Radiation reaction of betatron oscillation in plasma wakefield accelerators

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Abstract
A classical model of radiation reaction for the betatron oscillation of an electron in a plasma wakefield accelerator is presented. The maximum energy of the electron due to the longitudinal radiation reaction is found, and the betatron oscillation damping due to both the longitudinal and transverse radiation reaction effects is analyzed. Both theoretical and numerical solutions are shown with good agreements. The regime that the quantum radiation takes effect is also discussed. This model is important for designing future plasma based super accelerators or colliders.

1. Introduction

Plasma wakefield accelerators (PWA), driven by either short laser pulses or charged particle beams [1–3], have the electric (E) and magnetic (M) fields
\[ E_0 \sim 96 \sqrt{n_p} \left( \text{cm}^{-3} \right) \left( \text{V m}^{-1} \right) \]
\[ B_0 \sim 320 \sqrt{n_p} \left( 10^{18} \text{ cm}^{-3} \right) \left( \text{T} \right) \]
which are commonly more than 2 orders of magnitude larger than the EM-fields in conventional accelerators [4]. With the help of powerful wakefield drivers, people are developing compact high-energy accelerators around the world [5–9].

In a PWA, the transverse focusing force exerted on the trailing beam, due to the radial E-field and azimuthal B-field of the wake, exists along the whole accelerator. As a consequence, the normalized emittance of the beam may grow due to the betatron oscillation (BO) and finally saturate [10], with the saturation normalized emittance commonly ranging from 0.1 to 100 mm mrad, depending on the injection mechanism, e.g. self-injection, density gradient injection, optical injection, ionization injection and so on [11–20].

Radiation reaction (RR) is the friction on an electron due to EM radiation of itself. The RR effect has been studied for laser-plasma interactions, typically on energetic electrons colliding with high-intensity laser pulses. It is predicted that more than 35% of laser energy is converted to EM radiation at a laser radiance of \( I \lambda^2 > 10^{22} \text{ W \mu m}^2 \text{ cm}^{-2} \) in a duration of 20 fs [21]. The quantum correction in RR has been confirmed by recent experiments of the head-on collision between energetic electron bunches and high-intensity laser pulses [22, 23]. These researches focused on an extreme condition of highly intense EM-field in a short period of the fs-order to study the elementary process of a radiating electron. On the other hand, the long-time accumulation of the RR effect has also been studied, e.g. for runaway electrons in Tokamak to engineer good plasma conditions of magnetic confinement plasma [24].

For a regular PWA length of the order of 0.1 m or shorter, RR is negligible because the RR force is several orders of magnitude smaller than the acceleration and focusing forces of the wake. However, the magnitude of the RR force becomes significant at high beam energies. Moreover, the RR force is a damping force and its effect accumulates, which may finally become noticeable at long acceleration distances. Such effects are crucial for developing future super PWA and colliders which have acceleration distances of the order of 10 to 10\(^3\) meters [25].
In this work, we show a classical model of the RR effects for the single electron BO in a PWA. The theory is reinforced by numerical results obtained by the code PTracker [26], which uses 4th order Runge–Kutta method to solve the simplified equations of motion of an electron in a PWA. This paper is organized as follows. Section 2 gives the general form of BO of an electron in a PWA. Section 3 shows the BO amplitude scaling during acceleration or deceleration processes. Section 4 presents a classical model of RR during BO, and section 5 discusses the limit between classical and quantum RRs.

2. Betatron oscillations without radiation reaction

For a single electron trapped (with normalized longitudinal velocity \(\beta_z \approx 1\) and transverse velocity \(|\beta_y| < 1\) where speed of light in vacuum \(c\) is the normalizing unit) in a blowout plasma wakefield (either driven by charged beam or laser pulse) and moving in the \(z-x\) plane, the focusing force (in \(x\) direction) and accelerating force (in \(z\) direction) for \(|\xi| \lesssim 1\) and \(|\chi| \lesssim 1\) are [27, 28]

\[
\begin{align*}
    f_x^{\text{ext}} &= -\frac{1}{4} (1 + \beta_z) x, \\
    f_z^{\text{ext}} &= -\frac{1}{2} \zeta + \frac{1}{4} \beta_z x,
\end{align*}
\]

where \(\zeta = z - \beta_z t\) is the wake co-moving frame with \(\zeta = 0\) and \(x = 0\) being the center of the blowout bubble, \(\beta_w = \sqrt{1 - \gamma^2} / c\) is the wake phase velocity normalized to \(c\), and \(\gamma_w \gg 1\) is the Lorentz factor of the wake. Note we have adopted the normalized units where length is normalized to \(k_p^{-1}\), time is normalized to \((\epsilon k_p)^{-1}\), momentum is normalized to \(m_e c\), energy and work are normalized to \(m_e c^2\), force is normalized to \(m_e c^2 k_p\), with \(k_p = \sqrt{\mu_0 \epsilon \omega_p / m_e}\) being the wavenumber of plasma wake, \(\mu_0\) being the vacuum permeability, \(\epsilon\) being the elementary charge, \(n_p\) being the plasma density and \(m_e\) being the electron mass. Without RR, the equation of motion is

