Pullback attraction in $H^1_0$ for semilinear heat equation in expanding domains

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Abstract

In this article, we consider the pullback attraction in $H^1_0$ of pullback attractor for semilinear heat equation with domains expanding in time. Firstly, we establish higher-order integrability of difference about variational solutions; then, we prove the continuity of variational solution in $H^1_0(Ω)$. As application of continuity, we obtain the pullback $D_λ$ attraction in $H^1_0$-norm.

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1 Introduction

Let $\{Ω_t\}_{t∈R}$ be a family of nonempty bounded open subsets of $R^N$ such that

$$Ω_s ⊂ Ω_t, \quad s < t. \quad (1)$$

Define

$$Q_{t,T} := \bigcup_{t∈(τ,T)} Ω_t × \{t\}, \quad \tilde{Q}_{t,T} := \bigcup_{t∈(τ,T)} Ω_T × \{t\} \quad \text{for any } T > τ \quad (2)$$

and

$$Q_τ := \bigcup_{t∈(τ,∞)} Ω_t × \{t\}, \quad ∀τ ∈ R, \quad Ω_t := \bigcup_{t∈(τ,∞)} ∂Ω_t × \{t\}, \quad ∀τ < T.$$

We consider the following initial boundary value problem with homogeneous Dirichlet boundary condition:

$$\begin{align*}
\frac{∂u}{∂τ} - Δu + g(u) &= f(t) \quad \text{in } Q_τ, \\
u &= 0 \quad \text{on } Σ_τ, \\
u(τ,x) &= u_τ(x), \quad x ∈ Ω_τ,
\end{align*}$$  

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where \( u_\tau : \Omega_\tau \rightarrow \mathbb{R} \) and \( f : Q_\tau \rightarrow \mathbb{R} \) are given for \( \tau \in \mathbb{R} \), and \( g \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies the conditions: there exist nonnegative constants \( \alpha_1, \alpha_2, \beta, l, \) and \( p \geq 2 \) such that
\[
-\beta + \alpha_1|s|^p \leq g(s) \leq \beta + \alpha_2|s|^p, \quad \forall s \in \mathbb{R}
\]
(4)
and
\[
g'(s) \geq -l, \quad \forall s \in \mathbb{R}.
\]
(5)
For later observe that there exist nonnegative constants \( \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta} \) such that
\[
-\tilde{\beta} + \tilde{\alpha}_1|s|^p \leq G(s) \leq \tilde{\beta} + \tilde{\alpha}_2|s|^p, \quad \forall s \in \mathbb{R},
\]
(6)
where
\[
G(s) := \int_0^s g(r) \, dr.
\]
For each \( T > \tau \), consider the auxiliary problem
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + g(u) &= f(t) \quad \text{in } Q_{\tau, T}, \\
u &= 0 \quad \text{on } \Sigma_{\tau, T}, \\
u(\tau, x) &= u_\tau(x), \quad x \in \Omega_\tau,
\end{aligned}
\]
(7)
where \( u_\tau : \Omega_\tau \rightarrow \mathbb{R} \) for \( \tau \in \mathbb{R} \), \( g \) satisfies (4)–(5) and \( f \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega_\tau)) \).

The issue of non-cylindrical region usually refers to the problem that spatial region changes with time, also known as the problem of variable region. Variable region problems are applied widely in physics, chemistry, and cybernetics, so have been focused on relevant experts. Compared with the invariant regional system, the study of variable regional problem can vividly describe the actual phenomena. In addition, the problem defined in the variable region is essentially non-autonomous, so the discussion of variable regional problem adds vitality to the development and perfection of theory of non-autonomous system.

Based on the actual requirement, many mathematical researchers began to focus on the variable region problems, for example, see [1, 4–8, 15, 16] and so on. Recently, the existence and uniqueness of variational solution of system (3) have been considered in [6] with monotonic increase region, and then \( L^2, L^2 \) pullback \( \mathcal{D}_{\lambda_1} \) attractor has been established. In 2009, by means of differ-morphism, a similar conclusion of system (3) was obtained in [7]. Later, in [11], by the solution orbit being shifted via a fixed complete orbit, the authors obtained the pullback \( \mathcal{D}_{\lambda_1} \) attraction of \( L^2 \) pullback attractor in higher-order integrable spaces.

The continuity of solution plays an important role in the study of dynamic systems, especially in pullback attraction, fractal dimension, and so on. For the invariant region, the continuity of strong solution with respect to the initial data in \( H^1_0(\Omega) \) was considered for the space dimension \( N \leq 2 \), and the nonlinear term exponent \( p \geq 2 \), but \( p \leq 4 \) for \( N = 3 \) was required. For an autonomous system, in order to obtain continuity in \( H^1_0(\Omega) \) and
of non-autonomous system was studied in [10]. However, for a long time, the continuity of solution in $H^1_0(O_t)$ with respect to initial data has still been an open problem. Until 2008, when the nonlinear term $f$ of autonomous system satisfying (4) and (5) was introduced, the author obtained the uniform boundedness of $tu(t)$ by differentiating equation about time $t$, then considered the continuity of solution about initial data, see details in [14]. However, for a non-autonomous system, we cannot differentiate equation, so the method in [14] cannot be shifted to solve non-autonomous problems. In order to overcome the difficulties deriving from the non-autonomous character, in 2015, for the case of random equation, [2] discussed the continuity in $H^1_0(O_t)$ (Theorem 0.1) by studying the higher-order integrability of solutions difference near the initial time. Then, a natural problem arose: Does it still hold for variable domains? As far as the author knows, the continuity of solution in $H^1_0(O_t)$ about initial data is still unknown.

Enlightened by the above, we consider the continuity of variational solution in $H^1_0(O_t)$ with respect to initial data when the region of system (3) is monotonically increasing. As an application of continuity, we establish the pullback $\mathcal{P}_{\lambda_1}$ attraction in $H^1_0(O_t)$ for any $t \in \mathbb{R}$.

This paper is organized as follows. In Sect. 2, we recall some concepts and related results about variational solution. In Sect. 3, we prove higher-order integrability of difference of variational solutions near initial data (Theorem 3.3) and the continuity in $H^1_0(O_t)$ (Theorem 3.4), then establish the pullback $\mathcal{P}_{\lambda_1}$ attraction in $H^1_0(O_t)$ (Theorem 3.5).

## 2 Variational solutions

For each $t \in \mathbb{R}$, denoted by $(\cdot, \cdot)_t$ and $|\cdot|_t$ the usual inner product and related norm in $L^2(O_t)$ and by $((\cdot, \cdot))_t$ and $\|\cdot\|_t$ the usual gradient inner product and associated norm in $H^1_0(O_t)$. The usual duality product between $H^1_0(O_t)$ and $H^{-1}(O_t)$ is denoted by $(\cdot, \cdot)_t$. And $(\cdot, \cdot)$ and $\|\cdot\|_{L_p(O_t)}$ represent the duality product between $L^p(O_t)$ and $L^q(O_t)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and the associated norm.

We consider a process $U$ on a Banach space $X$, i.e., a family $\{U(t, \tau); -\infty < \tau \leq t < +\infty\}$ of continuous mappings $U(t, \tau): X \rightarrow X$ such that

$$U(t, \tau)x = x \quad \text{and} \quad U(t, \tau) = U(t, s)U(s, \tau) \quad \text{for all} \quad s \leq t \quad \text{and} \quad x \in X.$$ 

Suppose that $\mathcal{D}$ is a nonempty class of parameterized sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of $X$.

