Harnack Inequality for Distribution Dependent Stochastic Hamiltonian System

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Abstract

The dimension free Harnack inequality is established for the distribution dependent stochastic Hamiltonian system, where the drift is Lipschitz continuous in the measure variable under the distance induced by the Hölder-Dini continuous functions, which are \(\beta(\beta > \frac{2}{3})\)-Hölder continuous on the degenerate component and square root of Dini continuous on the non-degenerate one. The results extend the existing ones in which the drift is Lipschitz continuous in the measure variable under \(L^2\)-Wasserstein distance.

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1 Introduction

The stochastic Hamiltonian system, including the kinetic Fokker-Planck equation (see [14]), is a classical degenerate model. In [3] the authors study the regularity of stochastic kinetic equations. [4] investigates the Bismut formula, gradient estimate and Harnack inequality by using the method of coupling by change of measure. [19] and [20] focus on the derivative formula. [17] proves the hypercontractivity. One can refer to the references in the papers mentioned above for more related results.

On the other hand, the McKean-Vlasov stochastic differential equations (SDEs), presented in [9], can be used to characterize the nonlinear Fokker-Planck-Kolmogorov equations. Recently, there are plentiful results on McKean-Vlasov SDEs, such as the well-posedness,
Harnack inequality, the Bismut formula, exponential ergodicity, estimate of heat kernel, see [1, 2, 5, 6, 7, 10, 12, 13, 16, 18, 21] and references therein for more details. For the well-posedness, the drifts can be Lipschitz continuous in the measure variable under weighted variation distance, for instance [7, 13, 18] and so on. However, with respect to the Harnack inequality, most results concentrate on the case that the coefficients are Lipschitz continuous in the measure variable under $L^2$-Wasserstein distance, see [12] for the distribution dependent stochastic Hamiltonian system. In fact, for the well-posedness, the initial distributions (initial values) are assumed to be the same, while the Harnack inequality investigates the regularity of the nonlinear semigroup from different initial distributions, which will produce more difficulty than the study of well-posedness.

Quite recently, the first author and his co-author have established the log-Harnack inequality and Bismut derivative formula for non-degenerate McKean-Vlasov SDEs in [8], where for the log-Harnack inequality, the drifts are only assumed to be Lipschitz continuous under the distance induced by square root of Dini continuous functions, which allows the drifts even being not Dini continuous in the $L^2$-Wasserstein distance.

In this paper, we intend to study the Harnack inequality for distribution dependent stochastic Hamiltonian system, where the drift is Lipschitz continuous in the measure variable under the distance induced by the Hölder-Dini continuous functions. More precisely, the functions are assumed to be $\beta(\beta > \frac{2}{3})$-Hölder continuous on the degenerate component and square root of Dini continuous on the non-degenerate one. Compared with [8], we need to establish the gradient estimate of $P^\mu_t f$ with measure-valued curve parameter $\mu$ in non-degenerate and degenerate components respectively. Moreover, when $f$ only depends on the non-degenerate component, the gradient estimate on the degenerate component of $P^\mu_t f$ is also derived, which is crucial in the proof of the main result, see Theorem 3.1(2) below.

Fix $m, d \in \mathbb{N}^+$. Let $\mathcal{P}$ be the set of all probability measures in $\mathbb{R}^{m+d}$ equipped with the weak topology. For $k \geq 1$, define

$$\mathcal{P}_k := \{ \mu \in \mathcal{P} : \| \mu\|_k := \mu(|\cdot|^k)^{\frac{1}{k}} < \infty \}.$$

$\mathcal{P}_k$ is a Polish space under the $L^k$-Wasserstein distance

$$W_k(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left( \int_{\mathbb{R}^{m+d} \times \mathbb{R}^{m+d}} |x - y|^k \pi(dx, dy)^{\frac{1}{k}} \right),$$

where $\mathcal{C}(\mu, \nu)$ is the set of all couplings of $\mu$ and $\nu$.

For any $x \in \mathbb{R}^{m+d}$, let $x^{(1)}$ denote the first $m$ components and $x^{(2)}$ denote the last $d$ components, that is $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{m+d}$ with $x^{(1)} \in \mathbb{R}^m$ and $x^{(2)} \in \mathbb{R}^d$. Fix $T > 0$. Consider the following distribution dependent stochastic Hamiltonian system on $\mathbb{R}^{m+d}$:

$$\begin{align*}
\begin{cases}
    dX^{(1)}_t = MX^{(2)}_t dt, \\
    dX^{(2)}_t = B_t(X_t, \mathcal{L}_{X_t}) dt + \sigma_t dW_t,
\end{cases}
\end{align*}$$

(1.1)

where $W = (W_t)_{t \geq 0}$ is a $d$-dimensional standard Brownian motion with respect to a complete filtration probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $M$ is an $m \times d$ matrix, and $\sigma : [0, T] \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, $B : [0, T] \times \mathbb{R}^{m+d} \times \mathcal{P} \rightarrow \mathbb{R}^d$ are measurable.
Recall that for two probability measures $\mu, \nu \in \mathcal{P}$, the entropy and total variation norm are defined as follows:

$$\text{Ent}(\nu | \mu) := \begin{cases} \int_{\mathbb{R}^{m+d}} (\log \frac{d\nu}{d\mu}) d\nu, & \text{if } \nu \text{ is absolutely continuous with respect to } \mu, \\ \infty, & \text{otherwise}; \end{cases}$$

and

$$\|\mu - \nu\|_{\text{var}} := \sup_{|f| \leq 1} |\mu(f) - \nu(f)|.$$ 

By Pinsker’s inequality (see [11]),

$$\|\mu - \nu\|_{\text{var}}^2 \leq 2 \text{Ent}(\nu | \mu), \quad \mu, \nu \in \mathcal{P}.$$ 

Throughout the paper, we will use $C$ or $c$ as a constant, the values of which may change from one place to another. For a function $f$ on $\mathbb{R}^{m+d}$ and $i = 1, 2$, let $\nabla^{(i)} f(x)$ stand for the gradient with respect to $x^{(i)}$.

The paper is organized as follows: In Section 2, we state the main results, i.e. the Harnack inequality for distribution dependent stochastic Hamiltonian system and the proof is provided in Section 3; In Section 4, the well-posedness for degenerate McKean-Vlasov SDEs is investigated, where the drifts are assumed to be Lipschitz continuous in the measure variable under the weighted variation distance plus the $L^k$-Wasserstein distance.

## 2 Main Results

Let

$$\mathcal{A} := \left\{ \alpha : [0, \infty) \rightarrow [0, \infty) \text{ is increasing and concave}, \alpha(0) = 0, \int_0^1 \frac{\alpha(r)^2}{r} dr \in (0, \infty) \right\}.$$ 

For $\beta \in (0, 1], \alpha \in \mathcal{A}$, define

$$\rho_{\beta, \alpha}(x, y) = |x^{(1)} - y^{(1)}|^{\beta} + \alpha(|x^{(2)} - y^{(2)}|), \quad x, y \in \mathbb{R}^{m+d}.$$ 

Since $\alpha$ is concave and increasing, we conclude that $\rho_{\beta, \alpha}$ is a distance on $\mathbb{R}^{m+d}$. For a real valued function $f$ on $\mathbb{R}^{m+d}$, let

$$[f]_{\beta, \alpha} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho_{\beta, \alpha}(x, y)}.$$ 

Let

$$\mathcal{P}_{\beta, \alpha} := \left\{ \mu \in \mathcal{P} : \int_{\mathbb{R}^{m+d}} (|x^{(1)}|^{\beta} + \alpha(|x^{(2)}|)) \mu(dx) < \infty \right\}.$$ 

Define the Wasserstein distance induced by $\rho_{\beta, \alpha}$:

$$\mathbb{W}_{\beta, \alpha}(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^{m+d} \times \mathbb{R}^{m+d}} \rho_{\beta, \alpha}(x, y) \pi(dx, dy).$$
and \( \mathbb{W}_{\beta, \alpha} \) is a complete distance on \( \mathcal{P}_{\beta, \alpha} \). Moreover, we have the dual formula
\[
\mathbb{W}_{\beta, \alpha}(\mu, \nu) := \sup_{|f|_{\beta, \alpha} \leq 1} |\mu(f) - \nu(f)|, \quad \mu, \nu \in \mathcal{P}_{\beta, \alpha}.
\]
Noting that for any \( \mu, \nu \in \mathcal{P}_{\beta, \alpha} \), \( \{ f : f \in \mathcal{B}_b(\mathbb{R}^{m+d}), |f|_{\beta, \alpha} \leq 1 \} \) is dense in \( \{ f : |f|_{\beta, \alpha} \leq 1 \} \) under \( L^1(\mu + \nu) \), we have
\[
\mathbb{W}_{\beta, \alpha}(\mu, \nu) := \sup_{f \in \mathcal{B}_b(\mathbb{R}^{m+d}), |f|_{\beta, \alpha} \leq 1} |\mu(f) - \nu(f)|, \quad \mu, \nu \in \mathcal{P}_{\beta, \alpha}.
\]
Furthermore, it follows from the concavity of \( \alpha \) and \( \alpha(0) = 0 \) that
\[
(2.1) \quad \alpha(rt) \leq r\alpha(t), \quad t > 0, r \geq 1,
\]
By (2.1) for \( t = 1 \) and \( \alpha(t) \leq \alpha(1), t \in [0, 1] \), we conclude that
\[
(2.2) \quad \alpha(r) \leq \alpha(1)(1 + r), \quad r \geq 0.
\]
So, for any \( k \geq 1 \),
\[
(2.3) \quad \sup_{|f|_{\beta, \alpha} \leq 1} |f(x) - f(0)| \leq |x^{(1)}|^\beta + \alpha(|x^{(2)}|) \leq 2(\alpha(1) + 1)(1 + |x|^k), \quad x \in \mathbb{R}^{m+d}.
\]
Therefore \( \mathcal{P}_k \subset \mathcal{P}_{\beta, \alpha} \) for \( k \geq 1 \) and
\[
(2.4) \quad \frac{1}{2(\alpha(1) + 1)} \mathbb{W}_{\beta, \alpha}(\mu, \nu) \leq \mathbb{W}_{k, \text{var}}(\mu, \nu) := \sup_{|f|_{1+1+|k|^k}} |\mu(f) - \nu(f)|, \quad \mu, \nu \in \mathcal{P}_k.
\]
To obtain the Harnack inequality, we make the following assumptions.

