Abstract

In this note we illustrate how common matrix approximation methods, such as random projection and random sampling, yield projection-cost-preserving sketches, as introduced in [FSS13, CEM+15]. A projection-cost-preserving sketch is a matrix approximation which, for a given parameter $k$, approximately preserves the distance of the target matrix to all $k$-dimensional subspaces. Such sketches have applications to scalable algorithms for linear algebra, data science, and machine learning. Our goal is to simplify the presentation of proof techniques introduced in [CEM+15] and [CMM17] so that they can serve as a guide for future work. We also refer the reader to [CYD19], which gives a similar simplified exposition of the proof covered in Section 2.

1 Projection-Cost-Preserving Sketches

A projection-cost-preserving sketch is a matrix compression that preserves the distance of a matrix’s columns to any $k$-dimensional subspace. Let $\|M\|_F^2 = \sum_{i,j} M_{i,j}^2$ denote the squared Frobenius norm of a matrix $M$. Formally we define:

**Definition 1** (Projection-Cost-Preserving Sketch). $\tilde{A} \in \mathbb{R}^{n \times m}$ is an $(\epsilon, c, k)$-projection-cost-preserving sketch of $A \in \mathbb{R}^{n \times d}$ if, for any orthogonal projection matrix $P \in \mathbb{R}^{n \times n}$ with rank at most $k$,

\[
(1 - \epsilon)\|A - PA\|_F^2 \leq \|\tilde{A} - P\tilde{A}\|_F^2 + c \leq (1 + \epsilon)\|A - PA\|_F^2.
\]

Here $c$ is a constant that is independent of $P$ (but may depend on $A, \tilde{A}, \epsilon, k$).

In typical applications, $m \ll d$, so $\tilde{A}$ has fewer columns than $A$. It can serve as a surrogate in solving a number of low-rank optimization problems, such as PCA or $k$-means clustering, in which the goal is to chose a $k$-dimensional subspace from some set that is as close as possible to the input matrix. When $m \ll d$, using $\tilde{A}$ in place of $A$ can lead to significant computational savings in terms of runtime, memory, and communication cost.

1.1 Constrained Low-Rank Approximation

For example, a projection-cost-preserving sketch can be used to approximately solve any problem of the form:

**Problem 2** (Constrained Low-Rank Approximation). Let $\mathcal{S}_k$ be the set of all orthogonal projection matrices in $\mathbb{R}^{n \times n}$ with rank $\leq k$. Let $\mathcal{T}$ be any subset of $\mathcal{S}_k$. The constrained low-rank approximation problem over set $\mathcal{T}$ is to find: $P^* \in \arg\min_{P \in \mathcal{T}} \|A - PA\|_F^2$. 

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A simple manipulation of the bound of Definition 1 yields:

**Claim 1.** If $\tilde{A}$ is an $(\epsilon, c, k)$-PCP for $A$, then for any $T \subseteq S_k$, if $\hat{P} \leq \gamma \cdot \min_{P \in T} \| \tilde{A} - PA \|_F^2$ for some $\gamma \geq 1$, then:

$$\| A - \hat{P} A \|_F^2 \leq \frac{(1 + \epsilon) \gamma}{1 - \epsilon} \cdot \min_{P \in T} \| A - PA \|_F^2 + \frac{(1 - \gamma) c}{1 - \epsilon}.$$

Note that if $c$ is positive (as will typically be the case), $\frac{(1 - \gamma) c}{1 - \epsilon} < 0$, and thus $\hat{P}$ gives a relative error approximation to the optimum. This is also true if $c$ is negative and $\gamma \leq 1 + \frac{\min_{P \in T} \| A - PA \|_F^2}{c}$. Two important cases of Problem 2 are vanilla low-rank approximation, when $T = S_k$ and $k$-means clustering, when $T$ is the set of projections corresponding to the set of cluster indicator matrices. See [CEM+15] for details.

### 1.2 Sketch Constructions

It has been shown that a wide variety of dimensionality reduction methods can be used to obtain projection-cost-preserving sketches with positive $c \geq 0$ and dimension $m = \tilde{O}(k/\epsilon^q)$ for $q \in \{1, 2\}$. See Table 1 below.

In Section 2 we show how to prove that a dimensionality reduction method yields a projection-cost-preserving sketch by appealing to the well-studied matrix approximation guarantees of subspace embedding, approximate matrix multiplication, and Frobenius norm preservation. This proof mirrors the more general proof of [CEM+15] and the proof presented in [CYD19]. In Section 3 we show how to prove the projection-cost-preserving sketch guarantee in an alternative way: starting from a spectral approximation bound of the form $(1 - \epsilon)AA^T - M \preceq \tilde{A} \tilde{A}^T \preceq (1 + \epsilon)AA^T + \lambda I$, where $M \preceq N$ denotes that $N - M$ is positive semidefinite, $I$ is the $n \times n$ identity matrix, and $\lambda$ is an appropriately chosen regularization parameter. This proof mirrors that in [CMM17].

The two proofs are closely related and both follow a strategy of decomposing $A$ into the projections onto the singular vectors corresponding to its large (head) and small (tail) singular values. Error terms corresponding to these components are then bounded using well-studied matrix approximation guarantees (Section 2) or using the above spectral approximation bound (Section 3). [CYD19] further discusses how the two proof strategies can be unified under a general approach.

| Method                              | Dimension $m$  | Reference           |
|-------------------------------------|----------------|---------------------|
| SVD                                 | $[k/\epsilon]$ | Theorem 7 of [CEM+15] |
| Approximate SVD                     | $[k/\epsilon]$ | Theorems 8,9 of [CEM+15] |
| Random Projection                   | $O(k/\epsilon^2)$ | Theorem 12 of [CEM+15] |
| Non-Oblivious Random Projection$^1$ | $O(k/\epsilon)$  | Theorem 16 of [CEM+15] |
| Ridge Leverage Score Column Sampling| $O(k \log k/\epsilon^2)$ | Theorem 6 of [CMM17] |
| Leverage Score + Residual Column Sampling | $O(k \log k/\epsilon^2)$ | Theorem 14 of [CEM+15] |
| Deterministic Column Selection       | $O(k/\epsilon^2)$  | Theorem 15 of [CEM+15] |
| Frequent Directions Sketch          | $[k/\epsilon] + k$ | Theorem 31 of [Mus15] |

Table 1: Known projection-cost-preserving sketch constructions. All theorem references are to the arXiv versions of the cited papers. For randomized constructions, dependencies on success probability are hidden.

$^1$In this method, compute $Z \in \mathbb{R}^{d \times m}$ with orthonormal columns spanning the rows of $\Pi A$ where $\Pi \in \mathbb{R}^{m \times n}$ is a random projection matrix. Then let $\tilde{A} = AZ$. 
2 Proof Via Matrix Approximation Primitives

We start by defining three well-studied matrix approximation primitives:

**Definition 3** (Subspace Embedding). \( S \in \mathbb{R}^{d \times m} \) is an \( \epsilon \)-subspace embedding for \( M \in \mathbb{R}^{n \times d} \) if \( \forall x \in \mathbb{R}^n, \| x^T M \|_F^2 - \| x^T MS \|_F^2 \leq \epsilon \| x^T M \|_F^2 \).

