Space-Fractional Skellam Process

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Abstract

In this article, we propose space-fractional Skellam process and tempered space-fractional Skellam process via time changes in Poisson and Skellam processes by independent $\alpha$-stable subordinator and tempered stable subordinator, respectively. Further we derive the marginal probabilities, Lévy measure, governing difference-differential equations of the introduced processes. At last, we give the algorithm to simulate the sample paths of these processes.

Key words: Skellam process, subordination, Lévy measure, tempered stable subordinator.

1 Introduction

Skellam distribution is obtained by taking the difference between two independent Poisson distributed random variables which was introduced for the case of different intensities $\lambda_1, \lambda_2$ by Skellam (1946) and for equal means by Irwin (1937). For large values of $\lambda_1 + \lambda_2$, the distribution can be approximated by the normal distribution and if $\lambda_2$ is very close to 0, then the distribution tends to a Poisson distribution with intensity $\lambda_1$. Similarly, if $\lambda_1$ tends to 0, the distribution tends to a Poisson distribution with non-negative integer values. The Skellam random variable is infinitely divisible since it is the difference of two infinitely divisible random variables (see Steutel and Van Harn, 2004, Prop. 2.1, p.18). Therefore, one can define a continuous time Lévy process for Skellam distribution which is called Skellam process.

The Skellam process is the integer valued Lévy process and can also be obtained by taking the difference of two independent Poisson processes which marginal probability mass function (PMF) involves the modified Bessel function of the first kind. Skellam process has various applications in different areas such as to model the intensity difference of pixels in cameras (see Hwang et al 2007) and for modeling the difference of the number of goals of two competing teams in football game (see Karlis and Ntzoufras 2008). The model based on the difference of two point processes are proposed in (see Bacry et al. 2013; Barndroff-Neilsen et al. 2011; Carr 2011).

Recently, Kress et al. (2014) have studied time-fractional Skellam processes which is obtained by time-changing the Skellam process with an inverse stable subordinator. Further, they provided the application of time-fractional Skellam process in modeling of arrivals of jumps in high frequency trading data. It is shown that the inter arrival times between the positive and negative jumps follow Mittag-Leffler distribution rather then the exponential distribution. Similar observations are
observed in case of Danish fire insurance data (Kumar et al. 2019). Buchak and Sakhno (2018) also have proposed the governing equations for time-fractional Skellam processes.

In this paper, we introduce the space-fractional Skellam processes $S(D_\alpha(t)), t \geq 0$, where $D_\alpha(t)$ is the $\alpha$-stable subordinator with $\alpha \in (0, 1)$, independent of Skellam process $S(t)$ which is the difference of two independent Poisson processes. Space-fractional Skellam process is a Lévy process since it is the composition of two Lévy processes. However the time-fractional Skellam process is not a Lévy process. We extend the space-fractional Skellam process by considering a tempered space-fractional Skellam process. The tempering introduces a finite moments condition in space-fractional Skellam process. We obtain closed form expressions for the marginal distributions of the considered time-changed processes and their first and second order moments.

The remainder of this paper proceeds as follows: in section 2, we introduce all relevant definitions and results. We derive also the Lévy density for space- and tempered space-fractional Poisson processes. In Section 3, we determine the marginal PMF, governing equations for marginal PMF, Lévy densities and moment generating functions for Skellam processes time-changed by a $\alpha$-stable subordinator. In section 4, we introduce the tempered space-fractional Skellam process and its properties. Section 5 deals with the simulation of sample paths of introduced processes.

2 Preliminaries

In this section, we collect relevant definitions and some results on Skellam process, subordinators, space-fractional Poisson process and tempered space-fractional Poisson process. These results will be used to define the space-fractional Skellam processes and tempered space-fractional Skellam processes.

2.1 Skellam process

In this section, we revisit the Skellam process and also provide a characterization of it. Let $S(t)$ be a Skellam process, such that

$$S(t) = N_1(t) - N_2(t), \quad t \geq 0,$$

where $N_1(t)$ and $N_2(t)$ are two independent homogeneous Poisson processes with intensity $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. The Skellam process is defined in Barndorff-Nielsen et al. (2011) and the distribution has been introduced and studied in Skellam (1946), see also Irwin (1937). This process is symmetric only when $\lambda_1 = \lambda_2$. The PMF $s_k(t) = P(S(t) = k)$ of $S(t)$ is given by (see e.g. Skellam, 1946; Kress et al. 2014)

$$s_k(t) = e^{-t(\lambda_1 + \lambda_2)} \left( \frac{\lambda_1}{\lambda_2} \right)^{k/2} I_{|k|} \left( 2t \sqrt{\lambda_1 \lambda_2} \right), \quad k \in \mathbb{Z},$$

where $I_k$ is modified Bessel function of first kind (see Abramowitz and Stegun, p. 375),

$$I_k(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+k}}{n!(n+k)!}.$$
The PMF $s_k(t)$ satisfies the following differential difference equation (see Kress et al. 2014)

$$
\frac{d}{dt}s_k(t) = \lambda_1(s_{k-1}(t) - s_k(t)) - \lambda_2(s_k(t) - s_{k+1}(t)), \ k \in \mathbb{Z},
$$

with initial conditions $s_0(0) = 1$ and $s_k(0) = 0, \ k \neq 0$. The Skellam process is a Lévy process, its Lévy density $\nu_S$ is the linear combination of two Dirac delta function, $\nu_S(y) = \lambda_1\delta_{1}(y) + \lambda_2\delta_{-1}(y)$ and the corresponding Lévy exponent is given by

$$
\phi_{S(1)}(\theta) = \int_{-\infty}^{\infty} (1 - e^{-\theta y})\nu(y)dy.
$$

The moment generating function (MGF) of Skellam process is

$$
E[e^{\theta S(t)}] = e^{-t(\lambda_1 + \lambda_2 - \lambda_1 \theta e^{-\theta} - \lambda_2 e^{-\theta})}, \ \theta \in \mathbb{R}.
$$

