SOME ASPECTS OF RECENT WORKS ON LIMIT THEOREMS
IN ERGODIC THEORY WITH SPECIAL EMPHASIS ON THE
"CENTRAL LIMIT THEOREM"

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Abstract. This paper gives a redaction of a talk delivered at the “Ecole pluri-thématique de théorie ergodique” which took place at the CIRM of Marseille in May 2004.

1. Introduction. The aim of the following notes is to give an introduction to some recent developments in the domain of limit theorems in ergodic theory. The notes are based on a three hour lecture delivered at the “Ecole pluri-thématique de théorie ergodique” which took place in the CIRM of Marseille in May 2004. By no means the huge subject of “limit theorems” will be comprehensively dealt with. Only a few problems, methods and results that the author finds appealing shall be presented with a selected list of references.

This introduction starts from the observation which is now a classical fact, that there is no general statement of speed of convergence in the ergodic theorem (Part 1). Then the question of the validity, on an ergodic dynamical system, of the so-called “central limit theorem of the calculus of probability”, that is the appearance of a Gaussian distribution as the limit distribution of the fluctuations of some ergodic averages, shall be the subject of the rest of this article. This question subdivides into two aspects. The first one is an existence problem: on a nontrivial ergodic dynamical system is it always possible to find a function for which the central limit theorem holds? This problem, which was already considered in the 70’s, received a complete solution, which is positive, in the 90’s: this will be the topic of Part 2.

The second aspect is the problem of establishing sufficient conditions on a function defined on an ergodic dynamical system in order that the central limit theorem holds. In this direction a celebrated result is Sinai’s result for the geodesic flow on a negatively curved compact manifold (1960). Several methods are known: “mixing coefficients”, “Markovian partitions and spectral gap”, “accompanying Brownian path”... Part 3 will present the “martingale method” which was successively used recently, in the study of partially hyperbolic dynamical systems and also in a new approach to Sinai’s theorem. The paper will end with a few comments on other aspects of this broad domain.

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Part 1. On the absence of universal speed of convergence in the ergodic theorem. The basic limit theorem is the ergodic theorem: “Given a measure preserving transformation $T$ acting on a probability space $(X, \mathcal{F}, m)$ the sequence of the averages $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x)$ converges a.e. if $f \in L^1$. This convergence takes place also in $L^p$–norm if $f \in L^p$, for $p \geq 1$.”

This part gathers some remarks which are known by specialists but should be recalled here for a good understanding of the sequel of this paper.

1. About the statement of $L^2$–convergence (von Neumann’s ergodic theorem) the following easy remark can be made on the lack of speed of convergence: if there existed a real sequence $a_n$ increasing to $+\infty$ such that $\lim_{n \to \infty} \sup a_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k - \pi(f) \right\|_2 < +\infty$ for every $f \in L^2$, with $\pi$ denoting the orthogonal projector on the space of invariants of $T$, the Banach-Steinhaus theorem would yield the uniform convergence $\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k - \pi \right\| = 0$ that is impossible in a non-atomic ergodic system.

2. About the statement of a.e. convergence (Birkhoff’s ergodic theorem), the following complementary results for an arbitrary non-atomic ergodic system, have been obtained:

a) Let $a_n$ be an arbitrary real sequence increasing to $+\infty$. On the first hand there exists a bounded function $f$ with $\int_X f dm = 0$ such that $\lim_{n \to \infty} \sup a_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) \right\| = +\infty$ a.e. (Krengel, 1985). On the other hand, assuming $a_0 \geq 2$, there exists a set $B$ with $m(B) > 0$ such that

$$\left\| \sum_{k=0}^{n-1} \mathbb{1}_B(T^k x) - nm(B) \right\| \leq a_n$$

(Halasz, 1976).

b) For $f \in L^1$ with $\int_X f dm = 0$ , the averages $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x)$ change sign (weakly) infinitely often a.e. (Halasz, 1976; Petersen, 1983).

c) The space of measurable functions $f$ such that the sequence of averages $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x)$ converges a.e. contains strictly $L^1$. This space of measurable functions cannot be described by a condition bearing only on the distributions: if $f = f^+ - f^-$ with $\int_X f^+ dm = \int_X f^- dm = +\infty$ then there exists a function $f'$ that is an equimeasurable rearrangement of $f$ for which $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f' \circ T^k(x) = +\infty$ and $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f' \circ T^k(x) = -\infty$ a.e. (Broise and al. 1989). If the averages $\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x)$ converge a.e. the function $f$ can be written as $f = g + h - h \circ T$ a.e. where $g \in L^1$ and $h$ is measurable with $\lim_{n \to \infty} \frac{h \circ T^n}{n} = 0$ a.e. (Wos, 1986).
d) A result of another type about the absence of speed in the ergodic theorem was proved by Lesigne and Volny (2000): “let \((b_n)_{n \geq 0}\) be a real sequence converging to \(+\infty\); there exists a bounded measurable function \(f\) with \(\int f \, d\mu = 0\) such that

\[
\lim_{n \to \infty} b_n \mu\{S_n f \geq 1\} = +\infty
\]

3. The cohomological equation \(f = g + h - h \circ T\) is deeply linked to the ergodic theorem and to the "central limit theorem" considered below. Therefore it might be useful to recall a few properties of this equation (always assuming the ergodicity of the transformation):

a) “For \(f \in L^2\) there exists \(h \in L^2\) such that \(f = h - h \circ T\) if and only if the sequence \(\left\| \sum_{k=0}^{n-1} f \cdot T^k \right\|_2\) is bounded. In that case \(f\) is called a \(L^2\)-coboundary and the solution \(h\) of the cohomological equation, with \(\int_X h \, d\mu = 0\), is \(h = \lim \frac{1}{n} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} f \cdot T^j\).” This statement has been known for a long time; it is interesting to note that it follows directly from von Neumann’s ergodic theorem: if the sequence \(\left\| \sum_{k=0}^{n-1} f \cdot T^k \right\|_2\) is bounded, \(\frac{1}{n} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} f \cdot T^j\) has a weak limit point \(h\) in \(L^2\) hence \((I - T)h = w - \lim \frac{1}{n} \sum_{k=0}^{n-1} (I - T^k)f = f\); then the ergodic theorem yields \(h = \lim \frac{1}{n} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} f \cdot T^j\) (Lin and Sine, 1983).

