Finsler-Geometrical Approach to the Studying of Nonlinear Dynamical Systems

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Abstract

A two dimensional Finsler space associated with the differential equation
\[ y'' = Y_3y'^3 + Y_2y'^2 + Y_1y' + Y_0 \]
is characterized by a tensor equation and called the Douglas space. An application to the Lorenz nonlinear dynamical equation is discussed from the standpoint of Finsler geometry.

1 Introduction

The differential equation is usually the most appropriate mathematical tool for analyzing a dynamical system. In the 1960's E. Lorenz used a computer to model weather patterns, using a set of ordinary nonlinear equations
\[
\dot{x} = k(x - y), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz.
\]
This is perhaps the most celebrated set of nonlinear ordinary differential equations.

Since 1992 the first author has derived the second order differential equation \( y'' = f(x,y,y') \) from the Lorenz set and continued to study it from the standpoint of the differential geometry ([4], [5], [6]). Two of his results was very attractive to the second author:

1) The Lorenz equation coincides with the differential equation of geodesics of a two-dimensional space which belongs to the special class of Finsler spaces, called the Berwald (affinely connected) spaces [4], and

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2) The necessary condition (presented in invarianten under point transformations form) for determination of the Finsler metrics for equations type
\[ y'' = Y_3 y'^3 + Y_2 y'^2 + Y_1 y' + Y_0, \] [5].

We remember various geometrical investigations of the differential equation of the form
\[ y'' = Y_3(x, y) y'^3 + Y_2(x, y) y'^2 + Y_1(x, y) y' + Y(x, y), \]
(e.g., [3]). The Lorenz equation is just of this form. Recently the second author and S.Bacso [2] have succeeded in the characterization by a tensor equation of Finsler spaces whose geodesic equation is of this form; those spaces are called Douglas spaces. Therefore the remarkable results of the first author can be described as an interesting theory based on the Finsler-geometrical foundations.

2 Preliminaries

We consider an n-dimensional Finsler space \( F^n = (M^n, L(x, y)) \) on a smooth n-manifold \( M^n \) ([1], [8]). The fundamental function \( L(x, y) \), a real-valued function on the tangent bundle \( T M^n \), is usually supposed certain conditions from the geometrical standpoint, but only the homogeneity and the regularity are mainly important for our following considerations.

1. \( L(x, y) \) be positively homogeneous in \( y^i \) of degree one:
\[ L(x, py) = pL(x, y), \text{ for } p > 0. \]

2. \( L(x, y) \) be regular:
\[ g_{ij} = \hat{\partial}_i \hat{\partial}_j F \]
has non-zero \( g = \det g_{ij} \), where \( F = L^2/2 \) and \( \hat{\partial}_i = \frac{\partial}{\partial y^i} \).

Let \( (g^{ij}) \) be the inverse matrix of \( (g_{ij}) \) and construct
\[ 2\gamma^i_{jk}(x, y) = g^{ir}(\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{kj}), \]
\[ 2G^i(x, y) = g^{ij}\{(\partial_j \partial_i F) y^r - \partial_j F\}, \]
where \( \partial_i = \frac{\partial}{\partial x^i} \). Then we have \( \gamma^i_{jk}(x, y)y^j y^k = 2G^i(x, y) \).

The length \( s \) of an arc \( C: x^i = x(t) \) on \( M^n \) is defined by the integral \( s = \int L(x, \dot{x}) dt \), \( \dot{x}^i = \frac{dx^i}{dt} \), and the extremum of the integral, called the
geodesic, is given by the Euler differential equations \( d(\dot{\partial}L)/dt - \partial_i L = 0 \), which are written in the form

\[
\dot{x}^i \{ \ddot{x}^i + 2G(x, \dot{x}) \} - \dot{x}^j \{ \dddot{x}^j + 2G(x, \dot{x}) \} = 0.
\] (1)

In order to introduce geometrical quantities in \( F^n \), we are concerned with a Finsler connection \( F\Gamma = (F^i_{jk}(x, y), N^i_j(x, y), V^i_{jk}(x, y)) \) on \( F^n \). For a tensor field \( F\Gamma \) gives rise to the h and v - covariant differentiations: We treat of a tensor field \( X^i(x, y) \) of (1,0)-type, for brevity. Then we get two tensor fields as follows:

\[
\nabla^h_j X^i = \delta^i_j X^i + X^r F^i_{rj}(x, y),
\]

\[
\nabla^v_j X^i = \hat{\delta}^i_j X^i + X^r V^i_{rj}(x, y),
\]

where \( \delta^i_j = \partial^i_j - N^i_j(x, y) \hat{\partial}_r \). The h and v-covariant derivatives \( \nabla^h X \) and \( \nabla^v X \) are tensor fields of (1,1)-type.

In the following we needs the Berwald connection \( B\Gamma = (G^i_j(x, y), G^i_j(x, y), 0) \), where

\[
G^i_j = \hat{\partial}^i_j, \quad G^i_{jk} = \hat{\partial}^h_k G^i_j.
\]

"\( V^i_{jk} = 0" \) for \( B\Gamma \) means that \( \nabla^v_j = \hat{\partial}_i \) in \( B\Gamma \). We shall denote by (:) the h-covariant differentiation \( \nabla^h \) in \( B\Gamma \). Then we obtain the commutation formulae, called the Ricci identities:

\[
X^i_{\dot{j};k} - X^i_{\dot{k};j} = X^r H^i_{rjk} - X^r \hat{R}^i_{rjk},
\]

\[
\hat{\partial}^i_k(X^i_{\dot{j}}) - (\hat{\partial}^h_k X^i)_{\dot{j}} = X^r G^i_{rjk}.
\] (2)

\( R^i_{jk} \) is called the \((v)h\)-torsion tensor, defined by

\[
R^i_{jk} = \hat{\partial}^i_k G^i_j - G^i_{jr} G^r_k - [j, k],
\]

where \([j, k]\) denotes the interchange of indices \( j, k \) of the preceding terms. \( H^i_{jk} \) and \( G^h_{ijk} \) are called the h and hv- curvature tensors respectively, defined by

\[
H^h_{ij} = \dot{\partial}_i R^h_{jk}, \quad G^h_{ijk} = \hat{\partial}_i G^h_{jk}.
\] (3)

It is noted that \( G^h_{ijk} \) is symmetric in the subscripts. The contracted tensor \( \hat{G}^r_{jk} = G^i_{jk} \) called the hv-Ricci tensor.

