Recurrence relations for Wronskian Laguerre polynomials

Niels Bonneux\textsuperscript{1} and Marco Stevens\textsuperscript{2}

\textsuperscript{1,2}KU Leuven, Department of Mathematics, Celestijnenlaan 200B box 2400, 3001 Leuven, Belgium. E-mail: niels.bonneux@kuleuven.be and marco.stevens@kuleuven.be

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Abstract

The 3-term recurrence relation for Hermite polynomials was recently generalized to a recurrence relation for Wronskians of Hermite polynomials. In this note, a similar generalization for Laguerre polynomials is obtained.

1 Introduction

Hermite and Laguerre polynomials are well-studied classical orthogonal polynomials and can be defined via their 3-term recurrence relation [13]. Applying the Wronskian operator to a finite set of these polynomials yields Wronskian Hermite and Wronskian Laguerre polynomials. They appear in rational solutions of Painlevé equations, see for example [5, 14] and the references therein, and play a key role in the theory of exceptional orthogonal polynomials [3, 6, 8].

A recurrence relation for Wronskian Hermite polynomials was derived in [4] by direct computation of determinants. It generalizes the classical 3-term recurrence relation of Hermite polynomials. Subsequently, these results were shown to hold in general for Wronskians of Appell polynomials in [2], by using a connection with the theory of symmetric functions.

Modified Laguerre polynomials satisfy the Appell property and hence specifying the results in [2] to this case yields a recurrence relation for Wronskian Laguerre polynomials. However, the route of direct computations of determinants that was used in [4], a different recurrence relation is obtained and this is a proper generalization of the 3-term recurrence relation. It expresses Wronskian Laguerre polynomials of degree $n$ in terms of Wronskian Laguerre polynomials of degrees $n-1$ and $n-2$. This note is dedicated to this new recurrence relation, which is stated in Theorem 1.1, and cannot be obtained from symmetric function theory as in [2].

The rest of this section discusses the conventions of this note: Laguerre polynomials, the basics of the theory of partitions [10, 12], the definition of Wronskian Laguerre polynomials, the main result Theorem 1.1, a comparison with the result for exceptional Laguerre polynomials and with the result for Wronskian Hermite polynomials in [4], and a remark why the main result does not extend to Wronskian Jacobi polynomials. Subsequently, the proof of Theorem 1.1 is given in Section 2. The last section contains an alternative proof of the averaging property of Wronskian Laguerre polynomials with respect to the Plancherel measure. In contrast with the proof given in [2], Section 3 does not make use of the theory of symmetric functions. Instead, it is based on an inductive argument using Theorem 1.1.
1.1 Laguerre polynomials

Laguerre polynomials with parameter \( \alpha \) can be defined by their 3-term recurrence relation

\[
L_n^{(\alpha)}(x) = \left( 2 + \frac{-x + \alpha - 1}{n} \right) L_{n-1}^{(\alpha)}(x) - \left( 1 + \frac{\alpha - 1}{n} \right) L_{n-2}^{(\alpha)}(x)
\]

for \( n \geq 2 \), together with the initial conditions \( L_0^{(\alpha)}(x) = 1 \) and \( L_1^{(\alpha)}(x) = -x + \alpha + 1 \), see formula (5.1.10) in [13]. Using elementary identities for Laguerre polynomials, i.e., formulas (5.1.13) and (5.1.14) from [13], it is easy to show that Laguerre polynomials satisfy

\[
nL_n^{(\alpha)}(x) = (-x + \alpha + 1)L_{n-1}^{(\alpha+1)}(x) - xL_{n-2}^{(\alpha+2)}(x)
\]

for all \( n \geq 2 \). Therefore setting

\[
l_n^{(\alpha)}(x) := n! L_n^{(\alpha-n)}(-x)
\]

yields that (1.1) can be written as

\[
l_n^{(\alpha)}(x) = (x + \alpha - n + 1) l_{n-1}^{(\alpha)}(x) + x(n - 1) l_{n-2}^{(\alpha)}(x)
\]

for all \( n \geq 2 \), together with \( l_0^{(\alpha)}(x) = 1 \) and \( l_1^{(\alpha)}(x) = x + \alpha \). Moreover, \( l_n^{(\alpha)} \) is a monic polynomial of degree \( n \) and the sequence \((l_n^{(\alpha)})_{n=0}^{\infty}\) satisfies the Appell property, that is

\[
\frac{d}{dx} l_n^{(\alpha)}(x) = n l_{n-1}^{(\alpha)}(x)
\]

for all \( n \geq 1 \). It is noteworthy that even though the transformation from the sequence of polynomials \( L_n^{(\alpha)} \) to \( l_n^{(\alpha)} \) is a modification of the Laguerre polynomials, it are precisely these modified Laguerre polynomials that appear in the rational solutions of the third and fifth Painlevé equation, see [5, 14] and the references therein.

1.2 Partitions and degree vectors

Wronskian Laguerre polynomials are, similarly as in [2, 3], labelled by (integer) partitions. A brief overview of the necessary notions of the theory of partitions is given below. For a slightly more extended discussion (using the same notation), the reader is referred to [2], and for a thorough introduction to [10, 12].

