Endomorphisms of the projective plane

Oliver Röndigs

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Abstract

The endomorphism ring of the projective plane over a field $F$ of characteristic neither two nor three is slightly more complicated in the Morel-Voevodsky motivic stable homotopy category than in Voevodsky’s derived category of motives. In particular, it is not commutative precisely if there exists a square in $F$ which does not admit a sixth root. A byproduct of the computations is a proof of Suslin’s conjecture on the Suslin-Hurewicz homomorphism from Quillen to Milnor $K$-theory in degree four, based on work of Asok, Fasel, and Williams [AFW20].

1 Introduction

Automorphisms of geometric objects describe their symmetries, and hence important geometric information. It depends on the context which type of morphisms are considered useful. In the case of a projective space $\mathbb{P}^n$ over the complex numbers, one may consider its linear automorphisms (a group denoted $\text{PGL}_{n+1}(\mathbb{C})$), birational automorphisms (the Cremona group $\text{Cr}_n(\mathbb{C})$), diffeomorphisms, homeomorphisms, and self-homotopy-equivalences, just to name a few. The Morel-Voevodsky $\mathbf{A}^1$-homotopy theory provides an interesting way to consider self-homotopy-equivalences of varieties [MV99]. Although this setup is conceptionally very satisfying, concrete determinations of endomorphisms in the $\mathbf{A}^1$-homotopy category are hard to come by. For example, the endomorphism ring of $\mathbb{P}^1$ over a perfect field $F$ in the pointed $\mathbf{A}^1$-homotopy category is given by the Grothendieck ring of isomorphism classes of symmetric inner product spaces over $F$ with a chosen basis, where the isomorphisms preserve the inner product and have determinant 1 with respect to the chosen bases [Mor12, Remark 7.37].

Stabilization with respect to smashing with a projective line $\mathbb{P}^1 \wedge -$ provides a simpler categorical setting, the motivic stable homotopy category $\text{SH}(F)$, which is still richer than the corresponding derived category of motives [Voe98]. Part of the gain from leaving the unstable realm is an additive (in fact triangulated) structure, whence the set of endomorphisms of any object is always a ring. For example, it is a deep theorem of Morel’s that the endomorphism ring of the projective line in the motivic stable homotopy category
over a field $F$ is the Grothendieck-Witt ring of symmetric bilinear forms with coefficients in $F$ [Mor04]. The addition in the Grothendieck-Witt ring, whose elements are formal differences of symmetric bilinear forms, is induced by orthogonal sum, and the multiplication by tensor product of forms. By construction, it coincides with the endomorphism ring of the unit for the symmetric monoidal structure given by the smash product. Hence it has to be commutative. This is already different for the projective plane.

**Theorem.** Let $F$ be a field of characteristic neither 2 nor 3, with group of units $F^\times$. The endomorphism ring $[\mathbf{P}^2,\mathbf{P}^2]_{\text{SH}(F)}$ in the motivic stable homotopy category of $F$ has an underlying additive group isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus F^\times/(F^\times)^6$. The multiplication corresponds to the multiplication given by

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 y_1, x_1 y_2 + x_2 y_1, x_1 y_3 + x_3 y_1 + 2x_3 y_2).$$

In particular, the ring $[\mathbf{P}^2,\mathbf{P}^2]_{\text{SH}(F)}$ is non-commutative if and only if there exists a square in $F$ which does not admit a sixth root, or, equivalently, if the cube map $u \mapsto u^3$ is not surjective on $F$. Its group of units (which could be called the group of $\mathbf{P}^1$-stable self-$\mathbf{A}^1$-homotopy-equivalences of $\mathbf{P}^2$) consists of all triples $(x_1, x_2, x_3) \in \mathbb{Z} \oplus \mathbb{Z} \oplus F^\times/(F^\times)^6$ where either $x_1 = \pm 1$ and $x_2 = 0$, or $x_1 = \pm 1$ and $x_2 = -x_1$. It is as non-commutative as the endomorphism ring it belongs to. Along the way, the homotopy modules $\pi_1 \mathbf{P}^2$ and $\pi_2 \mathbf{P}^2$ will be determined, based on computations in [RSØ19] and [Rön20]. Moreover, a related computation of $\pi_3 \mathbf{P}^2 \wedge \mathbf{P}^2$ provides an ingredient to complete the program Aravind Asok, Jean Fasel and Ben Williams developed in [AFW20] to prove Suslin’s conjecture on the Suslin-Hurewicz homomorphism from Quillen to Milnor $K$-theory in degree four.

**Theorem.** Let $F$ be an infinite field of characteristic different from 2 and 3, and $A$ an essentially smooth local $F$-algebra. The image of the Suslin-Hurewicz homomorphism $K^\text{Quillen}_4(A) \to K^\text{Milnor}_4(A)$ coincides with $6K^\text{Milnor}_4(A)$.

## 2 Topology

Let $\mathbb{C}P^n$ denote complex projective space of complex dimension $n$. This section contains rather elementary calculations in the classical stable homotopy theory, which determine the endomorphism ring of $\mathbb{C}P^2$ in the stable homotopy category. These calculations are based on stable homotopy groups of spheres $\pi_m S$ in degree $m < 6$ and the action of the topological Hopf map $\eta: S^3 \to \mathbb{C}P^1 \cong S^2$, whose cofiber is $\mathbb{C}P^2$, on them. The standard reference here is [Tod62]. As is customary in stable homotopy theory, the notation for a map and its (de)suspensions coincide if the context allows it. The purpose of this section is not to present original results (there aren’t any), but instead to document the similarities and differences to the situation in the motivic stable homotopy category.

Choose a basepoint for $\mathbb{C}P^1$, and hence $\mathbb{C}P^2$, which will not appear in the notation. The cofiber sequence

$$S^3 \xrightarrow{\eta} \mathbb{C}P^1 \xrightarrow{i} \mathbb{C}P^2 \xrightarrow{q} S^4$$

(2.1)
induces a long exact sequence of stable homotopy groups

\[ \cdots \xrightarrow{\eta} \pi_m \mathbb{CP}^1 = \pi_{m-2} S \xrightarrow{i} \pi_m \mathbb{CP}^2 \xrightarrow{q} \pi_m S^4 = \pi_{m-4} S \xrightarrow{\eta} \pi_{m-1} \mathbb{CP}^1 = \pi_{m-3} S \to \cdots \]

terminating with \( \pi_2 \mathbb{CP}^1 = \pi_2 \mathbb{CP}^2 \). The induced short exact sequences

\[ 0 \to \pi_{m-2} S / \eta \pi_{m-3} S \to \pi_m \mathbb{CP}^2 \to \eta \pi_{m-4} S \to 0 \]

express the stable homotopy group of the complex projective plane as an extension of two groups, the subgroup annihilated by \( \eta \), and the cokernel of multiplication by \( \eta \), on the respective stable homotopy group of spheres. Since \( \eta: \pi_0 S \cong \mathbb{Z} \to \pi_1 S \cong \mathbb{Z}/2\mathbb{Z} \) is surjective, \( \eta: \pi_1 S \to \pi_2 S \) is an isomorphism, \( \eta: \pi_2 S \to \pi_3 S \cong \mathbb{Z}/24 \) is injective, and \( \pi_4 S \cong \pi_5 S \cong 0 \), the following table results, without any extension problem to solve.

| \( m \) | 2 | 3 | 4 | 5 | 6 | 7 |
|-------|---|---|---|---|---|---|
| \( \pi_m \mathbb{CP}^2 \) | \( \mathbb{Z} \) | 0 | 2\( \mathbb{Z} \) | \( \mathbb{Z}/12 \) | 0 | \( \mathbb{Z}/24 \) |

Here \( 2\mathbb{Z} \) denotes the abelian group of even integers under addition. The portion for \( m < 5 \) of this table implies that the cofiber sequence (2.1) induces a short exact sequence

\[ 0 \to \pi_4 \mathbb{CP}^2 = [S^4, \mathbb{CP}^2] \xrightarrow{q} [\mathbb{CP}^2, \mathbb{CP}^2] \xrightarrow{i^*} [\mathbb{CP}^1, \mathbb{CP}^2] = \pi_2 \mathbb{CP}^2 \to 0 \quad (2.2) \]

and hence the abelian group \([\mathbb{CP}^2, \mathbb{CP}^2]\) is an extension of two free abelian groups, each on one generator. Since \( \pi_2 \mathbb{CP}^2 \cong \mathbb{Z} \), the short exact sequence (2.2) splits. In order to describe the ring structure, it helps to be more specific. A generator for \( \pi_2 \mathbb{CP}^2 \) is the inclusion \( i: \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \). It is the image of \( \text{id}_{\mathbb{CP}^2} \) under \([\mathbb{CP}^2, \mathbb{CP}^2]\) \( i^* \) \([\mathbb{CP}^1, \mathbb{CP}^2]\). To describe a generator for \( \pi_4 \mathbb{CP}^2 \), observe that there exists a unique map \( \omega: S^4 \to \mathbb{CP}^2 \) such that \( q \circ \omega = 2\text{id}_{S^4} \). The short exact sequence (2.2) then implies that every element \( x \in [\mathbb{CP}^2, \mathbb{CP}^2] \) can uniquely be expressed as a sum \( x_1 \text{id}_{\mathbb{CP}^2} + x_2 (\omega \circ q) \), where \( x_1, x_2 \in \pi_0 S \cong \mathbb{Z} \). The ring structure is then given as

\[
\begin{align*}
x \circ y &= (x_1 \text{id}_{\mathbb{CP}^2} + x_2 (\omega \circ q)) \circ (y_1 \text{id}_{\mathbb{CP}^2} + y_2 (\omega \circ q)) \\
&= x_1 y_1 \text{id}_{\mathbb{CP}^2} + (x_1 y_2 + x_2 y_1) (\omega \circ q) + x_2 y_2 (\omega \circ q \circ \omega \circ q) \\
&= x_1 y_1 \text{id}_{\mathbb{CP}^2} + (x_1 y_2 + x_2 y_1 + 2x_2 y_2) (\omega \circ q)
\end{align*}
\]

and is in particular commutative. The group of units consists of the following 4 elements: \( \{\text{id}_{\mathbb{CP}^2}, -\text{id}_{\mathbb{CP}^2}, \text{id}_{\mathbb{CP}^2} - \omega \circ q, -\text{id}_{\mathbb{CP}^2} + \omega \circ q\} \) Of course knowledge does not stop at the dimension two. For example, [Muk93] determines the groups \([\mathbb{CP}^n, \mathbb{CP}^n]\) for \( n \leq 7 \). The complex dimension 7 is the smallest dimension where this group contains torsion; in fact, \([\mathbb{CP}^7, \mathbb{CP}^7]\) \( \cong \mathbb{Z}^7 \oplus \mathbb{Z}/2\mathbb{Z} \). The ring structure is commutative in all these dimensions.

