tr} \chi - 10\omega \text{tr} \chi - 8\rho - \frac{5}{2} (a)\text{tr}\chi(a)\text{tr}\chi\right)
\]

\[
= |q|^6\left((-14 + 18\cos^2 \theta)a^2 r^2 + 10a^4\cos^2 \theta - 6a^4\cos^4 \theta\right).
\]

We find a combination of \(\eta_1, \eta_2, \eta_3, \eta_4, e_1(\eta_1), \Lambda\eta_1\) which gives the above:

\[
|q|^6(n\eta_1 + m\eta_2 + pe_1(\eta_1) + q\Lambda\eta_1)
\]

\[
= |q|^6((m + p + (m - 2p - q)\cos^2 \theta)a^2 r^2 + (n - 2p - q)a^4\cos^2 \theta + (-n + p)a^4\cos^4 \theta))
\]

For every \(n\), if \(m = -n - 8\), \(p = n - 6\), \(q = 2 - n\) gives the desired expression, i.e.

\[
|q|^6(n\eta_1 + (-n - 8)\eta_2 + (n - 6)e_1(\eta_1) + (2 - n)\Lambda\eta_1)
\]

\[
= |q|^6\left((-14 + 18\cos^2 \theta)a^2 r^2 + 10a^4\cos^2 \theta - 6a^4\cos^4 \theta\right)
\]

which finally gives for every \(n\)

\[
|q|^6\Re(W) = |q|^6\left((4\Lambda^2 + \frac{1}{2} \text{tr} \chi \text{tr} \chi - 10\omega \text{tr} \chi - 8\rho - \frac{5}{2} (a)\text{tr}\chi(a)\text{tr}\chi\right)
\]

\[
+ n\eta_1 + (-n - 8)\eta_2 + (n - 6)e_1(\eta_1) + (2 - n)\Lambda\eta_1\right).
\]

In particular taking \(n = 8\) we can write

\[
\Re(W) = 4\Lambda^2 + \frac{1}{2} \text{tr} \chi \text{tr} \chi - 10\omega \text{tr} \chi - 8\rho - \frac{5}{2} (a)\text{tr}\chi(a)\text{tr}\chi
\]

\[
+ 2e_1(\eta_1) - 6\Lambda\eta_1 + 8\eta_1\eta_2 + 16\eta_2\eta_2.
\]

The imaginary part of the potential could contain terms like

\(\text{(a)tr}\chi\text{tr}\chi, \ (a)\text{tr}\chi\text{tr}\chi, \ \omega(a)\text{tr}\chi, \ (a)\rho, \ \eta_1\eta_2, \ e_1(\eta_2), \ \Lambda\eta_2\).
which are given by

\[
\begin{align*}
(n^{(a)} \text{tr} \chi \text{tr} \chi) &= \frac{a \cos \theta}{|q|^6} (-4r^3 + 8mr^2 - 4a^2r), \\
(n^{(a)} \text{tr} \chi \text{tr} \chi) &= \frac{a \cos \theta}{|q|^6} (4r^3 - 8mr^2 + 4a^2r), \\
(q^{(a)} \text{tr} \chi) &= \frac{a \cos \theta}{|q|^6} (2mr^2 - a^2(2 - 2 \cos^2 \theta)r - 2a^2 \cos^2 \theta m), \\
\ast \rho &= \frac{a \cos \theta}{|q|^6} (6mr^2 - 2ma^2 \cos^2 \theta), \\
\eta_1 \eta_2 &= \frac{a \cos \theta}{|q|^6} (a^2(1 - \cos^2 \theta)r), \\
\Lambda \eta_2 &= \frac{a \cos \theta}{|q|^6} (-r^3 - a^2r), \\
\epsilon_1(\eta_2) &= \frac{1}{|q|} \frac{\partial_6}{|q|^2} \left( -a \sin \theta r \right) = -a r \left( \cos \theta |q|^3 + 3a^2 |q| \cos \theta \sin^2 \theta \right) \\
&= \frac{a \cos \theta}{|q|^6} (-r^3 + 2a^2 \cos^2 \theta r - 3a^2 r).
\end{align*}
\]

We find a combination of the above which gives \( \Im(W) \). We have

\[
\begin{align*}
n^{(a)} \text{tr} \chi \text{tr} \chi + w^{(a)} \text{tr} \chi \text{tr} \chi + c^{(a)} \text{tr} \chi \text{tr} \chi + b \ast \rho + p\eta_1 \eta_2 + q\Lambda \eta_2 + te_1(\eta_2) \\
&= \frac{a \cos \theta}{|q|^6} \left( n(-4r^3 + 8mr^2 - 4a^2r) + w((4r^3 - 8mr^2 + 4a^2r) \\
&\quad + c(2mr^2 - a^2(2 - 2 \cos^2 \theta)r - 2a^2 \cos^2 \theta m) + b(6mr^2 - 2ma^2 \cos^2 \theta) \\
&\quad + pa^2(1 - \cos^2 \theta)r + q(-r^3 - a^2r) + t(-r^3 + 2a^2 \cos^2 \theta r - 3a^2 r) \right) \\
&= \frac{a \cos \theta}{|q|^6} \left( (-4n + 4w - q - t)r^3 + (8n - 8w + 2c + 6b)mr^2 + (-4n + 4w + p - q - 3t - 2c)a^2r \\
&\quad + (-p + 2t + 2c)a^2 \cos^2 \theta r + (-2c - 2b)a^2 \cos^2 \theta m \right).
\end{align*}
\]

From the last term we obtain \( c = -b - 6 \). From the \( mr^2 \) term we deduce \( 2n - 2w = 4 - b \). By comparing each term we obtain for every \( w, b \) and \( t \):

\[
\Im(W) = (w - \frac{1}{2}b + 2)^{(a)} \text{tr} \chi \text{tr} \chi + w^{(a)} \text{tr} \chi \text{tr} \chi + (-b - 6)\omega^{(a)} \text{tr} \chi + b \ast \rho + (2t - 2b)\eta_1 \eta_2 \\
+ (2b - t + 4)\Lambda \eta_2 + te_1(\eta_2).
\]

In particular for \( b = 4, t = 0 \) and \( w = 1 \), we obtain

\[
\Im(W) = ^{(a)} \text{tr} \chi \text{tr} \chi + ^{(a)} \text{tr} \chi \text{tr} \chi - 10\omega^{(a)} \text{tr} \chi + 4 \ast \rho + 12\Lambda \eta_2 - 8\eta_1 \eta_2. \tag{120}
\]

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A.3 The projection of the Teukolsky equation

The linear Teukolsky equation as obtained in our formalism is given by

\[-(c) \nabla_A (c) \nabla_3 A + \frac{1}{2} (c) D \otimes (c) \nabla_A (c) D \cdot A + \left( -\frac{1}{2} \text{tr} X - 2 \text{tr} X \right) (c) \nabla_3 A - \frac{1}{2} \text{tr} X \cdot (c) \nabla_4 A\]

\[+ \frac{1}{2} \nabla \cdot (c) D \cdot A + \left( 2H + \frac{1}{2} H \right) \cdot (c) D A + \left( -\text{tr} X \text{tr} X + 2 \text{P} \right) A + 2H \otimes (H \cdot A) = 0.\]

Recall that in Kerr

\[\omega = 0, \quad Z = -H.\]

We can therefore write out the conformal derivatives as

\[\nabla_4 \nabla_3 A = \nabla_3 A - 4\omega A, \quad (c) \nabla_4 A = \nabla_4 A,\]

\[-(c) \nabla_4 (c) \nabla_3 A = -\nabla_4 \nabla_3 A + 4\omega \nabla_4 A + (4 \nabla \omega A),\]

\[\frac{1}{2} (c) D \otimes (c) D \cdot A = \frac{1}{2} (D + 2Z) \otimes (D \cdot A - 2H \cdot A) = \frac{1}{2} (D - 2H) \otimes (D \cdot A - 2H \cdot A)\]

\[= \frac{1}{2} D \otimes (D \cdot A) - H \cdot D A - H \cdot D A + (-D \cdot H + 2H \cdot H) A,\]

\[\frac{1}{2} H \cdot (c) D A = \frac{1}{2} H \cdot (D + 2Z) A = \frac{1}{2} H \cdot D A - (H \cdot H) A,\]

\[\left( 2H + \frac{1}{2} H \right) \cdot (c) D A = \left( 2H + \frac{1}{2} H \right) \cdot (D + 2Z) A = \left( 2H + \frac{1}{2} H \right) \cdot D A - (4H \cdot H + H \cdot H) A.\]

This implies

\[-\nabla_4 \nabla_3 A + \frac{1}{2} D \otimes (D \cdot A) + \left( -\frac{1}{2} \text{tr} X - 2 \text{tr} X \right) \nabla_3 A + \left( -\frac{1}{2} \text{tr} X + 4\omega \right) \nabla_4 A\]

\[+ \frac{1}{2} H \cdot D A + \left( 2H + \frac{1}{2} H \right) \cdot D A\]

\[+ \left( -\text{tr} X \text{tr} X + 2 \text{P} + 2\omega \text{tr} X + 8\omega \text{tr} X + 4\nabla_4 \omega - D \cdot H + 4H \cdot H \right) A + 2H \otimes (H \cdot A) = 0.\]

Also writing

\[-\frac{1}{2} \text{tr} X + 4\omega = -\frac{1}{2} (\text{tr} \chi - i^{(a)} \text{tr} \chi) + 4\omega = \left( -\frac{1}{2} \text{tr} \chi + 4\omega \right) + i \left( \frac{1}{2} \text{tr} \chi \right),\]

\[-\frac{1}{2} \text{tr} X - 2 \text{tr} X = -\frac{1}{2} (\text{tr} \chi - i^{(a)} \text{tr} \chi) - 2 (\text{tr} \chi - i^{(a)} \text{tr} \chi) = \left( -\frac{5}{2} \text{tr} \chi \right) + i \left( -\frac{3}{2} \text{tr} \chi \right),\]

and writing \(-H \cdot D A - \frac{1}{2} H \cdot D A = -2\omega \cdot \nabla,\) we obtain

\[-\nabla_4 \nabla_3 A + \frac{1}{2} D \otimes (D \cdot A) + \left( -\frac{5}{2} \text{tr} \chi + i \frac{3}{2} \text{tr} \chi \right) \nabla_3 A + \left( -\frac{1}{2} \text{tr} \chi + 4\omega + i \frac{1}{2} \text{tr} \chi \right) \nabla_4 A\]

\[-2\omega \cdot \nabla A + 2H \cdot D A\]

\[+ \left( -\text{tr} X \text{tr} X + 2 \text{P} + 2\omega \text{tr} X + 8\omega \text{tr} X + 4\nabla_4 \omega - D \cdot H + 4H \cdot H \right) A + 2H \otimes (H \cdot A) = 0.\]
which gives

$$(\nabla_3 A)_{11} = e_3 (a) + i (a) \tr a,$$

$$(\nabla_4 A)_{11} = e_4 (a) + i (a) \tr a,$$

$$(\nabla_4 \nabla_3 A)_{11} = e_4 e_3 (a) + i (a) \tr e_4 (a) + i (a) \tr e_3 (a) + \left( - (a) \tr A + i e_4 (a) \tr \right) a,$$

$$\frac{1}{2} D \otimes (\nabla \cdot A)_{11} = \Delta a - \frac{1}{2} i (a) \tr e_3 (a) - \frac{1}{2} i (a) \tr e_4 (a) + 4i \Lambda e_2 (a) + \left( - 4 \Lambda^2 + (a) \tr + G_0 \right) a,$$

$$(2 \eta \cdot \nabla A)_{11} = 2 \eta_1 e_1 (a) + 2 \eta_2 e_2 (a) + 4i \Lambda \eta_2 a,$$

$$(2H \cdot \nabla A)_{11} = 4 \left( \eta_1 - i \eta_2 \right) e_1 (a) - 4 \left( \eta_2 + i \eta_1 \right) e_2 (a) + 8(\Lambda \eta_1 - i \Lambda \eta_2) a,$$

where $G_0 = \frac{1}{2} \tr A + \frac{1}{2} (a) \tr + 2 \rho$ is given by the Gauss equation.

**Proof.** Recall that in Kerr

$$g(D e_1, e_2) = \chi_{12} = \frac{1}{2} (a) \tr,$$

$$g(D e_1, e_2) = \chi_{12} = \frac{1}{2} (a) \tr,$$

which gives

$$\nabla_4 e_1 = \frac{1}{2} (a) \tr e_2,$$

$$\nabla_3 e_1 = \frac{1}{2} (a) \tr e_2.$$

We therefore have

$$(\nabla_3 A)_{11} = e_3 (A_{11}) - 2 A_{11} = e_3 (A_{11}) - (a) \tr A_{11} = e_3 (A_{11}) + i (a) \tr e_3 (A_{11}),$$

$$(\nabla_4 A)_{11} = e_4 (A_{11}) - 2 A_{11} = e_3 (A_{11}) - (a) \tr e_2 = e_4 (A_{11}) + i (a) \tr e_4 (A_{11}),$$

since $A_{12} = -i A_{11}$. This proves the first two identities.

We also have

$$(\nabla_3 \nabla_3 A)_{11} = e_4 (\nabla_3 A_{11}) - 2 \nabla_3 A_{11} = e_4 (\nabla_3 A_{11}) = e_4 (\nabla_3 A)_{11} = e_4 (\nabla_3 A_{11}) + i (a) \tr e_4 (\nabla_3 A)_{11}$$

which gives the third expression.
Recall the relation:

\[
\frac{1}{2} D \hat{\otimes} (D \cdot A) = \Delta_2 A - i(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) A
\]

as derived in Proposition 4.10. Using the Gauss formula (80):

\[
(\nabla_1 \nabla_2 - \nabla_2 \nabla_1) A = \frac{1}{2} \left( i^a \text{tr} \chi^i_3 + (a) \text{tr} \chi^i_4 \right) A + i \left( \frac{1}{2} \text{tr} \chi \chi + \frac{1}{2} i (a) \text{tr} \chi (a) \text{tr} \chi + 2 \rho \right) A
\]

one obtains

\[
\frac{1}{2} D \hat{\otimes} (D \cdot A) = \Delta_2 A - \frac{1}{2} i^a \text{tr} \chi^i_3 A - \frac{1}{2} i (a) \text{tr} \chi^i_4 A + \left( \frac{1}{2} \text{tr} \chi \chi + \frac{1}{2} (a) \text{tr} \chi (a) \text{tr} \chi + 2 \rho \right) A
\]

i.e.

\[
\frac{1}{2} D \hat{\otimes} (D \cdot A) = \Delta_2 A - \frac{1}{2} i (a) \text{tr} \chi^i_3 A + \left( \frac{1}{2} i (a) \text{tr} \chi^i_4 A + G_0 A \right)
\]

with \( G_0 = \frac{1}{2} \text{tr} \chi \chi + \frac{1}{2} (a) \text{tr} \chi (a) \text{tr} \chi + 2 \rho \), given by the Gauss equation.

Projecting to the 11 component, and using the previous relations we obtain

\[
\frac{1}{2} D \hat{\otimes} (D \cdot A)_{11} = (\Delta_2 A)_{11} - \frac{1}{2} i (a) \text{tr} \chi^i_3 A_{11} - \frac{1}{2} i (a) \text{tr} \chi^i_4 A_{11} + (G_0 + i G_1) A_{11}
\]

\[
= (\Delta_2 A)_{11} - \frac{1}{2} i (a) \text{tr} \chi^i_3 (e_3(a) + i (a) \text{tr} \chi a) - \frac{1}{2} i (a) \text{tr} \chi^i_4 (e_4(a) + i (a) \text{tr} \chi a) + (G_0 + i G_1) a
\]

\[
= (\Delta_2 A)_{11} - \frac{1}{2} i (a) \text{tr} \chi^i_3 (a) - \frac{1}{2} i (a) \text{tr} \chi^i_4 (a) + \left( \frac{1}{2} (a) \text{tr} \chi (a) \text{tr} \chi + G_0 + i G_1 \right) a.
\]

We now compute \((\Delta_2 A)_{11}\). Recall that in Kerr

\[
(A_{1})_{21} := g(D_1 e_1, e_2) = 0,
\]

\[
(A_{2})_{21} := g(D_2 e_1, e_2) = \frac{r^2 + a^2}{|q|^3} \cot \theta := \Lambda,
\]

\[
(A_{1})_{12} := g(D_1 e_2, e_1) = 0,
\]

\[
(A_{2})_{12} := g(D_2 e_2, e_1) = -\frac{r^2 + a^2}{|q|^3} \cot \theta = -\Lambda,
\]

i.e.,

\[
\nabla e_1 e_1 = \nabla e_1 e_2 = 0, \quad \nabla e_2 e_1 = \Lambda e_2, \quad \nabla e_2 e_2 = -\Lambda e_1.
\]

Therefore

\[
\nabla_1 A_{11} = e_1 A_{11} - 2 A \nabla_1 e_1 - e_1 a,
\]

\[
\nabla_2 A_{11} = e_2 (A_{11}) - 2 A \nabla_2 e_1 - e_2 (A_{11}) - 2 \Lambda A_{21} = e_2 (A_{11}) + 2 i \Lambda a.
\]

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Moreover,
\[ \nabla_1 \nabla_1 A_{11} = e_1 (\nabla_1 A_{11}) - \nabla \nabla_1 A_{11} - 2 \nabla_1 A \nabla_{11} = e_1(\eta_1 a), \]
\[ \nabla_2 \nabla_2 A_{11} = e_2 (\nabla_2 A_{11}) - \nabla \nabla_2 A_{11} - 2 \nabla_2 A \nabla_{11} \]
\[ = e_2 \left( e_2(a) + 2 i \Lambda a \right) + \Lambda \nabla_1 A_{11} - 2 \Lambda \nabla_2 A_{21} \]
\[ = e_2 e_2(a) + 2 i \Lambda e_2(a) + \Lambda e_1(a) + 2 i \Lambda \nabla_1 A_{11} \]
\[ = e_2 e_2(a) + 2 i \Lambda e_2(a) + \Lambda e_1(a) + 2 i \Lambda e_2(a) + 2 i \Lambda a \]
\[ = e_2 e_2(a) + 4 i \Lambda e_2(a) + \Lambda e_1(a) - 4 \Lambda^2 a. \]

We define the laplacian of a scalar \( a \) as
\[ \Delta a = g^{ab} \nabla_a \nabla_b a = \nabla_1 \nabla_1 a + \nabla_2 \nabla_2 a \]
\[ = e_1 e_1 a - D_{e_1} e_1 a + e_2 e_2 a - D_{e_2} e_2 a \]
\[ = e_1 e_1 a + e_2 e_2 a + \Lambda e_1 a. \]

We deduce,
\[ (\Delta_2 A)_{11} = \nabla_1 \nabla_1 A_{11} + \nabla_2 \nabla_2 A_{11} \]
\[ = e_1 (e_1 a) + e_2 e_2(a) + 4 i \Lambda e_2(a) + \Lambda e_1(a) - 4 \Lambda^2 a \]
\[ = \Delta a + 4 i \Lambda e_2(a) - 4 \Lambda^2 a. \]

From (122), we obtain
\[ \frac{1}{2} D * D(\nabla \cdot A)_{11} = \Delta a - \frac{1}{2} i^{(a)} \text{tr} \chi_2 (a) - \frac{1}{2} i^{(a)} \text{tr} \chi_2 (a) + 4 i \Lambda e_2(a) \]
\[ + \left( -4 \Lambda^2 + (a) \text{tr} \chi (a) \text{tr} \chi + G_0 \right) a \]
as desired.

