MODULAR PHENOMENA FOR REGULARIZED DOUBLE ZETA VALUES

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Abstract. In this paper, we investigate linear relations among regularized motivic iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ of depth two, which we call regularized motivic double zeta values. Some mysterious connections between motivic multiple zeta values and modular forms are known, e.g. Gangl–Kaneko–Zagier relation for the totally odd double zeta values and Ihara–Takao relation for the depth graded motivic Lie algebra. In this paper, we investigate so-called non-admissible cases and give many new Gangl–Kaneko–Zagier type and Ihara–Takao type relations for regularized motivic double zeta values. Specifically, we construct linear relations among a certain family of regularized motivic double zeta values from odd period polynomials of modular forms for the unique index two congruence subgroup of the full modular group. This gives the first non-trivial example of a construction of the relations among multiple zeta values (or their analogues) from modular forms for a congruence subgroup other than the $SL_2(\mathbb{Z})$.

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1. INTRODUCTION

Multiple zeta values (MZVs) are real numbers defined by

$$\zeta(k_1, \ldots, k_d) := \sum_{0 < m_1 < \cdots < m_d} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}$$

where $k_1, \ldots, k_d$ are positive integers with $k_d > 1$. Here $d$ is called the depth and $k_1 + \cdots + k_d$ is called the weight, and multiple zeta values of depth two are called double zeta values. They are periods in the sense of Kontsevich-Zagier, more specifically periods of mixed Tate motives over $\mathbb{Z}$, and are lifted to algebraic objects $\zeta^m(k_1, \ldots, k_d) \in P^m_{MT(\mathbb{Z})}$ called motivic multiple zeta values (or motivic double zeta values if $d = 2$). Here $P^m_{MT(\mathbb{Z})}$ is the ring of motivic periods of $MT(\mathbb{Z})$ and there exist a canonical ring morphism $\per: P^m_{MT(\mathbb{Z})} \to \mathbb{C}$ called the period map with the property $\per(\zeta^m(k_1, \ldots, k_d)) = \zeta(k_1, \ldots, k_d)$. We denote by $\mathcal{H} \subset P^m_{MT(\mathbb{Z})}$ the $\mathbb{Q}$-algebra generated by $1$ and all motivic multiple
Multiple zeta values also appear in other areas of pure and applied mathematics such as in the Kontsevich integrals of knots ([13], [14], [16]) in topology, or in the evaluation of scattering amplitudes ([4], [24]) in mathematical physics. Therefore they are already objects of significant interest and study.

Furthermore, there are many studies on the connection between (motivic) multiple zeta values and modular forms. The first indications of a connection between modular forms and MZV’s occurred in the formulation of the Broadhurst-Kreimer conjecture [3], when it was observed that the dimensions of the weight and depth graded multiple zeta values involves the term $S(x) = \sum_{n = 1}^{\infty} \frac{1}{n^{x+n}}$, which encodes the dimensions of $SL_2(\mathbb{Z})$-cusp forms. The depth 2 case of this conjecture is (essentially) the result of [10], which establishes the modular origin of $\zeta$(odd, odd) relations (See Theorem 1 below). The dual viewpoint is studied in [25], [2] for depth 2 and [6], [8], [21] for higher depths. Interpretation of conjectural dimension in Broadhurst-Kreimer conjecture is treated in [8], [17]. Furthermore, depth 2 and weight odd aspects are studied in [27], [20], and modular phenomena of another type for level 2 or level 3 modular forms are investigated in [13], [10]. Extensions to $q$-analogues or Eisenstein-series-analogue of multiple zeta values are studied in [1], [26].

The purpose of this paper is to establish new modular phenomena for regularized motivic iterated integrals on $\mathbb{P}^1 \setminus \{0,1,\infty\}$, which we call regularized motivic multiple zeta values. Regularized motivic multiple zeta values are generalization of usual motivic multiple zeta values, which are convergent integral cases of regularized motivic multiple zeta values. Regularized motivic multiple zeta values are very natural generalization of usual motivic multiple zeta values and used in various situations [4], however, there seems to be no work on their modular phenomena. In this paper, we give a lot of modular phenomena for regularized motivic double zeta values. Some of them are similar to the known cases of usual motivic double zeta values while others are not.

We denote by $V^w$ the space of homogeneous polynomials $p(X,Y)$ of degree $w$ with rational coefficients such that $p(-X,Y) = \pm p(X,Y)$. For an even $w$ and a modular form $f$ of weight $w+2$, an even (resp. odd) period polynomial $P_f^+(X,Y) \in \mathbb{C} \otimes V^w_+$ (resp. $P_f^-(X,Y) \in \mathbb{C} \otimes V^w_-$) is defined by using the special values of the $L$-function of $f$.

In this paper, we consider two types of modular phenomena:

**GKZ-type phenomena**: modular forms induce linear relations among regularized motivic multiple zeta values, 

**Ihara–Takao-type phenomena**: modular forms induce algebraic relations in the dual Hopf module of $H$.

### 1.1. Known GKZ-type phenomena for the usual motivic double zeta values

The eponymous result in the area of GKZ-type phenomena is the following well-known result due to Gangl-Kaneko-Zagier.

**Theorem 1** ([10], Gangl–Kaneko–Zagier). Let $w$ be a nonnegative even integer, $k = w + 2$, and $f$ a modular form for $SL_2(\mathbb{Z})$ of weight $k$. Define $a_0, \ldots, a_w \in \mathbb{C}$ by

$$\sum_{r=0}^{w} a_r x^r y^{w-r} = P_f^+(x+y,x).$$

Then

$$\sum_{r=0}^{w-2} a_r r!(w-r)\zeta^m(r + 1, w - r + 1) \equiv 0 \pmod{\zeta^m(k)}.$$

Furthermore, this gives a one-to-one correspondence between the modular forms for $SL_2(\mathbb{Z})$ of weight $k$ and the linear relations among

$$\{\zeta^m(r + 1, s + 1) \mid r + s = w, r \geq 0, s > 0, r : even\}$$

modulo $\zeta^m(k)$.

**Remark 2**. In fact, a more refined equality in the space of formal double zeta value is proved in [10]. The space of formal double zeta values is defined as the set of formal sums of indices modulo double shuffle relations, and motivic multiple zeta values satisfy double shuffle relations, thus Gangl–Kaneko–Zagier’s original result implies the former part of Theorem 1. If we assume that the relations among motivic double zeta values are exhausted by double shuffle relations of depth two and “Euler’s relations” $\zeta^m(a)\zeta^m(b) \in \mathbb{Q}\zeta^m(a + b) (a, b : even)$ (which seems to be reasonable), the latter part of Theorem 1 is also a consequence of their result.

**Remark 3**. In Theorem 1 the modulus is taken as $\mathbb{C}\zeta^m(k)$ rather than $\mathbb{Q}\zeta^m(k)$ so as to allow any modular forms. If we start with a modular form whose even period polynomial has rational coefficients, then Theorem 1 gives $Q$-linear relations among motivic MZVs. It is also known that any modular form can be expressed as a $C$-linear sum of such modular forms.

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1For example, regularized motivic multiple zeta values appear even in the calculation of the coproduct of usual motivic multiple zeta values. In addition, regularized motivic multiple zeta values appear in the coefficient of motivic KZ-associator.
Theorem 9 gives a complete description of the linear relations among motivic double zeta values of type $\zeta^m(\text{odd, odd})$ in terms of even period polynomials of modular forms. Thus this theorem gives rise to the following questions: How about the linear relations among $\zeta^m(\text{odd, even})'$s, $\zeta^m(\text{even, odd})'$s, or $\zeta^m(\text{even, even})'$s? The answers to these questions are also known as explained below.

For the first case, for an odd integer $k > 2$, Ma constructed linear relations among $\zeta^m(\text{odd, even})$'s of weight $k$ from the odd (resp. even) period polynomial of modular forms of weight $k - 1$ (resp. $k + 1$) for $SL_2(\mathbb{Z})$, and Li–Liu showed that all the linear relations among $\zeta^m(\text{odd, even})$'s are essentially exhausted by Ma’s relations. Hereafter, we use the expression such as $\frac{\partial}{\partial x}p(ax + by, cx + dy)$ for the partial derivative of $p(ax + by, cx + dy)$ with respect to $x$. We remind the reader that it should not be confused with $(\frac{\partial}{\partial y}) (ax + by, cx + dy)$ which is the partial derivative of $p(u, v)$ with respect to $u$ evaluated at $(u, v) = (ax + by, cx + dy)$.

**Theorem 9** ([27, Lemma 3], Ma). By (2.1), we can lift the first congruence to the one for motivic multiple zeta values. Here $\zeta^m(\text{odd, even})$ and $\zeta^m(\text{even, odd})$ are not introduced in [27]. However, we can easily extract a proof of Theorem 9 from Zagier’s original proof. More precisely, for an odd number $k > 2$, $10$ relations exhaust all the linear relations among $\zeta^m(\text{odd, odd})$.

**Theorem 11** (Ma). Let $w$ be an odd positive integer, $k = w + 2$, $f$ a cusp form for $SL_2(\mathbb{Z})$ of weight $k - 1$, and $g$ a cusp form for $SL_2(\mathbb{Z})$ of weight $k + 1$. Define $a_0, \ldots, a_w \in \mathbb{C}$ by

$$
\sum_{r=0}^{w} a_r X^r Y^{w-r} = XP_f^+(X + Y, X) + \frac{\partial}{\partial Y} P_g^+(X + Y, X).
$$

Then we have

$$
\sum_{r=0}^{w-1} (a_r - a_{w-r})! (w-r)! \zeta^m(r+1, w-r+1) \equiv 0 \pmod{\mathbb{C}\zeta^m(k)}.
$$

Furthermore, these linear relations are linearly independent, i.e., if $a_r - a_{w-r} = 0$ for all even $r$ then $f = g = 0$.

**Remark 5.** As in the case of Theorem 4, we can obtain a congruence modulo $\mathbb{Q}\zeta^m(k)$ if the coefficients of $P_f^+(X, Y)$ and $P_g^+(X, Y)$ are both rational, and any cusp form of weight $k - 1$ (resp. $k + 1$) is a $\mathbb{C}$-linear sum of the modular forms $f$ (resp. $g$) whose odd (resp. even) period polynomials are rational. Furthermore, Ma’s original theorem is stated as an equality in the formal double zeta space. Since all linear relations among odd weight motivic double zeta values are given by double shuffle relations, the result of Ma is equivalent to Theorem 4.

**Theorem 6** ([20], Ma). Let $k \geq 3$ be an odd integer. Then we have

$$2\zeta^m(k-2, 2) - (k-2)\zeta^m(1, k-1) + \sum_{1 \leq r \leq s \leq K-2, \text{odd}} (r-s)\zeta^m(r, s) \equiv 0 \pmod{\mathbb{Q}\zeta^m(k)}.
$$

**Remark 7.** Theorem 4 can be extend to the case where $f$ is an Eisenstein series for $SL_2(\mathbb{Z})$. One may guess that Theorem 6 is a multiple of the case $(f, g) = (E_{k-1}, 0)$ of Theorem 4 but this is not correct in general.

**Theorem 7** ([18], Li–Liu). Ma’s relations in Theorems 4 and 6 exhaust all the linear relations among $\zeta^m(\text{odd, even})'$s modulo $\mathbb{Q}\zeta^m(k)$.

For the second case, Zagier showed the following result.

**Theorem 9** ([27], Theorem 2, Lemma 3], Zagier). There are no linear relations among $\zeta^m(\text{even, odd})'$s and $\zeta^m(\text{odd, odd})'$s.

**Remark 10.** The original statement in [27, Theorem 2] is slightly different from Theorem 9 since a motivic setting is not introduced in [27]. However, we can easily extract a proof of Theorem 9 from Zagier’s original proof. More precisely, for an odd number $k = 2K + 1 \geq 3$, Zagier showed that

$$
\zeta(2r, k-2r) \equiv \sum_{s=1}^{K-1} \left( \frac{2K-2s}{2r-1} + \frac{2K-2s}{2K-2r} \right) \zeta(2s) \zeta(k-2s) \pmod{\mathbb{Q}\zeta(k)}
$$

([27, (36)]) and

$$
\det \begin{pmatrix}
\frac{2K-2s}{2r-1} + \frac{2K-2s}{2K-2r} \\
\frac{2K-2s}{2r-1} + \frac{2K-2s}{2K-2r}
\end{pmatrix}
_{1 \leq r, s \leq K-1} \neq 0
$$

([27, Lemma 3]). By (2.1), we can lift the first congruence to the one for motivic multiple zeta values. Here $\zeta^m(2s)\zeta^m(k-2s)$ with $1 \leq s \leq K - 1$ and $\zeta^m(k)$ are linearly independent. Thus the non-vanishing of the determinant implies Theorem 9.

For the third case, the harmonic product formula gives $\zeta^m(2a, 2b) + \zeta^m(2b, 2a) \in \mathbb{Q}\zeta^m(2a + 2b)$. In fact, these relations exhaust all the linear relations among $\zeta^m(\text{even, even})'$s modulo $\mathbb{Q}\zeta^m(\text{even})$.

**Theorem 11** (Tasaka, personal communication). The linear relations among $\zeta^m(\text{even, even})'$s modulo $\mathbb{Q}\zeta^m(\text{even})$ are exhausted by the relations $\zeta^m(2a, 2b) + \zeta^m(2b, 2a) \equiv 0 \pmod{\mathbb{Q}\zeta^m(2a + 2b)}$. 


Remark 12. Let \( \Gamma_B \subset \text{SL}_2(\mathbb{Z}) \) be the congruence subgroup defined in Section 2.3 Then the space of odd symmetric polynomials of degree \( w \) can be interpreted as the space of odd period polynomials for \( \Gamma_B \) (Proposition 1). Thus Theorem 11 says that

\[
\sum_{0 < r < w, r \text{ odd}} a_r r! (w - r)! \zeta^m(r + 1, w - r + 1) \equiv 0 \pmod{\zeta^m(w + 2)}
\]

if and only if there exists a modular form \( f \) for \( \Gamma_B \) of weight \( w + 2 \) such that \( P_f^j(X, Y) = \sum_{0 < r < w, r \text{ odd}} a_r X^r Y^{w-r} \).

Remark 13. The author learned Theorems 11, 17 and their proofs from Tasaka. We give a proof of Theorems 11 and 17 in Section 7.