\[
\hat{\vec{p}} = \vec{f}^{\text{ext}},
\]

with

\[
\begin{align*}
    p_x &= \gamma \dot{x}, \\
    p_z &= \gamma \left( \hat{\zeta} + \beta_w \right), \\
    \gamma^2 &= 1 + p_x^2 + p_z^2,
\end{align*}
\]

where a dot on top means time derivative, \(\vec{p}\) is the normalized momentum and \(\gamma\) is the Lorentz factor of the electron.

Assume \(|\chi|\) is a small quantity. Then the trajectory in the wake co-moving frame features a figure ‘8’ with the frequencies in \(x\)-axis being \(\omega_{\beta}\) and in \(z\)-axis being \(2\omega_{\beta}\) where \(\omega_{\beta} = 1/\sqrt{2\gamma}\) is the normalized BO frequency [29]

\[
\begin{align*}
    x &= x_1 \sin \omega_{\beta} t, \\
    \zeta &= \zeta_0 - \zeta_1 \sin 2\omega_{\beta} t,
\end{align*}
\]

where subscriptions 0/1 indicate ‘slow’/‘fast’ components with characteristic time scale much longer than/comparable to \(\omega_{\beta}^{-1}\). The normalized velocities are

\[
\begin{align*}
    \beta_x &= \dot{x} = x_1 \omega_{\beta} \cos \omega_{\beta} t, \\
    \beta_z &= \beta_w + \dot{\zeta} = \beta_{z0} - 2\zeta_1 \omega_{\beta} \cos 2\omega_{\beta} t,
\end{align*}
\]

where \(\beta_{z0} = \beta_w + \dot{\zeta}_0\) is \(\beta_z\) averaged in the \(\omega_{\beta}^{-1}\) time scale. Thus

\[
\gamma^{-2} = 1 - \beta_x^2 - \beta_z^2 = \gamma_{z0}^2 - \frac{x_1^2 \omega_{\beta}^2}{2} + \left( 4\beta_{z0} \zeta_1 \omega_{\beta} - \frac{x_1^2 \omega_{\beta}^2}{2} \right) \cos 2\omega_{\beta} t,
\]

where high order terms are neglected, and

\[
\gamma_{z0} = \frac{1}{\sqrt{1 - \beta_{z0}^2}}.
\]
Note \( \omega_\beta \ll 1 \) and \( |\zeta_0| \ll 1 \), the second term is negligible, thus

\[
\gamma = \gamma_0(t) + \left( \frac{\beta_0 \zeta_1}{16} \right) \cos 2\omega_\beta t,
\]

where

\[
\gamma_0(t) = \gamma_0(t_0) - \frac{1}{2} \int_{t_0}^{t} \zeta_0 \beta_0 \zeta_1 \, dt'.
\]

is \( \gamma \) averaged in the \( \omega_\beta^{-1} \) time scale.

We apply equation (14) to equation (11) and multiply both sides by \( \gamma_0^2 \). By collecting the ‘slowly’ varying parts one obtains

\[
\gamma_{\text{av}} = \frac{\gamma_0}{\sqrt{1 + \frac{\zeta_0^2}{4}}} = \frac{\gamma_0}{\sqrt{1 + \frac{x_0^2}{4}}}
\]

which can be applied to equation (12) for the averaged longitudinal velocity \( \beta_{\text{av}} \). Note

\[
\omega_\beta \approx \frac{1}{\sqrt{2\gamma_0}}
\]

because \( \gamma = \gamma_0 + \mathcal{O}(x_0^2) \). \( \gamma_{\text{av}} \) is regarded as the phase-locking Lorentz factor. If \( \gamma_w = \gamma_{\text{av}} \) and \( \zeta_0 = 0 \) initially, \( \zeta_0 \) will remain 0 and there will not be net acceleration or deceleration. By collecting the ‘fast’ oscillation parts one finds the longitudinal oscillation amplitude

\[
\zeta_1 = \frac{1 - 4\omega_\beta^2 \gamma_0^3}{1 - 8\omega_\beta^2 \gamma_0^3} \approx \frac{\omega_\beta x_0^2}{8} \approx \frac{x_0^2}{8\sqrt{2\gamma_0}}
\]

We show two numerical results obtained by PTracker (i.e. solving equations (1)–(6)) in figure 1. The initial parameters are \( \gamma_0 = 10^4, x_1 = 0.1, \zeta_0 = 0 \) and \( \zeta_0 = 161.16 = \gamma_{\text{av}} \) according to equation (16). We also confirm the phase-locking Lorentz factor equation (16) by comparing the two subplots, i.e. figure 1(a) shows a phase-locking case with \( \gamma_w = \gamma_{\text{av}} \) and figure 1(b) shows a phase drifting case with \( \gamma_w > \gamma_{\text{av}} \).
3. Betatron amplitude during acceleration or deceleration

