HOPF-GALOIS STRUCTURES ON NON-NORMAL EXTENSIONS OF DEGREE RELATED TO SOPHIE GERMAIN PRIMES

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Abstract. We consider Hopf-Galois structures on separable (but not necessarily normal) field extensions $L/K$ of squarefree degree $n$. If $E/K$ is the normal closure of $L/K$ then $G = \text{Gal}(E/K)$ can be viewed as a permutation group of degree $n$. We show that $G$ has derived length at most 4, but that many permutation groups of squarefree degree and of derived length 2 cannot occur. We then investigate in detail the case where $n = pq$ where $q \geq 3$ and $p = 2q + 1$ are both prime. (Thus $q$ is a Sophie Germain prime and $p$ is a safe prime). We list the permutation groups $G$ which can arise, and we enumerate the Hopf-Galois structures for each $G$. There are six such $G$ for which the corresponding field extensions $L/K$ admit Hopf-Galois structures of both possible types.

1. Introduction

Half a century ago, Chase and Sweedler [CS69] initiated Hopf-Galois theory, in part motivated by the study of inseparable field extensions. Their work nevertheless raises many interesting questions for separable extensions. Greither and Pareigis [GP87] showed that a separable extension may admit many Hopf-Galois structures, and that these can be described in group-theoretic terms.

Suppose first that $L/K$ is a field extension of degree $n$ which is normal as well as separable, so $L/K$ is a Galois extension in the classical sense. Let $G = \text{Gal}(L/K)$ be its Galois group. Then $L/K$ admits at least one Hopf-Galois structure, namely that given by the natural action of the group algebra $K[G]$. There may however be many other Hopf-Galois structures on $L/K$. If $H$ is the $K$-Hopf algebra acting on $L$ in one of these Hopf-Galois structures, then, for some group $\tilde{N}$ of order $n$, we have $L \otimes_K H \cong L[\tilde{N}]$ as $L$-Hopf algebras. We call the isomorphism class of $\tilde{N}$ the type of this Hopf-Galois structure. The
NIGEL P. BYOTT AND ISABEL MARTIN-LYONS

possible types, and the number of Hopf-Galois structures of each type, depend only on the Galois group $G$.

More generally, suppose now that $L/K$ is a separable (but not necessarily normal) extension of degree $n$. Let $E/K$ be the normal closure of $L/K$, and let $G = \text{Gal}(E/K)$ and $G' = \text{Gal}(E/L)$. Again, each Hopf-Galois structure on $L/K$ has a type $N$, which is (the isomorphism class of) a group of order $n$. The number of Hopf-Galois structures of each type depends only on $G$ together with its subgroup $G'$. Indeed, we may interpret the main result of [GP87] as saying that the Hopf-Galois structures on $L/K$ depend on the isomorphism class of $G$ as a permutation group acting transitively on the left coset space $X = G/G'$ (or, equivalently, acting on the set of $K$-linear embeddings of $L$ into $E$).

There is a substantial literature studying Hopf-Galois structures on various classes of separable field extensions; see for example [Koh98, Chi05, Tsa20] and the references therein. While some of this work, such as [Chi89, CRV16, CS20], treats non-normal extensions, most of it is concerned with Galois extensions. The study of Hopf-Galois structures on Galois extensions has recently become of wider interest due to a connection with (skew) braces and hence with set-theoretic solutions of the Yang-Baxter equation (see [Bac16] and the appendix to [SV18]).

When $n = pq$ is the product of two distinct primes, the Hopf-Galois structures on a Galois extension of degree $n$ were determined in [Byo04]. Provided that $p \equiv 1 \pmod{q}$, there are two groups of order $pq$, one cyclic and one non-abelian. A Galois extension with either Galois group admits Hopf-Galois structures of both types. Another case which has been investigated in several papers [Chi03, BC12, Koh13] is when $n = 2pq$ for odd primes $p$, $q$ with $p = 2q + 1$. (Primes $q$ such that $2q + 1$ is also prime are named after Sophie Germain in recognition of her work on Fermat’s Last Theorem for these exponents, while primes $p$ such that $(p - 1)/2$ is also prime are called safeprimes because of their significance in cryptography.) More generally, the Hopf-Galois structures on a Galois extension of arbitrary squarefree degree $n$ were investigated in [AB20]. In this case, the groups of order $n$ can be classified, and given any two such groups $G$, $N$, one can calculate the number of Hopf-Galois structures of type $N$ on a Galois extension $L/K$ with $\text{Gal}(L/K) \cong G$.

The question then arises of whether the results of [AB20] can be extended to non-normal (but separable) field extensions $L/K$ of squarefree degree $n$. The purpose of this paper is to take some initial steps in this direction. In this situation, the type $N$ of a Hopf-Galois structure is again a group of squarefree order $n$. The group $G = \text{Gal}(E/K)$, however, will have order a proper multiple of $n$ as soon as $L/K$ is
not normal, and we cannot expect this order to be squarefree. We are nevertheless able to obtain a few general results. A necessary condition for $L/K$ to admit any Hopf-Galois structures is that $G$ has derived length at most 4 (Theorem 3.3), but this condition is very far from sufficient (Theorem 3.4).

If we wish to determine all possible Hopf-Galois structures on extensions of a given squarefree degree $n$, we cannot proceed as in [AB20] and start with an arbitrary pair of groups $G$, $N$. This is because we have no a priori classification of permutation groups of degree $n$, and therefore we cannot specify in advance the groups $G$ we need to consider. A possible alternative strategy is take each group $N$ of order $n$ in turn, and find the permutation groups $G$ which can arise from Hopf-Galois structures of type $N$. We then have the new problem of determining when two such groups $G_1$ and $G_2$, arising from (possibly different) types $N_1$ and $N_2$, are isomorphic as permutation groups. If this occurs then field extensions realising this permutation group will admit Hopf-Galois structures of type $N_1$ and of type $N_2$.

It is not clear whether one can expect the above strategy to be viable for an arbitrary squarefree $n$. Our main goal in this paper is to carry out this strategy in the special case that $n = pq$ where $q \geq 3$ is a Sophie Germain prime and $p = 2q + 1$ is the associated safeprime. As there are conjecturally infinitely many Sophie Germain primes, we expect our results to hold for infinitely many values of $n$. For such $n$, we obtain a catalogue of the permutation groups of degree $n$ for which the corresponding field extensions $L/K$ admit at least one Hopf-Galois structure. We then enumerate all Hopf-Galois structures on these extensions $L/K$, and determine which permutation groups $G$ are realised by Hopf-Galois structures of both possible types. (Recall that we have two group $N$ of order $pq$, up to isomorphism.) There are six such $G$, including the two regular groups of degree $pq$, as shown in Table 6. Our catalogue by no means contains all permutation groups of degree $n$: those that do not occur are precisely those which cannot be realised by a Hopf-Galois structure.

2. Preliminaries

In this section, we recall some definitions and standard facts about Hopf-Galois structures, emphasising the connection to permutation groups. In particular, we outline the method of counting Hopf-Galois structures developed in [Byo96]. We also prove some technical results which will be useful later. For a more complete account of the theory of Hopf-Galois structures, we refer the reader to [Chi00, Chapter 2].
By a permutation group we mean a group $G$ together with an injective homomorphism $\pi: G \to \text{Perm}(X)$ into the group of permutations of a set $X$. We will also say that $G$ acts on $X$, and, for $g \in G$, $x \in X$, we write $g \cdot x$ for $\pi(g)(x)$. In this paper $X$ (and hence $G$) will always be finite. The degree of $G$ is the cardinality $|X|$ of $X$. We say that $G$ is transitive (respectively, regular) on $X$ if, for any $x, y \in X$, there is some $g \in G$ (respectively, a unique $g \in G$) with $g \cdot x = y$. Throughout this paper, all permutation groups will be assumed transitive.

The stabiliser of $x \in X$ is by definition the subgroup $G_x = \{ g \in G : g \cdot x = x \}$. Then the stabiliser of $g \cdot x$ is $gG_xg^{-1}$. As $G$ acts transitively on $X$, it is isomorphic as a permutation group to $G$ acting on the set of left cosets $G/G_x = \{ gG_x : g \in G \}$ via its left multiplication action $\lambda : G \to \text{Perm}(G/G_x)$, where $\lambda(g)(hG_x) = (gh)G_x$. Thus, up to isomorphism, a permutation group of degree $n$ can be taken to be an abstract group $G$ acting by left multiplication on $G/G'$, where $G'$ is a subgroup of index $n$ with trivial core. We define

$$\text{Aut}(G, G') = \{ \theta \in \text{Aut}(G) : \theta(G') = G' \}.$$ 

Then $\text{Aut}(G, G')$ is the group of automorphisms $\theta$ of $G$ as a permutation group satisfying the further condition that $\theta$ fixes the distinguished element $1_GG'$ of $G/G'$, where $1_G$ denotes the identity element of $G$.

Now let $L/K$ be a separable field extension of finite degree $n$, and let $E$ be its normal closure inside a fixed algebraic closure $K^c$ of $K$. Let $G = \text{Gal}(E/K)$ and $G' = \text{Gal}(E/L)$. Then $G$ acts transitively on the $K$-linear embeddings of $L$ into $E$ (or into $K^c$), and the stabiliser of the inclusion $L \hookrightarrow E$ is $G'$. Thus associated to $L/K$ we have the permutation group $G$ of degree $n$ acting on the set $X = G/G'$.

Next let $H$ be a cocommutative $K$-Hopf algebra. We say that $L$ is an $H$-module algebra if $H$ acts on $L$ as $K$-linear endomorphisms such that $h \cdot (xy) = \sum_{(h)} (h_{(1)} \cdot x)(h_{(2)} \cdot y)$ for $h \in H$ and $x, y \in L$, and $h \cdot k = \epsilon(h)k$ for $h \in H$ and $k \in K$. Here we write the comultiplication $H \to H \otimes_K H$ as $h \mapsto \sum_{(h)} h_{(1)} \otimes h_{(2)}$, and $\epsilon : H \to K$ is the augmentation of $K$.

