THE FRECKLED INSTANTONS

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Abstract

We study instanton-corrected renormalization group flow in the two dimensional sigma models and four dimensional gauge theory. In two dimensions we do that by replacing the non-linear supersymmetric CP\textsuperscript{N−1} model by the gauged linear sigma model which is in the same universality class. We compare the moduli spaces of the instantons in the non-linear model and that of BPS field configurations in the linear model. We reduce the problem of matching of the parameters of the two systems to the intersection theory on the compact moduli space of the latter model. In four dimensions we find that the analogue of the linear sigma model is the gauge theory on the non-commutative space-time. Its BPS moduli space is the space of torsion free sheaves. Both cases (2d and 4d) are unified by the notion of the freckled instantons. We also put an end to the discussion of the nature of the superpotentials \( W \sim \sigma \log \sigma \) in 2d and 4d and discover the surprising disconnectnessness of the effective target space.

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1 Introduction

The comparison between the two dimensional sigma models and four dimensional gauge theories is fruitful for both subjects [39]. Our paper continues this line of the research.

Motivation. Our original motivation for the study is the interest in the properties of the renormalization group flow in the four dimensional gauge theories. This problem is both rich and has some chances to be exactly soluble in the context of the $\mathcal{N} = 2$ supersymmetric gauge theories.

Consider for simplicity the case of the pure $SU(N)$ (twisted) supersymmetric gauge theory. The problem is formulated as follows. Let $\Phi$ be the complex adjoint scalar in the vector multiplet. For an irreducible representation $V$ of $SU(N)$ let $O_V = \text{Tr}_V \Phi$ be the local operator in the gauge theory.

It commutes with the certain supercharge $Q$. By acting on $O_V$ with other supercharges one can construct non-local gauge- and $Q$-invariant observables $\int_{C_i} O_V^{(i)}$, $i = 0, 1, 2, 3, 4$, where the superscript $i$ denotes the degree of differential form, which is being integrated over a closed $i$-dimensional submanifold $C_i$ of the space-time manifold.

The problem [30] is now the following. The theory has low-energy effective description in terms of the $\mathcal{N} = 2$ theory with $r = N - 1$ abelian vector multiplets, whose scalars $a^i$, $i = 1, \ldots, r$ are the special coordinates on the moduli space of vacua which is identified with the base of the family of the hyperelliptic curves $C_u$:

$$ z + \frac{\Lambda^{2N}}{z} = x^N + u_1 x^{N-2} + \ldots + (-1)^{r+1} u_r $$

(1)

The parameters $u_k$ are identified with the traces of $\Phi$ in the representations $V_k = \Lambda^{k+1} O^N$:

$$ u_k = \text{Tr}_{V_k} \Phi $$

(2)

The problem is to find a representative for the operators $O_V$ and their descendants $\int_{C_i} O_V^{(i)}$ in terms of $u_k$ and other data of the low-energy theory. On the general grounds we expect to find a relation like:

$$ O_V \sim P_V(u_1, \ldots, u_r; \Lambda) $$

(3)

where $P_V(u_1, \ldots, u_r; \Lambda)$ is the polynomial whose value at $\Lambda = 0$ coincides with the classical expression of the trace in the representation $V$
via the traces in the fundamentals. This correspondence is the four dimensional generalization of the well-known quantum cohomology rings of two dimensional supersymmetric sigma models.

Scenario. It was suggested in [37] to study the renormalization group flow with instantons present by integrating out all fast fluctuating fields a l`a Wilson, including the small size instantons. Now suppose we have a theory I which has instantons of all sizes, and a theory II which has both instantons of all sizes and some other type of topological defects which we shall call freckles whose characteristic size is bounded from above by some parameter $\rho$. Then if we integrate out all the fluctuations of the wavelengths smaller then $\rho$ then the remaining instantonic field configurations in both theories I and II become identical – we say that the theories are in the same universality class. If for some reason the theory II is simpler then the theory I then we can use the correlation functions in this theory to compute the correlation functions in the theory I. One needs to express the couplings $T_s(\lambda)$ of the intermediate theory at the scale $\lambda \gg \rho$ through the ultraviolet couplings of the theories I and II:

$$T_s(\lambda) = f_I(T_s^I; \lambda), \quad T_s(\lambda) = f_{II}(T_s^{II}; \lambda), \quad \lambda \gg \rho \quad (4)$$

Upon exclusion of $T_s(\lambda)$ we arrive at the relation:

$$T_s^I = F(T_s^{II}; \lambda) \quad (5)$$

Below we confine ourselves with the examples of topologically twisted supersymmetric sigma models in 2d and gauge theories in 4d. In these cases (for $T_s$ being the topological couplings, see below) we expect the function $F(\cdot; \lambda)$ to be independent of $\lambda$ as long as $\lambda$ is greater then the characteristic size $\rho$ of the freckles which are specific for the model.

Plan. In the present paper we start to carry out this program for the supersymmetric $\mathbb{CP}^{N-1}$ model in two dimensions and then generalize to the four dimensional gauge theory. The theories I and II in these cases are respectively: non-linear and gauged linear sigma model in two dimensions, non-abelian gauge theory and (conjecturally) the gauge theory on the non-commutative space in four dimensions. In the case of two dimensional theory we find a subtle nature of the effective description of the theory II, namely we find that it can be described as Landau-Ginzburg theory on the disconnected space. Then we observe that the similar phenomenon occurs in the compactification of the four dimensional theory down to two dimensions on a two-sphere.
2 Two dimensional instantons with freckles

Let us take for the theory I the two dimensional supersymmetric nonlinear $\mathbb{CP}^{N-1}$ model. It has instantons represented by the holomorphic maps $\varphi : \Sigma \to X$, where $\Sigma$ is the worldsheet and $X \approx \mathbb{CP}^{N-1}$ is the target space. The space $\mathcal{M}$ of such instantons is disconnected and non-compact. The former property is due to the existence of the maps of various degree $d \in \pi(\Sigma, X) \approx \mathbb{Z}$ and it leads to the possibility of adding a theta term to the action. The non-compactness of $\mathcal{M}$ is a serious albeit mostly computational problem.

The correlation functions of certain observables $O_i$ in the theory are the integrals over $\mathcal{M}$ of certain differential forms $\omega_i$. Was $\mathcal{M}$ compact we could change $\omega_i$ by the exact forms without affecting the integral over $\mathcal{M}$. This is no longer possible for non-compact $\mathcal{M}$. The situation is not as innocent as one could think, since the exterior derivative on $\mathcal{M}$ is the remnant of the (twisted) supersymmetry $Q$ of the model. Usually one discards the $Q$-exact terms in the correlations function when all the operators are $Q$-closed. This operation is dangerous when one has to regularize integrals, precisely because the virtual boundary contributions may not vanish.

A way to compactify $\mathcal{M}$ is provided by the theory II for which we take the gauged linear sigma model with the gauge group $U(1)$ and $N$ charged chiral multiplets.

More generally, a theory II for the sigma model with the target space $X \approx \mathbb{C}^N//G$ is the gauged linear sigma model with $N$ chiral multiplets which transform in a certain representation of the gauge group $G$ and $//\,$ denotes the symplectic quotient (the quotient of the space of zeroes of the $D$-terms by the action of the gauge group $G$).