The table above allows to determine \([\Sigma \mathbb{CP}^2, \mathbb{CP}^2]\) as well, which is useful, because this group contains the interesting element \( \eta \text{id}_{\mathbb{CP}^2} = \eta \wedge \mathbb{CP}^2 \). Since \( \pi_3 \mathbb{CP}^2 \) is the zero
group, there results an isomorphism $\pi_5 \mathbb{C}P^2/\eta \pi_4 \mathbb{C}P^2 \cong [\Sigma \mathbb{C}P^2, \mathbb{C}P^2]$. As $\pi_5 \mathbb{C}P^2$ is cyclic, generated by $i \circ \nu$, the abelian group $[\Sigma \mathbb{C}P^2, \mathbb{C}P^2]$ is generated by the map

$$\Sigma \mathbb{C}P^4 \xrightarrow{q} S^5 \xrightarrow{\nu} S^2 = \mathbb{C}P^1 \xrightarrow{i} \mathbb{C}P^2.$$ 

Its order can be determined by identifying $\omega \circ \eta \in \pi_5 \mathbb{C}P^2$, which is $\pm 6(i \circ \nu)$, as the Toda bracket $\langle \eta, 2, \eta \rangle = \{6\nu, -6\nu\}$ shows. Hence $[\Sigma \mathbb{C}P^2, \mathbb{C}P^2]$ is cyclic of order 6. The element $\eta \text{id}_{\mathbb{C}P^2}$ turns out to be the unique nonzero element of order 2, as the Toda bracket $\langle \eta, q = q \circ \omega, \eta \rangle$ also implies. The properties of Toda brackets supply an inclusion $\langle \eta, q = q \circ \omega, \eta \rangle \subset \langle \eta, q = q \circ \omega, \eta \rangle = \{6\nu, -6\nu\}$ which shows that it does not contain zero. More precisely, since the composition $\pi_3 S \xrightarrow{q^*} [\Sigma^2 \mathbb{C}P^2, S^3] \xrightarrow{\omega^*} \pi_3 S$ is multiplication with 2, one obtains $\langle \eta, q = q \circ \omega, \eta \rangle \subset \{3\nu \circ q, -3\nu \circ q\}$. Hence the identity $\eta \text{id}_{\mathbb{C}P^2} \in [\mathbb{C}P^2, \mathbb{C}P^2]$ satisfies

$$\eta \text{id}_{\mathbb{C}P^2} = \eta \wedge \mathbb{C}P^2 = \pm 3(i \circ \nu \circ q). \quad (2.3)$$ 

For comparison purposes with the motivic situation, it is instructive to look at the real case as well. Let $\mathbb{R}P^n$ denote real projective space. The cofiber sequence

$$S^1 \xrightarrow{2} \mathbb{R}P^1 \xrightarrow{i} \mathbb{R}P^2 \xrightarrow{q} S^2 \quad (2.4)$$

induces a long exact sequence of stable homotopy groups

$$\cdots \xrightarrow{2} \pi_m \mathbb{R}P^1 = \pi_{m-1} S \xrightarrow{i} \pi_m \mathbb{R}P^2 \xrightarrow{q} \pi_m S^2 = \pi_{m-2} S \xrightarrow{2} \pi_{m-1} \mathbb{R}P^1 = \pi_{m-2} S \to \cdots$$ 

terminating with $\pi_1 \mathbb{R}P^1 \to \pi_1 \mathbb{R}P^2$. The induced short exact sequences

$$0 \to \pi_{m-1} S/2\pi_{m-1} S \to \pi_m \mathbb{R}P^2 \to 2\pi_{m-2} S \to 0$$

express the stable homotopy group of the real projective plane as an extension of two groups, the 2-torsion subgroup, and the cokernel of multiplication by 2, on the respective stable homotopy group of spheres. The groups $\pi_3 \mathbb{R}P^2$ and $\pi_4 \mathbb{R}P^2$ are both extensions of $\mathbb{Z}/2 \mathbb{Z}$ by $\mathbb{Z}/2 \mathbb{Z}$. The Toda bracket $\langle 2, \eta, 2 \rangle = \{\eta^2\}$ implies that $\pi_3 \mathbb{R}P^2$ is given by the nontrivial extension, the extension for $\pi_4 \mathbb{R}P^2$ turns out to be trivial. The following table results.

| m   | 1    | 2    | 3    | 4    | 5    | 6    |
|-----|------|------|------|------|------|------|
| $\pi_m \mathbb{R}P^2$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | 0 |

\(^1\text{It would be better to use } -2 \text{ and not } 2, \text{ since the real realization of the algebraic Hopf map is negative.}\)
The portion for \( m < 3 \) of this table implies that the cofiber sequence (2.4) induces a short exact sequence

\[
0 \to \pi_2 \mathbb{RP}^2 = [S^2, \mathbb{RP}^2] \xrightarrow{q^*} [\mathbb{RP}^2, \mathbb{RP}^2] \xrightarrow{i^*} [\mathbb{RP}^1, \mathbb{RP}^2] = \pi_1 \mathbb{RP}^2 \to 0 \tag{2.5}
\]

and hence the abelian group \([\mathbb{RP}^2, \mathbb{RP}^2]\) is an extension of two groups of order two. As in the computation of \(\pi_3 \mathbb{RP}^2\), the extension is nontrivial, meaning that id_{\mathbb{RP}^2} \in [\mathbb{RP}^2, \mathbb{RP}^2]\) is an element of order 4, as already proven in [Bar55]. There cannot be any doubt whatsoever regarding the ring structure of \([\mathbb{RP}^2, \mathbb{RP}^2]\).

3 Over a field

Let \( F \) be a field, and let \( \text{SH}(F) \) denote the motivic stable homotopy category of \( F \) [Voe98]. For a motivic spectrum \( E \in \text{SH}(F) \) and integers \( s, w \in \mathbb{Z} \), let \( \pi_{s,w}E \) denote the abelian group \( [\Sigma^{s,w}1, E] \), where \( E \) is a motivic spectrum and \( 1_F = 1 \) is the motivic sphere spectrum. The grading conventions are such that the suspension functor \( \Sigma^{2,1} = \Sigma^{1+(1)} \) is suspension with \( \mathbb{P}^1 \), and \( \Sigma^{1,0} = \Sigma^{1+(0)} = \Sigma = \Sigma \) is suspension with the simplicial circle.

Set \( \pi_{s+(w)}E := \pi_{s+w,w}E \), and let

\[
\pi_{s+(*)}E = \bigoplus_{w \in \mathbb{Z}} \pi_{s+w,w}E
\]

denote the direct sum, considered as a \( \mathbb{Z} \)-graded module over the \( \mathbb{Z} \)-graded ring \( \pi_{0+(*)}1 \).

The notation \( \pi_{s-(*)}E := \pi_{s-(*-)}E \) will be used frequently. The strictly \( A^1 \)-invariant sheaf obtained as the associated Nisnevich sheaf of \( U \mapsto \pi_{s,w}E_U \) for \( U \in \text{Sm}_F \) is denoted \( \overline{\pi}_{s,w}E \), which gives rise to the homotopy module \( \pi_{s+(*)}E \). In the following, every occurrence of “\( \pi \)” can be replaced by “\( \overline{\pi} \)” without affecting the truth of the (suitably reinterpreted) statements. See [Mor04] for the following fundamental result.

**Theorem 3.1 (Morel).** Let \( F \) be a field. Then \( \pi_{0-(*)}1 \) is the Milnor-Witt \( K \)-theory of \( F \).

The Milnor-Witt \( K \)-theory of \( F \) is denoted \( K_{MW}(F) \), or simply \( K_{MW} \), following the convention that the base field or scheme may be ignored in the notation. The definition and some details regarding \( K_{MW} \) and modules over it are contained in the Appendix A. Theorem 3.1 implies that for every motivic spectrum \( E \) and for every integer \( s \), \( \pi_{s+(*)}E \) is a graded \( K_{MW} \)-module.

Choose a basepoint for \( \mathbb{P}^1 \), and hence \( \mathbb{P}^2 \), which will not appear in the notation. Neither will the base field \( F \) most of the time. The main cofiber sequence over a field is

\[
S^{1+(2)} \xrightarrow{\eta} \mathbb{P}^1 \xrightarrow{i} \mathbb{P}^2 \xrightarrow{q} S^{2+(2)}. \tag{3.1}
\]

It induces a long exact sequence of \( K_{MW} \)-modules

\[
\cdots \xrightarrow{\eta} \pi_{m+(*)}\mathbb{P}^1 \xrightarrow{i_*} \pi_{m+(*)}\mathbb{P}^2 \xrightarrow{q_*} \pi_{m+(*)}S^{2+(2)} \xrightarrow{\eta} \pi_{m-1+(*)}\mathbb{P}^1 \to \cdots
\]
terminating with \( \pi_{1+}(*) \mathbb{P}^2 \) by connectivity [Mor05]. The induced short exact sequences

\[
0 \to \pi_{m-1+}(*)1/\eta \pi_{m-1+}(*)1 \to \pi_{m+}(*) \mathbb{P}^2 \to \eta \pi_{m-2+}(*)1 \to 0 \tag{3.2}
\]

in which weight shifts are not displayed, express \( \pi_{m+}(*) \mathbb{P}^2 \) as an extension of two \( \mathbb{K}^{\text{MW}} \)-modules, the submodule of \( \pi_{m-2+}(*)1 \) annihilated by \( \eta \), and the cokernel of multiplication by \( \eta \) on \( \pi_{m-1+}(*)1 \). This justifies the relevance of the following statement, for which simpler proofs exist.