Moreover, we say that $L/K$ is $H$-Galois if in addition the $K$-linear map $\theta : L \otimes_K H \to \text{End}_K(L)$ given by $\theta(x \otimes h)(y) = x(h \cdot y)$ is bijective. A Hopf-Galois structure on $L/K$ consists of a cocommutative Hopf algebra $H$ and an action of $H$ on $L$ for which $L/K$ is $H$-Galois.

Greither and Paregis showed that the Hopf-Galois structures on $L/K$ depend only on the permutation group $G$. More precisely, the Hopf-Galois structures correspond bijectively to the regular subgroups $N$ of $\text{Perm}(X)$ which are normalised by the image $\lambda(G)$ of left multiplication.
For each such $N$, we have the $K$-Hopf algebra $H = E[N]^G$, where $G$ acts on $E[G]$ simultaneously as field automorphisms of $E$ and by conjugation (via $\lambda$) on $N$, and where $H$ acts on $L$ by Galois descent. We refer to the isomorphism type of $N$ as the type of this Hopf-Galois structure. We say that $L/K$ is almost classically Galois if the subgroup $G'$ of $G$ has a normal complement $C$. In that case, we obtain a Hopf-Galois structure by taking $N = C$.

Let $G$ be a permutation group, and let $G'$ be the stabiliser of a point. We say that a separable field extension $L/K$ realises $G$ if there is an isomorphism $\theta : G \rightarrow \text{Gal}(E/K)$ with $\theta(G') = \text{Gal}(E/L)$, where again $E$ is the normal closure of $L/K$. Moreover, we will say that $G$ is realised by a Hopf-Galois structure of type $N$ if $L/K$ admits a Hopf-Galois structure of type $N$. Given also an abstract group $N$ of order $n$, we would like to determine the number $e(G, N)$ of Hopf-Galois structures of type $N$ on a field extension $L/K$ which realises $G$. By the result of Greither and Pareigis, this is the number of regular subgroups $N^*$ of $\text{Perm}(X)$ which are isomorphic to $N$ and normalised by $\lambda(G)$, where $X = G/G'$. The following result simplifies the calculation of $e(G, N)$.

**Lemma 2.1.** [Byo96] Let $G$, $G'$ and $N$ be as above. Let $\text{Hol}(N) = N \rtimes \text{Aut}(N)$ be the holomorph of $N$, and let $e'(G, N)$ be the number of subgroups $M$ of $\text{Hol}(N)$ which are transitive on $N$ and isomorphic to $G$ via an isomorphism taking the stabiliser $M'$ of $1_N$ in $M$ to $G'$. Then

$$e(G, N) = \frac{|\text{Aut}(G, G')|}{|\text{Aut}(N)|} e'(G, N).$$

In particular, if $G$ is realised by a Hopf-Galois structure of type $N$, then $G$ is isomorphic to a transitive subgroup $M$ of $\text{Hol}(N)$.

The advantage of Lemma 2.1 is that it allows us to work with $\text{Hol}(N)$ rather than the much larger group $\text{Perm}(X)$.

We will write elements of $\text{Hol}(N) = N \rtimes \text{Aut}(N)$ as $[\eta, \alpha]$ with $\eta \in N$ and $\alpha \in \text{Aut}(N)$. Then the action of $\text{Hol}(N)$ as permutations of $N$ is given by $[\eta, \alpha] \cdot \mu = \eta \alpha(\mu)$. Thus the normal subgroup $N$ of $\text{Hol}(N)$ is identified with the group $\lambda(N)$ of left translations by $N$, the subgroup $\text{Aut}(N)$ is the stabiliser of $1_N$, and the group operation in $\text{Hol}(N)$ is

$$[\eta, \alpha][\mu, \beta] = [\eta \alpha(\mu), \alpha \beta].$$

To lighten notation, we will often write the elements $[\eta, \text{id}_N]$ and $[1_N, \alpha]$ in $\text{Hol}(N)$ as $\eta, \alpha$ respectively. Thus, for example, we have the identity $\alpha \eta = \alpha(\eta) \alpha$ in $\text{Hol}(N)$.

We end this section with some technical results concerning holomorphs.
Proposition 2.2. Let $N$ be an abelian group such that $\text{Aut}(N)$ is also abelian, and let $A, A'$ be subgroups of $\text{Aut}(N)$. Consider the subgroups $M = N \rtimes A$ and $M' = N \rtimes A'$ of $\text{Hol}(N)$. If there is an isomorphism $\phi : M \to M'$ with $\phi(N) = N$, then $M = M'$.

Proof. Let $\phi_N \in \text{Aut}(N)$ be the restriction of $\phi$ to $N$. For $g \in \text{Hol}(N)$, let $C_g \in \text{Aut}(N)$ be conjugation by $g$. Then $C_\alpha = \alpha$ for $\alpha \in \text{Aut}(N)$ by definition of the multiplication in $\text{Hol}(N)$, and $C_\eta = \text{id}_N$ for $\eta \in N$ since $N$ is abelian.

Let $\alpha \in A$. Then $\phi(\alpha) = \eta \alpha'$ for some $\eta \in N$ and $\alpha' \in A'$. We claim that $\alpha' = \alpha$. This will show that $M' \subseteq M$, and the same argument applied to $\phi^{-1}$ then gives equality.

To prove the claim, let $\mu \in N$ and apply $\phi_N$ to the relation $C_\alpha(\mu) = \alpha \mu \alpha^{-1}$. This gives
\[
\phi_N(C_\alpha(\mu)) = \phi_N(\alpha) \phi_N(\mu) \phi_N(\alpha)^{-1} = C_{\phi_N(\alpha)}(\phi_N(\mu)).
\]
Hence in $\text{Aut}(N)$ we have the equation $\phi_N C_\alpha = C_{\phi_N(\alpha)} \phi_N$. Since $\text{Aut}(N)$ is abelian, we conclude that $C_\alpha = C_{\phi_N(\alpha)}$, and so
\[
\alpha = C_\alpha = C_{\phi_N(\alpha)} = C_\eta C_{\alpha'} = C_{\alpha'} = \alpha'.
\]

Proposition 2.3. Let $N$ be any group and $A$ a subgroup of $\text{Aut}(N)$. Let $M$ be the subgroup $N \rtimes A$ of $\text{Hol}(N)$, and suppose that $N$ is characteristic in $M$. Then the group
\[
\text{Aut}(M, A) := \{ \theta \in \text{Aut}(M) : \theta(A) = A \}
\]
is isomorphic to the normaliser of $A$ in $\text{Aut}(N)$. In particular, if $\text{Aut}(N)$ is abelian then $\text{Aut}(M, A) \cong \text{Aut}(N)$.

Proof. An element $\theta$ of $\text{Aut}(M, A)$ is clearly determined by its restrictions $\theta_N \in \text{Aut}(N)$ and $\theta_A \in \text{Aut}(A)$.

Let $\eta \in N$, $\alpha \in A$, and write $\mu = \theta_N(\eta)$, $\beta = \theta_A(\alpha)$. Applying $\theta$ to the relation $\alpha \eta = \alpha(\eta) \alpha$ in $\text{Hol}(N)$ we get $\beta \mu = \theta_N(\alpha(\eta)) \beta$, so that
\[
\beta \mu \beta^{-1} = \theta_N(\alpha(\theta_N^{-1}(\mu))).
\]
As this holds for all $\mu \in N$, and as $\beta \mu \beta^{-1} = \beta(\mu)$, it follows that $\theta_A(\alpha) = \beta = \theta_N \alpha \theta_N^{-1}$ in $\text{Aut}(N)$. Hence $\theta_A$ is determined by $\theta_N$. Moreover, as $\theta_A(\alpha) \in A$ for all $\alpha \in A$, it follows that $\theta_N$ normalises $A$. Conversely, given any $\theta' \in \text{Aut}(N)$ which normalises $A$, we obtain an element of $\text{Aut}(M, A)$ by setting $\theta_N = \theta'$ and $\theta_A(\alpha) = \theta' \alpha \theta'^{-1}$. This proves the first statement, and the second follows. \qed
3. Extensions of squarefree degree: some general results

We first mention some old results. Childs [Chi89] showed that a permutation group $G$ of prime degree $p$ is realised by a Hopf-Galois structure if and only if it is soluble (so that $G$ is a subgroup of $C_p \rtimes C_p^{-1}$). More generally, let $n$ be a natural number satisfying the condition $\gcd(n, \varphi(n)) = 1$, where $\varphi$ is the Euler totient function. Then $n$ is squarefree and every group of order $n$ is cyclic. It follows from [Byo96, Theorem 2] that if a permutation group of degree $n$ is realised by a Hopf-Galois structure on a field extension $L/K$, then this is the unique Hopf-Galois structure admitted by $L/K$, and $L/K$ is almost classically Galois.

We now consider more general squarefree numbers $n$. The groups of order $n$ can be classified since every Sylow subgroup is cyclic.

Lemma 3.1. Let $N$ be a group of squarefree order $n$. Then

$$N = \langle \sigma, \tau : \sigma^e = \tau^d = 1, \tau \sigma = \sigma^k \tau \rangle$$

for some parameters $e$, $d$, $k$ such that $de = n$ and $k$ has order $d$ mod $e$. Thus $N = C_e \rtimes C_d$ is metacyclic.

Moreover, let $z = \gcd(e, k - 1)$ and $g = e/z$. Then $\Aut(N)$ contains a normal subgroup of order $g$ generated by $\theta : \sigma \mapsto \sigma, \tau \mapsto \sigma^s \tau$ and a complementary subgroup isomorphic to $\mathbb{Z}_e^*$ (the group of units in the ring of integers mod $e$) consisting of the automorphisms $\phi_s : \sigma \mapsto \sigma^s, \tau \mapsto \tau$ for $s \in \mathbb{Z}_e$. These satisfy the relations $\phi_s \theta \phi_s^{-1} = \theta^s$. Thus $\Aut(N) \cong C_g \rtimes \mathbb{Z}_e^*$ is metabelian.

