TEST VECTORS FOR LOCAL CUSPIDAL RANKIN–SELBERG INTEGRALS

ROBERT KURINCZUK AND NADIR MATRINGE

Abstract. Let $\pi_1, \pi_2$ be a pair of cuspidal complex, or $\ell$-adic, representations of the general linear group of rank $n$ over a nonarchimedean local field $F$ of residual characteristic $p$, different to $\ell$. Whenever the local Rankin–Selberg $L$-factor $L(X, \pi_1, \pi_2)$ is nontrivial, we exhibit explicit test vectors in the Whittaker models of $\pi_1$ and $\pi_2$ such that the local Rankin–Selberg integral associated to these vectors and to the characteristic function of $\sigma_F$ is equal to $L(X, \pi_1, \pi_2)$. As an application we prove that the $L$-factor of a pair of banal $\ell$-modular cuspidal representations is the reduction modulo $\ell$ of the $L$-factor of any pair of $\ell$-adic lifts.

§1. Introduction

The integral representation of local $L$-factors, of pairs of complex irreducible representations of general linear groups over a nonarchimedean local field $F$, was developed in the fundamental paper [5] of Jacquet–Piatetski-Shapiro–Shalika. These $L$-factors are Euler factors which are the greatest common divisors, in a certain sense, of families of integrals $I$ of Whittaker functions. For $n \geq m$, as a by-product of the definition, if $\pi_1$ and $\pi_2$ are irreducible smooth complex (or $\ell$-adic) representations of $GL_n(F)$ and $GL_m(F)$ respectively with Whittaker models $W(\pi_1, \psi)$ and $W(\pi_2, \psi^{-1})$, extended to all irreducible representations via the Langland’s classification, then it is known that there is a finite number $r$ of Whittaker functions $W_i \in W(\pi_1, \psi)$ and $W'_i \in W(\pi_2, \psi^{-1})$, and a finite number of Schwartz functions $\Phi_i$ on $F^m$ if $n = m$, such that the $L$-factor $L(X, \pi_1, \pi_2)$ can be expressed as $\sum_{i=1}^{r} I(X, W_i, W'_i)$, or $\sum_{i=1}^{r} I(X, W_i, W'_i, \Phi_i)$ when $n = m$. A natural question which thus arises is whether one can find an explicit family of such test vectors.

A famous instance of test vectors is the essential vectors for generic representations (cf. [4, 6, 9]). It is shown in these references that these

---

Received July 28, 2016. Revised August 8, 2017. Accepted August 13, 2017.
2010 Mathematics subject classification. 11F70.
vectors are test vectors for $L(X, \pi_1, \pi_2)$ when $\pi_1$ is a generic representation of $\mathrm{GL}_n(F)$, $\pi_2$ is an unramified standard module of $\mathrm{GL}_m(F)$, and $n > m$.

Interesting partial results have been obtained in [7], and, as indicated in [7], the theory of derivatives and its interpretation in terms of restriction of Whittaker functions (cf. [3, 9]) should reduce the general problem to the cuspidal case. Here, we establish the cuspidal case: that for pairs of cuspidal representations $\pi_1$ and $\pi_2$, we can choose $r = 1$, and moreover, we exhibit explicit test vectors, in the interesting case, whenever $L(X, \pi_1, \pi_2)$ is not equal to one. The fact that $r$ can be chosen to be 1 when $L(X, \pi_1, \pi_2) = 1$, for any pair of irreducible representations $\pi_1$ and $\pi_2$ of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, is explained in the proof of [5, Theorem 2.7] and follows from standard facts on Kirillov models. We do not provide completely explicit test functions in this case, possibly a quite technical problem, and we in fact do not consider this case in the remainder of this article, as it is not needed for our application to reduction modulo $\ell$.

Before we state our main theorem, we explain our normalization of Haar measure (Section 4), as for our application to reduction modulo $\ell$ some care needs to be taken with the normalization. Let $\mathfrak{o}_F$ denote the ring of integers of $F$ with unique maximal ideal $\mathfrak{p}_F$, and let $q$ denote the cardinality of the residue field $\mathfrak{o}_F/\mathfrak{p}_F$ and $p$ its characteristic. We fix our Haar measure on $\mathrm{GL}_n(F)$ to give the pro-$p$ unipotent radical $K_n^1$ of $\mathrm{GL}_n(\mathfrak{o}_F)$ volume one. It will turn out that this is a good choice of normalization for reduction modulo $\ell$ for primes $\ell$ not equal to $p$ because $K_n^1$ is a pro-$p$ subgroup. In particular, the volume of any pro-$p$ subgroup of $\mathrm{GL}_n(F)$ which occurs in our computation will be a power of $q$.

Now we state our main theorem. Let $\pi_1$ and $\pi_2$ be cuspidal complex, or $\ell$-adic, representations of $\mathrm{GL}_n(F)$ such that $L(X, \pi_1, \pi_2)$ is nontrivial, so that $\pi_2 \simeq \chi \pi_1^\vee$ for some unramified character $\chi$ of $F^\times$. Let $e$ denote the common ramification index of $\pi_1$ and $\pi_2$ (see Section 6). We denote by $W_1$ and $W_2$ the explicit Whittaker functions for $\pi_1$ and $\pi_2$, as constructed in [12], with respect to a suitable nondegenerate character of the standard maximal unipotent subgroup of $\mathrm{GL}_n(F)$ and suitable maximal extended simple types in $\pi_1$ and $\pi_2$.

**Theorem 9.1.** There is an integer $r$ such that

$$I(X, W_1, W_2, 1_{\mathfrak{o}_F^e}) = (q - 1)(q^{n/e} - 1)q^r \frac{1}{1 - (\nu(\pi_1) X)^{n/e}}$$

$$= (q - 1)(q^{n/e} - 1)q^r L(X, \pi_1, \pi_2).$$
The factor $q^r$ occurs in our computation as a product of volumes, with respect to certain quotient measures, of quotients of pro-$p$ subgroups related to the groups of Bushnell–Kutzko [2] in their explicit construction of $\pi_1, \pi_2$. Clearly, after our computation we could simply renormalize our measure by the factor $(q - 1)(q^{n/e} - 1)q^r$ and under the new normalization have an equality between the integral and the $L$-factor, hence $(W_1, W_2, 1_{\rho_p^n})$ is a test vector in the sense described earlier. However, it is important to keep track of these factors for our application to reduction modulo $\ell$.

We now describe the proof of this theorem. In Section 7, we carefully choose an appropriate basis of $F^n$ and simple types in our cuspidal representations, so that the subgroup of $GL_n(F)$ defined by these simple types decomposes well with respect to the Iwasawa decomposition and satisfies some other important properties (see Proposition 7.1). In Section 8, we analyze the support of the explicit Whittaker functions of Paskunas and Stevens in terms of this well chosen group (Proposition 8.4). This preparation, which constitutes a substantial amount of the path to our main result, then allows us to compute the integral in Section 9.

Our interest in test vectors originated in the study of $\ell$-modular Rankin–Selberg $L$-factors, for $\ell \neq p$, as introduced in [8]. Let $\pi_1$ and $\pi_2$ be integral cuspidal $\ell$-adic representations of $GL_n(F)$ and $GL_m(F)$, and $\tau_1 = r_\ell(\pi_1)$ and $\tau_2 = r_\ell(\pi_2)$ their reductions modulo $\ell$, which are cuspidal $\ell$-modular representations. By [8, Theorem 3.13], the local factor $L(X, \tau_1, \tau_2)$ always divides $r_\ell(L(X, \pi_1, \pi_2))$. In particular, $L(X, \tau_1, \tau_2) = r_\ell(L(X, \pi_1, \pi_2))$ whenever $L(X, \pi_1, \pi_2) = 1$. Hence the interesting case, where a strict division can happen is when $L(X, \pi_1, \pi_2)$ is not equal to 1, and, in particular, $n = m$. In [10], it was shown that for banal representations the $\ell$-modular Godement–Jacquet $L$-factor is equal to the reduction modulo $\ell$ of the $\ell$-adic Godement–Jacquet $L$-factor. It is thus natural to ask: if $\pi_1$ and $\pi_2$ are $\ell$-adic integral cuspidal representations of $GL_n(F)$ with banal reductions $\tau_1$ and $\tau_2$, does one have $L(X, \tau_1, \tau_2) = r_\ell(L(X, \pi_1, \pi_2))$? As a corollary of our main result on test vectors applied to $\ell$-adic Rankin–Selberg integrals, we answer this question in the affirmative.