(A1) \( \sigma_t \) is invertible and \( \|\sigma_t\| + \|\sigma_t^{-1}\| \) is bounded in \( t \in [0, T] \).

(A2) For any \( t \in [0, T], \gamma \in \mathcal{P}_2, \nabla B_t(\cdot, \gamma) \) is continuous. Moreover, there exist some constant \( K_B > 0, \alpha \in \mathcal{A} \) and \( \beta \in \left( \frac{2}{3}, 1 \right) \) such that
\[
|\nabla B_t(x, \gamma)| \leq K_B,
|B_t(x, \gamma) - B_t(x, \bar{\gamma})| \leq K_B(\mathbb{W}_2(\gamma, \bar{\gamma}) + \mathbb{W}_{\beta, \alpha}(\gamma, \bar{\gamma})),
|B_t(0, \delta_0)| \leq K_B, \quad t \in [0, T], x \in \mathbb{R}^{m+d}, \gamma, \bar{\gamma} \in \mathcal{P}_2.
\]

(A3) \( MM^* \) is invertible.

By (A2) and (2.4) for \( k = 1 \), there exist constants \( C_1, C_2 > 0 \) such that
\[
(2.5) \quad |B_t(x, \gamma)| \leq C_1(1 + |x| + \mathbb{W}_2(\gamma, \delta_0) + \mathbb{W}_{\beta, \alpha}(\gamma, \delta_0)) \leq C_2(1 + |x| + \|\gamma\|_2), \quad t \in [0, T], x \in \mathbb{R}^{m+d}, \gamma \in \mathcal{P}_2.
\]
So, according to Theorem 4.1 below for \( k = 2 \), under (A1)-(A2), (1.1) is well-posed in \( \mathcal{P}_2 \), and \( P_t\gamma := \mathcal{L}_{X_t}\gamma \) for the solution \( X_t\gamma \) to (1.1) with \( \mathcal{L}_{X_0} = \gamma \in \mathcal{P}_2 \) satisfy

\[
\|P_t\gamma\|_2^2 \leq C_1(1 + \|\gamma\|_2^2), \quad t \in [0, T]
\]

for some constant \( C_1 > 0 \). Define

\[
P_t f(\gamma) := \mathbb{E}[f(X_t\gamma)] = \int_{\mathbb{R}^{m+d}} f\{P_t\gamma\}. \]

For simplicity, we denote \( X_t^x = X_{t}^{\delta_x} \) and \( P_t f(x) = P_t f(\delta_x) \) for \( x \in \mathbb{R}^{m+d} \). The next result characterizes the Harnack inequality for (1.1).

**Theorem 2.1.** Assume (A1)-(A3). Then the following assertions hold.

1. There exists a constant \( c > 0 \) such that for any positive \( f \in \mathbb{B}_b(\mathbb{R}^{m+d}) \),

\[
P_t \log f(\tilde{\gamma}) \leq \log P_t f(\gamma) + \left( e^{c(1+\|\gamma\|_2+\|\tilde{\gamma}\|_2)^2} + \frac{c}{t^3} \right) \mathbb{W}_2(\gamma, \tilde{\gamma})^2, \quad t \in (0, T], \gamma, \tilde{\gamma} \in \mathcal{P}_2.
\]

Consequently, it holds

\[
\|P_t^*\gamma - P_t^*\tilde{\gamma}\|_{\text{var}}^2 \leq 2 \text{Ent}(P_t^*\gamma | P_t^*\tilde{\gamma})
\]

\[
\leq \left( 2 e^{c(1+\|\gamma\|_2+\|\tilde{\gamma}\|_2)^2} + \frac{2c}{t^3} \right) \mathbb{W}_2(\gamma, \tilde{\gamma})^2, \quad t \in (0, T], \gamma, \tilde{\gamma} \in \mathcal{P}_2.
\]

2. There exists \( c > 0 \) such that for any \( p > 1 \), \( t \in (0, T], \gamma, \tilde{\gamma} \in \mathcal{P}_2 \) and \( f \in \mathbb{B}_b^+(\mathbb{R}^{m+d}) \),

\[
(P_t f(\tilde{\gamma}))^p \leq P_t f_p^*(\gamma) \exp \left\{ \frac{cp}{(p-1)} e^{c(1+\|\gamma\|_2+\|\tilde{\gamma}\|_2)^2} \mathbb{W}_2(\gamma, \tilde{\gamma})^2 \right\}
\]

\[
\times \inf_{\pi \in \mathcal{C}(\gamma, \tilde{\gamma})} \int_{\mathbb{R}^{m+d} \times \mathbb{R}^{m+d}} \exp \left\{ \frac{cp}{(p-1)t^3} |x-y|^2 \right\} \pi(dx, dy).
\]

3. If in particular \( B \) is bounded, then (1) and (2) hold for some constant \( c > 0 \) replacing \( e^{c(1+\|\gamma\|_2+\|\tilde{\gamma}\|_2)^2} \).

### 3 Proof of Theorem 2.1

#### 3.1 The Bismut formula for stochastic Hamiltonian system with measure-valued curve parameter

The Bismut formula in the first assertion of the following theorem has been established in [4, 15, 19, 20], where the Malliavin calculus or coupling by change of measure plays an important role. For reader’s convenience, we will use the coupling by change of measure to
provide the proof. The second assertion in Theorem 3.1 below is new, which is crucial in the proof of Lemma 3.7 below.

For any \( \mu \in C([0,T], \mathcal{B}_2) \), consider the SDE with parameter \( \mu \):

\[
\begin{cases}
    d(X_t^{x,\mu})^{(1)} = M(X_t^{x,\mu})^{(2)}dt, \\
    d(X_t^{x,\mu})^{(2)} = B_t(X_t^{x,\mu}, \mu_t)dt + \sigma_t dW_t, \\
    X_0^{x,\mu} = x \in \mathbb{R}^{m+d}.
\end{cases}
\]

Observe that (3.1) is indeed a time non-homogeneous classical SDE. By (3.1) for \( k = 1, (3.1) \) is well-posed. In fact, by (A2) and (2.3) for \( k = 1 \), there exists a constant \( C > 0 \) such that

\[
\begin{align*}
    |B_t(x, \mu_t) - B_t(y, \mu_t)| &\leq C|x - y|, \\
    |B_t(0, \mu_t)| &\leq C(1 + \|\mu_t\|_2), \quad x, y \in \mathbb{R}^{m+d}, t \in [0,T].
\end{align*}
\]

Let \( P_t^{\mu} \) be the associated Markov semigroup, i.e.

\[
P_t^{\mu}f(x) := \mathbb{E}[f(X_t^{x,\mu})], \quad t \in [0,T], x \in \mathbb{R}^{m+d}, f \in \mathcal{B}_b(\mathbb{R}^{m+d}).
\]

**Theorem 3.1.** Assume (A1)-(A2). Then the following assertions hold.

1. Suppose that (A3) holds. Then for any \( f \in \mathcal{B}_b(\mathbb{R}^{m+d}) \), it holds

\[
\nabla_h P_t^{\mu}f(x) = \mathbb{E}[f(X_t^{x,\mu})N_t(h)], \quad x, h \in \mathbb{R}^{m+d}, t \in (0,T]
\]

with

\[
N_t(h) = \int_0^t \left\langle \sigma_s^{-1} \left[ \nabla B_s(X_s^{x,\mu}, \mu_s) \left( h^{(1)} + \int_0^s M\gamma_u(h, \gamma_s(h)) - \gamma_s'(h) \right) \right] , dW_s \right\rangle
\]

and

\[
\gamma_s(h) := \left[ \frac{(t-s)}{t} - \frac{3s(t-s)}{t^2}M^*(MM^*)^{-1}M \right] h^{(2)}
\]

\[
- \frac{6s(t-s)}{t^3}M^*(MM^*)^{-1}h^{(1)}, \quad s \in [0,t]
\]

satisfying

\[
|\gamma_s(h)| \leq c(|h^{(2)}| + t^{-1}|h^{(1)}|), \quad |\gamma_s'(h)| \leq c(t^{-1}|h^{(2)}| + t^{-2}|h^{(1)}|), \quad s \in [0,t]
\]

for some constant \( c > 0 \).

2. For any \( f \in \mathcal{B}_b(\mathbb{R}^{m+d}) \) with \( f(x) \) only depending on \( x^{(2)} \) and \( v \in \mathbb{R}^m \),

\[
\nabla_v^{(1)} P_t^{\mu}f(x) = \mathbb{E} \left[ f((X_t^{x,\mu})^{(2)}) \int_0^t \langle \sigma_s^{-1} \nabla_v^{(1)} B_s(X_s^{x,\mu}, \mu_s), dW_s \rangle \right], \quad t \in [0,T].
\]

Consequently, for any \( f \in \mathcal{B}_b(\mathbb{R}^{m+d}) \) with \( f(x) \) only depending on \( x^{(2)} \),

\[
|\nabla_v^{(1)} P_t f(x)| \leq C \left( \mathbb{E}|f((X_t^{x,\mu})^{(2)})|^2 \right)^{\frac{1}{2}} t^{\frac{1}{2}}, \quad t \in [0,T]
\]

for some constant \( C > 0 \).
Proof. (1) Fix $t \in (0,T]$. For any $\varepsilon \in (0,1)$, $h \in \mathbb{R}^{m+d}$, let $(X^\varepsilon_s)_{s \in [0,t]}$ solve the equation
\begin{align}
(3.7) \quad \begin{cases}
\text{d}(X^\varepsilon_s)^{(1)} = M(X^\varepsilon_s)^{(2)} \text{d}s, \\
\text{d}(X^\varepsilon_s)^{(2)} = B_s(X^\varepsilon_s, \mu_s) \text{d}s + \sigma_s \text{d}W_s + \varepsilon \gamma'_s(h) \text{d}s, \quad X^\varepsilon_0 = x + \varepsilon h.
\end{cases}
\end{align}

Then it is easy to see that
\begin{align}
(3.8) \quad X^\varepsilon_s = X^{x,\mu}_s + \left(\varepsilon h^{(1)} + \varepsilon \int_0^s M \gamma_u(h) \text{d}u, \varepsilon \gamma_s(h)\right), \quad s \in [0,t],
\end{align}
in particular, $X^\varepsilon_t = X^{x,\mu}_t$ due to (3.4). Let
\[\Phi^\varepsilon_s = \sigma_s^{-1}[B_s(X^\varepsilon_s, \mu_s) - B_s(X^{x,\mu}_s, \mu_s) - \varepsilon \gamma'_s(h)], \quad s \in [0,t].\]

