Kulish-Sklyanin type models: integrability and reductions

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Abstract

We start with a Riemann-Hilbert problem (RHP) related to a BD.I-type symmetric spaces $SO(2r+1)/S(O(2r-2s+1) \otimes O(2s))$, $s \geq 1$. We consider two Riemann-Hilbert problems: the first formulated on the real axis $\mathbb{R}$ in the complex $\lambda$-plane; the second one is formulated on $\mathbb{R} \oplus i\mathbb{R}$. The first RHP for $s = 1$ allows one to solve the Kulish-Sklyanin (KS) model; the second RHP is relevant for a new type of KS model. An important example for nontrivial deep reductions of KS model is given. Its effect on the scattering matrix is formulated. In particular we obtain new 2-component NLS equations. Finally, using the Wronskian relations we demonstrate that the inverse scattering method for KS models may be understood as a generalized Fourier transforms. Thus we have a tool to derive all their fundamental properties, including the hierarchy of equations and the hierarchy of their Hamiltonian structures.

Keywords – Symmetric spaces, Multi-component NLS equations, Lax representation, The group of reductions

1 Introduction

The Kulish-Sklyanin model [27] may be viewed as an alternative version of the famous Manakov model [29]. Both models have important applications in physics, of which we will briefly mention the Bose-Einstein condensates, see [24, 28, 2, 31, 23]. Both models are particular cases of the family of multi-component NLS (MNLS) equations, which are related to the symmetric
spaces [6]. They are integrable in the sense that they possess Lax representations whose Lax operator is linear in \( \lambda \) with potential \( Q(x, t) \) taking value in simple Lie algebra (generalized Zakharov-Shabat system). The solutions for the direct and the inverse scattering problem for such operators by now are well known: see [32, 5, 1, 8, 11, 15, 19, 21]. The solutions of these equations can be derived effectively using the dressing Zakharov-Shabat method [35, 33, 25], see also [8, 14, 15].

Another important trend in soliton theory is based on the derivation of new MNLS equations by applying Mikhailov reductions [30] to generic MNLS, see [14, 25, 15, 13, 10]. In particular, recently we have been able to derive two new 2-component MNLS [20]; below we find a third example of such equation which follows from a generic KSm.

Most of the techniques mentioned above can be applied also to Lax operators which are quadratic in \( \lambda \). Simplest cases of such Lax operators related to the algebra \( sl(2) \) have been considered before [18, 16, 17]. Recently an alternative approach based on the Riemann-Hilbert problem allowed one to generalize these results to any simple Lie algebra [9, 20]. Therefore we will start with a RHP and then will demonstrate that: i) it is equivalent to the traditional approach via the Lax representation; ii) it is a convenient tool to derive more general Lax representations in which both operators are polynomial in \( \lambda \) of order \( k \geq 2 \); iii) allows one to analyze the spectral properties not only of the Lax operators, but also of the relevant recursion operators. In conclusion we will demonstrate that the RHP can be viewed as most effective approach to the fundamental properties of the NLEE and especially to their hierarchies of Hamiltonian structures.

The family of Riemann-Hilbert problems (RHP) [9] is of the form:

\[
\begin{align*}
\xi^+(x, t, \lambda) &= \xi^-(x, t, \lambda)G(x, t, \lambda), \\
i \frac{\partial G}{\partial x} - \lambda^k [J, G(x, t, \lambda)] &= 0, \\
i \frac{\partial G}{\partial t} - \lambda^{2k} [J, G(x, t, \lambda)] &= 0,
\end{align*}
\]

(1.1)

where \( \xi^+(x, t, \lambda) \) take values in a simple Lie group \( G \) and \( J \) is an element to the corresponding simple Lie algebra \( g \). The specific choice of \( k, g \) and \( J \) determine the class and the type of NLEE under study.

The value of \( k \) determines the order of the Lax operator \( L \) as a polynomial in \( \lambda \). The majority of Lax operators \( L(\lambda) \) that have been used to study most important NLEE are linear in \( \lambda \), so \( k = 1 \). There have been also important examples of Lax operators that are quadratic in \( \lambda \), i.e. \( k = 2 \) [16, 17, 18]. Note, that the order \( p \) of the second Lax operators \( M(\lambda) \) is always \( p \geq k \). Below we will stick to these two values of \( k \) and will assume that \( p = 2k \), but the method outlined below can easily be extended to any finite values of \( k \) and \( p \). We will not touch here the other important classes of Lax operators that are rational functions of \( \lambda \) [33].

The choice of \( g \) and \( J \in \mathfrak{h} \) – the Cartan subalgebra of \( g \), determine the structure of the phase space of the NLEE; this includes such important things as the number of the independent functions entering the NLEE and the Poisson brackets between them. This has been well known for the generalized Zakharov-Shabat systems [32, 35]; the relevant Lax operators are linear in \( \lambda \) (\( k = 1 \)) and take the form:

\[
L \psi = i \frac{\partial \psi}{\partial x} + (Q(x, t) - \lambda J) \psi(x, t, \lambda) = 0.
\]

(1.2)
The phase space $\mathcal{M}$ of the relevant NLEE or, in other words, the space of allowed potentials is defined as:

$$\mathcal{M} \equiv \{ Q(x, t) = [J, X(x, t)], \quad X(x, t) \in \mathfrak{g} \},$$  \hspace{1cm} (1.3)

i.e. $Q(x, t)$ belongs to the co-adjoint orbit of $O_J \in \mathfrak{g}$ passing through $J$.

We note here that if $\text{rank} \mathfrak{g} > 1$ and we choose $J$ with different real eigenvalues then we will have NLEE on a homogeneous spaces. As typical representatives here we can mention the $N$-wave equations [32]. For other, more special choices of $J$ and $\mathfrak{g}$ our construction is on symmetric space. These facts have been well known for very long time [29, 6, 27]. They can also be generalized for $k \geq 2$.

In Section 2 below we give some preliminaries concerning the structure of the BD.I symmetric spaces, RHP and related Lax pairs and integrable equations, as well as the reduction group [30]. In Section 3 we provide the construction of the Lax representations via the RHP method for $k = 1$ and $k = 2$. The next Section 4 we give example of $\mathbb{Z}_6$-reduction of a generic KS model thus deriving new integrable 2-component NLS equations, see [20]. In Section 5 we formulate the consequences of these reductions on the scattering matrix and scattering data. In the last Section 6 we formulate the integrability properties of the KS type models. These are based on the Wronskian relations and the ‘squared’ solutions of the Lax operators. We prove that they are complete set of functions in the phase space $\mathcal{M}$ and derive the expansions of $Q(x, t)$ and its variation $\text{ad}_J^{-1}\delta Q$ over the squared solutions. These results allow us to formulate the fundamental properties of the NLEE and their Hamiltonian hierarchies.

2 Preliminaries

2.1 On BD.I-type symmetric spaces

For our specific purposes we will choose the simple Lie group $\mathcal{G} \simeq SO(2r + 1)$, its Lie algebra $\mathfrak{g} \simeq so(2r + 1)$. The orthogonality condition that we will use below is

$$X \in so(2r + 1) \quad \text{iff} \quad X + S_0X^TS_0^{-1} = 0, \quad S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1}E_{k,2r+2-k},$$  \hspace{1cm} (2.1)

where $E_{kn}$ is a $2r+1 \times 2r+1$-matrix with $(E_{kn})_{pj} = \delta_{kp}\delta_{nj}$. This choice ensures that the Cartan subalgebra $\mathfrak{h}$ consists of diagonal matrices. The element $J \in \mathfrak{h}$ is chosen as:

$$J = \sum_{k=1}^{s} H_{e_k} = \begin{pmatrix} \mathbb{I}_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathbb{I}_s \end{pmatrix}. \hspace{1cm} (2.2)$$

We will assume that the reader is familiar with the theory of simple Lie groups and algebras, see [22]. The system of positive roots of $so(2r + 1)$ is well known [22]:

$$\Delta^+ = \{ e_i \pm e_j, \quad 1 \leq i < j \leq r, \quad e_j, \quad 1 \leq j \leq r \}.$$
Here we also mention that using $J$ and the Cartan involution one can introduce a $\mathbb{Z}_2$-grading in $\mathfrak{g}$:

$$C_1 = \exp(\pi i J) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \mathbb{I}_{2r-2s+1} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)},$$

$$\mathfrak{g}^{(0)} \equiv \{ X \in \mathfrak{g}^{(0)} : C_1 X C_1^{-1} = X \}, \quad \mathfrak{g}^{(1)} \equiv \{ Y \in \mathfrak{g}^{(1)} : C_1 Y C_1^{-1} = -Y \}. \quad (2.3)$$

