Effects of the Encapsulating Membranes on the Translations of Pairs of Non-spherical Bubbles

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Abstract. In this paper, we perform a theoretical analysis on the translational motions of gas and encapsulated bubbles undergoing deformation. We verify our theoretical model for the case of a gas bubble collapsing near a wall. By comparison, the encapsulated bubble near the wall is less unstable in shape. In the second case, the translations of two gas bubbles subjected to an oscillating driving pressure agree with Bjerknes’ theory, that is, the two bubbles repel each other when the driving frequency is between the two natural frequencies \( \omega_0 < \omega_d < \omega_2 \); while the two bubbles move towards each other at \( \omega_d < \omega_0 \) or \( \omega_d > \omega_2 \). For encapsulated bubbles, the translational displacements are smaller than those of gas bubbles. The encapsulating membrane and the deformation affect the translational direction. The encapsulated bubbles are prone to experience repulsive motion at resonance.

1. Introduction
In the applications of contrast-enhanced agents and drug-carrier capsules, encapsulated bubbles do not exist independently. Surrounding bubbles affect the behavior of each bubble. The interaction of two gas bubbles without encapsulation has been well studied. The pioneer work traces back to that of Bjerknes in 1906 [1]. The famous secondary Bjerknes force is the reason of motion of a bubble due to fluid acceleration caused by the volume oscillations of another bubble. The coupled pulsation and translation were studied in recent decades [2,3]. The effects of the shape oscillation are included in the theoretical work of Takahira et al. [4] and in numerical work by Pelekasis et al. [5,6] and Chatzidai et al. [7]. Most of the previous work on encapsulated bubbles, which are widely used in medical ultrasound, focused on the single encapsulated bubble [8-10]. The interaction of two encapsulated bubbles is seldom studied. In this work, we study the translation of two encapsulated bubbles. We consider the deformation of the bubbles to investigate the effects of the shape oscillation. To this aim, we perform a linear analysis of the dynamics of the radial oscillation, the translational motion and the shape oscillations of two encapsulated bubbles.

2. Basic equations
The dynamics of two bubbles suspended in an unbounded fluid are studied. We establish an axisymmetric coordinate system with the symmetry axis connecting the centres of the two bubbles (figure 1). In the following equations, the subscript \( i \) denotes the one bubble, and \( j = 3 - i \) denotes the other one. Only the equations for the bubble \( i \) are written, while the equations for the bubble \( j \) are equivalent with the \( i \) and \( j \) exchanged.

The bubbles are coated by a viscoelastic membrane. We adopt the neo-Hookean law characterized by the surface modulus \( G_{si} \), the bending modulus \( G_{bi} \), the membrane viscosity \( \mu_{si} \), and the Poisson ratio \( \nu_{si} \).
The shape deformation is described by the radial and tangential disturbances \( a_k \) and \( b_k \), respectively, with respect to the \( k \)-th order (associated) Legendre polynomial. The dynamic balances in the normal and tangential directions yield the equations for \( a_k \) and \( b_k \). The normal dynamic balance equation is

\[
\frac{R_i}{k+1} \frac{d}{dt} \left( \frac{\dot{R}_i}{k+1} + \frac{2(k+2)}{\rho R_i} \right) + a_k \left[ \frac{k-1}{k+1} \frac{\dot{R}_i}{R_i} + (k+2)(k-1) \frac{\mu}{\rho R_i^2} \right] + \frac{4(k-1)\mu}{\rho R_i^2} \dot{R}_i + \frac{2\mu k}{\rho L} \int_0^{\infty} \left( \frac{\dot{R}_i}{R_i} \right)^3 - 1 \left( \frac{R_i}{s} \right)^3 - L^3 \left( \frac{\dot{R}_i}{R_i} \right) T_i(s,t) ds \right] = -2R_i \rho [\dot{v}_r]. 
\]

The second term is the Bjerknes force. The third term is the drag force of a sphere with radius \( R_i \) in the tangential directions yield the equations for \( v_r \).

The fourth and fifth terms are the effect of the other bubble and the translations.

\[
q_{ij} = -\frac{R_i}{R_j} \frac{R_j}{R_i} \text{ is a coefficient from velocity potential.} \]

\[
The last four terms on the left hand side of the equation represent the effect of the other bubble and the translations. \]

\[
q_{ij} = -\frac{R_i}{R_j} \frac{R_j}{R_i} \text{ is a coefficient from velocity potential.} \]

\[
The right hand side of the equation is the membrane force.
\]

\[
The equation of translational motion is:
\]

\[
\frac{d}{dt} \left( \frac{1}{2} \rho V_b \dot{v}_r \right) - \rho V_b \frac{d}{dt} \left( \frac{q_{ij}}{s} \right) = \frac{C_D}{2} \pi \rho R_i^3 \dot{v}_r + \rho V_b \frac{q_{ij}}{s} \int_0^{\infty} \left( \frac{R_i}{s} \right)^3 - 1 \left( \frac{R_i}{s} \right)^3 - L^3 \left( \frac{\dot{R}_i}{R_i} \right) T_i(s,t) ds \right] = -2R_i \rho [\dot{v}_r]. 
\]

In the above translation equation, the first term represents the inertial term of the added mass. The second term is the Bjerknes force. The third term is the drag force of a sphere with radius \( R_i \) at the velocity \( v_r \).

