Algorithms for general hard-constraint point processes via discretization

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Abstract

We study a general model for continuous spin systems with hard-core interactions. Our model allows for a mixture of $q$ types of particles on a $d$-dimensional Euclidean region $V$ of volume $\nu(V)$. For each type, particle positions are distributed according to a Poisson point process. The Gibbs distribution over all possible system states is characterized by the mixture of these point processes conditioned that no two particles are closer than some distance parameterized by a $q \times q$ matrix. This general model encompasses classical continuous spin systems, such as the hard-sphere model or the Widom–Rowlinson model.

We present sufficient conditions for approximating the partition function of this model, which is the normalizing factor of its Gibbs distribution. For the hard-sphere model, our method yields a randomized approximation algorithm with running time polynomial in $\nu(V)$ for the known uniqueness regime of the Gibbs measure. In the same parameter regime, we obtain a quasi-polynomial deterministic approximation algorithm for the hard-sphere model, which, to our knowledge, is the first rigorous deterministic approximation algorithm for a continuous spin system. We obtain similar approximation results for all continuous spin systems captured by our model and, in particular, the first explicit approximation bounds for the Widom–Rowlinson model. Additionally, we show how to obtain efficient approximate samplers for the Gibbs distributions of the respective spin systems within the same parameter regimes.

Key to our method is reducing the continuous model to a discrete instance of the hard-core model with size polynomial in $\nu(V)$. This generalizes existing discretization schemes for the hard-sphere model and, additionally, improves the required number of vertices of the generated graph from super-exponential to quadratic in $\nu(V)$, which we argue to be tight.

Keywords: continuous spin systems • partition function • hard-sphere model • Widom–Rowlinson model • sampling and approximation algorithms
1 Introduction

Statistical physics models complex systems of interacting particles as probability distributions. The main goal is to explain the macroscopic properties of such a system when it is only described by the microscopic interactions among its particles. Two classical models in this area include the hard-sphere model, central in the analysis of thermodynamics of liquids and liquid mixtures [4, 21], and the Widom–Rowlinson model, which explains the evaporation of liquids [39]. Each state of such a spin system consists of a finite set of points in Euclidean space, corresponding to centers of spherical particles. These sets are constrained by hard-core interactions, i.e., two particles are not allowed to occupy the same space. The probability distribution characterizing the valid states of a model is called its Gibbs distribution. A key part in better understanding this distribution is its normalizing factor, known as the partition function of the model.

Statistical physics studies spin systems in terms of their phase transitions, i.e., the change of the system’s macroscopic behavior as one of its parameters crosses a critical value. Rigorous proofs of phase transitions for continuous spin systems are rare. Concerning our previous examples, a phase transition has been proved to exist for the Widom–Rowlinson model [34], while it is still an open question for the hard-sphere model. Phase transitions are more commonly shown for discrete spin systems, where, instead of the Euclidean space, particles are placed on the vertices of a graph and interact only with particles on adjacent vertices.

One of the simplest and most well-studied discrete spin systems is the hard-core model. It is considered to be a universal model, as any discrete spin system can be mapped to a hard-core instance [35, Section 5.7]. For an undirected graph $G = (V, E)$ and a function $\lambda : V \to \mathbb{R}_{\geq 0}$, the hard-core model is defined by the tuple $(G, \lambda)$. Its set of valid states is the set of all independent sets $\mathcal{I}(G)$ of $G$. Each independent set $I \in \mathcal{I}(G)$ is assigned the weight $\prod_{v \in I} \lambda(v)$. The partition function $Z_{HC}(G, \lambda)$ of the hard-core model is the sum of weights of all independent sets and its Gibbs distribution $\mu_{HC}^{(G, \lambda)}$ assigns each independent set a probability proportional to its weight, normalized over the partition function. Formally,

$$Z_{HC}(G, \lambda) = \sum_{I \in \mathcal{I}(G)} \prod_{v \in I} \lambda(v) \quad \text{and, for all } I \in \mathcal{I}(G), \text{ we have } \mu_{HC}^{(G, \lambda)}(I) = \frac{\prod_{v \in I} \lambda(v)}{Z_{HC}(G, \lambda)}.$$ 

In the uniform case, where $\lambda$ is constant, classical results in statistical physics establish the existence of a threshold for which the hard-core model undergoes a phase transition on various graph classes, such as two-dimensional lattices [28] and regular trees [5, 26]. In a series of celebrated results, the tree threshold has been linked to a change in the computational behavior of the hard-core model. That is, for values of $\lambda$ where the Gibbs measure is unique on a $\Delta$-regular tree, there is a polynomial-time algorithm for approximating $Z_{HC}(G, \lambda)$ on general graphs of maximum degree $\Delta$ [38], while for the remaining values of $\lambda$, this is NP-hard [20, 37].

As discrete models are better understood, the hard-core model on lattices is often used as a coarse-grained version of the continuous hard-sphere model. Friedli and Velenik write for this discretization:

"Although it might appear as a significant departure from reality, we will see that it leads to satisfactory results and allows a good qualitative understanding of the corresponding phenomena." [17, Chapter 4.1]

In this article, we show for a broad class of continuous spin systems with hard-core interactions that an appropriately chosen discretization scheme based on the hard-core model leads not only to a qualitative but also to a quantitative understanding of the continuous model. Our main contributions are:
We introduce a hard-constraint point process of \( q \) different types of particles in the \( d \)-dimensional Euclidean space where each pair of particles is constrained by a hard-core interaction of different range, according to the particle types. This model captures classical continuous models, such as the hard-sphere model or the Widom–Rowlinson model.

We develop a discretization scheme where, for a given hard-constraint point process on a region \( V \subset \mathbb{R}^d \), we obtain an instance \((G, \lambda)\) of the hard-core model. We determine conditions such that the partition function \( Z_{HC}(G, \lambda) \) closely approximates the partition function of the hard-constraint point process and the number of vertices of \( G \) is polynomial in the volume of \( V \).

We obtain new algorithmic results for continuous spin systems by combining our discretization scheme with known algorithms for the hard-core model. We present sufficient conditions for randomized approximation algorithms of the partition function and sampling algorithms for the Gibbs distribution of the hard-constraint point process with running time polynomial in the volume of \( V \). Under the same conditions, we show the existence of deterministic approximation algorithms for the partition function with running time quasi-polynomial in the volume of \( V \).

### 1.1 Hard-constraint point processes

We define our model of general spin systems with hard-core interactions. Let \( V \subset \mathbb{R}^d \) be bounded and measurable, and let \( q \in \mathbb{N}_{\geq 1} \). The model represents the distribution of particles of \( q \) types, labeled by elements in \([q] := [1, q] \cap \mathbb{N}\), on \( V \). Particles of the same types are assumed to be indistinguishable. Let \( \lambda: [q] \to \mathbb{R}_{\geq 0} \) be a function that equips each particle type with a fugacity. For each particle type \( i \in [q] \), we assume that the positions of particles are distributed according to a (labeled) Poisson point process of intensity \( \lambda(i) \) on \( V \). To add the constraints to the model, let \( R \in \mathbb{R}^{d \times d \times q} \) be a symmetric \( q \times q \) matrix, called the interaction matrix. We condition the mixture of point processes by rejecting all configurations that contain particles at positions \( x_1, x_2 \in V \) with corresponding particle types \( \tau_1, \tau_2 \in [q] \) with \( d(x_1, x_2) < R(\tau_1, \tau_2) \). That is, the entries of \( R \) determine the minimum distance that particles of the respective types can have. Especially note that \( R(\tau_1, \tau_2) = 0 \) means that the particle types \( \tau_1, \tau_2 \in [q] \) are not subjected to any pairwise constraints. Thus, if all entries of \( R \) are set to 0, we obtain a simple mixture of Poisson point processes.

For any instance \((V, R, \lambda)\) of a hard-constraint point process, the above characterizes a Gibbs distribution as follows. For all \( k \in \mathbb{N} \) and all type assignments \( \tau: [k] \to [q] \), let \( D_{\tau}^{(R)}: (\mathbb{R}^d)^k \to \{0, 1\} \) be the function that indicates for a tuple of particles positions \( x = (x_i)_{i \in [k]} \in (\mathbb{R}^d)^k \) whether it forms a valid configuration, assuming that for each \( i \in [k] \), the particle at \( x_i \) is of type \( \tau(i) \). Formally,

\[
D_{\tau}^{(R)}(x) = \prod_{i,j \in [k], i < j} 1\{d(x_i, x_j) \geq R(\tau(i), \tau(j))\},
\]

noting that \( D_{\tau}^{(R)}(x) = 1 \) for the case that \( k = 0 \). For each tuple \((x, \tau)\) as above, the Gibbs distribution of \((V, R, \lambda)\) is defined via the probability density

\[
\mu^{(V, R, \lambda)}(x, \tau) = \frac{1}{Z(V, R, \lambda)} \frac{1}{\prod_{i \in [k]} \lambda^{(\tau(i))}} D_{\tau}^{(R)}(x),
\]

where

\[
Z(V, R, \lambda) = \prod_{i, j \in [k], i < j} \frac{\lambda^{(\tau(i))} \lambda^{(\tau(j))}}{R(\tau(i), \tau(j))}.
\]
where the normalizing constant $Z(\mathbb{V}, R, \lambda)$ is called the (grand canonical) partition function:

$$Z(\mathbb{V}, R, \lambda) = 1 + \sum_{k \in \mathbb{N}_{\geq 1}} \frac{1}{k!} \sum_{\tau : [k] \rightarrow [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \int_{\mathbb{V}^k} D_t^{(R)}(x) \, d\nu^{d \times k},$$

with $\nu^{d \times k}$ denoting the product of $k$ Lebesgue measures on $(\mathbb{R}^d)^k$.

The hard-sphere model, mentioned earlier, naturally falls into this framework.

**Example 1.1 (Hard-sphere model).** Given an instance of the hard-sphere model with particles of radius $r \in \mathbb{R}_{>0}$ and fugacity $\lambda \in \mathbb{R}_{\geq 0}$, let $q = 1$, and let $R_{\text{HS}}$ be a $1 \times 1$ matrix containing only the entry $2r$. The resulting hard-constraint point process $(\mathbb{V}, R_{\text{HS}}, \lambda)$ (where we slightly abuse notation and treat $\lambda$ as a constant function) describes a Poisson point process of intensity $\lambda$ on $\mathbb{V}$, conditioned on each pair of particles having at least a distance of twice the particle radius.

The second model we highlighted is the Widom–Rowlinson model. We present the most general version, where we allow an arbitrary number of particle types with different radii and fugacities.

**Example 1.2 (Widom–Rowlinson model).** Consider an instance of the Widom–Rowlinson model with $q \in \mathbb{N}_{\geq 1}$ particles types, each type $i \in [q]$ with radius $r_i \in \mathbb{R}_{>0}$ and fugacity $\lambda_i \in \mathbb{R}_{\geq 0}$. For all $i \in [q]$, let $\lambda(i) = \lambda_i$, and, for all $i, j \in [q]$, let

$$R_{\text{WR}}(i, j) = \begin{cases} 0 & \text{if } i = j, \\ r_i + r_j & \text{otherwise}. \end{cases}$$

In the resulting hard-constraint point process $(\mathbb{V}, R_{\text{WR}}, \lambda)$, particles of the same type are allowed to be arbitrarily close together, whereas particles of different types need to have at least a distance that is equal to the sum of their radii.

### 1.2 Reduction to a discrete hard-core model

Central to our proofs is the following approach. Given a (continuous) hard-constraint point process over $\mathbb{V}$ and a finite set of points $X \subset \mathbb{V}$, we define a (discrete) hard-core model which we call the hard-core representation based on $X$. We focus on sets $X$ that are the result of a function $\Phi : \mathbb{V} \rightarrow X$, which we call an allocation. Our main result is that the partition function of a discretization based on certain allocations closely approximates the partition function of the respective hard-constraint point process.

In detail, let $(\mathbb{V}, R, \lambda)$ be an instance of a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types. Further, let $\nu(\mathbb{V})$ denote the volume of $\mathbb{V}$. Given a finite and non-empty set of points $X \subset \mathbb{V}$, we construct an undirected graph $G_X = (V_X, E_X)$ and a function $\lambda_X : V_X \rightarrow \mathbb{R}_{\geq 0}$ as follows:

- For each point $x \in X$ and each type $i \in [q]$, we construct a vertex $v^{(i)}_x$. Furthermore, for each $i \in [q]$, we set $V^{(i)}_X = \{v^{(i)}_x \mid x \in X\}$, and we define $V_X = \cup_{i \in [q]} V^{(i)}_X$.

- For each $i, j \in [q]$ and $x, y \in X$, we connect $v^{(i)}_x, v^{(j)}_y \in V_X$ with an edge in $E_X$ if $\neg(x = y$ and $i = j)$ and $d(x, y) < R(i, j),$

where the condition $\neg(x = y$ and $i = j)$ prevents self-loops.
• For each $i \in [q]$ and each $x \in X$, we set $\lambda_X(v^{(i)}_x) = \frac{v(V)}{|X|} \lambda(i)$. Note that for all $x, y \in X$, this means that $\lambda_X(v^{(i)}_x) = \lambda_X(v^{(i)}_y)$, and we sometimes abuse notation and write $\lambda_X(i)$ instead.

Then $(G_X, \lambda_X)$ is the hard-core representation of $(V, R, \lambda)$ based on $X$.

We study allocations $\Phi : V \to X$ such that for each $x \in X$, the set $\Phi^{-1}(x)$ has roughly the same volume and a bounded diameter.

**Definition 1.3 ($\delta$-$\varepsilon$-allocation).** Let $V \subset \mathbb{R}^d$ be bounded and measurable, and assume $v(V) > 0$. For finite $X \subset V$, $\delta \in [0, 1)$, and $\varepsilon \in \mathbb{R}_{>0}$, we call a function $\Phi : V \to X$ a $\delta$-$\varepsilon$-allocation for $X$ if, for all $x \in X$,

1. $\Phi^{-1}(x)$ is measurable and

\[
(1 - \delta) \frac{v(V)}{|X|} \leq v(\Phi^{-1}(x)) \leq (1 + \delta) \frac{v(V)}{|X|},
\]

and,

2. for all $y \in \Phi^{-1}(x)$, it holds that $d(x, y) \leq \varepsilon$.

Our main result is the following theorem.

**Theorem 4.6.** Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types, and assume $V \subset \mathbb{R}^d$ is star-convex. Furthermore, set $R_{\min} = \inf_{i,j \in [q]} \{ R(i, j) \mid R(i, j) > 0 \}$, and set $\lambda_{\max} = \max_{i \in [q]} \lambda(i)$. Let $X \subset V$ with $4 \lambda_{\max} v(V) \leq |X| < \infty$, and let $(G_X, \lambda_X)$ be the hard-core representation of $(V, R, \lambda)$ based on $X$. Finally, let $\delta \in [0, \frac{1}{2}]$ and $\varepsilon \in [0, \frac{R_{\min}}{4}]$, and assume that $V$ has a $\delta$-$\varepsilon$-allocation for $X$. Then

\[
|Z_{HC}(G_X, \lambda_X) - Z(V, R, \lambda)| \leq \left( e^{\frac{q}{|X|} \sum_{i \in [q]} \lambda(i)^2 v(V)} e^\left( 2 \delta + \left( \frac{4\varepsilon}{R_{\min}} \right)^q \right) \sum_{i \in [q]} \lambda(i) v(V) - 1 \right) Z(V, R, \lambda).
\]

Since Theorem 4.6 is at the core of our techniques, we briefly overview its proof. Given $X \subset V$ and a $\delta$-$\varepsilon$-allocation $\Phi$, for each $x \in X$, the total weight that $\Phi^{-1}(x)$ contributes to $Z(V, R, \lambda)$ is assigned to the total weight the vertex of $G_X$ that corresponds to $x$ contributes to $Z_{HC}(G_X, \lambda_X)$. This weight allocation introduces three types of errors between the two partition functions, which we need to bound. The first one is due to the Poisson point process of the continuous model potentially generating more than one point in $\Phi^{-1}(x)$, for some $x \in X$, which cannot be represented as part of the hard-core partition function. To bound this error, we reduce the continuous model to an intermediate multiset version of the hard-core model, where valid configurations are allowed to contain multiple copies of the same vertex. Then, we bound the difference between the partition function of the multiset hard-core model and $Z_{HC}(G_X, \lambda_X)$. The second error type is due to the regions $\Phi^{-1}(x)$ not necessarily having the same volume. However, this error is small, due to $\delta$. The third error type is due to configurations that are valid for the hard-constraint point process potentially being allocated to configurations that are invalid for the hard-core model and vice versa. This happens because, e.g., two points $x_1, x_2 \in V$ of type $\tau_1, \tau_2$, respectively, with $d(x_1, x_2) \geq R(\tau_1, \tau_2)$ get allocated to $\Phi(x_1), \Phi(x_2) \in X$ with $d(\Phi(x_1), \Phi(x_2)) < R(\tau_1, \tau_2)$. The parameter $\varepsilon$ ensures that each point $x \in V$ is allocated to a close by point $\Phi(x) \in X$. This allows us to bound this error by the difference of two partition functions with slightly different values in their exclusion matrices $R$ and obtain Theorem 4.6.

Under a choice of $X$, with $|X|$ polynomial in $(v(V), \delta^{-1}, \varepsilon^{-1})$, Theorem 4.6 is an approximation-preserving reduction from $Z(V, R, \lambda)$ to $Z_{HC}(G_X, \lambda_X)$ that is polynomial in $v(V)$ and produces a graph $G_X$ of size

\footnote{We use the convention that $\inf \emptyset = \infty$, which in this setting means that $R_{\min}$ can be chosen arbitrarily large.}
polynomial in \(v(V)\). In Section 1.3, we discuss how to choose the point set \(X\) with \(|X| \in O((v(V))^2)\) in the case of cubic regions \(V = [0, \ell]^d\) for some \(\ell \in \mathbb{R}_{>0}\), and we apply Theorem 4.6 in order to use the known algorithmic results on the hard-core model. Our bound on the required number of points in this setting is asymptotically tight in \(v(V)\) for general hard-constraint point processes, as we show in Section 3.1.

For cubic regions, we show that there is a \(\delta\)-\(\varepsilon\)-allocation with \(\delta = 0\). However, allowing for a small margin of volume error \(\delta > 0\) in Theorem 4.6 results in interesting connections between the partition function of the hard-core model on random geometric graphs and the continuous partition function of the hard-constraint point process, as we discuss in Section 1.5.

We note that a discretization scheme similar to ours was used by Friedrich et al. [18] to obtain an approximation algorithm for the hard-sphere model with uniform fugacity. Their algorithm utilizes the succinct representation and structural information of the produced graph \(G\), as their discretization results in graphs whose number of vertices is super-exponential in \(v(V)\). Since our discretization yields much smaller graphs, we are able to use the results on the hard-core model as out-of-the-box algorithms and for a broader class of spin systems.

### 1.3 Approximation algorithms via canonical discretization

Our algorithmic results focus on cubic regions \(V = [0, \ell]^d, \ell \in \mathbb{R}_{>0}\). This suggests the following canonical discretization. The point set \(X \subset V\) consists of a \(d\)-dimensional grid of side length \(n\), and the allocation \(\Phi\) maps each point \(y = (y(\ell))_{\ell \in [d]} \in V\) to the closest point in \(x = (x(\ell))_{\ell \in [d]} \in X\) such that, for all \(\ell \in [d]\), it holds that \(x(\ell) \leq y(\ell)\). Since, for each \(x \in X\), the set \(\Phi^{-1}(x)\) has the same volume, this results in a \(0\)-\(\varepsilon\)-allocation. By Theorem 4.6, in order to approximate the partition function of the hard-constraint point process \(Z(V, R, \lambda)\), it suffices to do so for the partition function of the hard-core model on \((G_X, \lambda_X)\).

When restricting the hard-core model to the uniform case, i.e., there is a constant \(\lambda\) such that, for each \(v\), it holds that \(\lambda(v) = \lambda\), there is a selection of algorithmic results applicable to our setting. Note that all known approximation algorithms for the partition function of the uniform hard-core model, apply to instances \((G, \lambda)\), where \(G\) has maximum degree \(\Delta\) and \(\lambda < \lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta} \sim \frac{e}{\Delta}\). This parameter regime is tight, as approximating the partition function for a hard-core model is known to be NP-hard for instances \((G, \lambda)\) with \(\lambda > \lambda_c(\Delta)\) [20, 37]. We obtain our algorithms by bounding the maximum degree of \(G_X\) and using the known approximation algorithms for the hard-core model. Namely, by using the polynomial-time approximation algorithm of Anari et al. [2] and Chen et al. [12], we obtain a randomized approximation algorithm for the hard-constraint point process with running time polynomial in the volume of \(V\).

Furthermore, using the deterministic approximation algorithm of Weitz [38] with running time in \(O(n^{\log \Delta})\), we obtain a deterministic algorithm with running time quasi-polynomial in the volume of \(V\).

In order to formally state our algorithmic results, we introduce the volume exclusion matrix \(B \in \mathbb{R}_{\geq 0}^{q \times q}\) of a hard-constraint point process \((V, R, \lambda)\) with \(q \in \mathbb{N}_{\geq 1}\) particles types. For all \(i, j \in [q]\), it holds that \(B(i, j) = v(B(R(i, j)))\), where \(B(r)\) denotes the volume of a \(d\)-dimensional ball of radius \(r\). Intuitively, the entry \(B(i, j)\) gives an upper bound on the volume of the region around a point \(x \in V\) in which no particle of type \(j \in [q]\) can be placed, given that a particle of type \(i \in [q]\) is placed at \(x\). Our main algorithmic result is the following.

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2 Note that there is an algorithmic result with running time in \(O(\Delta^3 n \log n)\) [11], but the dependency on \(\Delta\) does not result in a polynomial-time algorithm for the hard-core model on \(G_X\).

3 We get the same result by combining the results of Peters and Regts [33] with the method of Barvinok [3].
Theorem 6.1. Let \((V, R, \lambda)\) be a hard-constraint point process with \(q \in \mathbb{N}_{\geq 1}\) particle types, \(V = [0, \ell]^d\) for some \(\ell \in \mathbb{R}_{>0}\), and let \(\lambda\) be a constant. Let \(B\) be the corresponding volume exclusion matrix and denote its \(L_1\)-norm by \(\|B\|_1\). If
\[
\lambda < \frac{e}{\|B\|_1},
\]
then for all \(\varepsilon_A \in (0, 1]\), there is a randomized \(\varepsilon_A\)-approximation algorithm for \(Z(V, R, \lambda)\) with running time in \(\text{poly}\left(\frac{\nu(V)}{\varepsilon_A}\right)\) and a quasi-polynomial deterministic approximation algorithm for \(Z(V, R, \lambda)\) with running time \(\left(\frac{\nu(V)}{\varepsilon_A}\right)^{\Theta(\ln(\nu(V)/\varepsilon_A))}\).

To the best of our knowledge, this is the first rigorous deterministic approximation algorithm for the partition function of a continuous spin system and, furthermore, extends the family of spin systems for which we can approximate their partition functions.

Applying Theorem 6.1 to the special case of the hard-sphere model, we obtain the following corollary.

Corollary 6.2. Let \(V = [0, \ell]^d\) for some \(\ell \in \mathbb{R}_{>0}\). Further, let \(r \in \mathbb{R}_{>0}\) and \(\lambda \in \mathbb{R}_{\geq 0}\). Denote by \(Z_{\text{HS}}(V, r, \lambda)\) the hard-sphere partition function on \(V\) with particles of radius \(r\) and fugacity \(\lambda\). If
\[
\lambda < \frac{e}{2^d \nu(B(r))},
\]
then for all \(\varepsilon_A \in (0, 1]\), there is a randomized \(\varepsilon_A\)-approximation algorithm for \(Z_{\text{HS}}(V, r, \lambda)\) with running time in \(\text{poly}\left(\frac{\nu(V)}{\varepsilon_A}\right)\) and a quasi-polynomial deterministic \(\varepsilon_A\)-approximation algorithm for \(Z_{\text{HS}}(V, r, \lambda)\) with running time \(\left(\frac{\nu(V)}{\varepsilon_A}\right)^{\Theta(\ln(\nu(V)/\varepsilon_A))}\).

Corollary 6.2 achieves the best known algorithmic bound in terms of \(\lambda\) \([18, 30]\), which corresponds to the known bound for the absence of a phase transition \([29]\). The quasi-polynomial deterministic approximation algorithm is a new result and partially answers an open question of Michelen and Perkins \([30]\).

