Research Article

Periodic Oscillating Dynamics for a Delayed Nicholson-Type Model with Harvesting Terms

Changjin Xu,1 Wei Zhang,2 and Peiluan Li3

1Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550004, China
2College of Economics and Statistics, Guangzhou University, Guangzhou 510006, China
3School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, China

Correspondence should be addressed to Wei Zhang; wzha19680317@126.com

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In this manuscript, a delayed Nicholson-type model with linear harvesting terms is investigated. Applying coincidence degree theory, we establish a sufficient condition which guarantees the existence of positive periodic solutions for the delayed Nicholson-type model. By constructing suitable Lyapunov functions, a new criterion for the uniqueness and global attractivity of the periodic solution of the Nicholson-type delay system is obtained. The derived results of this article are completely new and complement some previous investigations.

1. Introduction

In 1954, Nicholson [1] and later in 1980, Gurney et al. [2] established the following Nicholson’s blowfly model,

\[
\frac{dx}{dt} = -\delta x(t) + px(t-\tau) e^{-ax(t-\tau)}, \quad \delta, p, \tau, \quad a \in (0, \infty),
\]

(1)

to describe the population of the Australian sheep-blowfly Lucilia cuprina. In model (1), \(x(t)\) denotes the size of the population at time \(t\), \(p\) denotes the maximum per capita daily egg production rate, \(\delta\) stands for the per capita daily adult death rate, \((1/a)\) represents the size at which the blowfly population reproduces at its maximum rate, and \(\tau\) is the generation rate. Since then, model (1) and its revised versions have been extensively investigated. For example, So and Yu [3] analyzed the stability and uniform persistence of the discrete version of model (1), Kulenovic et al. [4] investigated the global attractivity of system (1), and Ding and Li [5] focused on the stability and bifurcation of the numerical discretization model of (1). For more details, we refer the reader to [6–21].

It is well known that oscillatory behavior of population densities is one characteristic phenomenon of the population [22]. Thus, there have been extensive results on the existence of periodic solutions for Nicholson’s blowfly models. We refer the reader to [7, 11, 21–29]. In recent years, Berezansky et al. [30] investigated the global dynamics of the following Nicholson-type delay model:

\[
\begin{align*}
\frac{dx_1}{dt} &= -a_1x_1(t) + b_1x_2(t) + c_1x_1(t-\tau)e^{-x_1(t-\tau)}, \\
\frac{dx_2}{dt} &= -a_2x_2(t) + b_2x_1(t) + c_2x_2(t-\tau)e^{-x_2(t-\tau)},
\end{align*}
\]

(2)

with the initial conditions

\[
x_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \quad \varphi_i(0) > 0,
\]

(3)

where \(\varphi_i(s) \in C([-\tau, 0], [0, +\infty))\) and \(a_i, b_i, c_i,\) and \(\tau\) are nonnegative constants, \(i = 1, 2.\) Taking into account the effect of periodically varying environment, Wang et al. [31] proposed the following nonautonomous Nicholson-type delay model:
and focused on the existence and exponential convergence of positive almost periodic solutions for (4). Here, $a_i, \beta_i, c_{ij}, \tau_{ij}, \sigma_i, H_i: R \rightarrow (0, +\infty)$ are almost periodic functions, and $i = 1, 2$ and $j = 1, 2, \ldots, m$. In 2011, Liu [22] studied the existence and uniqueness of positive periodic solutions of (4). In 2010, assuming that a harvesting function is a function of the delayed estimate of the true population, Berezansky et al. [32] established the following Nicholson-type delay system with a linear harvesting term:

$$
\frac{dx}{dt} = -\delta x(t) + \rho x(t - \tau)x(t - \sigma) - H x(t - \sigma), \delta, \rho, \tau,
$$

$$
a \in (0, \infty),
$$

where $\delta, \rho, a, H, \sigma \in (0, +\infty)$, $H x(t - \sigma)$ denotes a linear harvesting term, $x(t)$ represents the size of the population at time $t$, $\rho$ represents the maximum per capita daily egg production rate, $\delta$ represents the per capita daily adult death rate, $(1/a)$ represents the size at which the blowfly population reproduces at its maximum rate, and $\tau$ is the generation rate. Berezansky et al. [32] proposed an open problem: how about the dynamical behaviors of Nicholson’s blowfly model with a linear harvesting term?

