The fully non-linear post-Friedmann frame-dragging vector potential:
Magnitude and time evolution from N-body simulations

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Abstract
Newtonian simulations are routinely used to examine the matter dynamics on non-linear scales. However, even on these scales, Newtonian gravity is not a complete description of gravitational effects. A post-Friedmann approach shows that the leading order correction to Newtonian theory is the existence of a vector potential in the metric. This vector potential can be calculated from N-body simulations, requiring a method for extracting the velocity field. Here, we present the full details of our calculation of the post-Friedmann vector potential, using the Delauney Tesselation Field Estimator (DTFE) code. We include a detailed examination of the robustness of our numerical result, including the effects of box size and mass resolution on the extracted fields. We present the power spectrum of the vector potential and find that the power spectrum of the vector potential is $\sim 10^5$ times smaller than the power spectrum of the fully non-linear scalar gravitational potential at redshift zero. Comparing our numerical results to perturbative estimates, we find that the fully non-linear result can be more than an order of magnitude larger than the perturbative estimate on small scales. We extend the analysis of the vector potential to multiple redshifts, showing that this ratio persists over a range of scales and redshifts. We also comment on the implications of our results for the validity and interpretation of Newtonian simulations.
I. INTRODUCTION

On the largest scales in cosmology, theoretical calculations can be carried out using standard cosmological perturbation theory. These calculations fully encompass General Relativity (GR) but are limited to scales where the perturbations, in particular the density perturbation, are small. On smaller scales, where the focus is on non-linear structure formation, Newtonian N-body simulations are used. These simulations do not require that the density contrast is small, but they suffer from the limitations of being Newtonian rather than GR simulations. There is an entire field in cosmology dedicated to developing, running and analysing these Newtonian N-body simulations. There has been sporadic interest in understanding the use of Newtonian theory in cosmology [1–11], as well as examining the relativistic interpretation of the simulations [12–15]. These studies have predominantly focussed on whether the dynamics of density contrast and scalar potential accurately match those of GR.

In this paper, we are mostly interested in another important limitation of Newtonian simulations: Even if the matter dynamics are being computed correctly, there are cosmological quantities of interest on non-linear scales that have no counterpart in Newtonian theory. Examples of these quantities include the difference between the two scalar potentials, gravitational waves and the vector potential in the metric, all of which must exist on non-linear scales in a GR universe.

These extra quantities would naively be expected to be small if the Newtonian simulations are a good approximation to a GR universe. However, explicitly calculating these quantities has several advantages: To start with, it would be good to have a quantitative check of whether these quantities are small, and indeed how small they are. In particular, as we enter the era of precision cosmology, we need to check that these quantities will not affect the observables at the percent-level. Furthermore, checking that these quantities are negligible provides a quantitative check on the Newtonian approximation in a ΛCDM cosmology.

We will be working with the post-Friedmann formalism [8, 16]. This generalises to cosmology the weak-field (post-Minkowski) approximation, with a post-Newtonian style expansion [17–19] in inverse powers of the speed of light $c$ of the perturbative quantities. These expansions need to be performed differently in cosmology to in the Solar System due to the different situations and aims in the two cases. For example, the time-time and space-space
components of the metric need to be treated at the same order in cosmology in order for
the resulting equations to be a consistent solution of the Einstein equations.

The post-Friedmann formalism, when linearised, correctly reproduces conventional linear
perturbation theory and can thus describe structure formation on the largest scales. More
importantly, the leading order equations in the \(1/c\) expansion can be examined and are
expected to yield the non-linear Newtonian equations. Note that in this “Newtonian” regime,
the density contrast has not been assumed to be small. The equations in this regime will be
shown in section II essentially comprising the Newtonian equations, as expected, plus an
additional equation. This additional equation shows how the vector potential in the metric,
the lowest order beyond-Newtonian quantity, is generated by the matter dynamics. This
vector potential is the beyond-Newtonian quantity that we will examine in this paper, it is
the cosmological manifestation of the ubiquitous relativistic effect of frame dragging. This
effect has been measured in the Solar System by Gravity Probe B \[20\].

In this paper, we present a calculation of this vector potential based on extracting the density
and velocity fields from N-body simulations. We expand on the results of \[14\], which was the
first calculation of an intrinsically relativistic quantity on fully non-linear scales from large
scale cosmological matter fields, rather than from individual astrophysical occurrences. The
main focus in this paper is to present the method used to extract this vector potential from
N-body simulations. In particular, we examine the robustness of the numerical extraction
of the vector potential and present the tests we carried out to examine the numerical effects
of simulation parameters on the extraction, which were not presented in \[14\].

The main physical results of this paper are figures 24 and 29, showing the power spectrum of
the vector potential at redshift zero and its evolution with time respectively. Additionally,
we have presented the ratio of the vector potential power spectrum to that of the scalar
potential in figures 26 and 30. The results on the magnitude and evolution of the power
spectrum of the vector potential in this paper were used in \[21\] to examine the possible
weak-lensing consequences of the vector potential.

This paper is laid out as follows. In section II we present the pertinent details of the
post-Friedmann formalism and show the equation governing the vector potential. We will
also present our definitions and notation regarding vector power spectra. In section III, we
explain how the relevant fields were extracted from N-body simulations and examine the
robustness of this extraction. In section IV, we show the power spectrum of the vector
gravitational potential and its time evolution, as well as comparing it to the closest analytical results in the literature. We conclude in section V.

II. POST-FRIEDMANN FORMALISM

The post-Friedmann approach is developed in [8, 16], see there for the full details. This approach considers a dust (pressure-less matter) cosmology with a cosmological constant. The perturbed FLRW metric, in Poisson gauge, is expanded up to order $c^{-5}$, keeping the $g_{00}$ and $g_{ij}$ scalar potentials at the same order:

\begin{align*}
g_{00} &= -\left[1 - \frac{2U_N}{c^2} + \frac{1}{c^4} \left(2U_N^2 - 4U_P\right)\right] \\
g_{0i} &= -a\frac{B_N^i}{c^3} - a\frac{B_P^i}{c^5} \\
g_{ij} &= a^2 \left[\left(1 + \frac{2V_N}{c^2} + \frac{1}{c^4} \left(2V_N^2 + 4V_P\right)\right) \delta_{ij} + \frac{h_{ij}}{c^4}\right]
\end{align*}

(2.1)

The $g_{00}$ and $g_{ij}$ scalar potentials have been split into the Newtonian ($U_N, V_N$) and post-Friedmann ($U_P, V_P$) components. Similarly, the vector potential has been split up into $B_N^i$ and $B_P^i$. Since this metric is in the Poisson gauge, the three-vectors $B_N^i$ and $B_P^i$ are divergenceless, $B_{ni}^N = 0$ and $B_{ni}^P = 0$. In addition, $h_{ij}$ is transverse and tracefree, $h_i^i = h_{ij}^j = 0$. Note that at this order, $h_{ij}$ is not dynamical, so it does not represent gravitational waves. From a post-Friedmann viewpoint, there are two different levels of perturbations in the theory, corresponding to terms of order $c^{-2}$ and $c^{-3}$, or of order $c^{-4}$ and $c^{-5}$ respectively. Defining “resummed” variables, such as $\Phi = 2U_N + c^{-2} \left(2U_N^2 - 4U_P\right)$, then calculating the Einstein equations and linearising them, reproduces linear GR perturbation theory in Poisson gauge. Thus, this approach is capable of describing structure formation on the largest scales.

