Two–dimensional Conformal Sigma Models
and Exact String Solutions

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Abstract
We discuss two classes of exact (in $\alpha'$) string solutions described
by conformal sigma models. They can be viewed as two possibilities
of constructing a conformal model out of the non-conformal one
based on the metric of a $D$-dimensional homogeneous $G/H$ space.
The first possibility is to introduce two extra dimensions (one space-like and
one time-like) and to impose the null Killing symmetry condition
on the resulting $2+D$ dimensional metric. In the case when the “transverse”
model is $n = 2$ supersymmetric and the $G/H$ space is Kähler-Einstein the
resulting metric-dilaton background can be found explicitly. The second
possibility – which is realised in the sigma models corresponding
to $G/H$ conformal theories – is to deform the metric, introducing at the
same time a non-trivial dilaton and antisymmetric tensor backgrounds.
The expressions for the metric and dilaton in this case are derived
using the operator approach in which one identifies the equations for
marginal operators of conformal theory with the linearised (near a background)
expressions for the sigma model ‘$\beta$-functions’. Equivalent results are
then reproduced by the direct field-theoretical approach based on
computing first the effective action of the $G/H$ gauged WZNW model
and then solving for the $2d$ gauge field. Both the bosonic and the
supersymmetric cases are discussed.

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1. Introduction

As is well known, solutions of string field equations are usually represented in terms of conformal invariant 2\textit{d} theories\textsuperscript{1}. Interpreting string theory as a theory of quantum gravity one needs to study solutions which have a curved space-time part, i.e. which should be described by conformal theories with Minkowski signature of a target space. It is important to be able to go beyond the leading orders of expansion in \(\alpha'\) (the leading order solutions may not correctly describe the behaviour in the strong curvature regions near singularities, etc.). Since the string effective equations contain terms of arbitrarily high order in \(\alpha'\) their solutions should in general be non-trivial functions of \(\alpha'\). Given that the exact form of the string effective action is not known, it may seem unlikely that such solutions can be found directly in the perturbative sigma model framework. It is often assumed that in contrast to conformal field theory methods, the sigma model approach is useful only for obtaining perturbative solutions in first leading orders in \(\alpha'\).

Let us call ‘conformal sigma model’ (c.s.m.) a 2\textit{d} field theory with couplings that solve the ‘\(\beta\)-function’ conditions of sigma model conformal invariance, reserving the name ‘conformal field theory’ (c.f.t.) for a theory constructed using the operator algebra (conformal bootstrap) methods (in which case one usually knows not only the conformal point but also the spectrum of operators or perturbations). The correspondence between c.s.m.’s and c.f.t.’s is far from being one-to-one and is poorly understood in general. Taking any solution of the one-loop sigma model conformal invariance conditions, one can in principle ‘deform’ it order by order in \(\alpha'\) to get a c.s.m. for which in most cases the corresponding c.f.t. will be unknown or may not even exist (e.g. if the \(\alpha'\)-series does not converge, etc.). There are also explicitly known exact (all order in \(\alpha'\)) solutions of the conformal invariance conditions (in particular, with Minkowski signature) for which the existence of their c.f.t. counterparts is unclear. On the other hand, there are many c.f.t.’s which apparently do not correspond to any weak coupling c.s.m., i.e. do not have an obvious space-time interpretation. Ideally, one would like to have both the c.s.m. and c.f.t. descriptions of a string solution to have an obvious space-time interpretation as well as to know the spectrum of states and to be able to compute their scattering amplitudes on a given background.

We would like to describe here some recent work on establishing exact string solutions directly in 2\textit{d} field theory or sigma model framework. In Sec.2 we shall briefly review a class of solutions with Minkowski signature which generalise an arbitrary non-conformal sigma model in \(N\) dimensions to a conformal one in \(D = N + 2\) dimensions by replacing the ‘running’ couplings of \(N\)-dimensional theory by functions of a light-cone coordinate in \(N + 2\) dimensions\textsuperscript{2}. In particular, the explicit form of these exact solutions can be found in the supersymmetric case\textsuperscript{3}. Which conformal theories correspond to these solutions remains an open question.

In the rest of the paper we shall consider a large and important class of (Euclidean or Minkowski) string solutions for which the conformal field theory description is well known: we shall present a general construction of conformal sigma models corresponding to coset \(G/H\) conformal theories. Since coset conformal field theories\textsuperscript{4} can be represented at the Lagrangian level in terms of gauged WZNW models\textsuperscript{5} one can try to obtain a sigma model description by starting with a gauged WZNW model, integrating out the 2\textit{d} gauge field and fixing a gauge as was first discussed for the \(SL(2, R)/U(1)\) model in ref.6. This gives, however, only the leading order form of the sigma model action since the quantum \((1/k\) or \(\alpha'\)) corrections are implicitly ignored. In fact, one does find \(\alpha'\)-corrections to the leading order background\textsuperscript{7,6} by explicitly solving the ‘\(\beta\)-function’ equations\textsuperscript{8}. The exact form\textsuperscript{9}
of the background metric and dilaton can be inferred indirectly by using the ‘operator approach’ (i.e. by comparing the structure of the $L_0$-operator of the corresponding coset conformal theory with that of a Klein-Gordon equation in a background) and turns out to be consistent with the $\alpha'$-perturbation theory for the sigma model$^9,10$. This $SL(2, R)/U(1)$ solution$^9$ represents the first explicitly known example of an exact string solution which non-trivially depends on $\alpha'$.* The exact form of some more general ($D > 2$) non-trivial $G/H$ backgrounds was found in the operator approach in ref.13. We shall describe a direct field-theoretical approach to derivation of such solutions in the general $G/H$ case$^{14,15,16}$.

The main idea of this approach$^{14}$ is to first find the quantum effective action of the gauged WZNW theory and then eliminate the 2$d$ gauge field. After the resulting non-local action is identified with the effective action of the corresponding sigma model, the sigma model couplings can be determined by dropping out all the non-local terms$^{16}$. Our presentation will follow mainly ref.16 (especially in Secs.4 and 5), where more details and references can be found. In Sec.3 we shall discuss, from a rather general point of view, the operator approach to derivation of the exact background fields corresponding to a conformal theory based on an affine-Virasoro construction. In Sec.4 we shall present the expressions for the effective action in the gauged WZNW theory (in both bosonic and supersymmetric cases). The derivation of the sigma model couplings corresponding to the $G/H$ gauged WZNW theory will be given in Sec.5. In Sec.6 we shall establish the equivalence between the results obtained in the operator and the sigma model approaches and make some concluding remarks.

2. Conformal Sigma Models with Null Killing Vector

Consider a non-conformal sigma model in $N$ dimensions with a metric $g_{ij}$. One possible strategy of constructing a conformal model out of it could be to couple it to a 2$d$ gravity with the conformal factor of the 2$d$ metric becoming an extra $N + 1$ coordinate$^{17}$. The resulting conformal invariance equations are, however, second order in the new coordinate and their solution is ambiguous (an ‘initial value’ problem is not well defined). A much simpler and better defined construction of a (Minkowski) conformal sigma model from a (Euclidean) non-conformal one$^2,3$ is based on adding two extra coordinates (say, light-cone coordinates $u$ and $v$), at the same time imposing the condition of $v$-independence. The latter condition (more precisely, the condition of existence of a covariantly constant null Killing vector) can be thought of as effectively reducing the dimension to $N + 1$. The conformal invariance conditions in $2 + N$ dimensions then take the form of the standard first order RG equations in $N$-dimensional theory with $u$ playing the role of the RG time.

The general $D = N + 2$ dimensional Minkowski signature metric admitting a covariantly constant null Killing vector can be represented in the form

$$ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = -2 du dv + g_{ij}(u,x)dx^i dx^j ,$$

where $\mu$ and $\nu = 0, 1, ..., N + 1$, $i,j = 1, ..., N$. To establish the UV finiteness of the corresponding sigma model on a flat 2$d$ background one should check that there exists a

* As was recently observed$^{11}$ (see also refs.12,13) the causal structure of the exact solution is very different from that of the leading order one$^6$, in particular, it does not have the ‘black hole’ singularity. The exact solution in the supersymmetric case, however, still has the leading order ‘black hole’ form$^{10,13}$. 

