Dynamics of \(d\)-wave Vortices: Angle-Dependent Nonlinear Hall Effect

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We study the dynamics of vortices in \(d\)-wave superconductors using a phenomenological time-dependent Ginzburg-Landau equation with mixing of \(s\) and \(d\)-wave components. We present numerical simulations under an external driving current \(\mathbf{J}\) oriented with an angle \(\varphi\) with respect to the \(b\) crystal axis, calculating the vortex motion and induced electric fields for \(\kappa = \infty\). We find an intrinsic Hall effect for \(\varphi \neq 0\) which depends as \(\sim \sin(4\varphi)\), and increases non-linearly with \(\mathbf{J}\).

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The discovery of high-\(T_c\) superconductivity has renewed the interest in the vortex state of type II superconductors [1]. In conventional \(s\)-wave superconductors, the physics of the vortex state was built upon two basic results: (i) vortices form a triangular lattice structure in equilibrium as shown by Abrikosov [2], and (ii) when an external current is applied there is dissipation due to vortex motion [3]. There is now a growing experimental [4] and theoretical [5] evidence that the symmetry of the superconducting gap is most likely to be \(d_{x^2-y^2}\) wave. Therefore, it is now an issue of central interest how both the static and dynamic macroscopic properties of vortices are affected by the \(d\)-wave superconductivity.

The structure of a single vortex in a \(d\)-wave superconductor was recently studied by Volovik [7], Soininen [8] and Schopohl and Maki [9]. It was found in [8] that a \(s\)-wave component is nucleated near the vortex core with opposite winding of the phase relative to the \(\mathbf{E}\) field, and there is a four-lobe structure of \(d\)-wave vortices. Therefore, it is now an issue of central interest how both the static and dynamic macroscopic properties of vortices are affected by the \(d\)-wave superconductivity.

The GL equation studied in [8] has an order parameter with two components \(d(\mathbf{r})\) and \(s(\mathbf{r})\) and is given by:

\[
\bar{h} \frac{\partial \phi}{\partial t} = -\Gamma \frac{\partial^2 \phi}{\partial s^2}, \quad \bar{h} \left[ \partial_t + i \frac{\varepsilon^*}{\hbar} \phi \right] d = -\Gamma \frac{\partial^2 d}{\partial s^2},
\]

with \(\Gamma\) the relaxation parameter and \(\phi\) the scalar potential. We consider strongly type-II materials with \(\gamma_s < \gamma_d < \gamma_c\). For generality, we consider these constants to be independent with the constrain \(\gamma_d \gamma_s > \gamma_c\). We assume that the TDGL equations of motion for the order parameters \(d\) and \(s\) are simply given as [3]

\[
\bar{h} \left[ \partial_t + i \frac{\varepsilon^*}{\hbar} \phi \right] d = -\Gamma \frac{\partial d}{\partial s^2}, \quad \bar{h} \left[ \partial_t + i \frac{\varepsilon^*}{\hbar} \phi \right] s = -\Gamma \frac{\partial s}{\partial s^2},
\]

with \(\Gamma\) the relaxation parameter and \(\phi\) the scalar potential. We consider strongly type-II materials in the \(\kappa = \infty\) limit. Therefore, the magnetic field is uniform and time-independent, and we consider the case \(\mathbf{B} \parallel \mathbf{c}\). The system of equations is closed after considering current conservation: \(\nabla \cdot \mathbf{J} = 0\), with the total current density \(\mathbf{J} = \mathbf{J}_s + \mathbf{J}_n\). The supercurrent \(\mathbf{J}_s\) is given by

\[
\mathbf{J}_s = \mathbf{J}_0 \cos \varphi + \mathbf{J}_1 \sin \varphi,
\]
\[
\mathbf{J}_s = \frac{2e}{\hbar} \gamma_d \left[ d^* \langle \Pi d \rangle + \frac{2e}{\hbar} \gamma_s \left[ s^* \langle \Pi s \rangle \right] \right] + \frac{2e}{\hbar} \gamma_v \left\{ \left[ s^* \langle \Pi_0 d \rangle + \langle \Pi_0 s \rangle^* d \right] \mathbf{b} \right.
- \left. \left[ s^* \langle \Pi_0 d \rangle + \langle \Pi_0 s \rangle^* d \right] a \right\} + c.c.
\]

while the normal current is \( \mathbf{J}_n = \sigma_n \mathbf{E} \), with \( \sigma_n \) an effective normal-state conductivity.

