PRACTICAL SOLUTION OF SOME FAMILIES OF QUARTIC 
DIOPHANTINE HYPERELLIPTIC EQUATIONS 

KONSTANTINOS A. DRAZIOTIS 

Abstract. Using elementary number theory we study Diophantine equations 
over the rational integers of the following form, $y^2 = (x + a)(x + a + b)(x + 
b + k)$ and $y^2 = x^2 x^4 + ax^2 + b$. We express their integer solutions by 
means of the divisors of the discriminant of $f(x)$, where $y^2 = f(x)$. 

1. Introduction 

Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial which is not a square. We consider the 
hyperelliptic curve, 

$y^2 = f(x), \ 2 \deg f \text{ and } \deg f \geq 4.$ 

One way to study the integer points of curve (1.1) is to use the so called Runge's 
method [1, 4, 12, 14] (see Appendix A for a brief summary of this method). In fact, 
Runge proved the finiteness of the integer points of equation (1.1) in 1887 [12]. 

In [5, 19], using Runge’s method with a combination of an effective version of 
Eisenstein theorem [15], the authors provided a uniform upper bound for the size 
of their integral points on curves of the form (1.1). Also in [6], the bounds were 
further improved for some specific cases. 

In [18], the author considers more general equations of the form 

$F(x) = G(y), \ F, G \text{ monic, } F(x) - G(y) \text{ irreducible in } \mathbb{Q}[x, y], 

where \deg F = n, \ \deg G = m (m \geq n)$ are such that, $\gcd(n, m) > 1$. With $H(F)$ we 
denote the height of $F$ i.e. the maximum of the absolute values of the coefficients 
of polynomial $F$. Then (using big-O notation), 

$max(|x|, |y|) = O((2hm)^{4m^2}) \ (h = \max(H(F), H(G)), 

for some effective computable constant. For the full details of the bound see [18, 
Theorem 2.2.1] and for detailed references of Runge’s method see, [18, Section 2.1]. 

For the special case $\deg f = 4$, we get the quartic hyperelliptic curve, 

$y^2 = x^4 + ax^3 + bx^2 + cx + d.$ 

Masser showed that (see, [7]), 

$|x| \leq 26H(f)^3,$ 

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method. 

1A more general result was proved, where our curve is a special case. See Appendix C for the 
general curves that satisfy Runge’s condition.
2.1. Main idea. Let \( y^2 = f(x) \), with \( f(x) \in \mathbb{Z}[x] \) not a square, \( \deg f = 4 \), and \( D(z) = \text{Res}_x(f(x) + z, f'(x)) \), where with \( \text{Res}_x(\cdot, \cdot) \) we denote the resultant with respect to indeterminate \( x \). The polynomial \( D(z) \in \mathbb{Z}[z] \), has degree 3 and assume that there exists \( z_0 \in \mathbb{Q} \) which is a double root of the equation \( D(z) = 0 \). Then, under some plausible geometric conditions, \( f(x) + z_0 = r(x)^2 \), for some \( r(x) \in \mathbb{Q}[x] \).

In order this last identity to be useful to us, we need one further assumption, except the existence of a rational double root of \( D(z) = 0 \). We assume that \( z_0 = A/B^2 \), with \( \gcd(A, B) = 1 \). Let \( (x_0, y_0) \in \mathbb{Z}^2 \) such that, \( y_0^2 - f(x_0) = 0 \).

On the other hand, we have \( B^2(y_0^2 - f(x_0)) = 0 \iff B^2 y_0^2 - (r(x_0)^2 - A) = 0 \iff (r(x_0) - B y_0)(r(x_0) + B y_0) = A \).

Then, using factorization we can calculate explicit formulas for \( x_0 \) and \( y_0 \).
In order to shed more light about the geometric condition, we provide two examples. Let \( f(x) = 49x^4 - 15x^2 - 2 \), then
\[
D(z) = \text{Res}_x(f(x) + z, f'(x)) = 38416(z - 2)(196z - 617)^2,
\]
and \( f(x) + z_0 = f(x) + \frac{617}{196} = \frac{1}{196}(98x^2 - 15)^2 \).
If \( g(x) = x^4 + 225x^3 + 49 \), then
\[
D(z) = \text{Res}_x(g(x) + z, g'(x)) = (256z - 69198034331)(z + 49)^2,
\]
and \( g(x) + z_0 = g(x) - 49 = x^3(x + 225) \). I.e. both \( f(x), g(x) \) have a square in the decomposition of their resultant, but \( g(x) + z_0 \) does not contain a square in its decomposition. This is because the point \((0, -49)\) of the curve \( z + g(x) = 0 \) is inflexion point.

