On the canonical formalism of $f(R)$-type gravity using Lie derivatives

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Abstract
We present a canonical formalism of the $f(R)$-type gravity using the Lie derivatives instead of the time derivatives by refining the formalism of our group. The previous formalism is a direct generalization of the Ostrogradski's formalism. However the use of the Lie derivatives was not sufficient in that Lie derivatives and time derivatives are used in a mixed way, so that the expressions are somewhat complicated. In this paper, we use the Lie derivatives and foliation structure of the spacetime thoroughly, which makes the procedure and the expressions far more concise.

1 Introduction
The $f(R)$-type gravity is one of the generalized gravities and since its use by Caroll et al.[1] to explain the discovered accelerated expansion of the universe[2], the theory confronted with observations and has been attracting much attention and its various aspects and applications have been investigated[3, 4, 5, 6]. Before that the generalized gravity theories have mainly been interested in because of their theoretical advantages. Main advantageous points are: (i) The theory of gravitons can be renormalizable contrary to the Einstein gravity[7, 8]. (ii) It might be possible to avoid the initial singularity of the universe[9] proved by Hawking[10]. (iii) The inflation of the universe at early stage could be explained without introducing an ad hoc scalar field[11].

However, the canonical formalism of the $f(R)$-type gravity had not been so systematic since it is a somewhat complicated higher derivative theory. So in the previous paper[12], our group proposed a formalism by directly generalizing the canonical formalism of Ostrogradski[13] by using the Lie derivatives instead of the time derivatives. The generalization is necessary since the scalar curvature $R$ depends on the time derivatives of the lapse function and shift vector. So, if the Ostrogradski’s formalism is directly applied, these variables have to obey field equations. Then only the solutions to these equations are allowed. This, however, is in conflict with general covariance since a set of these variables specify a coordinate frame, so should be taken arbitrarily. One of the ways to resolve this problem had been given by Buchbinder and Lyakhovich(BL method)[14] which is sometimes referred to as the generalization of the Ostrogradski’s one[15]. However, BL method has an undesirable property that, when the generalized coordinates are transformed, the Hamiltonian is also transformed[12].

The formalism in [12] remedied the undesirable property of the BL method due to the

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direct generalization of the Ostrogradski’s one. However the Lie derivatives and time derivatives are used in a mixed way, so some expressions are complex. Here all the time derivatives are replaced by the Lie derivatives, which makes the procedure more transparent and the resulting expressions far more concise. We also note that, while the time derivatives depend on the coordinate frame so that the differentiated quantities are not tensors generally, Lie derivatives preserve the tensorial property, so would be favorable derivatives for relativistic treatment. Furthermore, in ADM formalism[16], which we use here, foliations of the d-dimensional spacelike hypersurfaces are connected by a one-parameter mapping, so that Lie derivatives are natural to express the rate of changes and in the definition of the Hamiltonian density.

Finally, we note that surface term, Gibbons-Hawking[17] term, which is necessary in Einstein gravity for the variational principle to be consistent, is not necessary in the f(R)-type gravity[18].

In section 2, we review the canonical formalism of Ostrogradski to help seeing that our method is a direct generalization of the Ostrogradski’ formalism. In section 3, we present our formalism. In section 4, we show the invariance of our Hamiltonian under the transformation of the generalized coordinates. Section 5 is devoted to the summary and discussions.

2 Review of the Ostrogradski’s canonical formalism

In this section, we review the canonical formalism of Ostrogradski which is the generalization of the formalism for usual systems to the system described by the Lagrangian with higher order time derivatives. We also expect that this section would be helpful in seeing that our formalism given in the next section is a direct generalization of the Ostrogradski’s one.

Let us consider a system with N degrees of freedom, the generalized coordinates of which will be denoted as $q^i, (i = 1, 2, \ldots, N)$. Its Lagrangian $L$ is assumed to be dependent on these generalized coordinates and their time derivatives up to n-th order. The $N(n + 1)$ dimensional space, which will be called as velocity phase space, has coordinates which are the arguments of the Lagrangian:

$$D^s q^i (s = 0, 1, \ldots, n; i = 1, \ldots, N),$$

where $D \equiv \frac{d}{dt}$, i.e., the Lagrangian is assumed to be defined in the velocity phase space[19];

$$L = L(q^i, q^i_1, \ldots, q^i_n) \equiv L(D^s q^i).$$

It is possible to generalize further that n is different for different i, but we do not think of this possibility for simplicity. Transition to the canonical formalism is given by the Ostrogradski mapping, which might be seen as the straightforward generalization of the Legendre mapping for systems with $n = 1$.

