Quark-like potentials in an extended Maxwell theory

Harry Schiff

Department of Physics, Theoretical Physics Institute, University of Alberta
Edmonton, Alberta, T6G 2J1.

Abstract

The exact Liénard-Wiechert solutions for the point charge in arbitrary motion are shown to be null fields everywhere. These are used as a basis to introduce extended electromagnetic field equations that have null field solutions with fractional charges that combine with absolute confining potentials.
I. INTRODUCTION

Some time ago I pointed out [1] that the Liénard-Wiechert solutions in the Lorentz gauge for a point charge $g$ in arbitrary motion, the following relation between fields and potentials holds everywhere

$$F_{\mu\nu}^2 = -2g^{-2}(A_\mu^2)^2,$$  \hfill (1.1)

where [2] $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, $x_4 = i$ it, so that

$$F_{\mu\nu}^2 = 2(H^2 - E^2), \quad A_\mu^2 = A^2 - \phi^2.$$  

By looking at (1.1) in the rest frame of $g$ it is clear that it applies only to coulomb solutions. In addition, the exact solutions satisfy $E \cdot H = 0$, i.e.,

$$F_{\mu\nu}F^*_{\mu\nu} = 0.$$  \hfill (1.2)

Writing (1.1) as

$$F_{\mu\nu}^2 + 2g^{-2}(A_\mu^2)^2 = 0,$$  \hfill (1.1)'

shows that it can be expressed as the square,

$$[F_{\mu\nu} + \sqrt{2}g^{-1}A_\mu A_\nu]^2 = 0$$  \hfill (1.3)

due to the opposite symmetries of $F_{\mu\nu}$ and $A_\mu A_\nu$. Defining the mixed tensor

$$G_{\mu\nu} \equiv F_{\mu\nu} + \sqrt{2}g^{-1}A_\mu A_\nu$$  \hfill (1.4)

and noting that $F_{\mu\nu}^* = G_{\mu\nu}^*$, (1.2) and (1.1)' can be written as

$$G_{\mu\nu}G^*_{\mu\nu} = 0,$$  \hfill (1.5)

$$G_{\mu\nu}G_{\mu\nu} = 0.$$  \hfill (1.6)

Consequently $G_{\mu\nu}$ satisfies the conditions for a null field, which hold everywhere for the exact solutions, compared to more general solutions of Maxwell’s equations for which only
the radiation field is null.

The null field $G_{\mu\nu}$ is interesting, stemming as it does from an exact solution to Maxwell’s equation, but is that its only significance or is there more, perhaps, that can be explored? This is the motivation for considering the non-gauge invariant equation,

$$\partial_\mu G_{\mu\nu} = 4\pi j_\nu \quad i.e.,$$

$$\partial_\mu (F_{\mu\nu} + \sqrt{2}g^{-1}A_\mu A_\nu) = 4\pi j_\nu,$$  \hspace{1cm} (1.7)

where $j_\nu$ is a localized 4-current. The total current thus consists of $j_\nu$ plus $-\sqrt{2}g^{-1}\partial_\mu A_\mu A_\nu$, an intrinsic contribution from the potentials, which are now physically significant since gauge invariance does not apply.

In this note I examine the simplest solution of (1.8), static and radially symmetric, which also satisfy the null conditions (1.5) and (1.6).

**II. SOLUTIONS OF (1.8)**

For all solutions considered here, $A = A_r(r)$, thus $\mathbf{H} = 0$ and the null condition (1.5) is identically satisfied. The magnetic part of (1.8) is then,

$$\sqrt{2}g^{-1}\nabla\cdot(AA) = 4\pi J(r),$$

with the simple solution of the homogeneous part,

$$A = k/r,$$  \hspace{1cm} (2.1)

$$A = k/r,$$  \hspace{1cm} (2.2)

where $k$ is arbitrary.

The linear electric field equation from (1.8) can be written

$$\nabla \cdot \mathbf{E} - \sqrt{2}g^{-1}\nabla \cdot (A\phi) = 4\pi \rho(r).$$

The homogeneous part of (2.3) becomes, with
\[ \gamma \equiv \sqrt{2}g^{-1}k, \]  
(2.4)

\[ A = \gamma g/\sqrt{2}r, \]  
(2.5)

\[ \frac{1}{r} \frac{d^2(r\phi)}{dr^2} + \frac{\gamma}{r^2} \frac{d(r\phi)}{dr} = 0. \]  
(2.6)

For any value of \( \gamma \), except \( \gamma = 1 \), there are two indicial solutions of (2.6),

\[ \phi_1 = \frac{b}{r}, \]  
(2.7)

\[ \phi_2 = c_2 r^{-\gamma}, \]  
(2.8)

consisting of a Coulomb potential and, for \( c_2 > 0, \gamma < 0 \), an absolute confining potential.