It turns out that the moduli space of the BPS field configurations $\Psi$ (i.e. the solutions to the equation $Q\Psi = 0$) in the gauged linear sigma model is the compact space $\overline{\mathcal{M}}$ which contains the space $\mathcal{M}$ as an open subspace.
2.1 BPS moduli spaces in theory I vs. that of II

In the case of $X \approx \mathbb{C}P^{N-1}$ the space $\overline{M} = \Pi_d \overline{M}_d$ looks as follows:

$$\overline{M}_d = \{(P_0, \ldots, P_{N-1})\}/\sim$$

where $P_k$ are the degree $d$ polynomials ($=$ holomorphic sections of the line bundle $\mathcal{O}(d)$ over $\Sigma$) which are not equal identically zero altogether and $\sim$ is the equivalence relation: $(P_0, \ldots, P_{N-1}) \sim (\lambda P_0, \ldots, \lambda P_{N-1})$, $\lambda \in \mathbb{C}^*$. Thus $\overline{M}_d \approx \mathbb{P}^{N(d+1)-1}$. The simplicity of this space is misleading. The catch is that not every $N$-tuple of the polynomials defines a map of $\mathbb{P}^1$ to $X$. Only polynomials without common divisors do that. Inside of $\overline{M}_d$ there is a space $\mathcal{M}_d$ of the polynomials without the common zeroes. Unfortunately in addition one finds a stratum $\mathcal{M}_{d-1} \times \mathbb{P}^1$ which consist of the polynomials with one common zero, one also finds a stratum $\mathcal{M}_{d-2} \times \mathbb{P}^2$ of polynomials with two common zeroes and so on, all the way down to the stratum $X \times \mathbb{P}^d$ which consists of the polynomials of the form $P_k(z) = a_k P(z)$ where $P(z)$ is an arbitrary degree $d$ polynomial and

$$(a_0 : a_1 : \ldots : a_{N-1})$$

is a point in $X$. The common zeroes of the polynomials $P_k$ are traditionally called vortices. However we shall also call them “freckles” for the reasons explained in the scenario.

In this way we arrive at the following stratification of $\overline{M}_d$:

$$\overline{M}_d = \mathcal{M}_d \cup \mathbb{P}^1 \times \mathcal{M}_{d-1} \cup \ldots \cup \mathbb{P}^d \times \mathcal{M}_0$$

Following [10] we refer to the points in $\overline{M}_d$ as to the quasimaps. However having in mind the four dimensional generalizations we suggest another name: the “freckled instanton”.

Notice that there are canonical maps “gluing of the point-like instantons”

$$v_l : \overline{M}_d \times \mathbb{P}^l \to \overline{M}_{d+l},$$

which actually add vortices-freckles to a quasimap:

$$v_l (P_0(z), \ldots, P_{N-1}(z)|x_1, \ldots, x_{l}) = (P_0(z)Q(z), \ldots, P_{N-1}(z)Q(z)),
\begin{equation}
Q(z) = (z-x_1)(z-x_2)\ldots(z-x_l)
\end{equation}$$
2.2 Great expectations: setup for the correlators

A typical computation in the (twisted) supersymmetric sigma model is the evaluation of the correlation function of the observables $O^{(0)}_\alpha$ and $\int_\Sigma O^{(2)}_\alpha$ where $O^{(0)}_\alpha$ are the zero-observables corresponding to the cohomology classes of $X$, and $O^{(2)}_\alpha$ are their two-descendants.

To be specific let us take $X \approx P^{N-1}$. The cohomology ring of $X$ is $N$ dimensional and is spanned by $\omega_r = \omega^r$, $r = 0, \ldots, N - 1$, where $\omega \in H^2(X)$ is the Kähler form. The interesting observables are the zero-observables $O^{(0)}_r \leftrightarrow \omega_r$ and the two-observables $\int_\Sigma O^{(2)}_r$, $r > 1$.

The two-observable of $\omega_1 \equiv \omega$ measures the degree of the holomorphic map:

$$\langle \cdots \int_\Sigma O^{(2)}_1 \rangle_d = d^{\langle \cdots \rangle}_d, \quad d = \int_\Sigma \varphi^* \omega$$

which is constant in the given instanton sector.

The correlation function

$$\langle O^{(0)}_{\alpha_1}(x_1) \cdots O^{(0)}_{\alpha_k}(x_k) \int_\Sigma O^{(2)}_{\beta_1} \cdots \int_\Sigma O^{(2)}_{\beta_l} \rangle$$

computes the number of the rigid holomorphic maps of $\Sigma$ to $X$ with the following conditions: the fixed points $x_1, \ldots, x_k$ on $\Sigma$ are mapped to the submanifolds $C^{\alpha_1}, \ldots, C^{\alpha_k} \subset X$ which represent the cycles which are Poincare dual to the cohomology classes $\omega_{\alpha_1}, \ldots, \omega_{\alpha_k}$ of $X$ while some points $y_1, \ldots, y_l$ whose position is not specified are mapped to the submanifolds $C^{\beta_1}, \ldots, C^{\beta_l} \subset X$ whose homology classes are dual to $\omega_{\beta_1}, \ldots, \omega_{\beta_l}$.

The answer is independent of the specific choice of $C^\alpha, C^\beta$ as long as everything remains in generic position.

In order to use the compactification $\mathcal{M}_d$ one has to formulate the computation of the correlation function (8) in terms of the linear gauged sigma model, more specifically, in terms of the $N$-tuples of the polynomials of degree $d$.

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3 This interpretation of the correlation function holds when one chooses the specific representatives for the cohomology classes $\omega_r$ of $X$, namely delta functions supported at $C^\alpha, C^\beta$. If one smoothes out the delta functions then at some point the supports of the smeared delta functions overlap and one should use more intricate arguments to show that the correlation function computes the same thing.
It is not hard to do that. The cycle in $\mathbb{P}^{N-1}$ which is Poincaré dual to $\omega_{r_\alpha}$ can be represented by a plane $C^{\alpha}$ which is the space of solutions to a system of linear equations $\ell_i^{\alpha} = 0$, $i = 1, \ldots, r_\alpha$ and each $\ell_i^{\alpha}$ is a section of $\mathcal{O}(1)$, i.e. the linear function in the homogeneous coordinates $w_0, \ldots, w_{N-1}$ on $X$:

$$\ell_i^{\alpha}(w_0, \ldots, w_{N-1}) = \sum_{\kappa=0}^{N-1} L_{i,\kappa}^{\alpha} w_\kappa, \quad L_{i,\kappa}^{\alpha} \in \mathbb{C}$$

Now the correlation function in the non-linear sigma model computes the number of such $N$-tuples $P_0, \ldots, P_{N-1}$ modulo the common multiple $\lambda \in \mathbb{C}^*$ and the points $y_1, \ldots, y_l \in \Sigma$ which obey the following equations:

$$\ell_i^{\alpha_p}(P_0(x_p), \ldots, P_{N-1}(x_p)) = 0, \quad i = 1, \ldots, r_{\alpha_p}, \quad p = 1, \ldots, k \quad (9)$$

$$\ell_i^{\beta_q}(P_0(y_q), \ldots, P_{N-1}(y_q)) = 0 \quad i = 1, \ldots, r_{\beta_q}, \quad q = 1, \ldots, l \quad (10)$$

If we relax the conditions on the polynomials $P_k$, i.e. allow them to have common divisors we replace the computation in the non-linear sigma model by that in the gauged linear sigma model.