2.2. The curve \( y^2 = (x + a)(x + a + k)(x + b)(x + k) \). We begin with the study of the equation:
\[
y^2 = f(x) = (x + a)(x + a + k)(x + b)(x + b + k).
\]
We shall prove the following Proposition.

Proposition 2.1. Let \( a, b, k, \) be three integers with \( a \neq b \) (if \( a = b \), then \( f(x) \) is a square). We set
\[
C = 2ab + a + b - (d_1 + d_2)/2 \quad \text{and} \quad \Delta = 4((a + b + 1)^2 - 2C),
\]
where \( d_1, d_2 \) are integers of the same parity and such that \( d_1d_2 = (ka - kb)^2 \). Assume that \( \Delta = \delta^2 \), for some \( \delta \in \mathbb{Z} \). Then, the integer solutions of Diophantine equation \( y^2 = f(x) \), are of the form:
\[
(x_0, |y_0|) = \left(-\frac{a + b + k}{2} \pm \frac{\delta}{4} \frac{|d_1 - d_2|}{4}\right).
\]

Proof. Let \( G(x, y) = y^2 - f(x) \) and
\[
D(z) = \text{Res}_x(f(x) + z, f'(x)) = M(z)\left((k(a - b))^2 - 4z\right)^2,
\]
for some linear polynomial \( M(z) \). We set \( z_0 = \frac{(k(a-b))^2}{4} \), then
\[
4(G(x, y) - z_0) = 4G - (k(a - b))^2 = -(r(x) - 2y)(r(x) + 2y),
\]
where
\[
r(x) = 2x^2 + 2x(a + b + k) + 2ab + ka + kb.
\]
Note that, \( k^2(a - b)^2 \) divides the discriminant of \( f(x) \). Let \( (x_0, y_0) \in \mathbb{Z}^2 \) such that, \( G(x_0, y_0) = 0 \). Then, there exist integers \( d_1 \) and \( d_2 \) such that,
\[
r(x_0) - 2y_0 = d_1, r(x_0) + 2y_0 = d_2, d_1d_2 = (k(a - b))^2,
\]
and so,
\[
r(x_0) = (d_1 + d_2)/2, y_0 = (d_2 - d_1)/4, d_1d_2 = (k(a - b))^2.
\]
We set
\[
B = 2(a + b + k) \quad \text{and}
\]
from the hypothesis,
\[
C = 2ab + ka + kb - (d_1 + d_2)/2,
\]
\[2\]For a symbolic computation of the resultant you can see: https://github.com/drazioti/simple_quartic/tree/main/resultant
thus we get,

\[ 2x_0^2 + Bx_0 + C = 0, \quad y_0 = \frac{(d_2 - d_1)}{4}, \quad d_1d_2 = (k(a - b))^2. \]

From the second equality we see that \(d_1, d_2\) must have the same parity. Since \(\Delta = \delta^2\), we get

\[ x_0 = \frac{-2(a + b + k) \pm \delta}{4} = \frac{-(a + b + k)}{2} \pm \frac{\delta}{4}. \]

The result follows. \(\square\)

Now, we easily get the following pseudocode.

**Input:** \(a, b, k\)

**Output:** The integer solutions of \(y^2 = (x + a)(x + a + k)(x + b)(x + b + k)\), with \(y \geq 0\).

01. \(\text{DIV} \leftarrow \{ n \in \mathbb{Z} : n|(ka - kb)^2\} \)
02. \(B \leftarrow 2(a + b + 1)\)
03. \(L = [\ ] \# \text{this is the list where we keep the solutions}\)
04. For \(d_1\) in \(\text{DIV}\)
05. \(d_2 \leftarrow (ka - kb)^2/d_1\)
06. \(C \leftarrow 2ab + a + b - (d_1 + d_2)/2\)
07. \(\Delta \leftarrow B^2 - 8C\)
08. If \((d_1 \equiv d_2 \mod 2)\) AND \((\Delta\) is a square, say \(\delta^2\)\) AND \((d_1 \leq d_2)\)
09. \(x_{1,2} \leftarrow -\frac{a + b + k}{2} \pm \frac{\delta}{4}\)
10. \(y \leftarrow \frac{d_2 - d_1}{4}\)
11. Append list \(L\) with \((x_1, y), (x_2, y)\)
12. return \(L\)

It is easy to implement the previous algorithm in order to find the integer solutions. The complexity is dominated by the complexity of the algorithm which computes divisors of \((ka - kb)^2\).

