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Curricular approaches to algebra in Estonia, Finland and Sweden – a comparative study

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ABSTRACT

The aim of the study is to investigate and compare approaches to algebra in the Estonian, Finnish, and Swedish national core curricula (Grades 1–9). Despite the similarities in the school systems of these neighboring countries, the analysis reveals three quite different curricular approaches. The Estonian approach shows influences of the Russian Davydov School. The Finnish approach to some extent resembles the traditional one whereby algebra is addressed first at the lower secondary level and then in a formal manner. However, there are also characteristics typical of the functional view, which dominates the Swedish curriculum. Here, as opposed to the documents from the other two countries, a transition to more formal sophisticated methods at the secondary level is not visible at all. We discuss the results in relation to earlier research and students’ different learning outcomes in light of international evaluations.

Introduction

A prerequisite for mathematics in many different areas is a profound knowledge and understanding of algebra. Several studies show that students with inadequate algebraic knowledge encounter serious problems when they begin to study mathematics at the tertiary level (Brandell et al., 2008; Hiebert et al., 2005). In Western countries, the introduction of algebra in school mathematics has traditionally been postponed until the lower secondary school level, based on the assumption that young pupils are not capable of thinking algebraically. However, this assumption has been challenged by several mathematics educators (Blanton et al., 2015; Kieran et al., 2016). Also, the idea that students’ development in learning algebraic concepts would be reflected in the historical development of algebra (e.g., Katz & Barton, 2007) has been questioned (Bråting & Pejlare, 2015; Schubring, 2011). Recent studies show that it is possible, and even beneficial, to start working with algebraic ideas and generalizations in parallel with arithmetic already in early grades (Blanton et al., 2011; Cai et al., 2005; Carraher et al., 2006).

At the same time, students’ difficulties with algebra have been widely documented in research literature on mathematics education (e.g., Brandell et al., 2008; Hemmi & Löfwall, 2011; Herscovics & Linchevski, 1994; Jupri et al., 2014). School algebra has been experienced as manipulations of symbols without any mathematical meaning or relevance to students’ everyday lives. Moreover, students’ understanding of these symbol manipulations is shown to often be “fragmented and separated from their earlier mathematical activities” (Baek, 2008, p. 141; Röj-Lindberg et al., 2017). The late and abrupt introduction of algebra in isolation from other branches of mathematics has apparently contributed to students’ difficulties (Kilpatrick & Izsak, 2008). Therefore, it has been
suggested that students in earlier grades should have experiences that better prepare them for a more formal study of algebra in later grades (e.g., Greenes & Rubenstein, 2008; Mason, 2018).

This recent development in the research field is also visible in steering documents, as several countries have revised their mathematics curricula in order to begin familiarizing students with algebraic thinking already at early ages (e.g., Greenes & Rubenstein, 2008; Mason, 2018). In recent decades, National Council of Teachers of Mathematics [NCTM], (2000) Standards have strongly promoted the “Algebra for All” movement, which demands that all students study algebra before graduating high school. This has also been a trend in Europe, and for instance, in Sweden, algebra has been mentioned as part of the national compulsory curriculum for all children for a number of decades (e.g., Hemmi et al., 2018). During the 1990s, a specific textbook called Algebra for all (Bergsten et al., 1997) was published and used in teacher education at several universities in Sweden. A new version of the textbook has recently been released (Häggström et al., 2019). In discussions of these reform efforts, researchers highlight the importance of learning algebra through understanding, and teaching it by making connections between children’s arithmetic problem solving and algebraic reasoning (Baek, 2008; Carpenter et al., 2003; Kaput, 2000; NCTM, 2000).

The curriculum has long been the focus of scholarly inquiry in mathematics education. Among other things, numerous studies investigate the possibilities to introduce algebraic content into elementary school curricula and examine ways to develop students’ algebraic thinking (e.g., Blanton et al., 2015; Cai, 2004; Confrey et al., 2017; Fong, 2004; Kilpatrick & Izsak, 2008). Typically, these studies concentrate on analyzing the mathematics curriculum of a single country (Greenes & Rubenstein, 2008; Kieran, 2018). Yet, cross-national studies of mathematics curricula can offer new perspectives and findings that simply cannot be gained through research conducted in one country alone (e.g., Lloyd et al., 2017). Such investigations enable us to understand different paths of how algebra-related topics have been intended to be introduced – and thus students’ algebraic thinking to be developed – and to recognize the unique features of particular curricular programs.

Nevertheless, there are few comparative studies examining the progression of algebra in different countries’ curricula. The existing international comparisons of school algebra have also mainly concerned differences between countries with significantly different cultures and school systems (e.g., Cai et al., 2005; Leung et al., 2014). In contrast, the present study investigates curricular approaches to algebra in Estonia, Finland and Sweden, three neighboring countries with quite similar school systems. We deem that the similarity of the school systems facilitates the identification of specific features concerning the particular views of expected student progression in algebra in these countries.

According to international evaluations, there are differences in students’ learning outcomes in the three countries. In Sweden, students’ results in algebra in TIMSS (Trends in International Mathematics and Science Study) and SIMS (the Second International Mathematics Study) have been low since the 1960s, regardless of the variation in other mathematical topics (Bråting et al., 2018; Murray & Liljefors, 1983; The Swedish National Agency for Education, 2008, 2012, 2016). This is despite the fact that algebra has been mentioned as part of the national compulsory curriculum for all children for a number of decades (e.g., Hemmi et al., 2018). Although Finnish students’ results in algebra have been better than the Swedish ones’, algebra has been the weakest mathematical area in TIMSS (Gronmo et al., 2014). Estonian students’ performance on TIMSS (2003) was significantly above the international average and students scored equally well in all content areas, including algebra (see Mullis et al., 2004). Contrary to the Swedish students’ poor results in PISA (the Programme for International Student Assessment) both the Estonian and Finnish students’ results have been over average. In a PISA ranking over mathematics performance from 2015 Estonia was at place 8, Finland at place 11 and Sweden at place 21 (Organization for Economic Co-operation and Development [OECD], 2019). These differences make it interesting to compare the countries’ mathematics curricula from different perspectives.

In the present study, the term “curriculum” is used to mean the intended curriculum (e.g., Valverde et al., 2002; Van den Akker, 2003) – the curriculum document representing official
intentions for instruction, national goals, and the description of content regarding what mathematics students are expected to learn and in what sequence. When analyzing and comparing the algebra approaches we draw on Blanton et al.’s (2015) ‘big ideas’ consisting of 1) equivalence, expressions, equations and inequalities; 2) generalized arithmetic; 3) functional thinking; 4) variables; and 5) proportional reasoning. The big ideas are further described in the sections Conceptualizing the development of algebraic thinking and Methods.

The paper contributes to our understanding of different curricular approaches to algebra when discussing results of different international comparisons even though we cannot draw any simple conclusions about the effects of national curricula on students’ performance (cf. Jablonka & Gellert, 2010). National curriculum documents are produced within certain educational traditions and are shaped by national perspectives on education as well as on the specific school subjects (cf. Andrews, 2007). In all three countries, that are the focus of the study, the national curriculum have the same role in the educational system and teachers are to follow these guidelines. We aim to answer the following question:

What are the specific characteristics of the approaches to algebra in Estonian, Finnish and Swedish national core curriculum for Grades 1-9?