1.2. Known Ihara–Takao-type phenomena for usual motivic double zeta values. Since \( A \) (resp. \( H \)) has a structure of a Hopf algebra (resp. \( A \)-comodule), the dual vector space \( A^\vee \) (resp. \( H^\vee \)) of \( A \) (resp. \( H \)) has a structure of non-commutative algebra (resp. \( A^\vee \)-module):

\[
A^\vee \times A^\vee \to A^\vee; \ (\sigma, \tau) \mapsto \sigma \tau,
\]

\[
A^\vee \times H^\vee \to H^\vee; \ (\sigma, \tau) \mapsto \sigma \tau.
\]

For an odd (resp. even) integer \( n \geq 2 \), let \( \sigma_n \) be any element of \( A^\vee \) (resp. \( H^\vee \)) such that \( (\sigma_n, 1) = 0 \) and \( (\sigma_n, \zeta^m(m)) = \delta_{n,m} \). We denote by \( \hat{H}_w \) the space spanned by all the linear relations among any \( \sigma \in A^\vee \) and \( \sigma \in H^\vee \).

Theorem 14 ([12, 25], Ihara–Takao, Schneps). Let \( w \) be a nonnegative even integer, \( k = w + 2 \), and \( a_2, a_4, \ldots, a_{w-2} \) be complex numbers. Then

\[
\sum_{2 \leq r \leq w-2, r \text{ even}} a_r \sigma_{r+1} \sigma_{w-r+1} \bigg|_{\hat{H}_w} = 0
\]

if and only if there exists \( f \in M_k(\text{SL}_2(\mathbb{Z})) \) such that

\[
P_f^j(X, Y) = \sum_{2 \leq r \leq w-2, r \text{ even}} a_r X^r Y^{w-r}.
\]

On the other hand, for odd \( w \), \( \sigma_n \sigma_m |_{\hat{H}_w} \)'s are linearly independent. For subsets \( N, N' \subset \mathbb{Z}_{\geq 0} \), we denote by \( \hat{H}_w(N, N') \) the subspace of \( \hat{H}_w \) spanned by

\[
\{ \zeta^m(r + 1, s + 1) : r + s = w, s > 0, r \in N, s \in N' \}.
\]

We denote by \( \text{odd} \) (resp. \( \text{even} \)) the set of nonnegative odd (resp. even) integers. Then it is known that \( \hat{H}_w(\text{even}, \text{even}) = \hat{H}_w \) for an even \( w \) ([10, Theorem 2]) and \( \hat{H}_w(\text{odd}, \text{even}) = \hat{H}_w \) for an odd \( w \) ([27, Theorem 2]). However \( \hat{H}_w(\text{even}, \text{odd}) \) and \( \hat{H}_w(\text{odd}, \text{odd}) \) do not coincide with \( \hat{H}_w \) in general. The following three theorems gives complete descriptions of all the linear relations among \( \sigma_n \sigma_m |_{\hat{H}_w(\text{even}, \text{odd})} \)'s and those among \( \sigma_n \sigma_m |_{\hat{H}_w(\text{odd}, \text{odd})} \)'s.

Theorem 15 ([27, Section 6], Zagier). Let \( w \) be an odd positive integer, \( w = k + 2 \), \( f \) a cusp form for \( \text{SL}_2(\mathbb{Z}) \) of weight \( k - 1 \), and \( g \) a cusp form for \( \text{SL}_2(\mathbb{Z}) \) of weight \( k + 1 \). Define \( a_0, \ldots, a_w \in \mathbb{C} \) by

\[
\sum_{r=0}^{w} a_r X^r Y^{w-r} = XP_f^j(X, Y) + \frac{\partial}{\partial Y} P_g^j(X, Y).
\]

Then we have

\[
\sum_{2 \leq r \leq w-1, r \text{ even}} a_r \sigma_{r+1} \sigma_{w-r+1} \bigg|_{\hat{H}_w(\text{even}, \text{odd})} = 0.
\]

Furthermore, these linear relations are linearly independent, i.e., if \( a_r = 0 \) for all even \( r \) then \( f = g = 0 \).

Theorem 16 ([13, Li–Liu]). Zagier’s relations in Theorem 15 exhaust all linear relations among \( \sigma_n \sigma_m |_{\hat{H}_w(\text{even}, \text{odd})} \)'s.

Theorem 17 (Tasaka, personal communication). Let \( w \) be a nonnegative even integer and \( k = w + 2 \). Then

\[
\sum_{0 < r < w, r \text{ even}} a_r \sigma_{r+1} \sigma_{w-r+1} \bigg|_{\hat{H}_w} = 0
\]

if and only if there exists a weight \( k \) modular forms \( f \) for \( \Gamma_0(2) \) such that

\[
P_f^j(X, Y) = \sum_{0 < r < w, r \text{ even}} a_{w-r} X^r Y^{w-r}.
\]
1.3. The first main theorem: GKZ-type modular phenomena for non admissible motivic double zeta values. Recall that a motivic multiple zeta value \( \zeta_m(k_1, \ldots, k_d) \) for \( k_d > 1 \) is defined by a motivic admissible iterated integral
\[
(-1)^d I_m(0; 10^{k_1-1} \ldots 10^{k_d-1}; 1).
\]
Let \( 0' \) and \( 1' \) be tangential base points at 0 and 1 respectively such that \( I_m(0'; 0; 1') = 0 \) and \( I_m(0'; 1; 1') = -T \) where \( T \) is an indeterminate. We define a regularized motivic multiple zeta value by
\[
J(k_0; k_1, \ldots, k_d) := I_m(0'; 0; 10^{k_0} 10^{k_1} \ldots 10^{k_d}; 1') \in H[T].
\]
We call \( J(k_0; k_1, \ldots, k_d) \) an admissible (resp. non admissible) motivic multiple zeta value if \( k_0 = 0 \) and \( k_d > 0 \) (resp. \( k_0 \neq 0 \) or \( k_d = 0 \)). By definition, an admissible motivic multiple zeta values is a usual motivic multiple zeta value, i.e.,
\[
J(0; k_1, \ldots, k_d) = (-1)^d \zeta_m(k_1 + 1, \ldots, k_d + 1).
\]
Note that an explicit expression of a regularized motivic multiple zeta value in terms of admissible motivic multiple zeta values is given by
\[
J(k_0; k_1, \ldots, k_d) = (-1)^{k_0 + d} \sum_{l_1 + \ldots + l_d = k_0} \binom{k_1 + l_1}{l_1} \cdots \binom{k_d + l_d}{l_d} \zeta_m(w(k_1 + 1, \ldots, k_d + l_d + 1; T)),
\]
where \( \zeta_m(w(k_1 + 1, \ldots, k_d + l_d + 1; T)) \) is the shuffle regularized polynomial (e.g. \( \zeta_m(1; T) = T \)). Note that Theorem \( \text{I} \) is concerned with \( J(0; \text{even}, \text{even})'s \), Theorems \( \text{II} \) \( \text{III} \) \( \text{IV} \) and \( \text{V} \) are concerned with \( J(0; \text{odd}, \text{even})'s \), Theorem \( \text{VI} \) is concerned with \( J(0; \text{odd}, \text{even})'s \), and Theorems \( \text{VII} \) \( \text{VIII} \) are concerned with \( J(0; \text{odd}, \text{odd})'s \), and Theorem \( \text{IX} \) is concerned with the space of all motivic double zeta values (which is equal to the space spanned by \( J(0; \text{even}, \text{even})'s \)). Thus the following 8 questions about the behavior of \( J(N; N'; N'') \) and \( J(N; N'; 0)'s \) for \( N, N' \in \{ \text{even}, \text{odd} \} \) then naturally arise. In this paper, we answer all these questions. Furthermore, we also give complete answers to the same questions about \( J(1; \text{even}, \text{even})'s, J(1; \text{even}, \text{odd})'s, J(1; \text{even}, \text{even})'s, J(1; \text{odd}, \text{odd})'s \) and \( J(1; \text{odd}, \text{odd})'s \).

We put \( \tilde{H} = H[T] \). We define the depth filtration \( \mathcal{D} \) on \( \tilde{H} \) by
\[
\mathcal{D}_d \tilde{H} = \left\{ J_d(n_1, \ldots, n_t; T) \mid i \leq d, (n_1, \ldots, n_t) \in \mathbb{Z}_{\geq 0}^l \right\}.
\]
We define the depth graded \( J \)-value \( J_d(k_0; k_1, \ldots, k_d) \) to be the image of \( J(k_0; k_1, \ldots, k_d) \) in \( \text{gr}_d^2 \tilde{H} = \mathcal{D}_d \tilde{H}/\mathcal{D}_{d+1} \tilde{H} \). We put \( \text{gr}^0 = \mathbb{Z} \) and \( \text{gr}^1 = \{ 1 \} \). For subsets \( N, N', N'' \subset \mathbb{Z}_{\geq 0} \), we denote by \( H_w(N; N', N'') \) the subspace of \( \mathcal{D}_d \tilde{H} \) spanned by
\[
\{ J_d(r; s, t) : r + s + t = w, r \in N, s \in N', t \in N'' \}.
\]
We put \( H_w := H_w(\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\geq 0}, \mathbb{Z}_{\geq 0}) \). We denote by \( W_w^{+} \subset V_w \) (resp. \( W_w^{-} \subset V_w \)) the space of even (resp. odd) periodic polynomials with rational coefficients for weight \( w + 2 \) modular (resp. cusp) forms for \( \Gamma \) (see Section \( \text{II} \) for detail). We put \( W_w^{+} := W_w^{+ \cdot SL_2(\mathbb{Z})} \). Then Theorem \( \text{I} \) can be restated as the existence of the exact sequence
\[
\begin{align*}
0 \to W_w^{+} & \xrightarrow{u_{\text{GKZ}}} V_w^{+} \xrightarrow{X^a Y^b \cdot \text{ad} \cdot J_d(0; a, b)} H_w(0; \text{even}, \text{even}) \to 0 \quad \text{ (w : even)}
\end{align*}
\]
where \( u_{\text{GKZ}} \) is a linear map defined by
\[
u_{\text{GKZ}}(p(X, Y)) := \frac{1}{2} (p(-X - Y, X) + p(X - Y, X)).
\]
Since the left-hand side of Theorem \( \text{II} \) is equal to
\[
2\zeta_m(k - 2, 2) - (k - 2)\zeta_m(1, k - 1) + \sum_{1 \leq r \leq k - 2, \text{odd}} (r - s)\zeta_m(r, s)
\]
\[
= 2J(0; w - 1, 1) - wJ(0; 0, w) + \sum_{0 \leq r \leq w - 1, \text{even}} (r - s)J(0; r, s) \quad (w := k - 2, r := r - 1, s := s - 1)
\]
\[
= \Psi \left( \frac{2X^{w-1}Y}{(w-1)!} - Y \frac{W}{(w-1)!} + X \sum_{0 \leq r \leq w-1, \text{even}} \frac{X^{r-1}Y^s}{(r-1)!s!} - Y \sum_{0 \leq r \leq w-1, \text{even}} \frac{X^r Y^{s-1}}{r!(s-1)!} \right) \text{ (}\Psi(X^a Y^b) := \text{ad} \cdot J_d(0; a, b)\text{)}
\]
\[
= \frac{1}{(w-1)!} \Psi \left( 2X^{w-1}Y - Y + X \frac{X + Y}{2} - \frac{X - Y}{2} \right) - Y \frac{X + Y}{2} + \frac{X - Y}{2}
\]
\[
= \frac{1}{2(w-1)!} \Psi \left( (X - Y)(X + Y)^{w-1} - (X + Y)(X - Y)^{w-1} - 2Y^w + 4X^{w-1}Y \right),
\]
where
Theorem 9 is equivalent to the exactness of the sequence

\[ 0 \to W_{w-1}^- \times \tilde{W}_{w+1}^+ \times \mathbb{Q} \xrightarrow{u_M^{\Gamma_A}} V_w^+ \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(0, a, b)} H_w(0; \text{even, odd}) \to 0 \quad (w: \text{odd}) \]

where

\[ \tilde{W}_{w+1}^+ := W_{w+1}^+ / \mathbb{Q}(X^{w+1} - Y^{w+1}) \]

and \( u_M \) is a linear map defined by

\[ u_M(p(X, Y), q(X, Y), c) = \frac{1}{2} \left( R(X, Y) - R(Y, X) + R(-X, Y) - R(Y, -X) \right) \]

\[ + c \left( (X - Y)(X + Y)^{w-1} - (X + Y)(X - Y)^{w-1} - 2Y^w + 4X^{w-1}Y \right) \]

with

\[ R(X, Y) := X p(X + Y, X) + \frac{\partial}{\partial Y} q(X + Y, X). \]

Theorem 11 is equivalent to the exactness of the sequence

\[ 0 \to V_w^- \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(0, a, b)} H_w(0; \text{odd, even}) \to 0 \quad (w: \text{odd}), \]

and Theorem 13 can be restated as the existence of the exact sequence

\[ 0 \to W_w^- \xrightarrow{\text{id}} V_w^- \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(0, a, b)} H_w(0; \text{odd, odd}) \to 0 \quad (w: \text{even}) \]

where \( \text{id} \) is just the inclusion map.

