A POLYGONAL DISCONTINUOUS GALERKIN METHOD WITH MINUS ONE STABILIZATION

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Abstract. We propose an Hybridized Discontinuous Galerkin method on polygonal tessellation, with a stabilization term penalizing locally in each element $K$ a residual term involving the fluxes in the norm of the dual of $H^1(K)$. The scalar product corresponding to such a norm is numerically realized via the introduction of a (minimal) auxiliary space of VEM type. Stability and optimal error estimates in the broken $H^1$ norm are proven, and confirmed by the numerical tests.

1. Introduction

Methods for solving PDEs based on polyhedral meshes are having a fast development and attracting more and more attention. They provide greater flexibility in mesh generation, can be exploited as transitional elements in finite element meshes, and are better suited than methods based on tetrahedral or hexahedral meshes for many applications on complicated and/or moving domains [1]. Many different approaches exist, just to quote the more recent we recall: the Agglomerated Finite Element method [4], the Virtual Element Method [5], the Hybrid High Order method [15].

A common ingredient to all of these method is the presence of some stabilization term that penalizes a residual in some mesh dependent norm [11], and dealing with such terms in the analysis relies on the use of some kind of inverse inequality, and results in suboptimal estimates when the factor stemming from such inequality does not cancel out with some small factor coming from the approximation properties of the involved space. This is the case when, for instance, the elements are not shape regular or when we want to obtain $hp$ estimates [14, 16]. This kind of problem arises when a mesh dependent norm is used to mimic the action of the norm of the space where the penalized residual naturally “lives”, usually a negative or fractionary norm. On the other hand, it has been observed that, at least theoretically, it is possible to design stabilization terms based on such a norm [3, 7], for which the analysis does not require the validity of any inverse inequality.

In the following we propose an hybridized discontinuous Galerkin method on a polyhedral tessellation with an element by element stabilization similar to the one proposed by [13], that penalizes the residual on the flux in the norm of the dual of $H^1$. The numerical realization of the $(H^1)'$ norm has been the object of several papers [10, 2], and we follow here the general approach proposed by [8]. Even though for the sake of simplicity in this paper we consider the case of a shape regular mesh, and we only perform the analysis of the convergence in $h$, we believe that this approach (which can, of course, be applied also to other formulations) has the potential to tackle more general cases.

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2. The DG method with minus one stabilization

2.1. Scaled norms, seminorms and duals. In the following, for $\phi \in V$ and $\lambda \in V'$ ($V$, $V'$ any couple of dual Hilbert spaces), we will indicate by $\langle \lambda, \phi \rangle$ the action of $\lambda$ on $\phi$. Depending on the context, $V$ and $V'$ will be different couples of dual Sobolev spaces. In the analysis that follows, it will be essential to carry over to the dual norms some scaling arguments developed for positive Sobolev norms. In order to do so, we introduce non standard forms for the norms of some Sobolev space. Letting $K \in \mathbb{R}^2$ denote a shape regular polygon, and letting $e$ be an edge of $K$ of length $h$, let us start by considering the space $H^{1/2}(e)$. For $\phi \in H^{1/2}(e)$ we let

$$\|\phi\|^{1/2}_{1/2,e} = \alpha_e |\bar{\phi}|^2 + |\phi|^{1/2}_{1/2,e},$$

where $\alpha_e$ is a weight to be chosen later on, and with

$$\bar{\phi} = \frac{1}{|e|} \int_e \phi, \quad |\phi|^{1/2}_{1/2,e} = \int_e \int_e dx \, dy \frac{|\phi(x) - \phi(y)|^2}{|x - y|^2}.$$ 

On the dual space $(H^{1/2}(e))'$ we also define both a norm and a seminorm. More precisely, we let

$$|\lambda|^{-1/2}_{-1/2,e} = \sup_{\phi \in H^{1/2}(e)} \frac{\langle \lambda, \phi \rangle}{|\phi|^{1/2}_{1/2,e}},$$

and

$$\|\lambda\|^{2}_{-1/2,e} = \beta_e |\bar{\lambda}|^2 + |\lambda|^{2}_{-1/2,e}, \quad \text{with} \quad \bar{\lambda} = \frac{\langle \lambda, 1 \rangle}{|e|},$$

where $\beta_e$ is also a weight to be chosen in the following. Recall that the seminorm (2.2) is scale invariant: letting $\hat{e} = [0, 1]$ and $e = [0, h]$, and setting $\hat{x} = h^{-1} x$, for $\hat{\phi} \in H^{1/2}(\hat{e})$ and $\phi(x) = \hat{\phi}(\hat{x}) \in H^{1/2}(e)$ we have the identity

$$|\hat{\phi}|^{1/2}_{1/2,\hat{e}} = |\phi|^{1/2}_{1/2,e}.$$ 

A corresponding scaling property also holds for the dual semi norm (2.3), as stated by the following proposition.

**Proposition 2.1.** Let $\hat{e} = [0, 1]$ and $e = [0, h]$ let $\hat{x} = h^{-1} x$ and, for $\hat{\lambda} \in L^2(\hat{e})$ let $\lambda(x) = \hat{\lambda}(\hat{x})$. Then

$$|\lambda|^{-1/2}_{-1/2,e} = h |\hat{\lambda}|^{-1/2}_{-1/2,\hat{e}}.$$ 

**Proof.**

$$|\lambda|^{-1/2}_{-1/2,e} = \sup_{\phi \in H^{1/2}(e)} \frac{\int_e \lambda \phi(x) \, dx}{|\phi|^{1/2}_{1/2,e}} = \sup_{\phi \in H^{1/2}(e)} \frac{\int_{\hat{e}} \hat{\lambda}(\hat{x}) \hat{\phi}(\hat{x}) h \, d\hat{x}}{|\phi|^{1/2}_{1/2,e}}.$$ 

□

The two norms defined by (2.1) and (2.4) satisfy a duality relation, as stated by the following lemma.
Lemma 2.2. Provided $\alpha_e, \beta_e = h^2$ we have
\[
\|\lambda\|_{-1/2,e} \simeq \sup_{\phi \in H^{1/2}(e)} \frac{\int_e \lambda \phi}{\|\phi\|_{1/2,e}}.
\]

Proof. We have
\[
\sup_{\phi \in H^{1/2}(e)} \frac{\int_e \lambda \phi}{\|\phi\|_{1/2,e}} = \sup_{\phi \in H^{1/2}(e)} \frac{\int_e \lambda^e \phi^e + \langle \lambda - \lambda^e, \phi - \phi^e \rangle}{\sqrt{\alpha_e|\phi^e|^2 + |\phi|_{1/2,e}^2}} \leq \sup_{\phi \in H^{1/2}(e)} \frac{h|\lambda^e||\phi^e| + |\lambda - \lambda^e|_{-1/2,e}|\phi - \phi^e|_{1/2,e}}{\sqrt{\alpha_e|\phi^e|^2 + |\phi|_{1/2,e}^2}} \leq \sup_{\phi \in H^{1/2}(e)} \frac{\sqrt{h^2 \alpha_e^{-1}|\lambda^e|^2 + |\lambda|_{-1/2,e}^2}}{\sqrt{\alpha_e|\phi^e|^2 + |\phi|_{1/2,e}^2}} = \|\lambda\|_{-1/2,e}.
\]

On the other hand, setting $\lambda^0 = \lambda - \lambda^e$ we start by observing that, by the definition of $|\cdot|_{-1/2,e}$, for each $\varepsilon > 0$ there exists $\phi^0_\varepsilon \in H^{1/2}(e)$ with $\int_e \phi^0_\varepsilon = 0$ and with $|\phi^0_\varepsilon|_{1/2,e} = |\lambda^0|_{-1/2,e}$ such that
\[
\int_e \lambda^0 \phi^0_\varepsilon \geq (1 - \varepsilon)|\lambda^0|^2_{-1/2,e}.
\]

