Comment on the dressed Polyakov loop in the Nambu–Jona-Lasinio model

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We argue that in spite of the fact that the dressed Polyakov loop in the Nambu–Jona-Lasinio (NJL) model shows a rapid change as a function of the temperature, this should not be interpreted as a confinement-deconfinement phase transition. In particular, it certainly does not mean that the NJL model itself is confining. Rather, we show that this behaviour is simply a remnant of the chiral transition.

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I. INTRODUCTION

The Polyakov loop [1] is a particular representation of a static source of color, propagating in compact Euclidean time $\tau$. When the source has a finite mass, the path in Euclidean space need not be any more a straight line in $\tau$ - a possibility to deviate in space directions goes as $\propto 1/m|l|$, where $m$ is the quark mass, and $|l|$ is the number of links in a loop [2–4]. In this case we say that the Polyakov loop is “dressed” [2–4].

Due to its transformation w. r. t. to the center of the color $SU(N_c)$ group of Quantum Chromodynamics (QCD) in the static limit $m \to \infty$, it can be used as an order parameter for confinement-deconfinement phase transition, just like the ordinary Polyakov loop [4]. The dressed Polyakov loop (dPL) has proven to be a valuable tool in continuum studies of QCD, with the confinement-deconfinement crossover studied in the Dyson-Schwinger framework [6–8].

Of particular concern to us is the usage of this quantity in NJL and Polyakov-NJL models, see e. g. [9–12]. For example, it has been shown [10] that even in the NJL model the dPL shows a rapid change when proceeding from low to high temperatures. Such a statement sounds puzzling - if interpreted literary one might claim that the NJL model is confining. This would be in complete disagreement with the common statement in the literature, i. e. that one of the main disadvantages of the NJL model is, in fact, lack of confinement. A simple reasoning behind this statement is that quark propagator in the NJL model is just the tree level fermion propagator, albeit with a constituent ($\sim 300$ MeV), rather than current quark mass ($\sim 5$ MeV). Therefore, a positive definite spectral representation is available, allowing the excitation of these states.

We believe that this puzzle merits a more careful analysis, to which this Comment is devoted. By simple Landau analysis we will understand how it comes to be that the dPL in the NJL model exhibits a rapid change as a function of the temperature in the first place. We will analytically show that the temperature at which the change is most pronounced is, irrespective of the model details, the chiral restoration temperature. With this result we are well motivated to understand that the crossover behaviour in the dPL calculated in the NJL model should be interpreted merely as an imprint of the chiral phase transition, and not as a confinement-deconfinement phase transition.

II. NAMBU–JONA-LASINIO MODEL WITH TWISTED BOUNDARY CONDITIONS

We work in the chiral limit with $N_f = 2$, and zero real chemical potential, $\mu = 0$. In order to calculate the dPL one has to distort the fermionic boundary conditions, by introducing a
twisted angle $\phi^*$. Alternatively, one can start from the imaginary chemical potential, so that the thermodynamic potential in NJL model is given as

$$\Omega = \Omega_{\text{cond}} + \Omega_{\text{kin}}^{\text{vac}} + \Omega_{\text{kin}}^{\text{th}}, \quad (1)$$

where the condensation potential is $\Omega_{\text{cond}} = \frac{\sigma^2}{2G}$, and the vacuum and thermal one-loop contribution read

$$\Omega_{\text{kin}}^{\text{vac}} = -\frac{d_q}{2} \int \frac{d^3p}{(2\pi)^3} \frac{E}{2}, \quad (2)$$

$$\Omega_{\text{kin}}^{\text{th}} = -\frac{d_q}{2} \int \frac{d^3p}{(2\pi)^3} [\log(1 + e^{-\beta(E + i\mu_I)}) + (\mu_I \to -\mu_I)], \quad (3)$$

where $E = \sqrt{p^2 + \sigma^2}$, and $d_q = 2 \times 2 \times N_f \times N_c$. Divergence of the vacuum energy (2) is regulated by a cutoff $\Lambda$. The mean field $\sigma$ is obtained by minimizing the thermodynamic potential (1). Non-zero value of the mean field $\sigma$ signals chiral symmetry breaking. Equivalently, one considers the quark condensate $\langle \bar{q}q \rangle = -\frac{\sigma}{G}$ as an order parameter.

