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Research Article

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Set-valued and fuzzy stochastic integral equations driven by semimartingales under Osgood condition

Abstract: We analyze the set-valued stochastic integral equations driven by continuous semimartingales and prove the existence and uniqueness of solutions to such equations in the framework of the hyperspace of nonempty, bounded, convex and closed subsets of the Hilbert space $L^2$ (consisting of square integrable random vectors). The coefficients of the equations are assumed to satisfy the Osgood type condition that is a generalization of the Lipschitz condition. Continuous dependence of solutions with respect to data of the equation is also presented. We consider equations driven by semimartingale $Z$ and equations driven by processes $A,M$ from decomposition of $Z$, where $A$ is a process of finite variation and $M$ is a local martingale. These equations are not equivalent. Finally, we show that the analysis of the set-valued stochastic integral equations can be extended to a case of fuzzy stochastic integral equations driven by semimartingales under Osgood type condition. To obtain our results we use the set-valued and fuzzy Maruyama type approximations and Bihari’s inequality.

Keywords: Set-valued stochastic integral equation, Set-valued stochastic integrals, Fuzzy stochastic integral equation, Fuzzy stochastic differential equation, Semimartingale, Maruyama approximation, Existence and uniqueness of solution, Osgood’s condition, Bihari’s inequality

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1 Introduction

Mathematical economics and optimal control are typical fields widely using the methods of the set-valued analysis (see e.g. [10, 27]). The set-valued mappings are fundamental for differential inclusions (cf. [7, 19, 29, 52]) and for certain research areas in physics, including theory of defects in crystal, magnetic monopoles, vortices in superfluids and superconductors (see e.g. [22]). Differential inclusions and fuzzy differential inclusions [6], a generalization of differential equations, find their application in dynamical systems with incomplete information and velocities that are not uniquely determined by the state of the considered system. These can also be used for modeling systems with uncertainty resulting from vagueness of human knowledge (for example, it very often happens that the perfect value of the initial condition is not known with only the set of initial values being determined). However, the real-world phenomena characterized by uncertainty are often subjected to random forces. Hence, the deterministic differential inclusions or stochastic differential equations cannot always serve as tools adequate enough to model such phenomena. In such situations stochastic differential inclusions can be helpful. Stochastic differential inclusions combine two ways of representing uncertainty: a stochastic uncertainty generated by random noise and a contingent uncertainty driven by set-valued mappings (see e.g. [5, 8, 9, 11, 12, 14, 17, 20, 28, 30, 45, 49–51, 53, 56–58]).

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In this paper we focus on a slightly different new approach that can be useful in dealing with phenomena subjected to stochastic and contingent uncertainties. It involves the so-called set-valued stochastic integral equations (SSIEs, for short) [38–42]. Such equations have solutions that are set-valued mappings taking on values in the hyperspace of nonempty, bounded, convex and closed subsets of the Hilbert space $L^2$ (consisting of square integrable random vectors). In contrast, the stochastic differential inclusions have solutions that are single-valued stochastic processes. However, SSIEs and stochastic differential inclusions both generalize ordinary stochastic differential equations. SSIEs extend the notion of the deterministic set-valued differential equations [1–4, 15, 18, 21, 24, 25, 33, 35–37, 55] that have coalesced into an independent research discipline [31]. SSIEs are also a basis for studies of the fuzzy stochastic integral equations [38–42].

In [38–42] both the existence and uniqueness of solutions to SSIEs are proved under the Lipschitz-type conditions. In this paper, given the Osgood type condition (for which the Lipschitz condition as stated in [38–42] is a special case), we perform an analysis of the problem of existence and uniqueness of solutions to SSIEs driven by semimartingales. In order to obtain the continuity of solutions we assume that the integrators have continuous sample paths. We show that solutions to SSIEs depend continuously on the data of the equation. To avoid future repetitions, we also investigate the problem of possible generalizations of SSIEs. Thus we arrive at the fuzzy stochastic integral equations under the Osgood type condition.

The paper is organized as follows. In Section 2 we recall the definitions and properties of the set-valued stochastic trajectory integrals with respect to the processes of finite variation and martingales. Section 3 is devoted to the analysis of SSIEs with two stochastic trajectory integrals with respect to processes $A, M$ (respectively) from decomposition $Z = A + M$ of semimartingale $Z$. In Section 4 we consider SSIEs with only one integral driven by semimartingale $Z$. The integrands $F, G$ in Section 3 come from wider classes than the integrand $F$ in Section 4, but integrator $Z$ in Section 4 is more general than integrators $A, M$ in Section 3. The equations considered in Sections 3 and 4 are not equivalent. Although the results in Sections 3 and 4 are similar we decided to present them separately, since these can be viewed as independent research areas. In Section 5 and 6 we describe the results concerning fuzzy stochastic integral equations with coefficients satisfying the Osgood type condition.

2 Preliminaries

Let $(\mathcal{X}, \| \cdot \|_X)$ be a separable Banach space, $K_b^c(\mathcal{X})$ the family of all nonempty, closed, bounded and convex subsets of $\mathcal{X}$. The Hausdorff metric $H_X$ in $K_b^c(\mathcal{X})$ is defined by

$$H_X(A, B) := \max \left\{ \sup_{a \in A} \text{dist}_X(a, B), \sup_{b \in B} \text{dist}_X(b, A) \right\},$$

where $\text{dist}_X(a, B) := \inf_{b \in B} \|a - b\|_X$. It is known (cf. [26]) that $(K_b^c(\mathcal{X}), H_X)$ is a complete and separable metric space. Also, the set $K_b^c(\mathcal{X})$ has a semilinear structure under addition and scalar multiplication defined as:

$$A + B = \{a + b : a \in A, b \in B\}, \quad \lambda A = \{\lambda a : a \in A\}, \quad A, B \in K_b^c(\mathcal{X}), \quad \lambda \in \mathbb{R}.$$ 

For nonempty subsets $A_1, A_2, B_1, B_2$ of $\mathcal{X}$ it holds

$$H_X(A_1 + A_2, B_1 + B_2) \leq H_X(A_1, B_1) + H_X(A_2, B_2).$$

Let $(U, \mathcal{U}, \mu)$ be a measure space. A set-valued mapping (multifunction) $F: U \rightarrow K_b^c(\mathcal{X})$ is said to be measurable if it satisfies:

$$\{u \in U : F(u) \cap O \neq \emptyset\} \in \mathcal{U} \quad \text{for every open set} \quad O \subset \mathcal{X}.$$

A measurable multifunction $F: U \rightarrow K_b^c(\mathcal{X})$ is said to be $L^p_\mu(\mu)$-integrally bounded ($p \geq 1$), if there exists $h \in L^p(U, \mathcal{U}, \mu; \mathbb{R})$ such that the inequality $H_X(F(u), \{\theta_X\}) \leq h$ holds $\mu$-a.e., where $\theta_X$ denotes the zero element in $\mathcal{X}$. It is known (see [23]) that $F$ is $L^p_\mu(\mu)$-integrally bounded if, and only if, $u \mapsto H_X(F(u), \{\theta_X\})$ belongs to $L^p(U, \mathcal{U}, \mu; \mathbb{R})$. 

Denote $I = [0, T]$, where $T < \infty$. Let $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t \in I}, \mathbb{P})$ be a complete filtered probability space satisfying the usual hypotheses, i.e. $\{\mathcal{A}_t\}_{t \in I}$ is an increasing and right-continuous family of sub-$\sigma$-algebras of $\mathcal{A}$ and $\mathcal{A}_0$ contains all $\mathbb{P}$-null sets. We will also assume that $\sigma$-algebra $\mathcal{A}_t$ is separable with respect to probability measure $\mathbb{P}$. Let $\mathcal{P}$ denote the $\sigma$-algebra of predictable elements in $I \times \Omega$, i.e. the smallest $\sigma$-algebra with respect to which every $\{\mathcal{A}_t\}$-adapted stochastic process with left-continuous paths is measurable. A stochastic process $f : I \times \Omega \to \mathbb{R}^d$ is called predictable if $f(\cdot, \cdot)$ is $\mathcal{P}$-measurable. A set-valued stochastic process $F : I \times \Omega \to \mathcal{K}^b_{\mathbb{R}}(\mathbb{R}^d)$ is predictable if $F(\cdot, \cdot)$ is a $\mathcal{P}$-measurable multifunction.

Let $Z : I \times \Omega \to \mathbb{R}$ be a semimartingale with the canonical representation

$$Z = A + M, \quad Z(0) = 0, \quad A(0) = 0, \quad M(0) = 0,$$

(1)

where $A : I \times \Omega \to \mathbb{R}$ is an $\{\mathcal{A}_t\}$-adapted cádlág stochastic process of finite variation, $M : I \times \Omega \to \mathbb{R}$ is a local $\{\mathcal{A}_t\}$-martingale. It is known that if $Z$ has continuous sample paths then $A, M$ have continuous sample paths, too. Also $A$ is predictable and representation (1) is unique. Obviously, each process $A$ and $M$ can be treated as a semimartingale.

Since $A$ is of finite variation, almost each (w.r.t. to $P$) sample path $A(\cdot, \omega) : I \to \mathbb{R}$ generates a measure $\Gamma_{A(\cdot, \omega)}$ with the total variation on the interval $[0, t]$ given by $|A(\omega)|_t = \int_0^t \Gamma_{A(\cdot, \omega)}(ds)$. For a local martingale $M$ one can define the quadratic variation process $[M] : I \times \Omega \to \mathbb{R}$ (cf. [16]). Now we denote by $H^2$ the set of all semimartingales $Z : I \times \Omega \to \mathbb{R}$ with finite norm $\|Z\|_{H^2}$, where

$$\|Z\|_{H^2} := \|[M]_T^{1/2}\|_{L^2} + \|\Gamma_{A(\cdot, \omega)}\|_{L^2},$$

where $L^2 := L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d)$. For further aims we denote also $L^2_t := L^2(\Omega, \mathcal{A}_t, \mathbb{P}; \mathbb{R}^d), t \in I$.

It is known (cf. [48] chap. II, sec. 6, Corollary 4) that for a continuous semimartingale $Z \in H^2$ the process $M$ in (1) is a continuous square integrable martingale and $\mathbb{E}|A|_T^2 < \infty$.

The processes $A, M$ from the representation (1) of the semimartingale $Z$ induce two measures $\mu_A, \mu_M$ defined on $(I \times \Omega, \mathcal{P})$. The measure $\mu_A$ is defined similarly as in [14]

$$\mu_A(C) := \int \int_{\Omega} 1_C(t, \omega)|A(\omega)|_T \Gamma_{A(\cdot, \omega)}(dt) P(d\omega) \quad \text{for} \quad C \in \mathcal{P}.$$

For $f \in L^2_T(\mu_A)$, where

$$L^2_T(\mu_A) := L^2(I \times \Omega, \mathcal{P}, \mu_A; \mathbb{R}^d)$$

one can define the single-valued stochastic Lebesgue–Stieltjes integral $\int_0^t f(s) dA(s)$ sample path by sample path (cf. [48]). Note that

$$\mathbb{E}\left(\int_0^t (f(s) dA(s))_s \right)^2 \leq \int \int \left(\int_0^t \|f(s, \omega)\|^2_{\mathbb{R}^d} d\Gamma_{A(\cdot, \omega)}(ds) \right)^2 P(d\omega)$$

$$\leq \int \int \left(\int_0^t (|A(\omega)|_s - |A(\omega)|_0) \int_0^t \|f(s, \omega)\|^2_{\mathbb{R}^d} d\Gamma_{A(\cdot, \omega)}(ds) \right) P(d\omega)$$

$$\leq \int \int \left(\int_0^t |A(\omega)|_T \int_0^t \|f(s, \omega)\|^2_{\mathbb{R}^d} d\Gamma_{A(\cdot, \omega)}(ds) \right) P(d\omega)$$

$$= \int \int |f|^2_{\mathbb{R}^d} d\mu_A.$$

(2)

The second measure $\mu_M$ is the well-known Doléan–Dade measure (cf. [16]), i.e. a measure such that

$$\mu_M(\{0\} \times A_0) = 0, \quad \mu_M(\{s, t\} \times A) = \mathbb{E}1_A(M(t) - M(s))^2,$$

where $A_0 \in \mathcal{A}_0, 0 \leq s < t \leq T, A \in \mathcal{A}_s$. For $f \in L^2_T(\mu_M)$, where

$$L^2_T(\mu_M) := L^2(I \times \Omega, \mathcal{P}, \mu_M; \mathbb{R}^d),$$

the measure $\mu_M$ is given by

$$\mu_M(C) := \int \int_{\Omega} 1_C(t, \omega)|M(\omega)|_T \Gamma_{M(\cdot, \omega)}(dt) P(d\omega) \quad \text{for} \quad C \in \mathcal{P}.$$
and \( t \in I \) one can define the single-valued stochastic integral \( f^t_0 f(s) dM_s \) and we have (cf. [16])

\[
\int_{[0,t] \times \Omega} f^2_{\mathbb{R}^d} d\mu_M = \mathbb{E} \int_{0}^{t} \| f(s) \|^2_{\mathbb{R}^d} d[M](s) = \mathbb{E} \left\| \int_{0}^{t} f(s) dM(s) \right\|^2_{\mathbb{R}^d}.
\]  

**Remark 2.1.** Let \( Z: I \times \Omega \to \mathbb{R} \) be a semimartingale with a unique representation (1). Then \( Z \in \mathcal{H}^2 \) if and only if the measures \( \mu_M, \mu_A \) are finite.