Take the following variation of the action $S$:

$$\delta S \equiv \int_{t_1 + \delta t_2}^{t_2 + \delta t_2} L(q^i + \delta q^i, \ldots, q^{i(n)} + \delta q^{i(n)})dt - \int_{t_1}^{t_2} L(q^i, \ldots, q^{i(n)})dt$$

\footnote{This section is essentially the first half of [12] which in turn depends heavily on a book by Kimura T. and Sugano R. [19]}
where

$$\delta q^i \equiv (q + \delta q)^i(t + \delta t) - q^i(t) \approx \delta^* q^i + q^i \delta t$$

with

$$\delta^* q^i \equiv (q + \delta q)^i(t) - q^i(t),$$

i.e., $\delta^* q^i$ is equal to the variation used in the variational principle. Then we have

$$\delta S = \left[ L \delta t \right]_{t_1}^{t_2} + \delta^* S,$$

where we used the approximation

$$\int_{t_k}^{t_k + \delta t_k} L(q^i, \ldots, q^{(n)}) dt = L \left( q^1(t_k), \ldots, q^{(n)}(t_k) \right) \delta t_k, \quad (k = 1, 2).$$

In $\delta^* S$, the effects of the variation of the time $\delta t$ are assumed to be negligible, so $\delta^* S$ reduces to the variation taken in the variational principle and is expressed as

$$\delta^* S = \int_{t_1}^{t_2} \delta^* L \ dt = \left[ \delta F \right]_{t_1}^{t_2} + \tilde{\delta} S.$$  \tag{2.7}

By arranging the sum in $\delta^* S$, we have

$$\delta F = \sum_{i=1}^{N} \sum_{s=0}^{n-1} \left\{ \sum_{r=s+1}^{n} (-1)^{r-s-1} D^{r-s-1} \left( \frac{\partial L}{\partial (D^s q^i)} \right) \delta^* q^i(s) \right\},$$

and

$$\tilde{\delta} S = \sum_{i=1}^{N} \int_{t_1}^{t_2} \sum_{s=0}^{n} (-1)^s D^s \left( \frac{\partial L}{\partial (D^s q^i)} \right) \delta^* q^i dt.$$ \tag{2.9}

Thus we have

$$\delta S = \left[ L \delta t + \delta F \right]_{t_1}^{t_2} + \tilde{\delta} S,$$ \tag{2.10}

where in the first term on the right hand side, $\delta^* q^i$ is written by $\delta q^i$ and $\delta t$ given in (2.4). The generalized coordinates of the phase space (often referred to as the new generalized coordinates for $s \geq 1$) $q^i_s$ are taken as

$$q^i_s \equiv D^s q^i, \quad (i = 1, \ldots, N; s = 0, \ldots, n - 1),$$ \tag{2.11}

and the momenta canonically conjugate to these coordinates, $p^s_i$, are defined to be the coefficients of $\delta D^s q^i = \delta q^i_s$ in $\delta F$:

$$p^s_i \equiv \sum_{r=s+1}^{n} \left[ (-1)^{r-s-1} D^{r-s-1} \left( \frac{\partial L}{\partial (D^r q^i)} \right) \right].$$ \tag{2.12}

The Hamiltonian, $H$, is defined as $(-1) \times$ (the coefficient of $\delta t$) in $L \delta t + \delta F$:

$$H = \sum_{i=1}^{N} \sum_{s=0}^{n-1} p^s_i D^s q^i_s - L.$$ \tag{2.13}

Note that for $s = n - 1$, equation (2.12) has simple expressions:

$$p^{n-1}_i = \frac{\partial L}{\partial (\dot{q}^i_{n-1})}.$$ \tag{2.14}

Thus, it is easily seen that the Ostrogradski mapping

$$(q^i, \dot{q}^i, \ldots, q^{(n)}; L) \rightarrow (q^i, \ldots, q^{(n-1)}; p_1^0, \ldots, p_{n-1}^0; H)$$ \tag{2.15}

is a generalization of the Legendre mapping. It is noted that the Hamiltonian is invariant under the transformation of generalized coordinates ; $q^i \rightarrow q^{ii}$.  

