A CA Hybrid of the Slow-to-start and the Optimal Velocity Models and its Flow–Density Relation

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Abstract. The s2s–OVCA is a cellular automaton (CA) hybrid of the optimal velocity (OV) model and the slow-to-start (s2s) model, which is introduced in the framework of the ultradiscretization method. Inverse ultradiscretization as well as the time continuous limit, which leads the s2s–OVCA to an integral-differential equation, is presented. Several traffic phases such as a free flow as well as slow flows corresponding to multiple metastable states are observed in the flow–density relations of the s2s–OVCA. Based on the properties of the stationary flow of the s2s–OVCA, the formulas for the flow–density relations are derived.

Key words: optimal velocity (OV) model; slow-to-start (s2s) effect; cellular automaton (CA); ultradiscretization, flow–density relation

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1 Introduction

Self-driven many-particle systems have provided a good microscopic point of view on the vehicle traffic [1, 2]. The optimal velocity model [3] gives a description of such a system with a set of ordinary differential equations (ODE). It is a car-following model describing an adaptation to the optimal velocity that depends on the distance from the vehicle ahead. Another way of describing such systems is provided by cellular automata (CA). For example, the elementary CA of Rule 184 (ECA184) [4], the Fukui–Ishibashi (FI) model [5] and the slow-to-start (s2s) model [6] are CA describing vehicle traffic as self-driven many-particle systems.

Studies of the self-driven many-particle systems have been wanting a framework that commands a bird’s eye view of both ODE and CA models in a unified manner. Ultradiscretization [7], which gives a link between the KdV equation and integrable soliton CA [8], is expected to provide such a framework, for ultradiscretization can also be applied to non-integrable systems. As a first step, an ultradiscretization of the OV model [9] was presented, which lead to the s2s–OVCA [10]. The s2s–OVCA is a CA-type hybrid of the OV model and the s2s model. As we shall show in section 2 the s2s–OVCA reduces to an ODE that is an extension of the OV model in the inverse-ultradiscrete and the time-continuous limits.

It was observed by numerical experiments that motion of the vehicles described by the s2s–OVCA went to stationary flow in the long run, irrespectively of the initial configuration [10, 11]. It was also observed by numerical experiments that the flow–density relation for the stationary flow of the s2s–OVCA was piecewise linear and flipped-\(\lambda\) shaped diagram with several metastable slow branches [10]. Exact expression for the flow–density relation was given by a set of exact solutions giving stationary flows of the s2s–OVCA [11]. The flipped-\(\lambda\) shaped diagram captures the characteristic of observed flow–density relations [1, 2]. We shall explain in section 3 the flow–density relation of the s2s–OVCA based on the properties of the stationary flow which was numerically observed [10].
2  s2s–OVCA and its Inverse Ultradiscretization

The s2s–OVCA is given by a set of difference equations below,

\[ x_{k+1}^n = x_k^n + \min\left(\min_{n'=0}^{n_0} (x_{k+1}^{n-n'} - x_k^{n-n'} - 1), v_0 \right), \]  

where the integers \( n_0 \geq 0, v_0 \geq 0 \) and \( x_k^n, k = 1, 2, \ldots, K, \) are the monitoring period, the top speed and the position of the car \( k \) at the \( n \)-th discrete time. Note that the definition of the symbol \( \min_{k=0}^{N} \) is

\[ \min(a_k) := \min(a_0, a_1, a_2, \cdots, a_N). \]

The equation (1) is called an ultra-discrete equation in the sense that it is a difference equation which is piecewise linear with respect to the dependent variables \( x_k^n \). The s2s–OVCA includes the ECA184 \((n_0 = 0, v_0 = 1)\) [4], the FI model \((n_0 = 0)\) [5] and the s2s model \((n_0 = 1, v_0 = 1)\) [6] as its special cases.