**Definition 2.1** ([3]) The family $\mathcal{A} = \{\mathcal{A}(t) : \mathcal{A}(t) \in \mathcal{P}(X), t \in \mathbb{R}\}$ is said to be a pullback $\mathcal{D}$-attractor for the process $U(\cdot, \cdot)$ if

1. $\mathcal{A}(t)$ is compact in $X$ for all $t \in \mathbb{R}$;
2. $\mathcal{A}$ is pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t, \tau)D(\tau), \mathcal{A}(t)) = 0 \quad \text{for all} \quad \hat{D} \in \mathcal{D} \quad \text{and all} \quad t \in \mathbb{R};$$

3. $\mathcal{A}$ is invariant, i.e.,

$$U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t) \quad \text{for any} \quad -\infty < \tau \leq t < \infty.$$
Fix $T > \tau$, for each $t \in [\tau, T]$ denoted by

$$H^1_0(O_\tau)^\perp = \{ v \in H^1_0(O_T): ((v, w)_T = 0, \forall w \in H^1_0(O_\tau) \}$$

is the orthogonal subspace of $H^1_0(O_\tau)$ with respect to the inner product in $H^1_0(O_T)$. We may identify $w$ with its null-expansion and by $P(t) \in L(H^1_0(O_T))$ the orthogonal projection operator from $H^1_0(O_T)$ to $H^1_0(O_\tau)^\perp$, which is defined as

$$P(t)v \in H^1_0(O_\tau)^\perp, \quad v - P(t)v \in H^1_0(O_\tau)$$

for each $v \in H^1_0(O_T)$. Consider the family $p(t; \cdot, \cdot)$ of symmetric bilinear forms on $H^1_0(O_T)$ defined by

$$p(t; v, w) := \left( (P(t)v, w) \right)_T, \quad \forall v, w \in H^1_0(O_T), \forall t \geq \tau.$$

It can be proved that the mapping $[\tau, +\infty) \ni t \mapsto p(t; v, w)$ is measurable for all $v, w \in H^1_0(O_T)$, for each integer $k \geq 1$ and $t \geq \tau$, define

$$p_k(t; v, w) := k \int_0^1 p(t + r; v, w) \, dr \quad \forall v, w \in H^1_0(O_T), \forall t \geq \tau,$$

and denote by $P_k(t) \in L(H^1_0(O_T))$ the associated operator defined by

$$\left( (P_k(t)v, w) \right) := p_k(t; v, w), \quad \forall v, w \in H^1_0(O_T), \forall t \geq \tau.$$

Then we know from the above that, for any integers $1 \leq h \leq k$, any $t \geq \tau$, and any $v, w \in H^1_0(O_T),$

$$0 \leq p_h(t; v, v) \leq p_k(t; v, v) \leq p(t; v, v) = \left\| P(t)v \right\|^2_T \leq \| v \|^2_T.$$  \hfill (8)

For each $T > \tau$, denote

$$U_{\tau, T} := \{ \phi \in L^2(\tau, T; H^1_0(O_\tau)) \cap L^p(\tau, T; L^p(O_\tau)), \phi' \in L^2(\tau, T; L^2(O_\tau)), \phi(T) = 0, \phi(t) \in H^1_0(O_\tau) \text{ for a.e. } t \in (\tau, T) \}.$$

**Definition 2.2** A variational solution of equation (7) is a function $u$ such that

- (C1) $u \in L^2(\tau, T; H^1_0(O_\tau)) \cap L^p(\tau, T; L^p(O_\tau));$
- (C2) $u(t) \in H^1_0(O_\tau)$ a.e. $t \in (\tau, T);$
- (C3) for all $\phi \in U_{\tau, T},$

$$\int_{\tau}^T \left[ -(u(t), \phi'(t))_T + ((u(t), \phi(t)))_T + (g(u(t), \phi(t)), \phi(t))_T \right] \, dt = \int_\tau^T (f(t), \phi(t))_T \, dt;$$

- (C4) $\lim_{\tau \to t} \frac{1}{T - \tau} \int_\tau^T \| u(s) - u(\tau) \|^2_T \, ds = 0.$

The existence and uniqueness of variational solution for equation (7) have been derived as follows.
Theorem 2.3 ([6]) Suppose that (1), (2), (4), and (5) hold; for \( f \in L^2(\tau; T; L^2(\Omega_T)) \) and \( u_0 \in L^2(\Omega_T) \), there exists a unique variational solution \( u \in L^2(\tau; H^1_0(\Omega_T)) \cap L^p(\tau; L^p(\Omega_T)) \) of equation (7), which satisfies energy equality a.e. \( t \in [\tau, T] \), that is,

\[
|u(t)|^2_T + 2 \int_\tau^T \|u(s)\|_{\tau}^2 \, ds + 2 \int_\tau^T \langle g(u(s)), u(s) \rangle_T \, ds = |u_{\tau}|^2_T + 2 \int_\tau^T \langle f(s), u(s) \rangle_T \, ds \tag{9}
\]

holds for a.e. \( t \in [\tau, T] \). In addition, \( u \in C([\tau, T]; L^2(\Omega_T)) \) and satisfies the energy equality for all \( t \in [\tau, T] \). Moreover, if \( u_\tau \in H^1_0(\Omega_T) \cap L^p(\Omega_T) \), then \( u \) also satisfies

\[
u \in L^\infty(\tau, T; H^1_0(\Omega_T)) \cap L^\infty(\tau, T; L^p(\Omega_T)), \quad u' \in L^2(\tau, T; L^2(\Omega_T)).
\]

Remark 2.4 ([6]) If \( T_2 > T_1 > \tau \) and \( u \) is a variational solution of (7) with \( T = T_2 \), then the restriction of \( u \) to \( Q_{\tau, T_1} \) is a variational solution of (7) with \( T = T_1 \).

We can also obtain the following result.

Theorem 2.5 Under the assumptions of Theorem 2.3, if \( u_\tau \in H^1_0(\Omega_T) \cap L^p(\Omega_T) \), then \( u \in L^2(\tau, T; H^2(\Omega_T)) \).

Proof One can take an orthonormal Hilbert basis \( \{\omega_j\} \) of \( L^2 \) and \( H^1_0 \) formed by the elements of \( H^1_0 \cap L^p \cap H^2 \) such that the vector space generated by \( \{\omega_j\} \) is dense in \( H^1_0 \) and \( L^p \). Then one takes a sequence \( \{u_{\tau n}\} \) such that \( u_{\tau n} \to u_\tau \) in \( H^1_0(\Omega_T) \) with \( u_{\tau n} \) in the vector space spanned by the \( n \) first \( \omega_j \).

Consider the equality

\[
\left( \frac{\partial u_{\tau n}(t)}{\partial t}, \omega_j \right)_T + \langle A_k(t)u_{\tau n}(t), \omega_j \rangle_T + \langle g(u_{\tau n}(t)), \omega_j \rangle_T = \langle f(t), \omega_j \rangle_T \tag{10}
\]

for a.e. \( t \in [\tau, T] \).

Multiplying (10) by \( \lambda_j \rangle^2_{\tau n}(t) \) and summing from \( j = 1 \) to \( n \), we know that

\[
\left( \frac{\partial u_{\tau n}(t)}{\partial t}, -\Delta u_{\tau n}(t) \right)_T + \langle A_k(t)u_{\tau n}, -\Delta u_{\tau n}(t) \rangle_T + \langle g(u_{\tau n}(t)), -\Delta u_{\tau n}(t) \rangle_T = \langle f(t), -\Delta u_{\tau n}(t) \rangle_T
\]

for a.e. \( t \in [\tau, T] \).

According to (5), it follows

\[
\int_{\Omega_T} g(u_{\tau n}(t))(\Delta u_{\tau n}(t)) \, dx = \int_{\Omega_T} g'(u_{\tau n}(t))|\nabla u_{\tau n}(t)|^2 \, dx \geq -l|\nabla u_{\tau n}(t)|^2_T, \tag{11}
\]

Combining (11) and Hölder’s inequality, we have

\[
\frac{d}{dt}|\nabla u_{\tau n}(t)|^2_T + |\Delta u_{\tau n}(t)|^2_T + 2k(\langle P_k(t)\nabla u_{\tau n}(t), \nabla u_{\tau n}(t) \rangle_T) \leq 2l \int_{\Omega_T} |\nabla u_{\tau n}(t)|^2 \, dx + |f(t)|^2_T
\]
a.e. \( t \in (\tau, T) \), integrate the inequality above from \( \tau \) to \( t \),

\[
\| \nabla u_{kn}(t) \|_{T}^2 + \int_{\tau}^{t} | \Delta u_{kn}(s) \|_{T}^2 ds + 2k \int_{\tau}^{t} \left( (P_{k}(s) \nabla u_{kn}(s), \nabla u_{kn}(s)) \right)_{T} ds
\]

\[
\leq 2l \int_{\tau}^{t} | \nabla u_{kn}(s) \|_{T}^2 ds + \int_{\tau}^{t} | f(s) \|_{T}^2 ds + | \nabla u_{kn}(\tau) \|_{T}^2 \quad \text{a.e.} \ t \in (\tau, T),
\]

