Large sets with multiplicity

Tuvi Etzion¹ · Junling Zhou²

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Abstract
Large sets of combinatorial designs has always been a fascinating topic in design theory. These designs form a partition of the whole space into combinatorial designs with the same parameters. In particular, a large set of block designs, whose blocks are of size $k$ taken from an $n$-set, is a partition of all the $k$-subsets of the $n$-set into disjoint copies of block designs, defined on the $n$-set, and with the same parameters. The current most intriguing question in this direction is whether large sets of Steiner quadruple systems exist and to provide explicit constructions for those parameters for which they exist. In view of its difficulty no one ever presented an explicit construction even for one nontrivial order. Hence, we seek for related generalizations. As generalizations, to the existence question of large sets, we consider two related questions. The first one is to provide constructions for sets on Steiner systems in which each block (quadruple or a $k$-subset) is contained in exactly $\mu$ systems. The constructions of such systems also yield secure protocols for the generalized Russian cards problem. The second question is to provide constructions for large set of H-designs (mainly for quadruples, but also for larger block size), which have applications in threshold schemes and in quantum jump codes. We prove the existence of such systems for many parameters using orthogonal arrays, perpendicular arrays, ordered designs, sets of permutations, and one-factorizations of the complete graph.

Keywords H-designs · Large sets · Latin squares · One-factorizations · Ordered designs · Permutations · Perpendicular arrays · Steiner systems

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Tuvi Etzion
etzion@cs.technion.ac.il

Junling Zhou
jlzhou@bjtu.edu.cn

¹ Department of Computer Science, Technion, Haifa 3200003, Israel
² Department of Mathematics, Beijing Jiaotong University, Beijing, China
1 Introduction

A Steiner system of order \( n \), \( S(t, k, n) \), is a pair \((Q, B)\), where \( Q \) is an \( n \)-set (whose elements are called points) and \( B \) is a collection of \( k \)-subsets (called blocks) of \( Q \), such that each \( t \)-subset of \( Q \) is contained in exactly one block of \( B \). A large set of Steiner systems \( S(t, k, n) \), on an \( n \)-set \( Q \), is a partition of all \( k \)-subsets of \( Q \) into Steiner systems \( S(t, k, n) \).

The interest in large sets is from block design point of view and also from graph theory point of view. A large set of Steiner systems \( S(1, k, n) \) is equivalent to a partition of the \( k \)-uniform complete hypergraph into disjoint perfect matchings. If \( k = 2 \) this large set is known as one-factorization of the complete graph \( K_n \). A comprehensive discussion on these one-factorizations was given in [51]. Pelsesohn [39] solved the problem for \( k = 3 \). A solution for \( k \geq 4 \) was given in the celebrated work of Baranayai [1] who proved the existence of such large sets using a network flow. Recently, an explicit and efficient construction for \( k = 4 \) was considered in [12].

A Steiner system \( S(2, 3, n) \) is known as Steiner triple system and the corresponding large set is known to exist for every admissible \( n \equiv 1 \) or \( 3 \) (mod 6), where \( n \neq 7 \). It was first proved by Lu [33,34], who left six open cases which were solved by Teirlinck [49]. An alternative shorter proof was given later by Ji [23]. The next interesting case is for Steiner system \( S(3, 4, n) \), which is also called a Steiner quadruple system and denoted by SQS\((n)\). A construction of a large set of SQS\((n)\) is known only for the trivial case when \( n = 4 \). For larger \( n \), the only known result is the existence proof of Keevash [26] who proved that a large set of Steiner systems \( S(t, k, n) \) exists if \( n \) is large enough and satisfies some necessary conditions (this \( n \) is beyond our imagination). The proof is nonconstructive and hence it does not throw any light on any explicit construction for these values of \( n \). It also does not provide any indication on the existence of such systems for smaller values of \( n \) (which can be very large).

In the absence of known explicit and efficient constructions for large sets of SQS\((n)\) the research can be done in two different directions. One direction is to find the maximum number of pairwise disjoint SQS\((n)\). The best results in this direction can be found in [15,16]. A second direction is to define large sets with multiplicity. A large set of \( S(t, k, n) \) with multiplicity \( \mu \), denoted by LS\((t, k, n; \mu)\), is a set of Steiner systems \( S(t, k, n) \) on an \( n \)-set \( Q \), such that each \( k \)-subset of \( Q \) is contained in exactly \( \mu \) systems. The goal is to find such large sets for any given positive integer \( \mu \). Note that a LS\((t, k, n; 1)\) is a large set which implies LS\((t, k, n; \mu)\) for any \( \mu \geq 1 \). Clearly, an LS\((t, k, n; \mu)\) consists of \( \mu {\binom{n-1}{k-t}} \) Steiner systems \( S(t, k, n) \). Large sets with multiplicity were considered in [14], where it is proved that a LS\((3, 4, 2^\mu; \mu)\) exists, for any \( r \geq 3 \) and \( \mu \geq 2 \). Such large sets with multiplicity imply the existence of another family of large sets (with multiplicity one), namely, large sets of H-designs, which are interesting designs for themselves. They have applications in threshold schemes [14] and in quantum jump codes [52].

Large sets with multiplicity are also important in the design of secure protocols for the generalized Russian cards problem (see the excellent exposition of Swanson and Stinson [45] for the problem and references for previous work; it also motivated some further work afterwards). The generalized Russian cards problem was stated in [45] as follows: Alice, Bob, and Cathy are dealt a deck of \( n \) cards, each given \( a, b, \) and \( c \) cards, respectively. The goal is for Alice and Bob to learn each other’s hand via public communication, without Cathy...
learning the fate of any particular card. Swanson and Stinson [45] gave secure deterministic protocols based on large sets of Steiner systems. They proved that if there exists a large set with multiplicity $\text{LS}(k - 1, k; n; \mu)$, then there exists a secure nondeterministic protocol. Further connections between the two concepts can be found in [45].

The goals of this paper are to construct large sets with multiplicity and large sets of H-designs. The fact that asymptotically, for very large $n$, they exist does not help for the applications, where usually we require a structure with relatively small parameters. In this paper we provide many constructions of these structures as well as many interesting supporting results. For the presentation of the main results in our paper we need to define the concept of a perpendicular array. A *perpendicular array* $\text{PA}_\lambda(k, \ell; n)$ is a $\lambda \cdot \binom{n}{k} \times \ell$ matrix $A$ with entries from an $n$-set, say $\mathbb{Z}_n$, such that each row has $\ell$ distinct entries and each submatrix of $A$ which consists of any $k$ columns contains each $k$-subset of $\mathbb{Z}_n$ exactly $\lambda$ times. There are many new results in this paper and they are summarized in Sect. 7. Two of the highlights proved in this paper (which motivates future research), are presented in Sects. 5 and 6:

**If there exist a large set $\text{LS}(3, 4; n; \mu)$ and a perpendicular array $\text{PA}_\gamma(2, n; n)$, $\mu = \frac{(n - 1)\gamma}{2}$, then there exists a large set $\text{LS}(3, 4, 2^m n; \mu)$, for each $m \geq 0$.**

**There exists an LS$\bigl(3, 4, 5 \cdot 2^m; 9\ell\bigr)$ for each integer $m \geq 1$ and each integer $\ell \geq 1$.**

The rest of this paper is organized as follows. In Sect. 2 we present the basic concepts required for our expositions. These include the definition of an H-design and the connection between a large set with multiplicity and a large set of H-designs. Other concepts include orthogonal arrays, one-factorizations, and arrays of permutations. Section 3 is devoted to large sets with multiplicity and large sets of H-designs with small parameters. In Sect. 3.1 large sets of H-designs are obtained recursively from an initial large set of H-designs with small parameters. All the large sets constructed in this section are new and were not known before. In Sect. 3.2 large sets with multiplicity are constructed using some structure of Steiner systems and sets of permutations. They yield new large sets of H-designs with small parameters. In Sect. 4 we present a few well-known constructions of pairwise disjoint SQS$\bigl(n\bigr)$ which will be adapted and used for our constructions of large sets with multiplicity of Steiner quadruple systems. These constructions contain doubling and quadrupling constructions. In Sects. 5 and 6 we present our main construction for large sets with multiplicity of Steiner quadruple systems. In Sect. 5 a quadrupling construction for large sets with multiplicity are presented. In Sect. 6 the quadrupling construction is generalized for multiplication by $2^m$ instead of multiplication by 4. In Sect. 7 we summarize all the new results of our work and suggest directions for future research.

### 2 Preliminaries

This section is devoted to define several concepts which are important in our exposition. In Sect. 2.1 we define the concepts of H-designs and large sets of H-designs whose construction is one of the goals of our work. In Sect. 2.2 we define the concepts of orthogonal arrays and large sets of orthogonal arrays. Finally, we prove a connection between large set with multiplicity, large set of orthogonal arrays, and large set of H-designs. In Sect. 2.3 we define a design used in many constructions of block designs, namely, one-factorization. In Sect. 2.4 we consider permutations and arrays of permutations such as Latin squares, ordered designs, and perpendicular arrays. Finally, in Sect. 2.5 we define the concept of configurations which enables us to categorize the different blocks of a design after some partition of the point set is made.
2.1 H-designs

Large sets with multiplicity have their own interest, but they are also important in constructions for large sets of H-designs, which are large sets of Steiner systems “with holes” [20,50]. This type of designs was introduced by Mills [36]. An \( H \)-design \( H(n, g, k, t) \) is a triple \((Q, G, B)\), which satisfies the following properties:

1. \( Q \) is a set with \( ng \) points.
2. \( G \) is a partition of \( Q \) into \( n \) subsets (called groups), each one with \( g \) points.
3. \( B \) is a set of \( k \)-subsets of \( Q \) (called blocks), such that a group and a block contain at most one common point, and any \( t \) points from any \( t \) distinct groups occur in exactly one block.

To simplify the constructions in the sequel, we will assume that for a given H-design \( H(n, g, k, t) \), the elements of the \((ng)\)-set \( Q \) are ordered and in a block \( \{x_1, x_2, \ldots, x_k\} \) the elements are ordered, i.e., \( x_1 < x_2 < \cdots < x_k \), by the order of \( Q \).

H-designs were defined and used first in [21,37]. The following necessary and sufficient conditions for the existence of H-designs for which \((k, t) = (4, 3)\) were proved in [24,25,37].

Theorem 1 The necessary and sufficient conditions for the existence of an \( H(n, g, 4, 3) \) are

\[ gn \equiv 0 \pmod{2}, \quad g(n - 1)(n - 2) \equiv 0 \pmod{3}, \quad n \geq 4, \quad \text{and} \quad (n, g) \neq (5, 2). \]

A large set of H-designs, denoted by \( LH(n, g, k, t) \), is a partition of all the \( k \)-subsets from the \( ng \) points of \( Q \) taken from any \( k \) distinct groups, into pairwise disjoint H-designs \( H(n, g, k, t) \). The number of H-designs in a large set is calculated in the following lemma.

Lemma 1 The size of an \( LH(n, g, k, t) \) is \( \binom{n-t}{k-t} g^{k-t} \).

Proof An H-design \( H(n, g, k, t) \) contains \( \binom{n}{k} g^t \) blocks and the number of \( k \)-subsets in \( Q \), where each \( k \)-subset contains at most one element from each group, is \( \binom{n}{k} g^k \). Hence, the number of H-designs in an \( LH(n, g, k, t) \) is

\[ \binom{n}{k} g^k / \binom{n-t}{k-t} g^t = \binom{n-t}{k-t} g^{k-t} = \binom{n-t}{k-t} g^{k-t}. \]

\( \square \)

An H-design \( H(n, g, 3, 2) \) is usually called a group divisible design of type \( g^n \) and denoted by \( GDD(g^n) \). The large sets of disjoint group divisible designs were first studied because of their connection with perfect threshold schemes [42,44]. Combining the existence result of large sets of Steiner triple systems and much work on large sets of GDDs by Chen et al [13] and Teirlinck [50], Lei [30] finally established that there exists a large set of GDD(\( g^n \)) if and only if

\[ n(n - 1)g^2 \equiv 0 \pmod{6}, \quad (n - 1)g \equiv 0 \pmod{2}, \quad \text{and} \quad (g, n) \neq (1, 7). \]

2.2 Orthogonal arrays

An orthogonal array \( OA(t, k, n) \) is an \( n^t \times k \) matrix \( C \), with entries from \( \mathbb{Z}_n \), such that any submatrix generated by any \( t \) columns of \( C \) contains each \( t \)-tuple from \( \mathbb{Z}_n \) exactly once as a row (note that \( t \)-tuples are always ordered). A large set of orthogonal arrays \( LOA(t, k, n) \) is a set of \( n^{k-t} \) orthogonal arrays \( OA(t, k, n) \), such that each vector of length \( k \) over \( \mathbb{Z}_n \) occurs as a row in exactly one of the orthogonal arrays. The first result is quite straightforward, but with no obvious reference and for completeness a proof is provided.

Theorem 2 If there exists an \( OA(t, k, n) \), then there exists an \( LOA(t, k, n) \).

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Proof Let $C$ be an OA $(t, k, n)$. Let $X$ be the set of $n^{k-t}$ vectors of length $k$ over $\mathbb{Z}_n$ such that the last $t$ entries in all the vectors of $X$ are zeroes. For each $x \in X$ define the set $C_x$ as follows:

$$C_x \triangleq x + C \triangleq \{x + c : c \in C\}.$$  

Clearly, $C_x$ is also an OA $(t, k, n)$.

Assume now that there exist two words $x_1, x_2 \in X$ and two words $c_1, c_2 \in C$ such that $x_1 + c_1 = x_2 + c_2$. Since the last $t$ entries in $x_1$ and $x_2$ are zeroes, it follows that the last $t$ entries of $c_1$ and $c_2$ are equal. Since $C$ is an OA $(t, k, n)$ and the last $t$ entries of $c_1$ and $c_2$ are equal, it follows that $c_1 = c_2$. Hence, we also have $x_1 = x_2$ and therefore $\{C_x : x \in X\}$ is an LOA $(t, k, n)$.

By Theorem 2 the existence of an OA $(t, k, n)$ implies the existence of an LOA $(t, k, n)$. In the following results we will use these orthogonal arrays to form large sets of H-designs. The related constructions will require the existence of other large sets of H-designs with smaller group size or the existence of some large sets with multiplicity.

Theorem 3 If there exists an LH $(n, g, k, t)$ and an OA $(t, k, u)$, then there exists an LH $(n, gu, k, t)$.