With the help of MGF, one can easily find the moments of Skellam process. In next result, we give a characterization of Skellam process, which is not available in literature as per our knowledge.

**Theorem 2.1.** Suppose an arrival process has the independent and stationary increments and also satisfies the following incremental condition, then the process is Skellam.

$$
\mathbb{P}(N(t, t + \delta) = k) \approx \begin{cases} 
\lambda_1 \delta & \text{if } k = 1 \\
\lambda_2 \delta & \text{if } k = -1 \\
1 - \lambda_1 \delta - \lambda_2 \delta & \text{if } k = 0 \\
0 & \text{otherwise.}
\end{cases}
$$

**Proof.** Consider the interval $[0,t]$ which is discretized with $n$ sub-intervals of size $\delta$ each such that $n\delta = t$. For $k \geq 0$, we have

$$
\mathbb{P}(N(0, t) = k) = \sum_{m=0}^{[n-k]} \frac{n!}{m!(m+k)!(n-2m-k)!} (\lambda_1 \delta)^{m+k} (\lambda_2 \delta)^m (1 - \lambda_1 \delta - \lambda_2 \delta)^{n-2m-k}
$$

$$
= \sum_{m=0}^{[n-k]} \frac{n!}{m!(m+k)!(n-2m-k)!} \left( \frac{\lambda_1 t}{n} \right)^{m+k} \left( \frac{\lambda_2 t}{n} \right)^m (1 - \frac{\lambda_1 t}{n} - \frac{\lambda_2 t}{n})^{n-2m-k}
$$

$$
= \sum_{m=0}^{[n-k]} \frac{(\lambda_1 t)^{m+k}(\lambda_2 t)^m}{m!(m+k)!(n-2m-k)!n^{2m+k}} (1 - \frac{\lambda_1 t}{n} - \frac{\lambda_2 t}{n})^{n-2m-k}
$$

$$
= e^{-(\lambda_1 + \lambda_2)t} \sum_{m=0}^{[n-k]} \frac{(\lambda_1 t)^{m+k}(\lambda_2 t)^m}{m!(m+k)!}
$$

by taking $n \to \infty$. The result follows now by using the definition of modified Bessel function of first kind $I_k$. Similarly, we prove when $k < 0$. \qed
2.2 Subordinators

In this section, we introduce some known results of $\alpha$-stable subordinator $D_\alpha(t)$ and tempered $\alpha$-stable subordinator (TSS) $D_{\alpha,\mu}(t)$. The class of stable distributions $D(\alpha, \beta, \xi, \sigma)$ is characterized by four parameters: the stability parameter $\alpha \in (0, 2]$, the skewness parameter $\beta \in [-1, 1]$, location parameter $\xi \in \mathbb{R}$ and the shape parameter $\sigma > 0$. Stable distributions are infinitely divisible and hence also provide a rich class of Lévy processes which are called stable Lévy processes. Stable process has self-similar property i.e.

$$D_\alpha(ct) = c^{1/\alpha}D_\alpha(t), \ c > 0.$$ 

In general, subordinator is a non-negative, non-decreasing Lévy process (see Applebaum, 2009). The one-sided stable Lévy process denoted by $D_\alpha(t)$, where the increments $D_\alpha(t) - D_\alpha(s)$ has a stable distribution with parameters $\alpha \in (0, 1)$, $\beta = 0$, $\xi = 0$ and $\sigma = (|t - s|)^{1/\alpha}$, is called an $\alpha$-stable subordinator. Let $f(x, 1)$ be the density of $D_\alpha(1)$ which has infinite-series form (see Feller, 1971, p. 583; Aletti, 2018),

$$f(x, 1) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{1}{x^{k\alpha+1}} \sin(\pi \alpha k) = \frac{1}{u} W_{-\alpha,0} \left( \frac{1}{x^\alpha} \right), \ x > 0, \quad (2.4)$$

where $W_{\gamma,\beta}(z)$ is the Wright’s generalized Bessel function (see Haubold et al. 2011),

$$W_{\gamma,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n)\Gamma(\beta + \gamma n)}, \ z \in \mathbb{C}, \ \gamma > -1, \ \beta \in \mathbb{R}.$$ 

Let $f(x, t)$ be the PDF of $D_\alpha(t)$, which has Laplace transform (LT) (see Samorodnitsky and Taqqu, 1994)

$$E(e^{-uD_\alpha(t)}) = L_x(f(x, t)) = \int_0^{\infty} e^{-ux}f(x, t)dx = e^{-tu^\alpha}, \ u > 0. \quad (2.5)$$