The $\mathbb{Z}_2$-grading means that

$$[X_1, X_2] \in \mathfrak{g}^{(0)}, \quad [X_1, Y_1] \in \mathfrak{g}^{(1)}, \quad [Y_1, Y_2] \in \mathfrak{g}^{(0)}, \quad (2.4)$$

and provides the local structure of the symmetric space of BD.I-class $SO(2r+1)/(SO(2r-2s+1) \otimes SO(2s))$. We will see that this construction is directly related to the KSm for $s = 1$. It is well known that the set of positive roots and the system of simple roots of $B_r$ are:

$$\Delta^+_{B_r} \equiv \{ e_i - e_j, \quad e_i + e_j, \quad 1 \leq i < j \leq r; \quad e_j, \quad 1 \leq j \leq r \} \quad (2.5)$$

Introducing $J$ as in (2.2) we split the system of positive roots into two subsets:

$$\Delta^+_1 = \{ \beta, \quad \beta(J) = 1 \} \quad \Delta^+_{0} = \{ \beta, \quad \beta(J) = 0 \mod 2 \}. \quad (2.6)$$

Below we will consider two cases with $s = 1$ and $s = 3$:

$$s = 1, r \geq 3 \quad \Delta^+_0 = \{ e_i \pm e_j, \quad 2 \leq i < j \leq r; \quad e_j, \quad 2 \leq i \leq r \},$$

$$\Delta^+_1 = \{ e_1 \pm e_j, \quad 2 \leq j \leq r; \quad e_1 \}; \quad (2.7)$$

$$s = 3, r = 4 \quad \Delta^+_0 = \{ e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3, e_1, e_2, e_3 \},$$

$$\Delta^+_1 = \{ e_1 \pm e_4, e_2 \pm e_4, e_3 \pm e_4, e_4 \};$$

The potential of the Lax operator as well as the coefficients $Q_{2s-1}$ are given by:

$$Q(x, t) = \sum_{\alpha \in \Delta^+_1} (q_{\alpha} E_{\alpha} + p_{\alpha} E_{-\alpha}) \in \mathfrak{g}^{(1)}. \quad (2.8)$$

2.2 The RHP and Lax representations

We will start by formulating the RHP (1.1). Given the sewing function $G(x, t, \lambda)$ find the functions $\xi^\pm(x, t, \lambda)$ taking values in the simple Lie group $\mathcal{G}$ and analytic for $\Im \lambda^k \gtrless 0$ such that eq. (1.1) holds. It is natural to impose also the normalization condition

$$\lim_{\lambda \to \infty} \xi^\pm(x, t, \lambda) = 1, \quad (2.9)$$

which ensures that the RHP has unique regular solution, on Figure 1 we show the analyticity regions and the contours that will be used below for $k = 1$ and $k = 2$. 

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In order to derive the relevant NLEE in terms of 

\[ \xi^{-}(x, t, \lambda) \]

equivalent ways, e.g.:

\[ \xi = \text{Obviously such choice for normalization (2.9). Indeed, in this case we can use the asymptotic expansion:} \]

\[ L\xi \equiv i \frac{\partial \xi}{\partial x} + U(x, t, \lambda)\xi(x, t, \lambda) - \lambda^{k}[J, \xi(x, t, \lambda)] = 0, \]

\[ M\xi \equiv i \frac{\partial \xi}{\partial x} + V(x, t, \lambda)\xi(x, t, \lambda) - \lambda^{2k}[J, \xi(x, t, \lambda)] = 0, \]

where \( U(x, t, \lambda) \) and \( V(x, t, \lambda) \) are polynomials in \( \lambda \) of order \( k - 1 \) and \( 2k - 1 \) respectively:

\[ U(x, t, \lambda) = \lambda^{k}J - (\lambda^{k}\xi_{J}(x, t, \lambda))_{+}, \quad V(x, t, \lambda) = \lambda^{2k}J - (\lambda^{2k}\xi_{J}(x, t, \lambda))_{+}, \]

**Idea of the proof.** Consider the functions

\[ g^{\pm}(x, t, \lambda) = i \frac{\partial \xi}{\partial x}(\xi^{-1}(x, t, \lambda)) + \lambda^{k}\xi^{\pm}(x, t, \lambda)J(\xi^{-1}(x, t, \lambda)), \]

\[ f^{\pm}(x, t, \lambda) = i \frac{\partial \xi}{\partial t}(\xi^{-1}(x, t, \lambda)) + \lambda^{2k}\xi^{\pm}(x, t, \lambda)J(\xi^{-1}(x, t, \lambda)), \]

and using the explicit \( x \) and \( t \)-dependence of \( G(x, t, \lambda) \) prove that \( g^{+}(x, t, \lambda) = g^{-}(x, t, \lambda) \) and \( f^{+}(x, t, \lambda) = f^{-}(x, t, \lambda) \). Then, using eq. (2.9) we find that

\[ \lim_{\lambda \to \infty} g^{+}(x, t, \lambda) = \lambda^{k}J, \quad \lim_{\lambda \to \infty} f^{+}(x, t, \lambda) = \lambda^{2k}J, \]

It remains to apply the great Liouville theorem that ensures that the functions \( g^{+}(x, t, \lambda) = g^{-}(x, t, \lambda) \) (resp. \( f^{+}(x, t, \lambda) = f^{-}(x, t, \lambda) \)) are analytic on the whole complex \( \lambda \)-plane and therefore are polynomial in \( \lambda \) of order \( k \) (resp. \( 2k \)). Of course the coefficients of these polynomials may depend on \( x \) and \( t \). The explicit relations (2.11) of these coefficients with the solution \( \xi^{\pm}(x, t, \lambda) \) has been proposed by Drinfeld-Sokolov [4], see also [9].

\[ \square \]

The explicit derivation of the Lax pairs is very effective if \( \xi^{\pm}(x, t, \lambda) \) satisfy the canonical normalization (2.9). Indeed, in this case we can use the asymptotic expansion:

\[ \xi^{\pm}(x, t, \lambda) = \exp(Q(x, t, \lambda)), \quad Q(x, t, \lambda) = \sum_{s=1}^{\infty} \lambda^{-2s+1}Q_{2s-1}. \]

Obviously such choice for \( \xi^{\pm}(x, t, \lambda) \) involves a \( \mathbb{Z}_{2} \) reduction, which can be formulated in several equivalent ways, e.g.:

\[ \begin{aligned}
  a) \quad & \xi^{+}(x, t, -\lambda) = (\xi^{-})^{-1}(x, t, \lambda), \quad Q_{2s-1} \in \mathfrak{g}^{(1)}, \\
  b) \quad & \xi^{+}(x, t, \lambda^{s}) = (\xi^{-})^{s}(x, t, \lambda) \quad \text{iff} \quad Q_{2s-1} = -Q_{2s-1}^{\dagger},
\end{aligned} \]

In order to derive the relevant NLEE in terms of \( Q_{2s-1} \) we will use the formulae:

\[ \xi^{\pm}J\xi^{\pm}(x, t, \lambda) = J + \sum_{p=1}^{\infty} \frac{1}{p!} \text{ad}_{Q}^{p}J, \quad \frac{\partial \xi^{\pm}}{\partial x}(x, t, \lambda) = \frac{\partial Q}{\partial x} + \sum_{p=1}^{\infty} \frac{1}{(p+1)!} \text{ad}_{Q}^{p} \frac{\partial Q}{\partial x}. \]

which allow us to express the Lax pair coefficients \( U_{s}(x, t) \) and \( V_{s}(x, t) \) in terms of \( Q_{2s-1} \) and their \( x \)-derivatives. In all our considerations we will need only the first few terms of these expansions; for more details see [9].

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2.3 The reduction group

Following Mikhailov [30] we will also impose additional reductions using the famous reduction group \( G_R \).