We provide two examples. If \(a = 1, b = 2, k = 41\), we have the equation,

\[ y^2 = x^4 + 88x^3 + 2063x^2 + 5588x + 3612 \]

and the integer solutions are (for \(y \geq 0\)):

\[ \{ (7, 420), (-51, 420), (-22, 420), (-1, 0), (-43, 0), (-2, 0), (-42, 0) \}. \]

For the equation,

\[ y^2 = x^4 + 20x^3 + 97x^2 - 30x - 504, \quad (a, b, k) = (3, -2, 9) \]

we get (for \(y \geq 0\)),

\[ \{ (9, 168), (-19, 168), (3, 30), (-13, 30), (-5, 14), (-4, 12), (-6, 12), (2, 0), (-12, 0), (-3, 0), (-7, 0) \}. \]

**Remark 2.2.** If \(f(x) = (x + a)(x + a + k)(x + b)(x + b - k)\), then the study of \(y^2 = f(x)\) can be treated in a similar way.

\(^3\text{For instance, see https://github.com/drazioti/simple_quartic/blob/main/1.py, for an implementation in sagemath [13].}\)
Remark 2.3. If we consider \( a = 0, \ b = 2^k \) and \( k = 1 \), and we use the method of [11], we get an interval that contains \([0, 2^k]\). In this interval we have to search for \( x \), checking one by one if it provides an integer solution \((x, y)\). So, this method is not practical for large \( \ell \). On the other hand, the number of divisors of \((ka - kb)^2 = 2^{2\ell}\) is \(2(2\ell + 1)\). So the complexity is \(O(\ell)\), whereas the complexity of [11] is \(O(2^\ell)\).

Corollary 2.4. If \((x, y)\) is an integer solution of \(y^2 = (x + a)(x + a + k)(x + b)(x + b + k)\), and \(M = \max\{|a|, |b|, |k|\}\), then

\[|x| < M^4 + \frac{3}{2}M.\]

Proof. Since \(|a|, |b|, |k| \leq M, \delta = k^2(a - b)^2 < 4M^4\) and \(|x| < \left\lfloor \frac{a + b + k}{2} \right\rfloor + \frac{1}{2}\) the result follows.

2.3. The curve \(y^2 = c^2x^4 + ax^2 + b\). We continue our study with the equation \(y^2 = f(x) = c^2x^4 + ax^2 + b\), where \(a, b, c\), are integers and \(f(x)\) is not a square.

Proposition 2.5. Let \(a, b, c\) be three integers with \(c \neq 0, \delta = a^2 - 4bc^2 \neq 0\). We set \(f(x) = c^2x^4 + ax^2 + b\) and \(G(x, y) = y^2 - f(x)\). Then, any integer solution \((x_0, y_0)\) of the Diophantine equation \(G(x, y) = 0\), is of the form:

\[(x_0, |y_0|) = \left( \pm \sqrt{\frac{d_1 + d_2 - 2a}{4c^2}}, \left\lfloor \frac{d_1 - d_2}{4c} \right\rfloor \right),\]

for some \(d_1, d_2\), integers of the same parity such that, \(d_1d_2 = \delta\) and assuming that the square root exists.

Proof. We have

\[D(z) = \text{Res}_z(f(x) + z, f'(x)) = 16c^4(z + b)(4c^2z - \delta)^2,\]

and we set \(z_0\) be the double root \(\frac{\delta}{4c^2}\). Then,

\[4c^2(G(x, y) - z_0) = 4c^2G - \delta = -(r(x) - 2cy)(r(x) + 2cy),\]

where \(r(x) = 2c^2x^2 + a\). Without loss of generality, with \(c\) we write the positive square root of \(c^2\).

Let \((x_0, y_0) \in \mathbb{Z}^2\) such that, \(G(x_0, y_0) = 0\). Then,

\[(r(x_0) - 2cy_0)(r(x_0) + 2cy_0) = \delta.\]

Thus, there exist two integers \(d_1, d_2\), such that,

\[r(x_0) - 2cy_0 = d_1, r(x_0) + 2cy_0 = d_2, d_1d_2 = \delta.\]

Therefore,

\[r(x_0) = \frac{d_1 + d_2}{2}, y_0 = \frac{d_2 - d_1}{4c}, d_1d_2 = \delta.\]

From the second equality we note that, \(d_1, d_2\) must have the same parity, else \(y_0 \notin \mathbb{Z}\). We conclude therefore,

\[x_0 = \pm \sqrt{\frac{d_1 + d_2 - 2a}{4c^2}}.\]

If \(x_0\) is integer, we end up with an integer solution of \(y^2 = f(x)\). \(\square\)
We provide the pseudocode.

