Method of experimentally identifying the complex mode shape of the self-excited oscillation of a cantilevered pipe conveying fluid

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Abstract The dynamics of a flexible cantilevered pipe conveying fluid have been researched for several decades. It is known that the flexible pipe undergoes self-excited vibration when the flow speed exceeds a critical speed. This instability phenomenon is caused by nonconservative forces. From a mathematical point of view, the system has a characteristic of non-selfadjointness and the linear eigenmodes can be complex and non-orthogonal to each other. As a result, such a mathematical feature of the system is directly related to the instability phenomenon. In this study, we propose a method of experimentally identifying the complex mode from experimentally obtained time histories and decomposing the linear mode into real and imaginary components. In nonlinear analysis, we show that the nonlinear effects of practical systems on the mode in the steady-state self-excited oscillation are small. The real and imaginary components identified using the proposed method for experimental steady-state self-excited oscillations are compared with those obtained in the theoretical analysis, thus validating the proposed identification method.

Keywords Pipe conveying fluid · Complex mode · Experiment

1 Introduction

Instability phenomena of a flexible pipe conveying fluid have been investigated for several decades. It is known that the dynamics of a pipe depend on the support condition. A simply supported pipe loses stability through divergence [9], whereas a cantilevered pipe is dynamically destabilized by flutter [13]. The latter phenomenon results from a nonconservative follower force, which is the fluid force of the internal flow acting on the pipe orthogonally to the cross section of the pipe [29]. It is well known that when the flow speed of the fluid in the pipe exceeds a critical speed, the trivial steady-state amplitude becomes unstable and Hopf bifurcation occurs [4]. After the transient state, the pipe exhibits steady-state self-excited oscillation with a constant amplitude. From a mathematical point of view, the nonconservative force makes the system non-selfadjoint [16] and the eigenmodes can then be complex and non-orthogonal to each other.

The first serious study on the dynamics of a pipe conveying fluid was Bourrieres’s work [7]. After this work, Ashley and Haviland [2] investigated vibrations of oil pipelines. A pioneering theoretical study of the dynamic destabilization phenomenon due to such fluid–solid interaction was conducted using an articulated pipe by Benjamin [5]. Gregory and Pai-
doussis [11] conducted a theoretical analysis of continuous pipes with an infinite number of degrees of freedom. Bishop and Fawzy investigated the dynamics of a pipe subjected to sinusoidal force in addition to fluid forces [6]. After these studies, many associated studies [22,23] were performed owing to the characteristic excitation mechanism and the specific mathematical features that the system can have non-selfadjointness and non-orthogonal and complex eigenmodes. For example, there have been studies of the effects of pulsating internal fluid flow on motion and stability [14,24,33,39]. Early works, such as [5,6,11], focused on the planar motion of the pipe. Lundgren et al. and Bajaj and Sethna conducted a theoretical analysis of the nonplanar motion of a plain pipe [3,17]. Following these works, the nonplanar motions of pipes with an end mass [18] and spring support [10,30,31] were examined.

Most of the studies mentioned above investigated in detail the change in the eigenvalues, which relate to the stability of the system directly, with respect to changes in system parameters, but there has been less consideration of the complex eigenmodes due to non-selfadjointness [36,38] and no research from an experimental point of view to the best of our knowledge. The pioneering work of Benjamin [5] noted the characteristics of the mode shape but only qualitatively, and there has been no investigation of the mode shape since.

Incidentally, in most theoretical studies, the Galerkin discretization scheme has been used to examine the equation of the motion of a flexible pipe conveying fluid [11,15,28]. In this analytical approach, the modes are assumed by the modes of the free oscillation in which the effect of the internal flow is not taken into account. This assumption neglects the non-selfadjointness of the system, that the eigenmodes can be complex and nonorthogonality. These mathematical characteristics result from nonconservative forces directly and must therefore be considered in not only theoretical investigations but also experimental studies. It is thus necessary to express the vibration mode using a complex mode and to capture the mode shape through the superposition of its real and imaginary components. This approach exposes an aspect of the instability phenomena due to non-selfadjointness.

Experimental studies on the mode shape of instability vibration have been limited to observing stroboscopic photographs of the mode shape, as done by Gregory and Païdoussis [12] and Yoshizawa [38]. We cannot capture the complex mode from these photographs. Unlike the case when the eigenmodes are real, when the eigenmodes are complex, the phases of vibration at each point on the pipe are expressed as a function of time and position, and different points on the pipe thus have different phases. This characteristic of the mode is thought to produce characteristic vibration shapes such as the “dragging” shape noted in [5,12]. Such a feature of the mode cannot presently be detected using conventional methods such as stroboscopic photography. The vibration with a complex mode can also be regarded as a traveling wave [8,25]. However, in the present study, in order to clarify the effect of the non-selfadjointness due to the follower force in comparison with the vibration in the system without follower and gyroscopic forces, which is selfadjoint and has real orthogonal stationary modes, we focus on the expression using the standing wave. Then, a new method of obtaining a complex mode and expressing the real and imaginary components experimentally is needed.

As another method to describe the mode shape, the method based on POD (proper orthogonal decomposition) has been proposed [1,27]. It is assumed that the mode shape consists of the finite eigenmodes of the approximate governing equation in which the effect of the follower force is neglected. Because the eigenmodes are orthogonal to each other, they do not reflect the non-orthogonality of the eigenmodes due to non-selfadjointness of the original governing equation. On the other hand, our proposed method takes into account the non-orthogonality and can describe the destabilized complex mode. The method with POD is applicable to dynamics with multifrequency components including chaos [1], but the application of our proposed method is limited to the behavior with a single frequency component.

In this study, first, using linear theory, we obtain the complex-valued eigenmode theoretically considering the non-selfadjointness of the system. The eigenmode is shown as the composition of the real and imaginary components. We then clarify that the phase of the lateral vibration at each point on the pipe is a function of the distance from the supporting point. Next, nonlinear analysis shows that the nonlinear elastic and inertia forces play an important role in determining the magnitude of the amplitude in steady-state self-excited oscillation, but do not have a large effect on the eigenmode.

Second, using this knowledge, we present an experimental identification method for detecting the real
and imaginary components of an imaginary eigenmode shape of self-excited vibration from experimentally obtained data of the time histories of some points. In practice, using the time history data of steady-state self-excited oscillation obtained for a cantilevered pipe conveying fluid in our experimental apparatus, we experimentally identify the complex eigenmode and decompose the real and imaginary components. The comparison of the components with those obtained theoretically demonstrates the validity of the proposed experimental identification method for the complex eigenmode of self-excited oscillation generated by the non-selfadjointness.

