Random Cyclic Quadrilaterals

STEVEN FINCH

October 3, 2016

Abstract. The circumcircle of a planar convex polygon \( P \) is a circle \( C \) that passes through all vertices of \( P \). If such a \( C \) exists, then \( P \) is said to be cyclic. Fix \( C \) to have unit radius. While any two angles of a uniform cyclic triangle are negatively correlated, any two sides are independent. In contrast, for a uniform cyclic quadrilateral, any two sides are negatively correlated, whereas any two adjacent angles are uncorrelated yet dependent.

To generate a cyclic triangle is easy: select three independent uniform points on the unit circle and connect them. To generate a cyclic quadrilateral is harder: select four such points and connect them in, say, a counterclockwise manner. Convexity follows immediately [1], as does the fact that opposite angles are supplementary [2, 3]. The inter-relationship of adjacent angles is more mysterious, as we shall soon see. Our initial focus, however, will be on adjacent sides, opposite sides and diagonals.

Let the four vertices be given by \( \exp(i \theta_k) \), where \( i \) is the imaginary unit, \( 0 \leq \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi \) are central angles relative to the horizontal axis, and \( 1 \leq k \leq 4 \). Define \( \theta_0 = \theta_4 - 2\pi \) and \( \theta_5 = \theta_1 + 2\pi \) for convenience, then polygonal sides \( s_k \) and polygonal angles \( \alpha_k \) are given by

\[
s_k = 2 \sin \left( \frac{\theta_k - \theta_{k-1}}{2} \right), \quad \alpha_k = \frac{\theta_{k+1} - \theta_{k-1}}{2}.
\]

Proof of the \( s_k \) expression comes from the Law of Cosines and a half angle formula:

\[
s_k^2 = 1 + 1 - 2 \cdot 1 \cdot 1 \cos(\theta_k - \theta_{k-1}) = 2 \left[ 1 - \cos(\theta_k - \theta_{k-1}) \right]
\]

\[
= 4 \frac{1 - \cos(\theta_k - \theta_{k-1})}{2} = 4 \sin^2 \left( \frac{\theta_k - \theta_{k-1}}{2} \right).
\]

Proof of the \( \alpha_k \) expression follows the fact that an inscribed angle is one-half the length of its intercepted circular arc. The polygonal diagonals \( d_k \) clearly satisfy \( d_k = 2 \sin(\alpha_k) \). Let also \( \omega \) denote the smaller of the two angles at the intersection point between the diagonals.

Our labor draws upon the distribution of the order statistics \( \theta_1, \theta_2, \theta_3, \theta_4 \). We must be careful in summarizing the results because, while \( s_2, s_3, s_4 \) possess the same...
Figure 1: Vertices, angles and sides of a cyclic quadrilateral.
distribution, the one corresponding to \( s_1 \) is different. Hence, to make statements regarding arbitrary sides \( s, t, u, v \) of the quadrilateral, we must use a \((3/4, 1/4)\)-mixture of densities. Likewise, \( \alpha_2 \) and \( \alpha_1 \) possess distinct distributions. Thus, to make statements regarding arbitrary adjacent angles \( \alpha, \beta \) of the quadrilateral, we must use a \((1/2, 1/2)\)-mixture of densities.

A probabilistic analysis of the perimeter \( s + t + u + v \) and area \( 2 \sin(\alpha) \sin(\beta) \sin(\omega) \) is beyond our current capabilities. Hopefully the groundwork established here will be a launching point for someone else’s research in the near future.