(12)

so, we have

\[
| \nabla u_{kn}(t) \|_{T}^2 \leq ce^{2(l-t)} \left( | \nabla u_{kn}(\tau) \|_{T}^2 + \int_{\tau}^{t} | f(s) \|_{T}^2 ds \right)
\]

(13)

and

\[
\int_{\tau}^{t} | \nabla u_{kn}(s) \|_{T}^2 ds \leq ce^{2(l-t)} \left( | \nabla u_{kn}(\tau) \|_{T}^2 + \int_{\tau}^{t} | f(s) \|_{T}^2 ds \right).
\]

(14)

Note (8), we have

\[
\int_{\tau}^{t} \left( (P_{k}(s) \nabla u_{kn}(s), \nabla u_{kn}(s)) \right)_{T} ds \geq 0.
\]

By (12)–(14), we obtain

\[
| \nabla u_{kn}(t) \|_{T}^2 + \int_{\tau}^{t} | \Delta u_{kn}(s) \|_{T}^2 ds \leq ce^{2(l-t)} \left( | \nabla u_{kn}(\tau) \|_{T}^2 + \int_{\tau}^{t} | f(s) \|_{T}^2 ds \right)
\]

for a.e. \( t \in (\tau, T) \), where \( c \) may be different from line to line, recall that \( u_{kn}(\tau) = u_{\tau n}, u_{\tau n} \to u_{\tau} \) in \( H_{0}^{1}(\Omega_{\tau}) \) as \( n \to \infty \). So, \( \{u_{kn}\} \) is bounded in \( L^{2}(\tau, T; H^{2}(\Omega_{T})) \cap L^{\infty}(\tau, T; H_{0}^{1}(\Omega_{T})) \), there exists a subsequence, denoted still by \( \{u_{kn}\} \), such that as \( n \to \infty \)

\[
\begin{align*}
& u_{kn} \to u_{k} \quad \text{weakly in} \ L^{2}(\tau, T; H^{2}(\Omega_{T})), \\
& u_{kn} \rightharpoonup u_{k} \quad \text{weakly star in} \ L^{\infty}(\tau, T; H_{0}^{1}(\Omega_{T})),
\end{align*}
\]

therefore,

\[
| \nabla u_{k}(t) \|_{T}^2 + \int_{\tau}^{t} | \Delta u_{k}(s) \|_{T}^2 ds \leq ce^{2(l-t)} \left( \| u_{\tau} \|_{T}^2 + \int_{\tau}^{t} | f(s) \|_{T}^2 ds \right),
\]

and also \( \{u_{k}\} \) is bounded in \( L^{2}(\tau, T; H^{2}(\Omega_{T})) \cap L^{\infty}(\tau, T; H_{0}^{1}(\Omega_{T})) \). There exists a subsequence, denoted still by \( \{u_{k}\} \), such that it is convergent weakly, convergent weakly star to the uniqueness variational solution \( u \) of (7) in \( L^{2}(\tau, T; H^{2}(\Omega_{T})) \cap L^{\infty}(\tau, T; H_{0}^{1}(\Omega_{T})) \) as \( k \to \infty \). \( \Box \)

**Theorem 2.6** ([15]) Suppose that \( u_{\tau} \in L^{\infty}(\Omega_{\tau}) \cap H_{0}^{1}(\Omega_{\tau}), f \in L^{\infty}(\bar{\Omega}_{T}, \tau) \) hold and \( g \) satisfies (4). Then there exists a positive constant \( K \) which depends on \( \| u_{\tau} \|_{L^{\infty}(\Omega_{\tau})}, \| f \|_{L^{\infty}(\bar{\Omega}_{T}, \tau)}, \beta \) and \( \alpha_{1} \) such that the variational solution \( u \) of (7) satisfies

\[
\| u \|_{L^{\infty}(\bar{\Omega}_{T}, \tau)} \leq K.
\]
3 Pullback $\mathcal{D}_1$, attraction in $H^1_0(\Omega)$

By Theorem 2.3 and Remark 2.4, we know that, for any $\tau \in \mathbb{R}$ and any $u_\tau \in L^2(\Omega_\tau)$, there exists a unique variational solution $u(\cdot; \tau, u_\tau)$ satisfying energy equality for a.e. $t \in (\tau, T)$ and any $T > \tau$. Moreover, $u \in C([\tau, T]; L^2(\Omega_T))$ satisfying energy equality for all $t \in (\tau, T)$ with any $T > \tau$.

Define

$$U(t, \tau)u_\tau := u(t; \tau, u_\tau), \quad -\infty < \tau \leq t < \infty, u_\tau \in L^2(\Omega_\tau). \quad (15)$$

By the uniqueness of variational solution for (7) and $u \in C([\tau, T]; L^2(\Omega_T))$ satisfying energy equality for all $t \in (\tau, T)$ with any $T > \tau$, we know $U(\cdot; \cdot)$ defined by (15) is a process for the family of Hilbert spaces $\{L^2(\Omega_t), t \in \mathbb{R}\}$.

To obtain main results, the following lemma is necessary.

Lemma 3.1 ([11]) For any $k > 0$ and any $\phi \in H^1_0(\Omega_\tau) \cap L^\infty(\Omega_\tau)$, the following equality holds:

$$\int_{\Omega_\tau} \nabla \phi \cdot \nabla (|\phi|^k \phi) \, dx = (k + 1) \left( \frac{2}{k + 2} \right)^2 \int_{\Omega_\tau} |\nabla |\phi|^{k+1} |^2 \, dx, \quad (16)$$

where $\cdot$ stands for the usual inner product in $\mathbb{R}^N$.

In the following, suppose

$$f \in L^2_{\text{loc}}(\mathbb{R}^{N+1}) \quad \text{and} \quad u_\tau, v_\tau \in L^2(\Omega_\tau). \quad (17)$$

Due to the density of $L^\infty(\Omega_\tau)$ in $L^2(\Omega_\tau)$, there exist sequences $\{u_{\tau m}\}, \{v_{\tau m}\} \subset L^\infty(\Omega_\tau), \{f_m\} \subset L^\infty(\bar{\Omega}_\tau, \tau)$ such that

$$u_{\tau m} \rightarrow u_\tau, \quad v_{\tau m} \rightarrow v_\tau \quad \text{in} \quad L^2(\Omega_\tau), m \rightarrow +\infty, \quad (18)$$

$$f_m \rightarrow f \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^{N+1}), m \rightarrow +\infty$$

and it can be done that, for each $m = 1, 2, \ldots,$

$$|u_{\tau m}|^2 \leq 2|u_\tau|^2 + 1, \quad |v_{\tau m}|^2 \leq 2|v_\tau|^2 + 1. \quad (19)$$

Based on the above, applying the interpolation inequality to estimate the $L^{2n-2}$-norm of approximation solution, we can establish the higher-order integrability near initial time $\tau$ for approximation solution as follows.

Theorem 3.2 Suppose that (1), (2), (4), and (5) hold, $f \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega_\tau)), u_\tau, v_\tau \in L^2(\Omega_\tau)$.

Then, for any $T \geq \tau$, any $k = 1, 2, \ldots$, there exists a positive constant $M_k = M(T - \tau, k, N, l, |u_\tau|_{L^2(\Omega_\tau)}, |v_\tau|_{L^2(\Omega_\tau)})$ such that

$$(t - \tau)^{N-k} \| (t - \tau)^{k} w_m(t) \|_{L^2(\mathbb{R}^{N+1})} \leq M_k \quad \text{for all} \ t \in [\tau, T] \quad (A_k)$$
and

\[
\int_{\tau}^{T} \left( \int_{\Omega_T} |(t - \tau)^{b_k} \cdot w_m(t)|^{\frac{2N}{N - 2}} dx \right) \frac{N - 2}{N} dt \leq M_k,
\]

where \( w_m(t) = u_m(t) - v_m(t) = U(t, \tau)u_{\tau m} - U(t, \tau)v_{\tau m}, \)

\[
b_1 = 1 + \frac{1}{2}, \quad b_2 = 1 + \frac{1}{2} + 1 \quad \text{and} \quad b_{k+1} = b_k + \frac{1 + \frac{N}{N - 2}}{2 \left( \frac{N}{N - 2} \right)^{k+1}} \quad \text{for} \ k = 2, 3, \ldots
\]

and all constants \( M_k (k = 1, 2, \ldots) \) are independent of \( m \).