Then (A1)-(A2) and (3.8) imply
\begin{align}
(3.9) \quad |\Phi^\varepsilon_s| \leq c_0 \left[ \varepsilon |h^{(1)}| + \varepsilon \|M\| \int_0^s |\gamma_u(h)| \text{d}u + \varepsilon |\gamma_s(h)| \right] + \varepsilon |\gamma'_s(h)|, \quad s \in [0,t]
\end{align}
for some constant $c_0 > 0$. In view of
\begin{align}
(3.10) \quad |\gamma_s(h)| \leq c(|h^{(2)}| + t^{-1}|h^{(1)}|), \quad |\gamma'_s(h)| \leq c(t^{-1}|h^{(2)}| + t^{-2}|h^{(1)}|), \quad s \in [0,t]
\end{align}
for some constant $c > 0$, it follows from Girsanov’s theorem that
\[\tilde{W}_s := W_s - \int_0^s \Phi^\varepsilon_u \text{d}u, \quad s \in [0,t]\]
is a $d$-dimensional Brownian motion on $[0,t]$ under $\mathbb{Q}^\varepsilon_t = R^\varepsilon_t \mathbb{P}$, where
\[R^\varepsilon_t = \exp \left[ \int_0^t \langle \Phi^\varepsilon_u, \text{d}W_u \rangle - \frac{1}{2} \int_0^t |\Phi^\varepsilon_u|^2 \text{d}u \right].\]

Then (3.7) reduces to
\begin{align*}
\begin{cases}
\text{d}(X^\varepsilon_s)^{(1)} = M(X^\varepsilon_s)^{(2)} \text{d}s, \\
\text{d}(X^\varepsilon_s)^{(2)} = B_s(X^\varepsilon_s, \mu_s) \text{d}s + \sigma_s \text{d}\tilde{W}_s, \quad X^\varepsilon_0 = x + \varepsilon h,
\end{cases}
\end{align*}
which yields that the law of $X^\varepsilon_t$ under $\mathbb{Q}^\varepsilon_t$ coincides with that of $X^{x,\varepsilon h,\mu}_t$ under $\mathbb{P}$. As a result, we get
\[P^\mu f(x + \varepsilon h) = \mathbb{E}^{\mathbb{Q}^\varepsilon_t} f(X^\varepsilon_t) = \mathbb{E}^{\mathbb{Q}^\varepsilon_t} f(X^{x,\mu}_t) = \mathbb{E} [R^\varepsilon_t f(X^{x,\mu}_t)], \quad f \in \mathcal{B} (\mathbb{R}^{m+d}).\]

Due to (3.9) and (3.10), it is not difficult to verify that $\left\{ \frac{R^\varepsilon_t - 1}{\varepsilon} \right\}_{\varepsilon \in (0,1)}$ is uniformly integrable and hence applying the dominated convergence theorem, (A1)-(A3) and (3.8), we have
\[\lim_{\varepsilon \to 0} \mathbb{E} \left| \frac{R^\varepsilon_t - 1}{\varepsilon} - N_t(h) \right| = 0,\]
which derives (3.3). This combined with (3.10) completes the proof.

(2) For any \( x \in \mathbb{R}^{m+d}, v \in \mathbb{R}^m \), let \( X^\varepsilon_t = ((X^\varepsilon_t)^{(1)} + \varepsilon v, (X^\varepsilon_t)^{(2)}) \). Then it is clear that
\[
\begin{align*}
\left\{ \begin{array}{l}
d(\hat{X}^\varepsilon_x)(1) = M(\hat{X}^\varepsilon_x)^{(2)} ds, \\
d(\hat{X}^\varepsilon_x)(2) = B_s(X^\varepsilon_s, \mu_s) ds + \sigma_s dW_s, \\
\end{array} \right. \\
\hat{X}^\varepsilon_0 = (x^{(1)} + \varepsilon v, x^{(2)}).
\end{align*}
\]
Rewrite it as
\[
\begin{align*}
\left\{ \begin{array}{l}
d(\hat{X}^\varepsilon_x)(1) = M(\hat{X}^\varepsilon_x)^{(2)} ds, \\
d(\hat{X}^\varepsilon_x)(2) = B_s(X^\varepsilon_s, \mu_s) ds + \sigma_s d\hat{W}_s, \\
\end{array} \right. \\
\hat{X}^\varepsilon_0 = (x^{(1)} + \varepsilon v, x^{(2)}),
\end{align*}
\]
with
\[
d\hat{W}_s = dW_s - \eta_s^\varepsilon ds, \quad \eta_s^\varepsilon = \sigma_s^{-1}[B_s(\hat{X}^\varepsilon_s, \mu_s) - B_s(X^\varepsilon_s, \mu_s)], \quad s \in [0, t].
\]
Then (A1)-(A2) gives
\[
|\eta_s^\varepsilon| \leq \sup_{s \in [0, T]} \|\sigma_s^{-1}\|K_B\varepsilon^{|v|}, \quad s \in [0, t].
\]
Let
\[
\hat{R}_t^\varepsilon = \exp \left[ \int_0^t \langle \eta_u^\varepsilon, dW_u \rangle - \frac{1}{2} \int_0^t |\eta_u^\varepsilon|^2 du \right].
\]
Girsanov’s theorem yields that \( (\hat{W}_s)_{s \in [0,t]} \) is a \( d \)-dimensional Brownian motion under \( \hat{Q}_t^\varepsilon = \hat{R}_t^\varepsilon \mathbb{P} \) and so (3.11) implies that the law of \( \hat{X}^\varepsilon_t \) under \( \hat{Q}_t^\varepsilon \) coincides with that of \( X^\varepsilon_t \) under \( \mathbb{P} \), which together with \( (\hat{X}^\varepsilon_x)^{(2)} = (X^\varepsilon_t)^{(2)} \) yields that for any \( f \in \mathcal{B}_b(\mathbb{R}^{m+d}) \) with \( f(x) \) only depending on \( x^{(2)} \),
\[
P_t^\varepsilon f(x^{(1)} + \varepsilon v, x^{(2)}) = \mathbb{E}^{\hat{Q}_t^\varepsilon} f((\hat{X}^\varepsilon_x)^{(2)}) = \mathbb{E}^{\hat{Q}_t^\varepsilon} f((X^\varepsilon_x)^{(2)}) = \mathbb{E}[\hat{R}_t^\varepsilon f((X^\varepsilon_t)^{(2)})].
\]
Similarly to (1), by (3.12), one may verify that \( \{\hat{R}_t^\varepsilon \} \) is uniformly integrable, which together with the dominated convergence theorem and (A1)-(A2) yields
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left| \hat{R}_t^\varepsilon - \frac{1}{\varepsilon} \int_0^t \langle \sigma_s^{-1} \nabla v^{(1)} B_s(X^\varepsilon_s, \mu_s), dW_s \rangle \right| = 0,
\]
This implies (3.5). Finally, (3.6) follows from (3.5), Cauchy-Schwarz’s inequality and (A1)-(A2).

\[\square\]

### 3.2 Proof of Theorem 2.1

For any \( \gamma \in \mathcal{P}_2 \), consider the decoupled SDE:
\[
\begin{align*}
\left\{ \begin{array}{l}
d(X^{\varepsilon, \gamma}_t)^{(1)} = M(X^{\varepsilon, \gamma}_t)^{(2)} dt, \\
d(X^{\varepsilon, \gamma}_t)^{(2)} = B_t(X^{\varepsilon, \gamma}_t, P_t^{\varepsilon, \gamma}) dt + \sigma_t dW_t, \\
X^{\varepsilon, \gamma}_0 = x \in \mathbb{R}^{m+d}.
\end{array} \right.
\]

(2.6) together with (A2) implies that (3.2) holds for $\mu_t = P_t^\gamma$, so that SDE (3.13) is well-posed and it is standard to derive that for any $p \geq 1$, there exists a constant $C > 0$ such that

\begin{equation}
\mathbb{E} \sup_{t \in [0, T]} |X_t^{x, \gamma}|^p \leq C(1 + |x|^p + \|\gamma\|_2^p).
\end{equation}

Let $P_t^\gamma$ be the associated Markov semigroup to (3.13), i.e.

\[ P_t^\gamma f(x) := \mathbb{E}[f(X_t^{x, \gamma})], \quad t \in [0, T], x \in \mathbb{R}^{m+d}, f \in \mathcal{B}_b(\mathbb{R}^{m+d}). \]

Then it holds

\[ P_t f(\gamma) := \int_{\mathbb{R}^{m+d}} f(x)(P_t^\gamma)(dx) = \int_{\mathbb{R}^{m+d}} P_t^\gamma f(x)\gamma(dx), \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}). \]

**Lemma 3.2.** Assume (A1)-(A2). Then for any $p \geq 1$, there exists a constant $c_p > 0$ such that

\begin{equation}
\mathbb{E}|(X_t^{x, \gamma})^{(1)} - x^{(1)} - tMx^{(2)}|^p 
\leq c_p(1 + |x|^p + \|\gamma\|_2^p)t^{\frac{p}{2}}, \quad t \in [0, T], x \in \mathbb{R}^{m+d}, \gamma \in \mathcal{P}_2.
\end{equation}

If $B$ is bounded, then

\begin{equation}
\mathbb{E}|(X_t^{x, \gamma})^{(1)} - x^{(1)} - tMx^{(2)}|^p 
\leq c_p t^{\frac{p}{2}}, \quad t \in [0, T], x \in \mathbb{R}^{m+d}, \gamma \in \mathcal{P}_2.
\end{equation}

\begin{equation}
\mathbb{E} \sup_{s \in [0,t]} |(X_s^{x, \gamma})^{(2)} - x^{(2)}|^p \leq c_p t^{\frac{p}{2}}, \quad t \in [0, T], x \in \mathbb{R}^{m+d}, \gamma \in \mathcal{P}_2.
\end{equation}

**Proof.** Observe that

\[
\begin{align*}
(X_t^{x, \gamma})^{(1)} &= x^{(1)} + \int_0^t M(X_s^{x, \gamma})^{(2)} ds, \\
(X_t^{x, \gamma})^{(2)} &= x^{(2)} + \int_0^t B_s(X_s^{x, \gamma}, P_s^\gamma) ds + \int_0^t \sigma_s dW_s.
\end{align*}
\]

We have

\[
(X_t^{x, \gamma})^{(1)} - x^{(1)} - tMx^{(2)} = \int_0^t M((X_s^{x, \gamma})^{(2)} - x^{(2)}) ds.
\]

So, it is sufficient to prove (3.16) and (3.18). When $B$ is bounded, it is easy to get (3.18) by the BDG inequality. Furthermore, it is not difficult to see from the BDG inequality, (2.5), (2.6) and (3.14) that (3.16) holds.