Inspired by Berezansky et al. [30], Wang et al. [31], and Berezansky et al. [32], Liu and Meng [33] proposed the following Nicholson’s blowfly model with linear harvesting terms:

$$
\frac{dx}{dt} = -\alpha_i(t)x_i(t) + \beta_i(t)x_{i-1}(t) + \sum_{j=1}^{m} c_{ij}(t)x_j(t - \tau_{ij})e^{-\gamma_{ij}(t)x_j(t - \tau_{ij})},
\frac{dx}{dt} = -\alpha_i(t)x_i(t) + \beta_i(t)x_{i-1}(t) + \sum_{j=1}^{m} c_{ij}(t)x_j(t - \tau_{ij})e^{-\gamma_{ij}(t)x_j(t - \tau_{ij})},
$$

where $\delta, \rho, a, H, \sigma \in (0, +\infty)$, $H x(t - \sigma)$ denotes a linear harvesting term, $x(t)$ represents the size of the population at time $t$, $\rho$ represents the maximum per capita daily egg production rate, $\delta$ represents the per capita daily adult death rate, $(1/a)$ represents the size at which the blowfly population reproduces at its maximum rate, and $\tau$ is the generation rate. Berezansky et al. [32] proposed an open problem: how about the dynamical behaviors of Nicholson’s blowfly model with a linear harvesting term?

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$$
\frac{dx}{dt} = -\alpha_i(t)x_i(t) + \beta_i(t)x_{i-1}(t) + \sum_{j=1}^{m} c_{ij}(t)x_j(t - \tau_{ij})e^{-\gamma_{ij}(t)x_j(t - \tau_{ij})},
\frac{dx}{dt} = -\alpha_i(t)x_i(t) + \beta_i(t)x_{i-1}(t) + \sum_{j=1}^{m} c_{ij}(t)x_j(t - \tau_{ij})e^{-\gamma_{ij}(t)x_j(t - \tau_{ij})},
$$

where $\alpha_i, \beta_i, c_{ij}, \tau_{ij}, \sigma_i, H_i: R \rightarrow (0, +\infty)$ are almost periodic functions and $i = 1, 2$ and $j = 1, 2, \ldots, m$. Liu and Meng [33] established some sufficient conditions to check the existence, uniqueness, and local exponential convergence of the positive almost periodic solution of (6).

Here, we would like to point out that periodic phenomenon plays an important role in characterizing the dynamical behavior of Nicholson’s blowfly models. Thus, it is worthwhile to investigate the periodic solution of Nicholson’s blowfly models. Up to now, there is no manuscript which handles this aspect on the periodic solution of model (6).

The principle objective of this manuscript is to find a set of sufficient conditions that guarantee the existence of at least a positive periodic solution for model (6) and by constructing a suitable Lyapunov function to investigate the stability of periodic solutions of model (6).

Let
The manuscript is planned as follows. In Section 2, we state the necessary preliminary results. We then establish, in Section 3, some simple criteria for the existence of positive periodic solutions of model (6) by coincidence degree theory [34]. The uniqueness and global attractivity of the positive periodic solution are displayed in Section 4. An example is given to illustrate our key results in Section 5.

2. Preliminaries

In this section, some related basic knowledge is displayed [34, 35]. Assume that $X, Y$ are normed vector spaces, $L: \text{Dom}L \subset X \rightarrow Y$ is a linear mapping, and $N: X \rightarrow X$ is a continuous mapping. We call the mapping $L$ a Fredholm mapping of index zero if

$$\dim \ker L = \text{codim} \text{Im} L < \infty$$

and $\text{Im} L$ is closed in $Y$. If $L$ is a Fredholm mapping of index zero and $\exists$ continuous projectors $P: X \rightarrow X$ and $Q: Y \rightarrow Y$ which satisfy $\text{Im} P = \ker L$, $\text{Im} Q = \ker L = (I - P)X$, then $L|\text{Dom} L \cap \ker P: (I - P)X \rightarrow \text{Im} L$ is invertible [34, 35]. Let $K_\varphi$ denote the inverse of $L|\text{Dom} L \cap \ker P$. If $\Omega$ is an open bounded subset of $X$, we call the mapping $N: X \rightarrow \overline{\Omega}$ compact on $\overline{\Omega}$ if $Q N(\overline{\Omega})$ is bounded and $K_\varphi (I - Q) N: \overline{\Omega} \rightarrow X$ is compact. Since $\text{Im} Q = \ker L$, $\exists$ isomorphisms $\mathcal{J}: \text{Im} Q \rightarrow \ker L$ [34, 35].

Lemma 1 (see [34]). Suppose that $L$ is a Fredholm mapping of index zero and $N$ is $L$-compact on $\overline{\Omega}$. If

(i) $\forall \lambda \in (0, 1)$, all solutions $x$ of $Lx = \lambda Nx$ satisfy the following condition: $x \notin \partial \Omega$,

(ii) $\exists \lambda > 0, \forall x \in \ker L \cap \partial \Omega$, and $\deg [\mathcal{J} \cap N, \Omega \cap \ker L, 0] \neq 0$, then $Lx = Nx$ possesses at least one solution, which stays in $\text{Dom} L \cap \overline{\Omega}$

3. Existence of Positive Periodic Solutions

Theorem 2. If (A1)–(A3) are satisfied, then system (6) has at least one positive $\omega$-periodic solution.