Let us denote

\[
\Delta_A := \mu_A(I \times \Omega), \quad \Delta_M := \mu_M(I \times \Omega).
\]

Let \( F, G: I \times \Omega \to \mathcal{K}^b_C(\mathbb{R}^d) \) be some predictable set-valued stochastic processes. We will assume throughout the paper that \( F \) is \( L^2_P(\mu_A) \)-integrally bounded and \( G \) is \( L^2_P(\mu_M) \)-integrally bounded. For such processes let us define the sets

\[
S^2_P(F, \mu_A) := \{ f \in L^2_P(\mu_A) : f \in F, \mu_A\text{-a.e.} \},
\]

\[
S^2_P(G, \mu_M) := \{ g \in L^2_P(\mu_M) : g \in G, \mu_M\text{-a.e.} \}.
\]

Due to the Kuratowski and Ryll-Nardzewski Selection Theorem we have \( S^2_P(F, \mu_A) \neq \emptyset, S^2_P(G, \mu_M) \neq \emptyset \), and we can define the set-valued stochastic trajectory integrals (see [41]).

**Definition 2.2.**

(a) For a predictable and \( L^2_P(\mu_A) \)-integrally bounded set-valued stochastic process \( F: I \times \Omega \to \mathcal{K}^b_C(\mathbb{R}^d) \) and for \( \tau, t \in \mathbb{R}_+, \tau < t \) the set-valued stochastic trajectory integral (over interval \([\tau, t]\)) of \( F \) with respect to the bounded variation process \( A \) is the following subset of \( L^2_I \)

\[
\int_{\tau}^{t} F(s) dA(s) := \{ f(s) dA(s) : f \in S^2_P(F, \mu_A) \}.
\]

(b) For a predictable and \( L^2_P(\mu_M) \)-integrally bounded set-valued stochastic process \( G: I \times \Omega \to \mathcal{K}^b_C(\mathbb{R}^d) \) and for \( \tau, t \in \mathbb{R}_+, \tau < t \) the set-valued stochastic trajectory integral (over interval \([\tau, t]\)) of \( G \) with respect to the martingale \( M \) is the following subset of \( L^2_I \)

\[
\int_{\tau}^{t} G(s) dM(s) := \{ g(s) dM(s) : g \in S^2_P(G, \mu_M) \}.
\]

It is known (see [41]) that \( \int_{\tau}^{t} F(s) dA(s) \) and \( \int_{\tau}^{t} G(s) dM(s) \) are nonempty, bounded, convex, closed and weakly compact subsets of \( L^2_I \). We will exploit the following properties.

**Proposition 2.3** ([41]). Let \( F_1, F_2: I \times \Omega \to \mathcal{K}^b_C(\mathbb{R}^d) \) be the predictable and \( L^2_P(\mu_A) \)-integrally bounded set-valued stochastic processes. Let \( G_1, G_2: I \times \Omega \to \mathcal{K}^b_C(\mathbb{R}^d) \) be the predictable and \( L^2_P(\mu_M) \)-integrally bounded set-valued stochastic processes. Then

(i) for every \( \tau, a, t \in I, \tau \leq a \leq t \)

\[
\int_{\tau}^{a} F_1(s) dA(s) = \int_{\tau}^{a} F_1(s) dA(s) + \int_{a}^{t} F_1(s) dA(s),
\]

\[
\int_{\tau}^{a} G_1(s) dM(s) = \int_{\tau}^{a} G_1(s) dM(s) + \int_{a}^{t} G_1(s) dM(s).
\]
(ii) for every \( t, \tau \in I, \tau < t \)

\[
H_{L^2}^2 \left( \int_{\tau}^{t} F_1(s) dA(s), \int_{\tau}^{t} F_2(s) dA(s) \right) \leq \int_{[\tau,t] \times \Omega} H_{\mathbb{R}^d}^2 (F_1(s, \omega), F_2(s, \omega)) \mu_A (ds, d\omega),
\]

\[
H_{L^2}^2 \left( \int_{\tau}^{t} G_1(s) dM(s), \int_{\tau}^{t} G_2(s) dM(s) \right) \leq \int_{[\tau,t] \times \Omega} H_{\mathbb{R}^d}^2 (G_1(s, \omega), G_2(s, \omega)) \mu_M (ds, d\omega),
\]

(iii) for every \( \tau \in I \) the mappings

\[
\tau, T \ni t \mapsto \int_{\tau}^{t} F_1(s) dA(s) \in K^b_{L^2}, \quad \tau, T \ni t \mapsto \int_{\tau}^{t} G_1(s) dM(s) \in K^b_{L^2}
\]

are \( H_{L^2} \)-continuous.

3 Set-valued equations driven by processes \( A, M \) from decomposition of semimartingale \( Z \)

We begin this section with a discussion on potential utility of the theory of set-valued stochastic integral equations. We indicate that such equations can be used in modeling dynamics of real-world phenomena subjected to uncertainty and also in modeling optimality problems.

Consider a situation when a pharmacist grows a population of a bacteria species that lives on a given bounded territory. Assume that the number of individuals is affected by random factors and the pharmacist can control the growth of the population by steerage of feeding. Then the number of individuals at the instant \( t \in I \), denoted by \( n(t) \), is random and could be modelled by the controlled stochastic integral equation

\[
n(t) = n_0 + \int_{0}^{t} f(n(s), u(s)) ds + \int_{0}^{t} g(n(s), u(s)) dW(s), \quad t \in I, \quad P\text{-a.e.},
\]

(4)

where \( n_0: \Omega \rightarrow \mathbb{R} \) denotes initial number of individuals, \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) symbolizes drift coefficient, \( g: \mathbb{R}^2 \rightarrow \mathbb{R} \) is a diffusion coefficient, \( u \) denotes a strategy of feeding, \( u \in U \), \( U \) is a set of feeding actions - controls, \( W \) denotes the Wiener process. Assuming that \( n(t) \in L^2 \) for \( t \in I \) we can transform (4) to an equation in the space \( L^2 \), i.e. to the equation

\[
n(t) = n_0 + \int_{0}^{t} \tilde{f}(s, n(s), u) ds + \int_{0}^{t} \tilde{g}(s, n(s), u) dW(s), \quad t \in I,
\]

(5)

where the mappings \( \tilde{f}, \tilde{g}: I \times \Omega \times L^2 \times U \rightarrow \mathbb{R} \) are defined as follows

\[
f(s, \omega, n, u) := f(n(\omega), u(s, \omega)) \text{ and } \tilde{g}(s, \omega, n, u) := g(n(\omega), u(s, \omega)).
\]

In most situations the pharmacist is not able to determine \( n_0 \) precisely. Suppose that it is only known that \( n_0 \) is an \( \mathcal{A}_0 \)-measurable random variable with values bounded by a constant \( \xi > 0 \). In this way, under presence of uncertainty, the initial number of individuals can be viewed as the following set

\[
X_0 := \{ n_0 \in L^2_0 : 0 \leq n_0 \leq \xi \} \in K^b_{L^2}.
\]

The uncertain number of individuals is governed by dynamics of set-valued solution to the set-valued stochastic integral equation

\[
X(t) = X_0 + \int_{0}^{t} F(s, X(s)) ds + \int_{0}^{t} G(s, X(s)) dW(s), \quad t \in I,
\]

(6)
where $F, G : I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R})$ are defined as follows

$$F(s, \omega, A) := \bar{c}(\bigcup_{n \in A} \bigcup_{u \in U} \tilde{f}(s, \omega, n, u)) \quad \text{and} \quad G(s, \omega, A) := \bar{c}(\bigcup_{n \in A} \bigcup_{u \in U} \tilde{g}(s, \omega, n, u)),$$

The symbol $\bar{c}(B)$ denotes the closed and convex hull of the set $B$.

The equation (6) is a set-valued stochastic integral equation driven by processes from decomposition of semimartingale $Z = A + M$ with $A(s) = s$ and $M(s) = W(s)$. A problem of finding its solution is a natural question. The set-valued solution can give informations on an approximate dynamics of population growth.

Throughout the paper we will assume that $A, M$ are such that $\mu_A^\Omega, \mu_M^\Omega$ are data of the equation.

Definition 3.1. An $H_{L^2}$-continuous mapping $X : I \to \mathcal{K}_c^b(L^2)$ is called the solution to (7) if $X$ satisfies (7) and $X(t) \in \mathcal{K}_c^b(L^2)$ for every $t \in I$. A solution $X$ is said to be unique if $X(t) = Y(t)$ for $t \in I$, where $Y$ is any solution to (7).

By $\mu_A^\Omega, \mu_M^\Omega$ we denote measures on $(I, \beta(I))$, where $\beta(I)$ denotes the Borel $\sigma$-algebra of subsets of $I$. They are defined as follows

$$\mu_A^\Omega(B) = \mu_A(B \times \Omega), \quad \mu_M^\Omega(B) = \mu_M(B \times \Omega) \quad \text{for} \quad B \in \beta(I).$$

Throughout the paper we will assume that $A, M$ are such that $\mu_A^\Omega, \mu_M^\Omega$ are absolutely continuous with respect to the Lebesgue measure $\lambda$, so that the Radon–Nikodem derivatives $\frac{d\mu_A^\Omega}{d\lambda}, \frac{d\mu_M^\Omega}{d\lambda}$ exist. Also we will assume that

$$S_A := \text{ess sup}_I \frac{d\mu_A^\Omega}{d\lambda}, \quad S_M := \text{ess sup}_I \frac{d\mu_M^\Omega}{d\lambda}$$

are finite numbers.

Remark 3.2. Let $Z : I \times \Omega \to \mathbb{R}$ be a semimartingale with representation (1) in which $A(t) = t$ and $M(t) = W(t)$, where $W$ is $\{A_t\}$-Wiener process. Then $\mu_A^\Omega = T\lambda, \mu_M^\Omega = \mu_W^\Omega = \lambda, S_A = T, S_M = S_W = 1$. Hence in this case the assumptions written above are fulfiled.

For our further aims we denote

$$v_A(t) := \mu_A^\Omega([0, t]), \quad v_M(t) := \mu_M^\Omega([0, t]), \quad t \in I.$$

It is clear that functions $v_A, v_M : I \to [0, \infty)$ are uniformly continuous, non-negative, nondecreasing and bounded. The function $v : I \to [0, \infty)$ defined as

$$v(t) := v_A(t) + v_M(t), \quad t \in I,$$
possesses these properties as well.

In this part of the paper we shall show the existence and uniqueness theorem for solutions to (7) in the case when $F$ and $G$ satisfy the Osgood type condition. Therefore we formulate the following assumptions:

(A1) for every $A \in \mathcal{S}_c(L^2)$ the set-valued stochastic processes $F(\cdot, A), G(\cdot, A) : I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ are predictable,

(A2) there exists a constant $C > 0$ such that for
\[ \mu_{\lambda} \text{-a.e. it holds } H_{\mathbb{R}^d}^2 \left( F(t, \omega, \{\theta_{\mathbb{R}^d}\}) \right) \leq C, \]
\[ \mu_{\lambda} \text{-a.e. it holds } H_{\mathbb{R}^d}^2 \left( G(t, \omega, \{\theta_{\mathbb{R}^d}\}) \right) \leq C, \]
(A3) there exists a continuous, nondecreasing, concave function $\kappa : [0, \infty) \to [0, \infty)$ satisfying:
\[ \int_0^1 \frac{1}{\kappa(s)} ds = +\infty, \kappa(0) = 0, \kappa(s) > 0 \text{ for } s > 0, \]
\[ \mu_{\lambda} \text{-a.e. it holds } H_{\mathbb{R}^d}^2 \left( F(t, \omega, A), F(t, \omega, B) \right) \leq \kappa \left( H_{L^2}^2(A, B) \right), A, B \in \mathcal{K}_c^b(L^2), \]
\[ \mu_{\lambda} \text{-a.e. it holds } H_{\mathbb{R}^d}^2 \left( G(t, \omega, A), G(t, \omega, B) \right) \leq \kappa \left( H_{L^2}^2(A, B) \right), A, B \in \mathcal{K}_c^b(L^2). \]

The assumption (A3) is called the Osgood type condition and it is weaker than the Lipschitz condition imposed in [38–42].

**Remark 3.3.** If the function $\kappa$ is of the form $\kappa(t) = Mt$, where $M$ is a positive constant, then condition (A3) reduces to the Lipschitz condition.

