MATCHED ASYMPTOTICS FOR LARGE SOLUTIONS TO THE GELFAND-LIOUVILLE PROBLEM IN TWO-DIMENSIONAL, DOUBLY CONNECTED DOMAINS

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Abstract. In this paper we provide a formal matched asymptotic analysis for large solutions to the Gelfand-Liouville problem in planar, doubly connected domains in the plane. Using these, we rigorously construct a good approximate solution to the problem.

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1. Introduction

1.1. The problem. We consider the Gelfand-Liouville problem:

\[
\begin{aligned}
\Delta u + \lambda^2 e^u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\tag{1.1}
\]

where \( \Omega \) is a bounded, smooth domain of \( \mathbb{R}^N, N \geq 2 \), and \( \lambda > 0 \) is a parameter.

1.2. Motivation.

1.2.1. Motivation from physics. This equation arises in the theories of thermionic emission (see [13]), isothermal gas spheres (see [5]), and gas combustion (see [25]). Furthermore, this type of equations appear in statistical mechanics (see [3], [4], [8]).

1.2.2. Motivation from geometry. This type of equations arise in the prescribed Gaussian and scalar curvature problems in a compact manifold (see [6], [7], [21]).

1.3. Known results. Problem (1.1) has been studied extensively. With no hope of being complete, let us mention the following results which are more related to the scope of the current paper.

1.3.1. General \( N \geq 2 \). It is well known that there exists a \( \lambda_* > 0 \) such that (1.1) admits a minimal solution \( u_\lambda \) if \( \lambda \in (0, \lambda_*) \) (here minimal means smallest); no solution if \( \lambda > \lambda_* \); admits a solution if \( \lambda = \lambda_* \) and \( N \leq 9 \) (see [9], [10], [12] and [25]). In fact, the minimal solution \( u_\lambda \) can be constructed by the method of upper and lower solutions and satisfies

\[
\|u_\lambda\|_{L^\infty(\Omega)} \to 0 \quad \text{as } \lambda \to 0.
\]

The simply connectedness of the domain plays an important role in the structure of solutions to (1.1). This can already be made clear by looking at the case of radially symmetric domains. In the case of a ball, a classical result of Gidas, Ni and Nirenberg [14] implies that every solution is radially symmetric and decreasing (it goes without saying that every solution in any domain is positive). Therefore, the problem reduces to an ordinary differential equation. Based on this observation, Joseph and Lundgren [20] were able to completely characterize the structure of solutions in all dimensions (see also [13] and [26]). Of particular interest is the relationship they observed between the multiplicity of solutions and the space dimension. On the other hand, in the case of an annulus, there exists a continuous curve of radial solutions along which infinitely many bifurcations to non-radial solutions takes place (see [23], [24], [30]). The aforementioned radial solutions are critical points of mountain-pass type for the associated energy in the natural energy space of radially symmetric functions (see [16]). In fact, if \( \lambda > 0 \) is sufficiently small, problem (1.1) has exactly two radial solutions, namely the minimal one and the mountain-pass in the class of radial solutions (see [23] and [29]). In particular, when \( N = 2 \), these solutions can be given explicitly by using their invariance through a transformation group (see [13], [23]).
1.3.2. **Behaviour as \( \lambda \to 0^+ \).** \( N = 2 \). In this case, thanks to the works \([2], [22], [29]\) and \([31]\), we can classify all possible solutions to \((1.1)\) by the limit of the quantity

\[
T_\lambda = \lambda^2 \int_\Omega e^u dx.
\]

Loosely speaking, if \( T_\lambda \) remains bounded as \( \lambda \to 0^+ \) then the solutions blow-up at a finite number of points in \( \Omega \). More precisely, they exhibit "bubbling" behaviour at a fixed finite number of points in \( \Omega \): after a proper rescaling near each such point, the solution resembles the unique solution of the following limit problem:

\[
\Delta u + e^u = 0 \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u dx < \infty.
\]

\[N \geq 2.\] In any dimension, the asymptotic behaviour of radial mountain-pass solutions in an annulus was investigated in \([16]\). It was shown there in that \( T_\lambda \) diverges to infinity as \( \lambda \to 0 \). Moreover, the solutions blow-up in the whole of \( \Omega \) and in fact concentrate their energy around a special hyper-sphere. After a proper rescaling, the solutions resemble that of the unique one-dimensional bubble

\[
u'' + e^u = 0 \text{ in } \mathbb{R}, \quad \int_{\mathbb{R}} e^u dx < \infty,
\]

transplanted along that hyper-sphere.

1.4. **The problem and the main result.** It would be of great interest to see if there exist in general domains analogous solutions to the radial ones of \([16]\). In this paper we make a modest step in this direction by carrying out successfully matched asymptotic expansions for the problem in the simplest possible nonradial situation: In the case where \( \Omega \) is two-dimensional doubly connected domain in the plane. We would like to point out that, to the best of our knowledge, our calculations are new even in the radial case (there they carry out considerably more easily and in any dimension).

In our main result, stated in Proposition 8.1 below, we use these asymptotic expansions to construct a good global approximate solution to \((1.1)\) for small \( \lambda > 0 \), which blows up in the whole of \( \text{Omega} \) and concentrates its energy along a special curve in \( \Omega \). Near that curve, and in the normal direction to it, that approximate solution given to main order by a properly rescaled solution of \((1.2)\). Moreover, using our detailed estimates, one can also calculate an asymptotic expansion for \( T_\lambda \) as \( \lambda \to 0^+ \).

It is natural to believe that there exists a genuine solution of \((1.1)\) near the approximate one. In the radial case, this can be made rigorous by a linearization argument in some carefully chosen weighted spaces and expanding on some ideas from \([1]\). However, in the nonradial setting at hand there are various issues which prevent a straightforward application of the techniques in the aforementioned reference. Further, in light of our previous discussion on nonradial bifurcation, a new difficulty is also expected to occur by the presence of subtle resonance phenomena (see also \([11]\)).

2. **The curve \( \gamma \) and its harmonic measures \( W_{\gamma}^{\pm} \)**

**Proposition 2.1.** There exists a smooth closed Jordan curve \( \gamma \) in \( \Omega \), dividing \( \Omega \) in two open domains \( \Omega^+ \) (the outer) and \( \Omega^- \) (the inner), with the following property:
Suppose that $u^\pm$ satisfy classically
\[
\begin{aligned}
\Delta u^\pm &= 0 \text{ in } \Omega^\pm, \\
u^\pm &= 0 \text{ on } \partial\Omega^\pm \cap \partial\Omega,
\end{aligned}
\tag{2.1}
\]
and
\[
\partial_t^k u^+ = (-1)^k \partial_t^k u^- \text{ on } \gamma,
\tag{2.2}
\]
for some integer $k \in \{0, 1\}$; where $\partial_t$ denotes derivation in the direction perpendicular to the curve $\gamma$ (see (4.1) below). Then, it holds that
\[
\partial_t^{1-k} u^+ = (-1)^{1-k} \partial_t^{1-k} u^- \text{ on } \gamma.
\tag{2.3}
\]

Proof. Since $\Omega$ is doubly connected, by Theorem 5.10h in [18] (see also [19, Chpt. 15]), there exists a one-to-one transformation $w = f(z)$ that maps $\Omega$ conformally onto the annulus
\[
\mathcal{A} = \{ a < r < b \},
\]
where $r = |w|$, for some $a, b > 0$. Here we have identified $\mathbb{R}^2$ with the complex plane. In fact, the ratio $b/a$ is unique. Furthermore, without loss of generality, we may assume that $\partial\Omega^+$ maps onto $r = b$. We will first verify the assertions of the proposition for the case of the annulus, exploiting that the corresponding solutions can be represented explicitly, and then argue that they continue to hold for the general case by transplanting back.

In the “ideal” case where $\Omega = \mathcal{A}$ and $\gamma$ is the circle
\[
\mathcal{C} = \{ r = \sqrt{ab} \}
\]
(this is not a guess but is based on the analysis in [16]), we have the following explicit formulas for the solutions $V^\pm$ of (2.1) (see [19, Chpt. 15] or [28, Chpt. 6]):
\[
V^-(r, \theta) = A_0^- \ln \left( \frac{r}{a} \right) + \sum_{n=1}^{\infty} A_n^- \left[ \left( \frac{r}{a} \right)^n - \left( \frac{r}{a} \right)^{-n} \right] \cos(n\theta) + \sum_{n=1}^{\infty} B_n^- \left[ \left( \frac{r}{a} \right)^n - \left( \frac{r}{a} \right)^{-n} \right] \sin(n\theta),
\]
a $\leq r \leq \sqrt{ab}$, $-\pi \leq \theta \leq \pi$, and
\[
V^+(r, \theta) = A_0^+ \ln \left( \frac{b}{r} \right) + \sum_{n=1}^{\infty} A_n^+ \left[ \left( \frac{b}{r} \right)^n - \left( \frac{b}{r} \right)^{-n} \right] \cos(n\theta) + \sum_{n=1}^{\infty} B_n^+ \left[ \left( \frac{b}{r} \right)^n - \left( \frac{b}{r} \right)^{-n} \right] \sin(n\theta),
\]
\[
\sqrt{ab} \leq r \leq b, -\pi \leq \theta \leq \pi,
\]
where $A^\pm_n$ and $B^\pm_n$, $n = 1, \ldots$ are arbitrary constants. In particular, we find that
\[
V^-(\sqrt{ab}, \theta) = A_0^- \ln \sqrt{\frac{b}{a}} + \sum_{n=1}^{\infty} A_n^- \left[ \left( \frac{b}{a} \right)^{\frac{n}{2}} - \left( \frac{b}{a} \right)^{-\frac{n}{2}} \right] \cos(n\theta) + \sum_{n=1}^{\infty} B_n^- \left[ \left( \frac{b}{a} \right)^{\frac{n}{2}} - \left( \frac{b}{a} \right)^{-\frac{n}{2}} \right] \sin(n\theta),
\]
\[
V^+(\sqrt{ab}, \theta) = A_0^+ \ln \sqrt{\frac{b}{a}} + \sum_{n=1}^{\infty} A_n^+ \left[ \left( \frac{a}{b} \right)^{\frac{n}{2}} - \left( \frac{a}{b} \right)^{-\frac{n}{2}} \right] \cos(n\theta) + \sum_{n=1}^{\infty} B_n^+ \left[ \left( \frac{a}{b} \right)^{\frac{n}{2}} - \left( \frac{a}{b} \right)^{-\frac{n}{2}} \right] \sin(n\theta),
\]
and
\[
V_{r}^-(\sqrt{ab}, \theta) = \frac{A_0^-}{\sqrt{ab}} + \sum_{n=1}^{\infty} \frac{n A_n^-}{a} \left[ \left( \frac{b}{a} \right)^{\frac{n-1}{2}} + \left( \frac{b}{a} \right)^{-\frac{n+1}{2}} \right] \cos(n\theta) + \sum_{n=1}^{\infty} \frac{n B_n^-}{a} \left[ \left( \frac{b}{a} \right)^{\frac{n-1}{2}} + \left( \frac{b}{a} \right)^{-\frac{n+1}{2}} \right] \sin(n\theta),
\]
\[
V_{r}^+(\sqrt{ab}, \theta) = -\frac{A_0^+}{\sqrt{ab}} + \sum_{n=1}^{\infty} \frac{n A_n^+}{b} \left[ \left( \frac{a}{b} \right)^{\frac{n-1}{2}} + \left( \frac{a}{b} \right)^{-\frac{n+1}{2}} \right] \cos(n\theta) + \sum_{n=1}^{\infty} \frac{n B_n^+}{b} \left[ \left( \frac{a}{b} \right)^{\frac{n-1}{2}} + \left( \frac{a}{b} \right)^{-\frac{n+1}{2}} \right] \sin(n\theta),
\]
for \(-\pi \leq \theta \leq \pi\). If \(V^\pm\) satisfy (2.2) for some \(k \in \{0, 1\}\) (with \(\gamma = C\)), it follows that 
\[
A_0^- = A_0^+, \quad A_n^- = -A_n^+, \quad B_n^- = -B_n^+, \quad n = 1, \cdots
\]
(we remark that \(\partial_t = -\partial_r\)). It is then straightforward to verify that this implies that (2.3) holds.