The drastic difference between the spaces $\mathcal{M}_d$ and $\bar{\mathcal{M}}_d$ shows up in the following: if the polynomials $P_0, \ldots, P_{N-1}$ have a common zero $x_*$ then the equations (10) have solution e.g. $y_a = y_b = y_c = x_*$ for some $a \neq b \neq c$. These solutions have nothing to do with the properties of the actual holomorphic map defined by the polynomials $P_k$ with the common factor divided out. In other words they do not contribute to the solution of the enumerative problem posed by the non-linear model. Nevertheless they contribute to the correlation function of the linear model.

The accurate subtraction of these extra contributions is the heart of the relation (4) between the non-linear and the linear sigma models. The most notorious example of this renormalization is known under the name of the mirror map in the case of sigma models with Calabi-Yau threefolds as the target space [47, 18, 34].

In case of the manifold $X \approx \mathbb{P}^{N-1}$ (more generally if $X$ is a $D$-dimensional Fano variety with $\int_X \omega^{D-1} \wedge c_1(X) > D$) the correlations functions of the zero-observables $\mathcal{O}^{(0)}_\alpha$ can be rather easily computed by replacing the space $\mathcal{M}_d$ of the holomorphic maps by the space
The proof goes as follows: Suppose that the ghost numbers \( \Delta_{\alpha_l} \) of the observables \( O_{\alpha_1}^{(0)}, \ldots, O_{\alpha_k}^{(0)} \) (which coincides with the degree of the cohomology classes of \( X \) which they represent) saturate the ghost number anomaly in the instanton sector \( d \), i.e.

\[
\sum_l \Delta_{\alpha_l} = N(d + 1) - 1
\]

Since the positions of the zero-observables \( x_1, \ldots, x_k \in \Sigma \) are distinct the vortices can contribute to the correlation function \( \langle O_{\alpha_1}^{(0)}(x_1) \ldots O_{\alpha_k}^{(0)}(x_k) \rangle \) only if their location coincides with some of the points \( x_1, \ldots, x_k \). The vortices with the fixed location \( x_{i_1}, \ldots, x_{i_p} \) on \( \Sigma \) form a submanifold \( v_p(\mathcal{M}_{d-p} \times \{x_{i_1}, \ldots, x_{i_p}\}) \subset \mathcal{M}_d \) of codimension \( Np \). Each class in \( H^*(X) \) has degree which is less then \( N \) (in complex units). The complement to the vortices in \( \Sigma \) is mapped to \( X \) with the degree \( d - p \) therefore the rest of the observables must be saturated by the zero modes which are present in the holomorphic maps of the degree \( d - p \), i.e. by \( N(d - p + 1) - 1 \) zero modes. But their total number is

\[
\sum_{l \neq i_1, \ldots, i_p} \Delta_l \geq N(d + 1) - 1 - p(N - 1) = N(d - p + 1) - 1 + p > N(d - p + 1) - 1 = \dim \mathcal{M}_{d-p}.
\]

Hence the dimensions do not match and the vortices do not contribute.

The counting changes as soon as we start computing the correlation functions of the two-observables. In that case the point of insertion of the two-observable is not fixed and may hit other observables at the same time with the vortex. If the total ghost number of the collided observables is greater then \( N - 1 \) then in general there is a contribution of the vortex to the correlation function. \textit{It is this phenomenon which induces the contact terms} \[4\] which enter the relation \( [\beta] \) and provides its microscopic explanation.

### 2.3 Computations in theory II: geometric story

Let us compute the generating function of the correlators of the zero- and two-observables in the gauged linear sigma model. The arguments of the equations \( [\beta] \), \( [\gamma] \) are the polynomials \( P_k \) and the points \( y_q \).

\[4\] It is well-known that the contact terms correspond to the change of couplings.\[10\]
The equations themselves are the conditions of the vanishing of certain sections of the vector bundles $\mathcal{E}_r^{0}, \mathcal{E}_r^{2}$ over $\mathcal{M}_d \times \Sigma \times \ldots \times \Sigma$. Clearly, inside of a correlator in the instanton sector with the instanton number $d$ the following relation holds:

$$\int_\Sigma \mathcal{O}_k^{(2)} = k \int_\Sigma \mathcal{O}_1^{(2)} \mathcal{O}_k^{(0)} = k \int_\Sigma \mathcal{O}_1^{(2)} \mathcal{O}_k^{(0)}$$

(this statement follows from the K"unneth formula [29], cf. [45]). After these preparations we are now in the position to write down the answer:

$$F_{0}^{\text{lin}}(t, T) := \sum_{N-1} \prod_{k=0}^{N} \frac{T_k^{m_k} m_k}{n_k!} \left( \prod_{k=0}^{m_k} \int_\Sigma \mathcal{O}_k^{(2)} \right) = (11)$$

$$= \sum_{d,l} \oint_{d} \exp \left( \sum_{k} t_k c_{\text{top}}(\mathcal{E}_k^{0}) + \sum_{q} T_{r_q} c_{\text{top}}(\mathcal{E}_r^{2}) \right)$$

$$= \sum_{d} \oint \frac{d\sigma}{\sigma^{N(d+1)}} \prod_{q=1}^{l} \frac{d\omega_q}{\omega_q^2} \exp \left( \sum_{k} t_k \sigma^{k} + \sum_{q \{r_q\}} T_{r_q} (\sigma + d \cdot \omega_q)^{r_q} \right)$$

$$= \sum_{d} \oint \frac{d\sigma}{\sigma^{N(d+1)}} \exp \left( \sum_{k} t_k \sigma^{k} + kd \cdot \sigma^{k-1} \right)$$

$$= \oint \frac{d\sigma}{\sigma^{N}} \exp \left( \sum_{r} \frac{t_r \sigma^{r}}{r} T_r \right)$$

(12)

This representation of the generating function of the correlation functions in the linear gauged sigma model is rather suggestive yet shows the crucial difference between the non-linear and the linear gauged models. The difference shows up as the asymmetry between the zero- and two-observables and the absence of the constant (in $T_k$) metrics on the space of zero-observables [21, 43]. The suggestive part of the

5 The bundles and their rôles are the following: the zero-observables $\mathcal{O}_r^{(0)}$ represented by (10) is the top degree Chern class $c_{\text{top}}(\mathcal{E}_r^{0})$ of the bundle $\mathcal{E}_r^{0} = \pi^* \mathcal{O}(1) \otimes C^k$, while the integrated two-observable $\int_\Sigma \mathcal{O}_r^{(2)}$ is the integral over the $q$-th copy $\Sigma$ in $\Sigma^l$ of the top degree Chern class of the bundle $\mathcal{E}_r^{2} = \pi^* \mathcal{O}(1) \otimes p_q^* \mathcal{O}(d) \otimes C^k$. Here $\pi, p_q$ are the projections $\mathcal{M}_d \times \Sigma^l \to \mathcal{M}_d$, and to the $q$-th copy of $\Sigma$ respectively.
The formula (12) is in its contour integral form, which looks very much like the correlation function in the topological Landau-Ginzburg theory \(^{41}\) (with the contact terms \(^{26}\) equal to zero). The latter must have the following structure:

\[
F_{0}^{LG}(t, T) = \oint \mu(X) dX^1 \wedge \ldots \wedge dX^D \exp \sum_k t_k \Phi_k(X) \tag{13}
\]

where \(X^i\) are the holomorphic coordinates on the target space \(X\) of the LG sigma model, \(dW_T(X) = dW_0(X) + \sum_k T_k d\Phi_k\) is the holomorphic one-differential (the derivative of the superpotential \(W_T\)), \(\Phi_k\) are the representatives of the local ring of the superpotential and \(\mu(X) dX^1 \wedge \ldots \wedge dX^D\) is the holomorphic top degree form on \(X\) \(^{41, 26, 46, 24}\).

