ADAPTIVE TESTING METHOD FOR ERGODIC DIFFUSION PROCESSES BASED ON HIGH FREQUENCY DATA

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Abstract. We consider parametric tests for multidimensional ergodic diffusions based on high frequency data. We propose two-step testing method for diffusion parameters and drift parameters. To construct test statistics of the tests, we utilize the adaptive estimator and provide three types of test statistics: likelihood ratio type test, Wald type test and Rao’s score type test. It is proved that these test statistics converge in distribution to the chi-squared distribution under null hypothesis and have consistency of the tests against alternatives. Moreover, these test statistics converge in distribution to the non-central chi-squared distribution under local alternatives. We also give some simulation studies of the behavior of the three types of test statistics.

1. Introduction

We consider a $d$-dimensional diffusion process satisfying the following stochastic differential equation:

$$
\begin{cases}
    dX_t = b(X_t, \beta) dt + a(X_t, \alpha) dW_t, & t \in [0, T], \\
    X_0 = x_0,
\end{cases}
$$

where $W_t$ is the $r$-dimensional standard Wiener process, $\alpha \in \Theta_\alpha \subset \mathbb{R}^{p_1}$, $\beta \in \Theta_\beta \subset \mathbb{R}^{p_2}$, $\theta = (\alpha, \beta)$, $\Theta := \Theta_\alpha \times \Theta_\beta$ being compact and convex parameter space, $a : \mathbb{R}^d \times \Theta_\alpha \to \mathbb{R}^{d \otimes \mathbb{R}^r}$ and $b : \mathbb{R}^d \times \Theta_\beta \to \mathbb{R}^d$ are known except for the parameter $\theta$. We assume the true parameter $\theta^*_0 = (\alpha^*_0, \beta^*_0)$ belongs to Int$(\Theta)$. The data are discrete observations $(X_{t^*_i})_{0 \leq i \leq n}$, where $t^*_i = ih_n$ for $i = 0, 1, \ldots, n$, and the discretization step $h_n$ satisfies $h_n \to 0$, $nh_n \to \infty$ and $nh_n^2 \to 0$ as $n \to \infty$.

Diffusion processes are used as the mathematical models to describe the random development of the phenomena depending on time in many fields such as physics, neuroscience, meteorology, epidemiology and finance. For these models, the data are discretely observed. The statistical inference for ergodic diffusion process based on discrete observations has been studied by many researchers; see Florens-Zmirou [4], Yoshida [15, 16], Genon-Catalot and Jacod [5], Kessler [8, 9], Uchida and Yoshida [13] and reference therein. In particular, Kessler [8, 9] proposed the adaptive maximum likelihood (ML) type estimator and joint ML type estimator which has asymptotic efficiency under $nh_n^{p} \to 0$, where $p$ is an arbitrary integer with $p \geq 2$. Uchida and Yoshida [13] presented the polynomial type large deviation of adaptive statistical random fields under $nh_n^{p} \to 0$ and moment convergence of the adaptive ML type estimator with asymptotic efficiency.

The parametric testing problem for ergodic diffusions has been studied as the following joint test:

$$
\begin{cases}
    H_0 : \alpha_1 = \cdots = \alpha_{r_1} = 0, \quad \beta_1 = \cdots = \beta_{r_2} = 0, \quad (1 \leq r_1 \leq p_1, \ 1 \leq r_2 \leq p_2) \\
    H_1 : \text{not } H_0.
\end{cases}
$$

Kitagawa and Uchida [10] proposed three kinds of test statistics (likelihood ratio type test, Wald type test and Rao’s score type test) and proved their asymptotic properties. De Gregorio and Iacus [1, 2] constructed the test statistics by means of $\phi$-divergence measure and the empirical $L^2$-distance when $r_1 = p_1$ and $r_2 = p_2$. 

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In this paper, we consider the following set of tests instead of (1):

\[
\begin{aligned}
H_0^{(1)} &: \alpha_1 = \ldots = \alpha_{r_1} = 0, \\
H_1^{(1)} &: \text{not } H_0^{(1)},
\end{aligned}
\quad
\begin{aligned}
H_0^{(2)} &: \beta_1 = \ldots = \beta_{r_2} = 0, \\
H_1^{(2)} &: \text{not } H_0^{(2)}.
\end{aligned}
\tag{2}
\]

The set of tests (2) give more information about parameters than the test (1). The test (1) provides only two interpretations: \(H_0^{(1)}\) is rejected or \(H_0^{(1)}\) is not rejected. On the other hand, the tests (2) gives the following four conclusions: (i) \(H_0^{(1)}\) is rejected and \(H_0^{(2)}\) is rejected; (ii) \(H_0^{(1)}\) is rejected and \(H_0^{(2)}\) is not rejected; (iii) \(H_0^{(1)}\) is not rejected and \(H_0^{(2)}\) is rejected; (iv) \(H_0^{(1)}\) is not rejected and \(H_0^{(2)}\) is not rejected. We utilize the adaptive ML type estimator of Uchida and Yoshida [13] and construct three types of test statistics. Furthermore, we prove that these test statistics converge in distribution to the chi-squared distribution under null hypothesis, have consistency of the tests under alternatives and converge in distribution to the non-central chi-squared distribution under local alternatives. For the asymptotic null distribution of adaptive test statistics based on local means for noisy ergodic diffusion processes and consistency of the tests under alternatives, see Nakakita and Uchida [12].

The paper is organized as follows. In Section 2 notation and assumptions are introduced. In Section 3 we state main results. Two quasi log likelihood functions are constructed and three kinds of adaptive test statistics are proposed. Moreover, their asymptotic properties are shown. In Section 4 we give some examples and simulation results of the asymptotic performance for three types of test statistics for 1-dimensional ergodic diffusion processes. Section 5 is devoted to the proofs of the results presented in Section 3.

2. Notation and assumptions

Let \(\partial_i := \partial/\partial x_i\), \(\partial_{a_i} := \partial/\partial a_i\), \(\partial_{b_i} := \partial/\partial b_i\), \(\partial_a := (\partial_{a_1}, \ldots, \partial_{a_{p_1}})^T\), \(\partial_{\beta} := (\partial_{\beta_1}, \ldots, \partial_{\beta_{p_2}})^T\), \(\partial_a^2 := \partial_{a_i}\partial_{a_j}\), \(\partial_{b_a}^2 := \partial_{b_i}\partial_{a_j}\), where \(T\) is the transpose of a matrix. For \((m \times n)\)-matrices \(A\) and \(B\), it is defined that \(A^{\otimes 2} := AA^T, \|A\|^2 := \text{tr}(A^{\otimes 2}), B[A] := \text{tr}(BA^T)\), and we set \(S(x, \alpha) := (a(x, \alpha))^{\otimes 2}\). For \(l \geq 1\) and \(m \geq 1\), let \(C^l_{m} (\mathbb{R}^d \times \Theta)\) be the set of functions \(f\) satisfying the following conditions: (i) \(f(x, \theta)\) is \(l\) times continuously differentiable with respect to \(x\); (ii) \(f(x, \theta)\) and all its \(x\)-derivatives up to order \(l\) are \(m\) times continuously differentiable with respect to \(\theta\); (iii) \(f(x, \theta)\) and all its derivatives are of polynomial growth in \(x\), uniformly in \(\theta\). Next, for any positive sequence \(u_n\), \(R : \Theta \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) denotes a function with a constant \(C > 0\) such that for all \(\theta \in \Theta\) and \(x \in \mathbb{R}^d\), \(|R(\theta, u_n, x)| \leq u_n C (1 + |x|)^C\). \(\xrightarrow{P}\) and \(d\) indicate convergence in probability and convergence in distribution, respectively. Let \(I(\theta; \theta_0)\) be the \((p_1 + p_2) \times (p_1 + p_2)\)-matrix defined as

\[
I(\theta; \theta_0) := \begin{pmatrix} I_a(\alpha; \alpha_0) & 0 \\ 0 & I_b(\theta; \theta_0) \end{pmatrix},
\]

where \(I_a(\alpha; \alpha_0) = (I_a^{(ij)}(\alpha; \alpha_0))_{1 \leq i, j \leq p_1}, I_b(\theta; \theta_0) = (I_b^{(ij)}(\theta; \theta_0))_{1 \leq i, j \leq p_2}\).

\[
I_a^{(ij)}(\alpha; \alpha_0) = \frac{1}{2} \int \text{tr} \left( S^{-1}(\partial_{a_i}, S) S^{-1}(\partial_{a_j}, S) S^{-1} + S^{-1}(\partial_{a_i}, S) S^{-1}(\partial_{a_j}, S) S^{-1} \right. \\
- S^{-1}(\partial_{a_i}, S) S^{-1}(\partial_{a_j}, S) S^{-1}, (x, \alpha) S(x, \alpha_0) \mu_{\alpha_0}(dx) \\
+ \frac{1}{2} \int \text{tr} \left( S^{-1}(\partial_{a_i}, S) - S^{-1}(\partial_{a_j}, S) S^{-1}(\partial_{a_i}, S) \right) (x, \alpha) \mu_{\alpha_0}(dx), \\
I_b^{(ij)}(\theta; \theta_0) = \int (b(x, \beta) - b(x, \beta_0))^T S^{-1}(x, \alpha) (\partial_{\beta_i} \partial_{\beta_j} b(x, \beta)) \mu_{\theta_0}(dx) \\
+ \int (\partial_{\beta_i} b(x, \beta))^T S^{-1}(x, \alpha) (\partial_{\beta_j} b(x, \beta)) \mu_{\theta_0}(dx).
\]

We make the following assumptions.

**A1** There exists \(C > 0\) such that for all \(x, y \in \mathbb{R}^d\),

\[
\sup_{\alpha \in \Theta_a} |a(x, \alpha) - a(y, \alpha)| + \sup_{\beta \in \Theta_\beta} |b(x, \beta) - b(y, \beta)| \leq C |x - y|.
\]

**A2** The diffusion process \(X\) is ergodic with its invariant measure \(\mu_{\theta_0}(dx)\).
A3  \( \inf_{x, \alpha} \det(S(x, \alpha)) > 0 \).

A4  For all \( p \geq 0 \), sup \( E_{\theta}[|X_t|^p] < \infty \).

A5  \( a \in C_{r, \gamma}(\mathbb{R}^d \times \Theta_\alpha), b \in C_{r, \gamma}(\mathbb{R}^d \times \Theta_\beta) \).

A6  \( b(x, \beta) = b(x, \beta_0) \) for \( \mu_{\theta_0} \) a.s. all \( x \rightarrow \beta = \beta_0 \).

\( S(x, \alpha) = S(x, \alpha_0) \) for \( \mu_{\theta_0} \) a.s. all \( x \rightarrow \alpha = \alpha_0 \).

A7  \( I(\theta; \theta_0) \) is non-singular.

3. Main Results

3.1. Asymptotic distribution under null hypothesis.

For \( 1 \leq r_1 \leq p_1 \) and \( 1 \leq r_2 \leq p_2 \), we consider the following set of tests:

\[ \begin{align*}
H_0^{(1)}: & \quad \alpha_1 = \cdots = \alpha_{r_2} = 0, \\
H_1^{(1)}: & \quad \text{not } H_0^{(1)},
\end{align*} \]

\[ \begin{align*}
H_0^{(2)}: & \quad \beta_1 = \cdots = \beta_{r_2} = 0, \\
H_1^{(2)}: & \quad \text{not } H_0^{(2)}.
\end{align*} \]

Let \( \hat{\Theta}_n := \{ \alpha \in \Theta_n \mid \alpha \) holds \( H_0^{(1)} \}, \hat{\Theta}_\beta := \{ \beta \in \Theta_\beta \mid \beta \) holds \( H_0^{(2)} \}, \hat{\Theta} := \hat{\Theta}_\alpha \times \hat{\Theta}_\beta \).

The quasi log likelihood functions are defined as follows:

\[ U_n^{(1)}(\alpha) := \frac{1}{2} \sum_{i=1}^{n} \left\{ h_i^{-1} S^{-1}(X_{t_1}, \alpha) \left[ (X_{t_1} - X_{t_1}) \right] + \log \det S(X_{t_1}, \alpha) \right\}, \]

\[ U_n^{(2)}(\beta) := \frac{1}{2} \sum_{i=1}^{n} \left\{ h_i^{-1} S^{-1}(X_{t_1}, \alpha) \left[ (X_{t_1} - X_{t_1} - h_i b(X_{t_1}, \beta)) \right] \right\}. \]

The adaptive ML type estimator under \( \Theta \), \( \hat{\beta}_n := (\hat{\alpha}_n, \hat{\beta}_n) \) and the adaptive ML type estimator under \( \hat{\Theta} \), \( \hat{\theta}_n := (\hat{\alpha}_n, \hat{\beta}_n) \) are defined as

\[ \hat{\alpha}_n := \text{arg sup}_{\alpha \in \Theta_n} U_n^{(1)}(\alpha), \quad \hat{\alpha}_n := \text{arg sup}_{\alpha \in \Theta_n} U_n^{(1)}(\alpha), \]

\[ \hat{\beta}_n := \text{arg sup}_{\beta \in \Theta_\beta} U_n^{(2)}(\beta), \quad \hat{\beta}_n := \text{arg sup}_{\beta \in \Theta_\beta} U_n^{(2)}(\beta). \]

Next we set

\[ I_{\alpha, \beta}(\alpha, \beta) := -\frac{1}{2} \partial_2^2 U_n^{(1)}(\alpha, \beta), \quad J_{\alpha, \beta}(\alpha, \beta) := \frac{1}{2} \partial_2^2 U_n^{(2)}(\beta, \alpha), \]

\[ I_{\alpha, \beta}(\alpha, \beta) := \{ \text{I}_n(\alpha) \) is non-singular \}, \quad J_{\alpha, \beta}(\alpha, \beta) := \{ \text{J}_n(\beta) \) is non-singular \}, \]

\[ \text{I}_{\alpha, \beta}(\alpha, \beta) := \begin{cases} \text{I}_{\alpha, \beta}^{-1}(\alpha) \) (\omega \in J_{\alpha, \beta}(\alpha)), \quad \text{I}_{\alpha, \beta}^{-1}(\beta) \) (\omega \in J_{\alpha, \beta}(\beta)), \\
\end{cases} \]

\[ \text{J}_{\alpha, \beta}(\alpha, \beta) := \begin{cases} \text{J}_{\alpha, \beta}^{-1}(\alpha) \) (\omega \in J_{\alpha, \beta}(\alpha)), \quad \text{J}_{\alpha, \beta}^{-1}(\beta) \) (\omega \in J_{\alpha, \beta}(\beta)). \\
\end{cases} \]

We define the likelihood ratio, Wald and Rao type test statistics \( \Lambda_n^{(1)}, W_n^{(1)}, R_n^{(1)} \) \((i = 1, 2)\) as follows:

\[ \Lambda_n^{(1)} := -2(U_n^{(1)}(\hat{\alpha}_n) - U_n^{(1)}(\alpha_n)), \quad \Lambda_n^{(2)} := -2(U_n^{(2)}(\hat{\beta}_n) - U_n^{(2)}(\beta_n)), \]

\[ W_n^{(1)} := n(\hat{\alpha}_n - \alpha_n) ^T I_n(\hat{\alpha}_n)(\hat{\alpha}_n - \alpha_n), \quad W_n^{(2)} := nh_n(\hat{\beta}_n - \beta_n) ^T I_n(\hat{\beta}_n)(\hat{\beta}_n - \beta_n), \]

\[ R_n^{(1)} := n^{-1}(\partial_\alpha U_n^{(1)}(\alpha_n)) ^T I_n(\hat{\alpha}_n)(\partial_\alpha U_n^{(1)}(\hat{\alpha}_n)), \quad R_n^{(2)} := (nh_n)^{-1}(\partial_\beta U_n^{(2)}(\beta_n)(\hat{\beta}_n - \beta_n) ^T I_n(\hat{\beta}_n)(\partial_\beta U_n^{(2)}(\hat{\beta}_n)(\hat{\beta}_n)), \]

where \( \Lambda_n^{(1)}, W_n^{(1)} \) and \( R_n^{(1)} \) are used in testing \( \alpha \) and \( \Lambda_n^{(2)}, W_n^{(2)} \) and \( R_n^{(2)} \) are used in testing \( \beta \).

The following gives asymptotic distributions of these test statistics under null hypothesis.

**Theorem 1** Assume A1-A7. If \( h_n \rightarrow 0 \), \( nh_n \rightarrow \infty \) and \( nh_n^2 \rightarrow 0 \), then

(i) \( \Lambda_n^{(1)} \rightarrow \chi_{r_1}^2 \) (under \( H_0^{(1)} \)), \( \Lambda_n^{(2)} \rightarrow \chi_{r_2}^2 \) (under \( H_0^{(2)} \)),

(ii) \( W_n^{(1)} \rightarrow \chi_{r_1}^2 \) (under \( H_0^{(1)} \)), \( W_n^{(2)} \rightarrow \chi_{r_2}^2 \) (under \( H_0^{(2)} \)),

(iii) \( R_n^{(1)} \rightarrow \chi_{r_1}^2 \) (under \( H_0^{(1)} \)), \( R_n^{(2)} \rightarrow \chi_{r_2}^2 \) (under \( H_0^{(2)} \)).
Remark 1 When we perform the parametric tests for $\theta = (\alpha, \beta)$, the testing procedure is as follows:

1. test $H_0^{(1)}$ v.s. $H_1^{(1)}$;
2. test $H_0^{(2)}$ v.s. $H_1^{(2)}$ regardless of the result of (1).

This testing method provides four interpretations, which gives more information about the parameters than the joint test by Kitagawa and Uchida [10].

3.2. Consistency of tests.

Next we consider under alternatives $H_1^{(1)}$ or $H_1^{(2)}$. To emphasize that we consider under alternatives, $\theta_i^* = (\alpha_i^*, \beta_i^*)$ denotes the true parameter under alternatives. We define an optimal parameter $\theta^* = (\alpha^*, \beta^*) \in \hat{\Theta}$ as

$$\alpha^* := \arg \sup_{\alpha \in \Theta_\alpha} U_1(\alpha; \alpha_i^*), \quad \beta^* := \arg \sup_{\beta \in \Theta_\beta} U_2(\alpha_i^*, \beta; \beta_i^*),$$

where

$$U_1(\alpha; \alpha_i^*) := -\frac{1}{2} \int (\text{tr} (S(x, \alpha_i^*) S^{-1}(x, \alpha)) + \log \det S(x, \alpha)) \mu_{\theta_1}(dx),$$

$$U_2(\alpha_i^*, \beta; \beta_i^*) := -\frac{1}{2} \int S^{-1}(x, \alpha) \left[ (b(x, \beta_i^*) - b(x, \beta))^\circ 2 \right] \mu_{\theta_2}(dx).$$

Remark 2 It always true that $\alpha_i^* \neq \alpha^*$. Since $\alpha_i^*$ is the true value under $H_1^{(i)}$, there exists $i \in \{1, \ldots, r_1\}$ such that $\alpha_i \neq 0$. On the other hand, $\alpha^*$ is the optimal value under $H_1^{(0)}$, hence $\alpha_i = 0$ ($i = 1, \ldots, r_1$). Similarly, it always holds $\beta_i^* \neq \beta^*$.

