Study of orbital transfers with time constraint and fuel optimization

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Abstract. Over the decades, with the advance in computational power and the development of new optimization techniques, a new opportunity arose for a reviewed approach and analysis of already existing solutions to orbital transfer problems. With that in mind, a study was proposed over Lambert's problem of bi-impulsive orbital transfers with time constraint for the execution of minimum fuel maneuvers between coplanar and non coplanar elliptical orbits. In order to achieve this goal, a genetic search algorithm was developed to determine the minimum fuel trajectory with an iterative variation of two Keplerian elements of the final orbit, one of them being the mean anomaly and the other a choice between the eccentricity, the inclination or the argument of periapsis. A number of six simulations were performed under these conditions, using different values - on each simulation - for the Keplerian elements of the final orbits not being iteratively varied. Remarkable sections were noted regarding the behaviour of fuel consumption values of the minimum fuel trajectories. One of which was a noted relation between the lowest cost minimum fuel transfers and the ratio of the semi-major axis and eccentricity of these transfers' initial and final orbits, considering both the orbits have the same value of argument of periapsis. Amongst all simulations, it was important to identify the choice between short or long way for the determination of the minimum fuel transfer, analysing the implications of that choice on the transfer time.

1. Lambert's Problem
The objective of this paper is the presentation of a study on methods of bi-impulsive orbital transfers with time constraint for the realization of fuel optimization between coplanar and non coplanar elliptical orbits. As an approach to calculate the transfer orbital maneuvers was considered the same methodology used in [1-5], where the equations were taken from the solution of the Lambert's problem presented in [6-8]. This method allows to determine the transfer orbit, with time constraint, that connects the terminal points within the initial and final orbits. This is known as the Two Point Boundary Value Problem (TPBVP) as shown in a simplified way in figure 1.

Initially, after defining the initial and final position vectors \( r_i \) and \( r_f \), it is necessary to determine their norm. With them, it is possible to determine the transfer angle for the short way, in equation 1, or the long way, in equation 2.
Figure 1. Two point boundary value problem

\[ \Delta \Phi = \cos^{-1}\left( \frac{r_0^* \cdot r_1^*}{r_0 r_1} \right) \]  

(1)

\[ \Delta \Phi = 2\pi - \cos^{-1}\left( \frac{r_0^* \cdot r_1^*}{r_0 r_1} \right) \]  

(2)

With the transfer angle, determine the \( A \) constant value as seen in equation 3.

\[ A = \sqrt{r_0 r_1 \left( 1 + \cos(\Delta \Phi) \right)} \]  

(3)

By iteratively estimating a \( z \) constant value inside the interval \([0, 4/a^2]\), it is possible to determine a \( C \) and \( S \), alongside a \( y \) and \( x \) constant, and calculate the estimated time, from equations 4 to 6.

\[ C = \frac{1 - \cos(\sqrt{z})}{x} \] \[ S = \frac{\sqrt{x - \sin(\sqrt{z})}}{\sqrt{2} a} \]  

(4)

\[ y = r_0 + r_1 - A \left( \frac{1 - x^2}{\sqrt{C}} \right) \] \[ x = \frac{y}{\sqrt{C}} \]  

(5)

\[ \Delta t = \frac{1}{\sqrt{\mu}} \left[ x^3 S + A \sqrt{y} \right] \]  

(6)

After the difference between the estimated time and the determined time fit inside a tolerance value, a set of constants \( f \), \( g \) and \( \dot{g} \) can be determined from use of equations 7 to 9.

\[ f = 1 - \frac{y}{r_0} \]  

(7)

\[ g = A \sqrt{\frac{y}{\mu}} \]  

(8)

\[ \dot{g} = 1 - \frac{y}{r_1} \]  

(9)

With these values, it is possible to calculate the velocity components on the initial and final points, with the use of equations 10, and the sub consequent velocity increment on those points, from equations 11.

\[ \frac{\dot{r}_A^*}{g} = \frac{1}{g} [r_1^* - f r_0^*] \] \[ \frac{\dot{r}_B^*}{g} = \frac{1}{g} [\dot{g} r_1^* - r_0^*] \]  

