Order-from-quantum disorder effects and Zeeman field tuned quantum phase transitions in a bosonic quantum anomalous Hall system

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We study possible many body phenomena in the recent experimentally realized weakly interacting quantum anomalous Hall system of spinor bosons by Wu, et.al, Science 354, 83-88 (2016). At a zero Zeeman field $h = 0$, by incorporating order from quantum disorder effects, we determine the quantum ground state to be a $N = 2$ XY-antiferromagnetic superfluid state and also evaluate its excitation spectra. At a finite small $h$, the competition between the Zeeman energy and the effective potential generated by the order from the quantum disorder leads to a canted antiferromagnetic superfluid state, then drives a second order transition to a spin-polarized superfluid state along the $z$ direction. The transition is in the same universality class as the zero density superfluid to Mott transition. Scaling behaviours of various physical quantities are derived. The ongoing experimental efforts to detect these novel phenomena are discussed.

Introduction. — The investigation and control of spin-orbit coupling (SOC) have become subjects of intensive research in both condensed matter and cold atom systems after the discovery of the topological insulators \cite{1,2}. In materials sides, the SOC plays crucial roles in various 2d or layered insulators, semi-conductor systems, metals and superconductors without inversion symmetry. The quantum anomalous Hall (QAH) effect was experimentally realized in Cr doped Bi(Sb)$_2$Te$_3$ thin films \cite{3,4}. In the cold atom side, using Raman schemes, several experimental groups \cite{5,7} generated 2d SOC for neutral cold atoms in both continuum and in optical lattices. Especially, the bosonic analog of the QAH for spinor bosons $^{87}$Rb was realized in \cite{7}, and the lifetime of SOC $^{87}$Rb Bose-Einstein condensation (BEC) have been improved from 300 ms to 900 ms recently.

Motivated by the recent experiment \cite{7}, we investigate possible quantum many body phenomena in the bosonic QAH model in a square lattice. Our main results are presented in Fig.1. At zero Zeeman field $h = 0$, the SOC leads to a spurious SU(2) symmetry which, in turn, leads to a classically infinitely degenerate ground state manifold. The order from quantum disorder (OFQD) effect generates an effective potential which selects the quantum ground state from such a manifold to be a $N = 2$ XY-antiferromagnetic (AFM) superfluid (SF) state where $N = 2$ stands for the number of BEC momenta. It also opens a gap to the spurious quadratic gapless roton mode. A finite small $h$ tends to offset the order from the disorder effects. The competition between the Zeeman energy and the effective potential first leads to a canted AFM SF state, then drives a second order transition to a spin-polarized SF state along the $z$ direction. The transition is in the same universality class as the zero density superfluid to Mott transition, therefore has the critical exponents $\nu = 1/2$, $\eta = 0$. The scaling behaviours of various physical quantities such as the roton gaps, the specific heat and transverse magnetization are derived. The ongoing experimental efforts to detect these novel phenomena are discussed.

The interacting Bosonic QAH model and order from quantum disorder effects. — The experimentally realized quantum anomalous Hall model of spinor bosons in a square lattice is described by the Hamiltonian \cite{7}

$$
\mathcal{H} = -t_0 \sum_{\langle ij \rangle} a_i^\dagger \sigma^z a_j + i t_{so} \sum_{\langle ij \rangle} a_{1,i}^\dagger \mathbf{d}_{ij} \cdot \vec{\sigma} a_j + \frac{U}{2} \sum_i n_i (n_i - 1) - \mu \sum_i n_i \tag{1}
$$

where $a_i = (a_{i\uparrow}, a_{i\downarrow})^\dagger$ and $\langle ij \rangle$ denotes a pair of nearest neighbor sites, and $\mathbf{d}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. $U$ is the repulsive onsite interaction energy, and $\mu$ is the chemical potential. $\sigma_{x,y,z}$ are three Pauli matrices, and $n_i = a_i^\dagger a_i$ is the number of particles on site-$i$. For the $^{87}$Rb atoms used in the recent experiment \cite{7}, the two pseudo-spin components $\sigma = \uparrow, \downarrow$ denote the two hyperfine states $|F = 1, m_F = 0 \rangle$ and $|F = 1, m_F = -1 \rangle$. In this Letter, we focus on the experimentally accessible regime $0 \leq t_{so} \leq t_0$ and $U/t_0 \ll 1$. The global phase diagram in most general parameter space will be worked out in a much longer version \cite{3}.

The Hamiltonian has the particle number conservation $U_c(1)$ symmetry, and a spin-orbital coupled $C_4$ symmetry. At $h = 0$, the Hamiltonian enjoys an extra anti-unitary mirror symmetry: $\mathcal{M} = (-1)^{4} \mathcal{R}_z(\pi/2) \mathcal{T}$, which is a composition of following operations: (1) a spin rotation $\mathcal{R}_z(\pi) : a_i \rightarrow i \sigma_z a_i$; (2) a time reversal $\mathcal{T}$; (3) a sublattice rotation $\mathcal{R}_x : a_i \rightarrow (-1)^{x+y} a_i$. Under $\mathcal{M}$, $h \rightarrow -h$, so we only need to focus on $h \geq 0$ cases.

