How universal is the one-particle Green’s function of a Luttinger liquid?

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The one-particle Green’s function of the Tomonaga-Luttinger model for one-dimensional interacting Fermions is discussed. Far away from the origin of the plane of space-time coordinates the function falls off like a power law. The exponent depends on the direction within the plane. For a certain form of the interaction potential or within an approximated cut-off procedure the different exponents only depend on the strength of the interaction at zero momentum and can be expressed in terms of the Luttinger liquid parameters \( K_\rho \) and \( K_\sigma \) of the model at hand. For a more general interaction and directions which are determined by the charge velocity \( v_\rho \) and spin velocity \( v_\sigma \) the exponents also depend on the “smoothness” of the interaction at zero momentum and the asymptotic behavior of the Green’s function is not given by the Luttinger liquid parameters alone. This shows that the physics of large space-time distances in Luttinger liquids is less universal than is widely believed.

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Over the past 20 years it has been shown that the low energy physics of a variety of models of one-dimensional (1D) correlated electrons can be described by the Luttinger liquid (LL) phenomenology [1]. Because of the progress in the experimental realization of quasi 1D systems and speculations about possible LL behavior in the normal state of the high-temperature superconductors LL phenomenology has lately attracted considerable attention. In Fermi liquids the low energy physics and low temperature thermodynamics is dominated by the excitation of quasi-particles which are in a one-to-one correspondence to the particle-hole excitations of the non-interacting system. The elementary excitations in LL’s are of collective bosonic nature. In LL’s the discontinuity of the momentum distribution function \( n(k) \) at the Fermi wave vector \( k_F \) vanishes and the density of states of the one-particle Green’s function at the Fermi energy is zero. Close to the Fermi surface both functions are dominated by power law behavior with exponents which are given by the so called LL parameters \( K_\rho \) and \( K_\sigma \). According to LL phenomenology thermodynamic quantities (e. g. the compressibility and spin susceptibility) and the non-analytic behavior of correlation functions can be expressed in terms of these two parameters and the velocities \( v_\rho \) and \( v_\sigma \) of charge and spin excitations. By calculating the one-particle Green’s function of the Tomonaga-Luttinger (TL) model [2,4] we will show that the asymptotic behavior of the Green’s function of a general LL at large space-time distances is nonetheless less universal than is widely believed. The exponents of the algebraic decay in the directions within the \( x-v_F t \) plane which are determined by the velocities \( v_\rho \) and \( v_F \) are not given by the LL parameters of the model alone. Here \( v_F \) denotes the Fermi velocity. We will furthermore discuss the implications for the spectral function.

The TL model is a continuum model of interacting 1D electrons. The linearization of the electron dispersion around the two Fermi points and neglecting the backscattering processes between electrons makes it feasible to determine the spectrum of the Hamiltonian and calculate all correlation functions. Following Luttinger [4] we introduce right- (\( \alpha = + \)) and left-moving (\( \alpha = - \)) Fermions with spin \( s \), creation operators \( a^\dagger_{k,\alpha,s} \), dispersion \( \xi_\alpha(k) = \alpha v_F (k - \alpha k_F) \), density operators (\( q \neq 0 \)) \( \rho_{\alpha,s}(q) = \sum_k a^\dagger_{k,\alpha,s} a_{k+q,\alpha,s} \) and particle number operators \( n_{k,\alpha,s} = a^\dagger_{k,\alpha,s} a_{k,\alpha,s} \). To simplify the mathematical treatment Luttinger added an infinite filled Fermi sea to the ground state [3]. The Hamiltonian for a system of length \( L \) is given by

\[
H = \sum_{k,\alpha,s} \xi_\alpha(k) \left[ n_{k,\alpha,s} - \langle n_{k,\alpha,s} \rangle_0 \right] + \frac{1}{2L} \sum_{q \neq 0, s, s'} \left[ g_{4,\|}(q) \delta_{s,s'} + g_{4,\perp}(q) \delta_{s,-s'} \right] \rho_{\alpha,s}(q) \rho^\dagger_{\alpha,s'}(q) + \frac{1}{L} \sum_{q \neq 0, s, s'} \left[ g_{2,\|}(q) \delta_{s,s'} + g_{2,\perp}(q) \delta_{s,-s'} \right] \rho_{+,s}(q) \rho^\dagger_{-,s'}(q). \tag{1}
\]