3
3 Canonical formalism of $f(R)$-type gravity

In this section, we present a canonical formalism of $(1 + d)$-dimensional $f(R)$-type gravity by refining the formalism proposed in [12] which is a direct generalization of Ostrogradski’s formalism. As the variables for gravity, we adopt the ADM variables[16], i.e., the metric $h_{ij}$ of the $d$-dimensional hypersurface $\Sigma_t$ which has the normal vector field $n^\mu = N^{-1}(1, -N^i)$, $N$, the lapse function and $N^i$, the shift vector. That is, we regard the spacetime to have the foliation structure[20]. Then the scalar curvature $R$ is expressed in terms of these variables as

$$ R = h^{ij} L_n^2 h_{ij} + \frac{1}{4} \left( h^{ij} L_n h_{ij} \right)^2 - \frac{3}{4} h^{ik} h^{jl} L_n h_{ij} L_n h_{kl} + dR - 2\Delta(\ln N). \quad (3.1) $$

$L_n$ represents the Lie derivative along the normal vector field $n^\mu$. $dR$ is the scalar curvature of $\Sigma_t$. It is noted that $R$ does not depend on $L_n N$ and $L_n N^i$.

3.1 Variation of the action of the $f(R)$-type gravity

Lagrangian density of the $f(R)$-type gravity, $L_G$, is taken as

$$ L_G = \sqrt{-g} f(R), \quad (3.2) $$

where $g \equiv \det g_{\mu \nu}$. Then the Lagrangian $L_G(\Sigma_t)$ and the action $S$ are expressed as follows:

$$ L_G(\Sigma_t) = \int_{\Sigma_t} L_G \, d^d x, \quad S = \int_{t_1}^{t_2} L_G(\Sigma_t) \, dt = \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^d x L_G. \quad (3.3) $$

From (3.1) and (3.3), $L_G$ depends on the ADM variables in the following way:

$$ L_G = L_G(N, h_{ij}, L_n h_{ij}, L_n^2 h_{ij}). \quad (3.4) $$

It is noted that $L_n h_{ij}$ is related to the extrinsic curvature $K_{ij}$ of $\Sigma_t$ as

$$ K_{ij} = \frac{1}{2} L_n h_{ij} = \frac{1}{2N} (\partial_\beta h_{ij} - N_{ij} - N_{ij}), \quad (3.5) $$

where a semicolon denotes the covariant derivative with respect to the metric $h_{ij}$.

Now we consider the following variation of the action

$$ \delta S \equiv \int_{t_1}^{t_2 + \delta t_2} dt \int_{\Sigma_t} d^d x (L_G + \delta L_G) - \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^d x L_G $$

$$ = \int_{t_2}^{t_2 + \delta t_2} dt \int_{\Sigma_t} d^d x L_G - \int_{t_1}^{t_1 + \delta t_1} dt \int_{\Sigma_t} d^d x L_G + \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^d x \delta L_G \quad (3.6) $$

where

$$ \delta L_G \equiv \frac{\delta L_G}{\delta N} \delta N + \frac{\delta L_G}{\delta h_{ij}} \delta h_{ij} + \frac{\partial L_G}{\partial (L_n h_{ij})} \delta (L_n h_{ij}) + \frac{\partial L_G}{\partial (L_n^2 h_{ij})} \delta (L_n^2 h_{ij}). \quad (3.7) $$

Here the first two terms on the right-hand side are not the partial derivatives but the functional derivatives since the scalar curvature $R$ depends on the derivatives of $N$ and $h_{ij}$ in $\Delta(\ln N)$ and $dR$ as seen in (3.1). Since the correspondence of points on each $\Sigma_t$ is made by one-parameter transformation, derivative with respect to $t$ is given by the Lie derivatives along the timelike curves, $L_t$ e.g.,

$$ h_{ij}(x, t + dt) = h_{ij}(x, t) + L_t h_{ij} \, dt, \quad (3.8) $$
to the first order in \( dt \). So variations of gravitational variables are as follows:

\[
\begin{align*}
\delta h_{ij} &\equiv (h_{ij} + \delta h_{ij})(t + \delta t) - h_{ij}(t) = \delta^* h_{ij}(t) + L_t h_{ij}(t) \delta t \\
\delta N &\equiv (N + \delta N)(t + \delta t) - N(t) = \delta^* N(t) + L_t N(t) \delta t
\end{align*}
\]  

