Quadratic harvesting in a fractional order scavenger model

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Abstract. This paper examines the dynamical nature of a prey - predator - scavenger involving harvesting on predator population. A discretization process is applied to the fractional order system and its discrete version is obtained. Existence of possible fixed points is established and the local stability analysis is carried out. It is also proved that the system undergoes Neimark-Sacker (NS) and Period-Doubling bifurcation (PDB) at certain parametric values for interior fixed point with the help of an explicit criterion for NS and PDB. Numerical simulations are performed for the model with quadratic harvesting on predator. Also the existence of limit cycles is shown and bifurcation diagrams are plotted for selected range of growth parameter.

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1. Introduction
Scavengers play an important role in ecosystems by consuming dead animals and usually they do not predate on living animals. In recent years, there are several mathematical models of food chain involving decomposer, detritivores and omnivores [2], [3], [5]. Mathematical models in population ecology are studied using partial differential equations, ordinary differential equations, fractional order differential equations, difference equations, and stochastic differential equations. In recent years, researchers prefer to use fractional differential equation for their ability to accommodate memory effect which is useful in biological phenomena [1].

2. Scavenger model with quadratic harvesting
We consider the following fractional order predator - prey - scavenger system with quadratic harvesting of predator:

\[
\begin{align*}
D^\alpha x(t) &= x(t) \left(1 - rx(t)\right) - x(t)y(t) - x(t)z(t) \\
D^\alpha y(t) &= x(t)y(t) - \mu y(t) - \eta_1 y^2(t) \\
D^\alpha z(t) &= ax(t)z(t) + by(t)z(t) - cz(t) - \eta_2 z^2(t).
\end{align*}
\]

Here \( r \) - growth rate associated with prey population \( x \), \( \mu \) and \( c \) - the natural death rates of scavenger and predator, \( a \) - the rate of change in the scavenger involving a prey population, \( b \) - the rate of change in the scavenger involving a predator, \( \eta_1 \) and \( \eta_2 \) - both carrying capacities of scavenger respectively but the term \( \eta_1 y^2 \) and \( \eta_2 z^2 \) - quadratic harvesting of scavenger population. Here \( t > 0 \) and \( \alpha \) is the fractional order satisfying \( \alpha \in (0,1] \). Now, applying the discretization process of a
fractional-order system outlined in [1], we obtain the discrete fractional order scavenger system as follows:

\[
x_{t+1} = x_t + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( x_t (1-rx_t) - x_t y_t - x_t z_t \right)
\]
\[
y_{t+1} = y_t + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( x_t y_t - \mu y_t - \eta y_t^2 \right)
\]
\[
z_{t+1} = z_t + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \left( ax_t z_t + by_t z_t - cz_t - \eta z_t^2 \right).
\]

3. Existence of fixed points
The fixed points of the scavenger system (1) are: (i) The trivial fixed point \( E_0 = (0,0,0) \) and the axial fixed point \( E_0 = \left( \frac{1}{r}, 0, 0 \right) \), always exist. (ii) The boundary fixed point in \( xy \)-plane \( E_{xy} = (x, y, 0) \) always exist. (iii) The second boundary fixed point in \( xz \)-plane \( E_{xz} = (x, 0, z) \) exist when \( rc < a \). (iv) The interior fixed point \( E_i = (x_*, y_*, z_*) \) with

\[
x_* = \frac{(b + \eta_2)(c + \eta_2)\mu + (a + \eta_2)\eta_1}{(a + \eta_1)(b + (r \eta_1 + 1)\eta_2)},
\]
\[
y_* = \frac{(c - a \mu) + (1 - r \mu)\eta_2}{(a + \eta_1)(b + (r \eta_1 + 1)\eta_2)},
\]
\[
z_* = \frac{(a \eta_1 + b) - c(1 + \eta_1) + (a - rb)\mu}{(a + \eta_1)(b + (r \eta_1 + 1)\eta_2)}
\]

4. Local stability analysis of discretization fractional order
This section investigates the dynamical behavior of the discretized fractional-order model (2) which is determined by the parameters \( r, s, a, b, c, \mu, \eta, \eta_2 \) and \( \alpha \). The Jacobian matrix \( J \) of (2) evaluated at a fixed point is

\[
J(x, y, z) = \\
\begin{bmatrix}
1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)} (1 - 2rx - y - z) & -\frac{s^\alpha}{\alpha \Gamma(\alpha)} x & -\frac{s^\alpha}{\alpha \Gamma(\alpha)} x \\
\frac{s^\alpha}{\alpha \Gamma(\alpha)} y & 1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)} (x - \mu - 2\eta y) & 0 \\
\frac{s^\alpha}{\alpha \Gamma(\alpha)} az & \frac{s^\alpha}{\alpha \Gamma(\alpha)} bz & 1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)} (ax + by - c - 2\eta z)
\end{bmatrix}
\]