Proof Let $A_1, A_2, \ldots, A_k$ be an LH $(n, g, k, t)$ defined on an $(ng)$-set $Q$ (taken as $\mathbb{Z}_n \times \mathbb{Z}_g$ with group set $\{(i) \times \mathbb{Z}_g : i \in \mathbb{Z}_n\}$, where $\delta = (n-\delta)g^{k-t}$. By Theorem 2 the existence of OA $(t, k, u)$ implies the existence of an LOA $(t, k, u)$. Let $B_1, B_2, \ldots, B_{\gamma}$ be an LOA $(t, k, u)$ defined on $\mathbb{Z}_u$, where $\gamma = u^{k-t}$. Define $\delta \gamma$ sets $C_{i,j}$, $1 \leq i \leq \delta$, $1 \leq j \leq \gamma$, where $C_{i,j}$ consists of \binom{\delta}{\gamma} (gu)^j blocks. For each $\{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\} \in A_i$, where $(x_\ell, y_\ell) \in \mathbb{Z}_n \times \mathbb{Z}_g$, and each $(b_1, b_2, \ldots, b_k) \in B_j$ the following block is defined:

$$\{(x_1, (y_1, b_1)), (x_2, (y_2, b_2)), \ldots, (x_k, (y_k, b_k))\}.$$  

It is easy to verify that each $C_{i,j}$ forms an H-design with group set $\{(i) \times (\mathbb{Z}_g \times \mathbb{Z}_u) : i \in \mathbb{Z}_n\}$. By Lemma 1, the number of H-designs in LH $(n, gu, k, t)$ is $\binom{\gamma}{\delta\gamma}(gu)^k = \delta \gamma$ and the size of $C \triangleq \{C_{i,j} : 1 \leq i \leq \delta, 1 \leq j \leq \gamma\}$ is $\delta \gamma$. Therefore, to prove that $C$ forms an LH $(n, gu, k, t)$, it is sufficient to show that each $k$-subset of $\mathbb{Z}_n \times (\mathbb{Z}_g \times \mathbb{Z}_u)$, meeting each group in at most one point, is contained in one H-design from $\{C_{i,j} : 1 \leq i \leq \delta, 1 \leq j \leq \gamma\}$.

Let $Z = \{(x_1, (y_1, b_1)), (x_2, (y_2, b_2)), \ldots, (x_k, (y_k, b_k))\}$, where $(x_1, (y_1, b_1)), (x_2, (y_2, b_2)), \ldots, (x_k, (y_k, b_k)) \in \mathbb{Z}_n \times (\mathbb{Z}_g \times \mathbb{Z}_u)$ and $x_1 < x_2 < \cdots < x_k$. The set $Z' = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$ is a $k$-subset of $\mathbb{Z}_n \times \mathbb{Z}_g$ such that $x_1 < x_2 < \cdots < x_k$ and hence $Z'$ is a $k$-subset in a unique H-design, say $A_i$. Also, $(b_1, b_2, \ldots, b_k)$ is a row in a unique orthogonal array $B_j$. Therefore, $Z$ is a $k$-subset of the unique set $C_{i,j}$. Thus, $\{C_{i,j} : 1 \leq i \leq \delta, 1 \leq j \leq \gamma\}$ is an LH $(n, gu, k, t)$.

The next result generalizes a related theorem for LH $(n, g, 4, 3)$ proved in [14].

Theorem 4 If there exist an OA $(t, k, g)$ and an LS $(t, k, n; g^{k-t})$, then there exists an LH $(n, g, k, t)$.

Proof Let $S_1, S_2, \ldots, S_{\delta}, \delta = \binom{n-\delta}{k-\delta}g^{k-t}$, be an LS $(t, k, n; g^{k-t})$, on the point set $\mathbb{Z}_n$, and let $C_1, C_2, \ldots, C_{\delta}$ be an LOA $(t, k, g)$ implied by Theorem 2. We construct an LH $(n, g, k, t)$ on the point set $\mathbb{Z}_n \times \mathbb{Z}_g$, i.e. group sets $\{(i) \times \mathbb{Z}_g, i \in \mathbb{Z}_n\}$, with H-designs $S_1^*, S_2^*, \ldots, S_{\delta}^*$.  

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Given any $k$-subset $\{x_1, x_2, \ldots, x_k\}$ of $\mathbb{Z}_n$ which appears in $g^{k-t}$ distinct Steiner systems, $S_{i1}, S_{i2}, \ldots, S_{ig^{k-t}}$, we form the following $g^t$ blocks for $S^*_i$,

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}, \quad (y_1, y_2, \ldots, y_k) \in C_j.$$

By Lemma 1, the number of H-designs in LH(n, g, k, t) is $\delta$ and hence to prove that $S^*_1, S^*_2, \ldots, S^*_g$ form an LH(n, g, k, t), it is sufficient to show that each $k$-subset of $\mathbb{Z}_n \times \mathbb{Z}_g$, meeting each group in at most one point, is contained in one of the H-designs from $S^*_1, S^*_2, \ldots, S^*_g$.

Let $Z = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$, where $(x_i, y_i) \in \mathbb{Z}_n \times \mathbb{Z}_g$ and $|\{x_1, x_2, \ldots, x_k\}| = k$. Since $X = \{x_1, x_2, \ldots, x_k\}$ is a $k$-subset of $\mathbb{Z}_n$, it follows that $X$ is contained in exactly $g^{k-t}$ Steiner systems $S_{i1}, S_{i2}, \ldots, S_{ig^{k-t}}$ of the large set with multiplicity $LS(t, k, n; g^{k-t})$. Since $Y = (y_1, y_2, \ldots, y_k)$ is a word of length $k$ over $\mathbb{Z}_g$, it follows that $Y$ is a row of an OA$(t, k, g)$ from the related large set. It follows by the construction implied by (1) that $Z$ is contained in one of the constructed blocks. □

Theorem 4 is the key for our construction of new large sets of H-designs which will be discussed in Sect. 3. In [14] we have the following application of Theorem 4.

**Theorem 5** For any integers $g \geq 2$ and $r \geq 2$, there exists an LH$(2^r, g, 4, 3)$.

Theorem 5 was proved in [14] without the explicit use of orthogonal arrays. The orthogonal arrays required to obtain this theorem and the ones which will be heavily used in our exposition are presented in the following trivial well-known theorem.

**Theorem 6** For any integer $t \geq 2$ and for any given $g \geq 2$, there exists an OA$(t-1, t, g)$.

**Proof** Define the following set of $t$-tuples

$$M \triangleq \{(x_1, x_2, \ldots, x_t) : x_i \in \mathbb{Z}_g, \sum_{i=1}^{t} x_i \equiv 0 \pmod{g}\}.$$

Consider the set $M$ as an array with $t$ columns with any order of the rows by the $t$-tuples of $M$. Clearly, each $(t-1)$-tuple appears exactly once in the projection of $t-1$ coordinates of $M$, which implies that the array $M$ is an OA$(t-1, t, g)$. □

To apply Theorems 3 and 4 to obtain a LS$(t, k, n; \mu)$ related orthogonal arrays are required. Parameters of such orthogonal arrays can be found in the excellent books [22,40].

### 2.3 One-factorizations

A one-factorization $F = \{F_0, F_1, \ldots, F_{v-2}\}$ of the complete graph $K_v$, where $v$ is an even positive integer, is a partition of all the edges of $K_v$ into perfect matchings. Each $F_i$, $0 \leq i \leq v-2$, is a perfect matching of $K_v$. This perfect matching is also called a one-factor. As stated before a one-factor is a Steiner system $S(1, 2, v)$ and a one-factorization is a related large set. Clearly, a one-factor contains $\frac{v}{2}$ pairs of vertices whose union is the set of $v$ vertices in $K_v$. One-factorizations were used as building blocks in many constructions of various block designs.
2.4 Arrays of permutations

A permutation acting on the point set of a Steiner system $S(t, k, n)$ yields another Steiner system $S(t, k, n)$. This is the obvious motivation to use arrays of permutations in constructions of large sets of Steiner systems with multiplicity and large sets of H-designs.

A $v \times v$ Latin square is a $v \times v$ array in which each row and each column is a permutation of a $v$-set $Q$. A Latin square has no $2 \times 2$ subsquares if any $2 \times 2$ subsquare restricted to two rows and two columns does not form a Latin square [27]. Such Latin squares were used in a doubling construction [31] to form a set of pairwise disjoint SQSs. The DLS Construction, which will be presented in Sect. 4, is based on the construction in [31], but the constructions which will be given in Sects. 5 and 6 do not require such Latin squares. We will make use of combinatorial designs called ordered designs and perpendicular arrays.

An ordered design $OD_\lambda(k, \ell, n)$ is a $\lambda \cdot \binom{n}{k} \times \ell$ matrix $A$ with entries from an $n$-set, say $Z_n$, such that

1. each row has $\ell$ distinct entries and
2. each submatrix of $A$ which consists of any $k$ columns contains each $k$-tuple of $Z_n$ exactly $\lambda$ times.

A perpendicular array $PA_\lambda(k, \ell, n)$ is a $\lambda \cdot \binom{n}{k} \times \ell$ matrix $A$ with entries from an $n$-set, say $Z_n$, such that

1. each row has $\ell$ distinct entries and
2. each submatrix of $A$ which consists of any $k$ columns contains each $k$-subset of $Z_n$ exactly $\lambda$ times.

Although probably known before, the first formal definition of perpendicular arrays is given in [38]. Ordered designs were defined first by Teirlinck [46–48]. Perpendicular arrays have found applications in authentication and secrecy codes [43]. This has motivated an extensive research, e.g. [2,3,5–9,18,19,28,35,38]. Some of the constructions which follow will make use of ordered designs and perpendicular arrays which can also be regarded as permutation sets.

An ordered design $OD_\lambda(k, n, n)$ or a perpendicular array $PA_\lambda(k, n, n)$ consists of a set of permutations from $S_n$, the set of all permutations on an $n$-set. A set $P \subseteq S_n$ of permutations is (uniformly) $k$-homogeneous if it is a $PA_\lambda(k, n, n)$; it is $k$-transitive if it is an $OD_\lambda(k, n, n)$. These are the combinatorial designs required for the constructions in the sequel.

A few types of ordered designs $OD_\lambda(k, n, n)$ and perpendicular arrays $PA_\lambda(k, n, n)$ are required, e.g. those with $k = 2$ or those with $k > 2$ and small $n$. For $k > 3$ these permutations sets are required for a simple construction of an LS$(t, k, n; \mu)$ from a given Steiner system $S(t, k, n)$. This construction is presented in the following theorem.

**Theorem 7** If there exists a Steiner system $S(t, k, n)$ and a perpendicular array $PA_\lambda(k, n, n)$, then there exist an LS$(t, k, n; \mu)$, where $\mu = \lambda \binom{n}{k} / \binom{k}{t}$.

**Proof** By applying all the permutations of a perpendicular array $PA_\lambda(k, n, n)$ on each block of a Steiner system $S(t, k, n)$, each $k$-subset of the $n$-set is produced exactly $\lambda$ times and hence the claim follows. \qed

To apply Theorem 7, it is required to have related Steiner systems and perpendicular arrays. Unfortunately, if $k > 3$ and $n \geq 25$ there is no subgroup of $S_n$ which forms such a permutation set, of a perpendicular array or an ordered designs, except for the group of all permutation $S_n$ and the alternating group $A_n$ which contains all the even permutation
A probabilistic proof for the existence of such a permutation set with a smaller number of permutations is given in [29]. A probabilistic construction with a smaller number of permutations is given in [17]. In both cases the number of permutations in the related perpendicular arrays and ordered designs is too large. Moreover, there is no concrete construction. This implies that we will not be able to use these ordered designs or perpendicular arrays with \( k > 3 \) to construct large sets with small multiplicity using such arrays in Theorem 7. For \( n \leq 24 \) there are some interesting ordered designs and perpendicular arrays which yield some interesting large sets. These constructions will be considered in Sect. 3 when large sets with small parameters will be discussed.

For \( k = 2 \), \( \text{PA}_3(2, n, n) \) will be used in our main constructions instead of Latin squares which were used in other constructions [14,16,31] (also \( \text{OD}_3(2, n, n) \) can be used for this purpose). Some parameters for these designs are given in the following theorem.

**Theorem 8** The following perpendicular arrays exist:

1. \( \text{PA}_2(2, q, q) \), \( q \) power of 2 [41].
2. \( \text{PA}_2(2, q, q) \), \( q \) odd prime power [41].
3. \( \text{PA}_2(2, 10, 10) \) [5].

For \( k > 2 \) and small \( n \), perpendicular arrays and ordered designs will be used in Sect. 3 to generate \( \text{LS}(t, k, n; \mu) \) for small values of \( n \). Related perpendicular arrays and ordered designs are given in the following theorem.

**Theorem 9** The following perpendicular arrays and ordered designs exist:

1. \( \text{PA}_3(3, 7, 7) \) [5].
2. \( \text{OD}_1(4, 11, 11) \) [4].

For more information on parameters of perpendicular arrays and ordered designs the reader is referred to the short survey in [4].

### 2.5 Configurations

Most of the constructions which will be described in the sequel and those which were already mentioned (e.g. see Theorem 4), are based on partitions of the point set. Most of our partitions will be with parts of equal size, unless when there are only two parts, where the two parts might not be of equal size.

Assume that the point set \( Q \) has size \( n \). This set can be partitioned into two subsets \( A \) and \( B \), where \( |A| = \ell \) and \( |B| = n - \ell \). Given such a partition and a \( k \)-subset \( X = \{x_1, x_2, \ldots, x_k\} \) of \( Q \), we say that \( X \) is a \( k \)-subset from configuration \((i, j)\), where \( i + j = k \), if \( |X \cap A| = i \) and \( |X \cap B| = j \). The definition of configuration is generalized for a partition of the point set into more than two parts. For example, if \( Q \) is partitioned into four sets \( A_1, A_2, A_3, \) and \( A_4 \) (usually of equal size), then the \( k \)-subset \( X \) is from configuration \((i_1, i_2, i_3, i_4)\), where \( i_1 + i_2 + i_3 + i_4 = k \), if \( |X \cap A_j| = i_j \) for \( 1 \leq j \leq 4 \).

The partition of the point set has a few advantages such as applying certain permutations on one set of points, or considering each configuration separately makes it easier to analyse the structure of the \( k \)-subsets.
3 New large sets with small parameters

In this section we present some basic constructions for large sets with multiplicity and for large sets of H-designs. The number of groups in these constructions of H-designs is small and the same is true for the number of points in the large sets with multiplicity. In Sect. 3.1 large sets of H-designs with small parameters are obtained by ad-hoc constructions. In Sect. 3.2 large sets with multiplicity and small parameters are obtained by applying sets of permutations on the coordinates of some Steiner systems. These large sets with multiplicity imply related large sets of H-designs.

3.1 Large sets of H-designs with small number of groups

This section is devoted to a few ad-hoc constructions presented in the proofs of the next three lemmas.

Lemma 2 There exists an LH(5, 4u, 4, 3) for any positive integer u.

