HARDY-SOBOLEV TYPE INEQUALITY AND SUPERCRITICAL EXTREMAL PROBLEM

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ABSTRACT. This paper deals with Hardy-Sobolev type inequalities involving variable exponents. Our approach also enables us to prove existence results for a wide class of quasilinear elliptic equations with supercritical power-type nonlinearity with variable exponent.

1. Introduction and main results. Let $B \subset \mathbb{R}^N$, $N \geq 3$, be the unit ball, and denote $H^1_{0, \text{rad}}(B)$ as the first order Sobolev space of radial functions, and $2^* = 2N/(N - 2)$ the corresponding critical Sobolev embedding exponent. In the recent paper [13], J.M do Ó et al. investigated Sobolev type embeddings for radial functions into variable exponent Lebesgue spaces proving that

$$\sup_{u \in H^1_{0, \text{rad}}(B), \|\nabla u\|_2 = 1} \int_B |u|^{\varphi(x)} \, dx < +\infty, \quad (1)$$

where $\varphi(r) = 2^* + |x|^\sigma$, with $\sigma > 0$. Furthermore, it was proved that the supremum (1) is attained if $0 < \sigma < \min \{N/2, N - 2\}$. It is a surprise because the growth of $\varphi(x)$ is strictly larger than $2^*$, except in the origin.

As we shall see below, the inequality (1) is related with the classical Hardy inequality [15]

$$\int_0^\infty \left( \frac{F(r)}{r} \right)^p \, dr \leq \left( \frac{p}{p - 1} \right)^p \int_0^\infty F(r)^p \, dr \quad (2)$$

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where \( p > 1, f \geq 0 \) is an integrable function and 
\[
F(r) = \int_0^r f(t) \, dt, \quad r > 0.
\]

The Hardy’s inequality (2) has many applications in analysis, geometry and in the theory of differential equations see e.g. [1, 21, 22]. There are a lot of papers which deals with the generalizations, improvements and variations of the Hardy’s inequality, see for instance [4, 16, 19, 20, 23], and more recently [2, 7, 25]. For instance, let \( AC_{\text{loc}}(0, R) \) be the space of all locally absolutely continuous functions on the interval \((0, R), 0 < R \leq \infty\); using the Hardy inequality (2), A. Kufner and B. Opic [19] established conditions under the which, for all \( u \in AC_{\text{loc}}(0, R) \) with \( \lim_{r \to R} u(r) = 0 \), the inequality
\[
\left( \int_0^R r^\theta |u|^q \, dr \right)^{1/q} \leq C \left( \int_0^R r^\alpha |u'|^p \, dr \right)^{1/p},
\]  
(3)
holds for some positive constant \( C > 0 \) depending only on \( p, q, \theta \) and \( R \), where \( 1 \leq p \leq q < \infty \) and \( \theta, \alpha \in \mathbb{R} \).

In fact, in the bounded case \( R < \infty \), the above inequality holds if the following conditions is fulfilled
\[
1 \leq q \leq p^*, \quad \text{with} \quad p^* = \frac{(\theta + 1)p}{\alpha - p + 1}
\]  
(4)
and
\[
\alpha - p + 1 > 0. \quad \text{(Sobolev condition)}
\]  
(5)
In the case \( R = \infty \), the inequality (3) it holds only if \( q = p^* \), under the Sobolev condition (5). The inequality (3) was used successfully to deal with a wide class of quasilinear elliptic operators. For instance, we cite [6,8–10,12,17,18] and references therein. As will be seen later our approach will be more in the line of [13,19]. Then, inspired by the above results, our goals are the following:

- we offer a version of (1) to \( W^{1,p}_{0,\text{rad}}(B) \), \( p > 2 \) and “fractional dimentions”;
- we investigate the associated extremal problem,
- we establish improvements for an one-dimensional Hardy-type inequality,
- we also apply our results to prove existence of solutions to supercritical elliptic equations including the Hessian operator.

In order to be precise our main results, let us introduce briefly some notations and preliminary results. Set \( X_R^{p}(\alpha, \beta) \) or simply \( X_R \) be the space of functions \( u \in AC_{\text{loc}}(0, R) \) with the right boundary condition \( u(R) = 0 \) and such that \( \max\{\|u'\|_{L^p_\theta}, \|u\|_{L^q_\theta}\} \) is finite, where \( L^q_\theta = L^q_\theta(0, R), \ \theta \geq 0 \) be the weighted Lebesgue space with norm
\[
\|u\|_{L^q_\theta} = \left( \int_0^R r^\theta |u(r)|^q \, dr \right)^{1/q} < \infty, \quad \text{for} \quad 1 \leq q < \infty.
\]
For \( 0 < R < \infty, p \geq 1 \) and \( \alpha, \beta > 0 \) real numbers, up to a completion, \( X_R \) is a Banach space endowed with the norm
\[
\|u\| = \|u'\|_{L^p_\theta}.
\]  
(6)
Further, as consequence of (3), it is possible to prove the continuous embeddings
\[
X_R^{1,p}(\alpha, \beta) \hookrightarrow L^q_\theta \quad \text{if} \quad q \in (1, p^*], \quad \text{and} \quad \min\{\theta, \beta\} \geq \alpha - p,
\]  
(7)
where \( p^* := p^*(\alpha, p, \theta) \) given by (4) is the critical exponent associated with \( X_R \), see [6]. Also, the embedding (7) are compact if \( q < p^* \).

Motivated by (1) and by Hardy-type inequality (3), in this paper we investigate conditions on the function \( \varphi = \varphi(r) \) and the real parameters \( \alpha, p \) and \( \theta \) under which the supremum

\[
U_\varphi := \sup_{u \in X_R, \|u\| = 1} \int_0^R r^\theta |u|^{\varphi(r)} \, dr
\]

is finite.

From now on, we assume the following conditions on \( p, \alpha \) and \( \theta \):

\[
p > 1, \quad \alpha - p + 1 > 0 \quad \text{ (Sobolev condition) \ and \ \theta > \alpha - p.} \quad (9)
\]

The next result extends the Hardy-type inequality (3) to supercritical growth and variable exponents.

**Theorem 1.1.** Let \( \varphi(r) = p^* + r^\sigma \) and \( \sigma > 0 \). Then the supremum \( U_\varphi \) is finite.

The above theorem has the following easy consequence. For a deepening on this kind of result we refer [11].

**Corollary 1.** The following embedding is continuous:

\[
X_R^{1,p} \hookrightarrow L_{\theta, \varphi}(0, R)
\]

where \( L_{\theta, \varphi}(0, R) \) is defined as follows

\[
L_{\theta, \varphi}(0, R) := \{ u : (0, R) \to \mathbb{R} \ \text{measurable} \mid \int_0^R r^\theta |u|^{\varphi(r)} \, dr < \infty \}
\]

with the norm

\[
\|u\|_{\theta, \varphi} = \inf \{ \lambda > 0, \int_0^R r^\theta |\lambda^{-1} u(r)|^{\varphi(r)} \, dr \leq 1 \}.
\]

Denote \( \Sigma_p \) the following best constant of Sobolev type for \( X_R \) (see [6, 9])

\[
\Sigma_p = \sup_{\|u\| = 1} \int_0^R r^\theta |u|^{p^*} \, dr. \quad (10)
\]

With this notation we will establish the following main results:

**Theorem 1.2.** Let \( \varphi(r) = p^* + r^\sigma \) and assume

\[
0 < \sigma < \min \{ (\theta + 1)/p, (\alpha - p + 1)/(p - 1) \} \]

then

\[
U_\varphi > \Sigma_p. \quad (11)
\]

It is well-known the supremum in (10) is not attained, see [6] for more details. The following theorem shows that \( U_\varphi \) is attained if it is larger than \( \Sigma_p \).

**Theorem 1.3.** Let \( \varphi(r) = p^* + r^\sigma \) and assume \( U_\varphi > \Sigma_p \), then the supremum \( U_\varphi \) is attained.