Twisted boundary conditions for the fermion field $\psi(x, \tau) = e^{i\phi} \psi(x, \tau + \beta)$, are equivalent to setting $\mu_I = T(\phi - \pi)$, with $\phi \in [0, 2\pi)$. The statistically correct fermion degrees of freedom are obtained with $\phi = \pi$. However, it is important to notice, that using $\phi = 0$ does not make these fields bosons. Only by altering the overall sign in the vacuum and thermal one-loop contribution does one obtain a true Bose potential.

### A. No symmetry restoration at the boundary

We employ a $\sigma/\Lambda \ll 1$, and a $\sigma/T \ll 1$ expansion in the vacuum and thermal parts, respectively. The relevant expressions are well known in the literature: the vacuum part can be found in [13], and the thermal part e.g. in [14]. To discuss second order chiral phase transition, quadratic part has to contain vacuum and thermal fluctuations, while for the quartic part one can stick just to the vacuum contribution

$$\Omega(\sigma) = -\frac{1}{2} a(T, \phi) \sigma^2 - \frac{d_q}{64\pi^2} \log \left( \frac{\sigma^2}{4\Lambda^2} \right) \sigma^4, \quad (4)$$

where

$$a(T, \phi) = a_0 + \frac{d_q T^2}{2} B_2 \left( \frac{\phi}{2\pi} \right) = a_0 + \frac{d_q T^2}{8\pi^2} \left[ (\phi - \pi)^2 - \frac{\pi^2}{3} \right]. \quad (5)$$

and $B_2(x)$ is the second Bernoulli polynomial. The factor $a_0$ is just the quadratic vacuum contribution

$$a_0 = \frac{1}{G_c} - \frac{1}{G},$$

where $G_c \Lambda^2 = 8\pi^2/d_q$. For fermion boundary conditions $\phi = \pi$, the usual role of thermal fluctuations is to flip the sign of the quadratic term, marking the critical temperature.

In the case of general $\phi$ it is interesting that the thermal contribution of $a(T, \phi)$ itself can change sign. This happens at

$$\phi_{\pm} = \pi \pm \frac{\pi}{\sqrt{3}}.$$

* Details are omitted for simplicity, interested reader is invited to consult [2] or [4].
Namely, in the region $\phi_- < \phi < \phi_+$, which we call fermion-like, the model can be subjected to standard symmetry breaking-restoration scenario provided that the symmetry is broken in the vacuum, i.e. $G > G_c$. This is the usual case in the NJL model. However, at boson-like twisted angles $0 \leq \phi < \phi_-, \phi_+ < \phi < 2\pi$, the quark condensate will not respond to arbitrary high thermal excitations. In other words, the critical temperature obtained from the condition $a(T, \phi) = 0$

$$T_c(\phi) = \frac{8\pi^2}{d_q a_0} \frac{1}{\frac{2\pi}{\pi} - (\phi - \pi)^2},$$

(6)
diverges at the boundaries. For convenience we denote $T_\chi = T_c(\pi)$.

Thus, the only way for the mean field $\sigma$ to be zero at boson-like angles is by altering the theory by hand. For example, if we choose $a_0 < 0$, i.e. $G < G_c$ - then we find ourselves in a weird position where the model has a restored phase at low temperature, and a broken phase at high temperature. The other possibility would be to acknowledge the fact that bosonic theories perfectly well break and restore symmetries as the fermionic ones. More precisely, if we understand the vacuum contribution in (1) as a potential term in a classical bosonic $Z(2)$ theory, then the thermal contribution has to have an ad hoc sign change if the thermal fluctuations are also understood as coming from bosonic fields. Only then this bosonic theory breaks the symmetry at low, and restores it at high temperatures.

