The Random Fuse Network as a Dipolar Magnet

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We introduce an approximate mapping between the random fuse network (RFN) and a random field dipolar Ising model (RFDIM). The state of the network damage is associated with a metastable spin configuration. A mean-field treatment, numerical solutions, and heuristic arguments support the broad validity of the approximation and yield a generic phase diagram. At low disorder, the growth of a single unstable ‘crack’ leads to an abrupt global failure. Beyond a critical disorder, the conducting network sustains significant damage before the coalescence of cracks results in global failure.

Heterogeneities are present in real materials, in a wide range of magnitudes and scales, and generate fluctuations in local toughness. The fracture mechanics then results from an interplay between disorder and stress fluctuations.\[1-3\] In attempting to elucidate the role of the disorder, a number of classifications and characterizations of rupture have been proposed \[3-10\]. In particular, the behavior of the damage as the applied force approaches its critical value (global rupture)—partial versus no damage, macroscopic abrupt versus continuously increasing damage—allowed to distinguish various types of fracture. Most studies were carried out either numerically or in the framework of extreme statistics (see Refs. \[1,2\] and references therein), while few satisfactory analytical approaches exist. In parallel, a number of authors \[11,12\] introduced the so-called random fuse network (RFN), less involved than fracture models because of its scalar nature but still capturing the main aspects of rupture mechanics \[3\]. Thus the RFN often is seen as a toy model for the investigation of mechanical rupture. The mechanical bonds are replaced by fuses burning if the current (or equivalently the electrical field) exceeds a random threshold. As the total applied current flowing through the network is increased, more and more fuses burn, thereby decreasing the effective conductivity of the network until a global failure at which no current can go through anymore. The similarity of the RFN with a driven random field Ising model at zero temperature \[14\] has been recently noted \[6\], but no systematic correspondence has been established.

In this letter, we present a mapping, valid at least at the initial stages of failure, between the RFN and a driven random field Ising model with dipolar coupling. This mapping associates the configuration of the burnt fuses at a given value of the applied electric field with a zero temperature metastable state of the random field dipolar Ising model (RFDIM). The mapping between fracture and a classical model of statistical mechanics, is supported by mean-field and heuristic arguments and by a preliminary numerical study, from which a generic phase diagram with a disorder induced transition emerges.

The RFN consists of a \(d\)-dimensional lattice of fuses, each with a conductivity \(\sigma_i(x) = \sigma_0\) initially, where \(x = am\), \(m \in \mathbb{Z}^d\) and \(a\) is the lattice spacing (we will assume the continuum limit \(a \to 0\)), and \(i = 1, \ldots, d\) corresponds to the orientation of the fuse. If the electric field along \(i\) at \(x\), \(E_i(x)\), exceeds a random threshold \(E_i(x)\), the fuse burns and \(\sigma_i(x) = 0\). We consider a scenario in which the electric field in the direction \(i = 1\) is slowly increased from zero and \(E_i = 0\) for \(i \neq 1\) at all times.

Assuming locality and analyticity, we can write a general equation for the time evolution of the conductivity field as

\[
\eta \partial_t \sigma_i(x,t) = f_i(\{\sigma_j(x,t)\}, \{\partial_j \sigma_k(x,t)\}, \ldots) \theta(-\Delta_i(x,t)),
\]

where \(\Delta_i = E_i(x)^2 - E_i(x,t)^2\) and the Heaviside function ensures that the conductivity of the fuse remains \(\sigma_0\) as long as the electric field it sustains does not exceed its threshold. The simplest choice for the function \(f_i\) within the constraints of positivity \((\sigma_i(x,t) \geq 0)\) and monotonicity \((\partial_t \sigma_i(x,t) \leq 0)\), \(f_i = -\text{constant} \times \sigma_i(x,t)\), yields

\[
\eta \partial_t \sigma_i(x,t) = -\sigma_i(x,t) \theta(-\Delta_i(x,t))
\]

after absorption of the constant in a redefinition of \(\eta\). Equation \[2\] incorporates two time scales: the relaxation time \(\eta\) and the characteristic time \(\tau = E_i(dE_i/dt)^{-1}\). We focus on the limit \(\eta/\tau \to 0\), in which the conductivity relaxes instantaneously from \(\sigma_0\) to 0 as the local field crosses the local threshold. To capture the sequential dynamics associated with the \(\eta \neq 0\) retardation effect, we discretize \[2\] Eq. \[2\] in time as

\[
\sigma_i(x,t + \delta t) = \sigma_i(x,t) \theta(\Delta_i(x,t)).
\]
Formally, this corresponds to setting \( \eta = \delta t \) and discretizing Eq. (2) in the usual fashion. As expected, Eq. (3) prescribes an irreversible dynamics such that \( \sigma_i(x) = 0 \) at time \( t \) implies \( \sigma_i(x) = 0 \) for all later times.