Theorem 6.1 also yields the first rigorous algorithmic result for the continuous Widom–Rowlinson model.

Corollary 6.3. Let \(V = [0, \ell]^d\), \(\ell \in \mathbb{R}_{>0}\), let and \(r \in \mathbb{R}_{>0}\) and \(\lambda \in \mathbb{R}_{\geq 0}\). Let \(Z_{\text{WR}}(V, r, \lambda)\) denote the Widom–Rowlinson partition function on \(V\) with \(q \in \mathbb{N}_{\geq 1}\) particle types, each of radius \(r\) and fugacity \(\lambda\). If
\[
\lambda < \frac{e}{(q - 1)2^d \nu(B(r))},
\]
then for all \(\varepsilon_A \in (0, 1]\), there is a randomized \(\varepsilon_A\)-approximation algorithm for \(Z_{\text{WR}}(V, r, \lambda)\) with running time in \(\text{poly}\left(\frac{\nu(V)}{\varepsilon_A}\right)\) and a quasi-polynomial deterministic \(\varepsilon_A\)-approximation algorithm for \(Z_{\text{WR}}(V, r, \lambda)\) with running time \(\left(\frac{\nu(V)}{\varepsilon_A}\right)^{\Theta(\ln(\nu(V)/\varepsilon_A))}\).

For non-uniform hard-constraint point processes, i.e., each particle type has its own fugacity, the resulting instance of the hard-core model is also non-uniform. There are some algorithmic results for the non-uniform hard-core model in the literature, but none of them is known to establish a tight condition. To our knowledge, the broadest parameter regime for the non-uniform hard-core model is achieved by a randomized approximation algorithm of Friedrich et al. \([19]\). Applying their approximation algorithm to the instance \((G_X, \lambda_X)\), we obtain the following algorithmic condition for the hard-constraint point process.
Then, for each vertex $v$, recall that each vertex
is not satisfied by an ordinary computer. However, for practical considerations, the floating-point precision
of common discrete computational models might be seen as sufficient for applying our sampling approach.

Assuming a computational model that can perform arithmetic operations of floating-point values with
arbitrary precision and uniformly sample a random floating-point number from an interval (see Assumption 7.1),
we use our discretization $G_X$ to recover an approximate sampler for $\mu(V, R, \lambda)$. Such an assumption
is not satisfied by an ordinary computer. However, for practical considerations, the floating-point precision
of common discrete computational models might be seen as sufficient for applying our sampling approach.

Given a $\delta$-allocation $\Phi$, our sampling algorithm (Algorithm 1) first samples an independent set $I$ from
the Gibbs distribution of the hard-core model on $(G_X, \lambda_X)$, using one of the known sampling algorithms
(see, e.g., [2, 12]). Recall that each vertex $v_X^{(i)}$ of $G_X$ corresponds to a point $x \in X$ of a particle of type $i$.
Then, for each vertex $v_X^{(i)}$ in $I$, our sampler chooses a position in $\Phi^{-1}(x)$ uniformly at random and places a
particle of type $i$ at this position. This way, we obtain the following sampling analogue of Theorem 6.1.

1.4 Sampling via random perturbations

The resulting hard-core representation used in Corollary 6.6 is a bipartite graph with different fugacities
for each vertex partition. Algorithmic results on such hard-core instances have been obtained by Cannon
and Perkins [7]. However, their results do not appear to give feasible parameter regimes in our setting.

For the non-uniform hard-core model faster randomized algorithms [9], as well as, deterministic algo-
rithms [24] have appeared in the literature. These results are also applicable to our setting. The stated
results are chosen to yield the broadest parameter regime possible.
with edge connection threshold

As seen in Sections 1.3 and 1.4, our algorithmic results use a 0-1 concentration of random discretizations on the computational behavior of the system—for more examples, see [6, 10, 16].

Our sampling procedure does not always result in a valid configuration for the hard-constraint point process. However, by bounding the total-variation distance between the two distributions, &mu; \((V, R, \lambda)\) and the distribution of our sampler, we bound the probability of the sampler returning an invalid configuration by a small constant. In this case, we reject this configuration and repeat the sampling procedure.

Observe that the sampling analogues of Corollaries 6.2 and 6.3 immediately follow from the above theorem. With Algorithm 1, we also obtain the following theorem for the non-uniform case.

\[\text{Theorem 7.6. Suppose a computational model that satisfies Assumption 7.1. Let } (V, R, \lambda) \text{ be a hard-constraint point process with } q \in \mathbb{N}_{\geq 1} \text{ particle types and } V = [0, \ell]^d \text{ for some } \ell \in \mathbb{R}_{>0}. \text{ Let } B \text{ be the corresponding volume exclusion matrix and denote its } L_1 \text{-norm by } \|B\|_1. \text{ If }
\lambda < \frac{e}{\|B\|_1},
\]
then for all \(\varepsilon_S \in (0, 1]\), there is an \(\varepsilon_S\)-approximate sampler for \(\mu^{(V, R, \lambda)}\) with running time in \(\text{poly}\left(\frac{\nu(V)}{\varepsilon_S}\right)\). \(\blacktriangleleft\)

As with the uniform case, the sampling analogue of Corollary 6.6 follows from the above theorem.

\[\text{Theorem 7.10. Suppose a computational model that satisfies Assumption 7.1. Let } (V, R, \lambda) \text{ be a hard-constraint point process with } q \in \mathbb{N}_{\geq 1} \text{ particle types and } V = [0, \ell]^d, \ell \in \mathbb{R}_{>0}. \text{ Further, let } B \text{ be the corresponding volume exclusion matrix. If there is a function } f: [q] \to \mathbb{R}_{>0} \text{ such that for all } i \in [q],
\]
\[f(i) > \sum_{j \in [q]} B(i, j)f(j)\lambda(j),
\]
then for all \(\varepsilon_S \in (0, 1]\), there is an \(\varepsilon_S\)-approximate sampler for \(\mu^{(V, R, \lambda)}\) that has a running time in \(\text{poly}\left(\frac{\nu(V)}{\varepsilon_S}, \ln\left(\frac{\max_{x \in [q]} f(i)}{\min_{x \in [q]} f(j)}\right)\right)\). \(\blacktriangleleft\)

1.5 Concentration of random discretizations

As seen in Sections 1.3 and 1.4, our algorithmic results use a 0-\(\varepsilon\)-allocation. However, Theorem 4.6 allows for a more general set of allocations, where \(\delta > 0\). Using this generality, we show a connection between the hard-core model on random geometric graph structures and continuous hard-constraint point processes.

\[\text{Corollary 8.7. Let } (V, R, \lambda) \text{ be a hard-constraint point process with } q \in \mathbb{N}_{\geq 1} \text{ particle types and } V = [0, \ell]^d \text{ for some } \ell \in \mathbb{R}_{>0}. \text{ Let } X \subset V \text{ with } |X| = n, \text{ chosen uniformly at random. For all } \ell_D \in (0, 1] \text{ and } p \in (0, 1], \text{ there exists an } n_{\ell_D, p} \in \Theta(\nu(V)^4 \ell_D^{-3} \ln(\frac{\nu(V)}{\varepsilon_S}))) \text{ such that for all } n \geq n_{\ell_D, p}, \text{ with probability at least } 1 - \varepsilon \text{ it holds that } e^{-\varepsilon D} Z(V, R, \lambda) \leq Z_{HC}(G_X, \lambda_X) \leq e^{\varepsilon D} Z(V, R, \lambda). \blacktriangleleft\]

An immediate consequence of Corollary 8.7 is that, for uniform random geometric graphs on \(V = [0, \ell]^d\) with edge connection threshold \(2\tau\) and appropriately chosen fugacities, the hard-core partition function concentrates with increasing number of vertices around the partition function of a hard-sphere model with particle radius \(r\) on \(V\). We believe this connection to be interesting in its own right.

1.6 Discussion and future directions

For discrete spin systems, change of macroscopic properties of the systems studied in statistical physics often relate to a change in the computational behavior of the system—for more examples, see [6, 10, 16].
Many discrete spin systems have been shown to exhibit sharp phase transitions at a specific threshold. However, there are only a few exceptions of continuous spin systems where the existence of a phase transition has been rigorously proved. One such exception where a sharp phase transition is known is for the two-particle continuous Widom–Rowlinson model [8, 14, 34]. For the hard-sphere model, this is not the case, as it remains open to prove the existence of a phase transition. Advancing the understanding of the computational behavior of such models helps understand their macroscopic behavior. Our universal treatment of the hard-sphere model and the Widom–Rowlinson model under the same framework could help us determine similarities and differences among the two models.

The phase transition results for the two-particle Widom–Rowlinson model are existential and only apply when \( \lambda_1 = \lambda_2 \) are some large value. Besides experimental results limited to the 2-dimensional case [23], no rigorous bounds for the phase transition point have been established. Our results suggest that the Gibbs measure is unique for the parameter regimes for which we obtain efficient approximation and sampling algorithms. This is consistent with the hard-sphere model, where the algorithmic results match the known regime of uniqueness of the Gibbs measure [29]. Our insights from the discretization method could be transferred to the continuous setting, as done by Helmuth et al. [22], to establish a rigorous bound for uniqueness of the Gibbs measure. We note that the regime of Corollary 6.6 is consistent with the conjecture that the phase transition of the Widom–Rowlinson model only occurs when \( \lambda_1 = \lambda_2 \) (see, e.g., [14]).

A challenging task is establishing an upper bound for the phase transition or a hardness-of-approximation result for a continuous spin system. To our knowledge, hardness of approximation is not known for any continuous spin system. The tight results for discrete spin systems rely on the behavior of the spin system on random bipartite graphs (see, e.g., [37]), which we are not able to generate in a straightforward manner within our construction of \( G_X \). There are discrete models with hard-core interactions for which the phase transition threshold on trees has been determined. However, the corresponding hardness of approximation is not known. For example, the tree threshold for the two-particle discrete Widom–Rowlinson model is known [27] and some hardness-of-approximation results have been shown [15], but it has not been shown whether the phase transition coincides with a change in computational behavior. When the support of the interaction matrix \( R \) is bipartite, then the graph \( G_X \) obtained by our discretization is also bipartite, which might be helpful for establishing hardness of approximation for continuous spin systems.

From an algorithmic point of view, it is interesting to determine which structural properties of the interaction matrix \( R \) or the volume exclusion matrix \( B \) yield a broader parameter regime for approximation and sampling. One such parameter is the largest eigenvalue of \( B \), which can be used to upper bound to the connective constant of \( G_X \) (see, e.g., [1]). This results in a broader parameter regime for efficient algorithms [36]. However, for hard-sphere model, this would not improve the bound, as the connective constant is asymptotically equivalent to the maximum degree of the graph (see [32] and [18, Section 1.4]).

A further interesting algorithmic question is whether the quasi-polynomial running time of the deterministic approximation algorithm can be improved to polynomial. The two techniques yielding deterministic algorithms for the hard-core model have a running time of \( O(n^{\log \Delta}) \) for graphs of maximum degree \( \Delta \), which corresponds in our setting to the quasi-polynomial running time. In the correlation decay method of Weitz [38], this running time comes from computing the self-avoiding walk tree of \( G \). In Barvinok’s interpolation method, this comes from enumerating induced connected subgraphs [31]. Constructing a deterministic algorithm for general graphs of maximum degree \( \Delta \) and better running-time dependency on \( \Delta \), if possible, seems to require significant conceptual insight. However, one could hope that the symmetric structure of \( G_X \), produced by our discretization, leads to faster running times for the computational tasks used in any of these two algorithmic techniques.
2 Preliminaries

Throughout the paper, denote by \( \mathbb{N} \) the set of all non-negative integers, including 0 and write \( \mathbb{N}_{\geq 1} \) for the set \( \mathbb{N} \setminus \{0\} \). For all \( n \in \mathbb{N} \) denote by \( [n] \) the set \( \{1, n \} \cap \mathbb{N} = \{1, 2, \ldots, n\} \). Furthermore, we use the convention that \( \inf \emptyset = \infty \).

Let \( S \) be some set and \( f : S \to \mathbb{R} \). For any \( \alpha \in \mathbb{R} \) we write \( \alpha f \) for the function \( s \mapsto \alpha f(s) \) for all \( s \in S \). Within this work, we occasionally encounter functions whose domain is the empty set. Recall that, following the definition of functions as subsets of the Cartesian product, there is exactly one function \( f : \emptyset \to S \) for all sets \( S \).

For any subset of \( d \)-dimensional Euclidean space \( A \subseteq \mathbb{R}^d \) we write \( \alpha A \) for the set \( \{\alpha x \mid x \in A\} \). Let \( d(x, y) \) denote the Euclidean distance between points \( x, y \in \mathbb{R}^d \). For \( x \in \mathbb{R}^d \) and \( r \in \mathbb{R}_{>0} \), we write \( B_x(r) \) for the open ball of radius \( r \) around \( x \), formally \( B_x(r) = \{y \in \mathbb{R}^d \mid d(x, y) < r\} \). When the center \( x \) of the ball is not important, we simply omit it and write \( B(r) \) instead. Further, we denote the Lebesgue measure on \( \mathbb{R}^d \) by \( \nu^d \). We extend this notation by writing \( \nu^{d \times k} \) for the product of \( k \) Lebesgue measures on \( (\mathbb{R}^d)^k \). Whenever \( d \) and \( k \) are clear from the context, we simply write \( \nu(\cdot) \) to simplify notation. This is, for all measurable sets \( A \subseteq \mathbb{R}^d \) we write \( \nu(A) = \nu^d(A) \), and for all measurable sets \( B \subseteq (\mathbb{R}^d)^k \) we write \( \nu(B) = \nu^{d \times k}(B) \). Finally, a set \( A \subseteq \mathbb{R}^d \) is called star-convex if there is a point \( x \in A \) such that for all \( y \in A \) and all \( \alpha \in [0, 1] \) it holds that \( \alpha x + (1 - \alpha)y \in A \). We say that any such point \( x \in A \) is a center of \( A \).

### 2.1 Notions of approximation

For \( x \in \mathbb{R}_{\geq 0} \) and some \( \varepsilon \in \mathbb{R}_{>0} \) we call \( x' \in \mathbb{R}_{\geq 0} \) an \( \varepsilon \)-approximation for \( x \) if \( e^{-\varepsilon}x \leq x' \leq e^\varepsilon x \). Further, we call a real-valued random variable \( X \) a randomized \( \varepsilon \)-approximation for \( x \) if \( \Pr[|e^{-\varepsilon}x \leq X \leq e^\varepsilon x|] \geq 3/4 \).

For approximate sampling we use the following formal notion. Let \( \xi \) be a probability distribution on a state space \( \Omega \). For \( \varepsilon \in (0, 1] \), we say that a distribution \( \xi' \) on \( \Omega \) is an \( \varepsilon \)-approximation of \( \xi \) if \( d_{\text{TV}}(\xi, \xi') \leq \varepsilon \), where \( d_{\text{TV}}(\cdot, \cdot) \) denotes the total-variation distance. Further, we say that we can \( \varepsilon \)-approximately sample from \( \xi \) if we can sample from any distribution \( \xi' \) such that \( \xi' \) is an \( \varepsilon \)-approximation of \( \xi \).

### 2.2 The hard-core model

We recall the definition of the hard-core model from the introduction. For an undirected graph \( G = (V, E) \) the multivariate hard-core model is defined as a tuple \( (G, \lambda) \) where \( \lambda : V \to \mathbb{R}_{\geq 0} \). Let \( I(G) \) denote the set of all independent sets in \( G \). We assign each independent set \( I \in I(G) \) a weight of \( \prod_{v \in I} \lambda(v) \). The multivariate hard-core partition function is now defined as the sum of weights of all independent sets. Further, the Gibbs distribution of the multivariate hard-core model assigns each independent set a probability proportional to its weight. Formally, that is

\[
Z_{\text{HC}}(G, \lambda) = \sum_{I \in I(G)} \prod_{v \in I} \lambda(v) \quad \text{and} \quad \mu_{\text{HC}}^{(G, \lambda)}(I) = \frac{\prod_{v \in I} \lambda(v)}{Z_{\text{HC}}(G, \lambda)} \quad \text{for all} \ I \in I(G).
\]

If \( \lambda \) is a constant, we call the hard-core model univariate (or uniform). Note that in most of the existing literature the term hard-core model refers to the univariate version. Further, the multivariate case appears sometimes as polymer models in the algorithmic context. However, in fact there are some conceptual differences between both terms, such as that polymer models are used to encode other computational problems and tend to be exponentially large in the original problem instance, which means that no explicit
representation of the polymer graph is assumed. We stick to this conceptual difference, and thus, we use the term hard-core model. Unless we explicitly state that any of our statements is limited to the univariate case, we will use the term hard-core model for the multivariate version. Further, whenever we deal with the univariate case, we will abuse notation and simply write $\lambda$ as a constant instead of $\lambda(v)$. Note that this simplifies the weight of an independent $I \in \mathcal{I}(G)$ to $\lambda[I]$, which directly carries over to the definitions of the partition function and the Gibbs distribution.

The following paragraphs summarize known algorithmic results for approximate sampling from the Gibbs distribution and approximating the partition function. For obtaining a deterministic approximation of the partition function, two main algorithmic approaches are known, namely the computational tree method, introduces by Weitz [38], and the polynomial interpolation method of Barvinok [3] and Patel and Regts [31]. To get a randomized approximation, usually a Markov chain Monte Carlo algorithm based on self-reducibility is used. For this approach, it is necessary to approximately sample from the Gibbs distribution. This is often done via Glauber dynamics, which is an ergodic Markov chain with state space $\mathcal{I}(G)$ and stationary distribution $\mu_{HC}^{(G,\lambda)}$. In the multivariate setting, the update rule of this Markov chain works as follows. Let $I_t$ be the state of the chain at time $t$. The configuration at time $t+1$ is obtained by first choosing a vertex $v \in V$ uniformly at random. With probability $\frac{1}{1 + \lambda(v)}$, we set $I_{t+1} = I_t \setminus \{v\}$ (note that nothing changes if $v \notin I_t$). Otherwise, with probability $\frac{\lambda(v)}{1 + \lambda(v)}$, we set $I_{t+1} = I_t \cup \{v\}$ if $I_t \cup \{v\} \in \mathcal{I}(G)$.

**Univariate approximation**: For the univariate hard-core model, the ability to efficiently approximate the partition function is closely related to the maximum degree of the graph. For a graph of maximum degree $\Delta \in \mathbb{N}_{\geq 2}$, the tree threshold is defined as

$$\lambda_c(\Delta) = \frac{(\Delta - 1)^{\Delta-1}}{(\Delta - 2)^\Delta}.$$  

The following theorems show some of the algorithmic results that are known below this threshold. The first statement deals with approximately sampling from the Gibbs distribution of the univariate hard-core model via Glauber dynamics.

**Theorem 2.1 ([12, Theorem 1])**. Let $G = (V, E)$ be an undirected graph with maximum vertex degree bounded by $\Delta \in \mathbb{N}_{\geq 2}$. For all $\lambda \in \mathbb{R}_{\geq 0}$ with $\lambda < \lambda_c(\Delta)$ the Glauber dynamics on the univariate hard-core model $(G, \lambda)$ are mixing in time $\text{poly}(|V|)$, and for all $\epsilon_S \in (0, 1]$ we can $\epsilon_S$-approximately sample from the Gibbs measure $\mu_{HC}^{(G,\lambda)}$ in time $\text{poly}\left(\frac{|V|}{\epsilon_S}\right)$.

Based on Theorem 2.1, a Markov chain Monte Carlo approximation for the partition function of the hard-core model can be obtained using self-reducibility.

**Theorem 2.2 ([12, Theorem 1])**. Let $G = (V, E)$ be an undirected graph with maximum vertex degree bounded by $\Delta \in \mathbb{N}_{\geq 2}$ and let $\lambda \in \mathbb{R}_{\geq 0}$ with $\lambda < \lambda_c(\Delta)$. For all $\epsilon_A \in (0, 1]$ there is a randomized $\epsilon_A$-approximation algorithm for the univariate hard-core partition function $Z_{HC}(G, \lambda)$ with running time $\text{poly}\left(\frac{|V|}{\epsilon_A}\right)$.

Besides this randomized algorithm, the following result can be obtained deterministically.

**Theorem 2.3 ([38, Theorem 2.7])**. Let $G = (V, E)$ be an undirected graph with maximum vertex degree bounded by $\Delta \in \mathbb{N}_{\geq 2}$ and let $\lambda \in \mathbb{R}_{\geq 0}$ with $\lambda < \lambda_c(\Delta)$. For all $\epsilon_A \in (0, 1]$ there is a deterministic
ε₂A-approximation algorithm for the univariate hard-core partition function \( Z_{HC}(G, \lambda) \) with running time 
\[
\left( \frac{|V|}{\varepsilon} \right)^{O(\ln(\Delta))}.
\]

Note that Theorem 2.2 is based on the computational tree method by Weitz [38]. A similar result can be obtained by combining the zero-freeness result by Peters and Regts [33] and the polynomial interpolation method as proposed by Patel and Regts [31].

**Remark 2.4.** In fact, Theorems 2.1 to 2.3 can also be applied in the non-uniform setting when the condition is replaced by 
\[ \lambda_{\text{max}} < \lambda_c(\Delta), \]
where \( \lambda_{\text{max}} = \max_{v \in V} \lambda(v) \).

**Multivariate approximation:** One of the least restrictive conditions that appeared in the literature for efficient approximation of multivariate hard-core partition functions is the clique dynamics condition, introduced by Friedrich et al. [19]. Initially, the condition was used for proving rapid mixing of clique dynamics for abstract polymer models. However Glauber dynamics can, in fact, be seen as a special case of clique dynamics for cliques of size 1. The result for Glauber dynamics is summarized in the following statement.

**Theorem 2.5 ([19, Theorem 2]).** Let \((G, \lambda)\) be a multivariate hard-core model. If there is a function \( f: V \to \mathbb{R}_{>0} \) such that for all \( v \in V \) it holds that
\[
f(v) \geq \sum_{w \in N(v)} f(w) \frac{\lambda(w)}{1 + \lambda(w)},
\]
where \( N(v) \) denotes the neighborhood of \( v \) in \( G \), then the Glauber dynamics on \((G, \lambda)\) are mixing in time \( \text{poly}\left( |V|, \ln\left( \frac{\max_{v \in V} f(v)}{\min_{v \in V} f(v)} \right) \right) \). Further, for all \( \varepsilon_S \in (0, 1] \) we can sample \( \varepsilon_S \)-approximately from the Gibbs distribution \( \mu_{HC}^{(G, \lambda)} \) in time \( \text{poly}\left( |V|, \ln\left( \frac{\max_{v \in V} f(v)}{\min_{v \in V} f(v)} \right) \right) \).

Further, the following approximation result is obtained via a Markov chain Monte Carlo approach and Theorem 2.5.

**Theorem 2.6 ([19, Theorem 3]).** Let \((G, \lambda)\) be a multivariate hard-core model. If there is a function \( f: V \to \mathbb{R}_{>0} \) such that for all \( v \in V \) it holds that
\[
f(v) \geq \sum_{w \in N(v)} f(w) \frac{\lambda(w)}{1 + \lambda(w)},
\]
then for all \( \varepsilon_A \in (0, 1] \) there is a randomized \( \varepsilon_A \)-approximation algorithm for the partition function \( Z_{HC}(G, \lambda) \) with running time in \( \text{poly}\left( \frac{|V|}{\varepsilon_A}, \ln\left( \frac{\max_{v \in V} f(v)}{\min_{v \in V} f(v)} \right) \right) \).

### 3 Point processes with general hard constraints

We recall the definition of a hard-constraint process from the introduction and give some alternative forms of the partition function. Let \( V \subseteq \mathbb{R}^d \) be bounded and measurable, and let \( q \in \mathbb{N}_{\geq 1} \). The model represents
the distribution of particles of \( q \) types, labeled by elements in \([q] := [1, q] \cap \mathbb{N}\), on \( \mathbb{V} \). Particles of the same types are assumed to be indistinguishable. Let \( \lambda: [q] \to \mathbb{R}_{\geq 0} \) be a function that equips each particle type with a fugacity. For each particle type \( i \in [q] \), we assume that the positions of particles are distributed according to a (labeled) Poisson point process of intensity \( \lambda(i) \) on \( \mathbb{V} \). To add the constraints to the model, let \( R \in \mathbb{R}^{q \times q} \) be a symmetric \( q \times q \) matrix, called the \textit{interaction matrix}. We condition the mixture of point processes by rejecting all configurations that contain particles at positions \( x_1, x_2 \in \mathbb{V} \) with corresponding particle types \( \tau_1, \tau_2 \in [q] \) whenever \( d(x_1, x_2) < R(\tau_1, \tau_2) \). That is, the entries of \( R \) determine the minimum distance that particles of the respective types can have. Especially note that \( R(\tau_1, \tau_2) = 0 \) means that the particle types \( \tau_1, \tau_2 \in [q] \) are not subjected to any pairwise constraints.