**Lemma 3.3.** Assume (A2). Then there exists a constant $c > 0$ such that

\begin{equation}
\mathbb{W}_2(P_t^\gamma, P_t^{\tilde{\gamma}}) \leq c\mathbb{W}_2(\gamma, \tilde{\gamma}) + c \int_0^t \mathbb{W}_{\beta,\alpha}(P_s^\gamma, P_s^{\tilde{\gamma}}) ds, \quad t \in [0, T], \gamma, \tilde{\gamma} \in \mathcal{P}_2.
\end{equation}
Proof. Take $\mathcal{F}_0$-measurable random variables $X_0^\gamma, X_0^\tilde{\gamma}$ such that
\begin{equation}
\mathbb{L}X_0^\gamma = \gamma, \quad \mathbb{L}X_0^\tilde{\gamma} = \tilde{\gamma}, \quad \mathbb{W}_2(\gamma, \tilde{\gamma})^2 = \mathbb{E}|X_0^\gamma - X_0^\tilde{\gamma}|^2.
\end{equation}
By (A2), we find a constant $c_1 > 1$ such that
\begin{align*}
\mathbb{E}\left[ \sup_{s \in [0,t]} |X_s^\tilde{\gamma} - X_s^\gamma|^2 \right] & \leq c_1 \mathbb{E}|X_0^\tilde{\gamma} - X_0^\gamma|^2 \\
& + c_1 \left( \int_0^t \left\{ \mathbb{W}_{\beta,\alpha}(P_s^*, \gamma, P_s^* \tilde{\gamma}) + \mathbb{W}_2(P_s^*, \gamma, P_s^* \tilde{\gamma}) \right\} ds \right)^2 \\
& + c_1 \mathbb{E} \int_0^t |X_s^\tilde{\gamma} - X_s^\gamma|^2 ds, \quad t \in [0, T].
\end{align*}
So, it follows from the inequality $\mathbb{W}_2(P_s^*, \gamma, P_s^* \tilde{\gamma})^2 \leq \mathbb{E}|X_s^\tilde{\gamma} - X_s^\gamma|^2$ and Gronwall’s inequality that (3.19) holds.

The following Hölder inequality for concave functions comes from [8, Lemma 2.4].

**Lemma 3.4.** Let $\alpha : [0, \infty) \to [0, \infty)$ be concave. Then for any non-negative random variables $\xi$ and $\eta$,
\begin{equation}
\mathbb{E}[\alpha(\xi)\eta] \leq \|\eta\|_{L^p(P)} \alpha\left( \|\xi\|_{L^{p/(p-1)}(P)} \right), \quad p \geq 1.
\end{equation}

The following lemma is crucial in the proof of the desired Harnack inequality.

**Lemma 3.5.** Assume (A1)-(A3). Then there exists a constant $c > 0$ such that
\begin{equation}
\mathbb{W}_{\beta,\alpha}(P_t^*, \gamma, P_t^* \tilde{\gamma}) \leq c\mathbb{W}_2(\tilde{\gamma}, \gamma)(1 + \|\gamma\|_2 + \|\tilde{\gamma}\|_2)
\end{equation}
\begin{equation}
\times \left\{ \frac{\alpha(t^{\frac{1}{2}})}{\sqrt{t}} + t^{\frac{3(\beta-1)}{2}} + e^{(1+\|\gamma\|_2+\|\tilde{\gamma}\|_2)^2} \right\}, \quad t \in (0, T], \gamma, \tilde{\gamma} \in \mathcal{P}_2.
\end{equation}
Consequently, there exists a constant $\tilde{c} > 0$ such that for any $\gamma, \tilde{\gamma} \in \mathcal{P}_2$,
\begin{equation}
\sup_{t \in [0, T]} \mathbb{W}_2(P_t^*, \gamma, P_t^* \tilde{\gamma}) \leq e^{\tilde{c}(1+\|\gamma\|_2+\|\tilde{\gamma}\|_2)^2} \mathbb{W}_2(\gamma, \tilde{\gamma}).
\end{equation}
If $B$ is bounded, we have
\begin{equation}
\mathbb{W}_{\beta,\alpha}(P_t^*, \gamma, P_t^* \tilde{\gamma}) \leq c\mathbb{W}_2(\tilde{\gamma}, \gamma) \left\{ \frac{\alpha(t^{\frac{1}{2}})}{\sqrt{t}} + t^{\frac{3(\beta-1)}{2}} \right\}
\end{equation}
and
\begin{equation}
\sup_{t \in [0, T]} \mathbb{W}_2(P_t^*, \gamma, P_t^* \tilde{\gamma}) \leq c\mathbb{W}_2(\tilde{\gamma}, \gamma).
\end{equation}
Let $X_0^\gamma$ and $X_0^{\tilde{\gamma}}$ be in (3.20). For any $\varepsilon \in [0, 2]$, let
\[X_0^{\gamma^\varepsilon} := X_0^\gamma + \varepsilon (X_0^{\tilde{\gamma}} - X_0^\gamma), \quad \gamma^\varepsilon := \mathcal{L}_{X_0^\gamma}.\]

By the definition of $\mathbb{W}_{\beta, \alpha}$, for any $\varepsilon, r \in [0, 1]$ and $t \in [0, T]$, we have
\[
\mathbb{W}_{\beta, \alpha}(P_t^{\gamma^\varepsilon + r}, P_t^{\gamma^\varepsilon})^2 = \sup_{f \in \mathcal{B}(\mathbb{R}^{m+d}), |f|_{\beta, \alpha} \leq 1} |P_t f(\gamma^\varepsilon + r) - P_t f(\gamma^\varepsilon)|^2 \\
\leq 2 \sup_{f \in \mathcal{B}(\mathbb{R}^{m+d}), |f|_{\beta, \alpha} \leq 1} |\gamma^\varepsilon + r (P_t^{\gamma^\varepsilon + r} f - P_t^{\gamma^\varepsilon} f)|^2 \\
+ 2 \sup_{f \in \mathcal{B}(\mathbb{R}^{m+d}), |f|_{\beta, \alpha} \leq 1} |\gamma^\varepsilon + r (P_t^{\gamma^\varepsilon} f) - \gamma^\varepsilon (P_t^{\gamma^\varepsilon} f)|^2 \\
=: I_1(t) + I_2(t).
\]

Therefore, to prove Lemma 3.5, it is sufficient to derive the estimates for $I_1(t)$ and $I_2(t)$, which will be provided in the following two lemmas.

**Lemma 3.6.** Assume (A1)-(A2). Then there exists a constant $c > 0$ such that
\[
I_1(t) \leq c(1 + \|\gamma\|_2 + \|\tilde{\gamma}\|_2)^2 \psi(\varepsilon, r) \\
\times \left( r^2 \mathbb{W}_2(\gamma, \tilde{\gamma})^2 + \int_0^t \mathbb{W}_{\beta, \alpha}(P_s^{\gamma^\varepsilon}, P_s^{\gamma^\varepsilon + r})^2 ds \right), \quad t \in [0, T].
\]

When $B$ is bounded, we conclude that
\[
I_1(t) \leq c \psi(\varepsilon, r) \left( r^2 \mathbb{W}_2(\gamma, \tilde{\gamma})^2 + \int_0^t \mathbb{W}_{\beta, \alpha}(P_s^{\gamma^\varepsilon}, P_s^{\gamma^\varepsilon + r})^2 ds \right), \quad t \in [0, T],
\]
where
\[
\psi(\varepsilon, r) := e^{cr^2 \mathbb{W}_2(\gamma, \tilde{\gamma})^2 + c \int_0^T \mathbb{W}_{\beta, \alpha}(P_s^{\gamma^\varepsilon}, P_s^{\gamma^\varepsilon + r})^2 ds}.
\]

**Proof.** Firstly, by the definition of $\gamma^\varepsilon$, we have
\[
\|\gamma^\varepsilon\|_2^2 \leq 8\|\gamma\|_2^2 + 8\|\tilde{\gamma}\|_2^2, \quad \varepsilon \in [0, 2],
\]
and
\[
\mathbb{W}_2(\gamma^\varepsilon, \gamma^\varepsilon + r)^2 \leq r^2 \mathbb{W}_2(\gamma, \tilde{\gamma})^2, \quad \varepsilon, r \in [0, 1].
\]

For any $\varepsilon \in [0, 2]$, consider the SDE
\[
\begin{aligned}
\left\{ \begin{array}{l}
d(X_t^{x, \gamma^\varepsilon})^{(1)} = M(X_t^{x, \gamma^\varepsilon})^{(2)} dt, \\
(\gamma_t^{x, \gamma^\varepsilon})^{(2)} = B_t(X_t^{x, \gamma^\varepsilon}, P_t^{\gamma^\varepsilon}) dt + \sigma_t dB_t,
\end{array} \right. \quad X_0^{x, \gamma^\varepsilon} = x \in \mathbb{R}^{m+d}, t \in [0, T].
\]

For any $r, \varepsilon \in [0, 1]$, define
\[
\eta_t^{\varepsilon, r} = \sigma_t^{-1} [B_t(X_t^{x, \gamma^\varepsilon}, P_t^{\gamma^\varepsilon + r}) - B_t(X_t^{x, \gamma^\varepsilon}, P_t^{\gamma^\varepsilon})], \quad t \in [0, T].
\]
By (A1)-(A2), (2.4) for \( k = 1, (2.6) \) and (3.30), there exist constants \( c_1, c_2 > 0 \) such that

\[
\sup_{t \in [0, T]} |\tilde{\eta}_t^{\varepsilon,r}| \leq c_1 \sup_{t \in [0, T]} \left\{ \mathbb{W}_{\beta,\alpha}(P_t^{\gamma, \varepsilon}, P_t^{\gamma, \varepsilon+r}) + \mathbb{W}_2(P_t^{\gamma, \varepsilon}, P_t^{\gamma, \varepsilon+r}) \right\} \\
\leq c_2 (1 + \|\gamma\|_2 + \|\tilde{\gamma}\|_2), \quad r, \varepsilon \in [0, 1].
\]