For a given configuration of the conductivities, Kirchoff’s laws may be recast into the integral equation

\[
E_i(x) = E_0 \delta_{i1} + \int d^d y \sum_j G_{ij}(x - y) \delta \sigma_j(y) E_j(y),
\]

where \( E_0 \) is the magnitude of the external field applied along the first axis, \( \delta \sigma_i(x) = \sigma_i(x) - \sigma_0 \), \( G_{ij}(x) = -\partial_i \partial_j \varphi(x) \) the dipolar tensor \([17a]\), and the Green’s function \( g \) is defined by \( \sigma_0 \nabla^2 g(x) = -\delta(x) \). To lowest non-trivial order, Kirchoff’s laws become

\[
E_i(x) = E_0 \left[ \delta_{i1} + \int d^d y G_{i1}(x - y) \delta \sigma_1(y) \right].
\]

As we shall see, this simple approximation captures much of the physics of network damage. First, due to the presence of the dipolar kernel \( \prod_{i=1,k} \delta \sigma_1(y_i) \) which vanishes with high probability, especially at the early stages of the damage. In addition, this approximation satisfies the necessary property that the field at the tip of a crack increases with crack growth. For one isolated burnt fuse, the field on the next parallel link in the direction perpendicular to the applied field is enhanced by a factor \( \alpha > 1 \). The approximation gives \( \alpha = 1 - G_{11}(0,1) \approx 2.14 \) on the \( 2d \) square lattice (and \( \alpha = 2 \) in the continuum limit \([17b]\)); the exact result on the square lattice is \( \alpha = 4/\pi \). When \( n \) such parallel fuses are broken, the field at the tip of the crack is known to scale as \( n^{1/2(d-1)} \) \([13]\), while in the approximation of Eq. (4)

\[
E_{\text{tip}} = E_1(0, \rho = n + 1) = E_0 (1 + \sum_{1<\rho<n+1} 1/\rho^d)
\]

(where \( \rho^2 = x^2 + \ldots + x_2^2 \) is the transverse square distance) converges to a finite value as \( n \) increases, corresponding formally to the \( d = \infty \) case. For uniform disorder distributed according to \( p(\mathcal{E}) = \theta(\mathcal{E}^2 - \mathcal{E}_{\text{th}}^2)\theta(\mathcal{E}_{\text{th}}^2 + w - \mathcal{E}^2)/w \) with \( w < 2 \) or a binary disorder \( E_{\text{tip}} \) can be larger than the largest threshold and we expect the same results as if \( E_{\text{tip}} \sim n^{1/2(d-1)} \). For a disorder distribution allowing with a high probability very large values of thresholds (such as distributions with fat tails), the scaling of \( E_{\text{tip}} \) with the size of the crack becomes relevant and we cannot use the approximation \([3]\). Here, we present results for the uniform disorder while similar results are obtained for binary disorder.

Squaring Eq. (2) we obtain, to lowest nontrivial order, the distance \( \Delta_i \) from the local threshold in terms of the damage \( \delta \sigma_i \), as

\[
\Delta_1(x,t) \approx \mathcal{E}_1(x)^2 - E_0^2 - 2E_0^2 \int d^d y G_{11}(x - y) \delta \sigma_1(y,t)
\]

\[
\Delta_j(x,t) \approx \mathcal{E}_j(x)^2 - E_0^2 \left[ \int d^d y G_{j1}(x - y) \delta \sigma_1(y,t) \right]^2,
\]

(with \( j \neq 1 \)) which, along with the update rule of Eq. (3) completely determines the state of the network under an applied field \( E_0 \).

In terms of the ‘Ising spins’ defined by \( s(x,t) = 2\delta \sigma_1(x,t)/\sigma_0 + 1 \), Eq. (5) simply stipulates that each spin \( s(x,t) \) will lie along a local field \( \Delta_1(x,t) \). Eq. (5) shows that this local field \( \Delta_1 \) results from a uniform applied component, a random component, and a spin-spin interaction mediated by the component \( G_{11} \) of the dipolar kernel. Therefore, as the applied field is switched from 0 to \( E_0 \), spins flip in avalanches until they reach the state \( s(x) = \text{sgn}[\Delta_1(x,t = \infty)] \) which minimizes the Hamiltonian

\[
\mathcal{H} = -\frac{1}{2} \int d^d x d^d y s(x) J(x - y) s(y) - \int d^d x h(x) s(x),
\]

where \( J(x) = -E_0^2 \sigma_0 G_{11}(x), h = H + \mathcal{E}_1(x)^2 \), and \( H = -E_0^2 [1 - \sigma_0 G_{11}(0)] = -E_0^2 (1 + 1/d) \). As the driving field \( |H| \) is increased, more and more spins flip downward, and the state of the RFN follows the metastable (nonequilibrium) spin state of \( \mathcal{H} \), connected by its history to the initial condition \( s(x) = +1 \) for all \( x \).