For smaller scales, in a dust cosmology, we are interested in the weak field, slow motion, sub-horizon, quasi-static and negligible pressure regime. This is simply derived by retaining only the leading order terms in the $c^{-1}$ expansion and upon doing so we recover Newtonian cosmology, albeit with a couple of subtleties. The first is that the space-time metric is a well-defined approximate solution of the Einstein equations. The second is that we have an additional equation, which is a constraint equation for the vector gravitational potential $B_N^i$. The full system of equations obtained from the Einstein and hydrodynamic equations...
As expected, we have the Newtonian continuity and Euler equations from the hydrodynamic equations as well as Poissons equation from the Einstein equations. Note that the time derivative here is the convective derivative, $\frac{dA}{dt} = \partial A/\partial t + v^i A_i/A$, for any quantity $A$. The Einstein equations yield two additional equations: The first is an equation forcing the scalar potentials $V_N$ and $U_N$ to be equal, consistent with there being only one scalar potential in Newtonian theory. Note that some approaches consider the potentials to be a priori equal at leading order whereas here we assumed the full generality of GR and the equality of the potentials arose naturally on taking the Newtonian regime. The second additional equation relates the leading order vector gravitational potential, $B^N_i$, to the momentum of the matter. Thus, even in the regime where the matter dynamics are correctly described by Newtonian theory, the frame-dragging potential $B^N_i$ should not be set to zero; this would correspond to putting an extra constraint on the Newtonian dynamics. We note that there is a similar equation in several other formalisms in the literature. We can see from equation (2.2e) that the potential $B^N_i$ is sourced by the vector part of the energy current $\rho \vec{v}$. This is made apparent by taking the curl of this equation, which gives

$$\nabla \times \nabla^2 \vec{B}^N = - \left( 16\pi G\rho_b a^2 \right) \nabla \times \left[ (1 + \delta) \vec{v} \right],$$

(2.3)

where the source term on the right hand side splits up into three terms: the vorticity $\nabla \times \vec{v}$ and then two further terms,

$$\nabla \times \left[ (1 + \delta) \vec{v} \right] = \nabla \times \vec{v} + \delta \nabla \times \vec{v} + \nabla \delta \times \vec{v}.$$

(2.4)

This is the equation that will be used in the rest of the paper. Since the matter dynamics are not affected at this order, i.e. they are described by the standard Newtonian equations
The density and velocity fields sourcing the vector potential are Newtonian and can be extracted from N-body simulations. Using the definitions of vector power spectra in appendix A, the power spectrum of the vector potential is given by

\[ P_{\vec{B}N}(k) = \left( \frac{16\pi G \rho_b a^2}{k^2} \right)^2 \frac{1}{k^2} P_{\delta v}(k), \quad (2.5) \]

with

\[ P_{\delta v} = P_{\nabla \times \vec{v}}(k) + P_{\delta \nabla \times \vec{v}}(k) + P_{(\nabla \delta) \times \vec{v}}(k) \]

\[ + P_{(\nabla \delta \times \vec{v})(\nabla \times \vec{v})}(k) + P_{(\delta \nabla \times \vec{v})(\nabla \times \vec{v})}(k) + P_{(\delta \nabla \times \vec{v})(\delta \nabla \times \vec{v})}(k). \quad (2.6) \]

### III. SIMULATIONS

Our simulations have all been run using the publicly available N-body code Gadget2 [22]. Many simulations have been run in order to quantify the effects of box size and mass resolution on the quantities that we are extracting, see table I for a full list of the simulations. All of the simulations were run with dark matter particles only, as the equation for the vector potential is derived for a pressureless matter and cosmological constant cosmology. To allow comparison to previous studies of vorticity [23], the simulations were run with the cosmological parameters \( \Omega_m = 0.27, \Omega_\Lambda = 0.73, \Omega_b = 0.046, h = 0.72, \tau = 0.088, \sigma_8 = 0.9 \) and \( n_s = 1 \). All of the simulations started at redshift 50 and had their initial conditions created using 2LPTic [24]. Our final result for the vector potential is taken from the three 160\( h^{-1}\)Mpc simulations with 1024\(^3\) particles, these will be referred to as the high-resolution (HR) simulations.

#### A. Tessellation

To extract the necessary fields from the simulations, the Delauney Tesselation Field Estimator (DTFE) code was used [25]. Standard methods of extracting fields from N-body simulations, such as Cloud-In-Cells (CIC) [26] work well for the density field, as the particles, by definition, sample the density field well. However, these methods have several shortcomings when applied to the extraction of velocity fields: One is that the field is only sampled where there are particles, so in a low density region the velocity field is artificially set to zero. In addition, the extracted field will be a mass-weighted, rather than volume-weighted...
TABLE I: Parameters for the simulations.

| Box Size ($h^{-1}$Mpc) | Particle number | Mass resolution ($10^8$ M$_\odot$) | # Realisations | Softening length ($h^{-1}$kpc) |
|------------------------|-----------------|------------------------------------|----------------|-------------------------------|
| 80                     | $512^3$         | 3.97                               | 8              | 6.25                          |
| 80                     | $512^3$         | 3.97                               | 1              | 4.0                           |
| 140                    | 768$^3$         | 6.31                               | 8              | 6.25                          |
| 140                    | 560$^3$         | 16.3                               | 8              | 6.25                          |
| 160                    | 1024$^3$        | 3.97                               | 3              | 6.25                          |
| 160                    | 880$^3$         | 6.26                               | 3              | 6.25                          |
| 160                    | 640$^3$         | 16.3                               | 8              | 6.25                          |
| 160                    | 640$^3$         | 16.3                               | 1              | 5.0                           |
| 160                    | 320$^3$         | 130                                | 8              | 15.0                          |
| 200                    | 1024$^3$        | 7.76                               | 2              | 6.25                          |
| 240                    | 960$^3$         | 16.3                               | 3              | 6.25                          |
| 240                    | 480$^3$         | 130                                | 8              | 15.0                          |
| 320                    | 640$^3$         | 130                                | 8              | 15.0                          |

TABLE II: Redshifts used to probe time evolution of quantities.

| Scale Factor | Redshift | Colour on time evolution plots |
|--------------|----------|--------------------------------|
| 0.33         | 2.0      | black                         |
| 0.4          | 1.5      | red                           |
| 0.5          | 1.0      | magenta                       |
| 0.6          | 0.67     | yellow                        |
| 0.7          | 0.43     | green                         |
| 0.8          | 0.25     | cyan                          |
| 0.9          | 0.11     | blue                          |
| 1.0          | 0.0      | brown                         |

field. A consequence of these shortcomings is that, as the grid size is increased, the velocity field will not converge. In fact, it will become zero in an increasing proportion of the grid cells as the grid size increases. Several authors have looked at using the Delauney tesselation [27, 29] for astrophysical applications including the examination of velocity fields. See also
for comparisons of extracting velocity fields with tesselations rather than more standard methods. The DTFE code constructs the Delauney tesselation of the set of particles, consisting of tetrahedra whose nodes are located at the particles positions. The tetrahedra are constructed such that the circumsphere of each tetrahedron does not contain any of the particles except for the particles located at the nodes of the tetrahedron in question. This makes the tesselation unique. The particles’ velocities are then linearly interpolated across each tetrahedron, yielding a value for the smoothed velocity field and its gradients at every point in the simulation volume. A regular $N_{\text{grid}}^3$ grid is laid down and the code samples $N_{\text{samples}}$ points at random in each grid cell and averages the field over these points, giving a value for the smoothed field in each grid cell. Once the fields are obtained on the regular grid, the power spectra are calculated using the standard process of averaging the modulus-squared of the fourier coefficients over a given range of $k$. For the analyses here, we used $N_{\text{grid}}/4$ bins, although varying this value does not affect the results (see appendix B 4).

B. Convergence/Tests

It is important to ensure that our numerical result for the vector potential is robust and independent of the simulation parameters. In this subsection we will present the results of our examination into the effects of different simulation parameters on the extracted vector power spectrum. Since the velocity and density fields both contribute to the source for the vector potential, we will examine the density, vorticity and velocity divergence spectra too: We will examine their behaviour individually, compare them to other studies and methods of extraction and also consider the consistency of the extracted fields through the relations

$$k^2 P_\delta(k) = P_{\nabla \delta}(k)$$

$$k^2 P_\nabla(\vec{v})(k) = P_{\nabla \cdot \vec{v}}(k) + P_{\nabla \times \vec{v}}(k).$$

The two main parameters to examine are whether the box size and mass resolution of the simulation have an effect on the extracted fields. In addition, we have examined the effect of varying the grid size and $N_{\text{samples}}$, which are both internal DTFE parameters. The parameters of the different simulations used are in table 1. We chose the softening lengths of the N-body simulations to be consistent with [23] in order to recreate their study of the
velocity divergence and vorticity, however varying the softening length did not influence the results, see appendix B3.

Although we did run some simulations with a box size below $140h^{-1}\text{Mpc}$, we have not included these in the analysis here as smaller box sizes have systematically less power. See appendix B1 for the results from these simulations and how they compare to the larger box sizes. For further results regarding the effects of a small simulation box on cosmological quantities, see [30–32].