\( G_R \) is a finite group which preserves the Lax representation (2.10), i.e. it ensures that the reduction constraints are automatically compatible with the evolution. \( G_R \) must have two realizations: i) \( G_R \subset \text{Aut} \, \mathfrak{g} \) and ii) \( G_R \subset \text{Conf} \, \mathbb{C} \), i.e. as conformal mappings of the complex \( \lambda \)-plane. To each \( g_k \in G_R \) we relate a reduction condition for the Lax pair as follows:

\[
C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda), \quad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda),
\]

(2.17)

where \( C_k \in \text{Aut} \, \mathfrak{g} \) and \( \Gamma_k(\lambda) \in \text{Conf} \, \mathbb{C} \) are the images of \( g_k \) and \( \eta_k = 1 \) or \(-1\) depending on the choice of \( C_k \). Since \( G_R \) is a finite group then for each \( g_k \) there exist an integer \( N_k \) such that \( g_k^{N_k} = \text{I} \). More specifically, below we will consider \( \mathbb{Z}_2 \)-reductions of the form:

a) \( B_1 U^+(\kappa_1(\lambda))B_1^{-1} = U(\lambda), \quad B_1(V^+(\kappa_1(\lambda))B_1^{-1} = V(\lambda), \)

(2.18)

b) \( B_2 U^T(\kappa_2(\lambda))B_2^{-1} = -U(\lambda), \quad B_2(V^T(\kappa_2(\lambda))B_2^{-1} = -V(\lambda), \)

(2.19)

c) \( B_3 U^*(\kappa_1(\lambda))B_3^{-1} = -U(\lambda), \quad B_3(V^*(\kappa_1(\lambda))B_3^{-1} = -V(\lambda), \)

(2.20)

d) \( B_4 U(\kappa_2(\lambda))B_4^{-1} = U(\lambda), \quad B_4(V(\kappa_2(\lambda))B_4^{-1} = V(\lambda), \)

(2.21)

where the automorphisms \( B_k \) must of finite order. In the cases (2.18), (2.19) and (2.20) \( B_k \) must be of even order, which in general could be bigger than 2.

Beside the \( \mathbb{Z}_2 \)-reductions we will impose additional \( \mathbb{Z}_p \)-reductions with \( p > 2 \):

\[
A_p U(x, t, \kappa_p(\lambda))A_p^{-1} = U(x, t, \lambda), \quad A_p V^*(x, t, \kappa_p(\lambda))A_p^{-1} = V(x, t, \lambda),
\]

\[
A_p(\xi^+)(x, t, \kappa_p(\lambda))A_p^{-1} = (\xi^+)^{-1}(x, t, \lambda), \quad \kappa_p(\lambda) = \lambda \omega, \quad \omega = e^{2\pi i / p},
\]

(2.22)

where \( A_p \) is an automorphism of \( \mathfrak{g} \) of order \( p \). Typically we will use a realization of \( A_p \) as an element of the Weyl group of \( \mathfrak{g} \).

3 Generic Kulish-Sklyanin type models

The sewing function of our RHP problem depends on two additional parameters \( x \) and \( t \) in a special way, which makes it convenient to analyze and solve special classes of nonlinear evolution equations (NLEE) in two-dimensional space-time, known also as soliton equations. Here we also assume that the relevant symmetric space is \( SO(2r+1)/(SO(2r-1) \times SO(2)) \).

3.1 The compatibility condition \( k = 1, \ s = 1 \).

In the simplest case \( k = 1 \) and \( s = 1 \) the Lax pair:

\[
U^{(1)}(x, t, \lambda) = U_1(x, t) - \lambda J, \quad V^{(1)}(x, t, \lambda) = iQ_{1,x} + V_2(x, t) + \lambda V_1(x, t) - \lambda^2 J,
\]

(3.1)
Figure 1: The continuous spectrum of a $L(\lambda)$ in thick blue, the analyticity regions $\Omega_s$ and the contours $\gamma_s$, $s = 1, \ldots, k$. Left panel: $k = 1$; $L(\lambda)$ is linear in $\lambda$ (1.2). Right panel: $k = 2$: $L(\lambda)$ is quadratic in $\lambda$ (3.2).

where

$$U_1(x,t) = [J, Q_1(x,t)] = Q(x,t), \quad V_1(x,t) = Q(x,t) \quad V_2(x,t) = \frac{1}{2} \text{ad}_Q Q(x,t),$$

(3.2)

$$Q_1(x,t) = \begin{pmatrix} 0 & q^T & 0 \\ -\vec{p} & 0 & s_0 \vec{q} \\ 0 & -\vec{p}^T s_0 & 0 \end{pmatrix}, \quad V_2(x,t) = \begin{pmatrix} (\vec{q}, \vec{p}) & 0 \\ 0 & s_0 \vec{q}^T \vec{p} s_0 - \vec{p}^T \vec{q}^T & 0 \\ 0 & 0 & -(\vec{q}, \vec{p}) \end{pmatrix}.$$ (3.3)

As a result we get that this Lax pair leads to the well known Kulish-Sklyanin model whose integrability has been known since 1981 [27]:

$$i \frac{\partial \vec{q}}{\partial t} + \frac{\partial^2 \vec{q}}{\partial x^2} + 2(\vec{q}^T, \vec{q})\vec{q} - (\vec{q}^T s_0 \vec{q}) s_0 \vec{q}^* = 0, \quad s_0 = \sum_{k=1}^{2r-1} (-1)^k E_{k,2r-k}.$$ (3.4)

where now $E_{kn}$ is a $2r-1 \times 2r-1$-matrix with $(E_{kn})_{pj} = \delta_{kp}\delta_{nj}$. For applications of this model to Bose-Einstein condensates and detailed analysis for the inverse spectral transform see [8].

3.2 The compatibility condition $k = 1$, $s > 1$.

For $k = 1$ and $s > 1$ the Lax pair takes the form:

$$U(x,t,\lambda) = Q(x,t) - \lambda J, \quad V^{(1)}(x,t,\lambda) = iQ_{1,x} + V_2(x,t) + \lambda Q(x,t) - \lambda^2 J,$$ (3.5)
where
\[
Q(x, t) = \begin{pmatrix} 0 & q & 0 \\ p & 0 & \tilde{q} \\ 0 & \tilde{p} & 0 \end{pmatrix}, \quad V_2(x, t) = \begin{pmatrix} -qp & 0 & 0 \\ 0 & pq - \tilde{q}p & 0 \\ 0 & 0 & \tilde{q}p \end{pmatrix}.
\]
(3.6)

The matrix \(S_0\) from orthogonality condition (2.1) in this case equals \(\begin{pmatrix} 0 & 0 & s_1 \\ s_1^{-1} & 0 & 0 \end{pmatrix}\). The blocks \(s_k, k = 1, 2\) are easily determined from (2.1) and satisfy \(s_2^2 = 1, s_2^{-1} = s_2\). Then \(\tilde{q} = -s_2q^T s_1\) and \(\tilde{p} = -s_2p^T s_1^{-1}\). As a result we get the generic Kulish-Sklyanin model [27]:
\[
i \frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + 2qpq - q\tilde{q}p = 0, \quad i \frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial x^2} - 2pqp + \tilde{q}\tilde{p}p = 0,
\]
(3.7)

After the additional reduction \(p = q^\dagger\) we get:
\[
i \frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + 2qq^\dagger q - q\tilde{q}q^\dagger = 0.
\]
(3.8)