Proof We construct an H(5, 4, 4, 3) on the point set \(\mathbb{Z}_8 \cup \{8, 9, \ldots, 19\}\) with group set \(\{(0, 2, 4, 6), \{1, 3, 5, 7\}\} \cup \{(i, i + 3, i + 6, i + 9) : i = 8, 9, 10\}\). Its block set \(\mathbb{B}_0\) is as follows:

| 0, 1, 8, 9 | 2, 5, 8, 9 | 3, 8, 9, 10 | 4, 8, 9, 13 | 6, 8, 9, 16 | 7, 8, 9, 19 |
| 7, 0, 8, 10 | 1, 4, 8, 10 | 2, 8, 10, 12 | 5, 8, 10, 15 | 6, 8, 10, 18 | 5, 6, 8, 13 |
| 6, 7, 8, 15 | 0, 3, 8, 15 | 7, 2, 8, 13 | 1, 8, 13, 15 | 0, 8, 13, 18 | 3, 8, 12, 13 |
| 4, 5, 8, 12 | 7, 8, 12, 16 | 4, 7, 8, 18 | 0, 8, 12, 19 | 5, 8, 18, 19 | 5, 0, 8, 16 |
| 6, 1, 8, 12 | 1, 2, 8, 19 | 8, 16, 18 | 2, 3, 8, 18 | 2, 8, 15, 16 | 3, 4, 8, 16 |
| 3, 6, 8, 19 | 4, 8, 15, 19 | 0, 1, 11, 13 | 7, 0, 12, 13 | 2, 5, 12, 13 | 4, 7, 11, 13 |
| 6, 11, 12, 13 | 1, 12, 13, 14 | 4, 12, 13, 17 | 1, 4, 13, 18 | 6, 7, 13, 18 | 1, 2, 9, 13 |
| 2, 3, 13, 17 | 5, 13, 17, 18 | 6, 1, 13, 17 | 0, 3, 9, 13 | 0, 13, 15, 17 | 3, 6, 13, 15 |
| 5, 0, 13, 14 | 5, 9, 11, 13 | 4, 5, 13, 15 | 3, 4, 13, 14 | 3, 11, 13, 18 | 2, 11, 13, 15 |
| 7, 13, 14, 15 | 6, 9, 13, 14 | 2, 13, 14, 18 | 7, 9, 13, 17 | 5, 0, 15, 19 | 2, 3, 15, 19 |
| 5, 0, 9, 10 | 5, 0, 12, 17 | 5, 0, 11, 18 | 1, 2, 11, 18 | 6, 7, 9, 10 | 0, 1, 10, 12 |
| 0, 1, 14, 15 | 0, 1, 16, 17 | 0, 1, 18, 19 | 7, 2, 18, 19 | 0, 9, 17, 19 | 7, 0, 17, 18 |
| 3, 6, 17, 18 | 5, 6, 9, 17 | 3, 9, 16, 17 | 0, 3, 10, 17 | 0, 10, 14, 18 | 0, 3, 16, 18 |
| 3, 14, 18, 19 | 0, 12, 14 | 0, 11, 12, 16 | 7, 0, 15, 16 | 0, 10, 11, 15 | 0, 3, 11, 19 |
| 7, 0, 9, 11 | 7, 0, 14, 19 | 0, 9, 14, 16 | 4, 5, 9, 16 | 5, 9, 14, 19 | 2, 5, 17, 19 |
| 5, 15, 16, 17 | 5, 6, 11, 15 | 2, 5, 11, 16 | 1, 9, 11, 16 | 3, 6, 9, 11 | 6, 1, 10, 11 |
| 6, 1, 15, 16 | 6, 1, 9, 19 | 1, 4, 14, 19 | 1, 9, 10, 14 | 1, 4, 9, 17 | 1, 4, 11, 15 |
| 1, 4, 12, 16 | 1, 2, 12, 17 | 3, 4, 15, 17 | 3, 10, 14, 15 | 2, 3, 10, 11 | 2, 3, 12, 16 |
| 3, 6, 14, 16 | 5, 12, 14, 16 | 6, 7, 12, 14 | 5, 6, 10, 14 | 5, 6, 16, 18 | 6, 7, 11, 16 |
| 6, 7, 17, 19 | 5, 6, 12, 19 | 5, 10, 11, 12 | 7, 10, 11, 18 | 1, 2, 10, 15 | 1, 10, 17, 18 |
| 2, 9, 10, 17 | 1, 2, 14, 16 | 6, 1, 14, 18 | 1, 11, 12, 19 | 7, 2, 11, 12 | 1, 15, 17, 19 |
| 2, 3, 9, 14 | 4, 7, 9, 14 | 4, 10, 12, 14 | 7, 10, 12, 17 | 7, 2, 15, 17 | 2, 16, 17, 18 |
| 2, 5, 10, 18 | 3, 4, 10, 18 | 2, 5, 14, 15 | 7, 2, 9, 16 | 7, 2, 10, 14 | 2, 9, 11, 19 |
| 2, 12, 14, 19 | 3, 4, 9, 19 | 3, 4, 11, 12 | 3, 6, 10, 12 | 3, 11, 15, 16 | 3, 12, 17, 19 |
| 4, 5, 10, 17 | 4, 5, 11, 19 | 4, 17, 18, 19 | 4, 5, 14, 18 | 4, 11, 16, 18 | 4, 7, 10, 15 |
| 4, 7, 12, 19 | 4, 7, 16, 17 | 4, 9, 10, 11 | 4, 14, 15, 16 | 6, 10, 15, 17 | 6, 11, 18, 19 |
| 6, 12, 16, 17 | 6, 14, 15, 19 | 7, 11, 15, 19 | 7, 14, 16, 18 |

We apply to \(\mathbb{B}_0\) the automorphism group \((\mathbb{Z}_8, +)\) (only to the points of \(\mathbb{Z}_8\), leaving each one of the points in the set \(\{8, 9, \ldots, 19\}\) unchanged) to obtain 8 mutually disjoint H(5, 4, 4, 3)s.
namely an LH(5, 4, 4, 3). Finally, an LH(5, 4μ, 4, 3) for any μ ≥ 1 is obtained by Theorem 3.

Lemma 3 There exists an LH(6, g, 4, 3) if and only if g is divisible by 3.

Proof First we construct an H(6, 3, 4, 3) on the point set \( \mathbb{Z}_9 \cup \{9, 10, \ldots, 17\} \) with group set \( \{i, i + 3, i + 6\} : i = 0, 1, 2, 9, 10, 11\). Its block set \( B_0 \) is as follows:

| 0, 1, 2, 9 | 8, 0, 1, 10 |
| 3, 5, 12, 14 | 4, 8, 12, 14 |
| 1, 2, 13, 17 | 2, 3, 4, 13 |
| 4, 5, 9, 13 | 7, 0, 9, 13 |
| 6, 8, 13, 15 | 7, 12, 13, 17 |
| 5, 7, 13, 14 | 1, 3, 5, 13 |
| 6, 7, 11, 13 | 4, 8, 11, 13 |
| 0, 4, 11, 16 | 4, 5, 0, 10 |
| 0, 2, 15, 17 | 5, 0, 15, 16 |
| 7, 0, 2, 10 | 0, 2, 12, 16 |
| 1, 2, 10, 12 | 6, 7, 8, 12 |
| 6, 8, 9, 11 | 8, 10, 14 |
| 8, 1, 11, 16 | 8, 1, 14, 15 |
| 4, 10, 11, 15 | 4, 6, 10, 12 |
| 3, 5, 7, 11 | 5, 10, 11, 12 |
| 6, 1, 11, 12 | 1, 3, 10, 11 |
| 2, 9, 10, 14 | 3, 5, 9, 10 |
| 1, 2, 11, 15 | 1, 3, 12, 17 |
| 2, 14, 15, 16 | 5, 6, 1, 9 |
| 6, 1, 10, 15 | 1, 9, 14, 16 |
| 2, 6, 10, 11 | 3, 4, 10, 14 |
| 8, 3, 1, 16 | 4, 6, 8, 16 |
| 7, 11, 12, 16 | 8, 10, 15, 1 |

We apply to \( B_0 \) the automorphism group \( (\mathbb{Z}_9, +) \) (only to the points of \( \mathbb{Z}_9 \)), leaving each one of the points in the set \( \{9, 10, \ldots, 17\} \) unchanged to obtain 9 mutually disjoint H(6, 3, 4, 3)s, namely an LH(6, 3, 4, 3). Finally, an LH(6, g, 4, 3) for any g divisible by 3 is obtained by Theorem 3. For g which is not divisible by 3, an LH(6, g, 4, 3) does not exist by Theorem 1.

Lemma 4 There exists an LH(7, g, 4, 3) if and only if g is even.

Proof First we construct an H(7, 2, 4, 3) on the point set \( \mathbb{Z}_{14} \) with groups \( \{i, i + 7\}, 0 \leq i \leq 6 \). Its block set \( B_0 \) is in Table 1.

We apply 8 permutations of the \( 8 \times 14 \) array in Table 2 on these blocks to \( B_0 \) to generate an LH(7, 2, 4, 3).

Finally, an LH(7, g, 4, 3) for any even g is obtained by applying Theorem 3 on this large set. For odd g, an LH(7, g, 4, 3) does not exist by Theorem 1 which completes the claim of the lemma.

The conclusion of Lemmas 2, 3, 4, and Theorem 1 is the following result.

Theorem 10 An LH(n, g, 4, 3) exists for all admissible parameters with \( n \in \{5, 6, 7\}, \) with possible exceptions for \( n = 5 \) and \( g \equiv 2 \) (mod 4).
Large sets with multiplicity

3.2 Large sets of H-designs from large sets with multiplicity

The goal of this section is to obtain large sets with multiplicity and then to obtain large sets of H-designs using Theorem 4. The first step in this direction is to find a large set LS(\(t, k; n; g^{k-t}\)), for different sets of parameters, where the multiplicity \(g^{k-t}\) is not too large. A special case of Theorem 4, which is the key to obtain large set of H-designs from large set with multiplicity, was applied in [14] to obtain large set LH(\(2^m, g, 4, 3\), \(m \geq 3\), for each \(g \geq 2\) (see Theorem 5). If a construction of an LS(\(t, k; n\)) is known then clearly there exists an LS(\(t, k; n; g^{k-t}\)) for any \(g \geq 1\) and Theorem 4 can be applied trivially. Hence, our first target is to construct such large sets LS(\(3, 4; n; \mu\)), where \(n\) is not a power of 2 and \(\mu\) small as possible. Our second target is to construct large sets LS(\(t, k; n; g^{k-t}\)) for \(k > t > 3\).

Let \(S\) be an S(\(t, k; n\)) on the point set \(\mathbb{Z}_n\). We would like to use this Steiner system to form an LS(\(t, k; n; \mu\)) on the point set \(\mathbb{Z}_n\). The main idea is to use a set of permutations on \(\mathbb{Z}_n\), to form isomorphic systems to \(S\), such that the union of the permutations restricted to the \(k\)-subsets of \(S\) yield each \(k\)-subset of \(\mathbb{Z}_n\) exactly \(\mu\) times. This is the point that we want to have OD(\(k, n, n\)) or PA(\(k, n, n\)) with \(\lambda\) small as possible. The simplest set of such permutations are all the \(n!\) permutations of \(S_n\). The outcome is \(n!\) Steiner systems S(\(t, k; n\)) such that each \(k\)-subset of \(\mathbb{Z}_n\) is contained in exactly \(\binom{\binom{n}{k}}{t}n!/(n^t)\) of these systems. This is a simple way to obtain a large set LS(\(t, k; n; \mu\)), where \(\mu = \binom{n}{n-k}(k-t)!/(n-t)^t\). The multiplicity \(\mu\) of this large set is very large and our target is to obtain such a large set with a much smaller multiplicity \(\mu\). The multiplicity can be cut to half if we use only the \(n!/2\) even permutations.

| Table 1 | Block set \(B_0\) in Lemma 4 |
|---------|-------------------------------|
| [0, 1, 2, 3] | [0, 1, 4, 5] | [0, 1, 6, 9] | [0, 1, 10, 11] | [0, 1, 12, 13] | [0, 2, 4, 6] | [0, 2, 8, 12] |
| [0, 2, 10, 13] | [0, 2, 5, 11] | [0, 4, 10, 12] | [0, 6, 8, 10] | [0, 3, 5, 6] | [0, 6, 11, 12] | [0, 3, 9, 12] |
| [0, 5, 9, 10] | [0, 5, 8, 13] | [0, 3, 4, 13] | [0, 3, 8, 11] | [0, 9, 11, 13] | [0, 4, 8, 9] | [1, 2, 4, 13] |
| [2, 3, 5, 13] | [2, 8, 11, 13] | [2, 7, 12, 13] | [2, 3, 4, 12] | [2, 3, 7, 8] | [2, 3, 6, 11] | [2, 5, 6, 8] |
| [2, 4, 8, 10] | [2, 4, 5, 7] | [1, 2, 5, 10] | [3, 4, 5, 8] | [1, 2, 6, 7] | [1, 2, 11, 12] | [2, 7, 10, 11] |
| [1, 5, 6, 11] | [6, 9, 10, 11] | [6, 7, 8, 11] | [4, 6, 8, 12] | [4, 7, 8, 13] | [2, 6, 10, 12] | [1, 4, 6, 10] |
| [1, 3, 6, 12] | [3, 7, 11, 12] | [3, 5, 9, 11] | [1, 3, 5, 7] | [3, 7, 9, 13] | [3, 4, 6, 7] | [1, 3, 4, 9] |
| [1, 4, 7, 12] | [1, 9, 10, 12] | [1, 5, 9, 13] | [4, 5, 6, 9] | [5, 7, 8, 9] | [5, 8, 10, 11] | [5, 7, 11, 13] |
| [4, 5, 10, 13] | [8, 9, 10, 13] | [4, 9, 12, 13] | [1, 3, 11, 13] | [1, 7, 9, 11] | [1, 7, 10, 13] | [3, 6, 8, 9] |
| [3, 8, 12, 13] | [4, 7, 9, 10] | [5, 6, 7, 10] | [6, 7, 9, 12] | [7, 8, 10, 12] | [8, 9, 11, 12] | [10, 11, 12, 13] |

| Table 2 | 8 permutations in Lemma 4 |
|---------|--------------------------|
| 0 1 2 3 4 5 6 7 8 9 10 11 12 13 |
| 0 8 6 9 3 12 4 7 1 13 2 10 5 11 |
| 0 12 11 8 13 3 2 7 5 4 1 6 10 9 |
| 1 3 9 7 4 6 12 8 10 2 0 11 13 5 |
| 1 4 2 5 10 13 7 8 11 9 12 3 6 0 |
| 1 12 4 0 10 13 9 8 5 11 7 3 6 2 |
| 2 0 4 5 3 6 1 9 7 11 12 10 13 8 |
| 2 3 5 1 13 11 7 9 10 12 8 6 4 0 |
instead of all the \( n! \) permutations of \( S_n \). As pointed out in Sect. 2.4, no such array, ordered design or perpendicular array, for \( k > 3 \), with a reasonable \( \lambda \), is known. Generally, to find a small set of permutations, for this purpose, is an interesting open problem for itself. For small parameters, to find large sets with small multiplicity, we can use at least three different strategies. The first one is to construct large sets with multiplicity using computer search. The second one is to apply ordered designs or perpendicular arrays with small parameters. The third one is to present ad-hoc constructions. A few examples for all these methods are given in this subsection.