Letting $\phi_\varepsilon = \frac{h}{\alpha_e} \lambda^e + \phi^0_\varepsilon$ we have
\[
\|\phi_\varepsilon\|_{1/2,e}^2 = \alpha_e^{-1} h^2 |\lambda^e|^2 + |\phi^0_\varepsilon|^2_{1/2,e} = \beta_e |\lambda^e|^2 + |\lambda|_{-1/2,e}^2 = \|\lambda\|^2_{-1/2,e}
\]
and
\[
\int_e \lambda \phi_\varepsilon = h \int_e \lambda^e \phi + \int_e \lambda^0 \phi_\varepsilon \geq (1 - \varepsilon)\|\lambda\|^2_{-1/2,e} = (1 - \varepsilon)\|\lambda\|_{-1/2,e}\|\phi_\varepsilon\|_{1/2,e}.
\]

The arbitrariness of $\varepsilon$ yields the thesis. \qed

By duality we also obtain:

Corollary 2.3. Provided $\alpha_e, \beta_e = h^2$ we have
\[
\|\phi\|_{1/2,e} \simeq \sup_{\lambda \in (H^{1/2}(e))^\prime} \frac{\int_e \lambda \phi}{\|\lambda\|_{-1/2,e}}.
\]

We can now choose $\alpha_e$ and $\beta_e$. Two choices are significant:

(2.5) \hspace{1cm} \alpha_e = |e|, \hspace{1cm} \beta_e = |e|

and

(2.6) \hspace{1cm} \alpha_e = 1, \hspace{1cm} \beta_e = |e|^2.

Observe that the choice (2.5) yields a norm for $H^{1/2}(e)$ equivalent, uniformly in $h$, to the natural norm
\[
\left( \int_e |u|^2 + |u|_{1/2,e}^2 \right)^{1/2},
\]
while the choice (2.6) yield a scaled $H^{1/2}(e)$ norm equivalent, uniformly in $h$, to the norm
\[
|e|^{-1} \left( \int_e |u|^2 + |u|_{1/2,e}^2 \right)^{2}.\]
which is scale invariant and quite commonly used in domain decomposition. Throughout this paper, we choose definition (2.6) and we set $\alpha = 1$, $\beta = h^2$.

We also define, in an analogous way, the norm and seminorm for $H^{1/2}(\partial K)$ and its dual $H^{-1/2}(\partial K)$. More precisely we set

$$
\|\phi\|^2_{1/2,\partial K} = |\bar{\phi}\partial K|^2 + |\phi|_{1/2,\partial K}, \quad |\phi|_{1/2,\partial K} = \int_{\partial K} d\sigma \left( \frac{|\phi(x) - \phi(y)|^2}{|x - y|^2} \right),
$$

and

$$
\|\lambda\|_{-1/2,\partial K} = h^2|\bar{\lambda}\partial K|^2 + |\lambda|_{-1/2,\partial K}, \quad |\lambda|_{-1/2,\partial K} = \sup_{\phi \in H^{1/2}(\partial K)} \frac{\langle \lambda, \phi \rangle}{|\phi|_{-1/2,\partial K}},
$$

with $\bar{\phi}_{\partial K} = |\partial K|^{-1} \int_{\partial K} \phi$ and $\bar{\lambda}_{\partial K} = |\partial K|^{-1} \langle \lambda, 1 \rangle$.

Finally, we define the norm for $H^1(K)$ as

$$
\|f\|^2_{1,K} = |\bar{f}\partial K|^2 + |f|_{1,K}^2, \quad \text{with} \quad |\bar{f}\partial K| = |\partial K|^{-1} \int_{\partial K} f, \quad |f|_{1,K} = \int_K |\nabla f|^2.
$$

Observe that a perhaps more natural definition of the norm for such a space would be the one where the average over $K$ is used in the place of the average over $\partial K$. In view of the validity of Poincaré inequality for the functions with zero average on $\partial K$, the two norms are equivalent, as stated by the following Proposition

**Proposition 2.4.** Letting $K$ be a shape regular polygon, and letting $\bar{f}^K = |K|^{-1} \int_K f$ we have

$$
\|f\|^2_{1,K} \simeq |\bar{f}^K|^2 + |f|_{1,K}^2.
$$

**Proof.** Letting $C_{K,\partial}$ denote the constant in the Poincaré Wirtinger inequality

$$
u \in H^1(K) \text{ and } \int_{\partial K} u = 0 \quad \text{implies} \quad \|u\|_{L^2(K)} \leq C_{K,\partial} |u|_{1,K},
$$

we can write

$$
|\bar{f}\partial K| \leq |\bar{f}^K - \bar{f}\partial K| + |\bar{f}^K|, \quad \text{and} \quad |\bar{f}^K| \leq |\bar{f}^K - \bar{f}\partial K| + |\bar{f}\partial K|.
$$

Now we have

$$
|\bar{f}^K - \bar{f}\partial K| \leq |K|^{-1/2} \|f - \bar{f}\partial K\|_{L^2(K)} \lesssim |K|^{-1/2} C_{K,\partial}(K) |f|_{1,K}.
$$

Recalling that for shape regular polygons it holds that $C_{K,\partial} \lesssim |K|^{1/2}$, we get the thesis \(\Box\)

**Remark 2.5.** An analogous result holds for any constant yielding a Poincaré inequality. Particularly important for our application $\bar{u}^e$ being defined as the average over an edge. Reasoning as in the proof of the previous proposition, thanks to the shape regularity of $K$ we have

$$
|\bar{u}^e|^2 + |u|_{1,K} \simeq |\bar{u}^K| + |u|_{1,K}.
$$

The classical trace Theorem rewrites as
Theorem 2.6. It holds that
\[ \inf_{u \in H^1(k) : u = \phi \text{ on } \partial K} \| u \|_{1,K} \simeq \| \phi \|_{1/2, \partial K}, \]
uniformly in \( h \).

Proof. Letting \( \bar{u}^{\partial K} \) denote the average of \( u \) on \( \partial K \) we can write
\[ \| u \|_{1/2, \partial K} = \| \bar{u}^{\partial K} \| + \| u \|_{1/2, \partial K} \lesssim \| \bar{u}^{\partial K} \|^2 + \| u \|_{1,K}^2. \]
By the previous proposition this implies the thesis. \( \square \)

Analogously to what we did for the space \( H^{1/2}(e)' \), we can now introduce a semi-norm for the dual of \( H^1(K) \), namely
\[ |G|_{-1,K} = \sup_{u \in H^1(K) \atop \bar{u} = 0} \frac{\langle G, u \rangle}{\| u \|_{1,K}}. \]
Observe that, letting \( \gamma_K : H^1(K) \to H^{1/2}(\partial K) \) denote the trace operator, and letting \( \gamma^*_K \) denote its adjoint, we have for \( \lambda \in H^{-1/2}(\partial K) \)
\[ |\gamma^*_K \lambda|_{-1,K} = \sup_{\phi \in H^1(K) \atop \phi = 0} \frac{\langle \lambda, \gamma_K \phi \rangle}{\| \phi \|_{1,K}} \lesssim |\lambda|_{-1/2,K}. \]
Remark also that for \( F \in (H^1(K))' \) with \( \langle F, 1 \rangle = 0 \) we have
\[ \sup_{\phi \in H^1(K) \atop \phi = 0} \frac{\langle F, \phi \rangle}{\| \phi \|_{1,K}} = \sup_{\phi \in H^1(K) \atop \phi = 0} \frac{\langle F, \phi \rangle}{\| \phi \|_{1,K}}. \]

