A New Method to Predict Meson Masses

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The Feynman–Hellmann theorem is used to show that vector meson energy eigenvalues are monotonically decreasing functions of the reduced masses of their constituent quarks. The experimental meson masses are used to put constraints on the values of quark masses and to predict the masses of some as yet undiscovered mesons. The mass of the $B_c^*$ meson is predicted to be $6320 \pm 10$ MeV, and, with less precision, the masses of a number of excited vector mesons are also predicted.

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Some years ago, Quigg and Rosner [1] applied the Feynman–Hellmann theorem [2,3] to obtain information about the binding energy of a quark-antiquark system. Specifically, they showed that if the quark-antiquark interaction does not depend on quark flavor, the energy eigenvalues of the Schrödinger equation are monotonically decreasing functions of the reduced mass of the system. Since then, several authors [4–6] have used this result to obtain constraints on quark and hadron masses, usually in the form of inequalities. The Feynman–Hellman theorem has also been applied to some relativistic wave equations [7] and used with other assumptions to obtain inequalities among quark masses [8]. In the present work, we use the Feynman–Hellmann theorem as a new method to predict the masses of some as yet undiscovered vector mesons.

Let us consider a Hamiltonian \( H \), which depends on a parameter \( \lambda \). Then the Feynman–Hellmann theorem states that

\[
\frac{\partial E}{\partial \lambda} = \langle \frac{\partial H}{\partial \lambda} \rangle,
\]

where \( E \) is an eigenvalue of \( H \) and the expectation value is taken with respect to the normalized eigenfunction belonging to \( E \).

The Quigg and Rosner result is obtained by applying (1) to the nonrelativistic Hamiltonian \( H = p^2/(2\mu) + V \), where \( \mu \) is the reduced mass and \( V \) is a flavor-independent interaction. Then

\[
\frac{\partial E}{\partial \mu} = -\langle \frac{p^2}{(2\mu^2)} \rangle < 0,
\]

i.e., \( E \) decreases monotonically as \( \mu \) increases because \( p^2 \) is a positive definite operator. Of course, if \( V \) depends on \( \mu \) but \( \partial V/\partial \mu \leq 0 \), then \( \partial E/\partial \mu < 0 \) remains valid.

In the relativistic case, the two-body Hamiltonian depends on the quark and antiquark masses \( m_1 \) and \( m_2 \) rather than just on \( \mu \). As an example [7], we consider the two-body Salpeter Hamiltonian \( H(m_1, m_2) \), given by

\[
H(m_1, m_2) = \sum_i \left[ \left(p_i^2 + m_i^2\right)^{1/2} - m_i \right] + V(m_1, m_2),
\]

where we have let the interaction \( V \) depend explicitly on the \( m_i \) \((i = 1, 2)\). Taking the partial derivative with respect to \( m_i \) and using (1), we obtain

\[
\frac{\partial E}{\partial m_i} = \langle m_i/(p_i^2 + m_i^2)^{1/2} \rangle - 1 + \langle \partial V/\partial m_i \rangle.
\]

We can see from Eq. (4) that if

\[
\langle \partial V/\partial m_i \rangle \leq 0,
\]

then,

\[
\frac{\partial E}{\partial m_i} < 0.
\]

Although we have considered a specific Hamiltonian with relativistic kinematics, it is interesting to examine the consequences of requiring that (6) be true provided (5) holds. The next step is to note that if \( \mu \) increases and neither \( m_1 \) nor \( m_2 \) decreases, it follows from (6) that

\[
\frac{\partial E}{\partial \mu} < 0.
\]

Thus, \( E \) decreases when \( \mu \) increases, provided neither of the quark masses decreases.

Let us assume isospin invariance and consider mesons with constituent quarks \( q (= u \text{ or } d) \), \( s \), \( c \), and \( b \) whose masses satisfy the inequalities \( m_q < m_s < m_c < m_b \). Then, from the discussion above, we expect

\[
E_{bb} < E_{cb} < E_{cc} < E_{sc} < E_{ss} < E_{qs} < E_{qq},
\]

where \( E_{ij} \) is a particular eigenvalue for quark \( i \) and antiquark \( j \). (The inequalities remain true if we replace \( E_{cc} \) by \( E_{sb} \), \( E_{sc} \) by \( E_{qb} \), and \( E_{ss} \) by \( E_{qc} \).) Thus, (7) holds for the mesons with eigenenergies
as in (8), whereas this may not be the case when one of the two quark masses increases and the other decreases.