A partition is a finite sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) of strictly positive integers such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \). The length of \( \lambda \) is \( \ell(\lambda) = r \) and the size is \( |\lambda| = \sum \lambda_i \). If \( |\lambda| = n \), then \( \lambda \vdash n \). Each partition can be identified with its Young diagram \( D_\lambda \), which consists of \( r \) rows of boxes, and row \( i \) contains \( \lambda_i \) boxes. The set of all partitions \( \mathcal{P} \) is partially ordered by \( \mu \leq \lambda \) if the Young diagram of \( \mu \) fits within the Young diagram of \( \lambda \), or equivalently, if \( \mu_i \leq \lambda_i \) for all \( i = 1, 2, \ldots, \ell(\mu) \). This partial ordering turns the set \( \mathcal{P} \) into a lattice, called the Young lattice.

If \( \mu \leq \lambda \), then the difference of Young diagrams \( D_\lambda \setminus D_\mu \) defines the skew partition \( \lambda / \mu \), and \( |\lambda / \mu| := |\lambda| - |\mu| \). In the case that \( \lambda / \mu \) is a border strip of size \( k \) [10, I.1], that is, it is connected and does not contain a \( 2 \times 2 \)-square, then \( \mu \in \mathcal{R}_k^{(\lambda)} \), and \( \text{ht}(\lambda / \mu) \) denotes the number of rows minus 1. Furthermore, \( \lambda \in \mathcal{R}_k^{(\mu)}(\mu) \) if and only if \( \mu \in \mathcal{R}_k^{(\lambda)}(\lambda) \). In the special case that \( |\lambda / \mu| = 1 \), the notation \( \mu < \lambda \) (or equivalently \( \lambda > \mu \)) is used. In this case, the Young diagram of \( \mu \) is obtained by removing precisely one box from the Young diagram of \( \lambda \). If this box is in the \( i \)th row, then

\[
c(\lambda / \mu) = \lambda_i - i
\]

denotes the content of the skew partition. This is in correspondence with the general notion of contents of partitions [10, 12]. If \( \lambda / \mu \) is a border strip of size 2, then the Young diagram
of μ is obtained from the Young diagram of λ by removing a domino tile. If this domino tile is horizontally placed, then \( \text{ht}(\lambda/\mu) = 0 \) and if it is vertically placed, then \( \text{ht}(\lambda/\mu) = 1 \). This also corresponds to the assignments of signs in [4].

Any partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) has a degree vector \( n_\lambda = (n_1, n_2, \ldots, n_r) \) associated to it, defined by \( n_i = \lambda_i + r - i \) for all \( i = 1, 2, \ldots, r \). Note that a partition is a weakly decreasing sequence while its degree vector is strictly decreasing.

The number of directed paths in the Young lattice from \( \emptyset \) (the unique partition of size 0) to a partition \( \lambda \), denoted by \( F_\lambda \), can be written in terms of the degree vector as
\[
F_\lambda = \frac{|\lambda|! \prod_{i<j}(n_i - n_j)}{\prod_i n_i!}
\]  
and is equal to the number of standard Young tableaux of shape \( \lambda \) [1, Corollary 3.31], as well as to the dimension of the irreducible representation of the symmetric group associated to \( \lambda \).

### 1.3 Wronskian Laguerre polynomials

Following the convention of [2] and [4], the Wronskian Laguerre polynomial with parameter \( \alpha \) associated to the partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) is defined by
\[
l^{(\alpha)}_\lambda = \frac{\text{Wr}[l^{(\alpha)}_1, l^{(\alpha)}_2, \ldots, l^{(\alpha)}_r]}{\prod_{i<j}(n_j - n_i)}
\]  
where the (modified) Laguerre polynomials \( l_n^{(\alpha)} \) are defined by (1.2) and \( n_\lambda = (n_1, n_2, \ldots, n_r) \) is the degree vector of \( \lambda \). From the definition it immediately follows that \( l^{(\alpha)}_\lambda \) is a monic polynomial of degree \( |\lambda| \). Taking the trivial partition \( \lambda = (n) \) yields \( l^{(\alpha)}_1 = l_n^{(\alpha)} \), and so the Wronskian Laguerre polynomials are a generalization of the Laguerre polynomials. The main result of this note, which is proven in Section 2, is the following recurrence relation for the Wronskian polynomials.

**Theorem 1.1.** If \( \lambda \) is a non-empty partition, then
\[
F_\lambda l^{(\alpha)}_\lambda(x) = \sum_{\mu \preceq \lambda} (x + \alpha - c(\lambda/\mu))F_\mu l^{(\alpha)}_\mu(x) + x(|\lambda| - 1) \sum_{\rho \in R^+_\lambda} (-1)^{\text{ht}(\lambda/\rho)} F_\rho l^{(\alpha)}_\rho(x)
\]  
where \( c(\lambda/\mu) \) is the content defined in (1.4).