While similarities between the RFN and the random field Ising model (RFIM) have been pointed out in the past, here a precise mapping is elucidated. We stress that, at odds with previous studies on the RFIM \([2]\), in which the
spin-spin coupling is taken to be short-ranged, constant, and isotropic, the coupling $J(x)$ is rather peculiar as its magnitude is proportional to the applied field $H$ and its modulation is dipolar. Even in the absence of disorder, the dipolar interaction in Ising spin models [21] leads to a rich behavior [22,23] not yet fully understood. In particular, a striped phase was identified [21,23] with alternating up and down spins. A similar scenario occurs in the RFN at very low (narrow) disorder. All the fuses remain intact until $E_0$ reaches the lowest threshold and the corresponding fuse burns. The dipolar coupling then relieves its longitudinal neighbors while further stressing its transverse neighbors, who burn in turn, and a stripe of burnt fuses develops, eventually spanning the whole system (the network then stops conducting and the formation of additional stripes is consequently prohibited). At high (wide) disorder, a resilient fuse in the passage of a burnt stripe will stop the growth of the crack, and new cracks will nucleate elsewhere. This suggest a transition, upon increase of the disorder [30], from an abrupt regime characterized by the growth of a single macroscopic unstable crack to a continuous regime in which the system is significantly damaged before its global failure.

For a qualitative understanding of the role of disorder, we first revert to a simpler version of the problem in which the dipolar kernel is replaced by a uniform infinite-range coupling $J(x) = E_0^2 J/N$, where $J > 0$ measures the strength of the interaction and $N$ is the total number of spins (equivalently, fuses). This ‘democratic’ model is mean-field like in that the evolution of the local field (Eq. (7)) acting on a spin $s(x)$ may be described in terms of a single degree of freedom, the fraction $n_-$ of down spins (burnt fuses),

$$\Delta_1(x) = \mathcal{E}_1(x)^2 - E_0^2 - 2JE_0^2 n_-,$$

and depends on the position only through the local random threshold. Upon increase of the applied field a number of spins flip, thus increasing $n_-$ which in turn causes more flips, and so on until this avalanching terminates at $n_-(E_0)$, the metastable fraction of downward spins at an applied field $E_0$. We follow the graphical scheme of Ref. [27] to obtain the resulting phase diagram. For the uniform disorder of width $w$, clearly $n_- = 0$ as long as $E_0^2 < \mathcal{E}_m$, the lowest threshold. If $w < 2J\mathcal{E}_m^2$, $n_-$ then jumps to 1 abruptly; if $w > 2J\mathcal{E}_m^2$, on the other hand, $n_- = (E_0^2 - \mathcal{E}_m^2)/(w - 2JE_0^2)$ increases continuously with $E_0$ up to $n_- = 1$ at $E_0^2 = (\mathcal{E}_m^2 + w)/(1 + 2J)$. As expected, a critical width $w_0 \equiv 2J\mathcal{E}_m^2$ of the disorder separates two regimes: at low disorder, a huge avalanche is leading to global failures, while for large disorder, tiny avalanches can be stable, until cracks coalesce. We note the broad validity of this picture, at least in the mean field. In contrast with the ‘democratic fiber bundle model’ [28,29] for which it was shown [27] that abrupt failures are but an artifact of a large discontinuity in the threshold distribution, here the phase diagram extends to distributions that are not continuous, e.g., to any uniform distribution on a support $[\mathcal{E}_m^2, \mathcal{E}_m^2 + w]$. These mean-field results are also recovered from the homogeneous saddle point of the partition function

$$Z = \sum_{\{s(x) = \pm 1\}} e^{-\beta H}$$

This saddle point corresponds to the stability condition for a microcrack: if the electric field at the tip of the crack is larger than the largest threshold, the crack is unstable. This implies the existence of the critical width $w_0$ found above. It is quite remarkable that this result—which was already found by probabilistic methods [30]—appears here naturally as the uniform saddle point or mean field.