4.1. Stability of \( E_0 = (0,0,0) \) The eigenvalues of the Jacobian matrix \( J \) for the system evaluated at the trivial fixed point \( E_0 \) are

\[
\lambda_1 = 1 + \frac{s^\alpha}{\alpha \Gamma(\alpha)} \mu, \quad \lambda_2 = 1 - \frac{s^\alpha}{\alpha \Gamma(\alpha)} \mu \quad \text{and} \quad \lambda_3 = 1 - \frac{s^\alpha}{\alpha \Gamma(\alpha)} c.
\]

Thus \( E_0 \) is stable when \( s < 0 \), \( 0 < s < \sqrt{\frac{2\alpha \Gamma(\alpha)}{\mu}} \) and \( 0 < s < \sqrt{\frac{2\alpha \Gamma(\alpha)}{c}} \). Otherwise \( E_0 \) is unstable.

4.2. Stability of \( E_1 = \left( \frac{1}{r}, 0, 0 \right) \)
The eigenvalues of $J$ for the system evaluated at the axial fixed point $E_i$ are $\lambda_i = 1 - \frac{s^r}{a \Gamma(\alpha)}$, $\lambda_2 = 1 + \frac{s^r}{a \Gamma(\alpha)} \left( \frac{1}{r} - \mu \right)$ and $\lambda_3 = 1 + \frac{s^r}{a \Gamma(\alpha)} \left( \frac{a}{r} - c \right)$. Hence $E_i$ is locally asymptotically stable if it satisfies the conditions $0 < s < \sqrt{2r\alpha(r')^{-1}}$, $0 < s < \sqrt{2r\alpha(r')^{-1}}$ and $0 < s < \sqrt{2r\alpha(r')^{-1}}$. The axial fixed point $E_i$ loses its stability when at least one of the above three conditions is violated and it becomes a saddle point.

4.3. Stability of $E_s = (x, y, 0)$

The eigenvalues of the Jacobian matrix $J$ evaluated at the first boundary fixed point $E_s$ are

$$\lambda_1 = 1 + \frac{s^s}{a \Gamma(\alpha)} (ax + by - c)$$

and

$$\lambda_{2,3} = 1 + \frac{s^s}{a \Gamma(\alpha)} \left( (\beta_1 + \beta_2) \pm \sqrt{(\beta_1 - \beta_2)^2 - 4xy} \right)$$

where $\beta_1 = 1 - 2r \bar{x} - \bar{y}$ and $\beta_2 = \bar{x} - \mu - 2\eta \bar{y}$. Hence $E_s$ is locally asymptotically stable when

$$0 < s < \sqrt{2r \alpha(\alpha')}$$

and

$$\left( \frac{s^s}{a \Gamma(\alpha)} \right)^2 \left[ (1 - 2r \bar{x} - \bar{y})(\bar{x} - \mu - 2\eta \bar{y}) - \bar{x} \bar{y} \right] < 0.$$ 

The fixed point $E_s$ becomes saddle if $s > \sqrt{2r \alpha(\alpha')}$$1 - 2r \bar{x} - \bar{z}$ and

$$\left( \frac{s^s}{a \Gamma(\alpha)} \right)^2 \left[ (1 - 2r \bar{x} - \bar{z})(\bar{x} - \mu - 2\eta \bar{z}) - \bar{a} \bar{z} \right] < 0.$$ 

4.4. Stability of $E_{sz} = (\bar{x}, 0, \bar{z})$

The eigenvalues of the Jacobian matrix evaluated at the second boundary fixed point $E_{sz}$ are

$$\lambda_1 = 1 + \frac{s^s}{a \Gamma(\alpha)} (\bar{x} - \mu)$$

and

$$\lambda_{2,3} = 1 + \frac{s^s}{a \Gamma(\alpha)} \left( (\beta_1 + \beta_4) \pm \sqrt{(\beta_1 - \beta_4)^2 - 4axz} \right)$$

where $\beta_3 = 1 - 2r \bar{x} - \bar{z}$ and $\beta_4 = \bar{x} - c - 2\eta \bar{z}$. Hence $E_{sz}$ is locally asymptotically stable when $0 < s < \sqrt{2r \alpha(\alpha')}$ and

$$\left( \frac{s^s}{a \Gamma(\alpha)} \right)^2 \left[ (1 - 2r \bar{x} - \bar{z})(\bar{x} - c - 2\eta \bar{z}) - \bar{a} \bar{z} \right] < 0.$$ 

