A robust and efficient method for calculating the magnification of extended sources caused by gravitational lenses

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Abstract. To determine the magnification of an extended source caused by gravitational lensing one has to perform a two-dimensional integral over point-source magnifications in general. Since the point-source magnification jumps to an infinite value on caustics, special care is required. For a uniformly bright source, it has been shown earlier that the calculation simplifies if one determines the magnification from the area of the images of the extended source by applying Green’s theorem so that one ends up with a one-dimensional integration over the image boundaries. This approach is discussed here in detail, and it is shown that it can be used to yield a robust and efficient method also for limb-darkened sources. It is also shown that the centroid shift can be calculated in a similar way.

Key words: Methods:numerical – gravitational lensing – planetary systems – binaries: general – dark matter

1. Introduction

For fitting light curves for the ongoing microlensing events, there is a need for robust and efficient methods for calculating the magnification of extended sources, which are not limited to point-lenses. Among the observed events, the presence of binary lenses is a reality (Dominik & Hirshfeld 1994, 1996; Udalski et al. 1994; Alard et al. 1995; Bennett et al. 1996), and planetary events involve a special case of a binary lens. In addition, for some configurations, the light curve for a limb-darkened source will differ significantly from that of a uniformly bright source. The limb-darkening effect has recently been observed in the galactic microlensing event MACHO 97-BLG-28, which involves both an extended source and a binary lens, by the PLANET collaboration (Albrow et al. 1998a,b); the fitting has been done by myself using the algorithm described in this letter.

If one wants to integrate the point-source magnification in two dimensions one has to take special care of the position of the caustics, where the point-source magnification becomes infinite. While this integration can be performed easily for a point-mass lens (e.g. Schneider et al. 1992, p. 313; Witt & Mao 1994; Sahu 1994; Dominik 1996), this would be a difficult task for a general lens (e.g. a binary lens), especially at a cusp singularity. In contrast, the area of the images of the extended source and therefore its magnification remains continuous when the source hits a caustic. The determination of the extended source magnification from the boundaries of the image areas has been used to analyze the images of background galaxies behind a cluster of galaxies (Dominik 1993). The image boundaries can be obtained with a contour plot of an implicit function describing the source boundary in the lens plane (Schramm & Kayser 1987). This method has been expanded with routines for correcting, testing and finally analyzing the contour line in order to produce an efficient and safe algorithm (Dominik 1993). In that paper, it is noted that it is easy to analyze the images from the contour line data, and an example is given, where quantities such as the area, width, length and curvature of the image have been determined.

Concerning microlensing light curves, it has been noted by Bennett & Rhie (1996) that it is advantageous to integrate in the lens plane rather than in the source plane to determine the magnification of an extended source. For uniformly bright sources, Gould & Gaucherel (1997) proposed applying Green’s theorem so that only one integration along the image boundary must be performed rather than two over the image area. This approach is identical to that used earlier (Dominik 1993). The contour plot method is the most convenient way to obtain data points on the image boundary from which the area can be calculated. In Sect. 2, this general approach is described, Sect. 3 gives details for a uniformly bright source, and Sect. 4 shows how this approach can also be used for limb-darkened sources, in which case an easy-to-perform two-dimensional integration remains. In Sect. 5, the calculation of the centroid shift is discussed.

2. Magnification and Green’s theorem

The magnification is given by the ratio of the area of the images to the area of the source in the absence of the lens. Let $(y_1, y_2)$ denote arbitrary cartesian source coordinates and $(x_1, x_2)$ denote corresponding image coordinates, where in the absence of the lens $y = x$. Let $I_s(y)$ denote the surface brightness of
the source and \( I(x) \) denote the surface brightness of the images. The conservation of photon number then requires (see e.g. Schneider et al. [1992], p. 33; Schramm & Kayser [1987]; Kayser & Schramm [1988]) that

\[
I(x) = I_s(y(x)),
\]

where \( y(x) \) gives the source position \( y \) related to the image position \( x \).