Let us assume now that \(\Omega \subset \mathbb{R}^2\) is a general bounded, doubly connected domain with smooth boundary and that \(f\) is a one-to-one conformal map that maps \(\Omega\) onto the annulus \(\mathcal{A}\), as described above. Set
\[
\gamma = f^{-1}(C),
\]
and define \(\Omega^\pm \subset \Omega\) accordingly. Assume that \(u^\pm\) satisfy (2.1) and let
\[
V^\pm(w) = u^\pm(f^{-1}(w)), \quad w \in \mathcal{A}, \tag{2.4}
\]
denote their corresponding conformal transplants under the mapping \(f\). As is well known, the functions \(V^\pm\) are still harmonic and thus satisfy (2.1) with \(\Omega = \mathcal{A}\) and \(\gamma = C\). If
\[
u_w^+ = u^- \quad \text{on} \quad \gamma,
\]
then clearly
\[
V^+ = V^- \quad \text{on} \quad C.
\]
As we showed previously, the above relation implies that
\[
(\nabla V^-(w), \nu_w) + (\nabla V^+(w), \nu_w) = 0 \quad \text{on} \quad C, \tag{2.5}
\]
where \(\nu_w\) stands for the inner unit normal to \(C\) at \(w \in C\), while \((\cdot, \cdot)\) stands for the Euclidean inner product in \(\mathbb{R}^2\). If
\[
z = f^{-1}(w) \in \gamma,
\]
and \(n_z\) denotes the inner unit normal to \(\gamma\) at \(z\), we have that
\[
\nu_w = \kappa Df(z)n_z \quad \text{for some} \quad \kappa \in \{-1, 1\}. \tag{2.6}
\]
In order not to interrupt the line of thought, we will show this at the end of the proof. Therefore, in view of (2.4) and (2.5), we obtain that
\[
(Df^{-1}(w)\nabla u^-(z), Df(z)n_z) + (Df^{-1}(w)\nabla u^+(z), Df(z)n_z) = 0 \quad \text{on} \quad \gamma.
\]
Using that \(Df^{-1}(w)Df(z) = I\) and the orthogonality of \(Df^{-1}(w)\) (by the fact that \(f^{-1}\) is conformal), we deduce that
\[
(\nabla u^-(z), n_z) + (\nabla u^+(z), n_z) = 0 \quad \text{on} \quad \gamma,
\]
as desired.

If \(\partial_t u^+ = -\partial_t u^- \quad \text{on} \quad \gamma\), we can show that \(u^+ = u^- \quad \text{on} \quad \gamma\) by reversing the above steps.

It remains to show (2.6). To this end, let \(\tau_z\) be a (nontrivial) tangent vector to \(\gamma\) at \(z\). Then, since \(f(\gamma) = C\), it follows that the vector
\[
T_w = Df(z)\tau_z
\]
is tangent to \(C\) at \(w\). Moreover, since the matrix \(Df(z)\) is orthogonal (by the conformality of \(f\), that is \(Df(z)Df(z) = I\)), we see that \(|T_w| = |\tau_z|\). The orthogonality of \(Df(z)\) implies in addition that the angle between \(\tau_z\) and \(n_z\) is the same as the one between \(T_w\) and \(Df(z)n_z\). Hence, the vector \(Df(z)n_z\) is normal to \(C\) at \(w\). So, since it has unit norm (again by the orthogonality of \(Df(z)\)), we infer that (2.6) holds. \(\square\)
Definition 2.2. [19, Chpt. 15.5] The unique functions \( W_+^\gamma, \ W_-^\gamma \) such that
\[
\begin{cases}
\Delta W_+^\gamma = 0 \text{ in } \Omega^+,
W_+^\gamma = 0 \text{ on } \partial\Omega^+ \cap \partial\Omega,
W_+^\gamma = 1 \text{ on } \gamma,
\end{cases}
\]
are called the harmonic measures of \( \gamma \) with respect to \( \partial\Omega^+ \cap \partial\Omega, \partial\Omega^- \cap \partial\Omega \) respectively.

It follows from Proposition 2.1 that
\[
\partial_n W_+^\gamma + \partial_n W_-^\gamma = 0 \text{ on } \gamma,
\]
where \( n \) denotes the unit normal vector to \( \gamma \) pointing in \( \Omega^- \). Furthermore, by the strong maximum principle, we deduce that
\[
0 < W_+^\gamma < 1 \text{ in } \Omega^+.
\]
Moreover, from Hopf’s boundary point lemma, we have that
\[
\pm \partial_n W_+^\gamma > 0 \text{ on } \gamma.
\]

3. The function \( V_0 \) and the associated linearized problem

The function
\[
V_0(x) = \ln \frac{4e^{\sqrt{2}x}}{(1 + e^{\sqrt{2}x})^2}
\]
solves
\[
v_{xx} + e^v = 0, \quad x \in \mathbb{R}.
\]
We see that \( V_0(0) = (V_0)_x(0) = 0 \), \( V_0 \) is even and
\[
\begin{cases}
V_0(x) = \sqrt{2}x + \ln 4 + \mathcal{O} \left( e^{-\sqrt{2}|x|} \right), \quad x \to -\infty, \\
(V_0)_x(x) = \sqrt{2} + \mathcal{O} \left( e^{-\sqrt{2}|x|} \right), \quad x \to -\infty.
\end{cases}
\]
Note that
\[
V(x) = 2\ln \mu + V_0(\mu(x - h))
\]
also solves (3.2) for every \( \mu > 0 \) and \( h \in \mathbb{R} \). It follows that the linear equation
\[
\psi_{xx} + e^{V_0} \psi = 0, \quad x \in \mathbb{R},
\]
has two linearly independent solutions given by
\[
(V_0)_x \text{ and } x(V_0)_x + 2.
\]

The variation of constants formula yields (see also [17]):

Lemma 3.1. Suppose that \( g \in C(\mathbb{R}) \) satisfies \(|g(x)| + |g_x(x)| \leq De^{-d|x|}, \ x \in \mathbb{R}, \) for some constants \( d, D > 0 \).

If \( g \) is even, then the linear equation
\[
\psi_{xx} + e^{V_0} \psi = g, \quad x \in \mathbb{R},
\]
has two linearly independent solutions given by
\[
(V_0)_x \text{ and } x(V_0)_x + 2.$$
Given a smooth function $\Delta$ for every $\alpha = \alpha(d, D > 0)$.

If $g$ is odd, then (3.6) has a one parameter family of odd solutions satisfying

$$
\psi(x) = -\sqrt{2} \Delta x + \frac{1}{\sqrt{2}} \int_{-\infty}^{0} (t(V_0)_t(t) + 2) g(t) dt - 2\Delta + \mathcal{O}(e^{-\alpha|x|}) + \Delta \mathcal{O}\left(e^{-\sqrt{2}|x|}\right), \quad x \to -\infty,
$$

$$
\psi_\ell(x) = -\sqrt{2} \Delta + \mathcal{O}(e^{-\alpha|x|}) + \Delta \mathcal{O}\left(e^{-\sqrt{2}|x|}\right), \quad x \to -\infty,
$$

for every $\Delta \in \mathbb{R}$, where $\alpha = \alpha(d, D > 0)$.

4. The inner approximation

4.1. The set up near the curve $\gamma$. Let $\gamma$ be the closed smooth curve in Proposition 2.1, and $\ell = |\Gamma|$ its total length. We consider the natural parametrization $\gamma = \gamma(s)$ of $\Gamma$ with positive orientation, where $s$ denotes an arc length parameter measured from a fixed point of $\Gamma$. Let $n(s)$ denote the inner unit normal to $\Gamma$. Points $y$ that are $\delta_0$-close to $\Gamma$, for sufficiently small $\delta_0$, can be represented in the form

$$
y = \gamma(s) + t n(s), \quad s \in [0, \ell], \quad |t| < \delta_0, \quad (4.1)
$$

where the map $y \mapsto (s, t)$ is a local diffeomorphism. Note that we have $0 < t < \delta_0$ in $\Omega^-$. For any smooth function $u$ that is defined in this region, identifying $u(y)$ with $u(s, t)$ (allowing some abuse of notation), we have that

$$
\nabla_y u = \left(\frac{u_s}{1 - k t}, u_t\right), \quad (4.2)
$$

(with the obvious interpretation). Therefore, equation (1.1) for $u$ expressed in these coordinates becomes

$$
S(u) := u_{tt} + \frac{1}{a} u_{ss} - \frac{\partial a}{2a^2} u_s + \frac{\partial a}{2a} u_t + \lambda^2 e^u = 0, \quad (4.3)
$$

in the region described in (4.3), where $a = (1 - tk(s))^2$, and $k$ is the curvature of $\gamma$.

Let $\mu, f \in C^2_{\text{per}}([0, \ell])$, the space of $\ell$-periodic, $C^2$-functions, with $\mu > 0$, and

$$
c \ln \frac{1}{\lambda} \leq \lambda \mu \leq C \ln \frac{1}{\lambda}, \quad \lambda \mu f = o(\ln(\lambda \mu)) \quad \text{as } \lambda \to 0 \quad \text{on } \gamma. \quad (4.4)
$$

Given a smooth function $u$, defined close to the curve $\gamma$, let

$$
u(s, t) = v(s, x) + 2 \ln \mu, \quad x = \lambda \mu(t - f), \quad s \in [0, \ell], \quad |t| < \delta_0. \quad (4.5)$$
We want to express equation (1.1) in terms of these new coordinates (in the region described in (4.5)). We compute:

\[
\begin{aligned}
    u_t &= \lambda \mu v_x, \\
    u_{tt} &= \lambda^2 \mu^2 v_{xx}, \\
    u_s &= v_s + (\mu' \mu^{-1} x - \lambda \mu f') v_x + 2 \mu' \mu^{-1}, \\
    u_{ss} &= v_{ss} + 2(\mu' \mu^{-1} x - \lambda \mu f') v_{sx} + (\mu' \mu^{-1} x - \lambda \mu f')^2 v_{xx} + (\mu'' \mu^{-1} x - 2 \lambda \mu f' - \lambda \mu f'') v_x + 2 \mu'' \mu^{-1} - 2(\mu')^2 \mu^{-2}.
\end{aligned}
\]  

(4.6)

A short calculation shows that \( u \) solves (4.3) if and only if \( v \), defined in (4.5), solves \( S(v + 2 \ln \mu) = 0 \), where

\[
S(v + 2 \ln \mu) = \lambda^2 \mu^2 v_{xx} + \frac{1}{1 - (\lambda^{-1} \mu^{-1} x + \ell)^k} \left[ v_{ss} + 2(\mu' \mu^{-1} x - \lambda \mu f') v_{sx} + (\mu' \mu^{-1} x - \lambda \mu f')^2 v_{xx} + (\mu'' \mu^{-1} x - 2 \lambda \mu f' - \lambda \mu f'') v_x + 2 \mu'' \mu^{-1} - 2(\mu')^2 \mu^{-2} \right] - \frac{k}{1 - (\lambda^{-1} \mu^{-1} x + \ell)^k \lambda \mu v_x} + \frac{\lambda^{-1} \mu^{-1} x + k'}{1 - (\lambda^{-1} \mu^{-1} x + \ell)^k} \left[ v_s + (\mu' \mu^{-1} x - \lambda \mu f') v_x + 2 \mu' \mu^{-1} \right] + \lambda^2 \mu^2 v.
\]  

(4.7)

4.2. The first order inner approximate solution. As a first order approximation, valid near the curve \( \gamma \), we consider \( u_0 \) as described in (4.5) with \( v = V_0(x) \), defined in (3.1), i.e.,

\[
u_0(s, t) = 2 \ln \mu + V_0(\lambda \mu(t - f)), \quad s \in [0, \ell], \quad |t| \leq \lambda^{-1} \mu^{-1} L.
\]  

(4.8)

Here \( L \) satisfies

\[
M \ln \left( \ln \frac{1}{\lambda} \right) \leq L \leq 2M \ln \left( \ln \frac{1}{\lambda} \right),
\]  

(4.9)

for some large constant \( M > 0 \) to be determined independently of \( \lambda \) (recall (4.4)).