### 3.3 The compatibility condition \(k = 2\)

Now \(U^{(2)}(x, t, \lambda)\) is quadratic in \(\lambda\), \(V^{(4)}(x, t, \lambda)\) is quartic in \(\lambda\) and we keep \(s = 1\):
\[
U^{(2)}(x, t, \lambda) = U_2(x, t) + \lambda Q(x, t) - \lambda^2 J, \\
V^{(4)}(x, t, \lambda) = V_4(x, t) + \lambda V_3(x, t) + \lambda^2 V_2(x, t) + \lambda^3 - \lambda^4 J,
\]
(3.9)

where \(Q(x, t)\) is given by (3.6) and \(U_2(x, t) = V_2(x, t)\) and in addition
\[
V_3(x, t) = \begin{pmatrix} 0 & V_{3,2} & 0 \\ V_{3,1} & 0 & s_0 V_{3,2} \\ 0 & V_{3,1} s_0 & 0 \end{pmatrix}, \quad V_{3,1} = \bar{w} + \frac{1}{3} (\bar{p}^T s_0 \bar{p}) s_0 \bar{q} - \frac{2}{3} (\tilde{q}, \bar{p}) \tilde{p},
\]
\[
V_{3,2} = s_0 \bar{v} + \frac{1}{3} (\tilde{q}^T s_0 \bar{q}) \bar{p} - \frac{2}{3} (\tilde{q}, \bar{p}) s_0 \bar{q}.
\]
(3.10)

\[
V_4 = \begin{pmatrix} V_{4,11} & 0 & 0 \\ 0 & V_{4,22} & 0 \\ 0 & 0 & V_{4,33} \end{pmatrix}, \quad Q_3(x, t) = \begin{pmatrix} 0 & \bar{v}^T & 0 \\ -\bar{w} & 0 & s_0 \bar{v} \\ 0 & -\bar{w}^T s_0 & 0 \end{pmatrix},
\]
(3.11)

\[
V_{4,11} = -V_{4,33} = i ((\tilde{q}, \bar{p}) - (\tilde{q}, \bar{p}_x)) + (\tilde{q}, \bar{p})^2 - \frac{1}{2} (\tilde{q}^T s_0 \bar{q}) (\bar{p}^T s_0 \bar{p}),
\]
\[
V_{4,22} = i (\tilde{p}_x \tilde{q}^T - \bar{p} \tilde{q}_x^T + s_0 (\tilde{q}_x \bar{p}^T - \tilde{q} \bar{p}_x^T) s_0) - (\tilde{q}, \bar{p}) (\bar{q}^T - s_0 \bar{q} \tilde{q}^T) s_0.
\]

The compatibility condition reads:
\[
\lambda^3 : \quad i \frac{\partial Q}{\partial x} + [U_2, Q] + [Q, V_2] = [J, V_3],
\]
\[
\lambda^2 : \quad i \frac{\partial V_2}{\partial x} + [U_2, V_2] + [Q, V_3] = [J, V_4],
\]
(3.12)
Since $U_2 = V_2$ the first of the equations (3.12) gives:

$$V_3(x, t) = \text{ad}^{-1} i \frac{\partial Q}{\partial x} = i \begin{pmatrix} 0 & \tilde{q}_x^T & 0 \\ -\tilde{p}_x & 0 & s_0 \tilde{q}_x \\ 0 & -\tilde{p}_x s_0 & 0 \end{pmatrix},$$

(3.13)

which in components give:

$$\tilde{v} = i \frac{\partial \tilde{q}}{\partial x} + \frac{2}{3} (\tilde{p}, \tilde{q}) \tilde{q} - \frac{1}{3} (\tilde{q}^T s_0 \tilde{q}) s_0 \tilde{p}, \quad \tilde{w} = -i \frac{\partial \tilde{p}}{\partial x} + \frac{2}{3} (\tilde{p}, \tilde{q}) \tilde{p} - \frac{1}{3} (\tilde{p}^T s_0 \tilde{p}) s_0 \tilde{q}.$$  

(3.14)

The second equation in (3.12) is identically satisfied as a consequence of (3.14).

Finally the equations of motion:

$$\lambda : \quad i \frac{\partial V_3}{\partial x} - i \frac{\partial Q}{\partial t} + [Q, V_4] + [U_2, V_3] = 0, \quad \lambda^0 : \quad i \frac{\partial V_4}{\partial x} - i \frac{\partial U_2}{\partial t} + [U_2, V_4] = 0.$$  

(3.15)

Since in addition we put $\tilde{p} = \epsilon \tilde{q}^*$ we get:

$$i \frac{\partial \tilde{q}}{\partial t} + \frac{\partial^2 \tilde{q}}{\partial x^2} + i \epsilon s_0 (\tilde{q} s_0 \tilde{q}) \frac{\partial \tilde{q}^*}{\partial x} + s_0 (\tilde{q}, \tilde{q}^*) (\tilde{q} s_0 \tilde{q}) s_0 \tilde{q}^* - \left( \frac{1}{2} |(\tilde{q} s_0 \tilde{q})|^2 + 2i \epsilon (\tilde{q}, \tilde{q}^*) \right) \tilde{q} = 0.$$  

(3.16)

One can check that the second equation in (3.15) holds identically as a consequence of (3.16).

### 3.4 The gauge equivalent systems $k = 2$

We fix up the gauge by requesting that $U(x, t, \lambda = 0) = 0$ and $V(x, t, \lambda = 0) = 0$. Thus

$$\tilde{L} \tilde{\psi} \equiv i \frac{\partial \tilde{\psi}}{\partial x} + (\lambda \tilde{Q}(x, t) - \lambda^2 J) \tilde{\psi}(x, t, \lambda) = 0,$$

$$\tilde{M} \tilde{\psi} \equiv i \frac{\partial \tilde{\psi}}{\partial t} + (\lambda \tilde{V}_3(x, t) + \lambda^2 \tilde{V}_2(x, t) + \lambda^3 \tilde{Q}(x, t) - \lambda^2 J) \tilde{\psi}(x, t, \lambda) = 0.$$  

(3.17)

Therefore $\tilde{\psi}(x, t, \lambda) = g_0(x, t) \psi(x, t, \lambda)$, i.e

$$i \frac{\partial g_0}{\partial x} - g_0(x, t) U_2(x, t) = 0, \quad i \frac{\partial g_0}{\partial t} - g_0(x, t) V_4(x, t) = 0.$$  

(3.18)

so $g_0(x, t)$ is a block-diagonal function. Then $\tilde{V}_k$ and $\tilde{Q}$ have the same block structure of $V_k$ and $Q$; we will denote their blocks by additional ‘tilde’. Then the NLEE get the form:

$$i \frac{\partial \tilde{q}}{\partial t} + \frac{\partial^2 \tilde{q}}{\partial x^2} + i \frac{\partial}{\partial x} \left( 2 \tilde{q} (\tilde{q}^T \tilde{p}) - s_0 \tilde{p} (\tilde{q}^T \tilde{q}) \right) = 0,$$

$$i \frac{\partial \tilde{p}}{\partial t} - \frac{\partial^2 \tilde{p}}{\partial x^2} - i \frac{\partial}{\partial x} \left( 2 \tilde{p} (\tilde{q}^T \tilde{q}) - s_0 \tilde{q} (\tilde{p}^T \tilde{p}) \right) = 0.$$  

(3.19)
4 Generic KS type models and their $\mathbb{Z}_n$-reductions

Here we derive KS type models related to generic BD.I symmetric spaces $SO(2r+1)/(SO(2k) \times SO(2r-2k+1))$. These are rather complicated systems of equations for $k(2r-2k+1)$ functions of $x$ and $t$. In the second subsection we consider special case with $r = 4$ and $k = 3$ apply to it a special $\mathbb{Z}_6$-reduction. The result is a new type of 2-component NLS.

4.1 $\mathbb{Z}_6$-reduction of KS models for $SO(9)/(SO(6) \times SO(3))$

Consider $SO(9)/(SO(6) \times SO(3))$. The the subset of positive roots is split into

$$\Delta_0^+ \equiv \{ e_1 \pm e_2, e_2 \pm e_3, e_1 \pm e_3, e_4 \}, \quad \Delta_1^+ \equiv \{ e_1 \pm e_4, e_2 \pm e_4, e_3 \pm e_4, e_1, e_2, e_3 \}, \quad (4.1)$$

The reduction is given by the Weyl-group element $w_4 = S_{e_1-e_2}S_{e_2-e_3}S_{e_4}$, i.e. $w_4^6 = \text{Id}$. Obviously this Weyl group element leaves invariant $\Delta_0$ and $\Delta_1$ and the roots $\Delta_1$ are split into four orbits:

$$\mathcal{O}_1^+ : \pm e_1 - e_4 \to \pm e_2 + e_4 \to \pm e_3 - e_4 \to \pm e_1 + e_4 \to \pm e_2 - e_4 \to \pm e_3 + e_4,$$
$$\mathcal{O}_3 : e_1 \to e_2 \to e_3, \quad \mathcal{O}_4 : - e_1 \to - e_2 \to - e_3. \quad (4.2)$$

