I review some recent results showing that the physics of negative energy densities, as predicted by relativistic quantum field theories, is more complicated than has generally been appreciated. On the one hand, in external potentials where there is a time-dependence, however slight, the Hamiltonians are unbounded below. On the other, there are limitations of quantum measurement in detecting or utilizing these negative energies.

1 Introduction

It has been known for a long time that in some sense negative energy densities are predicted by relativistic quantum field theory. However, it is only recently that we have begun to understand how often they occur, or indeed the precise sense in which they occur. There are a number of surprises in the area, and it may not be too much to say that we have been missing the essential physics of the situation.

The main surprise is that arbitrarily negative energy densities occur pervasively, even in tame, “everyday” situations. Much of this paper will outline how this occurs in a general quantum-field-theoretic setting. No one really knows, at present, how to resolve the difficulties pervasive negative energy densities might cause. I shall outline, at the end, some suggestions of mine, and their possible consequences, but this is very much an open area.

2 Instances of Negative Energies, Densities and Fluxes

Before discussing the recent general results, I recall some important instances of negative energy density phenomena.

Casimir was probably the first to find a negative energy density in quantum field theory. He showed that between two parallel perfect plane conductors separated by a distance \( l \) there was a renormalized energy

\[
E = -\frac{\pi^2 \hbar c}{720l^3}
\]  

per unit area. Although Casimir did not calculate the energy density, one can show on invariance grounds, given the very simple nature of the boundary
conditions, that the expectation of the energy density is constant, so

\[
\langle \hat{T}_{tt} \rangle_{\text{ren}} = -\frac{\pi^2 \hbar c}{720 l^4}
\]  

(2)

I should mention at this juncture that, while the associated force of attraction has been observed, this negative energy density has not. The force is essentially the component \(\hat{T}_{xz}\) of the stress–energy, whereas the energy density is \(\hat{T}_{tt}\). There is a relation between the two — the long–time average of the force is minus the gradient of the energy — but we do not at present have a measurement of the energy density operator, and it is hard to see how such a measurement might be made with current technology. Detecting the effect of the negative energy by, say, a Cavendish experiment would require plates of the order of a light–year on a side.

A second instance of negative energy density was found by Hawking, at the horizon of a black hole\(^2\). The argument here is indirect, but very interesting because one sees the physical assumptions: Hawking first computed the famous blackbody spectrum of radiation from a black hole, on the assumption that it had formed at some finite time in the past. Given that positive energy is being radiated to infinity, the energy must come from somewhere, and the only source is the hole. If the hole is losing energy, and we may apply classical general relativity at the horizon, then the Area Theorem (also due to Hawking) implies a negative effective energy density as the horizon\(^3\).

A third instance of negative energy, more particularly of negative energy fluxes, occurs in the radiation from moving mirrors, first studied by Davies and Fulling\(^4,5\). Here, in the case of one spatial dimension, one has excellent analytic control over the model, so the physical issues show up very clearly. One finds in particular correlations between negative and positive energy fluxes, and restrictions on the negative fluxes.

A fourth instance of negative energies occurs with “supercritical” external potentials: potentials strong enough to engender pair creation\(^6\).

Each of these examples is of interest in its own right, has stimulated a great deal of work, and has led to a deeper understanding of quantum field theory. On the other hand, it has not really been clear whether there is any unified theory of negative energy densities, as quantum–field–theoretic phenomena, although there has been some work in this direction. The seminal paper is by Epstein, Glaser and Jaffe\(^7\). It is a theorem in axiomatic field theory, and it asserts, essentially, that if a theory is Poincaré–invariant, possesses a ground

\(^a\)Recent computations confirm this, and suggest that the energy density is negative (for at least some observers) everywhere outside the hole, becoming unambiguously positive only at infinity.\(^8\)
state, and possesses an energy density operator $\hat{\rho}$, then $\hat{\rho}$ cannot be a non-negative operator. In other words, the total energy

$$\hat{H} = \int \hat{\rho} \, d^3x$$

(3)

is positive, but for any positive smooth compactly-supported (that is, vanishing outside a bounded set) function $f(x, y, z)$, the operator

$$\hat{H}(f) = \int \hat{\rho} f \, d^3x$$

(4)

has a spectrum including negative numbers.

