Application of the $\tau$-function theory of Painlevé equations to random matrices:
$P_V$, $P_{III}$, the LUE, JUE and CUE

P.J. Forrester
Department of Mathematics and Statistics,
University of Melbourne, Victoria 3010, Australia

AND

N.S. Witte
Department of Mathematics and Statistics, and School of Physics,
University of Melbourne, Victoria 3010, Australia

Abstract

With $\langle \cdot \rangle$ denoting an average with respect to the eigenvalue PDF for the Laguerre unitary ensemble, the object of our study is $\tilde{E}_N(I; a, \mu) := \langle \prod_{l=1}^N \chi_{(0,\infty) \setminus I}(\lambda_l - \lambda_l^0) \rangle$ for $I = (0, s)$ and $I = (s, \infty)$, where $\chi_I^{(l)} = 1$ for $\lambda_l \in I$ and $\chi_I^{(l)} = 0$ otherwise. Using Okamoto’s development of the theory of the Painlevé V equation, it is shown that $\tilde{E}_N(I; a, \mu)$ is a $\tau$-function associated with the Hamiltonian therein, and so can be characterised as the solution of a certain second order second degree differential equation, or in terms of the solution of certain difference equations. The cases $\mu = 0$ and $\mu = 2$ are of particular interest as they correspond to the cumulative distribution and density function respectively for the smallest and largest eigenvalue. In the case $I = (s, \infty)$, $\tilde{E}_N(I; a, \mu)$ is simply related to an average in the Jacobi unitary ensemble, and this in turn is simply related to certain averages over the orthogonal group, the unitary symplectic group and the circular unitary ensemble. The latter integrals are of interest for their combinatorial content. Also considered are the hard edge and soft edge scaled limits of $\tilde{E}_N(I; a, \mu)$. In particular, in the hard edge scaled limit it is shown that the limiting quantity $E^{\text{hard}}((0, s); a, \mu)$ can be evaluated as a $\tau$-function associated with the Hamiltonian in Okamoto’s theory of the Painlevé III equation.

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1 Introduction and summary

In a previous paper \cite{17} the quantities

\begin{equation}
\tilde{E}_N(\lambda; a) := \left\langle \prod_{l=1}^{N} \chi_{(-\infty, \lambda]}^{(l)}(\lambda - \lambda_l)^a \right\rangle_{\text{GUE}}, \quad \chi_{(-\infty, \lambda]}^{(l)} = \begin{cases}
1 & \lambda_l \in (-\infty, \lambda], \\
0 & \text{otherwise}
\end{cases}
\end{equation}

and

\begin{equation}
F_N(\lambda; a) := \left\langle \prod_{l=1}^{N} (\lambda - \lambda_l)^a \right\rangle_{\text{GUE}},
\end{equation}
where the averages are with respect to the joint eigenvalue distribution of the Gaussian unitary ensemble (GUE), were shown to be equal to the \( \tau \)-functions occurring in Okamoto’s theory \[29\] of the Painlevé IV equation. It was noted in \[17\] that we expect the analogous quantities for the Laguerre unitary ensemble (LUE) to be expressible in terms of the \( \tau \)-functions occurring in Okamoto’s theory \[30\] of the Painlevé V equation. It is the purpose of this article to verify this statement by giving the details of the correspondences between the multi-dimensional integrals defining the analogues of \( \tilde{E}_N(\lambda; a) \) and \( F_N(\lambda; a) \) for the LUE, and the \( \tau \)-functions from \[30\].

Let us first recall the definition of the LUE. Let \( X \) be a \( n \times N \) \((n \geq N)\) Gaussian random matrix of complex elements \( z_{jk} \), with each element independent and distributed according to the Gaussian density \( \frac{1}{\pi} e^{-|z_{jk}|^2} \) so that the joint density of \( X \) is proportional to

\[
\exp\left(-\operatorname{Tr} X^\dagger X \right).
\]

(1.3)

Denote by \( A \) the non-negative matrix \( X^\dagger X \). Because (1.3) is unchanged by the replacement \( X \mapsto UXV \) for \( U \) a \( n \times n \) unitary matrix and \( V \) a \( N \times N \) unitary matrix, the ensemble of matrices \( A \) is said to have a unitary symmetry. The probability density function (PDF) for the eigenvalues of \( A \) is given by

\[
\frac{1}{C} \prod_{l=1}^{N} \lambda_l^{a} e^{-\lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2, \quad \lambda_l > 0,
\]

(1.4)

where \( C \) denotes the normalization and \( a = n - N, n \geq N. \) (Throughout, unless otherwise stated, the symbol \( C \) will be used to denote some constant i.e. a quantity independent of the primary variables of the equation.) Because \( \lambda^a e^{-\lambda} \) is the weight function occurring in the theory of the Laguerre orthogonal polynomials, and the ensemble of matrices \( A \) has the aforementioned unitary symmetry, the eigenvalue PDF (1.4) is said to define the LUE.

Analogous to \( \tilde{E}_N(\lambda; a) \) specified by (1.1) for the GUE, we introduce
the quantities

\begin{align}
\hat{E}_N((0, s); a, \mu) &:= \left\langle \prod_{l=1}^{N} \chi_{(s, \infty)}^{(l)}(\lambda_l - s)^\mu \right\rangle_{\text{LUE}} \\
\hat{E}_N((s, \infty); a, \mu) &:= \left\langle \prod_{l=1}^{N} \chi_{(0, s)}^{(l)}(s - \lambda_l)^\mu \right\rangle_{\text{LUE}}
\end{align}

where the averages are with respect to (1.4) (the parameter $a$ in (1.1) has been denoted $\mu$ in (1.5), (1.6) to avoid confusion with the parameter $a$ in (1.4)). Explicitly

\begin{align}
\hat{E}_N((0, s); a, \mu) &= \frac{1}{C} \prod_{l=1}^{N} \int_{s}^{\infty} d\lambda_l \lambda_l^\mu e^{-\lambda_l} (\lambda_l - s)^\mu \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 \\
&= \frac{e^{-Ns}}{C} \prod_{l=1}^{N} \int_{s}^{\infty} d\lambda_l \lambda_l^\mu (\lambda_l + s)^\mu e^{-\lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 \\
&= \frac{e^{-Ns} s^{(a+\mu)N+N^2}}{C} \prod_{l=1}^{N} \int_{0}^{\infty} d\lambda_l \lambda_l^\mu (\lambda_l + 1)^\mu e^{-s\lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2
\end{align}

\begin{align}
\hat{E}_N((s, \infty); a, \mu) &= \frac{1}{C} \prod_{l=1}^{N} \int_{0}^{s} d\lambda_l \lambda_l^\mu e^{-\lambda_l} (s - \lambda_l)^\mu \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 \\
&= \frac{s^{(a+\mu)N+N^2}}{C} \prod_{l=1}^{N} \int_{0}^{1} d\lambda_l \lambda_l^\mu (1 - \lambda_l)^\mu e^{-s\lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2.
\end{align}

In the case $\mu = 0$ the first integrals in (1.7) and (1.8) are the definitions of the probability that there are no eigenvalues in the intervals $(0, s)$ and $(s, \infty)$ respectively of the Laguerre unitary ensemble. The case $\mu = 2$ also has significance in this context. To see this, first note from the definitions (using the first integral representation in each case) that

\begin{align}
\frac{d}{ds} \hat{E}_{N+1}((0, s); a, 0) &\propto s^a e^{-s} \hat{E}_N((0, s); a, 2) \\
\frac{d}{ds} \tilde{E}_{N+1}((s, \infty); a, 0) &\propto s^a e^{-s} \tilde{E}_N((s, \infty); a, 2).
\end{align}
On the other hand, with \( p_{\text{min}}(s; a) \) and \( p_{\text{max}}(s; a) \) denoting the distribution of the smallest and largest eigenvalue respectively in the \( N \times N \) LUE, we have

\[
\begin{align*}
\tag{1.11} p_{\text{min}}(s; a) &= -\frac{d}{ds} \tilde{E}_N((0, s); a, 0) \\
\tag{1.12} p_{\text{max}}(s; a) &= \frac{d}{ds} \tilde{E}_N((s, \infty); a, 0).
\end{align*}
\]

Under the replacement \( N \mapsto N + 1 \), \( p_{\text{min}}(s; a) \) and \( p_{\text{max}}(s; a) \) are determined by \( \tilde{E}_N((0, s); a, 2) \) and \( \tilde{E}_N((s, \infty); a, 2) \) respectively.

