MONADS AND THEORIES

JOHN BOURKE AND RICHARD GARNER

Abstract. Given a locally presentable enriched category $E$ together with a small dense full subcategory $A$ of arities, we study the relationship between monads on $E$ and identity-on-objects functors out of $A$, which we call $A$-pretheories. We show that the natural constructions relating these two kinds of structure form an adjoint pair. The fixpoints of the adjunction are characterised as the $A$-nervous monads—those for which the conclusions of Weber’s nerve theorem hold—and the $A$-theories, which we introduce here.

The resulting equivalence between $A$-nervous monads and $A$-theories is best possible in a precise sense, and extends almost all previously known monad–theory correspondences. It also establishes some completely new correspondences, including one which captures the globular theories defining Grothendieck weak $\omega$-groupoids.

Besides establishing our general correspondence and illustrating its reach, we study good properties of $A$-nervous monads and $A$-theories that allow us to recognise and construct them with ease. We also compare them with the monads with arities and theories with arities introduced and studied by Berger, Melliès and Weber.

1. Introduction

Category theory provides two approaches to classical universal algebra. On the one hand, we have finitary monads on $Set$ and on the other hand, we have Lawvere theories. Relating the two approaches we have Linton’s result [25], which shows that the category of finitary monads on $Set$ is equivalent to the category of Lawvere theories. An essential feature of this equivalence is that it respects semantics, in the sense that the algebras for a finitary monad coincide up to equivalence over $Set$ with the models of the associated theory, and vice versa.

There have been a host of generalisations of the above story, each dealing with algebraic structure borne by objects more general than sets. In many of these [30, 29, 21, 22], one starts on one side with the monads on a given category that preserve a specified class of colimits. This class specifies, albeit indirectly, the arities of operations that may arise in the algebraic structures encoded by such monads, and from this one may define, on the other side, corresponding notions of theory and model. These are subtler than in the classical setting, but

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once the correct definitions have been found, the equivalence with the given class of monads, and the compatibility with semantics, follows much as before.

The most general framework for a monad–theory correspondence to date involves the notions of monad with arities and theory with arities. In this setting, the permissible arities of operations are part of the basic data, given as a small, dense, full subcategory of the base category. The monads with arities were introduced first, in [33], as a setting for an abstract nerve theorem. Particular cases of this theorem include the classical nerve theorem, identifying categories with simplicial sets satisfying the Segal condition of [31], and also Berger’s nerve theorem [7] for the globular higher categories of [6]. More saliently, when Weber’s nerve theorem is specialised to the settings appropriate to the monad–theory correspondences listed above, it becomes exactly the fact that the functor sending the algebras for a monad to the models of the associated theory is an equivalence. This observation led [27] and [8] to introduce the theories with arities, and to prove their equivalence with the monads with arities using Weber’s nerve theorem. The monad–theory correspondence obtained in this way is general enough to encompass all of the instances from [30, 29, 21, 22].

Our own work in this paper has two motivations: one abstract and one concrete. The abstract motivation is a desire to explain the apparently ad hoc design choices involved in the monad–theory correspondences outlined above; for while these choices must be carefully balanced so as to obtain an equivalence, there is no guarantee that different careful choices might not yield more general or more expressive results. The concrete motivation comes from the study of the Grothendieck weak \( \omega \)-groupoids introduced by Maltsiniotis [26], which, by definition, are models of a globular theory in the sense of Berger [7]. Globular theories describe algebraic structure on globular sets with arities drawn from the dense subcategory of globular cardinals; see Example 6(v) below. However, globular theories are not necessarily theories with arities, and in particular, those capturing higher groupoidal structures are not. As such, they do not appear to one side of any of the monad–theory correspondences described above.

The first goal of this paper is to describe a new schema for monad–theory correspondences which addresses the gaps in our understanding noted above. In this schema, once we have fixed the process by which a theory is associated to a monad, everything else is forced. This addresses our first, abstract motivation. The correspondence obtained in this way is in fact best possible, in the sense that any other monad–theory correspondence for the same kind of algebraic structure must be a restriction of this particular one. In many cases, this best possible correspondence coincides with one in the literature, but in others, our correspondence goes beyond what already exists. In particular, an instance of our schema will identify the globular theories of [7] with a suitable class of monads on the category of globular sets. This addresses our second, concrete motivation.

The second goal of this paper is to study the classes of monads and of theories arising from our correspondence-schema. We do so both at a general level, where we will see that both the monads and the theories are closed under essentially all the constructions one could hope for; and also at a practical level, where we
will see how these general constructions allow us to give expressive and intuitive presentations for the structure captured by a monad or theory.

To give a fuller account of our results, we must first describe how a typical monad–theory correspondence arises. As in [33], such correspondences may be parametrised by pairs consisting of a category $E$ and a small, full, dense subcategory $K: A \to E$. For example, the Lawvere theory–finitary monad correspondence for finitary algebraic structure on sets is associated to the choice of $E = \text{Set}$ and $A = \mathbb{F}$ the full subcategory of finite cardinals.

Each of the correspondences associated to the pair $(A, E)$ will be an equivalence between a suitable category of $A$-monads and a suitable category of $A$-theories. $A$-monads are certain monads on $E$; while $A$-theories are certain identity-on-objects functors out of $A$. We are being deliberately vague about the conditions on each side, as they are among the seemingly ad hoc design choices we spoke of earlier. Be these as they may, the correspondence itself always arises through application of the following two constructions.

**Construction A.** For an $A$-monad $T$ on $E$, the associated $A$-theory $\Phi(T)$ is the identity-on-objects functor $J_T: A \to A_T$ arising from the (identity on objects, fully faithful) factorisation

\[
A \xrightarrow{J_T} A_T \xrightarrow{V_T} E_T
\]

of the composite $F_T K: A \to E \to E_T$. Here $F_T$ is the free functor into the Kleisli category $E_T$, so $A_T$ is equally the full subcategory of $E_T$ with objects those of $A$.

**Construction B.** For an $A$-theory $J: A \to T$, the associated $A$-monad $\Psi(T)$ is obtained from the category of concrete $T$-models, which is by definition the pullback

\[
\begin{array}{ccc}
\text{Mod}_c(T) & \to & [T^{op}, \text{Set}] \\
\downarrow \scriptstyle U^T & & \downarrow \scriptstyle [J^{op}, 1] \\
E & \xrightarrow{N_K=E(K-1)} & [A^{op}, \text{Set}].
\end{array}
\]

Since $U^T$ is a pullback of the strictly monadic $[J^{op}, 1]$, it will be strictly monadic so long as it has a left adjoint. When this is the case, as is ensured by local presentability of $E$, we can take $\Psi(T)$ to be the monad whose algebras are the concrete $T$-models.

There remains the problem of choosing the appropriate conditions to be an $A$-monad or an $A$-theory. Of course, these must be carefully balanced so as to obtain an equivalence, but this still seems to leave too many degrees of freedom; one might hope that everything could be determined from $E$ and $A$ alone. The main result of this paper shows that this is so: there are notions of $A$-monad and $A$-theory which require no further choices to be made, and which rather than being plucked from the air, may be derived in a principled manner.

The key observation is that Constructions A and B make sense when given as input any monad on $E$, or any “$A$-pretheory”—by which we mean simply an...
identity-on-objects functor out of $\mathcal{A}$. When viewed in this greater generality, these constructions yield an adjunction

$$
\begin{array}{c}
\text{Mnd}(\mathcal{E}) \\
\downarrow \Phi \\
\text{Preth}_{\mathcal{A}}(\mathcal{E})
\end{array}
$$

between the categories of monads on $\mathcal{E}$ and of $\mathcal{A}$-pretheories. Like any adjunction, this restricts to an equivalence between the objects at which the counit is invertible, and the objects at which the unit is invertible. Thus, if we define the $\mathcal{A}$-monads and $\mathcal{A}$-theories to be the objects so arising, then we obtain a monad–theory equivalence. By construction, it will be the largest possible equivalence whose two directions are given by Constructions A and B above.

Having defined the $\mathcal{A}$-monads and $\mathcal{A}$-theories abstractly, it then behooves us to give tractable concrete characterisations. In fact, we give a number of these, allowing us to relate our correspondence to existing ones in the literature. We also investigate further aspects of the general theory, and provide a wide range of examples illustrating the practical utility of our results.

Before getting started on the paper proper, we conclude this introduction with a more detailed outline of its contents. We begin in Section 2 by introducing our basic setting and notions. We then construct, in Theorem 5, the adjunction (1.3) between monads and pretheories. With this abstract result in place, we introduce in Section 3 a host of running examples of our basic setting. To convince the reader of the expressive power of our notions, we construct, via colimit presentations, specific pretheories for a variety of mathematical structures.

In Section 4 we obtain our main result by characterising the fixpoints of the monad–theory adjunction: the $\mathcal{A}$-monads and $\mathcal{A}$-theories described above. The $\mathcal{A}$-monads are characterised as what we term the $\mathcal{A}$-nervous monads, since they are precisely those monads for which Weber’s nerve theorem holds. The $\mathcal{A}$-theories turn out to be precisely those $\mathcal{A}$-pretheories for which each representable is a model. These characterisations lead to our main Theorem 17 describing the “best possible” equivalence arising between $\mathcal{A}$-theories and $\mathcal{A}$-nervous monads.

Section 5 develops some of the general results associated to our correspondence-schema. We begin by showing that our monad–theory correspondence commutes, to within isomorphism, with the taking of semantics on each side. We also prove that the functors taking semantics are valued in monadic right adjoint functors between locally presentable categories. The final important result of this section states that colimits of $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories are algebraic, meaning that the semantics functors send them to limits.

Section 6 is devoted to exploring what the $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories amount to in our running examples. In order to understand the $\mathcal{A}$-nervous monads, we prove the important result that they are equally the colimits, amongst all monads, of free monads on $\mathcal{A}$-signatures. We also introduce the notion of a saturated class of arities as a setting in which, like in [30, 29, 21, 22], the $\mathcal{A}$-nervous monads can be characterised in terms of a colimit-preservation property. With these results in place, we are able to exhibit many of these existing monad–theory correspondences as instances of our general framework.
In Section 7, we examine the relationship between the monads and theories of our correspondence, and the monads and theories with arities of \([33, 27, 8]\). In particular, we see that every monad with arities \(A\) is an \(A\)-nervous monad but that the converse implication need not be true: so \(A\)-nervous monads are strictly more general. Of course, the same is also true on the theory side. We also exhibit a further important point of difference: colimits of monads with arities, unlike those of nervous monads, are not necessarily algebraic. This means that there is no good notion of presentation for monads or theories with arities.

Finally, in Section 8, we give a number of proofs deferred from Section 6.

2. Monads and pretheories

2.1. The setting. In this section we construct the monad–pretheory adjunction

\[
\begin{array}{c}
\text{Mnd}(\mathcal{E}) \\
\downarrow \Phi \\
\downarrow \Psi \\
\text{Preth}_A(\mathcal{E})
\end{array}
\]

The setting for this, and the rest of the paper, involves two basic pieces of data:

(i) A locally presentable \(\mathcal{V}\)-category \(\mathcal{E}\) with respect to which we will describe the monad–theory adjunction; and

(ii) A notion of arities given by a small, full, dense sub-\(\mathcal{V}\)-category \(K: A \hookrightarrow \mathcal{E}\).

We will discuss examples in Section 2.1 below, but for now let us clarify some of the terms appearing above. While in the introduction, we focused on the unenriched context, we now work in the context of category theory enriched over a symmetric monoidal closed category \(\mathcal{V}\) which is locally presentable as in [12]. In this context, a locally presentable \(\mathcal{V}\)-category [17] is one which is cocomplete as a \(\mathcal{V}\)-category, and whose underlying ordinary category is locally presentable.

We recall also some notions pertaining to density. Given a \(\mathcal{V}\)-functor \(H: C \to D\) with small domain, the nerve functor \(N_K: D \to [C^{op}, \mathcal{V}]\) is defined by \(N_K(X) = D(K -, X)\). We call a presheaf in the essential image of \(N_K\) a \(K\)-nerve, and we write \(K-Ner(\mathcal{V})\) for the full sub-\(\mathcal{V}\)-category of \([C^{op}, \mathcal{V}]\) determined by these.

We say that \(K\) is dense if \(N_K\) is fully faithful; whereupon \(N_K\) induces an equivalence of categories \(D \simeq K-Ner(\mathcal{V})\). Finally, we call a small sub-\(\mathcal{V}\)-category \(A\) of a \(\mathcal{V}\)-category \(\mathcal{E}\) dense if its inclusion functor \(K: A \hookrightarrow \mathcal{E}\) is so.

2.2. Monads. We write \(\text{Mnd}(\mathcal{E})\) for the (ordinary) category whose objects are \(\mathcal{V}\)-monads on \(\mathcal{E}\), and whose maps \(S \to T\) are \(\mathcal{V}\)-transformations \(\alpha: S \Rightarrow T\) compatible with unit and multiplication. For each \(T \in \text{Mnd}(\mathcal{E})\) we have the \(\mathcal{V}\)-category of algebras \(U^T: \mathcal{E}^T \to \mathcal{E}\) over \(\mathcal{E}\), but also the Kleisli \(\mathcal{V}\)-category \(F_T: \mathcal{E} \to \mathcal{E}_T\) under \(\mathcal{E}\), arising from an (identity on objects, fully faithful) factorisation

\[
\begin{array}{c}
\mathcal{E} \\
\downarrow W_T \\
\mathcal{E}_T \\
\downarrow F_T \\
\mathcal{E}
\end{array}
\]

of the free \(\mathcal{V}\)-functor \(F^T: \mathcal{E} \to \mathcal{E}^T\); concretely, we may take \(\mathcal{E}_T\) to have objects those of \(\mathcal{E}\), hom-objects \(\mathcal{E}_T(A, B) = \mathcal{E}(A, TB)\), and composition and identities

\[
\begin{array}{c}
\mathcal{E}_T \\
\downarrow F_T \\
\mathcal{E} \\
\downarrow \Phi \\
\text{Preth}_A(\mathcal{E})
\end{array}
\]
derived using the monad structure of \( T \). Each monad morphism \( \alpha : S \to T \) induces, functorially in \( \alpha \), \( \mathcal{V} \)-functors \( \alpha^* \) and \( \alpha_! \) fitting into diagrams

\[
\begin{array}{ccc}
\mathcal{E}^T & \xrightarrow{\alpha^*} & \mathcal{E}^S \\
\downarrow U^T & & \downarrow U^S \\
\mathcal{E} & \xleftarrow{\alpha_!} & \mathcal{E}_T \\
\end{array}
\]

here \( \alpha^* \) sends an algebra \( a : TA \to A \) to \( a \circ \alpha_A : SA \to A \) and is the identity on homs, while \( \alpha_! \) is the identity on objects and has action on homs given by the postcomposition maps \( \alpha_B \circ (-) : \mathcal{E}_S(A, B) \to \mathcal{E}_T(A, B) \). In fact, every \( \mathcal{V} \)-functor \( \mathcal{E}^T \to \mathcal{E}^S \) over \( \mathcal{E} \) or \( \mathcal{V} \)-functor \( \mathcal{E}_S \to \mathcal{E}_T \) under \( \mathcal{E} \) is of the form \( \alpha^* \) or \( \alpha_! \) for a unique map of monads \( \alpha \)—see, for example, [28]—and in this way, we obtain fully faithful functors

\[
\begin{array}{ccc}
\text{Mnd}(\mathcal{E})^{\text{op}} & \xrightarrow{\text{Alg}} & \mathcal{V}\text{-CAT}/\mathcal{E} \\
\text{Mnd}(\mathcal{E}) & \xrightarrow{\text{Kl}} & \mathcal{E}/\mathcal{V}\text{-CAT} \\
\end{array}
\]

2.3. **Pretheories.** An \( \mathcal{A} \)-pretheory is an identity-on-objects \( \mathcal{V} \)-functor \( J : \mathcal{A} \to \mathcal{T} \) with domain \( \mathcal{A} \). We write \( \text{Preth}_\mathcal{A}(\mathcal{E}) \) for the ordinary category whose objects are \( \mathcal{A} \)-pretheories and whose morphisms are \( \mathcal{V} \)-functors commuting with the maps from \( \mathcal{A} \). While the \( \mathcal{A} \)-pretheory is only fully specified by both pieces of data \( \mathcal{T} \) and \( J \), we will often, by abuse of notation, leave \( J \) implicit and refer to such a theory simply as \( \mathcal{T} \).

Just as any \( \mathcal{V} \)-monad has a \( \mathcal{V} \)-category of algebras, so any \( \mathcal{A} \)-pretheory has a \( \mathcal{V} \)-category of models. Generalising (1.2), we define the \( \mathcal{V} \)-category of concrete \( \mathcal{T} \)-models \( \text{Mod}_\mathcal{V}(\mathcal{T}) \) by a pullback of \( \mathcal{V} \)-categories as below left; so a concrete \( \mathcal{T} \)-model is an object \( X \in \mathcal{E} \) together with a chosen extension of \( \mathcal{E}(K^-, X) : \mathcal{A}^{\text{op}} \to \mathcal{V} \) along \( J^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{T}^{\text{op}} \). The reason for the qualifier “concrete” will be made clear in Section 5.2 below, where we will identify a more general notion of model.

\[
\begin{array}{ccc}
\text{Mod}_\mathcal{V}(\mathcal{T}) & \xrightarrow{P_T} & [\mathcal{T}^{\text{op}}, \mathcal{V}] \\
U_T & \downarrow N_K & \downarrow \downarrow \\
\mathcal{E} & \xrightarrow{\text{Mod}_\mathcal{V}(\mathcal{S})} & [\mathcal{A}^{\text{op}}, \mathcal{V}]
\end{array}
\]

Any \( \mathcal{A} \)-pretheory map \( H : \mathcal{T} \to \mathcal{S} \) gives a functor \( H^* : \text{Mod}_\mathcal{V}(\mathcal{S}) \to \text{Mod}_\mathcal{V}(\mathcal{T}) \) over \( \mathcal{E} \) by applying the universal property of the pullback left above to the commuting square on the right. In this way, we obtain a semantics functor:

\[
\text{Preth}_\mathcal{A}(\mathcal{E})^{\text{op}} \xrightarrow{\text{Mod}_\mathcal{V}} \mathcal{V}\text{-CAT}/\mathcal{E} 
\]

However, unlike (2.4), this is not always fully faithful. Indeed, in Example 8 below, we will see that non-isomorphic pretheories can have isomorphic categories of concrete models over \( \mathcal{E} \).

2.4. **Monads to pretheories.** We now define the functor \( \Phi : \text{Mnd}(\mathcal{E}) \to \text{Preth}_\mathcal{A}(\mathcal{E}) \) in (2.1). As in Construction A of the introduction, this will take the \( \mathcal{V} \)-monad \( \mathcal{T} \) to the \( \mathcal{A} \)-pretheory \( J_{\mathcal{T}} : \mathcal{A} \to \mathcal{A}_{\mathcal{T}} \) arising as the first part of an (identity-on-objects,
fully faithful) factorisation of $F^T K : A \to E^T$, as to the left in:

\[
\begin{array}{c}
A \xrightarrow{J_T} A_T \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
E \xrightarrow{F_T} \mathcal{E}_T
\end{array}
\quad
\begin{array}{c}
A \xrightarrow{J_T} A_T \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
E \xrightarrow{F_T} \mathcal{E}_T
\end{array}
\]

Since the comparison $W_T : \mathcal{E}_T \to \mathcal{E}^T$ is fully faithful, we can also view $J_T$ as arising from an (identity-on-objects, fully faithful) factorisation as above right; the relationship between the two is that $K_T = W_T \circ V_T$. Both perspectives will be used in what follows, with the functor $K_T : A_T \to \mathcal{E}^T$ of particular importance.

