THE ASKEY-WILSON FUNCTION TRANSFORM

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Abstract. In this paper we present an explicit (rank one) function transform which contains several Jacobi-type function transforms and Hankel-type transforms as degenerate cases. The kernel of the transform, which is given explicitly in terms of basic hypergeometric series, thus generalizes the Jacobi function as well as the Bessel function. The kernel is named the Askey-Wilson function, since it provides an analytical continuation of the Askey-Wilson polynomial in its degree. In this paper we establish the $L^2$-theory of the Askey-Wilson function transform, and we explicitly determine its inversion formula.

1. Introduction.

In [9] several explicit function transforms are realized as Fourier transforms on the non-compact quantum $SU(1,1)$ group. In this paper we focus on the function theoretic aspects of the most general Fourier transform on the quantum $SU(1,1)$ group. Since the corresponding radial part of the Casimir operator is Askey’s and Wilson’s [1] second order $q$-difference operator $L$, we are led to the task of studying its spectral properties as a symmetric operator with respect to an explicit measure of unbounded support. This measure naturally arises in [9] as a Haar weight on the quantum $SU(1,1)$ group. The kernel of the corresponding generalized Fourier transform is an eigenfunction of $L$, and it provides an analytic continuation of the Askey-Wilson polynomial in its degree. Consequently, we name the kernel the Askey-Wilson function.

The Askey-Wilson function satisfies a beautiful symmetry property, which we refer to as duality in this paper. It essentially states that the geometric and the spectral parameter of the Askey-Wilson function are interchangeable (up to a certain involution on the parameters).

For a full exploitation of duality for the development of the $L^2$-theory of the corresponding generalized Fourier transform, it is necessary to add another degree of freedom. This can be done in the following natural way. The Askey-Wilson function can be expanded as a linear combination of the asymptotically free solutions of $L$ on an arbitrary $q$-lattice. The corresponding explicit coefficients, the so-called $c$-functions, can then be used to define an explicit measure whose support around infinity lies on the chosen $q$-lattice. This gives rise to a one parameter family of measures of unbounded support, in which the extra parameter labels the different $q$-lattices. It contains the measure arising from the quantum $SU(1,1)$ group as a special case.

The generalized Fourier transform with respect to this one parameter family of measures, and with the Askey-Wilson function as its kernel, is called the Askey-Wilson function transform. In view of its harmonic analytic interpretation on the

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quantum $SU(1,1)$ group, we may think of the Askey-Wilson function transform as a generalization of the classical Jacobi transform, whence in particular of the Mellin-Fock (or Legendre) transform. In fact, it was shown in [10] that the Askey-Wilson function transform is on top of an hierarchy consisting of several different generalizations of the Jacobi and Hankel transform. In particular, the $L^2$-theory for the Askey-Wilson function transform unifies the $L^2$-theory for the spherical Fourier transform on the (quantum) $SU(1,1)$ group as well as on the (quantum) group of plane motions.

The main objective of the paper is to show that the Askey-Wilson function transform extends to an isometric isomorphism in a natural way, and that its inverse is the Askey-Wilson function transform with respect to a different, "dual" choice of parameters.

We expect that the present theory on the Askey-Wilson function transform will turn out to be the special rank one case of a general theory on Macdonald function transforms associated with root systems, in analogy with the polynomial setting (see e.g. [12], [2]). From this point of view, it is natural to expect that the structure of the Askey-Wilson function transform, and eventually also the structure of the general Macdonald function transform, will be best understood from an extension of Cherednik’s affine Hecke algebraic approach to the non-polynomial setting. A first indication of the pivotal role of affine Hecke algebras in the structure of the Askey-Wilson function transform is the duality, which in the polynomial setting stems from an involution of the associated double affine Hecke algebra (see for instance [3], and [13], [14] for the rank one setting). We hope that future research will shed more light on these matters.

The set-up of the paper is as follows. In §2 we recall the Askey-Wilson polynomials and in §3 we define the Askey-Wilson functions. We emphasize in these sections the important concept of duality. In §4 the asymptotically free solutions of $L$ are discussed, as well as the corresponding $c$-function expansion of the Askey-Wilson function. We formulate our main result in §5, which states that the Askey-Wilson function transform extends to an isometric isomorphism in a natural way, and that its inverse is given by the Askey-Wilson function transform with respect to dual parameters.

The remaining sections occupy the proof of the main result. Duality plays a pivotal and simplifying role in the proof, since it allows us to avoid the machinery of spectral analysis of (unbounded) self-adjoint operators (in contrast with the analysis in some of its degenerate cases, see e.g. [1], [2] and [5]). The other main ingredient of the proof is the computation of the weak limit of the Wronskian of the Askey-Wilson functions. As an interesting by-product of this computation, we obtain in §7 explicit orthogonality relations for “low degree” Askey-Wilson polynomials with respect to the one-parameter family of measures of unbounded support.

Notations and conventions: We assume that $0 < q < 1$ is fixed throughout the paper. We use the standard notations for basic hypergeometric series, see for instance [6]. In particular, we write $(a_1, \ldots, a_r; q)_\infty = \prod_{i=1}^r (a_i; q)_\infty$ with $(a; q)_\infty = (aq^{-1}; q)_\infty$ and $(a; q)_\infty = \prod_{i=0}^\infty (1-aq^i)$ for (products) of $q$-shifted factorials, and we write

$$r+1\phi_r \left( \frac{a_1, a_2, \ldots, a_{r+1}}{b_1, b_2, \ldots, b_r}; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_k}{(q, b_1, \ldots, b_r; q)_k} z^k.$$
for the \( r+1 \phi_r \) basic hypergeometric series. The very well poised \( s\phi_7 \) is defined by
\[
sW_7(a; b, c, d, e, f; q, z) = \sum_{k=0}^{\infty} \frac{1 - aq^{2k}}{1 - a} \frac{(a, b, c, d, e, f; q)_k z^k}{(q, qa/b, qa/c, qa/d, qa/e, qa/f; q)_k}.
\]
We use the branch of the square root \( \sqrt{\cdot} \) which is positive on \( \mathbb{R}_{>0} \) throughout the paper.

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2. The Askey-Wilson polynomials.

In this section we briefly recall the basic properties of the Askey-Wilson polynomials. We formulate these properties in such a way that the connections with the non-polynomial setting (see section 5) will be as transparent as possible. In particular, we emphasize the important concept of duality.

The Askey-Wilson polynomials \( p_n(x) = p_n(x; a, b, c, d; q) \) \( n \in \mathbb{Z}_+ \) are defined by the series expansion
\[
p_n(x) = 4\phi_3 \left( q^{-n}, q^{n-1}abcd, ax; ax^{-1}; q, q \right),
\]
see [1]. If we associate dual parameters \( \tilde{a}, \tilde{b}, \tilde{c} \) and \( \tilde{d} \) with \( a, b, c \) and \( d \) by the formulas
\[
\begin{align*}
\tilde{a} &= \sqrt{q^{-1}abcd}, \quad \tilde{b} = ab/\tilde{a} = q\tilde{a}/cd, \\
\tilde{c} &= ac/\tilde{a} = q\tilde{a}/bd, \quad \tilde{d} = ad/\tilde{a} = q\tilde{a}/bc,
\end{align*}
\]
then it immediately follows from the explicit expression (2.1) that the Askey-Wilson polynomials satisfy the duality relation
\[
p_n(aq^m; a, b, c, d; q) = p_m(\tilde{a} q^n; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}; q), \quad m, n \in \mathbb{Z}_+.
\]

The deeper understanding of duality stems from affine Hecke algebraic considerations, see [13].

