Mathematical model of the elongated body vibrations to describe the elastic properties of the aerial vehicle

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Abstract. The article deals with the problem of building models of oscillations of an elongated body, which, in particular, can be imagined as an aerial vehicle. The general theoretical information of the aeroelastic vibrations of the aircraft is also considered.

1. Introduction

While studying different issues of the dynamics of various mechanical structures with large elongation: beams, arched systems of seismic constructions, etc., there is a need to analyze the stability of such systems.

Thus, for example, the elastic properties of the aerial vehicle (AV) can be significantly manifested in its motion dynamics [1, 2]. Transient processes in the stabilization circuit occurring under the influence of aerodynamic forces are accompanied by elastic deformations of the AV body, which affect the signals of the gauges. In the AV stabilization circuit there are additional inverse connections – in short, elastic components, which should be taken into account when analyzing the properties of the circuit. The mathematical description of these components, as well as the subsequent analysis of the stabilization circuit taking them into account is a very difficult task. Usually they are content with approximate models of the phenomenon in the form of one or several vibrational components corresponding to the basic tones of elastic vibrations [3-5]. The question of how accurate such approximations of the model are is solved mainly by experimental methods. In this paper, we consider mathematical models of elastic vibrations of the AV body suitable for further analysis of their influence on the stability of the AV motion stabilization system in a given direction.

2. The Equation of Vibrations of an Elongated Body

Firstly, we consider the plane vibrations of a straight beam [6, 7] – an elongated body that resists bending. The beam, as we know, is called straight when the centers of gravity of its cross sections areas lie on the same straight line – the axis of the beam. Very often such vibrations are caused by some concentrated forces acting on the beam resting on two props. In general, we can assume that the beam is under the influence of some system of forces distributed along its length, including the reaction of the props or some complex (e.g., elastic) base. In the absence of elastic deformations (that is, when the beam is absolutely rigid and cannot bend at all) under the impact of this system of forces, the beam is in equilibrium, that is, the main vector and the main moment of forces equals zero. If the beam as a whole body does not fit in the space, this condition remains for the vibrating beam, too.
Finally, we assume that the system of external forces at beam deformations does not change. Vibrations of the beam will be considered in the plane $xOz$, taking the left end of the beam as the beginning point, and the axis of the beam in the non-deformed state as the axis $Ox$, as shown in Fig. 1.

![Image](image_url)

**Figure 1.** The plane of beam vibrations.

The beam length is designated by letter $l$. Let us first consider the beam equation in statics, when there is no fluctuation. Let $\pi(x)$ be a function describing the distribution of forces along the beam axis. This means that the resulting vector of external forces acting on the part of the beam up to the section $aa'$ and the resulting moment relative to this section will be respectively:

$$S(x) = \int_{0}^{x} \pi(\xi)d\xi,$$  \hspace{1cm} (1)

$$M(x) = -\int_{0}^{x} (x-\xi)\pi(\xi)d\xi,$$  \hspace{1cm} (2)

It must be mentioned that the force $S(x)$ is called the cutting force, and the moment $M(x)$ is known as the bending moment. Mentally cutting off the part of the beam to the left of the section $aa'$, we can replace it in the equilibrium scheme of the remaining part with the cutting force $S(x)$ and the bending moment $M(x)$. Considering the equations (1), (2), when $x=l$ we get the equations

$$S(l) = \int_{0}^{l} \pi(\xi)d\xi$$ \hspace{1cm} (3)

$$M(l) = -\int_{0}^{l} (l-\xi)\pi(\xi)d\xi = \int_{0}^{l} \xi\pi(\xi)d\xi = 0$$ \hspace{1cm} (4)

as beam equilibrium conditions: the resulting vector of all forces and the resulting moment of these forces with respect to any point are zero.

Beam bending deformations are subject to the law relating the curvature of the beam axis to the bending moment. This law is expressed by the formula

$$\frac{1}{R} = \frac{M}{EJ}.$$ \hspace{1cm} (5)

In the expression (1.5) $R$ is the radius of the curvature, $J$ is the inertia moment of the cross section, determined by the equation

$$MJ = \int_{F} z^2d\delta,$$
where \( d\delta \) is an element of cross section area, and \( E \) is the so called Young modulus. The value \( EJ \) is a structural characteristic of the beam depending on the material and design of the beam. It characterizes the rigidity of the beam, its ability to resist bending.