We have
\[ (2 \eta \cdot \nabla A)_{11} = 2 \eta_1 \nabla_1 A_{11} + 2 \eta_2 \nabla_2 A_{11} = 2 \eta_1 e_1 a + 2 \eta_2 e_2(a) + 4 i \Lambda \eta_2 a. \]

Finally, we compute
\[ (2H \cdot \nabla A)_{11} = 2 H_1 \nabla_1 A_{11} + 2 H_2 \nabla_2 A_{11} \]
\[ = 2 H_1 (\nabla_1 - i \ast \nabla_1) A_{11} + 2 H_2 (\nabla_2 - i \ast \nabla_2) A_{11} \]
\[ = 2 H_1 (\nabla_1 - i \nabla_2) A_{11} + 2 H_2 (\nabla_2 + i \nabla_1) A_{11} \]
\[ = 2 (H_1 + i H_2) \nabla_1 A_{11} + 2 (H_2 - i H_1) \nabla_2 A_{11}. \]

We have
\[ H_1 + i H_2 = \eta_1 + i \ast \eta_1 + i (\eta_2 + i \ast \eta_2) \]
\[ = \eta_1 + i \eta_2 + i (\eta_2 - i \eta_1) \]
\[ = 2 \eta_1 + 2 i \eta_2. \]
and
\[ H_2 - i H_1 = -i (H_1 + i H_2) = 2 \eta_2 - 2i \eta_1. \]

This gives
\[
(2H \cdot D)_{11} = 4(\eta_1 + i \eta_2) \nabla_1 A_{11} + 4(\eta_2 - i \eta_1) \nabla_2 A_{11} \\
= 4(\eta_1 + i \eta_2) e_1(a) + 4(\eta_2 - i \eta_1)(e_2(a) + 2i \Lambda a) \\
= 4(\eta_1 + i \eta_2) e_1(a) + 4(\eta_2 - i \eta_1)e_2(a) + 8(\Lambda \eta_1 + i \Lambda \eta_2)a.
\]

Using that \( \eta_1 = \eta_1 \), and \( \eta_2 = -\eta_2 \), we have
\[
(2H \cdot D)_{11} = 4(\eta_1 - i \eta_2)e_1(a) + 4(-\eta_2 - i \eta_1)e_2(a) + 8(\Lambda \eta_1 - i \Lambda \eta_2)a
\]
which gives the final identity.

We can now use the above Lemma to project the linear Teukolsky equation (121) to the component 11, to obtain a wave equation for \( a \). We summarize the computation in the following proposition.

**Proposition A.2.** The complex scalar \( a \) verifies the following wave equation:
\[
\Box g_{Kerr} a = \left( -4 \omega + i \frac{(a) \text{tr} \chi}{2} \right) e_4(a) + \left( 2 \text{tr} \chi + 3i \frac{(a) \text{tr} \chi}{2} \right) e_3(a) \\
+ i 4 \eta_2 e_1(a) + \left( 8 \eta_2 - 4i(\Lambda - \eta_1) \right) e_2(a) + V a
\]
(124)

where
\[
V = \left( 4 \Lambda^2 + \frac{5}{2} \text{tr} \chi \text{tr} \chi - 10 \omega \text{tr} \chi - 8 \rho + 6e_1(\eta_1) - 6 \Lambda \eta_1 + 8 \eta_1 \eta_2 - 16 \eta_2 \eta_2 \right) \\
+ i \left( \frac{(a) \text{tr} \chi \text{tr} \chi}{2} + \frac{(a) \text{tr} \chi \text{tr} \chi}{2} - 10 \omega (a) \text{tr} \chi + 3e_1(\eta_2) + 15 \Lambda \eta_2 + 8 \eta_1 \eta_2 \right)
\]
is the potential.

**Proof.** From projecting to the 11 component the linear Teukolsky equation (121), we obtain
\[
-\nabla_4 \nabla_3 A_{11} - 2 \eta_1 \cdot \nabla A_{11} + \left( \left( -\frac{1}{2} \text{tr} \chi + 4 \omega \right) + i \left( \frac{(a) \text{tr} \chi}{2} \right) \right) \nabla_4 A_{11} \\
+ \left( \left( -\frac{5}{2} \text{tr} \chi \right) + i \left( -\frac{3}{2} \text{tr} \chi \right) \right) \nabla_3 A_{11} + \frac{1}{2} D \hat{\otimes} (D \cdot A)_{11} + 2H \cdot \nabla A_{11} \\
+ (-\text{tr} \chi \text{tr} \chi + 2 \rho + 2 \omega \text{tr} X + 8 \omega \text{tr} X + 4 \nabla \omega - D \cdot \nabla - 4H \cdot \nabla H) A_{11} + 2H \hat{\otimes} (H \cdot A)_{11} = 0.
\]
Using Lemma A.1, the first line of the above reduces to

\[-\nabla_4 \nabla_3 A_{11} - 2\eta \cdot \nabla A_{11} + \left( \left( -\frac{1}{2} \text{tr} \chi + 4\omega \right) + i \left( \frac{1}{2} (a) \text{tr}_\chi \right) \right) \nabla_4 A_{11} + \left( e_4(a) - i (a) \text{tr}_\chi e_4(a) - i (a) \text{tr}_\chi e_3(a) + \left( (a) \text{tr}_\chi (a) \text{tr}_\chi - i e_4(a) \text{tr}_\chi \right) \right) \nabla_4 A_{11} \]

\[= -e_4 e_3(a) - i (a) \text{tr}_\chi e_4(a) - i (a) \text{tr}_\chi e_3(a) + \left( (a) \text{tr}_\chi (a) \text{tr}_\chi - i e_4(a) \text{tr}_\chi \right) \nabla_4 A_{11} \]

\[\quad + \left( \left( -\frac{1}{2} \text{tr} \chi + 4\omega \right) + i \left( \frac{1}{2} (a) \text{tr}_\chi \right) \right) \left( e_4(a) + i (a) \text{tr}_\chi a - 2\eta_1 e_1 a - 2\eta_2 e_2(a) - 4i\Lambda\eta_2 a \right) \]

\[= -e_4 e_3(a) - i (a) \text{tr}_\chi e_3(a) + \left( \left( -\frac{1}{2} \text{tr} \chi + 4\omega \right) - i \left( \frac{1}{2} (a) \text{tr}_\chi \right) \right) e_4(a) - 2\eta_1 e_1 a - 2\eta_2 e_2(a) \]

\[\quad + \left( \frac{1}{2} (a) \text{tr}_\chi (a) \text{tr}_\chi + i \left( -e_4(a) \text{tr}_\chi - \frac{1}{2} (a) \text{tr}_\chi \text{tr}_\chi + 4\omega (a) \text{tr}_\chi - 4\Lambda\eta_2 \right) \right) a. \]

The second line reduces to

\[\left( \left( -\frac{5}{2} \text{tr} \chi \right) + i \left( -\frac{3}{2} (a) \text{tr}_\chi \right) \right) \nabla_4 A_{11} + \frac{1}{2} D \hat{\otimes} (\overline{D} \cdot A)_{11} + 2 H \cdot \overline{D} A_{11} \]

\[= \left( \left( -\frac{5}{2} \text{tr} \chi \right) + i \left( -\frac{3}{2} (a) \text{tr}_\chi \right) \right) \left( e_3(a) + i (a) \text{tr}_\chi a \right) \]

\[\quad + \nabla a - \frac{1}{2} i (a) \text{tr}_\chi e_3(a) - \frac{1}{2} i (a) \text{tr}_\chi e_4(a) + 4i \Lambda e_2(a) + \left( -4\Lambda^2 + (a) \text{tr}_\chi (a) \text{tr}_\chi + G_0 + iG_1 \right) a \]

\[\quad + 4 \left( \eta_1 - i\eta_2 \right) e_1(a) - 4 \left( \eta_2 + i\eta_1 \right) e_2(a) + 8(\Lambda\eta_1 - i\Lambda\eta_2)a \]

\[= \Delta a + \left( \left( -\frac{5}{2} \text{tr} \chi \right) + i \left( -\frac{3}{2} (a) \text{tr}_\chi \right) \right) e_3(a) - \frac{1}{2} i (a) \text{tr}_\chi e_4(a) \]

\[\quad + \left( \frac{4 \eta_1 - i4\eta_2}{} \right) e_1(a) + 4 \left( -\eta_2 + i(\Lambda - \eta_1) \right) e_2(a) \]

\[\quad + \left( \frac{5 (a) \text{tr}_\chi (a) \text{tr}_\chi - 4\Lambda^2 + 8\Lambda\eta_1 + G_0}{} \right) + i \left( -\frac{5}{2} \text{tr} \chi (a) \text{tr}_\chi - 8\Lambda\eta_2 \right) \right) a. \]

We compute

\[(2H \hat{\otimes} (\overline{H} \cdot A))_{11} = 2H_1(\overline{H} \cdot A)_1 - H \cdot (\overline{H} \cdot A) \]

\[= 2H_1(\overline{H} \cdot A)_1 - H_1(\overline{H} \cdot A)_1 - H_2(\overline{H} \cdot A)_2 \]

\[= H_1(\overline{H} \cdot A)_1 - H_2(\overline{H} \cdot A)_2. \]

We have, using that \( A_{12} = -iA_{11} \) and \( A_{22} = -A_{11} \):

\[\overline{H} \cdot A)_1 = \overline{H}_1 A_{11} + \overline{H}_2 A_{12} = (\overline{H}_1 - i\overline{H}_2) A_{11}, \]

\[\overline{H} \cdot A))_2 = \overline{H}_1 A_{12} + \overline{H}_2 A_{22} = -(i\overline{H}_1 + \overline{H}_2) A_{11}. \]

This gives

\[(2H \hat{\otimes} (\overline{H} \cdot A))_{11} = H_1(\overline{H}_1 - i\overline{H}_2) A_{11} + H_2(i\overline{H}_1 + \overline{H}_2) A_{11} \]

\[= (H_1 \overline{H}_1 + H_2 \overline{H}_2 - i(H_1 \overline{H}_1 - H_2 \overline{H}_1)) A_{11}. \]
We also have
\[ H \cdot \overline{H} = H_1 \overline{H}_1 + H_2 \overline{H}_2. \]
Therefore
\[ -4H \cdot \overline{H} A_{11} + 2H \otimes (\overline{H} \cdot A)_{11} = (-3H_1 \overline{H}_1 - 3H_2 \overline{H}_2 - i(H_1 \overline{H}_2 - H_2 \overline{H}_1))a. \]

Observe that in Kerr \( H_1 = \overline{H}_1, \quad H_2 = -\overline{H}_2, \) and therefore
\[ -4H \cdot \overline{H} A_{11} + 2H \otimes (\overline{H} \cdot A)_{11} = (-3 \overline{H}_1 \overline{H}_1 + 3 \overline{H}_2 \overline{H}_2 - 2i(\overline{H}_1 \overline{H}_2))a. \]

Using that \( \overline{H}_1 = -i \overline{H}_2, \) we have
\[ -4H \cdot \overline{H} A_{11} + 2H \otimes (\overline{H} \cdot A)_{11} = 4 \overline{H}_2 \overline{H}_2 a = 4(\eta_2 + i \eta_1)(\eta_2 + i \eta_1)a = 4(-\eta_1 \eta_1 + \eta_2 \eta_2 + 2i(\eta_1 \eta_2))a. \]

We also compute
\[ D \cdot \overline{H} = D_1 \overline{H}_1 + D_2 \overline{H}_2 \]
\[ = (\nabla_1 + i \ast \nabla_1) \overline{H}_1 + (\nabla_2 + i \ast \nabla_2) \overline{H}_2 \]
\[ = (\nabla_1 + i \nabla_2) \overline{H}_1 + (\nabla_2 - i \nabla_1) \overline{H}_2 \]
\[ = \nabla_1 \overline{H}_1 + i \nabla_2 \overline{H}_1 + \nabla_2 \overline{H}_2 - i \nabla_1 \overline{H}_2. \]

We therefore obtain
\[ D \cdot \overline{H} = e_1(\overline{H}_1) + e_2(\overline{H}_2) - \overline{H}_2 \nabla_2 + i(e_2(\overline{H}_1) - \overline{H}_2 \nabla_1 - e_1(\overline{H}_2)) \]
\[ = e_1(\eta_1 - i \eta_2) + \Lambda \overline{H}_1 + i(-\Lambda \overline{H}_2 - e_1(\eta_2 + i \eta_1)) \]
\[ = 2e_1(\eta_1) - 2ie_1(\eta_2) + \Lambda(\eta_1 - i \eta_2) - i\Lambda(\eta_2 + i \eta_1) \]
\[ = 2e_1(\eta_1) - 2ie_1(\eta_2) + 2\Lambda(\eta_1 - i \eta_2). \]

The third line reduces to
\[
\left( \begin{aligned}
(\text{tr}X \text{tr}X + 2P + 2\omega \text{tr}X + 8\omega \text{tr}X + 4\nabla_4 \omega - D \cdot \overline{H} - 4H \cdot \overline{H})A + 2H \otimes (\overline{H} \cdot A) \\
- (\text{tr} \chi + i^{(a)} \text{tr} \chi)(\text{tr} \chi - i^{(a)} \text{tr} \chi) + 2(\rho - i \ast \rho) + 2\omega(\text{tr} \chi - i^{(a)} \text{tr} \chi) + 8\omega(\text{tr} \chi + i^{(a)} \text{tr} \chi) + 4\nabla_4 \omega - (2e_1(\eta_1) - 2ie_1(\eta_2) + 2\Lambda(\eta_1 - i \eta_2)) + 4(-\eta_1 \eta_1 + \eta_2 \eta_2 + 2i(\eta_1 \eta_2)) \right) a \\
- \text{tr} \chi \text{tr} \chi - (a) \text{tr} \chi^{(a)} \text{tr} \chi + 10\omega \text{tr} \chi + 2(\rho - 4\omega - 2e_1(\eta_1) - 2\Lambda \eta_1 - 4\eta_1 \eta_1 + 4\eta_2 \eta_2 + i(-a) \text{tr} \chi \text{tr} \chi + (a) \text{tr} \chi \text{tr} \chi + 6\omega (a) \text{tr} \chi - 2 \ast \rho + 2e_1(\eta_2) + 2\Lambda \eta_2 + 8\eta_1 \eta_2) \right) a.
\]

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Putting the three lines together we obtain
\[-e_4 e_3(a) + \Delta a + \left( -\frac{1}{2} \chi \chi + 4 \omega \right) - i \left( a \chi \right) e_4(a) + \left( -\frac{5}{2} \chi \chi \right) + i \left( -3 \chi \chi \right) \] 
\[\left( 2 \eta_1 - i 4 \eta_2 \right) e_1(a) + \left( -6 \eta_2 + 4 i (A - \eta_1) \right) e_2(a) \]
\[-4 \Lambda^2 - tr \chi \chi + 2 \left( a \chi \right) tr \chi + 10 \omega \chi \chi + 2 \rho + G_0 + 4 \nabla_4 \omega - 2 e_1(\eta_1) + 6 \Lambda \eta_1 - 4 \eta_1 \eta_1 + 4 \eta_2 \eta_2 \]
\[+ i \left( -e_4(\chi) tr \chi - \frac{3}{2} \chi tr \chi - \frac{3}{2} \chi tr \chi + 10 \omega (a \chi) tr \chi - 2 * \rho + 2 e_1(\eta_2) - 10 \Lambda \eta_2 + 8 \eta_1 \eta_2 \right) \] \[= 0. \]

Recall that for a scalar, we can write
\[\square_{\text{gKerr}} a = -e_4 e_3 a - \frac{1}{2} tr \chi e_3 a - \frac{1}{2} tr \chi e_3 a + \Delta a + 2 \eta_1 e_1(a) + 2 \eta_2 e_2(a). \] (125)

We finally obtain
\[\square_{\text{gKerr}} a = \left( -4 \omega + i (a \chi) \right) e_4(a) + \left( 2 tr \chi + 3 i (a \chi) \right) e_3(a) \]
\[+ i 4 \eta_2 e_2(a) + \left( 8 \eta_2 - 4 i (A - \eta_1) \right) e_2(a) + V a \]

where
\[V = 4 \Lambda^2 + tr \chi \chi - 2 \left( a \chi \right) tr \chi - 10 \omega tr \chi + 2 \rho - G_0 + 4 \nabla \omega + 2 e_1(\eta_1) - 6 \Lambda \eta_1 + 4 \eta_1 \eta_1 - 4 \eta_2 \eta_2 \]
\[+ i \left( e_4(\chi) tr \chi + \frac{3}{2} \chi tr \chi + \frac{3}{2} \chi tr \chi - 10 \omega (a \chi) tr \chi + 2 * \rho - 2 e_1(\eta_2) + 10 \Lambda \eta_2 - 8 \eta_1 \eta_2 \right) \]

is the potential.

