Abstract: The geometric Langlands correspondence was described some years ago in terms of $S$-duality of $\mathcal{N} = 4$ super Yang-Mills theory. Some additional matters relevant to this story are described here. The main goal is to explain directly why an $A$-brane of a certain simple kind can be an eigenbrane for the action of ’t Hooft operators. To set the stage, we review some facts about Higgs bundles and the Hitchin fibration. We consider only the simplest examples, in which many technical questions can be avoided.
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1 Introduction

Some years ago, it was shown by A. Kapustin and the author [1] that the geometric Langlands correspondence (see for example [2] for an introduction) can be formulated in terms of S-duality of \( N = 4 \) super Yang-Mills theory in four dimensions. The basic idea was to consider, in the context of compactification to two dimensions, boundary conditions in this theory, the action of S-duality on those boundary conditions, and the behavior of supersymmetric line operators near a boundary. A number of generalizations have been developed subsequently. The story has been extended to encompass tame [3] and wild [4] ramifications, by including in the analysis surface operators and line operators that live on them. The simplest issues involving geometric Langlands duality for branes supported at singularities were explored in [5]. The study in [6, 7] of half-BPS boundary conditions and domain walls and their behavior under S-duality was partly motivated by potential applications to geometric Langlands. Some of the ingredients that enter in the gauge theory approach to geometric Langlands are also relevant for understanding Khovanov homology via gauge theory [8]. For informal explanations of some aspects of gauge theory and geometric Langlands, see [9–11].
The purpose of the present paper is to present some additional topics relevant to this subject. We consider only the unramified case, so we do not include surface operators. Our main goal is to explain concretely (without assuming electric-magnetic duality) why certain simple $A$-branes on the moduli space $M_H$ of Higgs bundles are “magnetic eigenbranes,” as predicted by duality. As described in [1], the branes in question are branes of rank 1 supported on a fiber of the Hitchin fibration and dual to zero-branes. In order to show directly that these branes are magnetic eigenbranes, we first need to review some facts about Higgs bundles and the Hitchin fibration.

Though we repeat some details to make this paper more nearly self-contained, the reader will probably need to be familiar with parts of the previous paper [1]. Our notation largely follows that previous paper, with some minor modifications. In particular, as in [1], we write $^G\lambda G$ for the Langlands or GNO dual of a compact Lie group $G$ (in some of the above-cited papers, the dual group is denoted as $G^\vee$).

As a preliminary, we review in sections 2 and 3 some basic facts about Higgs bundles and the Hitchin fibration. None of this material is new, but some may be relatively inaccessible. (In these sections, we explain a few useful details that are not strictly needed for our application.) In section 4, following [1], we describe the concept of an “eigenbrane,” describe the electric eigenbranes on $M_H(^G\lambda G)$, and describe the predictions of $S$-duality for magnetic eigenbranes on $M_H(G)$. The rest of the paper is devoted to showing directly that the branes in question really are magnetic eigenbranes. We do this only in the simplest cases, that is for those gauge groups and ’t Hooft operators for which this is most straightforward.

2 Compactification And Hitchin’s Moduli Space

2.1 Preliminaries

The basic reason that it is natural to derive the geometric Langlands correspondence from gauge theory in four dimensions is that $2 + 2 = 4$. As usually formulated mathematically, the geometric Langlands correspondence relates categories associated respectively to representations of the fundamental group or to holomorphic vector bundles on a Riemann surface $C$. From a physical point of view, a representation of the fundamental group or a holomorphic vector bundle is described by a gauge field on $C$. So we should expect to do some sort of gauge theory. However, we must take into account the fact that the geometric Langlands correspondence is a statement not about numbers, or vector spaces, associated to $C$, but about categories.

A $d$-dimensional quantum field theory associates a number – the partition function or the value of the path integral – to a $d$-manifold $M_d$. To a $d - 1$-manifold $M_{d-1}$, it associates a vector space, the space $\mathcal{H}_d$ of quantum states obtained in quantization of the theory on $M_{d-1}$. The next step in this hierarchy is slightly less familiar (see for example [14–16]). A $d$-dimensional quantum field theory associates a category to a manifold $M_{d-2}$ of dimension $d - 2$. For example, a $d = 2$ field theory associates a category to a 0-manifold, that is, to a point. Since any two points are isomorphic, this just means that a two-dimensional field theory determines a category – the category of boundary conditions. (The most familiar
examples are probably the categories of $A$-branes and $B$-branes in topologically twisted $\sigma$-models in two dimensions.)

If, therefore, we want a quantum field theory to associate a category to an arbitrary two-manifold $C$, we have to start with a quantum field theory in $2 + 2 = 4$ dimensions. If, moreover, the category is supposed to be associated to gauge fields on $C$, then we should start with gauge fields in four dimensions. Finally, if we are hoping to find a duality between a category associated in some way to a gauge group $G$ and a category associated in some way to the Langlands or GNO dual group $^L G$, then we should start with a theory in four dimensions that has a duality that exchanges these two groups. The gauge theory with the right properties is $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions.

For the application to geometric Langlands, the $\mathcal{N} = 4$ theory is topologically twisted so as to produce, formally, a topological field theory in four dimensions. For details of this twisting, and now it produces a pair of topological field theories, the $A$-model and the $B$-model, exchanged by $S$-duality, see [1].

A Riemann surface $C$ is introduced simply by considering compactification on $C$ from four to two dimensions. Thus we consider the four-dimensional $\mathcal{N} = 4$ theory on the four-manifold $\mathbb{R}^2 \times C$, or more generally on $\Sigma \times C$, where $\Sigma$ is a two-manifold that is assumed to be much larger than $C$. The (partial, as in the footnote) topological field theory on $\Sigma \times C$ reduces after compactification on $C$ to a topological field theory on $\Sigma$.

In studying this theory on $\Sigma$, we can always assume that $\Sigma$ is much larger than $C$ and work at large distances on $\Sigma$. This means, for many purposes, that we can replace the full four-dimensional theory on $\Sigma \times C$ by a two-dimensional theory on $\Sigma$ [17, 18]. This theory is a supersymmetric $\sigma$-model whose target space is the space of classical supersymmetric vacua that arise in compactification on $C$. Away from singularities at which some of the gauge symmetry is restored, this space is the Hitchin hyper-Kahler moduli space of “Higgs bundles” on $C$ with structure group $G$. We call this space $M_H$, or $M_H(G, C)$ if we wish to specify the gauge group $G$ and the Riemann surface $C$ on which we have compactified.

Although a description by a $\sigma$-model with target $M_H$ is very useful, this description is really only generically valid. That is because of the singularities of $M_H$ at which some of the gauge symmetry becomes restored and additional degrees of freedom must be included to

---

1 This is very likely only a partial topological field theory. It is not clear that its partition function is well-defined in general; to define it, one would have to grapple with integration over noncompact spaces of zero-modes. (On a four-manifold $M$, the space of classical minima of the action is the generically noncompact space of homomorphisms $\pi_1(M) \to G_C$, up to conjugation. The noncompactness of this space will cause difficulty in a proof of topological invariance and of cutting and gluing properties expected in a topological field theory.) However, observables for which the noncompactness of the field space do not come into play are well-defined. In particular, there are well-defined categories of branes, and other observables relevant to geometric Langlands are also well-defined. Technically, because $M_H$ is not compact, different conditions on the behavior of a brane at infinity are possible and a correct choice is needed to agree in detail with standard mathematical formulations of geometric Langlands.

2 In this statement, we assume that either the genus $g_C$ of $C$ is at least 2, or ramification is included in genus 0 or 1. In genus 0 or 1, in the absence of ramification, the solutions of Hitchin’s equations have abelian structure group and a more careful formulation is needed. Also, we assume that $G$ is simple or semi-simple. For $G = U(1)$ or $U(N)$, matters are different as there is always an unbroken $U(1)$ subgroup of $G$. See section 7.1.2 for a discussion of this point.
give a good description. Exactly how many additional degrees of freedom must be included for a good description depends on which question one asks. The universal description, valid for all purposes without any approximation, is the underlying four-dimensional gauge theory on $\Sigma \times C$, with no attempt to reduce to an effective two-dimensional theory. If one wants to describe this complete four-dimensional theory as a two-dimensional $\sigma$-model, one should call it a two-dimensional supersymmetric $\sigma$-model with target space the space $\mathcal{Y}$ of all gauge fields on $C$ (or more precisely, as we discuss later, the corresponding cotangent bundle $\mathcal{W} = T^*\mathcal{Y}$ to include the Higgs field), and gauge group the infinite-dimensional group $\mathcal{G}$ of all maps from $C$ to the finite-dimensional group $G$. If one topologically twists this $\sigma$-model to make either an $A$-model or a $B$-model, then $G$ is effectively replaced by its complexification, the group $G_C$ of maps from $C$ to the complexification $G_C$ of $G$.

To make contact between this picture and standard statements in geometric Langlands, we should specialize to either the $A$-model or the $B$-model side of the duality. For brevity, we consider here the $A$-model side; analogous statements hold for the $B$-model. Algebraic geometers formulate the $A$-model in terms of the “stack” of all holomorphic $G$-bundles over $C$, not necessarily stable.\(^3\) However, according to Atiyah and Bott \[19\], a concrete model of this stack is the infinite-dimensional space $\mathcal{Y}$ acted on by the infinite-dimensional group $\mathcal{G}_C$. Thus a physicist’s interpretation of the statement that a complete description involves the full “stack” is just to say that a full description involves a supersymmetric $\sigma$-model with target the full infinite-dimensional space $\mathcal{Y}$ (acted on by $\mathcal{G}_C$) or in other words the full four-dimensional gauge theory.

### 2.2 Hitchin’s Equations

In this paper, however, we largely consider questions for which it is adequate to consider a $\sigma$-model with target $\mathcal{M}_H$. $\mathcal{M}_H$ is defined by a familiar system of equations known as Hitchin’s equations. These are equations for a pair $A, \phi$, where $A$ is a connection on a $G$-bundle $E$ over the two-dimensional surface $C$, and $\phi$ is a one-form on $C$ with values in the adjoint representation, that is, in the adjoint bundle $\text{ad}(E)$ associated to $E$. In writing these equations, $C$ is assumed to be a Riemann surface, with a chosen complex structure.

Introducing a local complex coordinate $z$ on $C$, the equations can be written

$$
F_z - [\phi_z, \phi_z] = 0,
$$

$$
D_z \phi = D_\bar{z} \phi = 0. \quad (2.1)
$$

Here $D_\sigma = \partial_\sigma + [A_\sigma, \cdot], D_z = \partial_z + [A_z, \cdot]$.

Alternatively, we can combine $A$ and $\phi$ to the complex-valued connection $A = A + i\phi$. We view this as a connection on a $G_C$-bundle $E_C \to C$ that is obtained by complexifying $E \to C$. It has structure group $G_C$ and curvature $F = dA + A \wedge A$. The real and imaginary

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\(^3\)Thus according to \[12\], on one side of the duality, one must consider $\mathcal{D}$-modules (modules for the sheaf of differential operators) on the stack of $G$-bundles (and not just on the moduli space of stable $G$-bundles). The relation of the $A$-model to $\mathcal{D}$-modules depends on the fact that the moduli space of Higgs bundles can be approximated by a cotangent bundle and is explained in \[1\].
parts of $\mathcal{F}$ are $\text{Re}\mathcal{F} = F - \phi \wedge \phi$, $\text{Im}\mathcal{F} = D\phi$. Hitchin’s equations are equivalent to

$$
\mathcal{F} = 0
$$

$$
D \ast \phi = 0.
$$

Here $D \ast \phi = D\varphi_z + D_z\varphi$. These two ways to write Hitchin’s equations lead to two different complex structures on the moduli space $\mathcal{M}_H$ of solutions of those equations. Starting from eqn. \eqref{2.2}, we observe that the equation $\mathcal{F} = 0$ is invariant under complex-valued gauge transformations of $\mathcal{A}$, and that modulo such a gauge transformation, the space of solutions is the space of equivalence classes of homomorphisms $\pi_1(C) \to G_C$, up to conjugation. In particular, the equation $\mathcal{F} = 0$ is topologically-invariant; it does not really depend on the complex structure on $C$. A classic theorem \cite{20} says that given a mild and generically valid condition of semi-stability, it is equivalent to classify complex flat connections on $C$ up to $G_C$-valued gauge transformations, or to impose the “moment map” equation $D \ast \phi = 0$ and divide only by $G$-valued gauge transformations.

Thus, in one perspective, $\mathcal{M}_H$ can be interpreted as the moduli space of semi-stable homomorphisms $\pi_1(C) \to G_C$. Since $G_C$ is itself a complex manifold, this interpretation endows $\mathcal{M}_H$ with a complex structure. This complex structure has been called $J$ by Hitchin \cite{13}.

In another perspective, we view $\mathcal{A}_z$ and $\phi_z$ as holomorphic variables (and their complex conjugates $\overline{\mathcal{A}}_z$ and $\phi_{\overline{\mathcal{A}}}$ as antiholomorphic variables). Hitchin’s equations \eqref{2.1} thus split up as a holomorphic equation $D\overline{\mathcal{A}}_z\phi_z = 0$ (which implies the complex conjugate equation $D_z\phi_{\overline{\mathcal{A}}} = 0$) and the “moment map” equation $F - \phi \wedge \phi = 0$. To interpret the holomorphic equation, we observe first that for any $G$-valued connection $\mathcal{A}$ on a bundle $E \to C$, once a complex structure is picked on $C$, the corresponding $\overline{\partial}$ operator $D_z$ endows $E$ with a holomorphic structure. The equation $D\overline{\mathcal{A}}_z\phi_z = 0$ says that in this complex structure, $\phi_z$ can be viewed as a holomorphic section of $K \otimes \text{ad}(E)$, where $K$ is the canonical bundle of $E$. A Higgs bundle, in Hitchin’s terminology, is a pair consisting of a holomorphic $G_C$-bundle $E \to C$ and a holomorphic “Higgs field” $\varphi = \phi_z dz \in H^0(C, K \otimes \text{ad}(E))$, or in other words a solution of $D\overline{\mathcal{A}}_z\phi_z = 0$. We will call such a pair $(E, \varphi)$ a Hitchin pair.

$G_C$-valued gauge transformations act naturally on the space of pairs $A_z, \phi_z$ that obey $D\overline{\mathcal{A}}_z\phi_z = 0$. However, Hitchin proves that – given again a mild condition of semistability,$^4$

\begin{footnotesize}
\footnote{This condition says that if the monodromies of the flat connection $\mathcal{A}$ can be simultaneously conjugated to a block-diagonal form \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, then in fact they can be conjugated to the block-triangular form \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}. This condition is generically satisfied since (for $C$ of genus $\geq 2$) generically the monodromies of a complex flat connection on $C$ cannot be conjugated to a block-triangular form.}
\end{footnotesize}

$^5$For simplicity, we will state this condition only for $G = SU(N)$. In this case, the holomorphic $G_C$-bundle $E_C \to C$ is semi-stable if any subbundle has non-positive first Chern class. A Higgs bundle $(E, \varphi)$ is semi-stable if any $\varphi$-invariant subbundle has non-positive first Chern class. (The concept of $\varphi$-invariance is as follows. As $\varphi$ is a section of $K \otimes \text{ad}(E)$, it can be understood naturally as a map $E \to K \otimes E$. Picking a local trivialization of $K$, this gives a map $E \to E$, and a subbundle of $E$ is $\varphi$-invariant if it is invariant under this map. This criterion does not depend on the choice of local trivialization of $K$.) In particular, if $E$ is semi-stable, then $(E, \varphi)$ is semi-stable for any $\varphi \in H^0(C, K \otimes \text{ad}(E))$. In these statements, to

\begin{footnotesize}
\footnote{For $G_C$-valued gauge transformations act naturally on the space of pairs $A_z, \phi_z$ that obey $D\overline{\mathcal{A}}_z\phi_z = 0$. However, Hitchin proves that – given again a mild condition of semistability$^5$, this condition is generically satisfied since (for $C$ of genus $\geq 2$) generically the monodromies of a complex flat connection on $C$ cannot be conjugated to a block-triangular form.}
\end{footnotesize}
– it is equivalent to divide the space of such pairs by $G_C$-valued gauge transformations or to impose the “moment map” equation $D \star \phi = 0$ and divide only by $G$-valued gauge transformations.

Thus, from this point of view, $\mathcal{M}_H$ acquires another interpretation as the moduli space of stable Higgs bundles. This description makes manifest another complex structure that Hitchin calls $I$. Hitchin shows, moreover, that the complex structures $I$ and $J$ form part of a hyper-Kahler structure on $\mathcal{M}_H$, with in particular an action on the tangent space to $\mathcal{M}_H$ at any smooth point of the quaternion units $I, J,$ and $IJ = K$.

Many aspects of this picture that are relevant to geometric Langlands have been described in [1] and will not be repeated here. About the statement that $\mathcal{M}_H$ is hyper-Kahler, we remark only that it can be constructed as a hyper-Kahler quotient. The space $\mathcal{W}$ of pairs $A, \phi$ can be regarded as an infinite-dimensional hyper-Kahler manifold with flat hyper-Kahler metric

$$ds^2 = -\frac{1}{2\pi} \int_C |d^2 z| \text{Tr}(\delta A_z \otimes \delta A_T + \delta \phi_z \otimes \delta \phi_T), \quad |d^2 z| = idz d\overline{z}. \quad (2.3)$$

The group $\mathcal{G}$ of $G$-valued gauge transformations acts on $\mathcal{W}$, preserving its hyper-Kahler structure. Hitchin’s equations assert the vanishing of the hyper-Kahler moment map for the action of $\mathcal{G}$ on $\mathcal{W}$. The space $\mathcal{M}_H$ of solutions of Hitchin’s equations, modulo $\mathcal{G}$, can thus be interpreted as the hyper-Kahler quotient $\mathcal{W}///\mathcal{G}$. As such it naturally carries a hyper-Kahler structure.

We will focus here on aspects of the picture in complex structure $I$ that were not explained in [1] and that are important in a deeper understanding of geometric Langlands. For example, we will use some of this understanding later in discussing magnetic eigenbranes.

Since we will concentrate on just one complex structure, namely $I$, we will not see the full hyper-Kahler structure of $\Sigma$. However, there is an important part of the structure that is quite visible in complex structure $I$. In general, a hyper-Kahler manifold $X$ has a three-dimensional space of real symplectic structures. In any one of the complex structures on $X$, two of the three real symplectic structures can be combined as the real and imaginary parts of a holomorphic symplectic structure. In the case of the complex structure $I$ on $\mathcal{M}_H$, the holomorphic symplectic structure is

$$\Omega_I = \frac{1}{\pi} \int |d^2 z| \text{Tr} \delta \phi_z \wedge \delta A_T. \quad (2.4)$$

Here $-\text{Tr}$ is an invariant quadratic form on the Lie algebra $\mathfrak{g}$ of $G$. (We will use a standard normalization in which short roots have length squared 2.) In complex structure $I$, the hyper-Kahler quotient $\mathcal{W}///\mathcal{G} = \mathcal{M}_H$ can be understood as a complex symplectic quotient, that is a symplectic quotient with respect to the action of $\mathcal{G}_C$ on $\mathcal{W}$, viewed as a complex replace “semi-stable” by “stable,” one merely has to replace “non-positive” by “negative.” As a matter of terminology, we should note that what is usually called the moduli space of stable objects of any given kind is usually defined to parametrize objects of the given type that are stable or semi-stable. This is why one calls $\mathcal{M}_H$ the moduli space of stable Higgs bundles, even though in general some of the objects that it parametrizes are only semi-stable.
symplectic manifold with symplectic form $\Omega_I$. (Such a symplectic quotient is defined by setting to zero the complex moment map, which in the present case is $\mu = D\phi z$, and dividing by the complex symmetry group, here $G_C$. In the present instance, this is the operation that produces $M_H$.)

We can write $\Omega_I = \omega_J + i\omega_K$, where $\omega_J$ and $\omega_K$ are real symplectic forms that are Kahler forms for complex structure $J$ and $K$ respectively. In geometric Langlands, one is primarily interested in the $A$-model with symplectic form $\omega_K$; we call this more briefly the $A$-model of type $K$.

### 2.3 $M_H$ and the Cotangent Bundle

To begin our more detailed study of the geometry of $M_H$ in complex structure $I$, recall first that a Hitchin pair $(E, \varphi)$, where $E \to C$ is a holomorphic $G$ bundle and $\varphi \in H^0(C, K \otimes \text{ad}(E))$, is stable or semi-stable if the underlying bundle $E$ is stable or semi-stable. (The converse is not true.) So in particular, if $E$ is a stable bundle, the Hitchin pair $(E, 0)$ is always stable. This gives a natural embedding of $\mathcal{M}$, the moduli space of stable $G$-bundles on $C$, into the Hitchin moduli space $M_H$. $\mathcal{M}$ is a holomorphic submanifold of $M_H$ in complex structure $I$, since it is defined by the equation $\varphi = 0$, which is holomorphic in complex structure $I$. (In complex structures $J$ and $K$, $\mathcal{M}$ is not holomorphic. But it is Lagrangian with respect to $\Omega_I$ and hence also for $\omega_J$ and $\omega_K$.)

If $E$ is stable, then the Hitchin pair $(E, \varphi)$ is stable for every $\varphi \in H^0(C, K \otimes \text{ad}(E))$. The tangent space to $\mathcal{M}$ at the point represented by $E$ is $H^1(C, \text{ad}(E))$, and by Serre duality, the dual of this, or in other words the cotangent space to $\mathcal{M}$ at the point $E$, is $H^0(C, K \otimes \text{ad}(E))$. Since $\varphi$ takes values in this space, it follows that the space of all pairs $(E, \varphi)$ with stable $E$ is the cotangent bundle $T^*\mathcal{M}$. We thus actually get an embedding of $T^*\mathcal{M}$ in $M_H$. The holomorphic symplectic form $\Omega_I$ of $M_H$ in complex structure $I$ restricts on $T^*\mathcal{M}$ to its natural symplectic structure as a holomorphic cotangent bundle.

The image of $T^*\mathcal{M}$ in $M_H$ is not all of $M_H$ because a Hitchin pair $(E, \varphi)$ may be stable even if $E$ is unstable.\footnote{For $G = SU(2)$, this happens if $E$ has a holomorphic line sub-bundle $\mathcal{L}$ of positive first Chern class (so $E$ is not stable), but $\mathcal{L}$ is not $\varphi$-invariant (so $(E, \varphi)$ is stable). We give examples in section 2.7.} However, the stable Hitchin pairs $(E, \varphi)$ for which $E$ is unstable are of sufficiently high codimension to be unimportant for many applications. Upon throwing away this set, $M_H$ becomes isomorphic to $T^*\mathcal{M}$, and has a natural map to $\mathcal{M}$ by forgetting $\varphi$. (This map is used in [1] in understanding the relation of $A$-branes of type $K$ on $M_H(G, C)$ to $D$-modules on $M(G, C)$.)

Instead of making a projection from $M_H$ to $\mathcal{M}$, which is only generically defined, we can consider the foliation of $M_H$ by the holomorphic type of the bundle $E$ that underlies a stable Hitchin pair $(E, \varphi)$. This foliation is defined throughout the smooth part of $M_H$. It has middle-dimensional leaves, which are Lagrangian with respect to the complex symplectic structure $\Omega_I$.

### 2.4 The Hitchin Fibration

What is usually called the Hitchin fibration is not the map that sends $\varphi$ to 0 but another map that sends a Hitchin pair $(E, \varphi)$ to the characteristic polynomial of $\varphi$. For $G =
$SU(2)$, this characteristic polynomial is simply the quadratic differential $w = \text{Tr} \varphi^2$. $w$ is holomorphic since $\varphi$ is, so it takes values in $\mathcal{V} = H^0(C, K^2) \cong \mathbb{C}^{3g-3}$. The Hitchin fibration is the map $\pi : \mathcal{M}_H \to \mathcal{V}$ that sends $(E, \varphi)$ to $w = \text{Tr} \varphi^2$.

For any $G$, the Hitchin fibration is defined similarly, incorporating all of the independent Casimirs of $G$, and not just the quadratic Casimir. For example, for $G = SU(N)$, we define $w_n = \text{Tr} \varphi^n \in H^0(C, K^n)$, $n = 2, \ldots, N$, and let $\mathcal{V} = \bigoplus_{n=2}^{N} H^0(C, K^n)$. The Hitchin fibration is then defined to take $(E, \varphi)$ to $(w_2, w_3, \ldots, w_n) \in \mathcal{V}$. For any Lie group $G$ of rank $r$, the ring of invariant polynomials on the Lie algebra $\mathfrak{g}$ is a polynomial algebra with $r$ generators $\mathcal{P}_i$. The degrees $d_i$ of these polynomials obey

$$\sum_i (2d_i - 1) = \dim(G). \tag{2.5}$$

For example, for $G = SU(N)$, for the $\mathcal{P}_i$ we can take the polynomials $\text{Tr} \varphi^n$, $n = 2, \ldots, N$, of degree $n$, so that the identity (2.5) becomes $\sum_{n=2}^{N}(2n-1) = N^2 - 1 = \dim(G)$. For any $G$, the Hitchin fibration is defined to take $(E, \varphi)$ to the collection of invariant polynomials $\mathcal{P}_i(\varphi) \in H^0(C, K^{d_i})$. So the base of the Hitchin fibration is $\mathcal{V} = \bigoplus_i H^0(C, K^{d_i})$.

Since $\dim H^0(C, K^d) = (2d-1)(g-1)$, it follows from (2.5) that the complex dimension of $\mathcal{V}$ is $(g-1)\dim(G)$, which equals the dimension of $\mathcal{M}$, and one-half of the dimension of $\mathcal{M}_H$. The Hitchin fibration $\pi : \mathcal{M}_H \to \mathcal{V}$ is surjective, as we will discuss momentarily. A generic fiber $\mathfrak{F}$ of the Hitchin fibration therefore also has half the dimension of $\mathcal{M}_H$:

$$\dim \mathfrak{F} = \dim \mathcal{V} = \frac{1}{2} \dim \mathcal{M}_H = (g - 1) \dim G. \tag{2.6}$$

In section 2.7, we will construct explicitly, for each point $w \in \mathcal{V}$, a stable Hitchin pair $(E, \varphi)$ that projects to $w$ under the Hitchin fibration. In the meantime, we give a more qualitative argument that the Hitchin fibration is surjective. For example, take $G = SU(2)$. Pick a stable $SU(2)$ bundle $E$, and look for a Hitchin pair $(E, \varphi)$ that maps to a given point in $\mathcal{V}$, defined by a quadratic differential $w$. For this we need to find $\varphi \in H^0(C, K \otimes \text{ad}(E))$ with $\text{Tr} \varphi^2 = w$. That is a system of $3g - 3$ equations for $3g - 3$ unknowns so the generic number of solutions is $2^{3g-3}$. A similar counting is possible for other $G$.

2.5 Complete Integrability

One of Hitchin’s main results [21] is the statement that $\mathcal{M}_H$ is a completely integrable Hamiltonian system in the complex structure $I$. In fact, we can find $\frac{1}{2} \dim \mathcal{M}_H$ functions $H_a$ on $\mathcal{M}_H$ that are holomorphic in complex structure $I$, are algebraically independent, and commute in the Poisson brackets obtained from the holomorphic symplectic form $\Omega_I$.

In fact, we can take the $H_a$ to be linear functions on $\mathcal{V}$, since the dimension of $\mathcal{V}$ is the same as the desired number of functions. For $G = SU(2)$, we simply begin by picking a basis $\alpha_a$, $a = 1, \ldots, 3g - 3$ of the $(3g - 3)$-dimensional space $H^1(C, T)$, which is dual to $H^0(C, K^2) \cong \mathcal{V}$. (Here $T$ is the holomorphic tangent bundle to $C$.) We represent $\alpha_a$ by

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7By taking a real slice of $\mathcal{M}_H$, one can extract from this construction a conventional integrable system with a real phase space and real commuting Hamiltonians. See [22] for some examples and references.
We claim that these functions are Poisson-commuting with respect to the holomorphic symplectic form $\Omega_I$.

A natural proof uses the fact that the definition of the $H_a$ makes sense on the infinite-dimensional space $W$, before taking the hyper-Kahler quotient. Using the symplectic structure $\Omega_I$ on $W$ to define Poisson brackets, the $H_a$ are obviously Poisson-commuting. For in these Poisson brackets, given the form (2.4) of $\Omega_I$, $\varphi_z$ has vanishing Poisson brackets with itself (its Poisson brackets with $A_{\varpi}$ are of course nonzero). But the $H_a$ are functions of $\varphi_z$ only, not $A_{\varpi}$.

The $H_a$ can be restricted to the locus with $0 = D_{\varphi}\varphi_z$, and then, because they are invariant under the $G_C$-valued gauge transformations, they descend to holomorphic functions on $M_H$. A general property of symplectic reduction (in which one sets to zero a moment map, in this case $\nu_I = D_{\varphi}\varphi_z$, and then divides by the corresponding group, in this case the group of $G_C$-valued gauge transformations) is that it maps Poisson-commuting functions to Poisson-commuting functions. So the $H_a$ are Poisson-commuting as functions on $M_H$.

There are enough of them to establish the complete integrability of $M_H$.

The generalization from $G = SU(2)$ to arbitrary $G$ is made by simply using all the independent gauge-invariant polynomials $P_i$, not just $\text{Tr} \varphi^2$, to define the Hamiltonians. The commuting Hamiltonians are now $H_{a,i} = \int_C \alpha_{a,i} P_i(\varphi)$, where now $\alpha_{a,i} \in H^1(C, K^{1-d_i})$. These commuting Hamiltonians are the full set of linear functions on the base $V$ of the Hitchin fibration, and equal in number to one-half the dimension of $M_H$.

In this construction, we started with a particular choice of $(0,1)$-forms $\alpha_{a,i}$ that are used to construct the commuting Hamiltonians. But after restricting and descending to $M_H$, the functions we get on $M_H$ depend only on the cohomology classes of the $\alpha_a$. In fact, once we have $D_{\varphi} = 0$ and hence $\partial P_i(\varphi) = 0$, a simple integration by parts shows that the $H_{a,i}$ are invariant under $\alpha_{a,i} \to \alpha_{a,i} + \partial \alpha_{a,i}$.

The Poisson-commuting functions $H_a$ generate commuting flows on $M_H$ that are holomorphic in complex structure $I$. Moreover, these flows commute with the Hitchin fibration – since the commuting Hamiltonians are precisely the functions on the base of this fibration. So the flows act on the fibers of the Hitchin fibration, and in particular those fibers admit a maximal set of commuting flows.

Complex tori admit such a maximal set of commuting flows, and one might surmise that the orbits generated by the $H_a$ are complex tori at least generically. This follows from general arguments given the “properness” of the Hitchin fibration (the compactness of the fibers), but we will demonstrate it more directly, following [21], by using the theory of the spectral curve.

One easy and important consequence of complete integrability is that the fibers of the Hitchin fibration are Lagrangian submanifolds in the holomorphic symplectic structure $\Omega_I$. Indeed, a fiber of this fibration is defined by equations $H_{a,i} - h_{a,i} = 0$, where $H_{a,i}$ are the commuting Hamiltonians and $h_{a,i}$ are complex constants. In general, a middle-dimensional...
submanifold defined by the vanishing of a collection of Poisson-commuting functions, such as \(H_{a,i} - h_{a,i}\) in the present case, is Lagrangian.

2.6 The Spectral Curve

2.6.1 Basics

To describe the idea of the spectral curve, let us consider the case \(G = SU(N)\). We think of \(E\) as a rank \(N\) complex vector bundle. Because \(\varphi\) takes values in the adjoint representation, we can think of it locally as an \(N \times N\) matrix of holomorphic one-forms – which we can take to act on the fiber of \(E\). Then, fixing a point \(p \in C\), and denoting as \(\psi\) an element of the fiber of \(E\) at \(p\), we can consider the “eigenvalue problem”

\[
\varphi(p)\psi = y\psi.
\]  

(2.8)

Since the matrix elements of \(\varphi(p)\) take values in \(K|_p\) – the fiber at \(p\) of the canonical bundle \(K\) – we cannot interpret \(y\) as a number. But the eigenvalue problem makes sense if we interpret \(y\) as an element of \(K|_p\).

We know how to find the eigenvalues of an \(N \times N\) matrix. They are the zeroes of the characteristic polynomial. Thus, they obey the equation

\[
\det(y - \varphi) = 0.
\]  

(2.9)

For generic \(\varphi\) and \(p\), the equation (2.9) has \(N\) distinct roots. For \(N = 2\), the equation simplifies. Since the \(2 \times 2\) matrix \(\varphi\) is everywhere traceless, the equation reduces to

\[
y^2 - \frac{1}{2}\text{Tr} \varphi^2 = 0.
\]  

(2.10)

So far we have presented this at a single point \(p \in C\), but obviously we can consider the equation for all \(p\). As \(p\) varies, the \(N\) roots of (2.9) sweep out an algebraic curve \(D\) which is an \(N\)-fold cover of \(C\). \(D\) maps to \(C\) by “forgetting” \(y\). We let \(W\) be the algebraic surface which is the total space of the line bundle \(K\) over \(C\). The curve \(D\) naturally lies in the surface \(W\). We can think of \(y\) as parametrizing the fiber of the fibration \(W \to C\). The equation \(\det(y - \varphi) = 0\) singles out \(N\) points in each fiber, making up the spectral curve \(D\), with its \(N\)-fold covering map \(\psi: D \to C\).

Now let us consider the problem of describing the fiber \(F\) of the Hitchin fibration \(\pi: \mathcal{M}_H \to \mathcal{V}\). Choosing a particular fiber \(F\) means making a particular choice of \(\text{Tr} \varphi^2\) (for \(SU(2)\)) or of the characteristic polynomial \(\det(y - \varphi)\) of \(\varphi\) (for any unitary group). Hence the choice of a fiber determines a particular spectral curve \(D\); every fiber is associated with its own \(D\). If we pick a sufficiently generic fiber, the curve \(D\) is smooth and irreducible. For example, for \(SU(2)\), this is so precisely if the zeroes of the quadratic differential \(\text{Tr} \varphi^2\) are distinct.

These zeroes are the branch points of the cover \(D \to C\). A quadratic differential on \(C\) has \(4g_C - 4\) zeroes. The double cover of a curve of genus \(g_C\) branched over \(4g_C - 4\) points

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\(^8\)In understanding the following material, I was greatly assisted by explanations by R. Donagi in the period 2004-6.
The spectral curve is given near $z$.