**Theorem 3.2.** Let \( F \) be a field of characteristic not two. The submodule of \( \mathbb{K}^{\text{MW}} \) annihilated by \( \eta \) coincides with the image of multiplication by the hyperbolic plane on \( \mathbb{K}^{\text{MW}} \):

\[
\eta \mathbb{K}^{\text{MW}} = h \mathbb{K}^{\text{MW}}
\]

**Proof.** Consider \( kq \), the very effective cover of the motivic spectrum \( KQ \) representing hermitian \( K \)-theory in \( \text{SH}(F) \). By [ARØ20, Proposition 11], it fits into a cofiber sequence

\[
\Sigma^{(1)}kq \xrightarrow{\eta} kq \xrightarrow{\text{forget}} kgl \xrightarrow{\text{hyper}} \Sigma^{1+(1)}kq \tag{3.3}
\]

where the first map is multiplication by \( \eta \), the second map is induced on very effective covers by the forgetful map \( KQ \to KGL \) from hermitian to algebraic \( K \)-theory, and the connecting map “hyper” is induced by the hyperbolic map from algebraic to hermitian \( K \)-theory. The latter is basically defined by sending a vector bundle \( V \to X \) over a smooth \( F \)-scheme \( X \) to the bundle \( V \oplus \text{Dual}(V) \to X \), equipped with the canonical hyperbolic form. Moreover, the canonical unit map \( 1 \to kq \) induced by the unit of \( KQ \) induces an isomorphism of \( \mathbb{K}^{\text{MW}} \)-modules \( \pi_{0+}(*)1 \to \pi_{0+}(*)kq \) [ARØ20, Theorem 25]. Hence the cofiber sequence (3.3) induces a long exact sequence of \( \mathbb{K}^{\text{MW}} \)-modules, terminating with

\[
\cdots \to \pi_{1+}(*)kgl \xrightarrow{\text{hyper}} \pi_{0+}(*)kq \xrightarrow{\eta} \pi_{0+}(*)kq \xrightarrow{\text{forget}} \pi_{0+}(*)kgl \to 0.
\]

This gives an identification \( \pi_{0+}(*)kgl \cong K^M \). Moreover, the kernel of \( \pi_{0+}(*)kq \xrightarrow{\eta} \pi_{0+}(*)kq \), which is isomorphic to \( \eta \mathbb{K}^{\text{MW}} \), coincides with the image of \( \pi_{1+}(*)kgl \xrightarrow{\text{hyper}} \pi_{0+}(*)kq \). It remains to identify the latter, which will proceed via the slice spectral sequence. The first (very) effective cover of \( kgl \) is \( f_1kgl \simeq \Sigma^{1+(1)}kgl \), and the cofiber sequence

\[
f_1kgl \to kgl \to s_0kgl \simeq MZ \tag{3.4}
\]

defining the zero slice provides Voevodsky’s motivic Eilenberg-MacLane spectrum [Lev08], [Voe02]. The cofiber sequence (3.4) induces a short exact sequence

\[
\cdots \to \pi_{1+}(*)f_1kgl \to \pi_{1+}(*)kgl \to \pi_{1+}(*)s_0kgl \to 0.
\]

The motivic spectrum \( \Sigma^{1+(1)}kq \), to which \( kgl \) maps via the hyperbolic map, has a trivial zero slice by construction. Moreover, the composition

\[
\pi_{1+}(*)kq \xrightarrow{\text{forget}} \pi_{1+}(*)kgl \to \pi_{1+}(*)MZ
\]
is surjective by [ARØ20, Proposition 27]. Hence the image of the $K^{\text{MW}}$-module homomorphism $\pi_{1+(\ast)}kgl \to \pi_{1+(\ast)}\Sigma^{1+(\ast)}kq$ coincides with the image of its restriction to

$$\pi_{1+(\ast)}f_1kgl \cong \pi_{1+(\ast)}\Sigma^{1+(\ast)}kgl \cong \pi_{0+(\ast)}kgl \cong K^M.$$  

This $K^{\text{MW}}$-module is generated by a single element, for example $1 \in K^M_0$, the image of the unit in $K^{\text{MW}}_0$. Hence its image in $\pi_{1+(\ast)}\Sigma^{1+(\ast)}kq \cong \pi_{0+(\ast)}kq \cong K^{\text{MW}}$ is generated, as a $K^{\text{MW}}$-module, by the image of $1 \in K^M_0$. This image is the hyperbolic plane $h \in K^{\text{MW}}_0$, as the short exact sequence

$$0 \to \mathbb{Z}\{h\} \to K^{\text{MW}}_0 = \pi_{0+(0)}kq \xrightarrow{\eta} K^{\text{MW}}_1 = \pi_{0+(1)}kq \to 0,$$

in which $\eta$ induces the canonical projection from the Grothendieck-Witt ring to the Witt ring, shows.

This provides the exactness of the sequence mentioned in [CH, Remark 4.3]. It is quite special that the kernel of multiplication by $\eta$ on $K^{\text{MW}}$ is generated by a single element, but then the element $\eta$ is also quite special. As a consequence of Theorem 3.2, the short exact sequence

$$0 \to \pi_{1+(\ast)}1/\eta\pi_{1+(\ast)}1 \to \pi_{2+(\ast)}P^2 \to \eta\pi_{0+(\ast)}1 \to 0$$

specializes to an isomorphism $\pi_{1+(w-1)}1/\eta\pi_{1+(w-2)}1 \cong \pi_{2+(w)}P^2$ for $w > 2$. Therefore knowing $\pi_{1+(\ast)}1/\eta\pi_{1+(\ast)}1$ is essential.

**Theorem 3.3.** Let $F$ be a field of characteristic not two or three. The unit map $1 \to kq$ induces a surjection $\pi_{1+(\ast)}1/\eta\pi_{1+(\ast)}1 \to \pi_{1+(\ast)}kq/\eta\pi_{1+(\ast)}kq$ whose kernel is $K^M/12$ (generated in $\ast = 2$) after inverting the exponential characteristic of $F$.

**Proof.** This is a consequence of [RSØ19, Theorem 5.5] in the formulation given in [Rön20, Theorem 2.5]. First of all, the unit map $\pi_{1+(\ast)}1 \to \pi_{1+(\ast)}kq$ is surjective, whence the same is true for the induced map on the quotients. Set $e$ to be the exponential characteristic of $F$. Consider the following natural transformation

$$
\begin{array}{cccccc}
0 & \longrightarrow & K^M_{2-\ast}/24[\frac{1}{e}] & \longrightarrow & \pi_{1+(\ast)}1[\frac{1}{e}] & \longrightarrow & \pi_{1+(\ast)}kq[\frac{1}{e}] & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & \pi_{1+(\ast)}1[\frac{1}{e}]/\eta & \longrightarrow & \pi_{1+(\ast)}kq[\frac{1}{e}]/\eta & \longrightarrow & 0
\end{array}
$$

of short exact sequences. Applied to the lower transformation, the snake lemma implies that the map $K^M/24[\frac{1}{e}] \to A$ is surjective, because the map $\eta\pi_{1+(\ast)}1 \to \eta\pi_{1+(\ast)}kq$ on the kernels is surjective. The map $\eta: \Sigma^{(1)}1 \to 1$ factors by construction as $\eta: \Sigma^{(1)}1 \to f_11 \to 1$, where $f_11 \to 1$ denotes the first effective cover. The presentations given in [Rön20, Lemma 2.3, Theorem 2.5] then imply that the kernel of the map $\eta\pi_{1+(\ast)}1 \to \eta\pi_{1+(\ast)}kq$ is generated by $12\nu = \eta^2\eta_{\text{top}}$, where $\eta_{\text{top}} \in \pi_{1+(0)}1$ denotes the topological Hopf map.\footnote{This element appeared in Section 2 without the subscript.} Hence the snake lemma also implies that $A \cong K^M/12[\frac{1}{e}]$. \qed
Theorem 3.3 is valid in characteristic 3 as well in the sense that the kernel of
\[ \pi_{1+(*)}1/\eta\pi_{1+(*)}1 \to \pi_{1+(*)}kq/\eta\pi_{1+(*)}kq \]
is isomorphic to \( K^M/4[\frac{1}{3}] \) after inverting 3. Theorems 3.2 and 3.3 provide sufficient information about the outer terms in the short exact sequence
\[ 0 \to \pi_{1+(*)}1/\eta\pi_{1+(*)}1 \to \pi_{2+(*)}P^2 \to \eta\pi_{0+(*)}1 \to 0 \] (3.5)
of \( K^{MW} \)-modules. Actually the outer terms are \( K^M \)-modules in a natural way; \( \eta \) acts trivially on these. However, \( \eta \) acts nontrivially on the middle term. The reason is the Toda bracket \( \langle \eta, h, \eta \rangle = \{6\nu, -6\nu\} \) from [Rön20, Proposition 4.1]. It then follows essentially by construction – see [Rön20, Proposition 4.2] – that there exists an element \( g \in \pi_{2+(2)}P^2 \) which on the one hand maps to \( h \in \eta\pi_{0+(0)}1 \), and on the other hand is such that \( g\eta \) is the image of \( 6\nu \in \pi_{1+(2)}1 \). Inspecting the short exact sequence (3.5) in weight 3 gives an isomorphism \( \pi_{1+(2)}1/\eta\pi_{1+(1)}1 \cong \pi_{2+(3)}P^2 \) by Theorem 3.2, whence \( \pi_{2+(3)}P^2 \) is cyclic of order 12, with the image \( i\circ\nu \) of \( \nu \) as a generator. Hence \( g\eta \) is the unique nonzero element of order two in this group. Inspecting the short exact sequence (3.5) in weight 2 provides
\[ 0 \to \pi_{1+(1)}1/\eta\pi_{1+(0)}1 \to \pi_{2+(2)}P^2 \to \eta\pi_{0+(0)}1 \to 0. \]
In particular, the element \( g \in \pi_{2+(2)}P^2 \) defines a splitting of this short exact sequence of abelian groups.

**Lemma 3.4.** The extension
\[ 0 \to \pi_{1+(*)}1/\eta\pi_{1+(*)}1 \to \pi_{2+(*)}P^2 \to \eta\pi_{0+(*)}1 \to 0 \]
describing the \( K^{MW} \)-module \( \pi_{2+(*)}P^2 \) is determined by the fact that \( g\eta = 6(i\circ\nu) \), where \( g \in \pi_{2+(2)}P^2 \) is the chosen lift of \( h \in \pi_{0+(0)}1 \).

**Proof.** This essentially follows from the discussion above. To be more precise, the extension (3.5) is given by an element in \( \text{Ext}^1_{K^{MW}}(K^M, \pi_{1+(*)}1/\eta\pi_{1+(*)}1) \) by Theorem 3.2. Lemma A.3 implies that this group is isomorphic to
\[ h\pi_{1+(2)}1/\eta\pi_{1+(1)}1 \cong 2\pi_{1+(2)}1/\eta\pi_{1+(1)}1 \cong 2\mathbb{Z}/12 \]
where the last isomorphism follows from Theorem 3.3. The extension in question corresponds to the unique nonzero element in \( \pi_{1+(2)}1/\eta\pi_{1+(1)}1 \) of order two by the Toda bracket \( \langle \eta, h, \eta \rangle = \{6\nu, -6\nu\} \) from [Rön20, Proposition 4.1].