b) A measurable-coboundary is a function \(f\) such that there exists a measurable \(h\) which satisfies the equation \(f = h - h \circ T\); a necessary and sufficient condition for a measurable \(f\) to be a measurable-coboundary is that the sequence of the distributions of the sums \(\sum_{k=0}^{n-1} f \cdot T^k\) is tight (Schmidt, 1977).

c) If a function \(f \in L^1\) is a measurable-coboundary then necessarily \(\int_X f \, d\mu = 0\). A sufficient condition for \(f \in L^1\) to be a measurable-coboundary is the boundedness from above (or below) by a constant of the sequence \(\sum_{k=0}^{n-1} f \cdot T^k\) and \(\int_X f \, d\mu = 0\).

d) For \(f \in L^1\) with \(\int_X f \, d\mu = 0\) and every \(\epsilon > 0\), there exist measurable functions \(g\) and \(h\) such that \(|g| < \epsilon \text{ a.e.}\) and \(f = g + h - h \circ T\text{ a.e.}\). If the transformation \(T\) acts on a metric compact space \(X\) the function \(g\) can be chosen continuous (Kocergin, 1976).

As a momentary conclusion of this first part it appears clearly that any statement improving upon the ergodic theorem in any respect must be a statement based on the relationship which occurs between the given function \(f\) and the measure preserving transformation \(T\).

**Part 2. The weak convergence to a Gaussian distribution for ergodic sums: the existence problem.** Given an ergodic, non atomic, dynamical system \((X, \mathcal{F}, m, T)\) does there exist a real function \(f \in L^2(X, \mathcal{F}, m)\) such that \(\left\| \sum_{k=0}^{n-1} f \cdot T^k \right\|_2^2 \sim \cdots\)
With \( \sigma^2 > 0 \) and the sequence of the distributions of\[ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ T^k \]
converges weakly to the Gaussian distribution \( \mathcal{N}(0, \sigma^2) \)? Of course, the distributions of\[ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ T^k \]
are taken with respect to the invariant measure \( m \).

This second part is devoted to this question.

Following the expression fixed by usage in the calculus of probability we shall say that the central limit theorem holds for the function \( f \) when this Gaussian weak limit appears.

Sometimes the problem was put into the following form which is somewhat less precise: does there exist \( f \in L^2 \) such that the sequence \( \frac{1}{\sigma_n} \sum_{k=0}^{n-1} f \circ T^k \), with \[ \left\| \sum_{k=0}^{n-1} f \circ T^k \right\|_2 = \sigma_n , \]
weakly converges to the standard Gaussian distribution \( \mathcal{N}(0, 1) \)?

The question of the existence of the limit variance \( \sigma^2 \) is delicate and requires a separate study in each specific situation; we shall come back to this point in Part 3.

When the entropy of the system is positive there is a factor system which is a Bernoulli scheme, therefore the answer to the question is obviously positive by a direct application of the central limit theorem for a sequence of independent and identically distributed random variables defined on the factor. During the seventies the problem was set out for entropy zero systems.

As far as the author knows, the first example of an entropy zero system for which it was proved that this existence problem has a positive solution, was a Gaussian dynamical system. At first thought this might seem trivial since a limit of Gaussian distributions is Gaussian. But it might degenerate into a Dirac measure, hence it is interesting to recall this example due to Maruyama.

2.1. Example of Maruyama (1975). Let \((X_n)_{n \in \mathbb{Z}}\) be a real centered Gaussian stationary sequence, and \((\Omega, \mathcal{F}, P, T)\) the Gaussian dynamical system associated with it. It is well known that ergodicity of the system is equivalent to the property that the scalar spectral measure \( \eta \) of the process \((X_n)_{n \in \mathbb{Z}}\), defined on the one-dimensional torus has no atom. The mixing property is equivalent to the convergence to 0 of the sequence of the Fourier coefficients of \( \eta \). Furthermore the entropy of the system is zero if and only if \( \eta \) is singular with Lebesgue measure. So take for \( \eta \) a symmetric continuous and singular probability measure on the interval \((-1/2, +1/2)\) such that its sequence of Fourier coefficients satisfies:

\[ \int_{(-1/2, +1/2)} e^{int} \, d\eta(t) = E(X_k X_{k+n}) = O(|n|^{1/2 + \varepsilon}) , \]

with arbitrary small \( \varepsilon > 0 \) (existence of which is given by Zygmund, vol 2, p.146).

If limits exist:

\[ \lim_{n \to \infty} \frac{1}{\sqrt{n}} \left\| \sum_{k=0}^{n-1} X_k \right\|_2 = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \left[ \int_{(-1/2, +1/2)} \left| \frac{1 - e^{2\pi i nt}}{1 - e^{2\pi i t}} \right|^2 \, d\eta(t) \right]^{1/2} . \]

This value may be 0 if we assume, as we may, that 0 is a "Lebesgue point" of the singular measure \( \eta \); in this case the sequence \( X_n \) itself cannot yield the result we are looking for.

But let us introduce the stochastic Gaussian measure \( M \) associated with \( \eta \) giving the Wiener-Kolmogorov decomposition \( X_n = \int_{(-1/2, +1/2)} e^{2\pi i nx} M(dx) \). Then the
random variable

\[ Z = \int \int \int \int_{(-1/2, +1/2)^4} M(dx_1)M(dx_2)M(dx_3)M(dx_4) \]

belongs to \( L^2(\Omega, \mathcal{F}, P) \) and

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \left\| \sum_{k=0}^{n-1} Z.T^k \right\|_2 = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \left[ \int_{(-1/2, +1/2)} \left| \frac{1 - e^{2\pi i nt}}{1 - e^{2\pi i t}} \right|^2 d\eta_\ast \eta_\ast \eta_\ast \eta(t) \right]^{1/2}
\]

is strictly positive because of Fejér’s theorem, since the condition on the size of the Fourier coefficients of \( \eta \) implies that the convolution \( \eta_\ast \eta \) has an \( L^2 \) density, thus \( \eta_\ast \eta_\ast \eta_\ast \eta \) has a continuous density which is strictly positive at 0.