For $SU(N)$, let $\mathcal{P}(y) = \det(y - \varphi)$. The “discriminant” of a polynomial $\mathcal{P}$ with roots $\lambda_i$ is $\prod_{i<j} (\lambda_i - \lambda_j)^2$. In the present context, the discriminant is a gauge-invariant polynomial $\Delta(\varphi)$ homogeneous of degree $N(N-1)$, and thus an element of $H^0(C, K^{N(N-1)})$. $D$ is smooth and irreducible if $\Delta(\varphi)$ has only simple zeroes (and more generally if and only if $\varphi$ always has only a single Jordan block for each eigenvalue). A zero of the discriminant is the same as a point on $C$ over which there is a point on $D$ at which $\mathcal{P}(y) = \mathcal{P}'(y) = 0$. The equation $\mathcal{P}(y) = 0$ defines $D$, and the points on $D$ with $\mathcal{P}'(y) = 0$ are exactly the ramification points of the cover $D \to C$. The zero set of $\mathcal{P}'(y)$ thus defines a divisor $\mathcal{R}$ on $D$ that we will call the ramification divisor. It consists generically of $2N(N-1)(g_C - 1)$ distinct points (one for each zero of $\Delta(\varphi) \in H^0(C, K^{N(N-1)})$), and this implies that the genus of $D$ is $g_D = g_C + (N^2 - 1)(g_C - 1) = g_C + \dim(V) = g_C + \dim(\mathfrak{f})$. If the ramification points are distinct, this suffices to ensure that $D$ is smooth.

Now we will explain a key fact: the eigenvectors of $\varphi$ furnish a line bundle over $D$. Suppose that a Hitchin pair $(E, \varphi)$ is given. For some $p \in C$, we find a root $y$ of the characteristic equation (2.9). This corresponds to a point $p \in D$, one of the $N$ points $q_1, \ldots, q_N \in D$ that lie above $p$. For each such $y$, the eigenvector problem (2.8) has a solution for some nonzero $\psi$. To be more precise, assuming the roots of (2.9) are distinct, the eigenvector problem for given $y$ and thus given $q \in D$ has a one-dimensional space of solutions that we will call $\mathcal{N}_q$. As $q$ varies, $\mathcal{N}_q$ varies as the fiber of a complex line bundle $\mathcal{N}$ over $D$.

Does the definition of the line bundle $\mathcal{N}$ break down when the roots fail to be distinct? When the curve $D$ is smooth, it does not. Let us explain this just for $G = SU(2)$, the general case being similar. The roots fail to be distinct precisely where $\text{Tr} \varphi^2 = 0$. We pick a local complex coordinate $z$ on $C$ and assume that $\text{Tr} \varphi^2 = 0$ at $z = 0$. As we require $\text{Tr} \varphi^2$ to have simple zeroes, the behavior of $\varphi$ near $z = 0$ is, up to conjugacy,

$$\varphi \sim \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}. \tag{2.11}$$

The spectral curve is given near $z = 0$ by

$$0 = \det(y - \varphi) = y^2 - z. \tag{2.12}$$

The solutions of this equation are given, for given $z$, by $y = \pm \sqrt{z}$. However, a better way to describe the spectral curve near $z = 0$ is to regard $y$ as a local parameter, with $z = y^2$. Plugging $z = y^2$ into the local formula (2.11) for $\varphi$, the eigenvalue equation (2.8) becomes

$$\begin{pmatrix} 0 & y^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = y \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.13}$$
This has, for every $y$, a one-dimensional space of solutions, generated by

$$s = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} y \\ 1 \end{pmatrix}, \quad (2.14)$$

showing that the definition of the line bundle $\mathcal{N}$ works perfectly well at simple branch points of the spectral curve $D$.

In effect, we have defined $\mathcal{N}$ as the kernel of $\varphi - y$. We can define a related line bundle $\mathcal{L}$ as the cokernel of $\varphi - y$. As long as the eigenvalues of $\varphi$ are distinct, the linear transformation $\varphi - y$, regarded as a map from sections of $E \otimes K^{-1}$ to sections of $E$, has a one-dimensional kernel and therefore an $(N - 1)$-dimensional image $E' = \text{Im} \ (\varphi - y)$. The quotient $\mathcal{L} = E/E'$ is therefore one-dimensional away from the ramification points, so this gives us another holomorphic line bundle away from those points.

We can use the above local model to verify that $\mathcal{L}$ naturally extends as a line bundle over the ramification points. In the local model,

$$\varphi - y = \begin{pmatrix} -y & z \\ 1 & -y \end{pmatrix}, \quad (2.15)$$

so the image $E'$ of $\varphi - y$ is generated by

$$u = \begin{pmatrix} -y \\ 1 \end{pmatrix}, \quad (2.16)$$

Hence the quotient $\mathcal{L} = E/E'$ is generated by

$$t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (2.17)$$

that is, $u$ and $t$ give a basis for every $y$. This exhibits a natural extension of $\mathcal{L}$ across the point $z = 0$.

There is also an important relationship between $\mathcal{L}$ and $\mathcal{N}$. We defined $\mathcal{N}$ as the subbundle of $E$ (or more precisely, of $\psi^*(E)$, the pullback of $E$ to $D$) annihilated by $\varphi - y$, so there is a natural embedding $i : \mathcal{N} \to \psi^*(E)$. Also, $\mathcal{L}$ is a quotient of $\psi^*(E)$ given by reducing mod $E' = \text{Im} \ (\varphi - y)$, so there is a natural map $r : \psi^*(E) \to \mathcal{L}$. The composition

$$\mathcal{N} \xrightarrow{i} \psi^* E \xrightarrow{r} \mathcal{L} \quad (2.18)$$

gives a holomorphic map $\theta = ri : \mathcal{N} \to \mathcal{L}$.

Away from ramification points, $\varphi - y$ can be block-diagonalized with a 1-dimensional kernel and an invertible block of codimension 1:

$$\varphi - y = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}, \quad (2.19)$$

So the kernel and cokernel are generated by the upper element (the zero-mode) and the map $\theta : \mathcal{N} \to \mathcal{L}$ is an isomorphism.
To see what happens at a ramification point, we again use the local model for $N = 2$. $N$ is generated by the vector called $s$ in (2.14). To evaluate $\theta(s)$, we just reduce mod $u$, defined in (2.16), and express the result as a multiple of the generator $t$ of $\mathcal{L}$, found in (2.17). We have $s = u + 2yt$, or

$$s = 2yt \text{ mod } u.$$  

So a generator $s$ of $N$ maps to $2y$ times a generator of $\mathcal{L}$. As $y$ has a simple zero at the ramification divisor $R$, the result is that

$$\mathcal{L} \cong N(R),$$

in other words a section of $\mathcal{L}$ is equivalent to a section of $N$ that may have a pole at the ramification divisor.

The ramification divisor, as noted above, is precisely the zero set on $D$ of the polynomial $P'(y)$, which is a section of $\psi^* (K^{N-1})$ (since $y$ is a section of $\psi^* (K)$, and $P'(y)$ is of degree $N - 1$ in $y$). So we can alternatively write

$$\mathcal{L} \cong N \otimes \psi^* (K^{N-1}).$$

2.6.2 Abelianization

The key insight is now that one can reconstruct the bundle $E \to C$ (and the Higgs field $\varphi$) from the line bundle $\mathcal{L} \to D$. This represents progress, because $E$ is a $G$-bundle where $G$ is nonabelian, while a line bundle is the analog of a $G$-bundle with $G$ replaced by the abelian group $U(1)$. The price we pay for this abelianization is that instead of working on the curve $C$, we have to work on its spectral cover $D$.

Suppose first that a point $p \in C$ is not a branch point of the fibration $\psi : D \to C$. Then the fiber of $E$ at $p$, which we denote as $E_p$, can be decomposed as a sum of the $N$ one-dimensional eigenspaces of $\varphi$. These eigenspaces are the fibers of what we have called $N$ at the points $q_i$ that lie above $p$. Thus, as long as $p$ is not a branch point, we get

$$E_p = \oplus_{i=1}^N N_{q_i}.$$  

This certainly shows that we can recover $E$ from $N$ away from the branch points. Since $N$ and $\mathcal{L}$ are naturally isomorphic away from the branch points, we can equally well write

$$E_p = \oplus_{i=1}^N \mathcal{L}_{q_i}.$$  

away from branch points. This description turns out to extend more simply over the branch points.

The extension of this formula over the branch points involves another notion in algebraic geometry, the “push-forward.” This is defined for any map $\psi : D \to C$ and any sheaf on $D$. Our sheaves will be sheaves of sections of a line bundle or vector bundle, and we will not distinguish in the notation between a bundle and the corresponding sheaf of sections. If $\mathcal{L}$ is a line bundle on $D$, the push-forward $\psi_*(\mathcal{L})$ is the sheaf on $C$ defined by saying that sections of $\psi_*(\mathcal{L})$ over a sufficiently small open set $U \subset C$ are the same as sections of $\mathcal{L}$ over $\psi^{-1}(U)$. 

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The claim is that $E$ can be reconstructed from $\mathcal{L}$ as
\[ E = \psi_*(\mathcal{L}). \tag{2.25} \]
Away from the ramification points, this is just a fancy restatement of (2.24). Let us see what happens at ramification points, again using the local model. We do this just for $N = 2$, though the result is general.

A section of $\mathcal{L}$ is the same thing as a section of $\psi^*E$ except that we must reduce mod $u$, the generator of $\text{Im}(\varphi - y)$. A section of $\psi_*(\psi^*E)$ is, informally, the same as a section of $E$ except that it is allowed to depend on $y$. So a section of $\psi_*(\mathcal{L})$ is a section of $E$ which (i) may depend on $y$; (ii) is considered trivial if it is a multiple of $u$.

A section of $E$ takes the form
\[ \left( \begin{array}{c} a(z) \\ b(z) \end{array} \right), \tag{2.26} \]
with holomorphic functions $a(z)$, $b(z)$. A section of $\psi_*(\mathcal{L})$ can be written
\[ \left( \begin{array}{c} A(z) + yC(z) \\ B(z) + yD(z) \end{array} \right) \mod (G(z) + yH(z)) \left( \begin{array}{c} -y \\ 1 \end{array} \right), \tag{2.27} \]
where we allow the $y$ dependence and reduce mod $u$. For $G = -C$, $H = D$, we have
\[ (G(z) + yH(z)) \left( \begin{array}{c} -y \\ 1 \end{array} \right) = y \left( \begin{array}{c} C \\ D \end{array} \right) + \left( \begin{array}{c} -zD \\ -C \end{array} \right), \tag{2.28} \]
showing that the equivalence relation in (2.27) suffices to set $C = D = 0$ in a unique fashion, thus rendering (2.27) equivalent to (2.26) and showing that the statement $E = \psi_*(\mathcal{L})$ remains true at the ramification points.

One similarly can recover the Higgs field $\varphi$ by
\[ \varphi = \psi_*(y). \tag{2.29} \]
This formula means that if $f$ is a section of $\mathcal{L} \to D$ and $s = \psi_*(f)$ is the corresponding section of $E \to C$, then $\varphi(s) = \psi_*(yf)$, a condition that suffices to determine $\varphi$. To see that this is true, note that a section $\tilde{f}$ of $\mathcal{L}$ is the same as a section $\tilde{f}$ of $\psi^*(E)$ modulo the equivalence relation of setting to zero $(\varphi - y)\chi$ for any $\chi$. Modulo this equivalence relation, $y\tilde{f} = \varphi \tilde{f}$. Since this is true for all $\tilde{f}$, it pushes down to the relation (2.29) on $C$.

### 2.6.3 Which Line Bundles Appear?

So every Hitchin pair $(E, \varphi)$ comes from a line bundle $\mathcal{L}$ on the spectral cover $D$. But which line bundles appear this way? We will first consider the degree of $\mathcal{L}$ and then its moduli.

We start by computing the push-forward $\psi_* (\mathcal{O})$, where $\mathcal{O}$ is a trivial line bundle over $D$. A local holomorphic section $s$ of $\mathcal{O} \to D$ can be expanded in powers of $y$:
\[ s = a_0 + y a_1 + y^2 a_2 + \cdots + y^{N-1} a_{N-1}. \tag{2.30} \]
The series stops at $y^{N-1}$ because of the equation $\det(y - \varphi) = 0$. As $y$ is a section of $\psi^*(K)$, each $\alpha_i$ is a section of the line bundle $K^{-i}$ over $C$. So

$$\psi_*(\mathcal{O}) = \mathcal{O} \oplus K^{-1} \oplus K^{-2} \oplus \cdots \oplus K^{-(N-1)}.$$  

(2.31)

For future reference, we can also easily calculate $\psi_*(\mathcal{L})$ if $\mathcal{L} = \psi^*(\mathcal{L}_0)$ is the pullback of a line bundle $\mathcal{L}_0$ over $C$. In this case, a local section $s$ of $\mathcal{L}$ can be expanded just as in (2.30), but $\alpha_i$ is now a section of $\mathcal{L}_0 \otimes K^{-i}$. So

$$\psi_*(\psi^*(\mathcal{L}_0)) = \mathcal{L}_0 \otimes \left( \mathcal{O} \oplus K^{-1} \oplus K^{-2} \oplus \cdots \oplus K^{-(N-1)} \right).$$

(2.32)

As $K^{-i}$ has degree $-2i(g_C - 1)$, (2.31) implies that the first Chern class of $\psi_*(\mathcal{O})$ is $d = -N(N-1)(g_C - 1)$. If instead $\mathcal{L} \to D$ has degree $c$, then $\psi_*(\mathcal{L})$ will be a vector bundle on $C$ of first Chern class $c + d$.\footnote{One way to see this is to use the Riemann-Roch formula and the fact that the holomorphic Euler characteristic of $\mathcal{L} \to D$ equals that of $\psi_*(\mathcal{L}) \to C$. If the first Chern class of $\mathcal{L} \to D$ is increased by 1, the holomorphic Euler characteristic increases by 1, and hence the first Chern class of $\psi_*(\mathcal{L}) \to C$ must increase by 1.} To get an $SU(N)$ bundle, the first Chern class should vanish, so the degree of $\mathcal{L}$ must be

$$c_0 = N(N - 1)(g_C - 1).$$

(2.33)

Therefore, a line bundle $\mathcal{L}$ associated with an $SU(N)$ Hitchin pair is of this degree and represents a point in $\text{Pic}_{c_0}(D)$, the component of the Picard group of $D$ that parametrizes line bundles of degree $c_0$.

The dimension of each component of the Picard group is equal to the genus $g_D$ of $D$, which is $g_D = g_C + \dim(\mathfrak{F})$. So the number of continuous parameters upon which the line bundle $\mathcal{L}$ depends exceeds by $g_C$ the dimension of the fiber $\mathfrak{F}$ of the Hitchin fibration. To parametrize that fiber, we must impose $g_C$ conditions on $\mathcal{L}$.

It is clear what those conditions must be. The condition (2.33) on the degree of $\mathcal{L}$ ensures that the determinant line bundle $\det(E)$ is of degree zero and so is topologically trivial. To get an $SU(N)$ Hitchin pair $(E, \varphi)$, $\det(E)$ must also be trivial holomorphically. \textit{A priori}, $\det(E)$ takes values in $\text{Jac}(C)$, the Jacobian of $C$, which is $g_C$-dimensional. So asking for $\det(E)$ to be holomorphically trivial imposes $g_C$ conditions, reducing to the correct dimension. If we let $\Lambda$ be the map that takes a line bundle $\mathcal{L}$ over $D$ to the line bundle $\Lambda(\mathcal{L}) = \det \psi_*(\mathcal{L})$ over $C$, then the Hitchin fiber for $SU(N)$ consists of line bundles such that $\Lambda(\mathcal{L}) \cong \mathcal{O}$. In other words, this fiber is

$$\mathfrak{F}_{SU(N)} = \Lambda^{-1}(\mathcal{O}).$$

(2.34)

Away from the branch points, we can express the condition on $\mathcal{L}$ more simply. If $p$ is not a branch point, then $E_p$ has the decomposition (2.24), by virtue of which $\det(E)_p = \otimes_{i=1}^{N} \mathcal{L}_{q_i}$. Thus away from the branch points, there is a holomorphically varying isomorphism

$$\otimes_{i=1}^{N} \mathcal{L}_{q_i} \cong \mathbb{C}.$$  

(2.35)
In general, the kernel of a holomorphic map between complex tori, in this case $\Lambda : \text{Pic}_{c_0}(D) \to \text{Jac}(C)$, is a complex torus or a union of tori. We show in section 3.3 that $\mathfrak{F}_{SU(N)}$ is connected, so it is actually a complex torus. Is it in a natural way an abelian variety? To exhibit it as one, we must pick in a natural fashion a point in $\mathfrak{F}_{SU(N)}$, that is, we must pick a line bundle $L \to D$ of the appropriate degree that is in the kernel of $\Lambda$. For this, pick a square root $\mathcal{K}^{1/2}$ of the canonical bundle of $C$ and let $L_0 = \psi^*(\mathcal{K}^{(N-1)/2})$. This does have the appropriate degree ($\mathcal{K}^{(N-1)/2}$ has degree $(g_C - 1)(N - 1)$, and because $L_0$ is the pullback by a map of degree $N$, its degree is $N$ times greater). It is also true that $L_0$ is in the kernel of the map $\Lambda$. Indeed, from (2.32), we have in this case $\psi_*(L) = \mathcal{K}^{(N-1)/2} \otimes \left(\mathcal{O} \oplus \mathcal{K}^{-1} \oplus \mathcal{K}^{-2} \oplus \cdots \oplus \mathcal{K}^{-(N-1)}\right)$, from which it follows that $\det(\psi_*(L_0)) \cong \mathcal{O}$.

If $N$ is odd, $(N - 1)/2$ is an integer, and this construction did not really depend on choosing a square root of $\mathcal{K}$. So for odd $N$, we have exhibited $\mathfrak{F}_{SU(N)}$ as an abelian variety, with a distinguished representative $(E, \varphi)$ of each fiber of the Hitchin fibration. For even $N$, this is not quite the case, since the choice of $\mathcal{K}^{1/2}$ does matter. For even $N$, the best we can do is to construct a pair $(E, \varphi)$ that is natural up to the possibility of tensoring by a flat line bundle of order 2. This distinction was pointed out and generalized to other $G$ in [23]. See also our discussion below at the end of section 2.7.

2.6.4 Relation To K-Theory

We obtained the degree $c_0$ of the line bundle $L \to D$ in a rather technical fashion, but actually the answer (eqn. (2.33)) has a nice interpretation, which was essentially described in section 4.3 of [5].

Let us view $W = T^*C$ as a real symplectic manifold, with the symplectic form being the imaginary (or real) part of the holomorphic symplectic form $\Omega_I$. We consider the $A$-model of $W$ with this symplectic form. We view $C \subset W$ as a Lagrangian submanifold and consider $A$-branes supported on $C$.

Naively, a rank 1 $A$-brane supported on $C$ is endowed with a flat Chan-Paton line bundle $U \to C$. But actually, the $K$-theory interpretation of $D$-branes means that there is a twist that involves the square root of the normal bundle to $C$ in $W$. The upshot is that $\mathcal{U}$, rather than a flat line bundle, should be a flat Spin$_c$ structure on $C$. In terms of complex geometry, this means that the degree of $\mathcal{U}$ should be not zero but the degree of a square root of the canonical bundle $K \to C$, that is, it should be $g_C - 1$.

Similarly, the Chan-Paton bundle $E$ of a rank $N$ $A$-brane supported on $C$ should be $N(g_C - 1)$. This means that if we want the line bundle $L \to D$ to have the property that $E = \psi_*(L)$ is the Chan-Paton bundle of an $A$-brane supported on $C$, then its degree must be not $c_0$ as defined in eqn. (2.33) but

$$c'_0 = c_0 + N(g_C - 1) = N^2(g_C - 1) = g_D - 1.$$  \hspace{1cm} (2.36)

But $D$ is also a Lagrangian submanifold of $W$, and this value of the degree of $L \to D$ simply means that $D$, endowed with the Chan-Paton bundle $L$, can be interpreted as an $A$-brane.
Thus we can think of the Higgs bundle and spectral cover construction as giving a correspondence between an $A$-brane of rank 1 supported on $D$ and an $A$-brane of rank $N$ supported on $C$.

### 2.6.5 The Unitary Group

In a similar fashion, we can analyze the fiber of the Hitchin fibration for the related groups $U(N)$ and $PSU(N) = SU(N)/\mathbb{Z}_N$.

First we consider $G = U(N)$. For this example, we should modify the construction at the beginning by not requiring $\varphi$ to be traceless. Accordingly, in defining $V$, we include $\text{Tr} \varphi$ along with the higher traces, with the result that the dimension of $V$ is $(gC - 1)N^2 + 1$. Furthermore, for $U(N)$, we would not want to restrict the determinant of $E$ to be trivial, so $(gC - 1)N^2 + 1$ is the appropriate dimension of the fiber $\mathfrak{F}$. These are equal to each other and to $\frac{1}{2} \dim M_H$, which is $N^2(gC - 1) + 1$ for $G = U(N)$. Since the determinant of $E$ is not constrained to be trivial, $\psi_*(L)$ can be anything, and roughly speaking, $L$ can be any line bundle on $D$. Thus, the fiber of the Hitchin fibration for $U(N)$ is simply the Picard group $\text{Pic}(D)$ that parametrizes line bundles on $D$.

To be more precise, we need to observe that there are two related versions of the question. For $U(N)$, a Higgs bundle $(E, \varphi)$ has an integer topological invariant, which is the first Chern class of $E$, denoted $c_1(E)$. A stable Higgs bundle can have any first Chern class. The Hitchin fibration makes sense for any value of this invariant. The fiber of the Hitchin fibration for $c_1(E) = m$ is the moduli space $\text{Pic}_{c_0 + m}(D)$ parametrizing line bundles on $D$ of degree $c_0 + m$, where $c_0$ was given in (2.33). This follows by the same analysis as before.

However, the equivalence of stable Higgs bundles with solutions of Hitchin’s equations holds only when the first Chern class vanishes. One of Hitchin’s equations, $F - \phi \wedge \phi = 0$ reduces upon taking the trace to $\text{Tr} F = 0$, implying that the first Chern class must vanish. So the moduli space $M_H(U(N), C)$ of solutions of Hitchin’s equations (or classical vacua of our field theory) actually corresponds to the component of the moduli space of stable Higgs bundles with $c = 0$. The fiber of the Hitchin fibration of $M_H(U(N), C)$ is therefore

$$
\mathfrak{F}_{U(N)} = \text{Pic}_{c_0}(D). \tag{2.37}
$$

Precisely as we discussed above for $SU(N)$, this fiber is canonically an abelian variety if $N$ is odd, but it is not quite an abelian variety for even $N$. Rather, it becomes one once a spin structure $K^{1/2}$ is chosen on $C$.

For $U(N)$, we are mainly interested in Higgs bundles $(E, \varphi)$ of degree zero, because they represent points in the moduli space $M_H(U(N), C)$ of classical vacua of our $\sigma$-model. One can ask nevertheless whether Higgs bundles $(E, \varphi)$ of other degrees are relevant to geometric Langlands. The answer is that they can be relevant if one asks questions for which a reduction from the four-dimensional gauge theory to the two-dimensional $\sigma$-model with target $M_H$ is not adequate. See section 7.1.2 for more on this.
2.6.6 The Group $PSU(N)$

Finally, we will determine the fiber of the Hitchin fibration for the group $PSU(N) = SU(N)/Z_N$, where here $Z_N$ is the center of $SU(N)$.

A $PSU(N)$ bundle over $C$ can be topologically non-trivial. Let us first consider the topologically trivial case. A topologically trivial bundle $E'$ over $C$ with structure group $PSU(N)$ can be lifted to an $SU(N)$ bundle $E$ over $C$, but not quite uniquely. The failure of uniqueness is that $E$ can be twisted by a line bundle $S$ that is of order $N$, or alternatively by a flat line bundle whose monodromies lie in the center of $SU(N)$. Such a flat bundle of order $N$ disappears when one projects to $PSU(N)$. Let $\Gamma_N$ be the group of line bundles on $C$ of order $N$. Then for topologically trivial $PSU(N)$ bundles, the fiber of the Hitchin fibration is

$$\mathfrak{F}_{PSU(N)}^{(0)} = \mathfrak{F}_{SU(N)}/\Gamma_N = \Lambda^{-1}(\mathcal{O})/\Gamma_N. \quad (2.38)$$

(An element $S \in \Gamma_N$ acts on $\mathcal{L} \in \Lambda^{-1}(\mathcal{O})$ by $\mathcal{L} \rightarrow \mathcal{L} \otimes \psi^*(S)$.) $\mathfrak{F}_{PSU(N)}^{(0)}$ is always an abelian variety in a canonical way. For odd $N$, $\mathfrak{F}_{SU(N)}$ is already an abelian variety, even before dividing by $\Gamma_N$. For even $N$, to make $\mathfrak{F}_{SU(N)}$ into an abelian variety, we must pick a spin structure on $C$. But dividing by $\Gamma_N$ precisely cancels the dependence on the spin structure. Indeed, changing the spin structure amounts to twisting by a line bundle of order 2, but for even $N$ such a line bundle is an element of $\Gamma_N$.

We find another useful way to describe $\mathfrak{F}_{PSU(N)}^{(0)}$ if we observe that $PSU(N)$ coincides with $PU(N) = U(N)/U(1)$. For $S$ a line bundle on $C$ of degree zero, $\psi^*(S)$ is a line bundle on $D$ of degree zero. So for $\mathcal{L} \in \mathfrak{F}_{U(N)} = \text{Pic}_{c_0}(D)$, the tensor product $\mathcal{L} \otimes \psi^*(S)$ is also in $\text{Pic}_{c_0}(D)$. This gives a group action of $\text{Jac}(C)$ on $\text{Pic}_{c_0}(D)$, and we write $\text{Pic}_{c_0}(D)/\text{Jac}(C)$ for the quotient. Then

$$\mathfrak{F}_{PSU(N)}^{(0)} = \text{Pic}_{c_0}(D)/\text{Jac}(C). \quad (2.39)$$

The basis for this assertion is that, since $\psi_*(\mathcal{L} \otimes \psi^*(S)) = \psi_*(\mathcal{L}) \otimes S$ (according to (2.32)), and $\psi_*(\mathcal{L})$ is of rank $N$, we have $\det(\psi_*(\mathcal{L} \otimes \psi^*(S))) = \psi_*(\mathcal{L}) \otimes S^N$. Hence, $S$ can be chosen to set $\det(\psi_*(\mathcal{L} \otimes \psi^*(S))) = \mathcal{O}$, so that $\mathcal{L} \otimes \psi^*(S)$ is an element of $\mathfrak{F}_{SU(N)}$. This condition does not quite uniquely fix $S$. We are still free to twist $S$ by an element of $\Gamma_N$. After allowing for this, we see that (2.38) and (2.39) are equivalent.

This second description for the topologically trivial case is a useful starting point for analyzing the Hitchin fibration for topologically non-trivial $PSU(N)$ bundles. Topologically, a $PSU(N)$ bundle $E$ is classified by a characteristic class $\xi$ that takes values in $H^2(C,\mathbb{Z}) \cong \mathbb{Z}_N$. (For $N = 2$, $\xi$ is the second Stieffel-Whitney class $w_2(E)$.) This characteristic class measures the obstruction to “lifting” a $PSU(N)$ bundle to $SU(N)$. Let $M_H^{(d)}$ be the component of $M_H(PSU(N),C)$ that parametrizes Higgs bundles with characteristic class $\xi = d$.

Although a topologically non-trivial $PSU(N)$ bundle cannot be lifted to $SU(N)$, it can be lifted non-uniquely to $U(N)$. In fact, if $E$ is any $U(N)$ bundle, then the associated adjoint bundle $\text{ad}(E)$ is a $PSU(N)$ bundle, and any $PSU(N)$ bundle can be obtained this way for some $E$. If the first Chern class of $E$ is $d$, or more generally if it is congruent to $d$ modulo $N$, then the characteristic class of the $PSU(N)$ bundle $\text{ad}(E)$ is $\xi = d$. 

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If $E_1$ and $E_2$ are two different $U(N)$ bundles, then the corresponding $PSU(N)$ bundles $\text{ad}(E_1)$ and $\text{ad}(E_2)$ are isomorphic if and only if $E_1 \cong E_2 \otimes S$, for some line bundle $S \to C$. (Note that in this operation, the first Chern classes of $E_1$ and $E_2$ are equal modulo $N$, since $c_1(E_1) = c_1(E_2) + Nc_1(S)$. They reduce mod $N$ to $d = \xi(\text{ad}(E))$. This leads to an analog of (2.39). The Hitchin fiber $\hat{\mathcal{S}}_{PSU(N)}^{(d)}$ for $PSU(N)$ bundles of $\xi = d$ is

$$\hat{\mathcal{S}}_{PSU(N)}^{(d)} = \text{Pic}_{c_0 + d}(D)/\text{Jac}(C).$$

The idea here is that a point in $\text{Pic}_{c+d}(D)$ determines a $U(N)$ bundle $E$ over $C$ with $c_1(E) = c_1 = d$; acting by $S \in \text{Jac}(C)$ has the effect $E \to E \otimes S$, and does not affect the associated $PSU(N)$ bundle $\text{ad}(E)$.

There is also a useful, though less canonical-looking, analog of (2.38). Given a $PSU(N)$ bundle that we will call $\text{ad}(E)$, its lift $E$ to $U(N)$ is far from unique, because of the freedom $E \to E \otimes S$. The action on $\text{det}(E)$ is $\text{det}(E) \to \text{det}(E) \otimes S^N$. So, restricting ourselves to $S$ of degree zero, we can adjust $\text{det}(E)$ in an arbitrary fashion within its component of $\text{Pic}(C)$, and then we are still free to twist $E$ by a line bundle of order $N$, that is, by an element of $\Gamma_N$. We may pick any convenient line bundle $S_0$ over $C$ of degree $d = \xi(\text{ad}(E))$, and require $\text{det}(E) = S_0$. For example, we may pick a point $p \in C$ and choose $S_0 = \mathcal{O}(p)^d$. So we get the analog of (2.38):

$$\hat{\mathcal{S}}_{PSU(N)}^{(d)} = \Lambda^{-1}(S_0)/\Gamma_N,$$

expressing the fact that by the action of $J(C)$, we can fix the determinant of $E$, and we are then still free to act by $\Gamma_N$.

One last comment is that what we have determined in (2.38) and (2.41) is precisely the fiber of the Hitchin fibration of $\mathcal{M}_H(PSU(N), C)$, which is defined by dividing the space of all solutions of Hitchin’s equations by the group of all $PSU(N)$-valued gauge transformations. It is also sometimes convenient to work on the universal cover of $\mathcal{M}_H(PSU(N), C)$, which one achieves by dividing only by the subgroup of gauge transformations that can be continuously deformed to the identity. On the fiber of the Hitchin fibration, this has the effect precisely of not dividing by $\Gamma_N$ in the above constructions. So if $\mathcal{M}_H^{(d)}(PSU(N), C)$ is the cover of $\mathcal{M}_H(PSU(N), C)$ obtained by dividing only by connected gauge transformations, then the corresponding fiber of the Hitchin fibration is

$$\hat{\mathcal{S}}_{PSU(N)}^{(d)} = \Lambda^{-1}(\mathcal{L}_0).$$

If we set $d = 0$, $\mathcal{M}_H^{(d)}(PSU(N), C)$ reduces to $\mathcal{M}_H(SU(N), C)$. Indeed, for $d = 0$, we can take $\mathcal{L}_0 = \mathcal{O}$, and $\hat{\mathcal{S}}_{PSU(N)}^{(d)}$ reduces to $\hat{\mathcal{S}}_{SU(N)}$.

One might wonder what is the dual of these topologically non-trivial components of $\mathcal{M}_H(PSU(N))$. Here the relevant duality is the $S$-duality or mirror symmetry between $\mathcal{M}_H(PSU(N))$ and $\mathcal{M}_H(SU(N))$ that underlies geometric Langlands duality for these gauge groups; see [1] for a full explanation. As explained from a mathematical point of view in [24], the dual of $\mathcal{M}_H^{(d)}(PSU(N))$ is our friend $\mathcal{M}_H(SU(N))$, but endowed with a certain non-trivial gerbe. We elaborate on the relevant concept of a gerbe in section
6.3 below. See also section 7 of [1], where the gerbe in question is interpreted in terms of a discrete $B$-field of the $\sigma$-model and is related to a discrete version of electric-magnetic duality. And see [23] for an analysis of these issues for arbitrary $G$.

2.6.7 Spectral Covers For Other Gauge Groups

We have described the theory of the spectral cover only for $G = SU(N)$ and for the closely related groups $U(N)$ and $PSU(N)$. What happens for other $G$?

If $G$ has a convenient “small” representation, one can develop a somewhat similar story to what has been described above. This is practical for $SO(N)$, $Sp(N)$, and even $G_2$ [21, 26, 27].

In general – for example for $G = E_8$ – there is no conveniently “small” representation and more abstract methods are necessary. See [23].

2.7 The Distinguished Section

2.7.1 The Case Of $SU(N)$

For practice with these ideas, and because it turns out to be useful, we will look more closely at a certain family of distinguished Hitchin pairs $(E, \varphi)$.

We begin with $G = SU(2)$ and we take $E$ to be the rank two complex vector bundle $E = K^{1/2} \oplus K^{-1/2}$. This bundle is unstable; it contains the line subbundle $K^{1/2}$, which is of positive degree. Nonetheless, there are stable Hitchin pairs $(E, \varphi)$, and we will describe them, following [13].

One reflection of the fact that $E$ is unstable is that, unlike a stable bundle, it has non-constant automorphisms. If we write a section of $E$ as \( (s \ t) \), where $s$ is a section of $K^{1/2}$ and $t$ of $K^{-1/2}$, then $SL(2, \mathbb{C})$ automorphisms of $E$ act by

\[
\begin{pmatrix}
\lambda & \tau \\
0 & \lambda^{-1}
\end{pmatrix}.
\]

(2.43)

Here $\lambda$ is a nonzero complex number, and $\tau$ an element of $H^0(C, K)$. We want to classify stable Hitchin pairs $(E, \varphi)$ up to the action of these automorphisms on $\varphi$.

