Convergence and quasi-optimality of adaptive finite element methods for harmonic forms

Alan Demlow

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Abstract Numerical computation of harmonic forms (typically called harmonic fields in three space dimensions) arises in various areas, including computer graphics and computational electromagnetics. The finite element exterior calculus framework also relies extensively on accurate computation of harmonic forms. In this work we study the convergence properties of adaptive finite element methods (AFEM) for computing harmonic forms. We show that a properly defined AFEM is contractive and achieves optimal convergence rate beginning from any initial conforming mesh. This result is contrasted with related AFEM convergence results for elliptic eigenvalue problems, where the initial mesh must be sufficiently fine in order for AFEM to achieve any provable convergence rate.

Mathematics Subject Classification 65N15 · 65N30

1 Introduction

In this paper we prove convergence and rate-optimality of adaptive finite element methods (AFEM) for computing harmonic forms. Let $\Omega \subset \mathbb{R}^3$ be a polyhedral domain with Lipschitz boundary $\partial \Omega$. The space $\mathcal{H}^1$ of harmonic forms is

$$\mathcal{H}^1 = \{ v \in L^2(\Omega)^3 \mid \operatorname{curl} v = 0, \operatorname{div}(v) = 0, v \cdot n = 0 \text{ on } \partial \Omega \}.$$  \hspace{1cm} (1.1)

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Alan Demlow
demlow@math.tamu.edu

1 Department of Mathematics, Texas A&M University, College Station, TX 77843–3368, USA

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Here \( \mathbf{n} \) is the outward unit normal on \( \partial \Omega \). \( \beta^1 := \dim \mathcal{H}^1 < \infty \) is equal to the number of handles of the domain \( \Omega \), so \( \beta^1 = 0 \) if \( \Omega \) is simply connected. We denote by \( \{ q^1, \ldots, q^{\beta^1} \} \) a fixed \( L_2 \)-orthonormal basis for \( \mathcal{H}^1 \).

Computation of harmonic forms arises in applications including computer graphics and surface processing [18, 35] and numerical solution of PDE related to problems having Hodge–Laplace structure posed on domains with nontrivial topology. Important practical examples of the latter type arise in boundary and finite element methods for electromagnetic problems. There computation of harmonic fields may arise as a necessary precursor step which in essence factors nontrivial topology out of subsequent computations. We refer to [1, 5, 22, 23, 28, 34] for general discussion, topologically motivated algorithms for efficient computation of \( \mathcal{H}^1 \), and specific applications requiring such a step. In addition, harmonic fields on spherical subdomains play an important role in understanding singularity structure of solutions to Maxwell’s equations on polyhedral domains [11, Section 6.4]. Computation of harmonic forms is also an important part of the finite element exterior calculus (FEEC) framework. The FEEC framework provides a systematic exposition of the tools needed to stably solve Hodge–Laplace problems related to the de Rham complex and other differential complexes [2, 3]. It thus has provided a broader exposition of many of the tools needed for the numerical analysis of Maxwell’s equations. The mixed methods used in these works solve the Hodge–Laplace problem modulo harmonic forms, so as above their accurate computation is a prerequisite for accurate computation of Hodge–Laplace solutions. This is true of adaptive computation of solutions to Hodge–Laplace problems also [15, 26]. Our results thus serve as a necessary precursor to study of convergence of AFEM for solving the full Hodge–Laplace problem as posed within the FEEC framework.

While keeping in mind the concrete representation (1.1), outside of the introduction we will use the more general notation and tools that have been developed within the FEEC framework. The results described in the introduction below are valid essentially verbatim in arbitrary space dimension \( n \) and for arbitrary cohomology group \( \mathcal{H}^k \), \( 1 \leq k \leq n - 1 \).

We briefly describe our setting. Let \( T_\ell, \ell \geq 0, \) be a set of nested, adaptively generated simplicial decompositions of \( \Omega \). Let also \( V^0_\ell \subset H^1(\Omega) \) and \( V^1_\ell \subset H(\text{curl} ; \Omega) \) be conforming finite element spaces which together satisfy a complex property described in more detail below. For (1.1) we may take \( V^0_\ell \) to be \( H^1 \)-conforming piecewise linear functions and \( V^1_\ell \) to be lowest-order Nédélec edge elements. Higher-degree analogues also may be used. The space \( \mathcal{H}^1_\ell \) of discrete harmonic fields corresponding to \( \mathcal{H}^1 \) is then those fields \( q_\ell \in V^1_\ell \) satisfying

\[
\langle q_\ell, \nabla \tau_\ell \rangle = 0, \quad \tau_\ell \in V^0_\ell, \\
\langle \text{curl} q_\ell, \text{curl} v_\ell \rangle = 0, \quad v_\ell \in V^1_\ell.
\]

(1.2)

Note that while \( \text{curl} q_\ell = 0, q_\ell \) is not generally in \( H(\text{div}; \Omega) \). The first equation in (1.2) instead implies that \( \text{div}_h q_\ell = 0 \), where \( \text{div}_h \) is a weakly defined discrete divergence operator which is not generally the same as the restriction of \( \text{div} \) to \( V^1_\ell \). We denote by \( \{ q^j_\ell \}_{j=1}^{\beta^1} \) a (computed) orthonormal basis for \( \mathcal{H}^1_\ell \). Let also \( P_{\mathcal{H}^1_\ell} \) be the \( L_2 \) projection...
onto $\mathcal{S}_1^l$. Given $q \in \mathcal{S}_1^l$, a posteriori error estimates for controlling $\|q - P_{\mathcal{S}_1^l}q\|_{L^2(\Omega)}$ were given in [15]. From these we may generate an adaptive finite element method having the standard form solve $\rightarrow$ estimate $\rightarrow$ mark $\rightarrow$ refine for controlling the defect between $\mathcal{S}_1^l$ and $\mathcal{S}_1^l$.

There are important technical and conceptual parallels between AFEM for controlling harmonic forms and AFEM for elliptic eigenvalue problems, as both consider approximation of finite-dimensional (invariant) subspaces. Consider the eigenvalue problem $-\Delta u = \lambda u$ with Dirichlet boundary conditions. Assume that $\lambda$ is the $j$-th eigenvalue of the continuous problem. AFEM employing residual-type error indicators based on the $j$-th discrete eigenvalue $\lambda_\ell$ on the $\ell$-th mesh level yield an approximating sequence $(\lambda_\ell, u_\ell)$ for the $j$-th continuous eigenpair $(\lambda, u)$. Analyses of AFEM for elliptic eigenvalue problems have focused on two convergence regimes. The first is a preasymptotic regime in which the method converges, but with no provable rate. In [20] it was proved that starting from any initial mesh $T_0$, $(\lambda_\ell, u_\ell) \rightarrow (\lambda, u)$ in $\mathbb{R} \times H_0^1(\Omega)$, but with no guaranteed convergence rate. In the second convergence regime the eigenvalue problem is effectively linearized, and the convergence behavior is that of a source problem. More precisely, if the initial mesh $T_0$ is sufficiently fine, then the AFEM is contractive and achieves an optimal convergence rate [6,12,13,19,21]. The plain convergence results of [20] guarantee that such a state is initially reached from any initial mesh $T_0$.

Our results below indicate that the convergence behavior of harmonic forms essentially begins in a transition region in which the AFEM contracts, but with contraction constant that may improve as the overlap between $\mathcal{S}_1^l$ and $\mathcal{S}_1^l$ increases. A regime in which AFEM contracts with constants independent of essential quantities is then eventually reached, as for AFEM for eigenvalue problems. It is to be expected that AFEM for harmonic forms contract from any initial mesh as computation of harmonic forms reduces to solving $Au = 0$ with $A$ a linear operator. That is, harmonic forms are eigenfunctions with known eigenvalue and their computation is essentially a linear problem. On the other hand, we have set as our main goal the production of an orthonormal basis for $\mathcal{S}_1^l$ and do not assume any particular alignment or method of production for the basis. The problem of producing an orthonormal basis is mildly nonlinear, so it is reasonable that the contraction constant can improve as the mesh is refined. As we discuss in Sect. 7.4, alternate methods for producing and aligning the discrete and continuous bases may lead to AFEM with different properties. Our framework has the advantage of being completely generic with respect to the method used to compute $\mathcal{S}_1^l$.

There are two main challenges in proving contraction of AFEM for computing eigenspaces. These are lack of orthogonality caused by non-nestedness of the discrete eigenspaces, and lack of alignment of computed eigenbases at adjacent discrete levels. In the context of elliptic eigenvalue problems, suitable nestedness and orthogonality is recovered only on sufficiently fine meshes as the nonlinearity of the problem is resolved. In the case of harmonic forms a novel situation arises. The discrete spaces $\{\mathcal{S}_1^l\}$ of harmonic forms are not themselves nested. However, as we show below the Hodge decomposition nonetheless guarantees sufficient nestedness and orthogonality.
uniformly starting from any initial mesh. In essence, topological resolution of $\Omega$ by $T_0$ is sufficient to ensure some analytical resolution of $H^1$ by $H^1_0$.

Lack of alignment between discrete bases at adjacent mesh levels occurs in the case of multi-dimensional target subspaces, including multiple or clustered eigenvalues and harmonic forms when $\beta > 1$. Standard AFEM convergence proofs employ “indicator continuity” arguments which in our case would require comparing discrete basis members $q_j^\ell$ and $q_{j+1}^\ell$ on adjacent mesh levels. When $\beta^1 > 1$, $q_j^\ell$ and $q_{j+1}^\ell$ may not be meaningfully related. To overcome the difficulty of multiple eigenvalues, we follow [6,12,19] in using a non-computable error estimator $\mu^\ell$ calculated with respect to projections $P_q^j$ of a fixed basis for the continuous harmonic forms. Indicator continuity arguments apply to these theoretical error indicators, which must in turn be compared to the practical ones based on $\{q_j^\ell\}$. We follow [6] in establishing an equivalence between the theoretical and practical indicators with constants asymptotically independent of essential quantities.

We now briefly describe our results. We first show that there exists $\gamma > 0$ and $0 < \rho < 1$ such that

$$\beta^1 \sum_{j=1} \|q_j - P_{\ell+1}q_j\|^2 + \gamma \mu_{\ell+1}^2 \leq \rho \left( \sum_{j=1} \|q_j - P_{\ell}q_j\|^2 + \gamma \mu_\ell^2 \right).$$

(1.3)

While we may take $\gamma, \rho$ above independent of mesh level, our proof indicates that $\rho$ may in fact improve (decrease) as the overlap between $H^1$ and $H^1_\ell$ improves. Because a contraction occurs from the initial mesh with fixed constant $\rho < 1$, this improvement in overlap is guaranteed to occur. Thus our result lies between standard results for elliptic source problems in which contraction occurs from the initial mesh with fixed contraction constant, and AFEM convergence results for elliptic eigenvalue problems for which sufficient overlap between discrete and continuous eigenspaces is not guaranteed unless the initial mesh is sufficiently fine.