2 Complex linear normal mode due to non-selfadjointness

2.1 Governing equation of a cantilevered elastic pipe conveying fluid

We consider the analytical model for the vibration of a cantilevered elastic pipe conveying fluid as shown in Fig. 1. The pipe is hung vertically under the influence of gravity $g$ and the upper end is clamped. The pipe is assumed to be a flexible and inextensible beam with flexural rigidity $EI$. The length, mass per unit length and cross-sectional flow area of the pipe are, respectively, denoted $l$, $m$ and $S$. The pipe is sufficiently long compared with its diameter and conveys an incompressible fluid having a mass per unit length $M$. The flow in the pipe is assumed to be constant and one-dimensional, and the flow velocity $U$ relative to the pipe is assumed to be parallel to the pipe centerline. Under special conditions, there is nonplanar motion [3, 34], but to gain a fundamental understanding of the complex eigenmode, we consider the motion in a vertical $x$–$y$ plane. We introduce the $x$–$y$ coordinate system, whose origin $O$ is located at the upper end of the pipe. In addition, we let $s$ be the curvilinear coordinate along the centerline of the pipe.

Let $v$ be the displacement along the pipe centerline in the $y$ direction. Using $l$ as the representative length $L$ and $\sqrt{(m + M)L^4/EI}$ as the representative time $T$, we introduce the dimensionless parameters $\beta$, $\gamma$ and $U^*$ as

$$\beta = \frac{M}{m + M}, \quad \gamma = \frac{(m + M)gl^3}{EI}, \quad U^* = \sqrt{\frac{Ml^2}{EI}}U, \quad (1)$$

where $\beta$, $\gamma$ and $U^*$, respectively, denote the ratio of the fluid mass to the total mass, the ratio of the gravity force to the elastic restoring force of the pipe, and the dimensionless flow velocity. The equation of the pipe conveying fluid can be written with terms up to the third order of $v^*$ in non-dimensional form [20, 37] as

$$\begin{align*}
v^{***} + U^2v^{**} + 2\sqrt{\beta}Uv' + \gamma \left\{ v' - (1 - s)v'' \right\} + v^* & = 0, \\
\{ v' \frac{\partial^2}{\partial t^2} \int_0^s \left( \frac{-1}{2}v^{*2} \right) ds - U\sqrt{\beta}v^{*2}v' + \frac{1}{2}v^{*2}v^* & = 0, \\
- \frac{1}{2}U^2v^{*2}v'' - \left( \frac{3}{2}v^{*3} + 3v^{*2}v'' + \frac{1}{2}v^{*2}v''' \right) & = 0, \\
v'' + \int_s^1 \left( \frac{\partial^2}{\partial t^2} \int_0^1 (-1/2v^{*2}) ds + v^{*2}v'' \right) ds & = 0, \\
- \frac{1}{2}\gamma v^{*2}v'' + \frac{1}{2}v^{*2}v'' \gamma (1 - s) & = 0, \\
+ \frac{1}{2}v^{*2}v'' \bigg|_{s=1} & = 0. \quad (2)
\end{align*}$$

where $()$ and $(\cdot)$, respectively, denote the derivatives with respect to dimensionless time $t$ and position $s$. Hereafter, we omit the $(\cdot)$ notation for simplicity. On the left-hand side of Eq. (2), the second, third and fourth terms are, respectively, associated with centrifugal, Coriolis and gravitational forces.
The boundary conditions for the pipe are expressed as

\[ s = 0 : \ v(0) = 0, \ v'(0) = 0, \]
\[ s = 1 : \ v''(1) = 0, \ v'''(1) = 0. \]  
(3)

2.2 Linear modal analysis

By neglecting the nonlinear terms with respect to \( v \) in Eq. (2), we obtain the linear partial differential equation

\[ v''' + (U^2 - \gamma (1-s))v'' + 2\sqrt{\beta U} v' + \gamma v' + \dot{v} = 0 \]  
(4)

and the associated boundary conditions of Eq. (3). The eigenfrequencies and associated eigenmodes, which are obtained from Eqs. (3) and (4), have been well investigated [22] but we summarize the modulations of the eigenfrequencies and eigenmodes depending on the flow velocity because they relate to the main topic of complex eigenmodes. We here express the mode shapes \( \Phi(s) \) using a power series of the coordinate \( s \) [38].

Let \( v \) be

\[ v = \sum_{n=1}^{\infty} A_n \Phi_n(s)e^{\lambda ns} + C.C., \]  
(5)

where \( A_n, \Phi_n \) and \( \lambda_n = i\omega_n = i(\omega_{nr} + i\omega_{ni}) \) are, respectively, the complex amplitude, \( n \)th eigenmode and \( n \)th eigenvalue, and \( C.C. \) denotes the complex conjugate of the proceeding term. \( \omega_{nr} \) is the natural frequency and \( \omega_{ni} \) corresponds to the damping ratio. By substituting Eq. (5) in Eq. (4), we obtain the equation governing \( \Phi_n(n = 1, 2, \ldots) \) as

\[ \Phi_n''' + (U^2 - \gamma (1-s))\Phi_n'' + (2i\omega_n\sqrt{\beta U} + \gamma)\Phi_n' - \omega_n^2\Phi_n = 0, \]  
(6)

where \( \Phi_n \) can be complex. We can then express \( \Phi_n(s) \). From the boundary conditions given as Eq. (3), we have the associated boundary conditions

\[ \begin{cases} \Phi_n(0) = 0, \Phi_n'(0) = 0 \\ \Phi_n''(1) = 0, \Phi_n'''(1) = 0. \end{cases} \]  
(7)

We express \( \Phi_n \) using a power series [21,35,38] as

\[ \Phi_n = \begin{cases} \begin{array}{ll} 0\Phi_n = \sum_{j=0}^{\infty} \frac{a_j}{j!} s^j & (0 \leq s \leq 0.5) \\ 1\Phi_n = \sum_{j=0}^{\infty} \frac{b_j}{j!} (1-s)^j & (0.5 \leq s \leq 1). \end{array} \end{cases} \]  
(8)

From the boundary conditions, we have

\[ a_0 = 0, a_1 = 0, \]  
(9)

\[ b_2 = 0, b_3 = 0. \]  
(10)

We then determine the coefficients \( a_j \) and \( b_j \) \((j = 0, 1, 2, \ldots)\). We initially consider the mode in the range of \( 0 \leq s \leq 0.5 \). Substituting Eq. (8) in Eq. (6), we obtain

\[ \begin{array}{l} [a_{4s}^0 + a_{3s}^1 + \cdots + \frac{1}{j!}a_{j+4s}^j + \cdots] \\ + (U^2 - \gamma)[a_{2s}^0 + a_{3s}^1 + \cdots + \frac{1}{j!}a_{j+2s}^j + \cdots] \\ + \gamma[a_{2s}^1 + a_{3s}^2 + \cdots + \frac{1}{(j-1)!}a_{j+3s}^j + \cdots] \\ + (2i\omega_n\sqrt{\beta U} + \gamma) \\ [a_{1s}^0 + a_{2s}^1 + \cdots + \frac{1}{j!}a_{j+1s}^j + \cdots] \\ - \omega_n^2(a_0s^0 + a_1s^1 + \cdots + \frac{a_{j}s^j}{j!} + \cdots) = 0. \end{array} \]  
(11)