1. Sides

Let \( X_1 < X_2 < X_3 < X_4 \) denote the order statistics for a random sample of size 4 from the uniform distribution on \([0, 1]\). The density for \((X_1, X_2) = (x, y)\) is

\[
\begin{align*}
\begin{cases}
12(1-y)^2 & \text{if } 0 < x < y < 1, \\
0 & \text{otherwise};
\end{cases}
\end{align*}
\]

the density for \((X_1, X_3) = (x, y)\) is

\[
\begin{align*}
\begin{cases}
24(y-x)(1-y) & \text{if } 0 < x < y < 1, \\
0 & \text{otherwise};
\end{cases}
\end{align*}
\]

the density for \((X_1, X_4) = (x, y)\) is

\[
\begin{align*}
\begin{cases}
12(y-x)^2 & \text{if } 0 < x < y < 1, \\
0 & \text{otherwise};
\end{cases}
\end{align*}
\]

the density for \((X_2, X_4) = (x, y)\) is

\[
\begin{align*}
\begin{cases}
24x(y-x) & \text{if } 0 < x < y < 1, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

Consider the transformation \((x, y) \mapsto (y-x, y) = (u, v)\). Since this has Jacobian determinant 1 and since \( 0 < u < v < 1 \), it follows that the density for \( X_2 - X_1 \) is

\[
12 \int_{u}^{1} (1-v)^2 dv = -4(1-v)^3|_{u}^{1} = 4(1-u)^3;
\]

the density for \( X_3 - X_1 \) is

\[
24 \int_{u}^{1} u(1-v) dv = -12u(1-v)^2|_{u}^{1} = 12u(1-u)^2;
\]
the density for \( X_4 - X_1 \) is
\[
12 \int_u^1 u^2 dv = -12u^2(1-v)|_u^1 = 12u^2(1-u);
\]
the density for \( X_4 - X_2 \) is
\[
24 \int_u^1 u(v-u)dv = 12u(v-u)^2|_u^1 = 12u(1-u)^2.
\]
We disregard \( X_4 - X_2 \) further since its distribution is the same as that for \( X_3 - X_1 \).

Consider the scaling \( u \mapsto \pi u = x \). It follows that

the density for \( \frac{\theta_2 - \theta_1}{2} \) is \( \frac{4}{\pi} \left( \frac{1 - x}{\pi} \right)^3 \);
the density for \( \frac{\theta_3 - \theta_1}{2} \) is \( \frac{12}{\pi} \left( \frac{x}{\pi} \right) \left( 1 - \frac{x}{\pi} \right)^2 \);
the density for \( \frac{\theta_4 - \theta_1}{2} \) is \( \frac{12}{\pi} \left( \frac{x}{\pi} \right)^2 \left( 1 - \frac{x}{\pi} \right) \).

Next, the function \( x \mapsto \sin(x) = y \) possesses two preimages \( \arcsin(y) \) and \( \pi - \arcsin(y) \) in the interval \([0, \pi]\) and has derivative \( \cos(x) = \sqrt{1 - y^2} \). It follows that the three densities are \([6]\)

\[
\frac{4}{\pi \sqrt{1 - y^2}} \left[ \left( 1 - \frac{\arcsin(y)}{\pi} \right)^3 + \left( \frac{\arcsin(y)}{\pi} \right)^3 \right],
\]
\[
\frac{12}{\pi \sqrt{1 - y^2}} \left[ \left( \frac{\arcsin(y)}{\pi} \right) \left( 1 - \frac{\arcsin(y)}{\pi} \right)^2 + \left( 1 - \frac{\arcsin(y)}{\pi} \right) \left( \frac{\arcsin(y)}{\pi} \right)^2 \right],
\]
\[
\frac{12}{\pi \sqrt{1 - y^2}} \left[ \left( \frac{\arcsin(y)}{\pi} \right)^2 \left( 1 - \frac{\arcsin(y)}{\pi} \right) + \left( 1 - \frac{\arcsin(y)}{\pi} \right)^2 \left( \frac{\arcsin(y)}{\pi} \right) \right]
\]
respectively. The second and third expressions are identical. Finally, the scaling \( y \mapsto 2y = z \) and an algebraic expansion gives the density for \( s_2 \) as
\[
\frac{4}{\pi \sqrt{4 - z^2}} \left[ 1 - \frac{3 \arcsin\left( \frac{z}{2} \right) \left( \pi - \arcsin\left( \frac{z}{2} \right) \right)}{\pi^2} \right]
\]
and the density for both \( d_2 \) and \( s_1 \) as
\[
\frac{4}{\pi \sqrt{4 - z^2}} \left[ 0 + \frac{3 \arcsin\left( \frac{z}{2} \right) \left( \pi - \arcsin\left( \frac{z}{2} \right) \right)}{\pi^2} \right].
\]
We omit details for $s_3, s_4$ (same as $s_2$) and $d_1$ (same as $d_2$).