**Definition 4** (Approximate Matrix Multiplication). \( S \in \mathbb{R}^{d \times m} \) satisfies \( \epsilon \)-approximate matrix multiplication for \( M \in \mathbb{R}^{n \times d} \), \( N \in \mathbb{R}^{d \times p} \) if \( \| MN^T - MSST^T N \|_F^2 \leq \epsilon \cdot \| M \|_F \cdot \| N \|_F \).

**Definition 5** (Frobenius Norm Preservation). \( S \in \mathbb{R}^{d \times m} \) satisfies \( \epsilon \)-Frobenius norm preservation for \( M \in \mathbb{R}^{n \times d} \) if \( \| AS \|_F^2 - \| MS \|_F^2 \leq \epsilon \| A \|_F^2 \).

We will also define a useful notion of splitting any matrix into the part in the span of its top \( r \) singular vectors and the part outside this span:

**Definition 6** (Head-Tail Split). For any \( M \in \mathbb{R}^{n \times d} \) consider the singular value decomposition \( U \Sigma V^T = M \), where \( U \in \mathbb{R}^{n \times \text{rank}(M)}, V \in \mathbb{R}^{d \times \text{rank}(M)} \) have orthonormal columns (the left and right singular vectors of \( M \) respectively), and \( \Sigma \) is a nonnegative diagonal matrix with entries equal to \( M \)'s singular values \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\text{rank}(M)} > 0 \).

For any \( r \leq \text{rank}(M) \) let \( U_r \in \mathbb{R}^{n \times r}, V_r \in \mathbb{R}^{d \times r} \) denote the first \( r \) columns of \( U, V \) respectively and let \( M_r = U_r U_r^T M = MV_r V_r^T \) and \( M_{\overline{r}} = M - M_r \). Note that \( M_r \) is the optimal \( r \)-rank approximation of \( M \): \( M_r = \arg \min_{\text{rank}(C) = r} \| M - C \|_F^2 \).

We now have the following theorem, which is very similar to Theorem 2 of [CYD19]:

**Theorem 2** (Projection-cost-preserving sketch via matrix approximation). If \( S \in \mathbb{R}^{d \times m} \):

1. Is an \( \frac{c}{n} \)-subspace embedding (Definition 3) for \( A_k \).
2. Satisfies \( \frac{c}{d \sqrt{k}} \)-approximate matrix multiplication (Definition 4) for \( A_{\backslash k}, A_{\backslash k} \).
3. Satisfies \( \frac{c}{d \sqrt{k}} \)-approximate matrix multiplication (Definition 4) for \( A_{\backslash k}, V_k \).
4. Satisfies \( \frac{c}{d \sqrt{k}} \)-Frobenius norm preservation (Definition 5) for \( A_{\backslash k} \).

Then \( \tilde{A} = AS \) is an \( (\epsilon, 0, k) \)-projection-cost-preserving sketch of \( A \).

**Proof.** For any orthogonal projection matrix \( P \in \mathbb{R}^{n \times n} \) with rank at most \( k \), let \( Y = I - P \), where \( I \) is the \( n \times n \) identity matrix. To prove the theorem, it suffices to show that:

\[
\| YA \|_F^2 - \| YAS \|_F^2 \leq \epsilon \| YA \|_F^2. \tag{2}
\]

This immediately gives (1) with \( c = 0 \). To prove (2) we will decompose the error into head and tail, terms following Definition 6. We write \( A = A_k + A_{\backslash k} \) and then rewrite (2) as:

\[
\| Y(A_k + A_{\backslash k}) \|_F^2 - \| Y(A_k + A_{\backslash k})S \|_F^2 \leq \epsilon \| YA \|_F^2. \tag{3}
\]

Expanding out the left hand side, using that \( \| M \|_F^2 = \text{tr}(MM^T) \) and that \( \text{tr}(Y A_k A_k^T Y) = 0 \) since \( A_k \) and \( A_{\backslash k} \) have orthogonal row spans we can see that to show (3) it suffices to show:

\[
\begin{aligned}
\left| \text{tr}(Y A_k A_k^T Y) - \text{tr}(Y A_k SSS^T A_k^T Y) \right| + \left| \text{tr}(Y A_{\backslash k} A_{\backslash k}^T Y) - \text{tr}(Y A_{\backslash k} SSS^T A_{\backslash k}^T Y) \right| + 2 \left| \text{tr}(Y A_k SSS^T A_k^T A_k^T Y) \right| & \leq \epsilon \| YA \|_F^2. \tag{4}
\end{aligned}
\]

We now bound the three terms of (4) separately.
**Claim 3** (Head Bound - via subspace embedding). **Under the assumptions of Theorem 2, for any orthogonal projection matrix P ∈ ℝ^{n×n} with rank at most k and Y = I − P:**

\[ | \text{tr}(YA_k A_{k}^T Y) − \text{tr}(YA_k S S^T A_{k}^T Y) | \leq \frac{\epsilon}{3} \|YA\|^2_F. \]  

(5)

**Proof.** By the assumption that S is an \( \frac{\epsilon}{3} \)-subspace embedding for \( A_k \) (Definition 3) we have:

\[ | \text{tr}(YA_k A_{k}^T Y) − \text{tr}(YA_k S S^T A_{k}^T Y) | = \| Y A_k \|^2_F − \| Y A_k S \|^2_F \]

\[ \leq \frac{\epsilon}{3} \| Y A_k \|^2_F \leq \frac{\epsilon}{3} \| Y A \|^2_F. \]

The second to last inequality follows from the subspace embedding property. In particular, let \((YA_k)_i\) and \((YA_k S)_i\) denote the \( i \)th rows of \( YA_k \) and \( YA_k S \) respectively. Then by the subspace embedding property we have \( |\|(YA_k)_i\|^2_F − \| (YA_k S)_i\|^2_F | \leq \frac{\epsilon}{3} \| (YA_k)_i\|^2_F. \) \( \square \)

**Claim 4** (Tail Bound - via approximate matrix multiplication & Frobenius norm preservation). **Under the assumptions of Theorem 2, for any orthogonal projection matrix P ∈ ℝ^{n×n} with rank at most k and Y = I − P:**

\[ | \text{tr}(YA_{\setminus k} A_{k}^T Y) − \text{tr}(YA_{\setminus k} S S^T A_{k}^T Y) | \leq \frac{\epsilon}{3} \|YA\|^2_F. \]  

(6)

**Proof.** We rewrite the tail term as:

\[ | \text{tr}(YA_{\setminus k} A_{k}^T Y) − \text{tr}(YA_{\setminus k} S S^T A_{k}^T Y) | = | |(YA_{\setminus k})^2_F − \|YA_{\setminus k} S\|^2_F | \]

\[ = | ((I − P)A_{\setminus k})^2_F − \|I − P\|A_{\setminus k} S\|^2_F \]

\[ = | \|A_{\setminus k}\|^2_F − \|PA_{\setminus k}\|^2_F − \|A_{\setminus k} S\|^2_F + \|PA_{\setminus k} S\|^2_F | \]

\[ \leq \|A_{\setminus k}\|^2_F + \|PA_{\setminus k} S\|^2_F - \|A_{\setminus k} S\|^2_F - \|PA_{\setminus k} S\|^2_F. \]  

(7)

The third line follows from the Pythagorean theorem. By the assumption that S satisfies \( \frac{\epsilon}{6} \)-Frobenius norm preservation (Definition 5) for \( A_{\setminus k} \) we can bound the first term in (7) by:

\[ \|A_{\setminus k}\|^2_F − \|PA_{\setminus k} S\|^2_F | \leq \frac{\epsilon}{6} \|A_{\setminus k}\|^2_F \leq \frac{\epsilon}{6} \|YA\|^2_F, \]  

