Global symplectic potentials on the Witten covariant phase space for bosonic extendons

R. Cartas-Fuentevilla

Instituto de Física, Universidad Autónoma de Puebla, Apartado postal J-48 72570, Puebla Pue., México (rcartas@sirio.ifuap.buap.mx).

It is proved that the projections of the deformation vector field, normal and tangential to the worldsheet manifold swept out by Dirac-Nambu-Goto bosonic extendons propagating in a curved background, play the role of global symplectic potentials on the corresponding Witten covariant phase space. It is also proved that the presymplectic structure obtained from such potentials by direct exterior derivation, has not components tangent to the action of the relevant diffeomorphism group of the theory.

Running title: Global symplectic....

I. INTRODUCTION

The purpose of the present letter is to extend and remark our recent analysis [1] on the symplectic geometry of the Witten covariant phase space for Dirac-Nambu-Goto(DNG) bosonic extendons or p-branes. In that letter we have demonstrated, using a covariant description of the deformation dynamics of such extended objects, that the phase space \( Z \) is endowed with a (pre)symplectic structure. Such a geometrical structure is covariant in the strong sense of being expressed in terms of ordinary tensor fields defined with respect to background coordinate systems, and not referred to distinct worldsheet and external background quantities (a remarkable virtue of the Carter formalism for the deformation dynamics of extendons [2, 3]). Furthermore, in our analysis [1], a gauge-fixing condition on the deformation dynamics was imposed, specifically the so called orthogonal gauge,
which considers the deformations orthogonal to the worldsheet as the only physically relevant, and ignores the deformations tangent to the worldsheet, since can always be identified with the action of a worldsheet diffeomorphism. Another limitation of that analysis is that the existence of degenerate directions on the covariant phase space, and associated with the gauge transformations of the extendon theory, was not considered explicitly. Such gauge directions may lead to a possible degeneracy of the presymplectic form, which in turns, may affect the invertibility of such a differential form, in order to construct the Poisson brackets, which are essential for a possible quantum mechanical treatment of the theory.

Therefore, the purpose of the present analysis is twofold. First, we shall demonstrate that in the general case where no gauge-fixing condition on the deformation dynamics is imposed, there exists an exact two-form on the phase space obtained from certain one-forms by exterior derivation. Such a two-form will be, in particular, closed, as required for constructing a (pre)symplectic form on \( Z \). It is shown that the one-forms (called symplectic potentials), have a remarkably simple form in terms of fundamental fields describing the deformation dynamics. Additionally, it is also proved that the gauge directions of the extendon theory under consideration, are eliminated in passing from \( Z \) to the reduced phase space (the space of solutions modulo gauge transformations), obtaining finally a nondegenerate geometrical structure for the theory.

At this point, it is opportune to review briefly the main results of the letter [1], which will be useful in the present analysis. Working in the orthogonal gauge,

\[
\eta^{\mu\nu} \xi^{\nu} = 0, \tag{1}
\]

where \( \eta^{\mu\nu} \) is the fundamental tensor of the worldsheet (that together with the complementary orthogonal projection \( \perp^{\mu\nu} \) satisfy \( g^{\mu\nu} = \eta^{\mu\nu} + \perp^{\mu\nu} \), being \( g_{\mu\nu} \) the background metric), and \( \xi^{\nu} \equiv \delta X^{\nu} \) the deformation vector field of the worldsheet, the symplectic current that is worldsheet covariantly conserved,

\[
\tilde{F}^{\nu} = \xi^{\rho} \tilde{\nabla}^{\nu} \xi_{\rho}, \quad \tilde{\nabla}_{\nu} \tilde{F}^{\nu} = 0, \tag{2}
\]

permits to construct a two-form \( \omega \) on the phase space \( Z \) of DNG bosonic extendons, given by