1. A note on error bars

The error bars that we have included on our plots denote the standard error on the mean calculated from the respective realisations, i.e. they are calculated by dividing the standard deviation of the values from the realisations by the square root of the number of realisations. However, we note that the error bars are of limited utility in this work for several reasons. Firstly, for several of our simulation parameters, notably the HR simulations, we only have three realisations to work with, so the error on the error may be large. In addition, when dealing with the vector potential, there is a discrepancy between the two different methods of computing it, as examined in sections III B5 and IV A. On the largest scales, this discrepancy is larger than the error bars for each individual method. We have explicitly examined variation amongst realisations in appendix B2 for several quantities, notably the vorticity and vector potential. In particular, we note there that when considering the vector potential, cosmic variance on the largest scales affects smaller scales, as explained by a perturbative analysis [33, 34]. See B2 for more discussion of this. We also note that the variation of the vorticity amongst realisations seems to be larger than the variation of the density, although there seems to be no discussion of this in the literature. For these reasons, and due to the crowded nature of some of the plots in this paper, we have only included the error bars where we felt they were particularly instructive or important. Appendix B2 can be consulted to examine the variation amongst realisations and thus approximate errors for most of the plots in this paper.
2. Mass Resolution

We have examined the dependence of the density, velocity divergence, vorticity and vector potential on the mass resolution of the simulations. In figures 1 and 2, we show the effect of changing mass resolution on extracting the density and velocity divergence. On smaller scales, there is evidence for a mild dependence on mass resolution for both of these fields. This is likely to be due to the DTFE window function, which cannot be compensated for, rather than a mass-resolution dependence of the field itself. The effect of the mass-resolution dependence is negligible for our HR simulations, as seen later when comparing to alternative methods of calculating the density power spectrum.

The dependence of the vorticity power spectrum with mass resolution is shown in figure 3. The power spectrum shows spurious additional power when the mass resolution is insufficient. However, once the resolution is sufficient, of order $10^9 M_\odot$, there is no evidence for any systematic dependence on mass resolution. This dependence on mass resolution, followed by convergence around $\sim 10^9 M_\odot$, matches previous findings notably [23].

In figure 4, we show the dependence of the vector potential on mass resolution. There is a clear dependence of the vector potential on mass resolution, similar to that seen for the vorticity. However, there are several differences. In particular, the mass-resolution dependence seems to be less important for smaller scales, where there is a greater dependence on box size (see later). In addition, the dependence on mass-resolution is still apparent around $10^9 M_\odot$. However, once there mass resolution has improved to around $6 \times 10^8 M_\odot$, there is no evidence of a mass resolution dependence of the vector potential.

To show this further, figure 5 shows a high resolution sub-sample of the simulations complete with error bars. The y-axis is $k^2 P_B(k)$ in order to show the errors more clearly over the range of scales being considered. The cyan line shows the simulation in the sub-sample with the worst resolution ($16.3 \times 10^8 M_\odot$) and indeed this simulation shows a systematic deviation on the largest scales. The better resolution simulations show better convergence, with the $140h^{-1}$Mpc simulation with $768^3$ particles being consistent with the HR simulations for essentially the entire range under cosnideration. This convergence is examined further in appendix B2.
FIG. 1: The density power spectra extracted from simulations with varying box size and mass resolution. Lines with the same colour share the same mass resolution (in units of $10^8 M_\odot$: 3.97 (red), 6.26 (magenta), 6.31 (yellow), 7.76 (green), 16.3 (cyan), and 130 (blue). The black curve is the linear matter power spectrum for comparison.

FIG. 2: The velocity divergence power spectra extracted from simulations with varying box size and mass resolution. Lines with the same colour share the same mass resolution (in units of $10^8 M_\odot$: 3.97 (red), 6.26 (magenta), 6.31 (yellow), 7.76 (green), 16.3 (cyan), and 130 (blue). The black curve is the linear matter power spectrum for comparison.

3. Box Size

We have also considered the effect of varying the box size on the extracted power spectra. In figures 6, 7 and 8 we show the effect of changing box size on extracting the density,
FIG. 3: The vorticity power spectra extracted from simulations with varying box size and mass resolution. Lines with the same colour share the same mass resolution (in units of $10^8 M_\odot$: 3.97 (red), 6.26 (magenta), 6.31 (yellow), 7.76 (green), 16.3 (cyan), and 130 (blue). The black curve is the linear matter power spectrum for comparison.

Figure 9 shows the box size dependence of the vector potential. As mentioned above, the

FIG. 4: The vector potential power spectra extracted from simulations with varying box size and mass resolution. Lines with the same colour share the same mass resolution (in units of $10^8 M_\odot$: 3.97 (red), 6.26 (magenta), 6.31 (yellow), 7.76 (green), 16.3 (cyan), and 130 (blue).

velocity divergence and vorticity respectively. As expected, there is no evidence for any systematic dependence of these power spectra on the box size of the simulations. However, for sufficiently small boxes, a systematic deviation does arise, see appendix B 1.
FIG. 5: The vector potential power spectra extracted from the highest resolution simulations complete with errors. The simulations shown are $140h^{-1}$ Mpc $768^3$ particles (yellow), $160h^{-1}$ Mpc $640^3$ particles (cyan), $160h^{-1}$ Mpc $880^3$ particles (magenta), $160h^{-1}$ Mpc $1024^3$ particles (blue) and $200h^{-1}$ Mpc $1024^3$ particles (green). The y-axis is now $k^2 P_{\vec{B}}(k)$ in order to better show the errors. Note that, other than the worst resolution (cyan) simulation on largest scales, there is good agreement between the different simulations.

The vector potential does show some dependence on box size. The vector potential shows signs of a dependence on the box size on scales below $1 h^{-1}$Mpc, however this is difficult to entangle from the effects of mass resolution and the window function. For box sizes below $200 h^{-1}$Mpc, there is no systematic dependence of the vector potential power spectrum with box size. In appendix B2, we examine the variation between realisations for the vector potential, and relate it to the behaviour that might be expected from perturbative arguments. In particular, figure 44 shows how the variation between realisations is larger than the effects of box size and mass resolution for simulations with box sizes below $200 h^{-1}$Mpc and mass resolution of at least $6 \times 10^8 M_\odot$. Thus, we expect numerical effects from the simulation parameters to be a sub-dominant source of error as long as the parameters are within this range.

4. DTFE parameters

There are several internal DTFE parameters that are used when computing these fields on a regular grid. We investigate the effects of two of these parameters here, the grid size and
FIG. 6: The density power spectra extracted from simulations with varying box size and mass resolution. Lines with the same colour share the same box size: $140h^{-1}\text{Mpc}$ (red), $160h^{-1}\text{Mpc}$ (magenta), $200h^{-1}\text{Mpc}$ (yellow), $240h^{-1}\text{Mpc}$ (green), $320h^{-1}\text{Mpc}$ (blue). The black curve is the linear matter power spectrum for comparison.

FIG. 7: The velocity divergence power spectra extracted from simulations with varying box size and mass resolution. Lines with the same colour share the same box size: $140h^{-1}\text{Mpc}$ (red), $160h^{-1}\text{Mpc}$ (magenta), $200h^{-1}\text{Mpc}$ (yellow), $240h^{-1}\text{Mpc}$ (green), $320h^{-1}\text{Mpc}$ (blue). The black curve is the linear matter power spectrum for comparison.

the number of samples that are made in each grid cell, $N_{\text{samples}}$.

In figures [10] [11] and [12] we show the effect of varying grid size on the extracted density, velocity divergence and vorticity power spectra. These are the spectra extracted from one
FIG. 8: The vorticity power spectra extracted from simulations with varying box size and mass resolution. Lines with the same colour share the same box size: $140 h^{-1} \text{Mpc}$ (red), $160 h^{-1} \text{Mpc}$ (magenta), $200 h^{-1} \text{Mpc}$ (yellow), $240 h^{-1} \text{Mpc}$ (green), $320 h^{-1} \text{Mpc}$ (blue). The black curve is the linear matter power spectrum for comparison.