**Proof** For any fixed \( \tau \in (-\infty, T] \), denote

\[
w_m(t) := u_m(t) - v_m(t), \quad \tau \leq t \leq T,
\]

where \( u_m(t), v_m(t) \) are the variational solutions of equation (7) corresponding to the data \((u_{\tau m}, f_m), (v_{\tau m}, f_m)\) satisfying (18) respectively. By Theorem 2.3 and Theorem 2.6, we know

\[
w_m \in L^2 (\tau, T; H^1_0(\Omega_T)) \cap L^\infty(Q_T),
\]

and

\[
\int_{\tau}^{T} \left[ -\left( w_m(t), \phi'(t) \right)_T + \left( \left( w_m(t), \phi(t) \right)_T + \left( g(u_m(t)) - g(v_m(t)), \phi(t) \right)_T \right) \right] dt = 0
\]

for any \( \phi \in \mathcal{U}_{\tau, T} \).

For any \( \theta > 0 \), we have

\[
|w|^\theta w_m \in L^2 (\tau, T; H^1_0(\Omega_T)) \cap L^\infty(Q_T),
\]

and choose any \( \eta \in C^1_c(\tau, T) \) to get

\[
\left\{ \begin{array}{l}
\eta |w|^\theta w_m \in L^2 (\tau, T; H^1_0(\Omega_T)) \cap L^\infty(Q_T),
\frac{d}{dt} (\eta |w|^\theta w_m) \in L^2 (\tau, T; L^2(\Omega)),
\eta(T) |w|^\theta w_m (T) = \eta(\tau) |w|^\theta w_m (\tau) = 0,
\eta(t) |w|^\theta w_m (t) \in H^1_0(\Omega_t) \quad \text{a.e.} \quad t \in (\tau, T).
\end{array} \right.
\]

Hence, we can choose \( \eta |w|^\theta w_m \) as a test function to have

\[
\int_{\tau}^{T} \left[ -\left( w_m(t), (\eta(t) |w|^\theta w_m(t))' \right)_T + \left( \left( w_m(t), \eta(t) |w|^\theta w_m(t) \right)_T \right) + \left( g(u_m(t)) - g(v_m(t)), \eta(t) |w|^\theta w_m(t) \right)_T \right] dt = 0,
\]

note that

\[
\int_{\tau}^{T} \left( w_m'(t), |w_m(t)|^\theta w_m(t) \right)_T + \left( \left( w_m(t), |w_m(t)|^\theta w_m(t) \right)_T \right)
\]
\[ + \left( g(u_m(t)) - g(v_m(t)), \|w_m(t)\|_{L^2(\mathcal{O}_T)}^\theta \right) \eta(t) \, dt = 0 \]

for any \( \eta \in C^1_c(\tau, T) \) holds. Therefore, for a.e. \( t \in (\tau, T) \),

\[
\begin{align*}
&\left( w'_m(t), \|w_m(t)\|_{L^2(\mathcal{O}_T)}^\theta \right) + \left( (w_m(t), \|w_m(t)\|_{L^2(\mathcal{O}_T)}^\theta \right) \\
&\quad + \left( g(u_m(t)) - g(v_m(t)), \|w_m(t)\|_{L^2(\mathcal{O}_T)}^\theta \right) = 0.
\end{align*}
\]

By (5), for a.e. \( t \in (\tau, T) \), we have

\[
\frac{1}{\theta + 2} \frac{d}{dt} \|w_m(t)\|_{L^{\theta+2}(\mathcal{O}_T)}^{\theta+2} - \frac{1}{\theta + 2} \int_{\mathcal{O}_T} |\gamma w_m(t)|^{\theta+2} \, dS \\
+ (1 + \frac{2}{\theta + 2}) \int_{\mathcal{O}_T} |\nabla w_m(t)|^{1+\theta} \, dx \leq l \|w_m(t)\|_{L^{\theta+2}(\mathcal{O}_T)}^{\theta+2}
\]

thanks to \( w_m(t) \in H^1_0(\mathcal{O}_T) \) for a.e. \( t \in (\tau, T) \), so \( \gamma w_m(t) = 0 \) for a.e. \( t \in (\tau, T) \), it follows

\[
\frac{1}{\theta + 2} \frac{d}{dt} \|w_m(t)\|_{L^{\theta+2}(\mathcal{O}_T)}^{\theta+2} + (1 + \frac{2}{\theta + 2}) \int_{\mathcal{O}_T} |\nabla w_m(t)|^{1+\theta} \, dx \\
\leq l \|w_m(t)\|_{L^{\theta+2}(\mathcal{O}_T)}^{\theta+2}.
\]

In the following, we separate our proof into two steps.

**Step 1.** \( k = 1 \)

Firstly, taking \( \phi = w_m \) in (22), from the definition of variational solution and (5), we obtain that

\[
\frac{1}{2} \frac{d}{dt} \|w_m\|_{L^2(\mathcal{O}_T)}^2 + \int_{\mathcal{O}_T} |\nabla w_m(t)|^2 \, dx = -\int_{\mathcal{O}_T} (g(u_m) - g(v_m)) w_m \, dx \\
\leq l \|w_m(t)\|_{L^2(\mathcal{O}_T)}^2 \quad \text{a.e. } t \in (\tau, T),
\]

which implies that

\[
\|w_m(t)\|_{L^2(\mathcal{O}_T)}^2 \leq e^{2(t-\tau)} \|w_m(\tau)\|_{L^2(\mathcal{O}_T)}^2,
\]

and then,

\[
\int_\tau^T |\nabla w_m(t)|_{L^2(\mathcal{O}_T)}^2 \, dt \leq l \int_\tau^T \|w_m(s)\|_{L^2(\mathcal{O}_T)}^2 \, ds + \frac{1}{2} \|w_m(\tau)\|_{L^2(\mathcal{O}_T)}^2 \leq \frac{1}{2} \left( e^{2(T-\tau)} + 1 \right) \|w_m(\tau)\|_{L^2(\mathcal{O}_T)}^2.
\]

Consequently, combining with the embedding

\[
\left( \int_{\mathcal{O}_T} |v|^{2N/(N-2)} \, dx \right)^{\frac{N-2}{N}} \leq c_{N,T} \int_{\mathcal{O}_T} |\nabla v|^2 \, dx, \quad \forall v \in H^1(\mathcal{O}_T), \forall s \in [\tau, T],
\]

we can deduce that

\[
\int_\tau^T \left( \int_{\mathcal{O}_T} |(t-\tau)^{\beta_1} w_m(t)|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \, dt
\]
\[ (t - \tau)^{2b_1} \frac{c_{N,T}}{2} \left( e^{2(T-\tau)} + 1 \right) |w_m(t)|_T^2, \]  
(25)

Note that here the embedding constant \( c_{N,T} \) in (24) depends only on the domain \( O_T \).

Secondly, take \( \theta = \frac{N}{N-2} - 2 \) in (23), by Lemma 3.1, we have that

\[
\frac{1}{2} \left( \frac{N-2}{N} \right) \frac{d}{dt} \| w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)} + c_i \int_{O_T} |\nabla w_m(t)|_{L^{\frac{N}{N-2}}}^2 \, dx = \leq l \| w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)} \quad \text{a.e. } t \in (\tau, T).
\]

In the following we denote by \( c, c_i \) (\( i = 1, 2, \ldots \)) the constants which depend only on \( N, T - \tau \), and \( l \), which may differ from line to line. Then the above inequality can be written as

\[
\frac{d}{dt} \| w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)} + c_i \int_{O_T} |\nabla w_m(t)|_{L^{\frac{N}{N-2}}}^2 \, dx \leq c_2 \| w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)},
\]

and by multiplying both sides with \((t - \tau)^{\frac{2N}{N-2}}\), we obtain that

\[
\frac{d}{dt} \| (t - \tau)^{b_1} w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)} + c_i \int_{O_T} |\nabla (t - \tau)^{b_1} w_m(t)|_{L^{\frac{N}{N-2}}}^2 \, dx \leq c_2 \| (t - \tau)^{b_1} w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)},
\]

(26)

here \( b_1 = 1 + \frac{1}{2} \).