(3.33)

It follows from Girsanov’s theorem that

\[
W_t^{\varepsilon,r} = W_t - \int_0^t \tilde{\eta}_s^{\varepsilon,r} ds, \quad t \in [0, T]
\]

is a \( d \)-dimensional Brownian motion under the probability \( \mathbb{Q}^{\varepsilon,r} := R_t^{\varepsilon,r} \mathbb{P} \) with

\[
R_t^{\varepsilon,r} := \exp \left\{ \int_0^t \langle \tilde{\eta}_s^{\varepsilon,r}, dW_s \rangle - \frac{1}{2} \int_0^t |\tilde{\eta}_s^{\varepsilon,r}|^2 ds \right\}, \quad t \in [0, T].
\]

Therefore, (3.32) can be reformulated as

\[
\begin{align*}
\left\{ &d(X_t^{x, \gamma})^{(1)} = M(X_t^{x, \gamma})^{(2)} dt, \\
&d(X_t^{x, \gamma})^{(2)} = B_t(X_t^{x, \gamma}, P_t^{\gamma, \varepsilon+r}) dt + \sigma_t dW_t^{\varepsilon,r}, \quad X_0^{x, \gamma} = x \in \mathbb{R}^{m+d}, t \in [0, T].
\end{align*}
\]

So, for any \( f \in \mathcal{B}_b(\mathbb{R}^{m+d}) \), it holds

\[
P_t^{\gamma, \varepsilon+r} f(x) - P_t^{\gamma, \varepsilon} f(x) \\
= \mathbb{E} \left[ f(X_t^{x, \gamma}) (R_t^{\varepsilon,r} - 1) \right] \\
= \mathbb{E} \left[ (f(X_t^{x, \gamma}) - f(x^{(1)} + tM(x^{(2)})) (R_t^{\varepsilon,r} - 1)) \right], \quad \varepsilon, r \in (0, 1], t \in [0, T], x \in \mathbb{R}^{m+d}.
\]

Moreover, by (3.33), (3.19) and (3.31), we obtain

\[
\begin{align*}
\mathbb{E} |R_t^{\varepsilon,r} - 1|^2 &\leq \mathbb{E} [(R_t^{\varepsilon,r})^2 - 1] \leq \text{esssup}_\Omega \left( e^{\int_0^t |\tilde{\eta}_s^{\varepsilon,r}|^2 ds} \int_0^t |\tilde{\eta}_s^{\varepsilon,r}|^2 ds \right) \\
&\leq \text{esssup}_\Omega \left( e^{\int_0^t |\tilde{\eta}_s^{\varepsilon,r}|^2 ds} \int_0^t |\tilde{\eta}_s^{\varepsilon,r}|^2 ds \right) \\
&\leq \psi(\varepsilon, r) \int_0^t \{ \mathbb{W}_{\beta,\alpha}(P_s^{\gamma, \varepsilon}, P_s^{\gamma, \varepsilon+r})^2 + \mathbb{W}_2(P_s^{\gamma, \varepsilon}, P_s^{\gamma, \varepsilon+r})^2 \} ds \\
&\leq c_3 \psi(\varepsilon, r) \left( r^2 \mathbb{W}_2(\gamma, \tilde{\gamma})^2 + \int_0^t \mathbb{W}_{\beta,\alpha}(P_s^{\gamma, \varepsilon}, P_s^{\gamma, \varepsilon+r})^2 ds \right), \quad t \in [0, T]
\end{align*}
\]

(3.34)

for some constant \( c_3 > 0 \). By (3.33), we have

\[
\psi := \sup_{\varepsilon, r \in [0, 1]} \psi(\varepsilon, \gamma) < \infty.
\]

(3.35)

Combining (2.3) for \( k = 1 \), the Cauchy-Schwarz inequality and (3.34), we arrive at

\[
\sup_{f \in \mathcal{B}_b(\mathbb{R}^{m+d}), |||f|||_{\beta,\alpha} \leq 1} \mathbb{E} |\gamma^{\varepsilon+r} (P_t^{\gamma, \varepsilon+r} f - P_t^{\gamma} f)|^2
\]
\[
\begin{align*}
\leq \left( \int_{\mathbb{R}^{m+d}} \sup_{f \in \mathcal{B}_0(\mathbb{R}^{m+d}), \|f\|_{\beta, \alpha} \leq 1} \left| \mathbb{E} \left[ (f(X_t^x, \gamma^x) - f(x^{(1)} + tMx^{(2)}, x^{(2)}))(R_t^{x, r} - 1) \right] \right| \gamma^{x+r}(dx) \right)^2 \\
\leq \left( \int_{\mathbb{R}^{m+d}} \{2(2(\alpha(1) + 1)^2)2\mathbb{E}(1 + |X_t^{x, \gamma^x} - (x^{(1)} + tMx^{(2)}, x^{(2)})|^2)\}^{\frac{1}{2}} \gamma^{x+r}(dx) \right)^2 \\
\times c_3 \psi(\varepsilon, r) \left( r^2 \mathbb{W}_2(\gamma, \tilde{\gamma})^2 + \int_0^t \mathbb{W}_{\beta, \alpha}(P_s^{x, \gamma^x}, P_s^{x, \gamma^{x+r}})^2 ds \right), \quad t \in [0, T].
\end{align*}
\]

This together with (3.15), (3.16) and (3.30), we derive from Jensen’s inequality that (3.27) holds for some constant $c > 0$. When $B$ is bounded, applying (3.17) and (3.18) replacing (3.15) and (3.16) respectively, we derive (3.28). \hfill \Box

**Lemma 3.7.** Assume (A1)-(A3). Then there exists a constant $c > 0$ such that

\[
(3.36) \quad I_2(t) \leq cr^2 \mathbb{W}_2(\gamma, \tilde{\gamma})^2(1 + \|\gamma\|_2 + \|\tilde{\gamma}\|_2)^2 \left( t^{3(\beta-1)} + \frac{\alpha(t^2)}{t} \right), \quad t \in (0, T].
\]

When $B$ is bounded, it holds

\[
(3.37) \quad I_2(t) \leq cr^2 \mathbb{W}_2(\gamma, \tilde{\gamma})^2 \left( t^{3(\beta-1)} + \frac{\alpha(t^2)}{t} \right), \quad t \in (0, T].
\]

**Proof.** Note that by Theorem 3.1, for any $p > 1$, there exists some constant $c_1 > 0$ such that for $h \in \mathbb{R}^{m+d}$,

\[
(3.38) \quad \left( \mathbb{E}(N_t(h^{(1)}, 0)^p) \right)^{\frac{1}{p}} \leq c_1|h^{(1)}|t^{-\frac{1}{2}}, \quad \left( \mathbb{E}(N_t(0, h^{(2)})^p) \right)^{\frac{1}{p}} \leq c_1|h^{(2)}|t^{-\frac{1}{2}}, \quad t \in (0, T].
\]

For any $x_0 \in \mathbb{R}^{m+d}$ and any $f \in \mathcal{B}_0(\mathbb{R}^{m+d})$, let

\[
\begin{align*}
&f_x^{(1)}(x) = f(x) - f(x_0^{(1)} + Mtx_0^{(2)}, x^{(2)}), \\
&\tilde{f}_x^{(1)}(x) = f(x_0^{(1)} + Mtx_0^{(2)}, x^{(2)}) - f(x_0^{(1)} + Mtx_0^{(2)}, x_0^{(2)}), \\
&f_x^{(2)}(x) = f(x) - f(x_0^{(1)} + Mtx_0^{(2)}, x_0^{(2)}), \quad x \in \mathbb{R}^{m+d}, t \in [0, T].
\end{align*}
\]

By Theorem 3.1(1), (3.15), the first inequality in (3.38), (3.30) and Hölder’s inequality, we have

\[
(3.39) \quad \sup_{f \in \mathcal{B}_0(\mathbb{R}^{m+d}), \|f\|_{\beta, \alpha} \leq 1} |\nabla^{(1)} P_t^{x, \gamma^x} f_x^{(1)}(x_0)| \leq c_2(1 + |x_0| + \|\gamma\|_2 + \|\tilde{\gamma}\|_2)^2 t^{\frac{3(\beta-1)}{2}} \gamma^{x+r}(dx), \quad t \in (0, T]
\]

for some constant $c_2 > 0$. By (3.6), (3.16), (3.30) and (2.2), we can find constants $c_3, c_4 > 0$ such that

\[
(3.40) \quad \sup_{f \in \mathcal{B}_0(\mathbb{R}^{m+d}), \|f\|_{\beta, \alpha} \leq 1} |\nabla^{(1)} P_t^{x, \gamma} \tilde{f}_x^{(1)}(x_0)| \leq c_3 \{1 + \mathbb{E}(|X_t^{x_0, \gamma^x} - x_0^{(2)})^2\}^{\frac{1}{2}} + \frac{c_4(1 + |x_0| + \|\gamma\|_2 + \|\tilde{\gamma}\|_2), \quad t \in [0, T].
\]
Moreover, by Theorem 3.1(1), the second inequality in (3.38), (3.15), (3.16), (3.30), (3.21) and (2.2), we conclude

\[
\sup_{f \in \mathcal{B}_0(\mathbb{R}^{m+d}), |f|_{\beta, \alpha} \leq 1} |\nabla (2) \, P_t^{\gamma^e} f_{x_0}^{(2)}|(x_0)
\leq c_5 (1 + |x_0| + \|\gamma\|_2 + \|\tilde{\gamma}\|_2) \left( t^{\frac{3\beta-1}{2}} + \frac{\alpha(t^\frac{1}{2})}{t^\frac{1}{2}} \right), \quad t \in (0, T]
\]