Proof. Based on the practical significance of model (6), here we only discuss the positive solutions of model (6) $\forall t \geq 0$. Set

$$u_1 (t) = \ln[x_1 (t)],$$

$$u_2 (t) = \ln[x_2 (t)],$$

In view of (6) and (8), one has

$$\frac{du_1 (t)}{dt} = -a_1 (t) + b_1 (t) e^{u_2 (t) - u_1 (t)}$$

$$+ \sum_{j=1}^{m} c_{1j} (t) e^{\omega_1 (t - \tau_{1j} (t)) - u_1 (t - \tau_{1j} (t)) e^{\omega_2 (t - \tau_{1j} (t))}}$$

$$- \mathcal{H}_1 (t) e^{\omega_1 (t - \sigma_1 (t)) - u_1 (t)}.$$  \hfill (9)

$$\frac{du_2 (t)}{dt} = -a_2 (t) + b_2 (t) e^{u_1 (t) - u_2 (t)}$$

$$+ \sum_{j=1}^{m} c_{2j} (t) e^{\omega_2 (t - \tau_{2j} (t)) - u_2 (t - \tau_{2j} (t)) e^{\omega_2 (t - \tau_{2j} (t))}}$$

$$- \mathcal{H}_2 (t) e^{\omega_2 (t - \sigma_2 (t)) - u_2 (t)}.$$  \hfill (10)

Define $\mathcal{P}$ and $\mathcal{Q}$ as follows:

$$\mathcal{P} u = \frac{1}{\omega} \int_{0}^{\omega} u (t) dt,$$

$$\mathcal{Q} u = \frac{1}{\omega} \int_{0}^{\omega} u (t) dt, \quad u \in X, u \in X.$$

Hence, $\ker L = \{ u \in X | u = h \in R^2 \}$ and $\text{Im} L = \{ u \in L | \int_{0}^{\omega} u (t) dt = 0 \}$ are closed in $X$, and $\dim (\ker L) = 2 = \text{codim} (\text{Im} L)$. Then,
where

\[
\begin{align*}
\mathcal{F}_1(s) &= -\alpha_1(s) + \beta_1(t)u_2(s) - u_1(s) \\
&\quad + \sum_{j=1}^{m} c_{1j}(s)u_1\left(s - t_{1j}(s)\right) - u_1(s) - \gamma_{1j}(s)e^{\omega_1(t - t_{1j}(s))}
\end{align*}
\]

\[
\mathcal{F}_2(s) = -\alpha_2(s) + \beta_2(s)u_1(s) - u_2(s) + \sum_{j=1}^{m} c_{2j}(s)u_2\left(s - t_{2j}(s)\right) - u_2(s) - \gamma_{2j}(s)e^{\omega_2(t - t_{2j}(s))}
\]

Clearly, \(\mathcal{Q}, \mathcal{N}\) and \(K_{\mathcal{F}}(I - \mathcal{Q})\mathcal{N}\) are continuous. We can easily check that \(K_{\mathcal{F}}(I - \mathcal{Q})\mathcal{N}(\Omega)\) is compact \(\forall \Omega \subset X\). In addition, \(\mathcal{Q}(\Omega)\) is bounded. So, \(\mathcal{N}\) is \(\mathcal{L}\)-compact on \(\Omega \subset X\).

In view of \(\mathcal{L}u = \lambda\mathcal{N}u, \lambda \in (0, 1)\), one has

\[
\begin{align*}
\frac{du_1(t)}{dt} &= \lambda \left[-\alpha_1(t) + \beta_1(t)e^{\omega_1(t - t_{11})} - u_1(t)\right] \\
&\quad + \sum_{j=1}^{m} c_{1j}(t)e^{\omega_1\left(t - t_{1j}(t)\right)} - u_1(t) - \gamma_{1j}(t)e^{\omega_1(t - t_{1j}(t))}
\end{align*}
\]

\[
\begin{align*}
\frac{du_2(t)}{dt} &= \lambda \left[-\alpha_2(t) + \beta_2(t)e^{\omega_2(t - t_{21})} - u_2(t)\right] \\
&\quad + \sum_{j=1}^{m} c_{2j}(t)e^{\omega_2\left(t - t_{2j}(t)\right)} - u_2(t) - \gamma_{2j}(t)e^{\omega_2(t - t_{2j}(t))}
\end{align*}
\]

Suppose that \(u(t) = (u_1(t), u_2(t))^T \in \mathcal{X}\) is an arbitrary solution of system (15) for a certain \(\lambda \in (0, 1)\); according to (15), one has
Because \((x_1(t), x_2(t))^T \in \mathcal{X}\), \(\exists \theta_1, \epsilon_i \in [0, \omega]\) which satisfies
\[
\begin{align*}
&u_i(\epsilon_i) = \max_{\epsilon \in [0, \omega]} u_i(t), \\
&u_i(\epsilon_i) = \min_{\epsilon \in [0, \omega]} u_i(t), \\
&u'_i(\epsilon_i) = u'_i(\theta_1) = 0, \quad i = 1, 2.
\end{align*}
\]  
(21)

Let
\[
u_i(\epsilon_i) = \max_{j=1, 2, \ldots, m} u_i(\epsilon_i - \tau_{ij}(\epsilon_i)).
\]  
(22)