Finally, in the following, we will need to compare the \( H^{1/2}(e) \) norm to the \( H^{1/2}_{00}(e) \) norm, which, identifying \( e \) with the segment \((a, b)\) can be defined as
\[ \| \phi \|_{H^{1/2}_{00}(e)} = \int_a^b dx \int_a^b dy \frac{\| \phi(x) - \phi(y) \|}{|x - y|} + \int_a^b \frac{|\phi(x)|^2}{|x - a|} dx + \int_a^b \frac{|\phi(x)|^2}{|x - b|} dx. \]
Observe that \( \| \phi \|_{H^{1/2}_{00}(e)} \) is scale invariant, analogously to the \( H^{1/2}(e) \) seminorm. It is not difficult to prove that for \( \phi \in H^{1/2}_{00}(e) \) we have
\[ \| \phi \|_{1/2,e} \lesssim \| \phi \|_{H^{1/2}_{00}(e)}; \]
In fact, we have that
\[ |\bar{\phi}|^2 \leq |e| \int_a^b |\phi(x)|^2 dx \lesssim \int_e \frac{|\phi(x)|^2}{|x - a|} dx; \]
uniformly as \( |e| \) tends to 0. By duality (recall that \( H^{-1/2}(e) = H^{1/2}_{00}(e)' \)), we have that
\[ \| \lambda \|_{H^{-1/2}(e)} \lesssim \| \lambda \|_{-1/2,e}. \]
The scale invariance of the $H_{00}^{1/2}$ norms, implies, by duality, a scaling property for the dual norm: for $\lambda \in L^2(\hat{e})$, letting $\lambda \in L^2(\hat{e})$ defined as in Proposition 2.1 we have

\[
\|\lambda\|_{H^{-1/2}(\hat{e})} = \sup_{\phi \in H_{00}^{1/2}(\hat{e})} \int_{\hat{e}} \lambda(x)\phi(x) \, dx = \sup_{\hat{\phi} \in H_{00}^{1/2}(\hat{e})} h \int_{\hat{e}} \hat{\lambda}(\hat{x})\hat{\phi}(\hat{x}) \, d\hat{x} = h\|\lambda\|_{H^{-1/2}(\hat{e})}
\]

Thanks to the scaling property of both the $H^{-1/2}(\hat{e})$ norm and the $\| \cdot \|_{-1/2,\hat{e}}$ it is not difficult to prove that for all $\lambda \in \mathbb{P}_k(\hat{e})$ it holds that

\[
\|\lambda\|_{-1/2,\hat{e}} \lesssim \|\lambda\|_{H^{-1/2}(\hat{e})}.
\]

The following Lemma allows to compare the $H^{-1/2}(\partial K)$ norm with the $H^{-1/2}(\hat{e})$ and the $(H_{00}^{1/2}(\hat{e}))'$ norms.

**Lemma 2.7.** Let $\lambda \in L^2(\partial K)$. It holds

\[
\left( \sum_e \|\lambda\|_{H^{-1/2}(\hat{e})}^2 \right)^{1/2} \lesssim \sup_{u \in H^1(K)} \frac{\int_{\partial K} \lambda u}{\|u\|_{1,K}} \lesssim \left( \sum_e \|\lambda\|_{-1/2,\hat{e}}^2 \right)^{1/2}
\]

(where the sum is taken over the edges $e$ of $K$).

**Proof.** Let $\lambda \in L^2(\partial K)$. We have

\[
\sup_{u \in H^1(K)} \frac{\int_{\partial K} \lambda u}{\|u\|_{1,K}} \leq \sup_{u \in H^1(K)} \sum_e \int_e \lambda u \leq \sup_{u \in H^1(K)} \sum_e \|u\|_{1,K} \leq \sum_e \|\lambda\|_{-1/2,\hat{e}} \|u\|_{1,\hat{e}}
\]

Let us now compare $\|u\|_{1,\hat{e}}$ with $\|u\|_{1,K}$. We have, setting $\bar{u}^e = |e|^{-1} \int_e u$, thanks to the Poincaré Wirtinger inequality,

\[
\|u\|_{1,\hat{e}} = |\bar{u}^e| + |u|_{1,\hat{e}} \leq |\bar{u}^e| + |u|_{1,\partial K} \leq |\bar{u}^e| + |u|_{1,K} \lesssim \|u\|_{1,K},
\]

yielding

\[
\sup_{u \in H^1(K)} \frac{\int_{\partial K} \lambda u}{\|u\|_{1,K}} \lesssim \sum_e \|\lambda\|_{-1/2,\hat{e}} \lesssim \left( \sum_e \|\lambda\|_{-1/2,\hat{e}}^2 \right)^{1/2}.
\]

As far as the lower bound is concerned, we start by introducing the space $\Phi_0 \subset H^{1/2}(\Gamma^k)$ defined as

\[
\Phi_0 = \{ \phi \in H^{1/2}(\partial K) : \phi|_e \in H_{00}^{1/2}(e) \},
\]

and we observe that for all $\phi \in \Phi_0$ we have

\[
|\partial K|^{-1} \left| \int_{\partial K} \phi \right| \lesssim \sum_e |e|^{-1} \left| \int_e \phi \right| \lesssim \sum_e \|\phi\|_{H_{00}^{1/2}(e)} \lesssim \left( \sum_e \|\phi\|_{H_{00}^{1/2}(e)}^2 \right)^{1/2}.
\]

We can then write

\[
\|\lambda\|_{H^{-1/2}(\partial K)} = \sup_{\phi \in H^{1/2}(\partial K)} \frac{\int_{\partial K} \lambda \phi}{\|\phi\|_{1/2,\partial K}} \gtrsim \sup_{\phi \in \Phi_0} \frac{\sum_e \int_e \lambda \phi}{\|\phi\|_{H_{00}^{1/2}(e)}} \gtrsim \sum_e \sup_{\phi \in H_{00}^{1/2}(e)} \left| \int_e \lambda \phi \right| \geq \sum_e \|\lambda\|_{H^{-1/2}(e)} \gtrsim \left( \sum_e \|\lambda\|_{H^{-1/2}(e)}^2 \right)^{1/2}.
\]
were we used that the dual of product of spaces is the product of the duals. □

2.2. The model problem and its discretization. In the following we consider the simple model problem

**Problem 2.1.** Find $u$ solution to

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$ 

We look for a solution to Problem 2.1 by a discontinuous Galerkin method. More precisely, let $\mathcal{T}_h$ denote a quasi uniform tessellation of $\Omega$ into shape regular polygons of diameter $\simeq h$. Let $\mathcal{E}_h$ denote the set of edges of the tessellation, and let $\Sigma = \bigcup_{e \in \mathcal{E}_h} e$ denote the skeleton of the decomposition. For each element $K$, let $E_K$ denote the set of edges of $K$.

We set

$$V_h = \prod_K P_k(K), \quad \Lambda_h = \prod_K \Lambda_K, \text{ with } \Lambda_K = \{ \lambda \in L^2(\partial K) : \lambda|_e \in P_k(e) \text{ for all } e \in \mathcal{E}^K \},$$

as well as

$$\Phi_h = \{ \phi \in L^2(\Sigma) : \phi|_e \in P_k(e) \text{ for all } e \in \mathcal{E}_h, \phi|_{\partial\Omega} = 0 \},$$

where, for any one- or two-dimensional domain $D$, $P_k(D)$ denotes the space of uni- or bi-variate polynomials of degree less or equal than $k$ on $D$. We endow $V_h$ with the norm

$$\|u\|_{1,*} = \sum_K |u^K|_{1,K} + \sum_e |\bar{u}|^2,$$

where for all interior edges $e$ common to two elements $K^+$ and $K^-$ we set

$$|\bar{u}| = |u^{K^+} - u^{K^-}|$$

and for $e \subset \partial K \cap \partial\Omega$ we set

$$|\bar{u}| = |u^K|.$$

On $\Lambda_h$ we consider the two norms

$$\|\lambda\|_{-1/2,\Sigma}^2 = \sum_K \sum_{e \subset \partial K} \|\lambda^K_e\|_{-1/2,e}^2, \quad \text{and} \quad \|\lambda\|_{-1/2,\Sigma}^2 = \sum_K \|\gamma^*_K \lambda^K\|_{-1, K}^2.$$

Observe that for all $\lambda \in \prod_K L^2(\partial K)$ we have

$$\|\lambda\|_{-1/2,\Sigma} \lesssim \|\lambda\|_{-1/2,\Sigma},$$

while for all $\lambda \in \Lambda_h$ it holds that

$$\|\lambda\|_{-1/2,\Sigma} \simeq \|\lambda\|_{-1/2,\Sigma}.$$

On $\Phi$ we define the norm

$$\|\phi\|_{1/2,\Sigma}^2 = \sum_{e \in \mathcal{E}_h} \|\phi\|_{1/2,e}^2.$$

In order to define our discrete problem, we introduce, for all $K$, a bilinear form $s_K : (H^1(K))^' \times (H^1(K))^' \rightarrow \mathbb{R}$, satisfying the following assumptions.

