LARGE DEVIATIONS PRINCIPLE FOR SUB-RIEMANNIAN
RANDOM WALKS

MARIA GORDINA†, TAI MELCHER††, DAN MIKULINCER‡, AND JING WANG§

Abstract. We study large deviations for random walks on stratified (Carnot) Lie
groups. For such groups, there is a natural collection of vectors which generates their
Lie algebra, and we consider random walks with increments in only these directions.
Under certain constraints on the distribution of the increments, we prove a large
deviation principle for these random walks with a natural rate function adapted to
the sub-Riemannian geometry of these spaces.

Contents

1. Introduction 1
2. Background and setup 6
  2.1. Large deviations principles on metric spaces 6
  2.2. Carnot groups 8
3. Large deviations 21
  3.1. Vector space LDP 22
  3.2. Exponentially good approximations 24
  3.3. LDP for the random walk 38
  3.4. Solving the variational problem for Gaussian random walks 43
References 44
Appendix A. Proof of the upper bound 45
Appendix B. Random symplectic form 47

1. Introduction

The study of large deviations for random walks goes back to Cramér’s work in
1938. A general version of Cramér’s theorem says that, given an i.i.d. sequence of
random variables $Y_1,Y_2,\ldots$ taking values in a locally convex vector Polish space,
under suitable conditions on the tail distribution of the random variable $Y_1$, the

2020 Mathematics Subject Classification. Primary 60G50, 60F10; Secondary 53C17, 22E99.
Key words and phrases. large deviations, Carnot groups, random walks.
† Research was supported in part by NSF Grant DMS-2246549.
†† Research was supported in part by NSF Grant DMS-1255574.
‡ Research was supported in part by Simons Investigator award (622132, PI- Mossel).
§ Research was supported in part by NSF Grant DMS-1855523 and DMS-2246817.
random walks induced by \( \{Y_i\}_{i=1}^\infty \) satisfy a Large Deviations Principle (LDP) with a good rate function.

Due to its generality, Cramér’s theorem later became a starting point of various classical LDP results on path spaces (see for instance \[14,15\]). However, few results are known addressing the question of how such results depend on the underlying topological space, such as for a random walk on a Riemannian or a sub-Riemannian manifold. Recently Kraaij-Redig-Versendaal in \[23\] and Versendaal \[33,34\] obtained a Cramér-type LDP for geodesic random walks on Riemannian manifolds that were first introduced by E. Jorgensen in \[22\]. These authors were also able to obtain related Mogulskii’s theorem in this setting. Closer to our setting is a Cramér-type LDP for random matrix products considered by C. Sert in \[31\].

Our goal is to study Cramér-type theorems in a sub-Riemannian setting, or more precisely on Carnot groups. For such groups the Lie algebras have a stratified structure that ensures Hörmander’s condition is satisfied, and so induces a natural sub-Riemannian manifold structure. While we lack the structure of a vector space, the upshot is that the group structure affords natural constructions of random walks on these curved spaces equipped with a sub-Riemannian metric. For example, in \[28,30\] random walks have been constructed on nilpotent groups, while in \[2,9,20\] a very different construction has been introduced on more general sub-Riemannian manifolds. In this paper the random walks we consider shall rely on the Lie group structure of the underlying manifold. Such random walks were first considered by Pap in \[29\] to prove a central limit theorem on nilpotent Lie groups (see also \[7\]). Thus while the small deviations regime is well-understood and covered by central limit theorem type results, our aim is to cover the complementary large deviations regime.

To further motivate our results we note the growing interest in non-linear large deviations in recent years (see \[3,11,12,16,25\] for some prominent examples). A key element in these works is that when the non-linear functional is highly symmetric, such as subgraph counts in Erdős-Rényi graphs, precise and quantitative large deviations results can be obtained. As it will soon become apparent, our random walks can equivalently be regarded as non-linear, or even polynomial, functionals on appropriate product spaces. In this context, the non-linearity arises from the algebraic structure of the group operation. Furthermore, the group-theoretic aspects of these induced functionals entail numerous symmetries, which we can utilize to derive exact LDPs.

We now present our setting. Let \( G \) be a Carnot group with Lie algebra \( \mathfrak{g} \). That is, \( \mathfrak{g} = \mathcal{H} \oplus \mathcal{V} \) such that any basis \( \{ \mathcal{X}_1, \ldots, \mathcal{X}_d \} \) of \( \mathcal{H} \) satisfies Hörmander’s condition

\[
\text{span}\{ \mathcal{X}_i, [\mathcal{X}_i, \mathcal{X}_j], \ldots, [\mathcal{X}_i, [\mathcal{X}_j, \cdots, \mathcal{X}_k]] : i_k \in \{1, \ldots, d\}\} = \mathfrak{g} \text{ for some } r \in \mathbb{N}.
\]

We assume that \( \mathcal{H} \) is equipped with an inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \), and therefore the Carnot group \( G \) has a natural sub-Riemannian structure. Namely, one may use left translation to define a horizontal distribution \( \mathcal{D} \), a sub-bundle of the tangent bundle \( TG \), and a metric on \( \mathcal{D} \) as follows. First, we identify the space \( \mathcal{H} \subset \mathfrak{g} \) with \( \mathcal{D}_e \subset T_eG \). Then for \( g \in G \) let \( L_g \) denote the left translation \( L_g h := gh \), and define \( \mathcal{D}_g := (L_g)_* \mathcal{D}_e \).
for any \( g \in G \). A metric on \( D \) may then be defined by translating back to \( H \),

\[
\langle u, v \rangle_{D_g} := \langle (L_{g^{-1}})_*u, (L_{g^{-1}})_*v \rangle_{D_e} = \langle (L_{g^{-1}})_*u, (L_{g^{-1}})_*v \rangle_H
\]

for all \( u, v \in D_g \).

We will sometimes identify the horizontal distribution \( D \) with \( H \). Vectors in \( D \) are called horizontal, and we say that a path \( \gamma : [0, 1] \to G \) is horizontal if \( \gamma \) is absolutely continuous and \( \gamma'(t) \in D_{\gamma(t)} \) for a.e. \( t \). Equivalently, \( \gamma \) is horizontal if the Maurer-Cartan form \( c_{\gamma}(t) := (L_{\gamma(t)} - 1)_*\gamma'(t) \in H \) for a.e. \( t \). Such horizontal paths are used to define a left-invariant Carnot-Carathéodory distance \( G \) as one of left-invariant homogeneous distances on \( G \). We view the Carnot group \( G \) as a metric space with respect to one of such a distance.

Suppose \( \{X_n\}_{n=1}^{\infty} \) is a sequence of i.i.d. random variables with mean 0 taking values in \( H \). We consider a sub-Riemannian random walk on \( G \) defined by

\[
S_0 = e, \\
S_n := \exp(X_1) \cdots \exp(X_n), \quad n \in \mathbb{N}.
\]

One of the features of Carnot groups is a dilation which can be used to scale the random walk appropriately. We give more details in Section 2.2, in particular, (2.6) lists their main properties. For \( a > 0 \), we denote by \( D_a : G \to G \) the dilation homomorphism on \( G \) adapted to its stratified structure. If \( \{X_n\} \subset H \) are i.i.d. with mean 0, the law of large numbers says that almost surely

\[
\lim_{n \to \infty} D_{\frac{1}{n}} S_n = e;
\]

see for example [19], or [27].

To state our results, we denote

\[
\Lambda(\lambda) := \Lambda_X(\lambda) := \log \mathbb{E}[\exp(\langle \lambda, X \rangle_H)]
\]

and let \( \Lambda^* \) be the Legendre transform of \( \Lambda \) defined by (3.3). Let \( \mu_n \) be the distribution of \( D_{\frac{1}{n}} S_n \). Our main result is the following Cramér-type large deviations principle for \( \{\mu_n\}_{n=1}^{\infty} \).

**Theorem 1.1.** Suppose \( \{X_k\}_{k=1}^{\infty} \) are i.i.d. with mean 0 random variables in \( H \) such that \( \Lambda_X(\lambda) \) exists for all \( \lambda \in H \). Further assume that one of the following assumptions is satisfied.

(i) \( G \) is of step 2 and the distribution of each \( X_k \) is sub-Gaussian on \( H \);

(ii) \( G \) is of step \( \geq 3 \) and the distribution of each \( X_k \) is either a standard Gaussian on \( H \) or has bounded support.

Then for

\[
S_n := \exp(X_1) \cdots \exp(X_n),
\]

the measures \( \{\mu_n\}_{n=1}^{\infty} \) satisfy a large deviation principle with rate function

\[
J(g) := \inf \left\{ \int_0^1 \Lambda^*(c_{\gamma}(t)) \, dt : \gamma : [0, 1] \to G \text{ horizontal}, \gamma(0) = e, \gamma(1) = g \right\},
\]

where \( c_{\gamma} \) is the Maurer-Cartan form for the horizontal path \( \gamma \).
Remark 1.2. As usual, there exists a modification of the above statement to accommodate non-centered distributions. But for simplicity we will keep the mean 0 assumption.

As is usual for LDPs, Theorem 1.1 characterizes the large deviations rate function as the solution of a variational problem. Since the variational problem in (1.1) is defined on the path space, one can see the connection between the geometry of the Carnot group and the LDPs of the random walks. Keeping this connection in mind, it is interesting to determine whether the variational problem admits explicit solutions that express the underlying metric structure more clearly.

As a particular case, we are able to determine an explicit solution to (1.1) in the important case of Gaussian random walks. When \(\{X_n\}_{n=1}^{\infty}\) are i.i.d. normally distributed random vectors in \(H\), the rate function (1.1) has the following explicit expression

\[
J_N(g) = \inf \left\{ \frac{1}{2} \int_0^1 |\gamma'(t)|^2 \gamma(t) dt : \gamma \text{ horizontal}, \gamma(0) = e, \gamma(1) = g \right\}
\]

which gives the exact minimum energy to reach \(g\) from \(e\) at time 1. In particular, the rate function in (1.2) can be described in terms of the natural geometry on the group \(G\). As we recall in Section 2.2, Hörmander’s condition implies that any two points in \(G\) can be connected by a horizontal path by Chow–Rashevskii’s theorem. One may define the (finite) Carnot–Carathéodory distance \(\rho_{cc}(x, y)\) between two points \(x, y \in G\) to be the length of the shortest horizontal path connecting \(x\) and \(y\) (see Section 2.2.3 for details). With this definition we can state our results for Gaussian random walks.

Corollary 1.3. Let \(G\) be a homogeneous Carnot group and suppose \(\{X_n\}\) are independent \(\mathcal{N}(0, \text{Id}_H)\) random variables on \(H\). Then Theorem 1.1 holds for the associated random walk with the rate function

\[
J_N(x) = \frac{1}{2} \rho^2_{cc}(e, x), \quad x \in G.
\]

Since Carnot groups can be identified with copies of \(\mathbb{R}^N\) equipped with non-commutative group operations, Corollary 1.3 can be viewed as a broad generalization of the standard LDP for Gaussian random walks in Euclidean spaces, where the rate function is given by \(\frac{1}{2} \|x\|^2\). More generally, the rate function in (1.2) is in line with that for LDPs on path space, like the classical one for Brownian motion (that is, Schilder’s Theorem) and other diffusions on \(\mathbb{R}^n\). Such path space LDPs for continuous-time processes are sometimes accessible from Cramér-type results via finite-dimensional distributions, and it’s natural to ask if the results of the present paper may be used to prove path space LDPs for the associated hypoelliptic diffusions (à la Schilder) or polygonal and piecewise-linear paths (à la Mogulskii). While LDPs are known for some hypoelliptic diffusions (see for example [4]) proving path space LDPs from our starting point typically requires additional assumptions, like ellipticity of the generator, or Cameron-Martin-Maruyama-type quasi-invariance, or
Girsanov-type results. These conditions are generally not available in our present setting, except for a limited class of sub-Riemannian manifolds, see [6]. We also mention that Schilder-type LDPs are known from [4] for a large class of hypoelliptic diffusions with the rate function being the classical one for Brownian motion on $\mathbb{R}^n$.

At this point let us comment a bit about the proof of Theorem 1.1 and hence also Corollary 1.3. As mentioned above, the case of hypoelliptic diffusions poses a challenge in general, since the setting lacks the appropriate conditions required for most classical LDPs. Thus, our approach will need a new construct, and in particular, be tailored to the sub-Riemannian geometry of group. Indeed, our approach is different from previous proofs for LDPs: we treat the random walk as an intrinsic stochastic process on the Carnot group viewed as a geodesic metric space. As a result, and in contrast to previous works, we show that by taking horizontal (geodesic) paths connecting $S_n$ and $S_{n+1}$ instead of a piecewise linearization of the continuous-time processes, we recover geometrically natural rate functions. This procedure elucidates why the LDPs in this paper relate the random walks to the geometry of the underlying space, as in Theorem 1.1.

We now compare our results to two particular papers on LDPs for random walks in Lie groups. First consider [34], in which the author studies Cramér-type LDPs on Lie groups equipped with a Riemannian structure. While the main results in our paper and in [34] might appear similar, the LDPs are fundamentally different both in the way the random walks are constructed and the geometric structures of the underlying spaces. In addition, we are able to prove LDPs for a more general class of samples $\{X_n\}_{n=1}^{\infty}$ than the bounded distributions considered in [34]. This allows us to include the very natural case of standard normal sampling, the one and only case that leads to the LDP with the rate function in the energy form given by (1.2). One of the tools we rely on are concentration inequalities for polynomials of Gaussian random vectors.

Now consider [5], in which the authors study LDPs on nilpotent Lie groups. While in the present paper we focus on random walks taking steps only in horizontal directions, we see that the LDPs proved here should be comparable to those for random walks taking steps in arbitrary directions, as considered in [5]; see Remark 3.2.1. Significantly, the approach of the present paper allows us to show that the infimum is achieved specifically on horizontal paths, clearly tying the rate function to the underlying natural geometry of the space, as we see for example by identifying the rate function for Gaussian samples as described earlier in the introduction.

In [5] the authors use the fact that these groups can be identified with Euclidean spaces to describe the random walks as an end point process of solutions to stochastic differential equations. As mentioned earlier, LDPs for hypoelliptic diffusions in $\mathbb{R}^n$ that are solutions of SDEs are well-known due to work by Azencott [4]. This approach is a natural application of classical LDP results to spaces like nilpotent Lie groups identified with Euclidean space; however, it cannot be used for more general geometric structures. We also note that as a consequence of this approach the group $G$ is equipped with the Euclidean metric structure. There is a significant difference when one looks at the group equipped with a Riemannian or sub-Riemannian distance, as it should be the metric structure that determines the rate functions for the LDP, as we
show in the case of homogeneous Carnot groups. Working instead in the tangent space as in the present paper offers the potential to generalize to other sub-Riemannian manifolds.

Our paper is organized as follows. In Section 2 we provide the terminology needed to state and prove Theorem 1.1 precisely, and in Section 3 we prove Theorem 1.1. Note that it is only the arguments of Section 3.2, where we show we have exponentially good approximations of the walk, that require the additional assumptions (sub-Gaussian or Gaussian or bounded) on the distribution.

2. Background and setup

2.1. Large deviations principles on metric spaces. We first recall some basic definitions for large deviations principles. These can be found for example in [14, Section 1.2]. Suppose $M$ is a Hausdorff topological space.

**Definition 2.1.** A function $I : M \rightarrow [0, \infty]$ is called a rate function if $I$ is not identically $\infty$ and if $I$ is lower semi-continuous. That is, the set $\{x \in M : I(x) \leq a\}$ is closed for every $a \geq 0$. $I$ is called a good rate function if in addition $\{x \in M : I(x) \leq a\}$ is compact for every $a \geq 0$.

In our setting $M = (M, \rho)$ is a metric space, and therefore one can verify lower semi-continuity on sequences. I.e., $I$ is lower semi-continuous if and only if

$$\liminf_{x_n \to x} I(x_n) \geq I(x)$$

for all $x \in M$. This means that a good rate function on $M$ is a rate function that achieves its infimum over closed sets. We denote by $B$ the (complete) Borel $\sigma$-algebra over the metric space $M$.

**Definition 2.2.** A sequence of probability measures $\{\mu_n\}_{n=1}^\infty$ on $(M, B)$ satisfies the large deviation principle (LDP) with the rate function $I : M \rightarrow [0, \infty]$ if the function on $B$ defined as

$$\frac{1}{n} \log \mu_n(B), B \in B,$$

converges weakly to the function $-\inf_{x \in B} I(x)$.

Equivalently, for any open set $O \subset M$ and any closed set $F \subset M$

$$\liminf_{n \to \infty} \frac{1}{n} \log \mu_n(O) \geq -\inf_{x \in O} I(x),$$

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(F) \leq -\inf_{x \in F} I(x).$$

(2.1)

We say that $\{\mu_n\}_{n=1}^\infty$ satisfies the weak LDP if (2.1) holds for all compact sets $F \subset M$.

Similarly, for a sequence of $M$-valued random variables $\{Z_n\}_{n=1}^\infty$ defined on probability spaces $(\Omega, B_n, P_n)$, we say $\{Z_n\}_{n=1}^\infty$ satisfies the (weak) LDP with the rate function $I$ if the sequence of push-forward measures $\{P_n \circ Z_n^{-1}\}_{n=1}^\infty$ satisfies the (weak) LDP with the rate function $I$.

We will also need the following standard fact known as the contraction principle (see for example [14, Theorem 4.2.1]).
Theorem 2.3 (Contraction principle). Suppose $M$ and $N$ are Hausdorff topological spaces and $f : M \to N$ is a continuous map. Let $I : M \to [0, \infty]$ be a good rate function.

(i) For any $y \in N$, define

$$ J(y) := \inf \{ I(x) : x \in M \text{ and } y = f(x) \}, $$

then $J : N \to [0, \infty]$ is a good rate function on $N$, where as usual the infimum over empty set is taken as $\infty$.

(ii) If $I$ controls the LDP associated with a family of probability measures $\{\mu_n\}_{n \geq 1}$ on $M$, then $J$ controls the LDP associated with the push-forwards $\{\mu_n \circ f^{-1}\}_{n \geq 1}$ on $N$.

Also important in the theory of LDP are the notions of exponential equivalence and exponential approximation. For the following definition, see for example [14, Definition 4.2.10].

Definition 2.4. For $n \in \mathbb{Z}^+$, let $(\Omega, B_n, P_n)$ be probability spaces and $Z_n$ and $\tilde{Z}_n$ be sequences of $M$-valued random variables with joint laws $P_n$. Then $\{Z_n\}$ and $\{\tilde{Z}_n\}$ are called exponentially equivalent if for every $\delta > 0$ and $\Gamma_\delta := \{(x, y) : \rho(x, y) > \delta\} \subset M \times M$, the set $\{(\tilde{Z}_n, Z_n) \in \Gamma_\delta\} \in B_n$ and

$$ \limsup_{n \to \infty} \frac{1}{n} \log P_n(\Gamma_\delta) = -\infty. $$

The following theorem records the known relationship between the LDPs of exponentially equivalent families of random variables; see for example [14, Theorem 4.2.13].

Theorem 2.5 (Theorem 4.2.13 in [14]). Suppose that $\{Z_n\}$ is exponentially equivalent to $\{\tilde{Z}_n\}$ and satisfies an LDP with a good rate function. Then $\{\tilde{Z}_n\}$ also satisfies an LDP with the same rate function.

For the following see for example [14, Definition 4.2.14].

Definition 2.6. For $n, m \in \mathbb{Z}^+$, let $(\Omega, B_n, P_{n,m})$ be probability spaces, and $Z_n$ and $Z_{n,m}$ be sequences of $M$-valued random variables with joint laws $P_{n,m}$. Then $\{Z_{n,m}\}$ are called exponentially good approximations of $\{\tilde{Z}_n\}$ if for every $\delta > 0$ and $\Gamma_\delta := \{(x, y) : \rho(x, y) > \delta\} \subset M \times M$, the set $\{(\tilde{Z}_n, Z_{n,m}) \in \Gamma_\delta\} \in B_n$ and

$$ \lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log P_{n,m}(\Gamma_\delta) = -\infty. $$

The following theorem records the known relationship between the LDPs of families of exponentially good approximations; see for example [14, Theorem 4.2.16].

Theorem 2.7 (Theorem 4.2.16 in [14]). Suppose that for every $m$, the family of random variables $\{Z_{n,m}\}_{n=1}^\infty$ satisfies the LDP with the rate function $I_m$ and that $\{Z_{n,m}\}$ are exponentially good approximations of $\{\tilde{Z}_n\}$.
Then \( \{ \tilde{Z}_n \}_{n=1}^{\infty} \) satisfies a weak LDP with the rate function

\[
I(y) := \sup_{\delta > 0} \liminf_{m \to \infty} \inf_{z \in B_{y,\delta}} I_m(z)
\]

where \( B_{y,\delta} \) denotes the ball \( \{ z : \rho(y, z) < \delta \} \).