Existence of a strictly positive finite limit for the sequence \( \frac{1}{\sqrt{n}} \left\| \sum_{k=0}^{n-1} Z.T^k \right\|_2 \) is the decisive step. Then using the moment method it is easy to get that the sequence \( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} Z.T^k \) weakly converges to a non degenerate Gaussian distribution.

The first general result about the existence problem regarding the central limit theorem was obtained by Burton and Denker (1987) who proved that any ergodic non-atomic dynamical system carries a function \( f \) for which the central limit under the alternative form stated above, holds; actually they proved that the set of such functions is dense in \( L^2 \). It does not seem that the precise form of the central limit theorem, based on the typical growth of the variances \( \left\| \sum_{k=0}^{n-1} f.T^k \right\|_2^2 \) as \( n \), can be deduced from this result. However the linear growth of the variances is an essential characteristic of a sequence of sums of independent identically distributed random variables; it is a feature of the physical process of diffusion. Therefore the question of the existence of a centered function \( f \) for which the sequence \( \left\| \sum_{k=0}^{n-1} f.T^k \right\|_2^2 \) grows linearly is interesting in itself, maybe as interesting as the initial question about the validity of the central limit theorem.

The answer to the existence question about the central limit theorem under its precise statement was given by Lacey (1991). We shall now present a slightly simplified version of Lacey’s construction for an ergodic rotation. Let \( T_\alpha \) be an ergodic rotation on the one dimensional torus: we shall show the existence of a centered and square integrable function \( f \) generating a sequence of sums \( \sum_{k=0}^{n-1} f.T^k_\alpha \) whose variances grow linearly. The ergodic rotation is critical for the general question since it defines a rigid system with discrete spectrum, therefore without mixing factor (Maruyama’s example was a mixing system); moreover this situation allows a somewhat simpler exposition.

2.2. Lacey’s construction (1993). Let \( \alpha \) be an irrational number and \( T_\alpha : [0, 1] \to [0, 1] \) defined by \( T_\alpha(x) = x + \alpha \mod 1 \), the associated ergodic rotation.

Let us put \( K_n(t) = \sum_{k=0}^{n-1} e^{2\pi i k t} = \frac{1 - e^{2\pi i nt}}{1 - e^{2\pi i t}} \) for \( t \neq 0 \). We have \( |K_n(t)| \leq \frac{2}{|1 - e^{2\pi i t}|} \), therefore \( |K_n(t)| \leq \min(n, \frac{c}{\sqrt{t}}) \) for \( 0 < t < 1/2 \) where \( c \) is some positive constant and also \( n = 2 \int_0^{1/2} |K_n(t)|^2 \, dt \).
For a given $f \in L^2_{(0,1)}$ with $\int_{(0,1)} f(t) dt = 0$, we put $S_n(f) = \sum_{k=0}^{n-1} f \cdot T^k_n$ and $\sigma^2_n(f) = \int_{(0,1)} S_n(f)^2(t) dt$. Denoting the Fourier development of $f$ by $f(t) = \sum_{k=-\infty}^{+\infty} a_k e^{2\pi i k t}$ we get $S_n(f)(t) = \sum_{k=-\infty}^{+\infty} a_k K_n(k) e^{2\pi i k t}$ and $\sigma^2_n(f) = \sum_{k=-\infty}^{+\infty} |a_k|^2 |K_n(k)k|^2$.

We shall construct a sequence of coefficients $a_k$ analogous to a lacunary sequence such that $\sigma^2_n(f) = n + O(n^{1-\epsilon})$ for some $\epsilon > 0$.

For this we consider the sequence of integers

$$k_j = \min \left\{ k > 0; \frac{1}{j+1} \leq \frac{1}{j} \mod 1 \right\}$$

with $j > 1$; $k_j$ is the entrance time of the orbit of 0 under $T_n$ into the interval $I_{j} = [\frac{1}{j+1}, \frac{1}{j}]$. Then we put $a_{k_j} = a - k_j = \frac{1}{j}$ for every $j > 1$ and $a_k = 0$ otherwise (that is if $|k|$ does not belong to the values of the sequence of the entrance times $k_j$).

Hence the corresponding function $f$ is the real even function $f(t) = \sum_{j=2}^{+\infty} \frac{\cos 2\pi j k_j t}{j}.

We shall check that it yields what we are looking for: $\sigma^2_n(f) = n + O(n^{1-\epsilon})$ for some $\epsilon > 0$.

Let us introduce two parameters $q > 1 > r > 1/2$ and put $J_1 = [0, 1/n^q]$, $J_2 = [1/n^q, 1/n^r]$ and $J_3 = [1/n^r, 1/2]$ for $n$ large enough. The following estimates hold true:

i) $\int_{J_1} |K_n(t)|^2 dt \leq n^2 |J_1| = n^2 - q$, with the absolute value $|J_1|$ denoting the length of the interval.

ii) $\int_{J_2} |K_n(t)|^2 dt \leq c \int_{n^q}^{1/2} \frac{dt}{t^2} \leq cn^r$

iii) $\sum_{j \geq n^q} a^2_{k_j} |K_n(k_j)\alpha|^2 \leq n^2 \sum_{j \geq n^q} \frac{1}{j^2} = O(n^{2-\epsilon})$

iv) $\sum_{j \leq n^r} a^2_{k_j} |K_n(k_j)\alpha|^2 \leq c^2 \sum_{j \leq n^r} 1/j^4$ which is bounded.

These estimates are immediate consequences of the definitions and the inequality $|K_n(t)| \leq \min(n, \frac{c}{t})$.

It remains to compare $A_n = \int_{J_2} |K_n(t)|^2 dt$ and $B_n = \sum_{j \geq n^r} a^2_{k_j} |K_n(k_j)\alpha|^2$.