Relevant research

In this section, we first elaborate on the frameworks researchers have suggested for conceptualizing the development of algebraic thinking. Then, we exemplify various curricular approaches to school algebra identified by researchers in different contexts.

Conceptualizing the development of algebraic thinking

There is general agreement that the way to develop algebraic ideas in earlier grades is not to simply push the secondary school algebraic content into the elementary mathematics. Thus, there have emerged a number of studies investigating how to introduce algebraic content into elementary mathematics before the formal study of algebra (e.g., Blanton et al., 2015; Cai, 2004; Cai et al., 2005; Carpenter et al., 2003; Fong, 2004; Radford, 2018; Schifter, 2018). For example, there are studies suggesting that algebraic thinking could be developed at primary school level by means of problem solving, generalizing and functions (Fong, 2004) and by representing and understanding quantitative relationships (Cai, 2004). Several studies consider building foundations for more formal algebra by learning the fundamental properties of arithmetic in elementary grades (e.g., Carpenter et al., 2003).

Kieran (2004) suggests that a successful transition from arithmetic to algebra requires five adjustments:

1. A focus on relations and not merely on the calculation of numeric answers;
2. A focus on operations as well as their inverses and on the related idea of doing and undoing;
3. A focus on both representing and solving a problem instead of merely solving it;
4. A focus on both numbers and letters rather than on numbers alone; and
5. A refocus on the meaning of the equal sign.

These adjustments are influenced by, for example, Kilpatrick et al.’s (2001) ‘Adding it up’ whereby developing mathematical proficiency beyond number is taken into consideration.

Researchers have suggested several frameworks for conceptualizing the development of algebraic thinking. A number of these frameworks (e.g., Blanton et al., 2015; Carrara & Schliemann, 2018) draw on Kaput’s (2008) definitions regarding algebraic thinking. Kaput suggests that algebraic thinking involves two core practices: (a) making and expressing generalizations in increasingly formal and conventional symbol systems; and (b) reasoning with symbolic forms (e.g., Arcavi, 1994; Arcavi et al., 2017). These practices take place across the following three content strands: 1)
algebra as the study of structures and relations arising in arithmetic (sometimes referred to as
generalized arithmetic); 2) algebra as the study of functions; and 3) algebra as the application of
a cluster of modeling languages (Kaput, 2008). Thus, Kaput distinguishes between algebraic thinking
on the one hand and algebra as a discipline on the other. This distinction has formed a basis for
Blanton et al.’s (2015) big ideas of algebra (applied as analytical tool in this study).

Blanton et al. (2015) draw on Kaput’s (2008) definition of algebraic thinking, but also on the
interpretation of learning progression by Shin et al. (2009). The ‘big ideas’ of Blanton et al. (2015)
comprise important strands and content connected to the development of algebraic thinking in
earlier research. As they point out, this is only one of several possible ways of classifying algebraic
content, but the benefit is that the big ideas both “[…] provide rich contexts in which algebraic
thinking can occur (e.g., through generalized arithmetic) and they represent central components of
algebra as a discipline (e.g., variable)” (Blanton et al., 2015, p. 44). Therefore, we chose to use the ‘big
ideas’ as a starting point in our data analysis. In the methods section below, descriptions of the five
big ideas will be presented.

While several frameworks of the development of algebraic thinking are based on the idea of
moving from arithmetic to algebra, the Russian psychologist Davydov and his colleagues researched
and developed theories on the development of algebraic thinking in which algebra preceded
arithmetic (e.g., Davydov, 1990; Davydov & Rubtsov, 2018; Schmittau, 2005). Their ideas were
based on the socio-cultural theories of Vygotsky (Schmittau, 2005; Zinchenko, 2011). In contrast
to the conception that algebraic thinking develops by building generalizations on arithmetic, the idea
is to “ascend” from the theoretical and general to the specific (e.g., Davydov, 1990; Davydov &
Rubtsov, 2018). Thus, instead of thinking that children obtain pre-algebraic experiences that are
numerical, pre-numerical algebraic experiences are applied in specific situations (Schmittau &
Morris, 2004). From the first school grade, children are to develop algebraic thinking by exploring
and comparing quantities and part-whole relationships. For example, the relation between different
components in arithmetic operations is studied and, based on the general relations, children can
represent and solve equations already during the first school grades (e.g., Schmittau, 2004). Carraher
et al. (2006) regard Davydov’s work as “groundbreaking” as it highlighted “many of the means by
which algebraic concepts could be made accessible and meaningful to young children” (p. 93). Yet,
they claim that the downplay of arithmetic as a basis for algebraic knowledge excludes, for example,
students’ intuition about number lines based on metrics and basic knowledge of addition and
subtraction facts. They also point out that functions are not at all in focus in Davydov’s work and
that it therefore excludes the view of arithmetic operations as functions (Carraher et al., 2006).

Next, we report on earlier research on different curricular approaches in different contexts related
to the various theories about algebraic thinking elaborated on above.

Different curricular approaches to algebra

In recent decades, researchers have categorized school algebra with respect to algebraic content,
Sources of meaning, and models of algebraic activity (Arcavi et al., 2017; Cai et al., 2010; Kieran,
2007; Lagrange, 2014). The so-called structural perspective (sometimes called the traditional per-
spective) on algebra learning emphasizes the importance of developing the abilities to generalize,
work abstractly using symbols, and follow procedures in a systematic way (Cai et al., 2010). The
content typically includes the simplification of expressions, symbolic manipulations, factoring polyno-

mial and rational expressions, and solving equations, inequalities, and systems of equations using
formal methods (Kieran, 2007). Meanwhile, the functional perspective emphasizes the concepts of
change and variation, with the idea of representing various situations by means of relationships
between variables. The content is based on contextualized, real-world problems, and attempts to
solve these problems with methods other than manual symbolic manipulations. The use of technol-
yogy-supported methods, especially graphical and dynamic geometry software, is encouraged within
the functional perspective (Kieran, 2007; Lagrange, 2014).
Cai et al. (2010) compared different US curricula in relation to the functional/structural perspective. One of the differences between the curricula was how the concept of variable was considered. In curricula with a functional perspective, variables were regarded as quantities that change and were used to represent relationships, and were not formally defined until Grade 7. Meanwhile, from the structural perspective, variables were treated predominantly as placeholders and represented unknowns in expressions and equations. In Grade 6, the concept of variable was formally defined as a symbol used to represent a number (Cai et al., 2010, p. 77). Nevertheless, as Kieran (2007, p. 710) points out, most curricula take neither a strict functional nor a strict structural perspective.