Since the second term in the right hand side of eq. (1) gives the speed of the car \( k \) at the time \( n \), the s2s–OVCA describes many cars running on a single lane highway in one direction, which is driven by cautious drivers requiring enough headway to go on at least for \( n_0 \) time steps before they accelerate their cars. Without loss of generality, we can assume that the cars are arrayed in numerical order, \( x_1^n < x_2^n < \cdots < x_K^n \), which is also assumed throughout below. Then the number of empty cells between the cars \( k \) and \( k + 1 \) for any \( k \) is always non-negative, i.e.,

\[ x_{k+1}^n - x_k^n - 1 := \Delta x_k^n - 1 \geq 0. \]  

It is obvious that the inequality holds for \( n = 0 \). We assume that the inequality holds up to some \( n \), as the induction hypothesis. The induction hypothesis as well as the definition of \( \min \) assure the inequality

\[ 0 \leq \min_{n'=0}^{n_0} (\Delta x_k^{n-n'} - 1, v_0) \leq \Delta x_k^n - 1 \]

for any \( k \). Using equation (1), we get an expression of \( \Delta x_k^n \) as

\[ \Delta x_k^{n+1} - 1 = \Delta x_k^n - 1 + \min_{n'=0}^{n_0} (\min_{n'=0}^{n_0} (\Delta x_{k+1}^{n-n'} - 1, v_0)) - \min_{n'=0}^{n_0} (\min_{n'=0}^{n_0} (\Delta x_k^{n-n'} - 1), v_0) \]

\[ = \min_{n'=0}^{n_0} (\Delta x_{k+1}^{n-n'} - 1, v_0)) + \left[ \Delta x_k^n - 1 - \min_{n'=0}^{n_0} (\min_{n'=0}^{n_0} (\Delta x_k^{n-n'} - 1), v_0) \right]. \]  

The inequality (3) and the equation (4) show that the inequality (2) holds for \( n + 1 \). The inequality (2) means that both overtake and clash are prohibited by the s2s–OVCA.

We should note that the s2s–OVCA is obtained from a difference equation by a limiting procedure named ultradiscretization [7], which generates a piecewise-linear equation from a difference equation via the limit formula,

\[ \lim_{\delta x \to +0} \delta x \log \left( \sum_{k=0}^{N} b_k e^{a_k/\delta x} \right) = \max_{k=0}^{N} a_k = max(a_k), \]

where arbitrary numbers \( b_k \) must be positive. The equation (5) is rewritten as

\[ \lim_{\delta x \to +0} \delta x \log \left( \sum_{k=0}^{N} b_k e^{-a_k/\delta x} \right)^{-1} = \min_{k=0}^{N} a_k, \]
for \( \min(a_0, a_1, a_2, \ldots, a_N) = -\max(-a_0, -a_1, -a_2, \ldots, -a_N) \).

For the sake of convenience in the calculation below, we introduce two parameters in the s2s–OVCA,

\[
x_{k+1}^n = x_k^n + \min_{n'=0}^{n_0} (\Delta x_k^{n-n'}) - x_0, v_0 \delta t \quad =: x_k^n + v_{\text{opt}}^{n} (\Delta_{\text{eff}} x_k^{n}) \delta t,
\]

where \( \Delta_{\text{eff}} x_k^{n} := \min_{n'=0}^{n_0} (\Delta x_k^{n-n'}) \). The parameters \( x_0 \) and \( \delta t \) are the length of a cell and the discrete time-step, respectively. Since we have shown \( \Delta x_k^{n} - x_0 \geq 0 \), the effective headway \( \Delta_{\text{eff}} x_k^{n} - x_0 \) is also always non-negative, \( \Delta_{\text{eff}} x_k^{n} - x_0 \geq 0 \), for any \( k \). With the aid of the identity,

\[
\min(A, B) = A - \max(0, A - B) = \max(0, A) - \max(0, A - B)
\]

for any \( A \geq 0 \), the optimal velocity function \( v_{\text{opt}}^{n}(x) \delta t := \min(x - x_0, v_0 \delta t) \) in the s2s–OVCA is expressed as

\[
v_{\text{opt}}^{n}(x) \delta t = \max(0, x - x_0) - \max(0, x - x_0 - v_0 \delta t),
\]