Equating the coefficients of like powers of \( s \) yields

\[ \begin{array}{l} s^0: \ a_4 + (U^2 - \gamma)a_2 \\ + (2i\omega_n\sqrt{\beta U} + \gamma)a_1 - \omega_n^2a_0 = 0 \\ s^1: \ a_5 + (U^2 - \gamma)a_3 \\ + \gamma a_2 + (2i\omega_n\sqrt{\beta U} + \gamma)a_2 - \omega_n^2a_1 = 0 \\ \vdots \\ s^j: \ \frac{1}{j!}a_{j+4} + \frac{(U^2 - \gamma)}{j!}a_{j+2} + \frac{\gamma}{(j-1)!}a_{j+1} \\ + \frac{2i\omega_n\sqrt{\beta U} + \gamma}{j!}a_{j+1} - \frac{\omega_n^2}{j!}a_j = 0. \end{array} \]  
(12)

Considering the boundary conditions given as Eq. (9), we express \( a_j \) in Eq. (12) using only \( a_2 \) and \( a_3 \); we take the coefficients of \( a_2 \) and \( a_3 \) in \( a_j \) as \( ^{00}_{00}\Phi_n \) and \( ^1_0\Phi_n \), respectively. We then obtain \( ^{00}_{00}\Phi_n \) and \( ^1_0\Phi_n \) from Eq. (12) as follows. For \( 0 \leq j \leq 3 \):

\[ \begin{array}{l} ^{00}_{00}\Phi_n = 0, \ ^1_0\Phi_n = 0 \\ ^{00}_{00}\Phi_n = 0, \ ^1_0\Phi_n = 0 \\ ^{01}_{00}\Phi_n = 1, \ ^1_0\Phi_n = 0 \\ ^{02}_{00}\Phi_n = 0, \ ^1_0\Phi_n = 1. \end{array} \]  
(13)

For \( j \geq 4 \):

\[ ^{00}_{00}\Phi_n = (\gamma - U^2)^{00}_{j-2}\Phi_n \\ - 2i\omega_n\sqrt{\beta U} - \gamma(j-3)\Phi_n \]

\[ ^{00}_{00}\Phi_n + \omega_n^2^{00}_{j-4}\Phi_n. \]  
(14)
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\[ 0^1 \Phi_n = (\gamma - U^2) \left( j - 2 \right)^{0^1} \Phi_n \]
\[ - \left\{ 2i \omega_n \sqrt{\beta} U + \gamma (j - 3) \right\} \left( j - 3 \right)^{0^1} \Phi_n + \omega_n^2 \left( j - 4 \right)^{0^1} \Phi_n. \]  

(15)

Using \(0^0 \Phi_n\) and \(0^1 \Phi_n\), we can rewrite \(0^0 \Phi_n\) in Eq. (8) as

\[ 0^0 \Phi_n(s) = a_0 s^0 + a_1 s^1 + \cdots + \frac{1}{j!} a_j s^j + \cdots \]
\[ = a_2 \sum_{j=0}^{\infty} \frac{1}{j!} 0^0 \Phi_n s^j + a_3 \sum_{j=0}^{\infty} \frac{1}{j!} 0^1 \Phi_n s^j, \]

(16)

and taking \(\sum_{j=0}^{\infty} \frac{1}{j!} 0^0 \Phi_n s^j\) and \(\sum_{j=0}^{\infty} \frac{1}{j!} 0^1 \Phi_n s^j\) as \(0^0 \Phi_n(s)\) and \(0^1 \Phi_n(s)\), we can express the \(n\)th eigenmode \(\Phi_n(s)\) at \(0 \leq s \leq 0.5\) as \(0^\Phi_n\) with the simple form

\[ 0^\Phi_n(s) = a_2 0^0 \Phi_n(s) + a_3 0^1 \Phi_n(s). \]

(17)

We next consider the mode in the range of \(0.5 \leq s \leq 1\). We can determine the coefficients of the term \((1 - s)^j\) of the \(n\)th eigenmode \(\Phi_n\) at \(0.5 \leq s \leq 1\) as \(a_j\) in the same manner as for \(a_j\). In this case, by substituting Eq. (8) in Eq. (6), we obtain

\[ (1 - s)^0 : b_4 + U^2 b_2 \]
\[ - (2i \omega_n \sqrt{\beta} U + \gamma) b_1 - \omega_n^2 b_0 = 0 \]
\[ (1 - s)^1 : b_5 + U^2 b_3 \]
\[ - \frac{1}{j!} b_j + \frac{U^2}{j!} b_j + \frac{\gamma}{(j - 1)!} b_{j+1} \]
\[ - \frac{2i \omega_n \sqrt{\beta} U + \gamma}{j!} b_{j+1} - \frac{\omega_n^2}{j!} b_j = 0. \]

(18)

and taking into account the boundary conditions of Eq. (10), we can express \(b_j\) in Eq. (18) using only \(b_0\) and \(b_1\); we take the coefficients of \(b_0\) and \(b_1\) in \(b_j\) as \(10^0 \Phi_n\) and \(11^j \Phi_n\), respectively. Then we obtain \(10^j \Phi_n\) and \(11^j \Phi_n\) as \(0 \leq j \leq 3:\)

\[
\begin{align*}
10^0 \Phi_n = & \begin{cases} 1, & 11^0 \Phi_n = 0 \\
0, & 11^1 \Phi_n = 1 \\
0, & 11^2 \Phi_n = 0 \\
0, & 11^3 \Phi_n = 0 \\
\end{cases} \\
10^1 \Phi_n = & \begin{cases} 1, & 11^0 \Phi_n = 0 \\
0, & 11^1 \Phi_n = 1 \\
0, & 11^2 \Phi_n = 0 \\
0, & 11^3 \Phi_n = 0 \\
\end{cases} \\
10^2 \Phi_n = & \begin{cases} 1, & 11^0 \Phi_n = 0 \\
0, & 11^1 \Phi_n = 1 \\
0, & 11^2 \Phi_n = 0 \\
0, & 11^3 \Phi_n = 0 \\
\end{cases} \\
10^3 \Phi_n = & \begin{cases} 1, & 11^0 \Phi_n = 0 \\
0, & 11^1 \Phi_n = 1 \\
0, & 11^2 \Phi_n = 0 \\
0, & 11^3 \Phi_n = 0 \\
\end{cases} \\
\end{align*}
\]

(19)

\[ j \geq 4: \]
\[ 10^j \Phi_n = -U^2 10^j \Phi_n + \left\{ 2i \omega_n \sqrt{\beta} U + \gamma (j - 3) \right\} 10^j \Phi_n + \omega_n^2 10^j \Phi_n. \]

(20)

\[ 11^j \Phi_n = -U^2 11^j \Phi_n + \left\{ 2i \omega_n \sqrt{\beta} U + \gamma (j - 3) \right\} 11^j \Phi_n + \omega_n^2 11^j \Phi_n. \]

(21)

Letting \(\sum_{j=0}^{\infty} \frac{1}{j!} 10^j \Phi_n (1 - s)^j\) and \(\sum_{j=0}^{\infty} \frac{1}{j!} 11^j \Phi_n (1 - s)^j\) be \(10^j \Phi_n(s)\) and \(11^j \Phi_n(s)\), respectively, we can express the \(n\)th eigenmode \(\Phi_n\) at \(0.5 \leq s \leq 1\) as \(1^j \Phi_n(s)\) by

\[ 1^j \Phi_n(s) = b_0 10^0 \Phi_n(s) + b_1 11^1 \Phi_n(s). \]