One direct result of (27) is that

\[
(t - \tau) \frac{d}{dt} \| (t - \tau)^{b_1} w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)} \leq c \| (t - \tau)^{b_1} w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)},
\]

and so

\[
(t - \tau) \frac{d}{dt} \| (t - \tau)^{b_1} w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)}^2 \leq c \frac{N-2}{N} \| (t - \tau)^{b_1} w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)}^2.
\]

Consequently, for any \( t \in [\tau, T] \), integrating (28) over \([\tau, t] \), we obtain that

\[
(t - \tau) \int_{\tau}^{t} \| (s - \tau)^{b_1} w_m(s) \|_{L^{\frac{2N}{N-2}}(O_T)}^2 \, ds \leq c \| w_m(\tau) \|_{T}^2 \quad \text{by (25),}
\]

hence,

\[
(t - \tau)^{\frac{2N}{N-2}} \| (t - \tau)^{b_1} w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)} \leq c \| w_m(\tau) \|_{T}^{\frac{2N}{N-2}} \quad \text{for any } t \in [\tau, T].
\]

(29)

Then, multiplying (27) by \((t - \tau)^{\frac{2N}{N-2}}\), we obtain that for a.e. \( t \in (\tau, T) \),

\[
(t - \tau)^{\frac{2N}{N-2}} \frac{d}{dt} \| (t - \tau)^{b_1} w_m(t) \|_{L^{\frac{2N}{N-2}}(O_T)} + c_i \int_{O_T} |\nabla (t - \tau)^{b_1} w_m(t)|_{L^{\frac{N}{N-2}}}^2 \, dx \]

Let \( c(t - \tau)^{\frac{2N}{N-2}} \| (t - \tau)^{b_1} w_m(t) \|_{L^{\frac{2N}{N-2}}(\Omega_T)} \leq c \| w_m(\tau) \|_{L^{\frac{2N}{N-2}}(\Omega_T)}^{\frac{2N}{N-2}}. \)

Integrating the above inequality over \([\tau, T]\) with respect to \( t\), we obtain that

\[
\int_{\tau}^{T} \int_{\Omega_T} \| (t - \tau)^{b_2} w_m(t) \|_{L^{\frac{2N}{N-2}}}^2 \, dx \, dt \leq c \| w_m(\tau) \|_{L^{\frac{2N}{N-2}}(\Omega_T)}^{\frac{2N}{N}}. 
\]

Consequently, applying embedding (24) again, we can deduce that

\[
\int_{\tau}^{T} \left( \int_{\Omega_T} |(t - \tau)^{b_2} w_m(t)|_{L^{\frac{2N}{N-2}}}^2 \, dx \right)^{\frac{N-2}{N}} \, dt \leq c_{N,T} c_1 \| w_m(\tau) \|_{L^{\frac{2N}{N-2}}(\Omega_T)}^{\frac{2N}{N}}. \tag{30} 
\]

Therefore, noticing (18) and (19), from (29) and (30) we know that there is a positive constant \( M_1\), which depends only on \( N, \tau, T, I, |u_r|, \tau, |i_r|\), such that (A1) and (B1) hold.

Step 2. Assume (A_k) and (B_k) hold for \( k \geq 1\), in the following, we will show that (A_{k+1}) and (B_{k+1}) hold.

Take \( \theta = 2(N - \frac{N}{N-2}) k^{1-1} - 2 \) in (23), we obtain that

\[
\frac{d}{dt}\| w_m(t) \|_{L^{\frac{2N}{N-2}}}^{2(N - \frac{N}{N-2}) k^{1+1}} + c \int_{\Omega_T} \| \nabla w_m(t) \|_{L^{\frac{2N}{N-2}}}^{\frac{N}{N-2} k^{1+1}} \, dx 
\leq c_1 \| w_m(t) \|_{L^{\frac{2N}{N-2}}}^{2(N - \frac{N}{N-2}) k^{1+1}} \quad \text{a.e. } t \in (\tau, T). \tag{31} 
\]

Multiplying both sides of (31) with \((t - \tau)^{2(N - \frac{N}{N-2}) k^{1+1} b_{k+1}}, \) we deduce that

\[
\frac{d}{dt} \left( (t - \tau)^{2(N - \frac{N}{N-2}) k^{1+1} b_{k+1}} \| w_m \|_{L^{\frac{2N}{N-2}}}^{2(N - \frac{N}{N-2}) k^{1+1}} (\Omega_T) \right) + c \int_{\Omega_T} \| \nabla (t - \tau)^{b_{k+1} \cdot w_m(t)} \|_{L^{\frac{2N}{N-2}}}^{\frac{N}{N-2} k^{1+1}} \, dx 
\leq c_1 \| (t - \tau)^{b_{k+1} \cdot w_m(t)} \|_{L^{\frac{2N}{N-2}}}^{2(N - \frac{N}{N-2}) k^{1+1}} (\Omega_T) + c_2 \| (t - \tau)^{2(N - \frac{N}{N-2}) k^{1+1} b_{k+1+1}} \| w_m \|_{L^{\frac{2N}{N-2}}}^{2(N - \frac{N}{N-2}) k^{1+1}} (\Omega_T), 
\]

i.e.,

\[
\frac{d}{dt} \left( (t - \tau)^{b_{k+1} \cdot w_m(t)} \right)^{2(N - \frac{N}{N-2}) k^{1+1}} (\Omega_T) + c \int_{\Omega_T} \| \nabla (t - \tau)^{b_{k+1} \cdot w_m(t)} \|_{L^{\frac{2N}{N-2}}}^{\frac{N}{N-2} k^{1+1}} \, dx 
\leq \left( c_1 + \frac{c_2}{t - \tau} \right) \| (t - \tau)^{b_{k+1} \cdot w_m(t)} \|_{L^{\frac{2N}{N-2}}}^{2(N - \frac{N}{N-2}) k^{1+1}} (\Omega_T), \tag{32} 
\]

At first, from (32) we have

\[
(t - \tau) \frac{d}{dt} \| (t - \tau)^{b_{k+1} \cdot w_m(t)} \|_{L^{\frac{2N}{N-2}}}^{2(N - \frac{N}{N-2}) k^{1+1}} (\Omega_T) \leq c \| (t - \tau)^{b_{k+1} \cdot w_m(t)} \|_{L^{\frac{2N}{N-2}}}^{2(N - \frac{N}{N-2}) k^{1+1}} (\Omega_T), \tag{33} 
\]

and so,

\[
(t - \tau) \frac{d}{dt} \| (t - \tau)^{b_{k+1} \cdot w_m(t)} \|_{L^{\frac{2N}{N-2}}}^{\frac{N}{N-2} k^{1+1}} (\Omega_T) 
\]
\[
(\tau \leq c \frac{N - 2}{N} \| (t - \tau)^{\frac{b_{k+1}}{2}} \cdot w_m(t)^{\frac{N}{2}}\|^{\frac{2}{k+1}}_{L^\infty} (\partial_T).
\]  \hspace{1cm} (34)

Integrating (34) over $[\tau, t]$ and applying \(B_k\), we deduce that
\[
(t - \tau)\| (t - \tau)^{\frac{b_{k+1}}{2}} \cdot w_m(t)^{\frac{N}{2}}\|^{\frac{2}{k+1}}_{L^\infty} (\partial_T)
\leq \left( c \frac{N - 2}{N} + 1 \right) \int_\tau^T \| (s - \tau)^{\frac{b_{k+1}}{2}} \cdot w_m(s)^{\frac{N}{2}}\|^{\frac{2}{k+1}}_{L^\infty} (\partial_T) \, ds
\leq \left( c \frac{N - 2}{N} + 1 \right) M_k \quad \text{for all } t \in [\tau, T],
\]  \hspace{1cm} (35)

which implies that
\[
(t - \tau)^{\frac{N}{2N}} \| (t - \tau)^{\frac{b_{k+1}}{2}} \cdot w_m(t)^{\frac{N}{2}}\|^{\frac{2}{k+1}}_{L^\infty} (\partial_T)
\leq \left[ \left( c \frac{N - 2}{N} + 1 \right) M_k \right]^{\frac{N}{2N}} \quad \text{for all } t \in [\tau, T].
\]