(8)

while the last inequality follows from the fact that \( A_{\setminus k} = \arg \min_{\text{rank-k}} \|A - C\|^2_F \). Thus \( \|A_{\setminus k}\|^2_F = \|A - A_{\setminus k}\|^2_F \leq \|A - PA\|^2_F \) for any rank-k projection P. We write the second term of (7) as:

\[ \|PA_{\setminus k}\|^2_F − \|PA_{\setminus k} S\|^2_F | = | \text{tr}(PA_{\setminus k} A_{k}^T Y) − A_{\setminus k} S S^T A_{k}^T Y |. \]

\( PA_{\setminus k} A_{k}^T Y - A_{\setminus k} S S^T A_{k}^T Y \) P has rank at most k since P has rank at most k. Letting \( \lambda_1, \ldots, \lambda_k \) denote its eigenvalues we have:

\[ \|PA_{\setminus k}\|^2_F − \|PA_{\setminus k} S\|^2_F | = | \text{tr}(PA_{\setminus k} A_{k}^T Y) − A_{\setminus k} S S^T A_{k}^T Y | = \left| \sum_{i=1}^{k} \lambda_i \right| \leq \sum_{i=1}^{k} |\lambda_i| \leq \sqrt{k} \left( \sum_{i=1}^{k} \lambda_i^2 \right) \]

\[ = \sqrt{k} \|PA_{\setminus k} A_{k}^T Y - A_{\setminus k} S S^T A_{k}^T Y \|^2_F \]

\[ \leq \sqrt{k} \|PA_{\setminus k} A_{k}^T Y - A_{\setminus k} S S^T A_{k}^T Y \|^2_F. \]  

(9)
The last inequality follows since $P$ is a projection matrix and can only decrease the Frobenius norm. By our assumption that $S$ satisfies $\frac{\epsilon}{6\sqrt{k}}$-approximate matrix multiplication (Definition 4) for $A_{\setminus k}, A_{\setminus k}$ we thus can bound:

\[
\|PA_{\setminus k}\|^2_F - \|PA_{\setminus k}S\|^2_F \leq \frac{\epsilon}{6}\|A_{\setminus k}\|^2_F \leq \frac{\epsilon}{6}\|YA\|^2_F, \tag{10}
\]

where the second inequality again follows since $A_{\setminus k} = A - A_k$ is the error of the best rank-$k$ approximation to $A$. Plugging (8) and (10) back into (7) we have:

\[
|\text{tr}(YA_{\setminus k}A_{\setminus k}^TY) - \text{tr}(YA_{\setminus k}SS^TA_{\setminus k}^TY)| \leq \frac{\epsilon}{3}\|YA\|^2_F,
\]

which gives (6) and completes the claim. 

\[
\square
\]

**Claim 5** (Cross Term Bound - via approximate matrix multiplication). Under the assumptions of Theorem 2, for any orthogonal projection matrix $P \in \mathbb{R}^{n \times n}$ with rank at most $k$ and $Y = I - P$:

\[
2\left|\text{tr}(YA_kSS^TA_{\setminus k}^TY)\right| \leq \frac{\epsilon}{3}\|YA\|^2_F. \tag{11}
\]

**Proof.** Let $C = AA^T$ and let $C^+$ be its pseudoinverse. Writing $A$ in its SVD, $A = USV^T$, we have $C^+ = U\Sigma^{-2}V^T$. We let $C'^+ = U\Sigma^{-1}V^T$. We can bound the cross term as:

\[
2\left|\text{tr}(YA_kSS^TA_{\setminus k}^TY)\right| = 2\left|\text{tr}(YCC^+A_kSS^TA_{\setminus k}^TY)\right| (\text{Since the columns of } A_k \text{ fall within the column span of } C)
\]

\[
= 2\left|\text{tr}(Y^2CC^+A_kSS^TA_{\setminus k}^TY)\right| (\text{By the cyclic property of the trace})
\]

\[
= 2\left|\text{tr}(YCC^+A_kSS^TA_{\setminus k}^TY)\right| (\text{Since } Y = I - P \text{ is an orthogonal projection so } Y^2 = Y.)
\]

\[
\leq 2\sqrt{\text{tr}(YCC^+2C^{+2}CY)} \cdot \sqrt{\text{tr}(A_kSS^TA_{\setminus k}^TC^{+2}A_kSS^TA_{\setminus k}^TY)}
\]

\[
\leq 2\sqrt{\text{tr}(YCC^+2C^{+2}CY)} \cdot \sqrt{\text{tr}(A_kSS^TA_{\setminus k}^TC^{+2}A_kSS^TA_{\setminus k}^TY)} \cdot \sqrt{\text{tr}(A_kSS^TA_{\setminus k}^TC^{+2}A_kSS^TA_{\setminus k}^TY)} \cdot \sqrt{\text{tr}(A_kSS^TA_{\setminus k}^TC^{+2}A_kSS^TA_{\setminus k}^TY)} \cdot \sqrt{\text{tr}(A_kSS^TA_{\setminus k}^TC^{+2}A_kSS^TA_{\setminus k}^TY)}
\]

The last inequality follows from Cauchy-Schwarz. The first term of (12) can be bounded by:

\[
\sqrt{\text{tr}(YCC^+2C^{+2}CY)} = \sqrt{\text{tr}(YCY)} = \sqrt{\text{tr}(YAA^TY)} = \|YA\|_F. \tag{13}
\]

We bound the second term of (12) by using the SVD to write $A_k = U_k\Sigma_kV_k^T$, where $U_k, V_k$ are as in Definition 6 and $\Sigma_k \in \mathbb{R}^{k \times k}$ is the top left $k \times k$ submatrix of $\Sigma$. We have:

\[
\sqrt{\text{tr}(A_{\setminus k}SS^TA_{\setminus k}^TC^{+2}C^{+2}A_kSS^TA_{\setminus k}^TY)} \leq \sqrt{\text{tr}(A_{\setminus k}SS^TV_k\Sigma_kU_k^TU_k^TU_k\Sigma_kV_k^TSS^TA_{\setminus k}^TY)}
\]

\[
= \sqrt{\text{tr}(A_{\setminus k}SS^TV_kV_k^TSS^TA_{\setminus k}^TY)}
\]

\[
= \|A_{\setminus k}SS^TV_k\|_F = \|A_{\setminus k}SS^TV_k - A_{\setminus k}V_k\|_F. \tag{14}
\]

The last line follows from the fact that the rows of $A_{\setminus k}$ are orthogonal to the columns of $V_k$ and thus $A_{\setminus k}V_k = 0$. By the assumption that $S$ satisfies $\frac{\epsilon}{6\sqrt{k}}$-approximate matrix multiplication for $A_{\setminus k}, V_k$, we thus have

\[
\sqrt{\text{tr}(A_{\setminus k}SS^TA_{\setminus k}^TC^{+2}C^{+2}A_kSS^TA_{\setminus k}^TY)} \leq \frac{\epsilon}{4\sqrt{k}}\|A_{\setminus k}\|_F \cdot \|V_k\|_F
\]

\[
\leq \frac{\epsilon}{6\sqrt{k}}\|YA\|_F \cdot \sqrt{k} = \frac{\epsilon}{6}\|YA\|_F. \tag{15}
\]
Plugging (13) and (15) back into (12) we have:

$$2 \left| \text{tr}(YA_k SS^T A_{\ell, k}^T Y) \right| \leq \frac{\epsilon}{3} \|YA\|_F^2,$$

which gives (11) and completes the claim.