**Remark 3.5.** Regarding the unstable situation, the \( A^1 \)-fiber sequence
\[ A^1 \setminus \{0\} \to A^3 \setminus \{0\} \to P^2 \]
and the \( A^1 \)-discreteness of \( A^1 \setminus \{0\} \) provide an identification \( \pi_2^{A^1} \mathbb{P}^2 \cong \pi_2^{A^1}(A^3 \setminus \{0\}) \cong K_{MW}^{A^1} \) of (unstable) \( A^1 \)-homotopy sheaves, by [Mor12, Theorem 1.23]. The generator of \( \pi_2^{A^1+3} \mathbb{P}^2 \cong K_{MW}^0 \) is thus the class of the canonical map \( A^3 \setminus \{0\} \to \mathbb{P}^2 \). Stabilization with respect to \( \mathbb{P}^1 \) provides a homomorphism

\[
\pi_2^{A^1} \mathbb{P}^2 \to \pi_2+3 \mathbb{P}^2
\]

to the stable homotopy sheaf computed in Lemma 3.4. It sends the generator to \( m(i \circ \nu) \), where \( m \) is an integer unique up to multiples of 12, and \( i \circ \nu \) is the generator of the target. A comparison with the classical topological situation via complex or étale realization, which is possible since the target does not depend on the base field, shows that \( m = 2 \).

The computations provided by \( \pi_1+(\ast) \mathbb{P}^2 \cong K^M \) and Lemma 3.4 suffice to conclude the following statement.

**Theorem 3.6.** Let \( F \) be a field of characteristic not in \( \{2, 3\} \). The cofiber sequence (3.1) induces a short exact sequence

\[
0 \to \pi_2+(\ast) \mathbb{P}^2/\eta \pi_2+(\ast) \mathbb{P}^2 \to [\Sigma^{(\ast)} \mathbb{P}^2, \mathbb{P}^2] \to \pi_1+(\ast) \mathbb{P}^2 \to 0 \quad (3.6)
\]

of \( K_{MW} \)-modules described more explicitly above. In particular, after inverting the exponential characteristic, there is an isomorphism

\[
\mathbb{Z}/6 \cong \pi_2+(3) \mathbb{P}^2/\eta \pi_2+(3) \mathbb{P}^2 \cong [\Sigma^{(1)} \mathbb{P}^2, \mathbb{P}^2]
\]

with \( i \circ \nu \circ q \) as a generator, and an isomorphism

\[
\mathbb{Z} \oplus \mathbb{Z} \oplus K_1^M/6(F) \cong [\mathbb{P}^2, \mathbb{P}^2]
\]

of abelian groups, with \( \text{id}_{\mathbb{P}^2} \) a generator of one free summand. The equality \( \eta \cdot \text{id}_{\mathbb{P}^2} = \eta \wedge \text{id}_{\mathbb{P}^2} = 3(i \circ \nu \circ q) \) determines the \( K_{MW} \)-module structure on \( [\Sigma^{(\ast)} \mathbb{P}^2, \mathbb{P}^2] \).

**Proof.** As before, the cofiber sequence (3.1) induces a short exact sequence

\[
0 \to \pi_2+(\ast) \mathbb{P}^2/\eta \pi_2+(\ast) \mathbb{P}^2 \to [\Sigma^{(\ast)} \mathbb{P}^2, \mathbb{P}^2] \to \eta \pi_1+(\ast) \mathbb{P}^2 \to 0.
\]

The short exact sequence for \( \pi_1+(\ast) \mathbb{P}^2 \) specializes to the identification \( \pi_1+(\ast) \mathbb{P}^2 \cong K^M \) mentioned already above. Since \( \eta \) acts as zero on this \( K_{MW} \)-module, the short exact sequence (3.6) of \( K_{MW} \)-modules follows. The identity \( \text{id}_{\mathbb{P}^2} \) hits the canonical generator \( i \in \pi_1+(\ast) \mathbb{P}^2 \). The \( K_{MW} \)-module structure is then determined by specifying \( \eta \cdot \text{id}_{\mathbb{P}^2} \in [\Sigma^{(1)} \mathbb{P}^2, \mathbb{P}^2] \). Lemma A.3 identifies the corresponding extension group as

\[
\text{Ext}^1_{K_{MW}}(\pi_1+(\ast) \mathbb{P}^2 \cong K^M; \pi_2+(\ast) \mathbb{P}^2/\eta \pi_2+(\ast) \mathbb{P}^2) \cong \text{h} \pi_2+(3) \mathbb{P}^2/\eta \pi_2+(2) \mathbb{P}^2 \cong 2\mathbb{Z}/6.
\]

Here the description of \( \pi_2+(\ast) \mathbb{P}^2 \) as a \( K_{MW} \)-module from Theorem 3.4 supplies the last isomorphism in this sequence, as well as the first isomorphism mentioned in the statement.
of the theorem. Hence \( \eta \text{id}_{\mathbb{P}^2} = m(i \circ \nu \circ q) \) for some \( m \in \mathbb{Z} \) which is unique up to multiples of 6. The element \( \eta \text{id}_{\mathbb{P}^2} \) turns out to be the unique nonzero element of order 2, as the Toda bracket \( \langle \eta, h = q \circ g, \eta \rangle \) implies. The properties of Toda brackets supply an inclusion

\[
\langle \eta, q, \eta \land \mathbb{P}^2 \rangle \circ g \subset \langle \eta, q, \eta \land \mathbb{P}^2 \circ g = g \circ \eta \rangle \subset \langle \eta, q \circ g, \eta \rangle = \{6\nu, -6\nu\}
\]

which shows that it does not contain zero. This already suffices to conclude. More precisely, since the composition

\[
\pi_{1+(2)} \mathbb{1} \xrightarrow{g} [\Sigma^{1+1}(\mathbb{P}^2, S^{2+1})] \xrightarrow{g^*} \pi_{1+(2)} \mathbb{1}
\]

is multiplication with \( h \), one obtains \( \langle \eta, q, \eta \land \mathbb{P}^2 \rangle \subset \{3\nu \circ q, -3\nu \circ q\} \). Hence \( \text{id}_{\mathbb{P}^2} \in [\mathbb{P}^2, \mathbb{P}^2] \) satisfies \( \eta \land \mathbb{P}^2 = \pm (i \circ \nu \circ q) \).

Theorem 3.6 implies that every element \( x \in [\mathbb{P}^2, \mathbb{P}^2] \) can be expressed uniquely as a sum \( x_1 \text{id}_{\mathbb{P}^2} + x_2 (g \circ q) + x_3 (i \circ \nu \circ q) \), where \( x_1, x_2 \in \mathbb{Z} \) and \( x_3 \in K_1^M / 6 \). It would probably be more honest to think of the integers \( x_1, x_2 \) as the ranks of virtual quadratic forms. In particular, the hyperbolic form “h” corresponds to the integer “2”. Using that the composition \( q \circ i \) is the zero map, the ring structure is then given as

\[
x \circ y = (x_1 \text{id}_{\mathbb{P}^2} + x_2 (g \circ q) + x_3 (i \circ \nu \circ q)) \circ (y_1 \text{id}_{\mathbb{P}^2} + y_2 (g \circ q) + y_3 (i \circ \nu \circ q))
\]

\[
= x_1 y_1 \text{id}_{\mathbb{P}^2} + (x_1 y_2 + x_2 y_1 + 2x_3 y_2) (g \circ q) + (x_1 y_3 + x_3 y_1 + 2x_3 y_2) (i \circ \nu \circ q)
\]

and in particular is not commutative if \( 2K_1^M(F) / 6 \) is nonzero. The group of units in \( [\mathbb{P}^2, \mathbb{P}^2] \) consists of the elements

\[
\{ \pm \text{id}_{\mathbb{P}^2} + x_3 (i \circ \nu \circ q), \pm \text{id}_{\mathbb{P}^2} \oplus g \circ q + x_3 (i \circ \nu \circ q) : x_3 \in K_1^M / 6 \}.
\]

If \( F \subset \mathbb{C} \), then complex realization \( [\Sigma^{(1)} \mathbb{P}^2, \mathbb{P}^2] \to [\Sigma \mathbb{CP}^2, \mathbb{CP}^2] \) is an isomorphism, but \( [\mathbb{P}^2, \mathbb{P}^2] \to [\mathbb{CP}^2, \mathbb{CP}^2] \) is possibly only surjective, not injective. Nevertheless, every map in \( [\mathbb{P}^2, \mathbb{P}^2] \) such that its complex realization is a unit in \( [\mathbb{CP}^2, \mathbb{CP}^2] \) is already a unit in \( [\mathbb{P}^2, \mathbb{P}^2] \).

**Remark 3.7.** A different motivic type of endomorphisms of the projective plane occurs in the motivic stable homotopy category \( \text{SH}(\mathbb{P}^2_F) \) for \( \mathbb{P}^2_F \), with unit \( 1_{\mathbb{P}^2_F} = \varphi^*(1_F) \). Here \( \varphi : \mathbb{P}^2_F \to \text{Spec}(F) \) is the structure morphism. The choice of a rational point provides a splitting \( 1_F \to \varphi_1(1_{\mathbb{P}^2_F}) \to 1_F \) of the counit, whence \( \varphi_2(1_{\mathbb{P}^2_F}) \simeq \mathbb{P}^2_F \vee 1_F \). Then \( \varphi_2 \) induces a ring homomorphism

\[
\pi_{0+(*)} 1_{\mathbb{P}^2_F} \xrightarrow{\varphi_2} [\Sigma^{(*)} \mathbb{P}^2_+, \mathbb{P}^2_+] \cong [\Sigma^{(*)} \mathbb{P}^2, \mathbb{P}^2] \oplus \pi_{0+(*)} 1_F \oplus [\Sigma^{(*)} \mathbb{P}^2, 1]
\]

where the source

\[
\pi_{0+(*)} 1_{\mathbb{P}^2_F} \cong [\Sigma^{(*)} \varphi_2(1_{\mathbb{P}^2_F}), 1_F] \cong [\Sigma^{(*)} \mathbb{P}^2, 1_F] \oplus \pi_{0+(*)} 1_F
\]

is a commutative ring by definition. Using [Rön20, Theorem 2.7], the short exact sequence

\[
0 \to \pi_{2+(2)} 1_F / \eta \pi_{2+1} 1_F \cong K_2^M(F) \to [\mathbb{P}^2, 1] \to \eta \pi_{1+1} 1 \cong K_1^M(F) / 24 \to 0
\]

implies that \( \pi_{0+0} 1_{\mathbb{P}^2_F} \) is not isomorphic to the Grothendieck-Witt ring of \( \mathbb{P}^2_F \), which is isomorphic to \( K_0^{MW}(F) \oplus K_0^M(F) \) [Wal].
4 Suslin’s conjecture

An application of the computations performed in Section 3 is a proof of Suslin’s conjecture on the Hurewicz homomorphism from Quillen to Milnor K-theory in degree four, exploiting the beautiful work [AFW20]. Unstable homotopy sheaves will occur, as already in Remark 3.5. A preliminary computation which will be essential is $π_3 P^2 \wedge P^2$. The role of $P^2 \wedge P^2$ is that of the double cone on the Hopf map $η$. As Lemma 3.4 demonstrates, the Hopf map $η$ is still nonzero on $P^2$, whence it makes sense to cone it off again.