(ii) If \( I \) is a good rate function and for every closed set \( F \)

\[
\inf_{y \in F} I(y) \leq \limsup_{m \to \infty} \inf_{y \in F} I_m(y)
\]

then the full LDP holds for \( \{ \tilde{Z}_n \}_{n=1}^{\infty} \) with the rate function \( I \).

2.2. Carnot groups. In this paper we concentrate on a particular class of metric spaces, namely, homogeneous Carnot groups equipped with the Carnot-Carathéodory metric (by [8, Proposition 2.2.17, Proposition 2.2.18], the assumption about homogeneity is without loss of generality). We begin by recalling basic facts about Carnot (stratified) groups that we require in the sequel. For the uninitiated reader we have tried to be as comprehensive as possible in our exposition. Any missing details as well as further elaboration can be found in a number of references, see for example [8,32].

2.2.1. Carnot groups as Lie groups. We say that \( G \) is a Carnot group of step \( r \) if \( G \) is a connected and simply connected Lie group whose Lie algebra \( g \) is stratified, that is, it can be written as

\[
g = V_1 \oplus \cdots \oplus V_r,
\]

where

\[
[V_1, V_{i-1}] = V_i, \quad 2 \leq i \leq r,
\]

\[
[V_1, V_r] = \{0\}.
\]

To exclude trivial cases we assume that the dimension of \( g \) is at least 3. In addition we will use a stratification such that the center of \( g \) is contained in \( V_r \). In particular, Carnot groups are nilpotent. For \( \mathcal{X} \in g \) we will write

\[
\mathcal{X} = \mathcal{X}^{(1)} + \cdots + \mathcal{X}^{(r)} \in V_1 \oplus \cdots \oplus V_r.
\]

Notation 2.8. By \( \mathcal{H} := V_1 \) we denote the space of horizontal vectors that generate the rest of the Lie algebra with \( V_2 = [\mathcal{H}, \mathcal{H}], \ldots, V_r = \mathcal{H}^{(r)} \).

As usual, we let

\[
\exp : g \to G,
\]

\[
\log : G \to g
\]

denote the exponential and logarithmic maps, which are global diffeomorphisms for connected nilpotent groups, see for example [13, Theorem 1.2.1]. Also, for \( \mathcal{X} \in g \), we let \( \text{ad}_\mathcal{X} : g \to g \) denote the adjoint map defined by \( \text{ad}_\mathcal{X} \mathcal{Y} := [\mathcal{X}, \mathcal{Y}], \mathcal{Y} \in g \).
2.2.2. Identifying $G$ with a linear space. Since $\exp$ and $\log$ are global diffeomorphism between $G$ and $\mathfrak{g}$, we obtain a natural way to identify $G$ with a linear space, underlying its Lie algebra and equipped with some non-trivial group law. We now explain how this identification works. First, by identifying $\mathfrak{g}$ with $\mathbb{R}^N$ we can obtain the following notion of stratified coordinates.

**Definition 2.9.** A set $\{X_1, \ldots, X_N\} \subset \mathfrak{g}$ is a basis for $\mathfrak{g}$ adapted to the stratification if the subset $\{X_{d_0+d_1+\cdots+d_{i-1}+j}\}_{j=1}^{d_i}$ is a basis of $V_i$ for each $i \in [r]$, where we adopt the standard notation $[r] := \{1, \ldots, r\}$ for $r \in \mathbb{N}$.

We now recall the Baker-Campbell-Dynkin-Hausdorff formula, expressing the group product in terms of the Lie algebra. Since $\mathfrak{g}$ is nilpotent, the formula takes a particular appealing form and allows to present the group multiplication by polynomials. For the following version, see for example [8, p. 585, Equation (4.12)] or [13, p. 11].

**Notation 2.10.** For any $X, Y \in \mathfrak{g}$, the *Baker-Campbell-Dynkin-Hausdorff formula* is given by

$$BCDH(X, Y) := \log(e^X e^Y)$$

$$= X + Y + \sum_{k=1}^{r-1} \sum_{(n,m) \in \mathcal{I}_k} a_{n,m}^k \text{ad}^n_X \text{ad}^m_Y \cdots \text{ad}^k_X \text{ad}^k_Y X,$$

where

$$a_{n,m}^k := \frac{(-1)^k}{(k+1)m!n!(|n|+1)}.$$

$$\mathcal{I}_k := \{(n,m) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^k : n_i + m_i > 0 \text{ for all } i \in [k]\}, \text{ and for each multi-index } n \in \mathbb{Z}_+^k,$$

$$n! = n_1! \cdots n_k! \text{ and } |n| = n_1 + \cdots + n_k.$$

Since $\mathfrak{g}$ is nilpotent of step $r$ we have

$$\text{ad}^{n_1}_X \text{ad}^{m_1}_Y \cdots \text{ad}^{n_k}_X \text{ad}^{m_k}_Y X = 0 \text{ if } |n| + |m| \geq r$$

for $X, Y \in \mathfrak{g}$.

The Baker-Campbell-Dynkin-Hausdorff formula suggests it could be beneficial to lift coordinates from the Lie algebra $\mathfrak{g}$ to the group $G$.

**Definition 2.11.** A system of *exponential coordinates (of the first kind)*, relative to a basis $\{X_1, \ldots, X_N\}$ of $\mathfrak{g}$ adapted to the stratification, is a map from $\mathbb{R}^N$ to $G$ defined by

$$x \mapsto \exp \left( \sum_{i=1}^{N} x_i X_i \right), \text{ where } x = (x_1, \ldots, x_N) \in \mathbb{R}^N.$$
With the exponential coordinates we can now equip $\mathbb{R}^N$ with a group operation pulled back from $G$ by

$$z := x \ast y,$$

$$\sum_{i=1}^{N} z_i \mathcal{X}_i = \text{BCDH} \left( \sum_{i=1}^{N} x_i \mathcal{X}_i, \sum_{i=1}^{N} y_j \mathcal{X}_j \right).$$

In particular, in this identification $x^{-1} = -x$. Note that $\mathbb{R}^N$ with this group law is a Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$. Both $G$ and $(\mathbb{R}^N, \ast)$ are nilpotent, connected and simply connected, therefore the exponential coordinates give a diffeomorphism between $G$ and $\mathbb{R}^N$. Thus we identify both $G$ and $\mathfrak{g}$ with $\mathbb{R}^N$.

For $x = \exp(\mathcal{X}) \in G$ with $\mathcal{X} = \sum_{i=1}^{N} x_i \mathcal{X}_i$, we will write

$$(2.5) \quad x = (x^{(1)}, \ldots, x^{(r)}) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_r},$$

where $x^{(j)} = (x_{d_1+\cdots+d_{j-1}+1}, \ldots, x_{d_1+\cdots+d_j})$. We shall henceforth always identify a homogeneous Carnot group $G$ with the $(\mathbb{R}^N, \ast)$.

**Dilations:** Analogous to scaling in Euclidean normed spaces, a stratified Lie algebra is equipped with a natural family of dilations defined for any $a > 0$ by

$$\delta_a(\mathcal{X}) := a^i \mathcal{X}, \text{ for } \mathcal{X} \in V_i.$$  

For each $a > 0$, $\delta_a$ is a Lie algebra isomorphism, and the family of all dilations $\{\delta_a\}_{a>0}$ forms a one-parameter group of Lie algebra isomorphisms. We use the identification between $G$ and $\mathfrak{g}$ to define similar automorphisms $D_a$ on $G$. The maps $D_a := \exp \circ \delta_a \circ \log : G \rightarrow G$ satisfy the following properties.

$$(2.6) \quad D_a \circ \exp = \exp \circ \delta_a \quad \text{for any } a > 0,$$

$$D_{a_1} \circ D_{a_2} = D_{a_1 a_2}, \quad D_1 = I \quad \text{for any } a_1, a_2 > 0,$$

$$D_a(g_1) D_a(g_2) = D_a(g_1 g_2) \quad \text{for any } a > 0 \text{ and } g_1, g_2 \in G,$$

That is, the group $G$ has a family of dilations which is adapted to its stratified structure. Actually, $D_a$ is the unique Lie group automorphism corresponding to $\delta_a$ in the sense that $dD_a = \delta_a$. On a homogeneous Carnot group $\mathbb{R}^N$ the dilation $D_a$ can be described explicitly by

$$D_a(x_1, \ldots, x_N) := (a^{\sigma_1} x_1, \ldots, a^{\sigma_N} x_N),$$

where $\sigma_j \in \{1, \ldots, r\}$ is called the homogeneity of $x_j$, with

$$\sigma_j = i, \quad \text{for } d_0 + d_1 + \cdots + d_{i-1} < j \leq d_1 + \cdots + d_i,$$

with $i = 1, \ldots, r$ and recalling that $d_0 = 0$. That is, $\sigma_1 = \cdots = \sigma_{d_1} = 1, \sigma_{d_1+1} = \cdots = \sigma_{d_1+d_2} = 2$, and so on. In other words, $\sigma_j = i$, if and only if $x_j \in V_i$. 
Group operations as polynomials: A key observation for our analysis is that the group operation of a homogeneous Carnot group $G = (\mathbb{R}^N, \star)$ can be expressed component-wise as homogeneous polynomials. For example, by [17, Proposition 2.1] we have

$$x \star y = x + y + Q(x, y) \text{ for } x, y \in \mathbb{R}^N,$$

where $Q = (Q^{(1)}, \ldots, Q^{(r)}) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ with $Q^{(i)} = (Q_{d_1+\ldots+d_{i-1}+1}, \ldots, Q_{d_1+\ldots+d_i}) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^{d_i}$ and each $Q_j$ is a homogeneous polynomial of degree $\sigma_j$ with respect to the dilations $D_a$ of $G$,

$$Q_j(D_ax, D_ay) = a^{\sigma_j}Q_j(x, y), \quad \text{for } x, y \in \mathbb{R}^N.$$

Moreover, for any $x, y \in G$ we have

$$Q_1(x, y) = \ldots = Q_{d_1}(x, y) = 0,$$

$$Q_j(x, 0) = Q_j(0, y) = 0, Q_j(x, x) = Q_j(x, -x) = 0, \quad \text{for } d_1 < j \leq N,$$

and for $d_1 + \ldots + d_i < j \leq d_1 + \ldots + d_{i+1}$

$$Q_j(x, y) = Q_j((x^{(1)}, \ldots, x^{(i)}), (y^{(1)}, \ldots, y^{(i)})) = -Q_j(-y, -x).$$

In particular, (2.7) gives a direct argument to see that group operations on homogeneous Carnot groups are differentiable. Note that [17] uses a slightly different notation $h_i := d_1 + \ldots + d_i$.

In addition by [8, Proposition 2.2.22 (4)] we have for $j = d_1 + 1, \ldots, N$

$$Q_j(x, y) = \sum_{(k, \ell) \in I_j} (x_ky_\ell - x_\ell y_k) R^{k, \ell}_j(x, y),$$

where $I_j := \{(k, \ell) : k < \ell, \sigma_k + \sigma_\ell \leq \sigma_j\}$ (as described in [18, p. 195]) and $R^{k, \ell}_j$ are homogeneous polynomials of degree $\sigma_j - \sigma_k - \sigma_\ell$ with respect to the group dilations.

**Notation 2.12** (Symplectic form). For any $m$ and vectors $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ define

$$\omega_{k, \ell}(x, y) := x_k y_\ell - x_\ell y_k,$$

for $1 \leq k, \ell \leq m$.

Note that $\omega_{k, \ell}(x, y) = -\omega_{\ell, k}(x, y) = -\omega_{k, \ell}(-y, -x)$, and using this notation we can write (2.8) as

$$Q_j(x, y) = \sum_{(k, \ell) \in I_j} \omega_{k, \ell}(x, y) R^{k, \ell}_j(x, y),$$

for $j = d_1 + 1, \ldots, N$, where $R^{k, \ell}_j(x, y) = R^{k, \ell}_j(-y, -x)$.

**Example 2.13** (Step 2). Suppose $G \cong \mathbb{R}^{d_1 + d_2}$ is a homogeneous Carnot group of step 2. That is, $\mathfrak{g} = \mathcal{H} \oplus \mathcal{V}$ with dim $\mathcal{H} = d_1$, dim $\mathcal{V} = d_2$, and homogeneity $\sigma_j = 1$ if $1 \leq j \leq d_1$ and $\sigma_j = 2$ if $d_1 + 1 \leq j \leq d_1 + d_2$. Then

$$x \star y = x + y + (0_{d_1}, Q^{(2)}(x, y)) = x + y + (0_{d_1}, Q_{d_1+1}(x, y), \ldots, Q_{d_1+d_2}(x, y)).$$
Note that, for $j = d_1 + 1, \ldots, d_1 + d_2$, (2.8) becomes
\[ Q_j(x, y) = \sum_{1 \leq k < \ell \leq d_1} \omega_{k,\ell}(x, y) R_{j}^{k,\ell}(x, y), \]
where the polynomials $R_{j}^{k,\ell}$ are necessarily constant, since in this case $r = 2$ and $\sigma_k = \sigma_{\ell} = 1$, and therefore $\sigma_j - \sigma_k - \sigma_{\ell} = 0$ for $j = d_1 + 1, \ldots, d_1 + d_2$. Thus the components $Q_j$ are skew-symmetric bilinear forms on $\mathbb{R}^{d_1}$,
\[ (2.10) \quad Q_j(x, y) = \sum_{1 \leq k < \ell \leq d_1} \alpha_{j}^{k,\ell} \omega_{k,\ell}(x^{(1)}(y^{(1)})) \]
for some constants $\alpha_{j}^{k,\ell}$ and $d_1 + 1 \leq j \leq d_1 + d_2$.

To make it more transparent, we can write $x_1 = (h_1, v_1)$, $x_2 = (h_2, v_2)$, where $h_1, h_2 \in \mathbb{R}^{d_1}$ and $v_1, v_2 \in \mathbb{R}^{d_2}$, so that each $Q_j$ can be expressed in a matrix form as
\[ Q_j(x_1, x_2) = Q_j((h_1, v_1), (h_2, v_2)) = h_1^T A_j h_2, \]
with
\[ A_j := \left( \alpha_{j}^{k,\ell} \right)_{k,\ell=1}^{d_1}, \quad \alpha_{j}^{k,\ell} = -\alpha_{j}^{\ell,k}. \]

We can use this representation of the group multiplication to write the product of $n$ elements $x_1, \ldots, x_n \in G$, where $x_i = (h_i, v_i)$ with $h_i \in \mathbb{R}^{d_1}$ and $v_i = (v_i^{d_1+1}, \ldots, v_i^{d_1+d_2}) \in \mathbb{R}^{d_2}$, as follows.
\[ \prod_{i=1}^{n} x_i := x_1 \ast \cdots \ast x_n \]
\[ = \left( \sum_{i=1}^{n} h_i, \sum_{i=1}^{n} v_i^{d_1+1} + \sum_{1 \leq k < \ell \leq d_1} h_k^T A_{d_1 + 1} h_\ell, \ldots, \sum_{i=1}^{n} v_i^{d_1+d_2} + \sum_{1 \leq k < \ell \leq d_1} h_k^T A_{d_1 + d_2} h_\ell \right). \]
In particular, when $v_i = 0_{d_2}$ for all $i$
\[ (2.11) \quad \prod_{i=1}^{n} (h_i, 0_{d_2}) = \left( \sum_{i=1}^{n} h_i, \sum_{1 \leq k < \ell \leq d_1} h_k^T A_{d_1 + 1} h_\ell, \ldots, \sum_{1 \leq k < \ell \leq d_1} h_k^T A_{d_1 + d_2} h_\ell \right), \]
and in the case $d_2 = 1$ we have
\[ \prod_{i=1}^{n} (h_i, 0) = \left( \sum_{i=1}^{n} h_i, \sum_{1 \leq k < \ell \leq d_1} h_k^T A h_\ell \right) \]
for a skew-symmetric matrix $A$.

**Example 2.14 (Heisenberg groups).** The (Isotropic) Heisenberg group is an example of a step 2 group with $d_2 = 1$, $d_1 = 2d$ and $A$ being a $2d \times 2d$ matrix with the blocks
\[ \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \]
on the diagonal. Then for $(x_i, y_i, z_i) \in G \cong \mathbb{R}^{2n} \times \mathbb{R}, i \in [n]$, we have
\[
\prod_{i=1}^{n} (x_i, y_i, z_i) = \left( \sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i, \sum_{i=1}^{n} z_i + \sum_{1 \leq i < j \leq n} (x_i, y_i) A(x_j, y_j)^T \right).
\]

Example 2.15 (Step 3 Engel group). This is a group of step 3 that can be modeled on \(\mathbb{R}^4\), with \(\mathcal{H} = \mathbb{R}^2 \times \{0\}\). The multiplication is given by

\[
x \star y = x + y + \left( 0, 0, \frac{1}{2} \omega_{1,2}(x, y), \frac{1}{2} \omega_{1,3}(x, y) + \frac{1}{12} \omega_{1,2}(x, y)(x_1 - y_1) \right),
\]

so that

\[
(x_1, x_2, 0, 0) \star (y_1, y_2, 0, 0) = \left( x_1 + y_1, x_2 + y_2, \frac{1}{2} \omega_{1,2}(x, y), \frac{1}{12} \omega_{1,2}(x, y)(x_1 - y_1) \right)
\]

and

\[
(x_1, x_2, x_3, x_4) \star (-x_1, -x_2, 0, 0) = \left( 0, 0, x_3, x_4 + \frac{1}{2} x_1 x_3 \right).
\]

The following observation will be useful in the sequel and is a straightforward consequence of the Baker-Campbell-Hausdorff-Dynkin formula.

Lemma 2.16. Suppose \(G\) is a homogeneous Carnot group of step \(r\) and \(X_1, \ldots, X_k\) are any elements of \(\mathcal{H}\). Then for \(\ell = 2, \ldots, r\)

\[
(e^{X_1} \star \cdots \star e^{X_n})^{(\ell)} = \sum_{i=(i_1, \ldots, i_{\ell}) \in J_\ell} c_i \text{ad}_{X_{i_1}} \cdots \text{ad}_{X_{i_{\ell-1}}} X_{i_{\ell}}
\]

for some coefficients \(|c_i| < 1\), where \(J_\ell\) is some strict subset of \(\{1, \ldots, n\}^\ell\) and thus \(#J_\ell \leq n^\ell\).

Finally, we can also express some differentials of the group law as polynomials over the Lie algebra. Since we concern ourselves with differentials, we recall that the Lie algebra \(\mathfrak{g}\) is naturally identified with the tangent space \(T_x G\), for any \(x \in G\). Thus, we will still continue to use \(\mathcal{X}\) to denote elements of the Lie algebra which are being mapped to elements of the group \(x = e^x\), but we will also begin at this point to use more traditional vector notation \(u, v\), etc. to denote elements of the Lie algebra and more generally in the tangent spaces of \(G\).

We now introduce the multiplication operator and its differential.

Notation 2.17. For \(x \in G\), we denote by \(L_x : G \to G\) the left multiplication

\[
L_x y := x \star y, \text{ for } y \in G,
\]

and the corresponding pushforward (differential) \((L_x)_* : TG \to TG\) by

\[
(L_x)_* : T_y G \to T_{xy} G
\]

\[
v \mapsto (L_x)_* v.
\]

The Maurer-Cartan form \(\omega\) is the \(\mathfrak{g}\)-valued one-form on \(G\) defined by

\[
\omega : T_x G \to T_x G \cong \mathfrak{g}, \quad v \in T_x G,
\]

\[
\omega(v) := (L_{x^{-1}})_* v \in \mathfrak{g}.
\]
The next statement describes the pushforward of left multiplication on elements of the Lie algebra. It is a corollary of [26, Proposition 3.15], but we include a proof here for convenience.

