Dicke-Type Energy Level Crossings in Cavity-Induced Atom Cooling: Another Superradiant Cooling

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This paper is devoted to energy-spectral analysis for the system of a two-level atom coupled with photons in a cavity. It is shown that the Dicke-type energy level crossings take place when the atom-cavity interaction of the system undergoes changes between the weak coupling regime and the strong one. Using the phenomenon of the crossings we develop the idea of cavity-induced atom cooling proposed by the group of Ritsch, and we lay mathematical foundations of a possible mechanism for another superradiant cooling in addition to that proposed by Domokos and Ritsch.

The process of our superradiant cooling can function well by cavity decay and by control of the position of the atom, at least in (mathematical) theory, even if there is neither atomic absorption nor atomic emission of photons.

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I. INTRODUCTION

Laser cooling is one of attractive subjects of modern physics. It has been demonstrated with several experimental techniques such as the ion cooling [1], the Doppler cooling [2], the Sisyphus cooling [3], etc. Also it has enabled us to observe many fundamental phenomena in theoretical physics. One of typical instances of applications of the laser cooling is the observation of Bose-Einstein condensation [4, 5]. It has been about ten years since another type of laser cooling was proposed using the strong atom-photon interaction. Such a strong interaction between atom and photons is realized in the so-called cavity QED [6, 7, 8, 9, 10, 11].

Thus, the group of Ritsch has investigated the system of a two-level atom coupled with a laser, and then, they have found a mechanism for cooling the atom [12], which is similar to that of the Sisyphus cooling. The cooling mechanism is called cavity-induced atom cooling. In the process of their cavity-induced atom cooling, the method to carry away the energy from the atom coupled with photons is given not only by atomic decay (i.e., atomic spontaneous emission of photons) but also by cavity decay. It has experimentally been confirmed that the cavity decay works in the cooling system [13, 14]. Concerning the cooling methods using cavity QED, Domokos and Ritsch proposed a concept of superradiant cooling [15] based on the atomic self-organization and cooperation among many atoms in a cavity [16].

The atom-photon interaction in the strong coupling regime brings the situation amazingly different from ordinary atomic decay [10]. We will adopt this difference into our arguments on the cavity-induced atom cooling.

As well as the ensemble of many two-level atoms coupled with a laser has the possibility of making super-radiance as a cooperative effect in optics [17, 18], another superradiance may also appear in energy spectrum even for the system of a two-level atom coupled with a laser, provided that it is in the strong coupling regime [19, 21, 22, 23, 24, 25]. As far as atom-laser interaction in the cooling process for the Bose-Einstein condensation goes, a superradiance has been experimentally observed under a certain physical condition [24, 25, 26, 27], and this phenomenon has been theoretically shown [28, 29, 30]. It was pointed out that there is a possibility that superradiance causes the energy level crossing between the initial ground state energy and an initial excited state energy [16, 22, 23], namely, a kind of phase transition occurs. This is an optical phenomenon of light-induced phase transitions though it is not a cooperative effect in optics. The details of such an energy level crossing has precisely been studied, and then, this type of crossing is called the Dicke-type (energy level) crossing [31].

We will strictly define its meaning in Sec.II. The reason why we call the energy level crossing so is that it is basically caused by the mathematical mechanism [19, 22] of Dicke’s superradiance [32].

This paper is devoted to developing the cavity-induced atom cooling. Namely, we will show that the Dicke-type energy level crossings take place when the system undergoes changes between the weak coupling regime and the strong one. Using the crossings, we will propose a possibility of another superradiant cooling in terms of the energy spectrum from our point of view. In our proposal we will consider whether the followings are possible in theory for cooling the atom in a cavity: 1) can we use a laser only for controlling the strength of the atom-cavity interaction without throwing another laser to the atom for driving it to an excited state? 2) can we expect that the energy loss caused by cavity decay becomes much larger? To perform our research into the problems, we consider an ideal situation only to see the energy-spectral property for our system without considering, for example, the laser heating processes caused by diffusion of the atomic momentum. Some mathematical technique to make the
spectral analysis for such systems have been developed lately [31, 33, 34, 35, 36]. Thus, another purpose of this paper is to show the mechanism of the Dicke-type energy level crossing in the cavity-induced atom cooling as rigorously as in the works in Ref. [37] so that the process of our superradiant cooling can function well in (mathematical) theory.

Our paper is constructed in the following. In Sec. II we will generalize the Hamiltonian \( H(\Omega, \alpha; d) \) which Ritsch’s group handled, where \( \Omega \) is a function of space-time point and governs the atom-photon interaction, \( \alpha \) also a function of space-time and a generalization of the strength of the pump field, and \( d \) a parameter for non-linear coupling of the atom and photons. Moreover, we will define some notion to explain what the Dicke-type energy level crossing is. We will make energy-spectral analysis for the generalized Hamiltonian \( H(\Omega, \alpha; d) \) in and after Sec. III. In Sec. IV we will show that the Dicke-type energy level crossings take place for \( H(\Omega, \alpha; d) \) with \( \alpha = 0 \). In Sec. V we will show the existence of the superradiant ground state energy for \( H(\Omega, \alpha; d) \) with \( \alpha = 0 \) and \( d = 1 \) in the strong coupling regime. In Sec. VI we will argue the stability of the Dicke-type energy level crossing under the condition \( \alpha \neq 0 \).

II. HAMILTONIAN AND SOME NOTION

In Ref. [12] the group of Ritsch studied a Hamiltonian adopting dipole and rotating wave approximation. To write down their Hamiltonian, we define some operators: the atomic position (resp. momentum) operator is denoted by \( x \) (resp. \( p \)), the photon annihilation (resp. creation) operator by \( a \) (resp. \( a^\dagger \)), and the atomic operator is given by \( \sigma_{ij} = |i\rangle\langle j|, i, j = 0, 1 \). Then, the Hamiltonian is

\[
H = \frac{1}{2m} p^2 - \Delta \sigma_{11} - \Delta_c a^\dagger a \\
+ i\Omega(x) \left( \sigma_{01} a^\dagger - \sigma_{10} a \right) + i\alpha(a - a^\dagger),
\]

where two real numbers \( \Delta \) and \( \Delta_c \) with \( -\infty < \Delta < +\infty \) and \( \Delta_c < 0 \) are respectively the atom-pump detuning and the detuning of the empty cavity relative to the pump frequency, and \( \Omega(x) \) stands for the atom-cavity coupling constant, i.e., \( \Omega(x) = \Omega_0 \cos kx \) with the position \( x \) of the atom and the wave number \( k \) of photons of the laser. We note that in the case \( \alpha = 0 \) the Hamiltonian \( H \) is used to argue the resonant interaction of an atom with a microwave field [38]. In Hamiltonian \( H \), the part consisting of the first, the second, and the third terms (i.e., \( (2m)^{-1} p^2 - \Delta \sigma_{11} - \Delta_c a^\dagger a \)) is the free Hamiltonian of our system. Each of the fourth and fifth terms represents the Hamiltonian of interaction and the energy operator of the pump field respectively. In this section we generalize the above Hamiltonian. Our generalization is the following: (1) we consider not only the linear coupling but also non-linear coupling; (2) we introduce the time-dependence into the coupling constant \( \Omega(x) \); (3) we consider the general operator which represents not only the energy operator of the pump field but also, for instance, the energy operator of the pump field plus some error potential coming from the environment of the experiment for testing the system. Thus our Hamiltonian reads

\[
H(\Omega, \alpha; d) = \frac{1}{2m} p^2 - \Delta \sigma_{11} - \Delta_c a^\dagger a \\
+ i\Omega(x,t) \left( \sigma_{01} a^\dagger d - \sigma_{10} a d \right) + \alpha(x,t) W(x,t)
\]

for \( d = 1, 2, \cdots \), where \( \Omega(x,t) \) and \( \alpha(x,t) \) are continuous, real-valued functions of \( x \) with \( \Omega(x,0) = \Omega(x,0) = 0 \) for every position \( x \) of the atom, and \( \alpha(x,t) W(x,t) \) is the generalization of the energy operator of the pump field. As an example of \( \Omega(x,t) \), we often adopt \( \Omega(x,t) = \Omega_0(t) \gamma(x) \) in this paper. Here \( \Omega_0(t) \) is a continuous, real-valued function of time \( t \geq 0 \) with \( \Omega_0(0) = 0 \), and \( \gamma(x) \) a bounded, continuous, real-valued function of the position \( x \) of the atom. For instance, \( \gamma(x) = \cos kx \).

In the case where \( \Omega(x,t) = 0 \) and \( \alpha(x,t) = 0 \), we denote eigenvalues of \( H(0,0,0) := H(0,0,0,0) \) by \( E_0 < E_1 < \cdots < E_n < \cdots \). When either \( \Omega(x,t) \) or \( \alpha(t) \) is alive, we denote eigenvalues of \( H(\Omega, d) \) by \( E_n(\Omega, d) \) for \( n = 0, 1, \cdots \). If the interaction \( H_{\text{int}} := i\Omega(t,x) \left( \sigma_{01} a^\dagger d - \sigma_{10} a d \right) + \alpha(t) W(x,t) \) is a small perturbation for \( H(0,0,0) \), then each eigenvalue \( E_n(\Omega, d) \) sits near its original position \( E_n \), so that the primary order among eigenvalues is kept: \( E_0(\Omega, d) < E_1(\Omega, d) < \cdots < E_n(\Omega, d) < \cdots \). On the other hand, the phase transition of the superradiance [13, 22, 23] tells us about a possibility that \( E_1(\Omega, d) \) is less than \( E_0(\Omega, d) \) and thus becomes a new ground state energy provided that the interaction \( H_{\text{int}} \) has some strong strength. For our Hamiltonian, we can classify crossings into two types. One type is the crossing between an ascending eigenvalue \( E_n^+ (\Omega, d) \) and a descending one \( E_n^- (\Omega, d) \) as the strength of the interaction \( H_{\text{int}} \) grows enough. Another type is the crossing only among descending eigenvalues \( E_n^- (\Omega, d) \) (or ascending eigenvalues \( E_n^+ (\Omega, d) \)). We call the former type a trivial crossing, and the latter type a non-trivial crossing. For the non-trivial crossing, as the strength of the interaction becomes much stronger, even many \( E_n^- (\Omega, d) \) may be less than \( E_0^- (\Omega, d) \). We call such a non-trivial crossing the Dicke-type (energy level) crossing [31]. Moreover, \( E_n^- (\Omega, d) \)’s are capable of usurping the position of the ground state energy in turn. We call such a new ground state energy the super-radiant ground state energy. The Dicke-type energy level crossings and the appearance of the superradiant ground state energy can be used, together with cavity decay, for carrying away the energy from the system. Based on this idea, we construct mathematical foundations of the concept of superradiant cooling different from that proposed in Ref. [15] in and after the next section.
III. THE DICKE-TYPE ENERGY LEVEL CROSSINGS IN THE CASE $\alpha \equiv 0$