Corollary 10.1. Let $\tau_1$ and $\tau_2$ be two banal cuspidal $\ell$-modular representations of $GL_n(F)$, and $\pi_1$ and $\pi_2$ be any cuspidal $\ell$-adic lifts, then

$L(X, \tau_1, \tau_2) = r_\ell(L(X, \pi_1, \pi_2))$. 
In [8], this corollary plays a key role in the classification of $L$-factors of generic $\ell$-modular representations, and their relationship with $\ell$-adic $L$-factors via reduction modulo $\ell$.

It would be interesting to pursue the methods of this paper for integral representations of other $L$-factors, such as the Asai, exterior square, and symmetric square $L$-factors.

§2. Notations

Let $F$ be a nonarchimedean local field of residual characteristic $p$ and residual cardinality $q$. Throughout, $R$ will denote one of the fields $\mathbb{C}$, $\mathbb{Q}_\ell$, and $\overline{\mathbb{F}}_\ell$ and we assume that $\ell \neq p$. For $E$ any extension of $F$, we denote by $\mathfrak{o}_E$ the ring of integers of $E$; by $\varpi_E$ a uniformizer of $E$; by $p_E = (\varpi_E)$ the unique maximal ideal of $\mathfrak{o}_E$; by $q_E$ the residual cardinality of $E$; and let $||_{E,R}: E^\times \to R^\times$ denote the unramified character defined by $|\varpi_E|_{E,R} = q_E^{-1}$, thus $||_{E,C}$ is the restriction of the absolute value to $E^\times$ normalized in the usual way. When the field $R$ considered is clear we remove the index $R$ from $||_{E,R}$, and when $E = F$ we remove the index $F$ as well.

Let $G_n = \text{GL}_n(F)$, $K_n = \text{GL}_n(\mathfrak{o}_F)$, $K_n^1 = 1 + \text{Mat}_{n,n}(p_F)$, and let $Z_n$ be the center of $G_n$. For $g$ in $G_n$, by abuse of notation, we denote by $|g|$ the quantity $|\det(g)|$. Put $\eta_n = (0 \cdots 0 1) \in \text{Mat}_{1,n}(F)$, and let $P_n$ be the standard mirabolic subgroup of $G_n$, that is, the set of all matrices $g$ in $G_n$ such that $\eta_n g = \eta_n$. Let $N_n$ be the unipotent radical of the standard Borel subgroup of upper triangular matrices in $G_n$. For $k \in \mathbb{Z}$, let $G_n^{(k)} = \{g \in G_n : |g|_F = q^{-k}\}$. For any subset $X$ of $G_n$, let $X^{(k)} = X \cap G_n^{(k)}$, and let $1_X$ denote the characteristic function of $X$.

Let $\overline{\mathbb{Q}}_\ell$ denote an algebraic closure of the $\ell$-adic numbers, $\overline{\mathbb{F}}_\ell$ denote its ring of integers, and $\mathbb{F}_\ell$ denote its residue field which is an algebraic closure of the finite field of $\ell$-elements.

§3. Representations with coefficients in $R$

We only consider smooth $R$-representations, that is smooth representations with coefficients in $R$, and we use $\vee$ as an exponent to denote the contragredient. We call a representation on a $\overline{\mathbb{Q}}_\ell$-vector space an $\ell$-adic representation, and a representation on an $\overline{\mathbb{F}}_\ell$-vector space an $\ell$-modular representation. Let $(\pi, \mathcal{V})$ be an irreducible $\ell$-adic representation of $G_n$. We call $\pi$ integral if $\mathcal{V}$ contains a $G_n$-stable $\overline{\mathbb{Z}}_\ell$-lattice. Notice that for an $\ell$-adic character $\nu : G_n \to \overline{\mathbb{Q}}_\ell^\times$, this just means that $\nu$ takes values in $\overline{\mathbb{Z}}_\ell^\times$. 
An $R$-representation is called \textit{cuspidal} if it is irreducible and never appears as a quotient of a properly parabolically induced representation. By [13, II 4.12], a cuspidal $\ell$-adic representation is integral if and only if its central character is integral, hence the contragredient of a cuspidal $\ell$-adic representation $\pi$ is integral if and only if $\pi$ is integral. Let $\pi$ be an integral cuspidal $\ell$-adic representation and $\mathfrak{L}$ be a $G_n$-stable $\mathbb{Z}_\ell$-lattice in the space of $\pi$. Let $r_\ell(\pi)$ be the $\ell$-modular representation induced on the space $\mathfrak{L} \otimes_{\mathbb{Z}_\ell} \overline{F}_\ell$. This $\ell$-modular representation is also cuspidal and irreducible by [13, III 5.10], and hence independent of the choice of the lattice $\mathfrak{L}$ by the Brauer–Nesbitt principle [14, Theorem 1], we thus write $r_\ell(\pi)$ for $r_\ell(\pi)$ and call $r_\ell(\pi)$ the reduction modulo $\ell$ of $\pi$. We also say that $\pi$ lifts $r_\ell(\pi)$, and it follows from [13, III 5.10] that all cuspidal $\ell$-modular representations lift to cuspidal $\ell$-adic representations. Following [11, Remark 8.15], we call a cuspidal $\ell$-modular representation $\tau$ \textit{banal} if $\tau \not\simeq \tau \otimes ||_F$ (notice that the definition in [11, Remark 8.15] refers to a condition given in Proposition 8.9 of this reference, which in the cuspidal case reduces to the condition we give here). For $H$ a closed subgroup of $G$, we write $\text{Ind}^G_H$ for the functor of smooth induction taking representations of $H$ to representations of $G$, and write $\text{ind}^G_H$ for the functor of smooth induction with compact support.

§4. Normalization of Haar measures

We now discuss our normalization of Haar measures. The basic reference for $R$-Haar measures is [14, I 2], but we also refer the reader to [8, Section 2.2] for more details on the splitting of Haar measures with respect to standard decompositions. Let $dg$ be the Haar measure on $G_n$ normalized to give $K_n^1$ volume 1.

We normalize the right Haar measure on $P_n$ so that $dp(P_n \cap K_n^1) = 1$, on $N_n$ so that $dn(N_n \cap K_n^1) = 1$, and on $Z_n$ so that $dz(Z_n \cap K_n^1) = 1$. For the remainder of this section, let $G$ denote a closed subgroup of $G_n$ with Haar measure $d_Gg$. For any open subgroup $U$ of $G$, we define the Haar measure $d_Ug$ on $U$ as the restriction of $d_Gg$, in particular $d_Ug$ is normalized as soon as $d_Gg$ is.

If $H$ is a closed subgroup of $G$ with right Haar measure $d_Hh$, and such that the modulus character of $G$ restricts to $H$ as the modulus character of $H$, we descend $d_Gg$ to a right-invariant measure $d_{H \setminus G}g$ on $H \setminus G$ as explained in [14, I 2.8]. For $f$ a smooth map from $G$ to $R$ with compact support,
denoting by $f^H$ the map on $H \backslash G$ defined by

$$f^H(g) = \int_H f(hg) \, d_H h,$$

the usual relation is satisfied:

$$\int_{H \backslash G} f^H(g) \, d_{H \backslash G} g = \int_G f(g) \, d_G g.$$

This implies that $d_{H \backslash G} g$ is normalized as soon as $d_G g$ and $d_H g$ are.

Indeed, if $K$ is a compact subgroup of $G$, applying the equality above to $f = 1_K$, so that

$$f^H = d_H(K \cap H) 1_{H \backslash HK}$$

gives the relation

$$(1) \quad d_G(K) = d_{H \backslash G}(H \backslash HK) d_H(K \cap H).$$

This gives for example the normalization

$$d_{H \backslash G}(H \backslash HK_1^n) = d_H(H \cap K_1^n) \backslash d_G(G \cap K_1^n).$$

With these normalizations, we have the splitting

$$dg = |p|^{-1} \, dp \, dz \, dk.$$

This splitting descends on $N_n \backslash G_n$, in which case $dg$ denotes the normalized right-invariant measure on $N_n \backslash G_n$ and $dp$ the right-invariant measure on $N_n \backslash P_n$. Notice that with such normalizations, the volume of all pro-$p$ subgroups of $G_n$, of $P_n$ and of $Z_n$ will be (positive or negative) powers of $q$.