In this section we consider the effect of slowly varying $\gamma_0$. We only keep the first order variation. Within a time scale $t \sim \omega_\beta^{-1} \ll \gamma_0/\dot{\gamma}_0$, the change of $\gamma_0$ is

$$\Delta \gamma_0 = \dot{\gamma}_0 t. \quad (19)$$

As a result, the amplitude and frequency of BO also change

$$\Delta x_1 = \ddot{x}_1 t, \quad (20)$$
$$\Delta \omega_\beta = \frac{1}{2} \ddot{\omega}_\beta t. \quad (21)$$

Note the factor $1/2$ in equation (21) comes from the fact that a linear chirp contributes twice to the frequency shift. BO still follows equation (7) but with slow-varying $x_1$ and $\omega_\beta$

$$x = (x_1 + \Delta x_1) \sin (\omega_\beta + \Delta \omega_\beta) t$$
$$= (x_1 + \ddot{x}_1 t) \sin \phi, \quad (22)$$

where $\phi \equiv (\omega_\beta + \frac{1}{2} \ddot{\omega}_\beta t) t$. According to equations (1) and (3) with $\beta_\perp \approx 1$, we have

$$\dot{p}_x = -\frac{1}{2} x_1 \left(1 + \frac{\ddot{x}_1}{x_1} t\right) \sin \phi. \quad (23)$$

Meanwhile, we write down

$$p_x = (\gamma_0 + \Delta \gamma_0) \times \dot{x}$$
$$= \gamma_0 x_1 \omega_\beta \left(1 + \frac{\ddot{x}_0}{\gamma_0} t + \frac{\ddot{x}_1}{x_1} t + \frac{\ddot{\omega}_\beta}{\omega_\beta} t\right) \cos \phi + \gamma_0 \ddot{x}_1 \sin \phi, \quad (24)$$

where $\ddot{x}_0, \ddot{x}_1, t$ and $\ddot{\omega}_\beta t$ are in the same order, and higher order terms are neglected. Then we take derivative one more time

$$\dot{p}_x = -\gamma_0 x_1 \omega_\beta^2 \left(1 + \frac{\ddot{x}_0}{\gamma_0} t + \frac{\ddot{x}_1}{x_1} t + 2 \frac{\ddot{\omega}_\beta}{\omega_\beta} t\right) \sin \phi + \gamma_0 x_1 \omega_\beta \left(\frac{\ddot{x}_0}{\gamma_0} + 2 \frac{\ddot{x}_1}{x_1} + \frac{\ddot{\omega}_\beta}{\omega_\beta}\right) \cos \phi. \quad (25)$$

We substitute left-hand-side of equation (23) by equation (25), the 0th order of the ‘sin’ term retrieves equation (17). The 1st order of the ‘sin’ and ‘cos’ terms construct two equations

$$\frac{\ddot{x}_0}{\gamma_0} + 2 \frac{\ddot{\omega}_\beta}{\omega_\beta} = 0, \quad (26)$$
$$\frac{\ddot{\gamma}_0}{\gamma_0} + 2 \frac{\ddot{x}_1}{x_1} + \frac{\ddot{\omega}_\beta}{\omega_\beta} = 0, \quad (27)$$

which lead to

$$\frac{\ddot{\omega}_\beta}{\omega_\beta} = -\frac{1}{2} \frac{\ddot{x}_0}{\gamma_0}, \quad (28)$$
$$\frac{\ddot{x}_1}{x_1} = -\frac{1}{4} \frac{\ddot{\gamma}_0}{\gamma_0}. \quad (29)$$

By integral, we obtain the long-term dependencies of $\omega_\beta$ and $x_1$ on $\gamma_0$

$$\omega_\beta \propto \gamma_0^{-\frac{1}{2}}, \quad (30)$$
$$x_1 \propto \gamma_0^{-\frac{1}{4}}. \quad (31)$$

The amplitude of $p_x$ oscillation is

$$p_{x1} = x_1 \gamma_0 \omega_\beta = \frac{1}{\sqrt{2}} x_1 \gamma_0^{\frac{1}{4}}. \quad (32)$$

As a result, the area encircled by the trajectory of the particle in the $x$–$p_x$ phase space,

$$S = \pi x_1 p_{x1} = \frac{\pi}{\sqrt{2}} x_1^\frac{3}{2} \gamma_0^{\frac{1}{4}} = \frac{\pi}{\sqrt{2}} A^2, \quad (33)$$
is a constant with varying $\gamma_0$, where $A$ is defined as the normalized BO amplitude

$$A \equiv x_1 \gamma_0^{\frac{1}{4}}. \quad (34)$$

We check the scaling equation (31) using PTracker. Two cases with significant acceleration and deceleration are shown in figure 2. Because $\zeta_0 = 0$ and $\gamma_{z0} < \gamma_w$ at $t = 0$, the electron firstly drifts to the acceleration phase, and $\gamma_0$ increases so that $\gamma_{z0} > \gamma_w$ at some point. Later it crosses the $\zeta = 0$ point, enters the deceleration phase and $\gamma_0$ starts to decrease. The two numerical cases show exact scaling of $x_1 = A \gamma^{-1/4}$.

