Topological Speedups for Minimal Cantor Systems

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**Topological category**
Homeomorphism of a Cantor set $T : X \to X$.

*Minimal:* All orbits dense (no closed subsystems).

**Measure-theoretic category**
Measure-preserving automorphism of a non-atomic Lebesgue probability space $T : (X, \mu) \to (X, \mu)$.

*Ergodic:* $T^{-1}(A) = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$ (no non-trivial subsystems).

The space of Borel $T$-invariant measures with the weak* topology forms a Choquet simplex (convex metric space) where the extreme points are the ergodic measures. We will denote this as $\mathcal{M}_T$. 
Given two dynamical systems $T : X \rightarrow X$ and $S : Y \rightarrow Y$.

**Conjugacy:** There exists an invertible map $h : X \rightarrow Y$ such that

$$hT = Sh.$$ 

**Orbit equivalence:** Up to conjugacy, there exists $p, q : X \rightarrow \mathbb{Z}$ such that for all $x \in X$

$$T^{p(x)}(x) = S(x) \quad \text{and} \quad T(x) = S^{q(x)}(x).$$

**Speedup:** Up to conjugacy, there exists $p : X \rightarrow \mathbb{Z}^+$ such that for all $x \in X$

$$T^{p(x)}(x) = S(x).$$
**Theorem (Dye):** Any two ergodic systems are orbit equivalent in the measure-theoretic category.

In topological category, this cannot be the case. If $S$, $T$ have the same orbits then $\mathcal{M}_T = \mathcal{M}_S$.

If $S(x) = T^{p(x)}(x)$, sets of the form $A_r = \{x : p(x) = r\}$ are closed. If $x_n \in A_r$ and $x_n \to x$, then

$$S(x) = \lim S(x_n) = \lim T^{p(x_n)}(x_n) = \lim T^r(x_n) = T^r(x)$$

$$S(E) = \bigcup T^r(E \cap A_r)$$
Giordano-Putnam-Skau Theorem 1: Let $(X, T)$ and $(Y, S)$ be minimal Cantor systems. The following are equivalent.

1. $T$ and $S$ are orbit equivalent.
2. $(G(T)/\text{Inf}(T), (G(T)/\text{Inf}(T))_+, u_T) \cong (G(S)/\text{Inf}(S), (G(S)/\text{Inf}(S))_+, u_S)$.
3. There is a homeomorphism $F : X \to Y$ such that $F_* : \mathcal{M}_T \to \mathcal{M}_S$ is a bijection.

Where:

- $G(T) = \{f \in C(X, \mathbb{Z})\}/\{f - fT^{-1}\}$
- $\text{Inf}(T) = \{\int f \ d\mu = 0 \text{ for all } \mu \in \mathcal{M}_T\}$
- $G(T)_+ = \{[f] : f(x) \geq 0 \text{ for all } x \in X\}$
- $u_T = [1]$
**Giordano-Putnam-Skau Theorem 2:** Let \((X, T)\) and \((Y, S)\) be minimal Cantor systems. The following are equivalent.

1. \(T\) and \(S\) are strongly orbit equivalent.
2. \((G(T), G(T)_+, u_T) \cong (G(S), G(S)_+, u_S)\).
3. \(C(X) \times_T \mathbb{Z} \cong C(Y) \times_S \mathbb{Z}\)

Where:
- \(G(T) = \{f \in C(X, \mathbb{Z})\}/\{f - fT^{-1}\}\)
- \(G(T)_+ = \{[f] : f(x) \geq 0 \text{ for all } x \in X\}\)
- \(u_T = [1]\)

**Strong Orbit Equivalence:** \(T^{p(x)}(x) = S(x)\) and \(T(x) = S^{q(x)}(x)\) where \(p, q\) have at most one point of discontinuity each.
Remark

**Boyle:** If $T^{p(x)}(x) = S(x)$ and $T(x) = S^{q(x)}(x)$ where $p, q$ are continuous, then $T$ is conjugate to either $S$ or $S^{-1}$.

Key: $p(x) = 1 + a(x) - aS(x)$. Then set $h = T^a(\cdot)$. We have

\[
S(x) = T^{p(x)}(x)
\]

\[
S(x) = T^{1+a(x)-aS(x)}(x)
\]

\[
T^{aS(x)}S(x) = T^{1+a(x)}(x)
\]

\[
hS(x) = Th(x)
\]
Up to conjugacy

\[ S(x) = T^{p(x)}(x) \]

where \( p > 0 \).

**Measure-Theoretic Result**

**Arnoux-Ornstein-Weiss:** Any two ergodic system is the speedup of any other ergodic system.

Possible that \( T^2 \) is not minimal.

Even if \( S = T^2 \) is minimal, it can be the case that \( \mathcal{M}_T \not\subseteq \mathcal{M}_S \).
**Theorem (Ash-O):** Let \((X, T)\) and \((Y, S)\) be minimal Cantor systems. The following are equivalent.