Writing
\[\nabla \omega = \rho + (\eta - \eta) \cdot \zeta - \eta \cdot \eta = \rho - (\eta - \eta) \cdot \eta - \eta \cdot \eta = \rho + \eta \cdot \eta - 2 \eta \cdot \eta \]
\[= \rho + \eta_1 \eta_1 + \eta_2 \eta_2 - 2 \eta_1 \eta_1 - 2 \eta_2 \eta_2 \]
\[= \rho - \eta_1 \eta_1 + \eta_2 \eta_2 \]
\[= \rho - \eta_1 \eta_1 + 3 \eta_2 \eta_2 \]

and \[G_0 = 2 \frac{1}{2} tr \chi tr \chi + \frac{1}{2} (a \chi) tr \chi + 2 \rho, \] we obtain for the real part of the potential:
\[\Re(V) = 4 \Lambda^2 + 2 \frac{1}{2} tr \chi tr \chi - 5 \left( a \chi \right) tr \chi - 10 \omega tr \chi + 8 \rho + 2 e_1(\eta_1) - 6 \Lambda \eta_1 + 8 \eta_1 \eta_1 - 16 \eta_2 \eta_2 \]

which coincides with (119).

For the imaginary part of the potential, we write
\[e_4(\chi) tr \chi = - \frac{1}{2} (a \chi) tr \chi + tr \chi (a \chi) tr \chi + 2 \text{curl} \eta + 2 * \rho. \]

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Observe that
\[
\text{curl } \eta = \nabla_1 \eta_2 - \nabla_2 \eta_1 = e_1(\eta_2) - \eta_2 \nabla_1 \eta_2 - (e_2 \eta_1 - \eta_2 \nabla_2 \eta_1)
\]
which gives
\[
e_4 (a) \text{tr} \chi = -\frac{1}{2} (a) \text{tr} \chi \text{tr} \chi + \text{tr} \chi (a) \text{tr} \chi + 2 e_1(\eta_2) + 2 \Lambda \eta_2 + 2 \ast \rho.
\]
We therefore obtain
\[
\Im(V) = (a) \text{tr} \chi \text{tr} \chi - 10 \omega (a) \text{tr} \chi + 4 \ast \rho + 12 \Lambda \eta_2 - 8 \eta_1 \eta_2
\]
which coincides with (120).

\[\square\]

B Proof of Proposition 8.6

We write here the proof of Proposition 8.6, which consists in computing the commutator \([Q, \mathcal{L}] A\).

B.1 The commutators for \(Q\)

We first compute the following commutators for \(Q\), where
\[
Q(U) = (c) \nabla_3 (c) \nabla_3 U + \text{C}(c) \nabla_3 U + \text{D} U.
\]

Proposition B.1. Let \(U\) be a symmetric traceless two tensor of conformal type \(s\). Then we have:

- the following commutator with \((c) \nabla_3\) and \((c) \nabla_4\):

\[
[Q, (c) \nabla_3] U = (\nabla_3 (c) \nabla_3 U + (\nabla_3 (c) \nabla_3 U) - (c) \nabla_3 \text{tr} \chi (\eta - \eta) \cdot (c) \nabla_3 U
\]

\[
+ 2 \left( (c) \nabla_3 (\eta - \eta) + \left( -\frac{1}{2} \text{tr} \chi + \text{C} \right) (\eta - \eta) \right) \cdot (c) \nabla U - (a) \text{tr} \chi (\eta - \eta) \cdot (c) \nabla U
\]

\[
+ (c) \nabla_3 (\text{C}_{3,4}(U)) + \text{C}_{3,4} (c) \nabla_3 U) + \left( 2 (\eta - \eta) \cdot \eta - (c) \nabla_4 (\text{C}) \right) (c) \nabla_3 U
\]

\[
+ 2 (\eta - \eta) \cdot \text{C}_{3,4}(U) + \text{C} \text{C}_{3,4}(U) - (c) \nabla_4 (\text{D}) U + \frac{a}{p^2} \Gamma_g \delta^{1 \leq 1} U,
\]

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• the following commutator with \( (c)\nabla_a \):

\[
[Q, (c)\nabla_a]U_{bc} = 2\eta_a (c)\nabla_3 (c)\nabla_3 U_{bc} - tr_X (c)\nabla_a (c)\nabla_3 U_{bc} - (a) tr_X * (c)\nabla_a (c)\nabla_3 U_{bc} \\
+ \left( - (c)\nabla_a (C) + (c)\nabla_3 \eta_a + (C - \frac{1}{2} tr_X) \eta_a - \frac{1}{2} (a) tr_X * \eta_a \right) (c)\nabla_3 U_{bc} \\
+ (c)\nabla_3 C^0_{3,a} (U) + C^0_{3,a} (c)\nabla_3 U \\
+ \left( - \frac{C}{2} tr_X + \frac{1}{2} tr_X^2 - \frac{1}{2} (a) tr_X^2 \right) (c)\nabla_a U_{bc} \\
+ \left( - \frac{C}{2} (a) tr_X + tr_X (a) tr_X \right) * (c)\nabla_a U_{bc} \\
- \frac{1}{2} tr_X C^0_{3,a} (U) - \frac{1}{2} (a) tr_X * C^0_{3,a} (U) + C C^0_{3,a} (U) - (c)\nabla_a (D) U_{bc} \\
+ (c)\nabla_3 \left( \frac{a}{r} \Gamma_g U + \Gamma_g a \leq 1 U \right),
\]

• the following commutator with \( (c)\bar{\mathcal{D}} \):

\[
[Q, (c)\bar{\mathcal{D}}]U = 2\bar{H} \cdot (c)\nabla_3 (c)\nabla_3 U - tr_X (c)\nabla_3 U + \frac{1}{2} tr_X (tr_X - C) (c)\bar{\mathcal{D}} \cdot U \\
+ \left( (c)\nabla_3 \bar{H} + (- (s - 2) tr_X + C) \bar{H} - (c)\mathcal{D} (C) \right) \cdot (c)\nabla_3 U \\
+ \left( \frac{1}{2} (s - 2) tr_X \left( (c)\nabla_3 \bar{H} + (tr_X - C) \bar{H} \right) - (c)\bar{\mathcal{D}} (D) \right) \cdot U \\
+ (c)\nabla_3 \left( \frac{a}{r} \Gamma_g U + \Gamma_g a \leq 1 U \right).
\]

Let \( F \) be a one form of conformal type \( s \). Then we have

• the following commutator with \( (c)\bar{\mathcal{D}} \):

\[
[Q, (c)\bar{\mathcal{D}}]\hat{F} = 2\bar{H} \hat{\otimes} (c)\nabla_3 (c)\nabla_3 F - tr_X (c)\nabla_3 F + \frac{1}{2} (tr_X) (tr_X - C) (c)\bar{\mathcal{D}} \hat{\otimes} F \\
+ \left( (c)\nabla_3 \bar{H} + (- (s + 1) tr_X + C) \bar{H} - (c)\mathcal{D} (C) \right) \hat{\otimes} (c)\nabla_3 F \\
+ \left( \frac{1}{2} (s + 1) tr_X \left( (c)\nabla_3 \bar{H} + (tr_X - C) \bar{H} \right) - (c)\bar{\mathcal{D}} (D) \right) \hat{\otimes} F \\
+ (c)\nabla_3 \left( \frac{a}{r} \Gamma_g F + \Gamma_g a \leq 1 F \right).
\]

Proof. We compute

\[
[Q, (c)\nabla_3]U = (c)\nabla_3 (c)\nabla_3 + C (c)\nabla_3 + D) (c)\nabla_3 U - (c)\nabla_3 (c)\nabla_3 (c)\nabla_3 U + C (c)\nabla_3 U + D U \\
= ( - (c)\nabla_3 C) (c)\nabla_3 U + (- (c)\nabla_3 D) U
\]

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as stated. Similarly, we compute

\[ [Q, (c)\nabla_4]U = (c)\nabla_3[(c)\nabla_3, (c)\nabla_4]U + [(c)\nabla_3, (c)\nabla_4](c)\nabla_3U + C [(c)\nabla_3, (c)\nabla_4]U - (c)\nabla_4(C) (c)\nabla_3U - (c)\nabla_4(D) U. \]

Recall from Lemma 5.8,

\[ [(c)\nabla_3, (c)\nabla_4]U = 2((\eta - \eta) \cdot (c)\nabla)U + C^0_{3,4}(U) \]

where

\[ C^0_{3,4}(U) = ((s - 2)P + (s + 2)P - 2s\eta \cdot \eta)U - 4\eta \hat{\otimes}(\eta \cdot U) + 4\eta \hat{\otimes}(\eta \cdot U) + 1.0.t. \]

is a zero-th order term in \( U \). In particular, since \((c)\nabla_3U\) is conformal of type \( s - 1 \), we have

\[ \nabla_3(c)\nabla_4 | (c)\nabla_3U = 2((\eta - \eta) \cdot (c)\nabla) (c)\nabla_3U + C^0_{3,4}(c)\nabla_3U. \]

On the other hand we compute

\[ \nabla_3(c)\nabla_4 U = 2((\eta - \eta) \cdot (c)\nabla) U + (c)\nabla_3(C^0_{3,4}(U)) + \text{l.o.t.} \]

Using Lemma 5.8, we have

\[ [(c)\nabla_3, (c)\nabla_a]U_{bc} = \frac{1}{2} \text{tr} (c)\nabla_a U_{bc} - \frac{1}{2} (a)\text{tr} (c)\nabla_a U_{bc} + \eta_a (c)\nabla_3 U_{bc} + C^0_{3,a}(U) \]

where

\[ C^0_{3,a}(U) = -\frac{1}{2} \text{tr} (s(\eta_a)U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta \cdot U)_c - \delta_{ac}(\eta \cdot U)_b) \]

\[ -\frac{1}{2} (a)\text{tr} \left( s(\eta_a)U_{bc} + \eta_b \ast U_{ac} + \eta_c \ast U_{ab} - \varepsilon_{ab}(\eta \cdot U)_c - \varepsilon_{ac}(\eta \cdot U)_b \right) \]

\[ + \frac{a}{r} \Gamma_g U + \Gamma g \delta^{\leq 1} U \]

We can therefore write

\[ \nabla_3(c)\nabla_4 U = 2((\eta - \eta) \cdot (c)\nabla) U + (c)\nabla_3(C^0_{3,4}(U)) + \frac{a}{r} \left( B + \frac{a}{r} \Gamma_g \right) U + \frac{a}{r^2} \Gamma g \delta^{\leq 1} U. \]

Putting the above together, we obtain the desired formula.

We compute

\[ [Q, (c)\nabla_a]U_{bc} = (c)\nabla_3[((c)\nabla_3, (c)\nabla_a)U_{bc}) + [(c)\nabla_3, (c)\nabla_a] \nabla_3 U_{bc} + C [(c)\nabla_3, (c)\nabla_a]U_{bc} \]

\[ - (c)\nabla_a(C) (c)\nabla_3 U_{bc} - (c)\nabla_a(D) U_{bc}. \]
Using
\[
[(c)\nabla_3, (c)\nabla_a]U_{bc} = -\frac{1}{2}\text{tr} \chi (c)\nabla_a U_{bc} - \frac{1}{2}(a)\text{tr} \chi^* (c)\nabla_a U_{bc} + \eta_a (c)\nabla_c U_{bc} + C_{3,a}^0(U) + \frac{a}{r} \Gamma_g U + \Gamma_g \delta^{1} U
\]
we have
\[
[(c)\nabla_3, (c)\nabla_a] (c)\nabla_3 U_{bc} = -\frac{1}{2}\text{tr} \chi (c)\nabla_a (c)\nabla_3 U_{bc} - \frac{1}{2}(a)\text{tr} \chi^* (c)\nabla_a (c)\nabla_3 U_{bc} + \eta_a (c)\nabla_3 (c)\nabla_3 U_{bc} + C_{3,a}^0 (c)\nabla_3 U_{bc} + \frac{a}{r} \Gamma_g (c)\nabla_3 U_{bc} + \Gamma_g \delta^{1} (c)\nabla_3 U_{bc}
\]
and
\[
(c)\nabla_3 ([ (c)\nabla_3, (c)\nabla_a] U_{bc}) = -\frac{1}{2}\text{tr} \chi (c)\nabla_3 (c)\nabla_a U_{bc} - \frac{1}{2}(c)\nabla_3 \text{tr} \chi (c)\nabla_a U_{bc}
- \frac{1}{2}(a)\text{tr} \chi (c)\nabla_3 \ast (c)\nabla_a U_{bc} - \frac{1}{2}(c)\nabla_3 (a)\text{tr} \chi^* (c)\nabla_a U_{bc}
+ \eta_a (c)\nabla_3 (c)\nabla_3 U_{bc} + (c)\nabla_3 \eta_a (c)\nabla_3 U_{bc} + (c)\nabla_3 C_{3,a}^0 (U) + (c)\nabla_3 \left( \frac{a}{r} \Gamma_g U + \Gamma_g \delta^{1} U \right)
\]
\[
\]
\[
= -\frac{1}{2}\text{tr} \chi (c)\nabla_3 (c)\nabla_3 U_{bc} - \frac{1}{2}\text{tr} \chi (c)\nabla_3 U_{bc} - \frac{1}{2}(a)\text{tr} \chi^* (c)\nabla_3 U_{bc} + \eta_a (c)\nabla_3 U_{bc} + C_{3,a}^0(U))
- \frac{1}{2} \left( \frac{1}{2} \text{tr} \chi^2 + \frac{1}{2}(a)\text{tr} \chi^2 \right) (c)\nabla_a U_{bc}
- \frac{1}{2}(a)\text{tr} \chi^* (c)\nabla_3 (c)\nabla_3 U_{bc} - \frac{1}{2}\text{tr} \chi (c)\nabla_3 U_{bc} - \frac{1}{2}(a)\text{tr} \chi^* (c)\nabla_3 U_{bc} + \eta_a (c)\nabla_3 U_{bc} + C_{3,a}^0(U))
- \frac{1}{2} \left( \frac{1}{2} \text{tr} \chi^2 + \frac{1}{2}(a)\text{tr} \chi^2 \right) (c)\nabla_a U_{bc}
+ \eta_a (c)\nabla_3 (c)\nabla_3 U_{bc} + (c)\nabla_3 \eta_a (c)\nabla_3 U_{bc} + (c)\nabla_3 (c)\nabla_3 U_{bc} + (c)\nabla_3 \left( \frac{a}{r} \Gamma_g U + \Gamma_g \delta^{1} U \right)
\]
Putting the above together, we obtain the stated expression.

We compute
\[
[Q, (c)\nabla \cdot U] = (c)\nabla_3[(c)\nabla_3, (c)\nabla \cdot U] + [(c)\nabla_3, (c)\nabla \cdot U] (c)\nabla_3 U + C[(c)\nabla_3, (c)\nabla \cdot U]
- (c)\nabla \cdot (c)\nabla_3 U - (c)\nabla \cdot (c)\nabla_3 U \cdot U.
\]
Recall that for a two tensor $U$ of conformal type $s$:
\[
[(c)\nabla_3, (c)\nabla \cdot U] = -\frac{1}{2}\text{tr} \chi \left( (c)\nabla \cdot U + (s-2)(c)\nabla \cdot U \right) + (c)\nabla_3 U + \frac{a}{r} \Gamma_g U + \Gamma_g \delta^{1} U.
\]
In particular, since $(c)\nabla_3 U$ is conformal of type $s-1$, we have
\[
[(c)\nabla_3, (c)\nabla \cdot (c)\nabla_3 U] = -\frac{1}{2}\text{tr} \chi \left( (c)\nabla \cdot (c)\nabla_3 U + (s-3)(c)\nabla \cdot (c)\nabla_3 U \right) + (c)\nabla_3 (c)\nabla_3 U + \frac{a}{r} \Gamma_g (c)\nabla_3 U + \Gamma_g \delta^{1} (c)\nabla_3 U.
\]

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On the other hand we compute

\[
(c)\nabla_3\left( [ (c) \nabla_3, (c) \hat{D} ] U \right) = -\frac{1}{2} (c) \nabla_3 \text{tr} \left( (c) \hat{D} \cdot U + (s - 2) \hat{H} \cdot U \right)
\]

Putting the above together we obtain the desired formula.