The single-particle energy spectrum $U = 0$ of Eq.\((1)\)$ in the momentum space can be easily obtained: $e_{h \pm} k = \pm \sqrt{[h + 2 t_0 (\cos k_x + \cos k_y)]^2 + 4 t_{so}^2 (\sin^2 k_x + \sin^2 k_y)}$. 


At $h = 0$, its lower branch develops two degenerate minima at momentum $(0, 0)$ and $(\pi, \pi)$, and two eigen-spinors are $z_0 = (1, 0)^T$ and $z_\pi = (0, 1)^T$ respectively. There is a classically degenerate family of states:

$$\Psi_c = c_0 z_0 + (-1)^i z_\pi$$

(2)

where the two complex numbers $c_0$ and $c_\pi$ satisfy normalization condition $|c_0|^2 + |c_\pi|^2 = 1$. This classically degenerate manifold is due to the spurious SU(2) symmetry at the mean field level.

We write the spinor field as the condensation part $\Psi = \sqrt{N_0} \Psi_0 + \psi$, where $N_0$ denotes the total number of condensate particles. Again, the zero order term gives the classical ground state energy $E_0 = -\frac{1}{2} U n_0^2 N_s$, where $N_s$ denotes the total number of lattice site and $n_0 = N_0 / N_s$ denotes the condensate density. Setting the linear term vanish gives the value of the chemical potential $\mu = -4t + Un_0$. Diagonalizing $\mathcal{H}^{(2)}$ by a generalized $8 \times 8$ Bogoliubov transformation leads to:

$$\mathcal{H} = E_0 + E_{\text{OFQD}} + \sum_{l,k} \omega_{l,k} a_{l,k}^\dagger a_{l,k}$$

(3)

where $E_{\text{OFQD}} = - (4t_0 + \frac{1}{2} n_0 U) N_s + \frac{1}{2} \sum_{l,k} \omega_{l,k}$ is quantum correction to the mean field ground-state energy, and $\omega_{l,k}$ with $l = 1, \ldots, 4$ represent 4 Bogoliubov spectra.

It is convenient to parameterize $c_0$ and $c_\pi$ as

$$c_0 = e^{-i\phi/2} \cos(\theta/2), \quad c_\pi = e^{i\phi/2} \sin(\theta/2)$$

(4)

then we can plot the energy density $E_{\text{OFQD}}(\theta, \phi) = E_{\text{OFQD}}(\theta, \phi)/N_s$ as a function of $\theta$ and $\phi$ in Fig.2 where one can identify the quantum ground states as $\theta = \pi/2$ and $\phi_m = \pi/4 + m\pi/2$ ($m = 0, 1, 2, 3$). It has a uniform density $\langle n_i \rangle = n_0$ and a XY-AMF ordered spin structure: $\langle S_i \rangle = (-1)^{i+l} n_0 (\cos \phi_m, \sin \phi_m, 0)$ where $S_i = a_i^\dagger \sigma_i a_i$. It is 4 fold degenerate breaking the joint $C_4$ symmetry.

After identifying the quantum ground-state as the $N = 2$ XY-AMF state, we can also evaluate all $\omega_{1,2,3,4}(k)$ in Eq.(3). There are one linear $\omega_1 \propto k$ SF Goldstone mode and one $\omega_2 \propto k^2$ quadratic roton mode located at $(0, 0)$.

In fact, both analytical and numerical calculations show that

$$E_{\text{OFQD}} = \frac{1}{2} A(|c_0|^2 - |c_\pi|^2)^2 + \frac{1}{64} B (1 - \cos 2\theta)^2 (1 + \cos 4\phi + \cdots)$$

(5)

Appealingly, one can cast the energy density into a compact form

$$E_{\text{OFQD}} = \frac{1}{2} A (|c_0|^2 - |c_\pi|^2)^2 + \frac{1}{2} B (\cos c_0^2 + c_\pi^2)^2$$

(6)

where the coefficients $A$ and $B$ are positive numbers. Equation (6) can be viewed as an OFQD generated effective potential which will compete with the Zeeman energy at a small $h$.

Any small $h > 0$ breaks the mirror symmetry and split the degenerate single-particle state at momentum $(0, 0)$ and $(\pi, \pi)$. A direct analysis predicts the condensation occurs at either $(0, 0)$ or $(\pi, \pi)$ depending on the sign of $h$. So we obtain the superfluid with the many-body ground-state $|\Psi_{sf,\uparrow\downarrow}\rangle \sim (\sum_i a_i^\dagger)^N |0\rangle$ if $h > 0$ or $|\Psi_{sf,\downarrow}\rangle \sim (\sum_i (-1)^i a_i^\dagger)^N |0\rangle$ if $h < 0$. Due to its Z ferromagnetic (FM) like spin structure, we name it $Z - FM$ superfluid.