Cai et al. (2005) investigated and compared the algebraic approaches in the Chinese, Singaporean, and South Korean national elementary curricula, the specifically selected curricula in Russia (Davydov), and the US (Investigations). They related the curricula to Kieran’s five adjustments (see previous section) and the NCTM (2000) goals for the four algebra strands. These goals comprise 1) understanding patterns, relations, and functions; 2) representing and analyzing mathematical situations and structures using algebraic symbols; 3) using mathematical models to represent and understand quantitative relationships; and 4) analyzing change in various contexts. This study is interesting to us, as it is one of the few to compare algebra curricula developed in different contexts. All five curricula address the development of algebraic thinking and fulfill Kieran’s adjustments to great extent from the early school grades but in different manners. In all three Asian curricula it was common to employ reverse operations in informal equation solving while analyzing change in various contexts (NCTM, 2000) is not developed until the secondary-level mathematics curricula. The Russian Davydov curriculum employs all five of Kieran’s adjustments from the very beginning of Grade 1. It also deals extensively with three NCTM goals but does not include patterns and functions in the early grades curriculum. The five curricula take various approaches to formal algebraic representations related to generalizations and symbolism, with the Russian and US curricula representing quite opposite views. Contrary to the Davydov curriculum, the US Investigations curriculum is based on children’s intuitive understanding and informal learning, while the formal treatment of algebraic content with symbolic representations is postponed until secondary-level mathematics (Cai et al., 2005).

Yet another difference between the Russian and US curricula is how familiar, real-world situations are considered. Although the students engage with various problems from many different authentic contexts, the problems in the Russian curriculum are purposefully sequenced to help students develop a theoretical understanding of mathematical concepts and their ability to analyze problem situations at a theoretical level (Cai et al., 2005). Investigations engage students in mathematical problems embedded in authentic contexts, whereby the students are to explore them in depth, construct strategies and approaches utilizing a variety of tools, and communicate their mathematical reasoning through drawing, writing, and talking (Cai et al., 2005).

**Methods**

We first briefly describe the educational contexts and the character of the three countries’ national curricula, and then we give an account of the analytical tool used in the qualitative document analysis as well as the process of our data analysis.

**The educational contexts of Estonia, Finland, and Sweden**

The three countries have similar school systems in several aspects. In all the countries, pupils start school at the age of six or seven, and compulsory school lasts nine years. In compulsory school, pupils most often study in mixed-ability groups, as there is no tracking. Hence, there is only one steering document for all students in compulsory school. In this article, we have analyzed all these documents (Eesti Vabariigi Valitsuse määrus, 2011; Finnish National Board of Education, 2016; The Swedish National Agency for Education, 2011). These curriculum documents can only be
regarded as frameworks, as they do not suggest any textbook materials or teaching methods, lesson plans, or tests. Instead, teachers can freely choose the textbooks or even choose to not use a textbook (cf. Hemmi et al., 2017, 2013; Lepik et al., 2015). However, teachers in all three countries are obligated to follow the national curriculum. There are many similarities in the structures of the three curriculum documents. They are all written at the same level of generality (Hemmi et al., 2013), although the levels of detail concerning different mathematical topics vary to some extent, which would make a quantitative text analysis uninformative. Each document includes a section presenting the general aims of mathematics (some part of this applies to all the grades, 1–9) and another the mathematical contents. We have analyzed every sentence in all these parts. In addition, each document contains some kind of assessment criteria, which refer to the aims and to all the content sections. For instance, in the Swedish curriculum document there are assessment criteria for the competences of problem solving, conceptual understanding, usage of mathematical methods and, finally, mathematical reasoning and communication. These kinds of descriptions are offered for grade levels 1–3, 4–6, and 7–9 in the Estonian and Swedish documents, and for grade levels 1–2, 3–6, and 7–9 in the Finnish document. We have analyzed these parts as well. All three mathematics curricula are integrated by nature, meaning that there are no separate courses for geometry or algebra, for example.

An interesting difference is that, in the content description of the Swedish curriculum, Algebra as well as Relationships and Change are used as headings throughout the descriptions for all grade levels, while in the Finnish curriculum the heading Algebra is used for the middle grades and Algebra and Functions for the lower secondary level. In the Estonian curriculum, Algebra and data as a heading appears for the middle grades. Interestingly, here algebra is connected to statistics. In the lower secondary-level documents, Algebra, Functions, and Percentages all have their own sections.

The analytical tool

In order to identify the algebraic items in the material, we used the ‘big ideas’ of Blanton et al. (2015) as the basis for an analytical tool. The big ideas were developed from previous research on algebraic thinking (e.g., Kaput, 2008; Kieran, 2004) and are applicable to the grade levels relevant in our study, which was one important factor in our choice of analytical tool. Another important factor was that the big ideas take into consideration both algebraic thinking and algebra as a discipline. We started to test and develop the categories within the Swedish context, applying them in an initial analysis of Swedish mathematics curriculum and textbooks (Bråting et al., 2019). In the present study, we conducted a more fine-grained analysis, which we describe below. We commence by giving an account of the categories and how we interpreted them in our analysis:

(1) EEEI – Equivalence, Expressions, Equations, and Inequalities – includes relational understanding of the equal sign, representing and reasoning with expressions and equations, and relationships between and among generalized quantities (Blanton et al., 2015, p. 43). In our analysis, we have also included in this category modeling word problems by creating and solving equations.

(2) GA – Generalized Arithmetic – involves reasoning about structures of arithmetic expressions (rather than their computational value) as well as generalizations of arithmetical relationships, which includes fundamental properties of numbers and operations (e.g., the commutative property of addition) (Blanton et al., 2015). In our analysis, we have, for example, included relations between arithmetic operations within this category.

(3) FT – Functional Thinking – involves generalizations of relationships between co-varying quantities, and representations and reasoning with relationships through natural language, algebraic (symbolic) notation, tables, and graphs (Blanton et al., 2015, p. 43). For instance, this can mean generating linear data and organizing them in a table, identifying recursive patterns and function rules.
(4) **PR – Proportional Reasoning** – refers to opportunities for reasoning algebraically about two generalized quantities that are related in such a way that the ratio of one to the other is invariant (Blanton et al., 2015, p. 43). In our analysis, we have included some specific applications of proportional reasoning such as scaling, similarity, and congruence.

(5) **VAR – Variable** – refers to “symbolic notation as a linguistic tool for representing mathematical ideas in succinct ways and includes the different roles variable plays in different mathematical contexts” (Blanton et al., 2015, p. 43). One typical example within this category is the ability to use variables to represent arithmetic generalizations. In this category, we have distinguished between variable and unknown number. The former is a broader term that can correspond to variables, unknown numbers, placeholders, parameters etc. This distinction provided us with information about the balance between a functional approach and a more structural approach (e.g., Kieran, 2007; Lagrange, 2014).

During the data analysis, we created subthemes to the big ideas based on the findings in the three countries’ curricula (see Table 1). The development of the subthemes is described in the section **The data analysis** below. In the Results section, we further display the various topics we have included in each category.