1. \(S\) is a speedup of \(T\).
2. There is an exhaustive epimorphism

\[
\varphi : \left( \frac{G(S)}{\text{Inf}(S)}, \left( \frac{G(S)}{\text{Inf}(S)} \right)_+, u_S \right) \twoheadrightarrow \left( \frac{G(T)}{\text{Inf}(T)}, \left( \frac{G(T)}{\text{Inf}(T)} \right)_+, u_T \right).
\]

3. There is a homeomorphism \(F : X \rightarrow Y\) such that

\[
F^\ast : \mathcal{M}_T \hookrightarrow \mathcal{M}_S
\]

is an injection.
Exhaustive Property:
Given two unital ordered groups \((H, H_+, v)\) and \((G, G_+, u)\), an epimorphism
\[
\varphi : (H, H_+, v) \rightarrow (G, G_+, u)
\]
is called exhaustive if for every \(h \in H_+\) and \(g \in G\) satisfying \(0 < g < \varphi(h)\), there is an \(h' \in H\) such that \(\varphi(h') = g\) and \(0 < h' < h\).

Implicit in this is that \(\varphi(H_+) = G_+\).
Remark

In particular if two systems $S$, $T$ are orbit equivalent then each is a speedup of the other.

The converse is true in the case where space of invariant measures is finite dimensional.

For infinite dimension case, the converse is not true (Melleray).
Up to conjugacy

\[ S(x) = T^{p(x)}(x) \]

where \( p > 0 \) and \( p \) has at most one point of discontinuity.

**Theorem (Ash-O):** Let \((X, T)\) and \((Y, S)\) be minimal Cantor systems. The following are equivalent.

1. \( S \) is a strong speedup of \( T \).
2. There is an exhaustive epimorphism
\[
\varphi : (G(S), G(S)_+, u_S) \twoheadrightarrow (G(T), G(T)_+, u_T).
\]

**Question:** What is the \( C^* \)-algebra interpretation for (2)?
Given two unital ordered groups \((H, H_+, \nu)\) and \((G, G_+, u)\), an epimorphism

\[ \phi : (H, H_+, \nu) \rightarrow (G, G_+, u) \]

is called exhaustive if for every \(h \in H_+\) and \(g \in G\) satisfying \(0 < g < \phi(h)\), there is an \(h' \in H\) such that \(\phi(h') = g\) and \(0 < h' < h\).

In the case where \((H, H_+, u_H)\) has no infinitesimals, we can view the ordered group \(H\) as affine functions on the state space \(S_H\), a compact convex set.

With this perspective, the epimorphism corresponds to a restriction to a subspace.
Let $K$ be a convex, compact set.

Consider $A(K, \mathbb{R})$, the set of affine functions $\psi : K \rightarrow \mathbb{R}$.

$A(K, \mathbb{R})_+$ is the set of functions where $\psi(K) \subset (0, \infty)$ along with the 0 function.

Now consider a countable dense subset $H \subset A(K, \mathbb{R})$ with the same ordering.
Exhaustive Property

For \( L \subset K \) we have the restriction map \( \psi \mapsto \psi|_L \).

When is this map exhaustive?

One condition: \( L \) must be maximal in a certain sense.

Another condition that has to do with the way that the countable dense set \( H \) sits inside \( A(K, \mathbb{R}) \) and \( A(L, \mathbb{R}) \).
Suppose

- \( H = \{ (a\alpha + b\beta + c, a\alpha + d) : a, b, c, d \in \mathbb{Z} \} \) where \( \alpha, \beta \) are irrational, not rationally related.
- \( H_+ = \{ (x, y) \in H : x > 0, y > 0 \} \)
- \( u_H = (1, 1) \)

Project onto first coordinate. \( \pi : (H, H_+, u_H) \to (\mathbb{R}, \mathbb{R}_+, 1) \) and consider the image.

Fix \( b, c \) such that \( 0 < b\beta + c < 1 \). No element \( h' \) in \( H \) with \( (0, 0) < h' < (1, 1) \), and \( \pi(h) = b\beta + c \).
Exhaustive Property

More generally, in the case where $(H, H_+, u_H)$ has no infinitesimals, we can view the ordered group $H$ as affine functions on the state space $S_H$, a compact convex set.

With this perspective, the epimorphism corresponds to a restriction to a subspace.

Could this perspective lead to another characterization?
Joint work with Lori Alvin and Drew Ash
For speedups, if $S(x) = T^{p(x)}(x)$ with $p$ bounded then $p(x)$ is a constant $c$ plus a $T$-coboundary and every $T$-orbit is split into exactly $c$ $S$-orbits.

But $S$ is not necessarily conjugate to $T^c$.

Call $c$ the orbit number of the bounded speedup.

Example 1: Orbit number 2, $T^2$ is nonminimal, but $T^{p(\cdot)}$ is minimal.

Example 2: Orbit number 2, $T^{p(\cdot)}$ where $p(\cdot) - 2$ has nonzero integral for some $\mu \in \mathcal{M}_S$. 
**Theorem** If $\mathcal{M}_T = \mathcal{M}_S$, then

$$h(S) = c\ h(T).$$

More generally,

$$c\ h(T) \leq h(S) \leq \sup_{\nu \in \mathcal{M}_S} \left( \int p \ d\nu \right) h(T)$$

No hope to control entropy in the general case.
More results, questions
Bounded speedup of a subshift is a (different) subshift.

The bounded speedup of an odometer is a conjugate odometer.

The bounded speedup of a substitution is a substitution.

Is the bounded speedup of a Toeplitz system Toeplitz?
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