The Askey-Wilson polynomials \( \{p_n\}_{n \in \mathbb{Z}_+} \) form a basis of the polynomial algebra \( \mathbb{C}[x + x^{-1}] \) consisting of eigenfunctions of the Askey-Wilson second order \( q \)-difference operator
\[
L = \alpha(x)(T_q - 1) + \alpha(x^{-1})(T_q^{-1} - 1),
\]
\[
\alpha(x) = \frac{(1 - ax)(1 - bx)(1 - cx)(1 - dx)}{(1 - x^2)(1 - qx^2)},
\]
where \( (T_q^{\pm 1} f)(x) = f(q^{\pm 1} x) \). The eigenvalue of \( L \) corresponding to the Askey-Wilson polynomial \( p_n \) is \( \mu(\gamma_n) \), where \( \gamma_n = \tilde{a} q^n \) and
\[
\mu(\gamma) = -1 - \tilde{a}^2 + \tilde{a}(\gamma + \gamma^{-1}).
\]
We remark that the three term recurrence relations for the Askey-Wilson polynomials \( p_n(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}; q) \) \( m \in \mathbb{Z}_+ \) can be derived from the eigenvalue equations \( Lp_n = \mu(\gamma_n) p_n \) \( n \in \mathbb{Z}_+ \) by applying the duality (2.3), see e.g. [1] and [13].
For the remainder of this section, we restrict our attention to the case that the parameters $a, b, c$ and $d$ are positive and less than one. Then Askey and Wilson \[1\] proved the orthogonality relations

\[
\frac{1}{2\pi i C_0} \int_{x \in \mathbb{T}} p_n(x)p_m(x) \Delta(x) \frac{dx}{x} = \delta_{m,n} \frac{\text{Res}_{x = \gamma_n} \left( \Delta(x) \frac{dx}{x} \right)}{\text{Res}_{x = \gamma_0} \left( \Delta(x) \frac{dx}{x} \right)},
\]

where $\delta_{m,n}$ is the Kronecker delta and $\mathbb{T}$ is the counterclockwise oriented unit circle in the complex plane, with the weight function given by

\[
\Delta(x) = \frac{(a^2, 1/x^2; q)_{\infty}}{(ax, a/x, bx, b/x, cx, c/x, dx, d/x; q)_{\infty}},
\]

and with $\Delta(x)$ the weight function $\Delta(x)$ with respect to dual parameters. Here the positive normalization constant $C_0$ is given by the Askey-Wilson integral

\[
C_0 = \frac{1}{2\pi i} \int_{x \in \mathbb{T}} \Delta(x) \frac{dx}{x} = \frac{2(abcd; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}}.
\]

The orthogonality relations written in the form (2.6) exhibit the duality (2.3) of the Askey-Wilson polynomials on the level of the orthogonality relations, since it expresses the quadratic norms explicitly in terms of the dual weight function $\Delta$. This description of the quadratic norms was proved in \[13\] using affine Hecke algebraic techniques.

The above results describe the spectral properties of the Askey-Wilson second order $q$-difference operator $L$ as a unbounded symmetric operator on the weighted $L^2$-space with respect to the measure $\Delta(x) \frac{dx}{x}$ on the unit circle $\mathbb{T}$. For suitable parameter values, this is directly related to harmonic analysis on the compact quantum $SU(2)$ group, see e.g. \[11\].

In this paper we study the spectral analysis of $L$ as a symmetric operator with respect to a one-parameter family of measures of unbounded support. This setting naturally arises in the study of harmonic analysis on the non-compact quantum $SU(1,1)$ group, see \[9\]. Duality, which we have emphasized in this section on the polynomial level, will play a crucial role in the development of the corresponding $L^2$-theory.

### 3. The Askey-Wilson function.

In this section we consider a solution of the eigenvalue equation

\[
(Lf)(x) = \mu(\gamma)f(x),
\]

which reduces to the Askey-Wilson polynomial for $\gamma = \gamma_n (n \in \mathbb{Z}_+)$, and which enjoys the same symmetry properties as the Askey-Wilson polynomials.

Two linearly independent solutions of the eigenvalue equation (3.1) can be derived from Ismail’s and Rahman’s \[5, (1.11)–(1.16)\] solutions for the three term recurrence relation of the associated Askey-Wilson polynomials. The solutions are given in terms of very well poised $8\phi_7$ series. In this section we consider the solution...
The solution \( \phi_\gamma(x) = \phi_\gamma(x; a; b, c; d; q) \) of (3.1) given by

\[
\phi_\gamma(x) = \frac{(qax\gamma/d, qa\gamma/dx; q)_{\infty}}{(a\tilde{b}\tilde{c}\gamma, q\gamma/d, q\gamma/dx; q)_{\infty}} \times W_7(a\tilde{b}\tilde{c}\gamma/q; ax, a/x, \tilde{a}\gamma, b\gamma, \tilde{c}\gamma; q, q/d\gamma), \quad |q/d\gamma| < 1.
\]

This solution of (3.1) is also the subject of study in Suslov’s papers [15], [16] in which Fourier-Bessel type orthogonality relations are derived, see also remark 2.

Applying Bailey’s formula [4, (III.36)] shows that

\[
\phi_\gamma(x) = \frac{1}{(bc, qa/d, q/ad; q)_{\infty}} 4\phi_3(\frac{ax}{ab, ac, ad}; q, q) + \frac{(ax, a/x, \tilde{a}\gamma, \tilde{a}/\gamma, q\gamma/d, q\gamma/dx; q)_{\infty}}{(qx/d, q/dx, q\gamma/d, q/d\gamma, ab, ac, ba, qa/d, ad/q; q)_{\infty}} \times 4\phi_3(\frac{qx/d, q/dx, q\gamma/d, q/d\gamma}{q^2/ad}; q, q)
\]

cf. [13, (2.8)], hence \( \phi_\gamma(x) \) extends to a meromorphic function in \( x \) and \( \gamma \) for generic parameters \( a, b, c \) and \( d \), with possible poles at \( x^{\pm 1} = q^{1+k}/d \) (\( k \in \mathbb{Z}_+ \)) and \( \gamma^{\pm 1} = q^{1+k}/d \) (\( k \in \mathbb{Z}_+ \)). It follows from (3.1) that \( \phi_{\gamma \pm}(x^{\pm 1}) = \phi_\gamma(x) \) (all sign combinations possible), and that \( \phi_\gamma \) satisfies the duality relation

\[
\phi_\gamma(x; a; b, c; d; q) = \phi_\gamma(x; \tilde{a}; b; c; d; q).
\]

In the remainder of the paper we use the short-hand notation \( \tilde{\phi}_x(\gamma) \) for the right-hand side of (4.4).

Observe that the meromorphic continuation (3.3) of \( \phi_\gamma(x) \) implies that

\[
\phi_{\gamma_n}(x) = \frac{1}{(bc, qa/d, q/ad; q)_{\infty}} p_n(x), \quad n \in \mathbb{Z}_+,
\]

since the factor \( (a/\gamma; q)_{\infty} \) in front of the second \( 4\phi_3 \) vanishes when \( \gamma = \gamma_n = \tilde{a}q^n \) for \( n \in \mathbb{Z}_+ \). In particular, the duality (2.3) of the Askey-Wilson polynomials is a special case of the duality (4.3) of \( \phi_\gamma \).

**Definition 1.** The solution \( \phi_\gamma \) (see (3.2) and (3.3)) of the eigenvalue equation \( (Lf)(x) = \mu(\gamma)f(x) \) is called the Askey-Wilson function.
Furthermore, for generic $\gamma$ there exists a unique solution $\Phi_\gamma(x)$ of the eigenvalue equation (3.1) on $I$ of the form $\Phi_\gamma(x) = \Phi_\gamma^{free}(x)g(x)$, where $g$ has a convergent power series expansion around $|x| = \infty$ with constant coefficient equal to one. $\Phi_\gamma$ is the so-called asymptotically free solution of the eigenvalue equation (3.1).