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vector $M_\mu$ such that the $\beta$-function for the target space metric $G_{\mu\nu} = \hat{g}_{\mu\nu}$ (1) vanishes up to a $M_\mu$-reparametrisation term

$$\beta^G_{\mu\nu} + 2D(\mu M_\nu) = 0 .$$  \hfill (2)

It can be proved that (2) is indeed satisfied for such a $g_{ij}(x, u)$ as a function of $u$ that

$$\frac{dg_{ij}}{du} = \beta^g_{ij} ,$$  \hfill (3)

where $\beta^g_{ij}$ is the $\beta$-function of the ‘transverse’ ($N$-dimensional) theory. Namely, the following statement is true: if the metric $g_{ij}(x)$ depends on $u$ in such a way that it satisfies the standard RG equation of the $N$-dimensional sigma model then the $(N + 2)$-dimensional sigma model based on (1) is UV-finite to all orders of the loop expansion. This statement has a straightforward generalisation in the presence of non-trivial dilaton, i.e. for solutions of conformal invariance conditions on a curved 2d background: the metric (1) and the dilaton

$$\phi = v + \phi(u, x) ,$$  \hfill (4)

represent a solution of conformal invariance conditions in $N + 2$ dimensions if $g_{ij}(u, x)$ and $\phi(u, x)$ solve the first-order RG equations of the ‘transverse’ theory. Given a set of couplings $(g_{ij}(x), \phi(x))$ of a non-conformal theory in $N$ dimensions, one obtains a conformal theory in $N + 2$ dimensions by solving the RG equations in $N$ dimensions and replacing the RG parameter by the light cone coordinate $u$.

The metric (and dilaton) in $N + 2$ dimensions is thus determined by the $\beta$-function of the transverse theory. The explicit all-order expression for the latter is not known in bosonic sigma models. On the other hand, there are examples of $n = 2$ supersymmetric ($n$ is the number of $2d$ supersymmetries) sigma models with homogeneous Einstein-Kähler target spaces for which the exact $\beta$-function coincides with its one-loop expression\textsuperscript{18}. The above general statements have direct analogues in the case of supersymmetric sigma models\textsuperscript{3}.

For the metric with the null Killing vector (1) the action of the 2-dimensional ($n = 1$) supersymmetric sigma model can be represented in terms of the real superfields $U$, $V$ and $X^i$

$$I = \frac{1}{4\pi\alpha'} \int d^2zd^2\theta [-2DU\bar{D}V + g_{ij}(U, X)X^i\bar{X}^j] .$$  \hfill (5)

The solution $g_{ij}(u, x)$ of the finiteness condition is determined by the $\beta$-function of the ‘transverse’ part of (5), i.e. of the supersymmetric model with the metric $g_{ij}(u, x)$ for constant $u$. As is well known\textsuperscript{19}, if the transverse space is Kähler then the $N$-dimensional model is $n = 2$ supersymmetric. If it is also a compact homogeneous Einstein space (e.g. $S^2 = SO(3)/SO(2)$ or $CP^m$) then it is very plausible that its $\beta$-function is exactly calculable and is given by the one-loop expression\textsuperscript{18}. This was actually proved in ref.18 for the following classes of Einstein-Kähler manifolds: $M_1 = SO(m+2)/SO(m) \times SO(2)$, $M_2 = SU(m+k)/SU(m) \times SU(k) \times U(1)$, $M_3 = Sp(m)/SU(m) \times U(1)$, $M_4 = SO(2m)/SU(m) \times SO(2)$, $M_5 = SU(m+1)/(U(1))^m$. In that case the transverse part of the metric, the $\beta$-function and the solution of the RG equation are given simply by

$$g_{ij}(u, x) = f(u)\gamma_{ij}(x) , \quad \beta(f) = a , \quad f(u) = au .$$  \hfill (6)
The constant \( a > 0 \) is determined by the geometry of the transverse space \((a_1 = m, \text{ etc.})\) and can be absorbed into a redefinition of the coordinates \( u \) and \( v \). Then the final expression for the Minkowski signature metric of the finite \( 2 + N \)-dimensional supersymmetric sigma model is (we are assuming \( u > 0 \))*

\[
ds^2 = -2dudv + u\gamma_{ij}(x)dx^i dx^j .
\] (7)

As a reflection of IR singularity of the coupling of the transverse theory this metric has a curvature singularity at \( u = 0 \).

The simplest non-trivial example of the finite models we have constructed corresponds to the case when the transverse theory is represented by the \( O(3) \) supersymmetric sigma model\(^{20}\). The resulting metric is that of \( \text{four} \ (N = 2) \) dimensional space with the transverse part being proportional to the metric on \( S^2 \),

\[
ds^2 = -2dudv + (d\theta^2 + \sin^2 \theta d\phi^2) .
\] (8)

This space is conformal to the direct product of two-dimensional Minkowski space and two-sphere. To get a solution of the conformal invariance conditions one should supplement the metric by the following expression for the dilaton

\[
\phi(v, u) = \phi_0 + v + qu + \frac{1}{8}N \ln u , \quad q = \text{const} .
\]

The resulting backgrounds represent exact solutions of superstring effective equations with non-trivial dilaton\(^3\). It is interesting to note that the string coupling \( e^\phi = Au^{N/8} e^{(qu + pv)} \) goes to zero in the strong coupling region \( u \to 0 \) of the transverse sigma model, i.e. is small near the singularity \( u = 0 \).

3. Operator Approach to Derivation of Background Fields Corresponding to Coset Conformal Theories

A possible strategy of determining the geometry corresponding to a given conformal theory is to try to interpret the Virasoro condition \((L_0 + \bar{L}_0 - 2)F = 0\) on states as linear field equations in some background and to extract the expressions for background fields from the explicit form of the differential operators involved. The idea is that marginal operators \( F \) of conformal theory serve as ‘probes’ of geometry, so that one may be able to extract the corresponding metric, etc from their equations just as from geodesic equations or field equations in a curved space. In order to implement this program one is to make a number of important assumptions. First, one should specify which configuration (‘target’) space \( M \) (with coordinates \( x^\mu , \mu = 1, \ldots , D \)) should be used, so that \( F \) will be parametrised by fields on \( M \), and \( L_0 \) acting on \( F \) will reduce to differential operators on \( M \). Next, one should understand how to represent the resulting equations in terms of background fields. The expectation is that the conformal theory should correspond to a sigma model

\[
S = \frac{1}{4\pi\alpha'} \int d^2z \sqrt{\gamma} [\partial_m x^\mu \partial^m x^\nu G_{\mu\nu}(x) + i\epsilon^{mn} \partial_m x^\mu \partial_n x^\nu B_{\mu\nu}(x) .
\]

* Note that while the transverse model (with fixed constant \( u \)) is \( n = 2 \) supersymmetric, the full \((N + 2)\)-dimensional model apparently has only \( n = 1 \) supersymmetry.
+α′Rφ(x) + T(x) + ...]. \tag{9}

One is to invoke the knowledge of the structure of the sigma model conformal anomaly coefficients (‘β-functions’), or the effective action which generates them

\[ S = \int d^D x \sqrt{G} e^{-2φ} \left( \frac{2}{3} (D - 26) - α'[R + 4(∂μφ)^2 - \frac{1}{12} H^2_{\lambda\mu\nu}] + \frac{1}{16}[α'(∂μT)^2 - 4T^2] + ... \right). \tag{10} \]

The idea is start with this background-independent action, linearise the corresponding equations near an arbitrary background, and compare them with the equations for the corresponding states in conformal theory. The equations for the tachyon, graviton, dilaton and the antisymmetric tensor perturbations (\(t = T - T_*\), \(h = G - G_*\), \(ϕ = φ - φ_*\), \(b = B - B_*\); in what follows we shall omit the superscript * indicating background fields) take the following symbolic form (\(α' = 1\))

\[ (-Δ + 2G^{μν}∂_μφ∂_ν)t - 4t + ... = 0 , \quad Δ ≡ \frac{1}{\sqrt{G}}∂_μ(\sqrt{G}G^{μν}∂_ν), \tag{11} \]

\[ (-Δ + 2G^{μν}∂_μφ∂_ν)h + Rh + H∂b + ... = 0 , \tag{12} \]

\[ (-Δ + 2G^{μν}∂_μφ∂_ν)ϕ + H∂b + R∂^2h + ... = 0 , \tag{13} \]

\[ (-Δ + 2G^{μν}∂_μφ∂_ν)b + H∂h + ... = 0 . \tag{14} \]

Given a second order differential equation which follows from the \(L_0\)-condition for the lowest scalar ‘tachyonic’ state it should be possible to determine the corresponding background metric and dilaton by looking at the coefficients of the terms which are second and first order in derivatives and comparing them with (11). In an attempt to determine the antisymmetric tensor field strength one should compare the first-derivative terms in the equations for ‘massless’ perturbations with the corresponding terms in (12),(13),(14).