For stable subordinator, the integer order moments are infinite. To overcome this limitation the tempered stable subordinators (TSS) are introduced in Rosinski (2007). The TSS $D_{\alpha,\mu}(t)$ with tempering parameter $\mu > 0$ and stability index $\alpha \in (0, 1)$, is also a Lévy process. The probability density function $f_{\alpha,\mu}(x, t)$ for $D_{\alpha,\mu}(t)$ is given by (see Rosinski, 2007)

$$f_{\alpha,\mu}(x, t) = e^{-\mu x + \mu^\alpha t}f(x, t), \ x > 0, \ \mu > 0, \ \alpha \in (0, 1). \quad (2.6)$$

Further, the LT of $D_{\alpha,\mu}(t)$ (see Meerschaert et al., 2013)

$$L_x(f_{\alpha,\mu}(x, t)) = e^{-t((u+\mu)^\alpha - \mu^\alpha)}. $$

Due to the exponential tempering in the distribution $\alpha$-stable subordinator, all order moments of TSS are finite and it is infinitely divisible. However it does not have the self-similarity property. The Lévy density corresponding to TSS is given by (see e.g. Cont and Tankov, 2004, p. 115)

$$\nu_{D_{\alpha,\mu}}(x) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{e^{-\mu x}}{x^{\alpha+1}}, \ x > 0. \quad (2.7)$$
2.3 The space-fractional Poisson process

In this section, we discuss main properties of space-fractional Poisson process (SFPP). We also provide the Lévy density for SFPP which is not discussed in the literature. The SFPP $N_{\alpha}(t)$ was introduced by Orshinger and Polito (2012), as follows

$$N_{\alpha}(t) = \begin{cases} N(D_{\alpha}(t)), & t \geq 0, \quad 0 < \alpha < 1, \\ N(t), & t \geq 0, \quad \alpha = 1. \end{cases} \quad (2.8)$$

The probability generating function (PGF) of this process is of the form

$$G^\alpha(u,t) = \mathbb{E}u^{N_{\alpha}(t)} = e^{-\lambda^\alpha(1-u)^\alpha t}, \quad |u| \leq 1, \quad \alpha \in (0,1). \quad (2.9)$$

The PMF of SFPP is

$$P^\alpha(k,t) = \mathbb{P}\{N_{\alpha}(t) = k\} = \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha)^r t^r}{(r!)} \frac{\Gamma(ra + 1)}{\Gamma(ra - k + 1)}$$

$$= \frac{(-1)^k}{k!} \psi_1 \left[ \begin{array}{c} (1, \alpha); \\ (1-k, \alpha); \end{array} (-\lambda^\alpha t) \right], \quad (2.10)$$

where $\psi_1(z)$ is the Fox Wright function (see Kilbas et al. 2006, p. 56, formula (1.11.14)). It was shown in Orshinger and Polito (2012) that the PMF of the SFPP satisfies the following fractional differential-difference equations

$$\frac{d}{dt}P^\alpha(k,t) = -\lambda^\alpha(1-B)^\alpha P^\alpha(k,t), \quad \alpha \in [0,1], \quad k = 1, 2, \ldots \quad (2.11)$$

$$\frac{d}{dt}P^\alpha(0,t) = -\lambda^\alpha P^\alpha(0,t), \quad (2.12)$$

with initial conditions

$$P(k,0) = \delta_k(0) = \begin{cases} 0, & k \neq 0, \\ 1, & k = 0. \end{cases} \quad (2.13)$$

The fractional difference operator

$$(1-B)^\alpha = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j B^j \quad (2.14)$$

is defined in Beran (1994), where $B$ is the backward shift operator. The characteristic function of SFPP is

$$\mathbb{E}[e^{i\theta N_{\alpha}(t)}] = e^{-\lambda^\alpha(1-e^{i\theta})^\alpha t}. \quad (2.15)$$

**Proposition 2.1.** The Lévy density $\nu_{N_{\alpha}}(x)$ of SFPP is given by

$$\nu_{N_{\alpha}}(x) = \lambda^\alpha \sum_{n=1}^{\infty} (-1)^{n+1} \binom{\alpha}{n} \delta_{n}(x). \quad (2.16)$$
Proof. We use Lévy-Khintchine formula (see Sato, 1999),

\[
\begin{align*}
\int_{\{0\}^c} (e^{i\theta x} - 1) \lambda^\alpha \sum_{n=1}^\infty (-1)^{n+1} \left( \frac{\alpha}{n} \right) \delta_n(x) dx \\
= \lambda^\alpha \left[ \sum_{n=1}^\infty (-1)^{n+1} \left( \frac{\alpha}{n} \right) e^{i\theta n} + \sum_{n=0}^\infty (-1)^n \left( \frac{\alpha}{n} \right) - 1 \right] \\
= \lambda^\alpha \sum_{n=0}^\infty (-1)^{n+1} \left( \frac{\alpha}{n} \right) e^{i\theta n} = -\lambda^\alpha (1 - e^{i\theta})^\alpha,
\end{align*}
\]

which is the characteristic exponent of SFPP from equation (2.15).

