Abstract

Let $(\Omega, \mu)$ be a set of real numbers to which we associate a measure $\mu$. Let $\alpha \geq 0$, let $\Omega_\alpha = \{ x \in \Omega / \alpha(x) = \alpha \}$, where $\alpha$ is the concentration index defined by Halsey et al. [Halsey et al., 1986]. Let $f_H(\alpha)$ be the Hausdorff dimension of $\Omega_\alpha$. Let $f_L(\alpha)$ be the Legendre spectrum of $\Omega$, defined in [Halsey et al., 1986]. The task of comparing $f_H, f_C$, and $f_L$ for different measures $\mu$ was tackled by several authors ([Cawley and Mauldin, 1992], [Mandelbrot and Riedi, 1997], [Riedi and Mandelbrot, 1998]) working, mainly, on self-similar measures $\mu$. The Farey tree partition in the unit segment induces a probability measure $\mu$ on an universal class of fractal sets $\Omega$ that occur in Physics and other disciplines. This measure $\mu$ is the Hyperbolic measure $\mu^H$, fundamentally different from any self-similar one. In this paper we compare $f_H, f_C$, and $f_L$ for $\mu^H$.

1 Introduction

In recent papers [Cawley and Mauldin, 1992], [Mandelbrot and Riedi, 1997], [Riedi and Mandelbrot, 1998], a number of multifractal spectra $(\alpha, f(\alpha))$ of a certain object $\Omega$ were defined and their various properties studied. For instance $(\alpha, f_G(\alpha))$ [Riedi and Mandelbrot, 1998] is the mathematically correct version of the original spectrum [Halsey et al., 1986] defined in 1986 by Halsey, Procaccia and others. But there are other spectra, such as $f_H(\alpha)$, which is the Hausdorff dimension of the set of all elements in $\Omega$ sharing the same concentration $\alpha$; furthermore, we have spectra like $f_p$ [Riedi and Mandelbrot, 1998]... and so on.

Throughout this paper, $\Omega$ will be the unit segment. For the so-called "self-similar" measures of $\Omega$, "all reasonably well defined spectra coincide, and fulfill the Legendre equations" [Riedi and Mandelbrot, 1998].

1.1 The original spectrum $f(\alpha)$

Let us introduce this spectrum [Halsey et al., 1986] with a very well known example that will be relevant for our work below. Let $\mu$ be a probability measure on $\Omega$, given, say, in an experimental way: we are measuring, empirically, a certain variable that we will name $x$, $x \in [0, 1]$. We start the experiment from an initial value of $x$, namely $x_0$, and we determine another value of $x$, say $x_1 \in (0, 1)$. We will consider intervals $[0, x_1]$ and $[x_1, 1]$ as equiprobable, i.e. we will write $\mu([0, x_1]) = \mu([x_1, 1]) = \frac{1}{2}$. We repeat the experiment, now taking the value $x_1$ as the initial value, and we obtain another value of $x$, say $x_2$, and, after another iteration of the process—$x_2$ the new initial value—we obtain $x_3$. Suppose that we find, e.g., $x_2 \in (x_1, 1)$, whereas
$x_3 \in (0, x_1)$. Then, we state: $\mu([0, x_3]) = \mu([x_3, x_1]) = \mu([x_1, x_2]) = \mu([x_2, 1]) = \frac{1}{4} = \frac{1}{2^2}$; i.e. we defined $\mu$ to be equiprobable on the 4 segments of the second partition of the unit segment. Next, let us suppose that the reiteration of the process provides us with an $x_4$, $x_5$, $x_6$, and $x_7$ inside the interior of the four segments above, then we say that we have the third partition of $[0, 1]$, we obtain $8 = 2^3$ segments, unequal in length, and we define $\mu$ to be equiprobable over all eight of them, i.e. $\mu$ will be $1/2^3$ for each segment in the third partition. Partitions obtained in this way, by iterating a process, will be called "canonical". Suppose that, given the $k^{th}$ canonical partition, the nature of the laws underlying the experiment is such that the reiteration of the process will equidistribute values $x_i$ in each and all segments in it; then we can continue defining $\mu$ in each of the new $2^{k+1}$ segments as $\frac{1}{2^{k+1}}$. As $k \to \infty$, we obtain a probability measure on $[0, 1]$, which is, needless to say, singular with respect to the Lebesgue measure.

Next, given a convenient value of $\alpha \geq 0$, we take a certain $k$-partition whose segments will be called $I_i = I_i^{(k)}$, will be enumerated from left to right and will have lengths $l_i = l_i^{(k)}$. Now, we select those intervals $I_i$ for which $\mu(I_i) \cong l_i^{(k)}$; i.e. $\frac{\ln(\mu(I_i))}{\ln(l_i)} \cong \alpha$, and we write $N_{\alpha} = N_{\alpha}^{(k)} = \# \{ i / \frac{\ln(\mu(I_i))}{\ln(l_i)} \cong \alpha \}$ When it is not strictly necessary, we will drop the supra index $k$, writing $l_i$ or $I_i$ instead of $l_i^{(k)}$ ... etc. Then we define $f(\alpha)$ as the limit (if it exists), when $k \to \infty$ and the partition gets finer and finer, of the quotient $\frac{\ln(N_{\alpha})}{\ln(l_i)}$. Notice that, if all $\mu(I_i)$ in the same partition have the same value, then all the $I_i$ selected so that $\frac{\ln(\mu(I_i))}{\ln(l_i)} \cong \alpha$ have, albeit roughly, the same length ($l_i$).

This $(\alpha, f(\alpha))$ is obtained, when iterating the process referred to above, in purely a computational way: we ask the computer to select those $I_i$ such that $\frac{\ln(\mu(I_i))}{\ln(l_i)} \cong \alpha$, we ask the computer to count those indices $i$ ... We will, then, take the liberty of calling $f_G$ ("$C$" for computational) this originally first studied spectrum.

### 1.2 The spectrum $f_G(\alpha)$

While keeping all the structure just depicted for $f_C$, Riedi and Mandelbrot [Riedi and Mandelbrot, 1998] stress that each and all partitions considered of $[0, 1]$ should be composed of segments of equal length. This is a most sensible requirement, and it enables Riedi, Mandelbrot and others to prove, in a rigorous way, different properties about the corresponding spectrum, named $f_G$ ("$G$" for grid).

Now, let us consider the measure described in Section 1.1, equiprobable on each segment $I_i^{(k)}$ in canonical partition $k$. Let us consider, in the same unit segment another partition, a uniform one, composed of segments $I'_i$, all of equal length. We can always fill up each segment $I'_i$ exactly, with non-overlapping intervals $I^{(k)}$, with different values of $k$, even if it is necessary to use very large values of $k$ in order to do so. Then, we can obtain the exact value of $\mu(I'_i) \forall I'_i$ in the uniform partition. Once $\mu(I'_i)$ is known, the rest of the structure proposed by Riedi and Mandelbrot for $f_G$ is virtually the same as in 1.1 for $f_C$.

The (theoretical) distinction between the original $f_C$ and $f_G$ is mathematically important,
for \( \frac{\ln(N_\alpha)}{\ln(1/l)} \) has no theoretically requirement to have a limit independent of the nature of the partition.

### 1.3 The inversion formula \( \bar{f}(\alpha) \) of a spectrum \( f(\alpha) \)

Let \((\alpha, f_H(\alpha))\) be the Hausdorff spectrum of \((\Omega, \mu)\), \(f_H\) as defined in the Abstract. Let us consider finer and finer canonical partitions \( P_1, \ldots, P_k, \ldots \). For each segment, \( I_i^{(k)} \) we know its length and measure \( l_i^{(k)} \) and \( \mu(I_i^{(k)}) \) respectively. If we wanted to invert the roles of \( l_i \) and \( \mu(I_i) \) —i.e. change lengths with measures and vice versa— what would the new Hausdorff spectrum, denoted by \((\alpha, \bar{f}_H(\alpha))\), look like? The beautiful answer \cite{Riedi and Mandelbrot, 1997}, \cite{Mandelbrot and Riedi, 1997}, \cite{Riedi and Mandelbrot, 1998} is the inversion formula \( \bar{f}_H(\alpha) = \alpha f_H(\frac{1}{\alpha}) \).

The new segments in partition \( k \) would have lengths \( \mu(I_i^{(k)}) \) and measure \( l_i^{(k)} \), so the new spectrum should be very very different. Notice that the inversion formula was proved by Riedi and Mandelbrot for every probability measure!

Riedi and Mandelbrot \cite{Riedi and Mandelbrot, 1998} stress that if we replace \( f_H \) by \( f_G \), then the inversion formula might not hold. Now, since \( f_G = f_H \) \cite{Riedi and Mandelbrot, 1998} for the so called ”self-similar measures”, then the inversion formula holds, trivially, for \( f_G \) for such measures.

### 1.4 Legendre coordinates and transformations

Let us recall that, when depicting \( f_C \), we did not contemplate, necessarily, a uniform partition. In \cite{Halsey et al., 1986}, however, the authors see a very important advantage in working always with uniform partitions: if \( l = l^{(k)} \) is the length associated with the intervals of a certain fine partition \( P_k \), then the so-called partition function \( \sum \mu(I_i)^q \), \( q \) a certain parameter between \(-\infty \) and \(+\infty \), can be rewritten as

\[
\sum_i \mu(I_i)^q = \sum_\alpha N_\alpha(l^{(\alpha)})^q = \sum_\alpha N_\alpha l^{\alpha q}.
\]

If we can confirm that \( N_\alpha \sim l^{-f_C(\alpha)} \) (which will certainly be the case for the Cantor dust or any real self-similar fractal, for which \( f_C = f_H \)) we would have for the partition function:

\[
\sum_i \mu(I_i)^q = \sum_\alpha l^{-f_C(\alpha)} l^{\alpha q} = \sum_\alpha l^{(\alpha q - f_C(\alpha))},
\]

a quantity very well estimated by \( l^{\min_\alpha(\alpha q - f_C(\alpha))} \), which implies: \( q - f'_C(\alpha) = 0 \), and for such value of \( q \) we would have

\[
\frac{\ln \sum p_i^q}{\ln l} \approx \alpha q - f_C(\alpha),
\]

where \( p_i = \mu(I_i) \).

We call \( \tau(q) \) the limit, when the norm of the partition \( P_k \) tends to zero, of the logarithmic quotient above.
Needless to say, we terribly oversimplified all the above calculations — among other things it is not quite clear why $f_C$ would have a derivative.

The so-called Legendre transformations are then $\tau(q) \approx \alpha q - f_C(\alpha)$, for a value of $q = f'_C(\alpha)$.

Now, if the first of these two equations is really tight enough — such as to replace "\(\approx\)" by "\(=\)" — then we can differentiate both terms of it with respect to $q$:

$$\tau'(q) = \frac{d}{dq} \alpha q + \alpha - f'_C(\alpha), \quad \frac{d}{dq} \alpha q = \alpha$$

in our particular case of $q = f'_C(\alpha)$.

So a third equation $\tau'(f'_C(\alpha)) = \alpha$ holds.

### 1.5 Legendre’s reading in Quantum Mechanics

Identifying $q$ with the inverse temperature $T$ of a system (times the Boltzman constant $\beta$), $\alpha$ with its internal energy $E$, $f_C(\alpha)$ with its entropy $S$, and relating the partition function, via $\tau(q)$, with the free energy $F$ of the system, then

$$\tau(q) \approx \alpha q - f_C(\alpha) \quad \text{and} \quad f'_C(\alpha) = q$$

definitely becomes the classical relationships between $T, E, S, F$, and the partition function [Duong-Van, 1987]; ... so that, theoretically, one could study some problems of Quantum Mechanics as Fractal Geometry, and vice versa.

### 1.6 The hyperbolic metric

We will very briefly touch on this subject, referring the interested reader to standard texts. Let $\mathbb{H}$ be the open half plane $\{(x, y) \in \mathbb{R}^2 / y > 0\}$. In $\mathbb{H}$ we will define the rigid movements (and they will be the equivalent of our Euclidean translations, rotations, ... all congruences) as follows: Let $(x, y)$ and $(x', y')$ be in $\mathbb{H}$. If there is a $2 \times 2$ matrix $M$ with real entries $a_{ij}$ and unit determinant such that $\frac{y'}{x'} = \frac{a_{11}x + a_{12}y}{a_{21}x + a_{22}y}$, then we say that $(x', y')$ is $(x, y)$ moved by a hyperbolic rigid movement. The value of the trace $a_{11} + a_{22}$ determines the nature of this movement. When all entries $a_{ij}$ are in $\mathbb{Z}$ we define the set of these $2 \times 2$ matrices as the unimodular (multiplicative) group $U$ which keeps appearing all over Mathematics when one least expects it! To this $U$ there is a tiling of $\mathbb{H}$ associated: there is a tile — called fundamental tile — $R$ such that, if we apply each and all elements of $U$ to $R$ we obtain an infinity of non-overlapping tiles (i.e. intersecting only at their boundaries) covering all of $\mathbb{H}$.

There is a unique measure on $\mathbb{H}$ such that all these infinite tiles have the same measure. Without entering into detail let us call it $\mu^{\mathbb{H}}$, the hyperbolic measure. Each tile is a perfect polygon; all tiles are, if we look at them with hyperbolic spectacles, the same polygon, displaced, by rigid movements, from place to place.

Let us suppose that we have two tiles $T_1$ and $T_2$, and that each of them has two vertices $v_1^1, v_1^2$ and $v_2^1, v_2^2$, respectively, in $[0, 1]$. The unit segment inherits, then, from $\mu^{\mathbb{H}}$ a certain measure $\mu$: since $T_1$ and $T_2$ are the same for $\mu^{\mathbb{H}}$, in shape, size, and measure, then $\overline{v_1^1v_2^2}$ and
have to be, for the inherited measure \( \mu \), totally indistinguishable. *Par abús de langage* let us denote this new measure on the unit segment to be \( \mu^{H} \).

Now: the tiling associated with \( U \) is far from unique: Series [Series, 1985] performed a surgical operation on the fundamental tile \( R \), obtaining another fundamental \( \bar{R} \) and another tiling (with interesting properties) associated with \( U \), also covering exactly \( H \) with no overlapping, and inheriting another hyperbolic measure on the unit segment. Elsewhere [Grynberg and Piacquadio, 1995] we have operated on \( \bar{R} \) producing another \( \bar{\bar{R}} \) and another tiling of \( \bar{H} \) associated with \( U \), whose inheritance \( \mu^{H} \) on the unit segment is, precisely, the Farey-Brocot (F-B) \( P_{k} \) partitions (see below). The segments in \([0, 1]\) with the same \( \mu^{H} \) measure (such as \( v_{1}^{2}v_{2}^{2} \) and \( v_{1}^{2}v_{2} \) above) are now segments in the same \( P_{k} \) partition. The larger the \( k \), the smaller the segments.