For the later use we shall give the three classes of special Finsler spaces as follows:
1. Riemannian spaces, characterized by $g_{ij} = g_{ij}(x)$, that is, the C-tensor $C_{ijk} = (\partial_k g_{ij})/2$ vanishes.

2. Locally Minkowski spaces, characterized by the existence of the adapted coordinate system $(x^i)$ such that $L = L(y)$. Then $G_{ijk} = 0$ and $G^h_{ijk} = 0$. The tensorial characterization is $R^i_{jk} = 0$ and $G^h_{ijk} = 0$.

3. Berwald spaces, characterized by $G^i_{jk} = G^i_{jk}(x)$, that is, $G^h_{ijk} = 0$. The classes (1) and (2) are contained in the class (3).

We consider a change of Finsler metric: $F^n = (M^n, L(x, y)) \rightarrow \bar{F}^n = (M^n, \bar{L}(x, y))$. If any geodesic of $F^n$ coincides with a geodesic of $\bar{F}^n$ as a set of points and vice versa, then the change is called projective and $F^n$ is said to be projectively related to $\bar{F}^n$ ([1],[7]). $F^n$ is projectively related to $\bar{F}^n$, if and only if there exists a scalar field $P(x,y)$, positively homogeneous in $y^i$ of degree one, satisfying

$$\bar{G}^i(x, y) = G^i(x, y) + P(x, y)y^i.$$ 

If we put $P_i = \dot{\partial}_i P$, then we get

$$\bar{G}^i_j = G^i_j + P_j y^i + P \delta^i_j.$$ 

From these relations we obtain the invariant of projective change as follows:

$$Q^h = G^h - \frac{1}{n + 1} G^r_{r} y^h. \tag{4}$$

Consequently we are led to the following projective invariants by means of successive differentiation with respect to $y^i$:

$$Q^h_i = \dot{\partial}_i Q^h = G^h_i - \frac{1}{n + 1}(G^r_{ri} y^h + G^r_i \delta^h_i),$$

$$Q^h_{ij} = \dot{\partial}_j Q^h_i = G^h_{ij} - \frac{1}{n + 1}(G^r_{rji} y^h + G^r_{rj} \delta^h_i + G^r_{rj} \delta^h_j),$$

where $G_{ij}$ is the hv-Ricci tensor. Further we get the Douglas tensor

$$D^h_{ijk} = \dot{\partial}_k Q^h_{ij} = G^h_{ijk} - \frac{1}{n + 1}\{ (\dot{\partial}_k G_{ij}) y^h + G_{ij} \delta^h_k + G_{kj} \delta^h_i + G_{ik} \delta^h_j \}, \tag{5}$$

where $G^h_{ijk}$ is the hv-curvature tensor.

A Finsler space $F^n = (M^n, L(x, y))$ is said to be with rectilinear extremals, if $M^n$ is covered by coordinate neighborhoods in which any geodesic
is represented by n linear equations \( x^i = x^i_0 + t a^i \) in a parameter \( t \), where \( x^i_0 \) and \( a^i \) are constants.

Next a Finsler space is called projectively flat, if it has a covering by coordinate neighborhoods in which it is projectively related to a locally Minkowski space. A Finsler space is projectively flat, if and only if it is with rectilinear extremals. We have the well-known theorem as follows:

Theorem Pf.
A Finsler space of dimension \( n \) is projectively flat, if and only if
1. \( n \geq 3 \): \( W^i_{jk} = 0 \) and \( D^h_{ijk} = 0 \),
2. \( n=2 \): \( K_{ij} = 0 \) and \( D^h_{ijk} = 0 \),

here \( D^h_{ijk} \) is the Douglas tensor. "\( W^i_{jk} = 0 \)" is equivalent to the fact that \( F^n \) be of scalar curvature. On the other hand, "\( K_{ij} = 0 \)" for \( n = 2 \) is a differential equation satisfied by the h-scalar curvature \( R \) (or the Gauss curvature, cf. p. 4).

3 Douglas spaces

The present section is devoted to the short introduction to the recent theory which was given by [2].

We shall start our discussions from the equations (1) of geodesics of a two-dimensional Finsler space \( F^2 \). If we denote \((x^1, x^2)\) by \((x,y)\), take \( x \) as the parameter \( t \) and use the symbols \( y' = dy/dx \), \( y'' = dy'/dx \), then (1) \((i=1, j=2)\) for \( F^2 \) is written in the form

\[
y'' = f(x, y, y') = X_3 y'^3 + X_2 y'^2 + X_1 y' + X_0,
\]

where \( X_3 = G^1_{22} \), \( X_2 = 2G^1_{12} - G^2_{22} \), \( X_1 = G^1_{11} - 2G^2_{12} \), \( X_0 = -G^2_{11} \) and \( G^i_{jk} = G^i_{jk}(x, y, 1, y') \) [10].

If we are specially concerned with a Riemannian space \( F^2 \), then \( G^i_{jk} = \gamma^i_{jk} \) are usual Christoffel symbols, and hence \( X \)'s of (3) do not contain \( y' \). Next, if \( F^2 \) is a Berwald space, then \( G^i_{jk} \) do not contain \( y' \) by definition. Consequently \( f(x, y, y') \) of those spaces is a polynomial in \( y' \) of degree at most three.

This special property of \( f(x, y, y') \) is equivalent to the fact that \( \dot{x}^1 G^2(x, \dot{x}) - \dot{x}^2 G^1(x, \dot{x}) \) of (1) is a homogeneous polynomial in \( \dot{x}^1, \dot{x}^2 \) of degree three.

Generalizing this fact, we shall give

Definition
A Finsler space $F^n$ is said to be of Douglas type or called a Douglas space, if $D^{ij}(x, y) = G^i(x, y)y^j - G^j(x, y)y^i$ are homogeneous polynomials in $y^i$ of degree three.

Proposition 1.

A Berwald space is of Douglas type, where $G^i(x, y)$ of (1) are of the form $G^i_{jk}(x)y^jy^k/2$

Theorem 1.

A Finsler space $F^2$ of dimension two is of Douglas type, if and only if, in every local coordinate system $(x, y)$ the differential equation $y'' = f(x, y, y')$ of geodesics is such that $f(x, y, y')$ is a polynomial in $y'$ of degree at most three.

Example 1.