\[
\omega \equiv \int_{\Sigma} \sqrt{-\gamma} \tilde{F}^{\mu} d\Sigma_{\mu}, \tag{3}
\]
independent on the choice of $\Sigma$ (a spacelike section of the worldsheet corresponding to a Cauchy $p$-surface for the configuration of the extendon), and in particular Poincaré invariant. $d\bar{\Sigma}_\mu$ is the surface measure element of $\Sigma$, and is normal on $\Sigma$ and tangent to the worldsheet, and $\gamma$ is the determinant of the world sheet metric. In $[1]$, it is proved, by direct calculation of the exterior derivative of Eq. (3), that $\omega$ is identically closed on $Z$, $\delta \omega = 0$, representing then a (pre)symplectic structure on $Z$. However, in the general situation where a gauge-fixing condition is no imposed, and Eq. (1) is no necessarily satisfied, the symplectic current has the more complicated form $[1]$,

$$\tilde{J}^\nu = (\eta^{\mu\nu} \perp \sigma \rho + 2\eta^{\nu} [\sigma \eta_{\rho}] \xi^\sigma \nabla_\mu \xi^\rho, \tag{4}$$

and although is also worldsheet covariantly conserved $[1]$,

$$\nabla_\nu \tilde{J}^\nu = 0, \tag{5}$$

it remains the question whether $\omega$ in (3), with $\tilde{J}^\nu$ being of the general form (4), is closed. Although one can follow exactly the same procedure used in $[1]$ for answering this question, we shall face the problem using a different strategy, which will have the virtue of revealing a rich underlying structure of the phase space for DNG extendons.

In next Section the explicit forms and properties of the symplectic potentials on $Z$ are discussed. In Section III we discuss the issue of degenerate directions and the invariance properties of the relevant geometrical quantities on $Z$. We conclude in Section IV with some remarks and prospects for the future.

II. GLOBAL SYMPLECTIC POTENTIALS

As it is well known, on any manifold with symplectic structure $\omega$, locally one can introduce coordinates $p_i$ and $q_i$ such that $\omega = \delta(q_i \delta p_i)$, where $q_i \delta p_i$ is an one-form called the canonical symplectic potential. In general, such a potential exists only locally and is not unique.

In this section we shall demonstrate that, without invoking local canonical coordinates (which is in fact the spirit of the present covariant description of the phase space), there exists a global symplectic potential on $Z$, which generates by direct exterior derivation the symplectic current (4).
The demonstration is very simple, and consists in to rewrite $\tilde{J}^\nu$ in a way such that we can identify directly the corresponding symplectic potential.

Hence, using the definition $\tilde{\nabla}_\mu \equiv \eta_{\mu}^\nu \nabla_\nu$, the property $\eta^{\mu}_\rho \eta^{\nu}_\nu = \eta^{\mu}_\nu$, and the decomposition $\perp_{\rho\sigma} = g_{\rho\sigma} - \eta_{\rho\sigma}$, we can rewrite $\tilde{J}^\nu$ in (4), after suitably grouping, as

$$\tilde{J}^\nu = \eta^{\mu\alpha}[\xi_\sigma \nabla_\alpha \xi^\sigma - \eta^{\sigma}_\rho \xi^\sigma (\nabla_\alpha \xi_\rho + \nabla_\rho \xi_\alpha) + \eta^{\sigma}_\rho \xi_\alpha \nabla_\sigma \xi^\rho]; \tag{6}$$

the first terms can be identified directly in terms of variations of fundamental geometrical quantities, considering that $\delta \xi_\mu = -\xi^\nu \nabla_\mu \xi_\nu$ (see Eq. (23) in [1]), and that $\delta \eta^{\mu\nu} = -\eta^{\mu}_\rho \eta^{\nu}_\sigma \delta g^{\rho\sigma}$:

$$\eta^{\nu\alpha} \xi_\sigma \nabla_\alpha \xi^\sigma = -\eta^{\nu\alpha} \delta \xi_\alpha, \quad \xi^\sigma \eta^{\nu\alpha} \eta^{\sigma}_\rho (\nabla_\alpha \xi_\rho + \nabla_\rho \xi_\alpha) = -\xi_\sigma \delta \eta^{\nu\sigma}, \tag{7}$$