An explicit expression for $\Phi_\gamma$ can be obtained from Ismail’s and Rahman’s [8, (1.13)] solution of the three term recurrence relation for the associated Askey-Wilson polynomials. After application of the transformation formula [4, (III.23)] for very well poised $8\phi7$’s, it can be expressed as

$$\Phi_\gamma(x) = \frac{(qa/\gamma, q\theta/\gamma, q-c\gamma/d, dq/dx, d/q)}{(q/ax, q/bx, q/cx, q/dx, q^2\gamma^2/dx, q^3)} \times \Phi_\gamma(x)$$

for $x \in I$ with $|x| \gg 0$. We now expand the Askey-Wilson function $\phi_\gamma(x)$ as a linear combination of the asymptotically free solutions $\Phi_\gamma(x)$ and $\Phi_\gamma^{-1}(x)$ for $x \in I$ with $|x| \gg 0$. The coefficients in this expansion can be expressed in terms of the $c$-function $c(\gamma) = c(\gamma; a, b, c, d; q, t)$, which is defined by

$$c(\gamma) = \frac{1}{(ab, ac, bc, qa/d; q)_{\infty}} \frac{(a/\gamma, b/\gamma, c/\gamma; q)_{\infty}}{(q/\gamma, q/\gamma^2; q)_{\infty}}$$

where $\theta(x) = (x, q/x, q)_{\infty}$ is the renormalized Jacobi theta function. We call $\tilde{c}(\gamma) = c(\gamma; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}; q, \tilde{t})$ the dual $c$-function, with the dual parameter $\tilde{t}$ defined by

$$\tilde{t} = 1/qd\gamma.$$

**Proposition 1.** Let $x \in I$ with $|x| \gg 0$. Then we have the $c$-function expansion

$$\phi_\gamma(x) = \tilde{c}(\gamma)\Phi_\gamma(x) + \tilde{c}(\gamma^{-1})\Phi_\gamma^{-1}(x)$$

for generic $\gamma$.

**Proof.** Apply Bailey’s three term recurrence relation [8, (III.37)] with its parameters specialized as follows:

$$a \rightarrow q^2/\gamma dx, \quad b \rightarrow q/dx, \quad c \rightarrow q/\gamma \tilde{a}, \quad d \rightarrow q/\gamma \tilde{d}, \quad e \rightarrow \tilde{b} \gamma, \quad f \rightarrow \tilde{c} \gamma.$$

This gives an expansion of the form [8, (3.14)] for explicit coefficients $\tilde{c}(\gamma)$ and $\tilde{c}(\gamma^{-1})$, which at a first glance still depend on $x \in I$. Using the functional equation

$$\theta(q^k x) = q^{-k(1-k)}(-x)^{-k} \theta(x), \quad k \in \mathbb{Z}$$

for theta functions, it is easily seen that the coefficients are independent of $x$ and that they coincide with the dual $c$-functions $\tilde{c}(\gamma^{-1})$ as defined above. $\square$

5. The Askey-Wilson function transform.

In this section we define the Askey-Wilson function transform and we state the main result of this paper. At this stage we need to specify a particular parameter domain for the five parameters $(a, b, c, d, t)$ in order to ensure positivity of measures.

**Definition 2.** Let $V$ be the set of parameters $(a, b, c, d, t) \in \mathbb{R}^5$ satisfying the conditions

$$t < 0, \quad 0 < b, c \leq a < d/q, \quad bd, cd \geq q, \quad ab, ac < 1.$$
Observe that $b, c < 1$ and $d > q$ for all $(a, b, c, d, t) \in V$. The domain $V$ is self-dual in the following sense.

**Lemma 1.** The assignment $(a, b, c, d, t) \mapsto (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{t})$ defined by (2.2) and (4.4), is an involution on $V$.

**Proof.** Direct verification. \[ \square \]

We fix parameters $(a, b, c, d, t) \in V$ for the remainder of the paper. For the moment we furthermore assume that $x \mapsto 1/c(x)(x^{-1})$ only has simple poles. This imposes certain generic conditions on the parameters $(a, b, c, d, t)$, which will be removed later by a continuity argument.

It is convenient to renormalize the function $1/c(x)(x^{-1})$ as follows,

$$ W(x) = \frac{1}{c(x)c(x^{-1})c_0} \frac{(qx/d, q/dx, 1/x^2, q)_{\infty}}{(ax, a/x, bx, b/x, cx, c/x; q)_{\infty} \theta(dx)\theta(dt/x)} $$

where $c_0$ is the positive constant

$$ c_0 = \frac{(ab, ac, bc, qa/d; q^2_{\infty})}{a^2} \theta(adt)^2. $$

It follows from (5.1) that

$$ W(x) = \frac{\theta(dx)\theta(dx)}{\theta(dx)\theta(dx)} \Delta(x), $$

where $\Delta(\cdot)$ is the weight function (2.7) for the Askey-Wilson polynomials. By (4.6), the quotient of theta functions in (5.3) is a quasi-constant, i.e. it is invariant under replacement of $x$ by $qx$. In particular, the weight function $W(x)$ differs from $\Delta(x)$ only by a quasi-constant factor.

Let $S$ be the discrete subset

$$ S = \{ x \in \mathbb{C} \ | \ |x| > 1, c(x) = 0 \} = S_+ \cup S_-, $$

$$ S_+ = \{ aq^k \ | \ k \in \mathbb{Z}_+, \ aq^k > 1 \}, $$

$$ S_- = \{ dtq^k \ | \ k \in \mathbb{Z}, \ dtq^k < -1 \}. $$

We denote $\tilde{S}$ and $\tilde{S}_\pm$ for the subsets $S$ and $S_\pm$ with respect to dual parameters. We define a measure $d\nu(x)$ by

$$ \int f(x)d\nu(x) = \frac{K}{4\pi i} \int_{x \in \mathbb{T}} f(x)W(x) \frac{dx}{x} $$

$$ + \frac{K}{2} \sum_{x \in S} f(x)\text{Res}_{y=x} \left( \frac{W(y)}{y} \right) - \frac{K}{2} \sum_{x \in S^{-1}} f(x)\text{Res}_{y=x} \left( \frac{W(y)}{y} \right), $$

where the positive constant $K$ is given by

$$ K = (ab, ac, bc, qa/d, q; q^2_{\infty}) \sqrt[\infty]{\theta(qt)\theta(adt)\theta(bdt)\theta(cdt)} \sqrt{abcdt^2}. $$

This particular choice of normalization constant for the measure $\nu$ is justified in theorem 1.

In view of (5.3), we can relate the discrete masses $\nu(\{x\}) = -\nu(\{x^{-1}\})$ for $x \in S_+$ to residues of the weight function $\Delta(\cdot)$, which were written down explicitly
in [1] (see also [1, (7.5.22)]) in order to avoid a small misprint). Explicitly, we obtain for \( x = aq^k \in S_+ \) with \( k \in \mathbb{Z}_+ \) the expression

\[
\nu\{aq^k\} = \frac{(qa/d, q/ad, 1/a^2; q)_\infty}{(q, ab, b/a, ac, c/a; q)_\infty \theta(ad) \theta(dt/a)} \times \frac{(a^2, ab, ac, ad; q)_k}{(q, qa/b, qa/c, qa/d; q)_k} \left( 1 - a^2 q^{2k} \right) K \frac{1}{2a^{2k}}
\]

(5.7)

for the corresponding discrete weight. For fixed \( k \in \mathbb{Z}_+ \), the right hand side of (5.7) gives the unique continuous extension of the discrete weight \( \nu\{aq^k\} \) and \( -\nu\{a^{-1}q^{-k}\} \) to all parameters \((a, b, c, d, t)\in V\) satisfying \( aq^k > 1 \). Furthermore, the (continuously extended) discrete weight \( \nu\{aq^k\} \) is strictly positive for these parameter values.

A similar argument can be applied for the discrete weights \( \nu\{x\} = -\nu\{x^{-1}\} \) with \( x \in S_- \). Explicitly we obtain for \( x = dtq^k \in S_- \) with \( k \in \mathbb{Z} \),

\[
\nu\{dtq^k\} = \frac{(qt, q/d^2; q)_\infty}{(q, q/a, dt, b/dt, c/dt, adt, bdt, cdt; q)_\infty} \times \frac{(1/t, a/dt, b/dt, c/dt; q)_k}{(q/adt, q/bdt, q/cdt, q/d^2t; q)_k} \left( 1 - \frac{1}{d^2 t^2 q^{2k}} \right) K \frac{1}{2a^{2k}}
\]

(5.8)

As for \( \nu\{x\} \) with \( x \in S_+ \), we use the right hand side of (5.8) to define the strictly positive weight \( \nu\{dtq^k\} = -\nu\{d^{-1}t^{-1}q^{-k}\} \) for all \((a, b, c, d, t)\in V\) satisfying \( dtq^k < -1 \).