From the second formula in (1.7), we see that with \( F_N(s; a, \mu) := \left( \prod_{l=1}^{N} (\lambda_l - s)^\mu \right) \) \(_{\text{LUE}} \)
we have

\[
\tag{1.13} F_N(s; a, \mu) = \left( e^{Ns} \tilde{E}_N((0, s); \mu, a) \right) \bigg|_{s \to -s}
\]
(notice the dual role played by \( \mu \) and \( a \) on the different sides of (1.14)) so there is no need to consider \( F_N \) separately. Note that with \( \mu = 2 \), (1.13) multiplied by \( s^a e^{-s} \) is proportional to the definition of the density in the LUE with \( N \mapsto N + 1 \). We also remark that for \( \mu \) non-integer the definition (1.13) must be complemented by a definite choice of branch of the function \( (\lambda_l - s)^\mu \). But this case does not appear in our random matrix applications so we will not address the issue further.

Some quantities generalizing (1.7) and (1.8), and which include (1.13) are

\[
\tag{1.15} \tilde{E}_N((0, s); a, \mu; \xi) := \frac{1}{C} \prod_{l=1}^{N} \left( \int_{0}^{\infty} -\xi \int_{0}^{s} \right) d\lambda_l \lambda_l^\mu e^{-\lambda_l (\lambda_l - s)^\mu} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2
\]

\[
\tag{1.16} \tilde{E}_N((s, \infty); a, \mu; \xi) := \frac{1}{C} \prod_{l=1}^{N} \left( \int_{0}^{\infty} -\xi \int_{s}^{\infty} \right) d\lambda_l \lambda_l^\mu e^{-\lambda_l (s - \lambda_l)^\mu} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2.
\]
Thus
\[
\tilde{E}_N((0, s); a, \mu; 0) = F_N(s; a, \mu) \\
\tilde{E}_N((0, s); a, \mu; 1) = \tilde{E}_N((0, s); a, \mu) \\
\tilde{E}_N((s, \infty); a, \mu; 1) = \tilde{E}_N((s, \infty); a, \mu).
\]

Note that only one of the quantities (1.15), (1.16) is independent since
\[
\tilde{E}_N((s, \infty); a, \mu; \xi) = e^{-\pi i N \mu} \left( \frac{1}{(1 - \lambda)^N} \tilde{E}_N((0, s); a, \mu; \lambda) \right) \bigg|_{\lambda = \xi/(\xi - 1)}.
\]

Their interest stems from the facts that
\[
\frac{(-1)^n}{n!} \frac{\partial^n}{\partial \xi^n} \tilde{E}_N((0, s); a, 0, \xi) \bigg|_{\xi = 1} = E_N(n; (0, s)) \\
\frac{(-1)^n}{n!} \frac{\partial^n}{\partial \xi^n} \tilde{E}_N((s, \infty); a, 0, \xi) \bigg|_{\xi = 1} = E_N(n; (s, \infty))
\]
where \(E_N(n; I)\) denotes the probability that in the LUE the interval \(I\) contains precisely \(n\) eigenvalues.

The second integral in (1.8) is of interest for its relevance to the Jacobi unitary ensemble, which is specified by the eigenvalue PDF
\[
\frac{1}{C} \prod_{l=1}^{N} \lambda_l^a (1 - \lambda_l)^b \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2, \quad 0 < \lambda_l < 1.
\]

This ensemble is realised by matrices of the form \((A + B)^{-1}\), where \(A = X^\dagger X, B = Y^\dagger Y\), for \(X, Y\) an \(n_1 \times N\) \((n_2 \times N)\) complex Gaussian random matrix with joint density (1.3). The parameters \(a\) and \(b\) are then specified by \(a = n_1 - N, b = n_2 - N\) (c.f. the value of \(a\) in (1.4)). We see from (1.17) that
\[
\frac{1}{C} \prod_{l=1}^{N} \int_0^1 d\lambda_l \lambda_l^a (1 - \lambda_l)^b e^{s \lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2 = \left< e^{s \sum_{j=1}^{N} \lambda_j} \right>_{\text{JUE}},
\]
so substituting in (1.8) and compensating for the different normalizations in (1.8) and (1.17) shows
\[
\tilde{E}_N((s, \infty); a, \mu) = \frac{J_N(a, \mu)}{I_N(a)} \varepsilon^{(a+\mu)N+2} \left< e^{-s \sum_{j=1}^{N} \lambda_j} \right>_{\text{JUE}} \bigg|_{b \to \mu}
\]
where
\[
J_N(a, \mu) = \prod_{l=1}^{N} \lambda_l^a (1 - \lambda_l)^b \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2
\]
and
\[
I_N(a) = \int_0^1 \prod_{l=1}^{N} d\lambda_l \lambda_l^a (1 - \lambda_l)^b \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2.
\]
where

\[
I_N(a) := \prod_{l=1}^{N} \int_{0}^{1} d\lambda_l \, \lambda_l^a e^{-\lambda_l} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2
\]

(1.20)

\[
J_N(a, \mu) := \prod_{l=1}^{N} \int_{0}^{1} d\lambda_l \, \lambda_l^a (1 - \lambda_l)^\mu \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2.
\]

(1.21)

As is well known, the integrals \(I_N(a)\) and \(J_N(a, \mu)\) can be evaluated in the form \(N! \prod_{j=0}^{N-1} c_j^2\), where \(c_j\) is the normalization of the monic orthogonal polynomial of degree \(j\) associated with the weight functions \(\lambda^a e^{-\lambda}\) and \(\lambda^a (1 - \lambda)^\mu\) respectively.

One important feature of the JUE is that with the change of variables

\[
\lambda_j = \frac{1}{2} (\cos \theta_j + 1),
\]

(1.22)

the PDF (1.17) assumes a trigonometric form, which for appropriate \((N, a, b)\) coincides with the PDF for the independent eigenvalues \(e^{i\theta_j}\) of random orthogonal and random unitary symplectic matrices. In the case of orthogonal matrices, one must distinguish the two classes \(O_N^+\) and \(O_N^-\) according to the determinant equalling +1 or −1 respectively. The cases of \(N\) even and \(N\) odd must also be distinguished. For \(N\) odd all but one eigenvalue come in complex conjugate pairs \(e^{\pm i\theta_j}\), with the remaining eigenvalue equalling +1 for \(O_N^+\) and −1 for \(O_N^-\). For \(N\) even, all eigenvalues of \(O_N^+\) come in complex conjugate pairs, while for \(O_N^-\) all but two eigenvalues come in complex conjugate pairs, with the remaining two equalling ±1. Let us replace \(N\) in (1.17) by \(N^*\) and make the change of variables (1.22). Then the PDF for the independent eigenvalues of an ensemble of random orthogonal matrices is (see e.g. [10]) of the form (1.17) with

\[
(N^*, a, b) = \begin{cases} 
(N/2, -1/2, -1/2) & \text{for matrices in } O_N^+, \; N \text{ even} \\
((N - 1)/2, -1/2, 1/2) & \text{for matrices in } O_N^+, \; N \text{ odd} \\
((N - 1)/2, 1/2, -1/2) & \text{for matrices in } O_N^-, \; N \text{ odd} \\
(N/2 - 1, 1/2, 1/2) & \text{for matrices in } O_N^-, \; N \text{ even}
\end{cases}
\]

(1.23)
Matrices in the group $USp(N)$ are equivalent to $2N \times 2N$ unitary matrices in which each $2 \times 2$ block has a real quaternion structure. The eigenvalues come in complex conjugate pairs $e^{\pm i\theta_j}$, and the PDF of the independent elements is of the form (1.17) with the change of variables (1.22) and

\[(N^*, a, b) = (N, 1/2, 1/2).\]