In this section we show how the Dicke-type energy level crossing takes place for $H(\Omega, 0; d)$, that is, for $H(\Omega, \alpha; d)$ in the case where $\alpha(t) \equiv 0$. As in Ref.12 using the well-known identification so that the ground state $|0\rangle$ with the energy $\varepsilon_0$ and the 1st excited state $|1\rangle$ with the energy $\varepsilon_1$ are unitarily equivalent to $|0\rangle$ and $|1\rangle$ respectively, $H(\Omega, 0; d)$ approximately reads

$$H_0(z, t; d) := \left( \begin{array}{cc} -\Delta_c a^\dagger a + \varepsilon_1 - \Delta & -i\Delta \Omega(z, t) a d \\ -i\Delta \Omega(z, t) a^d & -\Delta_c a^\dagger a + \varepsilon_0 \end{array} \right).$$

(3.1)

Here we note $\sigma_{00} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$, $\sigma_{01} = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)$, $\sigma_{10} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$, and $\sigma_{11} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$. In their paper we always assume that

$$\varepsilon_1 - \Delta > \varepsilon_0.$$  

(3.2)

Therefore, the ground state energy of $H_0(z, 0; d)$ (i.e., $H_0(z, t; d)$ with $\Omega(z, t) = 0$) is always $\varepsilon_0$.

As shown in Sec. A with the method which is a generalization of that in Refs.22 and 13, since $H_0(z, t; d)$ is basically the Hamiltonian of the Jaynes-Cummings model 33, all the energy levels of $H_0(z, t; d)$ are perfectly determined. They are given by $-\Delta_c n + \varepsilon_0$ for non-negative integer $n$ with $n < d$, and $\Xi_n(d) \equiv \Upsilon_n(z, t; d)$ for non-negative integer $n$ with $n \geq d$, where

$$\Xi_n(d) = -\Delta_c n + \frac{1}{2} \left( \varepsilon_0 + \varepsilon_1 + d\Delta_c - \Delta \right),$$

and $\Upsilon_n(z, t; d)$ is the generalized Rabi frequency 44:

$$\Upsilon_n(z, t; d) = \frac{1}{2} \sqrt{\left( \varepsilon_1 - \varepsilon_0 + d\Delta_c - \Delta \right)^2 + 4|\Omega(z, t)|^2 \frac{n!}{(n - d)!}}.$$

(3.3)

Here we note all the energy levels of $H_0(z, 0; d)$ are $-\Delta_c n + \varepsilon_0$ and $-\Delta_c n + \varepsilon_1 - \Delta$, $n = 0, 1, \cdots$. Therefore, we can conclude that the energy levels of $H_0(z, t; d)$ are completely given by energies $E^0_n(z, t; d)$ and energies $E^+_{n}(z, t; d)$ of the generalized Jaynes-Cummings doublet 17, continuous functions of $(z, t)$, for each $n = 0, 1, \cdots$.

For non-negative integers with $n < d$

$$E^0_n(z, t; d) = -\Delta_c n + \varepsilon_0.$$ 

For non-negative integers $n$ with $n \geq d$, on the other hand,

$$E^+_{n}(z, t; d) = \begin{cases} \Xi_n(d) - \Upsilon_n(z, t; d) & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - d\Delta_c, \\ \Xi_{n+d}(d) - \Upsilon_{n+d}(z, t; d) & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - d\Delta_c, \end{cases}$$

and

$$E^-_{n}(z, t; d) = \begin{cases} \Xi_n(d) + \Upsilon_n(z, t; d) & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - d\Delta_c, \\ \Xi_{n+d}(d) + \Upsilon_{n+d}(z, t; d) & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - d\Delta_c. \end{cases}$$

It follows from these definitions that

$$E^-_{n}(z, 0; d) = \begin{cases} -\Delta_c n + \varepsilon_0 & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - d\Delta_c, \\ -\Delta_c n + \varepsilon_1 - \Delta & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - d\Delta_c, \end{cases}$$

and

$$E^+_{n}(z, 0; d) \geq E^+_{n}(z, 0; d) = \begin{cases} -\Delta_c n + \varepsilon_1 - \Delta & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - d\Delta_c, \\ -\Delta_c n + \varepsilon_0 & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - d\Delta_c. \end{cases}$$

The later inequality means that the candidates of super-radiant ground state energy are only $E^-_{n}(z, t; d)$s.

We define two spatiotemporal domains $D^w_{mn}(d)$ and $D^sc_{mn}(d)$ for non-negative integers $m$ and $n$ with $\max\{d, m\} < n$ as

$$D^w_{mn}(d) := \{(z, t) | E^0_m(z, t; d) < E^-_{n}(z, t; d) \text{ if } m < d; \ E^-_{m}(z, t; d) < E^-_{n}(z, t; d) \text{ if } m \geq d \}$$

and

$$D^sc_{mn}(d) := \{(z, t) | E^0_m(z, t; d) > E^-_{n}(z, t; d) \text{ if } m < d; \ E^-_{m}(z, t; d) > E^-_{n}(z, t; d) \text{ if } m \geq d \}$$

respectively.

A. In the case $d = 1$

In this subsection we investigate the behavior of the Dicke-type energy level crossings in the case $d = 1$. To do that, we introduce some positive numbers and some domains of the space-time. Then, we divide the whole space of the space-time into three classes, namely, the weak coupling regime, the strong coupling regime, and the critical regime:

For each natural number $n$, we define a positive number $C^0_{0n}$ by

$$C^0_{0n} := \begin{cases} \Delta^2 n - \Delta_c (\varepsilon_1 - \varepsilon_0 + \Delta_c - \Delta) & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - d\Delta_c, \\ \Delta^2 (n + 1) - \Delta_c (\varepsilon_1 - \varepsilon_0 + \Delta_c - \Delta) & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - d\Delta_c. \end{cases}$$

We define three domains $D^w_{0n}(1)$, $D^sc_{0n}(1)$, and $D^0_{0n}(1)$ of the space-time as follows: the spatiotemporal domain $D^w_{0n}(1)$ for the weak coupling regime is given by $D^w_{0n}(1) := \{(z, t) | |\Omega(z, t)|^2 < C^0_{0n}\}$, and the domain $D^sc_{0n}(1)$ for the strong coupling regime by
The domain $\mathcal{D}^{\text{wc}}_{mn}(1)$ for the critical regime is defined by

$$\mathcal{D}^{\text{wc}}_{mn}(1) := \{(z, t) \mid |\Omega(z, t)|^2 > C^0_{mn}\}.$$  

The domain $\mathcal{D}^{\text{wc}}_{mn}(1)$ for the critical regime is defined by

$$\mathcal{D}^{\text{wc}}_{mn}(1) := \{(z, t) \mid |\Omega(z, t)|^2 = C^0_{mn}\}.$$  

The following theorem says that how the Dicke-type energy level crossing takes place is completely determined: Let us suppose $1 < n$ now. Then, the spatiotemporal domain $\mathcal{D}^{\text{wc}}_{mn}(1)$ is equal to the domain $\mathcal{D}^{\text{wc}}_{mn}(1)$, and the spatiotemporal domain $\mathcal{D}^{\text{wc}}_{mn}(1)$ is defined by $\mathcal{D}^{\text{wc}}_{mn}(1) \subset \mathcal{D}^{\text{wc}}_{mn}(1)$, i.e., $\mathcal{D}^{\text{wc}}_{mn}(1) = \mathcal{D}^{\text{wc}}_{mn}(1)$ and $\mathcal{D}^{\text{wc}}_{mn}(1) = \mathcal{D}^{\text{wc}}_{mn}(1)$. Namely, the energy level crossing takes place as

$$E^0_n(z, t; 1) < E^-_n(z, t; 1) \text{ if and only if } (z, t) \text{ in } \mathcal{D}^{\text{wc}}_{mn}(1),$$  

$$E^0_n(z, t; 1) = E^-_n(z, t; 1) \text{ if and only if } (z, t) \text{ in } \mathcal{D}^{\text{wc}}_{mn}(1),$$  

$$E^0_n(z, t; 1) > E^-_n(z, t; 1) \text{ if and only if } (z, t) \text{ in } \mathcal{D}^{\text{wc}}_{mn}(1).$$  

These inequalities (3.4)–(3.6) guarantee the Dicke-type energy level crossing because $E^0_n(z, t; 1)$ and $E^-_n(z, t; 1)$ are continuous functions of the space-time point $(z, t)$.