Because \(u_i(\epsilon_i - \tau_{ij}(\epsilon_i)) - u_i(\epsilon_i) \geq 0, \ j = 1, 2, \ldots, m\), by (9) and (22), one has
\[
\alpha_i(\epsilon_i) + \mathcal{H}_1(\epsilon_i)e^{\alpha_i(\epsilon_i - \sigma_i(\epsilon_i)) - u_i(\epsilon_i)}
\]
\[
= \beta_1(\epsilon_i)e^{\alpha_i(\epsilon_i) - u_i(\epsilon_i)}
\]
\[
+ \sum_{j=1}^{m} c_{ij}(\epsilon_i)e^{\alpha_i(\epsilon_i - \tau_{ij}(\epsilon_i)) - u_i(\epsilon_i) - \gamma_{ij}(\epsilon_i)e^{\gamma_{ij}(\epsilon_i)}}
\]
\[
\geq e^{-\gamma_{ij}(\epsilon_i)}\sum_{j=1}^{m} c_{ij}(\epsilon_i),
\]  
(23)

which leads to
\[
\alpha_i^{M} + \mathcal{H}_1^{M}e^{\alpha_i(\epsilon_i - \sigma_i(\epsilon_i)) - u_i(\epsilon_i)} \geq e^{-\gamma_{ij}(\epsilon_i)}\sum_{j=1}^{m} c_{ij}(\epsilon_i).
\]  
(24)

By \((\mathcal{A}2)\), we have
\[
u_i(\epsilon_i - \tau_{ij}(\epsilon_i)) \geq \ln \left[ \frac{1}{M} \prod_{j=1}^{m} c_{ij} \right] \ln \left[ \frac{1}{M} \prod_{j=1}^{m} c_{ij} \right].
\]  
(25)

Set \(\epsilon_i - \tau_{ij}(\epsilon_i) = m\omega + \sigma\), where \(\sigma \in [0, \omega]\) is an integer. By (19) and (21), one has
\[
u_i(t) \geq u_i(\sigma) - \int_{0}^{\omega} |\dot{u}_i(\tau)| d\tau \geq \ln \left[ \frac{1}{M} \prod_{j=1}^{m} c_{ij} \right] \ln \left[ \frac{1}{M} \prod_{j=1}^{m} c_{ij} \right] - 2\pi \omega = \omega_1,
\]  
(26)

\[\forall t \in R.\] In a similar way, it follows from (20) and (21) that
\[
u_i(t) \geq \ln \left[ \frac{1}{M} \prod_{j=1}^{m} c_{ij} \right] \ln \left[ \frac{1}{M} \prod_{j=1}^{m} c_{ij} \right] - 2\pi \omega = \omega_2.
\]  
(27)

By (9) and (21), one gets
\[
\alpha_i(\epsilon_i) + \mathcal{H}_1(\epsilon_i)e^{\alpha_i(\epsilon_i - \sigma_i(\epsilon_i)) - u_i(\epsilon_i)} = \beta_1(\epsilon_i)e^{\alpha_i(\epsilon_i) - u_i(\epsilon_i)}
\]
\[
+ \sum_{j=1}^{m} c_{ij}(\epsilon_i)e^{\alpha_i(\epsilon_i - \tau_{ij}(\epsilon_i)) - u_i(\epsilon_i) - \gamma_{ij}(\epsilon_i)e^{\gamma_{ij}(\epsilon_i)}}
\]
\[
= \beta_1(\epsilon_i)e^{\alpha_i(\epsilon_i) - u_i(\epsilon_i)}
\]
\[
+ \sum_{j=1}^{m} c_{ij}(\epsilon_i)e^{\gamma_{ij}(\epsilon_i)}
\]  
(28)

Since \(\sup_{\nu \in \nu}(\nu e^{\nu}) = (1/e)\),
\[
\alpha_i(\epsilon_1) \leq \alpha_i(\epsilon_1) + \mathcal{H}_1(\epsilon_1)e^{\alpha_i(\epsilon_1 - \tau_1(\epsilon_1)) - u_i(\epsilon_1)}
\]
\[
\leq \beta_1(\epsilon_1)e^{\alpha_i(\epsilon_1 - \tau_1(\epsilon_1)) - u_i(\epsilon_1)} + \sum_{j=1}^{m} c_{ij}(\epsilon_1) \frac{1}{e} e^{\gamma_{ij}(\epsilon_1)},
\]  
(29)

which leads to
\[
\alpha_i(\epsilon_1) - \beta_1(\epsilon_1)e^{\alpha_i(\epsilon_1 - \tau_1(\epsilon_1)) - u_i(\epsilon_1)} \leq \sum_{j=1}^{m} c_{ij}(\epsilon_1)
\]  
(30)