To define $\Phi$ on morphisms, we make use of the orthogonality of identity-on-objects $V$-functors to fully faithful ones; this asserts that any commuting square of $V$-functors as below, with $F$ identity-on-objects and $G$ fully faithful, admits a unique diagonal filler $J$ making both triangles commute.

\[
\begin{array}{c}
A \xrightarrow{H} C \\
\downarrow \downarrow \downarrow \downarrow \\
B \xrightarrow{K} \mathcal{D}
\end{array}
\]

Explicitly, $J$ is given on objects by $Ja = Ha$, and on homs by

\[
B(a, b) \xrightarrow{K_a,b} \mathcal{D}(Ka, Kb) = \mathcal{D}(GHa, GHb) \xrightarrow{GHa,Hb^{-1}} \mathcal{C}(Ha, Hb).
\]

In particular, given a map $\alpha : S \to T$ of $\text{Mnd}(\mathcal{E})$, this orthogonality guarantees the existence of a diagonal filler in the diagram below, whose upper triangle we take to be the map $\Phi(\alpha) : \Phi(S) \to \Phi(T)$ in $\text{Preth}_A(\mathcal{E})$:

\[
\begin{array}{c}
A \xrightarrow{J_T} A_T \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\mathcal{A}_S \xrightarrow{V_S} \mathcal{E}_S \xrightarrow{\alpha} \mathcal{E}_T
\end{array}
\]

2.5. Pretheories to monads. Thus far we have not exploited the local presentability of $\mathcal{E}$. It will be used in the next step, that of constructing the left adjoint to $\Phi : \text{Mnd}(\mathcal{E}) \to \text{Preth}_A(\mathcal{E})$. We first state a general result which, independent of local presentability, gives a sufficient condition for an individual pretheory to have a reflection along $\Phi$. Here, by a reflection of an object $c \in \mathcal{C}$ along a functor $U : B \to \mathcal{C}$, we mean a representation for the functor $\mathcal{C}(c, U-) : B \to \text{Set}$.

**Theorem 1.** A pretheory $J : \mathcal{A} \to T$ admits a reflection along $\Phi$ whenever the forgetful functor $U_T : \text{Mod}_c(T) \to \mathcal{E}$ from the category of concrete models has a left adjoint $F_T$. In this case, the reflection $\Psi T$ is characterised by an isomorphism
\[ \mathcal{E}^{\Psi(T)} \cong \text{Mod}_c(T) \] over \( \mathcal{E} \), or equally, by a pullback square

\[
\begin{array}{ccc}
\mathcal{E}^{\Psi(T)} & \xrightarrow{U^{\Psi(T)}} & [\mathcal{T}^{\text{op}}, \mathcal{V}] \\
\downarrow & & \downarrow \\
\mathcal{E} & \xrightarrow{N_K} & [\mathcal{A}^{\text{op}}, \mathcal{V}] 
\end{array}
\] (2.8)

To prove this result, we will need a preparatory lemma, relating to the notion of discrete isofibration: this is a \( \mathcal{V} \)-functor \( U: \mathcal{D} \to \mathcal{C} \) such that, for each \( c \cong Ud \) in \( \mathcal{C} \), there is a unique \( f': c' \cong d \) in \( \mathcal{D} \) with \( U(f') = f \).

**Example 2.** For any \( \mathcal{V} \)-monad \( T \) on \( \mathcal{C} \), the forgetful \( \mathcal{V} \)-functor \( U^T: \mathcal{C}^T \to \mathcal{C} \) is a discrete isofibration. Indeed, if \( x: Ta \to a \) is a \( T \)-algebra and \( f: b \cong a \) in \( \mathcal{C} \), then \( y = f^{-1}.x.Tf: Tb \to b \) is the unique algebra structure on \( b \) for which \( f: (b, y) \to (a, x) \) belongs to \( \mathcal{C}^T \). In particular, for any identity on objects \( \mathcal{V} \)-functor \( F: \mathcal{A} \to \mathcal{B} \) between small \( \mathcal{V} \)-categories, the functor \( [F, 1]: [\mathcal{B}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}] \) has a left adjoint and strictly creates colimits, whence is strictly monadic. It is therefore a discrete isofibration by the above argument.

**Lemma 3.** Let \( U: \mathcal{A} \to \mathcal{B} \) be a discrete isofibration and \( \alpha: F \Rightarrow G: \mathcal{X} \to \mathcal{B} \) an invertible \( \mathcal{V} \)-transformation. The displayed projections give isomorphisms between liftings of \( F \) through \( U \), liftings of \( \alpha \) through \( U \), and liftings of \( G \) through \( U \): 

\[
\begin{array}{ccc}
F \times U & \xrightarrow{\text{dom}} & A \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\alpha} & \mathcal{B} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F} & \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\alpha} & \mathcal{B} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{G} & \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{U} & \mathcal{B} \\
\end{array}
\]

**Proof.** Given \( G: \mathcal{X} \to \mathcal{A} \) as to the right, there is for each \( x \in \mathcal{X} \) a unique lifting of the isomorphism \( \alpha_x: Fx \cong UGx \) to one \( \tilde{\alpha}_x: \tilde{F}x \cong \tilde{G}x \). There is now a unique way of extending \( x \mapsto \tilde{F}x \) to a \( \mathcal{V} \)-functor \( F: \mathcal{X} \to \mathcal{A} \) so that \( \tilde{\alpha}: \tilde{F} \cong \tilde{G} \); namely, by defining the action on homs \( \tilde{F}_{x,y} = A(\tilde{\alpha}_x, \tilde{\alpha}_y^{-1}) \circ \tilde{G}_{x,y}: \mathcal{X}(x, y) \to \mathcal{A}(\tilde{F}x, \tilde{F}y) \). In this way, we have found a unique lifting of \( \alpha \) through \( U \) with codomain the given lifting of \( G \) through \( U \). So the right-hand projection is invertible; the argument for the left-hand one is the same on replacing \( \alpha \) by \( \alpha^{-1} \). \( \square \)

We can now give:

**Proof of Theorem 1.** \( U_T \) has a left adjoint by assumption, and—as a pullback of the strictly monadic \([J^{\text{op}}, 1]: [\mathcal{T}^{\text{op}}, \mathcal{V}] \to [\mathcal{A}^{\text{op}}, \mathcal{V}]—strictly creates coequalisers for \( U_T \)-absolute pairs. It is therefore strictly monadic. Taking \( \Psi(T) = U_T F_T \) to be the induced monad, we thus have an isomorphism \( \mathcal{E}^{\Psi(T)} \cong \text{Mod}_c(T) \) over \( \mathcal{E} \).

It remains to exhibit isomorphisms \( \text{Mnd}(\mathcal{E})(\Psi(T), S) \cong \text{Preth}_A(\mathcal{E})(T, \Phi(S)) \) natural in \( S \). We do so by chaining together the following sequence of natural bijections. Firstly, monad maps \( \alpha_0: \Psi(T) \to S \) correspond naturally to functors
\( \alpha_1 : \mathcal{E}^S \to \mathcal{E}^{\psi T} \) rendering commutative the left triangle in

\[
\begin{array}{ccc}
\mathcal{E}^S & \xrightarrow{\alpha_1} & \mathcal{E}^{\psi T} \\
\downarrow U^S & & \downarrow U^{\psi T} \\
\mathcal{E} & \xrightarrow{\alpha_2} & [T^{op}, \mathcal{V}] \\
\end{array}
\]

(2.9)

Since \( \mathcal{E}^{\psi T} \) is defined by the pullback (2.14), such \( \alpha_1 \) correspond naturally to functors \( \alpha_2 \) rendering commutative the square above right. Next, we observe that there is a natural isomorphism in the triangle below left

\[
\begin{array}{ccc}
\mathcal{E}^S & \xrightarrow{U^S} & \mathcal{E} \\
\downarrow \cong & & \downarrow N_K \\
\mathcal{E}^{\psi T} & \xrightarrow{\alpha_3} & [T^{op}, \mathcal{V}] \\
\end{array}
\]

(2.10)

with components the adjointness isomorphisms \( \mathcal{E}(Ka, U^Sb) \cong \mathcal{E}^S(F^S Ka, b) \). Since \( J^{op} \) is identity-on-objects, \( [J^{op}, 1] \) is a discrete isofibration by Example 2, whence by Lemma 3 there is a natural bijection between functors \( \alpha_2 \) as in (2.9) and ones \( \alpha_3 \) as in (2.10). We should now like to transpose this last triangle through the legitimate isomorphisms

\[
\mathcal{V} \cdot \operatorname{CAT}(\mathcal{E}^S, [X^{op}, \mathcal{V}]) \cong \mathcal{V} \cdot \operatorname{CAT}(\mathcal{E}^{\psi T}, [\mathcal{E}^{\psi T}, \mathcal{V}])
\]

(2.11)

However, since \( \mathcal{E}^S \) is large, the functor category \( [\mathcal{E}^S, \mathcal{V}] \) will not always exist as a \( \mathcal{V} \)-category, and so (2.11) is ill-defined. To resolve this, note that \( N_{F^S K} \) is, by its definition, pointwise representable; whence so too is \( \alpha_3 \), since \( J \) is identity-on-objects. We may thus transpose the right triangle of (2.10) through the legitimate isomorphisms

\[
\mathcal{V} \cdot \operatorname{CAT}(\mathcal{E}^{\psi T}, [X^{op}, \mathcal{V}])_{pwr} \cong \mathcal{V} \cdot \operatorname{CAT}(\mathcal{E}^S, [\mathcal{E}^S, \mathcal{V}]_{\text{rep}})
\]

(2.12)

where on the left we have the category of pointwise representable \( \mathcal{V} \)-functors, and on the right, the legitimate \( \mathcal{V} \)-category of representable \( \mathcal{V} \)-functors \( \mathcal{E}^S \to \mathcal{V} \). In this way, we establish a natural bijection between functors \( \alpha_3 \) and functors \( \alpha_4 \) rendering commutative the left square in:

\[
\begin{array}{ccc}
\mathcal{A}^{op} & \xrightarrow{J^{op}} & \mathcal{T}^{op} \\
\downarrow (F^S K)^{op} & \xleftarrow{\kappa} & \downarrow (F^S K)^{op} \\
(F^S K)^{op} \Downarrow Y & \xleftarrow{\alpha_5^{op}} & \Downarrow \mathcal{E}^{\psi T} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{A}^{op} & \xrightarrow{J} & \mathcal{T} \\
\downarrow J_5 & \xleftarrow{\kappa} & \downarrow J_5 \\
\mathcal{A}_S & \xrightarrow{K_5} & \mathcal{E}^S \\
\end{array}
\]

(2.13)

Now orthogonality of the identity-on-objects \( J^{op} \) and the fully faithful \( Y \) draws the correspondence between functors \( \alpha_4 \) and functors \( \alpha_5 \) satisfying \( \alpha_5 \circ J = F^S K \) as left above. Finally, since \( \mathcal{A}_S \) fits in to an (identity-on-objects, fully faithful) factorisation of \( F^S K \), orthogonality also gives the correspondence, as right above, between functors \( \alpha_5 \) and functors \( \alpha_6 \) satisfying \( \alpha_6 \circ J = J_5 \), as required. \( \Box \)

We now show that the assumed local presentability of \( \mathcal{E} \) ensures that every pretheory has a reflection along \( \Phi : \operatorname{Mnd}(\mathcal{E}) \to \operatorname{Preth}(\mathcal{E}) \), which consequently
has a left adjoint. The key result about locally presentable categories enabling this is the following lemma.

**Lemma 4.** Consider a pullback square of \( \mathcal{V} \)-categories

\[
\begin{array}{ccc}
C' & \xrightarrow{G'} & D' \\
\downarrow{U} & \nearrow{V} & \downarrow{U} \\
C & \xrightarrow{G} & D
\end{array}
\]

in which \( G \) and \( V \) are right adjoints between locally presentable \( \mathcal{V} \)-categories and \( V \) is strictly monadic. Then \( U \) and \( G' \) are right adjoints between locally presentable \( \mathcal{V} \)-categories and \( U \) is strictly monadic.

**Proof.** Since \( V \) is strictly monadic, it is a discrete isofibration, and so its pullback against \( G \) is, by [13, Corollary 1], also a bipullback. By [9, Theorem 6.11] the 2-category of locally presentable \( \mathcal{V} \)-categories and right adjoint functors is closed under bilimits in \( \mathcal{V} \text{-CAT} \), so that both \( U \) and \( G \) are right adjoints between locally presentable categories. Finally, since \( U \) is a pullback of the strictly monadic \( V \), it strictly creates coequalisers for \( U \)-absolute pairs. Since it is already known to be a right adjoint, it is therefore also strictly monadic. \( \square \)

With this in place, we can now prove:

**Theorem 5.** Let \( \mathcal{E} \) be locally presentable. Then \( \Phi: \text{Mnd}(\mathcal{E}) \to \text{Preth}_A(\mathcal{E}) \) has a left adjoint \( \Psi: \text{Preth}_A(\mathcal{E}) \to \text{Mnd}(\mathcal{E}) \), whose value at the pretheory \( J: \mathcal{A} \to \mathcal{T} \) is characterised by an isomorphism \( \mathcal{E}^{\Psi(T)} \cong \text{Mod}_c(\mathcal{T}) \) over \( \mathcal{E} \), or equally, by a pullback square

\[
\begin{array}{ccc}
\mathcal{E}^{\Psi(T)} & \to & [\mathcal{T}^{\text{op}}, \mathcal{V}] \\
\downarrow{U^{\Psi(T)}} & \nearrow{[\mathcal{J}^{\text{op}}, \mathcal{V}]_{\Delta}} & \downarrow{[\mathcal{J}^{\text{op}}, \mathcal{V}]_{\Delta}} \\
\mathcal{E} & \xrightarrow{N_K} & [\mathcal{A}^{\text{op}}, \mathcal{V}].
\end{array}
\]

**Proof.** Let \( J: \mathcal{A} \to \mathcal{T} \) be a pretheory. The pullback square (2.5) defining \( \text{Mod}_c(\mathcal{T}) \) is a pullback of a right adjoint functor between locally presentable categories along a strictly monadic one: so it follows from Lemma 4 that \( U_T: \text{Mod}_c(\mathcal{T}) \to \mathcal{E} \) is a right adjoint, whence the result follows from Theorem 1. \( \square \)

### 3. Pretheories as presentations

In the next section, we will describe how the monad–pretheory adjunction (2.1) restricts to an equivalence between suitable subcategories of \( \mathcal{A} \)-theories and of \( \mathcal{A} \)-nervous monads. However, the results we have so far are already practically useful. The notion of \( \mathcal{A} \)-pretheory provides a tool for presenting certain kinds of algebraic structure, by exhibiting them as categories of concrete \( \mathcal{T} \)-models for a suitable pretheory in a manner reminiscent of the theory of sketches [5]. Equivalently, via the functor \( \Psi \), we can see \( \mathcal{A} \)-pretheories as a way of presenting certain monads on \( \mathcal{E} \).
3.1. **Examples of the basic setting.** Before giving our examples of algebraic structures presented by pretheories, we first describe a range of examples of the basic setting of Section 2.1 above.

**Examples 6.** We begin by considering the unenriched case where $\mathcal{V} = \text{Set}$.

(i) Taking $\mathcal{E} = \text{Set}$ and $\mathcal{A} = \mathcal{F}$ the full subcategory of finite cardinals captures the classical case of *finitary* algebraic structure borne by *sets*; so examples like groups, rings, lattices, Lie algebras, and so on.

(ii) Taking $\mathcal{E}$ a locally finitely presentable category and $\mathcal{A} = \mathcal{E}$ a skeleton of the full subcategory of finitely presentable objects, we capture *finitary* algebraic structure borne by $\mathcal{E}$-objects. Examples when $\mathcal{E} = \text{Cat}$ include finite product, finite colimit, and monoidal closed structure; for $\mathcal{E} = \text{CRng}$, we have commutative $k$-algebra, differential ring and reduced ring structure.

(iii) We can replace “finitary” above by “$\lambda$-ary” for any regular cardinal $\lambda$. For example, when $\lambda = \aleph_1$, this allows for $\omega$-cpo structure if $\mathcal{E} = \text{Set}$ and countable product structure if $\mathcal{E} = \text{Cat}$; while for suitable $\lambda$ it allows for sheaf or *sheaf of rings* structure when $\mathcal{E} = [\mathcal{O}(X)^{\text{op}}, \text{Set}]$ for a space $X$.

(iv) Let $\mathcal{G}_1$ be the category freely generated by the graph $0 \Rightarrow 1$, so that $\mathcal{E} = [\mathcal{G}_1^{\text{op}}, \text{Set}]$ is the category of directed multigraphs, and let $\mathcal{A} = \Delta_0$ be the full subcategory of $[\mathcal{G}_1^{\text{op}}, \text{Set}]$ on graphs of the form

$$\Delta_0 \colon [n] := 0 \longrightarrow 1 \longrightarrow \cdot \longrightarrow n \quad \text{for } n > 0.$$

$\Delta_0$ is dense in $[\mathcal{G}_1^{\text{op}}, \text{Set}]$ because it contains the representables $[0]$ and $[1]$. This example captures structure borne by graphs in which the operations build vertices and arrows from *paths* of arrows: for example, the structures of *categories*, *involutive categories*, and *groupoids*.

(v) The *globe category* $\mathbb{G}$ is freely generated by the graph

$$0 \begin{array}{c} \sigma \downarrow \tau \\ \sigma \downarrow \tau \end{array} 1 \begin{array}{c} \sigma \downarrow \tau \\ \sigma \downarrow \tau \end{array} 2 \begin{array}{c} \sigma \downarrow \tau \\ \sigma \downarrow \tau \end{array} \cdots$$

subject to the coglobular relations $\sigma \sigma = \sigma \tau$ and $\tau \sigma = \tau \tau$. This means that for each $m > n$, there are precisely two maps $\sigma^{m-n}, \tau^{m-n} : n \Rightarrow m$, which by abuse of notation we will write simply as $\sigma$ and $\tau$.