As the result of bending, let the beam axis take the form of a curve describing the bends of the beam relative to the non-deformed state

\[
z = z(x).
\] (6)

The curvature is known to be determined by the expression

\[
\frac{1}{R} = \frac{d^2 z(x)}{dx^2} \left[ 1 + \left( \frac{dz(x)}{dx} \right)^2 \right]^{-\frac{1}{2}},
\]

Which for small deviations \( |z'(x)| \approx 1 \) takes the form of

\[
\frac{1}{R} = \frac{d^2 z(x)}{dx^2}.
\]

That is why the equation (5) can be written for small deviations the following way:

\[
EJ \frac{d^2 z(x)}{dx^2} = -M.
\] (7)

Now, let us note that differentiating the equations (1), (2), we will get

\[
\frac{dS(x)}{dx} = \pi(x),
\] (8)

\[
\frac{dM(x)}{dx} = -S(x)
\] (9)

Twice differentiating the expression (7) and taking into consideration the equations (8), (9), we can easily define

\[
\frac{d^2}{dx^2} E(x)J(x) \frac{d^2 z(x)}{dx^2} = \pi(x)
\] (10)

The equation (10) is the differential equation of beam bending integration of which, under the conditions defining fastening the beam, allows to define the form of the bend (6). Based on the expression (10) it is easy to obtain the differential equation of beam vibrations for the bend \( z(x,t) \) as a function of two variables. To do this, we should apply the principle of d'Alembert, which states that the equations of dynamics can be considered as the equations of equilibrium of statics, if we include inertia forces into the system of forces. We shall define the acceleration of the beam point with the abscissa along the axis \( OZ \) and express:

\[
a_z(x) = \frac{\partial^2 z(x,t)}{\partial t^2}.
\]

Then the inertia force of the element between the cross sections \( aa', bb' \), will have the form of

\[-\rho(x) \frac{\partial^2 z(x,t)}{\partial t^2} dx,\]

where \( \rho(x)dx \) is the element mass, \( \rho(x) \) is the density of mass distribution along the beam length. Thus, in case of beam vibrations instead of the density of force distribution \( \pi(x) \) we will have the density of force distribution.
Given that in general the density of force distribution can vary over the time, the equation (10) can now be rewritten as

$$\frac{\partial^2}{\partial x^2} E(x) J(x) \frac{\partial^2 z(x,t)}{\partial t^2} + \rho(x) \frac{\partial^2 z(x,t)}{\partial t^2} = \pi(x,t)$$

(12)

It is clear that it is necessary to use here symbols of partial derivatives, as the bend is a function of two independent variables $x$ and $t$. Next, in order to simplify the writing, we will indicate derivatives in $x$ with a dash, and derivatives in time – with a dot.

Then, very useful for understanding the derivation of the equation of vibrations is the idea of the beam as an elastic non-inertial frame experiencing the impact of forces with density of distribution $\pi(x) - \rho(x) \frac{d^2 z(x,t)}{dt^2}$. This representation is especially evident when the beam is a supporting structure (aerial vehicle, vessel, girder, etc.), which houses a certain mass system. So at the point of attachment, for example, of a separate concentrated mass on the beam, the force $P - m \ddot{z}(x,t)$ acts, where $P$ is the concentrated external force at the point, and $m$ is the mass of placement in it.

Integration of the equation (12) under the given initial and boundary conditions allows to study the process of beam vibrations. Let us consider these conditions in more detail. Assuming that the time is counted from zero, the initial conditions can be assigned in the form

$$u(x,0) = \dot{u}(x), \quad \ddot{u}(x,0) = \nu(x)$$

(13)

where $u(x), \nu(x)$ are given functions. Thus, at the initial moment the bends of the beam and the speed of their change at each point are set.

The boundary conditions are determined by the methods of fixing the beam or, in general, by some previously known laws at the characteristic points of the beam. For example, for the left end of the beam there may be cases of:

1. swivel movable fastening - the fastening point is stationary and $z(0,t) = 0$; the bending moment also equals zero due to the equation (7) $z''(0,t) = 0$;

2. rigid fastening – the point of fastening is motionless and $z(0,t) = 0$; also $z'(0,t) = 0$ since the inclination angle of the beam in the sealing equals zero;

3. free ends, where $S = 0, M = 0$, and, consequently, due to the equations (7), (9) there are the equations $z''(0,t) = 0$ and $z'''(0,t) = 0$.

Similar cases of boundary conditions may occur for the right end of the beam as well. It is clear that there will be nine combinations of boundary conditions on the left and right ends of the beam. There may also be different cases of similar conditions at other points of the beam. Thus, the multiple boundary conditions are very diverse, and generally speaking, each case of boundary conditions brings its own peculiarities in making the picture of the beam vibrations. We will further consider the equation of vibrations on the example of a beam with free ends, that is, the boundary conditions of the beam are

$$z''(0,t) = 0, \quad z'''(0,t) = 0$$

$$z(l,t) = 0, \quad z''(l,t) = 0$$

(14)

(15)

Finally, ignoring the changes in rigidness along the beam axis, we assume that $E(x) J(x) = \text{const}$. Let us define as the mean value the coefficient $\rho(x)/E(x) J(x)$. It will allow to write the vibration equation in the following form
\[ z^{(iv)}(x,t) + k^4 \ddot{z}(x,t) = p(x,t) \]  