Hence,
\[
\alpha_i^{L}(\epsilon_1) - \beta_1^{L}e^{\alpha_i^{L}(\epsilon_1 - \tau_1(\epsilon_1)) - u_i(\epsilon_1)} \leq \sum_{j=1}^{m} c_{ij}(\epsilon_1)
\]  
(31)

In a similar way, one also gets
\[
\alpha_i^{2}(\epsilon_2) - \beta_{2}^{M}e^{\alpha_i^{2}(\epsilon_2 - \tau_2(\epsilon_2)) - u_i(\epsilon_2)} \leq \sum_{j=1}^{m} c_{ij}(\epsilon_2)
\]  
(32)

In view of (31) and (32), one has
\[
\begin{bmatrix}
\epsilon_{i_1}(\epsilon_1) \\
\epsilon_{i_2}(\epsilon_2)
\end{bmatrix} = \begin{bmatrix}
\alpha_i^{L} - \beta_i^{L} \\
\alpha_i^{2} - \beta_i^{2}
\end{bmatrix}^{-1} \begin{bmatrix}
\alpha_i^{L} - \beta_i^{L} \\
\alpha_i^{2} - \beta_i^{2}
\end{bmatrix} \begin{bmatrix}
\epsilon_{i_1}(\epsilon_1) \\
\epsilon_{i_2}(\epsilon_2)
\end{bmatrix}
\]  
(33)

Then, one has
\[
u_i(t) \leq u_i(\epsilon_i) \leq \theta_1
\]  
(34)

where
\[ \theta_1^* = \ln \left[ \frac{a_1}{a_1 - \beta_1^2 \rho_{12}^2} \sum_{j=1}^{M_2} c_{1j} \frac{e^{\alpha_1 l} - e^{-\beta_1^2 \rho_{12}^2}}{l} \right], \]
\[ \theta_2^* = \ln \left[ \frac{a_2}{a_2 - \beta_2^2 \rho_{12}^2} \sum_{j=1}^{M_2} c_{2j} \frac{e^{\alpha_2 l} - e^{-\beta_2^2 \rho_{12}^2}}{l} \right]. \]  
(35)

Let \( M_1 = \max \{|\theta_1^*|, |\theta_2^*|\} \) and \( M_2 = \max \{|\theta_2^*|, |\theta_2^*|\} \). Then, \( M_i (i = 1, 2) \) are independent of \( \lambda \). Consider the equation \( \mathcal{Q} \mathcal{N} \mathbf{u} = 0 \). If \( \mathbf{u} \in \mathbb{R}^2 \), then \( \exists u_j, (j = 1, 2, \ldots, m) \) and \( v_j, (j = 1, 2, \ldots, m) \) which satisfy

\[
\mathcal{Q} \mathcal{N} \mathbf{u} = \left[ \begin{array}{c}
\int_0^\infty f_1(t) \, dt \\
\int_0^\infty f_2(t) \, dt
\end{array} \right] = \left[ \begin{array}{c}
-\alpha_1 + \beta_1 e^{u_1 - u_1} + \sum_{j=1}^{M_1} c_{1j} e^{-y_j (v_1)} e^{v_1} - \mathcal{H}_1 \\
-\alpha_2 + \beta_2 e^{u_2 - u_2} + \sum_{j=1}^{M_2} c_{2j} e^{-y_j (v_2)} e^{v_2} - \mathcal{H}_2
\end{array} \right] = 0,
\]
(36)

where

\[
f_1(t) = -\alpha_1 (t) + \beta_1 (t) e^{u_1(t) - u_1 (t)} + \sum_{j=1}^{M_1} c_{1j} (t) e^{u_1(t) - y_j (v_1)} e^{v_1},
\]
\[
f_2(t) = -\alpha_2 (t) + \beta_2 (t) e^{u_2(t) - u_2 (t)} + \sum_{j=1}^{M_2} c_{2j} (t) e^{u_2(t) - y_j (v_2)} e^{v_2},
\]
(37)

Let \( \mathcal{M}^* > 0 \) be large enough. If \( \mathbf{u}^* = (u_1^*, u_2^*)^T \) is a solution of equation (36), then \( \sum_{i=1}^{2} |u_i^*| < \mathcal{M}^* \). We define \( \phi : \text{Dom} \mathcal{X} \times [0, 1] \rightarrow \mathcal{X} \) by

\[
\phi(u_1, u_2, \lambda) = \left[ \begin{array}{c}
-\bar{\alpha}_1 + \frac{\sum_{j=1}^{M_1} c_{1j} e^{-y_j (v_1)} e^{v_1}}{\mathcal{H}_1} \\
-\bar{\alpha}_2 + \frac{\sum_{j=1}^{M_2} c_{2j} e^{-y_j (v_2)} e^{v_2}}{\mathcal{H}_2} + \lambda \left[ \frac{\beta_1 e^{u_1 - u_1} - \mathcal{H}_1}{\beta_2 e^{u_2 - u_2} - \mathcal{H}_2} \right]
\end{array} \right].
\]
(38)