Taking \( \lambda = (n) \) in (1.7) precisely yields the 3-term recurrence (1.2). This is in contrast with the recurrence relation obtained in [2] for Wronskian Laguerre polynomials, which says
\[
F_\lambda l^{(\alpha)}_\lambda(x) = (x + \alpha) \sum_{\mu \preceq \lambda} F_\mu l^{(\alpha)}_\mu(x) + \alpha \sum_{k=2}^{n} (-1)^{k-1} \frac{|\lambda|! - 1}{(|\lambda| - k)!} \sum_{\nu \in R^+_k(\lambda)} (-1)^{\text{ht}(\lambda/\nu)} F_\nu l^{(\alpha)}_\nu(x)
\]  
for any partition \( |\lambda| \geq 1 \). Taking \( \lambda = (n) \) in this relation yields
\[
l^{(\alpha)}_n(x) = (x + \alpha) l^{(\alpha)}_{n-1}(x) + \alpha \sum_{k=2}^{n} (-1)^{k-1} \frac{(n-1)!}{(n-k)!} l^{(\alpha)}_{n-k}(x)
\]
which is fundamentally different in shape from the 3-term recurrence (1.2), but nevertheless equivalent with it by using formulas (5.1.13) and (5.1.14) from [13]. It is noteworthy that there are less terms in the right-hand side of (1.7) than in (1.8).

**Remark 1.2.** Both recurrence relations (1.7) and (1.8) generate the set of Wronskian Laguerre polynomials together with the initial conditions \( l^{(\alpha)}_0(x) = 1 \) and \( l^{(\alpha)}_1(x) = x + \alpha \).
Remark 1.3. Exceptional Laguerre polynomials [3, 6] are constructed using Wronskians of classical Laguerre polynomials $L_n^{(\alpha)}$ instead of the modified polynomials $\ell_n^{(\alpha)}$. More concretely, they make use of the polynomials

$$L^{(\alpha)}_n = c_\lambda \text{Wr}[L^{(\alpha)}_{n_1}, L^{(\alpha)}_{n_2}, \ldots, L^{(\alpha)}_{n_r}]$$

that are defined for every partition $\lambda$, with $c_\lambda$ being a normalization constant to obtain monic polynomials. Using modified polynomials (as in (1.6)) or classical Laguerre polynomials (as in (1.9)) turns out to yield different Wronskian polynomials, to such an extent that the recurrence relations in Theorem 1.1 or in (1.8) do not translate to recurrence relations for exceptional polynomials. Using modified polynomials (as in (1.6)) or classical Laguerre polynomials (as in (1.9)) turns out to yield different Wronskian polynomials, to such an extent that the recurrence relations in Theorem 1.1 or in (1.8) do not translate to recurrence relations for exceptional Laguerre polynomials such as those obtained in [7, Corollary 5.1]. Nevertheless, it is noteworthy that (1.6) and (1.9) do relate for partitions whose Young diagrams are rectangles. More precisely, if $\lambda = (n, n, \ldots, n)$ with $\ell(\lambda) = m$, then $L^{(\alpha)}_\lambda(x) = L^{(-\alpha-n)}_{\lambda'}(x)$ for any $\alpha$, where $\lambda' = (m, m, \ldots, m)$ with $\ell(\lambda') = n$ is the conjugate partition of $\lambda$.

Remark 1.4. The Hermite polynomials $(\text{He}_n)_{n=0}^\infty$ are defined by

$$\text{He}_n(x) = x \text{He}_{n-1}(x) - (n - 1) \text{He}_{n-2}(x)$$

for $n \geq 2$, together with $\text{He}_0(x) = 1$ and $\text{He}_1(x) = x$, see [9, Chapter 9]. The Wronskian Hermite polynomials, defined by

$$\text{He}_\lambda = \text{Wr}[\text{He}_{n_1}, \text{He}_{n_2}, \ldots, \text{He}_{n_r}]$$

satisfy

$$F_\lambda \text{He}_\lambda(x) = x \sum_{\mu \in \lambda} F_\mu \text{He}_\mu(x) - (|\lambda| - 1) \sum_{\rho \in R^*_\lambda(\lambda)} (-1)^{ht(\lambda/\rho)} F_\rho \text{He}_\rho(x)$$

for any non-empty partition $\lambda$, as was shown in [4] by the direct computation of determinants. Notice the similarity between (1.7) and (1.11) and the fact that (1.11) reduces to (1.10) when taking $\lambda = (n)$. The relation (1.11) also follows by specifying the general recurrence relation for Wronskian Appell polynomials in [2], which can be done since the Hermite polynomials form an Appell sequence.