By directly employ the Bogoliubov theory at any $h \neq 0$, we obtain a linear gapless mode and a gapped roton mode with the gap $\Delta_h = 2|h|$ (c.f. dash line in Fig.3a). As one decreases $h$ from a finite value to 0, the roton mode gets lower and lower, then touches zero at $h = 0$. This behaviour may signify a possible first order transition with a critical Zeeman field $h_c = 0$. However, as to be shown below, this physical picture holds when $h$ is sufficiently large $h > U$, but breaks down at a small $h < U$.

The Competition between the OFQD generated potential and the Zeeman energy.— Here we will take a dual
To explore the CAFM SF phase, we perform an expansion. For $0 < h < h_c$ the spin-bond structure of the ground-state as the Zeeman field $h$ from 0 to $h_c$ and beyond. (b) The Roton gap $\Delta_R$ due to the competition between the effective potential generated by the order from quantum disorder and the Zeeman energy. The dashed line of the roton dropping is before considering the order from quantum disorder effects.

The canted antiferromagnetic SF phase at $h < h_c$.— To explore the CAFM SF phase, we perform an expansion around the saddle point $(n = n_0, \chi = 0, \theta = 0, \phi = \phi_0)$. In the long-wave length limit, one reaches the Lagrangian density up to the quadratic order:

$$\mathcal{L}_{\text{CAF}} = \delta n \partial_t \delta \chi - \frac{1}{2} \cos \theta \delta n \partial_t \delta \phi + i \frac{1}{2} n_0 \sin \theta \delta n \partial_\tau \delta \phi + \frac{U}{2} \delta n^2 + \frac{1}{2} (n_0 h \cos \theta) \delta \phi^2$$

$+ \frac{1}{2} B \sin^4 \theta_0 \delta \phi^2 - n_0 \frac{v^2}{4} \frac{1}{n_0} (\nabla \delta n)^2 + 4 (\nabla \delta \chi)^2$

$+ (\nabla \delta \theta)^2 + (\nabla \delta \phi)^2 - 4 \cos \theta_0 (\nabla \delta \chi) \cdot (\nabla \delta \phi)$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\alpha/2} \cos(\beta/2) \\ e^{i\alpha/2} \sin(\beta/2) \end{pmatrix}.$

Now $\theta = 0$ corresponding to $(\alpha = 0, \beta = \pi/2)$. By applying similar procedures to the saddle point, $(n = n_0, \chi = 0, \alpha = 0, \beta = \pi/2)$, we obtain the Lagrangian:
density inside the Z-FM SF state:
\[
\mathcal{L}_{Z\text{-FM}} = i\delta n \partial_t \delta \chi + \frac{U}{2} (\delta n)^2 - n_0 v^2 \left[ \frac{1}{n_0^2}(\nabla \delta n)^2 + 4(\nabla \delta \chi)^2 \right]
\]
+ \( i\frac{1}{2} n_0 \delta \partial_t \partial_\lambda \alpha + \frac{1}{2}(n_0 h - A)(\delta \alpha)^2 + (\delta \beta)^2 \)

\[- n_0 \frac{v^2}{4} [(\nabla \delta \alpha)^2 + (\nabla \delta \beta)^2] \quad (14)\]

where one can see the superfluid mode remains the same as before, but the roton mode becomes

\[
\omega_{R,k} = 2(h - A/n_0) + v^2 k^2 \quad (15)
\]

For \( h > h_c \), the roton gap is \( \Delta_R = 2(h - A/n_0) \). In \( h \rightarrow h_c \) limit, the roton gap vanishes as \( \Delta_R \sim h - h_c \) and its critical dispersion becomes \( \omega_R \sim v^2 k^2 \). This matches those from the CAFM side indicating a possible second order quantum phase transition at \( h = h_c \).

**Quantum phase transition at \( h = h_c \) and scalings.** At zero temperature, the superfluid mode remains non-critical across the quantum critical point (QCP), we only need focus on the critical behaviour of the roton mode \( \omega_R \sim v^2 k^2 \). The order parameter is the magnetization in the XY plane \( \langle S_x S_y \rangle = 0 \) in the Z-FM phase, \( \langle S_x S_y \rangle \neq 0 \) in the CAFM phase. Setting \( h = h_c \) in Eq. (14) leads to the quantum critical action to the quadratic order:

\[
\mathcal{L}_{QC} = \frac{1}{2} n_0 \delta \partial_t \delta \alpha - n_0 v^2 [(\nabla \delta \alpha)^2 + (\nabla \delta \beta)^2] \quad (16)
\]

There is an emergent U(1) symmetry at the QCP which indicates the CAFM to the Z-FM transition is in the same universality class of the zero-density SF-mott transition with the critical exponents \( z = 2, \nu = 1/2, \eta = 0 \) which was already studied in Ref. [13].