We conclude that the definition of the measure \( \nu \) (see (5.5)) can be extended to arbitrary parameters \((a, b, c, d, t)\in V\) using the continuous extensions of its discrete weights given above. The resulting measure \( \nu \) is a positive measure for all \((a, b, c, d, t)\in V\).

**Definition 3.** Let \( \mathcal{H} \) be the Hilbert space consisting of \( L^2 \)-functions \( f \) with respect to \( \nu \) which satisfy \( f(x) = f(x^{-1}) \) \( \nu \)-almost everywhere.

We write \( \nu \) for the measure \( \nu \) with respect to dual parameters \((\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{t})\), and \( \tilde{\mathcal{H}} \) for the associated Hilbert space \( \mathcal{H} \).

Let \( D \subset \mathcal{H} \) be the dense subspace of functions \( f \) with compact support, i.e.

\[
D = \{ f \in \mathcal{H} \mid f(dtq^{-k}) = 0, \ k \gg 0 \},
\]

and define

\[
(\mathcal{F}f)(\gamma) = \int f(x)\phi_{\gamma}(x)d\nu(x), \quad f \in D
\]

(5.9)

for generic \( \gamma \in \mathbb{C} \setminus \{0\} \).

**Remark 1.** Observe that the analytic continuation (3.3) of \( \phi_{\gamma}(x) \) is not defined for parameters \((a, b, c, d, t)\in V\) satisfying \( \theta(ad) = 0 \). These apparent poles can be removed in view of the original definition (3.2) for \( \phi_{\gamma}(x) = \phi_{\gamma^{-1}}(x) \) in terms of very well poised \( \theta \)-transforms (observe that \( q/d < 1 \), so that either \( |q\gamma/d| < 1 \) or \( |q/d\gamma| < 1 \) for \( \gamma \in \mathbb{C} \setminus \{0\} \), hence \( \phi_{\gamma} \) can be expressed in terms of the original definition (3.2) for generic \( \gamma \)). In particular, the transform \( \mathcal{F} \) is well defined for parameters satisfying \( \theta(ad) = 0 \).
We write $\tilde{D} \subset \tilde{H}$ (respectively $\tilde{F}$) for the dense subspace $D$ (respectively the function transform $\mathcal{F}$) with respect to dual parameters $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{t})$. The main theorem of this paper can now be stated as follows.

**Theorem 1.** Let $(a, b, c, d, t) \in V$. The transform $\mathcal{F}$ extends to an isometric isomorphism $\mathcal{F}: H \to \tilde{H}$ by continuity. The inverse of $\mathcal{F}$ is given by $\tilde{\mathcal{F}}$.

We will discuss the proof of theorem 1 in detail in the remaining sections.

**Definition 4.** The isometric isomorphism $\mathcal{F}: H \to \tilde{H}$ is called the Askey-Wilson function transform.

Theorem 1 gives a simultaneous generalization of the $L^2$-theory for the Mellin-Fock transform and for the (rank one) Hankel transform, see [9] and [10]. In fact, the Askey-Wilson function transform is on top of an hierarchy of several different generalizations of the Mellin-Fock and Hankel transform, in analogy with the polynomial setting. This point of view is emphasized in [10]. In particular, the Askey-Wilson function generalizes the Jacobi function as well as the Bessel function to the level of very well poised $8\phi_7$ series.

Theorem 1 may be partially reformulated in terms of orthogonality relations for “low degree” Askey-Wilson polynomials with respect to the measures $\nu$ ($t < 0$). We say more about this point of view in section 7, where we prove the orthogonality relations for the Askey-Wilson functions $\phi_\gamma$ ($\gamma \in \tilde{S}$) with respect to the measure $\nu$ as an intermediate step of the proof of theorem 1.

### 6. The Wronskian.

We let $\chi_k \in D$ for $k \in \mathbb{Z}$ with $k \ll 0$ be the characteristic function which is zero at $dtq^m, d^{-1}t^{-1}q^{-m}$ for $m \in \mathbb{Z}$ with $m < k$ and which is equal to one otherwise. The sequence $\{\chi_k\}_k$ of characteristic functions is an approximation of the unit, in the sense that $\chi_k f \to f$ in $H$ as $k \to -\infty$ for all $f \in H$.

The starting point for the proof of theorem 1 is the following simple observation.

**Lemma 2.** Let $f, g \in \tilde{D}$ and $k \ll 0$, then

$$\int \chi_k(x)(\tilde{\mathcal{F}}f)(x)(\tilde{\mathcal{F}}g)(x)d\nu(x) =$$

$$= \int \int f(\gamma)g(\gamma') \left( \int \chi_k(x)\phi_\gamma(x)\phi_{\gamma'}(x)d\tilde{\nu}(x) \right) d(\tilde{\nu} \times \tilde{\nu})(\gamma, \gamma').$$

**Proof.** We substitute the definition of the dual Askey-Wilson function transform $\tilde{\mathcal{F}}$ into the left hand side of the desired identity, and we use that

$$\phi_\gamma(x) = \tilde{\phi}_\gamma(x) = \phi_\gamma(x)$$

for $x \in \text{supp}(\nu)$ and $\gamma \in \text{supp}(\tilde{\nu})$, see (3.4) for the second equality. We arrive at

$$\int \chi_k(x)(\tilde{\mathcal{F}}f)(x)(\tilde{\mathcal{F}}g)(x)d\nu(x) =$$

$$= \int \chi_k(x) \left( \int f(\gamma)\phi_\gamma(x)d\tilde{\nu}(\gamma) \right) \left( \int g(\gamma')\phi_{\gamma'}(x)d\tilde{\nu}(\gamma') \right) d\nu(x).$$
Since $f, g \in \mathcal{D}$, all integrations are over compact subsets, so we may use Fubini’s theorem to interchange the order of integration. This yields the desired result. □

The proof of theorem 1 now hinges on the explicit evaluation of the weak limit as $k \to -\infty$ of the function

$$
(\gamma, \gamma') \mapsto \int \chi_k(x)\phi_\gamma(x)\phi_{\gamma'}(x)d\nu(x)
$$

with respect to the product measure $\nu \times \nu$.

As a first step, we express the integral (6.1) in terms of the Wronskian of the Askey-Wilson functions $\phi_\gamma$ and $\phi_{\gamma'}$. Here the Wronskian $[f, g](x)$ for two functions $f, g : I \to \mathbb{C}$ is defined by

$$
[f, g](x) = 2\nu(\{x\})\alpha(x)(f(x)g(qx) - f(qx)g(x)), \quad x \in S_-, \quad (6.2)
$$

where $\alpha(x)$ is given by (2.4).

**Proposition 2.** Let $(a, b, c, d, t) \in V$. For generic spectral parameters $\gamma, \gamma' \in \mathbb{C}$ such that $\mu(\gamma) \neq \mu(\gamma')$, we have

$$
\int \chi_k(x)\phi_\gamma(x)\phi_{\gamma'}(x)d\nu(x) = \frac{[\phi_\gamma, \phi_{\gamma'}](dtq^{-1})}{\mu(\gamma) - \mu(\gamma')}
$$

for $k \in \mathbb{Z}$ sufficiently negative.

**Proof.** The proof of the desired identity simplifies when we slightly perturb the parameters $(a, b, c, d, t)$. The proof for parameters $(a, b, c, d, t) \in V$ can then be derived from the perturbed case using the fact that the left and right hand side of the identity depend continuously on the parameters $(a, b, c, d, t)$.