Lemma 4.1. Let $F$ be a field of exponential characteristic $e$ not in $\{2, 3\}$. After inverting $e$, the unit $1 \to kq$ induces a short exact sequence

$$0 \to K^M_{4-\gamma}/6 \to π_{3+(\ast)} P^2 \wedge P^2 \to π_{2+(\ast-1)} P^2 \wedge kgl \to 0$$

of $K^{MW}$-modules, in which the image of $K^M_{4-\gamma}/6$ in $π_{3+(\ast)} P^2 \wedge P^2$ is generated by $(i \wedge i) \circ ν$.

Proof. By [ARØ20, Proposition 11], $P^2 \wedge kq \simeq Σ^{1+(1)} kgl$ is a suspension of the connective algebraic $K$-theory spectrum. Hence the unit $1 \to kq$ induces a map

$$P^2 \wedge P^2 \to Σ^{1+(1)} P^2 \wedge kgl.$$ 

Its effect on $π_{3+(\ast)}$ can be described using the following commutative diagram

$$\begin{array}{cccccc}
0 & \to & π_{2+(\ast-2)} P^2/η & \to & π_{3+(\ast)} P^2 \wedge P^2 & \to & π_{1+(\ast-2)} P^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & π_{2+(\ast-2)} P^2 \wedge kq/η & \to & π_{2+(\ast-1)} P^2 \wedge kgl & \to & π_{1+(\ast-2)} P^2 \wedge kq \\
\end{array}$$

(4.1)

in which the horizontal short exact sequences are analogous to (3.2), and $e$ is implicitly inverted. The vertical map on the right hand side of diagram (4.1) is an isomorphism on $K^M$. Theorem 3.3 implies that the unit map $π_{2+(\ast)} P^2 \to π_{2+(\ast)} P^2 \wedge kq$ is surjective, with kernel isomorphic to $K^M_{2-\gamma}/12$, and generated by $i \circ ν$, at least after inverting $e$. As a consequence, the vertical map on the left hand side of diagram (4.1) is surjective. Its kernel can be determined by applying the snake lemma to a diagram similar to (4.1), using Lemma 3.4, which gives the result. □

Theorem 3.6 on the endomorphisms of $P^2$ provides another argument for Lemma 4.1 because the Spanier-Whitehead dual of $P^2$ is $Σ^{−3−(3)} P^2$, giving an isomorphism of $K^{MW}$-modules $π_{3+(\ast)} P^2 \wedge P^2 \simeq [Σ^{−3−3} P^2, P^2]$. To continue, comparison with unstable homotopy sheaves is essential, as in [AFW20]. Let

$$Q_{2n-1} := \{(a, b) \in A^n \times A^n : \sum_{j=1}^n a_j b_j = 1\}$$

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be the smooth affine quadric hypersurface which is weakly equivalent to $A^n \setminus \{0\}$ via projection to the first $n$ coordinates [ADF17]. The quotient scheme $Q_{2n-1}/GL_1$ with respect to the action $\lambda \cdot (a, b) := (\lambda a, \lambda^{-1} b)$ is then weakly equivalent via projection to the first $n$ coordinates to $P^{n-1}$. Given $(a, b, q) \in Q_{2n-1} \times A^1 \setminus \{0\}$, let $[a, b, q] \in GL_n$ denote the matrix whose entry at $(j, k)$ is $\delta_{jk} + (q^{-1} - 1) a_j b_k$, where $\delta_{jk}$ is the Kronecker symbol. Set $\psi_n(a, b, q)$ to be the product of the diagonal matrix whose nonzero entries are $(q^{-1}, 1, \ldots, 1)$ with $[a, b, q]$. This produces a map

$$\psi_n : Q_{2n-1} \times A^1 \setminus \{0\} \to SL_n$$

compatible with the given $GL_1$ action on the first factor (and trivial actions on the other factor and $SL_n$), sending $Q_{2n-1} \times \{1\} \cup \{((1, 0, \ldots, 0), (1, 0, \ldots, 0))\} \times A^1 \setminus \{0\}$ to the identity matrix. Let $\psi_n : \Sigma^{(1)} Q_{2n-1} / GL_1 \to SL_n$ denote the induced pointed map, a variant of the map to $GL_n$ constructed in [Wil12]. Its complex realization is denoted $j_n$ in [Muk82, p. 180], and $f_{SU(n)}$ in the even more classical source [Yok57, Section 4]. It is straightforward to check that the diagram

$$\begin{array}{ccc}
\Sigma^{(1)} Q_{2n-1} / GL_1 & \xrightarrow{\psi_n} & SL_n \\
\downarrow & & \downarrow \\
\Sigma^{(1)} Q_{2n+1} / GL_1 & \xrightarrow{\psi_n+1} & SL_{n+1}
\end{array}$$

(4.2)

with obvious inclusions as vertical maps commutes. It is less straightforward to check that the map $\psi_2 : \Sigma^{(1)} Q_3 / GL_1 \to SL_2$ is a weak equivalence. Note that the map $\psi_n$ is not injective. For example, $\psi_2((1, 0), (1, 2^{-1}), -1) = \psi_2((1, 0), (1, -1), 2)$. With the help of $\psi_3$, the cell structure of $SL_3$ looks as follows. (More general statements for $SL_n$, similar to [Yok57, Theorem 5.4], are possible, and will be pursued in further work.)

**Lemma 4.2.** Let 2 be invertible. The homotopy cofiber of $\psi_3$ is given by $S^{3+5}$. 

**Proof.** The following argument attempts to mimic the corresponding proof in [Yok57]; Jean Fasel has a much better unpublished proof, which works over any base. Let $H \subset SL_3$ be the preimage of the column vector $(0, 0, 1) \in A^3 \setminus \{0\}$ under the map $SL_3 \to A^3 \setminus \{0\}$ sending a matrix to its last column. It is actually a subgroup, and the inclusion $SL_2 \to H$ is an equivalence, since $H \cong SL_2 \times A^2$. Let $P$ be the pushout of

$$\Sigma^{(1)} Q_5 / GL_1 \leftarrow \Sigma^{(1)} Q_3 / GL_1 \xrightarrow{\psi_2} SL_2 \leftarrow H$$

so that $\psi_3$ induces a map $P \to SL_3$, using (4.2). The multiplication on $SL_3$ induces a diagram

$$\begin{array}{ccc}
H \times H \cup \{1\} \times P & \rightarrow & P \\
\downarrow & & \downarrow \\
H \times P & \rightarrow & SL_3
\end{array}$$

(4.3)
which is a homotopy pushout diagram, thereby completing the proof. In order to see the homotopy pushout property, observe that the diagram

\[
\begin{array}{ccc}
H \times H & \rightarrow & H \\
\downarrow & & \downarrow \\
H \times H \cup \{1\} \times P & \rightarrow & P
\end{array}
\]

is a homotopy pushout diagram, because it is a pushout diagram with vertical maps being inclusions. Hence it suffices to prove that the composed diagram

\[
\begin{array}{ccc}
H \times H & \rightarrow & H \\
\downarrow & & \downarrow \\
H \times P & \rightarrow & \text{SL}_3
\end{array}
\] (4.4)

is a homotopy pushout diagram. The vertical maps in diagram (4.4) are inclusions. In this situation, it would suffice to prove that every matrix in \(\text{SL}_3 \setminus H\) is in the image of \(H \times P \rightarrow \text{SL}_3\). By passing to suitable inverses, this amounts to find, for a given matrix \(C \in \text{SL}_3 \setminus H\), a point \((a, b, q) \in Q_5 \times A^1 \setminus \{0\}\) such that \(\psi_3(a, b, q) \cdot C \in H\). Since \(C \not\in H\), the last column \((c_{13}, c_{23}, c_{33})\) of \(C\) is different from \((0, 0, 1)\). Set \(a := (c_{13}, c_{23}, c_{33} - 1)\). It remains to find suitable elements \(b\) and \(q\). Without loss of generality, \(C \in \text{SL}_3(R)\), where \(R\) is a local ring in which 2 is invertible. If \(x \in R\) is invertible, then either \(1 - x\) or \(1 + x\) is invertible as well. Since \(\det(C) = 1\), at least one of the elements \(c_{13}, c_{23}, c_{33}\) is invertible.

Suppose \(c_{13}\) is invertible. If \(1 - c_{13}\) is invertible, set \(q := c_{13}\) and \(b := \left(\frac{1}{1-c_{13}}c_{13}, 0, \frac{c_{13}}{1-c_{13}}\right)\). If \(1 + c_{13}\) is invertible, set \(q := -c_{13}\) and \(b := \left(\frac{1}{1+c_{13}}c_{13}, 0, \frac{-c_{13}}{1+c_{13}}\right)\).

Similarly, if \(c_{23}\) is invertible such that \(1 \mp c_{23}\) is invertible, set \(q := \pm c_{23}\) and \(b := \left(0, \frac{1}{1 \mp c_{23}}c_{23}, \frac{\pm c_{23}}{1 \mp c_{23}}\right)\). If neither \(c_{13}\) nor \(c_{23}\) are invertible, then \(c_{33}\) is invertible. If additionally \(1 - c_{33}\) is invertible, set \(q := \frac{1}{c_{33}}\) and \(b := \left(0, 0, \frac{1}{c_{33}}\right)\). However, if \(1 - c_{33}\) is not invertible, there is no \(b \in R^3\) with \(c_{13}b_1 + c_{23}b_2 + (c_{33} - 1)b_3 = 1\). Instead, consider the \(A^1\)-homotopy

\[
H \times P \times A^1 \rightarrow \text{SL}_3, \quad (D, p, t) \mapsto \begin{pmatrix} 1 & 0 & -t \cdot (D \cdot \psi_3(p))_{33} (D \cdot \psi_3(p))_{13} \\ 0 & 1 & -t(D \cdot \psi_3(p))_{23} \\ 0 & 0 & 1 \end{pmatrix} \cdot D \cdot \psi_3(p)
\]

which restricts to the constant \(A^1\)-homotopy given by the multiplication on \(H \times H\). Evaluating the homotopy at \(t = 0\) gives the lower horizontal map in diagram (4.4). Evaluating this homotopy at \(t = 1\) provides a map whose image contains in particular the last group of matrices, choosing \(D = C^{-1}\) and \(p\) to be the class of \((0, 0, 1), (0, 0, 1), \frac{1}{c_{33}}\). The result follows.