The formula (12) is not exactly of the form (13) but it is easy to map it into this form: replace the summation over \(d\) by the integral using the Poisson resummation trick:

\[
\sum_{d \geq 0} A_d = \sum_{n \in \mathbb{Z}} \int_0^\infty e^{2\pi i x n} A_x dx
\]

assuming that the function \(A_d\) defined on \(\mathbb{Z}_+\) can be extended to the whole of \(\mathbb{R}_+\) as a continuous function. In order to apply this trick to the formula above (12) we rewrite the contour integral over \(\sigma\) as the integral over \(\phi\) from 0 to \(2\pi\) with \(\sigma = e^{i\phi}\).

Doing these manipulations we arrive at:

\[
F_0^{lin}(t, T) = \oint_{\mathcal{X}} \mu(X) \prod_{\phi} \left( \prod_{n=0}^{N-1} \frac{d\phi}{2\pi i n - i N \phi} e^{i\phi} + \sum_k kT_k e^{i(k-1)\phi} \right)
\]

\[
= \int_{-\infty}^{\infty} d\phi \prod_{\phi} \left( \prod_{n=0}^{N-1} \frac{d\phi}{2\pi i n - i N \phi} e^{i\phi} + \sum_k kT_k e^{i(k-1)\phi} \right)
\]

\[
= \oint_{\mathcal{X}} \mu^2(\phi) \prod_{\phi} \left( \prod_{n=0}^{N-1} \frac{d\phi}{2\pi i n - i N \phi} e^{i\phi} + \sum_k kT_k e^{i(k-1)\phi} \right)
\]

\[
= \int_{-\infty}^{\infty} d\phi \prod_{\phi} \left( \prod_{n=0}^{N-1} \frac{d\phi}{2\pi i n - i N \phi} e^{i\phi} + \sum_k kT_k e^{i(k-1)\phi} \right)
\]

\[
= \int_{-\infty}^{\infty} d\phi \prod_{\phi} \left( \prod_{n=0}^{N-1} \frac{d\phi}{2\pi i n - i N \phi} e^{i\phi} + \sum_k kT_k e^{i(k-1)\phi} \right)
\]

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= \int_{-\infty}^{\infty} d\phi \prod_{\phi} \left( \prod_{n=0}^{N-1} \frac{d\phi}{2\pi i n - i N \phi} e^{i\phi} + \sum_k kT_k e^{i(k-1)\phi} \right)
\]

\[
= \int_{-\infty}^{\infty} d\phi \prod_{\phi} \left( \prod_{n=0}^{N-1} \frac{d\phi}{2\pi i n - i N \phi} e^{i\phi} + \sum_k kT_k e^{i(k-1)\phi} \right)
\]

\[
= \int_{-\infty}^{\infty} d\phi \prod_{\phi} \left( \prod_{n=0}^{N-1} \frac{d\phi}{2\pi i n - i N \phi} e^{i\phi} + \sum_k kT_k e^{i(k-1)\phi} \right)
\]

\[
= \int_{-\infty}^{\infty} d\phi \prod_{\phi} \left( \prod_{n=0}^{N-1} \frac{d\phi}{2\pi i n - i N \phi} e^{i\phi} + \sum_k kT_k e^{i(k-1)\phi} \right)
\]

The superpotential on the \(n\)'th copy of \(\mathbb{C}\) is equal to:

\[
W_T(\phi) = -(N \phi - 2\pi n) e^{i\phi} + \sum_k \tilde{T}_k e^{i k \phi}, \quad i\tilde{T}_k = T_k + N \delta_{k,1} \tag{16}
\]
The contour integral in (15) is taken around infinity at each connected component of $X$. As usual the expression (15) must be understood perturbatively in $T_k$ for $k > 1$.

The requirement that $W_T$ must be linear in $T_k$ constrains it but does not fix it uniquely. We could interpret (14) in a slightly different way, by taking $\mu = e^{i(-N/2)\varphi}$ and $\tilde{W}_T = -\frac{1}{2} (N\varphi - 2\pi n)^2 + T_1\varphi + \sum_{k=2}^{N-1} \frac{k}{k-1} T_k e^{i(k-1)\varphi}$.

It is not possible to make a fair choice between all these options without further physical insight. We turn to it in the next subsection.

2.4 Physical derivation

Despite the exotic nature of the effective target space and the superpotential which we derived in the previous section using topological techniques applied to the moduli space of the BPS field configurations in the gauged linear sigma model it makes perfect sense! In fact, we shall now derive the same result using the effective field theory techniques.

Recall the field content of the two dimensional gauged linear $\mathcal{N} = 2$ supersymmetric sigma model. The chiral multiplets $\Phi^i = (\phi^i, \psi_+^i, F_i)$ take values in some complex vector space $W$ with the constant Kähler form $\omega$. The vector multiplets $V = (A_\mu, \sigma, \lambda_+, \bar{\lambda}_+, D)$ take values in the Lie algebra of the group $G$ which acts in $W$ preserving Kähler structure. The superfield $\Sigma$ which contains the field strength $F = dA + \frac{1}{2} [A, A]$ is the twisted chiral superfield with the quantized highest component $F$ (for abelian $G$).

The bosonic part of the Lagrangian is (in the absence of the tree level superpotential):

$$\int \|D_\mu \phi^i\|^2 + \frac{1}{2e^2} F_{\mu\nu}^2 + \frac{e^2}{2} \|\mu\|^2 + \||\sigma, \bar{\sigma}\||^2$$

where $\mu \sim T^a_{ij} \phi^i \bar{\phi}^j$ is the moment map for the $G$ action in $W$. If the gauge group $G$ contains $U(1)$ factors then we may deform the model by adding a constant per each $U(1)$ to $\mu$ (Fayet-Illiopoulos terms) thereby promoting $\mu$ to

$$\mu = T^a_{ij} \phi^i \bar{\phi}^j - r_i 1_i$$

Also, for each $U(1)$ factor there is a $\theta$ term: $\theta_i \int F_i$. Altogether we get a complex parameter $t_i = ir_i + \frac{\theta_i}{2\pi}$ per each $U(1)$. Correlation
functions of chiral observables are holomorphic in $t_i$ – an important constraint.

Finally, $e^2$ in (17) is the gauge coupling (for several $U(1)$ factors one may have different couplings $e_i^2$). The standard lore says that in the infrared $e_i^2 \to \infty$ and the gauged linear model looks more and more like the non-linear sigma model with the target

$$X = W//G = \mu^{-1}(0)/G$$

For $G = U(1)$, $W = \mathbb{C}^N$ being the $N$ times the standard charge one representation of $G$ the space $X$ is isomorphic to $\mathbb{C}P^{N-1}$ for $r > 0$, $X$ is a point for $r = 0$ and $X$ is empty for $r < 0$.