Remark 1.5. The Hermite and Laguerre polynomials are two types of classical orthogonal polynomials. The third (and last) type is formed by the class of Jacobi polynomials $P_n^{(\alpha, \beta)}$ [13], which (for fixed parameters $\alpha$ and $\beta$) do not satisfy the Appell property. However, assuming that $\alpha + \beta \notin \mathbb{Z}_{\leq 0}$, the modified Jacobi polynomials

$$A^{(\alpha, \beta)}_n(x) := \frac{2^n n!}{(\alpha + \beta - n + 1)_n} P_n^{(\alpha-n, \beta-n)}(x)$$

where $(a)_n = a(a + 1) \cdots (a + n - 1)$ denotes the Pochhammer symbol, do satisfy the Appell property and their 3-term recurrence is

$$(\alpha + \beta - n + 1)A^{(\alpha, \beta)}_n(x) = ((\alpha + \beta - 2n + 2)x + \alpha - \beta) A^{(\alpha, \beta)}_{n-1}(x) + (x^2 - 1)(n - 1) A^{(\alpha, \beta)}_{n-2}(x)$$

for $n \geq 2$. Hence, by the general results in [2], there is a generating recurrence relation for the Wronskian Jacobi polynomials

$$A^{(\alpha, \beta)}_\lambda = \text{Wr}[A^{(\alpha, \beta)}_{n_1}, A^{(\alpha, \beta)}_{n_2}, \ldots, A^{(\alpha, \beta)}_{n_r}]$$

$$\prod_{i<j} (n_i - n_j)$$
but this relation is not of the form
\[
F_{\lambda} A_{\lambda}^{(\alpha,\beta)}(x) = \sum_{\mu \leq \lambda} (a_{\mu,\lambda} x + b_{\mu,\lambda}) F_{\mu} A_{\mu}^{(\alpha,\beta)}(x) + \sum_{\rho \in R_{\lambda}^{-}} (c_{\rho,\lambda} x^2 + d_{\rho,\lambda} x + e_{\rho,\lambda}) F_{\rho} A_{\rho}^{(\alpha,\beta)}(x)
\]
where \(a_{\mu,\lambda}, b_{\mu,\lambda}, c_{\rho,\lambda}, d_{\rho,\lambda}, e_{\rho,\lambda} \in \mathbb{R}\). In fact, an implementation in the computer software Maple shows that a recurrence relation of this form does not exist. The structure of the proof of the recurrence relation for Wronskian Hermite and Laguerre polynomials, as given in Section 2, does not have an analogue for the Jacobi case. This is because both terms in the right-hand side of the 3-term recurrence relation (1.12) are polynomials of degree \(n\), whereas there is only one such term in (1.2) and (1.10).

\[\text{2 Proof of Theorem 1.1}\]

The proof of the recurrence relation (1.7) for Wronskian Laguerre polynomials has the same structure and ideas as the proof of the recurrence relation for Wronskian Hermite polynomials which is given in [4, Section 5]. This structure has three parts.

1. The first part is a necessary rewriting exercise of the Wronskian polynomial and works for every Appell sequence.

2. The rewriting exercise enables invoking the 3-term recurrence relation in the second part. Hence, the specifics of the second part depend on the sequence of polynomials.

3. Invoking the 3-term recurrence relation yields a few terms. For the Hermite case, as treated in [4], there are two terms, which can be rewritten as a sum over the partitions \(\mu \leq \lambda\) and \(\rho \in R_{\lambda}^{-}\), respectively. In the Laguerre case, there are three terms; two of those are the analogous terms of the Hermite case, and the third one vanishes, as shown below. To build further upon Remark 1.5: completing part 1 and 2 for the Jacobi case yields terms which cannot be rewritten in any known combination of Wronskian Jacobi polynomials.

**Part 1.** The polynomial \(F_{\lambda} l_{\lambda}^{(\alpha)}\) can be rewritten to make it suitable for invoking the 3-term recurrence relation in part 2. For this, fix a partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)\) with degree vector \(n_{\lambda} = (n_1, n_2, \ldots, n_r)\). For any permutation \(\sigma \in S_r\) and for any \(i \in \{1, 2, \ldots, r\}\), write
\[
\sigma(n_{\lambda})_i = n_i - \sigma(i) + 1 \quad (2.1)
\]
as in [4]. Now, use (1.5) for \(F_{\lambda}\), evaluate the determinant in (1.6) as a sum over permutations, and use the Appell property (1.3) repeatedly to arrive at
\[
F_{\lambda} l_{\lambda}^{(\alpha)}(x) = (-1)^{r(r-1)/2} |\lambda|! \sum_{\sigma \in S_r} \text{sgn}(\sigma) \prod_{i=1}^{r} \frac{l_{\lambda}^{(\alpha)}(x)}{\sigma(n_{\lambda})_i!} \quad (2.2)
\]
which is similar to [4, formula (5.7)]. Subsequently, for \(j = 1, 2, \ldots, r\) write
\[
n_{\lambda}[j] = (n_1, \ldots, n_{j-1}, n_j - 1, n_{j+1}, \ldots, n_r) \quad (2.3)
\]
as in [4]. Then part 4 of Lemma 5.4 in [4] says that
\[
|\lambda|! \prod_{i=1}^{r} \frac{1}{\sigma(n_{\lambda})_i!} = \sum_{j=1}^{r} (|\lambda| - 1)! \prod_{i=1}^{r} \frac{1}{\sigma(n_{\lambda}[j])_i!}.
\]
which can be applied in (2.2). It yields

\[ F_{\lambda}l^{(a)}_{\lambda}(x) = \sum_{j=1}^{r} (-1)^{\frac{r(r-1)}{2}}(|\lambda| - 1)! \sum_{\sigma \in S_r} \text{sgn}(\sigma) \prod_{i=1}^{r} \frac{j^{(a)}_{\sigma(\lambda)_i}}{\sigma(n_{\lambda}[j])_i} \]  

(2.4)

which is the suitable form for invoking the 3-term recurrence relation in part 2. As mentioned before, so far, nothing depends on the specific case of Laguerre polynomials: any other Appell sequence satisfies the same equation.