(16)

where \( p(x) = \frac{1}{EJ} \pi(x,t) \), \( k^4 = \left[ \frac{\rho(x)}{E(x)}j(x) \right]_{ip} \)

3. Aerial Vehicle Vibration Equations

Based on the obtained results, we will study the question of differential equations of elastic vibrations of a moving aerial vehicle. Let the movement of the apparatus be considered in the horizontal plane and the coordinate system \( XOZ \) is rigidly connected with the vehicle the length of which is \( l \). We assume that the coordinate system coincides with the rear section of the apparatus, axis \( Ox \) represents a longitudinal axis of the non-deformed apparatus, \( Oz \) is the axis perpendicular to it in the horizon plane, axis \( Oy \) is perpendicular to the horizon plane. The given coordinate system is shown in Fig. 2.

![Figure 2. Coordinate system of an AV vibration.](image)

On this scheme \( c \) is the center of the vehicle mass, \( z(x,t) \) is the bend of the body as a function of two variables. Thus, under study are the vibrations of the vehicle body with respect to its non-deformed position. Let \( \vec{i}, \vec{j}, \vec{k} \) be unit vectors of the axes \( Ox, Oy, Oz \) respectively. Differentiating these vectors, we will define

\[ \frac{d\vec{i}}{dt} = \vec{a} \times \vec{j} = \omega_j \vec{j} \times \vec{i} = \omega_j \vec{k} \]

\[ \frac{d\vec{k}}{dt} = \vec{a} \times \vec{k} = \omega_j \vec{j} \times \vec{k} = -\omega_j \vec{i} \]

where \( \vec{a} \times \vec{b} \) is vector multiplication of vectors, \( \vec{a} = \omega_j \vec{j} \) is angular velocity vector, \( \omega_j \) is angular velocity of rotation of the device around the axis \( Oy \).

Sequentially differentiating the radius vector of an arbitrary point of the vibrating apparatus

\[ \vec{r}(x) = r_c + (x - x_c) \vec{y} + \vec{k}z(x,t) \]

in relation to the fixed point in the space, now we can find the velocity

\[ \vec{V}(x) = \vec{V}_c + [(x - x_c)\omega_j + \dot{z}(x,t)]\vec{k} - z(x,t)\omega_j \vec{i} \]

and after that the acceleration of the point of the apparatus with the abscissa \( x \) :

\[ \vec{a}(x) = \vec{a}_c + [(x - x_c)\omega_j + \dot{z}(x,t)]\vec{k}^2 - \]

\[-[(x - x_c)\omega_j^2 + 2\ddot{z}(x,t)\omega_j + \ddot{z}(x,t)\omega_j^2] \vec{i} \]

where \( \vec{V}_c, \vec{a}_c \) are velocity and acceleration vectors, respectively, of the center of mass.

Now we can easily find the acceleration along axis \( Oz \)

\[ a_z(x) = a_{cz} + (x - x_c)\omega_j - z(x,t)\omega_j^2 + \ddot{z}(x,t) , \]
or, discarding a small value \( z(x,t)\omega_x^2 \),

\[
a_z(x) = a_e + (x - x_c)\dot{\omega}_y + \ddot{z}(x,t).
\]  

(17)

Thus, in the equation (11) the force distribution density, now is

\[
p(x,t) = \pi(x,t) - \rho(x)a_z(x),
\]

(18)

where \( \pi(x,t) \) is the density of external (aerodynamic) lateral forces distribution along the axis \( Ox \), \( \rho(x) \) is the density of mass distribution along the axis of the apparatus.

Let \( Z \) and \( M_y \) be the resulting vector and the resulting moment in regard to the center of external (aerodynamic) forces mass, typically changing in time. Now the equations are fair [20]:

\[
\begin{cases}
ma_z = Z, \\
J_y\dot{\omega}_y = M_y,
\end{cases}
\]

(19)

where \( m \) is overall mass, and \( J_y \) is inertial moment about the axis \( Oy \) of the vehicle, respectively. It is obvious that

\[
\begin{cases}
\int_0^l \pi(x,t)dx = Z, \\
\int_0^l (x - x_c)\pi(x,t)dx = M_y,
\end{cases}
\]

(20)

Due to (19), the expression (18) takes the form

\[
p(x,t) = \pi(x,t) - \frac{\rho(x)}{m}Z - \frac{(x-x_c)\rho(x)}{J_y}M_y - \rho(x)\ddot{Z}(x,t),
\]

or

\[
p(x,t) = P(x,t) - \rho(x)\ddot{z}(x,t)
\]

(21)

where

\[
P(x,t) = \pi(x,t) - \frac{\rho(x)}{m}Z - \frac{(x-x_c)\rho(x)}{J_y}M_y
\]

(22)

Thus, the equation of elastic vibrations of the apparatus can now be written this way

\[
\frac{d^2}{dx^2}E(x)J(x)\frac{d^2z(x,t)}{dt^2} + \rho(x)\frac{d^2z(x,t)}{dt^2} = P(x,t),
\]

(23)

where \( E(x)J(x) \) the rigidity of the apparatus in arbitrary cross-section with the abscissa \( x \), \( \rho(x) \) is the density of mass distribution along the axis \( Ox \), \( P(x,t) \) is the density of distribution of all forces (external and inertia forces from the motion of the device as a solid object) along the axis \( Ox \).