So we have four orbits. After applying another $\mathbb{Z}_2$-reductions, i.e. $Q = Q^\dagger$ one may expect a 2-component NLS. Realization of the automorphism $w_4$ is as follows $w_4(X) = A_1X A_1^{-1}$:

$$A_1 = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \quad (4.3)$$

The orthogonality condition is given by (2.1) with $S_0 = \begin{pmatrix} 0 & 0 & s_1 \\ 0 & -s_1 & 0 \\ s_1 & 0 & 0 \end{pmatrix}$, $s_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ The corresponding potential of the Lax operator is as in (3.3) with

$$q(x,t) = \begin{pmatrix} q_1 & q_2 & -q_1 \\ -q_1 & -q_2 & q_1 \\ q_1 & q_2 & -q_1 \end{pmatrix}, \quad p(x,t) = \begin{pmatrix} p_1 & -p_1 & p_1 \\ p_2 & -p_2 & p_2 \\ -p_1 & p_1 & -p_1 \end{pmatrix}. \quad (4.4)$$

The condition $Q = Q^\dagger$ reduces to $p_1 = q_1^*$ and $p_2 = q_2^*$. Then the NLEE becomes

$$i \frac{\partial q_1}{\partial t} + \frac{\partial^2 q_1}{\partial x^2} + 6(|q_1|^2 + |q_2|^2)q_1(x,t) - 3q_2^2q_1^*(x,t) = 0, \quad (4.5)$$
$$i \frac{\partial q_2}{\partial t} + \frac{\partial^2 q_2}{\partial x^2} + 3(4|q_1|^2 + |q_2|^2)q_2(x,t) - 6q_1^2q_2^*(x,t) = 0, \quad (4.5)$$

The corresponding Hamiltonian is:

$$H = 2 \left| \frac{\partial q_1}{\partial x} \right|^2 + \left| \frac{\partial q_2}{\partial x} \right|^2 - \frac{3}{2} \left( 2|q_1|^2 + |q_2|^2 \right)^2 + 3 \left( q_1q_2^* - q_1^*q_2 \right)^2. \quad (4.6)$$
The change of variables: \( v_1 = \sqrt{6}q_1, \ v_2 = \sqrt{3}q_2 \) leads to:

\[
\begin{align*}
\frac{i}{\partial t} \frac{\partial v_1}{\partial x} + \frac{\partial^2 v_1}{\partial x^2} + (|v_1|^2 + 2|v_2|^2)v_1(x, t) - v_2^*v_1(x, t) &= 0, \\
\frac{i}{\partial t} \frac{\partial v_2}{\partial x} + \frac{\partial^2 v_2}{\partial x^2} + (2|v_1|^2 + |v_2|^2)v_2(x, t) - v_1^*v_2(x, t) &= 0,
\end{align*}
\]

(4.7)

\[
q(x, t) = \frac{1}{\sqrt{6}} \begin{pmatrix} v_1 & \sqrt{2}v_2 - v_1 \\ -v_1 - \sqrt{2}v_2 & v_1 \end{pmatrix}, \quad p = q^\dagger.
\]

It is easy to check that assuming canonical Poisson brackets \( \{v_k(x), v_j^*(y)\} = \delta_{kj}\delta(x - y) \) for \( v_j \), the canonical Hamiltonian equations of motion:

\[
\frac{i}{\partial t} \frac{\partial v_j}{\partial x} = \frac{\delta H'}{\delta v_j^*}, \quad j = 1, 2,
\]

(4.8)

with

\[
H' = |\frac{\partial v_1}{\partial x}|^2 + |\frac{\partial v_2}{\partial x}|^2 - \frac{1}{2} (|v_1|^2 + |v_2|^2)^2 + \frac{1}{2} (v_1v_2^* - v_1^*v_2)^2,
\]

(4.9)

coincides with (4.7).

### 5 Spectral properties of Lax operators

#### 5.1 The case \( SO(2r + 1)/(SO(2r - 1) \times SO(2)) \)

Here we will outline the methods of solving the direct and the inverse scattering problem (ISP) for \( L \). We will use the Jost solutions which are defined by, see [8] and the references therein

\[
\lim_{x \to -\infty} \phi(x, t, \lambda)e^{i\lambda^kJx} = \mathbb{I}, \quad \lim_{x \to \infty} \psi(x, t, \lambda)e^{i\lambda^kJx} = \mathbb{I}
\]

(5.1)

and the scattering matrix \( T(\lambda, t) \equiv \psi^{-1}\phi(x, t, \lambda) \). Due to the special choice of \( J \) and to the fact that the Jost solutions and the scattering matrix take values in the group \( SO(2r + 1) \) we can use the following block-matrix structure of \( T(\lambda, t) \) and its inverse \( \hat{T}(\lambda, t) \):

\[
T(\lambda, t) = \begin{pmatrix} m_1^+ & \tilde{b}_+^T & c_1^- \\ \tilde{b}_+ & T_{22} & -s_0\tilde{b}_- \\ c_1^+ & \tilde{B}_{21}^T & m_1^- \end{pmatrix}, \quad \hat{T}(\lambda, t) = \begin{pmatrix} m_1^- & \tilde{b}_-^T & c_1^+ \\ -\tilde{B}_+ & T_{22} & s_0\tilde{B}_- \\ c_1^- & -\tilde{B}_{21}^T & m_1^+ \end{pmatrix},
\]

(5.2)

where \( \tilde{b}_+^\pm(\lambda, t) \) and \( \tilde{B}_+^\pm(\lambda, t) \) are \( 2r - 1 \)-component vectors, \( T_{22}(\lambda) \) and \( \hat{T}_{22}(\lambda) \) are \( 2r - 1 \times 2r - 1 \) blocks and \( m_1^\pm(\lambda) \), \( c_1^\pm(\lambda) \) are scalar functions satisfying \( c_1^\pm = 1/2(\tilde{b}_+^\pm \cdot s_0\tilde{b}_-^\pm)/m_1^\pm \).
Important tools for reducing the ISP to a Riemann-Hilbert problem (RHP) are the fundamental analytic solution (FAS) $\chi^\pm(x, t, \lambda)$. Their construction is based on the generalized Gauss decomposition of $T(\lambda, t)$

$$\chi^\pm(x, t, \lambda) = \phi(x, t, \lambda)S_\pm^x(t, \lambda) = \psi(x, t, \lambda)T_\pm^x(t, \lambda)D_\pm^x(\lambda). \quad (5.3)$$

Here $S_\pm^x$ and $T_\pm^x$ are upper- and lower-block-triangular matrices, while $D_\pm^x(\lambda)$ are block-diagonal matrices with the same block structure as $T(\lambda, t)$ above. Skipping the details we give the explicit expressions of the Gauss factors in terms of the matrix elements of $T(\lambda, t)$

$$S_\pm^x(t, \lambda) = \exp \left( \pm \sum_{\beta \in \Delta_2^i} \tau_\pm^\beta(\lambda, t)E_{\pm\beta} \right), \quad T_\pm^x(t, \lambda) = \exp \left( \mp \sum_{\beta \in \Delta_2^i} \rho_\pm^\beta(\lambda, t)E_{\pm\beta} \right), \quad (5.4)$$

$$D_+^x = \begin{pmatrix} m_1^+ & 0 & 0 \\ 0 & m_2^+ & 0 \\ 0 & 0 & 1/m_1^+ \end{pmatrix}, \quad D_-^x = \begin{pmatrix} 1/m_1^- & 0 & 0 \\ 0 & m_2^- & 0 \\ 0 & 0 & m_1^- \end{pmatrix}$$

where $\vec{b}^+ = (T_{1,2}, \ldots, T_{1,2r})^T$

$$\vec{\tau}^+(\lambda, t) = \frac{\vec{b}^-}{m_1^+}, \quad \vec{\tau}^-(\lambda, t) = \frac{\vec{b}^+}{m_1^-}, \quad \vec{m}_2^+ = \vec{T}_{22} + \frac{\vec{b}^+\vec{b}^-\cdot t}{2m_1^+} = \vec{T}_{22} + \frac{s_0\vec{b}^-\vec{b}^+\cdot t}{2m_1^+}, \quad \vec{m}_2^- = \vec{T}_{22} + \frac{\vec{b}^+\vec{b}^-\cdot t}{2m_1^-} = \vec{T}_{22} + \frac{s_0\vec{b}^-\vec{b}^+\cdot t}{2m_1^-}. \quad (5.5)$$