These TDGL equations are solved numerically using a finite-difference algorithm \([21]\) in a square grid along the \( x, y \) axis with discretization \( \Delta x = \Delta y = 0.2 \xi \) and system size \( 25.6\xi \times 25.6\xi \), with \( \xi = \sqrt{\gamma_d/|\alpha_d|} \). We take periodic boundary conditions for the physical quantities \( \mathbf{h}, \mathbf{E}, \mathbf{J}_s, |d| \) and \( |s| \). One convenient gauge choice is a time independent \( \mathbf{A} = n_v B_0 y \hat{x} \), where \( n_v \) is the number of vortices in the sample, \( B_0 = \Phi_0/S \) and \( \Phi_0 \) is the flux quantum, and \( \mathbf{E} = -\nabla \phi \). The scalar potential \( \phi \) can be decomposed as \( \phi(\mathbf{r}) = \phi_0(\mathbf{r}) - \mathbf{E}_0(\mathbf{r}) \cdot \mathbf{r} \) with \( \phi_0(\mathbf{r}) \) being periodic and satisfying \( \nabla^2 \phi_0 = \sigma_n^{-1} \nabla \cdot \mathbf{J}_s \) and \( \mathbf{E}_0(\mathbf{t}) = \sigma_n^{-1} \mathbf{J}_{ext} - 1/S \int d^2r \mathbf{J}_s(\mathbf{t}) \), since for an applied external current, from global current conservation, we have \( \mathbf{J}_{ext} = 1/S \int d^2r \mathbf{J}_s \). To ensure the periodicity of \( \mathbf{J}_s \) the phases of \( d \) and \( s \) must obey the boundary conditions: \( \theta(\mathbf{r} + L_x \hat{x}) = \theta(\mathbf{r}) + \Theta_x \) and \( \theta(\mathbf{r} + L_y \hat{y}) = \theta(\mathbf{r}) + g y + \Theta_y \) where \( g = n_v B_0 L_y / \partial_\Theta x = L_x E_{o2}(t) \) and \( \partial_\Theta y = L_y E_{o3}(t) \). The external current is taken to be along the discretization y-axis, and the angle \( \phi \) between \( \hat{a} \) and \( \hat{x} \) [see Fig.1(a)] is varied by “rotating” the GL equations. (We obtain similar results by fixing the numerical grid to \( (x, y) = (a, b) \) and rotating \( \mathbf{J}_{ext} \), but the effects of the discretization are stronger in this case). The equation are integrated with a time step \( \Delta t = 0.001 h/|\alpha_d| \) averaging over 5000 steps after a transient of \( 10^4 \) steps.

A vortex moving with a velocity \( \mathbf{v}_L \) induces an average electric field \( \mathbf{E} = -\mathbf{v}_L \times (\mathbf{B}) \) \([1]\). Since from current conservation we have that \( \nabla \cdot \mathbf{E} = -\sigma_n^{-1} \nabla \cdot \mathbf{J}_s = 4\pi \rho \), the exchange of supercurrents into normal currents occurring in the vortex core results in a dipolar charge distribution with momentum \( \mathbf{p} \perp \mathbf{v}_L \), such that \( \mathbf{E} \propto \mathbf{p} \). Under an external current \( \mathbf{J}_{ext} \) there is a Hall effect when the current is not parallel to \( \mathbf{E} \), defining a Hall angle \( \tan \theta_H = \langle E_L \rangle / \langle E_0 \rangle \). The TDGL equation has been used in the past to model vortex dynamics in conventional superconductors \([14,15]\). In this case, it has been shown by Dorsey \([15]\) that a Hall effect can be induced from two mechanisms: (i) by a complex relaxation parameter \( \Gamma = \Gamma_1 + i \Gamma_2 \) with \( \Gamma_2 \neq 0 \) (modeling a hydrodynamic contribution to the Hall effect), and (ii) by a normal state off-diagonal conductivity \( \sigma_{n,xy} \) (modeling a quasiparticle core contribution). We consider here the TDGL equations \([2]\) with \( \Gamma_2 = 0 \) and \( \sigma_{n,xy} = 0 \), to look for additional contributions to the Hall effect coming from the lack of rotational symmetry in d-wave superconductivity.