A general holomorphic section $\varphi$ of $H^0(C, K \otimes \text{ad}(E))$ can be written

\[
\begin{pmatrix}
v & w/2 \\
u & -v
\end{pmatrix},
\]

(2.44)

where $u$ is a complex number, $v$ an element of $H^0(C, K)$, and $w$ an element of $H^0(C, K^2)$. (The factor of $1/2$ multiplying $w$ is for convenience.) $u$ must be nonzero in order for the pair $(E, \varphi)$ to be stable, since if $u = 0$, the line bundle $K^{1/2} \subset E$, whose degree is positive, is $\varphi$-invariant. If $u \neq 0$, we can choose $\lambda$ and $\tau$ to set $u = 1$, $v = 0$ (in a way that is unique modulo the center $\{\pm 1\}$ of $SL(2, \mathbb{C})$), in which case

\[
\varphi = \begin{pmatrix} 0 & w/2 \\ 1 & 0 \end{pmatrix}.
\]

(2.45)
As shown in [13], any such pair \((E, \varphi)\) is stable (and \(E\) is the most unstable bundle for which there are such stable pairs). Clearly \(\text{Tr} \varphi^2 = w\), so with this particular \(E\), there is up to automorphism a unique stable pair \((E, \varphi)\) with given \(\text{Tr} \varphi^2\). So the family \(V_E\) of stable Hitchin pairs with this \(E\) intersects each fiber \(\mathfrak{F}\) of the Hitchin fibration in precisely one point. The family \(V_E\) thus gives, modulo the choice of \(K^{1/2}\), a distinguished section

\[ \zeta : V \to \mathcal{M}_H \quad (2.46) \]

of the Hitchin fibration. As we explained at the end of section 2.4, for generic stable \(E\), the number of intersections of \(V_E \) with \(F\) is \(2^{3g-3}\).

The spectral curve \(D\) is defined by the equation \(\det(y - \varphi) = 0\) or \(y^2 = w/2\). The line bundle \(N \to D\) is found by solving the eigenvector equation \(\varphi \psi = y \psi\) or

\[ \begin{pmatrix} 0 & w/2 \\ 1 & 0 \end{pmatrix} \psi = y \psi. \quad (2.47) \]

Locally this can be handily solved by picking

\[ \psi = \left( \frac{y \alpha}{\alpha} \right), \quad (2.48) \]

where \(\alpha\) is any local holomorphic section of \(K^{-1/2}\). This gives a natural map from sections of \(K^{-1/2}\) to sections of \(N\), implying that in this example, \(N\) coincides with \(\psi^* (K^{-1/2})\).

The “cokernel” of \(\varphi - y\) defines another line bundle \(L\), which according to (2.22) is \(L = N \otimes \psi^* (K) = \psi^* (K^{1/2})\). The bundle \(E\) is supposed to be \(\psi_* (L)\), which according to (2.32) is \(K^{1/2} \oplus K^{-1/2}\). This is indeed isomorphic to the bundle \(E\) with which we began.

The analog of this for \(G = SU(N)\) is to take \(E = K^{(N-1)/2} \oplus K^{(N-1)/2-1} \oplus \cdots \oplus K^{-(N-1)/2}\). We express a section of \(E\) as a column vector

\[ \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{pmatrix}, \quad (2.49) \]

where \(s_i\) is a section of \(K^{(N+1)/2-i}\). An analysis similar to that above shows that stable Hitchin pairs \((E, \varphi)\) with this \(E\) can be placed in the form

\[ \varphi = \begin{pmatrix} 0 & w_2/2 & w_3/3 & \cdots & w_{N-1}/(N-1) & w_N/N \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad (2.50) \]

with 1’s just below the main diagonal, and \(w_n, n = 2, \ldots, N\), taking values in \(H^0(C, K^n)\). We have \(\text{Tr} \varphi^n = w_n\). So for this \(E\), there is again a unique stable Hitchin pair on each
fiber $\mathcal{F}$ of the Hitchin fibration, giving a natural section of the Hitchin fibration. Similarly to what we found for $SU(2)$, by solving the equation for an eigenvector, one finds that $N \sim \psi^*(K^{-(N-1)/2})$, and therefore, using $(2.22)$, $L \sim \psi^*(K^{(N-1)/2})$. We expect to recover $E$ as $\psi^*(L)$. Indeed, from $(2.32)$, we do find that $\psi^*(L) = K^{(N-1)/2} \oplus K^{(N-1)/2-1} \oplus \cdots \oplus K^{-(N-1)/2} = E$.

### 2.7.2 Section Of The Hitchin Fibration For Any $G$

All of this has an analog for any simple Lie group $G$, though we will not aim to give a complete explanation.

The starting point in general is to embed the $SU(2)$ bundle $K^{1/2} \oplus K^{-1/2}$ in $G$, using Kostant’s principal $SU(2)$ embedding. This gives a special unstable $G$-bundle $E$. For this $E$, there is a unique stable Hitchin pair $(E, \varphi)$ on any given fiber of the Hitchin fibration. This is shown, in a slightly different formulation, in [1] using the theory of “opers.” Thus this construction gives for any $G$ a nearly canonical section of the Hitchin fibration. We call this section nearly canonical since in general it may depend on the choice of $K^{1/2}$.

In fact, the Kostant embedding is for any $G$ an embedding of the Lie algebra of $SU(2)$ in that of $G$. At the group level, what is embedded in $G$ may be either $SU(2)$ or $PSU(2) = SO(3)$. In the latter case, the dependence on a choice of $K^{1/2}$ disappears when we make the Kostant embedding, and we get a truly canonical section of the Hitchin fibration. In the former case, we get a section of the Hitchin fibration for each choice of $K^{1/2}$.

For example, let $\mathfrak{su}(N)$ denote the Lie algebra of $SU(N)$. The Kostant embedding of $\mathfrak{su}(2)$ in $\mathfrak{su}(N)$ is the one in which the $N$-dimensional representation of $\mathfrak{su}(N)$ is irreducible with respect to $\mathfrak{su}(2)$ (and hence transforms with spin $(N-1)/2$). At the group level, the image of the Kostant map for $SU(N)$ is $SO(3)$ for odd $N$, but is $SU(2)$ for even $N$. More generally, if $G = SU(N)/\Gamma$, where $\Gamma$ is a subgroup of the center of $SU(N)$, then the image of the Kostant map is $SO(3)$ except when $N$ is even and the element $-1$ of $G$ is not contained in $\Gamma$. For example, for $G = PSU(N)$, the image of the Kostant map is always $SO(3)$. These results are of course consistent with what we found in section 2.6.3.

In general, if $G$ is of adjoint type, the image of the Kostant map is always $SO(3)$. So for example, for $G = E_8$, the fiber of the Hitchin fibration is canonically an abelian variety.

### 3 Dual Tori And Hitchin Fibrations

As we have reviewed in section 2, for any simple Lie group, the fiber $\mathcal{F}$ of the Hitchin fibration is generically a complex torus, because of the complete integrability of the space $M_H(G, C)$. As has been argued physically [17, 18], S-duality in four dimensions acts by $T$-duality on the fiber $\mathcal{F}$ of the Hitchin fibration, along with a geometrical symmetry of the base $V$. As was originally explained in [24], the mirror symmetry between $M_H(\mathcal{L}G, C)$ and $M_H(G, C)$ is a rare example in which the SYZ approach to mirror symmetry [25] can be implemented in some detail, because the hyper-Kähler structure makes it possible

\[\text{For the unitary groups that we give as examples, the geometrical symmetry of the base is a simple rescaling that does not play an important role.}\]
to construct rather explicitly a fibration by Lagrangian tori. This interpretation of the duality between $M_H(\mathcal{L} \mathcal{G}, C)$ and $M_H(G, C)$ was important in [1]. Here we will give a more thorough explanation of some aspects than was provided there.

To be precise, the claim is that if $G$ and $\mathcal{L} \mathcal{G}$ are two dual groups, the Hitchin fibers $\mathfrak{F}$ and $\mathfrak{F}'$ over corresponding points in the base are dual complex tori. We recall that this means that $\mathfrak{F}$ parametrizes flat line bundles on $\mathcal{L} \mathfrak{F}$, and vice-versa. A duality between complex tori $\mathfrak{F}$ and $\mathfrak{F}'$ can be described most symmetrically by presenting a unitary line bundle $\mathcal{T}$ with connection over the product $\mathfrak{F} \times \mathfrak{F}'$, such that $\mathcal{T}$ is flat when restricted to $f \times \mathfrak{F}'$ for any $f \in \mathfrak{F}$, or to $\mathfrak{F} \times f'$ for any $f' \in \mathfrak{F}'$. (If $\mathfrak{F}$ and $\mathfrak{F}'$ have complex or algebraic structures, one wants $\mathcal{T}$ to be a holomorphic or algebraic line bundle.) Thus, letting $f$ vary, we get a family $\mathcal{Y}$ of flat line bundles over $\mathfrak{F}'$, and letting $f'$ vary, we get a family $\mathcal{Y}'$ of flat line bundles over $\mathfrak{F}$. If $\mathcal{Y}'$ is the moduli space of flat bundles over $\mathfrak{F}'$ and $\mathcal{Y}$ is the moduli space of flat bundles over $\mathfrak{F}$, then $\mathfrak{F}$ and $\mathfrak{F}'$ are called dual tori, and $\mathcal{T}$ is called a Poincaré line bundle for the pair.

An immediate consequence of the definition is that if $\mathfrak{F}$ and $\mathfrak{F}'$ are dual tori, then there is always a unique distinguished point $f \in \mathfrak{F}$ such that $\mathcal{T}$ is trivial when restricted to $f \times \mathfrak{F}'$; conversely, there is always a unique distinguished point in $f' \in \mathfrak{F}'$ such that $\mathcal{T}$ is trivial when restricted to $\mathfrak{F} \times f'$. As a result, if $\mathfrak{F}$ and $\mathfrak{F}'$ are complex algebraic tori that are dual in the above sense, they always are abelian varieties in a canonical way.

This section is devoted to discussing these issues in the context of Hitchin fibrations. On a number of points, we go into more detail than is needed in the rest of the paper.

### 3.1 Examples

For an elementary example of dual tori, let $S$ be a circle parametrized by an angular variable $x$, $0 \leq x \leq 2\pi$, and $S'$ another circle parametrized by an angular variable $y$, $0 \leq y \leq 2\pi$. Of course $S \times S' \cong \mathbb{R}^2/\mathbb{Z}^2$ where $\mathbb{Z}^2$ acts by $(x, y) \rightarrow (x + 2\pi n, y + 2\pi m)$. Let $\omega = dx \wedge dy/2\pi$. A line bundle $\mathcal{T} \rightarrow S \times S'$ with curvature $F = -i\omega$ can serve as a Poincaré line bundle. We can define such a $\mathcal{T}$ by picking the connection form $B = -ix dy/2\pi$ on a trivial line bundle $\mathcal{O}$ over $\mathbb{R}^2$, and then descending to $S \times S'$ by dividing by unit translations in the $x$ and $y$ directions. (We take $(x, y) \rightarrow (x + 2\pi n, y + 2\pi m)$ to act on $\mathcal{O}$ as multiplication by $e^{i2\pi n}$.) The holonomy of $\mathcal{T}$ around the circle $x \times S'$ is $e^{ix}$ and the holonomy around the circle $S \times y$ is $e^{iy}$. Each possible holonomy of a flat line bundle appears once in each family, so $\mathcal{T}$ is a Poincaré line bundle that establishes a duality between $S$ and $S'$.

Instead of picking an explicit connection form $B$ to construct the line bundle $\mathcal{T}$, we could accomplish the same by simply asking that $\mathcal{T}$ have curvature $F = -i\omega$ and trivial holonomy when restricted to $x = 0$ or to $y = 0$. These conditions completely characterize $\mathcal{T}$ up to isomorphism.

This elementary example has the following generalization. Let $\Gamma$ and $\Gamma^*$ be any two dual lattices, with a unimodular bilinear form $\langle \cdot, \cdot \rangle : \Gamma \times \Gamma^* \rightarrow \mathbb{Z}$. Define the vector spaces $V = \Gamma \otimes \mathbb{R}$, $V^* = \Gamma^* \otimes \mathbb{R}$ and the tori $\mathfrak{F} = V/\Gamma$, $\mathfrak{F}^* = V^*/\Gamma^*$. $V$ and $V^*$ are dual vector spaces, as the form $\langle \cdot, \cdot \rangle$ extends naturally to a bilinear pairing $V \otimes V^* \rightarrow \mathbb{R}$. Moreover, the tori $\mathfrak{F}$ and $\mathfrak{F}^*$ are always dual tori in a natural way. One simply makes the same
construction that we have just seen, beginning with the symplectic form \( \omega = \langle dx, dy \rangle / 2\pi \), where we write \( x \) or \( y \) for a lift of a point in \( F \) or \( F^* \) to \( V \) or \( V^* \).

For an example closer to our subject, let \( D \) be a compact two-dimensional oriented closed manifold – for the moment endowed with no complex structure – and let \( \mathcal{J} \) be the moduli space of flat unitary complex line bundles on \( D \). Then \( \mathcal{J} \) is a self-dual torus, because it is \( (\Gamma \otimes \mathbb{Z} \mathbb{R})/\Gamma \), where \( \Gamma \) is the self-dual lattice \( \Gamma = H_1(D, \mathbb{Z}) \). Consequently, the remarks of the last paragraph apply to generate a canonical self-duality of \( \mathcal{J} \).

Alternatively, it will be helpful in our later work to understand the self-duality of \( \mathcal{J} \) in the following way, using gauge theory. We define a line bundle \( \mathcal{T} \rightarrow \mathcal{J} \times \mathcal{J} \) as follows. Let \( \mathcal{L} \) be a flat line bundle over \( D \), with connection \( A \), and let \( \mathcal{M} \) be a second flat line bundle over \( D \), with connection \( B \). Being flat, the connections obey

\[
0 = F_A = dA, \quad 0 = F_B = dB.
\]

The pair \( \mathcal{L}, \mathcal{M} \) determines a point \( P \times P' \subset \mathcal{J} \times \mathcal{J} \). We define a two-form over \( \mathcal{J} \times \mathcal{J} \) by the formula

\[
\omega = \frac{1}{2\pi} \int_D \delta A \wedge \delta B.
\]

\( \omega \) vanishes if contracted with a vector field generating a gauge transformation of \( A \) or \( B \), since if one adds an exact form \( de \) to \( \delta A \) or \( \delta B \) and integrates by parts, the result vanishes upon using the flatness condition (3.1). So \( \omega \) descends to a two-form on \( \mathcal{J} \times \mathcal{J} \), and this form is nondegenerate by Poincaré duality. For any \( P, P' \subset \mathcal{J} \), \( \omega \) vanishes when restricted to \( P \times \mathcal{J} \) (which we can do by setting \( \delta A = 0 \)) or to \( \mathcal{J} \times P' \) (which we do by setting \( \delta B = 0 \)). Moreover, we have normalized \( \omega \) so that its periods are integer multiples of \( 2\pi \).

All this being so, there exists a unitary line bundle \( \mathcal{T} \) with connection over \( \mathcal{J} \times \mathcal{J} \), such that the curvature of \( \mathcal{T} \) is equal to \(-i\omega \). Moreover, \( \mathcal{T} \) is uniquely determined up to isomorphism if we ask that it is trivial (flat and vanishing monodromies) if restricted to \( 0 \times \mathcal{J} \) or \( \mathcal{J} \times 0 \), where \( 0 \in \mathcal{J} \) is the point corresponding to the trivial flat connection.

Such a \( \mathcal{T} \) establishes the self-duality of \( \mathcal{J} \), that is, \( \mathcal{J} \) is its own dual torus. In fact, if we pick a symplectic basis of one-cycles on \( D \), to reduce the intersection form to a sum of \( 2 \times 2 \) blocks of the form \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]
then this example reduces to a product of copies of the elementary example we considered first, involving a line bundle over \( S \times S' \) with curvature \(-idx \wedge dy/2\pi \).

This also makes it possible to construct \( \mathcal{T} \) explicitly by elementary formulas (as we did in the toy example of \( S \times S' \)). But a more elegant way to construct \( \mathcal{T} \) is to consider a Chern-Simons gauge theory over the three-manifold \( \mathbb{R} \times D \) in which the gauge group is \( U(1) \times U(1) \), the gauge field consisting of a pair of \( U(1) \) gauge fields that we call \( A, B \). We take the action to be

\[
I = \frac{1}{2\pi} \int_{\mathbb{R} \times D} A \wedge dB.
\]

The classical equations of motion tell us that \( dA = dB = 0 \), and the classical phase space is the moduli space of solutions up to gauge transformations, or \( \mathcal{J} \times \mathcal{J} \). The symplectic structure on the classical phase space is what we have called \( \omega \), and the first step in
quantization is to construct a line bundle $\mathcal{T}$ with the properties that we have claimed. This can be done directly using the gauge-invariance of the Chern-Simons action (for example, see [28, 29]).

As in many constructions that can be carried out in general for a smooth two-dimensional surface $D$, this one can be given an important alternative description in case a complex structure is picked on $D$. In this case, $\mathcal{J}$ becomes the Jacobian of $D$, which parametrizes holomorphic line bundles over $D$ of degree zero; we denote it as $\text{Jac}(D)$. This is a classic example of a self-dual abelian variety, and very relevant to Hitchin fibrations, as we will discuss momentarily. Apart from the above topological construction, the Poincaré line bundle of $\text{Jac}(D)$ can be conveniently constructed using holomorphic methods; we explain one way to do this in section 6.4.

### 3.2 The Case Of Unitary Groups

Here we will describe explicitly the duality between the fibers of dual Hitchin fibrations for the dual pair of groups $SU(N)$ and $PSU(N)$, and similarly the self-duality for $U(N)$.

In doing this, we meet immediately the fact that $\mathcal{M}_H(PSU(N))$ has $N$ components, labeled by the value of a discrete characteristic class $\xi$ (introduced in section 2.6.6), while $\mathcal{M}_H(SU(N))$ is connected. As explained in [24] and in section 7 of [1] (see also section 6.3 below), the dual of $\xi$ is a certain discrete $B$-field (or gerbe) that can be introduced on $\mathcal{M}_H(SU(N))$. We will postpone this issue and for now limit ourselves to the duality between $\mathcal{M}_H(SU(N))$ and the component $\mathcal{M}_H(0)(PSU(N))$ of $\mathcal{M}_H(PSU(N))$ with $\xi = 0$. Similarly, in the case of $U(N)$, we will for now consider Higgs bundles with vanishing first Chern class. (A formulation of the duality without these limitations is given in section 6.5.)

Even with these restrictions, a further conundrum presents itself. As we have explained above, if $F$ and $F'$ are dual complex algebraic tori, they are always abelian varieties in a canonical way. The fiber $\mathfrak{f}_{SU(N)}^{(0)}$ has this property for all $N$, but as we have seen, $\mathfrak{f}_{SU(N)}$ and $\mathfrak{f}_{U(N)}$ are only abelian varieties in a natural fashion if $N$ is odd. For $N$ even, they become abelian varieties in a natural fashion only once a spin structure $K^{1/2}$ is picked on $C$. Therefore, the claim that $\mathfrak{f}_{SU(N)}$ and $\mathfrak{f}_{PSU(N)}^{(0)}$ are dual (and similarly the claim that $\mathfrak{f}_{U(N)}$ is self-dual) cannot quite be correct for even $N$. A corrected statement was formulated in section 10 of [5], but will not be described here. For our purposes, we will just say that for even $N$, the duality statements about $\mathfrak{f}_{SU(N)}$, $\mathfrak{f}_{PSU(N)}^{(0)}$, and $\mathfrak{f}_{U(N)}$ hold once a choice of $K^{1/2}$ is made.\footnote{In brief, the formulation in [5] is that, for even $N$, $S$-duality generates a $B$-field (or gerbe) on $\mathcal{M}_H$ that is trivial but not canonically trivial; it can be trivialized by a choice of $K^{1/2}$. (This gerbe is described in section 6.3.) To state for even $N$ the duality between $\mathfrak{f}_{SU(N)}$ and $\mathfrak{f}_{PSU(N)}^{(0)}$ or the self-duality of $\mathfrak{f}_{U(N)}$ without making a choice of $K^{1/2}$, one has to incorporate this $B$-field on one side of the duality. A direct explanation from four-dimensional gauge theory of why $S$-duality generates this trivial but not canonically trivial $B$-field has not yet been given.}

We recall that a fiber $\mathfrak{f}_{SU(N)}$ is determined by a spectral cover $\psi : D \to C$, and is parametrized, as we found in section 2.6.3, by the choice of a line bundle $\mathcal{L} \to D$ of degree $N(N - 1)(g_C - 1)$. Once a choice of $K^{1/2}$ is made, $\mathfrak{f}_{SU(N)}$ contains the canonical point $\mathfrak{f}_{PSU(N)}^{(0)}$, and $\mathfrak{f}_{U(N)}$ holds.
\[ \psi^*(K^{(N-1)/2}), \text{ and is an abelian variety, conveniently parametrized by the line bundle} \]
\[ \mathcal{L}' = \mathcal{L} \otimes \zeta^*(K^{-(N-1)/2}) \]

which is of degree zero. To describe \( \hat{\mathfrak{f}}_{SU(N)} \) in terms of \( \mathcal{L}' \), one more concept is useful. Given a covering \( \psi : D \to C \) of Riemann surfaces, one defines a “norm” map \( \text{Nm} : \text{Pic}(D) \to \text{Pic}(C) \). If \( \mathcal{M} \to D \) has divisor \( \sum n_i q_i \), with integers \( n_i \) and \( q_i \in D \), then \( \text{Nm}(\mathcal{M}) \) has divisor \( \sum n_i \psi(q_i) \). The norm map differs from the pushforward operation \( \mathcal{M} \to \text{det} \psi^*(\mathcal{M}) \) that we considered before in that it does not have a correction for the ramification. For our \( N \)-fold covering \( \psi : D \to C \), the two are related by \( \text{det} \psi^*(\mathcal{M}) = \text{Nm}(\mathcal{M}) \otimes K^{-N(N-1)/2} \), for any \( \mathcal{M} \). (It is enough to verify this for \( \mathcal{M} = \mathcal{O} \); we have \( \text{Nm} (\mathcal{O}) = \mathcal{O} \), while (2.31) implies \( \text{det} \psi^*(\mathcal{O}) = K^{-N(N-1)/2} \).) So the description (2.34) of the fiber of the Hitchin fibration for \( SU(N) \) is equivalent to saying that \( \hat{\mathfrak{f}}_{SU(N)} \), when parametrized by \( \mathcal{L}' \), is the subgroup of \( \text{Jac}(D) \) defined by

\[ \hat{\mathfrak{f}}_{SU(N)} = \text{Nm}^{-1}(\mathcal{O}). \quad (3.5) \]

On the other hand, in terms of \( \mathcal{L}' \), the fiber \( \hat{\mathfrak{f}}_{PSU(N)}^{(0)} \) of the Hitchin fibration for topologically trivial \( PSU(N) \) bundles is

\[ \hat{\mathfrak{f}}_{PSU(N)}^{(0)} = \text{Jac}(D)/\text{Jac}(C). \quad (3.6) \]

That these are dual has been shown in [24] in a fairly elementary way.

A rough explanation is as follows. If \( \mathcal{J} \) is any self-dual torus with Poincaré line bundle \( \mathcal{P} \to \mathcal{J} \times \mathcal{J} \), and \( \mathcal{F} \) is any subtorus of \( \mathcal{J} \), then the dual of \( \mathcal{F} \) is \( \mathcal{J}/\mathcal{F}^\perp \), where \( \mathcal{F}^\perp \) consists of all \( f^\perp \in \mathcal{J} \) such that \( \mathcal{P} \) restricted to \( \mathcal{F} \times f^\perp \) is trivial. For \( \mathcal{J} = \text{Jac}(D) \), \( \mathcal{F} = \hat{\mathfrak{f}}_{SU(N)} = \text{Nm}^{-1}(\mathcal{O}) \), one has \( \mathcal{F}^\perp = \psi^*(\text{Jac}(C)) \), and the dual of \( \mathcal{F} \) is then \( \hat{\mathfrak{f}}_{PSU(N)}^{(0)} = \mathcal{J}/\mathcal{F}^\perp = \text{Jac}(D)/\text{Jac}(C) \). We explain more in section 3.3, and we describe another approach to this duality in section 6.5.

Similarly, when expressed in terms of \( \mathcal{L}' \) instead of \( \mathcal{L} \), (2.40) becomes

\[ \hat{\mathfrak{f}}_{PSU(N)}^{(d)} = \text{Pic}_d(D)/\text{Jac}(C). \quad (3.7) \]

An equivalent characterization can be found by restating (2.41):

\[ \hat{\mathfrak{f}}_{PSU(N)}^{(d)} = \text{Nm}^{-1}(\mathcal{L}_0)/\Gamma_N, \quad (3.8) \]

where \( \mathcal{L}_0 \) is any fixed degree \( d \) line bundle over \( C \). And likewise, (2.42) becomes

\[ \hat{\mathfrak{f}}_{PSU(N)}^{(d)} = \text{Nm}^{-1}(\mathcal{L}_0). \quad (3.9) \]

The analog for \( U(N) \) is simply that, when parametrized by the degree zero line bundle \( \mathcal{L}' \), the fiber of the Hitchin fibration is a copy of the Jacobian of \( D \):

\[ \hat{\mathfrak{f}}_{U(N)} = \text{Jac}(D). \quad (3.10) \]

This is certainly a self-dual abelian variety.
3.3 Topological Viewpoint

Here, for $G = SU(N)$, we consider the spectral curve more carefully from a topological viewpoint. We will fill in some gaps in previous arguments and obtain a few useful results. In addition, given the basic claim that the geometric Langlands program can be derived from topological field theory, we want to understand things topologically when possible.

In general, a map $\psi: D \to C$ from one Riemann surface to another determines an associated map $\psi_*: H_1(D, \mathbb{Z}) \to H_1(C, \mathbb{Z})$. We simply map the homology class of a loop $\gamma \subset D$ to the homology class of the corresponding loop $\psi(\gamma) \subset C$. We abbreviate the homology groups as $H_1(D)$ and $H_1(C)$ and denote this map as

$$\psi_*: H_1(D) \to H_1(C).$$

(3.11)

The map $\psi_*$ is surjective if every closed loop in $C$ can be lifted to a closed loop in $D$. This is not so for every map between Riemann surfaces; for example, it is not so for an unramified covering $D \to C$. However, the spectral cover $\psi: D \to C$ has plenty of ramification, and in this case $\psi_*$ is surjective. One way to prove this is to pick a convenient spectral cover, using the fact that the space of smooth spectral covers is connected, so that any two smooth spectral covers have the same topology. We pick a point $p \in C$ and pick a spectral cover for which $\text{Tr} \varphi^n$ vanishes at $p$ for $n = 2, \ldots, N$, with $\text{Tr} \varphi^N$ having only a simple zero at $p$. The equation of the spectral cover then looks near $p$ like $y^N - z = 0$, where $z$ is a local parameter on $C$ that vanishes at $p$. Hence, in this example, $D$ contains only a single point $q$ that lies above $p$. Now, any closed loop in $C$ can be deformed to a path that begins and ends at $p$. Such a path lifts in $D$ to a path that begins and ends at $q$, and so forms a closed loop. So the map $\psi_*$ is surjective.

We write $\Gamma$ for the kernel of $\psi_*$. Since $\psi_*$ is surjective, we get an exact sequence of lattices

$$0 \to \Gamma \to H_1(D) \xrightarrow{\psi_*} H_1(C) \to 0.$$  

(3.12)

Such a sequence can always be split, though not canonically:

$$H_1(D) = \Gamma \oplus H_1(C).$$

(3.13)

Relative to this splitting, the map $\psi_*$ is just

$$\psi_* = 0 \oplus 1.$$  

(3.14)

We can also define another map $\psi^*: H_1(C) \to H_1(D)$ which maps the homology class of a closed loop $\gamma \subset C$ to that of $\psi^{-1}(\gamma) \subset D$. For any $\gamma \subset C$, $\psi(\psi^{-1}(\gamma))$ is an $N$-fold cover of $\gamma$, which means at the level of homology that

$$\psi_* \circ \psi^* = N.$$  

(3.15)

Relative to the decomposition (3.13), we have therefore

$$\psi^* = m \oplus N,$$  

(3.16)
where \( m : H_1(C) \to \Gamma \) is some lattice map, and \( N : H_1(C) \to H_1(C) \) is multiplication by \( N \). Because the splitting (3.13) is not canonical, \( m \) is only uniquely determined modulo \( N \).

The dual of the exact sequence (3.12) is another exact sequence

\[
0 \to H_1(C)^* \xrightarrow{i} H_1(D)^* \to \Gamma^* \to 0. \tag{3.17}
\]

Here \( \Gamma^* = \text{Hom}(\Gamma, \mathbb{Z}) \) is the dual lattice of a lattice \( \Gamma \). The map \( i \) by definition maps \( x : H_1(C) \to \mathbb{Z} \) to \( x \circ \psi : H_1(D) \to \mathbb{Z} \). However, \( H_1(C) \) and \( H_1(D) \) are self-dual via their intersection pairings, which we denote \( \langle \cdot , \cdot \rangle_C \) and \( \langle \cdot , \cdot \rangle_D \). If \( x : H_1(C) \to \mathbb{Z} \) is \( y \to \langle x', y \rangle_C \) for some \( x' \), then \( i(x) \) takes \( z \in H_1(D) \) to \( \langle x', \psi(z) \rangle_C \). But, for any \( x' \in H_1(C) \), \( z \in H_1(D) \), we have \( \langle x', \psi(z) \rangle_C = \langle \psi^*(x'), z \rangle_D \) (as one can argue by counting intersections on \( C \) and on \( D \)). This means that under the self-duality of \( H_1(D) \), the element \( i(x) \in H_1(D)^* \) corresponds to \( \psi^*(x') \in H_1(D) \). So we can write the exact sequence (3.17) in a more useful way:

\[
0 \to H_1(C) \xrightarrow{\psi^*} H_1(D) \to \Gamma^* \to 0. \tag{3.18}
\]

This tells us that \( H_1(D)/\psi^*(H_1(C)) \) is a lattice, in fact

\[
H_1(D)/\psi^*(H_1(C)) = \Gamma^*. \tag{3.19}
\]

The moduli space of flat \( U(1) \) bundles on \( C \) is \( \mathcal{J}(C) = \text{Hom}(H_1(C), U(1)) \), which we abbreviate as \( H_1(C)^\vee \). Likewise, the moduli space of flat \( U(1) \) bundles on \( D \) is \( \mathcal{J}(D) = \text{Hom}(H_1(D), U(1)) = H_1(D)^\vee \). There is a natural map \( (\psi_\ast)^\vee : H_1(C)^\vee \to H_1(D)^\vee \), taking \( \phi : H_1(C) \to U(1) \) to \( \phi \circ \psi_\ast : H_1(D) \to U(1) \). The fiber of the Hitchin fibration for topologically trivial \( PSU(N) \) bundles is

\[
\mathfrak{F}_{PSU(N)}^{(0)} = \mathcal{J}(D)/\mathcal{J}(C) = H_1(D)^\vee/(\psi_\ast)^\vee(H_1(C)^\vee). \tag{3.20}
\]

In the context of the splitting in (3.13), \( \mathfrak{F}_{PSU(N)}^{(0)} \) parametrizes homomorphisms from \( H_1(D) \) to \( U(1) \), except that we do not care what such a homomorphism does to \( H_1(C) \). So

\[
\mathfrak{F}_{PSU(N)}^{(0)} = \text{Hom}(\Gamma, U(1)) = \Gamma^\vee. \tag{3.21}
\]

Likewise we define \( \psi^\vee : H_1(D)^\vee \to H_1(C)^\vee \), taking \( \phi : H_1(D) \to U(1) \) to \( \phi \circ \psi^* : H_1(C) \to U(1) \). This is a topological version of the norm map in algebraic geometry. The norm map can be defined in algebraic geometry for line bundles of any degree, but for degree zero it is equivalent to \( \psi^\vee \), whose definition does not use a complex structure. The fiber of the Hitchin fibration for \( SU(N) \) is according to (3.5) the kernel of the norm map, or

\[
\mathfrak{F}_{SU(N)} = \ker(\psi^\vee). \tag{3.22}
\]

Concretely, \( \ker(\psi^\vee) \) parametrizes homomorphisms \( \phi : H_1(D) \to U(1) \) that are trivial on \( \psi^*(H_1(C)) \). These are the same as homomorphisms to \( U(1) \) from \( H_1(D)/\psi^*(H_1(C)) = \Gamma^* \), so we have actually

\[
\mathfrak{F}_{SU(N)} = \text{Hom}(\Gamma^*, U(1)) = (\Gamma^*)^\vee. \tag{3.23}
\]
In particular, $\mathfrak{F}_{SU(N)}$ is connected, as promised in section 2.6.3, and is a torus.

Comparing (3.21) and (3.23), we see that $\mathfrak{F}_{PSU(N)}^{(0)}$ and $\mathfrak{F}_{SU(N)}$ are obtained as $\text{Hom}(\Gamma, U(1))$ and $\text{Hom}(\Gamma^*, U(1))$ for two dual lattices $\Gamma$ and $\Gamma^*$. Tori obtained in this way are always canonically dual, as we explained in section 3.1.

Since $\Gamma$ is a sublattice of $H_1(D)$, the unimodular intersection pairing on $H_1(D)$ restricts to an integer-valued bilinear form on $\Gamma$. This gives a natural injective map $\Gamma \to \Gamma^*$. If $r_N$ denotes reduction mod $N$, then we have a natural surjective map $r = r_N \circ \psi : H_1(D) \to \Gamma_N$, where $\Gamma_N = H_1(C, \mathbb{Z}N)$. $r_N$ annihilates $\psi^*(H_1(C))$, in view of (3.15), so it can be regarded as a map from $\Gamma^* = H_1(D)/\psi^*(H_1(C))$ onto $\Gamma_N$. It also annihilates $\Gamma$, since $\psi$ does, so it really gives a map from $\Gamma^*/\Gamma$ onto $\Gamma_N$. Finally, by taking discriminants, one can show that this map is an isomorphism. So we get an exact sequence

$$0 \to \Gamma \to \Gamma^* \to \Gamma_N \to 0. \quad (3.24)$$

Taking homomorphisms to $U(1)$, and using the fact that $\mathfrak{F}_{PSU(N)}^{(0)} = \Gamma^\vee$ while $\mathfrak{F}_{SU(N)} = (\Gamma^*)^\vee$, we have

$$\mathfrak{F}_{PSU(N)}^{(0)} = \mathfrak{F}_{SU(N)}/\Gamma_N. \quad (3.25)$$

This result was explained from the viewpoint of complex geometry in (2.38).

### 3.3.1 Characterization of $\mathfrak{F}_{SU(N)}$

Now, we return to complex geometry and see what we can deduce from the fact that $\mathfrak{F}_{SU(N)} = \text{Nm}^{-1}(\mathcal{O})$ is a complex torus.

Let $p$ be any point in $C$ and $q, q'$ two points lying above it in $D$. Then the line bundle $\mathcal{O}(q) \otimes \mathcal{O}(q')^{-1}$ over $D$ has norm $\mathcal{O}$ and so defines a point in $\mathfrak{F}_{SU(N)}$.