It should be emphasised that if this approach works at all, its consistency should not be too surprising. If the correspondence between a conformal sigma model and a conformal field theory exists in a given case, then solutions of the conformal invariance conditions that follow from (10) should represent the conformal point; their perturbations should correspond to marginal perturbations of a conformal theory. If there exists a sigma model (9) behind a given conformal field theory, then the equations for marginal perturbations must have the form of (11)–(13).

Let us now specialise to a large class of conformal theories based on generalised affine–Virasoro construction\(^4,21,22,23\). Here one starts with a finite-dimensional Lie group \(G\) and defines a holomorphic stress tensor of a conformal theory by a Sugawara-type relation

\[ T_{zz} = C^{AB} : J_A(z)J_B(z) : , \tag{15} \]

where \(J_A(z)\) are the generators of the affine (current) algebra determined by the structure constants \(f_{BC}^A\) and the Killing metric \(η_{AB}\) of \(G\) (with the central term proportional to \(κ_{AB} = kη_{AB}\)). The condition that \(T_{zz}\) should satisfy the Virasoro algebra imposes a ‘master equation’\(^21\) on the symmetric coefficients \(C^{AB}\)

\[ C^{AB} = 2C^{AC}κ_{CD}C^{DB} - C^{CD}C^{KL}f^A_{CK}f^B_{DL} - C^{CD}f^K_{CL}f^L_{DK}C^{B}L \tag{16} \]
(the central charge of the Virasoro algebra is $C = 2\kappa_{AB}C^{AB}$). The standard Sugawara-GKO $G/H$ coset conformal theory$^4$ corresponds to a particular solution of (16)

$$C^{AB} = \frac{1}{k + \frac{1}{2}c_G} \eta^{AB} - \frac{1}{k + \frac{1}{2}c_H} \eta^{AB}_H.$$  \hspace{1cm} (17)

Here $f^{ACD}f^B_{CD} = c_G \eta^{AB}$, $f^{abcd}f^e_{cd} = c_H \eta^{ab}$; $A, B, \ldots = 1, \ldots, \dim G = D_G$; $a, b, \ldots = 1, \ldots, \dim H = D_H$ and $\eta^{AB}_H$ denotes the projector on the Lie algebra of $H$.

In general, unless solutions of (16) have non-trivial extra symmetries (commuting operators), the only natural choice for a configuration space is the group space $G$ (with coordinates $x^M$)

$$J_A = E^M_A(x)\partial_M, \quad J_A = \bar{E}^M_A(x)\partial_M,$$ \hspace{1cm} (18)

$$G^{0MN} = \eta_{AB}E^A_ME^B_N = \eta_{AB}\bar{E}^A_M\bar{E}^B_N,$$ \hspace{1cm} (19)

$$[J_A, J_B] = f^C_{AB}J_C, \quad [J_A, \bar{J}_B] = f^C_{BA}\bar{J}_C, \quad [J_A, \bar{J}_B] = 0,$$ \hspace{1cm} (20)

$$(E^M_A \text{ and } \bar{E}^M_A \text{ are the left-invariant and right-invariant vielbeins on } G; \text{ indices are raised and lowered with } \eta_{AB} \text{ and } G_{0MN}) \text{ we get from the zero mode part of } (L_0 + \bar{L}_0 - 2)F = 0 \text{ the following equation for the lowest scalar state } t(x)$$

$$[-(G^{MN}\partial_M\partial_N + G^N\partial_N) - 2]t(x) = 0,$$ \hspace{1cm} (21)

Eq.(20) becomes equivalent to the sigma model equation (11) if there exists a $\phi$ such that

$$G^N = G^{MN}\partial_M \ln (\sqrt{G} \mathrm{e}^{-2\phi}).$$

In fact, such $\phi$ can be found explicitly by using the properties of $E^A_M$, or by observing that that since $J^A$ and $\bar{J}^A$ are anti-Hermitian with respect to the invariant scalar product on the group defined by $G_0$, $(f, g) = \int d^D x \sqrt{G_0} f^*(x)g(x)$, one has

$$\partial_ME^M_A = -E^M_A\partial_M \ln \sqrt{G_0}, \quad E^A_M\partial_ME^M_A = -E^M_A\partial_ME^A_N,$$

$$\partial_ME^A_N - \partial_NE^A_M = f^A_{BC}E^B_ME^C_N.$$

As a result,

$$\phi = \frac{1}{2} \ln \sqrt{G \over G_0}, \quad \text{i.e. } \sqrt{G} \mathrm{e}^{-2\phi} = \sqrt{G_0}.$$ \hspace{1cm} (22)

This result becomes obvious if one compares the effective action leading to (11) ($\int \sqrt{G} \mathrm{e}^{-2\phi}G^{MN}\partial_Mt\partial_Nt$) with the ‘expectation value’ $(t, Ht)$ of the zero-mode ‘Hamiltonian’ $H = \frac{1}{2}C^{AB}(J_A\bar{J}_B + J_A\bar{J}_B)$ and uses the anti-hermiticity of currents. The dilaton’s role is to compensate for the fact that the two scalar products have different measures.
It is clear that the dilaton field is non-trivial because the metric \( G_{MN} \) is, in general, different from the canonical Killing metric \( G_{0MN} \) on \( G \). A ‘deformation’ of the metric is directly related to the conformal invariance (Virasoro) condition (16). If there exists the corresponding Lorentz-invariant sigma model it should also contain the antisymmetric tensor field coupling (cf. ref. 24).

There seems to exist an interesting connection between algebraic and geometrical aspects of such conformal theories (and corresponding string solutions). The geometry is determined by the choice of the group and a choice of particular solution of the ‘master equation’. The question about a relation between group-theoretic and geometrical aspects of a similar construction was raised independently in the quantum mechanical context in refs. 22 and 24. We see that once the condition of conformal invariance is satisfied the geometry that appears is that of the corresponding string solutions described by conformal sigma models.