and the last term can be rewritten as

$$\eta^{\nu\alpha} \xi_\sigma \eta^{\sigma}_\rho \nabla_\sigma \xi^\rho = \frac{1}{2} (\eta^{\nu\alpha} \xi_\alpha) \eta^{\sigma}_\rho [\nabla_\sigma \xi_\rho + \nabla_\rho \xi_\sigma] = \eta^{\nu\alpha} \xi_\alpha \frac{\delta \sqrt{-\gamma}}{\sqrt{-\gamma}}, \tag{8}$$

where the last equality follows from the formula $\delta \sqrt{-\gamma} = \sqrt{-\gamma} \eta^{\mu\nu} \delta g^{\mu\nu}$ [3]. Therefore, from Eqs. (6), (7), and (8) we have that,

$$\sqrt{-\gamma} \tilde{J}^\nu = \sqrt{-\gamma} (\eta^{\alpha\nu} \delta \xi_\alpha + \delta \eta^{\alpha\nu} \xi_\alpha) + (\eta^{\alpha\nu} \xi_\alpha) \delta \sqrt{-\gamma} = \delta (-\sqrt{-\gamma} \eta^{\alpha\nu} \xi_\alpha), \tag{9}$$

where we have considered the Leibniz rule for $\delta$, and the fact that $\eta^{\alpha\nu} \xi_\alpha$ and $\delta \sqrt{-\gamma}$ correspond to one-forms on $Z$ and thus are anticommuting objects [3]. Therefore, Eq. (9) shows that the smooth one-form $\omega^\nu \equiv -\sqrt{-\gamma} \eta^{\alpha\nu} \xi_\alpha$ is a global symplectic potential density on $Z$ for DNG extendsos. Hence, Eq. (9) permits to write out $\omega$ (see paragraph after Eq. (5)) as

$$\omega = \int_\Sigma \delta \omega^\nu d\tilde{\Sigma}_\nu, \tag{10}$$

which shows that $\omega$ is an exact differential form, and in particular, an identically closed two-form on $Z$, since $\delta$ is nilpotent,

$$\delta \omega = 0, \tag{11}$$

as required for constructing a (pre)symplectic structure on the phase space, in this general case where no gauge-fixing condition (Eq. (1)) is imposed. Note that Eq. (5) guarantees that $\omega$ in (10) is independent on the choice of $\Sigma$ and, in particular, is Poincaré invariant.
It is worth pointing out some properties of the symplectic potential $\omega^\nu$. First, $\omega^\nu$ is not unique, and the form of the ambiguity is easily determined from Eq. (9), and using again the nilpotency of $\delta$: it is defined up to the exterior derivative of any (background) vector field, say $\lambda^\nu$, since

$$\sqrt{-\gamma}\tilde{J}^\nu = \delta(\omega^\nu + \delta\lambda^\nu). \tag{12}$$

Thus, one can think of $\omega^\nu$ and $\omega^\nu + \delta\lambda^\nu$ as gauge fields on the phase space, which correspond to the same field strength $\omega$, the only physically meaningful geometrical structure on $Z$. Furthermore, using again the decomposition $\eta^\nu_\alpha = g^\nu_\alpha - \perp^\nu_\alpha$, we can rewrite $\omega^\nu$ as

$$\omega^\nu = \sqrt{-\gamma}\perp^\nu_\alpha \xi^\alpha - \sqrt{-\gamma}\xi^\nu = \sqrt{-\gamma}\perp^\nu_\alpha \xi^\alpha - \delta(\sqrt{-\gamma}X^\nu), \tag{13}$$