FIG. 9: The vector potential power spectra extracted from simulations with varying box size and mass resolution. Lines with the same colour share the same box size: $140 h^{-1} \text{Mpc}$ (red), $160 h^{-1} \text{Mpc}$ (magenta), $200 h^{-1} \text{Mpc}$ (yellow), $240 h^{-1} \text{Mpc}$ (green), $320 h^{-1} \text{Mpc}$ (blue). Of the $160 h^{-1} \text{Mpc}$ simulations with $1024^3$ particles at redshift zero. The different lines show the different grid sizes used: $1024$ (blue), $950$ (cyan), $850$ (green), $750$ (magenta) and $640$ (red). The black line shows the linear density power spectrum for comparison. In all cases the agreement is very good, except on the smallest scales. A discrepancy on this scales is
FIG. 10: The density power spectra extracted from one of the $160 \, h^{-1} \text{Mpc}$ simulations with $1024^3$ particles.

FIG. 11: The velocity divergence power spectra extracted from one of the $160 \, h^{-1} \text{Mpc}$ simulations with $1024^3$ particles.

expected due to the change in the resolution of the grid and the effects of the DTFE window function. However, even on the smallest scales, the discrepancy is small. For our results, we have used the suggested value $N_{\text{grid}}^3 = N_{\text{part}} = 1024$.

Our analysis has all been carried out with $N_{\text{samples}} = 100$ points per grid cell, partly due to computing constraints; increasing the number of samples increases the run time and memory required when analysing a snapshot. However, in figure 13 we show the effect of increasing $N_{\text{samples}}$ to 1000 points per grid cell for one of the $160 \, h^{-1} \text{Mpc}$ simulations with $1024^3$ particles.
FIG. 12: The vorticity power spectra extracted from one of the $160h^{-1}$Mpc simulations with $1024^3$ particles.

FIG. 13: The ratios of the power spectra computed with $N_{\text{samples}} = 100$ and $N_{\text{samples}} = 1000$. The ratios shown are for the density (red), velocity divergence (blue), vorticity (cyan) and vector potential (black). The velocity divergence and vorticity spectra agree very well between the two different numbers of samples. The density power spectrum shows a deviation that increases towards smaller scales, however is within 5% for the range of scales under consideration here. The power spectrum of the vector potential shows more deviation, with decreasing deviation for smaller scales. However, the change in the vector potential is within 10% for every bin after the first and is within 5% for all scales $k > 0.3h^{-1}$Mpc.
5. Consistency Checks

There are a few consistency checks that can be performed on the different fields that we are interested in. The quantities that are used for the vector potential include the density field and its gradients as well as the velocity field and its gradients. There are two relations between these fields and their derivatives,

\[ k^2 P_\delta(k) = P_{\nabla \delta}(k) \]  \hspace{1cm} (3.2)

\[ k^2 P_{\vec{v}}(k) = P_{\nabla \cdot \vec{v}}(k) + P_{\nabla \times \vec{v}}(k). \]  \hspace{1cm} (3.3)

In figures 14 and 15, we show the ratios \( P_{\nabla \delta}(k)/k^2 P_\delta(k) \) and \( k^2 P_{\vec{v}}(k)/(P_{\nabla \cdot \vec{v}}(k) + P_{\nabla \times \vec{v}}(k)) \) respectively, for the average power spectra computed from the HR simulations. In both cases, two curves are plotted, corresponding to two different methods of calculating the ratio. The blue line shows the ratio exactly as suggested above, with the factor of \( k \) in equation 3.2 taken to be the value defining the centre of the bin. For the red curve, the exact \( k \)-value for each mode is used when computing the sum in each bin. For small bins, or fields where the values vary slowly as a function of \( k \), these two should agree and indeed they do for smaller scales where our (logarithmic) bins are smaller. There is a difference between the methods for the largest scales in our simulations, this will be discussed below for each test.

For the density field, the two methods for calculating the ratio do give different answers. However, for both methods, the deviation is within 2% for every bin except the first. Thus, this consistency check for the density field is well satisfied for all scales \( k \geq 0.2h\text{Mpc}^{-1} \).

The consistency check for the velocity field is less well satisfied: there is a sharp divergence in the power spectra on the smallest scales, such that the check is not satisfied within 10% at \( k \approx 8h^{-1}\text{Mpc} \). This shows the effect of the DTFE window function on the extracted fields. We will not consider the extracted vector potential for \( k \) larger than \( k \approx 8h^{-1}\text{Mpc} \) when presenting our results. Furthermore, the two methods show very different behaviour: the method using the average \( k \)-value for each bin causes the consistency test to fail on large scales. However, with the more exact method, the consistency check is very well satisfied on all of the largest scales. This suggests that the dominant contribution to the bins on the largest scales comes from the low-\( k \) end of each bin, hence the overestimation of \( k^2 P_{\vec{v}}(k) \)
FIG. 14: The ratio $P_{\nabla \delta}(k)/k^2 P_\delta(k)$, with $P_{\nabla \delta}(k)/k^2$ calculated using two different methods (the red and blue curves), see text for details. This consistency test is well satisfied, with little difference between the two methods.

when the average $k$-value for each bin is used. The strong effect here is partly caused by the relatively steep slope of the velocity power spectrum. We note that this effect would also come into play when calculating the dimensionless velocity power spectrum for binned data. Nonetheless, the good agreement of the consistency check when using the second method is strong evidence that the derivatives of the velocity field are being calculated correctly.

A further check that we can perform is to extract the complete momentum field, $\vec{p} = (1+\delta) \vec{v}$, and decompose it into its vector and scalar parts directly rather than dealing with derivatives. The power spectrum of the vector potential can then be calculated from the vector part of the momentum field, $\vec{p}^v$, using

$$P_{\vec{B}^v}(k) = \left( \frac{16\pi G \rho_0 a^2}{k^2} \right)^2 P_{\vec{p}^v}(k). \quad (3.4)$$

In figure 16 we show the ratio of the vector power spectrum calculated using the two methods. Figure 16 shows the ratio for the individual realisations. The vector potentials calculated from the two methods are broadly consistent, within 20% for most of the range under consideration, and agreeing to within a factor of 2 for $k \geq 0.2h\text{Mpc}^{-1}$. We are unsure what the causes of the difference between the two methods are. In particular, we checked for whether there is an effect coming from the use of $k$ averaged over the bin, as in the velocity
FIG. 15: The ratio \( k^2 P_{\vec{v}}(k)/(P_{\nabla \cdot \vec{v}}(k) + P_{\nabla \times \vec{v}}(k)) \), with \( k^2 P_{\vec{v}}(k) \) calculated using two different methods (the red and blue curves), see text for details. This consistency test is satisfied except for the smallest scales, however there is a substantial difference between the methods of calculating \( k^2 P_{\vec{v}}(k) \).

field consistency check, however this effect is negligible for the gravitomagnetic potential\(^1\). The difference between the methods is larger than the variation amongst realisations for either method.

We can also extract the momentum field directly using a standard cloud-in-cells (CiC) approach \(^2\), and compare this to the momentum field extracted using the DTFE code. The ratio between these fields is shown in figure 17. There is good agreement between the two methods of computing the momentum power spectrum on larger scales, but with a divergence between the two methods on smaller scales. It is unclear which method would be expected to be more accurate on these smaller scales: the DTFE method suffers from having a window function that cannot be deconvolved, however the CiC method will have cells with a zero momentum field, due to the lack of nearby particles, for a sufficiently large grid. In fact, the CiC method does not converge as the grid size is increased. We used a 512\(^3\) grid for the CiC code, although we checked that changing this to 256 or 512 does

\(^1\) As an aside, we note that we also calculated the momentum field by extracting the velocity field and density field separately at each grid point, before multiplying them together. The power spectrum calculated from this field agrees well with that calculated by extracting the momentum field as a single field. The same agreement is not obtained when extracting the field \( \delta^2 \) and comparing to squaring the density field, when using either the DTFE code or a CiC method.
FIG. 16: The ratio of the vector potential power spectra computed using the vector part of the momentum field and the curl of the momentum field. The blue, magenta and red curves show the ratio for the three realisations of the HR simulations, and the black curve shows the average over these three. There is reasonable agreement between the two power spectra for the smaller scales, however the two methods diverge for the largest scales and there is a difference of a factor of 5 at the largest scales. For most of the range of $k$ under consideration ($k \geq 0.2h\text{Mpc}^{-1}$), the two vector power spectra agree to within a factor of 2.

not significantly affect the results. Unlike the DTFE method, derivatives cannot be directly extracted with the CiC method, so the consistency checks performed earlier for the DTFE method cannot be applied to the CiC method. This also means that the first method of extracting the vector potential, using the curl of the momentum field, cannot be carried out with the CiC method.