Note we have replaced $\gamma_0$ by $\gamma$ here, because their difference is not distinguishable in these plots, and $\gamma$ can be obtained much more easily than $\gamma_0$ from numerical result.

4. Radiation reaction effects on the betatron oscillation

The Lorentz–Abraham–Dirac equation [30–32] in the same normalized units as in section 2 is

$$F_{\mu}^{rad} = \frac{2}{3} r_e \left[ \frac{d^2 p_\mu}{d\tau^2} - p_\mu \frac{dp_\nu}{d\tau} \frac{dp_\nu}{d\tau} \right]. \quad (35)$$

with the metric $(-1, 1, 1, 1)$, where $\tau$ is the proper time ($\gamma d\tau = dt$), $r_e$ is the classical electron radius (also normalized to $k^{-1}$), $p^\mu = (\gamma, \vec{p})$ is the four-momentum, and $F_{\mu}^{rad}$ is the four-force of RR. The space component of equation (35) is

$$F_{\mu=x,y,z}^{rad} = \gamma F_{x,y,z}^{rad} = \frac{2}{3} r_e \left[ \gamma \frac{d\gamma}{d\tau} \frac{\dot{\vec{p}}}{\gamma} - \frac{\vec{p}}{\gamma} \left( -\gamma^2 + |\vec{p}|^2 \right) \right]. \quad (36)$$

The three-dimensional (3D) RR force $\vec{F}^{rad}$ is decomposed to

$$\vec{F}^{rad1} = \frac{2}{3} r_e \frac{d\gamma}{d\tau} \frac{\dot{\vec{p}}}{\gamma}, \quad (37)$$

$$\vec{F}^{rad2} = -\frac{2}{3} r_e \vec{p} \gamma \left( |\vec{p}|^2 - \gamma^2 \right). \quad (38)$$

With the RR force, the equation of motion equation (3) is modified as

$$\dot{\vec{p}} = \vec{f}^{ext} + \vec{F}^{rad1} + \vec{F}^{rad2}. \quad (39)$$

4.1. Radiation reaction force of betatron oscillation

We assume RR is perturbation to the electron BO which has the form discussed in section 2,

$$\gamma = \gamma_0 + \frac{1}{32} x_1^2 \cos 2\omega_3 t, \quad (40)$$

$$p_x = x_1 \omega_3 \gamma_0 \cos \omega_3 t, \quad (41)$$

$$p_z = \beta_0 \gamma_0 - \frac{3}{32} x_1^2 \cos 2\omega_3 t, \quad (42)$$
\[ \dot{\gamma} = \gamma_0 - \frac{1}{16} x_1^2 \omega_\beta \sin 2 \omega_\beta t, \]  
(43)

\[ \dot{p}_x = -\frac{1}{2} x_1 \sin \omega_\beta t, \]  
(44)

\[ \dot{p}_z = -\frac{1}{2} \gamma_0 + \frac{3}{16} x_1^2 \omega_\beta \sin 2 \omega_\beta t. \]  
(45)

Note equation (15), as long as \(|\gamma_0| \sim |\varpi| \lesssim 1 \ll \omega_\beta^{-1}\), we have

\[ |\mathbf{p}|^2 - \gamma_0^2 \approx \mathbf{p}_x^2. \]  
(46)

Note \( \dot{\zeta}_0 \approx \gamma_0^2 / 2 - x_1^2 \gamma_0^{-1} / 8 \). According to equations (37) and (38), the RR force terms are

\[ f_{x_{\text{rad1}}} = -\frac{1}{3} r_x x_1 \gamma_0 \omega_\beta \cos \omega_\beta t = -\frac{1}{3} \gamma_0 \rho_x, \]  
(47)

\[ f_{x_{\text{rad2}}} = -\frac{1}{12} r_x x_1^2 (1 - \cos 2 \omega_\beta t), \]  
(48)

\[ f_{z_{\text{rad2}}} = -\frac{1}{12} r_x x_1^2 (1 - \cos 2 \omega_\beta t). \]  
(49)