Let us first consider the motion of a single d-wave vortex under a current \( \mathbf{J}_{ext} = 0.6 J_c \), with \( J_c = \frac{2e|\alpha_d| \gamma_d}{\hbar \delta_5 \gamma_5 \delta_5} \) (the critical current of the pure d-wave solution), and parameter values \( \alpha_s/|\alpha_d| = 1.25, \gamma_c/\gamma_d = 1, \gamma_s/\gamma_d = 2, \beta_d/\beta_d = 0.5, \Gamma_1 = 1, \sigma_n = (4e^2/h)(\gamma_d/2\beta_d \Gamma_1) \), and for different values of \( \varphi \), see Figure 2. We find that when \( \mathbf{J}_{ext} \) is in one of the directions of maximum symmetry (\( \varphi = 0^\circ, 45^\circ, 90^\circ \)) the vortex moves with \( \mathbf{v}_L \perp \mathbf{J}_{ext} \), and therefore \( \theta_H = 0 \). This is shown in Fig.2(a-c) for \( \varphi = 0 \). We see in Fig.2(a) that the amplitude \( |d| \) has a square-like shape aligned with the \( ab \) axes as seen in the static case \([1]\). We also see that \( |d| \) is depressed in the forward direction of motion where the vortex currents add to \( \mathbf{J}_{ext} \). The corresponding s-wave amplitude \( |s| \) shows the four-lobe structure corresponding to the four satellite vortices \([2]\), but with a strong deformation due to the vortex motion. Also the relative position of the satellite vortices is displaced backwards with respect to the static vortex. Furthermore, there is a finite \( |s| \) induced at long distances because even in the absence of vortices a current \( \mathbf{J}_{ext} \) induces a s-wave component \([21]\). The motion induced charge \( \rho(\mathbf{r}) = \frac{1}{4\pi} \nabla \cdot \mathbf{J}_s \) is shown in Fig.2(c). It corresponds to a dipole \( \mathbf{p} \propto \mathbf{E}_{o4} \) in the direction of \( \mathbf{J}_{ext} \), and therefore \( \theta_H = 0 \). The d-wave state gives only a small additional quadrupolar contribution to \( \mathbf{E} \) in this case. On the contrary, when the current \( \mathbf{J}_{ext} \) is not in a direction of symmetry there is always a finite Hall effect, which is maximum for \( \varphi = 22.5^\circ \), the case shown in Fig.2(d-f). In Fig.2(d) and (e) we see that the amplitudes \( |d| \) and \( |s| \) show now a motion-induced deformation that is determined both by the direction of \( \mathbf{J}_{ext} \) and by the orientation of the \( ab \) axes (i.e. amplitude depression in the direction perpendicular to \( \mathbf{J}_{ext} \) and four-lobe structure in the \( ab \) orientation). This results in vortex motion with a velocity \( \mathbf{v}_L \) in a direction that is not perpendicular to \( \mathbf{J}_{ext} \). In Fig.2(f), we see from the plot of \( \rho(\mathbf{r}) \) that the induced dipole \( \mathbf{p} \) is not collinear with \( \mathbf{J}_{ext} \). Therefore, there is a transverse component in \( \mathbf{E} \) giving \( \theta_H \neq 0 \).