Here we use “g-ology” notation [4] and \( \langle \ldots \rangle_0 \) denotes the (non-interacting) ground state expectation value. Contrary to many authors we keep the explicit \( q \) dependence of the coupling functions. We assume that the Fourier transforms \( g_i(q,\kappa) \) (\( i = 2, 4; \kappa = |\parallel|, \perp \)) of the two-particle interaction have only contributions for \( q^\perp < q_c \ll k_F \) with an interaction cut-off \( q_c \). At no stage of the discussion it will be necessary to introduce any further cut-offs “by hand” despite the infinite (filled) Fermi sea at negative energies. The model only belongs to the LL universality class if \( g_{2,\kappa}(q = 0) \) is finite for \( \kappa = |\parallel| \) and \( \perp \) and at least one of the two coupling constants is non-zero. Thus we restrict ourselves to these kind of interactions.

Bosonization of the Hamiltonian and a canonical transformation leads to [2]

\[
H = \sum_{q \neq 0} \sum_{\nu=\rho,\sigma} \varepsilon_{\nu}(q) \beta^\dagger_{\nu}(q) \beta_{\nu}(q), \tag{2}
\]

with bosonic operators \( \beta_{\nu}(q) \) describing charge (\( \nu = \rho \)) and spin (\( \nu = \sigma \)) excitations (spin-charge separation). The energies \( \varepsilon_{\nu,q} \) are given by
\[\varepsilon_{p,\nu}(q) = v_F \sqrt{\left(1 + \frac{g_{4,\nu}(q)}{\pi v_F}\right)^2 - \left(\frac{g_{2,\nu}(q)}{\pi v_F}\right)^2} \equiv v_\nu(q), \quad (3)\]

where we have introduced the renormalized charge and spin density velocities \(v_\nu(q)\) and interactions \(g_{i,\rho/s}(q) = \left[g_{i,\rho}(q) \pm g_{i,\sigma}(q)\right]/2\). The one-particle Green’s function \(iG_{\alpha,s}^<(x,t) = \langle \psi_{\alpha,s}(0,0)|\psi_{\alpha,s}(x,t)\rangle\), which after a double Fourier transformation leads to the spectral function \(\rho_{\alpha,s}^<(k,\omega)\) relevant for photoemission experiments, can be calculated using the bosonization of the fermion fields \(\psi_{\alpha,s}(x) = \frac{1}{\sqrt{L}} \sum_{k} e^{-i k x} a_{\alpha,s}^\dagger\). In the thermodynamic limit and at zero temperature we obtain \(G_+^<(x,t) = [G_+^>(x,t)\exp\{F(x,t)\}]\) with

\[F(x,t) = \frac{1}{2} \sum_{\nu=\rho,s} \int_0^\infty dq q \left\{ e^{-iq(x-vt)t} - e^{-iq(x-vt)} \right\} + 2\gamma_\nu(q) \left\{ \cos(qx)e^{iqv\sigma(x,t)} - 1 \right\} \quad (4)\]

and the non-interacting Green’s function

\[G_+^>(x,t) = \frac{1}{2\pi} \frac{e^{ikpx}}{x-vt-i0}. \quad (5)\]

Because the Green’s function is the same for both spin directions we have suppressed the spin index \(s\). By leaving out irrelevant particle number contributions in the Hamiltonian we effectively shifted the energy scale so that the chemical potential is zero. \(\gamma_\nu(q)\) in Eq. (4) is given by the eigenvectors of the canonical transformation and can be written as \(\gamma_\nu(q) = \left[K_\nu(q) + 1/K_\nu(q) - 2\right]/4, \quad \) with

\[K_\nu(q) = \sqrt{\left(1 + \frac{g_{4,\nu}(q)}{\pi v_F} - g_{2,\nu}(q)/\pi v_F\right)^2 + \left(\frac{g_{2,\nu}(q)}{\pi v_F}\right)^2}. \quad (6)\]

Due to the interaction cut-off the momentum integral in Eq. (4) is regular in the ultraviolet limit. The \(q \to 0\) limits of the four functions \(v_\nu(q)\) and \(K_\nu(q)\) define the two velocities and two LL parameters \((v_\rho,v_\sigma,K_\rho,K_\sigma)\).