Moreover, for such choices, reduction modulo $\ell$ commutes with integration (cf. [8, Remark 2.1]), that is, if $f \in C_c^\infty(X, \mathbb{Z}_\ell)$ for $X$ equal to $G_n$ or any of the homogeneous spaces $K \backslash L$ with $L$ a subgroup of $G_n$ considered above, then $\int_X f(x) \, dx \in \mathbb{Z}_\ell$, and

$$r_\ell \left( \int_X f(x) \, dx \right) = \int_X r_\ell(f(x)) \, dx.$$

For the rest of this section, we suppose that $R$ has characteristic zero, and we recall some classical equalities, which all follow from Relation (1).
For a finite set $A$, we let $|A|$ denote its cardinality in $R$. Suppose that $G = K$ compact, and $U$ is an open subgroup of $K$, then

\[(2) \quad d_{U \setminus K}(U \setminus K) = \frac{d_K(K)}{d_K(U)} = |U \setminus K| \in R.\]

Finally, if $V$ is a closed subgroup of $K$ (using the fact that $K$ is unimodular, hence that $d_K(UV) = d_K(V^{-1}U^{-1}) = d_K(VU)$), one obtains

\[(3) \quad d_{V \setminus K}(V \setminus VU) = \frac{d_K(U)}{d_V(V \cap U)} = d_{V \cap U \setminus V}(V \cap U \setminus V).\]

By convention, from now on, we use the same letter for the measure on $G$ and its descent to $H \setminus G$ (and when the context is clear for its restriction to an open subgroup as well).

§5. Rankin–Selberg integrals and local factors

Let $\psi$ be an additive character of $F$ which is trivial on $p_F$, but nontrivial on $o_F$. By abuse of notation, also denote by $\psi$ the nondegenerate character of $N_n$ defined for $x = (x_{i,j}) \in N_n$ by

\[\psi(x) = \psi \left( \sum_{i=1}^{n-1} x_{i,i+1} \right),\]

which is necessarily integral in the $\ell$-adic case because $N_n$ is exhausted by its pro-$p$ subgroups. If $\pi$ is a cuspidal representation of $G_n$, then it is generic (cf. [1] in the complex or $\ell$-adic case, and [13, III 5.10] for $R = \mathbb{F}_\ell$), meaning $\dim(\text{Hom}_{N_n}(\pi, \psi)) = 1$, and hence it has a unique Whittaker model $W(\pi, \psi)$, equal to the image of $\pi$ in $\text{Ind}_{N_n}^G(\psi)$. Suppose that $\pi$ is an integral cuspidal $\ell$-adic representation of $G_n$, then the $\mathbb{Z}_\ell$-submodule $W_e(\pi, \psi)$ of $W(\pi, \psi)$ consisting of all functions in $W(\pi, \psi)$ which take values in $\mathbb{Z}_\ell$ is a $G_n$-stable lattice in $\pi$ (cf. [14, Theorem 2]). Then by definition $r_\ell(\pi) \simeq W_e(\pi, \psi) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$, which is irreducible and cuspidal (cf. Section 2.1 and the references given there). Thus $W_e(\pi, \psi) \otimes_{\mathbb{Z}_\ell} \mathbb{F}_\ell$ is a space of Whittaker functions for $\pi$ with values in $\mathbb{F}_\ell$, hence equal to $W(r_\ell(\pi), r_\ell(\psi))$. For $W \in W_e(\pi, \psi)$, we write $r_\ell(W)$ for the image of $W$ in $W(r_\ell(\pi), r_\ell(\psi))$. 
Finally, we recall the definition of the Rankin–Selberg local $L$-factors for a pair of cuspidal $R$-representations of $G_n$. The construction is originally due to Jacquet–Piatetski-Shapiro–Shalika [5] for complex representations, and works equally well for $\mathbb{Q}_\ell$-representations. This construction was extended to a construction for representations over any algebraically closed field of characteristic prime to $p$ in [8]. As we are ultimately interested in $\mathbb{C}, \mathbb{Q}_\ell$ and $\mathbb{F}_\ell$ representations we give precise references to the construction in [8].

Let $\pi_1$ and $\pi_2$ be cuspidal representations of $G_n$, $W_1 \in W(\pi_1, \psi)$, $W_2 \in W(\pi_2, \psi^{-1})$, and $\Phi \in C_c^\infty(F^n)$ be a locally constant function from $F^n$ to $R$ with compact support. By [8, Proposition 3.3], for $k \in \mathbb{Z}$, the coefficients

$$c_k(W_1, W_2, \Phi) = \int_{N_n \backslash G_n^{(k)}} W_1(g)W_2(g)\Phi(\eta_n g) \, dg$$

are well defined and vanish for $k$ sufficiently negative. In fact, these coefficients vanish for $k$ sufficiently negative because both $W_1$ and $W_2$ vanish on $P_n^{(k)}$ for such $k$, as a consequence of [5, Proposition 2.2]. Hence the local Rankin–Selberg integral

$$I(X, W_1, W_2, \Phi) = \sum_{k \in \mathbb{Z}} c_k(W_1, W_2, \Phi)X^k$$

is a formal Laurent series with coefficients in $R$. In fact, by [8, Theorem 3.5], $I(X, W_1, W_2, \Phi) \in R(X)$ is a rational function, and as $W_1$ varies in $W(\pi_1, \psi)$, $W_2$ varies in $W(\pi_2, \psi^{-1})$, and $\Phi$ varies in $C_c^\infty(F^n)$, the $R$-submodule of $R(X)$ spanned by $I(X, W_1, W_2, \Phi)$ is a fractional ideal of $R[X^{\pm 1}]$, and has a unique generator $L(X, \pi_1, \pi_2)$ which is an Euler factor. We call $L(X, \pi_1, \pi_2)$ the local Rankin–Selberg $L$-factor, and note that it does not depend on the choice of the character $\psi$. If $R = \mathbb{Q}_\ell$, it is shown in [8, Corollary 3.6] that the $L$-factor is the inverse of a polynomial in $\mathbb{Z}_\ell[X]$, and thus it makes sense to talk of its reduction modulo $\ell$. Moreover, it follows from [8, Theorem 3.13], that if $\pi_1$ and $\pi_2$ are two integral cuspidal $\ell$-adic representations of $G_n$, then one has

$$L(X, r_\ell(\pi_1), r_\ell(\pi_2))|_{r_\ell(L(X, \pi_1, \pi_2))}.$$ 

Now by [5, Proposition 8.1, (ii)], the $L$-factor $L(X, \pi_1, \pi_2)$ is equal to 1 unless $\pi_2 \simeq \chi\pi_1^\vee$ for some unramified character $\chi$ of $F^\times$. Hence if $\pi_2 \not\cong \chi\pi_1^\vee$ then $L(X, r_\ell(\pi_1), r_\ell(\pi_2)) = r_\ell(L(X, \pi_1, \pi_2)) = 1$.

For our computations to come, we use a decomposition of the Rankin–Selberg integral in the special case where $\pi_2 \simeq \pi_1^\vee$, in particular their central
characters are inverse of each other. Thus we assume this is the case for the rest of this section. For \( k \in \mathbb{Z} \), we set

\[
b_k(W_1, W_2) = \int_{N_n \setminus P_n^{(k)}} W_1(p)W_2(p) \, dp,
\]

which, similarly to \( c_k \), vanishes for \( k \) sufficiently negative, and we put

\[
I_{(0)}(X, W_1, W_2) = \sum_{k \in \mathbb{Z}} b_k(W_1, W_2)q^k X^k.
\]

Let \( \Phi \in C_\infty_c (F^n) \) be a \( K_n \)-invariant function, for \( i \in \mathbb{Z} \), we set

\[
a_{ni}(\Phi) = \int_{z \in G_1^{(ni)}} \Phi(\eta_n z) \, dz,
\]

which vanishes for \( i \) sufficiently negative, and we put

\[
Z(X, \Phi) = \sum_{i \in \mathbb{Z}} a_{ni}(\Phi) X^{ni}.
\]

As \( G_n^{(k)} = \prod_{i \in \mathbb{Z}} P_n^{(k-ni)} Z_n^{(ni)} K_n \), from the splitting of Section 4 we find

\[
c_k(W_1, W_2, \Phi) = \sum_{i \in \mathbb{Z}} a_{ni}(\Phi) q^{k-ni} \int_{(K_n \cap P_n) \setminus K_n} b_{k-ni}(\rho(k)W_1, \rho(k)W_2) \, dk,
\]

from which we deduce

\[
I(X, W_1, W_2, \Phi) = Z(X, \Phi) \left( \int_{(K_n \cap P_n) \setminus K_n} I_{(0)}(X, \rho(k)W_1, \rho(k)W_2) \, dk \right).
\]