**Part 2.** The next step is to use the 3-term recurrence relation. More concretely, for each term in (2.4), the relation is applied to the \( j^{th} \) factor in the product for every \( \sigma \in S_r \). For this factor, the 3-term recurrence relation (1.2) says

\[ j^{(a)}_{\sigma(\lambda)_j}(x) = (x + \alpha - \lambda_j + j + \sigma(j) - r) j^{(a)}_{\sigma(\lambda)_{j-1}}(x) + x(\sigma(n_{\lambda})_j - 1) j^{(a)}_{\sigma(\lambda)_{j-2}}(x) \]

which follows from \( \sigma(n_{\lambda})_j = n_j - \sigma(j) + 1 \) and \( n_j = \lambda_j + r - j \). Therefore

\[ F_{\lambda}l^{(a)}_{\lambda}(x) = A + (-1)^{\frac{r(r-1)}{2}}(|\lambda| - 1)!B + C \]  

(2.5)

where

\[ A = \sum_{j=1}^{r} (-1)^{\frac{r(r-1)}{2}}(|\lambda| - 1)! (x + \alpha - \lambda_j + j) \sum_{\sigma \in S_r} \text{sgn}(\sigma) \frac{j^{(a)}_{\sigma(\lambda)_{j-1}}(x) \prod_{i \neq j} j^{(a)}_{\sigma(\lambda)_i}(x)}{\prod_{i} \sigma(n_{\lambda}[j])_i} \]  

(2.6)

\[ B = \sum_{j=1}^{r} \sum_{\sigma \in S_r} \text{sgn}(\sigma) (\sigma(j) - r) \frac{j^{(a)}_{\sigma(\lambda)_{j-1}}(x) \prod_{i \neq j} j^{(a)}_{\sigma(\lambda)_i}(x)}{\prod_{i} \sigma(n_{\lambda}[j])_i} \]  

(2.7)

\[ C = x \sum_{j=1}^{r} (-1)^{\frac{r(r-1)}{2}}(|\lambda| - 1)! \sum_{\sigma \in S_r} \text{sgn}(\sigma) (\sigma(n_{\lambda})_j - 1) \frac{j^{(a)}_{\sigma(\lambda)_{j-2}}(x) \prod_{i \neq j} j^{(a)}_{\sigma(\lambda)_i}(x)}{\prod_{i} \sigma(n_{\lambda}[j])_i} \]  

(2.8)

and these terms can now be treated separately.

**Part 3.** The decomposition (2.5) should be compared with Proposition 5.6 in [4]: in the Hermite case analogues of the terms \( A \) and \( C \) exist, but not for the term \( B \). In fact, completely similar as in the proof of [4, Theorem 3.1], the terms \( A \) and \( C \), see (2.6) and (2.8), are equal to

\[ A = \sum_{\mu = \lambda} (x + \alpha - c(\lambda/\mu)) F_{\mu}l^{(a)}_{\mu}(x) \quad C = x(|\lambda| - 1) \sum_{\rho \in R_{\Sigma}^{\text{c}}(\lambda)} (-1)^{h(\lambda/\rho)} F_{\rho}l^{(a)}_{\rho}(x) \]  

(2.9)

and hence it is now sufficient to prove that \( B = 0 \). For this, a symmetry argument that again does not depend on the specifics of the Laguerre polynomials suffices. Namely, define the set

\[ X = \{(j, \sigma) \in \{1, 2, \ldots, r\} \times S_r \mid \sigma(j) \neq r\} \]

so that

\[ B = \sum_{(j, \sigma) \in X} b_{j, \sigma} \]  

(2.10)

where \( b_{j, \sigma} \) denotes the full expression of the term in the double sum in \( B \), see (2.7). Next, define the map

\[ T : X \rightarrow X : (j, \sigma) \mapsto (k, \tau) \]

where \( k = \sigma^{-1}(\sigma(j) + 1) \) and \( \tau = \sigma(jk) \). Then \( \text{sgn}(\sigma) = -\text{sgn}(\tau) \) and \( \sigma(j) = \tau(k) \), and therefore \( T \) is well-defined. A direct computation yields \( T \circ T = \text{Id}_X \) so that \( T \) is bijective, and
moreover, $\sigma(n, j)_i = \tau(n, k)_i$ for all $i$, which follows directly from (2.1) and (2.3). Applying this to (2.10) yields

$$2B = \sum_{(j, \sigma) \in X} (b_{j, \sigma} + b_{T(j, \sigma)})$$

and so $B = 0$ as all individual terms in the previous sum vanish because $b_{j, \sigma} = -b_{T(j, \sigma)}$ for all $(j, \sigma) \in X$. Hence, combining (2.5) and (2.9), together with $B = 0$, yields (1.7).