The field configurations which preserve some of the supersymmetry obey the equations [47]:

$$\left(\bar{\partial} + \bar{A}\right) \phi^i = 0, \quad F = -e^2 \mu$$

(19)

It is possible to show that for the spherical worldsheet the space of solutions to these equations in a given instanton sector ($\int F = 2\pi i d$) coincides with the space $\overline{\mathcal{M}}_d$ of quasimaps[8]. The quasimaps with vortices correspond to the perfectly smooth solutions of (19) which are the generalizations of the Abrikosov-Nielsen-Olesen vortices[1]. They have the characteristic size of the order $1/e\sqrt{r}$ and therefore for $\rho > \frac{1}{e\sqrt{r}}$ are integrated out in (4). The fact that the vortices have the finite size in the microscopic theory II is another justification for the name “freckle” which we suggest.

To derive the relations (13), (16) within the physical theory one integrates out the $N$ chiral multiplets in the background of slowly varying fields of the vector multiplet. One induces the well-known twisted superpotential[14] (in our notations)

$$\tilde{W}_{\text{eff}}(\sigma) = N \frac{\sigma}{2\pi} \log \frac{\sigma}{\tilde{\Lambda}}$$

(20)

with $\tilde{\Lambda} = e^{-2\pi i t - 1}$. The complex scalar $\sigma$ enters the twisted chiral superfield $\Sigma$ which has a constrained $F$ component - it comes from the gauge field strength hence must be quantized:

$$\int F \in 2\pi i \mathbb{Z}$$

(21)

If this was not the case the superpotential (20) would not make any sense due to the branching of the logarithm[47]. The bosonic part of
the Lagrangian which contains the effective superpotential \( (20) \) has the form:

\[
\int \partial_\sigma \tilde{W}_{\text{eff}} (D + iF) + \overline{\partial_\sigma \tilde{W}_{\text{eff}}} (D - iF) \tag{22}
\]

To draw some physical conclusions from the shape of the effective superpotential we need to map the constrained superfield \( \Sigma \) to the unconstrained (twisted) chiral multiplet. Otherwise we cannot simply integrate out \( F \) in \( (22) \) to get an effective potential.

Instead we perform the following duality transformation. First of all we assume the worldsheet to have a spherical topology (it is not necessary but simplifies the discussion a bit). Let us choose an arbitrary point \( p \) on the worldsheet. The field \( \sigma \) taking values in \( \mathbb{C}^\star \) (the point \( \sigma = 0 \) is the singular point as there the matter fields become massless and this singularity is avoided dynamically as we shall see in a minute) is the same thing as the field \( \varphi \) taking values in \( \mathbb{C} \), \( \sigma = \exp i\varphi \) with the only condition:

\[
0 \leq \text{Re} \varphi(p) < 2\pi \tag{23}
\]

This condition is not holomorphic but this will not bother us, as we will get rid of it very soon. The change of variables from \( \sigma \) to \( \varphi \) with the constraint \( (23) \) maps all the fields in the supermultiplet \( \Sigma \) to those in the supermultiplet \( \Phi \). Now we can relax the condition \( (21) \) at the expense of adding to \( (22) \) a term

\[
2\pi i n \int F \tag{24}
\]

where \( n \in \mathbb{Z} \) is the integer-valued “field” which must be summed over in the path integral. The term \( (24) \) is equivalent to the shift of the twisted superpotential by the term, linear in \( \sigma \):

\[
\tilde{W} \rightarrow \tilde{W} + 2\pi i n \sigma = ie^{i\varphi} (N\varphi - N\tilde{t} - 2\pi n)
\]

Now, the summation over all \( n \in \mathbb{Z} \) with the path integral over \( \varphi \) obeying \( (24) \) is equivalent to the summation over \( n \in \mathbb{Z}_N \) with the path integral over all \( \varphi \) without any constraints, as promised. For the shift of \( \varphi \) by \( 2\pi \) is equivalent to the shift of \( n \) by \( N \). In this way we arrive at the formula \( (16) \) with \( T_1 = Nt, T_k = 0, k > 1 \).

To get the full set of “times” \( T_k \) we must start with the linear sigma model deformed by all two-observables. It is easy to see that this deformation is equivalent to the shift of the superpotential by the terms \( \sim T_k \sigma^k, k > 1 \).
2.5 More examples of disconnected targets

In this subsection we consider more examples of effective theories with disconnected target spaces. This story slightly takes us off the route so the reader who is interested in further details concerning the relations between the theories I and II may skip it and proceed directly to the next section.

Compactification of $\mathcal{N} = 2$ gauge theory. Consider four dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with the gauge group $SU(N)$. Let us keep the four dimensional coupling finite so that we allow the point-like instantons to contribute to the effective action. Then the effective two-dimensional theory is given simply by the Kaluza-Klein reduction of the effective four dimensional low-energy action.

Recall that the bosonic part of the latter is given by the following expression:

$$S_{\text{bos}} = \int \tau_{ij} F^-_i \wedge F^-_j - \tau_{ij} F^+_i \wedge F^+_j + \text{Im} \tau_{ij} d\bar{a}^i \wedge \ast d\bar{a}^j$$  \hspace{1cm} (25)

The couplings $\tau_{ij}$, $i,j = 1, \ldots, N - 1$ are determined with the help of the family of hyperelliptic curves $\mathcal{C}$. When we consider all possible configurations of the gauge field we must include those which have non-trivial fluxes through the sphere $S^2$. Naturally we get sectors labelled by $\vec{n} = (n_1, \ldots, n_{N-1})$, where

$$\int_{S^2} F_i = 2\pi i n_i, \quad n_i \in \mathbb{Z}$$ \hspace{1cm} (26)

Now, the resulting two dimensional theory still contains abelian gauge fields. As it is well-known the gauge fields are non-dynamical in two dimensions. It is perhaps less known that this non-dynamical nature of the gauge fields can be conveniently summarized by saying that instead of the path integral over the gauge equivalence classes of the gauge fields one can simply take the path integral over two-forms $F_i$ with the only constraint that the integral of $F_i$ over the (compact) space-time is quantized $[15, 7, 16]$: 

$$\int_{\text{space-time}} F_i = 2\pi i \tilde{m}_i, \quad \tilde{m}_i \in \mathbb{Z}$$

This description automatically takes care of the gauge fixing and the Faddeev-Popov determinants. As we did in the previous analysis it
is convenient to enforce the latter constraint by introducing another integer-valued field $\vec{m} = (m^1, \ldots, m^{N-1})$, $m_k \in \mathbb{Z}$ to be summed over, relaxing the condition on $F_i$ by allowing it to be any two-form, at the same time adding to the action an extra term

$$S \to S + \int_{\text{space-time}} m^i F_i$$

Let us compactify this theory on a two-sphere $S^2$. Since the sphere has no covariantly constant spinors the supersymmetry will be broken. In order to avoid that we consider the (partially) twisted theory, i.e. add certain curvature couplings to the Lagrangian. The simplest approach starts with the theory which is topologically twisted in four dimensions [44, 5]. The field content of the theory is: a vector $A$, a one-form fermion $\psi$, a self-dual two-form fermion $\chi$, a scalar fermion $\eta$, an auxiliary self-dual bosonic two-form $H$, a complex bosonic scalar $a$.