2.4 Tempered space-fractional Poisson process

The tempered space-fractional Poisson process (TSFPP) can be obtained by subordinating homogeneous Poisson process \( N(t) \) with the independent tempered stable subordinator \( D_{\alpha,\mu}(t) \) (see Gupta et al. 2019)

\[
N_{\alpha,\mu}(t) = N(D_{\alpha,\mu}(t)), \quad \alpha \in (0, 1), \quad \mu > 0.
\]

This process have finite integer order moments due to the tempered \( \alpha \)-stable subordinator. The PMF of TSFPP is given by (see Gupta et al. 2019)

\[
P_{\alpha,\mu}(k,t) = (-1)^k e^{t\mu^\alpha} \sum_{m=0}^\infty \frac{\mu^m}{m!} \lambda^\alpha \sum_{r=0}^\infty \frac{(-t)^r}{r!} \lambda^r (\alpha r - m) \binom{\alpha r - m}{k}
\]

\[
= e^{t\mu^\alpha} \frac{(-1)^k}{k!} \sum_{m=0}^\infty \frac{\mu^m \lambda^m}{m!} \left[ (1, \alpha); (1 - k - m, \alpha); (-\lambda^\alpha t) \right], \quad k = 0, 1, \ldots.
\]

The governing difference-differential equation is given by

\[
\frac{d}{dt} P_{\alpha,\mu}(k,t) = -((\mu + \lambda(1 - B))^\alpha - \mu^\alpha) P_{\alpha,\mu}(k,t).
\]

The characteristic function of TSFPP,

\[
E[e^{i\theta N_{\alpha,\mu}(t)}] = e^{-t((\mu + \lambda(1 - e^{i\theta}))^\alpha - \mu^\alpha)}.
\]

Using a standard conditioning argument, the mean and variance of TSFPP are given by

\[
E(N_{\alpha,\mu}(t)) = \lambda \alpha \mu^{\alpha-1} t, \quad \text{Var}(N_{\alpha,\mu}(t)) = \lambda \alpha \mu^{\alpha-1} t + \lambda^2 \alpha (1 - \alpha) \mu^{\alpha-2} t.
\]

Proposition 2.2. The Lévy density \( \nu_{N_{\alpha,\mu}}(x) \) of TSFPP is

\[
\nu_{N_{\alpha,\mu}}(x) = \sum_{n=1}^\infty \mu^{\alpha-n} \binom{\alpha}{n} \lambda^n \sum_{l=1}^n \binom{n}{l} (-1)^{l+1} \delta_l(x), \quad \mu > 0.
\]
Proof. Using \cite{220}, the characteristic exponent of TSFPP is given by $F(\theta) = ((\mu + \lambda (1 - e^{i\theta}))^\alpha - \mu^\alpha)$. We find the Lévy density with the help of Lévy-Khintchine formula (see Sato 1999),

$$
\int_{[0,\infty)} e^{i\theta x} - 1) \sum_{n=1}^{\infty} \mu^{\alpha-n} \left( \frac{\alpha}{n} \right) \lambda^n \left( \sum_{l=1}^{n} \left( \frac{n}{l} \right) (-1)^{l+1} \delta_l(x) dx \right.
\]

$$
= \sum_{n=1}^{\infty} \mu^{\alpha-n} \left( \frac{\alpha}{n} \right) \lambda^n \left( \sum_{l=1}^{n} \left( \frac{n}{l} \right) (-1)^{l+1} e^{i\theta x} - \sum_{l=1}^{n} \left( \frac{n}{l} \right) (-1)^{l+1} \right)
$$

$$
= \sum_{n=0}^{\infty} \mu^{\alpha-n} \left( \frac{\alpha}{n} \right) \lambda^n \sum_{l=0}^{n} \left( \frac{n}{l} \right) (-1)^{l+1} \delta_l(x) - \mu^\alpha
$$

$$
= -((\mu + \lambda (1 - e^{i\theta}))^\alpha - \mu^\alpha),
$$

hence proved.

\[ \square \]

3 The space-fractional Skellam processes

In this section, we introduce two types of space-fractional Skellam processes (SFSP), which we call (SFSP-I) and (SFSP-II). Further, for introduced processes, we study main results such as state probabilities and governing difference-differential equations of marginal PMF.

3.1 Space-fractional Skellam process type - I (SFSP-I)

Let $N_1(t)$ and $N_2(t)$ be two independent homogeneous Poisson processes with intensities $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. Let $D_{\alpha_1}(t)$ and $D_{\alpha_2}(t)$ be two independent stable subordinators with indices $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1)$ respectively. These subordinators are independent of the Poisson processes $N_1(t)$ and $N_2(t)$. The subordinated stochastic process

$$
S_{\alpha_1, \alpha_2}(t) = N_1(D_{\alpha_1}(t)) - N_2(D_{\alpha_2}(t))
$$

is called a SFSP-I. Next we derive the moment generating function (MGF) of SFSP-I. We use the expression for marginal (PMF) of SFPP given in (2.10) to obtain the marginal PMF of SFSP-I.

$$
M_\theta(t) = \mathbb{E}[e^{\theta S_{\alpha_1, \alpha_2}(t)}] = e^{\theta(N_1(D_{\alpha_1}(t)) - N_2(D_{\alpha_2}(t)))} = e^{-t(\lambda_1^{\alpha_1}(1-e^{i\theta})^{\alpha_1} + \lambda_2^{\alpha_2}(1-e^{-i\theta})^{\alpha_2})}, \ \theta \in \mathbb{R}.
$$

In the next result, we obtain the state probabilities of the SFSP-I.