### B. Qualitative behaviour of the dressed Polyakov loop

Strictly speaking, the dPL can be defined only when the quark mass is non-zero [4]. Naively, one can still calculate this quantity by using its definition [4] as a first Fourier mode of the quark condensate at non-trivial twisted angle

$$\Sigma_1(T) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi} \langle \bar{q}q \rangle(T, \phi).$$

(7)

We will now use arguments of the previous subsection to qualitatively understand that $\Sigma_1(T)$ has to rapidly change across the chiral phase boundary.

First of all, by letting $T \to 0$ $\langle \bar{q}q \rangle(T, \phi)$ does not depend on $\phi$. This is because the generalized boundary conditions modify only the thermal part [8]†. Therefore, at $T \simeq 0$, we conclude that $\Sigma_1 \simeq 0$.

However, in the high temperature phase $T \gtrsim T_\chi$, chiral symmetry is first restored in a small region around $\phi = \pi$. This allows for a non-trivial Fourier transform (7), establishing a non-zero $\Sigma_1$. Therefore, it appears that as long as chiral symmetry is broken in the vacuum, i.e. $a_0 > 0$, the dPL will inevitably display a significant change which one might be tempted to interpret as confinement-deconfinement phase transition.

### III. DIVERGENCE OF THE TEMPERATURE DERIVATIVE OF THE DRESSED POLYAKOV LOOP

From a general thermodynamical point of view it is known that the phase transition leaves its mark on all thermodynamic quantities calculated from the partition function. For example, a second order chiral phase transition leads to a non-analyticity in the the second derivatives of the thermodynamic potential, like e.g. the chiral or thermal susceptibility, heat capacity and so on.

† Actually, this is tantamount of saying that the Polyakov loop $\Phi$ itself will be zero strictly at $T = 0$ regardless of the fact that the theory is confining or not. That is, even if the free energy $F$ of a static quark is finite, we have that $\Phi = e^{-F/T} = 0$ since $T = 0$. 

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Therefore, it is not unreasonable to expect that also the dPL in the NJL model should display similar properties not because the NJL model can say something about confinement as well as the confinement-deconfinement phase transition, but simply because it does a good job in describing the chiral one.

Let us now look for the temperature where the value of \( d\Sigma_1/dT \) has a maximum. By acknowledging the fact that \( \langle \bar{q}q \rangle \) can be zero at temperatures \( T > T_\chi \), we have

\[
\Sigma_1(T) = \int_0^{\phi_c(T)} d\phi \frac{\cos \phi \langle \bar{q}q \rangle(T, \phi)}{\pi},
\]

where the upper limit of integration is depending on the temperature, the specific values given by solving \( a(T, \phi) = 0 \) for \( \phi \)

\[
\phi_c(T) = \pi - \frac{\pi}{\sqrt{3}} \left( 1 - \frac{T_\chi^2}{T^2} \right)^{1/2}. 
\]

Using simple algebra, the quantity \( d\Sigma_1(T)/dT \) is given as

\[
\frac{d\Sigma_1}{dT} = \int_0^{\phi_c(T)} \frac{d\phi}{\pi} \cos \phi \frac{\partial \langle \bar{q}q \rangle(T, \phi)}{\partial T} + \frac{d\phi_c(T)}{dT} \frac{\partial}{\partial \phi} \left[ \frac{1}{\pi} \langle \bar{q}q \rangle(T, \phi) \cos \phi \right]_{\phi=\phi_c(T)}.
\]

Here

\[
\frac{d\phi_c(T)}{dT} = -\frac{\pi}{\sqrt{3}} \frac{1}{T} \frac{1}{T_\chi^2} \left( 1 - \frac{T_\chi^2}{T^2} \right)^{-1/2}.
\]

We now realize that (11) diverges as \( T \to T_\chi \) from above. If we naively assume that the first term in (10), as well as the one multiplying (11), is smooth across the phase transition, then \( d\Sigma_1/dT \) would have a maximum, or – more precisely – would diverge, exactly at \( T = T_\chi \).