**Completing the Proof:**

Finally, we combine the head, tail and cross term bounds of Claims 3, 4, and 5 to give:

$$|\text{tr}(YA_k A^T_k Y) - \text{tr}(YA_k SS^T A_{\ell, k}^T Y)| + |\text{tr}(YA_k A^T_k Y) - \text{tr}(YA_k SS^T A^T_{\ell, k} Y)| + 2 |\text{tr}(YA_k SS^T A_{\ell, k}^T Y)|$$

$$\leq \frac{\epsilon}{3} \|YA\|_F^2 + \frac{\epsilon}{3} \|YA\|_F^2 + \frac{\epsilon}{3} \|YA\|_F^2 = \epsilon \|YA\|_F^2.$$

This yields (4), which completes the proof of Theorem 2.

**2.1 Constructions Satisfying Theorem 2**

Theorem 2 can be used to prove that a number of constructions of \(S\) give projection-cost-preserving sketches. A simple example is when \(S\) is a random projection matrix. In fact, any projection matrix satisfying a certain Johnson-Lindenstrauss moment property suffices.

**Definition 7** ((\(\epsilon, \delta, \ell\))-JL moment property, [KN14]). A matrix \(S \in \mathbb{R}^{d \times m}\) satisfies the \((\epsilon, \delta, \ell)\)-JL moment property if for any \(x \in \mathbb{R}^d\) with \(\|x\|_2 = 1\),

$$\mathbb{E}_{S} \|x^T S\|_2^2 - 1 \leq \epsilon \cdot \delta.$$

**Lemma 6** (Projection-cost-preservation from JL moment property). If \(S\) satisfies the \((\frac{\epsilon}{\sqrt{6 \sqrt{k}}}, \delta, \ell)\)-JL moment property and the \((\frac{\epsilon}{\sqrt{7}}, \frac{\delta}{\sqrt{9}}, \ell)\)-JL moment property for any \(\ell \geq 2\), then with probability \(\geq 1 - 4\delta\), \(\tilde{A} = AS\) is an \((\epsilon, 0, k)\)-projection-cost-preserving sketch of \(A\).

**Proof.** It is well known that if \(S\) satisfies the \((\frac{\epsilon}{\sqrt{7}}, \frac{\delta}{\sqrt{9}}, \ell)\)-JL moment property for any \(\ell > 0\), then with probability \(\geq 1 - \delta\), \(S\) is an \(\frac{\epsilon}{\sqrt{7}}\)-subspace embedding for \(A_k\) since \(A_k\) has rank \(k\). The proof follows from a net argument, as given in [Sar06] or Theorem 2.1 of [Woo14].

We also have from Theorem 2.8 in [Woo14] that if \(S\) satisfies the \((\frac{\epsilon}{6 \sqrt{k}}, \delta, \ell)\)-JL moment property for any \(\ell \geq 2\), then \(S\) satisfies the \(\frac{\epsilon}{6 \sqrt{k}}\)-approximate matrix multiplication property with probability \(\geq 1 - \delta\) for any pair of matrices.

Finally, we claim that if \(S\) satisfies the \((\frac{\epsilon}{\sqrt{7}}, \delta, \ell)\)-JL moment property for any \(\ell > 0\), then \(S\) satisfies the \(\frac{\epsilon}{\sqrt{7}}\)-Frobenius norm preservation condition for any matrix \(M \in \mathbb{R}^{n \times d}\), with probability \(1 - \delta\). In particular, let \(m_1, \ldots, m_n\) denote the rows of \(M\). Let \(\hat{\epsilon}\) denote \(\epsilon/6\). We have:

$$\Pr \left[ \left| \|MS\|_F^2 - \|M\|_F^2 \right| > \hat{\epsilon} \|M\|_F^2 \right] \leq \epsilon \hat{\epsilon} \|M\|_F^{-2\ell} \cdot \mathbb{E} \left[ \left| \|MS\|_F^2 - \|M\|_F^2 \right|^{\ell} \right]$$

$$= \epsilon \hat{\epsilon} \|M\|_F^{-2\ell} \cdot \mathbb{E} \left[ \sum_{i=1}^n \left( \|m_i^T S\|_2^2 - \|m_i\|_2^2 \right)^{\ell} \right]$$

$$\leq \epsilon \hat{\epsilon} \|M\|_F^{-2\ell} \cdot \left( \sum_{i=1}^n \left[ \mathbb{E} \left( \|m_i^T S\|_2^2 - \|m_i\|_2^2 \right)^{1/\ell} \right]^{\ell} \right)$$

The last inequality follows from Minkowski’s inequality. Then we use the JL-moment property:
\[ \tilde{c}^{-\ell} ||M||_F^{-2\ell} \left[ \sum_{i=1}^{n} \left( \mathbb{E} ||m_i^T S||_2^2 - ||m_i||_2^{2\ell} \right)^{1/\ell} \right]^{\ell} \leq \tilde{c}^{-\ell} ||M||_F^{-2\ell} \cdot \left[ \sum_{i=1}^{n} \left( \tilde{c}^\ell \cdot \delta \cdot ||m_i||_2^{2\ell} \right)^{1/\ell} \right]^{\ell} \]
\[ \leq \tilde{c}^{-\ell} ||M||_F^{-2\ell} \cdot \left[ \tilde{c} \cdot \delta^{1/\ell} \cdot ||M||_F^{\ell} \right]^{\ell} \leq \delta. \]

Applying a union bound, we have that all four condition of Theorem 2 hold with probability \(\geq 1 - 4\delta\), which completes the proof of Lemma 6.

Many standard random projection matrices, including classic random Gaussian matrices, random Rademacher matrices, and fast and sparse JL transforms can be shown to satisfy the requirements of Lemma 6 with varying embedding dimensions. For example, we have the following

Corollary 7 (Dense Random Projection). Let \( S \in \mathbb{R}^{d \times m} \) be a matrix with each entry set independently to \( S_{i,j} = \mathcal{N}(0,1) \) where \( \mathcal{N}(0,1) \) is a standard normal random variable. If \( m \geq c \cdot \frac{k+\log(1/\delta)}{\epsilon^2} \)
for a sufficiently large universal constant \( c \) then with probability \( \geq 1 - \delta \), \( \tilde{A} = AS \) is an \((\epsilon, 0, k)\)-projection-cost-preserving sketch of \( A \).

Proof. When \( S \) is a random Gaussian matrix, \( \|x^T S\|^2_2 \) is sum of independent Chi-squared random variables. We can thus directly apply a moment bound for sub-exponential random variables [Wai19, Ver18] to establish that when \( m \geq c \cdot \frac{k+\log(1/\delta)}{\epsilon^2} \), \( S \) satisfies both the \((\frac{\epsilon/2}{\sqrt{k}}, \frac{\delta}{4}, \ell)\)-JL moment property and the \((\frac{\epsilon/4}{\sqrt{k}}, \ell)\)-JL moment property. Applying Lemma 6 completes the proof.