Some another known (cf. [43]) examples of the function $\kappa : [0, \infty) \to [0, \infty)$ which can appear in the Osgood condition are $\kappa_1, \kappa_2$ described below
\[ \kappa_1(t) = \begin{cases} \log(t^{(1)}), & 0 \leq t \leq \delta, \\ \delta \log(\delta^{-1}) + \kappa_2'(\delta)(t-\delta), & t > \delta, \end{cases} \]
\[ \kappa_2(t) = \begin{cases} t \log(t^{(1)}), & 0 \leq t \leq \delta, \\ \delta \log(\delta^{-1}) \log(t^{(1)}), & t > \delta, \end{cases} \]
where $\delta \in (0, 1)$ is sufficiently small and $\kappa_2'(\delta)(k = 1, 2)$ denotes left-sided derivative of $\kappa_k$ at $\delta$.

Since in (A3) the function $\kappa$ is concave we can derive the following claim.

**Remark 3.4.** If $F, G : I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)$ satisfy (A2) and (A3), then there exists a constant $K > 0$ such that
\[ \mu_{\lambda} \text{-a.e. it holds } H_{\mathbb{R}^d}^2 \left( F(t, \omega, A), \{\theta_{\mathbb{R}^d}\} \right) \leq K \left( 1 + H_{L^2}^2(A, \{\theta_{\mathbb{R}^d}\}) \right) \text{ for } A \in \mathcal{K}_c^b(L^2), \]
\[ \mu_{\lambda} \text{-a.e. it holds } H_{\mathbb{R}^d}^2 \left( G(t, \omega, A), \{\theta_{\mathbb{R}^d}\} \right) \leq K \left( 1 + H_{L^2}^2(A, \{\theta_{\mathbb{R}^d}\}) \right) \text{ for } A \in \mathcal{K}_c^b(L^2). \]

In the derivation of existence of solution to (7) we will use the sequence $\{X_n\}_{n=1}^\infty$ of approximate solutions $X_n : I \to \mathcal{K}_c^b(L^2)$ of Maruyama type, i.e.
\[ X_n(0) = X_0, \]
\[ X_n(t) = X_0 \left( \frac{k-1}{n} \right) + \int_{\frac{k-1}{n}}^t F \left( s, X_n \left( \frac{k-1}{n} \right) \right) dA(s) + \int_{\frac{k-1}{n}}^t G \left( s, X_n \left( \frac{k-1}{n} \right) \right) dM(s) \]
for $t \in \left( \frac{k-1}{n}, \frac{k}{n} \right) \cap I, k = 1, 2, \ldots$

Note that each approximant $X_n$ appearing above is $H_{L^2}$-continuous and $X_n(t) \in \mathcal{K}_c^b(L^2)$ for every $t \in I$. Moreover if we define $\bar{X}_n : I \to \mathcal{K}_c^b(L^2)$ $(n \in \mathbb{N})$ by
\[ \bar{X}_n(t) = X_0 \mathbf{1}_{(0)}(t) + \sum_{k \geq 1} X_n \left( \frac{k-1}{n} \right) \mathbf{1}_{(\frac{k-1}{n}, \frac{k}{n})}(t) \text{ for } t \in I, \]
then we can write
\[ X_n(t) = X_0 + \int_0^t F(s, \tilde{X}_n(s))dA(s) + \int_0^t G(s, \tilde{X}_n(s))dM(s) \quad \text{for} \quad t \in I. \]

Before we formulate the main result concerning equation (7), we list some useful lemmata. We begin with Bihari’s inequality which will play a key role in derivations of our results.

**Lemma 3.5** ([13]). Let \( f : I \to [0, \infty) \) be continuous and \( w : I \to [0, \infty) \) be continuous and nondecreasing, and let \( g : [0, \infty) \to [0, \infty) \) be continuous nondecreasing function such that \( g(s) > 0 \) for \( s > 0 \). If \( f \) satisfies the following integral inequality
\[ f(t) \leq c + \int_0^t g(f(s))dw(s), \quad \text{for} \quad t \in I, \]
where \( c \) is a positive constant, then
\[ f(t) \leq J^{-1}(J(c) + w(t)) \quad \text{for all} \quad t \in I \quad \text{such that} \quad J(c) + w(t) \in \text{Dom}(J^{-1}), \]
where
\[ J(r) = \int_1^r \frac{ds}{g(s)}, \quad r > 0, \]
and \( J^{-1} \) is the inverse function of \( J \).

In our setting the role of function \( g \), which appears in Lemma 3.5, will be played by the function \( \kappa \) from assumption (A3). Then it is easy to observe that in our setting we have \( J(r) \to 0^+ \) as \( r \to 0^+ \) and \( J^{-1}(r) \to \infty \) as \( r \to \infty \).

**Lemma 3.6.** Assume that \( X_0 \in \mathcal{K}_{c}^h(L_2^0) \) and \( F, G \) satisfy (A1)–(A3). Then
\[ \sup_n \sup_{t \in I} H_{L^2}^2(X_n(t), \{\theta_{L^2}\}) \leq M_1, \]
where \( M_1 = \left(1 + 3H_{L^2}^2(X_0, \{\theta_{L^2}\})\right) \exp\{3K(\Delta_A + \Delta_M)\} - 1. \)

**Proof.** For \( n \in \mathbb{N} \) and \( t \in I \) we have, due to Proposition 2.3 (ii),
\[ H_{L^2}^2(X_n(t), \{\theta_{L^2}\}) \leq 3H_{L^2}^2(X_0, \{\theta_{L^2}\}) + 3H_{L^2}^2\left(\int_0^t F(s, \tilde{X}_n(s))dA(s), \{\theta_{L^2}\}\right) \]
\[ + 3H_{L^2}^2\left(\int_0^t G(s, \tilde{X}_n(s))dM(s), \{\theta_{L^2}\}\right) \]
\[ \leq 3H_{L^2}^2(X_0, \{\theta_{L^2}\}) + 3 \int_{[0, t] \times \Omega} H_{W^d}^2(F(s, \omega, \tilde{X}_n(s)), \{\theta_{W^d}\}) \mu_A(ds, d\omega) \]
\[ + 3 \int_{[0, t] \times \Omega} H_{W^d}^2(G(s, \omega, \tilde{X}_n(s)), \{\theta_{W^d}\}) \mu_M(ds, d\omega). \]

Applying Remark 3.4 we get
\[ H_{L^2}^2(X_n(t), \{\theta_{L^2}\}) \leq 3H_{L^2}^2(X_0, \{\theta_{L^2}\}) + 3K \int_{[0, t] \times \Omega} \left(1 + H_{L^2}^2(\tilde{X}_n(s), \{\theta_{L^2}\})\right) \mu_A(ds, d\omega) \]
\[ + 3K \int_{[0, t] \times \Omega} \left(1 + H_{L^2}^2(\tilde{X}_n(s), \{\theta_{L^2}\})\right) \mu_M(ds, d\omega). \]
\[ = 3H^2 \left( X_0, \{ \theta_{L^2} \} \right) + 3K \int_{[0,t]} \left( 1 + H^2 \left( \tilde{X}_n(s), \{ \theta_{L^2} \} \right) \right) \mu^Q_A (ds) \\
+ 3K \int_{[0,t]} \left( 1 + H^2 \left( \tilde{X}_n(s), \{ \theta_{L^2} \} \right) \right) \mu^Q_M (ds) \\
= 3H^2 \left( X_0, \{ \theta_{L^2} \} \right) + 3K \int_0^t \left( 1 + H^2 \left( \tilde{X}_n(s), \{ \theta_{L^2} \} \right) \right) dv_A (s) \\
+ 3K \int_0^t \left( 1 + H^2 \left( \tilde{X}_n(s), \{ \theta_{L^2} \} \right) \right) dv_M (s) \\
= 3H^2 \left( X_0, \{ \theta_{L^2} \} \right) + 3K \int_0^t \left( 1 + H^2 \left( \tilde{X}_n(s), \{ \theta_{L^2} \} \right) \right) dv(s). \]

Hence for every \( n \in \mathbb{N} \) and every \( t \in I \)

\[
\sup_{r \in [0,t]} H^2 \left( X_n(r), \{ \theta_{L^2} \} \right) \leq 3H^2 \left( X_0, \{ \theta_{L^2} \} \right) + 3K \int_0^t \left( 1 + \sup_{r \in [0,s]} H^2 \left( \tilde{X}_n(r), \{ \theta_{L^2} \} \right) \right) dv(s). \]

Further we can infer that

\[
1 + \sup_{r \in [0,t]} H^2 \left( X_n(r), \{ \theta_{L^2} \} \right) \leq 1 + 3H^2 \left( X_0, \{ \theta_{L^2} \} \right) \\
+ 3K \int_0^t \left( 1 + \sup_{r \in [0,s]} H^2 \left( \tilde{X}_n(r), \{ \theta_{L^2} \} \right) \right) dv(s). \]

Using Lemma 3.5 we obtain

\[
1 + \sup_{r \in [0,t]} H^2 \left( X_n(r), \{ \theta_{L^2} \} \right) \leq \left( 1 + 3H^2 \left( X_0, \{ \theta_{L^2} \} \right) \right) \exp \left\{ 3K \left( \mu^Q_A ([0,t]) + \mu^Q_M ([0,t]) \right) \right\} \]

for \( n \in \mathbb{N}, t \in I \).

**Lemma 3.7.** Let \( X_0 \in K^q_\mathbb{C}(L^0_\mathbb{R}) \) and \( F, G \) satisfy (A1)–(A3). Then for every \( n \in \mathbb{N} \) and every \( r, t \in I, r \leq t \)

\[
H^2 \left( X_n(t), X_n(r) \right) \leq M_2 (t - r), \]

where \( M_2 = 2K (1 + M_1) (S_A + S_M), M_1 \) is defined as in Lemma 3.6.

**Proof.** Due to Proposition 2.3 (i) and (ii), Remark 3.4, Lemma 3.6 we have: for \( n \in \mathbb{N} \) and \( r \leq t \)

\[
H^2 \left( X_n(t), X_n(r) \right) \leq H^2 \left( \int_r^t F(s, \tilde{X}_n(s)) dA(s) + \int_r^t G(s, \tilde{X}_n(s)) dM(s), \{ \theta_{L^2} \} \right) \\
\leq 2K \int_{[r,t]} \left( 1 + H^2 \left( \tilde{X}_n(s), \{ \theta_{L^2} \} \right) \right) \mu^Q_A (ds) + 2K \int_{[r,t]} \left( 1 + H^2 \left( \tilde{X}_n(s), \{ \theta_{L^2} \} \right) \right) \mu^Q_M (ds) \\
\leq 2K \int_{[r,t]} \left( 1 + \sup_{w \in I} H^2 \left( X_n(w), \{ \theta_{L^2} \} \right) \right) \mu^Q_A (ds) \\
+ 2K \int_{[r,t]} \left( 1 + \sup_{w \in I} H^2 \left( X_n(w), \{ \theta_{L^2} \} \right) \right) \mu^Q_M (ds) \\
\leq 2K (1 + M_1) (S_A + S_M) (t - r). \]

\[ \square \]
Corollary 3.8. Assume that \( X_0 \in \mathcal{K}_c^b(L^2_{\mathcal{A}}) \) and \( F, G \) satisfy (A1)–(A3). Then for \( n \in \mathbb{N} \) and \( t \in I \)
\[
H^2_{L^2}(X_n(t), \hat{X}_n(t)) \leq \frac{M_2}{n},
\]
where \( M_2 \) is as in Lemma 3.7.

Lemma 3.9. Let \( X_0 \in \mathcal{K}_c^b(L^2_{\mathcal{A}}) \) and \( F, G \) satisfy (A1)–(A3). Then
\[
\sup_{t \in I} H^2_{L^2}(X_n(t), X_m(t)) \to 0, \text{ as } n, m \to \infty.
\]

Proof. Observe that for \( n, m \in \mathbb{N} \) and for \( t \in I \) we have
\[
H^2_{L^2}(X_n(t), X_m(t)) \leq 2H^2_{L^2}\left( \int_0^t F(s, \hat{X}_n(s))dA(s), \int_0^t F(s, \hat{X}_m(s))dA(s) \right)
+ 2H^2_{L^2}\left( \int_0^t G(s, \hat{X}_n(s))dM(s), \int_0^t G(s, \hat{X}_m(s))dM(s) \right)
\leq 6 \int_{[0,t] \times \Omega} \left[ H^2_{\mathcal{R}^d}(F(s, \omega, \hat{X}_n(s)), F(s, \omega, X_n(s)))
+ H^2_{\mathcal{R}^d}(F(s, \omega, X_n(s)), F(s, \omega, X_m(s)))
+ H^2_{\mathcal{R}^d}(F(s, \omega, \hat{X}_m(s)), F(s, \omega, X_m(s)))
+ 6 \int_{[0,t] \times \Omega} \left[ \kappa \left( H^2_{L^2}(X_n(s), X_m(s)) \right) + \kappa \left( H^2_{L^2}(X_n(s), X_m(s)) \right) \right] \mu_A(ds, d\omega)
+ 6 \int_{[0,t] \times \Omega} \left[ \kappa \left( H^2_{L^2}(X_m(s), \hat{X}_m(s)) \right) \right] \mu_M(ds)
+ 6 \int_{[0,t] \times \Omega} \left[ \kappa \left( H^2_{L^2}(X_n(s), \hat{X}_m(s)) \right) \right] \mu_M(ds)
+ 6 \int_{[0,t] \times \Omega} \left[ \kappa \left( H^2_{L^2}(X_n(s), \hat{X}_m(s)) \right) \right] \mu_M(ds)
\]

Since \( \kappa \) is nondecreasing, using Corollary 3.8 we get
\[
H^2_{L^2}(X_n(t), X_m(t)) \leq A_{n,m} + 6 \int_{[0,t]} \kappa \left( H^2_{L^2}(X_n(s), X_m(s)) \right) \mu_M(ds)
+ 6 \int_{[0,t]} \kappa \left( H^2_{L^2}(X_n(s), X_m(s)) \right) \mu_M(ds)
= A_{n,m} + 6 \int_0^t \kappa \left( \sup_{w \in [0,s]} H^2_{L^2}(X_n(w), X_m(w)) \right) dv(s),
\]
where \( A_{n,m} = 6 \left[ \kappa \left( \frac{M_2}{n} \right) + \kappa \left( \frac{M_2}{m} \right) \right] (\Delta_A + \Delta_M) \). Hence for \( t \in I \)
\[
\sup_{w \in [0,t]} H^2_{L^2}(X_n(w), X_m(w)) \leq A_{n,m} + 6 \int_0^t \kappa \left( \sup_{w \in [0,s]} H^2_{L^2}(X_n(w), X_m(w)) \right) dv(s).
\]
Finally, by Bihari’s inequality we infer that for every \( t \in I \)
\[
\sup_{w \in [0,t]} H_{L^2}^2(X_n(w), X_m(w)) \leq J^{-1}(J(A_{n,m}) + 6\nu(t)) \leq J^{-1}(J(A_{n,m}) + 6(\Delta_A + \Delta_M)).
\]

Since \( J^{-1}(J(A_{n,m}) + 6(\Delta_A + \Delta_M)) \xrightarrow{n,m \to \infty} 0 \), we get the assertion. \( \square \)

Now we are in a position to prove a main result of this section.