To get the desired result on $f$ it is enough to show that $|A_n - B_n| = O(n^{1-\epsilon})$, thanks to the preceding estimates. Since $a^2_{k_j} = (1 + \frac{1}{j})|I_j|$ we get, using the disjointness of the intervals $I_j$:

$|A_n - B_n| \leq \sum_{j \geq n^r}^{n^{q+1}} \left\{ \int_{I_j} (|K_n(t)|^2 - |K_n(k_j)\alpha|^2) dt + \frac{1}{j^2(j+1)} |K_n(k_j)\alpha|^2 \right\}.$

Then using the elementary inequality $||z|^2 - |w|^2| \leq |z - w| (|z| + |w|)$, and $|K_n(s)|^2 \leq n$, $|K_n'(s)| \leq 2\pi n^2$, and the mean value inequality it is easy to get:

$|A_n - B_n| \leq O \left[ n^2 \left( \sum_{j \geq n^r}^{n^{q+1}} \frac{1}{j^2 j} \int_{I_j} (|K_n(t)| + |K_n(k_j)\alpha|) dt + n^{-2r} \right) \right].$

When $n^r - 1 \leq j < n$ we have, with some constant $C$, the inequality:

$\int_{I_j} (|K_n(t)| + |K_n(k_j)\alpha|) dt \leq \frac{C}{j}$
because $|K_n(t)| \leq c(j+1)$ when $t \in I_j$. On the other hand when $n \leq j \leq n^3 + 1$ we have:
\[ \int_{I_j} (|K_n(t)| + |K_n(k_j\alpha)|)dt \leq \frac{2n}{j^2}. \]
Thus easy summations yield $|A_n - B_n| \leq O(n^{2(1-\epsilon)})$ which is the desired result:
the function $f(t) = \sum_{j=2}^{\infty} \cos \frac{2\pi k_j t}{j}$ must satisfy $\sigma_n^2(f) = n + O(n^{1-\epsilon})$ for some $\epsilon > 0$.

Then the central limit theorem for this function is an easy byproduct of a classical
version of the central limit theorem for lacunary series: see Zygmund Vol 2, p.270, Th. 6.4.

Lacey’s paper also contains a discussion of the best possible regularity of the function $f$. More precise results are proved by Volny(1999) who shows that refined limit
theorems of the theory of probability may hold on any aperiodic ergodic dynamical
system.

**Part 3. Sufficient conditions for the ”central limit theorem”: the martingale method.** To prove that for a class of functions defined on a dynamical
system the central limit theorem holds, there are several methods. In this part
we would like to give a short introduction to one of them: the martingale method.

**Theorem.** Let $(Y_n)_{n \geq 1}$ be a sequence of square integrable real random variables
adapted to the increasing filtration $(\mathcal{F}_n)_{n \geq 0}$ of the probability space $(\Omega, \mathcal{F}, P)$. Let
$s_n^2$ be: $s_n^2 = \sum_{k=1}^{n} E(Y_k^2)$. Let us assume that the three following conditions hold:

1) the conditional expectations $E(Y_n | \mathcal{F}_{n-1}) = 0$ a.s. for every $n \geq 1$ (condition
”martingale difference”).

2) $\lim_{n \to \infty} \frac{1}{s_n^2} \sum_{k=1}^{n} E(Y_k^2 | \mathcal{F}_{k-1}) = \sigma^2$ in probability, with a constant $\sigma^2 \geq 0$.

3) for every $\epsilon > 0$, $\lim_{n \to \infty} \frac{1}{s_n^2} \sum_{k=1}^{n} E(Y_k^2 1_{|Y_k| \geq \epsilon s_n}) = 0$ (Lindeberg’s condition).

Then the sequence $(\frac{1}{s_n} \sum_{k=1}^{n} Y_k)_{n \geq 1}$ weakly converges to the Gaussian distribution
$N(0, \sigma^2)$ (that is the Dirac measure at 0 if $\sigma^2 = 0$).

**Remarks.** i) The application to the case where the sequence $(Y_n)_{n \geq 1}$ is stationary
is immediate.

ii) In the preceding statement the increasing filtration $(\mathcal{F}_n)_{n \geq 0}$ can be replaced by
a decreasing filtration $(\mathcal{G}_n)_{n \geq 0}$. Then the martingale condition must be written in
a reverse form: $E(Y_n | \mathcal{G}_{n+1}) = 0$ a.s. for every $n \geq 1$, and condition 2 must bear on $E(Y_n^2 | \mathcal{G}_{n+1})$.

In this reversed situation under the same three assumptions the conclusion of the
theorem remains valid.
Before going further a few general observations about the limit variance of the sums of a stationary sequence are in order. Let an ergodic dynamical system \((X, \mathcal{F}, m, T)\) be given and \(f \in L^2(X, \mathcal{F}, m)\) with \(\int f dm = 0\). By definition the limit variance is \(\sigma^2 = \lim \frac{1}{n} \left\| \sum_{k=0}^{n-1} f \circ T^k \right\|_2^2\) if this limit exists. A sufficient condition for its existence is the convergence of the series \(\sum_{k=0}^{\infty} \left| \int f(f \circ T^k) dm \right|\); then it is an easy exercise of summability to get

\[
\sigma^2 = \|f\|_2^2 + 2 \sum_{k=1}^{\infty} \left| \int f(f \circ T^k) dm \right|
\]

which can be equal to 0.

Under the stronger assumption that the series \(\sum_{k=0}^{\infty} k \left| \int f(f \circ T^k) dm \right|\) converges, the sequence \(\left( \left\| \sum_{k=0}^{n-1} f \circ T^k \right\|_2^2 - n\sigma^2 \right)_n\) remains bounded and \(\sigma^2 = 0\) if and only if \(f\) is a \(L^2\)-coboundary. For a proof of this last statement see Lemme 2.2 in Conze-Le Borgne (2001).

Remarks. i) The existence of the limit variance as it is defined above is not a necessary condition for the weak convergence of the sums \(\sum_{k=0}^{n} f \circ T^k\) to a non degenerate Gaussian distribution, with \(f \in L^2(X, \mathcal{F}, m)\) and \(\int f dm = 0\); this is not obvious (a counterexample was communicated to the author by J.P.Conze).

ii) Limit theorems for stationary processes make up an important chapter of probability theory. We only refer to the two classical books: Ibragimov, Linnik (1971) and Hall, Heyde (1980); Herrndorf (1983) gives an interesting counterexample for orthogonal and mixing processes.