It follows from the above revision that

\[\left< e^{s\text{Tr}(U)} \right>_{U \in G}\]

for $G = O_N^+, O_N^- \text{ or } USp(N)$ is a special case of the more general average (1.18). Explicitly,

\[\left< e^{s\text{Tr}(U)} \right>_{U \in G} = e^{-2sN^*}e^{\chi_{N^*}s}\left< e^{4s\sum_{j=1}^{N^*} \lambda_j} \right>_{\text{JUE}}\]

where on the RHS the dimension of the JUE is at first $N^*$, then the parameters $(N^*, a, b)$ are specified as in (1.23) or (1.24). Also, $\chi_{N^*} = 0$ for $G = USp(N)$ and $O_N^+, O_N^-$ ($N$ even), while $\chi_{N^*} = \pm 1$ for $G = O_N^+, O_N^-$ ($N$ odd) respectively. The averages (1.25) have independent importance due to their occurrence as generating functions for certain combinatorial problems. Let us quote one example. Set $f_{nl}^{(\text{inv})}$ equal to the number of fixed point free involutions of $\{1, 2, \ldots, 2n\}$ constrained so that the length of the maximum decreasing subsequence is less than or equal to $2l$, and introduce the generating function

\[P_l(t) := e^{-t^2/2} \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n n! (2n - 1)!!} f_{nl}^{(\text{inv})}.\]

According to a result of Rains [32] (see also [3]) we have

\[P_l(t) = e^{-t^2/2} \left< e^{l\text{Tr}(U)} \right>_{U \in USp(l)}.\]

A list of similar results is summarised in [2].

The second integral in (1.8) can be written in a trigonometric form distinct from that which results from the substitution (1.22). For this
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purpose we recall the integral identity \[13\]

\[(1.29) \quad \prod_{l=1}^{N} \int_{0}^{1} dt_l t_l^{N-1} f(t_1, \ldots, t_N) = \left(\frac{\pi}{\sin \pi \epsilon}\right)^N \prod_{l=1}^{N} \int_{-1/2}^{1/2} dx_l e^{2\pi i x_l} f(-e^{2\pi i x_1}, \ldots, -e^{2\pi i x_N})\]

valid for $f$ a Laurent polynomial. Applying (1.29) to (1.21) gives the trigonometric integral

\[(1.30) \quad M_N(a',b') := \prod_{l=1}^{N} \int_{-1/2}^{1/2} dx_l \, e^{\pi i x_l(a'-b')} |1 + e^{2\pi i x_l}|^{a'+b'} \times \prod_{1 \leq j<k \leq N} |e^{2\pi i x_k} - e^{2\pi i x_j}|^2.\]

Moreover, applying (1.29) to the LHS of (1.18) and using Carlson’s theorem it follows that

\[(1.31) \quad \left\langle e^{\sum_{j=1}^{N} \lambda_j}\right\rangle_{\text{JUE}} \big|_{b \rightarrow \mu} = \frac{M_N(0,0)}{M_N(a',b')} \left(\prod_{l=1}^{N} e^{\pi i x_l(a'-b')} |1 + e^{2\pi i x_l}|^{a'+b'} e^{se^{2\pi i x_l}} \right)_{\text{CUE}},\]

where $a' = N + a + \mu$, $b' = -(N + a)$ and the CUE (circular unitary ensemble) average is over the eigenvalue PDF

\[(1.32) \quad \frac{1}{C} \prod_{1 \leq j<k \leq N} |e^{2\pi i x_k} - e^{2\pi i x_j}|^2, \quad -1/2 \leq x_l \leq 1/2.\]

Recalling (1.19) then gives

\[(1.33) \quad \left\langle \prod_{l=1}^{N} e^{\pi i x_l(a'-b')} |1 + e^{2\pi i x_l}|^{a'+b'} e^{-se^{2\pi i x_l}} \right\rangle_{\text{CUE}} = \frac{M_N(a',b') I_N(a)}{M_N(0,0) J_N(a,\mu)} s^{-(a+\mu)N-N^2} \tilde{E}_N((s,\infty);a,\mu),\]

although the RHS is convergent for only a subset of the parameter values for which the LHS converges (explicitly the RHS requires Re$(a'+b') > -1$
and \( \text{Re}(-N-b') > -1 \) to be properly defined, while the LHS only requires \( \text{Re}(a' + b') > -1 \). Note however that by replacing the integrals over \( \lambda_l \in [0,1] \) in (1.8) by the Barnes double loop integral \([41]\) the quantity \( \tilde{E}_N((s,\infty);a,\mu) \) has meaning for general values of \( a \) and \( \mu \). In the case \( a' = 0, b' = k \in \mathbb{Z}_{>0} \), the LHS of (1.33) reads

\[
(1.34) \quad \left\langle \prod_{l=1}^{N}(1 + e^{-2\pi i x_l})^k e^{-\sigma_0 x_l} \right\rangle_{\text{CUE}} = \left\langle \det(1 + \bar{U})^k e^{-\sigma_{\text{Tr}(U)}} \right\rangle_{U \in \text{CUE}}.
\]

Like (1.28) this latter average with the dimension \( N \) replaced by \( l \) is the generating function for certain combinatorial objects \([38]\) (random words from an alphabet of \( k \) letters with maximum increasing subsequence length constrained to be less than or equal to \( l \)).

As we have already stated, our interest is in the evaluation of (1.8) and (1.7) as \( \tau \)-functions occurring in Okamoto’s theory of \( P_V \). In some cases such evaluations, or closely related evaluations, are already available in the literature. One such case is (1.8) and (1.7) with \( \mu = 0 \), for which Tracy and Widom \([35]\) (for subsequent derivations see \([1, 20, 7]\)) have shown

\[
(1.35) \quad \tilde{E}_N((0,s);a,0) = \exp \int_0^s \frac{\sigma(t)}{t} \, dt \\
(1.36) \quad \tilde{E}_N((s,\infty);a,0) = \exp \left( -\int_s^\infty \frac{\sigma(t)}{t} \, dt \right)
\]

where \( \sigma(t) \) satisfies the Jimbo-Miwa-Okamoto \( \sigma \)-form of the Painlevé V equation,

\[
(1.37) \quad (t\sigma'') - \left[ \sigma - t\sigma' + 2(\sigma')^2 + (\nu_0 + \nu_1 + \nu_2 + \nu_3)\sigma' \right]^2 \\
+ 4(\nu_0 + \sigma')(\nu_1 + \sigma')(\nu_2 + \sigma')(\nu_3 + \sigma') = 0,
\]

with

\[
(1.38) \quad \nu_0 = 0, \quad \nu_1 = 0, \quad \nu_2 = N + a, \quad \nu_3 = N
\]

(or any permutation thereof, since (1.37) is symmetrical in \( \{\nu_k\} \)), and subject to appropriate boundary conditions in each case. For general \( \mu \)
the second integral in (1.8) has been written in an analogous form to the right hand side of (1.35) by Adler and van Moerbeke [2], with the function corresponding to $\sigma$ presented as the solution of a certain third order differential equation. It is remarked in [2] that this equation can be reduced to a second order second degree equation, known to be equivalent to (1.37), using results due to Cosgrove [9, 8] (see also the discussion in the final section of [42]). The average (1.34) can be computed in terms of Painlevé transcendents by making use of an identity [11] implying

\[
\langle \det(1 + \bar{U}^k e^{-sT(U)}) \rangle_{U \in \text{CUE}} = e^{ks} \tilde{E}_k((0, s); N, 0),
\]

(note the dual role played by $k$ and $N$ on the different sides of (1.39)) then appealing to (1.35) (direct evaluations have been given by Tracy and Widom [38] and Adler and van Moerbeke [2]).