We compute

\[
[Q, (c) \hat{D} \hat{\otimes}] F = (c) \nabla_3 \left( [ (c) \nabla_3, (c) \hat{D} \hat{\otimes} ] F \right) + [ (c) \nabla_3, (c) \hat{D} \hat{\otimes} ] (c) \nabla_3 F + C [ (c) \nabla_3, (c) \hat{D} \hat{\otimes} ] F
\]

Recall that for a one form of conformal type $s$:

\[
[ (c) \nabla_3, (c) \hat{D} \hat{\otimes} ] F = -\frac{1}{2} \text{tr} \left( (c) \hat{D} \hat{\otimes} F + (s + 1) H \hat{\otimes} F \right) + H \hat{\otimes} (c) \nabla_3 F.
\]

In particular, since $(c) \nabla_3 F$ is conformal of type $s - 1$, we have

\[
[ (c) \nabla_3, (c) \hat{D} \hat{\otimes} ] (c) \nabla_3 F = -\frac{1}{2} \text{tr} \left( (c) \hat{D} \hat{\otimes} (c) \nabla_3 F + (s) H \hat{\otimes} (c) \nabla_3 F \right) + H \hat{\otimes} (c) \nabla_3 (c) \nabla_3 F.
\]

On the other hand we compute

\[
(c) \nabla_3 \left( [ (c) \nabla_3, (c) \hat{D} \hat{\otimes} ] F \right) = -\frac{1}{2} (c) \nabla_3 \text{tr} \left( (c) \hat{D} \hat{\otimes} F + (s + 1) H \hat{\otimes} F \right)
\]

Putting the above together we obtain the desired formula.
where
\[
I = -[Q, (c)\nabla_4 (c)\nabla_3]A, \\
J = \frac{1}{2}[Q, (c)D \otimes (c)\overline{D}]A, \\
K = [Q, \left(-\frac{1}{2} \text{tr}X - 2\text{tr}\overline{X}\right) (c)\nabla_3]A, \\
L = [Q, -\frac{1}{2} \text{tr}X (c)\nabla_4]A, \\
M = [Q, (4H + H + \overline{H}) \cdot (c)\nabla]A, \\
N = [Q, (-\overline{\text{tr}X} \overline{X} + 2\overline{\Pi})]A + 2[Q, H \otimes \overline{H}]A.
\]

B.1.1 Expression for \( I \)

We have
\[
I = -[Q, (c)\nabla_4] (c)\nabla_3 A - (c)\nabla_4 ([Q, (c)\nabla_3]A).
\]

From Proposition B.1, we deduce
\[
[Q, (c)\nabla_4] (c)\nabla_3 A = 4(\eta - \overline{\eta}) \cdot (c)\nabla (c)\nabla_3 (c)\nabla_3 A \\
+ 2\left((c)\nabla_3(\eta - \overline{\eta}) + \left(\frac{1}{2} \text{tr}X + C\right)(\eta - \overline{\eta})\right) \cdot (c)\nabla (c)\nabla_3 A \\
- (a)\text{tr}X(\eta - \overline{\eta}) \cdot (c)\nabla (c)\nabla_3 A \\
+ (c)\nabla_3(C_0^0 (c)\nabla_3 A)) + C_3^0 (c)\nabla_3 (c)\nabla_3 A \\
+ \left(2(\eta - \overline{\eta}) \cdot \eta - (c)\nabla_4(C)\right) (c)\nabla_3 (c)\nabla_3 A \\
+ 2(\eta - \overline{\eta}) \cdot C_{3,a}^0 (c)\nabla_3 A + C C_3^0 (c)\nabla_3 A - (c)\nabla_4(D) (c)\nabla_3 A \\
+ \frac{a}{r^2} \Gamma_9 \theta^{0 \leq 1} (c)\nabla_3 A.
\]

We also deduce
\[
(c)\nabla_4([Q, (c)\nabla_3]A) = (c)\nabla_4((- (c)\nabla_3 C) (c)\nabla_3 A + (- (c)\nabla_3 D) A) \\
= (- (c)\nabla_3 C) (c)\nabla_4 (c)\nabla_3 A + (- (c)\nabla_4 (c)\nabla_3 C) (c)\nabla_3 A + (- (c)\nabla_3 D) (c)\nabla_4 A \\
+ (- (c)\nabla_4 (c)\nabla_3 D) A.
\]

We therefore obtain
\[
I = -4(\eta - \overline{\eta}) \cdot (c)\nabla (c)\nabla_3 (c)\nabla_3 A + I_{43} (c)\nabla_4 (c)\nabla_3 A \\
+ \hat{I}_{33}(A) + I_{a3} \cdot (c)\nabla (c)\nabla_3 A + I \cdot a3 \cdot (c)\nabla (c)\nabla_3 A + I_4 (c)\nabla_4 A + \hat{I}_3(A) + I_0 A \\
+ \frac{a}{r^2} \Gamma_9 \theta^{0 \leq 1} (c)\nabla_3 A \\
+ \frac{a}{r^2} \Gamma_9 \theta^{0 \leq 1} (c)\nabla_3 A.
\]
where

\[ I_{33} = (c)\nabla_3 C, \]
\[ \tilde{I}_{33}(A) = - (c)\nabla_3 (c)_{3,4}(c)\nabla_3 A) - c_{3,4}^0 (c)\nabla_3 (c)\nabla_3 A - (2(\eta - \eta) \cdot \eta - (c)\nabla_4 (C) (c)\nabla_3 (c)\nabla_3 A, \]
\[ I_{e3} = -2 \left( (c)\nabla_3 (\eta - \eta) + \frac{1}{2} \tr \chi + C \right) (\eta - \eta), \]
\[ I_{43} = (a)\tr \chi (\eta - \eta), \]
\[ I_4 = (c)\nabla_3 D, \]
\[ \tilde{I}_3(A) = -2(\eta - \eta) \cdot C_{3,4}^0 (c)\nabla_3 A) - C c_{3,4}^0 (c)\nabla_3 A) + \left( (c)\nabla_4 (D) + (c)\nabla_4 (c)\nabla_3 C \right) (c)\nabla_3 A, \]
\[ I_0 = (c)\nabla_4 (c)\nabla_3 D. \]

We now compute \( \tilde{I}_{33}(A) \) and \( \tilde{I}_3(A) \). Using (126) we write

\[ \tilde{C}_{3,4}^0 (c)\nabla_3 A) = (-P + 3P - 2\eta \cdot \eta) (c)\nabla_3 A - 4\eta \hat{\otimes} (\eta \cdot (c)\nabla_3 A) + 4\eta \hat{\otimes} (\eta \cdot (c)\nabla_3 A) \]
\[ C_{3,4}^0 (c)\nabla_3 (c)\nabla_3 A) = (-2P + 2P) (c)\nabla_3 (c)\nabla_3 A - 4\eta \hat{\otimes} (\eta \cdot (c)\nabla_3 (c)\nabla_3 A) + 4\eta \hat{\otimes} (\eta \cdot (c)\nabla_3 (c)\nabla_3 A). \]

We therefore have

\[ \tilde{I}_{33}(A) = - (c)\nabla_3 (-P + 3P - 2\eta \cdot \eta) (c)\nabla_3 A - (P + 3P - 2\eta \cdot \eta) (c)\nabla_3 (c)\nabla_3 A + 4 (c)\nabla_3 (c)\nabla_3 \eta \hat{\otimes} (\eta \cdot (c)\nabla_3 A) + 4\eta \hat{\otimes} (\eta \cdot (c)\nabla_3 A) \]
\[-4 (c)\nabla_3 \eta \hat{\otimes} (\eta \cdot (c)\nabla_3 A) - 4\eta \hat{\otimes} (\eta \cdot (c)\nabla_3 (c)\nabla_3 A) - 4\eta \hat{\otimes} (\eta \cdot (c)\nabla_3 (c)\nabla_3 A) \]
\[ + (2P - 2P) (c)\nabla_3 (c)\nabla_3 A + 4\eta \hat{\otimes} (\eta \cdot (c)\nabla_3 (c)\nabla_3 A) - 4\eta \hat{\otimes} (\eta \cdot (c)\nabla_3 (c)\nabla_3 A) \]
\[-2(\eta - \eta) \cdot \eta - (c)\nabla_4 (C) (c)\nabla_3 A. \]

To simplify \( \tilde{I}_3(A) \) we make use of the following lemma.

**Lemma B.2.** For a one form \( \xi \) and a two tensor \( U \), we have

\[ \xi \cdot C_{3,a}^0(U) = -\frac{s}{2} \left( (tr \chi \xi - (a)tr \chi \ast \xi \cdot \eta) U - \frac{1}{2} tr \chi (\eta \hat{\otimes} (\xi \cdot U) - \xi \hat{\otimes} (\eta \cdot U) \right) \]
\[ -\frac{1}{2} (a)tr \chi \left( -\eta \hat{\otimes} (\ast \xi \cdot U) + \ast \xi \hat{\otimes} (\eta \cdot U) \right) + r^{-1}(\Gamma_y)^{2\delta \leq 1} U. \]

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Proof. Using (127), we have

\[
(\xi \cdot C_{3,a}^{ij}(U))_{bc} = \xi_a \left[ -\frac{1}{2} \text{tr}_X \left( s(\eta_b)U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta \cdot U)_c - \delta_{ac}(\eta \cdot U)_b \right) \right.
\]

\[
- \frac{1}{2} (a) \text{tr}_X \left( s(\mathfrak{a}\eta_b)U_{bc} + \mathfrak{a}\eta_b *U_{ac} + \eta_c *U_{ab} - \epsilon_{ab}(\eta \cdot U)_c - \epsilon_{ac}(\eta \cdot U)_b \right) \]

\[
+ \frac{1}{r} \Gamma_g U + \Gamma_g \mathfrak{d} \mathfrak{d} ^1 U \]

\[
= - \frac{s}{2} \left( \text{tr}_X(\xi \cdot \eta) + (a) \text{tr}_X(\xi \cdot \eta) \right) U_{bc}
\]

Observe that

\[
\xi_a \eta_b U_{ac} + \xi_a \eta_c U_{ab} - \xi_b(\eta \cdot U)_c - \xi_c(\eta \cdot U)_b = \eta_b(\xi \cdot U)_c + \eta_c(\xi \cdot U)_b - \xi_b(\eta \cdot U)_c - \xi_c(\eta \cdot U)_b
\]

and

\[
\xi_a \eta_b *U_{ac} + \xi_a \eta_c *U_{ab} + *\xi_b(\eta \cdot U)_c + *\xi_c(\eta \cdot U)_b
\]

\[
= \eta_b(\xi \cdot *U)_c + \eta_c(\xi \cdot *U)_b + *\xi_b(\eta \cdot U)_c + *\xi_c(\eta \cdot U)_b
\]

\[
= - \eta(\xi \cdot U)_c - \eta_c(\xi \cdot U)_b + *\xi_b(\eta \cdot U)_c + *\xi_c(\eta \cdot U)_b
\]

The lemma easily follows. \(\square\)

Using the lemma, we have

\[
\tilde{I}_3(A) = \left( (\text{tr}_X(\eta - \mathfrak{b}) - (a) \text{tr}_X(\eta - \mathfrak{b})) \cdot \right) ^{(c)} \nabla_3 A
\]

\[
+ \text{tr}_X \left( \eta \hat{\otimes}((\eta - \mathfrak{b}) \cdot ^{(c)} \nabla_3 A) - (\eta - \mathfrak{b}) \hat{\otimes}(\eta \cdot ^{(c)} \nabla_3 A) \right)
\]

\[
+ (a) \text{tr}_X \left( - \eta \hat{\otimes}(\eta - \mathfrak{b}) \cdot ^{(c)} \nabla_3 A \right) + (\eta - \mathfrak{b}) \hat{\otimes}(\eta \cdot ^{(c)} \nabla_3 A)
\]

\[
- C \left( (-P + 3\mathfrak{F} - 2\eta \cdot \mathfrak{b}) \cdot ^{(c)} \nabla_3 A - 4\eta \hat{\otimes}(\eta \cdot ^{(c)} \nabla_3 A) + 4\eta \hat{\otimes}(\eta \cdot ^{(c)} \nabla_3 A) \right)
\]

\[
+ \left( ^{(c)} \nabla_4 (\mathbf{D}) + ^{(c)} \nabla_4 ^{(c)} \nabla_3 C \right) ^{(c)} \nabla_3 A.
\]

Therefore we have

\[
\tilde{I}_{33}(A) + \tilde{I}_3(A) = I_{33} ^{(c)} \nabla_3 ^{(c)} \nabla_3 A + I_{3} ^{(c)} \nabla_3 A + I_{1} ^{(c)} \nabla_3 A + I_{3} ^{(c)} \nabla_3 A + I_{1} ^{(c)} A
\]

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From Proposition B.1, we deduce, recalling that
\[ I = 3P - 5P + 4\eta \cdot \tilde{\eta} - 2|\eta|^2 + (c)\nabla_4(C), \]
\[ I_3 = -(c)\nabla_3(-P + 3P - 2\eta \cdot \eta) + (\text{tr}_\chi(\eta \cdot \eta) - (a)\text{tr}_\chi^*(\eta - \tilde{\eta})(\eta - C(-P + 3P - 2\eta \cdot \eta) \]
\[ + (c)\nabla_4(D) + (c)\nabla_4(c)\nabla_3C, \]
and

\[
\begin{align*}
I_{33}^a(A) &= 8(\eta \cdot (c)\nabla_3 (c)\nabla_4 (c)\nabla_3 A) - 8\tilde{\eta}(\eta \cdot (c)\nabla_3 (c)\nabla_3 A), \\
I_3^a(A) &= 4(c)\nabla_3 \tilde{\eta}(\eta \cdot (c)\nabla_3 A) + 4\tilde{\eta}( (c)\nabla_3 \eta \cdot (c)\nabla_3 A) \\
&\quad - 4(c)\nabla_3 \eta(\eta \cdot (c)\nabla_3 A) - 4\tilde{\eta}( (c)\nabla_3 \eta \cdot (c)\nabla_3 A) \\
&\quad + (a)\text{tr}_\chi\left((\eta \cdot (c)\nabla_3 A) - (\eta - \tilde{\eta}) (\eta \cdot (c)\nabla_3 A)\right) \\
&\quad + (a)\text{tr}_\chi\left(-\eta/(\eta - \tilde{\eta})(c)\nabla_3 A + (\eta - \tilde{\eta}) (c)\nabla_3 A\right) \\
&\quad - C(-4\eta \cdot (c)\nabla_3 A) + 4\tilde{\eta}(\eta \cdot (c)\nabla_3 A).
\end{align*}
\]

We finally write
\[
I = -4(\eta - \tilde{\eta}) \cdot (c)\nabla_3 (c)\nabla_3 A + I_{33} (c)\nabla_4 (c)\nabla_3 A
+ I_{33} (c)\nabla_4 (c)\nabla_3 A + I_{33}^a(A) + I_3 (c)\nabla_3 A + I_3 (c)\nabla_3 A + I_4 (c)\nabla_3 A
+ I_0 A + \frac{a}{\nu^2} \Gamma_3 \delta \leq 1 (c)\nabla_3 A + \text{1.o.t.}
\]
with the above expressions.