The fixed point $E_{sz}$ becomes saddle if $s > \sqrt{2r \alpha(\alpha')}$$1 - 2r \bar{x} - \bar{z}$ and

$$\left( \frac{s^s}{a \Gamma(\alpha)} \right)^2 \left[ (1 - 2r \bar{x} - \bar{z})(\bar{x} - c - 2\eta \bar{z}) - \bar{a} \bar{z} \right] < 0.$$ 

5. Bifurcation analysis of prey, predator and scavenger

We will study the local stability and the parametric conditions for the existence of NSB and PDB at the interior fixed point $E_*$ of system (2). The Jacobian matrix (3) at $E_*(x_*, y_*, z_*)$ has the form

$$J(E_*) = \begin{bmatrix}
\frac{s^s}{a \Gamma(\alpha)} (\bar{x}_1) & -\frac{s^s}{a \Gamma(\alpha)} x_* & -\frac{s^s}{a \Gamma(\alpha)} x_* \\
-\frac{s^s}{a \Gamma(\alpha)} y_* & 1 - \frac{s^s}{a \Gamma(\alpha)} (\bar{x}_2) & 0 \\
\frac{s^s}{a \Gamma(\alpha)} z_* & \frac{s^s}{a \Gamma(\alpha)} b z_* & 1 - \frac{s^s}{a \Gamma(\alpha)} (\bar{x}_3)
\end{bmatrix}.$$
The characteristic equation of \( J(E_*) \) is \( \lambda^3 + \delta_1 \lambda^2 + \delta_2 \lambda + \delta_3 = 0 \) (4)

where \( \delta_1 = \frac{s^a}{\alpha}(\xi_1 + \xi_2 + \xi_3) - 3 \), \( \delta_2 = 3 - 2 \frac{s^a}{\alpha}(\xi_1 + \xi_2 + \xi_3) + \frac{s^a}{\alpha}(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_1) \), \( \delta_3 = \frac{s^a}{\alpha}(\xi_1 + \xi_2 + \xi_3)^2 - \frac{s^a}{\alpha}(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_1 + ax_z + x y) + \frac{s^a}{\alpha}(\xi_1 \xi_2 \xi_3 + x y z + a \xi_2 x z - 1) \) (5)

such that \( \xi_1 = 2r_x + y + z - 1 \), \( \xi_2 = \mu + 2 \eta_y y - x \), and \( \xi_3 = c + 2 \eta_z z - ax - by \). By Routh-Hurwitz criterion, the interior fixed point \( E_\ast \) is locally asymptotically stable if and only if \( \delta_1 > 0 \), \( \delta_2 > 0 \) and

\[
\left( \frac{s^a}{\alpha} \right)^3 \left[ \xi_1^2 (\xi_2 + \xi_3) + \xi_2^2 (\xi_1 + \xi_3) + \xi_3^2 (\xi_1 + \xi_2) + 2 \xi_1 \xi_2 \xi_3 + bx_z y + x y \right] + \frac{s^a}{\alpha} \left[ 2(\xi_1 + \xi_2 + \xi_3) \right] + 1 + \left( \frac{s^a}{\alpha} \right)^2 \left[ 3(\xi_1 \xi_2 \xi_3 + \xi_1 \xi_2 + \xi_2 \xi_3 + 2(\xi_1^2 + \xi_2^2 + \xi_3^2) \right] - (y + a x_z) \nu \] \]

5.1. Neimark-Sacker bifurcation

To study the NSB of the system (2), we need the following explicit criterion for Hopf bifurcation [7].

Theorem 5.1.1: The interior fixed point \( E_\ast \) of the system (2) undergoes NSB for \( c + \eta_y > (a + r \eta_y) \mu \), \( a (\mu + \eta_y) + b > rb \mu + c(1 + r \eta_y) \) if the following conditions hold: \( 1 - \delta_2 + \delta_1 (\delta_3 - \delta_2) = 0 \), \( 1 + \delta_3 - \delta_2 (\delta_3 - \delta_2) = 0 \), \( 1 + \delta_1 + \delta_2 + \delta_3 > 0 \) and \( 1 - \delta_1 + \delta_2 - \delta_3 > 0 \), where \( \delta_1 \), \( \delta_2 \) and \( \delta_3 \) are given in (5).