We are interested in the F-B partition of the unit segment, since it crops up in a variety of problems in Physics.

### 1.7 The Farey-Brocot partition in Physics

We propose to give here a variety of examples in which the F-B partition appears. But first, let us illustrate the canonical formations of the F-B partitions with a simple diagram:

**Diagram A:**

\[
P_{0} \quad \frac{0}{1} \quad \frac{1}{1} \quad \mu^{H}([0, 1]) = 1
\]

\[
P_{1} \quad \frac{0}{1} \quad \frac{0+1}{1+1} \quad \frac{1}{1} \quad \mu^{H}([0, \frac{1}{2}]) = \mu^{H}([\frac{1}{2}, 1]) = \frac{1}{2}
\]

\[
P_{2} \quad \frac{0}{1} \quad \frac{0+1}{1+1} \quad \frac{1}{1+2} \quad \frac{1+1}{1+2} \quad \frac{1}{1} \quad \mu^{H}([0, \frac{1}{3}]) = \mu^{H}([\frac{1}{3}, \frac{1}{2}]) =
\]

\[
\vdots
\]

\[
\mu^{H}([\frac{1}{2}, \frac{2}{3}]) = \mu^{H}([\frac{2}{3}, 1]) = \frac{1}{2}
\]

We trust the diagram is self-evident, and that, henceforth, we can replace \( \mu^{H} \) by a simplified notation \( \mu \), without confusion.

Let us now go to some physical examples.

#### 1.7.1 Physical examples in which the F-B or hyperbolic measure appears.

Let us develop three examples of physical phenomena where the Farey-Brocot sequence (the Farey tree in the Physics literature) appears, in the hope that these examples will convince the reader of the relevance of the F-B sequence in Physics and Chemistry.
1) Let us consider the forced pendulum, with internal frequency $\omega$. The angle $\theta$ formed by pendulum and vertical is expressed by $\theta_{n+1} = f(\theta_n, \omega)$. When plotting the winding number $W$ of $\theta_n$ as a function $g$ of $\omega$, we have that, for certain critical values of the parameters involved, $W = g(\omega)$ is a Cantor-like staircase [Halsey et al., 1986]. It means that $g(\omega)$ is constant in the so-called intervals of resonance $I_k$ ($k$ a natural number) of the variable $\omega$, each $I_k$ producing a step of the staircase.

Cvitanovic, Jensen, Kadanoff and Procaccia [Cvitanovic et al., 1985] discovered a property of the staircase $W = g(\omega)$: Let $\frac{p}{q}$ and $\frac{p'}{q'}$ be the values of $W$ for a pair of intervals of resonance $I$ and $I'$, such that all intervals of resonance in the gap between $I$ and $I'$ are smaller in size than both $I$ and $I'$. Then, there is an interval of resonance $I''$ in this gap such that the corresponding constant value of $W$ is $\frac{p''}{q''} = \frac{p+p'}{q+q'}$. This interval $I''$ is the widest of all intervals of resonance in the gap between $I$ and $I'$. This is a purely empirical finding.

2) Bruinsma and Bak [Bruinsma and Bak, 1983] studied the one-dimensional Ising model with long range antiferromagnetic interaction — of strength given by an exponent $\alpha > 1$ — in an applied magnetic field $H$. For zero temperature, the plot of $y = f_B(x)$, where $x = -H$, and $y$ is the ratio of up spins over the total number of spins, is a Cantor-like staircase. The steps of the staircase are the stability intervals $\Delta H$ for which $y$ is a rational number $p/q$. Again, these stability intervals [Piacquadio and Grynberg, 1998] are distributed following the Farey tree described above for the forced-pendulum Cantor-like staircase.

3) To study the general properties of fermion systems such as crystal formation, Falicov and Kimball proposed a simple model. Gruber, Ueltschi and Jedrzejwki [Gruber et al., 1994] have considered the one-dimensional Falicov-Kimball model. Let us suppose that we have $q$ sites marked on the real line, $p'$ ions to put in the $q$ sites and $p$ electrons hovering around the ions.

Once the sites where the ions lie are chosen, we have a configuration $\omega$ that we repeat periodically in order to obtain an infinite one-dimensional lattice. The electron density is $P_e = p/q$. Given $U$, the electron-ion interaction, and the electrochemical potentials $(\mu_e, \mu_i)$ there is a configuration $\omega$ which minimizes the free energy density $f(\omega, \mu_e, \mu_i)$. For small values of $U$, the phase plane $(\mu_e, \mu_i)$ is divided into connected regions $D_{P_e}$, such regions share the same $P_e$, and this $P_e$ is the electronic density of an $\omega$ that minimizes $f(\omega, \mu_e, \mu_i)$.

Numerical results show that between regions $D_{p/q}$ and $D_{p'/q'}$ there appears a smaller one corresponding to $P_e = \frac{p+p'}{q+q'}$. This region is the largest between $D_{p/q}$ and $D_{p'/q'}$.

Cvitanovic et al. [Cvitanovic et al., 1985], Halsey et al. [Halsey et al., 1986], and Rosen [Rosen, 1998] have different examples of physical phenomena exhibiting Cantor staircases with such (F-B) arrangements. We can also find this arrangement in some of the staircases shown in [Bak, 1986], including the chemical reaction of Belusov-Zabotinsky. Other examples can be found in [Arrowsmith et al., 1996] and [McGehee and Peckham, 1996].

1.8 The purpose of this paper
1.8.1 Self-similar measures and the hyperbolic measure

Reviewing a diversity of results, Riedi and Mandelbrot [Riedi and Mandelbrot, 1998] stress the following:

a) For the so-called "self-similar measures" all reasonably well defined spectra $f_\alpha$ coincide, and
b) fulfill the Legendre equations.

c) In the general case of other measures, $f_G$ was the largest, $f_H$ the smallest function.
d) $f = f_H$ fulfilled the inversion formula for every measure in $[0, 1]$: $ar{f}(\alpha) = \alpha f(1/\alpha)$, where $\bar{f}$ means, as we saw, that probability measures become lengths of intervals and vice versa.
e) $f = f_G$ may not, in the general case of an arbitrary measure, fulfill the inversion formula—and interesting examples are given.

The following diagram provides an example of what a "self-similar measure" is. Here, $a$ and $p$ are irrational numbers in $(0, 1)$.

Diagram B:

![Diagram B]

Notice that the structure $\{1\}; \{a, 1-a\}; \{a^2, a(1-a), (1-a)a, (1-a)^2\}$ etc. depicting the lengths of intervals $I_i$, is formally identical with that of $\{1\}; \{p, 1-p\}; \{p^2, p(1-p), (1-p)p, (1-p)^2\}$ etc., depicting the probability measures of said intervals $I_i$.

That is, when we replace the role of lengths of $I_i$ by its probability measures $\mu_i$ we do not alter the formal structure: we end up with

Diagam C:
\[ \bar{\mu}([0, p]) = a \quad \bar{\mu}(\{p, 1\}) = 1 - a \]

\[ \bar{\mu} = a^2 \quad \bar{\mu} = a(1 - a) \quad \bar{\mu} = (1 - a)a \quad \bar{\mu} = (1 - a)^2 \]

\[ \bar{\mu} \quad \bar{\mu} \quad \bar{\mu} \quad \bar{\mu} \]

\[ \begin{array}{cccc}
\mu & 1/4 & 1/3 & 2/5 & 1/2 \\
\mu &=& 1/4 & 1/3 & 2/5 & 1/2 \\
\end{array} \]

Now, when we have an F-B arrangement of \([0, 1]\) with the hyperbolic measure, and we exchange roles for lengths and measures, then the new structure is very very different.

For "self-similar measures" the case \( l(I_i^{(k)}) = (1 - a)^{k-j}a^j \); for values of \( j \) such that \( 0 \leq j \leq k \); \( \mu(I_i^{(k)}) = (1 - p)^{k-j}p^j \) implied

\[ \frac{\ln(l(I_i))}{\ln(\mu(I_i))} = \frac{1}{c + d \frac{1}{k}} + \frac{1}{j} + c + b, \]

where \( b = \frac{\ln(p/(1-p))}{\ln(a/(1-a))} \), \( c = \frac{\ln(1-p)}{\ln(1-a)} \), \( d = \frac{\ln(p/(1-p))}{\ln(1-a)} \) and \( e = \frac{\ln(1-p)}{\ln(a/(1-a))} \); so for the inversion \( l \leftrightarrow \mu \) we can write the new \( \bar{l}_i \) and their corresponding measures \( \bar{\mu}_i \) directly, by selecting the \((i,j)\) order of the segment in the partition.

For our F-B arrangement — in this case (F-B)\(_3\) only:

**DIAGRAM D:**

\[ \begin{array}{cccc}
0 & 1/4 & 1/3 & 2/5 & 1/2 \\
\mu &=& 1/4 & 1/3 & 2/5 & 1/2 \\
\end{array} \]

There is no such possible easy way of understanding \( \bar{\mu} \), once we reverse lengths and measures.

The reader is invited to develop (F-B)\(_k\) arrangements for large values of \( k \) and to check how irregular the hyperbolic measure is when compared to the Euclidean one. In fact, Series and Sinai [Series and Sinai, 1990], among others, devoted excellent articles in order to stress the irreconcilable differences — both physical and mathematical — between the hyperbolic and the ordinary Lebesgue measure. For instance, looking at \( \mathbb{H} \), one would say that, as a half plane, it has dimension 2. Nevertheless, a common plane can be tessellated in a specific and finite number of ways if all the convex tiles are to be the same in shape and size. Other tilings are equivalent to the latter. But the infinity of — not equivalent — corresponding tilings of \( \mathbb{H} \) allows us to show that \( \mathbb{H} \) behaves, in certain ways, as \( \mathbb{R}^\infty \).

Series and Sinai were also able to construct on \( \mathbb{H} \) a non-numerable set of non-equivalent Gibbs measures.
We propose to show that, despite their fundamental differences, the hyperbolic measure in $[0,1]$ behaves very much like the self-similar ones in the sense of Section 1.8.1; that is:

a') For the hyperbolic measure, it appears that $f_G$ and $f_H$ coincide. If, as in c), $f_G$ is the largest and $f_H$ is the smallest, then "all reasonably well defined spectra coincide" for the hyperbolic measure, and

b') this measure fulfills the Legendre equations.

Some proofs are analytical, others are numerical. Both a') and b') as well as other results shown in Section 6 show a powerful analogy between Euclidean and hyperbolic measures.

We trust to be able to give some conjecture(s), at the end of the paper, that could explain such analogies.

2 $f_C$ and $f_G$ for the hyperbolic measure and the inversion formula for $f_C$

2.1 $f_C$ and $f_G$ for the hyperbolic measure

Let us recall, once more, how the F-B sequences were constructed; let us consider them between $1/2$ and 1, due to the symmetry of the denominators. Henceforth we will write $\mu^H = \mu$ for short.

Diagram E:

$$
\begin{array}{c}
k = 0 \\
0 \quad 1
\end{array}
$$

$$
\begin{array}{c}
k = 1 \\
\frac{1}{2} \quad 1 \mu([\frac{1}{2}, 1]) = \frac{1}{2^k}
\end{array}
$$

$$
\begin{array}{c}
k = 2 \\
\frac{1}{2} \quad \frac{3}{2} \quad 1 \mu([\frac{1}{2}, \frac{3}{2}]) = \mu([\frac{3}{2}, 1]) = \frac{1}{2^k}
\end{array}
$$

Now, we want a uniform $k$-partition of $[\frac{1}{2}, 1]$, and we want to know the measure $\mu^H = \mu$ of each such interval in the uniform partition. We will take a relatively small $k$, $k = 11$, and we will divide $[0, 1]$ in equal segments of length $\frac{1}{2^{11}}$.

We stress that $k$ is rather small, since $[0, 1]$ would be divided into $2^{11} = 2048$ equal segments, and, as we will see throughout the last sections, we need partitions with well over 4 million segments in order to obtain a certain accuracy in our numerical results. We will take the segment $I = [\frac{m}{2^k}, \frac{m+1}{2^k}]$, $m = 1265$, in this partition, and we will try to calculate $\mu(I)$. For different F-B $k$-partitions, we will locate our interval $I$. Briefly, the situation is as follows:

Diagram F:
Proceeding in this way, we locate $a$ and $b$ for increasing values of $k$; and, as shown in the column at the right, we estimate $\mu(I)$ from below; but with increasing accuracy. Finally, in $k = 37$ (and not before) we find both $a = \frac{m}{210}$ and $b = \frac{m+1}{210}$, and we are able to give an exact answer to the question "how much is $\mu([\frac{1265}{210}, \frac{1266}{210}])$?" Now, the reality is that, in order to obtain the value of $\mu([\frac{m}{210}, \frac{m+1}{210}])$, the computer has to reach, at least, to the F-B 37-partition... and the fastest machine at our University gets stuck for days and days at $k = 24$.

Even if, with the help of some super computer (in some far away place) we could possibly bridge the gap between $k = 23$ and $k = 37$, we still would have a uniform partition (of intervals, together with their exact $\mu_I$ measures) of barely 2000 intervals... when we need over 4 million at the very very least.

... We are absolutely forced to work, not with the mathematically correct uniform partition, but with the F-B partitions as they come —and that means that we are forced to work with $f_C$ in lieu of the mathematically well defined $f_G$.

2.2 The inversion formula for $f_C$

Working directly with $f_C$ instead of the well defined $f_G$ has an advantage: the inversion formula holds for $f_C$, a result whose proof is rather short: We know that, if $k$ is large enough, then
\[ \bar{f}_C(\cdot) \approx \frac{\ln(\# \{ i / \ln(p_i^{(k)}) \approx 1/\alpha \})}{\ln(1/2^k)}, \]

where both signs "\( \approx \)" above tend to "\( = \)" as \( k \to \infty \). Therefore,

\[ \alpha \bar{f}_C(\frac{1}{\alpha}) \approx \alpha \frac{\ln(\# \{ i / \ln(1/2^k) \approx \frac{1}{\alpha} \})}{\ln(1/2^k)} \approx \]

\[ \approx \frac{\ln(1/2^k)}{\ln(p_i^{(k)})} \frac{\ln(\# \{ i / \ln(p_i^{(k)}) \approx \alpha \})}{\ln(1/2^k)} = \frac{\ln(\# \{ i / \ln(1/2^k) \approx \alpha \})}{\ln(p_i^{(k)})} = f_C(\alpha), \]

and, with "\( \approx \)" turning to "\( = \)" as \( k \to \infty \) we obtain \( \alpha \bar{f}_C(\frac{1}{\alpha}) = f_C(\alpha) \), or \( \bar{f}_C(\alpha) = \frac{1}{\alpha} f_C(\frac{1}{\alpha}) \), which is the inversion formula.