Taking \( \Phi \) equal to the characteristic function \( 1_{\sigma_{\mathbb{F}}^n} \), we obtain the formula

\[
(4) \ I(X, W_1, W_2, 1_{\sigma_{\mathbb{F}}^n}) = \frac{q - 1}{1 - X^n} \int_{(K_n \cap P_n) \setminus K_n} I_{(0)}(X, \rho(k)W_1, \rho(k)W_2) \, dk.
\]

The equality \( Z(X, 1_{\sigma_{\mathbb{F}}^n}) = (q - 1/1 - X^n) \) is standard (cf. \cite[Theorem 3.1]{}) except that in our setting, we get the extra constant \( q - 1 \) from our choice of normalization on \( Z_n \), as we set \( dz(Z_n \cap K_n^1) = 1 \) instead of the usual \( dz(Z_n \cap K_n) = 1 \).
§6. Simple types and reduction modulo \(\ell\)

For the beginning of this section we assume that \(R = \mathbb{C}\) or \(\overline{\mathbb{Q}}\). Let \(V\) be an \(n\)-dimensional \(F\)-vector space, let \(\text{End}_F(V)\) denote the \(F\)-algebra \(\text{End}_F(V)\) of \(F\)-endomorphisms of \(V\) and let \(G\) denote the group \(\text{Aut}_F(V)\) of \(F\)-automorphisms of \(V\). Hence \(G\) identifies with \(G_n\) as soon as we choose a basis of \(V\). In [2], every cuspidal \(R\)-representation of \(G\) is constructed explicitly as \(\text{ind}_G^J(\Lambda)\), where \(J\) is an open and compact-mod-center subgroup of \(G\), and \(\Lambda\) is an irreducible representation of \(J\) of finite dimension. The pairs \((J, \Lambda)\) are called extended maximal simple types, and for any such pair \(\text{ind}_G^J(\Lambda)\) is (irreducible and) cuspidal by [2, Chapter 6]. We briefly explain the construction of the group \(J\), focusing on the properties which we shall use.

An \(\sigma_F\)-lattice chain \(\mathcal{L}\) in \(V\) is a nonempty set of \(\sigma_F\)-lattices \(\{L_i : i \in \mathbb{Z}\}\) such that, for all \(i \in \mathbb{Z}\), \(L_{i+1} \subseteq L_i\) and there exists \(e(\mathcal{L}) \in \mathbb{Z}\) such that \(L_{i+e(\mathcal{L})} = \varpi_F L_i\). The construction of [2], starts with data \((\beta, \mathcal{L})\) called maximal simple strata consisting of

1. an element \(\beta \in \text{End}_F(V)\) which generates a simple field extension \(E = F[\beta]\);
2. an \(\sigma_F\)-lattice chain \(\mathcal{L}\) in \(V\) such that \(E^\times \mathcal{L} \subseteq \mathcal{L}\) (i.e., for any \(x \in E^\times\) and \(L \in \mathcal{L}\) we have \(xL \in \mathcal{L}\)); in particular \(\mathcal{L}\) is an \(\sigma_E\)-lattice chain, and it is required (as \((\beta, \mathcal{L})\) is maximal) that \(L_{i+1} = \varpi_E L_i\);

which satisfy a technical condition (cf. [2, 1.5.5] where the simple strata we consider are among those denoted \([\mathfrak{A}, -, 0, \beta]\)).

Let \((\beta, \mathcal{L})\) be a maximal simple strata. We denote by \(\mathfrak{A} = \mathfrak{A}(\mathcal{L})\) the \(\sigma_F\)-order in \(\text{End}_F(V)\) and \(\mathfrak{B} = \mathfrak{B}(\beta, \mathcal{L})\) the \(\sigma_E\)-order in \(\text{End}_E(V)\) defined by \(\mathcal{L}\),

\[
\mathfrak{A} = \text{End}_{\sigma_F}(\mathcal{L}) = \bigcap_k \text{End}_{\sigma_F}(L_k), \quad \mathfrak{B} = \mathfrak{B}(\beta, \mathcal{L}) = \text{End}_{\sigma_E}(\mathcal{L}) = \text{End}_{\sigma_E}(L_0).
\]

In [2, 3.1] Bushnell–Kutzko define compact open subgroups of \(G\) denoted by \(H^1 = H^1(\beta, \mathcal{L})\), \(J^1 = J^1(\beta, \mathcal{L})\), and \(J = J(\beta, \mathcal{L})\). The properties we need are:

1. the groups \(H^1 \leq J^1\) are pro-\(p\) (by definition), are normalized by \(E^\times\) and are normal subgroups of \(J\) by [2, 3.1.15], moreover \(J \subseteq \text{Aut}_{\sigma_F}(L_0)\) (by definition).
2. Put \(m = n/[E : F]\), by [2, 3.1.15] we have
\[ J = \mathfrak{B}^\times J^1, \quad \mathfrak{B}^\times \cap J^1 = 1 + \varpi_E\mathfrak{B} \quad \text{and} \]
\[ J/J^1 \simeq \mathfrak{B}^\times / (1 + \varpi_E\mathfrak{B}) \simeq G_m(k_E). \]

We then set \( J = J(\beta, \mathcal{L}) = E^\times J \), in particular \( J \) is compact mod \( E^\times \) and hence compact mod \( F^\times \). Notice that if \( \pi \simeq \text{ind}_J^G \Lambda \), the center \( F^\times \) of \( G \) acts by the central character \( \omega_\pi \) of \( \pi \) through \( \Lambda \). Finally, we note that the construction of \( \Lambda \) depends on our fixed additive character \( \psi \) (cf. [2, 3.2]).

The definitions above do not include the groups of the maximal simple types for level zero cuspidal representations (see [2, 5.5.10(b)]), although these can be considered formally as part of the construction described above for the maximal zero strata \((0, \mathcal{L})\) with \( \beta = 0 \) and \( e(\mathcal{L}) = 1 \). In this case, we put \( J = \mathfrak{A}^\times, \ J = F^\times J, H^1 = J^1 = 1 + \varpi_F\mathfrak{A}, \) and \( J/J^1 = \mathfrak{A}^\times / (1 + \varpi_F\mathfrak{A}) \simeq G_n(k_F) \).

Now we consider \( \overline{\mathbb{F}}_\ell \)-representations. It follows from [13, Chapitre IV] that the Bushnell–Kutzko classification of cuspidal \( \overline{\mathbb{Q}}_\ell \)-representations adapts well to \( \overline{\mathbb{F}}_\ell \)-representations. We only need to know the following facts:

Let \( \tau \) be a cuspidal \( \ell \)-modular representation of \( G \). As we recalled in Section 3, there exists an integral cuspidal \( \ell \)-adic representation \( \pi \) such that \( \tau = r_\ell(\pi) \). Choose an extended maximal simple type \((J, \Lambda)\) such that \( \pi \simeq \text{ind}_J^G \Lambda \), as in the beginning of this section. A cuspidal \( \ell \)-adic representation is integral if and only if its central character \( \omega_\pi \) is integral, by [13, II 4.13] (the direction integral implies integral central character being clear). We recall why this is true. First as \( J \) is compact mod \( F^\times \), we claim that the irreducible representation \( \Lambda \) is integral if and only if \( \omega_\pi \) is integral. Again, one direction is clear. For the other, suppose that \( \omega_\pi \) is integral and choose a random not necessarily \( J \)-stable lattice \( \mathfrak{L}_0 \) in the space \( V_\Lambda \) of \( \mathfrak{L} \). It is stabilized by a compact open subgroup \( U \) of \( J \), and choosing representatives \( c_1, \ldots, c_r \) of \( J/F^\times U \), one has \( \Lambda(J)(\mathfrak{L}_0) = \sum_{i=1}^r \Lambda(c_i)(\mathfrak{L}_0) \), hence \( \mathfrak{L}_\Lambda = \Lambda(J)(\mathfrak{L}_0) \) is a \( J \)-stable lattice in \( V_\Lambda \) by [13, 9.3]. The induced \( \overline{\mathbb{Z}}_\ell \)-representation \( \text{ind}_J^G(\mathfrak{L}_\Lambda) \) is then a lattice in \( \tau \) by [13, 9.3]. Moreover \( \tau = r_\ell(\pi) \simeq \text{ind}_J^G(\varpi_\ell(\Lambda)) \), and \( r_\ell(\Lambda) \) is an irreducible representation of \( J \) by irreducibility of \( \tau \).