**Example 1** We have used a computer search to find a large set \( \text{LS}(4, 5, 11; 2) \), on the point set \( \mathbb{Z}_{11} \), with 14 Steiner systems \( S(4, 5, 11) \). The first system \( S_1 \) has the following 66 blocks.

\[
\begin{align*}
[0, 1, 2, 3, 4] & \quad [0, 1, 2, 5, 6] & \quad [0, 1, 2, 7, 8] & \quad [0, 1, 2, 9, 10] & \quad [0, 1, 3, 5, 7] & \quad [0, 1, 3, 6, 9] \\
[0, 1, 3, 8, 10] & \quad [0, 1, 4, 5, 10] & \quad [0, 1, 5, 8, 9] & \quad [0, 1, 4, 6, 8] & \quad [0, 1, 4, 7, 9] & \quad [0, 1, 6, 7, 10] \\
[0, 2, 3, 5, 8] & \quad [0, 2, 3, 7, 9] & \quad [0, 2, 3, 6, 10] & \quad [0, 2, 4, 8, 10] & \quad [0, 2, 6, 8, 9] & \quad [0, 2, 4, 6, 7] \\
[0, 2, 5, 7, 10] & \quad [0, 5, 6, 8, 10] & \quad [0, 4, 5, 7, 8] & \quad [0, 3, 4, 7, 10] & \quad [0, 3, 5, 9, 10] & \quad [0, 3, 4, 5, 6] \\
[0, 2, 4, 5, 9] & \quad [0, 3, 4, 8, 9] & \quad [0, 3, 6, 7, 8] & \quad [0, 4, 6, 9, 10] & \quad [0, 5, 6, 7, 9] & \quad [0, 7, 8, 9, 10] \\
[1, 2, 3, 5, 9] & \quad [1, 2, 3, 7, 10] & \quad [1, 2, 6, 7, 9] & \quad [1, 4, 5, 6, 9] & \quad [1, 2, 4, 5, 7] & \quad [1, 2, 4, 8, 9] \\
[1, 4, 7, 8, 10] & \quad [2, 3, 4, 7, 8] & \quad [1, 2, 3, 6, 8] & \quad [1, 2, 4, 6, 10] & \quad [3, 4, 6, 8, 10] & \quad [3, 5, 7, 8, 10] \\
[3, 4, 5, 7, 9] & \quad [3, 6, 7, 9, 10] & \quad [1, 2, 5, 8, 10] & \quad [2, 5, 6, 9, 10] & \quad [1, 3, 4, 5, 8] & \quad [1, 3, 4, 6, 7] \\
[1, 3, 4, 9, 10] & \quad [2, 3, 4, 6, 9] & \quad [1, 3, 5, 6, 10] & \quad [1, 3, 7, 8, 9] & \quad [1, 5, 6, 7, 8] & \quad [2, 3, 5, 6, 7] \\
[1, 5, 7, 9, 10] & \quad [1, 6, 8, 9, 10] & \quad [2, 3, 4, 5, 10] & \quad [2, 3, 8, 9, 10] & \quad [2, 4, 5, 6, 8] & \quad [2, 4, 7, 9, 10] \\
[2, 5, 7, 8, 9] & \quad [2, 6, 7, 8, 10] & \quad [3, 5, 6, 8, 9] & \quad [4, 5, 6, 7, 10] & \quad [4, 5, 8, 9, 10] & \quad [4, 6, 7, 8, 9]
\end{align*}
\]

This system and the 13 systems obtained by applying the following 13 coordinate permutations

\[
\begin{align*}
0 & 1 2 3 4 6 5 9 10 7 8 \\
0 & 1 2 3 6 7 9 4 5 10 8 \\
0 & 1 2 3 6 8 9 7 10 4 5 \\
0 & 1 2 3 8 5 10 9 6 4 7 \\
0 & 1 2 3 9 5 8 4 10 6 7 \\
0 & 1 2 4 9 5 7 8 3 6 10 \\
0 & 1 2 5 10 4 8 7 6 3 9 \\
0 & 1 2 6 9 7 4 8 5 10 3 \\
0 & 1 2 7 5 4 9 10 3 8 6 \\
0 & 1 2 7 3 10 6 8 4 9 5 \\
0 & 1 2 8 6 10 4 5 9 7 3 \\
0 & 1 2 8 10 6 4 9 7 5 3 \\
0 & 1 2 9 8 4 6 5 3 7 10
\end{align*}
\]

yield the large set \( \text{LS}(4, 5, 11; 2) \).

Example 1 yields the following result.

**Lemma 5** There exist the following three large sets with multiplicity:

\[
\text{LS}(4, 5, 11; 2), \text{LS}(3, 4, 10; 2), \text{LS}(5, 6, 12; 2).
\]
Proof

1. The LS(4, 5, 11; 2) was constructed in Example 1 using $S_1$, the Steiner system S(4, 5, 11), and the given 13 permutations on $\mathbb{Z}_{11}$.
2. Each system of the large set LS(4, 5, 11; 2) has exactly 30 blocks containing the point 10. By considering these 30 blocks and removing the point 10 from each one of them yields an LS(3, 4, 10; 2). This LS(3, 4, 10; 2) consists of 14 Steiner systems S(3, 4, 10), on the point set $\mathbb{Z}_{10}$, which are the derived systems of the 14 Steiner systems S(4, 5, 11) of the LS(4, 5, 11; 2).
3. Let $S_1, S_2, \ldots, S_{14}$ be the 14 systems of the LS(4, 5, 11; 2). Let $S'_i \triangleq \{ X \cup \{11\} : X \in S_i \} \cup (\mathbb{Z}_{11} \setminus X : X \in S_i)$, for each $1 \leq i \leq 14$. It is well-known that each such $S'_i$ is a S(5, 6, 12). It implies that $\{ S'_i : 1 \leq i \leq 14 \}$ is a large set LS(5, 6, 12; 2).