We assume the following conditions.

**B1** (a) For all $\epsilon > 0$,

$$\sup_{\{\alpha \in \Theta_\alpha; |\alpha - \alpha^*| \geq \epsilon\}} (U_1(\alpha; \alpha_i^*) - U_1(\alpha^*; \alpha_i^*)) < 0.$$

(b) For all $\epsilon > 0$,

$$\sup_{\{\beta \in \Theta_\beta; |\beta - \beta^*| \geq \epsilon\}} (U_2(\alpha_i^*, \beta; \beta_i^*) - U_2(\alpha_i^*, \beta^*; \beta_i^*)) < 0.$$

**B2** (a) For all $\alpha \in \Theta_\alpha$, $I_0(\alpha; \alpha_i^*)$ is non-singular.

(b) For all $\beta \in \Theta_\beta$, $I_0(\alpha_i^*, \beta; \beta_i^*)$ is non-singular.

The following gives the consistency of tests.

**Theorem 2** Assume A1-A7 and B1. If $h_n \to 0$ and $nh_n \to \infty$, then for all $\epsilon \in (0, 1)$,

(i) $P(H_n^{(1)} \geq \chi^2_{r_1, \epsilon}) \to 1$ (under $H_1^{(1)}$), $P(H_n^{(2)} \geq \chi^2_{r_2, \epsilon}) \to 1$ (under $H_1^{(2)}$),

(ii) $P(W_n^{(1)} \geq \chi^2_{r_1, \epsilon}) \to 1$ (under $H_1^{(1)}$), $P(W_n^{(2)} \geq \chi^2_{r_2, \epsilon}) \to 1$ (under $H_1^{(2)}$),

(iii) $P(R_n^{(1)} \geq \chi^2_{r_1, \epsilon}) \to 1$ (under $H_1^{(1)}$ and B2-(a)), $P(R_n^{(2)} \geq \chi^2_{r_2, \epsilon}) \to 1$ (under $H_1^{(2)}$ and B2-(b)),

where $\chi^2_{r, \epsilon}$ denotes the upper $\epsilon$ point of chi-squared distribution with $p$ degrees of freedom.

3.3. Asymptotic distribution under local alternatives.

For $u_\alpha \in \mathbb{R}^{p_1}$ and $u_\beta \in \mathbb{R}^{p_2}$, we consider the following tests.

$$\begin{cases}
H_0^{(1)}: \alpha = \alpha_0, \\
H_1^{(1)}: \alpha = \alpha_1^* := \alpha_0 + \frac{u_\alpha}{\sqrt{n}}.
\end{cases} \quad \begin{cases}
H_0^{(2)}: \beta = \beta_0, \\
H_1^{(2)}: \beta = \beta_1^* := \beta_0 + \frac{u_\beta}{\sqrt{nh_n}}.
\end{cases}$$

Let $\theta_i^*_{1,n}$ denote the true parameter under local alternatives. We assume the following condition.

**C1** $P_{\theta_i^*_{1,n}}$ is contiguous with respect to $P_{\theta}$.

That is, for sequence of sets $A_n$,

$$\lim_{n \to \infty} P_{\theta_0}(A_n) = 0 \Rightarrow \lim_{n \to \infty} P_{\theta_i^*_{1,n}}(A_n) = 0.$$
Remark 3  (i) Let $\bar{\theta}$ denote the true parameter of $\alpha$ under $H_{0,n}^{(2)}$ or $H_{1,n}^{(2)}$, and $\hat{\beta}$ denote the true parameter of $\beta$ under $H_{0,n}^{(1)}$ or $H_{1,n}^{(1)}$. By using these symbols, C1 is applied as follows:

(I) If we test for $\alpha$, then $\theta_0 = (\alpha_0, \beta)$, $\theta_{1,n}^* = (\alpha_{1,n}^*, \beta)$.

(II) If we test for $\beta$, then $\theta_0 = (\bar{\alpha}, \beta_0)$, $\theta_{1,n}^* = (\bar{\alpha}, \beta_{1,n}^*)$.

(ii) A sufficient condition of C1 is local asymptotic normality (LAN), and a sufficient condition of LAN is as follows: A1-A5 and

(I) There exists $C > 0$ such that for all $x \in \mathbb{R}^d$ and $(\alpha, \beta) \in \Theta$,

$$|c(x, \beta)| \leq C(1 + |x|),$$

where $|a(x, \alpha)| = |\theta_0(x)|$. Let $\beta(x, \beta)$ denote the true parameter $\theta_0(x)$

(II) There exists $c_0 > 0$ and $K > 0$ such that for all $(x, \beta) \in \mathbb{R}^d \times \Theta$,

$$b(x, \beta) = -c_0|x|^2 + K.$$ 

For details of the relation between LAN and C1, see, for example, van der Vaart[14].

Theorem 3  Assume A1-A7 and C1. If $h_n \to 0$, $nh_n \to \infty$ and $nh_n^2 \to 0$, then

(i) $\Lambda_n^{(1)} \xrightarrow{d} \chi_1^2(u_n^\top I_a(\alpha_0; \alpha_0)u_n)$ (under $H_{1,n}^{(1)}$, $\Lambda_n^{(2)} \xrightarrow{d} \chi_2^2(u_n^\top I_\beta(\theta_0; \theta_0)u_\beta)$ (under $H_{1,n}^{(2)}$),

(ii) $W_n^{(1)} \xrightarrow{d} \chi_1^2(u_n^\top I_a(\alpha_0; \alpha_0)u_n)$ (under $H_{1,n}^{(1)}$, $W_n^{(2)} \xrightarrow{d} \chi_2^2(u_n^\top I_\beta(\theta_0; \theta_0)u_\beta)$ (under $H_{1,n}^{(2)}$),

(iii) $R_n^{(1)} \xrightarrow{d} \chi_1^2(u_n^\top I_a(\alpha_0; \alpha_0)u_n)$ (under $H_{1,n}^{(1)}$, $R_n^{(2)} \xrightarrow{d} \chi_2^2(u_n^\top I_\beta(\theta_0; \theta_0)u_\beta)$ (under $H_{1,n}^{(2)}$),

where $\chi_p^2(c)$ denotes the non-central chi-squared distribution with $p$ degrees of freedom and noncentrality parameter $c$.

4. Examples and simulations

4.1. Model 1.

We consider the following 1-dimensional Ornstein-Uhlenbeck process:

$$\begin{cases}
  dX_t = -(X_t - \beta)dt + \alpha dW_t, \\
  X_0 = 1.0.
\end{cases}$$  

(3)

We simulate the asymptotic performance of the three types of test statistics: likelihood ratio type, Wald type and Rao’s score type. In model [3], we deal with the following hypothesis tests:

$$\begin{align*}
  &H_0^{(1)} : \alpha = 1.0, \\
  &H_1^{(1)} : \alpha \neq 1.0, \\
  &H_0^{(2)} : \beta = 2.0, \\
  &H_1^{(2)} : \beta \neq 2.0.
\end{align*}$$

(4)

These tests derive the four kinds of results as follows:

Case 1. Neither $\alpha$ nor $\beta$ is rejected;

Case 2. $\alpha$ is not rejected, but $\beta$ is rejected;

Case 3. $\alpha$ is rejected, but $\beta$ is not rejected;

Case 4. Both $\alpha$ and $\beta$ are rejected;

We choose the true parameter $\theta^* = (\alpha^*, \beta^*)$ from $\{(1.0, 2.0), (1.0, 2.5), (1.1, 2.0), (1.1, 2.5)\}$, where $\theta^*$ corresponds to the true parameter of Cases 1-4. Let $n$ be fixed and $h_n = n^{-2/3}$, which satisfies the conditions $nh_n = n^{1/3} \to \infty$ and $nh_n^2 = n^{-1/3} \to 0$ as $n \to \infty$. In this simulation, we consider the cases of $n = 10^4, 10^5$ and $10^6$. Let the significance level denote $\varepsilon = 0.05$ and each test is rejected when the realization of each test statistic is greater than $\chi^2_{1,0.05}$. The simulation is repeated 1000 times.

Tables [14] show the number of counts of Cases 1-4 selected by the tests [4], where the true parameter $\theta^*$ corresponds to each case. In all of the tables, the true case is most often selected as $n$ increases. In Table 1, the percent of misidentification is about 10 percent, which is the sum of the significance levels of the considering tests. Figures [15] are simulation results of the histograms and the empirical distributions of the three types of test statistics in Theorem 1. Theoretically, these test statistics converge in distribution to $\chi^2_1$, and we see from all of the figures that these test statistics have good behavior. Tables [5-8] show the empirical sizes of the three test statistics when the null hypothesis is true.
Table 1. Results of test statistics in **Case 1**: \((\alpha^*, \beta^*) = (1.0, 2.0)\).

| \(n\)  | \(h_n\) | \(n_{h_n}\) | Test type  | Case 1 | Case 2 | Case 3 | Case 4 |
|--------|---------|-------------|------------|--------|--------|--------|--------|
| \(10^4\) | \(2.15 \times 10^{-3}\) | 21.5        | Likelihood | 890    | 53     | 53     | 4      |
|         |         |             | Wald       | 891    | 54     | 52     | 3      |
|         |         |             | Rao        | 891    | 52     | 52     | 5      |
| \(10^5\) | \(4.64 \times 10^{-4}\) | 46.4        | Likelihood | 897    | 50     | 50     | 3      |
|         |         |             | Wald       | 897    | 50     | 50     | 3      |
|         |         |             | Rao        | 897    | 50     | 50     | 3      |
| \(10^6\) | \(1.00 \times 10^{-4}\) | 100         | Likelihood | 900    | 49     | 47     | 4      |
|         |         |             | Wald       | 900    | 49     | 47     | 4      |
|         |         |             | Rao        | 901    | 49     | 46     | 4      |

Table 2. Results of test statistics in **Case 2**: \((\alpha^*, \beta^*) = (1.0, 2.5)\).

| \(n\)  | \(h_n\) | \(n_{h_n}\) | Test type  | Case 1 | Case 2 | Case 3 | Case 4 |
|--------|---------|-------------|------------|--------|--------|--------|--------|
| \(10^4\) | \(2.15 \times 10^{-3}\) | 21.5        | Likelihood | 352    | 590    | 23     | 35    |
|         |         |             | Wald       | 353    | 591    | 22     | 34    |
|         |         |             | Rao        | 352    | 590    | 23     | 35    |
| \(10^5\) | \(4.64 \times 10^{-4}\) | 46.4        | Likelihood | 74     | 881    | 3      | 42    |
|         |         |             | Wald       | 74     | 883    | 3      | 40    |
|         |         |             | Rao        | 74     | 880    | 3      | 43    |
| \(10^6\) | \(1.00 \times 10^{-4}\) | 100         | Likelihood | 1      | 950    | 0      | 49    |
|         |         |             | Wald       | 1      | 950    | 0      | 49    |
|         |         |             | Rao        | 1      | 950    | 0      | 49    |

Table 3. Results of test statistics in **Case 3**: \((\alpha^*, \beta^*) = (1.1, 2.0)\).

| \(n\)  | \(h_n\) | \(n_{h_n}\) | Test type  | Case 1 | Case 2 | Case 3 | Case 4 |
|--------|---------|-------------|------------|--------|--------|--------|--------|
| \(10^4\) | \(2.15 \times 10^{-3}\) | 21.5        | Likelihood | 0      | 0      | 956    | 44    |
|         |         |             | Wald       | 0      | 0      | 956    | 44    |
|         |         |             | Rao        | 0      | 0      | 956    | 44    |
| \(10^5\) | \(4.64 \times 10^{-4}\) | 46.4        | Likelihood | 0      | 0      | 954    | 46    |
|         |         |             | Wald       | 0      | 0      | 954    | 46    |
|         |         |             | Rao        | 0      | 0      | 954    | 46    |
| \(10^6\) | \(1.00 \times 10^{-4}\) | 100         | Likelihood | 0      | 0      | 955    | 45    |
|         |         |             | Wald       | 0      | 0      | 955    | 45    |
|         |         |             | Rao        | 0      | 0      | 955    | 45    |

Table 4. Results of test statistics in **Case 4**: \((\alpha^*, \beta^*) = (1.1, 2.5)\).

| \(n\)  | \(h_n\) | \(n_{h_n}\) | Test type  | Case 1 | Case 2 | Case 3 | Case 4 |
|--------|---------|-------------|------------|--------|--------|--------|--------|
| \(10^4\) | \(2.15 \times 10^{-3}\) | 21.5        | Likelihood | 0      | 0      | 425    | 575   |
|         |         |             | Wald       | 0      | 0      | 425    | 575   |
|         |         |             | Rao        | 0      | 0      | 425    | 575   |
| \(10^5\) | \(4.64 \times 10^{-4}\) | 46.4        | Likelihood | 0      | 0      | 114    | 886   |
|         |         |             | Wald       | 0      | 0      | 114    | 886   |
|         |         |             | Rao        | 0      | 0      | 114    | 886   |
| \(10^6\) | \(1.00 \times 10^{-4}\) | 100         | Likelihood | 0      | 0      | 2      | 998   |
|         |         |             | Wald       | 0      | 0      | 2      | 998   |
|         |         |             | Rao        | 0      | 0      | 2      | 998   |
Figure 1. Histograms of the three types of test statistics for $\alpha$ in Case 1. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the probability density function of $\chi^2_1$.

Figure 2. Histograms of the three types of test statistics for $\beta$ in Case 1. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the probability density function of $\chi^2_1$. 
Figure 3. Histograms of the three types of test statistics for $\alpha$ in Case 2. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the probability density function of $\chi^2_1$.

Figure 4. Histograms of the three types of test statistics for $\beta$ in Case 3. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the probability density function of $\chi^2_1$. 

Figure 5. Empirical distributions of the three types of test statistics for $\alpha$ in Case 1. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the cumulative distribution function of $\chi^2_1$.

Figure 6. Empirical distributions of the three types of test statistics for $\beta$ in Case 1. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the cumulative distribution function of $\chi^2_1$. 
Figure 7. Empirical distributions of the three types of test statistics for $\alpha$ in Case 2. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the cumulative distribution function of $\chi^2_1$.

Figure 8. Empirical distributions of the three types of test statistics for $\beta$ in Case 3. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the cumulative distribution function of $\chi^2_1$. 
In order to check consistency of the tests, we treat the three kinds of $\alpha^*$ (1.1, 1.01 and 1.001) in Case 3 and the three kinds of $\beta^*$ (3.0, 2.5 and 2.1) in Case 2. Tables 9 and 10 are simulation results of the empirical powers of the three test statistics. For $\alpha^* = 1.1$, 1.01 and $\beta^* = 3.0$, 2.5, the empirical powers increase and tend to 1 as $n$ increases.

Next, we consider the asymptotic distributions of the three test statistics under local alternatives. The hypothesis tests are defined as

\[
\begin{align*}
H_{0,n}^{(1)} & : \alpha = \alpha_0 = 1.0, \\
H_{1,n}^{(1)} & : \alpha = \alpha_1 = \alpha_0 + u_\alpha \sqrt{n}, \\
H_{0,n}^{(2)} & : \beta = \beta_0 = 2.0, \\
H_{1,n}^{(2)} & : \beta = \beta_1 = \beta_0 + u_\beta \sqrt{n},
\end{align*}
\]

where we set $u_\alpha = 5.0$ and $u_\beta = 2.0$. In the Ornstein-Uhlenbeck model (3), the invariant distribution is the normal distribution with mean $\beta$ and variance $\frac{\sigma^2}{\alpha^2}$. Therefore we calculate

\[
I_n(\alpha_0, \alpha_0) = \frac{2.0}{\alpha_0^2} = 2.0, \quad I_n(\theta_0, \theta_0) = \frac{1.0}{\alpha_0^2} = 1.0,
\]

\[
I_n(\alpha_0, \alpha_0) u_\alpha^2 = 50, \quad I_n(\theta_0, \theta_0) u_\beta^2 = 4.0.
\]

Figures 9-12 show the histograms and the empirical distributions of the three test statistics under local alternatives. Theoretically, the asymptotic distributions of the three test statistics for $\alpha$ are $\chi^2_{1}(50)$ and those for $\beta$ are $\chi^2(4)$. We see from Figures 9-12 that as $n$ increases, the three test statistics for $\alpha$ and $\beta$ are approximately distributed as $\chi^2_{1}(50)$ and $\chi^2(4)$, respectively. In particular, the Likelihood ratio test statistic for $\alpha$ has good performance.

| $\alpha^*$ | $n$ | $n_{nh}$ | Likelihood | Wald | Rao |
|------------|-----|---------|------------|------|-----|
| 1.1        | $10^4$ | 1.000 | 1.000 | 1.000 |
|            | $10^5$ | 1.000 | 1.000 | 1.000 |
|            | $10^6$ | 1.000 | 1.000 | 1.000 |
| 1.01       | $10^4$ | 0.326 | 0.316 | 0.349 |
|            | $10^5$ | 0.994 | 0.994 | 0.994 |
|            | $10^6$ | 1.000 | 1.000 | 1.000 |
| 1.001      | $10^4$ | 0.050 | 0.047 | 0.048 |
|            | $10^5$ | 0.095 | 0.093 | 0.098 |
|            | $10^6$ | 0.295 | 0.294 | 0.297 |

| $\beta^*$ | $n_{nh}$ | Likelihood | Wald | Rao |
|------------|---------|------------|------|-----|
| 3.0        | $10^4$ | 21.5 | 0.994 | 0.994 | 0.994 |
|            | $10^5$ | 46.4 | 1.000 | 1.000 | 1.000 |
|            | $10^6$ | 100 | 1.000 | 1.000 | 1.000 |
| 2.5        | $10^4$ | 21.5 | 0.625 | 0.625 | 0.625 |
|            | $10^5$ | 46.4 | 0.923 | 0.923 | 0.923 |
|            | $10^6$ | 100 | 0.999 | 0.999 | 0.999 |
| 2.1        | $10^4$ | 21.5 | 0.091 | 0.091 | 0.091 |
|            | $10^5$ | 46.4 | 0.098 | 0.098 | 0.098 |
|            | $10^6$ | 100 | 0.154 | 0.154 | 0.154 |
Figure 9. Histograms of the three types of test statistics for $\alpha$ under $H^{(1)}_{1,n}$. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the probability density function of $\chi^2_2(50)$.