(3.41)

for some constant \( c_5 > 0 \). Since

\[
\nabla (1) \, P_t^{\gamma^e} f = \nabla (1) \, P_t^{\gamma^e} f_{x_0}^{(1)} + \nabla (1) \, P_t^{\gamma^e} \tilde{f}_{x_0}^{(1)}, \quad \nabla (2) \, P_t^{\gamma^e} f = \nabla (2) \, P_t^{\gamma^e} f_{x_0}^{(2)}, \quad f \in \mathcal{B}_0(\mathbb{R}^{m+d}),
\]

we derive from (3.39) and (3.40) that

\[
\sup_{f \in \mathcal{B}_0(\mathbb{R}^{m+d}), |f|_{\beta, \alpha} \leq 1} |\nabla (1) \, P_t^{\gamma^e} f|(x_0) \leq c_6 (1 + |x_0| + \|\gamma\|_2 + \|\tilde{\gamma}\|_2) t^{\frac{3\beta-1}{2}}, \quad t \in (0, T]
\]

(3.42)

and from (3.41) that

\[
\sup_{f \in \mathcal{B}_0(\mathbb{R}^{m+d}), |f|_{\beta, \alpha} \leq 1} |\nabla (2) \, P_t^{\gamma^e} f|(x_0) \leq c_5 (1 + |x_0| + \|\gamma\|_2 + \|\tilde{\gamma}\|_2) \left( t^{\frac{3\beta-1}{2}} + \frac{\alpha(t^\frac{1}{2})}{t^\frac{1}{2}} \right)
\]

(3.43)

for some constant \( c_6 > 0 \). Observe that

\[
I_2(t) = 2 \sup_{f \in \mathcal{B}_0(\mathbb{R}^{m+d}), |f|_{\beta, \alpha} \leq 1} \left| \mathbb{E} \int_0^r \frac{d}{d\theta} P_t^{\gamma^e} f(X_0^{\gamma^e+\theta}) d\theta \right|^2
= 2 \sup_{f \in \mathcal{B}_0(\mathbb{R}^{m+d}), |f|_{\beta, \alpha} \leq 1} \left| \mathbb{E} \int_0^r \left\{ \nabla X_0^{\gamma^e} - X_0^{\gamma^e} \right\} P_t^{\gamma^e} f(X_0^{\gamma^e+\theta}) d\theta \right|^2.
\]

Combining this with (3.42), (3.43), (3.30) and (3.20), we find constants \( c_7, c_8 > 0 \) such that

\[
I_2(t) \leq c_7 \left\{ \int_0^r \left[ \|X_0^\gamma - X_0^{\tilde{\gamma}}\|_{L^2(P)} (1 + \|X_0^{\gamma^e+\theta}\|_{L^2(P)} + \|\gamma\|_2 + \|\tilde{\gamma}\|_2) \left( t^{\frac{3\beta-1}{2}} + \frac{\alpha(t^\frac{1}{2})}{t^\frac{1}{2}} \right) \right] d\theta \right\}^2
\leq c_8 r^2 \|2(\gamma, \tilde{\gamma})^2 (1 + \|\gamma\|_2 + \|\tilde{\gamma}\|_2)^2 \left( t^{3(\beta-1)} + \frac{\alpha(t^\frac{1}{2})^2}{t} \right).
\]

When \( B \) is bounded, repeating the above procedure and using (3.17) and (3.18) replacing (3.15) and (3.16) respectively, we can obtain (3.37).

Now, we are in the position to prove Lemma 3.5.
Proof of Lemma 3.5. Firstly, by $\alpha \in \mathcal{A}$ and $\beta > \frac{2}{3}$, we conclude that

\begin{equation}
\int_0^T \frac{\alpha(t^2)}{t} dt = 2 \int_0^{T^2} \frac{\alpha(s)}{s} ds < \infty, \quad \int_0^T t^{3(\beta - 1)} dt < \infty.
\end{equation}

(3.46) together with (3.27) and (3.26) yields

\begin{equation}
\mathbb{W}_{\beta,\alpha}(P_t^* \gamma^\varepsilon, P_t^* \gamma^\varepsilon + r)^2 \\
\leq c(1 + \|\gamma\|_2 + \|\bar{\gamma}\|_2)^2 \psi(\varepsilon, r) \int_0^t \mathbb{W}_{\beta,\alpha}(P_s^* \gamma^\varepsilon, P_s^* \gamma^\varepsilon + r)^2 ds \\
+ cr^2 \mathbb{W}_2(\gamma, \bar{\gamma})^2(1 + \|\gamma\|_2 + \|\bar{\gamma}\|_2)^2 \left( \psi(\varepsilon, r) + t^{3(\beta - 1)} + \frac{\alpha(t^\frac{1}{2})^2}{t} \right), \quad t \in (0, T].
\end{equation}

(3.47)

Let

\[ \Gamma_t(\varepsilon, r) := \int_0^t \mathbb{W}_{\beta,\alpha}(P_s^* \gamma^\varepsilon, P_s^* \gamma^\varepsilon + r)^2 ds. \]

So, it follows from (3.45) that

\begin{equation}
\Gamma_t(\varepsilon, r) \leq cr^2 \mathbb{W}_2(\gamma, \bar{\gamma})^2 H(\varepsilon, r) + c F(\varepsilon, r) \int_0^t \Gamma_s(\varepsilon, r) ds, \quad t \in [0, T],
\end{equation}

(3.48)

\[ H(\varepsilon, r) := [1 + \|\gamma\|_2 + \|\bar{\gamma}\|_2]^2 \int_0^T \left[ \psi(\varepsilon, r) + t^{3(\beta - 1)} + \frac{\alpha(t^\frac{1}{2})^2}{t} \right] dt, \]

\[ F(\varepsilon, r) := [1 + \|\gamma\|_2 + \|\bar{\gamma}\|_2]^2 \psi(\varepsilon, r), \quad \varepsilon, r \in [0, 1]. \]

By Gronwall’s inequality and (3.46), for any $\varepsilon, r \in [0, 1]$ we have

\begin{equation}
\Gamma_t(\varepsilon, r) \leq cr^2 \mathbb{W}_2(\gamma, \bar{\gamma})^2 e^{c F(\varepsilon, r) t} H(\varepsilon, r), \quad t \in [0, T].
\end{equation}

(3.49)

Substituting this into (3.45), we get

\begin{equation}
\mathbb{W}_{\beta,\alpha}(P_t^* \gamma^\varepsilon, P_t^* \gamma^\varepsilon + r)^2 \leq c(1 + \|\gamma\|_2 + \|\bar{\gamma}\|_2)^2 r^2 \mathbb{W}_2(\gamma, \bar{\gamma})^2 \\
\times \left[ c\psi(\varepsilon, r)e^{c F(\varepsilon, r) t} H(\varepsilon, r) + \psi(\varepsilon, r) + t^{3(\beta - 1)} + \frac{\alpha(t^\frac{1}{2})^2}{t} \right].
\end{equation}

(3.50)

Note that (3.35), (3.29), (3.46) and (3.47) imply that $\psi(\varepsilon, r)$ is bounded in $(\varepsilon, r) \in [0, 1]^2$ with $\psi(\varepsilon, r) \to 1$ as $r \to 0$, so that by (3.48) and the dominated convergence theorem we find a constant $C > 1$ such that

\begin{equation}
\limsup_{r \to 0} \frac{\mathbb{W}_{\beta,\alpha}(P_t^* \gamma^\varepsilon, P_t^* \gamma^\varepsilon + r)}{r} \\
\leq C \mathbb{W}_2(\bar{\gamma}, \gamma)(1 + \|\gamma\|_2 + \|\bar{\gamma}\|_2) \left\{ \frac{\alpha(t^\frac{1}{2})}{\sqrt{t}} + t^{\frac{3(\beta - 1)}{2}} + C(1 + \|\gamma\|_2 + \|\bar{\gamma}\|_2)^2 \right\}.
\end{equation}

(3.51)
where we have used the fact that for some constant $C > 1$,

$$(1 + \|\gamma\|_2 + \|\tilde{\gamma}\|_2)^2 e^{TC(1 + \|\gamma\|_2 + \|\tilde{\gamma}\|_2)^2} \leq e^{C(1 + \|\gamma\|_2 + \|\tilde{\gamma}\|_2)^2}.$$ 

By the triangle inequality,

$$\left| \mathcal{W}_{\beta,\alpha}(P^*_s \gamma, P^*_s \gamma^\varepsilon) - \mathcal{W}_{\beta,\alpha}(P^*_s \gamma^\varepsilon, P^*_s \gamma^{\varepsilon + r}) \right| \leq \mathcal{W}_{\beta,\alpha}(P^*_s \gamma^\varepsilon, P^*_s \gamma^{\varepsilon + r}), \quad \varepsilon, r \in [0, 1],$$

so that (3.49) implies that $\mathcal{W}_{\beta,\alpha}(P^*_s \gamma, P^*_s \gamma^\varepsilon)$ is Lipschitz continuous (hence a.e. differentiable) in $\varepsilon \in [0, 1]$ for any $t \in (0, T]$, and

$$\left| \frac{d}{d\varepsilon} \mathcal{W}_{\beta,\alpha}(P^*_s \gamma, P^*_s \gamma^\varepsilon) \right| \leq \limsup_{r \downarrow 0} \frac{\mathcal{W}_{\beta,\alpha}(P^*_s \gamma^\varepsilon, P^*_s \gamma^{\varepsilon + r})}{r} \leq C\mathcal{W}_2(\gamma, \tilde{\gamma})(1 + \|\gamma\|_2 + \|\tilde{\gamma}\|_2) \left\{ \frac{\alpha(t_\varepsilon^2)}{\sqrt{t}} + t^{3(\beta - 1)} + e^{C(1 + \|\gamma\|_2 + \|\tilde{\gamma}\|_2)^2} \right\}, \quad \varepsilon \in [0, 1].$$

Noting that $\gamma^1 = \tilde{\gamma}$, this implies the desired estimate (3.22), which combined with (3.19) yields (3.23).