The Okamoto $\tau$-function theory of $P_V$ provides a unification and extension of these results. Consider first (1.7). Proposition 3.2 below gives

\[
\tilde{E}_N((0, s); a, \mu) = \frac{I_N(a+\mu)}{I_N(a)} \exp \int_0^s \frac{V_N(t; a, \mu) + \mu N}{t} dt
\]

\[
= \frac{I_N(\mu)}{I_N(a)} s^N a e^{-Ns} \exp \left( - \int_s^\infty \frac{V_N(t; a, \mu) + Nt - N(a - \mu)}{t} dt \right)
\]

where $V_N(t; a, \mu)$ satisfies the Jimbo-Miwa-Okamoto $\sigma$ form of $P_V$ (1.37) with

\[
\nu_0 = 0, \quad \nu_1 = -\mu, \quad \nu_2 = N + a, \quad \nu_3 = N,
\]

and subject to the boundary condition

\[
V_N(t; a, \mu) \sim_{t \to \infty} -Nt + N(a - \mu) - \frac{N(N + \mu)a}{t} + O(\frac{1}{t^2}).
\]

Regarding (1.8), in Proposition 3.1 below we show

\[
\tilde{E}_N((s, \infty); a, \mu) = \frac{J_N(a, \mu)}{I_N(a)} s^{(a+\mu)N + N^2} \exp \int_s^\infty \frac{U_N(t; a, \mu) - aN - N^2}{t} dt
\]

\[
= s^{N\mu} \exp \left( - \int_s^\infty \frac{U_N(t; a, \mu)}{t} dt \right)
\]
where like \( V_N(t; a, \mu) \), \( U_N(t; a, \mu) \) satisfies (1.37) with parameters (1.41), but with the boundary condition

\[
U_N(t; a, \mu) \sim aN + N^2 - N \frac{a + N}{a + \mu + 2N} t.
\]

As well as \( V_N(t; a, \mu) \) and \( U_N(t; a, \mu) \) satisfying the differential equation (1.37), we show in Proposition 3.5 that they also satisfy third order difference equations in both the \( a \) and \( \mu \) variables.

By substituting the first equation of (1.43) in (1.19) and (1.33) we obtain the evaluations

\[
\left\langle e^{-s \sum_{j=1}^{N} \lambda_j} \right\rangle_{JUE} \bigg|_{b \rightarrow \mu} = \exp \int_0^s \frac{U_N(t; a, \mu) - aN - N^2}{t} dt,
\]

\[
\left\langle \prod_{i=1}^{N} e^{\pi i x_{i}(a'-b')} |1 + e^{2\pi i x_{i}} |a'+b'| e^{-se^{2\pi i x_{i}}} \right\rangle_{CUE} = \frac{M_N(a', b')}{M_N(0, 0)} \exp \int_0^s \frac{U_N(t; a, \mu) - aN - N^2}{t} dt \bigg|_{a=(N+b'), \mu=a'+b'}.
\]

Comparison of (1.45) with (1.26) gives that for \( G = O_N^{\pm}, USp(N) \),

\[
e^{2sN} \left\langle e^{-s \text{Tr}(U)} \right\rangle_{U \in G} = e^{2sN} \exp \int_0^{4s} \frac{U_N^*(t; a, b) - aN^* - N^{*2}}{t} dt
\]

with \((N^*, a, b)\) specified by (1.23) or (1.24) as appropriate. In particular, with \( G = USp(N) \) and thus \((N^*, a, b)\) given by (1.24), substitution into (1.28) shows

\[
P_1(s) = e^{-s^2/2} e^{2st} \exp \int_0^{4s} \frac{U(t; 1/2, 1/2) - 1/2 - l^2}{t} dt
\]

After recalling the statement below (1.12) and inserting the proportionality constants in (1.9) and (1.10), the evaluations (1.40) and (1.43)
give

\begin{equation}
\left. p_{\min}(s;a) \right|_{N \to N+1} = (N+1) \frac{I_N(a+2)}{I_{N+1}(a)} s^{a} e^{-s} \exp \int_{0}^{s} \frac{V_N(t; a, 2) + 2N}{t} dt
\end{equation}

(1.50)

\begin{equation}
\left. p_{\max}(s;a) \right|_{N \to N+1} = (N+1) \frac{I_N(a)}{I_{N+1}(a)} s^{a+2} e^{-s} \exp \left( - \int_{s}^{\infty} \frac{U_N(t; a, 2)}{t} dt \right).
\end{equation}

Also, recalling (1.14) and the sentence below that equation, we see from (1.40) that

\begin{equation}
\left. \rho(s) \right|_{N \to N+1} = (N+1) \frac{I_N(a+2)}{I_{N+1}(a)} s^{a} e^{-(N+1)s} \exp \left( - \int_{0}^{s} \frac{V_N(t; 2; a) + aN}{t} dt \right),
\end{equation}

where $\rho(s)$ denotes the eigenvalue density in the LUE.

According to Proposition 3.3

\begin{equation}
\bar{E}_N((0, s); a, 0; \xi) = \exp \int_{0}^{s} \frac{W_N(t; a, 0)}{t} dt
\end{equation}

where $W_N(t; a, 0)$ satisfies (1.37) with parameters (1.41), $\mu$ set equal to zero, and boundary condition

\begin{equation}
W_N(t; a, 0) \underset{t \to 0}{\sim} -\xi \frac{\Gamma(N+a+1)}{\Gamma(N)\Gamma(a+1)\Gamma(a+2)} t^{a+1}.
\end{equation}

This result is known from [35].

A generalization of the integral identity (1.39) is derived in the context of the Painlevé theory in Proposition 3.7 below. We remark that this generalization is in fact a special case of a still more general identity, known from an earlier study [11], involving an arbitrary parameter $\beta$. For $\beta = 1$ an average in the Laguerre orthogonal ensemble is related to an average in the circular symplectic ensemble, while for $\beta = 4$ an average in the Laguerre symplectic ensemble is related to an average in the circular orthogonal ensemble.
The Laguerre ensemble permits four scaled, large $N$ limits. These are

\begin{equation}
\label{eq:1.54}
 s \mapsto s/4N, \ N \to \infty
\end{equation}

\begin{equation}
\label{eq:1.55}
 s \mapsto 4N + 2(2N)^{1/3}s, \ N \to \infty
\end{equation}

\begin{equation}
\label{eq:1.56}
a = (\gamma - 1)N, \ s \mapsto N(1 - \sqrt{\gamma})^2 - \nu_-(N)s, \ N \to \infty,
\end{equation}

\begin{equation}
\label{eq:1.57}
a = (\gamma - 1)N, \ s \mapsto N(1 + \sqrt{\gamma})^2 + \nu_+(N)s, \ N \to \infty,
\end{equation}

where in (1.56), (1.57) we require $\gamma > 1$ and

\begin{equation}
\nu_-(N) = \left[N(\sqrt{\gamma} - 1)(1 - \frac{1}{\sqrt{\gamma}})\right]^{\frac{1}{3}}, \quad
\nu_+(N) = \left[N(\sqrt{\gamma} + 1)(1 + \frac{1}{\sqrt{\gamma}})\right]^{\frac{1}{3}}.
\end{equation}

The first gives the limiting distributions at the hard edge, so called because the eigenvalue density is strictly zero on one side. The second, third and fourth give the limiting distribution at the soft edge, so called because there is a non-zero density for all values of the new coordinates, but with a fast decrease on one side.

The soft edge limit for the quantities (1.1) and (1.2) in the GUE was studied in our work [17], where evaluations in terms of solutions of the general Jimbo-Miwa-Okamoto $\sigma$-form of $P_{II}$

\begin{equation}
(\nu')^2 + 4u'\left((u')^2 - tu' + u\right) - \alpha^2 = 0.
\end{equation}

were given. From either of the limiting procedures (1.55), (1.56) or (1.57) we reclaim the results of [17] (because no new functional forms appear we will not discuss this case further in subsequent sections). In particular, it follows from the second equation in (1.43) that

\begin{equation}
\label{eq:1.60}
\tilde{E}_{\text{soft}}(s; \mu) = \lim_{s \to 4N + 2(2N)^{1/3}s}
\lim_{N \to \infty} Ce^{-\mu^2/2} \tilde{E}_N((s, \infty); a, \mu)
\end{equation}

\begin{equation}
= \tilde{E}_{\text{soft}}(s_0; \mu) \exp \int_{s_0}^{s} u(t; \mu) \, dt,
\end{equation}