Let us now specialise to the case of the \( G/H \) coset conformal theory with \( C^{AB} \) given by (17) where the main assumption of existence of the sigma model description is satisfied given the existence of the Lagrangian formulation in terms of gauged WZNW models and since this assumption can be checked in the semiclassical approximation (see also refs. 9,13). In this case there exists an extra symmetry which makes it possible to subject the states to the \( H \)-invariance condition \((J_a - \bar{J}_a)F = 0\). In particular,

\[
(J_a - \bar{J}_a)t = 0, \quad Z^M_a \partial t = 0, \quad Z^M_a \equiv E^M_a - \tilde{E}^M_a. \tag{23}
\]

As a result, \( t \) can be restricted to depend only on \( D = D_G - D_H \) coordinates \( x^\mu \) of the coset space \( G/H \) which will thus play the role of the configuration space of the corresponding sigma model. The presence of the constraint (23) implies that the metric we will get from (20),(21) will be the ‘projected’ one. Let us define the projection operator on the subspace orthogonal (with respect to \( G_{0MN} \)) to \( Z^M_a \)

\[
\Pi^N_M \equiv \delta^N_M - Z^N_a (ZZ)^{-1ab} Z^M_b, \quad (ZZ)_{ab} = G_{0MN} Z^M_a Z^N_b, \quad \Pi^2 = \Pi. \tag{24}
\]

Then

\[
G^{MN} = \Pi^M_K \hat{G}^{KL} \Pi^L_N, \quad \hat{G}^{MN} = \frac{1}{k + \frac{1}{2}c_G} \eta^{AB} E^M_A E^N_B - \frac{1}{k + \frac{1}{2}c_H} \eta^{ab} E^M_a E^N_b, \tag{25}
\]

\[
\hat{G}^{MN} = \frac{1}{k + \frac{1}{2}c_G} (E^M_A E^{AN} - \gamma E^M_a E^{aN}) = \frac{1}{k + \frac{1}{2}c_G} [E^M_i E^{iN} - (\gamma - 1) E^M_a E^{aN}], \tag{26}
\]

\[
\gamma = \frac{k + \frac{1}{2}c_G}{k + \frac{1}{2}c_H}, \quad \gamma - 1 = \frac{c_G - c_H}{2(k + \frac{1}{2}c_H)}. \tag{27}
\]
\[ G_{\mu\nu} = \frac{1}{k + \frac{1}{2}c_G}(E^\mu A^\nu - \gamma E^{\mu a} E_a^\nu), \quad E_A^\mu \equiv H_M^\mu E_A^M. \] (29)

In the simplest case \( H_M^\mu = \delta_M^\mu \). More generally, one can choose any set of vectors \( H_M^\mu \) which are orthogonal to \( Z_a^M \). As a result, we will get again eqs. (20)–(22) with the tensor indices \( M, N, \ldots \), restricted to \( G/H \), i.e. replaced by \( \mu, \nu, \ldots \). Since in the present case (under the constraint) the operators \( J_A \) are anti-Hermitian with respect to the invariant metric on the coset

\[ G_{\mu\nu}^{(0)} = \eta_{ij} E_i^\mu E_j^\nu, \] (30)

we find as in (22) (cf. ref.22)

\[ \phi = \frac{1}{2} \ln \sqrt{G/G^{(0)}}. \] (31)

We conclude that the non-triviality of the dilaton can be attributed to the fact that the metric \( G_{\mu\nu} \) is a ‘deformed’ one, i.e. different from the standard metric on the coset. We have thus proved that the combination of the metric and dilaton in (22) is equal to the determinant of \( G^{(0)} \) (and, in particular, is \( k \)-independent \(^{13,15,25} \)).

It should certainly be possible also to compute the antisymmetric tensor background by comparing the equation for a massless \((1,1)\) state with (12)–(14). We shall find the antisymmetric tensor in Sec.5 by using the direct field-theoretical approach where one determines the sigma model action from the gauged WZNW theory (which provides a Lagrangian formulation of the coset conformal theory). In Secs.5,6 we shall reproduce the above expressions for the metric and dilaton and also, motivated by the field-theoretical derivation, will understand how to put them into a more explicit form.

4. Effective Action in Gauged WZNW theory

The classical gauged WZNW action

\[ S = kI(g, A), \quad I(g, A) = I(g) + \frac{1}{\pi} \int d^2z \ Tr \left( -A \partial g g^{-1} + \tilde{A} g^{-1} \partial g + g^{-1} A g \tilde{A} - A \tilde{A} \right), \] (32)

\[ I(g) = \frac{1}{2\pi} \int d^2z \ Tr (\partial g^{-1} \partial g) + \frac{i}{12\pi} \int d^3z \ Tr (g^{-1} dg)^3, \] (33)

is invariant under the standard vector \( H \)-gauge transformations (with the generator corresponding to (23))

\[ g \rightarrow u^{-1} gu, \quad A \rightarrow u^{-1}(A + \partial)u, \quad \tilde{A} \rightarrow u^{-1}(\tilde{A} + \bar{\partial})u, \quad u = u(z, \bar{z}). \]

The 2\(d\) gauge field (with values in the adjoint representation of a subgroup \( H \)) at the classical level plays the role of a Lagrange multiplier, which sets the \( H \)-components of the currents to zero. To obtain a sigma model action for gauge-invariant degrees of freedom out of (32) one needs to fix a gauge and to integrate out the non-propagating ‘Lagrange multiplier’ degrees of freedom \( A_m \). This is straightforward to do at the semiclassical level\(^8\). However, integrating out the gauge field \( A_m \) while keeping \( g \) as a background breaks down conformal invariance starting at two loops\(^8\). To preserve conformal invariance (which is manifest in the underlying coset conformal theory) one must treat \( A_m \) and \( g \) on an equal footing at the quantum level.
The idea of how to go beyond the semiclassical approximation is to derive first the quantum effective action for the theory (32), solve for the gauge field (and fix a gauge) and identify the result with the effective action of the underlying sigma model\textsuperscript{14}. The classical sigma model action (i.e. its couplings) are then found by separating the local (second-derivative) part of the effective action. Since the quantum theory of (32) can be formulated in an exactly conformally invariant way, the corresponding sigma model couplings should also represent an exact solution of the sigma model conformal invariance conditions.

As in the case of the partition function\textsuperscript{5}, the effective action of a gauged $G/H$ WZNW theory can be represented in terms of the effective actions of the ungauged theories for the group $G$ and subgroup $H$. Defining the effective action in the WZNW theory by
\begin{equation}
e^{-\Gamma(g)} = \int [d\tilde{g}] e^{-S(\tilde{g})} \delta[\bar{J}(\tilde{g}) - \bar{J}(g)] \ , \quad \bar{J} \equiv \bar{\partial}gg^{-1},
\end{equation}

one finds\textsuperscript{14,16}
\begin{equation}
\Gamma(g) = (k + \frac{1}{2} c_G) I(g') \ , \quad \bar{\partial}g'g'^{-1} = \frac{k}{k + \frac{1}{2} c_G} \bar{\partial}gg^{-1} ,
\end{equation}
i.e. up to the field renormalisation (which makes $\Gamma(g)$ non-local) the effective action is given by the classical WZNW action with the shifted $k$. This action has the right symmetries (conformal and chiral $G \times G$ invariance) one would like to preserve at the quantum level. Note that (35) coincides with $I(g)$ in the classical limit and is different from the usual Legendre transform of the generating functional $W$ of the currents (which contains the unshifted $k$ and is given by the classical contribution\textsuperscript{29}). The functional $\Gamma$ can be considered as a ‘quantum’ Legendre transform of $W$ in which almost all of quantum corrections except the ‘one-loop’ determinant (providing the shifts of $k$ in (35)) are absent (cf. ref.29).

Although it is not obvious that the functional (34) is equivalent to the standard generating functional of 1-PI correlators of the field $g$ itself, the resulting action (35) is perfectly consistent with the presence of the shifted $k$ in the quantum equations of motion and the stress tensor (cf.(15),(17)) in the operator approach to WZNW model as conformal theory\textsuperscript{27},
\begin{align*}
(k + \frac{1}{2} c_G)\partial g(z, \bar{z}) &= : J_A(z)g(z, \bar{z})T^A : , \\
(k + \frac{1}{2} c_G)\bar{\partial}g(z, \bar{z}) &= : J_A(\bar{z})T^A g(z, \bar{z}) : , \\
T_{zz} &= \frac{1}{k + \frac{1}{2} c_G} \eta^{AB} : J_A(z)J_B(z) : .
\end{align*}
The action (35) can be considered as a ‘classical’ representation of these quantum relations with the normal ordering suppressed (note that the stress tensor should be given by the variation of the effective action over the 2d metric and that the field renormalisation in (35) is actually the renormalisation of the current). An action with the same shift of $k$ as in (35) (also originating from a determinant) was discussed in ref.28 in connection with the free-field representation of the corresponding conformal theory.