**The data analysis**

In cross-cultural studies, it is important to have both insiders and outsiders participating in the data analyses in order to capture specific features from each context (cf. Clarke, 2013). In our study, the insiders (familiar with each context) first conducted the analysis of each curriculum document in the original language. The unit of analysis was a statement or part of a statement that addressed an issue connected to one or, in a few cases, several big ideas. For example, “Lots of time and energy is invested into detailed study of rational expressions- monomials, multinomials and operations with them; students should add, subtract, and multiply monomials and multinomials and divide monomials and multinomials by a monomial, simplify rational expressions” (Estonian curriculum, Grades 7–9) was categorized into EEEI. The Finnish expressions dealing with the same algebraic area were at the same phase coded as two units: “They [students] familiarize themselves with the concept of polynomial and practice the addition, subtraction and multiplication of polynomials.” and “They practice forming and reducing exponential expressions.” We analyzed all the parts of the mathematics curricula including contents, goals and assessment criteria. Therefore, it was hard to avoid blending process with content.

Next, we translated the items coded into different categories into English but kept the original wording. The two team members that do not know the Finnish language, could also check the analysis conducted by the Finnish-speaking team member as there is an English translation of the full document. This was also the case with the Estonian documents. The team member responsible for the analysis of the Finnish curriculum is fluent in Swedish so she checked the analysis of the Swedish documents as well.

After this procedure, we held a meeting discussing the alignment of each other’s analysis according to our interpretation of the big ideas. For example, some items connected to the decimal system were excluded because they were considered to belong to the development of number sense rather than algebra. We reached consensus concerning the categories. Sometimes, the context of the phrases in the curriculum document helped us interpret them. For example, Estonian EEEI sentences about word problems also refer to creating and solving equations, which is one reason why “Modeling and solving word problems” is a subtheme to EEEI. We sometimes categorized the same piece of data as representing two different big ideas or sub-themes. For instance, drawing and interpreting graphs were categorized in both the subthemes ‘Functional relations’ and ‘Tables and graphs’.
Thereafter, we looked more in-depth at the expressions categorized into the big ideas and marked similar items with a certain color in a table where we had gathered the expressions in columns for each country. In this way, we eventually created the subthemes characterizing various items found within the big ideas (see Table 1). We structured the subthemes in the same order within each country to be able to compare them. During this phase, we also merged similar phrases into one topic (and in some cases rephrased them to some extent to unify the wording). It was possible to do this, as we do not aim to quantify the occurrence of the same items. Sometimes one expression coded to a big idea was split into several sub-themes. For example, the authentic expression from the Swedish curriculum (Grades 4–6) “Simple algebraic expressions, formulas and equations in situations that are relevant for the students” coded to EEEI, was split into the sub-theme of Handling of expressions and Modeling and solving word problems. The expression from the Estonian upper secondary curriculum above was worded in the final table as three different topics of the sub-theme Handling of expressions, namely: “The concepts of monomial and multinomial”, “Add, subtract, multiply and divide monomials and multinomials” and “Simplify rational expressions”. The Finnish expressions were worded: “The concept of polynomial”, “Addition, subtraction and multiplication of polynomials and “Forming and reducing exponential expressions”.

The next step was to once again, systematically check the documents in relation to the themes and subthemes with the insider in each country, which led to some minor corrections concerning the formulation of the topics in the sub-themes. The topics presented in the tables in the result section below are still close to the wording of the original data sources making the analysis transparent.

Finally, we compared the topics from each country in different big ideas and the subthemes. Within this process, we identified specific features and gaps in each country’s intended curriculum regarding how algebra related issues have been introduced.

Results

Next, we present the results of the analysis separately for each big idea. Every section commences with a comparison of the three countries’ curricula with respect to how they address the big idea or subtheme in question at a certain school level, combined with a table displaying the categorization of the authentic expressions identified in the documents. However, we have omitted the tables of the subthemes that are relatively similar for the three countries.

**EEEI – equivalence, expressions, equations, and inequalities**

EEEI consists of the three subthemes Equivalence, inequalities and equations, Handling of expressions, and Modeling and solving word problems. Tables 2–4 show the distribution of these subthemes in the three countries. Both the Estonian and the Swedish curricula address the handling of simple mathematical equations or equalities already in the first years of school, Grades 1–3 (see Table 2). The Swedish document uses the notion of equality, and stresses the importance, and the use, of the equal sign, while the meaning of the equal sign is not an issue in any phase of the Estonian or Finnish curriculum. The Estonian document explicitly states that students should find the numerical
value of a *letter* in equations through the informal method of testing or analogy, referring to the use of the inverse property of arithmetic operations. The Finnish curriculum for the first stage (Grades 1–2) does not address anything connected to equalities or equations.

In Grades 4–6 in Estonia and Sweden, and 3–6 in Finland (Table 2), all three curricula prescribe the introduction of equations and solving simple equations. While both the Estonian and Finnish documents explicitly prescribe reasoning and experimentation as the solution methods (the Estonian one already at the earlier stage, Grades 1–3), the Swedish one only mentions that methods for solving simple equations should be presented but does not pinpoint their character. Based on the analysis of the big idea of generalized arithmetic below, the use of inverse operations seems to be important for the examinations and reasoning with equation solving in the Estonian and Finnish cases.

In all three curricula, the ability to solve equations is prescribed to be further developed in Grades 7–9 (Table 2). Here, we find clear differences between the Estonian and Finnish curricula on the one hand and the Swedish curriculum on the other. The expectation that students are to proceed to more sophisticated and formal methods in algebra is clearly targeted in both the Estonian and Finnish curricula. They address solving first- and second-degree equations as well as equation systems both analytically and graphically, while the Swedish curriculum only generally addresses methods for solving equations in a manner similar to that for Grades 4–6. The only difference is that the adjective ‘simple’ is not used with the word ‘equation’ for Grades 7–9. A special feature of the Finnish lower secondary curriculum is that it mentions solving first-degree inequalities, whereas inequalities are not an issue in the Estonian or Swedish curriculum at any stage.

As for the second subtheme of EEEI, *Handling of expressions*, the Estonian curriculum contains the most explicit description compared to the other two countries’ documents. Working with expressions is addressed already in the description for the first grades, when the focus is on determining the correct order of operations in expressions (Table 3). In Grades 4–6, Estonian students should learn to simplify one-variable expressions and calculate the value of letter expressions. The Swedish curriculum mentions simple algebraic expression for Grades 4–6 while the Finnish curriculum does not address expressions in the Grades 1–6 at all.

Looking at the goals and content for the upper grades, again, the Swedish curriculum clearly differs from the Estonian and Finnish ones, only mentioning algebraic expressions, indicating that they now can be somewhat more complicated than the simple ones in the earlier grades. Here, the Estonian and Finnish documents are more concrete and use concepts connected to algebra that are more formal. Both the Estonian and Finnish documents address operations with polynomials, but the Finnish one excludes the division of polynomials. Instead, it mentions forming and reducing exponential expressions, which is missing in the Estonian document. Finally, the Estonian
The curriculum includes statements concerning algebraic fractions, simplifying rational expressions as well as formulas for difference of squares, squares of sums, and squares of differences. The calculation of the value of a mathematical expression is introduced in Grades 4–6 in the Estonian document, Grades 7–9 in the Finnish document while it is lacking in the Swedish curriculum at all grade levels.