where

\begin{equation}
\label{eq:1.61}
u(t, \mu) = \lim_{N \to \infty} \frac{2(2N)^{1/3}}{4N} \left(U_N(t; a, \mu) + N\mu - \frac{\mu}{2}t\right)\bigg|_{t \to 4N + 2(2N)^{1/3}t}.
\end{equation}
Making the replacements
\[
\begin{align*}
\sigma &\mapsto \sigma - N\mu + \frac{\mu}{2}t, \\
t &\mapsto 4N + 2(2N)^{1/3}t, \\
\sigma(4N + 2(2N)^{1/3}t) &\mapsto (2N)^{2/3}u(t; \mu)
\end{align*}
\]
in (1.37) with parameters (1.41), and equating terms of order \( N^2 \) in the equation (which is the leading order) shows \( u(t; \mu) \) satisfies (1.59) with \( \alpha = \mu \). We know from \([17]\) that (1.59) is to be solved subject to the boundary condition
\[
(1.62) \quad u(t; \mu) \underset{t \to -\infty}{\sim} \frac{1}{4}t^2 + \frac{4\mu^2 - 1}{8t} + \frac{(4\mu^2 - 1)(4\mu^2 - 9)}{64t^4} + \ldots
\]
As an application, (1.61) can be used to compute the scaled limit
\[
P(s) := \lim_{l \to \infty} P_l\left(l - \frac{1}{2}(2l)^{1/3}s\right).
\]
Imposing the condition that \( P(s) \to 1 \) as \( s \to \infty \) (which fixes the constants), a short calculation using (1.61) in (1.48) shows
\[
(1.63) \quad P(s) = \exp\left( -\int_s^\infty (u(-t; 1/2) - t^2/4) \, dt \right).
\]
This is interesting because it has been proved by Baik and Rains \([4]\) that
\[
(1.64) \quad P(2^{1/3}s) = F_1(s),
\]
where \( F_1(s) \) is the cumulative distribution function for the largest eigenvalue in the scaled, infinite Gaussian orthogonal ensemble. Tracy and Widom \([37]\) (see \([15]\) for a simplified derivation) have shown that
\[
(1.65) \quad F_1(s) = e^{-\frac{1}{2} \int_s^\infty (t-s)q^2(t) \, dt} e^{\frac{1}{2} \int_s^\infty q(t) \, dt}
\]
where with \( \text{Ai}(t) \) denoting the Airy function, \( q(t) \) is the solution of the non-linear equation
\[
(1.66) \quad q'' = 2q^3 + tq,
\]
(Painlevé II equation with \( \alpha = 0 \)) subject to the boundary condition
\[
(1.67) \quad q(t) \sim -\text{Ai}(t) \quad \text{as} \quad t \to \infty.
\]
By equating (1.63) and (1.64) we obtain as an alternative to (1.65) the formula

\begin{equation}
F_1(2^{-1/3}s) = \exp \left( - \int_{s}^{\infty} \left( u(-t; 1/2) - t^2/4 \right) dt \right).
\end{equation}

However the boundary condition (1.62) gives \( u(-t; 1/2) - t^2/4 \sim 0 \) which is vacuous, whereas the boundary condition according to the requirement (1.67) is

\[ 2^{-1/3} \frac{d}{dt} (u(-t; 1/2) - t^2/4) \bigg|_{t \to 2^{-1/3} t} \sim \frac{1}{2} \left[ - \text{Ai}'(t) + \left( \text{Ai}(t) \right)^2 \right]. \]

The significance of (1.68) from the viewpoint of gap probabilities in random matrix ensembles as \( \tau \)-functions for Hamiltonians associated with Painlevé systems will be discussed in a separate publication [19].

For the hard edge scaling (1.54), define the scaled version of (1.7) by

\begin{equation}
\tilde{E}_{\text{hard}}(s; a, \mu) := \lim_{s \to \infty} \frac{I_N(a)}{I_N(a+\mu)} \tilde{E}_N((0, s); a, \mu).
\end{equation}

In the case \( \mu = 0 \), it is known from the work of Tracy and Widom [36] that

\begin{equation}
\tilde{E}_{\text{hard}}(s; a, 0) = \exp \left( - \int_{0}^{s} \frac{\sigma_B(t)}{t} dt \right)
\end{equation}

where \( \sigma_B(t) \) satisfies the Jimbo-Miwa-Okamoto \( \sigma \)-form of the Painlevé III equation

\begin{equation}
(t \sigma''')^2 - v_1 v_2 (\sigma')^2 + \sigma'(4\sigma' - 1)(\sigma - t \sigma') - \frac{1}{45} (v_1 - v_2)^2 = 0
\end{equation}

with

\begin{equation}
v_1 = a, \quad v_2 = a.
\end{equation}

It follows from Proposition 4.7 below that for general \( \mu \)

\begin{equation}
\tilde{E}_{\text{hard}}(s; a, \mu) = \exp \left( - \int_{0}^{s} \frac{\sigma(t) + \mu(\mu + a)/2}{t} dt \right)
\end{equation}
where $\sigma$ satisfies (1.71) with

$$v_1 = a + \mu, \quad v_2 = a - \mu$$

and subject to the boundary condition

$$\sigma(t) \sim t \to \infty \frac{t^{1/2}}{4} - \frac{at^{1/2}}{2} + \left(\frac{a^2}{4} - \frac{\mu^2}{2}\right).$$

In Proposition 4.8 we show that $v(t; a, \mu) = -\sigma(t) - \mu(\mu + a)/2$ satisfies third order difference equations in both $a$ and $\mu$.

One application of (1.73) is to the evaluation of

$$p_{\text{hard}}(s; a) := \lim_{N \to \infty} \frac{1}{4N} p_{\text{min}}\left(\frac{s}{4N}; a\right).$$

First we note that the explicit evaluation of (1.21) is

$$\prod_{j=1}^{N} \frac{\Gamma(1+j)}{\Gamma(a+j)},$$

which together with the asymptotic formula $\Gamma(x + a)/\Gamma(x) \sim x^a$ for $x \to \infty$ shows that

$$\lim_{N \to \infty} \frac{1}{4N} (N+1) \frac{I_N(a+2)}{I_{N+1}(a)} s^a e^{-s} \bigg|_{s \to s/4N} = \frac{s^a}{2^{a+2} \Gamma(a+1) \Gamma(a+2)}.$$

Using this formula and (1.73) we can take the large $N$ limit in (1.49) as required by (1.76) to conclude

$$p_{\text{hard}}(s; a) = \frac{s^a}{2^{a+2} \Gamma(a+1) \Gamma(a+2)} \exp\left(- \int_{0}^{s} \frac{\sigma(t) + a + 2}{t} \, dt\right)$$

where $\sigma(t)$ satisfies (1.71) with

$$v_1 = a + 2, \quad v_2 = a - 2$$

and is subject to the boundary condition (1.74) with $\mu = 2$. Similarly, it follows that the hard edge density is given by

$$\rho_{\text{hard}}(s) = \frac{s^a e^{-s/4}}{2^{a+2} \Gamma(a+1) \Gamma(a+2)} \exp\left(\int_{-\infty}^{0} \frac{\sigma(t) + a(a+2)/2}{t} \, dt\right)$$

where $\sigma(t)$ satisfies (1.71) with

$$v_1 = 2 + a, \quad v_2 = 2 - a.$$
and is subject to the boundary condition \((1.75)\) with \((a, \mu) = (2, a)\).

The result \((1.73)\) also has relevance to the CUE. This occurs via the scaled limit of \((1.39)\), which has been shown \([1]\) to imply for \(a \in \mathbb{Z}_{\geq 0}, \mu \in \mathbb{Z}\), the identity

\[
\tilde{E}_{\text{hard}}(t; a, \mu) \propto e^{-t/4} e^{-\mu a/2} \left\langle e^{\frac{1}{2} \sqrt{\text{Tr}(U+\bar{U})}} (\det U)^{-\mu} \right\rangle_{U \in \text{CUE}_a}
\]

(eq. \((4.32)\) below; the proportionality constant is specified in \((4.33)\)). It thus follows that this CUE\(_a\) average can be evaluated in terms of the transcendent \(\sigma(t)\) in \((1.73)\) (in the case \(\mu = 0\) this has been shown directly in the previous works \([39, 2]\)).