Figure 10. Histograms of the three types of test statistics for $\beta$ under $H^{(2)}_{1,n}$. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the probability density function of $\chi^2_2(4)$. 
Figure 11. Empirical distributions of the three types of test statistics for $\alpha$ under $H_{1,n}^{(1)}$. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the cumulative distribution function of $\chi_2^2(50)$.

Figure 12. Empirical distributions of the three types of test statistics for $\beta$ under $H_{1,n}^{(1)}$. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the cumulative distribution function of $\chi_2^2(4)$. 
4.2. Model 2.

Next, we consider a 1-dimensional diffusion process satisfying the following stochastic differential equation which is more complex than model 1:

\[
\begin{cases}
    dX_t = -\beta_1 (X_t - \beta_2) dt + \left( \alpha_1 + \frac{\alpha_2}{1 + X_t^2} + \alpha_3 \cos^2 X_t \right) dW_t, \\
    X_0 = 1.0.
\end{cases}
\]  

We deal with the following tests:

\[
\begin{align*}
    H_0^{(1)} : (\alpha_1, \alpha_2) &= (1.0, 1.0), \\
    &H_1^{(1)} : \text{not } H_0^{(1)}, \\
    H_0^{(2)} : (\beta_1, \beta_2) &= (2.0, 2.0), \\
    &H_1^{(2)} : \text{not } H_0^{(2)}.
\end{align*}
\]

We set the true parameter \(\alpha_3^* = 0.5\) and choice the true parameter \((\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*)\) from \(\{(1.0, 1.0, 2.0, 2.0), (1.0, 1.0, 2.5, 2.5), (1.05, 1.05, 2.0, 2.0), (1.05, 1.05, 2.5, 2.5)\}\), which corresponds to the true parameter of Cases 1-4 described in Section 4.1, respectively. Each test is rejected when the realization of each test statistics is greater than \(\chi^2_{0.05, n}\), and the other simulation settings are the same as in the model \(1\) of Section 4.1.

Tables 11-14 show the number of counts of Cases 1-4 selected by the tests \((6)\), where the true parameter \((\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*)\) corresponds to each case. In all of the tables, the true case is most often selected as \(n\) increases. However, the Rao type test statistic does not have good performance in Cases 2 and 4 when \(n = 10^4\). Figures 13-20 are simulation results of the histograms and empirical distributions of the three types of test statistics in Theorem 1. Theoretically, these test statistics converge in distribution to \(\chi^2_n\). Tables 15-18 show the empirical sizes of the three types of test statistics when null hypothesis is true. From these figures and tables, the Likelihood ratio type test statistic has the best behavior of the three types of test statistics.

**Table 11.** Results of test statistics in Case 1: \((\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*) = (1.0, 1.0, 2.0, 2.0)\).

| \(n\) | \(h_n\) | \(nh_n\) | Test type | Case 1 | Case 2 | Case 3 | Case 4 |
|------|--------|---------|-----------|--------|--------|--------|--------|
| \(10^4\) | \(2.15 \times 10^{-3}\) | 21.5 | Likelihood | 896 | 53 | 47 | 4 |
| | | | Wald | 878 | 68 | 48 | 6 |
| | | | Rao | 894 | 53 | 49 | 4 |
| \(10^5\) | \(4.64 \times 10^{-4}\) | 46.4 | Likelihood | 896 | 55 | 47 | 2 |
| | | | Wald | 884 | 68 | 46 | 2 |
| | | | Rao | 895 | 56 | 48 | 1 |
| \(10^6\) | \(1.00 \times 10^{-4}\) | 100 | Likelihood | 905 | 41 | 51 | 3 |
| | | | Wald | 902 | 44 | 51 | 3 |
| | | | Rao | 914 | 32 | 54 | 0 |

**Table 12.** Results of test statistics in Case 2: \((\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*) = (1.0, 1.0, 2.5, 2.5)\).

| \(n\) | \(h_n\) | \(nh_n\) | Test type | Case 1 | Case 2 | Case 3 | Case 4 |
|------|--------|---------|-----------|--------|--------|--------|--------|
| \(10^4\) | \(2.15 \times 10^{-3}\) | 21.5 | Likelihood | 55 | 888 | 5 | 52 |
| | | | Wald | 37 | 904 | 4 | 55 |
| | | | Rao | 325 | 613 | 24 | 38 |
| \(10^5\) | \(4.64 \times 10^{-4}\) | 46.4 | Likelihood | 0 | 947 | 0 | 53 |
| | | | Wald | 0 | 944 | 0 | 56 |
| | | | Rao | 11 | 936 | 0 | 53 |
| \(10^6\) | \(1.00 \times 10^{-4}\) | 100 | Likelihood | 0 | 949 | 0 | 51 |
| | | | Wald | 0 | 949 | 0 | 51 |
| | | | Rao | 0 | 946 | 0 | 54 |
Table 13. Results of test statistics in **Case 3**: \((\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*) = (1.05, 1.05, 2.0, 2.0)\).

| \(n\)  | \(h_n\)       | \(n h_n\) | Test type | Case 1 | Case 2 | Case 3 | Case 4 |
|-------|---------------|------------|-----------|--------|--------|--------|--------|
| \(10^4\) | \(2.15 \times 10^{-3}\) | 21.5 | Likelihood | 12 | 2 | 926 | 60 |
|       |               |       | Wald      | 13 | 2 | 908 | 77 |
|       |               |       | Rao       | 9  | 4 | 923 | 64 |
| \(10^5\) | \(4.64 \times 10^{-4}\) | 46.4 | Likelihood | 0  | 0 | 955 | 45 |
|       |               |       | Wald      | 0  | 0 | 946 | 54 |
|       |               |       | Rao       | 0  | 0 | 954 | 46 |
| \(10^6\) | \(1.00 \times 10^{-4}\) | 100 | Likelihood | 0  | 0 | 951 | 49 |
|       |               |       | Wald      | 0  | 0 | 946 | 54 |
|       |               |       | Rao       | 0  | 0 | 956 | 44 |

Table 14. Results of test statistics in **Case 4**: \((\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*) = (1.05, 1.05, 2.5, 2.5)\).

| \(n\)  | \(h_n\)       | \(n h_n\) | Test type | Case 1 | Case 2 | Case 3 | Case 4 |
|-------|---------------|------------|-----------|--------|--------|--------|--------|
| \(10^4\) | \(2.15 \times 10^{-3}\) | 21.5 | Likelihood | 2  | 76 | 71 | 851 |
|       |               |       | Wald      | 1  | 91 | 44 | 864 |
|       |               |       | Rao       | 21 | 47 | 412 | 520 |
| \(10^5\) | \(4.64 \times 10^{-4}\) | 46.4 | Likelihood | 0  | 0 | 0 | 1000 |
|       |               |       | Wald      | 0  | 0 | 0 | 1000 |
|       |               |       | Rao       | 0  | 0 | 12 | 988 |
| \(10^6\) | \(1.00 \times 10^{-4}\) | 100 | Likelihood | 0  | 0 | 0 | 1000 |
|       |               |       | Wald      | 0  | 0 | 0 | 1000 |
|       |               |       | Rao       | 0  | 0 | 0 | 1000 |

**Figure 13.** Histograms of the three types of test statistics for \(\alpha\) in **Case 1**. Each row of figures corresponds to the case of \(n = 10^4\) (upper), \(10^5\) (middle) and \(10^6\) (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the probability density function of \(\chi^2_2\).
Figure 14. Histograms of the three types of test statistics for $\beta$ in Case 1. Each row of figures corresponds to the case of $n = 10^3$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the probability density function of $\chi^2_2$.

Figure 15. Histograms of the three types of test statistics for $\alpha$ in Case 2. Each row of figures corresponds to the case of $n = 10^3$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the probability density function of $\chi^2_2$. 
Figure 16. Histograms of the three types of test statistics for $\beta$ in Case 3. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the probability density function of $\chi^2_2$.

Figure 17. Empirical distributions of the three types of test statistics for $\alpha$ in Case 1. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the cumulative distribution function of $\chi^2_2$. 
Figure 18. Empirical distributions of the three types of test statistics for $\beta$ in Case 1. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the cumulative distribution function of $\chi^2_2$.

Figure 19. Empirical distributions of the three types of test statistics for $\alpha$ in Case 2. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the cumulative distribution function of $\chi^2_2$. 
Figure 20. Empirical distributions of the three types of test statistics for $\beta$ in Case 3. Each row of figures corresponds to the case of $n = 10^4$ (upper), $10^5$ (middle) and $10^6$ (bottom) and each column of figures corresponds to Likelihood ratio (left), Wald (middle) and Rao (right) type test statistics. The red line is the cumulative distribution function of $\chi^2_2$.

Table 15. Empirical sizes of the test statistics for $\alpha$ in Case 1.

| $n$ | $nh_n$ | Likelihood | Wald | Rao |
|-----|--------|------------|------|-----|
| $10^4$ | 21.5 | 0.051 | 0.054 | 0.053 |
| $10^5$ | 46.4 | 0.049 | 0.048 | 0.049 |
| $10^6$ | 100 | 0.054 | 0.054 | 0.054 |

Table 16. Empirical sizes of the test statistics for $\beta$ in Case 1.

| $n$ | $nh_n$ | Likelihood | Wald | Rao |
|-----|--------|------------|------|-----|
| $10^4$ | 21.5 | 0.057 | 0.074 | 0.057 |
| $10^5$ | 46.4 | 0.057 | 0.070 | 0.057 |
| $10^6$ | 100 | 0.044 | 0.047 | 0.032 |

Table 17. Empirical sizes of the test statistics for $\alpha$ in Case 2.

| $n$ | $nh_n$ | Likelihood | Wald | Rao |
|-----|--------|------------|------|-----|
| $10^4$ | 21.5 | 0.057 | 0.059 | 0.062 |
| $10^5$ | 46.4 | 0.053 | 0.056 | 0.053 |
| $10^6$ | 100 | 0.051 | 0.051 | 0.054 |

Table 18. Empirical sizes of the test statistics for $\beta$ in Case 3.

| $n$ | $nh_n$ | Likelihood | Wald | Rao |
|-----|--------|------------|------|-----|
| $10^4$ | 21.5 | 0.062 | 0.079 | 0.068 |
| $10^5$ | 46.4 | 0.045 | 0.054 | 0.046 |
| $10^6$ | 100 | 0.049 | 0.054 | 0.044 |

Next, in order to check consistency of the tests, we treat the three kinds of $(\alpha_1^*, \alpha_2^*)$ ($(1.05, 1.0), (1.0, 1.05)$ and $(1.05, 1.05)$) in Case 3 and the three kinds of $(\beta_1^*, \beta_2^*)$ ($(2.5, 2.0), (2.0, 2.5)$ and $(2.5, 2.5)$) in Case 2. Tables 19 and 20 show the empirical powers of the three types of test statistics. In Table 19 when $n \leq 10^5$, since the asymptotic behavior of the estimator of $\alpha_1$ is stable compared with that of $\alpha_2$, the empirical power in the case of $(\alpha_1^*, \alpha_2^*) = (1.05, 1.0)$ is greater than that in the case of $(\alpha_1^*, \alpha_2^*) = (1.0, 1.05)$. In Table 20 when $nh_n \leq 100$, the empirical power in the case of $(\beta_1^*, \beta_2^*) = (2.0, 2.5)$ is greater than that in the case of $(\beta_1^*, \beta_2^*) = (2.5, 2.0)$ because the asymptotic performance of the estimator of $\beta_2$ is better than that of $\beta_1$. 
TABLE 19. Empirical powers of the three types of test statistics for $\alpha$ in Case 3.

| $(\alpha_1^*, \alpha_2^*)$ | $n$  | Likelihood | Wald | Rao |
|--------------------------|------|------------|------|-----|
|                         | $10^4$ | 0.898      | 0.892 | 0.907 |
| (1.05, 1.0)             | $10^5$ | 1.000      | 1.000 | 1.000 |
|                         | $10^6$ | 1.000      | 1.000 | 1.000 |
|                         | $10^4$ | 0.164      | 0.149 | 0.185 |
| (1.0, 1.05)             | $10^5$ | 0.918      | 0.915 | 0.921 |
|                         | $10^6$ | 1.000      | 1.000 | 1.000 |
|                         | $10^4$ | 0.986      | 0.985 | 0.987 |
| (1.05, 1.05)            | $10^5$ | 1.000      | 1.000 | 1.000 |
|                         | $10^6$ | 1.000      | 1.000 | 1.000 |

| $(\beta_1^*, \beta_2^*)$ | $nh_n$  | Likelihood | Wald | Rao |
|--------------------------|----------|------------|------|-----|
| (2.0, 2.5)               | $10^4$ | 21.5       | 21.5 | 21.5 |
|                         | $10^5$ | 46.4       | 46.4 | 46.4 |
|                         | $10^6$ | 100        | 100  | 100  |
| (2.5, 2.5)               | $10^4$ | 46.4       | 46.4 | 46.4 |
|                         | $10^5$ | 100        | 100  | 100  |

TABLE 20. Empirical powers of the three types of test statistics for $\beta$ in Case 2.

5. PROOFS

Let
\[ I_a(a_0; \alpha_0) = \begin{pmatrix} I_{a,1}(\alpha_0; \alpha_0) & I_{a,2}(\alpha_0; \alpha_0) \\ I_{a,2}^\top(\alpha_0; \alpha_0) & I_{a,3}(\alpha_0; \alpha_0) \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & -I_{a,3}^{-1}(\alpha_0; \alpha_0) \\ 0 & I_{a,3}^{-1}(\alpha_0; \alpha_0) \end{pmatrix}, \]
where $I_{a,1}(\alpha_0; \alpha_0)$, $I_{a,2}(\alpha_0; \alpha_0)$, $I_{a,3}(\alpha_0; \alpha_0)$ are the $(r_1 \times r_1)$, $(r_1 \times (p_1 - r_1))$ and $((p_1 - r_1) \times (p_1 - r_1))$ matrices, respectively. Set
\[ I_b(\theta_0; \theta_0) = \begin{pmatrix} I_{b,1}(\theta_0; \theta_0) & I_{b,2}(\theta_0; \theta_0) \\ I_{b,2}^\top(\theta_0; \theta_0) & I_{b,3}(\theta_0; \theta_0) \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & -I_{b,3}^{-1}(\theta_0; \theta_0) \\ 0 & I_{b,3}^{-1}(\theta_0; \theta_0) \end{pmatrix}, \]
where $I_{b,1}(\theta_0; \theta_0)$, $I_{b,2}(\theta_0; \theta_0)$, $I_{b,3}(\theta_0; \theta_0)$ are the $(r_2 \times r_2)$, $(r_2 \times (p_2 - r_2))$ and $((p_2 - r_2) \times (p_2 - r_2))$ matrices, respectively. Moreover, $Y_1$ and $Y_2$ are the normal random variables with mean 0 and covariance matrices $I_a(\alpha_0; \alpha_0)$ and $I_b(\theta_0; \theta_0)$, respectively, that is, $Y_1 \sim N(0, I_a(\alpha_0; \alpha_0))$ and $Y_2 \sim N(0, I_b(\theta_0; \theta_0))$.

In order to prove Theorem 1, we need the following lemma.

Lemma 1  Assume $A1$-$A7$. If $h_n \to 0$, $nh_n \to \infty$ and $nh_n^2 \to 0$, then

(i) $\sqrt{n}(\hat{\alpha}_n - \alpha_n) \overset{d}{\to} (I_a^{-1}(\alpha_0; \alpha_0) - H_1)Y_1$ (under $H_0^{(1)}$).

(ii) $\sqrt{nh_n}(\hat{\beta}_n - \beta_n) \overset{d}{\to} (I_b^{-1}(\theta_0; \theta_0) - H_2)Y_2$ (under $H_0^{(2)}$).