When the above Dicke-type energy level crossing between $E^0_n(z, t; 1)$ and $E^-_n(z, t; 1)$ takes place, there is certainly an energy level crossing between $E^0_n(z, t; 1)$ and $E^-_n(z, t; 1)$ for a natural number $m$ with $1 < m < n$. To show it, we introduce two positive constants and define two domains of the space-time:

For natural numbers $m, n$ with $m < n$, we set positive constants $C_{mn}$ and $C_{mn}$ as

$$_{mn} C_{mn} = \begin{cases} \Delta^2 \left\{ \frac{m + n}{2} + \sqrt{\frac{(m + n)^2}{4} + \frac{K^2}{2\Delta}} \right\} & \text{if } \epsilon_1 - \epsilon_0 \geq \Delta - \Delta_c, \\ 0 & \text{if } \epsilon_1 - \epsilon_0 < \Delta - \Delta_c, \end{cases}$$  

and

$$_{mn} C_{mn} = \begin{cases} \Delta^2 \left\{ \frac{m + n}{2} + \sqrt{\frac{mn + K^2}{2\Delta^2}} \right\} & \text{if } \epsilon_1 - \epsilon_0 \geq \Delta - \Delta_c, \\ 0 & \text{if } \epsilon_1 - \epsilon_0 < \Delta - \Delta_c, \end{cases}$$

where $K = \epsilon_1 - \epsilon_0 + \Delta_c - \Delta$. We give two spatiotemporal domains $\mathcal{D}^{\text{wc}}_{mn}(1)$ and $\mathcal{D}^{\text{wc}}_{mn}(1)$ for non-negative integers $m, n$ with $m < n$ in the following: The domain $\mathcal{D}^{\text{wc}}_{mn}(1)$ for the weak coupling regime is given by $\mathcal{D}^{\text{wc}}_{mn}(1) = \{(z, t) \mid 0 \leq |\Omega(z, t)|^2 < C^0_{mn}\}$, and the domain $\mathcal{D}^{\text{wc}}_{mn}(1)$ for the strong coupling regime by $\mathcal{D}^{\text{wc}}_{mn}(1) = \{(z, t) \mid |\Omega(z, t)|^2 > C^0_{mn}\}$. Then, as far as such $m$ goes, the following theorem gives a sufficient condition so that the crossing between $E^0_n(z, t; 1)$ and $E^-_n(z, t; 1)$ occurs: Let $1 < m < n$ now. Then, the spatiotemporal domain $\mathcal{D}^{\text{wc}}_{mn}(1)$ is included in the domain $\mathcal{D}^{\text{wc}}_{mn}(1)$, and the domain $\mathcal{D}^{\text{wc}}_{mn}(1)$ in the domain $\mathcal{D}^{\text{wc}}_{mn}(1)$, i.e., $\mathcal{D}^{\text{wc}}_{mn}(1) \subset \mathcal{D}^{\text{wc}}_{mn}(1)$ and $\mathcal{D}^{\text{wc}}_{mn}(1) \subset \mathcal{D}^{\text{wc}}_{mn}(1)$:

$$E^0_n(z, t; 1) < E^-_n(z, t; 1) \text{ for } (z, t) \text{ in } \mathcal{D}^{\text{wc}}_{mn}(1),$$  

$$E^0_n(z, t; 1) > E^-_n(z, t; 1) \text{ for } (z, t) \text{ in } \mathcal{D}^{\text{wc}}_{mn}(1).$$  

These two theorems will be proved at the end of this subsection. But, before proving them, we concretely see some physical situation that they tell us. We employ $\cos 2\pi z$ as $\gamma(z)$. For the position fixed at $z = 0$, the five energies $E^0_n(0, t; 1)$ and $E^-_n(0, t; 1)$, $n = 1, 2, 3, 4$, are numerically calculated as in Fig. 1. In the case where the strength $|\Omega_0(t)|$ is given by $|\Omega_0(t)| = 2\kappa; 4\kappa; 6\kappa; 8\kappa$, the two energies $E^0_n(z, t; 1)$ and $E^-_n(z, t; 1)$ are in Fig. 2.

![FIG. 1: Dicke-type energy level crossings for $\gamma(z)=\cos 2\pi z$.](image-url)
coming from the descent. Then, we can expect that the crossing between $E_n^{-}(z, t; 1)$ and $E_n^{+}(z, t; 1)$ makes the temperature go down at most $\Delta T_{m-n}$ estimated at:

$$
\Delta T_{m-n} \approx \frac{2(n-m)}{k_B} \frac{|\Omega(z, t)|^2}{T_m(z_0, t_0; 1)} - |\Delta c| + \frac{2m}{k_B} \frac{|\Omega(z, t)|^2 - |\Omega(z_0, t_0)|^2}{T_m(z_0, t_0; 1)} - |\Delta c| \tag{3.9}
$$

for the point $(z_0, t_0)$ in the spatiotemporal domain $D_{m-n}(1)$ and the point $(z, t)$ in the domain $D_{m-n}(1)$, of course, provided that there is nothing to obstruct the temperature loss. Here $k_B$ is the Boltzmann constant and $T_{\ell}(z, t; 1) = (1/2) \sqrt{K^2 + 4|\Omega(z, t)|^2}$ the generalized Rabi frequency for each natural number $\ell$.

Conversely, let our atom be in the state with the energy $E_n^{-}(z_0, t_0; 1)$ at an initial space-time point $(z_0, t_0)$ in the domain $D_{m-n}(1)$. Then, since a process reverse to the above energy level crossing takes place if the coupling regime recoils from the spatiotemporal domain $D_{m-n}(1)$ to the domain $D_{m-n}(1)$, this reverse crossing can also carry away the temperature $\Delta T_{m-n}$ of the atom so that $\Delta T_{n-m} = \Delta T_{m-n}$ for the point $(z_0, t_0)$ in the spatiotemporal domain $D_{m-n}(1)$ and the point $(z, t)$ in the domain $D_{m-n}(1)$. We illustrate this situation for $E_n^{+}(z, t; 1)$ and $E_n^{-}(z, t; 1)$ in Fig.3. The diagrammatic illustration about temperature loss is represented by the thick arrows in Fig.4. Therefore, our arguments say that there is a possibility of the following mechanism for superradiant cooling: Let a space-time point $(z_{2\ell+1}, t_{2\ell+1})$ be in the domain $D_{m-n}(1)$ and a space-time point $(z_{2\ell+2}, t_{2\ell+2})$ in the domain $D_{m-n}(1)$, respectively, for each $\ell = 0, 1, \cdots, N-1$ with a natural number $N$. The Dicke-type energy level crossings and their reverse crossings may carry away the temperature of the atom:

$$
\frac{2}{k_B} \sum_{\nu=1}^{2N} \frac{|\Omega(z_{\nu}, t_{\nu})|^2}{|K|^2 + 4|\Omega(z, t)|^2} - \frac{4N}{k_B} |\Delta c| \tag{3.10}
$$

at most by Eq.3.9 if $K > 0$. We can make similar argument when $K < 0$.

When the whole space of the space-time is $D_{m-n}(1)$, we find the Sisyphus-type mechanism in the energy spectrum as the group of Ritsch pointed out in Ref.12. To see the diagrammatic representation we consider a concrete example now. For the strength $|\Omega_0(t)| = 2\kappa$ we have the cavity-induced atom cooling as in Fig.4. Up-arrows in Fig.4 represent the energy that the atom coupled with photons gains by photon absorption, namely, we have to throw a driving laser to the atom for its excitement. Down-arrows mean the energy loss caused by cavity decay. In this domain $D_{m-n}(1)$ of the space-time we cannot find such a sequence $\{(z_{\nu}, t_{\nu})\}_{\nu=1}^{2L}$ of the space-time points. Thus, Eq.3.10 does not work. As shown in the
following concrete example, on the other hand, we can expect the sequence \( \{(z_{\nu}, t_{\nu})\}^{2N}_{\nu=1} \): If we take a strength as \( |\Omega_0| = 8\kappa \), then Fig. 4 changes to Fig. 5 and then, we can find the sequence \( \{(z_{\nu}, t_{\nu})\}^{2N}_{\nu=1} \) in Fig. 6. Compare Fig. 4 and Fig. 5. In Fig. 5 we can make only down-arrows without any uparrow. It means that the system loses energy only because of cavity decay. Namely, we do not have to throw the driving laser to the atom in the cavity for its excitement. We note this type emission comes from a kind of superradiance \([10, 22, 23]\). Thus, this energy-spectral property may give a mechanism for another superradiant cooling for a two-level atom in the strong coupling regime, as well as self-organized, cooperative atoms \([15, 16]\). As we can realize it from Figs. 2 and 3, the energy loss by cavity decay gets large as the strength \( |\Omega_0(t)| \) of the coupling grows large. We give the surfaces of \( E_0^\nu(z, t; 1) \) and \( E_1^\nu(z, t; 1) \) as functions of \( z, t \) in Figs. 6 and 7.

We prove our two theorems on the Dicke-type energy level crossing now: To make the calculations below simple, we set \( K := \epsilon_1 - \epsilon_0 + \Delta_c - \Delta \) again.

We give a proof of crossings \([8, 13]\) first. Let us suppose \( K \geq 0 \) now. It is easy to show the equation,

\[
E_0^\nu(z, t; 1) - E_1^\nu(z, t; 1) = \Delta_c n - \frac{1}{2} K + \Upsilon_n(z, t; 1), \quad (3.11)
\]

where \( \Upsilon_n(z, t; 1) \) is the generalized Rabi frequency \([8, 13]\). Let the symbol \( \# \) denote either \( >, = \), or \( < \). After multiplying both sides of the expression \( |\Omega(z, t)|^2 \# \Delta_c^2 n - \Delta_c K \) by \( 4n \), add the term \( K^2 \) to both sides of the multipliend expression. Then, we know that the expression \( |\Omega(z, t)|^2 \# \Delta_c^2 n - \Delta_c K \) is equivalent to the expression \( K^2 + 4n|\Omega(z, t)|^2 \# 4(\Delta_c^2 n^2 - \Delta_c Kn + K^2/4) \). Since \(-\Delta_c n + K/2 \geq 0\), we can take the square root of the last expression, and thus, it is equivalent to \( 2\Upsilon_n(z, t; 1) \# (\Delta_c n + K/2) \). Therefore, Eq. (3.11) says

![Energy Surfaces](image1.png)

**FIG. 6:** Surfaces of energies \( E_0^\nu(z, t; 1) = 0 \) (dashed line) and \( E_1^\nu(z, t; 1) \) (solid). The physical parameters are set as in Fig. 4.

![Energy Surfaces](image2.png)

**FIG. 7:** Surface of energy \( E_1^\nu(z, t; 1) \) (solid line), where \( E_0^\nu(z, t; 1) = 0 \). The physical parameters are set as in Fig. 1.