(3.9)

where we note that \( \delta^* \) variations are those given in (2.4). When we use (3.7) in (3.6), “partial integrationshave to be done for terms including \( L_n \delta h_{ij} \) and \( L_n^2 \delta h_{ij} \). This is done by using a relation for a scalar field \( \Phi \) and tensor fields \( T^{ij} \) and \( S_{ij} \):

\[
\begin{align*}
\mathcal{L}_n(\sqrt{h} N \Phi) &= \mathcal{L}_n(\sqrt{h} N) \Phi + \sqrt{h} N \mathcal{L}_n \Phi = \partial_\mu (n^\mu \sqrt{h} N \Phi), \\
\mathcal{L}_n(\sqrt{h} N T^{ij} S_{ij}) &= \partial_\mu (n^\mu \sqrt{h} N T^{ij} S_{ij}).
\end{align*}
\]

(3.10)

Then we have

\[
\delta \mathcal{L}_G = \frac{\delta \mathcal{L}_G}{\delta N} \delta N + \left[ \frac{\delta \mathcal{L}_G}{\delta h_{ij}} - \mathcal{L}_n \left( \frac{\partial \mathcal{L}_G}{\partial (\mathcal{L}_n h_{ij})} \right) + L_n^2 \left( \frac{\partial \mathcal{L}_G}{\partial (L_n^2 h_{ij})} \right) \right] \delta h_{ij} \\
+ \partial_\mu \left[ n^\mu \left\{ \left( \frac{\partial \mathcal{L}_G}{\partial (\mathcal{L}_n h_{ij})} - \mathcal{L}_n \left( \frac{\partial \mathcal{L}_G}{\partial (L_n^2 h_{ij})} \right) \right) \delta h_{ij} + \frac{\partial \mathcal{L}_G}{\partial (L_n^2 h_{ij})} \delta(\mathcal{L}_n h_{ij}) \right\} \right].
\]

(3.11)

On the right hand side of the second line of (3.6), the first two terms are approximated as

\[
\left[ \mathcal{L}_G(\Sigma_t) \delta t \right]_{t_1}^{t_2},
\]

(3.12)

and in the last term, it is assumed that effects of the variation of the time \( \delta t \) can be neglected as in the Ostrogradski’s formalism, so the \( \delta \)-variations can be replaced by the \( \delta^* \) variations. The situations are the same for Eq.(3.11). Thus the variation of the action \( \delta S \) is expressed as

\[
\delta S = \left[ \mathcal{L}_G(\Sigma_t) \delta t \right]_{t_1}^{t_2} + \tilde{\delta} S
\]

(3.13)

\[
+ \int_{\Sigma_t} d^d x \left[ n^0 \left\{ \frac{\partial \mathcal{L}_G}{\partial (\mathcal{L}_n h_{ij})} - \mathcal{L}_n \left( \frac{\partial \mathcal{L}_G}{\partial (L_n^2 h_{ij})} \right) \right\} \left( \delta h_{ij} - L_i h_{ij} \delta t \right) \right]_{t_1}^{t_2}
\]

\[
+ n^0 \left\{ \frac{\partial \mathcal{L}_G}{\partial (L_n^2 h_{ij})} \left( \delta(\mathcal{L}_n h_{ij}) - L_i (\mathcal{L}_n h_{ij}) \delta t \right) \right\} \right]_{t_1}^{t_2},
\]

where

\[
\tilde{\delta} S = \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^d x \left[ \frac{\delta \mathcal{L}_G}{\delta N} \delta^* N + \left\{ \frac{\delta \mathcal{L}_G}{\delta h_{ij}} - \mathcal{L}_n \left( \frac{\partial \mathcal{L}_G}{\partial (\mathcal{L}_n h_{ij})} \right) + L_n^2 \left( \frac{\partial \mathcal{L}_G}{\partial (L_n^2 h_{ij})} \right) \right\} \delta^* h_{ij} \right].
\]