(22)

In order to put them together as eigenmodes defined from \(0 \leq s \leq 1\), we need to connect them at \(s = 0.5\). Then, \(0^\Phi_n\) and \(1^\Phi_n\) in Eqs. (17) and (22) include the four unknown constants, \(a_2\), \(a_3\), \(b_0\) and \(b_1\). To obtain the ratio of these unknown constants, we impose the four conditions that \(0^\Phi_n\), \(1^\Phi_n\) and their derivatives from first to third order are the same at \(s = 0.5:\)

\[
\begin{align*}
0^\Phi_n(0.5) = & \ 1^\Phi_n(0.5) \\
0^\Phi_n'(0.5) = & \ 1^\Phi_n'(0.5) \\
0^\Phi_n''(0.5) = & \ 1^\Phi_n''(0.5) \\
0^\Phi_n'''(0.5) = & \ 1^\Phi_n'''(0.5) \\
\end{align*}
\]

(23)

These equations are expressed as homogeneous equations in terms of \(a_2\), \(a_3\), \(b_0\) and \(b_1:\)

\[
\begin{bmatrix}
0^0 \Phi_n(0.5) & 0^1 \Phi_n(0.5) & -1^1 \Phi_n(0.5) & -1^1 \Phi_n(0.5) \\
0^0 \Phi_n'(0.5) & 0^1 \Phi_n'(0.5) & -1^1 \Phi_n'(0.5) & -1^1 \Phi_n'(0.5) \\
0^0 \Phi_n''(0.5) & 0^1 \Phi_n''(0.5) & -1^1 \Phi_n''(0.5) & -1^1 \Phi_n''(0.5) \\
0^0 \Phi_n'''(0.5) & 0^1 \Phi_n'''(0.5) & -1^1 \Phi_n'''(0.5) & -1^1 \Phi_n'''(0.5)
\end{bmatrix}
\begin{bmatrix}
a_2 \\
a_3 \\
b_0 \\
b_1
\end{bmatrix}
\]

= 0.

(24)

To have nontrivial solutions for \(a_2\), \(a_3\), \(b_0\) and \(b_1\), the determinant of the coefficient matrix in Eq. (24) must be zero; because the imaginary and real parts must be zero, we have two real equations for the condition. By solving these two equations, we obtain \(\omega_n\) for the \(n\)th mode. We can then obtain the complex-valued coefficients of each power of \(s\) in Eqs. (17) and (22) and describe the real and imaginary components of the complex eigenmode.

Considering the terms up to 80th order of the power series, we obtain from Eq. (24) \(\omega_n\) for the first,
second and third modes. Figure 2 shows the changes in \( \omega_r \) and \( \omega_i \) depending on \( U \) for the first, second and third modes, where \( \beta = 0.388 \) and \( \gamma = 74.2 \) corresponding to the experimental parameter values of Pipe 1 in Sect. 5. The numbers in the figure are values of the non-dimensional flow velocity \( U \). The first and second modes are stable for any fluid velocity because \( \omega_i > 0 \) independently of \( U \). Unlike the case for the first and second modes, \( \omega_i \) of the third mode becomes zero from positive values at \( U = 11.8 \equiv U_{cri} \) when \( U \) increases. The system is destabilized and undergoes self-excited oscillation with the third mode at \( U > U_{cri} \).

At the critical point, we obtain \( 0^0 \Phi, 0^1 \Phi, 1^0 \Phi \) and \( 1^1 \Phi \) and a pair of \( a_2, a_3, b_0 \) and \( b_1 \) from Eq. (24). We obtain the theoretical third mode shape from these values. This mode is a complex function with respect to the arclength \( s \), and the real and imaginary components of the mode shape are presented in Fig. 3a and b, respectively, where the horizontal and vertical axes, respectively, give the normalized deflection and arclength \( s \). In the figure, the mode shapes are normalized by the absolute value \( \sqrt{\Phi_r^2(1) + \Phi_i^2(1)} \) of \( \Phi \) at \( s = 1 \).

3 Complex eigenmodes for nonlinear steady-state self-excited oscillation

Cantilevered elastic pipes conveying fluid essentially undergo self-excited oscillation due to the non-linear components of the restoring and inertia forces of a pipe necessarily exist, as are expressed by Eq. (2), there can be a limit cycle (i.e., a self-excited oscillation with a constant amplitude) after the transient state. In adopting the experimental identification method of complex modes proposed in Sect. 4, it is preferable to use the time history data for the steady state because the time history data for the transient state can include modes other than the targeted and destabilized mode.

This section examines the contribution of the cubic nonlinear components, which are responsible for the steady state in the self-excited oscillation, to the eigenmode. It is clarified that the mode in the steady-state self-excited oscillation resulting from the nonlinearities mentioned above can deviate from \( \Phi_0 \) in Sect. 2.2, which is obtained under the assumption of linearization.

To examine the steady-state self-excited oscillation with a constant amplitude, we consider the nonlinear effects expressed in Eq. (2), which are neglected in Sect. 2. To focus on the behavior in the neighborhood of the critical velocity for the \( n \)th mode, \( U = U_{ncr} \).
we introduce the detuning parameter $\Delta U$, which is assumed small, as

$$U = U_{ncr} + \Delta U_n = U_{ncr} + \epsilon \Delta \hat{U}_n,$$  \hspace{1cm} (25)

where $\epsilon$ is a small parameter and $\Delta \hat{U}_n = O(1)$. Additionally, we assume the solution to have a uniform expansion:

$$v = \epsilon^{\frac{1}{2}} \hat{v}_n + \epsilon \hat{v}_n + \cdots.$$  \hspace{1cm} (26)

Furthermore, we introduce the multiple time scales $t_0 = t$ and $t_1 = \epsilon t$. Substituting Eqs. (25) and (26) in Eqs. (2) and (3) and equating the coefficients of like powers of $\epsilon$ yields

$$\epsilon^{\frac{1}{2}} : \hat{v}_n = [U_{ncr} - \gamma(1-s)]\hat{v}_n'' + 2\sqrt{\beta}U_{ncr} D_0 \hat{v}_n + \gamma \hat{v}_n + D_0' \hat{v}_n = 0,$$  \hspace{1cm} (27)

$$s = 0 : \hat{v}_n(0) = 0, \quad \hat{v}_n'(0) = 0,$$  \hspace{1cm} (28)

$$s = 1 : \hat{v}_n''(1) = 0, \quad \hat{v}_n'''(1) = 0,$$  \hspace{1cm} (29)

where $D_0 = \partial / \partial t_0$ and $D_1 = \partial / \partial t_1$. In this paper, we consider the self-excited oscillation not with multiple modes but with a single nth mode. We let $\hat{v}_n$ and $\hat{v}_n$ be

$$\hat{v}_n = A_n(t) \Phi_n(s) e^{\lambda_n t_0} + C.C.,$$  \hspace{1cm} (30)

and by substituting these in Eqs. (27) to (30), we obtain equations that are satisfied by $\Phi_n$ and $\Phi_n$:

