A Unitary Extension of Exotic Massive 3D Gravity from Bi-gravity

Mehmet Ozkan, Yi Pang, and Utku Zorba

1Department of Physics, Istanbul Technical University, Maslak 34469 Istanbul, Turkey
2Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, U.K

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We obtain a new 3D gravity model from two copies of parity-odd Einstein-Cartan theories. Using Hamiltonian analysis, we demonstrate that the only local degrees of freedom are two massive spin-2 modes. Unitarity of the model in anti-de Sitter and Minkowski backgrounds can be satisfied for vast choices of the parameters without fine-tuning. The recent “exotic massive 3D gravity” model arises as a limiting case of the new model. We also show that there exist trajectories on the parameter space of the new model which cross the boundary between unitary and non-unitary regions. At the crossing point, one massive graviton decouples resulting in a unitary model with just one bulk degree of freedom but two positive central charges at odds with usual expectation that the critical model has at least one vanishing central charge. Given the fact that a suitable non-relativistic version of bi-gravity has been used as an effective theory for gapped spin-2 fractional quantum Hall states, our model may have interesting applications in condensed matter physics.

Three dimensional gravity possesses many surprises that are not shared by its higher dimensional cousins. There exists a particular combination of linear and quadratic curvature terms such that the theory is perturbatively unitary and power-counting renormalizable about the Minkowski background [1]. At linearized level, this theory reproduces the Fierz-Pauli theory propagating a pair of massive gravitons. It is general coordinate invariant, differing from the background dependent de Rham-Gabadadze-Tolley massive gravity [2] and thus gaining its name “New Massive Gravity” (NMG). The maximally symmetric vacua of NMG contains also the anti-de Sitter (adS) space. Unitarity in adS background requires in addition to the ghost-free dynamical degrees of freedom, positivity of the central charges present in the asymptotic symmetry algebra. This is a requirement from the correspondence between quantum gravity in adS3 background and two-dimensional conformal field theory (CFT) in which the central charges in the gravity model are identified with those in the dual CFT. NMG however, is not a satisfactory model to apply the adS3/CFT2 correspondence, as the condition under which the massive gravitons are unitary implies negative central charges.

Here comes another surprise in 3D gravity that many interesting models can be reformulated using frame field formalism as a multi-flavor Chern-Simons (CS) like theory [3]. Different from usual CS-theories, CS-like theories can bear dynamical degrees of freedom, as the structure constants disobey Jacobi identity. The resolution of the bulk/boundary clash for NMG in adS3 becomes straightforward, once adopting the frame field formalism and loosing the requirement that the massive gravitons interact only through finite number of curvature terms. As a consequence, NMG is extended to the “Zwei-Dreibein Gravity” (ZDG) [4] and further generalizations [5] which are also parity-even. The main building blocks of ZDG are two copies of parity-even cosmological Einstein-Cartan theories glued together by a cubic potential involving the two dreibeins. Thus ZDG can be viewed as a variant of bi-gravity. By properly choosing parameters, ZDG can satisfy both criteria for a unitary adS3 quantum gravity, thus serving as an attractive toy model of lower dimensional quantum gravity defined via adS3/CFT2 correspondence.

Besides the more mathematically orientated goal of constructing a well-defined lower dimensional quantum gravity model, 3D spatially covariant bi-gravity has recently been utilized as a tool to build effective theories for the spin-2 gapped collective excitations observed in certain fractional quantum Hall states (FQHS) [6]. This latest application of bi-gravity in condensed matter physics is of particular interest, as it directly links two broad areas in physics, gravity and condensed matter physics.

NMG and its two-frame fields extension ZDG are standard gravity models in the sense that the same parity property is shared by the equations of motion and the Lagrangian. Interestingly there also exist “exotic” 3D gravity models of which the equations of motion are parity-even, while the actions are parity-odd. The recently proposed “Exotic Massive Gravity” (EMG) [9] is of just such a kind whose perturbative spectrum coincides with that of the NMG, see e.g. [10] for recent discussions on solutions of EMG and generalizations. Similar to NMG, the full physical spectrum of EMG contains either ghost-like massive gravitons or negative entropy Bañados-Teitelboim-Zanelli (BTZ) black holes. Simple extensions of EMG by adding parity violating terms in the Lagrangian turn out to be not sufficient to recover unitarity [9]. However, just as the ZDG example, there could exist a unitary extension of EMG using two-frame fields. As we will show in later sections, such a unitary extension (in both adS3 and Minkowski background) indeed exists. Moreover, we have also found a trajectory...
are all parity odd. However, the odd parity implies that the interaction terms (4) singles out a unique invertible As we will see from the Hamiltonian analysis below, \{\gamma_1, \gamma_2\} refers to the parity of the action. 