(10)

\[ \Delta v_1 = r_A - r_0 \] \[ v_2 = r_1 - r_B \]  

(11)

2. Search parameters

This study starts from the analysis of multiple solutions to the Two Point Boundary Value Problem (TPBVP), by varying the given transfer time for given terminal points, all of them serving as inputs of the TPBVP. The output of the TPBVP is the velocity increment necessary to achieve such a transfer. To obtain the TPBVP solution, Holland’s canonical genetic algorithm [9-10] was implemented to find the \( z \) variable value, always contained in the \([0, 4/a^2]\) for elliptical problems. With this \( z \) variable value the velocity increment is determined.
It was evident from the results obtained from the iterative sweep of initial conditions for the TPBVP, that for each transfer time sweep there was a minimum value for the velocity increment. Therefore the genetic search algorithm was modified to search not for velocity increment of the given value of transfer time, but to search for the minimum velocity increment. Consequently, the transfer time is no more an input of the problem, it is an output. This approach made it easier to obtain the results for a minimum velocity increment. The graphical presentation of the results of these simulations, using the iterative sweep of the initial conditions for the TPBVP, facilitates their analysis.

The canonical genetic algorithm was implemented with a population of 50 individuals, which values were of a random variable \( z \), and a random arithmetic recombination within the 25 strongest individuals over 50 generations with a 15% chance of random mutation. This procedure was repeated 8 times, thus a total of 8 universes with randomly distinct initial populations inside the \([0,4\pi]\) interval were considered, taking the strongest individual within all the generations and universes iterated. With this approach, it was possible to analyze the variation of two Keplerian elements in each simulation. One of which was the mean anomaly of the final orbit, in order to sweep all possible final positions, whilst shifting a second Keplerian element in order to understand how these changes affected the behavior of the minimum fuel transfers. It is worthwhile to note that the algorithm calculates the minimum fuel consumption for the short and long way trajectory and chooses the actual minimum fuel trajectory, indicating the long way choices as blue dots in the graphs.

The initial and final orbit's Keplerian elements are presented by table 1 and table 2, respectively. Table 2 indicates the range of variation of the Keplerian elements being varied in each simulation.

### Table 1. Initial orbits’ Keplerian elements

| \( a \) (\( R_{\text{Earth}} \)) | \( e \) | \( i \) (°) | \( \Omega \) (°) | \( \omega \) (°) | \( M \) (°) |
|---|---|---|---|---|---|
| 1.5 | 0.0000001 | 0 | 0 | 0 | 0 |

### Table 2. Final orbits’ Keplerian elements

| Sim. | \( a \) (\( R_{\text{Earth}} \)) | \( e \) | \( i \) (°) | \( \Omega \) (°) | \( \omega \) (°) | \( M \) (°) |
|---|---|---|---|---|---|---|
| 1 | 2 | 0.0000001 – 0.9 | 0 | 0 | 0 | 0 – 360 |
| 2 | 2 | 0.3 | 0 | 0 | 0 | 0 – 360 |
| 3 | 2 | 0.0000001 | 0 – 90 | 0 | 0 | 0 – 360 |
| 4 | 2 | 0.0000001 | 0 – 90 | 45 | 0 | 0 – 360 |
| 5 | 2 | 0.8 | 0 | 0 | 0 – 360 | 0 – 360 |
| 6 | 5 | 0.3 | 0 | 0 | 0 – 360 | 0 – 360 |

3. Simulation results

The results of all the simulations can be seen in figure 2. Simulation 1 consists of 1000 points, simulations 2, 5 and 6 show 1200 results and simulations 3 and 4 determined 420 minimum fuel trajectories. Each point of the data plots displayed in figure 2 presents a minimum velocity increment for a transfer time determined by the genetic algorithm search. The multiple points are a consequence of the variation of the terminal points, which in turn are a function of the keplerian elements of the final orbit. For each sweep of keplerian elements, the minimum velocity increment found by the genetic algorithm presents a pattern of variation, indicated by the plots in figure 2. The analysis of theses plots may indicate regions of lower values of minimum velocity increment for different terminal points.