Roughly speaking the non-analyticities in spectral functions at low energies and small momenta are given by the behavior of \(G_+^<(x,t)\) at large arguments. To determine the non-analytic behavior of the momentum distribution \(n_+(k) = \int_{-\infty}^{\infty} dx e^{-ikx} iG_+^<(x,0)\) and the momentum integrated spectral function \(\rho_+^<(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} iG_+^<(0,t)/(2\pi)\) it is sufficient to discuss the behavior of \(G_+^<(x,t)\) along the \(x\) and \(vt\) axis. To emphasize the difference to the general case of non-vanishing \(x\) and \(vt\) we first focus on these two cases. With a little less mathematical rigor they have already been discussed several times. Using integration by parts in the first derivative of \(F(x,0)\) and \(F(0,t)\) we obtain the leading behavior \(F(x,0) \sim -\alpha \ln|x|\) and \(F(0,t) \sim -\alpha \ln|vt|\), with \(\alpha = g_{\rho}(0) + g_{\sigma}(0)\). In \(G_+^<(x,t)\) this leads to power law behavior along the \(x\) and \(vt\) axis with an exponent which only depends on the strength of the interaction at vanishing momentum and thus the LL parameter \(K_\rho\) and \(K_\sigma\) of the model at hand. This is in accordance with the LL phenomenology. From general theorems about Fourier transforms the by now well known LL behavior of the momentum distribution function follows for \(|k - k_F|/q_c \to 0\)

\[\frac{1}{2} - n_+(k) \sim \begin{cases} |k - k_F|^\alpha \text{sign}(k - k_F) & : \text{for } 0 < \alpha < 1 \\ (k - k_F) \ln |k - k_F| & : \text{for } \alpha = 1 \\ (k - k_F)^\alpha & : \text{for } \alpha > 1. \end{cases} \quad (7)\]

For \(\alpha > 1\) the leading behavior of \(n_+(k)\) is dominated by a linear term but a higher derivative still diverges at \(k_F\). Due to the different analytic properties of \(F(0,t)\) (analytic in the upper half of the complex \(t\) plane) and \(F(x,0)\) (not analytic in either the upper or the lower half of the complex \(x\) plane) \(\rho_+^<(\omega)\) is dominated by a non-analytic power law behavior even for \(\alpha > 1\). Thus anomalous dimensions \(\alpha > 1\) should in principle be observable in momentum integrated photoemission spectra. In the literature the mathematical reason for this important difference in the behavior of \(n_+(k)\) and \(\rho_+^<(\omega)\) has thus far not been properly pointed out. For \(\omega/(v_F q_c) \to 0\) we obtain

\[\rho_+^<(\omega) \sim \begin{cases} \Theta(-\omega)(-\omega)^\alpha & : \text{for } \alpha \notin \mathbb{N} \\ \Theta(-\omega)(-\omega)^\alpha \ln |\omega| & : \text{for } \alpha \in \mathbb{N}. \end{cases} \quad (8)\]

The prefactors and the range over which the leading behavior given in Eqs. (7) and (8) can be observed depend on the interaction at all \(q\). As a consequence numerically calculated curves for different interaction potentials \(g_{i,\rho}(q)\) but the same anomalous dimension might appear quite different. To find a more explicit form of the Green’s function the \(q\) integral in Eq. (4) has often been evaluated after replacing \(v_\nu(q) \to v_\nu(0), \gamma_\nu(q) \to \gamma_\nu(0)\) and multiplying the integrand by a factor \(\exp(-q\Lambda)\). The \(q\) integral can then be performed and one obtains

\[\left[G_+^>(x,t)\right]_A(x,t) = \frac{\pi}{x - vt - i0} \prod_{\nu=\rho,\sigma} \left[ x - vt - i0 \right]^{1/2} \times \left(\frac{\Lambda^2}{(x - vt - i\Lambda)(x + vt + i\Lambda)}\right)^{\gamma_\nu/2}. \quad (9)\]

For the Hamiltonian Eq. (4) there exists no special interaction potential so that this approximation becomes exact. As in the discussion of the general case below we will transform onto new variables \(s = x - ct\) and \(s' = x + ct\) with an arbitrary velocity \(c\) and discuss the behavior of \(\left[G_+^>(x,t)\right]_A(x[s,s'],t[s,s'])\) for large \(s\) with a fixed \(s'\) and vice versa and for different values of \(c\). For all velocities \(c\) but \(v_\rho\) and \(v_\sigma, G_+^>(x[s,s'],t[s,s'])\) falls off like \(s^{-1+\alpha}\) and \(s'^{-1+\alpha}\). This is the behavior we already found for \(vt = 0\) or \(x = 0\). For \(c = v_\rho\) the Green’s function falls off like
\[
\begin{align*}
[G^<_s]_A(x[s,s'],t[s,s']) &\sim s^{-(1+\gamma_\sigma+\gamma_\rho/2)}, \\
[G^<_s]_A(x[s,s'],t[s,s']) &\sim s^{-(1/2+\gamma_\rho+\gamma_\sigma/2)}. 
\end{align*}
\] (10)