In view of (4.7), we have

\[
S(u_0) = \frac{1}{1 - (\lambda^{-1} \mu^{-1} x + \ell)^k} \left[ - (\mu' \mu^{-1} x - \lambda \mu f')^2 e^{V_0} + (\mu'' \mu^{-1} x - 2 \lambda \mu f' - \lambda \mu f'') (V_0)_x + 2 \mu'' \mu^{-1} - 2(\mu')^2 \mu^{-2} \right] - k\lambda \mu (V_0)_x - k \left[ \frac{1}{1 - (\lambda^{-1} \mu^{-1} x + \ell)^k} - 1 \right] \lambda \mu (V_0)_x
\]

\[
+ \frac{\lambda^{-1} \mu^{-1} x + k'}{1 - (\lambda^{-1} \mu^{-1} x + \ell)^k} \left[ (\mu' \mu^{-1} x - \lambda \mu f') (V_0)_x + 2 \mu' \mu^{-1} \right].
\]  

(4.10)

4.3. The second order inner approximate solution. We search a refined approximate solution, valid near the curve \( \gamma \), in the form

\[
u_1(s, t) = 2 \ln \mu + V_0(\lambda \mu(t - f)) + \phi(s, \lambda \mu(t - f)), \quad s \in [0, \ell], \quad |t| \leq \lambda^{-1} \mu^{-1} L,
\]  

(4.11)

with \( \phi \) to be determined.
In view of (4.7), (4.10), we have
\[
S(u_1) = S(u_0) + \lambda^2 \mu^2 \left[ e^{V_0 + \phi} - e^{V_0} - e^{V_0} \phi \right] + \lambda^2 \mu^2 \left[ \phi_{xx} + e^{V_0} \phi \right] \\
+ \frac{1}{2-\lambda \mu} \left[ \phi_{ss} + 2(\mu' \mu^{-1}x - \lambda \mu f') \phi_{sx} + (\mu' \mu^{-1}x - \lambda \mu f')^2 \phi_{xx} \\
+ (\mu'' \mu^{-1}x - 2 \lambda \mu f' - \lambda \mu f'') \phi_x \right] - \frac{k}{2-\lambda \mu} \lambda \mu \phi_x \\
+ \frac{\lambda}{1-\lambda \mu} \left[ \phi_s + (\mu' \mu^{-1}x - \lambda \mu f') \phi_x \right].
\]
(4.12)

Since, at least formally,
\[
S(u_0) = -k \lambda \mu (V_0)_x + \text{lower order terms},
\]
we choose \( \phi = \phi_1(s, x) \) to satisfy, for fixed \( s \in [0, \ell] \),
\[
\phi_{xx} + e^{V_0} \phi = \lambda^{-1} \mu^{-1} k(V_0)_x, \quad x \in \mathbb{R}.
\]
(4.13)

**Lemma 4.1.** Equation (4.13) has a family of solutions such that
\[
\lambda \mu \phi_1(s, x) = \frac{1}{\sqrt{2}} k x^2 + 2 k x - \sqrt{2} \Delta_1 x - 2 \Delta_1 - E_1 + (1 + |\Delta_1| + |E_1|) O(e^{-c|x|}), \quad x \to -\infty,
\]
\[
\lambda \mu (\phi_1)_x(s, x) = \sqrt{2} k x + 2 k - \sqrt{2} \Delta_1 + (1 + |\Delta_1| + |E_1|) O(e^{-c|x|}), \quad x \to -\infty,
\]
\[
\lambda \mu \phi_1(s, x) = -\frac{1}{\sqrt{2}} k x^2 + 2 k x + \sqrt{2} \Delta_1 x - 2 \Delta_1 + E_1 + (1 + |\Delta_1| + |E_1|) O(e^{-c|x|}), \quad x \to \infty,
\]
\[
\lambda \mu (\phi_1)_x(s, x) = -\sqrt{2} k x + 2 k + \sqrt{2} \Delta_1 + (1 + |\Delta_1| + |E_1|) O(e^{-c|x|}), \quad x \to \infty,
\]
for every \( \Delta_1, \ E_1 \in C^2_{\text{per}}([0, \ell]) \).

**Proof.** Let
\[
Z_1(s, x) = k \frac{x^2}{2} (V_0)_x, \quad x \in \mathbb{R}.
\]
(4.14)

We see that
\[
(Z_1)_{xx} + e^{V_0} Z_1 = k (V_0)_x + k \frac{x^2}{2} (V_0)_{xxx} + 2 k x (V_0)_{xx} + k \frac{x^2}{2} e^{V_0} (V_0)_x \\
= k (V_0)_x - 2 k x e^{V_0}.
\]
We write
\[
\phi = \lambda^{-1} \mu^{-1} Z_1 + \psi.
\]
(4.15)

In terms of \( \psi \), equation (4.13) becomes
\[
\psi_{xx} + e^{V_0} \psi = 2 \lambda^{-1} \mu^{-1} k x e^{V_0}.
\]
(4.16)

We can now apply Lemma 3.1 to the above equation, since its righthand side decays exponentially to zero as \( x \to \pm \infty \) and is odd (plus the trivial even function). Note that
\[
\int_{-\infty}^{\infty} x e^{V_0}(V_0)_x dx = - \int_{-\infty}^{0} e^{V_0} dx = \int_{-\infty}^{0} (V_0)_{xx} dx = -\sqrt{2}.
\]
Hence, for every $\Delta_1$, $E_1 \in C^2_{\text{per}}([0,\ell])$, there exists a solution $\psi_1$ of (4.16) such that

$$\lambda \mu \psi_1(s,x) = 2kx + \sqrt{2} \Delta_1 x - 2 \Delta_1 + E_1 + (1 + |\Delta_1| + |E_1|)O\left(e^{-c|x|}\right), \quad x \to \infty,$$

$$\lambda \mu \psi_1(s,x) = 2kx - \sqrt{2} \Delta_1 x - 2 \Delta_1 - E_1 + (1 + |\Delta_1| + |E_1|)O\left(e^{-c|x|}\right), \quad x \to -\infty,$$

and the corresponding estimates for $\left(\psi_1\right)_x$.

Let $\phi_1$ be defined by relation (4.15) with $\psi = \psi_1$. Then $\phi_1$ solves equation (4.13) and has the desired asymptotic behavior as in the assertion of the lemma.

The proof of the lemma is complete. \qed

In view of the above lemma, and (4.11), we have

$$u_1(s,-\lambda^{-1} \mu^{-1} L) = 2 \ln(2\mu) - \sqrt{2}(L + \lambda \mu f) + \lambda^{-1} \mu^{-1} \frac{k}{\sqrt{2}}(L + \lambda \mu f)^2 - 2 \lambda^{-1} \mu^{-1} k(L + \lambda \mu f)$$

$$+ \sqrt{2} \Delta_1 \lambda^{-1} \mu^{-1} (L + \lambda \mu f) - 2 \lambda^{-1} \mu^{-1} \Delta_1 - \lambda^{-1} \mu^{-1} E_1$$

$$+ \lambda^{-1} \mu^{-1} (1 + |\Delta_1| + |E_1|) e^{-c(L+\lambda \mu f)}, \quad (4.17)$$

and

$$u_1(s,\lambda^{-1} \mu^{-1} L) = 2 \ln(2\mu) - \sqrt{2}(L - \lambda \mu f) - \lambda^{-1} \mu^{-1} \frac{k}{\sqrt{2}}(L - \lambda \mu f)^2 + 2 \lambda^{-1} \mu^{-1} k(L - \lambda \mu f)$$

$$+ \sqrt{2} \Delta_1 \lambda^{-1} \mu^{-1} (L - \lambda \mu f) - 2 \lambda^{-1} \mu^{-1} \Delta_1 + \lambda^{-1} \mu^{-1} E_1$$

$$+ \lambda^{-1} \mu^{-1} (1 + |\Delta_1| + |E_1|) e^{-c(L-\lambda \mu f)}. \quad (4.18)$$

4.4. **The third order inner approximate solution.** We search a refined approximate solution, valid near the curve $\gamma$, in the form

$$u_2(s,t) = u_1(s,t) + \phi(s,\lambda \mu(t-f))$$

$$= 2 \ln \mu + V_0(\lambda \mu(t-f)) + \phi_1(s,\lambda \mu(t-f)) + \phi(s,\lambda \mu(t-f)),$$  (4.19)

$s \in [0,\ell], \ |t| \leq \lambda^{-1} \mu^{-1} L$, with $\phi$ to be determined.
In view of (4.7), by a careful calculation and rearrangement of terms, we can write

\[
S(u_2) = \lambda^2 \mu^2 (\phi_{xx} + e^{V_0} \phi) + \frac{1}{\lambda^2 - 1} \left[ (\phi + \phi)_{ss} + 2(\mu' \mu^{-1} x - \lambda \mu f') (\phi + \phi)_{sx} \right. \\
+ (\mu' \mu^{-1} x - \lambda \mu f') (V_0 \phi + \phi)_{xx} + (\mu'' \mu^{-1} x - 2 \lambda \mu f' - \lambda \mu f'')(V_0 \phi + \phi)_{x} \\
+ \frac{1}{2} \mu'' \mu^{-1} - (2 \mu')^2 \mu^{-2} \left. - k \lambda \mu (\phi + \phi)_{x} - k^2 (x + \lambda \mu f)(V_0 \phi + \phi)_{x} \right] \\
\left. - k \lambda \mu \left[ \frac{1}{\lambda^2 - 1} \frac{1}{x + f} - 1 - (\lambda^{-1} \mu^{-1} x + f) k \right] (V_0 + \phi)_{x} \right] \\
+ \frac{1}{\lambda^2 - 1} \frac{1}{x + f} \left[ (\phi + \phi)_{s} + (\mu' \mu^{-1} x - \lambda \mu f')(V_0 + \phi)_{x} + 2 \mu' \mu^{-1} \right] + \lambda^2 \mu^2 \mu^2 \phi_1^2 \\
+ \lambda^2 \mu^2 (e^{V_0 + \phi_1} - e^{V_0} - e^{V_0} \phi_1 - e^{V_0} \phi_1^2 + \lambda^2 \mu^2 (e^{V_0 + \phi_1} - e^{V_0} - e^{V_0} \phi_1) \\
+ \lambda^2 \mu^2 (e^{V_0} + \phi_1 - e^{V_0}) \phi, \right. \\
(4.20)
\]

where we have underbraced higher order terms.

Motivated from the above relation, and recalling the equation satisfied by \( V_0 \), we choose \( \phi = \phi_2(s, x) \) to satisfy, for fixed \( s \in [0, \ell] \),

\[
\phi_{xx} + e^{V_0} \phi = \lambda^{-2} \mu^{-2} (\mu' \mu^{-1} x - \lambda \mu f')^2 e^{V_0} - \lambda^{-2} \mu^{-2} (\mu'' \mu^{-1} x - 2 \lambda \mu f' - \lambda \mu f'')(V_0)_{x} \\
- 2 \lambda^{-2} \mu'' \mu^{-3} + 2 \lambda^{-2} (\mu')^2 \mu^{-4} + \lambda^{-1} \mu^{-1} k(\phi)_{x} \\
+ \lambda^{-2} \mu^{-2} k^2 (x + \lambda \mu f)(V_0)_{x} - \frac{1}{2} e^{V_0} \phi_1^2,
\]

\( x \in \mathbb{R} \).

Arguing as in the proof of Lemma 4.1 we have:

**Lemma 4.2.** Equation (4.21) has a family of solutions such that

\[
\phi_2(s, x) = \left( -\frac{\sqrt{2}}{6} \lambda^{-2} \mu'' \mu^{-3} + \frac{\sqrt{2}}{3} \lambda^{-2} \mu^{-2} k^2 \right) x^3 + \left( -\lambda^{-2} \mu'' \mu^{-3} + \lambda^{-2} (\mu')^2 \mu^{-4} + k^2 \lambda^{-2} \mu^{-2} \right. \\
- \frac{1}{\sqrt{2}} \lambda^{-2} \mu^{-2} \Delta_{1} k + \sqrt{2} \lambda^{-1} \mu^{-2} f' + \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} f'' + \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} k^2 f \right) x^2 \\
+ \left( -\sqrt{2} \Delta_{2} + B_{2} \right) x + A_2 - 2 \Delta_{2} - E_2 + \sum_{i=1}^{2} (|\Delta_{i}| + |E_{i}| + 1)^2 \mathcal{O}(e^{-c|x|}),
\]

as \( x \to -\infty \), and

\[
\phi_2(s, x) = \left( \frac{\sqrt{2}}{6} \lambda^{-2} \mu'' \mu^{-3} - \frac{\sqrt{2}}{3} \lambda^{-2} \mu^{-2} k^2 \right) x^3 + \left( -\lambda^{-2} \mu'' \mu^{-3} + \lambda^{-2} (\mu')^2 \mu^{-4} + k^2 \lambda^{-2} \mu^{-2} \right. \\
+ \frac{1}{\sqrt{2}} \lambda^{-2} \mu^{-2} \Delta_{1} k - \sqrt{2} \lambda^{-1} \mu^{-2} f' - \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} f'' - \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} k^2 f \right) x^2 \\
+ \left( \sqrt{2} \Delta_{2} + B_{2} \right) x + A_2 - 2 \Delta_{2} + E_2 + \sum_{i=1}^{2} (|\Delta_{i}| + |E_{i}| + 1)^2 \mathcal{O}(e^{-c|x|}),
\]
as \( x \to \infty \), where
\[
A_2 = a_1(f')^2 + a_2 \lambda^{-2} \mu'' \mu^{-3} + a_3 \lambda^{-2} \mu^{-2} k^2 + a_4 \lambda^{-2} \mu^{-2} \Delta_1^2 + a_5 \lambda^{-2} \mu^{-2} E_1^2 + a_6 \lambda^{-2} \mu^{-2} k E_1,
\]
\[
B_2 = b_1 \lambda^{-1} \mu^{-1} f'' + b_2 \lambda^{-1} \mu^{-1} \mu' f' + b_3 \lambda^{-2} \mu^{-2} k \Delta_1 + b_4 \lambda^{-1} \mu^{-1} k^2 f + b_5 \lambda^{-2} \mu^{-2} \Delta_1 E_1,
\]
for some known constants \( a_i, b_i \) (independent of \( \lambda \)), determined by integrals of known functions, for every \( \Delta_i, E_i \in C^2_{\text{per}}([0, \ell]) \), \( i = 1, 2 \) (\( \Delta_1, E_1 \) determine \( \phi_1 \) by Lemma 4.2, note that here we opted not to place the coefficient \( \lambda^{-2} \mu^{-2} \) in front of \( \Delta_2, E_2 \)). Moreover, the above estimates can be differentiated.