We begin our construction by introducing the basic fields in the frame-field formalism. We choose the same pattern. The model we start with takes the form of EMG as its kinetic terms are composed by two copies of parity-odd exotic Einstein-Cartan theories \[16, 17\] in which the role of the curvature term is played by gravitational CS term. Given the connection between bi-gravity and FQHS, it is conceivable that the new model obtained here together with the parity-even models paves the way for a fully covariant non-linear theory describing the bulk gapped collective spin-2 excitations present in a hidden sector of quantum Hall states exhibiting parity-violating pattern.

We now proceed to counting the degrees of freedom. There are in total 4 Lorentz vector valued one-forms in \[2\]. Their temporal components are the Lagrange multipliers, thus the physical phase space is 24-dimensional spanned by the spatial components of the one-forms. As the Lagrangian is first order in time-derivative, the 24 phase space variables already form 12 canonical pairs. Varying the Lagrangian with respect to the 12 temporal components, one obtains 12 primary constraints. The integrability conditions of the first order equations of motion give rise to new algebraic conditions on the fields. We find that upon imposing \(\beta_3 = 0\) and the invertibility of \(e_2^a\), the integrability conditions lead to only two secondary constraints 

\[e^3 e_{1a} e_{2}^a = 0,\]  
\[e^3 e_{2a} (\omega_{a}^{2} - \omega_{a}^{1}) = 0,\]  

and solutions to the Lagrangian multipliers associated with the temporal components of \(e_1^a, \omega_1^a - \omega_2^a\) for generic choice of the parameters. Using the procedure given in \[3\], one can check that amongst the 14 constraints, 6 of them are first class while the rest are second class. We are then left with a 24−12−8 = 4 dimensional phase space (per space point) indicating 2 degrees of freedom in the usual sense. It should also be mentioned that the action \[2\] is symmetric under \(\{e_1^a, \omega_1^a\} \leftrightarrow \{e_2^a, \omega_2^a\}\) modulating the parameters. Thus there is a totally equivalent choice by imposing \(\beta_3 = 0\) and invertibility of \(e_1^a\). From now on, we will focus on the model \[2\] with \(\beta_3 = 0\) while keeping other parameters generic.

The linearised spectrum about the maximally symmetric adS vacuum should be compatible with the results from non-perturbative Hamiltonian analysis, namely, the model is free of the Boulware-Deser ghost \[19, 20\] propagating only two physical massive spin-2 modes. The field equations can be readily derived from \[2\] from which we can read off the adS vacuum 

\[e_I = a_I \tilde{e}, \quad \omega_I = \tilde{\omega} + b_I \tilde{e},\]  

provided that the parameters satisfy the relation 

\[\alpha_1 = \frac{2(\beta_1 a_1^2 + \beta_4 a_2^2)}{\Lambda + b_1^2}, \quad \alpha_2 = \frac{2\beta_2 a_2^2}{\Lambda + b_2^2}, \quad \gamma_1 = \frac{-2b_1 \beta_1}{3a_1}, \quad \gamma_2 = \frac{-2(b_1 \beta_1 + b_2 \beta_2)}{3a_2},\]  