The equation (23) together with the initial (13) and boundary conditions, e.g., (14), (1.15) defines the elastic vibrations of the moving apparatus. As

\[
\int_0^l \rho(x)dx = m, \quad \int_0^l x\rho(x)dx = mx_c,
\]

(24)
by the definition of the center of mass and

$$\int_0^l (x - x_c)^2 \rho(x) \, dx = J_y,$$

(25)

first of all, note that

$$\int_0^l \rho(x,t) \, dx = 0, \quad \int_0^l (x - x_c) \rho(x,t) \, dx = 0$$

(26)

due to the equation (20). Now note that according to the principle of d'Alembert the system of all external forces and the apparatus inertia forces when it moves forms an equilibrium system. This means that the sum of all these forces and their moments to any point is zero (meaning the forces acting along the axis $Oz$). Thus

$$\int_0^l \rho(x,t) \, dx = 0, \quad \int_0^l x\rho(x,t) \, dx = 0,$$

as $\rho(x)$ is the density of distribution of the sum of external and inertia forces. Substituting here the expressions (21), (22) and considering the equation (26), we can find

$$\int_0^l \rho(x)\ddot{z}(x,t) \, dx = 0, \quad \int_0^l x\rho(x)\ddot{z}(x,t) \, dx = 0,$$

or changing the order of integrating and differentiating

$$\frac{d^2}{dt^2} \int_0^l \rho(x)z(x,t) \, dx = 0, \quad \frac{d^2}{dt^2} \int_0^l x\rho(x)z(x,t) \, dx = 0.$$

Integrating the first equation and considering the initial conditions (13), we find

$$\int_0^l \rho(x)z(x,t) \, dx = \int_0^l x\rho(x)u(x) \, dx + t \int_0^l \rho(x)\nu(x) \, dx.$$  

(27)

Similarly integrating the second equation, we find

$$\int_0^l x\rho(x)z(x,t) \, dx = \int_0^l x\rho(x)u(x) \, dx + t \int_0^l x\rho(x)\nu(x) \, dx.$$  

(28)

Out of these equations with limited collectively variables of the function $(0 \leq x \leq l, t \geq 0)$, i.e. when vibrations are physically possible, it follows that

$$\int_0^l \rho(x)\nu(x) \, dx = 0, \quad \int_0^l x\rho(x)\nu(x) \, dx = 0.$$  

(29)

In other words, the functions determining the initial conditions of vibrations must be integrable with appropriate weights in the interval $[0,l]$, and $\nu(x)$ satisfy the equations as well (29). We should also pay attention to the equation (26). When forming a model of elastic vibrations, the exact aerodynamic forces distribution density is not always known. Therefore, it is necessary to accept certain assumptions about the nature of the distribution of these forces. In addition, we often have to apply various simplifications in the description of the problem. In the finally adopted model of forces, all its
elements must be consistent. Thus, the conditions (26), (29) can be considered as the conditions for the correctness of the beam vibration model formation.

It should also be noted that estimating the processes of elastic vibrations in the main, the change in rigidness $E(x)J(x)$ along the axis of the apparatus is neglected, and $\rho(x)/E(x)J(x)$ is taken as average value with the derivative $\dot{z}(x,t)$. Then the equations of vibrations can be written in the form

$$\frac{\partial^4 z(x,t)}{\partial x^4} + k^4 \frac{\partial^2 z(x,t)}{\partial t^2} = R(x,t),$$  \hspace{0.5cm} (30)

where $k^4 = \left[ \frac{\rho(x)}{E(x)J(x)} \right]_{cp}$,

$$R(x,t) = \frac{1}{EJ} P(x,t), \hspace{0.5cm} EJ = [E(x)J(x)]_{cp}$$  \hspace{0.5cm} (31)

and $P(x,t)$ is defined by the equation (22).

**Conclusion**

The practice of designing flight tests of modern aerial vehicles shows that it is completely inadequate to regard an aircraft as an absolutely solid body. Therefore, the dynamic characteristics of an aircraft as an elastic body must be taken into account in the early stages of developing automatic control systems. The general theoretical information of aeroelastic vibrations of an aircraft described in this paper is the basis for constructing a mathematical model of an aircraft as an elastic body.

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