Use equations (31) and (34), we have \( x_1^2 \gamma_0 \gg 1 \) and conclude \( |f_{x_{\text{rad1}}}^\text{rad1}| \gg |f_{x_{\text{rad1}}}^\text{rad2}| \). Use equation (17) and \( |\varpi_0| \lesssim 1 \), we conclude \( |f_{z_{\text{rad2}}}^\text{rad2}| \gg |f_{z_{\text{rad2}}}^\text{rad2}| \). We also have

\[ \frac{f_{z_{\text{rad2}}}^\text{rad2}}{f_{z_{\text{rad2}}}^\text{rad2}} = \beta_x = x_1 \omega_\beta \cos \omega_\beta t \propto x_1 \gamma_0^{-\frac{3}{2}} \propto \gamma_0^{-\frac{1}{2}} \]  
(51)

4.2. Perturbation correction and phase-locking with longitudinal forces

In section 2 we have obtained the phase-locking Lorentz factor \( \gamma_{\varpi_0} \) in equation (16). With RR, extra longitudinal deceleration force exists. Define the phase-locking \( \zeta_0 \) to be

\[ \zeta_0 \equiv -\frac{1}{6} r_x x_1^2 \gamma_0^2, \]  
(52)

the summation of the longitudinal external and RR forces is written by using equations (2), (7)–(9), (18) and (50)

\[ f_z = \frac{\zeta_0 - \zeta_0}{2} + \frac{3}{16} x_1^2 \gamma_0 \sin 2 \omega_\beta t + \frac{1}{12} r_x x_1^2 \gamma_0 \cos 2 \omega_\beta t. \]  
(53)

If \( \zeta_0 = \zeta_0 \), the longitudinal momentum gain averaged in the \( \omega_\beta^2 \) time scale is zero. If \( \gamma_0 = \gamma_{\varpi_0} \) is also satisfied, the phase-locking will occur and there will not be net acceleration or deceleration. The 2nd term, coming from BO, is already included in equations (43) and (45). In cases the 3rd term, coming from RR, is comparable or more significant than the 2nd term, i.e. \( r_x \gamma_0^{\frac{3}{2}} \gtrsim 1 \), equations (40), (42), (43) and (45) should be modified

\[ \gamma = \gamma_0 + \frac{1}{32} x_1^2 \cos 2 \omega_\beta t + \frac{1}{24} r_x x_1^2 \gamma_0 \sin 2 \omega_\beta t, \]  
(54)

\[ p_z = \beta_{\varpi_0} \gamma_0 - \frac{3}{32} x_1^2 \cos 2 \omega_\beta t + \frac{1}{24} r_x x_1^2 \gamma_0 \sin 2 \omega_\beta t, \]  
(55)

\[ \dot{\gamma} = \dot{\gamma}_0 - \frac{1}{16} x_1^2 \omega_\beta \sin 2 \omega_\beta t, \]  
(56)

\[ \dot{p}_z = -\frac{\zeta_0 - \zeta_0}{2} + \frac{3}{16} x_1^2 \gamma_0 \sin 2 \omega_\beta t + \frac{1}{12} r_x x_1^2 \gamma_0^2 \cos 2 \omega_\beta t, \]  
(57)

where the \( \gamma_0 \) definition equation (15) is modified as

\[ \gamma_0 (t) = \gamma_0 (t_0) = \frac{1}{2} \int \zeta_0 \left( \dot{\zeta}_0 - \zeta_0 \right) \beta_{\varpi_0} dt'. \]  
(58)

In the regime \( r_x \gamma_0 \ll 1 \), equation (46) still holds, and in equations (47)–(50), only equation (48) has to be modified

\[ f_z^\text{rad2} = \frac{2}{3} r_x \gamma_0^2 + \frac{1}{8} r_x x_1^2 \cos 2 \omega_\beta t - \frac{1}{9} r_x x_1^2 \gamma_0^3 \omega_\beta \sin 2 \omega_\beta t. \]  
(59)
Note that $|f^{\text{rad}2}| \gg |f^{\text{rad}1}|$ still holds, the above discussion is self-consistent. One may compare our equations (56) and (57) with equations (30) and (32) of reference [33] by taking $K^2 = k_0^2/2$ and find that our model keeps more details of the BO, while they have omitted the difference between $\dot{\gamma}$ and $\ddot{p}_x$. These details, as shown in section 4.3, are important for the long term RR effect.

We also estimate the maximum $\gamma_0$ achievable in PW A, due to the limited size of the blowout wakefield structure, by setting $\zeta_0 \sim -1$ in equation (52) and using equation (33):

$$\gamma_{0\text{max}} \sim \left( \frac{3\sqrt{2}\pi}{r_0S} \right)^\frac{1}{2} = \left( \frac{6}{r_0 A^2} \right)^\frac{1}{2}. \quad (60)$$

One should note that this maximum energy is not due to dephasing or pump-depletion effects. It is the power of radiation loss approximately equals to the maximum power of acceleration in the wakefield.