### 3 The theoretical spectrum \((\alpha, f_H(\alpha))\) for the hyperbolic measure

Let us recall the definition of the theoretical spectrum \((\alpha, f_H(\alpha))\) given in Section 1.

Let \( x \in [0, 1] \) and \( I^{(k)}(x) \) be the unique interval in the \( k^{th} \) step of the F-B partition where \( x \) lies; then

\[ \alpha(x) = \lim_{k \to \infty} \alpha_k(x) = \lim_{k \to \infty} \frac{\ln(\mu(I^{(k)}(x)))}{\ln(l(I^{(k)}(x)))} \]

when the limit exists. Here \( l(I^{(k)}(x)) \) is the length of the interval \( I^{(k)}(x) \), and as defined before, \( \mu(I^{(k)}(x)) = \frac{1}{2^k} \), for \( \mu \) is the hyperbolic measure.

Given any value \( \alpha \in \mathbb{R}^+ \), let \( \Omega_\alpha \) be the set of \( x \in [0, 1] \) which share the \( \alpha \)-index, \( \Omega_\alpha = \{ x / \alpha(x) = \alpha \} \). By definition \( f_H(\alpha) = d_H(\Omega_\alpha) \), where \( d_H(\Omega_\alpha) \) denotes the Hausdorff dimension of \( \Omega_\alpha \).

To estimate such a curve \((\alpha, f_H(\alpha))\) we have to obtain an adequate formula for the length of the intervals \( I^{(k)} \). Inevitably, we must introduce some formalism about continued fractions.

Any real number \( x \in (0, 1) \) can be expressed as a continued fraction:

\[ x = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \cdots}}} = [n_1, n_2, \ldots], \text{ with } n_i \in \mathbb{N}. \]

The sequence is finite if and only if \( x \) is rational.

If \( x \) is irrational, and we consider the \( N^{th} \) rational approximant to \( x \):
Lemma 3.1 Let $Q_n$ be an adequate estimate of dimension and Hausdorff measure. Any $\lambda$ partial quotients $F_K$, and $\alpha$ is well approximated by $\sqrt{\pi^2/6 - 1.2^{\lambda_1}}\ldots(K+1)^{\lambda_K}$. Thus we have to estimate $Q_n$ in order to obtain $\alpha(x)$. Obviously, $Q_n$ depends on the partial quotients $n_1, \ldots, n_N$. The following lemma (demonstrated in [Cesaratto, 1999]) gives us an adequate estimate of $Q_n$ as a function of $n_1, \ldots, n_N$.

**Lemma 3.1** Let $C_N$ be the set of rational numbers $[n_1, \ldots, n_N]$ for which $n_i$ does not exceed $K$. Let us partition $C_N$ into disjoint classes $C_{l_1,\ldots,l_K}^N$, where $l_1, \ldots, l_K$ are in $\mathbb{N}$, $0 \leq l_i \leq N$, $l_1 + \ldots + l_K = N$. We will say that $[n_1, \ldots, n_N] \in C_{l_1,\ldots,l_K}^N$ if $l_1$ elements $n_i$ are equal to 1, $l_2$ elements $n_i$ are equal to 2, \ldots etc. Let us define the frequency $\lambda_i = \frac{l_i}{N}$, and rename the $N$-classes $C_{l_1,\ldots,l_K}^N$ as $C_{\lambda_1,\ldots,\lambda_K}^N$. If $[n_1, \ldots, n_N] \in C_{\lambda_1,\ldots,\lambda_K}^N$, then $\sqrt{Q_n(n_1, \ldots, n_N)}$ is well approximated by $\sqrt{\frac{\pi^2}{6} - 1.2^{\lambda_1}}\ldots(K+1)^{\lambda_K}$.

### 3.1 A connection between $\alpha(x)$, $x = [n_1, \ldots, n_N \ldots]$ and $\lim_{N \to \infty} \frac{\sum_{i=1}^N n_i}{N}$

Returning to the $\alpha$-formula (1) we see that, with $k$, $K$, and $N$ as before, $\alpha(x)$ can be rewritten as

$$\alpha(x) = \lim_{k \to \infty} \frac{k \ln(2)}{\ln(Q_n Q_{N-1})} = \lim_{N \to \infty} \frac{\ln(2)}{2} \frac{\sum_{i=1}^N n_i}{\ln(Q_N)} = \lim_{N \to \infty} \frac{\ln(2)}{2} \frac{\sum_{i=1}^N n_i}{\ln(\sqrt{\frac{\pi^2}{6} - 1}) + \sum_{i=1}^K \lambda_i \ln(i+1)} = \frac{\ln(2)}{2} \frac{\sum_{i=1}^N n_i}{\ln(\sqrt{\frac{\pi^2}{6} - 1}) + \sum_{i=1}^K \lambda_i \ln(i+1)}$$

From (2) we notice that $\alpha(x)$, $x = [n_1, \ldots, n_N \ldots]$ depends on the average $\frac{\sum_{i=1}^N n_i}{N}$ of the partial quotients $n_i$. Therefore, we need to estimate the Hausdorff dimension $d_H(F_m)$, where $F_m = \{x \in [0, 1]/\lim_{N \to \infty} \frac{\sum_{i=1}^N n_i}{N} = m\}$.

We obtain [Cesaratto and Piacquadio, 1998], [Cesaratto, 1999]:

a) $d_H(F_m) \cong d_H(F_m \cap E_{K_m}) = 1 - \frac{6}{\pi^2 K_m} - \frac{72 \ln(K_m)}{\pi^2 K_m^2}$, where $E_K = \{x \in [0, 1]/1 \leq n_i \leq K\}$, and $K_m \cong e^{\left(\frac{\pi^2}{6}\right) - 1} m$.

b) $d_H(F_m)$ is the Hausdorff dimension of the set of elements $x$ which belong to the $N$-classes $C_{\lambda_1,\ldots,\lambda_K}^N$ for which

$$\lambda_i = \bar{\lambda}_i = \frac{1}{\sum_{i=1}^K \frac{1}{(i+1)^{2d_H(E_{K_m})}}}.$$
class of Hausdorff dimension inferior to \(d_H(F_m)\) — hence, of Hausdorff measure nil in such a dimension.

Let us go back to (2):

\[
\alpha(x) = \frac{\ln(2)}{2} \lim_{N \to \infty} \frac{\sum_{i=1}^{N} n_i}{N} \ln(\sqrt{\pi^2/6} - 1) + \sum_{i=1}^{K_m} \lambda_i \ln(i+1) = mC, \\
\]

for short. The average of the \(n_i\) is denoted by \(m\), and \(C = C(\lambda_1 \ldots \lambda_{K_m}, m)\).

Let us consider \(\lambda_i = \bar{\lambda}_i\). Then, the stability of \(C\) depends on the stability of \(\sum_{i=1}^{K_m} \bar{\lambda}_i \ln(i+1)\). Notice that, if \(m\) is not too small, then both sums \(\sum_{i=1}^{K_m}\) above are very much like \(\sum_{i=1}^{\infty} \frac{1}{(i+1)^2}\) and \(\sum_{i=1}^{\infty} \frac{\ln(i+1)}{(i+1)^2}\) respectively. Therefore, if \(m\) is not too small, the coefficient \(C(m)\) would be, essentially, independent of \(m\), and we can call it \(C\), for short. We would, then, obtain a connection — via \(C\) — between \(\alpha(x)\) and \(m\), when \(x = [n_1, \ldots n_N, \ldots]\) and \(\lim_{N \to \infty} \sum_{i=1}^{N} n_i = m\).

Let us consider other elements \(x = [n_1, n_2, \ldots]\) in \(F_m\). For instance, \(x_1 = [m, m, \ldots, m, \ldots]\) is, no doubt, in \(F_m\). But the corresponding \(C(x_1)\) is

\[
\frac{1}{\ln(\sqrt{\pi^2/6} - 1 + \ln(m+1))},
\]

a number as small as we want it, if \(m\) is large enough.

Again, let us consider \(x_2 = [1, 2m - 1, 1, 2m - 1, \ldots]\). We have \(\lim_{N \to \infty} \sum_{i=1}^{N} n_i = m\), but \(C(x_2) = \frac{1}{\ln(\sqrt{\pi^2/6} - 1 + \ln(2m)+\ln(m^2)\frac{1}{2})}\), a number quite different from \(C(x_1)\), if \(m\) is large.

Again, we can construct another \(x_3 = [n_1, n_2, \ldots]\) in \(F_m\) with strings \((m, \ldots, m)\) and strings \((1, 2m - 1, 1, 2m - 1, \ldots)\) large enough as to ensure that \(C(x_3)\) varies — with \(N\) — from \(C(x_1)\) to \(C(x_2)\) and back an infinity of times \(\ldots\). Fortunately, all such elements \(x \in F_m\) that destabilize the coefficient \(C(x)\) — moreover, such that possess a coefficient different from \(C\) — form a set of Hausdorff measure zero in the Hausdorff dimension \(d_H(F_m)\). Hence, with \(\alpha = mC\) we will say that \(\Omega_\alpha\) is, essentially, the set of elements \(x \in F_m \cap E_{K_m}\) with distribution \(\{\bar{\lambda}_i\}_{i=1}^{K_m}\), and that \(f_H(\alpha) = d_H(\Omega_\alpha) = d_H(F_m) = d_H(F_m \cap E_{K_m}) = 1 - \frac{6}{\pi^2e(\frac{6}{7})m} - \frac{72(\frac{6}{7})m}{\pi^4e^2(\frac{6}{7})m}\).

3.2 Small values of \(m = \sum_{i=1}^{N} n_i\)

Prior observations:

1) Let us consider variables \(m\) and \(K_m\). \(K_m\) is a discrete variable, whereas \(m\) is a continuous one. If both \(K_m\) and \(m\) are large enough, the difference between \(d_H(F_m \cap E_{K_m})\) and \(d_H(F_m)\) will not show. What really happens is that there are different values of the continuous variable \(m\) — a whole interval of such \(m\) — associated with the same \(K_m\).

If \(m\) and \(K_m\) are large enough, these \(m\) intervals are very small . . . but for small values of \(m\), say, for \(m \in [1, 1.39]\) we have \(K_m = 1\). We should take into account that these small values of \(m\) are of great importance to physicists — even if mathematicians tend to consider asymptotic behaviour — since, for \(m \in [1, 1.39]\) the corresponding \(d_H\) varies from 0 to 0.677, i.e. more than half of the whole range of \(f(\alpha)\)!
2) \( \lim_{K \to \infty} d_H(F_m \cap E_K) \) coincides with \( d_H(F_m \cap E_{K_m}) \) for \( m \geq 5 \), but again the former is very different from the latter for small values of \( m \). For such small values of \( m \), we obtain a better estimate of \( d_H(F_m) \) with \( \lim_{K \to \infty} d_H(F_m \cap E_K) \).

3) Finally, let us notice, in the last section, how many times we had to write "... if \( m \) is not small ..." or "... if \( m \) is large enough ...", All of which makes necessary a separate treatment for the case in which \( m \) is small.

The following theorem (we will not prove it here) gives us an estimate of \( d_H(F_m) \) via \( \lim_{K \to \infty} d_H(F_m \cap E_K) \), which allows us to treat the case \( m > 1, m \) small.

**Theorem 3.1** Let \( m \) and \( F_m \) be as before. Let \( K \in \mathbb{N} \), \( K \geq 2 \), let \( (y_{K,m}, z_{K,m}) \) be the solution in \( (e^{-2}, 1) \times [0, 1] \) of the following system of equations:

\[
\begin{align*}
 f^{m}_{K}(y, z) &= \sum_{i=0}^{K-1} (m-i-1)y^{a_i}z^i = 0 \\
 g^{m}_{K}(y, z) &= \frac{y}{z^m} \sum_{i=0}^{K-1} y^{a_i}z^i - 1 = 0
\end{align*}
\]

where \( a_i = (i-1) \ln(2) - i \ln(3) + \ln(i+2) \) and \( t = \ln(\sqrt{\pi^2/6-1}) - (m-2) \ln(2) + (m-1) \ln(3) \).

Let \( \bar{y}_m = \lim_{K \to \infty} y_{K,m} \). Then \( \lim_{K \to \infty} d_H(F_m \cap E_K) = -1/2 \ln(\bar{y}_m) \).

Moreover, let \( K_m \) be large enough as to ensure \( \bar{y}_m \equiv y_{K,m} \). Let us write \( \bar{\lambda}_{i,K} = y_{K,m}^{a_i+1}z_{K,m}^{i+1-m} \). Then, \( \lim_{K \to \infty} d_H(F_m \cap E_K) = -1/2 \ln(\bar{y}_m) \) agrees with the dimension of \( C^{\bar{\lambda}_{1,K}, \cdots, \bar{\lambda}_{K_m,K_m}} \), which we abbreviate as \( C^{\bar{\lambda}_{1,K}, \cdots, \bar{\lambda}_{K_m,K_m}} = \{ x = [n_1, \cdots, n_N, \cdots] / \lim_{N \to \infty} \frac{\bar{\lambda}_N}{N} = \bar{\lambda}_{i,K_m} \} \).

For all \( x \in C^{\bar{\lambda}_1, \cdots, \bar{\lambda}_{K_m}}_N \) we have

\[
\alpha = \alpha(x) = \frac{\ln(2)}{2} \left( \frac{\sum_i i \bar{\lambda}_i}{\ln(\sqrt{\pi^2/6-1}) + \sum_{i=1}^{K_m} \bar{\lambda}_i \ln(i+1)} \right) = \frac{\ln(2)}{2} \left( \frac{m}{\ln(\sqrt{\pi^2/6-1}) + \sum_{i=1}^{K_m} \bar{\lambda}_i \ln(i+1)} \right) \cong \alpha(m).
\]

For this value of \( \alpha = \alpha(m) \) we have that \( f_H(\alpha) = d_H(\Omega_\alpha) = d_H(F_m) = -1/2 \ln(\bar{y}_m) \). In Table 1 we show some numerical values obtained as we have just explained.
TABLE 1: Some numerical values for the theoretical spectrum.

| m  | K  | y_{K,m} | z_{K,m} | α   | f_H(α) |
|----|----|---------|---------|------|--------|
| 1.0 | 10 | 0.8892  | 0.0099  | 0.7325 | 0.0587 |
| 1.02 | 20 | 0.8152  | 0.0196  | 0.7336 | 0.1021 |
| 1.04 | 20 | 0.7074  | 0.0383  | 0.7358 | 0.1730 |
| 1.08 | 20 | 0.5689  | 0.0733  | 0.7404 | 0.2821 |
| 1.2  | 20 | 0.3744  | 0.1596  | 0.7561 | 0.4912 |
| 1.4  | 25 | 0.2582  | 0.2561  | 0.7863 | 0.6770 |
| 1.8  | 30 | 0.1863  | 0.3531  | 0.8562 | 0.8402 |
| 2    | 30 | 0.1717  | 0.3780  | 0.8942 | 0.8809 |
| 2.4  | 50 | 0.1560  | 0.4070  | 0.9736 | 0.9290 |
| 3    | 60 | 0.1456  | 0.4268  | 1.0988 | 0.9633 |
| 3.5  | 70 | 0.1416  | 0.4346  | 1.2061 | 0.9936 |
| 4    | 100| 0.1393  | 0.4386  | 1.3173 | 0.9854 |
| 5    | 300| 0.1371  | 0.4419  | 1.5528 | 0.9936 |
| 6    | 300| 0.1362  | 0.4437  | 1.6879 | 0.9968 |
| 7    | 400| 0.1358  | 0.4443  | 2.032  | 0.9983 |

The spectrum \((\alpha, f_H(\alpha))\) has the graph shown in Fig. 1:

![Graph of \(f_H(\alpha)\)](image)

Fig. 1: The graph of \(f_H(\alpha)\). Notice that \(\alpha_{min} = \ln(2) / \ln(1/\varphi)\); \(\varphi = \frac{\sqrt{5} - 1}{2}\) [Grynberg and Piacquadio, 1995].

