Dark Matter, Dark Energy and the Chaplygin Gas

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March 25, 2019

Abstract
We formulate a Zel’dovich-like approximation for the Chaplygin gas equation of state $P = -A/\rho$, and sketch how this model unifies dark matter with dark energy in a geometric setting reminiscent of $M$-theory.

1 Introduction

In the last few years improved observations [1] have forced a shift in our cosmological paradigm: the $\Omega_M = 1$ dust model has been swept aside and, in its place, we are faced with the problem of understanding a universe with an equation of state $\bar{W} = \bar{P}/\bar{\rho} < -1/3$. That is to say, on average, pressure is comparable with density and, moreover, negative.

Of course, parametrically, this is readily accommodated by a cosmological constant $\Lambda$ [2] with $\Omega_\Lambda \simeq 0.7$ and $\Omega_{DM} = 1 - \Omega_\Lambda \simeq 0.3$ (throughout the paper we neglect the small baryonic contribution). The well-known difficulty with $\Lambda$ is that a priori it seems an incredible accident that $\Omega_\Lambda \simeq \Omega_{DM}$ since

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\[ \rho_\Lambda / \rho_{DM} \sim a^3, \ a \text{ being the scale factor.} \] Hence, much attention has been devoted to quintessence \cite{3}, involving a real scalar field which tracks \cite{4} the background component until recently becoming dominant. However, simple tracking quintessence does not work \cite{5} and spintessence \cite{6}, where the scalar field is complex, suffers instabilities against the decay of dark energy into dark matter \cite{7}.

It is natural to conjecture that some of the aforementioned problems derive from treating dark matter and dark energy as separate issues. As an example, Barr and Seckel \cite{8} have pointed out that in axion dark matter models quantum gravity effects break the Pecci-Quinn symmetry leading to a universe trapped in a false vacuum with an effective \( \Lambda \) of the correct magnitude. In another approach, Wetterich \cite{9} has suggested that traditional WIMP dark matter should be replaced by quintessence lumps, thus unifying dark matter and dark energy. However, the radiation-matter transition and structure formation remain open questions in this scenario.

Herein we present a dark matter-energy unification model suggested by the observation of Kamenshchik et al. \cite{10} that a perfect fluid obeying the Chaplygin gas equation of state

\[ P = -\frac{A}{\rho} \] (1)

should lead to a homogenous cosmology with

\[ \bar{\rho}(a) = \sqrt{A + \frac{B}{a^6}}, \] (2)

with \( B \) being an integration constant, thus interpolating between dark matter, \( \bar{\rho}(a \to 0) \simeq \sqrt{B}/a^3 \) and dark energy \( \bar{\rho}(a \to \infty) \simeq \sqrt{A} \). Before doing so, we must first show why Eq. (1), aside from its interesting mathematical features \[\square\], might describe reality.

## 2 Brane New World

One of the most profound recent developments in fundamental physics has been the recognition that all of the extra dimensions required by string/M-theory do not have to be of the Planck length size: one (or more) could be as large as 0.1 mm provided that all standard-model fields except gravity are
confined to a 3-dimensional hypersurface or ‘brane’ in the higher dimensional
bulk (for a review see [12]). In this context, Kamenshchik et al. [13] obtained
Eq. (1) from the stabilization of branes in black hole bulks.

A simple way to see the connection between the Chaplygin gas and the
brane world is to follow Sundrum’s [14] effective field theory for the 3-
brane. The gauge fixed embedding of a 3+1 brane in a 4+1 bulk is described by
\( Y_M = (x^\mu, Y^4) \). With some nominal assumptions on the bulk metric
\( G_{MN} \), the induced metric on the 3-brane is
\[
\tilde{g}_{\mu\nu} = g_{\mu\nu} - \theta_{,\mu} \theta_{,\nu},
\]
with \( 0 \leq \theta \leq \ell_5 \), and the action for the brane reads
\[
S_{\text{brane}} = \int d^4x \sqrt{-\tilde{g}} \left[ -f^4 + \cdots \right] = \int d^4x \sqrt{-\tilde{g}} \sqrt{1 - g_{\mu\nu} \theta_{,\mu} \theta_{,\nu}} \left[ -f^4 - \cdots \right],
\]

where \( f^4 \) is the brane tension and the ellipsis includes standard-model fields
as well as higher-order terms in power counting. One estimates \( f \sim \ell_5^{-1} \sim \text{meV} \).

Retaining only the leading term in Eq. (4) and renaming \( f^4 = \sqrt{A} \), one
sees that its content is equivalent to
\[
\mathcal{L} = \frac{\phi^2}{2} g_{\mu\nu} \theta_{,\mu} \theta_{,\nu} - V \left( \frac{\phi^2}{2} \right),
\]
\[
V = \frac{1}{2} \left( \phi^2 + \frac{A}{\phi^2} \right)
\]
since \( \phi \) can be eliminated through its field equation
\[
g^{\mu\nu} \theta_{,\mu} \theta_{,\nu} = V'.
\]

We observe that \( \mathcal{L} \) corresponds to the Lagrangian for a complex field \( \Phi = \phi e^{-im\theta}/\sqrt{2m} \) in the ‘Thomas-Fermi’ approximation. The Thomas-Fermi ap-
proximation amounts to neglecting \( \phi_{,\mu}/m\phi \) compared to \( \sqrt{V'/\phi^2} \), i.e., the
scale of variation of \( \phi \) is large compared to the Compton wavelength. It is
also worth noting that dividing \( \mathcal{L} \) by \( \sqrt{A} \), the first term is the periodic Gauss-
ian model with coupling \( R = A^{1/4}/\phi \). The potential \( F = (R^2 + R^{-2})/2 \),
which is self-dual, can be interpreted as the mean field free-energy for ‘brane
cells’ filling a system of size $R$, in analogy to Rama’s [15] ‘string bit’ analysis of black holes.