**Theorem 3.10.** Assume that \( X_0 \in K_c^b(L^2_0) \) and \( F, G \) satisfy (A1)–(A3). Then there exists a unique solution \( X \) to (7). Moreover
\[
\sup_{t \in I} H_{L^2}^2(X(t), \{\theta_{L^2}\}) \leq M_1,
\]
where \( M_1 \) is as in Lemma 3.6.

**Proof.** Let \( C = C(I, K_c^b(L^2)) \) denote the set of all \( H_{L^2} \)-continuous mappings \( X : I \to K_c^b(L^2) \). It is clear that \( C \) endowed with the supremum metric becomes a complete metric space.

Due to Lemma 3.9 the Maruyama sequence \( \{X_n\} \) is a Cauchy sequence in \( C \). Therefore there exists \( X^* \in C \) such that
\[
\sup_{t \in I} H_{L^2}^2(X_n(t), X^*(t)) \to 0, \quad \text{as} \quad n \to \infty.
\]

Hence \( \sup_{t \in I} H_{L^2}^2(X^*(t), \{\theta_{L^2}\}) \leq M_1 \). Also, \( X_n(t) \in K_c^b(L^2_1) \) for every \( t \in I \), since \( X_n(t) \in K_c^b(L^2_1) \) and \( \{X_n(t)\} \) is a Cauchy sequence in a complete metric space \( (K_c^b(L^2_1), H_{L^2}) \). We shall show that \( X^* \) is the desired solution to (7). To this end let us note that for \( t \in I \) we have
\[
H_{L^2}^2\left(X^*(t), X_0 + \int_0^t F(s, X^*(s))dA(s) + \int_0^t G(s, X^*(s))dM(s)\right)
\]
\[
\leq 3H_{L^2}^2(\{X^*(t), X_n(t)\} + 3H_{L^2}^2\left(X_n(t), X_0 + \int_0^t F(s, \tilde{X}_n(s))dA(s) + \int_0^t G(s, \tilde{X}_n(s))dM(s)\right)
\]
\[
+ \int_0^t F(s, X^*(s))dA(s) + \int_0^t G(s, X^*(s))dM(s).\]

It is obvious that two of first summands converge to zero. For the third summand
\[
S_3 = H_{L^2}^2\left(\int_0^t F(s, \tilde{X}_n(s))dA(s) + \int_0^t G(s, \tilde{X}_n(s))dM(s),
\right)
\]
\[
\int_0^t F(s, X^*(s))dA(s) + \int_0^t G(s, X^*(s))dM(s)\right)
\]
we have
\[
S_3 \leq 4H_{L^2}^2\left(\int_0^t F(s, X_n(s))dA(s), \int_0^t F(s, X_n(s))dA(s)\right)
\]
\[
+ 4H_{L^2}^2\left(\int_0^t F(s, X_n(s))dA(s), \int_0^t F(s, X^*(s))dA(s)\right)
\]
Using Corollary 3.8 we obtain
\begin{align*}
S_3 &\leq 4 \kappa \left( \frac{M_2}{n} \right) (\Delta_A + \Delta_M) + 4 \int_{[0,t]} \kappa \left( H^2_{L^2}(X_n(s), X^*(s)) \right) \mu_A^2 (ds) \\
&\quad + 4 \int_{[0,t]} \kappa \left( H^2_{L^2}(X_n(s), X^*(s)) \right) \mu_M^2 (ds) \\
&\leq 4(\Delta_A + \Delta_M) \left[ \kappa \left( \frac{M_2}{n} \right) + \kappa \left( \sup_{s \in I} H^2_{L^2}(X_n(s), X^*(s)) \right) \right].
\end{align*}
Now it is easy to see that the last expression converges to zero. Hence we obtain
\begin{equation*}
X^*(t) = X_0 + \int_0^t F(s, X^*(s)) dA(s) + \int_0^t G(s, X^*(s)) dM(s) \quad \text{for} \quad t \in I.
\end{equation*}
What is left is to prove that solution $X^*$ is unique. Let us assume that $X^*, Y^*$ are two solutions to (7). Then for every $t \in I$ it holds
\begin{align*}
H^2_{L^2} (X^*(t), Y^*(t)) &\leq 2H^2_{L^2} \left( \int_0^t F(s, X^*(s)) dA(s), \int_0^t F(s, Y^*(s)) dA(s) \right) \\
&\quad + 2H^2_{L^2} \left( \int_0^t G(s, X^*(s)) dM(s), \int_0^t G(s, Y^*(s)) dM(s) \right) \\
&\leq 2 \int_{[0,t]} \kappa \left( H^2_{L^2}(X^*(s), Y^*(s)) \right) \mu_A^2 (ds) \\
&\quad + 2 \int_{[0,t]} \kappa \left( H^2_{L^2}(X^*(s), Y^*(s)) \right) \mu_M^2 (ds) \\
&\leq \varepsilon + 2 \int_0^t \kappa \left( H^2_{L^2}(X^*(s), Y^*(s)) \right) d\nu(s),
\end{align*}
where $\varepsilon \geq 0$. By Bihari's inequality
\begin{equation*}
H^2_{L^2} (X^*(t), Y^*(t)) \leq J^{-1} (J(\varepsilon) + 2\nu(t)) \leq J^{-1} (J(\varepsilon) + 2(\Delta_A + \Delta_M))
\end{equation*}
for every $t \in I$. Passing to the limit as $\varepsilon$ goes to zero, we obtain $H^2_{L^2} (X^*(t), Y^*(t)) = 0$ for $t \in I$. \hfill $\Box$

Now let us consider equation (7) and equation with another initial value $Y_0 \in \mathcal{K}_c^b (L^2_0)$, i.e.
\begin{equation}
Y(t) = Y_0 + \int_0^t F(s, Y(s)) dA(s) + \int_0^t G(s, Y(s)) dM(s), \quad \text{for} \quad t \in I.
\end{equation}
Theorem 3.11. Let \(X_0, Y_0 \in \mathcal{K}_c^b(L^2_0)\) and let \(F, G : I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)\) satisfy conditions (A1)–(A3). Then for the solutions \(X, Y\) of (7) and (8), respectively, the following estimation is true

\[
\sup_{t \in I} H_{L^2}(X(t), Y(t)) \leq J^{-1} \left( J(3H_{L^2}(X_0, Y_0)) + 3(\Delta_A + \Delta_M) \right).
\]

Proof. For \(t \in I\)

\[
H_{L^2}(X(t), Y(t)) \leq 3H_{L^2}(X_0, Y_0) + 3 \int_{[0, t] \times \Omega} H_{L^2}(F(s, \omega, X(s)), F(s, \omega, Y(s))) \mu_A(ds, d\omega)
\]

\[
+ 3 \int_{[0, t] \times \Omega} H_{L^2}(g(s, \omega, X(s)), g(s, \omega, Y(s))) \mu_M(ds, d\omega)
\]

\[
\leq 3H_{L^2}(X_0, Y_0) + 3 \int_0^t \kappa \left( H_{L^2}(X(s), Y(s)) \right) \mu_A(ds)
\]

\[
+ 3 \int_0^t \kappa \left( H_{L^2}(X(s), Y(s)) \right) \mu_M(ds) =
\]

\[
3H_{L^2}(X_0, Y_0) + 3 \int_0^t \kappa \left( H_{L^2}(X(s), Y(s)) \right) d\nu(s).
\]

By Bihari’s inequality, for every \(t \in I\) we get

\[
H_{L^2}(X(t), Y(t)) \leq J^{-1} \left( J(3H_{L^2}(X_0, Y_0)) + 3\nu(t) \right) \leq J^{-1} \left( J(3H_{L^2}(X_0, Y_0)) + 3(\Delta_A + \Delta_M) \right).
\]

This ends the proof. \(\square\)

Due to the above estimation we easily infer on continuous dependence of solution to (7) with respect to initial value.

Corollary 3.12. Let \(X_0 \in \mathcal{K}_c^b(L^2_0)\) and \(F, G\) satisfy (A1)–(A3). Let \(X\) denote the solution of (7) and for \(n \in \mathbb{N}\) let \(X_n\) denote the solution of equation

\[
X_n(t) = X_0^{(n)} + \int_0^t F(s, X_n(s))dZ(s) + \int_0^t F(s, X_n(s))dM(s), \quad t \in I,
\]

where \(X_0^{(n)} \in \mathcal{K}_c^b(L^2_0)\). If \(H_{L^2}(X_0^{(n)}, X_0) \to 0\), as \(n \to \infty\), then

\[
\sup_{t \in I} H_{L^2}(X_n(t), X(t)) \to 0, \quad \text{as} \quad n \to \infty.
\]

In the sequel, let us consider equation (7) and the equation

\[
Y(t) = X_0 + \int_0^t V(s, Y(s))dA(s) + \int_0^t Q(s, Y(s))dM(s) \quad \text{for} \quad t \in I,
\]

where nonlinearities \(V\) and \(Q\) are different from \(F, G\).

The next result presents a bound for the distance of \(X\) and \(Y\) which are solutions to (7) and (9), respectively.

Theorem 3.13. Assume that \(X_0 \in \mathcal{K}_c^b(L^2_0)\) and \(F, G, V, Q : I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)\) satisfy conditions (A1)–(A3). Then

\[
\sup_{t \in I} H_{L^2}(X(t), Y(t)) \leq \min\{J^{-1}(J(a_1 + a_2) + 4(\Delta_A + \Delta_M)), J^{-1}(J(b_1 + b_2) + 4(\Delta_A + \Delta_M))\}.
\]
where
\[ a_1 = 4 \int_{I \times \Omega} H^2_{\mathbb{R}^d} (F(s, \omega, X(s)), V(s, \omega, X(s))) \mu_A(ds, d\omega). \]
\[ a_2 = 4 \int_{I \times \Omega} H^2_{\mathbb{R}^d} (G(s, \omega, X(s)), Q(s, \omega, X(s))) \mu_M(ds, d\omega). \]
\[ b_1 = 4 \int_{I \times \Omega} H^2_{\mathbb{R}^d} (F(s, \omega, Y(s)), V(s, \omega, Y(s))) \mu_A(ds, d\omega). \]
\[ b_2 = 4 \int_{I \times \Omega} H^2_{\mathbb{R}^d} (G(s, \omega, Y(s)), Q(s, \omega, Y(s))) \mu_M(ds, d\omega). \]

Proof. Let us fix \( t \in I \). Then
\[
H^2_{L^2}(X(t), Y(t)) \leq 4 \int_{I \times \Omega} H^2_{\mathbb{R}^d} (F(s, \omega, X(s)), V(s, \omega, X(s))) \mu_A(ds, d\omega) \]
\[ + 4 \int_{[0, t] \times \Omega} H^2_{\mathbb{R}^d} (V(s, \omega, X(s)), V(s, \omega, Y(s))) \mu_A(ds, d\omega) \]
\[ + 4 \int_{I \times \Omega} H^2_{\mathbb{R}^d} (G(s, \omega, X(s)), Q(s, \omega, X(s))) \mu_M(ds, d\omega) \]
\[ + 4 \int_{[0, t] \times \Omega} H^2_{\mathbb{R}^d} (Q(s, \omega, X(s)), Q(s, \omega, Y(s))) \mu_M(ds, d\omega) \]
\[ \leq 4 \int_{I \times \Omega} H^2_{\mathbb{R}^d} (F(s, \omega, X(s)), V(s, \omega, X(s))) \mu_A(ds, d\omega) \]
\[ + 4 \int_{I \times \Omega} H^2_{\mathbb{R}^d} (G(s, \omega, X(s)), Q(s, \omega, X(s))) \mu_M(ds, d\omega) \]
\[ + 4 \int_0^t \kappa \left( H^2_{L^2}(X(s), Y(s)) \right) ds. \]