Let \( \Gamma_A \) be the congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \) defined by

\[ \Gamma_A := \left\{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \pmod{2} \right\}. \]

Our first main theorem is as follows:

**Theorem 18.** There exist the following 13 exact sequences.

**Case 1:**

\[ 0 \to W_w^- \xrightarrow{\text{id}} V_w^- \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(a, b, 0)} H_w(\text{odd; odd}, 0) \to 0 \quad (w: \text{even}) \]

**Case 2:**

\[ 0 \to W_w^+ \xrightarrow{\text{id}} V_w^+ \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(a, b, 0)} H_w(\text{even; even}, 0) \to 0 \quad (w: \text{even}) \]

**Case 3:**

\[ 0 \to W_w^+ \xrightarrow{u_{\text{GKZ}}} V_w^+ \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(1, a, b)} H_{w+1}(1; \text{even, even}) \to 0 \quad (w: \text{even}) \]

\[ 0 \to W_w^+ \xrightarrow{u_{\text{GKZ}}} V_w^+ \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(a, 0, b)} H_w(\text{even; 0, even}) \to 0 \quad (w: \text{even}) \]

\[ 0 \to W_w^- \xrightarrow{u_{\text{GKZ}}} V_w^- \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(a, 1, b)} H_{w+1}(\text{odd; 1, odd}) \to 0 \quad (w: \text{even}) \]

**Case 4:**

\[ 0 \to W_{w-1}^- \times \tilde{W}_{w+1}^+ \times \mathbb{Q} \xrightarrow{u_M} V_w^+ \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(1, a, b)} H_{w+1}(1; \text{even, odd}) \to 0 \quad (w: \text{odd}) \]

\[ 0 \to W_{w-1}^- \times \tilde{W}_{w+1}^+ \times \mathbb{Q} \xrightarrow{u_M} V_w^+ \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(a, 1, b)} H_{w+1}(\text{even; 1, odd}) \to 0 \quad (w: \text{odd}) \]

\[ 0 \to W_{w-1}^- \times \tilde{W}_{w+1}^+ \xrightarrow{u_M'} V_w^- \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(a, 0, b)} H_w(\text{odd; 0, even}) \to 0 \quad (w: \text{odd}) \]

where \( u_M' \) is a linear map defined by

\[ u_M'(p(X, Y), q(X, Y)) = \frac{1}{2} \left( R(X, Y) + R(Y, X) - R(-X, Y) - R(Y, -X) \right) \]

with \( R(X, Y) = X p(X + Y, X) + \frac{\partial}{\partial Y} q(X + Y, X). \)

**Case 5:**

\[ 0 \to W_w^- \xrightarrow{\text{id}} V_w^- \xrightarrow{X^a Y^b \mapsto \alpha b J_2^\bullet(a, 0, b)} H_w(\text{odd; 0, odd}) \to 0 \quad (w: \text{even}). \]
Case 6:

\[(1.14) \quad 0 \to \mathbb{Q}(X, Y) \xrightarrow{id} V_w^+ \xrightarrow{X^aY^b + abJ_D(a, b)} H_w(even; 0, odd) \to 0 \quad (w : odd)\]

\[(1.15) \quad 0 \to V_w^+ \xrightarrow{X^aY^b + abJ_D(a, b; 1, b)} H_{w+1}(even; 1, even) \to 0 \quad (w : even)\]

\[(1.16) \quad 0 \to V_w^+ \xrightarrow{X^aY^b + abJ_D(a, b; 0, 0)} H_w(even; odd, 0) \to 0 \quad (w : odd)\]

\[(1.17) \quad 0 \to V_w^+ \xrightarrow{X^aY^b + abJ_D(b, a, 0)} H_w(odd; even, 0) \to 0 \quad (w : odd)\]

where

\[A(X, Y) = (X + Y)^w + (-X + Y)^w.\]

Let us make some comments about this Theorem. Cases 1, 2 and 5 are the cases where whose second arrows are identity maps, but the modular groups differ. The most surprising case is Case 1, where the odd period polynomials appear. This seems to be the first non trivial example of constructing relations among multiple zeta values (or their analogues) from modular forms for a congruence subgroup other than the \(SL_2(\mathbb{Z})\). Cases 3, 4 and 5 deals with the exact sequences analogous to (1.1), (1.7) and (1.4), respectively. Finally, Case 6 is (almost) no relation case.

Theorem 17 can be restated as the existence of the exact sequence

\[(1.21)\]

while Theorem 16 can be restated as the existence of the exact sequence

\[(1.22)\]

Moreover, for even \(w\), we know that \(\tilde{H}_w(even, even) = \tilde{H}_w\). Hence (1.15) can be restated as the existence of the exact sequence

\[(1.20)\]

Theorems 15 and 16 can be restated as the existence of the exact sequence

\[(1.23)\]

Our second main theorem is as follows:

**Theorem 19.** There exist the following 13 exact sequences (each case corresponds to the one in Theorem 15).

Case 1:

\[(1.24) \quad 0 \to W_w^{-\Gamma_A} \xrightarrow{id} V_w^+ \xrightarrow{X^aY^b + abJ_D(a, b)} H_w(odd; odd; 0)^\vee \to 0 \quad (w : odd, w \geq 2)\]

Case 2:

\[(1.25) \quad 0 \to W_w^+ \xrightarrow{id} V_w^+ \xrightarrow{X^aY^b + abJ_D(a, b)} H_w(even; even; 0)^\vee \to 0 \quad (w : even)\]

The reason why we did not count (1.4) as the first example is because the linear relations given by (1.4) are nothing but the simple relation \(\zeta^n(2a + 2b) + \zeta^n(2b, 2a) \in \mathbb{Q} \zeta^n(2a + 2b)\). For the sake of uniform description, we related even such simple formulas with the space of period polynomials, but the author does not know whether this is a natural interpretation or not.
Case 3:

(1.26) \[ 0 \to W^+_w \xrightarrow{p \to f^Y_{p \to 0} \circ Y} V^+_w \xrightarrow{\lambda} H_{w+1(\text{even}; \text{even})} \to 0 \quad (w : \text{even}) \]

(1.27) \[ 0 \to W^+_w \xrightarrow{id} V^+_w \xrightarrow{\lambda} H_{w(\text{even}; 0; \text{even})} \to 0 \quad (w : \text{even}) \]

(1.28) \[ 0 \to (\partial_X^1 W^-_w) \xrightarrow{id} V^+_w \xrightarrow{\lambda} H_{w+1(\text{odd}; 1; \text{odd})} \to 0 \quad (w : \text{even}) \]

where

\[ \partial_X^1 W^-_w := \left\{ p \in V^+_w \mid \frac{\partial}{\partial Y} Y \in W^-_w \right\} \]

Case 4:

(1.29) \[ 0 \to (\partial_X^1 W^-_{w-1}) \times W^+_w \xrightarrow{(p,q) \to \cdot X \cdot Y \cdot p \cdot q} V^+_w \xrightarrow{\lambda} H_{w+1(1; \text{even}; \text{odd})} \to 0 \quad (w : \text{odd}) \]

(1.30) \[ 0 \to (\partial_X^1 W^-_{w-1}) \times W^+_w \xrightarrow{(p,q) \to \cdot Y \cdot p \cdot q} V^+_w \xrightarrow{\lambda} H_{w+1(\text{even}; 1; \text{odd})} \to 0 \quad (w : \text{odd}) \]

(1.31) \[ 0 \to W^-_{w-1} \oplus W^-_{w+1} \xrightarrow{(p,q) \to \cdot Y \cdot p \cdot q} V^+_w \xrightarrow{\lambda} H_{w(\text{odd}; 0; \text{even})} \to 0 \quad (w : \text{odd}) \]

where

\[ \partial_Y^1 W^-_{w-1} := \left\{ p \in V^-_w \mid \frac{\partial}{\partial Y} Y \in W^-_{w-1} \right\} \]

and

\[ \partial_X^1 W^-_{w-1} := \left\{ p \in V^-_w \mid \frac{\partial}{\partial Y} Y \in W^-_{w-1} \right\} \]

Case 5:

(1.32) \[ 0 \to W^-_{w-1} \times V^+_w \xrightarrow{id} V^+_w \xrightarrow{\lambda} H_{w(\text{odd}; 0; \text{odd})} \to 0 \quad (w : \text{even}, w \geq 2) \]

Case 6:

(1.33) \[ 0 \to QY^w \xrightarrow{id} V^+_w \xrightarrow{\lambda} H_{w(\text{even}; 0; \text{odd})} \to 0 \quad (w : \text{odd}) \]

(1.34) \[ 0 \to V^+_w \xrightarrow{\lambda} H_{w+1(\text{even}; 1; \text{even})} \to 0 \quad (w : \text{even}) \]

(1.35) \[ 0 \to V^+_w \xrightarrow{\lambda} H_{w(\text{even}; 0; \text{odd})} \to 0 \quad (w : \text{odd}) \]

(1.36) \[ 0 \to V^+_w \xrightarrow{\lambda} H_{w(\text{odd}; 0; \text{even})} \to 0 \quad (w : \text{odd}) \]

1.5. Contents of the paper. The paper is organized as follows. In Section 2 we define some notions and give some lemmas. In Sections 3, 4, 5, 6, 7, and 8 we discuss GKO and Ihara–Takao type formulas for the case \( J(\text{odd}; 0, 0) \), the case \( J(\text{even}; 0, 0) \), the cases similar to the case \( J(0; \text{even}, 0) \), the cases similar to the case \( J(0; \text{even}, \text{odd}) \), and the cases similar to the case \( J(\text{odd}; 0; \text{even}) \) respectively. In Section 8 we discuss the cases \( J(\text{even}; 1, 0) \), \( J(\text{even}; 0, 0) \), and \( J(\text{odd}; 0; 0) \), where there are almost no linear relations. In Appendix A we give explicit expressions for the spaces of period polynomials for various congruence subgroups that appear in this paper.

2. Some preliminaries

2.1. Motivic multiple zeta values. Let \( \text{MT}(\mathbb{Z}) \) be the category of mixed Tate motives over \( \mathbb{Z} \) which is a Tannakian category extracted from Voevodsky’s triangulated category of motives (see [9]). There exist two special functors \( \omega_B, \omega_{\text{DR}} : \text{MT}(\mathbb{Z}) \to \text{Vec}_{\mathbb{Q}}(\text{Betti and de Rham realizations}) \), and the ring of motivic periods of \( \text{MT}(\mathbb{Z}) \) is defined as the ring of functions on the scheme of tensor isomorphism from \( \omega_{\text{DR}} \) to \( \omega_B \)

\[ \mathcal{P}_{\text{MT}}^m := \mathcal{O}(\text{Isom}_{\text{MT}(\mathbb{Z})}(\omega_{\text{DR}}, \omega_B)) \]

and a special ring homomorphism \( \mathcal{P}_{\text{MT}}^m(\mathbb{Z}) \to \mathbb{C} \) called period map is naturally defined (see [1]). For \( x, y \in \{0, 1\} \) and a word \( w \in \{0, 1\} \), the motivic iterated integral \( I^m(x; w; y) \) is defined. They satisfy \( I^m(x; p_1 \cdots p_k; x) = \delta_{k, 0}, I^m(0; 0; 1) = I^m(0; 1; 1) = 0 \), the reversal formula \( I^m(x; p_1 \cdots p_k; y) = (-1)^k I^m(y; p_k \cdots p_1; x) \), and the shuffle product formula \( I^m(x; w_1 \shuffle w_2; y) = I^m(x; w_1; y) \cdot I^m(x; w_2; y) \). For an index \( (k_1, \ldots, k_d) \in \mathbb{Z}_{\geq 0}^d \) with \( d = 0 \) or \( k_d > 1 \), a motivic multiple zeta value \( \zeta^m(k_1, \ldots, k_d) \) is defined by \( (-1)^d I^m(0; 10^{k_1-1} \cdots 10^{k_d-1}, 1) \). Furthermore, \( \text{per}(\zeta^m(k_1, \ldots, k_d)) \) is equal to the multiple zeta value \( \zeta(k_1, \ldots, k_d) \). It is shown by Brown ([1]) that \( \mathcal{P}_{\text{MT}}^m(\mathbb{Z}) \) is spanned by the motivic multiple zeta values and the reciprocal of motivic 2πi. Let \( \mathcal{H} \subset \mathcal{P}_{\text{MT}}^m(\mathbb{Z}) \) be the subspace spanned by all weight \( k \) motivic multiple zeta values, and \( \mathcal{H} := \bigoplus_{k=0}^{\infty} \mathcal{H}_k \) the graded ring of motivic multiple zeta values. Put \( A := \mathcal{H} / \zeta^m(2) \mathcal{H} \). We denote the image of \( I^m(x; w; y) \) in \( A \) by \( I^m(x; w; y) \). By the Hopf structure of \( \mathcal{P}_{\text{MT}}(\mathbb{Z}) \), \( A \) becomes a commutative non-cocommutative Hopf algebra and \( \mathcal{H} \) becomes \( A \)-comodule. Its coproduct
We define an inner product in other words. Furthermore, we denote by Lemma 20.

2.2. Coaction and antipode formulas for motivic multiple zeta values. One of the main tools to study motivic multiple zeta values is their coproduct structure. Define a linear map \( \psi : Q[x_0, x_1, x_2] \to H \) by

\[
\psi(x_0^{k_0}x_1^{k_1}x_2^{k_2}) = k_0!k_1!k_2!J(k_0; k_1, k_2).
\]

We define an inner product \( \langle \cdot, \cdot \rangle \) on \( Q[X, Y] \) by

\[
\langle p(X, Y), q(X, Y) \rangle = \langle p(\partial X, \partial Y), q(X, Y) \rangle |_{X=Y=0},
\]

in other words

\[
\left\langle X^aY^b, X'^aY'^b \right\rangle = a!b!\delta_{a,a'}\delta_{b,b'}.
\]

Similarly, we also define an inner product \( \langle \cdot, \cdot \rangle \) on \( Q[x_0, x_1, x_2] \) by

\[
\langle p(x_0, x_1, x_2), q(x_0, x_1, x_2) \rangle = \langle p(\partial X, \partial Y, \partial Z), q(X, Y, Z) \rangle |_{X=Y=Z=0}.
\]

Furthermore, we denote by \( \langle \cdot \rangle_H : \tilde{H} \times \tilde{H}^* \to Q \) the natural pairing map.