Proof. The description of $N$ is well-known, and follows from [Rob96, 10.1.10] or [MM84, Lemma 3.5]. For the description of $\Aut(N)$, see [AB18, Lemma 4.1].

Corollary 3.2. Let $N$ be a group of squarefree order $n$, and let $p$ be the largest prime factor of $n$. Then $|\Hol(N)|$ is not divisible by $p^3$.

Proof. With the notation of the proof of Lemma 3.1 we have $|\Hol(N)| = n|\Aut(N)| = ng \varphi(e)$. Since $\varphi(e)$ divides $\varphi(n)$ whenever $e$ divides $n$, we have that $|\Hol(N)|$ divides $n^2 \varphi(n)$. As $p$ does not divide $\varphi(n)$, the result follows.

Theorem 3.3. Let $G$ be a transitive permutation group of squarefree degree $n$ which is realised by a Hopf-Galois extension. Then $G$ has derived length at most 4. In particular, $G$ is soluble.

Proof. Suppose that $G$ is realised by a Hopf-Galois structure of type $N$. Then, by Lemma 3.1 both $N$ and $\Aut(N)$ are metabelian, that is, they have derived length at most 2. Hence $\Hol(N) = N \rtimes \Aut(N)$ has
derived length at most 4. Now Lemma 2.1 tells us that $G$ is isomorphic to a subgroup of $\text{Hol}(N)$. Thus $G$ also has derived length at most 4, and is therefore soluble. \hfill \square

We next show that this necessary condition for $G$ to be realised by a Hopf-Galois structure is very far from sufficient.

**Theorem 3.4.** Let $n > 6$ be a composite squarefree number. Then there is a permutation group $G$ of degree $n$ with derived length 2 which is not realised by a Hopf-Galois extension.

**Proof.** It follows from Lemma 2.1 that any permutation group $G$ of degree $n$ which is realised by a Hopf-Galois structure must embed in $\text{Hol}(N)$ for some group $N$ of order $n$. Let $n = pm$ where $p$ is the largest prime dividing $n$. Then, by Corollary 3.2, it suffices to construct a permutation group $G$ of degree $n$ and derived length 2 whose order is divisible by $p^3$.

We first suppose $m \geq 3$. We construct a permutation group acting on the set $X = C_p \times C_m$ of size $pm = n$. We write $C_p$ and $C_m$ additively. For $0 \leq j \leq m - 1$, define $\sigma_j : X \to X$ by

$$
\sigma_j(a, b) = \begin{cases} 
(a + 1, b) & \text{if } b = j \\
(a, b) & \text{otherwise.}
\end{cases}
$$

Then $\sigma_j$ has order $p$, and $\sigma_j, \sigma_k$ commute for all $j, k$. Hence the $\sigma_j$ generate an elementary abelian group $V$ of order $p^m$, which has $m$ orbits on $X$, namely the sets $C_p \times \{j\}$ for $j \in C_m$. Next define $\tau$ by

$$
\tau(a, b) = \tau(a, b + 1).
$$

Then $\tau$ has order $m$ and $\tau\sigma_j\tau^{-1} = \sigma_{j+1}$. We set $G = \langle V, \tau \rangle = V \rtimes C_m$. Then $G$ acts transitively on $X$. As $G$ is non-abelian, but has an abelian normal subgroup $V$ with $G/V \cong C_m$, we see that $G$ has derived length 2. (In fact, $G$ is the wreath product $C_p \wr C_m$.) Also $|G| = p^m m^m$ with $m \geq 3$.

Finally, suppose that $m = 2$, so that $n = 2p$ for some odd prime $p$. We interchange the roles of $p$ and $m$ in the preceding construction to obtain a group $G$ of order $2^p p$ of derived length 2 acting on $C_2 \times C_p$. If $N$ is a group of order $2p$ then $N$ is either cyclic or dihedral. Thus $|\text{Hol}(N)| = 2p(p - 1)$ or $2p^2(p - 1)$. As $2^p > 2(p - 1)$ for $p \geq 3$, $\text{Hol}(N)$ cannot contain a subgroup of order $2^p p$. Again, $G$ cannot be realised as a Hopf-Galois structure. \hfill \square
4. Extensions of degree $pq$ with $p = 2q + 1$

For the remainder of the paper, we consider Hopf-Galois structures on separable extensions of degree $pq$, where $p = 2q + 1$ and $q$ are odd primes. Thus $q$ is a Sophie Germain prime and $p$ is a safe prime. We write $q - 1 = 2^r s$ with $r \geq 1$ and $s$ odd. We have $\gcd(s, 2pq) = 1$, but we make no further assumptions about the prime factorisation of $s$.

Up to isomorphism, there are two groups $N$ of order $pq$, namely the cyclic group $C_{pq}$ and the non-abelian group $C_p \rtimes C_q$. (We adopt the convention that the notation $A \rtimes B$ always refers to a semidirect product taken with respect to some faithful action of $B$ on $A$.) Thus we must determine the transitive subgroups of $\text{Hol}(C_{pq})$ and $\text{Hol}(C_p \rtimes C_q)$.

4.1. **Cyclic case.** Let $N$ be a cyclic group of order $pq$. We work with the presentation

$$N = \langle \sigma, \tau : \sigma^p = \tau^q = 1, \tau \sigma = \sigma \tau \rangle.$$  

As the subgroups $\langle \sigma \rangle$ and $\langle \tau \rangle$ are characteristic in $N$, we have

$$\text{Aut}(N) \cong \text{Aut}(\langle \sigma \rangle) \times \text{Aut}(\langle \tau \rangle)$$

where the factors are cyclic of order $p - 1 = 2q$ and $q - 1 = 2^r s$ respectively. Let $\alpha, \beta$ be automorphisms of $N$ of order $q$, $2$ respectively which fix $\tau$, and let $\gamma, \delta$ be automorphisms of order $2^r$, $s$ respectively fixing $\sigma$. Then $\text{Aut}(N)$ decomposes as the direct product $\langle \alpha \rangle \times \langle \beta, \gamma \rangle \times \langle \delta \rangle$, where the factors have coprime orders $q, 2^r, s$ respectively. A subgroup of $\text{Aut}(N)$ decomposes as a direct product of one subgroup from each of these factors. The subgroups of $\langle \alpha \rangle$ are $\langle \alpha \rangle$ and $\{\text{id}\}$, whereas $\langle \delta \rangle$ has one subgroup $\langle \delta^{s/d} \rangle$ of order $d$ for each divisor $d$ of $s$. We write $\sigma_0(s)$ for the number of divisors of $s$. The subgroups of $\langle \beta, \gamma \rangle$ are as follows:

(i) for $0 \leq c \leq r$, the group $\langle \beta, \gamma^{2^{r-c}} \rangle$ of order $2^{c+1}$, which is cyclic only when $c = 0$;

(ii) for $0 \leq c \leq r$, the cyclic group $\langle \gamma^{2^{r-c}} \rangle$ of order $2^c$;

(iii) for $1 \leq c \leq r$, the cyclic group $\langle \beta \gamma^{2^{r-c}} \rangle$ of order $2^c$.

**Proposition 4.1.** For $1 \leq t \leq q - 1$, let $J_t = \langle \sigma, [\tau, \alpha^t] \rangle$. Then $J_t$ is a non-abelian regular subgroup of $\text{Hol}(N)$. Moreover, the transitive subgroups $G$ of $\text{Hol}(N)$ are as shown in Table 1.

**Proof.** In $J_t$ we find that $[\tau, \alpha^t]$ has order $q$ (since $\alpha$ fixes $\tau$) and $[\tau, \alpha^t] \sigma = \alpha^t(\sigma) [\tau, \alpha^t]$, so $J_t$ is non-abelian of order $pq$ and regular on $N$.

As $\text{Hol}(N)$ contains a unique subgroup $H = \langle \sigma, \tau, \alpha \rangle$ of order $pq^2$ with index $2^r s$ coprime to $pq$, and any transitive subgroup $M$ has order
divisible by \( pq \), it follows that \( M \cap H \) must be transitive on \( N \). Thus either \( M \subset H \) or \( M \cap H \) is regular on \( N \). Now the subgroups of order \( pq \) in \( H \) are \( N \), the groups \( J_t \), and one further subgroup \( \langle \sigma, \alpha \rangle \) which is not regular. We then have \( M \cap H = H \) or \( N \) or \( J_t \) for some \( t \). In particular, every transitive subgroup \( M \) contains either \( N \) or some \( J_t \).

Now \( N \) is normal in \( \text{Hol}(N) \), so can be extended by any subgroup of \( \text{Aut}(N) \) to give a transitive subgroup \( M \). The normaliser of \( J_t \) in \( \text{Aut}(N) \) is \( \langle \alpha, \beta \rangle \) since if \( \phi \in \text{Aut}(N) \) and \( \phi(\tau) \neq \tau \), we have \( \phi[\tau, \alpha^t]\phi^{-1} = [\phi(\tau), \alpha^t] \notin J_t \). Hence if \( M \) is a transitive subgroup containing \( J_t \) but not \( N \) then \( M = J_t \) or \( J_t \not\cong \langle \beta \rangle \). The list of transitive subgroups then follows from the description of subgroups of \( \text{Aut}(N) \) given above.

The group \( N \) itself occurs in Table 1 as case (E) with \( (c, d) = (0, 1) \). We observe that, in all the subgroups in Table 1, the stabiliser of \( 1_N \) has a normal complement (either \( N \) or \( J_t \)), so that all the corresponding field extensions are almost classically Galois. The abstract isomorphism types of these groups are as shown in Table 1, where the groups in cases (C) and (F) have no non-trivial direct product decomposition. We write \( D_{2m} \) for the dihedral group \( C_m \times C_2 \) of order \( 2m \).

**Lemma 4.2.** Of the groups in Table 1, the \( q - 1 \) groups in case (G) are isomorphic as permutation groups, as are the \( q - 1 \) groups in case (H). There are no other isomorphisms, even as abstract groups.