With \( A_s \) being the region in the \((y_1, y_2)\)-plane subtended by the source and \( A_l \) being the region in the \((x_1, x_2)\)-plane subtended by the images, the magnification is given by

\[
\mu = \frac{\int_{A_l} I(x) \, dx_1 \, dx_2}{\int_{A_s} I_s(y) \, dy_1 \, dy_2}.
\]

In the limit of a point source, one obtains \( \mu = \sum \mu(x_j) \), where the sum runs over the \( n \) images \( x_j \) of the source, and \( \mu \) is given by

\[
\mu(x_j) = \frac{1}{\det \left( \frac{\partial y}{\partial x} \right)(x_j)}.
\]

The normalization of the brightness profile function \( I_s \) can be chosen so that the integral in the denominator (Eq. (3)) becomes the area of the source, and for a circular source of radius \( R_{src} \) one obtains

\[
\int_{A_s} I_s(y) \, dy_1 \, dy_2 = \pi R_{src}^2.
\]

For a uniformly bright source, one has \( I_s(y) = 1 \) for positions within the source, so that \( I(x) = 1 \) for positions within the images, and the magnification becomes

\[
\mu = \frac{1}{\pi R_{src}^2} \int_{A_l} dx_1 \, dx_2,
\]

where the remaining integral is just the area of the images.

Green’s theorem now states that for two functions \( P(x_1, x_2) \) and \( Q(x_1, x_2) \) which are continuous in a region \( A \) and whose partial derivatives \( \partial P/\partial x_2 \) and \( \partial Q/\partial x_1 \) are also continuous in \( A \),

\[
\int_{A} \left( \frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) \, dx_1 \, dx_2 = \int_{\partial A} P \, dx_1 + Q \, dx_2,
\]

where \( \partial A \) denotes the boundary of the region \( A \), which has to be piecewise-smooth. Note that the values of \( P \) and \( Q \) outside the region \( A \) do not play a role and can be chosen arbitrarily as long as \( P \) and \( Q \) satisfy the conditions above.

3. Uniformly bright sources

For a uniformly bright source, one can now choose \( P = -\frac{1}{2} x_2 \) and \( Q = \frac{1}{2} x_1 \) to obtain

\[
\mu = \frac{1}{2\pi R_{src}^2} \int_{\partial A} x_1 \, dx_2 - x_2 \, dx_1.
\]

4. Limb-darkened sources

Let us now consider a limb-darkened source at \( y^{(0)} \) with radius \( R_{src} \), with the brightness profile

\[
I_s(y) = f_1 \left( \frac{(y - y^{(0)})^2}{R_{src}^2} ; \tilde{u} \right),
\]

where

\[
f_1(r; \tilde{u}) = \frac{1}{1 - \tilde{u}/3} \left( \tilde{u} \sqrt{1 - r^2} + 1 - \tilde{u} \right)
\]

for \( 0 \leq r \leq 1 \), and \( f_1(r, \tilde{u}) = 0 \) for \( r > 1 \). This profile is normalized so that Eq. (3) is satisfied. The parameter \( \tilde{u} \) is chosen from the interval \([0, 1]\), where \( \tilde{u} = 0 \) corresponds to a uniformly bright source. This brightness profile is the simple ‘linear’ model\(^1\) widely used, though for some type of stars, one

\(^1\) The profile involves a linear term of \( \cos \theta \), where \( \theta \) is the angle between the normal to the surface and the direction to the observer.
should add other terms (e.g. Claret et al. [1995]). The general ideas of the approach discussed here do not depend on this special choice, especially terms like $(1 - r^2)^n$ (i.e. like $\cos^2 \theta$) can be treated in complete analogy to the $\sqrt{1 - r^2}$-term.\footnote{In fact, for the discussion of the MACHO 97-BLG-28 event (Al-brow et al. 1998a,b), a $\sqrt{1 - r^2}$-term has also been included in the brightness profile.}