For information on other random projection matrices that can be analyzed using Lemma 6, see [CEM+15] and [CYD19]. Some of these matrices can be applied faster or stored in less space than the dense random projection of Corollary 7 because they are sparse or structured.

Finally, we note that the conditions of Theorem 2 can also be satisfied by a simple sampling scheme, which was proposed in [CEM+15] and also analyzed in [CYD19]:

Corollary 8 (Leverage Score + Residual Sampling). Consider \( A \) with SVD \( A = USV^T \). For every \( i \in 1, \ldots, n \) let \( p_i = \frac{\|((U)_i)_i\|^2_2}{2k} + \frac{\|(A - (A)_i)_i\|^2_2}{2\|A - (A)_i\|_2^2} \). Let \( S \) be a sampling matrix selecting \( m \) columns of \( A \) where each column of \( S \) is set independently to \( \frac{1}{\sqrt{m}}p_i \) with probability \( p_i \), where \( e_i \) is the \( i^{th} \) standard basis vector. Then for \( m \geq \frac{c\log(k/\delta)}{\epsilon^2} \) for some universal constant \( c \), with probability \( \geq 1 - \delta \), \( S \) satisfies all four requirements of Theorem 2 and so \( \tilde{A} = AS \) is an \((\epsilon, 0, k)\)-projection-cost-preserving sketch of \( A \).

2.2 Proof Variants

We briefly mention a few variants on the proof of Theorem 2 that may be useful.

Head-Tail Split Using an Approximate Basis:

Corollary 8 requires sampling by the leverage scores of the rank-\( k \) subspace spanned by \( A \)'s true top \( k \) singular vectors. This is to ensure that \( S \) is an \( \frac{\epsilon}{3} \)-subspace embedding for \( A_k \) (requirement (1) of Theorem 2.) It can be shown that the leverage scores of any approximate subspace (computed e.g. via an input sparsity time sketching method) can be used instead (see Theorem 14 of [CEM+15], arXiv version). The proof requires splitting \( A \) into ‘approximate head and tail terms’ \( AZZ^T \) and \( A(I - ZZ^T) \) where \( Z \in \mathbb{R}^{d \times \Omega(k)} \) has orthonormal columns and where \( A - AZZ^T \) is a near optimal low-rank approximation of \( A \). See Lemma 10 of [CEM+15], arXiv version.
Allowing a Constant Error Term

Recall that Definition 1 allows $\|\tilde{A} - P\tilde{A}\|_F^2$ to approximate $\|A - PA\|_F^2$ up to an additive constant $c$, which can depend on $A, \tilde{A}, \epsilon, k$, but not on $P$. The use of such a term is useful, e.g., when $\tilde{A}$ is obtained by taking a low-rank approximation of $A$ and consistently underestimates the cost $\|A - PA\|_F^2$ (see Theorem 7, 8, 9 of [CEM+15], arXiv version).

Allowing a constant term can be useful also in loosening the requirements for Theorem 2. For example, requirement 4, that $S$ preserves $\|A_{i,k}\|_F^2$ up to $\frac{\epsilon}{6}$ error, can be relaxed: we need only require that $\|A_{i,k}S\|_F^2 \leq c_1\|A_{i,k}\|_F^2$ for some constant $c_1 \geq 1$. As long as $S$ preserves the norm in expectation, such a constant error bound is easy to show via Markov’s inequality. The Frobenius norm requirement is used in proving (8), which bounds the first term of (7). However, note that this term $||A_{i,k}||_F^2 - ||A_{i,k}S||_F^2$ is a constant independent of $P$ and is bounded by $c_1||A_{i,k}||_F^2$ as long as $||A_{i,k}S||_F^2 \leq c_1||A_{i,k}||_F^2$. Setting $c = -||A_{i,k}||_F^2 - ||A_{i,k}S||_F^2$, we can absorb this term into the projection-cost-preserving sketching bound. We can also see that this is sufficient to achieve a relative error bound in Claim 2 as long as $\gamma$ is a less than a constant sufficiently close to 1. See Lemma 7 of [MW17], arXiv version for an example application of this technique.

Head-Tail Split Using a Higher Dimension:

In some cases, we can relax the requirements of Theorem 2 or give a stronger guarantee (e.g., a projection-cost-preserving sketch where error is measured in the spectral rather than the Frobenius norm) by splitting $A = A_r + A_{\gamma}r$ for $r > k$ instead of $A = A_k + A_{\gamma,k}$. If we set $r = ck/\epsilon + k$ we have $||A_{\gamma,r}||_F^2 \leq \frac{c}{k^2}||A_{i,k}||_F^2$, which can be a useful bound. See Lemmas 7 and 8 of [MW17], arXiv version for an example of this technique. Also see Theorem 9 in Section 3.

3 Proof via Spectral Approximation

We now show an alternative strategy to proving that a sketching matrix $S \in \mathbb{R}^{d \times m}$ yields $\tilde{A} = AS$ which is a projection-cost-preserving sketch. This proof strategy was presented in [CMM17].

**Theorem 9** (Projection-cost-preserving sketch via spectral approximation). If $S \in \mathbb{R}^{d \times m}$:

1. Satisfies $(1 - \frac{\epsilon}{24}) AA^T - \lambda I \preceq ASS^T A^T \preceq (1 + \frac{\epsilon}{24}) AA^T + \lambda I$ for $\lambda = \frac{\epsilon \|A-A_k\|_F^2}{24k}$.

2. Satisfies $\frac{\epsilon}{12 \cdot \|A-A_k\|_F^2}$-Frobenius norm preservation (Definition 5) for $A_{i,p}$ where $p$ is the largest integer such that $\sigma^2_p \geq \frac{\|A-A_k\|_F^2}{k}$.

Then $\tilde{A} = AS$ is an $(\epsilon, 0, k)$-projection-cost-preserving sketch of $A$.

**Proof.** As in the proof of Theorem 2, letting $P \in \mathbb{R}^{n \times n}$ be any orthogonal projection matrix with rank at most $k$ and $Y = I - P$, to prove the theorem it suffices to show (2). Following Definition 6 we decompose $A = A_p + A_{\gamma}p$ where $p$ is the largest integer such that $\sigma^2_p \geq \frac{\|A-A_k\|_F^2}{k}$. Note that we always have $p \leq 2k$ since $\|A - A_k\|_F^2 = \sum_{i=k+1}^{\text{rank}(A)} \sigma^2_i \geq \sum_{i=k+1}^{2k} \sigma^2_i \geq k \cdot \sigma^2_{2k}$. Using this

---

2Equivalently, for any $x \in \mathbb{R}^n$, $(1 - \epsilon)\|x^T A\|_2^2 - \lambda \|x\|_2^2 \leq \|x^T \tilde{A}\|_2^2 \leq (1 + \epsilon)\|x^T A\|_2^2 + \lambda \|x\|_2^2$. 

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decomposition, we can see that to prove the theorem it suffices to prove an analogous bound to (4):

\[
\left| \text{tr}(YA_p A_p^T Y) - \text{tr}(YA_p SS^T A_p^T Y) \right| + \left| \text{tr}(YA_{\perp p} A_{\perp p}^T Y) - \text{tr}(YA_{\perp p} SS^T A_{\perp p}^T Y) \right|
\]

\[
\text{head term}
\]

\[
\text{tail term}
\]

\[
+ 2 \left| \text{tr}(YA_p SS^T A_{\perp p}^T Y) \right| \leq \epsilon \|YA\|_F^2.
\]

(16)

We now proceed as in the proof of Theorem 2, bounding the three terms of (16) separately.