The behavior of the Green's function for \(c = v_\sigma\) follows from Eqs. (10) and (11) by interchanging \(\gamma_\rho\) and \(\gamma_\sigma\). Within the above approximation the exponents of the asymptotic behavior for all \(c\) and thus for all directions within the \(x\)-\(v_F\) plane can be expressed in terms of \(K_\rho\) and \(K_\sigma\). The resulting spectral function displays power law singularities (bounded non-analyticities in case the \(\gamma_\nu\) are too large) at \(\omega = \pm v_\rho(k - k_F)\) for all \(\Lambda(-k + k_F) > 0\) (11).

Without using any approximations it has been shown that Eq. (1) gives the correct asymptotic behavior of \(G^<_s(x,t)\) for a box potential \(g_{i,n}(q) = g_{i,n}\Theta(q_c - |q|)\) (10). In this case the momentum range over which \(\rho^<_+(k,\omega)\) is dominated by two power law singularities at the charge and spin excitation energies which is limited to \(0 < (-k + k_F)/q_c < 1\). For other momenta further non-analyticities occur (10).

Next we will show that \([G^<_s]_A(x,t)\) does not give the correct asymptotic behavior for an arbitrary shape of the interaction. After transforming onto \(s\) and \(s'\) the Green's function is given by

\[
G^<_s(x[s,s'],t[s,s']) = [G^<_s]_A(0) \left(\frac{1}{2} - \frac{v_\rho(q)}{c}\right) + \frac{1}{2} \sum_{\nu = \rho, \sigma} \int_0^\infty dq \left\{ -i \frac{1}{2} \left(1 - \frac{v_\rho(q)}{c}\right) e^{-iqx} e^{-iqx} \right\}.
\] (12)

where \(\tilde{F}(s,s')\) follows from Eq. (1) and \((x,t) \rightarrow (s,s')\). As an example we will discuss the behavior of \(\tilde{F}(s,s')\) for large \(s'\) and fixed \(s\) in more detail and only present the results for the other case. We first take the derivative with respect to \(s'\). This gives

\[
\frac{d\tilde{F}(s,s')}{ds'} = \frac{1}{2} \sum_{\nu = \rho, \sigma} \int_0^\infty dq \left\{ -i \frac{1}{2} \left(1 - \frac{v_\rho(q)}{c}\right) e^{-iqx} e^{-iqx} \right\}
\]

\[
+ \frac{1}{2} \left(1 + \frac{v_F}{c}\right) e^{-iqx} e^{-iqx} + \frac{1}{2} \left(1 - \frac{v_F}{c}\right) e^{-iqx} e^{-iqx} + \frac{1}{2} \left(1 + \frac{v_F}{c}\right) e^{-iqx} e^{-iqx} + \frac{1}{2} \left(1 - \frac{v_F}{c}\right) e^{-iqx} e^{-iqx}.
\] (13)

To simplify the discussion we assume that \(v_\rho(q)\) are monotonic functions. Using integration by parts and the method of stationary phase in the asymptotic expansion of the integral we find for all velocities \(c\) but \(v_\rho(0)\), \(v_\sigma(0)\), and \(v_F\) that \(\tilde{F}(s,s')\) goes like \(-\ln(s')\) and thus

\[
\tilde{G}^<_s(s,s') \equiv G^<_s(x[s,s'],t[s,s']) \sim s^{-(1+\alpha)}. 
\] (14)

For large \(s\) and fixed \(s'\) we obtain the same behavior.