We will present the vector power spectrum from both the momentum field and the curl method in the results section. We note that the agreement between the two plots just presented suggests that our vector potential power spectrum is robust and correct to within a factor of 2. It is unclear to us which method should be trusted more; whilst the momentum field method is simpler, the derivative method allows us to examine the different components, notably the vorticity, and check that it behaves as expected. The differences between the two methods do not affect the observability of the vector potential, see [14, 21].
FIG. 17: The ratio of the vector potential power spectra computed using the vector part of the momentum field calculated using the Cloud-in-Cells method and the DTFE method. The blue, magenta and red curves show the ratio for the three realisations of the HR simulations, and the black curve shows the average over these three. The two methods agree very well on larger scales, but diverge for the smallest scales.

6. **Linear evolution**

A further check that can be performed is to examine how the time variation of our extracted density, velocity divergence and vorticity power spectra compares to the respective linear predictions, see figures 18, 19 and 20. In these figures, the power spectrum at each redshift has been divided by the linear growth factor for that redshift raised to the appropriate power. For the density and velocity divergence fields, this is $(D_+(z)/D_+(z = 0))^2$ as per the standard result. For the vorticity we have used $(D_+(z)/D_+(z = 0))^7$, as suggested by [23]. In [23], the authors include an approximate analytic derivation of the time evolution of the vorticity power spectrum, finding it to behave as $f_v^2(z)D_+^6(z)$, where $f_v(z)$ is the fraction of the volume that undergoes orbit crossing. Fitting to their simulations, they found $(D_+(z)/D_+(z = 0))^7 \pm 0.3$ to be the best fit value.

Figures 18 and 19 show the behaviour of the density and velocity divergence is as expected, with the linear theory prediction being best at the largest scales, and becoming increasingly inaccurate for lower redshifts due to the longer time for the non-linear effects to accumulate. The vorticity spectrum shows a similar scaling with the seventh power of the density growth factor to that found in [23]. However, in our simulations this scaling appears to break down
FIG. 18: The non-linear density power spectrum at selected redshifts, $z = 2$ to $z = 0$, each divided by the respective linear theory growth factor squared. See table II for details and an explanation of the colours. As expected, the linear theory prediction works best on the largest scales and is generally worse for smaller scales and later times.

for higher redshift, $z \geq 1$. We see a smaller vorticity spectrum at these times than expected from the $(D_+(z)/D_+(z = 0))^7$ scaling. Figure 20 shows this discrepancy along with the standard error on the mean. These errors do not appear sufficiently large to explain the discrepancy. However, it is worth noting that the variation amongst our realisations (see appendix B2) is large enough to explain the difference in the time evolution of the vorticity between our simulations and the single high resolution simulation in [23]. The variation between realisations was not considered in [23], however it seems likely that the function $f_v(z)$ varies between realisations. The range of the scaling of the vorticity with the linear growth factor has an upper value of 7.3 in [23]. In figure 21 we plot the vorticity power spectrum rescaled by this factor of 7.3, rather than the central value of 7.0 used in figure 20. Using this value reduces, but does not remove the discrepancy.

7. Comparison with the POWMES density power spectrum

The density and density gradient power spectra (the latter divided by $k^2$, see the consistency check above) that we have extracted can be compared to the density field extracted by
FIG. 19: The non-linear velocity divergence power spectrum at selected redshifts, $z = 2$ to $z = 0$, each divided by the respective linear theory density growth factor squared. See table II for details and an explanation of the colours. As expected, the linear theory prediction works well on the largest scales and is generally worse for smaller scales and later times.

FIG. 20: The non-linear vorticity power spectrum at selected redshifts, $z = 2$ to $z = 0$, each divided by the respective linear theory density growth factor to the seventh power, see [23]. See table II for details and an explanation of the colours. As expected, the linear theory prediction works well on the largest scales and is generally worse for smaller scales and later times, however the scaling as the seventh power of the density growth factor seems to break down at earlier times. The error bars on this plot show the standard error on the mean for each set of realisations.
FIG. 21: As for figure 20 except that the curves are rescaled by the growth factor to the power of 7.3, rather than 7.0. The error bars on this plot show the standard error on the mean for each set of realisations.

POWMES [35], a state of the art conventional density power spectrum estimator. For the HR simulations, the power spectra agree within 10% for $0.2h^{-1}\text{Mpc} \leq k \leq 7.0h^{-1}\text{Mpc}$, see figure 22 and within 5% for the majority of this range. A similar result is seen for the ratio of the DTFE gradient of the density spectrum (divided by $k^2$) to the POWMES density spectrum, see figure 23.

The agreement on the largest scales, in the first 4-5 bins, is affected by the choice of binning. If the number of bins used for the DTFE extraction is doubled, then the DTFE and POWMES extractions agree much more closely as the bins are then of a more similar size and location. As noted in appendix B 4, if we increase the number of bins then the number of $k$ modes contributing to the first few bins is much smaller, so we will continue to use $N_{\text{grid}}/4$ bins in our analysis. The agreement between the POWMES and DTFE methods is sufficient to support the robustness of our density and density gradient spectra.

8. Comparison to previous findings

There are several works in the literature to which we can compare our findings on the velocity field and its components. As mentioned above, the vorticity and velocity divergence power spectra were extracted from N-body simulations in [23] using an alternative implementation of the Delauney tesselation. They found a strong dependence on resolution of the extracted
FIG. 22: The ratio of the density power spectra extracted using the DTFE and POWMES methods. The red, green and blue lines show the ratio for the 3 individual realisations and the dashed black line denotes the average of these three curves.

FIG. 23: The ratio of the density gradient power spectrum extracted using the DTFE method and the density power spectrum extracted using the POWMES method. The red, green and blue lines show the ratio for the 3 individual realisations and the dashed black line denotes the average of these three curves.

vorticity power spectrum and an approximate scaling of the vorticity power spectrum with the seventh power of the linear growth factor.

The vorticity and velocity divergence power spectra in [23] are consistent with the spectra extracted for this paper and we found the same resolution dependence of the vorticity power
spectrum (see above). However, as already mentioned, we do not find the same scaling of the vorticity spectrum with the seventh power linear growth factor: Although this scaling seems to hold at low redshift, it no longer holds at redshift one and beyond. At these earlier times, the power spectrum is smaller than expected from the growth factor to the seven scaling, so the vorticity power spectrum must have grown by less at redshift two than expected.

Two recent publications \[36, 37\] have examined the velocity field from the point of view of redshift space distortions. In these works, a different method of extracting velocity fields is used, the nearest particle method. In this method, the velocity at a grid point is given by the velocity of the nearest particle to that grid point. See those works for comments on the differences between the nearest particle and Delauney tesselation methods of extracting the velocity power spectra. Here, we note that there appear to be pros and cons to both methods, with no clear “better” method. It would be interesting to examine how close the agreement between the vector potentials extracted by the DTFE and nearest particle methods is.

Nonetheless, there are some general observations that can be compared between these works. Notably, the magnitude of the velocity and vorticity spectra is found to be similar, considering the differences in cosmological parameters. Also, the onset of non-linearity is found to occur at lower \( k \) for the velocity divergence than for the density. In addition, \[36\] finds a strong dependence of the curl component of the velocity field on the resolution, similarly to both this paper and \[23\]. They also find a time dependence of this component that is approximately \( D_+^2 \) up to \( z = 2 \), although this relationship breaks down by up to a factor of two for certain redshifts and scales. As mentioned above, whilst our simulations also find this time dependence of the vorticity at low redshift, we find that the relationship breaks down for \( z > 1 \). There is no examination of multiple realisations in \[36\] and, similarly to the comments above regarding \[23\], the difference between our realisations is sufficient to explain the difference between our results and those of \[36\].