![Energy Surfaces](image3.png)

**FIG. 8:** Surface of energy \( E_1^\nu(z, t; 1) \) (solid line), where \( E_0^\nu(z, t; 1) = 0 \). The physical parameters are set as in Fig. 1.
that
\[ E_0^+(z, t; 1) - E_n^-(z, t; 1) \geq 0 \iff |\Omega(z, t)|^2 \geq \Delta^2_n - \Delta_c K, \]
where “RHS”\(\iff\)“LHS” means that the right hand side of the equation is equivalent to the left hand side. The equivalence (3.12) brings the crossings (3.4)–(3.6).

Let us suppose \(K < 0\) now. Then, in the same way we did in the case \(K > 0\), we have the equivalence:
\[ E_0^+(z, t; 1) - E_n^-(z, t; 1) \geq 0 \iff \Upsilon_{n+1}(z, t; 1) \geq \Delta_c (n + 1) + \frac{1}{2} K. \]  
(3.13)

The term \(-\Delta_c (n + 1) + K/2\) is positive because we assumed the condition (3.2) and \(\Delta_c < 0\). Thus, by taking the square of both sides of the right expression of the equivalence (3.13) and following the way we did in the case \(K > 0\), we know the expression \(E_0^+(z, t; 1) - E_n^-(z, t; 1) \geq 0\) is equivalent to the expression \(|\Omega(z, t)|^2 \geq \Delta^2_n - \Delta_c K\), which implies our desired result.

We consider the proofs of the crossing (3.7) and (3.8) next. Let us suppose \(d \leq m < n\). A direct calculation leads to
\[ E_m^+(z, t; 1) - E_n^-(z, t; 1) \geq 0 \iff \Delta_c (n - m) + \Upsilon_n(z, t; 1) - \Upsilon_m(z, t; 1) \geq (n - m) \left[ \Delta_c + \frac{|\Omega(z, t)|^2}{\Upsilon_n(z, t; 1) + \Upsilon_m(z, t; 1)} \right]. \]  
(3.14)

Since \(n > m\) and \(|\Delta_c| = -\Delta_c\), Eq. (3.14) says that the expression \(E_m^+(z, t; 1) - E_n^-(z, t; 1) \geq 0\) is equivalent to the expression \(|\Omega(z, t)|^2 (\Upsilon_n(z, t; 1) + \Upsilon_m(z, t; 1))^{-1} \geq |\Delta_c|\). Multiplying both sides of this by \(2|\Delta_c|^{-1}|\Omega(z, t)|^{-1} (\Upsilon_n(z, t; 1) + \Upsilon_m(z, t; 1))\), we realize that the later expression is equivalent to the expression \(2|\Omega(z, t)|^{-1}/|\Delta_c| \geq \sqrt{(K/|\Omega(z, t)|)^2 + 4n} + \sqrt{(K/|\Omega(z, t)|)^2 + 4m}\). Hence it follows from these equivalences that
\[ \sqrt{(K/|\Omega(z, t)|)^2 + 4m} \geq \sqrt{(K/|\Omega(z, t)|)^2 + 4n}. \]

(3.15)

Suppose \(K \geq 0\) now. Here we note a simple inequality for non-negative numbers \(A, B,\) and \(C:\)
\[(\sqrt{A + B} + \sqrt{A + C})^2 \geq A + B + A + C.\]
Using this, we can show that the expression \(4(|\Omega(z, t)|/|\Delta_c|)^2 < (K/|\Omega(z, t)|)^2 + 4n + (K/|\Omega(z, t)|)^2 + 4m\) implies the expression \(4(|\Omega(z, t)|/|\Delta_c|)^2 < \left\{ \sqrt{(K/|\Omega(z, t)|)^2 + 4n} + \sqrt{(K/|\Omega(z, t)|)^2 + 4m} \right\}^2\).

\[ 4 \left( \frac{|\Omega(z, t)|}{|\Delta_c|} \right)^2 < \left( \frac{K}{|\Omega(z, t)|} \right)^2 + 4n + \left( \frac{K}{|\Omega(z, t)|} \right)^2 + 4m \]
\[ \iff 2 |\Omega(z, t)| - \sqrt{\left( \frac{K}{|\Omega(z, t)|} \right)^2 + 4m} < \sqrt{\left( \frac{K}{|\Omega(z, t)|} \right)^2 + 4n}, \]
(3.16)

where “RHS”\(\iff\)”LHS” means that the right hand side of the equation implies the left hand side.

We define a polynomial \(g_{wc}(r)\) by \(g_{wc}(r) := 2r^2 - 2(m + n)r - K^2/|\Delta_c|^2\). Multiplying both sides of this by 2, the inequality \(g_{wc}\left(|\Omega(z, t)|/|\Delta_c|\right)^2 < 0\) is equivalent to the inequality \(4(|\Omega(z, t)|/|\Delta_c|)^4 < 4(m + n)(|\Omega(z, t)|/|\Delta_c|)^2 + 2(K/|\Delta_c|)^2\). Multiplying both sides of this newly obtained inequality by \(|\Delta_c|/|\Omega(z, t)|\)^2 we reach the following:
\[ \frac{\Omega(z,t)}{|\Delta_c|} > 0 \]

\[ 4 \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 < \left( \frac{K}{|\Omega(z,t)|} \right)^2 + 4m + \left( \frac{K}{|\Omega(z,t)|} \right)^2 + 4n. \]  

(3.17)

It follows from the implication (3.15) and equivalences (3.18) and (3.17) that

\[ g_{\text{wc}} \left( \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 \right) < 0 \]

\[ \iff \quad 4 \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 < \left( \frac{K}{|\Omega(z,t)|} \right)^2 + 4m + \left( \frac{K}{|\Omega(z,t)|} \right)^2 + 4n. \]

We prove the inequality (3.18) next. Multiply both sides of the inequality \((\Omega(z,t)/|\Delta_c|)^2 - (n-m) > (\Omega(z,t)/|\Delta_c|)\sqrt{4m}\) by 4, and add \((K/|\Omega(z,t)|)^2\) to both sides of the multiplied inequality. Then, we know that the inequality \((\Omega(z,t)/|\Delta_c|)^2 - (n-m) > (\Omega(z,t)/|\Delta_c|)\sqrt{4m}\) implies the inequality

\[ 4 \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 - 4 \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 \left( \frac{K}{|\Omega(z,t)|} \right)^2 + 4m \]

\[ > \left( \frac{K}{|\Omega(z,t)|} \right)^2 + 4n. \]

Taking the square root of both sides of the last inequality, we can reach the implication:

\[ \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 - (n-m) > \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 \sqrt{\left( \frac{K}{|\Omega(z,t)|} \right)^2 + 4m} \]

\[ \iff \quad 2 \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 - 4m > \frac{K}{|\Omega(z,t)|} \left( \frac{4}{\left( \frac{K}{|\Omega(z,t)|} \right)^2 + 4n} \right). \]  

(3.20)

We define a polynomial \(g_{\text{sc}}(r)\) by \(g_{\text{sc}}(r) := r^2 - 2(m+n)r + (n-m)^2 - K^2/|\Delta_c|^2\) this time. We note the inequality \(g_{\text{sc}} \left( \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 \right) > 0\) is equivalent to the inequality \((\Omega(z,t)/|\Delta_c|)^2 - 2(n-m)(\Omega(z,t)/|\Delta_c|)^2 + (n-m)^2 > (\Omega(z,t)/|\Delta_c|)^2 \{ (K/|\Omega(z,t)|)^2 + 4m \}.\) Here we added \(4m(\Omega(z,t)/|\Delta_c|)^2 + (K/|\Delta_c|)^2\) to both sides of \(g_{\text{sc}} \left( \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 \right) > 0,\) and then, the right hand side was factorized with the factor \((\Omega(z,t)/|\Delta_c|)^2\). Thus, taking the square root of both side of this inequality, we obtain the following implication:

\[ g_{\text{sc}} \left( \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 \right) > 0 \]

\[ \iff \quad \left( \frac{\Omega(z,t)}{|\Delta_c|} \right)^2 - (n-m) > \frac{K}{|\Delta_c|} \sqrt{\left( \frac{K}{|\Delta_c|} \right)^2 + 4m}. \]  

(3.21)
It follows from the implications (3.20) and (3.21), and equivalence (3.15) that
\[
\begin{align*}
gsc \left( \frac{1}{|\Delta_c|} \right)^2 \right) > 0 \\
\Rightarrow E^-_m(z,t;1) - E^-_n(z,t;1) > 0. & \quad (3.22)
\end{align*}
\]
Since \( \rho_0^{sc} := m + n + \sqrt{4mn + K^2/\Delta_c^2} \) satisfies \( gsc(\rho_0^{sc}) = 0 \), we have the inequality \( gsc(\rho) > gsc(\rho_0^{sc}) = 0 \) provided that \( \rho_0^{sc} < \rho \). This fact tells us that
\[
\left( \frac{1}{|\Delta_c|} \right)^2 > m + n + 2\sqrt{mn + \frac{K^2}{4\Delta_c^2}}
\]
\[
\Rightarrow gsc \left( \frac{1}{|\Delta_c|} \right)^2 > 0. & \quad (3.23)
\]
Therefore, we can conclude the inequality (3.3) from implications (3.22) and (3.23).

In the case where \( K < 0 \), we have
\[
\begin{align*}
E^-_m(z,t;1) - E^-_n(z,t;1) &= (n - m) \left( \Delta_c + \frac{\Omega(z,t)^2}{\Upsilon_n+1(z,t;1) + \Upsilon_m+1(z,t;1)} \right).
\end{align*}
\]
Hence it follows from this that we obtain the crossing (3.27) and (3.28) by using \( m + 1 \) and \( n + 1 \) instead of \( m \) and \( n \) respectively in the above argument for the case where \( K \geq 0 \).