The numerical solution for a case of phase-locking by PT racker with RR, i.e. equation (3) be replaced by equation (39), is shown in figure 3(a). With the parameters at $t = 0$ being $\gamma_0 = 10^5, x_1 = \gamma_{01}/4, \gamma_w = \gamma_0$, according to equation (16) and $\zeta_0 = \zeta_{00}$ according to equation (52), the numerical result shows a phase-locking with an oscillation amplitude $\zeta_1 = 8.84 \times 10^{-7}$ which confirms equation (18). The amplitude of the oscillation of $\dot{\gamma}$ is $7.43 \times 10^{-4}$ according to the 3rd term of equation (56), which is more significant than the 2nd term in this case.

4.3. Damping of betatron oscillation due to radiation fraction

The potential and the kinetic energy in the transverse direction transform to each other due to BO, with the maximum potential energy being

$$U = \frac{x_1^2}{4}. \quad (61)$$

Meanwhile, this potential energy is slowly lost due to the work of dissipation force averaged in the $\omega_\beta^{-1}$ time scale. The averaged power of the transverse force is

$$\langle \beta_p f_x \rangle = \langle \beta_p (x_1^{\text{ext}} + x_1^{\text{rad}1} + x_1^{\text{rad}2}) \rangle$$

$$= 0 - \frac{1}{3} r_0 \left( \frac{p_z^2}{\gamma} \right) - \frac{1}{12} r_0 x_1^2 \gamma_0 \left( 1 - \cos 2\omega_\beta t \right) \frac{p_z^2}{\gamma}$$

$$= -\frac{1}{12} r_0 x_1^2 \left( 1 + \frac{1}{8} x_1^2 \gamma_0 \right)$$

$$\approx -\frac{1}{96} r_0 x_1^4 \gamma_0. \quad (62)$$

Figure 3. Numerical results using PT racker with RR turned on. The length normalization unit is $k_0^{-1} = 10^{-5}$ m, thus the normalized classical electron radius is $r_0 = 2.82 \times 10^{-10}$. The parameters at $t = 0$ are $\gamma_0 = 10^5$, $x_1 = \gamma_{01}/4$, $\gamma_w = \gamma_0$, and $\zeta_0 = \zeta_{00}$. (a) The oscillation of $\dot{\gamma}$ and $\gamma$ at an early stage. The black thin lines at $\pm 8.84 \times 10^{-7}$ mark the theoretical oscillation range of $\dot{\gamma}$ according to equation (18), and the red thin lines at $\pm 7.43 \times 10^{-4}$ mark the theoretical oscillation range of $\gamma$ according to equation (56). (b) The damping of $\dot{\gamma}$ defined by equation (33), in a long time scale. Numerical solution of $\dot{\gamma}$ (black solid curve) is compared to equation (69) (red dashed curve) and (68) with numerical integral (blue dashed curve).
and the averaged power of the longitudinal forces by using equations (53)–(55) is

\[
\langle \beta f_z \rangle = \left( \beta_{\infty} - \frac{1}{8} \frac{x^2_{1}}{\gamma_0} \cos 2\omega_p t \right) \left( -\frac{\zeta_0 - \zeta_{\infty}}{2} + \frac{3}{16} \frac{x_{1}^2}{\omega_{p}} \sin 2\omega_{p} t + \frac{1}{12} \frac{r_{e} x_{1}^2}{\gamma_0} \cos 2\omega_{p} t \right)
\]

\[= -\frac{1}{2} (\zeta_0 - \zeta_{\infty}) \beta_{\infty} - \frac{1}{12} \frac{r_{e} x_{1}^4}{\gamma_0}, \tag{63} \]

where the angle brackets stands for averaging in the \(\omega_p^{-1}\) time scale. Note the 1st term in the right-hand-side of equation (63) contributes to the longitudinal acceleration or deceleration, while the 2nd term, coming from the coupling of the oscillation terms of \(\beta_z\) and \(f_z^{rad}\), contributes to the transverse damping of BO.

Thus

\[\hat{U} = \langle \beta f_z \rangle + \langle \beta f_z \rangle \mid_{\text{2nd term}} = -\frac{1}{64} \frac{r_e x_{1}^4}{\gamma_0}, \tag{64} \]

which can be applied to equation (61) for the damping rate of the BO amplitude due to RR

\[\frac{\dot{x}_{1}}{x_1} \mid_{\text{rad}} = -\frac{1}{32} \frac{r_e \gamma_0 x_{1}^2}{\gamma_0}. \tag{65} \]

Note equation (29), the total damping rate of \(x_1\) is

\[\frac{\dot{x}_{1}}{x_1} = -\frac{1}{32} \frac{r_e \gamma_0 x_{1}^2}{\gamma_0} - \frac{1}{4} \frac{\dot{\gamma}_0}{\gamma_0}. \tag{66} \]