Finally, the third subtheme within EEEI concerns Modeling and solving word problems as a method of working with word problems. As can be seen in Table 4, this subtheme is clearly emphasized in the Estonian curriculum throughout the grades. In fact, this curriculum differs from the other two in the sense that it explicitly and systematically describes the need to apply equation solving in the context of modeling and word problems. There is also a clear progression in the Estonian curriculum concerning the learning of solving word problems with the help of equations. The text in the content description does not explicitly mention equations, but our interpretation is based on the context. For example, the description for Grades 4–6 is found under the subtitle Algebra. In addition, the insider researcher in the research group stated that modeling of equations is included already in Grades 1–3, when students learn to represent internal quantitative relationships in word problems as well as to solve word problems with one and two operations. In Grades 4–6, students are expected to solve and compose word problems with several operations, and in Grades 7–9 they are to use both linear equations and equation systems to model and solve word problems.

There is an emphasis on problem solving throughout the grades in the Finnish and Swedish curricula, but in the Finnish document, problem solving is not connected to algebra at the elementary level (Grades 1–6). At this level, problems connected to familiar situations are addressed.

### Table 3. Distribution of the topics connected to the subtheme Handling of expressions across Grades 1–9.

|          | Estonia | Finland | Sweden |
|----------|---------|---------|--------|
| 1-2/3    | Determining correct order of operations in expressions (parentheses, mult/div, add/subtr). |      |        |
| 3/4-6    | Simplifying one-variable expressions. Calculating the value of letter expressions. |      |        |
| 7-9      | The concept of algebraic fraction. Operations with algebraic fractions. Formulas for difference of squares, squares of sums and differences. The concepts of monomial and multinomial. Add, subtract, multiply and divide monomials and multinomials. Simplify rational expressions. | The concept of variable. Calculating the value of a mathematical expression. Forming and reducing exponential expressions. The concept of polynomial. Addition, subtraction and multiplication of polynomials. | Algebraic expressions. Algebraic formulas. |

### Table 4. Distribution of the topics connected to the subtheme Modeling and solving word problems across Grades 1–9.

|          | Estonia | Finland | Sweden |
|----------|---------|---------|--------|
| 1-2/3    | Representing internal quantitative relationships in word problems and expressing these relationships numerically. Solving word problems with one and two operations. |      |        |
| 3/4-6    | Solving and composing word problems with several operations, and checking and evaluating the result. | Modeling and problem solving. Forming and solving first-degree equations and incomplete second-degree equations. Using algorithmic thinking and skills in applying mathematics and programming in problem solving. | Equations in situations relevant to students. |
| 7-9      | Using linear equations and equation systems to model and solve word problems. |      |        |

There is an emphasis on problem solving throughout the grades in the Finnish and Swedish curricula, but in the Finnish document, problem solving is not connected to algebra at the elementary level (Grades 1–6). At this level, problems connected to familiar situations are addressed.
and students are expected to present and discuss different solutions. Group work with investigations is also prescribed throughout the grades in the Finnish document. At lower secondary level, modeling and problem solving are addressed in the general aims, and students are to both form and solve first-degree equations and incomplete second-degree equations. Algorithmic thinking is also mentioned regarding applying mathematics and programming to problem solving. In the Swedish document, equations in situations relevant to the students are mentioned for both Grades 4–6 and 7–9, but nothing is said about modeling such equations (see Table 4).

**GA – generalized arithmetic**

GA consists of two subthemes: Relations between arithmetic operations and Arithmetic rules. Table 5 shows the distribution of these subthemes in the three countries.

All three curricula address the first subtheme, Relations between operations, in Grades 1–3. In the Estonian curriculum, the names of the terms/components and of the results of the arithmetic operations are also targeted, as is the concept of inverse operation. Both the Estonian and Finnish curricula focus on relations between operations still in the middle grades’ content, while the Swedish document only addresses them in the content for Grades 1–3.

When it comes to the second subtheme of GA, Arithmetic rules, all three countries’ documents for the first grades contain statements about the properties of operations. Here, the Finnish curriculum is the most explicit, mentioning the use of commutative and associative law in addition and multiplication in the content description for Grades 1–2. Both the Estonian and Finnish curricula continue targeting the use of the properties of operations in the middle grades, while the Swedish curriculum only mentions properties of operations without any specifications in the first grades’ content. There is a special statement in the Estonian document concerning the formulation of divisibility properties in the middle grades, which we include in this subtheme. Nothing is written about generalizations or the use of rules in the description of lower secondary mathematics in any of the three countries’ documents (see Table 5).

**FT – functional thinking**

FT consists of the three subthemes Patterns, Functional relations, and Tables and graphs. Table 6 shows the distribution of the first two of these three subthemes in the three countries’ documents.

Within the first subtheme of FT, Patterns, the Finnish curriculum emphasizes activities connected to regularities such as number sequences throughout all the grades, 1–9. Clearly, there is

| Table 5. Distribution of the topics connected to the subthemes of the big idea GA across Grades 1–9. |
|---|---|---|---|
| **Relations between operations** | **Estonia** | **Finland** | **Sweden** |
| 1-2/3 | Relation between addition and subtraction, subtraction defined as reverse of addition. Relations between multiplication and division, division defined as reverse of multiplication. | Principles and features of basic arithmetic operations. Relation between multiplication and division. | Relations between the four arithmetical operations. |
| 3/4-6 | Relation of multiplication to addition. Relations between components in arithmetic operations and the results of operations. | | |
| 7-9 | | Relations between operations. | |
| **Arithmetic rules** | **Estonia** | **Finland** | **Sweden** |
| 1-2/3 | Rules of mental and written arithmetic. | The commutative and the associative law in addition and multiplication. Mental and written arithmetic using the properties of operations. | Properties of arithmetic operations. |
| 3/4-6 | Properties of operations. Formulating and applying divisibility properties (div by 2, 3, 5, 9, and 10). | | |
| 7-9 | | | |
a progression across the grades: in Grades 1–2 students learn to find regularities; in Grades 3–6 they learn to continue number sequences by following rules; and in Grades 7–9 they deepen their skills in forming and examining number sequences. In the Swedish curriculum, number sequences and geometrical patterns are addressed at the primary and elementary level, but without any further descriptions. In the Estonian curriculum, this subtheme is lacking throughout all grade levels.

The second subtheme of FT, *Functional relations*, first appears in Grades 7–9 in all three curricula. For Grades 7–9, this subtheme is addressed in detail in the Estonian and Finnish curricula while the Swedish curriculum has a more general, shorter description. In all three curricula, linear functions are explicitly mentioned while quadratic functions are only mentioned in the Finnish and Estonian curricula. Since the Swedish curriculum uses the phrase “functions and linear equations” (Table 6), it is difficult to discern what kinds of functions are included merely from reading the curriculum document. Here, in a similar manner as within EEEI, we notice a difference between lower secondary expectations concerning more sophisticated methods and formal algebra in Estonia and Finland on the one hand and Sweden on the other. In the Estonian and Finnish curricula, zeros of a function as well as the dependence of position and form of functions’ graphs are emphasized. The involution rules are only mentioned in the Estonian curriculum, while direct and inverse proportionality as well as the concept of angular coefficient (slope) and constant term are only mentioned in the Finnish curriculum. In the Swedish curriculum, it is emphasized how functions can be used to investigate change and other relationships.