Theorem 3.1. The PMF $H_k(t) = \mathbb{P}(S_{\alpha_1, \alpha_2}(t) = k)$ of SFSP-I is given by

$$
H_k(t) = \sum_{n=0}^{\infty} \frac{(-1)^k}{n!(n+k)!} \left( \psi_1 \left( \begin{array}{c} (1, \alpha_1); \\ (1-n, \alpha_1); \end{array} \right) \left( \begin{array}{c} (1, \alpha_2); \\ (1-n, \alpha_2); \end{array} \right) \right) \mathbb{I}_{k \geq 0}
$$

$$
+ \sum_{n=0}^{\infty} \frac{(-1)^{|k|}}{n!(n+|k|)!} \left( \psi_1 \left( \begin{array}{c} (1, \alpha_1); \\ (1-n, \alpha_1); \end{array} \right) \left( \begin{array}{c} (1, \alpha_2); \\ (1-n, \alpha_2); \end{array} \right) \right) \mathbb{I}_{k < 0}
$$

(3.23)

for $k \in \mathbb{Z}$. 

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**Proof.** Note that \( N_1(D_{\alpha_1}(t)) \) and \( N_2(D_{\alpha_2}(t)) \) are independent, hence

\[
\mathbb{P}(S_{\alpha_1,\alpha_2}(t) = k) = \sum_{n=0}^{\infty} \mathbb{P}(N_1(D_{\alpha_1}(t)) = n + k)\mathbb{P}(N_2(D_{\alpha_2}(t)) = n)\mathbb{I}_{k \geq 0}
\]

\[
+ \sum_{n=0}^{\infty} \mathbb{P}(N_1(D_{\alpha_1}(t)) = n)\mathbb{P}(N_2(D_{\alpha_2}(t)) = n + |k|)\mathbb{I}_{k < 0}.
\]

Using (2.10), the result follows. \( \square \)

In the next theorem, we discuss the governing differential-difference equation of the marginal PMF of SFSP-I. Let \( B \) be the backward shift operator defined in (2.14) and \( F \) be the forward shift operator defined by \( F^j X(t) = X(t + j) \) such that \( (1 - F)^{\alpha} = \sum_{j=0}^{\infty} \binom{\alpha}{j} F^j \).

**Theorem 3.2.** The marginal distribution \( H_k(t) = \mathbb{P}(S_{\alpha_1,\alpha_2}(t) = k) \) of SFSP-I satisfy the following differential difference equations

\[
\frac{d}{dt}H_k(t) = -\lambda_1^{\alpha_1} (1 - B)^{\alpha_1} H_k(t) - \lambda_2^{\alpha_2} (1 - F)^{\alpha_2} H_k(t), \quad k \in \mathbb{Z} \tag{3.24}
\]

\[
\frac{d}{dt}H_0(t) = -\lambda_1^{\alpha_1} H_0(t) - \lambda_2^{\alpha_2} H_1(t), \tag{3.25}
\]

with initial conditions \( H_0(0) = 1 \) and \( H_k(0) = 0 \) for \( k \neq 0 \).

**Proof.** The proof follows by using probability generating function. \( \square \)

**Remark 3.1.** The MGF of the SFSP-I solves the differential equation

\[
\frac{dM_\theta(t)}{dt} = -M_\theta(t)(\lambda_1^{\alpha_1}(1 - e^{\theta})^{\alpha_1} + \lambda_2^{\alpha_2}(1 - e^{-\theta})^{\alpha_2}). \tag{3.26}
\]

**Proposition 3.1.** The Lévy density \( \nu_{S_{\alpha_1,\alpha_2}}(x) \) of SFSP-I is given by

\[
\nu_{S_{\alpha_1,\alpha_2}}(x) = \lambda_1^{\alpha_1} \sum_{n_1=1}^{\infty} (-1)^{n_1+1} \binom{\alpha_1}{n_1} \delta_{n_1}(x) + \lambda_2^{\alpha_2} \sum_{n_2=1}^{\infty} (-1)^{n_2+1} \binom{\alpha_2}{n_2} \delta_{n_2}(x).
\]

**Proof.** Substituting the Lévy densities \( \nu_{N_{\alpha_1}}(x) \) and \( \nu_{N_{\alpha_2}}(x) \) of \( N_1(D_{\alpha_1}(t)) \) and \( N_2(D_{\alpha_2}(t)) \), respectively from the equation (2.10), we obtain

\[
\nu_{S_{\alpha_1,\alpha_2}}(x) = \nu_{N_{\alpha_1}}(x) + \nu_{N_{\alpha_2}}(x),
\]

which gives the desired result. \( \square \)