We additionally prove rate-optimality, or more precisely that

$$\left( \sum_{j=1} \|q_j - P_{\ell}q_j\|^2 \right)^{1/2} \leq C (\#T_\ell - \#T_0)^{-s}$$

(1.4)

whenever systematic bisection is able to produce a sequence of meshes that similarly approximates $H^1$ with rate $s$. As is typical for AFEM results, our proof of (1.4) requires that the Dörfler (bulk) marking parameter that specifies the fraction of elements to be refined in each step of the algorithm be sufficiently small. We show that the threshold value for the Dörfler parameter is independent of all essential quantities, including the initial resolution of $H^1$ by $H^1_\ell$. This situation is typical of elliptic source problems; cf. [7]. In contrast to elliptic source problems, however, the constant $C$ may depend on the quality of approximation of $H^1$ by $H^1_\ell$. Finally, proof of rate optimality requires establishing a localized a posteriori upper bound for the defect between the target spaces on nested meshes. This step is substantially more involved for general problems...
of Hodge–Laplace type than for standard scalar problems. Proof of localized upper bounds has been previously given for Maxwell’s equations [36], but the necessary tools have not previously appeared in a form suitable for our purposes. Our approach below is valid for arbitrary form degree and space dimension and generically for problems of Hodge–Laplace type. In addition to being more general, our proof is also slightly simpler due to its use of the recently-defined quasi-interpolant of Falk and Winther [17].

The remainder of the paper is organized as follows. In Sect. 2 we describe continuous and discrete spaces of differential forms, interpolants into discrete spaces of differential forms, and existing a posteriori estimates for harmonic forms. In Sect. 3 we give a number of preliminary results concerning the Hodge decomposition and $L^2$ projections described above. Sections 5 and 6 contain statements, discussion, and proofs of our contraction and optimality results. Section 7 contains brief discussion of essential boundary conditions and harmonic forms in the presence of coefficients, and also of the effects of methods for computing harmonic fields on the resulting AFEM. Finally, in Sect. 8 we present numerical results illustrating our theory.

2 The de Rham complex and its finite element approximation

In this section we generalize the classical function space setting from the introduction to arbitrary space dimension and form degree and introduce corresponding finite element spaces and tools. We for the most part follow [3] in our notation and refer to that work for more detail.

2.1 The de Rham complex

Let $\Omega$ be a bounded Lipschitz polyhedral domain in $\mathbb{R}^n$, $n \geq 2$. Let $\Lambda^k(\Omega)$ represent the space of smooth $k$-forms on $\Omega$. The natural $L_2$ inner product is denoted by $\langle \cdot, \cdot \rangle$, the $L_2$ norm by $\| \cdot \|$, and the corresponding space by $L_2\Lambda^k(\Omega)$. We let $d$ be the exterior derivative, and $H^{\Lambda^k}(\Omega)$ be the domain of $d^k$ consisting of $L_2$ forms $\omega$ for which $d\omega \in L_2\Lambda^{k+1}(\Omega)$. We denote by $\|\cdot\|_H$ the associated graph norm; here one may concretely think of $H$ as $H(\text{curl})$, $H(\text{div})$, or $H^1$.

We denote by $W^{r,\Lambda^k}(\Omega)$ the corresponding Sobolev spaces of forms and set $H^r\Lambda^k(\Omega) = W^{r,\Lambda^k}(\Omega)$. Finally, for $\omega \subset \mathbb{R}^n$, we let $\|\cdot\|_{\omega} = \|\cdot\|_{L_2\Lambda^k(\omega)}$ and $\|\cdot\|_{H^r\omega} = \|\cdot\|_{H^r\Lambda^k(\omega)}$; in both cases we omit $\omega$ when $\omega = \Omega$.

Denote by $\delta$ the codifferential, that is, the adjoint of the exterior derivative $d$ with respect to $\langle \cdot, \cdot \rangle$. The space of harmonic $k$-forms is given by

$$\mathcal{H}^k = \{ q \in H^\Lambda^k(\Omega) : dq = 0, \delta q = 0, \text{tr} \star q = 0 \}. \quad (2.1)$$

Here $\text{tr}$ is the trace operator and $\star$ the Hodge star operator. We denote by

$$\beta^k = \dim \mathcal{H}^k \quad (2.2)$$
the $k$th Betti number of $\Omega$. When $n = 3$, $\beta^0 = 1$, $\beta^1$ is the number of holes in $\Omega$, $\beta^2$ the number of voids, and $\beta^3 = 0$. In general, $\beta < \infty$. We additionally let $\mathfrak{B}^k = dH^\Lambda^{k-1}$ be the range of $d^{k-1}$, $\mathfrak{Z}^k = \text{kernel} d^k$ the kernel of $d^k$, and by $\mathfrak{Z}^k \perp$ the orthogonal complement of $\mathfrak{Z}^k$ in $H^\Lambda^k$. We have $\mathfrak{Z}^k = \mathfrak{S}^k \oplus \mathfrak{B}^k$, and the Hodge decomposition is given by $H^\Lambda^k = \mathfrak{B}^k \oplus \mathfrak{S}^k \oplus \mathfrak{Z}^k \perp$. Note that $\mathfrak{B}^k \subset \mathfrak{Z}^k$, that is, $d \circ d = 0$.

2.2 Meshes and mesh properties

We employ, and now briefly review, the conforming simplicial mesh refinement framework commonly employed in AFEM convergence theory; cf. [33] for details. Let $T_0$ be a conforming, shape regular simplicial decomposition of $\Omega$. By fixing a local numbering of all vertices of all $T \in T_0$, all possible descendants $T$ of $T_0$ that can be created by newest vertex bisection are uniquely determined. By newest vertex bisection we mean either the refinement procedure as it was developed in two space dimensions or its generalization to any space dimension. The simplices in any of those partitions are uniformly shape regular, dependent only on the shape regularity parameters of $T_0$ and the dimension $n$. Generally a descendant of $T_0$ is non-conforming. Our AFEM will generate a nested sequence $T_0 \subset T_1 \subset T_2 \ldots$ of conforming meshes which we index by $\ell$. Given marked sets $M_\ell \subset T_\ell$, conforming bisection thus refines all $T \in M_\ell$ and then additional elements in order to ensure that $T_{\ell+1}$ is conforming. With a suitable numbering of the vertices in the initial partition, the total number of refinements needed to make the sequence $\{T_\ell\}$ conforming can be bounded by the number of marked elements. More precisely, for any $\ell \geq 0$, there is a constant $C_{\text{ref}, \ell}$ such that for $\ell > \ell,$

$$\#T_\ell - \#T_\ell \leq C_{\text{ref}, \ell} \sum_{i=\ell}^{\ell-1} \#M_i$$

(2.3)

with constant depending on $T_\ell$. Such an initial numbering always exists, possibly after an initial uniform refinement of $T_0$. Assuming such a numbering, we denote the set of all conforming descendants $T$ of $T_0$ by $T$. For $T, \tilde{T} \in T$, we write $T \subset \tilde{T}$ when $\tilde{T}$ is a refinement of $T$. Finally, for $T \in T \in T$ we let $h_T = |T|^{1/n}$.

2.3 Approximation of the de Rham complex and interpolants

Given $T \in T$, let $\ldots V_T^{k-1} \rightarrow V_T^k \rightarrow V_T^{k+1} \rightarrow \ldots$ be an approximating subcomplex of the de Rham complex with underlying mesh $T$. That is, $V_T^k \subset H^\Lambda^k(\Omega), v \in V_T^k$ is polynomial on each $T \in T$, and $dV_T^{k-1} \subset V_T^k$. When $T = T_\ell$ we use the abbreviation $V_{T_\ell} = V_\ell$ (where here we also depress the dependence on $k$). We refer to [3] for descriptions of the relevant spaces. In three space dimensions, we may think for example of standard Lagrange spaces approximating $H^\Lambda^0 = H^1$, Nédeléc spaces approximating $H(\text{curl})$, and Raviart–Thomas or BDM spaces approximating $H(\text{div})$. Springer
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In addition to the subcomplex property, we also require the existence of a commuting interpolant (or more abstractly, commuting cochain projection) with certain properties. Desirable properties include commutativity, boundedness on the function spaces $L_2 \Lambda$ and/or $H \Lambda$ that are natural for our setting, local definition of the operator, and projectivity, and a local regular decomposition property. Several recent papers have achieved constructions possessing some of these properties \cite{9,17,29,30}, though no one operator has been shown to possess all of them. We use two different such constructions below along with a standard Scott–Zhang interpolant. First, the commuting projection operator $\pi_{CW} : L_2 \Lambda(\Omega) \to V_T$ of Christiansen and Winther \cite{9} has the following useful properties:

$$d^k \pi_{CW}^k = \pi_{CW}^{k+1} d^k, \quad \pi_{CW}^2 = \pi_{CW}.$$  

(2.4)

The Christiansen–Winther interpolant also can be modified to preserve homogeneous essential boundary conditions with no change in its other properties.

Second, given $T \in \mathcal{T}$, there is a patch of elements surrounding $T$ such that

$$\#(T \subset \omega_T) \lesssim 1.$$  

(2.6)

and for $v \in H \Lambda(\Omega)$,

$$\|\pi_{FW} v\|_T \lesssim \|v\|_{\omega_T} + h_T \|d\pi_{FW} v\|_{\omega_T}, \quad \|\pi_{FW} v\|_{H \Lambda(T)} \lesssim \|v\|_{H \Lambda(\omega_T)}.$$  

(2.7)

The patch $\omega_T$ is not necessarily the standard patch of elements sharing a vertex with $T$, and its configuration may depend on form degree. The relationship (2.6) is however the essential one for our purposes. Given $v \in H^1 \Lambda(\omega_T)$, there also follows the interpolation estimate

$$h_T^{-1} \|v - \pi_{FW} v\|_T + |v - \pi_{FW} v|_{H^1 \Lambda(\omega_T)} \lesssim |v|_{H^1 \Lambda(\omega_T)}.$$  

(2.8)

Finally, $\pi_{FW}$ is a local projection in the sense that

$$u|_{\omega_T} \in V_T|_{\omega_T} \Rightarrow (u - \pi_{FW} u)|_T \equiv 0.$$  

(2.9)

In summary, $\pi_{FW}$ and $\pi_{CW}$ are both commuting projection operators. $\pi_{FW}$ is however locally defined, but only stable in $H \Lambda$, while $\pi_{CW}$ is globally defined but stable in $L_2 \Lambda$. Also, to our knowledge there is no detailed discussion in the literature of the modifications needed to construct a version of $\pi_{FW}$ which preserves essential...
boundary conditions, although a comment in [17] indicates that such an adaptation is natural.