Example 2 Let $S$ be the S(3, 4, 10) obtained as the derived system of the S(4, 5, 11) presented in Example 1, i.e. $\{ X \cup \{11\} : 11 \in X, X \in S_1 \}$. This system and the 20 systems obtained by applying the following 20 coordinates permutations

```
0  6  2  5  4  9  7  1  8  3
0  5  2  1  8  6  9  3  4  7
0  4  2  7  8  1  9  5  6  3
0  9  2  5  8  3  1  4  6  7
0  9  2  7  8  4  1  6  3  5
0  1  2  7  8  5  4  3  9  6
0  6  2  4  8  7  9  3  5  1
0  9  2  5  6  8  4  7  1  3
0  7  2  5  3  6  9  4  8  1
0  6  2  9  3  8  4  1  7  5
0  6  2  7  9  1  4  5  3  8
0  5  2  4  9  8  6  1  7  3
0  8  2  4  1  9  3  6  7  5
0  4  2  5  7  1  6  8  3  9
0  5  2  8  7  3  9  1  4  6
0  6  2  8  5  4  3  1  9  7
0  5  4  7  2  3  9  6  1  8
0  6  4  5  8  1  7  2  9  3
0  1  4  9  6  5  3  2  8  7
0  2  4  8  6  9  3  1  7  5
yield a large set LS(3, 4, 10; 3).

The LS(3, 4, 10; 3) of Example 2 and the LS(3, 4, 10; 2) obtained in Lemma 5 together with Theorem 4 imply the following theorem.

Theorem 11 For each $g \geq 2$ there exist an LS(3, 4, 10; g) and an LH(10, g, 4, 3).

Theorems 10, 5, 11, and 1 imply the following conclusion.

Corollary 1 An LH(n, g, 4, 3) exists for all admissible parameters with $n \in \{4, 5, 6, 7, 8, 10\}$, with possible exceptions for $n = 5$ and $g \equiv 2 \pmod{4}$.
Theorem 12  Let $S$ be an $S(3, 4, n)$ on the point set $\mathbb{Z}_n$, where $n \geq 14$, and let $A, B$ be a partition of $\mathbb{Z}_n$ into subsets of size 3 and $n - 3$, respectively. If there exists a PA$_\lambda$$(4, n - 3, n - 3)$, then there exists an LS$(3, 4, n; \mu)$, where $\mu = \lambda(n^3 - 4)/4$.

Proof First, we compute the number of blocks of $S$ in each configuration. There is exactly one block in configuration (3, 1), which implies that the unique subset from configuration (3, 0) is contained in one block of $S$. There are exactly $\frac{3(n-3)-3}{2} = \frac{3(n-4)}{2}$ blocks in configuration (2, 2), which implies that each one of the three subsets from configuration (2, 0) is contained in $\gamma_2 = \frac{n-4}{2}$ blocks of configuration (2, 2). There are exactly $\left(\binom{n-3}{2} - \frac{3(n-4)}{2}\right)/3 = \frac{(n-4)(n-5)}{2}$ blocks in configuration (1, 3), which implies that each one of the three subsets from configuration (1, 0) is contained in $\gamma_1 = \frac{(n-4)(n-5)}{6}$ blocks of configuration (1, 3). There are exactly $\gamma_0 = \left(\binom{n-3}{3} - \frac{(n-4)(n-5)}{2}\right)/4 = \binom{n-4}{3}/4$ blocks in configuration (0, 4).

Assume $M$ is a perpendicular array PA$_\lambda$$(4, n - 3, n - 3)$. Let $S'$ be the set of blocks obtained by applying the permutations of $M$ on the B-part of $S$. The computations done implies that each one of the $n - 3$ subsets of configuration (3, 1) is contained in $\lambda(n^3 - 4)/4$ subsets of $S'$. Each one of the $3(n-3)$ subsets of configuration (2, 2) is contained in $\gamma_2\lambda(n^4 - 3)/\binom{n-3}{2} = \lambda(n^3 - 4)/4$ subsets of $S'$. Each one of the $3(n-3)$ subsets of configuration (1, 3) is contained in $\gamma_1\lambda(n^4 - 4)/\binom{n-3}{3} = \lambda(n^3 - 4)/4$ subsets of $S'$. Finally, each one of the $(n-4)$ subsets of configuration (0, 4) is contained in $\gamma_0\lambda(n^4 - 3)/\binom{n-4}{3} = \lambda(n^3 - 4)/4$ subsets of $S'$.

Thus, $S'$ is an LS$(3, 4, n; \mu)$, where $\mu = \lambda(n^3 - 4)/4$. \hfill $\square$

Corollary 2  Let $S$ be an $S(3, 4, n)$ on the point set $\mathbb{Z}_n$, where $n \geq 14$, and let $A, B$ be a partition of $\mathbb{Z}_n$ into subsets of size 3 and $n - 3$, respectively. If there exists an OD$_\lambda$$(4, n - 3, n - 3)$, then there exists an LS$(3, 4, n; \mu)$, where $\mu = 6\lambda(n^3 - 4)/4$.

Theorem 12 and Corollary 2 can be generalized for other Steiner systems $S(t, t + 1, n)$. We omit these generalizations as there are no known perpendicular arrays or ordered design with a relatively small number of permutations. Theorem 12 and Corollary 2 can also be modified and generalized to other Steiner system $S(t, t + 1, n)$, where the point set is partitioned into two subsets of size $t$ and $n - t$, and also to the case when the point set is partitioned into two subsets of size $\ell$ and $n - \ell$, where $\ell < t$. In these case PA$_\lambda$$(t, n - \ell, n - \ell)$ or OD$_\lambda$$(t, n - \ell, n - \ell)$ have to be used. We omit the related discussion and theorems, since we do not have any example where a large set with relatively small multiplicity is obtained. However, Corollary 2 can be applied using OD$_1$(4, 11, 11) (see Theorem 9) to obtain the following result.

Corollary 3  There exists an LS$(3, 4, 14; 720)$.

Corollary 4  There exists an LH$(14, 720, 4, 3)$. 

The coordinate partitioning method which was mentioned above can be modified and applied to Steiner systems with specific parameters and related perpendicular arrays. We demonstrate this modified idea for $S$, a Steiner system $S(5, 6, 12)$. Assume that the system is constructed on a point set partitioned into two subsets $A_1$ and $A_2$ of size 5 and 7, respectively. First, we compute the number of blocks in $S$ for each configuration. Configuration (5, 1) has exactly one block, which implies that unique subset from configuration (5, 0) is contained in one block of $S$. Configuration (4, 2) has 15 blocks, which implies that each subset from configuration (4, 0) is contained in exactly 3 blocks of $S$. Configuration (3, 3) has 50 blocks,
which implies that each subset from configuration \((3, 0)\) is contained in exactly 5 blocks of \(S\). Configuration \((2, 4)\) has 50 blocks, which implies that each subset from configuration \((2, 0)\) is contained in exactly 5 blocks of \(S\). Configuration \((1, 5)\) has 15 blocks, which implies that each subset from configuration \((1, 0)\) is contained in exactly 3 blocks of \(S\). Finally, configuration \((0, 6)\) has exactly one block. Let \(M\) be a perpendicular array \(PA_3(3, 7, 7)\) (see Theorem 9). Let \(S'\) be the set of blocks obtained by applying the permutations of \(M\) on the part \(A_2\) of \(S\). The computations done imply that each subset of configuration \((5, 1)\) is contained in \(3 \binom{7}{1} / 7 = 15\) subsets of \(S'\). Each subset of configuration \((4, 2)\) is contained in \(15 \cdot 3 \binom{7}{1} / \binom{3}{2} = 15\) subsets of \(S'\). Each subset of configuration \((3, 3)\) is contained in \(50 \cdot 3 \binom{7}{1} / \left( \binom{3}{2} \binom{3}{2} \right) = 15\) subsets of \(S'\). Note, that the computations for configurations \((2, 4)\), \((1, 5)\), \((0, 6)\) are the same as for configurations \((3, 3)\), \((4, 2)\), \((5, 1)\), respectively. Thus, \(S'\) is an \(LS(5, 6, 12; 15)\). Combining this result and Lemma 5, we infer the following theorem.

**Theorem 13** For each \(\mu \geq 2\), there exist an \(LS(4, 5, 11; \mu)\), an \(LS(5, 6, 12; \mu)\), an \(LH(11, g, 5, 4)\), and an \(LH(12, g, 6, 5)\), with possible exceptions when \(\mu \in \{3, 5, 7, 9, 11, 13\}\).

### 4 The main ingredient constructions

In this section we will discuss the main recursive constructions for Steiner quadruple systems or more precisely, for a set of pairwise disjoint Steiner quadruple systems. The first construction is a doubling construction due to Lindner [31]. It will be called the DLS (for Doubling Lindner Systems) Construction. We will present a slightly different variant than the one given in [31]. This variant was already presented in [14]. The second construction is a folklore doubling construction which will be called the DB (for doubling) Construction. The third construction is a quadrupling construction. It is a variant of the construction presented in another paper of Lindner [32]. It will be called the QLS (for Quadrupling Lindner systems) Construction. The constructions will be presented first in their original form and later with the modifications required for the new constructed designs. While the two doubling constructions will be defined and discussed in details, for the quadrupling construction, only some properties will be discussed, while the exact definition of the construction for our setup will be given in Sect. 5. These constructions will be used later in a variant for a construction of Etzion and Hartman [15] which is a more complicated construction, in which instead of doubling, or quadrupling, the multiplication is by \(2^m\). As this will be the main construction and it is more complicated, it will be presented separately in Sect. 6.

#### 4.1 DLS construction (doubling)

Let \((\mathbb{Z}_n, B)\) be an \(SQS(n)\) and let \(A\) be a \(n \times n\) Latin square, on the point set \(\mathbb{Z}_n\), with no \(2 \times 2\) subsquares. Denote by \(\alpha_i\) the permutation on \(\mathbb{Z}_n\) defined by \(\alpha_i(j) = y\) if and only if \(A(i, j) = y\). For each \(i\), \(0 \leq i \leq n - 1\), we define a set of quadruples \(B_i\) on \(\mathbb{Z}_n \times \mathbb{Z}_2\) as follows:
1. For each quadruple \( \{x_1, x_2, x_3, x_4\} \in B_i \), the following 8 quadruples are contained in \( B_i \):

\[
\{ (x_1, 0), (x_2, 0), (x_3, 0), (\alpha_i(x_4), 1) \}, \quad \{ (x_1, 1), (x_2, 1), (x_3, 1), (\alpha_i(x_4), 0) \}
\]
\[
\{ (x_1, 0), (x_2, 0), (\alpha_i(x_3), 1), (x_4, 0) \}, \quad \{ (x_1, 1), (x_2, 1), (\alpha_i(x_3), 0), (x_4, 1) \}
\]
\[
\{ (x_1, 0), (\alpha_i(x_2), 1), (x_3, 0), (x_4, 0) \}, \quad \{ (x_1, 1), (\alpha_i(x_2), 0), (x_3, 1), (x_4, 1) \}
\]
\[
\{ (\alpha_i(x_1), 1), (x_2, 0), (x_3, 0), (x_4, 0) \}, \quad \{ (\alpha_i(x_1), 0), (x_2, 1), (x_3, 1), (x_4, 1) \}
\]

2. For each pair \( \{x_1, x_2\} \subset \mathbb{Z}_n \), the quadruple \( \{ (x_1, 0), (x_2, 0), (\alpha_i(x_1), 1), (\alpha_i(x_2), 1) \} \) is contained in \( B_i \).

This DLS Construction is a variant [14] of the Lindner Construction [31]. Each \( B_i \) constructed via the DLS Construction is an SQS(2n) and the set \( \{B_i : 0 \leq i \leq n - 1\} \) is a set of \( n \) pairwise disjoint SQS(2n). This was the goal when this construction was suggested.

The DLS construction is applied with a set of permutations defined by the \( n \) rows of the \( n \times n \) Latin square \( M \) with no \( 2 \times 2 \) subsquares (to avoid repeated quadruples). The DLS Construction can be applied also with Latin squares in which there is no requirements for the nonexistence of \( 2 \times 2 \) subsquares as was done in [14]. The constructions of large sets given in the sequel also do not require arrays of permutations with no \( 2 \times 2 \) subsquares since the multiplicity requires some repeated quadruples. The construction to achieve our goals will be applied with other sets (or multiset) of permutations of \( S_n \). In this case we have to make the following analysis. Assume that \( M \) is such a set (or multiset) of \( \gamma \frac{(n-1)n}{2} \) permutations, where \( \gamma \) is even. \( M \) can be viewed as a \( (\gamma \frac{(n-1)n}{2}) \times n \) matrix, where each row represents a permutation from \( M \). Assume further that in each \( (\gamma \frac{(n-1)n}{2}) \times 2 \) submatrix of \( M \) each unordered pair \( \{i, j\}, i, j \in \mathbb{Z}_n \), appears in exactly \( \gamma \) rows. By applying these permutations of \( M \) in the DLS Construction, instead of the \( n \times n \) Latin square, we obtain a set of \( \gamma \frac{(n-1)n}{2} \) SQS(2n) which contains each quadruple from configuration \( (3, 1) \) in exactly \( \frac{(n-1)n}{2} \) systems and the same is true for each quadruple from configuration \( (1, 3) \). Each quadruple from configuration \( (2, 2) \) is contained in \( \gamma \) of these systems. There are no quadruples from configurations \( (0, 4) \) and \( (4, 0) \) in all these systems. These quadruples required in our main constructions can be obtained with the next construction, namely the DB Construction. These observations will be used in our main construction and are summarized as follows.

**Lemma 6** If the DLS Construction is applied on a given SQS(n), on the point set \( \mathbb{Z}_n \), and a \( PA_{\gamma}(2, n, n) \), \( \gamma \) even, then the DLS Construction yields a set \( \mathcal{R} \) with \( \gamma \frac{(n-1)n}{2} \) systems (SQS(2n)) on the point set \( \mathbb{Z}_n \times \mathbb{Z}_2 \) (with two natural parts \( \mathbb{Z}_n \times \{0\} \) and \( \mathbb{Z}_n \times \{1\} \)).

1. Each quadruple from configuration \( (3, 1) \) is contained in exactly \( \frac{\gamma(n-1)}{2} \) systems of \( \mathcal{R} \).
2. Each quadruple from configuration \( (1, 3) \) is contained in exactly \( \frac{\gamma(n-1)}{2} \) systems of \( \mathcal{R} \).
3. Each quadruple from configuration \( (2, 2) \) is contained in exactly \( \gamma \) systems of \( \mathcal{R} \).

There are no quadruples from configurations \( (4, 0) \) and \( (0, 4) \) in any system of \( \mathcal{R} \).

### 4.2 DB construction (doubling)

Let \((\mathbb{Z}_n, B)\) be an SQS(n), let \( F = \{F_0, F_1, \ldots, F_{n-2}\} \) and \( F' = \{F'_0, F'_1, \ldots, F'_{n-2}\} \) be two one-factorizations (not necessarily distinct) of \( K_n \) on the vertex set \( \mathbb{Z}_n \) (\( F \) will be called the first one-factorization and \( F' \) the second one-factorization).

Let \( \alpha \) be any permutation on the set \( \{0, 1, \ldots, n - 2\} \). Define the collection of quadruples \( B' \) on \( \mathbb{Z}_n \times \mathbb{Z}_2 \) as follows.

\[ \text{Springer} \]
1. For each quadruple \(\{x_1, x_2, x_3, x_4\} \in B\), the following two quadruples are contained in \(B'\).

\[
\{(x_1, 0), (x_2, 0), (x_3, 0), (x_4, 0)\}, \{(x_1, 1), (x_2, 1), (x_3, 1), (x_4, 1)\}.
\]

2. For each \(i \in \{0, 1, \ldots, n-2\}\) and \(\{x_1, x_2\} \in F_i\), \(\{y_1, y_2\} \in F_j\), where \(j = \alpha(i)\), the following quadruple is contained in \(B'\).

\[
\{(x_1, 0), (x_2, 0), (y_1, 1), (y_2, 1)\}.
\]

Then, \(B'\) forms an SQS\((2n)\).

Now, assume that instead of the set \(B\), we apply the DB Construction with an LS\((3, 4, n; g(n-1))\). Instead of one permutation \(\alpha\), the construction is applied with \(n-1\) permutations on \(\{0, 1, \ldots, n-2\}\) taken from an \((n-1) \times (n-1)\) Latin square \(M\). An LS\((3, 4, n; g(n-1))\) contains \(g(n-1)(n-3)\) systems of SQS\((n)\), which implies that each permutation of \(M\) is applied \(g(n-3)\) times to obtain SQS\((2n)\) in the LS\((3, 4, n; g(n-1))\). The following lemma summarizes the configurations of the quadruples obtained in this construction.