for any \( x > 0 \), which is given by the ultradiscrete limit \( \delta x \to +0 \) of a function

\[
v_{\text{opt}}^{d}(x) \delta t = \delta x \log \left[ \frac{1}{\frac{1+e^{-(x-x_0)/\delta x}}{1+e^{-x_0/\delta x}}} \right] = \delta x \log \left( \sum_{n'=0}^{n_0} e^{-\Delta x_k^{n-n'}/\delta x} \right)^{-1}.
\]

which is an inverse-ultradiscretization of the optimal velocity function \( v_{\text{opt}}^{n} \). Note that we have introduced arbitrary coefficients so as to make \( v_{\text{opt}}^{d}(0) = 0 \). In a similar way to the above calculation, an inverse-ultradiscretization of the effective interval \( \Delta_{\text{eff}} x_k^{n} \) is also obtained as

\[
\Delta_{\text{eff}} x_k^{n} := \delta x \log \left( \sum_{n'=0}^{n_0} e^{-\Delta x_k^{n-n'}/\delta x} \right)^{-1}.
\]

Therefore an inverse-ultradiscretization of the us2s–OVCA is given by \( x_{k+1}^{n+1} = x_k^n + v_{\text{opt}}^{d}(\Delta_{\text{eff}} x_k^{n}) \delta t, \)

which is explicitly written as

\[
x_{k+1}^{n+1} = x_k^n + \delta x \left\{ \log \left[ 1 + \left( \sum_{n'=0}^{n_0} e^{-(\Delta x_k^{n-n'}-x_0)/\delta x} \right)^{-1} \right] - \log \left( 1 + e^{-x_0/\delta x} \right) \right. \\
\left. - \log \left[ 1 + \left( \sum_{n'=0}^{n_0} e^{-(\Delta x_k^{n-n'}-x_0-v_0 \delta t)/\delta x} \right)^{-1} \right] + \log \left( 1 + e^{-(x_0+v_0 \delta t)/\delta x} \right) \right\}.
\]

In other words, the s2s–OVCA is given by the ultradiscrete limit \( \delta x \to +0 \) of the above difference equation \( \text{10} \).

The continuum limit \( \delta t \to 0 \) of the above difference equation \( \text{10} \) goes to integral-differential equation

\[
\frac{dx_k}{dt} = v_0 \left( 1 + \frac{1}{t_0} \int_0^{t_0} e^{-\Delta x_k(t-t')-x_0}/\delta x \, dt' \right)^{-1} - v_0 \left( 1 + e^{x_0/\delta x} \right)^{-1},
\]

where \( t_0 := n_0 \delta t \) and \( \frac{dx_k}{dt} = \lim_{\delta t \to 0} \frac{x_{k+1}^{n+1}-x_k^n}{\delta t} \). In terms of an optimal velocity function and an effective distance,

\[
v_{\text{opt}}(x) := v_0 \left( \frac{1}{e^{-(x-x_0)/\delta x}} - \frac{1}{e^{x_0/\delta x}} \right)
\]

\[
\Delta_{\text{eff}} x_k(t) := \delta x \log \left( \frac{1}{t_0} \int_0^{t_0} e^{-\Delta x_k(t-t')-x_0}/\delta x \, dt' \right)^{-1},
\]
the above integral-differential equation is expressed as

\[ \frac{dx_k}{dt} = v_{\text{opt}}(\Delta_{\text{eff}} x_k(t)). \]

Since the effective distance \( \Delta_{\text{eff}} x_k(t) \) goes to \( \Delta x_k(t) \) in the limit below,

\[ \Delta x_k(t - t_0) = \lim_{h \to t_0} \delta x \log \left( \frac{1}{t_0 - h} \int_h^{t_0} e^{-(\Delta x_k(t' - t_0))/\delta x} dt' \right)^{-1}, \]

this integral-differential equation is an extension of the Newell model \[12\],

\[ \frac{dx_k}{dt} = v_{\text{opt}}(\Delta x_k(t - t_0)). \] (12)

which is a car-following model dealing with retarded adaptation to the optimal velocity determined by the headway in the past.