In view of the above lemma, and (4.19), we have
\[
\begin{align*}
u_2(s, -\lambda^{-1} \mu^{-1} L) & = u_1(s, -\lambda^{-1} \mu^{-1} L) - \frac{\sqrt{2}}{6}(2 \lambda^{-2} \mu^{-2} k^2 - \lambda^{-2} \mu^{-3} \mu'')(L + \lambda \mu f)^3 \\
& + \left(-\lambda^{-2} \mu'' \mu^{-3} + \lambda^{-2}(\mu')^2 \mu^{-4} + k^2 \lambda^{-2} \mu^{-2} - \frac{1}{\sqrt{2}} \lambda^{-2} \mu^{-2} \Delta_1 k + \sqrt{2} \lambda^{-1} \mu^{-1} \mu' f' \\
& + \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} f'' + \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} k^2 f \right) (L + \lambda \mu f)^2 \\
& + (\sqrt{2} \Delta_2 - B_2)(L + \lambda \mu f) + A_2 - 2 \Delta_2 - E_2 \\
& + \sum_{i=1}^{2} (|\Delta_i| + |E_i| + 1)^2 e^{-c(L+\lambda \mu f)},
\end{align*}
\]
and
\[
\begin{align*}
u_2(s, \lambda^{-1} \mu^{-1} L) & = u_1(s, \lambda^{-1} \mu^{-1} L) + \frac{\sqrt{2}}{6}(-2 \lambda^{-2} \mu^{-2} k^2 + \lambda^{-2} \mu^{-3} \mu'')(L - \lambda \mu f)^3 \\
& + \left(-\lambda^{-2} \mu'' \mu^{-3} + \lambda^{-2}(\mu')^2 \mu^{-4} + k^2 \lambda^{-2} \mu^{-2} + \frac{1}{\sqrt{2}} \lambda^{-2} \mu^{-2} \Delta_1 k - \sqrt{2} \lambda^{-1} \mu^{-1} \mu' f' \\
& - \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} f'' - \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} k^2 f \right) (L - \lambda \mu f)^2 \\
& + (\sqrt{2} \Delta_2 + B_2)(L - \lambda \mu f) + A_2 - 2 \Delta_2 + E_2 \\
& + \sum_{i=1}^{2} (|\Delta_i| + |E_i| + 1)^2 e^{-c(L-\lambda \mu f)},
\end{align*}
\]

5. The outer approximations

In this section we will construct outer approximations \( W^\pm \), valid in \( \Omega^\pm \) respectively, with the following properties: \( W^\pm \) are harmonic in \( \Omega^\pm \) with zero boundary conditions on \( \partial \Omega^\pm \cap \partial \Omega \), and they match, in the \( C^1 \)-sense with the inner approximation \( u_i \) of \( i + 1 \) order, close to the curve \( \gamma \).

5.1. The second order outer approximation. By virtue of (4.4), (4.9), we can expand
\[
\begin{align*}
W^+(s, -\lambda^{-1} \mu^{-1} L) & = W^+(s, 0) - (\partial_n W^+(s, 0)) \lambda^{-1} \mu^{-1} L + \frac{1}{2} W^{++}_n(s, 0) \lambda^{-2} \mu^{-2} L^2 \\
& - \frac{1}{6} W^{+++}_n(s, 0) \lambda^{-3} \mu^{-3} L^3 + \mathcal{O}(W^{+++}_n) \lambda^{-4} \mu^{-4} L^4,
\end{align*}
\]
and
\[ W^-(s, +\lambda^{-1}\mu^{-1}L) = W^-(s, 0) + (\partial_n W^-(s, 0)) \lambda^{-1}\mu^{-1}L + \frac{1}{2} W_{tt}^-(s, 0) \lambda^{-2}\mu^{-2}L^2 + \frac{1}{4} W_{ttt}^-(s, 0) \lambda^{-3}\mu^{-3}L^3 + \mathcal{O}(W_{ttt}^-) \lambda^{-4}\mu^{-4}L^4. \tag{5.2} \]

From (4.17) and (5.1), we find that
\[ (u_1 - W^+)(s, -\lambda^{-1}\mu^{-1}L) = 2 \ln(2\mu) - \sqrt{2}\lambda\mu f - W^+(s, 0) \]
\[ + \lambda^{-1}\mu^{-1} \left( \frac{k}{\sqrt{2}} \lambda^2 f^2 - 2k\lambda\mu f + \sqrt{2}\Delta_1 \lambda\mu f - 2\Delta_1 - E_1 \right) \]
\[ + \lambda^{-1}\mu^{-1}L \left( -\sqrt{2}\lambda\mu + \sqrt{2}k\lambda\mu f - 2k + \sqrt{2}\Delta_1 + \partial_n W^+(s, 0) \right) \]
\[ + \lambda^{-1}\mu^{-1}L^2 \left( -\frac{k}{\sqrt{2}} - \frac{1}{2} W_{tt}^+(s, 0) \lambda^{-1}\mu^{-1} \right) \]
\[ - \mathcal{O}(W_{ttt}^+) \lambda^{-3}\mu^{-3}L^3 + \lambda^{-1}\mu^{-1} (1 + |\Delta_1| + |E_1|) e^{-c(L+\lambda\mu f)}. \tag{5.3} \]

Similarly, via (4.18) and (5.2), we have
\[ (u_1 - W^-)(s, \lambda^{-1}\mu^{-1}L) = 2 \ln(2\mu) + \sqrt{2}\lambda\mu f - W^-(s, 0) \]
\[ + \lambda^{-1}\mu^{-1} \left( -\frac{k}{\sqrt{2}} \lambda^2 f^2 - 2k\lambda\mu f - \sqrt{2}\Delta_1 \lambda\mu f - 2\Delta_1 + E_1 \right) \]
\[ + \lambda^{-1}\mu^{-1}L \left( \sqrt{2}\lambda\mu + \sqrt{2}k\lambda\mu f + 2k - \sqrt{2}\Delta_1 - \partial_n W^-(s, 0) \right) \]
\[ + \lambda^{-1}\mu^{-1}L^2 \left( -\frac{k}{\sqrt{2}} - \frac{1}{2} W_{tt}^-(s, 0) \lambda^{-1}\mu^{-1} \right) \]
\[ - \mathcal{O}(W_{ttt}^-) \lambda^{-3}\mu^{-3}L^3 + \lambda^{-1}\mu^{-1} (1 + |\Delta_1| + |E_1|) e^{-c(L-\lambda\mu f)}. \tag{5.4} \]

We claim that there exist harmonic functions \( W^\pm \) in \( \Omega^\pm \), satisfying Dirichlet boundary conditions on \( \partial \Omega^\pm \cap \partial \Omega \), and smooth functions \( f, \mu > 0, \Delta_1, E_1 \) on \( \gamma \), satisfying (4.4), such that the following hold: The first two lines of the righthand side of (5.3), (5.4) respectively vanish, and the third and fourth vanish up to main order.

We seek suitable \( W^\pm \) in the form
\[ W_i^\pm = \Gamma_1 W_i^\pm + w_i^0 + w_i^1 + w_i^2, \]
with \( \Gamma_1 \in \mathbb{R} \), and \( w_i^\pm \) satisfying
\begin{align*}
\Delta w_i^\pm &= 0 \quad \text{in } \Omega^\pm, \\
w_i^\pm &= 0 \quad \text{on } \partial \Omega^\pm \cap \partial \Omega, \tag{5.5}
\end{align*}
i = 0, 1, 2. In view of the third lines of (5.3), (5.4), we choose \( w_0^\pm \) by imposing that
\[ \partial_n w_0^\pm = 2k \quad \text{on } \gamma. \tag{5.6} \]
By Proposition 2.1, we infer that
\[ w_0^+ = -w_0^- \quad \text{on } \gamma. \tag{5.7} \]
Therefore we can choose $f$ as

$$f_1 = \frac{1}{\sqrt{2}} \frac{w_0^\pm}{\lambda \mu} \text{ on } \gamma. \quad (5.8)$$

Writing equations $\Delta W^\pm_\gamma = 0$ in coordinates $(s, t)$, as in (4.3), and recalling that $W^\pm_\gamma(s, 0) = 1$, $s \in [0, \ell]$, we see that

$$(W^\pm_\gamma)_{tt} - k(W^\pm_\gamma)_t = 0 \text{ on } \gamma. \quad (5.9)$$

So,

$$\pm \frac{k}{\sqrt{2}} \frac{1}{2} W^\pm_\gamma \lambda^{-1} \mu^{-1} = \pm \frac{k}{\sqrt{2}} \frac{\Gamma_1}{2} k(\partial_n W^\pm_\gamma) \lambda^{-1} \mu^{-1} - \frac{1}{2} [(w_0^\pm)_t + (w_1^\pm)_t + (w_2^\pm)_t] \lambda^{-1} \mu^{-1} \text{ on } \gamma.$$ 

In view of the third and fourth line of the righthand side of (5.3), (5.4) respectively, and the above relation, given $\Gamma_1$, we can choose $\mu = \mu_1$ such that

$$\lambda \mu_1 = \pm \frac{\Gamma_1}{\sqrt{2}} (\partial_n W^\pm_\gamma) \text{ on } \gamma, \quad (5.10)$$

(keep in mind (2.10)). This is indeed possible by property (2.8). In view of the resulting first line of (5.3), (5.4), we choose $\Gamma_1$ such that

$$2 \ln \frac{\sqrt{2}}{\lambda} + 2 \ln \Gamma_1 = \Gamma_1, \quad (5.11)$$

and $w_1^\pm$ by imposing that

$$w_1^\pm = 2 \ln(\pm \partial_n W^\pm_\gamma) \text{ on } \gamma. \quad (5.12)$$

Clearly, if $\lambda > 0$ is sufficiently small, equation (5.11) has a unique solution

$$\Gamma_1 = 2 \ln \frac{\sqrt{2}}{\lambda} + 2 \ln \left( \ln \frac{\sqrt{2}}{\lambda} \right) + 2 \ln 2 + o(1) \text{ as } \lambda \to 0. \quad (5.13)$$

Note also that by Proposition 2.1, and (2.8), we deduce that

$$\partial_n w_1^+ + \partial_n w_1^- = 0 \text{ on } \gamma. \quad (5.14)$$

It remains to choose $w_2^\pm, \Delta_1, \text{ and } E_1$. In view of the second and third line of (5.3), (5.4) respectively, and relation (5.14), we choose

$$\Delta_1 = -k \lambda \mu_1 f_1 \mp \frac{1}{\sqrt{2}} \partial_n w_1^+, \quad (5.15)$$

$$E_1 = \frac{k}{\sqrt{2}} \lambda^2 \mu_1^2 f_1^2 + \sqrt{2} \Delta_1 \lambda \mu_1 f_1 = -\frac{1}{\sqrt{2}} k \lambda^2 \mu_1^2 f_1^2 \pm \lambda \mu_1 f_1 \partial_n w_1^\pm.$$ 

With the above choices, relations (5.3), (5.4) have simplified to

$$(u_1 - W_1^+)(s, -\lambda^{-1} \mu_1^{-1} L) = -w_2^+(s, 0) + \sqrt{2} \lambda^{-1} \mu_1^{-1} \partial_n w_1^+(s, 0) + \lambda^{-1} \mu_1^{-1} L \partial_n w_2^+(s, 0)$$

$$- \frac{1}{2} \lambda^{-2} \mu_1^{-2} L^2 [(w_0^+)_tt + (w_1^+)_tt + (w_2^+)_tt] (s, 0)$$

$$- \mathcal{O}(W_1^+tt)L^{-3} \mu_1^{-3} L^3 + \lambda^{-1} \mu_1^{-1} (1 + |\Delta_1| + |E_1|) e^{-c(L + \lambda \mu_1 f_1)}, \quad (5.16)$$
and
\[(u_1 - W^-)(s, \lambda^{-1} \mu^{-1} L) = -w_2^- (s, 0) - \sqrt{2} \lambda^{-1} \mu^{-1} \partial_t w_1^- (s, 0) - \lambda^{-1} \mu^{-1} L \partial_n w_2^- (s, 0)\]
\[= -\frac{1}{2} \lambda^{-2} \mu^{-2} L^2 \left[ (w_0^-)_{tt} + (w_1^-)_{tt} + (w_2^-)_{tt} \right] (s, 0)\]
\[-\mathcal{O}(W^-_{tt}) \lambda^{-3} \mu^{-3} L^3 + \lambda^{-1} \mu^{-1} (1 + |\Delta_1| + |E_1|) e^{-c(L - \lambda \mu)} , \]
respectively. Finally, we choose \(w_2^+\) by imposing that
\[w_2^+ = \pm \sqrt{2} \lambda^{-1} \mu^{-1} \partial_n w_1^+ = \frac{2}{\Gamma_1} \partial_n W_\gamma^+ \text{ on } \gamma. \] 

By Theorem 6.6 in [15] or Theorem 2I in [27], (5.12), (5.13), and the smoothness of the curve \(\gamma\), we infer that
\[\|w_2^+\|_{C^m(\bar{\Omega}^\pm)} \leq C_m \left( \ln \frac{1}{\lambda} \right)^{-1}, \quad m \geq 0. \] 

For future reference note that, thanks to Proposition 2.1, (5.14), and (5.18), we have
\[\partial_n w_2^+ + \partial_n w_2^- = 0 \text{ on } \gamma. \]

By (4.9), (5.8), (5.13), (5.16), (5.17), and (5.18), (keeping in mind (5.19)), we infer that
\[\left| (u_1 - W^\pm)(s, \mp \lambda^{-1} \mu^{-1} L) \right| \leq C \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right]^3 \left( \ln \frac{1}{\lambda} \right)^{-2}, \]
for \(\lambda > 0\) sufficiently small (having increased the value of \(M\) so that the exponential terms become negligible).