The category $\mathcal{E} = [\mathbb{G}^{\text{op}}, \text{Set}]$ is the category of *globular sets*; it has a dense subcategory $\mathcal{A} = \Theta_0$, first described by Berger [7], whose objects have been termed *globular cardinals* by Street [32]. These include the representables—the $n$-globes $Y_n$ for each $n$—but also shapes such as the globular set with distinct cells as depicted below.

$$\begin{array}{c} \bullet \ 
\begin{array}{c} \sigma \downarrow \tau \\ \sigma \downarrow \tau \end{array} \bullet \ 
\begin{array}{c} \sigma \downarrow \tau \\ \sigma \downarrow \tau \end{array} \bullet \ 
\begin{array}{c} \sigma \downarrow \tau \\ \sigma \downarrow \tau \end{array} \bullet \ 
\end{array}$$

The globular cardinals can be parametrised in various ways, for instance using trees [6, 7]; following [26], we will use *tables of dimensions*—sequences $\vec{n} = (n_1, \ldots, n_k)$ of natural numbers of odd length with $n_{2i-1} > n_{2i} < n_{2i+1}$. 

\[\text{(3.1)}\]
Given such a table $\vec{n}$ and a functor $D: G \to C$, we obtain a diagram

\[
\begin{array}{ccccccc}
D_{n_1} & \downarrow & D_{n_2} & \downarrow & D_{n_3} & \downarrow & \cdots & \downarrow & D_{n_k} \\
D_{n_1} & \downarrow & D_{n_2} & \downarrow & D_{n_3} & \downarrow & \cdots & \downarrow & D_{n_k} \\
& & & & & & & &
\end{array}
\]

whose colimit in $C$, when it exists, will be written as $D(\vec{n})$, and called the $D$-globular sum indexed by $\vec{n}$. Taking $D = Y: G \to [G^{op}, \mathbf{Set}]$, the category $\Theta_0$ of globular cardinals is now defined as the full subcategory of $[G^{op}, \mathbf{Set}]$ spanned by the $Y$-globular sums. For example, the globular cardinal in (3.1) corresponds to the $Y$-globular sum $Y(1, 0, 2, 1, 2)$.

This example captures algebraic structures on globular sets in which the operations build globes out of diagrams with shapes like (3.1); these include strict $\omega$-categories and strict $\omega$-groupoids, but also the (globular) weak $\omega$-categories and weak $\omega$-groupoids studied in [6, 24, 3].

We now turn to examples over enriched bases.

(vi) Let $V$ be a locally finitely presentable symmetric monoidal category whose finitely presentable objects are closed under the tensor product (cf. [17]). By taking $E = V$ and $A = V_f$ a skeleton of the full sub-$V$-category of finitely presentable objects, we capture $V$-enriched finitary algebraic structure on $V$-objects as studied in [30]. When $V = \mathbf{Cat}$ this means structure on categories $C$ built from functors and natural transformations $C^n \to C$ for finitely presentable $I$: which includes symmetric monoidal or finite limit structure, but not symmetric monoidal closed or factorization system structure. Similarly, when $V = \mathbf{Ab}$, it includes $A$-module structure but not commutative ring structure.

(vii) Taking $V$ as before, taking $E$ to be any locally finitely presentable $V$-category [17] and taking $A = E_f$ a skeleton of the full subcategory of finitely presentable objects in $E$, we capture $V$-enriched finitary algebraic structure on $E$-objects as studied in [29]. As before, there is the obvious generalization from finitary to $\lambda$-ary structure.

(viii) This example builds on [22]. Let $V$ be a locally presentable symmetric monoidal closed category, and consider a class of $V$-enriched limit-types $\Phi$ with the property that the free $\Phi$-completion of a small $V$-category is again small. A $V$-functor $F: C \to V$ with small domain is called $\Phi$-flat if its cocontinuous extension $\text{Lan}_y F: [C^{op}, V] \to V$ preserves $\Phi$-limits, and $a \in V$ is $\Phi$-presentable if $V(a, -): V \to V$ preserves colimits by $\Phi$-flat weights.

Suppose that if $C$ is small and $\Phi$-complete, then every $\Phi$-continuous $F: C \to V$ is $\Phi$-flat; this is Axiom A of [22]. Then by Proposition 3.4 and §7.1 of ibid., we obtain an instance of our setting on taking $E = V$ and $A = V_\Phi$ a skeleton of the full sub-$V$-category of $\Phi$-presentable objects.

A key example takes $V = E = \mathbf{Cat}$ and $\Phi$ the class of finite products; whereupon $V_\Phi$ is the subcategory $F$ of finite cardinals, seen as discrete categories. This example captures strongly finitary [18] structure on categories involving functors and transformations $C^n \to C$; this includes monoidal or finite product structure, but not finite limit structure.
(ix) More generally, we can take \( E = \Phi \text{-} \text{Cts}(C, V) \), the \( V \)-category of \( \Phi \)-continuous functors \( C \to V \) for some small \( \Phi \)-complete \( C \), and take \( A \) to be the full image of the Yoneda embedding \( Y : C^{\text{op}} \to \Phi \text{-} \text{Cts}(C, V) \). This example is appropriate to the study of \( \Phi \)-ary algebraic structure on \( E \)-objects—subsuming most of the preceding examples.

### 3.2. Pretheories as presentations.

We will now describe examples of pretheories and their models in various contexts; in doing so, it will be useful to avail ourselves of the following constructions. Given a pretheory \( A \to T \) and objects \( a, b \in T \), to adjoin a morphism \( f : a \to b \) is to form the \( V \)-category \( T[f] \) in the pushout square to the left of:

\[
\begin{array}{c}
2 \xrightarrow{2} T \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
[2] \xrightarrow{f} T[f]
\end{array}
\quad \quad \quad
\begin{array}{c}
2 + 2 \xrightarrow{2} T \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
[2] \xrightarrow{f = g} T[f = g]
\end{array}
\]

Here, \( \iota : 2 \to 2 \) is the inclusion of the free \( \mathcal{V} \)-category on the set \( \{0, 1\} \) into the free \( \mathcal{V} \)-category \( 2 = \{0 \to 1\} \) on an arrow. Since \( \iota \) is identity-on-objects, its pushout \( \bar{\iota} \) may also be chosen thus, so that we may speak of adjoining an arrow to a pretheory \( J : A \to T \) to obtain the pretheory \( J[f] = \bar{\iota} \circ J : A \to T[f] \).

Recall from (2.5) that a concrete \( T \)-model comprises \( X \in \mathcal{E} \) and \( F \in \{T^{\text{op}}, \mathcal{V}\} \) for which \( F \circ J^{\text{op}} = \mathcal{E}(K_-, X) : A \to \mathcal{V} \). Thus, by the universal property of the pushout (3.2), a concrete \( T[f] \)-model is the same as a concrete \( T \)-model \( (X, F) \) together with a map \( [f] : \mathcal{E}(Kb, X) \to \mathcal{E}(Ka, X) \) in \( \mathcal{V} \).

Similarly given parallel morphisms \( f, g : a \rightarrow b \) in the underlying category of \( T \) we can form the pushout above right. In this way we may speak of adjoining an equation \( f = g \) to a pretheory \( J : A \to T \) to obtain the pretheory \( J[f = g] = \iota \circ J : A \to T[f = g] \). In this case, we see that a concrete \( T[f = g] \)-model is a concrete \( T \)-model \( (X, F) \) such that \( FF = Fg : \mathcal{E}(Kb, X) \to \mathcal{E}(Ka, X) \).

**Example 7.** In the context of Examples 6(i) appropriate to classical finitary algebraic theories—namely, \( V = \mathcal{E} = \text{Set} \) and \( A = \mathbb{F} \)—we will construct a pretheory \( J : \mathbb{F} \to \mathcal{M} \) whose category of concrete models is the category of monoids.

We start from the initial pretheory \( \text{id} : \mathbb{F} \to \mathbb{F} \) whose concrete models are simply sets, and construct from it a pretheory \( J_1 : \mathbb{F} \to \mathcal{M}_1 \) by adjoining morphisms

\[
m : 1 \to 2 \quad \text{and} \quad i : 1 \to 0
\]

representing the monoid multiplication and unit operations, and also morphisms

\[
m1, 1m : 2 \rightrightarrows 3 \quad \text{and} \quad i1, 1i : 2 \rightrightarrows 1
\]

which will be necessary later to express the monoid equations. Note that our directional conventions mean that the input arity of these operations is in the **codomain** rather than the domain. It follows from the preceding remarks that a concrete \( \mathcal{M}_1 \)-model is a set \( X \) equipped with functions

\[
m : X^2 \to X \quad , \quad [i] : 1 \to X \quad , \quad [m1], [1m] : X^3 \rightrightarrows X^2 \quad , \quad [i1], [1i] : 1 \rightrightarrows X
\]
interpreting the morphisms adjoined above. We now adjoin to $\mathcal{M}_1$ the eight equations necessary to render commutative the following squares in $\mathcal{M}_1$:

\[
\begin{array}{cccc}
1 & \xrightarrow{m} & 2 & \\
\downarrow{\iota_1} & & \downarrow{\iota_1} & \\
1+1 & \xrightarrow{1+1} & 1+2 & \\
\downarrow{\iota_2} \downarrow{\iota_2} & & \downarrow{\iota_2} \downarrow{\iota_2} & \\
1 & \xrightarrow{id} & 1 & \\
\end{array}
\]

(3.5)

\[
\begin{array}{cccc}
1 & \xrightarrow{m} & 2 & \\
\downarrow{\iota_1} & & \downarrow{\iota_1} & \\
1+1 & \xrightarrow{1+1} & 1+2 & \\
\downarrow{\iota_2} \downarrow{\iota_2} & & \downarrow{\iota_2} \downarrow{\iota_2} & \\
1 & \xrightarrow{id} & 1 & \\
\end{array}
\]

A concrete model for the resulting theory $J : F \to \mathcal{M}$ is a concrete $\mathcal{M}_1$-model $(X,F)$ for which $F^{op} : \mathcal{M}_1 \to \textbf{Set}^{op}$ sends each diagram in (3.5) and (3.6) to a commuting one. Commutativity in (3.5) forces $|m1| = m \times 1 \times 1 : X^3 \to X^2$ and so on; whereupon commutativity of (3.6) expresses precisely the monoid axioms, so that concrete $\mathcal{M}$-models are monoids, as desired. Extending this analysis to morphisms we see that $\textbf{Mod}_c(\mathcal{M})$ is isomorphic to the category of monoids and monoid homomorphisms.

**Example 8.** In the same way we can describe $\mathbb{F}$-pretheories modelling any of the categories of classical universal algebra—groups, rings and so on. Note that the same structure can be presented by distinct pretheories: for instance, we could extend the pretheory $\mathcal{M}$ just constructed by adjoining a further morphism $m11 : 3 \to 4$ and two equations forcing it to become $|m1| \times 1 \times 1 : X^4 \to X^3$ in any model; on doing so, we would not change the category of concrete models. However, in $\mathcal{M}_1$, all of the maps $3 \to 4$ belong to $\mathbb{F}$ while in the new pretheory, $m11$ does not. This non-canonicity will be rectified by the theories introduced in Section 4 below; in particular, Corollary 22 implies that, to within isomorphism, there is at most one $\mathbb{F}$-theory which captures a given type of structure.

**Example 9.** In the situation of Examples 6(iv), where $\mathcal{E} = [\text{G}_0^{op}, \text{Set}]$ is the category of directed graphs and $\mathcal{A} = \Delta_0$, we will describe a pretheory $\Delta_0 \to C$ whose concrete models are categories. The construction is largely identical to the example of monoids above. Starting from the initial $\Delta_0$-pretheory, we adjoin composition and unit maps $m : [1] \to [2]$ and $i : [1] \to [0]$ as well as the morphisms $1m, m1 : [2] \xrightarrow{[3]}$ and $i1, li : [2] \xrightarrow{[1]}$ required to describe the category axioms.

We now adjoin the necessary equations. First, we have four equations ensuring that composition and identities interact appropriately with source and target:

\[
\begin{array}{cccc}
[0] & \xrightarrow{\sigma} & [1] & \\
\downarrow{\sigma} & & \downarrow{\tau} & \\
[1] & \xrightarrow{m} & [2] & \\
\downarrow{\iota_1} & & \downarrow{\iota_2} & \\
[0] & \xrightarrow{\sigma} & [1] & \\
\end{array}
\]

where $\sigma, \tau$ and ! are the images under $J_1$ of the relevant coproduct injections or maps from 0 in $\mathbb{F}$; together with three equations which render commutative:

\[
\begin{array}{cccc}
1 & \xrightarrow{m} & 2 & \\
\downarrow{\iota_1} & & \downarrow{\iota_1} & \\
1+1 & \xrightarrow{1+1} & 1+2 & \\
\downarrow{\iota_2} \downarrow{\iota_2} & & \downarrow{\iota_2} \downarrow{\iota_2} & \\
1 & \xrightarrow{id} & 1 & \\
\end{array}
\]

(3.6)

where $\iota_3, \iota_2$ and $!$ are the images under $J_1$ of the relevant coproduct injections or maps from 0 in $\mathbb{F}$; together with three equations which render commutative:

\[
\begin{array}{cccc}
1 & \xrightarrow{m} & 2 & \\
\downarrow{\iota_1} & & \downarrow{\iota_1} & \\
1+1 & \xrightarrow{1+1} & 1+2 & \\
\downarrow{\iota_2} \downarrow{\iota_2} & & \downarrow{\iota_2} \downarrow{\iota_2} & \\
1 & \xrightarrow{id} & 1 & \\
\end{array}
\]
where here we write $\sigma, \tau: [0] \Rightarrow [1]$ for the two endpoint inclusions, and $\iota_1, \iota_2$ for the two colimit injections into $[1]_{\tau + \sigma} [1] = [2]$. We also require analogues of the eight equations of (3.5) and three equations of (3.6). The modifications are minor: replace $n$ by $[n]$, the coproduct inclusions $\iota_1: n \to n + m \leftarrow m: \iota_2$ by the pushout inclusions $\iota_1: [n] \to [n]_{\tau + \sigma} [m] \leftarrow [m]: \iota_2$, the first appearance of $!: 0 \to 1$ by $\sigma: [0] \to [1]$ and its second appearance by $\tau: [0] \to [1]$. After adjoining these six morphisms and fifteen equations, we find that the concrete models of the resulting pretheory $\Delta_0 \to \mathcal{C}$ are precisely small categories.

We can extend this pretheory to one for groupoids. To do so, we adjoin a morphism $c: [1] \to [1]$ modelling the inversion plus the further maps $1c: [2] \to [2]$ and $c1: [2] \to [2]$ required for the axioms. Now four equations must be adjoined to force the correct interpretation of $1c$ and $c1$, plus the two equations for left and right inverses. On doing so, the resulting pretheory $\Delta_0 \to \mathcal{G}$ has as its concrete models the small groupoids.

**Example 10.** In the situation of Examples 6(v), where $\mathcal{E}$ is the category of globular sets and $\mathcal{A} = \Theta_0$ is the full subcategory of globular cardinals, one can similarly construct pretheories whose concrete models are strict $\omega$-categories or strict $\omega$-groupoids. For instance, one encodes binary composition of $n$-cells along a $k$-cell boundary (for $k < n$) by adjoining morphisms $m_{n,k}: Y(n) \to Y(n,k,n)$ to $\Theta_0$. In fact, all of the standard flavours of globular weak $\omega$-category and weak $\omega$-groupoid can also be encoded using $\Theta_0$-pretheories; see Examples 42(v) below.

**Example 11.** Consider the case of Examples 6(viii) where $\mathcal{V} = \mathcal{E} = \text{Cat}$ and $\mathcal{A} = \mathcal{F}$, the full subcategory of finite cardinals (seen as discrete categories). We will describe an $\mathcal{F}$-pretheory capturing the structure of a monoidal category. In doing so, we exploit the fact that our pretheories are no longer mere categories, but 2-categories; so we may speak not only of adjoining morphisms and equations between such, but also of adjoining an (invertible) 2-cell—by taking a pushout of the inclusion $2 + 2 \to D_2$ of the parallel pair 2-category into the free 2-category on an (invertible) 2-cell—and similarly of adjoining an equation between 2-cells.

To construct a pretheory for monoidal categories, we start essentially as for monoids: freely adjoining the usual maps $m, i, m1, 1m, i1, li$ to the initial pretheory, but now also morphisms $m11, 1m1, 11m: 3 \to 4$ and $11i: 3 \to 2$ needed for the monoidal category coherence axioms; thus, ten morphisms in all.

We now add the $8 \times 2 = 16$ equations asserting that each of the morphisms beyond $m$ and $i$ has the expected interpretation in a model, plus the equation $1m \circ m11 = m1 \circ 11m: 2 \to 4$. This being done, we next adjoin invertible 2-cells

\[
\begin{array}{ccc}
1 & \xrightarrow{m} & 2 \\
\downarrow m1 & & \downarrow m1 \\
2 & \xrightarrow{1m} & 3
\end{array}
\quad
\begin{array}{ccc}
m & \xrightarrow{1} & i1 \\
\downarrow m & & \downarrow m & \downarrow i1 \\
1 & \xrightarrow{1} & 1
\end{array}
\quad
\begin{array}{ccc}
m & \xrightarrow{1} & 2 \\
\downarrow m & & \downarrow m & \downarrow 1 \\
1 & \xrightarrow{1} & 1
\end{array}
\]

\[\text{It may be prima facie unclear why this is necessary; after all, if } 1m, m11, m1 \text{ and } 11m \text{ have the intended interpretations in a model, then it is certainly the case that they will verify this equality. Yet this equality is not forced to hold in the pretheory, and we need it to do so in order for (3.7) to type-check.}\]
expressing the associativity and unit coherences, as well as the invertible 2-cells

\[
\begin{array}{ccc}
2 & \xrightarrow{m} & 3 \\
\downarrow & \Downarrow & \downarrow \\
1 & \xrightarrow{l} & 4 \\
\end{array} \quad \begin{array}{ccc}
2 & \xrightarrow{1m} & 3 \\
\downarrow & \Downarrow & \downarrow \\
1 & \xrightarrow{l} & 4 \\
\end{array}
\]

which will be needed to express the coherence axioms. Finally, we must adjoin equations between 2-cells: the \(2 \times 4 = 8\) equations ensuring that \(\alpha, 1, \lambda, \rho\) and \(1\) have the intended interpretation in any model, plus two equations expressing the coherence axioms:

\[
\begin{array}{ccc}
2 & \xrightarrow{m} & 3 \\
\downarrow & \Downarrow & \downarrow \\
1 & \xrightarrow{l} & 4 \\
\end{array} \quad \begin{array}{ccc}
2 & \xrightarrow{1m} & 3 \\
\downarrow & \Downarrow & \downarrow \\
1 & \xrightarrow{l} & 4 \\
\end{array}
\]

All told, we have adjoined ten morphisms, seventeen equations between morphisms, seven invertible 2-cells, and nine equations between 2-cells to obtain a pretheory \(J : F \to MC\) whose concrete models are precisely monoidal categories.

4. The monad–theory correspondence

In this section, we return to the general theory and establish our “best possible” monad–theory correspondence. This will be obtained by restricting the adjunction (2.1) to its fixpoints: the objects on the left and right at which the counit and the unit are invertible. The categories of fixpoints are the largest subcategories on which the adjunction becomes an adjoint equivalence, and it is in this sense that our monad–theory correspondence is the best possible.

4.1. A pullback lemma. The following lemma will be crucial in characterising the fixpoints of (2.1) on each side. Note that the force of (2) below is in the “if” direction; the “only if” is always true.