4 The computational spectrum \((\alpha, f_C(\alpha))\)

From now on \(k\) will enumerate the F-B sequences, and \(p_i = p_i^{(k)}\) will denote \(I(I_i^{(k)})\).

We have \(f_C(\alpha) = \lim_{k \to \infty} f_C^{(k)}(\alpha)\) where \(f_C^{(k)}(\alpha) = \alpha f_C^{(k)}(1/\alpha); f_C^{(k)}\) the computational function obtained, via Legendre conditions, from \(\bar{\tau}^{(k)}(q) = \frac{\ln(\sum_{i=1}^{2k} p_i^{(k)})}{\ln(q)}\), where \(p_i = p_i^{(k)} = \frac{1}{Q_i^{(k)} Q_{i+1}^{(k)}}\) and \(I_i^{(k)} = \left[\frac{f_i^{(k)}}{Q_i^{(k)}}, \frac{f_i^{(k)}}{Q_{i+1}^{(k)}}\right]\)
Now, when plotting $f_C^{(k)}(\alpha)$ versus $\alpha$ we notice that, $\forall k$, the spectrum is a perfect type III of Tel [Tel, 1988], as seen in Fig. 2.

Fig. 2: The spectrum $f_C^{(k)}(\alpha)$.

$f_C^{(k)}(\alpha)$ always reaches the value unity at $\alpha = \alpha_k$, and then decreases to a minimum reached at $\alpha_{max}$. The distance $\Delta^{(k)}\alpha = \alpha_{max}^{(k+1)} - \alpha_{max}^{(k)}$ decreases to zero as $k$ increases, so we are tempted to conclude that $f_C = \lim_{k \to \infty} f_C^{(k)}$ is also type III. Yet, $\Delta^{(k)}\alpha$ goes to zero much slower than the harmonic $1/k$, as shown in Table 2:

### TABLE 2: Some numerical values for the $\Delta^{(k)}\alpha$.

| $k$ | $\alpha_{max}$ | $\Delta^{(k)}\alpha$ |
|-----|----------------|----------------------|
| 10  | 2.709          | 0.1814               |
| 11  | 2.8906         | 0.1777               |
| 12  | 3.0684         | 0.1745               |
| 13  | 3.2403         | 0.1716               |
| 14  | 3.4144         | 0.1690               |
| 15  | 3.5834         | 0.1666               |
| 16  | 3.75           | 0.1644               |
| 17  | 3.9144         | 0.1624               |
| 18  | 4.0768         | 0.1605               |
| 19  | 4.2374         | 0.1588               |
| 20  | 4.3962         | 0.1572               |
| 21  | 4.5534         | 0.1557               |
| 22  | 4.7091         |                      |

So, in fact, $\alpha_{max}^{(k)} \to +\infty$.

Let us examine the
4.1 Theoretical reasons responsible for $\alpha_{\text{max}}^{(k)} \to +\infty$

Let us first prove

**Theorem 4.1** No matter how large $\alpha$ is, $f_C(\alpha) > 0$.

**Sketch of the proof.**

We know that $f_C(\alpha) = \lim_{k \to \infty} \frac{\ln(\# \{i \mid \ln(\frac{1}{2 \ln(p_i^{(k)})}) \geq \alpha\})}{\ln(\frac{1}{p_i^{(k)}})} = \alpha \lim_{k \to \infty} \frac{\ln(\# \{i \mid \ln(Q_i^{(k)} Q_{i+1}^{(k)}) \geq \frac{k \ln(2)}{\alpha}\})}{k \ln(2)} = \frac{\alpha}{\ln(2)} \lim_{k \to \infty} \frac{\ln(\# \{i \mid \ln(Q_i^{(k)}) + \ln(Q_{i+1}^{(k)}) \geq \frac{\ln(2)}{\alpha}\})}{k}$,

so $f_C(\alpha)$ away from zero implies $\frac{\ln(\# \{i \mid \ln(Q_i^{(k)}) + \ln(Q_{i+1}^{(k)}) \geq \frac{\ln(2)}{\alpha}\})}{k} \approx e^{\varepsilon(\alpha)}(k)$ away from zero if $k \geq k(\alpha)$ is large enough.

Let us notice that, if $\alpha$ is very large, such quotient will be very small. Let us call it $\varepsilon(\alpha)$. Therefore, the numerator of $Q^{(\alpha)}(k)$ will behave as $\varepsilon(\alpha)k$ for $k$ large enough. Then $\# \{ \}$ behaves as $e^{\varepsilon(\alpha)k}$, which in turn implies

$$\frac{\# \{i \mid \ln(Q_i^{(k)}) + \ln(Q_{i+1}^{(k)}) \geq \frac{\ln(2)}{\alpha}\}}{\# \{i \mid \ln(Q_i^{(k)} Q_{i+1}^{(k)}) \geq \frac{\ln(2)}{\alpha}\}} \approx e^{\varepsilon(\alpha)(k+1)} = e^{\varepsilon(\alpha)k} = e^{\varepsilon(\alpha)} > 1,$$

$e^{\varepsilon(\alpha)}$ being independent of $k$.

But, if $\alpha$ is very large, even if we take large values of $k$, it could be that $\varepsilon(\alpha)$ is so small that $e^{\varepsilon(\alpha)}$, though greater than unity, will be so close to unity as to appear indistinguishable form it. Let us analyse the reasons behind the extreme smallness of $\varepsilon(\alpha)$ when $\alpha$ is large. The condition

$$\frac{\ln(Q_i^{(k)}) + \ln(Q_{i+1}^{(k)})}{k} \approx \frac{\ln(2)}{\alpha}$$

can be rewritten as

$$\frac{\ln(Q_i^{(k)})}{k} + \frac{\ln(Q_{i+1}^{(k)})}{k} \approx \frac{\pi}{M},$$

where $M$ is a suitable constant, and $\pi$ is the limit, as $k \to \infty$, of the average of the $2^k$ logarithms of the bases $Q^{(k)}(i)$ of the F-B denominators $Q_i^{(k)}$ considered as exponentials with exponent $k$, i.e. $Q^{(k)}_i = (Q^{(k)}(i))^k$. Such average

$$\frac{\sum_{i=1}^{2^k} \ln(Q^{(k)}(i))}{2^k} = \pi_k$$
increases with \( k \), \( \bar{\pi}_k \to \bar{\pi} \ (k \to \infty) \), a value that cannot exceed \( \ln(\varphi) \), since \( \varphi^k \) is a very good approximation of the largest F-B denominator in the \( k^{th} \) step. The constant \( M = \frac{\bar{\pi}}{\ln(\varphi)} \alpha \) is a measure of how large \( \alpha \) is. Now, one of the two values, \( Q_i^{(k)} \), \( Q_{i+1}^{(k)} \) is the largest of the pair, say \( Q_{i+1}^{(k)} \). Therefore, \( Q_k^{(k)}(i) \) varies growing from unity, in which case its logarithm is zero and \( \frac{\ln(Q_k^{(k)}(i))}{k} \approx \frac{\bar{\pi}}{M} \), reaching a value of \( Q_k^{(k)} \) very near \( Q_{i+1}^{(k)} \), in which case \( \frac{\ln(Q_{k+1}^{(k)})}{k} \approx \frac{\bar{\pi}}{M} \). In both cases, and in general, \( \frac{\ln(Q_k^{(k)}(i))}{k} \approx \lambda \frac{\bar{\pi}}{M} \), \( \lambda \in (\frac{2}{3}, 1] \) (and \( \frac{\ln(Q_k^{(k)}(i))}{k} \) is even smaller). If we have in mind that the average of such values \( \frac{\ln(Q_k^{(k)}(i))}{k} \) is \( \bar{\pi} \), then we understand that, to talk about
\[
\frac{\ln(Q_k^{(k)}(i))}{k} = \lambda \frac{\bar{\pi}}{M}
\]
is to talk about a minute quantity of indices \( i \), since \( \lambda \frac{\bar{\pi}}{M} \) is a tiny proportion of \( \bar{\pi} \).

Intuitively, it may be clear that such quantity of "\( i \)" will grow with \( k \), but such growth could be so slow as to be undetectable from \( k \) to \( k + 1 \).

We will introduce two simplifications, one of them quite coarse: we will be quite generous with the sign "\( \approx \)" in
\[
\frac{\ln(Q_k^{(k)}(i))}{k} + \ln(Q_{k+1}^{(k)}) \approx \frac{\bar{\pi}}{M} ;
\]
this will mean for us
\[
\frac{\ln(Q_k^{(k)}(i))}{k} + \ln(Q_{k+1}^{(k)}) \in \left[ \frac{\bar{\pi}}{M}(1 - \frac{1}{4}), \frac{\bar{\pi}}{M}(1 + \frac{1}{4}) \right].
\]

Even now, if \( \alpha \) (and therefore \( M \)) is very very large, the corresponding quantity of indices \( i \) verifying such condition is so minute that we cannot detect its growth from \( k \) to \( k + 1 \). Therefore, we will look for an \( m = m(\alpha) \) so large such that, from \( k \) to \( k + m(\alpha) \) we will detect such growth of the quantity of "\( i \)"; i.e. the quotient between \( \frac{\ln(Q_k^{(k)}(i)) + \ln(Q_{k+1}^{(k)})}{k} \) will be clearly seen as larger than unity.

In order to achieve that, it will be enough to take one F-B interval with denominators \( Q_k^{(k)} \) and \( Q_{k+1}^{(k)} \) — which we will denote, for simplicity, \( I_k,i = [Q_k^{(k)}(i), Q_{k+1}^{(k)}(i)] \) — which fulfills the condition
\[
\frac{\ln(Q_k^{(k)}(i)) + \ln(Q_{k+1}^{(k)})}{k} := q_{i,k} \in \left[ \frac{\bar{\pi}}{M}(1 - \frac{1}{4}), \frac{\bar{\pi}}{M}(1 + \frac{1}{4}) \right] = Int,
\]
and verify that, for a certain \( m = m(\alpha) \) sufficiently large we have, from the step \( k + m(\alpha) \) onwards, two intervals \( I_{k+m}(h,j) \) (in \( I_{k,i} \)) for which the corresponding \( q_{j,k+m(\alpha)} \) belong to \( Int \).

We start from \( q_{i,k} \in Int \), for a pair \( (Q_k^{(k)}(i), Q_{k+1}^{(k)}(i)) \) of denominators adjacent in F-B, say, of the order of \( e^{\frac{\pi}{2M} k} \) and \( e^{\frac{\pi}{2M} k} \) respectively: the corresponding \( q_{i,k} \) is \( \frac{3}{8} \frac{\pi}{M} + \frac{3}{8} \frac{\pi}{M} = \frac{\pi}{M} \) exactly.

We want to see if, for steps \( k + 1, k + 2, \ldots, k + h, \ldots \) it is easy to find an interval \( I_{k+h}(h,j) \) inside \( I_{k,i} \) that inherits \( q_{i,k+h} \in Int \).

Let us F-B interpolate in \( [Q_k^{(k)}(i), Q_{k+1}^{(k)}(i)] \); we obtain two intervals \( L \) and \( R \) (for left and right). From here on \( L^2 \) will be the interval "left of the left" \( LR \) "right of the left", \ldots and so on.

We start by considering \( L = \left[ e^{\frac{3}{2M} k}; e^{\frac{5}{2M} k} \right] + e^{\frac{1}{2M} k}; \) the interpolating value \( e^{\frac{3}{2M} k} + e^{\frac{5}{2M} k} \) can be rewritten as \( e^{\frac{1}{2M} k}(1 + \frac{1}{e^{\frac{3}{2M} k}}) \).
We choose now $k = k(\alpha)$, our starting value of $k$: we want it to be of the order of $\frac{4M}{\pi} \ln(\frac{4M}{\pi})$ (a huge number), so that $\frac{1}{e^{\frac{5\pi}{4M}}}$ is a very small number. Now $L = [e^{\frac{3\pi}{2\pi R}}; e^{\frac{3\pi}{2\pi R}}(1 + \frac{5\pi}{4M})]$ and the corresponding $q$ (with a slight change of notation that, we hope, simplifies the text) is

$$q_{L,k+1} = \frac{\frac{\frac{\pi}{2}k + \frac{5\pi}{4M} + \ln(1 + \frac{5\pi}{4M})}{k+1}}{\frac{\pi}{M(k+1)}} = \frac{\frac{\pi}{M} k + \ln(1 + \frac{5\pi}{4M})}{k+1} = \frac{\frac{\pi}{M}}{k+1} \left[ \frac{k + \ln(1 + \frac{5\pi}{4M})}{M} \right] \approx \frac{\frac{\pi}{M}}{k+1} \left[ k + \frac{5\pi}{4M} \right] = \frac{\frac{\pi}{M}}{(1 + \frac{1}{k+1})} = \frac{\frac{\pi}{M}}{k+1} + \frac{\pi}{4M(k+1)},$$

where $\frac{\pi}{4M(k+1)}$ is so very small that $q_{L,k+1}$ is almost indistinguishable from $\frac{\pi}{M}$: $q_{L,k+1} \in Int$.

Let us explore the situation in $L^2 = [e^{\frac{3\pi}{2\pi R} k}; e^{\frac{3\pi}{2\pi R} k} + e^{\frac{3\pi}{2\pi R} k}(1 + \frac{5\pi}{4M})]$. The interpolating value can be rewritten as $e^{\frac{3\pi}{2\pi R} k}(1 + \frac{5\pi}{4M})$.