More generally, let $p_i, i = 1, \ldots, k$ be any collection of points in $C$, and for each $i$ let $q_i$ and $q'_i$ lie above $p_i$ in $D$. We pick integers $n_i$ and set $\mathcal{D} = \sum_i n_i(q_i - q'_i)$. Then the line bundle $\mathcal{L} = \mathcal{O}(\mathcal{D}) = \otimes_{i=1}^k (\mathcal{O}(q_i) \otimes \mathcal{O}(q'_i)^{-1})^{n_i}$ has trivial norm and lies in $\mathfrak{F}_{SU(N)}$.

We claim that conversely, if $\text{Nm}(\mathcal{L}) = \mathcal{O}$, then $\mathcal{L} = \mathcal{O}(\mathcal{D})$ for a divisor $\mathcal{D}$ of this form.

The argument depends upon knowing that $\mathfrak{F}_{SU(N)}$ is a complex torus. This being so, we can divide by $N$, and find some $\mathcal{M} \in \mathfrak{F}_{SU(N)}$ with $\mathcal{M}^N \cong \mathcal{L}$. There is some divisor $\mathcal{D}' = \sum_i n_i q_i$ with $\mathcal{M} = \mathcal{O}(\mathcal{D}')$, and hence $\mathcal{L} = \mathcal{O}(N\mathcal{D}')$. Now for each $i$, let $q_{i,\alpha} = q_i^{}$, $\alpha = 1, \ldots, N$ be the points in $D$ with $\psi(q_{i,\alpha}) = \psi(q_i)$ (one of the $q_{i,\alpha}$ is equal to $q_i$; we allow for ramification by permitting some of the $q_{i,\alpha}$ to be equal). $\text{Nm}(\mathcal{M})$ is trivial since $\mathcal{M} \in \mathfrak{F}_{SU(N)}$, so $\psi^*(\text{Nm}(\mathcal{M}))$ is also trivial. But $\psi^*(\text{Nm}(\mathcal{M})) = \mathcal{O}(\mathcal{D}'')$, where $\mathcal{D}'' = \sum_{i,\alpha} n_{i,\alpha} q_{i,\alpha}$. So the fact that $\psi^*(\text{Nm}(\mathcal{M})) = \mathcal{O}$ means that we can equivalently characterize $\mathcal{L}$ as $\mathcal{O}(N\mathcal{D}' - \mathcal{D}'')$. But

$$N\mathcal{D}' - \mathcal{D}'' = \sum_{i,\alpha} n_{i}(q_{i} - q_{i,\alpha}). \quad (3.26)$$

This is a sum of divisors $q_i - q'_i$ where $\psi(q_i) = \psi(q'_i)$, so we have established that every $\mathcal{L} \in \text{Nm}^{-1}(\mathcal{O})$ is $\mathcal{O}(\mathcal{D})$ for a divisor $\mathcal{D}$ of that form.
3.4 The Symplectic Form

The results that we have just obtained are helpful in understanding another aspect of the geometry of \( \mathcal{M}_H \) in complex structure \( I \). At least for unitary groups, we want to describe in terms of spectral covers the holomorphic symplectic structure \( \Omega_I \), which was introduced via gauge theory in eqn. (2.4). (This detailed description will not be used in the rest of the paper.) For more general groups, one would hope to get similar results based on a more abstract approach to spectral covers.

Our approach will be in two steps. First, we construct directly a holomorphic symplectic structure on the space of pairs consisting of a spectral cover \( \psi : D \to C \) and a line bundle \( \mathcal{L} \to D \) of the appropriate degree. Then we will compare this formula to our expectations from the gauge theory definition.

First we recall the Abel-Jacobi map, which explicitly identifies the Jacobian of a Riemann surface \( D \) as a complex torus. Let \( g_D \) be the genus of \( D \) and pick a basis of \( g_D \) holomorphic one-forms \( \omega^a, a = 1, \ldots, g_D \). Map the first homology group \( H_1(D, \mathbb{Z}) \) to \( \mathbb{C}^{g_D} \) by mapping an integral one-cycle \( \gamma \), representing an element of \( H_1(D, \mathbb{Z}) \), to the periods \( (\int_\gamma \omega^1, \ldots, \int_\gamma \omega^{g_D}) \). (By definition, an integral one-cycle is a formal linear combination of oriented closed loops with integer coefficients.) The image of \( H_1(D, \mathbb{Z}) \) in \( \mathbb{C}^{g_D} \) is a lattice \( \Gamma_D \) of rank \( 2g_D \). The quotient \( \mathcal{T} = \mathbb{C}^{g_D}/\Gamma_D \) is a complex torus.

Now suppose that \( \mathcal{L} \) is a line bundle over \( D \) of degree zero. It is isomorphic to \( \mathcal{O}(D) \) for some divisor \( D = \sum_i n_i q_i \), with integers \( n_i \) such that \( \sum_i n_i = 0 \) and points \( q_i \in D \). Since \( \sum_i n_i = 0 \), we can find a one-dimensional “chain” \( \gamma \) (a formal linear combination of oriented paths, not necessarily closed, with integer coefficients) whose boundary \( \partial \gamma \) is equal to \( \sum_i n_i q_i \). Then we map \( \mathcal{L} \) to the point \( x \in \mathcal{T} \) with coordinates \( (\int_\gamma \omega^1, \ldots, \int_\gamma \omega^{g_D}) \). This gives a well-defined point in \( \mathcal{T} \), because \( \gamma \) is uniquely determined up to addition of a one-cycle \( \gamma' \), whose addition to \( \gamma \) will shift the coordinates of \( x \) by an element of the lattice \( \Gamma_D \). The Abel-Jacobi theorem says that the point \( x \) depends only on \( \mathcal{L} \) and not on the choice of a divisor \( D \) representing \( \mathcal{L} \), and moreover that this map gives an isomorphism of \( \text{Jac}(D) \) with the complex torus \( \mathcal{T} \).

For spectral curves, this can be implemented in a particularly nice way. We begin with the case of \( G = U(N) \), and then describe the minor variations needed for \( SU(N) \) and \( PSU(N) \). On the total space \( W \) of the cotangent bundle of \( C \), there is a natural holomorphic one-form \( \lambda \) which, in terms of a local coordinate \( z \) on \( C \), can be written \( \lambda = y dz \). Once we pick a spectral curve \( D \), \( \lambda \) can be restricted to a holomorphic differential on \( D \). Of course, \( D \) is defined by an equation \( \det(y - \varphi) = 0 \), which more explicitly takes the form \( \mathcal{P}(y, z; H_a) = 0 \), where \( \mathcal{P} \) is a polynomial that is of degree \( N \) in \( y \), and in which the commuting Hamiltonians \( H_a \) appear as parameters.

The cohomology class of the restriction of \( \lambda \) to \( D \) depends on the parameters \( H_a \). One can compute this dependence by simply differentiating \( \lambda \) with respect to \( H_a \) at fixed \( z \). Differentiating the equation \( \mathcal{P}(y, z; H_a) = 0 \) in this fashion, we learn that \( 0 = \mathcal{P}'(\partial y/\partial H_a) + \partial \mathcal{P}/\partial H_a \), where \( \mathcal{P}' \) is short for \( \partial \mathcal{P}/\partial y \). So

\[
\frac{\partial y}{\partial H_a} = -\frac{1}{\mathcal{P}'} \frac{\partial \mathcal{P}}{\partial H_a}
\]  

(3.27)
Hence if we set \( \omega^a = \partial \lambda / \partial H_a \), we get
\[
\omega^a = \frac{\partial y}{\partial H_a} \, dz = -\frac{\partial P}{\partial H_a} \, dz. \tag{3.28}
\]
The objects \( \omega^a \) are actually holomorphic one-forms on \( D \). The only point here that is not completely trivial is that the \( \omega^a \) have no pole at \( P' = 0 \). But in fact, for a smooth plane curve \( \mathcal{P}(y, z) = 0 \), the differential \( dz / (\partial \mathcal{P} / \partial y) \) is regular at points with \( \mathcal{P} = \partial \mathcal{P} / \partial y = 0 \), as it can also be written \(-dy / (\partial \mathcal{P} / \partial z)\).

For \( G = U(N) \), the number of commuting Hamiltonians is equal to the genus \( g_D \) of \( D \). The \( \omega^a \) defined in (3.28) are all linearly independent, since the polynomial \( P(y, z; H_a) \) does depend nontrivially on all of the \( H_a \). So they give a basis of holomorphic differentials on \( D \).

Now we can write down almost by inspection a holomorphic two-form on \( \mathcal{M}_H \). If \( D \) is a spectral curve, and \( \mathcal{L} \) a line bundle over \( D \) of degree given in (2.33), we write \( \mathcal{L} = \psi^* (K^{(N-1)/2}) \otimes \mathcal{L}' \), where \( \mathcal{L}' \) has degree zero. We represent \( \mathcal{L}' \) by a divisor \( D = \sum_{i=1}^k n_i q_i \), and find a one-chain \( \gamma \) with \( \partial \gamma = D \). Then we set
\[
\Omega = \sum_{a=1}^{g_D} dH_a \wedge d \int_\gamma \omega^a. \tag{3.29}
\]
To see that this does not depend on the choice of \( \gamma \), note that if we add a one-cycle \( \overline{\gamma} \) to \( \gamma \), the change in \( \Omega \) is
\[
\Delta \Omega = \sum_{a=1}^{g_D} dH_a \wedge d \int_\gamma \omega^a = \sum_{a,b=1}^{g_D} dH_a \wedge dH_b \int_\gamma \frac{\partial \omega^a}{\partial \lambda} = 0. \tag{3.30}
\]

Similarly, the choice of \( K^{1/2} \) in the definition of \( \mathcal{L}' \) does not matter. Changing \( K^{1/2} \) has the effect of changing \( \mathcal{L}' \) by a line bundle whose square is trivial. So it changes the vector of periods \( (\int_\gamma \omega^1, \ldots, \int_\gamma \omega^{g_D}) \) by half of a lattice vector, or in other words by 1/2 of \( (\int_\gamma \omega^1, \ldots, \int_\gamma \omega^{g_D}) \), for some one-cycle \( \overline{\gamma} \). The computation already performed in (3.30), but with an extra factor of 1/2, now serves to show that this operation does not change \( \Omega \).

Actually, we can write \( \Omega \) in a way that manifestly does not depend on the choice of one-chain \( \gamma \). In (3.29), when the exterior derivative acts on \( \int_\gamma \omega^a \), it may differentiate either \( \gamma \) or \( \omega^a \). However, the terms in which the one-forms \( \omega^a \) are differentiated do not contribute, again by the same reasoning as in (3.30). So we only have to differentiate the chain \( \gamma \), or more precisely, its endpoints, which are characterized by \( \partial \gamma = \sum_{i=1}^k n_i q_i \). So we have
\[
\Omega = \sum_{a=1}^{g_D} \sum_{i=1}^k n_i dH_a \wedge (\omega^a, dq_i). \tag{3.31}
\]
The meaning of the symbol \( dq_i \) is as follows. Since a tangent vector to \( \mathfrak{g}_{SU(N)} \) is represented concretely by a first order displacement of the \( q_i \), there is, for each \( i = 1, \ldots, N \), a natural
map from the tangent space to $\mathfrak{f}_{SU(N)}$ to the tangent space $TD_{q_i}$ to $D$ at $q_i$. Equivalently, for each $i$, there is a natural 1-form on $\mathfrak{f}_{SU(N)}$ with values in $TD_{q_i}$. This has been denoted $dq_i$. Also, $(\omega^a, \cdot)$ represents the pairing with $TD$ of the 1-form $\omega^a$ on $D$. So $(\omega^a, dq_i)$ is for each $i$ a 1-form on $\mathfrak{f}_{SU(N)}$. And thus $\Omega$ is a 2-form on $\mathcal{M}_H$, as desired.

If we denote the linear coordinates on the fibers of the Hitchin fibration as $X$, then $\Omega$ is schematically of the form $dX \wedge dH$. (The components of $X$ are functions of the $q_i$.) This ensures that the functions $H_a$ are Poisson-commuting and, moreover, by Poisson brackets they generate linear motion of the $X$’s. Furthermore, under a rescaling of the Higgs field $\varphi$, the form $\Omega$ is homogeneous of degree 1 (since $\lambda = y dz$ has this property and $dH_a \omega^a = dH_a \partial \lambda / \partial H_a$ scales in the same way as $\lambda$). These properties agree with those of the symplectic form $\Omega_I$ defined in the underlying gauge theory. In section 3.4.1, we aim to demonstrate directly that they coincide.

But first we consider the analogous formulas for $G = SU(N)$ and $PSU(N)$. On the Riemann surface $C$, whose genus is $g_C$, there are $g_C$ holomorphic differentials $w^\alpha$, $\alpha = 1, \ldots, g_C$. Of the $g_D$ holomorphic differentials on $D$, we can take $g_C$ of them to be pullbacks $\omega^a = \psi^*(w^\alpha)$ from $C$. There remain $g_D - g_C$ such differentials $\tilde{\omega}^a$, $a = 1, \ldots, g_D - g_C$ on $D$ that are not pullbacks from $C$. We can normalize the $\tilde{\omega}^a$ to require that $\psi_*(\tilde{\omega}^a) = 0$ for all $a$. Here, for a holomorphic differential $\omega = f(y, z) dz$ on $D$, $\psi_*(\omega)$ is obtained by pushing $\omega$ forward to $C$. Concretely, if $y_i, i = 1, \ldots, N$ are the roots of the equation $\mathcal{P}(y, z) = 0$ (regarded as an equation for $y$ with fixed $z$), then $\psi_*(\omega) = \sum_i f(y_i, z) dz$.

To go from $U(N)$ to $SU(N)$ gauge theory, we remove $N$ commuting Hamiltonians by setting $\text{Tr} \varphi = 0$. The roots $y_i$ of the characteristic polynomial $\mathcal{P}(y, z; H_a)$ are the same as the eigenvalues of $\varphi$, and the fact that $\text{Tr} \varphi = 0$ means that $\sum_i y_i = 0$. The holomorphic differential $\lambda = y dz$ therefore obeys $\psi_*(\lambda) = \sum_i y_i dz = 0$. Since $\psi_*(\lambda) = 0$ for all values of the commuting Hamiltonians of the $SU(N)$ gauge theory, its derivatives with respect to those Hamiltonians also vanish. So

$$0 = \frac{\partial}{\partial H_a} \psi_*(\lambda) = \psi_* \left( \frac{\partial \lambda}{\partial H_a} \right), \quad (3.32)$$

where here the $H_a$ are the commuting Hamiltonians of $SU(N)$, not $U(N)$. Thus, upon setting $\tilde{\omega}^a = \partial \lambda / \partial H_a$, $a = 1, \ldots, g_D - g_C$, we get precisely the differentials on $D$ that are annihilated by $\psi_*$. 

Now let us apply the Abel-Jacobi map to $\mathfrak{f}_{SU(N)}$, which parametrizes line bundles $L' \in \text{Jac}(D)$ such that $\text{Nm}(L') = O$. As we have explained in section 3.3.1, any such $L'$ is isomorphic to $\mathcal{O}(D)$ where we can take the divisor $\mathcal{D}$ to be $\mathcal{D} = \sum_i n_i (q_i - q'_i)$, with $\psi(q_i) = \psi(q'_i)$. We can take a chain $\gamma$ with $\partial \gamma = D$ to be a sum of paths from $q'_i$ to $q_i$, so that $\psi(\gamma)$ is a closed cycle in $C$. As we learned at the end of section 3.3.1, such a closed cycle in $C$ can be lifted to a closed cycle $\tilde{\gamma} \subset D$. By replacing $\gamma$ with $\gamma - \tilde{\gamma}$, we can assume that $\psi(\gamma) = 0$. The Abel-Jacobi map takes $L'$ to the point in $\mathbb{C}^{g_D} / \Gamma_D$ with coordinates

$$\left( \int_{\gamma} \tilde{\omega}^a, \int_{\gamma} \psi^*(w^\alpha) \right). \quad (3.33)$$
However, if \( w \) is a holomorphic differential on \( C \), then \( \int_\gamma \psi^*(w) = \int_\psi(\gamma) w = 0 \), as we have required \( \psi(\gamma) = 0 \). So the image of \( L' \) under the Abel-Jacobi map is

\[
\left( \int_\gamma \tilde{\omega}^\alpha, \, 0 \right).
\]  

(3.34)

Having set the last \( g_C \) coordinates to zero, we now are free to shift \( \gamma \) only by an element of the lattice \( \Gamma \) which is the kernel of \( \psi_\ast \) (recall eqn. (3.12)). Shifting \( \gamma \) by an element of \( \Gamma \) does not affect the vanishing of the last \( g_C \) entries of the Abel-Jacobi map, since if \( \psi_\ast(\gamma) = 0 \) and \( w \in H^0(C, K) \), then \( \int_{\psi_\ast(\gamma)} w = 0 \). Shifting \( \gamma \) by any other element of \( \Gamma \) would destroy the vanishing of those last \( g_C \) entries.

So the Abel-Jacobi map sends \( \tilde{\mathfrak{f}}_{SU(N)} \) to \( \mathbb{C}^{g_D-g_C}/\Gamma \). The map is injective because the Abel-Jacobi map is injective even on the larger space \( \tilde{\mathfrak{f}}_{U(N)} \), and it is surjective on dimensional grounds. So \( \tilde{\mathfrak{f}}_{SU(N)} \) is isomorphic to \( \mathbb{C}^{g_D-g_C}/\Gamma \). This is in complete accord with the assertion in (3.23) that \( \tilde{\mathfrak{f}}_{SU(N)} = \text{Hom}(\Gamma^*, U(1)) \), since \( \text{Hom}(\Gamma^*, U(1)) \) is canonically isomorphic to \( (\Gamma \otimes \mathbb{Z}/\mathbb{R})/\Gamma \), which from the point of view of the complex geometry of \( C \) is the same as \( \mathbb{C}^{g_D-g_C}/\Gamma \).

At this stage, we can imitate either of the two formulas for the symplectic form that we had in the case of \( U(N) \). For \( G = SU(N) \), we define a holomorphic symplectic form on \( M_H \) by

\[
\Omega = \sum_{a=1}^{g_D-g_C} dH_a \wedge d\tilde{\omega}^a = \sum_{a=1}^{g_D-g_C} \sum_{i=1}^k n_i \, dH_a \wedge (\tilde{\omega}^a, dq_i) .
\]  

(3.35)

Now what about \( G = PSU(N) \)? The component \( M_{H}^{(0)}(PSU(N)) \) of \( M_{H}(PSU(N)) \) that parametrizes Higgs bundles of \( PSU(N) \) in the topologically trivial case is just the quotient \( M_{H}(SU(N))/\Gamma_N \), where \( \Gamma_N \) is the group of line bundles of order \( N \) on \( C \). The symplectic form \( \Omega \) is invariant under twisting \( L' \) by a line bundle of order \( N \) (as one sees by repeating the argument in (3.30) with an extra factor of \( 1/N \), so the same formulas give a holomorphic symplectic form on \( M_{H}^{(0)}(PSU(N)) \).

The other components of \( M_{H}(PSU(N)) \) similarly are quotients by \( \Gamma_N \) of what we have called \( M_{H}^{(d)}(PSU(N)) \) (the moduli space of solutions of Hitchin’s equations with \( \xi = d \) up to gauge transformations that are homotopic to the identity), and again it suffices to construct the symplectic form over \( \tilde{M}_{H}^{(d)}(PSU(N)) \). We pick a point \( p \in C \), set \( L_0 = \mathcal{O}(p)^d \), and then, according to (3.9), the fiber of the Hitchin fibration is \( \text{Nm}^{-1}(L_0) \).

If \( \mathcal{L}' \) is a line bundle with \( \text{Nm}(\mathcal{L}') = L_0 \), we write \( \mathcal{L}' = \mathcal{O}(D') \) with some divisor \( D' \), and we set \( D = D' - dq \), where \( q \in D \) lies above \( p \in C \). Hence \( \text{Nm}(\mathcal{O}(D)) = 0 \), so the divisor \( D \) is linearly equivalent to \( D'' = \sum_i n_i (q_i - q'_i) \), where \( q_i, q'_i \in D \) have the same image in \( C \). Finding a one-chain \( \gamma \) with \( \partial \gamma = D'' \), we define the symplectic form by the same formulas (3.35).

### 3.4.1 Comparison To Gauge Theory

In gauge theory, the holomorphic symplectic form was defined in eqn. (2.4) as

\[
\Omega_I = \frac{1}{\pi} \int_C |d^2 z| \text{Tr} \delta \phi \cdot \delta A_{\mathbb{C}}.
\]  

(3.36)
Let us remember that the symbol $\delta$ denotes the exterior derivative on an infinite-dimensional function space, such as the space of connections. When we get down to finite dimensions, we will just write $d$ for the exterior derivative.

The element $\varphi_z \, dz \in H^0(C, \text{ad}(E) \otimes K)$ is obtained by pushing down the differential $y \, dz \in H^0(D, K_D)$ from the spectral cover $D$. In terms of data on $D$, the symplectic form can be written

$$
\Omega_I = \frac{1}{\pi} \int_D |d^2 z| \delta \alpha|_{\tau},
$$

where the complex structure of the line bundle $\mathcal{L}' \to D$ is defined by a $\overline{\partial}$ operator that can locally be written (wherever $z$ is a good local parameter, that is, away from ramification points) as $d\overline{\tau}(\partial \tau + a_\tau)$.

The main step in comparing this to our formulas such as (3.31) is to find a convenient choice of $a_\tau$. Of course, $a_\tau$ is only uniquely determined up to gauge transformation, and we want to find a convenient representative. If the line bundle $\mathcal{L}'$ is $\mathcal{O}(\mathcal{D})$, where $\mathcal{D} = \sum_i n_i r_i$, with $r_i \in D$ and $\sum_i n_i = 0$, then up to an additive constant, there is a unique solution on $D$ of the equation $\overline{\partial} \delta f = i\pi \sum_i n_i \delta r_i$. Here $\delta r_i$ is a distributional $(1,1)$-form on $D$, supported at $r_i$ and with $\int_D \delta r_i = 1$. If for simplicity $r_i$ is not a ramification point, so that $z$ is a good local parameter near $r_i$, then near $r_i$ we have $f \sim \frac{1}{r_i} \ln |z - r_i|^2$. We can define the complex structure of $\mathcal{L}'$ by $\delta a_\tau = \partial \delta f$. This is not a pure gauge because of the singularity in $f$. Now we can evaluate $\delta a_\tau$, where $\delta$ refers to the variation in a change in $a_\tau$, that is in the $r_i$. We have

$$
\delta a_\tau d\overline{\tau} = \sum_i d\tau_i \partial \tau_i a_\tau d\overline{\tau} = \sum_i d\tau_i d\overline{\tau} \partial \tau_i f = -i \sum_i \pi n_i \delta r_i \cdot d\tau_i,
$$

where the derivative $\partial \tau_i a_\tau$ was evaluated using the fact that $a_\tau$ is antiholomorphic away from the $r_i$ and has a known singular behavior at $z = r_i$. As in (3.31), we think of $d\tau_i$ as a 1-form on $\mathfrak{g}_{SU(N)}$ with values in $TD_{r_i}$. In (3.38), $d\tau_i$ is contracted with the $(1,1)$-form $\delta r_i$ on $D$ to make a 1-form on $\mathfrak{g}_{SU(N)}$ with values in $(0,1)$-forms on $D$. The contraction is denoted $\delta r_i \cdot d\tau_i$. Now when we reduce to $\mathcal{M}_H$, we have $\delta y \, dz = \sum_a dH_a \omega^a$ as in the derivation of (3.29). Using this fact and (3.38), and also recalling that $|d^2 z| = i \, dz \wedge d\overline{\tau}$, we can evaluate (3.37) to give

$$
\Omega_I = \sum_{a,i} n_i dH_a \wedge (\omega^a, d\tau_i),
$$

in agreement with (3.31).

4 't Hooft Operators And Hecke Modifications

4.1 Eigenbranes

Our main goal in the rest of this paper is to describe the properties of 't Hooft operators and Hecke modifications in more depth than was explained in [1]. We begin with a short introduction, relying on the reader to consult [1] for more detail.
Figure 1. A line operator $L$ approaching a boundary labeled by a brane $B$. This gives a new composite boundary condition $B' = L \cdot B$.

Starting in four dimensions, one twists $\mathcal{N} = 4$ super Yang-Mills theory in two possible ways, producing two (partial, as in footnote 1) topological field theories that we will call the $A$-model and the $B$-model. Each model has half-BPS line operators – Wilson operators in the $B$-model and ‘t Hooft operators in the $A$-model. The next step is compactification to two dimensions on a Riemann surface $C$. A four-dimensional line operator, supported at a point $p \in C$, descends to a line operator in two dimensions. One studies the effective two-dimensional theory that results from compactification on a two-manifold $\Sigma$ with boundary. The boundary condition is defined by a brane $B$. (We will use the same notation for the brane $B$ and the corresponding boundary condition.) The key is now to consider the behavior as a line operator approaches the boundary (fig. 1). Clearly, in a two-dimensional topological field theory, a line operator $L$ approaching a boundary with boundary condition $B$ makes a new composite boundary condition $LB$. This gives an operation of line operators on branes:

$$LB = B'.$$  \hspace{1cm} (4.1)

One can act on a brane with a succession of line operators that approach the boundary one by one (fig. 2), and clearly the action of line operators on branes is associative. Actually, in the present context, there is some commutativity as well. If $L, L'$ are line operators supported at distinct points $p, p' \in C$, then they can be passed through each other in $\Sigma$.
Figure 2. Line operators can be brought to the boundary one by one. This gives an associative action of line operators on branes.

without meeting any singularity and therefore

\[ LL'B = L'L'B. \]  

(4.2)

But actually, in the context of a four-dimensional topological field theory, a line operator supported at a given point \( p \in C \) is locally independent of \( p \) (globally there may be a nontrivial monodromy if \( p \) traverses a noncontractible loop in \( C \)). So in trying to more two line operators past each other, we can always assume that they are inserted at different points in \( C \). Therefore line operators in the 2d theory that originate from loop operators in four dimensions commute even without the restriction \( p \neq p' \).

Since the line operators of interest do commute, the question arises of whether in some sense they can be simultaneously “diagonalized.” To explain the relevant notion requires a few preliminaries. We think of a brane \( B \) as being represented in the effective two-dimensional description by a brane on \( \mathcal{M}_H \). This brane is an \( A \)-brane or a \( B \)-brane depending on which twist we start with in the underlying four-dimensional super Yang-Mills theory. It is described by a sheaf \( \mathcal{U} \) over \( \mathcal{M}_H \) (whose support may be all of \( \mathcal{M}_H \) or a submanifold, depending on the brane considered). There is a natural operation on branes of tensoring \( \mathcal{U} \) with a fixed vector space \( V \). Applied to a brane \( B \), this gives a new brane that we call \( B \otimes V \); it is associated to the sheaf \( \mathcal{U} \otimes V \) over \( \mathcal{M}_H \). If \( V \) is of dimension \( n \), then roughly speaking \( B \otimes V \) is the sum of \( n \) copies of \( B \) (this is a rough description as it does not take into account the \( GL(n, \mathbb{C}) \) group of automorphisms of \( V \), which enables one...
to construct families of branes in which $V$ varies nontrivially). We say that the brane $B$ is an eigenbrane of the line operator $L$ if

$$LB = B \otimes V$$

(4.3)

for some vector space $V$.

Unfortunately, it is difficult to find a convenient terminology for the vector space $V$ that appears in this definition. It plays the role of the eigenvalue $\lambda$ in an ordinary matrix equation $M\psi = \lambda\psi$, and it is a vector space. So one might think of referring to $V$ as an “eigenvector space,” but unfortunately this phrase has another and more elementary meaning.

Since the Wilson operators of the $B$-model, or the ’t Hooft operators of the $A$-model, commute with each other, for reasons explained above, it is possible to have a brane $B$ that is a simultaneous eigenbrane for all Wilson operators or all ’t Hooft operators. A simultaneous eigenbrane for the Wilson operators is what we call an electric eigenbrane, and a simultaneous eigenbrane for the ’t Hooft operators is what we call a magnetic eigenbrane.

The geometric Langlands correspondence, as formulated mathematically [15], was deduced in [1] from the duality between Wilson operators of $L_G$ and ’t Hooft operators of $G$. In particular, the duality maps electric eigenbranes of $L_G$ to magnetic eigenbranes of $G$. But what, concretely, are these branes?

### 4.2 The Electric Eigenbranes

It is straightforward to find the electric eigenbranes. In fact, more generally, it is straightforward to describe the action of Wilson operators on branes. We will give a brief explanation, referring to section 8.1 of [1] for more detail. Let $\mathcal{M}$ be the moduli space of $L_G$-bundles on $C$, so that for any $m \in \mathcal{M}$, there is a corresponding $L_G$-bundle $E_m \to C$. The “universal bundle” over $C$ is a bundle $\mathcal{E} \to C \times \mathcal{M}$ with connection $A$ with the property that for any $m \in \mathcal{M}$, the restriction of $\mathcal{E}$ to $C \times m$ is isomorphic to $E_m$. If $^LR$ is an irreducible representation of $L_G$, we write $\mathcal{E}_{LR}$ for the associated bundle in the representation $^LR$. Actually, if the center of $L_G$ acts nontrivially in the representation $^LR$, then $\mathcal{E}_{LR} \to C \times \mathcal{M}$ is not an ordinary vector bundle but a twisted bundle, twisted by a nontrivial $B$-field (or gerbe) over $\mathcal{M}$ of finite order. The physical meaning of this, as explained in [1] and also in section 6.3 below, is that a Wilson operator in the representation $^LR$ carries a discrete electric charge which is measured by the gerbe. This discrete electric charge is transformed by $S$-duality to a discrete magnetic charge carried by the dual ’t Hooft operator.

The universal Higgs bundle is an analogous concept. $\mathcal{M}_H$ be the moduli space of $L_G$ Higgs bundles over $C$. The universal Higgs bundle is an $L_G$-bundle $\mathcal{E}_H \to C \times \mathcal{M}_H$, with connection $A$, and endowed with a Higgs field $\phi$ that is a section of $T^*C \otimes \text{ad}(\mathcal{E}_H)$ (where here by $T^*C$ we mean really the pullback of $T^*C$ from $C$ to $C \times \mathcal{M}_H$) such that the restriction of $(A, \phi)$ to $C \times m$ for any $m \in \mathcal{M}_H$ is the Higgs bundle $(E, \phi)$ associated to $m$. (Again $\mathcal{E}_H$ in general must be understood as a twisted vector bundle.) By the universal Higgs bundle in representation $^LR$, we simply mean the bundle $\mathcal{E}_{H,^LR} \to C \times \mathcal{M}_H$ associated to $\mathcal{E}_H$ in the representation $^LR$. 

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Given this concept, the action of a Wilson operator on a $B$-brane can be described in general. Let $B$ be a $B$-brane described by a sheaf $\mathcal{U} \to \mathcal{M}_H$ and let $W(\mathcal{I}R, p)$ be a Wilson operator in the representation $^L R$, and supported at a point $p \in C$. Then the brane $LB$ is described by the sheaf $\mathcal{U} \otimes \mathcal{E}_{H, \mathcal{I}R}|_{p \times \mathcal{M}_H}$. In other words, the action of a Wilson operator $W(\mathcal{I}R, p)$ on the sheaf $\mathcal{U} \to \mathcal{M}_H$ that describes a $B$-brane is

$$\mathcal{U} \to \mathcal{U} \otimes \mathcal{E}_{H, \mathcal{I}R}|_{p \times \mathcal{M}_H}. \quad (4.4)$$

Here $\mathcal{E}_{H, \mathcal{I}R}|_{p \times \mathcal{M}_H}$ is simply the restriction of $\mathcal{E}_{H, \mathcal{I}R}$ to $p \times \mathcal{M}_H$; thus it is a vector bundle over $\mathcal{M}_H$ and it makes sense to tensor $\mathcal{U} \to \mathcal{M}_H$ with this vector bundle.

The intuition behind the claim in eqn. (4.4) is as follows. Let us think of the vertical direction in fig. 1 as the “time” direction. So a Wilson operator that runs along the boundary is a static, time-independent Wilson operator. It represents an impurity, in the representation $^L R$, that depends on the gauge coupling constant and the fiber $G$. Supported at a point $p \in C$, it represents an impurity, in the representation $^L R$, that depends on the gauge coupling constant and the fiber $G$. Supported at a point $p \in C$ it represents an impurity, in the representation $^L R$, that depends on the gauge coupling constant and the fiber $G$.

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Eqs. (4.4) shows that a generic $B$-brane is not an electric eigenbrane, because $\mathcal{E}_{H, \mathcal{I}R}|_{p \times \mathcal{M}_H}$ is a non-trivial vector bundle over $\mathcal{M}_H$, not a constant vector space. An electric eigenbrane will be a brane $B$ such that, when restricted to the support of the corresponding sheaf $\mathcal{U}$, $\mathcal{E}_{H, \mathcal{I}R}|_{p \times \mathcal{M}_H}$ becomes (holomorphically) trivial. This happens if and only if $B$ is a zero-brane supported at a point $m \in \mathcal{M}_H$. Then $\mathcal{U}$ is a skyscraper sheaf supported at $m$, and

$$\mathcal{U} \otimes \mathcal{E}_{H, \mathcal{I}R}|_{p \times \mathcal{M}_H} \cong \mathcal{U} \otimes \mathcal{E}_{H, \mathcal{I}R}|_{p \times m}. \quad (4.5)$$

On the right hand side, $\mathcal{E}_{H, \mathcal{I}R}|_{p \times m}$ is the restriction of $\mathcal{E}_{H, \mathcal{I}R}$ to $p \times m \in C \times \mathcal{M}_H$, and in particular is a constant vector space. Thus a zero-brane $B$ supported at $m$ is an electric eigenbrane:

$$W(\mathcal{I}R, p)B = B \otimes \mathcal{E}_{H, \mathcal{I}R}|_{p \times m}. \quad (4.6)$$

To get a magnetic eigenbrane, therefore, we simply must apply $S$-duality to a zero-brane. Consider a zero-brane $B$ supported at a point $m \in \mathcal{M}_H$ that lies in a particular fiber $^L \mathfrak{F}$ of the Hitchin fibration $^L G$. Since $S$-duality acts as $T$-duality on the fibers of the Hitchin fibration $\mathcal{M}_H \to \mathcal{V}$, the $S$-dual of $B$ will be a rank 1 $A$-brane $B'$ supported on the fiber $\mathfrak{F}$ of the Hitchin fibration $^L G$ that corresponds to $^L \mathfrak{F}$. $B'$ is described by a flat $\text{Spin}_c$ structure over $\mathfrak{F}$ that encodes the position of $m$ in $^L \mathfrak{F}$. We will refer to it as a brane of type $F$.