In the following, \(Q_{2n-1}/\text{GL}_1\) will be identified with \(\mathbb{P}^{n-1}\), whence maps may be defined only in the homotopy category. The simplicial suspension of \(\psi_3\) leads to a map

\[
\xi_3: \mathbb{P}^1 \wedge \mathbb{P}^2 \cong \Sigma^{1+(1)} \mathbb{P}^2 \xrightarrow{\Sigma \psi_3} \Sigma \text{SL}_3 \hookrightarrow \text{BSL}_3
\]

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Lemma 4.3. The map 

\[ \Psi : P^1 \land P^2 \to B^2 SL_3 \]

fits into a homotopy cofiber sequence

\[ S^{4+5} \to X \to \Sigma^2 SL_3^{\wedge 2} \]

which implies that \( X \) is \( A^1 \)-3-connected. Again [AF16, Theorem 4.1] then implies that \( \xi_3 : \pi_3^A P^1 \land P^2 \to \pi_3^A B^2 SL_3 \cong \pi_3^A BSL_3 \) is surjective. Alas, the map is not injective, which can be seen from its fourth contraction. In the Hopf fibration

\[ S^{1+(2)} \to S^{3+4} \to S^{2+2} \]

obtained from the Hopf construction on \( SL_2 \), the inclusion of the fiber is \( A^1 \)-homotopic to the constant map, giving splittings \( \pi_3^A S^{2+2} \cong \pi_3^A S^{3+4} \oplus \pi_3^A S^{1+2} \) for all \( n \). The injection \( \Sigma_3+(1)i \) induces a surjection \( \pi_3^A(\Sigma_3+(4)) \to \pi_3^A(\Sigma_3)+(4) \) \( P^1 \land P^2 \) in which the element \( 6(\Sigma_3+(1)i \circ \nu) \) is nonzero (as one can see using complex or étale realization). However, its image in \( \pi_3+(4)BLSL_3 \cong \mathbb{Z}/6 \), as computed in [AF14, Theorem 3.14, with Corollary 3.11], must be zero.

The failure of injectivity is measured by the image of the canonical map \( \pi_3^A X \to \pi_3^A P^1 \land P^2 \). The inclusion \( S^{4+4} \cong \Sigma^2 SL_3^{\wedge 2} \to \Sigma^2 SL_3^{\wedge 2} \) factors canonically over \( X \to \Sigma^2 SL_3^{\wedge 2} \). It will be relevant to add more cells to the source of \( P^1 \land P^2 \to B^2 SL_3 \) so that \( S^{4+4} \) is removed from the corresponding homotopy cofiber, as in the following statement.

**Lemma 4.3.** The map \( \xi_3 : P^1 \land P^2 \to BSL_3 \) extends over the inclusion \( i \land P^2 : P^1 \land P^2 \to P^2 \land P^2 \). The resulting extension \( P^2 \land P^2 \to BSL_3 \) can be chosen such that it factors over the simplicial 2-skeleton as \( P^2 \land P^2 \to B^2 SL_3 \to BSL_3 \), where the map \( \Psi : P^2 \land P^2 \to B^2 SL_3 \) fits into a commutative diagram

\[
\begin{array}{ccc}
P^2 \land P^2 & \xrightarrow{\Psi} & B^2 SL_3 \\
\downarrow & & \downarrow \\
S^{4+4} & \cong & \Sigma^2 SL_2 \land SL_2 \\
\end{array}
\]

with canonical quotient maps as vertical arrows.
Proof. Abbreviate $i \wedge \mathbb{P}^2$ as $i$, and similarly for $q \wedge \mathbb{P}^2$ and $\eta \wedge \mathbb{P}^2$. For the first statement, the long exact sequence

$$\cdots \to [\mathbb{P}^2 \wedge \mathbb{P}^2, \text{BSL}_3] \xra{i^*} [\mathbb{P}^1 \wedge \mathbb{P}^2, \text{BSL}_3] \xra{\eta^*} [\Sigma^{1+(2)} \mathbb{P}^2, \text{BSL}_3]$$

of sets (and eventually abelian groups) shows that it suffices to prove $\eta^*(\xi_3) = 0$. In the long exact sequence

$$\cdots \to [S^{3+(4)}, \text{BSL}_3] \xra{q^*} [\Sigma^{1+(2)} \mathbb{P}^2, \text{BSL}_3] \xra{i^*} [S^{2+(3)}, \text{BSL}_3] \xra{\eta^*} [S^{2+(4)}, \text{BSL}_3]$$

the group $[S^{2+(3)}, \text{BSL}_3] \cong [S^{1+(3)} \text{SL}_3] \cong 0$, because $\pi^A_1 \text{SL}_3 \cong K_2^M$, for example by [VW16] (which does not include the case of finite fields) or the much more recent preprint [MS] (which does include the case of finite fields). The group

$$[S^{3+(4)}, \text{BSL}_3] \cong [S^{2+(4)}, \text{SL}_3] \cong \mathbb{Z}/6$$

was computed in [AF14, Theorem 3.14, with Corollary 3.11] and does not depend on the base field. In order to determine the image of $\eta^*: [S^{3+(3)}, \text{BSL}_3] \to [S^{3+(4)}, \text{BSL}_3]$, a map defined over the integers, it thus suffices to compute its image after complex realization. Since the topological map $\eta_{\text{top}}: S^6 \to S^5$ has order two, the image of $\eta_{\text{top}}^*: \pi_5 \text{SU}_3 \cong \mathbb{Z} \to \pi_6 \text{SU}_3 \cong \mathbb{Z}/6$ can have at most two elements. It consists of exactly two by [MT64, Prop. 3.2]: If a generator $g \in \pi_5 \text{SU}_3$, having the property that $pg = 2 \in \pi_5 S^5$, satisfied $g \eta_{\text{top}} = 0$, then the inclusion $p \circ \langle g, \eta_{\text{top}}, 2 \rangle \subset \langle pg, \eta_{\text{top}}, 2 \rangle = \{\eta_{\text{top}}^2\} \subset \pi_7 S^5$ would lead to a contradiction, because $p_*: \pi_7 \text{SU}_3 \to \pi_7 S^5$ is the zero map. Hence $[\Sigma^{1+(2)} \mathbb{P}^2, \text{BSL}_3] \cong \mathbb{Z}/3$.

Having identified the group where $\eta^*(\xi_3)$ resides in, it remains to prove that it is zero. Again it suffices to prove this after complex realization. The topological Hopf map $\eta_{\text{top}}: S^3 \to S^2$ generates the free group $\pi_3 S^2$, and in particular is not a torsion element. The latter is also true for the element $\eta_{\text{top}} \wedge \mathbb{C}P^2 \in [\Sigma^3 \mathbb{C}P^2, \Sigma^2 \mathbb{C}P^2]$, a group which can be determined via the long exact sequence

$$\cdots \to \pi_6 \Sigma^2 \mathbb{C}P^2 \xra{\eta_{\text{top}}} \pi_7 \Sigma^2 \mathbb{C}P^2 \xra{q^*} [\Sigma^3 \mathbb{C}P^2, \Sigma^2 \mathbb{C}P^2] \xra{i^*} \pi_5 \Sigma^2 \mathbb{C}P^2 \xra{\eta_{\text{top}}} \cdots$$

and [Muk82, Propositions 8.1 and 8.2]. It follows that $[\Sigma^3 \mathbb{C}P^2, \Sigma^2 \mathbb{C}P^2] \cong \mathbb{Z} \oplus \mathbb{Z}/3$, with the free part generated by the composition $\Sigma^3 \mathbb{C}P^2 \xra{\eta_{\text{top}}} S^7 \xra{\nu \circ q} S^4 \xra{\Sigma^2 \nu} \Sigma^2 \mathbb{C}P^2$. It is possible to employ an unstable version of the Toda bracket argument used in the stable computation $\eta \wedge \mathbb{C}P^2 = \pm (i \circ \nu \circ q)$ from (2.3), but since the unstable generator hits the stable generator $i \circ \nu \circ q$, the complex realization of $\eta^*(\xi_3)$ is thrice another element in the group $\mathbb{Z}/3$, hence zero. As a consequence, there exists a map $\Psi: \mathbb{P}^2 \wedge \mathbb{P}^2 \to \text{BSL}_3$ such that the composition

$$\mathbb{P}^1 \wedge \mathbb{P}^2 \xra{\iota \wedge \nu} \mathbb{P}^2 \wedge \mathbb{P}^2 \to \text{BSL}_3$$

equals $\xi_3$. 

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The argument for the first statement of the proposition already provides information on the possible choices involved. The aforementioned connectivity of the inclusion $B^2SL_3 \hookrightarrow BSL_3$ implies that the induced map $[P^2 \wedge P^2, B^2SL_3] \rightarrow [P^2 \wedge P^2, BSL_3]$ is an isomorphism, whence any given extension $P^2 \wedge P^2 \rightarrow BSL_3$ of $\xi_3$ lifts uniquely to the 2-skeleton. Let $\alpha : B^2SL_3 \rightarrow \Sigma^2SL_3^{∧2}$ denote the canonical projection. A straightforward motivic variant of [Tod62, Proposition 1.9] shows that for every possible extension $\xi_3$ of $\xi_3$, there exists an element $\lambda \in \langle \alpha, \xi_3, \eta \wedge P^2 \rangle$ such that the diagram

$$
\begin{array}{ccc}
P^2 \wedge P^2 & \xrightarrow{\xi} & B^2SL_3 \\
\downarrow & & \downarrow \alpha \\
\Sigma^{2+(2)}P^2 & \xrightarrow{\lambda} & \Sigma^2SL_3^{∧2}
\end{array}
$$

commutes, and conversely. The Toda bracket $\langle \alpha, \xi_3, \eta \wedge P^2 \rangle$, being defined by the argument above, is a subset of

$$[\Sigma^{2+(2)}P^2, \Sigma^2SL_3^{∧2}] \cong [S^{4+(4)}, \Sigma^2SL_3^{∧2}]$$

(the isomorphism implied by 3-connectivity of $\Sigma^2SL_3^{∧2}$) and a coset of the subgroup $[\Sigma^{2+(1)}P^2, \Sigma^2SL_3^{∧2}] \circ \eta + \alpha \circ [\Sigma^{2+(2)}P^2, B^2SL_3]$. Lemma 4.2 gives that the induced map $\pi^1\Sigma^{2+(1)}P^2 \wedge SL_2 \rightarrow \pi^1\Sigma^2SL_3^{∧2}$ is an isomorphism, which implies $\pi^1\Sigma^2SL_3^{∧2} \cong K^M_0$. It follows that $[S^{4+(4)}, \Sigma^2SL_3^{∧2}] \cong K^M_0 \cong \mathbb{Z}$ is constant. It remains to show that the Toda bracket $\langle \alpha, \xi_3, \eta \wedge P^2 \rangle$, contains the element corresponding to $1 \in \mathbb{Z}$. For this purpose, it remains to refer to the straightforward version of [Muk82, Lemma 6.1] in the pointed unstable $A^1$-homotopy category. It implies in particular that the Toda bracket $\langle q \wedge P^2, i \wedge P^2, \eta \wedge P^2 \rangle$ contains the identity on $\Sigma^{2+(2)}P^2$. Letting $\gamma$ denote the canonical map

$$\Sigma^{2+(2)}P^2 \xrightarrow{\Sigma^{2+(2)}q} S^{4+(4)} \simeq \Sigma^2SL_2^{∧2} \rightarrow \Sigma^2SL_3^{∧2}$$

there results an inclusion

$$\gamma \circ \langle q \wedge P^2, i \wedge P^2, \eta \wedge P^2 \rangle \subset \langle \gamma \circ q \wedge P^2, i \wedge P^2, \eta \wedge P^2 \rangle \subset \langle \alpha, \xi_3, \eta \wedge P^2 \rangle$$

giving the result. 