$$\epsilon^\frac{1}{2} : \Phi_n'' + (U_{ncr} - \gamma(1-s))\Phi_n'' + (2i\omega_n \sqrt{\beta}U_{ncr} + \gamma)\Phi_n - \omega_n^2 \Phi_n = 0,$$  \hspace{1cm} (31)

where $n(\Phi_n)$ is given in Appendix A. Because Eq. (32) is equivalent to Eq. (6), $\Phi_n$ corresponds to $\Phi_n$ in Sect. 2. Equations (32) and (33) are non-selfadjoint, and thus, to obtain the solvability condition [19] of $\Phi_n$, we need to use the adjoint function $\Psi_n$ with respect to $\Phi_n$, which is discussed in detail in Appendix B. Multiplying Eq. (34) by the adjoint function $\Psi_n$ (see Appendix B) and integrating the result over the interval from 0 to 1 with respect to $s$ yields

$$\int_0^1 [\Phi_n''' + (U_{ncr}^2 - \gamma(1-s))\Phi_n'''] + (2i\omega_n \sqrt{\beta}U_{ncr} + \gamma)\Phi_n'' - \omega_n^2 \Phi_n] \Psi_n ds =$$

$$= \int_0^1 [ -2\Delta \hat{U}_n U_{ncr} \Phi_n' - 2i\omega_n \sqrt{\beta} \Delta \hat{U}_n A_n \Phi_n' + |A_n|^2 A_n n(\Phi_n) \right] \Psi_n ds.$$  \hspace{1cm} (36)

Considering the boundary conditions given in Eq. (B.3), Eq. (36) leads to

$$D_1 A_n = \frac{K_1}{K_0} \Delta \hat{U}_n A_n + \frac{K_2}{K_0} |A_n|^2 A_n,$$  \hspace{1cm} (37)

where

$$K_0 = -\int_0^1 (-2\sqrt{\beta}U_{ncr} \Phi_n' - 2i\omega_n \Phi_n) \Psi_n ds,$$  \hspace{1cm} (38)

$$K_1 = \int_0^1 (2U_{ncr} \Phi_n'' + 2i\omega_n \sqrt{\beta} \Phi_n') \Psi_n ds,$$  \hspace{1cm} (39)

$$K_2 = -\int_0^1 n(\Phi_n) \Psi_n ds.$$  \hspace{1cm} (40)

Substituting $A_n = \frac{1}{2} \hat{a} e^{i\phi}$ in Eq. (37), we obtain equations expressing the time variation of the absolute value and the argument of the complex amplitude $A_n$ as

$$\frac{da}{dt} = k_{1r} \Delta U_n a + \frac{1}{4} k_{2a} a^3,$$  \hspace{1cm} (42)

$$\frac{d\phi}{dt} = k_{1i} \Delta U_n + \frac{1}{4} k_{2a} a^2,$$  \hspace{1cm} (43)

where $a = \epsilon^{\frac{1}{2}} \hat{a}$ and

$$k_{1r} = \text{Re} \left( \frac{K_1}{K_0} \right), \quad k_{1i} = \text{Im} \left( \frac{K_1}{K_0} \right)$$
When \( \Delta U_n = 0 \), the trivial steady-state amplitude \( a_{st} = 0 \) is stable. When \( \Delta U_n > 0 \), the trivial steady-state amplitude becomes unstable and the stable nontrivial steady-state amplitude emerges. The amplitude reaches the nontrivial steady-state amplitude after the transient state \( a_{st} = \sqrt{-4k_1 r \Delta U_n / k_2 r} \).

\[
k_{2r} = \text{Re} \left( \frac{K_2}{K_0} \right), \quad k_{2i} = \text{Im} \left( \frac{K_2}{K_0} \right).
\]

Letting \( \frac{da}{dt} = 0 \) in Eq. (39) gives the steady-state amplitude as

\[
a_{st} = 0, \quad \sqrt{-4k_1 r \Delta U_n / k_2 r}.
\]

It is known that a supercritical Hopf bifurcation occurs if a pipe is sufficiently slender [4]. Thus, in this study, because we consider the pipe to be slender, \( k_1 r, k_2 r < 0 \) and we obtain the bifurcation diagram with the control parameter \( \Delta U \) shown in Fig. 4 using Eq. (42). The trivial steady-state amplitude \( a_{st} = 0 \) is stable when \( \Delta U_n \) is negative. When \( \Delta U_n \) becomes positive, the trivial steady-state amplitude becomes unstable and the amplitude increases until the magnitude reaches the nontrivial steady-state amplitude \( a_{st} = \sqrt{-4k_1 r \Delta U_n / k_2 r} \).

In the steady state, \( v \) is obtained from Eqs. (26) and (31) as

\[
v = \frac{a_{st}}{2} \Phi_{n1}(s) e^{i(\omega_r t + \phi)} + C.C. + O(\epsilon^2),
\]

where \( \Phi_{n1}(s) \) can be complex and expressed as \( \Phi_{n1}(s) = \Phi_{nr}(s) + i \Phi_{ni}(s) \). In the absence of a non-conservative force, \( \Phi_n(s) \) is real and \( \Phi_{ni}(s) = 0 \) at all \( s \).

The leading-order term is expressed by the linear mode derived in Sect. 2. Therefore, the contribution of the nonlinearity of the system to the mode is very small and \( O(\epsilon^2) \). We can regard the mode of self-excited vibration in the steady state, which is experimentally observed in the steady state, as the linear mode approximately.

Substituting \( \Phi_n(s) = \Phi_{nr}(s) + i \Phi_{ni}(s) \) in Eq. (43), we obtain

\[
v = a_{st} \left\{ \Phi_{nr}(s) \cos(\omega_r t + \phi) - \Phi_{ni}(s) \sin(\omega_r t + \phi) \right\} + O(\epsilon^3)
\]

\[
= a_{st} \left\{ |\Phi_n(s)| \cos(\omega_r t + \phi + \psi(s)) + O(\epsilon^3) \right\}
\]

\[
= a_{st} \left\{ |\Phi_n(s)| \cos \psi(s) \cos(\omega_r t + \phi) - |\Phi_n(s)| \sin \psi(s) \sin(\omega_r t + \phi) \right\},
\]

where

\[
\cos \psi(s) = \frac{\Phi_{nr}(s)}{\sqrt{\Phi_{nr}^2(s) + \Phi_{ni}^2(s)}},
\]

\[
\sin \psi(s) = \frac{\Phi_{ni}(s)}{\sqrt{\Phi_{nr}^2(s) + \Phi_{ni}^2(s)}},
\]

\[
|\Phi_n(s)| = \sqrt{\Phi_{nr}^2(s) + \Phi_{ni}^2(s)}.
\]

From Eq. (44), we can express \( \Phi_{nr}(s) \) and \( \Phi_{ni}(s) \) as

\[
\Phi_{nr}(s) = |\Phi_n(s)| \cos \psi(s),
\]

\[
\Phi_{ni}(s) = |\Phi_n(s)| \sin \psi(s).
\]

Therefore, because the mode is complex, as seen from Eq. (44), the phase \( \omega_r t + \phi + \psi(s) \) depends not only on time \( t \) but also arclength \( s \), that is, the position on the pipe. We cannot determine the eigenmode from the deflection shape of the pipe at a moment in time in stroboscopic photographs of the vibration. Figure 5 presents stroboscopic photographs of Pipe 1 obtained in the experiment described in Sect. 5; the vertical line corresponds to the \( x \) axis in theory and is the position of the pipe in its rest state. The photographs were taken at 1/6 intervals of the period. A blue mark was placed on the upper part of the pipe and a red mark on the lower part for observation of the motion. The figure shows that both move in the same direction in Fig. 5a–c and d–f. Meanwhile, in Fig. 5c and d, we see that the absolute value of the displacement of the upper mark decreases while that of the lower mark increases. Such motion cannot be explained by any real modes.
and indicates that the eigenmode is complex and the phase depends on the arclength $s$ as mentioned above; the characteristic motion corresponds to that reported by Gregory and Paidoussis [12]. It is impossible to identify the real and imaginary components of the complex eigenmode from experimentally obtained time histories of the steady-state self-excited oscillation in the next section.