Lemma 20. For \( p \in Q[x_0, x_1, x_2] \) and \( q \in V_{+}^* \), we have

\[
\langle \psi(p(x_0, x_1, x_2)), \lambda(q(X, Y)) \rangle_H = \langle p(x_0, x_1, x_2), q(x_1 - x_0, x_2 - x_0) + q(x_2 - x_1, x_1 - x_0) - q(x_2 - x_1, x_2 - x_0) \rangle
\]

\[
= \langle p(-X - Y, X, Y) + p(-Y, -X + Y, X) - p(-Y, -X, X + Y), q(X, Y) \rangle_2.
\]

Proof. Recall that \( J(k_0; k_1, \ldots, k_d) \in \tilde{H} \) is defined by a motivic iterated integral

\[
J(k_0; k_1, \ldots, k_d) = I^m(0; \theta^01^k \cdots 1^k; 1')
\]

Brown’s coproduct formula is for iterated integral with tangential base points \( \tilde{I}_0 \) and \( -\tilde{I}_1 \), but the same formula also holds for iterated integral with tangential base points \( 0' \) and \( 1' \), i.e.,

\[
\Delta I^m(a'_0; a_1, \ldots, a_k; a'_{k+1}) = \sum_{r=0}^k \sum_{0 \leq i_0 < i_1 < \cdots < i_r < i_r+1 = k+1} \prod_{j=0}^r I^m(a'_{i_j}; a_{i_j+1}, \ldots, a'_{i_{j+1}-1} a'_{i_{j+1}} \otimes I^m(a'_0; a_1, \ldots, a_r; a'_{r+1})
\]

for \( a_0, \ldots, a_{k+1} \in \{0, 1\} \). We put

\[
\tilde{J}(t_0; t_1, \ldots, t_d) := \sum_{k_0, \ldots, k_d=0}^\infty t_0^{k_0} \cdots t_d^{k_d} J(k_0; \ldots, k_d).
\]

Then by (2.2) we have

\[
\Delta \tilde{J}(x_0; x_1, x_2) = 1 \otimes \tilde{J}(x_0; x_1, x_2) + \tilde{J}(x_0; x_1, x_2) \otimes 1
\]

\[
+ \tilde{J}(x_0; x_1, x_2) \otimes \tilde{J}(x_0; x_1) + \tilde{J}(x_1; x_2) \otimes \tilde{J}(x_0; x_1) - \tilde{J}(x_2; x_1) \otimes \tilde{J}(x_0; x_2)
\]

where \( \tilde{J}(\cdot; \cdot) = (\tilde{J}(\cdot; \cdot) \bmod \zeta^m(2)) \). (See [11] Theorem 5.1 for detail, where Goncharov gives an explicit formula for the coproduct of such type generating function at the level of formal symbols for general depth, and the case \( m = 2 \).

(2.3)

\[
\gamma(\cdot, \cdot) := (\gamma(\cdot, \cdot) \bmod \zeta^m(2)).
\]

(2.4)

\[
\gamma(\cdot, \cdot) = (\gamma(\cdot, \cdot) \bmod \zeta^m(2)).
\]

Since \( \gamma(0; y - x, x) \sigma_{n+1} = \gamma(0; y - x, x) \sigma_{n+1} = (y - x)^n \) and \( \gamma(0; x_1, x_2, \sigma_{m+1} \sigma_{n+1}) = \gamma(0; x_1, x_2, \sigma_{m+1} \sigma_{n+1}) = \gamma(0; x_1, x_2, \sigma_{m+1} \sigma_{n+1}) \), the \( \gamma(x_0; x_1, x_2) = \sum_{j} f_j \otimes g_j \) gives

\[
\gamma(x_0; x_1, x_2, \lambda(q(X, Y)))_H = q(x_1 - x_0, x_2 - x_0) + q(x_2 - x_1, x_1 - x_0) - q(x_2 - x_1, x_2 - x_0).
\]
Thus,
\[ \langle \psi(p(x_0, x_1, x_2)), \lambda(q(X, Y)) \rangle_H = \langle \langle 3(x_0; x_1, x_2), p(x_0, x_1, x_2) \rangle_3, \lambda(q(X, Y)) \rangle_H \]
\[ = \langle p(x_0, x_1, x_2), \langle 3(x_0; x_1, x_2), \lambda(q(X, Y)) \rangle_H \rangle_3 \]
\[ = \langle p(x_0, x_1, x_2), q(x_1 - x_0, x_2 - x_0) + q(x_2 - x_1, x_1 - x_0) - q(x_2 - x_1, x_2 - x_0) \rangle. \]
Hence the first equality of the lemma is proved. The second equality is obvious by the definition.

It is known that for \( u \in \tilde{H}, \Delta(u) - 1 \otimes u - u \otimes 1 = 0 \) if and only if \( u \in \mathcal{D}_1\tilde{H} \). As corollaries of the above lemma, we have the following lemmas.

**Lemma 21.** Let \( p = p(x_0, x_1, x_2) \in \mathbb{Q}[x_0, x_1, x_2] \) be a homogeneous polynomial of degree \( w \). Then \( \psi(p) \in \mathbb{Q}[J(w + 1) \] if and only if \( p(-X - Y, X, Y) + p(-X, Y + X) + p(-Y, X, Y + X) \in V_w^- \).

**Lemma 22.** Let \( w \) be a positive integer, \( N_0, N_1, N_2 \) subsets of \( \mathbb{Z}_{\geq 0} \), and \( q(X, Y) \in V_w^+ \). Then \( \lambda(q) \mid_{H_w(N_0, N_1, N_2)} = 0 \) if and only if for all \( n_0 \in N_0, n_1 \in N_1, n_2 \in N_2 \), the coefficients of \( x_0^{n_0}x_1^{n_1}x_2^{n_2} \) in
\[ q(x_1 - x_0, x_2 - x_0) + q(x_2 - x_1, x_1 - x_0) - q(x_2 - x_1, x_2 - x_0) \]
vanish.

**Lemma 23.** Let \( w \) be a nonnegative even integer, \( p = p(x_0, x_1, x_2) \in \mathbb{Q}[x_0, x_1, x_2] \) be a homogeneous polynomial of degree \( w \), and \( q(X, Y) \in V_w^+ \). Then
\[ \langle \psi(p(x_0, x_1, x_2)), \lambda(q(X, Y)) \rangle_H = \langle \psi(p(x_0, x_0, x_2)), \lambda(q(Y, X)) \rangle_H. \]

**Lemma 24.** Let \( a \in 2\mathbb{Z}_{\geq 0} \) and \( b \in \mathbb{Z}_{\geq 0} \). Then for \( q \in \mathbb{Q}[X^2, Y] \),
\[ \langle J(1; a, b), \lambda(q(X, Y)) \rangle_H = \left\langle J(0; a, b), \lambda \left( \frac{\partial}{\partial Y} q(X, Y) \right) \right\rangle_H. \]

Recall that \( \tilde{A} = \tilde{H}/\mathbb{C}^m(2)\tilde{H} \) has a structure of a Hopf algebra. Let \( S : \tilde{A} \to \tilde{A} \) be the antipode of this Hopf algebra. Note that \( S \) is an automorphism of \( \tilde{A} \). By Lemma 23 we have the following lemma.

**Lemma 25.** For nonnegative integers \( a, b \) and \( c \) such that \( a + b + c \) is even, we have
\[ S(J_2(a; b, c)) = J_2(b; a, c). \]

### 2.3. Preliminaries for modular forms.

For a congruence subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \), we denote by \( S_k(\Gamma) \) (resp. \( M_k(\Gamma) \)) the \( \mathbb{C} \)-linear vector space of weight \( k \) cusps (resp. modular) forms for \( \Gamma \). Many modular phenomena for multiple zeta values occur via the period polynomials of modular forms, which are defined as follows.

Let \( w \) be a nonnegative even integer and \( k = w + 2 \). For a level \( m \) congruence subgroup \( \Gamma \) and \( f(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} a_n e^{2\pi inz} \in M_k(\Gamma) \), we put \( \tilde{L}(f, s) := \sum_{n \in \mathbb{Z}_{\geq 0}} a_n n^{-s} \) and \( \tilde{L}(f, s) := (2\pi)^{-\Gamma}(k - s)^{-1}L(f, s) \). Note that \( \tilde{L}(f, n) = 0 \) for integers \( n \) except for the case \( 0 \leq n \leq k \). We denote by \( V_w \subset \mathbb{Q}[X, Y] \) the space of homogeneous polynomial of degree \( w \) in \( X \) and \( Y \). We define an even (resp. odd) period polynomial \( P^+_w(X, Y) \) (resp. \( P^-_w(X, Y) \)) by the sum
\[ k! \sum_n \tilde{L}(f, n)X^{n-1}Y^{k-n-1} \]
where \( n \) runs all odd (resp. even) integers between 0 and \( k \). Then \( P^+_w(X, Y) \in V_w^{\alpha} \otimes \mathbb{C} \) and \( P^-_w(X, Y) \in \frac{1}{X^2}V_w^{\omega + 2} \otimes \mathbb{C} \). Furthermore, \( P^-_w(X, Y) \in V_w \otimes \mathbb{C} \) if and only if \( f \) vanishes at two cusps 0 and \( \infty \). We define the space of even (resp. odd) period polynomials with complex or rational coefficients by
\[ W_{w, \mathbb{C}}^{+, \Gamma} := \{ P^+_w(X, Y) \mid f \in M_{w+2}(\Gamma) \}, \quad W_{w, \mathbb{C}}^{-, \Gamma} := \{ P^-_w(X, Y) \mid f \in \tilde{S}_{w+2}(\Gamma) \}, \]
\[ W_{w, \mathbb{C}}^{+, \mathbb{Q}} := W_{w, \mathbb{C}}^{+, \Gamma} \cap \mathbb{Q}[X, Y], \quad W_{w, \mathbb{C}}^{-, \mathbb{Q}} := W_{w, \mathbb{C}}^{-, \Gamma} \cap \mathbb{Q}[X, Y] \]
where \( \tilde{S}_{w+2}(\Gamma) \) is the subspace of \( M_{w+2}(\Gamma) \) consisting modular forms \( f \) which vanish at two cusps 0 and \( \infty \). We simply write \( W_{w, \mathbb{C}}^{\pm} \) and \( W_{w, \mathbb{C}}^{\pm, \mathbb{Q}} \) as \( W_{w, \mathbb{C}}^{\pm} \) and \( W_{w, \mathbb{C}}^{\pm, \mathbb{Q}} \), respectively. We define the right action of \( \text{Z}[\text{GL}(2, \mathbb{Z})] \) to \( \mathbb{C}[X, Y] \) by
\[ p(X, Y) \gamma = p(aX + bY, cX + dY) \quad (\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)). \]
We put
\[ \epsilon = \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), \quad S = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \quad U = \left( \begin{array}{cc} 1 & -1 \\ 1 & 0 \end{array} \right), \quad T = U S^{-1} = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \]
\[ T' = U^2 S^{-1} = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right), \quad M = SUSU^2 S = \left( \begin{array}{cc} -1 & -1 \\ 2 & 1 \end{array} \right). \]
We denote by \( \Gamma_A, \Gamma_B \) and \( \Gamma_0(2) \) the congruence subgroups of \( \text{SL}_2(\mathbb{Z}) \) defined by
\[
\Gamma_A := \Gamma(2) \cup UT(2) \cup U^2\Gamma(2),
\]
\[
\Gamma_B := \Gamma(2) \cup ST(2),
\]
and
\[
\Gamma_0(2) := \Gamma(2) \cup TT(2)
\]
where \( \Gamma(2) \) is the principal congruence subgroup of level 2. Then we have
\[
W^+_w = \{ p \in V^+_w : p|_{1+SU} = p|_{1+U+U^2} = 0 \},
\]
\[
W^+_{\Gamma_0(2)} = \{ p \in V^+_w : p|_{1-T+U(1+M)} = 0 \},
\]
\[
W^+_{\Gamma_A} = \{ P \in V^+_w : p|_{1+U+U^2} = p|_{SU+SU^2} = 0 \},
\]
\[
W^- = \{ p \in V^-_w : p|_{1+U+U^2} = 0 \},
\]
\[
W^-_{\Gamma_A} = \{ P \in V^-_w : p|_{1+U+U^2} = p|_{SU+SU^2} = 0 \},
\]
and for these cases, \( W^{+,\Gamma} = W^{+,\Gamma} \otimes \mathbb{C} \) and the maps \( f \mapsto P^+ f \) are injective (see Lemma 31).

3. The case \( J(\text{odd};\text{odd},0) \)

The following two theorems (Theorems 26 and 27) says that \( \text{(24)} \) and \( \text{(25)} \) are exact.

**Theorem 26.** Let \( w \) be a positive even integer. Then there is an exact sequence
\[
0 \rightarrow W^+_{\Gamma_A} \xrightarrow{id} V^+_w \xrightarrow{\lambda} H_w(\text{odd};\text{odd},0)^\vee \rightarrow 0.
\]

**Proof.** It is enough to check that \( \ker \lambda = W^+_{\Gamma_A} \). Let \( p(X,Y) \in V^+_w \). By Lemma 22 \( \lambda(p(X,Y)) = 0 \) if and only if
\[
p(X,Y)|_{(T'\Gamma+T)(1-\iota)} = 0.
\]
Since \( p(X,Y)|_{\iota} = p(X,Y) \), we have
\[
p(X,Y)|_{(T'\Gamma+T)(1-\iota)} = p(X,Y)|_{(U+U^2-SU-SU^2)S}.
\]
Thus the condition \( p(X,Y)|_{(T'\Gamma+T)(1-\iota)} = 0 \) is equivalent to
\[
p(X,Y)|_{1+U+U^2} = p(X,Y)|_{1+(SU)+SU^2}. \]
Thus \( W^+_{\Gamma_A} \subset \ker \lambda \). On the other hand if \( p(X,Y) \in \ker \lambda \) then \( q = p(X,Y)|_{1+U+U^2} = p(X,Y)|_{1+(SU)+SU^2} \) vanishes since \( q|_U = q \) and \( q|_{SU} = q \). Thus \( \ker \lambda \subset W^+_{\Gamma_A}. \) Hence the theorem is proved. \( \square \)

**Theorem 27.** Let \( w \) be a nonnegative even integer. Then there is an exact sequence
\[
0 \rightarrow W^-_{\Gamma_A} \xrightarrow{id} V^-_w \xrightarrow{\phi} H_w(\text{odd};\text{odd},0) \rightarrow 0
\]
where \( \phi \) is a linear map defined by \( \phi(X^aY^b) = \text{Alt}_b J_{\overline{b}}(a,b),0) \).