**Proof.** Each of the groups in cases (A)–(F) has the form \( N \times A \) for some subgroup \( A \subset \text{Aut}(N) \). By Proposition 2.2, no two of these groups are isomorphic. Moreover, none of these groups can be isomorphic to a group in case (G) or (H), since the latter groups do not contain an

| Key | Order | Parameters | \# groups | Group |
|-----|-------|------------|-----------|-------|
| (A) | \( 2^{c+1}d_{pq}^2 \) | \( 0 \leq c \leq r, d \mid s \) | \( (r + 1)\sigma_0(s) \) | \( N \times \langle \alpha, \beta, \gamma^{2^c-c}, \delta^{s/d} \rangle \) |
| (B) | \( 2^c_{dpq}^2 \) | \( 0 \leq c \leq r, d \mid s \) | \( (r + 1)\sigma_0(s) \) | \( N \times \langle \alpha, \gamma^{2^c-c}, \delta^{s/d} \rangle \) |
| (C) | \( 2^c_{dpq}^2 \) | \( 1 \leq c \leq r, d \mid s \) | \( r\sigma_0(s) \) | \( N \times \langle \alpha, \beta, \gamma^{2^c-c}, \delta^{s/d} \rangle \) |
| (D) | \( 2^{c+1}_{dpq} \) | \( 0 \leq c \leq r, d \mid s \) | \( (r + 1)\sigma_0(s) \) | \( N \times \langle \beta, \gamma^{2^c-c}, \delta^{s/d} \rangle \) |
| (E) | \( 2^c_{dpq} \) | \( 0 \leq c \leq r, d \mid s \) | \( (r + 1)\sigma_0(s) \) | \( N \times \langle \gamma^{2^c-c}, \delta^{s/d} \rangle \) |
| (F) | \( 2^c_{dpq} \) | \( 1 \leq c \leq r, d \mid s \) | \( r\sigma_0(s) \) | \( N \times \langle \beta, \gamma^{2^c-c}, \delta^{s/d} \rangle \) |
| (G) | \( 2_{pq} \) | \( 1 \leq t \leq q - 1 \) | \( q - 1 \) | \( J_t \times \langle \beta \rangle \) |
| (H) | \( pq \) | \( 1 \leq t \leq q - 1 \) | \( q - 1 \) | \( J_t \) |

**Table 1.** Transitive subgroups for \( N \) cyclic
Table 2. Structures of transitive subgroups for $N$ cyclic

| Key | Restrictions | Order | Structure |
|-----|--------------|-------|-----------|
| (A) | $(c, d) \neq (0, 1), (1, 1)$, $(c, d) = (1, 1)$, $(c, d) = (0, 1)$ | $2^{c+1}pq^2d$ | $(C_p \times C_{2q}) \times (C_q \times C_{2d})$ |
|     |               | $4pq^2$ | $(C_p \times C_{2q}) \times D_{2q}$ |
|     |               | $2pq^2$ | $(C_p \times C_{2q}) \times C_q$ |
| (B) | $(c, d) \neq (0, 1), (1, 1)$, $(c, d) = (1, 1)$, $(c, d) = (0, 1)$ | $2^{c+1}pq^2d$ | $(C_p \times C_q) \times (C_q \times C_{2d})$ |
|     |               | $2pq^2$ | $(C_p \times C_q) \times D_{2q}$ |
|     |               | $pq^2$ | $(C_p \times C_q) \times C_q$ |
| (C) | $(c, d) \neq (0, 1), (1, 1)$, $(c, d) = (1, 1)$, $(c, d) = (0, 1)$ | $2^{c}pq^2d$ | $C_{pq} \times C_{2d,q}$ |
| (D) | $(c, d) \neq (0, 1), (1, 1)$, $(c, d) = (1, 1)$, $(c, d) = (0, 1)$ | $2^{c+1}pqd$ | $D_{2p} \times (C_q \times C_{2d})$ |
|     |               | $4pq$ | $D_{2p} \times D_{2q}$ |
|     |               | $2pq$ | $D_{2p} \times C_q$ |
| (E) | $(c, d) \neq (0, 1), (1, 1)$, $(c, d) = (1, 1)$, $(c, d) = (0, 1)$ | $2^{e}pqd$ | $C_p \times (C_q \times C_{2d})$ |
|     |               | $2pq$ | $C_p \times D_{2q}$ |
|     |               | $pq$ | $C_{pq}$ |
| (F) | $(c, d) \neq (0, 1), (1, 1)$, $(c, d) = (1, 1)$, $(c, d) = (0, 1)$ | $2^{e}pqd$ | $C_{pq} \times C_{2d}$ |
|     |               | $2pq$ | $D_{2pq}$ |
| (G) | $(c, d) = (1, 1)$ | $2pq$ | $C_p \times C_{2q}$ |
| (H) | $(c, d) = (1, 1)$ | $pq$ | $C_p \times C_q$ |

abelian subgroup of order $pq$. Thus we only need consider cases (G) and (H).

Let $1 \leq t \leq q - 1$ and let $\phi \in \text{Aut}(N)$ with $\phi(\tau) = \tau^t$. Then $\phi[\tau, \alpha^t]^{-1} = [\phi(\tau), \alpha^t] = [\tau, \alpha]^t$. Also $\phi\beta\phi^{-1} = \beta$. Thus conjugation by $\phi$ gives an isomorphism $J_t \times \langle \beta \rangle \to J_1 \times \beta$, and this is an isomorphism of permutation groups as it fixes the stabiliser $\langle \beta \rangle$ of $1_N$. Moreover, it restricts to an isomorphism $J_t \to J_1$. Hence all the groups in case (G) are isomorphic as permutation groups, and similarly for case (H). \[\square\]

**Lemma 4.3.** The numbers of Hopf-Galois structures are as in Table 2.

**Proof.** In each case, the stabiliser of $1_N$ in $M$ is $M = M' \cap \text{Aut}(N)$. In cases (A)–(F), $N$ is characteristic in $M$ since it is the unique abelian subgroup of order $pq$. As $\text{Aut}(N)$ is abelian, it follows from Proposition 2.3 that $\text{Aut}(M, M') = \text{Aut}(N)$. Thus $|\text{Aut}(M, M')| = (p - 1)(q - 1)$. Since there are no automorphisms between distinct groups in cases (A)–(F), the number of isomorphism classes is just the number of choices of parameters. (For example, in case (A) there are $r + 1$ ways to choose $c$ and $s_0(s)$ ways to choose $s$.) For each isomorphism class $M$, the
| Key | Order | $|\text{Aut}(M, M')|$ | # isom. classes | # HGS per isom. class |
|-----|--------|-----------------|-----------------|-----------------|
| (A) | $2^{c+1}dpq^2$ | $(p-1)(q-1)$ | $(r+1)\sigma_0(s)$ | 1 |
| (B) | $2^c dpq^2$ | $(p-1)(q-1)$ | $(r+1)\sigma_0(s)$ | 1 |
| (C) | $2^c dpq^2$ | $(p-1)(q-1)$ | $r\sigma_0(s)$ | 1 |
| (D) | $2^{c+1}dpq$ | $(p-1)(q-1)$ | $(r+1)\sigma_0(s)$ | 1 |
| (E) | $2^c dpq$ | $(p-1)(q-1)$ | $(r+1)\sigma_0(s)$ | 1 |
| (F) | $2^c dpq$ | $(p-1)(q-1)$ | $r\sigma_0(s)$ | 1 |
| (G) | $2pq$ | $p-1$ | 1 | 1 |
| (H) | $pq$ | $p(p-1)$ | 1 | $p$ |

Table 3. Hopf-Galois Structures for $N$ cyclic

The number of Hopf-Galois structures is $|\text{Aut}(M, M')|/|\text{Aut}(N)| = 1$ by Lemma 2.1.

We now consider case (H). The $q-1$ regular groups $J_t$ form a single isomorphism class. We have $|\text{Aut}(J_t)| = p(p-1)$, and for $M = J_t$ we have $|M'| = 1$. Thus the number of Hopf-Galois structures of cyclic type on a $J_t$-extension is

$$(q-1) \frac{|\text{Aut}(M, M')|}{|\text{Aut}(N)|} = \frac{(q-1)p(p-1)}{(p-1)(q-1)} = p.$$ 

(This calculation was already in [Byo04].)

Finally we consider case (G). Let $M = J_t \rtimes \langle \beta \rangle$. We apply Proposition 2.3 with $J_t$ in place of $N$ and $A = M' = \langle \beta \rangle$. Conjugation by $\beta$ inverts $\sigma$ and fixes the generator $\tau' = [\tau, \alpha^t]$ of order $q$. If $\phi \in \text{Aut}(J_t)$ we have $\phi(\sigma) = \sigma^a$ and $\phi(\tau') = \sigma^b\tau'$ for $1 \leq a \leq p-1$ and $0 \leq b \leq p-1$. Then $\phi$ normalises $M'$ in $\text{Aut}(J_t)$ if and only if $b = 0$. Thus $|\text{Aut}(M, M')| = p-1$, and the $q-1$ conjugate subgroups give $(q-1)(p-1)/|\text{Aut}(N)| = 1$ Hopf-Galois structure of this type. □

We summarise the results for cyclic $N$ in the following theorem.