We shall also employ an $L_2$-stable Scott–Zhang operator $I_{SZ} : L_2 \Lambda^k(\Omega) \to \tilde{V}_T^k \subset H^1 \Lambda^k(\Omega)$ [31]. Here $\tilde{V}_T^k$ is the smallest space of forms containing all forms with continuous piecewise linear coefficients. $I_{SZ}$ may be obtained by employing the standard scalar Scott–Zhang interpolant coefficientwise. We then have for $u \in L_2 \Lambda(\Omega)$

$$\|I_{SZ}u\|_{L_2 \Lambda(T)} \lesssim \|u\|_{L_2 \Lambda(\Omega)},$$

$$h_T^{-1}\|u - I_{SZ}u\|_{L_2 \Lambda(T)} + |u - I_{SZ}u|_{H^1 \Lambda(\Omega)} \lesssim |u|_{H^1 \Lambda(\Omega)}.$$(2.10)

We shall finally use a regular decomposition property, which is given in [15, Lemma 5] (this lemma is largely a reformulation of more general results given in [27]). Given a form $v \in H \Lambda^k(\Omega)$, there are $\phi \in H_1 \Lambda_{k-1}(\Omega)$ and $z \in H \Lambda^k(\Omega)$ such that

$$v = d\phi + z, \quad \|\phi\|_{H_1 \Lambda_{k-1}(\Omega)} + \|z\|_{H \Lambda^k(\Omega)} \lesssim \|v\|_{H \Lambda^k(\Omega)}.$$ (2.11)

2.4 Discrete harmonic forms

Let $\mathcal{H}^k_T \subset V_T^k$ be the set of discrete harmonic forms, which is more precisely defined as those $q_T \in V_T^k$ satisfying

$$\langle q_T, d\tau_T \rangle = 0, \forall \tau_T \in V_T^{k-1},$$

$$\langle dq_T, dv_T \rangle = 0, \forall v_T \in V_T^k.$$ (2.12)

The discrete Hodge decomposition $V_T^k = \mathfrak{B}_T^k \oplus \mathcal{H}_T^k \oplus \mathfrak{Z}_T^{k,\perp}$ ($\mathfrak{Z}_T^k = \mathfrak{B}_T^k \oplus \mathcal{H}_T^k = \ker d|_{V_T^k}$) is defined entirely analogously to the continuous version. Note however that while

$$\mathfrak{Z}_T^k \subset \mathcal{Z}^k \text{ and } \mathfrak{B}_T^k \subset \mathfrak{B}_T^k,$$ (2.13)

there holds

$$\mathcal{H}_T^k \not\subset \mathfrak{Z}_T^k \text{ and } \mathfrak{Z}_T^{k,\perp} \not\subset \mathcal{Z}_T^{k,\perp}.$$ (2.14)

We briefly summarize the index system we use. The integer $n$ is the dimension of the domain $\Omega$. The integer $k$, $0 \leq k \leq n$, is the order of differential form. The subscript $\ell$, $\ell = 0, 1, 2 \ldots$ is used for a sequence of triangulations of the domain $\Omega$ and the corresponding finite element spaces. For a fixed $k$, $0 \leq k \leq n$, we consider the convergence of the sequence $\{q_\ell, \ell = 0, 1, 2, \ldots\}$. Therefore we may suppress the index $k$ when no confusion can arise. $j$ is used to index the bases of $\mathfrak{R}$ and $\mathcal{R}_T$. 

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3 Properties of an $L_2$ projection

We first specify the error quantity that we seek to control. Let $\{q^j\}_{j=1}^\beta$ be an orthonormal basis for $\mathcal{S}_\beta$ and let $\{q^j_T\}_{j=1}^\beta$ be an orthonormal basis for $\mathcal{S}_\beta^T$. Denote by $P_T$ the $L_2$ projection onto $\mathcal{S}_\beta$ and by $P_T^T = P_{\mathcal{S}_\beta^T}$ the $L_2$ projection onto $\mathcal{S}_\beta^T$. As above we use the abbreviation $P_T^\ell = P_{\mathcal{T}_\ell}$ and similarly for $q^\ell_j$. We seek to control the subspace defect

$$E_T := \left( \sum_{j=1}^\beta \|q^j - P_T q^j\|^2 \right)^{1/2}.$$  

(3.1)

Proposition 1

$$\left( \sum_{j=1}^\beta \|q^j - P_T q^j\|^2 \right)^{1/2} = \left( \sum_{j=1}^\beta \|q^j_T - P_T q^j_T\|^2 \right)^{1/2}.$$  

(3.2)

Proof There holds $P_T q^j = \sum_{m=1}^\beta \langle q^j, q^m_T \rangle q^m_T$ and $\|P_T q^j\|^2 = \sum_{m=1}^\beta \langle q^j, q^m_T \rangle^2$, and similarly $\|P_T q^m_T\|^2 = \sum_{j=1}^\beta \langle q^j_T, q^j \rangle^2$. Also, $\|q^j - P_T q^j\|^2 = \langle q^j - P_T q^j, q^j - P_T q^j \rangle = 1 - \|P_T q^j\|^2$ and similarly for $\|q^m_T - P_T q^m_T\|^2$. Thus

$$\sum_{j=1}^\beta \|q^j - P_T q^j\|^2 = \sum_{j=1}^\beta \left( 1 - \sum_{m=1}^\beta \langle q^j, q^m_T \rangle^2 \right) = \sum_{m=1}^\beta \|q^m_T - P_T q^m_T\|^2.$$  

(3.3)

Previous error estimates for errors in harmonic forms have instead controlled the gap between $\mathcal{S}_\beta$ and $\mathcal{S}_\ell$. Given two finite-dimensional subspaces $A$, $B$ of the same ambient Hilbert space with $\text{dim } A = \text{dim } B$, we define

$$\delta(A, B) = \sup_{x \in A, \|x\|=1} \|x - P_B x\|, \quad \text{gap}(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$  

(3.4)

There also holds in this case

$$\delta(A, B) = \delta(B, A).$$  

(3.5)

Let $P_{\mathcal{Z}_T}$ be the $L_2$ projection onto $\mathcal{Z}_T$.

Lemma 1

$$P_T q = P_{\mathcal{Z}_T} q, \quad \text{for all } q \in \mathcal{S}_\beta.$$  

(3.6)
Our first goal is to control the error quantity $E$.

**Proposition 2** Given $T \subset T'$ and $q \in \mathcal{H}$, we have $\|P_T q\| \leq \|P_{T'} q\|$. In particular, $\|P_\ell q\| \leq \|P_{\ell+1} q\|$.

**Proof** For $q \in \mathcal{H}$, $P_T q = P_{\mathcal{T}} q$. Noting that $\mathcal{T} \subset \mathcal{T}'$ completes the proof. \qed

We next show that there is overlap between $\mathcal{H}$ and $\mathcal{H}_\ell$ on any conforming mesh $T_\ell$; cf. [26, Theorem 2.7].

**Proposition 3** The projection $P_\ell$ is injective. Namely, given $q \in \mathcal{H}$ and $q \neq 0$, $P_\ell q \neq 0$.

**Proof** Let $\pi$ be a commuting projection operator (either $\pi_{FW}$ or $\pi_{CW}$ will suffice). Suppose there exists a $0 \neq q \in \mathcal{H}$ with $P_\ell q = 0$. By (3.5), $\delta(\mathcal{H}, \mathcal{H}_\ell) = 1 = \delta(\mathcal{H}_\ell, \mathcal{H})$, so there is a non-zero form $q_\ell \in \mathcal{H}_\ell$ with $P_{\mathcal{H}} q_\ell = 0$. Note that $\mathcal{H}_\ell \subset \mathcal{H}_\ell \subset \mathcal{H} = \mathcal{B} \oplus \mathcal{H}$, so $P_{\mathcal{H}} q_\ell = 0$ implies $q_\ell \in \mathcal{B}$. That is, $Q_\ell = d v$ for some $v \in \mathcal{V}^{k-1}$. But $q_\ell = \pi^k q_\ell = \pi^k d v = d \pi^{k-1} v \in \mathcal{B}$. But $\mathcal{B} \perp \mathcal{H}_\ell$ implies that $q_\ell = 0$, which is a contradiction. \qed

Combining the above two propositions yields the following lemma.

**Lemma 2** $P_\ell : \mathcal{H} \rightarrow \mathcal{H}_\ell$ is an isomorphism. In addition, $\|P_\ell^{-1}\| \leq \|P_0^{-1}\| < \infty$.

**Proof** Since $\beta = \beta_\ell < \infty$ and $P_\ell$ is injective, we conclude $P_\ell$ is an isomorphism. Consider the constant $c_0 = \inf_{q \in \mathcal{H}, \|q\|=1} \|P_0 q\|$, which is positive because it is the infimum over a compact set of a continuous and positive function. Then $\|P_0^{-1}\| \leq c_0^{-1}$.

By Proposition 2, $\|P_0 q\| \leq \|P_1 q\| \leq \|P_2 q\| \leq \ldots$, and so we have $\|P_\ell^{-1}\| \leq \|P_{\ell-1}^{-1}\| \leq \cdots \leq \|P_0^{-1}\| \leq c_0^{-1}$. \qed

4 A posteriori error estimates

4.1 Previous estimates

Our first goal is to control the error quantity $\mathcal{E}_\ell$ defined in (3.1), which is a measure of the distance between $\mathcal{H}$ and $\mathcal{H}_\ell$.

We follow [15], where a posteriori error estimates were given for measuring the gap between $\mathcal{H}$ and $\mathcal{H}_\ell$. Let $T_\ell$ be the mesh on level $\ell$, and let $h_T = |T|^{1/n}$ for $T \in T_\ell$. Given $q_\ell \in \mathcal{H}_\ell$, let

\[
\eta_\ell(T; q_\ell) = h_T \|\delta T q_\ell\|_T + h_T^{1/2} \|\text{tr} \star q_\ell\|_{\partial T}
\]

(4.1)

and

\[
\eta_\ell(T)^2 = \sum_{j=1}^\beta \eta_\ell(T; q_{\ell}^j)^2, \quad \eta_\ell(T) = \left(\sum_{T \in T} \eta_\ell(T)^2\right)^{1/2}, \quad T \subset T_\ell.
\]

(4.2)
Here $[\cdot]$ denotes the jump in the given quantity across element interfaces. In the concrete case of the classical space given in (1.1), we have $\eta_\ell(T; q_\ell) = h_T \| \text{div} q_\ell \|_T + h_T^{1/2} \| [q_\ell \cdot n] \|_{\partial T}$. A slight modification of Lemma 9 and Lemma 13 of [15] yields

$$
\eta_\ell(T_0) \leq C_1 E_\ell \lesssim \eta_\ell(T_\ell).
$$

(4.3)

with $C_1$ and the constant hidden in $\lesssim$ depending on the shape regularity properties of $T_0$, but independent of essential quantities including especially the dimension $\beta$ of $\mathcal{H}$ and $\mathcal{H}_\ell$. From [15] we also have

$$
gap(\mathcal{H}, \mathcal{H}_\ell) \lesssim \eta_\ell(T_\ell) \lesssim \sqrt{\beta} \cdot \text{gap}(\mathcal{H}, \mathcal{H}_\ell).
$$

(4.4)

Employing the error notion $E_\ell$ thus allows us to obtain estimates with entirely nonessential constants.