Mixing the densities for $s_2$ (with weight $3/4$) and for $s_1$ (with weight $1/4$), the density for an arbitrary side $0 \leq s \leq 2$ emerges:

$$
\frac{3}{\pi \sqrt{4 - s^2}} \left[ 1 - \frac{2 \arcsin\left(\frac{s}{2}\right) \left(\pi - \arcsin\left(\frac{s}{2}\right)\right)}{\pi^2} \right]
$$

which implies that

$$
E(s) = \frac{6}{\pi} - \frac{24}{\pi^3}, \quad E(s^2) = 2 - \frac{3}{\pi^2}.
$$

The corresponding moments for a diagonal $0 \leq d \leq 2$ are $48/\pi^3$ and $2 + 6/\pi^2$. Joint moments are available via the joint density of $\theta_1, \theta_2, \theta_3, \theta_4$:

$$
\begin{cases}
\frac{4!}{(2\pi)^4} & \text{if } 0 \leq \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi, \\
0 & \text{otherwise.}
\end{cases}
$$

For example,

$$
E(s_2 s_3) = \frac{3}{2\pi^4} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ 2 \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \right] \left[ 2 \sin\left(\frac{\theta_3 - \theta_2}{2}\right) \right] d\theta_4 d\theta_3 d\theta_2 d\theta_1
$$

$$
= \frac{48}{\pi^2} - \frac{384}{\pi^4}
$$

(same for $E(s_3 s_4)$) and

$$
E(s_1 s_2) = \frac{3}{2\pi^4} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ 2 \sin\left(\frac{\theta_4 - \theta_1}{2}\right) \right] \left[ 2 \sin\left(\frac{\theta_2 - \theta_1}{2}\right) \right] d\theta_4 d\theta_3 d\theta_2 d\theta_1
$$

$$
= -\frac{24}{\pi^2} + \frac{384}{\pi^4}
$$

(same for $E(s_4 s_1)$) imply that, for arbitrary adjacent sides $s$ and $t$,

$$
E(st) = \frac{12}{\pi^2}, \quad \rho(s, t) \approx -0.183.
$$

We used

$$
\frac{\theta_1 - \theta_0}{2} = \frac{\theta_1 - (\theta_4 - 2\pi)}{2} = \frac{2\pi - (\theta_4 - \theta_1)}{2} = \frac{\pi - (\theta_4 - \theta_1)}{2}
$$

and $\sin(\pi - z) = \sin(z)$ in writing the preceding integral. The same value $12/\pi^2$ is also obtained for the expected product of arbitrary opposite sides $s$ and $t$. The proximity of quadrilateral sides is (evidently) immaterial when assessing their correlation.
Figure 2: Density function for side $s$ in Section 1.

Figure 3: Density function for diagonal $d$ in Section 1.
2. Angles

By our work starting with \( X_3 - X_1 \) and \( X_4 - X_2 \), it is clear that \( \alpha_2 \) and \( \alpha_3 \) are identically distributed. Since \( \alpha_1 = \pi - \alpha_3 \), the density of \( \alpha_2 \) is \( 12x(\pi - x)^2/\pi^4 \) while the density of \( \alpha_1 \) is \( 12x^2(\pi - x)/\pi^4 \). Mixing the densities for \( \alpha_2 \) and for \( \alpha_1 \) with equal weighting, the marginal density for an arbitrary angle \( 0 \leq \alpha \leq \pi \) becomes \( 6x(\pi - x)/\pi^3 \).