**Theorem 4.4.** There are in total $(6r+4)\sigma_0(s)+2$ isomorphism types of permutation groups $G$ of degree $pq$ which are realised by a Hopf-Galois structure of cyclic type. These include the two regular groups, i.e. the cyclic and non-abelian groups of order $pq$ (for which the corresponding Galois extensions have 1 and $p$ Hopf-Galois structures of cyclic type respectively). For all the remaining groups $G$, any field extension $L/K$ realising $G$ is almost classically Galois and admits a unique Hopf-Galois structure of cyclic type.
4.2. **Metacyclic case.** Now let $N$ be the non-abelian group of order $pq$:

\[ N = \langle \sigma, \tau : \sigma^p = 1 = \tau^q, \tau\sigma = \sigma^g\tau \rangle, \]

where $g$ has order $q \mod p$. (Thus $g$ is the square of a primitive root mod $p$.) Then, by Lemma 3.1 (or [Byo04]), $\text{Aut}(N)$ has order $p(p - 1) = 2pq$, and is generated by automorphisms $\alpha, \beta, \epsilon$, of orders $q, 2, p$ respectively, where

\[ \alpha(\sigma) = \sigma^g, \quad \alpha(\tau) = \tau; \]
\[ \beta(\sigma) = \sigma^{-1}, \quad \beta(\tau) = \tau; \]
\[ \epsilon(\sigma) = \sigma, \quad \epsilon(\tau) = \sigma\tau. \]

Note that $\alpha, \beta$ now depend on the choice of the generator $\tau$ of $N$. In $\text{Aut}(N)$ we have the commutation relations

\[ \beta\alpha = \alpha\beta, \quad \alpha\epsilon = \epsilon\alpha, \quad \beta\epsilon = \epsilon^{-1}\beta. \]

Thus $\text{Hol}(N) = \langle \sigma, \tau, \alpha, \beta, \epsilon \rangle$ is a group of order $2p^2q^2$.

By definition, $\text{Hol}(N)$ is the semidirect product $N \rtimes \text{Aut}(N)$. It is convenient, however, to work with a different description of $\text{Hol}(N)$ as a semidirect product. The subgroup $P = \langle \sigma, \epsilon \rangle \cong C_p \times C_p$ is the unique Sylow $p$-subgroup of $\text{Hol}(N)$, and has a complementary subgroup $R = \langle \tau, \alpha, \beta \rangle \cong C_q \times C_q \times C_2$. Thus we have the semidirect product decomposition $\text{Hol}(N) = P \rtimes R$. The element $\sigma\epsilon^g^{-1} \in P$ commutes with $\tau$:

\[ \sigma\epsilon^g^{-1}\tau = \sigma\epsilon^g^{-1}(\tau)\epsilon^g^{-1} = \sigma^g\tau\epsilon^g^{-1} = \tau\sigma\epsilon^g^{-1}. \]

We write $P$ additively, and identify it with $\mathbb{F}_p^2$, the vector space of dimension 2 over the field $\mathbb{F}_p$ of $p$ elements. We choose basis vectors

\[ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

corresponding to $\sigma, \sigma\epsilon^g^{-1}$ respectively. Then the generators $\tau, \alpha, \beta$ of $R$ are identified with the matrices

\[ T = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \]

We write $I$ for the identity matrix, and $0$ for the identity element of $P$. For later use, we note that

\[ (4.1) \quad I + T + T^2 + \cdots + T^{q-1} = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}, \quad I + A + A^2 + \cdots + A^{q-1} = 0. \]
We write an element of Hol(N) = P ⋊ R as [v, U] for v ∈ \( \mathbb{F}_p^2 \) and U a matrix corresponding to an element of R. The multiplication in Hol(N) is then given by

\[[v, U][w, V] = [v + Uw, UV].\]

As before, we often abbreviate [v, I], [0, U] to v, U.

We next give a preliminary description of the transitive subgroups of Hol(N) in this notation.

Lemma 4.5. A subgroup M of Hol(N) is transitive on N if and only if it satisfies the following two conditions:

(i) the image of M under the quotient map Hol(N) → R is one of the 2 + 2q groups \( \langle T, A \rangle, \langle T, A, B \rangle = R, \langle TA^u \rangle, \langle TA^u, B \rangle \) where \( 0 ≤ u ≤ q - 1 \);

(ii) \( M \cap P \) is one of the three subgroups \( \mathbb{F}_p^2, \mathbb{F}_p\mathbf{e}_1, \mathbb{F}_p\mathbf{e}_2 \), each of which is normalised by R.

Proof. Suppose that M is transitive. Then the orbit of 1\( _N \) under \( M \cap P \) must have size \( p \), so the projection of M into R cannot be contained in Aut(N). Hence this projection must be one of the groups listed in (i).

If \( P ⊈ M \) then \( M \cap P \) has order \( p \). Thus \( M \cap P = \mathbb{F}_p\mathbf{e}_1 \) or \( \mathbb{F}_p(\lambda\mathbf{e}_1 + \mathbf{e}_2) \) for some \( \lambda \in \mathbb{F}_p \). In the latter case, M contains an element of the form \([v, TA^aB^b]\) for some \( v \in P \), \( 0 ≤ a ≤ q - 1 \), \( 0 ≤ b ≤ 1 \), and

\[TA^aB^b(\lambda\mathbf{e}_1 + \mathbf{e}_2) = \left(\begin{array}{c}( -1)^bg^{a+1}\lambda \\ -1)^bg^a\end{array}\right) = ( -1)^bg^a(\lambda g\mathbf{e}_1 + \mathbf{e}_2).\]

The vector \( g\lambda\mathbf{e}_1 + \mathbf{e}_2 \) does not lie in \( M \cap P \) unless \( \lambda = 0 \). Thus \( M \cap P \) is one of the subgroups listed in (ii). It is clear that these are all normalised by R.

Conversely, it is easy to check that if M satisfies (i) and (ii) then M is transitive. \( \square \)

We now determine in detail the transitive subgroups M of Hol(N). If M is a transitive group of even order then it is generated by its unique subgroup \( M^* \) of index 2 (which is one of the transitive subgroups of odd order) together with an element of order 2 which normalises \( M^* \). The elements of Hol(N) of order 2 are precisely those of the form \([v, B]\) for \( v \in P \). We therefore consider the transitive subgroups of each possible odd order in turn, in each case then identifying the subgroups of twice that order.

Proposition 4.6. The transitive subgroups M of Hol(N) of order divisible by \( p^2q \) are as follows:
Proposition 4.7. The transitive subgroups $M$ of $\text{Hol}(N)$ of order $pq^2$ are as follows:

(i) $\langle e_1, T, [\mu e_2, A] \rangle$ for $\mu \in \mathbb{F}_p$;
(ii) $\langle e_2, [\mu e_1, T], [\mu e_1, A] \rangle$ for $\mu \in \mathbb{F}_p$.

Each of these $2p$ groups is non-abelian with structure $C_q \times (C_p \times C_q)$.

Proof. By Lemma 4.6 if $M$ is a transitive subgroup of order $pq^2$ then $M \cap P = \mathbb{F}_p e_1$ or $\mathbb{F}_p e_2$, and the projection of $M$ to $R$ is $(T, A)$.

If $M \cap P = \mathbb{F}_p e_1$ then $M$ is generated by elements $e_1$, $[v, T]$, $[w, A]$ for some $v, w \in \mathbb{F}_p^2$. Replacing $[v, T]$ by $(re_1)[v, T]$ for a suitable $r$, we may assume that $v = \lambda e_2$ for some $\lambda \in \mathbb{F}_p$. Similarly, we may assume that $w = \mu e_2$ for some $\mu \in \mathbb{F}_p$. Now, using (4.1), we have

$$[\lambda e_2, T]^q = [(1 + T + T^2 + \cdots + T^{q-1})\lambda e_2, T^q] = q\lambda e_2.$$ As $e_2 \not\in M$, it follows that $\lambda = 0$. However, $[\mu e_2, A]^q = 0$ for any choice of $\mu$. Then $M$ is generated by $e_1, T$ and $[\mu e_2, A]$, which have orders $p$, $q$, $q$ respectively. We check that $T$ and $[\mu e_2, A]$ commute, and that

$$Te_1 T^{-1} = g e_1 = [\mu e_2, A] e_1 [\mu e_2, A]^{-1}.$$ Thus $e_1, T$ and $[\mu e_2, A]$ do indeed generate a group of order $pq^2$, and its centre is generated by the element $[\mu e_2, AT^{-1}]$ of order $q$. Thus $M \cong C_q \times (C_p \times C_q)$. This gives the $p$ groups in (i).

If now $M \cap P = \mathbb{F}_p e_2$ then $M$ is generated by elements $e_2$, $[\lambda e_1, T]$, $[\mu e_1, A]$ for $\lambda, \mu \in \mathbb{F}_p$. In this case, the second and third generators have order $q$ for any choice of $\lambda, \mu$. We calculate

$$[\lambda e_1, T][\mu e_1, A][\lambda e_1, T]^{-1} = [(\mu - \lambda)(g - 1)e_1, I][\mu e_1, A].$$ Thus $(\mu - \lambda)(g - 1)e_1 \in M$, so $\lambda = \mu$ and $M$ is as in (ii). We then find that $[\mu e_1, T]$ is in the centre of $M$, while

$$[\mu e_2, A] e_2 [\mu e_2, A]^{-1} = g e_2.$$ This gives the $p$ groups in (ii). \qed

Proposition 4.8. The transitive subgroups $M$ of $\text{Hol}(N)$ of order $2pq^2$ are as follows:

(i) the $p$ groups $\langle e_1, T, [\mu e_2, A], [2(1 - g)^{-1} \mu e_2, B] \rangle$ for $\mu \in \mathbb{F}_p$;
(ii) the $p$ groups $\langle e_2, [\mu e_1, T], [\mu e_1, A], [2(1 - g)^{-1} \mu e_1, B] \rangle$ for $\mu \in \mathbb{F}_p$. 

These groups all have the structure $C_q \times (C_p \rtimes C_{2q})$.

Proof. Any such group is obtained by adjoining an element of order 2 to one of the groups in Proposition 4.7(i) or (ii). In the first case, we may take the generator of order 2 to be $[\nu e_2, B]$ for some $\nu \in \mathbb{F}_p$. Then

$$M = \langle e_1, T, [\mu e_2, A], [\nu e_2, B] \rangle.$$  

We require $[\nu e_2, B]$ to normalise the subgroup of order $pq^2$. We calculate

$$[\nu e_2, B][\mu e_2, A][\nu e_2, B]^{-1} = [\nu e_2 - \mu e_2, BA][\nu e_2, B]$$  

which lies in $M$ only if $\nu - \mu - g\nu = \mu$, so that $\nu = 2(1 - g)^{-1}\mu$, and then $[\mu e_2, A]$ and $[\nu e_2, B]$ commute. Similar (but simpler) calculations show that $[\nu e_2, B]$ commutes with $T$ and inverts $e_1$. Thus $M \cong C_q \times (C_p \rtimes C_{2q})$, and we obtain the groups in (i).