We also establish a existence result for a wide class of related elliptic equations with a supercritical nonlinearity:
Theorem 1.4. Let $\alpha, \theta > 0$ and $p > 1$ real numbers satisfying (9). Set $\varphi(r) = p^* + r^\sigma$, with $\sigma$ under the condition (11). Then the problem
\[
\begin{cases}
- r^{-\theta} (r^\alpha |u'|^{p-2} u)' = |u|^{p(r)-2} u, & \text{in } (0, R) \\
u(R) = 0
\end{cases}
\] (13)
has a nontrivial weak solution $u \in X_R$.

We emphasize that the class of operators in (13) includes, when acting in symmetric function defined in ball $B_R \subset \mathbb{R}^N$, the Laplacian, $p$-Laplacian and $k$-Hessian operators. This kind of problem has received attention of some authors in recent years [6, 8–10, 12, 17, 18]. In [6], Figueiredo et al. consider the Brezis-Nirenberg type problems and, in particular, for $\varphi(r) = p^*$ the above problem doesn’t admit non-trivial solution. In the above problem (13), following [13], we propose a new kind of perturbation which also allows us to produce existence of solution as well as the classical Brezis-Nirenberg’s perturbation [5].

2. Hardy-type inequality. We are going to use the following version of Strauss Radial Lemma [24].

Lemma 2.1. Assume $\alpha - p + 1 > 0$. Then, for any $0 < r \leq R$
\[
|u(r)| \leq \frac{C(R-r)^{\frac{p-1}{r-\alpha}}}{r-\alpha} \|u\|, \quad \forall \ u \in X_R^1(\alpha, \beta)
\] (14)
where $C$ is the dependent only on $\alpha, p$ and $R$. In particular, there is $c > 0$ such that
\[
|u(r)| \leq \frac{c\|u\|}{r^{\frac{\alpha - p + 1}{p}}}, \quad 0 < r \leq R.
\] (15)

Proof. For each $u \in X_R$, the Hölder inequality gives
\[
|u(r)| \leq \int_r^R |u'(s)| ds \leq \int_r^R s^{\frac{\alpha - p + 1}{p}} |u'(s)| s^{-\frac{\alpha - p + 1}{p}} ds
\]
\[
\leq \left( \int_r^R s^\alpha |u'|^p ds \right)^{\frac{1}{p}} \left( \int_r^R s^{-\alpha - p + 1} ds \right)^{\frac{p-1}{p}}
\]
\[
\leq \|u\| \left[ - \frac{p-1}{\alpha - p + 1} \left( R^{-\frac{\alpha - p + 1}{p}} - r^{-\frac{\alpha - p + 1}{p}} \right) \right]^{\frac{p-1}{p}}
\]
\[
= \left[ \frac{p-1}{\alpha - p + 1} \left( 1 - \left( \frac{r}{R} \right)^{-\frac{\alpha - p + 1}{p}} \right) \right]^{\frac{p-1}{p}} \|u\|^{\frac{p-1}{p}} \cdot R^{\frac{p-1}{p}}.
\]
Now, for any $a > 0$, we have $(1 - t^a)/(1 - t)$ goes to $a$, as $t \to 1$. Thus, the above inequalities imply
\[
|u(r)| \leq \frac{C(R-r)^{\frac{p-1}{p}}}{r^{\frac{\alpha - p + 1}{p}}} \|u\|,
\]
for some constant $C > 0$ depending only on $\alpha, p$ and $R$. \qed
2.1. Proof of Theorem 1.1.

Proof. For \( u \in X_R \) with \( \|u\| = 1 \), we can write

\[
\int_0^R r^\rho |u|^{p^* + r^\sigma} \, dr = \int_0^\rho r^\rho |u|^{p^* + r^\sigma} \, dr + \int_\rho^R r^\rho |u|^{p^* + r^\sigma} \, dr
\]

(16)

where \( \rho \) will be determined later. Since \( \|u\| = 1 \), the Lemma 2.1 implies

\[
|u(r)| \leq c r^{\frac{\alpha - p^* + 1}{p}}.
\]

(17)

We shall estimate each of these two terms in (16) separately. Firstly,

\[
\int_0^\rho r^\rho |u|^{p^*} (|u|^{r^\sigma} - 1) \, dr \leq \int_0^\rho r^\rho |u|^{p^*} (|u|^{r^\sigma} - 1) \, dr + \Sigma_p.
\]

Using (17) and the definition of \( p^* \), we get

\[
\int_0^\rho r^\rho |u|^{p^*} (|u|^{r^\sigma} - 1) \, dr \leq c^* \int_0^\rho r^\rho \exp \left( r^\sigma \log c - \frac{c}{r^{p^* - 1}} \right) \, dr.
\]

(18)

Let us denote

\[
g(r) = r^\sigma \left| \log c - \frac{c}{r^{p^* - 1}} \right|.
\]

It easy to show that there are positive constants \( c_1 \) and \( c_2 \) such that

\[
0 < g(r) \leq \frac{1}{|\log r|^\gamma} \left( c_1 + c_2 |\log r| \right), \quad \gamma > 2
\]

(19)

for \( r \) near of 0. In particular, \( g(r) \to 0 \), as \( r \to 0 \). Hence, given \( d > 1 \) there exist \( \rho = \rho(d) > 0 \) such that

\[
c^{g(r)} - 1 \leq dg(r), \quad \forall r \in (0, \rho).
\]

Thus, from (18) we obtain

\[
\int_0^\rho r^\rho |u|^{p^*} (|u|^{r^\sigma} - 1) \, dr \leq c^* d \int_0^\rho \frac{g(r)}{r} \, dr.
\]

(20)

Using (19) it easy to check that

\[
\int_0^\rho \frac{g(r)}{r} \, dr < +\infty,
\]

and thus

\[
\int_0^\rho r^\rho |u|^{p^*} (|u|^{r^\sigma} - 1) \, dr < +\infty.
\]
In order to estimate the second integral in (16), we proceed analogously. From (17), choosing \( c > 0 \) satisfying \( 1/c^\alpha - p + 1 \), we get

\[
\int_\rho^R r^\theta |u|^{p^* + r^\sigma} dr \leq \int_\rho^R r^\theta \left( \frac{c}{r^{\frac{n+\alpha}{p}}} \right)^{p^* + r^\sigma} dr = \frac{c^\sigma}{\rho^{\frac{n+\alpha}{p} + 1}} \left( \frac{r^\alpha - p + 1}{\rho^\alpha} \right)^{p^*} \cdot \frac{1}{r^{\frac{n+\alpha}{p} + 1}} \log(r) dr
\]

which is finite by continuity.

2.2. Proof of Corollary 1.

Proof. Consider the variable exponent Lebesgue space

\[ L_{\theta,p^*+r^\sigma}(0,R) = \{ u : (0,R) \to \mathbb{R} \text{ measurable} \mid \int_0^R r^\theta |u|^{p^* + r^\sigma} dr < \infty \} \]

endowed with the norm

\[
\|u\|_{\theta,p^*+r^\sigma} = \inf \{ \lambda > 0 \mid \int_0^R r^\theta |\lambda^{-1}u(r)|^{p^* + r^\sigma} dr \leq 1 \}.
\]

We shall prove that there is a constant \( C > 0 \) such that \( \|u\|_{\theta,p^*+r^\sigma} \leq C \), for \( u \in X_R \) with \( \|u\| = 1 \). Using Theorem 1.1, there is \( c > 0 \) such that

\[
\int_0^R r^\theta |u|^{p^* + r^\sigma} dr \leq c, \quad \forall u \in X_R \text{ with } \|u\| = 1.
\]

Thus, for \( \lambda_0 > 1 \) large enough such that \( c/\lambda_0^p \leq 1 \), we can write

\[
\int_0^R r^\theta |\lambda^{-1}u(r)|^{p^* + r^\sigma} dr \leq \frac{1}{\lambda_0^p} \int_0^R r^\theta |u(r)|^{p^* + r^\sigma} dr \leq \frac{c}{\lambda_0^p} \leq 1.
\]

Then \( \|u\|_{\theta,p^*+r^\sigma} \leq \lambda_0 \).