Let us indicate one class of possible perturbations of the parameters which is sufficient for our purposes. Let $(\alpha, \beta, \gamma, \delta, t) \in V$ with $\alpha, \beta, \alpha \delta, -\delta t \not\in q^2$. Let $\epsilon > 0$ be sufficiently small, then we set

$$
a = \alpha e^{\pi i \epsilon}, \quad b = \beta e^{2\pi i \epsilon}, \quad c = \gamma e^{3\pi i \epsilon}, \quad d = \delta e^{-6\pi i \epsilon},
$$

while we keep $t$ undisturbed. Observe that $|a| = \alpha$, $|b| = \beta$ etc., and that $abed = \alpha \beta \gamma \delta$. Note that the weight function $W(x)$ (see (5.4)) corresponding to the parameters $(a, b, c, d, t)$ only has simple poles. We can now define the (in general complex) measure $\mu$ with respect to the parameters $(a, b, c, d, t)$ by the same formulas (5.4) and (6.3) as before. We also keep the same notations for the other parameter-dependent objects we have encountered (such as e.g. $\phi_\gamma, \chi_k$ etc.). Then for spectral parameters $\gamma, \gamma' \in \mathbb{C} \setminus \{0\}$ with $\gamma \pm 1, (\gamma') \pm 1 \not= q^{1+l}/d$ ($l \in \mathbb{Z}_+$) we can write for $k \ll 0$,

$$
\int \chi_k(x)\phi_\gamma(x)\phi_{\gamma'}(x)d\nu(x) = \frac{K}{4\pi i} \int_{x \in C_k} \phi_\gamma(x)\phi_{\gamma'}(x) \left( \frac{W(x)}{x} \right) dx, \tag{6.3}
$$

where $C_k$ is a continuous, rectifiable Jordan curve in the complex plane, satisfying the following additional conditions:

- $C_k$ has a parametrization of the form $r_k(z)e^{2\pi iz}$ for $z \in [0, 1]$ with positive radial function $r_k : [0, 1] \to (0, \infty)$ (we orientate $C_k$ according to this parametrization);
- $C_k$ is invariant under inversion, i.e. $C_k^{-1} = C_k$,
- The sequences $(aq^l, bq^l, cq^l, q^{1+l}/d)_{l \in \mathbb{Z}_+}$ are in the interior of $C_k$;
- The intersection of the $q$-interval $I$ with the interior of $C_k$ is given by the sequence $\{dtq^m \mid m \in \mathbb{Z} : m \geq k\}$. 
The existence of $C_k$ is easy to prove with our choice of perturbed parameters, and \((6.3)\) is then a direct consequence of Cauchy’s theorem.

Now recall that the Askey-Wilson function $\phi_\gamma$ is an eigenfunction of the second-order $q$-difference operator $L$ with eigenvalue $\mu(\gamma)$, see section 3. This implies that

\[
(\mu(\gamma) - \mu(\gamma')) \int \chi_k(x)\phi_\gamma(x)\phi_{\gamma'}(x) d\nu(x) = \frac{K}{4\pi i} \int_{x \in C_k} \left( (L\phi_\gamma)(x)\phi_{\gamma'}(x) - \phi_\gamma(x)(L\phi_{\gamma'})(x) \right) \left( \frac{W(x)}{x} \right) dx. \tag{6.4}
\]

In view of the explicit expression for $L$ (see \((2.4)\)), we can write

\[
(L\phi_\gamma)(x)\phi_{\gamma'}(x) - \phi_\gamma(x)(L\phi_{\gamma'})(x) = \alpha(1/x)\xi_{\gamma,\gamma'}(x/q) - \alpha(x)\xi_{\gamma,\gamma'}(x) \tag{6.5}
\]

with

\[
\xi_{\gamma,\gamma'}(x) = \phi_\gamma(x)\phi_{\gamma'}(qx) - \phi_\gamma(qx)\phi_{\gamma'}(x).
\]

Furthermore, by the explicit expression for the weight function $W(x)$ (see \((2.1)\)) and for the function $\alpha(x)$ (see \((2.4)\)), we have the identity

\[
\alpha(1/x)W(x) = \alpha(x/q)W(x/q). \tag{6.6}
\]

Substitution of \((6.5)\) and \((6.6)\) into the right hand side of \((6.4)\) then shows that

\[
(\mu(\gamma) - \mu(\gamma')) \int \chi_k(x)\phi_\gamma(x)\phi_{\gamma'}(x) d\nu(x) = \frac{K}{4\pi i} \int_{x \in q^{-1}C_k - C_k} \alpha(x)\xi_{\gamma,\gamma'}(x) \left( \frac{W(x)}{x} \right) dx. \tag{6.7}
\]

Using the explicit form of the contour $C_k$, we can shrink $q^{-1}C_k$ back to $C_k$ in the right hand side of \((6.7)\) while picking up residues of the integrand at the simple poles $x = dtq^{k-1}$ and $x = d^{-1}t^{-1}q^{-k}$. By \((6.6)\) and by the identity $\xi_{\gamma,\gamma'}(x) = -\xi_{\gamma,\gamma'}(q^{-1}x^{-1})$, we can pull the two residues together, yielding

\[
(\mu(\gamma) - \mu(\gamma')) \int \chi_k(x)\phi_\gamma(x)\phi_{\gamma'}(x) d\nu(x) = K \frac{Res_{y=dtq^{k-1}} \left( \frac{W(y)}{y} \right) \alpha(dtq^{k-1})\xi_{\gamma,\gamma'}(dtq^{k-1})}{|\phi_\gamma, \phi_{\gamma'}|(dtq^{k-1})}, \tag{6.8}
\]

as desired. \hfill \Box

7. Orthogonality relations.

If we take in \((6.1)\) the values of the spectral parameters $\gamma, \gamma'$ in the discrete support $\tilde{S}$ of the dual measure $\tilde{\nu}$, then the limit of \((6.1)\) as $k \to -\infty$ can be taken point-wise. We evaluate these limits in this section, which lead to explicit orthogonality relations for the Askey-Wilson functions $\phi_\gamma$ ($\gamma \in \tilde{S}$) with respect to the measure $\nu$. This essentially establishes theorem 3.3 for the “completely discrete part” of the Askey-Wilson function transform.

**Lemma 3.** The Askey-Wilson functions $\phi_\gamma$ ($\gamma \in \tilde{S}$) are elements of the Hilbert space $\mathcal{H}$. Furthermore,

\[
\int \phi_\gamma(x)\overline{\phi_{\gamma'}(x)} d\nu(x) = 0, \quad \gamma, \gamma' \in \tilde{S}, \quad \gamma \neq \gamma'.
\]
that (2.4)). Combined with (7.1), it follows from the definition (6.2) of the Wronskian c removable. Hence we can extend the definition of the
In particular, the limit in the right hand side of (7.5) tends to zero, hence
for the coefficient
for all \((a, b, c, d, t) \in V\) in order to be able to apply the c-function expansion for \(\phi_\gamma (\gamma \in \tilde{S})\). We fix \(\gamma \in \tilde{S}\), so that \(\phi_\gamma (x) = \overline{c(\gamma^{-1})} \Phi_{\gamma^{-1}}(x)\) for \(x \in \mathcal{S}_-\) sufficiently negative by proposition \([\ref{prop:asymptotics}]\). Observe that the apparent poles of \(\overline{c(\gamma^{-1})}\) as function of the parameters \((a, b, c, d, t) \in V\) are removable. Hence we can extend the definition of the c-function \(\overline{c(\gamma^{-1})}\) to parameter values \((a, b, c, d, t) \in V\) by continuity and we obtain
for all \((a, b, c, d, t) \in V\) if \(x \in I\) is sufficiently negative. Since
and \(|\gamma^{-1}| < 1\) for \(\gamma \in \tilde{S}\), it follows from \((7.1)\) that \(\phi_\gamma \in \mathcal{H}\) for all \(\gamma \in \tilde{S}\).
Fix now \(\gamma, \gamma' \in \tilde{S}\) with \(\gamma \neq \gamma'\). Then \(\mu(\gamma) \neq \mu(\gamma')\) by \((2.4)\), hence by proposition \([\ref{prop:asymptotics}]\) and by the previous paragraph,
Here we have used the fact that \(\phi_{\gamma'}(\cdot)\) is real valued on \(\text{supp}(\nu)\) in order to get rid of the complex conjugate in the integrand.
It remains to prove that the limit in the right hand side of \((7.5)\) tends to zero. Let \(f\) and \(q\) be two functions in the Hilbert space \(\mathcal{H}\). By the asymptotics \((7.2)\), we have \(f(dtq^{-k}) = o(\overline{\alpha}k)\) as \(k \to \infty\), and similarly for \(g\). Furthermore, we have the asymptotics
for the coefficient \(\alpha(\cdot)\) of the Askey-Wilson second order \(q\)-difference operator \(L\) (see \((2.4)\)). Combined with \((7.4)\), it follows from the definition \((6.2)\) of the Wronskian that
In particular, the limit in the right hand side of \((7.5)\) tends to zero, hence \(\phi_\gamma\) is orthogonal to \(\phi_{\gamma'}\) in \(\mathcal{H}\).