**4 Method of experimentally identifying the complex eigenmode**

As shown in Sect. 3, the nonlinear effects play a role in determining the magnitude of a steady-state amplitude but hardly modify the linear mode shape. Therefore, using the experimentally obtained steady-state self-excited oscillations, we propose a method of experimentally identifying the real and imaginary components of the linear complex mode, that is, $\Phi_{nr}(s)$ and $\Phi_{ni}(s)$ in Eq. (44).

Equation (2) is autonomous, and thus without loss of generality, we can introduce a new time $t_e = t + [\phi + \psi(s_0)]/\omega$ so that the initial phase $\psi(s) + \phi$ at a point $s = s_0$ is zero. Substituting $t = t_e - [\phi + \psi(s_0)]/\omega$ in Eq. (44), we obtain

$$v = a|\Phi_n(s)|[\cos \psi(s) \cos(\omega t + \phi)] - \sin \psi(s) \sin(\omega t + \phi)$$

$$= a|\Phi_n(s)|[\cos \Delta \psi(s) \cos \omega t_e - \sin \Delta \psi(s) \sin \omega t_e]$$

$$= a|\Phi_n(s)| \cos(\omega t_e + \Delta \psi(s))$$

$$= a\{\Phi_{nr}(s_e) \cos(\omega t_e) - \Phi_{ni}(s_e) \sin(\omega t_e)\},$$

(47)

where $\Delta \psi(s) = \psi(s) - \psi(s_0)$, $\Phi_{nr}'(s) = |\Phi_n(s)| \cos \Delta \psi(s)$ and $\Phi_{ni}'(s) = |\Phi_n(s)| \sin \Delta \psi(s)$. Hereafter, we omit the ($'$) notation for simplicity. By placing $N$ measurement markers on the pipe, we obtain $N$ time histories in the experiment as shown in Sect. 5. The displacements at the $n$th ($1 \leq n \leq N$) marker $v_{en}$ are expressed as

$$v_{e1}(s_1, t_e) = a|\Phi_{en}(s_1)| \cos(\omega t_e + \Delta \psi_{1k}),$$

$$v_{e2}(s_2, t_e) = a|\Phi_{en}(s_2)| \cos(\omega t_e + \Delta \psi_{2k}),$$

$$\vdots$$

$$v_{eN}(s_N, t_e) = a|\Phi_{en}(s_N)| \cos(\omega t_e + \Delta \psi_{Nk}),$$

(48)

where $\Delta \psi_{ik} = \psi_e(s_i) - \psi_e(s_k)$ and $\Phi_{en}$, $\psi_e$ are the $n$th eigenmode and phase obtained from the experiment. We select $s = s_1$ as the reference point of the phase, that is, $\Delta \psi_{1k} = 0$. Then, letting $v_e = (v_{e1} \ v_{e2} \ \cdots \ v_{en})^t$, we obtain

$$v_e = a \begin{pmatrix} |\Phi_{en}(s_1)| \\ |\Phi_{en}(s_2)| \cos \Delta \psi_{21} \\ \vdots \\ |\Phi_{en}(s_N)| \cos \Delta \psi_{N1} \end{pmatrix} \cos \omega t_e.$$
-α\begin{pmatrix} 0 & |Φ_{en}(s_2)| \sin Δψ_{21} & \vdots & |Φ_{en}(s_N)| \sin Δψ_{N1} \\ |Φ_{en}(s_2)| \cos Δψ_{21} & 0 & \vdots & |Φ_{en}(s_N)| \cos Δψ_{N1} \end{pmatrix} \sin ωt_e. \quad (49)

Comparing Eq. (49) with Eq. (47), we can regard the coefficients of \cos ωt_e and \sin ωt_e in Eq. (49) as Φ_{nr}(s) and Φ_{ni}(s), respectively. We then express the real and imaginary components of the complex eigenmode obtained in the experiment, Φ_{er} and Φ_{ei}, as

\begin{align*}
Φ_{er} &= \begin{pmatrix} |Φ_{en}(s_1)| \\ |Φ_{en}(s_2)| \cos Δψ_{21} \\ \vdots \\ |Φ_{en}(s_N)| \cos Δψ_{N1} \end{pmatrix}, \\
Φ_{ei} &= \begin{pmatrix} 0 \\ |Φ_{en}(s_2)| \sin Δψ_{21} \\ \vdots \\ |Φ_{en}(s_N)| \sin Δψ_{N1} \end{pmatrix}. \quad (50)
\end{align*}

5 Experimental identification of the complex mode shape

5.1 Experimental setup

We show the experimental apparatus in Figs. 6 and 7. We used three pipes whose parameters are given in Table 1. The material of pipes is silicone rubber. Pipes 2 and 3 were the same pipe but Pipe 3 was attached to two brass rods with a diameter of 0.8 mm to increase its flexural rigidity. Pipes 1 and 2 underwent self-excited vibration in the third mode but had different critical speeds, whereas Pipe 3 underwent self-excited vibration in the second mode. Flexural rigidity was identified experimentally using the method described in Appendix C. To obtain the mode shape experimentally, we placed 18 or 19 markers on the pipe as shown in Fig. 8 and recorded the time history of the steady-state self-excited oscillation using a high-speed camera (Photron: FASTCAM APX RS) after sufficient time had passed, because in the steady state, the oscillation components relating to modes other than the focused self-excited mode decay. The number of frames per second, average marker interval, spatial resolution and magnitude of the normalized resolution are given in Table 2. From the recorded video, we derive the time history data of the motion at each marker and obtain a|Φ_{en}(s_n)| and ϕ + ψ_n from Sect. 4 by analyzing the time histories using a fast Fourier transformation. a|Φ_{en}(s_n)| expresses the amplitude at each point in Eq. (48), and the phase difference between markers is obtained as ϕ + ψ_n. We can therefore decompose the complex mode into real and imaginary components as in Eq. (51) experimentally.