Therefore, $q_{L^2,k+2} = \frac{\frac{\frac{\pi}{2}k + \frac{5\pi}{4M} + \ln(1 + \frac{5\pi}{4M})}{k+2}}{\frac{\pi}{M(k+2)}} = \frac{\frac{\pi}{M} k + \ln(1 + \frac{5\pi}{4M})}{k+2} = \frac{\frac{\pi}{M} k + \frac{5\pi}{4M} k + \ln(1 + \frac{5\pi}{4M})}{k+2} = \frac{\frac{\pi}{M} k + \frac{5\pi}{4M} (k + \frac{1}{2})}{k+2} = \frac{\pi}{M} + \frac{\pi}{2M(k+2)},$ again a value very much alike $\frac{\pi}{M}$, slightly larger than it.

Next, we need to know if these $q$'s are increasing, getting away from $\frac{\pi}{M}$ towards $\frac{\pi}{M}(1 + \frac{1}{4})$, i.e. is $\frac{\pi}{4M(k+1)} < \frac{\pi}{2M(k+2)}$? The answer is "yes". So, "$q \in Int''" for $L, L^2, \ldots$ but up to a point, since the corresponding $q$'s appear to increase —very slowly— and, at some distant future they could surpass $\frac{\pi}{M}(1 + \frac{1}{4})$: let us stop the process: we choose $i$ such that $q_{L^i,k+i}$ is of the order of $\frac{\pi}{4M}$. We have $i = i(k) = i(k(\alpha))$. From here we can conclude $\frac{M}{\pi} \ln(1 + i \frac{5\pi}{4M}) = \frac{3}{R} - \frac{k}{k+1}$, and $\frac{M}{\pi} \ln(1 + i \frac{5\pi}{4M}) = \frac{3}{2} (k + i) - k$, and we can replace $i$ by $i + 1$ (in this equality) with confidence.

Then $q_{L^iR,k+i+1} = \frac{\frac{\pi}{M}}{k+i+1} \left[ \frac{\frac{\pi}{2}k + \frac{5\pi}{4M} + \ln(1 + (i + 1) \frac{5\pi}{4M})}{k+i+1} \right] + \frac{\frac{\pi}{M}}{k+i+1} \left[ \frac{\frac{\pi}{2}k + \frac{5\pi}{4M} + \ln(1 + i \frac{5\pi}{4M})}{k+i+1} \right] = \frac{\frac{\pi}{M}}{k+i+1} \left[ 5k + 4\left( \frac{3}{2} (k + i + 1) - k \right) + \frac{3}{4} \right] > \frac{3}{4M} = q_{L^i,k+i}$.

For $L^1R, L^2R^2, \ldots, L^iR^h$, the corresponding $q$'s start to grow again, with the danger of surpassing, eventually, $(1 + \frac{1}{4}) \frac{\pi}{M}$.

However, $q_{L^iR^h,k+i+h} = \frac{\frac{\pi}{M}}{k+i+h} \left[ \frac{\frac{\pi}{2}k + \frac{5\pi}{4M} + \ln(1 + i \frac{5\pi}{4M})}{k+i+h} \right] + \frac{\frac{\pi}{M}}{k+i+h} \left[ \frac{\frac{\pi}{2}k + \frac{5\pi}{4M} + \ln(1 + h(1) \frac{5\pi}{4M})}{k+i+h} \right] = \frac{\frac{\pi}{M}}{k+i+h} \left[ \frac{\frac{\pi}{2}k + \frac{5\pi}{4M} + \ln(1 + i \frac{5\pi}{4M}) + \ln(h + (hi+1) \frac{5\pi}{4M})}{k+i+h} \right] + \frac{\frac{\pi}{M}}{k+i+h} \left[ \frac{\frac{\pi}{2}k + \frac{5\pi}{4M} + \ln(1 + h(1) \frac{5\pi}{4M})}{k+i+h} \right]$, and again, L'Hopital in the variable $h$ shows that, eventually, the last quotient diminishes until it reaches $1 - \frac{1}{4} = \frac{3}{4}, \ldots$. By choosing carefully the spelling, the letters are $L$ and $R$, of the word interval inside $L$ (notice that the spelling is far from unique), we can reasonably be assured of finding an interval inside $L$ for which the corresponding $q \in Int$, provided the step $k \geq k(\alpha)$.

**The interval $R$**

We will explore next the other segment inside $I_k, i.e. let us explore $R$ and the corresponding $q_{R,k+1}$. 

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We have \( R = \left[ e^{\frac{k}{M}} k (1 + \frac{5 q}{4 M}) , e^{\frac{k}{M}} k \right] \), and \( q_{R,k+1} = \frac{1}{k+1} \left[ \frac{5 q}{4 M} k + \ln(1 + \frac{5 q}{4 M}) + \frac{5 q}{4 M} k \right] = \frac{1}{k+1} \left[ \frac{5 q}{4 M} k + \ln(1 + \frac{5 q}{4 M}) \right] < \frac{1}{k+1} \left[ \frac{5 q}{4 M} k + \frac{5 q}{4 M} \right] \text{ (since } \ln(1 + \varepsilon) < \varepsilon) = \frac{5 q}{4 M} (k+1) \frac{1}{k+1} = \frac{5 q}{4 M} \) exactly. Therefore \( R \) does inherit (via \( \ln(1 + \varepsilon) < \varepsilon \)) the property \( q \in \text{Int} \).

Nevertheless, if we had approximated \( \ln(1 + \varepsilon) \) by \( \varepsilon \) (and \( \varepsilon = \frac{5 q}{4 M} \) is very small), we would have had \( q_{R,k+1} = (1 + \frac{1}{4}) \frac{5 q}{4 M} \), and if we had considered \( \text{Int} \) as open, then \( q_{R,k+1} \) would have been simply out of \( \text{Int} \). So \( R \) would be out, would not inherit \( q \in \text{Int} \) from \( I_{k,i} \).

Let us consider next \( RL \) and \( R^2 \). The corresponding F-B interpolating element separating both segments is \( e^{\frac{k}{M}} k (2 + \frac{5 q}{4 M}) \), and looking at the extremes of both segments \( RL \) and \( R^2 \), clearly the smallest possible corresponding \( q \) will be the one associated with \( R^2 \): \( q_{R^2,k+2} = \frac{5 q}{4 M} k + \frac{5 q}{4 M} \), larger still than \( q_{R,k+1} \approx \frac{5 q}{4 M} \).

...And yet, considering \( R^i \) we would have \( q_{R^i,k+i} = \frac{5 q}{4 M} k + \frac{5 q}{4 M} \ln(i + \frac{5 q}{4 M}) \), and, L'Hopital in variable \( i \) shows that the last quotient can be made as small as we want. Therefore, there is a value of \( i \) such that \( q_{R^i,k+i} \in \text{Int} \), say, \( q_{R^i,k+i} = \frac{5 q}{4 M} \); and this \( i \) is our \( m(\alpha) \). Then we change the spelling, by introducing letters \( L \) ... and so on: we feel that, for \( k \geq k(\alpha) + m(\alpha) \) we can safely be assured of finding two intervals inside the starting one \( I_{k,i} \) for which the corresponding \( q \) is in \( \text{Int} \) ... which is what we wanted.

4.2 The \( \alpha_k \) for which \( f^{(k)}_C(\alpha_k) = 1 \)

Let us go back to Fig. 2: \( \alpha_k \) is the value of \( \alpha \) for which \( f^{(k)}_C \) reaches unity. We do have that \( \Delta^{(k)} \alpha = \alpha_{k+1} - \alpha_k \to 0 \) when \( k \to +\infty \). Nevertheless \( \Delta^{(k)} \alpha \to 0 \) far slower than the harmonic \( \frac{1}{k} \), as seen in Table 3.

| \( k \) | \( \alpha_k / f(\alpha_k) = 1 \) | \( \Delta^{(k)} \alpha \) |
|-------|-----------------|-----------------|
| 10    | 1.2220          | 0.0178          |
| 11    | 1.2398          | 0.0167          |
| 12    | 1.2564          | 0.0157          |
| 13    | 1.2722          | 0.0149          |
| 14    | 1.2871          | 0.0141          |
| 15    | 1.3012          | 0.0135          |
| 16    | 1.3147          | 0.0129          |
| 17    | 1.3276          | 0.0123          |
| 18    | 1.3399          | 0.0118          |
| 19    | 1.3518          | 0.0114          |
| 20    | 1.3632          | 0.0110          |
| 21    | 1.3741          | 0.0106          |
| 22    | 1.3847          |                |

Therefore \( \alpha_k \to \infty \).

Let us explore the theoretical reasons underlying this fact.
Theorem 4.2 The value $\alpha_k$ for which $f^{(k)}_C$ is unity tends to infinity as $k \to \infty$.

Sketch of the proof

We know that $f^{(k)}_C(\alpha) = \alpha f^{(k)}_C(\frac{1}{\alpha})$, where $f^{(k)}_C$ is obtained by changing weights $\frac{1}{q}$ by lengths $p'_i = \frac{1}{Q_iQ_{i+1}}$, having now a uniform partition of the unit segment. For $\alpha = \alpha_k$ we have $f^{(k)}_C(\alpha_k) = 1$ and $f^{(k)}_C(\alpha_k) = 0$.

But

$$f^{(k)}_C(\alpha_k) = f^{(k)}_C(\frac{1}{\alpha}) - \frac{1}{\alpha} f^{(k)}_C(\frac{1}{\alpha}) = 0 \quad (4)$$

for $\alpha = \alpha_k$.

Being obtained from a uniform partition, $f^{(k)}_C$ is expressible through the Legendre equations via a certain $\tau(q) = \tau_k(q) = \frac{\ln\sum_{i=1}^{2k} (p_i^{(k)})^q}{\ln(\frac{1}{q})}$ (it makes no sense complicating the notation by writing $\tilde{q}$ or $\tilde{q}_k$ instead of $q$). Therefore, with $\frac{1}{\alpha} = \beta$ we have

$$\begin{cases}
\bar{\tau}_k(q) = \beta q - f^{(k)}_C(\beta) \\
q = f^{(k)}_C(\beta) \\
\bar{\tau}_k(q) = \beta
\end{cases} \quad (5)$$

which, from (4) and for $\beta_k = \frac{1}{\alpha_k}$ becomes

$$\begin{cases}
\bar{\tau}_k(q) = 0 \\
q = f^{(k)}_C(\beta_k) \\
\bar{\tau}'_k(q) = \beta_k
\end{cases} \quad (6)$$

Now, $\bar{\tau}_k(q) = 0$ means $\sum_{i=1}^{2k} (p_i^{(k)})^q = 1$, which happens when $q = 1$, and from (6) we obtain $f^{(k)}_C(\beta_k) = 1$ and $\bar{\tau}'_k(1) = \beta_k$.

Let us consider $\bar{\tau}'_k(q) = \frac{\sum_{i=1}^{2k} (p_i^{(k)})^q \sum_{i=1}^{2k} (p_i^{(k)})^q \ln(p_i^{(k)})}{\ln(\frac{1}{q})}$; when $q = 1$ we are left with

$$\bar{\tau}'_k(1) = \frac{\sum_{i=1}^{2k} (p_i^{(k)})^q \ln(p_i^{(k)})}{-k \ln(2)} = \frac{\sum_{i=1}^{2k} \ln(Q_i^{(k)}/Q_{i+1}^{(k)})}{k \ln(2)} = \beta_k.$$  

Now, if it happens that $\beta_k \to 0$ when $k \to \infty$, then $\alpha_k = \frac{1}{\beta_k} \to \infty$ when $k \to \infty$, which is what we wanted.

Now, $\beta_k$ is of the order of $\frac{1}{\ln(2)} \sum_{i=1}^{2k} \frac{\ln(Q_i^{(k)})}{kQ_i^{(k)}/Q_{i+1}^{(k)}}$, and we know that the average $\frac{\sum_{i=1}^{2k} \ln(Q_i^{(k)})}{2k} = \bar{\pi}_k$ increases to $\bar{\pi} < \ln(\varphi) \approx 0.4812$.

Next, we notice that $\bar{\pi}_{18} = 0.3914$, $\bar{\pi}_{20} = 0.3917$, $\bar{\pi}_{22} = 0.3921$, and if we study the type of growth of $\bar{\pi}_k$ we safely conclude that $\bar{\pi}$ is somewhere between 0.4 and 0.4812; $\bar{\pi}$, and therefore, $\bar{\pi}_k$, when $k$ is very large, is surprisingly near its maximal value $\ln(\varphi)$. Let us notice that this nearness of $\bar{\pi}_k$ to $\ln(\varphi)$ cannot be achieved on account of compensating pairs of very unequal numbers such as, e.g., $\frac{\ln(Q_k^{(k)})}{k} = \frac{\ln(1)}{k} = 0$ and $\frac{\ln(Q_k^{(k)})}{k} = 2\bar{\pi}_k$, since $2\bar{\pi}_k$ is considerably larger than $\ln(\varphi)$, and said $Q_j^{(k)}$ is, therefore, non-existing. This observation, and a moment of
reflexion on the nature of the growth of the \( Q_i^{(k)} \) when \( k \to \infty \), tell us that, if \( k \) is very large, we may assume that there is a certain percentage \( p \) of \( Q_i^{(k)} \) for which \( \frac{\ln(Q_i^{(k)})}{k} \) and \( \tilde{\pi}_k \) are quite close. The tighter the closeness, the smaller the percentage \( p \).

Then, if \( k \) is large enough, save for a constant \( c \in (p, 1) \), the order of \( \sum_{i=1}^{2^k} \frac{\ln(Q_i^{(k)})}{k} \) is given by \( 2^k \frac{\tilde{\pi}_k}{e^{2\pi k}} \), and therefore the order of \( \beta_k \) is given by \( \frac{2}{\ln(2)} \tilde{\pi}_k e^{2\pi k} = \frac{2\pi k}{\ln(2)} e^{2\pi k} \).

\( \tilde{\pi}_k \) is stable for \( k \) large. We are interested in what happens to \( \frac{2\pi k}{\ln(2)} e^{2\pi k} \) when \( k \) is large, i.e. we want to know if \( 2 < e^{2\pi k} \) or \( \ln(2) < 2\tilde{\pi}_k \) or \( \frac{2\pi k}{\ln(2)} > 1 \) if \( k \) is large. When \( k = 22 \), we do obtain a value of \( \tilde{\pi}_k \) for which \( \frac{2\pi k}{\ln(2)} = 1.1314 \), substantially larger than unity... But notice that the corresponding order of the F-B partition of the unit segment \([0, 1]\) is over 4 million segments! We have, then, \( \beta_k \to 0 \) and \( \alpha_k \to \infty \), as we wanted.

Next, let us see that

4.3 \( f''_C(\alpha) \leq 0 \) for \( \alpha \) in its domain

We know that \( f_C(\alpha) = \alpha f_C(\frac{1}{\alpha}) \), where \( f_C = \lim_{k \to \infty} f^{(k)}_C \), is a spectrum with uniform partitions on the unit interval, and therefore fulfills all the Legendre conditions, including \( f''_C(\alpha) \leq 0 \) for any \( \alpha \) in its domain.