We consider a Randers space $R^2$ of dimension two, that is, the metric being $L(x, y) = \alpha + \beta = a_{11}(x, y)\dot{x}^2 + 2a_{12}(x, y)\dot{x}\dot{y} + a_{22}(x, y)\dot{y}^2$, $\beta = b_1(x, y)\dot{x} + b_2(x, y)\dot{y}$. Suppose that the Riemannian $\alpha$ be positive-definite, and hence we can refer to an isothermal coordinate system $(x, y)$ such that

$$\alpha = aE, \quad a = a(x, y) > 0, \quad E = \sqrt{\dot{x}^2 + \dot{y}^2}.$$ 

Then the equation of geodesics of $R^2$ is written in the form

$$ay'' + (a_x y' - a_y)(1 + y'^2) = (b_{1y} - b_{2x})(1 + y'^2)^{3/2}.$$ 

Consequently $R^2$ is not of Douglas type in general; $R^2$ is of Douglas type, if and only if $b_{1y} - b_{2x} = 0$.

On the other hand, it is shown that a Kropina space of dimension two, whose metric is $L = \alpha^2/\beta$, is a Douglas space.

We treat of $D^{lm}_{ij} = G^l y^m - G^m y^l$ in the Definition. $D^{lm}$ are homogeneous polynomials in $y^i$ of degree three, if and only if $D^{lm}_{hijk} = \partial_k \partial_j \partial_i \partial_h D^{lm} = 0$. We have the relations between $D^{lm}_{hijk}$ and the Douglas tensor $D^h_{ijk}$ as follows:

$$D^{ln}_{hijk} = (n + 1)D^l_{hij}, \quad (7)$$

$$D^{lm}_{hijk} = (\partial_k D^l_{hij})y^m + D^l_{ijk} \partial^m_h + D^l_{kh} \partial^m_i + D^l_{kh} \partial^m_j + D^l_{hij} \partial^m_k - [l, m]. \quad (8)$$

Consequently, if $D^{lm}_{hijk} = 0$, then the first relation implies $D^l_{hij} = 0$, and if $D^l_{hij} = 0$, then the second relation implies $D^{lm}_{hijk} = 0$. Therefore we have

Fundamental Theorem
A Finsler space is of Douglas type, if and only if the Douglas tensor $D^h_{ijk}$ vanishes identically.

Thus the special property of geodesics of a two-dimensional Finsler space, stated in Theorem 1, has been characterized by the tensor equation $D^h_{ijk} = 0$ in the viewpoint of Finsler geometry. Since $D^h_{ijk} = \partial_k \partial_j \partial_i Q^h$ from (5), we have

Theorem 2.
A Finsler space is of Douglas type, if and only if $Q^h(x,y)$ of (4) are homogenous polynomials in $y^i$ of degree two.

Thus, for a Douglas space $F^n$, we can put

$$G^h = \frac{1}{n+1} G^h_{ij} y^i y^j + \frac{1}{2} Q^h_{ij}(x) y^i y^j,$$

which shows that (4) can be written in the form

$$\dot{x}^i \ddot{x}^j - \ddot{x}^j \dot{x}^i = \{Q^{ij}_{hk} \dot{x}^j - Q^{ji}_{hk} \dot{x}^i\} \dot{x}^h \dot{x}^k.$$  (9)

In the two-dimensional case (8) may be written as

$$y'' = Y_3 y'^3 + Y_2 y'^2 + Y_1 y' + Y_0,$$

$$Y_3 = Q^1_{22}, \quad Y_2 = 2Q^1_{12} - Q^2_{22}, \quad Y_1 = Q^1_{11} - 2Q^2_{12}, \quad Y_0 = -Q^2_{11},$$  (10)

where $Q^i_{jk} = Q^i_{jk}(x,y)$ do not contain $y'$.

4 Two-dimensional Douglas space

The present section is devoted to studying Douglas spaces of dimension two. Let $F^2 = (\pi(x,y), L(x,y;p,q))$ be a two-dimensional Finsler space, which is defined on the $(x,y)$ plane $\pi(x,y)$ and has the fundamental function $L(x,y;p,q)$. Since this L is positively homogeneous in $(p,q)$ of degree one, we can introduce

$$W = \frac{L_{pp}}{q^2} = -\frac{L_{pq}}{qp} = \frac{L_{qq}}{p^2},$$

called the Weierstrass invariant (\cite{10},\cite{12}). Then the Euler equation $d(\dot{\theta},L)/dt - \partial_i L = 0$ of geodesic can be written in the single equation

$$p\dot{q} - \dot{p} q + \frac{1}{W} (L_{xq} - L_{yp}) = 0,$$  (11)

and (4) shows $\frac{1}{W} (L_{xq} - L_{yp}) = 2(pG^2 - qG^1)$. Therefore we have
Theorem 3.
A two-dimensional Finsler space is a Douglas space, if and only if \( \frac{1}{W}(L_{xq} - L_{yp}) \) is a homogeneous polynomial in \((p,q)\) of degree three.

Example 2.
([11], (3.7b)). We deal with a two-dimensional Finsler space \( F^2 \) with the metric

\[
L = (q - p)\log |z - 1| - (q + p)\log |z + 1| - 2xq, \quad z = \frac{q}{p}.
\]

We have \( \frac{1}{W}(L_{xq} - L_{yp}) = p(p^2 - q^2) \). Thus \( F^2 \) is a Douglas space. The differential equation of geodesics is \( y'' = y'^2 - 1 \).

For \( F^2 = (\pi(x,y), L(x,y : p,q)) \) we introduce the associated fundamental function \( A(x, y, z) \) of three arguments by \( A(x,y,y') = L(x,y;1,z) \). Then we have the relation between \( L \) and \( A \):

\[
L(x,y;p,q) = pA(x,y,q\frac{q}{p}).
\] (12)

If we put \( A' = \partial A/\partial z \), then we get

\[
L_{xq} - L_{yp} = zA_y + A'_x - A_y, \quad W = \frac{A''}{p^3} (13)
\]

Consequently (11) is written in the form

\[
A''y'' + y'A'_y + A'_x - A_y = 0, \quad z = y',
\] (14)

which is called the Rashevsky form [4].