Obviously, \( \phi(u_1, u_2, 1) = \mathcal{Q} \mathcal{N} \mathbf{u} \). Similar to the analysis of model (15), if \( (u_1^*, u_2^*)^T \) is a solution of \( \phi(u_1, u_2, 0) = 0 \), then \( \exists \) two constants \( \mathcal{M}_1^*, \mathcal{M}_2^* > 0 \) which satisfy

\[
|u_1| < \mathcal{M}_1^*, |u_2| < \mathcal{M}_2^*,
\]
(39)

and \( \mathcal{M}_i^* (i = 1, 2) \) are independent of \( \lambda \). Set \( \mathcal{M}_0 = \sum_{i=1}^{2} M_i^* - M_0 + M^* + \mathcal{M}_0^* \), where \( \mathcal{M}_0 = \sum_{i=1}^{2} M_i \). Let \( \Omega = \left\{ (u_1, u_2)^T \in \mathbb{R}^2 : \sum_{i=1}^{2} |u_i^*| < \mathcal{M}_0^* \right\} \).

(40)

Next, we will check that all assumptions of Lemma 1 hold true.

By \( \mathcal{M} > \mathcal{M}_0 = \sum_{i=1}^{2} M_i \), one knows that if \( (u_1, u_2)^T \in \partial \Omega \cap \ker \mathcal{L} \), then \( \mathcal{L} \mathbf{u} \neq \lambda \mathcal{N} \mathbf{u}, \lambda \in (0, 1) \). So, assumption (a) of Lemma 1 holds true. By \( \mathcal{M} > \mathcal{M}_0 \), one knows that if \( (u_1, u_2)^T \in \partial \Omega \cap \ker \mathcal{L} \), then \( \phi(u_1, u_2, \lambda) \neq \lambda \), \( \lambda \in (0, 1) \). Then, \( \phi(u_1, u_2, 1) \neq 0 \), i.e., \( \mathcal{Q} \mathcal{N} \mathbf{u} \neq 0 \). Obviously, \( \mathcal{Q} \mathcal{N} \mathbf{u} = \phi(u_1, u_2, \lambda) = 0 \) has a unique solution \( (u_1^*, u_2^*)^T \). Set \( \mathcal{F} = \mathcal{L} \mathbf{u} - \lambda \mathcal{N} \mathbf{u} \). We define

\[
\phi(u_1, u_2, 0) = \left[ \begin{array}{c}
\sum_{j=1}^{M_1} C_{1j} e^{-y_j (v_1)} y_j (v_1) e^{v_1} \\
\sum_{j=1}^{M_2} C_{2j} e^{-y_j (v_2)} y_j (v_2) e^{v_2}
\end{array} \right] = \left[ \begin{array}{c}
\sigma_1^* \\
\sigma_2^*
\end{array} \right],
\]
(42)

where \( \sigma_1^* = -\sum_{j=1}^{M_1} C_{1j} e^{-y_j (v_1)} y_j (v_1) e^{v_1} \), \( \sigma_2^* = -\sum_{j=1}^{M_2} C_{2j} e^{-y_j (v_2)} y_j (v_2) e^{v_2} \), and \( (u_1^*, u_2^*)^T \) is a unique solution of \( \mathcal{Q} \mathcal{N} \mathbf{u} = \phi(u_1, u_2, 0) = 0 \). Then,

\[
\text{deg} \left\{ \mathcal{F} \mathcal{N} \mathbf{u} (u_1, u_2)^T ; \Omega \cap \ker \mathcal{L} ; 0 \right\} = \text{deg} \left\{ \phi(u_1, u_2, 0) ; \Omega \cap \ker \mathcal{L} ; 0 \right\} = \text{sign} \left\{ \det \left[ \begin{array}{cc}
\sigma_1^* & 0 \\
0 & \sigma_2^*
\end{array} \right] \right\},
\]
(43)

Then, assumption (b) of Lemma 1 is true. So, \( \mathcal{L} \mathbf{u} \equiv \mathcal{N} \mathbf{u} \) has at least one solution \( (u_1(t), u_2(t))^T \) in \( \text{Dom} \mathcal{X} \cap \Omega \). Thus, \( (x_1(t), x_2(t))^T = (e^{u_1(t)}, e^{u_2(t)})^T \) is an \( \omega \)-positive periodic solution of model (6). The proof ends.

### 4. Stability Behavior of Periodic Solutions

Assume that the varying delays become constants, i.e., \( \sigma_1 (t) = \sigma_1, \sigma_2 (t) = \sigma_2, \tau_{1j} (t) = \tau_{1j}, \) and \( \tau_{2j} (t) = \tau_{2j} (j = 1, 2, \ldots, m) \).