Multiplying both sides of (32) by \((t - \tau)^{\frac{N}{2N}}\), we obtain that
\[
(t - \tau)^{1 + \frac{N}{2N}} \frac{d}{dt} \| (t - \tau)^{\frac{b_{k+1}}{2}} \cdot w_m(t)^{\frac{N}{2}}\|^{\frac{2}{k+1}}_{L^\infty} (\partial_T)
+ \frac{c}{T} \int_{\partial_T} \nabla (t - \tau)^{\frac{b_{k+1}}{2}} \cdot w_m(t)^{\frac{N}{2}} \|^{\frac{2}{k+1}}_{L^\infty} (\partial_T)
\leq c_3 (t - \tau)^{\frac{N}{2N}} \| (t - \tau)^{\frac{b_{k+1}}{2}} \cdot w_m(t)^{\frac{N}{2}}\|^{\frac{2}{k+1}}_{L^\infty} (\partial_T).
\]

Then, from (35) and the definition of \(b_{k+2}\), we obtain that
\[
(t - \tau)^{1 + \frac{N}{2N}} \frac{d}{dt} \| (t - \tau)^{\frac{b_{k+1}}{2}} \cdot w_m(t)^{\frac{N}{2}}\|^{\frac{2}{k+1}}_{L^\infty} (\partial_T)
+ \frac{c}{T} \int_{\partial_T} \nabla |(t - \tau)^{\frac{b_{k+2}}{2}} \cdot w_m(t)|^{\frac{2}{k+1}}_{L^\infty} dx
\leq c_3 \left[ \left( c \frac{N - 2}{N} + 1 \right) M_k \right]^{\frac{N}{2N}} \quad \text{for all } t \in [\tau, T].
\]

Integrating the above inequality over $[\tau, T]$ and using (35) again, we deduce that
\[
\int_\tau^T \int_{\partial_T} \nabla |(t - \tau)^{\frac{b_{k+2}}{2}} \cdot w_m(t)|^{\frac{2}{k+1}}_{L^\infty} dx \, dt \leq c_4 \left[ \left( c \frac{N - 2}{N} + 1 \right) M_k \right]^{\frac{N}{2N}}.
\]  \hspace{1cm} (36)

Combining (36) with the embedding inequality (24), we obtain that
\[
\int_T \left( \int_{\partial_T} |(t - \tau)^{\frac{b_{k+2}}{2}} \cdot w_m(t)|^{\frac{N}{k+2}}_{L^\infty} dx \right)^{\frac{2}{N-2}} \, dt \leq c_5 \left[ \left( c \frac{N - 2}{N} + 1 \right) M_k \right]^{\frac{N}{2N}}.
\]  \hspace{1cm} (37)
Theorem 3.3 Suppose that \( L \) is a solution in variationalsolution of equation (7). This result shows some decay rate of variationalsolution satisfying (18), (19), where \( \tau \) is a constant.

Proof (35) and (37) implies that \((A_{k+1})\) and \((B_{k+1})\) hold respectively. \(\square\)

Next, we start to establish the higher-order integrability near the initial time \( \tau \) for the variational solution of equation (7). This result shows some decay rate of variationalsolution in \( L^{N-2}N^{k+1} \)-norm near the initial time \( \tau \).

Theorem 3.3 Suppose that (1), (2), (4), and (5) hold,\( f \in L^2_{\text{loc}}(\mathbb{R}; L^2(O_\tau)) \), \( u_t, v_t \in L^2(O_\tau) \). Then, for any \( T \geq \tau \), any \( k = 1, 2, \ldots \), there exists a positive constant \( M_k = M(T - \tau, k, N, l, |u_t|_{L^2(O_T)}, |v_t|_{L^2(O_T)}) \) such that

\[
(t - \tau)^N \left\| (t - \tau)^b w(t) \right\|_{L^2(N^{k+1})} \leq M_k \quad \text{for all } t \in [\tau, T],
\]

where \( w(t) = U(t, \tau)u_t - U(t, \tau)v_t \) and

\[
b_1 = 1 + \frac{1}{2}, \quad b_2 = 1 + \frac{1}{2} + 1 \quad \text{and}
\]

\[
b_{k+1} = b_k + \frac{1 + \frac{N}{N-2}}{2(N^{k+1})} \quad \text{for } k = 2, 3, \ldots .
\]

Proof For any (fixed) \( \tau \in \mathbb{R} \) and \( T \geq \tau \), choose two sequences \( (u_{\tau m}, f_m) \) and \( (v_{\tau m}, f_m) \) satisfying (18), (19).

Then from Theorem 3.2 \( (A_k) \) we have that, for any \( k = 1, 2, \ldots \), there exists a positive constant \( M_k = M(T - \tau, k, N, l, |u_t|_{L^2(O_T)}, |v_t|_{L^2(O_T)}) \) such that

\[
(t - \tau)^N \left\| (t - \tau)^b (u_{m}(t) - v_{m}(t)) \right\|_{L^2(N^{k+1})} \leq M_k \quad \text{for all } t \in [\tau, T],
\]

where \( u_m \) and \( v_m \) are the unique variational solutions of (3) corresponding to the regular data \( (u_{\tau m}, f_m) \) and \( (v_{\tau m}, f_m) \) on the interval \([\tau, T]\) respectively.

On the other hand, from [6] Proposition 11, there exist a subsequence \( \{u_{m_j}\} \) of \( \{u_m\} \) and \( \{v_{m_j}\} \) of \( \{v_m\} \) such that

\[ u_{m_j}(t) \to u(t) \quad \text{and} \quad v_{m_j}(t) \to v(t) \quad \text{a.e. on } O_T \text{ as } j \to \infty. \]

Hence, by applying Fatou’s lemma,

\[
(t - \tau)^N \left\| (t - \tau)^b (u(t) - v(t)) \right\|_{L^2(N^{k+1})} = \liminf_{j \to \infty} \int_{O_T} (t - \tau)^b (u_{m_j}(t) - v_{m_j}(t)) \left\| (t - \tau)^b (u_{m_j}(t) - v_{m_j}(t)) \right\|_{L^2(N^{k+1})} dx \]
\[
\leq \liminf_{j \to \infty} \int_{O_T} (t - \tau)^b (u_{m_j}(t) - v_{m_j}(t)) \left\| (t - \tau)^b (u_{m_j}(t) - v_{m_j}(t)) \right\|_{L^2(N^{k+1})} dx
\]
For any fixed \( t \), we have the following a priori estimates:

\[
H^1(0,1) \ni u_{n} \quad \text{where} \quad \| u_{n} \|_{H^1(0,1)} \leq c_{0,M_0} + c_{0,M_0,M_0,t-\tau,\beta,\theta} | u_{\tau} - v_{\tau} |_{H^1(0,1)}^{2\theta},
\]

where \( \theta \in (0,1) \) is the exponent of the interpolation \( \| \cdot \|_{L^{2\theta}} \leq \| \cdot \|_{L^{2\theta}}^{1-\theta} \| \cdot \|_{L^{2\theta}}^{\theta} \) with some \( k_0 \in \mathbb{N} \) satisfying \( 2(\frac{N}{N-2})k_0 > 2p - 2 \), and \( \rho_0 = \frac{N}{N-2} \frac{2-2\theta}{2-2\theta} \). The constant \( M_0 \) depends only on \( t - \tau, \Omega, \lambda_1, \int_{\tau}^{T} |f(s)|^2 \, ds, \beta, \alpha_1, |u_{\tau} - v_{\tau}|, p \), uniform bound of \( \{u_{\tau,n}\}_{n=1}^{\infty} \) in \( L^2(\Omega) \) and \( M_{\theta} \).