We need, however, to find the joint distribution for arbitrary adjacent angles \( \alpha \) and \( \beta \). A fresh approach for obtaining this involves the Dirichlet\((1, 1, 1; 1)\) distribution on a 3-dimensional simplex [7, 8, 9]:

\[
\begin{align*}
\xi_1 &< 1, \\
0 &< \xi_2 < 1, \\
0 &< \xi_3 < 1 \\
\xi_1 + \xi_2 + \xi_3 &< 1
\end{align*}
\]

and calculation of the joint density for \( \eta_1 = \xi_1 + \xi_2, \eta_2 = \xi_2 + \xi_3 \). The list \( \xi_1, \xi_2, \xi_3 \) can be thought of as duplicating any one of the eight lists given in Table 1, each weighted with probability \( 1/8 \). In words, up to the preservation of adjacency of angles \( \pi \eta_1, \pi \eta_2 \), any implicit ordering within \( \xi_1, \xi_2, \xi_3 \) has been removed. This formulation will simplify our work, removing the need to mix distributions (like before) as a concluding step.

Table 1. Eight possibilities for \( \xi_1, \xi_2, \xi_3 \).

| Candidate Lists | Resulting Angles |
|-----------------|-----------------|
| \( \frac{\theta_2 - \theta_1}{2\pi} \), \( \frac{\theta_3 - \theta_2}{2\pi} \), \( \frac{\theta_4 - \theta_3}{2\pi} \) | \( \pi \eta_1 = \alpha_2 \), \( \pi \eta_2 = \alpha_3 \) |
| \( \frac{\theta_4 - \theta_3}{2\pi} \), \( \frac{\theta_3 - \theta_2}{2\pi} \), \( \frac{\theta_2 - \theta_1}{2\pi} \) | \( \pi \eta_1 = \alpha_3 \), \( \pi \eta_2 = \alpha_2 \) |
| \( \frac{\theta_3 - \theta_2}{2\pi} \), \( \frac{\theta_4 - \theta_3}{2\pi} \), \( \frac{\theta_5 - \theta_4}{2\pi} \) | \( \pi \eta_1 = \alpha_3 \), \( \pi \eta_2 = \alpha_4 \) |
| \( \frac{\theta_5 - \theta_4}{2\pi} \), \( \frac{\theta_4 - \theta_3}{2\pi} \), \( \frac{\theta_3 - \theta_2}{2\pi} \) | \( \pi \eta_1 = \alpha_3 \), \( \pi \eta_2 = \alpha_3 \) |
| \( \frac{\theta_4 - \theta_3}{2\pi} \), \( \frac{\theta_5 - \theta_4}{2\pi} \), \( \frac{\theta_2 - \theta_1}{2\pi} \) | \( \pi \eta_1 = \alpha_4 \), \( \pi \eta_2 = \alpha_1 \) |
| \( \frac{\theta_2 - \theta_1}{2\pi} \), \( \frac{\theta_5 - \theta_4}{2\pi} \), \( \frac{\theta_4 - \theta_3}{2\pi} \) | \( \pi \eta_1 = \alpha_1 \), \( \pi \eta_2 = \alpha_4 \) |
| \( \frac{\theta_5 - \theta_4}{2\pi} \), \( \frac{\theta_2 - \theta_1}{2\pi} \), \( \frac{\theta_3 - \theta_2}{2\pi} \) | \( \pi \eta_1 = \alpha_1 \), \( \pi \eta_2 = \alpha_2 \) |
| \( \frac{\theta_3 - \theta_2}{2\pi} \), \( \frac{\theta_5 - \theta_4}{2\pi} \), \( \frac{\theta_5 - \theta_4}{2\pi} \) | \( \pi \eta_1 = \alpha_2 \), \( \pi \eta_2 = \alpha_1 \) |