**Assumption 2.1.** For all $F, G \in (H^1(K))^'$ we have

$$s_K(F, G) \lesssim |F|_{-1,K} |G|_{-1,K}.$$ 

**Assumption 2.2.** For all $\lambda \in \Lambda_K$

$$s_K(\gamma^*_K \lambda, \gamma^*_K \lambda) \gtrsim |\gamma^*_K \lambda|_{-1,K}^2.$$
We then consider the following discrete problem, where $\alpha > 0$ and $t \in \mathbb{R}$ are two mesh independent parameters.

**Problem 2.2.** Find $u = (u^K)_K \in V_h$, $\lambda = (\lambda^K)_K \in \Lambda_K$, $\phi \in \Phi_h$ such that, for all $v = (v^K)_K \in V_h$, $\mu = (\mu^K)_K \in \Lambda_K$, $\psi \in \Phi_h$ it hold that

$$
(2.7) \quad \int_{K} \nabla u^K \cdot \nabla v^K - \int_{\partial K} \lambda^K v^K + t \alpha s_K(Du^K - \gamma^K \lambda^K, Du^K) = \int_{K} f v^K + t \alpha s_K(f, Du^K),
$$

$$
(2.8) \quad \int_{\partial K} u^K \mu^K - \alpha s_K(Du^K - \gamma^K \lambda^K, \gamma^K \mu^K) - \int_{\partial K} \phi \mu^K = -\alpha s_K(f, \gamma^K \mu^K),
$$

$$
(2.9) \quad \sum_K \int_{\partial K} \lambda^K \psi = 0.
$$

where $D : H^1(K) \to (H^1(K))'$ is defined as

$$
\langle Du, v \rangle = \int_{K} \nabla u(x) \cdot \nabla v(x) \, dx.
$$

Observe that for each $K$, (2.7-2.8) yield a local discrete Dirichlet problem with Lagrange multiplier and a non standard stabilization term, while (2.9) imposes continuity of the fluxes $\lambda$. The coupling between the different local problems stem from the common Dirichlet data $\phi$, which is single valued on the interface, as well as from equation (2.9). Remark also that, due to the choice of discrete spaces, the discrete fluxes turn out to be strongly continuous.

It is interesting to give an interpretation of the stabilization term as resulting from defining a suitable numerical trace, in the ideal case where $s^K$ is the scalar product for the space $H^1_s(K) = \{ u \in H^1(K) : \int_{\partial K} u = 0 \}$ endowed with the norm $| \cdot |_{1,K}$. It is easy to check that letting $\mathcal{R} : (H^1_s(K))' \to H^1_s(K)$ denote the Riesz’s isomorphism, which, we recall, is defined in such a way that

$$
\langle s^K(F, G) \rangle = \int_{K} \nabla \mathcal{R} F \cdot \nabla \mathcal{R} G = \langle F, \mathcal{R} G \rangle = \langle G, \mathcal{R} F \rangle,
$$

we have $\mathcal{R} = D^{-1}$. Considering, for simplicity, the case $t = 0$, we then have

$$
s^K(D_{\mu^K} - \gamma^K \lambda^K, \gamma^K \mu^K) = \langle \gamma^K \mu^K, \mathcal{R}(D_{\mu^K} - \gamma^K \lambda^K) \rangle
$$

$$
= \langle \gamma^K \mu^K, u^K - \mathcal{R} \gamma^K \lambda^K \rangle = \int_{\partial K} (u^K - \mathcal{R} \gamma^K \lambda^K) \mu^K.
$$

The stabilized discrete local problems then become

$$
\int_{K} \nabla u^K \cdot \nabla v^K - \int_{\partial K} \lambda^K v^K = \int_{K} f v^K
$$

$$
\int_{\partial K} \hat{u}^K \mu^K = \int_{\partial K} \hat{\phi}^K \mu^K
$$

with

$$
\hat{u}^K = (1 - \alpha)u^K + \gamma^K \alpha \mathcal{R}(\gamma^K \lambda^K), \quad \text{and} \quad \hat{\phi}^K = \phi^K - \gamma^K \alpha \mathcal{R}(f).
$$

Observe that, unlike what happens in the standard DG framework, the numerical trace $\hat{u}^K$ is indeed the trace of an $H^1(K)$ function.
2.3. Stability and convergence analysis. In order to analyze Problem 2.2, we will rely on the following discrete Poincaré like inequality.

**Lemma 2.8.** Let $u \in \prod_K H^1(K)$ and let $\bar{u} = (\bar{u}^K)_K$ denote the piecewise constant function where $\bar{u}^K$ is the average of $u^K$ on $K$. Then

$$\|u\|_{0,\Omega}^2 \lesssim \sum_K h^2 |u^K|_{1,K}^2 + \sum_e \|\bar{u}\|^2.$$  

**Proof.** We have

(2.10)  

$$\|u\|_{0,\Omega}^2 \leq \|u - \bar{u}\|_{0,\Omega}^2 + \|\bar{u}\|_{0,\Omega}^2 \lesssim \sum_K h^2 |u^K|_{1,K}^2 + \|\bar{u}\|_{0,\Omega}^2.$$  

We then only need to bound the last term. Let $w$ be the solution of

(2.11)  

$$- \Delta w = \bar{u}, \text{ in } \Omega, \quad w = 0, \text{ on } \partial \Omega.$$  

Observe that $w \in H^2(\Omega)$ implies continuity of the normal derivative across the skeleton. Fixing for each edge a normal $\nu^e$, we can define

$$\tilde{\mu}^e = \frac{1}{|e|} \int_e \frac{\partial w}{\partial \nu_e}.$$  

Then, using (2.3) and integrating by part element by element, we write

$$\|\bar{u}\|_{0,\Omega}^2 = - \sum_K \int_{\partial K} \frac{\partial w}{\partial \nu^K} u^K = \sum_e \int_e \tilde{\mu}^e[\bar{u}] \leq \sum_e h |\tilde{\mu}^e||[\bar{u}]|  

\lesssim \left( \sum_e h^2 |\tilde{\mu}^e|^2 \right)^{1/2} \left( \sum_e \|\bar{u}\|^2 \right)^{1/2}.$$  

It only remains to bound the first factor in the product on the right hand side. We have

$$\sum_e h^2 |\tilde{\mu}^e|^2 \leq \sum_e h \left( \int_e |\tilde{\mu}^e|^2 \right) \leq \sum_e h \left( \int_e \left| \frac{\partial w}{\partial \nu^e} \right|^2 \right) \lesssim \sum_K h \left( \int_{\partial K} \left| \frac{\partial w}{\partial \nu_K} \right|^2 \right)$$  

Now, we observe that, since $w \in H^2(\Omega)$, we have that $\nabla w \in H^1(\Omega)$, so that we can bound (the dependence on $h$ derives from a scaling argument)

$$\int_{\partial K} \left| \frac{\partial w}{\partial \nu_K} \right|^2 \lesssim h^{-1} \int_K |\nabla w|^2 + \int_\Omega |\Delta w|^2 \, dx$$  

Adding up the contributions of the different elements we obtain (without loss of generality we can assume that $h \lesssim 1$)

$$\sum_e h^2 |\tilde{\mu}^e|^2 \lesssim \int_\Omega |\nabla w|^2 \, dx + \int_K |\Delta w|^2 \, dx \lesssim \|\bar{u}\|_{0,\Omega}^2$$  

which yields

$$\|\bar{u}\|_{0,\Omega}^2 \lesssim \|\bar{u}\|_{0,\Omega} \left( \sum_e \|\bar{u}^e\|^2 \right)^{1/2}.$$  

Dividing both sides by $\|\bar{u}\|_{0,\Omega}$ and combining with (2.10) we get the thesis. □
In order to prove the well posedness of Problem 2.2, we apply Theorem 1.1, Section II.1 of [12]. In order to do so, we introduce the space $V_h = V_h \times \Lambda_h$ and rewrite Problem 2.2 as follows: find $U = (u, \lambda) \in V_h$, $\phi \in \Phi_h$ such that for all $V = (v, \mu) \in V_h$, $\phi \in \Phi_h$ it holds that
\begin{equation}
(2.12)
a(U, V) + b(\phi, V) = F(V), \quad b(U, \psi) = 0,
\end{equation}
with
\begin{equation}
(2.13)
a_K(u^K, \lambda^K; v^K, \mu^K) = \sum_K a_K(u^K, \lambda^K; v^K, \mu^K), \quad b_K(\phi; v^K, \mu^K),
\end{equation}
where the local bilinear forms $a_K$ and $b_K$ are respectively defined as
\begin{equation}
(2.14)
b_K(v^K, \mu^K; \phi) = \int_{\partial K} \mu^K \phi.
\end{equation}

We start by observing that we have an inf-sup condition of the form
\begin{equation}
(2.15)
\inf_{\phi \in P_K(e)} \sup_{\lambda \in P_K(e)} \int_{\partial e} \phi \lambda \| \phi \|_{1/2, e} \| \lambda \|_{-1/2, e} \gtrsim 1.
\end{equation}
Then, proving the well posedness of Problem 2.2 reduces to proving the well posedness of the following problem.