In Gordin’s method the basic idea is now to try to approximate a given stationary sequence \(f \circ T^n\) defined on an ergodic dynamical system \((X, \mathcal{F}, m, T)\) by a sequence which is a “martingale difference” sequence and to deduce the central limit theorem for the given stationary sequence from the result for the martingale. The monotone filtration with respect to which the martingale condition should be written must be chosen in accordance with the transformation \(T\); in a given concrete system the construction of this filtration is difficult.

Let us consider first the problem in an abstract setting where the Hilbertian arguments of Gordin’s method are easy to explain.

As before we consider an ergodic system \((X, \mathcal{F}, m, T)\); the measure preserving transformation \(T\) is assumed to be invertible. Moreover a sub-\(\sigma\)-algebra \(A_0\) such that \(T^{-1}A_0 \subset A_0\) is given.

Then we can define the increasing sequence of sub-\(\sigma\)-algebras \(A_n = T^n A_0\), with \(n \in \mathbb{Z}\). The orthogonal projections of the space \(L^2(X, \mathcal{F}, m)\) onto the subspaces \(L^2(X, A_n, m)\) are the conditional expectations and denoted \(\pi_n = E(\cdot \mid A_n)\). They obey the following relations:

\[
\pi_n \pi_{n+1} = \pi_{n+1} \pi_n = \pi_n \text{ and } \pi_n T = T \pi_{n+1}.
\]

Here and in the sequel we denote just by \(T\) the linear operator \(T f = f \circ T\). These relations imply at once \((\pi_0 T^{-n})^n = \pi_0 T^{-n}\).
Proposition 1. Let \( f \in L^2(X, \mathcal{A}_0, m) \). If the series \( \varphi = \sum_{k=0}^{\infty} \pi_0 T^{-k} f \) is \( L^2 \)-convergent then there exists \( h \in L^2(X, \mathcal{A}_0, m) \) such that \( f = h + (I - T)(\varphi - f) \); \( f \) and \( h \) are then said to be \( L^2 \)-cohomologous.

Proof: Let \( h = \varphi - \pi_{-1}\varphi; \) obviously \( h \in L^2(X, \mathcal{A}_0, m) \). Using the \( L^2 \)-convergence of the series and the intertwining relations between \( T \) and \( \pi_n \) we get:

\[
    h = \sum_{k=0}^{\infty} (\pi_0 - \pi_{-1}) T^{-k} f = f + \sum_{k=1}^{\infty} (\pi_0 - T\pi_0) T^{-k} f
\]

hence \( f = h + (T - I)(\varphi - f) \).

Corollary. Under the same assumptions the sequence \( (h_n T^n)_{n \geq 0} \) is a "martingale difference" sequence and the sequence \( \left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f_k T^k \right)_{n \geq 1} \) weakly converges to the Gaussian distribution \( N(0, \sigma^2) \) with

\[
    \sigma^2 = \int h^2 dm = \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} f_k T^k \right\|_2^2.
\]

If \( \sigma^2 = 0 \) the limit is the Dirac measure at 0 and \( f = (I - T)\psi \) with \( \psi \in L^2 \) (in which case \( f \) is called an \( L^2 \)-coboundary for \( T \)).

Proof. The martingale property is immediate since \( h_n T^n \in L^2(X, \mathcal{A}_n, m) \) and \( \pi_{-n-1} T^n h = T^n \pi_{-1} h = 0 \). By the central limit theorem for a stationary "martingale difference" sequence (in reversed form) recalled above we get the weak convergence of

\[
    \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} h_k T^k \text{ to } N(0, \sigma^2)
\]

with \( \sigma^2 = \int h^2 dm \).

Since

\[
    \sum_{k=0}^{n-1} f_k T^k = \sum_{k=0}^{n-1} h_k T^k + (T^n - I)(\varphi - f)
\]

and

\[
    \lim_{n \to \infty} \frac{1}{\sqrt{n}} \left\| (T^n - I)(\varphi - f) \right\|_2 = 0,
\]

the sequence \( \left( \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f_k T^k \right)_{n \geq 1} \) must weakly converge too, with the same limit, and the formula for the variance holds.

This elegant argument is the first step of the martingale method. In spite of its great simplicity, it yields interesting results for symbolic ergodic dynamical systems. Let us give an example for the multiplication by 2 transformation.

3.1. Example. Let \( T \) be the transformation of the unit interval \([0,1]\) defined by \( Tx = 2x \mod 1 \). It is obviously expanding. With respect to the Lebesgue measure \( m \) it defines an ergodic system that is isomorphic with the one-sided Bernoulli shift \( (\Omega = \{0,1\}^\mathbb{N}, P = (\frac{1}{2}(\delta_{0} + \delta_{1}))^\otimes \mathbb{N}, \theta) \) which represents the infinite "Head and Tail" game; the isomorphism being the correspondence between the number \( x \) and its infinite dyadic development.

We put \( X_n : \Omega \to \{0,1\} \) with \( X_n(\omega) = \omega_n \), the \( n \)th symbol of the sequence \( \omega \in \Omega \); we have \( X_n(\theta \omega) = X_n+1(\omega) \). The transformations \( T \) or \( \theta \) are not invertible but the preceding framework is still applicable since we shall take for the \( \sigma \)-algebra \( \mathcal{A}_0 \) the Borel \( \sigma \)-algebra on \([0,1]\) or the infinite product \( \sigma \)-algebra on \( \Omega \), and all the
functions that we shall consider will be $\mathcal{A}_0$-measurable (the introduction of the natural invertible extension of the system is not essential).

The question we consider now is the following: which condition on a real square integrable function $f$ defined on $[0,1]$ with $\int_{[0,1]} f(x) dx = 0$, shall be sufficient in order to get the central limit theorem for the sequence $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f_s T^k$ or, with an obvious abuse of notation, for $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f_s \theta^k$ when $f$ is defined on $\Omega$?