Input$^4$: $c, a, b$

Output: The integer solutions of $y^2 = f(x) = c^2x^4 + ax^2 + b$, with $y \geq 0$.

01. $d \leftarrow a^2 - 4bc$
02. DIV$\leftarrow \{n \in \mathbb{Z} : n|d\}$
03. $L = []$
04. For $d_1$ in DIV
05.     $d_2 \leftarrow d/d_1$
06.     If $(d_1 \equiv d_2 \mod 2)$
07.         $K \leftarrow \frac{1}{2^c} \left( \frac{d_1 + d_2}{2} - a \right)$
08.         If ($K$ is a square integer, say $m_i^2$) AND ($d_1 \leq d_2$)
09.             $x_{1,2} \leftarrow \pm m_i$
10.             $y \leftarrow \frac{d_1 - d_2}{4c}$
11.             Append list $L$ with $(x_1, y), (x_2, y)$
12. return $L$

Remark 2.6. In magma [2], the command SIntegralLjunggrenPoints([D,A,B,C], [1]), provides the integral points on the curve $C : Dg^2 = Ax^4 + Bx^2 + C$, provided that $C$ is nonsingular. Furthermore, IntegralQuarticPoints([a,b,c,d,e]) provides the integral points on the curve $C : y^2 = ax^4 + bx^3 + cx^2 + dx + e$.

Remark 2.7. For $c = 1$ someone can apply the method of [11]. For instance, if we consider the equation $y^2 = x^4 - 2ℓx^2 + 1$, then following the method of [11] we have to search the integer solutions $x$ in the interval $[-2^\ell, 2^\ell]$, which is exponentially large. In our case we have to find the divisors of the integer $2^{2\ell} - 4$, which is feasible for all $\ell$, say $80 \leq \ell \leq 120$. So our method is practical, whereas method [11] is infeasible for this case$^5$. Furthermore, the case $c = 1$ can be treated by the original Masser’s method with factorization, see Appendix B.

Remark 2.8. In subsection 2.1 we demanded that the double root of the resultant is of the form $A/B^2$, with $\gcd(A, B) = 1$. In the proof of the previous Proposition, we prove that there is always a double root, $z_0 = \frac{x^2 - 4bc}{4c^2}$. But, here it may occur $\gcd(a^2 - 4bc^2, 4c^2) > 1$. We can see that always $\gcd(a^2 - 4bc^2, 4c^2)$ is a square, so after we delete the gcd, we get $z_0 = A'/B'^2$, with $\gcd(A', B') = 1$.

Now, we can easily study also the Diophantine equation $cy^2 = cx^4 + ax^2 + b$.

Corollary 2.9. Let $a, b, c, d$ be three integers with $ac \neq 0$ and $\Delta = a^2 - 4bc \neq 0$. We set $h(x) = cx^4 + ax^2 + b$. Then, the integer solutions of the Diophantine equation $cy^2 = h(x)$, are of the form:

$$(x_0, |y_0|) = \left( \pm \sqrt[4c]{\frac{d_1 + d_2 - 2a}{4c}}, \left| \frac{d_1 - d_2}{4c} \right| \right),$$

for some $d_1, d_2$ integers of the same parity such that $d_1d_2 = \Delta$ and assuming that the square root exists.

$^4$For instance see https://github.com/drazioti/simple_quartic/blob/main/2.py

$^5$In sagemath, with the following code we can compute extremely fast, the prime factorization of $2^{2\ell} - 4$, for all $\ell \in [80, 120] : \{[k, factor(2**(2*k)-4)] \text{ for } k \text{ in range(80,121)]\}$. In https://github.com/drazioti/simple_quartic/blob/main/4.txt we computed all the integer points of the curves $y^2 = x^4 - 2^\ell x^2 + 1$ for $80 \leq \ell \leq 120.$
Proof. It’s the same as in Proposition 2.5, by making the substitution \( c^2 \rightarrow c \). □

Similarly, as previous we get the following pseudocode.