Because of the form of the coupling in this simplified democratic model, we can carry out a similar investigation without recourse to the linear approximation of Eq. (3), but using Eq. (7) directly. Assuming $J < 1/2$ (without which the problem is ill-defined), we find not only a similar phase diagram but also, surprisingly, the same value for the critical disorder $w_0 = 2J\mathcal{E}_m^2$. At least in this mean field version of the original problem, the linear approximation is legitimate. This also suggests that when Eq. (9) is valid, the early stages of the damage dominate the whole breakdown process.

Evidently, a mean field phase diagram may be significantly modified by fluctuations governed by a dipolar coupling. On the one hand the dipolar kernel decays as $1/r^d$, on the other hand its angular dependence invests the model with a so-called nonmonotonicity: burning a fuse relieves the current on a fraction of neighboring fuses. Monotonicity is a simplifying feature of many driven systems, and its lack clearly introduces complications and a possibly richer phenomenology [26]. Furthermore, while a mean field treatment predicts the evolution of damage, it fails to capture the random fluctuations of the breakdown field $E_0$ as well as finite-size effects. These two features are however present in the approximation (3). In particular, it is easily shown that the maximum of the electric field scales as $\ln L$, as a consequence of the long range of the dipolar tensor. Investigation of finite size effects in the framework of this approximation present a complementary route to most of the approaches to finite size effects, which rely on extreme statistics [31] and neglect correlations.
For a better grasp of the full problem we obtained numerical solutions of the RFN (using Eqs. (3,4)) and the RFDIM (using the approximation Eq. (7)). We found (Fig. 1) that the RFDIM reproduces quite well the behavior of the RFN. In both cases, we observe a critical value \( w_0 \) of the disorder beyond which there is a large window of damage that precedes the global failure.

Before concluding, we propose a first attempt to describe the stability of cracks in the spirit of the Imry-Ma argument \([31,22]\). In three dimensions, the creation of a spherical crack of radius \( R \) modifies the total energy (cf. Eq. (9)) by

\[
\Delta \mathcal{H} \approx 2 \overline{h} R^3 - w R^{3/2} + g R^3
\]

where the last term comes from the dipolar interaction \( g > 0 \) and merely renormalizes the average of the random field \( \overline{h} \). If the applied field \( E_0 \) is small, the average \( \overline{h} \) is positive and the formation of a large crack is prohibited. In this case, the typical size of a crack will be \( R_0 \approx [w/(2\overline{h} + g)]^{2/3} \). On the other hand, if \( 2\overline{h} + g < 0 \) it is favorable for a crack to grow indefinitely. A more precise argument based on an oblate (‘penny-shaped’) crack perpendicular to the applied field leaves the above conclusions unchanged. As \( w \) is increased, the typical size \( R_0 \) of damaged regions becomes larger, up to a critical value \( w_0 \) at which the latter percolate. This occurs when \( R_0 \) becomes comparable to the typical distance between nucleation sites, \( a[w/(H-E_0^2)]^{1/3} \) for an uniform disorder. As long as \( w < w_0 \), breakdown results from a single crack that spans the system when \( 2\overline{h} + g \) becomes negative upon increase of the applied field \( H \). When \( w > w_0 \), localized cracks grow and percolate before \( 2\overline{h} + g \) changes sign, which allows significant damage while the network is still conducting. This heuristic argument also substantifies the broad validity of the approximation of Eq. (8) \( a \) priori only valid at the initial stages of the damage: the breakdown is controlled either by an instability or by the coalescence of (small) cracks.

In summary, we have presented an approximate mapping of the RFN to the RFDIM, and argued for its broad validity. The state of the RFN—in particular the extent of damage (number of burn fuses)—is mapped into a metastable spin state. From a mean-field investigation, numerical solutions, and heuristic arguments, a generic picture emerges, characterized by a critical value of the disorder. At low disorder, a macroscopic crack grows until it spans the system. At large disorder, threshold fluctuations stabilize (micro)cracks and a significant precursor damage develops before the latter coalesce into a percolating structure. Our investigation were carried out for a bounded disorder. By contrast, a slowly decaying threshold distribution may modify the character of the transition or suppress it altogether, an interesting question for future study.

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FIG. 1. Effective conductivity $\Sigma$ versus $E_0$ for uniform disorder for two values of the disorder width $w = 0.5 < w_0$ and $w = 1.8 > w_0$ (for a system size $50 \times 50$ and averaged over 100 configurations). (a) Calculation for the RFN by solving Eqs. (3,4). (b) Calculation for the RFDIM obtained from the numerical solution of Eqs. (6a,b). In both cases (a) and (b), for $w < w_0$, the breakdown is abrupt and there is no fluctuation in the breakdown field (for (a), it is not fully abrupt, due to finite-size effects). For $w > w_0$, the breakdown field fluctuates and there is some damage before complete breakdown of the system.