**Claim 10 (Head Bound).** *Under the assumptions of Theorem 9, for any orthogonal projection matrix \( P \in \mathbb{R}^{n \times n} \) with rank at most \( k \) and \( Y = I - P \):

\[
\left| \text{tr}(YA_p A_p^T Y) - \text{tr}(YA_p SS^T A_p^T Y) \right| \leq \frac{\epsilon}{12} \|YA\|_F^2.
\]

(17)

**Proof.** Since for any \( x \in \mathbb{R}^n \) we can write \( x^T A_p A_p^T x = y^T A A^T y \) for \( y = U_p U_p^T x \), by our spectral error assumption we have:

\[
(1 - \frac{\epsilon}{24}) x^T A_p A_p^T x - \frac{\epsilon \|A_{\perp k}\|_F^2}{24k} \|y\|_2^2 \leq x^T A_p SS^T A_p^T x \leq \left( 1 + \frac{\epsilon}{24} \right) x^T A_p A_p^T x - \frac{\epsilon \|A_{\perp k}\|_F^2}{24k} \|y\|_2^2.
\]

(18)

By our choice of \( p \), \( y \) is orthogonal to all singular directions of \( A \) except those with squared singular value greater than or equal to \( \frac{\|A_{\perp k}\|_F^2}{k} \). It follows that

\[
x^T A_p A_p^T x = y^T A A^T y \geq \frac{\|A_{\perp k}\|_F^2}{k} \cdot \|y\|_2^2,
\]

and plugging back into (18), that for any \( x \in \mathbb{R}^n \):

\[
(1 - \frac{\epsilon}{12}) x^T A_p A_p^T x \leq x^T A_p SS^T A_p^T x \leq \left( 1 + \frac{\epsilon}{12} \right) x^T A_p A_p^T x.
\]

(19)

This yields (17), completing the claim. \(\square\)

**Claim 11 (Tail Bound).** *Under the assumptions of Theorem 9, for any orthogonal projection matrix \( P \in \mathbb{R}^{n \times n} \) with rank at most \( k \) and \( Y = I - P \):

\[
\left| \text{tr}(YA_{\perp p} A_{\perp p}^T Y) - \text{tr}(YA_{\perp p} SS^T A_{\perp p}^T Y) \right| \leq \frac{\epsilon}{6} \|YA\|_F^2.
\]

(20)

**Proof.** As in the proof of Theorem 2 (see equation (7)) we bound the tail term as:

\[
\left| \text{tr}(YA_{\perp p} A_{\perp p}^T Y) - \text{tr}(YA_{\perp p} SS^T A_{\perp p}^T Y) \right| \leq \|A_{\perp p}\|_F^2 - \|A_{\perp p} S\|_F^2 + \|PA_{\perp p}\|_F^2 - \|PA_{\perp p} S\|_F^2.
\]

(21)

By the assumption that \( S \) satisfies \( \frac{\epsilon}{12} \cdot \frac{\|A_{\perp p}\|_F^2}{\|A_{\perp p}\|_F^2} \). Frobenius norm preservation (Definition 5) for \( A_{\perp p} \) we can bound the first term in (21) by:

\[
\|A_{\perp p}\|_F^2 - \|A_{\perp p} S\|_F^2 \leq \frac{\epsilon}{12} \|A_{\perp p}\|_F^2.
\]

(22)
We next bound the second term of (21), $\|PA_{\mathbf{p}}\|_F^2 - \|PA_{\mathbf{p}}S\|_F^2$. For any $x \in \mathbb{R}^n$ we can write $x^T A_{\mathbf{p}} A_{\mathbf{p}} x = y^T A A^T y$ where $y = \left(I - U_p U_p^T\right)x$. By our spectral error assumption we have:

$$
\left(1 - \frac{\epsilon}{24}\right) x^T A_{\mathbf{p}} A_{\mathbf{p}}^T x - \frac{\epsilon\|A_{\mathbf{k}}\|_F^2}{24k}\|y\|^2 \leq x^T A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T x \leq \left(1 + \frac{\epsilon}{24}\right) x^T A_{\mathbf{p}} A_{\mathbf{p}}^T x + \frac{\epsilon\|A_{\mathbf{k}}\|_F^2}{24k}\|y\|^2.
$$

Noting that $\|y\|^2 \leq \|x\|^2$ and that by definition of $p$, $\|A_{\mathbf{p}}\|_2^2 \leq \frac{\|A_{\mathbf{k}}\|_2^2}{k}$ and thus $x^T A_{\mathbf{p}} A_{\mathbf{p}}^T x \leq \|x\|_2^2 \cdot \frac{\|A_{\mathbf{k}}\|_2^2}{k}$ we obtain:

$$
\left| x^T (A_{\mathbf{p}} A_{\mathbf{p}}^T - A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T) x \right| \leq \frac{\epsilon \cdot \|x\|_2 \cdot \|A_{\mathbf{k}}\|_2^2}{12k}.
$$

(23)

Finally, since $P$ is a rank-$k$ projection matrix, we can write $P = Z Z^T$ where $Z \in \mathbb{R}^{n \times k}$ has orthonormal columns $z_1, \ldots, z_k$: Using (23) we can bound:

$$
\|PA_{\mathbf{k}}\|_F^2 - \|PA_{\mathbf{k}} S\|_F^2 = |\text{tr}(P[A_{\mathbf{k}} A_{\mathbf{k}}^T - A_{\mathbf{k}} S S^T A_{\mathbf{k}}^T]P)|
$$

$$
= |\text{tr}(Z Z^T[A_{\mathbf{k}} A_{\mathbf{k}}^T - A_{\mathbf{k}} S S^T A_{\mathbf{k}}^T]Z Z^T)|
$$

$$
= |\text{tr}(Z Z^T[A_{\mathbf{k}} A_{\mathbf{k}}^T - A_{\mathbf{k}} S S^T A_{\mathbf{k}}^T]Z)|
$$

$$
\leq \sum_{i=1}^k \frac{\epsilon \cdot \|z_i\|_2^2 \cdot \|A_{\mathbf{k}}\|_F^2}{12k} = \frac{\epsilon}{12} \cdot \|A_{\mathbf{k}}\|_2^2,
$$

(24)

where the second line follows from the cyclic property of trace and the fact that $Z^T Z = I$. Plugging (22) and (24) back into (21) gives:

$$
|\text{tr}(Y A_{\mathbf{p}} A_{\mathbf{p}}^T Y) - \text{tr}(Y A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T Y)| \leq \frac{\epsilon}{12} \|YA\|_F^2 + \frac{\epsilon}{12} \|A_{\mathbf{k}}\|_F^2 \leq \frac{\epsilon}{6} \|YA\|_F^2,
$$

which gives (20), completing the claim.

\[\square\]

**Claim 12 (Cross Term Bound).** Under the assumptions of Theorem 9, for any orthogonal projection matrix $P \in \mathbb{R}^{n \times n}$ with rank at most $k$ and $Y = I - P$:

\[2 \left| \text{tr}(Y A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T Y) \right| \leq \frac{\epsilon}{2} \|YA\|_F^2.\]

(25)

**Proof.** We follow equation (12) in the proof of Claim 5, writing $C = AA^T$ and bounding

$$
2 \left| \text{tr}(Y A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T Y) \right| \leq 2 \sqrt{\text{tr}(Y C^2 + C^T C^2)} \cdot \sqrt{\text{tr}(A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T C)^2 + C^T A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T C^2).}
$$