The third subtheme of FT, *Tables and graphs*, is somewhat overlapping with the second subtheme as, for example, the graphical solution of problems is often mentioned in connection to functional relationships, at least in the Estonian and Finnish lower secondary curricula. In Grades 1–6 in Estonian and Finnish as well as Swedish classrooms, tables and diagrams are to be dealt with in connection to statistics. The Swedish document especially connects tables and diagrams to

**Table 6. Distribution of the topics connected to the subthemes Patterns (highlighted in gray in the Table) and Functional relations across Grades 1–9.**

|          | Estonia                        | Finland                                      | Sweden                                      |
|----------|--------------------------------|----------------------------------------------|---------------------------------------------|
| 1-2/3    | Finding regularities.          | Describing and expressing simple patterns in number sequences and simple geometrical forms. | Continuing a number sequence following its rule. |
| 3/4-6    | Observing the regularities of number sequences. | Describing and expressing patterns in number sequences and simple geometrical forms. | Constructing patterns in number sequences and simple geometrical forms. |
| 7-9      | Introduction of the function concept. Linear and quadratic functions with their graphs. Dependence of position and form of functions’ graphs on the coefficient in the function’s expression. The meaning of zeros of a function and finding zeros on graphs and formulas. The involution rules. | Deepening skills in examining and forming number sequences. The concept of function. Interpreting and producing a graph of first- and second-degree functions. Correlations depicted both graphically and algebraically. Direct and inverse proportionality. The concept of angular coefficient (slope) and constant term. Determining the zeros of a function. | Functions and linear equations. Using functions to investigate change, rate of change and other relations. |
investigations throughout the grades. The coordinate system appears in Grades 4-6/3-6 and 7–9 in all three curricula, but interpreting and drawing graphs are only mentioned in the Estonian and Swedish curriculum for the middle level. For Grades 7–9, the Estonian and Finnish curricula emphasize drawing graphs of functions, with explicit mention of straight lines and parabolas. Similar to the description of the middle grades, the Swedish curriculum is more general compared to the Estonian and Finnish curricula at this stage as well. In addition, as mentioned earlier, graphs also appear in the previous subtheme on functional relations (Table 6) for Grades 7–9 in both the Estonian and Finnish curricula, and in the next big idea, proportional reasoning (Table 7).

**PR – proportional reasoning**

PR consists of the three subthemes *Proportional relationships* (see Table 7), *Specific applications of proportional reasoning*, and *Percentage*.

The Swedish curriculum addresses proportional relationship throughout the Grades. Simple proportional relationships including doubling and halving are introduced already in Grades 1–3 and students are expected to use and give examples of simple proportional relationships in “familiar situations”. The Estonian and Finnish curriculum address the idea of proportionality and proportional relationships first in Grades 7–9 and in addition introduce inverse proportional relationships, which is missing in the Swedish document (Table 7).

The second subtheme of PR concerns *specific applications of proportional reasoning*. One common approach in all three countries’ curricula is to use the scale as context to illustrate the concept of proportionality. In Sweden, the scale, with simple examples of enlargements and reductions, is introduced already in Grades 1–3 and in Grades 7–9 students are expected to be able to perform the enlargement and reduction of both two- and three-dimensional objects. The other two curricula prescribe the scale-based applications only in the content of the middle grades. Another application in which students’ proportional reasoning can be developed is the context of measurement. All three curricula mention unit conversions, but for different grades. The Estonian document addresses them already in Grades 1–3, Finland in Grades 3–6, and Sweden in Grades 7–9. Still another application is in the area of geometry. Both the Estonian and Finnish documents prescribe introduction of the concepts of congruence and similarity of polygons in Grades 7–9. The Estonian document also mentions similarity properties of triangles. In the Swedish curriculum, these content elements are not treated at all.

The treatment of the third subtheme, *Percentage*, is very similar in all three countries. The concept of percentage and calculations, with simple examples, is first mentioned in the middle grades in all three curricula. In the content for Grades 7–9, all three curricula prescribe the

| Table 7. The distribution of the topics connected to the subtheme *Proportional relationships* across Grades 1–9. |
|---------------------------------------------------------------|
| **Estonia** | **Finland** | **Sweden** |
|---|---|---|
| 1-2/3 | Different proportional relationships, including doubling and halving. Using and giving examples of simple proportional relationships in situations relevant to the students. Graphs for expressing different types of proportional relations in simple investigations. | |
| 3/4-6 | Proportional and inversely proportional relations. Proportional equations and decompositions. Meaning of proportional dependence based on real-life examples. | Using proportions in problem solving. Direct and inverse proportionality. |
| 7-9 | | |
development of students’ abilities to use and calculate percentages. Similarly, all curricula introduce the concept of percentage of change.

**VAR – variable**

VAR consists of the two subthemes *Unknown number* and *Variable* (Table 8). While the Finnish and the Swedish documents only consider the concept of the unknown (number) in Grades 1–6 and introduce the concept of variable in Grades 7–9, the Estonian curriculum includes both concepts from the very beginning. The Estonian document is also more explicit in addressing the *symbolic representation* of the unknown. It also connects the concept of variable to expressions as well as simplifying and calculating the value of expressions from Grades 4–6, and to the concept of function in Grades 7–9.

**Conclusions and discussion**

We have analyzed and compared the Estonian, Finnish, and Swedish national core curricula for Grades 1–9 as to how they address items connected to big ideas in algebra. In this section, we first state our main conclusions and thereafter, in more detail discuss the conclusions and the results they are based on in relation to the frameworks on development of algebraic thinking and different approaches to algebra identified in earlier research. Finally, we close this section by discussing the different approaches in relation to the differences in student achievements in international comparisons and suggest further research.