Proof.  (i) First of all, we will show that
\[ H_1 \frac{1}{\sqrt{n}} \partial_n U_n^{(1)}(\alpha_0) = \sqrt{n}(\hat{\alpha}_n - \alpha_0) + o_P(1) \quad \text{(under } H_0^{(1)}).. \]
From Taylor’s theorem,
\[ \frac{1}{\sqrt{n}} \partial_n U_n^{(1)}(\hat{\alpha}_n) - \frac{1}{\sqrt{n}} \partial_n U_n^{(1)}(\alpha_0) = \hat{I}_{a,n}(\hat{\alpha}_n, \alpha_0) \sqrt{n}(\hat{\alpha}_n - \alpha_0), \]
where
\[ \hat{I}_{a,n}(\hat{\alpha}_n, \alpha_0) := \int_0^1 \frac{1}{n} \partial_n^2 U_n^{(1)}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)) du. \]
Noting that $H_1 \partial_n U_n^{(1)}(\hat{\alpha}_n) = 0$, we obtain
\[ H_1 \frac{1}{\sqrt{n}} \partial_n U_n^{(1)}(\alpha_0) = -H_1 \hat{I}_{a,n}(\hat{\alpha}_n, \alpha_0) \sqrt{n}(\hat{\alpha}_n - \alpha_0). \]
On the other hand, we can show

\[
-\tilde{I}_{a,n}(\tilde{\alpha}_n, \alpha_0) \overset{P}{\to} I_a(\alpha_0; \alpha_0) \quad \text{(under } H_0^{(1)})
\]

\[
\frac{1}{\sqrt{n}} \partial_a U_n^{(1)}(\alpha_0) \overset{d}{\to} Y_1 \quad \text{(under } H_0^{(1)})
\]

\[
H_1 \frac{1}{\sqrt{n}} \partial_a U_n^{(1)}(\alpha_0) \overset{d}{\to} H_1 Y_1 \quad \text{(under } H_0^{(1)}).
\]

We set \(\tilde{\alpha}_n^{(p_1-r_1)} = (\tilde{\alpha}_{n,r_1+1} - \alpha_{n,r_1+1}, \ldots, \tilde{\alpha}_{n-r_1} - \alpha_{n-r_1})\). Noting that the first \(r_1\) components of \(\sqrt{n}(\tilde{\alpha}_n - \alpha_0)\) are all 0, one has that under \(H_0^{(1)}\),

\[
H_1 \tilde{I}_{a,n}(\alpha_0, \alpha_0) \sqrt{n}(\tilde{\alpha}_n - \alpha_0) = \left( I_{a,3}^{-1} \tilde{I}_{a,n,2}(\alpha_0, \alpha_0) \right) \left( \sqrt{n} \tilde{\alpha}_n^{(p_1-r_1)} \right)
\]

\[
= \left( I_{a,3}^{-1} \tilde{I}_{a,n,2}(\alpha_0, \alpha_0) \sqrt{n} \tilde{\alpha}_n^{(p_1-r_1)} \right).
\]

\[
H_1 I_a(\alpha_0; \alpha_0) \sqrt{n}(\tilde{\alpha}_n - \alpha_0) = \left( I_{a,3}^{-1} \tilde{I}_{a,n,2}(\alpha_0, \alpha_0) \right) \left( \sqrt{n} \tilde{\alpha}_n^{(p_1-r_1)} \right) = \sqrt{n}(\tilde{\alpha}_n - \alpha_0),
\]

where \(E_p\) is the \((p \times p)\) identity matrix. The above equations \(9, 10, 12\) and \(13\) implies that under \(H_0^{(1)}\),

\[
\sqrt{n} \tilde{\alpha}_n^{(p_1-r_1)} = O_P(1).
\]

Since the first \(r_1\) components of \(\sqrt{n}(\tilde{\alpha}_n - \alpha_0)\) are all 0, it follows that

\[
\sqrt{n}(\tilde{\alpha}_n - \alpha_0) = O_P(1) \quad \text{(under } H_0^{(1)}).
\]

Hence, by \(9, 10, 14\) and \(15\), we have that under \(H_0^{(1)}\),

\[
H_1 \frac{1}{\sqrt{n}} \partial_a U_n^{(1)}(\alpha_0) = H_1 (I_a(\alpha_0; \alpha_0) + o_P(1)) \sqrt{n}(\tilde{\alpha}_n - \alpha_0)
\]

\[
= H_1 I_a(\alpha_0; \alpha_0) \sqrt{n}(\tilde{\alpha}_n - \alpha_0) + o_P(1)
\]

\[
= \sqrt{n}(\tilde{\alpha}_n - \alpha_0) + o_P(1).
\]

This accords with \(7\). Next, it follows from \(8, 11, 15, 7\) and \(11\), that

\[
\frac{1}{\sqrt{n}} \partial_a U_n^{(1)}(\tilde{\alpha}_n) = \frac{1}{\sqrt{n}} \partial_a U_n^{(1)}(\alpha_0) + \tilde{I}_{a,n}(\tilde{\alpha}_n, \alpha_0) \sqrt{n}(\tilde{\alpha}_n - \alpha_0)
\]

\[
= \frac{1}{\sqrt{n}} \partial_a U_n^{(1)}(\alpha_0) - I_a(\alpha_0; \alpha_0) \sqrt{n}(\tilde{\alpha}_n - \alpha_0) + o_P(1)
\]

\[
= \frac{1}{\sqrt{n}} \partial_a U_n^{(1)}(\alpha_0) - I_a(\alpha_0; \alpha_0) H_1 \frac{1}{\sqrt{n}} \partial_a U_n^{(1)}(\alpha_0) + o_P(1)
\]

\[
= (E_{p_1} - I_a(\alpha_0; \alpha_0) H_1) \frac{1}{\sqrt{n}} \partial_a U_n^{(1)}(\alpha_0) + o_P(1)
\]

\[
\overset{d}{\to} (E_{p_1} - I_a(\alpha_0; \alpha_0) H_1) Y_1 \quad \text{(under } H_0^{(1)}).
\]

By noting that \(\partial_a U_n^{(1)}(\tilde{\alpha}_n) = 0\), it follows from Taylor’s theorem that

\[
-\frac{1}{\sqrt{n}} \partial_a U_n^{(1)}(\tilde{\alpha}_n) = \tilde{I}_{a,n}(\tilde{\alpha}_n, \tilde{\alpha}_n) \sqrt{n}(\tilde{\alpha}_n - \tilde{\alpha}_n)
\]

\[
= -I_a(\alpha_0; \alpha_0) \sqrt{n}(\tilde{\alpha}_n - \tilde{\alpha}_n) + o_P(1).
\]

Since \(I_a(\alpha_0; \alpha_0)\) is non-singular, by \(16\) and Slutsky’s theorem,

\[
\sqrt{n}(\tilde{\alpha}_n - \tilde{\alpha}_n) \overset{d}{\to} I_a^{-1}(\alpha_0; \alpha_0)(E_{p_1} - I_a(\alpha_0; \alpha_0) H_1) Y_1
\]

\[
= (I_a^{-1}(\alpha_0; \alpha_0) - H_1) Y_1 \quad \text{(under } H_0^{(1)}).
\]
This completes the proof of (i).
(ii) First, we will show that
\[ H_2 \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0|\alpha_0) = \frac{1}{\sqrt{n}h_n}(\tilde{\beta}_n - \beta_0) + o_P(1) \quad \text{(under } H_0^{(2)}). \]

From Taylor’s theorem with respect to \( \beta \),
\[ \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta|\alpha_n) = \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0|\alpha_0) + \tilde{I}_{b,n}(\tilde{\beta}_n, \beta_0) \sqrt{n}h_n(\tilde{\beta}_n - \beta_0), \]
where
\[ \tilde{I}_{b,n}(\tilde{\beta}_n, \beta_0) := \int_0^1 \frac{1}{nh_n} \partial_\beta^2 U_n^{(2)}(\beta_0 + u(\tilde{\beta}_n - \beta_0)|\alpha_n) du. \]
Moreover, by Taylor’s theorem with respect to \( \alpha \),
\[ \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta|\alpha_n) = \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0|\alpha_0) + \hat{I}_{a,b,n}(\hat{\alpha}_n, \alpha_0) \sqrt{n}(\hat{\alpha}_n - \alpha_0), \]
where
\[ \hat{I}_{a,b,n}(\hat{\alpha}_n, \alpha_0) := \int_0^1 \frac{1}{n\sqrt{h_n}} \partial_\alpha \partial_\beta U_n^{(2)}(\beta_0 + u(\hat{\alpha}_n - \alpha_0)|\alpha_0) du. \]
We can show \( \hat{I}_{a,b,n}(\hat{\alpha}_n, \alpha_0) \xrightarrow{P} 0 \) (under \( H_0^{(2)} \)). It follows from (18) and (19) that
\[ \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\tilde{\beta}_n|\alpha_n) = \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0|\alpha_0) + \tilde{I}_{b,n}(\tilde{\beta}_n, \beta_0) \sqrt{n}h_n(\tilde{\beta}_n - \beta_0) + o_P(1). \]

Since \( H_2 \partial_\beta U_n^{(2)}(\tilde{\beta}_n|\alpha_n) = 0 \), we obtain
\[ H_2 \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0|\alpha_0) = -H_2 \tilde{I}_{b,n}(\tilde{\beta}_n, \beta_0) \sqrt{n}h_n(\tilde{\beta}_n - \beta_0) + o_P(1). \]

On the other hand, we have
\[ \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0|\alpha_0) \xrightarrow{d} Y_2 \quad \text{(under } H_0^{(2)}), \]
\[ H_2 \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0|\alpha_0) \xrightarrow{d} H_2Y_2 \quad \text{(under } H_0^{(2)}). \]

If we set \( \tilde{\beta}_n^{(p_2-r_2)} = (\tilde{\beta}_{n,r_2+1} - \beta_{0,r_2+1}, \ldots, \tilde{\beta}_{n,p_2} - \beta_{0,p_2}) \), then
\[ H_2 \tilde{I}_{b,n}(\beta_0, \beta_0) \sqrt{n}h_n(\tilde{\beta}_n - \beta_0) = \begin{pmatrix} 0 \\ \bar{I}_{b,3}^{-1} \bar{I}_{b,n,3}(\beta_0, \beta_0) \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{n}h_n \tilde{\beta}_n^{(p_2-r_2)} \end{pmatrix}, \]
\[ = \begin{pmatrix} 0 \\ \bar{I}_{b,3}^{-1} \bar{I}_{b,n,3}(\beta_0, \beta_0) \sqrt{n}h_n(\tilde{\beta}_n - \beta_0) \end{pmatrix}. \]

In an analogous manner to the proof of (15), one has that
\[ \sqrt{n}h_n(\tilde{\beta}_n - \beta_0) = O_P(1) \quad \text{(under } H_0^{(2)}). \]
Hence, (21), (22), (27) and (26) implies that
\[ H_2 \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0 | \alpha_0) = H_2 (I_b(\theta_0; \theta_0) + o_P(1)) \sqrt{n}h_n(\hat{\beta}_n - \beta_0) + o_P(1) \]
\[ = H_2 I_b(\theta_0; \theta_0) \sqrt{n}h_n(\hat{\beta}_n - \beta_0) + o_P(1) \]
\[ = \sqrt{n}h_n(\hat{\beta}_n - \beta_0) + o_P(1). \]  
(28)

This accords with (17). It follows from (20), (22), (27), (28) and (23) that
\[ \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\hat{\beta}_n | \hat{\alpha}_n) = \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0 | \alpha_0) + \tilde{I}_{b,n}(\hat{\beta}_n, \hat{\beta}_n) \sqrt{n}h_n(\hat{\beta}_n - \beta_0) + o_P(1) \]
\[ = \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0 | \alpha_0) - I_b(\theta_0; \theta_0) \sqrt{n}h_n(\hat{\beta}_n - \beta_0) + o_P(1) \]
\[ = \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0 | \alpha_0) - I_b(\theta_0; \theta_0) H_2 \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0 | \alpha_0) + o_P(1) \]
\[ = (E_{p_2} - I_b(\theta_0; \theta_0) H_2) \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0 | \alpha_0) + o_P(1) \]
\[ \overset{d}{\to} (E_{p_2} - I_b(\theta_0; \theta_0) H_2) Y_2 \quad \text{(under } H_0^{(2)}). \]  
(29)

Since we have that under \( H_0^{(2)} \),
\[ \sqrt{n}h_n(\hat{\beta}_n - \tilde{\beta}_n) = o_P(1), \]
\[ \tilde{I}_{b,n}(\hat{\beta}_n, \tilde{\beta}_n) = -I_b(\theta_0; \theta_0) + o_P(1), \]
it follows from Taylor’s theorem that under \( H_0^{(2)} \),
\[ -\frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\hat{\beta}_n | \hat{\alpha}_n) = \tilde{I}_{b,n}(\hat{\beta}_n, \hat{\beta}_n) \sqrt{n}h_n(\hat{\beta}_n - \tilde{\beta}_n) \]
\[ = -I_b(\theta_0; \theta_0) \sqrt{n}h_n(\hat{\beta}_n - \tilde{\beta}_n) + o_P(1). \]
Since \( I_b(\theta_0; \theta_0) \) is non-singular, by (29) and Slutsky’s theorem,
\[ \sqrt{n}h_n(\hat{\beta}_n - \tilde{\beta}_n) \overset{d}{\to} I_b^{-1}(\theta_0; \theta_0)(E_{p_2} - I_b(\theta_0, \theta_0) H_2) Y_2 \]
\[ = (I_b^{-1}(\theta_0; \theta_0) - H_2) Y_2 \quad \text{(under } H_0^{(2)}). \]

This completes the proof of (ii).

\( \Box \)

\textbf{Proof of Theorem 1.} (i) By Taylor’s theorem, we obtain
\[ U_n^{(1)}(\hat{\alpha}_n) - U_n^{(1)}(\hat{\alpha}_n) = \left( \sqrt{n}(\hat{\alpha}_n - \hat{\alpha}_n) \right)^\top J_{a,n}(\hat{\alpha}_n, \hat{\alpha}_n) \sqrt{n}(\hat{\alpha}_n - \hat{\alpha}_n), \]
where
\[ J_{a,n}(\hat{\alpha}_n, \hat{\alpha}_n) = \int_0^1 (1 - u) \frac{1}{n} \partial_\alpha^2 U_n^{(1)}(\hat{\alpha}_n + u(\hat{\alpha}_n - \hat{\alpha}_n)) du. \]
It follows from Lemma \( \square \) that
\[ \sqrt{n}(\hat{\alpha}_n - \hat{\alpha}_n) \overset{d}{\to} (I_a^{-1}(\alpha_0; \alpha_0) - H_1) Y_1 \quad \text{(under } H_0^{(1)}). \]
Moreover, we can check that under \( H_0^{(1)} \),
\[ \hat{\alpha}_n \overset{P}{\to} \alpha_0, \quad \hat{\alpha}_n \overset{P}{\to} \alpha_0, \]
\[ J_{a,n}(\hat{\alpha}_n, \hat{\alpha}_n) \overset{P}{\to} -\frac{1}{2} I_a(\alpha_0; \alpha_0). \]
We set \( Y_1 = I_7^{\frac{1}{2}}(\alpha_0; \alpha_0) Z_1, \ Z_1 \sim N(0, E_{p_2}) \) and
\[ P_1 = I_7^{\frac{1}{2}}(\alpha_0; \alpha_0)(I_a^{-1}(\alpha_0; \alpha_0) - H_1) I_7^{\frac{1}{2}}(\alpha_0; \alpha_0). \]
It follows from the continuous mapping theorem that
\[ \Lambda_n^{(1)} = -2(U_n^{(1)}(\hat{\alpha}_n) - U_n^{(1)}(\hat{\alpha}_n)) \]
\[ \xrightarrow{d} Y_1^T (I_a^{-1}(\alpha_0; \alpha_0) - H_1) I_a(\alpha_0; \alpha_0)(I_a^{-1}(\alpha_0; \alpha_0) - H_1) Y_1 \]
\[ = Z_1^T I_\beta^2(\alpha_0; \alpha_0)(I_a^{-1}(\alpha_0; \alpha_0) - H_1) I_\beta(\alpha_0; \alpha_0) Z_1 = Z_1^T P_1 Z_1. \]
Since \( P_1 \) is the projection matrix with
\[ \text{rank } P_1 = \text{trace } P_1 = r_1, \]
we obtain
\[ \Lambda_n^{(1)} \xrightarrow{d} \chi^2_{r_1} \quad (\text{under } H_0^{(1)}), \]
which implies the first statement of (i).

Next, by Taylor's theorem with respect to \( \beta \), one has that under \( H_0^{(2)} \),
\[ U_n^{(2)}(\hat{\beta}_n|\hat{\alpha}_n) - U_n^{(2)}(\beta_n|\alpha_n) = \left( \sqrt{n\hat{h}_n}(\hat{\beta}_n - \hat{\beta}_n) \right)^T J_{b,n}(\hat{\beta}_n, \hat{\beta}_n) \sqrt{n\hat{h}_n}(\hat{\beta}_n - \hat{\beta}_n), \]
where
\[ J_{b,n}(\hat{\beta}_n, \hat{\beta}_n) := \int_0^1 (1 - u) \frac{1}{n\hat{h}_n} \partial^2_u U_n^{(2)}(\hat{\beta}_n + u(\hat{\beta}_n - \hat{\beta}_n)|\hat{\alpha}_n) du. \]
It follows from Lemma II that
\[ \sqrt{n\hat{h}_n}(\hat{\beta}_n - \hat{\beta}_n) \xrightarrow{d} (I_\beta^{-1}(\theta_0; \theta_0) - H_2) Y_2 \quad (\text{under } H_0^{(2)}), \]
and we can check
\[ \hat{\beta}_n \xrightarrow{P} \beta_0, \quad \hat{\beta}_n \xrightarrow{P} \beta_0, \]
\[ J_{b,n}(\hat{\beta}_n, \hat{\beta}_n) \xrightarrow{P} -\frac{1}{2} J_b(\theta_0; \theta_0). \]
Let \( Y_2 = I_\beta^\frac{1}{2}(\theta_0; \theta_0)Z_2, \ Z_2 \sim N(0, E_{\theta_2}) \) and
\[ P_2 = I_\beta^\frac{1}{2}(\theta_0; \theta_0)(I_\beta^{-1}(\theta_0; \theta_0) - H_2) I_\beta^\frac{1}{2}(\theta_0; \theta_0). \]
Note that \( P_2 \) is the projection matrix with
\[ \text{rank } P_2 = \text{trace } P_2 = r_2. \]
The continuous mapping theorem yields that
\[ \Lambda_n^{(2)} = -2(U_n^{(2)}(\hat{\beta}_n|\hat{\alpha}_n) - U_n^{(2)}(\hat{\beta}_n|\hat{\alpha}_n)) \]
\[ \xrightarrow{d} Y_2^T (I_\beta^{-1}(\theta_0; \theta_0) - H_2) J_b(\theta_0; \theta_0)(I_\beta^{-1}(\theta_0; \theta_0) - H_2) Y_2 \]
\[ = Z_2^T I_\beta^\frac{1}{2}(\theta_0; \theta_0)(I_\beta^{-1}(\theta_0; \theta_0) - H_2) I_\beta^\frac{1}{2}(\theta_0; \theta_0) Z_2 = Z_2^T P_2 Z_2 \]
\[ \sim \chi^2_{r_2} \quad (\text{under } H_0^{(2)}). \]
This completes the proof of the Likelihood type.