The quadratic norms of the Askey-Wilson functions \(\phi_\gamma (\gamma \in \tilde{S})\) in \(\mathcal{H}\) can be evaluated as follows.

**Lemma 4.** For all \(\gamma \in \tilde{S}\),

\[
\int |\phi_\gamma(x)|^2 d\nu(x) = \frac{1}{2\nu(\{\gamma\})}.
\]
Proof. As in the proof of lemma 3, we assume for the moment that \(ad \not\in q\mathbb{Z}\) and that \(\tilde{a}, \tilde{d} \not\in \pm q\mathbb{Z}\). In view of (7.3), (7.4) and (7.1), these generic conditions can be removed at the end of the proof by applying the dominated convergence theorem.

We fix \( \gamma \in \mathcal{S} \), then by lemma 3 and proposition 2,

\[
\int |\phi_\gamma(x)|^2 d\nu(x) = \lim_{k \to -\infty} \int \chi_k(x)\phi_\gamma(x)^2 d\nu(x)
\]

\[
= \lim_{k \to -\infty} \left( \lim_{\gamma' \to \gamma} \int \chi_k(x)\phi_\gamma(x)\phi_{\gamma'}(x) d\nu(x) \right)
\]

\[
= \lim_{k \to -\infty} \left( \lim_{\gamma' \to \gamma} \frac{[\phi_{\gamma}, \phi_{\gamma'}](dtq^{-k})}{(\mu(\gamma) - \mu(\gamma'))} \right)
\]

where we used that \(\phi_\gamma(\cdot)\) is real valued on \(\text{supp}(\nu)\) for the first equality.

It remains to evaluate the limits of the Wronskian in the last equality of (7.7).

We use the \(c\)-function expansion for the Askey-Wilson functions (see proposition 1 and (7.3)) to rewrite the Wronskian as

\[
[\phi_\gamma, \phi_{\gamma'}](dtq^{-k}) \to +\tilde{c}(\gamma)\tilde{c}(\gamma') [\Phi_{\gamma-1}, \Phi_{\gamma'}](dtq^{-k})
\]

\[
+ \tilde{c}(\gamma)\tilde{c}(\gamma'-1) [\Phi_{\gamma-1}, \Phi_{\gamma'-1}](dtq^{-k})
\]

for \(k \gg 0\). Now there exists open neighbourhoods \(U_\pm\) of \(\gamma \pm 1\) in the complex plane such that

\[
\Phi_{\gamma'}(x) = \Phi_{\gamma'}^{free}(x)(1 + f_{\gamma'}(x)), \quad x \in S_-, \quad x \ll 0
\]

where \(f_{\gamma'}(x)\) admits a convergent power-series expansion around \(x = -\infty\) with coefficients depending analytically on \(\gamma' \in U_\pm\) and with constant coefficient equal to zero. Furthermore, for sufficiently small neighbourhoods \(U_\pm\), differentiation with respect to \(\gamma'\) may be interchanged with summation in the power series expansion of \(f_{\gamma'}(x)\) around \(x = -\infty\) when \(|x| \geq N > 0\) for some \(U_\pm\) independent positive constant \(N\).

Combined with (7.9), we see that

\[
[\Phi_{\gamma-1}, \Phi_{\gamma\pm 1}](dtq^{-k}) = \sum_{\kappa = \pm 1} \tilde{c}(\gamma)\tilde{c}(\gamma)^{1-k}(\gamma^{-1}\gamma^{\kappa})(1 + O(q^k))
\]

as \(k \to \infty\), with \(O(q^k)\) uniform in \(\gamma' \in U_\pm\). Now we substitute (7.3) in (7.8), and we use the fact that \(\tilde{c}(\gamma')\) has a simple zero at \(\gamma' = \gamma\) and that

\[
\mu(\gamma) - \mu(\gamma') = \frac{\tilde{a}}{\tilde{\gamma}}(\gamma - \gamma')(\gamma - \gamma'^{-1})
\]

then using (7.10) and the fact that \(|\gamma| > 1\), we derive that

\[
[\phi_\gamma, \phi_{\gamma'}](dtq^{-k}) = 2\nu \{ dtq^{-k} \} \alpha(dtq^{-k})\tilde{a}^{2k-1}(\gamma^{-1}\gamma^{\kappa})(1 + O(q^k))
\]

with

\[
s_k(\gamma, \gamma') = \sum_{\kappa = \pm 1} \tilde{c}(\gamma')\tilde{c}(\gamma^{-1})\gamma^{\kappa(k-1)}\gamma^{1-k}(\gamma^{-1} - \gamma')
\]

and with a remainder term \(r_k(\gamma, \gamma')\) satisfying

\[
\lim_{\gamma' \to \gamma} \frac{r_k(\gamma, \gamma')}{\mu(\gamma) - \mu(\gamma')} = O(q^k), \quad k \to \infty.
\]
Proposition 3. Reformulate lemma 3 and lemma 4 in the following way.

With respect to dual parameters. In view of lemma 2 and lemma 1, we can now evaluate now the limits in the last equality of (7.7) by substituting (7.12) for the Wronskian and by using (7.1), (7.6), (7.13) and (7.14) together with the fact that |γ| > 1. This gives

$$\int |\phi_\gamma(x)|^2 d\nu(x) = \frac{\tilde{K} M}{c_0} \frac{1}{2\nu(\{\gamma\})}.$$  

By a direct computation one verifies that $\tilde{K} M / c_0 = 1$, which completes the proof of the lemma.

Let $D_d \subset D$ be the sub-space consisting of the functions in $D$ which are supported on the discrete support $S$ of the measure $\nu$, and let $D_d$ be the sub-space $D_d$ with respect to dual parameters. In view of lemma 3 and lemma 4, we can now reformulate lemma 3 and lemma 4 in the following way.

**Proposition 3.** Let $f, g \in D_d$, then $\mathcal{F} f, \mathcal{F} g \in \tilde{H}$ and

$$\int (\mathcal{F} f)(\gamma) (\mathcal{F} g)(\gamma) d\tilde{\nu}(\gamma) = \int f(x) g(x) d\nu(x).$$

Recall from (3.3) that $\phi_\gamma$ is a multiple of the Askey-Wilson polynomial $p_k$ when $\gamma = \gamma_k = a q^k \in \mathbb{S}_+$ ($k \in \mathbb{Z}_+$). So if $a > 0$, then lemma 3 and lemma 4 give explicit orthogonality relations for the “low degree” Askey-Wilson polynomials $p_k$ ($k \in \mathbb{Z}_+$, $k \leq k_0$) with respect to the one-parameter family of measures $\nu (t < 0)$, where $k_0$ is the largest positive integer such that $a q^{k_0} > 1$. The sub-space spanned by $\{p_k\}_{k=0}^{k_0}$ is exactly the sub-space of polynomials in $x + x^{-1}$ which are square integrable with respect to $\nu$ (this follows easily from (7.1)). Furthermore, the functions $\phi_\gamma \in \mathcal{H}$ with $\gamma \in \mathbb{S}_-$ constitute an explicit family of mutually orthogonal functions with respect to the measure $\nu$, which in addition are orthogonal to the finite set of Askey-Wilson polynomials $p_k$ ($k \in \mathbb{Z}_+$, $k \leq k_0$).

From this point of view, lemma 3 and lemma 4 bear close resemblance with indeterminate moment problems and non-extremal measures. In fact, the measures $\nu (t < 0)$ formally reduce to a genuine one-parameter family of non-extremal orthogonality measures for some indeterminate moment problem in certain degenerate cases of the Askey-Wilson function transform. This point of view was emphasized in [10], where it was for instance shown that a formal limit of the above orthogonality relations leads to a one-parameter family of non-extremal orthogonality measures for the continuous dual $q^{-1}$-Hahn polynomials, see also [8].