Now we consider the equation (3) of a geodesic. From (14) it follows that

\[
A''f + zA'_y + A'_x - A_y = 0, \quad z = y',
\] (15)

must be identically satisfied by \((x,y,z)\) [5], where \( f = f(x,y,z) \) and \( A = A(x,y,z) \). Differentiate (13) successively by \( z \): Putting \( S = \log |A''| \) and \( P = S_x + zS_y \), we obtain

\[
S'f + f' + P = 0, \quad (16)
\]

\[
S''f + S'f' + f'' + P' = 0, \quad (17)
\]

\[
S'''f + 2S''f' + S'f'' + f''' + P'' = 0, \quad (18)
\]

and

\[
S^{1V}f + 3S'''f' + 3S''f'' + S'f''' + f^{1V} + P''' = 0. \quad (19)
\]
Suppose that $F^2$ be a Douglas space. Then $f^{1V} = 0$ from Theorem 1., and hence (19) is reduced to

$$S^{1V}f + 3S'''f' + 3S''f'' + S'f''' + P''' = 0.$$  \hspace{1cm} (20)

Then the coefficients of $(f, f', f'', f''', 1)$ in the above five equations of (3.5) must satisfy

$$\Delta(A) = \begin{vmatrix}
A'' & 0 & 0 & 0 & \delta(A) \\
S' & 1 & 0 & 0 & P' \\
S'' & S' & 1 & 0 & P'' \\
S''' & 2S'' & S' & 1 & P'' \\
S^{1V} & 3S''' & 3S'' & S' & P'' \\
\end{vmatrix} = 0,$$

where $\delta(A) = zA'_y + A'_x - A_y$.

**Theorem 4**

A two-dimensional Finsler space is a Douglas space, if and only if the associated fundamental function $A(x, y, z)$ satisfies $\Delta(A) = 0$, where $S = \log |A''|$, $P = S_x + zS_y$ and $\delta(A) = zA'_y + A'_x - A_y$.

**Proof:** Only the sufficiency must be shown. From $\Delta(A) = 0$ it follows that the five linear equations

$$A''x_1 + (zA'_y + A'_x - A_y)x_5 = 0,$$  \hspace{1cm} (21)

$$S'x_1 + x_2 + Px_5 = 0,$$  \hspace{1cm} (22)

$$S'''x_1 + S''x_2 + x_3 + P'x_5 = 0,$$  \hspace{1cm} (23)

$$S'''x_1 + 2S''x_2 + S'x_3 + x_4 + P''x_5 = 0,$$  \hspace{1cm} (24)

$$S^{1V}x_1 + 3S'''x_2 + 3S''x_3 + S'x_4 + P'''x_5 = 0,$$  \hspace{1cm} (25)

has a non-trivial solution $(x_1, \ldots, x_5)$. Suppose that $x_5 = 0$. Then the first relation gives $x_1 = 0$ because of $A'' = pW \neq 0$ from (13). Hence the second leads to $x_2 = 0$, the third to $x_3 = 0$ and fourth to $x_4 = 0$, which is a contradiction. Thus we have non-zero $x_5$. Hence (1) and (13) lead to $f = x_1/x_5$. Then the second, comparing with (16), gives $f' = x_2/x_5$. Similarly we obtain $f'' = x_3/x_5$ and $f''' = x_4/x_5$. Consequently fifth gives (20), and, comparing with (13), $f^{1V} = 0$ is concluded. Therefore $f(x, y, z)$ is a polynomial in $z$ of degree three.

**Remark**

1) The determinant given in the previous papers ([5], [6]) for the differential equation such as (10) is necessary, but not sufficient. It must be corrected to $\Delta(A) = 0$ as above.

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(2) According to the Fundamental Theorem and Theorem 4, it is sure that $\Delta(A) = 0$ should coincide with vanishing of the Douglas tensor in the two-dimensional case. In fact, $\Delta(A)$ is constructed from $A(x,y,z)$ by the differentiation one time with respect to $(x,y)$ and six times with respect to $z$. The Douglas tensor is the set of components $D^h_{ijk}$, constructed from $L(x^i,y^j)$ in the same way, that is, by the differentiation one time with respect to $x^i$ and six times with respect to $y^j$.

Example 3 ([12], (4.12)). We treat of $F^2$ with

$$L = 2p \log \left| \frac{q}{p} \right| + qu(x,y),$$

where $u(x,y)$ is a function of $(x,y)$. From ([12] we have $A(x,y,z) = 2 \log |z| + zu$. Hence $S = \log 2 - 2 \log |z|$, $P = 0$ and it is easy to show $\Delta(A) = 0$ for any $u(x,y)$. Thus $F^2$ is a Douglas space. The geodesic equation is given by $2y'' = u_{x}y^2$.

The differential equation $\Delta(A) = 0$, which is expanded in the form

$$\begin{align*}
A''\left[S'''_x + zS'''_y + 3S''_y - \{3S'' - S'^2\}(S'_x + zS'_y + S_y) - \right. \\
S'(S''_x + zS''_y + 2S'_y) - (S_x + zS_y)\{3S''' - 5S'S'' + S'^3\} - \left. \delta(A)\{S^{1V} - 4S'S'' - 3S'^2 + 6S'^2S'' - S'^4\} = 0, \right.
\end{align*}$$

has fundamental importance on the Inverse Problem of the Calculus of Variations for ODE.

There are many types of reductions of this equation to differential equations of functions which are obtained from $A(x,y,z)$ by lessening the number of variables.

A remarkable example of such reduction corresponds to the choice of $A(x,y,z)$ in the form ([6], (8))

$$A(x,y,z) = x^{(\beta/\alpha)-2}\omega(\xi, \eta), \quad \xi = y^\alpha x^{-\beta}, \quad \eta = z^\alpha x^{\alpha-\beta}.$$ 

Then the initial equation is reduced to the differential equation of $\omega(\xi, \eta)$. Further the reduction

$$A(x,y,z) = x^\alpha y^{-\alpha-1} z^2 \omega(\xi), \quad \xi = \frac{zx}{y},$$

leads to the ordinary differential equation of $\omega(\xi)$ ([5], (40)).
Let us consider an example of solution. Let function $A(x, y, z)$ be in form
\[ A = \frac{\omega(z)}{x}, \quad A_{zz} = \frac{\omega''}{x}. \]
Then we get
\[ S = \log A_{zz} = \log \omega'' - \log x, \]
and
\[ A_{zx} = -\frac{\omega'}{x^2}, \quad S_z = \frac{\omega'''}{\omega'}, \quad S_x = -\frac{1}{x}. \]

The equation take the form
\[ A_{zz}\left[-S_x(S^3 + 3S_{zzz} - 5S_zS_{zz}) + A_{xx}[S^4_2 + 4S_zS_{zzz} - 6S^2_2S_z - S_{zzzz} - 3S^2_{zz}]\right] = 0, \]
or
\[ \omega''[S^3 + 3S'' - 5S'S'''] - \omega'[S^4 + 4S'S''' - 6S^2S'' - S^{1V} + 3S''^2] = 0. \]