When $B$ is bounded, we derive from (3.28), (3.37) and (3.36) that

$$\mathcal{W}_{\beta,\alpha}(P^*_s \gamma^\varepsilon, P^*_s \gamma^{\varepsilon + r})^2 \leq c\psi(\varepsilon, r) \int_0^t \mathcal{W}_{\beta,\alpha}(P^*_s \gamma^\varepsilon, P^*_s \gamma^{\varepsilon + r})^2 ds + cr^2\mathcal{W}_2(\gamma, \tilde{\gamma})^2 \left( \psi(\varepsilon, r) + t^{3(\beta - 1)} + \frac{\alpha(t_\varepsilon^2)}{t} \right), \quad t \in (0, T].$$

Then by (3.50), we have

$$\Gamma_t(\varepsilon, r) \leq cr^2\mathcal{W}_2(\gamma, \tilde{\gamma})^2 H(\varepsilon, r) + cF(\varepsilon, r) \int_0^t \Gamma_s(\varepsilon, r) ds, \quad t \in [0, T],$$

$$H(\varepsilon, r) := \int_0^T \left[ \psi(\varepsilon, r) + \frac{\alpha(t_\varepsilon^2)}{t} + t^{3(\beta - 1)} \right] dt,$$

$$F(\varepsilon, r) := \psi(\varepsilon, r), \quad \varepsilon, r \in [0, 1].$$

Repeating the same procedure by replacing (3.46) with (3.51), we derive (3.24) and (3.25).

Finally, we intend to prove Theorem 2.1.

**Proof of Theorem 2.1.** Consider

$$\begin{cases}
\text{d}(X_s^{x,\gamma})^{(1)} = M(X_s^{x,\gamma})^{(2)} \text{d}s, \\
\text{d}(X_s^{x,\gamma})^{(2)} = B_s(X_s^{x,\gamma}, P_s^\varepsilon \gamma) \text{d}s + \sigma_s \text{d}W_s, \quad X_0^{x,\gamma} = x.
\end{cases}$$
Recall that $\gamma_s(h)$ is defined in (3.4). Let $\tilde{X}_s$ solve

\begin{align}
&\left\{ \begin{aligned}
d\tilde{X}_s^{(1)} &= M\tilde{X}_s^{(2)} ds, \\
d\tilde{X}_s^{(2)} &= B_s(X_s^{x,\gamma}, P_s^{*\gamma}) ds + \sigma_s dW_s + \gamma'_s(y-x) ds, \quad \tilde{X}_0 = y.
\end{aligned} \right.
\end{align}

Then it holds

$$
\tilde{X}_s = X_s^{x,\gamma} + \left( y^{(1)} - x^{(1)} + \int_0^s M\gamma_u(y-x) du, \quad \gamma_s(y-x) \right), \quad s \in [0, t],
$$

in particular, $\tilde{X}_t = X_t^{x,\gamma}$ due to (3.4). Let

$$
\eta_s^{\gamma,\bar{\gamma}} := \sigma_s^{-1}(B_s(\tilde{X}_s, P_s^{*\gamma}) - B_s(X_s^{x,\gamma}, P_s^{*\gamma}) - \gamma'_s(y-x)), \quad s \in [0, t],
$$

\[ R_t^{\gamma,\bar{\gamma}} := e^{\int_0^t \eta_u^{\gamma,\bar{\gamma}} du} \int_0^t \eta_u^{\gamma,\bar{\gamma}}^2 ds. \]

(A1)-(A2) imply

\begin{align}
|\eta_s^{\gamma,\bar{\gamma}}| &\leq c_0 \left( |\gamma_s(y-x)| + |y^{(1)} - x^{(1)}| + \|M\| \int_0^s |\gamma_u(y-x)| du 
\right. \\
&\quad + \mathbb{W}_2(P_s^{\gamma,\bar{\gamma}}) + \mathbb{W}_{\beta,\alpha}(P_s^{\gamma,\bar{\gamma}}) + |\gamma'_s(y-x)| \left), \quad s \in [0, t] \right.
\end{align}

for some constant $c_0 > 0$. By (3.44) and Lemma 3.5, there exists a constant $c_1 > 0$ such that

$$
\int_0^T \{ \mathbb{W}_{\beta,\alpha}(P_s^{\gamma,\bar{\gamma}})^2 + \mathbb{W}_2(P_s^{\gamma,\bar{\gamma}}, P_s^{*\gamma})^2 \} ds \leq c_1 e^{c_1 (1+\|\gamma\|_2+\|\bar{\gamma}\|_2)^2} \mathbb{W}_2(\gamma, \bar{\gamma})^2.
$$

This together with (3.53) and (3.10) gives

\begin{align}
\int_0^t |\eta_s^{\gamma,\bar{\gamma}}|^2 ds &\leq \frac{c_2 |x-y|^2}{t^3} + c_2 e^{c_2 (1+\|\gamma\|_2+\|\bar{\gamma}\|_2)^2} \mathbb{W}_2(\gamma, \bar{\gamma})^2
\end{align}

for some constant $c_2 > 0$. As a result, Girsanov’s theorem implies that

$$
W_s^{\gamma,\bar{\gamma}} = W_s - \int_0^s \eta_u^{\gamma,\bar{\gamma}} du, \quad s \in [0, t]
$$

is a $d$-dimensional Brownian motion under the probability measure $\mathbb{Q}_t^{\gamma,\bar{\gamma}} = R_t^{\gamma,\bar{\gamma}} \mathbb{P}$. So, (3.52) can be rewritten as

\begin{align}
&\left\{ \begin{aligned}
d\tilde{X}_s^{(1)} &= M\tilde{X}_s^{(2)} ds, \\
d\tilde{X}_s^{(2)} &= B_s(\tilde{X}_s, P_s^{*\gamma}) ds + \sigma_s dW_s^{\gamma,\bar{\gamma}}, \quad \tilde{X}_0 = y,
\end{aligned} \right.
\end{align}

which derives

$$
P_t^{\gamma} f(y) = \mathbb{E}^{\mathbb{Q}_t^{\gamma,\bar{\gamma}}} f(\tilde{X}_t) = \mathbb{E}^{\mathbb{Q}_t^{\gamma,\bar{\gamma}}} f(X_t^{x,\gamma}) = \mathbb{E}[R_t^{\gamma,\bar{\gamma}} f(X_t^{x,\gamma})], \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}).
$$
By Young’s inequality, we have

\[ P^\gamma_t \log f(y) \leq \log P^\gamma_t f(x) + \mathbb{E}(R^{\gamma,\tilde{\gamma}}_t \log R^{\gamma,\tilde{\gamma}}_t) \]

(3.55)

\[ \leq \log P^\gamma_t f(x) + \frac{1}{2} \mathbb{E} \int_0^t |\eta_s^{\gamma,\tilde{\gamma}}|^2 ds, \quad f \in \mathcal{B}_b^+(\mathbb{R}^{m+d}), f > 0. \]

Hölder’s inequality yields that for any \( p > 1, \)

\[ (P^\gamma_t f(y))^p \leq P^\gamma_t f^p(x)(\mathbb{E}(R^{\gamma,\tilde{\gamma}}_t)^p)^{p-1} \]

(3.56)

\[ \leq P^\gamma_t f^p(x) \text{esssup}_\Omega \exp \left\{ \frac{p}{2(p - 1)} \int_0^t |\eta_u^{\gamma,\tilde{\gamma}}|^2 du \right\}, \quad f \in \mathcal{B}_b^+(\mathbb{R}^{m+d}). \]

Applying (3.54), taking expectation in (3.55) and (3.56) with respect to any \( \pi \in \mathcal{C}(\gamma, \tilde{\gamma}) \) and then taking infimum in \( \pi \in \mathcal{C}(\gamma, \tilde{\gamma}) \), the proof is completed by Jensen’s inequality and (1.2).

\[ \square \]

4 Well-posedness

In this section, we consider a general version of (1.1). Fix \( T > 0 \) and let \( k \geq 1 \). Consider the distribution dependent SDEs on \( \mathbb{R}^{m+d} \):

(4.1)

\[
\begin{align*}
\text{d}X^{(1)}_t &= b_t(X_t)dt, \\
\text{d}X^{(2)}_t &= B_t(X_t, \mathcal{L}X_t)dt + \sigma_t(X_t)dW_t,
\end{align*}
\]

where \( b : [0, T] \times \mathbb{R}^{m+d} \to \mathbb{R}^m, B : [0, T] \times \mathbb{R}^{m+d} \times \mathcal{P} \to \mathbb{R}^d, \sigma : [0, T] \times \mathbb{R}^{m+d} \to \mathbb{R}^d \otimes \mathbb{R}^d \) are measurable and \( W_t \) is a \( d \)-dimensional Brownian motion on some complete filtration probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). Let \( C([0, T]; \mathcal{P}_k) \) denote the continuous maps from \([0, T] \) to \((\mathcal{P}_k, \mathbb{W}_k)\).

**Definition 4.1.** The SDE (4.1) is called well-posed for distributions in \( \mathcal{P}_k \), if for any \( \mathcal{P}_0 \)-measurable initial value \( X_0 \) with \( \mathcal{L}X_0 \in \mathcal{P}_k \) (respectively any initial distribution \( \gamma \in \mathcal{P}_k \)), it has a unique strong solution (respectively weak solution) such that \( \mathcal{L}_X \in C([0, T]; \mathcal{P}_k) \).

We make the following assumptions.

**(C1)** For any \( t \in [0, T], x \in \mathbb{R}^{m+d}, \) \( \sigma_t(x) \) is invertible and \( \|\sigma^{-1}\|_\infty \) is finite.

**(C2)** There exists \( K > 0 \) such that

\[
\begin{align*}
|b_t(x) - b_t(\bar{x})| + |\sigma_t(x) - \sigma_t(\bar{x})| &\leq K|x - \bar{x}|, \\
|B_t(x, \gamma) - B_t(\bar{x}, \gamma)| &\leq K(|x - \bar{x}| + \mathbb{W}_k(\gamma, \bar{\gamma}) + \|\gamma - \bar{\gamma}\|_{k, var}), \\
|b_t(0)| + |\sigma_t(0)| &\leq K, \quad |B_t(0, \gamma)| \leq K(1 + \|\gamma\|_k), \quad x, \bar{x} \in \mathbb{R}^{m+d}, \gamma, \bar{\gamma} \in \mathcal{P}_k.
\end{align*}
\]
For any \( \mu \in C([0, T], \mathcal{P}_k) \), consider

\[
\begin{aligned}
\frac{dX_t^{(1)}}{dt} &= b_t(X_t)dt, \\
\frac{dX_t^{(2)}}{dt} &= B_t(X_t, \mu_t)dt + \sigma_t(X_t)dW_t.
\end{aligned}
\]

Under (C2), for any \( \mathcal{F}_0 \)-measurable random variable \( X_0 \) with \( \mathcal{L}_{X_0} \in \mathcal{P}_k \), let \( X_t^{X_0, \mu} \) be the unique solution to (4.2) with initial value \( X_0 \). It is standard to derive from (C2) that

\[
\mathbb{E}( \sup_{t \in [0,T]} |X_t^{X_0, \mu}|^n |\mathcal{F}_0) \leq c(n)(1 + |X_0|^n), \quad n \geq 1.
\]

Define the mapping \( \Phi_t^{X_0} : C([0, T], \mathcal{P}_k) \to C([0, T], \mathcal{P}_k) \) as

\[
\Phi_t^{X_0}(\mu) = \mathcal{L}_{X_t^{X_0, \mu}}, \quad t \in [0, T].
\]

The following theorem provides the well-posedness for (4.1) and the proof is similar to that in [18, Theorem 3.2].