Finally, we give another characterization of banal cuspidal representations: recall, from Section 3, by definition \( \tau \) is banal if and only if the cardinality of the cuspidal line \( \mathbb{Z}_\tau = \{ ||k\tau, \ k \in \mathbb{Z} \} \) is greater than 1. By [11, Lemme 5.3], this cardinality is the same as the integer \( o(\tau) \) introduced in [11, Section 5.2, (5.4)]. From [11, Section 5.2, (5.4)], \( o(\tau) \) is the order of \( q^{n/e} \) in \( \overline{\mathbb{F}}_\ell^\times \), where \( e = e(E/F) \) is the ramification index attached to \((J, \Lambda)\) which
in particular does not depend on the choice of extended maximal simple type. Hence \( \tau \) is banal if and only if \( q^{n/e} - 1 \neq 0 \) in \( \mathbb{F}_\ell \).

\section{The modified Paskunas–Stevens basis}

For this section \( R = \mathbb{C} \) or \( \mathbb{Q}_\ell \). Let \( \pi \) be a cuspidal \( R \)-representation of \( G \) and \( (J = J(\beta, \mathcal{L}), \Lambda) \) be an extended maximal simple type in \( \pi \). According to [12, Corollaries 3.4 and 4.13], there exists an \( F \)-basis \( \mathcal{B} = (v_1, \ldots, v_n) \) of \( V \) particularly suited to relating the Whittaker model of \( \pi \) and the model \( \text{ind}^G_J(\Lambda) \) defined via type theory. In particular, \( \mathcal{B} \) splits \( L_k \), that is,

\[
L_k = \bigoplus_{i=1}^n p_F^{a_i(k)} v_i \quad \text{with} \quad a_i(k) \in \mathbb{Z} \quad \text{for all} \quad k \in \mathbb{Z},
\]

and is such that if \( N \) is the maximal unipotent subgroup of \( G \) attached to the maximal flag defined by \( \mathcal{B} \), and if \( \psi \), by abuse of notation, denotes the nondegenerate character of \( N \) defined for \( x \in N \) by

\[
\psi(x) = \psi \left( \sum_{i=1}^{n-1} \text{Mat}_\mathcal{B}(x)_{i,i+1} \right),
\]

where \( \text{Mat}_\mathcal{B}(x) \) denotes the matrix of \( x \) with respect to the basis \( \mathcal{B} \), then the triple \( (J, \Lambda, \psi) \) satisfies

\[
\text{Hom}_{N \cap J}(\psi, \Lambda) \neq 0.
\]

Let \( P \) be the mirabolic subgroup defined by

\[
P = \{ g \in G, (g - \text{Id})V \subset \text{Vect}_F(v_1, \ldots, v_{n-1}) \}.
\]

We put \( \mathcal{M} = (P \cap J)J^1 \), which is a group as \( J^1 \) is normal in \( J \). It follows from [12] that the image of \( \mathcal{M} \) in \( J/J^1 \simeq G_m(k_E) \) is isomorphic to \( P_m(k_E) \).

We now explain how to extract this from [12]: in the notation of [12], our group \( P \) is denoted \( \mathcal{M} \) and [12, Corollary 4.8] shows that

\[
\mathcal{M} = (P \cap \mathfrak{B}^\times)J^1.
\]

In [12, Section 4.1], Paskunas–Stevens introduce another mirabolic group they denote by \( \mathcal{M}_E \) which satisfies \( P \cap \mathfrak{B}^\times = \mathcal{M}_E \cap \mathfrak{B}^\times \) by the equality just before [12, Corollary 4.7], and they also denote by \( \mathcal{M}_\mathfrak{B} \) the group \( (\mathcal{M}_E \cap \mathfrak{B}^\times)(1 + \varpi_E \mathfrak{B}) \). Hence Equation (5) gives \( \mathcal{M} = \mathcal{M}_\mathfrak{B}J^1 \) as \( (1 + \varpi_E \mathfrak{B}) = \mathfrak{B}^\times \cap J^1 \). Finally, from the discussion after the proof of [12, Lemma 4.10], the image of \( \mathcal{M}_\mathfrak{B} \) in \( \mathfrak{B}^\times/1 + \varpi_E \mathfrak{B} \simeq G_m(k_E) \) is isomorphic to \( P_m(k_E) \), hence the same is true for the image of \( \mathcal{M} \) in \( J/J^1 \simeq \mathfrak{B}^\times/1 + \varpi_E \mathfrak{B} \simeq G_m(k_E) \).
In particular, the following index will appear in our computation:

\[ |J/M| = |G_m(k_E)/P_m(k_E)| = q_E^m - 1 = q^{n/e} - 1. \]

For \( i \in \{1, \ldots, n\} \), the functions \( a_i : \mathbb{Z} \to \mathbb{Z} \) satisfy the relation \( a_{i+e}(k) = a_i(k) + 1 \). In particular, this holds for \( i = n \), and the map \( k \mapsto a_n(k) \) is increasing with values in \( \mathbb{Z} \), so there is \( k_0 \) between 1 and \( e \) such that \( a_n(k_0) = a_n(k_0 - 1) + 1 \), and then \( a_n(k_0 + i) = a_n(k_0) \) for \( i \in \{0, \ldots, e - 1\} \). Hence by reindexing the lattice chain \( L \) if necessary, by a translation, \( k \mapsto k - k_0 \), we can suppose that

\[ a_n(0) = a_n(-1) + 1 = 0, \quad \text{and} \quad a_n(1) = \cdots = a_n(e - 1) = 0. \]

We recall that \( L_0 = \bigoplus_{i=1}^{n} p_F^{a_i(0)} v_i \), and we set \( B' = (\varpi_F^{a_1(0)} v_1, \ldots, \varpi_F^{a_n(0)} v_n) \), which we write as \( B' = (w_1, \ldots, w_n) \).

We use this basis to identify \( G \) with \( G_n \). With this choice, one has \( J \subset K_n \) because \( J \subset \text{Aut}_{F_L}(L_0) \). The group \( P \) identifies with \( P_n \), the group \( N \) identifies with \( N_n \), and the character \( \psi \) of \( N_n \) identifies with

\[ \psi_n : n \mapsto \psi \left( \sum_{i=1}^{n-1} t_i n_i i+1 \right), \]

where \( t_i = \varpi_F^{a_i(0)} - a_i+1(0) \).

For our computation to come, it will be useful to notice the following property of \( B' \): one has

\[ L_0 = \bigoplus_{i=1}^{n} \mathfrak{o}_F w_i, \quad L_k = \bigoplus_{i=1}^{n-1} p_F^{a_i(k) - a_i(0)} w_i \oplus \mathfrak{o}_F w_n, \]

for \( k \in \{1, \ldots, e - 1\} \). As \( \varpi_k L_k = L_{k+1} \) for any \( k \in \mathbb{Z} \), the properties above and the fact that \( L_{k+e} = \varpi F L_k \), imply that the last row of \( \varpi_E^i \in G_n \) belongs to \( (\mathfrak{o}_F)^n - (p_F)^n \) for \( i = 0, \ldots, e - 1 \), and more generally that it belongs to \( (p_F^i)^n - (p_F^{i+1})^n \) if \( i = le + r \), with \( r \in \{0, \ldots, e - 1\} \). As an immediate consequence, if we write an Iwasawa decomposition of \( \varpi_E^i \),

\[ \varpi_E^i = p_i z_i k_i, \quad p_i \in P_n, \ z_i \in Z_n, \ k_i \in K_n, \]

we can choose \( z_i = I_n \) for \( i = 0, \ldots, e - 1 \), and more generally \( z_i = \varpi_F^i I_n \) for \( i = le + r \), with \( r \in \{0, \ldots, e - 1\} \). In particular \( |p_i| = q^{-in/e} \), for \( i = 0, \ldots, e - 1 \).