Introducing \( \eta_3 = \xi_3 \), we have

\[
\begin{align*}
\xi_1 &= \eta_1 - \eta_2 + \eta_3, \\
\xi_2 &= \eta_2 - \eta_3, \\
\xi_3 &= \eta_3
\end{align*}
\]
and calculate the Jacobian determinant to be equal to 1. From
\[ 0 < \eta_1 - \eta_2 + \eta_3 < 1, \]
\[ 0 < \eta_2 - \eta_3 < 1, \]
\[ 0 < \eta_3 < 1, \]
\[ 0 < \eta_1 + \eta_3 < 1 \]

it follows that
\[ -\eta_1 + \eta_2 < \eta_3 < 1 - \eta_1 + \eta_2, \]
\[ -1 + \eta_2 < \eta_3 < \eta_2, \]
\[ 0 < \eta_3 < 1, \]
\[ -\eta_1 < \eta_3 < 1 - \eta_1 \]

hence \( \max\{-\eta_1 + \eta_2, 0\} < \eta_3 < \min\{\eta_2, 1 - \eta_1\} \). There are four cases:

1. If \( 1 - \eta_2 < \eta_1 < \eta_2 \), then \(-\eta_1 + \eta_2 < \eta_3 < 1 - \eta_1 + \eta_2\)

2. If \( \eta_1 < \eta_2 < 1 - \eta_1 \), then \(-\eta_1 + \eta_2 < \eta_3 < \eta_2\)

3. If \( 1 - \eta_1 < \eta_2 < \eta_1 \), then \(0 < \eta_3 < 1 - \eta_1\)

4. If \( \eta_2 < \eta_1 < 1 - \eta_2 \), then \(0 < \eta_3 < \eta_2\)

giving rise to
\[ \int_{-\eta_1 + \eta_2}^{1-\eta_1} 6 \, d\eta_3 = 6 \left(1 - \eta_2\right), \]
\[ \int_{-\eta_1 + \eta_2}^{\eta_2} 6 \eta_1 \, d\eta_3 = 6 \eta_1, \]
\[ \int_{0}^{\eta_2} 6 \, d\eta_3 = 6 \left(1 - \eta_1\right), \]
\[ \int_{0}^{\eta_1} 6 \eta_3 \, d\eta_2 = 6 \eta_2 \]

and thus the joint density for \( \eta_1, \eta_2 \) is
\[
\begin{cases} 
6 \left(1 - \eta_2\right) & \text{if } 1 - \eta_2 < \eta_1 < \eta_2 \text{ and } 1/2 < \eta_2 < 1, \\
6\eta_1 & \text{if } \eta_1 < \eta_2 < 1 - \eta_1 \text{ and } 0 < \eta_1 < 1/2, \\
6 \left(1 - \eta_1\right) & \text{if } 1 - \eta_1 < \eta_2 < \eta_1 \text{ and } 1/2 < \eta_1 < 1, \\
6\eta_2 & \text{if } \eta_2 < \eta_1 < 1 - \eta_2 \text{ and } 0 < \eta_2 < 1/2.
\end{cases}
\]

The sought-after joint density for \( \alpha, \beta \) is therefore
\[
\begin{cases} 
6 \left(\pi - \beta\right)/\pi^3 & \text{if } \pi - \beta < \alpha < \beta \text{ and } \pi/2 < \beta < \pi, \\
6\alpha/\pi^3 & \text{if } \alpha < \beta < \pi - \alpha \text{ and } 0 < \alpha < \pi/2, \\
6 \left(\pi - \alpha\right)/\pi^3 & \text{if } \pi - \alpha < \beta < \alpha \text{ and } \pi/2 < \alpha < \pi, \\
6\beta/\pi^3 & \text{if } \beta < \alpha < \pi - \beta \text{ and } 0 < \beta < \pi/2. 
\end{cases}
\]

and we call this the bivariate tent distribution (as opposed to pyramid distribution, which already means something else [10]). It is clear that \( \rho(\alpha, \beta) = 0 \) yet \( \alpha \) and \( \beta \) are dependent.
Figure 4: Density function for angle $\alpha$ in Section 2.