We now discuss when (5) might be valid for quarkonium states (for a review of quark potential models, see [9]). The interaction \( V \) can be written as \( V_1 + V_2 \), where \( V_1 \) is independent of quark flavors and \( V_2 \) depends on flavor. The term \( V_1 \) is the static quark-antiquark potential, which is commonly assumed to contain a Coulomb-like term, an approximately linear confining term, and a constant term, all independent of flavor.

The term \( V_2 \) is much more uncertain than \( V_1 \). In the Fermi–Breit approximation, \( V_2 \) contains both spin-dependent and spin-independent terms which are explicitly functions of flavor through the quark masses. However, most phenomenological treatments of quarkonia have not needed the Fermi–Breit spin-independent term [9], and we shall neglect it here. In states with zero orbital angular momentum, the most important spin-dependent term is the colormagnetic interaction. In a relativistic treatment [10], this term, which we denote by \( V_{cm} \), has the form

\[
V_{cm} = f(r)\sigma_i \cdot \sigma_j / (m_i m_j),
\]

where \( \sigma_i \) are Pauli spin matrices and \( f(r) \) is a positive definite operator which in some approximation is proportional to a smeared-out delta function.

If \( V = V_1 + V_{cm} \), we obtain

\[
\langle \partial V / \partial m_i \rangle = - \langle f(r) \rangle \langle \sigma_i \cdot \sigma_j \rangle / (m_i m_j).
\]

As is well known, \( \langle \sigma_i \cdot \sigma_j \rangle \) is 1 for vectors and -3 for pseudoscalars. Then, because \( f(r) \) is a positive definite operator, we see from (10) that the vector mesons satisfy (5) but the pseudoscalars do not. We therefore expect the energy eigenvalues of the vectors to be monotonically decreasing functions of \( \mu \). On the other hand, the pseudoscalars violate this condition for small \( m_i \), where the contribution from (10) is large.

In addition to \( V_{cm} \), other terms are expected to contribute to \( V_2 \). Among them are instantons [11], which are apparently important primarily in states with spin and orbital angular momentum zero (pseudoscalar mesons). Instanton contributions depend on flavor in a way which is more subtle than just through quark masses. For example, they are different in meson states with isospin zero and one, even if they contain the same quarks. Instantons also tend to mix the wave functions of certain mesons, like the \( \eta \) and \( \eta' \) (which contain both \( q\bar{q} \) and \( s\bar{s} \) in their wave functions, and perhaps some glueball admixture as well). Such states are unsuitable for our scheme, as, in order to compute the reduced mass of a system, we must know its quark content. This is another reason why in this paper we focus on vector mesons.

We now turn to the experimental data [12–14] on ground-state vector mesons to see how well our expectations are borne out. We cannot directly use the data on meson masses \( M_{ij} \) to compute the energy eigenvalues \( E_{ij} \) because the latter are given by

\[
E_{ij} = M_{ij} - m_i - m_j,
\]

and the quark masses \( m_i \) are a priori considered unknown. However, we can test our ideas with a selection of constituent quark masses which have appeared in the literature [4,10,15–22]. In Table I we give various sets of quark masses (the list is by no means complete). For comparison, we give in the first row the set of quark masses which we use in this work.