Replacement of \( t \) with \( t + t_0 \) in eq. and the Taylor expansion of \( \dot{x}_k(t + t_0) = v_k(t + t_0) \) yield

\[ v_{\text{opt}}(\Delta x_k(t)) = v_k(t + t_0) \]

\[ = v_k(t) + \frac{dv_k}{dt} \cdot t_0 + \frac{1}{2} \frac{d^2v_k}{dt^2} \cdot t_0^2 + \cdots, \]

which is equivalent to

\[ \frac{dv_k}{dt} + \frac{1}{2} \frac{d^2v_k}{dt^2} \cdot t_0 + \cdots = \frac{1}{t_0} (v_{\text{opt}}(\Delta x_k(t)) - v_k(t)). \] (13)

The equation of motion of the OV model

\[ \frac{dv_k}{dt} = \frac{1}{t_0} (v_{\text{opt}}(\Delta x_k(t)) - v_k(t)) \]

is given by neglecting the higher order terms in the left hand side of the equation \[13\].

The discussion shown above in this section shows how the inverse ultradiscretization and the time continuous limit connect the s2s model and the Newell model, which approximates the OV model, through the s2s–OVCA.

3 Flow–Density Relation

Figure 1 gives typical examples of the spatio-temporal pattern showing jams and the flow–density relation of the s2s–OVCA \[10\]. In the numerical calculation, the periodic boundary condition is imposed and the length of the circuit \( L \), which is the same as the number of all the cells, is fixed at \( L = 100 \). The number of the cars \( K \) is set at \( K = 30 \). The maximum velocity \( v_0 \) and the monitoring period \( n_0 \) are \( v_0 = 3 \) and \( n_0 = 2 \).

The spatio temporal pattern shows the trajectories of the cars. As we can see, irregular motion of cars is observed in the early stage of the time evolution, \( 0 \leq n \leq 30 \), where \( n \) is the time. But after that, the flow of the cars become stationary in the sense that length of the jam is almost constant and that cars with intermediate speeds appear only temporarily.

The flows \( Q \) in the flow–density relation are computed by averaging over the time period \( 800 \leq n \leq 1000 \),

\[ Q := \frac{1}{(n_f - n_i + 1)L} \sum_{k=1}^{K} \sum_{n=n_i}^{n_f} v^n_k, \quad v^n_k := x^n_{k+1} - x^n_k, \] (14)
in which the traffic is expected to be stationary in the above mentioned sense. The car density \( \rho := \frac{K}{L} \). As we have mentioned before, the flow–density relation of the s2s–OVCA is piecewise linear and flipped-\( \lambda \) shaped diagram with several metastable slow branches.

The flow–density relation shown above is derived by admitting the features of the flow of the s2s–OVCA. Namely, the flow of the s2s–OVCA goes to one of the stationary flows in the long run. The stationary flows consist of the free flow in which all the cars run at the top speed \( v_0 \) and the slow flows that always contain slow cars running at the minimum speed \( v_{\text{min}}^\infty \), \( 0 \leq v_{\text{min}}^\infty < v_0 \), which remains constant. Formation of the line of slow cars corresponds to that of traffic jam. In the slow flows, lengths of the jams are almost constant and fluctuate periodically. Our previous paper [11] gives a set of such stationary flows.

First we shall deal with the free flow and its flow–density relation. Since all the cars run at the top speed, \( v_k^n = v_0 \) for \( \forall k \), all the headways respectively remain constant, \( \Delta x_k^{n+1} = \Delta x_k^n \geq v_0 + 1 \), \( \forall k \). Hence the flow \( Q \) is also constant in the future. Using the definition of the flow [14] with
\( n_i = n_f = n \), we have

\[ Q = \frac{1}{L} \sum_{k=1}^{K} v_0 = \rho v_0, \]

which gives the straight line with a positive inclination that is equal to the top speed \( v_0 \) in the flow–density relation. For example in fig. 2, the flow–density relation of the free flow which is labeled with \( v = 3 \) is \( Q = 3\rho \), since \( v_0 = 3 \) in this case.