(ii) By noting that
\[ I_{a,n}(\hat{\alpha}_n) \xrightarrow{P} I_a(\alpha_0; \alpha_0) \quad (\text{under } H_0^{(1)}), \]
it follows from Lemma II and the continuous mapping theorem that
\[ W_n^{(1)} = \left( \sqrt{\hat{m}(\hat{\alpha}_n - \hat{\alpha}_n)} \right)^T I_{a,n}(\hat{\alpha}_n) \sqrt{\hat{m}(\hat{\alpha}_n - \hat{\alpha}_n)} \]
\[ \xrightarrow{d} Y_1^T (I_a^{-1}(\alpha_0; \alpha_0) - H_1) I_a(\alpha_0; \alpha_0)(I_a^{-1}(\alpha_0; \alpha_0) - H_1) Y_1 \]
\[ \sim \chi^2_{r_1} \quad (\text{under } H_0^{(1)}). \]
Moreover, we check
\[ I_{b,n}(\hat{\beta}_n|\hat{\alpha}_n) \overset{P}{\to} I_b(\theta_0;\theta_0) \quad (\text{under } H_0^{(2)}), \]
and it holds
\[
W_n^{(2)} = \left( \sqrt{n} \left( \hat{\beta}_n - \tilde{\beta}_n \right) \right)^\top I_{b,n}(\hat{\beta}_n|\hat{\alpha}_n) \sqrt{n} \left( \hat{\beta}_n - \tilde{\beta}_n \right)
\]
\[
\overset{d}{\to} Y_2^\top (I_b^{-1}(\theta_0;\theta_0) - H_2) I_b(\theta_0;\theta_0) (I_b^{-1}(\theta_0;\theta_0) - H_2) Y_2
\]
\[ \sim \chi_{r_2}^2 \quad (\text{under } H_0^{(2)}). \]
This completes the proof of the Wald type.
(iii) From the proof of Lemma 1, it holds
\[
\frac{1}{\sqrt{n}} \partial_n U_n^{(1)}(\hat{\alpha}_n) \overset{d}{\to} (E_{p_1} - I_a(\alpha_0;\alpha_0) H_1) Y_1 \quad (\text{under } H_0^{(1)}).
\]
It follows that
\[
\tilde{I}_{a,n}(\hat{\alpha}_n) \overset{P}{\to} I_a^{-1}(\alpha_0;\alpha_0) \quad (\text{under } H_0^{(1)}).
\]
Hence we obtain from the continuous mapping theorem that
\[
R_n^{(1)} = \left( \frac{1}{\sqrt{n}} \partial_n U_n^{(1)}(\hat{\alpha}_n) \right)^\top \tilde{I}_{a,n}(\hat{\alpha}_n) \frac{1}{\sqrt{n}} \partial_n U_n^{(1)}(\hat{\alpha}_n)
\]
\[
\overset{d}{\to} Y_1^\top (E_{p_1} - I_a(\alpha_0;\alpha_0) H_1) I_a^{-1}(\alpha_0;\alpha_0) Y_1
\]
\[ = Y_1^\top (E_{p_1} - I_a(\alpha_0;\alpha_0) H_1) I_a^{-1}(\alpha_0;\alpha_0) I_a(\alpha_0;\alpha_0) I_a^{-1}(\alpha_0;\alpha_0) Y_1
\]
\[ \sim \chi_{r_1}^2 \quad (\text{under } H_0^{(1)}). \]
Furthermore, it follows from the proof of Lemma 1 that
\[
\frac{1}{\sqrt{n} \alpha_0} \partial \beta U_n^{(2)}(\tilde{\beta}_n|\hat{\alpha}_n) \overset{d}{\to} (E_{p_2} - I_b(\theta_0;\theta_0) H_2) Y_2 \quad (\text{under } H_0^{(2)}),
\]
and we can check
\[
I_{b,n}(\hat{\beta}_n|\hat{\alpha}_n) \overset{P}{\to} I_b^{-1}(\theta_0;\theta_0) \quad (\text{under } H_0^{(2)}).
\]
Thus, one has
\[
R_n^{(2)} = \left( \frac{1}{\sqrt{n} \alpha_0} \partial \beta U_n^{(2)}(\tilde{\beta}_n|\hat{\alpha}_n) \right)^\top \tilde{I}_{b,n}(\hat{\beta}_n|\hat{\alpha}_n) \frac{1}{\sqrt{n} \alpha_0} \partial \beta U_n^{(2)}(\tilde{\beta}_n|\hat{\alpha}_n)
\]
\[
\overset{d}{\to} Y_2^\top (E_{p_2} - I_b(\theta_0;\theta_0) H_2) I_b^{-1}(\theta_0;\theta_0) (E_{p_2} - I_b(\theta_0;\theta_0) H_2) Y_2
\]
\[ = Y_2^\top (I_b^{-1}(\theta_0;\theta_0) - H_2) I_b(\theta_0;\theta_0) I_b^{-1}(\theta_0;\theta_0) (E_{p_2} - I_b(\theta_0;\theta_0) H_2) Y_2
\]
\[ \sim \chi_{r_2}^2 \quad (\text{under } H_0^{(2)}). \]
This completes the proof of the Rao type. \(\square\)

Next, we will prove the following lemma to show Theorem 2.

**Lemma 2** Assume A1-A7 and B1. If \( h_n \to 0 \) and \( nh_n \to \infty \), then
\[
\text{(i) } \hat{\alpha}_n \overset{P}{\to} \alpha^* \quad (\text{under } H_1^{(1)}),
\]
\[
\text{(ii) } \hat{\beta}_n \overset{P}{\to} \beta^* \quad (\text{under } H_1^{(2)}).
\]
Proof. (i) From the definition of $\bar{U}_1(\alpha;\alpha_1^*)$, it holds
\[
\sup_{\alpha \in \Theta_n} \left| \frac{1}{n} U_n^{(1)}(\alpha) - \bar{U}_1(\alpha;\alpha_1^*) \right| \xrightarrow{P} 0 \quad \text{(under $H_1^{(1)}$)}.
\]

**B1-(a)** implies the following: for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\alpha \in \Theta_\alpha$,
\[
|\alpha - \alpha^*| \geq \varepsilon \implies \bar{U}_1(\alpha^*;\alpha_1^*) - \bar{U}_1(\alpha;\alpha_1^*) > \delta.
\]
Therefore we obtain
\[
0 \leq P \left( |\tilde{\alpha}_n - \alpha^*| \geq \varepsilon \right)
\leq P \left( \bar{U}_1(\alpha^*;\alpha_1^*) - \bar{U}_1(\tilde{\alpha}_n;\alpha_1^*) > \delta \right)
= P \left( \bar{U}_1(\alpha^*;\alpha_1^*) - \frac{1}{n} U_n^{(1)}(\alpha^*) + \frac{1}{n} U_n^{(1)}(\alpha^*) - \frac{1}{n} U_n^{(1)}(\tilde{\alpha}_n) + \frac{1}{n} U_n^{(1)}(\tilde{\alpha}_n) - \bar{U}_1(\tilde{\alpha}_n;\alpha_1^*) > \delta \right)
\leq 2P \left( \sup_{\alpha \in \Theta_n} \left| \frac{1}{n} U_n^{(1)}(\alpha) - \bar{U}_1(\alpha;\alpha_1^*) \right| > \frac{\delta}{3} \right) + P \left( \frac{1}{n} U_n^{(1)}(\alpha^*) - \frac{1}{n} U_n^{(1)}(\tilde{\alpha}_n) > \frac{\delta}{3} \right)
\to 0 \quad (n \to \infty),
\]
which implies the statement of (i).

(ii) For sake of simplicity, $\alpha_1^*$ denotes the true value of $\alpha$ under $H_1^{(2)}$. From the definition of $\bar{U}_2(\alpha,\beta;\beta_1^*)$, it holds
\[
\sup_{\beta \in \Theta} \left| L_n(\alpha,\beta) - \bar{U}_2(\alpha,\beta;\beta_1^*) \right| \xrightarrow{P} 0 \quad \text{(under $H_1^{(2)}$)},
\]
where
\[
L_n(\alpha,\beta) = \frac{1}{nh_n} L_n^{(2)}(\beta|\alpha) - \frac{1}{nh_n} U_n^{(2)}(\beta_1^*|\alpha).
\]

**B1-(b)** implies the following: for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\beta \in \Theta_\beta$,
\[
|\beta - \beta^*| \geq \varepsilon \implies \bar{U}_2(\alpha_1^*,\beta^*;\beta_1^*) - \bar{U}_2(\alpha_1^*,\beta;\beta_1^*) > \delta.
\]
Hence it holds
\[
0 \leq P \left( |\tilde{\beta}_n - \beta^*| \geq \varepsilon \right)
\leq P \left( \bar{U}_2(\alpha_1^*,\beta^*;\beta_1^*) - \bar{U}_2(\alpha_1^*,\tilde{\beta}_n;\beta_1^*) > \delta \right)
= P \left( \bar{U}_2(\alpha_1^*,\beta^*;\beta_1^*) - L_n(\tilde{\alpha}_n,\beta^*) + L_n(\tilde{\alpha}_n,\beta^*) - L_n(\tilde{\alpha}_n,\tilde{\beta}_n) + L_n(\tilde{\alpha}_n,\tilde{\beta}_n) - \bar{U}_2(\alpha_1^*,\tilde{\beta}_n;\beta_1^*) > \delta \right)
\leq 2P \left( \sup_{\beta \in \Theta_n} \left| L_n(\tilde{\alpha}_n,\beta) - \bar{U}_2(\alpha_1^*,\beta;\beta_1^*) \right| > \frac{\delta}{3} \right) + P \left( L_n(\tilde{\alpha}_n,\beta^*) - L_n(\tilde{\alpha}_n,\tilde{\beta}_n) > \frac{\delta}{3} \right)
= 2P \left( \sup_{\beta \in \Theta_n} \left| L_n(\tilde{\alpha}_n,\beta) - \bar{U}_2(\alpha_1^*,\beta;\beta_1^*) \right| > \frac{\delta}{3} \right),
\]
where
\[
P \left( \sup_{\beta \in \Theta_n} \left| L_n(\tilde{\alpha}_n,\beta) - \bar{U}_2(\alpha_1^*,\beta;\beta_1^*) \right| > \frac{\delta}{3} \right) \leq P \left( \sup_{\beta \in \Theta_n} \left| L_n(\tilde{\alpha}_n,\beta) - L_n(\alpha_1^*,\beta) \right| > \frac{\delta}{6} \right)
+ P \left( \sup_{\beta \in \Theta_n} \left| L_n(\alpha_1^*,\beta) - \bar{U}_2(\alpha_1^*,\beta;\beta_1^*) \right| > \frac{\delta}{6} \right)
\leq P \left( \sup_{\beta \in \Theta_n} \left| L_n(\tilde{\alpha}_n,\beta) - L_n(\alpha_1^*,\beta) \right| > \frac{\delta}{6} \right)
+ P \left( \sup_{\beta \in \Theta_n} \left| L_n(\alpha_1^*,\beta) - \bar{U}_2(\alpha_1^*,\beta;\beta_1^*) \right| > \frac{\delta}{6} \right). \tag{30}
\]
The second term on the right hand side of the inequality \([30]\) converges to 0 under \(H_0^{(2)}\). It follows from Taylor’s theorem that
\[
|L_n(\hat{\alpha}_n, \beta) - L_n(\alpha_1^*, \beta)| = \left| \int_0^1 \partial_n L_n(\alpha_1^* + u(\hat{\alpha}_n - \alpha_1^*), \beta) du (\hat{\alpha}_n - \alpha_1^*) \right|
\leq \int_0^1 |\partial_n L_n(\alpha_1^* + u(\hat{\alpha}_n - \alpha_1^*), \beta)| du |\hat{\alpha}_n - \alpha_1^*|
\leq \sup_{\theta \in \Theta} |\partial_n L_n(\alpha, \beta)| |\hat{\alpha}_n - \alpha_1^*|,
\]

and it is easy to check
\[
|\hat{\alpha}_n - \alpha_1^*| = o_P(1), \quad \sup_{\theta \in \Theta} |\partial_n L_n(\alpha, \beta)| = O_P(1).
\]

Therefore, the first term on the right hand side of the inequality \([30]\) converges to 0, which implies \(\hat{\beta}_n \overset{P}{\to} \beta^*\).

Proof of Theorem 2. (i) Since Lemma 2 implies that under \(H_1^{(1)}\),
\[
\hat{\alpha}_n \overset{P}{\to} \alpha_1^*, \quad \hat{\alpha}_n \overset{P}{\to} \alpha^* \neq \alpha_1^*,
\]
\[
\sup_{\alpha \in \Theta_n} \left| \frac{1}{n} U_n^{(1)}(\alpha) - \hat{U}_1(\alpha; \alpha_1^*) \right| \overset{P}{\to} 0,
\]
we obtain
\[
\frac{1}{n} \Lambda_n^{(1)} = \frac{2}{n} (U_n^{(1)}(\hat{\alpha}_n) - U_n^{(1)}(\hat{\alpha}_n))
\overset{P}{\to} 2(\hat{U}_1(\alpha_1^*; \alpha_1^*) - \hat{U}_1(\alpha^*; \alpha_1^*)) > 0.
\]

Therefore, for all \(\varepsilon \in (0, 1)\),
\[
0 \leq P \left( \Lambda_n^{(1)} \leq \chi^2_{1,\varepsilon} \right) = P \left( \frac{1}{n} \Lambda_n^{(1)} \leq \frac{\chi^2_{1,\varepsilon}}{n} \right) \to 0 \quad \text{(under } H_1^{(1)}, \: n \to \infty),
\]
which implies the first statement of (i).

Next, we consider \(\Lambda_n^{(2)}\) under \(H_1^{(2)}\). It holds
\[
\frac{1}{nh_n} \Lambda_n^{(2)} = \frac{2}{nh_n} (U_n^{(2)}(\hat{\beta}_n; \hat{\alpha}_n) - U_n^{(2)}(\hat{\beta}_n; \hat{\alpha}_n))
= \frac{2}{nh_n} (U_n^{(2)}(\hat{\beta}_n; \hat{\alpha}_n) - U_n^{(2)}(\beta_1^*; \hat{\alpha}_n)) + \frac{2}{nh_n} (U_n^{(2)}(\beta_1^*; \hat{\alpha}_n) - U_n^{(2)}(\hat{\beta}_n; \hat{\alpha}_n))
\geq \frac{2}{nh_n} (U_n^{(2)}(\beta_1^*; \hat{\alpha}_n) - U_n^{(2)}(\hat{\beta}_n; \hat{\alpha}_n)).
\]

Lemma 2 implies that under \(H_1^{(2)}\),
\[
\hat{\alpha}_n \overset{P}{\to} \alpha_1^*, \quad \hat{\beta}_n \overset{P}{\to} \beta^* \neq \beta_1^*,
\]
\[
\sup_{\theta \in \Theta} \left| L_n(\alpha, \beta) - \hat{U}_2(\alpha, \beta; \beta_1^*) \right| \overset{P}{\to} 0.
\]

Noting that \(\hat{U}_2(\alpha_1^*, \beta_1^*; \beta_1^*) = 0\) and \(L_n(\alpha, \beta) = \frac{1}{nh_n} (U_n^{(2)}(\beta(\alpha) - U_n^{(2)}(\beta_1^*; \alpha)))\), one has
\[
\frac{2}{nh_n} (U_n^{(2)}(\beta_1^*; \hat{\alpha}_n) - U_n^{(2)}(\beta_1^*; \hat{\alpha}_n)) \overset{P}{\to} -2\hat{U}_2(\alpha_1^*, \beta_1^*; \beta_1^*) > 0.
\]
Therefore, for all $\varepsilon \in (0, 1)$,
\[
0 \leq P \left( \Lambda_n^{(2)} \leq \chi^2_{r_2, \varepsilon} \right) = P \left( \frac{1}{nh_n} \Lambda_n^{(2)} \leq \frac{\chi^2_{r_2, \varepsilon}}{nh_n} \right) \leq P \left( \frac{2}{nh_n} (U_n^{(2)}(\beta_1^*|\hat{\alpha}_n) - U_n^{(2)}(\hat{\beta}_n|\hat{\alpha}_n)) \leq \frac{\chi^2_{r_2, \varepsilon}}{nh_n} \right) \rightarrow 0 \quad \text{(under } H_1^{(2)}, n \to \infty). \]
This completes the proof of the Likelihood type.

(ii) We have that under $H_1^{(1)}$,
\[
\hat{\alpha}_n - \tilde{\alpha}_n \overset{P}{\to} \alpha^*_1 - \alpha^* \neq 0, \\
I_{a,n}(\hat{\alpha}_n) \overset{P}{\to} I_a(\alpha^*_1; \alpha^*_1).
\]
Hence
\[
\frac{1}{n} W_n^{(1)} = (\hat{\alpha}_n - \tilde{\alpha}_n)^\top I_{a,n}(\hat{\alpha}_n)(\hat{\alpha}_n - \tilde{\alpha}_n) \overset{P}{\to} (\alpha^*_1 - \alpha^*)^\top I_a(\alpha^*_1; \alpha^*_1)(\alpha^*_1 - \alpha^*) > 0
\]
because of A7. Therefore, for all $\varepsilon \in (0, 1)$,
\[
P(W_n^{(1)} \geq \chi^2_{r_1, \varepsilon}) = P \left( \frac{1}{n} W_n^{(1)} \geq \frac{\chi^2_{r_1, \varepsilon}}{n} \right) \to 1 \quad \text{(under } H_1^{(1)}). \]
Moreover, Lemma 2 implies that under $H_1^{(2)}$,
\[
\hat{\beta}_n - \tilde{\beta}_n \overset{P}{\to} \beta^*_1 - \beta^* \neq 0, \\
I_{b,n}(\hat{\beta}_n|\hat{\alpha}_n) \overset{P}{\to} I_b(\beta^*_1; \beta^*_1).
\]
Hence
\[
\frac{1}{nh_n} W_n^{(2)} = (\hat{\beta}_n - \tilde{\beta}_n)^\top I_{b,n}(\hat{\beta}_n|\hat{\alpha}_n)(\hat{\beta}_n - \tilde{\beta}_n) \overset{P}{\to} (\beta^*_1 - \beta^*)^\top I_b(\beta^*_1; \beta^*_1)(\beta^*_1 - \beta^*) > 0.
\]
Therefore, for all $\varepsilon \in (0, 1)$,
\[
P(W_n^{(2)} \geq \chi^2_{r_2, \varepsilon}) = P \left( \frac{1}{nh_n} W_n^{(2)} \geq \frac{\chi^2_{r_2, \varepsilon}}{nh_n} \right) \to 1 \quad \text{(under } H_1^{(2)}). \]
This completes the proof of the Wald type.