Application of Bihari’s inequality leads us to the conclusion
\[
\sup_{t \in I} H^2_{L^2}(X(t), Y(t)) \leq J^{-1} \left( J(a_1 + a_2) + 4(\Delta_A + \Delta_M) \right). \]

Further, let us observe that
\[
H^2_{L^2}(X(t), Y(t)) \leq 4 \int_{[0, t] \times \Omega} H^2_{\mathbb{R}^d} (F(s, \omega, X(s)), F(s, \omega, Y(s))) \mu_A(ds, d\omega) \]
\[ + 4 \int_{[0, t] \times \Omega} H^2_{\mathbb{R}^d} (F(s, \omega, Y(s)), V(s, \omega, Y(s))) \mu_A(ds, d\omega) \]
\[ + 4 \int_{[0, t] \times \Omega} H^2_{\mathbb{R}^d} (F(s, \omega, Y(s)), Q(s, \omega, Y(s))) \mu_M(ds, d\omega) \]
\[ + 4 \int_{[0, t] \times \Omega} H^2_{\mathbb{R}^d} (F(s, \omega, Y(s)), V(s, \omega, Y(s))) \mu_M(ds, d\omega). \]

Proceeding similarly as above we infer that
\[
\sup_{t \in I} H^2_{L^2}(X(t), Y(t)) \leq J^{-1} \left( J(b_1 + b_2) + 4(\Delta_A + \Delta_M) \right). \]

Theorem 3.13 allows for easy deduction on continuous dependence of solution to (7) with respect to the nonlinearities \( F \) and \( G \).
Corollary 3.14. Let $X_0 \in \mathcal{K}_c^b(L_0^2)$ and $F, F_n, G, G_n : I \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{K}_c^b(\mathbb{R})$ satisfy (A1)–(A3). Let $X$ denote solution to (7) and $X_n$ denote solution to the equation

$$X_n(t) = X_0 + \int_0^t F_n(s, X_n(s))dA(s) \int_0^t G_n(s, X_n(s))dM(s), \quad t \in I,$$

$n \in \mathbb{N}$. Assume that for every $A \in \mathcal{K}_c^b(L^2)$

$$\int_{I \times \Omega} \mu_{A}(dt, d\omega) \to 0, \quad \text{as} \quad n \to \infty,$$

and

$$\int_{I \times \Omega} \mu_{A}(dt, d\omega) \to 0, \quad \text{as} \quad n \to \infty.$$

Then

$$\sup_{t \in I} H_{L^2}(X_n(t), X(t)) \to 0, \quad \text{as} \quad n \to \infty.$$

Let us notice that the results established for SSIEs have immediate consequences for single-valued stochastic integral equations driven by semimartingales [48]. Namely, consider $f, g : I \times \Omega \times L^2 \to \mathbb{R}^d$ and $x_0 \in L_0^2$. Then the following single-valued stochastic integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))dA(s) + \int_0^t g(s, x(s))dM(s) \quad (10)$$

can be viewed as SSIE (7) with $F = \{f\}, G = \{g\}$ and $X_0 = \{x_0\}$. Therefore rewriting (A1)–(A3) in terms of $f, g$ we obtain the following conditions

(A1’) for every $a \in L^2$ the stochastic processes $f(\cdot, \cdot, a), g(\cdot, \cdot, a) : I \times \Omega \to \mathbb{R}^d$ are predictable,

(A2’) there exists a constant $C > 0$ such that

$$\mu_A \text{-a.e. it holds } \|f(t, \omega, \theta_{L^2})\|_{L^d} \leq C,$$

$$\mu_M \text{-a.e. it holds } \|g(t, \omega, \theta_{L^2})\|_{L^d} \leq C,$$

(A3’) there exists a continuous, nondecreasing, concave function $\kappa : [0, \infty) \to [0, \infty)$ satisfying:

$$f_0 \frac{1}{\kappa(s)} ds = +\infty, \kappa(0) = 0, \kappa(s) > 0 \text{ for } s > 0, \text{ and such that }$$

$$\mu_A \text{-a.e. it holds } \|f(t, \omega, a) - f(t, \omega, b)\|_{L^d} \leq \kappa(\|a - b\|_{L^2}), \quad a, b \in L^2,$$

$$\mu_M \text{-a.e. it holds } \|g(t, \omega, a) - g(t, \omega, b)\|_{L^d} \leq \kappa(\|a - b\|_{L^2}), \quad a, b \in L^2.$$

Proceeding like in set-valued case we can get the results concerning single-valued equations. Although all the counterparts can be rewritten for single-valued equation, here we write only the main theorem on existence and uniqueness of solution.

Corollary 3.15. Assume that $x_0 \in L_0^2$ and $f, g$ satisfy (A1’)–(A3’). Then there exists a unique solution to single-valued equation (10).

4 Set-valued equations driven by semimartingale $Z$

In this section we will need a notion of a set-valued stochastic integral with respect to semimartingales. As before, let $Z : I \times \Omega \to \mathbb{R}$ be a continuous semimartingale with the canonical representation (1). If additionally $Z \in \mathcal{H}^2$, then one can define a finite measure $\mu_Z$ on $(I \times \Omega, \mathcal{P})$ as

$$\mu_Z(C) := \mu_A(C) + \mu_M(C), \quad C \in \mathcal{P},$$
where the measures $\mu_A, \mu_M$ are as in Section 3. Let us denote
\[
\Delta Z := \mu_Z(I \times \Omega), \quad L^2_p(\mu_Z) := L^2(I \times \Omega, P, \mu_Z; \mathbb{R}^d).
\]
For $f \in L^2_p(\mu_Z)$ one can define the single-valued stochastic integral $\int_0^t f(s)dZ(s)$ with respect to semimartingale $Z$ as follows
\[
\int_0^t f(s)dZ(s) := \int_0^t f(s)dA(s) + \int_0^t f(s)dM(s).
\]
Due to (2) and (3) we claim that:

**Corollary 4.1.** If $f \in L^2_p(\mu_Z)$ then for every $\eta, t \in I$, $\eta < t$, it holds
\[
\mathbb{E}\left\| \int_\eta^t f(s)dZ(s) \right\|^2_{\mathbb{R}^d} \leq 2 \int_{[\eta, t] \times \Omega} \| f \|^2_{\mathbb{R}^d} d\mu_Z.
\]

Let $F: I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ be a predictable set-valued stochastic process which is $L^2_p(\mu_Z)$-integrally bounded. For such a process let us define the set
\[
S^2_{p}(F, \mu_Z) := \{ f \in L^2_p(\mu_Z) : f \in F, \mu_Z \text{-a.e.} \).
\]
This set is nonempty, and similarly as e.g. in [38, 42] we can define the set-valued stochastic trajectory integral with respect to semimartingales.

**Definition 4.2.** For a predictable and $L^2_p(\mu_Z)$-integrally bounded set-valued stochastic process $F: I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ and for $\tau, t \in \mathbb{R}_+, \tau < t$ the set-valued stochastic trajectory integral (over interval $[\tau, t]$) of $F$ with respect to the semimartingale $Z$ is the following subset of $L^2_{\tau}$
\[
\int_\tau^t F(s)dZ(s) := \left\{ \int_\tau^t f(s)dZ(s) : f \in S^2_{p}(F, \mu_Z) \right\}.
\]

It is known (cf. [38, 42]) that $\int_\tau^t F(s)dZ(s)$ is a nonempty, bounded, convex, closed and weakly compact subset of $L^2_{\tau}$.

Proceeding similarly as in preceding section we can obtain (without any difficulties) assertions which are parallels of those presented in Section 3.

Now, in the derivations, we use the following properties.

**Proposition 4.3** ([38, 42]). Let $F_1, F_2: I \times \Omega \to \mathcal{K}_c^b(\mathbb{R}^d)$ be the predictable and $L^2_p(\mu_Z)$-integrally bounded set-valued stochastic processes. Then
(i) for every $\tau, a, t \in I$, $\tau \leq a \leq t$ it holds $\int_\tau^t F_1(s)dZ(s) = \int_\tau^a F_1(s)dZ(s) + \int_a^t F_1(s)dZ(s)$.
(ii) for every $t \in I$, $\tau < t$
\[
H^2_{\mathbb{R}^d} \left( \int_\tau^t F_1(s)dZ(s), \int_\tau^t F_2(s)dZ(s) \right) \leq \int_{[\tau, t] \times \Omega} H^2_{\mathbb{R}^d}(F_1(s, \omega), F_2(s, \omega)) d\mu_Z(ds, d\omega).
\]
(iii) for every $\tau \in I$ the mapping $[\tau, T] \ni t \mapsto \int_\tau^t F_1(s)dZ(s) \in \mathcal{K}_c^b(L^2)$ is $H_{L^2}$-continuous.

An SSIE driven by semimartingale $Z$ is the following relation in the metric space $(\mathcal{K}_c^b(L^2), H_{L^2})$:
\[
X(t) = X_0 + \int_0^t F(s, X(s))dZ(s) \quad \text{for} \quad t \in I,
\]
where $X_0 \in \mathcal{K}(L^2_{\mu}), F: I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d)$.

This equation is slightly different than (7). Here, the integrand $Z$ is more general, but $F$ comes from a narrower class than $\tilde{F}$ in (7). Indeed, here $F$ needs to be $L^2_p(\mu_A)$-integrally bounded and $L^2_p(\mu_Z)$-integrally bounded, simultaneously, i.e. $F$ needs to be $L^2_p(\mu_Z)$-integrally bounded.
An $H_{L^2}$-continuous mapping $X: I \rightarrow K^b_c(L^2)$ is called the solution to (11) if $X$ satisfies (11) and $X(t) \in K^b_c(L^2)$ for every $t \in I$. A solution $X$ is said to be unique if $X(t) = Y(t)$ for $t \in I$, where $Y$ is any solution to (11).

By $\mu^\Omega_\mathcal{Z}$ we denote a measure on $(I, \mathcal{B}(I))$ defined as

$$\mu^\Omega_{\mathcal{Z}}(B) := \mu_B(B \times \Omega) \quad \text{for} \quad B \in \mathcal{B}(I).$$

Obviously, $\mu^\Omega_{\mathcal{Z}} = \mu^\Omega_A + \mu^\Omega_M$. We will assume that semimartingale $Z$ is such that $\mu^\Omega_{\mathcal{Z}}$ is absolutely continuous with respect to the Lebesgue measure $\lambda$, and $S_{\mathcal{Z}} := \text{ess sup}_{t \in I} \frac{d\mu^\Omega_{\mathcal{Z}}}{d\lambda} < \infty$.

The assertions concerning equation (11) will be stated under following assumptions:

$(Z1)$ for every $A \in K^b_c(L^2)$ the mapping $F(\cdot, A): I \times \Omega \rightarrow K^b_c(\mathbb{R}^d)$ is predictable,

$(Z2)$ there exists a constant $C > 0$ such that $\mu_{\mathcal{Z}}$-a.e. it holds

$$H^2_{\mathbb{R}^d}(F(t, \omega, \{\theta_{\mathcal{Z}}\}), \{\theta_{\mathcal{Z}_d}\}) \leq C,$$

$(Z3)$ there exists a continuous, nondecreasing, concave function $\kappa: [0, \infty) \rightarrow [0, \infty)$ satisfying: $\int_0^1 \frac{1}{\kappa(s)} ds = +\infty$, $\kappa(0) = 0$, $\kappa(s) > 0$ for $s > 0$, and such that $\mu_{\mathcal{Z}}$-a.e. it holds

$$H^2_{\mathbb{R}^d}(F(t, \omega, A), F(t, \omega, B)) \leq \kappa(H^2_{L^2}(A, B)) \quad \text{for} \quad A, B \in K^b_c(L^2).$$

Due to concavity of $\kappa$ we obtain:

**Remark 4.4.** If $F: I \times \Omega \times K^b_c(L^2) \rightarrow K^b_c(\mathbb{R}^d)$ satisfies $(Z2)$ and $(Z3)$, then there exists a constant $K > 0$ such that $\mu_{\mathcal{Z}}$-a.e. it holds

$$H^2_{\mathbb{R}^d}(F(t, \omega, A), \{\theta_{\mathcal{Z}_d}\}) \leq K\left(1 + H^2_{L^2}(A, \{\theta_{\mathcal{Z}_I}\})\right) \quad \text{for} \quad A \in K^b_c(L^2).$$

The existence of solution to (11) is established with a help of approximate solutions $X_n: I \rightarrow K^b_c(L^2)$ ($n = 1, 2, \ldots$) of Maruyama type, i.e.

$$X_n(0) = X_0,$$

$$X_n(t) = X_n\left(\frac{k-1}{n}\right) + \int_{\frac{k-1}{n}}^t F\left(s, X_n\left(\frac{k-1}{n}\right)\right) dZ(s)$$

for $t \in \left(\frac{k-1}{n}, \frac{k}{n}\right] \cap I$, $k = 1, 2, \ldots$. For every $n$ the mapping $X_n$ is $H_{L^2}$-continuous and $X_n(t) \in K^b_c(L^2)$ for every $t \in I$. If we define $\tilde{X}_n: I \rightarrow K^b_c(L^2)$ ($n \in \mathbb{N}$) by

$$\tilde{X}_n(t) = X_nI_{\{0\}}(t) + \sum_{k \geq 1} X_n\left(\frac{k-1}{n}\right)I_{\left(\frac{k-1}{n}, \frac{k}{n}\right]}(t) \quad \text{for} \quad t \in I,$$

then we can write

$$X_n(t) = X_0 + \int_0^t F(s, \tilde{X}_n(s)) dZ(s) \quad \text{for} \quad t \in I.$$

Proceeding similarly as in preceding section we obtain the following results.