Upon Kaluza-Klein reduction (which amounts to keeping the internal parts of the fields harmonic) we will get: a vector $A$, a one-form fermion $\psi$, a two-form fermion $\chi$, a scalar fermion $\eta$, an auxiliary two-form bosonic field $H$, a complex bosonic scalar $a$. Upon the trick with the introduction of the auxiliary labels $\vec{m}$ described in the previous section the vector $A$ is replaced by another auxiliary bosonic two-form $F$. All fields have in addition a label $i$ which runs from 1 to $N-1$. Let us look at the supersymmetry transformations: the scalar supercharge $Q$ acts as follows:

$$Q F_i = d\psi_i, \quad Q \psi_i = da_i, \quad Q a_i = 0$$

$$Q \chi_i = H_i, \quad Q H_i = 0,$$

$$Q \bar{a}^i = \eta^i, \quad Q \bar{\eta}^i = 0$$

One can easily recognize here the $Q$-operator of the topological type $B$ sigma model. Usually the fermion $\chi_i$ is denoted as $\theta_i$, while the auxiliary field $H_i$ goes under the name $\tilde{F}_i$.

**What is the target space?** The immediate answer would be: “the space of $a_i$’s”, since these are the target space coordinates in the supersymmetry transformation laws. But we should remember that on the route to the sigma model description we were introducing some extra labels $\vec{n}, \vec{m}$ when we dealt with fluxes of the gauge field through the internal sphere as well as the two dimensional space-time itself.
The second guess would be that we should take as many copies of the space of $a_i$’s as there are labels $(\vec{n}, \vec{m})$. The truth sits in between. Recall that $a_i$’s are only local special coordinates on the moduli space $V$ of vacua of the gauge theory. As one goes around some non-contractible loops in $V$ one comes back with another set of $a_i$’s, transformed by a monodromy group. This very group also transforms the vectors $(\vec{n}, \vec{m})$.

After all these symmetries are taken into account we get the following statement:

The target space $V$ of the effective type B sigma model is the space of pairs: $(C_u, \gamma)$, where $C_u$ is one of the curves (1) and $\gamma \in H_1(C_u, \mathbb{Z})$.

The unusual property of this target space is its disconnectness: the cycles $\gamma_1, \gamma_2$ which are not in the same orbit of the monodromy group belong to the different connected components. The component where $\gamma = 0$ is isomorphic to the moduli space $V$ of the curves (1), while the other components are certain covers of $V$:

$$V \approx V \amalg \tilde{V} \amalg \ldots$$

For example, for $N = 2$, the component with $\gamma = 0$ is isomorphic to the complex line with the coordinate $u$, parameterizing the curves:

$$y^2 = (x^2 + u)^2 - 4\Lambda^4$$

while the other components are isomorphic to the strip $0 \leq \text{Re} \tau \leq 4, \text{Im} \tau > 0$. This space shows up in certain computations in Donaldson theory [33, 29].

Superpotential. The curves (1) are embedded by definition into a hyper-kähler manifold $T^* \mathbb{P}^1$, which has a holomorphic two-form $\omega$:

$$\omega = \frac{1}{2\pi i} dx \wedge \frac{dz}{z}$$

Our sigma model has a (twisted) superpotential $W$. In general, the superpotential needs not to be well-defined. Only the derivative $dW$ must be a well-defined holomorphic one-form on the target space.

In our case we can write a simple formula for the derivative $dW$:

$$dW = \oint_\gamma \omega$$

To prove the last assertion let us introduce a few more standard notations. Let $\lambda = \frac{1}{2\pi i} \tau \frac{dz}{z}$ be the meromorphic one-differential, $d\lambda = \omega$. 

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Its residues on the curve $C_u$ vanish for any $u$. Hence we can integrate (32) to get:

$$W = \oint \lambda = n_i a^i + m^i a_{D,i}$$  \hspace{2cm} (33)

where we expanded $\gamma$ in some (local) symplectic basis $(A_i, B^i)$ in $H_1(C_u, \mathbb{Z})$ and

$$a^i = \oint_{A_i} \lambda, \quad a_{D,i} = \oint_{B_i} \lambda$$

Now let us compute the potential which follows from the superpotential (33):

$$\left(n_i + \tau_{ik} m^k \right) \left( \text{Im} \tau^{-1} \right)^{ij} \left(n_j + \bar{\tau}_{jl} m^l \right)$$  \hspace{2cm} (34)

This is precisely the potential which one would get from the gauge theory action upon compactifying on the small sphere and performing the two dimensional duality transformation. Of course in this very setup one also has Kaluza-Klein modes along the two-sphere which have the same order of energy as the fluctuations in the potential (34). One can separate the two scales by adding the two-observable $t O_u^{(2)}$ whose support extends over all of the two dimensional space-time. This addition changes the superpotential by $tu$ and introduces another scale into the problem. More details on this issue can be found in [29].

**Remarks.** The effect of the unfolding of the moduli space of twisted chiral multiplets is very general. In fact, one can use the similar method of unfolding in four dimensional theories [22, 23] with effective superpotential of Veneziano-Yankielowicz type [42]. To the best of our knowledge the phenomenon of creating of disconnected components in the effective target space was not observed before neither in two- nor in four-dimensional models. This phenomenon makes the study of solitons connecting different vacua rather intricate. For the solitons are represented by the paths connecting the vacua sitting on different components of the target space. It means that the trajectory connecting the vacua should break at some point. The only way this breaking may be avoided is to assume that the metric on the effective target space is such that the components are actually glued together at infinity.

If this is not the case then the actual location of the breaking point on each sheet might be undetermined by means of the low energy theory alone. This is similar to the fact that the mass of the magnetic
monopole cannot be computed from the Maxwell theory alone. We shall elaborate on this and related issues elsewhere [32].

Notice that if the model is embedded into the string/M-theory, e.g. via brane realization then one gets a geometrical representation of the solitons [20]. Also note that the superpotential of the SW form (but without unfolded disconnected target space) was computed in [25] in the geometrical engineering approach to the compactifications on the Calabi-Yau fourfolds. We plan to return to this question elsewhere.

2.6 Comparison to the results of the XIX century

Despite the success of the previous two subsections we should warn the reader that their results where merely the groundwork for the solution of the real problem - the effective theory description of the non-linear sigma model. To see that the renormalization (4),(5) is non-trivial let us perform several numerical checks.

To be specific let us take $N = 3$, i.e. $\mathbf{CP}^2$ model. The genus zero correlation functions in the topologically twisted theory compute the numbers of rational planar curves which pass through a given number of points in $\mathbb{P}^2$ and intersect a given number of lines in generic position. In particular:

$$
\langle O_1^{(0)}(0) O_1^{(0)}(1) O_1^{(0)}(\infty) \int O_2^{(2)} \int O_2^{(2)} \rangle_{\text{non-lin}} = 1 \quad (35)
$$

is the number of degree one curves (lines) which pass through two distinct points and intersect three lines in generic position. From the elementary school we know that this number is equal to one. On the other hand, it is easy to see that

$$
\langle O_1^{(0)}(0) O_1^{(0)}(1) O_1^{(0)}(\infty) \int O_2^{(2)} \int O_2^{(2)} \rangle_{\text{lin}} = 2^2 = 4 \quad (36)
$$

This is the first in the series of numbers predicted by the gauged linear sigma model which differ from those computed by the last century algebraic geometers.

Of course, the difference happens precisely when we start looking at the correlation functions of two-observables, as we promised in the beginning of the section. We also said that the difference is due to
the contribution of the freckles. Let us now see explicitly how this happens.