**Remark 5.1.** In view of the first and third lines of (5.3), (5.4), one is at first tempted to choose \(\mu, w = W^- - w_0^\pm\) such that
\[2 \ln(2\mu) = w^\pm \text{ and } \sqrt{2} \lambda \mu = \pm \partial_n w^\pm \text{ on } \gamma. \]

This leads to the following semilinear boundary value problems:

\[
\begin{cases}
\Delta w^\pm = 0 & \text{in } \Omega^\pm, \\
w^\pm = 0 & \text{on } \partial \Omega^\pm \cap \partial \Omega, \\
\partial_n w^\pm = \pm \sqrt{2} e^{w^\pm} & \text{on } \gamma.
\end{cases}
\]

We are looking for “large” (maximal) solutions \(w^\pm\), so that (4.4) holds. This is a problem which seems to have its own independent interest, and we expect that it shares common features with the original problem (1.1).

5.2. **The third order outer approximation.** In this subsection, we will choose functions \(\mu, f\) satisfying (4.4), functions \(\Delta_i, E_i, i = 1, 2\) (\(\Delta_1, E_1\) possibly new, though they will turn out to be chosen the same as before), and harmonic functions \(W^\pm\), so that the inner approximation \(u_2\) matches, in the \(C^1\) sense, with the outer ones \(W^\pm\) at \(\mp \lambda^{-1} \mu^{-1} L\) respectively, with \(L\) satisfying (4.9), as \(\lambda \to 0\).
From (4.17), (4.22), and (5.1), we find that

\[
(u_2 - W^+)(s, -\lambda^{-1}\mu^{-1}L) = \\
\frac{\sqrt{2}}{6} \left( \lambda^{-2}\mu^{-3}\mu'' - 2k^2\lambda^{-2}\mu^{-2} + \frac{1}{\sqrt{2}}\lambda^{-3}\mu^{-3}W_{it}(s, 0) \right) L^3 \\
+ \left( \frac{1}{\sqrt{2}}\lambda^{-1}\mu^{-1}k - \frac{1}{\sqrt{2}}\lambda^{-1}\mu^{-1}k^2f + \frac{1}{\sqrt{2}}\lambda^{-1}\mu^{-2}f\mu - \mu''\lambda^{-2}\mu^{-3} + (\mu')^2\lambda^{-2}\mu^{-4} \\
+ \lambda^{-2}\mu^{-2}k^2 - \frac{1}{\sqrt{2}}\lambda^{-2}\mu^{-2}k\Delta_1 + \sqrt{2}\lambda^{-1}\mu^{-2}\mu'f' + \frac{1}{\sqrt{2}}\lambda^{-1}\mu^{-1}f'' - \frac{1}{2}\lambda^{-2}\mu^{-2}W_{1t}(s, 0) \right) L^2 \\
+ \left( -\sqrt{2} + \sqrt{2}kf + 2\lambda^{-1}\mu^{-1}k + \sqrt{2}\lambda^{-1}\mu^{-1}\Delta_1 + \sqrt{2}\lambda^{-1}\mu^{-1}f^2 - 2\lambda^{-1}\mu^{-2}f + 2(\mu')^2\lambda^{-1}\mu^{-3}f \\
+ 2\lambda^{-1}\mu^{-1}k^2f - \sqrt{2}\lambda^{-1}\mu^{-1}\Delta_1 k f + 2\sqrt{2}\mu^{-1}\mu'f' + \sqrt{2}f''f + \sqrt{2}\Delta_2 - b_1\lambda^{-1}\mu^{-1}f'' \\
- b_2\lambda^{-1}\mu^{-2}\mu'f' - b_3\lambda^{-2}\mu^{-2}\Delta_1 k - b_4\lambda^{-1}\mu^{-1}k^2f - b_5\lambda^{-2}\mu^{-2}\Delta_1 E_1 + \lambda^{-1}\mu^{-1}\lambda^{-1}E_1 - \frac{\sqrt{2}}{5}\lambda\mu k^2 f^3 \\
+ \frac{\sqrt{2}}{6}\lambda\mu f' - \mu''\mu^{-1}f^2 + (\mu')^2\mu^{-2}f^2 + k^2 f^2 - \frac{1}{\sqrt{2}}\Delta_1 k f^2 + \sqrt{2}\mu'f'f^2 + \frac{1}{\sqrt{2}}\lambda f''f^2 \\
+ \frac{1}{\sqrt{2}}\lambda\mu k^2 f^3 + \sqrt{2}\Delta_2 \lambda\mu f - b_1f''f - b_2\mu^{-1}f'f - b_3\lambda^{-1}\mu^{-1}\Delta_1 k f - b_4k^2 f^2 - b_5\lambda^{-1}\mu^{-1}\Delta_1 E_1 f \\
+ a_1(f')^2 + a_2\lambda^{-2}\mu^{-3}\mu'' + a_3\lambda^{-2}\mu^{-2}k^2 + a_4\lambda^{-2}\mu^{-2}\Delta_1^2 + a_5\lambda^{-2}\mu^{-2}E_1^2 + a_6\lambda^{-2}\mu^{-2}kE_1 - 2\Delta_2 \\
- E_2 - W^+(s, 0) + \mathcal{O}(W_{ittt}^+\lambda^{-4}\mu^{-4}L^4 + \sum_{i=1}^2(|\Delta_i| + |E_i| + 1)^2 e^{-c(L+\lambda\mu)}) \right) .
\]

(5.22)
Similarly, via (4.18), (4.23), and (5.2), we obtain that

\[(u_2 - W^-)(s, \lambda^{-1} \mu^{-1} L) = \]

\[\frac{\sqrt{2}}{6} \left( \lambda^{-2} \mu^{-3} \mu'' - 2k^2 \lambda^{-2} \mu^{-2} - \frac{1}{\sqrt{2}} \lambda^{-3} \mu^{-3} W'''_{ttt}(s, 0) \right) L^3 \]

\[+ \left( -\frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} k + \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} k^2 f - \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-2} \mu'' - \mu'' \lambda^{-2} \mu^{-3} + (\mu')^2 \lambda^{-2} \mu^{-4} \right) \]

\[+ \lambda^{-2} \mu^{-2} k^2 + \frac{1}{\sqrt{2}} \lambda^{-2} \mu^{-2} k \Delta_1 - \sqrt{2} \lambda^{-1} \mu^{-2} \mu' f + \frac{1}{\sqrt{2}} \lambda^{-1} \mu^{-1} f'' - \frac{1}{\lambda} \lambda^{-2} \mu^{-2} W''_{tt}(s, 0) \right) L^2 \]

\[+ \left( -\sqrt{2} + 2k f + 2 \lambda^{-1} \mu^{-1} k + \sqrt{2} \lambda^{-1} \mu^{-1} \Delta_1 + \frac{\sqrt{2}}{2} \mu^{-1} \mu'' f^2 + 2 \lambda^{-1} \mu^{-2} f'' - 2(\mu')^2 \lambda^{-1} \mu^{-3} f \right) \]

\[- 2\lambda^{-1} \mu^{-1} k^2 f - \sqrt{2} \lambda^{-1} \mu^{-1} \Delta_1 k f + 2 \sqrt{2} \mu^{-1} \mu' f f' + \sqrt{2} f'' f + \sqrt{2} \Delta_2 + b_1 \lambda^{-1} \mu^{-1} f'' \]

\[+ b_2 \lambda^{-1} \mu^{-2} \mu' f' + b_3 \lambda^{-2} \mu^{-2} \Delta_1 k + b_4 \lambda^{-1} \mu^{-1} k^2 f + b_5 \lambda^{-2} \mu^{-2} \Delta_1 E_1 - \lambda^{-1} \mu^{-1} \partial_n W^-(s, 0) \right) L \]

\[+ 2 \ln(2 \mu) + \frac{\sqrt{2}}{2} \mu^{-1} k f^2 - 2k f - \sqrt{2} \Delta_1 f - 2 \lambda^{-1} \mu^{-1} \Delta_1 + \lambda^{-1} \mu^{-1} E_1 + \frac{\sqrt{2}}{3} \lambda \mu^2 f^3 \]

\[- \frac{\sqrt{2}}{6} \lambda \mu^3 f^3 - \mu'' \mu^{-1} f^2 + (\mu')^2 \mu^{-2} f^2 + k^2 f^2 + \frac{1}{\sqrt{2}} \Delta_1 k f^2 - \sqrt{2} \mu' f' f^2 - \frac{1}{\sqrt{2}} \lambda \mu f'' f^2 \]

\[- \frac{1}{\sqrt{2}} \lambda \mu k^2 f^3 - \sqrt{2} \Delta_2 \lambda \mu f - b_1 f'' f - b_2 \mu^{-1} \mu' f f' - b_3 \lambda^{-1} \mu^{-1} \Delta_1 k f - b_4 k^2 f^2 - b_5 \lambda^{-1} \mu^{-1} \Delta_1 E_1 f \]

\[+ a_1 (f')^2 + a_2 \lambda^{-2} \mu^{-3} \mu'' + a_3 \lambda^{-2} \mu^{-2} k^2 + a_4 \lambda^{-2} \mu^{-2} \Delta_1^2 + a_5 \lambda^{-2} \mu^{-2} E_1^2 + a_6 \lambda^{-2} \mu^{-2} k E_1 - 2 \Delta_2 \]

\[+ E_2 - W^-(s, 0) + O(W'''_{ttt}) \lambda^{-4} \mu^{-4} L^4 + \sum_{i=1}^2 (|\Delta_i| + |E_i| + 1)^2 e^{-c(L - \lambda \mu f)} \]

\[= (5.23)\]

We seek suitable $W^\pm$ in the form

\[W^\pm_2 = \Gamma_2 W^\pm_{\gamma} + w^+_0 + w^+_1 + w^+_2 + w^+_3 + w^+_4, \]

\[= (5.24)\]

with $\Gamma_2 \in \mathbb{R}$, and $w^+_i, i = 0, \ldots, 4$, satisfying (5.5) but not necessarily equal to those determined in the previous subsection (as it turns out, $w^+_0, w^+_1$ and $w^+_2$ will be chosen the same as before). We choose

\[\partial_n w^+_0 = 2k, \]

\[= (5.25)\]

\[\Gamma_2 = 2 \ln \left( \frac{\sqrt{2}}{\lambda} \right) + 2 \ln \Gamma_2, \quad \text{i.e. } \Gamma_2 = \Gamma_1, \]

\[= (5.26)\]

recall (5.13),

\[\lambda \mu = \pm \frac{\sqrt{2}}{\Gamma} \partial_n W^\pm_{\gamma}, \]

\[= (5.27)\]

\[w^+_1 = 2 \ln(\pm \partial_n W^\pm_{\gamma}). \]

\[= (5.28)\]