**Remark 2.1.** In the above proof, the 3-term recurrence relation (1.2) was invoked in part 2, to obtain the terms $A$, $B$ and $C$. A careful analysis of these terms in part 3 yields the recurrence relation (1.7). However, it is not necessary that the invoked recurrence relation is a 3-term recurrence; as long as in part 3 the resulting terms can be rewritten in terms of Wronskian polynomials of lower degrees, this technique yields a way to derive recurrence relations for Wronskian polynomials.

It is in this way that the generating recurrence relation for Wronskian Appell polynomials, see [2, Theorem 6.3], can be derived without using results of symmetric function theory. Namely, let $(A_n)_{n=0}^{\infty}$ be an Appell sequence, that is $A_0(x) = 1$ and $A'_n(x) = nA_{n-1}(x)$ for all $n \geq 1$. Then, the exponential generating function is of the form

$$\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = e^{xt} f_A(t)$$

for some formal power series $f_A$. Next, set

$$z_n := A_n(0) \quad \text{log}(f_A(t)) := \sum_{n=1}^{\infty} c_n \frac{t^n}{n!}$$

such that

$$f_A(t) = \sum_{n=0}^{\infty} z_n \frac{t^n}{n!} \quad z_n = c_n + \sum_{i=1}^{n-1} \binom{n-1}{i} c_{n-i} z_i$$

(2.11)

see [2, Section 3.3].

**Lemma 2.2.** Any Appell sequence $(A_n)_{n=0}^{\infty}$ can be generated by

$$A_n(x) = xA_{n-1}(x) + \sum_{k=1}^{n} \binom{n-1}{k-1} c_k A_{n-k}(x)$$

(2.12)

for $n \geq 1$, along with the initial condition $A_0(x) = 1$.

**Proof.** Setting $x = 0$ in (2.12) yields, after some elementary rewriting, the expression given in (2.11). Therefore, it is sufficient to show that the polynomials on both sides of the equality (2.12) have the same derivative. Approaching by induction on $n$ and using the Appell property, this yields an easy exercise.

Invoking the recurrence relation (2.12) in part 2 of the above structure gives $n + 1$ terms, which can be rewritten to the terms that appear in [2, Theorem 6.3]. Since the details are similar as those in the proof of Theorem 3.1 in [4], they are left out.

## 3 Average Wronskian Laguerre polynomial

Wronskian Appell polynomials were constructed in [2] for any Appell sequence. Since the sequence of (modified) Laguerre polynomials is an Appell sequence by (1.3), the results obtained
there also hold specifically for Wronskian Laguerre polynomials as stated in [2, Section 7.3]. For example, for any \( n \geq 0 \),

\[
\sum_{\lambda \vdash n} \frac{F_{\lambda}^2}{n!} l_{\lambda}^{(\alpha)}(x) = (x + \alpha)^n
\]  

(3.1)

which describes the average Wronskian Laguerre polynomial of degree \( n \) with respect to the Plancherel measure [1, Definition 1.5], and follows from the weighted average property for Schur functions [12, Corollary 7.12.5]. As shown below, this result can also be proven by induction on \( n \) using the recurrence relation (1.7). This is analogous to the proof of the averaging result for Wronskian Hermite polynomials in [4, Theorem 3.4].

**Proof of (3.1).** By induction on \( n := |\lambda| \). For \( n = 0 \) or \( n = 1 \), the result is trivial because the average is taken of only 1 polynomial. Therefore, suppose that \( n > 1 \) and assume that the statement is true for \( n - 1 \). Applying (1.7) on the left-hand side of (3.1) yields

\[
\sum_{\lambda \vdash n} \frac{F_{\lambda}^2}{n!} l_{\lambda}^{(\alpha)}(x) = \frac{x + \alpha}{n!} \sum_{\lambda \vdash n} \sum_{\mu \vdash n} F_{\lambda} F_{\mu} l_{\mu}^{(\alpha)}(x) - \frac{1}{n!} \sum_{\lambda \vdash n} \sum_{\mu, \nu \vdash \lambda} c(\lambda/\mu) F_{\lambda} F_{\mu} l_{\mu}^{(\alpha)}(x)
\]

\[+ \frac{x(|\lambda| - 1)}{n!} \sum_{\lambda \vdash n} \sum_{\rho \in R^+_{\lambda}} (-1)^{ht(\lambda/\rho)} F_{\lambda} F_{\rho} l_{\rho}^{(\alpha)}(x) \]  