After calculation all derivatives we obtain the equation
\[ 3\omega - 14\frac{\omega''\omega^{1V}}{\omega''} + 12\frac{\omega'''^3}{\omega''} = \omega' \left[ 24\frac{\omega'''^4}{\omega'''^4} + 8\frac{\omega'''^5\omega^{1V}}{\omega'''^2} - 36\frac{\omega'''^6\omega^{1V}}{\omega'''^3} - \frac{\omega^{1V}}{\omega''} + 6\left(\frac{\omega^{1V}}{\omega''}\right)^2 \right]. \]

This equation may be solved by the means of simple transformations. Using the substitution
\[ \omega'' = T(\omega') = T(Q), \]
we get the equation
\[ T^3T' + 3T'''T^3 - 2T^2T'' = Q(T^4 - 3TT^2T'' + T^2T'T''' - T^3T^{1V} + 2T^2T''^2). \]

If we let
\[ T' = R(Q)T(Q), \quad T'' = (R' + R^2)T, \quad T''' = (R'' + 3RR'R + R^3)T, \]
\[ T^{1V} = (R''' + 3R^2 + 4RR'' + 6R^2R' + R^4)T, \]
we find that function $R$ satisfies the equation
\[ 3R'' + 7RR' + 2R^3 + Q(R''' + 3RR'' + R^2 + 2R^2R') = 0. \]
Then using the substitution
\[ R = \frac{1}{Q} U(\log Q), \]
we find the equation
\[ U'''' + 3(U - 1)U'' + U'^2 + 2(U - 1)^2 U' = 0, \]
or after the change of variable \( Z = U - 1 \)
\[ Z'''' + 3Z Z'' + Z'^2 + 2Z^2 Z' = 0. \]
For solution of this equation we present the function \( Z \) in form
\[ Z' = Y(Z) \]
from which is followed the equation
\[ YY'' + Y'^2 + 3Z Y' + Y + 2Z^2 = 0. \]
It has particular solutions
\[ Y = -\frac{1}{2} Z^2, \quad Y = -\frac{3}{2} Z^2, \]
and in general case can be reduce to the Abel’s type of equation using the substitutions.
\[ Y = Z^2 V(\log Z). \]
and
\[ V' = W(V). \]
Hence we get
\[ WW' + \frac{1}{V} W^2 + \left(\frac{3}{V} + 7\right) W + 6V + 7 + \frac{2}{V} = 0. \]
Or
\[ \chi \chi' + (3 + 7V) \chi + 6V^2 + 7V^2 + 2V = 0, \]
where
\[ W = \frac{\chi(V)}{V}. \]
The particular solution
\[ Y = -\frac{1}{2}Z^2 \]
lead to the function \( \omega \) in the form
\[ \omega' = A \exp\left[-\frac{1}{Bz + C}\right], \]
where \( A, B, C \) are constants. It is corresponded the equation
\[ y'' = \frac{1}{Bx}(By' + C)^2. \]

Remark
From the above equations we get the function \( f(x, y, z) \) in form
\[ f = \frac{P'''' - S'P'' + (S'^2 - 3S'')P' - (S'^3 + 3S''' - 5S'S'')P}{S'^3 + 4S'S'' - 6S'^2S'' - 3S''^2 - S'V}. \]
and corresponding expressions for its derivatives with respect to \( z \) \( f', f'' \) and \( f''' \). So, for determination of the Finsler metric for a given equation
\[ y'' = a_1(x, y) + a_2(x, y) y'^2 + a_3(x, y) y' + a_4(x, y) \]
we must solve corresponding system of nonlinear equations.

5 Geodesics of 1-form metrics

A Finsler metric \( L(x, y) \) of dimension \( n \) is called a 1-form metric, if there is a standard Minkowski metric \( L_{\alpha} \), where \( \alpha = 1, \cdots, n \), in a real vector \( n \)-space \( V^n \) with a fixed base and \( L(x, y) = L(a^\alpha) \), where \( a^\alpha = a^\alpha_i(x)y^i \) are \( n \) 1-forms in \( y^i \). These \( a^\alpha \) must be independent; \( d = \text{det}(a^\alpha_i) \neq 0. \)

Let \( b^i_{\alpha \beta} \) be the inverse matrix of \( (a^\alpha_i) \) and put
\[ F^i_{jk}(x) = b^i_{\alpha \beta} \partial_k a^\alpha_j. \] (27)

These give rise to the linear connection \( (F^i_{jk}(x)) \) ([1], 1.5.2.) and
\[ a^\alpha_{j;k} = \partial_j a^\alpha_k - a^\alpha_r F^r_{jk} = 0. \] (28)
Thus \( a^\alpha, \alpha = 1, \cdots, n \), are \( n \) covariant constant vector fields.

We introduce the Finsler connection \( F1 = (F^i_{jk}, F^i_{0j}, 0), \) \( F^i_{0j} = y^r F^r_{ij} \) called the 1-form connection. Putting \( L_\alpha = \partial L/\partial a^\alpha \), we get \( L_{(ij)}(= \hat{\partial}_j L) = \)
$L_\alpha a^\alpha_j$ and (28) gives $L_{(j);i} = 0$, which implies $L_{i(j)} = L_{(j)(r)}F_{0i}^r + L_r F_{ji}^r$, $L_i = \partial_i L$. Hence

$$L_{i(j)} - L_{(j)i} = L_{(j)(r)}F_{0i}^r - L_{(i)(r)}F_{0j}^r - L_r T_{ij}^r,$$

where $T_{ij}^r = F_{ij}^r - F_{ji}^r$ is the (h)h-torsion tensor of $F_1$.

Now we consider a two-dimensional Finsler space $F^2$ with 1-form metric $L(a^1, a^2)$. Then

$$M = L_{1(2)} - L_{2(1)} = L_{(2)(r)}F_{01}^r - L_{(1)(r)}F_{02}^r - L_\alpha T^\alpha,$$

where $T^\alpha = T_{12}^\alpha a^\alpha_i$.

Analogously to the Weierstrass invariant $W$, we can define from $L_{\alpha\beta}$, $\alpha, \beta = 1, 2$,

$$w = \frac{L_{11}}{(a^2)^2} = -\frac{L_{12}}{a^1 a^2} = \frac{L_{22}}{(a^1)^2},$$

called the intrinsic Weierstrass invariant of $F^2$. It is easy to show from $L_{(i)(j)} = L_{\alpha\beta} a^\alpha_i a^\beta_j$

$$W = wd^2. \tag{29}$$

Then $M = W(pF_{00}^2 - qF_{00}^1) - L_\alpha T^\alpha$ and (11) gives the equation of geodesics in the form

$$pq - p\dot{q} + pF_{00}^2 - qF_{00}^1 - \frac{1}{d^2 w} L_\alpha T^\alpha = 0. \tag{30}$$

Proposition 2.