### 3.2 Space-fractional Skellam process type - II (SFSP-II)

Let \( S(t) = N_1(t) - N_2(t), \quad t \geq 0 \) be a Skellam process and \( D_{\alpha}(t) \) be the \( \alpha \)-stable subordinator with \( \alpha \in (0, 1) \) independent of Poisson processes \( N_1(t) \) and \( N_2(t) \). We called the stochastic process

\[
S_{\alpha}(t) = S(D_{\alpha}(t))
\]
a space-fractional Skellam process of type 2 (SFSP-II). Let \( f(x, t) \) be the density function of \( \alpha \)-stable subordinator. The MGF of SFSP-II can be obtained as follows.

\[
L_\theta(t) = \mathbb{E}(e^{\theta S_\alpha(t)}) = \int_0^\infty \mathbb{E}[e^{\theta S(u)}]f(u, t)du
\]

\[
= \int_0^\infty e^{-u(\lambda_1 + \lambda_2 - \lambda_1 e^\theta - \lambda_2 e^{-\theta})} f(u, t)du
\]

\[
= e^{-t(\lambda_1 + \lambda_2 - \lambda_1 e^\theta - \lambda_2 e^{-\theta})}\alpha,
\]

which satisfies the following differential equation,

\[
\frac{d}{dt}L_\theta(t) = -L_\theta(t)(\lambda_1 + \lambda_2 - \lambda_1 e^\theta - \lambda_2 e^{-\theta})\alpha, \quad L_\theta(0) = 1.
\]

**Theorem 3.3.** The marginal PMF \( R_k(t) = \mathbb{P}(S_\alpha(t) = k) \) of SFSP-II is given by

\[
R_k(t) = \frac{1}{u} \left( \frac{\lambda_1}{\lambda_2} \right)^{k/2} \int_0^\infty e^{-u(\lambda_1 + \lambda_2)} I_{[k]} \left( 2u \sqrt{\lambda_1 \lambda_2} \right) W_{-\alpha,0} \left( \frac{-t}{u^\alpha} \right), \quad u > 0. \tag{3.27}
\]

**Proof.** We use the conditioning argument to write

\[
R_k(t) = \int_0^\infty s_k(u)f(u, t)du,
\]

where \( f(u, t) \) is the density of \( D_\alpha(t) \) and \( s_k(u) \) is the marginal PMF of the Skellam process. Using the PMF of Skellam process given in (2.1) and the probability density function of \( \alpha \)-stable subordinator from (2.4), the result follows. \( \square \)

**Remark 3.2.** The Lévy density \( \nu_{S_\alpha} \) of SFSP-II can be obtained by taking \( \alpha_1 = \alpha_2 = \alpha \) in equation (4.31), which leads to

\[
\nu_{S_\alpha}(x) = \lambda_1^\alpha \sum_{n_1=1}^{\infty} (-1)^{n_1+1} \left( \frac{\alpha}{n_1} \right) \delta_{n_1}(x) + \lambda_2^\alpha \sum_{n_2=1}^{\infty} (-1)^{n_2+1} \left( \frac{\alpha}{n_2} \right) \delta_{-n_2}(x).
\]

In the next section, we introduce the tempered versions of space fractional Skellam processes. We discuss the marginal distributions, moments and governing difference-differential equations which use tempered fractional shift operator instead of ordinary shift operator used in SFSP.

### 4 Tempered space-fractional Skellam processes

In this section, we present the TSFSP-I and TSFSP-II. We discuss the corresponding fractional difference-differential equations, marginal PMFs and moments of these processes.

#### 4.1 Tempered space-fractional Skellam process type-I (TSFSP-I)

The tempered space fractional Skellam process of type-I is obtained by taking the difference of two independent tempered space fractional Poisson processes. Let \( D_{\alpha_1, \mu_1}(t), D_{\alpha_2, \mu_2}(t) \) be two independent TSS and \( N_1(t), N_2(t) \) be two independent Poisson processes which are independent of TSS. Then the stochastic process

\[
S_{\alpha_1, \alpha_2}^{\mu_1, \mu_2}(t) = N_1(D_{\alpha_1, \mu_1}(t)) - N_2(D_{\alpha_2, \mu_2}(t))
\]
is called the TSFSP-I.

**Theorem 4.1.** The PMF $H_{k}^{\mu_1,\mu_2}(t) = \mathbb{P}(S_{\alpha_1,\alpha_2}^{(1)}(t) = k)$ is given by

$$H_{k}^{\mu_1,\mu_2}(t) = \sum_{n=0}^{\infty} \frac{(-1)^k}{n!(n+k)!} e^{(\mu_1^{1]+\mu_2^{1}])(t)} \left( \sum_{m=0}^{\infty} \frac{\mu_1^{m} \lambda_1 - m}{m!} \psi_1 \left[ (1, \alpha_1); (1-n-k-m, \alpha_1); (-\lambda_1^{\alpha_1} t) \right] \right) \times \left( \sum_{l=0}^{\infty} \frac{\mu_2^{l} \lambda_2 - l}{l!} \psi_1 \left[ (1, \alpha_2); (1-l-k, \alpha_2); (-\lambda_2^{\alpha_2} t) \right] \right), \quad (4.28)$$

where $k \geq 0$ and similarly for $k < 0$,

$$H_{k}^{\mu_1,\mu_2}(t) = \sum_{n=0}^{\infty} \frac{(-1)^{|k|}}{n!(n+|k|)!} e^{(\mu_1^{1]+\mu_2^{1}])(t)} \left( \sum_{m=0}^{\infty} \frac{\mu_1^{m} \lambda_1 - m}{m!} \psi_1 \left[ (1, \alpha_1); (-\lambda_1^{\alpha_1} t) \right] \right) \times \left( \sum_{l=0}^{\infty} \frac{\mu_2^{l} \lambda_2 - l}{l!} \psi_1 \left[ (1, \alpha_2); (1-l-n-|k|, \alpha_2); (-\lambda_2^{\alpha_2} t) \right] \right). \quad (4.29)$$