From Eqs. (3) and (4), one obtains the magnification as

$$ \mu = \frac{1}{\pi R_{\text{src}}^2} \int_{A_i} I(x) \, dx_1 \, dx_2 , $$

(12)

To apply Green’s theorem one has to find functions $P$ and $Q$ which satisfy

$$ \frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} = I(x_1, x_2) = I_s(y(x_1, x_2)) . $$

(13)

Such functions are given by

$$ P(x_1, x_2) = -\frac{1}{2} \int_{x_2}^{x_2} I_s(y(x_1, x_2')) \, dx_2' , $$

(14)

$$ Q(x_1, x_2) = \frac{1}{2} \int_{x_1}^{x_1} I_s(y(x_1', x_2)) \, dx_1' . $$

(15)

Since the brightness profile $I_s$ and the lens equation $y(x)$ are given in analytical form, the integrals can be evaluated numerically in general.

Since the brightness profile has an infinite slope at the limb of the source, it is advantageous to use another continuation for $r > 1$.\footnote{As mentioned earlier, the value for $r > 1$ does not play a role.} Let us write the brightness profile in the form

$$ f_1(r; \tilde{u}) = C(\tilde{u}) (\tilde{u} B(r) + 1 - 2\tilde{u}) , $$

(16)

where

$$ C(\tilde{u}) = \frac{1}{1 - \tilde{u}/3} $$

(17)

and

$$ B(r) = \begin{cases} 1 + \sqrt{1 - r^2} & \text{for } 0 \leq r < 1 \\ 1 & \text{for } r = 1 \\ 1 - \sqrt{1 - 1/r^2} & \text{for } r > 1 \end{cases} $$

(18)

This definition is identical to the previous one except for $r > 1$.\footnote{One may also use other continuations for $r > 1$ which decrease differently for $r \to \infty$.}

The functions $P$ and $Q$ read

$$ P(x_1, x_2) = -C/2 \left[ (1 - 2\tilde{u}) (x_2 - x_2^{(0)}) + \tilde{u} \int_{x_2}^{x_2} B(r(x_1, x_2')) \, dx_2' \right] , $$

(19)

$$ Q(x_1, x_2) = C/2 \left[ (1 - 2\tilde{u}) (x_1 - x_1^{(0)}) + \tilde{u} \int_{x_1}^{x_1} B(r(x_1', x_2)) \, dx_1' \right] , $$

(20)

where

$$ r(x_1, x_2) = \frac{(y(x_1, x_2) - y(0))^2}{R_{\text{src}}^2} . $$

(21)

The function $B(r)$ has been chosen, so that for all $r$, $B(r) > 0$ in order to avoid contributions of different sign in the numerical integration process. The tail of the function is limited to

$$ \int_1^{\infty} B(r) \, dr = \frac{\pi}{2} - 1 \approx 0.571 , $$

(22)

so that the integral is not dominated by the tail contribution.

One remaining point of interest is the choice of the lower integration bound $x^{(0)}$. To avoid unnecessary integration in the tail region of $B(r)$ that would result for a fixed choice for $x^{(0)}$ such as $(0, 0)$, the lower integration bound can be chosen as the center of each image which can be determined as shown in the next section.

With these functions $P$ and $Q$ one can approximate the magnification by the expression

$$ \mu = \frac{1}{2\pi R_{\text{src}}^2} \sum_{i=1}^{n} \left[ Q \left( x_1^{(i)} \frac{1}{2} (x_2^{(i)} + x_2^{(i+1)}) \right) + Q \left( x_1^{(i+1)} \frac{1}{2} (x_2^{(i)} + x_2^{(i+1)}) \right) \right] P \left( \frac{1}{2} (x_1^{(i)} + x_1^{(i+1)}), x_2^{(i)} \right) + P \left( \frac{1}{2} (x_1^{(i)} + x_1^{(i+1)}), x_2^{(i+1)} \right) \right] P \left( \frac{1}{2} (x_1^{(i+1)} - x_1^{(i)}), x_2^{(i+1)} \right) \right] , $$

(23)

which uses the symmetries and reduces to Eq. (9) with $x^{(0)} = (0, 0)$ for the case $\tilde{u} = 0$ (uniformly bright source). An example for a light curve for a limb-darkened source and one with a uniformly bright source behind a binary lens is shown in Fig. 2.