Here the operator $\pi_0 T^{-1}$ is given by $\pi_0 T^{-1}(f)(x) = \frac{1}{2} [f(x) + f(x + \theta)]$ on $[0,1]$ or by $\pi_0 T^{-1}(f) = E(f \theta^{-1} | \mathcal{A}_0)$ on $\Omega$; it is the adjoint of $T$. We have $\|\pi_0 T^{-k}(f)\|_2 = \|E(f | \mathcal{A}_-k)\|_2$ where as before $\mathcal{A}_-k = \theta^{-k} \mathcal{A}_0 = \sigma(X_k, X_{k+1}, ...)$, because of the invariance of the measure. The independance of the random variables $(X_n)_{n \geq 0}$ and Pythagoras theorem yield

$$\|E(f | \mathcal{A}_-k)\|_2^2 \leq \|f - E(f | X_0, ..., X_{k-1})\|_2^2.$$ 

Coming back to the interval $[0,1]$ the modulus of continuity $w_f$ of the function $f$ allows us to majorize this last quantity. If $w_f(\epsilon) = \sup_{|t-s| < \epsilon} |f(t) - f(s)|$ we get $\|\pi_0 T^{-k}(f)\|_2 \leq w_f(2^{-k+1})$.

The previous proposition requires the convergence of the series $\sum_{k=0}^{\infty} \pi_0 T^{-k}(f)$. Therefore the existence of a modulus of continuity of the kind $w_f(\epsilon) = O(|\log \epsilon|^{-\gamma})$ with $\gamma > 1$ together with the condition that $f$ is not an $L^2$-coboundary imply the validity for $f$ of the central limit theorem.

During the past last twenty years the preceding reasoning was extended by several authors in several directions. One recent improvement, still using the idea of the approximating martingale, yields a sufficient condition for the central limit theorem to hold for an abstract ergodic Markov chain without any nonsingularity assumption. Here is the precise statement “Let $P$ be a Markov operator on the probability space $(X, \mathcal{F}, m)$. Let us assume that $m$ is $P$-invariant and ergodic. Let $f \in L^2(X, F, m)$ with $\int f dm = 0$. If $\left\|\sum_{k=0}^{n-1} P_k f\right\|_2 = O(n^{\gamma-\epsilon})$ for some $\epsilon > 0$, then the sequence $(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(X_k))_n$ weakly converges to a Gaussian distribution, with respect to the stationary probability law of the Markov chain $(X_n)_{n \geq 0}$ induced by $P$ and $m$. This weak convergence holds also with respect to the probability law of the chain starting at $x$ for $m$-almost every $x \in X$.” (this is due to Maxwell and Woodroofe(2000) for the stationary part, to Lin and the present author(2003) for the a.e. part).

In the previous context the operator $\pi_0 T^{-1}$, associated to the choice of the sub-$\sigma$-algebra $\mathcal{A}_0$, defines a Markov operator $P$ on the probability space $(X, \mathcal{A}_0, m)$; it is ergodic. Moreover one can easily see, using a shift on the bilateral natural extension of the system $(X, \mathcal{A}_0, m, T)$, that the two sequences $(\sum_{k=0}^{n-1} f_s T^k)_{n \geq 0}$ and $(\sum_{k=0}^{n-1} f(X_k))_{n \geq 0}$ with $(X_k)_k$ the Markov chain associated to $P$, have the same probability distribution with respect to $m$ and therefore the central limit theorem holds.
for both if it holds for one. Using this result in the above example of the transformation $Tx = 2x \mod 1$ of $[0,1]$ we get that the central limit theorem can be valid for the function $f$ under the improved condition on the modulus of continuity: $w_f(\epsilon) = O(\log \epsilon^{-\gamma})$ with $\gamma > 1/2$.

**Remark.** In the preceding example, as in the general situation, it is important to know whether the function $f$ is a coboundary or not. This question is far from obvious.

A classical theorem of Kac says: "For the transformation $Tx = 2x \mod 1$, if the function $f$ has Fourier coefficients $c_n(f) \sim |n|^{-\alpha}$ with $\alpha > 1/2$ then $f$ is a coboundary if and only if $\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} f \circ T^k \right\|^2 = 0". This statement appears now as a direct corollary of the preceding reasoning.

There is also the classical criterion given by Livshits(1971): in the preceding example, or more generally on an hyperbolic system, if $f$ is a Hölder continuous function and if on every periodic orbit $f$ does sum to $0$, then $f$ is a coboundary (with a continuous cobounding function).

On a "concrete" dynamical system the martingale method can be applied only after the construction of the sub-$\sigma$-algebra $\mathcal{A}_0$ that will generate the increasing filtration with respect to which the martingale condition will be written. As a rule it is not an easy task.

In the sequel we will try to describe, without giving the detailed calculations, the application of this method to a non-uniformly hyperbolic automorphism of a torus following the work of Le Borgne (1999).

First of all an improved version of Proposition 1 is needed.

**Proposition 2.** With the same framework and notations as in Proposition 1, let $f \in L^2(X,\mathcal{F},m)$ be such that the two series $\sum_{n=0}^{\infty} \|\pi_n(f) - f\|_2$ and $\sum_{n=0}^{\infty} \|\pi_{-n}(f)\|_2$ converge. Then there exists $h \in L^2(X,\mathcal{A}_0,m) \oplus L^2(X,\mathcal{A}_{-1},m)$, the orthogonal complement of $L^2(X,\mathcal{A}_{-1},m)$ in $L^2(X,\mathcal{A}_0,m)$, and $\psi \in L^2(X,\mathcal{F},m)$ such that $f = h + (I-T)\psi$.