The equation of geodesics of a two-dimensional Finsler space with 1-form metric $L(a^1, a^2)$ is written as (30), where $F_{jk}^i$ are defined by (27), $d = det(a^\alpha_i)$, ”$w”$ is the intrinsic Weierstrass invariant and $\alpha = (F_{12}^i - F_{21}^i)a^\alpha_i$.

Since $F_{00}^i = F_{jk}^i(x)y^j y^k$ are polynomials in $y^i$ of degree two and $d$ does not contain $y^i$, (30) leads to

Theorem 5.

A two-dimensional Finsler space with 1-form metric is a Douglas space, if and only if $L_\alpha T^\alpha/w$ is a homogeneous polynomial in $y^i = (p, q)$ of degree three.

We shall use the following symbols for $(a^1, a^2)$ for brevity,

$$a^1 = a_1 p + a_2 q, \quad a^2 = b_1 p + b_2 q,$$

$$a_{ik} = \partial_k a_i, \quad b_{ik} = \partial_k b_i,$$
where \( a_i \) and \( b_i \) are functions of \((x,y)\). Then the connection coefficients \( F^i_{jk} \) are written as

\[
F^i_{jk} = -\frac{1}{d}(a_2 b_{jk} - b_2 a_{jk}), \quad F^2_{jk} = \frac{1}{d}(a_1 b_{jk} - b_1 a_{jk}).
\] (31)

Now we are concerned with Berwald spaces, specially simple Douglas spaces. In the two-dimensional case we refer to the Berwald frame \((1,m)\) in order to discuss such spaces ([1], 3.5). Then the \((v)\)h-torsion tensor \( R^* \) and the C-tensor \( C_{ijk} \) are written in the form \( R^i_{jk} \) and C-tensor \( C_{ijk} \) are written in the form

\[
R^i_{jk} = \epsilon L R m^i(l_j m_k - j_k m_j), \quad L C_{ijk} = I m_i m_j m_k,
\]

where \( \epsilon = \pm 1 \) is the signature; \( g_{ij} = l_i l_j + \epsilon m_i m_j \). The scalar \( R \) and \( I \) are called the h-scalar curvature (or the Gauss curvature) and the main scalar respectively.

All the Berwald spaces of dimension two are divided into three classes as follows:

1. \( R = 0 \) and \( I \neq \text{const} \).
2. \( R = 0 \) and \( I = \text{const} \).
3. \( R \neq 0 \) and \( I = \text{const} \).

A Berwald space belonging to the class (1) or (2) is a locally Minkowski space, and hence its geodesics is written as \( y'' = 0 \) in an adapted coordinate system \((x,y)\).

We shall deal with two-dimensional Berwald spaces with the constant main scalar \( I \). All of them has the 1-form metric \( L(a^1, a^2) \) and are divided into three classes, according as the signature \( \epsilon \) and the main scalar \( I \) as follows:

\[
B(1): \epsilon = +1, \quad I^2 = 4 \frac{J^2}{(1 + J^2)} < 4, \quad L = \sqrt{(a^1)^2 + (a^2)^2} \exp(J \arctan \frac{a^2}{a^1}),
\]

\[
B(2): \epsilon = +1, \quad I^2 = 4, \quad L = a^1 \exp \frac{a^2}{a^1},
\]

\[
B(3): L = (a^1)^p (a^2)^{1-p}, \quad p \neq 0, 1,
\]

(a) \( \epsilon = +1, \quad p < 0 \) or \( p > 1 \), \( I^2 = (2p - 1)^2/p(p - 1) > 4 \),

(b) \( \epsilon = -1, \quad 0 < p < 1 \), \( I^2 = (2p - 1)^2/p(1 - p) \).
\( L_\alpha T^\alpha /w \) of \( B(1), B(2) \) and \( B(3) \), appearing in \( (30) \) respectively given as follows:

\[
B(1) : \frac{1}{w} L_\alpha T^\alpha = \frac{(a^1)^2 + (a^2)^2}{1 + J^2} [(a^1 - Ja^2)T^1 + (Ja^1 + a^2)T^2],
\]

\[
B(2) : \frac{1}{w} L_\alpha T^\alpha = (a^1)^2 [(a^1 - a^2)T^1 + a^1 T^2],
\]

\[
B(3) : \frac{1}{w} L_\alpha T^\alpha = -a^1 a^2 [\frac{a^2}{1 - p} T^1 + \frac{a^1}{p} T^2].
\]

(32)

6 Lorenz dynamical system and Finsler metrics

Lorenz’s nonlinear dynamical system is given by

\[
\dot{x} = k(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz,
\]

where \( k, b \) are positive constant and \( r \) is a parameter. This is equivalent to the following second order differential equations:

\[
k(y - x)y'' + ky'^2 - \frac{1}{x}[ky - (b + 1)x]y' - \frac{1}{x} + \frac{1}{k(y - x)}[x^2 y + b(y - rx)] = 0.
\]

If we transform \((x,y)\) to \((u = x, v = 1/(y - x))\) and write \((u,v)\) as \((x,y)\) again, then the above is rewritten as \((4), (5), (6)\)

\[
y'' = \frac{3}{y} y'^2 + \left( \frac{1}{x} - my \right) y' + \left( nx^3 - lx \right) y^4 + (nx^2 + t) y^3 - \frac{s}{x} y^2,
\]

(33)

where we put

\[
m = 1 + \frac{b + 1}{k}, \quad n = \frac{1}{k^2}, \quad l = \frac{b(r - 1)}{k^2}, \quad t = \frac{b(k + 1)}{k^2}, \quad s = 1 + \frac{1}{k}.
\]

(34)

The purpose of the present section is to find the two-dimensional Finsler spaces \( F^2 \) whose geodesics are give by \((33)\), To do so, we shall pay attention to Berwald spaces \( F^2 \) belonging to the class \( B(3) \):

\[
L = (a^1)^p (a^2)^{1-p}, \quad p \neq 0, 1,
\]

\[
a^1 = a_1 x + a_2 y, \quad a^2 = b_1 x + b_2 y,
\]