Similar to the homotopy cofiber $X$ of $P^1 \wedge P^2 \hookrightarrow B^2SL_3$, let $Y$ denote the homotopy cofiber of the composition $P^1 \wedge P^2 \cup_{P^1 \wedge P^1} P^2 \wedge P^1 \hookrightarrow P^2 \wedge P^2 \xrightarrow{\psi} B^2SL_3$, and let $Z$ denote the homotopy cofiber of $P^2 \wedge P^2 \xrightarrow{\psi} B^2SL_3$, the map provided by Lemma 4.3. The homotopy cofiber sequence

$$S^{2+(3)} \xrightarrow{\eta \wedge i} P^1 \wedge P^2 \hookrightarrow P^1 \wedge P^2 \cup_{P^1 \wedge P^1} P^2 \wedge P^1$$

induces a homotopy cofiber sequence

$$S^{3+(3)} \rightarrow X \rightarrow Y$$

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whence $Y \cong X \vee S^{4+(3)}$, as $X$ is $\mathbb{A}^1$-3-connected. Hence so is $Y$, and $\pi_4^A Y \cong \pi_4^A X \oplus \pi_4^A S^{4+(3)}$. In fact, already the attaching map $i \wedge \eta: S^{2+(3)} \to \mathbb{P}^1 \wedge \mathbb{P}^2$ is $\mathbb{A}^1$-homotopic to a constant map, as one may see from the identification $\pi_4^A \mathbb{P}^1 \wedge \mathbb{P}^2 \cong K^M_2$. A splitting
\[ \mathbb{P}^1 \wedge \mathbb{P}^2 \cup_{\mathbb{P}^1 \wedge \mathbb{P}^1} \mathbb{P}^2 \wedge \mathbb{P}^1 \cong \mathbb{P}^1 \wedge \mathbb{P}^2 \vee S^{3+(3)} \] follows. Similarly, the homotopy cofiber sequence
\[ \mathbb{P}^1 \wedge \mathbb{P}^2 \cup_{\mathbb{P}^1 \wedge \mathbb{P}^1} \mathbb{P}^2 \wedge \mathbb{P}^1 \to \mathbb{P}^2 \wedge \mathbb{P}^2 \to S^{4+(4)} \]
induces a homotopy cofiber sequence
\[ S^{4+(4)} \to Y \to Z \]
allowing to exert some control over $\pi_4^A Z$. The homotopy cofiber sequence (4.5) supplies a short exact sequence
\[ K^M_{5/6} \cong \pi_4^A S^{4+(5)} \to \pi_4^A X \to \pi_4^A \Sigma^2 \text{SL}_3 \wedge 2 \to 0 \]
in which $\pi_4^A \Sigma^2 \text{SL}_3 \wedge 2 \cong K^M_4$ induced by the map $\Sigma^{2+(1)} \text{SL}_2 \wedge \mathbb{P}^2 \to \Sigma^2 \text{SL}_3 \wedge 2$, as already computed in the proof of Lemma 4.3. The first component of the composition
\[ \pi_4^A S^{4+(4)} \to \pi_4^A Y \cong \pi_4^A X \oplus K^M_{3/6} \to K^M_4 \oplus K^M_{3/6} \]
coincides with the canonical projection $K^M_4 \to K^M_4$ by Lemma 4.3. The second component is $\eta$, as one may deduce from the splitting (4.7) and the attaching map for the top cell in $\mathbb{P}^2 \wedge \mathbb{P}^2$. There results an identification
\[ \pi_4^A Z \cong L_5 \oplus K^M_3 \] of homotopy sheaves, where $L_5$ is a quotient of $K^M_{5/6}$.

**Theorem 4.4.** Let $F$ be an infinite field of characteristic different from 2 and 3, and $A$ an essentially smooth local $F$-algebra. The image of the Suslin-Hurewicz homomorphism $K^\text{Quillen}_4(A) \to K^M_4(A)$ coincides with $6K^M_4(A)$.

**Proof.** By [AFW20, Theorem 2.18] it suffices to prove that a certain canonical surjection $K^M_4/6 \to S_4$, obtained from Suslin’s Hurewicz homomorphism, is an isomorphism. For this purpose, a map $S_4 \to K^M_4/6$ will be constructed. Doing so requires some information on Suslin’s Hurewicz homomorphism in degree 4. The $\mathbb{A}^1$-fiber sequence
\[ S^{3+(4)} \to \text{BSL}_3 \to \text{BSL}_4 \]
of motivic spaces induces a long exact sequence of unstable homotopy sheaves terminating with
\[ \cdots \to \pi_3^A S^{3+(4)} \to \pi_3^A \text{BSL}_3 \to \pi_3^A \text{BSL}_4 \to 0. \]
The sheaf $S_4$ is isomorphic to the kernel of $\pi_3^A^1 \text{BSL}_3 \to \pi_3^A^1 \text{BSL}_4$. Homotopical stability provides that $\pi_3^A^1 \text{BSL}_4 \cong \pi_3^A \text{BSL}_\infty \cong K_3$ is the third algebraic $K$-theory sheaf, as explained in [AFW20]. Morel’s unstable computations provide $\pi_3^A S^{3+(4)} \cong K_4^{\text{MW}}$ [Mor12, Theorem 1.23]. Hence there exists a surjection $K_4^{\text{MW}} \cong \pi_3^A S^{3+(4)} \to S_4$, which, as is also explained in [AFW20], factors over $K_4^M/6$. Thus it suffices to provide a map $S_4 \to K_4^M/6$ such that the composition $K_4^M/6 \to S_4 \to K_4^M/6$ is the identity. Consider the composition

$$S^{3+(4)} \xrightarrow{\nu} S^{2+(2)} \cong P^1 \wedge P^1 \xrightarrow{i \wedge i} P^2 \wedge P^2$$

of maps of motivic spaces. It induces a map $\pi_3^A S^{3+(4)} \xrightarrow{(i \wedge i) \circ \nu} \pi_3^A P^2 \wedge P^2$ of homotopy sheaves. Stabilization with respect to $P^1$ provides a canonical map $\pi_3^A P^2 \wedge P^2 \to \pi_3(0) P^2 \wedge P^2$ of homotopy sheaves. The unit map $1 \to kq$ induces a map

$$P^2 \wedge P^2 \to P^2 \wedge P^2 \wedge kq \cong \Sigma^{1+(1)} P^2 \wedge kgl$$

of motivic spectra, using the Wood cofiber sequence from [ARØ20, Proposition 11]. Lemma 4.1 provides a short exact sequence

$$0 \to K_4^M/6 \to \pi_3^{+(*)} P^2 \wedge P^2 \to \pi_2^{+(*)} P^2 \wedge kgl \to 0 \quad (4.9)$$

of $K^M$-modules, where the kernel is generated by $(i \wedge i) \circ \nu$. Hence the image of the map

$$\pi_3^A S^{3+(4)} \xrightarrow{\pi_3^A (i \wedge i \circ \nu)} \pi_3^A P^2 \wedge P^2 \xrightarrow{P^1 \to \pi_3(0) P^2 \wedge P^2}$$

coincides with the kernel $K_4^M/6$ of the short exact sequence (4.9). It remains to see that the resulting map $\pi_3^A S^{3+(4)} \to K_4^M/6$ factors over the surjection $\pi_3^A S^{3+(4)} \to S_4$. For this purpose, Lemma 4.3 provides a map $P^2 \wedge P^2 \xrightarrow{\Psi} B^2 \text{SL}_3 \to \text{BSL}_3$ of motivic spaces. One may check using complex or étale realization that the composition

$$S^{3+(4)} \xrightarrow{\nu} S^{2+(2)} \xrightarrow{i \wedge i} P^2 \wedge P^2 \xrightarrow{\Psi} B^2 \text{SL}_3 \to \text{BSL}_3$$

coincides with the canonical map $S^{3+(4)} \to \text{BSL}_3$, at least up to a sign. More precisely, the map $\eta \wedge CP^2 : \Sigma^3 CP^2 \to \Sigma^2 CP^2$ sends a generator (which maps to $2 \in \pi_7 S^7$ under the quotient map) to the element $\pm i \Sigma \nu \in \pi_7 \Sigma^2 CP^2$, by [Tod62, (5.4)]. Hence the homotopy group $\pi_7 CP^2 \wedge CP^2$ (which is computed by the cofiber sequence, thanks to the classical Blakers-Massey theorem and connectivity) is free cyclic, generated by $(i \wedge i) \circ \nu$, using [Muk82, Proposition 8.2]. Since the map $\pi_7 CP^2 \wedge CP^2 \to \pi_7 BSU_3$ is surjective by cellular approximation, the claimed commutativity follows topologically, and hence motivically.