For clarity, we list the properties of the data \((J, \Lambda, \psi_n)\) that we use.
Proposition 7.1. With the above choice of basis we have:

1. The inclusion $J \subset K_n$.
2. The space $\text{Hom}_{N_n \cap J}(\psi_t, \Lambda) \neq 0$.
3. Set $\mathcal{M} = (P_n \cap J)J^1$, then $|J/\mathcal{M}| = q^{n/e} - 1$.
4. The element $\varpi_i^j E \in P_n K_n$ if and only if $i \in \{0, \ldots, e - 1\}$ and, in this case, if we choose $p_i \in P_n$ and $k_i \in K_n$, such that $\varpi_i^j E = p_i k_i$, then we have $|p_i| = |\varpi_i^j E| = q^{-in/e}$.

For the remainder, we consider the $k_i \in K_n$ and $p_i \in P_n$ chosen in Proposition 7.1 Statement (4) as fixed.

As $P_n \cap J^1$ is a pro-$p$ subgroup of $P_n$, and $J^1$ is a pro-$p$ subgroup of $G_n$, the volume

$$dk(P_n \cap J^1 \setminus J^1) = \frac{dk(J^1)}{dp(P_n \cap J^1)}$$

is a power of $q$ thanks to our normalization of measures, and we write

$$dk(P_n \cap J^1 \setminus J^1) = q^{r_1}.$$ 

A certain volume will appear in our later computation, we compute it in the next lemma.

Lemma 7.2. For any $i \in \{0, \ldots, e - 1\}$, we have

$$dk((P_n \cap K_n) \setminus (P_n \cap K_n)k_i J) = q^{r_1}(q^{n/e} - 1)q^{-in/e}.$$ 

Proof. We have

$$dk((P_n \cap K_n) \setminus (P_n \cap K_n)k_i J) = dk((P_n \cap K_n) \setminus (P_n \cap K_n)k_i Jk_i^{-1})$$

$$= dk((P_n \cap k_i Jk_i^{-1}) \setminus k_i Jk_i^{-1}),$$

the last equality thanks to Relation (3). Now, $dk(k_i Jk_i^{-1}) = dk(J)$. We also notice that

$$p_i(P_n \cap k_i Jk_i^{-1})p_i^{-1} = P_n \cap \varpi_i^j E J \varpi_i^{-j} = P_n \cap J,$$

hence

$$P_n \cap k_i Jk_i^{-1} = p_i^{-1}(P_n \cap J)p_i.$$ 

As for any compact open subset $A$ of $P_n$, one has $dp(pAp^{-1}) = |p|dp(A)$, as is easily seen by writing $dp = dgdu$, with $dg$ on $G_{n-1}$ and $du$ on $U_n$, we
obtain the relation
\[ dp(P_n \cap k_iJk_i^{-1}) = |p_i|^{-1} dp(P_n \cap J) = q^{\text{in/e}} dp(P_n \cap J). \]

We then obtain from Relations (1) and (2):
\[ dk((P_n \cap k_iJk_i^{-1}) \backslash k_iJk_i^{-1}) = \frac{dk(k_iJk_i^{-1})}{dp(P_n \cap k_iJk_i^{-1})} = q^{-\text{in/e}} \frac{dk(J)}{dp(P_n \cap J)} = q^{-\text{in/e}} dk((P_n \cap J) \backslash J). \]

Now by Relations (1) and (2) again, one has
\[ dk((P_n \cap J) \backslash J) = \frac{dk(J)}{dp(P_n \cap J)} = \frac{dk(J)}{dk(M) dp(P_n \cap J)} = |J \backslash M| dk(P_n \cap J) \backslash M). \]

Finally, because \( M = (P_n \cap J)J^1 \), applying Relation (3) gives:
\[ dk((P_n \cap J) \backslash J) = |J \backslash M| dk(P_n \cap J^1 \backslash J^1) = q^{r_1}(q^{\text{in/e}} - 1) \]
by Proposition 7.1(3) and our definition of \( r_1 \). This concludes the proof.

§8. Explicit Whittaker functions of Paskunas–Stevens

In this section we continue to assume that \( R = \mathbb{C} \) or \( \mathbb{Q}_\ell \). We now recall the definition and some properties of the explicit Whittaker functions of [12]. We set
\[ \mathcal{U} = (N_n \cap J)H^1. \]

We extend \( \psi_t \) to the group \( \mathcal{U} \) as in [12, Definition 4.2], and, by abuse of notation, denote this extension by \( \psi_t \). We fix a normal compact open subgroup \( \mathcal{N} \) of \( \mathcal{U} \) contained in \( \ker(\psi_t) \). We also denote by \( \rho \) the trace character of \( \Lambda \) and \( \rho^\vee \) that of \( \Lambda^\vee \).

**Definition 8.1. (Bessel functions)** For \( j \in J \), we define
\[ J(j) = |N \backslash \mathcal{U}|^{-1} \sum_{N \backslash \mathcal{U}} \psi_t(u)^{-1} \rho(ju), \quad \text{and} \]
\[ J^\vee(j) = |N \backslash \mathcal{U}|^{-1} \sum_{N \backslash \mathcal{U}} \psi_t(u) \rho^\vee(ju). \]
The Bessel functions enjoy the following properties:

**Proposition 8.2.**

1. We have the equality \( J(1) = 1 \).
2. \( J(uj) = J(ju) = \psi_t(u)J(j) \) for \( u \in U \) and \( j \in J \).
3. For all \( j \in J \), we have the relation
   \[
   J^{\vee}(j) = J(j^{-1}).
   \]
4. For all \( j_1 \) and \( j_2 \) in \( J \), we have
   \[
   \sum_{m \in U \setminus M} J(j_1 m^{-1})J(mj_2) = J(j_1 j_2).
   \]

**Proof.** See [12, Proposition 5.3 and Theorem 5.6]. The third property follows from a simple change of variables, and the relation \( \rho^{\vee}(ab) = \rho(b^{-1}a^{-1}) \) for any \( a \) and \( b \) in \( J \). The final property follows from [12, Proposition 5.3, Property (v)], thanks to the bijection \( m \leftrightarrow m^{-1} \) between \( M/U \) and \( U \setminus M \).

We can now define the explicit Whittaker functions \( W \) and \( W^{\vee} \) of Paskunas–Stevens following [12, Section 5.2] and recall a first property.

**Definition 8.3.** Both \( W \) and \( W^{\vee} \) are supported on \( N_nJ \), and

\[
W(nj) = \psi_t(n)J(j)
\]

for \( n \in N_n \) and \( j \in J \), whereas

\[
W^{\vee}(nj) = \psi_t^{-1}(n)J^{\vee}(j) = \psi_t^{-1}(n)J(j^{-1})
\]

for \( n \in N_n \) and \( j \in J \). Moreover, \( W \) belongs to \( W(\pi, \psi_t) \) and \( W^{\vee} \) belongs to \( W(\pi^{\vee}, \psi_t^{-1}) \).

We now prove further properties of \( W \) and \( W^{\vee} \).

**Proposition 8.4.** For \( l \geq 0 \), let \( W_l = 1_{G_n}W \), and \( W^{\vee}_l = 1_{G_n}W^{\vee} \).

1. The functions \( (W_l) |_{P_nK_n} \) and \( (W_l)^{\vee} |_{P_nK_n} \) are zero unless \( l = in/e \) for some \( i \in \{0, \ldots, e-1\} \), and in this case
   \[
   (W_l) |_{P_nK_n} = 1_{N_nE_j}W |_{P_nK_n}, \quad \text{and} \quad (W_l^{\vee}) |_{P_nK_n} = 1_{N_nE_j}W^{\vee} |_{P_nK_n}.
   \]
2. If \( W_{in/e}(pk) \neq 0 \), then \( i \in \{0, \ldots, e-1\} \), \( k \in P_nE_j \), and, in fact, \( k \in (P_n \cap K_n)E_j \).
3. If \( W_{in/e}(p\omega_l^j) \neq 0 \) with \( p \in P_n \) and \( j \in J \), then \( p \in N_n(P_n \cap J) \).
§9. Test vectors

Again, we assume that $R = \mathbb{C}$, or $\overline{\mathbb{Q}}_\ell$, and $\pi_1$ and $\pi_2$ are cuspidal $R$-representations of $G_n$. We denote by $(J, \Lambda)$ the extended maximal simple type of $\pi_1$, by $e = e(E/F)$ the ramification index of the field extension associated to $(J, \Lambda)$, and by $W, W^\vee$ the explicit Whittaker functions associated to $\pi_1$ (see Definition 8.3). This section is dedicated to proving our main result on test vectors.