(iii) From Taylor’s theorem, it holds
\[
\frac{1}{n} \partial_{\alpha} U_n^{(1)}(\hat{\alpha}_n) = - \int_0^1 \frac{1}{n} \partial^2_{\alpha} U_n^{(1)}(\hat{\alpha}_n + u(\hat{\alpha}_n - \tilde{\alpha}_n)) du(\hat{\alpha}_n - \tilde{\alpha}_n).
\]
It is shown that under $H_1^{(1)}$,
\[
\hat{\alpha}_n - \tilde{\alpha}_n \overset{P}{\to} \alpha^*_1 - \alpha^* \neq 0,
\]
\[
- \int_0^1 \frac{1}{n} \partial^2_{\alpha} U_n^{(1)}(\hat{\alpha}_n + u(\hat{\alpha}_n - \tilde{\alpha}_n)) du \overset{P}{\to} \int_0^1 I_a(\alpha^*_1 + u(\alpha^* - \alpha^*_1); \alpha^*_1) du.
\]
Hence it follows from B2-(a) that
\[
\frac{1}{n} \partial_{\alpha} U_n^{(1)}(\hat{\alpha}_n) \overset{P}{\to} \int_0^1 I_a(\alpha^*_1 + u(\alpha^* - \alpha^*_1); \alpha^*_1) du(\alpha^*_1 - \alpha^*) =: c_1 \neq 0,
\]
Furthermore, it follows from Taylor’s theorem with respect to Lemma 3

\[ \frac{1}{n} R_n^{(1)} = \left( \frac{1}{n} \partial_\alpha U_n^{(1)}(\hat{\alpha}_n) \right) \uparrow \bar{I}_{\alpha,n}(\hat{\alpha}_n) \frac{1}{n} \partial_\alpha U_n^{(1)}(\hat{\alpha}_n) \]

\[ \xrightarrow{P} c_1 \uparrow I_n^{-1}(\alpha_1^*, \alpha_1^*)c_1 > 0. \]

Therefore, for all \( \varepsilon \in (0,1) \),

\[ P(R_n^{(1)} \geq \frac{\chi^2_{1,\varepsilon}}{n}) = P \left( \frac{1}{n} R_n^{(1)} \geq \frac{\chi^2_{1,\varepsilon}}{n} \right) \rightarrow 1 \quad \text{(under } H_n^{(1)}). \]

Furthermore, it follows from Taylor’s theorem with respect to \( \beta \) that under \( H_n^{(2)} \),

\[ \frac{1}{nh_n} \partial_\beta U_n^{(2)}(\hat{\beta}_n|\hat{\alpha}_n) = -\int_0^1 \frac{1}{nh_n} \partial^2_\beta U_n^{(2)}(\hat{\beta}_n + u(\hat{\beta}_n - \hat{\beta}_n)|\hat{\alpha}_n) du \hat{\beta}_n - \hat{\beta}_n, \]

and we have

\[ \hat{\beta}_n - \hat{\beta}_n \xrightarrow{P} \beta_1^* - \beta_* \neq 0, \]

\[ -\int_0^1 \frac{1}{nh_n} \partial^2_\beta U_n^{(2)}(\hat{\beta}_n + u(\hat{\beta}_n - \hat{\beta}_n)|\hat{\alpha}_n) du \xrightarrow{P} \int_0^1 I_b((\alpha_1^*, \beta_1^* + u(\beta_* - \beta_1^*)); \theta_1^*) du. \]

Hence it follows from B2-(b) that

\[ \frac{1}{nh_n} \partial_\beta U_n^{(2)}(\hat{\beta}_n|\hat{\alpha}_n) \xrightarrow{P} \int_0^1 I_b((\alpha_1^*, \beta_1^* + u(\beta_* - \beta_1^*)); \theta_1^*) du =: c_2 \neq 0, \]

and

\[ \frac{1}{nh_n} R_n^{(2)} = \left( \frac{1}{nh_n} \partial_\beta U_n^{(2)}(\hat{\beta}_n|\hat{\alpha}_n) \right) \uparrow \bar{I}_{\beta,n}(\hat{\beta}_n|\hat{\alpha}_n) \frac{1}{nh_n} \partial_\beta U_n^{(2)}(\hat{\beta}_n|\hat{\alpha}_n) \]

\[ \xrightarrow{P} c_2 \uparrow I_n^{-1}(\theta_1^*, \theta_1^*)c_2 > 0. \]

Therefore, for all \( \varepsilon \in (0,1) \),

\[ P(R_n^{(2)} \geq \frac{\chi^2_{2,\varepsilon}}{nh_n}) = P \left( \frac{1}{nh_n} R_n^{(2)} \geq \frac{\chi^2_{2,\varepsilon}}{nh_n} \right) \rightarrow 1 \quad \text{(under } H_n^{(2)}). \]

This completes the proof of the Rao type. \( \square \)

Finally, we will prove Theorem 3. In order to prove this theorem, we use the following lemma.

**Lemma 3** Assume A1-A7 and C1. If \( h_n \to 0 \), \( nh_n \to \infty \) and \( nh_n^2 \to 0 \), then

(i) \( \sqrt{n}(\hat{\alpha}_n - \alpha_0) \xrightarrow{d} Y_1' \sim N(u_\alpha, I_n^{-1}(\alpha_0; \alpha_0)) \) (under \( H_n^{(1)}\)),

(ii) \( \sqrt{nh_n}(\hat{\beta}_n - \beta_0) \xrightarrow{d} Y_2' \sim N(u_\beta, I_n^{-1}(\theta_0; \theta_0)) \) (under \( H_n^{(2)}\)).

**Proof.** (i) First of all, it is shown that

\[ \left( \frac{1}{\sqrt{n}} \partial_\alpha U_n^{(1)}(\alpha_0) \right) \xrightarrow{d} L_1 \sim N(I_\alpha(\alpha_0; \alpha_0)u_\alpha, I_\alpha(\alpha_0; \alpha_0)) \quad \text{(under } H_n^{(1)}\). \hspace{1cm} (31) \]

For sake of simplicity, we use the following symbols:

\[ X_{i_n} = (X_{i_1}, \ldots, X_{i_d})^\top, \quad S(X_{i_{n-1}}, \alpha) = S_{i-1}(\alpha) = (S_{i-1,p,q}(\alpha))_{1 \leq p, q \leq d}, \quad u_\alpha = (u_{\alpha,1}, \ldots, u_{\alpha,p_1})^\top, \]

\[ R_{i-1}(h_n) := R(\theta, h_n, X_{i_{n-1}}), \quad G_i^\alpha := \sigma(W_s, s \leq t_i^\alpha). \]
It follows that
\[
U_n^{(1)}(\alpha) = -\frac{1}{2} \sum_{i=1}^{n} \left\{ h_n^{-1} S^{-1}(X_{t_i^n, \alpha}) \left[ (X_{t_i^n} - X_{t_{i-1}^n}) \right]^{\otimes 2} \right\} + \log \det S(X_{t_{i-1}^n, \alpha}) \right\} \\
= -\frac{1}{2h_n} \sum_{i=1}^{n} \sum_{p,q} \{ S_{i-1,pq}^{-1}(\alpha)(X_{i,p} - X_{i-1,p})(X_{i,q} - X_{i-1,q}) \} - \frac{1}{2} \sum_{i=1}^{n} \log \det S_{i-1}(\alpha).
\]

For \( l_1 = 1, \ldots, p_1 \), we have
\[
(\partial_{\alpha_i} S^{-1})_{i-1}(\alpha) = -(S^{-1}(\partial_{\alpha_i} S)S^{-1})_{i-1}(\alpha), \quad (\partial_{\alpha_i} \log \det S)_{i-1}(\alpha) = \text{tr} \left( (S^{-1}\partial_{\alpha_i} S)_{i-1}(\alpha) \right),
\]
and we set
\[
\partial_{\alpha_i} U_n^{(1)}(\alpha_0) := \sum_{i=1}^{n} (\xi_{i,1}^{l_1} + \xi_{i,2}^{l_1}),
\]
\[
\xi_{i,1}^{l_1} := \frac{1}{2h_n} \sum_{p,q} (S^{-1}(\partial_{\alpha_i} S)S^{-1})_{i-1,pq}(\alpha_0)(X_{i,p} - X_{i-1,p})(X_{i,q} - X_{i-1,q}),
\]
\[
\xi_{i,2}^{l_1} := -\frac{1}{2} \text{tr} \left( (S^{-1}\partial_{\alpha_i} S)_{i-1}(\alpha_0) \right).
\]

From Theorems 3.2 and 3.4 of Hall and Heyde \[7\], it is sufficient to show that under \( H_1^{(1)} \),
\[
\sum_{i=1}^{n} \mathbb{E}_{\theta_i^n} \left[ \frac{1}{\sqrt{n}} (\xi_{i,1}^{l_1} + \xi_{i,2}^{l_1}) \right] \to_P \sum_{j=1}^{p_1} I_j^{(l_1,j)}(\alpha_0; \alpha_0) u_{\alpha,j}, \quad \text{(32)}
\]
\[
\sum_{i=1}^{n} \mathbb{E}_{\theta_i^n} \left[ \frac{1}{n} (\xi_{i,1}^{l_1} + \xi_{i,2}^{l_1})(\xi_{i,1}^{l_2} + \xi_{i,2}^{l_2}) \right] \to_P I_0^{(l_1,l_2)}(\alpha_0; \alpha_0), \quad \text{(33)}
\]
\[
\sum_{i=1}^{n} \mathbb{E}_{\theta_i^n} \left[ \frac{1}{\sqrt{n}} (\xi_{i,1}^{l_1} + \xi_{i,2}^{l_2}) \right] \mathbb{E}_{\theta_i^n} \left[ \frac{1}{\sqrt{n}} (\xi_{i,1}^{l_2} + \xi_{i,2}^{l_2}) \right] \to_P 0, \quad \text{(34)}
\]
\[
\sum_{i=1}^{n} \mathbb{E}_{\theta_i^n} \left[ \frac{1}{n^2} (\xi_{i,1}^{l_1} + \xi_{i,2}^{l_2})^4 \right] \to_P 0, \quad \text{(35)}
\]
where \( l_1, l_2 = 1, \ldots, p_1 \).

Proof of (32). From the Itô-Taylor expansion and Lemma 7 of Kessler \[9\], we have
\[
\mathbb{E}_{\theta_i^n} [(X_{i,p} - X_{i-1,p})(X_{i,q} - X_{i-1,q})] = h_n S_{i-1,pq}(\alpha_1^n) + R_{i-1}(h_n^n),
\]
and it holds
\[
\mathbb{E}_{\theta_i^n}[\xi_{i,1}^{l_1} | G_{i-1}^n] = \frac{1}{2h_n} \sum_{p,q} (S^{-1}(\partial_{\alpha_i} S)S^{-1})_{i-1,pq}(\alpha_0) \mathbb{E}_{\theta_i^n}[ (X_{i,p} - X_{i-1,p})(X_{i,q} - X_{i-1,q}) | G_{i-1}^n]
\]
\[
= \frac{1}{2} \sum_{p,q} (S^{-1}(\partial_{\alpha_i} S)S^{-1})_{i-1,pq}(\alpha_0) (S_{i-1,pq}(\alpha_1^n) + R_{i-1}(h_n^n))
\]
\[
= \frac{1}{2} \sum_{p,q} (S^{-1}(\partial_{\alpha_i} S)S^{-1})_{i-1,pq}(\alpha_0) (\alpha_1^n) + R_{i-1}(h_n^n). \quad \text{(36)}
\]
By Taylor’s theorem, we obtain
\begin{align*}
S_{i-1,pq}(\alpha_{1,n}^*) &= S_{i-1,pq}(\alpha_0) + (\partial_\alpha S)_{i-1,pq}(\alpha_0)(\alpha_{1,n}^* - \alpha_0) \\
&\quad + (\alpha_{1,n}^* - \alpha_0)^\top \left( \int_0^1 u(\partial_\alpha^2 S)_{i-1,pq}(\alpha_0 + u(\alpha_{1,n}^* - \alpha_0))du \right) (\alpha_{1,n}^* - \alpha_0) \\
&= S_{i-1,pq}(\alpha_0) + \frac{1}{\sqrt{n}}(\partial_\alpha S)_{i-1,pq}(\alpha_0)u_\alpha \\
&\quad + \frac{1}{n}u_\alpha^\top \left( \int_0^1 u(\partial_\alpha^2 S)_{i-1,pq}(\alpha_0 + u(\alpha_{1,n}^* - \alpha_0))du \right) u_\alpha \\
&= S_{i-1,pq}(\alpha_0) + \frac{1}{\sqrt{n}}(\partial_\alpha S)_{i-1,pq}(\alpha_0)u_\alpha + \frac{1}{n}R_{i-1}(1).
\end{align*}

Note that \( \frac{h_n}{n} = nh_n > 1 \) and \( R_{i-1}(h_n) + \frac{1}{n}R_{i-1}(1) = R_{i-1}(h_n) \), it follows from (36) that
\begin{align*}
\mathbb{E}_{\theta_{1,n}^*}[\xi^{t_{i,1}}_1|\xi^{t_{i,2}}_1] &= \frac{1}{2} \sum_{p,q} (S^{-1}(\partial_{\alpha_{i,1}} S)^{-1})_{i-1,pq}(\alpha_0)S_{i-1,pq}(\alpha_0) \\
&\quad + \frac{1}{2\sqrt{n}} \sum_{p,q} (S^{-1}(\partial_{\alpha_{i,1}} S)^{-1})_{i-1,pq}(\alpha_0)(\partial_{\alpha} S)_{i-1,pq}(\alpha_0)u_\alpha + R_{i-1}(h_n) \\
&= \frac{1}{2} \text{tr} \left( (S^{-1}(\partial_{\alpha_{i,1}} S)^{-1})_{i-1}(\alpha_0) \right) \\
&\quad + \frac{1}{2\sqrt{n}} \sum_{p,q} (S^{-1}(\partial_{\alpha_{i,1}} S)^{-1})_{i-1,pq}(\alpha_0) \left( \sum_{j=1}^{p_1} (\partial_{\alpha_j} S)_{i-1,pq}(\alpha_0) \right) u_{\alpha,j} \\
&\quad + R_{i-1}(h_n) \\
&= \frac{1}{2} \text{tr} \left( (S^{-1}(\partial_{\alpha_{i,1}} S))_{i-1}(\alpha_0) \right) \\
&\quad + \frac{1}{2\sqrt{n}} \sum_{j=1}^{p_1} \text{tr} \left( (S^{-1}(\partial_{\alpha_{i,1}} S)^{-1}(\partial_{\alpha_j} S))_{i-1}(\alpha_0) \right) u_{\alpha,j} + R_{i-1}(h_n).
\end{align*}

On the other hand, we have
\begin{align*}
\mathbb{E}_{\theta_{1,n}^*}[\xi^{t_{i,1}}_1|\xi^{t_{i,2}}_1] &= -\frac{1}{2} \text{tr} \left( (S^{-1}\partial_{\alpha_{i,1}} S)_{i-1}(\alpha_0) \right).
\end{align*}

Hence, it follows from Lemma 8 of Kessler [2] that under \( H_{\alpha_0,n}^{(1)} \),
\begin{align*}
\sum_{i=1}^{n} \mathbb{E}_{\theta_{1,n}^*} \left[ \frac{1}{\sqrt{n}}(\xi^{t_{i,1}}_1 + \xi^{t_{i,2}}_1)|\xi^{t_{i,2}}_1 \right] &= \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{1}{2} \sum_{j=1}^{p_1} \text{tr} \left( (S^{-1}(\partial_{\alpha_{i,1}} S)^{-1}(\partial_{\alpha_j} S))_{i-1}(\alpha_0) \right) u_{\alpha,j} \right\} \\
&\quad + \frac{1}{n} \sum_{i=1}^{n} R_{i-1}(\sqrt{nh_n^2}) \\
&\quad + \frac{1}{n} \sum_{i=1}^{n} \int \text{tr} \left( (S^{-1}(\partial_{\alpha_{i,1}} S)^{-1}(\partial_{\alpha_j} S))(x,\alpha_0) \right) \mu_\alpha(dx)u_{\alpha,j} \\
&= \sum_{j=1}^{p_1} \int I^{(i,j)}(\alpha_0;\alpha_0)u_{\alpha,j}.
\end{align*}

By setting
\begin{align*}
Q_n := \sum_{i=1}^{n} \mathbb{E}_{\theta_{1,n}^*} \left[ \frac{1}{\sqrt{n}}(\xi^{t_{i,1}}_1 + \xi^{t_{i,2}}_1)|\xi^{t_{i,2}}_1 \right] - \sum_{i=1}^{p_1} \int I^{(i,j)}(\alpha_0;\alpha_0)u_{\alpha,j},
\end{align*}
the above convergence implies that for all \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} P_{\theta_n}(|Q_n| \geq \varepsilon) = 0.
\]
Hence it follows from C1 that
\[
\lim_{n \to \infty} P_{\theta_{1,n}}(|Q_n| \geq \varepsilon) = 0,
\]
which implies (32).