**B. In the case \( d \geq 2 \)**

We assume \( d \geq 2 \) in this subsection. Let \( m \) and \( n \) be non-negative integers satisfying \( m < d \leq n \). We set a positive number \( C_{mn}^0(d) \) by
\[
C_{mn}^0(d) := \begin{cases} 
\frac{(n - m)!}{n!} \left( \frac{n - m}{|\Delta_c|} \right) & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - d\Delta_c, \\
\frac{(n + d - m)!}{(n + m)!} \left( \frac{n + d - m}{|\Delta_c|} \right) \times & \left( |\Delta_c| - n + m \right) K_d \\
& \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - d\Delta_c,
\end{cases}
\]
where \( K_d := \varepsilon_1 - \varepsilon_0 + d\Delta_c - \Delta \). Then, we define three domains \( D^{sc}_{mn}(d), \) \( D^{cr}_{mn}(d), \) and \( D^{wc}_{mn}(d) \) of the space-time in the following: the domain \( D^{wc}_{mn}(d) \) for the weak coupling regime is given by \( D^{wc}_{mn}(d) := \{ (z, t) : |\Omega(z, t)|^2 < C_{mn}^0 \} \), and the domain \( D^{sc}_{mn}(d) \) for the strong coupling regime by \( D^{sc}_{mn}(d) := \{ (z, t) : |\Omega(z, t)|^2 > C_{mn}^0 \} \). The domain \( D^{cr}_{mn}(d) \) for the critical regime is defined by \( D^{cr}_{mn}(d) := \{ (z, t) : |\Omega(z, t)|^2 = C_{mn}^0 \} \).

In the case \( d \geq 2 \) we can show the following theorem: the domain \( D^{wc}_{mn}(d) \) is equal to the domain \( D^{sc}_{mn}(d) \), and the domain \( D^{sc}_{mn}(d) \) to the domain \( D^{cr}_{mn}(d) \).
since $\Delta_c < 0$, $m < n$, and $0 \leq K_d$. Finally, by the equivalences \((3.27)\) and \((3.28)\), we reach the equivalence:

$$E_{m}^{0}(z, t; d) - E_{m+1}^{0}(z, t; d) \geq 0 \iff |\Omega(z, t)|^2 \geq C_{mn}^{0}(d),$$

which says that the desired energy level crossing takes place when $K_d \geq 0$.

We see the energy level crossing in the case where $K_d < 0$ next. Let us suppose $K_d < 0$ now. Note the inequalities $-\Delta_c(n + d - m) + K_d/2 \geq (\varepsilon_1 - \varepsilon_0 - \Delta_c - \Delta)/2 > (\varepsilon_1 - \varepsilon_0 - \Delta)/2 > 0$ because of the assumption \((3.2)\) and the condition $-\Delta_c > 0$. Thus, in the same way we had the equivalence \((3.28)\), we obtain the equivalence,

$$|\Omega(z, t)|^2 \geq C_{mn}^{0}(d) \quad \iff \quad Y_{n+d}(z, t; d) \geq \Delta_c(n + d - m) + \frac{K_d}{2}.$$

Then, the equivalences \((3.27)\) and \((3.30)\) bring the equivalence \((3.29)\), which secures our statement in the case $K_d < 0$.

In the case where $d \geq 2$, there is not always a ground state of $H_0(z, t; d)$. Namely, there is a case where the minimum energy of $H_0(z, t; d)$ does not exist. To see this fact, we assume $\varepsilon_1 - \varepsilon_0 \approx -d \Delta_c$ for simplicity. Then, since we have

$$E_{n}^{0}(z, t; d) - E_{n+1}^{0}(z, t; d) = \Delta_c + |\Omega(z, t)|\sqrt{n(n-1)\cdots(n-d+2)} \times \{n+1 - n-d+1\} > \Delta_c + |\Omega(z, t)|\sqrt{n(n-1)\cdots(n-d+2)} \times \{n+1 - n-1\} = \Delta_c + 2|\Omega(z, t)|\sqrt{d+1}!,$$

where we used the inequalities, $n > d \geq 2$ and $\sqrt{1+n} > \sqrt{1+1/n} + \sqrt{1-1/n} \geq 2$. The last inequality says that the Hamiltonian $H_0(z, t; d)$ does not have a ground state for $(z, t)$ satisfying $|\Omega(z, t)| > |\Delta_c|/\sqrt{(d-1)!}$ because

$$\cdots < E_{n+1}^{0}(z, t; d) < E_{n}^{0}(z, t; d) < \cdots < E_{d+1}^{0}(z, t; d)$$
in the case where $d \geq 2$.

IV. SUPERRADIANT GROUND STATE ENERGY IN THE CASE $\alpha \equiv 0$.

As we knew at the end of the previous section, once the Dicke-type crossing takes place in the case $d \geq 2$, there is every possibility that $H_0(z, t; d)$ is not bounded from below. Thus, to make sure of the existence of the superradiant ground state energy, we consider only the case $d = 1$ in this section.

Set a positive number $\Theta_n$ for $n = 1, 2, \cdots$ as

$$\Theta_n = \begin{cases} 2n\Delta_c^2 + |\Delta_c|\sqrt{4n^2\Delta_c^2 + K^2} & \text{if } \varepsilon_1 - \varepsilon_0 \geq \Delta - \Delta_c, \\ 2(n + 1)\Delta_c^2 + |\Delta_c|\sqrt{4(n + 1)^2\Delta_c^2 + K^2} & \text{if } \varepsilon_1 - \varepsilon_0 < \Delta - \Delta_c, \end{cases}$$

where $K = \varepsilon_1 - \varepsilon_0 + \Delta_c - \Delta$. Using this number, we define a domain $G_n(1)$ of the time-space by

$$G_n(1) = \{(z, t) \mid |\Theta_n| < |\Omega(z, t)| < |\Theta_{n+1}|\}$$

for each $n = 1, 2, \cdots$. We can prove the following theorem: (i) when a space-time point $(z, t)$ is in $G_n(1)$, the point $(z, t)$ is always in $D_0^{\infty}(1)$. Namely, the Dicke-type energy level crossing takes place for that point $(z, t)$ in $G_n(1)$; (ii) the superradiant ground state energy $\inf \{\text{Spec}(H_0(z, t; 1))\}$ appears as:

$$\inf \{\text{Spec}(H_0(z, t; 1))\} = \min \{E_{n}^{-}(z, t; 1), E_{n+1}^{-}(z, t; 1)\}$$

for $(z, t)$ in $G_n(1)$.

To prove our theorem we set $K := \varepsilon_1 - \varepsilon_0 + \Delta_c - \Delta$ again. Part (i) follows from the following easy inequalities:

$$C_{0n} < 2n\Delta_c^2 + |\Delta_c|K \leq \Theta_n \text{ if } K \geq 0; \quad C_{0n} < 2(n + 1)\Delta_c^2 + |\Delta_c|K \leq \Theta_n \text{ if } K < 0.$$

For a non-negative number $r$, define a function $g(r)$ by $g(r) := |\Delta_c|^r L - (1/2)^r |\Omega(z, t)|^2 r$, where

$$L = (\varepsilon_0 + \varepsilon_1 + \Delta_c - \Delta).$$

Then, for every positive number $r$ the expression $g(r) \geq 0$ is equivalent to the expression

$$\Delta_c^2(K^2 + 4|\Omega(z, t)|^2 r)^2 > |\Omega(z, t)|^4,$$

Hence it follows from this that

$$g'(r) \equiv 0 \iff r \equiv r_0 := 4|\Delta_c|K - 4\Delta_c^2|\Omega(z, t)|^2,$$

provided that $r_0 > 0$. It is evident that the number $r_0$ is positive if and only if the inequality $|\Omega(z, t)|^2 > |\Delta_c|K$ holds. By the equivalence \((4.2)\) together with this fact, we have the implication:

$$|\Omega(z, t)|^2 > |\Delta_c|K \iff \inf_{r \geq 0} g(r) = g(r_0).$$

Simple calculation leads to the equivalence:

$$\begin{align*}
0 & \leq \frac{|\Omega(z, t)|^2}{4\Delta_c^2} - \frac{K^2}{4|\Omega(z, t)|^2} < n + 1 \\
\iff & \quad \begin{cases}
|\Omega(z, t)|^4 - 4n\Delta_c^2|\Omega(z, t)|^2 - \Delta_c^2K^2, \\
|\Omega(z, t)|^4 - 4(n + 1)\Delta_c^2|\Omega(z, t)|^2 - \Delta_c^2K^2 < 0.
\end{cases}
\end{align*}$$

We define two functions $g_j(r)$, $j = 1, 2$, for positive number $r$ by $g_1(r) := r^2 - 4n\Delta_c^2r - \Delta_c^2K^2$ and $g_2(r) := r^2 - 4(n + 1)\Delta_c^2r - \Delta_c^2K^2$. Then, setting $r_{10}$ as $r_{10} := 2n\Delta_c^2 + |\Delta_c|\sqrt{K^2 + 4n^2\Delta_c^2}$ and $r_{20} := 2(n + 1)\Delta_c^2 + |\Delta_c|\sqrt{K^2 + 4(n + 1)^2\Delta_c^2}$ respectively, we have
\[ g_1(r) = 0 \quad \text{if} \quad r > r_{10} \quad \text{and} \quad g_2(r) < g_2(r_{20}) = 0 \quad \text{if} \quad 0 < r < r_{20}. \] 

Set a non-negative number \( \theta_n \) for each \( n = 1, 2, \ldots \), with \( \theta_n := 2n\Delta_c^2 + |\Delta_c| \sqrt{K^2 + 4n^2\Delta_c^2}. \) Then, we have \( \theta_n \geq |\Delta_c| K. \) So, inserting \( |\Omega(z, t)|^2 \) into \( r \) of the above inequalities, we reach the implication:

\[
\theta_n < |\Omega(z, t)|^2 < \theta_{n+1}
\]

implies the equivalence \( (4.4) \) is satisfied. \( (4.5) \)

Combining the results \( (4.3), (4.4), \) and \( (4.5), \) we obtain the implication:

\[
\theta_n < |\Omega(z, t)|^2 < \theta_{n+1}
\]

implies \( g(n) \) or \( g(n+1) \) is located nearest \( \inf_{r \geq 0} g(r). \) \( (4.6) \)

Noting \( E_\alpha^-(z, t; 1) = g(n) \) if \( K \geq 0; \) \( E_\alpha^-(z, t; 1) = g(n+1) \) if \( K < 0, \) and \( \Theta_\alpha = \theta_n \) if \( K \geq 0; \) \( \Theta_\alpha = \theta_{n+1} \) if \( K < 0, \) we obtain part (ii).