If the kernel of $\pi_3^A P^2 \wedge P^2 \to \pi_3^A \text{BSL}_3$ and the image of $\pi_3^A S^{3+(4)} \to \pi_3^A P^2 \wedge P^2$ of the map $i \wedge i$, the produced map $\pi_3^A S^{3+(4)} \to K_4^M/6$ factors over the surjection $\pi_3^A S^{3+(4)} \to S_4$. With [AF16, Theorem 4.1], the kernel of $\pi_3^A P^2 \wedge P^2 \to \pi_3^A \text{BSL}_3$ coincides with the image of $\pi_3^A Z \to \pi_3^A P^2 \wedge P^2$, where $Z$ is the homotopy cofiber of
The image of $\pi_3^{A_1} S^{3+(4)} \to \pi_3^{A_1} P^2 \land P^2$ is a quotient of $K_4^{MW}$. In fact, it is even a quotient of $K_4^M$, because the composition

$$
\pi_3^{A_1} S^{4+(5)} \xrightarrow{\eta} \pi_3^{A_1} S^{3+(4)} \xrightarrow{\nu} \pi_3^{A_1} S^{2+(2)} \xrightarrow{i \land i} \pi_3^{A_1} P^2 \land P^2
$$

is zero. Consider the identification (4.8) describing $\pi_3^{A_1} Z$. Any element in the image of the composition $K_5^{MW} \to L_5 \subset \pi_3^{A_1} Z \to \pi_3^{A_1} P^2 \land P^2$ can be factored as a composition $S^3 \to S^{3+(5)} \to P^2 \land P^2$. Since the fifth contraction of the image of $K_4^M$ is zero, the contribution of $L_5$ intersects the image of $\pi_3^{A_1} S^{3+(4)}$ in $\pi_3^{A_1} P^2 \land P^2$ trivially. Similarly, any element in the image of $\pi_3^{A_1} S^{3+(4)} \to \pi_3^{A_1} P^2 \land P^2$ can be factored as a composition $S^3 \to S^{3+(4)} \to P^2 \land P^2$. Since the fourth contraction of $K_3^M$ is zero, the image of $\pi_3^{A_1} S^{3+(4)} \to \pi_3^{A_1} P^2 \land P^2$ intersects the image of $K_3^M \subset \pi_3^{A_1} Z \to \pi_3^{A_1} P^2 \land P^2$ trivially. Thus the image of $\pi_3^{A_1} Z \to \pi_3^{A_1} P^2 \land P^2$ injects into $\pi_3^{A_1} BSL_3$. As a consequence, $\pi_3^{A_1} S^{3+(4)} \to \pi_3^{A_1} P^2 \land P^2$ factors over the surjection $\pi_3^{A_1} S^{3+(4)} \to S_4$, completing the proof. \qed

A Modules over Milnor-Witt $K$-Theory

Let $F$ be a field, usually suppressed from the notation. The Milnor-Witt $K$-theory of $F$, as defined by Hopkins-Morel [Mor04], is the unital graded associative algebra generated by $\eta \in K_1^{MW} \cong \pi_{1,1}^1$ and the symbols $[u] \in K_1^{MW}(F) \cong \pi_{-1,-1}^1 1_F$ for every unit $u \in F^\times$, subject to the following relations.

Steinberg relation $[u][v] = 0$ whenever $u + v = 1$.

$\eta$-twisted logarithm $[uv] = [u] + [v] + \eta[u][v]$.

Commutativity $[u] \eta = \eta [u]$.

Hyperbolic plane $\eta + \eta^2[-1] = -\eta$.

If $u_1, \ldots, u_m \in F$ are units, then the element $(1 + \eta[u_1]) + \cdots + (1 + \eta[u_m]) \in K_0^{MW}(F)$ corresponds to the quadratic form $\langle u_1, \ldots, u_m \rangle$ given by the appropriate diagonal matrix under the identification of $K_0^{MW}(F)$ with the Grothendieck-Witt ring $GW(F)$ of $F$. Milnor $K$-theory [Mil70] is expressed as the quotient $K_*^M \cong K_0^M/\eta K_0^M$. Often the grading will be suppressed from the notation. Its reduction modulo 2 is denoted $K^M$ for brevity. If $A$ is a (graded) $K^M$-module, its degree $d$ part is $A_d$. For any $x \in K_d^M$, the kernel and cokernel of multiplication with $x$ on $A_\ell$ are denoted $x A_\ell$ and $A_\ell/x A_{\ell-d}$. As a warm-up, $K^M$-modules will be treated first.

Theorem A.1. There is an equality $\{-1\} K^M = 2K^M$ of $K^M$-modules.
Proof. This follows as an equality of zero modules from [Izh91] if \( F \) has characteristic 2. Suppose now that the characteristic of \( F \) is different from 2. Since \( 2\{-1\} = \{1\} = 0 \in K_1^M \), there is an inclusion \( \{-1\}K^M \subset 2K^M \) of \( K^M \)-modules. The cofiber sequence

\[
\text{MZ} \xrightarrow{2} \text{MZ} \xrightarrow{pz^\infty}{\xrightarrow{2}} \text{MZ}/2 \xrightarrow{\partial_2}{\xrightarrow{2}} \Sigma \text{MZ}
\]

induces a short exact sequence

\[
0 \to \pi_{1+(⋆)} \text{MZ}/2\pi_{1+(⋆)} \text{MZ} \to \pi_{1+(⋆)} \text{MZ}/2 \xrightarrow{\partial_2}{\xrightarrow{2}} \pi_{0+(⋆)} \text{MZ} \to 0
\]

def K^M-modules. Let \( \tau \in \pi_{1-(1)} \text{MZ}/2 = h^{0,1} \) denote the class of \(-1 \in F\). Then \( \partial_2(\tau) = \{-1\} \in \pi_{0-(1)} \text{MZ} = K^M_1 \). Voevodsky’s solution of the Milnor conjecture on Galois cohomology provides that multiplication with \( \tau \) is an isomorphism \( \pi_{0+(⋆)} \text{MZ}/2 \cong \pi_{1+(⋆)} \text{MZ}/2 \) of \( K^M \)-modules. It follows that multiplication with \( \{-1\} \) on \( K^M \) factors as

\[
\text{K}^M \xrightarrow{pz^\infty} \pi_{0+(⋆)} \text{MZ}/2 \xrightarrow{\tau} \pi_{1+(⋆)} \text{MZ}/2 \xrightarrow{\partial_2}{\xrightarrow{2}} \pi_{0+(⋆)} \text{MZ} = 2\text{K}^M \subset \text{K}^M
\]

in which the first three maps are surjective. The result follows.

\[\square\]

Lemma A.2. Let \( A \) be a \( K^M \)-module. There is an isomorphism

\[
\text{Ext}_{K^M}^1(k^M, A) \cong \{-1\}A_0/2A_0
\]

which is natural in \( A \).

Proof. The short exact sequence

\[
0 \to 2K^M \to K^M \to k^M \to 0
\]

induces an exact sequence

\[
0 \to \text{Hom}_{K^M}(k^M, A) \to \text{Hom}_{K^M}(K^M, A) \to \text{Hom}_{K^M}(2K^M, A) \to \text{Ext}_{K^M}^1(k^M, A) \to 0.
\]

The short exact sequence

\[
0 \to 2K^M \to K^M \to 2K^M \to 0
\]

induces an exact sequence

\[
0 \to \text{Hom}_{K^M}(2K^M, A) \to \text{Hom}_{K^M}(K^M, A) \to \text{Hom}_{K^M}(2K^M, A) \to \text{Ext}_{K^M}^1(2K^M, A) \to 0.
\]

The composition \( K^M \to 2K^M \to K^M \) coincides with multiplication by 2, whence naturally \( \text{Hom}_{K^M}(k^M, A) = 2A_0 \). In order to identify \( \text{Hom}_{K^M}(2K^M, A) \) in a similar way, observe the equality

\[
2K^M = \{-1\}K^M
\]

of \( K^M \)-modules from Theorem A.1. The composition \( K^M \to \{-1\}K^M = 2K^M \to K^M \) coincides with multiplication by \( \{-1\} \), whence naturally \( \text{Hom}_{K^M}(2K^M, A) = \{-1\}A_0 \). The statement follows.

\[\square\]
Lemma A.3. Let $A$ be a $K^{MW}$-module. There is a natural isomorphism

$$\text{Ext}^1_{K^{MW}}(K^M, A) \cong \eta A_{-1}/\eta A_0.$$ 

Proof. Abbreviate $\text{Hom}(A, B) := \text{Hom}_{K^{MW}}(A, B)$. The short exact sequence

$$0 \to \eta K^{MW} \to K^{MW} \to K^M \to 0$$

induces an exact sequence

$$0 \to \text{Hom}(K^M, A) \to \text{Hom}(K^{MW}, A) \to \text{Hom}(\eta K^{MW}, A) \to \text{Ext}^1_{K^{MW}}(K^M, A) \to 0.$$ 

The short exact sequence

$$0 \to \eta K_{*+1}^{MW} \to K_{*+1}^{MW} \to \eta K^{MW} \to 0$$

induces an exact sequence

$$0 \to \text{Hom}(\eta K^{MW}, A) \to \text{Hom}(K_{*+1}^{MW}, A) \to \text{Hom}(\eta K_{*+1}^{MW}, A) \to \text{Ext}^1_{K^{MW}}(\eta K^{MW}, A) \to 0.$$ 

The composition $K^{MW}_{*+1} \to \eta K^{MW} \to K^{MW}$ coincides with multiplication by $\eta$, whence naturally $\text{Hom}(K^M, A) = \eta A_0$. In order to identify $\text{Hom}(\eta K^{MW}, A)$ in a similar way, observe the existence of a further short exact sequence

$$0 \to hK^{MW} \to K^{MW} \to hK^{MW} = \eta K^{MW} \to 0$$

of $K^{MW}$-modules, the surjection being induced by multiplication with $h$. Note that $hK^{MW} = hK^M = \eta K^{MW}$ by Theorem 3.2. The composition $K^{MW}_{*+1} \to \eta K^{MW} \to K_{*+1}^{MW}$ coincides with multiplication by $h$, whence naturally $\text{Hom}(\eta K^{MW}, A) = hA_{-1}$. The statement follows.

Lemma A.4. Let $A$ be a $K^{MW}$-module. There is a natural exact sequence

$$0 \to \{-1\}(\eta A_0)/2(\eta A_0) \to \text{Ext}^1_{K^{MW}}(k^M, A) \to hA_{-1}/\eta A_0$$

Proof. Abbreviate $\text{Hom}(A, B) := \text{Hom}_{K^{MW}}(A, B)$ and $\text{Ext}(A, B) := \text{Ext}_{K^{MW}}(A, B)$. The short exact sequence

$$0 \to 2K^M \to K^M \to k^M \to 0$$

of $K^{MW}$-modules induces an exact sequence

$$0 \to \text{Hom}(k^M, A) \to \text{Hom}(K^M, A) \to \text{Hom}(2K^M, A) \to \text{Ext}^1(k^M, A) \to \text{Ext}^1(K^M, A) \to \cdots .$$

The natural map $\text{Hom}_{K^{MW}}(B, A) \to \text{Hom}_{K^M}(B, \eta A)$ is an isomorphism whenever $\eta B = 0$. The result then follows from Lemma A.2 and Lemma A.3. 

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