Proof: From [4], for three dimensional system \( n = 3 \), we have the characteristic polynomial (4) of system (2) evaluated at the interior fixed point \( E_\ast \). Thus we obtain the following equalities and inequalities: \( \Delta_1(r) = 1 - \delta_2 + \delta_1 (\delta_3 - \delta_2) = 0 \), \( \Delta_2(r) = 1 + \delta_3 - \delta_1 (\delta_3 - \delta_2) > 0 \), \( P_1(1) = 1 + \delta_1 + \delta_2 + \delta_3 > 0 \) and \( (-1)^3 P_3(-1) = 1 - \delta_1 + \delta_2 - \delta_3 > 0 \).

5.2. Period-doubling bifurcation

An explicit critical criterion for the existence of PDB is proposed for higher dimensional discrete time systems [6].

Theorem 5.2.1: The interior fixed point \( E_\ast \) of the system (2) undergoes PDB for \( c + \eta_y > (a + r \eta_y) \mu \), \( a (\mu + \eta_y) + b > rb \mu + c(1 + r \eta_y) \) if the following conditions hold: \( 1 - \delta_2 + \delta_1 (\delta_3 - \delta_2) > 0 \), \( 1 + \delta_3 - \delta_2 (\delta_3 - \delta_2) > 0 \), \( 1 + \delta_1 + \delta_2 + \delta_3 > 0 \) and \( -1 + \delta_1 - \delta_2 + \delta_3 = 0 \), where \( \delta_1 \), \( \delta_2 \) and \( \delta_3 \) are given in (5).

Proof: According to [4], for three dimensional system \( n = 3 \), we have the characteristic polynomial (4) of system (2) evaluated at the interior fixed point \( E_\ast \). Then we obtain the following equalities and inequalities: \( \Delta_1(r) = 1 - \delta_2 + \delta_1 (\delta_3 - \delta_2) > 0 \), \( \Delta_2(r) = 1 + \delta_3 - \delta_1 (\delta_3 - \delta_2) > 0 \), \( \Delta_3(r) = 1 + \delta_2 (\delta_1 - \delta_2) + \delta_1 = 0 \), \( P_1(1) = 1 + \delta_1 + \delta_2 + \delta_3 > 0 \) and \( P_3(-1) = -1 + \delta_1 - \delta_2 + \delta_3 = 0 \).

6. Numerical simulations and discussions of interior fixed point \( E_\ast \)

Let us consider the fractional order predator – prey – scavenger system in the absence of harvesting of predator as follows:
$$D^\alpha x(t) = x(t)\left(1 - rx(t)\right) - x(t)y(t) - x(t)z(t)$$  
$$D^\alpha y(t) = x(t)y(t) - \mu y(t)$$  
$$D^\alpha z(t) = ax(t)z(t) + by(t)z(t) - cz(t).$$  

Discrete version of fractional-order model in the absence of harvesting of predator is:

$$x_{t+1} = x_t + \frac{s^\alpha}{\alpha t(\alpha)} \left(x_t(1-rx_t) - x_t y_t - x_t z_t\right)$$

$$y_{t+1} = y_t + \frac{s^\alpha}{\alpha t(\alpha)} \left(x_t y_t - \mu y_t\right)$$

$$z_{t+1} = z_t + \frac{s^\alpha}{\alpha t(\alpha)} \left(ax_t z_t + by_t z_t - cz_t\right).$$

Local stability of discretization fractional order model with and without quadratic harvesting in (2) and (7) are illustrated. Figures 1 and 2, present the time series and phase trajectories of prey, predator and scavenger densities in the system studied above.

We observe that when the predator population is subject to quadratic harvesting, the local stability of the interior fixed point \(E^*\) can move from stable to unstable under the harvesting constant \(\eta_1 = 0.035\) and \(\eta_2 = 1.371, 1.491\) (see figures 1 and 2). On the other hand, the corresponding fixed point change from unstable to stable and the trajectory spirals inwards \(\forall\) outwards but does not approach a point. Finally the trajectory settles down as a limit cycle (see table 1).
Figure 2. Phase Portrait for stable and unstable solutions in the positive $xyz$-octant.

Figure 3. Bifurcation of system (2) w.r.t. $\alpha$ (a) $r = 1.349$ and (b) $r = 1.391$. 