3. Proof of Theorem 1.2. This section is devoted to prove the Theorem 1.2. The proof is based in the modified Bliss function introduced by [6] and follows some ideas in [5,13]. Firstly, for each \( 0 < R \leq \infty \), we define

\[
S(p^*, R) = \inf_{u \in X_R, \|u\|_{L_{\theta,p}^p} = 1} \|u\|^p. \tag{21}
\]

It is known that \( S(p^*, R) \) is independent of \( R \), and that it is achieved when \( R = +\infty \) (see [6], for more details). Furthermore, for each \( \epsilon > 0 \), the function

\[
u^*_\epsilon(r) = \frac{\epsilon \epsilon^n}{(\epsilon^n + r^n)^{\frac{n}{p}}}, \tag{22}\]
where
\[
\begin{align*}
 s &= \frac{\alpha - p + 1}{p^2 - p} \\
n &= \frac{\theta - \alpha + p}{p - 1} \\
m &= \frac{\theta - \alpha + p}{\alpha - p + 1},
\end{align*}
\]

satisfies
\[
S^{\frac{s+1}{\alpha-n+p}} = \int_0^\infty r^\alpha |(u^*_\epsilon)|^p \, dr \quad \text{and} \quad S^{\frac{s+1}{\alpha+p}} = \int_0^\infty r^\theta |u^*_\epsilon|^p \, dr,
\]

where \( S \) denotes the value of \( S(p^*, R) \) for all \( R = +\infty \), and then for any \( R > 0 \).

Let \( \eta \in C_0^\infty(0, R) \) be a fixed cut-off function satisfying
\[
\eta(r) = 1, \quad \forall \, r \in (0, r_0) \quad \text{and} \quad \eta(r) = 0, \quad \forall \, r \in [2r_0, R],
\]

for some \( 0 < r_0 < 2r_0 < R \). So we have the following:

**Claim 1.** Let \( \eta u^*_\epsilon \), for \( \eta \) and \( u^*_\epsilon \) given by (22) and (25), respectively. Then

(a) \( \|\eta u^*_\epsilon\|_{L^p_{\alpha}}^p = S^{\frac{s+1}{\alpha-n+p}} + O(\epsilon^p) \), as \( \epsilon \to 0 \);

(b) \( \|\eta u^*_\epsilon\|_{L^p_{\alpha}}^p = S^{\frac{s+1}{\alpha+p}} + O(\epsilon^p) \), as \( \epsilon \to 0 \).

**Proof.** From (24), an easy calculation shows that
\[
\int_0^R r^\alpha |(\eta u^*_\epsilon)|^p \, dr = \int_0^{r_0} r^\alpha |(u^*_\epsilon)|^p \, dr + \int_{r_0}^{2r_0} r^\alpha |(\eta u^*_\epsilon)|^p \, dr
\]

\[
= \int_0^\infty r^\alpha |(u^*_\epsilon)|^p \, dr - \int_{r_0}^{2r_0} r^\alpha |(u^*_\epsilon)|^p \, dr + O(\epsilon^p)
\]

\[
= S^{\frac{s+1}{\alpha-n+p}} + O(\epsilon^p).
\]

Also, we have
\[
|\eta u^*_\epsilon|^p = (|\eta|^{p^*} - 1)|u^*_\epsilon|^p^* + |u^*_\epsilon|^p^*
\]

and thus
\[
\int_0^R r^\theta |\eta u^*_\epsilon|^p \, dr = O(\epsilon^{p^*}) + \int_0^\infty r^\theta |u^*_\epsilon|^p^* \, dr = O(\epsilon^{p^*}) + S^{\frac{s+1}{\alpha+p^*}}.
\]

This completes the proof. \( \square \)

Let now
\[
u_\epsilon(r) = B \eta(r) u^*_\epsilon(r) = A \eta(r) \frac{\epsilon^s}{(\epsilon^n + r^n)^m}
\]

where
\[
B = S^{-\frac{1}{p} \frac{s+1}{\alpha-n+p}} \quad \text{and} \quad A = B \hat{c}.
\]

It is easily to check that
\[
\|u^*_\epsilon\|_{L^p_{\alpha}} = 1 + O(\epsilon^p)
\]
and
\[
\int_0^R r^{\theta} |u_\epsilon|^p \, dr = B^{p^*} \int_0^R r^{\theta} |\eta u_\epsilon^*|^p \, dr = S - \frac{\epsilon}{p} + O(\epsilon^{p^*}) = \Sigma_p + O(\epsilon^{p^*}).
\] (28)

Finally, using (27) and (28) we have
\[
\int_0^R r^{\theta} \left( \frac{|u_\epsilon(r)|}{\|u_\epsilon^*\|_{L^p}} \right)^{p^*} \, dr = \Sigma_p + O(\epsilon^{p^*}).
\] (29)

The proof of (12) relies on the following estimate of which the proof shall be postponed.

**Claim 2.** There exists a constant \( C > 0 \) such that for all \( \epsilon > 0 \) small enough
\[
\int_0^R r^{\theta} |u_\epsilon|^p + r^\sigma \, dr \geq \int_0^R r^{\theta} |u_\epsilon|^p \, dr + C |\log \epsilon| e^\sigma + O(\epsilon^{\frac{\sigma + 1}{p}}).
\] (30)

Assuming (30) we get
\[
U_\varphi = \sup_{u \in X_{\mu}, \|u\|=1} \int_0^R r^{\theta} |u|^p + r^\sigma \, dr \geq \int_0^R r^{\theta} |u_\epsilon|^p \, dr + C |\log \epsilon| e^\sigma + O(\epsilon^{\frac{\sigma + 1}{p}}) + O(\epsilon^{\frac{\sigma + 1}{p}})
\]
\[
> \Sigma_p,
\]
which concludes the proof of Theorem 1.2.

**Proof of Claim 2.** Firstly, we highlight some useful relations on the parameters in (23).

\[
\begin{align*}
sm &= \frac{1}{n} \quad \frac{1}{p} \\
n - sm &= \frac{\theta - \alpha + p}{p} \\
sp^* &= \frac{\theta + 1}{p - 1} \\
\theta - \frac{np^*}{m} + 1 &= -\frac{\theta + 1}{p - 1} \\
s - \frac{n}{m} &= -\frac{\alpha - p + 1}{p} \\
(s - \frac{n}{m})p^* &= -(\theta + 1).
\end{align*}
\] (31)

Note that
\[
Bu_\epsilon^* \leq 1 \iff r \geq (A^m e^{snm} - \epsilon^n)^{\frac{1}{p}} = \epsilon^{\frac{n}{p}} (A^m - \epsilon^{n - snm})^{\frac{1}{p}} := a_\epsilon.
\] (32)

In particular, using (9) and (31), for any fixed \( R > 0 \), we have
\[
0 < a_\epsilon < R \quad \text{and} \quad \frac{a_\epsilon}{\epsilon} > 1,
\] (33)
for \( \epsilon > 0 \) small enough. Now, we write

\[
\int_0^R r^\theta |u_\epsilon|^p r^\sigma \, dr = \int_0^{a_\epsilon} r^\theta |u_\epsilon|^p r^\sigma \, dr + \int_{a_\epsilon}^R r^\theta |u_\epsilon|^p r^\sigma \, dr. \tag{34}
\]

Note that (see (31))

\[
\int_{a_\epsilon}^R r^\theta |u_\epsilon|^p r^\sigma \, dr = \int_{a_\epsilon}^R r^\theta |B^\epsilon u_\epsilon|^p r^\sigma \, dr \\
\leq \int_{a_\epsilon}^R r^\theta \left( \frac{A\epsilon^s}{(\epsilon^n + r^n)^{\frac{\rho}{\sigma}}} \right)^p \, dr \\
\leq A\epsilon^p \epsilon^{sp} \int_{a_\epsilon}^R r^{\theta - \frac{np^*}{\sigma}} \, dr \\
= O(\epsilon^{\frac{\theta + 1}{p^*}}).
\]