In general, we find that both components of the electric field, longitudinal and transversal to the current, oscillate with \( \varphi \). We find that this results in a Hall angle with a dependence of \( \tan \theta_H \propto \sin(4\varphi) \). In Fig. 3(a) we show the dependence \( \tan \theta_H \) with the angle \( \varphi \) of the current for \( \gamma_c/\gamma_d = 1, \gamma_s/\gamma_d = 2, \alpha_s/|\alpha_d| = 5, \beta_s/\beta_d = 0.5 \) and \( \mathbf{J}_{ext} = J_c = 0.6 J_c \). In conventional superconductors, \( \tan \theta_H \) should be current independent for small \( J_c \). In Fig. 3(b) we show the current dependence of \( \tan \theta_H \) for \( \varphi = 22.5^\circ \). We see that the Hall angle depends with the external current as \( \tan \theta_H \propto (J_c/j_c)^2 \). This means that the Hall effect vanishes in linear response theory when \( \Gamma_2 = 0, \gamma_{n,xy} = 0 \), as shown by Dorsey \([1]\). In a rotationally symmetric system, it should also vanish for all orders of \( J_c \). Here, we find that in the d-wave case there is a strong contribution to the Hall effect in the non-linear regime. We have also added an imaginary component to the dissipation parameter, \( \Gamma_2 \neq 0 \). We have chosen \( \Gamma_2 \) such to give a finite \( \tan \theta_H \) for \( \varphi = 0 \) similar to the typical experimental values \([15]\). We see in the insets of Figures 3(a) and (b) that the d-wave induced Hall effect
is additive to the Hall angle induced by $\Gamma_2$. We have also studied the motion of a vortex lattice for different values of the field $Bt^2/\Phi_0 = 0.007, 0.021, 0.063$, and we find $\tan \theta_H$ is nearly field independent [23].

A simple understanding of this angle-dependent non-linear Hall effect can be obtained using the generalized London theory of Affleck et al. [22]. The supercurrent $J_s$ can be written in terms of the superfluid velocity $v_s \equiv \mathbf{v} = \nabla \phi_0 - (2e/hc)\mathbf{a}$ as:

$$J_s = \frac{2e}{h}g^2\gamma_d \left\{ \mathbf{v} - 2\xi^2 \gamma_d \left[ (\mathbf{b}_b - \mathbf{a}_a)(v_b^2 - v_a^2) - (\mathbf{b}_b - \mathbf{a}_a)(\partial_b v_b - \partial_a v_a) \right] \right\},$$  

with $g^2 = |\alpha_d|^2/2\beta_d$ and the small parameter $\epsilon = (|\alpha_d|/\alpha_s)(\gamma_v/\gamma_d)^2$. The main result here is that the $v_s$ is not collinear to $J_s$, except when oriented along the directions of symmetry of the $ab$ crystal. In a purely dissipative dynamics the balance of forces acting on a vortex line is given as $\eta \mathbf{v}_L = 2e\mu_0 \mathbf{v}_s \times \hat{n}$, with $\eta$ a dissipation parameter (determined by $\xi$ and $\Gamma_1$ and $\Gamma_2$) and $\hat{n}$ the direction of $\mathbf{B}$. Here $v_s$ is the supervelocity far away from the vortex core induced by the driving current $J_{ext} \equiv J$. From (4) we obtain that the external force acting on the vortex line is, at first order in $\epsilon$,

$$f_L \approx \frac{\Phi_0}{e} \mathbf{J} \times \hat{n} + \frac{\Phi_0}{e} \frac{2e}{h} \frac{\xi^2}{2g^2\gamma_d} (J_b^2 - J_a^2) (\mathbf{b}_b - \mathbf{a}_a) \times \hat{n}.$$

Therefore, the resulting “Lorentz force” is not perpendicular to $\mathbf{J}$. Since the motion induced electric field is $\mathbf{E} = \mathbf{v}_L \times \mathbf{B}$, we obtain a Hall angle given as

$$\tan \theta_H = A\epsilon \left( \frac{J}{J_c} \right)^2 \sin 4\phi,$$

with $A$ a constant of the order of unity. Even when this analysis is very crude, we have obtained the same result by a generalization of the Bardeen-Stephen model within a perturbative treatment of the London theory of Affleck et al. in the small parameter $\epsilon$ [23]. We have verified the Eq.(4) by fitting our numerical results of Fig. 3. In Fig.3(a) we have fitted the $\sin 4\phi$ dependence and in Fig.3(b) we have fitted the $(J/J_c)^2$ dependence for low currents. In Fig.3(c) we show the dependence with $\epsilon = (|\alpha_d|/\alpha_s)(\gamma_v/\gamma_d)^2$ for different values of $\alpha_d/\alpha_s$ and $\gamma_v/\gamma_d$ taken independently. We see that for small $\epsilon$ this is an adequate parameter combination and that the Hall effect is linear in $\epsilon$. From our numerical fits we obtain $A \approx 0.28$. Therefore, a measurement of the angle and current dependence of $\tan \theta_H$ will give a direct estimate of the parameters of the GL model of Eq.(3).