\[ \text{Input}^6: c, a, b \]
\[ \text{Output: The integer solutions of } cy^2 = cx^4 + ax^2 + b, \text{ with } y \geq 0. \]

\[ \begin{align*}
01. \ & \Delta \leftarrow a^2 - 4bc \\
02. \ & \text{DIV} \leftarrow \{ n \in \mathbb{Z} : n|\Delta \} \\
03. \ & L = [ ] \\
04. \ & \text{For } d_1 \text{ in DIV} \\
05. \ & d_2 \leftarrow \Delta/d_1 \\
06. \ & \text{If } (d_1 \equiv d_2 \mod 2) \text{ AND } (d_1 \leq d_2) \\
07. \ & K \leftarrow d_1+2a-2d_2 \text{ AND } (K \text{ is a square integer, say } m^2) \\
08. \ & x_{1,2} \leftarrow \pm m^4 \\
09. \ & y \leftarrow d_2/d_4 \text{ AND } (y \geq 0) \\
10. \ & \text{Append list } L \text{ with } (x_1, y), (x_2, y) \\
11. \ & \text{return } L \\
\end{align*} \]

For instance, for \( c = 6, a = 13, b = 2 \) we get the curve\(^7 6y^2 = 6x^4 + 13x^2 + 2 \). We compute its integer points \((y \geq 0) : (\pm 2, 5)\). For \( c = 12, a = -30, b = -24 \) we get \((\pm 2, 2)\).

### 2.4. One example of sextic hyperelliptic curve.

The following example concerns a sextic hyperelliptic equation, where our method can easily be applied. Let the curve \( C : y^2 = f(x) \), \( f(x) = (x^2 - 1)(x^2 - 4)(x^2 - 9) \). We remark that
\[
\text{Res}_x(f(x) + Z, f'(x)) = 64(27Z + 400)^2(Z - 36)^3.
\]

Although the denominator of \( z_0 \) (the double root) is not square we can work with the other factor, i.e. \((Z - 36)^3\). Then, \( F = y^2 - f(x) \), is such that
\[
F(x, y) - 36 = -(x^3 - 7x + y)(x^3 - 7x - y).
\]
Let \((a, b) \in C(\mathbb{Z})\), then
\[
(a^3 - 7a + b)(a^3 - 7a - b) = 36.
\]
We get
\[
a^3 - 7a + b = d_1, \ a^3 - 7a - b = d_2
\]
where \((|d_1|, |d_2|) = (1, 36), (36, 1), (2, 18), (18, 2), (3, 12), (12, 3), (4, 9), (9, 4), (6, 6)\).
So,
\[
2(a^3 - 7a) = d_1 + d_2 \in \{\pm 37, \pm 20, \pm 15, \pm 14, \pm 12\},
\]
therefore
\[
a^3 - 7a = \pm 10, \pm 7, \pm 6.
\]

\(^6\)For an implementation in sagemath see, https://github.com/drazioti/simple_quartic/blob/main/3.py.

\(^7\)Also, you can try the following code in magma:
\[ C := 6; A := 13; B := 2; \text{IntegralLjunggrenPoints([C,C,A,B],[])} \].

You may use the online calculator http://magma.maths.usyd.edu.au/calc/
Only the equation $a^3 - 7a = \pm 6$ has integer solutions, $a = \pm 1, \pm 2, \pm 3$. So, we get only the trivial solutions.

This example suggests that the equation

$$y^2 = (x^2 - 1)(x^2 - a^2)(x^2 - (a + 1)^2) = M(x),$$

can be treated with a similar way. Indeed, if $G(x, y) = y^2 - M(x)$, then the triple root of the resultant is,

$$z_0 = a^4 + 2a^3 + a^2$$

and

$$G(x, y) - z_0 = (-x^3 + xa^2 + xa + y)(-x^3 + xa^2 + xa + x + y).$$

We continue as in the example.

**Remark 2.10.** In [16] the author provides an algorithm for finding the integer points in the more general Diophantine equation (1.1) i.e. of the form,

$$y^2 = x^{2k} + a_{2k-1}x^{2k-1} + \cdots + a_1x + a_0,$$

where $k$ is a positive integer. This method again has its roots in the paper of Masser [7] and in [11]. Therefore, it is not based on Runge’s method. A further generalization was given in [17].