where \(\Lambda \equiv -1/\ell^2\) is the cosmological constant, \(\tilde{e}\) and \(\tilde{\omega}\) are the dreibein and spin connection of the unit-radius
\( \text{adS}_3 \text{ metric, and } a_t \text{ and } b_t \text{ are constants. Fluctuations about the adS}_3 \text{ vacuum are characterized by a small expansion parameter } \kappa \text{ as follows}
\)
\[
\omega_t = \omega + b_te + \kappa \nu_t, \quad e_t = a_t(e + \kappa k_t).
\]
\[
(9)
\]
Diagonalizing the quadratic action \([2]\) about adS vacuum is straightforward and results in
\[
\mathcal{L}^{(2)}_{\text{GEZDG}} = -\frac{K_-}{M} (\phi_- a D_\phi^a - M - \epsilon a b c e \phi_- D_\phi^c) + \frac{K_+}{M} (\phi_+ a D_\phi^a + M + \epsilon a b c e \phi_+ D_\phi^c) + a_+ (f_- a D_\phi^a + \ell^{-1} \epsilon a b c e \phi_- D_\phi^c) - a_- (f_- a D_\phi^a - \ell^{-1} \epsilon a b c e \phi_- D_\phi^c),
\]
\[
(10)
\]
where \( \{\phi_-, \phi_+\} \) form a pair of massive gravitons, and \( \{f_-, f_+\} \) are the usual massless gravitons. The coefficients in front of the kinetic terms read
\[
K_\pm = M_\pm^2 (b_1 \pm b_2)(a M_\pm^2 - 1) - a_\pm = \frac{\alpha_1 b_1 \ell + \alpha_2 b_2 \ell + \alpha_1 \pm \alpha_2}{\ell^2},
\]
\[
\Delta = (b_2 - b_1)(M^2 - M_\pm^2) \beta_2^2 \ell^2.
\]
\[
(11)
\]
The mass eigenvalues \( M_\pm \) can of course be solved in terms of the parameters in the Lagrangian. However, the expressions are not convenient for further analysis. Instead, we find that it is more handy to treat \( \{M_\pm, b_1, b_2, a_1, a_2, \beta_2, \ell\} \) as free parameters, recasting the original parameters \( \{\alpha_1, \alpha_2, \beta_1, \beta_3, \gamma_1, \gamma_2\} \) in the Lagrangian as functions of them using \([8]\) and the eigenvalue equation from which \( M_\pm \) is solved. In adS vacuum, no-tachyon and no-ghost conditions imply
\[
(\ell M_\pm^2) > 1, \quad K_\pm > 0.
\]
\[
(12)
\]
Choosing Brown-Henneaux boundary condition \([23]\) in adS\(_3\), the necessary condition for non-perturbative unitarity is the positivity of central charges appearing in the asymptotic Virasoro \( \oplus \) Virasoro symmetry algebra. The central charges can be computed from the CS-like action using the method given in \([15]\). The final results are
\[
c_\pm = \frac{3\ell}{2G} \left( b_1 \alpha_1 + b_2 \alpha_2 \pm \frac{1}{\ell} (\alpha_1 \pm \alpha_2) \right),
\]
\[
(13)
\]
which implies that
\[
c_+ = \frac{3\ell}{2G} a_+, \quad c_- = \frac{3\ell}{2G} a_-
\]
\[
(14)
\]
In terms of the new set of parameters, \( a_\pm \) take the form
\[
a_\pm = \frac{2(b_1 - b_2)(\ell M_- \pm 1)(\ell M_+ \mp 1)\beta_2^2 a_2}{(b_2 \ell \mp 1)\delta},
\]
\[
(15)
\]
\[
\delta = (b_1 - b_2)(1 - \ell^2 M_- M_+) + (M_- - M_+)(1 - \ell^2 b_1 b_2).
\]
The tachyon-free, ghost-free and \( c_\pm > 0 \) conditions can be simultaneously satisfied by various choices of the parameters. However, we will leave the systematic study for future work. Instead, we present one such region to stimulate more interesting in-coming discussion
\[
M_\pm \ell > 1, \quad M_- > M_+ +\ell, \quad b_1 > b_2\]
\[
b_1 + M_+ < 0, \quad \delta < 0, \quad \beta_2 > 0,
\]
\[
(16)
\]
where \( a_1 \) and \( a_2 \) are unconstrained. One representative from this unitary region takes the form
\[
M_- = 4, \quad M_+ = 2, \quad b_1 = -2.5, \quad b_2 = -15, \quad \beta_2 = 1, \quad a_1 = a_2 = 1.
\]
\[
(17)
\]
where various values are given in adS units with \( \ell = 1 \). The spectrum analysis about Minkowski vacuum can be obtained from the adS\(_3\) results by taking \( \ell \to \infty \lim \). As there is no BTZ black hole in Minkowski space, the unitarity condition becomes less stringent. Only \( K_\pm > 0 \) is required which can be easily satisfied.