The space $\overline{\mathcal{M}}_1$ contains the subspace $\mathbb{P}^1 \times \mathbb{P}^2$ of vortices, where $\mathbb{P}^1$ parameterizes the location $x_\ast$ of the vortex and $\mathbb{P}^2$ is the set of images $r$ in the target space of the complement to the vortex. The equations (10), (9) are obeyed both by the proper map of degree one and by the vortex configurations with $x_\ast = y_1 = y_2 = x_p$ where $x_p$ assumes one of the three available values: 0, 1, $\infty$. The two remaining equations state that the image $r$ must belong to two lines $\ell_{p'}, p' \neq p$ in $\mathbb{P}^2$, i.e. to their intersection point. Altogether we found three vortex configurations, each contributes one to the correlation function, hence:

$$4^{\text{lin}} = 1^{\text{non-lin}} + 3^{\text{vortex}}$$  \hfill (37)

The similar (but much more tricky) counting works for higher instanton charges. The effect of subtraction of the vortex contributions is to replace the effective superpotential (16) by the infinite series

$$(N \varphi - 2\pi n - T_1) e^{i\varphi} + \sum_l a_l(T) e^{ilN(1-\varphi)}$$  \hfill (38)

The computation of the coefficients $a_l(T)$ is a complicated problem yet related problems are studied in the enumerative geometry [9, 15]. If we were interested in the actual values of the correlators in the theory I we could use the techniques of [21]. Our goal is slightly different.

Remark. Consider the trivial non-linear sigma model with the target space $\mathbb{C} \mathbb{P}^0 = \text{a point}$. It becomes rather interesting once replaced by the linear gauged sigma model. The latter has the moduli space

$$\overline{\mathcal{M}}_d = \mathbb{P}^d = \text{Sym}^d \mathbb{P}^1$$  \hfill (39)

3 Back to future: four dimensional instantons with freckles

So far we have seen that adding some kind of point-like topological defects – “freckles” – may lead to the compactification of the moduli

\footnote{We should mention that the similar effects were observed more then a hundred years before the discovery of supersymmetry [18].}
space of instantons in a theory \( I: \mathcal{M} \hookrightarrow \overline{\mathcal{M}} \). We have also seen that these point-like defects may be perfectly smooth field configurations in the theory \( \Pi \), but of some characteristic size \( \rho \). The natural question is: can this work in four dimensions?

In four dimensions we study gauge theories with instantons. The moduli space of instantons \( \mathcal{M} \) is non-compact due to the well-known phenomenon of shrinking of instantons. Suppose that the euclidean space-time is a compact Kähler surface \( S \), with Kähler form \( \omega \). It is well-known that the moduli space of instantons in a given gauge bundle \( \mathcal{E} \) with the characteristic classes \( c_1(\mathcal{E}), c_2(\mathcal{E}) \) is isomorphic to the moduli space of \( \omega \)-semistable holomorphic bundles \( \mathcal{E} \) over \( S \) with the same Chern classes as \( \mathcal{E} \). We recall the notion of (semi)stability below. Algebraic geometers replace the holomorphic bundle \( \mathcal{E} \) by the sheaf \( \mathcal{E} \) of its holomorphic sections. More specifically, over each open set \( U \) one considers the abelian group \( \Gamma(\mathcal{E}|_U) \) of the holomorphic sections of \( \mathcal{E} \) over \( U \). The elements \( s \) of this group can be multiplied by the holomorphic functions \( f \) on \( U \), \( f \in \mathcal{O}_U \). This operation makes \( \Gamma(\mathcal{E}|_U) \) a module over the ring \( \mathcal{O}_U \) of holomorphic functions in \( U \). For sufficiently small \( U \) one can find a basis \( s_i \) in the space of holomorphic sections of \( \mathcal{E}|_U \) such that every section \( s \in \Gamma(\mathcal{E}|_U) \) can be uniquely expanded as:

\[
s = \sum_{i=1}^r f_is_i, \quad f_i \in \mathcal{O}_U
\]  

(40)

The sheaves with this property are called locally free – if the sheaf is locally free then it is a sheaf of sections of some holomorphic bundle. One can relax the condition (40) to the property of being torsion free:

\[
\text{If } fs = 0 \Rightarrow \text{either } f = 0 \text{ or } s = 0
\]  

(41)

The sheaf which is torsion free in general does not come from a holomorphic bundle. Nevertheless the beautiful property of the torsion free sheaves (in complex dimension two) is that every torsion free sheaf \( \mathcal{F} \) is almost a bundle, in fact over a complement to a finite number of points in \( S \) it is a bundle! One can always find (\cite{18}, lemma 1.1.8) a holomorphic vector bundle \( \mathcal{E} \) such that \( \mathcal{E}/\mathcal{F} = \mathcal{S}_Z \), in other words there is an exact sequence:

\[
0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{S}_Z \to 0
\]  

(42)

where \( \mathcal{S}_Z \) is a skyscraper sheaf supported at points, i.e. \( \dim Z = 0 \),

\[
\Gamma(\mathcal{S}_Z|_U) = \mathbb{C}^{\#Z \cap U}
\]
We see the first similarity between the torsion free sheaves and vortices in two dimensions: both differ from the honest instanton (the face of the freckled instanton) only at finite number of points (freckles). More quantitatively this similarity is supported by the fact that these points carry instanton charge:

\[ c_2(\mathcal{F}) = c_2(\mathcal{E}) - c_2(\mathcal{S}_Z) = c_2(\mathcal{E}) + \# \mathcal{Z} \]  

(43)

The importance of the torsion free sheaves in the studies of S-duality was first advocated in [27]. One can show that the natural backgrounds for the higher dimensional bc-systems studied in [28] are again the sheaves [32] rather then holomorphic bundles alone.

In order for the torsion free sheaves to be useful we need a way to construct their moduli space and make sure that it is compact. It turns out that if one takes the space of \( \omega \)-(semi)stable torsion free sheaves\(^7\) then the resulting space \( \mathcal{M}_{c_*} \) is compact (and even quasiprojective [17]). Moreover, it can be described rather explicitly in the case of the manifolds \( S \approx \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1 \) using ADHM techniques.

To make further contact with the two dimensional story we need:

a) the supersymmetric gauge theory whose moduli space of BPS fields is \( \mathcal{M}_{c_*} \),

b) an analogue of the computation (12).

The answer to the point a) is not completely known. In the light of [36] the natural conjecture would be to take the theory on the “non-commutative space” \( \hat{S} \) which is a quantization of \( S \) with the \( \omega^{-1} \) being the Poisson structure.