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where \( a_i \) and \( b_i \), \( i = 1, 2 \), are functions of \( (x, y) \). The equation of geodesics is written from (30), (31) and (32) as
\[
y'' = Y_3 y^3 + Y_2 y'^2 + Y_1 y' + Y_0,
\]
where we put
\[
Y_3 = F_{11}^1 - \frac{1}{d^2}\left(\frac{b_2}{1-p}T^1 + \frac{a_2^2}{p}T^2\right)a_2 b_2, \\
Y_0 = -F_{12}^2 - \frac{1}{d^2}\left(\frac{b_1}{1-p}T^1 + \frac{a_1}{p}T^2\right)a_1 b_1,
\]
\[
Y_2 = F_{12}^1 + F_{21}^1 - \frac{1}{d^2}\left(\frac{b_2}{p-1}(a_1 b_2 + 2a_2 b_1)T^1 + \frac{a_2^2}{p}(a_2 b_1 + 2a_1 b_2)T^2\right), \\
Y_1 = F_{11}^1 - F_{12}^2 - F_{21}^2 - \frac{1}{d^2}\left(\frac{b_1}{p-1}(a_2 b_1 + 2a_1 b_2)T^1 + \frac{a_1}{p}(a_1 b_2 + 2a_2 b_1)T^2\right).
\]

Since (33) has no term containing \( y'^3 \), we first pay attention to \( Y_3 \): (31) gives
\[
Y_3 = -\frac{1}{d}(a_2 b_2 y - b_2 a_2 y) - \frac{1}{d^2}\left(\frac{b_2}{1-p}T^1 + \frac{a_2}{p}T^2\right)a_2 b_2.
\]

It is observed that \( a_2 = 0 \) or \( b_2 = 0 \) implies \( Y_3 = 0 \) at once. Consequently, assume that \( a_2 = 0 \) in the following, and hence \( a_1 = 1 \) be assumed according to the special form of \( L \). Thus we put
\[
a_1 = \dot{x}, \quad a^2 = b(x, y)\dot{x} + a(x, y)\dot{y}.
\]

Consequently we have from (35) and (31)
\[
Y_2 = -\frac{a_y}{a}, \quad Y_1 = \frac{1}{pa}\left[(1-p)a_x - (1+p)b_y\right], \\
Y_0 = \frac{1}{pa^2}\left[b(a_x - b_y) - ab_x]\right].
\]

Consequently, comparing (33) with (38), our problem is to find two functions \( a(x, y) \) and \( b(x, y) \) which satisfy
\[
\frac{a_y}{a} = -\frac{3}{y}, \tag{39}
\]
\[
\frac{1}{pa}\left[(1-p)a_x - (1+p)b_y\right] = \frac{1}{x} - my, \tag{40}
\]
\[
\frac{1}{pa^2}\left[b(a_x - b_y) - ab_x]\right] = (nx^2 - lx)y^4 + (nx^2 + t)y^3 - \frac{s}{x}y^2. \tag{41}
\]
Suppose that $p \neq -1$. Then above relations lead to $a(x,y)$ and $b(x,y)$ as follows:

\[ a(x, y) = \frac{f(x)}{y^3}, \]  
\[ b(x, y) = \frac{p - 1}{2(p + 1)} \frac{f'(x)^2}{y} + \frac{p}{p + 1} \left( \frac{1}{2xy} - m \right) \frac{f(x)}{y} + g(x), \]

where $f(x)$ and $g(x)$ are functions of $x$ alone. Substituting (42) in (40), we obtain the equation of the form

\[ c_4 y^4 + c_3 y^3 + c_2 y^2 + c_1 y = 0, \]

where $c_i$'s are functions of $x$ alone. Since $m \neq 0$ from (34), $c_i = 0$ give the following four equations:

\[ g = \frac{p + 1}{m} (1 - nx^2)xf, \]

\[ (p + 1)xy' - (1 + 2x \frac{f'}{f})g = \left[ \frac{m^2 p}{p + 1} - (p + 1)(nx^2 + t) \right] xf, \]

\[ (2p - 1)(p - 1)mx f' + [2(p + 1)^2 s - 3mp] f = 0, \]

\[ (p^2 - 1) f'' = (p - 1) f'(\frac{2 f'}{f} - \frac{p - 1}{x}) + p(p + 2) \frac{f}{x^2}. \]

It is observed that $g(x)$ is given from $f(x)$ by the firsts of these relations is rewritten in the form

\[ (p - 1)(nx^2 - 1) xf' + [(3p + 2 - m)nx^2 + \frac{pm^3}{(p + 1)^2} - pl - mt] f = 0. \]

Thus we shall find $f(x)$ to satisfy (47), and previous relations. They are of the form

\[ A_i f' + B_i f = 0, \quad i = 1, 2. \]

From $f \neq 0$ it follows that $A_1 B_2 - A_2 B_1 = 0$ should be satisfied:

\[ (2p - 1)(p - 1)mx [(3p + 2 - m)nx^2 + \frac{pm^3}{(p + 1)^2} - pl - mt] - \]

\[ (p - 1)(nx^2 - l)x [2(p + 1)^2 s - 3pm] = 0. \]

Dividing by $(p - 1)x$, this gives the equation on form $Cx^2 + D = 0$, which implies $C = D = 0$:

\[ 2(p + 1)^2 s - 3pm - (2p - 1)(3p + 2 - m)m = 0, \]

\[ (2p - 1)\left[ \frac{pm^3}{(p + 1)^2} - pl - mt + (3p + 2 - m)]l \right] = 0. \]
We shall return to (45). Assume that \(2p - 1 \neq 0\). Then (48) enables us to define
\[
q = \frac{2(p + 1)^2 s - 3mp}{(2p - 1)(p - 1)m} = \frac{3p + 2 - m}{p - 1},
\]
and (45) is written as
\[
f' = -\frac{q}{x} f,
\]
which implies \(f'' = (q + 1)qf/x^2\). Hence (46) is now reduced to
\[
[(p - 1)q - p][(p - 1)q + p + 2] = 0.
\]
therefore we divide our discussion into the following two cases:

(i) \(q = \frac{p}{p + 1}\),

(ii) \(q = -\frac{p + 2}{p - 1}\).

We deal with the case (i).

(50) and (49) give \(m = 2(p + 1), s = 2p, t = 4p\). Hence (34) leads to \(k = 1/(2p - 1), b = 2/(2p - 1), n = (2p - 1)^2, l = 2(r - 1)(2p - 1)\). Consequently (51) and (44) give \(f\) and \(g\) respectively. Thus \(a(x,y)\) and \(b(x,y)\) are given by (42) as
\[
a(x, y) = cx^{p + 2} y^{-3},
\]
\[
b(x, y) = cx^{p - 1} [8p(2p - 1)x^2 - \frac{(3 - 4p)^2}{4} x^4 + \frac{1}{y^2} - \frac{4px}{y}],
\]
where \(c\) is a non-zero constant. Since \(k\) and \(b\) are positive, we have \(p > 1/2\).