(3.14)

### 3.2 Ostrogradski mapping

Ostrogradski mapping is defined by defining (i) the generalized coordinates of the phase space, (ii) momenta canonically conjugate to them which are, of course, the coordinates of the phase space and the (iii) Hamiltonian density. These are defined as follows:

#### 3.2.1 Generalized coordinates

As the generalized coordinates of the phase space, we take the components of the \( d \)-dimensional metric, \( h_{ij} \), which will be referred to as original generalized coordinates and those corresponding to \( D^{(1)}q^i \), as in [14], (a half of) the Lie derivatives of the original generalized coordinates which is equal to the extrinsic curvature \( K_{ij} \), Eq.(3.5), and will be referred to as new generalized coordinates and will be denoted by \( Q_{ij} \).
3.2.2 Conjugate momenta

Moments canonically conjugate to the original and new generalized coordinates, denoted as $p^{ij}$ and $P^{ij}$ respectively, are defined to be the coefficient of their variations in the total time derivative terms in (3.13) and are expressed as follows:

$$
p^{ij} = n^0 \left[ \frac{\partial L_G}{\partial (\dot{L}_n h_{ij})} - L_n \left( \frac{\partial L_G}{\partial (L_n^2 h_{ij})} \right) \right], \quad P^{ij} = 2n^0 \frac{\partial L_G}{\partial (L_n^2 h_{ij})}. \tag{3.15}
$$

Using (3.1) and (3.2), we have the following concrete expressions

$$
\begin{cases}
    p^{ij} = -\sqrt{h} \left[ L_n f'(R) h^{ij} + f'(R) Q^{ij} \right], \\
    P^{ij} = 2\sqrt{h} f'(R) h^{ij},
\end{cases} \tag{3.16}
$$

where, of course, $L_n f'(R)$ is also expressed as $f''(R) L_n R$. Decomposing a tensor $T^{ij}$ into traceless and trace parts, $T^{ij}_\perp$ and $T^{ij}$ respectively, as $T^{ij} = T^{ij}_\perp + 1/d h^{ij} T$, we see from (3.16) that $P^{ij}$ has only the trace part $P \equiv h^{ij} P^{ij} = 2d \sqrt{h} f'(R)$. Solving for $R$, we have

$$
R = f'^{-1}(P/2d\sqrt{h}) \equiv \psi(P/2d\sqrt{h}). \tag{3.17}
$$

In other words, we have constraints $P^{ij}_\perp = 0$, so only new independent canonical pair is

$$(Q, \frac{P}{d}) \equiv (Q, \Pi), \quad i.e. \quad \Pi \equiv \frac{P}{d}. \tag{3.18}$$

3.2.3 Hamiltonian density

Hamiltonian density, $H_G(x,t)$, is defined as the coefficient of $(-1) \times \delta t$ in the time boundary terms of (3.13):

$$
H_G = p^{ij} \dot{L}_t h^{ij} + P^{ij} \dot{L}_t Q^{ij} - L_G. \tag{3.19}
$$

Concrete expression is also given from Eqs. (3.1) and (3.2) as follows:

$$
H_G = \mathcal{N} H_0 + N^i \mathcal{H}_i + \text{divergent terms}, \tag{3.20}
$$

where, after a canonical transformation $(Q, \Pi) \rightarrow (\bar{Q}, \bar{\Pi}) \equiv (\Pi, -Q)$, $\mathcal{H}_0$ and $\mathcal{H}_i$ are expressed as[18]

$$
\begin{cases}
    \mathcal{H}_0 = \frac{2}{Q} \left( p^{ij}_\perp p^{ij}_\perp - \frac{1}{d} p^{ij}_\perp \right) - \frac{2}{d} p \Pi + \frac{1}{2} Q \psi(Q/2\sqrt{h}) - \frac{d}{2d} Q \Pi^2 - \frac{1}{2} d \bar{Q} + \Delta \bar{Q} - \sqrt{h} f \left( \psi(Q/2\sqrt{h}) \right) \\
    \mathcal{H}_i = 2 \left( p^{ij}_\perp - \frac{2}{d} p^{ij}_\perp \right) - Q \Pi_i + \frac{2}{d} (Q \Pi)_{;i}.
\end{cases} \tag{3.21}
$$
3.2.4 Canonical equations of motion

