General Eigensolutions for 2D Viscoelastic Materials

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Abstract—In recent years, viscoelastic materials have been widely used in industry, and the study of its mechanical properties has become the key of modern science and engineering. In this paper, based on Laplace integral transformation, a new analytical method is proposed to solve the mechanical problems related to viscoelastic materials. According to the variational principle and the method of separating variables, the governing equations of the basic problems are established, and all the eigensolutions of analytical forms are obtained. In the numerical calculation, the relationship between these eigensolutions and boundary conditions is studied systematically, and the time-dependent property of viscoelastic materials is well described.

1. INTRODUCTION

With the rapid development of science and technology, many new materials, various types of polymer, synthetic rubber, paint, glass, ceramics have been widely used in modern industry [1,2]. These materials not only have the characteristics of elastic solid, but also have the characteristics of viscous fluid, namely viscoelastic materials. With the emergence of these new materials, more and more attention has been paid to the material properties, especially the viscoelastic properties, which are not considered in the classical material mechanics and fluid mechanics. According to statistics, the volume of polymer materials in the world has greatly exceeded that of metal materials. These materials have the advantages of high strength, easy processing and forming. As structural materials, they are more and more widely used in chemical industry, machinery, aerospace, construction and marine development and other industrial fields. Statics and dynamics analysis of polymers, their composites and structures under various loading and service conditions has become an important research content in the field of mechanics [3-5]. In fact, the vigorous development of geomechanics, geomechanics, earthquake prediction, biomechanics and so on has also greatly stimulated and promoted the viscoelastic theory. It is of great theoretical significance and practical value to study this kind of material and its mechanical behavior in structure by using viscoelastic theory.

Viscoelasticity is one of the basic contents of solid mechanics, which usually includes two aspects: one is the reasonable description of material properties and the accurate expression of constitutive relationship; the other is the establishment, theoretical analysis and solution of various mathematical models related to viscoelasticity [6,7]. For a long time, people have made a lot of explorations and put forward various methods to solve viscoelastic problems. Due to the strong time, temperature and frequency effects of viscoelastic materials, theoretical analysis and quantitative calculation are very...
difficult, which can not keep up with the needs of application, so it has attracted great attention of scholars at home and abroad.

2. SOLUTION METHOD
Let’s consider a viscoelastic media of strip plane domain shown in Figure 1.

The constitutive equations viscoelasticity can be written in the following form:

\[
\begin{align*}
\sigma_y &= \frac{1}{3} \varepsilon_y \sigma_{mm} \\
\varepsilon_y &= \varepsilon_y \quad (1)
\end{align*}
\]

Generally, the viscoelastic stress-strain relations can be transformed into a set of corresponding elastic ones by applying Laplace transformation. As a result, the transformed Laplace domain stress-strain relations are written as

\[
\begin{align*}
\bar{\sigma}_{ij} (r, s) &= 2\bar{G}(s) \bar{\varepsilon}_{ij} (r, s) \\
\bar{\sigma}_{mm} (r, s) &= 3\bar{K}(s) \bar{\varepsilon}_{mm} (r, s)
\end{align*}
\]

where a bar over a variable designates its Laplace transform, and is the transform parameter. The displacement and stress vectors are

\[
\bar{q} = \{\bar{u}, \bar{\varepsilon}\}^{T}
\]

and

\[
\bar{p} = \begin{bmatrix}
(\lambda^* + 2\bar{G}^*)\bar{u} + \lambda^* \bar{\varepsilon} \\
\bar{G}^* (\bar{\varepsilon}_y \bar{u} + \bar{v})
\end{bmatrix} = \begin{bmatrix}
\bar{\sigma} \\
\bar{\tau}
\end{bmatrix}
\]

respectively. Based on the separation of variable method, the dual equations is obtained as

\[
\psi = \mathbf{H} \psi
\]

Solution of basic governing equation can be expressed as

\[
\psi = \psi_j (y) e^{\mu_j y}
\]

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The general eigensolutions can be obtained by using the variable separation method. These solutions are

\[ \psi_1 = \{1, 0, 0, 0\}^T, \]

\[ \psi_2 = \{0, 1, 0, 0\}^T, \]

\[ \psi_3 = \{-x, 0, 0, 0\}^T, \]

\[ \psi_4 = \{0, -vx, E, 0\}^T. \]

\[ \psi_5 = \{0, -vx, E, 0\}^T \]

\[ \psi_6 = \{x^3 - (1 + v)\mu^2 x, 0, 0, 3E\}^T. \]

### 3. Numerical Results

In this section, the parameters and geometry data take is taken as

\[ \frac{2G}{(3K)} = 0.4, \quad \frac{b}{l} = 0.2 \]

Based on the variational principle and the method of component variable and the solution method proposed in this paper, we find all the eigensolutions of the basic eigensolutions. By using these basic solutions, the viscoelastic boundary conditions can be easily solved. Figures 2-5 show the displacement of the first two eigensolutions in horizontal and vertical directions, and figures 6-9 describe the distribution of normal stress and shear stress distributions corresponding to the displacement components. It can be seen from these figures that the eigensolutions increase or decrease rapidly to a certain limit position with the increase of time. The results show that the non-zero eigensolutions only play an obvious role near the end, which is consistent with the famous Saint Venant's principle.

![Figure 2. The first order eigensolution of displacement u distribution.](image-url)
Figure 3. The first order eigensolution of displacement $v$ distribution.

Figure 4. The second order eigensolution of displacement $u$ distribution.

Figure 5. The second order eigensolution of displacement $v$ distribution.
Figure 6. The first order eigensolution of normal stress distribution.

Figure 7. The first order eigensolution of shear stress distribution.

Figure 8. The second order eigensolution of normal stress distribution.
4. CONCLUSION
According to Laplace integral transformation, the mechanical problems related to viscoelastic materials can be transformed into the form of elastic mechanics, ensuring the conservation of energy, and the separation variable method can be implemented smoothly. By using this method, the general solutions of the basic governing equations can be described as the eigensolutions in analytical form.

ACKNOWLEDGMENTS
This research was financially supported by Major Natural Science Research Projects in Colleges and Universities of Jiangsu Province (17KJA430012).

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