If the subgroup of $M$ of index 2 is as in Proposition 4.7(ii), we have

$$M = \langle e_2, [\mu e_1, T], [\mu e_1, A], [\nu e_1, B] \rangle$$  

for some $\nu \in \mathbb{F}_p$. Consideration of $[\nu e_1, B][\mu e_1, A][\nu e_1, B]$ shows that again $\nu = 2(1 - g)^{-1}\mu$, and then $[\nu e_1, B]$ and $[\mu e_1, A]$ commute. Similarly $[\nu e_1, B]$ and $[\mu e_1, T]$ commute, and conjugation by $[\nu e_1, B]$ inverts $e_2$. We then get the groups in (ii). \hfill \square

Proposition 4.9. The transitive subgroups $M$ of $\text{Hol}(N)$ of order $pq$ are as follows:

- (i) the $p(q - 2)$ non-abelian groups $\langle e_1, [\lambda e_2, TA^u] \rangle$ for $\lambda \in \mathbb{F}_p$ and $1 \leq u \leq q - 2$;
- (ii) the $p$ cyclic groups $\langle e_1, [\lambda e_2, TA^{-1}] \rangle$ for $\lambda \in \mathbb{F}_p$;
- (iii) the non-abelian group $\langle e_1, T \rangle = N$;
- (iv) the $p(q - 2)$ non-abelian groups $\langle e_2, [\lambda e_1, TA^u] \rangle$ for $\lambda \in \mathbb{F}_p$ and $1 \leq u \leq q - 2$;
- (v) the $p$ cyclic groups $\langle e_2, [\lambda e_1, T] \rangle$ for $\lambda \in \mathbb{F}_p$;
- (vi) the non-abelian group $\langle e_2, TA^{-1} \rangle$.

Proof. The transitive subgroups of order $pq$ are regular, and were already found in [Byo04]. For completeness, we determine them here. Any such group $M$ contains either $e_1$ or $e_2$.

If $M$ contains $e_1$ then $M = \langle e_1, [\lambda e_2, TA^u] \rangle$ for $\lambda \in \mathbb{F}_p$ and $0 \leq u \leq q - 1$. As

$$TA^u = \begin{pmatrix} g^{u+1} & 0 \\ 0 & g^u \end{pmatrix},$$
we have
\[ [\lambda e_2, TA^u]^q = \begin{cases} 0 & \text{if } u \neq 0, \\ q\lambda e_2 & \text{if } u = 0. \end{cases} \]
Hence if \( u = 0 \) we must have \( \lambda = 0 \). In either case, \([\lambda e_2, TA^u]\) has order \( q \). Moreover, we have
\[ [\lambda e_2, TA^u]e_1[\lambda e_2, TA^u]^{-1} = g^{u+1} e_1. \]
Thus \( M \) is abelian if and only if \( u = q - 1 \). Taking \( 1 \leq u \leq q - 2 \), and any \( \lambda \), we get the \( p(q-2) \) groups in (i). Taking \( u = q - 1 \) and \( \lambda \) arbitrary, we get the \( p \) groups in (ii). Taking \( u = 0 \) and \( \lambda = 0 \), we get the group in (iii).

Similarly, a transitive subgroup of order \( pq \) containing \( e_2 \) is of the form \( M = \langle e_2, [\lambda_1, TA^u] \rangle \). We have \([\lambda e_1, TA^u]^q = q\lambda e_1 \) if \( u = q - 1 \), so we must have \( \lambda = 0 \) in this case. The two generators commute if and only if \( u = 0 \). Thus we get the groups in (iv), (v) and (vi).

**Proposition 4.10.** The transitive subgroups \( M \) of \( \text{Hol}(N) \) of order \( 2pq \) are as follows:

1. the \( p(q-2) \) groups \( \langle e_1, [\lambda e_2, TA^u], [2(1-g^{-1})^{-1}\lambda e_2, B] \rangle \) of type \( C_p \times C_{2q} \)
for \( \lambda \in \mathbb{F}_p \) and \( 1 \leq u \leq q - 2 \);
2. the \( p \) groups \( \langle e_1, [\lambda e_2, TA^{-1}], [2(1-g^{-1})^{-1}\lambda e_2, B] \rangle \) of type \( D_{2p} \times C_q \)
for \( \lambda \in \mathbb{F}_p \);
3. the \( p \) groups \( \langle e_1, T, \mu e_2, B \rangle \) of type \( C_p \times C_{2q} \) for \( \mu \in \mathbb{F}_p \);
4. the \( p(q-2) \) groups \( \langle e_2, [\lambda e_1, TA^u], [2(1-g^{-1})^{-1}\lambda e_2, B] \rangle \) of type \( C_p \times C_{2q} \)
for \( \lambda \in \mathbb{F}_p \) and \( 1 \leq u \leq q - 2 \);
5. the \( p \) groups \( \langle e_2, [\lambda e_1, T], [2(1-g)^{-1}\lambda e_2, B] \rangle \) of type \( D_{2p} \times C_q \)
for \( \lambda \in \mathbb{F}_p \);
6. the \( p \) groups \( \langle e_2, TA^{-1}, [\mu e_2, B] \rangle \) of type \( C_p \times C_{2q} \) for \( \mu \in \mathbb{F}_p \).

**Proof.** First suppose that \( e_1 \in M \), so \( M \) contains one of the subgroups in Proposition 4.9 (i), (ii) or (iii). Then
\[ M = \langle e_1, [\lambda e_2, TA^u], [\mu e_2, B] \rangle \]
for some \( \mu \in \mathbb{F}_p \), where \( \lambda = 0 \) if \( u = 0 \), and the first two generators commute if and only if \( u = q - 1 \). A calculation similar to (4.2) gives
\[ [\mu e_2, B][\lambda e_2, TA^u][\mu e_2, B] = ([\mu - \lambda - g^u\mu]e_2, TA^u], \]
so that \( \mu - \lambda - g^u\mu = \lambda \). Then \([\mu e_2, B]\) commutes with \([\lambda e_2, TA^u]\) while \([\mu e_2, B][e_1][\mu e_2, B] = -e_1 \). If \( u \neq 0 \), we have \( \mu = 2(1-g^{-1})^{-1}\lambda \), and the resulting group is of type \( C_p \times C_{2q} \) if \( u \neq q - 1 \) and \( D_{2pq} \) if \( u = q - 1 \). This gives the \( p(q-2) \) groups in (i) and the \( p \) groups in (ii). If \( u = 0 \), we have \( \lambda = 0 \) but we may choose \( \mu \) arbitrarily. Thus we obtain the \( p \) groups in (iii).
If $e_2 \in M$, similar calculations give cases (iv), (v) and (vi).

The results of Propositions 4.6 and 4.10 are summarised in Table 4.

| Order | Parameters | Structure | Group |
|-------|------------|-----------|-------|
| $p^2q^2$ | $\mathbb{F}_p^2 \rtimes (C_q \times C_q)$ | $P \times \langle T, A \rangle = N \rtimes \langle \alpha, \epsilon \rangle$ | $\text{Hol}(N)$ |
| $2p^2q^2$ | $\mathbb{F}_p^2 \rtimes (C_q \times C_{2q})$ | $\langle e_2, [\mu e_1, T], [\mu e_1, A] \rangle$ | $\langle e_2, [\mu e_1, T], [\mu e_1, A], [2(1-g)^{-1}\mu e_1, B] \rangle$ |
| $pq$ | $\mu \in \mathbb{F}_p$ | $C_q \times (C_p \rtimes C_q)$ | $\langle e_1, T, [\mu e_2, A] \rangle$ |
| $pq$ | $\mu \in \mathbb{F}_p$ | $C_q \times (C_p \rtimes C_q)$ | $\langle e_1, T, [\mu e_2, A], [2(1-g)^{-1}\mu e_2, B] \rangle$ |
| $2pq$ | $\mu \in \mathbb{F}_p$ | $(C_p \rtimes C_{2q})$ | $\langle e_2, [\mu e_1, T], [\mu e_1, A], [2(1-g)^{-1}\mu e_1, B] \rangle$ |
| $2pq$ | $\mu \in \mathbb{F}_p$ | $(C_p \rtimes C_{2q})$ | $\langle e_2, [\mu e_1, T], [\mu e_1, A], [2(1-g)^{-1}\mu e_1, B] \rangle$ |

**Table 4.** Transitive subgroups for $N$ metabelian

For each of the transitive subgroups $M$ in Table 4 we wish to compute $|\text{Aut}(M, M')|$, where $M'$ is the stabiliser of $1_N$ in $M$. We also want to know when any two of these groups $M$ are isomorphic, either as abstract groups or as permutation groups. We observe that $P \cap \text{Aut}(N) = \langle \epsilon \rangle$. This subgroup is also generated by $e^{1-g} = \sigma(\sigma e^{-1})^{-1}$, which corresponds to the vector

$$f = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

**Proposition 4.11.** For the two groups $M$ in Proposition 4.6(i) and (ii), we have $|\text{Aut}(M, M')| = 2p(p-1)$.

**Proof.** First let $M = P \rtimes \langle T, A \rangle$ be the group of order $p^2q^2$ in Proposition 4.6(i). Then $M$ has exactly two normal subgroups $\mathbb{F}_p e_1$ and $\mathbb{F}_p e_2$ of order $p$, so any $\theta \in \text{Aut}(M, M')$ must either preserve or swap these
two subgroups. Thus the restriction of $\theta$ to $P$ acts as a matrix of the form
\[
\begin{pmatrix}
x & 0 \\
0 & y
\end{pmatrix}
or
\begin{pmatrix}
0 & x \\
y & 0
\end{pmatrix}
\]
with $x, y \in \mathbb{F}_p^\times$. However, $\theta$ must also fix the group $M' = M \cap \text{Aut}(N)$ of order $pq$, and hence must fix its unique Sylow $p$-subgroup. Hence $\theta(f) \in \mathbb{F}_p f$. Since
\[
\begin{pmatrix}
x & 0 \\
0 & y
\end{pmatrix} f = \begin{pmatrix}
x - y \\
y
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & x \\
y & 0
\end{pmatrix} f = \begin{pmatrix}
-x \\
y
\end{pmatrix},
\]
we have $y = x$ in either case. Let us define $\pi$ to be the identity map on $\{1, 2\}$ in the first case, and the unique transposition on $\{1, 2\}$ in the second, so that $\theta(e_i) = xe_{\pi(i)}$ in either case.