(a) Bifurcation diagram w.r.t. $\alpha$ with $r=1.349$

(b) Bifurcation diagram w.r.t. $\alpha$ with $r=1.391$
Table 1. Scavenger harvesting may stabilize or destabilize the systems (2) and (7).

| Figure | $\alpha$ | $s$ | $r$ | $a$ | $b$ | $c$ | $\mu$ | $\eta_1$ | $\eta_2$ | Fixed point | Initial |
|--------|---------|-----|-----|-----|-----|-----|-------|---------|---------|-------------|---------|
| 1(a) & 2(a) | 0.91 | 0.96 | 2.271 | 0.062 | 2.76 | 0.637 | 0.049 | 0.035 | 1.491 | Stable | (0.4,0.3,0.2) |
| 1(b) & 2(b) | 0.95 | 0.686 | 2.181 | 0.699 | 2.017 | 1.296 | 0.099 | 0 | 0 | Unstable | (0.2,0.3,0.4) |
| 1(c) & 2(c) | 0.91 | 0.96 | 2.271 | 0.062 | 2.176 | 0.637 | 0.049 | 0.035 | 1.371 | Unstable | (0.4,0.3,0.2) |
| 1(d) & 2(d) | 0.95 | 0.686 | 2.861 | 0.699 | 2.017 | 1.296 | 0.099 | 0 | 0 | Stable | (0.2,0.3,0.4) |

We use some documented data for some parameter like $s = 0.96$, $a = 2.89$, $b = 0.77$, $c = 0.75$, $\mu = 1.39$, $\eta_1 = 0.79$, $\eta_2 = 0.42$ and the initial state is $x = 0.4$, $y = 0.3$ and $z = 0.2$. Other parameter will be $\alpha$ in the $[0,1]$ with (a) $r = 1.349$ and (b) $r = 1.391$; step size $\Delta \alpha = 0.001$ are shown in figure 3. Figure 3 shows the bifurcation diagrams for prey and predator densities of the system (2) in the presence of quadratic harvesting. We can see that, whenever the fractional order $\alpha$ is varied and $r$ is fixed, the system moves from stability to chaotic. When $\alpha$ increases through certain values, for example $\alpha = 0.8$, the solutions of the system (2) converges and becomes stable. Various dynamic behaviors associated with figure 3 are shown in figures 4 and 5 by phase portraits.

Figure 4. Phase Portrait of system (2) for different values of $\alpha$ with $r = 1.349$.

For the fractional order $\alpha = 0.05$, $\alpha = 0.26$, $\alpha = 0.67$ and $\alpha = 1$, the interior fixed point $E_*$ moves from stable to unstable and unstable to stable (see figure 4(a)-4(b)). On the other hand, when the parameter value increases as $\alpha = 0.3$, $\alpha = 0.45$, $\alpha = 0.55$ and $\alpha = 1$ and keeping all other values are same, fixed point $E_*$ change from oscillation to convergence (see figure 5(a)-5(d)).
Figure 5. Phase Portrait of system (2) for different values of $\alpha$ with $r = 1.391$.

7. Conclusion
We have studied effect of quadratic harvesting of predator and scavenger in a discrete fractional order scavenger model with linear functional response. The stability, period doubling and Neimark - Sacker bifurcations of interior fixed point have been investigated with the help of an explicit criterion for NS, PDB and bifurcation theory. Numerical simulations are performed for the dynamical aspects of the system with quadratic harvesting on predator.

References
[1] Elsadany A A and Matouk A E 2014 Dynamical behaviors of fractional - order Lotka - Volterra predator - prey model and its discretization J. Appl. Math. Comput. pp 16
[2] George Maria Selvam A, Dhineshbabu R and Maria Thangaraj J 2014 Allee effect in discrete prey-predator interactions with harvesting on prey Journal of Applied Engineering 27 pp 143-150
[3] George Maria Selvam A, Janagaraj R and Dhineshbabu R 2016 Dynamical analysis of a discrete fractional order prey-predator 3D system International Journal of Research and Development Organisation 21 pp 24-31
[4] Irfan Ali, Umer Saeed and Qamar Din 2018 Bifurcation analysis and chaos control in discrete-time system of three competing species Arab. J. Math. pp 14
[5] Gupta R P and Peeyush Chandra 2017 Dynamical properties of a prey-predator-scavenger model with quadratic harvesting Communications in Nonlinear Science and Numerical Simulation, doi: 10.1016/j.cnsns.2017.01.026
[6] Wen G, Chen S and Jin Q 2008 A new criterion of period doubling bifurcation in maps and its application to an intertial impact shaker J. Sound Vib. 3111-2 pp 212-223
[7] Yao S 2012 New bifurcation critical criterion of flip-neimark-sacker bifurcations for two-parameterized family of dimensional discrete systems Discrete Dynamics in Nature and Society.