The case (ii) is analogously treated and we obtain \(k = 1/(3 - 4p), b = 8p/(3 - 4p), r = (4p - 1)/(3 - 4p)\). Since \(k\) and \(b\) are positive, we have \(0 < p < 3/4, p \neq 1/2\).

\[
a(x, y) = cx^{\frac{p - 2}{r - 1}} y^{-3},
\]
\[
b(x, y) = cx^{\frac{3}{p - 1}} [8p(2p - 1)x^2 - \frac{(3 - 4p)^2}{4} x^4 + \frac{1}{y^2} - \frac{4px}{y}],
\]
where \(c\) is a non-zero constant.

Further we consider the case \(p = 1/2\). Then (48) gives \(m = 3s\) and (46) is reduced to
\[
3x^2 f f'' - 4(xf')^2 - xf f' + 5f^2 = 0.
\]
\[(47) \text{ gives } f' = fh(x), \text{ where}
\]
\[h(x) = \frac{ux^2 - v}{(nx^2 - l)x}, \quad u(x) = (7 - 2m)n, \quad v = l + 2mt - \frac{4}{9}m^3. \quad (57)\]

Then (refdo) is rewritten as \(3x^2h' - (xh)^2 - xh + 5 = 0\). Substituting from (57), this is written in the second order equation in \(x^2\). Equating the three coefficients to zero, we obtain

\[(u - n)(u + 5n) = 0, \quad (u + 5n)(v - l) = 0, \quad (v - l)(v + 5l) = 0.\]

From \(u = n\) it follows that \(m = 3\) from (57), and hence \(s = 1\), which contradicts to (34). Thus the above leads to the following two cases:

(i) \(u = -5n, \quad v = l, \quad (ii) \ u = -5n, \quad v = -5l.\)

We treat of the case (i). Then (57) gives \(m = 6, \quad t = 8, \quad s = 2\), and hence (34) yields \(k = 1, \quad b = 4, \quad n = 1, \quad l = 4(r - 1)\). (57) gives \(h(x) = -\frac{5x^2 + 4(r - 1)}{x^2 - 4(r - 1)}x\) and \(f' = fh\) yields \(f = cx/E^3, \quad E = x^2 - 4(r - 1)\).

Therefore we obtain

\[a(x, y) = \frac{cx}{(Ey)^3}, \quad E = x^2 - 4(r - 1), \quad (58)\]
\[b(x, y) = \frac{cx}{E^2} \frac{x}{(Ey)^2} - \frac{2}{Ey} - \frac{x}{4}. \quad (59)\]

Next, it is easy to show that the case (ii) leads only to a special case where \(r = 1\) of (i).

Finally we deal with the case \(p = -1\). Since (40) give \(a = f(x)/y^3\) and \(2a_x/a = my - 1/x\) respectively, we have \(my - 1/x = 2f'(x)/f(x)\), which implies \(m = 0\), which contradicts to (34). Summarizing up all the above, we have

Theorem 6

The Lorenz equation (33) with (34) is regarded as the equation of geodesics of the two-dimensional Berwald space with the fundamental function

\[L(x, y; \dot{x}, \dot{y}) = \dot{x}^p(b\dot{x} + a\dot{y})^{1-p}, \text{ if } k, \ b \text{ and } r \text{ satisfy the following conditions, where } a(x, y) \text{ and } b(x, y) \text{ are given as follows:}
\]

(1) \(k = 1/(2p-1), \ b = 2/(2p-1), \ r = \text{ arbitrary}, \text{ where } p > 1/2, \ p \neq 1, \ a(x, y) \text{ and } b(x, y) \text{ are given by (52).}

(2) \(k = 1/(3-4p), \ b = 8p/(3-4p), \ r = (4p-1)/(3-4p), \text{ where } 0 < p < 3/4, \ p \neq 1/2. \ a(x, y) \text{ and } b(x, y) \text{ are given by (54).}

(3) \(k = 1, \ b = 4, \ r = \text{ arbitrary}, \text{ where } p = 1/2. \ a(x, y) \text{ and } b(x, y) \text{ are given by (58).} \)
Remark 1
The metric \( L = \sqrt{x(b\dot{x} + a\dot{y})} \) of the case (3) is a Riemannian metric with the signature \( \epsilon = -1 \).

Conclusion
The values of parameters plays a crucial role in geometry of equation. At the change of parameters the behaviour of its integral curves (and, correspondingly, its Geometry!) may be change radically. The Finsler-Geometrical approach to studying nonlinear dynamical systems the equivalent to ODE’s of type \( y'' = Y_3y'^3 + Y_2y'^2 + Y_1y' + Y_0 \) makes possible to investigate all kinds of Geometries connected with such type of equations. Fundamental partial differential equation \( \Delta = 0 \) for the Finsler metrics is basis of this approach and it gives the hope to understand the nature of chaos from geometrical point of view. It will be the object of next work.

Acknowledgement
This work has been supported partial by grant INTAS-93-0166 and the first author is very grateful to Physical Department of the Lecce University (Italy) for financial support, which allowed him to continue his scientific activity during the last years.

7 References
1. P.L.Antonelli, R.S.Ingarden and M.Matsumoto: The theory of Sprays and Finsler Spaces with Applications in Physics and Biology, Kluwer Acad. Publishers, Dordrecht, 1993.

2. S.Bacso and M.Matsumoto: Publ- Math. Debrecen 51 (1997), 385.

3. E.Cartan: Lecons sur la theorie des espaces a connexion projective, Gauthier-Villars, 1937.

4. V.S.Dryuma: Rep. of 1X Internl. Conf. on Topology and its Applications, 12-16, Oct., 1992, Ukraine, Kiev, p.70.

5. V.S.Dryuma: Theor. Mat. Fiz. 99 (1994), 241.
6 V.S.Dryuma: Proceedings of the First Workshop on Nonlinear Physics, 1995, p.83.

7 M.Matsumoto: Tensor, N.S. 34 (1980), 303.

8 M.Matsumoto: Foundations of Finsler Geometry and Special Finsler Spaces, Kaiseisha Press, Otsu, Japan, 1986.

9 M.Matsumoto: J.Math. Kyoto Univ. 29 (1989), 489.

10 M.Matsumoto: Open syst. and Inform. Dynamics 3 (1995), 291.

11 M.Matsumoto: J. Math. Kyoto Univ. 35 (1995), 357.

12 M.Matsumoto: Tensor, N.S., to appear.

14 M.Matsumoto and H.-S.Park: Rev. Roum. Math., to appear.