Figure 5: Density function for bivariate tent distribution on $[0, \pi] \times [0, \pi]$. 
3. Looking Back

Given a uniform cyclic triangle, the joint density for two arbitrary angles $\alpha$, $\beta$ is
\[
\begin{cases}
  2/\pi^2 & \text{if } 0 < \alpha < \pi, 0 < \beta < \pi \text{ and } \alpha + \beta < \pi, \\
  0 & \text{otherwise}
\end{cases}
\]
and trivially $\rho(\alpha, \beta) = -1/2$. Let $\Delta$ denote the isosceles triangular support of this distribution. Let $a$ denote the side opposite $\alpha$ and $b$ denote the side opposite $\beta$. From
\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} 2 \sin(\alpha) \\ 2 \sin(\beta) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},
\]
we have Jacobian determinant $4 \cos(\alpha) \cos(\beta)$ and preimages
\[
\begin{pmatrix} \arcsin \left( \frac{a}{2} \right) \\ \arcsin \left( \frac{b}{2} \right) \end{pmatrix}, \quad \begin{pmatrix} \pi - \arcsin \left( \frac{a}{2} \right) \\ \arcsin \left( \frac{b}{2} \right) \end{pmatrix}
\]
if $b < a$ and
\[
\begin{pmatrix} \arcsin \left( \frac{a}{2} \right) \\ \arcsin \left( \frac{b}{2} \right) \end{pmatrix}, \quad \begin{pmatrix} \arcsin \left( \frac{b}{2} \right) \\ \pi - \arcsin \left( \frac{a}{2} \right) \end{pmatrix}
\]
if $a < b$. Reason: if $b < a$, then $\arcsin(b/2) < \arcsin(a/2)$ and hence both preimages fall in $\Delta$ because $[\pi - \arcsin(a/2)] + \arcsin(b/2) < \pi$. No other preimages exist when $b < a$ because $\arcsin(a/2) + [\pi - \arcsin(b/2)] > \pi$ and $[\pi - \arcsin(a/2)] + [\pi - \arcsin(b/2)] > \pi$. Likewise for $a < b$.

The joint density for $a$ and $b$ is thus
\[
\begin{cases}
  \frac{4 \pi^2}{\sqrt{4 - a^2} \sqrt{4 - b^2}} & \text{if } 0 < a < 2 \text{ and } 0 < b < 2, \\
  0 & \text{otherwise}
\end{cases}
\]
which implies that sides $a$, $b$ are independent even though they are related so easily (via the sine function) to the dependent angles $\alpha$, $\beta$. As far as is known, this observation is new. We mention that the remaining side $c$ satisfies
\[
c = \begin{cases}
  \frac{1}{2} \left( a \sqrt{4 - b^2} + b \sqrt{4 - a^2} \right) & \text{with probability } 1/2, \\
  \frac{1}{2} \left( a \sqrt{4 - b^2} - b \sqrt{4 - a^2} \right) & \text{with probability } 1/2
\end{cases}
\]
for completeness’ sake.

4. Looking Forward

The polygonal angles $\alpha$, $\beta$, $\gamma$, $\delta$ associated with a uniform cyclic 5-gon can be studied via the Dirichlet$(1, 1, 1, 1; 1)$ distribution on a 4-dimensional simplex $[7, 8, 9]$:}
\[
\begin{cases}
  24 & \text{if } 0 < \xi_1 < 1, 0 < \xi_2 < 1, 0 < \xi_3 < 1, 0 < \xi_4 < 1 \text{ and } \xi_1 + \xi_2 + \xi_3 + \xi_4 < 1, \\
  0 & \text{otherwise}
\end{cases}
\]
and calculation of the joint density for \( \eta_1 = \xi_1 + \xi_2 + \xi_3, \eta_2 = \xi_2 + \xi_3 + \xi_4, \eta_3 = 1 - \xi_1 - \xi_2, \eta_4 = 1 - \xi_2 - \xi_3 \). Omitting elaborate details, we obtain the density to be 24 when