V. STABILITY OF DICKE-TYPE CROSSINGS IN THE CASE \( \alpha \neq 0. \)

In this section we consider the case where \( d = 1 \) too. We show the stability of the Dicke-type energy level crossings under the effect of the generalized energy operator \( \alpha(t)W(z, t) \) of the pump field for sufficiently small strength \( |\alpha(t)|. \) Our \( W(z, t) \) includes \( \alpha(a - a^\dagger) \), of course. To show this fact we employ the method for proving Theorem 4.3(v) of Ref.31. In the same way we did in Sec. III through the well-known identification as in Ref.12, \( H(\Omega, \alpha; 1) \) reads \( H_\alpha(z, t; 1) := H_0(z, t; 1) + \alpha(z, t)W(z, t) \). We recall \( \alpha(z, 0) = 0 \) for all the position \( z. \)

For the Hilbert space representing the state space in which \( H_\alpha(z, t; 0) \) acts, we denote the inner product of the Hilbert space by \( \langle \Psi | \Phi \rangle. \)

For every positive number \( \epsilon, \) we denote \( C_\epsilon(z, t) \) by

\[
C_\epsilon(z, t) := (1 + \epsilon) \left\{ 1 + \left( \frac{\epsilon}{1 + \epsilon} \right)^2 \left( \epsilon_0 + |\epsilon_1 - \Delta| \right) \right. \\
+ \left( \frac{1 + \epsilon}{\epsilon} \right)^2 \frac{|\Omega(z, t)|^2}{4\Delta_c^2} \right\}^{1/2}.
\]

Set a positive number \( C_0^0[\theta] \) for each natural number \( n \) and a non-negative number \( \theta \) as:

\[
C_0^0[\theta] := \begin{cases}
(\theta - \Delta_c)^2n + (\theta - \Delta_c)(\epsilon_1 - \epsilon_0 + \Delta_c - \Delta) & \text{if } \epsilon_1 - \epsilon_0 \geq \Delta - \Delta_c, \\
(\theta - \Delta_c)^2(n + 1) + (\theta - \Delta_c)(\epsilon_1 - \epsilon_0 + \Delta_c - \Delta) & \text{if } \epsilon_1 - \epsilon_0 < \Delta - \Delta_c,
\end{cases}
\]

and set a positive number \( C[n] \) for each natural number \( n \) as:

\[
C[n] := \begin{cases}
\Delta_c^2n + \frac{\epsilon_0(\epsilon_1 + \Delta_c - \Delta)}{n} & \text{if } \epsilon_1 - \epsilon_0 \geq \Delta - \Delta_c, \\
\Delta_c^2(n + 1) + \frac{\epsilon_0(\epsilon_1 + \Delta_c - \Delta)}{n + 1} & \text{if } \epsilon_1 - \epsilon_0 < \Delta - \Delta_c.
\end{cases}
\]

Moreover, for every positive function \( f(z, t), \) and positive numbers \( b \) and \( \epsilon', \) we define a domain \( D(\epsilon, \epsilon'; b, f) \) of the space-time by

\[
D(\epsilon, \epsilon'; b, f) := \{ (z, t) \mid |\alpha(z, t)| < \frac{\epsilon'}{2b} \text{ and } |\epsilon(z, t)| \left( bC_\epsilon(z, t) + f(z, t) \right) < \frac{(1 + \epsilon')}{2} \}
\]

Also the domain \( D^0_{\epsilon} \) of the space-time is defined by

\[
D^0_{\epsilon} := \{ (z, t) \mid |\Omega(z, t)|^2 > C^0_{\epsilon}[\theta] \text{ and } |\Omega(z, t)|^2 > C[n] \}
\]

We prove the following theorem concerning the stability of the Dicke-type energy level crossings: Let our \( W(z, t) \) satisfy the conditions (A1)–(A3):

(A1) For every space-time point \( (z, t), W(z, t) \) is a symmetric operator so that it can act the all states on which \( H_0(z, 0; 1) \) acts;

(A2) there is a positive constant \( b_1 \) so that

\[
\langle W(z, t)\Psi|W(z, t)\Psi \rangle^{1/2} \leq b_1(H_0(z, 0; 1)\Psi|H_0(z, 0; 1)\Psi)^{1/2} + b_2(z, t)(\Psi|\Psi)^{1/2}
\]

for all states \( \Psi \) on which \( H_0(z, 0; 1) \) acts and every space-time point \( (z, t), \) where \( b_2(z, t) \) is some positive function of \( (z, t); \)

(A3) there is a constant \( \epsilon \) with \( 0 < \epsilon < 1 \) so that

\[
|\alpha(z, t)| < \left( b_1(1 + \epsilon) \right)^{-1} \quad \text{for all the space-time points } (z, t).
\]

Then, the Hamiltonian \( H_\alpha(z, t; 1) \) has an eigenvalue \( \mathcal{E}^\alpha_\epsilon(z, t; 1) \) near \( \mathcal{E}^\alpha_0(z, t; 1) \) for each non-negative integer \( n, \) where \( z = 0, \pm. \) Moreover, there is a constant \( \kappa_0 \) with \( 0 < \kappa_0 < 1/4 \) so that the Dicke-type energy level crossing takes place between the eigenvalues \( \mathcal{E}^\alpha_0(z, t; 1) \) and \( \mathcal{E}^\alpha_\epsilon(z, t; 1) \) in the process from the space-time point \( (z, 0) \) to the space-time point \( (z, t_*), \) provided that the
latter point \((z_*, t_*)\) is in \(D(c, \kappa_0; b_1, b_2) \cap D_{\text{on}}(1; \theta)\) for a number \(\theta > 0\). Therefore, \(H_{n}(z_*, t_*; 1)\) has the superradiant ground state energy.

We prove this theorem below. Set \(H_1 = H_{0}(z, t; 1) - H_{0}(z, 0; 1)\) for simplicity. The symbol \(H_0\) stands for \(H_{0}(z, 0; 1)\) from now on. Here we recall \(\Omega(z, 0) = 0\) for every position \(z\).

In the same way that we proved Lemma 4.2 of Ref. [31], we can estimate \(\langle W(z, t) \Psi | W(z, t) \Psi \rangle\) from above by using \(\langle H_0(z, t; d) \Psi | H_0(z, t; d) \Psi \rangle\) and \(\langle \Psi | \Psi \rangle\). Using the canonical commutation relation \([a, a^\dagger]\) = 1 leads to the equation:

\[
\langle H_1 \Psi | H_1 \Psi \rangle = |\Omega(z, t)|^2 \left( \langle \Psi | a^\dagger a \Psi \rangle + |\sigma_{11}|^2 |\sigma_{11}|^2 \right),
\]
which easily implies the inequality

\[
\langle H_1 \Psi | H_1 \Psi \rangle \leq |\Omega(z, t)|^2 \left( \langle \Psi | a^\dagger a \Psi \rangle + \langle \Psi | \Psi \rangle \right).
\]

By using the Schwarz inequality and the inequality \(XY \leq (\eta X + Y/4\eta)^2\) for every \(X, Y \leq 0\) and arbitrary \(\eta > 0\), we obtain the inequality:

\[
\langle \Psi | a^\dagger a \Psi \rangle \leq \frac{(\eta^2 \langle \Psi | a^\dagger a \Psi \rangle + \langle \Psi | \Psi \rangle)}{(4\eta)}
\]
for arbitrary \(\eta > 0\). Simple calculation yields

\[
\langle H_0 \Psi | H_0 \Psi \rangle = \frac{\Delta_z}{2} \langle a^\dagger a \Psi | a^\dagger a \Psi \rangle + (\varepsilon_1 - \Delta) |\sigma_{11}|^2 |\sigma_{11}|^2
\]
so that

\[
\langle a^\dagger a \Psi | a^\dagger a \Psi \rangle = \frac{1}{2 \Delta_z} \left\{ \langle H_0 \Psi | H_0 \Psi \rangle - (\varepsilon_1 - \Delta) |\sigma_{11}|^2 |\sigma_{11}|^2 
\right.

\left. - \varepsilon_0 |\sigma_{00}|^2 |\sigma_{00}|^2 \right\}.
\]

Combining these inequalities and setting \(\eta := |\Delta_z| / (\sqrt{2} |\Omega(z, t)|)\) for arbitrary \(\delta > 0\), we reach the inequality:

\[
\langle H_1 \Psi | H_1 \Psi \rangle^{1/2} \leq |\Omega(z, t)| \left\{ \eta^2 \frac{\langle a^\dagger a \Psi | a^\dagger a \Psi \rangle + \langle \Psi | \Psi \rangle}{8 \eta^2 + 1} \right\}^{1/2}
\]
\[
\leq \{\delta^2 |\Delta_z| |\Omega(z, t)|^2 \theta |\Omega(z, t)|^2 \}^{1/2}
\]
\[
\leq \frac{\delta |\Delta_z| |\Omega(z, t)|^2 \theta |\Omega(z, t)|^2 \}^{1/2}
\]
\[
\leq \delta |\Delta_z| |\Omega(z, t)|^2 \theta |\Omega(z, t)|^2 \}^{1/2}
\]
where

\[
\Theta(\delta; \Omega) = \sqrt{1 + \delta^2 (\varepsilon_0 + |\varepsilon_1 - \Delta|) + \frac{|\Omega(z, t)|^2}{4 \Delta_z^2 \delta^2}}.
\]

Applying the triangle inequality \(\langle (A + B) \Psi | (A + B) \Psi \rangle^{1/2} \leq \langle A \Psi | A \Psi \rangle^{1/2} + \langle B \Psi | B \Psi \rangle^{1/2}\) to the term \(\langle H_0(z, t; 1) - H_1 \Psi | (H_0(z, t; 1) - H_1 \Psi) \}^{1/2}\), we have the inequality:

\[
\langle H_0 \Psi | H_0 \Psi \rangle^{1/2} \leq \langle H_0(z, t; 1) | H_0(z, t; 1) \Psi \rangle^{1/2}
\]
\[
+ \langle H_1 \Psi | H_1 \Psi \rangle^{1/2}.
\]