**Lemma 12.** A commuting square in \(\mathcal{V}\cdot\text{CAT}\)

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow & \Downarrow & \downarrow \\
C & \xrightarrow{G} & D
\end{array}
\]
with $G$ fully faithful and $H,K$ discrete isofibrations is a pullback just when:

1. $F$ is fully faithful; and
2. An object $b \in \mathcal{B}$ is in the essential image of $F$ if and only if $Kb$ is in the essential image of $G$.

Proof. If the square is a pullback, then $F$ is fully faithful as a pullback of $G$. As for (2), if $Kb \cong Gc$ in $\mathcal{D}$ then since $K$ is an isofibration we can find $b \cong b'$ in $\mathcal{B}$ with $Kb' = Gc$; now by the pullback property we induce $a \in \mathcal{A}$ with $Fa = b'$ so that $b \cong Fa$ as required. Suppose conversely that (1) and (2) hold. We form the pullback $\mathcal{P}$ of $K$ along $G$ and the induced map $L$ as below.

\[
\begin{array}{ccc}
A & \xrightarrow{L} & P \\
\downarrow & & \downarrow \\
H & \xrightarrow{P} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{G} & D \\
\end{array}
\]

$P$ is fully faithful as a pullback of $G$, and $F$ is so by assumption; whence by standard cancellativity properties of fully faithful functors, $L$ is also fully faithful.

In fact, discrete isofibrations are also stable under pullback, and also have the same cancellativity property; this follows from the fact that they are the exactly the maps with the unique right lifting property against the inclusion of the free $\mathcal{V}$-category on an object into the free $\mathcal{V}$-category on an isomorphism. Consequently, in (4.1), $Q$ is a discrete isofibration as a pullback of $K$, and $H$ is so by assumption; whence by cancellativity, $L$ is also a discrete isofibration.

If we can now show $L$ is also essentially surjective, we will be done: for then $L$ is a discrete isofibration and an equivalence, whence invertible. So let $(b,c) \in \mathcal{P}$. Since $Kb = Gc$, by (2) we have that $b$ is in the essential image of $F$. So there is $a \in \mathcal{A}$ and an isomorphism $\beta : b \cong Fa$. Now $K\beta : Gc = Kb \cong KFa = GHa$ so by full fidelity of $G$ there is $\gamma : c \cong Ha$ with $G\gamma = K\beta$; and so we have $(\beta, \gamma) : (b,c) \cong La$ exhibiting $(b,c)$ as in the essential image of $L$, as required. \qed

4.2. $\mathcal{A}$-theories. We first use the pullback lemma to describe the fixpoints of (2.1) on the pretheory side.

Definition 13. An $\mathcal{A}$-pretheory $J : \mathcal{A} \to \mathcal{T}$ is said to be an $\mathcal{A}$-theory if each $\mathcal{T}(\mathcal{J}_-, a) \in [\mathcal{A}^{\text{op}}, \mathcal{V}]$ is a $K$-nerve. We write $\text{Th}_\mathcal{A}(\mathcal{E})$ for the full subcategory of $\text{Preth}_\mathcal{A}(\mathcal{E})$ on the $\mathcal{A}$-theories.

In the language of Section 5.2 below, a pretheory $\mathcal{T}$ is an $\mathcal{A}$-theory just when each representable $\mathcal{T}(-, a) : \mathcal{T}^{\text{op}} \to \mathcal{V}$ is a (non-concrete) $\mathcal{T}$-model.

Theorem 14. An $\mathcal{A}$-pretheory $J : \mathcal{A} \to \mathcal{T}$ is an $\mathcal{A}$-theory if and only if the unit component $\eta_\mathcal{T} : \mathcal{T} \to \Phi \Psi \mathcal{T}$ of (2.1) is invertible.

Proof. The unit $\eta_\mathcal{T} : \mathcal{T} \to \Phi \Psi \mathcal{T}$ is obtained by starting with $\alpha_0 = 1 : \Psi \mathcal{T} \to \Psi \mathcal{T}$ and chasing through the bijections of Theorem 5 to obtain $\alpha_0 = \eta_\mathcal{T}$. Doing this,
we quickly arrive at \( \alpha_2 \) equal to \( P \), the projection in the depicted pullback square

\[
\begin{array}{ccc}
\mathcal{E}^{\Psi T} & \xrightarrow{P} & \mathcal{T}^{op}, V \\
U^{\Psi T} \downarrow & & \downarrow \alpha_4 \\
\mathcal{E} & \xrightarrow{N_K} & \mathcal{A}^{op}, V
\end{array}
\]

(4.2)

\[
\begin{array}{ccc}
\mathcal{A}^{op} & \xrightarrow{J^{op}} & \mathcal{T}^{op} \\
(A^{\Psi T})^{op} \downarrow \alpha_6^{op} & & \downarrow \alpha_4 \\
(K^{\Psi T})^{op} \circ (\mathcal{E}^{\Psi T})^{op} \xrightarrow{\alpha_5^{op}} Y & & \xrightarrow{\alpha_5^{op}} (\mathcal{E}^{\Psi T}, V)_{rep}
\end{array}
\]

defining \( \mathcal{E}^{\Psi T} \). Now \( \alpha_3 : \mathcal{E}^{\Psi T} \rightarrow \mathcal{T}^{op}, V \) is obtained by lifting an isomorphism through \( [J^{op}, 1] \) and so we have \( \alpha_3 \cong P \). We obtain \( \alpha_4 \) by transposing \( \alpha_3 \) through the isomorphism \((-)^t : \mathcal{V} \cdot \text{CAT}(\mathcal{E}^{\Psi T}, [\mathcal{T}^{op}, V])_{pwr} \cong \mathcal{V} \cdot \text{CAT}(\mathcal{T}^{op}, [\mathcal{E}^{\Psi T}, V]_{rep}) \) displayed in (2.12). The relationships between \( \alpha_4 \), \( \alpha_5 \) and the unit component \( \eta_T = \alpha_6 \) are depicted in the commutative diagram above right.

The identity on objects unit \( \eta_T = \alpha_6 \) will be invertible just when it is fully faithful which, since \( K^{\Psi T} \) is fully faithful, will be so just when \( \alpha_5 \) is fully faithful. Now, since \( P \cong \alpha_3 = (\alpha_4)^t = (Y \circ \alpha_5^{op})^t = N_{\alpha_5} \), and \( P \) is fully faithful, as the pullback of the fully faithful \( N_K \), it follows that \( N_{\alpha_5} : \mathcal{E}^{\Psi T} \rightarrow \mathcal{T}^{op}, V \) is also fully faithful. As a consequence, \( \alpha_5 \) is fully faithful just when there exists a factorisation to within isomorphism:

\[
Y \cong N_{\alpha_5} \circ G : \mathcal{T} \rightarrow \mathcal{E}^{\Psi T} \rightarrow \mathcal{T}^{op}, V .
\]

Indeed, in one direction, if \( \alpha_5 \) is fully faithful then the canonical natural transformation \( Y \Rightarrow N_{\alpha_5} \circ \alpha_5 \) is invertible. In the other, given a factorisation as displayed, \( G \) is fully faithful since \( N_{\alpha_5} \) and \( Y \) are. Moreover we have isomorphisms

\[
\mathcal{E}^{\Psi T}(\alpha_5 b, -) \cong [\mathcal{T}^{op}, V](Y b, N_{\alpha_5} -) \cong [\mathcal{T}^{op}, V](N_{\alpha_5} G b, N_{\alpha_5} -) \cong \mathcal{E}^{\Psi T}(G b, -)
\]

natural in \( b \). So by Yoneda, \( \alpha_5 \cong G \) and so \( \alpha_5 \) is fully faithful since \( G \) is so.

This shows that \( \eta_T \) is invertible just when there is a factorisation (4.3). Since \( N_{\alpha_5} \) is fully faithful this in turn is equivalent to asking that each \( Y b = T(-, b) \) lies in the essential image of \( \alpha_5 \), or equally in the essential image of the isomorphic \( P \). As the left square of (4.2) is a pullback, Lemma 12 asserts that this is, in turn, equivalent to each \( [J^{op}, 1](Y b) = T(J-, b) \) being in the essential image of \( N_K \); which is precisely the condition that \( J \) is an \( \mathcal{A} \)-theory.

\[\Box\]

4.3. \( \mathcal{A} \)-nervous monads. We now characterise the fixpoints on the monad side. In the following definition, \( \mathcal{A}_T, J_T \) and \( K_T \) are as in (2.7).

**Definition 15.** A \( \mathcal{V} \)-monad \( T \) on \( \mathcal{E} \) is called \( \mathcal{A} \)-nervous if

(i) The fully faithful \( K_T : \mathcal{A}_T \rightarrow \mathcal{E}^T \) is dense;

(ii) A presheaf \( X \in [\mathcal{A}_T^{op}, V] \) is a \( K_T \)-nerve if and only if \( X \circ J_T^{op} \) is a \( K \)-nerve.

We write \( \text{Mnd}_A(\mathcal{E}) \) for the full subcategory of \( \text{Mnd}(\mathcal{E}) \) on the \( \mathcal{A} \)-nervous monads.
Note that the adjointness isomorphisms $E^T(KTJTX, Y) = E^T(F^TKX, Y) \cong E(KX, UTY)$ for the adjunction $F^T \downarrow UT$ give a pseudo-commutative square

$$E^T \xrightarrow{N_{KT}} [A_{op}, V] \quad \xrightarrow{\cong} \quad [J_{op}, 1] \quad \xrightarrow{\cong} \quad E^T \xrightarrow{N_K} [A_{op}, V];$$

as a result of which, $[J_{op}, 1]$ maps $KT$-nerves to $K$-nerves. Thus the force of clause (ii) of the preceding definition lies in the if direction.

**Theorem 16.** The counit component $\varepsilon_T : \Psi \Phi T \to T$ of (2.1) at a monad $T$ on $E$ is invertible if and only if $T$ is $A$-nervous.

**Proof.** $\varepsilon_T$ is obtained by taking $\alpha_6 = 1 : J_T \to J_T$ and proceeding in reverse order through the series of six natural isomorphisms in the proof of Theorem 5. Doing this, we quickly reach $\alpha_3 = N_{KT}$. Then $\alpha_2 : E^T \rightarrow [(A_{op}, V)]$ is obtained by lifting the natural isomorphism $\varphi$ of (4.4) through the discrete isofibration $[J_{op}, 1]$, yielding a commutative square as left below.

$$E^T \xrightarrow{\alpha_2} [(A_{op}, V)] \quad \xrightarrow{\cong} \quad \Psi \Phi T \xrightarrow{\alpha_1} E^T \xrightarrow{\varepsilon_T} U^{\Psi \Phi T} \xrightarrow{\varepsilon_T} E.$$  

The map $\alpha_1 : E^T \rightarrow E^{\Psi \Phi T}$ is the unique map to the pullback, and $\alpha_0 = \varepsilon_T$ the corresponding morphism of monads. It follows that $\varepsilon_T$ is invertible if and only the square to the left of (4.5) is a pullback. Both vertical legs are discrete isofibrations and $N_K$ is fully faithful, so by Lemma 12 this happens just when, firstly, $\alpha_2$ is fully faithful, and, secondly, $X \in [A_{op}, V]$ is in the essential image of $\alpha_2$ if and only if $XJ_T$ is a $K$-nerve. But as $\alpha_2 \cong N_{KT}$, and natural isomorphism does not change either full fidelity or essential images, this happens just when $T$ is $A$-nervous.  

4.4. The monad–theory equivalence. Putting together the preceding results now yields the main result of this paper.

**Theorem 17.** The adjunction (2.1) restricts to an adjoint equivalence

$$Mnd_A(E) \xrightarrow{\Psi} Th_A(E) \xrightarrow{\Phi} Th_A(E)$$

between the categories of $A$-nervous monads and $A$-theories.

**Proof.** Any adjunction restricts to an adjoint equivalence between the objects with invertible unit and counit components respectively, and Theorems 14 and 16 identify these objects as the $A$-theories and the $A$-nervous monads.

Note that there is an asymmetry between the conditions found on each side: for while the condition characterising the $A$-theories among $A$-pretheories is
intrinsic, and easy to check in practice, the condition defining an $\mathcal{A}$-nervous monad refers to the associated pretheory, and is non-trivial to check in practice; indeed, one of the main points of [33, 8] is to provide a general set of sufficient conditions under which a monad can be shown to be $\mathcal{A}$-nervous.

In the sections which follow, we will provide a number of more tractable characterisations of the $\mathcal{A}$-theories and $\mathcal{A}$-nervous monads; the crucial fact which drives all of these is that the adjunction (2.1) is in fact idempotent. Recall that an adjunction $L \dashv R : D \to C$ is idempotent if the monad $RL$ on $C$ is idempotent, and that this is equivalent to asking that the comonad $LR$ is idempotent, or that any one of the natural transformations $R\varepsilon$, $\varepsilon L$, $\eta R$ and $L\eta$ is invertible.

**Theorem 18.** The adjunction (2.1) is idempotent.

*Proof.* We show for each $T \in \text{Mnd}(\mathcal{E})$ that the unit $\eta_{\Phi T} : \Phi T \to \Phi \Psi \Phi T$ is invertible. By Theorem 14, this is equally to show that $J_T : A \to \mathcal{A}_T$ is an $\mathcal{A}$-theory, i.e., that each $\mathcal{A}_T(J_T-, J_T a) \in [\mathcal{A}^{\text{op}}, \mathcal{V}]$ is a $K$-nerve. But $\mathcal{A}_T(J_T-, J_T a) \cong \mathcal{E}_T(F^T K-, F^T K a) \cong \mathcal{E}(K-, U^T F^T K a) = \mathcal{E}(K-, T K a)$ as required. □

Exploiting the alternative characterisations of idempotent adjunctions listed above, we immediately obtain the following result, which tells us in particular that a monad $T$ is $\mathcal{A}$-nervous if and only if it can be presented by some $\mathcal{A}$-pretheory.

**Corollary 19.** A monad $T$ on $\mathcal{E}$ is $\mathcal{A}$-nervous if and only if $T \cong \Psi T$ for some $\mathcal{A}$-pretheory $J : A \to T$; while an $\mathcal{A}$-pretheory $J : A \to T$ is an $\mathcal{A}$-theory if and only if $T \cong \Phi T$ for some monad $T$ on $\mathcal{E}$.

The following is also direct from the definition of idempotent adjunction.

**Corollary 20.** The full subcategory $\text{Mnd}_A(\mathcal{E}) \subseteq \text{Mnd}(\mathcal{E})$ is coreflective via $\Psi \Phi$, while the full subcategory $\text{Th}_A(\mathcal{E}) \subseteq \text{Preth}_A(\mathcal{E})$ is reflective via $\Phi \Psi$.

5. **Semantics**

In the next section, we will explicitly identify the $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories for many of the examples listed in Section 2.1, but before doing this, we study further aspects of their general theory related to the taking of semantics.

5.1. **Interaction with the semantics functors.** We begin by examining the interaction of our monad–pretheory adjunction with the semantics functors of Section 2. In fact, we begin at the level of the monad–pretheory adjunction (2.1).

**Proposition 21.** There is a natural isomorphism $\theta$ as on the left in:

\[
\begin{array}{ccc}
\text{Preth}_A(\mathcal{E})^{\text{op}} & \xrightarrow{\Psi^{\text{op}}} & \text{Mnd}(\mathcal{E})^{\text{op}} \\
\text{Mod}_c \downarrow & \theta^{\text{op}} \downarrow & \text{Alg} \\
\mathcal{V}^{\text{CAT}/\mathcal{E}} & \xrightarrow{\Phi^{\text{op}}} & \text{Preth}_A(\mathcal{E})^{\text{op}} \\
\text{Mod}_c \downarrow & \Phi^{\text{op}} \downarrow & \text{Alg} \\
\mathcal{V}^{\text{CAT}/\mathcal{E}} & .
\end{array}
\]

Its mate $\bar{\theta}$ under the adjunction $\Phi^{\text{op}} \dashv \Psi^{\text{op}}$, as right above, has component at $T \in \text{Mnd}(\mathcal{E})$ invertible if and only if $T$ is $\mathcal{A}$-nervous.
Proof. For the first claim, Theorem 5 provides the necessary natural isomorphisms \( \theta_T : \mathcal{E}^\Psi \to \text{Mod}_\omega(\mathcal{T}) \) over \( \mathcal{E} \). For the second, if we write as before \( \varepsilon_T : \Psi \Phi T \to T \) for the counit component of (2.1) at \( T \in \text{Mnd}(\mathcal{E}) \), then the \( T \)-component of \( \bar{\theta} \) is the composite \( \theta_{T \Psi} \circ (\varepsilon_T)^* : \mathcal{E}^T \to \mathcal{E}^{\Psi \Phi T} \to \text{Mod}_\omega(\Phi \mathcal{T}) \) over \( \mathcal{E} \). Since \( \theta_{T \Psi} \) is invertible and since Alg is fully faithful, \( \bar{\theta}_T \) will be invertible just when \( \varepsilon_T \) is so; that is, by Theorem 16, just when \( T \) is \( \mathcal{A} \)-nervous. \( \square \)

From this and the fact that each monad \( \Psi \mathcal{T} \) is \( \mathcal{A} \)-nervous, it follows that an \( \mathcal{A} \)-pretheory \( \mathcal{T} \) and its associated theory \( \Phi \Psi \mathcal{T} \) have isomorphic categories of concrete models. By contrast, the passage from a monad \( T \) to its \( \mathcal{F} \)-nervous coreflection \( \Psi \Phi T \) may well change the category of algebras. For example, the power-set monad on \( \text{Set} \), whose algebras are complete lattices, has its \( \mathcal{F} \)-nervous coreflection given by the finite-power-set monad, whose algebras are \( \vee \)-semilattices.

On the other hand, if we restrict to the case of \( \mathcal{A} \)-nervous monads and \( \mathcal{A} \)-theories, then the subtle distinctions just noted disappear:

**Corollary 22.** The monad–theory equivalence (4.6) commutes with the semantics functors; that is, we have natural isomorphisms:

\[
\begin{align*}
\text{Th}_\mathcal{A}(\mathcal{E})^{\text{op}} & \xrightarrow{\Phi^{\text{op}}} \text{Mnd}_\mathcal{A}(\mathcal{E})^{\text{op}} & \text{Mnd}_\mathcal{A}(\mathcal{E})^{\text{op}} & \xrightarrow{\Phi^{\text{op}}} \text{Th}_\mathcal{A}(\mathcal{E})^{\text{op}} \\
\text{Mod}_\omega & \xrightarrow{\bar{\theta}} \text{Alg} & \text{Alg} & \xrightarrow{\bar{\theta}} \text{Mod}_\omega
\end{align*}
\]

Moreover, both of these semantics functors are fully faithful. 