As seen above, one also has

\[
\int_{a_\epsilon}^R r^\theta |u_\epsilon|^p \, dr = O(\epsilon^{\frac{\theta + 1}{p}}).
\]

Hence, using that \( (|B^\epsilon u_\epsilon(r)|)^{p^*} - |B^\epsilon u_\epsilon(r)|^p \geq 0 \), for \( r \in [\epsilon, a_\epsilon] \) (see (32) and (33)) and the two last estimates, we get

\[
\int_0^R r^\theta |u_\epsilon|^p r^\sigma \, dr \\
= \int_0^R r^\theta |u_\epsilon|^p \, dr + \int_0^{a_\epsilon} r^\theta \left( |u_\epsilon|^p r^\sigma - |u_\epsilon|^p \right) \, dr + O(\epsilon^{\frac{\theta + 1}{p^*}}) \\
\geq \int_0^R r^\theta |u_\epsilon|^p \, dr + \int_0^\epsilon r^\theta \left( |B^\epsilon u_\epsilon|^p r^\sigma - |B^\epsilon u_\epsilon|^p \right) \, dr + O(\epsilon^{\frac{\theta + 1}{p^*}}).
\]

Set

\[
I_{1,\epsilon} := \int_0^\epsilon r^\theta \left( |B^\epsilon u_\epsilon|^p r^\sigma - |B^\epsilon u_\epsilon|^p \right) \, dr.
\]

Setting \( d = 2^{\frac{1}{n}} A \), we can write

\[
B^\epsilon u_\epsilon(r) \geq d\epsilon^{s-\frac{n}{m}}, \quad \text{for any} \quad 0 \leq r \leq \epsilon.
\]

Thus,

\[
I_{1,\epsilon} = \int_0^\epsilon r^\theta |B^\epsilon u_\epsilon|^p \left( |B^\epsilon u_\epsilon|^p r^\sigma - 1 \right) \, dr \\
\geq d^{p^*} \epsilon^{(s-\frac{n}{m})p^*} \int_0^\epsilon r^\theta \left( e^{c\sigma \log d + (\frac{n}{m} - s) \log \epsilon} - 1 \right) \, dr.
\]

Using \( e^x \geq 1 + x, \) for \( x \geq 0, \) we have

\[
I_{1,\epsilon} \geq d^{p^*} \epsilon^{(s-\frac{n}{m})p^*} \left( \log d + \left( \frac{n}{m} - s \right) \log \epsilon \right) \int_0^\epsilon r^{\theta + \sigma} \, dr \\
= d^{p^*} \left( \log d + \left( \frac{n}{m} - s \right) \log \epsilon \right) \frac{\epsilon^\sigma}{\theta + \sigma + 1} \tag{35}
\]

for suitable \( C > 0 \) and \( \epsilon > 0 \) small enough. This proves our claim. \( \square \)
4. Existence of extremal function.

4.1. Upper bound for the concentrated levels.

**Definition 4.1.** A sequence \((u_j) \subset X_R\) is a normalized concentrating sequence at origin if
\[
\|u_j\| = 1, \quad u_j \rightharpoonup 0 \text{ in } X_R \quad \text{and} \quad \int_{r_0}^R r^\alpha |u_j'|^p \, dr \to 0, \quad \forall \, r_0 > 0.
\]

**Proposition 1.** For any \((u_j) \subset X_R\) normalized concentrating sequence at origin, we have
\[
\limsup_{j \to \infty} \int_{r_0}^R r^\theta |u_j|^{p^* + r^\sigma} \, dr \leq \Sigma_p. \tag{36}
\]

**Proof.** It suffices to show the following: Given \(\epsilon > 0\) there are \(\eta > 0\) and \(j_0 \in \mathbb{N}\) such that
\[
\begin{align*}
(a) \quad & \int_{r_0}^\eta r^\theta |u_j|^{p^* + r^\sigma} \, dr \leq \Sigma_p + \epsilon/2; \\
(b) \quad & \int_{\eta}^R r^\theta |u_j|^{p^* + r^\sigma} \, dr \leq \epsilon/2,
\end{align*}
\]
for any \(j \geq j_0\). In order to prove \((a)\), we first use the Lemma 2.1 to get
\[
\int_{r_0}^\eta r^\theta |u_j|^{p^* \left(|u_j| r^\sigma - 1\right)} \, dr \leq \int_{r_0}^\eta r^\theta |u_j|^{p^* \left(e^{r^\sigma \left(\frac{1}{e-1} \frac{\alpha - p + 1}{p} - 1\right)} - 1\right)} \, dr.
\]

Noticing that
\[
r^\sigma \log \left(e^{-1} \frac{\alpha - p + 1}{p}\right) = -r^\sigma \log \left(e^{-1} \frac{\alpha - p + 1}{p}\right) \searrow 0, \quad \text{as } r \to 0,
\]
and
\[
\lim_{x \to 0^+} \frac{e^x - 1}{x} = 1,
\]
by choosing \(\eta > 0\) small enough, we can write
\[
\int_{r_0}^\eta r^\theta |u_j|^{p^* \left(|u_j| r^\sigma - 1\right)} \, dr \leq C \int_{r_0}^\eta r^\theta |u_j|^{p^* \left(e^{-1} \frac{\alpha - p + 1}{p}\right)} \, dr \leq C_1 \eta^\sigma \left| \log \left(e^{-1} \eta^{\frac{\alpha - p + 1}{p}}\right) \right| \int_{r_0}^\eta r^\theta |u_j|^{p^*} \, dr \leq C_1 \eta^\sigma \left| \log \left(e^{-1} \eta^{\frac{\alpha - p + 1}{p}}\right) \right| \Sigma_p.
\]
Hence, taking \(\eta = \eta(\epsilon) > 0\) small enough such that
\[
C_1 \eta^\sigma \left| \log \left(e^{-1} \eta^{\frac{\alpha - p + 1}{p}}\right) \right| \Sigma_p \leq \frac{\epsilon}{2},
\]
we obtain
\[
\int_{r_0}^\eta r^\theta |u_j|^{p^* + r^\sigma} \, dr \leq \int_{r_0}^\eta r^\theta |u_j|^{p^*} \, dr + \int_{r_0}^\eta r^\theta |u_j|^{p^* \left(|u_j| r^\sigma - 1\right)} \, dr \leq \Sigma_p + \frac{\epsilon}{2},
\]
which proves \((a)\).
To get (b) we argue as in Lemma 2.1. For any \( r \in (\eta(\epsilon), R) \), we obtain
\[
|u_j(r)| \leq \int_r^R |u_j'(s)| ds = \int_r^R s^\frac{p}{\alpha} |u_j'(s)| s^{-\frac{p}{\alpha}} ds
\]
\[
\leq \left( \int_\eta^R s^\alpha |u_j'|^p ds \right)^{\frac{1}{p}} \left( \int_r^R s^{-\frac{\alpha}{p-\alpha}} ds \right)^{\frac{p-1}{p}} \leq \delta_j \frac{1}{r^{\frac{\alpha}{p-\alpha}}},
\]
(37)
where
\[
\delta_j = C \left( \int_\eta^R s^\alpha |u_j'|^p ds \right)^{\frac{1}{p}},
\]
for some \( C = C(\alpha, p) \). Note that
\[
\delta_j \to 0, \quad \text{as} \quad j \to \infty.
\]
Then we can write
\[
\int_\eta^R r^\theta |u_j|^{p^*+r^*} dr \leq \int_\eta^R r^\theta \left( \frac{\delta_j}{r^{\frac{\alpha}{p-\alpha}}} \right)^{p^*+r^*} dr \leq \delta_j^{p^*} \int_\eta^R r^\theta \left( r^\frac{\alpha}{p-\alpha} - r^\frac{\alpha}{p-\alpha} \right) dr = \delta_j^{p^*} c(\eta) \leq \frac{\epsilon}{2},
\]
for \( j \) sufficiently large.