For any instance \((\mathbb{V}, R, \lambda)\) of a hard-constraint point process that satisfies the requirements above, this description characterizes a \textit{Gibbs distribution} as follows. For all \( k \in \mathbb{N} \) and all type assignments \( \tau: [k] \to [q] \) let \( D^{(R)}_\tau: (\mathbb{R}^d)^k \to \{0, 1\} \) be the function that indicates for a tuple of particles positions \( x = (x_1, \ldots, x_k) \in (\mathbb{R}^d)^k \) if it forms a valid configuration, assuming that for each \( i \in [k] \) the particle at \( x_i \) is of type \( \tau(i) \). Formally, this is

\[
D^{(R)}_\tau(x) = D^{(R)}_\tau(x_1, \ldots, x_k) = \prod_{i,j \in [k]: i < j} \mathbb{I}\{d(x_i, x_j) \geq R(\tau(i), \tau(j))\},
\]

where \( D^{(R)}_\tau(x) = 1 \) for the case that \( k = 0 \). For each tuple \((x, \tau)\) as above, the Gibbs distribution of \((\mathbb{V}, R, \lambda)\) is defined via the probability density

\[
\mu^{(\mathbb{V}, R, \lambda)}(x, \tau) = \frac{\frac{1}{k!} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) D^{(R)}_\tau(x)}{Z(\mathbb{V}, R, \lambda)},
\]

where the normalizing constant \( Z(\mathbb{V}, R, \lambda) \) is called the \textit{partition function}. This is

\[
Z(\mathbb{V}, R, \lambda) = 1 + \sum_{k \in \mathbb{N}_{\geq 1}} \frac{1}{k!} \sum_{\tau: [k] \to [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \int_{\mathbb{V}^k} D^{(R)}_\tau(x) \, dv^{d \times k}.
\]

(1)

The main goal of this paper is to investigate condition for an efficient (randomized) approximation of \( Z(\mathbb{V}, R, \lambda) \). Occasionally it will be convenient to include the case \( k = 0 \) in the sum in equation (1). To this end, we use the convention \( \int_{\mathbb{V}^0} D^{(R)}_\tau(x) \, dv^{d \times 0} = 1 \) to rewrite equation (1) as

\[
Z(\mathbb{V}, R, \lambda) = \sum_{k \in \mathbb{N}} \frac{1}{k!} \sum_{\tau: [k] \to [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \int_{\mathbb{V}^k} D^{(R)}_\tau(x) \, dv^{d \times k}.
\]

We can further simplify this notation by defining a weight

\[
w^{(R, \lambda)}(x, \tau) = \frac{1}{k!} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) D^{(R)}_\tau(x)
\]

for each \( x = (x_1, \ldots, x_k) \in \mathbb{V}^k \) and \( \tau: [k] \to [q] \). Using this weight function, we can rewrite the Gibbs
density and the partition function slightly more compact as
\[
\mu^{(V, R, \lambda)}(x, \tau) = \frac{w^{(R, \lambda)}(x, \tau)}{Z(V, R, \lambda)} \quad \text{and} \quad Z(V, R, \lambda) = \sum_{k \in \mathbb{N}} \sum_{\tau: [k] \to [q]} \int_{V^k} w^{(R, \lambda)}(x, \tau) \, d\nu^{d^k}.
\]

For formalizing our conditions for efficient approximation of the partition function of a hard-constraint point process \((V, R, \lambda)\) it will come in handy to introduce the volume exclusion matrix \(B \in \mathbb{R}^{q \times q}\) the model. It is defined by the entries \(B(i, j) = \nu(B(R(i, j)))\) for all \(i, j \in [q]\). Intuitively speaking, this means that the entry \(B(i, j)\) gives an upper bound on the volume of the region around a point \(x \in V\) in which no particle of type \(j \in [q]\) can be placed, given we place a particle of type \(i \in [q]\) at \(x\).

For our algorithmic results, we investigate asymptotic behavior in the volume \(\nu(v(V))\), while we assume the parameters \(q, d\), and the entries of \(R\) as well as the range of \(\lambda\) to be constant.

### 3.1 Scaling, bounds and monotonicity

We start by collecting some useful properties of the partition function of general hard-constraint point processes, which will come in handy when bounding the speed of convergence of our discretization. Our first lemma gives a trivial upper bound on the partition function.

**Lemma 3.1.** For every hard-constraint point process \((V, R, \lambda)\) with \(q \in \mathbb{N}_{\geq 1}\) particle types it holds that
\[
Z(V, R, \lambda) \leq e^{\sum_{i \in [q]} \lambda(i) \nu(V)}.
\]

**Proof.** Fix some \(k \in \mathbb{N}_{\geq 0}\) and observe that for any \(\tau: [k] \to [q]\) it holds that
\[
\int_{V^k} D^{(R)}_\tau(x) \, d\nu^{d^k} \leq \nu(V^k) = \nu(V)^k.
\]

Now, fix some \((k_1, \ldots, k_q) \in \mathbb{N}^q\) such that \(\sum_{i \in [q]} k_i = k\). Observe that there are exactly \(\frac{k!}{\prod_{i \in [q]} k_i!}\) type assignments \(\tau: [k] \to [q]\) such that \(|\tau^{-1}(i)| = k_i\) for all \(i \in [q]\). Further, for each such type assignment, it holds that
\[
\prod_{i \in [k]} \lambda(\tau(i)) = \prod_{i \in [q]} \lambda(i)^{k_i}.
\]

Thus, we obtain
\[
Z(V, R, \lambda) \leq 1 + \sum_{k \in \mathbb{N}_{\geq 1}} \frac{1}{k!} \sum_{\tau: [k] \to [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \nu(V)^k
\]
\[
= 1 + \sum_{k \in \mathbb{N}_{\geq 1}} \nu(V)^k \sum_{(k_1, \ldots, k_q) \in \mathbb{N}^q: \sum_{i \in [q]} k_i = k} \frac{\lambda(i)^{k_i}}{k_i!}
\]
\[
= 1 + \sum_{k \in \mathbb{N}_{\geq 1}} \nu(V)^k \frac{\left( \sum_{i \in [q]} \lambda(i) \right)^k}{k!},
\]

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where the last equality is obtained via the multinomial theorem. The claim follows from the Taylor expansion of the exponential function around 0.

Our next observation is concerned with the monotonicity of the partition function of hard-constraint point processes with respect to $V$, $R$ and $\lambda$.

**Observation 3.2.** Fix some $q \in \mathbb{N}_{\geq 1}$. Let $V, V_1, V_2 \subset \mathbb{R}^d$, let $R, R_1, R_2 \in \mathbb{R}^d$ be symmetric and let $\lambda: [q] \to \mathbb{R}_{\geq 0}, \lambda_1: [q] \to \mathbb{R}_{\geq 0}, \lambda_2: [q] \to \mathbb{R}_{\geq 0}$. It holds that:

(a) If $V_1 \subseteq V_2$ then $Z(V_1, R, \lambda) \leq Z(V_2, R, \lambda)$.

(b) If $R_1(i, j) \leq R_2(i, j)$ for all $i, j \in [q]$ then $Z(V, R_1, \lambda) \geq Z(V, R_2, \lambda)$.

(c) If $\lambda_1(i) \leq \lambda_2(i)$ for all $i \in [q]$ then $Z(V, R, \lambda_1) \leq Z(V, R, \lambda_2)$.

It is a folklore that the partition function the hard-sphere model and the Widom–Rowlinson model are log-submodular in the considered region. The following lemma formalizes this idea for general hard-constraint point processes.

**Lemma 3.3.** Let $V_1 \subset \mathbb{R}^d$ and $V_2 \subset \mathbb{R}^d$ be bounded and measurable, and let $q \in \mathbb{N}_{\geq 1}$. For symmetric $R \in \mathbb{R}_{\geq 0}^d$ and all functions $\lambda: [q] \to \mathbb{R}_{\geq 0}$ it holds that

$$Z(V_1 \cup V_2, R, \lambda) \leq Z(V_1, R, \lambda) \cdot Z(V_2, R, \lambda)$$

**Proof.** First, note that it is sufficient to prove the claim for disjoint sets $V_1$ and $V_2$. For non-disjoint $V_1, V_2$ the claim then follows from partitioning $V_1 \cup V_2$ into the disjoint sets $V_1$ and $V_2 \setminus V_1$, and by noting that $V_2 \setminus V_1 \subseteq V_2$, which by Observation 3.2 implies $Z(V_2 \setminus V_1, R, \lambda) \leq Z(V_2, R, \lambda)$.

Thus, assume $V_1 \cap V_2 = \emptyset$ and set $V = V_1 \cup V_2$. Fix some $k \in \mathbb{N}$. For $x = (x_1, \ldots, x_k) \in V^k$, let $k_1(x)$ denote the number from $V_1$. Formally, this is $\{i \in [k] \mid x_i \in V_1\}$. As a first step, we rewrite the partition function on $V$ as

$$Z(V, R, \lambda) = \sum_{k \in \mathbb{N}} \frac{1}{k!} \sum_{\tau: [k] \to [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \int_{V^k} D^{(R)}_\tau(x) \, dV^{d \times k}$$

$$= \sum_{k \in \mathbb{N}} \sum_{k_1 \in \mathbb{N}} \frac{1}{k!} \sum_{\tau: [k] \to [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \int_{V^k} \mathbb{1}_{\{k_1(x) = k_1\}} D^{(R)}_\tau(x) \, dV^{d \times k}$$

$$= \sum_{k \in \mathbb{N}} \sum_{k_1 \in \mathbb{N}} \frac{1}{k!} \int_{V^k} \mathbb{1}_{\{k_1(x) = k_1\}} \sum_{\tau: [k] \to [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) D^{(R)}_\tau(x) \, dV^{d \times k}.$$ 

Fix some $k \in \mathbb{N}$ and let $\pi: [k] \to [k]$ be a bijective (i.e., $\pi$ a permutation of $[k]$). For every type assignment $\tau: [k] \to [q]$ we write $\pi \circ \tau$ for the composition of both functions. That is, for all $i \in [k]$ we have $(\pi \circ \tau)(i) = \tau(\pi(i))$. Further, for all $x = (x_1, \ldots, x_k) \in V^k$ we abuse notation and write $\pi(x) = (x_{\pi(1)}, \ldots, x_{\pi(k)})$. Observe that for all permutations $\pi: [k] \to [k]$ it holds that

$$D^{(R)}_\tau(x) = D^{(R)}_{\pi \circ \tau}(\pi(x)).$$
Next, observe that for all $x \in \mathbb{V}^k$ there is exactly one permutation $\pi_x : [k] \rightarrow [k]$ such that $\pi_x(x) \in \mathbb{V}^{k_1(x)} \times \mathbb{V}^{k_2-k_1(x)}$ and the order among particle centers from $\mathbb{V}_1$ and from $\mathbb{V}_2$ is preserved. Formally, that means $\pi_x(x) \in \mathbb{V}^{k_1(x)} \times \mathbb{V}^{k_2-k_1(x)}$, for all $i, j \in [k_1(x)]$ it holds that $\pi_x(i) < \pi_x(j)$ if and only if $i < j$, and for all $i, j \in [k] \setminus [k_1(x)]$ it also holds that $\pi_x(i) < \pi_x(j)$ if and only if $i < j$. We can now rewrite

$$
\sum_{\tau : [k] \rightarrow [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) D^{(R)}_\tau(x) = \sum_{\tau : [k] \rightarrow [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) D^{(R)}_{\pi_x \circ \tau}(\pi_x(x))
$$

$$
= \sum_{\tau : [k] \rightarrow [q]} \left( \prod_{i \in [k]} \lambda(\pi_x(\tau(i))) \right) D^{(R)}_{\pi_x \circ \tau}(\pi_x(x)),
$$

where the last equality is due to the fact that applying the permutation $\pi_x$ does only change the order of the finite product.

For the next step, note that for every fixed permutation $\pi_x$ and every fixed type assignment $\tau$ there is exactly one type assignment $\tau'$ such that $\tau' = (\pi_x \circ \tau)$. This means, applying $\pi_x$ to $\tau$ does only change the order of summation. We obtain

$$
\sum_{\tau : [k] \rightarrow [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) D^{(R)}_\tau(x) = \sum_{\tau : [k] \rightarrow [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) D^{(R)}_\tau(\pi_x(x)).
$$

To proceed, we need to introduce an additional bit of notation. Let $k_1, k_2 \in \mathbb{N}$, and let $\tau_1 : [k_1] \rightarrow [q]$ and $\tau_2 : [k_2] \rightarrow [q]$. We write $\tau_1 \oplus \tau_2$ to denote the function $([k_1 + k_2] : [k_1 \rightarrow [q]] \times [k_2 \rightarrow [q]] \rightarrow [k_1 + k_2] \rightarrow [q]$ such that for all $i \in [k_1 + k_2]$ we have

$$(\tau_1 \oplus \tau_2)(i) = \begin{cases} 
\tau_1(i) & \text{if } i \leq k_1 \\
\tau_2(i - k_1) & \text{otherwise}.
\end{cases}
$$

Note that for each type assignment $\tau : [k] \rightarrow [q]$ and every fixed $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 = k$, there is exactly one pair of assignments $\tau_1 : [k_1] \rightarrow [q]$ and $\tau_2 : [k_2] \rightarrow [q]$ such that $\tau = (\tau_1 \oplus \tau_2)$. Further, for each such decomposition and all $(x_1, \ldots, x_k) \in \mathbb{V}^k$ it holds that

$$
D^{(R)}_\tau(x_1, \ldots, x_k) = D^{(R)}_{\tau_1 \oplus \tau_2}(x_1, \ldots, x_k) \leq D^{(R)}_{\tau_1}(x_1, \ldots, x_{k_1}) \cdot D^{(R)}_{\tau_2}(x_{k_1+1}, \ldots, x_k).
$$

For $x = (x_1, \ldots, x_k) \in \mathbb{V}^k$, let $q_1(x) = (x_{\pi_x(1)}, \ldots, x_{\pi_x(k_1(x))})$, and let $q_2(x) = (x_{\pi_x(k_1(x)+1)}, \ldots, x_{\pi_x(k)}))$. This is, $q_1$ and $q_2$ project $x$ to those component that come from $\mathbb{V}_1$ and $\mathbb{V}_2$ respectively while preserving the order among them. We obtain

$$
\sum_{\tau : [k] \rightarrow [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) D^{(R)}_\tau(\pi_x(x)) \leq \sum_{\tau_1 : [k_1(x)] \rightarrow [q]} \left( \prod_{i \in [k_1(x)]} \lambda(\tau_1(i)) \right) \sum_{\tau_2 : [k-k_1(x)] \rightarrow [q]} \left( \prod_{i \in [k-k_1(x)]} \lambda(\tau_2(i)) \right) D^{(R)}_{\tau_1}(q_1(x)) \cdot D^{(R)}_{\tau_2}(q_2(x))
$$

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and by substituting this into the definition of the partition function we get
\[
Z(V, R, \lambda) \leq \sum_{k \in \mathbb{N}} \frac{1}{k!} \sum_{k_1 \in [k]} \sum_{\tau_1: [k_1] \rightarrow [q]} \left( \prod_{i \in [k_1]} \lambda(\tau_1(i)) \right) \sum_{\tau_2: [k-k_1] \rightarrow [q]} \left( \prod_{i \in [k-k_1]} \lambda(\tau_2(i)) \right) \cdot \int_{V^k} 1\{k_1(x) = k_1\} D^{(R)}_{\tau_1}(\varphi_1(x)) \cdot D^{(R)}_{\tau_2}(\varphi_2(x)) \, d\nu^{d \times k}.
\]

Now, note that for all \( k \in \mathbb{N} \) and \( k_1 \in [k] \) and for all \( y \in V_{1}^{k_1} \) and \( z \in V_{2}^{k-k_1} \) it holds that there are exactly \( \frac{k!}{k_1!(k-k_1)!} \) configurations \( x \in V^k \) such that \( \varphi_1(x) = y \) and \( \varphi_2(x) = z \). Thus, it holds that
\[
Z(V, R, \lambda) \leq \sum_{k \in \mathbb{N}} \sum_{k_1 \in [k]} \frac{1}{k_1!} \sum_{\tau_1: [k_1] \rightarrow [q]} \left( \prod_{i \in [k_1]} \lambda(\tau_1(i)) \right) \int_{V_{1}^{k_1}} D^{(R)}_{\tau_1}(y) \, d\nu^{d \times (k-k_1)} \cdot \int_{V_{2}^{k-k_1}} D^{(R)}_{\tau_2}(z) \, d\nu^{d \times (k-k_1)}
\]

which concludes the proof.

To provide a sufficient error bound for our discretization, it will be helpful to translate between scaling the particle radii \( R \), the fugacities \( \lambda \) and the space itself \( V \). In Euclidean space, this can be done as stated in the following lemma.

**Lemma 3.4.** Let \((V, R, \lambda)\) be a hard-constraint point process with \( q \in \mathbb{N}_{\geq 1} \) particle types in \( d \in \mathbb{N}_{\geq 1} \) dimensions. For all \( \alpha \in \mathbb{R}_{>0} \) it holds that
\[
Z(\frac{1}{\alpha}V, \frac{1}{\alpha}R, \lambda) = Z(\alpha V, \frac{1}{\alpha^d}R, \frac{1}{\alpha^d} \lambda).
\]

**Proof.** The key observation to prove this claim is that for all pairs of points \( x, y \in V \) and types \( i, j \in [q] \) it holds that \( d(x, y) \geq \frac{1}{\alpha} R(i, j) \) if and only if \( d(\alpha x, \alpha y) \geq R(i, j) \). Thus, we get for all \( k \in \mathbb{N}_{\geq 1} \) and all type
assignments \( \tau: [k] \to [q] \) that

\[
\int_{V^k} D^{(R)}_\tau(x_1, \ldots, x_k) \, d\nu^{d \times k} = \int_{V^k} D^{(R)}_\tau(\alpha x_1, \ldots, \alpha x_k) \, d\nu^{d \times k}
\]

\[
= \frac{1}{\alpha^{dk}} \int_{(\alpha V)^k} D^{(R)}_\tau(x_1, \ldots, x_k) \, d\nu^{d \times k},
\]

where the additional factor of \( \frac{1}{\alpha^{dk}} \) compensates for the scaling of the Lebesgue measure under linear transformation.

Substituting this back into the definition of the partition function we obtain

\[
Z\left(V, \frac{1}{\alpha} R, \lambda \right) = 1 + \sum_{k \in \mathbb{N}_{\geq 1}} \frac{1}{k!} \sum_{\tau: [k] \to [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \frac{1}{\alpha^{dk}} \int_{(\alpha V)^k} D^{(R)}_\tau(x) \, d\nu^{d \times k}
\]

\[
= 1 + \sum_{k \in \mathbb{N}_{\geq 1}} \frac{1}{k!} \sum_{\tau: [k] \to [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \frac{1}{\alpha^d} \int_{(\alpha V)^k} D^{(R)}_\tau(x) \, d\nu^{d \times k}
\]

\[
= Z\left(\alpha V, R, \frac{1}{\alpha^d} \lambda \right).
\]

which proves the claim.

Based on Lemma 3.1 and observation 3.2 and Lemmas 3.3 and 3.4 we can now prove the following bound on the difference of partition functions on star-convex regions \( V \) with differently scaled particle interaction matrices \( R \), which will be the very core of our convergence result.

**Lemma 3.5.** Let \( \alpha \in [0, 1] \), and let \( V \subset \mathbb{R}^d \) be bounded, measurable and star-convex. For any hard-constraint point process \((V, R, \lambda)\) with \( q \in \mathbb{N}_{\geq 1} \) particle types it holds that

\[
Z(V, (1 - \alpha)R, \lambda) - Z(V, (1 + \alpha)R, \lambda) \leq \left( e^{(2\alpha)^d \sum_{i \in [q]} \lambda(i)\nu(V)} - 1 \right) Z(V, R, \lambda).
\]

**Proof.** Let \( x_0 \in V \) denote the center of \( V \). Note that the partition function is invariant under translation. Thus, we can assume that \( x_0 = 0 \). Note that this especially implies for all \( \beta_1 \in [0, 1] \) and \( \beta_2 \in [1, \infty) \) that \( \beta_1 V \subseteq V \subseteq \beta_2 V \).

We proceed by rewriting the difference based on Lemma 3.4 as

\[
Z(V, (1 - \alpha)R, \lambda) - Z(V, (1 + \alpha)R, \lambda)
\]

\[
= Z\left(\frac{1}{1 - \alpha} V, R, (1 - \alpha)^d \lambda \right) - Z\left(\frac{1}{1 + \alpha} V, R, (1 + \alpha)^d \lambda \right).
\]

(2)

Using the fact that \( \frac{1}{1 + \alpha} V \subseteq V \subseteq \frac{1}{1 - \alpha} V \) and Lemma 3.3 we obtain

\[
Z\left(\frac{1}{1 - \alpha} V, R, (1 - \alpha)^d \lambda \right) \leq Z\left(\frac{1}{1 - \alpha} V \setminus \frac{1}{1 + \alpha} V, R, (1 - \alpha)^d \lambda \right) Z\left(\frac{1}{1 + \alpha} V, R, (1 - \alpha)^d \lambda \right).
\]
Because $(1 - \alpha)^d \lambda(i) \leq (1 + \alpha)^d \lambda(i)$ for all $i \in [q]$ we can further bound this by Observation 3.2 to get

$$Z\left(\frac{1}{1 - \alpha} V, R, (1 - \alpha)^d \lambda\right) \leq Z\left(\frac{1}{1 - \alpha} V \setminus \frac{1}{1 + \alpha} V, R, (1 - \alpha)^d \lambda\right)Z\left(\frac{1}{1 + \alpha} V, R, (1 + \alpha)^d \lambda\right).$$

Combining equation (2) and inequality (3) yields

$$Z(V, (1 - \alpha)R, \lambda) - Z(V, (1 + \alpha)R, \lambda)
\leq Z\left(\frac{1}{1 - \alpha} V \setminus \frac{1}{1 + \alpha} V, R, (1 - \alpha)^d \lambda\right)Z\left(\frac{1}{1 + \alpha} V, R, (1 + \alpha)^d \lambda\right)
- Z\left(\frac{1}{1 + \alpha} V, R, (1 + \alpha)^d \lambda\right).
$$

Using Lemma 3.1 and the facts that $\frac{1}{1 - \alpha} V \leq \frac{1}{1 - \alpha} V$, $\nu\left(\frac{1}{1 + \alpha} V\right) = \left(\frac{1}{1 + \alpha}\right)^d \nu(V)$ and $\nu\left(\frac{1}{1 - \alpha} V\right) = \left(\frac{1}{1 - \alpha}\right)^d \nu(V)$ we further get

$$Z\left(\frac{1}{1 - \alpha} V \setminus \frac{1}{1 + \alpha} V, R, (1 - \alpha)^d \lambda\right) \leq e^{(1 - \alpha)^d \sum_{i \in [q]} \lambda(i)} \nu\left(\frac{1}{1 - \alpha} V \setminus \frac{1}{1 + \alpha} V\right)
\leq e^{(1 - \alpha)^d \left(\frac{1}{1 - \alpha} - \frac{1}{1 + \alpha}\right)^d \sum_{i \in [q]} \lambda(i) \nu(V)}
\leq e^{\left(1 - \frac{1 - \alpha}{1 + \alpha}\right)^d \sum_{i \in [q]} \lambda(i) \nu(V)}.
$$

Further, for $0 \leq \alpha$, it holds that

$$\left(1 - \frac{1 - \alpha}{1 + \alpha}\right)^d \leq (1 - (1 - 2\alpha))^d = (2\alpha)^d.
$$

Thus, we obtain

$$Z\left(\frac{1}{1 - \alpha} V \setminus \frac{1}{1 + \alpha} V, R, (1 - \alpha)^d \lambda\right) \leq e^{(2\alpha)^d \sum_{i \in [q]} \lambda(i) \nu(V)}.
$$

Substituting inequality (5) into inequality (4) now gives us

$$Z(V, (1 - \alpha)R, \lambda) - Z(V, (1 + \alpha)R, \lambda) \leq \left(e^{(2\alpha)^d \sum_{i \in [q]} \lambda(i) \nu(V)} - 1\right)Z\left(\frac{1}{1 + \alpha} V, R, (1 + \alpha)^d \lambda\right).
$$

Finally, because Lemma 3.4, Observation 3.2 and the fact that $(1 + \alpha)R(i, j) \geq R(i, j)$ for all $i, j \in [q]$ we have

$$Z\left(\frac{1}{1 + \alpha} V, R, (1 + \alpha)^d \lambda\right) = Z(V, (1 + \alpha)R, \lambda) \leq Z(V, R, \lambda).
$$

This gives us

$$Z(V, (1 - \alpha)R, \lambda) - Z(V, (1 + \alpha)R, \lambda) \leq \left(e^{(2\alpha)^d \sum_{i \in [q]} \lambda(i) \nu(V)} - 1\right)Z(V, R, \lambda),
$$

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and concludes the proof.