Alternatively, one may start with the assumption that the effective action $\Gamma(g)$ in the WZNW theory must satisfy conformal and chiral $G \times G$ invariance conditions. Then a natural (and probably unique) choice for such $\Gamma$ is the classical action itself, up to renormalisations of $k$ and the current, $\Gamma(g) = k'I(g')$, $\bar{\partial}g'g'^{-1} = Z\bar{\partial}gg^{-1}$. Correspondence
with the c.f.t. approach then fixes \( k' = k + \frac{1}{2} c_G \) (and probably fixes also \( Z = \frac{k}{k + \frac{1}{2} c_G} \)). A possibility to find an exact expression for the effective action of the WZNW theory should not be surprising, given its solubility in the operator approach.

Maintaining equivalence between the local field theory and operator conformal theory results is rather subtle and depends on a choice of a particular regularisation prescription (which should correspond to a normal ordering prescription in c.f.t.). As in the case of the 3d Chern-Simons theory\(^{39}\) the one-loop shift of \( k \) in the effective action may happen in one regularisation and not happen in another one (see e.g. ref.\(^{40}\)). The absence of a renormalisation of \( k \) in the standard Legendre transform of the generating functional for correlators (which does not receive loop corrections\(^{29}\)) and its presence in the ‘quantum’ Legendre transform (34) seems related to an observation of ref.\(^{41}\) that similar ‘quantum’ Legendre transform in \( SL(2, R) \) Chern-Simons theory relates two representations (in terms of affine and Virasoro conformal blocks) with ‘bare’ and renormalised values of \( k \).

Observing that the gauged action (32) can be put into the form

\[
I(g, A) = I(h^{-1}gh) - I(h^{-1}\bar{h}) , \quad A = h\partial h^{-1} , \quad \bar{A} = \bar{h}\partial h^{-1} ,
\]

(36)
taking into account the Jacobian of the change of variables, using (35), dropping out the non-local terms introduced by field renormalisations (which are irrelevant for our problem of deriving the corresponding sigma model couplings) and expressing the result back in terms of the original fields \( g \) and \( A, \bar{A} \) we get the following effective action\(^{14}\)

\[
\Gamma'(g, A) = (k + \frac{1}{2} c_G)I(g, A) + \frac{1}{2}(c_G - c_H)\Omega(A) .
\]

Here \( \Omega(A) \) is a non-local gauge invariant functional of \( A \) and \( \bar{A} \),

\[
\Omega(A) \equiv I(h^{-1}h) = \omega(A) + \bar{\omega}(-\bar{A}) + \frac{1}{\pi} \int d^2z \, \text{Tr} \, (A\bar{A}) ,
\]

(38)
\[
\omega(A) = I(h^{-1}) = -\frac{1}{\pi} \int d^2z \, \text{Tr} \, \left\{ \frac{1}{2} A \frac{\partial}{\partial A} A - \frac{1}{3} A [\frac{1}{2} A, \frac{\partial}{\partial A}] + O(A^4) \right\} ,
\]

\[
\bar{\omega}(-\bar{A}) = I(\bar{h}) = -\frac{1}{\pi} \int d^2z \, \text{Tr} \, \left\{ \frac{1}{2} \bar{A} \frac{\partial}{\partial \bar{A}} \bar{A} - \frac{1}{3} \bar{A} [\frac{1}{2} \bar{A}, \frac{\partial}{\partial \bar{A}}] + O(\bar{A}^4) \right\} .
\]

(39)

It may be possible to solve the equations for \( A, \bar{A} \) and eliminate them from the action directly starting with (37). However, one can simplify the problem by observing that since we are not interested in the non-local terms, it is possible first to truncate the action \( \Gamma' \) by dropping out terms that are of cubic and higher order in \( A, \bar{A} \). As a result, we obtain the following ‘truncated’ effective action\(^{16}\)

\[
\Gamma_{tr}(g, A) = (k + \frac{1}{2} c_G)I(g, A) + \frac{1}{2}(c_G - c_H)\Omega_0(A) ,
\]

(40)
\[
\Omega_0(A) \equiv \frac{1}{\pi} \int d^2z \, \text{Tr} \, (A\bar{A} - \frac{1}{2} A \frac{\partial}{\partial A} A - \frac{1}{2} \bar{A} \frac{\partial}{\partial \bar{A}} \bar{A}) = \frac{1}{2\pi} \int \text{Tr} \, F \frac{1}{\partial \partial} F , \quad F \equiv \partial A - \partial \bar{A} .
\]

The action (40) is exactly equal to (37) in the case when the subgroup \( H \) is abelian. In contrast to (37) the action (40) is invariant only under the abelian gauge transformations,
\( A \rightarrow A + \partial \epsilon, \ A \rightarrow \tilde{A} + \tilde{\partial} \epsilon, \) but it is sufficient for our purposes to know that the full gauge invariance can be restored by re-introducing the higher order non-local terms. The truncated action (40) can be represented also in the form

\[
\Gamma_{tr}(g, A) = (k + \frac{1}{2}c_G) \left[ I(g) + \Delta I(g, A) \right], \quad (41)
\]

\[
\Delta I(g, A) \equiv \frac{1}{\pi} \int d^2 z \ Tr \left[ (-A \tilde{\partial} gg^{-1} + \tilde{A} g^{-1} \partial g + g^{-1} Ag \tilde{A} - A \tilde{A}) + \frac{1}{2} b \ (AQA + \tilde{A}Q^{-1} \tilde{A} - 2A \tilde{A}) \right], \quad (42)
\]

\[
b \equiv -\frac{(c_G - c_H)}{2(k + \frac{1}{2}c_G)}, \quad Q \equiv \frac{\tilde{\partial}}{\partial}, \quad Q^{-1} \equiv \frac{\partial}{\tilde{\partial}}. \quad (43)
\]

If we formally set here \( \partial = \tilde{\partial} \) (i.e. \( Q = 1 \)) we obtain the \( d = 1 \) action of ref.15 which is the dimensional reduction of the full action (37) (the higher-order commutator terms in (39) do not contribute in the \( d = 1 \) limit). It is easy to see (on Lorentz-invariance grounds) that the \( d = 1 \) action is, in principle, sufficient in order to extract the metric and dilaton couplings of the corresponding sigma model\textsuperscript{15}. However, to derive the antisymmetric tensor coupling one should use the direct 2d approach based on (41).

To obtain the sigma model action from (41) one should first solve for the gauge field and then discard all the non-local terms. This will be discussed in Sec.5. Note that it is not correct just to omit the terms with the operator \( Q \) insertions (since \( Q \) has dimension zero and since this would break the gauge invariance of (42)); it is also not correct to replace \( Q \) by 1 (this would break the Lorentz invariance).

The above results for the effective actions have generalisations\textsuperscript{16} to the case of \( n = 1 \) supersymmetric (gauged) WZNW theory. Computation of the effective action in the supersymmetric WZNW theory can be reduced to that in the bosonic WZNW theory by using the observation\textsuperscript{30} that by a formal field redefinition the supersymmetric action can be represented as a sum of the bosonic action and the action of the free Majorana fermions in the adjoint representation of the group \( G \). The transformation of fermions which is needed to decouple them from \( g \) is, however, chiral and therefore produces a non-trivial Jacobian. The logarithm of the fermionic determinant gives a contribution proportional to the bosonic WZNW action, leading to the shift of the coefficient \( k \) in the bosonic part of the action\textsuperscript{31,32}: \( k \rightarrow \hat{k} \equiv k - \frac{1}{2} c_G \).\textsuperscript{*} As a result, the effective action in the ungauged supersymmetric WZNW theory is obtained by replacing \( k \) by \( k - \frac{1}{2}c_G \) in (35)\textsuperscript{16}

\[
\Gamma(g) = kI(g') , \quad \tilde{\partial}g'g'^{-1} = (1 - \frac{c_G}{2k})\tilde{\partial}gg^{-1}, \quad (44)
\]

i.e. is equal to the classical WZNW action with \textit{unshifted} \( k \): the shift of \( k \) in \( \Gamma \) produced by integrating out fermions is exactly cancelled out by the bosonic contribution.

In the case of the gauged supersymmetric WZNW theory one may use a superfield formalism to maintain manifest supersymmetry.\textsuperscript{**} Then the path integral quantisation of

\* This shift of \( k \) is consistent with sigma model perturbation theory\textsuperscript{33}.