### B.1.2 Expression for $J$

We have
\[
J = \frac{1}{2}[Q, (c)\nabla_3 (c)\nabla_3 A] + \frac{1}{2}(c)\nabla_3 (Q, (c)\nabla_3 A).
\]

From Proposition B.1, we deduce, recalling that $(c)\nabla_3 A$ is a one form of conformal type 2:
\[
\begin{align*}
[Q, (c)\nabla_3 (c)\nabla_3 A] &= 2H \tilde{\otimes} (c)\nabla_3 (c)\nabla_3 (c)\nabla_3 A - \text{tr}_X (c)\nabla_3 (c)\nabla_3 (c)\nabla_3 A \\
&\quad + \frac{1}{2}(\text{tr}_X)(\text{tr}_X - C) (c)\nabla_3 (c)\nabla_3 (c)\nabla_3 A \\
&\quad + \left((c)\nabla_3 H + (-3\text{tr}_X + C)H - (c)\nabla_3 (c)\nabla_3 (c)\nabla_3 A \\
&\quad + \left(\frac{3}{2}\text{tr}_X \left(- (c)\nabla_3 H + (\text{tr}_X - C)H\right) - (c)\nabla_3 (c)\nabla_3 A
\right).}
\end{align*}
\]

We write
\[
(c)\nabla_3 (c)\nabla_3 A = (c)\nabla_3 A - \frac{1}{2}\text{tr}_X \left((c)\nabla_3 A\right) + H \cdot (c)\nabla_3 A,
\]

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\[(c)D \otimes (c)\nabla_3 (c)D \cdot A = (c)\nabla_3 (c)D \otimes (c)D \cdot A - [(c)\nabla_3, (c)D \otimes (c)D \cdot A]
\]
\[= (c)\nabla_3 (c)D \otimes (c)D \cdot A + \frac{1}{2} \text{tr}X (c)D \otimes (c)D \cdot A
\]
\[-H \otimes (c)\nabla_3 (c)D \cdot A + \frac{3}{2} \text{tr}X H \otimes (c)D \cdot A
\]
\[= (c)\nabla_3 (c)D \otimes (c)D \cdot A + \frac{1}{2} \text{tr}X (c)D \otimes (c)D \cdot A - H \otimes (c)D \otimes (c)\nabla_3 A
\]
\[-H \otimes (H \cdot (c)\nabla_3 A) + \left( \frac{3}{2} \text{tr}X + \frac{1}{2} \text{tr}X \right) H \otimes (c)D \cdot A,
\]
and
\[(c)\nabla_3 (c)\nabla_3 (c)D \cdot A = (c)\nabla_3 (c)D \cdot (c)\nabla_3 A + (c)\nabla_3 [(c)\nabla_3, (c)D \cdot A]
\]
\[= (c)D \cdot (c)\nabla_3 (c)\nabla_3 A + [ (c)\nabla_3, (c)D \cdot (c)\nabla_3 A + (c)\nabla_3 [(c)\nabla_3, (c)D \cdot A]
\]
\[= (c)D \cdot (c)\nabla_3 (c)\nabla_3 A
\]
\[-\frac{1}{2} \text{tr}X \left( (c)D \cdot (c)\nabla_3 A - H \cdot (c)\nabla_3 A \right) + H \cdot (c)\nabla_3 (c)\nabla_3 A
\]
\[+ \frac{1}{2} \text{tr}X \left( (c)D \cdot (c)\nabla_3 A + H \cdot (c)\nabla_3 A \right)
\]
\[= (c)D \cdot (c)\nabla_3 (c)\nabla_3 A + H \cdot (c)\nabla_3 (c)\nabla_3 A
\]
\[+ \frac{1}{2} \text{tr}X \cdot (c)D \cdot A + (c)\nabla_3 (c)\nabla_3 A,
\]
where we used the intermediate computations in Proposition B.1. Putting the above together we obtain
\[
\left[ Q, (c)D \otimes (c)D \cdot A \right] = -\text{tr}X (c)\nabla_3 (c)D \otimes (c)D \cdot A + 2H \otimes (c)D \cdot (c)\nabla_3 (c)\nabla_3 A
\]
\[-\frac{C}{2} \left( \text{tr}X \right) (c)D \otimes (c)D \cdot A + 4H \otimes (H \cdot (c)\nabla_3 (c)\nabla_3 A)
\]
\[+ \left( (c)\nabla_3 H + (-2\text{tr}X - 2\text{tr}X + C) H - (c)D (C) \right) \otimes (c)D \cdot (c)\nabla_3 A
\]
\[+ 2H \otimes (c)\nabla_3 H \cdot (c)\nabla_3 A)
\]
\[+ \left( (c)\nabla_3 H + (-2\text{tr}X + C) H - (c)D (C) \right) \otimes (H \cdot (c)\nabla_3 A)
\]
\[+ \left( \frac{3}{2} \text{tr}X + \frac{1}{2} \text{tr}X \right) \left( (c)\nabla_3 H - CH \right) + \left( \text{tr}X \text{tr}X + (\text{tr}X)^2 \right) H
\]
\[+ \frac{1}{2} \text{tr}X \cdot (c)D \cdot (c)D \cdot (c)D \cdot A.
\]
Using that
\[
F \otimes (c)D \cdot U = (F \cdot (c)D) U = 2F \cdot (c)\nabla U,
\]
\[
E \otimes (E \cdot U) = (E \cdot E) U,
\]
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the above becomes
\[
[Q, \langle \sigma \rangle D \otimes (\langle \sigma \rangle D \cdot A) = -\text{tr}X \langle \sigma \rangle \nabla_3 \langle \sigma \rangle D \otimes \langle \sigma \rangle D \cdot A + 4H \cdot \langle \sigma \rangle \nabla_3 \langle \sigma \rangle \nabla_3 A
\]
\[
- \frac{C}{2} (\langle \sigma \rangle D \otimes \langle \sigma \rangle D \cdot A + 4(H \cdot \overline{H}) \langle \sigma \rangle \nabla_3 \langle \sigma \rangle \nabla_3 A
\]
\[
+ 2 \left( \langle \sigma \rangle \nabla_3 H + (-2\text{tr}X - 2\text{tr}X + C) H - \langle \sigma \rangle D(C) \right) \cdot \langle \sigma \rangle \nabla_3 \nabla_3 A
\]
\[
+ 2H \otimes (\langle \sigma \rangle \nabla_3 H \cdot \langle \sigma \rangle \nabla_3 A)
\]
\[
+ \left( \langle \sigma \rangle \nabla_3 H + (-2\text{tr}X + C) H - \langle \sigma \rangle D(C) \right) \otimes (\overline{H} \cdot \langle \sigma \rangle \nabla_3 A)
\]
\[
+ 2 \left( \frac{3}{2} \text{tr}X + \frac{1}{2} \text{tr}X \right) \left( \langle \sigma \rangle \nabla_3 H - CH \right) + (\text{tr}X \text{tr}X + (\text{tr}X)^2) H
\]
\[
+ \frac{1}{2} \text{tr}X \langle \sigma \rangle D(C) - \langle \sigma \rangle D(D) \right) \cdot \langle \sigma \rangle \nabla A.
\]

From Proposition B.1, we deduce
\[
\langle \sigma \rangle D \otimes (\langle \sigma \rangle D \cdot A) = \langle \sigma \rangle D \otimes \left( 2H \cdot \langle \sigma \rangle \nabla_3 \langle \sigma \rangle \nabla_3 A - \text{tr}X \langle \sigma \rangle D \cdot \langle \sigma \rangle \nabla_3 A + \frac{1}{2} \text{tr}X (\text{tr}X - C) \langle \sigma \rangle D \cdot A
\]
\[
+ \left( \langle \sigma \rangle \nabla_3 H + C \overline{H} - \langle \sigma \rangle D(C) \right) \cdot \langle \sigma \rangle \nabla_3 A + \left( -\langle \sigma \rangle D(D) \right) \cdot A \right)
\]

which gives
\[
\langle \sigma \rangle D \otimes (\langle \sigma \rangle D \cdot A) = 2 \langle \sigma \rangle D \otimes (H \cdot \langle \sigma \rangle \nabla_3 \langle \sigma \rangle \nabla_3 A - \text{tr}X \langle \sigma \rangle D \otimes \langle \sigma \rangle D \cdot \langle \sigma \rangle \nabla_3 A - \langle \sigma \rangle D \text{tr}X \otimes \langle \sigma \rangle D \cdot \langle \sigma \rangle \nabla_3 A
\]
\[
+ \frac{1}{2} \text{tr}X (\text{tr}X - C) \langle \sigma \rangle D \otimes \langle \sigma \rangle D \cdot A + \langle \sigma \rangle D \left( \frac{1}{2} \text{tr}X (\text{tr}X - C) \right) \langle \sigma \rangle D \cdot A
\]
\[
+ \langle \sigma \rangle D \otimes \left( \langle \sigma \rangle \nabla_3 H + C \overline{H} - \langle \sigma \rangle D(C) \right) \cdot \langle \sigma \rangle \nabla_3 A + \langle \sigma \rangle D \otimes \left( -\langle \sigma \rangle D(D) \right) \cdot A \right).
\]

Writing
\[
\langle \sigma \rangle D \otimes \langle \sigma \rangle D \cdot \langle \sigma \rangle \nabla_3 A = \langle \sigma \rangle D \otimes \langle \sigma \rangle \nabla_3 \langle \sigma \rangle D \cdot A - \langle \sigma \rangle D \otimes \langle \sigma \rangle \nabla_3, \langle \sigma \rangle D \cdot A
\]
\[
= \langle \sigma \rangle \nabla_3 \langle \sigma \rangle D \otimes \langle \sigma \rangle D \cdot A + \frac{1}{2} (\text{tr}X + \text{tr}X) \langle \sigma \rangle D \otimes \langle \sigma \rangle D \cdot A - H \otimes (\langle \sigma \rangle D \cdot \langle \sigma \rangle \nabla_3 A)
\]
\[
- H \otimes (\langle \sigma \rangle D \cdot A) \left( \frac{3}{2} \text{tr}X + \frac{1}{2} \text{tr}X \right) H \otimes \langle \sigma \rangle D \cdot A
\]
\[
+ \frac{1}{2} \langle \sigma \rangle D(\text{tr}X) \otimes (\langle \sigma \rangle D \cdot A) - \langle \sigma \rangle D \otimes (\overline{H} \cdot \langle \sigma \rangle \nabla_3 A)
\]
we have
\[
(c)D \hat{\otimes} ([Q, \overline{(c)D}]A) = -\text{tr}X (c)D \hat{\otimes} \overline{(c)D} \cdot A + \frac{1}{2} \text{tr}X(-\text{tr}X - C)(c)D \hat{\otimes} 
\]
\[
+2(c)D \hat{\otimes} (\overline{H} \cdot (c)D \hat{\otimes} \overline{H}) \cdot (c)\nabla_3 A + \text{tr}X(c)D \hat{\otimes} (\overline{H} \cdot (c)\nabla_3 A) + \text{tr}XH \hat{\otimes} (\overline{H} \cdot (c)\nabla_3 A)
\]
\[
+ \left( \frac{1}{2} \text{tr}XH - (c)D \text{tr}X \right) \hat{\otimes} (\overline{(c)D} \cdot (c)\nabla_3 A)
\]
\[
+ \left( \left( -\frac{3}{2} \text{tr}X \text{tr}X - \frac{1}{2} \text{tr}X^2 \right) H + \frac{1}{2} \text{tr}X(c)D(\text{tr}X) - \frac{1}{2} (c)D(\text{tr}X) \right) \hat{\otimes} (\overline{(c)D} \cdot A)
\]
\[
+ (c)D \hat{\otimes} \left( (c)\nabla_3 \overline{H} + C \overline{H} - (c)\overline{D}(C) \right) \cdot (c)\nabla_3 A - (c)D \hat{\otimes} (\overline{(c)D}(D) \cdot A).
\]

Using Leibniz rules, the above simplifies to
\[
(c)D \hat{\otimes} ([Q, \overline{(c)D}]A)
\]
\[
= -\text{tr}X (c)D \hat{\otimes} \overline{(c)D} \cdot A + 4\overline{H} \cdot (c)\nabla \cdot (c)\nabla_3 A
\]
\[
+ \frac{1}{2} \text{tr}X(-\text{tr}X - C)(c)D \hat{\otimes} \overline{(c)D} \cdot A + 2\left( (c)D \cdot \overline{H} \right)(c)\nabla_3 A
\]
\[
+ \left( \frac{1}{2} \text{tr}X \eta - 2\overline{(c)D} \text{tr}X + 2\left( (c)\nabla_3 \overline{H} + \overline{C} \overline{H} - (c)\overline{D}(C) \right) \right) \cdot (c)\nabla \cdot (c)\nabla_3 A
\]
\[
+ \left( \frac{1}{2} \text{tr}X \overline{(c)D} \cdot \overline{H} + \overline{H} \cdot \overline{(c)D} \cdot (c)\nabla_3 \overline{H} + C \overline{H} - (c)\overline{D}(C) \right) \cdot (c)\nabla_3 A
\]
\[
+ 2 \left( \left( -\frac{3}{2} \text{tr}X \text{tr}X - \frac{1}{2} \text{tr}X^2 \right) H + \frac{1}{2} \text{tr}X(c)D(\text{tr}X) - \frac{1}{2} (c)D(\text{tr}X) \right) \cdot (c)\nabla A - \overline{(c)D} \cdot (\overline{(c)D}(D) \cdot A).
\]

Putting the above together we finally obtain
\[
J = -\left( \text{tr}X + \text{tr}X \right)(c)\nabla_3 \left( \frac{1}{2} (c)D \hat{\otimes} \overline{(c)D} \cdot A \right)
\]
\[
+ \left( -\frac{1}{2} \text{tr}X \text{tr}X - \frac{C}{2} (\text{tr}X + \text{tr}X) \right) \left( \frac{1}{2} (c)D \hat{\otimes} \overline{(c)D} \cdot A \right)
\]
\[
+ 4\eta \cdot (c)\nabla \cdot (c)\nabla_3 \cdot (c)\nabla_3 A + \tilde{J}_3 (c)\nabla_3 \cdot (c)\nabla_3 A + \tilde{J}_{31} (c)\nabla \cdot (c)\nabla_3 A
\]
\[
+ \tilde{J}_3 (c)\nabla_3 A + \tilde{J}_3 (c)\nabla A + \tilde{J}_0 A
\]
\[
+ (c)\nabla_3 \left( \frac{a}{r^2} \Gamma_0 \delta \leq 1 \cdot A + \frac{1}{r} \Gamma_0 \delta \leq 2 \cdot A \right)
\]
where
\[
\begin{align*}
\tilde{J}_{33} &= (c)D \cdot \bar{H} + 2H \cdot \bar{H}, \\
\tilde{J}_{30} &= (c)\nabla_3 H + (-2\text{tr } X - 2\text{tr } X + C) H - (c)D(C) + 2\text{tr } X \eta - (c)D \bar{X} + (c)\nabla_3 \bar{H} + C \bar{H} - (c)D(C), \\
\tilde{J}_3 &= \frac{1}{2}\text{tr } X ((c)D \cdot \bar{H} + H \cdot \bar{H}) + \frac{1}{2} (c)D \cdot ((c)\nabla_3 \bar{H} + C \bar{H} - (c)D(C)), \\
\tilde{J}_3^3 (A) &= H \hat{\otimes} ((c)\nabla_3 \bar{H} \cdot (c)\nabla_3 A) \\
&+ \frac{1}{2} \left( (c)\nabla_3 H + (-2\text{tr } X + C) H - (c)D(C) \right) \hat{\otimes} (\bar{H} \cdot (c)\nabla_3 A), \\
\tilde{J}_a &= \left( \frac{3}{2} \text{tr } X + \frac{1}{2} \text{tr } X \right) \left( - (c)\nabla_3 H - CH \right) + (\text{tr } X \bar{X} + (\text{tr } X)^2) H \\
&+ \frac{1}{2} \text{tr } X (c)D(C) - (c)D(D) + \left( - \frac{3}{2} \text{tr } X \text{tr } X - \frac{1}{2} \text{tr } X \right) H \\
&+ \frac{1}{2} \text{tr } X (c)D(\text{tr } X) - \frac{1}{2} (c)D(\text{tr } X C) - (c)D(D), \\
\tilde{J}_0 &= - \frac{1}{2} (c)D \cdot (c)D(D).
\end{align*}
\]

Recalling the definition (88) of \( \mathcal{L}(A) \), we write
\[
\frac{1}{2} (c)D \hat{\otimes} ((c)D \cdot A) = (c)\nabla_4 (c)\nabla_3 A + \left( \frac{1}{2} \text{tr } X + 2\text{tr } X \right) (c)\nabla_3 A + \frac{1}{2} \text{tr } X (c)\nabla_4 A \\
- (4H + H + \bar{H}) \cdot (c)\nabla A + (\text{tr } X \text{tr } X - 2\mathcal{P}) A - 2H \hat{\otimes} (\bar{H} \cdot A)
\]
which gives
\[
(c)\nabla_3 \left( \frac{1}{2} (c)D \hat{\otimes} ((c)D \cdot A) \right)
\]
\[
= (c)\nabla_3 (c)\nabla_4 (c)\nabla_3 A + \left( \frac{1}{2} \text{tr } X + 2\text{tr } X \right) (c)\nabla_3 (c)\nabla_3 A \\
+ (c)\nabla_3 \left( \frac{1}{2} \text{tr } X + 2\text{tr } X \right) (c)\nabla_3 A + \frac{1}{2} \text{tr } X (c)\nabla_3 (c)\nabla_4 A + \frac{1}{2} (c)\nabla_3 \text{tr } X (c)\nabla_4 A \\
- (4H + H + \bar{H}) \cdot (c)\nabla_3 (c)\nabla A - (c)\nabla_3 (4H + H + \bar{H}) \cdot (c)\nabla A \\
+ (\text{tr } X \text{tr } X - 2\mathcal{P}) (c)\nabla_3 A + (c)\nabla_3 (\text{tr } X \text{tr } X - 2\mathcal{P}) A \\
- 2 (c)\nabla_3 H \hat{\otimes} (\bar{H} \cdot A) - 2H \hat{\otimes} ((c)\nabla_3 \bar{H} \cdot A) - 2H \hat{\otimes} (\bar{H} \cdot (c)\nabla_3 A).
\]