**Proof.** It is enough to check \( \ker \phi = \text{Im}u \). Let \( p \in V^-_w \). By Lemma 24 \( \phi(p) = 0 \) is and only if
\[
0 = p|_{(T'\Gamma+T)(1-\iota)} = p|_{1+U+U^2} - p|_{1+(SU)+SU^2}.
\]
Thus \( W^-_{\Gamma_A} \subset \ker \phi \). On the other hand if \( p \in \ker \phi \) then \( q = p|_{1+U+U^2} = p|_{1+(SU)+SU^2} \) vanishes since \( q|_U = q \) and \( q|_{SU} = q \). Thus \( \ker \phi \subset W^-_{\Gamma_A}. \) Hence the theorem is proved. \( \square \)

**Example 28.** By Theorem 27, for a nonnegative even integer \( w \), an odd period polynomial of a weight \( w+2 \) modular form for \( \Gamma_A \) gives a linear relation among \( \{ J_{\overline{b}}(a,b,0) \mid a,b \in 1+2\mathbb{Z}_{\geq 0}, a+b = w \} \). For example \( w = 4 \), the space of weight \( w+2 \) cusp forms for \( \Gamma_A \) is generated by the square roots of Ramanujan delta \( \sqrt{\Delta} \), and its odd period polynomial is \( c(X^3-Y^3) \) where \( c \in \mathbb{C}. \) Thus \( 13! J(1;3,0)-3!!J(3;1,0) \in \mathbb{Q} \cup \mathbb{Q} \). In fact,
\[
13! J(1;3,0) = 3!!J(3;1,0) = 11\zeta^m(6).
\]
See Table 1 for other weights.
Example 29. The even period polynomials for $\sqrt{\Delta} \in M_{4+2}(\Gamma_A)$ and $E_4 \sqrt{\Delta} \in M_{8+2}(\Gamma_A)$ are constant multiples of $Y^4 - 4X^2Y^2 + X^4$ and $3Y^8 - 16X^2Y^6 + 20X^4Y^4 - 16X^6Y^2 + 3X^8$, respectively. Thus by Theorem 26 we have
\[
\langle J(a; b; 0), 3\sigma_1\sigma_9 - 16\sigma_3\sigma_7 + 20\sigma_5\sigma_5 - 16\sigma_7\sigma_3 + 3\sigma_9\sigma_1 \rangle = 0 \quad (a, b : \text{odd})
\]
and
\[
\langle J(a; b; 0), 3\sigma_1\sigma_9 - 16\sigma_3\sigma_7 + 20\sigma_5\sigma_5 - 16\sigma_7\sigma_3 + 3\sigma_9\sigma_1 \rangle = 0 \quad (a, b : \text{odd}).
\]

4. THE CASE $J(\text{even}; \text{even}, 0)$

Let $w$ be a nonnegative even integer. We put
\[
H^\text{sym}_{w, \text{odd}} := \langle J_2(a; w - a, 0) + J_2(w - a; a, 0) | 0 \leq a \leq w \rangle,
\]
\[
H^\text{sym}_{w, \text{even}} := \langle J_2(a; w - a, 0) + J_2(w - a; a, 0) | 0 \leq a \leq w, \ a : \text{even} \rangle,
\]
\[
H^\text{sym}_{w, \text{odd}, \text{even}} := \langle J_2(a; w - a, 0) + J_2(w - a; a, 0) | 0 \leq a \leq w, \ a : \text{odd} \rangle.
\]

Lemma 30. We have
\[
H^\text{sym}_{w, \text{odd}, \text{even}} \subset H^\text{sym}_{w, \text{even}}.
\]

Proof. We denote by $V^\text{sym}_w$ the set of symmetric polynomials in $V^-_w$. The lemma is equivalent to the congruence
\[
\psi(g(X, Y)) \equiv 0 \pmod{H^\text{sym}_{w, \text{even}}}
\]
for $g(X, Y) \in V^\text{sym}_w$. We prove this congruence by using the double shuffle relation. For a degree $w$ homogeneous polynomial $p(X, Y, Z)$, define $\psi(p) \in H_w$ by linearity and
\[
\psi(X^aY^bZ^c) = a!b!c!J_2(a; b, c).
\]
Let $a$ and $b$ be nonnegative integer such that $a + b = w$. By the harmonic relation
\[
J_2(a)J_2(b) = J_2(0; a, b) + J_2(b; a, a),
\]
and the shuffle relation
\[
a!b!J(a)J(b) = \sum_{a_1 + a_2 = a} \binom{a}{a_1} a_1!b!(a + b_2)!J(0; a_1, b + a_2) + \sum_{b_1 + b_2 = b} \binom{b}{b_1} b_1!(a + b)!J(0; b_1, a + b_2),
\]
we have
\[
\psi(Y^aZ^b + Z^aY^b - (Y + Z)^aZ^b - (Y + Z)^bZ^a) = 0.
\]
Since $\psi(p(X, Y, Z)) = \psi(p(X, Y, -X - Y))$ for all $p$, we have
\[
\psi((X + Y)^a(X^a - (-Y)^b) + (X + Y)^b(X^b - (-Y)^a)) = 0.
\]
Note that gives linear relations among the generators of $H^\text{sym}_{w, \text{even}}$ (resp. $H^\text{sym}_{w, \text{odd}}$) for odd (resp. even) $a$ and $b$. Hence, let us assume that $a$ and $b$ are odd. Let $q(X, Y)$ be a symmetric odd polynomial $X^aY^b + X^bY^a$. Then (4.1) says that
\[
\psi(q(X, Y)) |_{U_{S+UZ}} = 0.
\]
Since the odd part of $q(X, Y)|_{U_{S+UZ}}$ is given by
\[
\frac{1}{2} q(X, Y) |_{(U_{S+UZ})^{(1-2)}} = \frac{1}{2} q(X, Y) |_{U_{S+UZ}SU^{2-2} \subset U_{S+UZ}} = \frac{1}{2} q(X, Y) |_{(U_{S+UZ})^{(S-1)}} = 0 \pmod{H^\text{sym}_{w, \text{even}}},
\]
we have
\[
\psi(q(X, Y)) |_{(U_{S+UZ})^{(S-1)}} = 0 \pmod{H^\text{sym}_{w, \text{even}}}.
\]
We denote by $\alpha : V_{w}^{\text{sym}} \rightarrow V_{w}^{\text{sym}}$ the map defined by $\alpha(p) = p|_{(1+U+U^2)(S-1)}$. Thus if $\alpha$ is surjective then the lemma is proved. For $p(X,Y) \in V_{w}^{\text{sym}} \cap \mathbb{Z}[X,Y]$, by direct calculation, we have
\[
\alpha(\alpha(p(X,Y))) \equiv p(X, Y) \pmod{3\mathbb{Z}[X,Y]).
\]
Thus $\alpha$ is injective and thus surjective.

**Lemma 31.** We have
\[
H_{w}^{\text{sym}} = \bigoplus_{m=0}^{\lfloor w/4 \rfloor} J_{\mathcal{D}}(m) J_{\mathcal{D}}(w-2m).
\]

**Proof.** Let $S : H_{w} \rightarrow H_{w}$ be the antipode map. Then $H_{w}$ is factored to $\pm$-part by the action of $S$; $H_{w} = H_{w}^{+} \oplus H_{w}^{-}$. Then we can easily show that
\[
H_{w}^{+} = \bigoplus_{m=0}^{\lfloor w/4 \rfloor} J_{\mathcal{D}}(2m) J_{\mathcal{D}}(w-2m).
\]
By Lemma 21, $S(v) = v$ for $v \in H_{w}^{\text{sym}}$ and $S(v) = -v$ for $v \in H_{w}^{\text{asym}}$. Thus $H_{w}^{\text{sym}} \subseteq H_{w}^{+}$ and $H_{w}^{\text{asym}} \subseteq H_{w}^{-}$. Since $H_{w}$ is spanned by $H_{w}^{\text{sym}}$ and $H_{w}^{\text{asym}}$, we have $H_{w}^{\text{sym}} = H_{w}^{-}$.

Thus the lemma is proved.

**Theorem 32.** The sequences (1.6) and (1.25) are exact.

**Proof.** Define $\phi : V_{w}^{+} \rightarrow V_{w}^{+}$ by
\[
\phi(p) := p|_{(1+U+U^2)(1-S)}.
\]
By Lemmas 21 and 22, the sequences (1.6) and (1.25) are exact if and only if $W_{w}^{+} = \{p \in V_{w}^{+} \mid \phi(p) = 0\}$.

Since we can easily check
\[
W_{w}^{+} \subseteq \{p \in V_{w}^{+} \mid \phi(p) = 0\},
\]
and it is enough to prove $p \in W_{w}^{+}$ for all $p \in \ker(\phi)$. Decompose $p$ as $p = p^{+} + p^{-}$ where $p^{\pm} = \frac{1}{2} p|_{1_{\pm}S}$. Since
\[
\phi(q)|_{S} = \phi(q)|_{S}
\]
for any $q \in V_{w}^{+}$, we have $\phi(p^{+}) = \phi(p^{-}) = 0$.

By definition, we have $0 = \phi(p^{-}) = \phi(p^{-}|_{(1+U+U^2)(1-S)})$. Put $q = p^{-}|_{1_{+}U+U^2}$. Then $q|_{U} = q$ and $q|_{S} = q$. Thus $q = 0$ or $w = 0$. Therefore $p^{-} \in W_{w}^{+}$.

Let us prove $p^{+} = 0$ by contradiction. Suppose that $p^{+} \neq 0$. Since $\phi(p^{+}) = 0$, we have
\[
\sum_{m=0}^{w/2} a_{m}(2m)! (w-2m)! J_{\mathcal{D}}(2m; w-2m, 0) = 0
\]
where $p^{+} = \sum_{m=0}^{w/2} a_{m} X^{2m} Y^{w-2m}$. This gives a non-trivial linear relation among $(J_{\mathcal{D}}(2m; w-2m, 0) + J_{\mathcal{D}}(w-2m; 2m, 0))$’s, and thus
\[
\dim H_{w, \text{even}}^{\text{sym}} \leq \{ m \in \mathbb{Z} \mid 0 \leq 2m \leq w - 2m \} = 1 + \left\lfloor \frac{w}{4} \right\rfloor.
\]
Since $H_{w, \text{even}}^{\text{sym}} = H_{w}^{\text{sym}}$ by Lemma 30, we have
\[
\dim H_{w}^{\text{sym}} < 1 + \left\lfloor \frac{w}{4} \right\rfloor.
\]
This contradicts Lemma 31. Hence, $p^{+} = 0$.

Therefore $p = p^{-} \in V_{w}^{+}$, and the theorem is proved.

**Example 33.** Let $w$ be a positive even integer. Then even period polynomial for the weight $w+2$ Eisenstein series is a constant multiple of $X^{w} - Y^{w}$. Thus by exactness of (1.5),
\[
J(w; 0, 0) - J(0; w, 0) \in \mathbb{Q} \zeta_{w}^{m}(w+2).
\]
In fact, $J(w; 0, 0) - J(0; w, 0) = \sum_{a=0}^{w-1} \zeta_{w}(a+1, w-a+1)$
\[
= \zeta_{w}(w+2).
\]
Example 34. Let \( w = 10 \). The linear relations among \( \{ J(0; 10, 0), J(2; 8, 0), \ldots, J(10; 0, 0), \zeta^m(12) \} \) are spanned by \( J(10; 0, 0) - J(0; 10, 0) = \zeta^m(12) \) and
\[
14J(2; 8, 0) - 9J(4; 6, 0) + 9J(6; 4, 0) - 14J(8; 2, 0) = \frac{6248}{691} \zeta^m(12).
\]
By multiplying a some rational number and taking a quotient by \( \mathbb{Q}\zeta^m(12) \), we obtain
\[
\text{(4.3)} \quad 1 \times 2!J(2; 8, 0) - 3 \times 4!J(4; 6, 0) + 3 \times 6!J(6; 4, 0) - 1 \times 2!J(8; 2, 0) \equiv 0 \pmod{\mathbb{Q}\zeta^m(12)}.
\]
Then the coefficients 1, −3, 1 in (4.3) coincide with the coefficients of an even period polynomial
\[
X^2Y^8 - 3X^4Y^6 + 3X^6Y^4 - X^8Y^2 \in W^+_{10}.
\]

5. The cases similar to the case \( J(0; \text{even, even}) \): \( J(1; \text{even, even}), J(\text{even, 0, even}), J(\text{odd, 1, odd}) \)

In this section, we prove the exactness of (1.7), (1.8), (1.9), (1.26), (1.27), (1.28). For the convenience of the reader, we also give a proof of the exactness of (1.1) and (1.20) because it can be proved simultaneously without any additional effort.

Lemma 35. Let \( w \) be a nonnegative even integer. Then \( u_{\text{GKZ}} : W^+_w \to V^+_w \) is injective.

Proof. Note that
\[
u_{\text{GKZ}}(p) = \frac{1}{2} p|_{SU^2S+U}.
\]
Assume that \( u_{\text{GKZ}}(p) = 0 \) for some \( p \in W^+_w \). Then
\[
0 = p|_{(SU^2S+U)U^2} = p|_{SU^2SU^2+1}.
\]
Thus
\[
p|_{(SU^2)^4} = p.
\]
Since \( (SU^2)^4 = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \), \( p \in \mathbb{Q}Y^w \). Furthermore, since \( p|_{1+S} = 0 \), \( p = 0 \). Thus the lemma is proved. \qed

Lemma 36. Let \( w \) be a nonnegative even integer. Let \( p(X, Y) \in V^+_w \). Then
\[
p(X, Y) - p(X + Y, Y) \pm p(X + Y, X) \in V^+_w,
\]
if and only if \( p(X, Y) \in W^+_w \).

Proof. We can assume \( w \geq 2 \) since the case \( w = 0 \) is obvious. The ‘if’ part is obvious, so it is enough to consider the ‘only if’ part. Assume that
\[
p(X, Y) - p(X + Y, Y) \pm p(X + Y, X) \in V^+_w.
\]
By swapping \( X \) and \( Y \) of (5.1), we have
\[
p(Y, X) - p(X + Y, X) \pm p(X + Y, Y) \in V^+_w.
\]
By (5.1) and (5.2), we have \( p(X, Y) \pm p(Y, X) \in V^+_w \). Furthermore, since \( p(X, Y) \in V^+_w \), \( p(X, Y)|_{1+S} = 0 \). Thus
\[
0 = p(X, Y)|_{(1-U^2T)(1-U)} = p(X, Y)|_{1-U^2S+1-SU^2} = p(X, Y)|_{(1-U+U^2)(1-U)} = p(X, Y)|_{(1-U+U^2)(1-U^2S)}.
\]
Thus \( p(X, Y)|_{1+U+U^2} = 0 \) since there is no nonzero polynomial annihilated by both \( 1 - US \) and \( 1 - U^2S \). \qed

Theorem 37. The sequences (1.1), (1.7), (1.8), (1.9), (1.20), (1.26), (1.27) and (1.28) are exact.

Remark 38. The exactness of (1.11) and (1.20) is not new (and others are new results).