**Theorem 9.1.** Suppose that $L(X, \pi_1, \pi_2)$ is nontrivial, so that $\pi_2 \simeq \chi \pi_1^\vee$ for some unramified character $\chi$ of $F^\times$. Then there is an integer $r$ such that

$$I(X, W, \chi W^\vee, 1_{F^n}) = \frac{q^r(q - 1)(q^{n/e} - 1)}{1 - (\chi(\varpi_F)X)^{n/e}} = q^r(q - 1)(q^{n/e} - 1)L(X, \pi_1, \pi_2).$$

We are now ready to prove the following crucial proposition. We recall that for all integers $l \geq 0$, the restriction $W_l$ has been defined in Proposition 8.4.

**Proposition 9.2.** Let $F_l : (K_n \cap P_n) \setminus K_n / J^1 \to R$ be defined by

$$F_l(k) = \int_{j \in J^1} \int_{N_n \setminus P_n} W_l(pkj) W_l^\vee(pkj) \, dp \, dj.$$

Then $F_l$ is nonzero if and only if $l = in/e$ and $i \in \{0, \ldots, e - 1\}$, and in this case, it is supported on $(K_n \cap P_n) k_i J$. Moreover, for $i \in \{0, \ldots, e - 1\}$, and for $k \in (K_n \cap P_n) k_i J$, there is an integer $r_2$ independent of $i$ such that

$$F_{in/e}(k) = q^{r_2}.$$

**Proof.** If $F_l(k)$ is nonzero, then $W_l(pkj)$ is nonzero at least for some $p \in P_n$ and $j \in J$, but then according to Statements (1) and (2)
of Proposition 8.4, this implies that $l$ is of the form $l = \text{in}/e$ with $i \in \{0, \ldots, e - 1\}$, and $k \in (K_n \cap P_n)k_iJ$. Moreover, from Statement (2) of the same proposition, we can write $k = p_0\varpi_E^i j_0$ for $p_0 \in P_n$ and $j_0 \in J$. But now notice that for such a $k$, we have

$$F_l(k) = \int_{j \in J^1} \int_{N_n \setminus P_n} W_l(pp_0\varpi_E^i j_0) W_l^\vee(pp_0\varpi_E^i j_0) \, dp \, dj$$

$$= \int_{j \in J^1} \int_{N_n \setminus P_n} W_l(p\varpi_E^i j_0) W_l^\vee(p\varpi_E^i j_0) \, dp \, dj.$$

Hence by Statement (3) of Proposition 8.4

$$F_l(k) = \int_{j \in J^1} \int_{N_n \setminus (P_n \cap J)} W_l(m\varpi_E^i j_0) W_l^\vee(m\varpi_E^i j_0) \, dm \, dj$$

$$= \int_{j \in J^1} \int_{N_n \setminus (J \cap P_n \cap J)} \mathcal{J}(m\varpi_E^i j_0) \mathcal{J}(j_0^{-1} \varpi_E^{-i} m^{-1}) \, dm \, dj,$$

the last equality according to Proposition 8.2(3). Now, as $J$ normalizes $J^1$, and for any $t \in G_n$ normalizing $J^1$, the automorphism $j \mapsto tj^{-1}$ of $J^1$ has modulus character equal to 1, because $J^1$ is an open subgroup of the unimodular group $G_n$, we have

$$F_l(k) = \int_{j \in J^1} \int_{N_n \setminus (N \cap P_n \cap J)} \mathcal{J}(m\varpi_E^i j_0) \mathcal{J}(j_0^{-1} \varpi_E^{-i} m^{-1}) \, dm \, dj$$

$$= \int_{J \setminus \mathcal{M}} \mathcal{J}(m\varpi_E^i j_0) \mathcal{J}(j_0^{-1} \varpi_E^{-i} m^{-1}) \, dm.$$

We write

$$dm(N_n \cap J \setminus (N_n \cap J)H^1) = dm(N_n \cap H^1 \setminus H^1) = q^{r_2},$$

which is indeed a power of $q$ as $H^1$ is pro-$p$. Moreover, as $H^1$ is normal in $J$, and as the integrand is invariant under $\mathcal{U}$ thanks to Property (2) in Proposition 8.2

$$F_l(k) = q^{r_2} \int_{\mathcal{U} \setminus \mathcal{M}} \mathcal{J}(m\varpi_E^i j) \mathcal{J}(j^{-1} \varpi_E^{-i} m^{-1}) \, dm = q^{r_2},$$

the last equality thanks to Statement (4) of Proposition 8.2.
Proposition 9.3. The coefficient
\[ b_l = \int_{P_n \cap K_n \setminus K_n} \int_{N_n \setminus P_n} W_l(pk) W_l^\vee(pk) \, dp \, dk \]
is zero unless \( l = i n/e \) for some \( i \in \{0, \ldots, e-1\} \), in which case there is an integer \( r \) such that
\[ b_l = q^r (q^{n/e} - 1) q^{-in/e}. \]

Proof. By definition, \( b_l \) is equal to
\[ \int_{P_n \cap K_n \setminus K_n \setminus K_n / J^1} F_l(k) \, dk = q^{r_3} \int_{P_n \cap K_n \setminus K_n} F_l(k) \, dk \]
with \( dk(J^1) = q^{r_3} \) (\( J^1 \) is pro-\( p \)). So according to Proposition 9.2, this is zero if \( l \neq in/e \) for \( i \in \{0, \ldots, e-1\} \), and if \( l = in/e \) for \( i \in \{0, \ldots, e-1\} \), it is equal to
\[ q^{r_3} \int_{P_n \cap K_n \setminus (P_n \cap K_n) k_i J} F_l(k) \, dk = q^{r_2 + r_3} dk(P_n \cap K_n \setminus (P_n \cap K_n) k_i J) \]
\[ = q^r (q^{n/e} - 1) q^{-in/e}, \]
where we write \( r = r_1 + r_2 + r_3 \), from Lemma 7.2. \( \square \)

If \( \pi \) is a cuspidal \( R \)-representation of \( G_n \) of ramification index \( e \), we denote by \( R(\pi) \) its ramification group, that is the group of unramified characters \( \nu \) of \( F^\times \) which satisfy \( \nu \pi \simeq \pi \). It follows from [2, 6.2.5], that \( R(\pi) \) is isomorphic to the group of \( n/e \)th roots of unity in \( R^\times \), via \( \nu \mapsto \nu(\varpi_F) \).

Proof of Theorem 9.1. We first suppose that \( \pi_2 \simeq \pi_1^\vee \). By Equation (4), the integral \( I(X, W, W^\vee, 1_{e^e_F}) \) is equal to
\[ \frac{q - 1}{1 - X^n} \int_{(K_n \cap P_n) \setminus K_n} I_{(0)}(X, \rho(k) W, \rho(k) W^\vee) \, dk. \]
Now, as \( W W^\vee = \sum_{l \in \mathbb{Z}} W_l W_l^\vee \), by Statement (1) of Propositions 8.4 and 9.3, we have
\[ \int_{(K_n \cap P_n) \setminus K_n} I_{(0)}(X, \rho(k) W, \rho(k) W^\vee) \, dk = \sum_{i=0}^{e-1} b_{in/e} q^{in/e} X^{in/e} \]
\[ = q^r (q^{n/e} - 1) \sum_{i=0}^{e-1} X^{in/e} = q^r (q^{n/e} - 1) \frac{1 - X^n}{1 - X^{n/e}}. \]
This gives the equality

\[ I(X, W, W^\vee, 1_{\sigma_F^\ell}) = (q - 1)(q^{n/e} - 1) \frac{q^r}{1 - X^{n/e}}. \]

On the other hand, and by [5, Proposition 8.1], the factor \( L(X, \pi, \pi^\vee) \) is equal to

\[ L(X, \pi, \pi^\vee) = \prod_{\nu \in R(\pi)} \frac{1}{1 - \nu(\varpi_F)X} = \frac{1}{1 - X^{n/e}}. \]

Now in general, as we supposed that \( L(X, \pi_1, \pi_2) \) is not equal to 1, we have \( \pi_2 \simeq \chi \pi_1^\vee \) for some unramified character \( \chi \) of \( F^\times \). However, we have

\[ L(X, \pi_1, \pi_2) = L(X, \pi_1, \chi \pi_1^\vee) = L(\chi(\varpi_F)X, \pi_1, \pi_1^\vee). \]

On the other hand, we have

\[ I(X, W, \chi W^\vee, 1_{\sigma_F^\ell}) = I(\chi(\varpi_F)X, W, W^\vee, 1_{\sigma_F^\ell}) = (q - 1)(q^{n/e} - 1) \frac{q^r}{1 - (\chi(\varpi_F)X)^{n/e}}. \]

However,

\[ L(X, \pi_1, \pi_2) = L(\chi(\varpi_F)X, \pi, \pi^\vee) = \frac{1}{1 - (\chi(\varpi_F)X)^{n/e}}, \]

and we are done.

\[ \square \]

**§10.** \( L \)-factors of banal cuspidal \( \ell \)-modular representations

In this section, we consider the cases \( R = \mathbb{F}_\ell \), and \( R = \mathbb{Q}_\ell \). In the \( \mathbb{Q}_\ell \) setting, we continue with the notations of the last section, and note that as \( \psi \) is integral, so are \( \psi_t \) and \( \psi_t^{-1} \). Our main theorem has the following interesting corollary.