The broad agreement between different methods, including agreement regarding resolution dependence and convergence, is promising. Details of the vorticity field and its evolution require further study, but the vorticity is a sub-dominant contribution to the vector potential. As the simulations and snapshots used in the papers mentioned in this section are different to ours, it is not possible to compare the methods and extracted fields any more precisely. We note that the three works mentioned here do not have multiple realisations of their high
resolution simulations, so we are unable to determine if the variation in vorticity between realisations found by us is reproduced (see appendix B.2).

As this manuscript was being prepared, [38] appeared on the arxiv. This paper investigates the properties of velocity divergence and vorticity and confirms many of the findings of [23]. In particular, they agree with our results regarding the convergence of the DTFE code for sufficient mass resolution and our finding of a resolution dependence of the velocity divergence, which did not appear in [23]. They use a different method to compute the vorticity and velocity divergence power spectra, which agrees with the DTFE code for sufficient resolution. However, as with the previous papers, there seems to be no examination of multiple realisations with the same resolution, in order to compare our findings. In addition, there is no examination of the time dependence and thus no confirmation or rejection of the $D_+^7$ scaling of the vorticity spectrum at higher redshifts.

IV. RESULTS

In this section we present the power spectrum of the post-Friedmann vector potential as calculated from N-body simulations. We show the power spectrum at $z = 0$ and the different components of the source, as well as the evolution of the power spectrum between $z = 2$ and $z = 0$. In addition, we show the ratio between the vector and scalar power spectra, and examine the time evolution of this quantity as well. The power spectra plotted for the scalar and vector gravitational potentials are the dimensionless power spectra. The closest analytic result to our calculation is the second order perturbative vector potential calculated in [34]. We will compare our results to theirs at redshift $z = 0$, as well as comparing the time evolution.

A. Results at redshift zero

In figures 24 and 25, we show the power spectrum of the post-Friedmann vector potential as well as the standard Newtonian scalar potential, at $z = 0$, for the curl and momentum field methods of extraction respectively. As expected, both methods show that the scalar potential is small over all scales and the vector potential is subdominant. There is a quantitative difference between the two methods on the largest scales, but this difference is not sufficient
FIG. 24: The scalar (dashed red line) and vector (solid blue line) gravitational potential power spectra at redshift zero, with the vector potential calculated using the curl method. The linear theory scalar potential is shown for comparison (dotted black line).

FIG. 25: The scalar (dashed red line) and vector (solid blue line) gravitational potential power spectra at redshift zero, with the vector potential calculated using the momentum field method. The linear theory scalar potential is shown for comparison (dotted black line).

to alter the expected qualitative behaviour. Notably, the effect of the vector potential on weak-lensing power spectra, as examined in [21], will remain negligible, regardless of which method is used to calculate the vector potential. We have been unable to determine the reason for this discrepancy and it is unclear to us which method should, a priori, be expected to be more accurate.
In figures 26 and 27, we show the ratio between the power spectra of the vector and scalar gravitational potentials at redshift zero, for the two methods of extracting the vector potential. In addition, we include the standard error on the mean. For the curl method, as shown in [14], this ratio is approximately $2.5 \times 10^{-5}$. This ratio does not vary significantly over the range of scales considered, although there is a slight increase towards smaller scales. However, for the momentum field method, the ratio is not approximately constant due to the decreased power on large scales. We will compare this behaviour to the analytic second order perturbative behaviour shortly, here we just note that the curl method produces qualitative behaviour that is closer to the analytic prediction.

In figure 28, we show the power spectra of the three sources of the vector potential using the curl method, see equation (2.4). The power spectra plotted here are given by $P(k) / (f^2 \mathcal{H}^2 (2\pi)^3)$, where $\mathcal{H}$ is the conformal time Hubble constant and $f = d \ln D / d \ln a$ is the logarithmic derivative of the linear growth factor $D$. These units are chosen such that the power spectrum of the velocity divergence agrees with the density power spectrum on linear scales and have the same units as the matter power spectrum, following [23]. The vorticity, although often ignored in perturbation theory, is the only one of these three quantities that is linear in perturbations. This figure shows that it is negligible compared to the other two components, so the vector potential is being predominantly generated by non-linear effects. Since this vector potential is the first correction to Newtonian theory, this calculation is
FIG. 27: The ratio at redshift zero between the vector potential, calculated using the momentum field method, and the scalar potential. The error bars denote the standard error on the mean.

FIG. 28: The power spectra of the three source terms for the vector potential in equation (2.4), the vorticity (dashed black line), $\nabla \delta \times \vec{v}$ (dot-dashed blue line) and $\delta \nabla \times \vec{v}$ (solid red line). The power spectra plotted here are given by $P(k)/\left(f^2\mathcal{H}^2(2\pi)^3\right)$, such that the power spectrum of the velocity divergence agrees with the density power spectrum on linear scales and ensuring that all of the power spectra have the same units, following [23]. The linear matter power spectrum is shown as a dotted magenta line for comparison.

the first quantitative check of the relationship between Newtonian simulations and GR on fully non-linear scales. The small magnitude of the vector potential suggests that running Newtonian simulations is sufficiently accurate for cosmological purposes, whereas a larger
calculated value for the vector potential would suggest that the approximations taken in deriving the fully non-linear Newtonian equations do not hold sufficiently well. As far as relating Newtonian and relativistic cosmologies goes, in the language of [13], the smallness of this vector potential allows the use of the abridged dictionary in [12], rather than the dictionary proposed in [13]. We note that the analysis here is for a ΛCDM cosmology, further work is required to determine the validity of Newtonian simulations in general dark energy cosmologies.

B. Time evolution

In this section we will examine the time evolution of the vector potential, and its ratio to the scalar potential, for the redshifts listed in table II. The vector potential is this section has been computed using the curl method. In figure 29 we show the vector gravitational potential computed at the eight different redshifts. Similarly to the scalar potential, there is relatively little variation in the vector potential over time, particularly on larger scales. In fact, for a given scale, the vector potential varies by less than a factor of 3 from $z = 2$ to $z = 0$. As for the scalar gravitational potential, the vector potential is not monotonic over time on non-linear scales.

In figure 30, we plot the ratio of the vector potential to the scalar potential for the eight different redshifts. The ratio is fairly constant in time, varying by no more than a factor of 2 for a given scale. Across the entire range of times and scales under consideration, the ratio varies by less than a factor of 4. Again, this quantity is not monotonic over the红shift range under consideration.

C. Comparison to perturbative calculation

In [34], an analytic calculation of the vector potential was performed using perturbation theory. As a perturbative analysis, it is unclear how large a value of $k$ this calculation should be extended to. Here we will assume it is valid on all of the scales of overlap between this method and ours.

For the curl method of computing the vector power spectrum, there is similar qualitative behaviour between the two methods, with the ratio of the power spectra of the vector and
FIG. 29: The vector potential plotted for 8 different redshifts between $z = 0$ and $z = 2$, see table II for details.

FIG. 30: The ratio of the vector potential to the scalar potential plotted for 8 different redshifts between $z = 0$ and $z = 2$, see table II for details. For clarity, the error bars have not been included, however they do not vary significantly with redshift. See figure 27 for the errors at redshift zero.

scalar potentials being fairly constant and of order $10^{-5}$ in both methods. The difference between the two methods being that the ratio in [31] is between the vector and the linear theory scalar potential, whereas the our ratio is between the vector and the fully non-linear scalar potential. This means that despite this similar qualitative behaviour, the power spectrum of the vector potential in [34] underestimates the fully non-linear value on these
scales by up to two orders of magnitude, the same factor by which the linear theory scalar potential power spectrum underestimates the power spectrum of the fully non-linear scalar potential.

The momentum field method of calculating the vector power spectrum results in less-similar qualitative behaviour. It is unclear how well the gravitomagnetic potential would be expected to match the perturbative prediction on these scales as the velocity field differs from the linear theory at larger scales than the density.