In practice, the markers on the pipe have displacement in not only the y direction but also the x direction. We let u be the displacement at s in the x direction, and express u as
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Table 1 Parameters of the three pipes used in the experiment

| Parameter | Pipe 1 | Pipe 2 | Pipe 3 |
|-----------|--------|--------|--------|
| Length [m] | 0.393 | 0.396 | 0.396 |
| $\beta$    | 0.388 | 0.289 | 0.263 |
| $\gamma$   | 74.2  | 53.3  | 14.6  |

Table 2 Shooting conditions for the three pipes

| Parameter                        | Pipe 1 | Pipe 2 | Pipe 3 |
|----------------------------------|--------|--------|--------|
| Number of frames [fps]           | 1500   | 1500   | 1500   |
| Average marker interval [mm]     | 2.16   | 2.18   | 2.05   |
| Spatial resolution [mm]          | 0.800  | 0.581  | 0.699  |
| Normalized resolution [$\times 10^3$] | 7.69   | 5.13   | 9.01   |

Fig. 8 Photograph of the pipe. Black markers on the pipe were used to obtain the mode shape

$u = \int_0^s \left( \sqrt{1 - \left( \frac{\partial v}{\partial s} \right)^2} - 1 \right) ds$

$\approx \frac{1}{2} \int_0^s \left( \frac{\partial v}{\partial s} \right)^2 ds.$  \hspace{1cm} (51)

From Eq. (51), we have $u = O(v^2)$. The displacement in the $x$ direction is much smaller than that in the $y$ direction and is considered negligible. Therefore, the lateral displacement in the $y$ direction at the point on the pipe whose distance from the supporting point along the curvilinear coordinate is $s$ is almost the same as that at the point on the pipe whose $x$ coordinate in Fig. 1 is $s$.

5.2 Preliminary experiments

In a preliminary investigation, we compared the theoretically and experimentally obtained critical flow speed and response frequency. These values are given in Table 3. Here, we compared the values for the self-excited oscillations of the second mode in Pipes 1 and 2 and the values for the self-excited oscillations of the third mode in Pipe 3. The differences between theoretical and experimental results are less than 6% for Pipes 1 and 2 and less than 10% for Pipe 3. In this study, we use the analytical model neglecting the internal and viscous damping. Referring to the method described in [32], we experimentally identify the non-dimensional values for Pipe 3. The method is summarized as follows. At first, we obtain the damping ratio $\gamma_1$ corresponding to the external damping from the free vibration of the pipe containing the aluminum rod in order not to be bent. Next, we obtain the damping ratio $\gamma_2$ including both effects of viscous and internal damping from the free vibration of the pipe without aluminum rod. We can obtain the non-dimensional coefficients of the viscous and internal damping from $\gamma_1$ and $\gamma_2 - \gamma_1$. As a result, the non-dimensional viscous and internal damping are $2.7 \times 10^{-2}$ and $2.2 \times 10^{-3}$, respectively. The discrepancies between theoretical and experimental results in Table 3 depend on their neglect in the analysis.

5.3 Experimentally identified complex mode shapes

In comparing the shapes of $\Phi_{er}(s)$ and $\Phi_{ei}(s)$ obtained experimentally using the method proposed in Sect. 4 with the shapes of $\Phi_r(s)$ and $\Phi_i(s)$ obtained theoreti-

Table 3 Comparison of the frequencies of self-excited vibration and critical speed between experimental results and theoretical results

|                     | Exp./Cal. | Pipe 1 | Pipe 2 | Pipe 3 |
|---------------------|-----------|--------|--------|--------|
| Frequency           |           | 3.45   | 3.45   | 4.8    | Hz     |
|                     | Exp.      | 3.38   | 3.67   | 5.26   | Hz     |
| Critical speed      |           | 4.24   | 4.79   | 8.25   | m/s    |
|                     | Exp.      | 4.31   | 5.02   | 8.74   | m/s    |
|                     | Cal.      |        |        |        |        |
from Eq. (44), we need to match the phase $\psi(s)$ of theoretical modes with that of experimental modes without loss of generality. Using the method described in Sect. 2, we obtain the theoretical complex modes as a function of $s$. Thus, using Eq. (45), we obtain the phase $\psi(s)$ at all $s (0 \leq s \leq 1)$ and match the phase of the theoretical mode with that of the experimental mode. In the present study, we select the moving marker closest to the upper end of the pipe as the reference point.

First, the mode shapes of Pipe 1 obtained theoretically and experimentally are shown in Fig. 9, where (a) and (b), respectively, show the real and imaginary components of the complex mode obtained experimentally using the method proposed in Sect. 4 and (c) and (d) show those obtained theoretically using the method presented in Sect. 2. A comparison of (a) with (c) and a comparison of (b) with (d) shows that the shapes are in agreement qualitatively. That is to say, in a comparison of the real components (i.e., (a) and (c)), both shapes have a node and antinode on the lower pipe; the positions of the node and antinode are $x = 0.869, 0.598$ in (a) and $s = 0.820, 0.570$ in (c), respectively. In these figures, the nodes and antinodes are shown by the yellow and green triangles, respectively; the positions of node and antinode are obtained by the interpolation of the plot. The mode shapes are thus in good agreement qualitatively. In the comparison of the imaginary components, there is a node in the experimental mode shape (b) at $x = 0.089$, whereas there are two nodes $s = 0.090$ and 0.340 in the theoretical mode shape. In these figures, the nodes are shown by the blue triangle. On the basis of the results in Sect. 4, we introduce a new time $t_e$ such that the phases of the experimental and theoretical modes, $\psi_e + \phi$ and $\psi + \phi$, at the reference point are zero. The positions of the nodes of the experimental and theoretical modes should then be the same, but the positions of the experimental and theoretical nodes $x = 0.089$ and $s = 0.09$ deviate slightly. This slight deviation arises because we calculate the theoretical mode shape with a step size of 0.01. That there is no node at $s = 0.340$ in the experimental mode shape can be explained by a lack of measurement accuracy, that is, the gradient of the mode shape at $s = 0.340$ is very small and, because the normalized pixel resolution is 0.00769, the normalized deflection at $0.134 \leq x \leq 0.361$ is $0.003 - 0.020$. In the middle of the mode shapes, there is a point at which the normalized deflection changes rapidly. The positions are $x = 0.413$ in (b) and $s = 0.390$ in (d), that is, the positions are similar.

Next, we show the mode shapes of Pipe 2 in Fig. 10. The experimental real and imaginary components of the mode shapes agree with the theoretical components qualitatively. That is to say, real components (i.e., (a) and (c)) have an antinode and a node at the middle of the mode shapes. The positions of the antinode and node are $x = 0.373$ and 0.521 in (a) and $s = 0.310$ and 0.490 in (c), that is, the positions are similar. The yellow and green triangles have the same meaning as those in Fig. 9. Regarding the imaginary components, there is an antinode near the bottom of the mode shapes at $x = 0.737$ in (b) and $x = 0.690$ in (d), where the antinodes are shown by the orange triangle. There is a node at $x = 0.100$ in (b), and there are two nodes at $s = 0.100$ and 0.980 in (d). The blue triangles have the same meaning as those in Fig. 9. There is no node at $s = 0.980$ in the experimental mode shape. As seen from (b) and (d), this absence is considered to be the effect of a slight error in the displacement of the lower markers.