The most important insight from our study is that there is a clear difference between the Estonian approach on the one hand and the Finnish and Swedish approaches on the other. We find a clear influence of the Russian Davydov School on the Estonian curriculum document. Another important result from our study is that both the Estonian and Finnish lower secondary curricula address more formal and sophisticated algebraic methods, including symbolic manipulations within several big ideas, while this kind of formal treatment is lacking in the Swedish curriculum. In terms of the distinction between the structural and functional reformist approach to algebra learning (cf. Cai et al., 2010; Kieran, 2007), the Finnish and the Estonian curriculum, in many cases resemble the structural approach although the Finnish curriculum addresses some items connected to functional thinking already at the lower grades. In contrast, the Swedish curriculum has more similarities with

| Table 8. Distribution of the topics connected to the subthemes of VAR across Grades 1–9. |
|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| Unknown number                              | Unknown numbers and their properties.       | Unknown numbers and their properties.       |
| 1-2/3                                       | The symbolic representation of the unknown. | The concept of the unknown.                 |
| 3/4-6                                       |                                              | Situations in which there is a need to represent an unknown number with a symbol. |
| 7-9                                         |                                              |                                              |
| Variable                                    | The concepts of number and variable as its representation. |                                              |
| 1-2/3                                       | Using letters to express the relationships between quantities. |                                              |
| 3/4-6                                       | Simplifying expressions with one variable.   | Calculating the value of letter expressions. |
| 7-9                                         | Variable quantity and function.             |                                              |
|                                              | The concept of variable.                    | The meaning of the variable concept.         |
|                                              |                                              | The use of variable in algebraic expressions, formulas and equations. |
the functional reformist approach than the other two countries. In Figure 1 below, we have summarized the results presented in the tables in the previous section.

Next, we will discuss our conclusions in more detail with respect to the relevant literature considered in the previous sections. The discussion is organized following Blanton et al.’s (2015) five big ideas, one at a time. We have created two figures to summarize the number of different topics within EEEI (Figure 2) on the one hand, and the subthemes Patterns, Functional relations and Proportional relationships (Figure 3) on the other hand. We have to bear in mind that our analysis is qualitative and the topics gathered in the figures from the tables are written at different level of generality. For example, we do not know what “Solving equation” exactly means in the Swedish lower secondary curriculum. However, the lack of specification shows that there is no emphasis on more formal issues in the Swedish context.

EEEI was divided into the three subthemes Equivalence, inequalities and equations, Handling of expressions, and Modeling and solving word problems. In Figure 2 below, we illustrate the number of different topics within EEEI at different grade levels in each country. Concerning the first and third subtheme, the Estonian and Swedish curricula attempt to create a basis for understanding and solving equations already from the first grades.

However, only the Swedish curriculum explicitly addresses the importance of understanding the meaning of the equal sign which has been stressed as utmost important by several researchers (e.g., Blanton et al., 2015; Kieran, 2004; Prediger, 2010; Röj-Lindberg et al., 2017), especially in order to prevent later problems when moving from an operational to a relational understanding of the equal sign (Jupri et al., 2014; Kieran, 1981). The emphasis of using the inverse properties of operations in the Estonian curriculum in Grades 1–3 is also typical in the curricula for elementary grades in China, Singapore, and South Korea (Cai et al., 2005) as well as within the Davydov curriculum (Davydov, 1990; Schmittau, 2004). The ability to move between real-world problems and mathematics is important for students to develop, in order to model and solve problems with the help of algebra (e.g., Van den Heuvel-panhuizen, 2003). This is clearly developed in the Estonian curriculum already from the first grades, in line with the Davydov curriculum (e.g., Davydov, 1990; Schmittau, 2005). In contrast, we identify a similarity between the Finnish and Swedish elementary-level approach to the

![Figure 1. Summarizing Tables 2–8.](image-url)
US Investigations, whereby problem solving is not systematically connected to modeling but rather to students’ investigations (Cai et al., 2005). At the secondary level, only the Estonian and Finnish curricula expect students to develop sophisticated methods for solving equations that are more complex. This could be due to the Swedish attempts to make algebra available for all students and avoid manipulations of symbols without relevance to students’ everyday life (Hemmi et al., 2013).

The second subtheme of EEEI, Handling of expressions, is an important ability for students to have in order to solve algebraic problems (e.g., Bokhove & Drijvers, 2010; Jupri et al., 2014) and it is possible to develop students’ ability to work with complex expressions already during the early grades (e.g., Hewitt, 2014). The Estonian curriculum displays a clear path concerning students’ development of handling expressions throughout Grades 1–9, and shows higher expectations for
students’ learning in all grades than the two other curricula do. A focus on both representing and solving a problem instead of merely solving it (cf. Kieran, 2004) is fulfilled in the Estonian curriculum from the early grades. Furthermore, the Estonian curriculum document has a strong emphasis on describing activities connected to symbolic manipulations. In terms of the distinction between the structural and functional approaches to algebra learning (cf. Cai et al., 2010; Kieran, 2007) the Estonian curriculum, in this case, is certainly closer to the former. The Finnish document reflects a norm that has been previously used in many Western countries, targeting algebraic expressions only in the goals and content for the upper grades, and in a quite formal manner. This is also typical of the so-called structural approach to algebra learning (Cai et al., 2010; Kieran, 2007). In the Swedish curriculum document, the term algebraic expression is addressed for Grades 4–9 but descriptions regarding the handling of the expressions are missing. As we are about to see, in connection with the big idea FT below, the Swedish curriculum is more related to the functional reformist approach on algebra learning (cf. Cai et al., 2010; Kieran, 2007) compared to the other two countries’ curricula.

Generalized Arithmetic was divided into the two subthemes Relations between arithmetic operations and Arithmetic rules. Regarding the former, all three curricula to some extent focus on relations and not only calculations (cf. Kieran, 2004). The Estonian curriculum addresses this in the most explicit manner. All three countries prescribe the relationship between different arithmetic operations from the first grades. The Estonian curriculum even focuses on the names of the components of the terms in the operations, which we interpret as generalizing the reverse relationships. This again is similar to the Davydov curriculum (Cai et al., 2005; Schmittau & Morris, 2004), in which theorizing from examples was the aim already during the first school grades (Davydov, 1990). In connection to the second subtheme of GA, a common cause of problems experienced by 12-14-year-old students is their lack of ability to apply properties of operations in their calculations (e.g., Jupri et al., 2014). While all three curricula mention properties of operations, only the Finnish curriculum explicitly addresses the use of properties of operations for achieving fluency in mental and written arithmetic throughout the elementary level. In line with the study by Bråting et al. (2019), generalized arithmetic is poorly developed in the Swedish curriculum although it is emphasized by several researchers as one of the most relevant topics within early algebra (e.g., Kaput, 2008; Kieran, 2018; Kieran et al., 2016). In fact, the term generalization and the use of rules are missing through all grade levels in the Swedish curriculum document.

Regarding the next two big ideas, Functional Thinking and Proportional Reasoning, there are several differences between the three countries. In connection to the subthemes Patterns, Functional relations and Proportional relationships (see Figure 3), we find it particularly convenient to discuss our results in relation to the structural and functional approaches to algebra learning (cf. Cai et al., 2010; Kieran, 2007). Regarding patterns, the Estonian curriculum contains no items at all at any grade level, which, again, is in line with the Davydov curricula (see Cai et al., 2005; Schmittau & Morris, 2004). In contrast, the Finnish and Swedish curricula address patterns already from the first grades.