The power spectrum of the perturbative vector potential is given in [34] as

\[ P_s(k) = \left( \frac{2\Delta R}{5g_\infty} \right)^4 \left( \frac{3g_g' + \mathcal{H}}{\Omega m \mathcal{H}^2} \right)^2 k^2 \Pi(u^2) \]  

(4.1)

where \( P_s \) is the dimensionless power spectrum of the vector potential, \( \Delta R \) is the primordial power of the curvature perturbation, \( g \) is the growth factor for the scalar potential, \( g_\infty \) is a normalisation parameter chosen so \( g(0) = 1 \), \( \Pi \) is a function of the transfer function, \( \Omega_m \) is the time dependent matter density and \( \mathcal{H} \) is the conformal Hubble constant. The second term in parentheses contains all of the time dependence of the vector potential power spectrum and essentially acts as the growth factor for the vector potential. We compare this perturbative prediction for the growth factor of the vector potential to the growth measured in the simulations in figure 31. Comparing this plot to figure 29 shows that the analytic prediction is not the main source of the time evolution of the vector potential. Is there a better way to examine or display this?

V. CONCLUSION/DISCUSSION

In this paper we have presented the post-Friedmann frame-dragging vector potential calculated on non-linear scales from N-body simulations. We have presented this vector potential at redshift zero, as well as examining its evolution with redshift. We have also presented the tests we have performed in order to establish the robustness of our result, including tests of simulation parameters and different methods of extracting the source of the vector potential.

We have shown that our density, velocity divergence and vorticity spectra are consistent with the literature and show similar behaviour regarding convergence tests, particularly mass resolution. We do not see the vorticity scaling with the seventh power of the linear
FIG. 31: The vector potential plotted for 8 different redshifts between $z = 0$ and $z = 2$, see table II for details. The lower black curve is the curve for $z = 2$. The dashed line is the power spectrum at $z = 0$; if the analytic growth is correct then the other curves should overlay this curve on the scales where the analytic approximation is valid.

growth factor $D_+$ [23] beyond $z = 1$, however the differences between our results and others’ are within the variance between realisations. We have noted a larger variation of the vorticity than the density and velocity divergence fields between different realisations, a result that does not seem to have been studied in the literature.

We have shown that there is no evidence for a systematic dependence of the vector potential spectrum on box size for boxes smaller than $200h^{-1}\text{Mpc}$, or on mass resolution with mass resolution better than $6 \times 10^8 M_\odot$. There is also no evidence that the vector potential is sensitive to the softening length, binning, number of samples (an internal DTFE parameter) or the grid size used in the analysis. There is a reasonable agreement between the different methods (curl and momentum field) of extracting the vector potential, although there is an unresolved discrepancy between the two methods on the largest scales. We do however note the importance of the variation of the vector potential between realisations, this issue is discussed more fully in appendix B 2.

Figures 24 and 29 comprise the main physical results of this paper, showing the magnitude of the vector potential power spectrum at redshift zero and its evolution with time respectively. The magnitude of the vector potential power spectrum can also be expressed in terms of its ratio to the power spectrum of the scalar potential, as shown in figures 26.
and [30]. We have shown that the power spectrum of the vector potential is around $10^5$ times smaller than the power spectrum of the scalar potential, over a range of scales and redshifts. These values were used in [14] and [21] when examining the observability of the vector potential, showing that it is negligible for currently planned weak-lensing surveys. The small magnitude of the vector potential found here is the first quantitative check of the validity of Newtonian simulations compared to GR on fully non-linear scales and supports the use of Newtonian simulations for computing cosmological observables. In terms of interpreting the simulations, the small value of this vector potential seems to justify the use of the abridged dictionary in [12], rather than the dictionary proposed in [13], for relating GR and Newtonian cosmoologies.

The work carried out so far considers a ΛCDM cosmology, so this conclusion may no longer be true for a dark energy or modified gravity cosmology. The post-Friedmann approach would need to be expanded to include modified Einstein equations and/or a fluid with pressure in order to examine alternative cosmologies and determine whether the use of Newtonian-type N-body simulations is still valid in those cosmologies. Work is currently underway to apply the post-Friedmann expansion to f(R) gravity and extract the vector potential from f(R) N-body simulations. We hope that this, and further extensions to the work in this paper, will allow us to understand how generic the findings in this paper are, and thus justify one of the most widely used tools in cosmology, N-body simulations.

Whilst this manuscript was being prepared for submission, [39] appeared on the ArXiv. Their preliminary results seem to agree with the results of this work. It will be interesting to perform a more in-depth comparison once the details of their work are available.

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Appendix A: Vector power spectra

We will be dealing with vector quantities, for which there are different ways to define the power spectrum. Our power spectrum for a generic vector \( \vec{v} \) is defined as

\[
\langle \vec{v}(\vec{k}) \cdot \vec{v}^*(\vec{k}) \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P_\vec{v}(k)
\]

(A1)

Note that for a divergenceless vector, such as \( \vec{B} \), \( k^2 P_{\vec{B}N}(k) = P_{\vec{v} \times \vec{B}N}(k) \). With our Fourier transform convention, the dimensionless power spectrum for a field \( X \) is given by

\[
P_X(k) = k^3 P_X(k)/2\pi^2.
\]

Using equation 2.3,

\[
\langle \nabla \times \vec{P}N(\vec{k}) \cdot \nabla \times \vec{P}N^*(\vec{k}') \rangle = \left( \frac{16\pi G \rho_0 a^2}{k^2} \right)^2 \left( \langle \vec{P}N(\vec{k}) \rangle \cdot \langle \vec{P}N^*(\vec{k}') \rangle \right) = \left( \frac{16\pi G \rho_0 a^2}{k^2} \right)^2 \left( \langle \nabla \times \vec{v} \rangle \cdot \nabla \times \vec{v}^* \right)
\]

(A2)

Noting that \( A \cdot B^* = (A^* \cdot B)^* \),

\[
\langle [\nabla \times \vec{v}] \cdot [\nabla \delta \times \vec{v}^*] \rangle + \langle [\nabla \delta \times \vec{v}] \cdot [\nabla \times \vec{v}^*] \rangle = 2\text{Re} \left( \langle (\nabla \times \vec{v}) \cdot (\nabla \delta \times \vec{v}) \rangle \right) = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P_{(\nabla \delta \times \vec{v})(\nabla \times \vec{v})}(k)
\]

(A3)

And therefore

\[
P_{\vec{B}N}(k) = \left( \frac{16\pi G \rho_0 a^2}{k^2} \right)^2 \frac{1}{k^2} \left( P_{\nabla \times \vec{v}}(k) + P_{\delta \nabla \times \vec{v}}(k) + P_{(\nabla \delta) \times \vec{v}}(k) + P_{(\nabla \delta \times \vec{v})(\nabla \times \vec{v})}(k) + P_{(\nabla \delta \times \vec{v})(\delta \nabla \times \vec{v})}(k) + P_{(\delta \nabla \times \vec{v})(\nabla \times \vec{v})}(k) \right)
\]

(A4)
FIG. 32: Extracted density spectra, colour-coded as in figure 6. The dashed black line shows the spectrum extracted from the the $80h^{-1}$Mpc simulations, note how this spectrum is significantly smaller than the others.

Appendix B: Additional robustness information

1. Smaller Boxes

Here we show some additional plots that demonstrate some of the comments made during the main text. Firstly, we show our results for smaller boxes. The effect of using (too) small boxes when running N-body simulations has been examined in the past, see e.g. [30–32]. We ran a set of 8 simulations with $512^3$ particles in an $80h^{-1}$Mpc box and extracted the same power spectra as from our other simulations. In figures 32, 33 and 34, we show the density, vorticity and vector potential power spectra as extracted from the different simulations at redshift zero, colour coded to match box size as in figures 6-9. The spectra extracted from the $80h^{-1}$Mpc simulations have been added as a dashed black line. It is clear that the spectra extracted from these boxes are systematically smaller, irrespective of any other dependence on box size and resolution. [30] suggests that an important quantity is the ratio between the box size and the scale of no-linearity. As a result, for our simulations, the smallest boxes we have run that we consider trustworthy are the $140h^{-1}$Mpc simulations. It remains to be seen whether smaller boxes, such as the $100h^{-1}$Mpc simulations used in [36] are a robust source of spectra such as the density and vorticity.
FIG. 33: Extracted vorticity spectra, colour-coded as in figure 8. The dashed black line shows the spectrum extracted from the the 80 $h^{-1}$Mpc simulations, note how this spectrum is significantly smaller than the others.

FIG. 34: Extracted vector potential spectra, colour-coded as in figure 9. The dashed black line shows the spectrum extracted from the the 80 $h^{-1}$Mpc simulations, note how this spectrum is significantly smaller than the others.