(26)

As in (13) we have:

$$
\sqrt{\text{tr}(Y C^2 + C^T C^2)} = \|YA\|_F.
$$

(27)

It remains to bound the second term in the product. Following (14) we have: Let $\Sigma_p \in \mathbb{R}^{p \times p}$ be the top left $p \times p$ submatrix of $\Sigma$ (with diagonal entries $\sigma_1, \ldots, \sigma_p$). We have $C^2 = U \Sigma^{-1} U^T$ and recalling that $A_{\mathbf{p}} = U_p U_p^T A$ can write:

$$
\text{tr}(A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T C^2) = \|A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T C^2\|_F^2
$$

$$
= \|A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T U_p U_p^T U \Sigma^{-1} U^T\|_F^2
$$

$$
= \|A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T U_p \Sigma^{-1}\|_F^2
$$

$$
= \sum_{i=1}^p \|A_{\mathbf{p}} S S^T A_{\mathbf{p}}^T u_i\|_2^2 \cdot \sigma_i^{-2}.
$$

(28)
We prove that the summand is small for every \( i \). Take \( p_i \) to be a unit vector:

\[
p_i = \frac{1}{\|A_{\setminus p}SS^T A u_i\|_2} \cdot A_{\setminus p}SS^T A u_i.
\]

Note that \( p_i \) falls within the column span of \( A_{\setminus p} \) and thus \( p_i^T A = p_i^T A_{\setminus p} \). Analogously, \( u_i^T A = u_i^T A_{\setminus p} \).

Using the first fact we can write:

\[
\|A_{\setminus p}SS^T A u_i\|_2^2 = (p_i^T A_{\setminus p}SS^T A u_i)^2 = (p_i^T ASS^T A u_i)^2.
\]

Now, suppose we construct the vector \( m = \left( \sigma_i^{-1} u_i + \frac{\mathbf{e}_1}{\|A_{\setminus k}\|_F} p_i \right) \). By our spectral error assumption we have that:

\[
m^T ASS^T A^T m \leq \left( 1 + \frac{\epsilon}{24} \right) m^T A A^T m + \frac{\epsilon \|A_{\setminus k}\|_F^2}{24k} \|m\|_2^2,
\]

which expands to give:

\[
\sigma_i^{-2} u_i^T ASS^T A u_i + \frac{k}{\|A_{\setminus k}\|_F} p_i^T ASS^T A p_i + \frac{2\sqrt{k}}{\sigma_i \|A_{\setminus k}\|_F} p_i^T ASS^T A u_i
\leq \left( 1 + \frac{\epsilon}{24} \right) \sigma_i^{-2} u_i^T A A^T u_i + \left( 1 + \frac{\epsilon}{24} \right) \frac{k}{\|A_{\setminus k}\|_F} p_i^T A A^T p_i + \frac{\epsilon \|A_{\setminus k}\|_F^2}{24k} \|m\|_2^2.
\]

There is no cross term on the right side since \( p_i^T A A^T u_i = p_i^T A_{\setminus p} A_{\setminus p}^T u_i = 0 \). From (19), and using that \( u_i^T A = u_i^T A_{\setminus p} \), we have:

\[
u_i^T A S S^T A^T u_i \geq \left( 1 - \frac{\epsilon}{12} \right) u_i^T A A^T u_i = \left( 1 - \frac{\epsilon}{12} \right) \sigma_i^2.
\]

Additionally, from (23), the fact that \( p_i^T A = p_i^T A_{\setminus p} \), and that \( p_i \) is a unit vector:

\[
p_i^T A S S^T A p_i \geq p_i^T A A^T p_i - \frac{\epsilon \|A_{\setminus k}\|_F^2}{12k}.
\]

Plugging (31) and (32) back into (30) gives:

\[
\left( 1 - \frac{\epsilon}{12} \right) + \frac{k}{\|A_{\setminus k}\|_F} p_i^T A A^T p_i - \frac{\epsilon}{12} + \frac{2\sqrt{k}}{\sigma_i \|A_{\setminus k}\|_F} p_i^T A S S^T A u_i
\leq \left( 1 + \frac{\epsilon}{24} \right) + \left( 1 + \frac{\epsilon}{24} \right) \frac{k}{\|A_{\setminus k}\|_F} p_i^T A A^T p_i + \frac{\epsilon \|A_{\setminus k}\|_F^2}{24k} \|m\|_2^2.
\]

Noting that \( p_i^T A A^T p_i \leq \frac{\|A_{\setminus k}\|_F^2}{k} \) since \( p_i \) lies in the column span of \( A_{\setminus p} \), rearranging (33) gives:

\[
\frac{2\sqrt{k}}{\sigma_i \|A_{\setminus k}\|_F} p_i^T A S S^T A u_i \leq \frac{\epsilon}{4} + \frac{\epsilon \|A_{\setminus k}\|_F^2}{24k} \cdot \|m\|_2^2 \leq \frac{\epsilon}{3}.
\]

The second inequality above follows from the fact that for \( i \leq p \), \( \sigma_i^{-2} \leq \frac{k}{\|A_{\setminus k}\|_F^2} \) and that \( u_i^T p_i = 0 \) so \( \|m\|_2^2 = \sigma_i^{-2} u_i^2 + \frac{k}{\|A_{\setminus k}\|_F^2} \|p_i\| \leq \frac{2k}{\|A_{\setminus k}\|_F} \). Squaring (34) gives:

\[
(p_i^T A S S^T A u_i)^2 \leq \frac{\epsilon^2}{36} \frac{\sigma_i^2 \|A_{\setminus k}\|_F^2}{k}.
\]
Plugging into (28) using (29) and that $p \leq 2k$ then gives:

$$\text{tr}(A_p S S^T A^T C + C^T A_p S S^T A^T Y) \leq \sum_{i=1}^{p} \frac{\epsilon^2}{36} \cdot \frac{\sigma_i^2 \|A^T \|_F^2}{k} \cdot \sigma_i^{-2} \leq \frac{\epsilon^2}{18} \|A^T \|_F^2$$

(35)

Finally, plugging (27) and (35) back into (26) gives:

$$2 \left| \text{tr}(Y A_p S S^T A^T Y) \right| \leq 2 \|YA\|_F \cdot \sqrt{\frac{1}{18}} \cdot \epsilon \|A^T \|_F \leq \frac{\epsilon}{2} \|YA\|_F^2,$$

(36)

which gives (25), completing the claim. □

**Completing the Proof:**

Finally, we combine the head, tail and cross term bounds of Claims 10, 11, and 12 to give:

$$|\text{tr}(Y A_p A^T Y) - \text{tr}(Y A_p S S^T A^T Y)| + |\text{tr}(Y A_p A^T Y) - \text{tr}(Y A_p S S^T A^T Y)| + 2 |\text{tr}(Y A_p S S^T A^T Y)|$$

$$\leq \frac{\epsilon}{12} \|YA\|_F^2 + \frac{\epsilon}{6} \|YA\|_F^2 + \frac{\epsilon}{2} \|YA\|_F^2 \leq \epsilon \|YA\|_F^2.$$  

(37)

This yields (4), completing the theorem. □

**3.1 Constructions Satisfying Theorem 9**

The spectral approximation and Frobenius norm preservation requirements of Theorem 9 are satisfied by many sketching methods. The are particularly natural in proving projection-cost-preserving sketch properties for column selection methods, two of which we use as examples below.