### 4.2 Non-computable error estimators

Convergence of adaptive FEM for multiple and clustered eigenvalues has been studied in [6, 12, 19]. Our problem is similar in that our AFEM approximates a subspace rather than a single function. The estimators defined above with respect to $\{q_j^\ell\}$ are problematic when viewed from the standpoint of standard proofs of AFEM contraction, which require continuity between estimators at adjacent mesh levels. Because the bases $\{q_j^\ell\}$ and $\{q_j^{\ell+1}\}$ are not generally aligned, such continuity results are not meaningful.

Following [12], we employ a non-computable intermediate estimator which solves this alignment problem and is equivalent to $\eta_\ell(T)$. Let $\{q_j^\ell\}_{j=1}^\beta$ be a fixed orthonormal basis for $\mathcal{H}$. We define

$$
\mu_\ell(T)^2 = \sum_{j=1}^\beta \eta_\ell(T; P_\ell q_j^\ell)^2, \quad \mu_\ell(T) = \left( \sum_{T \in \mathcal{T}} \mu_\ell(T)^2 \right)^{1/2}, \quad T \subset T_\ell.
$$

(4.5)

We next establish that approximation of $\mathcal{H}$ is sufficiently good on the initial mesh to guarantee that the estimators $\eta_\ell(T)$ and $\mu_\ell(T)$ are equivalent.

**Theorem 1**

$$
\mu_\ell(T) \leq \| P_\ell \| \eta_\ell(T) \leq \| P_\ell \| \| P_\ell^{-1} \| \mu_\ell(T) \leq \| P_\ell^{-1} \| \| \mu_\ell(T) \|, \quad T \in \mathcal{T}_\ell, \quad 0 \leq \ell \leq \ell.
$$

**Proof** Recall that $\mu_\ell$ is defined using $\{P_\ell q_j^i\}$ with $\{q_j^i\}_{j=1}^\beta$ a fixed orthonormal basis for $\mathcal{H}$. Let $q = (q^1, q^2, \ldots, q^\beta)^T$ and $q_\ell = (q_\ell^1, q_\ell^2, \ldots, q_\ell^\beta)^T$. The matrix $\{\langle q^i, q_j^\ell \rangle\} =: M : \mathbb{R}^\beta \to \mathbb{R}^\beta$ satisfies $P_\ell q = M q_\ell$.

Following the proof of [6, Lemma 3.1], let $B := MM^T = \{P_\ell q_i^j, P_\ell q_j^i\}$. $MM^T$ and $M^T M$ are isospectral and thus have the same 2-norm and $\|MM^T\|^2 = \|M\|^2 = \ldots$
\[ \| M^T M \| = \| B \|. \] We thus compute \( \| M^T \|. \) Given \( v \in \mathbb{R}^\beta \) with \( |v| = 1 \), let \( \tilde{v} = \sum_{j=1}^\beta v_j q_j \) so that \( \tilde{v} \in \mathcal{S} \) with \( \| \tilde{v} \| = 1 \). Then \( |M^T v| = |\sum_{j=1}^\beta (q_{\ell}^j, v_j q_j)| = \|(q_{\ell}^j, \tilde{v})\| = \| P_{\ell} \tilde{v} \| \leq \| P_{\ell} \|. \) Thus \( \| M \| = \| M^T \| \leq \| P_{\ell} \| \leq 1 \). Similarly, \( \| M^{-1} \| = \|(M^T)^{-1}\| = \sup_{y \in \mathbb{R}^\beta \setminus \{0\}} \frac{|M^{-T} y|}{|y|} = \sup_{v \in \mathbb{R}^\beta \setminus \{0\}} \frac{|M^{-T} M T v|}{\| M T v \|} = \sup_{v \in \mathbb{R}^\beta \setminus \{0\}} \frac{1}{\| M^T v \|} = \left( \inf_{v \in \mathbb{R}^\beta \setminus \{0\}} |M^T v| \right)^{-1} \left( \inf_{\tilde{v} \in \mathcal{S}, \| \tilde{v} \| = 1} \| P_{\ell} \tilde{v} \| \right)^{-1} = \| P_{\ell}^{-1} \|.

Thus \( \| M^{-1} \| = \| P_{\ell}^{-1} \|. \)

As \( \delta \) is linear and commutes with \( M \), we have
\[
\sum_{j=1}^\beta \| \delta T P_{\ell} q_j \|_T^2 = \sum_{j=1}^\beta \| (M \delta_T q_{\ell})_j \|_T^2 \leq \| M \|^2 \sum_{j=1}^\beta \| \delta_T q_{\ell}^j \|_T^2 \leq \| P_{\ell} \|^2 \sum_{j=1}^\beta \| \delta_T q_{\ell}^j \|_T^2.
\]

Similarly
\[
\sum_{i=1}^\beta \| \delta_T q_{\ell}^i \|_T^2 \leq \| M^{-1} \|^2 \sum_{i=1}^\beta \| \delta_T P_{\ell} q_{\ell}^i \|_T^2 \leq \| P_{\ell}^{-1} \|^2 \sum_{i=1}^\beta \| \delta_T P_{\ell} q_{\ell}^i \|_T^2.
\]

Similar inequalities hold for the boundary term. \( \square \)

Theorem 1 and (4.3) immediately imply an a posteriori bound using \( \mu_{\ell} \).

**Corollary 1**

\[ \mathcal{E}_{\ell}^2 \leq C_1 \| P_{\ell}^{-1} \|^2 \mu_{\ell}(T_{\ell})^2 \leq C_1 \| P_{\ell}^{-1} \|^2 \mu_{\ell}(T_{\ell})^2, \quad 0 \leq \ell \leq \ell, \quad (4.6) \]

where \( C_1 \) is independent of essential quantities.

### 4.3 Localized a posteriori estimates

As is common in AFEM optimality proofs, we require a localized upper bound for the difference between discrete solutions on nested meshes. More precisely, let \( \mathcal{R}_{T_{\ell} \to \bar{T}} \) be the set of elements refined in passing from \( T_{\ell} \) to \( \bar{T} \). A standard estimate for elliptic problems with finite element solutions \( u_{\ell} \) and \( \bar{u} \) on \( T_{\ell}, \bar{T} \) is \( \| u_{\ell} - \bar{u} \| \lesssim (\sum_{T \in \mathcal{R}_{T_{\ell} \to \bar{T}}} \xi(T)^2)^{1/2}, \) where \( \xi(T) \) is a standard elliptic residual indicator and \( \| \cdot \| \) is the energy norm. The estimate we prove is not as sharp but suffices for our purposes.
Lemma 3 Let $T$ be a refinement of $T_\ell$ so that $V_\ell \subset V_T \subset H^A$. Then there exists a set $\hat{R}_{T_\ell \rightarrow T}$ with $R_{T_\ell \rightarrow T} \subset \hat{R}_{T_\ell \rightarrow T}$ and

$$\# \hat{R}_{T_\ell \rightarrow T} \lesssim \# R_{T_\ell \rightarrow T}$$

such that

$$\sum_{j=1}^\beta \|(P_\ell - P_T)q^j\|^2 \leq C_2^2 \sum_{T \in \hat{R}_{T_\ell \rightarrow T}} \eta(T)^2,$$  \hspace{1cm} (4.8)

where $C_2$ is independent of essential quantities.

Proof Following the notation used in the proof of Theorem 1, denote by $q$, $q_\ell$, and $q_T$ the column vectors of orthonormal basis functions for $\delta_\ell$, $\delta_T$, and $\delta_T$, respectively. Let $M = (q^j, q^j_T)$ be the matrix satisfying $P_T q = M q_T$; recall that $\|M\| \leq \|P_T\| \leq 1$, with $\|M\|$ the matrix (operator) 2-norm. Recall also from (3.6) that $P_\ell q^j = P_\ell q^j_\ell$ for harmonic forms $q^j \in \delta_\ell$, so that $P_\ell q = P_\ell q_\ell$ and $P_T q = P_\ell q_T$. Because $V_\ell \subset V_T$ and $\delta_\ell \subset \delta_T$ we also have $P_\ell q_T = P_\ell q_T_\ell$, $q_T \in \delta_T$. Thus $P_\ell q = P_\ell P_T q$. We then compute

$$\sum_{j=1}^\beta \|(P_\ell - P_T)q^j\|^2 = \|(P_\ell - P_T)q\|^2 = \|(P_\ell P_T - P_T)q\|^2$$

$$= \|P_\ell M q_T - M q_T\|^2 = \|M(P_\ell q_T - q_T)\|^2$$

$$\leq \|M\|^2 \|P_\ell q_T - q_T\|^2 \leq \|q_\ell - P_T q_\ell\|^2,$$  \hspace{1cm} (4.9)

where in the last inequality above we employ $\|M\| \leq 1$ along with (3.2).

Following [15, (2.12) and Lemma 9], we note that $q^j_\ell - P_T q^j_\ell \in \mathcal{B}_T \perp \delta_T$, and $q^j_\ell \perp \mathcal{B}_\ell$. Thus for $1 \leq j \leq \beta$ and some $v_T \in V_T^{k-1}$ with $\|v_T\|_{H^A(\Omega)} \approx 1$,

$$\|q^j_\ell - P_T q^j_\ell\| \lesssim \sup_{v \in V_T^{k-1}, \|v\|_{H^A(\Omega)} = 1} \langle q^j_\ell - P_T q^j_\ell, d v \rangle \lesssim \langle q^j_\ell - P_T q^j_\ell, v_T \rangle.$$  \hspace{1cm} (4.10)

We next apply the regular decomposition (2.11) to find $z \in H^1 \Lambda^{k-1}(\Omega)$ with $d z = d v_T$ (note that $\varphi$ as in (2.11) plays no role here since $v_T = d \varphi + z$ implies $d v_T = d z$). We now denote by $\pi_T$ and $\pi_\ell$ the Falk-Winther interpolants $\pi_{FW}$ on $T$ and $T_\ell$, respectively. The commutativity of $\pi_T$ implies that $d v_T = \pi_T d v_T = d \pi_T z$, so that using orthogonality properties as above we have

$$\|q^j_\ell - P_T q^j_\ell\| \lesssim \langle q^j_\ell, d (I - \pi_\ell) \pi_T z \rangle.$$  \hspace{1cm} (4.11)

In addition, by (2.9) we have for some $\hat{R}_{T_\ell \rightarrow T}$ satisfying (4.7)

$$\text{supp}(d (I - \pi_\ell) \pi_T z) \subset \bigcup_{T \in \hat{R}_{T_\ell \rightarrow T}} T.$$  \hspace{1cm} (4.12)
Integrating by parts elementwise the last inner product in (4.11) and carrying out standard manipulations yields

\[ \|q_j^\ell - P_T q_j^\ell\| \lesssim \sum_{T \in \mathcal{R}_T \rightarrow T} \eta_{\ell}(T; q_j^\ell) \cdot (h_T^{-1}\|(I - \pi_{\ell})\pi_T z\|_T + h_T^{-1/2}\|(I - \pi_{\ell})\pi_T z\|_{\partial T}). \]  