**Theorem 4.1.** Assume (C1)-(C2). Then (4.1) is well-posed in \( \mathcal{P}_k \). Moreover, there exists a constant \( C > 0 \) such that

\[
\| P_t^* \gamma \|_k^k \leq C(1 + \| \gamma \|_k^k), \quad t \in [0, T].
\]

**Proof.** Since (4.4) is standard by (C2) and the BDG inequality, it is sufficient to prove that (4.1) is well-posed in \( \mathcal{P}_k \). It follows from (C2) that

\[
\begin{aligned}
|X_t^{X_0, \nu} - X_t^{X_0, \mu}| &\leq C_0 \left( \int_0^t [W_k(\mu_s, \nu_s) + \| \mu_s - \nu_s \|_{k, var}]ds \right)^k + C_0 \int_0^t |X_s^{X_0, \nu} - X_s^{X_0, \mu}|^k ds \\
&\quad + C_0 \left\| \int_0^t [\sigma_s(X_s^{X_0, \nu}) - \sigma_s(X_s^{X_0, \mu})]dW_s \right\|^k
\end{aligned}
\]

for some constant \( C_0 > 0 \). By (C2) and the BDG inequality, there exist constants \( C_1, C_2 > 0 \) such that

\[
\begin{aligned}
C_0 &\mathbb{E} \sup_{t \in [0, r]} \left| \int_0^t [\sigma_s(X_s^{X_0, \nu}) - \sigma_s(X_s^{X_0, \mu})]dW_s \right|^k \\
&\leq C_1 \mathbb{E} \left( \int_0^r |X_s^{X_0, \mu} - X_s^{X_0, \nu}|^2 ds \right)^{\frac{k}{2}} \\
&\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0, r]} |X_t^{X_0, \mu} - X_t^{X_0, \nu}|^k + C_2 \mathbb{E} \int_0^r |X_s^{X_0, \mu} - X_s^{X_0, \nu}|^k ds.
\end{aligned}
\]

(4.6) together with (4.5) and Gronwall’s inequality yields

\[
\mathbb{W}_k(\Phi_t^{X_0}(\mu), \Phi_t^{X_0}(\nu)) \leq \left( \mathbb{E} \sup_{s \in [0, t]} |X_s^{X_0, \mu} - X_s^{X_0, \nu}|^k \right)^{\frac{1}{k}} \leq C_3 \int_0^t [\mathbb{W}_k(\mu_s, \nu_s) + \| \mu_s - \nu_s \|_{k, var}]ds
\]

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for some constant $C_3 > 0$. Therefore, for any $\lambda > 0$, we have

\begin{equation}
\sup_{t \in [0,T]} e^{-\lambda t} \mathbb{W}_k(\Phi^X_t(\mu), \Phi^X_t(\nu)) \leq \frac{C_3}{\lambda} \sup_{t \in [0,T]} e^{-\lambda t}[\mathbb{W}_k(\mu_t, \nu_t) + \|\mu_t - \nu_t\|_{k,\text{var}}].
\end{equation}

Next, let

\[ \zeta_s = \sigma_s^{-1}(X^{X_0,\mu}_s) \mathbb{E}(B_s(X^{X_0,\mu}_s, \nu_s) - B_s(X^{X_0,\mu}_s, \mu_s)), \quad s \in [0,T], \]

\[ R(t) = \exp \left\{ \int_0^t \langle \zeta_s, dW_s \rangle - \frac{1}{2} \int_0^t |\zeta_s|^2 ds \right\}, \quad t \in [0,T], \]

\[ W_t^{\mu,\nu} = W_t - \int_0^t \zeta_s ds, \quad t \in [0,T]. \]

Then we have

\[ \begin{cases}
    d(X^{X_0,\mu}_t) = b_t(X^{X_0,\mu}_t) dt, \\
    d(X^{X_0,\mu}_t) = B_t(X^{X_0,\mu}_t, \nu_t) dt + \sigma_t(X^{X_0,\mu}_t) dW_t^{\mu,\nu}.
\end{cases} \]

Noting that $|\zeta_s| \leq K\|\sigma^{-1}\|_\infty(\mathbb{W}_k(\mu_s, \nu_s) + \|\mu_s - \nu_s\|_{k,\text{var}})$ due to (C1)-(C2) and (4.3), Girsanov’s theorem yields

\[ \Phi^X_t(\nu)(f) = \mathbb{E}(R(t)f(X^{X_0,\mu}_t)), \quad f \in \mathcal{B}_b([\mathbb{R}^{m+d}], t \in [0,T]). \]

Therefore, by the Cauchy-Schwarz inequality for conditional expectation, we obtain

\begin{equation}
\begin{aligned}
|\Phi^X_t(\nu) - \Phi^X_t(\mu)|_{k,\text{var}} \\
= \mathbb{E}[R(t) - 1|\mathcal{F}_0] \\
\leq \mathbb{E}\left[ (\mathbb{E}(\mathbb{E}|R(t) - 1|^2|\mathcal{F}_0))^\frac{1}{2} (\mathbb{E}((1 + |X^{X_0,\mu}_t|^k)^2|\mathcal{F}_0))^{\frac{1}{2}} \right] \\
\leq \mathbb{E}\left[ (\mathbb{E}(\mathbb{E}|R(t) - 1|^2|\mathcal{F}_0))^{\frac{1}{2}} (\mathbb{E}((1 + |X^{X_0,\mu}_t|^k)^2|\mathcal{F}_0))^{\frac{1}{2}} \right] \\
\leq \exp \left\{ c \int_0^t \mathbb{W}_k(\mu_s, \nu_s) + \|\mu_s - \nu_s\|_{k,\text{var}}^2 ds \right\} - 1 \right\}^{\frac{1}{2}} \\
\leq \exp \left\{ \frac{c}{2} \int_0^t \mathbb{W}_k(\mu_s, \nu_s) + \|\mu_s - \nu_s\|_{k,\text{var}}^2 ds \right\} \\
\times \sqrt{c} \left( \int_0^t \mathbb{W}_k(\mu_s, \nu_s) + \|\mu_s - \nu_s\|_{k,\text{var}}^2 ds \right)^{\frac{1}{2}}
\end{aligned}
\end{equation}

for some constant $c > 0$. For any $N \geq 1$, let

\begin{equation}
\mathcal{P}^{N,T}_{k,X_0} = \{ \mu \in C([0,T], \mathcal{P}_k), \mu_0 := \mathcal{L}_{X_0}, \sup_{t \in [0,T]} e^{-\lambda t} (1 + \mu_t(|\cdot|^k)) \leq N \}.
\end{equation}

Then it is clear that as $N \uparrow \infty$,

\[ \mathcal{P}^{N,T}_{k,X_0} \uparrow \mathcal{P}^T_{k,X_0} = \{ \mu \in C([0,T], \mathcal{P}_k), \mu_0 = \mathcal{L}_{X_0} \}. \]
So, it remains to prove that there exists a constant $N_0 > 0$ such that for any $N \geq N_0$, $\Phi^{X_0}$ is a contractive map on $\mathcal{P}_{k, X_0}$.

Firstly, it follows from (C2) and the BDG inequality that there exists a constant $c_1 > 0$ such that for any $\mu \in \mathcal{P}_{k, X_0}$,

$$
e^{-Nt} \mathbb{E}(1 + |X_t^{X_0, \mu}|^k) \leq \mathbb{E}(1 + |Z_0|^k) + c_1 e^{-Nt} \int_0^t \mathbb{E}(1 + |Z_s|^{k\mu}) ds$$

$$+ c_1 e^{-Nt} \int_0^t (1 + \mu_s(|\cdot|^k)) ds$$

$$\leq \mathbb{E}(1 + |Z_0|^k) + \frac{c_1}{N} \sup_{s \in [0, t]} e^{-N s} \mathbb{E}(1 + |Z_s^{X_0, \mu}|^k) + c_1.$$

As a result, there exists a constant $N_0 > 1$ such that for any $N \geq N_0$, $\Phi^{X_0}$ maps $\mathcal{P}_{k, X_0}$ to $\mathcal{P}_{k, X_0}$. Next, we derive from (4.8) and (4.9) that

$$\|\Phi^X_t(\nu) - \Phi^X_t(\mu)\|_{k, var} \leq C_0(N) \left( \int_0^t (\mathbb{W}_k(\mu_s, \nu_s) + \|\mu_s - \nu_s\|_{k, var})^2 ds \right)^{\frac{1}{2}}, \quad \mu, \nu \in \mathcal{P}_{k, X_0}$$

for some constant $C_0(N) > 0$, which implies that

$$\sup_{t \in [0, T]} e^{-\lambda t} \|\Phi^X_t(\nu) - \Phi^X_t(\mu)\|_{k, var} \leq \frac{C(N)}{\sqrt{\lambda}} \mathbb{W}_{k, \lambda}(\mu, \nu),$$

(4.11)

here for any $\lambda > 0$,

$$\mathbb{W}_{k, \lambda}(\mu, \nu) := \sup_{t \in [0, T]} e^{-\lambda t} (\|\nu_t - \mu_t\|_{k, var} + \mathbb{W}_k(\mu_t, \nu_t)), \quad \mu, \nu \in \mathcal{P}_{k, X_0}.$$

Combining (4.11) with (4.7), we conclude that for any $N \geq N_0$, there exists a constant $\lambda(N) > 0$ such that $\Phi^{X_0}$ is a strictly contractive map on $(\mathcal{P}_{k, X_0}, \mathbb{W}_{k, \lambda(N)})$. Therefore, the proof is completed by the Banach fixed point theorem and (4.10).

\[\square\]

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