\** This possibility was first mentioned in ref.32 where, however, the component approach (in the particular case of \( G = H \)) was discussed.
the supersymmetric theory becomes parallel to that of the bosonic theory (see (36) etc.) with fields replaced by superfields. The theory is reduced to that of the two ungauged supersymmetric WZNW theories for \( G \) and \( H \) (with the only difference with respect to the bosonic case that now the Jacobian of the change of variables is trivial). Applying (44) one finds the following expression for the (bosonic part of the) effective action\(^ {16} \)

\[
\Gamma_{susc}(g, A) = kI(\tilde{g}') - kI(\tilde{h}') ,
\]

\[
\bar{\partial}\tilde{g}'\tilde{g}'^{-1} = (1 - \frac{CG}{2k})\bar{\partial}\tilde{g}\tilde{g}^{-1} , \quad \bar{\partial}\tilde{h}'\tilde{h}'^{-1} = (1 - \frac{CH}{2k})\bar{\partial}\tilde{h}\tilde{h}^{-1}
\]

(here \( \tilde{g} = h^{-1}g\bar{h} \), \( \tilde{h} = h^{-1}\bar{h} \); cf.(36)). As in the ungauged supersymmetric WZNW theory but in contrast with the bosonic gauged WZNW case there are no shifts in the overall coefficients of the \( G \)- and \( H \)-terms in \( \Gamma_{susc} \). Ignoring the non-local corrections introduced by the field renormalisations we conclude that the local part of the effective action of the gauged supersymmetric WZNW theory is equal to the \textit{classical} action of the bosonic gauged WZNW theory

\[
\Gamma'_{susc}(g, A) = kI(g, A) = k\left[I(g) + \frac{1}{\pi} \int d^2z \operatorname{Tr} \left(-A \bar{\partial}gg^{-1} + \bar{A}g^{-1}\partial g \right. \right.
\]

\[
\left. + g^{-1}Ag\bar{A} - A\bar{A}\right] \right), \quad (46)
\]

i.e. in contrast with the bosonic case (37), it does not contain the quantum correction term proportional to \( b = -\frac{(cG - cH)}{2(k + \frac{1}{k})} \). As a consequence, the exact form of the corresponding sigma model will be equivalent to the ‘semiclassical’ form of the sigma model corresponding to the bosonic theory (with no shift of \( k \)). This conclusion is the same as the one reached in the operator approach in ref.10 (in the case of the \( SL(2, R)/U(1) \) supersymmetric model) and in ref.13 (in the case of a general \( G/H \) supersymmetric theory).

5. Derivation of Sigma Model Couplings

in Field-Theoretical Approach

As explained above, our starting point will be the truncated effective action (41),(42). Since it is quadratic in the gauge potentials \( A, \bar{A} \) it is straightforward to integrate them out. Representing (42) as

\[
\Delta I(g, A) = \frac{1}{\pi} \int d^2z \left[ ( -\bar{A}\bar{J} + \bar{A}J ) + AN\bar{A} + \frac{1}{2}b \left( AQA + \bar{A}Q^{-1}\bar{A} \right) \right] , \quad (47)
\]

\[
J_a = \operatorname{Tr} (T_a g^{-1}\partial g) , \quad \bar{J}_a = \operatorname{Tr} (T_a \bar{\partial}gg^{-1}) , \quad N_{ab} \equiv M_{ab} - b\eta_{ab} , \quad M_{ab} \equiv C_{ab} - \eta_{ab} , \quad C_{ab} \equiv \operatorname{Tr} (T_a gT_b g^{-1}) , \quad \operatorname{Tr} (T_a T_b) = \eta_{ab} , \quad (48)
\]

and solving for \( A, \bar{A} \) we obtain (after omitting the non-local terms in which \( Q \) or \( Q^{-1} \) are acting on \( N \))

\[
\Delta I(g) = \frac{1}{2\pi} \int d^2z \left[ J\bar{V}^{-1}(N^T\bar{J} + bQJ) + J\bar{V}^{-1T}(NJ + bQ^{-1}\bar{J}) \right] , \quad (49)
\]

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we can put the local part of the action (41),(49) into the sigma model form

\[ V \equiv NN^T - b^2 = MM^T - 2bM_S , \]

\[ \bar{V} \equiv N^T N - b^2 = M^T M - 2bM_S , \quad M_S \equiv \frac{1}{2} (M + M^T) \].

Ignoring the non-local terms, we can replace \( Q^{-1}\bar{J} \) by \( \text{Tr} (T_a \partial gg^{-1}) \) and \( QJ \) by \( \text{Tr} (T_a g^{-1} \partial g) \). Using the parametrisation in terms of the group-space coordinates \( x^M \) (we shall fix the gauge by restricting \( x^M \) to coset space coordinates \( x^\mu \) later)

\[ g^{-1} \partial g = T_A E^A_M (x) \partial x^M , \quad g^{-1} \partial g = T_A T_A E^A_M (x) \partial x^M , \quad \partial gg^{-1} = T_A \tilde{E}^A_M (x) \partial x^M , \]

\[ \tilde{g}g^{-1} = T_A \tilde{E}^A_M (x) \partial x^M , \quad \tilde{E}^A_M = C^A_B (x) E^B_M , \quad C_{AB} = \text{Tr} (T_A g T_B g^{-1}) \],

we can put the local part of the action (41),(49) into the sigma model form\(^{16}\)

\[ \Gamma_{loc}(g) = -\frac{1}{\mu^2} \int d^2 x \; G_{MN} (x) \partial x^M \partial x^N , \quad \alpha' = \frac{2}{k + 1/2} c_G \text{,} \]

\[ G_{MN} = G_{0MN} - b(\bar{V}^{-1})_ab E^a_M E^b_N - b(V^{-1})_ab \bar{E}^a_M \bar{E}^b_N \]

\[ -2(\bar{V}^{-1}N^T)_ab E^a_M \tilde{E}^b_N , \]

\[ B_{MN} = G_{[MN]} = B_{0MN} - 2(\bar{V}^{-1}N^T)_ab E^a_M \tilde{E}^b_N \].

Here \( G_{0MN} \), \( B_{0MN} \) stand for the original WZNW couplings,

\[ G_{0MN} = \eta_{AB} E^A_M E^B_N = \eta_{AB} \tilde{E}^A_M \tilde{E}^B_N , \]

\[ 3 \partial_{[K} B_{0MN]} = E^A_K E^B_M E^C_N f_{ABC} = \tilde{E}^A_K \tilde{E}^B_M \tilde{E}^C_N f_{ABC} \].

As in refs.6,14, the local part of the determinant of the matrix in the quadratic \((A, \bar{A})\) term in (47) gives the dilaton coupling

\[ \phi = \phi_0 - \frac{1}{4} \ln \det V \text{.} \]

The same expressions for the metric (53) and dilaton (56) were found using the 1d reduction of the action (37) in ref.15. The result for the antisymmetric tensor coupling (54) is equivalent to the expression also suggested in ref.15 on the basis of an analogy with the expression for the metric.*

The expressions for the metric and the antisymmetric tensor (53),(54) are yet in rather abstract form. To give a more explicit and useful representation for the metric one should first express \( \tilde{E}^a_M \) in terms of \( E^a_M \) and \( E^i_M \) with the help of (51)

\[ \tilde{E}^a_M = C_{ab} E^b_M + C_{ai} E^i_M \text{,} \quad C_{ad} C_{b}^d + C_{ai} C_{bi}^i = \eta_{ab} \text{,} \quad E^A_M E^M_A = \delta^A_B \text{,} \quad E^A_M E^N_A = \delta^N_M \text{.} \]

* Ref.15 contains also a derivation (without assuming the 1d reduction) of the antisymmetric tensor coupling in a particular case of the \( SL(2,R) \times SO(1,1)/SO(1,1) \) \((D = 3 \text{ ‘black string’})\) model.
The inverse of the metric (62) thus has a much simpler structure than the metric itself.

\[ G_{MN} = h_{AB}E^A_ME^B_N = h_{ij}E^i_ME^j_N + \ldots, \quad h_{ij} = \eta_{ij} - bV^{-1}_aC^a_iC^b_j. \] (58)

It is now straightforward to check that the metric (53),(58) is degenerate, having \( D_H \) null vectors

\[ Z^a_a = E^a_a - \tilde{E}^a_a = -M_{ab}E^{Nb} - C_{ai}E^{Ni}, \quad G_{MN}Z^a_a = 0. \] (59)

These vectors are recognised as being the generators of the vector subgroup \( H \) of the \( G \times G \) symmetry of the WZNW action which was gauged in (32) (cf.(23)).