4.2. Attainability of the best constant. In this section we prove the existence of extremal function stated in Theorem 1.3. In view of Proposition 1, we only need to prove the following:

**Proposition 2.** Let \( \varphi(r) = p^* + r^* \). Assume that \( U_\varphi \) is not attained, then any maximizing sequence for \( U_\varphi \) is concentrated at origin.

**Proof.** Assume that \( U_\varphi \) is not attained. Let \( (u_j) \) be a sequence in \( X_R \) satisfying
\[
\|u_j\| = 1 \quad \text{and} \quad U_\varphi = \lim_{j \to \infty} \int_0^R r^\theta |u_j|^{p^*+r^*} dr. \quad (38)
\]
Then, up to a subsequence, we have
\[
u_j \rightharpoonup u \quad \text{in} \quad X_R.
\]
Of course, we have \( \|u\| \leq \lim \inf \|u_j\| \leq 1 \). Now, we claim that \( u = 0 \). If not, we have
\[
\int_0^R r^\alpha |u'|^p dr > 0.
\]
By Brezis-Lieb type argument (see, [3]), we obtain
\[
\int_0^R r^\theta |u_j|^{p^*+r^*} dr = \int_0^R r^\theta |u_j - u|^{p^*+r^*} dr + \int_0^R r^\theta |u|^{p^*+r^*} dr + o(1), \quad (39)
\]
and
\[
1 = \int_0^R r^\alpha |u_j'|^p dr = \int_0^R r^\alpha |u_j' - u'|^p dr + \int_0^R r^\alpha |u'|^p dr + o(1), \quad (40)
\]
where \( o(1) \to 0 \), as \( j \to \infty \). From (40), if \( \|u\| = 1 \), we get \( u_j \to u \) strongly in \( X_R \) and hence \( \mathcal{U}_\varphi \) is attained by continuity. This contradicts our assumption. Hence, we can assume \( \|u\| < 1 \). Setting \( w_j = u - u \) and using (38), (39) and (40), we can write

\[
\mathcal{U}_\varphi = \int_0^R r^\theta |w_j|^{p^*r^*} dr + \int_0^R r^\theta |u|^{p^*r^*} dr + o(1)
\]

\[
= \int_0^R r^\theta \left( \frac{|w_j|}{\|w_j\|} \right)^{p^*r^*} |w_j|^{p^*r^*} dr + \int_0^R r^\theta \left( \frac{|u|}{\|u\|} \right)^{p^*r^*} |u|^{p^*r^*} dr + o(1)
\]

\[
\leq \mathcal{U}_\varphi \|w_j\|^{p^*} + \mathcal{U}_\varphi \|u\|^{p^*} + o(1)
\]

\[
= \mathcal{U}_\varphi \left( (1 - \|u\|^p + o(1))^{\frac{p^*}{p^*}} + (\|u\|^p)^{\frac{p^*}{p^*}} \right) + o(1)
\]

\[
< \mathcal{U}_\varphi,
\]

which is contradiction. Thus, we must have \( u \equiv 0 \). It remains to show that \((u_j)\) satisfies

\[
\int_{r_0}^R r^\alpha |u_j|^p dr \to 0, \forall r_0 > 0.
\]

(41)

Set \( X^1_{R,p}([r_0, R]) \) the space \( X_R \) modified to the interval \([r_0, R]\) instead \((0, R)\). We claim that the embedding

\[
X^1_{R,p}([r_0, R]) \hookrightarrow L_q^q([r_0, R])
\]

is compact for any \( q \geq p \). To prove (42), we consider the operator \( H_R : L^p_\alpha [r_0, R] \to L^q_\alpha [r_0, R] \) defined by

\[
H_R(f)(r) = \int_r^R f(s) ds.
\]

Using [19, Theorem 7.4], for \( 1 \leq p \leq q \), the operator \( H_R \) is compact if and only if

(i) \( B_R < \infty \)

(ii) \( \lim_{r \to r_0^+} F_R(r) = \lim_{r \to R^-} F_R(r) = 0 \)

where

\[
F_R(r) = \|s^{\theta/q}L_s^\|_{L^q([r_0, r])}\|s^{-\alpha/p}L_r^\|_{L^p_\alpha([r, R])} \quad \text{and} \quad B_R = \sup_{r_0 < r < R} F_R(r).
\]

An straightforward computation shows that (i) and (ii) are satisfied and, thus \( H_R \) is compact. Now, the embedding in (42) can be seen as \( T \circ H_R \), where \( T : X^1_{R,p}([r_0, R]) \to L^p_\alpha [r_0, R] \) is given by \( Tu = -u' \). Since \( T \) is a continuous operator, we conclude the compact embedding (42).

Now, under the conditions (9), we have \( p^* > p \) and with help of (42) and Lemma 2.1, we can write

\[
\int_{r_0}^R r^\theta |u_j|^{p^*r^*} dr = \int_{r_0}^R r^\theta |u_j|^{p^*}|u_j|^{r^*} dr \leq C \int_{r_0}^R r^\theta |u_j|^{p^*} dr \to 0.
\]

(43)

From the Ekeland’s principle [14, Theorem 3.1], since \((u_j)\) is a maximizing sequence we have

\[
\lambda_j \int_{r_0}^R r^\alpha |u_j|^{p-2} u_j' v' dr = \int_{r_0}^R r^\theta (p^* + r^*)|u_j|^{p^*-2+r^*} u_j v dr + o(1), v
\]

(44)
for some multiplier $\lambda_j$. Choosing $v = u_j$ we obtain
\[
\lambda_j \int_0^R r^\alpha |u_j'|^p \, dr \geq p^\ast \int_0^R r^\theta |u_j|^{p^\ast + r^\sigma} \, dr + \langle o(1), u_j \rangle \to p^\ast U_{\varphi}.
\]
Thus, we have $\lim \inf_j \lambda_j \geq p^\ast U_{\varphi}$.

Now, we choose a smooth cut-off function
\[
\eta(r) = \begin{cases} 
0, & \text{if } r \leq r_0/2 \\
1, & \text{if } r \geq r_0.
\end{cases}
\] (45)

and choose $v = \eta u_j$ in (44). Thus, we obtain from (43)
\[
\int_{r_0/2}^R r^\alpha |u_j'|^p \, dr = \frac{1}{\lambda_j} \int_{r_0/2}^R r^\theta (p^\ast + r^\sigma) |u_j|^{p^\ast + r^\sigma} \, dr + \langle o(1), \eta u_j \rangle \to 0.
\]

Using the compact embedding (7) to conclude
\[
\int_{r_0}^R r^\alpha |u_j|^p \, dr \leq C \int_{r_0}^R r^\theta |u_j|^p \, dr \to 0,
\]
and thus we get
\[
o(1) = \int_{r_0/2}^R r^\alpha |u_j'|^{p-2} u_j' (\eta u_j)' \, dr
= \int_{r_0/2}^R r^\alpha \eta |u_j'|^p \, dr + \int_{r_0/2}^R r^\alpha |u_j'|^{p-2} u_j u_j' \eta' \, dr
\geq \int_{r_0}^R r^\alpha |u_j'|^p \, dr - ||\eta'||_\infty ||u_j'||_{L^p_R}^{p-1} \left( \int_{r_0}^R r^\alpha |u_j|^p \, dr \right)^{\frac{1}{p}}
= \int_{r_0}^R r^\alpha |u_j'|^p \, dr + o(1)
\] (46)

and thus (41) is proved.