In conclusion, we have shown that in a $d$-wave superconductor, the four-lobe structure of vortices leads to a non-linear angle dependent Hall effect. The effect can be explained from the fact that the supercurrent and the supervelocity are not collinear as shown in the generalized London model of [22]. Therefore, this result does not depend on the detail of the quasiparticle physics in the vortex core, which is not fully described in a TDGL treatment. Instead, this effect is clearly a consequence of the breaking of rotational invariance in $d$-wave superconductivity. Close to $T_c$, in mean field, the depairing current vanishes as $J_c \sim t^{3/2}$, with $t = |T_c - T|/T_c$, and $\epsilon \sim t$, then we have that $\tan \theta_H \sim t^{-2}$ at constant $J$. Therefore, the effect should be more noticeable close to $T_c$, where the nonlinearity in currents is stronger. An experimental setup as shown schematically in the Fig.1(b) would allow a measurement of this angle-dependent Hall effect in an untwinned crystal. As we have found, one will have $\theta_H = \theta_H^0 + \theta_H^1(\varphi)$. Typical experiments in YBa$_2$Cu$_3$O$_{6+y}$ [18], which are at $\varphi = 0$, give $\tan \theta_H^0 \approx 0.5$ close to $T_c$. For reasonable values of the parameters of the GL theory [10] we have $\epsilon \approx 0.2$, which for $J/J_c = 0.5$ and $\varphi = 22.5^\circ$ Eq.(1) gives $\tan \theta_H^1 \approx 0.02$, therefore a dependence with $\varphi$ could be observable. In principle, the anisotropy between the $a$ and $b$ axis should also be considered, but this can lead only to dependences $\sim \sin 2\varphi$, at most. Therefore, we predict that a measurement of a Hall effect with an angular dependence $\sim \sin 4\varphi$ would be a direct evidence of the $d$-wave character of the High-$T_c$ superconductivity in the transport properties of vortices.

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FIG. 1. Schematic view of (a) the vortex with the crystalline axis and the frame of reference (see text) and (b) the experimental setup proposed to measure the angle dependent Hall effect.

FIG. 2. Contour plot of the amplitudes of d-wave ((a) and (d)) and s-wave ((b) and (e)) components of the order parameter of a moving vortex for a current \( J_{ext} = 0.6J_c \). The panels (c) (f) show the contour plot of the induced electric charge. The parameters are \( 0.5\gamma_s = \gamma_d = \gamma_v, \alpha_s = 1.25|\alpha_d| \) and \( \beta_d = 2\beta_{sd} \). In (a), (b) and (c) the angle between \( J_{ext} \) and the crystal b axis is \( \varphi = 0 \) and in (d),(e) and (f) is \( \varphi = 22.5^\circ \).

FIG. 3. Plot of the Hall angle \( \tan \theta_H \) (a) as function of the angle of the current \( \varphi \) for \( J_{ext} = 0.6J_c \), (b) as a function of \( (J_{ext}/J_c)^2 \) for \( \varphi = 22.5^\circ \). The parameters are \( \gamma_s = \gamma_d = \gamma_v, \alpha_s = 5|\alpha_d| \) and \( \beta_d = 2\beta_{sd} \). The continuous lines are the fittings with Eq.(6). In the insets the squares, circles and triangles correspond to \( \Gamma_2/\Gamma_1 = 0, -0.02, -0.3 \). In (c) \( \tan \theta_H \) as a function of \( \epsilon = (\gamma_v/\gamma_d)^2|\alpha_d|/\alpha_s \) for \( J_{ext} = 0.6J_c \) and \( \varphi = 22.5^\circ \).
Figure 1  J. J. Vicente Alvarez et al.
Figure 2
J. J. Vicente Alvarez et al.