**Proposition 2.18.** Let $X \in \mathfrak{g}$, $x := e^X \in G$, and $v \in T_e G = \mathfrak{g}$. Then

$$(L_x)_*v = \sum_{n=0}^{r-1} A_n \text{ad}_X^n v$$

where $A_0 = 1$ and for $n = 1, \ldots, r - 1$

$$A_n = -\sum_{k=1}^{n} \sum_{(m,n) \in I_k} \alpha_{k,m}^n$$

where the second sum is over $(n,m) \in \mathcal{I}_k$ so that $m_1 = \cdots = m_{k-1} = 0$, $m_k = 1$, and $|n| + |m| < r$ and $\alpha_{n,m}^k$ are as in (2.4). Equivalently, there exist polynomials $C_{j,\ell}^k$ in $x$ so that

$$(L_x)_*v = v + \left( \sum_{(k,\ell) \in I_{d_1+1}} C_{k,\ell}^{d_1+1}(x)\omega_{k,\ell}(x,v), \ldots, \sum_{(k,\ell) \in I_N} C_{k,\ell}^N(x)\omega_{k,\ell}(x,v) \right)$$

where again $I_j := \{(k,\ell) : k < \ell, \sigma_k + \sigma_{\ell} \leq \sigma_j \}$. In particular, when $G$ is step 2, these polynomials are constants and

$$(L_x)_*v = v + \left( \sum_{1 \leq k < \ell \leq d_1} \alpha_{k,\ell}^{d_1+1} \omega_{k,\ell}(x^{(1)},v^{(1)}), \ldots, \sum_{1 \leq k < \ell \leq d_1} \alpha_{k,\ell}^N \omega_{k,\ell}(x^{(1)},v^{(1)}) \right).$$

**Proof.** Let $\gamma(t) := e^{tv}$, so that $(L_x)_*v = \frac{d}{dt} \bigg|_0 x \ast \gamma(t)$. Using (2.3) we may write

$$x \ast \gamma(t) = \left( X + tv + \sum_{k=1}^{r-1} \sum_{(n,m) \in \mathcal{I}_k} \alpha_{n,m}^k \text{ad}_X^n \text{ad}_v^m \cdots \text{ad}_X^m \text{ad}_v^m X \right).$$

Then the first expression follows from noting that, for each term in the sum,

$$\frac{d}{dt} \bigg|_0 \text{ad}_X^n \text{ad}_v^m \cdots \text{ad}_X^m \text{ad}_v^m X$$

is

$$= \left\{ \begin{array}{ll} \text{ad}_X^n \text{ad}_v X & \text{if } m_k = 1 \text{ and } m_1 = \cdots = m_{k-1} = 0 \\ 0 & \text{otherwise} \end{array} \right.$$.

Alternatively, following (2.7) we may write

$$\frac{d}{dt} \bigg|_0 x \ast \gamma(t) = \frac{d}{dt} \bigg|_0 (x + \gamma(t) + Q(x,\gamma(t)))$$

$$= v + \frac{d}{dt} \bigg|_0 Q(x,\gamma(t)), \quad$$
and using (2.9) note that the only non-zero terms in the second summand are
\[ \frac{d}{dt} \bigg|_0 Q_j(x, \gamma(t)) = \sum_{(k,\ell) \in I_j} \omega_{k,\ell} (x, v) R^k_{j,\ell} (x, 0), \]
where \( R^k_{j,\ell} (x, 0) \) is a polynomial in \( x \) depending only on the structure of \( G \). The step 2 case follows from (2.10).

\[ \square \]

### 2.2.3. Carnot groups as metric spaces.

As discussed in the introduction, the inner product \( \langle \cdot, \cdot \rangle_{H} \) induces a natural sub-Riemannian structure on \( G \). We identify the horizontal space \( H \subset g \) with \( D_e \subset T_e G \), and then define \( D_x := (L_x)_0 \) for any \( x \in G \). Vectors in \( D \) are called horizontal. Recall that we introduced the \( g \)-valued Maurer-Cartan form in Notation 2.17.

**Definition 2.19.** A path \( \gamma : [0, 1] \to G \) is said to be **horizontal** if \( \gamma \) is absolutely continuous and \( \gamma'(t) \in D_{\gamma(t)} \) for a.e. \( t \), that is, the tangent vector to \( \gamma(t) \) at a.e. point of \( \gamma(t) \) is horizontal. Equivalently, \( \gamma \) is horizontal if the (left) Maurer-Cartan form \( c_{\gamma}(t) := (L_{\gamma(t)^{-1}})_* \gamma'(t) \) is in \( H \) for a.e. \( t \).

The metric on \( D \) is defined by
\[ \langle u, v \rangle_x := \langle (L_{x^{-1}})_* u, (L_{x^{-1}})_* v \rangle_H \quad \text{for all } u, v \in D_x. \]

The length of a horizontal path \( \gamma \) may be computed as
\[ \ell(\gamma) := \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}} \, dt = \int_0^1 \sqrt{\langle c_{\gamma}(t), c_{\gamma}(t) \rangle_H} \, dt := \int_0^1 |c_{\gamma}(t)|_H \, dt. \]  

**Example 2.20.** For a Carnot group of step 2 we can describe horizontal paths as follows. Suppose \( \gamma(t) = (A(t), a(t)) \) is an absolutely continuous path in \( G \) with \( A(t) \in \mathbb{R}^{d_1} \times \{0\} \) and \( a(t) \in \{0\} \times \mathbb{R}^{d_2} \). By Proposition 2.18

\[ (L_{\gamma(t)^{-1}})_* \gamma'(t) = \gamma'(t) \]
\[ + \left( 0_{d_1}, - \sum_{1 \leq k < \ell \leq d_1} \alpha^k_{d_1+1} \omega_{k,\ell} (\gamma(t), \gamma'(t)), \ldots, - \sum_{1 \leq k < \ell \leq d_1} \alpha^k_{d_1+d_2} \omega_{k,\ell} (\gamma(t), \gamma'(t)) \right) \]

Recall that the path \( \gamma \) is horizontal if \( (L_{\gamma(t)^{-1}})_* \gamma'(t) \in H \times \{0\} \), and thus we have

\[ a(t) = (a_{d_1+1}(t), \ldots, a_{d_1+d_2}(t)), \]
\[ a'_j(t) = \sum_{1 \leq k < \ell \leq d_1} \alpha^k_{d_1} \omega_{k,\ell} (A(t), A'(t)), \quad j = d_1 + 1, \ldots, d_1 + d_2, \]

for a.e. \( t \in [0, 1] \). That is, a path \( \gamma \) with \( \gamma(0) = e \) is horizontal in a stratified group \( G \) of step 2 if it is of the form...
\begin{equation}
\gamma(t) = \left( A(t), \int_0^t Q^{(2)}(A(s), A'(s)) \, ds \right)
\end{equation}

where \( A(0) = 0 \), \( Q^{(2)} = (Q_{d_1+1}, \ldots, Q_{d_1+d_2}) \) and
\[
Q_j(x, y) = \sum_{1 \leq k < \ell \leq d_1} \alpha_{j,k,\ell} \omega_{j,k,\ell}(x^{(1)}, y^{(1)}).
\]

The length of \( \gamma \) is then given by
\[
\ell(\gamma) = \int_0^1 |\gamma'(s)|_{\gamma(s)} \, ds = \int_0^1 |c_\gamma(t)|_H \, dt = \int_0^1 |A'(s)|_H \, ds.
\]

The group \( G \) as a sub-Riemannian manifold may then be equipped with a natural left-invariant Carnot-Carathéodory distance.

**Definition 2.21.** For any \( x_1, x_2 \in G \) the Carnot-Carathéodory distance is defined as
\[
\rho_{cc}(x_1, x_2) := \inf \{ \ell(\gamma) : \gamma : [0, 1] \to G \text{ is horizontal, } \gamma(0) = x_1, \gamma(1) = x_2 \}.
\]

We denote by
\[
d_{cc}(x) := \rho_{cc}(e, x)
\]
the corresponding norm.

The assumption that \( H \) generates the Lie algebra in (2.2) means that any basis of \( H \) will satisfy Hörmander’s condition. Therefore any two points in \( G \) can be connected by a horizontal path by the Chow–Rashevskii theorem, and the Carnot-Carathéodory distance is finite on \( G \). By [8, Theorem 5.15] the Carnot-Carathéodory distance is realized, that is, for any two points in \( G \) there is a horizontal path connecting those points which is a geodesic, so the infimum in Definition 2.21 is actually a minimum.

The Carnot-Carathéodory distance is just one of the distances on \( G \) which is left-invariant and homogeneous with respect to dilations.

**Definition 2.22** (Homogeneous distances and norms). A homogeneous distance on \( G \) is a continuous, left-invariant distance \( \rho : G \times G \to [0, \infty) \) such that
\[
\rho(D_a x, D_a y) = a \rho(x, y)
\]
for any \( a > 0 \) and \( x, y \in G \). The corresponding homogeneous norm will be denoted by \( d(x) := \rho(e, x) \).

It may be shown that all homogeneous norms on \( G \) are equivalent.

**Proposition 2.23.** [8, Proposition 5.1.4] Let \( d \) be any homogeneous norm on \( G \). Then there exists a constant \( c > 0 \) so that
\[
c^{-1}|x|_G \leq d(x) \leq c|x|_G
\]
where
\[
|x|_G := \left( \sum_{j=1}^r \|x^{(j)}\|^{2r!/j}_{\mathbb{R}^{d_j}} \right)^{1/2r!}.
\]
Similarly, all homogeneous distances satisfy the following.

**Proposition 2.24.** [8, Proposition 5.15.1] Let $\rho$ be any left-invariant homogeneous distance on $G$. Then, for any compact set $K \subset G$, there exists a constant $c_K > 0$ such that

$$c_K^{-1} \| x - y \|_{\mathbb{R}^N} \leq \rho(x, y) \leq c_K \| x - y \|_{\mathbb{R}^N}^{1/r}$$

where $r$ is the step of $G$.

Therefore the topology of a homogeneous Carnot group $(\mathbb{R}^N, \ast)$ with respect to the Carnot–Carathéodory distance (or any other homogeneous distance) coincides with the Euclidean topology of $\mathbb{R}^N$. More precisely, the categories of open, closed, bounded, or compact sets coincide in these two topologies [8, Proposition 5.15.4]. There is a huge literature on the subject, starting with Chow and Rashevsky. More details on homogeneous distances can be found in [8, Sections 5.1 and 5.2], and many references can be found in the bibliography of that text.

Recall that if $G$ is a general Lie group with a Lie algebra $\mathfrak{g}$ equipped with an inner product $\langle \cdot, \cdot \rangle$, we can define the corresponding left-invariant Riemannian distance on $G$. Then the map $(L_x)_* v = dL_x(v) : G \times \mathfrak{g} \rightarrow TG$ as introduced in Notation 2.17

$$dL. (\cdot) : G \times \mathfrak{g} \rightarrow TG,$$

$$(x, v) \mapsto (L_x)_* v \in T_x G$$

is smooth, and thus $(L_x)_* v$ is locally Lipschitz as a mapping in $(x, v)$ with respect to the product topology on $TG$.

For the present paper, we assume that only $\mathcal{H}$ is equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ with $| \cdot |_{\mathcal{H}}$ being the associated norm on $\mathcal{H} \cong \mathbb{R}^{d_h}$. We will need an analogous Lipschitz property in this setting. We use Proposition 2.18 to prove the following statement.

**Proposition 2.25.** For any compact domain $D \subset G \times \mathcal{H}$, there exists a constant $C_D > 0$ such that for any $(x, v), (y, u) \in D$,

$$\|(L_x)_* u - (L_y)_* v\|_{\mathbb{R}^N} \leq C_D (|u - v|_{\mathcal{H}} + \rho(x, y)), \tag{2.14}$$

where $\rho$ can be the Euclidean norm on $\mathbb{R}^N$ or any left-invariant homogeneous distance, and the constant $C_D$ depends only on $D$ and the choice of $\rho$. Moreover, for any $x, y \in K$, a compact subset of $G$, and $v \in \mathcal{H}$

$$\|(L_x)_* v - (L_y)_* v\|_{\mathbb{R}^N} \leq C_K |v|_{\mathcal{H}} \| x - y \|_{\mathbb{R}^N}. \tag{2.15}$$
Proof. By Proposition 2.18

\[(L_x)_*(u) - (L_y)_*(v) = u - v + \left(0_{d_1}, \sum_{(k, \ell) \in I_{d_1+1}} C^{k,\ell}_{d_1+1}(x) \omega_{k,\ell}(x, u), \ldots, \sum_{(k, \ell) \in I_N} C^{k,\ell}_N(x) \omega_{k,\ell}(x, u) \right) - \left(0_{d_1}, \sum_{(k, \ell) \in I_{d_1+1}} C^{k,\ell}_{d_1+1}(y) \omega_{k,\ell}(y, v), \ldots, \sum_{(k, \ell) \in I_N} C^{k,\ell}_N(y) \omega_{k,\ell}(y, v) \right).\]

We have that

\[
|\omega_{k,\ell}(x, u) - \omega_{k,\ell}(y, v)| \leq \|v\|_{\mathbb{R}^N} \|x - y\|_{\mathbb{R}^N} + \|x\|_{\mathbb{R}^N} \|u - v\|_{\mathbb{R}^N}
\]

and so

\[
\|(L_x)_*(u) - (L_y)_*(v)\|_{\mathbb{R}^N} \leq \|u - v\|_{\mathbb{R}^N} + \max\{\|C^{k,\ell}_j\|, \|C^{k,\ell}_j\|\} (\|v\|_{\mathbb{R}^N} \|x - y\|_{\mathbb{R}^N} + \|x\|_{\mathbb{R}^N} \|u - v\|_{\mathbb{R}^N}).
\]

For \(u, v \in \mathcal{H}\) we have \(\|u - v\|_{\mathbb{R}^N} = \|u - v\|_{\mathbb{R}^N} \leq C|u - v|_{\mathcal{H}}\), and so (2.14) (with \(C = \|\cdot\|_{\mathbb{R}^N}\)) and (2.15) follow. Inequality (2.14) for a general left-invariant homogeneous distance \(\rho\) follows from Proposition 2.24. \(\square\)

The next lemma is a version of Grönwall’s lemma in the sub-Riemannian setting, which says that if \(\sigma\) and \(\gamma\) are horizontal paths starting at the origin whose Maurer-Cartan forms \(c_\sigma\) and \(c_\gamma\) are close in \(L^1\), then the paths cannot get too far away from each other.

Lemma 2.26 (Grönwall’s lemma). Let \(G\) be a homogeneous Carnot group modeled on \(\mathbb{R}^N\). Suppose \(\varepsilon > 0\), and \(\sigma, \gamma : [0, 1] \to G\) are horizontal paths such that \(\sigma(0) = \gamma(0) = 0\),

\[
\int_0^1 |c_\sigma(t)|_{\mathcal{H}} dt < \infty,
\]

and

\[
\int_0^1 |c_\sigma(t) - c_\gamma(t)|_{\mathcal{H}} dt < \varepsilon.
\]

Then there exists a constant \(C = C(\|c_\sigma\|_{L^1}) < \infty\) such that

\[
(2.16) \quad \rho_{cc}(\sigma(1), \gamma(1)) < C\varepsilon.
\]

Proof. First note that from the assumptions there exists a compact set \(K\) that contains both paths \(\gamma([0, 1])\) and \(\sigma([0, 1])\) entirely. By Proposition 2.24, it then suffices to prove that \(\|\sigma(1) - \gamma(1)\|_{\mathbb{R}^N} \leq C'\varepsilon\) for some constant \(C' < \infty\). In the following estimates \(C\) will be a constant that depends only on \(K\), which in turn depends on \(\|c_\sigma\|_{L^1}\) but may vary from line to line.

To begin, taking the derivative of \(\gamma\) and \(\sigma\) in the ambient Euclidean space \(\mathbb{R}^N\), we have that
\[ \gamma'(t) - \sigma'(t) = (L_{\gamma(t)} \ast c_\gamma(t) - L_\sigma(t) \ast c_\sigma(t) \]
\[ = (L_{\gamma(t)} \ast (c_\gamma(t) - c_\sigma(t))) + (L_{\gamma(t)} - L_\sigma(t)) \ast c_\sigma(t). \]

Hence
\[ \|\gamma'(t) - \sigma'(t)\|_{\mathbb{R}^N} \leq \| (L_{\gamma(t)} \ast (c_\gamma(t) - c_\sigma(t))) \|_{\mathbb{R}^N} + \| (L_{\gamma(t)} - L_\sigma(t)) \ast c_\sigma(t) \|_{\mathbb{R}^N} \]
\[ \leq C (|c_\gamma(t) - c_\sigma(t)|_{\mathcal{H}} + |c_\sigma(t)|_{\mathcal{H}} \|\sigma(t) - \gamma(t)\|_{\mathbb{R}^N}), \]

where in the second inequality we have applied (2.14) and (2.15).

Now let
\[ F(t) := \|\gamma(t) - \sigma(t)\|_{\mathbb{R}^N}^2. \]

Then for a.e. \( t \)
\[ \frac{dF}{dt} = 2\langle \gamma'(t) - \sigma'(t), \gamma(t) - \sigma(t) \rangle_{\mathbb{R}^N} \leq 2\|\gamma'(t) - \sigma'(t)\|_{\mathbb{R}^N} \|\gamma(t) - \sigma(t)\|_{\mathbb{R}^N} \]
\[ \leq 2C (|c_\gamma(t) - c_\sigma(t)|_{\mathcal{H}} + |c_\sigma(t)|_{\mathcal{H}} \|\sigma(t) - \gamma(t)\|_{\mathbb{R}^N}) \|\gamma(t) - \sigma(t)\|_{\mathbb{R}^N}. \]

Let \( t_0 := \sup\{t \in [0,1] \mid \sigma(t) = \gamma(t) \} \). If \( t_0 = 1 \), then we have that \( \sigma(1) = \gamma(1) \) and (2.16) holds automatically. If \( t_0 \in [0,1) \), we can consider new paths \( \sigma[t_0,1] \) and \( \gamma[t_0,1] \) starting at \( \sigma(t_0) = \gamma(t_0) \) and show that their endpoints are close. With this argument we can then assume that \( F(t) \neq 0 \) for all \( t \in (0,1) \). Consider \( G(t) := \|\gamma(t) - \sigma(t)\|_{\mathbb{R}^N} = \sqrt{F(t)} \), then
\[ \frac{dG}{dt} = \frac{1}{2G(t)} F'(t) \leq C|c_\gamma(t) - c_\sigma(t)|_{\mathcal{H}} + C|c_\sigma(t)|_{\mathcal{H}} G(t). \]

We have
\[ (2.17) \]
\[ \frac{dG}{dt} \leq A(t) + B(t) G(t), \]
where \( A(t) = C|c_\gamma(t) - c_\sigma(t)|_{\mathcal{H}} \) and \( B(t) = C|c_\sigma(t)|_{\mathcal{H}} \). Let \( b(t) = \int_0^t B(s) ds = C \int_0^t |c_\sigma(s)|_{\mathcal{H}} \, ds \). Then (2.17) can be written as
\[ \frac{d}{dt} \left( e^{-b(t)} G(t) \right) \leq A(t)e^{-b(t)}. \]

Clearly \( G(0) = 0 \). It follows that
\[ e^{-b(t)} G(t) \leq \int_0^t A(s)e^{-b(s)} \, ds. \]

In particular for \( t = 1 \) we have that
\[ G(1) = \|\gamma(1) - \sigma(1)\|_{\mathbb{R}^N} \leq C e^{b(1)} \int_0^1 |c_\gamma(t) - c_\sigma(t)|_{\mathcal{H}} e^{-b(t)} \, dt \]
\[ \leq C e^{C\|\sigma\|_{L^1}} \int_0^1 |c_\gamma(t) - c_\sigma(t)|_{\mathcal{H}} \, dt < C\varepsilon. \]

\[ \Box \]
Remark 2.27. We consider the specific homogeneous distance $\rho_{cc}$ in this version of Lemma 2.26, but in light of its proof and Proposition 2.24, it is clear that this result could be stated with $\rho_{cc}$ replaced by any left-invariant homogeneous distance on $G$.

Remark 2.28. This estimate appears in various places in the literature (see for example [10, Lemma 6.7] or [24, Lemma 4.2.5]) for general Lie groups. However, in the general Lie group setting, it is difficult to make sense of comparing vectors in different tangent spaces in the absence of the unifying context of the ambient space $\mathbb{R}^N$.

That being said, the same proof as the one given here works for matrix Lie groups equipped with a left-invariant Riemannian distance $\rho$ under the (standard) assumption that the Lie bracket satisfies the continuity assumption

$$|[A,B]|_g \leq M|A|_g|B|_g$$

for any $A, B \in \mathfrak{g}$.

Proposition 2.29. Given a Carnot group $G$, define

$$L : G \longrightarrow \mathcal{H} := \mathcal{V}_1,$$

$$x \mapsto \int_0^1 c_\gamma(t) \, dt,$$

where $\gamma$ is any horizontal path such that $\gamma(0) = e$ and $\gamma(1) = x$. Then

$$L(x) = P_{\mathcal{H}}(\log(x)),$$

where $P_{\mathcal{H}}$ is the projection onto $\mathcal{H}$, and in particular $L$ is well-defined independent of the choice of $\gamma$. Additionally, $L$ is continuous as a map from $(G, \rho_{cc})$ to $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$.