**Proof.** Proposition 1 applies to $\pi_0(f)$ hence $\pi_0(f) = h_0 + (I-T)\psi_0$ with $h_0 \in L^2(\mathcal{A}_0) \oplus L^2(\mathcal{A}_{-1})$ and $\psi_0 \in L^2(X,\mathcal{F},m)$. Now $\pi_n(f) - \pi_{-n}(f)$ for $n > 0$. Since $T^j(\pi_n(f) - \pi_{-n}(f)) = (\pi_0 - \pi_{-1})T^j f \in L^2(\mathcal{A}_0) \oplus L^2(\mathcal{A}_{-1})$ we get $\pi_n(f) = h_n + (I-T)\psi_n$ with $h_n = h_0 + \sum_{i=1}^{n} (\pi_0 - \pi_{-1})T^i f \in L^2(\mathcal{A}_0) \oplus L^2(\mathcal{A}_{-1})$ and $\psi_n = \psi_0 + \sum_{i=1}^{n} T^i(\pi_0 - \pi_{-1})f \in L^2$. Since $L^2 \lim_{n \to +\infty} \pi_n(f) = f$ by assumption, it is enough to prove that the sequence $(\psi_n)_{n>0}$ is converging in $L^2$ to get the desired result: $f = h + (I-T)\psi$ with $\psi = \lim_{n \to +\infty} \psi_n$ and $h = \lim_{n \to +\infty} h_n$.

A direct computation yields

$$\psi_{n+j+1} - \psi_n = \sum_{k=0}^{n-1} T^k(\pi_{n+j} - \pi_{n-1})f + \sum_{k=n}^{n+j} T^k(\pi_{n+j} - \pi_k)f$$
with \( j > 0 \). The operator \( T \) is an isometry therefore we get
\[
\|\psi_{n+j} - \psi_n\|_2 \leq \sum_{k=n}^{n+j} \|f - \pi_k(f)\|_2 + j \|f - \pi_{n+j}(f)\|_2 + n \|\pi_{n+j} - \pi_{n-1}\|_2.
\]
Since the sequence \( \|f - \pi_n(f)\|_2 \) is decreasing and \( \sum_{n=0}^{\infty} \|\pi_n(f) - f\|_2 < \infty \) by assumption, we have \( \lim_{n \to +\infty} n \|f - \pi_n(f)\|_2 = 0 \) therefore this inequality proves that the sequence \( (\psi_n)_{n>0} \) is a Cauchy sequence.

For such an \( f \) the corollary of proposition 1 remains obviously valid. A direct corollary of Proposition 2 is a central limit theorem for the baker’s transformation. Let us recall that it is the transformation \( T \) of \([0, 1] \times [0, 1]\) defined by:
\[
T(x, y) = \begin{cases} 
(2x \mod 1, y/2) & \text{for } 0 \leq x \leq 1/2 \\
(2x \mod 1, (y+1)/2) & \text{for } 1/2 \leq x \leq 1
\end{cases}
\]
It is isomorphic to the two-sided infinite Bernoulli shift; one can see it as the invertible natural extension of the transformation \( 2x \mod 1 \). The horizontal segments are dilated, the vertical ones are contracted. Taking for \( A_0 \) the \( \sigma \)-algebra of the Borel sets that depend only on the first coordinate \( x \) which are the sets parallel to the contracted direction, Proposition 2 applies. The central limit theorem follows for Hölderian functions as before.

### 3.2. Le Borgne’s example of a central limit theorem for a partially hyperbolic automorphism of a torus (1999).
To be specific we consider the automorphism \( T \) of the four-dimensional torus \( X = \mathbb{R}^4 / \mathbb{Z}^4 \) induced by the matrix
\[
M = \begin{pmatrix}
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 2
\end{pmatrix}
\]
with \( \lambda > 2, 3 \) and \( \lambda' = \lambda^{-1} \) and two conjugate complex eigenvalues of modulus 1 which are not roots of unity. Hence \( T \) leaves the Lebesgue measure \( m \) on \( X \) invariant and is ergodic (even mixing). The space \( \mathbb{R}^4 \) splits up into the direct sum of the three subspaces \( F_u, F_s, F_c \) which are respectively the one dimensional eigenspace associated to \( \lambda \), the one dimensional eigenspace associated to \( \lambda' \), and the two dimensional \( M \)-invariant subspace corresponding to the two complex eigenvalues in which \( M \) acts like an irrational rotation. The same notations will denote the quotient subspaces of the torus \( X = \mathbb{R}^4 / \mathbb{Z}^4 \). In the \( F_c \)-direction \( T \) is contracting, in the \( F_u \)-direction it is expanding. The lines parallel to \( F_s \) will be called contracting leaves.

The \( \sigma \)-algebra \( A_0 \) is built to describe the future of the trajectories of \( T \) with respect to a finite partition \( \mathcal{P} \) of \( X \) adapted to the splitting into expanding, contracting or neutral directions. An important point is that we do not require the partition to be Markovian. This partition \( \mathcal{P} \) is made of parallelepipeds whose sides are parallel to the subspaces \( F_u, F_s, F_c \) and of small diameters (the size can be arbitrary small).

Then the sequence of partitions of \( X \) defined by \( \mathcal{P}_0 = \mathcal{P} \vee T^{-1} \mathcal{P} \vee \ldots \vee T^{-n} \mathcal{P} \) is increasing, each one of them being made of parallelepipeds whose sides are parallel to the subspaces \( F_u, F_c, F_c \) because these subspaces are \( M \)-invariant. Taking the limit \( \mathcal{P}_0^\infty = \bigvee_0^\infty T^{-n} \mathcal{P} \) we get a “measurable partition”; the atom \( \mathcal{P}_0^\infty(x) \) of this partition containing a given \( x \in X \) is contained in the contracting leaf \( x + F_s \) passing through \( x \) because the component in the \( F_u \)-direction is squeezed by the factor \( \lambda^{-1} \) at each step backwards, and the component in the \( F_c \)-direction disappears because of the
ergodicity of the "rotation" acting on $F_x$. It is proved that this atom $P_0^\infty(x)$ is an "open interval" on the line $x + F_x$ for $m$-a. e. $x \in X$. The sub-$\sigma$-algebra $A_0$ is the $\sigma$-algebra generated by this measurable partition and $A_{-n} = T^{-n}A_0$ is generated by $P_n^\infty = \sqrt{n}T^{-k}P$ for $n \in \mathbb{Z}$; each atom $P_n^\infty(x)$ is an "open interval" on the line $x + F_x$ for $m$-a. e. $x \in X$. To get the conditional expectation of a function $f$ on $X$ with respect to the $\sigma$-algebra $A_{-n}$ we need the disintegration of the Lebesgue measure $m$; it is given by the one dimensional Lebesgue measure on the line $F_x$ according to the formula:

$$E[f \mid A_{-n}](x) = \frac{1}{|P_n^\infty(x)|} \int_{P_n^\infty(x)} f(t)dt$$

(1.9) $f$ (the absolute value and $dt$ refer to the Lebesgue measure on the line $F_x$).