**Problem 2.3.** find $(u, \lambda) \in \ker b = \{(u, \lambda) \in V_h : b(u, \lambda; \psi) = 0, \, \forall \psi \in \Phi_h\}$ such that for all $(v, \mu) \in \ker b$
\begin{equation}
(2.16)
a(u, \lambda; v, \mu) = f(v, \mu).
\end{equation}
Remark that $\ker b$ is the set of couples $(u, \lambda) \in V_h$ such that $\{\lambda\} = 0$.

The following lemma holds.

**Lemma 2.9.** It holds
\begin{equation}
\inf_{(u, \lambda) \in \ker b} \sup_{(v, \mu) \in \ker b} \frac{a(u, \lambda; v, \mu)}{\|u, \lambda\|_V \|v, \mu\|_V} \gtrsim 1,
\end{equation}
with
\begin{equation}
(2.17)
\|u, \lambda\|^2_V := \|u\|^2_{1, \Sigma} + \|\lambda\|^2_{-1/2, \Sigma}.
\end{equation}

The proof of Lemma 2.9 is quite long and technical, and we postpone it to Section 2.4.
while for \( e \subset \partial K \cap \partial \Omega \) we define \( \{ \lambda \}_e = 0 \). We observe that, for \( \lambda \) with \( \{ \lambda \} = 0 \) we can write
\[
\sum_k \int_{\partial K} \lambda^K v^K = \sum_k \int_{\partial K} \lambda^K (v^K - \bar{v}^K) + \sum_e \int_e \lambda_e \| \bar{v} \| =
\sum_k \int_{\partial K} \lambda^K (v^K - \bar{v}^K) + \sum_e h \lambda_e \| \bar{v} \| \leq \sum_k |\gamma^K| \lambda^K \|_{-1,K} v^K \|_{1,K} + \sum_e h |\lambda_e| \| \bar{v} \|
\leq \left( \sum_k |\gamma^K| \lambda^K \|_{-1,K} \right)^{1/2} \left( \sum_k \| v^K \|_{1,K} \right)^{1/2} + \left( \sum_e h^2 |\lambda_e|^2 \right)^{1/2} \left( \sum_e \| \bar{v} \|^2 \right)^{1/2},
\]
where, for \( e \) common edge to \( K^+ \) and \( K^- \)
\[
\lambda_e = \lambda^{K^+} = -\lambda^{K^-} \text{ and } \| u \| = u^{K^+} - u^{K^-},
\]
and where \( \bar{\lambda}_e \) denotes the average of \( \lambda_e \) on \( e \). Now we can write (with \( \lambda^K \) denoting the average of \( \lambda \) on \( \partial K \))
\[
\sum_{e \subset \partial K} h^2 |\bar{\lambda}_e|^2 \leq \sum_{e \subset \partial K} h^2 \| \lambda - \bar{\lambda} \|_{-1/2,e}^2 + h^2 |\bar{\lambda}^K|^2 \leq |\gamma^K| \lambda \|_{-1,K} + h^2 |\bar{\lambda}^K|^2,
\]
yielding
\[
\sum_k \int_{\partial K} \lambda^K v^K \leq \| u, \lambda \|_v \| v, \mu \|_v.
\]

Remarking that
\[
\| Du \|_{-1,K} = \sup_{v \in H^1(K)} \frac{\int_K \nabla u(x) \cdot \nabla v(x) \, dx}{\| v \|_{1,K}} \leq \| u \|_{1,K},
\]
the continuity of the bilinear form \( a \) with respect to the norm \( \| \cdot \|_v \) is then not difficult to prove for all \( v \in \prod_K H^1(K) \) and all \( \lambda \in \prod_K L^2(\partial K) \) with \( \{ \lambda \} = 0 \). Problem 2.3 is then well posed, and admits a solution continuously depending on the data. By Theorem 1.1, Section II.1 of [12] it follows that Problem 2.2 admits a unique solution also depending continuously on the data.

Let now \( u \) be the continuous solution of Problem 2.1 and let \( \lambda = (\lambda^K) \), with \( \lambda^K = \partial u^K / \partial v^K \) denoting the outer normal derivative of \( u \) on \( \partial K \). Let \((v_h, \mu_h) \in \ker b\) be approximations to \( u \) and \( \lambda \). We can write
\[
\| u_h - v_h, \lambda_h - \mu_h \|_v \leq a(\lambda_h - \mu_h; z_h, \zeta_h)
\]
for some element \((z_h, \zeta_h) \in \ker b \) with \( \| z_h, \zeta_h \|_v = 1 \). We have
\[
a(\lambda_h; z_h, \zeta_h) = a(\lambda_h; \zeta_h),
\]
yielding
\[
\| u_h - v_h, \lambda_h - \mu_h \|_v \leq a(u - v_h, \lambda - \mu_h; z_h, \zeta_h) \leq \| u - v_h, \lambda - \mu_h \|_v,
\]
and, by triangular inequality
\[
\| u - u_h, \lambda - \lambda_h \|_v \leq \inf_{(v_h, \mu_h) \in \ker b} \| u - v_h, \lambda - \mu_h \|_v.
\]
It only remains to bound the term on the right-hand side of expression (2.18). Assuming that \( u \in H^{k+1}(\Omega) \), let, for each \( K, v^K_h \in P_k(K) \) denote the solution to
\[
\int_K (v^K_h - u) = 0, \quad \int_K \nabla (v^K_h - u) \cdot \nabla w_h = 0, \quad \text{for all } w_h \in P_k(K).
\]
We observe that
\[
\|u - v^K_h\|_{1,K}^2 + \sum_e \|\bar{u} - \bar{v}_h\|^2 = \sum_K |u - v^K_h|_{1,K}^2 \lesssim h^k |u|_{k+1,\Omega}.
\]
On the other hand, letting \( \mu^K_h \) be defined on each edge \( e \) of \( K \) as the \( L^2(e) \) projection of \( \lambda^K \), it is immediate to check that \( u \in H^{k+1}(\Omega) \subseteq H^2(\Omega) \) implies that \( \{\lambda\} = 0 \) and, thus, \( (v_h, \mu_h) \in \ker b \). It is also not difficult to check that
\[
\int_{\partial K} \mu^K_h = \int_{\partial K} \lambda^K,
\]
so that the contribution of \( \lambda - \mu_h \) to the \( \nabla \) norm of the error is
\[
\sum_K |\lambda^K - \mu^K_h|_{1/2,\partial K}^2 + \sum_K h^2 |\lambda^K - \mu^K_h|^2 = \sum_K |\lambda^K - \mu^K_h|_{1/2,\partial K}^2.
\]
By standard approximation estimates we have that
\[
\|\lambda^K - \mu^K_h\|_{L^2(\partial K)} \lesssim h^{k-1/2} |\partial_{\mu,K} u^K|_{k-1/2,\partial K} \lesssim h^{k-1/2} |u^K|_{k+1,K}.
\]
By a duality argument we can then bound \( \phi_h \in P(e) \) denoting the \( L^2(e) \) projection of \( \phi \)
\[
|\lambda^K - \mu^K_h|_{1/2,\partial K} = \sup_{\phi \in H^{1/2}(\partial K) \atop \|\phi\|_{1/2,\partial K} = 1} \int_{\partial K} (\lambda^K - \mu^K_h) \phi
\]
\[
= \sup_{\phi \in H^{1/2}(\partial K) \atop \|\phi\|_{1/2,\partial K} = 1} \sum_e \int_e (\lambda^K - \mu^K_h) (\phi - \phi_h)
\]
\[
\lesssim \sup_{\phi \in H^{1/2}(\partial K) \atop \|\phi\|_{1/2,\partial K} = 1} \sum_e \|\lambda^K - \mu^K_h\|_{L^2(e)} \|\phi - \phi_h\|_{L^2(e)}
\]
\[
\lesssim \sum_e h^{1/2} \|\lambda^K - \mu^K_h\|_{L^2(e)} \lesssim h \|\lambda^K - \mu^K_h\|_{L^2(\partial K)}.
\]
Adding up the different contributions we obtain that
\[
\|u - u_h, \lambda - \lambda_h\|_{V} \lesssim h^k |u|_{k+1,\Omega}.
\]