We will eventually understand more or less explicitly why a rank 1 $A$-brane supported on a fiber of the Hitchin fibration $\mathcal{M}_H \to \mathcal{V}$ is a magnetic eigenbrane. But we will first develop a better understanding of many properties of ’t Hooft operators in general.

---

Footnote 13: Fibers of the Hitchin fibration are parametrized by the values of invariant polynomials in the Higgs field $\varphi$. As there is a natural correspondence between invariant polynomials on the Lie algebras of $^L G$ and of $G$, there is a natural correspondence between the fibers of the two Hitchin fibrations. To be more precise, this correspondence involves a rescaling of the Higgs field that depends on the gauge coupling constant and will play no essential role.
4.3 ‘t Hooft Operators and Hecke Transformations

A half-BPS Wilson operator is represented by the holonomy of the complex connection $A = A + i\phi$. In the case of a closed loop $S$, the half-BPS Wilson operator in a representation $L^R$ of $^4G$ is

$$W(L^R, S) = \text{Tr}_{LR} P \exp \left( - \oint_S (A + i\phi) \right). \quad (4.7)$$

Upon $S$-duality to a magnetic description in gauge group $G$, the $A$-dependent part of the holonomy operator is replaced by the Dirac monopole singularity that defines an ‘t Hooft operator. The $\phi$-dependent part remains, and can be interpreted classically as creating a singularity in $\phi$. The upshot is that a half-BPS ‘t Hooft operator can be described by a half-BPS solution of the Bogomolny equations. To be precise, we consider the four-dimensional spacetime $M = \mathbb{R}^3 \times \mathbb{R}$, where $\mathbb{R}$ is the “time” direction, parametrized by $s$, and $\mathbb{R}^3$ is parametrized by a three-vector $\vec{x}$. Then an ‘t Hooft operator at rest at $\vec{x} = 0$ is defined by specifying the singularity that the fields should have near $\vec{x} = 0$. For gauge group $U(1)$ and an ‘t Hooft operator of charge 1, the solution is characterized by

$$F = \frac{i}{2} \ast_3 \frac{1}{|\vec{x}|} \quad \phi = \frac{i}{2|\vec{x}|} ds. \quad (4.8)$$

In other words, this is a Dirac monopole singularity for $F = dA$, extended to a solution of the Bogomolny equations by including an analogous point singularity in $\phi$.

In general [30], an arbitrary half-BPS ‘t Hooft operator for gauge group $G$ is defined by picking a homomorphism

$$\rho : U(1) \to G, \quad (4.9)$$

and using this homomorphism to embed the abelian solution (4.8) in $G$. In Langlands or GNO duality, the following statement is very fundamental: homomorphisms $U(1) \to G$ are classified, up to conjugacy, by dominant highest weights of the dual group $^L G$, or equivalently by irreducible representations $L^R$ of $^4G$. Thus ‘t Hooft operators of $G$ are in natural correspondence with Wilson operators of $^4G$.

$A$-model observables are basically evaluated by solving equations for field configurations with suitable unbroken supersymmetry and then counting the solutions, or suitably quantizing the space of solutions. To understand the action of an ‘t Hooft operator on a brane, the basic problem to consider is as follows. The four-dimensional spacetime is $\Sigma \times C$ where $\Sigma$ is a two-manifold with boundary. The ‘t Hooft operator of interest runs parallel to the boundary of $\Sigma$, as in fig. 1. Near its boundary, we can factorize $\Sigma$ as $I \times \mathbb{R}$ where $\mathbb{R}$ is the “time” direction, running along the boundary, and $I$ is a one-manifold with boundary. The supersymmetric fields that are relevant to understand the action of the ‘t Hooft operator on a brane are time-independent, so they are solutions of a gauge theory equation in three dimensions, namely on $I \times C$. Though a compact one-manifold $I$ with boundary inevitably has two ends, we focus attention on just one end, which we denote $\partial I$, as we wish to study an ‘t Hooft operator near one given boundary of $\Sigma$. 

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We will give a brief description of the supersymmetric equations that have to be satisfied, referring to [1] for more detail. The relevant equations are most familiar in the case that the part of the Higgs field $\phi$ that is tangent to $C$ vanishes along $\partial I$. In this case, we can assume that the only nonzero component of $\phi$ is the component $\phi_s$ in the “time” direction, which is forced to be nonzero because it actually has a singular behavior at the position of the ’t Hooft operator (eqn. (4.8)). One can show a vanishing theorem saying that in a well-behaved solution, the “time” component $A_s$ of $A$ vanishes. The equations then reduce to equations on the three-manifold $I \times C$ for a connection $A$ and adjoint-valued scalar field $\phi_s$. These equations prove to be the Bogomolny equations:

$$F = \star D \phi_s.$$  \hfill (4.10)

In the presence of an ’t Hooft operator, the relevant solutions of the Bogomolny equations will have Dirac monopole singularities. (Most of the literature on the Bogomolny equations deals with smooth solutions, but there is also a substantial literature on solutions with such singularities [31–38].)

In the more general case that the tangent part of $\phi$ does not vanish, one can still show a vanishing theorem both for $A_s$ and for the “normal” component of $\phi$, that is the component along $I$. It is convenient to parametrize $I$ by a real variable $y$ and also to pick a local coordinate $z$ on $C$. (For $C = \mathbb{R}^2$, $y$ and $z$ can be related to Euclidean coordinates on $T \times C = \mathbb{R}^3$ by $y = x^1$, $x = x^2 + i x^3$.) The vanishing theorem says that we can set $A_s = \phi_y = 0$. The supersymmetric equations (at $t = 1$ in the notation of eqn. (3.29) of [1]; this is a convenient value for studying the $A$-model) in general read

$$0 = F - \phi \wedge \phi + \star D_A \phi = D_A \star \phi, \hfill (4.11)$$

where $D_A = d + [A, \cdot]$ and $\star$ is the Hodge star. When we specialize to the case $A_s = \phi_y = 0$, the equations for the remaining fields can be formulated as follows (this description was introduced in [8], section 3.6, and exploited to find some interesting solutions of the equations). One defines the three operators

$$D_1 = \frac{D}{Dz} = \partial_z + [A_z, \cdot],$$  
$$D_2 = D_y - i[\phi_s, \cdot],$$  
$$D_3 = [\phi_z, \cdot], \hfill (4.12)$$

and also the “moment map”

$$\mu = \sum_{i=1}^{3} [D_i, D_i^\dagger]. \hfill (4.13)$$

The equations for a supersymmetric configuration can then be written as a “complex equation”

$$[D_i, D_j] = 0, \quad 1 \leq i < j \leq 3, \hfill (4.14)$$

and a “moment map condition”

$$\mu = 0. \hfill (4.15)$$
It will be convenient to refer to the combined system of equations as the extended Bogomolny equations – extended to include the Higgs field $\varphi = \phi_z \, dz$.

The complex equation $[D_i, D_j] = 0$ is invariant under $G_C$-valued gauge transformations. Under suitable conditions of semi-stability, which are satisfied in reasonably simple applications (such as we will consider in this paper), one expects that a solution of the complex equation modulo complex-valued gauge transformations is equivalent to the full system $[D_i, D_j] = \mu = 0$ modulo $G$-valued gauge transformations. Thus to determine the moduli space of solutions, one mainly has to understand the complex equations $[D_i, D_j] = 0$ modulo $G$-valued gauge transformations. Thus to determine the moduli space of solutions, one mainly has to understand the complex equations $[D_i, D_j] = 0$.

If $\varphi = 0$, then trivially $D_3 = 0$. The remaining equations $[D_1, D_2] = [D_1, D_1'] + [D_2, D_2'] = 0$ are the Bogomolny equations (4.10), written in a possibly unfamiliar way.

However, this way of writing the Bogomolny equations is very convenient for application to geometric Langlands.

First we must recall that the $(0, 1)$ part of any connection on a $G$-bundle $E \to C$ over a Riemann surface $C$ defines a $\overline{\partial}$ operator and turns $E$ (or more precisely its complexification) into a holomorphic $G_C$-bundle. Thus in particular, at any given value of $y$, the operator $D_1$ is such a $\overline{\partial}$ operator and gives the bundle $E$ a holomorphic structure. The equation $[D_1, D_2] = 0$ tells us that up to conjugation, the operator $D_1 = D_\varphi$ is independent of $y$, since

$$\frac{\partial}{\partial y} D_\varphi = -[A_y + i\phi_y, D_\varphi].$$

(4.16)

Thus as long as this equation is obeyed, the holomorphic structure of the bundle $E$ is independent of $y$.

Now suppose that there is an ’t Hooft operator at some point $y = y_0$, $z = z_0$. At that point, there is a delta-function source in the Bogomolny equations, and the condition $[D_1, D_2] = 0$ is not satisfied. The holomorphic type of the bundle $E$ actually does jump in crossing $y = y_0$; it has one type for $y > y_0$ and another for $y < y_0$. However, the singularity associated with the ’t Hooft operator occurs only at one point $z = z_0$ in $C$. If we omit this one point from $C$, then the bundle $E$ is unchanged holomorphically even in crossing the past $y = y_0$.

A modification of a holomorphic $G$-bundle $E \to C$ that is trivial if a single point $p$ is removed from $C$ is called in the context of the geometric Langlands program a Hecke modification of $E$ at $p$. (The terminology is based on an analogy with Hecke operators in number theory.) Thus an ’t Hooft operator induces a Hecke modification of the holomorphic $G$-bundle $E$ at the point in $C$ at which it is inserted.

An important fact in mathematical approaches to geometric Langlands is that there are different types of possible Hecke modifications for a $G$-bundle, and that these possible types are classified by the choice of an irreducible representation $LR$ of the dual group $L^G$. What we have described is a physical interpretation of this: the possible types of Hecke modifications are determined by the choice of an ’t Hooft operator, which determines the precise nature of the singularity in the solution of the Bogomolny equations. Moreover the ’t Hooft operators are indeed classified by the choice of $LR$.

It is not difficult to include the Higgs field $\varphi$ in this discussion. At fixed $y$, we have the equation $[D_1, D_3] = 0$, which tells us that $D_\varphi = 0$, in other words the pair $A_\varphi, \varphi$...
is a Hitchin pair and determines a Higgs bundle, which we regard as a pair consisting of a holomorphic bundle $E$ along with $\varphi \in H^0(C, K \otimes \text{ad}(E))$. The equations $[D_2, D_3] = 0$ or
\[
[D_y - i\phi_s, D_z] = [D_y - i\phi_s, \varphi] = 0
\]
tell us that, away from the position of a possible ’t Hooft operator, the holomorphic type of the Higgs bundle $(E, \varphi)$ is independent of $y$. In the presence of an ’t Hooft operator at $y = y_0$ and at a point $p \in C$, the holomorphic type of the pair $(E, \varphi)$ will jump at $y = y_0$, but in a way that is trivial if we omit the point $p$ from $C$. We can describe this by saying that in crossing $y = y_0$, the pair $(E, \varphi)$ undergoes a Hecke modification at the point $p \in C$.

If we simply forget $\varphi = D_3$ and the equations it enters, and remember only $D_1, D_2$ and the condition $[D_1, D_2] = 0$, we see that a Hecke modification of $(E, \varphi)$ consists, in particular, of an ordinary Hecke modification of $E$. Now remembering $\varphi$, the equations $[D_1, D_1] = [D_2, D_3] = 0$ imply that the holomorphic type of $(E, \varphi)$ is independent of $y$ for $y \neq y_0$. But what happens when we cross $y = y_0$? The condition $0 = [D_1, D_2] = [D_y - i\phi_s, \varphi]$ determines what happens to $\varphi$ in crossing $y = y_0$. If $\varphi$ is chosen generically for, say, $y > y_0$, it will have a pole at $z = z_0$ for $y < y_0$. This important fact will be explained in section 4.5. Given this fact, the possible Hecke modifications of a Hitchin pair $(E, \varphi)$ at a specified point $p \in C$ are a subset of the possible Hecke modifications of $E$ at $p$. We will refer to the Hecke modifications of $(E, \varphi)$ as $\varphi$-invariant Hecke modifications of $E$. The rationale for this terminology will become clear. Hecke modifications of Higgs bundles $(E, \varphi)$ have been considered mathematically [39].

Before understanding Hecke modifications of the pair $(E, \varphi)$, one should first be familiar with Hecke modifications of $E$ by itself, in the special case $\varphi = 0$. The reader may want to consult sections 9 and 10 of [1], where an introduction can be found. Here we will just recall a few facts which are the minimum that we will need for our study of magnetic eigenbranes.

4.4 Basic Examples

4.4.1 $G = U(1)$

The most basic case to consider is the group $G = U(1)$. In this case, $E$ is a complex line bundle that we denote as $L$. The isomorphism type of $L$ is going to be constant for $y \neq y_0$, for some $y_0$. We denote $L$ as $L_-$ for $y < y_0$ and as $L_+$ for $y > y_0$.

In crossing $y = y_0$, $L$ will be modified in a way that is trivial except at a point $p \in C$. We can pick a trivialization of $L_-$ in a neighborhood of $p$ and thus identify it with a trivial line bundle $\mathcal{O}$. Away from $p$, the trivialization of $L_-$ determines a natural trivialization of $L_+$ by parallel transport in the $y$ direction, using the connection $D_y - i\phi_s$. In other words, if for $y < y_0$ a holomorphic section $s$ of $L_-$ gives a trivialization of $L_-$ in a neighborhood $U$ of $p$, then parallel transport of $s$ to $y > y_0$ will give a trivialization of $L_+$ over $U \setminus p$ (that is, over $U$ with the point $p$ omitted). Hence, over $U \setminus p$, $L_+$ is naturally identified with $\mathcal{O}$.

But the trivialization of $\mathcal{O}$ does not necessarily extend over the point $p$. The general possibility is that, after being parallel transported to $y > y_0$, the section $s$ may have a zero of order $m$ (or a pole of order $-m$) at $p$ for some integer $m$. Thus, after identifying
\( \mathcal{L} \) with \( \mathcal{O}, \mathcal{L}_+ \) may be identified with \( \mathcal{O}(mp) \). Here \( \mathcal{O}(mp) \) is the line bundle whose local sections near \( p \) are holomorphic functions that are allowed to have a pole of order \( m \) (or required to have a zero of order \( -m \)) at \( p \).

It is explained in [1], section 4.5, that the Hecke transformation \( \mathcal{O} \to \mathcal{O}(mp) \) is the result of inserting at the point \( p \times y_0 \in C \times \mathbb{R} \) an ’t Hooft operator \( T(m) \) of magnetic charge \( m \). (We denote this ’t Hooft operator as \( T(m; \mathcal{O}) \) if we wish to specify the point \( p \in C \) at which it is inserted.) The basic idea in showing this is that such an ’t Hooft operator creates \( m \) units of magnetic flux. The operation that creates \( m \) units of magnetic flux at a point \( p \in C \) is described in algebraic geometry as the Hecke transformation \( \mathcal{O} \to \mathcal{O}(mp) \).

4.4.2 \( G = U(2) \)

For a second example, let us take \( G = U(2) \). The bundle \( E \) is now a complex vector bundle of rank 2 and we denote it as \( E_- \) or \( E_+ \) for \( y < y_0 \) or \( y > y_0 \). Since a holomorphic vector bundle is locally trivial, we can pick an identification of \( E_- \) as \( \mathcal{O} \oplus \mathcal{O} \) for \( y < y_0 \). A basic Hecke operation now transforms \( E_- \) to

\[
E_+ = \mathcal{O}(p) \oplus \mathcal{O}
\]  

for \( y > y_0 \). As explained in [1], this is the Hecke transformation implemented by an ’t Hooft operator dual to the natural two-dimensional representation of \( ^L U(2) \cong U(2) \).

We can think of this as the representation of \( U(2) \) with highest weight \( \lambda (1,0) \). We denote the corresponding ’t Hooft operator as \( T(\lambda) \). We refer to a Hecke transformation induced by \( T(\lambda) \) as a Hecke transformation of type \( \lambda \).

What is exhibited in eqn. (4.18) is a special case of a Hecke transformation induced by \( T(\lambda) \) for \( \lambda = (1,0) \). The reason that it is a special case is that a particular decomposition of \( E_- \) as \( \mathcal{O} \oplus \mathcal{O} \) was used. More generally, instead of saying that a section of \( E_+ \) is a section of \( \mathcal{O} \oplus \mathcal{O} \) may have a simple pole in the first component, we can allow a simple pole in a specified linear combination of the two components. For this, we pick a pair of complex numbers \( (u,v) \), not both 0, and we say that a section of \( E_+ \) is a pair \( (f,g) \) defining a section of \( \mathcal{O} \oplus \mathcal{O} \) away from \( p \), such that \( f \) and \( g \) are allowed to have a simple pole at \( p \), but the residue of this pole must be a multiple of \( (u,v) \). In formulas, if \( z \) is a holomorphic function on \( C \) with a simple zero at \( p \), we require

\[
(f,g) = (f_0,g_0) + \lambda \frac{z}{z}(u,v),
\]  

where \( f_0 \) and \( g_0 \) are holomorphic at \( p \), and \( \lambda \) is a possibly nonzero complex number. The bundle \( E_+ \) that is defined by this procedure clearly depends on the pair \( (u,v) \) only up to overall scaling, and thus the family of bundles \( E_+ \) that can be built this way, starting from \( E_- \) and making a Hecke transformation of type \( (1,0) \), is parametrized by a copy of \( \mathbb{C}P^1 \).

For future reference, it is convenient to rewrite eqn. (4.19) with the sections regarded as column vectors rather than row vectors. Thus a holomorphic section \( s \) of \( E_+ \) takes the form

\[
s = s_0 + \lambda \frac{z}{z} \begin{pmatrix} u \\ v \end{pmatrix},
\]
where $s_0$ is a holomorphic section of $E_-$ and the column vector \( \begin{pmatrix} u \\ v \end{pmatrix} \) now represents an element of $\mathbb{CP}^1$ with homogeneous coordinates $u, v$.

We could reach this result more intrinsically without ever picking a local trivialization of $E_-$. We write $E_{-,p}$ for the fiber of $E_-$ at $p$, and pick a nonzero vector $b \in E_{-,p}$. Then we characterize $E_+$ by saying that a section of $E_+$ near $p$ has the form

\[
  s = s_0 + \frac{b}{z} \lambda.
\]

Thus, we allow a simple pole, but its residue must be a multiple of $b$. The bundle obtained this way depends on $b$ only up to scaling, so the family of such bundles is a copy of $\mathbb{CP}^1$ that is obtained by projecting the two-dimensional vector space $E_{-,p}$. We denote this projectivization of $E_{-,p}$ as $\mathbb{P}(E_{-,p})$.

Thus, there is a natural space of Hecke modification of type $(1,0)$ of given bundle $E_- \rightarrow C$ at a specified point $p \in C$. This space is a compact smooth manifold, which is a copy of $\mathbb{CP}^1$, naturally isomorphic to $\mathbb{P}(E_{-,p})$.

Up to a certain point, we can treat arbitrary ’t Hooft operators of $U(2)$ in a similar way. The highest weight of an arbitrary representation of $^4U(2) \cong U(2)$ is given by a pair of integers $(n, m)$. The Weyl group acts by exchanging the two weights, so by a Weyl transformation, we can take $n \geq m$.

An example of a Hecke modification of type $(n, m)$ is the one that maps $E_- = O \oplus O$ to

\[
  E_+ = O(np) \oplus O(mp).
\]

This is a special case of a Hecke modification at $p$ of type $(n, m)$. The full family of such Hecke modifications has complex dimension $n - m$. However, there is an important complication compared to the case $n - m = 1$ that was treated above. The space of Hecke modifications of type $(n, m)$ for $n - m \geq 2$ is not compact, or better, it has a natural compactification that involves allowing Hecke modifications of lower weights (of weight $(n - k, m + k)$ for $1 \leq k \leq (n - m)/2$). From the point of view of ’t Hooft operators, this compactification involves “monopole bubbling,” in which a smooth BPS monopole, in the field of an ’t Hooft operator, shrinks down near the ’t Hooft operator and disappears.\(^{14}\)

The compactified space of Hecke modifications of type $(n, m)$ for $n - m \geq 2$ has singularities associated to monopole bubbling. In the literature on geometric Langlands, the space of Hecke modifications of a given type is called a Schubert cell, and its compactification is called a Schubert cycle (in the affine Grassmannian).

An introduction to these matters can be found in section 10 of [1]. Here, however, we prefer to avoid the complications associated to monopole bubbling and the singularities of the compactified space of Hecke modifications. Accordingly, we will limit ourselves to ’t Hooft operators $T(n, m)$ with $n - m \leq 1$.

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\(^{14}\)Such bubbling is relatively familiar for Yang-Mills instantons in four dimensions, but may be unfamiliar for BPS monopoles as it does not occur in the absence of ’t Hooft operators. More precisely, for gauge group $U(2)$, it does not occur except in the presence of ’t Hooft operators whose weights obey $n - m \geq 2$. 
Now let us consider the cases that $G$ be minuscule.) The group $SU(2)$ only has 2 elements, a representation of $L$ is that for a bundle. To make a rigorous statement, instead of a rank 2 complex vector bundle representation, which is the 2-dimensional representation. (As the Weyl group of $G$ acts by multiplication by $(\det g)^m$. The representation with highest weight $\langle m + 1, m \rangle$ is a 2-dimensional representation in which (regarding $g$ as a $2 \times 2$ matrix) $g$ acts by multiplication by $g(\det g)^m$. These are minuscule representations, since for example the 2-dimensional representation just mentioned has precisely 2 weights, which are exchanged by a Weyl transformation.

In general, let $^L R$ be an irreducible representation of any compact group $^L G$, with highest weight $^L w$. The space of Hecke modifications of type $^L w$ is compact, or equivalently there is no monopole bubbling in the field of an 't Hooft operator $T(^L w)$, if and only if the representation $^L R$ is minuscule. If $^L G$ is semi-simple, minuscule representations are in 1-1 correspondence with non-trivial characters of the center of $^L G$. Indeed, the smallest representation of $^L G$ that transforms as a given character of the center of $^L G$ is always minuscule. For example, $^L G = SU(2)$ has precisely one non-trivial minuscule representation, which is the 2-dimensional representation. (As the Weyl group of $SU(2)$ has only 2 elements, a representation of $SU(2)$ with dimension greater than 2 cannot possibly be minuscule.) The group $^L G = SO(3)$ has no non-trivial minuscule representation.

4.4.3 $G = SO(3)$ or $SU(2)$

Now let us consider the cases that $G$ is $SO(3)$ or $SU(2)$.

The dual group of $G = SO(3)$ is $^L G = SU(2)$. Before specializing to $SU(2)$, we make some remarks about $U(N)$ and $SU(N)$, which will be used in section 4.4.4.

The weight lattice of $U(N)$ is spanned by $N$-plets $(m_1, m_2, \ldots, m_N)$ of integers $m_1, \ldots, m_N$. The Weyl group acts by permutations, so up to a Weyl transformation we can impose a dominant weight condition $m_1 \geq m_2 \geq \cdots \geq m_N$.

The weight lattice of $SU(N)$ is spanned by similar $N$-plets $(m_1, m_2, \ldots, m_N)$, but now we take $m_i \in \mathbb{Z}/N$, but with $m_i - m_j \in \mathbb{Z}$, and we require $\sum_i m_i = 0$. The Weyl group still acts by permutations, and a dominant weight still obeys $m_1 \geq m_2 \geq \cdots \geq m_N$.

Specializing this description to $^L G = SU(2)$, this means that a dominant weight has the form $^L w = (n/2, -n/2)$, with an integer $n$. Now let us consider the Hecke modification associated to an 't Hooft operator $T(^L w)$. Naively speaking, a typical such Hecke modification at $p$ maps a bundle $E_\pm = \mathcal{O} \oplus \mathcal{O}$ to

$$E_+ = \mathcal{O}(np/2) \oplus \mathcal{O}(-np/2).$$

(4.23)

However, we should ask what this means if $n$ is odd so that $n/2 \notin \mathbb{Z}$. The answer is that for $^L G = SU(2)$, $G = SO(3)$. The group $SO(3)$ does not have a two-dimensional representation, so we should not try to think of a $G$-bundle as a rank 2 complex vector bundle. To make a rigorous statement, instead of a rank 2 complex vector bundle $E$ we should consider the associated bundle in the adjoint representation of $SO(3)$; this is $V = \text{ad}_0(E)$. (For a rank $N$ complex vector bundle $E$ with dual $E^*$, we write $\text{ad}(E)$ for
\( E \otimes E^* \) and \( \text{ad}_0(E) \) for the traceless part of \( E \otimes E^* \).) Note that \( V \) is endowed with a holomorphic, nondegenerate bilinear form, coming from \((v, v') = \text{Tr} vv'\), for \( v, v' \in \text{ad}_0(E) \).

For \( E_- = O \oplus O \), we have \( V_- = \text{ad}(E_-) = O \oplus O \oplus O \), and for \( E_+ \) defined informally as in eqn. (4.23), we have \( V_+ = \text{ad}_0(E_+) = O \oplus O(np) \oplus O(-np) \). Thus a typical example of the action of \( T^{(L-w)} \) on the \( SO(3) \) bundle \( V \) is

\[
O \oplus O \oplus O \rightarrow O \oplus O(np) \oplus O(-np). \tag{4.24}
\]

The quadratic form on the \( SO(3) \) bundle \( O \oplus O(np) \oplus O(-np) \) pairs \( O \) with itself and \( O(np) \) with \( O(-np) \), so in particular \( O(np) \) and \( O(-np) \) are null subspaces.

The only nontrivial minuscule representation of \( ^tG = SU(2) \) corresponds to \( n = 1 \), so that \( L_w = (1/2, -1/2) \) is the highest weight of the 2-dimensional representation of \( SU(2) \).

Then eqn. (4.24) gives the local form of the action of the corresponding \(^t\)Hooft operator \( T^{(L-w)} \) on an \( SO(3) \) bundle. However, a description in terms of the \( SO(3) \) bundle \( V \), although rigorous, tends to be lengthy, and it is simpler to relate an \( SO(3) \) bundle \( V \) to a rank 2 complex vector bundle \( E \), possibly of nontrivial determinant, via \( V = \text{ad}_0(E) \). In doing this, tensoring \( E \) with a line bundle \( L \) does not matter (since \( \text{ad}_0(E) \) is naturally isomorphic to \( \text{ad}_0(E \otimes L) \)). Instead of saying that \( T^{(L-w)} \) maps \( E_- = O \oplus O \) to the formal expression written in eqn. (4.23), we could just as well tensor formally with \( L = O(np/2) \) and say that

\[
E_+ = O(np) \oplus O, \tag{4.25}
\]

again with the understanding that we are really interested not in \( E_+ \) but in \( V_+ = \text{ad}(E_+) \).

Thus the most convenient way to describe the action of the \(^t\)Hooft operator \( T^{(L-w)} \), for \( L-w \) the minuscule weight \((1/2, -1/2)\) of \( SU(2) \), will be to say that locally it maps \( E_- = O \oplus O \) to

\[
E_+ = O(p) \oplus O. \tag{4.26}
\]

When we say this, we always bear in mind that this transformation from \( E_- \) to \( E_+ \) is a shorthand way to describe the Hecke modification from \( V_- = \text{ad}_0(E_-) = O \oplus O \oplus O \) to \( V_+ = \text{ad}_0(E_+) = O \oplus O(p) \oplus O(-p) \). The description by \( E_\pm \) is very useful, but not completely canonical, since without changing \( V_\pm \), we could replace \( E_\pm \) by \( E_\pm \otimes L \), where \( L \) is a line bundle, for instance \( L = O(kp) \) for some \( k \in \mathbb{Z} \).

For \( G = SU(2) \), we have \(^tG = SO(3) \). A highest \( L-w \) weight of \( SO(3) \) is just a highest weight of \( SU(2) \) that is divisible by 2, so it has the form \((k, -k)\) for some integer \( k \). Thus the generic local action of \( T^{(L-w; p)} \) on an \( SU(2) \) bundle is

\[
O \oplus O \rightarrow O(kp) \oplus O(-kp). \tag{4.27}
\]

This makes sense as a transformation of rank 2 bundles, in keeping with the fact that for \(^tG = SO(3) \), we have \( G = SU(2) \). However, the group \( SO(3) \) has no nonzero minuscule weights (since its center is trivial), so in studying Hecke modifications for \( G = SU(2) \), the complications due to monopole bubbling are inescapable.
4.4.4 $G = U(N)$, $PSU(N)$, or $SU(N)$

The cases that $G$ is $U(N)$, $PSU(N)$, or $SU(N)$ are quite similar to what we have just described for $N = 2$.

For $G = U(N)$, the dual group is also $^tG = U(N)$. As already remarked, a highest weight of $U(N)$ is a sequence of integers $^t w = (n_1, n_2, \ldots, n_N)$ with $n_1 \geq n_2 \geq \cdots \geq n_N$. A typical Hecke modification at $p$ of type $^t w$ acts by

$$\mathcal{O} \oplus \mathcal{O} \oplus \cdots \mathcal{O} \rightarrow \mathcal{O}(n_1 p) \oplus \mathcal{O}(n_2 p) \oplus \cdots \oplus \mathcal{O}(n_N p). \quad (4.28)$$

The representation of weight $^t w$ is minuscule if and only if $n_1 - n_N \leq 1$, or equivalently if and only if the integers $n_1, \ldots, n_N$ take at most two values. The case $(n_1, \ldots, n_N) = (m, m, \ldots, m)$ corresponds to a 1-dimensional representation in which $g \in U(N)$ acts as multiplication by $(\det g)^m$. More interesting is the case $(n_1, n_2, \ldots, n_N) = (m + 1, m + 1, \ldots, m + 1, m, m, \ldots, m)$, where we will write $k$ for the number of $m + 1$'s. This corresponds to the $k^{th}$ copy $\wedge^k W$ of the fundamental $N$-dimensional representation $W$ of $U(N)$, tensored with a one-dimensional representation, such that $g \in U(N)$ acts by $(\det g)^m \wedge^k g$. (Here $\wedge^k g$ is the matrix by which $g$ acts on $\wedge^k W$.)

Let us describe in more detail the Hecke modifications dual to the minuscule representation $\wedge^k V$, which corresponds to the weight $^t w = (1, 1, \ldots, 1, 0, \ldots, 0)$, with $k$ 1's. Specializing eqn. (4.28) to this case, we see that for some decomposition as $E_-$ as a direct sum of trivial line bundles, the transformation will be

$$\mathcal{O} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O} \rightarrow \mathcal{O}(p) \oplus \mathcal{O}(p) \oplus \cdots \oplus \mathcal{O}(p) \oplus \cdots \oplus \mathcal{O}, \quad (4.29)$$

with $k$ summands $\mathcal{O}(p)$. However, we can describe this in a more invariant way and thereby describe the space of all Hecke modifications at $p$ that are of type $^t w$ for this weight. Let $W$ be an arbitrary $k$-dimensional subspace of $E_{-p}$, the fiber at $p$ of $E_-$. Then we describe $E_+$ by saying that a holomorphic section of $E_+$ near $p$ takes the form

$$s = s_0 + \frac{b}{z}, \quad (4.30)$$

where $s_0$ is a local holomorphic section of $E_-$ near $p$, and $b$ is a local holomorphic section of $E_-$ such that $b(p) \in W$. In other words, $s$ is allowed to have a simple pole at $p$, but the residue of the pole must lie in $W$. Since $W$ is an arbitrary $k$-dimensional subspace of the $N$-dimensional vector space $E_{-p}$, the family of Hecke modifications of this type is a copy of $\text{Gr}(k, N)$, the Grassmannian of $k$-planes in $\mathbb{C}^N$.

For $G = PSU(N)$, we have $^tG = SU(N)$. A highest weight of $^tG$ is an $N$-tuple $^t w = (n_1, n_2, \ldots, n_N)$ now with $n_i \in \mathbb{Z}/N$ and $n_i - n_j \in \mathbb{Z}$, $\sum n_i = 0$. A generic Hecke modification at $p$ of a rank $N$ bundle $E_- = \mathcal{O} \oplus \cdots \mathcal{O}$ of type $^t w$ still maps it, formally, to

$$E_+ = \mathcal{O}(n_1 p) \oplus \mathcal{O}(n_2 p) \oplus \cdots \oplus \mathcal{O}(n_N p), \quad (4.31)$$

but now, because of the fractions in this formula, we should interpret this as a Hecke modification from $V_- = \text{ad}_0(E_-)$ to $V_+ = \text{ad}_0(E_+)$. The other remarks that we made in the $N = 2$ case also have close analogs. The nonzero minuscule weights of $SU(N)$ have
the form \( Lw = (1 - k/N, 1 - k/N, \ldots, 1 - k/N, -k/N, -k/N, \ldots, -k/N) \), with \( k \) copies of \( 1 - k/N \). This is the highest weight of the \( k^{th} \) exterior power of the \( N \)-dimensional representation of \( SU(N) \). The space of Hecke modifications dual to this representation is again a copy of \( \text{Gr}(k, N) \).

For \( G = SU(N) \), and thus \( ^tG = PSU(N) \), a highest weight \( Lw \) of \( ^tG \) is an \( N \)-plet \((n_1, n_2, \ldots, n_N)\) with \( n_i \in \mathbb{Z} \), \( \sum_i n_i = 0 \). The generic local action of \( T(Lw; p) \) is

\[
\mathcal{O} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O} \to \mathcal{O}(n_1p) \oplus \mathcal{O}(n_2p) \oplus \cdots \oplus \mathcal{O}(n_Np).
\]

(4.32)

There are no nonzero minuscule weights.

### 4.5 Incorporation Of The Higgs Field

#### 4.5.1 Generalities

In our application, we are really interesting in Hecke transformations of a Higgs bundle \((E, \varphi)\), with \( \varphi \in H^0(C, \text{ad}(E)) \), and not just of the bundle \( E \). In the presence of an ’t Hooft operator at a point \( p \times y_0 \in C \times \mathbb{R} \), the holomorphic type of the Higgs bundle will jump in crossing \( y = y_0 \), but in a way that is trivial if we omit the point \( p \) from \( C \). We write \((E_-, \varphi_-)\) for the Higgs bundle at \( y < y_0 \) and \((E_+, \varphi_+)\) for the Higgs bundle at \( y > y_0 \).

To understand what happens, we simply have to consider the implications of eqn. (4.17). This equation tells us that away from the point \( p \in C \), parallel transport in the \( y \) direction using the connection \( D_y - i\phi_s \) gives a holomorphic isomorphism from \((E_-, \varphi_-)\) to \((E_+, \varphi_+)\). We already know that this statement does not uniquely determine \( E_+ \) in terms of \( E_- \): a Hecke transformation may be made at the point \( p \), and in general the charge of the ’t Hooft operator at the point \( p \times y_0 \) does not uniquely determine what this Hecke transformation will be. But away from the point \( p \in C \), parallel transport in the \( y \) direction gives a distinguished isomorphism \( \Theta : E_- \to E_+ \). Moreover, since \( \varphi \) is invariant under that parallel transport, the relation between \( \varphi_+ \) and \( \varphi_- \) is just

\[
\varphi_+ = \Theta \varphi_- \Theta^{-1}.
\]

(4.33)

In these statements, \( \Theta \) is a gauge transformation away from the point \( p \in C \), but may have a singularity at \( p \).