Proof. First recall that log and exp are global diffeomorphisms in this setting. Therefore we can use the Baker-Campbell-Dynkin-Hausdorff formula (2.3) to see that for any $x_1, x_2 \in G$ we can find $X_1, X_2 \in \mathfrak{g}$ such that $x_1 = e^{X_1}$ and $x_2 = e^{X_2}$, and thus

$$P_{\mathcal{H}}(\log(x_1 x_2)) = P_{\mathcal{H}}(\log(e^{X_1} e^{X_2})) = P_{\mathcal{H}}(X_1) + P_{\mathcal{H}}(X_2) = P_{\mathcal{H}}(\log(e^{X_1})) + P_{\mathcal{H}}(\log(e^{X_2})) = P_{\mathcal{H}}(\log(x_1)) + P_{\mathcal{H}}(\log(x_2)).$$

In particular, this implies that

$$P_{\mathcal{H}}(\log(x_1 x_2)) = P_{\mathcal{H}}(\log(x_2 x_1)).$$

Additionally,

$$P_{\mathcal{H}}(\log(x^{-1})) = P_{\mathcal{H}}(\log(e^{-X})) = -P_{\mathcal{H}}(X) = -P_{\mathcal{H}}(\log(x)).$$

Now let $\gamma$ be any horizontal path such that $\gamma(0) = e$ and $\gamma(1) = x$. Using the above observations for $x_1 = \gamma(t)$ and $x_2 = \gamma(t + \varepsilon)$, we see that

$$P_{\mathcal{H}}(\log(\gamma(t + \varepsilon))) - P_{\mathcal{H}}(\log(\gamma(t))) = P_{\mathcal{H}}(\log(\gamma(t)^{-1})) + P_{\mathcal{H}}(\log(\gamma(t + \varepsilon))) = P_{\mathcal{H}}(\log(\gamma(t)^{-1} \gamma(t + \varepsilon))) - P_{\mathcal{H}}(\log(\gamma(t)^{-1} \gamma(t))).$$
since $P_{\mathcal{H}}(\log(e)) = 0$. Therefore

$$\frac{d}{dt} P_{\mathcal{H}}(\log(\gamma(t))) = \left| \frac{d}{d\varepsilon} P_{\mathcal{H}}(\log(\gamma(t + \varepsilon))) \right|_{\varepsilon=0} = (P_{\mathcal{H}} \circ \log)_t \left( (L_{\gamma(t)})^{-1}_t \gamma'(t) \right) = (P_{\mathcal{H}} \circ \log)_t(c_\gamma(t)) = c_\gamma(t),$$

for a.e. $t$ in $[0, 1]$, since $P_{\mathcal{H}} \circ \log : G \to \mathcal{H}$, and its differential (pushforward) $d(P_{\mathcal{H}} \circ \log) : \mathfrak{g} \to \mathcal{H}$ is the identity on the horizontal space $\mathcal{H}$. In particular,

$$P_{\mathcal{H}}(\log(x)) = \int_0^1 \frac{d}{dt} P_{\mathcal{H}}(\log(\gamma(t))) dt = \int_0^1 c_\gamma(t) dt \in \mathcal{H}$$

for any horizontal path $\gamma$ connecting $e$ and $x$.

Similarly, if we have $x_1, x_2 \in G$ and any horizontal path $\gamma$ connecting $x_1$ and $x_2$, we see that

$$P_{\mathcal{H}}(\log(x_1)) - P_{\mathcal{H}}(\log(x_2)) = \int_0^1 c_\gamma(t) dt.$$

Thus,

$$|P_{\mathcal{H}}(\log(x_1)) - P_{\mathcal{H}}(\log(x_2))|_{\mathcal{H}} = \left| \int_0^1 c_\gamma(t) dt \right|_{\mathcal{H}} \leq \int_0^1 |c_\gamma(t)|_{\mathcal{H}} dt = \ell(\gamma),$$

and taking the infimum over all such horizontal paths $\gamma$ gives

$$|L(x_1) - L(x_2)|_{\mathcal{H}} \leq \rho_{cc}(x_1, x_2),$$

which implies continuity.

The significance of the map $L$ is that unlike in the case of Lie groups equipped with a Riemannian metric, we only have metric on the horizontal space $\mathcal{H}$, but $\log$ does not respect the horizontal structure. This is a fundamental difference from the techniques used in [34, Proposition 5.2], and one can think of the map $L$ as a horizontal logarithmic map.

3. Large deviations

Now we return to our main result, Theorem 1.1. Suppose $\{X_i\}_{i=1}^\infty$ is a sequence of i.i.d. random variables in $\mathcal{H}$ and define

$$S_n := \exp(X_1) \ast \cdots \ast \exp(X_n).$$

Note that for a dilation $D_a : G \to G$ by (2.6) we have

$$D_a S_n = \prod_{i=1}^n D_a \exp(X_i) = \prod_{i=1}^n \exp(\delta_a X_i).$$

The proof consists of several steps. First in Section 3.1 we will consider a partial linearization of the $n$-fold group multiplication by partitioning $D_\frac{1}{n} S_n$ into a fixed number of blocks. This linearization naturally lives in the product space $\mathcal{H}^n$, and
so we will prove a large deviations principle for sequences of random vectors in $H^m$. In Section 3.2 we will use the linearization to find a family of exponentially good approximations to $\{D_n S_n\}$. Then in Section 3.3 we combine these results with the contraction principle, Theorem 2.3, to prove Theorem 1.1.

3.1. Vector space LDP. For a fixed $m \in \{1, \ldots, n\}$, we partition $D_n S_n$ into $m$ pieces as follows. For $k = 0, \ldots, m - 1$, let $n_k = k \lfloor n/m \rfloor$ and $n_m = n$, and for $k = 1, \ldots, m$ we define

$$S_n^{m,k} := \prod_{i=n_{k-1}+1}^{n_k} \star \exp(\delta_n X_i).$$

Now let

$$Y_n^{m,k} := L(S_n^{m,k}) = L\left(\exp(\delta_n X_{n_{k-1}+1}) \star \cdots \star \exp(\delta_n X_{n_k})\right) = \frac{1}{n} (X_{n_{k-1}+1} + \cdots + X_{n_k}) \in H,$$

where $L$ is the map defined in Proposition 2.29 and take

$$Y_n^m := (Y_n^{m,1}, \ldots, Y_n^{m,m}) \in H^m.$$

It will be useful later to note that, taking $d = \lfloor n/m \rfloor$ so that $n = dm + r$ for some $r \in \{0,1,\ldots,m-1\}$, we have that for $k \in [m-1]$, each $Y_n^{m,k}$ consists of $d$ steps, and $Y_n^{m,m}$ consists of $d + r$ steps. Now we prove the following large deviation principle for $\{Y_n^m\} \in H^m$.

**Proposition 3.1.** Suppose $\{X_i\}_{i=1}^\infty$ are i.i.d. mean $0$ random variables in $H$ such that

$$\Lambda(\lambda) := \Lambda_X(\lambda) := \log \mathbb{E}[\exp(\langle \lambda, X_1 \rangle_H)]$$

exists for all $\lambda \in H$. Fix $m \in \mathbb{N}$. Then for any closed $F \subset H^m$ and open $O \subset H^m$ we have that

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n^m \in F) \leq - \inf_{u \in F} I_m(u)$$

and

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n^m \in O) \geq - \inf_{u \in O} I_m(O),$$

where for $u = (u_1, \ldots, u_m) \in H^m$

$$I_m(u) := \frac{1}{m} \sum_{k=1}^m \Lambda^*(muk)$$

and $\Lambda^*$ is the Legendre transform,

$$\Lambda^*(u) = \sup_{v \in H} (\langle v, u \rangle_H - \Lambda(v)).$$
Proof. To prove the upper bound, we first note that by following the proof of the upper bound in the classical Cramér’s theorem (see for example [14, p. 37], or Proposition A.1 in the appendix) one may show that for any closed set $F \subset \mathcal{H}^m$

\begin{equation}
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y^m_n \in F) \leq - \inf_{u \in F} \sup_{\lambda \in \mathcal{H}^m} \left\{ \langle \lambda, u \rangle_{\mathcal{H}^m} - \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ \exp \left( n \langle \lambda, Y^m_n \rangle_{\mathcal{H}^m} \right) \right] \right\}.
\end{equation}

Now for $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{H}^m$

\[ \mathbb{E} \left[ \exp \left( n \langle \lambda, Y^m_n \rangle_{\mathcal{H}^m} \right) \right] = \prod_{k=1}^{m} \mathbb{E} \left[ \exp \left( n \langle \lambda_k, Y^m_{n,k} \rangle_{\mathcal{H}} \right) \right] \]
\[ = \prod_{k=1}^{m} \prod_{i=n_{k-1}+1}^{n_k} \mathbb{E} \left[ \exp \left( \langle \lambda_k, X_i \rangle_{\mathcal{H}} \right) \right] = \prod_{k=1}^{m} \exp(\Lambda(\lambda_k))^{n_k-n_{k-1}}. \]

Again letting $d = \left\lfloor \frac{n}{m} \right\rfloor$ so that $n = dm + r$ for some $r \in \{0, 1, \ldots, m-1\}$, we have that $n_k - n_{k-1} = d$ for all $k \in [m-1]$ and $n_m - n_{m-1} = n - (m-1)d = n-md + d = d + r$. Thus we may write

\[ \mathbb{E} \left[ \exp \left( n \langle \lambda, Y^m_n \rangle_{\mathcal{H}^m} \right) \right] = \left( \prod_{k=1}^{m} \exp(\Lambda(\lambda_k)) \right)^d \exp(\Lambda(\lambda_m))^r \]

and so

\[ \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ \exp \left( n \langle \lambda, Y^m_n \rangle_{\mathcal{H}^m} \right) \right] = \limsup_{n \to \infty} \frac{n/m}{n} \sum_{k=1}^{m} \Lambda(\lambda_k) + \frac{r}{2n} \Lambda(\lambda_m) \frac{r}{n} \Lambda(\lambda_m) \]
\[ = \frac{1}{m} \sum_{k=1}^{m} \Lambda(\lambda_k). \]

So for all $u = (u_1, \ldots, u_m) \in \mathcal{H}^m$ we have

\[ \sup_{\lambda \in \mathcal{H}^m} \left\{ \langle \lambda, u \rangle_{\mathcal{H}^m} - \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ \exp \left( n \langle \lambda, Y^m_n \rangle_{\mathcal{H}^m} \right) \right] \right\} \]
\[ = \sup_{\lambda \in \mathcal{H}^m} \left\{ \sum_{k=1}^{m} \langle \lambda_k, u_k \rangle_{\mathcal{H}} - \frac{1}{m} \sum_{k=1}^{m} \Lambda(\lambda_k) \right\} \]
\[ = \frac{1}{m} \sup_{\lambda \in \mathcal{H}^m} \left\{ \sum_{k=1}^{m} \langle \lambda_k, mu_k \rangle_{\mathcal{H}} - \sum_{k=1}^{m} \Lambda(\lambda_k) \right\} \]
\[ \leq \frac{1}{m} \sum_{k=1}^{m} \sup_{\lambda_k \in \mathcal{H}} \{ \langle \lambda_k, mu_k \rangle_{\mathcal{H}} - \Lambda(\lambda_k) \} = \frac{1}{m} \sum_{k=1}^{m} \Lambda^*(mu_k). \]
For the lower bound, as usual it suffices to prove that for any \( u = (u_1, \ldots, u_m) \in \mathcal{H}^m \) and \( \varepsilon > 0 \) we have
\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n^m \in B(u, \varepsilon)) \geq -\frac{1}{m} \sum_{k=1}^{m} \Lambda^*(mu_k).
\]
Choose \( \delta > 0 \) sufficiently small so that
\[
B(u_1, \delta) \times \cdots \times B(u_m, \delta) \subset B(u, \varepsilon).
\]
Then
\[
\log \mathbb{P}(Y_n^m \in B(u, \varepsilon)) \geq \log \mathbb{P}(Y_n^{m,1} \in B(u_1, \delta), \ldots, Y_n^{m,m} \in B(u_m, \delta))
\]
\[
= \log \prod_{k=1}^{m} \mathbb{P}(Y_n^{m,k} \in B(u_k, \delta))
\]
\[
= \sum_{k=1}^{m} \log \mathbb{P}(Y_n^{m,k} \in B(u_k, \delta)).
\]
Recall that for each \( k \in [m-1] \), \( Y_n^{m,k} = \frac{1}{n}(X_{n_{k-1}+1} + \cdots + X_{n_{k-1}+\lfloor n/m \rfloor}) \), and so again by the classical Cramér’s theorem
\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n^{m,k} \in B(u_k, \delta)) \geq -\frac{1}{m} \Lambda^*(mu_k),
\]
and the \( k = m \) case can be dealt with similarly as was done in the upper bound case, yielding the desired result.

3.2. Exponentially good approximations. For a fixed \( m \in \mathbb{N} \), define the map \( \Psi_m : \mathcal{H}^m \to G \) by
\[
(3.5) \quad \Psi_m(u_1, \ldots, u_m) := \exp(u_1) \ast \cdots \ast \exp(u_m).
\]

Example 3.2 (Example 2.14 revisited). In this case \( \Psi_m : \mathcal{H}^m \to G \) can be viewed as the map \( \Psi_m : (\mathbb{R}^{2d})^m \to \mathbb{R}^{2d} \times \mathbb{R} \) with \( u_i = (x_i, y_i) \)
\[
\Psi_m((x_1, y_1), \ldots, (x_m, y_m)) = \left( \sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i, \sum_{1 \leq i < j \leq n} (x_i, y_i) A(x_j, y_j)^T \right).
\]

In this section we show that \( \{\Psi_m(Y_n^m)\}_{m \in \mathbb{N}} \) are exponentially good approximations of \( \{D_n S_n\} \), as in Definition 2.6, considering the following cases separately: Carnot groups of step 2 with random vectors \( X_i \) having a sub-Gaussian distribution on \( \mathcal{H} \), general Carnot groups where \( X_i \) are bounded almost surely, and general Carnot groups where \( X_i \) are Gaussian. We will rely on the standard argument that if
\[
(3.6) \quad \mathbb{E}[e^{\pm \lambda Z}] \leq C(|\lambda|),
\]
for some $\lambda \in \mathbb{R}$, then for any $t > 0$

$$
P(|Z| \geq t) = P(Z \geq t) + P(-Z \geq t)
$$

(3.7)

$$
= P(e^{\lambda Z} \geq e^{\lambda t}) + P(e^{-\lambda Z} \geq e^{\lambda t}) \leq 2e^{-\lambda t}C(|\lambda|)
$$

by Markov’s inequality.

As it turns out, the general Gaussian case is the most technically challenging. This is due to the combination of the general group setting and the unbounded support of the Gaussian. We thus start with the other cases, which will allow us to introduce some of the necessary ideas.

3.2.1. Step 2 groups with sub-Gaussian distributions. Recall that a mean zero random variable $X$ is sub-Gaussian if there is a $k > 0$ such that

$$
\mathbb{E}[e^{\lambda X}] \leq e^{k\lambda^2}
$$

for all $\lambda \in \mathbb{R}$.

A random vector in $\mathbb{R}^n$ is sub-Gaussian if the one-dimensional marginals $\langle x, X \rangle$ are sub-Gaussian for all $x \in \mathbb{R}^n$.

Before proceeding further, we take the opportunity to further expand on some comments made in the introduction of this paper.

Remark 3.3. We look here at the step two case to illustrate our comments in Section 1 comparing our results with [5]. Let $X_1, X_2, \ldots$ and $Z_1, Z_2, \ldots$ be independent random variables on $\mathcal{H}$ and $\mathcal{V}$, respectively, and take

$$
S_n := \exp(X_1, Z_1) \star \cdots \star \exp(X_n, Z_n).
$$

Then

$$
D_{1/n}S_n \star (D_{1/n}S_n)^{-1} = \left(0, \frac{1}{n^2}(Z_1 + \cdots + Z_n)\right)
$$

and so by Proposition 2.24

$$
P(d_{cc}(D_{1/n}S_n \star (D_{1/n}S_n)^{-1}) > \delta) \sim P\left(\frac{c}{n^2}\|Z_1 + \cdots + Z_n\| > \delta\right),
$$

where on the right hand side this is just the standard Euclidean distance from the center. Suppose the distribution of each random variable $Z$ is sub-Gaussian with parameter $k$. Then $Z_1 + \cdots + Z_n$ is sub-Gaussian with parameter $nk$, and so

$$
P\left(\frac{c}{n^2}\|Z_1 + \cdots + Z_n\| > \delta\right) = P\left(\|Z_1 + \cdots + Z_n\| > \frac{\delta n^2}{c}\right) \leq 2e^{-\delta^2/2}e^{nk\lambda^2}
$$

for arbitrary $\lambda$. Keeping $\lambda$ a fixed constant, we may say that

$$
\lim_{n \to \infty} \frac{1}{n} \log P\left(\frac{c}{n^2}\|Z_1 + \cdots + Z_n\| > \delta\right) \leq \lim_{n \to \infty} \frac{1}{n} \log 2e^{-\delta^2/2}e^{nk\lambda^2}
$$

$$
= \lim_{n \to \infty} \frac{1}{n} \left(\log 2 - \frac{\delta^2}{c} + nk\lambda^2\right) = -\infty
$$
and thus \( \{D_{1/n}S_n\} \) and \( \{D_{1/n}S_n\} \) are exponentially equivalent in the sense of Definition 2.4. Therefore, Theorem 2.3 implies that, when \( \{D_{1/n}S_n\} \) satisfies an LDP, \( \{D_{1/n}S_n\} \) will satisfy an LDP with the same rate function. Thus, we may compare the application of Theorem 1.4 to the Heisenberg group to [2, Example 1].

It will also be useful to recall that we say a random variable \( X \) is sub-exponential with parameters \( \nu^2 \) and \( \alpha \) if
\[
\mathbb{E}[e^{\lambda X}] \leq e^{\nu^2 \lambda^2/2}, \quad \text{for any } |\lambda| < \frac{1}{\alpha}.
\]

We will write \( X \in SE(\nu^2, \alpha) \). Similarly, a random vector is called sub-exponential if it has all sub-exponential marginals. For Carnot groups of step 2 as in Example 2.13, the group operation is given by quadratic polynomials. Thus, if the distribution of the \( X_i \) is sub-Gaussian, we will observe a random walk with sub-exponential distributions. This will allow us to apply standard concentration results concerning quadratic forms and sub-exponential distributions in general. For convenience we record the following basic result.