The canonical equations of motion derived from (3.19) are expressed as

\[ \mathcal{L}_t h_{ij} = \delta \mathcal{H}_G / \delta p^{ij}, \quad \mathcal{L}_t p^{ij} = -\delta \mathcal{H}_G / \delta h_{ij}, \]  

(3.22a)

and

\[ \mathcal{L}_t Q_{ij} = \delta \mathcal{H}_G / \delta p_{ij}, \quad \mathcal{L}_t P^{ij} = -\delta \mathcal{H}_G / \delta Q_{ij}. \]  

(3.22b)\(^3\)

Since the f(R)-type gravity is massive, or, the graviton has more than two polarizations\(^2\), we would not modify (3.22a). However, (3.22b) contain dependent variables, we would extract from them the equations for independent variables (3.18). Since

\[ P^{ij} \mathcal{L}_t Q_{ij} = \Pi \mathcal{L}_t Q + (-2p^{ij} + \frac{1}{d}Q\Pi h^{ij})\mathcal{L}_t h_{ij}, \]  

(3.23)

we have

\[ \mathcal{L}_G = p^{ij} \mathcal{L}_t h_{ij} + \Pi \mathcal{L}_t Q - \tilde{\mathcal{H}}_G, \]  

(3.24)

where

\[ \tilde{\mathcal{H}}_G \equiv \mathcal{H}_G - N[8Q(p^{ij}p_{ij} - \frac{1}{d}p^2) + 2dQ\Pi^2] - N^i[4p_{ij}^{\ i} - \frac{4}{d}p^{i} + \frac{2}{d}(Q\Pi)^i] \]  

+ divergent terms.

Then the equations for independent variables in (3.22b) are as follows:

\[ \mathcal{L}_t Q = \frac{\delta \tilde{\mathcal{H}}_G}{\delta \Pi}, \quad \mathcal{L}_t \Pi = -\frac{\delta \tilde{\mathcal{H}}_G}{\delta Q}. \]  

(3.26)

We can obtain explicit expressions of these equations by using (3.21). In addition, noting \( \tilde{\mathcal{H}}_G \) as \( \tilde{\mathcal{H}}_G = NH_0 + N^i\tilde{H}_i \) + divergent terms as (3.21), we have

\[ \begin{cases} 
\tilde{H}_0 = -\frac{6}{Q}(p^{ij}p_{ij} - \frac{1}{d}p^2) - \frac{1}{d}(2p\Pi + \frac{d+1}{2}Q\Pi^2) + \frac{1}{2}Q\psi(\bar{Q}/2\sqrt{h}) - \frac{1}{2}\frac{dR}{\bar{Q}} \\
+ \Delta \bar{Q} - \sqrt{h}f(\psi(\bar{Q}/2\sqrt{h})) \\
\tilde{H}_i = -(2p_{ij}^{\ i} + \bar{Q}\Pi^i). 
\end{cases} \]  

(3.27)

4 Invariance of the Hamiltonian

We consider the following transformations of the generalized coordinates \( h_{ij} \):

\[ h_{ij} \rightarrow \phi_{ij} \equiv F_{ij}(h_{kl}) \quad \text{or inversely} \quad h_{ij} \equiv G_{ij}(\phi_{kl}), \]  

(4.1)

and show that the Hamiltonian is invariant under this transformation as in the case of Ostrogradski formalism. New generalized coordinates \( \Phi_{ij} \) are defined as in (3.5), i.e.,

\[ \Phi_{ij} \equiv \frac{1}{2} \mathcal{L}_n \phi_{ij}. \]  

(4.2)

\(^3\)Precisely, Eqs.(3.22b) are derived by using the Lagrange multiplier method, since they include the constraint equations.
Hamiltonian density $\mathcal{H}_G$ expressed in the transformed variables is defined to be

$$\mathcal{H}_G \equiv \pi^{ij} \mathcal{L}_t \phi_{ij} + \Pi^{ij} \mathcal{L}_t \Phi_{ij} - \bar{\mathcal{L}}_G(N, \phi_{ij}, \mathcal{L}_n \phi_{ij}, \mathcal{L}_n^2 \phi_{ij}),$$