Now the matrix $A$ corresponds to $\alpha \in \text{Aut}(N)$ of order $q$, so $\theta(A) = [\lambda f, A^c]$ with $\lambda \in \mathbb{F}_p$ and $1 \leq c \leq q - 1$. From the relations $Ae_i A^{-1} = ge_i$ for $i = 1, 2$ we obtain
\[
[\lambda f, A^c] e_{\pi(i)} [\lambda f, A^c]^{-1} = ge_{\pi(i)},
\]
which holds if and only if $g^c e_{\pi(i)} = ge_{\pi(i)}$. Hence $c = 1$. Also, we must have
\[
\theta(T) = [v, T^d A^m]
\]
for some vector $v$ and some $d, m \in \mathbb{F}_q$ with $d \neq 0$. As $T$ and $A$ commute, we have
\[
[\lambda f, A] [v, T^d A^m] = [v, T^d A^m] [\lambda f, A],
\]
which is equivalent to
\[
\lambda f + gv = v + \lambda T^d A^m f,
\]
and so to
\[
(4.3) \quad v = (g - 1)^{-1} \lambda (T^d A^m - I) f.
\]
Thus $v$ is determined by $\lambda$, $m$ and $d$. If $\pi$ is the identity map, the relations
\[
(4.4) \quad Te_1 T^{-1} = ge_1, \quad Te_2 T^{-1} = e_2
\]
give
\[
[v, T^d A^m] e_1 [v, T^d A^m]^{-1} = g x e_1, \quad [v, T^d A^m] e_2 [v, T^d A^m]^{-1} = x e_2,
\]
which are equivalent to
\[
g^{m+d} x e_1 = g x e_1, \quad g^m x e_2 = x e_2,
\]
so that $m = 0, d = 1$. If $\pi$ is the transposition, then (4.4) gives

$$[v, T^d A^m]xe_2 [v, T^d A^m]^{-1} = gx e_2, \quad [v, T^d A^m]xe_1 [v, T^d A^m]^{-1} = xe_1,$$

so that

$$g^m x e_2 = gx e_2, \quad g^{d+m} x e_1 = xe_1,$$

and $m = 1, d = q - 1$. Hence for $\theta \in \text{Aut}(M, M')$ we have either

$$\theta(e_1) = xe_1, \quad \theta(e_2) = xe_2, \quad \theta(A) = [\lambda f, A], \quad \theta(T) = [v, T]$$

with $v = \lambda e_1$, or

$$\theta(e_1) = xe_2, \quad \theta(e_2) = xe_1, \quad \theta(A) = [\lambda f, A], \quad \theta(T) = [v, T^{q-1} A]$$

with $v = -\lambda e_2$, where $x \in \mathbb{F}_p^\times$ and $\lambda \in \mathbb{F}_p$. Hence $|\text{Aut}(M, M')| = 2p(p - 1)$.

We next consider the case where $M$ is the full group Hol$(N)$ of order $2p^2q^2$ in Proposition 4.6(ii). If $\theta \in \text{Hol}(M, M')$ then the restriction of $\theta$ to the unique subgroup of index 2 must be one of the $2p(p - 1)$ automorphisms just described, and $\theta(B) \in \text{Aut}(N)$ has order 2, so that $\theta(B) = [\mu f, B]$ for some $\mu \in \mathbb{F}_p$. Since $\theta(A)$ and $\theta(B)$ must commute, we have

$$\lambda f + g \mu f = \mu f - \lambda f,$$

so

(4.5) $\mu = 2(1 - g)^{-1} \lambda.$

Finally, as $\theta(T)$ and $\theta(B)$ must commute, we require

$$[\mu f, B][v, T^d A^m] = [v, T^d A^m][\mu f, B],$$

which is equivalent to

$$\mu f - v = v + \mu T^d A^m f.$$

By (4.3) and (4.5), this last condition is automatically satisfied. Thus each of the $2p(p - 1)$ automorphisms $\theta$ of the subgroup of $M$ of index 2 extends uniquely to an automorphism of $M$. Again we have $|\text{Aut}(M, M')| = 2p(p - 1)$. \hfill \Box

Proposition 4.12. Let

$$M_u = P \rtimes (TA^u), \quad 0 \leq u \leq q - 1$$

be the $q$ groups of order $p^2q$ in Proposition 4.6(iii), and let

$$\widehat{M}_u = P \rtimes (TA^u, B), \quad 0 \leq u \leq q - 1$$

be the $q$ groups of order $2p^2q$ in Proposition 4.6(iv). Then $M_u$ and $M_v$ are isomorphic (as abstract groups or permutation groups) if and
only if \( u + v + 1 = 0 \) (mod \( q \)). Hence the groups \( M_u \) fall into \( \frac{1}{2}(q + 1) \) isomorphism classes. Similarly for the groups \( \widehat{M}_u \). Moreover

\[
|\text{Aut}(M_u, M'_u)| = \begin{cases} 
p^2(p - 1) & \text{if } u \neq \frac{1}{2}(q - 1), \\
2p^2(p - 1) & \text{if } u = \frac{1}{2}(q - 1) 
\end{cases}
\]

and

\[
|\text{Aut}(\widehat{M}_u, \widehat{M}'_u)| = \begin{cases} 
p(p - 1) & \text{if } u \neq \frac{1}{2}(q - 1), \\
2p(p - 1) & \text{if } u = \frac{1}{2}(q - 1) 
\end{cases}
\]

Proof. The generator \( TA^u \) of \( M_u \) acts on \( P \) with two distinct eigenvalues \( g^{u+1}, g^u \). Any element of order \( q \) in \( M_u \) has the form \([v, (TA^u)^c]\) for some \( v \in P \) and \( c \in \mathbb{F}_q^\times \), and this element acts on \( P \) with eigenvalues \( g^{(u+1)c}, g^{uc} \). Thus if \( M_u \) and \( M_v \) are isomorphic as abstract groups, then \( \{g^{(u+1)c}, g^{uc}\} = \{g^{v+1}, g^v\} \) for some \( c \). Thus either \((u + 1)c = v + 1, uc = v \) in \( \mathbb{F}_q \), or \((u + 1)c = v, uc = v + 1 \). In the first case \( c = 1 \) so \( u = v \). In the second case \( c = -1 \) so \( u + v + 1 = 0 \), and then \( v \neq u \) unless \( u = \frac{1}{2}(q - 1) \). We then have an isomorphism \( \phi : M_u \to M_v \) given by \( \phi(e_1) = e_2, \phi(e_2) = e_1, \phi(TA^u) = (TA^v)^{-1} \). As \( \phi(f) = -f \), it follows that \( \phi \) is an isomorphism of permutation groups. Thus the groups \( M_u \) fall into \( \frac{1}{2}(q + 1) \) isomorphism classes.

The argument for the groups \( \widehat{M}_u \) is similar, using the fact that the element \( TA^uB \) of order \( 2q \) acts with eigenvalues \( -g^{u+1}, -g^u \).

Now let \( \theta \in \text{Aut}(M_u, M'_u) \). Again, \( \theta \) must either fix or swap the two eigenspaces \( \mathbb{F}_p e_1, \mathbb{F}_p e_2 \), and must also fix the stabiliser \( \mathbb{F}_p f \) of the identity element of \( N \). Hence

\[
\theta(e_i) = xe_{\pi(i)} \text{ for } i = 1, 2,
\]

where \( x \in \mathbb{F}_p^\times \) and \( \pi \) is either the identity map or the transposition on \( \{1, 2\} \). Moreover \( \theta(TA^u) = [v, (TA^u)^c] \) for some \( v \in P \) and some \( c \neq 0 \). If \( \pi \) is the identity map, the commutation relations for \( e_1 \) and \( e_2 \) reduce to \( g^{(u+1)c} = g^{u+1}, g^{uc} = g^u \), so that \( c = 1 \) and we obtain no restriction on \( v \). If \( \pi \) is the transposition, we obtain \( g^{(u+1)c} = g^u, g^{uc} = g^{u+1} \), so that \( c = -1 \) and \( u = (q - 1)/2 \), again with no restriction on \( v \). Thus we have

\[
|\text{Aut}(M_u, M'_u)| = \begin{cases} 
p^2(p - 1) & \text{if } u \neq \frac{1}{2}(q - 1), \\
2p^2(p - 1) & \text{if } u = \frac{1}{2}(q - 1) 
\end{cases}
\]

Finally, let \( \theta \in \text{Aut}(\widehat{M}_u, \widehat{M}'_u) \). Arguing as above, we have \( \theta(TA^uB) = [v, (TA^uB)^c] \), where \( c = \pm 1 \) if \( u = -(q - 1)/2 \) and \( c = 1 \) otherwise. However, since \( (TA^uB)^q = B \) lies in the stabiliser \( \langle f, B \rangle \) of the identity element of \( N \), we have the additional constraint that \([v, (TA^uB)^c]^q \) also
lies in this group. Writing $S = (TA^u B)\mathbf{c}$, this means that $(I + S + \cdots + S^{q-1}) \mathbf{v} \in \mathbb{F}_p^\times$. Now $S$ is a diagonal matrix whose diagonal entries have order $2q$ or $2$ in $\mathbb{F}_p^\times$. It follows that the matrix $I + S + \cdots + S^{q-1}$ is invertible, so that there are only $p$ possibilities for $\mathbf{v}$. Hence $|\Aut(\widehat{M}_u, M'_u)| = p^{-1}|\Aut(M_u, M'_u)|$. \hfill $\Box$

**Remark 4.13.** We describe in more detail the structure of the groups $M_u$ and $\widehat{M}_u$, all of which were denoted rather loosely in Table 4 by $\mathbb{F}_p^\times \rtimes C_q$ or $\mathbb{F}_p^\times \rtimes C_{2q}$. We need only consider $0 \leq u \leq \frac{1}{2}(q - 1)$. When $u = 0$, the generator $TA^u = T$ commutes with $\mathbf{e}_2$, so $M_0 \cong C_p \times (C_p \rtimes C_q)$. Thus $M_0$ contains a normal subgroup of order $pq$, which is a complement to $M'_u = \mathbb{F}_p^\times$. Similarly, the group $\widehat{M}_0$ contains a normal complement to $M'_u$. When $1 \leq u \leq \frac{1}{2}(q - 1)$, the generator $TA^u$ acts on $P$ with the two distinct eigenvalues $g^u$, $g^{u+1}$ of order $q$. (In the exceptional case $u = \frac{1}{2}(q - 1)$, these eigenvalues are mutually inverse.) A normal complement in $M_u$ to $M'_u = \mathbb{F}_p^\times$ would be a transitive subgroup of order $pq$, and since $u \neq 0$, $q - 1$, it would be as in Proposition 4.11(i) or (iv). However, none of these groups is normalised by $P$. Hence $M'_u$ does not have a normal complement in $M_u$. A similar argument applies to $\widehat{M}_u$.