**Lemma 4.5.** Assume that $X_0 \in K^b_c(L^2)$ and $F$ satisfies $(Z1)$–(Z3). Then

$$\sup_{n} \sup_{t \in I} H^2_{L^2}(X_n(t), \{\theta_{\mathcal{Z}_I}\}) \leq M_1,$$

where $M_1 = \left(1 + 2H^2_{L^2}(X_0, \{\theta_{\mathcal{Z}_I}\})\right) \exp\{4K\Delta_Z\}.$
Lemma 4.6. Let $X_0 \in \mathcal{K}_c^b(L^2_0)$ and $F$ satisfy (Z1)–(Z3). Then for every $n \in \mathbb{N}$ and every $t, r \in I$, $r \leq t$

$$H^2_{L^2}(X_n(t), X_n(r)) \leq M_2(t - r),$$

where $M_2 = (1 + M_1)S_Z$. $M_1$ is defined as in Lemma 4.5.

Corollary 4.7. Let $X_0 \in \mathcal{K}_c^b(L^2_0)$ and $F$ satisfy (Z1)–(Z3). Then for $n \in \mathbb{N}$ and $t \in I$

$$H^2_{L^2}(X_n(t), \dot{X}_n(t)) \leq \frac{M_2}{n},$$

where $M_2$ is as in Lemma 4.6.

Lemma 4.8. Let $X_0 \in \mathcal{K}_c^b(L^2_0)$ and $F$ satisfy (Z1)–(Z3). Then

$$\sup_{t \in I} H^2_{L^2}(X_n(t), X_m(t)) \to 0, \quad \text{as} \quad n, m \to \infty.$$

Theorem 4.9. Let $X_0 \in \mathcal{K}_c^b(L^2_0)$ and $F$ satisfy (Z1)–(Z3). Then there exists a unique solution $X$ to (11) and

$$\sup_{t \in I} H^2_{L^2}(X(t), \{0, \{\theta_{L^2}\}\}) \leq M_1,$$

where $M_1$ is as in Lemma 4.5.

To infer on continuous dependence of solution to (11) with respect to initial value, let us consider equation (11) and another one with different initial value $Y_0 \in \mathcal{K}_c^b(\mathbb{R}^d)$, i.e.

$$Y(t) = Y_0 + \int_0^t F(s, Y(s))dZ(s), \quad t \in I. \tag{12}$$

Theorem 4.10. Let $X_0, Y_0 \in \mathcal{K}_c^b(L^2_0)$ and $F$ satisfy (Z1)–(Z3). For the solutions $X, Y$ of (11) and (12), respectively, the following estimation is true

$$\sup_{t \in I} H^2_{L^2}(X(t), Y(t)) \leq J^{-1}\left(J\left(2H^2_{L^2}(X_0, Y_0) + 2\Delta Z\right)\right).$$

Corollary 4.11. Let $X_0 \in \mathcal{K}_c^b(L^2_0)$ and $F$ satisfy (Z1)–(Z3). Let $X$ denote the solution of (11) and for $n \in \mathbb{N}$ let $X_n$ denote the solution of equation

$$X_n(t) = X_0^{(n)} + \int_0^t F(s, X_n(s))dZ(s), \quad t \in I,$$

where $X_0^{(n)} \in \mathcal{K}_c^b(L^2_0)$. If $H^{L^2}(X_0^{(n)}, X_0) \to 0$, as $n \to \infty$, then

$$\sup_{t \in I} H^2_{L^2}(X_n(t), X(t)) \to 0, \quad \text{as} \quad n \to \infty.$$

Now we consider two equations with two different nonlinearities $F$ and $V$, i.e. equation (11) and

$$Y(t) = X_0 + \int_0^t V(s, Y(s))dZ(s) \quad \text{for} \quad t \in I. \tag{13}$$

Theorem 4.12. Assume that $X_0 \in \mathcal{K}_c^b(L^2_0)$ and $F, V : I \times \Omega \times \mathcal{K}_c^b(L^2)$ satisfy conditions (Z1)–(Z3). Then

$$\sup_{t \in I} H^2_{L^2}(X(t), Y(t)) \leq \min\{a, b\},$$

where

$$a = J^{-1}\left(J\left(2\int_{I \times \Omega} H^2_{\mathbb{R}^d}(F(s, \omega, X(s)), V(s, \omega, X(s)))d\omega\right) + 2\Delta Z\right),$$

$$b = J^{-1}\left(J\left(2\int_{I \times \Omega} H^2_{\mathbb{R}^d}(F(s, \omega, Y(s)), V(s, \omega, Y(s)))d\omega\right) + 2\Delta Z\right).$$
Corollary 4.13. Let \( X_0 \in \mathcal{K}_b^b(L_0^2) \) and \( F, F_n: I \times \Omega \times \mathcal{K}_c^b(L^2) \to \mathcal{K}_c^b(\mathbb{R}^d) \) satisfy (Z1)–(Z3). Let \( X \) denote solution to (11) and \( X_n \) denote solution to the equation

\[
X_n(t) = X_0 + \int_0^t F_n(s, X_n(s))dZ(s), \quad t \in I,
\]

for every \( n \in \mathbb{N} \). Assume that for every \( A \in \mathcal{K}_b^b(L^2) \)

\[
\int_{I \times \Omega} H_{\mathbb{R}^d}^2 \left( F_n(t, \omega, A), F(t, \omega, A) \right) \mu_Z(dt, d\omega) \to 0, \quad n \to \infty.
\]

Then

\[
\sup_{t \in I} H_{L^2}^2(X_n(t), X(t)) \to 0, \quad n \to \infty.
\]

Similarly like in Section 3, the results concerning SSIE (11) imply the corresponding results for single-valued stochastic integral equations driven by semimartingales. Indeed, considering \( f: I \times \Omega \times L^2 \to \mathbb{R}^d \) and \( x_0 \in L_0^2 \) and the equation

\[
x(t) = x_0 + \int_0^t f(s, x(s))dZ(s)
\]

(14)

together with the assumptions

\[
(Z1') \quad \text{for every } a \in L^2 \text{ the stochastic process } f(\cdot, \cdot, a): I \times \Omega \to \mathbb{R}^d \text{ is predictable,}
\]

\[
(Z2') \quad \text{there exists a constant } C > 0 \text{ such that } \mu_Z-\text{a.e. it holds}
\]

\[
\|f(t, \omega, \theta_{L^2})\|_{\mathbb{R}^d}^2 \leq C,
\]

\[
(Z3') \quad \text{there exists a continuous, nondecreasing, concave function } \kappa: [0, \infty) \to [0, \infty) \text{ satisfying:}
\]

\[
f(t, \omega, a) - f(t, \omega, b)\|_{\mathbb{R}^d}^2 \leq \kappa(\|a - b\|_{L^2}^2), \quad a, b \in L^2,
\]

we can arrive to the following existence and uniqueness result.

Corollary 4.14. Assume that \( x_0 \in L_0^2 \) and \( f \) satisfies (Z1')–(Z3'). Then the single-valued equation (14) has a unique solution.

5 Fuzzy stochastic equations driven by processes \( A, M \) from decomposition of semimartingale \( Z \)

In this section we present results parallel to those established for set-valued equations. However, here the equations will be more general. Namely, here the values of the integrands will be some fuzzy sets. Such equations are called the fuzzy stochastic integral equations. They can be viewed as equations in a metric space After setting an appropriate framework (see also [38, 41] for more details), we formulate assertions for fuzzy stochastic equations driven by processes \( A, M \) from decomposition of \( Z \). These results are generalizations of those included in Section 3. Their derivations are omitted, since they are similar to those presented in set-valued cases.

Let \( F(\mathcal{X}) \) denote the set of all functions \( u: \mathcal{X} \to [0, 1] \), where \( (\mathcal{X}, \| \cdot \|) \) is a separable Banach space. Such set is a fundamental notion in fuzzy systems analysis (in industrial mathematics and engineering) and it is called the set of fuzzy sets of \( \mathcal{X} \). The function \( u: \mathcal{X} \to [0, 1] \) is called then a membership function of a fuzzy set \( u \in F(\mathcal{X}) \) and its
value \( u(x) \) is interpreted as a degree of membership of \( x \) in the fuzzy set \( u \). Note that every ordinary subset \( A \) of \( X \) is a fuzzy set of \( X \). Indeed, the power set of \( X \) can be embedded into \( \mathcal{F}(X) \) by means of the characteristic function of ordinary set. Fuzzy sets play a key role in systems with imprecisely described states (not defined by a single value), in systems with incomplete information (e.g., with uncertain initial data) etc. Such systems often appear in realistic world. Therefore modeling uncertain systems with mathematical tools focuses much attention.

In our framework we will use the following class \( \mathcal{F}_c^b(X) \) of functions \( u: X \to [0, 1] \)

\[
\mathcal{F}_c^b(X) := \{ u \in \mathcal{F}(X) : u \text{ satisfies conditions (i)-(iv) given below} \}
\]

(i) \( u \) is upper semicontinuous,
(ii) there exists \( a \in X \) such that \( u(a) = 1 \),
(iii) \( u \) is quasiconcave, i.e., \( u(aa + (1 - \alpha)b) \geq \min\{u(a), u(b)\} \) for \( a, b \in X \) and \( \alpha \in [0, 1] \),
(iv) \( \text{supp}(u) = \text{cl}_X \{a \in X : u(a) > 0\} \) is bounded and convex.

For \( \alpha \in (0, 1] \) the symbol \([u]^\alpha\) denotes the so-called \( \alpha \)-level set of \( u \in \mathcal{F}_c^b(X) \), i.e.,

\[
[u]^\alpha = \{a \in X : u(a) \geq \alpha \},
\]

and \([u]^0\) will be used to denote \( \text{supp}(u) \). It is clear that for \( u \in \mathcal{F}_c^b(X) \) we have \([u]^\alpha \in \mathcal{K}_c^b(X) \) for every \( \alpha \in [0, 1] \).

Addition and scalar multiplication in \( \mathcal{F}_c^b(X) \) is defined level-wise, i.e., the sum of \( u, v \in \mathcal{F}_c^b(X) \) is element \( u \oplus v \in \mathcal{F}_c^b(X) \) such that \([u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha, \alpha \in [0, 1] \), and the product of \( r \in \mathbb{R} \) and \( u \in \mathcal{F}_c^b(X) \) is an element \( ru \in \mathcal{F}_c^b(X) \) such that \([ru]^\alpha = r[u]^\alpha, \alpha \in [0, 1] \). The set \( \mathcal{F}_c^b(X) \) with these operations becomes a semilinear space. Also \( (\mathcal{F}_c^b(X), D_X) \) is a complete metric space, where

\[
D_X(u, v) = \sup_{\alpha \in (0, 1]} H_X([u]^\alpha, [v]^\alpha).
\]

The metric \( D_X \) has the property

\[
D_X(u_1 + v_1, u_2 + v_2) \leq D_X(u_1, u_2) + D_X(v_1, v_2).
\]

An fuzzy stochastic process \( f: I \times \Omega \to \mathcal{F}_c^b(X) \) is called:

(a) predictable, if \([f(\cdot, \cdot)]^\alpha: I \times \Omega \to \mathcal{K}_c^b(X) \) is a predictable set-valued stochastic process for every \( \alpha \in [0, 1] \),
(b) \( L^p(\mu_A) \)-integrally bounded (\( L^p(\mu_Z) \)-integrally bounded), if the stochastic process \( (t, \omega) \mapsto H_X([f(t, \omega)]^0, \{\theta_X\}) \) belongs to \( L^p(I \times \Omega, \mathcal{P}, \mu_A; \mathbb{R}) \), \((t, \omega) \mapsto H_X([f(t, \omega)]^0, \{\theta_X\}) \) belongs to \( L^p(I \times \Omega, \mathcal{P}, \mu_Z; \mathbb{R}) \), respectively.