To complete the connection to the Chaplygin gas, we point out a field-fluid duality: for $V’ > 0$, Eq. (7) defines a fluid 4-velocity $U_{\mu}U^\mu = 1$,

$$U^\mu = g^\mu\nu\theta_{,\nu}/\sqrt{V’}$$  \hspace{1cm} (8)

and then the energy-momentum tensor derived from $\mathcal{L}$ takes the perfect fluid form with

$$\rho = \frac{\phi^2}{2}V’ + V, \quad P = \frac{\phi^2}{2}V’ - V.$$  \hspace{1cm} (9)

In particular, for $V$ of Eq. (6), the equation of state (1) follows, and the energy density $\rho$ is given by

$$\rho = \phi^2 = \frac{\sqrt{A}}{\sqrt{1 - g^\mu\nu\theta_{,\mu}\theta_{,\nu}}},$$  \hspace{1cm} (10)

which is to say that matter corresponds to a wrinkled brane.

Finally, it must be said that the procedure can be reversed [16]. The equations

$$d\ln(\phi^2) = \frac{d\rho - dP}{\rho + P}, \quad V = \frac{1}{2}(\rho - P),$$  \hspace{1cm} (11)

obtained from Eq. (9), allow one to construct $\phi^2$ and $V$ given the equation of state. As an example, starting from $0 \geq W = P/\rho \geq -1$ and $0 \leq c_s^2 = dP/d\rho \leq 1$, with the relativistic limits coinciding, Eqs. (1), (6), and (4) follow.

### 3 The Inhomogeneous Chaplygin Gas

As yet, we have not dealt with the $\theta$ field equation; in the fluid language, it reads

$$\left(\sqrt{-g}\phi^2(\rho + P)U_{\mu}\right)_{,\mu} = 0.$$  \hspace{1cm} (12)

In comoving coordinates, $U_{\mu} = (1/\sqrt{g_{00}}, 0)$, the solution for the Chaplygin gas is [16]

$$\rho = \sqrt{A + \frac{B}{\gamma}}.$$  \hspace{1cm} (13)
Here $\gamma = -g/g_{00}$ is the determinant of the induced metric $\gamma_{ij} = g_{00}g_{ij}/g_{00} - g_{ij}$ which measures physical distances, and $B = B(\vec{x})$ can be taken as constant on the scales of interest.

The generalization \(13\) of Eq. \(2\) allows us to implement the geometric version \(17\) of the Zel’dovich approximation \(18\): the transformation from Lagrange to Euler (comoving) coordinates induces $\gamma_{ij}$ as

$$\gamma_{ij} = \delta_{kl}D_i^kD_j^l,$$  

(14)  

with $D_i^j$ the deformation tensor, $\varphi$ the velocity potential. Inserting this ansatz in the 0-0 Einstein equation to first order in $\varphi$ yields the evolution equation for $b(a)$

$$\frac{2}{3}a^2b'' + a(1 - \bar{w})b' = (1 + \bar{w})(1 - 3\bar{w})b,$$  

(15)  

$$\bar{w}(a) = -\frac{\Omega_\Lambda a^6}{1 - \Omega_\Lambda + \Omega_\Lambda a^6},$$  

(16)  

where we match the parameters $A, B$ to the $\Lambda$ model.

In Fig. 1 we show the evolution of $b(a)$ for the Chaplygin gas and for $\Lambda$ cold dark matter, the latter following by omitting the $(1 - 3\bar{w})$ factor in Eq. \(15\) and changing $a^6$ to $a^3$ in Eq. \(16\). In either case, the growth $b \propto a$ ceases near $a = 1$ and although $b$ remains constant, the perturbative density contrast $\delta_{\text{pert}} = b(1 + \bar{w})\varphi_{,i}^i$ thereafter vanishes as $\delta_{\text{pert}}(a \gg 1) \sim a^{-6}$.

Of course the value of the Zel’dovich approximation is that it offers a means of extrapolation into the nonperturbative regime via Eqs. \(13\) and

$$\sqrt{\gamma} = a^3(1 - \lambda_1b)(1 - \lambda_2b)(1 - \lambda_3b),$$  

(17)  

where the $\lambda_i$ are the eigenvalues of $\varphi_{,i}^i$. When one (or more) of the $\lambda$’s is positive, a caustic forms on which $\gamma \rightarrow 0$ and $w \rightarrow 0$, i.e., at the locations where structure forms the Chaplygin gas behaves as dark matter. Conversely, when all of the $\lambda$’s are negative, a void forms, $\rho$ is driven to its limiting value $\sqrt{A}$, and the Chaplygin gas behaves as dark energy driving accelerated expansion.

4 Discussion and Conclusions

A shortcoming of the Zel’dovich approximation is that at the caustic matter flows through unimpeded so that structures quickly dissolve \(19\). This may
Figure 1: Evolution of $b(a)/b(a_{eq})$ from $a_{eq} = 1.0 \times 10^{-4}$ for $\Omega_{\Lambda} = 0.7$ and $b'(a_{eq}) = 0$, for the Chaplygin gas (solid line) and $\Lambda$CDM (dashed line).

be circumvented via the truncated Zel’dovich approximation [19]. A preferable alternative would be an extension of the adhesion approximation [20] which also allows the extraction of mass functions.

Approximation technicalities aside, the case is made that the Chaplygin gas offers a realistic unified model of dark matter and dark energy. That this is achieved in a geometric (brane world) setting rooted in string/$M$ theory makes this model all the more remarkable.

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