The GEZDG \([2]\) model with \( \beta_3 = 0 \) is in fact related to the generalized EMG by taking a scaling limit in both the fields and the parameters. Specifically, in taking the scaling limit, we redefine the fields
\[
e_1 = e, \quad \omega_1 = \omega - \frac{1}{m^2} \ell, \quad e_2 = e + \frac{\lambda}{2} h, \quad \omega_2 = \omega,
\]
\[
(18)
\]
and the parameters
\[
\alpha_1 = \frac{\nu}{m^2}, \quad \alpha_2 = -1, \quad \beta_1 = \frac{\nu}{2} - \frac{1}{\lambda}, \quad \beta_2 = \frac{\nu}{2}, \quad \beta_4 = \frac{1}{\lambda}, \quad \gamma_1 + \gamma_2 = \frac{\nu m^2}{3\mu}.
\]
\[
(19)
\]
The \( \lambda \to 0 \) limit then reproduces the generalized EMG model \([9]\).

In fact, one can construct infinitely many trajectories connecting a point in the unitary region, such as the one given in \([16]\), to a tachyon-free generalized EMG model. One exemplary trajectory is exhibited here on which \( \ell = 1 \) and \( \{M_\pm, b_1, b_2, a_1, a_2, \beta_2\} \) are parametrized with the dependence on the flow parameter \( \lambda \)
\[
M_\pm = (1 - \lambda^2) M_\pm + \lambda^3 M_\pm^3, \quad b_1 = b_1^* \lambda^3 + (1 - \lambda^3)(b_2 - (\beta_2 \lambda + 1)(M_+ - M_-)), \quad b_2 = b_2^* \lambda^3 + (1 - \lambda^2)(M_+ - M_- + 1), \quad a_1 = a_1^*, \quad a_2 = a_1 (1 - \lambda^2) \left( 1 - \frac{\nu \lambda}{4} \right) + a_2^* \lambda^2, \quad \beta_2 = \beta_2^* \lambda^2 - \frac{\nu}{2} (1 - \lambda^2),
\]
\[
(20)
\]
so that near \( \lambda = 0 \), the leading behaviors of parameters above take the form \([19]\) up to terms linear or higher order in \( \lambda \). When \( \lambda = 1 \), the trajectory reaches the the unitary model defined by the parameters \([17]\). The starred parameters represent the values in a unitary region and \( M_\pm \) are adjustable parameters. In the \( \lambda \to 0 \) limit, we find that along the above trajectory, GEZDG reaches a
generalized EMG whose defining parameters \( \{\nu, m^2, \mu\} \) are expressed in terms of \( \{M_\pm, a_1\} \) as

\[
m^2 = \frac{(\hat{M}_- - 1)(\hat{M}_+ + 1)}{a_1^2}, \quad \mu = \frac{2(\hat{M}_- - 1)(\hat{M}_+ + 1)}{a_1(2\hat{M}_- - 2\hat{M}_+ - 1)},
\]

\[
\nu = -\frac{(\hat{M}_+ - \hat{M}_-)(\hat{M}_+ - \hat{M}_- + 2)}{a_1^2}.
\]  

We recall that the cosmological constant of the generalized EMG is given by [9]

\[
\hat{\Lambda} = -\left(\nu + \frac{m^4}{\mu^2}\right) = \frac{12(\hat{M}_+ - \hat{M}_-) - 1}{4a_1^2},
\]  

which is negative as long as we choose \( \hat{M}_+ > \hat{M}_- \). This means the limiting generalized EMG model admits adS vacuum about which the tachyon-free condition can also be satisfied if

\[
m^2|\mu| - m^2 \sqrt{-\hat{\Lambda} + |\mu|\hat{\Lambda}} > 0.
\]  

Upon setting \( \hat{M}_+ = \hat{M}_- - c \) with \( c > 0 \) and substituting in [21], the above condition becomes

\[
-4c\hat{M}_- - 8c + (2c + 1)\sqrt{12c + 1} + 4M_+^2 - 5 > 0,
\]  

which can be easily obeyed for large enough \( \hat{M}_- \). For instance, if \( c = 3 \), the required condition is satisfied for any value of \( M_- \). With these results in hand, we may now see explicitly how the central charges \( c_\pm \) and \( K_\pm \) change along the trajectory as \( \lambda \) runs between 0 and 1. In Fig. 1, Fig. 2 and Fig. 3, the parameters take the values given by [21] and

\[
\hat{M}_- = 12, \quad \hat{M}_+ = 9.
\]  

In Fig. 1, it is evident that central charges are smooth along the trajectory and are always positive. In Fig. 2, we plot \( K_- \) weighted by \( K_- \). The reason for this is that the magnitude of \( K_ \) becomes quite large near \( \lambda = 0 \) while their ratio is still modest. This ratio makes sense only when \( K_- \) stays positive along the trajectory which we have also confirmed. We also notice that \( K_+ \) crosses zero at \( \lambda \sim 0.85 \). At this point, \( K_+ \) and \( c_+ \) are still positive. This is very different from usual intuition that the unitary and non-unitary models are separated by critical points at which one of the central charges vanishes [15]. This intriguing feature guides us to take a closer look at the crossing point on the trajectory.