The coefficient of drag force \( C_D \) is consistent with that of a rigid sphere. The fourth and fifth terms are the effect of viscous toroidal field. The sixth term is the perturbed term related to the shape oscillation and the translational motion of the other bubble. The right hand side is the membrane force due to the movement of the material points along the surface.

The shape deformation is described by the radial and tangential disturbances \( a_k \) and \( b_k \), respectively, with respect to the \( k \)-th order (associated) Legendre polynomial. The dynamic balances in the normal and tangential directions yield the equations for \( a_k \) and \( b_k \). The normal dynamic balance equation is

\[
\frac{R_i}{k+1} \frac{d}{dt} \left( \frac{\dot{R}_i}{k+1} + \frac{2(k+2)}{\rho R_i} \right) + a_k \left[ \frac{k-1}{k+1} \frac{\dot{R}_i}{R_i} + 2(k+2)(k-1) \frac{\mu}{\rho R_i^2} \right] + \frac{4(k-1)\mu}{\rho R_i^2} \dot{R}_i + \frac{2\mu k}{\rho L} \int_0^{\infty} \left( \frac{\dot{R}_i}{R_i} \right)^3 - 1 \left( \frac{R_i}{s} \right)^3 - L^3 \left( \frac{\dot{R}_i}{R_i} \right) T_i(s,t) ds \right] = -2R_i \rho [\dot{v}_r]. 
\]
where

\[ k = 2, \quad G_k = \frac{3}{4} v_i \left[ \dot{a}_{k-1,i} + \frac{2k - 2}{2k - 1} \frac{\dot{R}_i a_{k-1,i}}{R_i} - \frac{(k - 1) R_i^{k-2} q_{01}}{L_k^k} \right] \]  

(5)

And the tangential dynamic balance equation is

\[ \frac{2 \mu_1}{R_i} \left[ k + 1 + \frac{\dot{a}_{k,i}}{R_i} \right] - \frac{k - 1}{R_i} \frac{\dot{R}_i a_{k,i}}{R_i} + \frac{2k + 1}{k + 1} \frac{R_i^{k-2} q_{01}}{R_i^{k+1}} + \frac{3k(2k + 1)}{2(2k - 1)} \frac{a_{k-1,i} \dot{v}_i}{R_i} - R_i^{k-2} \int_{R_i}^{\infty} s^{-k} T_i(s, t) ds \]

\[ \frac{\dot{G}_{k,i}}{R_i^3} \left[ k^2 + k - 1 + v_i \right] \left( a_{k,i} - b_{k,i} \right) + \frac{2 \mu_0}{R_i} \left[ 2 \dot{R}_i a_{k,i} - R_i \dot{a}_{k,i} + (k^2 + k - 1) \left( R_i b_{k,i} - \dot{R}_i b_{k,i} \right) \right]. \]  

(6)

In addition, the continuity in tangential velocity is needed:

\[ \frac{1}{k + 1} \left[ \dot{a}_{k,i} + 2 \frac{\dot{R}_i a_{k,i}}{R_i} - (2k + 1) q_{01} R_i^{k-1} \right] - \frac{3k \dot{v}_i a_{k-1,i}}{2(2k - 1) R_i} + \frac{R_i^{k-1}}{R_i} \int_{R_i}^{\infty} s^{-k} T_i(s, t) ds = b_{k,i} - \frac{\dot{R}_i b_{k,i}}{R_i} \]  

(7)

In the above equations, the toroidal field \( T_i \) is solved by

\[ \rho \frac{\partial T_i}{\partial t} + \rho \frac{\partial}{\partial r_i} \left[ \dot{R}_i (R_i/r_i)^2 T_i \right] - \mu_0 \frac{\partial^2 T_i}{\partial r_i^2} + \mu_0 k (k + 1) r_i^{-2} T_i = 0, \]  

(8)

with the boundary condition \( T_i \to 0 \) as \( r_i \to \infty \). \( T_i \) on the bubble surface is determined by satisfying the tangential force balance.

For the above equations, the readers are referred to [11] for a detailed derivation.

3. Results

In this part, we will show some results of both gas bubbles and encapsulated bubbles. For gas bubbles, the coefficient of surface tension is chosen as \( \gamma = 0.0729 \) N/m. For encapsulated bubbles, the elastic modulus is \( G_s = 0.5 \) N/m, the bending modulus is \( G_b = 2 \times 10^{-13} \) N m, and the membrane viscosity is \( \mu_{m} = 10^{-8} \) kg/s.