**Lemma 3.4.** If \( Z_i \) are sub-exponential random variables with parameters \( \nu_i^2 \) and \( \alpha_i \), then the random variable \( Z := \sum_{i=1}^n Z_i \) is also sub-exponential with parameters \( \alpha = \max_{1 \leq i \leq n} \alpha_i \) and
\[
(i) \quad \nu^2 = \sum_{i=1}^n \nu_i^2 \text{ if the } Z_i \text{s are independent.}
\]
\[
(ii) \quad \nu^2 = (\sum_{i=1}^n \nu_i)^2 \text{ if the } Z_i \text{s are dependent.}
\]

**Proof.** The first statement is obvious since \( X \) and \( Y \) independent implies that
\[
\mathbb{E}[e^{\lambda(X+Y)}] = \mathbb{E}[e^{\lambda X}]\mathbb{E}[e^{\lambda Y}].
\]
The second statement follows from an application of Hölder’s inequality
\[
\mathbb{E}[e^{\lambda(X+Y)}] \leq (\mathbb{E}[e^{p\lambda X}])^{1/p} \left( \mathbb{E}[e^{p\lambda Y/(p-1)}] \right)^{(p-1)/p} \leq e^{\nu^2 \lambda^2/2} e^{\nu^2 \lambda^2/2(p-1)}
\]
followed by optimization over \( p \). \( \square \)

**Proposition 3.5.** Suppose \( G \) is step 2 and that \( \{X_i\}_i \) are \( \mathcal{H} \)-valued i.i.d. sub-Gaussian random vectors with mean 0. Let \( Y_n \) be as in (3.1) and \( \Psi_m \) as given in (3.5). Then for all \( \delta > 0 \)
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\rho_{cc}(\Psi_m(Y_n^m), D_{1/n}S_n)) > \delta) = -\infty.
\]

**Proof.** Fix \( \delta > 0 \). Recall that by the left invariance of the distance
\[
\rho_{cc}(\Psi_m(Y_n^m), D_{1/n}S_n) = \rho_{cc}(e, (\Psi_m(Y_n^m))^{-1}D_{1/n}S_n) = d_{cc}((\Psi_m(Y_n^m))^{-1}D_{1/n}S_n).
\]
Consider first the case \( d_2 = \dim[\mathcal{H}, \mathcal{H}] = 1 \), which corresponds to the Heisenberg group, as in Example 2.14. Then by (2.11)
\[
D_{1/n}S_n = e^{X_1} \star \cdots \star e^{X_n} = \left( \frac{1}{n} \sum_{k=1}^n X_k, \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} X_i^T A_{d_{i+1}} X_j \right)
\]
and
\[ \Psi_m(Y^m_n) = e^{Y^m_1} \cdots e^{Y^m_n} \]
\[ = \left( \frac{1}{n} \sum_{k=1}^{n} X_k, \frac{1}{n^2} \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \sum_{i=n_{k-1}+1}^{n_k} \sum_{j=n_{\ell-1}+1}^{n_{\ell}} X_i^T A_{d_1+1} X_j \right). \]

Thus
\[ (\Psi_m(Y^m_n))^{-1} \ast D_{\frac{1}{n}} S_n = \left( 0, \frac{1}{n^2} \sum_{k=1}^{m} \sum_{n_{k-1}<i<j\leq n_k} X_i^T A_{d_1+1} X_j \right). \]

So in light of Proposition 2.23 it is sufficient to show that
\[ \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{n^2} \sum_{k=1}^{m} \sum_{n_{k-1}<i<j\leq n_k} X_i^T A_{d_1+1} X_j > \delta \right) = -\infty. \]

Now, by the comparison lemma [35, Lemma 6.2.3], for independent mean zero sub-Gaussian random vectors \( X \) and \( X' \) in \( \mathbb{R}^{d_1} \) with \( \|X\| \psi_2, \|X'\| \psi_2 \leq K \), and any \( \lambda \in \mathbb{R} \) and \( d_1 \times d_1 \) matrix \( A \)
\[ \mathbb{E}[\exp(\lambda X^T A X') \leq \mathbb{E}[\exp(C_1 K^2 \lambda G^T A G')] \]
where \( G \) and \( G' \) are independent \( \mathcal{N}(0, I_{d_1}) \) random vectors and \( C_1 \) is an absolute constant. (Here, \( \| \cdot \| \psi_2 \) is the sub-Gaussian norm, which is necessarily finite when \( X \) is sub-Gaussian.) Furthermore, by [35, Lemma 6.2.2] there are absolute constants \( C_2 \) and \( c \)
\[ \mathbb{E}[\exp(C_1 K^2 \lambda G^T A G')] \leq \exp(C_2 C_1^2 K^4 \lambda^2 \|A\|_{HS}^2) \]
for all \( \lambda \) satisfying \( |\lambda| \leq c/C_1 K^2 \|A\|_{op} \).

We denote by \( C \) and \( c \) constants that do not depend on the structure of the group or distribution, but might change from one bound to another. Putting the above together we see that for \( X \) and \( X' \) as given
\[ \mathbb{E}[\exp(\lambda X^T A X') \leq \exp(C K^4 \|A\|_{HS}^2 \lambda^2)], \quad \text{for any } |\lambda| \leq \frac{c}{K^2 \|A\|_{op}}. \]

This means that \( X^T A X' \) is sub-exponential with parameters \( \nu^2 := 2CK^4 \|A\|_{HS}^2 \lambda^2 \) and \( \alpha := \frac{K^2 \|A\|_{op}}{c} \).

In particular, the above is true for \( X = X_i \) and \( X' = X_j \) for any \( i \neq j \) with \( A = A_{d_1+1} \). Now for any \( \ell \geq 2 \) consider
\[ Z := \sum_{1 \leq i < j \leq \ell} X_i^T A X_j = \sum_{a=3}^{2\ell-1} \sum_{1 \leq i < j \leq \ell \atop i+j=a} X_i^T A X_j =: \sum_{a=3}^{2\ell-1} Z(a), \]
where the second double summation is the same as summing along the antidiagonals of the array \((X_i^T A X_j)_{1 \leq i < j \leq \ell}\). We will apply Lemma 3.4 to recognize this sum as having a sub-exponential distribution and to bound its parameters. Note in particular that each \( Z(a) \) is a sum of independent sub-exponential random variables \( X_i^T A X_j \) all with
common parameters $\nu_{ij}^2 = \nu^2$ and $\alpha_{ij} = \alpha$, where $\nu^2$ and $\alpha$ are as given above. Thus, $Z(a)$ is sub-exponential with parameters $\alpha_a = \alpha$ and $\nu_a^2 = \sum_{1 \leq i < j \leq \ell} \nu_{ij}^2 \leq \frac{\ell}{2} \nu^2$.

Therefore, as a sum of dependent sub-exponential random variables, $Z = \sum_{a=3}^{2\ell-1} Z(a)$ is sub-exponential with parameters $\alpha_Z = K^2 \|A\|_{op}/c$, and we may make the following (rough) estimate

$$\nu_a^2 = \left( \sum_{a=3}^{2\ell-1} \nu_a \right)^2 \leq (2\ell - 3)^2 \frac{\ell}{2} \nu^2 \leq \ell^3 C K^4 \|A\|_{HS}^2.$$ 

Applying this now to $Z_k := \sum_{n_{k-1} < i < j < n_k} X_i^T A X_j$, for any $k \in [m]$, we have $Z_k \in SE(d^3 C K^4 \|A\|_{HS}^2, K^2 \|A\|_{op}/c)$, where again $d = \lceil n/m \rceil$ (and $n_k - n_{k-1} = d$ for $k \in [m-1]$ and $n_m - n_{m-1} = d + r$ for some $r \in [m-1]$). Since $Z_k$ and $Z_{k'}$ for $k \neq k'$ are sums over non-overlapping subsets of indices, we have that the $Z_k$'s are also independent, so

$$\sum_{k=1}^m Z_k \in SE \left( m d^3 C K^4 \|A\|_{HS}^2, \frac{K^2 \|A\|_{op}}{c} \right).$$

Thus by (3.7) for any $\delta > 0$ and $0 < \lambda \leq \frac{c}{K^2 \|A\|_{op}}$

$$\mathbb{P} \left( \frac{1}{n^2} \sum_{k=1}^m Z_k \left| \mathbb{P} \right| > \delta \right) \leq 2 e^{-\lambda m n^2} e^{md^3 C K^4 \|A\|_{HS}^2 \lambda^2}.$$ 

In particular, this is true for any $\lambda = 1/d = 1/\lceil n/m \rceil$ for sufficiently large $n$ and thus

$$\lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{n^2} \sum_{k=1}^m \sum_{n_{k-1} < i < j < n_k} X_i^T A X_j \left| \mathbb{P} \right| > \delta \right) = \lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \frac{1}{n^2} \sum_{k=1}^m Z_k > \delta \right) \leq \lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \left( 2 e^{-m \delta n^2} e^{md^3 C K^4 \|A\|_{HS}^2} \right) = \lim_{m \to \infty} \limsup_{n \to \infty} (-\delta m + C K^4 \|A\|_{HS}^2) = -\infty.
This essentially completes the proof, since in the case that $d_2 = \dim[H, H] > 1$ and $(\Psi_m(Y^m_n))^{-1} \ast D_n^{1}S_n = (0, Z_1^n, \ldots, Z_{d_2}^n)$ with

$$Z_n^\ell = \frac{1}{n^2} \sum_{k=1}^{m} \sum_{n_{k-1} < i < j \leq n_k} X_i^T A_{d_1+i}X_j,$$

we have that

$$\mathbb{P}\left(\| (Z_1^n, \ldots, Z_{d_2}^n) \|_{\mathbb{R}^{d_2}} > \delta \right) = \mathbb{P}\left( \sum_{\ell=1}^{d_2} (Z_n^\ell)^2 > \frac{\delta^2}{d_2} \right) \leq \sum_{\ell=1}^{d_2} \mathbb{P}\left( (Z_n^\ell)^2 > \frac{\delta^2}{d_2} \right) = \sum_{\ell=1}^{d_2} \mathbb{P}\left( |Z_n^\ell| > \frac{\delta}{\sqrt{d_2}} \right)$$

and the result follows by applying the previous estimates to each term. □

Note that Appendix B describes properties of random quadratic forms. If $G$ is of step 3 or higher, we need to rely on concentration inequalities for polynomials of random vectors of higher order which are not easily available.

### 3.2.2. Higher step groups with bounded distributions

Suppose $G$ is a homogeneous Carnot group of step $r$ as described in Section 2.2. In particular, as before we identify both the Lie group $G$ and its Lie algebra $g$ with $\mathbb{R}^N$.

First we will need the next simple lemma for our estimates.

**Lemma 3.6.** Recall the notation (2.5). For $\ell = 2, \ldots, r$

$$(D_n^{1}S_n - \Psi_m(Y^m_n))^{(\ell)} = \frac{1}{n^{\ell}} \sum_{i_1, \ldots, i_\ell} c_{i_1, \ldots, i_\ell} \prod_{i=1}^{\ell} X_{i_i}$$

for some coefficients $|c_i| < 1$, where $\mathcal{J}_\ell'$ is some strict subset of $\{1, \ldots, n\}^\ell$ satisfying $\# \mathcal{J}_\ell' \leq C \frac{n^\ell}{m}$ where $C$ is a constant that only depends on $\ell$.

**Proof.** As with Lemma 2.16 the form of $(D_n^{1}S_n - \Psi_m(Y^m_n))^{(\ell)}$ follows from the Baker-Campbell-Hausdorff-Dynkin formula (2.3) along with the definition of the dilation. So we only need to prove the bound on $\# \mathcal{J}_\ell'$. However, we may note that

$$\mathcal{J}_\ell' \subseteq \{i = (i_1, \ldots, i_\ell) : i_1, \ldots, i_\ell \in \{n_{k-1} + 1, \ldots, n_k\} \text{ for some } k \in [m]\},$$

and so $\# \mathcal{J}_\ell' \leq m \cdot C(n/m)^\ell \leq C \frac{n^\ell}{m}$. □

Clearly the estimate in the lemma above on the number of terms appearing in the sum is rough, but it is sufficient for our purposes. We are now able to prove that under these conditions $\{\Psi_m(Y^m_n)\}$ are exponentially good approximations to $\{D_n^{1}S_n\}$.

**Proposition 3.7.** Suppose that $\{X_i\}_{i=1}^\infty$ are i.i.d. mean 0 bounded random vectors in $H$. Then for all $\delta > 0$

$$\lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\rho_{cc}(\Psi_m(Y^m_n), D_n^{1}S_n) > \delta) = -\infty.$$
Proof. Suppose \(|X_i|, i \geq 1\) are a.s. bounded by some constant \(M > 1\). First, note that by Lemma 2.16
\[
\|D_n S_n\|_{R^N} \leq \sum_{\ell=1}^{r} \frac{1}{n^\ell} \|S_n^{(\ell)}\|_{R^d}\ell \leq \sum_{\ell=1}^{r} \frac{1}{n^\ell} \sum_{i \in J_{\ell}} \|c_i\| \|\text{ad}_{X_{i_1}} \cdots \text{ad}_{X_{i_{\ell-1}}} X_{i_{\ell}}\|_{R^d}\ell
\]
\[
\leq \sum_{\ell=1}^{r} \frac{1}{n^\ell} n^\ell (2M)^\ell \leq r(2M)^r.
\]
Thus for all \(n\), \(D_n S_n\) is in some compact subset of \(R^N\) with diameter depending only on \(M\) and \(r\), and similarly for \(\Psi_m(Y_m^n)\). Thus, by Proposition 2.24, it suffices to prove that
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\|D_n S_n - \Psi_m(Y_m^n)\|_{R^N} > \delta) = -\infty,
\]
or rather that, for each \(\ell = 2, \ldots, r\),
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\|D_n S_n - \Psi_m(Y_m^n)\|^{(\ell)}_{R^d} > \delta) = -\infty.
\]
So fix \(\ell \in \{2, \ldots, r\}\). By Lemma 3.6 we have that
\[
\mathbb{P}(\|D_n S_n - \Psi_m(Y_m^n)\|^{(\ell)}_{R^d} > \delta) = \mathbb{P} \left( \frac{1}{n^{\ell}} \left\| \sum_{i \in J_{\ell}'} c_i \text{ad}_{X_{i_1}} \cdots \text{ad}_{X_{i_{\ell-1}}} X_{i_{\ell}} \right\|_{R^d} > \delta \right)
\]
\[
\leq \mathbb{P} \left( \frac{1}{n^{\ell}} \sum_{i \in J_{\ell}'} \|\text{ad}_{X_{i_1}} \cdots \text{ad}_{X_{i_{\ell-1}}} X_{i_{\ell}}\|_{R^d} > \delta \right)
\]
\[
\leq \sum_{i \in J_{\ell}'} \mathbb{P} \left( \frac{1}{n^{\ell}} \|\text{ad}_{X_{i_1}} \cdots \text{ad}_{X_{i_{\ell-1}}} X_{i_{\ell}}\|_{R^d} > \frac{\delta m}{Cn^{\ell}} \right)
\]
Since the distribution of the \(X_i\)'s is bounded in \(\mathcal{H}\), there exists an \(M\) such that \(\mathbb{P}(\|X_i\|_{\mathcal{H}} \geq M) = 0\). Since \(\|\text{ad}_X Y\| \leq 2|X|_{\mathcal{H}}|Y|_{\mathcal{H}}\), this implies that for any multi-index \(i \in J_{\ell}'\)
\[
\mathbb{P} \left( \|\text{ad}_{X_{i_1}} \cdots \text{ad}_{X_{i_{\ell-1}}} X_{i_{\ell}}\|_{R^d} \geq \frac{M^{\ell}}{2^\ell} \right) = 0.
\]
Thus \(\|\text{ad}_{X_{i_1}} \cdots \text{ad}_{X_{i_{\ell-1}}} X_{i_{\ell}}\|_{R^d}\) satisfies (3.6) with \(C(\lambda) = e^{M^\ell \lambda}\) for any \(\lambda > 0\) for each \(i \in J_{\ell}'\), and it follows by (3.7) that
\[
\sum_{i \in J_{\ell}'} \mathbb{P} \left( \|\text{ad}_{X_{i_1}} \cdots \text{ad}_{X_{i_{\ell-1}}} X_{i_{\ell}}\|_{R^d} > \frac{\delta m}{C} \right) \leq C \frac{n^{\ell}}{m} \cdot 2 \exp \left( -\lambda \frac{\delta m}{C} \right) \exp(M^\ell \lambda).
\]
In particular, this is true for \(\lambda = n\) for any \(n\). Thus,
\[
\frac{1}{n} \log \mathbb{P}(\|D_n S_n - \Psi_m(Y_m^n)\|^{(\ell)}_{R^d} > \delta) \leq \frac{1}{n} \log \left( 2C \frac{n^{\ell}}{m} \exp \left( -\lambda \frac{\delta m}{C} \right) \exp(M^\ell n) \right).
\]
Putting this all together gives
\[
\lim sup_{n \to \infty} \frac{1}{n} \log P\left( \| (D_{1/n}S_n - \Psi_m(Y_m^n))^{(\ell)} \|_{d_\ell} > \delta \right) \leq \lim sup_{n \to \infty} \frac{1}{n} \left( -\frac{\delta m}{C} + M^\ell n \right) = -\frac{\delta m}{C} + M^\ell
\]
and taking \( m \to \infty \) completes the proof. \( \square \)

3.2.3. **Higher step groups with a Gaussian distribution.** Again, we let \( G \) be a homogeneous Carnot group of step \( r \) and identify it and its Lie algebra \( \mathfrak{g} \) with \( \mathbb{R}^N \). We’ll now prove that, when \( \{X_i\}_{i=1}^\infty \) are i.i.d. standard Gaussian random variables on \( \mathcal{H} \), \( \{\Psi_m(Y_m^n)\} \) form exponentially good approximations of \( \{D_{1/n}S_n\} \).

**Proposition 3.8.** Suppose that \( \{X_i\}_{i=1}^\infty \) are i.i.d. \( \mathcal{N}(0, \text{Id}_H) \). Then for all \( \delta > 0 \)
\[
\lim_{m \to \infty} \lim sup_{n \to \infty} \frac{1}{n} \log P\left( \rho_{cc}\left( D_{1/n}S_n, \Psi_m(Y_m^n) \right) \geq \delta \right) = -\infty.
\]

Before explaining the proof of Proposition 3.8 we first introduce some specific conventions and notation. If \( E \) is an inner product space, \( \| \cdot \|_E \) will always stand for the Hilbertian norm on \( E \). When the space is clear from the context we will abbreviate to \( \| \cdot \| \). The notations \( C, C', C_1, C_2, \ldots \) will stand for constants which may depend on \( G \), but not on \( n \) and \( m \).

We also introduce the following projection operators which somewhat streamline and refine the projections induced by the stratification, as in (2.2).

**Notation 3.9.** For any step \( r \) homogeneous Carnot group \( G \cong \mathbb{R}^N \) and element \( x = (x^{(1)}, \ldots, x^{(r)}) \in G \), if \( \ell \in [r] \), we denote by \( \Pi_\ell : \mathbb{R}^N \to \mathbb{R}^{d_\ell} \) the projection onto the \( \ell \)th step of \( G \)
\[
\Pi_\ell x := x^{(\ell)} = (x_{d_1 + \cdots + d_{\ell-1} + 1}, \ldots, x_{d_1 + \cdots + d_\ell}).
\]
Moreover, for any \( j \in [d_\ell] \), \( \Pi^j_\ell : \mathbb{R}^N \to \mathbb{R} \) will be the projection onto the \( j \)th coordinate of \( \text{Image}(\Pi_\ell) \),
\[
\Pi^j_\ell x := x_{d_1 + \cdots + d_{\ell-1} + j}.
\]

With the above conventions, our proof of Proposition 3.8 relies on the following result, specializing to separate Euclidean approximations on each step of \( G \).

**Proposition 3.10.** Let \( \delta > 0 \) and \( \ell \in [r] \). It holds that
\[
\lim_{m \to \infty} \lim sup_{n \to \infty} \frac{1}{n} \log P\left( \left\| \Pi_\ell \left( (\Psi_m(Y_m^n))^{-1} * D_{1/n}S_n \right) \right\| \geq \delta \right) = -\infty.
\]

Given Proposition 3.10 we may now prove Proposition 3.8.

**Proof of Proposition 3.8.** First, the Carnot-Carathéodory metric is left-invariant, thus
\[
\rho_{cc}\left( D_{1/n}S_n, \Psi_m(Y_m^n) \right) = \rho_{cc}\left( (\Psi_m(Y_m^n))^{-1} * D_{1/n}S_n, 0 \right).
\]
Second, since all homogeneous norms on $G$ are equivalent by Proposition 2.23, we have

$$\rho_{cc}\left(\left(\Psi_m(Y_m^m)\right)^{-1} \ast D_{1/n}^n S_n, 0\right) \leq \left(C \sum_{\ell=1}^{r} \left\| \Pi_\ell \left(\left(\Psi_m(Y_m^m)\right)^{-1} \ast D_{1/n}^n S_n\right)\right\|^2 \right)^{\frac{1}{2r}}.$$ 

So,

$$\mathbb{P}\left(\rho_{cc}\left(D_{1/n}^n S_n, \Psi_m(Y_m^m)\right) \geq \delta\right) \leq \mathbb{P}\left(\sum_{\ell=1}^{r} \left\| \Pi_\ell \left(\left(\Psi_m(Y_m^m)\right)^{-1} \ast D_{1/n}^n S_n\right)\right\|^2 \geq \delta^{2r}/C\right) \leq \sum_{\ell=1}^{r} \mathbb{P}\left(\left\| \Pi_\ell \left(\left(\Psi_m(Y_m^m)\right)^{-1} \ast D_{1/n}^n S_n\right)\right\| \geq \delta^{\ell} \left(\frac{C' \ell}{\delta}\right)^{\frac{1}{2r}}\right).$$

The result follows by applying Proposition 3.10 to each summand separately with $\delta$ replaced by $\delta_{\ell}^{\ell}$ (with $C' \ell / \delta$), and proceeding in a similar fashion to the proofs of Propositions 3.5 and 3.7.

So we henceforth focus our attention on the proof of Proposition 3.10. This requires understanding the group operation on each step of $G$. In light of (2.8) we introduce the following polynomial functionals in Carnot groups.