2.4. Proof of Lemma 2.9. For simplicity, let us consider the case \( t = 0 \) (for \( t \neq 0 \) we get an extra term that we can bound essentially by the same arguments). Let \( (u, \lambda) \in \ker b \), and let \( \hat{v} = u - \kappa \hat{v} \), with \( \hat{v} = (\hat{u}^K)_K, \hat{u}^K = \int_{\partial K} \lambda^K, \) and \( \mu = \lambda + \beta \hat{u} \) with, on the common edge \( e \) to \( K^+ \) and \( K^- \), \( \hat{u}^{K^+} = -\hat{u}^{K^-} = h^{-1}_e (\bar{u}^{K^+} - \bar{u}^{K^-}) \), where \( \bar{u} \) denote the piecewise constant
function assuming on each $K$ the value of the average on $K$ of $u^K$. We have
\[
a(u, \lambda; v, \mu) = \sum_K |u^K|_{1,K}^2 + \kappa \sum_K \int_{\partial K} \lambda^K \bar{v}^K + \beta \sum_e h_e^{-1} \int_{\partial K} \|u\|\|\bar{u}\| - \alpha \sum_K s_K(Du^K - \gamma_*^K \lambda^K, \gamma_*^K(\lambda^K + \beta \bar{\mu}^K)).
\]

We now observe that
\[
h_e^{-1} \int_e \|u\|\|\bar{u}\| = h_e^{-1} \int_e \|\bar{u}\|^2 + h_e^{-1} \int_e (\|\bar{u}^e\| - \|\bar{u}\|) \|\bar{u}\| \geq \|\bar{u}\|^2 - \|\bar{u}^e\| - \|\bar{u}\| \|\bar{u}\| \geq \|\bar{u}\|^2 - \frac{1}{2} \|\bar{u}\|^2 - \frac{1}{2} \|\bar{u}^e\| - \|\bar{u}\|^2,
\]
where on $e$ we let $\|\bar{u}^e\|$ denote the average on edge of $\|u\|$.

We can bound the last term as follows:
\[
\|\bar{u}^e\| - \|\bar{u}\|^2 = - \left| e \right|^{-1} \int_e \|u - \bar{u}\|^2 \lesssim \left| e \right|^{-1} \int_e \|u - \bar{u}\|^2 \lesssim \left| e \right|^{-1} \int_e |u^{K^+} - \bar{u}^{K^+}|^2 + \left| e \right|^{-1} \int_e |u^{K^-} - \bar{u}^{K^-}|^2.
\]

Now we have that, for $e$ edge of $K$
\[
\int_e |u^K - \bar{u}^K|^2 \lesssim \|u^K - \bar{u}^K\|^2_{L^2(\partial K)} \lesssim h_e^{-1}\|u^K - \bar{u}^K\|^2_{L^2(K)} + h\|u^K - \bar{u}^K\|^2_{1,K} \lesssim h\|u^K\|_{1,K},
\]
finally yielding, for some positive constant $c$,
\[
h_e^{-1} \int_e \|u\|\|\bar{u}\| \gtrsim \|\bar{u}\|^2 - c(\|u^{K^+}\|_{1,K} + \|u^{K^-}\|_{1,K}).
\]

We also observe that
\[
\int_{\partial K} \lambda^K \bar{v}^K \simeq |\partial K|^2 |\bar{\lambda}^K|^2, \text{ with } \bar{\lambda}^K = |\partial K|^{-1} \int_{\partial K} \lambda^K.
\]

Moreover, we have that
\[
\|\bar{\mu}\|_{1/2, e}^2 = \|\bar{u}\|.
\]

Then we can write
\[
a(u, \lambda; v, \mu) \geq \sum_K |u^K|_{1,K}^2 + \kappa \sum_K |\partial K|^2 |\bar{\lambda}^K|^2 + \frac{\beta}{2} \sum_e \|\bar{u}\|^2 - c' \beta \sum_K |u^K|_{1,K}^2 - \alpha s_K(Du^K - \gamma_*^K \lambda^K, \gamma_*^K(\lambda^K + \beta \bar{\mu}^K)) = \sum_K |u^K|_{1,K}^2 + \kappa \sum_K |\partial K|^2 |\bar{\lambda}^K|^2 + \frac{\beta}{2} \sum_e \|\bar{u}\|^2 - c' \beta \sum_K |u^K|_{1,K}^2 + \alpha \sum_K s_K(\gamma_*^K \lambda^K, \gamma_*^K \lambda^K) + \alpha \beta \sum_K s_K(\gamma_*^K \lambda^K, \gamma_*^K \lambda^K) + \alpha \sum_K s_K(Du^K, \gamma_*^K \lambda^K) - \alpha \beta \sum_K s_K(Du^K, \gamma_*^K \bar{\mu}^K).
\]
We separately bound the four last terms on the right hand side. By Assumption 2.2, we have
\[ \sum_K s_K(\gamma^*_K \lambda^K, \gamma^*_K \lambda^K) \geq c_1 \sum_K |\gamma^*_K \lambda^K|^2_{-1,K}. \]
Using Assumption 2.1 as well as Lemma 2.7, we also have
\[ \sum_K s_K(\gamma^*_K \lambda^K, \gamma^*_K \mu^K) \leq c \sum_K |\gamma^*_K \lambda^K|_{-1,K} |\gamma^*_K \mu^K|_{-1,K} \leq c \left( \sum_K |\gamma^*_K \lambda^K|^2_{-1,K} \right)^{1/2} \left( \sum_K |\mu|^2_{-1,e} \right)^{1/2} \]
\[ \leq \varepsilon_2 \sum_K |\gamma^*_K \lambda^K|^2_{-1,K} + c(\varepsilon_2) \sum_e \|\tilde{u}\|^2, \]
and
\[ s_K(Du^K, \gamma^*_K \mu^K) \leq c \sum_K |u^K|_{1,K} |\gamma^*_K \mu^K|_{-1,K} \leq c_3 \sum_K \|\tilde{u}\|^2 + \sum_K |u^K|^2_{1,K}, \]
while, thanks to (2.17), we have
\[ \sum_K s_K(Du^K, \gamma^*_K \lambda^K) \leq c \sum_K |u^K|_{1,K} |\gamma^*_K \lambda^K|_{-1,K} \leq \varepsilon_3 \sum_K |\gamma^*_K \lambda^K|^2_{-1,K} + c(\varepsilon_3) \sum_K |u^K|^2_{1,K}, \]
where \( \varepsilon_2 \) and \( \varepsilon_3 \) are arbitrary positive constants and \( c(\varepsilon_2) \) and \( c(\varepsilon_3) \) are positive constants depending on \( \varepsilon_2 \) and \( \varepsilon_3 \) respectively. Combining the previous bounds we obtain
\[ a(u, \lambda; v, \mu) \geq \sum_K |u^K|^2_{1,K} (1 - c' \beta - ac(\varepsilon_3) - \alpha \beta c_3) + \beta \sum_K \|\tilde{u}\|^2 (\frac{1}{2} - ac(\varepsilon_2) - \alpha c_3) + \]
\[ \alpha \sum_K \|\gamma^*_K \lambda^K\|^2_{-1,K} (c_1 - \beta \varepsilon_2 - \varepsilon_3) + \kappa \sum_K |\partial K|^2 |\bar{\lambda}|^2. \]

We now set \( \beta = 1/(2c') \), and we choose \( \varepsilon_1 \) and \( \varepsilon_2 \) in such a way that \( (\varepsilon_2/(2c') + \varepsilon_3) \leq c_1 \).