4 Discretization and convergence

We recall the construction of the graph $G_X$ from the introduction. Let $(\mathbb{V}, R, \lambda)$ be an instance of a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types. For any finite and non-empty set of points $X \subseteq \mathbb{V}$, we construct an undirected graph $G_X = (V_X, E_X)$ and a function $\lambda_X : V_X \rightarrow \mathbb{R}_{\geq 0}$ as follows:

- For each point $x \in X$ and each type $i \in [q]$, we construct a vertex $v^{(i)}_x$. Further, for each $i \in [q]$, we set $V_X^{(i)} = \{v^{(i)}_x : x \in X\}$, and we set $V_X = \bigcup_{i \in [q]} V_X^{(i)}$.

- For each $i, j \in [q]$ and $x, y \in X$, we connect $v^{(i)}_x, v^{(j)}_y \in V_X$ with an edge in $E_X$ if and only if $\neg(x = y$ and $i = j)$ and $d(x, y) < R(i, j)$, where the condition $\neg(x = y$ and $i = j)$ prevents self-loops.

- For each $i \in [q]$ and each $x \in X$, we set $\lambda_X(v^{(i)}_x) = \frac{\nu(V_X^{(i)})}{|X|} \lambda(i)$. Note that for all $x, y \in X$ this means that $\lambda_X(v^{(i)}_x) = \lambda_X(v^{(i)}_y)$, and we sometimes abuse notation and write $\lambda_X(i)$ instead.

We note that the multivariate hard-core representation $(G_X, \lambda_X)$ does not only depend on the set of points $X$ but also on the hard-constraint point process $(\mathbb{V}, R, \lambda)$. Since we fix the hard-constraint point process in advance, our notation is unambiguous in each context.

4.1 Properties of the hard-core model and independent multisets

We prove important properties of the multivariate hard-core model that we use in our main proofs. Our first statement gives a trivial upper bound for the partition function.

- **Lemma 4.1.** Let $(G, \lambda)$ be a multivariate hard-core instance. It holds that
  $Z_{HC}(G, \lambda) \leq e^{\sum_{x \in V} \lambda(v)}$.

  **Proof.** The claim immediately follows by observing that
  $Z_{HC}(G, \lambda) \leq \sum_{S \subseteq V} \prod_{v \in S} \lambda(v) = \prod_{v \in V} (1 + \lambda(v)) \leq \prod_{v \in V} e^{\lambda(v)}$.

  Next, we show that the multivariate hard-core partition function is log-subadditive in $\lambda$.
  - **Lemma 4.2.** Let $(G, \lambda_1)$ and $(G, \lambda_2)$ be two multivariate hard-core models on the same graph $G$. Further, let us write $\lambda_1 + \lambda_2$ for the function $\lambda_1 + \lambda_2 : v \mapsto \lambda_1(v) + \lambda_2(v)$. It holds that
  $Z_{HC}(G, \lambda_1 + \lambda_2) \leq Z_{HC}(G, \lambda_1) \cdot Z_{HC}(G, \lambda_2)$.
We note that we require for all $v$ which proves the claim.

We define the multiset hard-core partition function $\mathcal{I}$ vertices that have an independent set as support. Formally, we describe such a multiset by a tuple $\mathcal{I}$. For each independent set $I \in \mathcal{I}(G)$, there is at most one function $c : I \rightarrow \mathbb{N}_{\geq 1}$ such that the empty set contributes exactly $1$. Note that there is always exactly one function $c : I \rightarrow \mathbb{N}_{\geq 1}$ and that the empty product evaluates to 1. This ensures that the empty set contributes exactly 1 to the partition function.

We obtain the following upper and lower bound on $Z_{HC}(G, \lambda)$ in terms of $Z_{HC}(G, \lambda)$.

**Lemma 4.3.** Let $(G, \lambda)$ be a multivariate hard-core instance, such that for all $v \in \mathcal{V}$, it holds that $\lambda(v) \leq \frac{1}{2}$. Further, let $\frac{1}{1-\lambda} : v \mapsto \frac{\lambda(v)}{1-\lambda(v)}$ and $2\lambda^2 : v \mapsto 2\lambda(v)^2$. Then

$$Z_{HC}(G, \lambda) \leq Z_{MHC}(G, \lambda) = Z_{HC}(G, \frac{\lambda}{1-\lambda}) \leq Z_{HC}(G, 2\lambda^2) \cdot Z_{HC}(G, \lambda) \leq e^{2\sum_{v \in \mathcal{V}} \lambda(v)^2} Z_{HC}(G, \lambda).$$

**Proof.** First observe that

$$Z_{HC}(G, \lambda_1 + \lambda_2) = \sum_{I \in \mathcal{I}(G)} \prod_{v \in I} (\lambda_1(v) + \lambda_2(v)) = \sum_{I \in \mathcal{I}(G)} \sum_{I \leq I_2} \prod_{v \in I_1} \lambda_1(v) \prod_{v \in I_2} \lambda_2(v).$$

For every independent set $I \in \mathcal{I}(G)$, it holds that all subsets of vertices $I_1 \subseteq I$ are also independent sets $I_1 \in \mathcal{I}(G)$. Note that this also implies that $I \setminus I_1 \in \mathcal{I}(G)$. Further, note that for every pair of independent sets $I_1, I_2 \in \mathcal{I}(G)$, there is at most one $I \in \mathcal{I}(G)$ such that $I_1 \cup I_2 = I$. Thus, we get

$$\sum_{I \in \mathcal{I}(G)} \sum_{I \leq I_1} \prod_{v \in I_1} \lambda_1(v) \prod_{v \in I_2} \lambda_2(v) \leq \sum_{I_1 \in \mathcal{I}(G)} \sum_{I_2 \in \mathcal{I}(G)} \prod_{v \in I_1} \lambda_1(v) \prod_{v \in I_2} \lambda_2(v)$$

$$= \left( \sum_{I \in \mathcal{I}(G)} \prod_{v \in I_1} \lambda_1(v) \right) \left( \sum_{I \in \mathcal{I}(G)} \prod_{v \in I_2} \lambda_2(v) \right)$$

$$= Z_{HC}(G, \lambda_1) \cdot Z_{HC}(G, \lambda_2),$$

which proves the claim.

To bound the error between the partition function of a hard-constraint point process and its hard-core representation, we need to deal with the following problem: The continuous process is capable to place particles very close to each other, as long as the respective particle types as not subjected to interactions according to the interaction matrix $H$. On the other hand, in the hard-core representation, each vertex cannot be multiple times in the same independent set. This means each combination of a point $x \in \mathcal{X}$ and a type $i \in [q]$ can be represented at most once in every valid configuration.

In order to overcome the difference between both configuration spaces, we consider a modified version of the multivariate hard-core model, which assigns weights to independent multisets, that is, multisets of vertices that have an independent set as support. Formally, we describe such a multiset by a tuple $(I, c)$ where $I \in \mathcal{I}(G)$ and $c : I \rightarrow \mathbb{N}_{\geq 1}$. Given a hard-core instance $(G, \lambda)$ where for all $v \in \mathcal{V}$, we have $\lambda(v) < 1$, we define the multiset hard-core partition function as

$$Z_{MHC}(G, \lambda) = \sum_{I \in \mathcal{I}(G)} \sum_{c : I \rightarrow \mathbb{N}_{\geq 1}} \prod_{v \in I} \lambda(v)^{c(v)}.$$
\[ I = \emptyset, \text{in which case the contribution to both partition functions is exactly 1.} \]

To derive the identity \( Z_{\text{MHHC}}(G, \lambda) = Z_{\text{HC}}\left(G, \frac{\lambda}{1 - \lambda}\right) \), we start by rewriting

\[
Z_{\text{MHHC}}(G, \lambda) = \sum_{I \in I(G)} \sum_{\nu : I \to \mathbb{N}_{\geq 1}} \prod_{v \in I} \lambda(v)^{c(v)} = \sum_{I \in I(G)} \prod_{\nu \in I} \lambda(v) \sum_{c : I \to \mathbb{N}} \prod_{v \in I} \lambda(v)^{c(v)}.
\]

As, for all \( v \in V \), we have \( \lambda(v) \leq \frac{1}{2} < 1 \), we further get

\[
\sum_{I \in I(G)} \prod_{\nu \in I} \lambda(v) \sum_{c : I \to \mathbb{N}} \prod_{v \in I} \lambda(v)^{c(v)} = \sum_{I \in I(G)} \prod_{\nu \in I} \lambda(v) \prod_{v \in I} \lambda(v)^{k} = \sum_{I \in I(G)} \prod_{\nu \in I} \frac{\lambda(v)}{1 - \lambda(v)}.
\]

which is exactly the desired result.

Next, for all \( v \in V \), observe that \( \lambda(v) \leq \frac{1}{2} \) implies \( \frac{\lambda(v)}{1 - \lambda(v)} \leq (1 + 2\lambda(v))\lambda(v) = 2\lambda(v)^{2} + \lambda(v) \). Using Lemma 4.2, we obtain

\[
Z_{\text{HC}}\left(G, \frac{\lambda}{1 - \lambda}\right) \leq Z_{\text{HC}}(G, 2\lambda^{2}) \cdot Z_{\text{HC}}(G, \lambda).
\]

Finally, applying Lemma 4.1 to \( Z_{\text{HC}}(G, 2\lambda^{2}) \) yields the last inequality.

We note that while we primarily make use of the upper bound \( e^{2\sum_{v \in V} \lambda(v)^{2}} Z_{\text{HC}}(G, \lambda) \), we believe that the intermediate bounds are of independent interest. Note that the choice of \( \frac{1}{2} \) as an upper bound of \( \lambda \) in Lemma 4.3 is rather arbitrary. In fact, for any \( a > 0 \), the upper bound \( 1 - \frac{1}{a} \) yields

\[
Z_{\text{MHHC}}(G, \lambda) \leq Z_{\text{HC}}(G, a\lambda^{2}) \cdot Z_{\text{HC}}(G, \lambda) \leq e^{a\sum_{v \in V} \lambda(v)^{2}} Z_{\text{HC}}(G, \lambda).
\]

For our purpose, the bound of \( \frac{1}{2} \) is sufficient.

### 4.2 Bounding the discretization error

Our bound for the discretization error for any given finite non-empty point set \( X \subseteq \mathbb{V} \) is closely related to the existence of a \( \delta, \varepsilon \)-allocation, defined as follows.

\textbf{Definition 1.3 (\( \delta, \varepsilon \)-allocation).} Let \( \mathbb{V} \subseteq \mathbb{R}^{d} \) be bounded and measurable, and assume \( \nu(\mathbb{V}) > 0 \). For finite \( X \subseteq \mathbb{V} \), \( \delta \in [0, 1) \), and \( \varepsilon \in \mathbb{R}_{>0} \), we call a function \( \Phi : \mathbb{V} \rightarrow X \) a \( \delta, \varepsilon \)-allocation for \( X \) if, for all \( x \in X \),

1. \( \Phi^{-1}(x) \) is measurable and

\[
(1 - \delta)\frac{\nu(\mathbb{V})}{|X|} \leq \nu(\Phi^{-1}(x)) \leq (1 + \delta)\frac{\nu(\mathbb{V})}{|X|}, \text{ and,}
\]

2. for all \( y \in \Phi^{-1}(x) \), it holds that \( d(x, y) \leq \varepsilon \).

For \( X \subseteq \mathbb{V} \) and a corresponding allocation \( \Phi \), we relate the hard-core representation \( (G_{X}, \lambda_{X}) \) to the continuous model \( (\mathbb{V}, R, \lambda) \). To this end, we consider the following intermediate continuous model. Let
Further, define the partition function and Gibbs density corresponding to $w_{\Phi}^{(G_X, \lambda_X)}$, respectively, as

$$Z_{\Phi}(\mathbb{V}, G_X, \lambda_X) = \sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{\mathbb{V}^k} w_{\Phi}^{(G_X, \lambda_X)}(x, \tau) \, dv^{d \times k} \quad \text{and}$$

$$\mu_{\Phi}^{(V, G_X, \lambda_X)}(x, \tau) = \frac{w_{\Phi}^{(G_X, \lambda_X)}(x, \tau)}{Z_{\Phi}(\mathbb{V}, G_X, \lambda_X)}.$$  

In Section 7, we give a detailed probabilistic interpretation of $\mu_{\Phi}^{(V, G_X, \lambda_X)}$. For this section and the approximation results in Section 6, we are mostly interested in $Z_{\Phi}(\mathbb{V}, G_X, \lambda_X)$.

The following bound is central to basically all results in this paper.

**Lemma 4.4.** Let $(\mathbb{V}, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types, and assume $\mathbb{V} \subset \mathbb{R}^d$ is star-convex. Further, set $R_{\min} = \inf_{i,j \in [q]} \{ R(i, j) \mid R(i, j) > 0 \}$, and set $\lambda_{\max} = \max_{i \in [q]} \lambda(i)$. Let $X \subseteq \mathbb{V}$ with $4\lambda_{\max} \nu(\mathbb{V}) \leq |X| < \infty$, and let $(G_X, \lambda_X)$ be the hard-core representation of $(\mathbb{V}, R, \lambda)$ based on $X$. Last, let $\delta \in \left[0, \frac{1}{2}\right]$ and $\varepsilon \in \left[0, \frac{R_{\min}}{2}\right]$, and assume that $\Phi$ is a $\delta$-$\varepsilon$-allocation for $X$. Then

$$\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{\mathbb{V}^k} \left| w_{\Phi}^{(R, \lambda)}(x, \tau) - w_{\Phi}^{(G_X, \lambda_X)}(x, \tau) \right| \, dv^{d \times k} \leq \left( 2 e^{2\delta} + \frac{1}{2 e^{2\delta}} \right) \left( 1 - e^{(2\delta + \frac{1}{2 e^{2\delta}})^d} \right) Z(\mathbb{V}, R, \lambda).$$

**Proof.** For all $x = (x_i)_{i \in [k]} \in \mathbb{V}^k$, we abuse notation and write $\Phi(x)$ for the tuple of points $(\Phi(x_i))_{i \in [k]} \in X^k$. By the triangle inequality, we have that

$$\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{\mathbb{V}^k} \left| w_{\Phi}^{(R, \lambda)}(x, \tau) - w_{\Phi}^{(G_X, \lambda_X)}(x, \tau) \right| \, dv^{d \times k} \leq \sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{\mathbb{V}^k} \left| w_{\Phi}^{(R, \lambda)}(x, \tau) - w_{\Phi}^{(R, \lambda)}(\Phi(x), \tau) \right| \, dv^{d \times k} + \sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{\mathbb{V}^k} \left| w_{\Phi}^{(R, \lambda)}(\Phi(x), \tau) - w_{\Phi}^{(G_X, \lambda_X)}(x, \tau) \right| \, dv^{d \times k}. \quad (6)$$

(7)
We conclude that

\[
\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \to [q]} \int_{V^k} \left| w^{(R, \lambda)}(x, \tau) - w^{(R, \lambda)}(\Phi(x), \tau) \right| \, \nu^{d \times k} \leq \left( e^{\frac{4}{\min \{q\}}} \sum_{i \in [q]} \lambda(i) \nu(V) - 1 \right) Z(V, R, \lambda).
\]

We do so by showing that

\[
\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \to [q]} \int_{V^k} \left| w^{(R, \lambda)}(x, \tau) - w^{(R, \lambda)}(\Phi(x), \tau) \right| \, \nu^{d \times k} \leq Z(V, R, \lambda) - Z(V, R, \lambda),
\]

where \( R_- \) and \( R_+ \) are slightly scaled versions of \( R \), and then applying Lemma 3.5.

Second, we show that

\[
\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \to [q]} \int_{V^k} \left| w^{(R, \lambda)}(\Phi(x), \tau) - w^{(R, \lambda)}(x, \tau) \right| \, \nu^{d \times k}
\leq \left( e^{\frac{4}{\min \{q\}}} \sum_{i \in [q]} \lambda(i) \nu(V) - 1 \right) e^{2\delta \sum_{i \in [q]} \lambda(i) \nu(V)} Z(V, R, \lambda).
\]

To this end, we argue that

\[
\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \to [q]} \int_{V^k} \left| w^{(R, \lambda)}(\Phi(x), \tau) - w^{(R, \lambda)}(x, \tau) \right| \, \nu^{d \times k}
\leq Z_{\text{MHC}}(G_X, (1 + \delta) \lambda_X) - Z_{\text{HC}}(G_X, (1 - \delta) \lambda_X)
\]

and apply Lemmas 4.1 to 4.3.

Combining both parts proves the claim.

**Bounding (6):** Note that for \( k \leq 1 \), it holds for all \( x \in V^k \) and \( \tau: [k] \to [q] \) that \( w^{(R, \lambda)}(x, \tau) = w^{(R, \lambda)}(\Phi(x), \tau) \). Fix \( k \geq 2 \) as well as \( \tau: [k] \to [q] \). Let \( N_\tau \subseteq V^k \) be the set of all points \( x = (x_i)_{i \in [k]} \in V^k \) such that \( D_\tau^{(R)}(\Phi(x)) \neq D_\tau^{(R)}(x) \). Observe that

\[
\left| w^{(R, \lambda)}(x, \tau) - w^{(R, \lambda)}(\Phi(x), \tau) \right| \leq \frac{1}{k!} \prod_{i \in [k]} \lambda(\tau(i)) 1\{x \in N_\tau\}.
\]

We proceed by characterizing a superset of \( N_\tau \) that is easier to analyze. Observe that, as \( \Phi \) is a \( \delta, \varepsilon \)-allocation, it holds for all \( x \in V \) that \( d(x, \Phi(x)) \leq \varepsilon \). Let \( R_- = \left( 1 - \frac{2 \varepsilon}{\min \{q\}} \right) R \) and \( R_+ = \left( 1 + \frac{2 \varepsilon}{\min \{q\}} \right) R \). Thus, for all \( i, j \in [q] \) with \( R(i, j) > 0 \), it holds that \( R_-(i, j) \leq R(i, j) - 2\varepsilon \) and \( R_+(i, j) \geq R(i, j) + 2\varepsilon \).

Let \( x = (x_i)_{i \in [k]} \in V^k \) such that \( D_\tau^{(R_\tau)}(x) = 0 \). Note that this implies that there are \( i, j \in [k] \) with \( i \neq j \) such that \( R(\tau(i), \tau(j)) > 0 \) and

\[
d(x_i, x_j) < R_-(\tau(i), \tau(j)) \leq R(\tau(i), \tau(j)) - 2\varepsilon.
\]

We conclude that \( d(x_i, x_j) < R(\tau(i), \tau(j)) \) and by triangle inequality \( d(\Phi(x_i), \Phi(x_j)) < R(\tau(i), \tau(j)) \). Thus,
we have $D_t^{(R)}(x) = D_t^{(R)}(\Phi(x)) = 0$ and consequently $x \notin N_r$.

Next, let $x = (x_i)_{i \in [k]} \in V^k$ with $D_t^{(R_+)}(x) = 1$. Note that this implies for all $i, j \in [k]$ with $i \neq j$ and $R(\tau(i), \tau(j)) > 0$ that

$$d(x_i, x_j) \geq R_+(\tau(i), \tau(j)) \geq R(\tau(i), \tau(j)) + 2\varepsilon.$$  

We conclude that $d(x_i, x_j) \geq R(\tau(i), \tau(j))$ and by triangle inequality $d(\Phi(x_i), \Phi(x_j)) \geq R(\tau(i), \tau(j))$. Thus, we have $D_t^{(R)}(x) = D_t^{(R)}(\Phi(x)) = 1$ and consequently $x \notin N_r$.

Finally, observe that, because for all $i, j \in [k]$, it holds that $R_-(i, j) \leq R_+(i, j)$, we have that $D_t^{(R_-)}(x) = 0$ implies $D_t^{(R_+)}(x) = 0$, and $D_t^{(R_+)}(x) = 1$ implies $D_t^{(R_-)}(x) = 1$. Thus, we have

$$\mathbb{1}\{x \in N_r\} \leq D_t^{(R_-)}(x) - D_t^{(R_+)}(x).$$

Substituting this into term (6) yields

$$\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{V^k} \left| w^{(R_+)}(x, \tau) - w^{(R_-)}(\Phi(x), \tau) \right| \, dv^{d \times k}$$

$$\leq \sum_{k \in \mathbb{N}_{\geq 2}} \frac{1}{k!} \sum_{\tau: [k] \rightarrow [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \int_{V^k} \mathbb{1}\{x \in N_r\} \, dv^{d \times k}$$

$$\leq \sum_{k \in \mathbb{N}_{\geq 2}} \frac{1}{k!} \sum_{\tau: [k] \rightarrow [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \int_{V^k} D_t^{(R_-)}(x) \, dv^{d \times k}$$

$$- \sum_{k \in \mathbb{N}_{\geq 2}} \frac{1}{k!} \sum_{\tau: [k] \rightarrow [q]} \left( \prod_{i \in [k]} \lambda(\tau(i)) \right) \int_{V^k} D_t^{(R_+)}(x) \, dv^{d \times k}$$

$$\leq Z(V, R_-, \lambda) - Z(V, R_+, \lambda).$$

Recalling the definitions of $R_-$ and $R_+$ as well as that $V$ is star-convex, we apply Lemma 3.5 with $\alpha = \frac{2\varepsilon}{R_{\min}} \in [0, 1]$ and obtain

$$\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{V^k} \left| w^{(R_+)}(x, \tau) - w^{(R_-)}(\Phi(x), \tau) \right| \, dv^{d \times k} \leq \left( e^\left( \frac{2\varepsilon}{R_{\min}} \right)^d \sum_{i \in [q]} \lambda(i) \nu(V) - 1 \right) Z(V, R, \lambda), \quad (8)$$

which concludes the first part of the proof.