**Remark 2.** Fix $k \in \mathbb{Z}$ sufficiently negative, and let $\gamma_1(k), \gamma_2(k)$ be two zeros of the function $\gamma \mapsto \phi_\gamma (dt q^{k-1})$ such that $\mu(\gamma_1(k)) \neq \mu(\gamma_2(k))$. Then proposition 2 implies that the Askey-Wilson functions $\phi_{\gamma_1(k)}$ and $\phi_{\gamma_2(k)}$ are mutually orthogonal with
respect to the measure $\chi_k(x)dv(x)$ of compact support. These Fourier-Bessel type orthogonality relations for the Askey-Wilson functions were derived and studied by Suslov in [13] and [14].

8. The continuous part of the Askey-Wilson function transform.

In this section we first evaluate the weak limit of (5.1) as $k \to -\infty$ with respect to the product measure $\tilde{\nu}|\mathbb{T} \times \tilde{\nu}|\mathbb{T}$. We directly formulate the result in terms of suitable unitarity properties of the Askey-Wilson function transform (compare with proposition 3 in the square integrable setting).

Let $D_c \subset \mathcal{D}$ be the sub-space consisting of the functions in $\mathcal{D}$ which are supported on $\mathbb{T}$, and let $\mathcal{D}_c$ be the sub-space $D_c$ with respect to dual parameters. Observe that $\mathcal{D} = D_c \oplus D_d$ is an orthogonal direct sum decomposition of the dense sub-space $\mathcal{D} \subset \mathcal{H}$.

**Proposition 4.** Let $f, g \in D_c$, then $\mathcal{F}f, \mathcal{F}g \in \mathcal{H}$ and

$$
\int (\mathcal{F}f)(\gamma)(\mathcal{F}g)(\gamma)d\tilde{\nu}(\gamma) = \int f(x)g(x)dv(x).
$$

**Proof.** The proof is similar to the proof of [8, prop 7.7] and of [7, prop 6.1], where the analogous statement was derived for the big and the little $q$-Jacobi function transforms, respectively. Since some care has to be taken in order to match the constants, we repeat here the proof in some detail.

We prove the proposition with respect to dual parameters (cf. lemma 3). It suffices to prove it for functions $f, g \in \tilde{D}_c$ which are continuous on $\mathbb{T}$ and supported within $\mathbb{T} \setminus \{\pm 1\}$.

We use lemma 3, proposition 3 and the explicit form of the measure $\tilde{\nu}|\mathbb{T}$ (see (5.1) and (5.3)), together with the invariance of the Askey-Wilson function $\phi_\gamma$ and the measure $d\tilde{\nu}(\gamma)$ under $\gamma \to \gamma^{-1}$, to write

$$
\int \chi_{1-k}(x)(\tilde{\mathcal{F}}f)(x)(\tilde{\mathcal{F}}g)(x)dv(x) = \frac{\tilde{K}^2}{4\pi^2c^2} \int_0^\pi \int_0^\pi f(e^{i\theta})g(e^{i\theta'})[\phi_{e^{i\theta}}, \phi_{e^{i\theta'}}](dtq^{-k})\frac{d\theta}{(\mu(e^{i\theta}) - \mu(e^{i\theta'}))} \frac{d\theta'}{[\tilde{c}(e^{i\theta'})]^2}
$$

for $k \in \mathbb{Z}$ sufficiently large. It suffices to prove that the right hand side of (8.1) tends to $\int f(\gamma)g(\gamma)d\tilde{\nu}(\gamma)$ as $k \to \infty$.

In order to compute the limit $k \to \infty$ of the right hand side of (8.1), we first observe that the factor $\mu(e^{i\theta}) - \mu(e^{i\theta'})$ occurring in the integrand can be rewritten as

$$
\mu(e^{i\theta}) - \mu(e^{i\theta'}) = 4\tilde{a} \sin\left(\frac{\theta + \theta'}{2}\right) \sin\left(\frac{\theta - \theta'}{2}\right),
$$

(8.2)

cf. (7.11). Similarly as in the proof of lemma 3, we derive now from the first equality in (8.2), the c-function expansion (see proposition 3) and the mean value theorem, that

$$
[\phi_{e^{i\theta}}, \phi_{e^{i\theta'}}](dtq^{-k}) = 2\nu\{dtq^{-k}\}\alpha(dtq^{-k})\tilde{\alpha}^{2k-1}(s_k(\theta, \theta') + r_k(\theta, \theta'))
$$

(8.3)
Lemma 5. 

We substitute the expression (8.3) for the Wronskian in the right-hand side of (8.1). In view of the asymptotics (7.1) and (7.8), it then suffices to calculate the limit $k \to \infty$ of

$$
\int_0^\pi f(\theta) g(\epsilon \theta) \sum_{\delta \leq \theta \neq \theta' \leq \pi - \delta} \left| \frac{r_k(\theta, \theta')}{(\mu(\epsilon \theta) - \mu(\epsilon \theta'))} \right| \frac{d\theta}{\epsilon c(e^{i\theta})} \to 0.
$$

The asymptotic behaviour (8.5) shows that the limit $k \to \infty$ of the integral (8.6) gives zero by dominated convergence. By the Riemann-Lebesgue lemma, the integral (8.6) of the remainder term $r_k(\theta, \theta')$ tends to zero as $k \to \infty$. It thus remains to calculate the limit $k \to \infty$ of (8.6) in which the factor $(s_k(\theta, \theta') + r_k(\theta, \theta'))$ of the integrand is replaced by

$$
t_k(\theta, \theta') = \sum_{\epsilon, \epsilon' = \pm 1, \epsilon \epsilon' = -1} \bar{c}(e^{i\epsilon \theta}) \bar{c}(e^{i\epsilon' \theta}) (e^{i(k-1)\epsilon \theta} e^{i(k-1)\epsilon' \theta'} (e^{i\epsilon \theta} - e^{i\epsilon' \theta'}).
$$

It follows now by yet another application of the Riemann-Lebesgue lemma that it remains to calculate the limit $k \to \infty$ of

$$
\int_0^\pi f(\theta) g(\epsilon \theta) \sum_{\theta = 0}^\pi \left| \frac{r_k(\theta, \theta')}{(\mu(\epsilon \theta) - \mu(\epsilon \theta'))} \right| \frac{d\theta}{\epsilon c(e^{i\theta})} \frac{d\theta'}{\epsilon c(e^{i\theta'})}.
$$

Now by the well-known $L^2$-properties of the Dirichlet kernel, the limit $k \to \infty$ of (8.7) exists, and it equals

$$
\int_0^\pi f(\theta) g(\epsilon \theta) \frac{d\theta}{\epsilon c(e^{i\theta})} = \frac{K}{c_0} \int f(\gamma) g(\gamma) d\nu(\gamma).
$$

The proposition follows now directly from the fact that $K M \epsilon c_0 = 1$ (compare with the proof of lemma 3).

In order to completely understand the weak limit of [8.7] as $k \to -\infty$ with respect to the product measure $\nu \times \tilde{\nu}$, we still have to deal with the mixed continuous-discrete case. This case is covered by the following lemma.

Lemma 5. Let $f \in D_c$ and $g \in D_d$, then

$$
\int (F f)(\gamma) (F g)(\gamma) d\tilde{\nu}(\gamma) = 0.
$$
Proof. We establish the desired identity with respect to dual parameters. Let \( f \in D_c \) and \( g \in D_d \). By proposition \( \ref{prop3} \) and proposition \( \ref{prop4} \) we have \( \tilde{F} f \in \mathcal{H} \) and \( \tilde{F} g \in \mathcal{H} \) respectively. In particular, we may assume without loss of generality that \( f \) is continuous on \( \mathbb{T} \), and supported within \( \mathbb{T} \setminus \{ \pm 1 \} \).