Writing
\[
(c)\nabla_3 (c)\nabla_4 A = (c)\nabla_4 (c)\nabla_3 A + 2(\eta - \eta) \cdot (c)\nabla A + C_{3,4}^0 (A),
\]
\[
(c)\nabla_3 (c)\nabla_4 (c)\nabla_3 A = (c)\nabla_4 (c)\nabla_3 (c)\nabla_3 A + 2(\eta - \eta) \cdot (c)\nabla (c)\nabla_3 A + C_{3,4}^0 ((c)\nabla_3 A),
\]
\[
(c)\nabla_3 (c)\nabla A = (c)\nabla (c)\nabla_3 A - \frac{1}{2} \text{tr } X (c)\nabla A - \frac{1}{2} (a) \text{tr } X \cdot (c)\nabla A + \eta (c)\nabla_3 A + C_{3,4}(A),
\]
\[
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\]
we have
\[
\begin{align*}
\nabla_3 \left( \frac{1}{2} \mathcal{D} \otimes (\mathcal{D} \cdot A) \right) &= \nabla_4 (\mathcal{D} \otimes (\mathcal{D} \cdot A) + \frac{1}{2} \text{tr} X (\mathcal{D} \otimes (\mathcal{D} \cdot A) - \frac{1}{4} \text{tr} X^2 (\mathcal{D} \cdot A) \\
&+ \hat{J}_a \cdot (\mathcal{D} \cdot A) + \nabla_3 (\mathcal{D} \cdot A) + \hat{J}_3 (\mathcal{D} \cdot A) + \hat{J}_g (A) \\
&+ \hat{J}_a \cdot (\mathcal{D} \cdot A) + \nabla_3 (\mathcal{D} \cdot A) + \hat{J}_0 (A) + \hat{J}_g (A)
\end{align*}
\]
where
\[
\begin{align*}
\hat{J}_a &= 2(\eta - \eta) - (4H + H + \overline{H}) \\
\hat{J}_3 &= \frac{1}{2} \text{tr} X + 2\text{tr} X_3 \\
\hat{J}_g (A) &= -4\eta \hat{\otimes} (\eta \cdot \nabla_3 A) + 4\eta \hat{\otimes} (\eta \cdot \nabla_3 A) - 2H \hat{\otimes} (H \cdot \nabla_3 A) \\
\hat{J}_a &= \text{tr} X (\eta - \eta) - (\mathcal{D} \cdot A) + \frac{1}{2} \text{tr} X (4H + H + \overline{H}) \\
\hat{J}_g (A) &= \frac{1}{2} \text{tr} X (4H + H + \overline{H}) \\
\hat{J}_0 (A) &= \frac{1}{2} \text{tr} X (4H + H + \overline{H}) \cdot (4H + H + \overline{H}) \hat{\otimes} (\eta \cdot A) \\
&+ \frac{1}{2} \text{tr} X (4H + H + \overline{H}) \hat{\otimes} (\eta \cdot A) \\
&+ \frac{1}{2} \text{tr} X (4H + H + \overline{H}) \cdot (4H + H + \overline{H}) \hat{\otimes} (\eta \cdot A)
\end{align*}
\]
We therefore obtain
\[
J = - (\text{tr} X + \text{tr} X_3) (\mathcal{D} \otimes (\mathcal{D} \cdot A) + \mathcal{D} \cdot \mathcal{D} \cdot A + 4\eta \cdot \mathcal{D} \cdot \mathcal{D} \cdot A + \mathcal{D} \cdot \mathcal{D} \cdot \mathcal{D} \cdot A + \mathcal{D} \cdot \mathcal{D} \cdot \mathcal{D} \cdot A)
\]
where
\[
\begin{align*}
J_{43} &= -\frac{1}{2} \text{tr} X \text{tr} X - \frac{C + \text{tr} X}{2} (\text{tr} X + \text{tr} X_3), \\
J_4 &= \frac{1}{4} (\text{tr} X)^2 (\text{tr} X) - \frac{C}{4} \text{tr} X (\text{tr} X + \text{tr} X_3),
\end{align*}
\]
where

\[
\begin{align*}
J_{a3} &= - (\text{tr}X + \text{tr}\overline{X}) J_{a3} + \tilde{J}_{a3}, \\
J_{33} &= - (\text{tr}X + \text{tr}\overline{X}) J_{33} + \tilde{J}_{33}, \\
J_3 &= - (\text{tr}X + \text{tr}\overline{X}) J_3 + \left( -\frac{1}{2} \text{tr}X \text{tr}X - \frac{C}{2} (\text{tr}X + \text{tr}\overline{X}) \right) \left( \frac{1}{2} \text{tr}X + 2 \text{tr}X \right) + \tilde{J}_3, \\
J^{\alpha}_0(A) &= - (\text{tr}X + \text{tr}\overline{X}) (J^{\alpha}_0(A) + \tilde{J}^{\alpha}_0(A)), \\
J_a &= - (\text{tr}X + \text{tr}\overline{X}) J_a - \left( -\frac{1}{2} \text{tr}X \text{tr}X - \frac{C}{2} (\text{tr}X + \text{tr}\overline{X}) \right) (4H + H + \overline{H}) + \tilde{J}_a, \\
J^{\ast}_a &= - (\text{tr}X + \text{tr}\overline{X}) J^{\ast}_a, \\
J_0 &= - (\text{tr}X + \text{tr}\overline{X}) J_0 + \left( -\frac{1}{2} \text{tr}X \text{tr}X - \frac{C}{2} (\text{tr}X + \text{tr}\overline{X}) \right) (\text{tr}X \overline{X} - 2\overline{P}) + \tilde{J}_0, \\
J^{\ast}_0(A) &= - (\text{tr}X + \text{tr}\overline{X}) J^{\ast}_0(A) - \left( -\frac{1}{2} \text{tr}X \text{tr}X - \frac{C}{2} (\text{tr}X + \text{tr}\overline{X}) \right) (2H \hat{\otimes} (H \cdot A)).
\end{align*}
\]

**B.1.3 Expression for K**

Observe that

\[
Q(fg) = Q(f)g + fQ(g) + 2 (^{(c)}\nabla_3 f) (^{(c)}\nabla_3 g - D f g). \tag{131}
\]

We therefore obtain, for a scalar \( \mathcal{F} \):

\[
[Q, \mathcal{F} (^{(c)}\nabla_3) A] = (Q(\mathcal{F}) - D \mathcal{F})(^{(c)}\nabla_3) A + \mathcal{F}[Q, (^{(c)}\nabla_3) A] + 2 (^{(c)}\nabla_3 \mathcal{F})(^{(c)}\nabla_3)(^{(c)}\nabla_3) A
\]

\[
= \left( 2 (^{(c)}\nabla_3 \mathcal{F}) (^{(c)}\nabla_3) A + (Q(\mathcal{F}) - D \mathcal{F} - \mathcal{F}(^{(c)}\nabla_3 C)(^{(c)}\nabla_3) A + (- \mathcal{F}(^{(c)}\nabla_3 D)) A. \right)
\]

In particular,

\[
K = [Q, \mathcal{F} (^{(c)}\nabla_3) A \quad \text{for} \quad \mathcal{F} = -\frac{1}{2} \text{tr}X - 2\text{tr}X.
\]

We therefore obtain\(^{21}\)

\[
K = K_{33} (^{(c)}\nabla_3 (^{(c)}\nabla_3) A + K_3 (^{(c)}\nabla_3) A + K_0 A
\]

where

\[
K_{33} = 2 (^{(c)}\nabla_3) \left( -\frac{1}{2} \text{tr}X - 2\text{tr}X \right),
\]

\[
K_3 = Q \left( -\frac{1}{2} \text{tr}X - 2\text{tr}X \right) - D \left( -\frac{1}{2} \text{tr}X - 2\text{tr}X \right) - \left( -\frac{1}{2} \text{tr}X - 2\text{tr}X \right) \nabla_3 C,
\]

\[
K_0 = - \left( -\frac{1}{2} \text{tr}X - 2\text{tr}X \right) \nabla_3 D.
\]

\(^{21}\)The expression for \( K \) does not have error terms.
B.1.4 Expression for $L$

Using (131), we obtain for a scalar $\mathcal{E}$,

$$
[Q, \mathcal{E}_i \nabla_4]A = (Q(\mathcal{E}) - D \mathcal{E}) (c) \nabla_4 A + \mathcal{E}[Q, (c) \nabla_4]A + 2(c) \nabla_3 \mathcal{E}_i (c) \nabla_3 (c) \nabla_4 A
$$

which gives

$$
[Q, \mathcal{E}_i \nabla_4]A = \left(2(c) \nabla_3 \mathcal{E}_i\right) (c) \nabla_4 (c) \nabla_3 A + 4\mathcal{E}(\eta - \eta) \cdot (c) \nabla (c) \nabla_3 A
$$

$$
+ (Q(\mathcal{E}) - D \mathcal{E}) (c) \nabla_4 A
$$

$$
+ \left[2\mathcal{E} \left((c) \nabla_3 (\eta - \eta) + \left(\frac{1}{2} \text{tr} \chi + C\right) (\eta - \eta)\right) + 4\nabla_3 \mathcal{E}_i (\eta - \eta)\right] \cdot (c) \nabla A
$$

$$
- \mathcal{E}_i (c) \text{tr} \chi (\eta - \eta) \cdot (c) \nabla A
$$

$$
+ \mathcal{E}_i (c) \nabla_3 \left(C^{0}_{3,4}(A)\right) + \mathcal{E}_{3,4} (c) \nabla_4 A + \mathcal{E} \left(2(\eta - \eta) \cdot (\eta - \eta) \nabla_4 (C)\right) \cdot (c) \nabla_3 A
$$

$$
+ 2\mathcal{E}(\eta - \eta) \cdot C_{3,4}^{0}(A) + \left(2\nabla_3 \mathcal{E} + \mathcal{E} C\right) C_{3,4}^{0}(A) - \mathcal{E}_i (c) \nabla_4 (D) A + \mathcal{E}_i \frac{a}{r^3} \Gamma_y \varphi \leq 1 A.
$$

In particular,

$$
L = [Q, \mathcal{E}_i \nabla_4]A \quad \text{for} \quad \mathcal{E} = -\frac{1}{2} \text{tr} \chi.
$$

We therefore obtain

$$
L = L_{43} (c) \nabla_4 (c) \nabla_3 A + L_{a3} \cdot (c) \nabla (c) \nabla_3 A + L_4 \nabla_4 (c) A + L_a \cdot (c) \nabla A + L_{\ast a} \cdot (c) \nabla A
$$

$$
+ L_4 (c) \nabla_3 A + L_{4}^{0}(A) + L_0 A + L_0^{0}(A) + \frac{a}{r^3} \Gamma_y \varphi \leq 1 A.
$$

(132)

where

$$
L_{43} = 2(c) \nabla_3 \mathcal{E} = -\frac{1}{2} (\text{tr} \chi)^2,
$$

$$
L_{a3} = 4\mathcal{E}(\eta - \eta) = -2\text{tr} \chi (\eta - \eta),
$$

$$
L_4 = Q(\mathcal{E}) - D \mathcal{E} = -\frac{1}{4} (\text{tr} \chi)^3 + \frac{C}{4} (\text{tr} \chi)^2,
$$

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and

\[ L_a = -\text{tr}X \left( (c)\nabla_3 (\eta - \eta) + \left( \frac{1}{2} \text{tr} \chi + C \right) (\eta - \eta) \right) + \text{tr}X^2 (\eta - \eta), \]

\[ L_{*a} = \frac{1}{2} \text{tr}X (a) \text{tr} (\eta - \eta), \]

\[ L_3 = -\frac{1}{2} \text{tr}X \left( -P + 7P - 6\eta \cdot \eta + 2(\eta - \eta) \cdot \eta - (c)\nabla_4 (C) \right), \]

\[ L_0 = \text{tr}X \left( (\text{tr} \chi (\eta - \eta) - (a)\text{tr}_X (\eta - \eta) \cdot \eta) + \left( \frac{1}{2} \text{tr}X^2 - \frac{1}{2} \text{tr}XC \right) (4P - 4\eta \cdot \eta) \right. \]

\[ + \frac{1}{2} \text{tr}X (c)\nabla_4 (D) - \frac{1}{2} \text{tr}X (c)\nabla_4 (4P - 4\eta \cdot \eta), \]

\[ L_0^*(A) = -\text{tr}X \left[ -\frac{1}{2} \text{tr}X \left( \eta \bigotimes ((\eta - \eta) \cdot A) - (\eta - \eta) \bigotimes (\eta \cdot A) \right) \right. \]

\[ - \frac{1}{2} (a)\text{tr}_X \left( -\eta \bigotimes \big( (\eta - \eta) \cdot A \big) + \big( \eta - \eta \big) \bigotimes (\eta \cdot A) \right) \]

\[ \left. + \left( \frac{1}{2} \text{tr}X^2 - \frac{1}{2} \text{tr}XC \right) (4\eta \bigotimes (\eta \cdot A) + 4\eta \bigotimes (\eta \cdot A)) \right) \]

\[ - \frac{1}{2} \text{tr}X (-4(c)\nabla_3 \eta \bigotimes (\eta \cdot A) - 4\eta \bigotimes (c)\nabla_3 \eta \cdot A - 4(c)\nabla_3 \bigotimes (\eta \cdot A)) + 4\eta \bigotimes (c)\nabla_3 \eta \cdot A). \]

B.1.5 Expression for \( M \)

Observe that

\[ Q(F \cdot G) = Q(F) \cdot G + F \cdot Q(G) + 2(c)\nabla_3 F \cdot (c)\nabla_3 G - D \cdot F \cdot G. \tag{133} \]

We therefore obtain

\[ M = Q(4H + \bar{H} + \bar{H}) \cdot (c)\nabla A + (4H + \bar{H} + \bar{H}) \cdot \left[ Q, (c)\nabla \right] A \]

\[ + 2(c)\nabla_3 (4H + \bar{H} + \bar{H}) \cdot (c)\nabla \cdot (c)\nabla A - D \cdot (4H + \bar{H} + \bar{H}) \cdot (c)\nabla A \]

which gives

\[ M = M_{33} (c)\nabla_3 (c)\nabla_3 A + M_{a3} \cdot (c)\nabla (c)\nabla_3 A + M_{a3} \cdot (c)\nabla (c)\nabla_3 A + M^g_{\alpha} (A) \]

\[ + M_a \cdot (c)\nabla A + M_{*a} \cdot (c)\nabla A + M^g_{\alpha} (A) + r^{-2} (c)\nabla_3 \left( \frac{a}{r} \Gamma_5 A + \Gamma_5 \delta^{51} A \right), \tag{134} \]

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where
\[ M_{33} = 2(4H + H + \overline{H}) \cdot \eta, \]
\[ M_{a3} = 2(c) \nabla_3(4H + H + \overline{H}) - \text{tr} \chi(4H + H + \overline{H}), \]
\[ M_{*a3} = -(a) \text{tr} \chi(4H + H + \overline{H}), \]
\[ M_{3}^\alpha(A) = (4H + H + \overline{H}) \cdot \left( -\frac{1}{2} \nabla(C) + (c) \nabla_3 \eta + (C - \frac{1}{2} \text{tr} \chi) \eta - \frac{1}{2} (\text{tr} \chi) \eta \right) (c) \nabla_3 A \]
\[ + (4H + H + \overline{H}) \cdot \left( (c) \nabla_3 C_{3,a}^0(A) + C_{3,a}^0 (c) \nabla_3 A \right) + 2(c) \nabla_3 (4H + H + \overline{H}) \cdot \eta (c) \nabla_3 A, \]
\[ M_{a} = Q(4H + H + \overline{H}) + \left( -\frac{1}{2} \text{tr} \chi + \frac{1}{2} \text{tr} \chi^2 - \frac{1}{2} (a) \text{tr} \chi^2 \right) (4H + H + \overline{H}) \]
\[ - \text{tr} \chi (c) \nabla_3 (4H + H + \overline{H}) - D(4H + H + \overline{H}), \]
\[ M_{*a} = \left( -\frac{1}{2} (a) \text{tr} \chi + \frac{1}{2} \text{tr} \chi^2 \right) (4H + H + \overline{H}) - (a) \text{tr} \chi (c) \nabla_3 (4H + H + \overline{H}), \]
\[ M_{0}^a(A) = (4H + H + \overline{H}) \cdot \left( -\frac{1}{2} \text{tr} \chi C_{3,a}^0(A) - \frac{1}{2} (a) \text{tr} \chi C_{3,a}^0(A) + C C_{3,a}^0(A) - (c) \nabla_a(D) A_{bc} \right) \]
\[ + 2(c) \nabla_3 (4H + H + \overline{H}) \cdot C_{3,a}^0(A). \]

B.1.6 Expression for \( N \)

Recall that
\[ N = [Q, (-\text{tr} X \text{tr} X + 2\overline{P})] A + 2[Q, H \otimes \overline{H}] A. \]

Using (131) and the fact that
\[ Q(E \otimes (F \cdot U)) = Q(E) \otimes (F \cdot U) + E \otimes (Q(F) \cdot U) + E \otimes (F \cdot Q(U)) \]
\[ + 2(c) \nabla_3 E \otimes (c) \nabla_3 F \cdot U + 2(c) \nabla_3 E \otimes (c) \nabla_3 U \]
\[ + 2E \otimes (c) \nabla_3 F \cdot (c) \nabla_3 U - 2DE \otimes (F \cdot U) \]

we obtain
\[ N = Q \left( -\text{tr} X \text{tr} X + 2\overline{P} \right) A + 2(c) \nabla_3 \left( -\text{tr} X \text{tr} X + 2\overline{P} \right) \nabla_3 A - D \left( -\text{tr} X \text{tr} X + 2\overline{P} \right) A \]
\[ + 2Q(H) \otimes (\overline{H} \cdot A) + 2H \otimes (Q(H) \cdot A) + 4(c) \nabla_3 H \otimes (c) \nabla_3 \overline{H} \cdot A + 4(c) \nabla_3 H \otimes (\overline{H} \cdot (c) \nabla_3 A) \]
\[ + 4H \otimes (c) \nabla_3 \overline{H} \cdot (c) \nabla_3 A) - 4DH \otimes (\overline{H} \cdot A). \]

Therefore\(^{22}\)
\[ N = N_3 (c) \nabla_3 A + N_3^\alpha (A) + N_0 (A) \]
\[ + N_0^\alpha (A). \]