This concludes the summary of our results. In the next section (Section 2) an overview of the Okamoto \(\tau\)-function theory of the Painlevé V equation is given. In particular a sequence of \(\tau\)-functions \(\tau[n]\), each of which can be characterised as the solution of a certain second order second degree equation, is shown to satisfy a Toda lattice equation. The fact that a special choice of initial parameters in the sequence permits the evaluations \(\tau[0] = 1\) and \(\tau[1]\) an explicit function of \(t\) (a confluent hypergeometric function) then implies that the general member of the sequence \(\tau[n]\) is given explicitly as a Wronskian type determinant depending on \(\tau[1]\). In Section 3 the Wronskian type determinants from Section 2 are evaluated in terms of the multiple integrals \((1.7)\) and \((1.8)\), thus identifying them as \(\tau\)-functions and implying their characterization as the solution of a non-linear second order second degree equation.

In Section 4 the Okamoto \(\tau\)-function theory of the Painlevé III equation is revised, and again a \(\tau\)-function sequence is identified which can be shown to coincide with a quantity in the LUE (explicitly \(\tilde{E}_{\text{hard}}(s; a, \mu)\) for general \(\mu\) and \(a \in \mathbb{Z}_{\geq 0}\)). By scaling results from Section 3 it is shown that for general \(\mu\) and general \(a\), \(\tilde{E}_{\text{hard}}(s; a, \mu)\) as specified in \((1.69)\) is a \(\tau\)-function in the \(P_{III}\) theory. Concluding remarks relating to the boundary conditions \((1.75)\) and \((1.62)\) are given in Section 5.

2 Overview of the Okamoto \(\tau\)-function theory of \(P_V\)
2.1 The Jimbo-Miwa-Okamoto $\sigma$-form of $P_V$

As formulated in [30], the Okamoto $\tau$-function theory of the fifth Painlevé equation $P_V$ is based on the Hamiltonian system $\{Q, P, t, K\}$

\[ tK = Q(Q - 1)^2 P^2 - [(v_2 - v_1)(Q - 1)^2 - 2(v_1 + v_2)Q(Q - 1) + tQ]P + (v_3 - v_1)(v_4 - v_1)(Q - 1) \]

where the parameters $v_1, \ldots, v_4$ are constrained by

\[ v_1 + v_2 + v_3 + v_4 = 0. \]

The relationship of (2.1) to $P_V$ can be seen by eliminating $P$ in the Hamilton equations

\[ Q' = \frac{\partial K}{\partial P}, \quad P' = -\frac{\partial K}{\partial Q}. \]

One finds that $Q$ satisfies the equation

\[ y'' = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) (y')^2 - \frac{1}{t} y' + \frac{(y - 1)^2}{t^2} \left( \alpha y + \beta \right) + \gamma \frac{y}{t} + \delta \frac{y(y + 1)}{y - 1} \]

with

\[ \alpha = \frac{1}{2}(v_3 - v_4)^2, \quad \beta = -\frac{1}{2}(v_2 - v_1)^2, \quad \gamma = 2v_1 + 2v_2 - 1, \quad \delta = -\frac{1}{2}. \]

This is the general $P_V$ equation with $\delta = -\frac{1}{2}$ (recall that the general $P_V$ equation with $\delta \neq 0$ can be reduced to the case with $\delta = -\frac{1}{2}$ by the mapping $t \mapsto \sqrt{-2\delta t}$). Note that the first of the Hamilton equations (2.3) implies

\[ tQ' = 2Q(Q - 1)^2 P - \left\{ (v_2 - v_1)(Q - 1)^2 - 2(v_1 + v_2)Q(Q - 1) + tQ \right\}, \]

which is linear in $P$, so using this equation to eliminate $P$ in (2.1) shows that $tK$ can be expressed as an explicit rational function of $Q$ and $Q'$. 
Of fundamental importance in the random matrix application is the fact that $tK$ (or more conveniently a variant of $tK$ obtained by adding a term linear in $t$) satisfies a second order second degree equation.

**Proposition 2.1** \[21, 30\] With $K$ specified by (2.1), define the auxiliary Hamiltonian $h$ by

\begin{equation}
(2.7) \quad h = tK + (v_3 - v_1)(v_4 - v_1) - v_1 t - 2v_1^2.
\end{equation}

The auxiliary Hamiltonian $h$ satisfies the differential equation

\begin{equation}
(2.8) \quad (th'')^2 - \left(h - th' + 2(h')^2\right)^2 + 4 \prod_{k=1}^{4} (h' + v_k) = 0.
\end{equation}

**Proof:** Following \[30\], we note from (2.1), (2.2) and the Hamilton equations (2.3) that

\begin{equation}
(2.9) \quad h' = -QP - v_1 \quad \text{and} \quad th'' = Q(Q^2 - 1)P^2 - \left[(v_3 + v_4 - 2v_1)Q^2 - v_2 + v_1\right]P
\end{equation}

\begin{equation}
+ (v_3 - v_1)(v_4 - v_1)Q.
\end{equation}

We see from these equations that

\begin{equation}
(2.10) \quad th'' = Q(h' + v_3)(h' + v_4) + P(h' + v_2),
\end{equation}

while the first of these equations together with (2.1) and (2.7) shows

\begin{equation}
(2.11) \quad h - th' + 2(h')^2 = Q(h' + v_3)(h' + v_4) - P(h' + v_2).
\end{equation}

Squaring both sides of (2.10) and (2.11) and subtracting, and making further use of the first equation in (2.3) gives (2.8).}

Actually the differential equation obtained by Jimbo and Miwa \[21\] is not precisely (2.8), but rather a variant obtained by writing

\begin{equation}
 h = \sigma^{(j)} - v_j t - 2v_j^2, \quad j = 1, \ldots, 4,
\end{equation}
or equivalently
\begin{equation}
\sigma^{(j)} = tK + (v_3 - v_1)(v_4 - v_1) - (v_1 - v_j)t - 2(v_1^2 - v_j^2).
\end{equation}

With this substitution (2.8) coincides with the Jimbo-Miwa-Okamoto \( \sigma \) form of \( PV \) (1.37) with
\begin{equation}
\{ \nu_0, \nu_1, \nu_2, \nu_3 \} = \{ v_1 - v_j, v_2 - v_j, v_3 - v_j, v_4 - v_j \}
\end{equation}
(because (1.37) is symmetrical in \( \{ \nu_k \} \) the ordering in the correspondence (2.13) is arbitrary).

The \( \tau \)-function is defined in terms of the Hamiltonian by
\begin{equation}
K =: \frac{d}{dt} \log \tau.
\end{equation}

It then follows from (2.12) that
\begin{equation}
\sigma^{(j)} = t \frac{d}{dt} \log \left( e^{-(v_1 - v_j)t} e^{(v_3 - v_1)(v_4 - v_1) - 2(v_1^2 - v_j^2)} \tau \right).
\end{equation}

Now, according to the Okamoto theory the \( \tau \)-function corresponding to some particular sequences of parameter values can be calculated recursively in an explicit determinant form. The above theory tells us that these \( \tau \)-functions can also be characterised as the solutions of a non-linear differential equation. The determinant solutions are a consequence of a special invariance property of the \( PV \) system which we will now summarise.

### 2.2 Bäcklund transformations

A Bäcklund transformation of (2.1) is a mapping, in fact a birational canonical transformation
\[ T(\mathbf{v}; q, p, t, H) = (\mathbf{v}; \bar{q}, \bar{p}, \bar{t}, \bar{H}), \]
where \( \mathbf{v}, \bar{q}, \bar{p}, \bar{t}, \bar{H} \) are functions of \( \mathbf{v}, q, p, t, H \), such that the \( PV \) coupled system (2.3) is satisfied in the variables \( \mathbf{v}; \bar{q}, \bar{p}, \bar{t} \). Formally
\[ dp \wedge dq - dH \wedge dt = d\bar{p} \wedge d\bar{q} - d\bar{H} \wedge d\bar{t}. \]
In general, the significance of such a transformation is that it allows an infinite family of solutions of the $P_V$ system to be obtained from one seed solution. From our perspective a key feature is the property

$$T^{-1}(H) = H \bigg|_{v \rightarrow T \cdot v}$$

which holds for some particular operators $T$ which have a shift action on the parameter vector $v = (v_1, v_2, v_3, v_4)$ whose components are referred to the standard basis. Explicitly, following [30], we are interested in the transformation $T_0$ which has the property (2.16) with the action on the parameters

$$T_0 \cdot v = \left(v_1 - \frac{1}{4}, v_2 - \frac{1}{4}, v_3 - \frac{1}{4}, v_4 + \frac{3}{4}\right).$$

Although the existence of $T_0$ was established in [30] (termed the parallel transformation $l(v)$), as was the fact that $T_0(t) = t$, explicit formulas for $T_0(P)$ and $T_0(Q)$ were not presented.