\[
\max\{1 - \eta_1, 1 - \eta_2\} < \eta_4 < \min\{2 - \eta_1 - \eta_2, 2 - \eta_1 - \eta_3\} \quad \text{and} \quad 1 < \eta_1 + \eta_3 < 2
\]

and 0 otherwise. It follows that

\[
\rho(\alpha, \beta) = \frac{1}{6}, \quad \rho(\alpha, \gamma) = -\frac{2}{3}, \quad \rho(\alpha, \delta) = -\frac{2}{3}, \quad \rho(\alpha, \varphi) = \frac{1}{6}
\]

where \( \varphi = 3\pi - \alpha - \beta - \gamma - \delta \). In particular, adjacent angles are positively correlated and non-adjacent angles are negatively correlated.

For a uniform cyclic 6-gon, we conjecture that

\[
\rho(\alpha, \beta) = \frac{1}{4}, \quad \rho(\alpha, \gamma) = -\frac{1}{2}, \quad \rho(\alpha, \delta) = -\frac{1}{2}, \quad \rho(\alpha, \varphi) = \frac{1}{4}
\]

where \( \varphi = 2\pi - \alpha - \gamma \) and \( \psi = 2\pi - \beta - \delta \). Again, adjacent angles are positively correlated and non-adjacent angles are negatively correlated. The fact that \( \delta \) is opposite \( \alpha \) seems not to affect its correlation with \( \alpha \), relative to either \( \gamma \) or \( \varphi \).

5. Area

Given a uniform cyclic triangle, moments of area \( 2 \sin(\alpha) \sin(\beta) \sin(\alpha + \beta) \) are computed by use of the joint angle density:

\[
\frac{2}{\pi^2} \int_0^\pi \int_0^{\pi - \beta} 2 \sin(\alpha) \sin(\beta) \sin(\alpha + \beta) d\alpha d\beta = \frac{3}{2\pi},
\]

\[
\frac{2}{\pi^2} \int_0^\pi \int_0^{\pi - \beta} 4 \sin^2(\alpha) \sin^2(\beta) \sin^2(\alpha + \beta) d\alpha d\beta = \frac{3}{8}.
\]

The density for area itself is \( 8xK(4x^2) \), where

\[
K(y) = \frac{1}{4\pi^3} \frac{1}{\sqrt{y}} \left\{ \Gamma\left(\frac{1}{3}\right) \right\}^3 \left(\frac{4y}{27}\right)^{-1/6} \quad 2F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; \frac{4y}{27}\right)
\]

\[
\quad 3 \Gamma\left(\frac{2}{3}\right) \left(\frac{4y}{27}\right)^{1/6} \quad 2F_1\left(\frac{2}{3}, \frac{2}{3}, \frac{4y}{27}\right)
\]

\( 2F_1 \) is the Gauss hypergeometric function and \( 0 < y < 27/4 \). This formula corrects that which appears in Case III of [13].
Given a uniform cyclic quadrilateral, we conjecture that the joint density for angles \( \alpha, \beta, \omega \) is

\[
 f(\alpha, \beta, \omega) = \begin{cases} 
 \frac{3}{\pi^3} & \text{if } \alpha + \beta > \omega, \alpha + \omega > \beta, \beta + \omega > \alpha \text{ and } \alpha + \beta + \omega < 2\pi, \\
 0 & \text{otherwise.}
\end{cases}
\]

It can be shown that, assuming the formula for \( f \) is valid, any two angles from the list \( \alpha, \beta, \omega \) are distributed according to the bivariate tent density. Our conjecture is consistent with computer simulation, but a rigorous proof is open. From this, we obtain area moments

\[
\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} 2 \sin(\alpha) \sin(\beta) \sin(\omega) f(\alpha, \beta, \omega) \, d\alpha \, d\beta \, d\omega = \frac{3}{\pi},
\]

\[
\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} 4 \sin^2(\alpha) \sin^2(\beta) \sin^2(\omega) f(\alpha, \beta, \omega) \, d\alpha \, d\beta \, d\omega = \frac{1}{2} + \frac{105}{16\pi^2}
\]

which again is consistent with experiment. The mean area for quadrilaterals is twice that for triangles. No formula for the density of area itself is known.