**Proposition 5.1** ([41]). Let \( f, g: I \times \Omega \to \mathcal{F}_c^b(\mathbb{R}^d) \) be the predictable fuzzy stochastic processes. Assume that \( f \) is \( L^2(\mu_A) \)-integrally bounded and \( g \) is \( L^2(\mu_Z) \)-integrally bounded. Then for every \( t, t' \in I \), \( t < t' \) there exists a unique element in \( \mathcal{F}_c^b(L_2^2) \) denoted by \( f_t^f(s) dA(s) \) such that for every \( \alpha \in (0, 1] \) it holds

\[
\int_0^1 f_t^f(s) dA(s) = \frac{\int_0^1 f(s)^\alpha dA(s)}{\int_0^1 (f(s))^\alpha dA(s)}, \quad \text{and} \quad \int_0^1 f(s) dA(s) \in \mathcal{F}_c^b(L_2^2)
\]

Also, there exists a unique element in \( \mathcal{F}_c^b(L_2^2) \) denoted by \( f_t^g(s) dM(s) \) such that for every \( \alpha \in (0, 1] \) it holds

\[
\int_0^1 g(s)^\alpha dM(s) = \frac{\int_0^1 g(s)^\alpha dM(s)}{\int_0^1 (g/s)^\alpha dM(s)}, \quad \text{and} \quad \int_0^1 g(s) dM(s) \in \mathcal{F}_c^b(L_2^2)
\]

The following properties are counterparts of those listed in Proposition 2.3 and are needed in derivations of presented results concerning fuzzy stochastic equations.
Proposition 5.2 ([41]). Let $f_1, f_2, g_1, g_2: \mathbb{R}_+ \times \Omega \to \mathcal{F}_t^b(\mathbb{R}^d)$ be the predictable fuzzy stochastic processes. Assume that processes $f_1, f_2$ are $L^2_\mu(\mu_A)$-integrally bounded, and $g_1, g_2$ are $L^2_\mu(\mu_M)$-integrally bounded. Then

(i) for every $\tau, t \in I$, $\tau \leq a \leq t$

\[
\int_\tau^t f_1(s)dA(s) = \int_\tau^t f_1(s)dA(s) \oplus \int_\tau^t f_1(s)dA(s),
\]

\[
\int_\tau^t g_1(s)dM(s) = \int_\tau^t g_1(s)dM(s) \oplus \int_\tau^t g_1(s)dM(s).
\]

(ii) for every $\tau, t \in I$, $\tau < t$

\[
D_{L^2}^2 \left( \int_\tau^t f_2(s)dA(s), \int_\tau^t f_2(s)dA(s) \right) \leq \int_{[\tau,t] \times \Omega} D_{\mathbb{R}_+^d}^2 (f_1(s, \omega), f_2(s, \omega))\mu_A(ds, d\omega),
\]

\[
D_{L^2}^2 \left( \int_\tau^t g_2(s)dM(s), \int_\tau^t g_2(s)dM(s) \right) \leq \int_{[\tau,t] \times \Omega} D_{\mathbb{R}_+^d}^2 (g_1(s, \omega), g_2(s, \omega))\mu_M(ds, d\omega).
\]

(iii) for every $\tau \in I$ the mappings $[\tau,T] \ni t \mapsto \int_\tau^t f_1(s)dA(s) \in \mathcal{F}_t^b(L^2)$ and $[\tau,T] \ni t \mapsto \int_\tau^t g_1(s)dM(s) \in \mathcal{F}_t^b(L^2)$ are continuous with respect to the metric $D_{L^2}$.

By a fuzzy stochastic equation driven by processes $A, M$ from decomposition of semimartingale $Z$ we mean the following relation in the metric space $(\mathcal{F}_t^b(L^2), D_{L^2})$

\[
x(t) = x_0 + \int_0^t f(s, x(s))dA(s) \oplus \int_0^t g(s, x(s))dM(s) \quad \text{for} \quad t \in I,
\]

(15)

where $f, g: I \times \Omega \times \mathcal{F}_t^b(L^2) \to \mathcal{F}_t^b(\mathbb{R}^d)$ and $x_0 \in \mathcal{F}_t^b(L^0_\mu)$.

Definition 5.3. By a solution to (15) we mean a $D_{L^2}$-continuous mapping $x: I \to \mathcal{F}_t^b(L^2)$ that satisfies (15) and such that $x(t) \in \mathcal{F}_t^b(L^2)$ for $t \in I$. A solution $x$ is unique if $x(t) = y(t)$ for every $t \in I$, where $y: I \to \mathcal{F}_t^b(L^2)$ is any solution to (15).

To obtain a result on existence and uniqueness of solution to (15) with the coefficients $f, g$ satisfying the Osgood type condition we will assume that:

(a1) the processes $f(\cdot, \cdot, u), g(\cdot, \cdot, u): I \times \Omega \to \mathcal{F}_t^b(\mathbb{R}^d)$ are predictable for every $u \in \mathcal{F}_t^b(L^2)$,

(a2) there exists a constant $C > 0$ such that

$\mu_A$-a.e. it holds $D_{\mathbb{R}_+^d}^2 (f(t, \omega, \theta_{L^2}), \theta_{L^2}) \leq C,
\mu_M$-a.e. it holds $D_{\mathbb{R}_+^d}^2 (g(t, \omega, \theta_{L^2}), \theta_{L^2}) \leq C$, where $(\theta_{L^2}), (\theta_{L^2})$ denote the characteristic functions of singletons $\{\theta_{L^2}\}$ and $\{\theta_{L^2}\}$, respectively.

(a3) there exists a continuous, nondecreasing, concave function $\kappa: [0, \infty) \to [0, \infty)$ satisfying $\int_0^1 \frac{1}{\kappa(s)} ds = +\infty, \kappa(0) = 0, \kappa(s) > 0$ for $s > 0$, and such that

$\mu_A$-a.e. it holds $D_{\mathbb{R}_+^d}^2 (f(t, \omega, u), f(t, \omega, v)) \leq \kappa(D_{L^2_{\mu}}^2(u, v))$ for $u, v \in \mathcal{F}_t^b(L^2)$,

$\mu_M$-a.e. it holds $D_{\mathbb{R}_+^d}^2 (g(t, \omega, u), g(t, \omega, v)) \leq \kappa(D_{L^2_{\mu}}^2(u, v))$ for $u, v \in \mathcal{F}_t^b(L^2)$.

Notice that if $\kappa(t) = Mt$, where $M$ is a positive constant, then (a3) becomes the Lipschitz condition.

Remark 5.4. If $f, g: I \times \Omega \times \mathcal{F}_t^b(L^2) \to \mathcal{F}_t^b(\mathbb{R}^d)$ satisfy (a2) and (a3), then there exists a constant $K > 0$ such that

$\mu_A$-a.e. it holds $D_{\mathbb{R}_+^d}^2 (f(t, \omega, u), \theta_{L^2}) \leq K \{1 + D_{L^2_{\mu}}^2(u, \theta_{L^2})\}$ for $u \in \mathcal{F}_t^b(L^2)$.
\( \mu -a.e. \) it holds \( D^2_{\mathbb{E}_{\mathbb{F}}} \left( g(t, \omega, u), (\theta_{\mathbb{F}}) \right) \leq K \left( 1 + D^2_{\mathbb{L}}(u, (\theta_{\mathbb{L}})) \right) \) for \( u \in \mathcal{F}(L^2) \).

A sequence \( \{x_n\}_{n=1}^{\infty} \) of approximate solutions \( x_n : I \rightarrow \mathcal{F}(L^2) \) of Maruyama type is as follows:

\[
x_n(0) = x_0,
\]

\[
x_n(t) = x_n \left( \frac{k-1}{n} \right) + \int_{k/n}^{t} f\left( s, x_n \left( \frac{k-1}{n} \right) \right) dA(s) + \int_{k/n}^{t} g\left( s, x_n \left( \frac{k-1}{n} \right) \right) dM(s)
\]

for \( t \in \left( \frac{k-1}{n}, \frac{k}{n} \right) \cap I, k = 1, 2, \ldots \). In the case \( A(t) = t \) and \( M(t) = W(t) \), where \( W \) is \{A\}_{t}-Wiener process we have shown [40] that the Euler-Maruyama approximations converge to the unique solution and this is achieved in [40] with the Lipschitz condition.

Note that each approximant \( x_n \) is \( D_{L^2} \)-continuous and \( x_n \in \mathcal{F}(L^2) \) for every \( t \in I \), and if we define \( \tilde{x}_n : I \rightarrow \mathcal{F}(L^2) \) (\( n \in \mathbb{N} \)) by

\[
\tilde{x}_n(t) = x_0 1_{\{0\}}(t) + \bigoplus_{k \geq 1} x_n \left( \frac{k-1}{n} \right) 1_{\left( \frac{k-1}{n}, \frac{k}{n} \right]}(t) \quad \text{for} \quad t \in I,
\]

then we can write

\[
x_n(t) = x_0 + \int_{0}^{t} f(s, \tilde{x}_n(s)) dA(s) + \int_{0}^{t} g(s, \tilde{x}_n(s)) dM(s) \quad \text{for} \quad t \in I.
\]

Proceeding similarly as in Section 3 with application of \( D_{L^2} \) and \( D_{\mathbb{F},d} \) instead of \( H_{L^2} \) and \( H_{\mathbb{F},d} \), and applying Proposition 5.2 instead of Proposition 2.3, and applying Remark 5.4 instead of Remark 3.4, we get the following counterparts of results placed in Section 3.

**Lemma 5.5.** Assume that \( x_0 \in \mathcal{F}(L^2) \) and \( f, g \) satisfy (a1)–(a3). Then

\[
\sup_{n} \sup_{t \in I} D^2_{L^2}(x_n(t), (\theta_{L^2})) \leq M_1,
\]

where \( M_1 = \left( 1 + 3D^2_{L^2}(x_0, (\theta_{L^2})) \right) \exp \{3K(A + M)\} - 1 \).

**Lemma 5.6.** Let \( x_0 \in \mathcal{F}(L^2) \) and \( f, g \) satisfy (a1)–(a3). Then for every \( n \in \mathbb{N} \) and every \( r, t \in I, r \leq t \)

\[
D^2_{L^2}(x_n(t), x_n(r)) \leq M_2(t-r).
\]

where \( M_2 = 2K(1 + M_1)(S_A + S_M) \), \( M_1 \) is defined as in Lemma 5.5

**Corollary 5.7.** Assume that \( x_0 \in \mathcal{F}(L^2) \) and \( f, g \) satisfy (a1)–(a3). Then for every \( n \in \mathbb{N} \) and \( t \in I \)

\[
D^2_{L^2}(\tilde{x}_n(t), \tilde{x}_n(t)) \leq \frac{M_2}{n},
\]

where \( M_2 \) is as in Lemma 5.6.

**Lemma 5.8.** Assume that \( x_0 \in \mathcal{F}(L^2) \) and \( f, g \) satisfy (a1)–(a3). Then

\[
\sup_{t \in I} D^2_{L^2}(x_n(t), x_m(t)) \rightarrow 0, \quad \text{as} \quad n, m \rightarrow \infty.
\]

The assertions written above are useful in derivation of existence of solution to (15).

**Theorem 5.9.** Assume that \( x_0 \in \mathcal{F}(L^2) \) and \( f, g \) satisfy (a1)–(a3). Then there exists a unique solution \( x \) to (15). Moreover \( \sup_{t \in I} D^2_{L^2}(x(t), (\theta_{L^2})) \leq M_1 \), where \( M_1 \) is as in Lemma 5.5.
Now let us consider equation (15) and equation
\[
y(t) = y_0 \oplus \int_0^t f(s, y(s))dA(s) \oplus \int_0^t g(s, y(s))dM(s) \quad \text{for} \quad t \in I
\]
with another initial value \(y_0 \in \mathcal{F}_c^b(L^2)\).

**Theorem 5.10.** Assume that \(x_0, y_0 \in \mathcal{F}_c^b(L^2)\) and \(f, g\) satisfy (a1)-(a3). Then for the solutions \(x, y\) of (15) and (16), respectively, the following estimation is true
\[
\sup_{t \in I} D^2_{L^2}(x(t), y(t)) \leq J^{-1}\left(J \left(3D^2_{L^2}(x_0, y_0) + 3(\Delta_A + \Delta_M)\right)\right).
\]
Hence, continuous dependence of solution \(x\) to (15) with respect to initial value follows easily.

**Corollary 5.11.** Let \(x_0 \in \mathcal{F}_c^b(L^2)\) and \(f, g\) satisfy (a1)-(a3). Let \(x\) denote the solution of (15) and for \(n \in \mathbb{N}\) let \(x_n\) denote the solution of equation
\[
x_n(t) = x_0^{(n)} \oplus \int_0^t f(s, x_n(s))dA(s) \oplus \int_0^t g(s, x_n(s))dM(s), \quad t \in I,
\]
where \(x_0^{(n)} \in \mathcal{F}_c^b(L^2)\). If \(D_{L^2}(x_0^{(n)}, x_0) \xrightarrow{n \to \infty} 0\), then \(\sup_{t \in I} D_{L^2}(x_n(t), x(t)) \xrightarrow{n \to \infty} 0\).