Note that, by Proposition 2.1, we have

\[w^+_0 + w^-_0 = 0 \quad \text{and} \quad \partial_n w^+_1 + \partial_n w^-_1 = 0 \quad \text{on } \gamma.\]
Given \( f \), keeping in mind the above relations, we determine \( \Delta_1, E_1, E_2 \), and \( w_2^\pm \) from the following relations on \( \gamma \):
\[
\sqrt{2}k f + \sqrt{2}\lambda^{-1}\mu^{-1}\Delta_1 \pm \lambda^{-1}\mu^{-1}\partial_n w_i^\pm = 0, \tag{5.29}
\]
\[
E_1 = \frac{1}{\sqrt{2}}\lambda^2 \mu^2 k f^2 + \sqrt{2}\Delta_1 \lambda \mu f, \tag{5.30}
\]
\[
E_2 = -\sqrt{2} \lambda \mu k^2 f^3 + \frac{\sqrt{2}}{6}\lambda \mu'' f^3 - \frac{1}{\sqrt{2}}\lambda^{-1}\mu^{-1}k f^2 + \sqrt{2}\lambda \mu f' f^2 + \lambda \mu ' f'' f^2 + \frac{\sqrt{2}}{2}\lambda \mu k^2 f^3 + \sqrt{2}\Delta_2 \lambda \mu f, \tag{5.31}
\]
and
\[
w_2^\pm = \pm\sqrt{2}\lambda^{-1}\mu^{-1}\partial_n w_i^\pm = \frac{2}{\Gamma_1} \frac{\partial_n w_i^\pm}{\partial_n W_\gamma^\pm}. \tag{5.32}
\]
We remark that the above choices of \( w_0^\pm, w_1^\pm, w_2^\pm, \Gamma_2, \Delta_1 \) and \( E_1 \) are the same as in the previous subsection. Again, from Proposition 2.1, we find that
\[
w_2^+ = w_2^- \text{ on } \gamma.
\]
Hence, we can choose
\[
\Delta_2 = -\frac{1}{2}\lambda^{-1}\mu^{-1} f^2 + \lambda^{-1}\mu^{-1}\Delta_1 k f - 2\lambda^{-1}\mu^{-1} f' f - f'' f + \frac{1}{\sqrt{2}}\lambda^{-1}\mu^{-1}\partial_n w_i^\pm. \tag{5.33}
\]
Next, given \( w_3^\pm \) such that
\[
w_3^+ + w_3^- = 0 \text{ on } \gamma, \tag{5.34}
\]
we take
\[
f = \pm\frac{\lambda^{-1}\mu^{-1}}{\sqrt{2}} w_0^\pm \mp \frac{\lambda^{-1}\mu^{-1}}{\sqrt{2}} w_3^\pm. \tag{5.35}
\]
Ideally, we would like \( w_3^\pm \) to satisfy
\[
0 = -2\lambda^{-1}\mu^{-2} \mu'' f + 2(\mu')^2 \lambda^{-1}\mu^{-3} f + 2\lambda^{-1}\mu^{-1} k^2 f - b_1 \lambda^{-1}\mu^{-1} f''
-b_2 \lambda^{-1}\mu^{-2} f' - b_3 \lambda^{-2}\mu^{-2} \Delta_1 k - b_4 \lambda^{-1}\mu^{-1} k^2 f - b_5 \lambda^{-2}\mu^{-2} \Delta_1 E_1
+ \lambda^{-1}\mu^{-1} \partial_n w_i^\pm. \tag{5.36}
\]
Observe that the above problem is actually a \emph{nonlocal} differential equation on \( \gamma \) (keep in mind (5.35)). Fortunately, the approximate solutions \( w_3^\pm \), determined by setting \( f = f_1 = \pm\frac{\lambda^{-1}\mu^{-1}}{\sqrt{2}} w_0^\pm = \frac{1}{\Gamma_1} \frac{w_0^\pm}{\partial_n w_\gamma} \) in the above equation, turn out to be satisfactory for our purposes.

Namely, we choose \( w_3^\pm \) such that
\[
\lambda^{-1}\mu^{-1} \partial_n w_3^\pm = 2\lambda^{-1}\mu^{-2} \mu'' f_1 - 2(\mu')^2 \lambda^{-1}\mu^{-3} f_1 - 2\lambda^{-1}\mu^{-1} k^2 f_1 + b_1 \lambda^{-1}\mu^{-1} f''_1
+b_2 \lambda^{-1}\mu^{-2} f_1' + b_3 \lambda^{-2}\mu^{-2} k \left( \pm \frac{1}{\sqrt{2}} \partial_n w_i^\pm - k \lambda \mu f_1 \right) + b_4 \lambda^{-1}\mu^{-1} k^2 f_1
+b_5 \lambda^{-2}\mu^{-2} \left( \frac{1}{\sqrt{2}} \lambda \mu f_1 (\partial_n w_1^\pm)^2 \pm \frac{3}{2} k \lambda^2 \mu^2 f_1^2 \partial_n w_1^\pm + \frac{1}{\sqrt{2}} k^2 \lambda^2 \mu^3 f_1^3 \right). \tag{5.37}
\]
Note that (5.34) holds (recall Proposition 2.1). Then, denoting the righthand side of (5.36) by $R(f, w_3^\pm)$, the above choice of $w_3^\pm$ yields that

$$R(f, w_3^\pm) = -2\lambda^{-1}\mu^{-2}\mu''h + 2(\mu')^2\lambda^{-1}\mu^{-3}h + 2\lambda^{-1}\mu^{-1}k^2h - b_1\lambda^{-1}\mu^{-1}h''$$

$$-b_2\lambda^{-1}\mu^{-2}\mu'h' + b_3\lambda^{-1}\mu^{-1}k^2h - b_4\lambda^{-1}\mu^{-1}k^2h$$

$$\mp \frac{3}{2} b_5 k(2f_1h + h^2)\partial_n w_1^\pm - \frac{b_5}{\sqrt{2}} k^2 \lambda \mu(h^3 + 3f_1h^2 + 3f_1^2h),$$

where

$$h \equiv \mp \frac{1}{\sqrt{2}} \lambda^{-1}\mu^{-1}w_3^\pm. \tag{5.39}$$

At this point, let us make a small de-tour. Given $g \in C^{2,\alpha}(\gamma), 0 < \alpha < 1$, we define the Dirichlet to Neumann mappings

$$T_{DN}^\pm : C^{2,\alpha}(\gamma) \to C^{1,\alpha}(\gamma),$$

by $T_{DN}^\pm g = \partial_n w^\pm$ on $\gamma$, where $w^\pm \in C^{2,\alpha}(\bar{\Omega}^\pm)$ satisfy (5.5) and $w^\pm = g$ on $\gamma$. It is easy to see that $T_{DN}^\pm$ are well defined linear bounded operators. Furthermore, we have that Kernel ($T_{DN}^\pm$) = 0 and thus $T_{DN}^\pm$ are invertible with bounded inverses (by the closed graph theorem). In other words, the Neumann to Dirichlet mappings

$$T_{ND}^\pm \equiv (T_{DN}^\pm)^{-1} : C^{1,\alpha}(\gamma) \to C^{2,\alpha}(\gamma),$$

are well defined linear bounded operators.

Now, it is easy to see that

$$\|\partial_n w_3^\pm\|_{C^2(\gamma)} \leq C \left( \ln \frac{1}{\lambda} \right)^{-1}. \tag{5.40}$$

Thus,

$$\|w_3^\pm\|_{C^{2,\alpha}(\gamma)} = \|T_{ND}^\pm(\partial_n w_3^\pm)\|_{C^{2,\alpha}(\gamma)} \leq C\|\partial_n w_3^\pm\|_{C^{1,\alpha}(\gamma)} \leq C \left( \ln \frac{1}{\lambda} \right)^{-1}. \tag{5.41}$$

By the same argument leading to (5.19), we infer that

$$\|w_3^\pm\|_{C^m(\bar{\Omega}^\pm)} \leq C_m \left( \ln \frac{1}{\lambda} \right)^{-1}, \quad m \geq 0. \tag{5.40}$$

In turn, via (5.39), this implies that

$$\|h\|_{C^m(\gamma)} \leq C_m \left( \ln \frac{1}{\lambda} \right)^{-2}, \quad m \geq 0. \tag{5.41}$$

Hence, from (5.38), we obtain that

$$\|R(f, w_3^\pm)\|_{C^m(\gamma)} \leq C_m \left( \ln \frac{1}{\lambda} \right)^{-3}, \quad m \geq 0. \tag{5.41}$$
Finally, we choose

\[ w_i^\pm = -\mu''\mu^{-1}f^2 + (\mu')^2\mu^{-2}f^2 + k^2f^2 \]

\[ -b_1f''f - b_2\mu^{-1}\mu'f'f - b_3\lambda^{-1}\mu^{-1}\Delta_1k f - b_4k^2f^2 - b_5\lambda^{-1}\mu^{-1}\Delta_1E_1f \]

\[ + a_1(f')^2 + a_2\lambda^{-2}\mu^{-3}\mu'' + a_3\lambda^{-2}\mu^{-2}k^2 + a_4\lambda^{-2}\mu^{-2}\Delta_1^2 + a_5\lambda^{-2}\mu^{-2}E_1^2 + a_6\lambda^{-2}\mu^{-2}kE_1 - 2\Delta_2. \]

(5.42)

Recalling (5.9), (5.27), we see that

\[ -\frac{1}{2}\lambda^{-2}\mu^{-2}\Gamma_2(W_\gamma^\pm)_{tt} = \mp\frac{1}{\sqrt{2}}\lambda^{-1}\mu^{-1}k \text{ on } \gamma. \]  

(5.43)

Since \( W_\gamma^\pm \) are harmonic in \( \Omega^\pm \) respectively, in view of (4.3), near the curve \( \gamma \), we have

\[ (W_\gamma^\pm)_{tt} + \frac{1}{(1-tk(s))^2}(W_\gamma^\pm)_{ss} + \frac{tk'(s)}{(1-tk(s))^3}(W_\gamma^\pm)_s - \frac{k(s)}{1-tk(s)}(W_\gamma^\pm)_t = 0, \]

for \( \pm t \geq 0 \) small, along \( \gamma \). Differentiating the above relation with respect to \( t \), we arrive at

\[ (W_\gamma^\pm)_{ttt} + \frac{2k}{(1-tk)^4}(W_\gamma^\pm)_{ss} + \frac{1}{(1-tk)^2}(W_\gamma^\pm)_{sst} + \frac{3tk'}{(1-tk)^3}(W_\gamma^\pm)_s + \frac{k'}{(1-tk)^4}(W_\gamma^\pm)_t \]

\[-\frac{k^2}{(1-tk)^4}(W_\gamma^\pm)_t - \frac{k}{1-tk}(W_\gamma^\pm)_{tt} + \frac{tk'}{(1-tk)^4}(W_\gamma^\pm)_{st} = 0, \]

for \( \pm t \geq 0 \) small, along \( \gamma \). Setting \( t = 0 \), and making use of (5.27), we obtain that

\[ \Gamma_2(W_\gamma^\pm)_{ttt} = \mp\sqrt{2}\lambda\mu'' \pm 2\sqrt{2}k^2\lambda\mu \text{ on } \gamma. \]

(5.44)

From the fact that \( w_0^\pm \) are harmonic functions, (4.3), (5.25), and (5.35), we obtain that

\[ -\frac{1}{2}\lambda^{-2}\mu^{-2}(w_0^\pm + w_3^\pm)_{tt} = \mp\frac{1}{\sqrt{2}}\lambda^{-1}\mu^{-2}\mu''f \mp \frac{1}{\sqrt{2}}\lambda^{-1}\mu^{-1}f'' \mp \sqrt{2}\lambda^{-1}\mu^{-2}\mu'f' - \lambda^{-2}\mu^{-2}k^2 \]

\[ + \frac{\lambda^{-2}\mu'^2}{2} \left[ -2k\mu^{-1}\mu'f + 2k\mu^{-2}(\mu')^2f + 2k^3f - b_1k\mu^{-1}f' - b_2k\mu^{-1}\mu'f' 
\]

\[-b_3\lambda^{-1}\mu^{-1}\Delta_1k^2 - b_4k^3f - b_5\lambda^{-1}\mu^{-1}\Delta_1E_1k - kR(f, w_3^\pm) \right] \]

(5.45)

on \( \gamma \). From the fact that \( w_1^\pm \) are harmonic functions, (4.3), (5.27), (5.28), and (5.29) we obtain that

\[ -\frac{1}{2}\lambda^{-2}\mu^{-2}(w_1^\pm)_{tt} = \mu''\mu^{-3}\lambda^{-2} - (\mu')^2\mu^{-4}\lambda^{-2} \pm \frac{1}{\sqrt{2}}k^2\lambda^{-1}\mu^{-1}f \pm \frac{1}{\sqrt{2}}k\lambda^{-2}\mu^{-2}\Delta_1 \text{ on } \gamma. \]

(5.46)
Combining all the above, we find that

\[ (u_2 - W_2^+)(s, -\lambda^{-1}\mu^{-1}L) = \]

\[
\frac{1}{6} \lambda^{-3} \mu^{-3} \sum_{i=0}^{4} (w_i^+)_{ttt} L^3 \\
- \frac{1}{2} \lambda^{-2} \mu^{-2} (w_2^+ + w_4^+) u L^2 \\
+ \left[ -2k\mu^{-1}\mu'' + 2k\mu^{-2}(\mu')^2 f + 2k^3 f - b_1 k f'' - b_2 k\mu^{-1}\mu' f' \right. \\
- b_3 \lambda^{-1} \mu^{-1} \Delta_1 k^2 - b_4 k^3 f - b_5 \lambda^{-1} \mu^{-1} \Delta_1 E_1 k - kR(f, w_3^+) \right] \frac{\lambda^{-2} \mu^{-2}}{2} L^2 \\
+ \left( R(f, w_3^+) + \lambda^{-1} \mu^{-1} \partial_n w_4^+ \right) L \\
+ \mathcal{O}(W_{ttt}^+) \lambda^{-4} \mu^{-4} L^4 + \sum_{i=1}^{2} (|\Delta_i| + |E_i| + 1)^2 e^{-c(L+\lambda\mu f)} ,
\]

and

\[ (u_2 - W_2^-)(s, \lambda^{-1}\mu^{-1}L) = \]

\[
\frac{1}{6} \lambda^{-3} \mu^{-3} \sum_{i=0}^{4} (w_i^-)_{tt} L^3 \\
- \frac{1}{2} \lambda^{-2} \mu^{-2} (w_2^- + w_4^-) u L^2 \\
+ \left[ -2k\mu^{-1}\mu'' + 2k\mu^{-2}(\mu')^2 f + 2k^3 f - b_1 k f'' - b_2 k\mu^{-1}\mu' f' \right. \\
- b_3 \lambda^{-1} \mu^{-1} \Delta_1 k^2 - b_4 k^3 f - b_5 \lambda^{-1} \mu^{-1} \Delta_1 E_1 k - kR(f, w_3^-) \right] \frac{\lambda^{-2} \mu^{-2}}{2} L^2 \\
- \left( R(f, w_3^-) + \lambda^{-1} \mu^{-1} \partial_n w_4^- \right) L \\
+ \mathcal{O}(W_{ttt}^-) \lambda^{-4} \mu^{-4} L^4 + \sum_{i=1}^{2} (|\Delta_i| + |E_i| + 1)^2 e^{-c(L-\lambda\mu f)}
\]

along \( \gamma \).