2. Realisations

In this section we show how the extracted spectra vary amongst realisations. We will illustrate this with the HR simulations, for which there are 3 realisations, and with the 160 $h^{-1}$Mpc 640$^3$ particle simulations for which there are 8 realisations. In all cases we consider the variation at redshift zero. In figures 35, 40 and 36, 41 we show the variation of the
density and velocity divergence respectively. Each solid line is an individual realisation and in all plots in this section a dashed black line denotes the average over the realisations. In figures [39 and 46] we plot the variation in the density spectra extracted using the POWMES code. These plots demonstrate that the variation amongst realisations for these spectra behaves as expected: The variation between the extracted density fields is very similar for the POWMES and DTFE methods and the spectra show reduced variance between realisations on smaller scales, where there are more modes. Quantitative measure of the expected variance?

In figures [37 and 42] we show the variation of the vorticity field amongst realisations. These plots show that the variation amongst realisations is greater for the vorticity than for the density. The variation amongst realisations is less, but still greater than for the density field, on smaller scales. We are not aware of this being previously noted in the literature, and the works [23, 36, 38] that we compare our vorticity spectrum to in the main text do not have multiple realisations in order to have seen this effect.

In figures [38 and 43] we show the variation amongst realisations of the vector potential. On large scales, the variation between realisations is very similar to that between the vorticity spectra. However, the variation does not appear to reduce on smaller scales. According to perturbative results [33, 34], the vector is generated most efficiently by coupling between two different $k$ modes, particularly if one of them is entering the horizon. Given the similar qualitative behaviour of the fully non-linear vector potential, it is reasonable to assume that this is also generated by coupling between large scale modes and small scale modes. Thus, the large scale variance between realisations will be affecting the vector power spectrum on smaller scales, resulting in the variance between realisations not decreasing on small scales.

In figure [44] we show how the value of the vector potential from the individual realisations of the HR sims compares to the average over realisations of simulations with different parameters. Note that the variation between the HR realisations is greater than the variation between the average over realisations for different simulation parameters. In figure [45] we show the same simulations, but showing the standard error on the HR simulations rather than showing the values in the separate realisations.

As mentioned above, the increased variance between realisations may be an unavoidable feature of the vector potential. As such, this represents the dominant source of error in calculating the vector potential, as long as the simulation parameters are sufficiently good.
FIG. 35: The density power spectra as extracted from the 8 $160h^{-1}\text{Mpc}$ simulations with $640^3$ particles. The dashed black line denotes the average of the 8 simulations.

FIG. 36: The velocity divergence power spectra as extracted from the 8 $160h^{-1}\text{Mpc}$ simulations with $640^3$ particles. The dashed black line denotes the average of the 8 simulations.

If an observational test of the vector potential was found, then many more realisations than the number carried out for this paper would be required, in order to more carefully investigate this affect and determine more precisely what the observational prediction would be for a $\Lambda\text{CDM}$ cosmology.
3. Softening length

In this paper we have chosen our softening lengths following [23] in order to compare to their results. In figure 47, we show how a 160\,h^{-1}\text{Mpc} simulation with 640^3 particles and the same initial conditions varies if the softening length changes from 6.5\,kpc to 5kpc. This is a 20\% change in the softening length. The variation between the density, velocity divergence and vorticity spectra is very small for this change. The power spectrum of the vector potential varies more, but is within 5\% of the value for nearly the entire range under consideration.
FIG. 39: The density power spectra as extracted using POWMES from the 8 $160h^{-1}$Mpc simulations with $640^3$ particles. The dashed black line denotes the average of the 8 simulations.

FIG. 40: The density power spectra as extracted from the 3 $160h^{-1}$Mpc simulations with $1024^3$ particles. The dashed black line denotes the average of the 3 simulations.

Since this 5% variation is significantly smaller than the 20% variation in the softening length, we do not think the choice of softening length significantly impacts our results for a sensible choice of softening length.

4. Number of bins

We considered the effect on our extracted power spectra of varying the number of bins. In figures 48, 49 and 50 we show the effect of varying the bin number on the density, velocity...
FIG. 41: The velocity divergence power spectra as extracted from the 3 $160h^{-1}$Mpc simulations with $1024^3$ particles. The dashed black line denotes the average of the 3 simulations.

FIG. 42: The vorticity power spectra as extracted from the 3 $160h^{-1}$Mpc simulations with $1024^3$ particles. The dashed black line denotes the average of the 3 simulations.

divergence and vorticity power spectra respectively. In each of these plots, the 256 bins used for the analysis in this paper is shown by the black line, the blue lines denote the use of 512 bins and the red lines are for 1024 bins. As expected, increasing the number of bins increases the noise of the power spectra and there is no systematic deviation. We have used 256 bins for our analysis to ensure that the low $k$ bins contain a sufficient number of $k$-modes. For the 256 bins, the first two $k$ bins contain 58 and 218 $k$-modes respectively, whereas these numbers are 12 and 41 for the corresponding bins when 1024 bins are used. Note that, as
FIG. 43: The vector potential power spectra as extracted from the 3 $160h^{-1}$Mpc simulations with $1024^3$ particles. The dashed black line denotes the average of the 3 simulations.

FIG. 44: The vector potential power spectra from different simulations, divided by the average vector potential from the three $160h^{-1}$Mpc simulations with $1024^3$ particles. The three red curves show the vector potential from the three realisations of the $160h^{-1}$Mpc simulations with $1024^3$ particles. The cyan and magenta curves show the vector potential from the average of the $160h^{-1}$Mpc simulations with $640^3$ and $880^3$ particles respectively. The yellow curve shows the average of the $140h^{-1}$Mpc simulations with $768^3$ particles and the green curves shows the average of the $200h^{-1}$Mpc simulations with $1024^3$ particles. Note that the variation between the high resolution simulations is greater than the variation between the average values from simulations with different parameters.
FIG. 45: The vector potential power spectra from different simulations, divided by the average vector potential from the three $160h^{-1}\text{Mpc}$ simulations with $1024^3$ particles, complete with the standard error on the HR simulations. The cyan and magenta curves show the vector potential from the average of the $160h^{-1}\text{Mpc}$ simulations with $640^3$ and $880^3$ particles respectively. The yellow curve shows the average of the $140h^{-1}\text{Mpc}$ simulations with $768^3$ particles and the green curves shows the average of the $200h^{-1}\text{Mpc}$ simulations with $1024^3$ particles.

FIG. 46: The density power spectra as extracted using POWMES from the 3 $160h^{-1}\text{Mpc}$ simulations with $1024^3$ particles. The dashed black line denotes the average of the 3 simulations.

mentioned in the POWMES section, the variation between the 256 bin and 512 bin power spectra is similar to the variation between the POWMES method and the DTFE method using 256 bins. This is due to the number and location of bins in the POWMES method.
FIG. 47: The ratio between the power spectra extracted at redshift zero for the same initial conditions run with two different softening lengths. The different spectra plotted are density (red), velocity divergence (blue), vorticity (cyan) and the vector potential (black), being very similar to the DTFE method with 512 bins.

In addition, figure 51 shows the variation of the vector potential power spectrum with the number of bins. Again, the change in the number of bins is negligible. In this plot, the dashed lines show the power spectra computed with the extra factors of $k$ explicitly included whilst summing over the modulus squared values of the field, see the velocity consistency check for more information. As expected, this change affects things the most in the largest bins and therefore on large scales and for the smallest number of bins, however it does not affect our results.

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FIG. 48: The density power spectrum binned into 256 bins (black), 512 bins (blue) and 1024 bins (red). The yellow curve shows the linear matter power spectrum for reference.

FIG. 49: The velocity divergence power spectrum binned into 256 bins (black), 512 bins (blue) and 1024 bins (red). The yellow curve shows the linear matter power spectrum for reference.

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FIG. 50: The vorticity power spectrum binned into 256 bins (black), 512 bins (blue) and 1024 bins (red). The yellow curve shows the linear matter power spectrum for reference.

FIG. 51: The vector potential power spectrum binned into 256 bins (black), 512 bins (blue) and 1024 bins (red), each as an average over the three realisations in the HR simulations. The dashed lines have the 2 extra factors of k (due to the curl) included whilst binning the modulus-squared values, see the velocity consistency check for more information.

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