Consider equation (15) and equation
\[
y(t) = x_0 \oplus \int_0^t p(s, y(s))dA(s) \oplus \int_0^t q(s, y(s))dM(s) \quad \text{for} \quad t \in I
\]
with another nonlinearities \(p, q\).

**Theorem 5.12.** Assume that \(x_0 \in \mathcal{F}_c^b(L^2)\) and \(f, g, p, q: I \times \Omega \times \mathcal{F}_c^b(L^2)\) satisfy conditions (a1)-(a3). Then
\[
\sup_{t \in I} D^2_{L^2}(x(t), y(t)) \leq \min\{J^{-1}(J(a_1 + a_2) + 4(\Delta_A + \Delta_M)), J^{-1}(J(b_1 + b_2) + 4(\Delta_A + \Delta_M))\},
\]
where
\[
a_1 = 4 \int_{I \times \Omega} D^2_{\mathcal{F}_c^b}(f(s, \omega, x(s)), p(s, \omega, x(s)))\mu_A(ds, d\omega),
\]
\[
a_2 = 4 \int_{I \times \Omega} D^2_{\mathcal{F}_c^b}(g(s, \omega, x(s)), q(s, \omega, x(s)))\mu_M(ds, d\omega),
\]
\[
b_1 = 4 \int_{I \times \Omega} D^2_{\mathcal{F}_c^b}(f(s, \omega, y(s)), p(s, \omega, y(s)))\mu_A(ds, d\omega),
\]
\[
b_2 = 4 \int_{I \times \Omega} D^2_{\mathcal{F}_c^b}(g(s, \omega, y(s)), q(s, \omega, y(s)))\mu_M(ds, d\omega).
\]
Hence we can infer on continuous dependence of solution \(x\) with respect to the nonlinearities \(f, g\).

**Corollary 5.13.** Let \(x_0 \in \mathcal{F}_c^b(L^2)\) and \(f, g, f_n, g_n: I \times \Omega \times \mathcal{F}_c^b(L^2) \to \mathcal{F}_c^b(R^d)\) satisfy (a1)-(a3). Let \(x\) denote solution to (15) and \(x_n\) denote solution to the equation
\[
x_n(t) = x_0 \oplus \int_0^t f_n(s, x_n(s))dA(s) \oplus \int_0^t g_n(s, x_n(s))dM(s), \quad t \in I,
\]
\(n \in \mathbb{N}\). Assume that for every \(u \in \mathcal{F}_c^b(L^2)\)
\[
\int_{I \times \Omega} D^2_{\mathcal{F}_c^b}(f_n(t, \omega, u), f(t, \omega, u))\mu_A(dt, d\omega) \to 0, \quad \text{as} \quad n \to \infty.
\]
Then
\[
\sup_{t \in I} D_{L^2}(x_n(t), x(t)) \to 0, \quad \text{as} \quad n \to \infty.
\]

6 Fuzzy stochastic equations driven by semimartingale \( Z \)

In this section we present results parallel to those presented in Section 4. First, we need to recall a notion of fuzzy stochastic trajectory integral.

**Proposition 6.1** (cf. [38, 42]). Assume that \( f: I \times \Omega \to \mathcal{F}_c^b(\mathbb{R}^d) \) is a predictable and \( L^2_{\mu_{\mathcal{Z}}}(\mu_{\mathcal{Z}}) \)-integrally bounded fuzzy stochastic process. Then for every \( \tau, t \in I, \tau \leq t \) there exists a unique element in \( \mathcal{F}_c^b(\mathcal{L}_2^\mu) \) denoted by \( \int_{\tau}^t f(s)dZ(s) \) such that for every \( \alpha \in (0, 1] \) it holds
\[
\left[ \int_{\tau}^t f(s)dZ(s) \right]^\alpha = \int_{\tau}^t [f(s)]^\alpha dZ(s), \quad \text{and} \quad \left[ \int_{\tau}^t f(s)dZ(s) \right]^0 \in \int_{\tau}^t [f(s)]^0 dZ(s),
\]
where \( \int_{\tau}^t [f(s)]^\alpha dZ(s) \) is set-valued integral defined in Section 4.

**Definition 6.2.** The element \( f^\tau f(s)dZ(s) \in \mathcal{F}_c^b(\mathcal{L}_2^\mu) \) from Proposition 6.1 is said to be the fuzzy stochastic trajectory integral (over interval \( [\tau, t] \)) of \( f \) with respect to the semimartingale \( Z \).

Since \( \mathcal{F}_c^b(\mathcal{L}_2^\mu) \subseteq \mathcal{F}_c^b(\mathcal{L}_2^\mu) \), we have \( \int_{\tau}^t f(s)dZ(s) \in \mathcal{F}_c^b(\mathcal{L}_2^\mu) \).

The following properties are counterparts of those listed in Proposition 4.3 and are needed in derivations of presented results concerning fuzzy stochastic equations.

**Proposition 6.3** ([38, 42]). Let \( f_1, f_2: I \times \Omega \to \mathcal{F}_c^b(\mathbb{R}^d) \) be the predictable and \( L^2_{\mu_{\mathcal{Z}}}(\mu_{\mathcal{Z}}) \)-integrally bounded fuzzy stochastic processes. Then
(i) for every \( \tau, a, t \in I, \tau \leq a \leq t \) it holds \( \int_{\tau}^a f_1(s)dZ(s) = \int_{\tau}^a f_1(s)dZ(s) + \int_{a}^t f_1(s)dZ(s) \),
(ii) for every \( \tau, t \in I, \tau < t \)
\[
D_{L^2}^2 \left( \int_{\tau}^t f_1(s)dZ(s), \int_{\tau}^t f_2(s)dZ(s) \right) \leq 2 \int_{[\tau,t] \times \Omega} D^2_{\mathcal{L}^2} (f_1(s, \omega), f_2(s, \omega)) \mu_{\mathcal{Z}}(ds, d\omega),
\]
(iii) for every \( \tau \in I \) the mapping \([\tau, T] \ni t \mapsto \int_{\tau}^t f_1(s)dZ(s) \in \mathcal{F}_c^b(\mathcal{L}_2^\mu) \) is continuous with respect to the metric \( D_{L^2} \).

By a fuzzy stochastic integral equation driven by semimartingale \( Z \) we mean the following relation in the metric space \( (\mathcal{F}_c^b(\mathcal{L}_2^\mu), D_{L^2}) \)
\[
x(t) = x_0 + \int_{0}^{t} f(s, x(s))dZ(s) \quad \text{for} \quad t \in I,
\]
where \( f: I \times \Omega \times \mathcal{F}_c^b(\mathcal{L}_2^\mu) \to \mathcal{F}_c^b(\mathbb{R}^d) \) and \( x_0 \in \mathcal{F}_c^b(\mathcal{L}_2^\mu) \).

The definitions of solution to (18) and its uniqueness are similar to those presented in Definition 5.3.

To obtain a result on existence and uniqueness of solution to (18) we will assume that:

(z1) \( f(\cdot, u): I \times \Omega \to \mathcal{F}_c^b(\mathbb{R}^d) \) is predictable for every \( u \in \mathcal{F}_c^b(\mathcal{L}_2^\mu) \),
(z2) there exists a constant \( C > 0 \) such that \( \mu_{\mathcal{Z}} \)-a.e. it holds
\[
D^2_{\mathcal{L}^2} (f(t, \omega, (\theta_{L^2})), (\theta_{L^2})) \leq C,
\]
(z3) there exists a continuous, nondecreasing, concave function \( \kappa: [0, \infty) \to [0, \infty) \) satisfying: \( \int_{0}^{+} \frac{1}{\kappa(s)} ds = +\infty, \)
\( \kappa(0) = 0, \kappa(s) > 0 \) for \( s > 0 \), and such that \( \mu_{\mathcal{Z}} \)-a.e. it holds
\[
D^2_{\mathcal{L}^2} (f(t, \omega, u), f(t, \omega, v)) \leq \kappa(D^2_{L^2}(u, v)) \quad \text{for} \quad u, v \in \mathcal{F}_c^b(\mathcal{L}_2^\mu).
Assume that Lemma 6.5. As before, to obtain the result on existence of solution to (18) we apply the following lemmata.

Now we can derive the result on existence and uniqueness of solution to (18).
In the sequel, let us consider equation (18) and equation

\[ y(t) = y_0 + \int_0^t f(s, y(s))dZ(s) \quad \text{for} \quad t \in I \quad (19) \]

with another initial value \( y_0 \in \mathcal{F}_c^b(L^2) \).

**Theorem 6.10.** Assume that \( x_0, y_0 \in \mathcal{F}_c^b(L^0_0) \) and \( f \) satisfies (z1)–(z3). Then, for the solution \( x \) to (18) and solution \( y \) to (19), it holds

\[
\sup_{t \in I} D_{L^2}^2(x(t), y(t)) \leq J^{-1}\left(J(2D_{L^2}^2(x_0, y_0) + 2\Delta Z)\right).
\]

**Corollary 6.11.** Let \( x_0 \in \mathcal{F}_c^b(L^2_0) \) and \( f \) satisfy (z1)–(z3). Let \( x \) denote the solution of (18), and for \( n \in \mathbb{N} \) let \( x_n \) denote the solution of

\[
x_n(t) = x_0^{(n)}(t) + \int_0^t f(s, x_n(s))dZ(s), \quad t \in I,
\]

where \( x_0^{(n)} \in \mathcal{F}_c^b(L^2_0) \). If \( D_{L^2}^2(x_0^{(n)}, x_0) \to 0 \), then

\[
\sup_{t \in I} D_{L^2}^2(x_n(t), x(t)) \to 0.
\]

Now, let us consider two equations with two different nonlinearities \( f \) and \( g \), i.e. equation (18) and

\[
y(t) = x_0 + \int_0^t g(s, y(s))dZ(s), \quad t \in I. \quad (20)
\]

A bound for the distance of solution \( x \) to (18) and solution \( y \) to (20) is as follows:

**Theorem 6.12.** Assume that \( x_0 \in \mathcal{F}_c^b(L^2_0) \) and \( f, g: I \times \Omega \times \mathcal{F}_c^b(L^2) \) satisfy conditions (z1)–(z3). Then

\[
\sup_{t \in I} D_{L^2}^2(x(t), y(t)) \leq \min(a, b),
\]

where

\[
a = J^{-1}\left(J(2 \int_{I \times \Omega} D_{\mathbb{R}^d}^2(f(s, \omega, x(s)), g(s, \omega, x(s)))\mu_Z(ds, d\omega) + 2\Delta Z)\right),
\]

\[
b = J^{-1}\left(J(2 \int_{I \times \Omega} D_{\mathbb{R}^d}^2(f(s, \omega, y(s)), g(s, \omega, y(s)))\mu_Z(ds, d\omega) + 2\Delta Z)\right).
\]

**Corollary 6.13.** Let \( x_0 \in \mathcal{F}_c^b(L^2_0) \) and \( f, g_n: I \times \Omega \times \mathcal{F}_c^b(L^2) \to \mathcal{F}_c^b(\mathbb{R}^d) \) satisfy (z1)–(z3). Let \( x \) denote solution to (18) and \( x_n \) denote solution to the equation

\[
x_n(t) = x_0 + \int_0^t f_n(s, x_n(s))dZ(s), \quad t \in I,
\]

\( n \in \mathbb{N} \). Assume that for every \( u \in \mathcal{F}_c^b(L^2) \)

\[
\int_{I \times \Omega} D_{\mathbb{R}^d}^2(f_n(t, \omega, u), f(t, \omega, u))\mu_Z(dt, d\omega) \to 0, \quad \text{as} \quad n \to \infty.
\]

Then

\[
\sup_{t \in I} D_{L^2}^2(x_n(t), x(t)) \to 0, \quad \text{as} \quad n \to \infty.
\]
7 Concluding remarks

In this paper, we study the set-valued and fuzzy stochastic integral equations. These equations generalize the classical single-valued stochastic differential equations [47, 48], random differential equations [54], deterministic set-valued and fuzzy differential equations [31, 32]. Our analysis concerns the equations driven by semimartingales that constitute the largest class of integrators with respect to which stochastic integrals can be reasonably defined. Up until now, the investigations have been undertaken in a framework of the Lipschitzian coefficients of the equations [38–42]. In this paper we relax the assumptions imposed on coefficients and require only that they satisfy an Osgood type condition, which is weaker than the Lipschitz one [46]. In this way, a class of admissible integrands is extended.

The fixed point theorems and Gronwall’s lemma [38, 39, 41, 42] are the main tools in derivations concerning the set-valued and fuzzy stochastic integral equations under the Lipschitz condition. For the equations with coefficients satisfying Osgood’s type condition we use Maruyama’s successive approximations method and the Bihari’s inequality. We show that the set-valued and fuzzy Maruyama’s approximations sequences are in fact certain Cauchy sequences with their limits being the desirable solutions. The uniqueness of the solution is showed with a help of Bihari’s inequality, which is also applied in proving the continuous dependence of solutions with respect to initial value and nonlinearities of the equation.

The results established for the set-valued and fuzzy stochastic integral equations have immediate implications for the single-valued stochastic integral equations driven by semimartingales. The latter are studied in [48] under the Lipschitz condition.

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