**Remark 5.2.** In the rest of the paper, the functions \( \mu \) and \( f \) will be given by \((5.27)\) and \((5.35)\) respectively. Furthermore, in order to avoid confusion, we remind to the reader that the functions \( \Delta_1 \) and \( E_1 \) will be as in this subsection. The same remark also applies to the harmonic functions \( w_i^\pm \).

### 6. The remainder of the third order inner approximate solution

In this subsection, we will estimate the remainder \( S(u_2) \) that is left in the equation by the third order inner approximate solution \( u_2 \), defined in \((4.19)\) with \( \phi = \phi_2 \) as in Lemma 4.2,
around the curve $\gamma$. In view of (4.20) and (4.21), the aforementioned remainder reduces to

$$
S(u_2) = \frac{1}{1-(\lambda^{-1} \mu^{-1}x+f)k_{\lambda}} \left[
(\phi_1 + \phi)_{ss} + 2(\mu' \mu^{-1}x - \lambda \mu f')(\phi_1 + \phi)_{sx}
\right]

+(\mu' \mu^{-1}x - \lambda \mu f')^2(\phi_1 + \phi)_{xx} + (\mu'' \mu^{-1}x - 2\lambda \mu f' - \lambda \mu f'')(\phi_1 + \phi)_x]

\left[-(\mu' \mu^{-1}x - \lambda \mu f')^2 e^{\lambda V_0} + (\mu'' \mu^{-1}x - 2\lambda \mu f' - \lambda \mu f'')(V_0)_x
\right]

+2\mu'' \mu^{-1} - 2(\mu')^2 \mu^{-2} - k\lambda \mu \phi_x - k^2(x + \lambda \mu f)(\phi_1 + \phi)_x

- k\lambda \mu \left[\frac{1}{1-(\lambda^{-1} \mu^{-1}x+f)k_{\lambda}} - 1 - (\lambda^{-1} \mu^{-1}x + f)k\right] (V_0 + \phi_1 + \phi)_x

+ \frac{(\lambda^{-1} \mu^{-1}x+f)k_{\lambda}'}{1-(\lambda^{-1} \mu^{-1}x+f)k_{\lambda}} \left[(\phi_1 + \phi)_s + (\mu' \mu^{-1}x - \lambda \mu f')(V_0 + \phi_1 + \phi)_x + 2\mu' \mu^{-1}\right]

+ \lambda^2 \mu^2 \left(e^{\lambda V_0 + \phi_1} - e^{V_0} - e^{\lambda V_0} \phi_1 - e^{\lambda V_0} \phi_1^2\right) + \lambda^2 \mu^2 \left(e^{\lambda V_0 + \phi_1 + \phi} - e^{V_0} \phi_1 - e^{V_0} \phi_1\right)

+ \lambda^2 \mu^2 \left(e^{V_0 + \phi_1} - e^{V_0}\right) \phi,

(6.1)

It follows from (5.13) and (5.27), keeping in mind that $\Gamma_2 = \Gamma_1$, that there exist constants $C_k > 0$ such that

$$
|\partial_s^k \mu(s)| \leq C_k \frac{1}{\lambda} \ln \frac{1}{\lambda}, \quad s \in [0, \ell], \quad k \geq 0,

(6.2)
$$

for $\lambda > 0$ sufficiently small (recall also the argument leading to (5.19)). On the other side, recalling (2.10), we find that

$$
\mu(s) \geq c \frac{1}{\lambda} \ln \frac{1}{\lambda}, \quad s \in [0, \ell],

(6.3)
$$

for $\lambda > 0$ sufficiently small. It then follows from (5.35) and (5.40) that

$$
|\partial_s^k f(s)| \leq C_k \left(\ln \frac{1}{\lambda}\right)^{-1}, \quad s \in [0, \ell], \quad k \geq 0,

(6.4)
$$

for $\lambda > 0$ sufficiently small (having increased the values of the generic constants $C_k$ if needed, something that we will do in the sequel without explicitly mentioning). Therefore, in view of (5.29)-(5.33), we obtain that

$$
|\partial_s^k \Delta_1(s)| + |\partial_s^k E_1(s)| \leq C_k,

(6.5)
$$

and

$$
|\partial_s^k \Delta_2(s)| + |\partial_s^k E_2(s)| \leq C_k \left(\ln \frac{1}{\lambda}\right)^{-2},

(6.6)
$$

$s \in [0, \ell], k \geq 0$, for $\lambda > 0$ small (keep in mind (5.19), (5.26) and (5.28)). In turn, the functions $A_2$ and $B_2$, as defined in Lemma 4.2, satisfy

$$
|\partial_s^k A_2(s)| + |\partial_s^k B_2(s)| \leq C_k \left(\ln \frac{1}{\lambda}\right)^{-2},
$$
s ∈ [0, ℓ), k ≥ 0, for λ > 0 small. It follows from the above and Lemma 4.2 that
\begin{equation}
|\phi_1(s, x)| ≤ C \left( \ln \frac{1}{\lambda} \right)^{-1} (x^2 + 1), \quad |x| ≤ C \ln \left( \ln \frac{1}{\lambda} \right), \quad s ∈ [0, ℓ).
\end{equation}
In fact, this estimate can be differentiated arbitrary many times, that is
\begin{equation}
|\left(\phi_1\right)_x| ≤ C \left( \ln \frac{1}{\lambda} \right)^{-1} (|x| + 1), \quad |\left(\phi_1\right)_s| ≤ C \left( \ln \frac{1}{\lambda} \right)^{-1} (x^2 + 1),
\end{equation}
and so on. Similarly, we have the estimate
\begin{equation}
|\phi_2(s, x)| ≤ C \left( \ln \frac{1}{\lambda} \right)^{-2} (|x|^3 + 1), \quad |x| ≤ C \ln \left( \ln \frac{1}{\lambda} \right), \quad s ∈ [0, ℓ),
\end{equation}
which can also be differentiated arbitrary many times.

Armed with the above information, and keeping the asymptotic behavior of \( V_0 \) in mind, it is easy to verify the following proposition which represents the main result of this section.

**Proposition 6.1.** The inner approximation \( u_2 \), defined in (4.19) (with \( \phi = \phi_2 \) as in Lemma 4.2), satisfies
\begin{equation}
\left| \Delta u_2(y) + \lambda^2 e^{u_2(y)} \right| ≤ C \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right]^2 \left( \ln \frac{1}{\lambda} \right)^{-1},
\end{equation}
provided that there is a (different) constant \( C > 0 \) such that
\begin{equation}
dist(y, \gamma) ≤ C \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1},
\end{equation}
for \( \lambda > 0 \) sufficiently small.

7. THE REMAINDER OF THE OUTER APPROXIMATIONS

Here, we will estimate the remainder that is left in the equation of (1.1) by the outer approximations \( W_2^\pm \).

**Proposition 7.1.** The outer approximations \( W_2^\pm \), defined in (5.24), satisfy
\begin{equation}
\left| \Delta W_2^\pm(y) + \lambda^2 e^{W_2^\pm(y)} \right| ≤ C \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1},
\end{equation}
if
\begin{equation}
y ∈ \Omega^\pm \quad \text{and} \quad \dist(y, \gamma) ≥ 2M \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1},
\end{equation}
where \( c > 0 \) is independent of both \( \lambda > 0 \) and \( M > 1 \), provided that \( \lambda > 0 \) is sufficiently small.

**Proof.** In view of (2.9) and (2.10), we have that
\begin{equation}
W_\gamma^\pm ≤ 1 - cM \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1},
\end{equation}
in the corresponding regions that are described by (7.1), with the constant \( c > 0 \) independent of both \( \lambda > 0 \) and \( M > 1 \), provided that \( \lambda > 0 \) is sufficiently small. So, in these regions, recalling (5.13) and that \( \Gamma_1 = \Gamma_2 \), we obtain that

\[
\Gamma_2 W_\gamma^\pm \leq 2 \ln \frac{1}{\lambda} - cM \ln \left( \ln \frac{1}{\lambda} \right),
\]

for a possibly different constant \( c > 0 \) (still independent of both \( \lambda \) and \( M \)), provided that \( \lambda > 0 \) is sufficiently small.

Since \( W_2^\pm \) are harmonic and all the \( w_i^\pm \)'s are uniformly bounded in \( \lambda \) (and independent of \( M \)), we find that in each respective region of (7.1) it holds that

\[
\left| \Delta W_2^\pm + \lambda^2 e^{W_2^\pm} \right| = \lambda^2 e^{W_2^\pm} \leq C\lambda^2 e^{\Gamma_2 W_2^\pm} \leq C \left( \ln \frac{1}{\lambda} \right)^{-cM},
\]

for some constants \( c, C > 0 \) that are independent of both \( \lambda \) and \( M \), provided that \( \lambda > 0 \) is sufficiently small, as desired. \( \square \)

### 8. Patching the inner and outer approximations

In this section, we will construct a global smooth approximate solution to the problem (1.1), for \( \lambda > 0 \) small, by interpolating between the inner approximation \( u_2 \) and the outer ones \( W_2^\pm \) at a distance of order \( \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1} \) from the curve \( \gamma \). To this aim, we will need to estimate the differences \( u_2 - W_2^\pm \), \( (u_2 - W_2^\pm)_i \), and \( \Delta \left( u_2 - W_2^\pm \right) = \Delta u_2 \) in the respective interpolation regions on each side of \( \gamma \).

#### 8.1. The estimate for \( u_2 - W_2^\pm \)

In view of (5.25), (5.26), (5.27), (5.28), (5.19), (5.40), (5.41), (6.2), (6.3), (6.4) and (6.5), it follows from (5.47) and (5.48) that

\[
\left| (u_2 - W_2^\pm) (s, \mp t) \right| \leq C \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right]^4 \left( \ln \frac{1}{\lambda} \right)^{-3}, \quad (8.1)
\]

for \( s \in [0, t] \) and \( M \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1} \leq t \leq 2M \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1} \), provided that \( \lambda > 0 \) is sufficiently small (having increased the value of \( M \) if necessary).