While this result proves, for example, that the energy density operator for the Klein–Gordon field in Minkowski space, if it exists, is not positive, it bears no direct link to any of the instances of negative energies above. This is because none of those configurations is Poincaré-invariant.

These various occurrences of negative energies are not the same physically. We shall have a better idea of this when we examine the structures of the Hamiltonians, below.

3 Some General Results

In the past few years, some constructive results on the stress–energy have become available. These build on the work of many authors; I shall only name Wald as one of the prime movers in this area, and direct the reader to his recent book for other references.

Brunetti, Fredenhagen and Köhler showed that, for a linear Bose field in curved space–time, the stress–energy operator was well-defined (modulo a well-known c-number ambiguity) as an operator–valued distribution. What this means is that for any tensor–valued test function $f^{ab}$, smooth on space–time and with compact supports, the quantity

$$\int \hat{T}^{ab} f^{ab} \, d\text{vol}$$

(5)

was a self–adjoint operator. We shall see below that this result is quite important. At first blush, however, it seems not quite one wants for the measurement of energy or the evolution of the quantum fields.

One does not usually think of integrating the stress–energy over a space–time volume; it has a more direct interpretation as determining Hamiltonian operators

$$\hat{H}(\xi, \Sigma) = \int \hat{T}_{ab} \xi^a d\Sigma^b.$$  

(6)
This operator is the generator of evolution along the vector field $\xi$ at the Cauchy surface $\Sigma$; it is a sort of weighted energy (or momentum, or angular momentum, according to the character of $\xi^a$) operator at $\Sigma$. For these operators, the results are negative.

Suppose that $\xi^a$ is not identically a Killing vector field at $\Sigma$. Then

(a) $\hat{H}(\xi, \Sigma)$ does not exist as a self-adjoint operator. It does exist as a Hermitian form (that is, its expectation values are well-defined on a dense set of states); but

(b) $\hat{H}(\xi, \Sigma)$ is unbounded below; the set of states on which its expectation value is $-\infty$ is dense in the physical Hilbert space;

(c) $\hat{H}(\xi, \Sigma)$ does in a well-defined way generate an evolution of the field algebra, but that evolution is not unitarily implementable.

All these phenomena occur whenever there is a departure, no matter how slight, from $\xi^a$ being the generator of a space-time symmetry. In particular, if a time-dependent gravitational field is present, no matter how small, there are no timelike Killing vectors, and so all of these phenomena are unavoidable. They are all local, and can be manifested as certain ultraviolet divergences.

In the next section, I will give a sketch of how such things can occur in apparently innocuous situations. For the moment, though, I want to be quite explicit about what these results mean. To some extent, they relate to odd phenomena which have previously been encountered in quantum field theory, but whose significances have not always been thought out.

Property (a) means that the Hamiltonian cannot be interpreted as an observable within conventional quantum theory. (It is the self-adjoint operators which have spectral representations; Hermiticity, a weaker property for unbounded operators, does not suffice.) Operators which exist only as forms are known in other contexts in quantum field theory, for example in current algebra. However, it is a rather serious modification of the theory to say that the Hamiltonian exists only weakly.

Property (b), that the expectations of the energy are unbounded below, is perhaps the most serious. In the ordinary course of physics, one would suspect this to lead to instabilities and a breakdown of the theory. Indeed, something like this is known to happen in “supercritical” external potentials, where the naïve vacuum becomes unstable against pair creation. While there is more than a passing connection between the present situation and supercritical

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\[ ^{\text{a}} \text{A slightly weaker statement was proved in ref. 10; a full proof is in ref. 11.} \]
potentials, it should be emphasized that \( \hat{H}(\xi, \Sigma) \) is unbounded below even in the presence of arbitrarily weak external fields.

A full discussion of the non–unitarity (property (c)) will be given elsewhere; here I shall only make precise the sense in which it occurs.

In quantum theory, as is well–known, there are two sorts of evolution: that of the state vectors, and that of the fields. The state vectors are required to evolve by unitary motions (except when reduction occurs). However, the evolution of the fields is determined by the field equations, and may or may not be unitarily implementable (that is, achievable by conjugation with a unitary operator on the Hilbert space). It is often thought that these two sorts of evolution are simply two sides of the same coin, and unitarity in one sense implies it in the other, but this is incorrect.