If $Q(x, t)$ evolves according to (3.15) then the scattering matrix and its elements satisfy the following linear evolution equations

$$i\frac{d\vec{B}^\pm}{dt} = \pm \lambda^{2k}\vec{B}^\pm(t, \lambda) = 0, \quad i\frac{d\vec{b}^\pm}{dt} = \pm \lambda^{2k}\vec{b}^\pm(t, \lambda) = 0, \quad i\frac{dm_1^\pm}{dt} = 0, \quad i\frac{dm_2^\pm}{dt} = 0, \quad (5.6)$$

so the block-diagonal matrices $D^\pm(\lambda)$ are generating functionals of the integrals of motion. The fact that all $(2r - 1)^2$ matrix elements of $m_2^\pm(\lambda)$ for $\lambda \in \mathbb{C}_\pm$ generate integrals of motion reflects the super-integrability of the model and is due to the degeneracy of the dispersion law determined by $\lambda^{2k}J$. We remind that $D_\pm^x(\lambda)$ allow analytic extension for $\lambda \in \mathbb{C}_\pm$ and that their zeroes and poles determine the discrete eigenvalues of $L$. We will use also another set of FAS:

$$\chi'^\pm(x, t, \lambda) = \chi^\pm(x, t, \lambda)\hat{D}_\pm^x(\lambda). \quad (5.7)$$

The FAS for real $\lambda^k$ are linearly related

$$\chi^+(x, t, \lambda) = \chi^-(x, t, \lambda)G_J(\lambda, t), \quad \chi'^+(x, t, \lambda) = \chi'^-(x, t, \lambda)G'_J(\lambda, t), \quad G_{0,J}(\lambda, t) = S_J^x(\lambda, t)S_J^x(t, \lambda), \quad (5.8)$$

One can rewrite eq. (5.8) in an equivalent form for the FAS $\xi^\pm(x, t, \lambda) = \chi^\pm(x, t, \lambda)e^{i\lambda Jx}$ and $\xi'^\pm(x, t, \lambda) = \chi'^\pm(x, t, \lambda)e^{i\lambda Jx}$ which satisfy also the relation

$$\lim_{\lambda \to \infty} \xi^\pm(x, t, \lambda) = 1, \quad \lim_{\lambda \to \infty} \xi'^\pm(x, t, \lambda) = 1. \quad (5.9)$$
Then for Im $\lambda^k = 0$ these FAS satisfy
\[
\xi^+(x, t, \lambda) = \xi^-(x, t, \lambda)G_J(x, \lambda, t), \quad G_J(x, \lambda, t) = e^{-i\lambda^k Jx}G_{0,J}(\lambda, t)e^{i\lambda^k Jx},
\]
\[
\xi'^+(x, t, \lambda) = \xi'^-(x, t, \lambda)G'_J(x, \lambda, t), \quad G'_J(x, \lambda, t) = e^{-i\lambda^k Jx}G'_{0,J}(\lambda, t)e^{i\lambda^k Jx}.
\]

(5.10)

Obviously the sewing function $G_J(x, \lambda, t)$ is uniquely determined by the Gauss factors of $T(\lambda, t)$. In view of eq. (5.4) we arrive to the following

**Lemma 5.1.** Let the potential $Q(x, t)$ be such that the Lax operator $L$ has no discrete eigenvalues. Then as minimal set of scattering data which determines uniquely the scattering matrix $T(\lambda, t)$ and the corresponding potential $Q(x, t)$ one can consider either one of the sets $\mathfrak{T}_i$, $i = 1, 2$

\[
\mathfrak{T}_1 \equiv \{ \vec{\rho}^+(\lambda, t), \vec{\rho}^-(\lambda, t), \lambda^k \in \mathbb{R} \}, \quad \mathfrak{T}_2 \equiv \{ \vec{\tau}^+(\lambda, t), \vec{\tau}^-(\lambda, t), \lambda^k \in \mathbb{R} \}.
\]

(5.11)

**Proof.** i) From the fact that $T(\lambda, t) \in SO(2r + 1)$ one can derive that
\[
\frac{1}{m_1 m_1^*} = 1 + (\rho^+, \rho^-) + \frac{1}{4}(\rho^+, s_0 \rho^-)(\rho^-, s_0 \rho^-)
\]
for $\lambda \in \mathbb{R}$. Using the analyticity properties of $m_1^\pm$ we can recover them from eq. (5.12) using Cauchy-Plemelj formulae. Given $\mathfrak{T}_i$ and $m_1^\pm$ one easily recovers $\vec{b}_i^\pm(\lambda)$ and $c_1^\pm(\lambda)$. In order to recover $\mathfrak{T}_i$ one again uses their analyticity properties, only now the problem reduces to a RHP for functions on $SO(2r + 1)$. The details will be presented elsewhere.

ii) Given $\mathfrak{T}_i$ one uniquely recovers the sewing function $G_J(x, t, \lambda)$. In order to recover the corresponding potential $Q(x, t)$ one can use the fact that the RHP (5.10) with canonical normalization has unique regular solution $\chi^\pm(x, t, \lambda)$. Given $\chi^\pm(x, t, \lambda)$ we recovers $Q(x, t)$ via:

\[
Q(x, t) = \lim_{\lambda \to \infty} \lambda \left( J - \chi^\pm J\hat{\chi}^{\pm}(x, t, \lambda) \right) = \lim_{\lambda \to \infty} \lambda \left( J - \chi'^\pm J\hat{\chi}'^{\pm}(x, t, \lambda) \right).
\]

(5.13)

which is well known.

We impose also the standard reduction, namely assume that $Q(x, t) = Q^t(x, t)$, or in components $p_k = q^*_k$. As a consequence we have $\vec{\rho}^-(\lambda, t) = \vec{\rho}^{+, *}(\lambda, t)$ and $\vec{\tau}^-(\lambda, t) = \vec{\tau}^{+, *}(\lambda, t)$.

### 5.2 The case $SO(9)/(SO(3) \times SO(6))$

Effects on the scattering data:

- $T(\lambda)$ belongs to $SO(9)$, therefore $T^{-1} = S_0 T^T(\lambda) S_0$;
- $T(\lambda)$ is unitary matrix $T^\dagger(\lambda^*) = T^{-1}(\lambda)$;
- $T(\lambda)$ is invariant with respect to the automorphism $A_1$.
We parametrize $T(t, \lambda)$ using the same block-matrix structure as for $Q(x, t)$ and $J$ (3.6):

$$T(\lambda) = \begin{pmatrix} m^+ - b^- & c^- \\ b^+ T_{22} - B^- \\ c^+ B^+ + m^- \end{pmatrix}, \quad T^{-1}(\lambda) = \begin{pmatrix} s_1 m^{-T} s_1 & s_1 B^{-T} s_1 & s_1 c^{-T} s_1 \\ -s_1 B^{+T} s_1 & s_1 T_{22}^{T} s_1 & s_1 b^{-T} s_1 \\ s_1 c^{+T} s_1 & -s_1 b^{+T} s_1 & s_1 m^{-T} s_1 \end{pmatrix},$$

$$T^{-1} = S_0 T^T(\lambda) S_0, \quad T^\dagger(\lambda^*) = T^{-1}(\lambda), \quad T(\lambda) = A_1 T(\lambda) A_1^{-1},$$

i.e.

$$m^{+,\dagger}(\lambda^*) = s_1 m^{-T}(\lambda) s_1, \quad c^{+,\dagger}(\lambda^*) = s_1 c^{-T}(\lambda) s_1,$$

$$b^{\pm,\dagger}(\lambda^*) = s_1 B^{\pm, T}(\lambda) s_1, \quad B^{\pm,\dagger}(\lambda^*) = s_1 b^{\pm, T}(\lambda) s_1,$$

and

$$m^+ = a_1 m^+ a_1^{-1}, \quad b^- = a_1 b^- a_2^{-1}, \quad c^- = a_1 c^- a_3^{-1},$$

$$b^+ = a_2 b^+ a_1^{-1}, \quad T_{22} = a_2 T_{22} a_2^{-1}, \quad B^- = a_2 B^- a_3^{-1},$$

$$c^+ = a_3 c^+ a_1^{-1}, \quad B^+ = a_3 B^+ a_2^{-1}, \quad m^- = a_3 m^- a_3^{-1}.$$

### 6 Integrability properties of Kulish-Sklyanin models

#### 6.1 The Wronskian relations and minimal sets of scattering data

The analysis of the mapping $\mathcal{F}: \mathcal{M} \to \mathcal{T}$ between the class of allowed potentials $\mathcal{M}$ and the scattering data of $L$ starts with the Wronskian relations, see [26, 3] for $sl(2)$-case and [8, 11]. For higher rank algebras and symmetric spaces (the block-matrix case) one should use [7, 11].