Proof. The exactness of (1.20) follows from Lemmas 22 and 66. The exactness of (1.26) and (1.27) follows from that of (1.20) and Lemmas 24 and 25.

Let us check the exactness of (1.28):
\[
0 \to (\delta_X^{-1}W^+_w) \xrightarrow{id} V^+_w \xrightarrow{\lambda} H^+_{\text{w} (\text{odd; 1; odd})} \to 0,
\]
where \( w \) is a nonnegative even integer. For \( p(X, Y) \in V^+_w \), by Lemma 22, \( \lambda(p) = 0 \) if and only if
\[
p_1(Y - X, -X) - p_1(-X, Y) + p_2(Y, -X) + p_3(Y - X, Y) \in V^+_w
\]
where \( p_1(Y,X) = \frac{\partial}{\partial X} p(X,Y) \) and \( p_2(X,Y) = \frac{\partial}{\partial Y} p(X,Y) \). Thus, \( \lambda(p) = 0 \) if and only if
\[
p_1(X,Y) - p_1(X + Y, Y) - p_1(X + Y, X) \in V_w^+.
\]
Note that \( p_1(X,Y) \in V_w^- \). Therefore, by Lemma 39 \( \lambda(p) = 0 \) if and only if \( p_1(X,Y) \in W_w^- \). Thus \( \lambda(p) = 0 \) if and only if \( \frac{\partial}{\partial X} p(X,Y) \in W_w^- \). Thus (1.25) is exact.

Let us check the exactness of (1.1) and (1.9):
\[
0 \to W_w^+ \xrightarrow{u_{\text{GKZ}}} V_w^+ \xrightarrow{X^a Y^b \mapsto aH^0} H_w(0;\text{even, even}) \to 0,
\]
\[
0 \to W_w^- \xrightarrow{u_{\text{GKZ}}} V_w^- \xrightarrow{X^a Y^b \mapsto aH^1} H_{w+1}(\text{odd};1,\text{odd}) \to 0.
\]

By Lemma 21 and direct calculation, we can show that these sequences are chain complex. By definition and Lemma 36, the left and right parts of these sequences are exact, i.e., \( 0 \to W_w^+ \xrightarrow{u_{\text{GKZ}}} V_w^+ \to W_w^- \xrightarrow{u_{\text{GKZ}}} V_w^- \), \( V_w^+ \to H_w(0;\text{even, even}) \to 0 \), and \( V_w^- \to H_{w+1}(\text{odd};1,\text{odd}) \to 0 \) are exact. Furthermore, by exactness of (1.20) and (1.28),
\[
\dim H_w(0;\text{even, even}) = \frac{W}{2} + 1 - \dim W_w^+,
\]
\[
\dim H_{w+1}(\text{odd};1,\text{odd}) = \frac{W}{2} - \dim W_w^-.
\]
Thus (1.1) and (1.9) are exact.

By Lemmas 24 and 25 the exactness of (1.7) and (1.8) follows from that of (1.1).

\[\qed\]

**Example 39.** In [10] Example of Theorem 3], the authors give two examples of linear relations coming from the GKZ-relation:
\[
28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12),
\]
\[
66\zeta(3,13) + 375\zeta(5,11) + 686\zeta(7,9) + 675\zeta(9,7) + 396\zeta(11,5) = \frac{78967}{3617}\zeta(16).
\]
The exactness of (1.6) and (1.7) implies that \( J(\text{even};\text{even}, 0)'s \) and \( J(1;\text{even}, \text{even})'s \) also satisfy very similar linear relations where the coefficients perfectly coincide (except for those of single zetas). In fact,
\[
28J(2;0,8) + 150J(4;0,6) + 168J(6;0,4) = \frac{118492}{691}\zeta^m(12),
\]
\[
28J(1;2,8) + 150J(1;4,6) + 168J(1;6,4) = -69\zeta^m(13),
\]
\[
66J(2;0,12) + 375J(4;0,10) + 686J(6;0,8) + 675J(8;0,6) + 396J(10;0,4) = \frac{3961222}{3617}\zeta^m(16),
\]
\[
66J(1;2,12) + 375J(1;4,10) + 686J(1;6,8) + 675J(1;8,6) + 396J(1;10,4) = -283\zeta^m(17).
\]

**Example 40.** Let us see the \( w = 10 \) case of (1.9). The linear relations among \( \{J(1;1,9), J(3;1,7), \ldots, J(9;1,1), \zeta^m(13)\} \) are given by
\[
48J(1;1,9) + 119J(3;1,7) + 10J(5;1,5) - 144J(7;1,3) = 640\zeta^m(13).
\]
By multiplying \(-30240\) and taking a quotient by \( \mathbb{Q}\zeta^m(13) \), we obtain
\[
(5.3) \quad -4 \times 19! J(1;1,9) - 119 \times 37! J(3;1,7) - 21 \times 55! J(5;1,5) + 144 \times 7! J(7;1,3) \equiv 0 \pmod{\mathbb{Q}\zeta^m(13)}.
\]
On the other hand, the space \( W_{10} \) is spanned by
\[
p(X,Y) := 4XY^9 - 25X^3Y^7 + 42X^5Y^5 - 25X^7Y^3 + 4X^9Y.
\]
Then
\[
u_{\text{GKZ}}(p(X,Y)) = \frac{p(-X - Y, X) + p(X - Y, X)}{2} = -4XY^9 - 119X^3Y^7 - 21X^5Y^5 + 144X^7Y^3.
\]
Thus the same coefficients \(-4, -119, -21, 144\) appears in both (5.3) and (5.4).

6. The cases similar to the case \( J(0;\text{even}, \text{odd}) \): \( J(1;\text{even}, \text{odd}), J(\text{even};1,\text{odd}), J(\text{odd};0,\text{even}) \)

In this section, we prove the exactness of (1.10), (1.11), (1.12), (1.24), (1.30), (1.31). For the convenience of the reader, we also give a proof of the exactness of (1.14) and (1.24) because it can be proved simultaneously without any additional effort.

**Lemma 41.** Let \( w \in 1 + 2\mathbb{Z}_{\geq 0} \), \( p(X,Y) \in W_{w-1}^+ \) and \( q(X,Y) \in W_{w+1}^+ \). If \( Yp(X,Y) + \frac{\partial}{\partial X} q(X,Y) = 0 \) then \( p(X,Y) = q(X,Y) = 0 \).
Thus by Lemma 42, we have

\[ Yp(X, Y) + \frac{\partial}{\partial X}q(X, Y) = 0. \]

By swapping X and Y, we have

\[ 0 = Xp(Y, X) + \frac{\partial}{\partial Y}q(Y, X) \]

(6.2)

\[ = \pm Xp(X, Y) + \frac{\partial}{\partial Y}q(X, Y). \]

By (6.1) and (6.2), we have

\[ 0 = X \frac{\partial}{\partial X}q(X, Y) + Y \frac{\partial}{\partial Y}q(X, Y) \]

(6.4)

\[ = (w + 1)q(X, Y). \]

Thus \( q(X, Y) = 0 \), and \( p(X, Y) = -\frac{1}{w + 1}q(X, Y) = 0 \). Hence the lemma is proved. \( \square \)

**Lemma 42.** Let \( w \in 1 + 2\mathbb{Z}_{\geq 0} \) and \( f(X, Y) \in W^\pm_w \). Put \( R(X, Y) := f(X + Y, X) \). If

\[ R(X, Y) \pm R(Y, X) \mp R(-X, Y) - R(Y, -X) = 0 \]

then

\[ f(X, Y) \in \mathbb{Q}X^w. \]

**Proof.** By assumption, we have

\[ 0 = f(X + Y, X) \pm f(X + Y, Y) \mp f(-X + Y, -X) - f(-X + Y, Y). \]

(6.3)

By swapping X and Y, we have

\[ 0 = f(X + Y, Y) \pm f(X + Y, X) \mp f(X - Y, -Y) - f(X - Y, X) \]

\[ = f(X + Y, Y) \pm f(X + Y, X) \mp f(-X + Y, Y) + f(-X + Y, -X) \]

(6.4)

\[ \pm 0 = f(X + Y, X) \pm f(X + Y, Y) + f(-X + Y, Y) \mp f(-X + Y, -X). \]

By adding (6.3) and (6.4), we have

\[ 0 = f(X + Y, X) \pm f(X + Y, Y) \]

\[ = f(X, Y) - f(X, -X + Y) \quad (X \mapsto Y, Y \mapsto X - Y). \]

Thus

\[ f(X, Y) \in \mathbb{Q}X^w. \]

\( \square \)

**Lemma 43.** Let \( w \in 1 + 2\mathbb{Z}_{\geq 0} \), \( p(X, Y) \in W^\pm_{w-1} \) and \( q(X, Y) \in W^\mp_{w+1} \). Put \( R(X, Y) := Xp(X + Y, X) + \frac{\partial}{\partial Y}q(X + Y, X) \). If

\[ R(X, Y) \pm R(Y, X) \mp R(-X, Y) - R(Y, -X) = 0 \]

then \( p(X, Y) = 0 \) and \( q(X, Y) \in \mathbb{Q}(X^{w+1}). \)

**Proof.** Put

\[ f(X, Y) := Yp(X, Y) + \frac{\partial}{\partial X}q(X, Y) \in V^\pm_w. \]

Then by definition,

\[ R(X, Y) = f(X + Y, X). \]

Thus by Lemma 42,

\[ f(X, Y) = \begin{cases} 0 & V^+_{w-1} = V^+_w \ 
tw \end{cases} \ 
v^\pm_w = v^\mp_w \]

where \( c \in \mathbb{Q} \). Thus

\[ Yp(X, Y) + \frac{\partial}{\partial X}q(X, Y) = 0 \]

where

\[ \bar{q}(X, Y) = \begin{cases} q(X, Y) & q(X, Y) - \frac{1}{w+1}(X^{w+1} - Y^{w+1}) \ 
q(X, Y) - \frac{1}{w+1}(X^{w+1} - Y^{w+1}) \quad V^+_w = V^+_w \ 
v^\mp_w = v^\mp_w \]

Thus by Lemma 41, we have \( p(X, Y) = 0 \) and \( \bar{q}(X, Y) = 0 \). Hence the lemma is proved. \( \square \)
Lemma 44. Let $w \in 1 + 2\mathbb{Z}_{\geq 0}$ and $C(X, Y) \in V_w^\pm$. Put $D(X, Y) = \frac{\partial}{\partial X} C(X, Y) + \frac{\partial}{\partial Y} C(Y, X)$, $L(X, Y) = \pm C(X + Y, X) + C(X + Y, Y) - C(X, Y)$, $S(X, Y) = \frac{1}{2} (L(X, Y) \pm L(-X, Y))$, $S_1(X, Y) = \frac{\partial}{\partial X} (S(X, Y) \pm S(Y, X))$, $S_2(X, Y) = \frac{\partial}{\partial Y} (S(X, Y) \mp S(Y, X))$. Then

$$\pm D(X + Y, X) + D(X + Y, Y) - D(X, Y) = S_1(X, Y) - S_2(X, X + Y).$$

Proof. By linearity, it is enough to only consider the case $C(X, Y) = X^a Y^b$ where $(-1)^a = \pm 1$ and $(-1)^b = \mp 1$. Then

$$p(X, Y) = a(X^{a-1}Y^b \pm Y^{a-1}X^b),$$

and thus

$$\pm D(X + Y, X) + D(X + Y, Y) - D(X, Y) = a \left( \pm (X + Y)^{a-1}X^b + X^{a-1}(X + Y)^b + (X + Y)^{a-1}Y^b \pm Y^{a-1}(X + Y)^b - X^{a-1}Y^b \mp Y^{a-1}X^b \right).$$

On the other hand,

$$L(X, Y) = \pm (X + Y)^a X^b + (X + Y)^a Y^b - X^a Y^b,$$

$$S(X, Y) = \frac{1}{2} \left( \pm (X + Y)^a X^b + (X + Y)^a Y^b - 2X^a Y^b - (X - Y)^a X^b + (X - Y)^a Y^b \right),$$

$$S(X, Y) \pm S(Y, X) = \pm (X + Y)^a X^b + (X + Y)^a Y^b - X^a Y^b \mp Y^a X^b,$$

$$S(X, Y) \mp S(Y, X) = -X^a Y^b \mp Y^a X^b - (X - Y)^a X^b + (X - Y)^a Y^b.$$

$$S_1(X, Y) = \frac{\partial}{\partial X} (S(X, Y) \pm S(Y, X)) = \pm (X + Y)^{a-1}X^b (aX + bX + bY) + a(X + Y)^{a-1}Y^b - aX^{a-1}Y^b \mp bY^a X^{b-1}.$$

$$S_2(X, Y) = -((X - Y)^{a-1}X^b - (aX + bX - bY)) + a((X - Y)^{a-1}Y^b - aX^{a-1}Y^b \pm bY^a X^{b-1}).$$

$$S_2(X, X + Y) = -((X - Y)^{a-1}X^b - (aX - bY)) + a((X - Y)^{a-1}Y^b - aX^{a-1}(X + Y)^b - aX^{a-1}X^b - b(X + Y)^a X^{b-1},$$

$$S_1(X, Y) - S_2(X, X + Y) = a \left( \pm (X + Y)^{a-1}X^b + X^{a-1}(X + Y)^b + (X + Y)^{a-1}Y^b \pm Y^{a-1}(X + Y)^b - X^{a-1}Y^b \mp Y^{a-1}X^b \right).$$

Thus the lemma is proved. \hfill \square

Lemma 45. Let $w \in 1 + 2\mathbb{Z}_{\geq 0}$ and $q(X, Y) \in V^\pm_w$. Assume that $q(X, Y) = \mp q(Y, X)$ and

$$\left( \frac{\partial}{\partial Y} - \frac{\partial}{\partial X} \right) (q(X, Y) - q(X + Y, Y) \pm q(X + Y, X)) \in V^\mp_w.$$

Then $q(X, Y) \in \tilde{W}^\pm_{w+1}$ where

$$\tilde{W}^+_{w+1} = W^+_{w+1}, \quad \tilde{W}^-_{w+1} = W^-_{w+1} \oplus \mathbb{Q}(X^w + Y^w).$$

Proof. By changing $X$ and $Y$,

$$\left( \frac{\partial}{\partial Y} - \frac{\partial}{\partial X} \right) (q(X, Y) - q(X + Y, Y) \pm q(X + Y, X)) \in V^\pm_w.$$