Now, for a trigonometric monomial $f(x) = e^{2i\pi\langle p,x \rangle}$ where $p \in \mathbb{Z}^d$ and $\langle p,x \rangle$ is the scalar product, the two conditions of Proposition 2 have to be checked. They will depend on a quantitative estimate of the equirrepartment of the contracting leaves.

The atoms of $A_n$ are almost surely pieces of contracting leaves. Hence the length of the interval $P_n^\infty(x)$ on the line $x + F_x$ is $O(\lambda^{-n})$ when $n \to +\infty$, and the mean value inequality for $f(x) = e^{2i\pi\langle p,x \rangle}$ yields $\|E(f \mid A_n) - f\|_\infty = O(\lambda^{-n})$; thus $\sum_{n=0}^\infty \|\pi_n(f) - f\|_2 < \infty$.

To get the convergence $\sum_{n=0}^\infty \|\pi_n(f)\|_2 < \infty$ is somewhat more delicate. First of all using the irrationality of the lattice $F_x$ in $\mathbb{R}^d$, it is proved that for the distance in $\mathbb{R}^d$, $d(p, F_x) \geq K |p|^{-\xi}$ for every $p \in \mathbb{Z}^d$ with some constant $K$, $|p|$ denoting a norm on $\mathbb{R}^d$. Using this inequality it is possible to find a constant $0 < \gamma < 1$ such that for $f(x) = e^{2i\pi\langle p,x \rangle}$ and for every bounded interval $C$ in $x + F_x$:

$$\frac{1}{|T^{-n}C|} \int_{T^{-n}C} f(t)dt < \frac{K}{|C|} |p|^{3\gamma n}$$

with some other constant $K$. Since the atom of $P_n^\infty$ containing $x$ is $P_n^\infty(x) = T^{-n} P_0^\infty(T^n x)$ we get:

$$|E(f \mid A_{-n})(x)| \leq \frac{K}{|P_0^\infty(T^n x)|} |p|^{3\gamma n}.$$

By a geometric analysis of the size of the atoms $P_0^\infty(T^n x)$ showing that for most of the atoms $|P_0^\infty(T^n x)| \geq K^{\beta n}$ with $\beta > \gamma$, it is possible to deduce $\|E(f \mid A_{-n})\|_2 = O(\gamma^n)$.

Therefore proposition 2 applies: the central limit theorem holds for $f(x) = e^{2i\pi\langle p,x \rangle}$. Lifshits' criterion recalled in the remark above, shows that the limiting variance is strictly positive because $f$ is not a coboundary since $f(0) = 1$.

Some further estimations show that the same result holds for a Fourier series $f(x) = \sum_{p \in \mathbb{Z}^d} c_p e^{2i\pi\langle p,x \rangle}$ when $|c_p| \leq K(1 + |p|)^{-2} \ln^{-\theta}(|p|)$ for some $K > 0$ and $\theta > 2$.

We only gave a sketch of the reasoning. The full details are given by Le Borgne (1999). The result extends to any partially hyperbolic automorphism of a torus of any dimension.

Some additional comments. The value of Le Borgne’s example lies not only in the result but also in the method itself. When the automorphism is uniformly
hyperbolic like the well known example induced by \( \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \) in dimension 2, it is possible to define a coding of the system based on a Markov partition. Then other methods to get the central limit theorem become available, especially the method of the spectral gap (operator-characteristic function method) for which we refer to Guivarc’h and Hardy(1988); see also Hennion and Hervé(2001). But in the case of partially hyperbolic automorphism it is not known whether a Markov partition exists. The method of martingales can also be easier to apply even when Markov partitions do exist. Recently Conze and Le Borgne(2001) exploited this approach to give a new proof of the central limit theorem for Hölder continuous functions for the geodesic flow on a surface of constant negative curvature. Their paper might well give the most explicit and easiest access to Sinai’s theorem. However when it applies, the spectral gap method, based on a quasi-compactness argument, gives more precise limit theorems than just the central limit theorem. The functional form of the central limit theorem (also called ”invariance principle”) can be proved by both methods, but the spectral gap method gives a direct access to the speed of convergence (Berry-Esseen type theorem). About the problem of the speed of convergence we refer to the recent publications of Rio(2000) and Pène(2002).

Another method was devised by Le Jan(1994) to prove the central limit theorem for the geodesic flow on manifolds of constant negative curvature: its basic idea is to replace a geodesic line by a path of a Brownian motion on the manifold. A somewhat similar quite elementary idea appeared in the study of limit theorems for the random walk on the non abelian free group: there the path of the random walk is comparable to the geodesic line having the same end point.

Still about the central limit theorem the deep work of Petit(1996) on the transformations defined by the multiplication mod 1 by a non integral \( \theta \) should be mentioned.

The list is too long of the important questions we did not have the ability to write about. We just indicate a few important recent references. Chernov (1998) uses the notion of Markov approximation to obtain bounds on the decay of correlations for Anosov flows. Young (1998) introduces a representation of hyperbolic systems with singularities as factors of Markov towers to prove, under quite general assumptions, the exponential decay of correlations, which was somewhat unexpected. One dimensional maps with a neutral fixed point at 0 of the form \( x + x^{1+\alpha} \) with \( 0 < \alpha < 1 \), examples of abstract Markov maps, are studied by Liverani, Saussol and Vaienti (1999) and also more recently by Gouëzel (2004); for such a map, with \( 1/2 < \alpha < 1 \), and a Hölder observable Gouëzel proves a central limit theorem or a convergence to a stable distribution of index \( 1/\alpha \), according to the observable. Very recently Dolgopyat (2004) introduces the notion of Markov family, more flexible than the notion of Markov partition, to get limit theorems for general partially hyperbolic systems.

The expansion of the domain is quite fast; the publication during the last past years of these remarkable papers gives of it ample evidence.

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