It is now realised [40, 25] that the required formulas are more readily forthcoming by developing the $P_V$ theory starting from the symplectic coordinates and Hamiltonian \{\(q, p, t, H\)\} specified by

$$tH := q(q - 1)p(p + t) - (v_2 - v_1 + v_3 - v_4)qp + (v_2 - v_1)p + (v_1 - v_3)tq.$$  

The coordinates and Hamiltonians constituting the two charts (2.1) and (2.18) are related by the coordinate transformation [10]

\[
(q - 1)(Q - 1) = 1
\]

\[
(q - 1)p + (Q - 1)P = v_3 - v_1
\]

\[
tK = tH + (v_3 - v_1)(v_2 - v_4)
\]

so in particular $1 - 1/q$ satisfies the Painlevé V equation (2.4).

The Hamiltonian (2.18) is associated with a set of four coupled first order equations in symmetric variables [3, 28] which admit an extended type $A_3^{(1)}$ affine Weyl group $\tilde{W}_a = W_a(A_3^{(1)}) \rtimes Z_4$ as Bäcklund transformations. The generators of the group $\tilde{W}_a = \langle s_0, s_1, s_2, s_3, \pi \rangle$ obey the
algebraic relations

\[ s_i^2 = 1, \quad s_is_{i\pm 1}s_i = s_{i\pm 1}s_is_{i\pm 1}, \quad s_is_j = s_js_i \quad (j \neq i, i \pm 1), \]
\[ \pi s_i = s_{i+1}\pi, \quad \pi^4 = 1, \quad (i, j = 0, \ldots, 3, \ s_4 := s_0). \]  

(2.20)

The parameters of \( P_V \) can also be characterised by the level sets of the root vectors \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) (with \( \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \delta \)) of the \( A_3^{(1)} \) root system, and are dual to the vector parameter \( \mathbf{v} \) in the following way

\[ \alpha_1(\mathbf{v}) := v_2 - v_1, \quad \alpha_2(\mathbf{v}) := v_1 - v_3, \quad \alpha_3(\mathbf{v}) := v_3 - v_4. \]

In general by the convention of [27] the action of a group operation \( T \) on the roots is given in terms of its action on the components of \( \mathbf{v} \) by

\[ T(\alpha) \cdot \mathbf{v} = \alpha(T^{-1} \cdot \mathbf{v}). \]  

(2.21)

Following [25] the action of the Bäcklund transformations on the root system \( \alpha \) and coordinates \( q, p \) is given in Table 2.1. Note that with \( T_0 \) defined by (2.17),

\[ T_0(\alpha) = (\alpha_0 - 1, \alpha_1, \alpha_2, \alpha_3 + 1), \]

so according to Table 2.1 we have

\[ T_0 = s_3s_2\pi, \quad T_0^{-1} = \pi^{-1}s_1s_2s_3. \]  

(2.22)

As is the case for the dynamical system associated with \( P_{IV} \) there is a symmetric form for the system describing the \( P_V \) transcendent [28], and is expressed in terms of the four variables

\[ f_0 = \frac{p + t}{\sqrt{t}}, \]
\[ f_1 = \sqrt{t}q, \]
\[ f_2 = -\frac{p}{\sqrt{t}}, \]
\[ f_3 = \sqrt{t}(1 - q). \]  

(2.23)
with the constraints \( f_0 + f_2 = f_1 + f_3 = \sqrt{7} \). Using these variables the Bäcklund transformations take the simple and generic forms of

\[
\begin{align*}
  s_i(\alpha_j) &= \alpha_j - a_{ij}\alpha_i, \\
  s_i(f_j) &= f_j + u_{ij} \frac{\alpha_i}{f_i}, \\
  \pi(\alpha_j) &= \alpha_{j+1}, \\
  \pi(f_j) &= f_{j+1},
\end{align*}
\]

(2.24)

where the Cartan matrix \( A = (a_{ij}) \) and orientation matrix \( U = (u_{ij}) \) are defined by

\[
A = \begin{bmatrix}
  2 & -1 & 0 & -1 \\
  -1 & 2 & -1 & 0 \\
  0 & -1 & 2 & -1 \\
  -1 & 0 & -1 & 2
\end{bmatrix}, \quad U = \begin{bmatrix}
  0 & 1 & 0 & -1 \\
  -1 & 0 & 1 & 0 \\
  0 & -1 & 0 & 1 \\
  1 & 0 & -1 & 0
\end{bmatrix}.
\]

(2.25)

Let us now compute the action of \( T_{0}^{-1} \) on the Hamiltonian \( tH \). First we can easily verify from Table 2.1 that

\[
\begin{align*}
  s_0(tH) &= tH + \alpha_0 \frac{t}{p+t} + \alpha_0 (\alpha_2 - 1) \\
  s_1(tH) &= tH + \alpha_1 t + \alpha_1 \alpha_3 \\
  s_2(tH) &= tH - \alpha_2 t + \alpha_2 (\alpha_0 - 1) \\
  s_3(tH) &= tH + \alpha_1 \alpha_3 \\
  \pi(tH) &= tH + (q - 1)p - \alpha_2 t.
\end{align*}
\]

(2.26)

It follows from these formulas, Table 2.1 and (2.22) that

\[
T_{0}^{-1}(tH) = tH \bigg|_{v \rightarrow T_{0}^{-1}\cdot v} = tH \bigg|_{\alpha \rightarrow T_{0}^{-1}\cdot \alpha} = tH + qp.
\]

(2.27)

According to the relations in (2.18) between \( tH \) and \( tK \), \( q,p \) and \( Q,P \), we therefore have

\[
T_{0}^{-1}(tK) = tK - Q(Q - 1)P + (v_3 - v_1)(Q - 1) \bigg|_{v \rightarrow T_{0}^{-1}\cdot v} = tK \bigg|_{\alpha \rightarrow T_{0}^{-1}\cdot \alpha},
\]

(2.28)

which is the result deduced indirectly in \([30]\). Note from (2.19) that

\[
T_{0}^{-1}(Q) = 1 + \frac{1}{T_{0}^{-1}(q) - 1},
\]

(2.29)

\[
T_{0}^{-1}(P) = \frac{(v_3 - v_1) - (T_{0}^{-1}(q) - 1)T_{0}^{-1}(p)}{T_{0}^{-1}(q) - 1}.
\]
APPLICATION OF THE $\tau$-FUNCTION THEORY

| $s_0$ | $-\alpha_0$ | $\alpha_1 + \alpha_0$ | $\alpha_2$ | $\alpha_3 + \alpha_0$ | $p$ | $q + \frac{\alpha_0}{p + t}$ |
|---|---|---|---|---|---|---|
| $s_1$ | $\alpha_0 + \alpha_1$ | $-\alpha_1$ | $\alpha_2 + \alpha_1$ | $\alpha_3$ | $p - \frac{\alpha_1}{q}$ | $q$ |
| $s_2$ | $\alpha_0$ | $\alpha_1 + \alpha_2$ | $-\alpha_2$ | $\alpha_3 + \alpha_2$ | $p$ | $q + \frac{\alpha_2}{p}$ |
| $s_3$ | $\alpha_0 + \alpha_3$ | $\alpha_1$ | $\alpha_2 + \alpha_3$ | $-\alpha_3$ | $p - \frac{\alpha_3}{q - 1}$ | $q$ |
| $\pi$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_0$ | $t(q - 1)$ | $-\frac{p}{q}$ |

Table 2.1. Bäcklund transformations relevant for the $P_V$ Hamiltonian (2.18).

where $T_0^{-1}(q)$ and $T_0^{-1}(p)$ can be deduced from Table 2.1 using (2.22).