At \( \lambda \sim 0.85 \), we checked numerically that \( \beta_1 \) and \( \gamma_1 \) are both 0 as they are correlated via relation given in [8]. If setting \( \beta_1 = \gamma_1 = 0 \) in the GEZDG action with \( \beta_3 = 0 \), we see that \( \epsilon_{1a} \) decouples and the resulting theory is a three-flavor model given by

\[
\mathcal{L}_{3f} = \alpha_1 L_{CS}(\omega_1) + \alpha_2 L_{CS}(\omega_2) + \beta_2\epsilon_{2a} T_2^a(\omega_2) + \beta_4\epsilon_{2a} T_2^a(\omega_1) + \gamma_{abc}\epsilon_{2a} \epsilon_{2b} \epsilon_{2c}.
\]  

\[
\epsilon^{ij} \epsilon_{2i} (\omega_{2j} \gamma_{1}) = 0,
\]  

indicating that the model propagates a single massive graviton. We have also checked that the new three-flavor model is inequivalent to “Topologically Massive Gravity” [21] or “Minimal Massive Gravity” [22]. The adS vacuum is now given by

\[
\epsilon_2 = \alpha_2 \epsilon, \quad \omega_1 = \omega + b_1 \epsilon, \quad \omega_2 = \omega + b_2 \epsilon,
\]  

together with parameter relations similar to those in [8].
upon setting $\beta_1 = \gamma_1 = 0$. The quadratic action for fluctuations about the adS$_3$ background takes the form
\begin{align}
L_{3f}^{(2)} = -\frac{A}{M} (\phi_a D_\phi f^a_+ - M \epsilon_{abc} \phi_b \phi_c) + a_+ (f_+ D_\ell f^+ - \epsilon^{-1} \epsilon_{abc} f^a_+ f^b_+) \\
- a_- (f_- D_\ell f^- - \epsilon^{-1} \epsilon_{abc} f^a_- f^b_-),
\end{align}
where $a_\pm$ are still related to central charges via (14). In terms of the new parameters $\{M, b_1, b_2, a_2, \beta_2, \ell\}$, $a_\pm$ and $A$ are given as
\begin{align}
a_\pm &= \frac{2a_2^2 b_2 (b_2 - b_1) (M \ell \pm 1)}{(b_2 \ell \mp 1)(b_1 \ell \pm 1)(b_2 + M)}, \\
A &= M (b_1 - b_2)(b_1 + M)(\ell^2 M^2 - 1). \tag{30}
\end{align}
The bulk/boundary unitarity is achieved when
\begin{equation}
A > 0, \quad a_\pm > 0, \tag{31}
\end{equation}
which can be satisfied for various choices of parameters. In adS units, one example is given for the choice of parameters below
\begin{equation}
b_1 = -3, \quad b_2 = -12, \quad M = 5, \quad \beta_2 = a_2 = 1. \tag{32}
\end{equation}

In this letter we report a new unitary four-flavor CS-like model obtained from merging together two copies of parity-odd exotic Einstein-Cartan theories. The generalized EMG model appears as a limiting case of the new model. Regarding the connection between bi-gravity and effective theory for the gapped spin-2 FQHS, the exotic nature of the new model may describe certain novel phenomenon in fractional quantum Hall effects once a non-relativistic limit is properly taken. From the effective field theory point of view, it is crucial to understand the boundary on the parameter space separating the unitary models from the non-unitary ones. Therefore it should be interesting to carry out a systematic study of all unitary four-flavor CS-like theories with bi-gravity origin. Besides the standard and exotic four-flavor CS-like theories, there exists a third kind of mixed nature by coupling a parity-even theory to a parity-odd one. Unitary models of this type have not been investigated up to date. Finally, incorporating supersymmetry in the CS-like theory is also interesting. It has been proposed that global supersymmetry may emerge in certain condensed matter systems, e.g. [24]. If in certain FQHS emergent local supersymmetry can be realized, its effective theory must be a supersymmetric bi-gravity model, thereby revitalizing supergravity in a new arena.

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