Next we shall consider the slow flows and their flow–density relations. Let us see a specific solution of the s2s–OVCA starting from the following initial configuration,

\[ 0 : 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots, 0 \ldots. \quad (15) \]

Note that the number 0 at the leftmost shows the time. The digits and the blank symbols \( \square \) in the above configuration mean the indices of the cars and the empty cells, respectively. Thus the number of the cars \( K \) is 10 and the length of the circuit \( L \) is 38 in this case. We set the monitoring period \( n_0 \) at 2. The speed of the cars 4, 9 and 0 is 3, which is the top speed \( v_0 \) of this case. The speed of the car 5 is 2, whose headway is also 2. All the other cars’ speeds are 1, whose headways are also 1 except for the car 3. Thus the headways of the cars in the past have nothing to do with the motion of the cars in the future except for the car 3. The headway of the car 3 at the time \( -1 \) is set to be 1.

Out of the above initial configuration \((15)\), the equation \((1)\) generates flow of vehicles as follows,

\[
\begin{align*}
0 & : 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots, 0, \\
1 & : \square 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots, 0, \\
2 & : 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots, 9, \\
3 & : \square 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots, 9, \\
4 & : \square \square 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots, 9, \\
5 & : \square \square \square, 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots, 9, \\
6 & : \square \square \square, \square, 0, 1, 2, 3, 4, 5, 6, 7, \ldots, 8, \ldots.
\end{align*}
\]

Note that the minimum speed of the cars \( v_{\text{min}}^\infty \) is 1 in the flow above. We notice that the configuration at the time 3 is obtained by moving all the cells of the initial configuration one cell rightward as well as changing the car indices \( k \) to \( k - 1 \) modulo 10. The configuration at the time 6 is also obtained by doing the same shifts and changes of car indices to the configuration at the time 3. In this sense, the above flow is a periodic motion of cars whose period is 3 in this case. The length of the jam, or the number of the cars running at the minimum speed, is thus almost constant. Intermediate speeds also appear only temporarily. That is why we call them stationary flows of the s2s–OVCA. Roughly speaking, the slow flows we shall deal with is the stationary flow of the type shown above. The density of the cars \( \rho \) and the average flow \( Q \) over the period, or the \( n_0 + 1 = 3 \) steps, are calculated as

\[
\rho = \frac{K}{L} = \frac{5}{19},
\]

\[
Q = \frac{1}{(n_0 + 1)L} \sum_{k=0}^{n_0} \sum_{n'=0}^{n_0} v_{k'}^{n'} = \frac{6 + 3 + 3 + 5 + 9 + 4 + 3 + 3 + 3 + 9}{3 \times 38} = \frac{8}{19},
\]

which will be verified with the formula we shall derive shortly.

Let us consider such slow flows as we have seen above as the specific solutions in a more general manner. Figure 3 shows configurations of a slow flow at times \( n \) and \( n + n_0 + 1 \). Since
we employ the periodic boundary condition, two cells containing the car \( K \) are identified. As a property of the slow flow, we assume that the slow flow is periodic in the sense that the configuration at the time \( n + n_0 + 1 \) in the box is given by the rightward displacement of the entire configuration at the time \( n \) in the box by \( n_0 v_{\text{min}}^\infty - 1 \) cells. The flow provided by this displacement of the entire configuration in \( n_0 + 1 \) time steps is \( \frac{n_0 v_{\text{min}}^\infty - 1}{n_0 + 1} \rho \). For example, the rightward displacement mentioned above for the slow flow in fig. 3 is \( 2 \times 1 - 1 = 1 \), which agrees with the observation before. The set of stationary flows given in [11] has the property of the slow flow we here assume.