The standard scaling shows the roton gap should scale as \( \Delta_R \sim |h - h_c|^{z\nu} \sim |h - h_c| \) which is consistent with our specific calculations shown above. The specific heat should scale as \( C_v \sim T^{d/z} \sim T \). Our specific calculations show that the specific heat in the spin sector \( C_{spin} = \frac{T^2}{3\pi^2} \) which is consistent with the scaling analysis. Applying the scaling result for the \( U(1) \) conserved quantity in [13] to the order parameter leads to:

\[
\langle \left( \frac{1}{N_s} \sum_i \left[ (S_i^x)^2 + (S_i^y)^2 \right] \right) \rangle = \frac{2mT}{4\pi} \left( \frac{1}{[\ln(\Lambda^2/(2mT))]^4} \right) \quad (17)
\]

where \( m = 1/(2v^2) = t_0/(2t_0^2 - t_0^2) \) and \( \Lambda \) is a momentum upper cutoff.

Despite the QCP is in the same universality class as that of the zero density SF to Mott transition, the two phases on the two sides are very much different than the SF and Mott. In the CAFM, there is cos \( \phi \) clock term in Eq [10] which is dangerously irrelevant near the QCP: it is irrelevant near the QCP, but controls the quantum phase \( h < h_c \). It leads to the CAFM phase, breaks the presumably \( U(1) \) symmetry to the joint \( C_4 \) symmetry and also opens a gap to the roton mode. There is a melting transition \( T_M \) above the CAFM phase (Fig.1).

The universality class of melting process belongs to 2D \( q = 4 \) state clock transition [11].

The SF sector stays un-critical across the QCP. Integrating out the quantum fluctuations in \( c_\theta \) and \( c_\pi \) and then integrating out \( \delta n \) lead to the linear SF mode:

\[
\mathcal{L}_{sf} = \frac{1}{2\pi} (\partial_\theta \delta \chi)^2 - n_0 v^2 (\nabla \delta \chi)^2 \quad (14)
\]

Thus we obtain the finite KT transition temperature (Fig.1). \( T_{KT} = \frac{2}{\pi} \rho_s \sim \pi n_0 v^2 \). There is also a specific heat contribution from the superfluid sector \( C_{sf} = \frac{\pi^2}{30} T^2 \) which is subleading to that from the critical roton mode near the QCP in the spin sector.

**Experimental detections.** In the ongoing experiment at USTC, the BEC has \( N \sim 3 \times 10^5 \) atoms and trapped within the diameter \( d = 80 \mu m \), so one lattice site has about \( n = 10 \) atoms. For typical experiment parameters, optical lattice potential is \( 4 \times 10^3 \) mG, and Raman potential is \( 1.32E_r \), where \( E_r = 375 \) nK denotes the recoil energy, so the tight binding model parameter can be estimated as \( t_0 \sim 100 \) nK, \( t_{so} \sim 30 \) nK, so \( t_{so}/t_0 \sim 1/3 \). The short-range Hubbard like interaction \( U = \frac{4\pi^2a_s}{m} \int d^2 r |\psi(r)|^4 \), where \( s \)-wave scattering length of the \(^{87}\)Rb atoms \( a_s = 103a_0 \) and \( a_0 \) is the Bohr radius and the mass of the bosons \(^{87}\)Rb lead to \( U \sim 10 \) nK, so \( n_0U \sim 100 \) nK.

Based on these experimental parameters, one can estimate \( T_{KT} \sim 100 \) nK, the roton gap away from the critical point \( \Delta_R \sim 1 \) nK, so the meting transition \( T_M \sim 1 \) nK and the critical Zeeman field \( h_c \sim 1 \) mG. So far, the experiments are operating at \( T \sim 20 \) nK, so may not be able to get into the CAFM phase yet. However, the quantum critical regime is always reachable and the critical scalings for the specific heat, transverse magnetization and spin-spin correlation functions can be such as dynamic or elastic, energy or momentum resolved, longitudinal or transverse Bragg spectroscopies [14–16], specific heat measurements [17, 18] and in-situ measurements [19].

**Conclusion.** The order from quantum disorder phenomena were originally discovered in the context of quantum magnetism in frustrated lattices. Here, we discuss its importance in the context of superfluid in a bipartite lattice due to the spin orbit coupling. So the SOC provides a completely new mechanism leading to novel frustrated SF. We demonstrate that the effective potential generated from the OFQD effect at least have 3 important impacts in the bosonic QAH system: (a) leads to a quantum ground state selection rule, (b) give quantum corrected excitation spectra, (c) its competition with the Zeeman energy due to a small finite \( h \) leads to a second order quantum phase transition. We expect that these novel effects are quite general and could happen in many other systems with SOC.