Bounding (7): For $x = (x_i)_{i \in [k]} \in V^k$ and $\tau: [k] \rightarrow [q]$, let

$$\tilde{D}_t^{(G_x, \Phi)}(x) = \prod_{i, j \in [k], i < j} \mathbb{1}\left( v^{(\tau(i))}_{\Phi(x_i)} v^{(\tau(j))}_{\Phi(x_j)} \notin E_x \text{ or } v^{(\tau(i))}_{\Phi(x_i)} = v^{(\tau(j))}_{\Phi(x_j)} \right).$$

Note that for all $x \in V^k$ and all type assignments $\tau: [k] \rightarrow [q]$, it holds that

$$D_t^{(G_x, \Phi)}(x) \leq D_t^{(R)}(\Phi(x)) \leq \tilde{D}_t^{(G_x, \Phi)}(x).$$

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Further, as \( \Phi \) is a \( \delta \)-\( \varepsilon \)-allocation for \( X \), it holds for all \( x \in X \) that

\[
(1 - \delta) \frac{\nu(V)}{|X|} \leq \nu(\Phi^{-1}(x)) \leq (1 + \delta) \frac{\nu(V)}{|X|}.
\]

For all \( x \in X \) and \( i \in [q] \), because \( \lambda_X \left( v^{(i)}_x \right) = \lambda(i) \frac{\nu(V)}{|X|} \), we have

\[
(1 - \delta) \frac{\nu(V)}{|X|} \leq \lambda(i) \leq (1 + \delta) \frac{\nu(V)}{|X|}.
\]

This implies for all \( x \in V^k \) and \( \tau: [k] \to [q] \) that

\[
\frac{1}{k!} \left( \prod_{i \in [k]} (1 - \delta) \frac{\lambda_X \left( v^{(r(i))}_{\phi(x)} \right)}{\nu(\Phi^{-1}(\Phi(x)))} \right) D^{(G_X, \Phi)}_\tau(x) \leq w^{(R, \lambda)}(\Phi(x), \tau)
\]

\[
\leq \frac{1}{k!} \left( \prod_{i \in [k]} (1 + \delta) \frac{\lambda_X \left( v^{(r(i))}_{\phi(x)} \right)}{\nu(\Phi^{-1}(\Phi(x)))} \right) \tilde{D}^{(G_X, \Phi)}_\tau(x)
\]

and

\[
\frac{1}{k!} \left( \prod_{i \in [k]} (1 - \delta) \frac{\lambda_X \left( v^{(r(i))}_{\phi(x)} \right)}{\nu(\Phi^{-1}(\Phi(x)))} \right) D^{(G_X, \Phi)}_\tau(x) \leq w^{(G_X, \lambda_X)}(x, \tau)
\]

\[
\leq \frac{1}{k!} \left( \prod_{i \in [k]} (1 + \delta) \frac{\lambda_X \left( v^{(r(i))}_{\phi(x)} \right)}{\nu(\Phi^{-1}(\Phi(x)))} \right) \tilde{D}^{(G_X, \Phi)}_\tau(x).
\] (9)

Substituting this into term (7) yields

\[
\sum_{k \in \mathbb{N}} \sum_{r: [k] \to [q]} \int_{V^k} \left| w^{(R, \lambda)}(\Phi(x), \tau) - w^{(G_X, \lambda_X)}(x, \tau) \right| d\nu^{d \times k}
\]

\[
\leq \sum_{k \in \mathbb{N}} \sum_{r: [k] \to [q]} \int_{V^k} \frac{1}{k!} \left( \prod_{i \in [k]} (1 + \delta) \frac{\lambda_X \left( v^{(r(i))}_{\phi(x)} \right)}{\nu(\Phi^{-1}(\Phi(x)))} \right) \tilde{D}^{(G_X, \Phi)}_\tau(x) d\nu^{d \times k}
\]

\[
- \sum_{k \in \mathbb{N}} \sum_{r: [k] \to [q]} \int_{V^k} \frac{1}{k!} \left( \prod_{i \in [k]} (1 - \delta) \frac{\lambda_X \left( v^{(r(i))}_{\phi(x)} \right)}{\nu(\Phi^{-1}(\Phi(x)))} \right) D^{(G_X, \Phi)}_\tau(x) d\nu^{d \times k}.
\] (10)

Next, note that due to \( \delta \leq 1 \) and \( 4\lambda_{\max} \nu(V) \leq |X| \), it holds for all \( v^{(i)}_x \in V_X \) that

\[
(1 + \delta) \frac{\nu(V)}{|X|} \leq 2 \frac{\nu(V)}{|X|} \lambda(i) \leq \frac{1}{2}.
\] (11)
Thus, $Z_{\text{MHC}}(G_X, (1 + \delta)\lambda_X)$ is well defined. We proceed by rewriting both functions, $Z_{\text{HC}}(G_X, (1 - \delta)\lambda_X)$ and $Z_{\text{MHC}}(G_X, (1 + \delta)\lambda_X)$, by expressing each in terms of tuples of vertices. For $Z_{\text{HC}}(G_X, (1 - \delta)\lambda_X)$, this yields

$$Z_{\text{HC}}(G_X, (1 - \delta)\lambda_X) = \sum_{k \in \mathbb{N}} \sum_{\tau: [k] \to [q]} \sum_{x \in X^k} \frac{1}{k!} \left( \prod_{i \in [k]} (1 - \delta)\lambda_X \left( v^{(\tau(i))}_{x_i} \right) \right) D_\tau^{(G_X, \Phi)}(x)$$

$$= \sum_{k \in \mathbb{N}} \sum_{\tau: [k] \to [q]} \int_{\mathcal{V}^k} \frac{1}{k!} \left( \prod_{i \in [k]} (1 - \delta)\lambda_X \left( v^{(\tau(i))}_{\Phi(x)_i} \right) \right) D_\tau^{(G_X, \Phi)}(x) \, d\nu^{d \times k}. \quad (12)$$

Similarly, for $Z_{\text{MHC}}(G_X, (1 + \delta)\lambda_X)$, we get

$$Z_{\text{MHC}}(G_X, (1 + \delta)\lambda_X) = \sum_{k \in \mathbb{N}} \sum_{\tau: [k] \to [q]} \sum_{x \in X^k} \frac{1}{k!} \left( \prod_{i \in [k]} (1 + \delta)\lambda_X \left( v^{(\tau(i))}_{x_i} \right) \right) \widehat{D}_\tau^{(G_X, \Phi)}(x)$$

$$= \sum_{k \in \mathbb{N}} \sum_{\tau: [k] \to [q]} \int_{\mathcal{V}^k} \frac{1}{k!} \left( \prod_{i \in [k]} (1 + \delta)\lambda_X \left( v^{(\tau(i))}_{\Phi(x)_i} \right) \right) \widehat{D}_\tau^{(G_X, \Phi)}(x) \, d\nu^{d \times k}. \quad (13)$$

Note that the multiset version always allows arbitrarily many copies of the same vertex, independent of $R$, which is captured by the indicator function $\widehat{D}_\tau^{(G_X, \Phi)}$.

Substituting equations (12) and (13) into inequality (10) yields

$$\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \to [q]} \int_{\mathcal{V}^k} \left| w^{(R, \lambda)}(\Phi(x), \tau) - w^{(G_X, \lambda_X)}_{\Phi}(x, \tau) \right| \, d\nu^{d \times k} \leq Z_{\text{MHC}}(G_X, (1 + \delta)\lambda_X) - Z_{\text{HC}}(G_X, (1 - \delta)\lambda_X).$$

We proceed by bounding both terms of the difference. To this end, we start by upper-bounding the partition function $Z_{\text{MHC}}(G_X, (1 + \delta)\lambda_X)$ in terms of $Z_{\text{HC}}(G_X, (1 + \delta)\lambda_X)$ using Lemma 4.3 (note that we can do so by equation (11)). We obtain

$$Z_{\text{MHC}}(G_X, (1 + \delta)\lambda_X) \leq e^{2 \sum_{i \in [q]} \sum_{x \in X^i} (1 + \delta)^2 \lambda_X} Z_{\text{HC}}(G_X, (1 + \delta)\lambda_X)$$

$$\leq e^{2(1 + \delta)^2 \sum_{i \in [q]} \lambda_X} \sum_{x \in X^i} v(x)^2 \sqrt{\frac{|X|}{2}} Z_{\text{HC}}(G_X, (1 + \delta)\lambda_X)$$

$$\leq e^{\frac{8}{\lambda}} \sum_{i \in [q]} \lambda_X v(X)^2 Z_{\text{HC}}(G_X, (1 + \delta)\lambda_X).$$

Further, by lemmas 4.1 and 4.2 we obtain

$$Z_{\text{HC}}(G_X, (1 + \delta)\lambda_X) \leq Z_{\text{HC}}(G_X, 2\delta \lambda_X) Z_{\text{HC}}(G_X, (1 - \delta)\lambda_X)$$

$$\leq e^{\sum_{i \in [q]} \sum_{x \in X^i} 2\delta \lambda_X} Z_{\text{HC}}(G_X, (1 - \delta)\lambda_X)$$

$$= e^{2\delta} \sum_{i \in [q]} \lambda_X v(X)^2 Z_{\text{HC}}(G_X, (1 - \delta)\lambda_X).$$
Thus, we have

\[
Z_{\text{MHC}}(G_X, (1 + \delta)\lambda_X) - Z_{\text{HC}}(G_X, (1 - \delta)\lambda_X)
\leq \left( e^{\frac{2\delta}{\min\{\lambda\}}} \sum_{i \in [q]} \lambda(i)^2 \nu(V)^2 e^{2\delta \sum_{i \in [q]} \lambda(i) \nu(V)} - 1 \right) Z_{\text{HC}}(G_X, (1 - \delta)\lambda_X).
\]

Next, by Inequality (9) and equation (12), we know that

\[
Z_{\text{HC}}(G_X, (1 - \delta)\lambda_X) \leq \sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{\mathbb{V}^k} w^{(R,\lambda)}(\Phi(x), \tau) \, dv^{d \times k}
\]

and by Inequality (8)

\[
\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{\mathbb{V}^k} \left| w^{(R,\lambda)}(\Phi(x), \tau) - \Phi_{\mathcal{H}, X}(x, \tau) \right| \, dv^{d \times k}
\leq \left( e^{\frac{2\delta}{\min\{\lambda\}}} \sum_{i \in [q]} \lambda(i)^2 \nu(V)^2 e^{2\delta \sum_{i \in [q]} \lambda(i) \nu(V)} - 1 \right) e^{\left( \frac{4w}{\min\{\lambda\}} \right)^d \sum_{i \in [q]} \lambda(i) \nu(V)} Z(\mathbb{V}, R, \lambda)
\]

and concludes the second part of the proof.

**Putting the pieces together:** Adding inequality (8) to inequality (14) yields

\[
\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{\mathbb{V}^k} \left| w^{(R,\lambda)}(x, \tau) - \Phi_{\mathcal{H}, X}(x, \tau) \right| \, dv^{d \times k}
\leq \left( e^{\frac{2\delta}{\min\{\lambda\}}} \sum_{i \in [q]} \lambda(i)^2 \nu(V)^2 e^{2\delta \sum_{i \in [q]} \lambda(i) \nu(V)} - 1 \right) Z(\mathbb{V}, R, \lambda),
\]

which proves the claim.

As Lemma 4.4 immediately implies a bound on \( |Z(\mathbb{V}, R, \lambda) - Z_{\Phi}(\mathbb{V}, G_X, \lambda_X)| \), this leads us to the question how to compute \( Z_{\Phi}(\mathbb{V}, G_X, \lambda_X) \) efficiently. To this end, the following identity will come in handy.

**Lemma 4.5.** Let \( (\mathbb{V}, R, \lambda) \) be a hard-constraint point process with \( q \in \mathbb{N}_{\geq 1} \) particle types and \( \mathbb{V} \subset \mathbb{R}^d \). Further, let \( X \subseteq \mathbb{V} \) be finite and non-empty, and let \( (G_X, \lambda_X) \) be the hard-core representation of \( (\mathbb{V}, R, \lambda) \) based on \( X \). For all \( \delta \in [0, 1) \) and \( \varepsilon \in \mathbb{R}_{>0} \), and all \( \delta, \varepsilon \)-allocations \( \Phi \) for \( X \), we have

\[
Z_{\Phi}(\mathbb{V}, G_X, \lambda_X) = Z_{\text{HC}}(G_X, \lambda_X).
\]
Proof. We prove this claim by rewriting $Z_{HC}(G_X, \lambda_X)$ in terms of tuples of vertices, yielding

$$Z_{HC}(G_X, \lambda_X) = \sum_{k \in \mathbb{N}} \sum_{i \in [k]} \sum_{x \in X^k} \frac{1}{k!} \left( \prod_{i \in [k]} \lambda_X \left( v^{(\tau(i))}_{x_i} \right) \right) D_{\tau}^{(G_X, \Phi)}(x)$$

$$= \sum_{k \in \mathbb{N}} \sum_{i \in [k]} \int \frac{1}{k!} \left( \prod_{i \in [k]} \lambda_X \left( v^{(\tau(i))}_{\Phi^{-1}(\Phi(x_i))} \right) \right) D_{\tau}^{(G_X, \Phi)}(x) \, dv^{d \times k}$$

$$= \sum_{k \in \mathbb{N}} \sum_{i \in [k]} \int w_{\Phi}^{(G_X, \lambda_X)}(x, \tau) \, dv^{d \times k}$$

$$= Z_{HC}(V, G_X, \lambda_X).$$

Note that $Z_{HC}(G_X, \lambda_X)$ is simply the partition function of the discrete hard-core model, which, as we discussed in the introduction, is well studied from a computational perspective. Combining Lemmas 4.4 and 4.5 immediately yields the following bound on the difference between $Z(V, R, \lambda)$ and $Z_{HC}(G_X, \lambda_X)$.

**Theorem 4.6.** Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types, and assume $V \subset \mathbb{R}^d$ is star-convex. Furthermore, set $R_{\min} = \inf_{i, j \in [q]} \{R(i, j) \mid R(i, j) > 0\}$, and set $\lambda_{\max} = \max_{i \in [q]} \lambda(i)$. Let $X \subset V$ with $4\lambda_{\max} v(V) \leq |X| < \infty$, and let $(G_X, \lambda_X)$ be the hard-core representation of $(V, R, \lambda)$ based on $X$. Finally, let $\delta \in \left[0, \frac{1}{2}\right]$ and $\epsilon \in \left[0, \frac{R_{\min}}{2}\right]$, and assume that $V$ has a $\delta$-$\epsilon$-allocation for $X$. Then

$$|Z_{HC}(G_X, \lambda_X) - Z(V, R, \lambda)| \leq \left( e^{8 \sum_{i \in [q]} \lambda(i) \nu(v(V))^2} e^{2\delta (\frac{\epsilon}{R_{\min}})^d} \sum_{i \in [q]} \lambda(i) v(V) - 1 \right) Z(V, R, \lambda).$$

We would like to add a remark about the error term, given in Theorem 4.6. While it seems to depend on three quantities, namely $\delta$, $\epsilon$, and $|X|$, it usually suffices to focus on $\delta$ and $\epsilon$. This is because decreasing $\epsilon$ requires to increase $|X|$. One way to see this is to observe that for any $\epsilon \in \mathbb{R}_{>0}$, a lower bound for the minimum size of $X \subset V$ such that a $\delta$-$\epsilon$-allocation exists is $\frac{\nu(V)}{v(B(2\delta))}$. Thus, choosing $\epsilon$ sufficiently small typically results in sufficiently large $|X|$.

Another interesting aspect is the role of $\Phi$. To construct the hard-core representation for the continuous model and apply Theorem 4.6, it is sufficient to know a set of points $X \subset V$ such that a suitable $\delta$-$\epsilon$-allocation exists. However, the allocation itself is not required to be known for constructing the discretization. This is because, even though Lemma 4.4 seems to depend on the specific allocation $\Phi$, Lemma 4.5 shows that $Z_{HC}(X, G_X, \lambda_X) = Z_{HC}(G_X, \lambda_X)$ independent of $\Phi$. For our algorithmic results in Section 6, we focus on the case where $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{>0}$ (i.e., $V$ is a hypercube). In this case, we provide an explicit allocation. Nevertheless, noting that an explicit construction is not required might be of its own interest when generalizing the results to different classes of continuous regions $V \subset \mathbb{R}^d$.

## 5 Canonical discretization for cubic regions

As mentioned earlier, our algorithmic results focus on cubic regions $V = [0, \ell]^d$, $\ell \in \mathbb{R}_{>0}$. Note that such regions are star-convex. In what follows, we show how to construct, for any given $\ell \in \mathbb{R}_{>0}$, a set of points

4 We use the convention that $\inf \emptyset = \infty$, which in this setting means that $R_{\min}$ can be chosen arbitrarily large.
where the product is meant to be the cartesian product. Note that

\[ y \]

That is,

\[ \rho \]

Thus, for

\[ X \]

\[ \nu \]

\[ p \]

end, let

\[ \text{Proof.} \]

We prove our claim by constructing an allocation

\[ X = \{ x^{(1)}, \ldots, x^{(d)} \} \in \mathbb{V} \]

\[ \forall i \in [d] \exists m \in \mathbb{N}: x^{(i)} = \frac{m}{\rho} \].

For a hard-constraint point process \((V, R, \lambda)\) we now define the canonical discretization to be the hard-core representation \((G_{X^\rho}, \lambda_{X^\rho})\) based on \(X_\rho\). Note that in the special case of the hard-sphere model, this canonical discretization coincides to the procedure proposed in [18]. However, our approach works for a much more general class of point processes and, as the rest of this section shows, we obtain much stronger error bounds on the partition function.

The following statement shows how the existence of an allocation for a canonical point set \(X_\rho\) depends on the resolution \(\rho\).

\[ \text{Lemma 5.1.} \]

Let \(\ell \in \mathbb{R}_{>0}\), let \(V = [0, \ell]^d\), and let \(\epsilon \in \mathbb{R}_{>0}\). For all feasible resolutions \(\rho \geq \frac{\sqrt{d}}{\epsilon}\) there is a \(0,\epsilon\)-allocation of \(V\) for the canonical point set \(X_\rho\).

\[ \text{Proof.} \]

We prove our claim by constructing an allocation \(\Phi: \mathbb{V} \rightarrow X_\rho\) with the desired properties. To this end, let \(y = (y^{(1)}, \ldots, y^{(d)}) \in \mathbb{V}\). We set

\[ \Phi(y) = \left( \left\lfloor \frac{\rho y^{(1)}}{\rho} \right\rfloor, \ldots, \left\lfloor \frac{\rho y^{(d)}}{\rho} \right\rfloor \right). \]

That is, \(y\) gets mapped to the closest point in \(x = (x^{(1)}, \ldots, x^{(d)}) \in X_\rho\) such that \(x^{(i)} \leq y^{(i)}\) for all \(i \in [d]\).

Now, let \(x = (x^{(1)}, \ldots, x^{(d)}) \in X_\rho\) and observe that

\[ \Phi^{-1}(x) = \prod_{i \in [d]} \left[ x^{(i)}, x^{(i)} + \frac{1}{\rho} \right), \]

where the product is meant to be the cartesian product. Note that \(\Phi^{-1}(x)\) is measurable. Further, we have \(|X_\rho| = \rho^d \nu(V)\). Thus, we have

\[ \nu(\Phi^{-1}(x)) = \frac{1}{\rho^d} = \frac{\nu(V)}{\rho^d \nu(V)} = \frac{\nu(V)}{|X_\rho|}. \]

Finally, observe that for all \(x \in X_\rho\) and all \(y \in \Phi^{-1}(x)\), it holds that

\[ d(x, y) \leq \sqrt{d \frac{1}{\rho^2}} = \frac{\sqrt{d}}{\rho}. \]

Thus, for \(\rho \geq \frac{\sqrt{d}}{\epsilon}\) we have \(d(x, y) \leq \epsilon\), which proves that \(\Phi\) is a \(0,\epsilon\)-allocation for \(X_\rho\) and concludes the proof.
We can now use the hard-core partition function of the canonical discretization as an approximation of the partition function of the hard-constraint point process as stated in the following result.

**Theorem 5.2.** Let \((\mathbb{V}, R, \lambda)\) be a hard-constraint point process with \(q \in \mathbb{N}_{\geq 1}\) particle types and \(\mathbb{V} = [0, \ell]^d\) for some \(\ell \in \mathbb{R}_{>0}\). For all \(\varepsilon_D \in (0, 1]\) there exists \(\rho_{\varepsilon_D} \in \Theta\left(\varepsilon_D^{-1/d} \nu(\mathbb{V})^{1/d}\right)\) such that for all feasible resolutions \(\rho \geq \rho_{\varepsilon_D}\) we have

\[
e^{-\varepsilon_D} Z(\mathbb{V}, R, \lambda) \leq Z_{HC}\left(G_{X_\rho}, \lambda_{X_\rho}\right) \leq e^{\varepsilon_D} Z(\mathbb{V}, R, \lambda).
\]

**Proof.** Let \(R_{\min} = \inf_{i,j \in [q]} \{R(i, j) \mid R(i, j) > 0\}\) and \(\lambda_{\max} = \max_{i \in [q]} \lambda(i)\). We set

\[
\rho_{\varepsilon_D} = \sqrt{d} \left(\frac{48q \max\{\lambda_{\max}, \lambda_{\max}^2\} \nu(\mathbb{V})}{\varepsilon_D}\right)^{\frac{d}{2}} \cdot \max\left\{1, \frac{4}{R_{\min}}\right\}.
\]

First observe that \(\rho_{\varepsilon_D}\) satisfies the required asymptotic behavior.

Now, for all feasible resolutions \(\rho \geq \rho_{\varepsilon_D}\) we know by Lemma 5.1 that there is a \(\delta, \varepsilon\)-allocation for the canonical point set \(X_\rho\) with \(\delta = 0\) and

\[
\varepsilon = \sqrt{\frac{d}{\rho}} \leq \left(\frac{\varepsilon_D}{48q \max\{\lambda_{\max}, \lambda_{\max}^2\} \nu(\mathbb{V})}\right)^{\frac{d}{2}} \cdot \min\left\{1, \frac{R_{\min}}{4}\right\}.
\]

To prove our claim, we aim for applying Theorem 4.6 based on the canonical point set \(X_\rho\) and the \(\delta, \varepsilon\)-allocation obtained from Lemma 5.1. We start by checking that \(\delta, \varepsilon, \) and \(|X_\rho|\) satisfy the conditions of Theorem 4.6.

To this end, note that it trivially holds that \(\delta = 0 \in \left[0, \frac{1}{2}\right]\). Further, observe that for sufficiently large \(\nu(\mathbb{V})\), it holds that \(\varepsilon \in [0, R_{\min}/2]\). Finally, note that

\[
|X_\rho| = \rho^{d} \left(\frac{48q \max\{\lambda_{\max}, \lambda_{\max}^2\} \nu(\mathbb{V})}{\varepsilon_D}\right)^{\frac{d}{2}} \cdot \max\left\{1, \left(\frac{4}{R_{\min}}\right)^{d}\right\}.
\]

Thus, for \(\nu(\mathbb{V}) \geq 1\) we have

\[
|X_\rho| \geq 4\lambda_{\max} \nu(\mathbb{V}).
\]

We can now apply Theorem 4.6. Recall that we have \(\delta = 0\). Further, by

\[
\varepsilon \leq \left(\frac{\varepsilon_D}{48q \max\{\lambda_{\max}, \lambda_{\max}^2\} \nu(\mathbb{V})}\right)^{\frac{d}{2}} \cdot \min\left\{1, \frac{R_{\min}}{4}\right\} \leq \left(\frac{\varepsilon_D}{6 \sum_{i \in [q]} \lambda(i) \nu(\mathbb{V})}\right)^{\frac{d}{2}} \frac{R_{\min}}{4}
\]

we get

\[
\left(\frac{4\varepsilon}{R_{\min}}\right)^{d} \sum_{i \in [q]} \lambda(i) \nu(\mathbb{V}) \leq \frac{\varepsilon_D}{6}.
\]
Observe that for
\[ |X_\rho| \geq d^2 48q \max_{\varepsilon_D} \{ \lambda_{\max}^2, \lambda_{\max}^2 \} v(\mathbb{V})^2 \cdot \max \left\{ 1, \left( \frac{4}{R_{\min}} \right)^d \right\} \geq 48 \sum_{i \in [q]} \lambda(i)^2 v(\mathbb{V})^2 \]

it holds that
\[ \frac{8}{|X_\rho|} \sum_{i \in [q]} \lambda(i)^2 v(\mathbb{V})^2 \leq \frac{\varepsilon_D}{6}. \]

Thus, using Theorem 4.6, we get
\[ \left| Z_{HC} \left( G_{X_\rho}, \lambda_{X_\rho} \right) - Z(\mathbb{V}, R, \lambda) \right| \leq \left( e^{\frac{\varepsilon_D}{2}} - 1 \right) Z(\mathbb{V}, R, \lambda). \]

Note that \( e^{\frac{\varepsilon_D}{2}} \leq e^{\varepsilon_D} \) and \( 2 - e^{\frac{\varepsilon_D}{2}} \geq e^{-\varepsilon_D} \) for \( \varepsilon_D \in (0, 1] \). Therefor, we have
\[ e^{-\varepsilon_D} Z(\mathbb{V}, R, \lambda) \leq Z_{HC} \left( G_{X_\rho}, \lambda_{X_\rho} \right) \leq e^{\varepsilon_D} Z(\mathbb{V}, R, \lambda), \]

which concludes the proof.

Note that the above statement works for \( \delta = 0 \), meaning that we do not make full use of the power of Theorem 4.6. However, we use the theorem to its full extend when investigating discretizations based on (uniformly) random point sets in Section 8. This does not directly lead to any new algorithmic results, but it points out an interesting relationship between the hard-core partition functions of random geometric graphs and continuous hard-constraint point processes, which we believe to be interesting in its own right.

For the rest of this section, we prove some properties of the canonical discretization that come in handy when deriving sufficient conditions for an efficient approximation of \( Z(\mathbb{V}, R, \lambda) \) via \( Z_{HC} \left( G_{X_\rho}, \lambda_{X_\rho} \right) \). We start with the following observation that bounds the number of vertices in \( G_{X_\rho} \).