Now \( f'_C(\alpha) = \tilde{f}_C(\frac{1}{\alpha}) + \alpha f''_C(\frac{1}{\alpha}) \frac{1}{\alpha^2} = \tilde{f}_C(\beta) - \beta f''_C(\beta) \); therefore \( f''_C(\alpha) = \frac{df''_C(\alpha)}{d\beta} \frac{d\beta}{d\alpha} = [f''_C(\beta) - \beta f''_C(\beta)]'(-\frac{1}{\alpha}) = -\frac{1}{\alpha^2} (f''_C(\beta) - \beta f''_C(\beta)) = \frac{1}{\alpha^2} \beta f''_C(\beta) = \frac{1}{\alpha^2} f''_C(\beta) \leq 0 \) since \( \alpha \geq 0 \) always.

4.4 What we know about \( f_C(\alpha) \):

1) It is zero in \( \alpha = \frac{\ln(2)}{\ln(p^2)} \) [Grynberg and Piacquadio, 1995].

2) For any \( \alpha \) as large as we want it, it is strictly larger than zero.

3) \( f_C(\alpha) = \lim_{k \to \infty} f^{(k)}_C(\alpha) \), and \( f^{(k)}_C(\alpha_k) = 1 \), with \( \alpha_k \to \infty \) when \( k \to \infty \).

4) \( f''_C(\alpha) \leq 0 \) in its domain.

1) to 4) indicate that, qualitatively, the shape of the curve \((\alpha, f_C(\alpha))\) is the one for \((\alpha, f_H(\alpha))\): both curves have, qualitatively, the same shape. Let us compare then now computationally.

5 The Legendre Equations and the hyperbolic measure

We know ([Cawley and Mauldin, 1992], [Riedi and Mandelbrot, 1998]) that, for a so called "self similar measure" we have \( f_C(\alpha) = f_H(\alpha) \), and both are equal to \( \min_q (q\alpha - \tau(q)) \) —an expression that Mandelbrot defined as \( f_L \).
Having a uniform partition in mind, \( l \) being the length of each little segment in the partition, and \( \tau(q) \) being \( \lim_{k \to \infty} \frac{\ln(\sum_{i=1}^{k} p_i^k)}{\ln(l)} \), let us recall that the first Legendre transformation is [Halsey et al., 1986]

\[
\tau(q) = \min_\alpha (q\alpha - f_C(\alpha))
\]  

which implies \( \frac{d}{d\alpha}(q\alpha - f_C(\alpha)) = 0 \), i.e. \( q = f_C'(\alpha) \), and therefore \( \tau(q) = q\alpha - f_C(\alpha) \) for \( q = f_C'(\alpha) \). Now, if we can differentiate both sides of this equality we have \( \tau'(q) = \alpha + q \frac{d\alpha}{dq} - f_C'(\alpha) \frac{dq}{d\alpha} = \alpha \), since \( q = f_C'(\alpha) \).

Then \( f_C'(\alpha) = q\alpha - \tau(q) \) for \( q = f_C'(\alpha) \), and \( \tau'(q) = \alpha \) for the same value of \( q \).

Let us now consider the other Legendre transformation: \( f_L(\alpha) = \min_q (q\alpha - \tau(q)) \). This entails \( \frac{d}{dq}(q\alpha - \tau(q)) = 0 \), i.e. \( \alpha = \tau'(q) \). Therefore

\[
f_L(\alpha) = q\alpha - \tau(q)
\]

where now \( \alpha \) and \( q \) are related through \( \alpha = \tau'(q) \). If we can differentiate equation (8), we obtain

\[
f'_L(\alpha) = q + \tau'(q) \frac{dq}{d\alpha} = q + q - \alpha \frac{dq}{d\alpha} = q,
\]

and therefore we can write \( f_L(\alpha) = q\alpha - \tau(q) \), where \( \alpha \) and \( q \) are related through \( f'_L(\alpha) = q \), or else \( \tau(q) = q\alpha - f_L(\alpha) \) for \( q = f'_L(\alpha) \); which means that, if we can differentiate equalities (3) and (8)—both relating a spectrum \( f(\alpha) \) with \( \tau(q) \)—then both Legendre transformations are equivalent.

Now, the expression "differentiate an equality" has the following sense here: originally [Halsey et al., 1986], equation (3) was a good approximation: \( \tau(q) \) was very well approximated by \( q\alpha - f_C(\alpha) \) with \( q = f_C'(\alpha) \), and two functions can be very much alike... not so their derivatives. We will have—Section 6—opportunity to stress the relevance of this point.

### 5.1 The spectrum \((\alpha, f_H(\alpha))\) of F-B does fulfill the Legendre transformation

In order to refer to \( f_L \) at all, we have to have a uniform partition in mind, so we will work with \( f_H(\alpha) = \alpha f_H(\frac{1}{\alpha}) \), and verify that, for the corresponding \( \bar{\tau}(q) = \lim_{k \to \infty} \frac{\ln(\sum_{i=1}^{k} p_i^k)}{\ln(1/2^k)} \) we have \( f_H(\alpha) = \min_q (q\alpha - \bar{\tau}(q)) \) or, as we just saw,

\[
\begin{cases}
\bar{\tau}(f_H'(\alpha)) = f_H'(\alpha)\alpha - f_H(\alpha) \\
\tau'(f_H'(\alpha)) = \alpha
\end{cases}
\]

We prefer to write the Legendre transformation in the form (9) rather than (8) because of the following reason: The RHS of (9) is obtained through purely theoretical means ([Cesaratto and Piaquadio, 1998], [Cesaratto, 1999] and Theorem (3.1) above). However, to know the value of \( \tau(q) \) entails knowing the expression of \( p_i = p_i^{(k)} \) for every \( i \) and \( k \), \( p_i^{(k)} = \frac{1}{Q_i^{(k)} Q_{i+1}^{(k)}} \); therefore we have to know the value of \( Q_i^{(k)} \) for every \( i \) and \( k \)—or, at least, a good approximation to such a value. Since this is a yet unsolved problem, the LHS of (9) is obtained through purely numerical methods: one finds the value of the \( Q_i^{(k)} \) for the largest \( k \) that our computer can handle, and then one obtains the best possible approximation to \( \bar{\tau}(q) \) for chosen different values of \( q = f_H'(\alpha) \)—which are theoretically obtained.
For short, let us denote \( \phi(\alpha) = f_H(\frac{1}{2}) \), a function we will deal with in Section 6. What we know about \( \phi \): the domain of \( \phi \) is \([0, \frac{\ln(\phi^2)}{\ln(2)}]\); \( \phi(0) = 1 \); \( \phi' \) is negative if \( \alpha \neq 0 \); \( \phi'(\frac{\ln(\phi^2)}{\ln(2)}) \) is a huge negative number \(-M\) (possibly \(-\infty\)); \( \phi \) decreases from 1 to zero as \( \alpha \) goes from 0 to \( \alpha_{\max} = \frac{\ln(\phi^2)}{\ln(2)} \); \( \phi'(0) = \phi''(0) = \ldots = \phi^{(k)}(0) = \ldots = 0 \); \( \phi \) is not analytic at \( \alpha = 0 \).

Now, \( f_H(\alpha) = \alpha \phi(\alpha) \); \( f'_H(\alpha) = \phi(\alpha) + \alpha \phi'(\alpha) \).

We want to compare

\[
\frac{\ln(\sum_{i=1}^{2k} f_{H}^{i}(\alpha))}{-k \ln(2)} \quad \text{with} \quad \alpha f_{H}^{i}(\alpha) - f_{H}(\alpha)
\]

for a high value of \( k \). Recall that the left hand side of (10) is numerically obtained, whereas the RHS is theoretically obtained; \( f_{H}^{i}(\alpha) = \phi(\alpha) + \alpha \phi'(\alpha) \), and \( \alpha f_{H}^{i}(\alpha) - f_{H}(\alpha) = \alpha \phi(\alpha) + \alpha^2 \phi'(\alpha) - \alpha \phi(\alpha) = \alpha^2 \phi'(\alpha) \).

With \( X(\alpha) \) we denote the LHS and with \( Y(\alpha) \) the RHS of (10), and we plot the curve \((X(\alpha), Y(\alpha))\) for a diversity of values of \( \alpha \). The curve is indistinguishable from the line \( Y = X \), and, in order to make a more analytical comparison, we present Table 4:

**TABLE 4 :** Some numerical values for LHS and RHS of Eq. (10).

To calculate \( Y(\alpha_n) \) we considered \( \bar{\alpha}_n = \frac{\alpha_n + \alpha_{n+1}}{2} \) and \( f_H(\bar{\alpha}_n) = \frac{f_H(\alpha_n) + f_H(\alpha_{n+1})}{2} \), thereby ending with 14 values of \( X(\alpha_n) \) and \( Y(\alpha_n) \).

| n | \( \alpha_n \) | \( f_H(\alpha_n) \) | \( \phi(\alpha_n) \) | \( \phi'(\alpha_n) \) | LHS = \( X(\alpha_n) \) | RHS = \( Y(\alpha_n) \) |
|---|---|---|---|---|---|---|
| 1 | 0.4921 | 0.4913 | 0.9983 | -0.015 | -0.0079 | -0.044 |
| 2 | 0.5925 | 0.5906 | 0.9968 | -0.0621 | -0.0330 | -0.0237 |
| 3 | 0.6440 | 0.6339 | 0.9936 | -0.0712 | -0.0467 | -0.0348 |
| 4 | 0.7591 | 0.7480 | 0.9854 | -0.1157 | -0.0877 | -0.0728 |
| 5 | 0.8291 | 0.8103 | 0.9773 | -0.1729 | -0.1477 | -0.1305 |
| 6 | 0.9101 | 0.8767 | 0.9633 | -0.2931 | -0.2945 | -0.2740 |
| 7 | 1.0271 | 0.9542 | 0.9290 | -0.5274 | -0.6261 | -0.6058 |
| 8 | 1.183 | 0.9851 | 0.8809 | -0.82 | -1.0871 | -1.0711 |
| 9 | 1.1680 | 0.9813 | 0.8402 | -1.5718 | -2.3704 | -2.3348 |
| 10 | 1.2718 | 0.8610 | 0.6770 | -3.6577 | -6.1776 | -6.1523 |
| 11 | 1.3226 | 0.6496 | 0.4912 | -7.9559 | -13.3682 | -13.3185 |
| 12 | 1.3506 | 0.3810 | 0.2821 | -12.9209 | -23.8365 | -23.7174 |
| 13 | 1.3591 | 0.2351 | 0.1730 | -17.3957 | -32.4206 | -32.2273 |
| 14 | 1.3631 | 0.1392 | 0.1021 | -21.2014 | -39.7022 | -39.4545 |
| 15 | 1.3652 | 0.081 | 0.0587 | |

The relative errors \( \Delta_n = \frac{Y(\alpha_n) - X(\alpha_n)}{\min\{Y(\alpha_n), X(\alpha_n)\}} \) are shown in Table 5.

For values \( n = 1, 2, 3, \) and \( 4 \) the magnitudes of both \( X(\alpha_n) \) and \( Y(\alpha_n) \) are very small, and the estimate of \( \Delta_n \) is always misleading; i.e. let us suppose that \( X(\alpha_n) = \varepsilon \) and \( Y(\alpha_n) = \varepsilon^2 \). Hence \( \Delta_n \), estimated from below, is \( \frac{\varepsilon - \varepsilon^2}{\varepsilon} \approx 1 \); and, estimated by excess —as we did in our table— \( \Delta_n \) is \( \frac{\varepsilon^2}{\varepsilon} \approx \frac{1}{\varepsilon} \). 

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We will deal separately with the values of $n$ for which $\alpha_n$ is small — i.e. for which $X(\alpha_n)$ and $Y(\alpha_n)$ are very small.

| $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $\Delta_n$ | 0.8190 | 0.3952 | 0.3420 | 0.2043 | 0.1320 | 0.0750 | 0.0335 |
|     | 8   | 9   | 10  | 11  | 12  | 13  | 14  |
|     | 0.0150 | 0.0153 | 0.0041 | 0.0037 | 0.005 | 0.006 | 0.0063 |

So we can safely conclude that equality (10) does follow.

5.2 Theoretical aspects of equation (10)

Equality (10) can be rewritten as

$$\frac{\ln(\sum_{i=1}^{2^k} p_i^{\phi(\alpha)+\alpha\phi'(\alpha)})}{-k \ln(2)} = \alpha^2 \phi'(\alpha)$$

(11)

where the value of $k$ in the LHS is the largest one we can handle numerically. Nevertheless, if the exponent $\phi(\alpha) + \alpha \phi'(\alpha) = q$ takes certain key values, one would be able to see why, analytically, equation (10) holds when $k \to \infty$:

1) Let $\alpha = 0$. Then $\phi(\alpha) = 1$ and $\phi'(\alpha) = 0$: the corresponding exponent $q$ is unity, then $\sum_{i=1}^{2^k} p_i^q = \sum_{i=1}^{2^k} p_i^{(k)} = 1$, and its log is zero, hence $\bar{\tau}(q) = 0$. The RHS is $\alpha^2 \phi'(\alpha)$, obviously zero.

2) Let $\alpha = \frac{\ln(\varphi^2)}{\ln(2)} = \bar{\alpha}_{max}$; $\phi(\bar{\alpha}_{max}) = 0$; $\phi'(\bar{\alpha}_{max}) = -M$, $M$ a huge number, the largest —in absolute value— that $\phi'$ can have, i.e. that $f_H^\beta$ can have. Now, let us recall that this largest possible negative value of $q$ (i.e. of $f_H^\beta$) selects [Halsey et al., 1986] the smallest $p_i = p_i^{(k)}$ in the partition function, which corresponds to $p_i^{(k)} = \frac{1}{Q_i^{(k)}} \approx \frac{1}{\varphi^2}$.

The LHS of (11), therefore, is

$$\frac{\ln\left(\frac{1}{\varphi^2\pi}^{0+\frac{\ln(\varphi^2)}{\ln(2)}}(\varphi^2\pi M)^{(-M)}\right)}{-k \ln(2)} = -M\ln(\varphi^2)\frac{\ln(\varphi^2)}{\ln(2)} = -M\left[\frac{\ln(\varphi^2)}{\ln(2)}\right]^2,$$

which is, exactly, $f_H^\beta(\bar{\alpha}_{max})\bar{\alpha}_{max}^2$ i.e. the RHS of (11).

3) We want now the value of $\alpha$ for which $q = 0$. Let us call $\beta$ that particular value of $\alpha$ for which $q = f_H^\beta(\alpha) = 0$. We have $f_H(\beta) = 1$. On the other hand $f_H(\alpha) = \alpha f_H^\beta$, so $f_H^\beta(\alpha) = f_H(\frac{1}{\alpha}) - \frac{1}{\alpha}f_H^\beta(\frac{1}{\alpha})$. Then, for $\frac{1}{\alpha} = \beta$ we have $f_H^\beta(\frac{1}{\beta}) = f_H(\beta) - \beta f_H^\beta(\beta) = 1$. Since the line $y = x$ is tangent to $f_H(\alpha) = f_C(\alpha)$ we have that $f_H(\frac{1}{\beta}) = \frac{1}{\beta}$.