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[1] M. Z. Hasan and C. L. Kane, Colloquium: Topological insulators, Rev. Mod. Phys. 82, 3045 (2010).
[2] X. L. Qi and S. C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. 83, 1057 (2011).
[3] Rui Yu, Wei Zhang, Hai-Jun Zhang, Shou-Cheng Zhang, Xi Dai1, Zhong Fang, Quantized Anomalous Hall Effect in Magnetic Topological Insulators, Science 329, 61 (2010).
[4] Cui-Zu Chang, Jinsong Zhang, Xiao Feng, Jie Shen, Zuocheng Zhang, Minghua Guo, Kang Li, Yunbo Ou, Fang Wei, Li-Li Wang, Zhong-Qing Ji, Yang Feng, Shuaihua Ji, Xi Chen, Jinfeng Jia, Xi Dai, Zhong Fang, Shou-Cheng Zhang, Ke He, Yau-Wang, Li Lu, Xu-Cun Ma, Qi-Kun Xue, Experimental Observation of the Quantum Anomalous Hall Effect in a Magnetic Topological Insulator, Science 340, 167 (2013).
[5] Lianghui Huang, et.al, Experimental realization of a two-dimensional synthetic spin-orbit coupling in ultracold Fermi gases, Nature Physics 12, 540-544 (2016).
[6] Zengming Meng, et.al, Experimental observation of topological band gap opening in ultracold Fermi gases with two-dimensional spin-orbit coupling, arXiv:1511.08492.
[7] Zhan Wu, Long Zhang, Wei Sun, Xiao-Tian Xu, Bao-Zong Wang, Si-Cong Ji, Youjin Deng, Shuai Chen, Xiong-Jun Liu, Jian-Wei Pan, Realization of two-Dimensional spin-orbit coupling for Bose-Einstein condensates, Science 354, 83 (2016).
[8] Fadi Sun, Junsen Wang, Jinwu Ye, Shuai Chen, and Youjin Deng, Symmetry protected bosonic topological phase transitions: Quantum Anomalous Hall system of weakly interacting spinor bosons in a square lattice, arXiv:1711.11580v1.
[9] D. van Oosten, P. van der Straten, and H. T. C. Stoof, Quantum phases in an optical lattice, Phys. Rev. A 63, 053601 (2001).
[10] See the supplementary materials for more details to establish the effective potential Eq.M4 and M5 generated by the order from quantum disorder. The analytical approach is based on a perturbation theory in $t_s/t$, and the numerical approach is a direct numerical evaluation which confirms the analytical calculations on $\omega_{l,k}$ and $E_{odl}(\theta, \phi)$ defined in Eq.(2).
[11] G. Ortiz, E. Cobanera, Z. Nussinov, Dualities and the phase diagram of the p-clock model, Nuc. Phys. B 854, 780 (2012).
[12] Subir Sachdev, Quantum Phase Transitions, 2nd ed. (Cambridge University Press, Cambridge, 2011).
[13] Subir Sachdev, T. Senthil, and R. Shankar, Finite-temperature properties of quantum antiferromagnets in a uniform magnetic field in one and two dimensions, Phys. Rev. B 50, 258 (1994).
[14] Si-Cong Ji, Long Zhang, Xiao-Tian Xu, Zhan Wu, Youjin Deng, Shuai Chen, and Jian-Wei Pan, Softening of Roton and Phonon Modes in Bose-Einstein Condensate with Spin-Orbit Coupling, Phys. Rev. Lett. 114, 105301 (2015).
[15] Jinwu Ye, J.M. Zhang, W.M. Liu, K.Y. Zhang, Yan Li, W.P. Zhang, Light scattering detection of various quantum phases of ultracold atoms in optical lattices, Phys. Rev. A 83, 051604 (R) (2011).
[16] Jinwu Ye, K.Y. Zhang, Yan Li, Yan Chen and W.P. Zhang, Optical Bragg, atom Bragg and cavity QED detections of quantum phases and excitation spectra of ultracold atoms in bipartite and frustrated optical lattices, Ann. Phys. 328 (2013), 103-138.
[17] Kinast, J. et al. Heat Capacity of a Strongly Interacting Fermi Gas. Science 307, 1296 (2005).
[18] Ku, M. J. H. et al. Revealing the Superfluid Lambda Transition in the Universal Thermodynamics of a Unitary Fermi Gas. Science 335, 563 (2012).
[19] Gemelke, N., Zhang X., Huang C. L., and Chin, C. In situ observation of incompressible Mott-insulating domains in ultracold atomic gases, Nature (London) 460, 995 (2009).
Supplementary Materials for “Order-from-disorder effects and Zeeman field tuned quantum phase transitions in a bosonic Quantum Anomalous Hall model”

In the main text we present an order-from-disorder calculation on a bosonic Quantum Anomalous Hall model. Here we provide more detailed calculation of order-from-disorder generated potential $E_{\text{odd}}$ in Sec. S1.

S1. ORDER-FROM-DISORDER GENERATED POTENTIAL

A. Analytical approach

The analytical approach is based on a perturbation theory.