The relation (4.33) ensures that \( \varphi_+ \) is holomorphic away from \( p \), since \( \varphi_- \) is holomorphic and \( \Theta \) and \( \Theta^{-1} \) are holomorphic away from \( p \). The extension of \( \varphi_+ \) over \( p \) as a holomorphic section of \( K \otimes \text{ad}(E) \) is unique if it exists. But generically this extension will not exist: \( \varphi_+ = \Theta \varphi_- \Theta^{-1} \) will have a pole at \( p \). Thus the possible Hecke transformations of a pair \((E_-, \varphi_-)\) are simply Hecke transformations of \( E_- \) that obey a condition such that \( \varphi_+ \) will not have a pole.

Even if \( \Theta \) is singular, the conjugacy (4.33) implies that invariant polynomials in \( \varphi_+ \) equal the corresponding invariant polynomials in \( \varphi_- \). For example, for \( G \) a unitary group,

\[
\text{Tr} \varphi^s_+ = \text{Tr} \varphi^s_- \quad \text{for all } s.
\]

(4.34)

The case that \( ^tG = G = U(1) \) is simple to describe, but too simple to really illustrate some of what we have just explained. In this case, \( \text{ad}(E) \) is a trivial line bundle over \( C \)
and $\varphi$ is simply a holomorphic 1-form, acted on trivially by $\Theta$. So $\varphi_+ = \varphi_-$, and if $\varphi_-$ is holomorphic at $p$, then so is $\varphi_+$. Thus for $U(1)$, a Hecke transformation of the pair $(E, \varphi)$ is simply a Hecke transformation of $E$, with no change in $\varphi$. To illustrate the implications of requiring that $\varphi_+$ has no pole at $p$, we need a nonabelian gauge group such as $U(2)$.

### 4.5.2 Minuscule Representation of $U(2)$

The simplest example that really illustrates the general story is $^L G = U(2)$, where for simplicity – and also because this example is important in our application – we take $^L w$ to be the minuscule weight $(1,0)$. The local action of $T(^L w)$, in some basis, is

$$O \oplus O \to O(p) \oplus O,$$

and this corresponds to

$$\Theta = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

Now consider for $y < y_0$ a Higgs field $\varphi \in H^0(C, K \otimes \text{ad}(E_-))$. With respect to the local trivialization of $E_-$ that is used in eqn. (4.35), we have

$$\varphi_- = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c$ and $d$ are local holomorphic sections of $K$. So

$$\varphi_+ = \Theta \varphi \Theta^{-1} = \begin{pmatrix} a & zb \\ z^{-1}c & d \end{pmatrix}.$$

Thus $\varphi_+$ has a pole at $p$ unless $c(p) = 0$.

The condition $c(p) = 0$ has a simple interpretation. We recall from eqn. (4.30) that an arbitrary Hecke modification of $E_-$ of type $^L w = (1,0)$ produces a bundle $E_+$ of which a local holomorphic section is

$$s = s_0 + \frac{\lambda}{z} \begin{pmatrix} u \\ v \end{pmatrix},$$

where $b = \begin{pmatrix} u \\ v \end{pmatrix}$ represents a point in $\mathbb{C}P^1$. In making the decomposition (4.35) and arriving at the form (4.36) for $\Theta$, we have taken $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. With this choice of $b$, the condition $c(p) = 0$ is equivalent to

$$\varphi_-(p) \cdot b = 0 \mod b.$$  

In other words, $\varphi_-(p) \cdot b$ is a multiple of $b$. This criterion for $\varphi_+$ to be holomorphic at $p$ holds for any $b$, not necessarily of the form $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with which we began.

We recall that the space of Hecke modification of $E_-$ at $p$ of type $^L w = (1,0)$ is a copy of $\mathbb{C}P^1$ obtained by projectivizing $E_{-,p}$. The Lie algebra $\text{ad}(E_{-,p})$ acts on this $\mathbb{C}P^1$; its
generators correspond to holomorphic vector fields on $\mathbb{CP}^1$. The condition (4.40) asserts that the point in $\mathbb{CP}^1$ given by $b = \left( \begin{array}{c} u \\ v \end{array} \right)$ is invariant under the symmetry of $\mathbb{CP}^1$ generated by $\varphi_{-}(p)$. The action of $\varphi_{-}(p)$ just rescales the homogeneous coordinates $u,v$ of this point.

We can summarize this as follows. In order for a Hecke transformation of a Higgs bundle $(E_{-},\varphi_{-})$ not to produce a pole in $\varphi_{+}(p)$, the Hecke modification of $E_{-}$ must be $\varphi_{-}$-invariant, meaning that it must be invariant under the symmetry generated by $\varphi_{-}$. This is a general condition that holds not just for the particular group $G = SU(2)$ and weight $t_{w} = (1,0)$ that we have considered, but for any group and representation. The criterion is easiest to understand and implement in the case of a minuscule weight, for then the symmetry of the space of Hecke modifications at $p$ that is generated by $\varphi_{-}$ depends only on $\varphi_{-}(p)$, the value of $\varphi_{-}$ at $p$. For a representation that is not minuscule, the analysis of the symmetry of the space of Hecke modifications generated by $\varphi$ is more complicated.

We have carried out this discussion in a way that treats $(E_{-},\varphi_{-})$ and $(E_{+},\varphi_{+})$ asymmetrically. Starting with $(E_{-},\varphi_{-})$, an ‘t Hooft operator $T(t_{w})$, inserted at some point $p \times y_{0} \in C \times \mathbb{R}$, induces a Hecke transformation with $(E_{+},\varphi_{+})$ as output. Looking at the same picture backwards, one can view $(E_{+},\varphi_{+})$ as input and $(E_{-},\varphi_{-})$ as output. If $\varphi_{-}$ and $\varphi_{+}$ are both free of poles at $p$, then the ‘t Hooft operator produces a $\varphi_{-}$-invariant or $\varphi_{+}$-invariant Hecke transformation depending on how one looks at it. To avoid committing ourselves to one point of view, we sometimes just say that the Hecke transformation is $\varphi$-invariant.

Now we consider a Higgs bundle $(E_{-},\varphi_{-})$ and ask how many $\varphi_{-}$-invariant Hecke modifications of type $(1,0)$ are possible at a given point $p$. In answering this question, we will assume that the $2 \times 2$ matrix $\varphi_{-}(p)$ has distinct eigenvalues; an equivalent statement is that $\text{Tr} \varphi(p)^{2} - \frac{1}{2} \text{Tr} \varphi(p)^{2} \neq 0$. In view of eqn. (4.34), this condition is satisfied by $\varphi_{+}$ if and only if it is satisfied by $\varphi_{-}$.

If an $N \times N$ complex matrix has $N$ distinct eigenvalues, we say that it is regular and semisimple. Such a matrix can be diagonalized by a complex-valued linear transformation. So if $\varphi_{-}$ is regular and semisimple, then in the right basis, it can be written near $p$ as

$$\varphi_{-} = \begin{pmatrix} \lambda_{1} & 0 \\ 0 & \lambda_{2} \end{pmatrix},$$

(4.41)

where $\lambda_{1}$ and $\lambda_{2}$ are sections of $K$. In this basis, a $\varphi_{-}(p)$-invariant Hecke transformation at $p$ simply corresponds to a point in $\mathbb{CP}^1$ with homogeneous coordinates $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So there are precisely two $\varphi_{-}(p)$-invariant Hecke modifications at $p$ if $\varphi_{-}(p)$ has distinct eigenvalues.

Finally, we come to a point that will be crucial in our application. Under the assumption that we have made, the two $\varphi(p)$-invariant Hecke modifications of type $(1,0)$ have a

\[ \text{Since } \varphi \text{ is valued not in the Lie algebra } \text{ad}(E) \text{ but in } \text{ad}(E) \otimes K, \text{ we have to pick a trivialization of } K \text{ near } p \text{ to think of } \varphi_{-}(p) \text{ as generating a symmetry of } \mathbb{CP}^1. \text{ But the choice of trivialization does not affect the condition of } \varphi_{-}(p) \text{ invariance.} \]
simple interpretation in terms of spectral covers. We recall that the $U(2)$ Higgs bundle $(E_-, \varphi_-)$ over $C$ can be derived from a branched double cover $\psi : D \to C$, together with a line bundle $\mathcal{L} \to D$. According to eqn. (2.25), $E_-$ is reconstructed from this data as $\psi_*(\mathcal{L})$, and $\varphi_-$ is similarly reconstructed as in eqn. (2.29). Let $q', q''$ be the two points in $D$ that lie above $p \in C$. They are distinct points since we have assumed that $\varphi(p)$ has distinct eigenvalues, and so the relation $E = \psi_*(\mathcal{L})$ gives simply $E_p = \mathcal{L}_{q'} \oplus \mathcal{L}_{q''}$.

We can extend this decomposition of $E$ at $p$ to a decomposition in a small neighborhood of $p$. In a neighborhood of $p \in C$, there are two sections $s_1 : C \to D$ and $s_2 : C \to D$, with $s_1(p) = q'$, $s_2(p) = q''$. In a neighborhood of $p$, we have

$$E \cong s_1^*(\mathcal{L}) \oplus s_2^*(\mathcal{L}).$$

To make a Hecke modification of $E$ of type $(1,0)$, we begin with a general decomposition of $E$ as $\mathcal{O} \oplus \mathcal{O}$ and replace one of the summands with $\mathcal{O}(p)$. However, if we want to get a $\varphi$-invariant Hecke modification, the summand that we modify must be one of the two $\varphi$-invariant summands in eqn. (4.43).

For example, a Hecke modification in which the first summand in eqn. (4.43) is modified by allowing a pole at $p$ will replace $s_1^*(\mathcal{L})$ with $s_1^*(\mathcal{L}) \otimes \mathcal{O}(p) = s_1^*(\mathcal{L}(q'))$. Thus it has the same effect as replacing $\mathcal{L}$ by $\mathcal{L}(q')$ on the spectral cover $D$. An analogous modification of the second summand in eqn. (4.43) is equivalent to replacing $\mathcal{L}$ with $\mathcal{L}(q'')$.

Thus the two possible $\varphi$-invariant Hecke modifications of type $L^w = (1,0)$ at a point $p$ at which $\varphi$ is regular and semisimple have a very simple interpretation in terms of the spectral cover $D \to C$. They correspond to replacing $\mathcal{L} \to D$ with either $\mathcal{L}(q')$ or $\mathcal{L}(q'')$, where $q'$ and $q''$ are the two points in $D$ that lie over $p$.

$$\mathcal{L} \to \begin{cases} \mathcal{L}(q') & \text{or} \\ \mathcal{L}(q''). \end{cases}$$

### 4.5.3 Nonminuscule Weights

The analysis of $\varphi$-invariant Hecke modifications involves much more technicality if $L^w$ is not a minuscule weight. In this case, the moduli space of Hecke modifications of type $L^w$ (the Schubert cell) is not compact, and has a natural compactification (the Schubert cycle) that involves monopole bubbling. One really needs to consider $\varphi$-invariant points on the compactification, and some of these points do lie at infinity on the Schubert cycle. So in the non-minuscule case, an analysis of Hecke modifications of Higgs bundles cannot be made without incorporating monopole bubbling.

In this paper, we will only aim to analyze the eigenbranes of ’t Hooft operators under the simplest conditions in which most of the technicalities do not arise. To this aim, we will consider only the comparatively elementary case that $L^w$ is a minuscule weight. We also make the assumption that has already been introduced above: we assume that $\varphi_-(p)$ is generic, meaning that it has distinct eigenvalues, or equivalently we assume that the point $p \in C$ is not a branch point of the spectral cover.
For $LG = U(N)$ or $PU(N) = SU(N)/\mathbb{Z}_N$, there are enough minuscule weights of $LG$ so that ’t Hooft operators dual to minuscule weights generate the ring of all ’t Hooft operators. This is far from true for other groups.

4.5.4 Analog For $SO(3)$ And For $SU(2)$

For $LG = SO(3)$, we can take $L_w$ to be the nonzero minuscule weight $(1/2, -1/2)$ of $G = SU(2)$. In this case, again assuming that $\varphi_-(p)$ has distinct eigenvalues (which for $SO(3) = PU(2)$ is equivalent to saying that it is not nilpotent), the analysis of $\varphi$-invariant Hecke modifications at $p$ of type $L_w$ is the same as for $U(2)$. There are two of them, and they can be described on the spectral cover as replacing $L$ by $L(q')$ or $L(q'')$.

For $LG = SU(2)$, the dual group $G = SO(3)$ has no nonzero minuscule weights, so the technicalities mentioned in section 4.5.3 are inescapable. Accordingly, we will not analyze magnetic eigenbranes for this group.

We should note that the technicalities associated with monopole bubbling are still relatively manageable if the representation $^R\mathfrak{r}$ has the property that each of its weight spaces has dimension at most 1. This condition is satisfied for an arbitrary representation of $SU(2)$ or $SO(3)$, so for these groups one can expect to extend the analysis we give of magnetic eigenbranes to arbitrary ’t Hooft operators without too much technicality. The more serious technicalities arise for groups of higher rank.

4.5.5 Generalization To $U(N)$

The discussion of section 4.5.2 can be readily adapted to $LG = U(N)$. Suppose that $\varphi(p)$ is regular and semisimple, meaning that it has $N$ distinct eigenvalues. Then near $p$, there are $N$ distinct solutions $\psi_i$ of the eigenvector equation

$$\varphi \cdot \psi_i = y_i \psi_i. \quad (4.45)$$

We can pick these to vary holomorphically in a neighborhood of $p$. Consider Hecke modifications of weight $L_w = (m_1, m_2, \ldots, m_N)$, where as usual $m_1 \geq m_2 \geq \cdots \geq m_N$. The general framework for such a Hecke modification was described in section 4.4.4, but to find a $\varphi$-invariant Hecke modification of this type, we have to be more precise. A $\varphi$-invariant Hecke modification $E_+$ of the given type can be described by saying that $E_+$ coincides with $E_-$ away from $p$, and near $p$ a holomorphic section of $E_+$ takes the form

$$s = \sum_{i=1}^{N} z^{-m_i} h_i \psi_i, \quad (4.46)$$

with $z$ a local parameter at $p$, and the functions $h_i$ being holomorphic at $p$. Choosing the basis functions $\psi_i$ to be eigenfunctions of $\varphi$ ensures that this particular Hecke modification is $\varphi$-invariant.

We can find $N!$ Hecke modifications that are $\varphi$-invariant by picking a permutation $\pi$ of the set $\{1, 2, \ldots, N\}$, and writing instead

$$s = \sum_{i=1}^{N} z^{-m_i} h_i \psi_{\pi(i)}. \quad (4.47)$$
By arguments similar to those that we have given already for \( U(2) \), one can show that these are the only \( \varphi \)-invariant Hecke modifications of weight \( Lw \). They are all inequivalent if the \( m_i \) are pairwise distinct, but if, say, \( m_i = m_j \), then exchanging \( s_i \) and \( s_j \) does not change the Hecke modification. So the number of \( \varphi \)-invariant Hecke modifications of \( E_- \) is \( N!/\#\Gamma \), where \( \#\Gamma \) is the order of the group \( \Gamma \) of permutations of the set \( \{1,2,\ldots,N\} \) that leaves the weights \( m_i \) invariant.

Alternatively, the number of \( \varphi \)-invariant Hecke modifications can be described as follows. The Weyl group \( W \) of \( U(N) \) is the group of permutations of the weights, which has \( N! \) elements. The weights of \( LR \) come in Weyl orbits. The number of weights in the orbit containing the highest weight is \( N!/\#\Gamma \), where \( \Gamma \) is the subgroup of \( W \) that leaves fixed the highest weight. This is the same result found in the last paragraph.

In the case that \( \varphi \) has \( N \) distinct eigenvalues, the \( \varphi \)-invariant Hecke modifications have the same sort of interpretation on the spectral curve that was described for \( U(2) \) in section 4.5.2. This may be seen as follows.

We recall from section 2.6 that the Higgs pair \((E,\varphi)\) is determined by a line bundle \( \mathcal{L} \to D \), where \( \pi : D \to C \) is the spectral cover. In particular, \( E = \pi_*(\mathcal{L}) \). We recall also that a Hecke modification changes \( E \) without changing the spectral curve. The change in \( E \) will come from a change in \( \mathcal{L} \), which we claim is as follows. Let \( q_1,\ldots,q_N \) be the points on \( D \) that lie above \( p \in C \). Then for a suitable ordering of the \( q_i \), the effect of the Hecke modification is

\[
\mathcal{L} \to \mathcal{L} \otimes (\otimes_{i=1}^N \mathcal{O}(q_i)^{m_i}).
\]

The different \( \varphi \)-invariant Hecke modifications of weight \( Lw \) come from the different orderings of the \( q_i \), or equivalently of the \( m_i \).

The idea behind this formula is very simple. At a point \( p \) at which the spectral cover \( \psi : D \to C \) is unramified, the inverse image of a small neighborhood \( U \) of \( p \) in \( C \) is a union of small open sets \( U_i \subset D \), each containing one point \( q_i \) lying over \( p \). The whole idea of the spectral cover is that locally, away from ramification points, it reduces \( U(N) \) gauge theory to \( U(1)^N \), with one \( U(1) \) on each sheet. A \( \varphi \)-invariant Hecke modification of \( E \to C \) corresponds on the spectral cover to a \( U(1) \) Hecke modification on the \( i^{th} \) sheet of weight \( m_i \) (or more generally of weight \( m_{\pi(i)} \) for some permutation \( \pi \)). Thus the Hecke modification acts on the \( i^{th} \) sheet by

\[
\mathcal{L} \to \mathcal{L} \otimes \mathcal{O}(mq_i) = \mathcal{L} \otimes \mathcal{O}(q_i)^{m_i}.
\]

This is the claim in eqn. (4.48).

The description that we have just given of \( \varphi \)-invariant Hecke modifications of type \( Lw \) is valid for any \( Lw \), but as usual it is less useful if the weight \( Lw \) is not minuscule, for in that case that are additional \( \varphi \)-invariant points on the compactification of the Schubert cell associated to monopole bubbling. Indeed, if the representation \( LR \) is not minuscule, then weights of this representation that are not on the Weyl orbit of the highest weight are associated to \( \varphi \)-invariant Hecke modifications whose description involves monopole bubbling.
5 Magnetic Eigenbranes

5.1 Preliminaries

In chapter 4.2, we analyzed the action of Wilson lines on branes and identified zero-branes as electric eigenbranes. Here, we will use the understanding of the action of ’t Hooft operators gained in chapter 4 to make a similar analysis of magnetic eigenbranes. We do so only under simplifying assumptions that were stated in section 4.5.3: we consider only ’t Hooft operators that are related to minuscule weights of the dual group, and we insert such an operator at a point that is not a branch point of the spectral cover.

There is a very important preliminary point about how we will make this analysis. In the geometric Langlands correspondence, one is really interested in $B$-branes on $M_H(G,C)$ in complex structure $J$, and their duality with $A$-branes on $M_H(L^C G,C)$ in symplectic structure $\omega_K$. One further wants to compare the action of Wilson operators on $B$-branes of $M_H(G,C)$ to the action of ’t Hooft operators on $A$-branes of $M_H(L^C G,C)$.

However, $M_H(G,C)$ is a hyper-Kahler manifold. The electric eigenbranes, as described in section 4.2, are zero-branes supported at a point $x \in M_H(G,C)$. A point is a complex submanifold in every complex structure, and in particular a zero-brane is a brane of type $(B,B,B)$ – that is, it is a $B$-brane in any of the complex structures on $M_H(G,C)$ that are part of its hyper-Kahler structure. Moreover, the natural Wilson operators of $\mathcal{N} = 4$ super Yang-Mills theory that act in the $B$-model of type $J$ are actually half-BPS Wilson operators that map branes of type $(B,B,B)$ to themselves. In particular, a zero-brane on $M_H(G,C)$ is not just an electric eigenbrane of type $J$, but is simultaneously an electric eigenbrane of type $I$, $J$, and $K$ (and more generally it is an electric eigenbrane in the $B$-model of any complex structure on $M_H(G,C)$ that is a linear combination of $I$, $J$, and $K$).

The $S$-dual of a half-BPS Wilson operator is a half-BPS ’t Hooft operator. Moreover, $S$-duality maps a brane of type $(B,B,B)$ to a brane of type $(B,A,A)$, that is a brane that is a $B$-brane in complex structure $I$ and simultaneously an $A$-brane in symplectic structure $\omega_J$ or $\omega_K$ (or in a linear combination of those two symplectic structures); and the half-BPS ’t Hooft operators map branes of type $(B,A,A)$ to branes of the same type. For further details on these assertions, see [1]. We refer to the $A$-model with symplectic structure $\omega_J$ or $\omega_K$ as the $A$-model of type $J$ or $K$. In particular, since a zero-brane on $M_H(G,C)$ is of type $(B,B,B)$, its dual will be a brane on $M_H(L^C G,C)$ that is an eigenbrane of type $(B,A,A)$ – that is, it will be an eigenbrane in each of the three indicated structures.

Concretely, since $S$-duality of $\mathcal{N} = 4$ super Yang-Mills theory acts by $T$-duality of the fibers of the Hitchin fibration, we can make a simple prediction for what the magnetic eigenbranes must be. $T$-duality maps a zero-brane supported on a particular fiber $L^C \mathfrak{g}$ of the Hitchin fibration of $L^C G$ to a rank 1 brane supported on the the corresponding fiber of the Hitchin fibration of $G$, which we call $\mathfrak{g}$. This brane has a flat Chan-Paton bundle $\mathcal{L} \rightarrow \mathfrak{g}$ of rank 1. A brane supported on a fiber of the Hitchin fibration and endowed with a flat Chan-Paton line bundle is what we will call a brane of type $F$. Thus, we expect that the magnetic eigenbranes will be branes of type $F$. 

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In geometric Langlands, one really cares about the magnetic eigenbranes as branes in the $A$-model of type $K$. But if $B$ is an $A$-brane of type $K$ that is actually a brane of type $(B, A, A)$, and if $T$ is an ’t Hooft operator that maps branes of type $(B, A, A)$ to themselves, then to determine the product $T \cdot B$, it suffices to identify this product as a brane in the $B$-model of type $I$. This is much simpler than trying to directly describe $T \cdot B$ as an $A$-brane of type $K$, and it is the way we will proceed.

Thus our method of identifying magnetic eigenbranes will use in an essential way the hyper-Kahler structure of $M_{H}(G, C)$ and $M_{H}(L_{G}, C)$. We will show that a brane of type $F$ is an eigenbrane in the $B$-model of type $I$, with an “eigenvalue” that is of the form required to match expectations from the dual description. It automatically follows that such a brane is an eigenbrane in the $A$-model of type $K$, with the same “eigenvalue.” For if some line operator $T$ and brane $B$ of type $(B, A, A)$ satisfy $T \cdot B = B \otimes V$, then we can read this statement equally either in the $B$-model of type $I$ or in the $A$-model of type $J$ or $K$.

### 5.2 Action On A Fiber Of The Hitchin Fibration

In eqn. (4.34), we showed that the characteristic polynomial of the Higgs field $\varphi$ is preserved by a Hecke modification – that is, by the action of an ’t Hooft operator. The fibers of the Hitchin fibration are labeled by this characteristic polynomial, and therefore an ’t Hooft operator will map a brane supported on a given fiber $\mathcal{F}$ of the Hitchin fibration to another brane supported on the same fiber.

We will consider only the case of branes supported on a smooth fiber $\mathcal{F}$ of the Hitchin fibration. Such a fiber is a complex torus. Moreover, it is a complex Lagrangian submanifold of $M_{H}(L_{G}, C)$, and therefore can readily be the support of a brane of type $(B, A, A)$. We simply endow $\mathcal{F}$ with a rank 1 Chan-Paton bundle that, from the point of view of complex structure $I$, is a holomorphic line bundle $L$ of zero\footnote{Because the normal bundle to $\mathcal{F}$ in $M_{H}$ is trivial, there is no analog of the shift due to the $K$-theory interpretation of branes that was described in section 2.6.4 for $A$-branes supported on $T^{*}C$. To be more exact, the closest such analog is the subtlety involving choice of a spin structure that is summarized in footnote 11.} first Chern class. $\mathcal{F}$ endowed with such a line bundle is a $B$-brane $B$ of type $I$, but since $L$ admits a natural flat connection, the brane $B$ also has the natural structure of an $A$-brane of type $J$ or $K$.

Incidentally, the fact that Hecke modifications map a fiber $\mathcal{F}$ of the Hitchin fibration to itself means that in a certain sense, Hecke modifications of Higgs bundles $(E, \varphi)$ are better-behaved than Hecke modifications of bundles alone. Starting with any bundle $E$, possibly stable, repeated Hecke modifications can produce an arbitrarily unstable bundle – in fact, they can produce an arbitrary bundle. However, it can be shown that if the characteristic polynomial of $\varphi$ is associated to a generic fiber of the Hitchin fibration, then every Higgs bundle $(E, \varphi)$ is stable.\footnote{A Higgs bundle $(E, \varphi)$ is stable if any non-trivial $\varphi$-invariant subbundle of $E$ obeys a condition that was stated in footnote 5. This condition is satisfied if the characteristic polynomial of $\varphi$ is irreducible, for then $E$ has no nontrivial $\varphi$-invariant subbundles at all.} Thus as long as we start on a generic fiber $\mathcal{F}$ of the Hitchin fibration, repeated Hecke modifications can be made without leaving the moduli space $M_{H}$ of stable Higgs bundles. Indeed, they can be made without leaving the fiber $\mathcal{F}$.
We can be much more precise than merely saying that a Hecke transformation maps a fiber $\mathfrak{F}$ to itself. In eqn. (4.44) for an ’t Hooft operator dual to a minuscule weight $(1,0)$ of $U(2)$, and in eqn. (4.48) more generally for any ’t Hooft operator of $U(N)$, we have determined precisely how the Hecke transformation acts on $\mathfrak{F}$. For instance, for the basic case of the minuscule weight of $U(2)$, the action is by $L \to L \otimes O(q')$ or $L \to L \otimes O(q'')$, where $q'$ and $q''$ are the two points in the spectral cover $\psi: D \to C$ that lie over $p \in C$. Tensoring with a fixed line bundle $O(q')$ or $O(q'')$ is an automorphism on the fiber $\mathfrak{F}$ of the Hitchin fibration, and this automorphism preserves the complex symplectic structure of $\mathfrak{F}$. That is concretely why the action of an ’t Hooft operator will map a brane of type $(B,A,A)$ supported on $\mathfrak{F}$ to another brane of the same type, also supported on $\mathfrak{F}$.

There is some subtlety in this last statement, because the line bundle $O(q')$ or $O(q'')$ has degree 1, and tensoring with this line bundle permutes the different components of Pic($D$), the group of holomorphic line bundles over $D$. We will take this into account in a more detailed analysis in section 7, but the basic conclusion holds that because tensoring with a fixed line bundle preserves the complex symplectic structure of Pic($D$), the action of an ’t Hooft operator will map a brane of type $(B,A,A)$ to another brane of the same type.

The basic setup in trying to identify how the ’t Hooft operator acts on the Chan-Paton bundle is that of fig. 1 of section 4, but now the vertical line $L$ represents an ’t Hooft operator parallel to the boundary of the worldsheet $\Sigma$. The ’t Hooft operator does not change the characteristic polynomial of the Higgs field $\varphi$, so it does not change the spectral cover $D$. But it changes the line bundle $L \to D$ that determines a given Higgs bundle $(E,\varphi)$.

Now consider an arbitrary brane $B$, with Chan-Paton bundle $U$, supported on a chosen fiber $\mathfrak{F}$ of the Hitchin fibration. We want to act with an ’t Hooft operator $T^{(tR,p)}$ and determine the Chan-Paton bundle $\hat{U}$ of the resulting brane $T^{(tR,p)} \cdot B$. For simplicity, we explain this first for the case that $tR$ corresponds to the minuscule weight $(1,0)$ of $tG = U(2)$.

If this line bundle over $D$ that determines a point on the fiber $\mathfrak{F}$ is $L$ to the left of the line $L$ in the figure, then it is $L \otimes O(q')$ or $L \otimes O(q'')$ near the boundary. This means that the sheaf $\hat{U}$, evaluated in a small neighborhood of a point on $\mathfrak{F}$ corresponding to $L$, is the direct sum of the sheaf $U$ evaluated at $L \otimes O(q')$ or $L \otimes O(q'')$. A more succinct way to say that is that
\[ \hat{U} = \Phi^\prime_\ast(U) \oplus \Phi''_\ast(U). \] (5.1)
Here $\Phi'$ and $\Phi''$ are respectively the automorphisms of $\mathfrak{F}$ that correspond to $L \to L \otimes O(q')$ and $L \to L \otimes O(q'')$.

The generalization of eqn. (5.1) for a minuscule weight of $G = U(N)$ is
\[ \hat{U} = \sum_i \Phi^*_i(U), \] (5.2)
where $\Phi_i: \mathfrak{F} \to \mathfrak{F}$ corresponds to the $\varphi$-invariant Hecke transformation $L \to L \otimes O(\sum_i m_i q_i)$ of eqn. (4.48). (As usual, for a non-minuscule weight, the analogous formula has additional contributions associated to monopole bubbling.)
For $G = G = U(1)$, matters are more simple. In this case, the spectral cover $D \to C$ is trivial; $D$ simply coincides with $C$. A charge $n$ 't Hooft operator inserted at $p$ acts on the fiber of the Hitchin fibration by $L \to L \otimes O(np)$. This gives an automorphism $\Phi$ of $\text{Pic}(C)$ and the action of the 't Hooft operator is by

$$U \to \tilde{U} = \Phi^*(U).$$  \hfill (5.3)

### 5.3 Translation Eigenbundles

To analyze the above formulas, we will need to understand the following situation. $F$ is a complex torus equipped with a flat line bundle $L \to F$. $\Phi : F \to F$ is a constant translation.

We are not quite in this situation in eqns. (5.1) and (5.3) because the operations $\Phi$, $\Phi'$, and $\Phi''$ involve tensoring $L$ with a line bundle over $D$ of nonzero degree. We will be in this situation, however, if we act with a product of 't Hooft operators carrying no net magnetic flux. (We explain in sections 6.5 and 7 how to think about the case that the 't Hooft operators do carry net magnetic flux.)

First consider simply the case of a torus $F$ (with no complex structure assumed) endowed with a flat line bundle $L$. Topologically, such a flat line bundle can be specified by giving the holonomies around one-cycles in $F$. These holonomies are obviously invariant under translations on $F$, so we conclude that if $\Phi$ is such a translation, then $\Phi^*(L)$ is isomorphic to $L$ as a flat line bundle.

This remains so if $F$ is a complex torus and $L$ is a holomorphic line bundle whose first Chern class vanishes. Such an $L$ admits a flat connection compatible with its holomorphic structure (compatibility means that the part of the connection of type $(0,1)$ is the $\partial$ operator determining the holomorphic structure of $L$), and the reasoning of the last paragraph applies.

We should, however, formulate this carefully. The group of translations of $F$ is a complex torus $\tilde{F}$. $\tilde{F}$ is isomorphic to $F$ once a base point $f_0$ in $F$ is picked. Let us denote a translation of $F$ additively as $f \to f + \tilde{f}$, with $f \in F$, $\tilde{f} \in \tilde{F}$. The statement now that $L$ is translation-invariant means that for $\tilde{f} \in \tilde{F}$,

$$\tilde{f}^*(L) = L \otimes N_{\tilde{f}},$$  \hfill (5.4)

where $N_{\tilde{f}}$ is a one-dimensional vector space that depends holomorphically on $\tilde{f}$. In other words, $N_{\tilde{f}}$ is the fiber at $\tilde{f}$ of a holomorphic line bundle $N \to \tilde{F}$. This holomorphic bundle is itself non-trivial.

We might describe (5.4) by saying that a flat line bundle on a complex torus is a “translation eigenbundle.” Its pullback under translation by $\tilde{f}$ is isomorphic to itself, but not canonically; the possible isomorphisms correspond to nonzero vectors in the 1-dimensional vector space $N_{\tilde{f}}$.

Now let us ask what holomorphic vector bundles $Y \to F$ are translation eigenbundles in the same sense:

$$\tilde{f}^*(Y) = Y \otimes N_{\tilde{f}}.$$  \hfill (5.5)
Writing $\mathcal{Y}_f$ for the fiber of $\mathcal{Y}$ at $f$, the condition is that there should be a holomorphically varying isomorphism

$$\mathcal{Y}_{f + \bar{f}} = \mathcal{Y}_f \otimes \mathcal{N}_{f + \bar{f}}. \quad (5.6)$$

Setting $f = f_0$, $f' = f + \bar{f}$, we have

$$\mathcal{Y}_{f'} = \mathcal{Y}_{f_0} \otimes \mathcal{N}_{f' - f_0}, \quad (5.7)$$

where $f' - f_0$ is the unique element of the translation group $\bar{F}$ that maps $f_0$ to $f'$.

Eqn. (5.7) tells us that a general translation eigenbundle $\mathcal{Y}$ is the tensor product of a fixed vector space (namely $\mathcal{Y}_{f_0}$) with a line bundle (whose fiber at $f'$ is $\mathcal{N}_{f' - f_0}$). In more physical terms, it means that any bundle of rank greater than 1 that is supported on $F$ and is a translation eigenbundle is a direct sum of identical copies of a fixed rank 1 eigenbundle supported on $F$.

These statements also have a partial converse. A line bundle $\mathcal{L} \to F$ with $c_1(\mathcal{L}) \neq 0$ does not admit a translation-invariant connection. A connection with translation-invariant curvature and holonomies must be flat (since curvature forces holonomies around non-contractible loops not to be translation-invariant). $\mathcal{L}$ admits such a connection if and only if $c_1(\mathcal{L}) = 0$.

Not only does a translation-invariant line bundle $\mathcal{L} \to F$ have $c_1(\mathcal{L}) = 0$, but the line bundle $\mathcal{N} \to \bar{F}$ that measures its translation “eigenvalue” likewise has $c_1(\mathcal{N}) = 0$. Indeed, as we see upon setting $\mathcal{Y} = \mathcal{L}$ in eqn. (5.7), $\mathcal{L}$ and $\mathcal{N}$ are essentially isomorphic, up to picking an identification between $F$ and $\bar{F}$ and tensoring with a fixed one-dimensional vector space.