In fact, the physical occurrence of non–unitarily implementable evolutions has been known in some sense since the study of infrared effects in quantum electrodynamics by Bloch and Nordsieck. The general picture that emerged from their analysis was that a scattering involving only a finite exchange of energy could give rise to a final state with infinitely many soft photons, and this infinity could be “big” enough to take the final state out of the initial Hilbert space. In modern parlance, the evolution is not unitarily implementable. However, notice that this same argument would rule out any ultraviolet non–unitarity. What we have found is that this last conclusion is wrong. The argument for it is wrong because energy is not a classical quantity, but an operator, and there are in general complicated interference effects between positive– and negative–energy states.

4 Structure of the Hamiltonians

In this section, I shall examine the structure of the Hamiltonians of linear bose theories to show how the phenomena discussed above appear. A rigorous treatment of this is technical, and the reader is referred to the literature for that. Here the goal is conceptual.

Consider the usual energy operator for a Klein–Gordon field in Minkowski space. We may write this operator in the form

\[
\hat{H} = \int \hat{T}_{ab}^i \Sigma^b = B^i_j \hat{a}_i^* \hat{a}_j,
\]

where \( \Sigma^b = t^b dx dy dz \), the \( \hat{a}_i \), \( \hat{a}_i^* \) are annihilation and creation operators, and \( B^i_j \) are coefficients forming a self–adjoint matrix. (One can allow “continuous matrices,” too; this distinction will be unimportant. So in a momentum representation \( B^i_j \rightarrow B(k, q) = \delta(k - q)(m^2 + k^2)^{1/2} \), as usual.)
Now imagine perturbing this situation, but staying within the category of linear field theories. Such a perturbation might include:

- Allowing a small time- and space-dependent gravitational field;
- Allowing other sorts of time-dependent external potentials;
- Measuring not exactly the energy, but some other weighted average of the stress-energy operator.

The last might be interesting, for example, in problems in quantum measurement theory. Suppose we want to take into account the fact that we can never perfectly synchronize clocks at different spatial locations. Then it seems we ought to allow perturbations of the vector field $t^a$ (and of the surface $\Sigma$).

If we allow such perturbations, then we find that the Hamiltonian will have the general form

$$\hat{H} = A_{ij} \hat{a}_i^* \hat{a}_j + B^i_j \hat{a}_i^* \hat{a}_j + A_{ij} \hat{a}_i^* \hat{a}_j^* + c - \text{number term}.$$  \hfill (8)

Here $A_{ij}$ and $B^i_j$ are coefficients (symmetric and self-adjoint, respectively). The operators $\hat{a}$ and $\hat{a}^*$ are still annihilation and creation operators, although in general situations they might not refer to particles, but to other field modes. (The interpretation can only be determined by analysis of the particular system and mode decomposition.) The c-number term can be quite interesting — it includes Casimir–type effects — but I shall have nothing more to say about it here. Thus the new features that we are concerned with are the $A_{ij}$ terms. The presence of these terms indicates that evolution does not preserve the decomposition of field operators into annihilation and creation parts.

In the case of perturbing a Klein–Gordon energy operator, one would expect the coefficients $A_{ij}$ to be small, and of $B^i_j$ to be close to those of the unperturbed case. In all of the cases considered here, this is true: in a suitable sense, the $A_{ij}$ are uniformly small, and the $B^i_j$ uniformly close to their unperturbed values. However, just having the coefficients uniformly small is not enough to rule out pathologies, because in a field theory there are infinitely many modes.