5. The centroid shift

The position of the centroid of light can be determined in a similar way. For $m$ images of the source, the position of the centroid of light $x_c$ is given by

$$ x_c = \frac{\sum_{j=1}^{m} \int_{A_j} x I(x) \, dx_1 \, dx_2}{\sum_{j=1}^{m} \int_{A_j} I(x) \, dx_1 \, dx_2} $$

(24)

where an integration has to be performed over the regions subtended by the images $A_j$. The integral in the denominator is just that discussed in the previous section and the integral in the numerator differs only by an additional factor of $x$. The integral can be solved in the way described in the last section by only replacing $I(x)$ by $x \, I(x)$ ($i = 1, 2$). For a uniformly bright source, the corresponding functions $P$ and $Q$ are

$$ P(x_1, x_2) = -\frac{1}{2} x_1 x_2 , \quad Q(x_1, x_2) = \frac{1}{4} x_1^2 $$

(25)
for the $x_1$-component and

$$P(x_1, x_2) = -\frac{1}{4} x_2^2, \quad Q(x_1, x_2) = \frac{1}{2} x_1 x_2$$

(26)

for the $x_2$-component, so that one obtains with Eq. (24) the expressions

$$x_{c,1} = \frac{1}{8A} \sum_{i=1}^{n} \left\{ \left( x_1^{(i)} + x_1^{(i+1)} \right) (x_2^{(i+1)} - x_2^{(i)}) + \right.$$ \left. + \left( x_1^{(i)} - x_1^{(i+1)} \right) (x_2^{(i+1)} + x_2^{(i)}) \right\},

$$x_{c,2} = -\frac{1}{8A} \sum_{i=1}^{n} \left\{ \left( x_2^{(i)} - x_2^{(i+1)} \right) (x_1^{(i+1)} + x_1^{(i)}) + \right.$$ \left. + \left( x_2^{(i)} + x_2^{(i+1)} \right) (x_1^{(i+1)} - x_1^{(i)}) \right\},

(27)

(28)

where $A$ denotes the area of the image, which can be determined as shown in Sect. 3.

For a point source, the centroid’s position can be written as

$$x_c = \frac{\sum_{j=1}^{m} \tilde{\mu}(x_j) x_j}{\sum_{j=1}^{m} \tilde{\mu}(x_j)}.$$  

(29)

If one compares this expression with that given by Eq. (24), one sees that the (finite) source size has cancelled out in Eq. (24).

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Fig. 2. The light magnification for a limb-darkened source with $\tilde{\mu} = 1$ (curve with larger peak amplification) and a uniformly bright source behind a binary lens. The angular source radius corresponds to 0.03 $\theta_E$, where $\theta_E$ denotes the angular Einstein radius. The mass ratio between the two lens components is 4, their angular separation is 0.68 $\theta_E$. The source center passes perpendicularly to the line connecting the lens objects at a minimal impact of 0.004 $\theta_E$ from the midpoint towards the lighter component. $t_E$ is the time in which the angular source position relative to the lens changes by $\theta_E$, $t = 0$ corresponds to the point of time when the center of the source crossing the line connecting the lens components. The inset shows the caustics, the position of the lens objects (crosses), and the size and trajectory of the source which sweeps over a cusp caustic; the coordinates are multiples of $\theta_E$. For a lens at a distance $D_d$ and a source at a distance $D_s$ from the observer, the angular Einstein radius for a lens of total mass $M$ is given by $\theta_E = \left( \frac{4GMc^2}{D_dD_s} \left( D_s - D_d \right) \right)^{1/2}$. 

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Figure 1
Figure 2