We next choose \( \alpha = 1/4 \min\{c(\varepsilon_3) + c_3/(2c'), c(\varepsilon_2) + c_3\} \). Observe that neither \( \beta \) nor \( \alpha \) depend on \( h \). With such a choice, for a constant \( c_0 \) independent of \( h \), it holds that
\[ a(u, \lambda; v, \mu) \geq c_4 \left( \sum_K |u^K|^2_{1,K} + \sum_K \|\tilde{u}\|^2 + \sum_K |\gamma^*_K \lambda^K|^2_{-1,K} + \kappa \sum_K |\partial K|^2 |\bar{\lambda}|^2 \right). \]

We now observe that
\[ \sum_K |e^K|^2_{1,K} + \sum_e \|\tilde{v}\|^2 \leq \sum_K |u^K|^2_{1,K} + \sum_K \|\tilde{u}\|^2 + \sum_e |h| |\tilde{\lambda}|^2 \]
\[ \leq \sum_K |u^K|^2_{1,K} + \sum_K \|\tilde{u}\|^2 + \sum_e h^2 |\bar{\lambda}|^2. \]
Moreover, thanks to Lemma 2.7 we have that
\[ |\gamma^*_K \mu^K|^2_{-1,K} \leq |\mu^K|^2_{-1/2,\partial K} \leq \sum_{e \subset \partial K} \|\tilde{\mu}\|^2_{-1/2,e} = \sum_{e \subset \partial K} \|\tilde{u}\|^2, \]
and
\[ \sum_K |\gamma_K \mu K|^2 \leq \sum_K |\gamma_K |^2 + \sum_K |\partial K|^2 |\lambda K|^2 + \sum_e \|u\|^2. \]

We then get that
\[ \sup_{(v, \mu)} \frac{a(u, \lambda; v, \mu)}{\|v, \mu\|_V} \geq \frac{a(u, \lambda; u, \lambda + \beta \lambda)}{\|u, \lambda\|_V} \gtrsim \|u, \lambda\|_V. \]

which concludes the proof.

3. Realizing a computable stabilizing term

In order for the proposed method to be practically feasible, we need to construct a computable bilinear form \( s_K \) satisfying (2.1) and (2.2). The numerical realization of scalar product for negative Sobolev spaces has been the object of several papers [10, 9, 2]. In particular, this can be done using the approach of [8]. Following such paper, we introduce an auxiliary space \( W_K \subseteq H^1(K) \), satisfying
\[ \inf_{\lambda \in \Lambda} \sup_{w \in W_K} \int_K \lambda w = \sup_{\lambda \in \Lambda} \int_{\partial K} \lambda \|w\|_{1,K} \gtrsim 1. \]

We let \( \phi_i, i = 1, \cdots, N \) denote a basis for \( W_K \), and we let \( S \) denote the stiffness matrix (or a preconditioner) relative to the operator \( s : H^1(K) \times H^1(K) \to \mathbb{R} \)
\[ s(w, v) = \int_K \nabla w \cdot \nabla v. \]

We can now introduce the bilinear form \( s_K : (H^1(K))^\prime \to (H^1(K))^\prime \) defined as follows:
\[ s_K(f, g) = \tilde{f}^T S^{-1} \tilde{g}, \quad \text{with} \quad \tilde{f} = (\langle f, \phi_i \rangle)_{i=0}^N, \quad \tilde{g} = (\langle g, \phi_i \rangle)_{i=0}^N. \]

It is possible to prove that the bilinear form \( s_K \) satisfies Assumption 2.1 and, provided (3.1) holds, (2.2) (actually, (3.1) is a necessary and sufficient condition for Assumption 2.2 to hold).

Observe that for \( u, v \in (H^1(K))^\prime \) and \( \lambda, \mu \in H^{-1/2}(\partial K) \) we have
\[ s_K(Du - \gamma_K^* \lambda, tDv - \gamma_K^* \mu) = \tilde{\eta}^T S^{-1} \tilde{\zeta}, \quad s_K(f, tDv - \gamma_K^* \mu) = \tilde{f}^T S^{-1} \tilde{\zeta} \]
with
\[ \eta_i = \int_K \nabla u \cdot \nabla \phi_i - \int_{\partial K} \lambda \phi_i, \quad \zeta_i = t \int_K \nabla v \cdot \nabla \phi_i - \int_{\partial K} \mu \phi_i, \quad f_i = \int_K f \phi_i. \]

We then only need to choose a (small) space \( W_K \) satisfying (3.1) (remark that \( W_K \) is not required to satisfy any approximation property). We choose a suitable subspace of the Virtual Element space of order \( k + 2 \) [3]. More precisely, we set
\[ W_K = \{ w \in H^1(K) : w|_e \in P_{k+2}(e), \forall \text{ edge } e \text{ of } K, -\Delta w \in P_k(K), \int_K wp = 0 \forall p \in P_k(K) \}. \]

The space \( W_K \) does indeed satisfy the inf-sup condition (3.1), as stated by the following Lemma.
Lemma 3.1. It holds that
\[
\inf_{\lambda \in \Lambda_K} \sup_{w \in W_K} \frac{\int_K \lambda w}{|\lambda|_{-1/2,K} |w|_{1,K}} \gtrsim 1.
\]

Proof. It is not difficult to check that
\[
\inf_{\lambda \in \Lambda_K} \sup_{w \in W_K} \frac{\int_{\partial K} \lambda w}{|\lambda|_{-1/2,\partial K} |w|_{1/2,\partial K}} \gtrsim 1.
\]
We have, for \( w \in W_K \) with \( \int_{\partial K} w = 0 \)
\[
|w|^2_{1,K} = \int_K |\nabla w|^2 = -\int_K w \Delta w + \int_{\partial K} w \partial_\nu w = \int_{\partial K} w \partial_\nu w \leq |\partial_\nu w|_{-1/2,\partial K} |w|_{1/2,\partial K}.
\]
Now we have
\[
|\partial_\nu w|_{-1/2,\partial K} \lesssim |\gamma_K^* (\partial_\nu w)|_{-1,K} = \sup_{\phi \in H^1(K)} \frac{\int_K \nabla w \cdot \nabla \phi + \int_K \phi \Delta w}{|\phi|_{1,K}} \lesssim \|w\|_{1,K},
\]
where we use Poincaré inequality to bound \( \|\phi\|_{L^2(K)} \) and an inverse inequality to bound \( \|\Delta w\|_{L^2(K)} \), thus obtaining
\[
|w|^2_{1,K} \lesssim |w|_{1,K} |w|_{1/2,\partial K}
\]
whence \( |w|_{1,K} \lesssim |w|_{1/2,\partial K} \) and, consequently
\[
\inf_{\lambda \in \Lambda_K} \sup_{w \in W_K} \frac{\int_{\partial K} \lambda w}{|\lambda|_{-1/2,\partial K} |w|_{1,K}} \gtrsim \inf_{\lambda \in \Lambda_K} \sup_{w \in W_K} \frac{\int_{\partial K} \lambda w}{|\lambda|_{-1/2,\partial K} |w|_{1/2,\partial K}} \gtrsim 1.
\]