\(^{22}\)The expression for \( N \) does not have error terms.
where

\[
N_3 = 2 (c) \nabla_3 \left( -\frac{1}{\text{tr}X} \text{tr}X + 2\bar{P} \right),
\]

\[
N_3^2 (A) = 4 (c) \nabla_3 H \otimes (\bar{H} \cdot (c) \nabla_3 A) + 4 H \otimes (c) \nabla_3 \bar{H} \cdot (c) \nabla_3 A,
\]

\[
N_0 = Q \left( -\frac{1}{\text{tr}X} \text{tr}X + 2\bar{P} \right) - D \left( -\frac{1}{\text{tr}X} \text{tr}X + 2\bar{P} \right),
\]

\[
N_0^2 (A) = 2Q(H) \otimes (c) \nabla_3 \bar{H} \cdot A + 2H \otimes (Q(\bar{H}) \cdot A)
\]

\[
+ 4 (c) \nabla_3 H \otimes (c) \nabla_3 \bar{H} \cdot A - 4DH \otimes (\bar{H} \cdot A).
\]

### B.1.7 The commutator

Putting the above expressions together we obtain

\[
[Q, \mathcal{L}]A = I + J + K + L + M + N
\]

\[
= -4(\eta - \eta) \cdot (c) \nabla_3 (c) \nabla_3 A + 4\eta \cdot (c) \nabla_3 (c) \nabla_3 A - \left( \text{tr}X + \frac{1}{\text{tr}X} \right) (c) \nabla_3 (c) \nabla_3 A
\]

\[
+ (I_{43} + J_{43} + L_{43}) (c) \nabla_4 (c) \nabla_3 A + (I_{4} + J_{4} + L_{4}) (c) \nabla_4 A
\]

\[
+ (I_{33} + J_{33} + K_{33} + M_{33}) (c) \nabla_3 (c) \nabla_3 A + \left( I_{33} \right)
\]

\[
\]

\[
+ (I_{a3} + J_{a3} + L_{a3} + M_{a3}) \cdot (c) \nabla_3 (c) \nabla_3 A + (I \ast a3 + M \ast a3) \cdot \ast (c) \nabla_3 A
\]

\[
+ (I_{3} + J_{3} + K_{3} + L_{3} + N_{3}) (c) \nabla_3 A + (I_{3}^2 (A) + I_{3}^2 (A) + I_{3}^2 (A) + M_{0} (A) + M_{0} (A) + N_{0} (A))
\]

\[
+ (J_{a} + L_{a} + M_{a}) \cdot (c) \nabla A + (J \ast a + L \ast a + M \ast a) \cdot \ast (c) \nabla A
\]

\[
+ (I_{0} + J_{0} + K_{0} + L_{0} + N_{0}) \cdot A + J_{0}^2 (A) + L_{0}^2 (A) + M_{0}^2 (A) + N_{0}^2 (A)
\]

\[
+ \ast (c) \nabla_3 \left( \frac{1}{r} \Gamma \phi \delta \leq 2 A \right) + \text{l.o.t.}
\]

which gives

\[
[Q, \mathcal{L}]A = 4\eta \cdot (c) \nabla_3 (c) \nabla_3 A - \left( \text{tr}X + \frac{1}{\text{tr}X} \right) (c) \nabla_3 (c) \nabla_3 A
\]

\[
+ (I_{43} + J_{43} + L_{43}) (c) \nabla_4 (c) \nabla_3 A + (I_{4} + J_{4} + L_{4}) (c) \nabla_4 A
\]

\[
+ (I_{33} + J_{33} + K_{33} + M_{33}) (c) \nabla_3 (c) \nabla_3 A + \left( I_{33} \right)
\]

\[
\]

\[
+ (I_{a3} + J_{a3} + L_{a3} + M_{a3}) \cdot (c) \nabla_3 (c) \nabla_3 A + (I \ast a3 + M \ast a3) \cdot \ast (c) \nabla_3 A
\]

\[
+ (I_{3} + J_{3} + K_{3} + L_{3} + N_{3}) (c) \nabla_3 A + (I_{3}^2 (A) + I_{3}^2 (A) + I_{3}^2 (A) + M_{0} (A) + M_{0} (A) + N_{0} (A))
\]

\[
+ (J_{a} + L_{a} + M_{a}) \cdot (c) \nabla A + (J \ast a + L \ast a + M \ast a) \cdot \ast (c) \nabla A
\]

\[
+ (I_{0} + J_{0} + K_{0} + L_{0} + N_{0}) \cdot A + J_{0}^2 (A) + L_{0}^2 (A) + M_{0}^2 (A) + N_{0}^2 (A) + \text{Err}[Q, \mathcal{L}]A.
\]

Recalling the definition of \(Q(A)\):

\[
(c) \nabla_3 (c) \nabla_3 A = Q(A) - C (c) \nabla_3 A - D A
\]

we write

\[
(c) \nabla_3 (c) \nabla_3 A = (c) \nabla Q(A) - C (c) \nabla_3 A - (c) \nabla C (c) \nabla_3 A - D (c) \nabla A - (c) \nabla D A,
\]

\[
(c) \nabla_4 (c) \nabla_3 A = (c) \nabla_4 Q(A) - C (c) \nabla_4 (c) \nabla_3 A - D (c) \nabla_4 A - (c) \nabla_4 C (c) \nabla_3 A - (c) \nabla_4 D A.
\]

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Hence,
\[
[Q, \mathcal{L}]A = 4 \eta \cdot (^c\nabla Q(A) - (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) \ (^c\nabla \text{tr} \mathbf{X} ) + (I_{43} + J_{43} + L_{43} + C \ (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) \ (^c\nabla \text{tr} \mathbf{X} ) + (I_{4} + J_{4} + L_{4} + D \ (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) \ (^c\nabla A + C_{\alpha} \cdot (^c\nabla A) \ + \ C_{0} \ A + C_{0}^\alpha (A) + \text{Err}[Q, \mathcal{L}]A
\]

where

\begin{align*}
C_{Q} &= I_{33} + J_{33} + K_{33} + M_{33}, \\
C_{a3} &= I_{a3} + J_{a3} + L_{a3} + M_{a3} - 4C\eta, \\
C_{\ast a3} &= I_{\ast a3} + M_{\ast a3}, \\
C_{3} &= I_{3} + J_{3} + K_{3} + L_{3} + N_{3} - 4\eta \cdot (^c\nabla \mathbf{C} + (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) - C (I_{33} + J_{33} + K_{33} + M_{33}), \\
C_{3}^\alpha (A) &= I_{3}^\alpha (A) + J_{3}^\alpha (A) + L_{3}^\alpha (A) + M_{3}^\alpha (A) + N_{3}^\alpha (A) - 8\eta \cdot (^c\nabla \mathbf{A}) + 8\eta \cdot (^c\nabla \mathbf{A}), \\
C_{a} &= J_{a} + L_{a} + M_{a} - 4D\eta, \\
C_{\ast a} &= I_{\ast a} + L_{\ast a} + M_{\ast a}, \\
C_{0} &= I_{0} + J_{0} + K_{0} + L_{0} + N_{0} - 4\eta \cdot (^c\nabla \mathbf{D} + (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) - D (I_{33} + J_{33} + K_{33} + M_{33}), \\
C_{0}^\alpha (A) &= J_{0}^\alpha (A) + L_{0}^\alpha (A) + M_{0}^\alpha (A) + N_{0}^\alpha (A) - 8\eta \cdot (^c\nabla \mathbf{A}) + 8\eta \cdot (^c\nabla \mathbf{A}).
\end{align*}

Observe that the coefficients of \(^c\nabla \text{tr} \mathbf{X} \) and \(^c\nabla A \) are respectively given by

\begin{align*}
I_{43} + J_{43} + L_{43} + C \ (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) &= (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) - \frac{C}{2} (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) - tr \mathbf{X} tr \mathbf{X}, \\
I_{4} + J_{4} + L_{4} + D \ (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) &= (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) - \frac{C}{4} tr \mathbf{X} (\text{tr} \mathbf{X}).
\end{align*}

Therefore if the above are \( \Gamma_{g} \) and \( r^{-1}\Gamma_{g} \) respectively, as in the assumption of Proposition 8.6, i.e. in (99) and (100), the commutator becomes

\[
[Q, \mathcal{L}]A = 4 \eta \cdot (^c\nabla Q(A) - (\text{tr} \mathbf{X} + \overline{\text{tr} \mathbf{X}}) \ (^c\nabla \text{tr} \mathbf{X} ) + (C_{0} (Q(A)) + 8\eta \cdot (^c\nabla \mathbf{A}) + 8\eta \cdot (^c\nabla \mathbf{A}),
\]

which proves the highest order terms of (101).

**B.1.8 The linear lower order terms**

We now show that the coefficients of \(^c\nabla \mathbf{X} \) and \( A \) are \( O(|a|) \). We denote by \( O \left( \frac{|a|}{r^c} \right) \) any function which vanishes in Schwarzschild, as multiples of \( \eta, \eta \cdot (^c\text{tr} \mathbf{X} \cdot \rho, \) and has a \( r^{-c} \) fall-off
in $r$. In particular observe that, according to (102) and (103), we can write

$$C = 2tr \, \chi + O \left( \frac{|a|}{r^2} \right), \quad D = \frac{1}{2} tr \, \chi^2 + O \left( \frac{|a|}{r^3} \right).$$

We therefore have

$$(c) \, \nabla_4 C = -tr \, \chi tr \, \chi + 4\rho + O \left( \frac{|a|}{r^3} \right),$$

$$(c) \, \nabla_3 C = -tr \, \chi^2 + O \left( \frac{|a|}{r^3} \right),$$

$$(c) \, \nabla_4 (c) \, \nabla_3 C = tr \, tr \, \chi^2 - 4tr \, \chi^2 + O \left( \frac{|a|}{r^4} \right),$$

$$(c) \, \nabla_4 D = -\frac{1}{2} tr \, \chi^2 + 2tr \, \chi^2 + O \left( \frac{|a|}{r^4} \right),$$

$$(c) \, \nabla_3 D = -\frac{1}{2} tr \, \chi^3 + O \left( \frac{|a|}{r^4} \right),$$

$$(c) \, \nabla_4 (c) \, \nabla_3 D = \frac{3}{4} tr \, \chi^3 - 3tr \, \chi^2 \rho + O \left( \frac{|a|}{r^5} \right).$$

In what follows we omit to write the error terms since they can all be included in the above expression for $Err[[Q, L]|A]$.

We compute $C_3$. We compute

$$I_3 = - (c) \, \nabla_3 (-P + 3P - 2\eta \cdot \eta) + (tr \, \chi (\eta - \eta) - (a) \, \chi^* (\eta - \eta)) \cdot \eta - C (-P + 3P - 2\eta \cdot \eta) + (c) \, \nabla_4 (D) + (c) \, \nabla_4 (c) \, \nabla_3 C$$

$$= - (c) \, \nabla_3 (2\rho) - 2tr \, \chi (2\rho) - \frac{1}{2} tr \, \chi^2 - 2tr \, \chi^2 + tr \, \chi tr \, \chi^2 - 4tr \, \chi^2 + O \left( \frac{|a|}{r^4} \right)$$

$$= \frac{1}{2} tr \, \chi^2 - 3tr \, \chi^2 + O \left( \frac{|a|}{r^4} \right),$$

$$J_3 = - (tr \, \chi + tr \, \chi) \, J_3 + \left( -\frac{1}{2} tr \, \chi tr \, \chi - \frac{C}{2} (tr \, \chi + tr \, \chi) \right) \left( \frac{1}{2} tr \, \chi + 2tr \, \chi \right) + \dot{J}_3$$

$$= -2tr \, \chi \left( (c) \, \nabla_3 \left( \frac{5}{2} tr \, \chi \right) + (tr \, \chi) (tr \, \chi) \right) + \left( -\frac{5}{2} tr \, \chi^2 \right) \left( \frac{5}{2} tr \, \chi \right) + O \left( \frac{|a|}{r^4} \right)$$

$$= -\frac{23}{4} tr \, \chi tr \, \chi^2 - 10tr \, \chi^3 + O \left( \frac{|a|}{r^4} \right),$$

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\[ K_3 = Q \left( -\frac{1}{2} \mathrm{tr} X - 2 \mathrm{tr} \bar{X} \right) - D \left( -\frac{1}{2} \mathrm{tr} X - 2 \mathrm{tr} \bar{X} \right) + \left( -\frac{1}{2} \mathrm{tr} X - 2 \mathrm{tr} \bar{X} \right) (c) \nabla_3 C \]

\[ = (c) \nabla_3 (c) \nabla_3 \left( -\frac{5}{2} \mathrm{tr} \chi \right) + 2 \mathrm{tr} \chi (c) \nabla_3 \left( -\frac{5}{2} \mathrm{tr} \chi \right) + \frac{5}{2} \mathrm{tr} \chi (-\mathrm{tr} \chi^2) + O \left( \frac{|a|}{r^4} \right) \]

\[ = -\frac{5}{2} (c) \nabla_3 \left( -\frac{1}{2} \mathrm{tr} \chi \right) + 5 \mathrm{tr} \chi (c) \nabla_3 \left( -\frac{1}{2} \mathrm{tr} \chi \right) + 5 \mathrm{tr} \chi (-\mathrm{tr} \chi^2) + O \left( \frac{|a|}{r^4} \right) \]

\[ = -\frac{5}{4} \mathrm{tr} \chi^2 + 2 \mathrm{tr} \chi \left( -\frac{1}{2} \mathrm{tr} \chi + 2 \rho \right) - 3 \mathrm{tr} \chi \rho - 10 \mathrm{tr} \chi \rho + O \left( \frac{|a|}{r^4} \right) \]

\[ L_3 = -\frac{1}{2} \mathrm{tr} \chi \left( -P + 7 \overline{P} - 6 \eta \cdot \eta + 2(\eta - \eta) \cdot \eta - (c) \nabla_4 (C) \right) \]

\[ = \frac{1}{2} \mathrm{tr} \chi \left( 6 \rho - (\mathrm{tr} \chi \mathrm{tr} \chi + 4 \rho) \right) + O \left( \frac{|a|}{r^4} \right) \]

\[ = \frac{1}{2} \mathrm{tr} \chi \mathrm{tr} \chi^2 - \mathrm{tr} \chi \rho + O \left( \frac{|a|}{r^4} \right) \]

\[ N_3 = 2 (c) \nabla_3 \left( -\mathrm{tr} \chi \mathrm{tr} \chi + 2 \overline{P} \right) \]

\[ = 2 \left( - \mathrm{tr} \chi \nabla_3 \mathrm{tr} \chi + (c) \nabla_3 \mathrm{tr} \chi + 2 \nabla_3 \chi \rho \right) + O \left( \frac{|a|}{r^4} \right) \]

\[ = 2 \mathrm{tr} \chi \mathrm{tr} \chi^2 - 10 \rho \mathrm{tr} \chi + O \left( \frac{|a|}{r^4} \right) . \]

We compute

\[ I_{33} = 3P - 5 \overline{P} + 4 \eta \cdot \eta - 2 |\eta|^2 + (c) \nabla_4 (C) = -\mathrm{tr} \chi \mathrm{tr} \chi + 2 \rho + O \left( \frac{|a|}{r^3} \right) , \]

\[ J_{33} = -(\mathrm{tr} \chi + \overline{\mathrm{tr} \chi}) \left( \frac{1}{2} \mathrm{tr} X + 2 \overline{\mathrm{tr} X} \right) + (c) \nabla_3 \nabla_3 + 2 \nabla_3 \nabla_3 = -5 \mathrm{tr} \chi \mathrm{tr} \chi + O \left( \frac{|a|}{r^3} \right) , \]

\[ K_{33} = 2 (c) \nabla_3 \left( -\frac{1}{2} \mathrm{tr} X - 2 \overline{\mathrm{tr} X} \right) = \frac{5}{2} \mathrm{tr} \chi \mathrm{tr} \chi - 10 \rho + O \left( \frac{|a|}{r^3} \right) , \]

\[ M_{33} = 2 \left( \overline{4H} + H + \overline{H} \right) \cdot \eta = O \left( \frac{|a|}{r^4} \right) . \]
We finally obtain
\[ C_3 = I_3 + J_3 + K_3 + L_3 + N_3 - 4 \eta.\]

\[
C_3 = \frac{1}{2} \text{tr} \chi \text{tr} \chi^2 - 3 \text{tr} \chi \rho - \frac{23}{4} \text{tr} \chi \text{tr} \chi^2 - 10 \text{tr} \chi \rho - \frac{5}{4} \text{tr} \chi^2 \text{tr} \chi - \frac{1}{2} \text{tr} \chi \text{tr} \chi^2 - \text{tr} \chi \rho \\
+ 2 \text{tr} \chi \text{tr} \chi^2 - 10 \rho \text{tr} \chi + 2 \chi (-\text{tr} \chi \text{tr} \chi + 4 \rho) \\
- 2 \text{tr} \chi \left(-\text{tr} \chi \text{tr} \chi + 2 \rho - 5 \text{tr} \chi \text{tr} \chi + \frac{5}{2} \text{tr} \chi \text{tr} \chi - 10 \rho\right) + O \left(\frac{|a|}{r^5}\right)
\]

\[ = O \left(\frac{|a|}{r^5}\right).\]