(4.3)

where $\pi^{ij}$ and $\Pi^{ij}$ are momenta canonically conjugate to $\phi_{ij}$ and $\Phi_{ij}$, respectively. Since

$$\mathcal{L}_n h_{ij} = \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n \phi_{kl}, \quad \mathcal{L}_n^2 h_{ij} = \mathcal{L}_n \left( \frac{\partial G_{ij}}{\partial \phi_{kl}} \right) \mathcal{L}_n \phi_{kl} + \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n^2 \phi_{kl},$$

(4.4)

$\bar{\mathcal{L}}_G$ is defined as

$$\bar{\mathcal{L}}_G(N, \phi_{ij}, \mathcal{L}_n \phi_{ij}, \mathcal{L}_n^2 h_{ij}) \equiv \mathcal{L}_G \left( N, G_{ij}(\phi_{kl}), \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n \phi_{kl}, \frac{\partial G_{ij}}{\partial \phi_{kl}} \mathcal{L}_n^2 \phi_{kl} \right).$$

(4.5)

$\pi^{ij}$ and $\Pi^{ij}$ satisfy relations similar to (3.15), and from these relations, we have

$$\pi^{ij} = p^{jk} \frac{\partial G_{kl}}{\partial \phi_{ij}}, \quad \Pi^{ij} = P^{jk} \frac{\partial G_{kl}}{\partial \phi_{ij}},$$

(4.6a)

or inversely

$$p^{ij} = \pi^{jk} \frac{\partial F_{kl}}{\partial h_{ij}}, \quad P^{ij} = \Pi^{jk} \frac{\partial F_{kl}}{\partial h_{ij}}.$$

(4.6b)

With help of (4.6a,b), we have

$$p^{ij} \mathcal{L}_t h_{ij} = \pi^{jk} \frac{\partial F_{kl}}{\partial h_{ij}} \frac{\partial G_{ij}}{\partial \phi_{mn}} \mathcal{L}_t \phi_{mn} = \pi^{ij} \mathcal{L}_t \phi_{ij}.$$  

(4.7)

Similar relation holds between $P^{ij}$ and $\Pi^{ij}$, so we have

$$\mathcal{H}_G = \bar{\mathcal{H}}_G.$$  

(4.8)

It is noted that the transformation (4.1) includes the coordinate transformation on $\Sigma_t$.

5 Summary and discussions

We presented a canonical formalism of the $f(R)$-type gravity by generalizing the Ostrogradski’s formalism. Present formalism refines the previous one by our group which remedied the undesirable points of BL method, i.e., the Hamiltonian is not invariant under the transformation of the generalized coordinates, here the metric $h_{ij}$ of the hypersurface $\Sigma_t$. In addition, we derived canonical equations of motion for independent variables $Q$ and $\Pi$.

The formalism has an important application to the problem of the equivalence theorem between the $f(R)$-type gravity and Einstein gravity coupled to a scalar field, a kind of the scalar-tensor gravity theories. To prove the theorem, a conformal transformation depending on the curvature is used. Therefore the transformation is not necessarily restricted to the canonical one in the phase space. It was in fact shown that it is not the canonical transformation in that the fundamental Poisson brackets before and after the transformation are not consistent although the explicit expressions after the transformation are not uniquely determined by the original ones[23]. Thus if the $f(R)$-type gravity were quantized canonically, transformed theory should be quantized noncanonically. This coincides with the result that noncanonical quantization would stabilize the extra-dimensional space[24, 25, 26].


Present formalism would also be helpful in the investigation of quantum gravity. One version of the theory is that of gravitons which are thought of as the duality partner of the gravitational waves which obey the usual wave equation. So it is reasonable that the theory is the canonical quantum theory. The other version is the quantum cosmology which is the canonical quantum mechanics of spacetime and the basic equation is the Wheeler-DeWitt (WDW) equation. However, no dual partner of spacetime is known definitely. Finally, in order to avoid the probabilistic problem concerning the WDW equation, the third quantization is investigated [27, 28] which is similar to the quantum field theory of the metric, if both theories were true ones, since both theories can treat the creation and annihilation of the universe. However, the confrontation with the observation is very difficult. Therefore we should first establish a reliable theory and then seek the confrontation of the theory with observations.

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