**Corollary 13** (Ridge Leverage Score Sampling). Let $a_i \in \mathbb{R}^n$ be the $i^{th}$ column of $A$. The $i^{th}$ $\lambda$-ridge leverage score of $A$ is given by

$$\tau_i(A) \overset{\text{def}}{=} a_i^T (AA^T + \lambda I)^{-1} a_i.$$

For every $i$, let $\tau_i(A)$ be an overestimate for the $i^{th}$ $\lambda$-ridge leverage score with $\lambda = \frac{\|A - A_k\|_F^2}{k}$. Let $p_i = \frac{\tau_i}{\sum_{i=1}^{d} \tau_i}$ and let $t = \frac{c \log(k/\delta)}{\epsilon^2} \sum_{i=1}^{d} \tau_i$ for any $\epsilon, \delta \in (0, 1)$ and some sufficiently large constant $c$. Let $S \in \mathbb{R}^{d \times t}$ be a sampling matrix selecting $t$ columns of $A$, where each column of $S$ is set independently to $\frac{1}{\sqrt{p_i}} e_i$ with probability $p_i$, where $e_i$ is the $i^{th}$ standard basis vector. Then, with probability $\geq 1 - \delta$, $S$ satisfies the conditions of Theorem 9 and hence $\tilde{A} = AS$ is an $(\epsilon, 0, k)$-projection-cost-preserving sketch of $A$.

Note that $\sum_{i=1}^{d} \tau_i(A) \leq 2k$ (see e.g., Lemma 4 of [CMM17]) and thus if the approximate ridge leverage scores are within a constant factor of the true ones, $\sum_{i=1}^{d} \tau_i = O(k)$ and so $\tilde{A}$ has $O(k \log(k/\delta)/\epsilon^2)$ columns.

**Proof.** The spectral approximation guarantee can be proven with a matrix Bernstein inequality. See Theorem 5 of [CMM17]. The Frobenius norm preservation guarantee can be proven with a standard scalar Chernoff bound. See Lemma 20 of [CMM17]. □
Corollary 14 (Deterministic Column Selection). There is a deterministic poly-time algorithm that, given \( A \in \mathbb{R}^{n \times d} \), \( \epsilon \in (0, 1) \) returns sampling matrix \( S \in \mathbb{R}^{d \times O(k/\epsilon^2)} \) satisfying the conditions of Theorem 9 and thus that \( \hat{A} = AS \) is an \((\epsilon,0,k)\)-projection-cost-preserving sketch of \( A \).

Proof. This corollary follows easily from a stable-rank approximate matrix multiplication result given in [CNW16]:

Theorem 15 (Theorem 5 of [CNW16]). For any \( k > 0 \) and \( \epsilon \in (0, 1) \) there is a deterministic polynomial-time algorithm that, given \( B \in \mathbb{R}^{n \times d} \) with \( \|B\|_2^2 \leq 1 \) and \( \|B\|_F^2 \leq k \) returns sampling matrix \( S \in \mathbb{R}^{d \times O(k/\epsilon^2)} \) satisfying

\[
\|BSS^T B^T - BB^T\|_2 \leq \epsilon. \tag{38}
\]

For \( \lambda = \frac{\|A - A_k\|_F^2}{k} \), we set \( B_1 = (AA^T + \lambda I)^{-1/2} A \) and let \( b_2 \in \mathbb{R}^d \) be the vector whose \( i \)th entry is equal to \( \frac{\|A_i\|_2}{\|A_i\|_F} \), where \( p \) is as defined in Theorem 9 and \( (A_i)_i \) is the \( i \)th column of \( A_i \). Let \( B = \frac{1}{2} [B_1; b_2] \) (that is, \( B \in \mathbb{R}^{n+1 \times d} \) is half \( B_1 \) with \( \frac{1}{2} b_2 \) appended as a final row.) Note that \( B \) can be computed in polynomial time. Additionally we have:

\[
\|B\|_2^2 \leq \frac{1}{4} (2\|B_1\|_2^2 + 2\|b_2\|_2^2) \leq \frac{1}{4}(2 + 2) = 1 \tag{39}
\]

and

\[
\|B\|_F^2 = \frac{1}{4}(\|B_1\|_2^2 + \|b_2\|_2^2) \leq \frac{1}{4}(2k + 1) \leq k \tag{40}
\]

where the second to last inequality follows since

\[
\|B\|_F^2 = \sum_{i=1}^{\text{rank}(A)} \sigma_i^2(B) = \sum_{i=1}^{\text{rank}(A)} \frac{\sigma_i^2(A)}{\sigma_i^2(A) + \lambda} \leq \sum_{i=1}^{\text{rank}(A)} \frac{\sigma_i^2(A)}{\sigma_i^2(A)} + \sum_{i=k+1}^{\text{rank}(A)} \frac{\sigma_i^2(A)}{\|A - A_k\|_F^2/k} = 2k.
\]

(39) and (40) allow us to apply Theorem 15 to \( B \) with error parameter \( \epsilon/48 \) obtaining \( S \) with \( O(k/\epsilon^2) \) rows satisfying:

\[
\|BSS^T B^T - BB^T\|_2 \leq \frac{\epsilon}{96}. \tag{41}
\]

(41) implies first that

\[
\frac{1}{4} |b_2^T SS^T b_2 - b_2^T b_2| = \frac{1}{4\|A_i\|_F^2} : \|A_i\|_P^2 \leq \frac{\epsilon}{96},
\]

which gives that

\[
\|A_i\|_P^2 - \|A_i\|_F^2 \leq \frac{\epsilon}{12} \|A_i\|_F^2
\]

where the last inequality follows from that fact that \( \|A_i\|_P^2 \leq 2\|A_i\|_F^2 \). If \( p \geq k \) this is true immediately. Otherwise, if \( p < k \) it follows since:

\[
\|A_i\|_P^2 - \|A_i\|_F^2 = \sum_{i=p+1}^{k} \sigma_i^2 \leq k \cdot \sigma_{p+1}^2 \leq \|A_i\|_F^2
\]

Each column of \( S \) is a scaled standard basis vector so \( \hat{A} = AS \) consists of a subset of reweighted columns of \( A \)
and so \( \|A \|_F^2 \leq 2\|A_k\|_F^2 \). (42) gives the Frobenius norm preservation condition of Theorem 9. From (41) we can also conclude that

\[
B_1B_1^T - \frac{\epsilon}{24} I \preceq B_1SS^TB_1^T \preceq B_1B_1^T + \frac{\epsilon}{24} I
\]

which after multiplying by \((AA^T + \lambda I)^{1/2}\) on the right and left (recalling that \(B_1 = (AA^T + \lambda I)^{-1/2}A\)) gives:

\[
AA^T - \frac{\epsilon}{24}(AA^T + \lambda I) \preceq ASS^TA^T \preceq AA^T + \frac{\epsilon}{24}(AA^T + \lambda I),
\]

which gives the spectral approximation condition of Theorem 9, completing the proof.

We note that [CNW16] proves that a number of sketching methods satisfy the stable-rank approximation matrix multiplication result of Theorem 15. We can use an analogous proof to that of Corollary 14 to prove that all these methods yield projection-cost-preserving sketches via Theorem 9.

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