In an actual calculation, it turns out that the critical behaviour itself is “one level” milder. The solution of the gap equation \( \partial \Omega/\partial \sigma = 0 \) with the truncated thermodynamic potential \( \Omega \) is given in terms of the well known Lambert \( W_{-1} \)-function [13, 15]. Thus, the thermal behavior of the condensate for general \( \phi \) can be approximated as

\[
\langle \bar{q}q \rangle(T, \phi) \simeq -\frac{2\Lambda}{\sqrt{G}} \exp \left[ -\frac{1}{4} - \frac{1}{2} W_{-1} \left( -\frac{4\pi^2 e^{1/2} a(T, \phi)}{d_q \Lambda^2} \right) \right] ,
\]
which is to be used only in the fermionic-like region. In the bosonic-like region the mass gap is finite, so the $\sigma/\Lambda, \sigma/T \ll 1$ expansion is no longer applicable, but we might just approximate the true solution in this region with its vacuum value. This is simply (12) with a replacement $a(T, \phi) \rightarrow a_0$. Then we can use this in order to integrate (5). We use the parameters of Ref. [10], where $GA^2 = 4.636$ and $\Lambda = 602.472$ MeV. Fig. 1 shows the result, where in the derivative of $dPL$ instead of the naive divergence, there is a sharp cusp structure.

It is important to stress that a similar cusp behaviour was seen in lattice QCD calculations in the strong coupling limit [17]. Thanks to the fact that the system is strongly coupled, deconfinement does not occur, so the change in the Polyakov loop, and in particular the cusp is indeed interpreted as an imprint of the chiral transition, see Fig. 2 and Fig. 3 in [17].

IV. CONCLUSIONS

Due to the fact that the NJL interaction dresses the quark with a momentum independent mass, the singularity structure of the quark propagator is very simple, which is usually interpreted as a lack of confinement †. However, calculation of the $dPL$ within NJL leads to a order parameter-like behaviour [10]. We interpret this apparent “confinement” as an imprint of the chiral transition. The temperature where $d\Sigma_1/dT$ diverges and the chiral transition temperature coincide exactly. Although this result was obtained numerically already in [10], the intention of this short Comment is to help clarify the meaning of this result.

We thus conclude that the thermal behaviour of the $dPL$ in the NJL model should not be taken as an indicator of the confinement-deconfinement phase transition. In fact, we have shown that, in the NJL model, this quantity carries no additional information.

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[1] A. M. Polyakov, Phys. Lett. B 72 (1978) 477.
[2] C. Gattringer, Phys. Rev. Lett. 97 (2006) 032003
[3] F. Synatschke, A. Wipf and C. Wozar, Phys. Rev. D 75 (2007) 114003
[4] E. Bilgici, F. Bruckmann, C. Gattringer and C. Hagen, Phys. Rev. D 77 (2008) 094007
[5] C. S. Fischer, J. Phys. G 32 (2006) R253
[6] C. S. Fischer, Phys. Rev. Lett. 103 (2009) 052003
[7] C. S. Fischer and J. A. Mueller, Phys. Rev. D 80 (2009) 074029
[8] M. Hopfer, M. Mitter, B.-J. Schaefer and R. Alkofer,
[9] K. Kashiwa, H. Kouno and M. Yahiro, Phys. Rev. D 80 (2009) 117901
[10] T. K. Mukherjee, H. Chen and M. Huang, Phys. Rev. D 82 (2010) 034015
[11] R. Gatto and M. Ruggieri, Phys. Rev. D 82 (2010) 054027
[12] A. Flachi, arXiv:1305.5848 [hep-th].
[13] K. Yagi, T. Hatsuda and Y. Miike, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. 23 (2005) 1.
[14] A. Actor, Nucl. Phys. B 265 (1986) 689.

† For example, in NJL model calculations this lack of confinement usually leads to the $\rho$ meson mass lying above the kinematic threshold for $q\bar{q}$ decay.
[15] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth, Adv. Comput. Math. 5 (1996) 329.
[16] H. Grigorian, Phys. Part. Nucl. Lett. 4 (2007) 223
[17] M. Fromm, J. Langelage, O. Philipsen, P. de Forcrand, W. Unger and K. Miura, PoS LATTICE 2011 (2011) 212