Before performing perturbation calculations, it is convenient to apply transformation $a_{i\downarrow} \rightarrow (-1)^i a_{i\downarrow}$ to the original Hamiltonian Eq.(1) and obtain

$$\mathcal{H} = -t \sum_{\langle ij \rangle} a_i^\dagger a_j + it_{\text{so}} \sum_{\langle ij \rangle} (-1)^i a_i^\dagger (d_{ij} \cdot \vec{\sigma}) a_j - h \sum_i a_i^\dagger \sigma^z a_i + \frac{U}{2} \sum_i n_i(n_i-1) - \mu \sum_i n_i \tag{S1}$$

Since the ground-state energy is a gauge invariant quantity, we can use the Hamiltonian Eq.(S1) for the ground-state energy calculation. In the absence of Zeeman field, we first separate Hamiltonian Eq.(S1) into two parts

$$\mathcal{H}_0 = -t \sum_{\langle ij \rangle} a_i^\dagger a_j + \frac{U}{2} \sum_i n_i(n_i-1) - \mu \sum_i n_i, \quad \mathcal{H}_{\text{so}} = it_{\text{so}} \sum_{\langle ij \rangle} (-1)^i a_i^\dagger (d_{ij} \cdot \vec{\sigma}) a_j \tag{S2}$$

It is clear the Hamiltonian $\mathcal{H}_0$ enjoys a spin SU(2) symmetry, the weak coupling $U \ll t$ ground-state is a superfluid with many-body wavefunction $|\Psi_{\text{sf}}\rangle \sim (\sum |\Psi_i a_i^\dagger|^2 ) |\text{Vac}\rangle$ where $|\Psi_i \rangle = c_0 \Psi_0 + c_\pi \bar{\Psi}_\pi$ and $|c_0|^2 + |c_\pi|^2 = 1$.

Within Bogoliubov approximation, the original boson operators replaced by condensate part plus fluctuations $a_i = \sqrt{N_0} \Psi_i + \delta a_i$, thus one can expand $\mathcal{H}_0$ and keep terms up to second order in the fluctuations. The quadratic theory is $\mathcal{H}_{0,\text{Bog}}$, which can be diagonalized by following Bogoliubov transformation

$$\delta a_{k\uparrow} = -\bar{c}_\pi \alpha_{1,k} + c_0 (\cosh \xi_k \alpha_{2,k} + \sinh \xi_k \alpha_{2,-k}^\dagger), \quad \delta a_{k\downarrow} = +\bar{c}_0 \alpha_{1,k} + c_\pi (\cosh \xi_k \alpha_{2,k} + \sinh \xi_k \alpha_{2,-k}^\dagger) \tag{S3}$$

and

$$\mathcal{H}_{0,\text{Bog}} = \text{const.} + \sum_k (\omega_{1,k} \alpha_{1,k}^\dagger \alpha_{1,k} + \omega_{2,k} \alpha_{2,k}^\dagger \alpha_{2,k}) \tag{S4}$$

The Bogoliubov ground-state is $|0\rangle$, which is the vacuum of $\alpha_{1,k}$ and $\alpha_{2,k}$ such that $\alpha_{1,k}|0\rangle = 0$ and $\alpha_{2,k}|0\rangle = 0$.

Notice the spurious spin-orbital SU(2) symmetry becomes an exact symmetry at $t_{\text{so}} = 0$ and $h = 0$, so $c_0$ and $c_\pi$ can take arbitrary value as long as they satisfy normalization condition. Consider $\mathcal{H}_{0,\text{Bog}}$ is perturbed by $\mathcal{H}_{\text{so}}$ which takes following form in Bogoliubov approximation

$$\mathcal{H}_{\text{so}} = 2t_{\text{so}} \sum_k (\gamma_k \delta a_{k\uparrow}^\dagger \delta a_{k\downarrow} + h.c.) \tag{S5}$$

where $\gamma_k = \sin k_x - i \sin k_y$ and $Q = (\pi, \pi)$. In order to apply the perturbation, it is convenient to reexpress $\mathcal{H}_{\text{so}}$ in terms of $\alpha_k$ and obtain

$$\mathcal{H}_{\text{so}} = t_{\text{so}} \sum_k [c_{0,k}^2 O_{1,k} + \bar{c}_0 c_\pi O_{2,k} + c_{\pi,k}^2 O_{3,k} + h.c.] \tag{S6}$$

where $O_{1,k} = \alpha_{1,k} + Q (v_k \alpha_{2,k}^\dagger + v_k \alpha_{2,-k}^\dagger)$ and we skip detailed formula for other $O_n,k$.