**Definition 3.11 (Polynomials in Carnot groups).** Let $G$ be a Carnot group of step $r$.

- Let $\beta, \beta' \in [r]$. A function $\omega : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is called a $(\beta, \beta')$-symplectic form if there exist $j_\beta \in [d_\beta]$ and $j_{\beta'} \in [d_{\beta'}]$ such that

\begin{equation}
\omega(x, y) = (\Pi_{j_\beta} x)(\Pi_{j_{\beta'}} y) - (\Pi_{j_\beta} y)(\Pi_{j_{\beta'}} x).
\end{equation}

- Let $\alpha \in [r]$. A function $P : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is called an $\alpha$-monomial if it is of the form

\begin{equation}
P(x, y) := \prod_{i=1}^{r'} \Pi_{j_i}^i z_i,
\end{equation}

for some $r' \leq r$, and where for each $i, z_i \in \{x, y\}, \ell_i \in [r], \text{ and } j_i \in [d_{\ell_i}]$. Moreover, we have the degree constraint $\sum \ell_i = \alpha$.

- A pair $(\omega, P)$ of a $(\beta, \beta')$-symplectic form and an $\alpha$-monomial is called an $\ell$-homogeneous pair if

\begin{equation}
\ell = \alpha + \beta + \beta'.
\end{equation}

With these definitions, the main observation is that the group operation can be decomposed into a linear part and a homogeneous part. Formally, for every $\ell \in [r], j \in [d_\ell], \text{ and } x, y \in \mathbb{R}^N$, we rewrite (2.7) and (2.8) as,

\begin{equation}
\Pi_{j_\ell}^j (y^{-1} \ast x) = \Pi_{j_\ell}^j (x - y) + \sum_{k=1}^{r^2} \omega^k(x, -y) P^k(x, -y).
\end{equation}
Here $K^j_\ell$ is some fixed number and for every $k \in [K^j_\ell]$, $(\omega^k, P^k)$ is an $\ell$-homogeneous pair.

We shall handle the linear part and the polynomial part in (3.11) separately. Towards this we require the following concentration inequalities for polynomials in Gaussian variables, which follow from hypercontractivity. The proof of which can be found in [21, Chapters 5 and 6].

**Lemma 3.12** (Theorem 5.10 and Theorem 6.7 with Remark 6.8 in [21]). Let $X_1, \ldots, X_n$ be i.i.d. standard Gaussian random variables in some Euclidean space $E$ and let $Q : E \to \mathbb{R}$ be a degree-$p$ polynomial. Then

1. For any $q \geq 1$,
   \[
   (\mathbb{E} \|Q(X_1, \ldots, X_n)\|^q)^{\frac{1}{q}} \leq C_{q,p} \sqrt{\mathbb{E} \|Q(X_1, \ldots, X_n)\|^2},
   \]
   where $C_{q,p} > 0$ depends only on $p$ and $q$.

2. For and $\delta > 0$,
   \[
   \mathbb{P}(\|Q(X_1, \ldots, X_n)\| \geq \delta) \leq \exp\left(-C_p \left(\frac{\delta}{\sqrt{\mathbb{E} \|Q(X_1, \ldots, X_n)\|^2}}\right)^{\frac{2}{p}}\right),
   \]
   where $C_p > 0$ depends only on $p$.

**Bounding the linear part:** Our task of bounding the elements appearing in (3.11) starts with the linear part, for which we will need the following second moment estimates.

**Lemma 3.13.** Let $\ell \in [r]$. It holds that

1. \[
   \mathbb{E} \left[\|\Pi_\ell D_{\frac{1}{n}} S_n\|^2\right], \mathbb{E} \left[\|\Pi_\ell \Psi_m(Y^m_n)\|^2\right] \leq \frac{C}{n^\ell},
   \]

2. \[
   \mathbb{E} \left[\|\Pi_\ell \left(D_{\frac{1}{n}} S_n - \Psi_m(Y^m_n)\right)\|^2\right] \leq \frac{C}{mn^\ell}.
   \]

**Proof.** For the first claim, by the Baker-Campbell-Hausdorff-Dynkin formula, (2.3), we have

\[
\Pi_\ell D_{\frac{1}{n}} S_n = \frac{1}{n^\ell} \sum_{I \in [n]^\ell} c_I \text{ad}_{X_{I_1}} \ldots \text{ad}_{X_{I_{\ell-1}}} X_{I_\ell},
\]

where $|c_I| \leq 1$ and $I = (I_1, \ldots, I_\ell)$. For $I \in [n]^\ell$ we abbreviate

\[
X_I := \text{ad}_{X_{I_1}} \ldots \text{ad}_{X_{I_{\ell-1}}} X_{I_\ell},
\]

so that,

\[
(3.12) \quad \mathbb{E} \left[\|\Pi_\ell D_{\frac{1}{n}} S_n\|^2\right] \leq \frac{1}{n^{2\ell}} \sum_{I,J \in [n]^\ell} |\mathbb{E}[X_I, X_J]|.
\]
Now, for $I, J \in [n]^\ell$, consider the multiset,

$$I \cup J := \{I_1, \ldots, I_\ell, J_1, \ldots, J_\ell\},$$

and suppose that there exists $k \in I \cup J$ which appears an odd number of times. In this case, by the symmetry of standard Gaussian random variables, and the bi-linearity of the Lie brackets

$$\mathbb{E}[X_I \cdot X_J] = -\mathbb{E}[X_I \cdot X_J] \implies \mathbb{E}[X_I \cdot X_J] = 0.$$

Thus, every non-zero summand in (3.12) must satisfy that every element in $I \cup J$ appears at least twice. Since there are at most $Cn^\ell$ such pairs we conclude

$$\mathbb{E}\left[\|\Pi_\ell D_{\frac{n^\ell}{n}} S_n\|^2\right] \leq \frac{1}{n^{2\ell}} Cn^\ell = \frac{C}{n^\ell}.$$

In the above we have used the fact that the distribution of $X_I \cdot X_J$ only depends on the number of identical elements and their positions. Since there is a finite, in $\ell$, number of such combinations, we have that

$$\mathbb{E}[X_I \cdot X_J] \leq C,$$

where $C > 0$ depends only on $\ell$. This concludes the bound of $\mathbb{E}\left[\|\Pi_\ell D_{\frac{n^\ell}{n}} S_n\|^2\right]$. The proof for $\mathbb{E}\left[\|\Pi_\ell \Psi_m(Y_{m^n})\|^2\right]$ is completely identical, with less relevant pairs in (3.12), and we omit it.

For the second part of the claim, we apply the Baker-Campbell-Hausdorff-Dynkin formula to $\Pi_\ell \left(D_{\frac{n^\ell}{n}} S_n - \Psi_m(Y_{m^n})\right)$, as in Lemma 3.6

$$\Pi_\ell \left(D_{\frac{n^\ell}{n}} S_n - \Psi_m(Y_{m^n})\right) = \frac{1}{n^\ell} \sum_{I \in \mathcal{I}_\ell} c_I \text{ad}_X_{i_1} \cdots \text{ad}_X_{i_{\ell-1}} X_{i_\ell},$$

where again $|c_I| \leq 1$, and $\mathcal{I}_\ell \subset [n]^\ell$ is such that for every $i \in [n],

$$|\{I_{\ell-1} : I \in \mathcal{I}_\ell \text{ and } I_\ell = i\}| \leq \frac{n}{m}.$$

This property says that, once the last element is chosen, there are at most $\frac{n}{m}$ different choices for the next element, and in particular $|\mathcal{I}_\ell| \leq \frac{n^\ell}{m}$. As before, it will be enough to count the number of pairs $I, J \in \mathcal{I}_\ell$ such that every element in $I \cup J$ appears an even number of times. Thus, we need to choose at most $n^\ell$ elements. There are $n$ choices for $I_1$ and once this element is chosen there are only $\frac{n}{m}$ for $I_{\ell-1}$ and necessarily $I_\ell \neq I_{\ell-1}$. The rest of the elements have at most $n$ choices, and there is a finite, in $\ell$, number of ways to arrange them. Altogether, there are at most $C\frac{n^\ell}{m}$ pairs in $\mathcal{I}_\ell$, for which

$$\mathbb{E}[X_I \cdot X_J] \neq 0.$$

The conclusion of the proof is identical to the previous part. \qed

We can now use Lemma 3.13 along with appropriate Gaussian concentration results to bound the linear part in (3.11).
Lemma 3.14. Let $\delta > 0$ and $\ell \in [r]$. It holds that,

$$
P \left( \| \Pi_\ell \left( D_{1/\sqrt{n}} S_n - \Psi_m (Y_n^m) \right) \| \geq \delta \right) \leq \exp \left( -C_1 (\delta^2 m)^{1/\ell n} \right),$$

for a constant $C_1 > 0$ that may depend on $\ell$ and $G$, but not on $n$ and $m$.

Proof. Observe that, by the Baker-Campbell-Hausdorff-Dynkin formula, (2.3), every entry of $\Pi_\ell \left( D_{1/\sqrt{n}} S_n - \Psi_m (Y_n^m) \right)$ is a degree $\ell$ polynomial in the Gaussian random variables $X_1, \ldots, X_n$. Thus, by Lemma 3.12 we have

$$
P \left( \| \Pi_\ell \left( D_{1/\sqrt{n}} S_n - \Psi_m (Y_n^m) \right) \| \geq \delta \right) \leq \exp \left( -C_1 \left( \frac{\delta}{\sqrt{E \left[ \| \Pi_\ell \left( D_{1/\sqrt{n}} S_n - \Psi_m (Y_n^m) \right) \|^2 \right]}} \right)^{2/\ell} \right) \leq \exp \left( -C_1 (\delta^2 m)^{1/\ell n} \right),$$

where the second inequality is due to Lemma 3.13 (2). □

Bounding the polynomial part: To bound the polynomial part in (3.11), we focus on a single summand. Thus, let $\omega$ be a $(\beta, \beta')$-symplectic form, as in (3.8), and let $P$ be an $\alpha$-monomial, in the form of (3.9). Assume further that $(\omega, P)$ form an $\ell$-homogeneous pair so that, as in (3.10), $\alpha + \beta + \beta' = \ell$.

As before, we shall require a second moment estimate, this time for monomials.

Lemma 3.15. Let $P$ be an $\alpha$-monomial, as in (3.9),

$$
P(X, Y) = \prod_{i=1}^{r'} \Pi_{\ell_i} Z_i,$$

with $\sum \ell_i = \alpha$, and $r' \leq \alpha$. Then for Gaussian random variables $X_1, \ldots, X_n$,

$$
E \left[ P(D_{1/\sqrt{n}} S_n - \Psi_m (Y_n^m))^2 \right] \leq \frac{C}{n^{\alpha}},
$$

for a constant $C > 0$ which may depend on $\alpha$. Consequently, for any $\delta > 0$,

$$
P \left( \left| P(D_{1/\sqrt{n}} S_n - \Psi_m (Y_n^m)) \right| \geq \delta \right) \leq \exp \left( -C \delta^{2/\alpha n} \right).
$$

Proof. By Hölder’s inequality,

$$
E \left[ P(D_{1/\sqrt{n}} S_n - \Psi_m (Y_n^m))^2 \right] \leq \left( \prod_{i=1}^{r'} E \left[ \| \Pi_{\ell_i} Z_i \|^2 r' \right] \right)^{1/r'},
$$

where for each $i$, $Z_i \in \{ D_{1/\sqrt{n}} S_n - \Psi_m^m \}$. Now, by the Baker-Campbell-Hausdorff-Dynkin formula, (2.3), for each $i$, both $\Pi_{\ell_i} D_{1/\sqrt{n}} S_n$ and $\Pi_{\ell_i} \Psi_m^m$ are degree-$\ell$ polynomials in the Gaussian variables $X_1, \ldots, X_n$. Applying Lemma 3.12 we obtain,

$$
E \left[ \| \Pi_{\ell_i} Z_i \|^2 r' \right] \leq C_{\alpha} \left( \frac{n^{\alpha}}{r' \ell_i} \right),
$$
where the second inequality is due to Lemma 3.13 (1), and $C_\alpha, C'_\alpha > 0$ are constants which can be chosen to only depend on $\alpha$. Thus,

$$\mathbb{E} \left[ P(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m))^2 \right] \leq C_\alpha^\frac{1}{\beta} \left( \prod_{i=1}^{r'} \frac{1}{m_i} \right)^\frac{1}{\beta'} = C_\alpha^\frac{1}{\beta} \frac{1}{n^\alpha}.$$

Above we have used $\sum \ell_i = \alpha$. This relation also shows that $P(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m))$ is a degree-$\alpha$ polynomial in the Gaussian variables, and hence by Lemma 3.12,

$$\mathbb{P} \left( \left| P(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m)) \right| \geq \delta \right) \leq \exp \left( -C'' \alpha \delta \left( \frac{\delta}{\sqrt{\mathbb{E} \left[ P(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m))^2 \right]}} \right)^\frac{2}{\alpha} \right) \leq \exp(-C\delta^\frac{2}{\alpha} n).$$

\[\square\]

With Lemma 3.13 we can now bound the homogeneous part in (3.11).

**Lemma 3.16.** Let $\omega$ and $P$ be as above with $\alpha + \beta + \beta' = \ell$. Then,

$$\mathbb{P} \left( \left| \omega(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m)) P(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m)) \right| \geq \delta \right) \leq \exp(-C(\delta^2 m)^\frac{1}{r} n)$$

for some $C > 0$, which can depend on $\alpha, \beta, \beta'$, and $G$, but not on $n$ and $m$.

**Proof.** Let us write $S = \Pi^{\beta}_\beta D_{\frac{1}{n}} S_n, S' = \Pi^{\beta'}_{\beta'} D_{\frac{1}{n}} S_n$, and similarly for $\Psi, \Psi'$. So,

$$\omega(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m)) = S\Psi' - S'\Psi = (S - \Psi)\Psi + (\Psi' - S')\Psi.$$

Let us fix $M, M' > 0$ to be chosen later and apply the union bound,

$$\mathbb{P} \left( \left| \omega(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m)) P(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m)) \right| \geq \delta \right)$$

$$\leq \mathbb{P} \left( \left| S - \Psi \right| \left| \Psi' P(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m)) \right| \geq \delta \right) + \mathbb{P} \left( \left| S' - \Psi' \right| \left| \Psi P(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m)) \right| \geq \delta \right)$$

$$\leq \mathbb{P} \left( \left| \Pi_{\beta}(D_{\frac{1}{n}} S_n - \Psi_m(Y_n^m)) \right| \geq \frac{\delta}{2M} \right) \quad + \mathbb{P} \left( \left| \Pi_{\beta'}(D_{\frac{1}{n}} S_n - \Psi_m(Y_n^m)) \right| \geq \frac{\delta}{2M^'} \right)$$

$$\quad + \mathbb{P} \left( \left| \Psi' P(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m)) \right| \geq M \right) \quad + \mathbb{P} \left( \left| \Psi P(D_{\frac{1}{n}} S_n, -\Psi_m(Y_n^m)) \right| \geq M' \right).$$

By Lemma 3.14,

$$\mathbb{P} \left( \left| \Pi_{\beta}(D_{\frac{1}{n}} S_n - \Psi_m(Y_n^m)) \right| \geq \frac{\delta}{2M} \right) \quad + \mathbb{P} \left( \left| \Pi_{\beta'}(D_{\frac{1}{n}} S_n - \Psi_m(Y_n^m)) \right| \geq \frac{\delta}{2M^'} \right)$$

(3.13) \quad \leq \exp \left( -C \left( \frac{\delta^2}{M^2 m} \right)^\frac{1}{\beta} n \right) \quad + \exp \left( -C \left( \frac{\delta^2}{M^2 m} \right)^\frac{1}{\beta'} n \right).$$
The other terms will be bounded using Lemma 3.15. Indeed, observe that \( \Psi P(D_{\frac{1}{n}} S_n; -\Psi_m(Y_n^m)) \) is a monomial of degree \( \alpha + \beta \) and that \( \Psi' P(D_{\frac{1}{n}} S_n; -\Psi_m(Y_n^m)) \) is a monomial of degree \( \alpha + \beta' \). Since, necessarily, \( \alpha + \beta, \alpha + \beta' \leq \ell \), Lemma 3.15 gives,

\[
\mathbb{P} \left( \left| \Psi P(D_{\frac{1}{n}} S_n; -\Psi_m(Y_n^m)) \right| \geq M' \right) \leq \exp \left( -CM' \frac{2}{\alpha + \beta} n \right),
\]

(3.14) \[
\mathbb{P} \left( \left| \Psi' P(D_{\frac{1}{n}} S_n; -\Psi_m(Y_n^m)) \right| \geq M \right) \leq \exp \left( -CM' \frac{2}{\alpha + \beta'} n \right).
\]

We now choose \( M = (\delta^2 m)^{-\frac{\beta + \alpha}{2\alpha}} \) and \( M' = (\delta^2 m)^{-\frac{\beta' + \alpha}{2\alpha}} \). Plugging these choices into (3.13) and (3.14), and recalling \( \alpha + \beta + \beta' = \ell \), we get

\[
\mathbb{P} \left( \left| \omega(D_{\frac{1}{n}} S_n; -\Psi_m(Y_n^m)) P(D_{\frac{1}{n}} S_n; -\Psi_m(Y_n^m)) \right| \geq \delta \right) \leq \exp( -C(\delta^2 m)^{\frac{1}{2}} n).
\]

\[\square\]

**Finishing the proof:** With Lemma 3.14 and Lemma 3.16 we can now prove Proposition 3.10.

**Proof of Proposition 3.10.** Applying the decomposition in (3.11) to each coordinate in \( \Pi_\ell \left( (\Psi_m(Y_n^m))^{-1} \ast D_{\frac{1}{n}} S_n \right) \), shows that there exists some numbers \( K, K' > 0 \) such that

\[
\mathbb{P} \left( \left\| \Pi_\ell \left( (\Psi_m(Y_n^m))^{-1} \ast D_{\frac{1}{n}} S_n \right) \right\| \geq \delta \right) \leq \mathbb{P} \left( \left\| \Pi_\ell \left( D_{\frac{1}{n}} S_n - \Psi_m(Y_n^m) \right) \right\| \geq \frac{\delta}{K} \right)
\]

\[
+ \sum_{k=1}^{K'} \mathbb{P} \left( \left| \omega^k(D_{\frac{1}{n}} S_n; -\Psi_m(Y_n^m)) P^k(D_{\frac{1}{n}} S_n; -\Psi_m(Y_n^m)) \right| \geq \frac{\delta}{K} \right),
\]

where for each \( k \in [K'] \), \( (\omega^k, P^k) \) are an \( \ell \)-homogeneous pair. Applying Lemmas 3.14 and 3.16 we see

\[
\mathbb{P} \left( \left\| \Pi_\ell \left( (\Psi_m(Y_n^m))^{-1} \ast D_{\frac{1}{n}} S_n \right) \right\| \geq \delta \right) \leq (K' + 1) \exp \left( -C \left( \frac{\delta^2}{K^2 m} \right)^{\frac{1}{2}} n \right).
\]

Thus,

\[
\lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \left| \Pi_\ell \left( (\Psi_m(Y_n^m))^{-1} \ast D_{\frac{1}{n}} S_n \right) \right| \geq \delta \right)
\]

\[
\leq \lim_{m \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \left( (K' + 1) \exp \left( -C \left( \frac{\delta^2}{K^2 m} \right)^{\frac{1}{2}} n \right) \right)
\]

\[
= \lim_{m \to \infty} -C \left( \frac{\delta^2}{K^2 m} \right)^{\frac{1}{2}} = -\infty.
\]

\[\square\]
3.3. LDP for the random walk. Before proceeding to the proof of Theorem 1.1, we introduce the following notation for some particular piecewise linear paths in $G$.

**Notation 3.17.** Fix $m \in \mathbb{N}$, and let $u \in \mathcal{H}^m$. We let $\sigma_{m,u} : [0,1] \to G$ denote the horizontal path such that $\sigma_{m,u}(0) = e$ and

$$c_{\sigma_{m,u}}(t) = (L_{\sigma_{m,u}(t)-1})_{*} \sigma'_{m,u}(t) = mu_k$$

for $\frac{k-1}{m} < t < \frac{k}{m}$ for $k \in [m]$.