For \( \gamma \in \tilde{S} \cup \tilde{S}^{-1} \) we now define \( f_\gamma \in D_c \) by
\[
f_\gamma(x) = f(\gamma(x)/(\mu(\gamma) - \mu(\gamma'))).
\]
Observe that \( f_\gamma \) is continuous on \( \mathbb{T} \) and supported within \( \mathbb{T} \setminus \{ \pm 1 \} \). By lemma \( \ref{lem2} \) and proposition \( \ref{prop2} \) we can now write
\[
\int (\tilde{F} f)(x)(\tilde{F} g)(x)d\tilde{\nu}(x) = 2 \sum_{\gamma \in \tilde{S}} g(\gamma)\tilde{\nu}(\{ \gamma \}) \lim_{k \to \infty} \int_{\mathbb{T}} f_\gamma(\gamma')[\phi_{\gamma'}, \phi_{\gamma'}](dtq^{-k})d\tilde{\nu}(\gamma'). \tag{8.8}
\]

Now we observe that \( \ref{eq7.3} \), \( \ref{eq7.4} \), proposition \( \ref{prop1} \) and the asymptotic behaviour
\[
\sup_{\gamma' \in \text{supp}(f_\gamma)} |\Phi_{\gamma' \pm 1}(dtq^{-k})| = O(\bar{a}^k), \quad k \to \infty
\]
imply that
\[
\sup_{\gamma' \in \text{supp}(f_\gamma)} |[\phi_{\gamma'}, \phi_{\gamma'}](dtq^{-k})| = 2\nu(\{ dtq^{-k} \})\alpha(dtq^{-k})\bar{a}^{2k-1}|\gamma|^{-k}O(1) = O(|\gamma|^{-k})
\]
as \( k \to \infty \), where the second equality is a consequence of \( \ref{eq7.4} \) and \( \ref{eq7.6} \). But \( |\gamma|^{-1} < 1 \) for \( \gamma \in \tilde{S} \), hence we may use Lebesgue’s dominated convergence theorem to interchange limit and integration in the right hand side of \( \ref{eq8.8} \). It follows that the right hand side of \( \ref{eq8.8} \) tends to zero, which completes the proof of the lemma.

9. Completion of the proof of theorem \( \ref{thm1} \).

In this section we complete the proof of theorem \( \ref{thm1} \).

First of all, we observe that the results of section \( \ref{sec7} \) and section \( \ref{sec8} \) immediately imply that the Askey-Wilson function transform \( F \) extends to an isometry \( F : \mathcal{H} \to \mathcal{H} \). Indeed, it follows from proposition \( \ref{prop3} \), proposition \( \ref{prop4} \) and lemma \( \ref{lem2} \) together with the orthogonal direct sum decomposition \( D = D_c \oplus D_d \), that
\[
\int (F f)(\gamma)(F g)(\gamma)d\bar{\nu}(\gamma) = \int f(x)g(x)d\nu(x), \quad f, g \in D.
\]
Since \( D \subset \mathcal{H} \) is dense, it follows that the Askey-Wilson function transform \( F \) uniquely extends to an isometry \( F : \mathcal{H} \to \mathcal{H} \) by continuity. In particular, the dual Askey-Wilson function transform \( \tilde{F} : \mathcal{H} \to \mathcal{H} \) is an isometry in view of lemma \( \ref{lem2} \).

Fix now arbitrary \( f \in \mathcal{H} \) and \( g \in \mathcal{H} \), and write \( \chi_k \in D \) \( (k \in \mathbb{Z} \) sufficiently negative) for the characteristic function \( \chi_k \) with respect to dual parameters (see the beginning of section \( \ref{sec3} \)). Since \( F : \mathcal{H} \to \mathcal{H} \) and \( \tilde{F} : \mathcal{H} \to \mathcal{H} \) are continuous, we
have
\[
\int (\tilde{F}f)(x)g(x)d\nu(x) = \lim_{k,m \to -\infty} \int \left( \int \chi_k(\gamma)f(\gamma)\phi_\gamma(x)d\nu(\gamma) \right) \chi_m(x)g(x)d\nu(x)
\]
\[
= \lim_{k,m \to -\infty} \int \chi_k(\gamma)f(\gamma) \left( \int \chi_m(x)g(x)\phi_\gamma(x)d\nu(x) \right) d\nu(\gamma)
\]
\[
= \int f(\gamma)(\tilde{F}g)(\gamma)d\nu(\gamma),
\]
where we used Fubini’s theorem, the duality (3.4) of the Askey-Wilson function, and the fact that the Askey-Wilson function \(\phi_\gamma(x)\) is real-valued for \(x \in \text{supp}(\nu)\) and \(\gamma \in \text{supp}(\tilde{\nu})\). But then we have for all \(f, g \in \mathcal{H}\),
\[
\int (\tilde{F}(\tilde{F}f))(x)g(x)d\nu(x) = \int (\tilde{F}f)(\gamma)(\tilde{F}g)(\gamma)d\nu(\gamma) = \int f(x)g(x)d\nu(x)
\]
since \(\mathcal{F} : \mathcal{H} \to \tilde{\mathcal{H}}\) is an isometry. It follows that \(\tilde{\mathcal{F}} \circ \mathcal{F} = \text{Id}_\mathcal{H}\). By a similar argument, we obtain \(\mathcal{F} \circ \tilde{\mathcal{F}} = \text{Id}_{\tilde{\mathcal{H}}}\) (or simply replace the parameters by the corresponding dual parameters in \(\tilde{\mathcal{F}} \circ \mathcal{F} = \text{Id}_\mathcal{H}\) and use lemma [4].

We conclude that \(\mathcal{F} : \mathcal{H} \to \tilde{\mathcal{H}}\) and \(\tilde{\mathcal{F}} : \tilde{\mathcal{H}} \to \mathcal{H}\) are isometric isomorphisms, and that \(\tilde{\mathcal{F}} = \mathcal{F}^{-1}\). This completes the proof of theorem [4].

References

1. R. Askey, J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Mem. Amer. Math. Soc. 54 (1985), no. 319.
2. I. Cherednik, Double affine Hecke algebras and Macdonald’s conjectures, Ann. Math. 141 (1995), pp. 191–216.
3. I. Cherednik, Macdonald’s evaluation conjectures and difference Fourier transform, Invent. Math. 122 (1995), pp. 119–145.
4. G. Gasper, M. Rahman, Basic hypergeometric series, Encyclopedia of Mathematics and its Applications 35, Cambridge University Press (1990).
5. M. Ismail, M. Rahman, The associated Askey-Wilson polynomials, Trans. Amer. Math. Soc. 328 (1991), pp. 201–237.
6. T. Kakehi, Eigenfunction expansion associated with the Casimir operator on the quantum group SU(1,1), Duke Math. J. 80 (1998), pp. 535–573.
7. T. Kakehi, T. Masuda, K. Ueno, Spectral analysis of a q-difference operator which arises from the quantum SU(1,1) quantum group, J. Operator Theory 33 (1995), pp. 159–196.
8. E. Koelink, J.V. Stokman, The big q-Jacobi function transform, preprint (1999).
9. E. Koelink, J.V. Stokman, Fourier transforms on the quantum SU(1,1) group (with an appendix by M. Rahman), preprint (1999).
10. E. Koelink, J.V. Stokman, The Askey-Wilson function transform scheme, preprint (1999). To appear in the proceedings of the International Workshop on Special Functions, Asymptotics, Harmonic Analysis and Mathematical Physics, Hong Kong (1999).
11. T.H. Koornwinder, Askey-Wilson polynomials as zonal spherical functions on the SU(2) quantum group, SIAM J. Math. Anal. 24 (1993), pp. 795–813.
12. I.G. Macdonald, Affine Hecke algebras and orthogonal polynomials, Sémin Bourbaki, Vol. 1994/95. Astérisque No. 237 (1996), Exp. No. 797, pp. 189–207.
13. M. Noumi, J.V. Stokman, Askey-Wilson polynomials: an affine Hecke algebraic approach, preprint (2000).
14. S. Sahi, Some properties of Koornwinder polynomials, in press.
15. S.K. Suslov, Some orthogonal very well poised \(\phi_7\)-functions, J. Phys. A 30 (1997), pp. 5877–5885.
16. S.K. Suslov, Some orthogonal very well poised \(\phi_7\)-functions that generalize Askey-Wilson polynomials, preprint (1997).
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