Proof of (33). From the Itô-Taylor expansion, it holds that for \( 1 \leq p, q, r, s \leq p_1 \),
\[
\mathbb{E}_{\theta_{1,n}}[(X_{i,p} - X_{i-1,p})(X_{i,q} - X_{i-1,q})(X_{i,r} - X_{i-1,r})(X_{i,s} - X_{i-1,s}) | G^n_{i-1}]
= h_n^2(S_{i-1,pq}(\alpha^*_{1,n})S_{i-1,rs}(\alpha^*_{1,n}) + S_{i-1,pr}(\alpha^*_{1,n})S_{i-1,qs}(\alpha^*_{1,n}) + S_{i-1,ps}(\alpha^*_{1,n})S_{i-1,qr}(\alpha^*_{1,n}))
+ R_{i-1}(h_n^3).
\]
Noting that \( \frac{h_n}{\sqrt{n}} = \sqrt{n}h_n^2 < 1 \), we obtain
\[
\mathbb{E}_{\theta_{1,n}}[\ell^1_{i,1}\ell^2_{i,1} | G^n_{i-1}] = \frac{1}{4h_n^2} \sum_{p,q,r,s} (\partial_{\alpha_{12}} S^{-1})_{i-1,pq}(\alpha_0)(\partial_{\alpha_{12}} S^{-1})_{i-1,rs}(\alpha_0)
+ \mathbb{E}_{\theta_{1,n}}[(X_{i,p} - X_{i-1,p})(X_{i,q} - X_{i-1,q})(X_{i,r} - X_{i-1,r})(X_{i,s} - X_{i-1,s}) | G^n_{i-1}]
= \frac{1}{4} \sum_{p,q,r,s} (\partial_{\alpha_{12}} S^{-1})_{i-1,pq}(\alpha_0)(\partial_{\alpha_{12}} S^{-1})_{i-1,rs}(\alpha_0)
\times (S_{i-1,pq}(\alpha^*_{1,n})S_{i-1,rs}(\alpha^*_{1,n}) + S_{i-1,pr}(\alpha^*_{1,n})S_{i-1,qs}(\alpha^*_{1,n})
+ S_{i-1,ps}(\alpha^*_{1,n})S_{i-1,qr}(\alpha^*_{1,n}))
+ R_{i-1}(1).
\]

Since
\[
\sum_{p,q,r,s} (\partial_{\alpha_{12}} S^{-1})_{i-1,pq}(\alpha_0)(\partial_{\alpha_{12}} S^{-1})_{i-1,rs}(\alpha_0)S_{i-1,pq}(\alpha_0)S_{i-1,rs}(\alpha_0)
= \text{tr} \left( (\partial_{\alpha_{12}} S^{-1})_{i-1}(\alpha_0) \right) \text{tr} \left( (\partial_{\alpha_{12}} S^{-1})_{i-1}(\alpha_0) \right),
\]
\[
\sum_{p,q,r,s} (\partial_{\alpha_{12}} S^{-1})_{i-1,pq}(\alpha_0)(\partial_{\alpha_{12}} S^{-1})_{i-1,rs}(\alpha_0)S_{i-1,pr}(\alpha_0)S_{i-1,qs}(\alpha_0)
= \sum_{p,q,r,s} (\partial_{\alpha_{12}} S^{-1})_{i-1,pq}(\alpha_0)(\partial_{\alpha_{12}} S^{-1})_{i-1,rs}(\alpha_0)S_{i-1,pr}(\alpha_0)S_{i-1,qs}(\alpha_0)
= \text{tr} \left( (\partial_{\alpha_{12}} S^{-1})_{i-1}(\alpha_0) \right),
\]

it follows from (37) that
\[
\mathbb{E}_{\theta_{1,n}}[\ell^1_{i,1}\ell^2_{i,1} | G^n_{i-1}] = \frac{1}{4} \text{tr} \left( (\partial_{\alpha_{12}} S^{-1})_{i-1}(\alpha_0) \right) \text{tr} \left( (\partial_{\alpha_{12}} S^{-1})_{i-1}(\alpha_0) \right)
+ \frac{1}{2} \text{tr} \left( (\partial_{\alpha_{12}} S S^{-1})(\partial_{\alpha_{12}} S S^{-1})_{i-1}(\alpha_0) \right) + \frac{1}{\sqrt{n}} R_{i-1}(1).
\]
Moreover, we have

\[ E_{\theta_i,n}[\xi^{i_1}_1,\xi^{i_2}_2|\theta_i^n] = \frac{1}{4}\tr((S^{-1}\partial_{\alpha_1}S)_{i-1}(\alpha_0)) \tr((S^{-1}\partial_{\alpha_2}S)_{i-1}(\alpha_0)), \]

\[ E_{\theta_i,n}[\xi^{i_1}_{i_1},\xi^{i_2}_{i_2}|\theta_i^n] = \frac{1}{4h_n}\tr((S^{-1}\partial_{\alpha_2}S)_{i-1}(\alpha_0)) \sum_{p,q}(S^{-1}(\partial_{\alpha_1}S)S^{-1})_{i-1,pq}(\alpha_0) \]

\[ \quad \times E_{\theta_i,n}[(X_{i,p} - X_{i-1,p})(X_{i,q} - X_{i-1,q})|\theta_i^n] \]

\[ = \frac{1}{4}\tr((S^{-1}\partial_{\alpha_2}S)_{i-1}(\alpha_0)) \sum_{p,q}(S^{-1}(\partial_{\alpha_1}S)S^{-1})_{i-1,pq}(\alpha_0)S_{i-1,pq}(\alpha_0) \]

\[ + \frac{1}{\sqrt{n}}R_{i-1}(1) \]

\[ = \frac{1}{4}\tr((S^{-1}\partial_{\alpha_2}S)_{i-1}(\alpha_0)) \tr((S^{-1}\partial_{\alpha_2}S)_{i-1}(\alpha_0)) + \frac{1}{\sqrt{n}}R_{i-1}(1). \]

Therefore, it holds that under \( H^{(1)}_{0,n} \),

\[ \sum_{i=1}^{n} E_{\theta_i,n}\left[ \frac{1}{n}(\xi^{i_1}_{i_1} + \xi^{i_2}_{i_2})(\xi^{i_1}_{i_1} + \xi^{i_2}_{i_2})|\theta_i^n \right] = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2}\tr((\partial_{\alpha_1}S^{-1})S(\partial_{\alpha_2}S^{-1})S)_{i-1}(\alpha_0)) \]

\[ + \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{n}}R_{i-1}(1) \]

\[ \xrightarrow{P} \frac{1}{2} \int \tr((\partial_{\alpha_1}S^{-1})S(\partial_{\alpha_2}S^{-1})S)(x, \alpha_0) \mu_{\theta_0}(dx) \]

\[ = \frac{1}{2} \int \tr((S^{-1}(\partial_{\alpha_1}S)S^{-1}(\partial_{\alpha_2}S))(x, \alpha_0) \mu_{\theta_0}(dx) \]

\[ = f_{\alpha_1}(1)(\alpha_0; \alpha_0). \]

From C1, this convergence also holds under \( H^{(1)}_{1,n} \).

Proof of (34). From the proof of (32), we have

\[ E_{\theta_i,n}\left[ \frac{1}{\sqrt{n}}(\xi^{i_1}_{i_1} + \xi^{i_2}_{i_2})|\theta_i^n \right] = \frac{1}{n} R_{i-1}(1), \]

\[ E_{\theta_i,n}\left[ \frac{1}{\sqrt{n}}(\xi^{i_2}_{i_1} + \xi^{i_2}_{i_2})|\theta_i^n \right] = \frac{1}{n} R_{i-1}(1). \]

Therefore, it holds that under \( H^{(1)}_{0,n} \),

\[ \sum_{i=1}^{n} E_{\theta_i,n}\left[ \frac{1}{\sqrt{n}}(\xi^{i_1}_{i_1} + \xi^{i_2}_{i_2})|\theta_i^n \right] E_{\theta_i,n}\left[ \frac{1}{\sqrt{n}}(\xi^{i_2}_{i_1} + \xi^{i_2}_{i_2})|\theta_i^n \right] = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{n}}R_{i-1}(1) \xrightarrow{P} 0, \]

and this also holds under \( H^{(1)}_{1,n} \).

Proof of (35). It is easy to show

\[ E_{\theta_i,n}\left[ \frac{1}{n^2}(\xi^{i_1}_{i_1} + \xi^{i_2}_{i_2})^4|\theta_i^n \right] \leq \frac{2^3}{n^2} \left( E_{\theta_i,n}\left[ (\xi^{i_1}_{i_1})^4|\theta_i^n \right] + E_{\theta_i,n}\left[ (\xi^{i_2}_{i_2})^4|\theta_i^n \right] \right). \]
We can evaluate

$$\mathbb{E}_{\theta_{t,n}} \left[ (\xi_{i,1}^4 | G^n_{t-1} \right] = \frac{1}{16h_n^4} \mathbb{E}_{\theta_{t,n}} \left[ (\sum_{p,q} (\partial_{\alpha_{i1}} S^{-1})_{i-1,pq}(\alpha_0) \right.
\times (X_{i,p} - X_{i-1,q})(X_{i,q} - X_{i-1,q}))^4 | G^n_{t-1} \right]$$

$$\leq \frac{C}{16h_n^4} (1 + |X_{t-1,n}|)^C \sum_{1 \leq p_1, \ldots, p_8 \leq d} \mathbb{E}_{\theta_{t,n}} \left[ \prod_{j=1}^8 (X_{i,j} - X_{i-1,j}) | G^n_{t-1} \right]$$

$$\leq R_{i-1}(h_n^{-4}) \sum_{1 \leq p_1, \ldots, p_8 \leq d, j=1}^8 \left( \mathbb{E}_{\theta_{t,n}} \left[ (X_{i,j} - X_{i-1,j})^8 \right] \right)^{\frac{1}{8}}$$

$$\leq d^8 R_{i-1}(h_n^{-4}) \mathbb{E}_{\theta_{t,n}} \left[ |X_{t,n} - X_{t-1,n}|^8 \right] | G^n_{t-1}$$

$$\leq R_{i-1}(1),$$

$$\mathbb{E}_{\theta_{t,n}} \left[ (\xi_{i,1}^4 | G^n_{t-1} \right] = \frac{1}{16} \left( \text{tr} \left( (S^{-1}\partial_{\alpha_{i1}} S)_{i-1}(\alpha_0) \right) \right)^4$$

$$= R_{i-1}(1).$$

Hence

$$\sum_{i=1}^n \frac{\phi^3}{n^2} \left( \mathbb{E}_{\theta_{t,n}} \left[ (\xi_{i,1}^4 | G^n_{t-1} \right] + \mathbb{E}_{\theta_{t,n}} \left[ (\xi_{i,2}^4 | G^n_{t-1} \right] \right) = \frac{1}{n} \sum_{i=1}^n \frac{1}{n} R_{i-1}(1) \rightarrow 0,$$

and

$$\sum_{i=1}^n \mathbb{E}_{\theta_{t,n}} \left[ \frac{1}{n^2} \left( \xi_{i,1}^4 + \xi_{i,2}^4 \right) | G^n_{t-1} \right] \rightarrow 0 \quad \text{(under } H_{0,n}^{(1)}).$$

From C1, the last convergence also holds under $H_{1,n}^{(1)}$. By using (32)-(35), we get (31).

Next we prove (i). It follows from Taylor’s theorem that

$$\left. -\frac{1}{\sqrt{n}} \partial_{\alpha} U_n^{(1)}(\alpha_0) \right| = \int_0^1 \frac{1}{n} \partial_{\alpha}^2 U_n^{(1)}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)) du \sqrt{n}(\hat{\alpha}_n - \alpha_0).$$

It holds

$$\int_0^1 \frac{1}{n} \partial_{\alpha}^2 U_n^{(1)}(\alpha_0 + u(\hat{\alpha}_n - \alpha_0)) du \rightarrow -I_\alpha(\alpha_0; \alpha_0) \quad \text{(under } H_{0,n}^{(1)}),$$

which also holds under $H_{1,n}^{(1)}$. By noting that $I_\alpha(\alpha_0; \alpha_0)$ is non-singular, it follows from Slutsky’s theorem that

$$\sqrt{n}(\hat{\alpha}_n - \alpha_0) \overset{d}{\rightarrow} I^{-1}_\alpha(\alpha_0; \alpha_0) L_1 = Y_1 \quad \text{(under } H_{1,n}^{(1)}).$$

This completes the proof of (i).

(ii) First of all, we will prove that

$$\left. \frac{1}{\sqrt{nh_n}} \partial_{\beta} U_n^{(2)}(\beta_0 | \alpha_0) \right| \overset{d}{\rightarrow} L_2 \sim N(I_\beta(\theta_0; \theta_0) u_\beta, I_\beta(\theta_0; \theta_0)) \quad \text{(under } H_{1,n}^{(2)}). \quad (38)$$

Letting $b(X_{t-1,n}, \beta) = (b_{1-1,1}(\beta), \ldots, b_{1-1,d}(\beta))^\top$ and $u_\beta = (u_{\beta,1}, \ldots, u_{\beta,p_2})^\top$, one has that

$$U_n^{(2)}(\beta | \alpha) = -\frac{1}{2} \sum_{i=1}^n \left\{ h_n^{-1} S^{-1}(X_{t-1,n}, \alpha) \left[ (X_{t,n} - X_{t-1,n} - h_n b(X_{t-1,n}, \beta))^2 \right] \right\}$$

$$= -\frac{1}{2h_n} \sum_{i=1}^n \sum_{p,q} S_{t-1,pq}^{-1}(\alpha)(X_{i,p} - X_{i-1,p} - h_n b_{i-1,p}(\beta))(X_{i,q} - X_{i-1,q} - h_n b_{i-1,q}(\beta)).$$
For $m_1 = 1, \ldots, p_2$, we have
\[
\partial_{\beta_{m_1}} U_n^{(2)}(\beta_0|\alpha_0) = \sum_{i=1}^{n} \eta_{i,1}^{m_1}
\]
\[
\eta_{i,1}^{m_1} = \sum_{p,q} S_{i-1,pq}^{-1}(\alpha_0)(\partial_{\beta_{m_1}} b)_{i-1,q}(\beta_0)(X_{i,p} - X_{i-1,p} - h_n b_{i-1,p}(\beta_0)).
\]

From Theorems 3.2 and 3.4 of Hall and Heyde [7] and C1, it is sufficient to show that under $H_0^{(2)}$,
\[
\sum_{i=1}^{n} \mathbb{E}_{\theta_i^n} \left[ \frac{1}{\sqrt{nh_n}} \eta_{i,1}^{m_1} | G_{i-1}^n \right] \xrightarrow{P} \sum_{j=1}^{p_2} I_b^{(m_1)}(\theta_0; \theta_0) u_{\beta,j}, \tag{39}
\]
\[
\sum_{i=1}^{n} \mathbb{E}_{\theta_i^n} \left[ \frac{1}{nh_n} \eta_{i,1}^{m_1} \eta_{i,1}^{m_2} | G_{i-1}^n \right] \xrightarrow{P} f_b^{(m_1m_2)}(\theta_0; \theta_0), \tag{40}
\]
\[
\sum_{i=1}^{n} \mathbb{E}_{\theta_i^n} \left[ \frac{1}{\sqrt{nh_n}} \eta_{i,1}^{m_1} | G_{i-1}^n \right] \mathbb{E}_{\theta_i^n} \left[ \frac{1}{\sqrt{nh_n}} \eta_{i,1}^{m_2} | G_{i-1}^n \right] \xrightarrow{P} 0, \tag{41}
\]
\[
\sum_{i=1}^{n} \mathbb{E}_{\theta_i^n} \left[ \frac{1}{(nh_n)^2} (\eta_{i,1}^{m_1})^2 | G_{i-1}^n \right] \xrightarrow{P} 0, \tag{42}
\]
where $m_1, m_2 = 1, \ldots, p_2$.

Proof of (39) By the Itô-Taylor expansion, we have
\[
\mathbb{E}_{\theta_i^n}[X_{i,p} - X_{i-1,p}|G_{i-1}^n] = h_n b_{i-1,p}(\beta_{1,n}) + R_{i-1}(h_n^2),
\]
and
\[
\mathbb{E}_{\theta_i^n}[\eta_{i,1}^{m_1} | G_{i-1}^n] = \sum_{p,q} S_{i-1,pq}^{-1}(\alpha_0)(\partial_{\beta_{m_1}} b)_{i-1,q}(\beta_0) \left( \mathbb{E}_{\theta_i^n}[X_{i,p} - X_{i-1,p}|G_{i-1}^n] - h_n b_{i-1,p}(\beta_0) \right)
\]
\[
= h_n \sum_{p,q} S_{i-1,pq}^{-1}(\alpha_0)(\partial_{\beta_{m_1}} b)_{i-1,q}(\beta_0)(b_{i-1,p}(\beta_{1,n}) - b_{i-1,p}(\beta_0)) + R_{i-1}(h_n^2). \tag{43}
\]

Since it follows from Taylor’s theorem that
\[
b_{i-1,p}(\beta_{1,n}) - b_{i-1,p}(\beta_0) = (\partial_{\beta} b)_{i-1,p}(\beta_0)(\beta_{1,n} - \beta_0)
\]
\[
= (\beta_{1,n} - \beta_0)^T \left( \int_0^1 u(\partial_{\beta} b)_{i-1,p}(\beta_0 + u(\beta_{1,n} - \beta_0)) du \right) (\beta_{1,n} - \beta_0)
\]
\[
= \frac{1}{\sqrt{nh_n}} (\partial_{\beta} b)_{i-1,p}(\beta_0) u_{\beta} + \frac{1}{nh_n} R_{i-1}(1), \tag{44}
\]
(43) implies that
\[
\mathbb{E}_{\theta_i^n}[\eta_{i,1}^{m_1} | G_{i-1}^n] = \sqrt{\frac{h_n}{n}} \sum_{p,q} S_{i-1,pq}^{-1}(\alpha_0)(\partial_{\beta_{m_1}} b)_{i-1,q}(\beta_0)(\partial_{\beta} b)_{i-1,p}(\beta_0) u_{\beta} + \frac{1}{n} R_{i-1}(1)
\]
\[
= \sqrt{\frac{h_n}{n}} \sum_{j=1}^{p_2} \sum_{p,q} S_{i-1,pq}^{-1}(\alpha_0)(\partial_{\beta_{m_1}} b)_{i-1,q}(\beta_0)(\partial_{\beta} b)_{i-1,p}(\beta_0) u_{\beta,j} + \frac{1}{n} R_{i-1}(1)
\]
\[
= \sqrt{\frac{h_n}{n}} \sum_{j=1}^{p_2} (\partial_{\beta_{m_1}} b)_{i-1}(\beta_0) S_{i-1}^{-1}(\alpha_0)(\partial_{\beta} b)_{i-1}(\beta_0) u_{\beta,j} + \frac{1}{n} R_{i-1}(1). 
\]
Therefore, it follows from Lemma 8 of Kessler [9] that under $H_{0,n}^{(1)}$,
\[
\sum_{i=1}^{n} \mathbb{E}_{\theta_{i}} \left[ \frac{1}{\sqrt{n} h_{n}} \eta_{i,1}^{m_1} \eta_{i,1}^{m_2} \right] = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{p_2} \left( (\partial_{\beta_{m_1}} b_{i-1}(\beta_0))^{\top} S_{i-1}^{-1}(\alpha_0) (\partial_{\beta_{m_2}} b_{i-1}(\beta_0) u_{\beta,j} + \frac{1}{\sqrt{n} R_{i-1}(1)} \right)
\]
\[
\sum_{j=1}^{p_2} \int \left( (\partial_{\beta_{m_1}} b_{i-1}(\beta_0))^{\top} S_{i-1}^{-1}(x, \alpha_0) (\partial_{\beta_{m_2}} b_{i-1}(\beta_0)) \mu_{\theta_0}(dx) u_{\beta,j}
\]
\[
= \sum_{j=1}^{p_2} I_{(m_1)}^{(1)} (\theta_0; \theta_0) u_{\beta,j}.
\]

Proof of (40). By the Itô-Taylor expansion, one has that
\[
\mathbb{E}_{\theta_{i}} \left[ (X_{i,p} - X_{i-1,p} - h_{n} b_{i-1,p}(\beta_0)) (X_{i,q} - X_{i-1,q} - h_{n} b_{i-1,q}(\beta_0)) | \mathcal{G}_{i-1} \right] = h_{n} S_{i-1,pq}(\alpha_0) + \frac{1}{\sqrt{n}} R_{i-1}(h_{n}).
\]