It is known [5] that a function \( \phi \in W_K \) is uniquely determined by a) the value of \( \phi \) at the vertices of the polygon \( K \); b) for each edge \( e \), the values of \( \phi \) at the \( k + 1 \) internal points of the \( k + 3 \)-points Gauss-Lobatto quadrature rule on \( e \) (the other degrees of freedom for the standard VEM space of order \( k + 2 \) are the moments up to order \( k \), which we fixed to be zero in the definition of \( W_K \)). Each one of these degrees of freedom corresponds to a basis function \( \phi_i \) (the unique function in \( W_K \) for which such degree of freedom assumes value 1, while all the other vanish). The basis functions are not explicitly known, but the knowledge of the degrees of freedom is sufficient to compute the vectors \( \vec{\eta} \) and \( \vec{\zeta} \). In fact, for \( u \in P_k(K) \) and \( \lambda \in \Lambda_K \) we have
\[
\eta_i = \int_K \nabla u \cdot \nabla \phi_i - \int_{\partial K} \lambda \phi_i = -\int_K \Delta u \phi_i + \int_{\partial K} \left( \frac{\partial u}{\partial \nu_K} - \lambda \right) \phi_i = \int_{\partial K} \left( \frac{\partial u}{\partial \nu_K} - \lambda \right) \phi_i
\]
where we used that \( \phi_i \) is orthogonal to all polynomials of degree less or equal than \( k \) and hence to \( \Delta u \). The last term is computable since it is the integral of a known piecewise polynomial. The fact that \( \phi \) is orthogonal to polynomials also allows us to approximate \( f_i \approx 0 \). As far as the matrix \( S \) is concerned, we recall that in general the stiffness matrix entries are not exactly computable. However, the virtual element method takes advantage of the observation that, by same technique used above to compute \( \eta_i \), it is possible to compute
the $H^1$ projection of the functions in $W_K$ onto the space $\mathbb{P}_{k+2}$. This allows to design a recipe for constructing a computable bilinear form $a^K$ satisfying $a^K(w, w) \simeq |w|^2_{1,K}$ (we refer to [3] for the details), and we take $S$ to be the stiffness matrix relative to such bilinear form.

Remark 3.2. A more natural choice for the auxiliary space $W_K$ would be

$$W_K = \{w \in H^1(K) : w|_e \in \mathbb{P}_{k+2}(e), \text{ for all edge } e \text{ of } K, \ -\Delta w = 0 \text{ in } K\}.$$

Remark that, however, for such a choice the consistency term $f_i$ would not be 0 and it would not be computable.

Also in this case we can give an interpretation of the stabilization as a suitable numerical trace. In fact, once again in the ideal case that $S = (s_{ij})$ with $s_{ij} = \int_K \nabla \phi_i \cdot \nabla \phi_j$, it is not difficult to realize that the vector $\vec{x} = (x_j) = S^{-1}\vec{y}$ is the vector of coefficient of the function $w_h = \sum_i x_i \phi_i \in W_K$ verifying for all $z_h \in W_K$

$$\int_K \nabla w_h \cdot \nabla z_h = \int_K \nabla u^K_h \cdot \nabla z_h - \int_{\partial K} \lambda^K_h z_h.$$

Letting $\hat{\Pi}^K : H^1(K) \to W_K$ denote the Galerkin projection onto $W_K$, it is not difficult to verify that the stabilized problem can be rewritten as

$$\int_K \nabla u^K_h \cdot \nabla v^K_h - \int_{\partial K} \lambda^K_h = \int_K f v^K_h$$

(3.3)

$$\int_{\partial K} \hat{u}^K_h \mu^K_h = \int_{\partial K} \hat{\phi}^K_h \mu^K_h$$

(3.4)

with

$$\hat{u}^K_h = u^K_h - \alpha \hat{\Pi}^K R(Du^K_h - \gamma^K \lambda^K_h), \quad \text{and} \quad \hat{\phi}^K_h = \phi - \alpha \hat{\Pi}^K R f.$$

Replacing the stiffness matrix $S$ with an approximation or a preconditioner, reduces to replacing the Galerkin projection operator $\hat{\Pi}^K$ with a spectrally equivalent projector $\tilde{\Pi}_W$ and setting

$$\tilde{\hat{u}}^K_h = u^K_h - \alpha \tilde{\Pi}_W R(Du^K_h - \gamma^K \lambda^K_h), \quad \text{and} \quad \tilde{\hat{\phi}}^K_h = \phi - \alpha \tilde{\Pi}_W R f.$$

4. Numerical Results

We take the domain $\Omega$ to be the unit square $[0, 1] \times [0, 1]$. We solve Problem 2.1 with Dirichlet boundary conditions and load term chosen in such a way that

$$u = \frac{1}{2\pi^2} \sin(\pi x) \sin(\pi y)$$

is the exact solution. The stabilization parameters are chosen to be $\alpha = t = 1$. We test our method on a deformed hexagonal mesh (Test case 1.) and on a shape regular Voronoi mesh (Test case 2.). For the first test case, Tables 2–4 show the relative errors $e^0_u = ||u - u_h||_{L^2}/||u||_{L^2}$, $e^1_u = ||u - u_h||_{H^1}/||u||_{H^1}$ and the estimated convergence rates (ecr) for several values of the polynomial degree $k$ on the eight meshes in Table 1. The results for such tests are also displayed in Figure 2.

For the second Test case, the results are shown in Tables 5 and 6 and summarized in Figure 4.
| Name                        | Nb. of elements | Nb. of faces | $h$  |
|-----------------------------|-----------------|--------------|------|
| Deformed hexagons 8x10      | 94              | 283          | 0.2190 |
| Deformed hexagons 18x20     | 389             | 1168         | 0.1033 |
| Deformed hexagons 26x30     | 822             | 2467         | 0.0711 |
| Deformed hexagons 34x40     | 1415            | 4246         | 0.0542 |
| Deformed hexagons 44x50     | 2270            | 6811         | 0.0423 |
| Deformed hexagons 52x60     | 3203            | 9610         | 0.0357 |
| Deformed hexagons 60x70     | 4296            | 12889        | 0.0308 |
| Deformed hexagons 70x80     | 5711            | 17134        | 0.0266 |

Table 1. Data for the eight meshes used in Test case 1.

| $k = 1$               | $k = 2$               |
|-----------------------|-----------------------|
| $e_{0}^{u}$ $ecr$     | $e_{1}^{u}$ $ecr$    |
| $5.7354e-03$ $ecr$    | $1.2437e-02$ $ecr$   |
| 2.0699e-02 $ecr$      | 1.4656e-01 $ecr$      |
| 4.7983e-03 $ecr$      | 7.0802e-02 $ecr$      |
| 2.2425e-03 $ecr$      | 4.8468e-02 $ecr$      |
| 1.2939e-03 $ecr$      | 3.6843e-02 $ecr$      |
| 8.0139e-04 $ecr$      | 2.9007e-02 $ecr$      |
| 5.6644e-04 $ecr$      | 2.4393e-02 $ecr$      |
| 4.2152e-04 $ecr$      | 2.1047e-02 $ecr$      |
| 3.1623e-04 $ecr$      | 1.8233e-02 $ecr$      |

Table 2. Errors and estimated convergence rates ($ecr$) for the first test case, $k = 1, 2$.

| $k = 3$               | $k = 4$               |
|-----------------------|-----------------------|
| $e_{0}^{u}$ $ecr$     | $e_{1}^{u}$ $ecr$    |
| $5.7354e-03$ $ecr$    | $1.2437e-02$ $ecr$   |
| 2.0699e-02 $ecr$      | 1.4656e-01 $ecr$      |
| 4.7983e-03 $ecr$      | 7.0802e-02 $ecr$      |
| 2.2425e-03 $ecr$      | 4.8468e-02 $ecr$      |
| 1.2939e-03 $ecr$      | 3.6843e-02 $ecr$      |
| 8.0139e-04 $ecr$      | 2.9007e-02 $ecr$      |
| 5.6644e-04 $ecr$      | 2.4393e-02 $ecr$      |
| 4.2152e-04 $ecr$      | 2.1047e-02 $ecr$      |
| 3.1623e-04 $ecr$      | 1.8233e-02 $ecr$      |

Table 3. Errors and estimated convergence rates ($ecr$) for the first test case, $k = 3, 4$. 
As we can see the results confirm the theoretical estimate, with the correct order of convergence for the $H^1$ norm of the error as $h$ tends to zero. We remark that, in the second Test case we seem to have some kind of superconvergence. We believe that this is rather due
Figure 1. meshes

Figure 2. Test case 1. Hexagonal meshes, H1 and L2 errors for different values of $k$

Figure 3. Voronoi meshes

to the the fact that, since the grids are unstructured, there mesh size parameter is non as clearcut defined as in a structured case.

References

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[2] M. Arioli and D. Loghin. Discrete interpolation norms with applications. SIAM J. Numer. Anal., 47:2924–2951, 2009.
Figure 4. Test case 2. Voronoi meshes, H1 and L2 errors for different values of $k$.