As to the subtheme Functional relations, the Estonian and Finnish curricula address it in detail at the secondary level and, in a similar way as within EEEI, the expectations regarding sophisticated methods and formal algebra are high. This is typical for the structural perspective of algebra learning (Cai et al., 2010; Kieran, 2007). In the Swedish curriculum, we can instead identify some tendency toward the functional reformist approach (Cai et al., 2010; Kieran, 2007). The concept of change is clearly emphasized and a separate heading “Relationships and Change” is used throughout the Swedish content descriptions for all grade levels. Moreover, the Swedish curriculum stresses the importance of familiar real-world connections, and we interpret that the real-world problems are to be solved using other methods than manual symbolic manipulations, because there are no items referring to them. This is typical for the functional approach (c.f. Cai et al., 2010; Kieran, 2007). In addition, the heavy emphasis on the development of proportional relationships from the very beginning in the Swedish curriculum, different from the other two countries, can be connected to
the functional approach (cf. Cai et al., 2010; Kieran, 2007). Regarding the Estonian and Finnish curricula, proportional relationships are not mentioned until Grades 7–9, which is similar to the former tradition in the Western countries where algebraic content, such as proportional relationships, was not introduced until the upper grades (cf. Kieran, 2007).

VAR consists of the two subthemes Unknown number and Variable. Several researchers state that students need various experiences of using symbols in different kinds of expressions in order to develop their symbol sense (e.g., Arcavi, 1994). A symbolic representation is addressed in both Estonian (Grades 1–3) and Swedish (Grades 4–6) curriculum while it is not explicitly focused on in the Finnish curriculum. Focusing on both numbers and letters from the beginning is important for the transition from arithmetic to algebraic thinking (e.g., Arcavi, 1994; Kieran, 2004). Furthermore, a poor understanding of the notion of variable causes problems for students when working with algebra (e.g., Jupri et al., 2014; Kilhamn, 2014; Usiskin, 1988). In the Estonian curriculum, the progression is displayed through a description of the contexts in which both the concept of the unknown and variable are dealt with from Grades 1–3. The Estonian approach clearly reflects the Davydov curriculum idea to ascend from the general to the specific (Davydov, 1990; Davydov & Rubtsov, 2018). We find it interesting that here, the Swedish curriculum differs from the functional reformist approach that considers the notion of variable as “quantities that change” already at the elementary level (Cai et al., 2010). Instead, both the Swedish and the Finnish curriculum here resemble a typical structural approach, whereby variables are treated as unknown placeholders in expressions and equations (cf. Cai et al., 2010; Kieran, 2007).

We close with a brief discussion about the results in relation to the different student outcomes according to international and national evaluations and suggest further research. As earlier stated, we cannot draw any simple connections between the results of our study and student learning as there are other factors influencing the efficiency of education. Yet, our study offers one piece of a puzzle for the field and together with other studies may help us understand the complex relation between the intended and attained curriculum (e.g., Valverde et al., 2002) in different cultural-educational contexts. It also shows the versatile nature of algebraic thinking and the importance of investigating different strands of curriculum more deeply.

Swedish students’ algebra results in international evaluations have been poor despite the efforts to include algebra in school curriculum from the very beginning (Hemmi et al., 2018). This could be partly explained by the low expectations when moving from informal methods to more sophisticated algebraic methods at the secondary level and the lack of focus in generalizations. Obviously, it is not enough to develop students’ understanding of the meaning of the equal sign if other parts of algebraic thinking are not connected to it. As for PISA, in which concrete real-life problems are solved, in contrast to the situation in Sweden, Estonian and Finnish students have had good results (OECD, 2019). We find this interesting, considering the traditional features in the Estonian and Finnish curricula at lower secondary level and the strong emphasis on investigations in familiar situations connected to equations in the Swedish curriculum. However, several abilities connected to the development of algebraic thinking are visible in the PISA test level framework (OECD, 2018). While it is possible to manage the problems at the lowest level with investigating and testing in specific contexts, the higher levels demand that students can work with explicit models for complex concrete situations. Moreover, students are to use different representations, including symbolic and formal characterizations linking them to aspects of real-world situations. At the highest level ability, to conceptualize, generalize, and utilize information based on investigations and modeling of complex problem situations is needed (OECD, 2018). Hence, concerning the Estonian success, the explanation could lie in the systematic hypothetic learning progression within the modeling of word problems. The ability to translate back and forth between the problem situation and mathematics is a challenge for students (e.g., Bell, 1996; Jupri et al., 2014) and the early focus on it could help students also to develop generalizations. The Finnish success in PISA cannot easily be explained by the core curriculum. At least, there are high demands of learning more sophisticated methods at lower secondary level and problem solving is also addressed at all school levels. The textbook studies
we have conducted so far (Hemmi et al., 2019) also show that inverse properties of operations and expressions are stressed in the Finnish textbooks from Grade 1 although they are not addressed in the national core curriculum (for a further discussion of the Finnish students’ success in mathematics, see for example, Andrews et al., 2014).

It is tempting to consider the results of this study in terms of East and West. We know that the Swedish national curriculum is influenced by the same reform ideas as expressed in the NCTM (1989, 2000) Standards and Adding it up (Kilpatrick et al., 2001) (Bergqvist & Bergqvist, 2017). This could explain the focus on the meaning of the equal sign and the functional approach. We find it surprising that several features in the Estonian approach resemble the ideas of Davydov School, since the Estonian curriculum does not officially lean on Davydov’s learning theories. However, Estonia was part of the Soviet Union at the time Davydov and colleagues developed their theories and it is possible that Davydov’s ideas have survived in mathematics educators’ beliefs and practices, and therefore still affect the formulation of national goals and content. However, this finding should be investigated more deeply in further studies. Finnish and Estonian languages are related to each other and they belong to another language group than Swedish and English. However, there is a Swedish speaking minority in Finland, and Finland has a long history of being a part of Sweden. Hence, it is possible that Finland has been influenced by both Western and Eastern educational cultures.

We have to bear in mind that all three curriculum documents are quite general descriptions of content that mathematics students are expected to learn, and we need to look at curriculum materials (such as textbooks) more in depth to refine, and possibly challenge, the results of the present study. This study is part of a larger project (Hemmi et al., 2018), and at the time of writing we have also started analyzing mathematics textbooks for the first grades for some of the big ideas (Bråting et al., 2019; Hemmi et al., 2019). This initial analysis confirms the specificity of the Estonian approach and challenges parts of the Finnish results (see above) but raises other questions connected to intra-national variation in the approaches to algebra, which we will further investigate.

Notes
1. The Davydov curriculum (sometimes referred as Elkonin and Davydov Curriculum) was developed in the 1960s by a research group led by Davydov in the Soviet Union. The Investigations curriculum was developed with support from the National Research Council [NRC] (Cai et al., 2005) at a time when there was a growing awareness in the US of making the essential concepts within school algebra accessible to students before secondary-level education (Nathan & Koellner, 2007). The effects of these special curricula on the two countries’ school mathematics is outside the scope of our study.
2. https://eperusteet.opintopolku.fi/#/fi/perusopetus/419550/sisallot/466344https://www.riigiteataja.ee/aktilisa/1140/1201/1001/VV1_lisa3.pdfhttps://tinyurl.com/yawp2bay.
3. We use the term ‘big idea’ as we draw on Blanton et al. (2015) work. Yet, these big ideas could be considered as constructs or concepts.

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