**Example 3.18.** It is a useful exercise to write these paths explicitly, at least in the step 2 case. We follow the notation in Example 2.13. For a given $m$ and $u = (u_1, \ldots, u_m) \in \mathcal{H}^m$, we may use the expression for horizontal paths given by (2.13) to explicitly describe the path $\sigma = \sigma_{m,u}$. For $0 < t < \frac{1}{m}$,

$$\sigma(t) = \left( tmu_1, \int_0^t Q^{(2)}(smu_1, mu_1) \, ds \right) = (tmu_1, 0),$$

for $\frac{1}{m} < t < \frac{2}{m}$, writing $\sigma(t) = (A(t), a(t))$,

$$A(t) = u_1 + \left( t - \frac{1}{m} \right) mu_2$$

$$a(t) = \left( \int_0^{1/m} Q^{(2)}(smu_1, mu_1) \, ds + \int_0^t Q^{(2)} \left( u_1 + \left( s - \frac{1}{m} \right) mu_2, mu_2 \right) \, ds \right)$$

$$= \left( t - \frac{1}{m} \right) Q^{(2)}(u_1, mu_2),$$

and generally for $\frac{k-1}{m} < t < \frac{k}{m}$, $k \in [m]$,

$$A(t) = \sum_{j=1}^{k-1} u_j + \left( t - \frac{k-1}{m} \right) mu_k$$

$$a(t) = \sum_{\ell=2}^{k-1} \int_{(\ell-1)/m}^{\ell/m} Q^{(2)} \left( \sum_{j=1}^{\ell-1} u_j + \left( s - \frac{\ell-1}{m} \right) mu_\ell, mu_\ell \right) \, ds$$

$$+ \int_{(k-1)/m}^{t} Q^{(2)} \left( \sum_{j=1}^{k-1} u_j + \left( s - \frac{k-1}{m} \right) mu_k, mu_k \right) \, ds$$

$$= \sum_{\ell=2}^{k-1} \sum_{j=1}^{\ell-1} Q^{(2)}(u_j, mu_\ell) + \left( t - \frac{k-1}{m} \right) \sum_{j=1}^{k-1} \omega(u_j, mu_k).$$

Note in particular that, for $\sigma(1) = (A(1), a(1))$, we have $A(1) = u_1 + \cdots + u_m$ and

$$a(1) = \sum_{\ell=2}^{m} \sum_{j=1}^{\ell-1} Q^{(2)}(u_j, u_\ell) = \sum_{j=1}^{m} \sum_{\ell=j+1}^{m} Q^{(2)}(u_j, u_\ell),$$

and thus $\sigma(1) = \exp(u_1) \ast \cdots \ast \exp(u_m)$. 

We are now ready to prove Theorem 1.1, which follows immediately from the following proposition coupled with the exponentially good approximations established in Propositions 3.5, 3.7, and 3.8.

Proposition 3.19. Let \( \{X_k\}_{k=1}^{\infty} \) be i.i.d. mean 0 random variables in \( \mathcal{H} \), and set
\[
\Lambda(\lambda) := \Lambda_X(\lambda) := \log \mathbb{E}[\exp(\langle \lambda, X_k \rangle_{\mathcal{H}})].
\]
Consider \( Y^m_n \), as in Proposition 3.1. If \( \Psi_m(Y^m_n) \) is an exponentially good approximation for \( D_1^n S_n \). Then for \( S_n := \exp(X_1) \ast \cdots \ast \exp(X_n) \) the measures \( \{\mu_n\}_{n=1}^{\infty} \) satisfy a large deviations principle with rate function given by
\[
J(x) := \inf \left\{ \int_0^1 \Lambda^*(c_\sigma(t)) \, dt : \sigma \text{ horizontal with } \sigma(0) = e \text{ and } \sigma(1) = x \right\},
\]
Proof. For a fixed \( m \in \mathbb{N} \), recall the map \( \Psi_m : \mathcal{H}^m \rightarrow G \) defined by \( 3.5 \).

Note that, as the composition of group products with the exponential map, \( \Psi_m \) is continuous. Thus, by Theorem 2.3 (the contraction principle) and Proposition 3.1, for each \( m \in \mathbb{N} \), an LDP holds for \( \Psi_m(Y^m_n) = S_n^m \ast \cdots \ast S_n^{m,m} \) with the rate function
\[
J_m(x) := I_m(\Psi_m^{-1}(\{x\})) := \inf \{ I_m(u) : u \in \mathcal{H}^m \text{ and } \Psi_m(u) = x \}
\]
where \( I_m \) is as given in \( 3.2 \). Note that for each \( m \) and \( u \in \mathcal{H}^m \)
\[
I_m(u) = \int_0^1 \Lambda^*(c_{\sigma_{m,u}}(t)) \, dt,
\]
where \( \sigma_{m,u} : [0, 1] \rightarrow G \) is the particular horizontal path introduced in Notation 3.17.

Note also that \( \Psi_m(u) = \sigma_{m,u}(1) \). Thus we may write
\[
J_m(x) = \inf \left\{ \int_0^1 \Lambda^*(c_{\sigma_{m,u}}(t)) \, dt : u \in \mathcal{H}^m \text{ and } \sigma_{m,u}(1) = x \right\}.
\]

For \( x \in G \), let
\[
\Sigma_x := \{ \sigma : \sigma \text{ horizontal with } \sigma(0) = e \text{ and } \sigma(1) = x \}
\]
and
\[
\Sigma_x^m := \{ \sigma_{m,u} : u \in \mathcal{H}^m \text{ with } \sigma_{m,u}(1) = x \}.
\]

With this notation
\[
J_m(x) = \inf \left\{ \int_0^1 \Lambda^*(c_\sigma(t)) \, dt : \sigma \in \Sigma_x^m \right\}
\]
and
\[
J(x) = \inf \left\{ \int_0^1 \Lambda^*(c_\sigma(t)) \, dt : \sigma \in \Sigma_x \right\}.
\]

Since we know that \( \Psi_m(Y^m_n) \) is an exponentially good approximation to \( D_1^n S_n \), by Theorem 2.7 we also know that a weak LDP holds for \( D_1^n S_n \) with the rate function
\[
\sup \liminf_{m \to \infty} \inf_{x' \in B(x, \varepsilon)} J_m(x').
\]
Thus, we wish to show that this expression is exactly $J(x)$. Moreover, noting that $J$ is a good rate function, if we further show that for every closed set $F$

$$\inf_{x \in F} J(x) \leq \limsup_{m \to \infty} \inf_{x \in F} J_m(x),$$

then the full LDP holds for $D_n S_n$ with the rate function $J$. But we get the latter estimate essentially for free, since for all $m$ and $x \in G$, $J(x) \leq J_m(x)$ as $\Sigma_x \supset \Sigma_x^m$ for all $m$.

So now note that, since $\varepsilon < \varepsilon'$ implies

$$\inf_{x' \in B(x,\varepsilon)} J_m(x') \geq \inf_{x' \in B(x,\varepsilon')} J_m(x'),$$

we have

$$\sup_{\varepsilon > 0} \liminf_{m \to \infty} \inf_{x' \in B(x,\varepsilon)} J_m(x') = \lim_{\varepsilon \downarrow 0} \liminf_{m \to \infty} \inf_{x' \in B(x,\varepsilon)} J_m(x').$$

Since $J_m(x') \geq J(x')$ for all $m$ and $x'$,

$$\lim_{\varepsilon \downarrow 0} \liminf_{m \to \infty} \inf_{x' \in B(x,\varepsilon)} J_m(x') \geq \lim_{\varepsilon \downarrow 0} \liminf_{m \to \infty} \inf_{x' \in B(x,\varepsilon)} J(x') = \lim_{\varepsilon \downarrow 0} \inf_{x' \in B(x,\varepsilon)} J(x') = J(x).$$

Thus, if $J(x) = \infty$ then we are done.

So assume that $J(x) < \infty$. Then, given any $\delta > 0$, there exists $\gamma \in \Sigma_\varepsilon$ such that

$$\int_0^1 \Lambda^*(c_\gamma(t)) \, dt < J(x) + \delta/2.$$

The proof will conclude by showing that we can approximate the left hand side arbitrarily well with curves from $\Sigma_\varepsilon^m$, when $x'$ is close to $x$. Indeed, below we prove Lemma 3.20 and Proposition 3.22 and together these imply that for all $\varepsilon > 0$ and $\ell \in \mathbb{N}$, there exist $m \geq \ell$, $x' \in B(x, (C+1)\varepsilon)$ (where $C < \infty$ is the constant in Lemma 2.20, and $\gamma_m \in \Sigma_\varepsilon^m$) such that

$$\left| \int_0^1 \Lambda^*(c_\gamma(t)) \, dt - \int_0^1 \Lambda^*(c_{\gamma_m}(t)) \, dt \right| < \varepsilon.$$

We then get

$$\lim_{\varepsilon \downarrow 0} \liminf_{\ell \to \infty} \inf_{m \geq \ell} \inf_{x' \in B(x,\varepsilon)} J_m(x') \leq J(x)$$

which completes the proof. □

**Lemma 3.20.** Suppose $c_\gamma \in L^1([0, 1], \mathcal{H})$ and $\Lambda^*(c_\gamma) \in L^1([0, 1], [0, \infty))$. Then given $\varepsilon > 0$ there exists a horizontal Lipschitz path $\sigma : [0, 1] \to G$ so that $\rho_{c_\gamma}(\gamma(1), \sigma(1)) < C\varepsilon$, where $C < \infty$ is a constant depending on $\gamma$ as in Lemma 2.20 and

$$\int_0^1 \Lambda^*(c_\gamma(t)) \, dt - \int_0^1 \Lambda^*(c_{\sigma}(t)) \, dt < \varepsilon.$$
Proof. Since \( c_\gamma \in L^1([0, 1], \mathcal{H}) \) and \( \Lambda^*(c_\gamma) \in L^1([0, 1], [0, \infty)) \), there exists \( \delta > 0 \) such that if \( E \subset [0, 1] \) is a measurable set with Lebesgue measure \( |E| < \delta \), then

\[
\int_0^1 |c_\gamma(t)|_{\mathcal{H}} 1_E(t) \, dt < \varepsilon
\]

and

\[
\int_0^1 \Lambda^*(c_\gamma(t)) 1_E(t) \, dt < \varepsilon.
\]

Also, \( c_\gamma \in L^1([0, 1], \mathcal{H}) \) implies that there exists \( R < \infty \) such that \( |\{t \in [0, 1] : \|c_\gamma(t)\|_{\mathcal{H}} > R\}| < \delta \) by Markov’s inequality.

Let \( \sigma := \gamma_R \) be the continuous path such that \( \gamma_R(0) = e \) and

\[
c_{\gamma_R}(t) = c_\gamma(t) 1_{\{|c_\gamma(t)|_{\mathcal{H}} < R\}}.
\]

Then \( \sigma = \gamma_R \) is a horizontal path and

\[
\int_0^1 |c_\gamma(t) - c_{\gamma_R}(t)|_{\mathcal{H}} \, dt = \int_0^1 |c_\gamma(t)|_{\mathcal{H}} 1_{\{|c_\gamma(t)|_{\mathcal{H}} > R\}} \, dt < \varepsilon.
\]

Thus by Grönwall’s Lemma \( 2.26 \), \( \rho(\gamma(1), \gamma_R(1)) < C\varepsilon \). We also have that

\[
\int_0^1 \Lambda^*(c_\gamma(t)) \, dt - \int_0^1 \Lambda^*(c_{\gamma_R}(t)) \, dt = \int_0^1 (\Lambda^*(c_\gamma(t)) - \Lambda^*(0)) 1_{\{|c_\gamma(t)|_{\mathcal{H}} > R\}} \, dt
\]

\[
\leq \int_0^1 2\Lambda^*(c_\gamma(t)) 1_{\{|c_\gamma(t)|_{\mathcal{H}} > R\}} \, dt < \varepsilon
\]

for sufficiently large \( R \), since \( \Lambda^* \) is a convex non-negative function and thus for \( R \) sufficiently large, \( |v| \geq R \) implies that \( \Lambda^*(v) \geq \Lambda^*(u) \) for any \( |u| \leq |v| \). \( \square \)

**Lemma 3.21.** Suppose \( a : [0, T] \to \mathbb{R}^N \) is a bounded measurable function and \( \{\pi_m\} \) is a sequence of partitions \( \pi_m = \{0 = \pi^0_m \leq \pi^1_m \leq \cdots \leq \pi^m_m = T\} \) such that mesh \( \pi_m \to 0 \). Given any \( \varepsilon > 0 \), there exists \( m \) sufficiently large, a simple function \( a_m : [0, T] \to \mathbb{R}^N \) defined on \( \pi_m \), and a measurable subset \( E_m \subset [0, T] \) such that \( |E_m^c| < \varepsilon \), where \( |\cdot| \) denotes the Lebesgue measure, and \( |a_m - a| < \varepsilon \) on \( E_m \).

**Proof.** Without loss of generality we may assume \( T = 1 \) and \( |a| \leq 1 \). It suffices to consider the case where \( N = 1 \) and \( a \geq 0 \). To deal with \( a \) taking values in \( \mathbb{R} \), we would simply construct the same approximations to the positive and negative parts of \( a \), and similarly for \( N > 1 \) we would consider \( a \) componentwise.

Fix \( k \in \mathbb{N} \) sufficiently large that \( \frac{1}{2^k} < \varepsilon \). Let \( T_k^j := a^{-1}\left(\left(\frac{j}{2^k}, \frac{j+1}{2^k}\right]\right) \) for \( j = 0, \ldots, 2^k - 1 \).

Let \( U_m^\ell := [\pi^\ell_m, \pi^{\ell+1}_m] \) for \( \ell = 0, \ldots, m - 1 \). Since the \( T_k^j \)'s are measurable, we may find sufficiently large \( m = m(k) \) so that for \( j = 0, 1, \ldots, 2^k - 1 \) there exist disjoint subsets \( I^k_j \subset \{1, \ldots, m\} \) which partition \( \{1, \ldots, m\} \) such that, for each \( j \),

\[
\Delta^j_k := T_k^j \Delta \left( \bigcup_{\ell \in I^k_j} U_m^\ell \right)
\]
has Lebesgue measure as small as one wants, for example, $|\Delta^j_k| < \frac{1}{2^k}$. These can be chosen so that $\ell \in I^j_k$ implies that $U^\ell_m \cap T^j_k \neq \emptyset$. Taking $\Delta_k := \cup_{j=0}^{2^k-1} \Delta^j_k$, we have that $|\Delta_k| < 2^k \cdot \frac{1}{2^k} = \frac{1}{2^k}$. Take $E_m := (\Delta_k)^c$.

For each $\ell = 0, \ldots, m - 1$, we have $\ell \in I^j_k$ for some $j$ and we can fix some $t^\ast_{\ell} \in U^\ell_m \cap T^j_k$, and define

$$a_m(t) := \sum_{\ell=0}^{m-1} a(t^\ast_{\ell}) 1_{U^\ell_m}(t).$$

Recalling that $|a(t) - \frac{j}{2^k}| < \frac{1}{2^k}$ for all $t \in T^j_k$ including $t^\ast_{\ell}$, we have that, for $t \in T^j_k \cap U^\ell_m$,

$$|a_m(t) - a(t)| \leq |a_m(t) - \frac{j}{2^k}| + \left| \frac{j}{2^k} - a(t) \right| < \left| a(t^\ast_{\ell}) - \frac{j}{2^k} \right| + \frac{1}{2^k} < 2 \cdot \frac{1}{2^k}.$$

Thus

$$|a_m - a| < 2\varepsilon \quad \text{on} \quad E_m = \bigcup_{j=0}^{2^k-1} \left( T^j_k \cap \left( \cup_{\ell \in I^j_k} U^\ell_m \right) \right).$$

**Proposition 3.22.** Suppose $\gamma$ is a horizontal path such that $\gamma$ is Lipschitz, and the Legendre transform satisfies $\Lambda^\ast(c_\gamma) \in L^1([0,1],[0,\infty))$. Given $\varepsilon > 0$, there exist $m \in \mathbb{N}$, $x' \in B(x,\varepsilon)$, and $\gamma_m \in \Sigma^m_x$, such that

$$\left| \int_0^1 \Lambda^\ast(c_\gamma(t)) \, dt - \int_0^1 \Lambda^\ast(c_{\gamma_m}(t)) \, dt \right| < \varepsilon.$$

**Proof.** Since $\gamma$ is Lipschitz, there exists $R < \infty$ such that $|c_\gamma(t)|_H \leq R$ for a.e. $t$. Since $\Lambda^\ast$ is convex on $\mathcal{H}$, $\Lambda^\ast$ is Lipschitz continuous on compact subsets. Fix $K \subset \mathcal{H}$ to be the closed ball of radius $2R$ centered at 0, and let $C' = C'_{\Lambda^\ast}(R)$ denote the Lipschitz coefficient of $\Lambda^\ast$ on $K$.

By Lemma 3.21 we may define a sequence of simple functions $\varphi_m$ on the partitions $\pi_m = \{0 < \frac{1}{m} < \cdots < \frac{m-1}{m} < 1\}$ such that $\varphi_m \rightarrow c_\gamma$ in $L^1([0,1],[0,1])$. We may choose $m$ sufficiently large that $|c_\gamma - \varphi_m|_{L^1([0,1],[0,1])} < \delta$, where $\delta < \min\{\varepsilon/C',\varepsilon/C\}$ where $C$ is the constant appearing in the statement of Lemma 2.20. Define $\gamma_m : [0,1] \rightarrow G$ to be the horizontal piecewise linear path so that $\gamma_m(0) = e$ and $c_{\gamma_m}(t) = \varphi_m(t)$ for all $t \in [0,1]$ where $\varphi_m$ is continuous. Note that $\gamma_m = \sigma_{m,u}$ where $u = (u_1, \ldots, u_m)$ is given by $u_k = \varphi_m(t)$ for $\frac{k-1}{m} < t < \frac{k}{m}$. By Lemma 2.20 $\gamma_m(1) \in B(x,\varepsilon)$, and

$$\left| \int_0^1 \Lambda^\ast(c_\gamma(t)) \, dt - \int_0^1 \Lambda^\ast(c_{\gamma_m}(t)) \, dt \right| \leq \int_0^1 |\Lambda^\ast(c_\gamma(t)) - \Lambda^\ast(c_{\gamma_m}(t))| \, dt \leq C' \int_0^1 |c_\gamma(t) - c_{\gamma_m}(t)|_H \, dt < C' \delta < \varepsilon.$$
Remark 3.23. Thus we see that, for the rate function $J$, the infimum is attained over Lipschitz paths. It is also standard that the horizontal distance is defined taking the infimum over horizontal Lipschitz paths, rather than just over horizontal paths. As in the case of the rate function, these definitions are equivalent which we may see as follows.

As before, we have

$$d_{cc}(x) = \inf \{ \ell(\gamma) : \gamma : [0, 1] \to G \text{ horizontal and } \gamma(0) = e, \gamma(1) = x \},$$

and also take

$$d_L(x) := \inf \{ \ell(\gamma) : \gamma : [0, 1] \to G \text{ horizontal, Lipschitz, and } \gamma(0) = e, \gamma(1) = x \}.$$

Clearly one has $d_L \geq d_{cc}$. Now, any horizontal curve of positive length is an absolutely continuous reparameterization of an arclength parameterized horizontal curve, that is, for any horizontal $\gamma : [0, 1] \to G$ with $\ell(\gamma) > 0$, there exists a (Lipschitz) $\tilde{\gamma} : [0, \ell(\gamma)] \to G$ with $|c_{\tilde{\gamma}}|_{H} = 1$ so that $\gamma = \tilde{\gamma} \circ \varphi$ for $\varphi : [0, 1] \to [0, \ell(\gamma)]$ given by $\varphi(t) = \int_0^t |c_{\gamma}(s)| \, ds$; see for example Lemma 3.71 of [1]. So for any horizontal path $\gamma$ with $\ell(\gamma) < \infty$, we have the Lipschitz horizontal path $\hat{\gamma} : [0, 1] \to G$ given by $\hat{\gamma}(t) := \tilde{\gamma}(\ell(\gamma)t)$ which satisfies $\hat{\gamma}(0) = e, \hat{\gamma}(1) = x$. Finally, the length of an absolutely continuous curve is invariant under an absolutely continuous reparameterization (this is just a change of variables in the integral, or see for example Lemma 3.70 of [1]), and so $\ell(\hat{\gamma}) = \ell(\gamma)$. Thus $d_L \leq d_{cc}$.

3.4. Solving the variational problem for Gaussian random walks. We now consider the implications of Theorem 1.1 when $X_n$ are i.i.d. $\mathcal{N}(0, \text{Id}_H)$ random variables on $H$. In this case, it is straightforward to see $\Lambda(\lambda) = \frac{1}{2} |\lambda|_H^2$, $\Lambda^*(u) = \frac{1}{2} |u|_H^2$, and so Theorem 1.1 gives an LDP for the associated sub-Riemannian random walk with the rate function

$$J_N(x) := \inf \left\{ \frac{1}{2} \int_0^1 |c_{\gamma}(t)|_H^2 \, dt : \gamma \text{ horizontal, } \gamma(0) = e, \gamma(1) = x \right\}$$

$$= \inf \left\{ \frac{1}{2} E(\gamma) : \gamma \text{ horizontal, } \gamma(0) = e, \gamma(1) = x \right\},$$

where $E(\gamma)$ is the so-called energy of the path $\gamma$. Recall that the length of a horizontal path is given by (2.12). Using an argument similar to [1, Lemma 3.64] we have the following lemma.