Using four different sets of input quark masses from Table I, we show in Fig. 1 how \( E \) varies as a function of \( \mu \) for the ground-state vector mesons. For completeness, we include mesons containing \( q\bar{b}, q\bar{c}, \) and \( s\bar{b} \) as well as \( sc, s\bar{s}, \) and \( c\bar{c} \) in order to see how \( E \) varies with \( \mu \) when \( m_1 \) increases and \( m_2 \) decreases. We include the \( \rho, K^*, \phi, D^*, D_s^*, B^*, B_s^*, J/\psi, \) and \( \Upsilon \). We omit the \( \omega \) meson in Fig. 1, because in our scheme it is degenerate with the \( \rho \), although it is actually 15 MeV heavier. (Instead of choosing the \( \rho \), we could choose the \( \omega \) or an average of both without appreciably affecting our results.) All the masses are taken from the Particle Data Group [12], except the mass of the \( B_s^* \),
which comes from two recent measurements [13,14] of the mass of the $B_s$ plus a measurement of $m(B^+_c) - m(B_s)$ (which needs confirmation) quoted in [12].

We see from Fig. 1 (a) and (b) that, with some choices of quark masses, $E$ appears indeed to be a monotonically decreasing function of $\mu$ for the vector mesons. In Fig. 1 (c) and (d), we see that with other choices of quark masses, $E$ does not behave smoothly as a function of $\mu$. Although the sets of quark masses used in (c) and (d) seem a priori as reasonable as those of sets (a) and (b), the Feynman-Hellmann theorem tells us that they are poor choices.

We next discuss how we obtain constraints on the quark masses and at the same time predict the masses of as yet undiscovered mesons. We assume that $E$ is a monotonically decreasing function of $\mu$. We also assume that $E(\mu)$ is smooth enough to lie on a curve containing only a few parameters, and we attempt to fit the data on ground-state vector mesons with simple three-parameter curves (the quark masses are four additional parameters). We vary these 7 parameters so as to obtain a best fit to the data. We have tried several different three-parameter curves in making our fits to get some idea of the errors involved in a particular choice. Specifically, we have used exponential, displaced-hyperbolic, and parabolic functions, and found that we obtain comparable fits to the data with all three. Furthermore, the meson energies are quite stable to our choice of functions.

We show in Fig. 2 our fit to the vector meson energies with an exponential curve. These quark masses are based on a somewhat arbitrary choice of 300 MeV for the mass of the $u$ and $d$ quarks. We can get comparable fits to the data for quark masses which differ from our choices by 100 MeV or more, but the mass differences are much more constrained.

The fit of the vector meson energies to the curve $E(\mu)$ allows us to predict the $B^*_c$ mass. We find

$$m(B^*_c) = 6320 \pm 10 \text{ MeV},$$

(12)

where we have estimated the theoretical error from the spread in values obtained using the above mentioned different functional forms for $E(\mu)$.

We next turn to excited meson states, confining ourselves to vector mesons for the reasons we have already discussed. The data are considerably poorer for the excited states. We use only the data shown in Table II, taken from the Particle Data Group [12]. Some of the quantum numbers of mesons in Table II are not known experimentally. We omit the excited $\rho$ and $\omega$ states, because they differ considerably in mass, and this may indicate appreciable mixing. Because the data for the excited vector meson states are sparse, it is adequate to use a linear fit to predict the masses of missing states, again using quark masses in the first row of Table I.

We show in Table III our predictions for the masses of vector meson excited states as well as the mass of the $B^*_c$ ground state. The errors are conservative estimates, based not only on the errors in the data but also on the error incurred by assuming a linear fit. In addition, the fact that the eigenenergy of the $D^*_s(2470)$, whose quantum numbers have not been measured, lies close to our linear fit lends support to the assumption that this state is a vector.

In conclusion, although the Feynman-Hellmann theorem gives us only an inequality $(\partial E/\partial \mu < 0)$, we are able to use it, in conjunction with the assumption that $E(\mu)$ behaves smoothly, to make quantitative predictions about the masses of as yet undiscovered mesons. In order to make these predictions, we have not assumed any specific functional form for the quark-antiquark interaction, but only some mild and general characteristics about its flavor dependence.