**Observation 5.3.** Let \( (\mathbb{V}, R, \lambda) \) be a hard-constraint point process with \( q \in \mathbb{N}_{\geq 1} \) particle types and \( \mathbb{V} = [0, \ell)^d \) for some \( \ell \in \mathbb{R}_{>0} \). For every feasible resolution \( \rho \in \mathbb{R}_{>0} \) it holds that
\[ |V_{X_\rho}| = q \cdot (\rho \ell)^d = q \rho^d v(\mathbb{V}). \]

Note that, by Observation 5.3, Theorem 5.2 implies that we only need \( \Theta \left( \frac{v(\mathbb{V})^2}{\varepsilon_D} \right) \) vertices in order to ensure that \( Z_{HC} \left( G_{X_\rho}, \lambda_{X_\rho} \right) \) is an \( \varepsilon_D \)-approximation of \( Z(\mathbb{V}, R, \lambda) \). Considering for instance the hard-sphere model, this improves the required number of vertices from super-exponential in \( v(\mathbb{V}) \) to quadratic, compared to [18].

Another important property of the canonical discretization are the vertex degrees. Especially upper bounding those degrees is closely related to our ability to approximate the hard-core partition function. To analyze how the degrees increase in terms of the resolution \( \rho \), we apply the following lemma.

**Lemma 5.4.** For \( d \in \mathbb{N}_{\geq 1} \) and \( s \in \mathbb{R}_{>0} \), let \( b_d(s) \) denote the number of integer points in the ball \( B_0(s) \), where \( 0 \) is the origin of \( \mathbb{R}^d \). Further, let \( \gamma \in (0, 1] \). For all \( s \geq 2d^{\gamma/2} \gamma \) it holds that
\[ b_d(s) \leq (1 + \gamma) v(B(s)). \]
Proof. Consider \( \mathbb{B}_0 \left( s + \sqrt{d} \right) \) and observe that for every integer point \((x^{(1)}, \ldots, x^{(d)}) \in \mathbb{B}_0(s) \cap \mathbb{Z}^d\) it holds that the hypercube \([x^{(1)}, x^{(1)} + 1) \times \cdots \times [x^{(d)}, x^{(d)} + 1)\) of volume 1 is completely contained in \( \mathbb{B}_0 \left( s + \sqrt{d} \right) \). Further, note that for different integer points, the corresponding hypercubes are disjoint. Thus, we obtain

\[
\nu(s) \leq v \left( \mathbb{B} \left( s + \sqrt{d} \right) \right).
\]

It remains to bound the volume \( v \left( \mathbb{B} \left( s + \sqrt{d} \right) \right) \) to get the desired result. First, note that

\[
v \left( \mathbb{B} \left( s + \sqrt{d} \right) \right) = v(\mathbb{B}(s)) \left(1 + \frac{\sqrt{d}}{s}\right)^d \leq e^{\frac{d\sqrt{2} \gamma}{s}} v(\mathbb{B}(s)).
\]

By \( s \geq 2 \frac{d\sqrt{2}}{\gamma} \) we have \( \frac{d\sqrt{2}}{s} \leq 1 \) and \( \frac{2d\sqrt{2}}{s} \leq \gamma \). Thus, we get

\[
e^{\frac{d\sqrt{2}}{s}} \leq 1 + \frac{2d\sqrt{2}}{s} \leq 1 + \gamma
\]

which concludes the proof.

Before we use this lemma to bound the vertex degrees in \( G_{X_p} \), we need to introduce some additional notation. Let \((\mathbb{V}, R, \lambda)\) be a hard-constraint point process with \( q \in \mathbb{N}_{\geq 1} \) particle types and \( \mathbb{V} = [0, \ell)^d \) for some \( \ell \in \mathbb{R}_{>0} \). For any feasible resolution \( \rho \in \mathbb{R}_{>0} \) and any vertex \( v_x^{(i)} \in V_{X_p} \), we write \( \Delta_{X_p} \left( v_x^{(i)} \right) \) for the number of neighbors of \( v_x^{(i)} \) in \( G_{X_p} \). Further, for any \( j \in [q] \), we write \( \Delta_{X_p} \left( v_x^{(i)} \right) \) to denote the number of neighbors of \( v_x^{(i)} \) that are from \( V^{(j)}_{X_p} \). The following lemma gives a bound on the vertex degrees in the canonical discretization in terms of the resolution \( \rho \) and the volume exclusion matrix \( B \) of the model.

**Lemma 5.5.** Let \((\mathbb{V}, R, \lambda)\) be a hard-constraint point process with \( q \in \mathbb{N}_{\geq 1} \) particle types and \( \mathbb{V} = [0, \ell)^d \) for some \( \ell \in \mathbb{R}_{>0} \). Let \( B \) be the corresponding volume exclusion matrix. Set \( R_{\min} = \inf_{i,j \in [q]} \{ R(i, j) \mid R(i, j) > 0 \} \) and let \( \gamma \in (0, 1] \). For all feasible resolutions \( \rho \geq 2 \frac{d\sqrt{2}}{\gamma R_{\min}} \) we have for all \( i, j \in [q] \) and \( x \in X_p \) that

\[
\Delta_{X_p}(v_x^{(i)}) \leq (1 + \gamma) \rho^d B(i, j)
\]

and consequently

\[
\Delta_{X_p}(v_x^{(i)}) \leq (1 + \gamma) \rho^d \sum_{j \in [q]} B(i, j).
\]

Proof. First, note that the bound on \( \Delta_{X_p}(v_x^{(i)}) \) follows immediately from the bound on \( \Delta_{X_p}(v_x^{(i)}) \) as the vertex sets \( V^{(j)}_{X_p} \) for \( j \in [q] \) are a partitioning of \( V_{X_p} \).

We proceed by bounding \( \Delta_{X_p}(v_x^{(i)}) \). Let \( y \in X_p \) and observe that there is only an edge between \( v_x^{(i)} \) and \( v_y^{(j)} \) in \( G_{X_p} \) if \( d(x, y) < R(i, j) \). Further, note that for every fixed \( i, j \in [q] \) and every \( x \in X_p \), there are at
most \( b_d(\rho R(i, j)) \) points \( y \in X_\rho \) such that \( d(x, y) < R(i, j) \). This is
\[
\Delta_{X_\rho}^{(j)}(v_x^{(l)}) \leq b_d(\rho R(i, j)).
\]

Further, note that for \( \rho \geq \frac{2d^{1/2}}{\sqrt{R_{\min}}} \) we have \( \rho R(i, j) \geq \frac{2d^{1/2}}{\sqrt{y}} \). Thus, using Lemma 5.4, we get
\[
\Delta_{X_\rho}^{(j)}(v_x^{(l)}) \leq (1 + y) v(B(\rho R(i, j))) = (1 + y) \rho^d B(i, j),
\]
which proves the claim.

\section{5.1 Tightness of the error bound}

In what follows, we argue that the bound for the number of points, required for the canonical discretization, might be considered as asymptotically tight in the volume \( \nu \) and the discretization error \( \epsilon_D \). Recall that, by Theorem 5.2, we require a resolution of \( \rho \in \Theta(\epsilon_D^{-d} \nu^{-1/d}) \), resulting in \( |X_\rho| \in \Theta(\epsilon_D^{-d} \nu^{-1}) \) points to ensure that
\[
e^{-\epsilon_D} Z(V, R, \lambda) \leq Z_{HC}(G_{X_\rho}, \lambda_{X_\rho}) \leq e^{\epsilon_D} Z(V, R, \lambda).
\]

To argue that this is in general optimal, we construct a simple case in which we actually require \( \Omega(\epsilon_D^{-d} \nu^{-1/d}) \). Consider a hard-constrain point process \( (V, R_0, \lambda) \) with \( q = 1 \) particle type and \( R_0 \) being the \( 1 \times 1 \)-matrix with entry 0. That is, \( (V, R_0, \lambda) \) is a simple Poisson point process on \( V \) with intensity \( \lambda \) (we treat \( \lambda \) as a constant here and omit the function notation). The partition function of \( (V, R_0, \lambda) \) is then
\[
Z(V, R_0, \lambda) = e^{\lambda \nu(V)}.
\]

Note that for all \( n \in \mathbb{N}_{\geq 1} \) and all \( X \subset V \) with \( |X| = n \), it holds that
\[
Z_{HC}(G_X, \lambda_X) = (1 + \lambda_X)^{|X|} = \left(1 + \frac{\lambda \nu(V)}{n}\right)^n,
\]
which is independent of the specific set of points \( X \).

We proceed by lower-bounding the multiplicative error between these two partition functions. The following elementary bound helps us to do so.

\begin{lemma}
For all \( x, y \in \mathbb{R}_{>0} \) with \( y \geq x \), it holds that
\[
\left(1 + \frac{x}{y}\right)^y \leq e^{\frac{e^x - 1}{6y}} \cdot e^x.
\]
\end{lemma}

\begin{proof}
First note that
\[
\left(1 + \frac{x}{y}\right)^y = e^{y \ln(1 + \frac{x}{y})}.
\]
Further, as \( 0 \leq \frac{x}{y} \leq 1 \), we use the Taylor expansion at 1 and obtain
\[
\ln \left(1 + \frac{x}{y}\right) = \sum_{k \in \mathbb{N}_{\geq 1}} (-1)^{k+1} \frac{x^k}{ky^k} \leq \frac{x}{y} - \frac{x^2}{2y^2} + \frac{x^3}{3y^3} \leq \frac{x}{y} - \frac{x^2}{6y^2}.
\]
Thus, we conclude that
\[
\left(1 + \frac{x}{y}\right)^y \leq e^{y\left(\frac{x}{y} - \frac{x^2}{6y^2}\right)} = e^{-\frac{x^2}{6y}} \cdot e^x.
\]

The following statement is an implication of Lemma 5.6.

**Lemma 5.7.** Let \(x > 6\) and \(\varepsilon_D \in (0, 1]\). For all \(y \in \left(0, \frac{x^2}{6\varepsilon_D}\right)\), it holds that
\[
\left(1 + \frac{x}{y}\right)^y < e^{-\varepsilon_D} \cdot e^x.
\]

**Proof.** Let \(y_0 \in \left[x, \frac{x^2}{2\varepsilon_D}\right]\) and observe that by Lemma 5.6 we have
\[
\left(1 + \frac{x}{y_0}\right)^{y_0} \leq e^{-\frac{x^2}{6y_0}} \cdot e^x < e^{-\varepsilon_D} \cdot e^x.
\]

Further, let \(y \in (0, x)\) and note that \(y < y_0\). Thus, by Bernoullis inequality, we have
\[
1 + \frac{x}{y} \leq \left(1 + \frac{x}{y_0}\right)^{y_0}.
\]
It follows that
\[
\left(1 + \frac{x}{y}\right)^y \leq \left(1 + \frac{x}{y_0}\right)^{y_0} < e^{-\varepsilon_D} \cdot e^x,
\]
which proves the claim.

We obtain the following result.

**Proposition 5.8.** Let \((V, R_0, \lambda)\) be a hard-constraint point process with \(q = 1\) particle type and \(R_0\) being the \(1 \times 1\)-matrix with entry \(0\). Assume that \(v(V) > \frac{\lambda}{2}\). For all \(\varepsilon_D \in (0, 1]\) and all non-empty \(X \subset V\) with \(|X| = n < \frac{\lambda^2 v(V)^2}{6\varepsilon_D}\), it holds that
\[
Z_{HC}(G_X, \lambda_X) < e^{-\varepsilon_D} Z(V, R_0, \lambda),
\]
meaning that \(Z_{HC}(G_X, \lambda_X)\) is not an \(\varepsilon_D\)-approximation of \(Z(V, R_0, \lambda)\).

**Proof.** Recall that
\[
Z(V, R_0, \lambda) = e^{\lambda v(V)} \quad \text{and} \quad Z_{HC}(G_X, \lambda_X) = \left(1 + \frac{\lambda v(V)}{n}\right)^n.
\]
The claim follows from applying Lemma 5.7 with \(x = \lambda v(V) > 6\) and \(y = n \in \left(0, \frac{x^2}{6\varepsilon_D}\right)\).

We would like to add a few remarks about Proposition 5.8. Firstly, note that it applies to all measurable and bounded regions \(V\) with sufficiently large volume. Thus, it especially holds for cubic regions \(V = [0, \ell]^d\) as investigated in Theorem 5.2. Furthermore, the considered hard-constraint point process is the most simple one that can be constructed, namely a point process without any constraints. This means, Proposition 5.8 does not give a lower bound for the number of points, required for discretizing any given hard-constraint point process. However, it gives a lower bound for the general class of models, and intuition suggests that additional constraints should not decrease the number of points that are required. Finally, the result only
applies to our method of discretization. That is, it proves that our bounds for the number of points are tight for our discretization method, but it does not exclude the existence of more efficient discretizations.

6 Efficient approximation

In this section, we investigate sufficient conditions for approximating the partition function of hard-constraint point processes on cubic regions $V = [0, \ell]^d$ with $\ell \in \mathbb{R}_{>0}$, using the canonical discretization proposed in Section 5. The characterization of cases for which we give an efficient approximation of the partition function is based on the volume exclusion matrix $B$, defined in Section 3. We split this up into two main parts.

First, we investigate a condition that is especially useful in the setting of uniform fugacities (i.e., $\lambda$ is a constant function on $[q]$). In this setting, the canonical discretization results in a univariate hard-core model, which allows us to apply the variety of algorithmic tools that are known for this model.

In the second part, we focus on the case of non-uniform fugacities. Our algorithms in this setting are based on tools that were initially developed for abstract polymer models, which are multivariate hard-core models.

6.1 Uniform fugacity

Consider a hard-constraint point process $(V, R, \lambda)$ with $q \in \mathbb{N}_{\geq 1}$ particle types where the fugacity $\lambda$ is a constant function on $[q]$. To simplify notation, we use $\lambda$ to denote the value of that function and omit the function notation here.

Note that for any fixed finite non-empty set of points $X \subseteq V$ with corresponding hard-core representation $(G_X, \lambda_X)$ this implies that $\lambda_X(v^{(i)}_x) = \lambda \frac{v(V)}{|X|}$ for all $x \in X$ and $i \in [q]$. That is, $\lambda_X$ is a constant function on $V_X$ and $(G_X, \lambda_X)$ is a univariate hard-core model. For the sake of brevity, we write $\lambda_X = \lambda \frac{v(V)}{|X|}$ to denote that constant.

Based on Theorem 2.2 and Theorem 2.3 we can now obtain our main approximation result for the partition function of hard-constraint point processes with uniform fugacities.

Theorem 6.1. Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types, $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{>0}$, and let $\lambda$ be a constant. Let $B$ be the corresponding volume exclusion matrix and denote its $L_1$-norm by $\|B\|_1$. If

$$\lambda < \frac{e}{\|B\|_1},$$

then for all $\varepsilon_A \in (0, 1]$, there is a randomized $\varepsilon_A$-approximation algorithm for $Z(V, R, \lambda)$ with running time in $\text{poly} \left( \frac{v(V)}{\varepsilon_A} \right)$ and a quasi-polynomial deterministic approximation algorithm for $Z(V, R, \lambda)$ with running time $\left( \frac{v(V)}{\varepsilon_A} \right) \Theta(\ln(v(V)/\varepsilon_A))$.

Proof. Let

$$\lambda = (1 - \alpha) \frac{e}{\|B\|_1},$$

for some $\alpha \in (0, 1]$. Set $\gamma = \alpha' = \frac{\alpha}{2}$.
By Theorem 5.2 we know that we can choose a resolution $\rho \in \Theta\left(\varepsilon_A^{-1/d} \nu(V)^{-1/d}\right)$ such that the canonical discretization $(G_{X_\rho}, \lambda_{X_\rho})$ satisfies
\[
e^{-\frac{\varepsilon_A}{\rho^d}} Z(V, R, \lambda) \leq Z_{\text{HC}}(G_{X_\rho}, \lambda_{X_\rho}) \leq e^{\frac{\varepsilon_A}{\rho^d}} Z(V, R, \lambda).
\]

To approximate the hard-core partition function of $(G_{X_\rho}, \lambda_{X_\rho})$ we aim for applying Theorem 2.2 and Theorem 2.3. Thus, we need to bound the maximum degree of $G_{X_\rho}$ and compare $\lambda_{X_\rho}$ with the corresponding tree threshold.

Set $R_{\min} = \min_{i,j \in [q]} \{ R(i, j) \mid R(i, j) > 0 \}$ and note that, for $\nu(V)$ sufficiently large, we can assume that $\rho \geq \frac{2d^{1/2}}{\varepsilon_A R_{\min}}$. Thus, by Lemma 5.5 we have
\[
\Delta_{X_\rho}(v_X^{(i)}) \leq (1 + \gamma) \rho^d \sum_{j \in [q]} B(i, j).
\]

Let $\Delta^*_{X_\rho}$ denote the maximum degree in $G_{X_\rho}$ and observe that
\[
\Delta^*_{X_\rho} = \max_{x \in X_\rho, i \in [q]} \Delta_{X_\rho}(v_X^{(i)}) \leq (1 + \gamma) \rho^d \|B\|_1.
\]

Further, note that
\[
\lambda_{X_\rho} = \lambda \frac{\nu(V)}{|X_\rho|} = \lambda \frac{1}{\rho^d}
\]
and so we get
\[
\lambda_{X_\rho} = (1 - \alpha) \frac{e}{\rho^d \|B\|_1} \leq (1 - \alpha') \frac{e}{(1 + \gamma) \rho^d \|B\|_1} \leq (1 - \alpha') \frac{e}{\Delta^*_{X_\rho}}.
\]

Now, observe that $\lambda_c(\Delta)$ converges to $\frac{e}{\Delta}$ from above for large $\Delta$. Thus, we obtain
\[
\lambda_{X_\rho} \leq (1 - \alpha') \lambda_c(\Delta^*_{X_\rho}),
\]
where $\alpha' > 0$.

Using Observation 5.3, we know that
\[
|V_{X_\rho}| = q \rho^d \nu(V) \in \Theta\left(\frac{\nu(V)^2}{\varepsilon_A}\right).
\]

By Theorem 2.2 we can immediately conclude the existence of a randomized approximation within the desired running time bounds. Further, note that
\[
\Delta^*_{X_\rho} \leq (1 + \gamma) \rho^d \|B\|_1 \in \Theta\left(\frac{\nu(V)}{\varepsilon_A}\right).
\]

Thus, applying Theorem 2.3 we obtain the deterministic approximation, which concludes the proof. ■

We now demonstrate which bounds can be obtained based on Theorem 6.1 for models commonly studied in statistical physics. We start with the monoatomic hard-sphere model.
Corollary 6.2. Let $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{>0}$. Further, let $r \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{>0}$. Denote by $Z_{HS}(V, r, \lambda)$ the hard-sphere partition function on $V$ with particles of radius $r$ and fugacity $\lambda$. If
\[ \lambda < \frac{e}{2^d \nu(B(r))}, \]
then for all $\epsilon_A \in (0, 1]$, there is a randomized $\epsilon_A$-approximation algorithm for $Z_{HS}(V, r, \lambda)$ with running time in $\text{poly}\left(\frac{\nu(V)}{\epsilon_A}\right)$ and a quasi-polynomial deterministic $\epsilon_A$-approximation algorithm for $Z_{HS}(V, r, \lambda)$ with running time $\left(\frac{\nu(V)}{\epsilon_A}\right)^{\Theta(\ln(1/\epsilon_A))}$.

Corollary 6.2 follows from Theorem 6.1 by observing that the corresponding interaction matrix is a $1 \times 1$ matrix containing the value 1 (see Example 1.1) and that $\|B\|_1 = \nu(B(2r)) = 2^d \nu(B(r))$. Note that the bound is exactly the known region for uniqueness of the Gibbs measure for this model, proved by Michelen and Perkins [29]. A randomized algorithm for that regime was previously achieved by Friedrich et al. [18], using the same discretization technique. However, due to the fact that we improved the required number of points from super-exponential in $\nu(V)$ to quadratic, we can run Glauber dynamics in order to obtain the approximation instead of clique dynamics. This greatly simplifies the randomized algorithm. Further, we present the first efficient deterministic approximation. This partially answers an open question, posed by Michelen and Perkins [30].

Corollary 6.3. Let $V = [0, \ell]^d$, $\ell \in \mathbb{R}_{>0}$, let and $r \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{>0}$. Let $Z_{WR}(V, r, \lambda)$ denote the Widom–Rowlinson partition function on $V$ with $q \in \mathbb{N}_{\geq 1}$ particle types, each of radius $r$ and fugacity $\lambda$. If
\[ \lambda < \frac{e}{(q-1)2^d \nu(B(r))}, \]
then for all $\epsilon_A \in (0, 1]$, there is a randomized $\epsilon_A$-approximation algorithm for $Z_{WR}(V, r, \lambda)$ with running time in $\text{poly}\left(\frac{\nu(V)}{\epsilon_A}\right)$ and a quasi-polynomial deterministic $\epsilon_A$-approximation algorithm for $Z_{WR}(V, r, \lambda)$ with running time $\left(\frac{\nu(V)}{\epsilon_A}\right)^{\Theta(\ln(1/\epsilon_A))}$.

Corollary 6.3 follows from Theorem 6.1 by observing that the interaction matrix is a $q \times q$ matrix, which is 0 on the diagonal and 1 otherwise (see Example 1.2). To the best of our knowledge, no rigorous computational result for this model has been known before.

Remark 6.4. In fact, based on Remark 2.4 we can apply Theorem 6.1 also in the non-uniform setting when the condition is replaced by
\[ \lambda_{\text{max}} < \frac{e}{\|B\|_1}, \]
where $\lambda_{\text{max}} = \max_{i \in [q]} \lambda(i)$.

6.2 Non-uniform fugacity

We proceed by looking at which conditions for efficient approximation we can get for hard-constraint point processes with non-uniform fugacities based on Theorem 2.6.

Theorem 6.5. Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types and $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{>0}$. Further, let $B$ be the corresponding volume exclusion matrix. If there is a function
\( f: [q] \to \mathbb{R}_{>0} \) such that for all \( i \in [q] \), it holds that

\[
f(i) > \sum_{j \in [q]} B(i, j) f(j) \lambda(j),
\]

then for all \( \varepsilon_A \in (0, 1] \), there is a randomized \( \varepsilon_A \)-approximation algorithm for \( Z(\mathcal{V}, R, \lambda) \) with running time in \( \text{poly}(\frac{\nu(\mathcal{V})}{\varepsilon_A}, \ln(\frac{\max_{i \in [q]} f(i)}{\min_{i \in [q]} f(i)}) \).

**Proof.** We prove the theorem by applying Theorem 2.6 to the canonical discretization \( (G_{X, \rho}, \lambda_{X, \rho}) \) with a sufficiently large resolution \( \rho \). To this end, we construct a function \( g: V_{X, \rho} \to \mathbb{R}_{>0} \) that satisfies the condition of Theorem 2.6.

Assume that for all \( i \in [q] \) we have

\[
f(i) \geq \alpha_i \sum_{j \in [q]} B(i, j) f(j) \lambda(j)
\]

for some \( \alpha_i > 1 \) and set \( \alpha = \min_{i \in [q]} \alpha_i \). By Theorem 5.2, we know that we can choose a resolution in \( \Theta\left(\varepsilon_A^{-1/d} \nu(\mathcal{V})^{1/d}\right) \) such that

\[
e^{-\frac{\varepsilon_A}{2}} Z(\mathcal{V}, R, \lambda) \leq Z_{HC}(G_{X, \rho}, \lambda_{X, \rho}) \leq e^{\frac{\varepsilon_A}{2}} Z(\mathcal{V}, R, \lambda).
\]

Set \( R_{\min} = \min_{i \in [q]} \{ R(i, j) \mid R(i, j) > 0 \} \) and note that, for \( \nu(\mathcal{V}) \) sufficiently large, it holds that \( \rho \geq \frac{2^{d/2}}{(\alpha-1) R_{\min}} \). Thus, by Lemma 5.5, we know that for all \( x \in X_{\rho} \) and \( i, j \in [q] \) it holds that

\[
\Delta^{(j)}_{X_{\rho}}(v^{(i)}_x) \leq \alpha \rho^d B(i, j).
\]

Define a function \( g: V_{X, \rho} \to \mathbb{R}_{>0} \) with \( g(v^{(i)}_x) = f(i) \) for all \( i \in [q] \) and all \( x \in X_{\rho} \). For all \( i \in [q] \) and all \( x \in X_{\rho} \), we have

\[
\sum_{w \in N(v^{(i)}_x)} g(w) \frac{\lambda_{X_{\rho}}(w)}{1 + \lambda_{X_{\rho}}(w)} \leq \sum_{j \in [q]} \Delta^{(j)}_{X_{\rho}}(v^{(i)}_x) f(j) \lambda(j) \rho^{-d} \]

\[
\leq \alpha \sum_{j \in [q]} B(i, j) f(j) \lambda(j) \]

\[
\leq f(i) \]

\[
\leq g(v^{(i)}_x).
\]

Thus, we can apply Theorem 2.6 to obtain the desired approximation.