**Proof.** Since $N_1(D_{\alpha_1,\mu_1}(t))$ and $N_2(D_{\alpha_2,\mu_2}(t))$ are independent,

$$\mathbb{P}(S_{\alpha_1,\alpha_2}^{(1,2)}(t) = k) = \sum_{n=0}^{\infty} \mathbb{P}(N_1(D_{\alpha_1,\mu_1}(t)) = n + k) \mathbb{P}(N_2(D_{\alpha_2,\mu_2}(t)) = n) \mathbb{I}_{k \geq 0}$$

$$+ \sum_{n=0}^{\infty} \mathbb{P}(N_1(D_{\alpha_1,\mu_1}(t)) = n) \mathbb{P}(N_2(D_{\alpha_2,\mu_2}(t)) = n + |k|) \mathbb{I}_{k < 0},$$

which gives the marginal PMF of TSFPP by using (2.18).

**Remark 4.1.** We use this expression to calculate the marginal distribution of tempered space fractional Skellam process. The MGF is obtained by using the conditioning argument. Let $f_{\alpha,\mu}(x,t)$ be the density function of $D_{\alpha,\mu}(t)$ which is defined in (2.6). Then

$$\mathbb{E}[e^{\theta N(D_{\alpha,\mu}(t))}] = \int_{0}^{\infty} \mathbb{E}[e^{\theta N(u)}] f_{\alpha,\mu}(u,t) du = e^{-t(\lambda(1-e^{\theta})+\mu)^{\alpha_1} - \mu^{\alpha}}. \quad (4.30)$$

Using (4.30), the MGF of TSFSP-I is

$$\mathbb{E}[e^{\theta S_{\alpha_1,\alpha_2}^{(1,2)}(t)}] = \mathbb{E}[e^{\theta N_1(D_{\alpha_1,\mu_1}(t))}] \mathbb{E}[e^{\theta N_2(D_{\alpha_2,\mu_2}(t))}] = e^{-t[(\lambda_1(1-e^{\theta})+\mu_1)^{\alpha_1} - \mu_1^{\alpha_1}] + ([\lambda_2(1-e^{\theta})+\mu_2)^{\alpha_2} - \mu_2^{\alpha_2}]].$$

**Remark 4.2.** We have $\mathbb{E}[S_{\alpha_1,\alpha_2}^{(1,2)}(t)] = t(\alpha_1^{\alpha_1} - \alpha_2^{\alpha_2})$. Further, the covariance of TSFSP-I can be obtained by using (2.21) and

$$\text{Cov} [S_{\alpha_1,\alpha_2}^{(1,2)}(t), S_{\alpha_1,\alpha_2}^{(1,2)}(s)] = \text{Cov}[N_1(D_{\alpha_1,\mu_1}(t)), N_1(D_{\alpha_1,\mu_1}(s))] + \text{Cov}[N_2(D_{\alpha_2,\mu_2}(t)), N_2(D_{\alpha_2,\mu_2}(s))]$$

$$= \text{Var}[N_1(D_{\alpha_1,\mu_1}(\min(t,s))) + \text{Var}[N_2(D_{\alpha_2,\mu_2}(\min(t,s)))].$$

**Proposition 4.1.** The Lévy density $\nu_{S_{\alpha_1,\alpha_2}^{(1,2)}}(x)$ of TSFSP-I is given by

$$\nu_{S_{\alpha_1,\alpha_2}^{(1,2)}}(x) = \sum_{n_1=1}^{\infty} \mu_1^{\alpha_1-n_1} \binom{\alpha_1}{n_1} \lambda_1^{n_1} \sum_{l_1=1}^{n_1} \binom{n_1}{l_1} (-1)^{l_1+1} \delta_{l_1}^{(1)}(x)$$

$$+ \sum_{n_2=1}^{\infty} \mu_2^{\alpha_2-n_2} \binom{\alpha_2}{n_2} \lambda_2^{n_2} \sum_{l_2=1}^{n_2} \binom{n_2}{l_2} (-1)^{l_2+1} \delta_{l_2}^{(2)}(x), \mu_1, \mu_2 > 0.$$
Proof. By adding Lévy densities $\nu_{N_{\alpha_1,\mu_1}}(x)$ and $\nu_{N_{\alpha_2,\mu_2}}(x)$ of $N_1(D_{\alpha_1,\mu_1}(t))$ and $N_2(D_{\alpha_2,\mu_2}(t))$ respectively from the equation (2.22), which leads to

$$\nu_{S_{\alpha_1,\alpha_2}}(x) = \nu_{N_{\alpha_1,\mu_1}}(x) + \nu_{N_{\alpha_2,\mu_2}}(x).$$