**Proof.** For any fixed \( \tau \in (-\infty, T] \), denote

\[
w_n(s) := u_n(s) - v_n(s), \quad \tau \leq t \leq T,
\]

where \( u_n(s), v_n(s) \) are the variational solutions of equation (7) corresponding to data \((u_{\tau,n}, f_n), (v_{\tau,n}, f_n) \) satisfying (18). Then the following holds:

\[
\int_{\tau}^{T} \left[ -\left( w_n(s), \phi'(s) \right)_T + \left( \left( w_n(s), \phi(s) \right) \right)_T + \left( g(u_n(s)) - g(v_n(s)), \phi(s) \right)_T \right] \, ds = 0
\]

for any \( \phi \in C^1(\Omega) \).

Noticing \( u_n \in L^2(\tau, T; H^1_0(\Omega)) \cap L^\infty(\Omega, T) \), so \( \eta |u_n|^\theta u_n (\eta \in C^1(\tau, T), \theta > 0) \) can be selected as a test function, hence, for a.e. \( s \in (\tau, T) \),

\[
\left( u_n'(s), |u_n(s)|^\theta u_n(s) \right)_T + \left( (u_n(s), |u_n(s)|^\theta u_n(s)) \right)_T
+ \left( g(u_n(s)), |u_n(s)|^\theta u_n(s) \right)_T = \left( f_n(s), |u_n(s)|^\theta u_n(s) \right)_T.
\]

By (4) and the standard energy estimate (e.g., see [9]), we have the following a priori estimates:

\[
\int_{\tau}^{t} \int_{\Omega} |u_n(s)|^{2p-2} \, dx \, ds + \int_{\tau}^{t} \int_{\Omega} |v_n(s)|^{2p-2} \, dx \, ds \leq M,
\]

where the constant \( M \) depends only on \( g, T - \tau, \Omega, \int_{\tau}^{T} |f(s)|^2 \, ds, \alpha_1, \beta, p, u_{\tau,n}, v_{\tau,n} \) and \( \lambda_1; T \);

\[
|w_n(s)|^2 \leq e^{2(t-\tau)} |w_n(\tau)|^2, \quad \forall t \geq \tau;
\]
and
\[
\int_{\tau}^{T} |\nabla w_n(s)|^2 ds \leq \frac{1}{2\lambda_{1,T}} |w_n(\tau)|^2 + \frac{l}{\lambda_{1,T}} \int_{\tau}^{T} |w_n(s)|^2 ds, \quad \forall t \geq \tau, \quad (43)
\]
recall that $\lambda_{1,T}$ is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega_T)$ and the constant $l$ comes from (5).

Noticing $u_n \in L^2(\tau, T; H^2(\Omega))$ in Theorem 2.5, let $\phi = -\eta \Delta w_n \ (\eta \in C^1_c(\tau, T))$ in (40), then
\[
\int_{\tau}^{T} \left[ (w_n'(s), -\Delta w_n(s))_{\Omega_T} + |\Delta w_n(s)|^2_{2, r} + (g(u_n(s)) - g(v_n(s)), -\Delta w_n(s))_{\Omega_T} \right] \eta \, ds = 0
\]
for any $\eta \in C^1_c(\tau, T)$. Hence,
\[
- \int_{\Omega_T} w_n' \Delta w_n \, dx + \int_{\Omega_T} |\Delta w_n(s)|^2 \, dx = \int_{\Omega_T} (g(u_n(s)) - g(v_n(s))) \Delta w_n \, dx,
\]
\[- \int_{\Omega_T} w_n' \Delta w_n \, dx - \frac{1}{2} \frac{d}{dt} \int_{\Omega_T} |\nabla w_n(s)|^2 \, dx - \int_{\Gamma_T} |\nabla w_n(s)|^2 w \cdot n_s \, d\sigma,
\]
where $n_s$ is the outside unit normal vector, $w$ is a velocity field. By trace theory and interpolation, for all $\delta \geq \frac{1}{2}$ (reference [13]),
\[
\left| \int_{\Gamma_T} |\nabla w_n(s)|^2 w \cdot n_s \, d\sigma \right| \leq C_2 \left( \int_{\Omega_T} |\Delta w_n(s)|^2 \, dx \right)^{\delta} \left( \int_{\Omega_T} |\nabla w_n(s)|^2 \, dx \right)^{1-\delta}. \quad (45)
\]
In particular, let $\delta = \frac{1}{2}$ and by Cauchy’s inequality, for all $s \in [\tau, T]$, we have
\[
\left| \int_{\Gamma_T} |\nabla w_n(s)|^2 w \cdot n_s \, d\sigma \right| \leq \frac{1}{4} \int_{\Omega_T} |\Delta w_n(s)|^2 \, dx + c^2 \int_{\Omega_T} |\nabla w_n(s)|^2 \, dx. \quad (46)
\]
On the other hand, by using (4) we have that
\[
\left| \int_{\Omega_T} (g(u_n(s)) - g(v_n(s))) \Delta w_n \, dx \right|
\leq c \int_{\Omega_T} \left( 1 + |u_n(s)|^{p-2} + |v_n(s)|^{p-2} \right) |w_n(s)| \Delta w_n(s) \, dx
\leq c \int_{\Omega_T} |w_n(s)| \Delta w_n(s) \, dx + c \int_{\Omega_T} \left( |u_n(s)|^{p-2} + |v_n(s)|^{p-2} \right) |w_n(s)| \Delta w_n(s) \, dx
\leq \frac{1}{4} \int_{\Omega_T} |\Delta w_n(s)|^2 \, dx + c |w_n(s)|^2
+ c \left( \|u_n(s)\|_{L^{2p-2}(\Omega_T)} + \|v_n(s)\|_{L^{2p-2}(\Omega_T)} \right) \|w_n(s)\|_{L^{2p-2}(\Omega_T)}^2. \quad (47)
\]
Combining (44)–(47), we obtain that
\[
\frac{d}{ds} |\nabla w_n|^2 \leq c |w_n(t)|^2 + c \left( \|u_n(t)\|_{L^{2p-2}(\Omega_T)}^{2p-4} + \|v_n(t)\|_{L^{2p-2}(\Omega_T)}^{2p-4} \right) \|w_n(t)\|_{L^{2p-2}(\Omega_T)}^2 \quad \text{a.e.} \ (r, t).
\]
Since \(2(\frac{N}{N-2})^k \to \infty\) as \(k \to \infty\), there is \(k_0 \in \mathbb{N}\) such that
\[
2\left(\frac{N}{N-2}\right)^{k_0} > 2p - 2.
\]
For this \(k_0\), by interpolation, we have
\[
\|w_n\|_{L^{2p-2}(\Omega)} \leq \|w_n\|_{L^{2p-2}(\Omega)}^{1-\theta} \|w_n\|_{L^2(\Omega)}^\theta,
\]
where \(\theta \in (0,1)\) depends only on \(p, k_0\).

Therefore, we have that
\[
\left(s - \frac{t + \tau}{2}\right)^m \frac{d}{ds}|\nabla w_n|^2 \leq c|w_n|^2 + c\left(\|u_n\|_{L^{2p-2}(\Omega)}^{2p-4} \|v_n\|_{L^{2p-2}(\Omega)}^{2p-4}\right) \cdot |w_n|^{2p},
\]
where \(b_{k_0}\) is given by (20).