(3.2)

where the first term of the recurrence relation (1.7) is separated into two parts. Next, consider each double sum separately. Interchanging sums leads to the equalities

\[
\sum_{\lambda \vdash n} \sum_{\mu \vdash \lambda} F_{\lambda} F_{\mu} l_{\mu}^{(\alpha)}(x) = \sum_{\mu \vdash \lambda} \sum_{\lambda \vdash n} F_{\lambda} F_{\mu} l_{\mu}^{(\alpha)}(x)
\]

\[
\sum_{\lambda \vdash n} \sum_{\mu \vdash \lambda} c(\lambda/\mu) F_{\lambda} F_{\mu} l_{\mu}^{(\alpha)}(x) = \sum_{\mu \vdash \lambda} \sum_{\lambda \vdash n} F_{\mu} l_{\mu}^{(\alpha)}(x) \sum_{\lambda \vdash n} F_{\lambda} c(\lambda/\mu)
\]

\[
\sum_{\lambda \vdash n} \sum_{\rho \in R^+_{\lambda}} (-1)^{ht(\lambda/\rho)} F_{\lambda} F_{\rho} l_{\rho}^{(\alpha)}(x) = \sum_{\rho \in R^+_{\lambda}} \sum_{\lambda \vdash n} F_{\rho} l_{\rho}^{(\alpha)}(x) \sum_{\lambda \vdash n} (-1)^{ht(\lambda/\rho)} F_{\lambda}
\]

where for each equality, the last sum can be calculated explicitly. Namely, Lemma 7.1 and Lemma 7.2 in [4] state that

\[
\sum_{\lambda \vdash \mu} F_{\lambda} = (|\mu| + 1) F_{\mu} \quad \sum_{\lambda \vdash n} (-1)^{ht(\lambda/\rho)} F_{\lambda} = 0
\]

while Proposition 3.2 below shows that

\[
\sum_{\lambda \vdash n} F_{\lambda} c(\lambda/\mu) = 0
\]

whence only the first double sum in (3.2) does not vanish. So

\[
\sum_{\lambda \vdash n} \frac{F_{\lambda}^2}{n!} l_{\lambda}^{(\alpha)}(x) = \frac{x + \alpha}{n!} \sum_{\mu \vdash \lambda} \frac{F_{\mu}^2}{(n - 1)!} l_{\mu}^{(\alpha)}(x) = (x + \alpha)^n
\]

where the last equality is obtained by the induction hypothesis. This ends the proof. \( \square \)

The following lemma is used in Proposition 3.2.

**Lemma 3.1.** Let \( \mu \) be a partition. Then

\[
\sum_{\lambda \vdash \mu} c(\lambda/\mu) = \sum_{\rho \vdash \mu} c(\mu/\rho)
\]

(3.3)

where \( c(\lambda/\mu) \) is defined in (1.4).
Proof. Write $\mu = (\mu_1^{k_1}, \mu_2^{k_2}, \ldots, \mu_m^{k_m})$, meaning that $\mu_t$ is repeated $k_t$ times for $t = 1, 2, \ldots, m$, such that $\mu_1 > \mu_2 > \cdots > \mu_m > 0$. For every $t$, the last box of the last row with $\mu_t$ elements can be removed to obtain a partition $\rho \preceq \mu$. The content of that box is $\mu_t - \sum_{s=1}^{k_t} k_s$ by (1.4), such that

$$\sum_{\rho \preceq \mu} c(\mu/\rho) = \sum_{t=1}^{m} \left( \mu_t - \sum_{s=1}^{k_t} k_s \right)$$

(3.4)

since these $m$ boxes are the only boxes that can be removed to yield a $\rho \preceq \mu$. Similarly, the left-hand side of (3.3) has $m + 1$ terms. Namely, for every $t = 1, 2, \ldots, m$, a box can be added to the first row of length $\mu_t$, and also a new row can be created at the end. Therefore,

$$\sum_{\lambda \succ \mu} c(\lambda/\mu) = \sum_{t=1}^{m} \left( (\mu_t + 1) - \left( 1 + \sum_{s=1}^{k_t} k_s \right) \right) + 1 - \left( 1 + \sum_{s=1}^{k_t} k_s \right)$$

(3.5)

which follows directly from working out the contents of the boxes that are involved. Finally, it is straightforward to observe that the right-hand sides of (3.4) and (3.5) coincide and hence (3.3) is established.

Proposition 3.2. Let $\mu$ be a partition. Then

$$\sum_{\lambda \succ \mu} F_{\lambda} c(\lambda/\mu) = 0$$

(3.6)

where $c(\lambda/\mu)$ is defined by (1.4).

Proof. The proof is by induction on $n := |\mu|$. If $n = 0$, then $\mu = \emptyset$ and the sum in (3.6) only consists of the term $\lambda = (1)$ and hence the result is trivial. Therefore, let $n > 0$ and assume that the result holds for all partitions of size $k < n$. Note that

$$\sum_{\lambda \succ \mu} F_{\lambda} c(\lambda/\mu) = \sum_{\lambda \succ \mu} \sum_{\gamma \preceq \lambda} F_{\gamma} c(\lambda/\mu)$$

since $F_{\lambda} = \sum_{\gamma \prec \lambda} F_{\gamma}$. The claim is now that

$$\sum_{\lambda \succ \mu} \sum_{\gamma \preceq \lambda} F_{\gamma} c(\lambda/\mu) = \sum_{\rho \preceq \mu} \sum_{\gamma \preceq \rho} F_{\gamma} c(\gamma/\rho)$$