Thus

$$\left( \frac{\partial}{\partial Y} - \frac{\partial}{\partial X} \right) (q(X, Y) - q(X + Y, Y) \pm q(X + Y, X)) = 0.$$

Thus there exists $c \in \mathbb{Q}$ such that

$$q(X, Y) - q(X + Y, Y) \pm q(X + Y, X) = c(X + Y)^{w+1}.$$  \hfill (6.5)

By changing $X$ and $Y$, we have

$$-q(X, Y) \mp q(X + Y, X) + q(X + Y, Y) = \pm c(X + Y)^{w+1}.$$  \hfill (6.6)

By considering (6.5) plus (6.6), we have

$$c(1 \pm 1) = 0.$$

Thus the lemma is proved. \hfill \square

Lemma 46. Let $w \in 1 + 2\mathbb{Z}_{\geq 0}$ and $C(X, Y) \in V^\pm_w$. Then

$$C(X, Y) = C(X + Y, Y) \mp C(X + Y, X) \in V^\mp_w$$

if and only if there exists $p(X, Y) \in W^\mp_{w-1}$ and $q(X, Y) \in W^\mp_{w+1}$ such that

$$C(X, Y) = Xp(X, Y) + \frac{\partial}{\partial Y}q(X, Y).$$
Proof. It is enough to prove the ‘only if’ part. Put
\[ p(X, Y) := \frac{1}{w + 1} \left( \frac{\partial}{\partial X} C(X, Y) \pm \frac{\partial}{\partial Y} C(Y, X) \right) \in V^\pm_w. \]
\[ q(X, Y) := \frac{1}{w + 1} (Y C(X, Y) \mp X C(Y, X)) \in V^\pm_w. \]
Then by definition,
\[ (6.7) \quad X p(X, Y) + \frac{\partial}{\partial Y} q(X, Y) = C(X, Y). \]
By Lemma 45, (6.7)
\[ p(X, Y) - p(X + Y, Y) \equiv p(X + Y, X) = 0. \]
Thus
\[ p(X, Y) \in W^\pm_w. \]
By (6.7), by modulo \( V^\pm_w \), we have
\[ 0 \equiv C(X, Y) - C(X + Y, Y) \mp C(X + Y, X) \]
\[ = X p(X, Y) + (X + Y) p(X + Y, Y) \mp p(X + Y, X) \]
\[ + \frac{\partial}{\partial Y} q(X, Y) - \left( \frac{\partial}{\partial Y} - \frac{\partial}{\partial X} \right) q(X + Y, Y) \mp \left( \frac{\partial}{\partial X} - \frac{\partial}{\partial Y} \right) q(X + Y, X) \]
\[ \equiv (X + Y) (p(X, Y) - p(X + Y, Y) \mp p(X + Y, X)) \]
\[ + \left( \frac{\partial}{\partial Y} - \frac{\partial}{\partial X} \right) (q(X, Y) - q(X + Y, Y) \mp q(X + Y, X)) \]
\[ \equiv \left( \frac{\partial}{\partial Y} - \frac{\partial}{\partial X} \right) (q(X, Y) - q(X + Y, Y) \mp q(X + Y, X)) \mod V^\pm_w. \]
By Lemma 45, \( q(X, Y) \in \tilde{W}^\pm_{w+1} \). Furthermore, we can easily show that \( q(X, Y) \in W^\pm_{w+1} \). Thus the lemma is proved.

\[ \square \]

Theorem 47. The sequences \( (1.2), (1.7), (1.10), (1.12), (1.13), (1.21), (1.24), (1.30) \) and \( (1.31) \) are exact.

Remark 48. The exactness of \( (1.2) \) and \( (1.12) \) is not new (and others are new results).

Proof. Let us check the exactness of \( (1.21) \) and \( (1.31) \):
\[ 0 \to W^-_{w-1} \oplus W^+_w \xrightarrow{(p,q) \mapsto X p + Y q} V^+_w \xrightarrow{\lambda_1} H_w(0; \text{even}; \text{odd}) \to 0 \]
\[ 0 \to W^+_{w-1} \oplus W^-_{w+1} \xrightarrow{(p,q) \mapsto Y p + X q} V^-_w \xrightarrow{\lambda_2} H_w(\text{odd}; 0; \text{even}) \to 0 \]
where \( \lambda_1 \) and \( \lambda_2 \) are linear maps induced from \( \lambda : V^+_w \to H^w_w \). By Lemma 41, the left parts \( 0 \to W^-_{w-1} \oplus W^+_w \to V^+_w \) and \( 0 \to W^+_{w-1} \oplus W^-_{w+1} \to V^-_w \) are exact. The exactness of the right parts is obvious. By Lemma 42, \( C(X, Y) \in \ker(\lambda_1) \) if and only if
\[ C(X, Y) - C(X + Y, Y) - C(X + Y, X) \in V^-_w. \]
Thus \( (1.21) \) is exact by Lemma 46. On the other hand, by Lemma 42, \( C(Y, X) \in \ker(\lambda_2) \) if and only if
\[ C(X, Y) - C(X + Y, Y) + C(X + Y, X) \in V^+_w. \]
Thus \( (1.31) \) is exact by Lemma 46.

The exactness of \( (1.20) \) and \( (1.30) \) follows from the exactness of \( (1.21) \) and Lemmas 21 and 23.

Let us check the exactness of \( (1.21) \) and \( (1.31) \):
\[ 0 \to W^-_{w-1} \times \tilde{W}^+_w \times \mathbb{Q} \xrightarrow{\eta_{w+1}} V^+_w \xrightarrow{X a Y b + a b i j} H_w(0; \text{even}; \text{odd}) \to 0 \]
\[ 0 \to W^+_{w-1} \times W^-_{w+1} \xrightarrow{\eta_{w-1}} V^-_w \xrightarrow{X a Y b + a b i j} H_w(\text{odd}; 0; \text{even}) \to 0. \]
By Lemma 21 and direct calculation, we can show that these sequences are chain complex. By Lemma 42, the left parts of these sequences are exact. The exactness of the right parts is obvious. Furthermore, by exactness of \( (1.21) \) and \( (1.31) \),
\[ \dim H_w(0; \text{even}; \text{odd}) = \frac{w + 1}{2} - \dim W^-_{w-1} - \dim W^+_w, \quad \dim H_w(\text{odd}; 0; \text{even}) = \frac{w + 1}{2} - \dim W^+_{w-1} - \dim W^-_{w+1}. \]
Thus \( (1.1) \) and \( (1.9) \) are exact.

By Lemmas 24 and 23, the exactness of \( (1.7) \) and \( (1.7) \) follows from that of \( (1.1) \).
Theorem 49 (Tasaka). The sequences \((1.13)\) and \((1.32)\) are exact.

Proof. Let \(w\) be a nonnegative even integer and \(\lambda' : V'_w \to H'_w(0; \text{odd}; \text{odd})\) the linear map induced from \(\lambda : V'_w \to H'_w\). Then, by Lemma 22, we have

\[
\ker(p) = 0 \iff p \mid U - SU^2 - SU^2 S + US = 0.
\]

Since

\[
p \mid U - SU^2 - SU^2 S + US = p \mid (1-T)(1+M)SU;
\]

the sequence \((1.22)\) is exact. The exactness of \((1.4)\) follows from that of \((1.22)\) and the calculation of the dimension. \(\square\)

Theorem 50. The sequences \((1.13)\) and \((1.32)\) are exact.

Proof. It follows from Lemma 23 and Theorem 49. \(\square\)

8. Almost no relation cases: \(J(\text{even}; 1, \text{even}), J(\text{even}; 0, \text{odd}), J(\text{odd}; \text{even}, 0)\)

We put \(J(m_0; m_1, m_2) := m_0!m_1!m_2! J_D(m_0; m_1, m_2)\) and \(J(n) := n! J_D(0; n)\). By Lemma 21, for an odd integer \(w\) and \(m_0 + m_1 + m_2 = w\), we have

\[
J(m_0, m_1, m_2) = \sum_{n=0}^{(w-1)/2} a_n J(2n) J(w-2n)
\]

where \(a_n\) is the coefficient of \(X^{2n} Y^{w-2n}\) in

\[
(-X-Y)^{m_0} X^{m_1} Y^{m_2} + (-Y)^{m_0} (-X+Y)^{m_1} X^{m_2} - (-Y)^{m_0} (-X)^{m_1} (X+Y)^{m_2}.
\]

Note that \(\{J(2n) J(w-2n)\}\) is linearly independent.

Theorem 51. There are no linear relations among \(J_D(\text{even}; 1, \text{even})\)’s.

Proof. Let \(w\) be a positive odd integer. By \((8.1)\),

\[
J(w-1-2m; 1, 2m) = \sum_{n=0}^{(w-1)/2} a_{m,n} J(2n) J(w-2n)
\]

for \(0 \leq m \leq (w-1)/2\) where

\[
a_{m,n} = \delta_{m,n} + \left( \frac{w-1-2m}{2n-1} \right) + \left( \frac{2m}{2n-1} \right).
\]

Since \(a_{m,n} \equiv \delta_{m,n} \pmod{2}\), the matrix \((a_{m,n})_{0 \leq m,n \leq w/2}\) is invertible. Thus there are no linear relations among \(J_D(w-2m-1; 1, 2m)\)’s. \(\square\)

Theorem 52. There are no linear relations among \(J_D(\text{even}; \text{odd}, 0)\)’s.

Proof. Let \(w\) be a positive odd integer. By \((8.1)\),

\[
J(2m; w-2m, 0) = \sum_{n=0}^{(w-1)/2} a_{m,n} J(2n) J(w-2n)
\]

for \(0 \leq m \leq (w-1)/2\) where

\[
a_{m,n} = \left( \frac{2m}{w-2n} \right) + \left( \frac{w-2m}{2n} \right).
\]

Then the matrix \(M_w = (a_{m,n})_{0 \leq m,n \leq (w-1)/2}\) is invertible since

\[
M_w \equiv \begin{pmatrix} (w-2m) & \cdots & 1 \\ 2n & \cdots & 0 \end{pmatrix} (\text{mod } 2).
\]

Thus there are no linear relations among \(J_D(2m; w-2m, 0)\)’s. \(\square\)
Remark 53. We found following observation concerning the determinant of $M_w$:

$$\det(M_w) \equiv \prod_{n=1, n \text{ odd}}^w c_n$$

where

$$c_n = (-1)^{(n-1)/2} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n \begin{cases} 1 & n \equiv 1, 5 \pmod{6} \\ \frac{1}{4} & n \equiv 3 \pmod{6}. \end{cases}$$

**Theorem 54.** There are no linear relations among $J_D(2m+1; 2n, 0)$'s.

**Proof.** Let $w$ be a positive odd integer. By (6.1),

$$J_l(w - 2m; 2m, 0) = \sum_{n=0}^{(w-1)/2} \left( - \left( \frac{w - 2m}{2n - 2m} \right) - \left( \frac{2m}{2n} \right) + \delta_{n,m} \right) J_l(2n)J_l(w - 2n).$$

for $0 \leq m \leq (w-1)/2$. Thus the claim is equivalent to

$$A_w := \det \left( - \left( \frac{w - 2m}{2n - 2m} \right) - \left( \frac{2m}{2n} \right) + \delta_{n,m} \right)_{0 \leq m, n \leq (w-1)/2} \neq 0.$$  

Define $\eta_w : V^+_w \to V^+_w$ by

$$\eta_w(p(X, Y)) = \frac{1}{2} (p(X, X + Y) - p(X, -X + Y) - p(X - Y, Y) - p(X + Y, Y) + 2p(X, Y)).$$

Then

$$\det \eta_w = A_w.$$  

Define $\Phi_w : V^+_w \to V_w$ and $\Phi_w : V^+_w \to V^+_w$ by

$$\Phi_w(p(X, Y)) = p(Y, X + Y) - p(X + Y, X) + p(X, Y)$$

and

$$\Phi_w(p(X, Y)) = \Phi'_w(p(X, Y) + p(-X, Y)).$$

Then

$$2\Phi'_w(\eta_w(p(X, Y))) \equiv (p(Y, X - 2Y) + 2p(Y, X + Y))$$

$$- 3p(X, X + Y) - 3p(X + Y, Y)$$

$$+ (-p(X + 2Y, X + Y) + p(X - Y, -2X + Y))$$

$$+ (-p(-2X + Y, X) - 2p(X + Y, X))$$

$$+ 2p(X, Y) \pmod{V^+_w}$$

for $p(X, Y) \in V^+_w$. We define the integral part of $V^+_w$ by $V^+_w, \mathbb{Z} := V^+_w \cap \mathbb{Z}[X, Y]$. Then for $p(X, Y) \in V^+_w, \mathbb{Z}$,

$$\Phi_w \circ \eta_w(p(X, Y)) \equiv p(X, Y) \pmod{3V^+_w, \mathbb{Z}}.$$  

Thus

$$A_w \neq 0 \pmod{3}.$$  

Therefore, the theorem is proved. \qed

**Theorem 55.** There are no linear relations among $J_D(\text{even}; 0, \text{odd})$'s except for

$$\sum_{m=0}^{(w-1)/2} J_D(2m; 0, w - 2m) = 0$$

where $w$ is a positive odd integer.