These ideas will be worked out for $L$ (1.2) with $s = 1$. With it one can associate:

$$i \frac{d\hat{\psi}}{dx} - \hat{\psi}(x, t, \lambda) U(x, t, \lambda) = 0, \quad U(x, \lambda) = Q(x) - \lambda J,$$

$$i \frac{d\delta\hat{\psi}}{dx} + \delta U(x, t, \lambda) \psi(x, t, \lambda) + U(x, t, \lambda) \delta \hat{\psi}(x, t, \lambda) = 0$$

$$i \frac{d\hat{\psi}}{dx} - \lambda J \psi(x, t, \lambda) + U(x, t, \lambda) \hat{\psi}(x, t, \lambda) = 0$$

where $\delta \psi$ corresponds to a given variation $\delta Q(x, t)$ of the potential, while by dot we denote the derivative with respect to the spectral parameter. We start with the identity:

$$(\hat{\chi} J \chi(x, \lambda) - J)|_{x=-\infty}^{x=\infty} = i \int_{-\infty}^{\infty} dx \hat{\chi}[J, Q(x)] \chi(x, \lambda),$$

where $\chi(x, \lambda)$ can be any fundamental solution of $L$. For convenience we use both choices $\chi(x, \lambda) = \chi^+(x, \lambda)$ and $\chi(x, \lambda) = \chi^\pm(x, \lambda)$.

The left hand side of (6.4) can be calculated explicitly by using the asymptotics of $\chi^\pm(x, \lambda)$ for $x \to \pm \infty$, (5.3). It would be expressed by the matrix elements of the scattering matrix $T(\lambda)$, i.e., by the scattering data of $L$. 

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Indeed, let us multiply both sides of eq. (6.4) by $E_{\pm\beta}$, $\beta \in \Delta_1^+$ and take the Killing form. In the right hand side of this equation we can use the invariance properties of the trace and rewrite it in the form:

\[
\langle (\hat{\chi}^\pm J\hat{\chi}^\pm(x,\lambda) - J) E_\beta \rangle |_{x=-\infty}^{\infty} = i \int_{-\infty}^{\infty} dx \langle ([J,Q(x)] | e_{\beta}^\pm(x,\lambda)) \rangle,
\]

\[
\langle (\hat{\chi}'^\pm J\hat{\chi}'^\pm(x,\lambda) - J) E_\beta \rangle |_{x=-\infty}^{\infty} = i \int_{-\infty}^{\infty} dx \langle ([J,Q(x)] | e_{\beta}'^\pm(x,\lambda)) \rangle,
\]

where

\[
e_{\beta}^\pm(x,\lambda) = \chi^\pm E_\beta \hat{\chi}^\pm(x,\lambda), \quad e_{\beta}'^\pm(x,\lambda) = \pi_0 J(\chi^\pm E_\beta \hat{\chi}^\pm(x,\lambda)),
\]

\[
e_{\beta}''^\pm(x,\lambda) = \chi'^\pm E_\beta \hat{\chi}'^\pm(x,\lambda), \quad e_{\beta}'''^\pm(x,\lambda) = \pi_0 J(\chi'^\pm E_\beta \hat{\chi}'^\pm(x,\lambda)),
\]

are the natural generalization of the ‘squared solutions’ introduced first for the $sl(2)$-case by Kaup [26] and generalized to any simple Lie algebra in [7, 8, 11]. By $P_{0J}$ we have denoted the projector $\pi_0 J = \text{ad} \, j^{-1} \text{ad} \, J$ on the block-off-diagonal part of the corresponding matrix-valued function.

The right hand sides of eq. (6.6) can be written down with the skew–scalar product:

\[
\langle X, Y \rangle = \int_{-\infty}^{\infty} dx \langle X(x), [J, Y(x)] \rangle,
\]

where $\langle X, Y \rangle$ is the Killing form; in what follows we assume that the Cartan-Weyl generators satisfy $\langle E_\alpha, E_{-\beta} \rangle = \delta_{\alpha,\beta}$ and $\langle H_j, H_k \rangle = \delta_{jk}$. The product is skew-symmetric $\langle X, Y \rangle = -\langle Y, X \rangle$ and is non-degenerate on the space of allowed potentials $\mathcal{M}$. Thus we find

\[
\rho_{\beta}^\pm = -i \langle Q(x), e_{\beta}'^\pm \rangle, \quad \tau_{\beta}^\pm = -i \langle Q(x), e_{\pm \beta}^\pm \rangle.
\]

Thus the mappings $\mathfrak{F} : Q(x,t) \to \Sigma_i$ can be viewed as generalized Fourier transform in which $e_{\beta}^\pm(x,\lambda)$ and $e_{\beta}'^\pm(x,\lambda)$ can be viewed as generalizations of the standard exponentials. In what follows we will show that the same ‘squared solutions’ appear in the analysis of the mapping between the variations $\delta Q(x,t)$ and $\delta \Sigma_i$.

The second type of Wronskian relations relate the variation of the potential $\delta Q(x)$ to the corresponding variations of the scattering data. To this purpose we start with the identity:

\[
\hat{\chi}^\pm \delta \hat{\chi}^\pm(x,\lambda) |_{x=-\infty}^{\infty} = i \int_{-\infty}^{\infty} dx \hat{\chi} \delta Q(x) \hat{\chi}(x,\lambda),
\]

which follows from eqs. (6.1) and (6.361). We apply ideas similar to the ones above and get:

\[
\delta \rho_{\beta}^\pm = \mp i \text{ad} - j^{-1} \delta Q(x), e_{\pm \beta}^\pm \rangle, \quad \delta \tau_{\beta}^\pm = \pm i \text{ad} j^{-1} \delta Q(x), e_{\pm \beta}^\pm \rangle,
\]

where $\beta \in \Delta_1^+$. These relations are basic in the analysis of the related NLEE and their Hamiltonian structures. Below we shall use them assuming that the variation of $Q(x)$ is due to its time evolution, and consider variations of the type:

\[
\delta Q(x, t) = Q_t \delta t + \mathcal{O}((\delta t)^2).
\]

Keeping only the first order terms with respect to $\delta t$ we find:

\[
\frac{d\rho_{\beta}^\pm}{dt} = \mp i \text{ad} j^{-1} Q_t(x), e_{\pm \beta}^\pm \rangle, \quad \frac{d\tau_{\beta}^\pm}{dt} = \pm i \text{ad} j^{-1} Q_t(x), e_{\pm \beta}^\pm \rangle.
\]
6.2 The generalized Fourier transforms and the completeness of the ‘squared solutions’

It is known that the ‘squared solutions’ $e^{\pm\alpha}(x, \lambda) = \pi_0 \chi^\pm(x, t, \lambda)$, form complete set of functions in the space of allowed potentials $q(x)$, see [8, 11]. For brevity and simplicity below we assume that $L$ has no discrete eigenvalues. Let us introduce the sets of ‘squared solutions’

$$\{\Psi\} \equiv \{e^{+\alpha}(x, \lambda), e^{-\alpha}(x, \lambda), \lambda \in \mathbb{R}, \alpha \in \Delta^+_1\},$$

$$\{\Phi\} \equiv \{e^{+\alpha}(x, \lambda), e^{-\alpha}(x, \lambda), \lambda \in \mathbb{R}, \alpha \in \Delta^-_1\}. \quad (6.13)$$

**Theorem 6.1** (see [8, 11]). The sets $\{\Psi\}$ and $\{\Phi\}$ form complete sets of functions in $\mathcal{M}_1$. The corresponding completeness relation has the form:

$$\delta(x - y)\Pi_{0J} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda (G^+_1(x, y, \lambda) - G^-_1(x, y, \lambda)), \quad (6.14)$$

where

$$\Pi_{0J} = \sum_{\alpha \in \Delta^+_1} (E_{\alpha} \otimes E_{-\alpha} - E_{-\alpha} \otimes E_{\alpha}),$$

$$G^+_1(x, y, \lambda) = \sum_{\alpha \in \Delta^+_1} e^{\pm\alpha}(x, \lambda) \otimes e^{\pm\alpha}(y, \lambda). \quad (6.15)$$

**Idea of the proof.** Apply the contour integration method to the Green function

$$G^\pm(x, y, \lambda) = G^+_1(x, y, \lambda)\theta(y - x) - G^-_1(x, y, \lambda)\theta(x - y),$$

$$G^\pm_2(x, y, \lambda) = \sum_{\alpha \in \Delta_0 \cup \Delta^-_1} e^{-\pm\alpha}(x, \lambda) \otimes e^{-\pm\alpha}(y, \lambda) + \sum_{j=1}^r h^\pm_j(x, \lambda) \otimes h^\pm_j(y, \lambda), \quad (6.16)$$

where $h^\pm_j(x, \lambda) = \pi_0 \chi^\pm(x, \lambda)H_j^\pm \chi^\pm(x, \lambda)$, and calculate the integral

$$J_G(x, y) = \frac{1}{2\pi i} \oint_{\gamma^+} d\lambda G^+(x, y, \lambda) - \frac{1}{2\pi i} \oint_{\gamma^-} d\lambda G^-(x, y, \lambda), \quad (6.17)$$

in two ways: i) via the Cauchy residue theorem and ii) integrating along the contours, see [8, 11].