Setting $\delta_{ps} := \begin{cases} 1 & (p = s), \\ 0 & (p \neq s), \end{cases}$ we obtain that
\[
\mathbb{E}_{\theta_{i}} \left[ [\eta_{i,1}^{m_1} \eta_{i,1}^{m_2} | \mathcal{G}_{i-1}^{n}] \right] = \sum_{p,q,r,s} S_{i-1,pq}(\alpha_0) S_{i-1,r,s}(\alpha_0) (\partial_{\beta_{m_1}} b_{i-1,q}(\beta_0)) (\partial_{\beta_{m_2}} b_{i-1,s}(\beta_0))
\]
\[
\times \mathbb{E}_{\theta_{i}} \left[ [(X_{i,p} - X_{i-1,p} - h_{n} b_{i-1,p}(\beta_0)) (X_{i,q} - X_{i-1,q} - h_{n} b_{i-1,q}(\beta_0)) | \mathcal{G}_{i-1}^{n}] \right]
\]
\[
= h_{n} \sum_{p,q,r,s} S_{i-1,pq}(\alpha_0) S_{i-1,r,s}(\alpha_0) S_{i-1,pr}(\alpha_0)
\]
\[
\times (\partial_{\beta_{m_1}} b_{i-1,q}(\beta_0)) (\partial_{\beta_{m_2}} b_{i-1,s}(\beta_0)) + \frac{1}{\sqrt{n}} R_{i-1}(h_{n})
\]
\[
= h_{n} \sum_{p,q,s} S_{i-1,pq}(\alpha_0) \left( \sum_{r=1}^{d} \sum_{i-1,r,s}(\alpha_0) S_{i-1,rp}(\alpha_0) \right)
\]
\[
\times (\partial_{\beta_{m_1}} b_{i-1,q}(\beta_0)) (\partial_{\beta_{m_2}} b_{i-1,s}(\beta_0)) + \frac{1}{\sqrt{n}} R_{i-1}(h_{n})
\]
\[
= h_{n} \sum_{p,q} S_{i-1,pq}(\alpha_0) \delta_{ps} (\partial_{\beta_{m_1}} b_{i-1,q}(\beta_0)) (\partial_{\beta_{m_2}} b_{i-1,s}(\beta_0)) + \frac{1}{\sqrt{n}} R_{i-1}(h_{n})
\]
\[
= h_{n} \sum_{p,q} S_{i-1,pq}(\alpha_0) (\partial_{\beta_{m_1}} b_{i-1,q}(\beta_0)) (\partial_{\beta_{m_2}} b_{i-1,p}(\beta_0)) + \frac{1}{\sqrt{n}} R_{i-1}(h_{n})
\]
\[
= h_{n} \left( (\partial_{\beta_{m_1}} b_{i-1}(\beta_0))^{\top} S_{i-1}^{-1}(\alpha_0) ((\partial_{\beta_{m_2}} b_{i-1}(\beta_0)) + \frac{1}{\sqrt{n}} R_{i-1}(h_{n}).
\]

Hence it holds
\[
\sum_{i=1}^{n} \mathbb{E}_{\theta_{i}} \left[ \frac{1}{\sqrt{n} h_{n}} \eta_{i,1}^{m_1} \eta_{i,1}^{m_2} | \mathcal{G}_{i-1}^{n} \right] = \frac{1}{n} \sum_{i=1}^{n} \left( (\partial_{\beta_{m_1}} b_{i-1}(\beta_0))^{\top} S_{i-1}^{-1}(\alpha_0) ((\partial_{\beta_{m_2}} b_{i-1}(\beta_0))
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{n}} R_{i-1}(1)
\]
\[
P \int (\partial_{\beta_{m_1}} b(x, \beta_0))^{\top} S_{i-1}^{-1}(x, \alpha_0) (\partial_{\beta_{m_2}} b(x, \beta_0)) \mu_{\theta_0}(dx)
\]
\[
= I_{b_{m_1 m_2}}^{(1)} (\theta_0; \theta_0).
\]
Proof of (41). From the proof of (39), we have
\[ \mathbb{E}_{\theta_{1,n}^*}\left[ \frac{1}{\sqrt{n}h_n} \eta_{1,1}^{m_1} | \mathcal{G}_{i-1} \right] = \frac{1}{n} R_{i-1}(1), \]
\[ \mathbb{E}_{\theta_{1,n}^*}\left[ \frac{1}{\sqrt{n}h_n} \eta_{1,1}^{m_2} | \mathcal{G}_{i-1} \right] = \frac{1}{n} R_{i-1}(1). \]
Hence
\[ \sum_{i=1}^{n} \mathbb{E}_{\theta_{1,n}^*}\left[ \frac{1}{\sqrt{n}h_n} \eta_{1,1}^{m_1} | \mathcal{G}_{i-1} \right] \mathbb{E}_{\theta_{1,n}^*}\left[ \frac{1}{\sqrt{n}h_n} \eta_{1,1}^{m_2} | \mathcal{G}_{i-1} \right] = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n} R_{i-1}(1) \xrightarrow{P} 0. \]

Proof of (42). It follows from the Itô-Taylor expansion that
\[ \mathbb{E}_{\theta_{1,n}^*}\left[ \prod_{j=1}^{4} (X_{i,p_j} - X_{i-1,p_j} - h_n b_{i-1,p_j} (\beta_0)) | \mathcal{G}_{i-1} \right] = R_{i-1}(h_n^2). \]
Hence we can evaluate
\[ \mathbb{E}_{\theta_{1,n}^*}\left[ (\eta_{1,1}^{m_1})^4 | \mathcal{G}_{i-1} \right] = \mathbb{E}_{\theta_{1,n}^*}\left[ \left( \sum_{p,q} S_{i-1,pq} (\alpha_0) (\partial_{b_{m_1}} b_{i-1,q} (\beta_0)) \times (X_{i,p} - X_{i-1,p} - h_n b_{i-1,p} (\beta_0))^4 | \mathcal{G}_{i-1} \right) \right] \]
\[ \leq R_{i-1}(1) \sum_{1 \leq p_1, \ldots, p_4 \leq d} \mathbb{E}_{\theta_{1,n}^*}\left[ \prod_{j=1}^{4} (X_{i,p_j} - X_{i-1,p_j} - h_n b_{i-1,p_j} (\beta_0)) | \mathcal{G}_{i-1} \right] \]
\[ \leq R_{i-1}(h_n^2). \]
Therefore it holds
\[ 0 \leq \sum_{i=1}^{n} \mathbb{E}_{\theta_{1,n}^*}\left[ \frac{1}{(n h_n)^2} (\eta_{1,1}^{m_1})^4 | \mathcal{G}_{i-1} \right] \leq \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n} R_{i-1}(1) \xrightarrow{P} 0, \]
and
\[ \sum_{i=1}^{n} \mathbb{E}_{\theta_{1,n}^*}\left[ \frac{1}{(n h_n)^2} (\eta_{1,1}^{m_1})^4 | \mathcal{G}_{i-1} \right] \xrightarrow{P} 0. \]

Using (39)-(42), we obtain (38).

Next, one shows (ii). From Taylor’s theorem with respect to \( \beta \), we have
\[ -\frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0) | \hat{\alpha}_n \right) = \int_{0}^{1} \frac{1}{n h_n} \partial_\beta^2 U_n^{(2)}(\beta_0 + u(\hat{\beta}_n - \beta_0)) \right) \right) du \sqrt{n h_n} (\hat{\beta}_n - \beta_0), \] (44)

and Taylor’s theorem with respect to \( \alpha \) yields that
\[ \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0) | \alpha_0 \right) = \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0) | \alpha_0 \right) \]
\[ + \int_{0}^{1} \frac{1}{n \sqrt{n}h_n} \partial_\alpha^2 U_n^{(2)}(\beta_0 + u(\hat{\alpha}_n - \alpha_0)) \right) \right) du \sqrt{n} (\hat{\alpha}_n - \alpha_0). \] (45)

One has that under \( H_{0,n}^{(2)} \)
\[ \int_{0}^{1} \frac{1}{n h_n} \partial_\beta^2 U_n^{(2)}(\beta_0 + u(\hat{\beta}_n - \beta_0)) \right) \right) du \xrightarrow{P} -I_0(\theta_0, \theta_0), \]
\[ \int_{0}^{1} \frac{1}{n \sqrt{n}h_n} \partial_\alpha^2 U_n^{(2)}(\beta_0 + u(\hat{\alpha}_n - \alpha_0)) \right) \right) du \xrightarrow{P} 0, \]
which also hold under \( H_{1,n}^{(2)} \) by C1. Therefore, it follows from (44) and (45) that
\[ -\int_{0}^{1} \frac{1}{n h_n} \partial_\beta^2 U_n^{(2)}(\beta_0 + u(\hat{\beta}_n - \beta_0)) \right) \right) du \sqrt{n h_n} (\hat{\beta}_n - \beta_0) = \frac{1}{\sqrt{n}h_n} \partial_\beta U_n^{(2)}(\beta_0) | \alpha_0 \right) + o_P(1). \]
Moreover, \( C_1 \) is non-singular, by Slutsky’s theorem, one has that
\[
\sqrt{n}h_n(\hat{\beta}_n - \beta_0) \xrightarrow{d} I_0^{-1}(\theta_0; \theta_0)L_2 = Y_2' \sim N(u_\beta, I_0^{-1}(\theta_0; \theta_0)) \quad \text{(under } H_{1,n}^{(2)}). \]
This completes the proof of (ii).

**Proof of Theorem 3.** (i) By Taylor’s theorem,
\[
U_n^{(1)}(\alpha_0) - U_n^{(1)}(\hat{\alpha}_n) = \left(\sqrt{n}(\hat{\alpha}_n - \alpha_0)\right)^\top J_{a,n}(\alpha_0, \hat{\alpha}_n) \sqrt{n}(\hat{\alpha}_n - \alpha_0).
\]
It follows from Lemma 2 that
\[
\sqrt{n}(\hat{\alpha}_n - \alpha_0) \xrightarrow{d} Y_1' \sim N(u_\alpha, I_0^{-1}(\alpha_0; \alpha_0)) \quad \text{(under } H_{1,n}^{(1)}). \]
Moreover, C1 and
\[
J_{a,n}(\alpha_0, \hat{\alpha}_n) \xrightarrow{P} -\frac{1}{2} I_a(\alpha_0; \alpha_0) \quad \text{(under } H_{0,n}^{(1)}), \]
implies
\[
J_{a,n}(\alpha_0, \hat{\alpha}_n) \xrightarrow{P} -\frac{1}{2} I_a(\alpha_0; \alpha_0) \quad \text{(under } H_{1,n}^{(1)}). \]
By noting that \( Y_1' = I_a^{-\frac{1}{2}}(\alpha_0; \alpha_0)Z_1' \), \( Z_1' \sim N(I_0^{\frac{1}{2}}(\alpha_0; \alpha_0)u_\alpha, E_{p_1}) \), it follows from the continuous mapping theorem that
\[
\Lambda_n^{(1)} = -2(U_n^{(1)}(\alpha_0) - U_n^{(1)}(\hat{\alpha}_n)) \xrightarrow{d} Y_1^\top I_a(\alpha_0; \alpha_0)Y_1' = Z_1^\top Z_1' \sim \chi^2_{p_1}(u_{\alpha}^\top I_a(\alpha_0; \alpha_0)u_\alpha) \quad \text{(under } H_{1,n}^{(1)}). \]
Moreover, by Taylor’s theorem with respect to \( \beta \), we have
\[
U_n^{(2)}(\beta_0|\hat{\alpha}_n) - U_n^{(2)}(\hat{\beta}_n|\hat{\alpha}_n) = \left(\sqrt{n}h_n(\hat{\beta}_n - \beta_0)\right)^\top J_{b,n}(\beta_0, \hat{\beta}_n) \sqrt{n}h_n(\hat{\beta}_n - \beta_0).
\]
It follows from Lemma 2 that
\[
\sqrt{n}h_n(\hat{\beta}_n - \beta_0) \xrightarrow{d} Y_2' \sim N(u_\beta, I_0^{-1}(\theta_0; \theta_0)) \quad \text{(under } H_{1,n}^{(2)}). \]
Next, we can check
\[
J_{b,n}(\beta_0, \hat{\beta}_n) \xrightarrow{P} -\frac{1}{2} I_b(\theta_0; \theta_0) \quad \text{(under } H_{1,n}^{(2)}). \]
Note that \( Y_2' = I_b^{-\frac{1}{2}}(\theta_0; \theta_0)Z_2' \), \( Z_2' \sim N(I_b^{\frac{1}{2}}(\theta_0; \theta_0)u_\beta, E_{p_2}) \). By the continuous mapping theorem, one has that
\[
\Lambda_n^{(2)} = -2(U_n^{(2)}(\beta_0|\hat{\alpha}_n) - U_n^{(2)}(\hat{\beta}_n|\hat{\alpha}_n)) \xrightarrow{d} Y_2^\top I_b(\theta_0; \theta_0)Y_2' = Z_2^\top Z_2' \sim \chi^2_{p_2}(u_{\beta}^\top I_b(\theta_0; \theta_0)u_{\beta}) \quad \text{(under } H_{1,n}^{(2)}). \]
This completes the proof of the Likelihood type.

(ii) From C1, we have
\[
I_{a,n}(\hat{\alpha}_n) \xrightarrow{P} I_a(\alpha_0; \alpha_0) \quad \text{(under } H_{1,n}^{(1)}). \]
Therefore, it follows from Lemma 2 and the continuous mapping theorem that
\[
W_n^{(1)} = \left(\sqrt{n}(\hat{\alpha}_n - \alpha_0)\right)^\top I_{a,n}(\hat{\alpha}_n) \sqrt{n}(\hat{\alpha}_n - \alpha_0) \xrightarrow{d} Y_1^\top I_a(\alpha_0; \alpha_0)Y_1' \sim \chi^2_{p_1}(u_{\alpha}^\top I_a(\alpha_0; \alpha_0)u_{\alpha}) \quad \text{(under } H_{1,n}^{(1)}). \]
Furthermore, we obtain
\[ I_{b,n}(\hat{\beta}_n | \hat{\alpha}_n) \overset{P}{\to} I_b(\theta_0; \theta_0) \quad \text{(under } H_{0,n}^{(2)}) \]
and
\[ W_{n}^{(2)} = \left( \sqrt{n} h_n(\hat{\beta}_n - \beta_0) \right) ^\top I_{b,n}(\hat{\beta}_n | \hat{\alpha}_n) \sqrt{n} h_n(\hat{\beta}_n - \beta_0) \]
\[ \overset{d}{\to} Y_2^\top I_b(\theta_0; \theta_0) Y_2' \]
\[ \sim \chi^2_{p_2}(u_{\beta}^\top I_b(\theta_0; \theta_0)u_{\beta}) \quad \text{(under } H_{1,n}^{(2)}) \]

This completes the proof of the Wald type.

(iii) From the proof of Lemma 3, it is shown that
\[ \frac{1}{\sqrt{n}} \partial_\alpha U_{n}^{(1)}(\alpha_0) \overset{d}{\to} L_1 = I_a(\alpha_0; \alpha_0) Y'_1 \sim N(I_a(\alpha_0; \alpha_0)u_\alpha, I_a(\alpha_0; \alpha_0)) \quad \text{(under } H_{1,n}^{(1)}) \]
On the other hand, we can check
\[ I_{a,n}(\hat{\alpha}_n) \overset{P}{\to} I_a^{-1}(\alpha_0; \alpha_0) \quad \text{(under } H_{1,n}^{(1)}) \]

Therefore, it follows from the continuous mapping theorem that
\[ R_{n}^{(1)} = \left( \frac{1}{\sqrt{n}} \partial_\alpha U_{n}^{(1)}(\alpha_0) \right) ^\top I_{a,n}(\hat{\alpha}_n) \frac{1}{\sqrt{n}} \partial_\alpha U_{n}^{(1)}(\alpha_0) \]
\[ \overset{d}{\to} L_1^\top I_a^{-1}(\alpha_0; \alpha_0) L_1 \]
\[ = Y'_1^\top I_a(\alpha_0; \alpha_0) Y'_1 \]
\[ \sim \chi^2_{p_1}(u_{\alpha}^\top I_a(\alpha_0; \alpha_0)u_{\alpha}) \quad \text{(under } H_{1,n}^{(1)}) \]

Moreover, it follows from the proof of Lemma 3 that
\[ \frac{1}{\sqrt{n} h_n} \partial_\beta U_{n}^{(2)}(\beta_0 | \alpha_0) \overset{d}{\to} L_2 = I_b(\theta_0; \theta_0) Y'_2 \sim N(I_b(\theta_0; \theta_0)u_\beta, I_b(\theta_0; \theta_0)) \quad \text{(under } H_{1,n}^{(2)}) \]

By C1 and Taylor’s theorem with respect to \( \alpha \), we have
\[ \frac{1}{\sqrt{n} h_n} \partial_\beta U_{n}^{(2)}(\beta_0 | \hat{\alpha}_n) = \frac{1}{\sqrt{n} h_n} \partial_\beta U_{n}^{(2)}(\beta_0 | \alpha_0) \]
\[ + \int_0^1 \frac{1}{n \sqrt{h_n}} \partial_\alpha \partial_\beta U_{n}^{(2)}(\beta_0 | \alpha_0 + u(\hat{\alpha}_n - \alpha_0)) du \sqrt{n}(\hat{\alpha}_n - \alpha_0) \]
\[ = \frac{1}{\sqrt{n} h_n} \partial_\beta U_{n}^{(2)}(\beta_0 | \alpha_0) + o_P(1) \quad \text{(under } H_{1,n}^{(2)}) \]

Next, we can show that
\[ I_{b,n}(\hat{\beta}_n | \hat{\alpha}_n) \overset{P}{\to} I_b^{-1}(\theta_0; \theta_0) \quad \text{(under } H_{1,n}^{(2)}) \]

Therefore, by the continuous mapping theorem, it holds
\[ R_{n}^{(2)} = \left( \frac{1}{\sqrt{n} h_n} \partial_\beta U_{n}^{(2)}(\beta_0 | \hat{\alpha}_n) \right) ^\top I_{b,n}(\hat{\beta}_n | \hat{\alpha}_n) \frac{1}{\sqrt{n} h_n} \partial_\beta U_{n}^{(2)}(\beta_0 | \hat{\alpha}_n) \]
\[ \overset{d}{\to} L_2^\top I_b^{-1}(\theta_0; \theta_0) L_2 \]
\[ = Y'_2^\top I_b(\theta_0; \theta_0) Y'_2 \]
\[ \sim \chi^2_{p_2}(u^\top \beta I_b(\theta_0; \theta_0)u_\beta) \quad \text{(under } R_{1,n}^{(2)}) \]

which completes the proof of the Rao type.  \( \square \)
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