Use equations (33) and (66), we find the damping rate of the area encircled by the trajectory of the particle in the \(x-p_x\) phase space

\[\frac{\dot{S}}{S} = -\frac{1}{16} \frac{r_e \gamma_0 x_{1}^2}{\gamma_0} = -\frac{1}{8\sqrt{2\pi}} \frac{r_e \gamma_0}{\gamma_0} \frac{\dot{\gamma}_0}{S}, \tag{67} \]

or in the integrated form

\[\frac{1}{S(t)} \mid_{t_0}^t = \frac{1}{S(t)} - \frac{1}{S(t_0)} = \frac{1}{8\sqrt{2\pi}} \int_{t_0}^t \frac{\dot{\gamma}_0}{\gamma_0} \, dt'. \tag{68} \]

In the time scale that \(\gamma_0\) does not change significantly, it can be simplified as

\[\frac{1}{S} \mid_{t_0}^t = \frac{1}{8\sqrt{2\pi}} \frac{r_e \gamma_0}{\gamma_0} \cdot (t - t_0). \tag{69} \]

Thus the length that \(S\) reduces by a half is

\[L_S = \frac{8\sqrt{2\pi}}{r_e \sqrt{\gamma_0} S} = \frac{16}{r_e \sqrt{\gamma_0} A^2}, \tag{70} \]

or in the unnormalized form (note \(S\) and \(A\) remain in normalized units)

\[k_p L_S = \frac{8\sqrt{2\pi}}{k_p r_e \sqrt{\gamma_0} S} = \frac{16}{k_p r_e \sqrt{\gamma_0} A^2}. \tag{71} \]

For an electron beam which has a distribution of \(S\), the damping rates are different for different \(S\). The reduction rate of the normalized emittance \(\epsilon_n \approx S/k_p\) is estimated by equation (67), but with \(S\) replaced by the largest \(S\) among the electrons. Thus, we obtain the engineering formula of the normalized emittance reduction length

\[L_{\epsilon_n} [m] = \frac{1.89 \times 10^6}{\sqrt{\gamma_0 (n_p [10^{18} \text{ cm}^{-3}])^3 \cdot \epsilon_n \text{ [mm mrad]}}. \tag{72} \]

The numerical solution for the S damping in a long time scale (\(\Delta t = 10^6 > L_S = 1.8 \times 10^6\)) with the same parameters as figure 3(a) is shown in figure 3(b). Equation (69) predicts the numerical result perfectly until at about \(t = 10^8\), because \(\gamma_0\) starts to change due to the breaking of phase-locking condition while \(x_1\) is damped. Equation (68) with numerical integral matches the numerical result even better at an extremely long time scale, but the integral of \(\gamma_0^{1/2}\) is replaced by the integral of \(\gamma^{1/2}\) because \(\gamma\) can be much more easily obtained than \(\gamma_0\) from numerical result.
5. From classical to quantum radiation domain

The correction by nonlinear quantum electrodynamics (QED) has been studied for high-intensity laser-electron interactions [34–36]. The radiation power with nonlinear QED correction is

\[
P = q(\chi)P_{\text{classical}},
\]

(73)

where \(P_{\text{classical}}\) is the classical radiation power, and \(q(\chi)\) is the QED correction factor [37–39]. \(q(\chi) \approx 1\) with \(\chi \ll 1\) is the classical regime of the radiation process, while \(q(\chi) < 1\) with \(\chi \gtrsim 1\) is the QED regime. We calculate the significance of this correction in our model.

RR is effective when a background force is strong enough and perpendicular to the electron momentum in the relativistic regime. This perpendicular direction is primarily the \(x\)-direction in our case. We write down the effective vector potential amplitude of the wakefield

\[
\mathcal{W}_0 = \mathcal{O}(p_{\text{ext}}) \sim x_1 \gamma \frac{\alpha}{\gamma},
\]

(74)

which is an analogy of the normalized laser vector potential amplitude \(a_0\) in laser-electron interactions.

When we consider a radiation process to be nonlinear Compton scattering by a laser pulse, the formation phase (or the formation length) \(a_0^{-1}\) is important for the locally constant field approximation (LCFA) in the semi-classical model [34–37, 39–41]. Namely, a single photon is emitted instantaneously in a short phase interval \(a_0^{-1}\) at high intensities. By analogy, the formation phase in the wakefield is \(\mathcal{O}(\mathcal{W}_0^{-1})\) which is short enough if \(\gamma \gg 1\). Thus, LCFA of a radiation process can be employed for the high-energy electron case.

Also, the quantum correction \(q(\chi)\) can be applied, with the dimensionless quantum parameter

\[
\chi = \frac{r_e}{\alpha} |F_\mu^\text{ext}| = \frac{r_e}{\alpha} \sqrt{-\left(\gamma f^\text{ext} \cdot \beta\right)^2 + |f^\text{ext}|^2} \\
= \frac{r_e}{2\alpha} \sqrt{\gamma^2(1-\beta_z^2) + \chi^2(1-\beta_z^2)} \\
\approx \frac{r_e}{2\alpha} \sqrt{\frac{\chi}{\gamma}},
\]