**Corollary 10.1.** Let \( \tau_1 \) and \( \tau_2 \) be two banal cuspidal \( \ell \)-modular representations of \( G_n \), and \( \pi_1 \) and \( \pi_2 \) be any cuspidal \( \ell \)-adic lifts, then

\[ L(X, \tau_1, \tau_2) = r_\ell(L(X, \pi_1, \pi_2)). \]

**Proof.** We already noticed in Section 5 that if \( L(X, \pi_1, \pi_2) \) is equal to 1, then

\[ L(X, \tau_1, \tau_2) = r_\ell(L(X, \pi_1, \pi_2)) = 1, \]**
whether $\tau_1$ and $\tau_2$ are banal or not. Hence we only need to focus on the case when $L(X, \pi_1, \pi_2)$ is not equal to 1, that is $\pi_2 \simeq \chi \pi_1^\vee$ for some unramified character $\chi$. Let $W$ be the Stevens–Paskunas explicit Whittaker function associated to an extended maximal simple type of $\pi_1$ as in the statement of Theorem 9.1.

**Lemma 10.2.** The explicit Whittaker functions $W$ and $\chi W^\vee$ lie in the $\mathbb{Z}_\ell$-submodules $W_e(\pi_1, \psi_t)$ and $W_e(\pi_2, \psi_t^{-1})$ respectively.

**Proof.** As in the proof of Theorem 9.1, the representation $\pi_1$ contains an extended maximal simple type $(J_1, \Lambda_1)$ and $W$ is chosen to be the Paskunas–Stevens Whittaker function of Definition 8.3 relative to this data. As $\pi_1$ is integral, $\Lambda_1$ is integral by the end of Section 6. This implies that the trace character $\rho_{\Lambda_1}$ of $\Lambda_1$ has values in $\mathbb{Z}_\ell$. In particular the Bessel function $J_1$ (see Definition 8.1) associated to the pair $(J_1, \Lambda_1)$ takes values in $\mathbb{Z}_\ell$. Hence, as $\psi_t$ is integral, $W \in W_e(\pi_1, \psi_t)$ (see Definition 8.3). Now, $\pi_2$ is of the form $\chi \pi_1^\vee$ with $\chi$ an unramified character of $F^\times$ (which is integral as $\chi$ is unramified), so Proposition 8.2(3) implies that the Bessel function $\chi J_1^\vee$ is integral. We conclude that $\chi W^\vee$ belongs to $W_e(\pi_2, \psi_t^{-1})$ (see Definition 8.3 again).

 Granted $W \in W_e(\pi_1, \psi_t)$ and $\chi W^\vee \in W_e(\pi_2, \psi_t)$, we have

$$r_\ell(q^r(q^{n/e} - 1))r_\ell(L(X, \pi_1, \pi_2)) = r_\ell(I(X, W, \chi W^\vee, 1_{B_1}))$$

$$= I(X, r_\ell(W), r_\ell(\chi W^\vee), r_\ell(1_{B_1})).$$

Notice that $r_\ell(q^r(q - 1)(q^{n/e} - 1))$ is nonzero if and only if $\pi_1$ (hence $\pi_2$) is banal by the end of Section 6. As the integral $I(X, r_\ell(W), r_\ell(\chi W^\vee), r_\ell(1_{B_1}))$ belongs to the fractional ideal $(L(X, \tau_1, \tau_2))$ of $\mathbb{F}_\ell[X^\pm 1]$, we deduce that $r_\ell(L(X, \tau_1, \tau_2))$ divides $L(X, \tau_1, \tau_2)$. As in any case, thanks to [8, Theorem 3.13], the $L$-factor $L(X, \tau_1, \tau_2)$ divides $r_\ell(L(X, \pi_1, \pi_2))$, we deduce the desired equality.

**Remark 10.3.** As noticed in the introduction and Section 5, the analogue of Corollary 10.1 is also true when $\pi_1$ and $\pi_2$ are cuspidal representations of general linear groups of different ranks as the $L$-factors are all trivial.

**Acknowledgments.** We thank Paul Broussous and Shaun Stevens for stimulating discussions, and Jim Cogdell for bringing the test vector problem to our attention. We thank the referee for a very thorough review which improved greatly the presentation and clarified certain parts of
the arguments. This work has benefited from the support of the Heilbronn Institute for Mathematical Research, GDRI 571 “French–British–German network in Representation Theory”, and from the grant ANR-13-BS01-0012 FERPLAY.

References

[1] I. N. Bernstein and A. V. Zelevinsky, Induced representations of reductive p-adic groups, I, Ann. Sci. Éc. Norm. Supér. (4) 10(4) (1977), 441–472.
[2] C. J. Bushnell and P. C. Kutzko, The Admissible Dual of GL(N) via Compact Open Subgroups, Annals of Mathematics Studies 129, Princeton University Press, Princeton, NJ, 1993.
[3] J. W. Cogdell and I. I. Piatetski-Shapiro, Derivatives and L-functions for GL(n). Available at https://people.math.osu.edu/cogdell.1/ (To appear in the volume in honor of R. Howe’s 70th birthday.)
[4] H. Jacquet, I. I. Piatetski-Shapiro and J. Shalika, Conducteur des représentations du groupe linéaire, Math. Ann. 256(2) (1981), 199–214.
[5] H. Jacquet, I. I. Piatetski-Shapiro and J. A. Shalika, Rankin–Selberg convolutions, Amer. J. Math. 105(2) (1983), 367–464.
[6] H. Jacquet, A correction to conducteur des représentations du groupe linéaire [MR620708], Pacific J. Math. 260(2) (2012), 515–525.
[7] K.-M. Kim, Test vectors of Rankin–Selberg convolutions for general linear groups, Ph.D. thesis, The Ohio State University, 2010. Available at https://etd.ohiolink.edu/.
[8] R. Kurinczuk and N. Matringe, Rankin–Selberg local factors modulo ℓ, Selecta Math. (N.S.) 23(1) (2017), 767–811.
[9] N. Matringe, Essential Whittaker functions for GL(n), Doc. Math. 18 (2013), 1191–1214.
[10] A. Mínguez, Fonctions zêta ℓ-modulaires, Nagoya Math. J. 208 (2012), 39–65.
[11] A. Mínguez and V. Sécherre, Représentations lisses modulo ℓ de GLm(D), Duke Math. J. 163(4) (2014), 795–887.
[12] V. Paskunas and S. Stevens, On the realization of maximal simple types and epsilon factors of pairs, Amer. J. Math. 130(5) (2008), 1211–1261.
[13] M.-F. Vignéras, Représentations ℓ-modulaires d’un groupe réductif p-adique avec ℓ ≠ p, Progress in Mathematics 137, Birkhäuser Boston Inc., Boston, MA, 1996.
[14] M.-F. Vignéras, “On highest Whittaker models and integral structures”, in Contributions to Automorphic Forms, Geometry, and Number Theory, Johns Hopkins University Press, Baltimore, MD, 2004, 773–801.

Robert Kurinczuk
Heilbronn Institute for Mathematical Research
Department of Mathematics
Imperial College London
UK
robkurinczuk@gmail.com
Nadir Matringe
*Université de Poitiers*
*Laboratoire de Mathématiques et Applications*
*Téléport 2 - BP 30179*
*Boulevard Marie et Pierre Curie*
*86962 Futuroscope Chasseneuil Cedex*
*France*
*Nadir.Matringe@math.univ-poitiers.fr*