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TABLE I. Values of quark constituent masses in MeV for calculating meson energy eigenvalues from experimental values of their masses. We show in the first row the values of the quark masses used in this work and, for comparison, values used by some other authors in subsequent rows.

| Reference | $m_q$ | $m_s$ | $m_c$ | $m_b$ |
|-----------|-------|-------|-------|-------|
| This work | 300   | 440   | 1590  | 4920  |
| [4]       | 310   | 620   | 1910  | 5270  |
| [10]      | 220   | 419   | 1628  | 4977  |
| [15]      | 335   | 450   | 1840  | 5170  |
| [16]      | 350   | 500   | 1500  | 4700  |
| [17]      | 330   | 550   | 1650  | 4715  |
| [18]      | 150   | 366   | 1320  | 4749  |
| [19]      | 337   | 600   | 1870  | 5259  |
| [20]      | 336   | 510   | 1680  | 5000  |
| [21]      | 270   | 600   | 1700  | 5000  |
| [22]      | 300   | 500   | 1800  | 5200  |

TABLE II. Input masses of excited vector mesons from the tables of the Particle Data Group [12]. We do not include the $\psi(3770)$ because we believe it to be a state with orbital angular momentum 2.

| Name | Quark content | $n\, ^{2S+1}L_J$ | Mass (MeV) |
|------|---------------|-----------------|------------|
| $K^*$ | $\bar{q}s$    | $2\, ^3S_1$     | 1412 ± 12  |
| $\phi$ | $\bar{s}s$   | $2\, ^3S_1$     | 1680 ± 50  |
| $D^*$ | $\bar{q}c$    | $2\, ^3S_1$     | 2469 ± 10  |
| $\psi$ | $\bar{c}c$   | $2\, ^3S_1$     | 3686 ± 1   |
| $\Upsilon$ | $b\bar{b}$ | $2\, ^3S_1$     | 10023 ± 1  |
| $K^*$ | $\bar{q}s$    | $3\, ^3S_1$     | 1714 ± 20  |
| $\psi$ | $\bar{c}c$   | $3\, ^3S_1$     | 4040 ± 10  |
| $\Upsilon$ | $b\bar{b}$ | $3\, ^3S_1$     | 10355 ± 1  |

$^\dagger$Not in main meson table of the Particle Data Group, and needs confirmation. The spin and parity of this state have not been measured.
TABLE III. Predicted masses of as yet unobserved vector mesons.

| Name     | Quark content | n 2S+1L_j | Mass (MeV) |
|----------|---------------|------------|------------|
| $B_c^*$  | bc            | $1\ ^3S_1$ | 6320 ± 10  |
| $B^*$    | qb            | $2\ ^3S_1$ | 5830 ± 30  |
| $D^*_s$  | $\bar{s}c$    | $2\ ^3S_1$ | 2620 ± 30  |
| $B_s^*$  | $bs$          | $2\ ^3S_1$ | 5940 ± 30  |
| $B_c^*$  | $bc$          | $2\ ^3S_1$ | 6940 ± 30  |
| $\phi$   | $ss$          | $3\ ^3S_1$ | 1860 ± 30  |
| $D^*$    | $\bar{q}c$   | $3\ ^3S_1$ | 2860 ± 30  |
| $B^*$    | $q\bar{b}$   | $3\ ^3S_1$ | 6190 ± 30  |
| $D^*_s$  | $\bar{s}c$    | $3\ ^3S_1$ | 2980 ± 30  |
| $B_s^*$  | $s\bar{b}$   | $3\ ^3S_1$ | 6300 ± 30  |
| $B_c^*$  | $\bar{b}c$   | $3\ ^3S_1$ | 7290 ± 30  |

Figure captions

FIG. 1. Energy eigenvalues of vector mesons using four sets of quark masses from Table I: (a) from Ref. [4], (b) from Ref. [10], (c) from Ref. [16], and (d) from Ref. [17]. The experimental values of the meson masses are from Refs. [12–14].

FIG. 2. Energy eigenvalues of vector mesons using our quark masses from the first row of Table I and the experimental masses from Refs. [12–14]. The open circles are the known vectors and the solid circle is our prediction for the $B_c^*$. The solid line is a fit to the vector meson data with an exponential form, resulting in $E(\mu) = 754 \ exp(-\mu/1375) - 506$, where all constants are in MeV.
This figure "fig1-1.png" is available in "png" format from:

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