If \(c = v_\rho(0)\) the phase \(q \{1 - v_\rho(q)/v_\rho(0)\}\) in the first and fourth term of Eq. (13) becomes stationary at \(q = 0\). At the stationary point also the prefactor \(\sqrt{1 - v_\rho(q)/v_\rho(0)}\) vanishes and thus we have to generalize the method of stationary phase. The details of this generalization will be given elsewhere and here we will only present the results. The leading contribution of the fourth term of Eq. (13) to the large \(s'\) behavior is

\[
\frac{\gamma_\rho}{2} \left(1 + \frac{1}{2p_\rho + 2} + \frac{\gamma_\rho}{2p_\rho + 2}\right) \ln(s').
\] (19)
Fits of the data for \( s' > 10^3 \) reproduce the expected prefactors of the \( \ln(s') \) within a relative error of less than 0.02%. For increasing \( s \) we have to go to larger
\[ s' \] to find the asymptotic behavior Eq. (13). In an intermediate regime of \( s' \), which increases with increasing \( s \), \( \text{Re} \{ \tilde{F}(s,s') \} \) displays the behavior of the \( p_\rho = \infty \) case. This is illustrated in Fig. 2. It shows \( \text{Re} \{ \tilde{F}(s,s') \} \) as a function of \( s' \) on a log-linear scale for the above form of coupling functions, \( c = v_\rho(0) \) and different \( s \). The parameters are \( 2 g/(\pi v_F) = 5 \), i.e. \( v_\rho(0)/v_F = \sqrt{5} \), \( \gamma_\rho = 0.2144 \) and \( p_\rho = 2 \). A fit for \( s_0 = 100 \) again reproduces the expected behavior Eq. (13) with a very high accuracy. The data for \( s_0 = 10 \) only show this behavior for \( s'q_\rho > 10^5 \). For \( s_0 = 1000 \) a fit for \( 10^6 < s'q_\rho < 10^7 \) gives 0.3934 \( \ln(s') \) which is very close to 0.3929 \( \ln(s') \), the expected behavior for \( p_\rho = \infty \). Similar to the \( s_0 = 100 \) curve for arguments between \( 10^6 \) and \( 10^7 \), the curve for \( s_0 = 1000 \) shows a cross over at very large \( s'q_\rho \) and the prefactor of the logarithmic term is again given by the \( p_\rho = 2 \) value 0.1904.

For the approximated Green’s function Eq. (9) and the box potential the two-dimensional Fourier transformation which leads to the momentum resolved spectral function can be performed analytically [10]. Thus far we have not succeeded in calculating \( \rho_\rho^A(k,\omega) \) for an interaction with \( p_\rho < \infty \). From the Fourier transformation of the approximated Green’s function it is known, that the exponents of the algebraic decay along the special directions (\( c = v_\nu \)) determine the exponents of the non-analyticities at \( \omega = \pm v_\nu(k-k_F) \). As we have shown above the exponents of the algebraic decay of \( G^<_\nu(x,t) \) along the special directions are different from the exponents of \( \{ G^<_\nu \} \), thus we have no reason to believe that \( \rho_\rho^A(k,\omega) \) shows power law singularities with the same exponents as \( \{ \rho_\rho^A \} \). It is not even obvious that the exact spectral function shows power law singularities at all. Certainly we expect the two peak structure of spin and charge excitations, but it is not clear if, for any non-vanishing \( k-k_F \), these peaks are given by algebraic singularities. The extended region in which \( G^<_\nu(x,t) \) displays the asymptotic behavior of \( \{ G^<_\nu \} \) on the other hand indicates that the resulting spectral functions might look very similar at least for very small \( |k-k_F| \). For \( |k-k_F| \to 0 \) we expect to find growing regions in which the exact spectral function resembles the power law behavior of the approximation (with the same exponents as the approximation) up to energies very close to \( \pm v_\nu(k-k_F) \) but not exactly at these energies. A comparison of broadened spectral functions for a finite system, i.e. of spectral functions with no “real” algebraic singularities, for a box and Gaussian potential is presented in Ref. [10] and indeed shows a prominent similarity between both spectra. A more detailed comparison will be given elsewhere.

The results presented here are important for the interpretation of numerically calculated Green’s functions and spectra of microscopic models. They show that it is impossible to determine the LL parameters of the considered model from the asymptotic behavior of the Green’s function within the special directions. The numerical evaluation of the asymptotic behavior of the Green’s function gives on the other hand a possibility to confirm our predictions. Furthermore it should be possible to confirm the predictions by analyzing the finite size scaling of the weight of single peaks in the spectral function of a microscopic model of finite length. For energies close to but different from \( \pm v_\nu(k-k_F) \) we expect to find power law behavior with exponents given by the LL parameters as in the approximation. Not so for the scaling of the peaks exactly at \( \pm v_\nu(k-k_F) \). Our results have also consequences for the comparison of angle resolved and angle integrated spectral functions which have been measured by high resolution photoemission [9].

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