### 5.4 Branes of type $F$ as Magnetic Eigenbranes

Now let us consider a brane $\mathcal{B}$ whose support is a fiber $\mathfrak{F}$ of the Hitchin fibration, and ask if it can be a magnetic eigenbrane.

Suppose first that $\mathcal{U}$ is of rank 1. Then $\mathcal{B}$ is what we have called a brane of type $F$, and we expect it to be a magnetic eigenbrane. From section 5.3, a flat bundle of rank 1 over $\mathfrak{F}$ is an eigenbundle for all translations, and in particular for the $\Phi_i$. Thus, we have $\Phi^*_i(\mathcal{U}) = \mathcal{U} \otimes \mathcal{N}_i$, for some one-dimensional complex vector spaces $\mathcal{N}_i$. By virtue of (5.2), it follows that

$$T(\mathcal{L}_w, p)\mathcal{B} = \mathcal{B} \otimes (\oplus_i \mathcal{N}_i), \quad (5.8)$$

so that $\mathcal{B}$ is a magnetic eigenbrane. We will compute the “eigenvalue” $\oplus_i \mathcal{N}_i$ in section 7, after some preliminaries.

Branes of higher rank supported on $\mathfrak{F}$ do not give any essentially new magnetic eigenbranes. The ’t Hooft operators are plentiful enough that a joint eigenbrane of the ’t Hooft operators is actually a full translation eigenbrane, and so as in eqn. (5.7), the Chan-Paton bundle of a general eigenbrane is just the tensor product of some line bundle over $\mathfrak{F}$ with a fixed vector space.
6 Determinant Line Bundles

We will now develop some mathematical techniques that will be helpful in computing the Hecke “eigenvalue” of a brane of type $F$.

Associated with a line bundle $L$ over a Riemann surface $D$ are the cohomology groups $H^0(D, L)$ and $H^1(D, L)$. The determinant line of $L$ is defined as

$$\det H^*(L) = \det H^0(D, L)^{-1} \otimes (\det H^1(D, L)).$$  \hfill (6.1)

As $L$ varies, its determinant line varies as the fiber of a line bundle $\text{Det}$ over $\text{Pic}(D)$ that is known as the determinant line bundle. The determinant line bundle is most familiar to physicists for its role in the study of two-dimensional chiral fermions.

6.1 Pairing Of Line Bundles

For our purposes, the utility of the determinant line bundle is that it enables one to define a sort of bilinear pairing of line bundles. If $L$ and $M$ are two line bundles over $D$, one defines

$$\langle L, M \rangle = \det H^*(D, L) \otimes \det H^*(D, M).$$  \hfill (6.2)

(If $V$ and $W$ are two one-dimensional vector spaces, we write $V \otimes W^{-1}$ as a fraction $V/W$.)

There are obvious natural isomorphisms

$$\langle L, M \rangle \cong \langle M, L \rangle$$ \hfill (6.3)

$$\langle L, O \rangle \cong \langle O, L \rangle \cong \mathbb{C}.$$ \hfill (6.4)

Here and in the rest of this analysis, the symbol $\cong$ refers to an isomorphism that can be defined in a universal way and which therefore also gives an isomorphism in families.

The most important property of the symbol $\langle \ , \ \rangle$ is that it is bilinear in the sense that for any three line bundles $L$, $M$, and $N$, we have a canonical isomorphism

$$\langle L, M \otimes N \rangle \cong \langle L, M \rangle \otimes \langle L, N \rangle.$$  \hfill (6.5)

Any line bundle $N$ takes the form $\otimes_i \mathcal{O}(q_i)^{n_i}$ for some divisor $\mathcal{D} = \sum_i n_i q_i$, and (6.4) follows by induction in $\sum_i |n_i|$ if it is true for the special cases $N = \mathcal{O}(q)^{\pm 1}$.

For $N = \mathcal{O}(q)^{-1}$, we have $L \otimes N = L(-q)$, the line bundle whose sections are sections of $L$ that vanish at $q$. We can compare the cohomology of $L$ and $L(-q)$ using the exact sequence of sheaves

$$0 \rightarrow L(-q) \rightarrow L \rightarrow L|_q \rightarrow 0$$ \hfill (6.6)

Our notation is a bit informal. We write $L(-q)$ and $L$ for either the indicated line bundle or the corresponding sheaf of sections, while $L|_q$ means the fiber of $L$ at $q$ and also the
corresponding skyscraper sheaf at \( q \). The map \( r \) in (6.5) is defined by evaluating a section of \( L \) at the point \( q \). From (6.5) we get a long exact sequence of cohomology groups

\[
0 \to H^0(D, L(-q)) \to H^0(D, L) \to L|_q \\
\to H^1(D, L(-q)) \to H^1(D, L) \to 0.
\]

(6.6)

Now in general, a long exact sequence of vector spaces

\[
0 \to A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} \ldots \xrightarrow{d_{n-1}} A_n \to 0
\]

(6.7)
determines an isomorphism

\[
\bigotimes_{i=0}^n (\det A_i)^{(-1)^i} \cong \mathbb{C}.
\]

(6.8)

In the present case, from (6.6), we get an isomorphism

\[
\det H^*(D, L(-q)) \cong \det H^*(D, L) \otimes L|_q.
\]

(6.9)

This is equivalent to

\[
\langle L, \mathcal{O}(-q) \rangle \cong L|_q^{-1} \otimes \frac{\det H^*(D, \mathcal{O}(-q))}{\det H^*(D, \mathcal{O})}.
\]

(6.10)

The special case of this with \( L = \mathcal{O} \) tells us that

\[
\frac{\det H^*(D, \mathcal{O}(-q))}{\det H^*(D, \mathcal{O})} \cong \mathbb{C},
\]

(6.11)

so actually

\[
\langle L, \mathcal{O}(-q) \rangle \cong L|_q^{-1}.
\]

(6.12)

If in (6.9) we replace \( L \) by \( M \) or by \( L \otimes M \), we learn that

\[
\det H^*(D, M(-q)) \cong \det H^*(D, M) \otimes M|_q
\]

(6.13)

\[
\det H^*(D, L \otimes M(-q)) \cong \det H^*(D, L \otimes M) \otimes (L \otimes M)|_q.
\]

Now if one writes out the definition of \( \langle L, M \otimes N \rangle \) for \( N = \mathcal{O}(-q) \), uses (6.13) to everywhere eliminate \( M(-q) \) in favor of \( M \), and evaluates \( L|_q^{-1} \) via (6.12), then one arrives at

\[
\langle L, M(-q) \rangle \cong \langle L, M \rangle \otimes \langle L, \mathcal{O}(-q) \rangle,
\]

(6.14)

which is (6.4) for \( N = \mathcal{O}(-q) \).

If we set \( M = \mathcal{O}(q) \) in (6.14), we learn that

\[
\langle L, \mathcal{O}(q) \rangle \cong \langle L, \mathcal{O}(-q) \rangle^{-1}.
\]

(6.15)

Replacing \( M \) by \( M(q) \) in (6.14), and using (6.15), one arrives at (6.4) for \( N = \mathcal{O}(q) \). So by induction, this result holds in general. We should say, however, that the explanation we have given is a little naive, as we have not proved that we get the same isomorphism regardless of how we choose to represent \( N \) as a tensor product of elementary factors \( \mathcal{O}(q_i)^{\pm 1} \). A complete treatment can be found in [40].
Finally, let us note that this result can be written more symmetrically. For any line bundle $\mathcal{L} \to D$, let us abbreviate $\det H^*(D, \mathcal{L})$ as $[\mathcal{L}]$. As a further abbreviation, let us omit the symbol for a tensor product. Then (6.4) amounts to the statement that for any three line bundles $\mathcal{L}, \mathcal{M}, \mathcal{N}$, one has canonically

$$\frac{[\mathcal{L}, \mathcal{M}, \mathcal{N}] [\mathcal{L}] [\mathcal{M}] [\mathcal{N}]}{[\mathcal{L}, \mathcal{M}] [\mathcal{M}, \mathcal{N}] [\mathcal{L}, \mathcal{N}] [\mathcal{O}]} \cong \mathbb{C}. \quad (6.16)$$

This has been called the theorem of the cube. The name reflects the fact that the eight factors on the left hand side can conveniently be arranged on the corners of a cube.

### 6.2 Interpretation

Now we can evaluate $\langle \mathcal{L}, \mathcal{M} \rangle$ in general. Suppose that $\mathcal{M} = \bigotimes_{i=1}^k \mathcal{O}(q_i)^{n_i}$, or in other words $\mathcal{M} = \mathcal{O}(D)$, where $D = \sum_i n_i q_i$ is a divisor of degree $d = \sum_i n_i$. An induction in $\sum_i |n_i|$ based on (6.4) and using the special cases (6.12) and (6.15) gives

$$\langle \mathcal{L}, \mathcal{M} \rangle = \bigotimes_{i=1}^k \mathcal{L}|_{n_i} \otimes \mathcal{M}|_{q_i}, \quad (6.17)$$

with as usual $\mathcal{L}|_{q_i}$ the fiber of $\mathcal{L}$ at the point $q_i$.

Suppose that $\mathcal{L}$ is of degree $c$. We want to keep $\mathcal{M}$ fixed and let $\mathcal{L}$ vary. $\mathcal{L}$ determines a point $x \in \text{Pic}_c(D)$, and to make this explicit, we will write $\mathcal{L}$ as $\mathcal{L}_x$. As $x$ varies, we want to interpret $\langle \mathcal{L}_x, \mathcal{M} \rangle$ as the fiber of a line bundle over $\text{Pic}_c(D)$. There is a crucial subtlety here, which one often encounters when one considers moduli spaces of line bundles. This is that a point $x \in \text{Pic}_c(D)$ determines a line bundle $\mathcal{L}_x \to D$ only up to isomorphism. $\mathcal{L}_x$ could be replaced by $\mathcal{L}_x \otimes \mathcal{R}$, where $\mathcal{R}$ is a fixed one-dimensional vector space. This is the general freedom to change $\mathcal{L}_x$, in the following sense. If $\mathcal{U}$ and $\mathcal{U}'$ are isomorphic line bundles on a Riemann surface $D$, there is no canonical isomorphism between them (since either one has a $\mathbb{C}^*$ group of automorphisms, which can be composed with any proposed isomorphism between $\mathcal{U}$ and $\mathcal{U}'$). However, there is a canonical isomorphism $\mathcal{U}' \cong \mathcal{U} \otimes \mathcal{R}$, where $\mathcal{R}$ is the one-dimensional vector space $H^0(D, \mathcal{U}' \otimes \mathcal{U}^{-1})$. (The isomorphism is made by observing that a vector in $\mathcal{R}$ gives, by definition of $H^0(D, \mathcal{U}' \otimes \mathcal{U}^{-1})$, a holomorphic map from $\mathcal{U}$ to $\mathcal{U}'$. So, once the isomorphism class of $\mathcal{L}_x$ is given, it is unique up to $\mathcal{L}_x \to \mathcal{L}_x \otimes \mathcal{R}$.

In view of (6.17), we have

$$\langle \mathcal{L}_x \otimes \mathcal{R}, \mathcal{M} \rangle = \langle \mathcal{L}_x, \mathcal{M} \rangle \otimes \mathcal{R}^d. \quad (6.18)$$

So $\langle \mathcal{L}_x, \mathcal{M} \rangle$ is canonically determined by the isomorphism class of $\mathcal{L}_x$ if and only if $d = 0$.

A related remark is that an automorphism of $\mathcal{L}_x$ that acts as multiplication of a complex number $\lambda \in \mathbb{C}^*$ acts on $\langle \mathcal{L}_x, \mathcal{M} \rangle$ as multiplication by $\lambda^d$. So automorphisms of $\mathcal{L}_x$ act trivially on $\langle \mathcal{L}_x, \mathcal{M} \rangle$ if and only if $\mathcal{M}$ has degree zero.

For $d = 0$, therefore, we do get, for each $\mathcal{M} = \bigotimes_i \mathcal{O}(q_i)^{n_i}$, a line bundle $\mathcal{N}$ over $\text{Pic}_c(D)$. This is the line bundle whose fiber at $x \in \text{Pic}_c(D)$ is

$$\mathcal{N}|_x = \bigotimes_{i=1}^k \mathcal{L}_x|_{n_i} \otimes \mathcal{M}|_{q_i}, \quad (6.19)$$


This is an elementary definition that we could have made without first discussing determinant line bundles. What we have learned from that discussion is that, up to a natural isomorphism, the line bundle $N$ defined in this way depends only on the isomorphism class of the line bundle $M = \mathcal{O}(D)$, and not on the specific divisor $D = \sum_i n_i q_i$ by which we represent this line bundle.

What happens if the degree $d$ of $M$ is not zero? To make a similar definition, we have to first pick a universal bundle line over $D \times \text{Pic}_c(D)$. We recall this notion from section 4.2: such a universal line bundle is a line bundle $W \rightarrow D \otimes \text{Pic}_c(D)$ whose restriction to $D \times x$, for any $x \in \text{Pic}_c(D)$, is of the isomorphism type corresponding to $x$. Such a universal line bundle exists, but is unique only up to

$$W \rightarrow W \otimes \pi_2^*(\mathcal{R}),$$

(6.20)

where $\mathcal{R} \rightarrow \text{Pic}_c(D)$ is some line bundle, and $\pi_2 : D \times \text{Pic}_c(D) \rightarrow \text{Pic}_c(D)$ is the projection.

Eqn. (6.19) can be nicely rewritten in terms of a universal bundle $W$. Given $M = \otimes \mathcal{O}(q_i)^{n_i}$, the corresponding line bundle over $\text{Pic}_c(D)$ is

$$N = \otimes_i W|_{q_i \times \text{Pic}_c(D)}.$$  

(6.21)

To recover eqn. (6.19) from this formula, one simply identifies $W|_{q_i \times x}$ with $L_x|_{q_i}$. This is the best we can do to generalize eqn. (6.19) for the case that $M$ has a nonzero degree $d$. However, if $W$ is transformed to $W \otimes \mathcal{R}$, for some line bundle $\mathcal{R} \rightarrow \text{Pic}_c(D)$, then $N$ is transformed by

$$N \rightarrow N \otimes \mathcal{R}^d.$$  

(6.22)

The dependence on the choice of a universal bundle means that $N$ is not really naturally defined as a line bundle over $\text{Pic}_c(D)$, but is more naturally understood as a twisted line bundle, twisted by a certain gerbe. (This is a tautology: one can define a gerbe over $\text{Pic}_c(D)$ that is naturally trivialized by any choice of a universal line bundle over $D \times \text{Pic}_c(D)$, and then eqn. (6.22) means that $N \rightarrow \text{Pic}_c(D)$ is best understood as a twisted line bundle, twisted by the $d^{th}$ power of that gerbe. See section 6.3.) The motivation for this formulation will hopefully be clear in section 7.

Our discussion so far has been asymmetric. We have considered the symbol $\langle L, M \rangle$, with $L$ and $M$ being respectively of degree $c$ and degree $d$. We have kept $M$ fixed and let $L$ vary, and found that in this case, $\langle L, M \rangle$ varies as a twisted line bundle of degree $d$ over $\text{Pic}_c(D)$. Obviously, by symmetry, if we keep $L$ fixed and let $M$ vary, we will get a twisted line bundle of degree $c$ over $\text{Pic}_d(D)$. So when we let both $L$ and $M$ vary, we get a twisted line bundle over $\text{Pic}_c(D) \times \text{Pic}_d(D)$ of bidegree $(d, c)$. After a digression on gerbes, we will reformulate this assertion in the language of duality.

### 6.3 Gerbes

“Gerbes” have figured in this paper at several points. We will give a minimal explanation of this concept, explaining only the simplest points that might be helpful for understanding this paper (and in particular as background for section 6.4).
For our purposes, a “gerbe” is associated to the group $\mathbb{C}^*$ or a subgroup $U(1)$ or $\mathbb{Z}_n$. A gerbe $\mathcal{G}$ over a space $X$ is trivial locally but possibly not globally. (Most gerbes of interest to us are trivial globally but not canonically so.) Two local trivializations of a gerbe over an open set $U \subset X$ differ by tensoring by a line bundle $\mathcal{L} \to U$. We will not explain this statement in an abstract way, but in examples the meaning will be clear. For a $U(1)$ gerbe, $\mathcal{L}$ has always a hermitian metric (so its structure group reduces to $U(1)$). For a $\mathbb{Z}_n$ gerbe, $\mathcal{L}$ always possesses an isomorphism $\mathcal{L}^n \cong \mathcal{O}$. If a gerbe has a connection, then the line bundle $\mathcal{L}$ associated to a local change of trivialization will also have a connection.

Here are the examples that have arisen in this paper. If $\mathcal{C}$ is a Riemann surface, there is a $\mathbb{Z}_2$ gerbe over $\mathcal{C}$, mentioned in footnote 11, whose trivializations are spin structures on $\mathcal{C}$. It is subtle to describe what is a spin structure, but two spin structures differ by twisting by a line bundle $\mathcal{L}$ of order 2, so this is a $\mathbb{Z}_2$ gerbe. Global spin structures exist, so this gerbe is trivial, but not canonically. For a second example, consider universal line bundles over $\mathcal{C} \times \text{Pic}_c(\mathcal{C})$ for some $c \in \mathbb{Z}$. Any two such universal line bundles differ by tensoring with a line bundle $\mathcal{R} \to \text{Pic}_c(\mathcal{C})$, which tells us that there is a gerbe over $\text{Pic}_c(\mathcal{C})$ that is trivialized by any choice of a universal line bundle over $\mathcal{C} \times \text{Pic}_c(\mathcal{C})$. Such universal line bundles exist, so the gerbe in question is trivial, but not canonically. To give a similar example with a nontrivial gerbe, let $\mathcal{G}$ be a simple nonabelian group with nontrivial center $\mathbb{Z}_n$. Then there is a gerbe over the moduli space $\mathcal{M}(\mathcal{G}, \mathcal{C})$ of $\mathcal{G}$ bundles\(^{19}\) that is trivialized over an open set $U \subset \mathcal{M}_H(\mathcal{G}, \mathcal{C})$ by the choice of a universal $\mathcal{G}$-bundle over $\mathcal{C} \times U$. Such universal bundles exist locally, and any two choices differ by twisting by a line bundle $\mathcal{L} \to U$ that is of order $n$. So the problem of finding a universal bundle in this situation defines a $\mathbb{Z}_n$ gerbe over $\mathcal{M}_H(\mathcal{G}, \mathcal{C})$. This gerbe is topologically non-trivial, since a universal bundle does not exist globally. For more on this example, see section 7 of [1]. This is the only nontrivial gerbe that is relevant in the present paper.

If a gerbe $\mathcal{G}$ over a space $X$ is globally trivial, then a trivialization of it is called a twisted line bundle. The motivation for this terminology is as follows. The difference between two trivializations is a line bundle $\mathcal{L} \to X$, but any one trivialization is not such a line bundle, so we call it instead a twisted line bundle,\(^{20}\) twisted by $\mathcal{G}$.

Now let us discuss connections on a $U(1)$ gerbe. If a $U(1)$ gerbe $\mathcal{G}$ is trivialized, then a connection can be represented simply by a two-form $B$. (This is analogous to the fact that if a line bundle $\mathcal{L} \to X$ is trivialized, then a connection on $\mathcal{L}$ can be represented by a 1-form $A$. Connections make sense more generally on non-trivial gerbes, but we will not need this notion.) A change of trivialization is accomplished by twisting by a line bundle $\mathcal{L} \to X$ that has a connection $A$. Let $F = dA$ be the corresponding curvature. Under the change of trivialization by $\mathcal{L}$, the two-form $B$ is shifted by $B \to B + F$. Here $B + F$ is the \(^{19}\)All statements we are about to make apply if $\mathcal{M}(\mathcal{G}, \mathcal{C})$ is replaced by the corresponding Higgs bundle moduli space $\mathcal{M}_H(\mathcal{G}, \mathcal{C})$.

\(^{20}\)If $\mathcal{G}$ is a non-trivial gerbe, then by definition a $\mathcal{G}$-twisted line bundle does not exist. However, there is a notion of a $\mathcal{G}$-twisted vector bundle, and a non-trivial gerbe may admit a twisted vector bundle. For instance, given a representation $\mathcal{R}$ of $\mathcal{G}$ on which the center of $\mathcal{G}$ acts non-trivially, a universal bundle $\mathcal{E}_{LR} \to C \times \mathcal{M}(\mathcal{G}, \mathcal{C})$ it does not exist as a vector bundle, but it does exist as a $\mathcal{G}$-twisted vector bundle, where $\mathcal{G}$ is the gerbe that is trivialized locally by a choice of a universal bundle. This statement is a fancy tautology. See section 7 of [1] for an elementary explanation.
connection form on $\mathcal{G}$ relative to the new trivialization. By a flat $\mathcal{G}$-twisted line bundle (or equivalently a flat trivialization of $\mathcal{G}$) we mean a trivialization such that the new connection form $B + F$ is identically 0. In order for a flat $\mathcal{G}$-twisted line bundle to exist, the three-form curvature $H = dB$ (which is invariant under $B \to B + F$) must vanish – in which case we call $\mathcal{G}$ a flat gerbe – and the periods of the two-form $B/2\pi$ must be integers. If flat $\mathcal{G}$-twisted line bundles exist, there may not be a canonical way to pick one, but any two differ by twisting by an ordinary flat line bundle.

6.4 Duality

In section 3, we introduced the notion of a duality between complex tori $F$ and $F'$. Such a duality means that $F$ parametrizes flat line bundles over $F'$, and vice-versa. It is most usefully expressed by exhibiting a Poincaré line bundle, which is a line bundle $T \to F \times F'$ whose main property is that its restriction to $f \times F'$, for $f \in F$, is the line bundle over $F'$ labeled by $f$, and similarly with the roles of $F$ and $F'$ reversed. Thus, the restriction to $f \times F'$ gives, as $f$ varies, a universal family of flat line bundles over $F'$, and similarly with the two factors exchanged.

We can extend this to a notion of a twisted duality. Suppose that two tori $F$ and $F'$ are endowed with flat $U(1)$ gerbes $\mathcal{G}$ and $\mathcal{G}'$ and suppose that these flat gerbes are trivial (as flat gerbes, not just topologically). So $\mathcal{G}$ or $\mathcal{G}'$ can be trivialized, but not canonically, by choosing a twisted line bundle, which moreover we can choose to be flat. What we will call a twisted duality between $F$ and $F'$ of type $(\mathcal{G}, \mathcal{G}')$ identifies $F'$ as the moduli space of flat trivializations of $\mathcal{G}$, and $F$ as the moduli space of flat trivializations of $\mathcal{G}'$. As in the ordinary case, the most natural way to describe a twisted duality is to exhibit a twisted Poincaré line bundle, twisted by the gerbe $\mathcal{G} \otimes \mathcal{G}'$ over $F \times F'$, whose restriction to $F \times f'$ (or $f \times F'$) gives, as $f'$ (or $f$) varies, a universal family of flat trivializations of $\mathcal{G}$ (or $\mathcal{G}'$).

We claim that the symbol $\langle \mathcal{L}, \mathcal{M} \rangle$ gives a canonical twisted Poincaré line bundle of bidegree $(d, c)$ over $\text{Pic}_c(D) \times \text{Pic}_d(D)$. Since the association $\mathcal{L}, \mathcal{M} \to \langle \mathcal{L}, \mathcal{M} \rangle$ is certainly canonical, and is twisted in the appropriate way, all we have to show is that it is a duality. The general case can be mapped, noncanonically, to the case $c = d = 0$; we simply map $\text{Pic}_c(D)$ and $\text{Pic}_d(D)$ to $\text{Pic}_0(D) = \text{Jac}(D)$ by picking basepoints (that is, by picking particular flat line bundles of degree $c$ or $d$ that we map to the origin in $\text{Jac}(D)$) and trivializations of the gerbes. So to show that $\langle \mathcal{L}, \mathcal{M} \rangle$ defines a canonical twisted duality of degree $(d, c)$, we just have to show that it does define a duality for $c = d = 0$.

As this is a significant fact, we will explain it in two ways. First, in differential geometry, to describe a family of holomorphic line bundles over a Riemann surface $D$, one can consider a fixed smooth complex line bundle $\mathcal{U}$ with a family of unitary connections. If we write the connection on $\mathcal{U}$ as $U$, then (6.23) the determinant line bundle over a family of complex line bundles obtained by letting $U$ vary has a natural hermitian metric and a natural unitary connection whose curvature is

$$
\frac{i}{4\pi} \int_D \delta U \wedge \delta U.
$$

(6.23)
Now we recall the definition of $\langle \mathcal{L}, \mathcal{M} \rangle$:

$$
\langle \mathcal{L}, \mathcal{M} \rangle = \frac{\det H^*(D, \mathcal{L}) \otimes \det H^*(D, \mathcal{M})}{\det H^*(D, \mathcal{L} \otimes \mathcal{M}) \otimes \det H^*(D, \mathcal{O})}
$$

We think of $\mathcal{L}$ and $\mathcal{M}$ as fixed smooth line bundles with connections $A$ and $B$, respectively. Then $\mathcal{L} \otimes \mathcal{M}$ is a fixed smooth line bundle with connection $A + B$. The determinant line bundles of $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{L} \otimes \mathcal{M}$ have natural connections whose curvatures are obtained, respectively, by substituting $A$, $B$, or $A+B$ for $U$ in (6.23). Then by taking the appropriate linear combination of these curvatures in view of the definition of $\langle \mathcal{L}, \mathcal{M} \rangle$, we learn that the line bundle given by $\langle \mathcal{L}, \mathcal{M} \rangle$ has a natural connection of curvature

$$
- \frac{i}{2\pi} \int_D \delta A \wedge \delta B.
$$

(6.25)

So far we have a line bundle over $\mathcal{A} \times \mathcal{A}'$, where the two factors are, respectively, the spaces of all connections on $\mathcal{L}$ and on $\mathcal{M}$. We want to descend to a line bundle over $\text{Jac}(D) \times \text{Jac}(D)$. The group of gauge transformations of $\mathcal{L}$ and $\mathcal{M}$ acts on $\mathcal{A} \times \mathcal{A}'$, and this action lifts to an action on the line bundle $\langle \mathcal{L}, \mathcal{M} \rangle$. We want to divide by the gauge group. For this, we have to take $\mathcal{L}$ and $\mathcal{M}$ to have degree zero. Otherwise, the constant gauge transformations, which act trivially on $A$ and $B$, will act nontrivially on $\langle \mathcal{L}, \mathcal{M} \rangle$, as was explained following (6.18), and there will be no reasonable quotient. Once we restrict $\mathcal{L}$ and $\mathcal{M}$ to have degree zero, we can restrict the connections $A$ and $B$ to be flat, and then the curvature form (6.25) descends to the quotient by the group of gauge transformations, as we explained following eqn. (3.2). At this point, we can take the quotient by the gauge transformations acting on the line bundle $\langle \mathcal{L}, \mathcal{M} \rangle$ and its connection, to get a line bundle over $\text{Jac}(D) \times \text{Jac}(D)$ whose curvature can be represented by the same formula (6.25). As we noted in discussing eqn. (3.2), a unitary line bundle over $\text{Jac}(D) \times \text{Jac}(D)$ with this curvature is a Poincaré line bundle.

For an alternative approach closer to algebraic geometry, pick a basepoint $Q \in D$ and consider the embedding $\theta : D \to \text{Jac}(D)$ that maps $R \in D$ to the point in $\text{Jac}(D)$ corresponding to the degree zero line bundle $O(R) \otimes O(Q)^{-1}$. Since this map gives an isomorphism $H_1(D, \mathbb{Z}) \cong H_1(\text{Jac}(D), \mathbb{Z})$, and flat line bundles are classified by homomorphisms of $H_1$ into $U(1)$, there is a one-to-one correspondence $W \to \theta^*(W)$ between flat line bundles $W \to \text{Jac}(D)$ and flat line bundles $\mathcal{M} \to D$, or equivalently holomorphic line bundles $\mathcal{M} \to D$ of degree zero.

To establish the duality, we think of the association $\mathcal{L} \to \langle \mathcal{L}, \mathcal{M} \rangle$, with $\mathcal{M}$ held fixed and $\mathcal{L}$ allowed to vary, as defining a line bundle $W_\mathcal{M}$ over $\text{Jac}(D)$. We want to show that, up to isomorphism, each flat line bundle $W \to \text{Jac}(D)$ is isomorphic to precisely one of the $W_\mathcal{M}$. But in view of the remark in the last paragraph, it suffices to show that $\theta^*(W) = \theta^*(W_\mathcal{M})$ for some unique $\mathcal{M}$. We claim that, for any degree zero line bundle $\mathcal{M} \to D$, we have

$$
\theta^*(W_\mathcal{M}) \cong \mathcal{M}.
$$

(6.26)

So we will have $\theta^*(W) \cong \theta^*(W_\mathcal{M})$ if and only if we take $\mathcal{M} = \theta^*(W)$.
To establish the claim (6.26), we must understand \( \langle \mathcal{L}, \mathcal{M} \rangle \) for \( \mathcal{L} = \mathcal{O}(R) \otimes \mathcal{O}(Q)^{-1} \). For this, we simply use (6.17), but now with the roles of \( \mathcal{L} \) and \( \mathcal{M} \) reversed, to learn that \( \langle \mathcal{O}(R) \otimes \mathcal{O}(Q^{-1}), \mathcal{M} \rangle \cong \mathcal{M}|_R \otimes \mathcal{M}|_{Q^{-1}}^{-1} \). We want to keep \( Q \) fixed (so \( \mathcal{M}|_{Q^{-1}}^{-1} \) is an inessential fixed one-dimensional vector space \( \mathcal{R} \)) and let \( R \) vary. The line bundle whose fiber at \( R \) is \( \mathcal{M}|_R \otimes \mathcal{M}|_{Q^{-1}}^{-1} \) is simply \( \mathcal{M} \otimes \mathcal{R} \). In other words, it is isomorphic to \( \mathcal{M} \), as we aimed to show.

6.5 More On Duality Of Hitchin Fibers

In the context of the spectral cover \( \psi : D \to C \), self-duality of the Jacobian of \( D \) means that the fiber of the Hitchin fibration is self-dual for the self-dual group \( U(N) \). Now let us reconsider duality of the Hitchin fibers for the groups \( SU(N) \) and \( PSU(N) \), using the spectral cover \( \psi : D \to C \).

We have already explained that the symbol \( \langle \mathcal{L}, \mathcal{M} \rangle \), for \( \mathcal{L} \in \text{Pic}_c(D) \), \( \mathcal{M} \in \text{Pic}_d(D) \), gives a twisted duality of degrees \( (d, c) \) between \( \text{Pic}_c(D) \) and \( \text{Pic}_d(D) \). This is the self-duality of the Hitchin fibers for \( G = U(N) \), generalized to allow electric and magnetic fluxes. (This interpretation is explained in section 7.1.) Now we claim that if we restrict the first variable to lie in the Hitchin fiber for \( SU(N) \), the second will naturally project to \( PSU(N) \), and we will get the desired twisted duality between \( SU(N) \) and \( PSU(N) \). So we require \( \mathcal{L} \) to take values in \( \mathfrak{g}_{SU(N)} \); in other words, we take \( \mathcal{L} \in \text{Jac}(D) \) and require \( \text{Nm}(\mathcal{L}) \) to be trivial. As we will show shortly, if \( \mathcal{M}_0 \) is a line bundle over \( C \) of degree zero, then \( \langle \mathcal{L}, \psi^*(\mathcal{M}_0) \rangle \) (where we allow \( \mathcal{L} \) to vary while keeping fixed \( \text{Nm}(\mathcal{L}) \) and \( \mathcal{M}_0 \)) is trivial as a line bundle over \( \mathfrak{g}_{SU(N)} \). Given this, the bilinearity of the pairing \( \langle \ , \ \rangle \) implies that \( \langle \mathcal{L}, \mathcal{M} \rangle \cong \langle \mathcal{L}, \mathcal{M} \otimes \psi^*(\mathcal{M}_0) \rangle \) for \( \mathcal{M}_0 \in \text{Jac}(C) \). Hence with \( \mathcal{L} \) so restricted, we can consider \( \mathcal{M} \) to take values in \( \mathfrak{g}_{PSU(N)} = \text{Pic}_d(D)/\text{Jac}(C) \). Then the symbol \( \langle \mathcal{L}, \mathcal{M} \rangle \) defines a twisted line bundle of degree \( (d, 0) \) over \( \mathfrak{g}_{SU(N)} \times \mathfrak{g}_{PSU(N)}. \) Thus, \( \mathfrak{g}_{SU(N)} \) parametrizes a family of ordinary line bundles over \( \mathfrak{g}_{PSU(N)} \), and \( \mathfrak{g}_{PSU(N)} \) parametrizes a family of twisted line bundles of degree \( d \) over \( \mathfrak{g}_{SU(N)}. \) This is the expected duality. The fact that it is a duality, and not just a pairing, depends upon the fact that the symbol \( \langle \mathcal{L}, \mathcal{M} \rangle \) has no additional symmetry in the second variable except what we have already accounted for; we discuss this briefly at the end of this section.

We can generalize this to let \( \mathcal{L} \) have degree \( c \). Now we pick a fixed line bundle \( \mathcal{L}_0 \to C \) of degree \( c \) and consider \( \mathcal{L} \in \text{Pic}_c(D) \) with \( \text{Nm}(\mathcal{L}) = \mathcal{L}_0 \). We claim that \( \langle \mathcal{L}, \mathcal{M} \rangle \) is still invariant to twisting \( \mathcal{M} \) by the pullback of a degree zero line bundle \( \mathcal{M}_0 \to C \). This being so, the symbol \( \langle \mathcal{L}, \mathcal{M} \rangle \) gives the expected duality of degree \( (d, c) \) between \( \mathfrak{g}_{PSU(N)}(c) \) and \( \mathfrak{g}_{PSU(N)}(d). \)

In these statements, the integers \( c \) and \( d \) correspond\(^{22}\) to the discrete electric and magnetic charges that can arise in gauge theory of \( PSU(N) \) or \( SU(N) \). We recall that

\(^{21}\)As explained in section 3.2 (see footnote 11), if \( N \) is even, identifying the fiber of the Hitchin fibration with the Jacobian depends upon a choice of square root of the canonical bundle \( K \). We likewise use below a description of the fiber for \( SU(N) \) that for even \( N \) depends on a choice of spin structure.

\(^{22}\)The reader may wish to return to this paragraph after reading a more detailed explanation of the same question for the case of \( G = U(1) \) in section 7.
for these groups, discrete $\mathbb{Z}_N$-valued electric and magnetic charges are possible. $\text{Pic}_c(D)$ parametrizes Higgs bundles with a value $c$ of the discrete magnetic charge, and a twisted line bundle over $\text{Pic}_c(D)$ that is twisted by the $d^{th}$ power of the universal bundle can be the Chan-Paton bundle of a brane whose discrete electric charge is $d$. In the duality between $PSU(N)$ and $SU(N)$, the roles of $c$ and $d$ are exchanged, as one would expect.