(4.13)

A standard scaled trace inequality may be applied to the term \(\|d(I - \pi_{\ell})\pi_T z\|_{\partial T}\) only if extra care is taken. We first write \((I - \pi_{\ell})\pi_T z = [\pi_T z - ISZ z] + [\pi_{\ell}(ISZ z - \pi_T z)] := I + II + III\), where \(ISZ\) is the Scott–Zhang interpolant on \(T\). For \(T' \in T\), we apply a standard scaled trace inequality \(\|v\|_{L_2(\partial T')} \lesssim h_{T'}^{-1}\|v\|_{T'} + h_{T'}|v|_{H^1(T')}\), an inverse inequality, and the approximation properties (2.8) and (2.10) to find

\[ h_{T'}^{-1}\|I\|_{\partial T}^2 \lesssim \sum_{T' \in T, T' \subset T} (h_T h_{T'})^{-1}\|I\|_{T'}^2, \]

\[ \lesssim \sum_{T' \in T, T' \subset T} (h_T h_{T'})^{-1}(\|z - \pi_T z\|_T + \|z - ISZ z\|_T)^2 \]

(4.14)

where we have used the fact that \(h_{T'} \leq h_T\) when \(T' \subset T\) as above. Next, \(III \in V_{\ell}^{k-1}\), so we apply the trace inequality on \(T \in \mathcal{T}_\ell\), the stability estimate (2.7) and an inverse inequality, and then approximation properties as before to find

\[ h_T^{-1}\|III\|_{\partial T}^2 \lesssim \sum_{T' \subset T} (h_T h_{T'})^{-1}\|(\pi_T - ISZ)z\|_{\partial T'}^2 \]

\[ \lesssim \sum_{T' \subset T} |z|_{H^1(\omega_{T'})}^2 \lesssim |z|_{H^1(\Lambda(\omega_{T'}))}^2. \]

(4.15)

Here we let \(\omega_{T'} = \bigcup_{T \in \omega_T} \omega_T\). Finally, we have \(II \in H^1(\Lambda(T), T \in \mathcal{T}_\ell\), so we directly apply a scaled trace inequality, approximation properties of \(\pi_\ell\), and \(H^1\) stability of \(ISZ\) on \(T\) to find that

\[ h_T^{-1}\|III\|_{\partial T}^2 \lesssim |z|_{H^1(\Lambda(\omega_{T'}))}^2. \]

(4.16)

Similar arguments yield

\[ h_T^{-2}\|(I - \pi_\ell)\pi_T z\|_{L_2(T)} \lesssim |z|_{H^1(\Lambda(\omega_{T'}))}^2. \]

(4.17)

Inserting the previous inequalities into (4.13), applying the Cauchy-Schwartz inequality over \(\mathcal{T}_\ell\), employing finite overlap of the patches \(\omega_T\), and finally using (2.11) along with \(\|v_T\|_{H^1(\Omega)} \lesssim 1\), we find that
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In the “mark” step we use a bulk (Dörfler) marking. That is, we fix $0 < \theta \leq 1$ and choose a minimal set $\mathcal{M}_\ell \subset T_\ell$ such that

$$\eta_\ell(\mathcal{M}_\ell) \geq \theta \eta_\ell(T_\ell).$$

(5.2)

The following is an easy consequence of Theorem 1 and $\|P_\ell^{-1}\| \leq \|P_0^{-1}\|$.

**Proposition 4** Let $0 < \theta \leq 1$, and let $\mathcal{M}_\ell \subset T_\ell$ satisfy (5.2). Let also $\bar{\theta} = \theta(\|P_\ell\|\|P_\ell^{-1}\|)^{-2}$ and $\tilde{\theta}_\ell = \theta\|P_\ell^{-1}\|^{-2}$. Then for $\ell \geq 0$ and $0 \leq \ell \leq \ell$,

$$\mu_\ell(\mathcal{M}_\ell)^2 \geq \bar{\theta}_\ell \mu_\ell(T_\ell)^2 \geq \tilde{\theta}_\ell \mu_\ell(T_\ell)^2.$$  

(5.3)

**Remark 3** Note that $\|P_\ell\|\|P_\ell^{-1}\| = 1$ when $\beta = 1$, and for $\beta > 1$ the deviation of $\|P_\ell\|\|P_\ell^{-1}\|$ from 1 is dependent on the isotropy of $P_\ell$. Thus if $P_\ell q^j \approx P_\ell^{-1}q^j$ for $1 \leq i, j \leq \beta$, then $\|P_\ell\|\|P_\ell^{-1}\| \approx 1$ even if $\mathcal{H}_\ell$ and $\mathcal{H}_{\ell+1}$ do not overlap strongly. If $\beta = 1$ or $P_\ell$ is isotropic, the theoretical and practical AFEM’s will therefore mark the same elements for refinement.

We next establish continuity of the theoretical indicators (also known as an estimator reduction property). The proof is standard and is thus omitted.

**Lemma 4** Given $0 < \theta \leq 1$, let $\mathcal{M}_\ell \subset T_\ell$ satisfy (5.2). Assume also that each $T \in \mathcal{M}_\ell$ is bisected at least once in passing from $T_\ell$ to $T_{\ell+1}$. Then there are constants $C_2$ and $1 > \lambda > 0$ independent of essential quantities such that for any $\alpha > 0$, any $\ell \geq 0$, and any $0 \leq \ell \leq \ell$,

$$\mu_{\ell+1}(T_{\ell+1})^2 \leq (1 + \alpha)(1 - \lambda \bar{\theta}_\ell)\mu_\ell(T_\ell)^2 + C_2 \left(1 + \frac{1}{\alpha}\right) \sum_{j=1}^{\beta} \|P_\ell - P_{\ell+1}\|q^j \|q^j\|^2$$

$$\leq (1 + \alpha)(1 - \lambda \tilde{\theta}_\ell)\mu_\ell(T_\ell)^2 + C_2 \left(1 + \frac{1}{\alpha}\right) \sum_{j=1}^{\beta} \|P_\ell - P_{\ell+1}\|q^j \|q^j\|^2.$$  

(5.4)

Here $\bar{\theta}_\ell$ and $\tilde{\theta}_\ell$ are as defined in Proposition 4.

Although $\mathcal{H}_\ell \not\subset \mathcal{H}_{\ell+1}$, the Hodge decomposition and complex-conforming structure of the finite element spaces nonetheless yields the following essential orthogonality result.

**Theorem 2** For $q \in \mathcal{H}_\ell$,

$$\|q - P_{\ell+1}q\|^2 = \|q - P_\ell q\|^2 - \|P_\ell - P_{\ell+1}\|q\|^2.$$  

(5.5)

**Proof** It suffices to prove $(q - P_{\ell+1}q, (P_\ell - P_{\ell+1})q) = 0$. This is a consequence of $(P_\ell - P_{\ell+1})q \in \mathcal{H}_{\ell+1}$, which holds due to the nestedness $\mathcal{H}_\ell \subset \mathcal{H}_{\ell+1}$ and the fact that $P_\ell : \mathcal{H}_\ell \to \mathcal{H}_\ell$ also acts as the $L_2$ projection from $\mathcal{H}_\ell$ to $\mathcal{H}_\ell$. \qed
Assembling the above estimates yields the following contraction result. The proof is standard, except we must track dependence of constants on the mesh level \( \ell \).

**Theorem 3** For each \( \ell > 0 \), there exist \( 0 < \rho_\ell < 1 \) and \( \gamma_\ell > 0 \) such that for \( \ell \geq \ell_0 \),

\[
E_{\ell+1}^2 + \gamma_\ell \mu_{\ell+1}(T_{\ell+1})^2 \leq \rho_\ell \left( E_\ell^2 + \gamma_\ell \mu_\ell(T_\ell)^2 \right). \tag{5.6}
\]

Here \( 1 > \rho_0 \geq \rho_1 \geq \cdots \geq \rho := \lim_{\ell \to \infty} \rho_\ell > 0 \), \( \rho_\ell \) depends on \( \| P_\ell^{-1} \| \) but not on other essential quantities, and \( \rho \) is independent of essential quantities. Finally, \( 0 < \gamma_\ell < C_2^{-1} \) with \( C_2 \) as in (5.4).

**Proof** Given \( \ell \geq 0 \) and \( \alpha \) as in (5.4), let \( \gamma_\ell = \frac{1}{C_2(1+\alpha^{-1})} \). (Here we suppress the dependence of \( \alpha \) on \( \ell \).) Then \( 0 < \gamma_\ell < C_2^{-1} \), as asserted. Applying (5.5) to \( q^j \) (\( j = 1, \ldots, \beta \)) and summing the resulting equalities, then adding the result to (5.4) multiplied through by \( \gamma_\ell \), yields

\[
E_{\ell+1}^2 + \gamma_\ell \mu_{\ell+1}^2 \leq E_\ell^2 + \gamma_\ell (1 + \alpha)(1 - \lambda \theta_\ell) \mu_\ell^2. \tag{5.7}
\]

Let now \( 0 < \rho_\ell < 1 \). From (4.6) we have \( (1 - \rho_\ell) E_\ell^2 \leq C_1 (1 - \rho_\ell) \| P_\ell^{-1} \|^2 \mu_\ell^2 \), so that

\[
E_{\ell+1}^2 + \gamma_\ell \mu_{\ell+1}^2 \leq \rho_\ell E_\ell^2 + \gamma_\ell \left[ (1 + \alpha)(1 - \lambda \theta_\ell) + \gamma_\ell^{-1} C_1 (1 - \rho_\ell) \| P_\ell^{-1} \|^2 \right] \mu_\ell^2. \tag{5.8}
\]

We next set \( \rho_\ell = (1 + \alpha)(1 - \lambda \theta_\ell) + \gamma_\ell^{-1} C_1 (1 - \rho_\ell) \| P_\ell^{-1} \|^2 \) and solve for \( \rho_\ell \). Before doing so, we introduce the shorthand \( K_\ell = 1 - \lambda \theta_\ell \) and \( M_\ell = C_1 C_2 \| P_\ell^{-1} \|^2 \). Then

\[
\rho_\ell = \frac{\alpha^2 K_\ell + \alpha(K_\ell + M_\ell) + M_\ell}{\alpha(1 + M_\ell) + M_\ell}. \tag{5.9}
\]

Recalling that \( \alpha > 0 \) is arbitrary, we minimize the above expression with respect to \( \alpha \) to obtain

\[
\rho_\ell = \frac{2\sqrt{M_\ell K_\ell (1 + M_\ell - K_\ell) + M_\ell^2 + M_\ell + K_\ell - K_\ell M_\ell}}{(1 + M_\ell)^2}. \tag{5.10}
\]