To obtain a non-degenerate metric one should restrict \( G_{MN} \) to the subspace orthogonal (with respect to \( G_{0MN} \)) to the null vectors \( Z^a_a \). One can change the original basis \( E^M_A = (E^M_i, E^a_a) \) to a new one \( (H^M_i, Z^a_a) \) with \( H^M_i \) being orthogonal to \( Z^a_a \). Then the degenerate metric (58) takes the form \( (H^M_i \equiv H^N_j \eta^{ij}G_{0MN}) \)

\[ G_{MN} = g_{ij}H^i_MH^j_N, \quad G_{0MN}H^M_iZ^a_a = 0, \quad \Pi^N_MH^M_i = H^N_i, \] (60)

where the projection operator \( \Pi \) is the same as in (24) and the expression for \( g_{ij} \) depends on a particular choice of the vectors \( H^M_i \) (there is a freedom of making a transformation \( H^M_i \rightarrow \Lambda^j_iH^M_j \)). A simple choice of \( H^M_i \) is (we shall use bars to denote objects corresponding to this basis)

\[ \bar{H}^i_M = E^i_M - M^{-1}_{ab}C^{bi}E^a_M = p^i_jE^j_M + M^{-1}_{ab}C^{bi}M^{-1}_{ca}Z^c_M, \] (61)

\[ p_{ij} \equiv \eta_{ij} + (MM^T)^{-1}_{ab}C^a_iC^b_j. \]

Then

\[ G_{MN} = \bar{g}_{ij}\bar{H}^i_M\bar{H}^j_N, \quad \bar{g}_{ij} = h_{ij} = \eta_{ij} - bV^{-1}_aC^a_iC^b_j. \] (62)

Since the inverse and determinant of the \( D \times D \) matrices of generic form \( m_{ij} = \eta_{ij} + f_{ab}C^a_iC^b_j \) are given by

\[ m^{-1}_{ij} = \eta_{ij} + f^{-1}_{ab}(C^a_iC^b_j), \quad f^{-1} = -[f^{-1} + (1 - CC^T)]^{-1}, \]

\[ \det m = \det [1 + f(1 - CC^T)], \] (63)

we find (cf.(25)–(27))

\[ G^{-1MN} = \Pi^K_M\hat{G}^{-1KL}\Pi^N_L, \quad G^{-1MN}G_{NK} = \Pi^K_M , \]

\[ \hat{G}^{-1KL} = E^M_AE^AN - \gamma E^M_AE^iN = E^M_iE^iN - (\gamma - 1)E^M_aE^aN, \] (64)

\[ \gamma = (b + 1)^{-1} = \frac{k + \frac{1}{2}c_G}{k + \frac{1}{2}c_H}, \]

\[ \det \bar{g}_{ij} = \det [(1 + b)V^{-1}] \det M^2. \] (65)

The inverse of the metric (62) thus has a much simpler structure than the metric itself.
Since the sigma model on the whole group space \( \int d^2z \ G_{MN}(x) \partial x^M \partial x^N + \ldots \) has the gauge invariance (generated by \( Z^M_a \)) the final step is to fix a gauge, e.g. restricting coordinates \( x^M \) on \( G \) to coordinates \( x^\mu \) on \( G/H \). Let \( R^a_M(x^M) = 0 \), \( R^a_M \delta x^M = 0 \), \( R^a_M \equiv \partial_M R^a \) be a gauge condition (the corresponding ghost determinant that should be included in the measure is \( \det R^a_M Z^M_b \)). One may either add a gauge term into the sigma model action (which will then depend on all \( x^M \) coordinates) or solve explicitly the gauge condition, expressing \( x^M \) in terms of \( D \) coordinates \( x^\mu \) on \( G/H \), \( x^M = x^M(x^\mu) \). * In the latter case we will get a sigma model on the \( D \)-dimensional space with \( E^A_M \) replaced by the \( D \times D \) matrix \( E^A_M = E^A_M \partial_x x^M \), i.e. \( H^i_M \) replaced by the \( D \times D \) matrix \( H^i_M \) (i.e. a vielbein), \( G_{MN} \) replaced by \( G_{\mu\nu} \), etc. The expressions for the sigma model metric and antisymmetric tensor then are

\[
G_{\mu\nu} = \bar{g}_{ij} H^i_M H^j_M, \quad \bar{H}^i_M = H^i_M \partial_x x^M, \quad (66)
\]

\[
B_{\mu\nu} = B_{0\mu\nu} - 2(\bar{V}^{-1} N^T C)_{ab} E^a_{\mu} E^b_{\nu} - 2(\bar{V}^{-1} N^T)_{ab} C^b_{i} E^a_{i} E^i_{\mu} E^i_{\nu}. \quad (67)
\]

6. Discussion

Let us now compare the above expressions for the background metric (66) and dilaton (56) corresponding to the gauged WZNW model with the results (29), (31) which were found in Sec.3 by identifying the operator \( L_0 \) of the \( G/H \) coset conformal theory with a ‘Klein-Gordon’ operator in a background. Using (64) we conclude that the inverse of the metric (62), (66) is equivalent to the inverse metric (25), (29) found in the operator approach (up to the overall factor \( k + \frac{1}{2}c_G \) which we absorbed in \( \alpha' \) in (52)). We have thus proved that both the operator and the field-theoretical approaches lead to the same expression for the target space metric.

Though expected, this equivalence is realised in a rather non-trivial way. While in the operator approach one obtains naturally the inverse of the metric, the field-theoretical derivation or sigma model approach gives the metric itself. The two approaches are in a sense ‘dual’ like momentum and coordinate representations or Hamiltonian and Lagrangian formalisms. The procedure of elimination of the \( H \)-gauge field is analogous to that of integration over the momentum (this analogy can probably be made more precise by rewriting the WZNW term in the effective action (37) in terms of an auxiliary ‘momentum’ or \( G \)-gauge field variable and extracting the inverse of the metric from the term quadratic in all components of the gauge field).

The metric (6.1) can be considered as a ‘deformation’ of the ‘round’ metric on \( G/H \). The latter corresponds to the sigma model which is found by integrating out the gauge field with values in the algebra of \( H \) in the action invariant under the \( H \)-gauge transformations generated by \( E^a_M \), i.e. \( g' = gu \)

\[
I = \int d^2z \ \text{Tr} \ (g^{-1} \partial_m g + A_m)^2. \quad (68)
\]

* If one uses the formulation in terms of all \( D_G \) coordinates \( x^M \) one should also impose as usual the gauge invariance (BRST invariance) condition on the observables. Adding a gauge-fixing term in the action one obtains the following non-degenerate metric on \( G \): \( \bar{G}_{MN} = G_{MN} + q_{ab} R^a_M R^b_N \). The determinant of the degenerate metric \( G_{MN} \) is defined as \( (\det G_{MN})^{-1/2} = (\det \bar{G}_{MN})^{-1/2} \det (R^a_M Z^M_b) (\det q_{ab})^{1/2} \).
This action (and hence the resulting sigma model metric) has also global $G$-invariance which is *absent* in the gauged WZNW action (46) (being broken by the $g^{-1}AgĀ$-term). Before gauge fixing, we get a degenerate metric $G^{(0)}_{\mu\nu}$ on the full $G$ (with null vectors $E^a_M$). Solving a gauge condition $R^a(x^M) = 0$ and expressing $x^M$ in terms of $x^\mu$ we obtain the metric $G^{(0)}_{\mu\nu}$ on the $D$-dimensional coset space $G/H$,

$$G^{(0)}_{MN} = \eta_{ij} E^i_M E^j_N, \quad G^{(0)}_{\mu\nu} = \eta_{ij} E^i_\mu E^j_\nu, \quad E^i_\mu = E^i_M \partial_\mu x^M. \quad (69)$$