The problem with angles is that we do not know a suitable way of relating \( \omega \) with parameters \( \theta_1, \theta_2, \theta_3, \theta_4 \). For a cyclic quadrilateral with successive sides \( a, b, c, d \), formulas like [14]

\[
\tan \left( \frac{\alpha}{2} \right) = \sqrt{\frac{(a + b + c + d)(a - b + c + d)}{(a - b - c + d)(a + b - c - d)}} \quad \text{where } \alpha \text{ is angle between } a \text{ and } b,
\]

\[
\tan \left( \frac{\beta}{2} \right) = \sqrt{\frac{(a - b + c + d)(a + b - c + d)}{(-a + b + c + d)(a + b + c - d)}} \quad \text{where } \beta \text{ is angle between } b \text{ and } c,
\]

\[
\tan \left( \frac{\omega}{2} \right) = \sqrt{\frac{(a - b + c + d)(a + b + c - d)}{(-a + b + c + d)(a + b - c + d)}} \quad \text{where } \omega \text{ is angle between diagonals}
\]
suggest an alternative approach to solution, but the path seems very complicated.

6. Acknowledgements

I am indebted to Chi Zhang for her hand calculations in Sections 2 and 4 (specifically, those involving \( \xi \)s and \( \eta \)s). I am also grateful to Guo-Liang Tian, Serge Provost and Paul Kettler for helpful discussions.
Figure 6: Tetrahedral support for $f$, with vertices $(0, 0, 0)$, $(0, \pi, \pi)$, $(\pi, 0, \pi)$, $(\pi, \pi, 0)$.
References

[1] I. Pinelis, Cyclic polygons with given edge lengths: existence and uniqueness, *J. Geom.* 82 (2005) 156–171; MR2161821.

[2] R. Morris, The cyclic quadrilateral, a recreation, *School Science and Mathematics* 24 (1924) 296–300.

[3] E. E. Moise, *Elementary Geometry from an Advanced Standpoint*, Addison-Wesley, 1963, pp. 192–196; MR0149339 (26 #6829).

[4] J. D. Gibbons, *Nonparametric Statistical Inference*, McGraw-Hill, 1971, pp. 26–30; MR0286223 (44 #3437).

[5] H. A. David and H. N. Nagaraja, *Order Statistics*, 3rd ed., Wiley, 2003, pp. 11–13; MR1994955.

[6] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, 1965, pp. 125–127, 201–205; MR0176501 (31 #773).

[7] J. S. Rao, Some tests based on arc-lengths for the circle, *Sankhya Ser. B* 38 (1976) 329–338; MR0652731 (58 #31571).

[8] S. B. Provost and Y.-H. Cheong, On the distribution of linear combinations of the components of a Dirichlet random vector, *Canad. J. Statist.* 28 (2000) 417–425; MR1792058.

[9] K. W. Ng, G.-L. Tian and M.-L. Tang, *Dirichlet and Related Distributions: Theory, Methods and Applications*, Wiley, 2011, pp. 37–96; MR2830563.

[10] P. C. Kettler, The pyramid distribution, unpublished note (2006), http://www.paulcarislekettler.net/academics/.

[11] R. E. Miles, The various aggregates of random polygons determined by random lines in a plane, *Adv. Math.* 10 (1973) 256–290; MR0319232 (47 #7777).

[12] T. Moore, RE: Random triangle problem (long summary), http://mathforum.org/kb/plaintext.jspa?messageID=86196.

[13] A. M. Mathai and D. S. Tracy, On a random convex hull in an n-ball, *Comm. Statist. A - Theory Methods* 12 (1983) 1727–1736; MR0704849 (85c:60013).

[14] C. V. Durell and A. Robson, *Advanced Trigonometry*, Bell, 1937, pp. 24–27.
Steven Finch  
MIT Sloan School of Management  
Cambridge, MA, USA  
steven_finch@harvard.edu