The total charge of the source in (2.3), using (2.7) and (2.8) with an application of the divergence theorem, is

\[ \int \rho d^3x = (1 - \gamma)b \]  
(2.9)

and is independent of the confining amplitude \( c_2 \).

Also, for any value of \( \gamma \) the total source current in (2.1), using the divergence theorem, is

\[ \int J(r)d^3x = \gamma^2 g/\sqrt{2}. \]  
(2.10)

For \( \gamma = 1 \) the two indicial solutions of (2.6) merge to a single Coulomb solution, so a second solution is needed. This is easily seen to be given by \( r\phi \sim logr \). The two independent solutions for \( \gamma = 1 \) are thus,

\[ \phi_3 = \frac{b}{r}, \]  
(2.11)

\[ \phi_4 = \frac{c_4 log(r/\alpha)}{r}. \]  
(2.12)

The logarithmic solution (2.12) can represent two possible confining potentials for \( c_4 > 0 \) and \( c_4 < 0 \), but not absolute ones. For \( c_4 > 0 \), \( \phi_4 \to -\infty \) as \( r \to 0 \) and peaks at \( r = e\alpha \),
while for $c_4 < 0$, $\phi_4 \to \infty$ as $r \to 0$, then forms a well of finite depth. For both cases $\phi_4 \to 0$ as $r \to \infty$. Although the Coulomb potential (2.11) is shown separately it can be absorbed in (2.12) by a redefinition of $\alpha$.

The total charge of the source (2.3) for $\gamma = 1$, using (2.11) and (2.12), with the application of the divergence theorem, yields

$$\int \rho \, d^3x = -c_4$$

(2.13)

and is independent of the Coulomb potential. However, due to the fact that (2.12) is not absolutely confining as well as the unsatisfactory charges resulting from $\gamma = 1$, indicated below, (2.12) will not be considered further.

Below, the null condition (1.6), or equivalently (1.1)', is applied using (2.5) and (2.7) for $\gamma < 0$, which is possible because both $A$ and $\phi$ go as $1/r$.

III. NULL FIELDS FOR $\gamma < 0$

For the static, radially symmetric solutions of (1.8) the null condition (1.1)’ becomes

$$E^2 + g^{-2}(A^2 - \phi^2)^2 = 0,$$

(3.1)

leading to the quadratic equation for the Coulomb charge $b$ in (2.7),

$$b^2 \pm gb - \gamma^2 g^2 / 2 = 0.$$

(3.2)

To avoid one set of $\pm$ signs from (3.2), the minus sign is chosen for solutions shown below; since $b$ changes sign with $g$ the remaining solutions are obtained directly for the $+$ sign in (3.2). Thus,

$$b_-^2 - gb_--\gamma^2 g^2 / 2 = 0$$

(3.3)

with solutions

$$b_-(1,2) = g/2[1 \pm \sqrt{(1 + 2\gamma^2)}].$$

(3.4)
For the two solutions \( b_- (1) \) and \( b_- (2) \) of (3.4) the following relations are seen to apply,

\[
b \propto g
\]

\[
b_- (1) + b_- (2) = g, \tag{3.5}
\]

\[
b_- (1) - b_- (2) = g \sqrt{1 + 2\gamma^2}, \tag{3.6}
\]

\[
b_- (1)/b_- (2) = \left(1 + \frac{1}{\sqrt{(1 + 2\gamma^2)}}\right)/\left(1 - \frac{1}{\sqrt{(1 + 2\gamma^2)}}\right). \tag{3.7}
\]

Choosing \( b_- (1)/b_- (2) = -2 \) and \( b_- (1) = 2/3 \) one gets \( \gamma^2 = 4, \ g = 1/3 \) and obviously \( b_- (2) = -1/3 \). Hence all solutions of (3.2) are,

\[
b_- (1, 2) = \pm(2/3, -1/3), \quad b_+ = \mp(2/3, -1/3) \tag{3.8}
\]

the results of the null fields solutions of (1.8) with \( g = \pm 1/3 \) and \( \gamma = -2 \), corresponding to a quadratic confining potential (2.8).

Null fields solutions also exist for which the charges \( \pm 2/3, \pm 1/3 \) appear singly, paired with charges of varying values. Below is a list of some of these results for corresponding values of \( g \) and \( \gamma \).

\[
g = \pm 1/3, \quad \gamma = -2\sqrt{3}, \quad b = (\mp/\pm)(1, -2/3)
\]

\[
g = \pm 2/3, \quad \gamma = -\sqrt{3}/2, \quad b = (\mp/\pm)(1, -1/3)
\]

\[
g = \pm 2/3, \quad \gamma = -2 \quad b = (\mp/\pm)(4/3, -2/3)
\]

\[
g = \pm 1, \quad \gamma = -2\sqrt{3/2}, \quad b = (\mp/\pm)(4/3, -1/3) \tag{3.9}
\]

Additional null fields exist for charges \( \pm 2/3, \pm 1/3 \) with paired charges greater than 4/3.