We now analyze the dependence of \( \rho_\ell \) on \( \ell \). Recall that \( M_\ell = C_1 C_2 \| P_\ell^{-1} \|^2 \) and \( K_\ell = 1 - \lambda \theta \| P_\ell^{-1} \|^{-2} \) are decreasing in \( \ell \). Tedious but elementary calculations also show that for \( 0 < K_\ell < 1 \) and \( M_\ell > 0 \), \( \rho_\ell \) is increasing in both \( M_\ell \) and \( K_\ell \). Thus (5.6) holds for all \( \ell \geq 0 \) with \( \rho_\ell = \rho_0 \). We in turn see that \( \| P_\ell \|, \| P_\ell^{-1} \| \to 1 \) as \( \ell \to \infty \), so \( M_\ell \downarrow C_1 C_2 := M_\infty \). In addition, \( K_\ell \downarrow 1 - \lambda \theta := K_\infty \) as \( \ell \to \infty \). Thus \( \rho_\ell \leq \rho_0 \) and \( \rho_\ell \) decreases to
\[ \rho = \frac{2\sqrt{C_1C_1(1-\lambda\theta)(C_1C_2 + \lambda\theta) + C_2^2C_2^2 + C_1C_2 + (1-\lambda\theta)(1-C_1C_2)}}{(1+C_1C_2)^2} \]

(5.11)

as \( \ell \to \infty \). This completes the proof. \( \square \)

**Remark 4** In Remark 3 we established that the theoretical and practical AFEM mark the same elements for refinement when \( P_\ell \) is isotropic, and in particular when \( \beta = 1 \). We could sharpen the proof of Theorem 3 to take advantage of this fact by redefining \( K_\ell = 1 - \lambda\theta_\ell = 1 - \lambda\theta(\|P_\ell^{-1}\|\|P_\ell\|)^{-2} \). However, doing so would compromise monotonicity of the sequence \( \{\rho_\ell\} \), and \( M_\ell \) depends on \( \|P_\ell^{-1}\| \) and not on the product \( \|P_\ell^{-1}\|\|P_\ell\| \) in any case.

**Remark 5** Dependence of \( \rho_\ell \) on \( \|P_\ell^{-1}\| \) seems unavoidable in our proofs. We prove a contraction by combining the orthogonality relationship (5.5), the estimator reduction inequality (5.4), and the a posteriori estimate (4.6) in a canonical way [7]. The orthogonality relationship (5.5) indicates that the error \( E_2^\ell \) is reduced by \( \sum_{j=1}^\beta \| (P_{\ell+1} - P_\ell) q_j \|^2 \) at each step of the AFEM algorithm. This quantity is directly related to the theoretical indicators \( \mu_\ell \) in the estimator reduction inequality (5.4). Combining these relationships leads to an error reduction on the order of \( \mu_\ell(T_\ell) \). On the other hand, \( E_\ell \) is uniformly equivalent to the practical estimator \( \eta_\ell \). Because the theoretical and practical estimators are related by \( \|P_\ell^{-1}\|^{-1} \eta_\ell \leq \mu_\ell \leq \|P_\ell\| \eta_\ell \), reducing \( E_\ell \) by \( \mu_\ell(T_\ell) \) is equivalent to a reduction lying between \( \frac{1}{\|P_\ell^{-1}\|} E_\ell \) and \( \|P_\ell\| E_\ell \).

**Remark 6** For elliptic source problems, a contraction is obtained from the initial mesh with contraction factor independent of essential quantities [7]. On the other hand, for elliptic eigenvalue problems such a contraction result holds only if the initial mesh is sufficiently fine [6, 19]. The situation for harmonic forms is intermediate between those encountered in source problems and eigenvalue problems. No initial mesh fineness assumption is needed to guarantee a contraction, but we only show that the contraction constant is independent of essential quantities on sufficiently fine meshes. In the case of eigenvalue problems a similar transition state likely exists in which AFEM can be proved to be contractive, but with contraction constant improving as resolution of the target invariant space improves.

### 6 Quasi-optimality

Our proof of quasi-optimality follows a more or less standard outline, simplified somewhat by lack of data oscillation but made more complicated by improvement in the contraction factor as the mesh is refined.
6.1 Approximation classes

Given rate $s > 0$ and $r \in [\mathcal{H}]^\beta$, we let

$$|r|_{\mathcal{A}_s} = \sup_{N > 0} \inf_{\# T - \# T_0 \leq N} N^{-s} \left( \sum_{j=1}^{\beta} \|r^j - P_T r^j\|^2 \right)^{1/2}.$$  \hspace{1cm} (6.1)

$\mathcal{A}_s$ is then the class of all $r \in [\mathcal{H}]^\beta$ such that $|r|_{\mathcal{A}_s} < \infty$.

If we applied $\mathcal{A}_s$ to arbitrary $r \in [H^\Lambda(\Omega)]^\beta$, it would be natural to replace the projection $P_T$ onto $\mathcal{H}_T$ used in (6.1) by the $L_2$ projection onto the full discrete space $V_T$. We show in Proposition 5 that best approximation over the full finite element space is equivalent up to a constant to best approximation over $\mathcal{H}_T$ only. Thus our definition of $\mathcal{A}_s$ makes use of the full approximation strength of the finite element space $V_T$, even though at first glance it may not seem that this is the case.

**Proposition 5** Given $T \in \mathcal{T}$ and $q \in \mathcal{H}$,

$$\|q - P_T q\| \lesssim \inf_{q_T \in V_T} \|q - q_T\|.$$ \hspace{1cm} (6.2)

**Proof** From (27) of [3], we have $\|q - P_{\mathcal{H}_T} q\| \leq \|(I - \pi_T)q\|$ with $\pi_T$ a commuting cochain projection. Thus for $q \in \mathcal{H}$, $q_T \in V_T$, we may use (2.4) to obtain

$$\|q - P_T q\| \leq \|(q - q_T) - \pi_{CW}(q - q_T)\| \leq (1 + C)\|q - q_T\|,$$ \hspace{1cm} (6.3)

where $C$ is the $L_2$ stability constant for $\pi_{CW}$. \hspace{1cm} $\square$

6.2 Rate optimality

We first state and prove two lemmas which are more or less standard in this context. It is important, however, that the constants in these two lemmas are entirely independent of essential quantities.

**Lemma 5** Let $C_1$ and $C_2$ be the constants from (4.3) and (4.8), respectively, and assume that $\theta < \frac{1}{C_1 C_2}$. Then for $T_\ell \subset T$ with

$$\|q - P_T q\| \leq [1 - \theta C_1 C_2]\|q - P_\ell q\|,$$ \hspace{1cm} (6.4)

there holds that

$$\eta_\ell(\hat{R}_{T_\ell \to T}) \geq \theta \eta_\ell(T_\ell).$$ \hspace{1cm} (6.5)
Proof Employing in turn (4.3), (6.4), the triangle inequality, and (4.8) yields

\[
\theta C_2 \eta(\mathcal{T}_\ell) \leq \theta C_1 C_2 \| q - P_\ell q \|
\]
\[
\leq \| q - P_\ell q \| - \| q - P_T q \|
\]
\[
\leq \| P_\ell q - P_T q \|
\]
\[
\leq C_2 \eta(\hat{R}_{\mathcal{T}_\ell \to \mathcal{T}}).
\]

(6.6)

Dividing through by \( C_2 \) completes the proof. \( \square \)

Lemma 6 The collection of marked elements \( \mathcal{M}_\ell \subseteq \mathcal{T}_\ell \) defined by the marking strategy (5.2) satisfies

\[
\# \mathcal{M}_\ell \lesssim |q|_{A_s}^{1/s} \mathcal{E}_\ell^{-1/s}.
\]

(6.7)

Proof By definition of \( A^s \) there exists a partition \( T' \in \mathbb{T} \) such that

\[
\# T' - \# T_0 \lesssim |q|_{A_s}^{1/s} \left( (1 - \theta C_1 C_2) \mathcal{E}_\ell \right)^{-1/s}
\]

(6.8)

and

\[
\| q - P_{T'} q \| \leq (1 - \theta C_1 C_2) \mathcal{E}_\ell.
\]

The smallest common refinement \( T \) of \( T_\ell \) and \( T' \) is in \( \mathbb{T} \) with \( \# T - \# T_\ell \leq \# T' - \# T_0 \) (cf. [32] last lines of the proof of Lemma 5.2). Since \( V_{T'} \subset V_T \), (6.2) and the last equation above yield

\[
\| q - P_{V_T} q \| \leq \| q - P_{V_{T'}} q \| \leq (1 - \theta C_1 C_2) \mathcal{E}_\ell.
\]

Thus \( \eta(\hat{R}_{\mathcal{T}_\ell \to \mathcal{T}}) \geq \theta \eta_\ell \) by Lemma 5. Since \( \mathcal{M}_\ell \) is the smallest subset of \( T_\ell \) with \( \eta(\mathcal{M}_\ell) \geq \theta \eta_\ell \), we conclude that

\[
\# \mathcal{M}_\ell \leq \# \hat{R}_{\mathcal{T}_\ell \to \mathcal{T}} \lesssim \# T - \# T_\ell \lesssim \# T' - \# T_0.
\]

(6.9)

We finally state our optimality result.

Theorem 4 Assume as in Lemma 5 that \( \theta < \frac{1}{C_1 C_2} \), and assume that \( q \in A_s \). Given \( \ell \geq 0 \), there exists a constant \( C_\ell \) depending on \( \| P^{-1}_\ell \| \) and the constant \( C_{\text{ref,} \ell} \) from (2.3) but independent of other essential quantities such that

\[
\mathcal{E}_\ell \leq C_\ell (\# T_\ell - \# L_\ell)^{-1/s} |q|_{A_s}, \quad \ell \geq \ell.
\]

(6.9)
Proof We first compute using Theorem 1, (4.3), and the fact that $\gamma_{\ell} \leq C_2^{-1}$ that for $k \geq \ell$,

$$E_k^2 + \gamma_{\ell}\mu_k(T_k)^2 \leq E_k^2 + \gamma_{\ell}\eta_k(T_k)^2 \lesssim (1 + \gamma_{\ell})E_k^2 \lesssim E_k^2. \quad (6.10)$$

Thus

$$E_k^{-1/s} \lesssim (E_k^2 + \gamma_{\ell}\mu_k(T_k)^2)^{-1/2s}. \quad (6.11)$$

We then use (5.6) to obtain

$$E_k^{-1/s} \lesssim \rho(\ell - k)^{2s}/2\ell(\gamma_{\ell}\mu_{\ell}(T_{\ell})^2)^{-1/2s}. \quad (6.12)$$

From (2.3), it then follows that

$$\#T_{\ell} - \#T_{\ell} = \sum_{k=\ell}^{\ell-1} \#R_{T_k \to T_{k+1}} \leq C_{ref,\ell} \sum_{k=\ell}^{\ell-1} \#M_k \lesssim C_{ref,\ell} |q|_{A_{\ell}}^{1/s} \sum_{k=\ell}^{\ell-1} E_k^{-1/s}$$

$$\leq C_{ref,\ell} |q|_{A_{\ell}}^{1/s} \sum_{k=\ell}^{\ell-1} \rho_{\ell}^{(\ell-k)/2s} (E_{\ell}^2 + \gamma_{\ell}\mu_{\ell}(T_{\ell})^2)^{-1/2s}$$

$$\leq C_{ref,\ell} \left(1 - \rho_{\ell}^{1/2s}\right)^{-1} |q|_{A_{\ell}}^{1/s} (E_{\ell}^2 + \gamma_{\ell}\mu_{\ell}(T_{\ell})^2)^{-1/2s}. \quad (6.13)$$

Setting $C_{\ell} := C_{ref,\ell} \left(1 - \rho_{\ell}^{1/2s}\right)^{-1}$ and rearranging the above expression completes the proof. \qed

Remark 7 As is standard in AFEM optimality results, $\theta$ is required to be sufficiently small in order to ensure optimality. $\theta$ must however only be sufficiently small with respect to $C_1/C_2$, which is entirely independent of the dimension $\beta$ of $\tilde{H}$, $\|P_{\ell}^{-1}\|$, and other essential quantities. In contrast to elliptic eigenvalue problems [6,19], we do not require an initial fineness assumption on $T_0$ in order to guarantee that the threshold value for $\theta$ is independent of essential quantities.