**Lemma 3.24.** A horizontal curve $\gamma : [0, 1] \to G$ is a minimizer of $\int_0^1 |c_{\gamma}(t)|_H^2 \, dt$ among the set of horizontal curves joining $e$ and $x$ if and only if it is a minimizer of the length functional $\ell(\gamma)$ among the horizontal curves joining $e$ and $x$.

**Proof.** We know that

$$\ell^2(\gamma) = \left( \int_0^1 |c_{\gamma}(t)|_H \, dt \right)^2 \leq \int_0^1 |c_{\gamma}(t)|_H^2 \, dt,$$
where equality holds if and only if $|c_\gamma(t)|_H$ is constant for all $t \in [0, 1]$. The result follows since any horizontal curve is an absolutely continuous reparametrization of an arc length-parameterized horizontal path \cite[Lemma 3.71]{1} and $\ell(\gamma)$ is invariant under absolutely continuous reparameterizations \cite[Lemma 3.70]{1}.

Thus, we can conclude that the minimizer of $J_N(x)$ (which necessarily satisfies $|J_N(x)| < \infty$) is indeed a minimizer of $d_{cc}(x)$. As a nice consequence we obtain Corollary \cite{13} a precise LDP for the normal sampling random walk, in terms of the sub-Riemannian metric.

**Proof of Corollary** \cite{13} This is a direct consequence of Chow–Rashevskii's Theorem and Lemma \cite{24}.

**References**

1. Andrei Agrachev, Davide Barilari, and Ugo Boscain, *A comprehensive introduction to sub-Riemannian geometry*, Cambridge Studies in Advanced Mathematics, vol. 181, Cambridge University Press, Cambridge, 2020, From the Hamiltonian viewpoint, With an appendix by Igor Zelenko. MR 3971262
2. Andrei Agrachev, Ugo Boscain, Robert Neel, and Luca Rizzi, *Intrinsic random walks in riemannian and sub-riemannian geometry via volume sampling*, ESAIM: COCV 24 (2018), no. 3, 1075–1105.
3. Fanny Augeri, *Nonlinear large deviation bounds with applications to Wigner matrices and sparse Erdős-Rényi graphs*, Ann. Probab. 48 (2020), no. 5, 2404–2448. MR 4152647
4. R. Azencott, *Grandes déviations et applications*, Eighth Saint Flour Probability Summer School—1978 (Saint Flour, 1978), Lecture Notes in Math., vol. 774, Springer, Berlin, 1980, pp. 1–176. MR 590626
5. Paolo Baldi and Lucia Caramellino, *Large and moderate deviations for random walks on nilpotent groups*, J. Theoret. Probab. 12 (1999), no. 3, 779–809. MR 1702883
6. Fabrice Baudoin, Qi Feng, and Maria Gordina, Integration by parts and quasi-invariance for the horizontal wiener measure on foliated compact manifolds, Journal of Functional Analysis 277 (2019), no. 5, 1362 – 1422.
7. Timothée Bénard, Drift of random walks on abelian covers of finite volume homogeneous spaces, Bull. Soc. Math. France 151 (2023), no. 3, 407–434. MR 4715578
8. A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, Stratified Lie groups and potential theory for their sub-Laplacians, Springer Monographs in Mathematics, Springer, Berlin, 2007. MR 2363343
9. Ugo Boscain, Robert Neel, and Luca Rizzi, Intrinsic random walks and sub-Laplacians in sub-Riemannian geometry, Adv. Math. 314 (2017), 124–184. MR 3658714
10. Emmanuel Breuillard, Geometry of locally compact groups of polynomial growth and shape of large balls, Groups Geom. Dyn. 8 (2014), no. 3, 669–732. MR 3267520
11. Sourav Chatterjee and Amir Dembo, Nonlinear large deviations, Adv. Math. 299 (2016), 396–450. MR 3519474
12. Sourav Chatterjee and S. R. S. Varadhan, The large deviation principle for the Erdős-Rényi random graph, European J. Combin. 32 (2011), no. 7, 1000–1017. MR 2825532
13. Lawrence J. Corwin and Frederick P. Greenleaf, *Representations of nilpotent Lie groups and their applications. Part I*, Cambridge Studies in Advanced Mathematics, vol. 18, Cambridge University Press, Cambridge, 1990, Basic theory and examples. MR 1070979 (92b:22007)
14. Amir Dembo and Ofer Zeitouni, *Large deviations techniques and applications*, Stochastic Modelling and Applied Probability, vol. 38, Springer-Verlag, Berlin, 2010, Corrected reprint of the second (1998) edition. MR 2571413
Appendix A. Proof of the upper bound

This section provides the proof of the upper bound (3.4). For fixed $m$, let $\{Y^m_n\}_{n \geq 1}$ be as described in Section 3. The following result arises naturally in the usual proof of the upper bound of Cramér’s Theorem. Indeed, here we follow the proof in [14].
However, the expression we leave for the upper bound is perhaps not completely standard. So for clarity we provide the proof here.

**Proposition A.1.** Fix $m \in \mathbb{N}$. For any closed $F \subset \mathcal{H}^m$ we have that

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} (Y^m_n \in F) \leq - \inf_{x \in F} I_m(x)$$

where

$$I_m(x) = \sup_{\lambda \in \mathcal{H}^m} \left\{ \langle \lambda, x \rangle_{\mathcal{H}^m} - \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ \exp \left( n \langle \lambda, Y^m_n \rangle_{\mathcal{H}^m} \right) \right] \right\}.$$

**Proof.** We use the argument for proving the upper bound in the classical Cramér’s theorem, see for example [14, p. 37]. First we assume that $F$ is compact, and note that it suffices to prove that, for any $\delta > 0$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} (Y^m_n \in F) \leq \delta - \inf_{x \in F} I^\delta_m(x)$$

where $I^\delta_m(x) := \min \left\{ \frac{1}{\delta}, I_m(x) - \delta \right\}$.

and

$$I_m(x) = \sup_{\lambda \in \mathcal{H}^m} \left\{ \langle \lambda, x \rangle_{\mathcal{H}^m} - \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ \exp \left( n \langle \lambda, Y^m_n \rangle_{\mathcal{H}^m} \right) \right] \right\}.$$

So fix $\delta > 0$ and let $\tilde{\Lambda}(\lambda) := \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left[ \exp \left( n \langle \lambda, Y^m_n \rangle_{\mathcal{H}^m} \right) \right]$. For every $q \in F$, let $\lambda_q \in \mathcal{H}^m$ be such that

(A.1) \[ \langle \lambda_q, q \rangle_{\mathcal{H}^m} - \tilde{\Lambda}(\lambda_q) \geq I^\delta_m(q), \]

and choose $r_q > 0$ such that $r_q |\lambda_q|_{\mathcal{H}^m} \leq \delta$. Denote by $B_q$ the ball centered at $q$ of radius $r_q$. By Markov’s inequality

$$\mathbb{E} \left[ \exp \left( n \langle \lambda_q, Y^m_n \rangle_{\mathcal{H}^m} \right) \right] \geq \mathbb{E} \left[ \exp \left( n \langle \lambda_q, Y^m_n \rangle_{\mathcal{H}^m} \right) 1_{Y^m_n \in B_q} \right]$$

$$\geq \exp \left[ n \inf_{x \in B_q} \left\{ \langle \lambda_q, x \rangle_{\mathcal{H}^m} \right\} \right] \mathbb{P} (Y^m_n \in B_q).$$

Thus we have

$$\mathbb{P} (Y^m_n \in B_q) \leq \mathbb{E} \left[ \exp \left( n \langle \lambda_q, Y^m_n \rangle_{\mathcal{H}^m} - n \inf_{x \in B_q} \left\{ \langle \lambda_q, x \rangle_{\mathcal{H}^m} \right\} \right) \right].$$

Moreover,

$$- \inf_{x \in B_q} \left\{ \langle \lambda_q, x \rangle_{\mathcal{H}^m} \right\} \leq r_q |\lambda_q|_{\mathcal{H}^m} - \langle \lambda_q, q \rangle_{\mathcal{H}^m} \leq \delta - \langle \lambda_q, q \rangle_{\mathcal{H}^m}.$$

Thus, for any $q \in F$,

$$\frac{1}{n} \log \mathbb{P} (Y^m_n \in B_q) \leq - \inf_{x \in B_q} \left\{ \langle \lambda_q, x \rangle_{\mathcal{H}^m} \right\} + \frac{1}{n} \log \mathbb{E} \left[ \exp \left( n \langle \lambda_q, Y^m_n \rangle_{\mathcal{H}^m} \right) \right],$$

hence

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P} (Y^m_n \in B_q) \leq \delta - \langle \lambda_q, q \rangle_{\mathcal{H}^m} + \tilde{\Lambda}(\lambda_q).$$
Now by compactness of $F$ we can extract a finite covering $\cup_{j=1}^{\infty} B_{q_j}$, and hence
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n^m \in F) \leq \frac{1}{n} \log N + \delta - \min_{1 \leq j \leq N} \{(\lambda_{q_j}, q_j) \mathcal{H}^m - \tilde{\Lambda}(\lambda_{q_j})\}
\]
By (A.1) we obtain that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n^m \in F) \leq \delta - \inf_{x \in F} I_m^\delta(x).
\]
Let $\delta \to 0$ we then have
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n^m \in F) \leq - \inf_{x \in F} I_m(x).
\]
At last we extend the above upper bound to any closed sets $F \subset \mathcal{H}^m$ by using the fact that $\mathbb{P}(Y_n^m \in \cdot)$ is an exponentially tight family of probability measures and by [14, Lemma 1.2.18]. Let $H_r := [-r, r]^d m$, then $H_r^c = \cup_{j=1}^{\infty} \{x : |x_j| > r\}$, hence
\[
\mathbb{P}(Y_n^m \in H_r^c) \leq \sum_{j=1}^{m} \sum_{i=1}^{d_j} \mathbb{P}(Y_{n,j,i}^m \geq r) + \mathbb{P}(Y_{n,j,i}^m \leq -r),
\]
where $Y_{n,j,i}^m$ represent the coordinates of $Y_{n,j}^m$, that is, $Y_{n,j}^m = (Y_{n,j,i}^m, \ldots, Y_{n,j,d_j}^m) \in \mathcal{H}$. Again by the Markov inequality we know that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_{n,j,i}^m \geq r) \leq - \Lambda^*(r), \quad \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_{n,j,i}^m \leq -r) \leq - \Lambda^*(-r),
\]
where $\Lambda^*(r) = \sup_{\lambda \in \mathbb{R}} \{r \lambda - \limsup_{n \to \infty} \log \mathbb{E}(\exp(n \lambda Y_{n,j,i}^m))\}$, which tends to $\infty$ as $r \to \infty$. Hence we have that
\[
\lim_{r \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n^m \in H_r^c) = -\infty.
\]
Since for any closed $F \subset \mathcal{H}^m$, we have
\[
\mathbb{P}(Y_n^m \in F) \leq \mathbb{P}(Y_n^m \in F \cap H_r) + \mathbb{P}(Y_n^m \in H_r^c),
\]
and hence
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n^m \in F) \leq \lim_{r \to \infty} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n^m \in F \cap H_r)
\]
\[
\quad \leq - \lim_{r \to \infty} \inf_{x \in F \cap H_r} I_m(x) = - \inf_{x \in F} I_m(x).
\]
\[\square\]

**APPENDIX B. RANDOM SYMPLECTIC FORM**

Suppose $\omega$ is a symplectic form on $\mathbb{R}^N$ and define
\[\omega_{ij} := \omega(v^i, v^j)\]
for any $v^i, v^j \in \mathbb{R}^N$. By antisymmetry of the symplectic form $\omega$ we see that for $k \leq \ell \leq n$ and $v = \{v^i\}_{i=1}^{n}$, $v^i \in \mathbb{R}^N$ we have
\[ W_{k,\ell}(v) := \omega \left( \sum_{i=1}^{k} v^i, \sum_{j=1}^{\ell} v^j \right) = \sum_{i=1}^{k} \sum_{j=i+1}^{\ell} \omega_{ij}. \]

Suppose now that \( v^1 = (x^1, y^1), \ldots, v^n = (x^n, y^n) \) is a collection of random variables in \( \mathbb{R}^N \times \mathbb{R}^N \), then we can consider the distribution of the random symplectic form \( W_{k,\ell} \). Let \( k \leq \ell \leq n \), and \( u_k^i = (x_k^i, y_k^i), k \in [N], i \in [n] \), then

\[
W_{k,\ell} = \sum_{i=k+1}^{\ell} \sum_{j=i+1}^{\ell} \omega_{ij} = X^T A Y, k \leq \ell,
\]

where

\[
X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n),
\]

\[
A = \{a_{ij}\}_{i,j=1}^{n} \text{ with } -a_{ji} = a_{ij} = 1 \text{ for } k+1 \leq i < j \leq \ell \text{ and } 0 \text{ otherwise,}
\]

so \( A \) is a skew-symmetric Toeplitz matrix. Observe that the Hilbert-Schmidt norm and operator norms of \( A \) are

\[
\|A\|_{HS} = \left( \sum_{i=k+1}^{\ell} \sum_{j=i+1}^{\ell} |a_{ij}|^2 \right)^{1/2} = \sqrt{(\ell - k + 1) (\ell - k)},
\]

(B.1) \[ \|A\| \leq \ell - k. \]

Before we describe the distribution of \( W_{k,\ell} \), we recall the definition of a sub-Gaussian random variable. Several equivalent characterizations of sub-Gaussian random variables are given in [35, Proposition 2.5.2]. We denote the sub-Gaussian norm (Orlicz norm) by

\[
\|X\|_{\psi_2} := \inf \left\{ s > 0 : \mathbb{E} e^{(X/s)^2} - 1 \leq 1 \right\}.
\]

We say that \( X \) is a (centered) sub-Gaussian random variables, if \( \mathbb{E}X = 0 \), and the moment generating function of \( X \) satisfies

\[
\mathbb{E} \exp (\lambda X) \leq \exp \left( C \lambda^2 \|X\|^2_{\psi_2} \right)
\]

for any \( \lambda \in \mathbb{R} \) and some absolute constant \( C > 0 \). If \( X \) and \( Y \) are i.i.d. sub-Gaussian random variables, then by [35, Lemma 2.7.7] \( XY \) is sub-exponential with

\[
\|XY\|_{\psi_1} := \inf \left\{ s > 0 : \mathbb{E} e^{\frac{|XY|}{s}} \leq 2 \right\} \leq \|X\|_{\psi_2} \|Y\|_{\psi_2}.
\]

This means that by [35, Proposition 2.7.1] there is a constant \( K > 0 \) such that

\[
\mathbb{E} e^{\lambda |XY|} \leq \exp (K \lambda)
\]

for any \( 0 \leq \lambda \leq 1/K \).

**Proposition B.1** (Properties of random symplectic forms). Let \((x_n, y_n)_{n=1}^{\infty}\) be a sequence of \( \mathbb{R}^2 \)-valued i.i.d. random variables with mean zero and variance \((1, 1)\).

1. Suppose \( X = (x_1, \ldots, x_n) \) and \( Y = (y_1, \ldots, y_n) \) are i.i.d. standard normal random variables. Then the random variable \( W_{k,\ell} \) is distributed as a linear combination
\(\chi\)-square distributed variables with coefficients depending on the eigenvalues of the matrix \(W\). In addition, there are universal constants \(c > 0\) and \(C > 0\) such that
\[
E e^{\lambda W_{k,\ell}} \leq e^{C\lambda^2(l-k+1)(l-k)}
\]
for all \(\lambda\) satisfying \(|\lambda| \leq c\).

(2) Suppose \(X = (x_1, \ldots, x_n)\) and \(Y = (y_1, \ldots, y_n)\) are i.i.d. sub-Gaussian random variables. Then there is a universal constant \(D > 0\) such that
\[
E e^{\lambda W_{k,\ell}} \leq e^{CD^2||X||^4_{\psi_2} \lambda^2(l-k+1)(l-k)}
\]
for all \(|\lambda| \leq \frac{c}{D||X||^2_{\psi_2}}\).

Here \(C\) and \(c\) are the universal constants in (1).

Proof. First observe that there is an orthogonal matrix \(R\) such that \(R^T AR\) is block diagonal with blocks being
\[
\begin{pmatrix}
0 & \theta \\
-\theta & 0
\end{pmatrix}.
\]
Note that \(\pm i\theta\) are eigenvalues of \(A\), and they can be found as roots of the characteristic polynomial. We denote by \(B\) the matrix
\[
B = \{b_{ij}\}_{i,j=1}^N, -b_{ji} = b_{ij} = 1 \text{ for } 1 \leq i < j \leq N \text{ and } 0 \text{ otherwise},
\]
whose characteristic polynomial is
\[
p_{2N}(\lambda) = \sum_{l=0}^{N} \binom{2N}{2l} \lambda^{2l},
\]
\[
p_{2N+1}(\lambda) = -\sum_{l=0}^{N} \binom{2N+1}{2l+1} \lambda^{2l+1},
\]
since \(p_{N+1}(\lambda) = -(\lambda - 1)p_N(\lambda) + (\lambda + 1)^N\).

Now we assume that \(X = (x_1, \ldots, x_n)\) and \(Y = (y_1, \ldots, y_n)\) are i.i.d. standard normal random variables. Recall that the original matrix \(A\) has a copy of a matrix \(B\) as a block on the diagonal with all other entries being 0. Note that the distribution of random vectors \(X\) and \(Y\) is invariant under orthogonal transformations, so we only need to determinate the distribution \(X^T AY\) when \(A\) is block diagonal. Moreover, it is enough to look at one block
\[
(x_1, x_2) \begin{pmatrix}
0 & \theta \\
-\theta & 0
\end{pmatrix} (y_1, y_2)^T = \theta (x_1y_2 - x_2y_1),
\]
where \(x_1, x_2, y_1, y_2\) are i.i.d. standard normal random variables.

First consider \(U\) and \(V\) which are i.i.d. standard normal random variables. Then \(U - V\) and \(U + V\) are two independent variables distributed as \(\mathcal{N}(0, 2)\), and
\[
UV = \frac{(U + V)^2}{4} - \frac{(U - V)^2}{4}
\]
thus $UV$ is distributed as the difference of two $\chi$-square distributed variables. This has a variance-gamma distribution which can be shown by using the moment generating functions. Thus the non-zero components of $X^TAY$ are linear combinations of two $\chi$-square distributed variables.

To prove the estimate we use [35, Lemma 6.2.2] giving an estimate of the moment generating function of Gaussian chaos. Namely, if $X$ and $Y$ are independent standard Gaussian vectors, $A$ is a matrix, then

$$E \exp (\lambda X^TAY) \leq \exp \left( C\lambda^2 \|A\|^2_{HS} \right)$$

for all $\lambda$ satisfying $|\lambda| \leq \frac{c}{\|A\|}$. Here the constant $C$ is a universal constant for a product of standard Gaussian variables and $c$ is a universal constant for sub-exponential variables. Thus by (B.1)

$$E \exp (\lambda W_{k,\ell}) \leq \exp \left( C\lambda^2 (\ell - k + 1) (\ell - k) \right)$$

for all $\lambda$ satisfying $|\lambda| \leq c$. Note that we can get a similar result from the previous argument as $\chi$-square distributed random variables are sub-exponential and the constants in [35, Lemma 6.2.2] use singular numbers of the matrix $A$ which can be estimated by using the Hilbert-Schmidt norm.

To prove the second part of the statement we will use [35, Lemma 6.2.3] which allows a comparison of the moment generating functions of sub-Gaussian and Gaussian random variables. Suppose $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ are i.i.d. sub-Gaussian random variables, $A$ is a matrix, and $U$ and $V$ are two independent standard Gaussian vectors. Denote $K := \|X\|_{\psi^2} = \|Y\|_{\psi^2}$, then by [35, Lemma 6.2.3] there is a universal constant $D > 0$ such that

$$E \exp (\lambda X^TAY) \leq \exp \left( DK^2 \lambda U^TAV \right)$$

for any $\lambda \in \mathbb{R}$. Then we can use the first part applied to $U^TAV$ and $\tilde{\lambda} = D\|X\|_{\psi^2}^2 \lambda$ to see that

$$E \exp (\lambda X^TAY) \leq \exp \left( D\|X\|_{\psi^2}^2 \lambda U^TAV \right)$$

$$\leq \exp \left( C\lambda^2 (\ell - k + 1) (\ell - k) \right) = \exp \left( CD^2\|X\|_{\psi^2}^4 \lambda^2 (\ell - k + 1) (\ell - k) \right)$$

for all $|\tilde{\lambda}| \leq c$, that is, $|\lambda| \leq c/D\|X\|_{\psi^2}^2$. \qed