We now compute \( C_0 \). We compute

\[
I_0 = (c) \nabla_4 (c) \nabla_3 D = \frac{3}{4} \text{tr} \chi \text{tr} \chi^3 - 3 \text{tr} \chi^2 \rho + O \left(\frac{|a|}{r^5}\right),
\]

\[
J_0 = - (\text{tr} X + \text{tr} X^2) \left((c) \nabla_3 (\text{tr} X \text{tr} X - 2 \mathcal{P}) + \frac{1}{2} \text{tr} X (4 \mathcal{P} - 4 \eta \cdot \eta)\right) \\
+ (\text{tr} \chi \left(4 \mathcal{H} + H + \mathcal{M}\right) - (a) \text{tr} \chi^* \left(4 \mathcal{H} + H + \mathcal{M}\right) \cdot \eta) \\
+ \left(-\frac{1}{2} \text{tr} X \text{tr} X - \frac{C}{2} (\text{tr} X + \text{tr} X^2) (\text{tr} X \text{tr} X - 2 \mathcal{P}) - \frac{1}{2} (c) D \cdot (c) D (D)\right) \]

\[
= - 2 \text{tr} \chi \left((c) \nabla_3 (\text{tr} \chi \text{tr} \chi - 2 \rho) + \frac{1}{2} \text{tr} \chi (4 \rho)\right) + \left(-\frac{5}{2} \text{tr} \chi^2\right) (\text{tr} \chi \text{tr} \chi - 2 \rho) + O \left(\frac{|a|}{r^5}\right)
\]

\[
= - \frac{1}{2} \text{tr} \chi \text{tr} \chi^3 - 9 \rho \text{tr} \chi^2 + O \left(\frac{|a|}{r^5}\right),
\]

\[
K_0 = - \left(-\frac{1}{2} \text{tr} X - 2 \text{tr} X^2\right) (c) \nabla_3 D = - \frac{5}{4} \text{tr} \chi \text{tr} \chi^3 + O \left(\frac{|a|}{r^5}\right),
\]

\[
L_0 = \text{tr} X \left((\text{tr} \chi (\eta - \eta) - (a) \text{tr} \chi^* (\eta - \eta)) \cdot \eta\right) + \left(\frac{1}{2} \text{tr} X^2 - \frac{1}{2} \text{tr} X C\right) (4 \mathcal{P} - 4 \eta \cdot \eta) \\
+ \frac{1}{2} \text{tr} X (c) \nabla_4 (D) - \frac{1}{2} \text{tr} X (c) \nabla_3 (4 \mathcal{P} - 4 \eta \cdot \eta) \\
= - 2 \text{tr} \chi^2 \rho + \frac{1}{2} \text{tr} \chi \left(-\frac{1}{2} \text{tr} \chi \text{tr} \chi^2 + 2 \text{tr} \chi \rho\right) - 2 \text{tr} \chi (c) \nabla_3 (\rho) + O \left(\frac{|a|}{r^5}\right)
\]

\[
= - \frac{1}{4} \text{tr} \chi \text{tr} \chi^3 + 2 \text{tr} \chi^2 \rho + O \left(\frac{|a|}{r^5}\right),
\]

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\[ N_0 = Q \left(-\overline{\text{tr}}X\text{tr}X + 2\overline{P}\right) - D \left(-\overline{\text{tr}}X\text{tr}X + 2\overline{P}\right) \]
\[ = (c)\nabla_3 (c)\nabla_3 (\text{tr} \chi \text{tr} \chi + 2\rho) + 2\text{tr} \chi (c)\nabla_3 (\text{tr} \chi \text{tr} \chi + 2\rho) \]
\[ = (c)\nabla_3 (\text{tr} \chi \text{tr} \chi^2 - 5\rho \text{tr} \chi) + 2\text{tr} \chi (\text{tr} \chi \text{tr} \chi^2 - 5\rho \text{tr} \chi) + O \left(\frac{|a|}{r^5}\right) \]
\[ = \left(-\frac{1}{2} \text{tr} \chi \text{tr} \chi + 2\rho\right) \text{tr} \chi^2 + 2\text{tr} \chi \text{tr} \chi \left(-\frac{1}{2} \text{tr} \chi^2\right) - 5 \left(-\frac{3}{2} \text{tr} \chi \rho\right) \text{tr} \chi - 5\rho \left(-\frac{1}{2} \text{tr} \chi^2\right) \]
\[ + 2\text{tr} \chi (\text{tr} \chi \text{tr} \chi^2 - 5\rho \text{tr} \chi) + O \left(\frac{|a|}{r^5}\right) \]
\[ = \frac{1}{2} \text{tr} \chi \text{tr} \chi^3 + 2\rho \text{tr} \chi^2 + O \left(\frac{|a|}{r^5}\right) . \]

We finally obtain
\[ C_0 = I_0 + J_0 + K_0 + L_0 + N_0 - 4\eta_1 (c)\nabla D + (c)\nabla_4 D (\text{tr} \chi + \overline{\text{tr}X}) - D (I_{33} + J_{33} + K_{33} + M_{33}) \]
\[ = \frac{3}{4} \text{tr} \chi \text{tr} \chi^3 - 3\text{tr} \chi^2 \rho - \frac{1}{2} \text{tr} \chi \text{tr} \chi^3 - 9\rho \text{tr} \chi^2 - \frac{5}{4} \text{tr} \chi \text{tr} \chi^3 + 2\text{tr} \chi^2 \rho - \frac{1}{4} \text{tr} \chi \text{tr} \chi^3 \]
\[ + 2\rho \text{tr} \chi^2 + \frac{1}{2} \text{tr} \chi \text{tr} \chi^3 + 2\text{tr} \chi \left(-\frac{1}{2} \text{tr} \chi \text{tr} \chi^2 + 2\text{tr} \chi \rho\right) \]
\[ - \frac{1}{2} \text{tr} \chi^2 \left(\text{tr} \chi \text{tr} \chi + 2\rho - 5\rho \text{tr} \chi + \frac{5}{2} \text{tr} \chi \text{tr} \chi - 10\rho\right) + O \left(\frac{|a|}{r^5}\right) \]
\[ = O \left(\frac{|a|}{r^5}\right) . \]

This implies that all the terms involving \((c)\nabla_3 A\) and \(A\) are \(O(|a|)\). We can write
\[ C_3 (c)\nabla_3 A + C_3^a (A) + C_0 A + C_0^a (A) = a \left(d_2 (c)\nabla_3 A + d_4 A\right) \]
with
\[ d_2 = O \left(\frac{1}{r^4}\right), \quad d_4 = O \left(\frac{1}{r^5}\right) . \]

**B.2 The structure of the linear lower order terms: proof of Lemma 8.7**

Observe that in Kerr
\[ \eta_1 = \Re(H_1) = -\Re(Z_1) = -\Re(Z_1) = \eta_1, \]
\[ \eta_2 = \Re(H_2) = \Re(Z_2) = \Re(Z_2) = -\eta_2, \]
therefore
\[ (\eta - \eta) \cdot (c)\nabla = \left(\eta_1 - \eta_1\right) (c)\nabla_1 + \left(\eta_2 - \eta_2\right) (c)\nabla_2 = 2\eta_2 (c)\nabla_2 = a d(r, \theta) (c)\nabla_2, \]

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with $d(r, \theta)$ is $O \left( \frac{1}{r} \right)$. Similarly,

$$( \ast \eta + \ast \eta ) \cdot (c) \nabla = \alpha a d(r, \theta) (c) \nabla_2.$$ 

We hereby show that there is a choice of $C$, which satisfies the assumptions of Proposition 8.6, such that the coefficient of the term $\ast (c) \nabla (c) \nabla_3 A$ are multiple of $(\eta - \eta)$ or $(\ast \eta + \ast \eta)$. This implies that

$$C_{a3} \cdot (c) \nabla (c) \nabla_3 A + C_{a3} \ast \ast (c) \nabla (c) \nabla_3 A$$

where $d(r, \theta)$ are generic functions of $r$ and $\theta$. In particular, there is no term of the form $(c) \nabla 1 (c) \nabla_3 A$. In what follows we omit to write the error terms since they can all be included in the above expression for $Err[Q, \mathcal{L}] A$.

We denote $O(\eta + \eta)$ and $O(\ast \eta - \ast \eta)$ any expression which is linear combination of $\eta + \eta$ and $\ast \eta - \ast \eta$.

We have

$$I_{a3} = \left( (a) \text{tr}_X (\eta - \eta) = (a) \text{tr}_X (\eta + \eta - 2\eta) = -2 (a) \text{tr}_X \eta + O(\eta + \eta), \right.$$ 

$$M_{a3} = \left( - (a) \text{tr}_X (4H + H + H) \right.$$

This gives

$$C_{a3} = I_{a3} + M_{a3}$$

Therefore

$$C_{a3} \ast (c) \nabla 3 A = \left( - 4i (a) \text{tr}_X \ast \eta + O(\eta + \eta) + O(\ast \eta - \ast \eta) \right) \ast (c) \nabla (c) \nabla_3 A$$

To compute $C_{a3}$, we recall

$$I_{a3} = -2 \left( (c) \nabla_3 (\eta - \eta) + \left( \frac{1}{2} \text{tr}_X + 2 \text{tr}_X \right) (\eta - \eta) \right) = O(\eta - \eta).$$
\[ J_{a3} = - (\text{tr}X + \text{tr}X^\ast) (2(\eta - \eta) - (4H + H + H)) + (c) \nabla_3 H + (-2\text{tr}X - 2\text{tr}X^\ast + C) H \\
- (c) D(C) + 2\text{tr}X^\ast - (c) D\text{tr}X + (c) \nabla_3 H + C H - (c) D(C) \]
\[ = 2\text{tr}X (4\eta + 4i \ast \eta + 2\eta) + 2(c) \nabla_3 \eta + (-4\text{tr}X)(\eta + i \ast \eta) + 2(\text{tr}X + i (a) \text{tr}X)\eta - (c) D\text{tr}X + 2C \eta - 2(c) \nabla(C) + O(\eta - \eta). \]

Using Codazzi equation
\[ (c) D\text{tr}X = (\text{tr}X - \text{tr}X^\ast) H + O(\epsilon) = -2i (a) \text{tr}X(\eta + i \ast \eta) \]
and writing
\[ (c) \nabla_3 \eta = (c) \nabla_3 (\eta - \eta) + (c) \nabla_3 \eta \]
\[ = \frac{1}{2} (a) \text{tr}X(\ast \eta - \eta) + O(\eta - \eta) \]
\[ = (a) \text{tr}X \ast \eta + O(\eta - \eta) + O(\ast \eta + \eta) \]
we have
\[ J_{a3} = 2\text{tr}X (3\eta + 2i \ast \eta + 2\eta) + 2i (a) \text{tr}X(\eta + \eta) + 2C \eta - 2(c) \nabla(C) + O(\eta - \eta) + O(\ast \eta + \eta) \]
\[ = 2\text{tr}X (5\eta - 2i \ast \eta) + 4i (a) \text{tr}X \eta + 2C \eta - 2(c) \nabla(C) + O(\eta - \eta) + O(\ast \eta + \eta). \]

We also have
\[ L_{a3} = -2\text{tr}X(\eta - \eta) + O(\eta - \eta) \]
and since \(4H + H + H = 6\eta - 4i \ast \eta + O(\eta - \eta) + O(\ast \eta + \eta) \)
\[ M_{a3} = 2(c) \nabla_3 (4H + H + H) - \text{tr}X (4H + H + H) \]
\[ = 2(c) \nabla_3 (6\eta - 4i \ast \eta) - \text{tr}X (6\eta - 4i \ast \eta) + O(\eta - \eta, \ast \eta + \eta). \]

Using that
\[ (c) \nabla_3 \ast \eta = -\frac{1}{2} \text{tr}X(\ast \eta - \eta) + \frac{1}{2} (a) \text{tr}X(-\eta + \eta) \]
\[ = -\text{tr}X \ast \eta + O(\eta - \eta) + O(\ast \eta + \eta) \]
we have
\[ M_{a3} = 12(a) \text{tr}X \ast \eta + 8i(\text{tr}X \ast \eta) - \text{tr}X (6\eta - 4i \ast \eta) + O(\eta - \eta, \ast \eta + \eta) \]
\[ = 12(a) \text{tr}X \ast \eta + 12i(\ast \eta) - 6\text{tr}X \ast \eta + O(\eta - \eta) + O(\ast \eta + \eta). \]

Therefore, the coefficient of \( (c) \nabla(C) \nabla A \) is given by
\[ C_{a3} = L_{a3} + J_{a3} + M_{a3} - 4C \eta \]
\[ = 2\text{tr}X (2\eta + 4i \ast \eta) + 12(a) \text{tr}X \ast \eta + 4i(a) \text{tr}X \eta - 2C \eta - 2(c) \nabla(C) + O(\eta - \eta) + O(\ast \eta + \eta). \]
Putting together with the \( C \cdot \alpha_3 \) we obtain

\[
C_{\alpha_3} + C \cdot \alpha_3 = 2 \text{tr} \chi (2 \eta + 4i \ast \eta) + 12 (a) \text{tr} \chi \ast \eta + 8i (a) \text{tr} \chi \eta - 2C \eta - 2 (c) \nabla(C)
+ O(\eta - \eta) + O( \ast \eta + \ast \eta).
\]

We now use Lemma 5.4 and the form of \( C = 2 \text{tr} \chi + i \epsilon (a) \text{tr} \chi \) to compute the above coefficient. We have

\[
(c) \nabla \text{tr} \chi = \nabla \text{tr} \chi - \text{tr} \chi \zeta = \nabla \text{tr} \chi + \text{tr} \chi \eta = \frac{1}{2} \text{tr} \chi \eta - \frac{3}{2} (a) \text{tr} \chi (\ast \eta - \ast \eta),
\]

\[
(c) \nabla (a) \text{tr} \chi = \nabla (a) \text{tr} \chi - (a) \text{tr} \chi \zeta = \nabla (a) \text{tr} \chi + (a) \text{tr} \chi \eta = \frac{1}{2} (a) \text{tr} \chi \eta - \frac{3}{2} (a) \text{tr} \chi \eta + \frac{1}{2} \text{tr} \chi (\ast \eta - \ast \eta).
\]

This gives

\[
(c) \nabla C = 2 (c) \nabla \text{tr} \chi + i \epsilon (c) \nabla (a) \text{tr} \chi
= 2 \left( \frac{1}{2} \text{tr} \chi \eta - \frac{3}{2} \text{tr} \chi \eta - \frac{1}{2} (a) \text{tr} \chi (\ast \eta - \ast \eta) \right) + i \epsilon \left( \frac{1}{2} (a) \text{tr} \chi \eta - \frac{3}{2} (a) \text{tr} \chi \eta + \frac{1}{2} \text{tr} \chi (\ast \eta - \ast \eta) \right)
= (-\text{tr} \chi \eta - 3 \text{tr} \chi \eta - (a) \text{tr} \chi (\ast \eta - \ast \eta)) + i \left( \frac{\epsilon}{2} (a) \text{tr} \chi \eta - \frac{3\epsilon}{2} (a) \text{tr} \chi \eta + \frac{\epsilon}{2} \text{tr} \chi (\ast \eta - \ast \eta) \right).
\]

Going back to

\[
C_{\alpha_3} + C \cdot \alpha_3 = 2 \text{tr} \chi (2 \eta + 4i \ast \eta) + 12 (a) \text{tr} \chi \ast \eta + 8i (a) \text{tr} \chi \eta - 2C \eta - 2 (c) \nabla(C)
+ O(\eta - \eta) + O( \ast \eta + \ast \eta)
\]

\[
= 2 \text{tr} \chi \eta + 6 \text{tr} \chi \eta + 2 (a) \text{tr} \chi \ast \eta + 10 (a) \text{tr} \chi \eta
+ i(( - \epsilon + 8) (a) \text{tr} \chi \eta + 3 \epsilon (a) \text{tr} \chi \eta - \text{ctr} \chi \ast \eta + (\epsilon + 8) \text{tr} \chi \ast \eta)
+ O(\eta - \eta) + O( \ast \eta + \ast \eta)
\]

which can again be simplified to

\[
C_{\alpha_3} + C \cdot \alpha_3 = 8 \text{tr} \chi \eta + 8 (a) \text{tr} \chi \ast \eta + i((2\epsilon + 8) (a) \text{tr} \chi \eta + (2\epsilon + 8) \text{tr} \chi \ast \eta)
+ O(\eta - \eta) + O( \ast \eta + \ast \eta).
\]

From (69), we deduce

\[
0 = \frac{1}{2} \text{tr} \chi (\eta + \eta) - \frac{1}{2} (a) \text{tr} \chi (\ast \eta - \ast \eta) = \text{tr} \chi \eta + (a) \text{tr} \chi \ast \eta + O(\eta - \eta) + O( \ast \eta + \ast \eta).
\]

Therefore we can write \( (a) \text{tr} \chi \ast \eta + \text{tr} \chi \eta = O(\eta - \eta) + O( \ast \eta + \ast \eta) \), and obtain

\[
C_{\alpha_3} + C \cdot \alpha_3 = i((2\epsilon + 8) (a) \text{tr} \chi \eta + (2\epsilon + 8) \text{tr} \chi \ast \eta) + O(\eta - \eta) + O( \ast \eta + \ast \eta).
\]

Choosing \( \epsilon = -4 \), we obtain

\[
C_{\alpha_3} + C \cdot \alpha_3 = O(\eta - \eta) + O( \ast \eta + \ast \eta).
\]

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and therefore

\[ C_{a_3} \cdot (c) \nabla (c) \nabla_3 A + C \cdot a_3 \cdot * (c) \nabla (c) \nabla_3 A = a \cdot d_1 (c) \nabla_2 (c) \nabla_3 A \quad \text{for} \ d_1 = O \left( \frac{1}{r^3} \right). \]

By performing similar computations we obtain

\[ C_a \cdot (c) \nabla A + C \cdot a \cdot * (c) \nabla A = a \cdot d_3 (c) \nabla A \quad \text{for} \ d_3 = O \left( \frac{1}{r^4} \right). \]

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