**Definition 1** (see [35]). A bounded positive solution \( (x_1^*(t), x_2^*(t))^T \) of model (6) is said to be globally
asymptotically stable if ∀ positive bounded solutions \((x_1(t), x_2(t))^T\) of model (6), the following equality holds:

\[
\lim_{t \to +\infty} \left[ \sum_{i=1}^{2} x_i(t) - x'_i(t) \right] = 0. \tag{44}
\]

**Definition 2** (see [35, 36]). Assume that \(g \in \mathbb{R}\) and \(f\) is a nonnegative function defined on \([g, +\infty)\), integrable on \([g, +\infty)\), and uniformly continuous on \([g, +\infty)\); then, \(\lim_{t \to +\infty} f(t) = 0\).

**Theorem 1.** If \((\mathscr{A}1)-(\mathscr{A}4)\) hold, then model (6) has a unique positive \(\omega\)-periodic solution \((x^*_1(t), x^*_2(t))^T\), and this solution is global attractivity.

**Proof.** According to Section 3, we know that there exists \(\mathcal{T} > 0\) such that

\[
ed_i < x^*_1(t) \leq e^{\delta_i}, \tag{45}
\]

\[
ed_i < x^*_2(t) \leq e^{\delta_i}, t > \mathcal{T}.
\]

Define

\[
\mathscr{T}_1(t) = \sum_{i=1}^{2} \left| x^*_i(t) - x_i(t) \right|, \tag{46}
\]

Then, \(\forall t \geq T\),

\[
\mathscr{D} \mathcal{T}_1(t) = \sum_{i=1}^{2} \left[ (x^*_i(t) - x_i(t)) \mathrm{sgn}(x^*_i(t) - x_i(t)) + \sum_{j=1}^{3} c_{ij}(t) \left( x_i^*(t) - x_j(t) \right) e^{-\gamma_{ij}(t)} \right]
\]

\[
= \sum_{j=1}^{3} \left[ c_{ij}(t) \left( x_i^*(t - \tau_{ij}) - x_j(t - \tau_{ij}) \right) e^{-\gamma_{ij}(t)} \right]
\]

\[
- \mathcal{H}_1(t) \left( x^*_i(t - \tau_1(t)) - x_i(t - \tau_1(t)) \right) + \sum_{j=1}^{3} \left[ c_{ij}(t) \left( x_i^*(t - \tau_{ij}) - x_j(t - \tau_{ij}) \right) e^{-\gamma_{ij}(t)} \right]
\]

\[
- \mathcal{H}_2(t) \left( x^*_i(t - \tau_2(t)) - x_i(t - \tau_2(t)) \right) \leq - \alpha_1 x^*_1(t) + \beta_1 x^*_2(t) - x_1(t)
\]

\[
+ \sum_{j=1}^{3} \left( c_{ij} \left( x_i^*(t - \tau_{ij}) - x_j(t - \tau_{ij}) \right) e^{-\gamma_{ij}(t)} \right)
\]

\[
+ \mathcal{H}_1 \left[ x^*_i(t - \tau_1(t)) - x_i(t - \tau_1(t)) \right] - \alpha_2 x^*_2(t) + \beta_2 x^*_1(t) - x_2(t)
\]

\[
+ \sum_{j=1}^{3} \left( c_{ij} \left( x_i^*(t - \tau_{ij}) - x_j(t - \tau_{ij}) \right) e^{-\gamma_{ij}(t)} \right)
\]

\[
+ \mathcal{H}_2 \left[ x^*_i(t - \tau_2(t)) - x_i(t - \tau_2(t)) \right].
\]
According to [38], define \( \forall t \in [\tau(t)], \| x(t) - x_i(t) \| = \sup_{t \geq 0} \| x(t) - x_i(t) \| \) for all \( t \geq \tau \), which satisfy

\[
\eta_i \leq \alpha_i - \beta_i - \sum_{j=1}^{m} \frac{c_{ij}}{e^j} - \mathcal{H}^j_1 - \mathcal{H}^j_2.
\]

By (53), one has

\[
\mathcal{V}''(t) + \sum_{i=1}^{2} \int_{\mathcal{T}} \eta_i(t) | x_i''(t) - x_i(t) | ds \leq \mathcal{V}'(\mathcal{T}^*) + \mathcal{V}'(\mathcal{T}^*)
\]

By (53) and (54), one gets

\[
\sum_{i=1}^{2} \int_{\mathcal{T}} \eta_i(t) | x_i''(t) - x_i(t) | ds \leq \mathcal{V}(\mathcal{T}^*) < \infty, \quad \text{for } t \geq \mathcal{T}^*.
\]

Remark 1. Zhou [17] considered the positive periodic solution of the Nicholson-type delay model which is a special form of (6), Liu [22] investigated the existence and uniqueness of positive periodic solutions of the Nicholson-type delay model without linear harvesting terms, and Liu and Meng [33] studied the positive almost periodic solution of system (6). All the results in [17, 22, 33] and the references therein cannot be applicable to prove that system (6) has a unique positive \( \omega \)-periodic solution which is globally attractive. This implies that the results of this article are new and that they complement earlier investigations.