The point b) is addressed using the monad description [3, 11, 3] of the moduli space of the torsion free sheaves (a useful review is [32]). For example, for \( S = \mathbb{CP}^2 \) the \( \omega \)-semistable torsion free sheaves \( \mathcal{F} \) arise as follows. Let \( V_0, V_1, V_2 \) be the complex vector spaces of dimensions \( v_{0,1,2} \) respectively. Consider the complex of bundles over \( S \):

\[ 0 \to V_0 \otimes \mathcal{O}(-1) \xrightarrow{a} V_1 \otimes \mathcal{O} \xrightarrow{b} V_2 \otimes \mathcal{O}(1) \to 0 \]  

(44)

In down-to-earth terms this sequence has the following meaning. The maps \( a, b \) in the homogeneous coordinates \( (z^0 : z^1 : z^2) \) are the matrix-valued linear functions: \( a(z) = z^\alpha a_\alpha, b(z) = z^\alpha b_\alpha \). The words “complex” mean that

\[ b(z) \cdot a(z) = z^\alpha z^\beta b_\alpha a_\beta = 0 \iff b_\alpha a_\alpha = 0, \quad \alpha = 0, 1, 2, \quad b_\alpha a_\beta + b_\beta a_\alpha = 0, \quad \alpha \neq \beta \]  

(45)

\(^7\)The sheaf \( \mathcal{F} \) is \( \omega \)-semistable if for any subsheaf \( \mathcal{F}' \) one has \( \int_S \omega \wedge \left( \frac{c_1(\mathcal{F})}{1+k} - \frac{c_1(\mathcal{F}')}{1+k} \right) \leq 0 \).
For the pair \((b, a)\) of the maps between the sheaves obeying (45) we can define a sheaf \(F\) over \(S\), whose space of sections over an open set \(U\) is
\[
\Gamma(F|U) = \ker b(z)/\text{im} a(z), \quad \text{for} \quad (z^0 : z^1 : z^2) \in U
\]
The space of monads is the space \(M_{\text{mon}}\) of triples of matrices \(a_\beta \in \text{Hom}(V_0, V_1)\), \(b_\alpha \in \text{Hom}(V_1, V_2)\) obeying (45). This space is acted on by the group \(G_{\text{mon}} = (\text{GL}(V_0) \times \text{GL}(V_1) \times \text{GL}(V_2))/\mathbb{C}^*\):
\[
(b, a) \mapsto g \cdot (b, a) = (g_2 b g_1^{-1}, g_1 a g_0^{-1}), \quad \text{for} \quad (g_0, g_1, g_2) \in G_{\text{mon}}^c \quad (46)
\]
The sheaves defined by the pairs \((b, a)\) and \(g \cdot (b, a)\) are isomorphic. The maximal compact subgroup of \(G_{\text{mon}}^c\)
acts in \(M_{\text{mon}}\) preserving its natural symplectic structure
\[
\Omega = \frac{1}{2i} \sum_\beta \text{Tr} \delta a_\beta \wedge \delta a_\beta^\dagger + \frac{1}{2i} \sum_\alpha \text{Tr} \delta b_\alpha^\dagger \wedge \delta b_\alpha \quad (47)
\]
Fix the real numbers \(r_0, r_1, r_2\), such that \(\sum_\alpha v_\alpha r_\alpha = 0, \ r_0, r_2 > 0\). Write the moment maps:
\[
\mu_1 = -r_0 \mathbf{1}_{v_0} + \sum_\beta a_\beta^\dagger a_\beta
\]
\[
\mu_2 = -r_1 \mathbf{1}_{v_1} + \sum_\alpha b_\alpha^\dagger b_\alpha - \sum_\beta a_\beta^\dagger a_\beta
\]
\[
\mu_3 = -r_2 \mathbf{1}_{v_2} + \sum_\alpha b_\alpha b_\alpha^\dagger \quad (48)
\]
Then the moduli space of the semistable sheaves is
\[
\mathcal{M}_{s} = \left(\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)\right)/G_{\text{mon}} \quad (49)
\]
The compactness of the space (49) is obvious: if we first perform a reduction with respect to the groups \(U(V_0) \times U(V_2)\) then the resulting space is the product of two Grassmanians: \(\text{Gr}(v_0, 3v_1) \times \text{Gr}(v_2, 3v_1)\) which is already compact. The subsequent reduction does not spoil this.
The Chern classes of the sheaf \( F \) determined by the pair \((b, a)\) are:

\[
r = v_1 - v_0 - v_2, \quad c_1 = (v_0 - v_2) \omega, \quad c_2 = \frac{1}{2} \left( (v_2 - v_0)^2 + v_0 + v_2 \right)
\]

Let \((i\psi, i\phi, i\chi)\) denote the elements of the Lie algebra of \(G_{\text{mon}}\), i.e. \(i\psi \in u(V_0), i\phi \in u(V_1), i\chi \in u(V_2)\) and \((\psi, \phi, \chi) \sim (\psi + 1_{V_0}, \phi + 1_{V_1}, \chi + 1_{V_2})\). We are interested in computing certain integrals over \(\overline{M}_{c_*}\). This can be accomplished by computing an integral over \(M_{c_*}\) with the insertion of the delta function in \(\mu_i\) and dividing by the volume of \(G_{\text{mon}}\) provided that the expression we integrate is \(G_{\text{mon}}\)-invariant:

\[
\int_{\overline{M}_{c_*}} (...) = \frac{1}{\text{Vol}(G_{\text{mon}})} \int_{\text{Lie}G_{\text{mon}}} d\psi d\phi d\chi e^{i \text{Tr}_\psi \mu_1 + i \text{Tr}_\phi \mu_2 + i \text{Tr}_\chi \mu_3} (...) \tag{51}
\]

The useful fact is that the observables of the gauge theory we are interested in are the gauge-invariant functions on \((\psi, \phi, \chi)\) only. More specifically, there is a universal sheaf \(U\) over \(M_{c_*} \times S\), whose Chern character is represented by:

\[
\text{Ch}(U) = \text{Tr} e^\phi - \text{Tr} e^{i \omega} - \text{Tr} e^{i + \omega} \tag{52}
\]

In particular (cf. (2)):

\[
\mathcal{O}^{(0)}_{u_1} = \frac{1}{2} \left( \text{Tr} \chi^2 + \text{Tr} \psi^2 - \text{Tr} \phi^2 \right),
\]

\[
\int_S \omega \wedge \mathcal{O}^{(2)}_{u_1} = \text{Tr} \chi - \text{Tr} \psi
\]

Since the observables are expressed through \(\psi, \phi, \chi\) only we can integrate out \(a_\beta, b_\alpha\) in (51) to obtain:

\[
\langle \exp t_1 \mathcal{O}^{(0)}_{u_1} + T_1 \int_S \omega \wedge \mathcal{O}^{(2)}_{u_1} \rangle_{\text{torsion free}} = \oint \prod_{i,j,k} d\psi_i d\chi_j d\psi_k \frac{\Pi_{\nu < \nu'} (\psi_{\nu'} - \psi_{\nu})^2 \Pi_{j' < j''} (\phi_{j''} - \phi_{j'})^2 \Pi_{k' < k''} (\chi_{k''} - \chi_{k'})^2 \Pi_{i,k} (\chi_k - \psi_i)^6}{\Pi_{i,j} (\phi_j - \psi_i + i0)^3 \Pi_{j,k} (\chi_k - \phi_j + i0)^3} \\
\times e^{t_1 \frac{1}{2} \left( \sum_k \chi_k^2 + \sum_i \psi_i^2 - \sum_j \phi_j^2 \right) + T_1 \left( \sum_k \chi_k - \sum_i \psi_i \right) + i \sum_i \psi_i + i \sum_j \phi_j + i \sum_k \chi_k} \tag{53}
\]

the universal sheaf is defined again as \(\text{Ker}(b(z))/\text{Im}(a(z))\) but now the space of parameters contains \((b, a)\) in addition to \(z\).
More elaborated answer to the point b) together with the explanation of the relation of this work to the attempts of computing ADHM integrals in \[12\] will be presented elsewhere \[32\].

As a final remark notice that the freckled instantons are present even for the gauge group $U(1)$, in which case the moduli space $\overline{M}_d$ is the resolution of singularities of the $d$-th symmetric power of the manifold $S$:

$$\overline{M}_d = \tilde{\text{Sym}}^d S$$

very much like in \[13\].

In both cases the freckled instantons have only freckles, but no face.

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