where we have considered that $\delta\sqrt{-\gamma} = 0$, corresponds to the first order action variation, and the definition $\xi^\nu = \delta X^\nu$ for rewriting the second term as the exterior derivative of the zero-form $\sqrt{-\gamma}X^\nu$, which can now be identified, in particular, with the above vector field $\lambda^\nu$. Therefore, from Eqs. (9), and (13) we can see that both the tangential projection $\eta^\nu_\alpha \xi^\alpha$ to the world-sheet of the infinitesimal deformation $\xi^\nu$ (identified with the action of a dynamical world-sheet infinitesimal diffeomorphism) and the orthogonal projection $\perp^\nu_\alpha \xi^\alpha$ (identified with the physically observable measure of the deformation), play the role of global symplectic potentials on the phase space $Z$, with the background coordinate field $(\sqrt{-\gamma})X^\nu$ playing the role of the generating function of a transformation from the (global) variables $(\eta^\nu_\nu, X^\nu)$ to $(\perp^\alpha_\nu, X^\nu)$, and conversely, in accordance to Eq. (13). Furthermore, note that specifically in the case of the tangential projection, which can be ignored since is not physically observable as a deformation, its exterior derivative on the phase space is, in accordance with Eq. (10), physically meaningful. Hence, a quantity that is “pure gauge” in the conventional deformation scheme, generates by direct exterior derivation a physically relevant geometrical structure $\omega$ on the (nonconventional) phase space $Z$.

**III. THE SYMPLECTIC STRUCTURE ON THE REDUCED PHASE SPACE**

Another aspect well known of the covariant phase space formulation is that, by defining the phase space $Z$ as the space of solutions dynamically allowed by the classical equations of motion, there exist
naturally degenerate directions, which will be revealed in the degeneracy of the geometrical structure \(\omega\). More specifically, the motions along such directions correspond to the gauge transformations of the theory. In the present case of DNG extendons embedded in a curved spacetime (such as any theory formulated in terms of a spacetime manifold and tensor fields defined on it), the degenerate directions will correspond to spacetime infinitesimal diffeomorphisms, and particularly to worldsheet infinitesimal reparametrizations. In this sense, the geometrical structure \(\omega\) previously constructed is actually a \textit{presymplectic} form. Therefore, our purpose in this section is to demonstrate that \(\omega\) has not effectively components along such gauge directions. The argument is again very simple and consists essentially in proving the invariance of the deformation vector field \(\xi^\nu\) under spacetime infinitesimal diffeomorphisms,

\[
X^\mu \rightarrow X^\mu + \delta X^\mu. \tag{14}
\]

Note that physically \(\xi^\nu \equiv \delta X^\nu\) represents deformations of the worldsheet geometry, whereas \(\delta X^\mu\) in (14), a diffeomorphism. Under such a transformation \(\xi^\nu\) changes by

\[
\xi^\mu \rightarrow \delta(X^\mu + \delta X^\mu) = \xi^\mu + \delta^2 X^\mu, \tag{15}
\]

we mean, by a second order term (as it must!), which is negligible in a first order deformation scheme. However, \(\delta^2 X^\mu\) vanishes strictly on the phase space, in virtue of the nilpotency property, and therefore the deformation \(\xi^\nu\), an one-form on \(Z\), has not components along the gauge directions (as awaited!, since \(\xi^\nu\) is “the physically observable measure” of the deformations of the worldsheet geometry). As a consequence, any object on \(Z\) that depends on \(\xi^\nu\), also will have vanishing components along such directions. Hence, if \(Z\) is the space of solutions of the DNG extendon dynamics, and \(\hat{Z}\) is the space of solutions modulo gauge transformations (we mean, \(\hat{Z}\) is the \textit{quotient} space \(Z/G\), or reduce phase space, with \(G\) being the group of diffeomorphisms), then \(\hat{J}^\nu\) and \(\omega\) have vanishing components along the \(G\) orbits, and specifically \(\omega\) will be a nondegenerate two-form on \(\hat{Z}\). Note that, among all degenerate directions on \(Z\), there will be some directions associated to worldsheet reparametrizations, and then the diffeomorphism in Eq. (14) can take, in particular, the form of a worldsheet diffeomorphism,

\[
\delta X^\mu = \epsilon^\alpha \partial_\alpha X^\mu, \tag{16}
\]
where $\epsilon^\alpha$ is a infinitesimal shift in the worldsheet coordinates, and $\partial_\alpha$ denotes derivation with respect to them. Then $\tilde{\mathcal{J}}^\nu$ and $\omega$ are, in particular, invariant under (16).