4.2 Tempered space-fractional Skellam process type-II (TSFSP-II)

Let $S_\alpha(t)$ and $D_{\alpha,\mu}(t)$ be independent Skellam process and TSS respectively. Then the time-changed stochastic process

$$S_{\alpha,\mu}(t) = S(D_{\alpha,\mu}(t))$$

is called a tempered space fractional Skellam process of type - II (TSFSP-II). The MGF is given by

Theorem 4.2. The marginal distribution $R^\mu_k(t) = \mathbb{P}(S_{\alpha,\mu}(t) = k)$ of TSFSP-II is

$$R^\mu_k(t) = \frac{1}{u} \left( \frac{\lambda_1}{\lambda_2} \right)^{k/2} \int_0^{\infty} e^{-u\mu + \mu^\alpha t} e^{-u(\lambda_1 + \lambda_2)} I_k \left( 2u \sqrt{\lambda_1 \lambda_2} \right) W_{-\alpha,0} \left( \frac{-t}{u^\alpha} \right), \quad u > 0.$$  (4.31)

Proof. We again use the standard conditioning argument to write

$$R^\mu_k(t) = \int_0^{\infty} s_k(u) f_{\alpha,\mu}(u,t) du,$$  (4.32)

where $f_{\alpha,\mu}(u,t)$ is the density (2.6) of TSS $D_{\alpha,\mu}(t)$. Using the PMF of Skellam process $s_k(t)$ and the probability density function $f_{\alpha,\mu}(u,t)$ of TSS, the proof concludes.

5 Simulation

We present the algorithm to simulate the sample trajectories for space-fractional Skellam processes and tempered space fractional Skellam processes.

5.1 Simulation of space fractional Skellam processes

Step-1: generate independent and uniformly distributed in $[0, 1]$ rvs $U, V$ for fix values of parameters;

Step-2: generate the increments of the $\alpha$-stable subordinator $D_{\alpha}(t)$ (see Cahoy et. al., 2010) with pdf $f_{\alpha}(x,t)$, using the relationship $D_{\alpha}(t + dt) - D_{\alpha}(t) \overset{d}{=} D_{\alpha}(dt) \overset{d}{=} (dt)^{1/\alpha} D_{\alpha_1}(1)$, where

$$D_{\alpha_1}(1) = \frac{\sin(\alpha \pi U) [\sin((1 - \alpha) \pi U)]^{1/\alpha - 1}}{[\sin(\pi U)]^{1/\alpha} |\log V|^{1/\alpha - 1}};$$

Step-3: generate the increments of Poisson distributed rv $N(D_{\alpha}(dt))$ with parameter $\lambda(dt)^{1/\alpha} D_{\alpha_1}(1)$;

Step-4: cumulative sum of increments gives the space fractional Poisson process $N(D_{\alpha}(t))$ sample trajectories;
**Step-5:** generate $N_1(D_{\alpha_1}(t))$, $N_2(D_{\alpha_2}(t))$ and subtract these to get the SFSP-I $S_{\alpha_1, \alpha_2}(t)$. Choosing $\alpha_1 = \alpha_2 = \alpha$ gives the sample trajectories of SFSP-II $S_{\alpha, \mu}(t)$.

We next present the algorithm for generating the sample trajectories of tempered space fractional Skellam processes.

### 5.2 Simulation of tempered space fractional Skellam processes

Use the first two steps of previous algorithm for generating the increments of $\alpha$-stable subordinator $D_{\alpha}(t)$.

**Step-3:** for generating the increments of TSS $D_{\alpha, \mu}(t)$ with pdf $f_{\alpha, \mu}(x, t)$, we use the following steps called “acceptance-rejection method”;

(a) generate the stable random variable $D_{\alpha}(dt)$;

(b) generate uniform $(0,1)$ rv $W$ (independent from $D_{\alpha}$);

(c) if $W \leq e^{-\mu D_{\alpha}(dt)}$, then $D_{\alpha, \mu}(dt) = D_{\alpha}(dt)$ (“accept”); otherwise go back to (a) (reject). Note that, here we used that $f_{\alpha, \mu}(x, t) = e^{-\mu x + \mu^{\alpha}t} f_{\alpha}(x, t)$, which implies $f_{\alpha, \mu}(x, t)(x, dt) / c f(x, dt) = e^{\mu x}$ for $c = e^{\mu^{\alpha}dt}$ and the ratio is bounded between 0 and 1;

**step 4:** generate Poisson distributed rv $N(D_{\alpha, \mu}(dt))$ with parameter $\lambda D_{\alpha, \mu}(dt)$

**step 5:** cumulative sum of increments gives the tempered space fractional Poisson process $N(D_{\alpha, \mu}(dt))$ sample trajectories;

**step 6:** generate $N_1(D_{\alpha_1, \mu_1}(t))$, $N_2(D_{\alpha_2, \mu_2}(t))$, then take difference of these to get the sample paths of the TSFSP-I $S_{\alpha_1, \alpha_2}^{\mu_1, \mu_2}(t)$. For generating the sample paths of TSFSP-II, we take choose $\alpha_1 = \alpha_2$ and $\mu_1 = \mu_2$.

![Figure 1: The sample trajectories of SFSP-I and TSFSP-II](image-url)
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