Skipping the details we write down the expansions of $q(x)$ and $\text{ad}^{-1}_J \delta q(x)$ assuming $L$ has no discrete spectrum:

$$Q(x) = -\frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta^+_1} (\tau^+_\alpha(\lambda)e^+_\alpha(x, \lambda) - \tau^-_\alpha(\lambda)e^-_\alpha(x, \lambda)), \quad (6.18)$$
\[
\text{ad}_J^{-1} \delta Q(x) = \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \sum_{\alpha \in \Delta_+^1} (\delta \tau_+^\alpha(\lambda)e_\alpha^+(x,\lambda) + \delta \tau_-^\alpha(\lambda)e_\alpha^-(x,\lambda)).
\] (6.19)

These expansions can be viewed as a tool to establish the one-to-one correspondence between \(q(x)\) (resp. \(\text{ad}_J^{-1} \delta q\)) and each of the minimal sets of scattering data \(T_i\) (resp. \(\delta T_i\)), \(i = 1, 2\).

To complete the analogy between the standard Fourier transform and the expansions over the ‘squared solutions’ we need the generating operators \(\Lambda_{\pm}\):

\[
\Lambda_{\pm} X(x) \equiv \text{ad}_J^{-1} \left( i \frac{dX}{dx} + i \left[ q(x), \int_{\pm \infty}^x dy [q(y), X(y)] \right] \right). \quad (6.20)
\]

for which the ‘squared solutions’ are eigenfunctions:

\[
(\Lambda_+ - \lambda)e_\alpha^+(x,\lambda) = 0, \quad (\Lambda_+ - \lambda)e_\alpha^-(x,\lambda) = 0,
\]

\[
(\Lambda_- - \lambda)e_\alpha^+(x,\lambda) = 0, \quad (\Lambda_- - \lambda)e_\alpha^-(x,\lambda) = 0. \quad (6.21)
\]

### 6.3 Fundamental properties of the KS type equations

The expansions (6.18), (6.19) and the explicit form of \(\Lambda_{\pm}\) and eq. (6.21) are basic for deriving the fundamental properties of all MNLS type equations related to the Lax operator \(L\). Each of these NLEE is determined by its dispersion law which we choose to be of the form \(F(\lambda) = f(\lambda)J\), where \(f(\lambda)\) is polynomial in \(\lambda\). The corresponding NLEE becomes:

\[
i \text{ad}_J^{-1} \frac{\partial Q}{\partial t} + f(\Lambda)Q(x,t) = 0. \quad (6.22)
\]

**Theorem 6.2.** The NLEE (6.22) are equivalent to: i) the equations (5.6) and ii) the following evolution equations for the generalized Gauss factors of \(T(\lambda)\):

\[
i \frac{dS_j}{dt} + [F(\lambda), S_j] = 0, \quad i \frac{dT_j}{dt} + [F(\lambda), T_j] = 0, \quad \frac{dD_j}{dt} = 0. \quad (6.23)
\]

The principal series of integrals is generated by the asymptotic expansion of \(\ln m_1^\pm(\lambda) = \sum_{k=1}^{\infty} I_k \lambda^{-k}\). The first three integrals of motion:

\[
I_1 = -\frac{i}{2} \int_{-\infty}^{\infty} dx \langle Q(x), Q(x)\rangle, \quad I_2 = \frac{1}{2} \int_{-\infty}^{\infty} dx \langle Q_x, \text{ad}_J^{-1} Q(x)\rangle, \quad I_3 = -\frac{i}{2} \int_{-\infty}^{\infty} dx \left( \langle \text{ad}_J^{-1} Q_x, \text{ad}_J^{-1} Q_x \rangle - \langle [\text{ad}_J^{-1} Q, Q(x)], [\text{ad}_J^{-1} Q, Q(x)] \rangle \right). \quad (6.24)
\]

Now \(iI_1\) can be interpreted as density of the particles, \(I_2\) is the momentum and \(I_3 = 2iH_{MNLS}\). Indeed, the Hamiltonian equations of motion provided by \(H_{(0)} = -iI_3/2\) with the Poisson brackets

\[
\{q_\alpha(y,t), p_\beta(x,t)\} = i\delta_{\alpha,\beta} \delta(x-y), \quad \alpha, \beta \in \Delta_+^1, \quad (6.25)
\]
coincide with the MNLS equations (3.8). The above Poisson brackets are dual to the canonical symplectic form:

\[ \Omega_0 = i \int_{-\infty}^{\infty} dx \text{tr} (\delta \vec{p}(x) \wedge \delta \vec{q}(x)) = \frac{1}{2i} [\text{ad}_{\vec{J}}^{-1} \delta \vec{q}(x) \wedge \text{ad}_{\vec{J}}^{-1} \delta \vec{q}(x)], \]

where \( \wedge \) means that taking the scalar or matrix product we exchange the usual product of the matrix elements by wedge-product. The Hamiltonian formulation of eq. (3.8) with \( \Omega_0 \) and \( H_0 \) is just one member of the hierarchy of Hamiltonian formulations provided by:

\[ \Omega_k = \frac{1}{i} [\text{ad}_{\vec{J}}^{-1} \delta \vec{Q} \wedge \Lambda^k \text{ad}_{\vec{J}}^{-1} \delta \vec{Q}], \quad H_k = i^k + 3 I_{k+3}. \] (6.26)

where \( \Lambda = \frac{1}{2}(\Lambda_+ + \Lambda_-) \). We can also calculate \( \Omega_k \) in terms of the scattering data variations. Imposing the reduction \( q(x) = q^I(x) \) we get:

\[ \Omega_k = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\lambda \lambda^k \left( \Omega_+^I(\lambda) - \Omega_-^I(\lambda) \right) \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \lambda^k \text{Im} \left( m_1^+(\lambda) \left( \vec{m}_2^+ \delta \vec{p}^+(\lambda) \wedge \delta \vec{r}^+(\lambda) \right) \right). \]

This allows one to prove that if we are able to cast \( \Omega_0 \) in canonical form, then all \( \Omega_k \) will also be cast in canonical form and will be pair-wise equivalent.

7 Conclusions

Using RHP formulated on the real axis of the complex \( \lambda \)-plane and compatible with the BD.II-type symmetric spaces \( SO(2r+1)/S(O(2r-2s+1) \otimes O(2s)), s \geq 1 \) we have derived Lax pairs for KS type models; the proper KS model is obtained for \( s = 1 \).

Another Riemann-Hilbert problems: formulated on \( \mathbb{R} \oplus i\mathbb{R} \) is relevant for a new type of KS model. We find nontrivial deep reductions of these systems and formulate their effects on the scattering matrix. In particular we obtain new 2-component NLS equations whose Hamiltonian depends not only on \( |q_1| \) and \( |q_2| \), but also on \( q_1^* q_1 + q_2^* q_2 \). Thus our example comes out of the scope of Zakharov-Schulman theorem [34].

Finally, using the Wronskian relations we demonstrate that the inverse scattering method for KS models may be understood as a generalized Fourier transforms. Thus we have a tool to derive all their fundamental properties, including the hierarchy of equations and he hierarchy of their Hamiltonian structures.

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