The constant $C_{\ell}$ does however depend on $\|P_{\ell}^{-1}\|$, and it is not clear that this constant will improve (decrease) as the mesh $\ell$ increases even though $\|P_{\ell}^{-1}\| \to 1$. The factor $(1 - \rho_{\ell}^{1/2s})^{-1}$ is nonincreasing, but the factor $C_{ref,\ell}$ arising from (2.3) is not uniformly bounded in $\ell$. According to [33, Theorem 6.1] it may in essence depend on the degree of quasi-uniformity of the mesh $T_{\ell}$ and thus may degenerate as the mesh is refined. In order to guarantee a uniform constant, we apply Theorem 4 with $\ell = 0$ and obtain a constant $C_0$ which depends on $\|P_0^{-1}\|$ in addition to $T_0$.

7 Extensions

In this section we briefly discuss possible extensions of our work.
7.1 Essential boundary conditions

Many of our results extend directly to the case of essential boundary conditions in which the requirement $\text{tr} \star q = 0$ in (2.1) is replaced by $\text{tr} q = 0$, with the latter condition imposed directly on the finite element spaces. The major hurdle in obtaining an immediate extension is the availability of quasi-interpolants which possess the necessary properties. The Christiansen–Winther interpolant in Sect. 2.3 has been defined and analyzed also for essential boundary conditions, while the Falk-Winther interpolant was fully analyzed in [17] only for natural boundary conditions. We used the properties of the Falk-Winther interpolant only to obtain the localized a posteriori upper bound (Lemma 3), which is necessary to obtain a quasi-optimality result but not a contraction. The contraction result given in Theorem 3 thus extends immediately to the case of essential boundary conditions. There is indication given in [17] that the properties of the Falk-Winther interpolant transfer naturally to essential boundary conditions, in particular by simply setting boundary degrees of freedom to 0. Assuming this extension our quasi-optimality results also hold for homogeneous essential boundary conditions. A suitable interpolant is also defined and analyzed in the paper [36] for the practically important case $d = 3, k = 1$.

7.2 Harmonic forms with coefficients

In electromagnetic applications the magnetic permeability $A$ is a symmetric, uniformly positive definite matrix having entries in $L^\infty(\Omega)$. If $A$ is nonconstant, then the space

$$
\mathcal{H}_A(\Omega) = \{v \in L^2(\Omega)^3 : \text{curl } v = 0, \ \text{div } A v = 0, \ A v \cdot n = 0 \text{ on } \partial \Omega \} 
$$

(7.1)

is the natural space of harmonic forms, but differs from that in (1.1). It is similarly possible to modify the more general definition (2.1) to include coefficients. As is pointed out in [3, Section 6.1], the finite element exterior calculus framework applies essentially verbatim to this situation once the inner products used in all of the relevant definitions are modified to include coefficients. Our a posteriori estimates and the contraction result of Sect. 5 similarly apply with minimal modification. Extension of the optimality results of Sect. 6 is possible but complicated by the presence of oscillation of the coefficient $A$ in the analysis; cf. [6,7,12].

7.3 Approximation of cohomologous forms

In some applications it is of interest to compute $P_\mathcal{H} f$ for a given (non-harmonic) form $f$. This is for example the case in the finite element exterior calculus framework for solving problems of Hodge–Laplace type. There $f$ is the right-hand-side data. Because the Hodge–Laplace problem is only solvable for data orthogonal to $\mathcal{H}$, it is necessary to compute $P_\mathcal{H} f$ and solve the resulting system with data $f - P_\mathcal{H} f$; cf. [25] for other applications. A possible adaptive approach to this problem is to adaptively reduce the defect between $\mathcal{H}$ and $\mathcal{H}_\ell$ as we do above and then project $f$ onto $\mathcal{H}_\ell$. However, this approach requires computation of a multidimensional space, while the
original problem only requires computation of a one-dimensional space (that spanned by $P_{\mathcal{H}}f$). An alternate method would be to approximate the full Hodge decomposition. More precisely, one could compute $P_{\mathcal{B}}f$ and $P_{3\perp}f$ by solving two constrained elliptic problems, but expense is an obvious disadvantage of this method also.

It may be desirable to instead adaptively compute $P_{\mathcal{H}}f$ directly. It might for example be the case that some members of $\mathcal{H}_\ell$ have singularities which are not shared by $P_{\mathcal{H}}f$ (such situations arise in eigenfunction computations). The task of constructing an AFEM for approximating only $P_{\mathcal{H}}f$ appears difficult, however. Assume for the sake of argument the extreme case where $P_{\mathcal{H}}f \neq 0$, but $f \perp \mathcal{H}_\ell$. The indirect approach of first controlling the defect between $\mathcal{H}$ and $\mathcal{H}_\ell$ and then projecting $f$ would continue to function with no problems in this case. On the other hand, it is not clear how to directly construct an a posteriori estimate for $\|P_{\mathcal{H}}f - P_{\ell}f\|$ which would be nonzero when $\mathcal{H}_\ell \perp f$. In particular, it is not difficult to construct an estimator for $\|P_{\ell}f - P_{\mathcal{H}}P_{\ell}f\|$ (cf. [15]), but such an estimator would be 0 and thus not reliable in this case.

7.4 Alternate methods for computing harmonic forms: cutting surfaces

We have largely ignored the actual method for obtaining $\mathcal{H}_\ell$ by simply assuming that we in some fashion produced an orthonormal basis for this space. This viewpoint is consistent with the finite element exterior calculus framework that we have largely followed. It also fits well with eigenvalue or SVD-based methods for computing $\mathcal{H}_\ell$, which are general with respect to space dimension and form degree and which may be easily implemented using standard linear algebra libraries. Discussion of methods for producing such a basis consistent with the FEEC framework may be found in [25]. However, the process of producing $\mathcal{H}_\ell$ is in and of itself not entirely straightforward, and the method for producing it may affect the structure and properties of the resulting adaptive algorithm. Different methods with potentially advantageous properties have been explored especially in three space dimensions [16, 28]. The method of cutting surfaces is such an example.

Let $\Omega \subset \mathbb{R}^3$, and assume that there exist $\beta$ regular and nonintersecting cuts (two-dimensional hypersurfaces) $\sigma_j$ such that $\Omega_0 = \Omega \setminus \bigcup_{j=1}^\beta \sigma_j$ is simply connected. The assumption that such cuts exist is nontrivial with respect to domain topology and excludes for example the complement of a trefoil knot in a box [4]. The methodology we discuss here thus does not apply in all situations where our theory above applies. A domain for which such a set of cutting surfaces exist is called a Helmholtz domain. Determination of cutting surfaces on finite element meshes is also a nontrivial and possibly computationally expensive problem [16], although in an adaptive setting one could potentially compute the cutting surfaces at low cost on a coarse mesh and transfer them to subsequent refinements.

Assuming that $\Omega$ is Helmholtz, let $\varphi^j$ solve

$$
-\Delta \varphi^j = 0 \text{ in } \Omega_0, \quad \nabla \varphi^j \cdot n = 0 \text{ on } \partial \Omega, \quad \llbracket \varphi^j \rrbracket_{\sigma_i} = \delta_{ij} \text{ and } \llbracket \nabla \varphi \cdot n \rrbracket_{\sigma_i} = 0, \quad 1 \leq i \leq \beta.
$$

(7.2)
The set \( \{ \nabla \varphi_j \}_{j=1}^\beta \) then serves as a basis for the space \( \mathbb{S}_1^1 \) of vector fields defined in (1.1). That is, the harmonic fields may be reclaimed from potentials consisting of \( H^1 \) functions.

Next assume that the cuts \( \sigma_j \) each consist of unions of faces in a simplicial mesh \( T \). Let now \( V^0 \) be piecewise linear Lagrange elements on \( \Omega \), and let \( V^0_j \) be the set of functions which are continuous and piecewise linear in \( \Omega \setminus \sigma_j \) and which satisfy \( [v_h]_{\sigma_j} = 1 \). The canonical finite element approximation to \( \varphi^j \) is to find \( \varphi^j_T \in V^0_j \) such that \( \int_{\Omega \setminus \sigma_j} \nabla \varphi^j_T \cdot \nabla v = 0 \) for all \( v \in V^0 \). \( \varphi^j_T \) is only unique up to a constant, but since we are only interested in \( \nabla \varphi^j_T \) this has little effect on our discussion. One can also verify that \( \nabla \varphi^j_T \) is a discrete harmonic field lying in the lowest-degree Nédélec edge space. Thus as in the continuous case, the discrete harmonic fields can be recovered from potentials. The same procedure may be applied with other complex-conforming pairs of spaces. An analogous procedure also exists for essential boundary conditions.

An obvious adaptive procedure for approximating \( \mathbb{S}_1^1 \) is to adaptively compute \( \varphi^j_T, j = 1, \ldots, J \), using standard AFEM for scalar Laplace problems. Convergence and optimality follows from standard results such as those found in [7] with slight modification to account for boundary conditions. The basis \( \{ \nabla \varphi^j \}_{j=1}^\beta \) that we thus approximate is not orthonormal, but has the advantage of being fixed using criterion that are passed on to the discrete approximations. Also note that here the approximation map \( \varphi^j \rightarrow \varphi^j_T \) is clearly linear. If we however orthonormalized the vectors \( \{ \nabla \varphi^j_T \} \) in order to produce \( \{ q^j_T \} \) as in our previous assumptions, then the approximation \( \{ \nabla \varphi^j \} \rightarrow \{ q^j_T \} \) would be nonlinear. This discussion indicates that while producing an orthonormal basis of forms is a nonlinear procedure, the nonlinearity is mild and not intrinsic to the task of finding some basis for the space of harmonic forms.