In the case of uniform fugacities, the approximation result obtained from Theorem 6.5 is worse than the one obtained from Theorem 6.1. However, in the non-uniform case, Theorem 6.5 can lead to interesting parameter regimes that are not covered by Theorem 6.1. To demonstrate that, we look into a Widom–Rowlinson model with two particle types of identical radii but different fugacities.
Corollary 6.6. Let $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{> 0}$. Further, let $r \in \mathbb{R}_{> 0}$, $\lambda_1, \lambda_2 \in \mathbb{R}_{> 0}$. Denote by $Z_{WR}(V, r, \lambda_1, \lambda_2)$ the Widom–Rowlinson partition function on $V$ with 2 particle types, both with the same radius $r$ but (possibly) different fugacities $\lambda_1, \lambda_2$. If

$$\lambda_1 \lambda_2 < \frac{1}{4d v(B(r))^2},$$

then for all $\varepsilon_A \in (0, 1]$, there is a randomized $\varepsilon_A$-approximation algorithm for $Z_{WR}(V, r, \lambda_1, \lambda_2)$ with running time in $\text{poly}(\frac{v(V)}{\varepsilon_A})$. □

Proof. Assume

$$\lambda_1 \lambda_2 \leq (1 - \alpha) \frac{1}{4d v(B(r))^2}$$

for some $\alpha \in (0, \frac{1}{2}]$. We construct the equivalent hard-constraint point process $(V, R, \lambda)$ with $q = 2$ particle types, where $H$ is a $2 \times 2$ matrix with 0 on the diagonal and 1 for all off-diagonal entries and $\lambda(i) = \lambda_i$ for all $i \in \{1, 2\}$. Note that inequality (15), $2^d v(B(r)) = v(B(2r))$, and the definition of $B$ imply that

$$B(2, 1) \lambda(1) B(2, 2) \lambda(2) \leq 1 - \alpha.$$

Set $f(1) = 1$ and $f(2) = (1 + \beta) B(2, 1) \lambda(1)$ for some $\beta \in (0, \frac{\alpha}{1 - \alpha})$. Observe that

$$B(2, 1) f(1) \lambda(1) = B(2, 1) \lambda(1) < f(2)$$

and

$$B(2, 1) f(2) \lambda(2) = (1 + \beta) B(2, 1) \lambda(1) B(1, 2) \lambda(2) \leq (1 + \beta)(1 - \alpha) < f(1).$$

Applying Theorem 6.5 concludes the proof. □

In the balanced case $\lambda_1 = \lambda_2 = \lambda$, Corollary 6.6 gives a bound on $\lambda$ that is worse by a factor of $e$ compared to what we can get from Theorem 6.1 and Remark 6.4 (compare Corollary 6.3). However, by using Theorem 6.5 we see that we can make one of the two fugacities much larger, as long as we scale down the other one appropriately. For example, according to Corollary 6.6 we can get a polynomial time randomized approximation for $\lambda_1 = e^{\frac{e^\varepsilon}{2d v(B(r))}}$ and $\lambda_2 < \frac{1}{e^{2e^\varepsilon d v(B(r))}}$, which is obviously not possible using Theorem 6.1 and Corollary 6.3. This leads to an efficient approximation as long as the fugacities are sufficiently unbalanced.

7 Sampling via random perturbations

So far, we only presented approximation results for the partition function $Z(V, R, \lambda)$ of a hard-constraint point process $(V, R, \lambda)$. As we briefly discussed in the introduction, when it comes to sampling from the Gibbs distribution $\mu(V, R, \lambda)$, a natural barrier is that outputting a sample (i.e., a tuple of points $x \in V^k$ and a type assignment $\tau: [k] \rightarrow [q]$ that represents a valid configuration) would, in fact, require infinite floating-point precision. If we allow ourselves to use a computational model that can handle floating-point values with arbitrary precision, we can actually use our hard-core representation $(G_X, \lambda_X)$ to recover an
approximate sampler for $\mu^{(V,R,\lambda)}$, given that we know a suitable allocation $\Phi$ for $X$. This is captured by the following assumption, required for the algorithmic results of this section.

► **Assumption 7.1.** The computational model can, in constant time,

- output floating-point numbers with arbitrary precision,
- perform basic arithmetic operations (addition, subtraction, ...) on floating-point numbers with arbitrary precision,
- compare floating-point numbers with arbitrary precision, and
- produce, for all $a, b \in \mathbb{R}$ with $a < b$, a uniformly random floating-point number from the intervals $[a, b]$, $[a, b)$, $(a, b]$, and $(a, b)$.

This assumption is only required to give meaningful theoretical error bounds in terms of total-variation distance, which are impossible to achieve otherwise.

We proceed by describing our sampling procedure for a given hard-constraint point process $(V, R, \lambda)$. Let $X \subset V$ be finite and non-empty and let $\Phi$ be an allocation for $X$. From Section 4, recall the definition of the intermediate continuous model, defined via the weight function $w_{\Phi}^{(G_X, \lambda_X)}$ as well as the corresponding Gibbs density $\mu_{\Phi}^{(V, G_X, \lambda_X)}$ and partition function $Z_{\Phi}(V, G_X, \lambda_X)$. Instead of directly sampling from $Z(V, R, \lambda)$, we approximately sample from $\mu_{\Phi}^{(V, G_X, \lambda_X)}$. To this end, we first bound the total-variation distance between these two distributions. The following lemma helps us use Lemma 4.4 for this purpose.

► **Lemma 7.2.** Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types. Further, let $X \subset V$ be finite and non-empty and, for $\delta \in [0, 1)$ and $\varepsilon \in \mathbb{R}_{>0}$, let $\Phi$ be a $\delta$-$\varepsilon$-allocation for $X$. It holds that

$$d_{TV}(\mu^{(V,R,\lambda)}, \mu_{\Phi}^{(V,G_X,\lambda_X)}) \leq \frac{1}{Z(V, R, \lambda)} \sum_{k \in \mathbb{N}} \sum_{[k] \rightarrow [q]} \int_{V^k} \left| w^{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X,\lambda_X)}(x, \tau) \right| d\nu^{d\times k}.$$

**Proof.** First, note that the total-variation distance can be written as

$$d_{TV}(\mu^{(V,R,\lambda)}, \mu_{\Phi}^{(V,G_X,\lambda_X)}) = \frac{1}{2} \sum_{k \in \mathbb{N}} \sum_{[k] \rightarrow [q]} \int_{V^k} \left| \mu^{(V,R,\lambda)}(x, \tau) - \mu_{\Phi}^{(V,G_X,\lambda_X)}(x, \tau) \right| d\nu^{d\times k}. \quad (16)$$

For all $x \in V^k$ and $\tau : [k] \rightarrow [q]$, observe that

$$\left| \mu^{(V,R,\lambda)}(x, \tau) - \mu_{\Phi}^{(V,G_X,\lambda_X)}(x, \tau) \right| = \frac{w^{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X,\lambda_X)}(x, \tau)}{Z(V, R, \lambda)}.$$

We proceed by describing our sampling procedure for a given hard-constraint point process $(V, R, \lambda)$. Let $X \subset V$ be finite and non-empty and let $\Phi$ be an allocation for $X$. From Section 4, recall the definition of the intermediate continuous model, defined via the weight function $w_{\Phi}^{(G_X, \lambda_X)}$ as well as the corresponding Gibbs density $\mu_{\Phi}^{(V, G_X, \lambda_X)}$ and partition function $Z_{\Phi}(V, G_X, \lambda_X)$. Instead of directly sampling from $Z(V, R, \lambda)$, we approximately sample from $\mu_{\Phi}^{(V, G_X, \lambda_X)}$. To this end, we first bound the total-variation distance between these two distributions. The following lemma helps us use Lemma 4.4 for this purpose.

► **Lemma 7.2.** Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types. Further, let $X \subset V$ be finite and non-empty and, for $\delta \in [0, 1)$ and $\varepsilon \in \mathbb{R}_{>0}$, let $\Phi$ be a $\delta$-$\varepsilon$-allocation for $X$. It holds that

$$d_{TV}(\mu^{(V,R,\lambda)}, \mu_{\Phi}^{(V,G_X,\lambda_X)}) \leq \frac{1}{Z(V, R, \lambda)} \sum_{k \in \mathbb{N}} \sum_{[k] \rightarrow [q]} \int_{V^k} \left| w^{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X,\lambda_X)}(x, \tau) \right| d\nu^{d\times k}.$$

**Proof.** First, note that the total-variation distance can be written as

$$d_{TV}(\mu^{(V,R,\lambda)}, \mu_{\Phi}^{(V,G_X,\lambda_X)}) = \frac{1}{2} \sum_{k \in \mathbb{N}} \sum_{[k] \rightarrow [q]} \int_{V^k} \left| \mu^{(V,R,\lambda)}(x, \tau) - \mu_{\Phi}^{(V,G_X,\lambda_X)}(x, \tau) \right| d\nu^{d\times k}. \quad (16)$$

For all $x \in V^k$ and $\tau : [k] \rightarrow [q]$, observe that

$$\left| \mu^{(V,R,\lambda)}(x, \tau) - \mu_{\Phi}^{(V,G_X,\lambda_X)}(x, \tau) \right| = \frac{w^{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X,\lambda_X)}(x, \tau)}{Z(V, R, \lambda)}.$$

### Proof

Let $\delta \in [0, 1)$ and $\varepsilon \in \mathbb{R}_{>0}$, let $\Phi$ be a $\delta$-$\varepsilon$-allocation for $X$. It holds that

$$d_{TV}(\mu^{(V,R,\lambda)}, \mu_{\Phi}^{(V,G_X,\lambda_X)}) \leq \frac{1}{Z(V, R, \lambda)} \sum_{k \in \mathbb{N}} \sum_{[k] \rightarrow [q]} \int_{V^k} \left| w^{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X,\lambda_X)}(x, \tau) \right| d\nu^{d\times k}.$$

**Proof.** First, note that the total-variation distance can be written as

$$d_{TV}(\mu^{(V,R,\lambda)}, \mu_{\Phi}^{(V,G_X,\lambda_X)}) = \frac{1}{2} \sum_{k \in \mathbb{N}} \sum_{[k] \rightarrow [q]} \int_{V^k} \left| \mu^{(V,R,\lambda)}(x, \tau) - \mu_{\Phi}^{(V,G_X,\lambda_X)}(x, \tau) \right| d\nu^{d\times k}. \quad (16)$$

For all $x \in V^k$ and $\tau : [k] \rightarrow [q]$, observe that

$$\left| \mu^{(V,R,\lambda)}(x, \tau) - \mu_{\Phi}^{(V,G_X,\lambda_X)}(x, \tau) \right| = \frac{w^{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X,\lambda_X)}(x, \tau)}{Z(V, R, \lambda)}.$$
where the last inequality follows from the triangle inequality. Substituting into equation (16), we have

\[
\begin{align*}
\frac{d_{TV}(\mu_{(V,R,\lambda)}^{\mu_{(V,G_X,\lambda_X)}}, \mu_{\Phi}^{(V,G_X,\lambda_X)})}{2Z(V, R, \lambda)} &\leq \frac{1}{2Z(V, R, \lambda)} \left( \sum_{k \in \mathbb{N}} \sum_{\tau : [k] \to [q]} \int_{V^k} \left| \omega_{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X,\lambda_X)}(x, \tau) \right| d\nu^{\delta \lambda k} \\
+ |Z(V, R, \lambda) - Z_{\Phi}(V, G_X, \lambda_X)| \right) \\
&\leq \frac{1}{Z(V, R, \lambda)} \sum_{k \in \mathbb{N}} \sum_{\tau : [k] \to [q]} \int_{V^k} \left| \omega_{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X,\lambda_X)}(x, \tau) \right| d\nu^{\delta \lambda k},
\end{align*}
\]

where the last inequality comes from the fact that

\[
|Z(V, R, \lambda) - Z_{\Phi}(V, G_X, \lambda_X)| \leq \sum_{k \in \mathbb{N}} \sum_{\tau : [k] \to [q]} \int_{V^k} \left| \omega_{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X,\lambda_X)}(x, \tau) \right| d\nu^{\delta \lambda k}.
\]

Note that under the conditions of Lemma 4.4, Lemma 7.2 immediately gives a bound for the total-variation distance between \(\mu_{(V,R,\lambda)}\) and \(\mu_{\Phi}^{(V,G_X,\lambda_X)}\).

In order to use \(\mu_{\Phi}^{(V,G_X,\lambda_X)}\) as an approximation for \(\mu_{(V,R,\lambda)}\), we use Algorithm 1. This algorithm adds random perturbations to a sample from the discretized distribution \(\mu_{(G_X,\lambda_X)}^{(G_X,\lambda_X)}\), based on \(\Phi^{-1}\). Applying Algorithm 1 shows a connection between \(\mu_{HC}^{(G_X,\lambda_X)}\) and \(\mu_{\Phi}^{(V,G_X,\lambda_X)}\).

**Observation 7.3.** Let \((V, R, \lambda)\) be a hard-constraint point process with \(q \in \mathbb{N}_{\geq 1}\) particle types. Further, let \(X \subset V\) be finite and non-empty and, for \(\delta \in [0, 1)\) and \(\epsilon \in \mathbb{R}_{>0}\), let \(\Phi\) be a \(\delta\)-\(\epsilon\)-allocation for \(X\). Let \((G_X, \lambda_X)\) be the hard-core representation of \((V, R, \lambda)\) based on \(X\), and let \(I \in \mathcal{I}(G_X)\) be drawn according to \(\mu_{HC}^{(G_X,\lambda_X)}\). Consider a tuple of points \(x \in V^k\) and a type assignment \(\tau : [k] \to [q]\) drawn uniformly at random. Applying Algorithm 1 to \(I\). Then \((x, \tau)\) is distributed according to the density \(\mu_{\Phi}^{(V,G_X,\lambda_X)}\).

The observation follows directly from writing down the resulting density of \((x, \tau)\).
In practice, we might not have the ability to exactly sample an independent set of $G_X$ according to $\mu_{HC}^{G_X, \lambda_X}$. The following lemma extends Observation 7.3 to the setting where we have access to an approximate sampler for $\mu_{HC}^{G_X, \lambda_X}$.

**Lemma 7.4.** Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{>1}$ particle types. Further, let $X \subset V$ be finite and non-empty and, for $\delta \in [0, 1)$ and $\epsilon \in \mathbb{R}_{>0}$, let $\Phi$ be a $\delta$-$\epsilon$-allocation for $X$. Let $(G_X, \lambda_X)$ be the hard-core representation of $(V, R, \lambda)$ based on $X$. For $\epsilon_S \in [0, 1]$, let $I \in I(G_X)$ be drawn according to an $\epsilon_S$-approximation of $\mu_{HC}^{G_X, \lambda_X}$. Consider a tuple of points $x \in \mathbb{V}^k$ and a type assignment $\tau : [k] \to [q]$ produced by applying Algorithm 1 to $I$. Then $(x, \tau)$ is distributed according to an $\epsilon_S$-approximation of $\mu_{HC}^{G_X, \lambda_X}$.

**Proof.** Let $\tilde{\mu}_{HC}^{G_X, \lambda_X}$ be the given $\epsilon_S$-approximation for $\mu_{HC}^{G_X, \lambda_X}$, and let $\tilde{\mu}_{\Phi}^{G_X, \lambda_X}$ be the resulting distribution over configuration $(x, \tau)$. We prove our claim by constructing a coupling of $\mu_{\Phi}^{G_X, \lambda_X}$ and $\tilde{\mu}_{\Phi}^{G_X, \lambda_X}$ such that they produce the same $(x, \tau)$ with probability at least $1 - \epsilon_S$. This results in the desired bound on the total-variation distance by the coupling lemma (see [13, Theorem 2.4]).

To construct this coupling of $\mu_{\Phi}^{G_X, \lambda_X}$ and $\tilde{\mu}_{\Phi}^{G_X, \lambda_X}$, we start by considering an optimal coupling between $\tilde{\mu}_{HC}^{G_X, \lambda_X}$ and $\tilde{\mu}_{HC}^{G_X, \lambda_X}$ (see [13, Theorem 2.12]). For a tuple $(I, I') \in I(G_X) \times I(G_X)$ drawn from that optimal coupling, it holds that $I = I'$ with probability $1 - \epsilon_S$. If this is the case, we produce $(x, \tau)$ by applying Algorithm 1 to $I = I'$, and we return the tuple of configurations $((x, \tau), (x, \tau))$. However, if we draw $I \neq I'$, then we produce corresponding configurations $(x, \tau)$ by applying Algorithm 1 to $I$, and $(x', \tau')$ by applying Algorithm 1 to $I'$, and return $((x, \tau), (x', \tau'))$.

Note that the description above gives a valid coupling of $\mu_{\Phi}^{G_X, \lambda_X}$ and $\tilde{\mu}_{\Phi}^{G_X, \lambda_X}$, as it preserves the marginal distributions. Further, note that for a tuple $((x, \tau), (x', \tau'))$ produced by this coupling, it holds that $(x, \tau) = (x', \tau')$ if $I = I'$ in the first step. This happens with probability at least $1 - \epsilon_S$, which proves the desired bound on the total-variation distance.

To state some explicit approximate sampling results while keeping our statements more simple, we restrict ourselves once again to cubic regions $\mathbb{V} = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{>0}$. First, we derive a bound on the resolution $\rho$ that we need for the canonical discretization to obtain a desired total-variation distance between $\mu_{HC}^{G_X, \lambda_X}$ and $\mu_{\Phi}^{G_X, \lambda_X}$.

**Lemma 7.5.** Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{>1}$ particle types and $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{>0}$. For all $\epsilon_S \in (0, 1]$, there exists $\rho_{\epsilon_S} \in \Theta(\epsilon_S^{-1/d} v(V)^{1/d})$ such that for all feasible resolutions $\rho \geq \rho_{\epsilon_S}$, there is an allocation $\Phi$ for $X_\rho$ such that

$$d_{TV}\left(\mu^{G_X, \lambda_X}, \mu_{\Phi}^{G_X, \lambda_X}\right) \leq \epsilon_S.$$
and observe that $\rho_{\varepsilon}$ satisfies the asymptotic assumptions of the statement.

For all feasible resolutions $\rho \geq \rho_{\varepsilon}$, let $\Phi$ be an allocation for the canonical point set $X_\rho$ as constructed in the proof of Lemma 5.1. This especially means that $\Phi$ is a $\delta$-$\varepsilon$-allocation for $X_\rho$ with $\delta = 0$ and

$$
\varepsilon = \sqrt[d]{\frac{d}{\rho}} \leq \left( \frac{\varepsilon_S}{32q \max\{\lambda_{\max}, \lambda_{\max}^2\} v(\mathbb{V})} \right)^{\frac{1}{d}} \cdot \min\left\{ 1, \frac{R_{\min}}{4} \right\}.
$$

We prove our claim, by combining Lemmas 7.2 and 4.4 based on the canonical point set $X_\rho$ and the $\delta$-$\varepsilon$-allocation $\Phi$. First, we check that $\delta$, $\varepsilon$, and $X_\rho$ satisfy the conditions of Theorem 4.6.

It trivially holds that $\delta = 0$ and $\varepsilon = \frac{p_\rho}{d} \lambda_{\max} \frac{q_{\max}}{\max(\lambda_{\max}, \lambda_{\max}^2)}$. Moreover, we have $\varepsilon \geq \frac{4}{R_{\min}}$ for $\lambda_{\max}$ sufficiently large.

Observe that $X_\rho = \rho^d \ell^d \geq d^d \frac{32q \max\{\lambda_{\max}, \lambda_{\max}^2\} v(\mathbb{V})^2}{\varepsilon_S} \cdot \max\left\{ 1, \left( \frac{4}{R_{\min}} \right)^d \right\}$.

Thus, we have $|X_\rho| \geq 4\lambda_{\max} v(\mathbb{V})$.

We proceed by upper-bounding

$$
\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{\mathbb{V}^k} \left| w^{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X, \lambda_X)}(x, \tau) \right| d\nu^{d \times k}
$$

via Theorem 4.6. Recall that we have $\delta = 0$. Further, note that

$$
\varepsilon \leq \left( \frac{\varepsilon_S}{32q \max\{\lambda_{\max}, \lambda_{\max}^2\} v(\mathbb{V})} \right)^{\frac{1}{d}} \cdot \min\left\{ 1, \frac{R_{\min}}{4} \right\} \leq \left( \frac{\varepsilon_S}{4 \sum_{i \in [q]} \lambda(i) v(\mathbb{V})} \right)^{\frac{1}{d}} R_{\min} \frac{1}{4}
$$

and thus

$$
\left( \frac{4\varepsilon}{R_{\min}} \right)^d \sum_{i \in [q]} \lambda(i) v(\mathbb{V}) \leq \frac{\varepsilon_S}{4}.
$$

Moreover, for

$$
|X_\rho| \geq d^d \frac{32q \max\{\lambda_{\max}, \lambda_{\max}^2\} v(\mathbb{V})^2}{\varepsilon_S} \cdot \max\left\{ 1, \left( \frac{4}{R_{\min}} \right)^d \right\} \geq \frac{32 \sum_{i \in [q]} \lambda(i)^2 v(\mathbb{V})^2}{\varepsilon_S},
$$

it holds that

$$
\frac{8}{|X_\rho|} \sum_{i \in [q]} \lambda(i)^2 v(\mathbb{V})^2 \leq \frac{\varepsilon_S}{4}.
$$

By Theorem 4.6, we obtain

$$
\sum_{k \in \mathbb{N}} \sum_{\tau: [k] \rightarrow [q]} \int_{\mathbb{V}^k} \left| w^{(R,\lambda)}(x, \tau) - w_{\Phi}^{(G_X, \lambda_X)}(x, \tau) \right| d\nu^{d \times k} \leq \left( e^{\varepsilon_S} - 1 \right) Z(\mathbb{V}, R, \lambda),
$$

and, by Lemma 7.2, this implies $d_{TV}(\mu^{(\mathbb{V}, R, \lambda)}, \mu_{\Phi}^{(\mathbb{V}, G_X, \lambda_X)}) \leq e^{\varepsilon_S} - 1.$
Finally, observe that, for $\varepsilon_S \in (0, 1]$, it holds that $e^{\varepsilon_S} - 1 \leq \varepsilon_S$, which concludes the proof.

We now state the sampling analogue of Theorem 6.1 and Theorem 6.5. We start with the case of uniform fugacities.

**Theorem 7.6.** Suppose a computational model that satisfies Assumption 7.1. Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types, $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{>0}$, and let $\lambda$ be a constant. Let $B$ be the corresponding volume exclusion matrix and denote its $L_1$-norm by $\|B\|_1$. If

$$\lambda < \frac{e}{\|B\|_1},$$

then for all $\varepsilon_S \in (0, 1]$, there is an $\varepsilon_S$-approximate sampler for $\mu_{(V,R,\lambda)}$ with running time in poly($\frac{v(V)}{\varepsilon_S}$).

**Proof.** The rough proof sketch is to choose the resolution $\rho$ and the canonical point set $X_\rho$ with allocation $\Phi$ that such $\mu_{\Phi(V,G_{X_\rho,\lambda X_\rho})}$ is an $\varepsilon_S$-approximation of $\mu_{(V,R,\lambda)}$. We then sample $\varepsilon_S$-approximately from $\mu_{HC}(V,G_{X_\rho,\lambda X_\rho})$ to obtain an $\varepsilon_S$-approximation of $\mu_{\Phi(V,G_{X_\rho,\lambda X_\rho})}$. Obviously, the resulting error is then bounded by $\frac{\varepsilon_S}{2} + \frac{\varepsilon_S}{2} = \varepsilon_S$.