It remains to show that if $\text{Nm}(\mathcal{L})$ is fixed, then $\langle \mathcal{L}, \psi^*(\mathcal{M}_0) \rangle$ is constant as $\mathcal{L}$ varies. For this, we need to know a few facts. First, for $\psi : D \to C$ a map between Riemann surfaces, we have simply $H^*(D, \mathcal{L}) = H^*(C, \psi_* \mathcal{L})$, so

$$\det H^*(D, \mathcal{L}) = \det H^*(C, \psi_* \mathcal{L}).$$  \hfill (6.27)

The right hand side, of course, is $\det H^*(C, E)$, where $E = \psi_* (\mathcal{L})$. Furthermore, for any vector bundle $E$ over a Riemann surface $C$, we have

$$\det H^*(C, E) = \det H^*(C, \det E).$$  \hfill (6.28)

This can be proved by induction in the rank of $E$. Any bundle $E \to C$ has a holomorphic line sub-bundle $\mathcal{M}$, and so appears in an exact sequence

$$0 \to \mathcal{M} \to E \to U \to 0.$$  \hfill (6.29)

From this, we get (as in the discussion of (6.7)) an isomorphism $\det E \cong \det \mathcal{M} \otimes \det U$. In addition, we can derive from (6.29) a long exact sequence of cohomology groups, which, as in the derivation of (6.9), gives an isomorphism $\det H^*(C, E) \cong \det H^*(C, \mathcal{M}) \otimes \det H^*(C, U)$. Combining these, we obtain (6.28) by induction in the rank of $E$.

Now for line bundles $\mathcal{L} \to D$, $\mathcal{M}_0 \to C$, we have $\det H^*(D, \mathcal{L} \otimes \psi^*(\mathcal{M}_0)) = \det H^*(C, \psi_* (\mathcal{L} \otimes \psi^*(\mathcal{M}_0))) = \det H^*(C, E \otimes \mathcal{M}_0) = \det H^*(C, \det (E \otimes \mathcal{M}_0))$. Similarly, $\det H^*(D, \mathcal{L}) = \det H^*(C, \det E)$. If we vary $\mathcal{L}$ keeping $\text{Nm}(\mathcal{L})$ fixed, then $\det E$ is also fixed, so finally $\det H^*(D, \mathcal{L} \otimes \mathcal{M}) \otimes \det H^*(D, \mathcal{L})^{-1}$ is also fixed. But in the definition of $\langle \mathcal{L}, \psi^*(\mathcal{M}_0) \rangle$, this is the factor that depends on $\mathcal{L}$. So $\langle \mathcal{L}, \psi^*(\mathcal{M}_0) \rangle$ remains fixed, as claimed, when $\mathcal{L}$ varies keeping fixed $\text{Nm}(\mathcal{L})$ and $\mathcal{M}_0$.

Finally, let us show that if, for $\mathcal{M}$ of degree zero, $\langle \mathcal{L}, \mathcal{M} \rangle$ is trivial for $\mathcal{L} \in \mathfrak{H}_{SU(N)}$, then $\mathcal{M}$ is a pullback from $C$. We showed in section 3.3.1 than if $\text{Nm}(\mathcal{L}) = \mathcal{O}$, then $\mathcal{L}$ is a tensor product of line bundles $\mathcal{O}(\mathfrak{q}) \otimes \mathcal{O}(\mathfrak{q}')^{-1}$, where $\psi(\mathfrak{q}) = \psi(\mathfrak{q}')$. By the bilinearity of the pairing, $\langle \mathcal{L}, \mathcal{M} \rangle$ is trivial for $\mathcal{L}$ such a tensor product if and only if it is trivial for a single factor. So we set $\mathcal{L} = \mathcal{O}(\mathfrak{q}) \otimes \mathcal{O}(\mathfrak{q}')^{-1}$, and get $\langle \mathcal{L}, \mathcal{M} \rangle = \mathcal{M}(\mathfrak{q}) \otimes \mathcal{M}(\mathfrak{q}')^{-1}$. So a trivialization of $\langle \mathcal{L}, \mathcal{M} \rangle$ as $\mathcal{L}$ varies gives an identification of $\mathcal{M}(\mathfrak{q})$ with $\mathcal{M}(\mathfrak{q}')$ whenever $\mathfrak{q}$ and $\mathfrak{q}'$ lie over the same point in $C$. Existence of such an identification implies that $\mathcal{M}$ is a pullback from $C$.

We have implicitly used here the fact that the variety parametrizing pairs $\mathfrak{q}, \mathfrak{q}' \in D$ with $\psi(\mathfrak{q}) = \psi(\mathfrak{q}')$ is irreducible. One can show this by finding a local model of the spectral curve with sufficiently large monodromy, for example $y^N + \epsilon y z + z = 0$, with $z$ a local parameter on $C$ and $\epsilon$ a complex constant.
7 Computing The Hecke Eigenvalue

Finally we will use what we have learned to compute the “eigenvalue” with which an ’t Hooft operator acts on a magnetic eigenbrane. We begin with the abelian case in section 7.1. This is the obvious place to start, and certainly leads to the most straightforward calculations, though we will run into some subtle questions of interpretation because of the elementary fact that – unlike a simple non-abelian group – the group $U(1)$ has a center and a fundamental group that are both of infinite order.

Geometric Langlands duality in abelian gauge theory is most commonly described by a somewhat different argument, attributed to Deligne, that can be found for example in [2]. We follow a route that will give a useful starting point for the nonabelian generalization, to which we turn in section 7.2.

7.1 The Abelian Case

7.1.1 Calculation

For $G = U(1)$, a Higgs bundle over a Riemann surface $C$ is just a pair $(\mathcal{L}, \varphi)$, where $\mathcal{L}$ is a complex line bundle, and $\varphi \in H^0(C, K)$ is a holomorphic differential. $\varphi$ will actually play little role in the analysis, the reason being that as $U(1)$ is abelian, Hitchin’s equations are linear, the gauge field and Higgs field are decoupled, $\varphi$ is invariant under Hecke transformations, and the interesting action of $S$-duality is just on the gauge field.

One important point, however, also discussed in section 2.6.5, is that Hitchin’s moduli space $M_H(U(1), C)$ parametrizes Higgs pairs $(\mathcal{L}, \varphi)$ for which the line bundle $\mathcal{L}$ has degree zero. This condition is not preserved, in general, by the action of ’t Hooft operators, so some discussion is required.

We consider $U(1)$ gauge theory in a Hamiltonian framework on a three-manifold $W = I \times C$, where $I$ is an interval. Boundary conditions on the right hand end of $W$ are defined by a brane that we choose to be an electric or magnetic eigenbrane. We act on an electric eigenbrane $B$ with Wilson operators $W^{(d_i)}(y_i, p_i)$ of charges $d_i$, inserted at points $y_i \times p_i \in I \times C$. Or dually we act on a magnetic eigenbrane $\hat{B}$ with ’t Hooft operators $T^{(d_i)}(y_i, p_i)$. The two-dimensional effective picture was sketched in fig. 1 in section 4.

Concretely, we pick an electric eigenbrane $B$ that is a zerobrane supported at some point $x \in M_H$ that corresponds to a Higgs bundle $(\mathcal{L}, \varphi)$. And we consider a product of Wilson operators acting on $B$:

$$\prod_{i=1}^n W^{(d_i)}(y_i, p_i) B.$$  \hfill (7.1)

As we explained in section 4.2, the zerobrane $B$ is an electric eigenbrane, with

$$\prod_{i=1}^n W^{(d_i)}(y_i, p_i) B = B \otimes \left( \otimes_{i=1}^n \mathcal{L}_{p_i}^{d_i} \right).$$  \hfill (7.2)

(Momentarily we will rewrite this in a way that is closer to eqn. (4.6) of section 4.2.)

An important subtlety here springs from the familiar fact that a point $x \in M_H$ only determines a corresponding line bundle $\mathcal{L}$ up to isomorphism. We are free to tensor $\mathcal{L}$ with

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a fixed one-dimensional vector space $\mathcal{R}$. This causes no trouble if $d = \sum_i d_i$ is equal to zero, but in general the “eigenvalue” with which the given product of Wilson operators acts on $\mathcal{B}$ is tensored with $\mathcal{R}^d$. Of course, this ambiguity is independent of the choice of points $p_i$, and we encountered it in a different guise in section 6.2. As in that discussion, the most illuminating way to proceed is to pick a universal bundle $\mathcal{W} \rightarrow C \times \text{Jac}(C)$ and express eqn. (7.2) in terms of this universal bundle:

$$\prod_{i=1}^{n} W^{(d_i)}(y_i, p_i) \mathcal{B} = \mathcal{B} \otimes \left( \otimes_{i=1}^{n} W|_{p_i \times x}^{d_i} \right).$$  \hspace{1cm} (7.3)

For $d \neq 0$, this result does depend on the choice of the universal bundle $\mathcal{W} \rightarrow C \times \text{Jac}(C)$. If we transform $\mathcal{W}$ to $\mathcal{W} \otimes \mathcal{R}$, where $\mathcal{R}$ is the pullback to $C \times \text{Jac}(C)$ of a line bundle over the second factor, then the right hand side of eqn. (7.3) is tensored with $\mathcal{R}^d$.

The physical meaning is as follows. The integer $d$ is the total electric charge carried by the product of Wilson operators $\prod_{i=1}^{n} W^{(d_i)}(y_i, p_i)$ with which we are acting. If the initial brane $\mathcal{B}$ is electrically neutral, then the brane that results from acting with this product of Wilson operators has an electric charge $23d$. The $\sigma$-model with target $\mathcal{M}_H(U(1), C)$ only describes neutral degrees of freedom, and to describe a brane that carries charge, we cannot just use this $\sigma$-model; to describe a charged brane, we need to include the unbroken $U(1)$ gauge multiplet in the effective two-dimensional description. We say more on this in section 7.1.2.

In general, a neutral brane over $\mathcal{M}_H(U(1), C)$ has a Chan-Paton bundle $\mathcal{U}$ that is an ordinary vector bundle or sheaf. A brane of electric charge $d$ has instead a Chan-Paton bundle $\mathcal{U}$ that is a twisted vector bundle or sheaf. The twisting is most easily described by saying that $\mathcal{U}$ can be described as an ordinary vector bundle or sheaf once a universal bundle $\mathcal{W} = C \times \text{Jac}(C)$ is picked, but $\mathcal{U}$ depends on the choice of $\mathcal{W}$: transforming $\mathcal{W}$ to $\mathcal{W} \otimes \mathcal{R}$ will transform $\mathcal{U}$ to $\mathcal{U} \otimes \mathcal{R}^d$. Clearly with this statement of what sort of object is the Chan-Paton bundle of a brane that carries a net electric charge, eqn. (7.3) is consistent with the way that we expect acting with a Wilson operator to transform the electric charge of a brane.

The dual of an electric eigenbrane $\mathcal{B}$ is a brane $\hat{\mathcal{B}}$ of type $\mathbf{F}$ that we expect to be a magnetic eigenbrane. It is supported on a fiber $\hat{\mathcal{F}}$ of the Hitchin fibration for which the Higgs field $\tilde{\varphi}$ is a multiple of the original Higgs field $\varphi$ on the electric side. $\hat{\mathcal{F}}$ is a copy of the Jacobian $\text{Jac}(C)$. The brane $\hat{\mathcal{B}}$ is characterized by a flat Chan-Paton line bundle that we will discuss shortly. We want to calculate the action of a product of ’t Hooft operators on $\hat{\mathcal{B}}$:

$$\prod_{i=1}^{n} T^{(d_i)}(y_i, p_i) \hat{\mathcal{B}}. \hspace{1cm} (7.4)$$

---

23In the effective $1 + 1$-dimensional description, a brane is a boundary condition at the end of a string. Saying that a brane $\mathcal{B}$ has charge $d$ simply means that with this boundary condition, the electric charge operator receives a contribution $d$ at the end of the string.

24The multiple is $\text{Im} \tau$, where $\tau = \theta/2\pi + 4\pi i/\epsilon^2$ is the usual gauge coupling parameter. It is usually just scaled out and plays no essential role in what follows.
Again, there is a subtlety; the product of the 't Hooft operators shifts the degree of the line bundle by \( d = \sum_i d_i \), so if \( d \) is nonzero, we are mapped out of \( \mathcal{M}_H \). As in the electric case, this means that a brane can carry magnetic flux, which is not incorporated in the two-dimensional \( \sigma \)-model with target \( \mathcal{M}_H \).

For simple or semi-simple nonabelian \( G \), the analogous notion of magnetic flux is a discrete topological invariant \( \xi \) (introduced in section 2.6.6) of a Higgs bundle. Because it is a discrete invariant, it can be carried by a flat bundle, or by a solution of the Hitchin equations for \( G \). Thus \( \mathcal{M}_H(G,C) \) for simple or semi-simple \( G \) has components that are labeled by \( \xi \). But for \( G = U(1) \), the magnetic flux is not conveniently described in a \( \sigma \)-model with target \( \mathcal{M}_H(G,C) \); to describe it, one has to retain the unbroken \( U(1) \) gauge symmetry in the low energy description. We will explain in section 7.1.2 what sort of brane carries magnetic flux for \( G = U(1) \).

Leaving aside questions of interpretation, to calculate the action of the 't Hooft operators on \( \hat{B} \) is actually a simple exercise. According to our reassessment of duality in section 6.5, a zerobrane supported at a point in \( \text{Jac}(C) \) corresponding to a line bundle \( \mathcal{L} \to C \) should have for its dual a brane \( \hat{B} \) of type \( F \) whose Chan-Paton bundle is \( \langle \mathcal{L}, \mathcal{M} \rangle \). Here this expression is viewed as a line bundle over a copy of \( \text{Jac}(C) \) parametrized by \( \mathcal{M} \), with \( \mathcal{L} \) kept fixed. According to eqn. (5.3), the Chan-Paton bundle of the brane \( \prod_{i=1}^n T^{(d_i)}(y_i, p_i) \hat{B} \) obtained by acting on \( \hat{B} \) with the indicated product of 't Hooft operators is \( \tilde{U} = \Phi^* (U) \), where \( \Phi \) is the automorphism \( \mathcal{M} \to \mathcal{M} \otimes \mathcal{O}(p_i)^{d_i} \) of \( \text{Jac}(C) \).

We have \( \Phi^* (\langle \mathcal{L}, \mathcal{M} \rangle) = \langle \mathcal{L}, \mathcal{M} \otimes \mathcal{O}(p_i)^{d_i} \rangle \). Using the bilinearity of the \( \langle , \rangle \) symbol, this is the same as \( \langle \mathcal{L}, \mathcal{M} \rangle \otimes \langle \mathcal{L}, \otimes \mathcal{O}(p_i)^{d_i} \rangle \). In other words,

\[
\tilde{U} = U \otimes \langle \mathcal{L}, \otimes \mathcal{O}(p_i)^{d_i} \rangle. \tag{7.5}
\]

This confirms that the brane \( \hat{B} \) is a magnetic eigenbrane with “eigenvalue” the one-dimensional vector space \( \langle \mathcal{L}, \otimes \mathcal{O}(p_i)^{d_i} \rangle = \otimes_i \mathcal{L}^{|p_i|} \), where we have made use of eqn. (6.17) to evaluate the \( \langle , \rangle \) symbol. Thus the brane \( \hat{B} \) is a magnetic eigenbrane with the “eigenvalue” that one would expect from the electric formula (7.2):

\[
\prod_{i=1}^n T^{(d_i)}(y_i, p_i) \hat{B} = \hat{B} \otimes \left( \otimes_{i=1}^n \mathcal{L}^{|p_i|} \right). \tag{7.6}
\]

This establishes the duality. The 't Hooft operators act on \( \hat{B} \) in the same way that the Wilson operators act on \( B \). It remains only to explain what the calculations mean in case \( d = \sum_i d_i \) is nonzero.

### 7.1.2 Electric And Magnetic Fields

Higgs pairs \( (\mathcal{L}, \varphi) \) with \( c_1(\mathcal{L}) \neq 0 \) inevitably appear when we act with 't Hooft operators. But they do not correspond to points in \( \mathcal{M}_H \). How then do they enter the gauge theory? They correspond to new branes of a sort that we have not yet considered.

The basic subtlety was actually pointed out in section 2.1. The argument that a four-dimensional gauge theory compactified on a Riemann surface \( C \) reduces at low energies to a two-dimensional \( \sigma \)-model with target \( \mathcal{M}_H(G,C) \) assumes that the gauge symmetry is
completely broken in situations of interest. For a simple nonabelian gauge group $G$, this is so (apart from a finite group, the center of $G$) as long as we avoid singularities of $\mathcal{M}_H$ – at which new degrees of freedom become relevant. However, $G = U(1)$ is different since constant gauge transformations act trivially and every adjoint-valued field has a continuous group of gauge symmetries. In any compactification of $\mathcal{N} = 4$ super Yang-Mills theory with $U(1)$ gauge group from four to two dimensions, the $U(1)$ gauge group remains unbroken and it should be considered in the low energy description.

As a result, the reduction to a low energy $\sigma$-model is not quite valid for $G = U(1)$. The low energy theory is a product of a $\sigma$-model of target $\mathcal{M}_H$ – at which new degrees of freedom become relevant. However, $G = U(1)$ is different since $\sigma$-model gauge transformations act trivially and every adjoint-valued field has a continuous group of gauge symmetries. In any compactification of $\mathcal{N} = 4$ super Yang-Mills theory with $U(1)$ gauge group from four to two dimensions, the $U(1)$ gauge group remains unbroken and it should be considered in the low energy description.

Let us consider the extended Bogomolny equations\(^{25}\) (4.14) and (4.15) on the three-manifold $W = I \times C$. We view these as equations that describe time-independent supersymmetric configurations relevant to quantization on $W$. We endow $W$ with a metric $dy^2 + d\Omega^2$, where $y = x^4$ is a Euclidean coordinate on $I$ and $d\Omega^2$ is a $y$-independent Kahler metric on $C$. For gauge group $U(1)$, these equations have solutions of a kind quite different than what we have considered so far. Let $\omega$ be a multiple of the Kahler form of $C$, normalized so that $\int_C \omega = 1$. We can solve the extended Bogomolny equations by picking a connection $A$ on a degree $d$ line bundle $L \to C$ whose curvature is\(^{26}\) $F = 2\pi d \omega$. Then we pull back $L$ and $A$ to $W$ and take\(^{27}\) $\phi_0 = 2\pi y \cdot d$. We take all other fields to vanish. In this fashion, for $G = U(1)$, we get a supersymmetric configuration with a line bundle on $C$ of any degree.

Returning to the question of what happens in eqn. (7.4) if $d = \sum_i d_i$ is nonzero, the answer is that in that case, to the left of all of the ’t Hooft operators, the line bundle $L$ has nonzero degree and a supersymmetric configuration will be created based on the solution that we have just described. Hence, in acting on $\hat{B}$, a product of ’t Hooft operators with nonzero total $d$ creates a brane different from those we have so far considered. This is a supersymmetric brane for which the boundary conditions require $c_1(L) = d$ and a normal derivative of $\phi_0$ equal to $2\pi d$. We might call this a brane with magnetic charge $d$.

This construction does not have a close analog for semi-simple nonabelian $G$, as long as we keep away from singularities of $\mathcal{M}_H$, because the extended Bogomolny equations would force $\phi_0$ to commute with all the other fields and to generate a symmetry of the solution. To get an analog for simple non-abelian $G$ of what we have found for $U(1)$, we can take $C$ to have genus zero or one, in which case a generic stable Higgs bundle $(E, \varphi)$ can leave a continuous group of unbroken gauge symmetries. For genus $0$, the semistable Higgs bundle with $E$ trivial and $\varphi = 0$ is generic (it has no deformations as a semistable Higgs bundle) and leaves the gauge symmetry completely unbroken. Thus the description

\(^{25}\) Equivalently, we consider the supersymmetric equations (3.29) of [1] at $t = 1$, a convenient value for studying the $A$-model of type $K$.

\(^{26}\) To avoid unnatural-looking factors of $i$, we here will take $F$ and $\phi$ to be real-valued. So the curvature of a unitary connection $D$ on a line bundle is $F = -iD^2$.

\(^{27}\) Here $\phi_0$ is just the “time” component of the Higgs field $\phi = \sum_{i=0}^3 \phi_i dx_i$. 

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of branes in genus\(^{28}\) 0 will be based on a low energy theory that is a gauge theory – with gauge group \(G\) – with no accompanying \(\sigma\)-model. In genus 1, the generic stable Higgs bundle for the case that the bundle \(E\) is topologically trivial leaves unbroken a subgroup of \(G\) that is isomorphic to its maximal torus. So in genus 1, the situation for a simple nonabelian \(G\) is somewhat like what happens for \(G = U(1)\): the generic description of branes will use an abelian gauge theory combined with a \(\sigma\)-model. In genus > 1, a generic brane can be described in the \(\sigma\)-model language, but this description will break down for branes supported at a singularity of \(\mathcal{M}_H\), at which there is enhanced gauge symmetry. (The case that the enhanced gauge symmetry is a finite group was studied in [5].)

Going back to \(G = U(1)\), the \(S\)-dual of a brane with magnetic flux \(d\) can be described classically as another novel kind of brane, which we will call a brane with electric flux \(d\). For this sort of brane, \(\phi_0\) is still a linear function of \(x^1\), but now there is a constant electric field instead of a constant magnetic field. For supersymmetry, such fields must obey the appropriate supersymmetric equations (eqn. (3.29) of [1], which should be taken at \(t = i\), which is the appropriate value for studying Wilson lines). In abelian gauge theory, those equations reduce (in Euclidean signature) to \(F + i d\phi = 0\). On a four-manifold \(M = \mathbb{R}^2 \times C\), with Euclidean coordinates \(x^0, x^1\) on \(\mathbb{R}^2\), these equations can be solved with \(\phi_0\) a linear function of \(x^1\), \(F_{01}\) constant, and everything else vanishing. In Euclidean signature, \(F_{01}\) has to be imaginary, but in Lorentz signature, which of course is the real home of physics, the factor of \(i\) disappears.

The field \(\phi_0\) that was important in this analysis is part of the supersymmetric multiplet – known as a vector multiplet – that contains the \(U(1)\) gauge field. The branes that carry electric and magnetic charge can be described in the combined low energy theory consisting of the \(\sigma\)-model coupled to the \(U(1)\) vector multiplet.

### 7.2 Nonabelian Generalization

#### 7.2.1 Calculation

Now we move on to the nonabelian generalization. To consider the basic idea, we take \(L^G = G = U(N)\), and we take a magnetic weight \(L^w = (1, 0, 0, \ldots, 0)\) that is the highest weight of the \(N\)-dimensional representation \(V\).

We let \(B\) be a zero-brane supported at a point \(x \in \mathcal{M}_H\) that corresponds to a Higgs bundle \(\mathcal{E} = (E, \varphi)\). The Wilson operator \(W(L^w; y, p)\), inserted at a point \(y \times p \in I \times C\), acts by

\[
W(L^w; y, p)B = B \otimes \mathcal{E}_{H,Y}|_{p \times x}
\]  

(7.7)

Here \(\mathcal{E}_{H,Y} \to C \times \mathcal{M}_H\) is the universal Higgs bundle in the representation \(V\) (the notion was introduced in section 4.2), and \(\mathcal{E}_{H,Y}|_{p \times x}\) is its restriction to \(p \times x \in C \times \mathcal{M}_H\). So \(\mathcal{E}_{H,Y}|_{p \times x}\) is a fixed vector space, and (7.7) expresses the fact that \(B\) is an electric eigenbrane.

We pick \(p\) and \(\mathcal{E}\) generically so that \(\varphi(p)\) is regular and semisimple. There consequently are \(N\) distinct solutions to the eigenvalue equation near \(p\),

\[
\varphi \psi_i = y_i \psi_i, \quad i = 1, \ldots, N.
\]  

(7.8)

\(^{28}\)Here and in the following discussion in genus 1, we consider – as throughout this paper – the unramified case of geometric Langlands. Ramification points will reduce the gauge symmetry, as in [3].
Correspondingly, $\mathcal{E}_{H,\mathcal{V}}|_{p \times x}$ decomposes as a sum of the eigenspaces of $\varphi$, which we denote as $\mathcal{E}_{H,\mathcal{V}}^{(i)}|_{p \times x}$. These eigenspaces have a familiar interpretation in terms of the spectral cover $\psi : D \to C$. The Higgs bundle $\mathcal{E} = (E, \varphi)$ is determined by a degree zero line bundle $\mathcal{M} \to D$. The points $q_i \in D$ that lie above $p \in C$ are in one-to-one correspondence with the eigenvectors of $\varphi(p)$, and the definition of $\mathcal{M}$ is such that $\mathcal{E}_{H,\mathcal{V}}^{(i)}|_{p \times x}$ is the same as $\mathcal{M}|_{q_i}$, the fiber of $\mathcal{M}$ at $q_i$. And so the eigenbrane equation (7.7) is more explicitly

$$W^{(Lw)}(y, p)\mathcal{B} = \mathcal{B} \otimes (\oplus_{i=1}^{N} \mathcal{M}|_{q_i}) \, .$$ (7.9)

The magnetic eigenbrane $\widehat{\mathcal{B}}$ that is dual to $\mathcal{B}$ is supported on the fiber $\mathcal{G}$ of the Hitchin fibration that contains the point $x$. Its Chan-Paton bundle $\mathcal{U}$ is determined by the analysis of duality in section 6. The fiber of $\mathcal{U}$ at a point in $\mathcal{G}$ corresponding to a degree zero line bundle $\mathcal{L} \to D$ is $\mathcal{U}|_{\mathcal{L}} = (\mathcal{L}, \mathcal{M})$.

Now we can determine the action of the ’t Hooft operator $T^{(Lw)}(y, p)$. The operator $T^{(Lw)}(y, p)$ can act by any of $N$ possible $\varphi$-invariant Hecke modifications of $\mathcal{E} = (E, \varphi)$. They correspond to holomorphic maps $\Phi_i : \mathcal{G} \to \mathcal{G}$, and are in natural one-to-one correspondence with the points $q_i \in D$ that lie over $p$. $\Phi_i$ acts by $\mathcal{L} \to \mathcal{L}(q_i)$, so it maps $W$ to a new Chan-Paton bundle whose fiber at $\mathcal{L}$ is $(\mathcal{L}(q_i), \mathcal{M})$, which, by virtue of the bilinearity of the pairing $\langle \cdot, \cdot \rangle$, is the same as $\langle \mathcal{L}, \mathcal{M} \rangle \otimes \mathcal{M}|_{q_i}$. So $\Phi_i$ maps $\widehat{\mathcal{B}} \otimes \mathcal{M}|_{q_i}$, and after summing over all choices of $\Phi_i$, we get

$$T^{(Lw)}(y, p)\widehat{\mathcal{B}} = \widehat{\mathcal{B}} \otimes (\oplus_{i=1}^{N} \mathcal{M}|_{q_i}) \, .$$ (7.10)

This is in good parallel with (7.7), so we have verified the expected duality in this example.

Other minuscule weights of $L^G = U(N)$ can be considered in a similar fashion. As in the discussion of eqn. (4.29), let $Lw(k) = (1, 1, \ldots, 1, 0, \ldots, 0)$ (with the number of 1’s being $k$) be the highest weight of the representation $\wedge^k \mathcal{V}$ of $U(N)$. First let us consider the action of the Wilson operator $W^{(Lw(k))}(y, p)$. It acts on the zerobrane $\mathcal{B}$ by

$$W^{(Lw(k))}(y, p)\mathcal{B} = \mathcal{B} \otimes \mathcal{E}_{H,\wedge^k \mathcal{V}}|_{p \times x}$$ (7.11)

$\mathcal{E}_{H,\wedge^k \mathcal{V}}$ is the universal Higgs bundle in the representation $\wedge^k \mathcal{V}$, and $\mathcal{E}_{H,\wedge^k \mathcal{V}}|_{p \times x}$ is its restriction to $p \times x$. Once $\mathcal{V}$ is decomposed as the direct sum of the one-dimensional eigenspaces $\mathcal{M}|_{q_i}$, $\wedge^k \mathcal{V}$ has an analogous decomposition as $\oplus_{\alpha} \mathcal{M}|_{\alpha}$, where $\alpha$ is a subset of the set $\{1, 2, \ldots, N\}$ of cardinality $k$, and $\mathcal{M}|_{\alpha} = \oplus_{i \in \alpha} \mathcal{M}|_{q_i}$. So we can write the eigenvalue equation as

$$W^{(Lw(k))}(y, p)\mathcal{B} = \mathcal{B} \otimes (\oplus_{\alpha} \mathcal{M}|_{\alpha}) \, .$$ (7.12)

On the magnetic side, the ’t Hooft operator $T^{(Lw(k))}(y, p)$ acts by a $\varphi$-invariant Hecke transformation. For each $\alpha$, there is a $\varphi$-invariant Hecke transformation $\Phi_{\alpha}$ that acts by $\mathcal{L} \to \mathcal{L}_{\alpha} = \mathcal{L} \otimes (\oplus_{i \in \alpha} \mathcal{O}(q_i))$. The Chan-Paton bundle $(\mathcal{L}, \mathcal{M})$ is mapped by $\Phi_{\alpha}$ to $(\mathcal{L}_{\alpha}, \mathcal{M})$, which, again using the bilinearity of the pairing, is the same as $(\mathcal{L}, \mathcal{M}) \otimes \mathcal{M}|_{\alpha}$. Thus, once we sum over $\alpha$, the ’t Hooft operator acts by

$$T^{(Lw(k))}(y, p)\widehat{\mathcal{B}} = \widehat{\mathcal{B}} \otimes (\oplus_{\alpha} \mathcal{M}|_{\alpha}) \, .$$ (7.13)
Again, in comparing (7.12) and (7.13), we see the expected duality.

The most general minuscule weight for $G = U(N)$ is obtained from a slight generalization of this. We introduce an integer $r$ and let $L^w(k; r) = (r + 1, r + 1, \ldots, r + 1, r, \ldots, r)$, with $k$ weights equal to $r + 1$ and the rest equal to $r$. For example, if $r = -1$ and $k = N - 1$, we get $L^w = (0, 0, \ldots, 0, -1)$, which is the highest weight of the representation $\mathcal{V}^r$ that is dual to $\mathcal{V}$. In general, $L^w(k; r)$ is the highest weight of the representation $\wedge^k \mathcal{V} \otimes (\det \mathcal{V})^r$.

On the electric side, the effect of including $r$ is that the right hand side of (7.12) must be tensored with $(\det \mathcal{E}|_{\mathcal{P} \times \mathcal{X}})^r$, which is the same as $(\otimes_{i=1}^N \mathcal{M}|_{\mathcal{A}_i})^r$:

$$W(L^w(k; r); y, p) \mathcal{B} = \mathcal{B} \otimes (\otimes_{\alpha} \mathcal{M}|_{\alpha}) \otimes (\otimes_{i=1}^N \mathcal{M}|_{\mathcal{A}_i})^r. \quad (7.14)$$

On the magnetic side, the $\varphi$-invariant Hecke modifications now act by $\mathcal{L} \rightarrow \mathcal{L} \otimes (\otimes_{i=1}^N \mathcal{O}(\mathcal{Q}_i))^r$. Repeating the derivation of (7.13), we now get on the magnetic side

$$T(L^w(k; r)) \hat{\mathcal{B}} = \hat{\mathcal{B}} \otimes (\otimes_{\alpha} \mathcal{M}|_{\alpha}) \otimes (\otimes_{i=1}^N \mathcal{M}|_{\mathcal{A}_i})^r. \quad (7.15)$$

This again exhibits the duality.

### 7.2.2 Interpretation

The sum of the weights of $L^w(k; r)$ is $k + rN$; this can be either positive or negative. Consider acting with a product of ’t Hooft operators determined by the minuscule weights $L^w(k_\sigma; r_\sigma)$, $\sigma = 1, \ldots, s$. This product of ’t Hooft operators changes $c_1(E)$ by $\Delta = \sum_\sigma (k_\sigma +Nr_\sigma)$. Now we recall that although a Higgs bundle $\mathcal{E} = (E, \varphi)$ for $G = U(N)$ may have any value of $c_1(E)$, the hyper-Kähler manifold $\mathcal{M}_H$ which is the target of the $\sigma$ model parametrizes precisely the Higgs bundles of $c_1(E) = 0$.

If $\Delta = 0$, the given product of ’t Hooft operators keeps us within this class. Otherwise, we run into the same phenomenon that we discussed for $U(1)$ gauge theory in section 7.1.2. For $G = U(N)$, a generic Higgs bundle has $U(1)$ symmetry, simply because the center of $U(N)$ is $U(1)$. So the low energy theory is the product of a $\sigma$-model with target $\mathcal{M}_H$ and a supersymmetric $U(1)$ gauge theory. As in section 7.1.2, when the $U(1)$ theory is included, there are branes with nonvanishing electric or magnetic charge. In the context of $G = U(N)$, these are branes in which $\text{Tr} \phi_0$ is a linear function of $x^1$.

A product of ’t Hooft operators with $\Delta \neq 0$ shifts the magnetic charge of a brane and maps a brane that can be described purely in the $\sigma$-model to a brane whose description really requires us to include the $U(1)$ vector multiplet. Similarly, a product of Wilson operators with $\Delta \neq 0$ shifts the electric charge of a brane. As always, a brane carrying electric charge has a Chan-Paton bundle that is a twisted line bundle (or twisted vector bundle), not an ordinary one, over $\mathcal{M}_H$. This is reflected in the above analysis in the fact that the “eigenvalue” by which a Wilson operator acts on a zerobrane $\mathcal{B}$ depends on the line bundle $\mathcal{M}$, not just its isomorphism class. This dependence cancels out if we act with a product of Wilson operators of $\Delta = 0$. As usual, the branes whose Chan-Paton wavefunctions depend on a choice of universal bundle (which in the present context means a choice of $\mathcal{M}$) are the ones that carry electric charge. But for the groups $U(N)$ or $U(1)$ whose center has positive dimension, a proper description of the branes carrying electric or magnetic charge requires including the $U(1)$ vector multiplet in the description.
These issues arise entirely because $U(N)$ has a center $U(1)$ that is not of finite order. We can avoid these issues, accordingly, while maintaining the spirit of the above demonstration of geometric Langlands duality, if we take $\mathbb{L}G = SU(N)$ rather than $U(N)$. This corresponds, of course, to $G = SU(N)/\mathbb{Z}_N$. Geometric Langlands duality for minuscule representations of $SU(N)$ can be analyzed in precisely the way that we have done for $U(N)$.

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