For $1 \leq u \leq \frac{1}{2}(q - 1)$, we denote the isomorphism classes of the groups $M_u$ and $\widehat{M}_u$ by $\mathbb{F}_p^\times \rtimes_u C_q$ and $\mathbb{F}_p^\times \rtimes_u C_{2q}$.

**Proposition 4.14.** The $2p$ groups $M$ of order $pq^2$ in Proposition 4.7 are all isomorphic as permutation groups, and $|\Aut(M, M')| = (p - 1)(q - 1)$ for these groups.

The same holds for the $2p$ groups of order $2pq^2$ in Proposition 4.8.

**Proof.** There is only one isomorphism class of abstract groups of the form $G = C_q \times (C_p \rtimes C_q)$. Such a group contains $pq$ non-normal subgroups of order $q$, and these form a single orbit under $\Aut(G)$. Hence all the groups of order $pq^2$ in Proposition 4.7 are isomorphic as permutation groups. A similar argument applies to the groups of order $2pq^2$.

To find $|\Aut(M, M')|$, we may therefore suppose $M = \langle \mathbf{e}_1, T, A \rangle$ of order $pq^2$ or $M = \langle \mathbf{e}_1, T, A, B \rangle$ of order $2pq^2$. In the first case, if $\theta \in \Aut(M, M')$ then $\theta(\mathbf{e}_1) = x\mathbf{e}_1$ and $\theta(A) = A^y$ for some $x \in \mathbb{F}_p^\times$, $y \in \mathbb{F}_q^\times$. As $\theta(A)\theta(\mathbf{e}_1)\theta(A)^{-1} = g\mathbf{e}_1$, we have $y = 1$. Then $\theta(T) = [z\mathbf{e}_1, T^c A^d]$ with $z \in \mathbb{F}_p^\times$, $c \in \mathbb{F}_q^\times$, $d \in \mathbb{F}_q$. As $\theta(A)$ and $\theta(T)$ commute, we have $gz = z$ and hence $z = 0$. As $\theta(T)\theta(\mathbf{e}_1)\theta(T)^{-1} = g\mathbf{e}_1$, we have $c + d = 1$ in $\mathbb{F}_q$. Thus we have $p - 1$ choices for $x$ and $q - 1$ choices for $c$, and
these choices determine $\theta$. Hence $|\text{Aut}(M, M')| = (p - 1)(q - 1)$ for each of the $2p$ groups of order $pq^2$.

Finally, if $M = \langle e_1, T, A, B \rangle$ then the above automorphism $\theta$ on its subgroup of index 2 extends to an element of $\text{Aut}(M, M')$ by taking $\theta(B) = B$, and this extension is unique as $B$ is the only element of order 2 in $M'$. □

The groups of order $pq$ in Proposition 4.9 are regular, so any two of them are isomorphic as permutation groups if they are isomorphic as abstract groups. We already know $\text{Aut}(M)$ in these cases.

**Proposition 4.15.** The $2p(q - 1)$ groups in Proposition 4.10 of isomorphism type $C_p \rtimes C_{2q}$ are all isomorphic as permutation groups. For these groups, $|\text{Aut}(M, M')| = p - 1$.

The $2p$ groups $M$ in Proposition 4.10 of isomorphism type $D_{2p} \times C_q$ are isomorphic as permutation groups. For these groups, $|\text{Aut}(M, M')| = (p - 1)(q - 1)$.

**Proof.** In both cases, the stabiliser of $1_N$ has order 2, and all subgroups of order 2 in $M$ are conjugate. Hence the groups of the same isomorphism type as abstract groups are isomorphic as permutation groups. Moreover, $C_p \rtimes C_{2q}$ (respectively, $D_{2p} \times C_q$) has $p - 1$ (respectively, $(p - 1)(q - 1)$) automorphisms fixing a given element of order 2. □

The information in Propositions 4.11–4.15 is summarised in Table 5, where we have also used Lemma 2.1 to find the number of Hopf-Galois structures in each case.

We summarise our results for Hopf-Galois structures of non-abelian type in the following theorem.

**Theorem 4.16.** There are in total $q + 9$ isomorphism types of permutation groups $G$ of degree $pq$ which are realised by Hopf-Galois structures of non-abelian type $C_p \rtimes C_q$, as listed in Table 5. These include the two regular groups, i.e. the cyclic and non-abelian groups of order $pq$ (for which the corresponding Galois extensions have $2(q - 1)$ and $2p(q - 2) + 2$ Hopf-Galois structures of non-abelian type respectively). For $q - 1$ of these permutation groups (more precisely, for all but one of the groups of each of the orders $p^2q$, $2p^2q$), the corresponding field extensions fail to be almost classically Galois. In the remaining 10 cases, the extensions are almost classically Galois.

**Proof.** Everything except the statements about almost classically Galois extensions follows from Table 5. For these, see Remark 4.13 for the groups of order $p^2q$ and $2p^2q$. In all the other cases, it is clear from the structure of $G$ that $G$ contains a regular normal subgroup. □
| Order | Structure | # groups | $|\text{Aut}(M, M')|$ | # HGS |
|-------|-----------|----------|-----------------|-------|
| $p^2q^2$ | $N \rtimes (C_p \rtimes C_q)$ | 1 | $2p(p-1)$ | 2 |
| $2p^2q^2$ | Hol($N$) | 1 | $2p(p-1)$ | 2 |
| $p^2q$ | $C_p \times (C_p \rtimes C_q)$ | 2 | $p^2(p-1)$ | 2p |
| | $F_p^2 \rtimes u C_q$, $1 \leq u \leq \frac{1}{2}(q-3)$ | 2 | $p^2(p-1)$ | 2p |
| | $F_p^2 \rtimes \frac{1}{2}(q-1) C_q$ | 1 | $2p^2(p-1)$ | 2p |
| $2p^2q$ | $(C_p \times (C_p \rtimes C_q)) \rtimes C_2$ | 2 | $p(p-1)$ | 2 |
| | $F_p^2 \rtimes u C_{2q}$, $1 \leq u \leq \frac{1}{2}(q-3)$ | 2 | $p(p-1)$ | 2p |
| | $F_p^2 \rtimes \frac{1}{2}(q-1) C_{2q}$ | 1 | $2p(p-1)$ | 2p |
| $pq^2$ | $C_q \rtimes (C_p \rtimes C_q)$ | 2$p$ | $(p-1)(q-1)$ | $2(q-1)$ |
| $2pq^2$ | $C_q \rtimes (C_p \rtimes C_{2q})$ | 2$p$ | $(p-1)(q-1)$ | $2(q-1)$ |
| $pq$ | $C_p \rtimes C_q$ | $2p(q-2) + 2$ | $p(p-1)$ | $2p(q-2) + 2$ |
| | $C_{pq}$ | $2p$ | $(p-1)(q-1)$ | $2(q-1)$ |
| $2pq$ | $C_p \rtimes C_{2q}$ | $2p(q-1)$ | $p-1$ | $2(q-1)$ |
| | $D_{2p} \rtimes C_q$ | $2p$ | $(p-1)(q-1)$ | $2(q-1)$ |

Table 5. Hopf-Galois structures for $N$ metabelian

**Remark 4.17.** In general, any Hopf-Galois structure of non-abelian type has an “opposite” Hopf-Galois structure of the same type; see [GP87, Lemma 2.4.2]. This explains why the number of Hopf-Galois structures in each row of Table 5 is even.

4.3. **Comparing the two types.** We can read off from Tables 2 and 5 which permutation groups $G$ are realised by Hopf-Galois structures of both types. The number of Hopf-Galois structures of each type are shown in Tables 3 and 5. This gives our final result.

**Theorem 4.18.** There are six permutation groups $G$ of degree $n$ which are realised by Hopf-Galois structures of both cyclic and non-abelian types. These are as shown in Table 6. In all these cases, the corresponding field extensions are almost classically Galois.

The last two rows of Table 6 recover the main results of [Byo04].

5. **Concluding Comments**

We have obtained some general results on the permutation groups of squarefree degree $n$ which are realised by Hopf-Galois structures, and we have listed all such groups in the case that $n = pq$ where $q$ is a Sophie Germain prime and $p$ is the associated safeprime. We have also determined those $G$ realised by Hopf-Galois structures of both possible types. In this special case, the field extensions admitting a Hopf-Galois
structure of cyclic type are always almost classically Galois, whereas (for large $q$) most of those admitting a Hopf-Galois structure of non-abelian type are not. Moreover, we found no cases where two distinct permutation groups, both realised by Hopf-Galois structures, had the same underlying abstract group. It would be interesting to investigate whether similar behaviour occurs for more general squarefree degrees $n$.

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