By Lemma 7.5, there is a resolution $\rho \in \Theta\left(\frac{1}{\varepsilon_S v(V)^{1/d}}\right)$ such that

$$d_{TV}\left(\mu_{(V,R,\lambda)}, \mu_{\Phi(V,G_{X_\rho,\lambda X_\rho})}\right) \leq \frac{\varepsilon_S}{2},$$

where, as described in Lemma 5.1,

$$\Phi(x) = \prod_{i \in [d]} \left[x^{(i)} + x^{(i)} + \frac{1}{\rho}\right].$$

Note that the corresponding graph $G_{X_\rho}$ has $\Theta\left(\frac{v(V)}{\varepsilon_S}\right)$ vertices.

Further, analogously to the proof of Theorem 6.1, if $\lambda < \frac{e}{\|B\|_1}$, constructing the hard-core representation $(G_{X_\rho}, \lambda X_\rho)$ can be done in such a way that $\lambda X_\rho < \lambda \left(\Delta_{X_\rho}\right)$. Thus, by Theorem 2.1, we can sample $\frac{\varepsilon_S}{2}$-approximately from $\mu_{HC}(G_{X_\rho,\lambda X_\rho})$ in time poly($\frac{v(V)}{\varepsilon_S}$), which is also in poly($\frac{v(V)}{\varepsilon_S}$).

We now obtain our sampler by drawing an independent set $I \in I(G_{X_\rho})$ according to this $\frac{\varepsilon_S}{2}$-approximation of $\mu_{HC}(G_{X_\rho,\lambda X_\rho})$ and applying Algorithm 1 to it. To this end, note that for all $y \in X_\rho$, a uniform sample $x \in \Phi^{-1}y$ is obtained by generating, for each $i \in [d]$, a uniformly random floating-point number $z^{(i)} \in \left[0, \frac{1}{\rho}\right)$ and setting $x = y + (z^{(i)})_i \in [d]$, where the addition is component-wise. Lemma 7.4 yields the desired bound on the total-variation distance.

The sampling analogues of Corollaries 6.2 and 6.3 follow immediately.

**Corollary 7.7.** Suppose a computational model that satisfies Assumption 7.1. Let $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{>0}$. Further, let $r \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{\geq 0}$. Denote by $\mu_{HC}(V,R,\lambda)$ the Gibbs distribution of the hard-sphere
model on \( V \) with particles of radius \( r \) and fugacity \( \lambda \). If
\[
\lambda < \frac{e}{2^d \nu(B(r))},
\]
then, for all \( \varepsilon_S \in (0, 1] \), there is an \( \varepsilon_S \)-approximate sampler for \( \mu_{\text{HS}}^{(V, r, \lambda)} \) with running time in \( \text{poly}\left( \frac{\nu(V)}{\varepsilon_S} \right) \).

**Corollary 7.8.** Suppose a computational model that satisfies Assumption 7.1. Let \( V = [0, \ell]^d \) for some \( \ell \in \mathbb{R}_{>0} \). Further, let \( r \in \mathbb{R}_{>0}, \lambda \in \mathbb{R}_{\geq 1} \) and \( q \in \mathbb{N}_{\geq 1} \). Denote by \( \mu_{\text{WR}}^{(V, r, \lambda)} \) the Gibbs distribution of the Widom–Rowlinson model on \( V \) with \( q \) particle types, each of radius \( r \) and fugacity \( \lambda \). If
\[
\lambda < \frac{e}{(q-1)2^d \nu(B(r))},
\]
then, for all \( \varepsilon_S \in (0, 1] \), there is an \( \varepsilon_S \)-approximate sampler for \( \mu_{\text{WR}}^{(V, r, \lambda)} \) with running time in \( \text{poly}\left( \frac{\nu(V)}{\varepsilon_S} \right) \).

**Remark 7.9.** Analogously to Theorem 6.1, letting \( \lambda_{\text{max}} = \max_{i \in [q]} \lambda(i) \), Theorem 7.6 extends to the non-uniform setting when the condition is replaced by
\[
\lambda_{\text{max}} < \frac{e}{\|B\|_1}.
\]

Just as for Theorem 6.1, we also state a sampling version of Theorem 6.5 under Assumption 7.1.

**Theorem 7.10.** Suppose a computational model that satisfies Assumption 7.1. Let \((V, R, \lambda)\) be a hard-constraint point process with \( q \in \mathbb{N}_{\geq 1} \) particle types and \( V = [0, \ell]^d, \ell \in \mathbb{R}_{>0} \). Further, let \( B \) be the corresponding volume exclusion matrix. If there is a function \( f: [q] \to \mathbb{R}_{>0} \) such that for all \( i \in [q] \),
\[
\sum_{j \in [q]} B(i, j)f(j)\lambda(j),
\]
then for all \( \varepsilon_S \in (0, 1] \), there is an \( \varepsilon_S \)-approximate sampler for \( \mu_{\text{WR}}^{(V, r, \lambda)} \) that has a running time in \( \text{poly}\left( \frac{\nu(V)}{\varepsilon_S}, \ln \left( \frac{\max_{i \in [q]} f(i)}{\min_{i \in [q]} f(i)} \right) \right) \).

**Proof:** As in the proof of Theorem 7.6, we start by using Lemma 7.5, by which there exists a resolution \( \rho \in \Theta\left( \varepsilon_S^{-1/d} \nu(V)^{1/d} \right) \) such that
\[
d_{\text{TV}}\left( \mu_{\text{WR}}^{(V, R, \lambda)}, \mu_{\Phi, \emptyset}^{(V, G_{X_s}, \lambda_{X_s})} \right) \leq \frac{\varepsilon_S}{2},
\]
where again
\[
\Phi(x) = \prod_{i \in [d]} x^{(i)} x^{(i)} + \frac{1}{\rho}.
\]
The corresponding graph \( G_{X_s} \) consists of \( \Theta\left( \frac{\nu(V)}{\varepsilon_S} \right) \) vertices.

Analogously to the proof of Theorem 6.5, there is a function \( f: [q] \to \mathbb{R}_{>0} \) such that for all \( i \in [q] \), it holds that
\[
f(i) > \sum_{j \in [q]} B(i, j)f(j)\lambda(j),
\]
which implies that there is also a function \( g: V_{X_s} \to \mathbb{R}_{>0} \) that satisfies the condition of Theorem 2.5. Thus, we can sample \( \frac{\varepsilon_S}{2} \)-approximately from \( \mu_{\text{HC}}^{(G_{X_s}, \lambda_{X_s})} \) in the desired
running time. Similarly to the proof of Theorem 7.6, applying Algorithm 1 to independent sets drawn from this $\epsilon_S$-approximation of $\mu_{HC}^{(G_y, \lambda_y)}$ yields the desired sampler, and by Lemma 7.4 we obtain the required bound on the total-variation distance.

The sampling version of Corollary 6.6 follows.

**Corollary 7.11.** Suppose a computational model that satisfies Assumption 7.1. Let $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{>0}$. Further, let $r \in \mathbb{R}_{>0}$, $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$. Denote by $\mu_{WR}^{(V, r, \lambda_1, \lambda_2)}$ the Gibbs distribution of the Widom–Rowlinson model on $V$ with 2 particle types, both with the same radius $r$ but (possibly) different fugacities $\lambda_1, \lambda_2$. If

$$\lambda_1 \lambda_2 < \frac{1}{4d \nu(B(r))^2}$$

then, for all $\epsilon_S \in (0, 1]$, there is an $\epsilon_S$-approximate sampler for $\mu_{WR}^{(V, r, \lambda_1, \lambda_2)}$ with running time in $\text{poly} \left( \frac{\nu(V)}{\epsilon_S} \right)$.

Note that the role of the allocation $\Phi$ in our sampling algorithms is fundamentally different than in our approximations for the partition function. For approximating the partition function, it is sufficient to know that a suitable allocation for a given point set $X$ exists. In contrast, Lemma 7.4, which is a core component of our sampling method, actually requires an explicit representation of $\Phi$ in order to sample for each $x \in X$ from $\Phi^{-1}(x)$. This raises the question whether there is a way to circumvent this.

## 8 Concentration of random discretizations

In this section we investigate the behavior of discretizations of a hard-constraint point process $(V, R, \lambda)$ based on uniformly random finite point sets $X \subseteq V$. We show sufficient conditions for the partition function of such random discretizations to get concentrated around the partition function of the continuous point process for a polynomial number of points. Our sufficient condition for the concentration is closely related to the concept of a specific partitioning of $V$.

**Definition 8.1 (\(\delta, \epsilon\)-partitioning).** Let $V \subseteq \mathbb{R}^d$ be bounded and measurable and assume $\nu(V) > 0$. For $\delta \in (0, 1]$ and $\epsilon \in \mathbb{R}_{>0}$ we say that a finite partitioning $(V_k)_{k \in [m]}$ of $V$ is a $\delta, \epsilon$-partitioning of size $m$ if and only if for all $k \in [m]$ it holds that

1. $V_k$ is measurable and
   $$\frac{\delta \nu(V)}{m} \leq \nu(V_k)$$
2. for all $x, y \in V_k$ it holds that
   $$d(x, y) \leq \epsilon.$$

the following lemma relates the notion of a $\delta, \epsilon$-partitioning with our previously introduced concept of a $\delta, \epsilon$-allocation.

**Lemma 8.2.** Let $V \subseteq \mathbb{R}^d$ be bounded and measurable with $\nu(V) > 0$. Let $\delta_1 \in (0, 1]$ and $\epsilon \in \mathbb{R}_{>0}$ and assume there is a $\delta_1, \epsilon$-partitioning of $V$ with size $m$. Further, let $\delta_2 \in (0, 1]$, $p \in (0, 1]$, and draw $X \subseteq V$ with $|X| = n \geq 48 \left( \frac{1}{\delta_2} \right)^2 \frac{1}{\delta_1} m \ln \left( \frac{2m}{p} \right)$ uniformly at random. With probability at least $1 - p$ there is a $\delta_2, \epsilon$-allocation for $X$. 

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Proof. Let \((\mathbb{V}_i)_{i \in [m]}\) be a \(\delta_1\)-partitioning of \(\mathbb{V}\) with size \(m\). Further, let \(X \subset \mathbb{V}\) with \(n = 48\left(\frac{1}{\delta_1}\right)^2 \ln \left(\frac{2m}{p}\right)\) be drawn uniformly at random.

The main idea of the proof is to show that with probability at least \(1 - \rho\) it holds that for all partitions \(\mathbb{V}_i\) for \(i \in [m]\) the number of points from \(X\) that are in \(\mathbb{V}_i\) is close to \(\frac{v(\mathbb{V}_i)}{v(\mathbb{V})} \cdot |X|\), and especially all partitions contain at least one point from \(X\). If this is the case, we can construct an allocation by fairly allocating each partition \(\mathbb{V}_i\) to the points in \(X \cap \mathbb{V}_i\) while ignoring any distance constraints. As we argue below, this always results in an allocation with the desired properties.

For all \(i \in [m]\), let \(X_i = X \cap \mathbb{V}_i\) and let \(Y_i = |X_i|\). We now show that, with probability at least \(1 - \rho\), it holds that

\[
(1 - \frac{\delta_2}{4})E[Y_i] \leq Y_i \leq (1 + \frac{\delta_2}{4})E[Y_i] \tag{17}
\]

for all \(i \in [m]\). To prove this claim, note that for each \(i \in [m]\) the random variable \(Y_i\) follows a binomial distribution with \(n\) trials and success probability \(\frac{v(\mathbb{V}_i)}{v(\mathbb{V})}\). Applying Chernoff’s inequality yields

\[
\Pr\left[|Y_i - E[Y_i]| \geq \frac{\delta_2}{4}\right] \leq 2 \cdot e^{-\frac{\delta_2^2}{8E[Y_i]}}.
\]

Further, note that we have \(E[Y_i] = n \cdot \frac{v(\mathbb{V}_i)}{v(\mathbb{V})} \geq \delta_1 \frac{n}{m}\), which gives us

\[
\Pr\left[|Y_i - E[Y_i]| \geq \frac{\delta_2}{4}\right] \leq 2 \cdot e^{-\frac{\delta_2^2}{8\delta_1 \frac{n}{m}}}.
\]

By our choice of \(n\), we obtain

\[
\Pr\left[|Y_i - E[Y_i]| \geq \frac{\delta_2}{4}\right] \leq \frac{p}{m}.
\]

Finally, applying union bound over \(i \in [m]\) proves that inequality (17) fails with probability at most \(p\).

Now, let us assume inequality (17) is satisfied. Note that this especially implies that \(Y_i > 0\). We now construct our allocation \(\Phi\) as follows. For each \(i \in [m]\), partition \(\mathbb{V}_i\) into sets \(\left(\mathbb{V}^{(j)}_i\right)_{j \in [Y_i]}\) such that all \(\mathbb{V}^{(j)}_i\) are measurable and have the same volume. Note that such a division of \(\mathbb{V}_i\) always exists. Now, for each such \(\mathbb{V}^{(j)}_i\), fix exactly one \(x \in X_i\) and set \(\Phi(y) = x\) for all \(y \in \mathbb{V}^{(j)}_i\). It remains to argue that \(\Phi\) is a \(\delta_2\)-allocation. To this end, choose some \(x \in X\) and assume that \(x \in X_i\) for some \(i \in [m]\). Further, let \(y \in \Phi^{-1}(x)\) and observe that by construction both, \(x\) and \(y\), are in \(\mathbb{V}_i\). Thus, we have

\[
d(x, y) \leq \epsilon.
\]

Moreover, because \(E[Y_i] = n \cdot \frac{v(\mathbb{V}_i)}{v(\mathbb{V})}\) we have that \(v(\mathbb{V}_i) = E[Y_i] \cdot \frac{v(\mathbb{V})}{n}\). Thus, by inequality (17), we obtain

\[
\Phi^{-1}(x) = \frac{v(\mathbb{V}_i)}{Y_i} = E[Y_i] \cdot \frac{v(\mathbb{V})}{n} \cdot \frac{1}{1 - \delta_2/4} \cdot \frac{v(\mathbb{V})}{n} \leq (1 + \delta_2) \cdot \frac{v(\mathbb{V})}{n}
\]

and analogously

\[
\Phi^{-1}(x) \geq \frac{1}{1 + \delta_2/4} \cdot \frac{v(\mathbb{V})}{n} \geq (1 - \delta_2) \cdot \frac{v(\mathbb{V})}{m},
\]

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which concludes the proof.

We obtain the following probabilistic bound on the difference between the partition function of a hard-constraint point process and the hard-core partition function of a random discretization.

**Corollary 8.3.** Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types, and assume $V \subset \mathbb{R}^d$ is star-convex. Further, set $\lambda_{\max} = \max_{i \in [q]} \lambda(i)$ and set $R_{\min} = \min_{i,j \in [q]} \{ R(i, j) \mid R(i, j) > 0 \}$. Let $\epsilon \in (0, R_{\min}/2]$ and $\delta_1 \in (0, 1]$ and assume there is a $\delta_1$-$\epsilon$-partitioning of $V$ size $m$. For any $\delta_2 \in (0, \frac{1}{2}]$, $p \in (0, 1]$ and $n \in \mathbb{N}$ with

$$\max \left\{ 4\lambda_{\max}^{v(V)}, 48 \left( \frac{1}{\delta_2} \right)^2 \frac{1}{\delta_1} m \ln \left( \frac{2m}{p} \right) \right\} \leq n$$

draw $X \subset V$ with $|X| = n$ uniformly at random and let $(G_X, \lambda_X)$ be the corresponding discretization of $(V, R, \lambda)$. It holds that

$$|Z_{HC}(G_X, \lambda_X) - Z(V, R, \lambda)| \leq \left( e^{\frac{1}{\delta_1} \sum_{i \in [q]} \lambda(i) v(V)} e^{\frac{2\delta_1 + \left( \frac{\epsilon}{\delta_1} \right)^d \sum_{i \in [q]} \lambda(i) v(V)} - 1 \right) Z(V, R, \lambda)$$

with probability at least $1 - p$.

Corollary 8.3 follows immediately from Theorem 4.6 and Lemma 8.2. Note that Corollary 8.3 means that, if we can construct $\delta_1$-$\epsilon$-partitionings for some constant $\delta_1 > 0$ and arbitrarily small $\epsilon > 0$, we can use this to derive a concentration result for the hard-core partition function of discretizations based on random point sets. A general statement for this is given in the following corollary.

**Corollary 8.4.** Let $(V, R, \lambda)$ be a hard-constraint point process with $q \in \mathbb{N}_{\geq 1}$ particle types, and assume $V \subset \mathbb{R}^d$ is star-convex. Further, assume there is some $\delta \in (0, 1]$ such that for all $\epsilon \in \mathbb{R}_{> 0}$ there is a $\delta$-$\epsilon$-partitioning of $V$ of size $\text{poly} \left( \frac{v(V)}{\epsilon} \right)$. For all $\epsilon_D \in (0, 1]$ and $p \in (0, 1]$ there is some $n_{\epsilon_D, p} \in \text{poly} \left( \frac{v(V)}{\epsilon_D} \ln \left( \frac{v(V)}{\epsilon_D p} \right) \right)$ such that for all $n \geq n_{\epsilon_D, p}$ it holds for $X \subset V$ with $|X| = n$ uniformly at random that

$$e^{-\epsilon_D} Z(V, R, \lambda) \leq Z_{HC}(G_X, \lambda_X) \leq e^{\epsilon_D} Z(V, R, \lambda)$$

with probability at least $1 - p$.

A more precise statement can be obtained when looking at specific regions of $d$-dimensional Euclidean space. To demonstrate this, we once again consider cubic regions $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{> 0}$. For such regions, we have the following observation.

**Lemma 8.5.** Let $V = [0, \ell]^d$ for some $\ell \in \mathbb{R}_{> 0}$. For all $\epsilon \in \mathbb{R}_{> 0}$ there is a $1$-$\epsilon$-partitioning of $V$ of size $\left\lfloor \sqrt{\frac{\ell^d}{\epsilon}} \right\rfloor$.

**Proof.** Set $a = \frac{\ell}{d}$ and divide $V$ into disjoint hypercubes of side length $a' = a \left\lceil \frac{\ell}{a} \right\rceil$. Note that $\frac{\ell}{a'} \in \mathbb{N}$. Formally, this gives the partitioning $V_{i_1, \ldots, i_d} = \prod_{j \in [d]} [i_j a', (i_j + 1)a')$. 

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for \((i_1, \ldots, i_d) \in \{0, \ldots, \frac{d}{2} - 1\}^d\). Obviously, this results in a partitioning of size \(\sqrt{|a^*|}^d\). Further, each partition has the same volume and their diameter is upper bounded by \(\sqrt{a'} \leq \sqrt{a} = \epsilon\), which proves that the partitioning is a 1-\(\epsilon\)-partitioning.

\[\begin{align*}
\text{Proof.} & \quad \text{Set} \\
& \quad \text{Due to Lemma 8.2 we know that for} \, \sqrt{a} \leq \sqrt{a} = \epsilon, \text{which proves that the partitioning is a 1-\(\epsilon\)-partitioning.}
\end{align*}\]

**Remark 8.6.** Note that the partitioning constructed in the proof of Lemma 8.5 can in fact be generated by taking the canonical point set \(X_p\) for the smallest feasible resolution \(\rho \geq \sqrt{a}^d\) and taking the pre-image of the 0-\(\epsilon\)-allocation \(\Phi\), constructed in the proof of Lemma 5.1, for each point in \(X_p\). This is not a coincidence. In fact, for every point set \(X\) with a \(\delta\)-\(\epsilon\)-allocation we can construct a \((1 - \delta)\)-2\(\epsilon\)-partitioning, which, together with Lemma 8.5, shows an interesting equivalence between both concepts. The results in this section could as well be stated in terms of allocations. However, as we are not interested in discretizations for specific point sets, it is more natural to state our results in terms of partitionings.

**Corollary 8.7.** Let \((V, R, \lambda)\) be a hard-constraint point process with \(q \in \mathbb{N}_{\geq 1}\) particle types and \(V = [0, \ell]^d\) for some \(\ell \in \mathbb{R}_{>0}\). Let \(X \subset V\) with \(|X| = n\), chosen uniformly at random. For all \(\ell_D \in (0, 1)\) and \(\rho \in (0, 1)\), there exists an \(n_{\ell_D, \rho} \in \Theta(\nu(V)^4\ell_D^{-3} \ln(\nu(V)))\) such that for all \(n \geq n_{\ell_D, \rho}\), with probability at least \(1 - \rho\) it holds that \(e^{-\rho}Z(V, R, \lambda) \leq ZHC(G_X, \lambda X) \leq e^{\rho}Z(V, R, \lambda)\).

**Proof.** Set \(\lambda_{\max} = \max_{i \in [q]} \lambda(i)\) and \(R_{\min} = \min_{i, j \in [q]} \{R(i, j) \mid R(i, j) > 0\}\). Note that by Lemma 8.5 we know that for

\[\begin{align*}
m &= \left[\sqrt{d} t \left(\frac{4q \max\{\lambda_{\max}, \lambda_{\max}^2\} \nu(V)}{\ell_D}\right)^{\frac{1}{d}} \max\left\{1, \frac{4}{R_{\min}}\right\}\right]^d
\end{align*}\]

there is a 1-\(\epsilon\)-partitioning of \(V\) with size \(m\) for some

\[\begin{align*}
\epsilon & \leq \left(\frac{\ell_D}{4q \max\{\lambda_{\max}, \lambda_{\max}^2\} \nu(V)}\right)^{\frac{1}{d}} \min\left\{1, \frac{R_{\min}}{4}\right\}.
\end{align*}\]

Set

\[\begin{align*}
\delta &= \frac{\ell_D}{8q \lambda_{\max} \nu(V)}
\end{align*}\]

and let

\[\begin{align*}
n & \geq \left(\frac{1}{\delta}\right)^2 m \ln \left(\frac{2m}{\Phi}\right) \in \Theta\left(\frac{\nu(V)^4}{\ell_D^3} \ln\left(\frac{\nu(V)}{\ell_D\rho}\right)\right).
\end{align*}\]

Due to Lemma 8.2 we know that for \(X \subset V\) with \(|X| = n\) uniformly at random there is a \(\delta\)-\(\epsilon\)-allocation for \(X\) with probability at least \(1 - \rho\).

The rest of the proof is mostly similar to the proof of Theorem 5.2. Let \(X \subset V\) with \(|X| = n\) such that a \(\delta\)-\(\epsilon\)-allocation for \(X\) exists. For \(\nu(V)\) sufficiently large we have that \(\epsilon \leq \frac{R_{\min}}{2}\), \(\delta \leq \frac{1}{2}\) and \(|X| = n \geq 4\lambda_{\max} \nu(V)\). Thus, we can apply Theorem 4.6 to \(X\). Now, observe that

\[\begin{align*}
\frac{8}{|X|} \sum_{i \in [q]} \lambda(i)^2 \nu(V) & \leq \frac{\ell_D}{6} \\
\left(\frac{\epsilon}{4R_{\min}}\right)^d \sum_{i \in [q]} \lambda(i) \nu(V) & \leq \frac{\ell_D}{12}
\end{align*}\]
\[2\delta \sum_{i \in [q]} \lambda(i) \nu(V) \leq \frac{\varepsilon_D}{12}.

This yields
\[
|Z_{HC}(G_X, \lambda_X) - Z(V, R, \lambda)| \leq \left( e^{\varepsilon_D} - 1 \right) Z(V, R, \lambda)
\]
for such point sets \(X\). Now, note that \(e^{\varepsilon_D} \leq e^{\varepsilon_D} \) and \(2 - e^{\varepsilon_D} \geq e^{\varepsilon_D} \) for \(\varepsilon_D \in (0, 1]\). Therefore, we have
\[
e^{-\varepsilon_D} Z(V, R, \lambda) \leq Z_{HC}(G_X, \lambda_X) \leq e^{\varepsilon_D} Z(V, R, \lambda),
\]
which concludes the proof.

Although Corollary 8.7 does not directly yield additional algorithmic results beyond what was already proved in Section 6, it shows an interesting relationship between the hard-core model on random geometric graph structures and continuous hard-constraint point process. An immediate consequence of Corollary 8.7 is that the hard-core partition functions of uniformly random geometric graphs on \(V = [0, \ell]^d\) with edge connection threshold \(2r\) for appropriately chosen fugacities concentrates around the partition function of a hard-sphere model with particle radius \(r\) on \(V\) as the number of vertices is increased. We believe that this connection between hard-core models on random geometric graphs and the continuous hard-sphere model is of its own interest.

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