### 7.5 Alternate methods for computing harmonic forms: adaptive correction of a cohomology basis

We next describe another possible option for computing the space \( \mathbb{S}_1^1 \) of forms given in (1.1). Given a coarse initial mesh \( T_0 \), one may first compute a basis \( \{ q^j_0 \}_{j=1}^\beta \) for the space \( \mathbb{S}^1_0 \) of harmonic forms on \( T_0 \). (More broadly, one could use as a starting point any basis for the first cohomology group on \( T_0 \), that is, a linearly independent set \( \{ \tilde{q}^j_0 \}_{j=1}^\beta \) of curl-free finite element vector fields that are not gradients of scalar potentials.) Assuming that \( \#T_0 \) is small, computation of \( \mathbb{S}^1_0 \) can be achieved relatively cheaply using a brute-force linear algebra technique such as an svd solver. Let then \( \phi^j \in H^1(\Omega) \) solve

\[
\langle \nabla \phi^j, \nabla \tau \rangle = \langle q^j_0, \nabla \tau \rangle, \quad \tau \in H^1(\Omega).
\]  

(7.3)

Because \( \mathbb{S}^1_0 \subset \mathbb{Z}^1 \), we have \( P_{\mathbb{S}} q^j_0 = q^j_0 - P_{\mathbb{S}} q^j_0 = q^j_0 - \nabla \phi^j \). In addition, Lemma 3 guarantees that \( \{ q^j_0 - \nabla \phi^j \}_{j=1}^\beta \) is a basis for \( \mathbb{S}^1_0 \). This suggests an alternative and
potentially cheaper method for adaptive computation of $\mathcal{H}^1$, namely to first find a basis for the discrete harmonic forms on a coarse mesh, then adaptively compute discrete approximations \( \{ \phi^j_\ell \}_{j=1}^\beta \) to the $\beta$ source problems in (7.3). The set \( \{ q^0_0 - \nabla \phi^j_\ell \}_{j=1}^\beta \) is then a basis for the discrete harmonic forms $\mathcal{H}^1$.

Adaptive finite element approximation of the corrections $\phi^j$ is nonstandard because the right-hand-side data for the problem (7.3) is only in $H^{-1}(\Omega)$. Convergence analysis of such AFEM is thus more difficult than for $L_2$ data, but was carried out under broad assumptions in [10] for appropriately defined error estimators and AFEM. Besides the potential use of a slightly nonstandard AFEM that is less likely to be available in standard codes, there are two main possible disadvantages of this approach. The first is that a straightforward application would employ $\beta$ meshes, as in the case of cutting surfaces. Depending on the application, using multiple meshes could be costly and inconvenient if $\beta > 1$. This can easily be remedied by summing the elementwise error indicators from the $j$ source problems (7.3) as we do in (4.2), but a direct application of this fix also has potential drawbacks. The set \( \{ q^0_0 - \nabla \phi^j_\ell \}_{j=1}^\beta \) is guaranteed to be a basis for $\mathcal{H}^1$, but may not be close to orthonormal even for large $\ell$ since it is an approximation to the basis \( \{ q^0_0 - \nabla \phi^j \} \). Thus error indicators from this basis (or from the associated corrections $\phi^j_\ell$) may artificially over- or underestimate the influence of features such as singularities from the space $\mathcal{H}^1$. An orthonormal basis, on the other hand, should in the aggregate equally weight all important features from $\mathcal{H}^1$.

It thus may be most advantageous to compute the harmonic forms at each level using the source problems (7.3), but adapt the mesh using error estimators derived from an orthonormal basis as we have proposed. It is relatively cheap to solve the above source problems, and it is then also straightforward to orthonormalize the basis \( \{ q^0_0 - \nabla \phi^j_\ell \}_{j=1}^\beta \) to obtain a set \( \{ q^j_\ell \}_{j=1}^\beta \) which is in the form we assume as input for our algorithm and theory.

We end by noting that extension of this method and its analysis to other form degrees and space dimensions is less clear. Consider computation of the space $\mathcal{H}^2$ in three space dimensions. Here the space $\mathcal{H}^2$ of discrete harmonic forms consists of strongly divergence-free and weakly curl-free fields. Assuming we have calculated $\mathcal{H}^2_0$, the corrections $\phi^j_\ell$ would lie in an $H(\text{curl})$-conforming finite element space $V^1_\ell$ and satisfy

$$
\langle \text{curl} \phi^j_\ell, \text{curl} \tau_\ell \rangle = \langle q^0_0, \text{curl} \tau_\ell \rangle, \quad \tau_\ell \in V^1_\ell.
$$

This problem is more complicated to solve than (7.3) because it is underdetermined, i.e., it only specifies $\phi^j_\ell$ up to an element of $\mathcal{Z}^1_\ell = \mathcal{B}^1_\ell \oplus \mathcal{H}^1_\ell$. Analysis of the resulting AFEM is also unclear because the right-hand-side data lies in the dual space of $H(\text{curl})$. In contrast to the case of standard elliptic scalar problems discussed above, analysis of AFEM for $H(\text{curl})$ systems with right-hand-side data in such negative-order spaces has not to our knowledge been carried out.
8 Numerical experiments

In this section we give brief numerical experiments which illustrate our theoretical results. We employed the MATLAB-based finite element toolbox iFEM [8]. Harmonic fields for two-dimensional domains were computed by first constructing the full system matrix for the Hodge (vector) Laplacian and then finding its kernel using the MATLAB svds command. This approach may be relatively inefficient from a computational standpoint, but allowed accurate computation of the discrete harmonic basis using off-the-shelf components. A more sophisticated algorithm for computing discrete harmonic fields, motivated by algebraic topology but only formulated for lowest-degree (Whitney) finite element forms, is available in [28].

We adaptively computed the space $H^1$ of harmonic forms on domains $\Omega \subset \mathbb{R}^2$. The de Rham complex in two space dimensions may be realized as

$$H^1(\Omega) \xrightarrow{\nabla} H(\text{rot}; \Omega) \xrightarrow{\text{rot}} L^2(\Omega),$$

(8.1)

where the rotation operator is given by $\text{rot} \mathbf{v} = \partial_y v_1 - \partial_x v_2$. The adjoint of $d = \text{rot}$ is the two-dimensional curl operator $\delta = \text{curl} \varphi = (-\partial_y, \partial_x)$. The corresponding space $\mathcal{H}^1$ of harmonic forms consists of rotation- and divergence-free vector fields with vanishing normal component $\mathbf{v} \cdot \mathbf{n}$ on $\partial \Omega$.

Our computations were performed on (non-simply connected) polygonal domains. We briefly recall some standard facts about singularities of solutions to elliptic PDE on polygonal domains. Assume that $-\Delta u = f$ with homogenous Neumann boundary conditions, that is, assume that $u$ solves a 0-Hodge Laplace problem for the complex (8.1). At a given vertex $v_i \in \partial \Omega$ with interior opening angle $\omega_i$, there generally holds $u(x) \sim r_i(x)^{\pi/\omega_i}$ with $r_i = \text{dist}(x, v_i)$. Solutions to the 1-Hodge Laplace problem $(d\delta + \delta d)u = (-\nabla \text{div} + \text{curl rot})u = f$ may generically be expected to have singularities one exponent stronger, that is, of the form $r_i^{\pi/\omega_i - 1}$; cf. [11,14] for related discussion of Maxwell’s equations in three space dimensions.

Because harmonic forms satisfy $(-\nabla \text{div} + \text{curl rot})u = 0$, one would expect a singularity structure similar to that for the corresponding Hodge Laplace problem. For $q^j \in H^1$, we thus expect

$$q^j \sim r_i^{\pi/\omega_i - 1}$$

(8.2)

Another way to see this is to consider the Hodge decomposition $\mathbf{v} = \nabla \varphi + q + z$ of a smooth vector field $\mathbf{v}$. $\nabla \varphi$ solves $-\Delta \varphi = \text{div} \mathbf{v}$, so we generally expect $\nabla \varphi \sim r_i^{\pi/\omega_i - 1}$ near $v_i$. Because $\mathbf{v}$ is smooth, $q \in \mathcal{H}^1$ and $z \in \mathcal{Z}^\perp$ may be expected to have offsetting singularities. As a final note, if $\omega_i > \pi$ (that is, if the opening at $v_i$ is nonconvex), then the exponent of $r_i$ in (8.2) is negative and thus we expect $q^j$ to be unbounded near $r_i$.

In all of our experiments below we compute harmonic forms on polygonal domains in $\mathbb{R}^2$ using rotated Raviart–Thomas elements of lowest degree. These elements give an a priori convergence rate of order $O(h)$ for $\|q - P_h q\|_{L^2(\Omega)}$ assuming sufficiently smooth $q$. When $q$ has vertex singularities such as those described above, AFEM is
able to recover a convergence rate of $(\#T_\ell - \#T_0)^{-1/2}$, just as for standard piecewise linear Lagrange AFEM for computing gradients of solutions to $-\Delta u = f$.

8.1 Experiment 1: $\beta = 1$

Our goals in this experiment are to demonstrate improved convergence rates for AFEM versus quasi-uniform mesh refinement and also to confirm that harmonic forms blow up at reentrant corners, as predicted by (8.2). Here $\Omega$ is a simple square annulus with reentrant corners having opening angle $\omega_j = 3\pi/2$, so we expect $q^1 \sim r^{-1/3}$ near reentrant corners. We correspondingly expect an a priori convergence rate of order $h^{2/3-\epsilon} \sim DOF^{-1/3+\epsilon}$ on sequences of quasi-uniform meshes, and an adaptive convergence rate of order $DOF^{-1/2}$. These are in fact observed in the left plot of Fig. 1. In the right plot of Fig. 1 we show the increase in $\|q_\ell^1\|_{L^\infty(\Omega)}$ as the mesh is refined, which provides confirmation that $q^1$ is singular as expected. In Fig. 2 we display an
In this experiment we investigate the case $\beta > 1$. In Fig. 3, an adaptively computed basis is displayed on a relatively coarse mesh $\mathcal{T}_\ell$ on a domain with $\beta = 3$, while Fig. 4 displays the computed discrete harmonic basis on a finer mesh $\tilde{\mathcal{T}}_{\tilde{\ell}}$ ($\tilde{\ell} > \ell$). Comparing the two bases provides an illustration of the alignment problem discussed in the introduction. There is no correspondence between $q^1_\ell$ and $q^1_{\tilde{\ell}}$. Comparing two such forms in an estimator reduction inequality is not meaningful, as in the case of elliptic eigenvalue problems. Our second comment concerns the discussion in Sect. 7.3. There we noted that forms in a given cohomology class may not be singular at all reentrant corners. This is illustrated by for example $q^1_\ell$ in the upper left of Fig. 3. The support of
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Fig. 4 Discrete harmonic basis elements $q^1_\ell$ (upper left), $q^2_\ell$ (upper right), and $q^3_\ell$ (lower) computed on a finer mesh

$q^1_\ell$ is mainly localized to the upper right quarter of $\Omega$. Here this localization occurred somewhat randomly, but can also be manufactured by design (cf. [25]).

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