Let us go back to $f_H(\alpha)$. Our value $\beta$ is obtained from

$$\begin{cases} q = f_H^\beta(\beta) = 0 \\ \bar{\tau}'(q) = \bar{\tau}'(0) = \beta \end{cases}$$

(12)
Now, \( \bar{\tau}'(q) = \lim_{k \to \infty} \frac{1}{\sum_{i=1}^{2^k} p_i^{\ln(p_i)}} \sum_{i=1}^{2^k} \frac{p_i^{\ln(p_i)}}{k \ln(2)} \), which, for \( q = 0 \) yields \( \bar{\tau}'(0) = \beta = \frac{2\pi}{\ln(2)} \). Therefore, the LHS of \( \lim_{k \to \infty} \ln(\sum_{i=1}^{2^k} p_i^{\phi + \alpha \phi'}) - k \ln(2) = \alpha \) —when \( \alpha = \beta, \phi = \bar{f}'_H(\beta) = 0, \) and \( \bar{f}'_H(\alpha) = \bar{f}'_H(\beta) = 1 — is \( \frac{\ln(2^k)}{-k \ln(2)} = -1 \), whereas the RHS is \( \beta \) —when \( \alpha = \beta, \bar{f}'_H(\beta) = 0, \) and \( \bar{f}'_H(\alpha) = \bar{f}'_H(\beta) = 1 — is \( \frac{\ln(2^k)}{-k \ln(2)} = -1 \), so again we have strict equality.

5.3 The derivative of equation (11)

We differentiate now both sides of equation (11) —\( \alpha \) is our variable— and we obtain

\[
\frac{1}{\sum_{i=1}^{2^k} p_i^{\phi + \alpha \phi'}} \sum_{i=1}^{2^k} \frac{p_i^{\phi + \alpha \phi'} \ln(p_i)}{-k \ln(2)} (\phi + \alpha \phi')' \neq 2\alpha \phi' + \alpha^2 \phi''
\]

\[
= \alpha (2\phi' + \phi'') = \alpha (\phi + \alpha \phi')'.
\]

Notice that, we can —due to the nature of \( \phi — safely cancel \( (\phi + \alpha \phi')' \), except for \( \alpha = 0 \) and some other occasional value of \( \alpha \) in \((0, \alpha_{\text{max}}]\). If we do so, we are left with the interrogation \( \bar{\tau}'(q) = \alpha \) for \( q = \bar{f}'_H(\alpha) \). We use the values of \( \alpha_n, \phi(\alpha_n), \) and \( \phi'(\alpha_n) \) in Table 4, in order to obtain Table 6,

| n   | 1    | 2    | 3    | 4    | 5    | 6    |
|-----|------|------|------|------|------|------|
| \( \bar{\tau}'(\bar{f}'_H(\alpha_n)) \) = \( X(\alpha_n) \) | 0.7439 | 0.7749 | 0.7906 | 0.8332 | 0.8847 | 0.9719 |
| \( \alpha_n = Y(\alpha_n) \) | 0.5423 | 0.6182 | 0.7016 | 0.7941 | 0.8696 | 0.9686 |

and the corresponding relative errors \( \Delta_n \), calculated by excess as before are in Table 7. Again the curve \((X(\alpha_n), Y(\alpha_n))\) is indistinguishable from the line \( Y = X \).

| n   | 1    | 2    | 3    | 4    | 5    | 6    | 7    |
|-----|------|------|------|------|------|------|------|
| \( \Delta_n \) | 0.3718 | 0.2534 | 0.1269 | 0.0492 | 0.0174 | 0.0034 | -0.001 |
| 8   | 9    | 10   | 11   | 12   | 13   | 14   |
|      | 1.0726 | 1.141 | 1.2223 | 1.3007 | 1.3474 | 1.3696 | 1.3742 | 1.3753 |
|      | 1.0727 | 1.1431 | 1.2199 | 1.2972 | 1.3366 | 1.3548 | 1.3611 | 1.3642 |

So we can safely conclude that equality \( \bar{\tau}'(q) = \alpha \) for \( q = \bar{f}'_H(\alpha) \) holds.
5.4 Theoretical aspects of equality $\bar{\tau}'(q) = \alpha$

1) As before, let us choose $\alpha = \bar{\alpha}_{\max} = \frac{\ln(\phi^2)}{\ln(2)}$, hence $\phi(\bar{\alpha}_{\max}) = 0$ and $\phi'$ is $-M$, with the corresponding selection of $p_i = p_i^{(k)} \equiv \frac{1}{\frac{1}{r_i} \cdot 2 \pi}$. We have then

$$
\bar{\tau}'(0 + \frac{\ln(\phi^2)}{\ln(2)} (-M)) = \frac{1}{\frac{\ln(\phi^2)}{\ln(2)} (-M)} \frac{\ln(\phi^2)}{\ln(2)}
$$

which makes $\bar{\tau}'(q) = \alpha$ for $q = \bar{\tau}_H'(\alpha)$ analytically correct.

2) Again, let us try $\alpha = 0$, for which $\phi$ is unity and $\phi'$ zero. Then the exponent $q$ is unity, and for $\bar{\tau}_K'(q)$, i.e. the expression for $\bar{\tau}'(q)$ minus "$\lim_{k \to \infty}$", we have: $\bar{\tau}_K'(q) = \bar{\tau}_K'(1) = \frac{1}{\sum_{i=1}^{2^k} p_i \ln(p_i)} \sum_{i=1}^{2^k} p_i \ln(p_i)$.

We want to understand why $\sum_{i=1}^{2^k} p_i \ln(p_i) \to 0 = \alpha$ as $k \to \infty$, a fact that we can observe in the screen of our computer. Again, we work with $k$ so large as to ensure that, for a certain percentage $p$, sensibly larger than zero, we have $\ln(Q^{(k)}(i)) \approx \bar{\pi}_k \equiv \bar{\pi}$. The tighter this resemblance, the smaller $p$ will be.

$$
\sum_{i=1}^{2^k} p_i \ln(p_i) = \frac{\sum_{i=1}^{2^k} \frac{\ln(Q^{(k)}_i)}{Q^{(k)}_i Q^{(k+1)}_i}}{k}
$$

which is of the order of $2(\sum_{i=1}^{2^k} \frac{\ln(Q^{(k)}_i)}{Q^{(k)}_i Q^{(k+1)}_i}) \frac{1}{k}$, whose order is given by $2^k \ln(e^{x_{\alpha}}) \frac{1}{k}$.

This value is $\bar{\pi}_k [\frac{2^k}{2 \pi \ln(2)}]$; $\bar{\pi}_k$ is a stable multiplicative constant; it remains to see if $\frac{2^k}{2 \pi \ln(2)} < 1$ if $k \to \infty$, i.e. if $2^k < 2 \pi \ln(2)$ or if $\ln(2) < 2 \pi \ln(2)$...but at the end of Section 4.2 we saw that $\frac{2^k}{2 \pi \ln(2)} > 1$ if $k$ is large — for $k = 22$ we had obtained $\frac{2^k}{2 \pi \ln(2)} = 1.1314$, and $\bar{\pi}$ was very slowly increasing as $k$ grew. So, again, for $\alpha = 0$ we have $\bar{\tau}'(q) = \alpha$.

3) Finally, let us consider the value of $\alpha$ for which $q = 0$, i.e. $\alpha = \frac{2^k}{2 \pi \ln(2)}$. Therefore, $\bar{\tau}'(q) = \lim_{k \to \infty} \bar{\tau}_k'(q) = \lim_{k \to \infty} \frac{1}{\sum_{i=1}^{2^k} p_i \ln(p_i)} \sum_{i=1}^{2^k} p_i \ln(p_i) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{2^k} \frac{\ln(Q^{(k)}_i Q^{(k+1)}_i)}{Q^{(k)}_i Q^{(k+1)}_i} = \frac{2^k}{2 \pi \ln(2)} = \alpha$, so again $\bar{\tau}'(q) = \alpha$ in a strict way, when $\alpha = \frac{2^k}{2 \pi \ln(2)}$.

6 The function $\phi(\alpha)$. Its Legendre Transformation

Let us consider $\phi(\alpha) = f_H(\frac{\bar{\alpha}}{\alpha}) = d_H(\Omega_{\frac{\bar{\alpha}}{\alpha}})$, where

$$
\Omega_{\frac{\bar{\alpha}}{\alpha}} = \{x \in [0, 1] / \forall k \in \mathbb{N} : x \in I_i^{(k)} ; I(I^{(k)}_i) = p_i^{(k)} = \frac{1}{Q_i^{(k)} Q_i^{(k+1)}}\};
$$
\[ \mu(I_i^{(k)}) = \frac{1}{2^k}; \text{ and } \frac{\ln(1/2^k)}{\ln(p_i^{(k)})} \to \frac{1}{\alpha} \text{ when } k \to \infty. \]

In other words, \( \phi(\alpha) = d_H(\{x \in [0,1]/\ln(1/2^k) \to \frac{1}{\alpha}; x \in I_i^{(k)}\forall k\}) \) for short. Here "\( x \in I_i^{(k)}\)" means: \( x \) belongs to a sequence of nested intervals of length \( p_i^{(k)} = \frac{1}{Q_i^{(k)}/Q_{i+1}^{(k)}} \). Notice that we can write

\[ \phi(\alpha) = d_H(\{x \in [0,1]/\ln(p_i^{(k)}) \to \alpha \text{ when } k \to \infty \}). \] (13)

Let us take some \( x \sim RLRLRL \ldots \) in the F-B left-right system in diagram G below. We trust the notation is obvious. At the left of diagram G we have our F-B system of segments of length \( p_i^{(k)} \) and \( \mu \) measure \( \frac{1}{2^k} \). At the right the roles of length and \( \mu \) measure are inverted, and we will denote this with \( \tilde{p} \) for lengths and \( \tilde{\mu} \) for measure. Notice that, in the \( \tilde{p}, \tilde{\mu} \) system at the right of the diagram we do have some element representative of the spelling \( RLRLRL \ldots \) in a sequence of nested intervals:

**DIAGRAM G:**

\[ \frac{1}{2} \quad p_{RL}^{(2)} = \frac{1}{2}; \quad \mu_{RL}^{(2)} = \frac{2}{3} \quad \frac{1}{2} \quad p_{RL}^{(2)} = \frac{1}{2}; \quad \mu_{RL}^{(2)} = \frac{3}{4} \quad \frac{1}{2} \]

\[ \frac{1}{2} \quad p_{RLR}^{(3)} = \frac{1}{5,3}; \quad \mu_{RLR}^{(3)} = \frac{2}{3} \quad \frac{1}{2} \quad p_{RLR}^{(3)} = \frac{1}{2}; \quad \mu_{RLR}^{(3)} = \frac{3}{5} \quad \frac{1}{2} \]

\[ \frac{1}{2} \quad p_{RLRL}^{(4)} = \frac{1}{5,8}; \quad \mu_{RLRL}^{(4)} = \frac{2}{3} \quad \frac{1}{2} \quad p_{RLRL}^{(4)} = \frac{1}{2}; \quad \mu_{RLRL}^{(4)} = \frac{3}{5} \quad \frac{1}{2} \]

\[ \frac{1}{2} \quad p_{RLRL}^{(4)} = \frac{1}{5,8}; \quad \mu_{RLRL}^{(4)} = \frac{2}{3} \quad \frac{1}{2} \quad p_{RLRL}^{(4)} = \frac{1}{2}; \quad \mu_{RLRL}^{(4)} = \frac{3}{5} \quad \frac{1}{2} \]

The spelling \( RL, RL, RLR, \ldots \) at the right of Diagram G gives a sequence of nested intervals of length \( \frac{1}{2^k} \). Now, if one did not know better (if one did not know about the inverse function), from (13), from the nestedness of the intervals of length \( \frac{1}{2^k} \) and measure \( p_i^{(k)} \), and from the sameness of the grammar, one would be tempted to conjecture:
\[ \phi(\alpha) = d_H(\tilde{\Omega}_\alpha), \]

where \( \tilde{\Omega}_\alpha \) refers to the system at the right in diagram G.

This is nonsense, since, in the left system, the element

\[ x \sim RLRLRL \ldots \]

is \( \frac{\sqrt{5}-1}{2} \), i.e. the number in the unit interval with the largest Markov irrationality coefficient (that is, the "most irrational number" in \([0,1]\)), whereas the corresponding \( RLRLRL \ldots \) in the right system \( \bar{p}, \bar{\mu} \) in diagram G is \( \frac{2}{3} \) — a very rational number! It means that \( \Omega_\alpha \) and \( \tilde{\Omega}_\alpha \) have little or nothing in common.

What would it take for these two sets to be alike — at least, so that they would share the same Hausdorff dimension? It would suffice, say, that the \( \frac{1}{2^k} \)-length segments and the \( p_i^{(k)} \)-length segments were very much alike, i.e. that the measures Euclidean and hyperbolic were very much alike; then, we could conjecture \( \phi(\alpha) \neq d_H(\tilde{\Omega}_\alpha) \), where the corresponding measure is indeed taken over a uniform partition \( \frac{1}{2^k} \) in the \( k^{th} \) step. If that were the case, we would have:

\[
\lim_{k \to \infty} \frac{\ln(\sum_{i=1}^{2^k} p_i^{(k)} \phi(\alpha))}{\ln(\frac{1}{2^k})} = \alpha \phi'(\alpha) - \phi(\alpha) \tag{14}
\]

and, if this equality was so sharp that we could differentiate both sides of it — \( \alpha \) our variable — and still obtain equality, we would then have

\[
\tau'(\phi'(\alpha)) = \alpha. \tag{15}
\]

The astonishing fact is that (14) is true: again, the LHS of (14) is considered numerically, and the RHS theoretically. Again the elements displayed in Table 8 are taken from Table 4.

\[
X(\alpha_n) = \ln(\sum_{i=1}^{2^k} p_i^{(\alpha_n)}) \text{ and } Y(\alpha_n) = \alpha_n \phi'(\alpha_n) - \phi(\alpha_n).
\]

| Table 8: | \( X(\alpha_n) = \frac{\ln(\sum_{i=1}^{2^k} p_i^{(\alpha_n)})}{\ln(\frac{1}{2^k})} \); \( Y(\alpha_n) = \alpha_n \phi'(\alpha_n) - \phi(\alpha_n) \). |
|---|---|---|---|---|---|---|
| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| \( X(\alpha_n) \) | -1.0169 | -1.0703 | -1.0807 | -1.1315 | -1.1970 | -1.3360 | -1.6107 |
| \( Y(\alpha_n) \) | -1.0057 | -1.0336 | -1.0395 | -1.0733 | -1.1207 | -1.23 | -1.4707 |

| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|
| -1.9597 | -2.8786 | -5.5259 | -10.5229 | -17.8914 | -23.9972 | -29.2117 |
| -1.7979 | -2.6760 | -5.3288 | -10.3522 | -17.7334 | -23.8149 | -29.0026 |

The plotting of the curve \( (X(\alpha_n), Y(\alpha_n)) \) is in Fig. 3:
Fig. 3: The curve $\mathbf{X}(\alpha_n), \mathbf{Y}(\alpha_n))$; $\mathbf{X}(\alpha_n) = \frac{\ln(\sum p_i \phi'(\alpha_n))}{-k\ln(2)}$; $\mathbf{Y}(\alpha_n) = \alpha_n\phi'(\alpha_n) - \phi(\alpha_n)$.