Last, we show the mode shapes of Pipe 3 in Fig. 11. Pipe 3 has the parameters given in Table 1 and its desta-
Method of experimentally identifying the complex mode shape

A flexible cantilevered pipe conveying fluid is self-excited by a nonconservative force generated by the fluid. From a mathematical point of view, the destabilization phenomenon relates to the non-selfadjointness of the governing equation, which makes the eigenmodes nonorthogonal to each other and converts the real eigenmodes to complex eigenmodes. The proposition of a method of identifying the complex modes experimentally is important in revealing a feature of the self-excited vibration due to non-selfadjointness.

In the present study, we first clarified theoretically the complex eigenmodes of the self-excited oscillation and investigated the effect of the nonlinearity of systems on the eigenmodes. We then proposed a method of experimentally identifying the complex mode and decomposing the mode into real and imaginary components using the time histories of some points on the pipe recorded using a high-speed video camera. For three pipes, two being self-excited in the third mode and the other in the second mode, the real and imaginary components of the self-excited eigenmode were compared between

Fig. 10 Comparison of the experimental mode shape with the theoretical mode shape for Pipe 2: (a) and (b) shapes of the experimental real and imaginary modes and (c) and (d) shapes of the theoretical real and imaginary modes

Fig. 11 Comparison of the experimental mode shape with the theoretical mode shape for Pipe 3: (a) and (b) shapes of the experimental real and imaginary modes and (c) and (d) shapes of the theoretical real and imaginary modes

6 Conclusion
the theoretical and experimental results. The validity of the proposed identification method was thus confirmed for the complex eigenmode of the self-excited oscillation due to the non-selfadjointness of the system.

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Data availability statement The manuscript has no associated data.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A Nonlinear term

The nonlinear term in Eq. (34) is expressed as

\[
n(\Phi_{n1}) = \left\{ -2i\omega_n^2 \overline{\Phi}_{n1} \int_0^1 \Phi_n^2 \, ds - iU_{ncr} \sqrt{\beta} \omega_n \Phi_n^2 \overline{\Phi}_{n1} \right. \\
\left. - \frac{1}{2} U_{ncr}^2 (\Phi_n^2 \overline{\Phi}_{n1}'' + 2 \Phi_n' \overline{\Phi}_{n1}''') + \omega_n \overline{\Phi}_{n1}'' + 2 \overline{\Phi}_{n1}'' \Phi_n' \overline{\Phi}_{n1}' + \gamma \Phi_n'' \int_0^1 \Phi_n \overline{\Phi}_{n1} \, ds \right. \\
\left. + \frac{1}{2} \Phi_n''' \int_0^1 \left( \int_0^s \Phi_n'' \, ds \right) \, ds + \omega_n^2 \int_0^1 \Phi_n \overline{\Phi}_{n1} \, ds \right. \\
\left. + \frac{1}{2} \gamma \left( \overline{\Phi}_{n1}'' \int_0^s \Phi_n'' \, ds + 2 \Phi_n'' \int_0^1 \Phi_n \overline{\Phi}_{n1} \, ds \right) \\
\left. + \frac{1}{2} \left( \overline{\Phi}_{n1}'' \Phi_n' \Phi_n'' \int_0^s \Phi_n'' \, ds + 2 \Phi_n'' \Phi_n'' \int_0^1 \Phi_n \overline{\Phi}_{n1} \, ds \right) \right\} |A_n|^2 A_n.
\]

(A.1)

B Non-selfadjointness of the system

We introduce \( \Psi_{n1} \) as the adjoint function of \( \Phi_{n1} \), which satisfies

\[
\int_0^1 [\Phi_{n1}''' + (U_{ncr}^2 - \gamma (1 - s)) \Phi_{n1}'] \\
+ (2i\omega_n \sqrt{\beta} U_{ncr} + \gamma) \Phi_{n1} - \omega_n^2 \Phi_{n1} \overline{\Psi}_{n1} \, ds = 0.
\]

(B.1)

Considering the boundary conditions of Eq. (33), we note that \( \Psi_{n1} \) is the solution to the boundary value problem

\[
\Psi_{n1}''' + (U_{ncr}^2 - \gamma (1 - s)) \overline{\Psi}_{n1}'' \\
+ (-2i\omega_n \sqrt{\beta} U_{ncr} + \gamma) \overline{\Psi}_{n1} - \omega_n^2 \overline{\Psi}_{n1} = 0,
\]

(B.2)

\[
\begin{align*}
\Psi_{n1}(0) &= 0 \\
\overline{\Psi}_{n1}(0) &= 0 \\
\Psi_{n1}(1) &= -U_{ncr}^2 \overline{\Psi}_{n1}(1) \\
\overline{\Psi}_{n1}(1) &= 2i\omega_n \sqrt{\beta} U_{ncr} \overline{\Psi}_{n1}(1) - U_{ncr}^2 \overline{\Psi}_{n1}'(1).
\end{align*}
\]

(B.3)

According to Eqs. (B.2) and (B.3), the differential equation satisfied by \( \Psi_{n1} \) does not correspond to Eqs. (32) and (33), which \( \Phi_{n1} \) satisfies. Therefore, \( \Psi_{n1} \) is not the eigenmode and this system has non-selfadjointness. When we assume the flow velocity to be zero in Eqs. (B.2) and (B.3), these equations correspond to Eqs. (32) and (33). Therefore, the fluid conveying in the pipe makes this system non-selfadjoint.

\( \Phi_{n1} \) and \( \Psi_{n1} \) satisfy \( \int_0^1 \Phi_{n1} \overline{\Psi}_{m1} \, ds = \delta_{nm} \), where \( \delta_{nm} \) is Kronecker’s delta. Hence, from this property, the adjoint function plays an important role in the analysis of non-selfadjoint system and is used to obtain the eigenvalue [26]. Also, the application of the orthonormality yields the solvability condition related to the amplitude equation in nonlinear non-selfadjoint systems [19]. In this manuscript, we use the adjoint function in order to derive the solvability condition of \( \Phi_{n3} \) as in Sect. 3. But, the adjoint function is not necessary for obtaining the complex mode shape experimentally as mentioned in Sect. 4.

C Identification of flexural rigidity

We need the flexural rigidity \( EI \) as a parameter when we calculate the eigenmode theoretically. In this study, we determined the flexural rigidity \( EI \) of the pipe so that the natural frequency of free vibration without internal flow for the first mode obtained experimentally is equal to that obtained theoretically. Using this flexural rigidity \( EI \), we find that the frequency of the second mode obtained experimentally is similar to that calculated theoretically as shown in Table 4.
| Table 4 | Comparison of the frequencies of free vibration of the first and second modes in the experimental results with those in the theoretical results |
|---------|--------------------------------------------------|
| Exp./Cal. | Pipe 1 | Pipe 2 | Pipe 3 | Unit |
| Frequency (first mode) | | | | |
| Exp. | 1.08 | 1.08 | 1.3 | Hz |
| Cal. | 1.07 | 1.09 | 1.30 | Hz |
| Frequency (second mode) | | | | |
| Exp. | 3.52 | 3.76 | 5.9 | Hz |
| Cal. | 3.48 | 3.67 | 5.81 | Hz |

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