Proof. The first statement follows from Proposition 21. For the second, note that \( \text{Alg} : \text{Mnd}_\mathcal{A}(\mathcal{E})^{\text{op}} \to \mathcal{V} \text{-CAT} / \mathcal{E} \) is obtained by restricting the fully faithful \( \text{Alg} : \text{Mnd}(\mathcal{E})^{\text{op}} \to \mathcal{V} \text{-CAT} / \mathcal{E} \) along a full embedding, and so is itself fully faithful. It follows that \( \text{Mod}_\omega \cong \text{Alg} \circ \Phi^{\text{op}} : \text{Th}_\mathcal{A}(\mathcal{E})^{\text{op}} \to \mathcal{V} \text{-CAT} / \mathcal{E} \) is also fully faithful. \( \square \)

In particular, full fidelity of \( \text{Mod}_\omega : \text{Th}_\mathcal{A}(\mathcal{E})^{\text{op}} \to \mathcal{V} \text{-CAT} / \mathcal{E} \) means that an \( \mathcal{A} \)-theory is determined to within isomorphism by its category of concrete models over \( \mathcal{E} \); this rectifies the non-uniqueness of pretheories noted in Example 8 above.

### 5.2. Non-concrete models.

In Section 2.3 we defined a concrete model of an \( \mathcal{A} \)-pretheory \( \mathcal{T} \) to be an object \( X \in \mathcal{E} \) endowed with an extension of \( \mathcal{E}(K-, X) : \mathcal{A}^{\text{op}} \to \mathcal{V} \) to a functor \( \mathcal{T}^{\text{op}} \to \mathcal{V} \). In the literature, one often encounters a looser notion of model for a theory which does not have an underlying object in \( \mathcal{E} \). Such a notion is available also in our setting: by an (unqualified) \( \mathcal{T} \)-model, we mean a functor \( F : \mathcal{T}^{\text{op}} \to \mathcal{V} \) whose restriction \( F : \mathcal{A}^{\text{op}} \to \mathcal{V} \) is a \( K \)-nerve.

The \( \mathcal{T} \)-models span a full sub-\( \mathcal{V} \)-category \( \text{Mod}(\mathcal{T}) \) of \([ \mathcal{T}^{\text{op}}, \mathcal{V} ]\). Recalling from Section 2.1 that \( K \text{-Ner}(\mathcal{V}) \) denotes the full sub-\( \mathcal{V} \)-category of \([ \mathcal{A}^{\text{op}}, \mathcal{V} ]\) on the
$K$-nerves, we may also express $\mathrm{Mod}(\mathcal{T})$ as a pullback as to the right in:

\begin{align*}
\begin{array}{c}
\mathrm{Mod}_c(\mathcal{T}) \\
\downarrow U_T \\
\mathcal{E}
\end{array} & \quad \begin{array}{c}
\rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \\
\downarrow W_T \\
\rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \\
\mathrm{Mod}(\mathcal{T}) & \quad [\mathcal{T}^{\text{op}}, \mathcal{V}] \\
\downarrow N_K & \quad \downarrow [\mathcal{T}^{\text{op}}, 1] \\
\mathrm{K-}\text{Ner}(\mathcal{V}) & \quad \rightarrow [\mathcal{A}^{\text{op}}, \mathcal{V}].
\end{array}
\end{align*}

(5.2)

On the other hand, $\mathrm{Mod}_c(\mathcal{T})$ is the pullback around the outside, and so there is a canonical induced map $\mathrm{Mod}_c(\mathcal{T}) \to \mathrm{Mod}(\mathcal{T})$ as displayed. By the usual cancellativity properties, the left square above is now also a pullback. Moreover, $W_T$ is an isofibration, as a pullback of the discrete isofibration $[\mathcal{J}^{\text{op}}, 1]$, and $N_K: \mathcal{E} \to \text{K-}\text{Ner}(\mathcal{V})$ is an equivalence. Since equivalences are stable under pullback along isofibrations, we conclude that:

**Proposition 23.** The comparison $\mathrm{Mod}_c(\mathcal{T}) \to \mathrm{Mod}(\mathcal{T})$ in (5.2) is an equivalence.

Taking non-concrete models gives rise to a semantics functor landing in $\mathcal{V}\text{-CAT}/\text{K-}\text{Ner}(\mathcal{V})$ which, like before, is not fully faithful on $\mathcal{A}$-pretheories, but is so on the subcategory of $\mathcal{A}$-theories. Note that the “underlying $K$-nerve” of a $\mathcal{T}$-model is more natural than it might seem, being the special case of the functor $\mathrm{Mod}(\mathcal{T}) \to \mathrm{Mod}(\mathcal{S})$ induced by a morphism of $\mathcal{A}$-pretheories for which $\mathcal{S}$ is the initial pretheory. However, in the following result, for simplicity, we view the semantics functors for $\mathcal{T}$-models as landing simply in $\mathcal{V}\text{-CAT}$.

**Theorem 24.** The monad–theory equivalence (4.6) commutes with the non-concrete semantics functors in the sense that we have natural transformations

\[
\begin{array}{ccc}
\text{Th}_{\mathcal{A}}(\mathcal{E})^{\text{op}} & \xrightarrow{\Psi^{\text{op}}} & \text{Mnd}_{\mathcal{A}}(\mathcal{E})^{\text{op}} \\
\downarrow \theta & & \downarrow \Phi^{\text{op}} \\
\mathcal{V}\text{-CAT} & \xrightarrow{\sim} & \text{Alg} \\
\text{Mod} & & \text{Alg} \\
\downarrow & & \downarrow \\
\mathcal{V}\text{-CAT} & \xrightarrow{\sim} & \text{Mod}
\end{array}
\]

whose components are equivalences.

**Proof.** Postcompose the natural isomorphisms (5.1) with the forgetful functor $\mathcal{V}\text{-CAT}/\mathcal{E} \to \mathcal{V}\text{-CAT}$, and paste with the natural transformation $\text{Mod}_c \Rightarrow \text{Mod}: \text{Th}_{\mathcal{A}}(\mathcal{E})^{\text{op}} \to \mathcal{V}\text{-CAT}$ coming from the previous proposition. \hfill $\Box$

### 5.3. Local presentability and algebraic left adjoints.

Next in this section, we consider the categorical properties of the $\mathcal{V}$-categories and $\mathcal{V}$-functors in the image of the semantics functors. For pretheories, we have:

**Proposition 25.** (i) If $J: \mathcal{A} \to \mathcal{T}$ is an $\mathcal{A}$-pretheory then $\mathrm{Mod}_c(\mathcal{T})$ is locally presentable and $U_T: \mathrm{Mod}_c(\mathcal{T}) \to \mathcal{E}$ is a strictly monadic right adjoint.

(ii) If $H: \mathcal{T} \to \mathcal{S}$ is a map of $\mathcal{A}$-pretheories, then $H^*: \mathrm{Mod}_c(\mathcal{S}) \to \mathrm{Mod}_c(\mathcal{T})$ is a strictly monadic right adjoint.

**Proof.** (i) follows from Lemma 4 and the description in (2.5) of $\mathrm{Mod}_c(\mathcal{T}) \to \mathcal{E}$ as a pullback. For (ii), applying the cancellativity properties of pullbacks to those
defining $\text{Mod}_c(S)$ and $\text{Mod}_c(T)$ yields a pullback square

\[
\begin{array}{c}
\text{Mod}_c(S) \\
\downarrow H^* \\
\text{Mod}_c(T)
\end{array} \xrightarrow{P_S} \begin{array}{c}
[S^{op}, V] \\
\downarrow [H^{op}, 1] \\
[T^{op}, V]
\end{array}
\]

Since $[H^{op}, 1]$ is strictly monadic and $P_T$ is a right adjoint between locally presentable categories, the result follows again from Lemma 4.

Composing with the equivalences $\text{Mod}(T) \simeq \text{Mod}_c(T)$ of Proposition 23, this immediately implies the local presentability of the categories $\text{Mod}(T)$ of non-concrete models, and their monadicity over $E$. On the other hand, taken together with Proposition 21, it immediately implies the corresponding result for nervous monads, which we state here as:

**Proposition 26.**

(i) If $T$ is an $A$-nervous monad then $E^T$ is locally presentable, and $U^T : E^T \to E$ is a strictly monadic right adjoint.

(ii) If $\alpha : T \to S$ is a map of $A$-nervous monads, then $\alpha^* : E^S \to E^T$ is a strictly monadic right adjoint.

**5.4. Algebraic colimits of monads and theories.** To conclude this section, we examine the interaction of the semantics functors with colimits of monads or (pre)theories, beginning with the more-or-less classical case of the category of monads $\text{Mnd}(E)$.

In general, $\text{Mnd}(E)$ need not be cocomplete. Indeed, when $V = E = \text{Set}$, it does not even have all binary coproducts; see [4, Proposition 6.10]. However many colimits of monads do exist, and an important point about these is that, in the terminology of [15], they are algebraic. That is, they are sent to limits by the semantics functor $\text{Alg} : \text{Mnd}(E)^{op} \to V\text{-CAT}/E$.

To prove this, we use the following lemma, which is a mild variant of the standard result that right adjoints preserve limits.

**Lemma 27.** Let $C$ be a complete (ordinary) category with a strongly generating class of objects $X$ and consider a functor $U : A \to C$. If each $x \in X$ admits a reflection along $U$ then $U$ preserves any limits that exist in $A$.

**Proof.** As $X$ is a strong generator, the functors $C(x, -)$ with $x \in X$ jointly reflect isomorphisms, and so jointly reflect limits. Accordingly $U$ preserves any limits that are preserved by $C(x, U-)$ for each $x \in X$. But each $C(x, U-)$ is representable and so preserves all limits; whence $U$ preserves any limits that exist.

In the setting of $\text{Set}$-enriched categories the following result, expressing the algebraicity of colimits of monads, is a special case of Proposition 26.3 of [15].

**Proposition 28.** $\text{Alg} : \text{Mnd}(E)^{op} \to V\text{-CAT}/E$ preserves limits.

**Proof.** The $V$-functors $F : X \to E$ with small domain form a strong generator for $V\text{-CAT}/E$. Moreover, it is shown in [11, Theorem II.1.1] that each such $F$ has a reflection along $\text{Alg} : \text{Mnd}(E)^{op} \to V\text{-CAT}/E$ given by its codensity monad $\text{Ran}_F(F) : E \to E$. The result thus follows from Lemma 27.

□
We now adapt the above results concerning \( \text{Mnd}(\mathcal{E}) \) to the cases of \( \text{Preth}_A(\mathcal{E}) \), \( \text{Mnd}_A(\mathcal{E}) \) and \( \text{Th}_A(\mathcal{E}) \). In Theorem 36 below, we will see that these categories are locally presentable; in particular, and by contrast with \( \text{Mnd}(\mathcal{E}) \), they are cocomplete. It is also not difficult to prove the cocompleteness directly.

**Proposition 29.** Each of the semantics functors \( \text{Alg}: \text{Mnd}_A(\mathcal{E})^{\text{op}} \to \mathcal{V}-\text{CAT}/\mathcal{E} \), \( \text{Mod}_c: \text{Preth}_A(\mathcal{E})^{\text{op}} \to \mathcal{V}-\text{CAT}/\mathcal{E} \) and \( \text{Mod}_c: \text{Th}_A(\mathcal{E})^{\text{op}} \to \mathcal{V}-\text{CAT}/\mathcal{E} \) preserves limits.

**Proof.** These three functors are isomorphic to the respective composites:

\[
\begin{align*}
\text{Mnd}_A(\mathcal{E})^{\text{op}} & \xrightarrow{\text{incl}^{\text{op}}} \text{Mnd}(\mathcal{E})^{\text{op}} \xrightarrow{\text{Alg}} \mathcal{V}-\text{CAT}/\mathcal{E} \\
\text{Preth}_A(\mathcal{E})^{\text{op}} & \xrightarrow{\Psi^{\text{op}}} \text{Mnd}(\mathcal{E})^{\text{op}} \xrightarrow{\text{Alg}} \mathcal{V}-\text{CAT}/\mathcal{E} \\
\text{Th}_A(\mathcal{E})^{\text{op}} & \xrightarrow{\Psi^{\text{op}}} \text{Mnd}(\mathcal{E})^{\text{op}} \xrightarrow{\text{Alg}} \mathcal{V}-\text{CAT}/\mathcal{E};
\end{align*}
\]

for (5.3) this is clear, while for (5.4) and (5.5) it follows from Proposition 21. The common second functor in each composite is limit-preserving by Proposition 28, while the first functor is limit-preserving in each case since it is the opposite of a left adjoint functor—by Corollary 20, Theorem 5 and Theorem 17 (taken together with Corollary 20) respectively. \( \square \)

We leave it to the reader to formulate this result also for non-concrete models.

### 6. The monad–theory correspondence in practice

In this section, we give explicit descriptions of the \( A \)-theories and their models, and of the \( A \)-nervous monads for the examples described in Section 2.1 above. In particular, we will see how our results allow us to re-find many of the monad–theory correspondences existing in the literature. We will obtain these explicit descriptions using further characterisation results for \( A \)-theories and \( A \)-nervous monads in particular situations, and so we begin this section by describing these.

#### 6.1. Theories in the presheaf context

A number of the examples of our basic setting described in Section 3.1 arise in the following manner. We take \( \mathcal{E} = [\mathcal{C}^{\text{op}}, \mathcal{V}] \) a presheaf category, and take \( A \) to be any full subcategory of \( \mathcal{E} \) containing the representables. In this situation, we then have a factorisation

\[
\mathcal{C} \xrightarrow{L} A \xrightarrow{K} [\mathcal{C}^{\text{op}}, \mathcal{V}] = \mathcal{E}
\]

of the Yoneda embedding. The Yoneda lemma implies that \( Y: \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathcal{V}] \) is dense, whence, by Theorem 5.13 of [16], so too are both \( I \) and \( K \). In particular, \( K \) provides an instance of our basic setting; we will call this the *presheaf context*. Each of Examples 6(i), (iv), (v), (vi), and (viii) arise in this way.

**Lemma 30.** In the presheaf context, we have \( N_I \cong K \) and \( N_K \cong \text{Ran} I^{\text{op}} \). Moreover, a functor \( F: A^{\text{op}} \to \mathcal{V} \) is a \( K \)-nerve just when it is the right Kan extension of its restriction along \( I^{\text{op}}: \mathcal{C}^{\text{op}} \to A^{\text{op}} \).

**Proof.** For the first isomorphism we calculate that

\[
N_I(x) = A(I- \cdot, x) \cong [\mathcal{C}^{\text{op}}, \mathcal{V}](KI- \cdot, Kx) = [\mathcal{C}^{\text{op}}, \mathcal{V}](Y- \cdot, Kx) \cong Kx
\]
by full fidelity of \( K \) and the Yoneda lemma. For the second, since \( \text{Lan}_Y K \dashv N_K \) and \([I^{op}, 1] \dashv \text{Ran}_{I^{op}}\) it suffices to show \( \text{Lan}_Y K \cong [I^{op}, 1]: [A^{op}, Y] \to [C^{op}, Y]\). Since both are cocontinuous, it suffices to show \((\text{Lan}_Y K) Y \cong [I^{op}, 1] Y\); but \((\text{Lan}_Y K) Y \cong K \cong N_I = [I^{op}, 1] Y\) using full fidelity of \( Y \) and (6.2). Finally, since \( I^{op} \) is fully faithful, \( F: A^{op} \to Y \) is a right Kan extension along \( I^{op} \) just when it is the right Kan extension of its own restriction. Thus the final claim follows using the isomorphism \( N_K \cong \text{Ran}_{I^{op}}\).

In this setting, we have practically useful characterisations of the \( A \)-theories and their (non-concrete) models.

**Proposition 31.** Let \( J: A \to T \) be an \( A \)-pretheory in the presheaf context (6.1).

(i) A functor \( F: T^{op} \to V \) is a \( T \)-model just when \( F.J^{op}: A^{op} \to V \) is the right Kan extension of its restriction along \( I^{op}: C^{op} \to A^{op}\).

(ii) \( J: A \to T \) is itself an \( A \)-theory just when it is the pointwise left Kan extension of its restriction along \( I: C \to A \).

**Proof.** (i) follows immediately from Lemma 30 since, by definition, \( F \) is a \( T \)-model just when \( F.J^{op} \) is a \( K \)-nerve. For (ii), note that by Proposition 4.46 of [16], \( J: A \to T \) is the pointwise left Kan extension of its restriction along \( I \) just when, for each \( x \in T \), the functor \( T(J-, x): A^{op} \to V \) is the right Kan extension of its restriction along \( I^{op} \). By Lemma 30, this happens just when each \( T(J-, x) \) is a \( K \)-nerve—that is, just when \( J \) is an \( A \)-theory.

We can sharpen these results using Day’s notion of density presentation [10]. Above, we defined a dense functor \( K: C \to D \) as one for which \( N_K: D \to [C^{op}, V] \) is fully faithful. In the unenriched case, density is often instead introduced as the assertion that each object of \( D \) is the colimit of a certain diagram in the image of \( K \); it is this perspective that the notion of density presentation generalises.

A family of colimits \( \Phi \) in the ordinary category \( D \) is a class of diagrams \((D_i: J_i \to D)_{i \in I}\) which each admits a colimit in \( D \). In the enriched case, a family of colimits \( \Phi \) in the \( V \)-category \( D \) is a class of pairs \((W_i: J_i^{op} \to V, D_i: J_i \to D)_{i \in I}\) such that each weighted colimit \( W_i \star D_i \) exists in \( D \). In either case, a full replete subcategory \( B \) of \( D \) is said to be closed in \( D \) under \( \Phi \)-colimits if colim \( D_i \in B \) whenever \( D_i \) takes values in \( B \). Finally, we say that \( D \) is the closure of \( B \) under \( \Phi \)-colimits if the only replete full subcategory of \( D \) containing \( B \) and closed under \( \Phi \)-colimits is \( D \) itself.

Now suppose we are given a fully faithful \( K: C \to D \). We say that a colimit in \( D \) is \( K \)-absolute if it is preserved by \( N_K\); equivalently, by each representable \( D(Kx, -): D \to V \). Now if \( D \) is the closure of \( C \) under a family \( \Phi \) of \( K \)-absolute colimits then \( \Phi \) is said to be a density presentation for \( K \). The nomenclature is justified by Theorem 5.19 of [16], which, among other things, says that the fully faithful \( K \) has a density presentation just when it is dense.

We will make of density presentations in the presheaf context (6.1) with respect not to the dense \( K \), but to the dense \( I \). By Lemma 30 we have \( N_I \cong K \), and so the \( I \)-absolute colimits are in this case those preserved by \( K: A \to \mathcal{E} \).

**Examples 32.**
(i) Examples 6(i) corresponds to the presheaf context
\[
1 \xrightarrow{I} \mathbb{F} \xrightarrow{K} \text{Set},
\]
wherein \(I\) has a density presentation given by all finite copowers of 1 \(\in \mathbb{F}\); these are \(I\)-absolute since \(K\) preserves them. In fact, \(\mathbb{F}\) has, and \(K\) preserves, all finite coproducts, whence there is a larger density presentations given by all finite coproducts in \(\mathbb{F}\).

(ii) Examples 6(iv) yields the presheaf context below, wherein \(I\) has a density presentation given by the wide pushouts \([n] \sim \left[1\right] + \left[0\right] \left[1\right] + \left[0\right] \ldots + \left[0\right] \left[1\right]\):
\[
\begin{array}{c}
G_1 \\
\hspace{1cm} \xrightarrow{I} \\
\hspace{1cm} \Delta_0 \\
\hspace{1cm} K \\
\hspace{1cm} [G_1^{\text{op}}, \text{Set}]
\end{array}
\]

We will see further examples of density presentations in the presheaf context in Section 6.3 below. The reason we care about density presentations is the following result, which comprises various parts of Theorem 5.29 of [16].