Remark 2. In [38], the authors dealt with the periodic solution of the ratio-dependent food-chain system with delays by applying coincidence degree theory. In this paper, we investigate the periodic solution of the continuous delayed Nicholson-type model with harvesting terms, and some inequality techniques are different from those in [38]. In [33, 39], the authors dealt with almost periodic solutions for Nicholson’s blowfly model. They did not involve the periodic...
solution of the delayed Nicholson-type model. So, our works supplement the previous publications.

5. Software Simulations

Give the following model:

\[
\begin{align*}
\frac{dx_1}{dt} &= -e^{-3}(0.1 + 0.3 \sin^2 10t)x_1(t) + e^{-4}(0.2 + 0.1 \cos^2 10t)x_2(t) \\
&\quad + e^{-2}(0.1 + 0.2 \cos^2 10t)x_1(t - 0.003)e^{-0.02x_1(t - 0.003)} \\
&\quad + e^{-2}(0.2 + 0.1 \sin^2 10t)x_1(t - 0.002)e^{-0.01x_1(t - 0.002)} \\
&\quad - e^{-3}(0.2 + 0.1 \cos^2 10t)x_1(t - 0.004), \\

\frac{dx_2}{dt} &= -e^{-3}(0.1 + 0.2 \cos^2 10t)x_2(t) + e^{-4}(0.2 + 0.2 \sin^2 10t)x_1(t) \\
&\quad + e^{-2}(0.1 + 0.3 \sin^2 10t)x_1(t - 0.002)e^{-0.01x_1(t - 0.002)} \\
&\quad + e^{-2}(0.1 + 0.2 \sin^2 10t)x_1(t - 0.001)e^{-0.02x_1(t - 0.002)} \\
&\quad - e^{-3}(0.3 + 0.2 \sin^2 10t)x_2(t - 0.004).
\end{align*}
\]
Then,
\[\begin{align*}
\alpha_1(t) &= e^{-\frac{3}{2}} (0.1 + 0.3 \sin^2 10t), \\
\alpha_2(t) &= e^{-\frac{3}{2}} (0.1 + 0.2 \cos^2 10t), \\
\beta_1(t) &= e^{-\frac{4}{3}} (0.2 + 0.1 \cos^2 10t), \\
\beta_2(t) &= e^{-\frac{4}{3}} (0.2 + 0.2 \sin^2 10t), \\
c_{11}(t) &= e^{-\frac{3}{2}} (0.1 + 0.2 \cos^2 10t), \\
c_{12}(t) &= e^{-\frac{2}{3}} (0.2 + 0.1 \sin^2 10t), \\
c_{21}(t) &= e^{-\frac{2}{3}} (0.1 + 0.3 \sin^2 10t), \\
c_{22}(t) &= e^{-\frac{3}{2}} (0.1 + 0.2 \sin^2 10t), \\
\mathcal{H}_1(t) &= e^{-\frac{3}{2}} (0.2 + 0.1 \cos^2 10t), \\
\mathcal{H}_2(t) &= e^{-\frac{3}{2}} (0.3 + 0.2 \sin^2 10t), \\
\gamma_{11}(t) &= 0.02, \\
\gamma_{12}(t) &= 0.01, \\
\gamma_{21}(t) &= 0.01, \\
\gamma_{22}(t) &= 0.02, \\
\sigma_1(t) &= 0.004, \\
\sigma_2(t) &= 0.004, \\
\tau_{11}(t) &= 0.003, \\
\tau_{12}(t) &= 0.002, \\
\tau_{21}(t) &= 0.002, \\
\tau_{22}(t) &= 0.001.
\end{align*}\]

By direct computation, one has
\[\begin{align*}
\alpha_1^M &= e^{-\frac{3}{2}} 0.4, \\
\alpha_1^L &= e^{-\frac{4}{3}} 0.1, \\
\alpha_2^M &= e^{-\frac{3}{2}} 0.3, \\
\alpha_2^L &= e^{-\frac{4}{3}} 0.1, \\
\beta_1^M &= e^{-\frac{4}{3}} 0.3, \\
\beta_2^M &= e^{-\frac{4}{3}} 0.4, \\
c_{11}^L &= e^{-\frac{3}{2}} 0.1, \\
c_{12}^L &= e^{-\frac{3}{2}} 0.2, \\
c_{21}^L &= e^{-\frac{3}{2}} 0.1, \\
c_{22}^L &= e^{-\frac{3}{2}} 0.1, \\
\mathcal{H}_1^M &= e^{-\frac{3}{2}} 0.3, \\
\mathcal{H}_2^M &= e^{-\frac{3}{2}} 0.5, \\
\gamma_{11}^M &= 0.02, \\
\gamma_{12}^M &= 0.01, \\
\gamma_{21}^M &= 0.01, \\
\gamma_{22}^M &= 0.02.
\end{align*}\] (58)

One can easily check that all the hypotheses (\(\mathcal{H}_1\))–(\(\mathcal{H}_4\)) are fulfilled. Thus, one can know that system (57) has a unique positive \((\pi/10)\)-periodic solution which is uniformly asymptotically stable. These results are displayed in Figures 1 and 2.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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