Here we should note that the leftward displacement of the car \( K \) by \( L \) cells, namely whole the circuit length, which is fictitiously introduced to make the shifted initial configuration from the real configuration at the time \( n_0 + 1 \) in the sense that the numerical order of the car arrays is maintained. In order to compensate the underestimation of the flow brought about by this leftward displacement, we have to add the flow corresponding to the rightward displacement of the car \( K \) by \( L \) cells in \( n_0 + 1 \) time steps, \( 1 \cdot \frac{1}{n_0 + 1} \cdot L = \frac{1}{n_0 + 1} \). Thus the flow of the slow flow with the minimum speed \( v_{\text{min}}^\infty \) is given by

\[
Q = \frac{n_0 v_{\text{min}}^\infty - 1}{n_0 + 1} \rho + \frac{1}{n_0 + 1}, \quad 0 \leq v_{\text{min}}^\infty < v_0. \tag{16}
\]

For example, substitution of \( \rho = \frac{5}{19} \), \( n_0 = 2 \) and \( v_{\text{min}}^\infty = 1 \) into eq. (16) yields

\[
Q = \frac{2 \times 1 - 1}{2 + 1} \times \frac{5}{19} + \frac{1}{2 + 1} = \frac{8}{19},
\]

which agrees with the flow \( Q = \frac{8}{19} \) for the slow flow given above as an specific solution. The formula (16) agrees with the flow–density relation given by numerical experiments, as we can see in fig. 2. Three branches labeled with \( v = 2, 1 \) and 0 are the flow–density relations with the minimum speeds \( v_{\text{min}}^\infty = v \) in fig. 2.

The maximum density \( \rho_{\text{max}}(v_{\text{min}}^\infty) \) that allows the minimum speed to be \( v_{\text{min}}^\infty \) is

\[
\rho_{\text{max}}(v_{\text{min}}^\infty) = \frac{1}{v_{\text{min}}^\infty + 1}. \tag{17}
\]

The flow \( Q(\rho_{\text{max}}(v_{\text{min}}^\infty)) \) corresponding to the maximum density \( \rho_{\text{max}}(v_{\text{min}}^\infty) \) is then given by

\[
Q(\rho_{\text{max}}(v_{\text{min}}^\infty)) = \rho_{\text{max}}(v_{\text{min}}^\infty) v_{\text{min}}^\infty. \tag{18}
\]

Since the two equations (17) and (18) holds at the same time, they leads to \( Q(\rho_{\text{max}}(v_{\text{min}}^\infty)) + \rho_{\text{max}}(v_{\text{min}}^\infty) = 1 \). Thus all the end points of the branches must be on the line

\[
Q + \rho = 1. \tag{19}
\]
The branching point, or the minimum density, of the flow–density relation of the slow flow corresponding to the minimum speed $v_\infty^{\min}$ is determined by the intersection of the flow density relations of the free flow and the slow flow,

$$\rho_{\min}(v_\infty^{\min}) = \frac{1}{n_0(v_0 - v_\infty^{\min}) + v_0 + 1}. \quad (20)$$

In fig. 2 the branching points corresponding to $v_\infty^{\min} = 2, 1$ and 0 are encircled with small circles, which agree with the above formula (20). The density of the cars $\rho$ needs to be sufficiently large so as to form the slow flow with the minimum speed $v_\infty^{\min}$. The branching point gives the lower bound of such density.

4 Summary

We have shown an inverse ultradiscretization from the s2s-OVCA (1) to an integral-differential equation (11), which is an extension of the Newell model (2). Since the Newell model (2) and the s2s-OVCA (10) are extended models of the OV [3] and the s2s models [6] respectively, the s2s-OVCA is interpreted as a CA-type hybrid of the OV and the s2s models.

Using the features of the stationary flows observed in the numerical experiments, we have derived the flow–density relations of the stationary flow of the s2s-OVCA. The flow–density relations of the s2s-OVCA were numerically obtained [10] and then derived by use of a set of stationary flows [11].

The s2s-OVCA has several types of monotonicity in its time evolution, which extend the results shown for the $n_0 = 1$ case [13]. We expect that the monotonicity determines the relaxation to the stationary flow from the initial configuration as well as the property of the stationary flow we assume here. We hope that results on the relaxation to stationary flows and the monotonicity in the time evolution of the s2s-OVCA will be reported soon.

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