By treating $t_{\text{so}}/t_0$ as a small parameter, one can apply perturbation theory to obtain

$$E_{\text{odd}} = \delta E^{(1)} + \delta E^{(2)} + \delta E^{(3)} + \delta E^{(4)} + O(t_{\text{so}}^5) \tag{S7}$$

where we denote $\delta E^{(n)} \propto t_{\text{so}}^n$. Since $\mathcal{H}_{\text{so}}$ has odd parity, all odd order term vanish and

$$\delta E^{(2)} = \langle 0 | \mathcal{H}_{\text{so}} g \mathcal{H}_{\text{so}} | 0 \rangle \tag{S8}$$

$$\delta E^{(4)} = \langle 0 | \mathcal{H}_{\text{so}} g \mathcal{H}_{\text{so}} g \mathcal{H}_{\text{so}} g \mathcal{H}_{\text{so}} | 0 \rangle - \langle 0 | \mathcal{H}_{\text{so}} g \mathcal{H}_{\text{so}} | 0 \rangle \langle 0 | \mathcal{H}_{\text{so}} g^2 \mathcal{H}_{\text{so}} | 0 \rangle \tag{S9}$$
where \( g = \sum_{n \neq 0} \frac{|n\rangle \langle n|}{E_n} \). From the formula of \( \delta E^{(n)} \), the \( n \)-th order perturbation acquires structure

\[
\delta E^{(n)} = p_{2n}(c_0, c_\pi)
\]

(S10)

where \( p_{2n}(c_0, c_\pi) \) denotes \( 2n \)-th order polynomial in \( c_0, c_\pi \). It is clear \( \langle 0| O_{n,k}^{(n)} |0 \rangle = 0 \) \( (\forall n = 1, 2, 3, \cdots) \), one can immediately conclude \( \delta E^{(n)} \) do not contain \( c_0^{2n} \), similar reasons hold for \( c_\pi^{2n} \), \( \tilde{c}_0^{2n} \), and \( \tilde{c}_\pi^{2n} \). With a little bit of effort, one can also generalize it to the following claim.

**Claim:** \( \delta E^{(n+m)} \) do not contain any terms like \( c_0^{2n} | c_\pi^{2m}, \tilde{c}_0^{2m} | c_\pi^{2m}, \tilde{c}_\pi^{2m} | c_\pi^{2m} \), and \( \tilde{c}_\pi^{2m} | c_\pi^{2m} \).

Taking parametrization of \( c_0 \) and \( c_\pi \) in Eq.(3) of main text, we have \( E_{\text{old}}(c_0, c_\pi) = E_{\text{old}}(\theta, \phi) \). From symmetries we know \( E_{\text{old}}(\theta, \phi) \) is periodic

\[
E_{\text{old}}(\theta, \phi) = E_{\text{old}}(\theta + \pi, \phi) = E_{\text{old}}(\theta, \phi + \pi/2)
\]

(S11)

and even

\[
E_{\text{old}}(\theta, \phi) = E_{\text{old}}(-\theta, \phi) = E_{\text{old}}(\theta, -\phi)
\]

(S12)

thus one can conclude \( E_{\text{old}} \) must satisfy

\[
E_{\text{old}}(\theta, \phi) = \sum_{mn} C_{mn} \cos(2n\theta) \cos(4m\phi)
\]

(S13)

Combining the general property of perturbation result Eq.(S10) and symmetry constrain from Eq.(S10), we obtain the 2nd order perturbation \( \delta E^{(2)} \)

\[
\delta E^{(2)} = a_2 + b_2 \cos 2\theta
\]

(S14)

and the 4th order perturbation \( \delta E^{(4)} \),

\[
E^{(4)} = (a_4 + b_4 \cos 2\theta + c_4 \cos 4\theta) \cos 4\phi + a'_4 + b'_4 \cos 2\theta + c'_4 \cos 4\theta
\]

(S15)

where undetermined coefficients \( a_4, b_4 \), and \( c_4 \) can be fixed by following observations:

I. No \( \phi \) dependence of \( E_{\text{old}} \) is allowed for \( \theta = 0 \), so the coefficient of \( \cos 4\phi \) vanish when \( \theta = 0 \).

II. As \( \theta \to 0 \) one can prove that the coefficient of \( \cos 4\phi \) vanish as \( \theta^4 \).

Notice the fact II is a direct result from the Claim.

Consider a Taylor expansion of \( c_0 \) and \( c_\pi \) around \( \theta = 0 \),

\[
c_0 = e^{-i\phi/2} \cos(\theta/2) \sim e^{-i\phi/2}(1 - \theta^2/8), \quad c_\pi = e^{+i\phi/2} \sin(\theta/2) \sim e^{+i\phi/2}\theta/2
\]

(S16)

notice \( c_0^4 \) is excluded in \( E^{(4)} \), the \( \cos 4\phi \) can only obtained from \( (c_0 c_\pi)^4 \) and \( (\tilde{c}_0 c_\pi)^4 \) which granted the leading term in \( \phi \) has coefficient \( \theta^4 \) when \( \theta \) is small. Notice, even including high order corrections such as \( \delta E^{(6)} \) and \( \delta E^{(8)} \), the coefficient of \( \cos 4\phi \) still vanish as \( \theta^4 \) as \( \theta \to 0 \).