**Proposition 33.** Let \(K : \mathcal{C} \to \mathcal{D}\) be fully faithful and dense. The following are equivalent:

(i) \(F : \mathcal{D} \to \mathcal{E}\) is the pointwise left Kan extension of its restriction along \(K\);
(ii) \(F\) sends \(\Phi\)-colimits to colimits for any density presentation \(\Phi\) of \(K\);
(iii) \(F\) sends \(\mathcal{K}\)-absolute colimits to colimits.

Combined with Proposition 31, this yields the desired sharper characterisation of the \(\mathcal{A}\)-theories and their models.

**Theorem 34.** Let \(J : \mathcal{A} \to \mathcal{T}\) be an \(\mathcal{A}\)-pretheory in the presheaf context (6.1), and let \(\Phi\) be a density presentation for \(I\).

(i) A functor \(F : \mathcal{T}^{\text{op}} \to \mathcal{V}\) is a \(\mathcal{T}\)-model just when \(FJ^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{V}\) sends \(\Phi\)-colimits in \(\mathcal{A}\) to limits in \(\mathcal{V}\);
(ii) \(J : \mathcal{A} \to \mathcal{T}\) is itself an \(\mathcal{A}\)-theory just when it sends \(\Phi\)-colimits to colimits.

6.2. Nervous monads, signatures and saturated classes. We now turn from characterisations for \(\mathcal{A}\)-theories to characterisations for \(\mathcal{A}\)-nervous monads. We know from Corollary 19 that a monad is \(\mathcal{A}\)-nervous just when it is isomorphic to \(\Psi\mathcal{T}\) for some \(\mathcal{A}\)-pretheory \(J : \mathcal{A} \to \mathcal{T}\), and from the examples in Section 3, it is intuitively reasonable to think that these are the monads which can be “presented by operations and equations with arities from \(\mathcal{A}\)”. Our first characterisation result makes this intuitive idea precise by exhibiting the category of \(\mathcal{A}\)-nervous monads as monadic over the following category of signatures.

**Definition 35.** The category \(\text{Sig}_A(\mathcal{E})\) of signatures is the category \(\mathcal{V}\text{-CAT}(\text{ob}\mathcal{A}, \mathcal{E})\). We write \(V : \text{Mnd}(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})\) for the functor sending \(\mathcal{T}\) to \((T_a)_{a \in \mathcal{A}}\).

The proof of the following characterisation result is deferred to Section 8.

**Theorem 36.** \(V : \text{Mnd}(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})\) has a left adjoint \(F : \text{Sig}_A(\mathcal{E}) \to \text{Mnd}(\mathcal{E})\) taking values in \(\mathcal{A}\)-nervous monads. Moreover:

(i) The restricted functor \(V : \text{Mnd}_A(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})\) is monadic;
(ii) A monad \(T \in \text{Mnd}(\mathcal{E})\) is \(\mathcal{A}\)-nervous if and only if it is a colimit in \(\text{Mnd}(\mathcal{E})\) of monads in the image of \(F\).
(iii) Each of \( \text{Mnd}_A(\mathcal{E}) \), \( \text{Preth}_A(\mathcal{E}) \) and \( \text{Th}_A(\mathcal{E}) \) is locally presentable.

The idea behind this result originates in [19]. A signature \( \Sigma \in \text{Sig}_A(\mathcal{E}) \) specifies for each \( a \in A \) an \( \mathcal{E} \)-object \( \Sigma a \) of “operations of input arity \( a \)”. The free monad \( F\Sigma \) on this signature has as its algebras the \( \Sigma \)-structures: objects \( X \in \mathcal{E} \) endowed with a function \( \mathcal{E}(a, X) \to \mathcal{E}(\Sigma a, X) \) for each \( a \in A \). The above result implies that a monad \( T \in \text{Mnd}(\mathcal{E}) \) is \( A \)-nervous just when it admits a presentation as a coequaliser \( FT \rightrightarrows F\Sigma \to T \)—that is, a presentation by a signature \( \Sigma \) of basic operations together with a family \( \Gamma \) of equations between derived operations.

We now turn to our second characterisation result for \( A \)-nervous monads. This is motivated by the fact, noted in the introduction, that in many monad–theory correspondences the class of monads can be characterised by a colimit-preservation property. To reproduce this result in our setting, we require a closure property of the arities in the subcategory \( A \) which, roughly speaking, says that substituting \( A \)-ary operations into \( A \)-ary operations again yields \( A \)-ary operations.

**Definition 37.** An endo-\( V \)-functor \( F: \mathcal{E} \to \mathcal{E} \) is called \( A \)-induced if it is the pointwise left Kan extension of its restriction along \( K \). We call \( A \) a saturated class of arities if \( A \)-induced endofunctors of \( \mathcal{E} \) are closed under composition.

**Example 38.** In the case of \( K: \mathbb{F} \hookrightarrow \mathbb{Set} \), there is a density presentation for \( K \) given by all filtered colimits in \( \mathbb{Set} \), so that by Proposition 33, an endofunctor \( \mathbb{Set} \to \mathbb{Set} \) is \( \mathbb{F} \)-induced just when it preserves filtered colimits. Thus \( \mathbb{F} \hookrightarrow \mathbb{Set} \) is a saturated class of arities.

**Example 39.** More generally, if \( \Phi \) is an class of enriched limit-types and \( K: \mathcal{A} \to \mathcal{E} \) exhibits \( \mathcal{E} \) as the free cocompletion of \( \mathcal{A} \) under \( \Phi \)-colimits, then there is a density presentation of \( K \) given by all \( \Phi \)-colimits, and an endofunctor of \( \mathcal{E} \) is \( K \)-induced just when it preserves \( \Phi \)-colimits. Thus \( \mathcal{A} \) is a saturated class of arities.

**Example 40.** Let \( K: \mathcal{A} \hookrightarrow \mathbb{Set} \) be the inclusion of the one-object full subcategory \( \mathcal{A} \) on the two-element set \( 2 = \{0, 1\} \). Since the dense generator 1 of \( \mathbb{Set} \) is a retract of 2, and taking retracts does not change categories of presheaves, \( \mathcal{A} \) is dense in \( \mathbb{Set} \). We claim it does not give a saturated class of arities.

To see this, note first that \((-)^2: \mathbb{Set} \to \mathbb{Set} \) is \( \mathcal{A} \)-induced, being a left Kan extension along \( K \) of the representable \( \mathcal{A}(2, -): \mathcal{A} \to \mathbb{Set} \). We claim that \((-)^2 \circ (-)^2 \) is not \( \mathcal{A} \)-induced. For indeed, by the Yoneda lemma, any \( X \in [\mathcal{A}, \mathbb{Set}] \) has an epimorphic cover by copies of the unique representable \( \mathcal{A}(2, -) \). Since left Kan extension preserves epimorphisms, each \( \text{Lan}_K(X) \) admits an epimorphic cover by copies of \((-)^2 \). But \((-)^2 \circ (-)^2 \cong (-)^4 \) can admit no such cover, since the identity map on 4 does not factor through 2, and so cannot be \( \mathcal{A} \)-induced.

The proof of the following result will again be deferred to Section 8 below.

**Theorem 41.** Let \( \mathcal{A} \) be a saturated class of arities in \( \mathcal{E} \). The following are equivalent properties of a monad \( T \in \text{Mnd}(\mathcal{E}) \):

(i) \( T \) is \( \mathcal{A} \)-nervous;
(ii) \( T: \mathcal{E} \to \mathcal{E} \) is \( \mathcal{A} \)-induced;
(iii) \( T: \mathcal{E} \to \mathcal{E} \) preserves \( \Phi \)-colimits for any density presentation \( \Phi \) of \( K \).
6.3. The monad–theory equivalence in practice. With our characterisation results in place, we now apply them to the examples of Section 2.1 to obtain explicit descriptions of the $\mathcal{A}$-theories and their models and the $\mathcal{A}$-nervous monads. In many cases, we will see on doing so that we have reconstructed a familiar monad–theory correspondence from the literature.

Examples 42. As before, we begin with the unenriched examples where $\mathcal{V} = \text{Set}$.

(i) The case $\mathcal{E} = \text{Set}$ and $\mathcal{A} = F$ corresponds to the instance of the presheaf context described in Examples 32(i). With the density presentations for $I$ given there, we see by Theorem 34 that an $F$-pretheory $J : F \to T$ is an $F$-theory when it preserves finite coproducts of $1$, or equally (using the larger density presentation) all finite coproducts. So the $F$-theories are the Lawvere theories of [23]. Moreover a functor $F : T^{\text{op}} \to \text{Set}$ is a $T$-model if and only if $FJ^{\text{op}} : F^{\text{op}} \to \text{Set}$ preserves finite products; since in this case, $J$ also reflects finite coproducts, this happens just when $F : T^{\text{op}} \to \text{Set}$ is itself finite-product-preserving: that is, a model of the Lawvere theory $T$.

On the other hand, by Example 38, $F$ is a saturated class of arities, and the $F$-induced endofunctors are the finitary ones, so that by Theorem 41 a monad on $\text{Set}$ is $F$-nervous just when it is finitary. Theorem 17 thus specialises to the classical finitary monad–Lawvere theory correspondence, while Theorem 24 recaptures its compatibility with semantics.

(ii) When $\mathcal{E}$ is locally finitely presentable and $\mathcal{A} = \mathcal{E}_f$ the category of $K$-nerves comprises by [12, Kollar 7.9] precisely the finite-limit-preserving functors $\mathcal{E}_f^{\text{op}} \to \text{Set}$. So an $\mathcal{E}_f$-pretheory $J : \mathcal{E}_f \to T$ is an $\mathcal{E}_f$-theory just when each $T(J(\cdot, a)) : \mathcal{E}_f^{\text{op}} \to \text{Set}$ preserves finite limits; equally, by Yoneda, just when $J$ preserves finite colimits. So the $\mathcal{E}_f$-theories are [29]’s Lawvere $\mathcal{E}$-theories.

The concrete $T$-models in this setting are precisely the models of [29, Definition 2.2]. The general $T$-models are functors $F : T^{\text{op}} \to \text{Set}$ for which $FJ^{\text{op}} : \mathcal{E}_f^{\text{op}} \to \text{Set}$ is a $K$-nerve, i.e., finite-limit-preserving; these are the more general models of [21, Definition 12], and the correspondence between the two notions in Proposition 23 recaptures Proposition 15 of ibid.

On the monad side, since $K : \mathcal{E}_f \to \mathcal{E}$ exhibits $\mathcal{E}$ as the free filtered-colimit completion of $\mathcal{E}_f$, Example 39 and Theorem 41 imply that $\mathcal{E}_f$ is a saturated class and that the $\mathcal{E}_f$-nervous monads are the finitary ones. So Theorem 17 and Corollary 22 in this case reconstruct (the unenriched case of) the monad–theory correspondence and its compatibility with semantics in [29, Theorem 5.2].

(iii) More generally, when $\mathcal{E}$ is locally $\lambda$-presentable and $\mathcal{A} = \mathcal{E}_\lambda$ is a skeleton of the full subcategory of $\lambda$-presentable objects, the $\mathcal{E}_\lambda$-theories are pretheories which preserve $\lambda$-small colimits; the $T$-models are functors $F : T^{\text{op}} \to \text{Set}$ for which $FJ^{\text{op}}$ preserves $\lambda$-small limits; and the $\mathcal{E}_\lambda$-nervous monads are those whose underlying endofunctor preserves $\lambda$-filtered colimits.

(iv) When $\mathcal{E} = [[G_1^{\text{op}}, \text{Set}]]$ and $\mathcal{A} = \Delta_0$, we are in the presheaf context of Examples 32(ii). With the density presentation for $I$ given there we see by Theorem 34 that a pretheory $J : \Delta_0 \to T$ is a $\Delta_0$-theory when it preserves the wide pushouts $[n] \cong [1] + [0] \{1\} + [0] \ldots + [0] \{1\}$. Moreover, a functor
F: T^{op} \to \textbf{Set} is a \mathcal{T}-model just when it sends each of these wide pushouts to a limit in \textbf{Set}; equivalently, when each canonical map to the limit

\begin{equation}
X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1
\end{equation}

is invertible. This is precisely the \textit{Segal condition} of [31].

In Corollary 47 below we will see that \Delta_0 is \textit{not} a saturated class of arities, and so we have no more direct characterisations of the \Delta_0-nervous monads than those given by Corollary 19 or Theorem 36. However, Example 9 provides us with natural examples of \Delta_0-nervous monads: namely, the monads T and T_g for categories and for groupoids on \([G_1^{op}, \textbf{Set}]\). As was already noted in [33], the nervosity of T recaptures the classical nerve theorem relating categories and simplicial sets. Indeed, the \Delta_0-theory associated to T is the first part of the (bijective-on-objects, fully faithful) factorisation

\[
\Delta_0 \xrightarrow{J_T} \Delta \xrightarrow{K_T} \textbf{Cat}
\]

of the composite \(F \cdot K: \Delta_0 \to \textbf{Cat}\). The interposing object here is the topologist’s simplex category \(\Delta\), with \(K_T\) the standard inclusion into \(\textbf{Cat}\). Thus, to say that \(T\) is \(\Delta_0\)-nervous is to say that:

(a) The classical nerve functor \(N_{K_T}: \textbf{Cat} \to [\Delta^{op}, \textbf{Set}]\) is fully faithful;
(b) The essential image of \(N_{K_T}\) comprises those \(X \in [\Delta^{op}, \textbf{Set}]\) for which \(X_J^T\) is a \(K\)-nerve.

This much is already done in [33], but our use of density presentations allows for a small improvement. To say that \(X_J^T\) is a \(K\)-nerve in (b) is equally to say that \(X\) is a \(\mathcal{T}\)-model, or equally that \(X\) satisfies the Segal condition expressed by the invertibility of each (6.3). This is a mild sharpening of [33], where the “Segal condition” is left in the abstract form given in (b) above.

In a similar way, the nervosity of the monad \(T_g\) for small groupoids captures the “symmetric nerve theorem”. This states that the functor \(\textbf{Gpd} \to [\mathcal{F}^{op}, \textbf{Set}]\) sending a groupoid to its symmetric nerve—indexed by the category of non-empty finite sets—is fully faithful, and characterises the essential image once again as the functors satisfying the Segal condition (6.3).

(v) With \(E = [G^{op}, \textbf{Set}]\) and \(\mathcal{A} = \Theta_0\), we are now in the presheaf context

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{I} & \Theta_0 \xrightarrow{K} [G^{op}, \textbf{Set}].
\end{array}
\]

\(I\) has a density presentation given by the \(I\)-\textit{globular sums} \((n_1, \ldots, n_k) \cong (n_1) + (n_2) + (n_3) + \cdots + (n_{k-1}) (n_k)\) in \(\Theta_0\); whence by Theorem 34, a pretheory \(J: \Theta_0 \to \mathcal{T}\) is a \(\Theta_0\)-theory when it preserves these \(I\)-globular sums—that is, when it is a \textit{globular theory} in the sense of [7]\footnote{The definition of globular theory in [7] has the extra condition, satisfied in most cases, that \(J\) be a faithful functor.}. A functor \(F: T^{op} \to \textbf{Set}\) is a \(\mathcal{T}\)-model when it sends \(I\)-globular sums to limits, thus when each map

\[
X^n \longrightarrow X_{n_1} \times X_{n_2} X_{n_3} \times X_{n_4} \cdots \times X_{n_{k-1}} X_{n_k}
\]

is invertible. Once again, \(\Theta_0\) is \textit{not} a saturated class of arities, and so there is no direct characterisation of the \(\Theta_0\)-nervous monads; however,
their interaction with $\Theta_0$-theories is important in the literature on globular approaches to higher category theory—as we now outline.

Globular theories can describe structures on globular sets such as strict or weak $\omega$-categories and $\omega$-groupoids. For the strict variants, we pointed out in Section 3.2 that these may be modelled by $\Theta_0$-pretheories; and since reflecting a pretheory $T$ into a theory $\Phi \Psi T$ does not change the models, it is immediate that there are $\Theta_0$-theories modelling these structures too.

The original definition of globular weak $\omega$-category, due to Batanin [6], expresses them as globular sets equipped with algebra structure for certain globular operads—which can be understood as certain cartesian monads on globular sets. Berger [7] introduced globular theories and described the passage from a globular operad $T$ to a globular theory $\Theta_T$ just as in Section 2.4 above. His Theorem 1.17—in our language—asserts exactly that each globular operad $T$ is $\Theta_0$-nervous, so that algebras for the globular operad are the same as models of the associated theory $\Theta_T$. In particular, Batanin’s weak $\omega$-categories are the models of a globular theory.

We now proceed on to our examples over a more general base for enrichment $\mathcal{V}$.

(vi) With $\mathcal{V} = \mathcal{E}$ a locally finitely presentable symmetric monoidal category and with $\mathcal{A} = \mathcal{V}_f$, we are in the presheaf context

$$
\mathcal{I} \xrightarrow{I} \mathcal{V}_f \xrightarrow{K} \mathcal{V},
$$

wherein $I$ has a density presentation given by the class of all finite tensors—tensors by finitely presentable objects of $\mathcal{V}$. Thus by Theorem 34, the $\mathcal{V}_f$-theories are the pretheories $J: \mathcal{V}_f \to \mathcal{T}$ which preserve finite tensors, which are precisely the Lawvere $\mathcal{V}$-theories of [30, Definition 3.1]. Furthermore, like in (i), a functor $F: \mathcal{T}^{\text{op}} \to \mathcal{V}$ is a $\mathcal{T}$-model just when it preserves finite cotensors, just as in Definition 3.2 of ibid. On the other hand, $\mathcal{V}_f \to \mathcal{V}$ exhibits $\mathcal{V}$ as the free filtered-colimit completion of $\mathcal{V}_f$, whence by Example 39 it is a saturated class of arities, and by Theorem 41 the $\mathcal{V}_f$-nervous monads are again the finitary ones. So Theorems 17 and 24 specialise to Theorems 4.3, 3.4 and 4.2 of [30].

(vii) Now taking $\mathcal{E}$ to be any locally finitely presentable presentable $\mathcal{V}$-category and $\mathcal{A} = \mathcal{E}_f$, we may argue as in (ii) to recapture the fully general enriched monad–theory correspondence of [29], and its interaction with semantics.

(viii) Now suppose we are in the situation of Examples 6(viii), provided with a class $\Phi$ of enriched colimit-types satisfying Axiom A of [22]. With $\mathcal{E} = \mathcal{V}$ and $\mathcal{A} = \mathcal{V}_\Phi$, we are now in the presheaf context

$$
\mathcal{I} \xrightarrow{I} \mathcal{V}_\Phi \xrightarrow{K} \mathcal{V}.
$$

As an aside, we note that a complete understanding of those globular theories corresponding to globular operads was obtained in Theorem 6.6.8 of [2]. See also Section 3.12 of [8].
By [16, Theorem 5.35], \( I \) has a density presentation given by \( \Phi \)-tensors (i.e., tensors by objects in \( \Phi \)) while by [22, Theorem 3.1], \( K \) exhibits \( \mathcal{V} \) as the free \( \Phi \)-flat cocompletion of \( \mathcal{V}_\Phi \). Arguing as in the preceding parts, we see that \( \mathcal{V}_\Phi \)-theories are pretheories \( J: \mathcal{A}^{\text{op}} \to \mathcal{T} \) which preserve \( \Phi \)-tensors, that \( \mathcal{T} \)-models are \( \Phi \)-tensor-preserving functors \( F: \mathcal{T}^{\text{op}} \to \mathcal{V} \), and that a monad is \( \mathcal{V}_\Phi \)-nervous if its underlying endofunctor preserves \( \Phi \)-flat colimits. This sharpens slightly the results obtained in [22] in the special case \( \mathcal{E} = \mathcal{V} \).