The case of two bubbles with the same size is equivalent to that of a single bubble near a wall which is located at the midpoint between the centres of the two bubbles. In order to compare with the experimental result of Lauterborn & Bolle[12], we choose the same parameters. The initial radius is \( R_{01} = 2.6 \times 10^{-3} \) m. The initial distance to the wall is \( 1.5 R_{01} \). The pressure outside the bubbles is set to be \( p_{\infty} = 1 \times 10^5 \) Pa, and the pressure inside the bubble is \( p_{01} = 2337 \) Pa initially. The solid line in Fig. 2 (a) shows the bubble shapes when \( t = t/[R_{01}(\rho/\Delta \rho)^{1/2}] = 0.725, 0.825, 0.961, 0.991, 1.016 \), progressing in inwards direction. The dots denote the experimental results. The bubble becomes elongated in the direction perpendicular to the wall and moves toward the wall. The third-order shape mode appears in the final time step, predicting a jet pointing towards the wall.

Figure 2 (b) shows the process of shrinking for an encapsulated bubble near a wall. We choose the same parameters as those in Fig. 2 (a), except for the parameters of the membrane which are added. Coincidentally, the natural frequency of this encapsulated bubble is 1258 Hz, close to that of the gas bubble in Fig. 2 (a) 1255 Hz. The deformation of the encapsulated bubble at \( t = 1.016 \) is smaller than that of the gas bubble, implying that the encapsulated bubble is more stable.

According to Bjerknes’ theory, the two gas bubbles will move away from each other when the driving frequency is between the two natural frequencies \( \omega_{01} < \omega_d < \omega_{02} \); otherwise, the two bubble will attract to each other at \( \omega_d < \omega_{01} \) or \( \omega_d > \omega_{02} \). To validate this theory, we choose two bubbles with \( R_{01} = 10 \) \( \mu \) m and \( R_{02} = 5 \) \( \mu \) m. We plot the final distances between the two bubbles after 20 driving cycles versus the driving frequencies in Fig. 3. For gas bubbles, the zeroth-order natural frequencies are \( \omega_{01} = 2 \pi \times 0.34 \) MHz and \( \omega_{02} = 2 \pi \times 0.72 \) MHz, respectively. We can see that the two bubbles attract each other when the driving frequency is smaller than both of the natural frequencies. When the driving frequency approaches to the smaller natural frequency, i.e. \( \omega_d = \omega_{01} \), the distance has a minimum. After this minimum, the attraction becomes repulsive. When the driving frequency approaches the other natural frequency, i.e. \( \omega_d = \omega_{02} \), the
two bubbles repel until a maximal distance. After this maximum, the translational motion of the bubbles shift to a weak attraction. For encapsulated bubbles, the zeroth-order natural frequencies are $\omega_{01} = 2\pi \times 0.51$ MHz and $\omega_{02} = 2\pi \times 1.28$ MHz, respectively. We can see that the repulsive and attractive motions are different from those of the Bjerknes theory (Fig. 3(b)). We predict the effects of the membrane force, and shape oscillations play an important role in these translations. A further investigation is in progress.

4. Summary
We investigated a bubble with and without an encapsulating membrane when collapsing near a wall by means of theoretical analysis. The encapsulated bubble behaves more stably in shape than the gas bubble when going towards the wall. The translations of the two bubbles subjected to a pressure wave at different frequencies were also studied. The behavior of gas bubbles agrees with Bjerknes’ theory, that is, the two bubbles repel from each other when the driving frequency is between the two natural frequencies $\omega_{01} < \omega_d < \omega_{02}$, otherwise, at $\omega_d < \omega_{01}$ or $\omega_d > \omega_{02}$, the two bubbles will move towards each other. However, the encapsulated bubbles show a different translational direction due to the effects of the membranes and the shape oscillations.

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References
[1] Bjerknes V F K 1906 *Fields of Force*, Columbia University Press
[2] Barbat T, Ashgriz N and Liu C 1999 *J. Fluid Mech.*, 389 137
[3] Harkin A, Kaper T J and Nadim A 2001 *J. Fluid Mech.*, 445 377
[4] Takahira H, Akamatsu T and Fujikawa S 1991 *Trans. JSME B* 57 447
[5] Pelekasis N A and Tsamopoulos 1993 *J. Fluid Mech.*, 254 467
[6] Pelekasis N A and Tsamopoulos 1993 *J. Fluid Mech.*, 254 501
[7] Chatzidai N, Dimakopoulos Y and Tsamopoulos 2011 *J. Fluid Mech.*, 673 513
[8] Tsiglifis K and Pelekasis N A 2011 *Phys. Fluids* 23 012102
[9] Liu Y, Sugiyama K, Takagi S and Matsumoto Y 2011 *Phys. Fluids* 23 041904
[10] Liu Y, Sugiyama K, Takagi S and Matsumoto Y 2012 *J. Fluid Mech.*, 691 315
[11] Liu Y, Sugiyama K and Takagi S in preparation
[12] Lauterborn W and Bolle H 1975 *J. Fluid Mech.* 72 391