Inequalities (5.2) and (5.3) tell us that the term \(\langle H_0 \Psi | H_0 \Psi \rangle^{1/2}\) is bounded from above as:

\[
\langle H_0 \Psi | H_0 \Psi \rangle^{1/2} \leq \langle H_0(z, t; 1) | H_0(z, t; 1) \Psi \rangle^{1/2}
\]
\[
+ \delta \langle H_0 \Psi | H_0 \Psi \rangle^{1/2} + \Theta(\delta; \Omega) |\Psi|^2,\]

which implies the inequality,

\[
(1 - \delta) \langle H_0 \Psi | H_0 \Psi \rangle^{1/2} \leq \langle H_0(z, t; 1) | H_0(z, t; 1) \Psi \rangle^{1/2}
\]
\[
+ \delta \langle H_0 \Psi | H_0 \Psi \rangle^{1/2} + \Theta(\delta; \Omega) |\Psi|^2,\]

Set \(\delta = 0 < \delta := (1 + \epsilon)^{-1} < 1\) for arbitrary number \(\epsilon\) with \(0 < \epsilon < 1\) now. Then, combining the inequalities (5.4) and (5.5) we can conclude that

\[
\langle W(z, t) \Psi | W(z, t) \Psi \rangle^{1/2}
\]
\[
\leq b_1 (1 + \epsilon) \langle H_0(z, t; 1) \Psi | H_0(z, t; 1) \Psi \rangle^{1/2}
\]
\[
+ (b_1 C_1 (z, t) + b_2 (z, t)) |\Psi|^2.
\]

Thus, applying Lemma 4.1 of Ref. [31] and Theorem 6.29 in III §6 of Ref. [42] to our Hamiltonian \(H_{\kappa}(z, t; 1) = H_0 + H_1\), we know that the Hamiltonian \(H_{\kappa}(z, t; 1)\) has an eigenvalue \(E_n(z, t; 1)\) for each \(n = 0, 1, 2, \ldots\).

Define an operator \(T(\kappa)\) as the closure of \(H_0(z, t; 1) + \kappa W(z, t)\) for every complex number \(\kappa\). By applying Theorem 2.6 and Remark 2.7 in VII §2 of Ref. [42] to the inequality (5.6), the operator \(T(\kappa)\) is an analytic family of type (A) for every \(\kappa \in \mathbb{C}\) with \(|\kappa| < |b_1 (1 + \epsilon)|^{-1}\). Thus, taking \(\alpha(z, t)\) as this \(\kappa\), i.e., \(H_{\kappa}(z, t; 1) = T(\alpha(z, t))\), Theorem 3.9 in VII §3 of Ref. [42] says that \(E_n(z, t; 1)\) is a continuous function of \((z, t)\) by the assumption (A3), and it sits near \(E_n(z, t; 1)\)\). So, as shown in Sec. III candidates which makes the Dicke-type energy level crossing are \(E_n(z, t; 1)\)\). Namely, for \(H_{\kappa}(z, t; 1)\) candidates which makes the Dicke-type energy level crossing are \(E_n(z, t; 1)\).

Let \(E_n\) and \(E_n(\kappa)\) be respectively eigenvalues of \(H_0(z, t; 1) + \kappa W(z, t)\) and \(T(\kappa)\) satisfying \(E_n(0) = E_n\). Following the regular perturbation theory (see XII of Ref. [43], the eigenvalue \(E_n(\kappa)\) has the expression,

\[
E_n(\kappa) = \frac{\langle \Phi_n | T(\kappa) \Psi_n | \Phi_n \rangle}{\langle \Phi_n | P_n(\kappa) \Phi_n \rangle}
\]
\[
= E_n + \kappa \langle \Phi_n | W(z, t) \Psi_n | \Phi_n \rangle
\]
where \(P_n(\kappa)\) is the orthogonal projection given by

\[
P_n(\kappa) = -\frac{1}{2\pi i} \int_{E_n - \epsilon} (T(\kappa) - E)^{-1} dE
\]
with a sufficiently small $\bar{c} > 0$, and $\Phi_n$ is a normalized eigenvector of $H_0(z, t; 1)$ satisfying $H_0(z, t; 1)\Phi_n = E_n\Phi_n$.

Since $W(z, t)$ is symmetric, we use the Schwarz inequality to make the inequality:

$$
\left| \frac{\langle \Phi_n | W(z, t)P_n(\kappa)\Phi_n \rangle}{\langle \Phi_n | P_n(\kappa)\Phi_n \rangle} \right| \leq \frac{\langle W(z, t)\Phi_n | W(z, t)\Phi_n \rangle^{1/2}}{\langle P_n(\kappa)\Phi_n | P_n(\kappa)\Phi_n \rangle}.
$$

(5.7)

To get the right hand side of the above inequality, we used the equations $P_n(\kappa)^2 = P_n(\kappa)$ in the denominator, and the inequality $(P_n(\kappa)\Phi_n | P_n(\kappa)\Phi_n) \leq 1$ in the numerator. By the inequality (5.5), we can estimate $\langle W(z, t)\Phi_n | W(z, t)\Phi_n \rangle^{1/2}$ as $\langle W(z, t)\Phi_n | W(z, t)\Phi_n \rangle^{1/2} \leq b_1(1 + \epsilon)E_n + b_1C_n(z, t) + b_2(z, t)$. Here we note that $0 \leq \langle P_n(\kappa)\Phi_n | P_n(\kappa)\Phi_n \rangle \rightarrow 1$ as $|\kappa| \rightarrow 0$, so that we have $1/2 < (P_n(\kappa)\Phi_n | P_n(\kappa)\Phi_n)$ for sufficiently small $|\kappa|$. Thus, combining these with the inequality (5.7), there is a positive constant $\kappa_0$ so that

$$
|\mathcal{E}_n(\kappa) - E_n| \leq 2|\kappa| \{b_1(1 + \epsilon)E_n + b_1C_n(z, t) + b_2(z, t)\}
$$

(5.8)

if $|\kappa| \leq \kappa_0$.

Now we take the coupling strength $|\alpha(z, t)|$ for space-time point $(z, t) \in D(\epsilon, \kappa_0; b_1, b_2)$ as the coupling parameter $\kappa$. Let us define a positive number $\kappa_1$ as $\kappa_1 := \kappa_0(1 + \epsilon)$. Then, it is easy to check that $2|\alpha(z, t)|b_1(1 + \epsilon) < \kappa_1$ and that $2|\alpha(z, t)| \{b_1C_n(z, t) + b_2(z, t)\} < \kappa_1$. Combining these inequalities with the inequality (5.8), we obtain the inequality:

$$(1 - \kappa_1)E_n - \kappa_1 \leq \mathcal{E}_n(\alpha(z, t)) \leq (1 + \kappa_1)E_n + \kappa_1
$$

(5.9)

for space-time point $(z, t) \in D(\epsilon, \kappa_0; b_1, b_2)$. From now on, we take the $\kappa_0$ and $\epsilon$ so that $0 < \kappa_1 < 1/2$.

Let us set a number $L$ as $L := \varepsilon_0 + \varepsilon_1 + \Delta_e - \Delta$. Then it is easy to show that $\Omega_n(z, t; 1)^2 - \Xi_n(1)^2 = -\Delta_e^2n^2 + (|\Omega_n(z, t)|^2 + \Delta_e^2L)n - \varepsilon_0(\varepsilon_1 + \Delta_e - \Delta)$, and thus, we obtain the equivalence:

$$
\Omega_n(z, t; 1) = |\Omega_n(z, t; 1)| \geq |\Xi_n(1)| \geq \Xi_n(1)
$$

$$
\Leftrightarrow \Omega_n(z, t; 1) \geq \Delta_e^2n^2 - \Delta_e - L + \frac{\varepsilon_0(\varepsilon_1 + \Delta_e - \Delta)}{n}.
$$

(5.10)

Hence it follows from this that $E_n^\alpha(z, t; 1)$ is negative for the point $(z, t) \in D_{\alpha(1)}^\infty(1; \theta)$.

In the same way we did to get the equivalence (5.12), we obtain the equivalence in the following way. Since $K \equiv \varepsilon_1 - \varepsilon_0 + \Delta_e - \Delta$ is non-negative, for every $\theta \geq 0$ and each natural number $n$ we have

$$
E_0^\alpha(z, t; 1) = \mathcal{E}_n(z, t; 1) + n\theta
$$

$$
\Leftrightarrow \Omega(z, t)^2 \geq (\theta - \Delta_e)n + (\theta - \Delta_e)K \equiv C_{\alpha(1)}^\infty[\theta],
$$

(5.11)

Thus, we obtain that $E_0^\alpha(z, t; 1) > (1 + \theta)E_n^\alpha(z, t; 1) + \theta$ since $1 < 1 + \theta$ and $E_n^\alpha(z, t; 1) < 0$ for every point $(z, t) \in D_{\alpha(1)}^\infty(1; \theta)$. We take the $\theta$ defined by the equation $\kappa_1 = \theta/(2 + \theta)$ now. Then, the following inequality holds:

$$
E_0^\alpha(z, t; 1) > (1 + \kappa_1)E_n^\alpha(z, t; 1)/(1 - \kappa_1) + 2\kappa_1/(1 - \kappa_1),
$$

(5.12)

which implies

$$
(1 - \kappa_1)E_0^\alpha(z, t; 1) - \kappa_1 > (1 + \kappa_1)E_n^\alpha(z, t; 1) + \kappa_1
$$

(5.13)

for every $(z, t) \in D_{\alpha(1)}^\infty(1; \theta)$.

Combining the inequalities (5.9) and (5.12) leads to the inequality:

$$
E_n^\alpha(z, t; 1) \leq (1 + \kappa_1)E_n^\alpha(z, t; 1) + \kappa_1
$$

$$
< (1 - \kappa_1)E_0^\alpha(z, t; 1) - \kappa_1 < E_0^\alpha(z, t; 1)
$$

(5.13)

for every $(z, t) \in D(\epsilon, \kappa_0; b_1, b_2) \cap D_{\alpha(1)}^\infty(1; \theta)$.

Since $E_0^\alpha(z, t; 1)$ is a continuous function of $(z, t)$ and $E_0^\alpha(z, 0; 1) = E_0^\alpha(z, 0; 1) < E_n^\alpha(z, 0; 1) = \mathcal{E}_n(z, 0; 1)$, the inequality (5.13) means that the Dicke-type energy level crossing takes place.