(ix) Finally, in the situation of Examples 6(ix), we find that the \( \mathcal{A} \)-theories are the \( \Phi \)-colimit preserving pretheories \( J: \mathcal{A} \to \mathcal{T} \); that the \( \mathcal{T} \)-models are functors \( F: \mathcal{T}^{\text{op}} \to \mathcal{V} \) such that \( FJ^{\text{op}} \) preserves \( \Phi \)-limits; and that a monad is \( \mathcal{A} \)-nervous just when it preserves \( \Phi \)-flat colimits. In this way, our Theorems 17 and 24 reconstruct Theorems 7.6 and 7.7 of [22].

7. Monads with arities and theories with arities

In the introduction, we mentioned the general framework for monad–theory correspondences obtained in [33, 8]. Similar to this paper, the basic setting involves a category \( \mathcal{E} \) and a small, dense subcategory \( \mathcal{K}: \mathcal{A} \hookrightarrow \mathcal{E} \); given these data, one defines notions of monad with arities \( \mathcal{A} \) and theory with arities \( \mathcal{A} \), and proves an equivalence between the two that is compatible with semantics.

In this section, we compare this framework with ours by comparing the classes of monads and of theories. We will see that our setting yields strictly larger classes of monads and theories which are better-behaved in practically useful ways. On the other hand, in the more restrictive setting of [33, 8], checking that a monad is in the required class gives greater combinatorial insight into the structure which it describes.

7.1. Monads with arities versus nervous monads.

In [33, 8] the authors work in the unenriched setting; the introduction to [8] states that the results “should be applicable” also in the enriched one. To ease the comparison to our results, we take it for granted that this is true, and transcribe their framework into the enriched context without further comment.

Another difference is that we assume local presentability of \( \mathcal{E} \) while [33] assumes only cocompleteness, and [8] not even that. Given a small dense subcategory, there is no readily discernible difference between cocompleteness and local presentability; however, cocompleteness is substantively different from nothing, so that in this respect [8]’s results are more general than ours. However all known applications are in the context of a locally presentable \( \mathcal{E} \), and so we do not lose much in restricting to this context. In conclusion, when we make our comparison we will work in exactly the same general setting as in Section 2.1, and now have:

**Definition 43.** [33, Definition 4.1] An endofunctor \( T: \mathcal{E} \to \mathcal{E} \) is said to have arities \( \mathcal{A} \) if the composite \( \mathcal{V} \)-functor \( N_K T: \mathcal{E} \to [\mathcal{A}^{\text{op}}, \mathcal{V}] \) is the left Kan extension of its own restriction along \( K \). A monad \( T \in \text{Mnd}(\mathcal{E}) \) is a monad with arities \( \mathcal{A} \) if its underlying endofunctor has arities \( \mathcal{A} \).

---

---

4Indeed, if there were, then it would negate the large cardinal axiom known as Vopěnka’s principle [1, Chapter 6].
We consider the following way of restating this to be illuminating.

**Proposition 44.** An endofunctor \( T : \mathcal{E} \to \mathcal{E} \) has arities \( A \) if and only if it sends \( K \)-absolute colimits to \( K \)-absolute colimits. In particular, each endofunctor with arities \( A \) is \( A \)-induced.

**Proof.** By Proposition 33, \( T \) has arities \( A \) just when \( N_K T \) sends \( K \)-absolute colimits to colimits. Since \( N_K \) is fully faithful, it reflects colimits, and so \( T \) has arities \( A \) just when \( T \) sends \( K \)-absolute colimits to colimits which are preserved by \( N_K \)—that is, to \( K \)-absolute colimits.

For the second claim, recall from Definition 37 that an endofunctor \( T : \mathcal{E} \to \mathcal{E} \) is \( A \)-induced if it is the left Kan extension of its own restriction to \( A \), or equivalently, by Proposition 33, when it sends \( K \)-absolute colimits to colimits. \( \square \)

Recall also that we call a class of arities \( A \) saturated when \( A \)-induced endofunctors are closed under composition. Example 40 shows that this condition is not always satisfied. In light of the preceding result, the notion of endofunctor with arities \( A \) can be seen a natural strengthening of \( A \)-determination for which composition-closure is always verified.

The reason that Weber introduced monads with arities was in order to prove his nerve theorem [33, Theorem 4.10], which in our language may be restated as:

**Theorem 45.** Monads with arities \( A \) are \( A \)-nervous.

One may reasonably ask whether the classes of monads with arities and \( A \)-nervous monads in fact coincide. In many cases, this is true; in particular, in the situation of Example 39, where \( K : A \to \mathcal{E} \) exhibits \( \mathcal{E} \) as the free \( \Phi \)-cocompletion of \( A \) for some class of colimit-types \( \Phi \). Indeed, this condition implies that a monad \( T \) is \( A \)-nervous precisely when \( T \) sends \( \Phi \)-colimits to \( \Phi \)-colimits; since \( \Phi \)-colimits are \( K \)-absolute, this in turn implies that \( N_K T \) sends \( \Phi \)-colimits to colimits, and so is the left Kan extension of its own restriction along \( K \). So in this case, every \( A \)-nervous monad has arities \( A \); so in particular, the two notions coincide in each of Examples 6(i), (ii), (iii), (vi), (vii), (viii) and (ix).

However, they do not coincide in general. That is, in some instances of our basic setting, there exist monads which are \( A \)-nervous but do not have arities \( A \). We give three examples of this. The first two arise in the setting of Example 6(iv), and concern the monads for groupoids and involutive graphs respectively.

**Proposition 46.** The monad \( T \) on \( \text{Grph} := [G_1^{op}, \text{Set}] \) whose algebras are groupoids is \( \Delta_0 \)-nervous but does not have \( \Delta_0 \)-induced underlying endofunctor. It follows that \( T \) does not have arities \( \Delta_0 \).

**Proof.** From Example 9 we know that \( T \) is \( \Delta_0 \)-nervous. To see that \( T \) is not \( \Delta_0 \)-induced, consider the graph \( X \) with vertices and arrows as to the left in:

\[
\begin{align*}
(7.1) & \quad a \xrightarrow{r} b \xleftarrow{\xi} c \\
& \quad [0] \xrightarrow{\tau} [1] \\
& \quad \downarrow s \\
& \quad [1] \xrightarrow{r} X.
\end{align*}
\]
This $X$ is equally the $K$-absolute pushout right above; so if $T$ were $\Delta_0$-induced then it would preserve this pushout. But $T[1] +_{T[0]} T[1]$ is the graph

\[
\begin{array}{c}
1_a \\ \\
\Rightarrow & r^{-1} & \Rightarrow & s^{-1} \\ \\
\Rightarrow & r & \Rightarrow & s \\ \\
\Rightarrow & b \\ \\
\Rightarrow & c \\
\end{array}
\]

wherein, in particular, there is no edge $a \to c$; while in $TX$ we have $s^{-1} \circ r: a \to c$. So the pushout is not preserved. This shows that $T$ is not $\Delta_0$-induced and so, by Proposition 44, that $T$ does not have arities $\Delta_0$.

Since the above result exhibits a $\Delta_0$-nervous monad whose underlying endofunctor is not $\Delta_0$-induced, we can apply Theorem 41 to deduce:

**Corollary 47.** $K: \Delta_0 \hookrightarrow \text{Grph} \text{ is not a saturated class of arities.}$

Our second example, originally due to Melliès [27, Appendix III], shows that even monads with $\Delta_0$-induced endofunctor need not have arities $\Delta_0$. In this example, we call a graph $s,t: X_1 \Rightarrow X_0$ involutive if it comes endowed with an order-2 automorphism $i: X_1 \to X_1$ reversing source and target, i.e., with $si = t$ (and hence also $ti = s$).

**Proposition 48.** The monad $T$ on $\text{Grph} := [\text{G}_1^{\text{op}}, \text{Set}]$ whose algebras are involutive graphs is $\Delta_0$-nervous and has $\Delta_0$-induced underlying endofunctor, but does not have arities $\Delta_0$.

**Proof.** The value of $T$ at $s,t: X_1 \Rightarrow X_0$ is given by $(s,t), (t,s): X_1 + X_1 \Rightarrow X_0$. It follows that $T$ is cocontinuous and so certainly $\Delta_0$-induced. To see it does not have arities $\Delta_0$, consider again the graph (7.1) and its $K$-absolute pushout presentation. If this were preserved by $N_K T: \text{Grph} \to [\Delta_0^{\text{op}}, \text{Set}]$ then, on evaluating at $[2]$, the maps $\text{Grph}([2], T[1]) \Rightarrow \text{Grph}([2], TX)$ given by postcomposition with $Tr$ and $Ts$ would be jointly surjective. To show this is not so, consider the map $f: [2] \to TX$ picking out the composable pair $(r: a \to b, i(s): b \to c)$. Since neither $Tr$ nor $Ts$ are surjective on objects, the bijective-on-objects $f$ cannot factor through either of them. This shows that $T$ does not have arities $\Delta_0$. □

Our final example shows that not even free monads on $A$-signatures—which are $A$-nervous by Theorem 36 above—need necessarily have arities $A$.

**Proposition 49.** Let $V = E = \text{Set}$ and let $A$ be the one-object full subcategory on a two-element set. The free monad on the terminal $A$-signature does not have $A$-induced underlying endofunctor and therefore does not have arities $A$.

**Proof.** The algebras for the free monad $T$ on the terminal signature are sets equipped with a binary operation. Elements of the free $T$-algebra on $X$ are binary trees with leaves labelled by elements of $X$, yielding the formula

$$TX = \sum_{n \in \mathbb{N}} C_n \times X^{n+1}$$

where $C_n$ is the $n$th Catalan number. In particular, $T$ contains at least one coproduct summand $(-)^4$ and so, as in Example 40, is not $A$-induced; in particular, by Proposition 44, it does not have arities $A$. □
7.2. Theories with arities $\mathcal{A}$ versus $\mathcal{A}$-theories. The paper [8] introduced theories with arities $\mathcal{A}$. These are $\mathcal{A}$-pretheories $J: \mathcal{A} \to \mathcal{T}$ for which the composite

\[
(7.2) \quad [\mathcal{A}^{\text{op}}, \mathcal{V}] \xrightarrow{\text{Lan}_J} [\mathcal{T}^{\text{op}}, \mathcal{V}] \xrightarrow{[J^{\text{op}}, 1]} [\mathcal{A}^{\text{op}}, \mathcal{V}]
\]

takes $K$-nerves to $K$-nerves. This functor takes the representable $\mathcal{A}(\_ , x)$ to $\mathcal{T}(J\_ , x)$, so that in this language, we may describe the $\mathcal{A}$-theories as the pretheories for which (7.2) takes each representable to a $K$-nerve. It follows that:

**Proposition 50.** Theories with arities $\mathcal{A}$ are $\mathcal{A}$-theories.

**Proof.** It suffices to observe that each representable $\mathcal{A}(\_ , x)$ is a $K$-nerve since $\mathcal{A}(\_ , x) \cong \mathcal{E}(K\_ , Kx) = N_K(Kx)$.

Theorem 3.4 of [8] establishes an equivalence between the categories of monads with arities $\mathcal{A}$ and of theories with arities $\mathcal{A}$. The functor taking a monad with arities to the corresponding theory with arities is defined in the same way as the $\Phi$ of Section 2.4, and so it follows that:

**Proposition 51.** The equivalence of monads with arities $\mathcal{A}$ and theories with arities $\mathcal{A}$ is a restriction of the equivalence between $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories.

In particular, there exist $\mathcal{A}$-theories which are not theories with arities $\mathcal{A}$; it is this statement which was verified in [27, Appendix III].

7.3. Colimits of monads with arities. In Theorem 36 we saw that the $\mathcal{A}$-nervous monads are the closure of the free monads on $\mathcal{A}$-signatures under colimits in $\text{Mnd}(\mathcal{E})$. Since colimits of monads are algebraic, this allows us to give intuitive presentations for $\mathcal{A}$-nervous monads as suitable colimits of frees. The pretheory presentations of Section 3 can be understood as particularly direct descriptions of such colimits.

Since not every $\mathcal{A}$-nervous monad has arities $\mathcal{A}$, the monads with arities are not the colimit-closure of the frees on signatures. We already saw one explanation for this in Proposition 49: the free monads on signatures need not have arities. However, this leaves open the possibility that the monads with arities $\mathcal{A}$ are the colimit-closure of some smaller class of basic monads—which would allow for the same kind of intuitive presentation as we have for $\mathcal{A}$-nervous monads. The following result shows that even this is not the case.

**Theorem 52.** Monads with arities $\mathcal{A}$ need not be closed in $\text{Mnd}(\mathcal{E})$ under colimits.

**Proof.** We saw in Proposition 48 that, when $\mathcal{E} = \text{Grph}$ and $\mathcal{A} = \Delta_0$, the monad $\mathcal{T}$ for involutive graphs does not have arities $\Delta_0$. To prove the result it will therefore suffice to exhibit $\mathcal{T}$ as a colimit in $\text{Mnd}(\text{Grph})$ of a diagram of monads with arities $\Delta_0$. This diagram will be a coequaliser involving a pair of monads $\mathcal{P}$ and $\mathcal{Q}$, whose respective algebras are:

- For $\mathcal{P}$: graphs $X$ endowed with a function $u: X_1 \to X_0$;
- For $\mathcal{Q}$: graphs $X$ endowed with an order-2 automorphism $i: X_1 \to X_1$. 
We construct this coequaliser of monads in terms of the categories of algebras. The category $\text{Graph}^T$ of involutive graphs is an equaliser in $\text{CAT}$ as to the left in:

$$
\begin{array}{ccc}
\text{Graph}^T & \xrightarrow{E} & \text{Graph}^Q & \xrightarrow{F} & \text{Graph}^P \\
\text{Grph} & \xrightarrow{\varphi} & Q & \xrightarrow{\varepsilon} & T
\end{array}
$$

where the functors $F$ and $G$ send a $Q$-algebra $(X, i)$ to the respective $P$-algebras $(X, si)$ and $(X, t)$. Since each of these functors commutes with the the forgetful functors to $\text{Graph}$, we have an equaliser of forgetful functors in $\text{CAT}/\text{Grph}$. Since the functor $\text{Alg}: \text{Mnd}(\text{Graph}) \to \text{CAT}/\text{Grph}$ is fully faithful, this equaliser must be the image of a coequaliser diagram in $\text{Mnd}(\text{Graph})$ as right above.

It remains to show that in this coequaliser presentation both $P$ and $Q$ have arities $\Delta_0$. By Proposition 33, this means showing that $N_K P$ and $N_K Q$ send $K$-absolute colimits to colimits, or equally, that each $\text{Graph}([n], P\dashv)$ and $\text{Graph}([n], Q\dashv)$ sends $K$-absolute colimits to colimits. To see this, we calculate $P$ and $Q$ explicitly. On the one hand, the free $P$-algebra on a graph $X$ is obtained by freely adjoining an element $u(f)$ to $X_0$ for each $f \in X_1$. On the other hand, the free $Q$-algebra on $X$ is obtained by freely adjoining an element $i(f) \in X_1$ for each $f \in X_1$. Thus we have $PX = X + X_1 \cdot [0]$ and $QX = X + X_1 \cdot [1]$ where we use $\cdot$ to denote copower. Since each $[n] \in \text{Graph}$ is connected, and since each hom-set $\text{Graph}([n], [m])$ has cardinality $\max(0, m - n + 1)$, we conclude that

$$
\text{Graph}([n], PX) = \begin{cases}
\text{Graph}([0], X) + \text{Graph}([1], X) & \text{if } n = 0; \\
\text{Graph}([n], X) & \text{if } n > 0.
\end{cases}
$$

$$
\text{Graph}([n], QX) = \begin{cases}
\text{Graph}([0], X) + 2 \cdot \text{Graph}([1], X) & \text{if } n = 0; \\
\text{Graph}([1], X) + \text{Graph}([1], X) & \text{if } n = 1; \\
\text{Graph}([n], X) & \text{if } n > 1.
\end{cases}
$$

Now by definition, $N_K$ sends $K$-absolute colimits to colimits, whence also each $\text{Graph}([n], -): \text{Graph} \to \text{Set}$. The functors with this property are closed under colimits in $[\text{Graph}, \text{Set}]$, and so (7.3) ensures that each $\text{Graph}([n], P\dashv)$ and $\text{Graph}([n], Q\dashv)$ sends $K$-absolute colimits to colimits as desired. □

It is not even clear to us if the category of monads with arities $A$ is always cocomplete. The argument for local presentability of $\text{Mnd}_A(\mathcal{E})$ in Theorem 36 does not seem to adapt to the case of monads with arities, and no other obvious argument presents itself. In any case, the preceding result shows that, even if the category of monads with arities does have colimits, they do not always coincide with the usual colimits of monads, and, in particular, are not always algebraic. This dashes any hope we might have had of giving a sensible notion of presentation for monads with arities.

8. Deferred proofs

8.1. Identifying the monads. In this section, we complete the proofs of the results deferred from Section 6 above, beginning with Theorem 36. Recall that the
category $\text{Sig}_A(\mathcal{E})$ of signatures is the (ordinary) category $\mathcal{V}\text{-CAT}(\text{ob } \mathcal{A}, \mathcal{E})$, and that $V: \text{Mnd}(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})$ is the functor sending $T$ to $(Ta)_{a \in \mathcal{A}}$.

**Proposition 53.** $V: \text{Mnd}(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})$ has a left adjoint $F$ which takes values in $\mathcal{A}$-nervous monads.

**Proof.** We can decompose $V$ as the composite

$$\text{Mnd}(\mathcal{E}) \xrightarrow{V_1} \mathcal{V}\text{-CAT}(\mathcal{E}, \mathcal{E}) \xrightarrow{V_2} \text{Sig}_A(\mathcal{E})$$

where $V_1$ takes the underlying endofunctor, and $V_2$ is given by evaluation at each $a \in \text{ob } \mathcal{A}$. Since $V_2$ is equally given by restriction along $\text{ob } \mathcal{A} \to \mathcal{A} \to \mathcal{E}$, it has a left adjoint $F_2$ given by pointwise left Kan extension, with the explicit formula:

$$F_2(\Sigma) = \sum_{a \in \mathcal{A}} \mathcal{E}(Ka, -) \cdot \Sigma a: \mathcal{E} \to \mathcal{E},$$

where $\cdot$ denotes $\mathcal{V}$-enriched copower. So it suffices to show that the free monad on each endofunctor $F_2(\Sigma)$ exists and is $\mathcal{A}$-nervous. By [15, Theorem 23.2], such a free monad $T$ is characterised by the property that $\mathcal{E}^{F_2(\Sigma)} \cong \mathcal{E}^T$ over $\mathcal{E}$, where on the left we have the $\mathcal{V}$-category of algebras for the mere endofunctor $F_2(\Sigma)$.