There exists another form for demonstrating the invariance of $\tilde{\mathcal{J}}^\nu$ and $\omega$ under diffeomorphisms, and consists in realizing the role of the field $\lambda^\nu$ in (12). As we have seen, $\omega^\nu$ corresponds to projections of the variation of the background coordinate field $X^\nu$, and then $\delta \lambda^\nu$ in (12) can be understood, in particular, as an infinitesimal diffeomorphism, $\delta \lambda^\nu = \delta X^\nu$, and thus the invariance of $\tilde{\mathcal{J}}^\nu$ and $\omega$ is guaranteed from (12). Thus, the indeterminacy directions of the symplectic potential can correspond, in particular, to the gauge directions of the theory. Therefore, $\omega$ is finally our covariant and gauge invariant description of the canonical formalism for DNG bosonic extendons.

IV. REMARKS AND PROSPECTS

As mentioned, locally $\omega = \delta q_i \delta p_i$, and can be described in terms of a canonical symplectic potential in the forms $\omega = \delta(q_i \delta p_i)$ or $\omega = \delta(-p_i \delta q_i)$, with $p_i$ and $q_i$ being the conventional canonically conjugate variables. Similarly, in the present covariant formulation, $\tilde{\mathcal{J}}^\nu$ (and hence $\omega$) can be written as

$$
\tilde{\mathcal{J}}^\nu = \delta(\eta_\alpha^\nu \xi^\alpha) = \delta(\eta_\alpha^\nu \delta X^\alpha) = \delta(-X^\alpha \delta \eta_\alpha^\nu),
$$

(17)

in virtue again of the Leibniz rule and the nilpotency property, and similarly for the pair $(\perp_{\nu,\alpha}, X^\alpha)$. Elucidating, we can think of $(\eta_\alpha^\nu, X^\alpha)$ as conjugate variables in this covariant description of the phase space for DNG extendons. This question, and other relevant aspects of the transition between the classical and quantum domains, require of course of a deeper research, and will be the subject of forthcoming communications.

Although we have studied only the bosonic case, physically more interesting theories such as superextendons, will also be a problem for the future.

In a recent paper [4] we have obtained, using a different scheme for the deformations dynamics of extended objects, similar results to those presented in [1]; however, that scheme is weakly covariant.
ACKNOWLEDGMENTS

This work was supported by CONACyT and the Sistema Nacional de Investigadores (México). The author wants to thank H. Garcia Compean for drawing my interest to the study of extendons, and Dr. G. F. Torres del Castillo for discussions.

References

[1] R. Cartas-Fuentevilla, *Identically closed two-form for covariant phase space quantization of Dirac-Nambu-Goto p-branes in a curved spacetime*, to be published, Phys. Lett. B, (2002).

[2] B. Carter, Phys. Rev. D 48, 4835 (1993).

[3] B. Carter, 1997 *Brane dynamics for treatment of cosmic strings and vortons*, in *Recent Developments in Gravitation and Mathematics, Proc. 2nd Mexican School on Gravitation and Mathematical Physics (Tlaxcala, 1996)* [http://kaluza.physik.uni-konstanz.de/2MS] ed. A. Garcia, C. Lammerzahl, A. Macias and D. Nuñez (Konstanz: Science Network).

[4] R. Cartas-Fuentevilla, *Towards a covariant canonical quantization for closed topological defects without boundaries*, submitted to Phys. Lett. B, (2002).