5. **Application to a class of supercritical equations.** In this section we prove existence of a nontrivial weak solution to problem (13). In order to get this, we apply variational arguments to the functional $I : X_R \to \mathbb{R}$ defined by
\[
I(u) = \frac{1}{p} \int_0^R r^\alpha |u'|^p \, dr - \int_0^R \frac{r^\theta}{p^\ast + r^\sigma} |u|^{p^\ast + r^\sigma} \, dr.
\] (47)

Precisely, we use the mountain-pass theorem. Initially, we observe the mountain pass geometry of functional.

**Lemma 5.1.** The functional $I$ satisfies:

(a) $I(0) = 0$,

(b) For any $u \in X_R \setminus \{0\}$, with $u \geq 0$ we have $I(tu) \to -\infty$, if $t \to \infty$,

(c) There are $\delta, \rho > 0$ such that $I(u) \geq \delta$, if $\|u\| = \rho$. 

\[\square\]
Proof. Of course, (a) holds. Also, there are constants \(c_1, c_2 > 0\) depending only on \(p, \theta, \sigma, R\) and \(u\) such that

\[ I(tu) \leq c_1 t^p - c_2 t^{p^*}, \quad t \geq 1. \]

Because \(p^* > p\), we get (b) holds. Finally, suppose \(u \in X_R\), with \(\|u\| < 1\). The Theorem 1.1 implies

\[ \int_0^R r^\theta \frac{\alpha^{p^*+r^*}}{p^*+r^*} |u|^{p^*+r^*} \, dr \leq C \|u\|^{p^*} \leq \frac{\|u\|^{p^*}}{p^*}. \]

It follows that

\[ I(u) \geq \frac{\|u\|^p}{p} - \frac{C \|u\|^{p^*}}{p^*}. \]

Thus, (c) follows if \(\|u\| = \rho\) is small enough.

From now on, we denote \(w_\epsilon = \eta u_\epsilon^{*}\) with \(\eta\) as in (25) and \(u_\epsilon^{*}\) given by (22).

Let us consider the mountain pass level

\[ 0 < c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} I(u), \]

where

\[ \Gamma = \{ \gamma \in C([0,T], X_R) : \gamma(0) = 0, \gamma(T) = Tw_\epsilon \} \]

with \(T > 0\) large enough, so that \(I(Tw_\epsilon) \leq 0\). With this notation we have

**Lemma 5.2.** Assume \(0 < \sigma < \min \{(\theta + 1)/p, (\alpha - p + 1)/(p - 1)\}\). Then mountain pass level \(c\) satisfies

\[ 0 < c < \frac{S^{\theta+1}}{p}. \]

**Proof.** Consider the path \(\gamma_\epsilon(t) = tw_\epsilon\), \(t \in [0,T]\) belongs to \(\Gamma\). For each \(\epsilon > 0\), there exists \(t_\epsilon > 0\) with

\[ c \leq \max_{t \in [0,T]} I(tw_\epsilon) = I(t_\epsilon w_\epsilon). \]

Firstly, we analyze the behavior of \((t_\epsilon)_{\epsilon > 0}\). From \(\frac{d}{dt} I(tw_\epsilon)|_{t=t_\epsilon} = 0\), we obtain

\[ t_\epsilon^{p-1} \int_0^R r^\alpha |w_\epsilon'|^p \, dr = \int_0^R t_\epsilon^{p^*+r^*} - r^\theta |w_\epsilon|^{p^*+r^*} \, dr. \]

Combining the Claim 1 and above expression we see that

\[ S^{\theta+1}/p + O(\epsilon^p) = t_\epsilon^{p^*} - p \left[ S^{\theta+1}/p + O(\epsilon^{p^*}) + A_\epsilon \right], \]

where

\[ A_\epsilon := \int_0^R r^\theta \left( r^\sigma |w_\epsilon|^{p^*+r^*} - |w_\epsilon|^{p^*} \right) \, dr. \]

In order to analyze the behavior of \(A_\epsilon\), as in (32), we set

\[ \alpha_\epsilon = (d^m \epsilon^m - \epsilon^m)^{\frac{1}{2}} = \epsilon^m \left( d^m - \epsilon^{n-sm} \right)^{\frac{1}{2}}, \text{ with } d = T \hat{c}. \]
It is easy to see that $t_\varepsilon w_\varepsilon (r) \leq 1$, for any $r \geq \alpha_\varepsilon$. The above expression and (31) give
\begin{equation}
\alpha_\varepsilon^C |\log \varepsilon| \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.
\end{equation}
Moreover, for all $0 < r \leq \alpha_\varepsilon$ and $\varepsilon > 0$ small enough
\begin{equation}
e^{\varepsilon^C r} \log d + (\frac{m}{s} - r) \log |\log \varepsilon| - 1 = e^{\varepsilon^C |\log \varepsilon|} \left( \frac{\log d}{1 + (\frac{m}{s} - r) \log |\log \varepsilon|} \right) - 1 \leq e^{\varepsilon^C |\log \varepsilon|} - 1
\end{equation}
for some $c > 0$. Thus, combining (53) and (e - 1)/x → 1, as $x \rightarrow 0$
\begin{equation}
e^{\varepsilon^C r} \log d + (\frac{m}{s} - r) \log |\log \varepsilon| - 1 \leq C_\varepsilon^C |\log \varepsilon|, \forall 0 < r \leq \alpha_\varepsilon.
\end{equation}
Since $t_\varepsilon w_\varepsilon (r) \leq 1$, for any $r \geq \alpha_\varepsilon$, we get
\[
A_\varepsilon \leq \int_0^{\alpha_\varepsilon} r^\theta |w_\varepsilon|^p \left(e^{\varepsilon^C \log d_1 (\frac{m}{s} - r) |\log \varepsilon|} - 1 \right) dr
\]
Thus, (54) and the definition of $w_\varepsilon$ imply
\begin{equation}
A_\varepsilon \leq C_1 \int_0^{\alpha_\varepsilon} r^\theta (s - \frac{m}{s}) p_\varepsilon r^\sigma |\log \varepsilon| dr + C_2 \int_{\alpha_\varepsilon} \int_0^{\alpha_\varepsilon} r^\theta e^{\varepsilon^C r - \frac{2n}{m} r^\sigma} |\log \varepsilon| dr
\end{equation}
\[
\leq C e^{\varepsilon^C |\log \varepsilon|},
\]
where we have used the condition (11).

On the other hand, by definition of $I$ we have
\[
p^{-1} t_\varepsilon^p \| w_\varepsilon \|^p \geq I(t_\varepsilon w_\varepsilon) > c,
\]
and, thus the Claim 1-(a) gives $\delta_1 > 0$ such that $\delta_1 \leq t_\varepsilon \leq T$, provided that $\varepsilon > 0$ is chosen small enough. Now, we set
\[
\beta_\varepsilon := (d_1^m s n - \varepsilon)^\frac{1}{n} = e^\varepsilon^C \left( d_1^m - \varepsilon^{n - s m} \right)^\frac{1}{n}, \text{ with } d_1 = \delta_1 ^C
\]
Using (31), we have $\beta_\varepsilon \rightarrow 0$, as $\varepsilon \rightarrow 0$. Thus, taking $\varepsilon > 0$ small such that $0 < \beta_\varepsilon \leq \rho_0$ (see (25)), we obtain
\[
t_\varepsilon w_\varepsilon \geq 1, \text{ if } 0 < r \leq \beta_\varepsilon.
\]
Hence,
\[
A_\varepsilon \geq \int_{\beta_\varepsilon}^R r^\theta |w_\varepsilon|^p \left(|t_\varepsilon w_\varepsilon|^r - 1 \right) dr \geq - \int_{\beta_\varepsilon}^R r^\theta |w_\varepsilon|^p dr,
\]
and thus
\begin{equation}
A_\varepsilon \geq -c \int_{\beta_\varepsilon}^R e^{\varepsilon^C r} r^\theta - \frac{2n}{m} dr \geq -c e^{\frac{2n+1}{p}}.
\end{equation}
Combining (11), (55) and (57), we obtain
\[
A_\varepsilon = O(e^{\frac{2n+1}{p}}) + O(e^{|\log \varepsilon|}).
\]
Therefore, from (51), we get (see, (31))
\begin{equation}
S^\frac{\theta+1}{\sigma-a+p} + O \left( e^{\frac{\theta+1}{\sigma-a+p}} \right)
\end{equation}
\begin{equation}
e^p - p \left[ S^\frac{\theta+1}{\sigma-a+p} + O(e^{\frac{\theta+1}{\sigma-a+p}}) + O(e^{|\log \varepsilon|}) + O \left( e^{\frac{\theta+1}{p}} \right) \right]
\end{equation}
from which we have $t_\varepsilon \rightarrow 1$. Thus,
\[
1 + O \left( e^{\frac{\theta+1}{\sigma-a+p}} \right) + O(e^{|\log \varepsilon|})
\]
\begin{equation}
= (1 + (p^*-p)(t_\varepsilon - 1)) \left[ 1 + O(e^{|\log \varepsilon|}) + O \left( e^{\frac{\theta+1}{p}} \right) \right]
\end{equation}
and using (11), we have
\[ R_\epsilon := t_\epsilon - 1 = O \left( \epsilon^{\frac{p+1}{p-r}} \right) + O \left( \epsilon |\log \epsilon| \right) + O \left( \epsilon^{\frac{p+1}{p}} \right) = O \left( \epsilon^{\frac{p+1}{p}} \log \epsilon \right). \]