There is a very simple example of this. If $|0\rangle$ is the vacuum state (that is, the state annihilated by all $\hat{a}$), then

$$\hat{H} |0\rangle = \mathcal{A}^{ij} \hat{a}_i^* \hat{a}_j |0\rangle.$$  \hfill (9)

Another difficulty is that the terms $A_{ij} \hat{a}_i^* \hat{a}_j$, $\mathcal{A}^{ij} \hat{a}_i^* \hat{a}_j$ are unbounded operators.
It is easy to see that this state is normalizable iff $A_{ij}A^{ij} < \infty$. However, this condition can very well be violated, even if the $A_{ij}$ are uniformly small. It turns out that, if one calculates $A_{ij}$ in any of the situations listed above, one does find $A_{ij}\overline{A}^{ij} = \infty$, and the divergence is an ultraviolet one. This means that the vacuum cannot be in the domain of $\hat{H}$. (By itself, this does not prove that the Hamiltonian cannot exist as a self-adjoint operator. One has to worry about whether there might be some recondite domain of “dressed” states on which the Hamiltonian turns out to be o.k. However, it turns out that this does not occur.)

Now suppose for the moment we had a finite-dimensional system instead of a field theory. Then we would know that at the classical level there was a canonical transformation taking the system to a direct sum of harmonic oscillators. This means that there is a canonical basis of the classical phase space relative to which the classical Hamiltonian vector field, which

$$V = i \begin{bmatrix} B & 2A \\ -2A & -B \end{bmatrix}$$

(10)

in a natural complex basis associated to the creation and annihilation operators, breaks down into a sum of $2 \times 2$ blocks, each of the form $\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, with $\omega$ the angular frequency of the oscillator. If $V$ is the classical vector field, then, the ground-state energy in terms of this new canonical basis is simply $(1/2)\sum \omega = (1/4)\text{tr}_R |V|$, where $|V| = \sqrt{-V^2}$. Now, the ground-state energy in the original basis is just $(1/2)\text{tr}_C B = (1/4)\text{tr}_R B$. Thus the renormalized ground state energy is

$$E_{\text{ground}} = \text{tr}_R (|V| - B) / 4.$$  

(11)

In the case of fields, this formula is much more delicate to justify rigorously, and indeed needs to be qualified in various ways. Still, we might take the attitude that the cases in which this fails (and so the field theory cannot be well-approximated by a sequence of quantum mechanical theories) as in themselves pathological. It turns out that in many interesting cases, the $A$’s are “small” and their commutators with $B$ are negligible to first order, and then one finds simple approximations $|V| \approx \sqrt{B^2 - 4AA}$, and $E_{\text{ground}} \approx -(1/2)\text{tr}_R B^{-1}\overline{A}A$. This trace is ultraviolet divergent in cases of interest.

The foregoing are sketches of how the lack of self-adjoint implementability of $\hat{H}$ and unboundedness–below arise. The proof of non-unitary implementability of evolution is a bit less direct (since one has to make a statement about finite evolutions, but it is too hard to integrate the equations of motion).
I will close this section with a few words about how these negative–energy phenomena relate to others. As mentioned above, Casimir–type effects are subsumed in the $c$–number term, and can be viewed as a kinematic rather than a dynamic effect. In all known examples, they are locally finite. The moving–mirror models are fields with (singular) time–dependent potentials; they in effect have $A_{ij} \neq 0$. The case of Hawking radiation is subtle, as the time–dependence is encoded in the assumption that the black hole formed at some finite time; this model has $A_{ij} \neq 0$ as well. For “supercritical” potentials, the modification of the theory is severe enough that the present Hamiltonian structure breaks down entirely. This is clearest in the case of time–independent potentials. For these, when one tries to construct the one–particle Hilbert space, one finds modes with imaginary fundamental frequencies $\omega$. Formally speaking, in the time–independent case, the coefficients $A_{ij} = 0$ and $B_{ij}$ acquires a non–self–adjoint part. However, this formal behavior is only a signal that the model has broken down; one must look at the physics of the situation to understand what actually happens. (One must quantize in another representation of the CCR, with a “dressed” vacuum.)

5 Towards a Resolution of the Difficulties

The most disturbing features of the negative energies uncovered above are their unboundedness–below and their pervasiveness. Given these features, why have we not already seen negative energy–density phenomena? Why do we not see ordinary particles absorb negative energies and become tachyons? Why do not small perturbations (which are always present) send the field cascading through increasingly negatively energetic states, while emitting positive–energy radiation? There must be some mechanism restricting negative energies, their production, their duration, or their interaction with ordinary fields.