Noting that according to (61) $\bar{H}^i_M = E^i_M + O(E^a_M)$ and choosing the gauge condition such that $\partial_M R^a = E^a_M$ one can show that

$$\det G_{\mu\nu} = \det G^{(0)}_{\mu\nu} \det g_{ij} (\det M)^{-2}, \quad (70)$$

$$\det G^{(s)}_{\mu\nu} = \det G^{(0)}_{\mu\nu} (\det M)^{-2}, \quad G^{(s)}_{\mu\nu} = \eta_{ij} \bar{H}^i_\mu \bar{H}^j_\nu, \quad (71)$$

where $G^{(s)}_{\mu\nu}$ is the semiclassical ($b = 0$) limit of $G_{\mu\nu}$ and $\det M$ is the corresponding ‘ghost determinant’ ($E^a_M Z^M_b = -M^a_b$, see (59)).

Using (70),(71),(65) and the expression for the dilaton (56) one finds

$$\sqrt{\det G_{\mu\nu}} \ e^{-2\phi} = \sqrt{\det G^{(s)}_{\mu\nu}} \sqrt{\det g_{ij}} \ e^{-2\phi}$$

$$= a \sqrt{\det G^{(s)}_{\mu\nu}} \det M = a \sqrt{\det G^{(0)}_{\mu\nu}}, \quad (72)$$

$$a = (\sqrt{1 + b})^D e^{-2\phi_0}.$$

The constant $a$ can be made equal to 1 by a choice of $\phi_0$. Therefore, in agreement with the result of the operator approach (31), the dilaton is given by the logarithm of the ratio of the determinants of the metric and the invariant metric on the coset space, i.e. the ‘measure factor’ (72) is nothing but the canonical measure on $G/H.$

Corresponding to the coset $G/H$ conformal theory, the sigma model with the metric (66), antisymmetric tensor (67) and dilaton (56) should be conformally invariant to all orders in the loop expansion, i.e. should represent a large class of exact solutions of string equations. Depending on $b$, the fields $G_{\mu\nu}, \bar{B}_{\mu\nu}$ and $\phi$ are non-trivial functions of the parameter $k$ or $\alpha'$ (see (43),(52)), $k + \frac{1}{2} c_G = \frac{2}{\alpha'}$, $b = \frac{1}{4}(c_H - c_G)\alpha'$, (the semiclassical limit is $b \to 0$). The dependence of the metric (66),(62) on $\alpha'$ can be represented in the following symbolic way: $G = G^{(s)} + \frac{\alpha' F_1}{F_2 + \alpha' F_3}$. At the same time, the inverse metric (64),(26) which appears naturally in the operator approach has simpler dependence: $G^{-1} = (G^{(s)})^{-1} + \frac{\alpha'}{\alpha' + q} F_4$, $q = 4/(c_H - c_G)$.

In spite of the fact that the underlying conformal theory has $G/H$ structure, the corresponding symmetry is not explicit at the sigma model level. It should be emphasised

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* The fact that the product $\sqrt{G} \ e^{-2\phi}$ is $k$-independent was first observed in the $SL(2, R)/U(1)$ case$^{9,25}$. It was further checked$^{13}$ on a number of non-trivial $G/H$ models. Refs.13,15 formulated this fact as a general statement and gave arguments supporting it using path integral measure considerations.
that it is the condition of conformal invariance that makes the resulting metric a ‘deformed’ (non-symmetric) one.* In fact, the standard $G/H$ sigma model (68) is not conformal since the metric $G^{(0)}$ of the homogeneous space has a non-vanishing Ricci tensor, giving a non-vanishing one-loop $\beta$-function. One possibility to make a conformal sigma model out of the homogeneous space metric is to introduce two extra dimensions (one space-like and one time-like) and impose the null Killing symmetry condition on the resulting $(2 + D)$-oddimensional metric as discussed in Sec.2.** Another possibility – which is realised in the models corresponding to the coset conformal theories – is to deform the metric, introducing at the same time a non-trivial dilaton (and antisymmetric tensor) background.***

As we have found in Sec.4, the local part of the bosonic term in the effective action of the gauged $n = 1$ supersymmetric WZNW theory is equal to the classical bosonic gauged WZNW action (46) with unshifted $k$ and no ‘quantum’ $b$-term. Thus in the supersymmetric case $\alpha' = 2/k$ and the corresponding exact sigma model couplings are given by the ‘semiclassical’ limit $b = 0$ of the bosonic WZNW theory expressions (66),(67),(56)

$$
G_{\mu\nu}^{(s)} = \eta_{ij} \bar{H}_i^{\mu} \bar{H}_j^{\nu} = \eta_{ij} (E_i^{\mu} - M^{-1}_{ab} C^{bi} E^{a \mu})(E_j^{\nu} - M^{-1}_{ab} C^{bj} E^{a \nu}) \ , \quad (73)
$$

$$
B_{\mu\nu}^{(s)} = B_{0\mu\nu} - 2(M^{-1})_{ab} E_{[\mu}^{a} E_{\nu]}^{b} - 2(M^{-1})_{ab} C^{b\mu} E_{[\mu}^{a} E_{\nu]}^{i} \ , \quad \phi^{(s)} = \phi_0 - \frac{1}{2} \det M .
$$

The non-triviality of $G_{\mu\nu}^{(s)}$ is ‘hidden’ in the choice of $\bar{H}_i^{\mu}$ (see (61)). For example, in the case when $G/H$ is Kähler, eq.(73) gives the couplings of the sigma model corresponding to a class of $n = 2$ superconformal theories36,13. The fact that the sigma model couplings in the supersymmetric case do not depend on $\alpha'$ means that they solve the sigma model conformal invariance conditions at each order of the loop expansion. Namely, once the one-loop conditions are satisfied, all higher-loop corrections to the $\beta$-functions should vanish (up to a field redefinition ambiguity) on the corresponding background.**** It is interesting to note that these models have, in general, only $n = 1$ (and not $n = 2$ or $n = 4$) supersymmetry (as

* Similarly, in the operator approach even though one uses the affine $G$-symmetric framework it is the condition of conformal (Virasoro) invariance (16) that, in general, constrains $C^{AB}$ in such a way that the corresponding target space metric is non-symmetric.

** One can also add just one extra dimension (time) and compensate the conformal anomaly by the time evolution of the scale factor (and dilaton). Such ‘cosmological’ solutions (with $G/H$ being a sphere) were found to exist in the order $\alpha'$ approximation34 but it is not clear how to extend them to exact solutions.

*** It is an interesting question if there are other exact solutions of the sigma model conformal invariance conditions for which the homogeneous space metric remains undeformed but the conformal anomaly is cancelled out by the contribution of the antisymmetric tensor (cf. ref.35).

**** This was checked explicitly up to the 5-loop order in the supersymmetric $SL(2, R)/U(1)$ model10. While the two - and three - loop terms in the $\beta$-function of the $n = 1$ supersymmetric sigma model are known to vanish37 (in the minimal subtraction scheme), the four-loop term does not vanish in general38. However, there exists such a renormalisation scheme in which it vanishes for the ‘one-loop’ $D = 2$ background of refs.7,6.
we have discussed in Sec.2, the models of ref.3 represent another example of finite $n = 1$
 supersymmetric models).

Though particular examples of geometries corresponding to coset conformal theories
look complicated and unusual (see ref.13 and refs. there), they may actually be more
characteristic to string theory than more symmetric spaces (which are not string solutions).
Having behind them a deep algebraic structure of coset conformal theory, they may have
some interesting universal features. The representation of the background fields in terms
of the group vielbeins (66)(67) generalises similar expressions for the group spaces and
may serve as a basis for a study of their geometrical properties.

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