Now, we return the estimate of the level \( c \), see (50).

**Claim 3.** We have
\[
I(t_\epsilon w_\epsilon) < \frac{S^{\frac{p+1}{p-r}}}{p}, \quad \text{for any } \epsilon > 0 \text{ sufficiently small.}
\]

Firstly, for each \( R \in (0, R) \), there is \( 0 < h = h(\epsilon, r) < 1 \) such that
\[
(1 + R_\epsilon)^{p^r + r^r} = 1 + (p^r + r^r)R_\epsilon + (p^r + r^r)(p^r + r^r - 1)(1 + hR_\epsilon)p^{r^r - 2}R_\epsilon^2.
\]

Since \( R_\epsilon \to 0 \), as \( \epsilon \to 0 \), there are positive constants \( a_1, a_2, b_1 \) and \( b_2 \) such that
\[
\begin{align*}
    a_1 &\leq 1 + (p^r + r^r)R_\epsilon + a_2 R_\epsilon^2 \\
    &\leq (1 + R_\epsilon)^{p^r + r^r} \\
    &\leq 1 + (p^r + r^r)R_\epsilon + b_2 R_\epsilon^2 \\
    &\leq b_1,
\end{align*}
\]

for any \( 0 \leq r \leq R \) and \( \epsilon > 0 \) small enough. Analogously, we can write
\[
(1 + R_\epsilon)^p = 1 + pR_\epsilon + O(R_\epsilon^2).
\]

Thus, using the Claim 1, the equation (61) and the definition of \( R_\epsilon \), we have
\[
I(t_\epsilon w_\epsilon) = \frac{S^{\frac{p+1}{p-r}}}{p} + O \left( \epsilon^{\frac{p+1}{p-r}} \right) + R_\epsilon S^{\frac{p+1}{p-r}} + O \left( R_\epsilon \epsilon^{\frac{p+1}{p-r}} \right) + O \left( R_\epsilon^2 \epsilon^{\frac{p+1}{p-r}} \right) - B_\epsilon,
\]

where
\[
B_\epsilon = \int_0^R \int_0^R t_\epsilon^{p^r + r^r} \frac{r^\theta}{p^r + r^r} |w_\epsilon|^{p^r + r^r} \, dr.
\]

Next, we proceed to analyze \( B_\epsilon \). First, we split as follows
\[
B_\epsilon = \int_0^R t_\epsilon^{p^r + r^r} \frac{r^\theta}{p^r + r^r} |w_\epsilon|^{p^r} \, dr + J_{1,\epsilon},
\]

where
\[
J_{1,\epsilon} = \int_0^R t_\epsilon^{p^r + r^r} \frac{r^\theta}{p^r + r^r} (|w_\epsilon|^{p^r + r^r} - |w_\epsilon|^{p^r}) \, dr.
\]

With help of (60), we obtain
\[
\int_0^R t_\epsilon^{p^r + r^r} \frac{r^\theta}{p^r + r^r} |w_\epsilon|^{p^r} \, dr \geq \int_0^R \frac{r^\theta}{p^r + r^r} |w_\epsilon|^{p^r} \, dr + R_\epsilon \int_0^R r^\theta |w_\epsilon|^{p^r} \, dr + a_2 R_\epsilon^2 \int_0^R \frac{r^\theta}{p^r + r^r} |w_\epsilon|^{p^r} \, dr.
\]

and thus, from Claim 1
\[
\int_0^R t_\epsilon^{p^r + r^r} \frac{r^\theta}{p^r + r^r} |w_\epsilon|^{p^r} \, dr \geq \frac{S^{\frac{p+1}{p-r}}}{p^r} + O(\epsilon^{\frac{p+1}{p}}) + R_\epsilon S^{\frac{p+1}{p-r}} + O(R_\epsilon) - J_{2,\epsilon},
\]

for any \( 0 \leq r \leq R \) and \( \epsilon > 0 \) sufficiently small.
where
\[ J_{2,\epsilon} = \int_0^R r^\theta \left( \frac{1}{p^*} - \frac{1}{p^* + r^\sigma} \right) |w_\epsilon|^p \, dr. \]

Thus, from (63)
\[ B_\epsilon \geq \frac{S^{\frac{\theta + 1}{\theta - \sigma}}}{p^*} + O(\epsilon^{\frac{\theta + 1}{\theta - \sigma}}) + R_c S^{\frac{\theta + 1}{\theta - \sigma}} + O(R_c \epsilon^{\frac{\theta + 1}{\theta - \sigma}}) + O(R_c^2) - J_{2,\epsilon} + J_{1,\epsilon}. \]

Leading it to (62), we get
\[
I(t_\epsilon w_\epsilon) \leq \frac{S^{\frac{\theta + 1}{\theta - \sigma}}}{p} + O \left( \epsilon^{\frac{\theta + 1}{\theta - \sigma}} \right) + O \left( R_c \epsilon^{\frac{\theta + 1}{\theta - \sigma}} \right) + O \left( R_c^2 \epsilon^{\frac{\theta + 1}{\theta - \sigma}} \right) + O \left( \epsilon^{\frac{\theta + 1}{\theta - \sigma}} \right) + J_{2,\epsilon} - J_{1,\epsilon}. \tag{64}
\]

Now, we are going to estimate \( J_{2,\epsilon} \) and \( J_{1,\epsilon} \). Initially,
\[ J_{2,\epsilon} \leq \frac{1}{p^2} \left( \int_0^\epsilon r^\theta + \sigma |w_\epsilon|^p \, dr + \int_\epsilon^R r^\theta + \sigma |w_\epsilon|^p \, dr \right) \]
from which and by (31)
\[ J_{2,\epsilon} \leq C_1 \int_0^\epsilon \epsilon^{(s - \frac{\sigma}{m})p^*} r^\theta + \sigma \, dr + C_2 \int_\epsilon^R \epsilon^{p^*} r^\theta + \sigma - \frac{\sigma n}{m} \, dr = O(\epsilon^\sigma). \]

Also, as in (56), there exists \( b_\epsilon \), with \( 0 < b_\epsilon \leq r_0 \) such that \( |w_\epsilon|^2 \geq 1 \), if \( 0 < r \leq b_\epsilon \). Namely,
\[ b_\epsilon := \left( \epsilon^m \epsilon^{sm} - \epsilon^n \right)^{\frac{1}{\beta}} = \epsilon^{\frac{m}{\beta}} \left( \epsilon^m - \epsilon^{n-sm} \right)^{\frac{1}{\beta}}. \]