**Lemma 7** Assume that the DB Construction is applied on an \((n-1) \times (n-1)\) Latin square \(M\) to an LS\((3, 4, n; g(n-1))\), where each permutation of \(M\) is applied on \(g(n-3)\) systems of the LS\((3, 4, n; g(n-1))\). Then the DB Construction yields a set \(\mathcal{R}\) with \(g(n-3)(n-1)\) systems (SQS\((2n)\)) on the point set \(\mathbb{Z}_n \times \mathbb{Z}_2\).

1. Each quadruple from configuration \((4, 0)\) is contained in exactly \(g(n-1)\) systems of \(\mathcal{R}\).
2. Each quadruple from configuration \((0, 4)\) is contained in exactly \(g(n-1)\) systems of \(\mathcal{R}\).
3. Each quadruple from configuration \((2, 2)\) is contained in exactly \(g(n-3)\) systems of \(\mathcal{R}\).

There are no quadruples from configurations \((3, 1)\) and \((1, 3)\) in any system of \(\mathcal{R}\).

By combining the DLS Construction (Lemma 6) and the DB Construction (Lemma 7) we infer the following result which yields a doubling construction for large sets with multiplicity.

**Theorem 14** If the DLS Construction is applied on an SQS\((n)\) and a PA\(\gamma(2, n, n)\); and the DB Construction is applied on an LS\((3, 4, n; g(n-1))\), where \(\gamma = 2g\), and an \((n-1) \times (n-1)\) Latin square \(M\), then the union of SQS\((2n)\)’s of the two constructions is an LS\((3, 4, 2n; g(n-1))\).

**Proof** By Lemma 6, if the DLS Construction is applied to the SQS\((n)\) with a PA\(\gamma(2, n, n)\), then each quadruple from configuration \((3, 1)\) or configuration \((1, 3)\) is contained once in exactly \(\mu = \frac{\gamma(n-1)}{2}\) systems. Each quadruple from configuration \((2, 2)\) is contained once in exactly \(\gamma\) systems. By Lemma 7 is applied to an LS\((3, 4, n; \frac{\gamma(n-1)}{2})\), then each quadruple from configuration \((2, 2)\) is contained in exactly \(\frac{\gamma(n-3)}{2} = \mu - \gamma\) systems. Finally, each quadruple from configuration \((4, 0)\) or configuration \((0, 4)\) is contained in exactly \(\mu\) of the systems. Thus, there exists an LS\((3, 4, 2n; \mu)\).

Applying Theorem 14 with PA\(2(2, 10, 10)\) (see Theorem 8) and LS\((3, 4, 10; 9)\) (see Theorem 11) we have

**Corollary 5** There exists an LS\((3, 4, 20; 9m)\) for each \(m \geq 1\).
4.3 QLS construction (quadrupling)

The QLS Construction is a quadrupling construction, which was presented first by Lindner [32], while another variant was introduced by Etzion and Hartman [15]. In these two papers the purpose of the construction was to obtain pairwise disjoint Steiner quadruple systems. This construction can have more applications, e.g. recently, another simpler variant was presented to obtain SQS(4n) with good sequencing in [10]. In this section, the structure of the systems obtained in the QLS construction will be described. The exact formulation and the formal steps of this construction, in the variant required for our constructions, will be described in details when it will be used in Sects. 5.1 and 5.2.

In the doubling construction, we consider only five configurations (4, 0), (3, 1), (2, 2), (1, 3), and (0, 4). In the quadrupling construction we have to consider thirty five configurations as follows. The 4n points of $\mathbb{Z}_n \times \mathbb{Z}_4$ are partitioned into four equal parts, $\mathbb{Z}_n \times \{i\}$, $i \in \mathbb{Z}_4$. The possible configurations of quadruples are categorized into five groups.

**Group 1**: In this group there are four configurations (4, 0, 0, 0), (0, 4, 0, 0), (0, 0, 4, 0), and (0, 0, 0, 4).

**Group 2**: In this group there are twelve configurations (3, 1, 0, 0), (1, 3, 0, 0), (3, 0, 1, 0), (1, 0, 3, 0), (3, 0, 0, 1), (0, 1, 0, 3), (0, 3, 1, 0), (0, 1, 3, 0), (0, 3, 0, 1), (0, 1, 0, 3), (0, 0, 3, 1), and (0, 0, 1, 3).

**Group 3**: In this group we have the six configurations (2, 2, 0, 0), (2, 0, 2, 0), (2, 0, 0, 2), (0, 2, 2, 0), (0, 2, 0, 2), and (0, 0, 2, 2).

**Group 4**: In this group there are twelve configurations (2, 1, 1, 0), (2, 1, 0, 1), (2, 0, 1, 1), (1, 2, 1, 0), (1, 2, 0, 1), (1, 1, 2, 0), (1, 0, 2, 1), (0, 1, 2, 1), (1, 1, 0, 2), (1, 0, 1, 2), and (0, 1, 1, 2).

**Group 5**: In this group there is one configuration (1, 1, 1, 1).

The QLS Construction is based on quadruples from Groups 2, 3, 4, and 5. We start with an SQS(n) defined on $\mathbb{Z}_n$ and an $n \times n$ Latin square $M$ with no $2 \times 2$ subsquares. The DLS Construction is applied to obtain a set $T$ of $n$ pairwise disjoint SQS(2n). We are now in the position to describe the framework of our variant of the quadrupling construction to construct 3n pairwise disjoint SQS(4n). An SQS(4n) in this construction will be one of three types, named Type A1, Type A2, and Type A3. Each one of these three type is constructed in a very similar way. We start with the quadruples of group 5. There are a total of $n^4$ quadruples from configuration (1, 1, 1, 1) in this group. From these $n^4$ quadruples, a set with $3n^3$ quadruples from configuration (1, 1, 1, 1) will be chosen, and partitioned into three subsets of size $n^3$ with properties which will be defined in the sequel. One subset will be in Type A1, one in Type A2, and one in Type A3. In each type there will be $n$ pairwise disjoint SQS(4n).

The configurations for Type A1 are defined as follows. Each SQS(2n) from $T$ is embedded in the point set $\mathbb{Z}_n \times \{0, 1\}$ and also embedded in the point set $\mathbb{Z}_n \times \{2, 3\}$. These two related Steiner quadruple systems $S(3, 4, 2n)$, obtained by the DLS Construction are part of an SQS(4n). Therefore, this SQS(4n) will contain quadruples from configurations (3, 1, 0, 0), (1, 3, 0, 0), (0, 0, 3, 1), and (0, 0, 1, 3). It contains $\binom{n}{3}$ quadruples from each one of these four configurations. It also contains quadruples from configurations (2, 2, 0, 0), and (0, 0, 2, 2), $\binom{n}{2}$ quadruples from each one of these two configurations. The SQS(4n) of Type A1 also contains quadruples from configurations (2, 0, 1, 1), (0, 2, 1, 1), (1, 1, 2, 0), and (1, 1, 0, 2). There will be $\binom{n}{3}$ quadruples in each one of these configurations. The last configuration in Type A1 is (1, 1, 1, 1) and there are $n^2$ quadruples from this configuration in each system, i.e. the $n^3$ quadruples chosen from configuration (1, 1, 1, 1) will be partitioned into $n$ sets, $n^2$ quadruples in each set.
Similarly an SQS(4n) in Type A2 is constructed. It has quadruples from configurations (3, 0, 0, 1), (1, 0, 0, 3), (0, 1, 3, 0), (0, 3, 1, 0), (2, 0, 0, 2), (0, 2, 2, 0), (2, 1, 1, 0), (0, 1, 1, 2), (1, 2, 0, 1), (1, 0, 2, 1), and (1, 1, 1, 1). Type A3 has quadruples from configurations (3, 0, 1, 0), (1, 0, 3, 0), (0, 1, 0, 3), (0, 3, 0, 1), (2, 0, 2, 0), (0, 2, 0, 2), (2, 1, 0, 1), (0, 1, 2, 1), (1, 2, 1, 0), (1, 0, 1, 2), and (1, 1, 1, 1).

This kind of quadrupling construction was described in [15,32]. In this paper, a perpendicular array PAγ(2, n, n) will be used, as indicated in Lemma 6, instead of the n × n Latin square M without no 2 × 2 subsquares. Such a perpendicular array has \( \gamma(n) = \frac{\gamma(n-1)}{2} n = \mu n \) permutations, i.e., the construction is applied \( \frac{\gamma(n-1)}{2} \) times compared to one application with an n × n Latin square. To conclude, in the construction presented in Sect. 5, the total number of systems in Type A1 will be \( \gamma(n) = \mu n \) (and not n as in the original QLS Construction). The same number of systems will be in Type A2 and in Type A3.

5 LS(3, 4, 4n; g), for \( n \equiv 2 \) or 4 (mod 6), \( n \geq 10 \)

In this section we prove the first result which leads to the main result of this work.

**Theorem 15** If there exist an LS(3, 4, n; \( \mu \)) and a PAγ(2, n, n), where \( \mu = \frac{(n-1)\gamma}{2} \), then there exists an LS(3, 4, 4n; \( \mu \)).

Theorem 15 is generalized by the main result of this work which will be proved in the next section.

**Theorem 16** If there exists a large set LS(3, 4, n; \( \mu \)) and a PAγ(2, n, n), \( \mu = \frac{(n-1)\gamma}{2} \), then there exists an LS(3, 4, 2mn; \( \mu \)), for each \( m \geq 0 \).

The first step in the proof of Theorem 15 is to apply the two doubling constructions, the DLS Construction and the DB Construction, in the right combination to obtain a large set of SQS(2n) with multiplicity. This was summarized in Theorem 14. The rest of the proof of Theorem 15 will be based on the QLS Construction, using the consequences of Lemmas 6 and 7 which are partially summarized in Theorem 14. The quadrupling construction in this section, which is a variant of the QLS Construction, will be called the (4n)-Construction. This step of the construction is presented in Sects. 5.1 and 5.2. In Sect. 5.1, the ideas of the construction are presented and in Sect. 5.2 the formal definition for the blocks of the construction and the proofs for the correctness of the construction, are given. In these steps the QLS Construction is adapted to obtain a large set with multiplicity which will be presented in the sequel.

### 5.1 The (4n)-construction: introduction

Assume that on the point set \( \mathbb{Z}_n \) there exists an LS(3, 4, n; \( \mu \)), and a PAγ(2, n, n), where \( \mu = \frac{(n-1)\gamma}{2} \) (which implies that \( \gamma \) is even). In the recursive construction to form an LS(3, 4, n·2mn; \( \mu \)) presented in Sect. 6, the first step is a construction of an LS(3, 4, 4n; \( \mu \)) which is presented in this section.

Such a LS(3, 4, 4n; \( \mu \)) consists of \( \mu(4n-3) \) systems of SQS(4n). We start by applying a variant of the QLS Construction to obtain 3\( \mu n \) systems of SQS(4n). After these 3\( \mu n \) systems of SQS(4n) are obtained we continue with a variant of the DB Construction to obtain \( \mu(n-3) \) systems of SQS(4n). This part of the construction is recursive and is based
on another quadrupling construction in which the LS$(3, 4, n; \mu)$ is used. We start with the
construction of the LS$(3, 4, 4n; \mu)$ on the point set $\mathbb{Z}_n \times \mathbb{Z}_4$. Let $M$ be the PA$_\gamma$(2, $n$, $n$). We
apply the DLS Construction with any SQS($n$) and the $n\mu$ permutations of $M$ to obtain a set $R$
with $n\mu = \frac{n(n-1)\gamma}{2}$ systems of SQS(2$n$) on the point set $\mathbb{Z}_n \times \mathbb{Z}_2$. Let $\mathcal{R}_i$, $1 \leq i \leq \frac{n(n-1)\gamma}{n\mu}$
be the $i$th such SQS(2$n$) in $\mathcal{R}$. The $(4n)$-construction for the LS$(3, 4, 4n; \mu)$ has two types
of SQS(4$n$), Type A and Type B.

**Type A:**
In this type there are quadruples from the configurations in Groups 2, 3, 4, and 5. The
systems in Type A are of the three sub-types, Type A1, Type A2, and Type A3, as described
in Sect. 4.3. Note that the total number of systems (SQS(4$n$)) of Type A1 is $n\mu = \gamma(\frac{3}{2})$,
which is the same as the number of systems (SQS(2$n$)) in $\mathcal{R}$. The same number of systems
are in Type A2 and the same number is also in Type A3. Thus, the total number of systems
in Type A is $3n\mu$.

**Type B:**
Each system of Type B contains quadruples from Groups 1, 3, and 5. It contains $n^3$ quadru-
ples of configuration $(1, 1, 1, 1), (n-1)^2$, from each one of the configurations $(2, 2, 0, 0),
(2, 0, 2, 0), (2, 0, 0, 2), (0, 2, 2, 0), (0, 2, 0, 2)$, and $(0, 0, 2, 2)$. It contains also $\binom{n}{3}/4$
quadru-
ples from each one of the configurations $(4, 0, 0, 0), (0, 4, 0, 0), (0, 0, 4, 0)$, and $(0, 0, 0, 4)$.

The total number of systems in Type B will be $\gamma(n-1)^2(n-3) = (n-3)\mu$.

**Summary:**
To summarize, the total number of systems in Type A and Type B is $(4n - 3)\mu$ which
is the number of systems required in an LS$(3, 4, 4n; \mu)$. In the next subsection the formal
definition of the blocks in these systems of all types are presented.

**5.2 The $(4n)$-construction: definitions and proofs**

In this subsection the formal definition for all the blocks in Type A and in Type B will
be presented. Proofs that the defined blocks yield an LS$(3, 4, 4n; \mu)$ are also given in this
subsection. The blocks are formed on the point set $\mathbb{Z}_n \times \mathbb{Z}_4$; recall also that $\mu = \frac{n(n-1)\gamma}{2}$. For
the construction, the following structures are required as input:

- Let $F = \{F_0, F_1, \ldots, F_{n-2}\}$ be a one-factorization of $K_n$ on the point set $\mathbb{Z}_n$.
- A set $\{S_{(i,j)} : 1 \leq i \leq n, 0 \leq j \leq \mu - 1\} = \{R_i : 1 \leq i \leq n\mu\}$ with $n\mu$ systems of
SQS(2$n$) obtained from the PA$_\gamma$(2, $n$, $n$) via the DLS Construction.

**Type A1:**
Let $T_{(i,j)}$, be the $(i, j)$-th system of Type A1, $1 \leq i \leq n, 0 \leq j \leq \mu - 1$, which is defined
as follows.

The SQS(2$n$) $S_{(i,j)}$, constructed on the point set $\mathbb{Z}_n \times \mathbb{Z}_2$, is embedded in the point set
$\mathbb{Z}_n \times \{0, 1\}$ and and also embedded in the point set $\mathbb{Z}_n \times \{2, 3\}$. These sets of quadruples
contain quadruples from Group 2 and from Group 3. They form the first set, denoted by
A1(1), of quadruples which are constructed in $T_{(i,j)}$.

Next, we form the following four sets of quadruples from Group 4 in $T_{(i,j)}$.

\[
\begin{align*}
\{(a, 0), (b, 0), (c, 2), (c + i + r + j, 3)\} & : 0 \leq r \leq n - 2, \{a, b\} \in F_r, \ c \in \mathbb{Z}_n, \\
\{(a, 1), (b, 1), (c, 2), (c + i + r + j, 3)\} & : 0 \leq r \leq n - 2, \{a, b\} \in F_r, \ c \in \mathbb{Z}_n, \\
\{(a, 2), (b, 2), (c, 0), (c + i + r, 1)\} & : 0 \leq r \leq n - 2, \{a, b\} \in F_r, \ c \in \mathbb{Z}_n, \\
\{(a, 3), (b, 3), (c, 0), (c + i + r, 1)\} & : 0 \leq r \leq n - 2, \{a, b\} \in F_r, \ c \in \mathbb{Z}_n.
\end{align*}
\]
These sets of quadruples form the second set of quadruples in $T_{(i,j)}$ and it will be denoted by $A_1(2)$. 

The last (third) set, denoted by $A_1(3)$, of quadruples in $T_{(i,j)}$ is from group 5:

$$\{((a, 0), (a + i + n - 1, 1), (b, 2), (b + i + n - 1 + j, 3)) : a, b \in \mathbb{Z}_n\}.$$ 