If we take advantage of these observations and require the Taylor expansion of \( E_{\text{old}} \) around \( \theta = \pi/2 \) and \( \pi = \pi/4 \) fit the form

\[
E_{\text{old}} = E_{\text{old},0} + \frac{A}{2} (\delta \theta)^2 + \frac{B}{2} (\delta \phi)^2
\]

(S17)

a set of equation is obtained

\[
a_4 + b_4 + c_4 = 0, \quad -(2b_4 + 8c_4) = 0, \quad 4b_2 = A, \quad 16(a_4 - b_4 + c_4) = B
\]

(S18)

thus one can solve for \( b_2, a_4, b_4, c_4 \) and obtain

\[
b_2 = \frac{A}{4}, \quad a_4 = \frac{3B}{128}, \quad b_4 = \frac{-B}{32}, \quad c_4 = \frac{B}{128}
\]

(S19)

By keeping leading terms in \( \theta \) and \( \phi \), we obtain

\[
E_{\text{old}} = E_{\text{old},0} + \frac{A}{4} (1 + \cos 2\theta) + \frac{B}{16} \sin^4 \theta (1 + \cos 4\phi)
\]

(S20)

where we also obtain analytical express for \( A \) and \( B \), which tell both \( A \) and \( B \) are positive define. \( E_{\text{old}} \) reaches its minima at \( \theta = \pi/2 \) and \( \phi = \pi/4 \).
B. Numerical approach

We will show that although the form Eq. (S20) is obtained from perturbation theory, but it fits numerical data very well. The data is obtained from numerical evaluation $\omega_{\text{I},k}$ and $E_{\text{odd}}(\theta, \phi)$ defined in Eq.(2).

The comparison consist of two parts: global property and local property.

For the global property, we plot numerical results from Eq.(2) and analytical result from Eq.(S20) in Fig.S1. It is clear two results fit quite well. However, if we only keep the form from perturbation theory and treat coefficients $A$ and $B$ as fitting parameters, then the relative error can be controlled below 1%. Figure S2 shows the form work very well even for $t_{\text{so}}$ is not small, i.e. $t_{\text{so}}/t = 1$.

![Figure S1. Plot of $E_{\text{odd}}$ for varies parameter regimes (a) $0 < \theta < \pi$ and $\phi = \pi/4$, (b) $\theta = \pi/2$ and $0 < \phi < \pi$, (c) $\theta = \pi/3$ and $0 < \phi < \pi$, (d) $\theta = \pi/4$ and $0 < \phi < \pi$. The numerical results is represent by red line, and analytical results from perturbation theory is represent by blue lines. The other parameters are $t_{\text{so}}/t = 1/3, n_0 U/t_0 = 1$.](image)

![Figure S2. The same as Fig. S1 except using different parameters $t_{\text{so}}/t = 1, n_0 U/t_0 = 1$. The numerical results are represent by red + symbol, analytical results from perturbation theory results are represent by blue lines, and fitting results are represent by green lines. The difference between numerical results and fitting results $\Delta E = E_{\text{numerical}} - E_{\text{fit}}$ are listed below each sub figure.](image)

For the local property, we focus on the derivative with respect to $\theta$ and $\phi$:

$$A'(\theta_0) = \frac{\partial^2}{\partial \theta^2} E_{\text{odd}}(\theta, \phi)\bigg|_{\theta=\theta_0, \phi=\pi/4}, \quad B'(\theta_0) = \frac{\partial^2}{\partial \phi^2} E_{\text{odd}}(\theta, \phi)\bigg|_{\theta=\theta_0, \phi=\pi/4}$$

Analytical result Eq. (S20) predicts $A'_{\text{an}} = -A \cos 2\theta_0$ and $B'_{\text{an}} = B \sin^4 \theta_0$, and numerical derivatives can be evaluated from finite differences of Eq.(2). We plot both results in Fig.S3 (a) and (b), and find the differences are quite small. The insert of Fig.S3 (b) also confirms the $B' \sim \theta^4$ behaviour. As a final comparison, we evaluate $A'$ and $B'$ as a function of $t_{\text{so}}$ and plot them in Fig.S3 (c) and (d). When extrapolating the data to $t_{\text{so}} \to 0$, they reach agreement with perturbation results with high accuracy. The insert of Fig.S3 (c) and (d) also confirms that the leading behaviours of $A' \sim t_{\text{so}}^2$, and $B' \sim t_{\text{so}}^4$.

In conclusion, we prove that the $E_{\text{odd}}$ obtained from perturbation theory fit data from direct numerical evaluation, even at $t_{\text{so}}/t = 1$. 
FIG. S3. (a) Coefficient $A'$ as a function of $\theta_0$. (b) Coefficient $B'$ as a function of $\theta_0$. The insert shows $B' \sim \theta_0^4$ when $\theta_0$ is small. (c) Coefficient $A'$ as a function of $t_{so}$. The insert shows $A' \sim t_{so}^2$ when $t_{so}$ is small. (d) Coefficient $B'$ as a function of $t_{so}$. The insert shows $B' \sim t_{so}^4$ when $t_{so}$ is small.