(75)

where \(\alpha\) is the fine structure constant, \(F_\mu^\text{ext} = (-\gamma f^\text{ext} \cdot \beta, \gamma f^\text{ext})\) is the four-force of the wakefield, \(\beta_z \sim x_1 \gamma^{-1/2} \ll 1\) and \(1 - \beta_z^2 \approx \gamma a_0^2 \sim x_1^2 / \gamma \ll x^2\). Note \(r_e\) is normalized to \(k_p^{-1}\) in the expression. The maximum \(\chi\) in one BO cycle is

\[
\chi_{\text{max}} \approx \frac{r_e}{2\alpha} x_1 \gamma = \frac{r_e}{2\alpha} A \gamma \frac{\gamma}{\gamma} = \frac{r_e}{\alpha} \sqrt{\frac{\gamma}{\gamma}}.
\]

(76)

For \(\chi_{\text{max}} \ll 1\), RR is classical, while for \(\chi_{\text{max}} \gtrsim 1\), quantum correction may appear in equation (73).

Commonly, PWA with internal injections has \(A \sim 1\). Thus for \(k_p \sim 10^3\ \text{m}^{-1}\) (\(r_e \sim 10^{-10}\) as a consequence), RR is classical if \(\gamma \ll 10^{10}\).

6. Summary and discussion

We have developed a classical model of the electron BO with RR in a plasma wakefield accelerator. We found a phase-locking condition \(\gamma_w = \gamma_{\text{init}}\) and \(\gamma_0 = \gamma_{\text{init}}\) using equations (16) and (52), under which the electron does not gain or loss energy. The maximum \(\gamma_0\) achievable in a plasma wakefield accelerator due to RR is estimated by equation (60). We also found the damping rate of \(S\), defined as the area encircled by the electron trajectory in the \(x-p_x\) phase space, in equation (67) and the length that \(S\) reduces by a half in equation (70). The quantum parameter characterizing the classical and quantum radiation domain is given by equation (76).

Some examples with different plasma densities \(n_p\), initial \(S\) and \(\gamma_0\) are listed in table 1. We can see that the reduction of \(S\) (and thus the transverse cooling) is positively related to (1) the plasma density, (2) the Lorentz factor of the electron, (3) the initial betatron amplitude and (4) the total length of the plasma wakefield accelerator. We also note that the quantum parameter \(\chi_{\text{max}}\) is sufficiently smaller than unity for all these cases.

In order to decrease \(L_0\) without an extremely large \(\gamma_0\), we can either increase \(n_p\) or \(S\). Although an internally injected electron beam has \(S \sim 2\), a large \(S\) is achievable by an external injection. In the two-stage scheme where the first stage with a smaller \(n_p\) is used for acceleration and the second stage with a larger \(n_p\) is used for radiation [42], both \(n_p\) and \(S\) are large for the second stage. Thus RR takes effect in a much
Table 1. Some examples of $\gamma_{\text{max}}$, $L_z$ and $\chi_{\text{max}}$ with varying $n_p$, $S$ and $\gamma_0$.

| Case No. | $n_p$ (cm$^{-3}$) | $k_p$ (m$^{-1}$) | $S$ | $\gamma_{\text{max}}$ | $\gamma_0$ | $L_z$ (m) | $\chi_{\text{max}}$ |
|----------|-------------------|-----------------|-----|------------------------|-----------|---------|---------------|
| 1        | $10^{18}$         | $1.88 \times 10^5$ | 2   | $5.4 \times 10^6$     | $5 \times 10^6$ | 563.2   | $1.9 \times 10^{-4}$ |
| 2        | $10^{19}$         | $5.95 \times 10^6$ | 2   | $2.1 \times 10^6$     | $2 \times 10^6$ | 31.5    | $3.7 \times 10^{-3}$ |
| 3        |                   |                 | 8   | $1.2 \times 10^7$     | $1 \times 10^7$ | 5632    | $6.1 \times 10^{-3}$ |
| 4        |                   |                 | 8   | $4.6 \times 10^8$     | $4 \times 10^8$ | 222.6   | $2.0 \times 10^{-3}$ |

shorter distance in this case. However, $\chi$ may be not negligible for large $S$ values and quantum RR should be taken into consideration.

The actual trajectory of an electron in a 3D wakefield could be an ellipse in the transverse space coordinate plane ($x$–$y$ plane). In this case, the direction of the major (minor) axis of the ellipse should be interpreted as the $x$ ($y$) axis. We introduce $S_x$ ($S_y$) to be the area encircled by the trajectory in the $x$–$p_x$ ($y$–$p_y$) plane. For most cases in a wakefield, the angular momentum of an electron is negligible so that $S_x \gg S_y$, thus the RR in $y$ direction can be neglected, and the present model is applied well. In some special cases $S_x \approx S_y$, which means the electron has considerable angular momentum and performs helical motion in 3D space [43], a similar model can be established as a future work. The present model has neglected the evolution of the bubble wakefield and the plasma non-uniformity [44, 45], which can also be considered in the future.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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