#### 8.2. The estimate for \( (u_2 - W_2^\pm)_i \)

It follows from (3.3), Lemma 4.1, (4.19), and Lemma 4.2 that

\[
(u_2)_x = (V_0)_x + (\phi_1)_x (s, x) + (\phi_2)_x (s, x)
\]

\[
= 3 \left( -\frac{\sqrt{2}}{6} \lambda \mu^\prime \mu'' \mu^{-3} + \frac{\sqrt{2}}{3} \lambda^{-2} \mu^{-2} k^2 \right) x^2 + 2 (-\lambda^{-2} \mu'' \mu^{-3} + \lambda^{-2} (\mu')^2 \mu^{-4} + k^2 \lambda^{-2} \mu^{-2}
\]

\[
+ \frac{\sqrt{2}}{2} \lambda \mu^{-1} - \frac{1}{\sqrt{2}} \lambda^{-2} \mu^{-2} \Delta_1 k + \frac{\sqrt{2} \lambda^{-1} \mu^{-2} \mu' f'}{\sqrt{2} \lambda^{-1} \mu^{-1} f'} + \frac{1}{\sqrt{2} \lambda^{-1} \mu^{-1} f'} + \frac{1}{\sqrt{2} \lambda^{-1} \mu^{-1} k^2 f} \right) x
\]

\[
+ 2 \lambda \mu^{-1} - \sqrt{2} \lambda^{-1} \mu^{-1} \Delta_1 - \sqrt{2} \lambda^{-1} E_1 + B_2 + \sqrt{2} + \sum_{i=1}^2 (|\Delta_i| + |E_i| + 1)^2 O(e^{cx}),
\]
as $x \to -\infty$. In turn, it follows from the definition of the coordinate $x$ in (4.5) that

$$(u_2)_t = \left( -\frac{1}{\sqrt{2}} \lambda \mu'' + \sqrt{2} \lambda \mu k^2 \right) t^2 + \left( \sqrt{2} \lambda \mu'' f - 2 \sqrt{2} \lambda \mu f k^2 - 2 \mu'' \mu^{-1} + 2 (\mu')^2 \mu^{-2} + 2k^2 \right) t - \frac{1}{\sqrt{2}} \lambda \mu'' f^2 + \sqrt{2} \lambda \mu f^2 k^2$$

$$+ 2 \lambda \mu'' \mu^{-1} - 2 f(\mu')^2 \mu^{-2} - 2k^2 f - \sqrt{2} k \lambda \mu f + \sqrt{2} f \Delta_1 k - 2 \sqrt{2} \lambda f \mu f' - \sqrt{2} \lambda \mu f''$$

$$- \sqrt{2} \lambda \mu k^2 f^2 + 2k - \sqrt{2} \Delta_1 - \sqrt{2} \Delta_2 \lambda \mu + B_2 \lambda \mu + \sqrt{2} \lambda \mu$$

$$+ \lambda \mu \sum_{i=1}^2 (|\Delta_i| + |E_i| + 1)^2 O(e^{-c\mu(t-f)}),$$

for $s \in [0, \ell)$ and $-C \left[ \ln \left( \frac{1}{x} \right) \right] \left( \ln \frac{1}{x} \right)^{-1} \leq t \leq -c \left[ \ln \left( \frac{1}{x} \right) \right] \left( \ln \frac{1}{x} \right)^{-1}$. Analogously, we have that

$$(u_2)_t = \left( \frac{1}{\sqrt{2}} \lambda \mu'' - \sqrt{2} \lambda \mu k^2 \right) t^2 + \left( \sqrt{2} \lambda \mu'' f + 2 \sqrt{2} \lambda \mu f k^2 - 2 \mu'' \mu^{-1} + 2 (\mu')^2 \mu^{-2} + 2k^2 \right) t + \frac{1}{\sqrt{2}} \lambda \mu'' f^2 - \sqrt{2} \lambda \mu f^2 k^2$$

$$- \sqrt{2} k \lambda \mu + \sqrt{2} \Delta_1 k - 2 \sqrt{2} \lambda \mu f' - \sqrt{2} \lambda \mu f'' - \sqrt{2} \lambda \mu k^2 f) t + \frac{1}{\sqrt{2}} \lambda \mu'' f^2 - \sqrt{2} \lambda \mu f^2 k^2$$

$$+ 2 \lambda \mu'' \mu^{-1} - 2 f(\mu')^2 \mu^{-2} - 2k^2 f + \sqrt{2} k \lambda \mu f - \sqrt{2} f \Delta_1 k + 2 \sqrt{2} \lambda f \mu f' + \sqrt{2} \lambda \mu f''$$

$$+ \sqrt{2} \lambda \mu k^2 f^2 + 2k + \sqrt{2} \Delta_1 + \sqrt{2} \Delta_2 \lambda \mu + B_2 \lambda \mu - \sqrt{2} \lambda \mu$$

$$+ \lambda \mu \sum_{i=1}^2 (|\Delta_i| + |E_i| + 1)^2 O(e^{-c\mu(t-f)}),$$

for $s \in [0, \ell)$ and $c \left[ \ln \left( \frac{1}{x} \right) \right] \left( \ln \frac{1}{x} \right)^{-1} \leq t \leq C \left[ \ln \left( \frac{1}{x} \right) \right] \left( \ln \frac{1}{x} \right)^{-1}$. On the other side, it follows from (5.24), (5.25), (5.27), (5.29), (5.33), (5.43), (5.44), (5.45) and (5.46) that

$$(W_2^\pm)_t(s, t) = \Gamma_2(W_?^\pm)_t(s, t) + \sum_{i=0}^4 \Gamma_2^\pm (w_i^\pm)_t(s, t)$$

$$= \pm \sqrt{2} \lambda \mu + 2k - \sqrt{2} k \lambda \mu f + 2 \sqrt{2} \lambda \mu \Delta_1 + 2 \sqrt{2} \lambda \mu \Delta_2 + \frac{1}{\sqrt{2}} \lambda \mu'' f^2 \pm 2 \Delta_1 k f$$

$$\mp 2 \sqrt{2} \lambda \mu f' f \mp 2 \sqrt{2} \lambda \mu f'' f + \partial_\nu w_?^\pm(s, 0) \pm \partial_\nu w_?^\pm(s, 0) + t \left( \pm \sqrt{2} k \lambda \mu \pm \sqrt{2} \lambda \mu'' f \right)$$

$$\pm \sqrt{2} \lambda \mu f'' f + 2 \sqrt{2} \lambda \mu f' + 2k^2 + 2 \mu'' \mu^{-1} + 2 (\mu')^2 \mu^{-2} + \sqrt{2} k \Delta_1$$

$$+ (w_?^\pm)_{tt}(s, 0) + (w_?^\pm)_{tt}(s, 0)) + \frac{\nu}{2} \left( \mp \sqrt{2} \lambda \mu'' \pm 2 \sqrt{2} k \lambda \mu + \sum_{i=0}^4 (w_i^\pm)_{ttt}(s, 0) \right)$$

$$+ O \left( t^3 \|W_2^\pm\|_{L^\infty} \right),$$

for $s \in [0, \ell)$ and $0 \leq t \leq C \left[ \ln \left( \frac{1}{x} \right) \right] \left( \ln \frac{1}{x} \right)^{-1}$. 


By the above relations and (5.37)-(5.38), we deduce that
\[
(u_2 - W_2^\pm)_{i} = -R(f, w^\pm_2)\lambda \mu + \partial_n w^\pm_2(s, 0) + t (-2k\mu^{-1}\mu''f + 2(\mu')^2k\mu^{-2}f + 2k^2f - b_1k\mu'')
- b_2k\mu^{-1}\mu'f' - b_3\lambda^{-1}\mu^{-1}\Delta_1k^2 - b_4k^3f - b_5\lambda^{-1}\mu^{-1}\Delta_1E_1k - kR(f, w^\pm_2) \\
- (w^\pm_2)_{i}(s, 0) - (w^\pm_2(s, 0)) - \frac{t^2}{\pi} \sum_{i=0}^{4} (w^\pm_2)_{i}(s, 0)
+ \mathcal{O} \left( t^2 \| (W_2^\pm)_{i}(s, 0) \|_{L^\infty(\Omega)} \right) + \lambda \mu \sum_{i=1}^{2} (|\Delta_i| + |E_i| + 1)^2 \mathcal{O}(e^{-c\lambda\|t\|_{L^\infty}}),
\]
for \( s \in [0, \ell] \) and \( c \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1} \leq \pi t \leq C \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1}.

Before proceeding further, let us complete the estimates for the harmonic functions \( w^\pm_i \), \( i = 0, \cdots, 4 \), by estimating \( w^\pm_4 \). It follows from (5.42), (6.2), (6.3), (6.4), (6.5) and (6.6) that
\[
|w^\pm_4| \leq C \left( \ln \frac{1}{\lambda} \right)^{-2} \text{ on } \gamma.
\]
In turn, by the same argument leading to (5.19), we find that
\[
\|w^\pm_4\|_{C^m(\Omega)} \leq C_m \left( \ln \frac{1}{\lambda} \right)^{-2}, \quad m \geq 0. \tag{8.2}
\]
As before, it follows readily that
\[
|(u_2 - W_2^\pm)_i(s, \pi t)| \leq C \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right]^3 \left( \ln \frac{1}{\lambda} \right)^{-2}, \tag{8.3}
\]
for \( s \in [0, \ell] \) and \( M \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1} \leq t \leq 2M \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1} \), provided that \( \lambda > 0 \) is sufficiently small (having increased the value of \( M \) if necessary).

8.3. The estimate for \( \Delta (u_2 - W_2^\pm) = \Delta u_2 \). It follows immediately from (4.19), (3.3), Lemmas 4.1-4.2, (6.3), (6.4) and Proposition 6.1 that
\[
|\Delta u_2(y)| \leq C \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right]^2 \left( \ln \frac{1}{\lambda} \right)^{-1}, \tag{8.4}
\]
if
\[
M \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1} \leq \text{dist}(y, \gamma) \leq 2M \left[ \ln \left( \ln \frac{1}{\lambda} \right) \right] \left( \ln \frac{1}{\lambda} \right)^{-1}, \tag{8.5}
\]
for \( \lambda > 0 \) sufficiently small (having possibly increased \( M \), since in this region \( \lambda^2 e^{u_2} \) is of order \( (\ln \frac{1}{\lambda})^{-cM} \) with \( c > 0 \) independent of both \( M \) and \( \lambda \)).

8.4. The global approximate solution \( u_{ap} \). We are now in position to construct a smooth global approximate solution to the problem (1.1), by interpolating between the inner and outer approximations in the region described by (8.5), and to be able to estimate the remainder that is left by it in the equation. As expected, this task will require us to use some cutoff functions.

Consider a fixed smooth cutoff function such that
\[
\eta(\tau) = \begin{cases} 
0, & |\tau| \leq 1, \\
1, & |\tau| \geq 2.
\end{cases} \tag{8.6}
\]
Then, let
\[ \eta_\lambda(t) = \eta\left(\frac{t}{M \left[\ln \left(\frac{1}{\exp(\lambda)}\right)\right]\left(\ln \frac{1}{\exp(\lambda)}\right)^{-1}}\right). \]

We can now define our global approximate solution, using the local coordinates \((s,t)\), as
\[ u_{ap}(y) = \begin{cases} 
  u_2, & \text{dist}(y, \gamma) \leq M \left[\ln \left(\frac{1}{\exp(\lambda)}\right)\right]\left(\ln \frac{1}{\exp(\lambda)}\right)^{-1}, \\
  u_2 + \eta_\lambda(t)(W_2^\pm - u_2), & \exists \ t \in \left(\left[\left[\ln \left(\frac{1}{\exp(\lambda)}\right)\right]\left(\ln \frac{1}{\exp(\lambda)}\right)^{-1}, 2M \left[\ln \left(\frac{1}{\exp(\lambda)}\right)\right]\left(\ln \frac{1}{\exp(\lambda)}\right)^{-1}\right), \\
  W_2^\pm, & y \in \Omega^\pm \text{ and dist}(y, \gamma) \geq 2M \left[\ln \left(\frac{1}{\exp(\lambda)}\right)\right]\left(\ln \frac{1}{\exp(\lambda)}\right)^{-1}.
\end{cases} \]

The main result of this section is the following.

**Proposition 8.1.** We can choose a large \(M\) such that the global approximation \(u_{ap}\) satisfies
\[ \|\Delta u_{ap} + \lambda^2 e^u_{ap}\|_{L^\infty(\Omega)} \leq C \left[\ln \left(\frac{1}{\exp(\lambda)}\right)\right]^2\left(\ln \frac{1}{\exp(\lambda)}\right)^{-1}, \]
if \(\lambda > 0\) is sufficiently small.

**Proof.** By virtue of Propositions 6.1 and 7.1, it remains to consider the intermediate regions in (8.5). There, using (4.2) and the fact that \(W_2^\pm\) are harmonic, the remainder under consideration reduces to
\[ \Delta u_2 + (\Delta \eta_\lambda)(W_2^\pm - u_2) + 2\partial_t \eta_\lambda(W_2^\pm - u_2)_t - \eta_\lambda \Delta u_2 + \lambda^2 e^{u_2 + \eta_\lambda(W_2^\pm - u_2)}\]
The sought after estimate now follows readily by using (8.1), (8.3), (8.4), the direct estimates
\[ |\partial_\gamma \eta_\lambda| \leq C \left[\ln \left(\frac{1}{\exp(\lambda)}\right)\right]^{-1}\left(\ln \frac{1}{\exp(\lambda)}\right), \quad |\Delta \eta_\lambda| \leq C \left[\ln \left(\frac{1}{\exp(\lambda)}\right)\right]^{-2}\left(\ln \frac{1}{\exp(\lambda)}\right)^2 \quad (\text{keep in mind (4.3))}, \]
and the comment below (8.5) (to estimate the last term). \qed

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