An important such restriction was discovered by Ford, and elaborated by him and co–workers.\footnote{One can also average over spatial directions; taking this into account would strengthen all the arguments below. However, the key thing turns out to be temporal averaging.} Starting from the physical observation that one does not measure energies (or energy densities) at mathematical points, but always with some finite averaging, he demonstrated that the stress–energy operator for the Klein–Gordon field satisfies a quantum inequality,

$$\langle \Psi | \int_{-\infty}^{\infty} T_{tt}(t, 0, 0, 0) b(t) \, dt | \Psi \rangle / \langle \Psi | \Psi \rangle \geq -3\hbar c / (32\pi^2 (ct_0^4)),$$

(12)

where the sampling function $b(t) = (t_0/\pi) / (t^2 + t_0^2)$ has characteristic width $\sim t_0$ and area unity. Thus one can have very negative energy densities for
short times, but, averaged over longer times, the energy density must be more nearly zero.

These inequalities point up an important issue. When we are considering the quantum stress–energy, and especially whenever negative energy–density phenomena are concerned, we should speak of the energy in a regime, where this term includes not only a spatial extent but a scale of temporal averaging. It is generally wrong to suppose that there is a well–defined energy density (even as a quantum operator) independent of the temporal averaging scale.

I want now to combine the quantum inequality, which provides a mathematical restriction on the stress–energy, with limitations of quantum measurement theory. Suppose an isolated device measures or traps the energy of a quantum field in a regime. If the device is to trap a negative energy, then according to the quantum inequalities it must turn on and off on a finite time scale, say \( \sim t_0 \), and so it must contain a clock resolving times over order \( \sim t_0 \).

By causality and the quantum inequality, the magnitude of any negative energy detected or trapped is bounded by

\[
|E_{\text{neg}}| \leq \left[ \frac{3\hbar c}{2\pi^2 (ct_0^4)} \right] \cdot \left( \frac{4\pi}{3} \right) (ct_0)^4 = \frac{\hbar}{8\pi t_0}.
\]  

Now there is an old argument, going back at least to the Bohr–Einstein dialog, that a clock which can resolve times of order \( t_0 \) must have a rest energy \( \gtrsim \hbar/t_0 \). This is about 25 times as great as the negative energy detected!

This suggests a general principle, which I shall refer to by analogy with the energy conditions of classical general relativity: Operational Weak Energy Condition (OWEC): The energy of a quantum field in a regime, plus the energy of an isolated device in that regime measuring or trapping the field’s energy, must be non–negative.

I would like to emphasize that while the argument for the OWEC is good, it is not a rigorous proof. Nevertheless, the form of the condition is so suggestive, and the factor of \( 8\pi \) so in excess of unity, that the OWEC seems worthwhile at least investigating as a hypothesis.

The OWEC would on its face rule out the conversion of ordinary particles to tachyons by absorption of negative energies. It is not yet known whether the OWEC precludes other pathologies; one must investigate each in turn.

\*Here are the caveats. First, the inequality \( E_{\text{clock}} \gtrsim \hbar/t_0 \) is only known as an order–of–magnitude relation. Second, we do not at present have mathematically proved quantum inequalities in curved space–time, although there are excellent reasons for thinking that at a local level there are similar results. Third, it is not possible to say anything rigorous about non–linear quantum fields. Finally, the OWEC could in principle be violated if there were a sufficiently large number of elementary field species.
Whether the OWEC rules out all negative–energy pathologies or not, it is interesting to investigate as a candidate quantum analog of the classical weak energy condition (WEC) in general relativity. This condition is a foundation of the most important results in classical relativity: the Area Theorem, the singularity theorems, and the positivity–of–energy theorems.

If the OWEC holds, and some version of Einstein’s equation holds with the stress–energy operator $\hat{T}_{ab}$ as a source, then in a negative–energy regime, one could never measure the geometry of space–time sufficiently accurately by direct local means to establish unambiguously the negativity of the energy density via Einstein’s equation. In other words, in a negative–energy regime, there are quantum limitations on the measurement of space–time geometry. Given the pervasiveness of negative energies, this suggests that “quantum space–time” phenomena may be accessible at ordinary scales. The potential significance of this would be hard to overstate.

These ideas are discussed more extensively in refs. 15, 16.

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†The WEC asserts that the energy density measured by any observer is non–negative.