By knowing that all the keplerian elements, except for the semi-major axis of orbit, have a finite well determined interval of possible values, the sweep was determined in order to cover all of the values the keplerian element being varied could achieve. Therefore, by covering all the values that
could be physically achieved, the lower value regions for minimum velocity increment were identified as the minimum values for the interval of terminal points analyzed.

In summary, the genetic algorithm determined the value of the minimum velocity increment for an already given terminal point. These terminal points were then iteratively varied, by sweeping the final orbit's keplerian elements. This process is proven useful in mission definition where the final orbit is open to determination. By identifying the lower value regions of minimum velocity increment, the final orbit's keplerian elements can be determined more easily. Nevertheless, it is imperative to say that mission determination is a multi objective search, where a minimum velocity increment is desirable but not the only target to be achieved. A multiobjective approach that could be applied in this case, to the search for a compromise solution that considers the simultaneous optimization of the fuel consumption and the orbital transfer time, can be found in [11-13]. However, the application of a multiobjective approach is beyond the scope of this work.

Additionally, careful considerations can be made for each of the simulation results. For example, if analyzed attentively, simulation 4 results in figure 1 shows two high fuel consumption boundary regions between the choices of short and long way. Figure 3 illustrates a remarkable trajectory close to these boundary regions. It is noticeable that a difference in the right ascension of the ascending node of the initial and final orbits raises conditions in which the final orbit can only be achieved, for transfer angles near 180°, by transfer orbits with very high inclination. This condition creates the highest Δv values of the simulation, explaining the source of the verified transition regions in figure 1.

4. Conclusion
A direct search for the minimum fuel consumption using Lambert's approach to the Two Point Boundary Value Problem, whilst calculating the transfer time relative to the minimum fuel trajectory determined, eases human analysis of results for simulations varying two keplerian elements simultaneously. Considering this type of simulation, it is important to note the choice made by the search algorithm between the short and long way. It must be able to analyze both options in order to be able to choose the actual minimum fuel trajectory, instead of imposing a reversion in the direction of the maneuver to achieve the short way and calculate a considerably larger fuel consumption.

The variation of the final orbit's mean anomaly and consequent sweep of multiple final points inside this orbit, with the variation of other of its keplerian elements and change on its shape and orientation, leads to surprising results. For example, a variation of the final orbit's eccentricity value, with a constant semi-major axis ratio between the initial and final orbits, added to equal values of argument of periapsis between them and quasi-circular initial orbit, leads to a lower value region of minimum fuel consumption trajectories for a specific eccentricity value. This region depends on the initial and final orbits' semi-major axis ratio. Another example is the change in the right ascension of the ascending node of the final orbit for non coplanar orbits leads to higher value regions of minimum fuel consumption trajectories for maneuvers with angles of approximately 180°. This is the opposite effect, as shown by the Hohmann transfer, displayed by initial and final orbits with equal right ascension of the ascending node.

The analysis must be extremely cautious in regions of angles near 180° and 0° or 360°. As the angles approach these values, numerical errors are far greater, making it advisable to discard a group of results that fell under these conditions. Nevertheless, the inclusion of these regions in the search parameters is interesting in order to help understand the global behavior of the simulation when approaching these extremes values.

A broad variety of possible simulations is still open to further studies. A total of six simulations were implemented. Amongst these, three of them were codependent simulations with the variation of the same pair of keplerian elements but with a change in a third element of the final orbit, providing independent results with different standards of behavior. A proposed new change of these values can be made to identify new standards of behavior or reinforce the ones already identified, or the variation of other pairs of keplerian elements, keeping a similarity to other simulations. Such analysis requires a great computational power, a great amount of time and the development of a faster and more efficient
algorithm, perhaps written in C language, but will unveil a broader understanding over Lambert's problem and all its possible approaches.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.pdf}
\caption{Graphical results of the \( \Delta v \) values of the minimum fuel transfers in function of \( e_2, i_2, \omega_2 \) and \( M_2 \).}
\end{figure}
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