VI. CONCLUSION

We have showed that the system of a two-level atom coupled with a laser in a cavity has the Dicke-type energy level crossing in the process that the atom-cavity interaction of the system undergoes changes between the weak coupling regime and the strong one. By using the Dicke-type energy level crossing, we have found the following two possibilities in (mathematical) theory for the cavity-induced atom cooling. We can use a laser only for controlling the strength of the atom-cavity interaction without throwing another laser to the atom for driving it to the excited state, and moreover, we can obtain much larger energy loss caused by cavity decay, if we obtain the cavity that implements the domain $D_{\alpha(1)}^\infty(1; \theta)$ of the space-time. Based on these results, we can say that we lay mathematical foundations for the concept of another superradiant cooling in addition to that proposed by Domokos and Ritsch. Adding the mathematical foundations to the idea of the cavity-induced atom cooling by Ritsch et al., we can also say that the process of our superradiant cooling requires only cavity decay and control of the position of the atom, without atomic absorption and emission of photons. Therefore, whether the mechanism of our superradiant cooling can primarily be demonstrated or not depends on whether we can make such a fine cavity that the spatiotemporal domain $D_{\alpha(1)}^\infty(1)$ for the strong coupling regime can be implemented or not.
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APPENDIX A: THE EIGENVALUE PROBLEM FOR Ho(z; t; d)

In this appendix, to solve the eigenvalue problem:

\[ H_0(z, t) |\psi(\omega)\rangle = E |\psi(\omega)\rangle, \]

we adopt the way that we did in §3 and §6 of Ref. [31] into our calculations. Set \( \lambda := i\Omega(z, t) \) for simplicity.

Let \( n < d \) for a while. For a complex constant \( g \) set

\[ |\psi(g)\rangle := \left( \begin{array}{c} 0 \\ g a^d |\psi_0\rangle \end{array} \right), \]

where \( |\psi_0\rangle \) is the vacuum state of the photon field of our laser. It follows immediately that the condition, \( H(d; \Omega, 0) |\psi(g)\rangle = E |\psi(g)\rangle \), is equivalent to the condition, \( E = \omega_n \).

Let \( n \geq d \) now. Set \( |\psi(g)\rangle := \left( \begin{array}{c} a^d |\psi_0\rangle \\ g a^d |\psi_0\rangle \end{array} \right) \) this time. In the same way as in Ref. [31], we conclude that the condition, \( H(d; \Omega, 0) |\psi(g)\rangle = E |\psi(g)\rangle \), is equivalent to the conditions,

\[ \begin{cases} (n - d)\omega + \lambda^* \left( \frac{n}{d} \right)d!g = E - \mu, \\ \lambda + ng\omega = Eg. \end{cases} \]

Solving these equations, we obtain

\[ g = \frac{d!\mu - 2\lambda^* \left( \frac{n}{d} \right)d!\lambda^*}{2\lambda^* \left( \frac{n}{d} \right)d!}. \]

and

\[ E = \omega_n + \frac{\mu - d\omega}{2} \pm \frac{1}{2} \sqrt{(\mu - d\omega)^2 + 4 \left( \frac{n}{d} \right)d!\lambda^2}. \]

References

[1] W. Neuhauser, M. Hohenstatt, P. E. Toschek, and H. Dehmelt, Phys. Rev. A 22, 1137 (1980).
[2] S. Chu, L. W. Hollberg, J. E. Bjorkholm, A. Cable, and A. Ashkin, Phys. Rev. Lett. 55, 48 (1985).
[3] C. Cohen-Tannoudji and W. D. Phillips, Physics Today 43, 33 (1990).
[4] E. A. Cornell and C. E. Wieman, Rev. Mod. Phys. 74, 875 (2002).
[5] W. Ketterle, Rev. Mod. Phys. 74, 1131 (2002).
[6] C. J. Hood, M. S. Chapman, T. W. Lynn, and H. J. Kimble, Phys. Rev. Lett. 80, 4157 (1998).
[7] J. M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. 75, 565 (2001).
[8] H. Mabuchi and A. C. Doherty, Science 298, 1372 (2002).
[9] J. McKeever, A. Boca, A. D. Boozer, J. R. Buck, and H. J. Kimble, Nature 425, 268 (2003).
[10] S. M. Dutra, Cavity Quantum Electrodynamics (Wiley-Interscience Publication, New York 2005).
[11] S. Haroche and J.-M. Raimond, Exploring the Quantum: Atoms, Cavities, and Photons (Oxford University Press, Oxford, 2006).
[12] P. Horak, G. Hechenblaikner, K. M. Gheri, H. Stecher, and H. Ritsch, Phys. Rev. Lett. 79, 4974 (1997).
[13] C. J. Hood, T. W. Lynn, A. C. Doherty, A. S. Parks, and H. J. Kimble, Science 287, 1447 (2000).
[14] T. Fischer, P. Maunz, P. W. H. Pinkse, T. Puppe, and G. Rempe, Phys. Rev. Lett. 88, 163002 (2002).
[15] P. Domokos and H. Ritsch, J. Opt. Soc. Am. B 20, 1098 (2003).
[16] P. Domokos and H. Ritsch, Phys. Rev. Lett. 25, 253003 (2002).
[17] C. C. Gerry and P. L. Knight, Introductory Quantum Optics (Cambridge University Press, Cambridge, 2005).
[18] A. V. Andreev, V. I. Emel’yanov, and Yu. A. Il’inskii, Cooperative Effects in Optics (Institute of Physics Publishing, Bristol, 1993).
[19] J.-S. Peng and G. X. Li, Introduction to Modern Quantum Optics (World Scientific, Singapore, 1998).
[20] G. Preparata, QED Coherence in Matter (World Scientific, Singapore, 1995).
[21] C. P. Enz, Helv. Phys. Acta 70, 141 (1997).
[22] M. Hirokawa, Rev. Math. Phys. 13, 221 (2001).
[23] M. Hirokawa, Phys. Lett. A 294, 13 (2002).
[24] J. Stenger, S. Inouye, D. M. Stamper-Kurn, A. P. Chikkatur, D. E. Pritchard, and W. Ketterle, Appl. Phys. B 69, 347 (1999).
[25] S. Inouye, A. P. Chikkatur, D. M. Stamper-Kurn, J. Steger, D. E. Pritchard, and W. Ketterle, Science 285, 571 (1999).
[26] D. Schneble, Y. Torii, M. Boyd, E. W. Streed, D. E. Pritchard, and W. Ketterle, Science 300, 475 (2003).
[27] L. Fallani, C. Fort, N. Piovella, M. Cola, F. S. Cataliotti, M. Inguscio, and R. Bonifacio, Phys. Rev. A 71, 033612 (2005).
[28] J. V. Pulè, A. F. Verbeure, and V. A. Zagrebnov, J. Phys. A: Math. Gen. 38, 5173 (2005).
[29] J. V. Pulè, A. F. Verbeure, and V. A. Zagrebnov, J. Phys. A: Math. Gen. 37, L321 (2004).
[30] J. V. Pulè, A. F. Verbeure, and V. A. Zagrebnov, J. Stat. Phys. 119, 309 (2005).
[31] M. Hirokawa, to appear in Indiana Univ. Math. J.
[32] R. H. Dicke, Phys. Rev. 93, 99 (1954).
[33] A. Parmeggiani and M. Wakayama, Forum Math. 14, 539 (2002); ibid. 14, 669 (2002); ibid. 15, 955 (2003).
[34] K. Nagatou, M. T. Nakao, and M. Wakayama, Numer. Funct. Anal. Optim. 23, 633 (2003).
[35] A. Parmeggiani, Kyushu J. Math. 58, 277 (2004); Comm. Math. Phys. 279, 285 (2008); Introduction to the spectral theory of non-commutative harmonic oscillators (COE Lecture Note vol.8, Kyushu University, The 21st Century COE Program “DMHF”, Fukuoka, 2008).
[36] T. Ichinose and M. Wakayama, Commun. Math. Phys. 256, 697 (2005); Rep. Math. Phys. 59, 421 (2007).
[37] K. Hepp and E. H. Lieb, Ann. Phys. (N.Y.) 76, 360 (1973); Phys. Rev. A 8, 2517 (1973); Helv. Phys. Acta 46, 573 (1973).

[38] M. O. Scully and M. S. Zubairy, Quantum Optics (Cambridge Univ. Press, 2006).

[39] E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963).

[40] H. J. Metcalf and P. van der Straten, Laser Cooling and Trapping (Springer-Verlag, New York, 1999).

[41] F. Bardou, J.-P. Bouchaud, A. Aspect, and C. Cohen-Tannoudji, Lévy Statistics and Laser Cooling (Cambridge University Press, Cambridge, 2002).

[42] T. Kato, Perturbation theory for linear operators (Springer-Verlag, New York, 1995).

[43] M. Reed and B. Simon, Methods of modern Mathematical Physics IV. Analysis of Operators (Academic Press, San Diego, 1978).

[44] P. W. Milonni and J. H. Eberly, Lasers (Wiley Interscience Publication, New York, 1988).

[45] S. M. Barnett and P. M. Radmore, Methods in Theoretical Quantum Optics (Oxford University Press, Oxford, 2002).

[46] R. J. Thompson, G. Rempe, and H. J. Kimble, Phys. Rev. Lett. 68, 1132 (1992).

[47] M. Brune, P. Nussenzveig, F. Schmidt-Kaler, F. Bernardot, A. Maali, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 72, 3339 (1994).

[48] M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, E. Hagley, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 76, 1800 (1996).

[49] J. J. Childs, K. An, M. S. Otteson, R. R. Dasari, and M. S. Feld, Phys. Rev. Lett. 77, 2901 (1996).

[50] P. Grangier, Science 281, 56 (1998).

[51] H. Nha, Y.-T. Chough, and K. An, J. Korean Phys. Soc. 37, 693 (2000).

[52] W. T. M. Irvine, K. Hennessy, and D. Bouwmeester, Phys. Rev. Lett. 96, 057405 (2006).