A line indistinguishable from $\mathbf{Y} = \mathbf{X}$.

The corresponding relative errors $\Delta_n$, estimated by excess are shown in Table 9.

| $n$ | 1  | 2  | 3  | 4  | 5  | 6  |
|-----|----|----|----|----|----|----|
| $\Delta_n$ | 0.0110 | -0.0343 | -0.0382 | -0.0510 | -0.0638 | -0.0793 |

Moreover, if we differentiate both sides of equation (14) we still obtain "equality"!

Once we differentiate both sides of equation (14), the corresponding —tentative— equation now reads

$$\tau'(\phi'(\alpha))\phi''(\alpha) \neq \alpha\phi''(\alpha) \quad (16)$$

The corresponding values —with $\phi''(\alpha_n)$ estimated discretely from our values of $\alpha_n$ and $\phi'(\alpha_n)$ in Table 4— and the corresponding plot $(\mathbf{X}(\alpha_n), \mathbf{Y}(\alpha_n))$ are shown in Table 10 and Fig. 4.

| $n$ | 1  | 2  | 3  | 4  | 5  | 6  |
|-----|----|----|----|----|----|----|
| $\mathbf{X}(\alpha_n) =$LHS of (16) | -0.7 | -0.1 | -0.5 | -0.9 | -1.4 | -2.6 |
| $\mathbf{Y}(\alpha_n) =$RHS of (16) | -0.40 | -0.1 | -0.4 | -0.6 | -1.1 | -2.3 |

TABLE 9: The relative errors $\Delta_n$ corresponding to the values in Table 8.

| $n$ | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 |
|-----|----|----|----|----|----|----|----|----|
| 0.0869 | -0.0825 | 0.0704 | 0.0357 | 0.0162 | 0.0088 | 0.0076 | 0.0072 |

TABLE 10: The RHS and LHS of (16).
2.5 3.5 4.8 6 6.9 -2.1 -0.6 0.34 1.6

-1.02 2.4 3.5 4.8 6 6.9 -2.1 -1 -0.6 0.11 1.5

$\ln(|X|)$

$\ln(|Y|)$

Fig. 4: The curve $(\ln(|X|)(\alpha_n), \ln(|Y(\alpha_n)|))$; $X(\alpha_n) = \text{LHS of (16)}$; $Y(\alpha_n) = \text{RHS of (16)}$. The points at the left of the curve correspond to small values of $\alpha$, and will be dealt with in Section 6.1.

Notice that we did *not* cancel $\phi''(\alpha)$ in both sides of (16), so that we are not really testing the (impossible) truthfulness of (15).

Before we go any further, let us explore the

6.1 Theoretical aspects of equations (14), (16) and (15)

Let us consider first equation (14)

1) for $\alpha = 0$. Then $\phi'(\alpha) = 0$ and $\phi(\alpha) = 1$. Clearly $\bar{\tau}(\phi'(\alpha)) = \bar{\tau}(0) = \lim_{k \to \infty} \frac{\sum_{i=1}^{2^k} p_i^0}{\ln(2^k)} = \lim_{k \to \infty} \frac{\ln(2^k)}{\ln(2^k)} = -1$, whereas $\alpha \phi'(\alpha) - \phi(\alpha)$ is reduced to $-1$ as well.

2) Let us pose $\alpha = \frac{\ln(\varphi^2)}{\ln(2)}$. Then $\phi(\alpha) = 0$ and $\phi'(\alpha) = -M$, $M$ a huge number. As before, the only surviving term in $\sum_{i=1}^{2^k} p_i^0$ is the one corresponding to the smallest possible $p_i^{(k)}$, i.e. $\frac{1}{\varphi^{2^k}}$ times a certain constant $c - k$ has to be large. Therefore,

$$\bar{\tau}(\alpha) = \lim_{k \to \infty} \frac{\ln((\frac{1}{\varphi^{2^k}})^{-M})}{-k \ln(2)} = (-M) \lim_{k \to \infty} \frac{-2k \ln(\varphi)}{-k \ln(2)}$$

$$= \frac{\ln(\varphi^2)}{\ln(2)} (-M) = \alpha \phi'(\alpha) - \phi(\alpha),$$

since $\phi(\alpha) = 0$.

Therefore, for the initial and final value of the domain of $\phi$, Eq. (14) is tight.

Let us consider equation (17)
1) for $\alpha = 0$. Now $\phi'(\alpha) = \phi''(\alpha) = 0$. Let us calculate

$$
\bar{\tau}'(\phi'(\alpha)) = \bar{\tau}'(0) = \lim_{k \to \infty} \frac{\sum_{i=1}^{2^k} p_i^0 \ln(p_i)}{\sum_{i=1}^{2^k} p_i^0 (-k \ln(2))}
$$

$$
= \lim_{k \to \infty} \frac{\sum_{i=1}^{2^k} \ln(p_i)}{2^k (-k \ln(2))} = \lim_{k \to \infty} -\frac{1}{\ln(2)} \frac{\sum_{i=1}^{2^k} \ln(p_i)/k}{2^k}
$$

$$
= \frac{1}{\ln(2)} \lim_{k \to \infty} \frac{\sum_{i=1}^{2^k} \ln(Q_{(i)}^{(k)}) + \sum_{i=1}^{2^k} \ln(Q_{(i+1)}^{(k)})}{2^k}
$$

$$
= \frac{1}{\ln(2)} \lim_{k \to \infty} 2\pi_k = \frac{2\pi}{\ln(2)}, \text{ a finite number. The fact: } \phi''(0) = 0, \text{ therefore, implies the validity of (16) for } \alpha = 0. \text{ Notice that, if } \phi''(0) \neq 0, \text{ we would cancel it in equation (16), and we would be left with the interrogation}
$$

$$
\bar{\tau}'(\phi'(\alpha)) \not\equiv \alpha \text{ when } \alpha = 0. \quad (17)
$$

Notice that the answer is "no", for $\bar{\tau}'(0) = \frac{2n}{\ln(2)} \neq 0 = \alpha$. Therefore $\bar{\tau}'(q) = \alpha$ for $q = f'(\alpha)$, i.e. (13) is false if $f = \phi$.

2) Let $\alpha = \frac{\ln(\phi^2)}{\ln(2)}$. Now, $\phi'' \neq 0$ and we can cancel it in both sides of equation (13): we are testing (15). We have now $\phi'(\alpha) = q = -M, M$ a huge value, and we know that, as before, sums $\sum_i p_i^q$ and $\sum_i p_i^q \ln(p_i)$ are replaced by the term corresponding to the smallest $p_i^{(k)}$ in the formula. We obtain, proceeding as above,

$$
\bar{\tau}'(-M) = \lim_{k \to \infty} \frac{1}{\phi^2(\frac{1}{\alpha})} \frac{\ln(\frac{1}{\phi^2})}{\ln(2)}
$$

$$
= \lim_{k \to \infty} \frac{\ln(\frac{1}{\phi^2})}{\ln(2)} = \alpha. \quad (18)
$$

Eq. (13) for $f = \phi$ is, therefore, tight for $\bar{\alpha}_{max}$.

**The second Legendre Equation for $\phi(\alpha)$**

Notice that equation (16) is essentially another form of (13): $\phi'(\alpha) = -\frac{1}{\alpha \phi''(\alpha)}$, i.e. the quotient of two monotonic functions; therefore $\phi''(\alpha)$ is quite free to change signs a number of times... yet (with perhaps the exception of $\alpha = 0$) each of these occasions will be an isolated value of $\alpha$. Nevertheless we did notice that, for $\alpha = 0$, we have $\bar{\tau}'(\phi'(\alpha)) \neq \alpha$, but, as $\phi''(\alpha) = 0$ for $\alpha = 0$, then (14) held, even if (13) did not: the value of $\phi''$ acted as a "leveler" for $\alpha = 0$ —i.e. when $\bar{\tau}'(q)$ was different from $\alpha$, $\bar{\tau}'\phi''$ was equal to $\alpha\phi''$. Now, from Table 10 and the graph in Fig. 4 corresponding to (13), we can infer that $\phi''$ is the "leveler" for every $\alpha$!... a mystery that implies, somehow, that although (13) is not true, a certain relaxed form of it is still true.

From bits and pieces in the last few sections, the reader can put together a rather lengthy and technical proof of equation (13) for a value of $\alpha$ strictly smaller than $\bar{\alpha}_{max}$ (for which equation (13) holds), i.e. for $\alpha = \frac{2n}{\ln(2)}$ —a value slightly larger than unity. We can then conjecture that, for the segment $[\frac{2n}{\ln(2)}, \bar{\alpha}_{max}]$ we do have equation (13). But, in fact, though this is not a big interval, its left extreme is difficult to reach ($k$ should be very large). Therefore,
we take a slightly smaller interval, and, for the largest value of \( k \) our fastest computer can handle we write: \( X(\alpha_n) = \bar{\tau}'(\phi'(\alpha_n)); Y(\alpha_n) = \alpha_n \), and we obtain Table 11.

| \( n \) | 1     | 2     | 3     | 4     | 5     | 6     | 7     |
|--------|-------|-------|-------|-------|-------|-------|-------|
| \( \bar{\tau}'(\phi'(\alpha_n)) \) | 1.2047 | 1.2410 | 1.2936 | 1.3352 | 1.3617 | 1.3709 | 1.3743 |
| \( \alpha_n \) | 1.1431 | 1.2199 | 1.2972 | 1.3366 | 1.3548 | 1.3611 | 1.3649 |

The relative errors, computed, as usual, by excess, are displayed in Table 12.

| \( n \) | 1     | 2     | 3     | 4     | 5     | 6     | 7     |
|--------|-------|-------|-------|-------|-------|-------|-------|
| \( \Delta n \) | 0.0538 | 0.0173 | -0.0028 | -0.0010 | 0.0051 | 0.0072 | 0.0074 |

These facts tell us that a segment contained in the line \( Y = X \) is contained in the graph \( (X(\alpha), Y(\alpha)), X(\alpha) = \bar{\tau}'(\phi'(\alpha)); Y(\alpha) = \alpha. \) In fact, plotting such graph from values taken from all tables above, we do notice a smooth curve, smoothly glued to a segment contained in \( Y = X, \alpha \in [\frac{2\bar{\alpha}}{m(2)}, \bar{\alpha}_{max}] \). It means that \( \langle 13 \rangle \), without \( \phi'' \) as a "leveler", holds in a substantial part of the domain of \( \phi \).

7 Conclusion

The mystery of \( \phi(\alpha) \) posed by equations \( \langle 14 \rangle \) and \( \langle 15 \rangle \) can be explained by conjecturing that the Lebesgue and the hyperbolic measures are not so terribly different as they seem. Another way of saying this: \([0, 1]\), as a fractal with \( \mu^H \) as a probability measure, is more self-similar than it seems. Yet another way: \( \mu^H \) has much more in common with the so called "self-similar measures" than what it looks like. In order to clarify this, let us consider certain examples of fractals contained in \([0, 1]\), given as the limit of an iterative process. The step of the iterations is given, as usual, by \( k \in \mathbb{N} \). The \( k^{th} \) iteration of the process is associated with a \( k \)-partition covering the fractal. We will assume that all segments in the \( k \)-partition are equiprobable.

Let us consider, first, the usual ternary Cantor set as \( \Omega \). The \( k^{th} \) partition consists of \( 2^k \) segments. Their length is \( \frac{1}{3^k} \). Their reciprocal length is \( 3^k \): an exponential with 3 as a base. The average \( \bar{\pi}_k \) of all the logarithms of these bases is \( \ln(3) \), and so is the limit \( \bar{\pi} \) of \( \bar{\pi}_k \) as \( k \to \infty \).

Let us now consider \( \Omega \) as the unit segment; the corresponding measure gives equiprobability to the two segments \([0, a]\) and \([a, 1]\), when \( a \in (0, 1), a \neq 1/2, a \) can be irrational. We have two contractions with factors \( a \) and \( b = 1 - a \) involved. When \( k = 2 \) we will have four equiprobable segments of lengths \( a^2, ab, ba \) and \( b^2, \ldots \) and so on. In the \( k^{th} \) equiprobable iteration of the process we have \( 2^k \) segments of length \( a^ib^{k-i}, 0 \leq i \leq k \). If we express \( \frac{1}{a^{i}b^{k-i}} \) as an exponential with exponent \( k \), then the logarithm of the base of the exponential, \( \ln(Q(i)^{k}) \), will be a number obviously bigger than \( \frac{1}{2} \min\{\ln\left(\frac{1}{a}\right), \ln\left(\frac{1}{b}\right)\} = A \) and smaller than \( \ln\left(\frac{1}{a}\right) + \ln\left(\frac{1}{b}\right) = B \.

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The corresponding average $\bar{\pi}_k$, therefore, will be in interval $(A, B)$, and so will be its limit $\bar{\pi}$—which can be proved to exist.

We can continue in this way and convince ourselves that, given any finite number of contractions, the corresponding average limit $\bar{\pi} = \bar{\pi}(\Omega)$ will exist.

Now, when we look at the $k^{th}$ F-B partition we find segments whose lengths vary from $\frac{1}{k}$ to $\frac{1}{F(k)F(k+1)}$, where $F(k)$ is the $k^{th}$ Fibonacci number, of the order of $\varphi^k$. If we consider, for simplicity, the reciprocal lengths, we find that they go from $k$ to $\varphi^{2k}$, passing through, say, $3k^2, 5k^3, \ldots$ you name it: $\Omega$ looks strongly non self-similar, it looks the opposite of having a finite number of contractors... Yet, the corresponding $\bar{\pi}_k$ grows with $k$ and reaches a value $\bar{\pi}$ very much alike its theoretical maximum: it means that segments of non-exponential reciprocal length, such as $k, 3k^2, \ldots$ etc., are, on the whole, negligible in number, and $\mu$ seems to behave, asymptotically, as a “self-similar measure”, despite all the irreconcilable differences between both measures referred to in the Introduction.

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