**Lemma 8** Each set $T_{(i,j)}$, $1 \leq i \leq n$, $0 \leq j \leq \mu - 1$, is an $SQS(4n)$.

**Proof** Since $S_{(i,j)}$ is an $SQS(2n)$, it follows that all triples from configurations $(2, 1, 0, 0), (1, 2, 0, 0), (0, 0, 2, 1), (0, 0, 1, 2), (3, 0, 0, 0), (0, 3, 0, 0), (0, 0, 3, 0), (0, 0, 0, 3)$ are contained in quadruples of $T_{(i,j)}$. Hence, the total number of quadruples in $T_{(i,j)}$ is $2^{2n/3} + 4(4/3)n + n^2 = \binom{4n}{3}/4$. Since this is the number of quadruples in an $SQS(4n)$ and each triple is contained in at least one quadruple, it follows that each triple is contained in exactly one quadruple of $T_{(i,j)}$, which completes the proof.

**Lemma 9** The quadruples from configuration $(1, 1, 1, 1)$ (Group 5) which are contained in the $n\mu$ systems of $SQS(4n)$ from Type A1, form the following $\mu$ sets $L_j$, $0 \leq j \leq \mu - 1$, each of size $n^3$:

$$L_j \triangleq \{((x, 0), (y, 1), (z, 2), (z + y - x + j, 3)) : x, y, z \in \mathbb{Z}_n\}.$$ 

**Proof** By the definition of the quadruples from configuration $(1, 1, 1, 1)$ in one system of Type A1. For a given $j$, $0 \leq j \leq \mu - 1$, in all the related systems of Type A1, we have the set of quadruples

$$\bigcup_{i=1}^{n}\{(a, 0), (a + i + n - 1, 1), (b, 2), (b + i + n - 1 + j, 3)\} : a, b \in \mathbb{Z}_n\}

= \bigcup_{i=1}^{n}\{(a, 0), (y = a + i + n - 1, 1), (b, 2), (b + y - a + j, 3)\} : a, b \in \mathbb{Z}_n\}

= \{((x, 0), (y, 1), (z, 2), (z + y - x + j, 3)) : x, y, z \in \mathbb{Z}_n\}.$$

Clearly, each such set is of size $n^3$ which completes the proof.

**Type A2:**

Let $T_{(i,j)}$, be the $(i, j)$-th system of Type A2, $1 \leq i \leq n$, $0 \leq j \leq \mu - 1$, which is defined as follows.

The $SQS(2n)$ $S_{(i,j)}$, constructed on the point set $\mathbb{Z}_n \times \{0, 2\}$, is embedded in the point set $\mathbb{Z}_n \times \{0, 2\}$ and also embedded in the point set $\mathbb{Z}_n \times \{1, 3\}$. These sets of quadruples contain...
quadruples from Group 2 and from Group 3. They form the first set, denoted by A2(1), of quadruples which are constructed in $T'_{(i,j)}$.

Next, we form the following four sets of quadruples from Group 4 in $T'_{(i,j)}$.

\[
\begin{align*}
\{(a, 0), (b, 0), (c, 1), (c + i + r + j, 3)\} & : 0 \leq r \leq n - 2, \quad \{a, b\} \in F_r, \quad c \in \mathbb{Z}_n, \\
\{(a, 2), (b, 2), (c, 1), (c + i + r + j, 3)\} & : 0 \leq r \leq n - 2, \quad \{a, b\} \in F_r, \quad c \in \mathbb{Z}_n, \\
\{(a, 1), (b, 1), (c, 0), (c + i + r, 2)\} & : 0 \leq r \leq n - 2, \quad \{a, b\} \in F_r, \quad c \in \mathbb{Z}_n, \\
\{(a, 3), (b, 3), (c, 0), (c + i + r, 2)\} & : 0 \leq r \leq n - 2, \quad \{a, b\} \in F_r, \quad c \in \mathbb{Z}_n.
\end{align*}
\]

These sets of quadruples form the second set of quadruples in $T'_{(i,j)}$ and it will be denoted by A2(2).

The last (third) set, denoted by A2(3), of quadruples in $T'_{(i,j)}$ is from group 5.

\[
\{(a, 0), (b, 1), (a + i + n - 1, 2), (b + i + n - 1 + j, 3)\} : a, b \in \mathbb{Z}_n.
\]

Similar to the proofs of Lemmas 8 and 9 we prove the following two lemmas.

**Lemma 10** Each set $T'_{(i,j)}$, $1 \leq i \leq n$, $0 \leq j \leq \mu - 1$, is an SQS(4n).

**Lemma 11** The quadruples from configuration $(1, 1, 1)$ (Group 5) which are contained in the $n \mu$ systems of SQS(4n) from Type A2, form the following $\mu$ sets $L'_j$, $0 \leq j \leq \mu - 1$, each one of size $n^3$:

\[
L'_j \triangleq \{(x, 0), (y, 1), (z, 2), (y + z - x + j, 3)\} : x, y, z \in \mathbb{Z}_n.
\]

**Type A3:**

Let $T''_{(i,j)}$, be the $(i,j)$-th system of Type A3, $1 \leq i \leq n$, $0 \leq j \leq \mu - 1$, which is defined as follows.

The SQS(2n) $S_{(i,j)}$, constructed on the point set $\mathbb{Z}_n \times \mathbb{Z}_2$, is embedded in the point set $\mathbb{Z}_n \times \{0, 3\}$ and also embedded in the point set $\mathbb{Z}_n \times \{1, 2\}$. These set of quadruples contains quadruples from Group 2 and from Group 3. They form the first set, denoted by A3(1), of quadruples which are constructed in $T''_{(i,j)}$.

Next, we form the following four sets of quadruples from Group 4 in $T''_{(i,j)}$.

\[
\begin{align*}
\{(a, 0), (b, 0), (c, 1), (c + i + r, 2)\} & : 0 \leq r \leq n - 2, \quad \{a, b\} \in F_r, \quad c \in \mathbb{Z}_n, \\
\{(a, 3), (b, 3), (c, 1), (c + i + r, 2)\} & : 0 \leq r \leq n - 2, \quad \{a, b\} \in F_r, \quad c \in \mathbb{Z}_n, \\
\{(a, 1), (b, 1), (c, 0), (c + i + r + j, 3)\} & : 0 \leq r \leq n - 2, \quad \{a, b\} \in F_r, \quad c \in \mathbb{Z}_n, \\
\{(a, 2), (b, 2), (c, 0), (c + i + r + j, 3)\} & : 0 \leq r \leq n - 2, \quad \{a, b\} \in F_r, \quad c \in \mathbb{Z}_n.
\end{align*}
\]

These sets of quadruples form the second set of quadruples in $T''_{(i,j)}$ and it will be denoted by A3(2).

The last (third) set, denoted by A3(3), of quadruples in $T''_{(i,j)}$ is from group 5.

\[
\{(a, 0), (b, 1), (b + i + n - 1, 2), (a + i + n - 1 + j, 3)\} : a, b \in \mathbb{Z}_n.
\]

Similar to the proofs of Lemmas 8 and 9 we prove the following two lemmas.

**Lemma 12** Each set $T''_{(i,j)}$, $1 \leq i \leq n$, $0 \leq j \leq \mu - 1$, is an SQS(4n).

**Lemma 13** The quadruples from configuration $(1, 1, 1)$ (Group 5) which are contained in the $n \mu$ systems of SQS(4n) from Type A3, form the following $\mu$ sets $L''_j$, $0 \leq j \leq \mu - 1$, each one of size $n^3$:

\[
L''_j \triangleq \{(x, 0), (y, 1), (z, 2), (x + z - y + j, 3)\} : x, y, z \in \mathbb{Z}_n
\]
The next step for Type A is to calculate the number of times that each quadruple from each configuration is contained in the $3\mu n$ SQS($4n$) of Type A. Recall, for the next theorem, that in the SQS($2n$) $S_{i,j}$, $1 \leq i \leq n$, $0 \leq j \leq \mu - 1$, is constructed in the DLS Construction, by using a $PA_\gamma(2, n, n)$, where $\gamma = \frac{2\mu}{n-1}$, i.e. $\mu = \frac{(n-1)\gamma}{2}$.

**Lemma 14** The systems of Type A in the $(4n)$-construction have the following containment properties.

1. Each quadruple from each configuration of Group 2 is contained in exactly $\mu = \frac{(n-1)\gamma}{2}$ systems of Type A.
2. Each quadruple from Group 3 is contained in exactly $\gamma = \frac{2\mu}{n-1}$ systems of Type A.
3. Each quadruple from Group 4 is contained in exactly $\mu$ systems.
4. The total number of quadruples from configuration $(1, 1, 1, 1)$, which are contained in the systems of Type A is $3\mu n^5$. These quadruples are the $3\mu n^5$ quadruples defined in the $3\mu$ systems $L_j$, $L'_j$, and $L''_j$, $0 \leq j \leq \mu - 1$.

**Proof**

1. is an immediate consequence from Lemma 6 and the definitions of quadruples of Type A1, Type A2, and Type A3.
2. is also an immediate consequence from Lemma 6 and the definitions of quadruples of Type A1, Type A2, and Type A3.
3. By the definition of Group 4, the total number of quadruples in a given configuration of Group 4 is $\frac{n(n-1)}{2} n^2$. Each SQS($4n$) of Type A contains $4\frac{n(n-1)}{2} n$ such quadruples from four distinct configurations, $\frac{n(n-1)}{2} n$ quadruples from each configuration. Clearly, there is no intersection between the configurations used in Type A1 to those used in Type A2 (and similarly between Type A1 and Type A3, and between Type A2 and Type A3). In each such type there are four configurations from the twelve configurations of Group 4. For example, in Type A1 there are quadruples from configurations $(2, 0, 1, 1)$, $(0, 2, 1, 1)$, $(1, 1, 2, 0)$, and $(1, 1, 0, 2)$. Consider for example, the quadruples from type $A1$ for fixed $i$ and $j$,

$$\{(a, 0), (b, 0), (c, 2), (c + i + r + j, 3)\},$$

for each $r$, $0 \leq r \leq n - 2$, $(a, b) \in F_r$, and $c \in \mathbb{Z}_n$. There are $(n - 1)n$ distinct ways to choose a pair $(r, c)$ and $\frac{n}{2}$ pairs from $F_r$. Hence, Type A1 contains exactly $\frac{(n-1)n^2}{2}$ quadruples from configuration $(2, 0, 1, 1)$. The same calculation holds for each configuration in Type A1, Type A2, and Type A3. Clearly for $1 \leq i_1 < i_2 \leq n$, the quadruples in (2) for $i = i_1$ and for $i = i_2$ are distinct which implies that each quadruple from configuration $(2, 0, 1, 1)$ is contained exactly once in Type A1 for a fixed $j$. Thus, in the $3n\mu$ systems of Type A, each quadruple from Group 4 is contained in exactly $\mu$ systems.

4. is an immediate consequence from Lemmas 9, 11, and 13.

□

**Type B:** For the systems of Type B, to be defined in the $(4n)$-construction, the following structures are required as input:

- Let $F = \{F_0, F_1, \ldots, F_{n-2}\}$ be a one-factorization of $K_n$ on the point set $\mathbb{Z}_n$.
- Let $M$ be an $(n - 1) \times (n - 1)$ Latin square on the points set $\{0, 1, \ldots, n - 2\}$. 

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A set $\{S^*_i : 1 \leq i \leq \mu(n-3)\}$, on the point set $\mathbb{Z}_n$, which form an LS$(3, 4; n; \mu)$.

Let $\mathcal{P}_i$, $1 \leq i \leq \mu(n-3)$, be the $i$-th system of Type B. Its blocks are defined on the point set $\mathbb{Z}_n \times \mathbb{Z}_4$ as follows.

$$\{(x_1, j), (x_2, j), (x_3, j), (x_4, j) \} : \{x_1, x_2, x_3, x_4 \} \in S^*_i, j \in \mathbb{Z}_4\}.$$

Given $j$, $0 \leq j \leq n-2$, for $\mathcal{P}_i$, $j/\gamma < i \leq (j+1)/\gamma$, the following six blocks from Group 3 are defined.

$$\{(x, 0), (y, 0), (z, 1), (v, 1) \} : \{x, y \} \in F_r, \{z, v \} \in F_{M(j, r)}, 0 \leq r \leq n-2\},$$

$$\{(x, 0), (y, 0), (z, 2), (v, 2) \} : \{x, y \} \in F_r, \{z, v \} \in F_{M(j, r)}, 0 \leq r \leq n-2\},$$

$$\{(x, 0), (y, 0), (z, 3), (v, 3) \} : \{x, y \} \in F_r, \{z, v \} \in F_{M(j, r)}, 0 \leq r \leq n-2\},$$

$$\{(x, 1), (y, 1), (z, 2), (v, 2) \} : \{x, y \} \in F_r, \{z, v \} \in F_{M(j, r)}, 0 \leq r \leq n-2\},$$

$$\{(x, 1), (y, 1), (z, 3), (v, 3) \} : \{x, y \} \in F_r, \{z, v \} \in F_{M(j, r)}, 0 \leq r \leq n-2\},$$

$$\{(x, 2), (y, 2), (z, 3), (v, 3) \} : \{x, y \} \in F_r, \{z, v \} \in F_{M(j, r)}, 0 \leq r \leq n-2\}.$$

**Remark 1** Note, that $(n-1)/(\mu - \gamma) = (n-3)/\mu$ and hence there is no ambiguity in the definition of the $\mathcal{P}_i$’s.

The most challenging design part of Type B is to construct the quadruples from configuration $(1, 1, 1, 1)$ to accommodate the large set. There are $n^4$ quadruples from configuration $(1, 1, 1, 1)$, each one should be contained in exactly $\mu$ of the $\mu(4n-3)$ systems of the LS$(3, 4; 4n; \mu)$ which is constructed. The sets of quadruples from configuration $(1, 1, 1, 1)$ for our system of Type B are defined as follows.

Let $t$ be the smallest positive integer such that $tn \geq \mu$. For each $j$, $0 \leq j \leq tn-1$, we form the set

$$D_j \triangleq \{(x, 0), (y, 1), (z, 2), (x + z - y + j, 3) \} : x, y, z \in \mathbb{Z}_n\}$$

For each $j$, $0 \leq j \leq (\mu - t)n - 1$, we form the set

$$E_j \triangleq \{(x, 0), (y, 1), (z, 2), (z + y - x + j, 3) \} : x, y, z \in \mathbb{Z}_n\}$$

Note, that two $D_j$’s are either the same or disjoint and the same holds for the $E_j$’s. Moreover, the $D_j$’s and the $E_j$’s might have nonempty intersection. Nevertheless, in the sequel if $i \neq j$ then $D_i$ and $D_j$ will be considered as distinct sets. The same is true for $E_i$ and $E_j$.

**Lemma 15** Each $D_j$ and each $E_j$ contains exactly $n^3$ quadruples from configuration $(1, 1, 1, 1)$. Each quadruple from configuration $(1, 1, 1, 1)$ is contained in exactly $\mu$ of these $D_j$’s and $E_j$’s of the set $\{D_j : 0 \leq j \leq tn - 1\} \cup \{E_j : 0 \leq j \leq (\mu - t)n - 1\}$.

**Proof** There are $n^3$ distinct ways to choose $x, y, z \in \mathbb{Z}_n$ and hence each $D_j$ and each $E_j$ contains exactly $n^3$ quadruples from configuration $(1, 1, 1, 1)$. Moreover, each such quadruple is an element in one of the $D_j$’s and one of the $E_j$’s. The set $\{D_j : 0 \leq j \leq tn - 1\}$ contains $tn$ subsets of quadruples, where $D_j = D_{n+j}$ for $0 \leq j \leq (t - 1)n - 1$. Therefore, each quadruple from configuration $(1,1,1,1)$ is contained in exactly $t$ of the $D_j$’s. The set $\{E_j : 0 \leq j \leq (\mu - t)n - 1\}$ contains $(\mu - t)n$ subsets of quadruples, where $E_j = E_{n+j}$ for $0 \leq j \leq (\mu - t)n - 1$. Therefore, each quadruple from configuration $(1,1,1,1)$ is contained in exactly $\mu - t$ of the $E_j$’s. \qed
Lemma 16

\[ \{ L_j, L'_j, L''_j : 0 \leq j \leq \mu - 1 \} \subset \{ D_j : 0 \leq j \leq tn - 1 \} \cup \{ E_j : 0 \leq j \leq (\mu - t)n - 1 \}. \]

Proof By the definition \( L_j = L'_j = E_j \) for each \( 0 \leq j \leq \mu - 1 \) and \( L''_j = D_j \) for each \( 0 \leq j \leq \mu - 1 \).

Corollary 6 The number of subsets with \( n^3 \) quadruples from configuration \((1, 1, 1, 1)\) in the multiset \( \{ D_j : 0 \leq j \leq tn - 1 \} \cup \{ E_j : 0 \leq j \leq (\mu - t)n - 1 \} \setminus \{ L_j, L'_j, L''_j : 0 \leq j \leq \mu - 1 \} \) is \((n - 3)\mu\).

By Corollary 6 we have a partition of the quadruples (with repetitions) from configuration \((1, 1, 1, 1)\) which are not contained in Type A into \((n - 3)\mu\) subsets, each one of size \(n^3\). These \((n - 3)\mu\) subsets are distributed arbitrarily among the \(P_i\)'s to complete the quadruples from Group 5 of the \(P_i\)'s.

Lemma 17 Each system of Type B is an SQS(4n).

Proof One can easily verify that each triple from \(\mathbb{Z}_n \times \mathbb{Z}_4\) is contained in one of the blocks for each system of type B. Hence, to complete the proof it is sufficient to show that the number of blocks in a system of Type B is \(\binom{4^n}{3}/4\) as required for SQS(4n). By their definition, in Type B, there are \(\binom{n}{3}\) quadruples of Group 1 in each system. The number of quadruple in each system from Group 3 is \(6(n - 1)\binom{n}{2}\) and from Group 5 this number is \(n^3\). Since \(\binom{n}{3} + 6(n - 1)\binom{n}{2} + n^3 = \binom{4^n}{3}/4\), it follows that each system of Type B is an SQS(4n).

Lemma 18 The systems of Type B in the \((4n)\)-construction have the following containment properties.

1. Each quadruple from each configuration of Group 1 is contained in exactly \(\mu = \frac{(n - 1)\gamma}{2}\) of the \((n - 3)\mu\) systems of Type B.
2. Each quadruple from Group 3 is contained in exactly \(\mu - \gamma\) systems of Type B.