(With regard to \( \gamma = 1 \) the ratio of charges from (3.7) has the unsatisfactory values of \( \pm(1 + \sqrt{3})/(1 - \sqrt{3}). \))
IV. ANOTHER ‘SOLUTION’

In this section, another solution asymptotic in nature, is considered. Referring to (2.3), with \( \rho = 0 \), a solution evidently exists with \( \phi \) a constant and \( \nabla \cdot \mathbf{A} = 0 \). With the possibility that this is an asymptotic form to a more general solution for which \( \nabla \cdot \mathbf{A} = 0 \), a solution to (2.3) is sought with

\[
A = \beta / r^2, \tag{4.1}
\]

where \( \beta \) has dimension of length times charge.

Although (4.1) is not a solution of the homogeneous part of (2.1), it will approach it asymptotically. Inserting it into the left side of (2.1) gives

\[
\sqrt{2} g^{-1} \nabla \cdot (\mathbf{A} \mathbf{A}) = \sqrt{2} g^{-1} \beta^2 [-2/r^5 + 4\pi \delta(r)/r^2]. \tag{4.2}
\]

With (4.1) the electric field equation (2.3) becomes

\[
\frac{d}{dr} \left( \frac{d\phi}{dr} \right) + \frac{2}{r} \frac{d\phi}{dr} = -\frac{\sqrt{2} g^{-1} \beta}{r^2} \frac{d\phi}{dr} - 4\pi \delta(0) \phi(0) - 4\pi \rho(r). \tag{4.3}
\]

Dividing the homogeneous part by \( d\phi/dr \) and integrating, one finds

\[
-\frac{d\phi}{dr} = E = \frac{q}{r^2} \exp \left( \frac{\sqrt{2} g^{-1} \beta}{r} \right), \tag{4.4}
\]

where \( q \) is a constant of integration giving the potential

\[
\phi = \frac{gq}{\sqrt{2}\beta} \exp \left( \frac{\sqrt{2} g^{-1} \beta}{r} \right), \tag{4.5}
\]

making the obvious choice \( \sqrt{2} g^{-1} \beta < 0 \).

Asymptotically

\[
\phi \simeq \frac{gq}{\sqrt{2}\beta} + \frac{q}{r} + \frac{\sqrt{2} g^{-1} q \beta}{r^2} + \ldots, \tag{4.6}
\]

exhibiting a Coulomb potential \( q/r \), plus the constant \( gq/\sqrt{2}\beta \).

Since \( \phi(0) = 0 \) in (4.5), the corresponding expression in the inhomogeneous term of (4.3) is 0. Also, applying the divergence theorem to (2.3) with the asymptotic expression (4.6), shows consistency with the charge density \( \rho \) being 0.
The third term in (4.6), (which stretches the approximation) suggests a possible interpretation of a charge distribution, representing perhaps a penetration at the edge of a particle.

It is interesting that a charge from the ‘quark’ region, $g$, is reflected in a constant that appears in the asymptotic region; for a particle anti-particle pair the two associated constants would naturally cancel. One could raise the question, should the asymptotic constant appear in some particle reaction, whether it could influence the results of the reaction.

V. CONCLUSIONS

The extended electromagnetic equations (1.8) have the important property of having Coulomb solutions combined with absolute confining potentials. In addition, the Coulomb fields in the subset of null field solutions have charges with quark-like fractional values $\pm 2/3$ and $\pm 1/3$, which occur with specific values of the index in the confining potential. Thus the non-gauge invariant mixed tensor $G_{\mu\nu}$, through the extended Maxwell equations (1.8), is seen to be more significant than being only a null field of the Liénard-Wiechert solutions. The value of unity for the index has been used phenomenologically; the results obtained here analytically though, have definite values of the index. It would be of interest to see whether these values are significant.

A feature of the approximate asymptotic solution (4.6) is the appearance of a constant in the asymptotic region, which involves the basic charge $g$ in (1.8). A constant potential is irrelevant in the Maxwell case, but in a transition between the ‘quark’ region and the Maxwell domain, as seems to be suggested by the asymptotic solution, the effect of a constant may be relevant in some reactions.

It is an open question at present why the null fields are the important ones in the solutions of (1.8) for obtaining the familiar fractional charges, a more detailed treatment involving the dynamics of the system and its quantization may shed some light on this question.
ACKNOWLEDGMENTS

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REFERENCES

[1] H. Schiff, Can. Jour. of Phys. 47, 2387 (1969).

[2] See also G. Rosen, Int. Jour. of Theor. Phys. 18, No. 4, 305, (1979) for an application of (1.1).