3. Conclusion

In the present work we provided explicit formulas for the integer solutions of quartic hyperelliptic curves of the form $y^2 = (x + a)(x + a + k)(x + b)(x + b + k)$ and $y^2 = c^2 x^4 + ax^2 + b$. The formulas depend on the divisors of the discriminant of $f(x)$, where $y^2 = f(x)$ is our curve. We used suitable factorization of the previous equations, and elementary number theory to find their integer solutions. The other methods used to practically solve the previous Diophantine equations, apply brute force in a suitable interval. Furthermore, we studied the sextic $y^2 = (x^2 - 1)(x^2 - \alpha^2)(x^2 - \beta^2)$, with $\beta - \alpha = 1$.

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Appendix

A. Runge’s Method

Runge’s method uses Puiseux series to study some classes of Diophantine equations (equations of the form (1.1) are of this type). We carry out the following steps:

1. Using Puiseux expansion theorem we can find a polynomial $g(x)$ and a power series $S(T)$, such that

$$y - g(x) = S\left(\frac{1}{x}\right), \text{ where } S(T) \in \mathbb{Q}[[T]]$$

and $g(x) \in \mathbb{Q}[x]$.

2. From the form of $S(T)$ we can find a positive constant $A = A(F)$ such that, if $|x| > A(F)$, then $|S(\frac{1}{x})| < c|x|^{-\rho}$, for some positive integer $\rho$ and a positive real number $c$.

3. If $(a, b) \in C(\mathbb{Z})$, then for $|a| > \max\{A(F), \sqrt{c}\}$, we get $|b - g(a)| < 1$.

4. Now, either we apply an effective version of Eisenstein’s theorem [15] and so we shall get a uniform bound for $|a|$ or with some add hoc method we explicit calculate the denominators of the coefficients of $g(x)$. Thus, multiplying say by $w$, we conclude with the inequality $|wb -wg(x)| < w$. We set $\mu(x) = wg(x) \in \mathbb{Z}[x]$. Then, we solve the finitely many equations $(wb)^2 = (\mu(x) + r)^2$, with $r = -w + 1, \ldots, w - 1$. Since $b^2 = f(a)$, we end up with the equations (with one unknown)

$$w^2f(a) - (\mu(a) + r)^2 = 0, \text{ for } r = -w + 1, \ldots, w - 1.$$
be appropriate in order to get a practical algorithm for the integer points.

B. Masser’s Method for $y^2 = x^4 + bx^2 + d$.

We follow [7]. Let $y^2 = f(x)$, where $f(x) = x^4 + ax^3 + bx^2 + cx + d \in \mathbb{Z}[x]$. Put,

$$c = 4b - a^2, \quad C = 64c - 8ae, \quad D = 64d - e^2, \quad \text{and} \quad Q(x) = 8x^2 + 4ax + e.$$ 

Then, the following identity holds,

$$64f(x) - (Q(x))^2 = Cx + D.$$ 

In our case, $a = c = 0$, so $e = 4b, C = 0, D = 64d - 16b^2$ and $Q(x) = 4(2x^2 + b)$.

Thus, the previous identity is written,

$$4f(x) - (2x^2 + b)^2 = 4d - b^2.$$ 

If $(x_0, y_0)$ is an integer point, then

$$4f(x_0) - (2x_0 + b)^2 = 4d - b^2, \quad \text{so} \quad (2y_0)^2 - (2x_0 + b)^2 = 4d - b^2.$$ 

Now, using elementary number theory we can find $(x_0, y_0)$.

C. Runge’s Condition.

Let $F \in \mathbb{Z}[x, y]$ be an irreducible polynomial and

$$F(x, y) = \sum_{0 \leq i \leq m, \ 0 \leq j \leq n} a_{ij}x^iy^j = 0.$$ 

If one of the following conditions does not hold, then we say that Runge’s condition is satisfied and we can apply the Runge’s finiteness result [12].

(i). $a_{in} = a_{mj} = 0$, for all non zero indexes $i, j$
(ii). $a_{ij} = 0$, for all $i, j$ such that $in + jm \geq mn$.
(iii). The leading term

$$\sum_{in + mj = mn} a_{ij}x^iy^j$$

is constant power of an irreducible polynomial in $\mathbb{Z}[x, y]$.

(iv). The algebraic function $y = y(x)$ defined by the equation $F(x, y) = 0$, has only one class of conjugate Puiseux expansions.

The curves we study in the present paper do not satisfy condition (iii), so they satisfy Runge’s condition. Indeed, the leading terms are one of the following form: $y^2 - x^4$ and $y^2 - c^2x^4$ which are reducible polynomials in $\mathbb{Z}[x, y]$. 

K. A. Draziotis, Department of Informatics, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece

Email address: drazioti@csd.auth.gr