3. The total number of quadruples from configuration \((1, 1, 1, 1)\) (Group 5), which are contained in the systems of Type B is \((n - 3)\mu n^3\).

Proof The enumeration is a straightforward result from the definitions.

1. It follows from the immediate observation that each quadruple from \(\mathbb{Z}_n\) is contained in exactly \(\mu\) systems of the LS(3, 4, \(n; \mu\)), \(\{ S^n_i : 1 \leq i \leq (n - 3)\mu \}\).
2. It follows immediately from the definition that for each \(j, 0 \leq j \leq n - 2\), for each \(i, j(\mu - \gamma) < i \leq (j + 1)(\mu - \gamma)\), the related \(P_i\)'s contain the same quadruples from Group 3.
3. The number of \(P_i\)'s is \((n - 3)\mu\). Each one contains either one of the \(D_j\)'s or one of the \(E_j\)'s, where each one contains \(n^3\) quadruples. Thus, the total number of quadruples from configuration \((1, 1, 1, 1)\), which are contained in the systems of Type B is \((n - 3)\mu n^3\).

Proof of Theorem 15 There are no quadruples from configurations of Group 1 in Type A, while by Lemma 18 each quadruple from configurations of Group 1 is contained in exactly \(\mu\) systems or Type B.

By Lemma 14 each quadruple from each configuration of Group 2 or Group 4 is contained in exactly \(\mu\) systems of Type A, while in Type B there are no quadruples from these groups.
By Lemma 14 each quadruple from each configuration of Group 3 is contained in \( \gamma \) systems of Type A and by Lemma 18 each quadruple from each configuration of Group 3 is contained in \( \mu - \gamma \) systems of Type B. Thus, each quadruple from each configuration of Group 3 is contained in exactly \( \mu \) systems of Type A or Type B.

By Lemmas 14, 15, 16, and Corollary 6, each quadruple from configuration \((1,1,1,1)\) is contained in exactly \( \mu \) systems of Type A or Type B.

6 LS\((3, 4, 2^m n; g)\), for \( n \equiv 2 \text{ or } 4 \pmod{6}, m \geq 3 \)

The quadrupling construction of Sect. 5 implies an LS\((3, 4, 4n; \mu)\) from an LS\((3, 4, n; \mu)\). The goal in this section is to continue and prove Theorem 16. Given the LS\((3, 4, 4n; \mu)\) constructed by the \((4n)\)-construction we amend it to a construction for an LS\((3, 4, 2^m n; \mu)\), where \( m \geq 3 \). First, note that in order to apply the \((4n)\)-construction recursively we need an appropriate perpendicular array and it might not exist (in fact it probably does not exist for most parameters).

The construction which will be used here is based on the idea given in [15]. The first \((2^m - 1)n\mu\) systems are based on the first \(3\mu n\) systems of Type A and are constructed similarly to the systems as explained in [15].

First, an order on \(\mathbb{Z}_2^m\) is induced by identifying \(x \in \mathbb{Z}_2^m\) with a nonnegative integer smaller than \(2^m\) whose binary representation is \(x\). This is the usual lexicographic order. Next, a set of \(2^m - 1\) SQSs on the point set \(\mathbb{Z}_2^m\) and block set \(B_i\) with \(i \in \mathbb{Z}_2^m \setminus \{0\}\) is defined. These sets are called the Boolean Steiner quadruple systems. The block set \(B_i, i \in \mathbb{Z}_2^m \setminus \{0\}\) is defined to be the union of the blocks of Types (B.1) and (B.2) specified below

\[
\begin{align*}
(B.1) \quad \{x, y, z, w\} : & \quad x + y + z + w = i, \quad |\{x, y, z, w\}| = 4 \\
(B.2) \quad \{x, y, z, w\} : & \quad x + y = z + w = i, \quad |\{x, y, z, w\}| = 4
\end{align*}
\]

The following result which can be easily verified was proved in [15].

**Lemma 19** For each \(i, i \in \mathbb{Z}_2^m \setminus \{0\}\) the set \(B_i\) is an SQS\((2^m)\).

- Each quadruple of \(\mathbb{Z}_2^m\) is contained in at least one of these \(2^m - 1\) SQS\((2^m)\).
- Each block of Type (B.1) is contained in exactly one of the \(2^m - 1\) SQS\((2^m)\).
- Each block of Type (B.2) is contained in exactly three of the \(2^m - 1\) SQS\((2^m)\).

We continue to define the blocks of each system in the \((2^m n)\)-Construction. The point set of each SQS\((2^m n)\) is \(\mathbb{Z}_n \times \mathbb{Z}_2^m\). The first \((2^m - 1)n\mu\) systems of Type A are constructed similarly to the systems constructed in [15].

The \((i, j)\)-th system \(P_{(i,j)}, i \in \mathbb{Z}_2^m \setminus \{0\}, 0 \leq j \leq n\mu - 1\), is defined as follows. Recall that \(R_z, 1 \leq s \leq n\mu\), is a set of SQS\((2n)\) on the point set \(\mathbb{Z}_n \times \mathbb{Z}_2^m\) defined via the DLS Construction using a PA\(_y\)(2, \(n, n\)). This set will be defined now on the point set \(\mathbb{Z}_n \times \{x, y\}\), where \(x, y \in \mathbb{Z}_n^m\) and \(x + y = i\).

Now, assume that \(x + y + z + w = 0\), where \(|\{x, y, z, w\}| = 4\) and \(x, y, z, w \in \mathbb{Z}_2^m\), where \(x < y, z < w, x + y = i, x + z = r, x + w = \ell\). Let \(g\) be a bijection \(g : \mathbb{Z}_4 \rightarrow \{x, y, z, w\}\). On the point set \(\mathbb{Z}_n \times \{g(0), g(1), g(2), g(3)\}\) we embed the \(n\mu\) systems of Type A1 for \(P_{(i,j)}, 0 \leq j \leq n\mu - 1\). On the point set \(\mathbb{Z}_n \times \{g(0), g(1), g(2), g(3)\}\) we embed the \(n\mu\) systems of Type A2 for \(P_{(r,j)}, 0 \leq j \leq n\mu - 1\). On the point set \(\mathbb{Z}_n \times \{g(0), g(1), g(2), g(3)\}\) we embed the \(n\mu\) systems of Type A3 for \(P_{(\ell,j)}, 0 \leq j \leq n\mu - 1\).
Remark 2 Note, that \( R_s \) is embedded only once for each point set \( \mathbb{Z}_n \times \{ x, y \} \) for each \( x, y \in \mathbb{Z}_2^m \) such that \( x + y = i \). Similarly, the quadruples from Group 4 and the quadruples from configuration \((1, 1, 1, 1)\) are embedded only once on the point set \( \mathbb{Z}_n \times \{ x, y, z, w \} \) for each \( x, y, z, w \in \mathbb{Z}_2^m \) such that \( x + y + z + w = i \).

Assume now, that \( x + y + z + w = i \), where \(|\{ x, y, z, w \}| = 4\) and \( x, y, z, w \in \mathbb{Z}_2^m \). We form the following last set of blocks in \( \mathcal{P}_{(i,r)} \), \( 0 \leq r \leq n\mu - 1 \).

\[
\{(a, x), (b, y), (c, z), (a + b + c + r, w) : a, b, c \in \mathbb{Z}_n\}.
\]

Lemma 20 Each \( \mathcal{P}_{(i,j)} \) is an SQS(\(2^{m}\)).

Proof It is straightforward to prove by the definition based on the DLS Construction and the \((4n)\)-construction that each triple of \( \mathbb{Z}_n \times \mathbb{Z}_2^m \) is contained in at most one block. Hence, to complete the proof it is sufficient to prove that each system \( \mathcal{P}_{(i,j)} \), \( i \in \mathbb{Z}_2^m \setminus \{0\} \), \( 0 \leq j \leq n\mu - 1 \), contains \((\binom{2n}{3})/4\) blocks as is the number of blocks in an SQS(\(2^{m}\)).

The number of blocks originated from Type A in a system is as follows. There are \(2^{m-1}(\binom{2n}{3})/4\) blocks for all SQS(\(2n\)) embedded in the set of points \( \mathbb{Z}_n \times \{ x, y \} \) such that \( x + y = i \). The number of blocks induced from Type A from Group 4 is \( \binom{2n}{3} \cdot \binom{2n}{3} \cdot \binom{2n}{3} \cdot \binom{2n}{3} \) where \( n \in \mathbb{Z}_2^m \). Therefore, the total number of blocks in \( \mathcal{P}_{(i,j)} \) is

\[
2^{m-1}\left(\frac{2n}{3}\right)^4 + \left(\frac{2n}{3}\right)^4 - \left(\frac{2n}{3}\right)^4 + \left(\frac{2n}{3}\right)^4 - \left(\frac{2n}{3}\right)^4 n^3.
\]

which equals \((\binom{2n}{3})/4\) as required.

Now, we define the last \((n - 3)\mu\) systems, named Type C. For each quadruple \( \{ x, y, z, v \} \subset \mathbb{Z}_2^m \) such that \( x + y + z + v = 0 \) we form these \((n - 1)(\mu - \gamma) = (n - 3)\mu\) systems of Type C from the \((4n)\)-construction on the point set \( \mathbb{Z}_n \times \{ x, y, z, v \} \). These \((n - 3)\mu\) systems of Type C (SQS(\(2^{m}\))) are constructed based on the \((n - 3)\mu\) systems (SQS(\(4n\))) of Type B constructed in the \((4n)\)-construction. Let \( \mathcal{P}_i \) be the \(i\)-th (SQS(\(4n\))) \( 1 \leq i \leq (n - 3)\mu \) of Type B. The first set of blocks of the \(i\)-th system of Type C \(((2^{m} - 1)n\mu + i)\)-th system of the whole large set) for the \((2^{m}\))-construction are defined as follows. Recall that \( F = \{ F_0, F_1, \ldots, F_{n-2} \} \) is a one-factorization on \( K_n \) and \( M \) is an \((n - 1) \times (n - 1)\) Latin square on the points set \( \{0, 1, \ldots, n - 2\} \). Finally, \( S^x_i, 1 \leq i \leq (n - 3)\mu \), is the \(i\)-th SQS(\(n\)) in an LS(\(3, 4, n; \mu\)).

For each \( j \in \mathbb{Z}_2^m \) and each block \( \{ a, b, c, d \} \in S^x_i \) construct the block

\[
\{(a, j), (b, j), (c, j), (d, j)\}.
\]

Given \( j, 0 \leq j \leq n - 2 \), and \( i, j(\mu - \gamma) < i \leq (j + 1)(\mu - \gamma) \), the following blocks form the second set of blocks in the \(i\)-th system (from the \((n - 3)\mu\) systems of Type C).

\[
\{(a, \ell), (b, \ell), (c, t), (d, t) : \ell, t \in \mathbb{Z}_2^m, \ell < t, \{ a, b \} \in F_r, \{ c, d \} \in F_{M(j,r)} \}, 0 \leq r \leq n - 2 \}
\]

For each block \( \{(a, 0), (b, 1), (c, 2), (d, 3)\} \in \mathcal{P}_i \) and for each four distinct values \( x, y, z, w \in \mathbb{Z}_2^m \) such that \( x < y < z < w \) and \( x + y + z + w = 0 \) construct the block

\[
\{(a, x), (b, y), (c, z), (d, w)\}.
\]

These blocks form the third set of blocks in the construction.
Lemma 21 Each one of the \((n - 3)\mu\) systems of Type C is an SQS\((2^m n)\).

Proof The number of blocks in the first set of each system is \(2^m \binom{n}{3}/4\), in the second set is \(\binom{2^m n}{2}(n - 1)\mu^2/4\), and in the third set is \((2^m n)^3/4\). Hence, the total number of blocks in each system is \((2^m n)^3/4\) which is the number of blocks in an SQS\((2^m n)\). One can easily continue and verify that no triple of \(Z_n \times Z_2^m\) is contained in more than one block of the system, which completes the proof. \(\square\)

The proof of the following lemma is identical and follows from the proof of Lemma 18.

Lemma 22 The \((n - 3)\mu\) systems of Type C, of SQS\((2^m n)\), have the following properties
1. Each quadruple \(\{(a, j), (b, j), (c, j), (d, j)\}, \{a, b, c, d\} \subset Z_n, j \in Z_2^m\) is contained in exactly \(\mu\) systems.
2. Each quadruple \(\{(a, i), (b, i), (c, j), (d, j)\}, \{a, b, c, d\} \subset Z_n, i < j, and \{a, b\} \subset Z_n, \{c, d\} \subset Z_n\), is contained in exactly \(\mu - \gamma\) systems.
3. For given four distinct values \(x, y, z, w \in Z_2^m\), such that \(x + y + z + w = 0\), the number of quadruples of the form \(\{(a, x), (b, y), (c, z), (d, w)\}, \{a, b, c, d\} \subset Z_n\), which are contained in the \((n - 3)\mu\) systems of Type C is \((n - 3)\mu n^3\).

Proof of Theorem 16: The existence of an LS\((3, 4, n; \mu)\) is given in the theorem. Hence, by Theorem 14 there exists an LS\((3, 4, 2n; \mu)\) and by Theorem 15 there exists an LS\((3, 4, 4n; \mu)\).

The rest of the proof is induced from the proof of Lemma 15. By Lemmas 20 and 21, each one of the systems of the \((2^m n)\)-Construction is an SQS\((2^m n)\). The number of such systems in the construction is \((2^m - 1)n\mu + (n - 3)\mu m = 2^m n\mu - 3\mu\) as required in an LS\((3, 4, 2^m n; \mu)\), \(m \geq 3\).

The proof that each quadruple of \(Z_n \times Z_2^m\) is contained in exactly \(\mu\) of the systems is similar to the one in Theorem 15 and it is left to the reader. \(\square\)

Theorems 3, 4, 8, 11, and 16 imply the following results.

Corollary 7 There exists an LS\((3, 4, 5 \cdot 2^m; 9)\) for each integer \(m \geq 1\).

Corollary 8 For each \(m \geq 1\) and each \(\ell \geq 1\) there exist an LS\((3, 4, 5 \cdot 2^m; 9\ell)\) and an LH\((5 \cdot 2^m, 9\ell, 4, 3)\).

7 Conclusion and problems for future research

The lack of known constructions for large sets of Steiner systems \(S(t, k, n)\), where \(2 < t < k < n\) has motivated the definition of a large set of Steiner systems with multiplicity. In such a system each \(t\)-subset of the \(n\)-set is contained in exactly \(\mu\) systems of the large set. The existence of such large sets with multiplicity implies the existence of large sets for H-designs with related parameters. A recursive construction for large set of Steiner quadruple systems with multiplicity was given. For small parameters some ad-hoc constructions for large sets with multiplicity were given using perpendicular arrays and ordered designs. Except for the large sets of H-designs derived from large sets with multiplicity, some ad-hoc constructions for large sets of H-designs with blocks of size four and small number of groups were also presented. The main new results in this paper can be summarized as follows:

1. If there exists an LH\((n, g, k, t)\) and an OA\((t, k, u)\), then there exists an LH\((n, gu, k, t)\) (Theorem 3).
2. If there exist an OA\((t, k, g)\) and an LS\((t, k, n; g^\gamma)\), then there exists an LH\((n, g, k, t)\) (Theorem 4).

3. For each \(g \geq 2\) there exist an LS\((3, 4, 10; g)\) and an LH\((10, g, 4, 3)\) (Theorem 11).

4. An LH\((n, g, 4, 3)\) exists for all admissible parameters with \(n \in \{4, 5, 6, 7, 8, 10\}\), with possible exceptions for \(n = 5\) and \(g \equiv 2 \pmod{4}\) (Corollary 1).

5. For each \(g \geq 2\), there exist an LS\((5, 6, 12; g)\), an LH\((10, 5, 5, 4)\), and an LH\((12, g, 6, 5)\), with possible exceptions when \(g \in \{3, 5, 7, 9, 11, 13\}\) (Theorem 13).

6. If there exist an LS\((3, 4, 3; \mu)\) and a PA\(_g\)\((2, n, n)\), \(\mu = \frac{(n-1)^2}{2}\), then there exists an LS\((3, 4, 2^m n; \mu)\), for each \(m \geq 0\) (Theorems 15 and 16).

7. For each integer \(m \geq 1\) and each integer \(\ell \geq 1\), there exist an LS\((3, 4, 5 \cdot 2^{m}; 9\ell)\) and an LH\((5 \cdot 2^{m}, 9\ell, 4, 3)\) (Corollary 8).

The exposition of this paper raises many open problems and in particular one would like to see constructions of large sets for larger range of parameters and small multiplicity as possible. Furthermore, we would like to see constructions for large sets and perpendicular arrays on which the results in this paper, and especially Theorems 15 and 16, can be applied.

Finally, the application of large sets with multiplicity to the Russian cards problem is very intriguing and deserves further research.

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