On the other hand, thanks to Theorem 3.2, we know that there is a constant \(M_{k_0}\), which depends only on \(t - \tau, N, k_0\), and the \(H^1_0 \cap L^p\)-bounds of \(u_{\tau n}, v_{\tau n}\) such that
\[
(s - t)^{\frac{N}{N-2}} \|w_n\|_{L^2(\Omega)}^{\frac{2(N-2)}{N-2}k_0} \leq M_{k_0}^{2p} \text{ for all } n = 1, 2, \ldots, s \in [\tau, t].
\]

Therefore, we have the following estimate for any \(n = 1, 2, \ldots\):
\[
\left(s - \frac{t + \tau}{2}\right)^m \frac{d}{ds}|\nabla w_n|^2 \leq c\left(s - \frac{t + \tau}{2}\right)^\theta (|\nabla w_n|^2 + |w_n|^2)
\]
\[
+ cM_{k_0}^{2p} \left(\|u_n(s)\|_{L^{2p-2}(\Omega)}^{2p-4} + \|v_n(s)\|_{L^{2p-2}(\Omega)}^{2p-4}\right) \cdot |w_n(s)|^{2p}, \text{ a.e. } s \in \left[\frac{t + \tau}{2}, t\right].
\]
To ensure that the exponent \( r_0 \) is strictly larger than 1, we may multiply both sides by 
\[
\left( s - \frac{t + \tau}{2} \right)^{1 + r_0} \frac{d}{ds} |\nabla w_n(s)|^2
\]
and then we obtain that
\[
\left( s - \frac{t + \tau}{2} \right)^{1 + r_0} \left( |\nabla w_n|^2 + |w_n|^2 \right) + c \left( s - \frac{t + \tau}{2} \right) M_{k_0}^{2 - 20} \left( \| u_n(s) \|^2_{2^p - 2(O_\lambda)} + \| v_n(s) \|^2_{2^p - 2(O_\lambda)} \right) \cdot |w_n(s)|^{20} \leq c \left( s - \frac{t + \tau}{2} \right) M_{k_0}^{2 - 20} \theta \kappa \left( \| u_n(s) \|_{2^p - 2(O_\lambda)}^2 + \| v_n(s) \|_{2^p - 2(O_\lambda)}^2 \right) \cdot |w_n(s)|^{20} \]
a.e. \( s \in \left[ \frac{\tau + t}{2}, t \right] \). (48)

Integrating the inequality above over \( \left[ \frac{\tau + t}{2}, t \right] \) with respect to \( s \), we finally obtain that, for any \( n = 1, 2, \ldots \)
\[
\left( \frac{t - \tau}{2} \right)^{1 + r_0} \left( |\nabla w_n(t)|^2 \right) \leq \left( 1 + r_0 \right) \left( \frac{t - \tau}{2} \right)^{r_0} \int_{\frac{\tau + t}{2}}^{t} |\nabla w_n(s)|^2 ds + c \int_{\frac{\tau + t}{2}}^{t} \left( \| u_n(s) \|^2_{2^p - 2(O_\lambda)} + \| v_n(s) \|^2_{2^p - 2(O_\lambda)} \right) \cdot |w_n(s)|^{20} ds \]
\[
+ c \left( \frac{t - \tau}{2} \right) M_{k_0}^{2 - 20} \int_{\frac{\tau + t}{2}}^{t} \left( \| u_n(s) \|^2_{2^p - 2(O_\lambda)} + \| v_n(s) \|^2_{2^p - 2(O_\lambda)} \right) \cdot |w_n(s)|^{20} ds \]
\[
:= I_1 + I_2 + I_3. \quad (49)
\]

Note that, from (42)–(43), we have that
\[
I_1 + I_2 \leq c r_0, t - \tau \left| w_n(\tau) \right|_{\tau}^2. \quad (50)
\]

For the estimate of \( I_3 \), by Hölder’s inequality, we have
\[
I_3 \leq c \left( \frac{t - \tau}{2} \right) M_{k_0}^{2 - 20} 2M_0^{2 - 20} \left( \int_{\frac{\tau + t}{2}}^{t} \left| w_n(s) \right|^{2(p-1)} ds \right)^{\frac{2}{p-2}} \leq c M_{k_0}^{2 - 20} \theta \kappa \left( \| u_n(\tau) \|^2_{2^p - 2(O_\lambda)} \right) \cdot \left| w_n(\tau) \right|_{\tau}^{20}. \quad (51)
\]

Combining with (42)–(49), it implies that
\[
|\nabla w_n(t)|^2 \leq c r_0, t - \tau \left| w_n(\tau) \right|_{\tau}^2 + c M_{k_0}^{2 - 20} \theta \kappa \left( \| u_n(\tau) \|^2_{2^p - 2(O_\lambda)} \right) \cdot \left| w_n(\tau) \right|_{\tau}^{20}. \quad (52)
\]

From (52) we know \( w_n \) is bounded in \( H^1_0(O_\lambda) \), so there exists a subsequence \( \{ w_{nj} \} \) such that
\[
w_{nj} \rightharpoonup \chi \quad \text{in} \quad H^1_0(O_\lambda), \quad \text{as} \quad j \to \infty. \quad (53)
\]

By [6] Proposition 11 again, it follows
\[
w_{nj}(t) \to u(t) - v(t) \quad \text{in} \quad L^2(O_\lambda), \quad \text{as} \quad j \to \infty,
\]
hence, \( \chi = u(t) - v(t) \).

Combining (52), (53), (21), and (19), we deduce that

\[
\left| \nabla (u(t) - v(t)) \right|^2 \leq \liminf_{j \to \infty} \left| \nabla w_{\eta_j}(t) \right|^2 \\
\leq c_{\eta_j,t-1} |u_\tau - v_\tau|^2 + c_{\eta_j,M_{\eta_j},p,M_{\eta_j}-t-\eta_j} |u_\tau - v_\tau|^{2\theta}.
\]

\( \square \)

In [15], the existence of pullback \( \mathcal{D}_{\lambda_1} \) attractor defined in time varying domains has been considered. Then we can establish the regularity attraction of \((L^2, L^2)\) pullback \( \mathcal{D}_{\lambda_1} \) attractor.

**Theorem 3.5** Suppose that \( U(t, \tau) \) is the process corresponding to a variational solution of (3), \( \mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\} \) is the \((L^2, L^2)\) pullback \( \mathcal{D}_{\lambda_1} \) attractor associated with \( U(t, \tau) \) and \( f \in L^2_{\text{loc}}(\mathbb{R}, L^2(\Omega)) \). Then \( \mathcal{A} \) is pullback \( \mathcal{D}_{\lambda_1} \) attraction in \( H^1_\text{loc} \).

That is, for any \( t \in \mathbb{R} \), any \( \hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_{\lambda_1} \),

\[
\text{dist}_{H^1_\text{loc} (\Omega)}(U(t, \tau)D(\tau), \mathcal{A}(t)) \to 0, \quad \tau \to -\infty.
\]

**Proof** For each \( t \in \mathbb{R} \), from the definition of the \((L^2, L^2)\) pullback \( \mathcal{D}_{\lambda_1} \) attractor \( \mathcal{A} \), we know that \( \mathcal{A}(t - 1) \) is compact in \( L^2(\Omega) \).

By \( B(t) \) being the 1-neighborhood of \( \mathcal{A}(t) \) for each \( t \in \mathbb{R} \) under the \( L^2(\Omega) \) norm, \( B(t) \) is bounded in \( L^2(\Omega) \). By (39), let \( t \) be fixed, \( \tau = t - 1 \), and \( u_{i\tau} \in B(t - 1) \) \((i = 1, 2)\), we have

\[
\| U(t, t-1)u_{\tau 1} - U(t, t-1)u_{\tau 2} \|_{H^1_\text{loc} (\Omega)}^2 \\
\leq c_1 |u_{\tau 1} - u_{\tau 2}|_{L^2}^2 + c_2 |u_{\tau 1} - u_{\tau 2}|^{2\theta},
\]

where \( c_1, c_2 \) are two constants. Now, for this fixed \( t \) and for any \( \epsilon > 0 \), by the definition of the \((L^2, L^2)\) pullback \( \mathcal{D}_{\lambda_1} \) attractor again, for any \( \hat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_{\lambda_1} \), there is a time \( \tau_0 \) \((t - 1) \) which depends only on \( t, \epsilon, \) and \( \hat{D} \) such that

\[
U(t - 1, \tau)D(\tau) \subset B(t - 1) \quad \text{for all } \tau \leq \tau_0,
\]

\[
\text{dist}_{L^2(\Omega)}(U(t - 1, \tau)D(\tau), \mathcal{A}(t - 1)) \leq \epsilon \quad \text{for all } \tau \leq \tau_0.
\]

Consequently,

\[
\text{dist}_{H^1_\text{loc} (\Omega)}(U(t, \tau)D(\tau), \mathcal{A}(t)) \\
= \text{dist}_{H^1_\text{loc} (\Omega)}(U(t, t-1)U(t-1, \tau)D(\tau), U(t, t-1)\mathcal{A}(t - 1)) \\
\leq c_1 \epsilon^2 + c_2 \epsilon^{2\theta} \quad \text{for all } \tau \leq \tau_0.
\]

Noticing the arbitrariness of \( \epsilon \) and \( \hat{D} \), the conclusion is proved. \( \square \)

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