To complete the proof, it suffices by Theorem 5 to exhibit $\mathcal{E}^{F_2(\Sigma)}$ as isomorphic to the $\mathcal{V}$-category of concrete models of some $\mathcal{A}$-pretheory.

To this end, we let $B$ be the collage of the $\mathcal{V}$-functor $N_K: \text{ob } \mathcal{A} \to [\mathcal{A}^{\text{op}}, \mathcal{V}]$. Thus $B$ is the $\mathcal{V}$-category with object set $\text{ob } \mathcal{A} + \text{ob } \mathcal{A}$ and the following hom-objects, where we write $\ell, r: \text{ob } \mathcal{A} \to \text{ob } \mathcal{B}$ for the two injections:

$$B(\ell a', \ell a) = \mathcal{A}(a', a) \quad B(r a', r a) = (\text{ob } \mathcal{A})(a', a)$$

Let $\ell: \mathcal{A} \to \mathcal{B}$ and $r: \text{ob } \mathcal{A} \to \mathcal{B}$ be the two injections into the collage, and now form the pushout $J: \mathcal{A} \to \mathcal{T}$ of $(\ell, r): \mathcal{A} + \text{ob } \mathcal{A} \to \mathcal{B}$ along $(1, \iota): \mathcal{A} + \text{ob } \mathcal{A} \to \mathcal{A}$. Since $(\ell, r)$ is identity-on-objects, so is $J: \mathcal{A} \to \mathcal{T}$, and so we have an $\mathcal{A}$-pretheory. To conclude the proof, it now suffices to show that $\mathcal{E}^{F_2(\Sigma)} \cong \text{Mod}_c(\mathcal{T})$ over $\mathcal{E}$.

By the universal property of the collage and the pushout, to give a functor $H: \mathcal{T} \to \mathcal{X}$ is equally to give a functor $F = HJ: \mathcal{A} \to \mathcal{X}$ together with $\mathcal{V}$-natural transformations $\alpha_a: \mathcal{E}(K-, \Sigma a) \Rightarrow \mathcal{X}(F-, Fa)$ for each $a \in \text{ob } \mathcal{A}$. In particular, taking $\mathcal{X} = \mathcal{V}^{\text{op}}$ and $F = \mathcal{E}(K-, X)$, we see that a concrete $\mathcal{T}$-model structure on $X \in \mathcal{E}$ is given by an ob $\mathcal{A}$-indexed family of $\mathcal{V}$-natural transformations

$$\alpha_a: \mathcal{E}(K-, \Sigma a) \Rightarrow [\mathcal{E}(Ka, X), \mathcal{E}(K-, X)]$$

or equally under transpose, by a family of maps

$$\mathcal{E}(Ka, X) \to [\mathcal{A}^{\text{op}}, \mathcal{V}](\mathcal{E}(K-, \Sigma a), \mathcal{E}(K-, X)).$$

By full fidelity of $N_K$, the right-hand side above is isomorphic to $\mathcal{E}(\Sigma a, X)$, and so concrete $\mathcal{T}$-model structure on $X$ is equally given by a family of maps $\mathcal{E}(Ka, X) \to \mathcal{E}(\Sigma a, X)$. Finally, using the universal properties of copowers and coproducts, this is equivalent to giving a single map

$$\bar{\alpha}: \sum_{a \in \mathcal{A}} \mathcal{E}(Ka, X) \cdot \Sigma a \to X$$

exhibiting $X$ as an $F_2(\Sigma)$-algebra. We thus have a bijection over $\mathcal{E}$ between objects of $\mathcal{E}^{F_2(\Sigma)}$ and objects of $\text{Mod}_c(\mathcal{T})$. 


A similar analysis shows that a morphism \( A \to \mathcal{E}(X, Y) \) in \( \mathcal{V} \) lifts through the monomorphism \( \text{Mod}_c(\mathcal{T})((X, \alpha), (Y, \beta)) \to \mathcal{E}(X, Y) \) if and only if it lifts through the monomorphism \( \mathcal{E}^{\mathcal{F}_2}(\Sigma)((X, \bar{\alpha}), (Y, \bar{\beta})) \to \mathcal{E}(X, Y) \). It follows that we have an isomorphism of \( \mathcal{V} \)-categories \( \mathcal{E}^{\mathcal{F}_2}(\Sigma) \cong \text{Mod}_c(\mathcal{T}) \) over \( \mathcal{E} \) as desired. \( \square \)

In proving the rest of Theorem 36, the following lemma will be useful.

**Lemma 54.** Let \( C_1 \subseteq C_2 \) be replete, full, colimit-closed sub-\( \mathcal{V} \)-categories of \( C \); for example, they could be coreflective. If \( V : C \to D \) has a left adjoint \( F \) taking values in \( \mathcal{V} \), we have a pullback square as to the right in:

\[
\begin{array}{ccc}
\text{Mod}_c(\mathcal{T}) & \xrightarrow{\Phi} & \text{Th}_A(\mathcal{E}) \\
\downarrow & & \downarrow H \\
\mathcal{V}(\text{ob } A, [\mathcal{A}^{\text{op}}, \mathcal{V}]) & \cong & \mathcal{V}(\text{ob } A, [\mathcal{A}^{\text{op}}, \mathcal{V}])
\end{array}
\]

\( (\mathcal{V}(\text{ob } A, [\mathcal{A}^{\text{op}}, \mathcal{V}])) \xrightarrow{\cong} \mathcal{V}(\text{ob } A, [\mathcal{A}^{\text{op}}, \mathcal{V}]) \).

Since \( K \)-\text{Ner} \hookrightarrow [\mathcal{A}^{\text{op}}, \mathcal{V}] \) is replete, this square is a pullback along a discrete isofibration, and so by [13, Corollary 1] also a bipullback. On the other hand, to the left, we have a pseudocommuting square as witnessed by the isomorphisms:

\[
(PJ_T)(A) = \mathcal{A}_T(J^{-1}_T, J_T A) = \mathcal{E}^T(F^T K^{-1}, F^T K A) \cong \mathcal{E}(K^{-1}, TK A) = N_K(TK A) .
\]

Since both horizontal edges of this square are equivalences, it is also a bipullback.

To show the required monadicity, we must prove that \( V \) creates \( \mathcal{V} \)-absolute coequalizers. Since the large rectangle is a bipullback—as the pasting of two bipullbacks—it suffices to show that \( H \) creates \( \mathcal{V} \)-absolute coequalizers. As the definition of \( H \) depends only on \( A \) and not \( \mathcal{E} \), we lose no generality in proving this if we assume that \( \mathcal{E} = [\mathcal{A}^{\text{op}}, \mathcal{V}] \) and \( K = Y \). In this case, every presheaf on \( A \) is a \( K \)-nerve, and so the horizontal composites in (8.1) are equivalences; and so, finally, it suffices to prove that \( V \) is monadic when \( \mathcal{E} = [\mathcal{A}^{\text{op}}, \mathcal{V}] \) and \( K = Y \).

Note that, in this case, \( A \) is a saturated class of arities: for indeed, by the universal property of free cocompletion, a functor \( F : [\mathcal{A}^{\text{op}}, \mathcal{V}] \to [\mathcal{A}^{\text{op}}, \mathcal{V}] \) is \( A \)-induced if and only if it is \( \text{cocontinuous} \). It thus follows from Proposition 56 below that the
restriction $V_c: \text{Mnd}_c(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})$ of $V$ to cocontinuous monads is monadic; so we will be done if $\text{Mnd}_c(\mathcal{E}) = \text{Mnd}_A(\mathcal{E})$. In this case, $\Psi: \text{Preth}_A(\mathcal{E}) \to \text{Mnd}_A(\mathcal{E})$ sends $J: A \to T$ to a monad which is isomorphic to that induced by the adjunction $\text{Lan}_J: [A^{op}, \mathcal{V}] \rightleftarrows [T^{op}, \mathcal{V}]: [J^{op}, 1]$, and so $\text{Mnd}_A(\mathcal{E}) \subseteq \text{Mnd}_c(\mathcal{E})$. To obtain equality, we apply Lemma 54. We have that:

- $\text{Mnd}_A(\mathcal{E})$ and $\text{Mnd}_c(\mathcal{E})$ are coreflective in $\text{Mnd}(\mathcal{E})$ by Corollary 20 and Lemma 55 respectively;
- $V: \text{Mnd}(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})$ has a left adjoint taking values in $\text{Mnd}_A(\mathcal{E})$;
- The restriction $V_c: \text{Mnd}_c(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})$ is monadic;

and so $\text{Mnd}_A(\mathcal{E}) = \text{Mnd}_c(\mathcal{E})$. This proves monadicity of $V$ in the special case $\mathcal{E} = [A^{op}, \mathcal{V}]$, whence also, by the preceding argument, in the general case.

In order to prove (ii), we let $C_1$ be the colimit-closure in $\text{Mnd}(\mathcal{E})$ of the image of $F$. Since $\text{Mnd}_A(\mathcal{E})$ contains this image and is colimit-closed, we have $C_1 \subseteq \text{Mnd}_A(\mathcal{E}) \subseteq \text{Mnd}(\mathcal{E})$. Thus, applying Lemma 54 to this triple and $V: \text{Mnd}_A(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})$ gives $\text{Mnd}_A(\mathcal{E}) = C_1$ as desired.

Finally we prove (iii). The monadicity of $V$ above implies that of $P$ and hence also of $H$ (by taking $\mathcal{E} = [A^{op}, \mathcal{V}]$). Since filtered colimits of $A$-pretheories can be computed at the level of underlying graphs, the forgetful $H$ preserves them; which is to say that $\text{Preth}_A(\mathcal{E})$ is finitarily monadic over the locally presentable $\mathcal{V}$-$\text{CAT}(\text{ob}, A, [A^{op}, \mathcal{V}])$, whence locally presentable by [12, Satz 10.3]. So in the right-hand and the large bipullback squares in (8.1), the bottom and right sides are right adjoints between locally presentable categories. Since by [9, Theorem 2.18], the 2-category of locally presentable categories and right adjoints is closed under bilimits in $\text{CAT}$, we conclude that each $\text{Th}_A(\mathcal{E})$ and each $\text{Mnd}_A(\mathcal{E})$ is also locally presentable. \hfill $\Box$

8.2. Saturated classes. We now turn to the deferred proof of Theorem 41. Recall the context: an endo-$\mathcal{V}$-functor $F: \mathcal{E} \to \mathcal{E}$ is called $A$-induced when the pointwise left Kan extension of its restriction along $K$, and $A$ is a saturated class of arities if $A$-induced endofunctors of $\mathcal{E}$ are composition-closed.

We begin by recording the basic properties of this situation. We write $\text{A-End}(\mathcal{E})$ and $\text{A-Mnd}(\mathcal{E})$ for the full subcategories of $\text{End}(\mathcal{E}) = \mathcal{V}$-$\text{CAT}(\mathcal{E}, \mathcal{E})$ and $\text{Mnd}(\mathcal{E})$ on, respectively, the $A$-induced endofunctors, and the monads with $A$-induced underlying endofunctor.

**Lemma 55.** $\text{A-End}(\mathcal{E})$ is coreflective in $\text{End}(\mathcal{E}) = \mathcal{V}$-$\text{CAT}(\mathcal{E}, \mathcal{E})$ via the coreflector $R(F) = \text{Lan}_K(FK)$, as on the left in:

\[
\begin{array}{ccc}
\text{A-End}(\mathcal{E}) & \xrightarrow{R} & \text{End}(\mathcal{E}) \\
\downarrow & \\n\text{A-Mnd}(\mathcal{E}) & \xrightarrow{R} & \text{Mnd}(\mathcal{E})
\end{array}
\]

If $A$ is a saturated class, then $\text{A-End}(\mathcal{E})$ is right-closed monoidal, and the coreflection left above lifts to the corresponding categories of monads as on the right.

**Proof.** Restriction and left Kan extension along the fully faithful $K$ exhibits $\text{A-End}(\mathcal{E})$ as equivalent to $\mathcal{V}$-$\text{CAT}(A, \mathcal{E})$, whence locally presentable. Since restriction along $K$ is a coreflector of $\text{End}(\mathcal{E})$ into $\mathcal{V}$-$\text{CAT}(A, \mathcal{E})$, it follows that $R(F) = \text{Lan}_K(FK)$ is a coreflector of $\text{End}(\mathcal{E})$ into $\text{A-End}(\mathcal{E})$. \hfill $\Box$
If \( \mathcal{A} \) is saturated then \( \mathcal{A} \)-\textit{End}(\( \mathcal{E} \)) is monoidal under composition. Since each endofunctor \((-) \circ F\) of \( \text{End}(\mathcal{E})\) is cocontinuous, and \( \mathcal{A} \)-\textit{End}(\( \mathcal{E} \)) is closed in \( \text{End}(\mathcal{E}) \) under colimits, each endofunctor \((-) \circ F\) of \( \mathcal{A} \)-\textit{End}(\( \mathcal{E} \)) is cocontinuous, and so has a right adjoint by local presentability. Thus \( \mathcal{A} \)-\textit{End}(\( \mathcal{E} \)) is right-closed monoidal.

Furthermore, the inclusion of \( \mathcal{A} \)-\textit{End}(\( \mathcal{E} \)) into \( \text{End}(\mathcal{E}) \) is strict monoidal, whence by [14, Theorem 1.5] the coreflection to the left of (8.2) lifts to a coreflection in the 2-category \textit{MONCAT} of monoidal categories, lax monoidal functors and monoidal transformations. Applying the 2-functor \textit{MONCAT}(1,–): \textit{MONCAT} \to \textit{CAT} yields the coreflection to the right of (8.2). □

The key step towards establishing Theorem 41 above is now:

**Proposition 56.** The left adjoint \( F\) of \( V: \text{Mnd}(\mathcal{E}) \to \text{Sig}_A(\mathcal{E}) \) takes values in \( \mathcal{A} \)-induced monads; furthermore, the restriction of \( V\) to \( \mathcal{A} \)-\textit{Mnd}(\( \mathcal{E} \)) is monadic.

**Proof.** For any \( T \in \text{Mnd}(\mathcal{E})\), its \( \mathcal{A} \)-induced coreflection \( \varepsilon_T: \text{IR}(T) \to T\) has as underlying map in \( \text{End}(\mathcal{E})\) the component \( \text{Lan}_K(TK) \to T\) of the counit of the adjunction given by restriction and left Kan extension along \( K\). Since \( K\) is fully faithful, the restriction of this map along \( K\) is invertible, whence in particular, \( V \varepsilon: V \text{IR} \Rightarrow V: \text{Mnd}(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})\) is invertible. So \( \eta: \text{id} \Rightarrow VF\) factors through \( V \varepsilon_F: V \text{IR}F \Rightarrow VF\) whence, by adjointness, \( \text{id}: F \Rightarrow F\) factors through \( \varepsilon_F\). Therefore each \( F(\Sigma)\) is a retract of \( \text{IRF}(\Sigma)\); since \( \mathcal{A} \)-\textit{Mnd}(\( \mathcal{E} \)) is closed under colimits in \( \text{Mnd}(\mathcal{E})\), it is retract-closed and so each \( F(\Sigma)\) belongs to \( \mathcal{A} \)-\textit{Mnd}(\( \mathcal{E} \)).

It remains to prove that the restriction of \( V\) to \( \mathcal{A} \)-\textit{Mnd}(\( \mathcal{E} \)) is monadic. To do so, we decompose this restriction as

\[
\mathcal{A} \text{-Mnd}(\mathcal{E}) \xrightarrow{V_1} \mathcal{A} \text{-End}(\mathcal{E}) \xrightarrow{V_2} \text{Sig}_A(\mathcal{E}),
\]

where \( V_1\) forgets the monad structure and \( V_2\) is given by precomposition with \( \text{ob}\mathcal{A} \to \mathcal{A} \to \mathcal{E}\), and apply the following result, which is [20, Theorem 2]:

**Theorem.** Let \( M\) be a right-closed monoidal category, and \( V_2: M \to N\) a monadic functor for which there exists a functor \( \diamond: M \times N \to N\) with natural isomorphisms \( X \diamond Y \cong V(X \otimes Y)\). If the forgetful functor \( V_1: \text{Mon}(M) \to M\) has a left adjoint, then the composite \( V_2V_1: \text{Mon}(M) \to N\) is monadic.

Indeed, by Lemma 55, \( \mathcal{A} \)-\textit{End}(\( \mathcal{E} \)) is a right-closed monoidal category, and \( \mathcal{A} \)-\textit{Mnd}(\( \mathcal{E} \)) the category of monoids therein. Under the equivalence \( \mathcal{A} \)-\textit{End}(\( \mathcal{E} \)) \simeq \mathcal{V} \text{-CAT}(\mathcal{A}, \mathcal{E}),\) we may identify \( V_2\) with precomposition along \( \text{ob}\mathcal{A} \to \mathcal{A}\). It is thus cocontinuous, and has a left adjoint given by left Kan extension; whence is monadic. Now since \( V_2V_1\) has a left adjoint and \( V_2\) is monadic, it follows that \( V_1\) also has a left adjoint. Finally, we have a functor

\[
\diamond: \mathcal{A} \text{-End}(\mathcal{E}) \times \text{Sig}_A(\mathcal{E}) \to \text{Sig}_A(\mathcal{E})
\]
defined by \( (F,G) \mapsto FG\), and this clearly has the property that \( M(FG) = F \diamond M(G)\). So applying the above theorem yields the desired monadicity. □

We are now ready to prove:
Theorem 41. Let $\mathcal{A}$ be a saturated class of arities in $\mathcal{E}$. The following are equivalent properties of a monad $T \in \text{Mnd}(\mathcal{E})$:

(i) $T$ is $\mathcal{A}$-nervous;
(ii) $T: \mathcal{E} \to \mathcal{E}$ is $\mathcal{A}$-induced;
(iii) $T: \mathcal{E} \to \mathcal{E}$ preserves $\Phi$-colimits for any density presentation $\Phi$ of $K$.

Proof. For (i) $\Leftrightarrow$ (ii), the monadicity of $V: \mathcal{A}\text{-Mnd}(\mathcal{E}) \to \text{Sig}_\mathcal{A}(\mathcal{E})$ verified in the previous proposition implies, as in the proof of Theorem 36(iii), that $\mathcal{A}\text{-Mnd}(\mathcal{E})$ is the colimit-closure in $\text{Mnd}(\mathcal{E})$ of the free monads on signatures. Since $\text{Mnd}_\mathcal{A}(\mathcal{E})$ is also this closure, we have $\text{Mnd}_\mathcal{A}(\mathcal{E}) = \mathcal{A}\text{-Mnd}(\mathcal{E})$ as desired. For (ii) $\Leftrightarrow$ (iii), we apply Proposition 33. □

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Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno 60000, Czech Republic
E-mail address: bourkej@math.muni.cz

Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: richard.garner@mq.edu.au