(3.7)

and hence, by applying the induction hypothesis for every $\rho \preceq \mu$ in the last sum, the right-hand side vanishes. This then establishes that (3.6) holds. To prove (3.7), define for any partition $\mu$ the set

$$S(\mu) := \{ \gamma \vdash |\mu| \mid \exists \lambda \succ \mu \text{ such that } \gamma \prec \lambda \text{ and } \gamma \neq \mu \}$$

and for any two partitions $\gamma$ and $\nu$ define the partition $\gamma \land \nu$ (respectively $\gamma \lor \nu$) identified with the intersection (respectively union) of both Young diagrams of $\gamma$ and $\nu$. Then the left-hand side of (3.7) is

$$\sum_{\gamma \in S(\mu)} F_{\gamma} c((\gamma \lor \mu)/\mu) + F_{\mu} \sum_{\lambda \succ \mu} c(\lambda/\mu)$$

(3.8)

whereas the right-hand side equals

$$\sum_{\gamma \in S(\mu)} F_{\gamma} c(\gamma/(\mu \land \gamma)) + F_{\mu} \sum_{\rho \preceq \mu} c(\mu/\rho)$$

(3.9)

because $S(\mu)$ can alternatively be written as

$$S(\mu) = \{ \gamma \vdash |\mu| \mid \exists \rho \preceq \mu \text{ such that } \gamma \succ \rho \text{ and } \gamma \neq \mu \}$$
since the Young lattice is a 1-differential poset [11]. By Lemma 3.1, the last sum in expressions (3.8) and (3.9) coincide. Moreover, for any \( \gamma \in S(\mu) \), the skew partitions \((\gamma \vee \mu)/\mu \) and \( \gamma/(\mu \wedge \gamma) \) are the same, and hence the first sums in (3.8) and (3.9) are equal term by term. This establishes identity (3.7) and therefore ends the proof.

\[ \text{Remark 3.3.} \] The identity in Proposition 3.2 can be expressed in terms of the degree vector via (1.5) and \( \lambda_i = n_i - r + i \) for \( i = 1, 2, \ldots, r \). Simplifying the equation yields the identity

\[
\sum_{k=1}^{r} n_k + 1 - r \prod_{j \neq k} \frac{n_k + 1 - n_j}{n_k - n_j} = r \prod_{k=1}^{r} \frac{n_k}{n_k + 1}
\]

which can be proven by induction on \( r \), i.e., the number of elements, and using the same ideas as the proof of Lemma 8 in [3]. This gives an alternative way to prove identity (3.6).

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\[ \text{References} \]

[1] Baik J., Deift P., Suidan T., Combinatorics and random matrix theory, Graduate Studies in Mathematics, Volume 172, American Mathematical Society, Providence, Rhode Island, 2016.
[2] Bonneux N., Hamaker Z., Stembridge J., Stevens M., Wronskian Appell polynomials and symmetric functions, Advances in Applied Mathematics 111 (2019), 101932.
[3] Bonneux N., Kuijlaars A.B.J., Exceptional Laguerre polynomials, Studies in Applied Mathematics 141 (2018), 547–595.
[4] Bonneux N., Stevens M., Recurrence relations for Wronskian Hermite polynomials, Symmetry Integrability and Geometry: Methods and Applications 14 (2018), 048.
[5] Clarkson P.A., Special polynomials associated with rational solutions of the Painlevé equations and applications to soliton equations, Computational Methods and Function Theory 6 (2006), 329–401.
[6] Durán A.J., Exceptional Meixner and Laguerre orthogonal polynomials, Journal of Approximation Theory 184 (2014), 176–208.
[7] Durán A.J., Higher order recurrence relation for exceptional Charlier, Meixner, Hermite and Laguerre orthogonal polynomials, Integral Transforms and Special Functions 26 (2015), 357–376.
[8] Gómez-Ullate D., Grandati Y., Milson R., Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials, Journal of Physics A: Mathematical and Theoretical 47 (2013), 015203.
[9] Jackson D., Fourier series and orthogonal polynomials, Carus Mathematical Monographs, Volume 6, Mathematical Association of America, Dover publications, Mineola, New York, 1941.
[10] Macdonald I.G., Symmetric functions and Hall polynomials, Oxford Mathematical Monographs, Oxford University Press, Second edition, New York, 1995.
[11] Stanley R.P., Differential posets, Journal of the American Mathematical Society 1 (1988), 919–961.
[12] Stanley R.P., Enumerative combinatorics, Cambridge Studies in Advanced Mathematics, Volume 2, Cambridge University Press, Cambridge, 1999.
[13] Szegő G., Orthogonal polynomials, American Mathematical Society, Colloquium Publications, Volume 23, 4th edition, American Mathematical Society, Providence, Rhode Island, 1975.
[14] Van Assche W., Orthogonal polynomials and Painlevé equations, Australian Mathematical Society Lecture Series, Volume 27, Cambridge University Press, Cambridge, 2018.