**Proof.** Let $w$ be a positive odd integer. By (6.1),

$$J_l(2m; 0, w - 2m) = \sum_{n=1}^{(w-1)/2} \left( \left( \frac{2m}{2n} \right) - \left( \frac{w - 2m}{2n} \right) \right) J_l(2n)J_l(w - 2n)$$

for $0 \leq m \leq (w-1)/2$. Thus the identity $\sum_{m=0}^{(w-1)/2} J_0(2m; 0, w - 2m) = 0$ can be checked easily from this, and the claim is equivalent to

$$\det \left( \begin{array}{cc} \frac{2m}{2n} & \frac{w - 2m}{2n} \\ \end{array} \right)_{1 \leq m, n \leq (w-1)/2} \neq 0.$$
For \( 1 \leq n \leq (w - 1)/2 \), since
\[
\left( \frac{2m}{2n} \right) - \left( \frac{w - 2m}{2n} \right) = \left( \frac{w + x}{2n} \right) - \left( \frac{w - x}{2n} \right) \quad (x := 2m - w/2),
\]
there exists a polynomial \( P_n \) of degree \( n - 1 \) such that
\[
\left( \frac{2m}{2n} \right) - \left( \frac{w - 2m}{2n} \right) = \left( \frac{2m - w}{2} \right) P_n \left( \left( \frac{2m - w}{2} \right)^2 \right).
\]
Since
\[
xP(x^2) = \frac{1}{(2n)!} \left( \prod_{j=0}^{2n-1} \left( \frac{w}{2} - j + x \right) - \prod_{j=0}^{2n-1} \left( \frac{w}{2} - j - x \right) \right).
\]
the coefficient of \( x^{n-1} \) in \( P_n \) is
\[
\frac{2}{(2n)!} \sum_{j=0}^{2n-1} \left( \frac{w}{2} - j \right) = \frac{w - 2n + 1}{(2n - 1)!}.
\]
Thus, by the Vandermonde’s determinant formula, we have
\[
\det \left( \frac{2m}{2n} - \left( \frac{w - 2m}{2n} \right) \right)_{1 \leq m, n \leq (w-1)/2} = \det \left( \left( \frac{2m - w}{2} \right) P_n \left( \left( \frac{2m - w}{2} \right)^2 \right) \right)_{1 \leq m, n \leq (w-1)/2}
\]
\[
= \left( \prod_{m=1}^{(w-1)/2} \left( \frac{2m - w}{2} \right) \right) \left( \prod_{1 \leq i < j \leq (w-1)/2} \left( \frac{2j - w/2}{2} - \left( \frac{2i - w}{2} \right)^2 \right) \right)
\]
\[
\times \left( \prod_{n=1}^{(w-1)/2} \frac{w - 2n + 1}{(2n - 1)!} \right)
\]
\[
= \left( \frac{w - 1}{2} \right)! \times \begin{cases} -1 & w \equiv 5 \pmod{8} \\ 1 & \text{otherwise} \end{cases} 
\]
\[\neq 0.\]
Therefore the theorem is proved. \( \square \)

9. Remarks

In this section, we make some comment for the cases not treated in Theorems 18 and 19. For the cases \( J(1; \text{odd, odd}) \) and \( J(\text{odd, odd, 1}) \), we have following conjectures.

Conjecture 56. There are no linear relations among \( J_D(1; \text{odd, odd})'s. \)

Conjecture 57. There are no linear relations among \( J_D(\text{odd, odd, 1})'s. \)

Let \( w \) be a positive even integer. By \[51,\]
\[
J_1(1; w - 2m - 1, 2m + 1) = \sum_{n=0}^{w/2-1} \left( -\delta_{m,n} + \left( \frac{w - 2m - 1}{2n} \right) - \left( \frac{2m + 1}{2n} \right) \right) J(w - 2n)J(2n + 1)
\]
and
\[
J(2m + 1; w - 2m - 1, 1) = \sum_{n=0}^{w/2-1} \left( -\delta_{m,n} + \left( \frac{w - 2m - 1}{2n - 2m} \right) - \left( \frac{2m + 1}{2n} \right) \right) J(w - 2n)J(2n + 1)
\]
by Lemma 21 \((n = w/2 \text{ terms vanish})\). Thus Conjectures 56 and 57 are equivalent to
\[
\det \left( -\delta_{m,n} + \left( \frac{w - 2m - 1}{2n} \right) + \left( \frac{2m + 1}{2n} \right) \right)_{0 \leq m, n \leq w/2-1} \neq 0
\]
and
\[
\det \left( -\delta_{m,n} + \left( \frac{w - 2m - 1}{2n - 2m} \right) - \left( \frac{2m + 1}{2n} \right) \right)_{0 \leq m, n \leq w/2-1} \neq 0.
\]
Thus one might be able to prove Conjectures 56 and 57 by giving a good approximation of the value of these determinants.

For other cases, numeric calculation suggests that the dimensions of the linear relations among $J_D$ (even; even, 1)'s, $J_D$ (1; odd, even)'s, $J_D$ (odd; 1, even)'s, $J_D$ (odd; even, 1)'s, and $J_D$ (even; odd, 1)'s are given by 1, $\dim S_k(\SL_2(\Z))$, $\dim S_k(\SL_2(\Z))$.

Note that the action of $\epsilon \Gamma$ does not preserve $\hat{\Phi}_w$. We can define the following spaces

$$W^\Gamma_w = \{ P \in \hat{\Phi}_w : P|_{1+2^S} = \hat{P}|_{1+2^S} = 0 \},$$

$$W^\Gamma_w = \{ P \in \hat{\Phi}_w : P(C) \in V_w \text{ for all } C \in \Gamma \}.$$

Note that the action of $\epsilon$ preserves $W^\Gamma_w$ and $W^\Gamma_w$. Thus we can get the decomposition $W^\Gamma_w = W^\Gamma_w \oplus W^\Gamma_w$ and $W^\Gamma_w = W^\Gamma_w \oplus W^\Gamma_w$.

For $f \in M_{w+2}(\Gamma)$ and $C \in \Gamma \backslash \SL_2(\Z)$, define $f|_{C} \in M_{w+2}(C^{-1}\Gamma C)$ by $f|_{C}(z) = (cz+d)^{-w-2} f(\frac{az+b}{cz+d})$ where $(a \ b \ c \ d) \in C$. Furthermore, define $\hat{P}_f \in \hat{\Phi}_w$ by $\hat{P}_f(C) = P|_{C}$. Then we can show $\hat{P}_f \in W^\Gamma_w$ for $f \in M_{w+2}(\Gamma)$ and $\hat{P}_f \in W^\Gamma_w$ for $f \in S_{w+2}(\Gamma)$. We put $\hat{\rho}^\pm_f = \frac{1}{2} (\hat{\rho}_f + \hat{\rho}_f)$ and $\hat{\rho}^\pm(f) = \hat{\rho}^\pm_f$.

**Theorem 58** ([22] Theorem 2.1), Popa–Pasol, Eichler–Shimura. Let $C^\Gamma_w$ be the subspace of $W^\Gamma_w$ defined in [22]. Then the composite map

$$S_{w+2}(\Gamma) \xrightarrow{\hat{\rho}^\pm} W^\Gamma_w \xrightarrow{W^\Gamma_w/C^\Gamma_w}$$

is an isomorphism.

Here we do not explain the definition of $C_w$ because we only use the injectivity of $S_{w+2}(\Gamma) \xrightarrow{\hat{\rho}^\pm} W^\Gamma_w$.

**Theorem 59** ([22] Proposition 8.4]). The spaces $M_{w+2}(\Gamma)$ and $W^\Gamma_w$ have same dimensions. If the extended Peterson Scalar product on $M_{w+2}(\Gamma)$ defined in [23] is nondegenerate then the maps $\hat{\rho}^\pm : M_k(\Gamma) \rightarrow W^\Gamma_w$, $f \mapsto \hat{\rho}^\pm_f$ are isomorphisms.

**Theorem 60** ([23] Theorem 4.4]). For $k > 2$ and $\Gamma \subseteq \{ \Gamma_1(1), \Gamma_0(1) \}$, the extended Peterson Scalar product on $M_k(\Gamma)$ is nondegenerate. Furthermore, for $k = 2$ and $\Gamma \subseteq \{ \Gamma_1(p), \Gamma_0(p) \}$ with $p$ prime, the extended Peterson Scalar product on $M_k(\Gamma)$ is also nondegenerate.
A.2. Period polynomials for $\text{SL}_2(\mathbb{Z}), \Gamma_0(2)$, $\Gamma_A, \Gamma_B$. In this subsection, we show the following.

**Proposition 61.** The even period maps

$$M_k(\Gamma) \to W^+_{\omega, \Gamma} : f \mapsto P_f^+$$

for $\Gamma = \text{SL}_2(\mathbb{Z})$, $\Gamma_0(2)$, and $\Gamma_A$, and the odd period maps

$$\tilde{S}_k(\Gamma) \to W^-_{\omega, \Gamma} : f \mapsto P_f^-$$

for $\Gamma = \text{SL}_2(\mathbb{Z})$, $\Gamma_A$, and $\Gamma_B$ are injective.

**Proof.** By Theorems 59 and 60, $\rho^\pm : M_k(\Gamma_0(2)) \to \tilde{W}^\Gamma_{\omega, \Gamma}$ are isomorphisms. We denote by $C, C', C''$ the cosets represented by $1, U, U^2$ respectively. Then by definition, $\tilde{W}^\Gamma_{\omega, \Gamma}$ is the space of maps $P$ from $\{C, C', C''\}$ to $\tilde{V}_w$ such that

$$0 = P(C)|S + P(C') = P(C''|_{1+S} = P(C) + P(C')|_{U^2} + P(C'')|_{U} = P(C)|_{1-\epsilon} = P(C'')|_{1-\epsilon}.$$ 

Thus $P \in \tilde{W}^\Gamma_{\omega, \Gamma}$ is determined by $P(C)$ since $P(C') = -P(C)|_S$ and $P(C'') = P(C)|_{SU-U^2}$. Thus the even period map $M_k(\Gamma_0(2)) \to W^+_{\omega, \Gamma}$ is injective. Let $P \in \tilde{W}^\Gamma_{\omega, \Gamma}$. By $P(C'') = P(C)|_{-U^2+SU}$ and $P(C'')|_{1+S} = 0$,

$$0 = P(C)|_{(SU-U^2)(1+S)} = P(C)|_{(1-T)SU(1+S)} = P(C)|_{(1-T)(1+M)SU}.$$ 

Thus $P(C)|_{(1-T)(1+M)} = 0$. Therefore,

$$W^+_{\omega, \Gamma} \subset \{ p \in V_w : p_{1-\epsilon} = p_{(1-T)(1+M)} = 0 \}.$$ 

Let $P \in \tilde{W}^\Gamma_{\omega, \Gamma}$ and $p = P(C'')$. Then $P(C'') = p_{SU-U^2}$. Thus

$$\{ p \in V_w : p_{1-\epsilon} = p_{(1-T)(1+M)} = 0 \} \subset W^+_{\omega, \Gamma}.$$ 

Thus the lemma is proved.

**Lemma 63.** The odd period map $\tilde{S}_k(\Gamma_B) \to W^-_{\omega, \Gamma}$ is injective (and thus an isomorphism) and

$$W^-_{\omega, \Gamma} = \{ p \in V_w : p_{1+\epsilon} = p_{1+S} = 0 \}.$$ 

**Proof.** Let $C, C'$ and $C''$ be the same as in the proof of Lemma 62. We remark that

$$M_k(\Gamma_B) = \{ f|_{C''} : f \in M_k(\Gamma_0(2)) \}.$$ 

Note that

$$W^-_{\omega, \Gamma} = \{ P(C'') : P \in \tilde{W}^\Gamma_{\omega, \Gamma} \cap V_w \}.$$ 

Let $P \in \tilde{W}^\Gamma_{\omega, \Gamma}$ and $p = P(C'')$. Then

$$P(C)|_{1+\epsilon} = p_U.$$ 

Since $P(C)|_{1+\epsilon} = 0$, $P(C)$ is uniquely determined by $p$. Thus $\tilde{S}_k(\Gamma_B) \to W^-_{\omega, \Gamma}$ is injective. The inclusion

$$W^-_{\omega, \Gamma} \subset \{ p \in V_w : p_{1+\epsilon} = p_{1+S} = 0 \}$$

is obvious from the definition.

Let $p \in \{ p \in V_w : p_{1+\epsilon} = p_{1+S} = 0 \}$. Then there is unique $q \in \tilde{V}_w$ such that

$$q_{(1-T)} = p_U.$$ 

Then

$$q_{1+\epsilon} = p_{U^2} = 0.$$
Then \( q_{1+e} \in \mathbb{Q} Y_w \). Since \( q \in \mathbb{Q} V_w \), \( q_{1+e} = 0 \). Now, if we define \( P : \{ C, C', C'' \} \to V_w \) by
\[
P(C) = q, \quad P(C') = -q_{|S}, \quad P(C'') = p,
\]
then \( P \in W_{\mathcal{W}_w}^{(2)} \). Thus
\[
\{ p \in V_w : p|_{1+e} = p|_{1+\delta} = 0 \} \subset W_{\mathcal{W}_w}^{-, \mathcal{M}}.
\]
Hence the lemma is proved.

**Lemma 64.** Then even (resp. odd) period map \( M_k(\Gamma_\lambda) \to W_{\mathcal{W}_w}^{+, \mathcal{A}} \) (resp. \( S_k(\Gamma_\Lambda) \to W_{\mathcal{W}_w}^{-, \mathcal{A}} \)) is injective (and thus an isomorphism) and
\[
W_{\mathcal{W}_w}^{\pm, \mathcal{A}} = \{ p \in V_w^\pm : p|_{1+U+U^2} = p|_{S+SU+SU^2} = 0 \}.
\]

**Proof.** First, let us prove the injectivity of \( \hat{\rho}^{\pm} : M_k(\Gamma_\lambda) \to \hat{W}_{\mathcal{W}_w}^{+, \mathcal{A}} \). Let \( f \in M_k(\Gamma_\lambda) \) and assume that \( \hat{\rho}^{\pm}(f) = 0 \). Then \( f \) can be written as \( f = f' + f'' \) where \( f' \in M_k(\text{SL}_2(\mathbb{Z})) \) and \( f'' \in \sqrt{\Delta} M_{k-\delta}(\text{SL}_2(\mathbb{Z})) \). Then \( \hat{\rho}^{\pm}(f') = \hat{\rho}^{\pm}(f'') = 0 \) since \( \hat{\rho}^{\pm}(f) = 0 \), \( f' = \frac{1}{i} f|_1|_S^+ \), and \( f'' = \frac{1}{i} f|_1|_S^- \). Thus \( f' = 0 \) by the injectivity of the period map for \( \text{SL}_2(\mathbb{Z}) \), and \( f'' = 0 \) by the injectivity of \( \hat{\rho}^{\pm} : S_k(\Gamma_\Lambda) \to \hat{W}_{\mathcal{W}_w}^{\pm, \mathcal{A}} \) (Theorem 58). Thus \( \hat{\rho}^{\pm} : M_k(\Gamma_\lambda) \to \hat{W}_{\mathcal{W}_w}^{\pm, \mathcal{A}} \) is injective and therefore an isomorphism. The claim of the lemma easily follows from this isomorphism.

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