Thus, there is \( c_2 > 0 \) satisfying
\[
J_{1,\epsilon} \geq c_2 \int_0^\epsilon r^\theta |w_\epsilon|^{p^*} \left( |w_\epsilon|^\sigma - 1 \right) \, dr + c_2 \int_{b_\epsilon}^R r^\theta |w_\epsilon|^{p^*} \left( |w_\epsilon|^\sigma - 1 \right) \, dr \tag{65}
\]
\[ \geq c_2 \int_0^\epsilon r^\theta |w_\epsilon|^{p^*} \left( |w_\epsilon|^\sigma - 1 \right) \, dr - c_2 \int_{b_\epsilon}^R r^\theta |w_\epsilon|^{p^*} \, dr \]

Next, we shall estimate each of these two terms separately. First, using (22), (31) and the definition of \( b_\epsilon \), we have
\[
-c_2 \int_{b_\epsilon}^R r^\theta |w_\epsilon|^{p^*} \, dr \geq -C_1 \int_{b_\epsilon}^R \epsilon^{p^*} r^\theta - \frac{\sigma n}{m} \, dr \tag{66}
\]
\[ = -C_1 \epsilon^{\frac{\theta + 1}{\theta - \sigma}} \left( \epsilon^m - \epsilon^{n-sm} \right)^{-\frac{\theta + 1}{\theta - \sigma}} + C_2 \epsilon^{\frac{\theta + 1}{\theta - \sigma}}, \]
for some positive constants \( C_1 \) and \( C_2 \). On the other hand, setting \( \tilde{d} = 2^{-\frac{1}{\beta}} \hat{c} \) and arguing as (35), we have
\[
c_2 \int_0^\epsilon r^\theta |w_\epsilon|^{p^*} \left( |w_\epsilon|^\sigma - 1 \right) \, dr \geq c_2 \tilde{d} \epsilon^{\theta(\sigma - \frac{\sigma n}{m})p^*} \int_0^\epsilon r^\theta \left( \epsilon^{\sigma \log \tilde{d} + (\frac{\sigma n}{m} - s) |w_\epsilon|^\sigma - 1 \right) \, dr \geq C \epsilon^\sigma |\log \epsilon| \tag{67}
\]
for suitable $C > 0$ and $\epsilon > 0$ sufficiently small. Combining (65), (66) and (67), we can write
\[
J_{1, \epsilon} \geq C \epsilon^\sigma |\log \epsilon| - C_1 \epsilon^{\frac{a+1}{p+1}} (\epsilon^m - \epsilon^{n+sm})^{\frac{a+1}{p+1}} + C_2 \epsilon^{\frac{a+1}{p+1}} \geq c \epsilon^\sigma |\log \epsilon|
\]
for $\epsilon > 0$ small enough and appropriated $c > 0$. Finally, using (11), (31) and (64), we obtain
\[
I(t, w, \epsilon) \leq \frac{S^{\frac{a+1}{p+\epsilon}}}{p} + \epsilon^\sigma |\log \epsilon| \left[ -c + O \left( \frac{\epsilon^{\frac{a-\sigma+1}{\sigma+1}}}{\epsilon^\sigma |\log \epsilon|} \right) + O \left( \frac{R^2}{\epsilon^\sigma |\log \epsilon|} \right) \right]
\]
\[
+ \epsilon^\sigma |\log \epsilon| \left[ O \left( \frac{\epsilon^{\frac{a+1}{p+1}}}{\epsilon^\sigma |\log \epsilon|} \right) + O \left( \frac{\epsilon^\sigma}{\epsilon^\sigma |\log \epsilon|} \right) \right]
\]
\[
< \frac{S^{\frac{a+1}{p+\epsilon}}}{p},
\]
for $\epsilon > 0$ small enough. It proves the Claim 3 and completes the proof of Lemma.

Using the mountain pass theorem, there is a Palais-Smale sequence $(u_j) \subset X_R$ such that
\[
I(u_j) \to \epsilon < \frac{S^{\frac{a+1}{p+\epsilon}}}{p},
\]
where $c$ is given by (49). Also
\[
I'(u_j) \phi = \int_0^R r^\alpha |u_j'|^{p-2} u_j' \phi' dr - \int_0^R r^\alpha |u_j|^{p^*+r-1} \phi dr \to 0. \quad (68)
\]
We observe that the function
\[
F(r, t) = \frac{t^{p^*+r}}{p^*+r}, \quad t \geq 0 \text{ and } 0 \leq r \leq R,
\]
satisfies the following (AR) condition:

there is $\xi \in (p, p^*)$ such that $\xi F(r, t) \leq f(r, t) t', \ t \geq 0$, where $f(r, t) = F_t(r, t)$. In view of this, the Palais-Smale sequence $(u_j) \subset X_R$ is bounded in $X_R$, and hence there is a subsequence that converges weakly to $u \in X_R$ which solves weakly the equation
\[
- r^{-\theta} (r^\alpha |u'|^{p-2} u)' = |u|^{p^*+r-2} u \quad \text{in } (0, R),
\]
see [8, 12], for more details. It remains to prove that $u \not\equiv 0$. By contradiction, assume that $u \equiv 0$. Analogous to (43), we have
\[
\int_{r_0}^R r^\alpha |u_j|^{p^*+r} dr \to 0, \ \text{for any } r_0 > 0. \quad (69)
\]
Choosing $\eta$ as in (45) and taking $\phi = \eta u_j$ in (68), we obtain
\[
\int_0^R r^\alpha |u_j'|^{p-2} u_j' (\eta u_j)' dr = \int_0^R r^\alpha |u_j|^{p^*+r-1} (\eta u_j) dr + (o(1), \eta u_j)
\]
from which, arguing as in (46), we get
\[
\int_{r_0}^R r^\alpha |u_j'|^p dr \to 0, \ \text{for any } r_0 > 0.
\]
We proceed to show that
\[ I(u_j) = I_0(u_j) + o(1) \] (70)
where
\[ I_0(w) = \frac{1}{p} \int_0^R r^{\alpha} |w'|^p dr - \frac{1}{p^*} \int_0^R r^{\theta} |w|^p dr. \]
Indeed, arguing as in Proposition 1, we get
\[ |I(u_j) - I_0(u_j)| \leq C \left| \int_0^\eta r^\theta |u_j|^p (|u_j|^\sigma - 1) dr \right| + C \left| \int_\eta^R r^\theta |u_j|^p (|u_j|^\sigma - 1) dr \right|. \]
Now, we observe that
\[ \left| \int_0^\eta r^\theta |u_j|^p (|u_j|^\sigma - 1) dr \right| \leq \int |u_j| \geq 1 \cap (0, \eta) r^\theta |u_j|^p (|u_j|^\sigma - 1) dr \]
\[ + \int |u_j| < 1 \cap (0, \eta) r^\theta |u_j|^p (1 - |u_j|^\sigma) dr \]
\[ \leq o(1) + \int |u_j| < 1 \cap (0, \eta) r^\theta |u_j|^p (1 - |u_j|^\sigma) dr \]
\[ \leq o(1) + \int |u_j| < 1 \cap (0, \eta) r^\theta dr \]
\[ \leq \frac{\eta^{\sigma+1}}{\sigma+1} + o(1). \]
Thus, we have concluded (70). Analogously, one shows that
\[ I'(u_j) \varphi = I_0'(u_j) \varphi + o(1). \]
Hence, we conclude that \((u_j)\) is a Palais-Smale sequence also to the functional \(I_0\). However, it is known that for \(I_0\) the Palais-Smale condition holds for any
\[ 0 < c < \frac{S^{\sigma+1}}{p}. \]
Thus, up to a subsequence, we have \(u_j \to 0\) strongly in \(X_R\) but it implies \(I(u_j) \to 0\), which contradicts \(I(u_n) \to c > 0\).

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