2.3 Toda lattice equation

Introduce the sequence of Hamiltonians

\begin{equation}
K[n] := K\bigg|_{v \mapsto (v_1 - n/4, v_2 - n/4, v_3 - n/4, v_4 + 3n/4)}
\end{equation}

and let $\tau[n]$ denote the corresponding $\tau$-functions so that

\begin{equation}
K[n] = \frac{d}{dt} \log \tau[n].
\end{equation}

We have already remarked that a crucial feature of $K[n]$ from the viewpoint of application to random matrix theory is the fact that it (or more precisely the quantity (2.12)) satisfies the differential equation (1.37). An equally crucial feature for application to random matrix theory is the recurrence satisfied by $\{\tau[n]\}$.

**Proposition 2.2** [30, 23] The $\tau$-function sequence $\tau[n]$, corresponding to the parameter sequence $(v_1 - n/4, v_2 - n/4, v_3 - n/4, v_4 + 3n/4)$, obeys the Toda lattice equation

\begin{equation}
\delta^2 \log \tau[n] = \frac{\tau[n - 1]\tau[n + 1]}{\tau^2[n]}, \quad \delta := t \frac{d}{dt}
\end{equation}
where

\[
\bar{\tau}[n] := t^{n^2/2}e^{(v_4-v_1+n)t}\tau[n].
\]

Proof: Following \cite{25} we consider the sequence of Hamiltonians \(H[n]\) associated with the given parameter sequence and the Hamiltonian (2.18). Now, with \(\hat{\tau}[n]\) denoting the corresponding \(\tau\)-function, it follows from (2.19) that

\[
\tau[n] = t^{(v_3-v_1)(v_2-v_4-n)}\hat{\tau}[n].
\]

Since (2.32) is unchanged by the replacement \(\bar{\tau}[n] \mapsto t^{a+bn}\tau[n]\), it suffices to show that (2.32) is satisfied with \(\hat{\tau}[n]\) replacing \(\tau[n]\) in (2.33).

From the definitions

\[
\delta \log \frac{\hat{\tau}[n-1]\hat{\tau}[n+1]}{\hat{\tau}^2[n]} = \left(T_0^{-1}(tH[n]) - tH[n]\right) - \left(tH[n] - T_0(tH[n])\right).
\]

On the other hand, it follows from (2.27) that

\[
(T_0^{-1}(tH[n]) - tH[n]) - (tH[n] - T_0(tH[n])) = q[n]p[n] - T_0(q[n])T_0(p[n]).
\]

With \(T_0^{-1}\) specified from (2.22), explicit formulas for \(T_0(q[n])\) and \(T_0(p[n])\) can be deduced from Table 2.1 and it can be verified using the Hamilton equations for \(H\) that the RHS of (2.33) is equal to

\[
\delta \log \left(q[n](q[n]-1)p[n] + (v_1 - v_3)q[n] + (v_4 - v_1 + n)\right)
= \delta \log \frac{d}{dt}(tH[n] + (v_4 - v_1 + n)t) = \delta \log \frac{d}{dt}(e^{(v_4-v_1+n)t}\hat{\tau}[n]).
\]

Equating this with the LHS of (2.34) gives a formula equivalent to (2.32).
2.4 Classical solutions

The Toda lattice equation (2.32) is a second order recurrence, and so requires the values of \( \bar{\tau}[0] \) and \( \bar{\tau}[1] \) for the sequence members \( \bar{\tau}[n], \ (n \geq 2) \) to be specified. It was shown by Okamoto \[30\] that for special choices of the parameters, corresponding to the chamber walls in the underlying \( A_3 \) root lattice, the \( P_V \) system admits a solution with \( \tau[0] = 1 \) and \( \tau[1] \) equal to a confluent hypergeometric function.

**Proposition 2.3** \[30\] For the special choice of parameters

\begin{equation}
(2.36) \quad v_1 = v_4
\end{equation}

it is possible to choose \( \tau[0] = 1 \). Furthermore, the first member \( \tau[1] \) of the \( \tau \)-function sequence (2.34) then satisfies the confluent hypergeometric equation

\begin{equation}
(2.37) \quad t(\tau[1])'' + (v_3 - v_2 + 1 + t)(\tau[1])' + (v_3 - v_1)\tau[1] = 0.
\end{equation}

**Proof:** Write \( Q = Q[0], \ P = P[0] \). In the case \( v_1 = v_4 \) we see from (2.1) that it is possible to choose

\begin{equation}
(2.38) \quad P = 0, \ K = 0,
\end{equation}

the latter allowing us to take \( \tau[0] = 1 \). With (2.38) the equations (2.28), (2.1) and (2.31) then give

\begin{equation}
(2.39) \quad t\frac{d}{dt} \log \tau[1](t) = (v_3 - v_1)(Q - 1).
\end{equation}

The quantity \( Q \) must satisfy the Hamilton equation

\begin{equation}
(2.40) \quad tQ' = \frac{\partial K}{\partial P} \bigg|_{p = v_4} = -(v_2 - v_1)(Q - 1)^2 - (v_3 - v_2)Q(Q - 1) - tQ
\end{equation}

where use has been made of (2.2). Substituting (2.39) in (2.40) gives (2.37).
The two linearly independent solutions of (2.37) are
\[
\tau^a[1](t) = \frac{\Gamma(v_3 - v_2 + 1)}{\Gamma(v_3 - v_1)\Gamma(v_1 - v_2 + 1)} \int_0^1 e^{-tu} u^{v_3-v_1-1}(1-u)^{v_1-v_2} du
\]
and
\[
\tau^s[1](t) = e^{-t}\psi(1 + v_1 - v_2, v_3 - v_2 + 1; t)
\]
(2.42)

(2.41)

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continuous ones. Here we briefly demonstrate this and find a difference equation for the Hamiltonians.

**Proposition 2.4** The Schlesinger transformation of the $P_V$ system for the shift operator $T_{0}^{-1}$ generating the parameter sequence $(\alpha_0 + n, \alpha_1, \alpha_2, \alpha_3 - n)$ with $n \in \mathbb{Z}$, corresponds to the second order difference equation of the discrete Painlevé type, $dP_{IV}$, namely

\[
x_n + x_{n-1} = \frac{t}{y_n} + \frac{\alpha_3 - n}{1 - y_n}
\]

(2.46)

\[
y_n y_{n+1} = t \left( x_n + \frac{1}{4} \alpha_1 + \alpha_0 + n \right) \frac{x_n^2 - \frac{1}{4} \alpha_1^2}{x_n^2 - 1} , \quad n \geq 1,
\]

where $x_n = f_0[n] f_1[n] - \frac{1}{2} \alpha_1$ and $y_n = \sqrt{t}/f_1[n]$. With $qp[n] = tH[n+1] - tH[n]$, the Hamiltonian times $t$, $tH[n]$, satisfies the third order difference equation

\[
-tqp[n] = \left\{ (\alpha_0 + n + qp[n]) [tH[n] + (\alpha_3 - n)qp[n] - \alpha_2 t] \right. \\
+ (\alpha_2 + qp[n+1]) [tH[n] + (\alpha_3 - n)qp[n] + (\alpha_0 + \alpha_1 + n) t] \left\} \\
\div \left\{ tH[n] + (\alpha_3 - n)qp[n] + (\alpha_3 - n - 1)(\alpha_2 + qp[n+1]) - \alpha_2 t \right\} \\
\times \left\{ (qp[n] - \alpha_3 + n) [tH[n] + (1 - \alpha_0 - n)qp[n] + \alpha_1 t] \\
+ (qp[n-1] - \alpha_1) [tH[n] + (1 - \alpha_0 - n)qp[n] - (\alpha_2 + \alpha_3 - n)t] \right\} \\
\div \left\{ tH[n] + (1 - \alpha_0 - n)qp[n] + (1 - \alpha_0 - n)(qp[n-1] - \alpha_1) + \alpha_1 t \right\} .
\]

**Proof:** The Schlesinger transformations for the shift operator $T_0$
that are relevant for the identification are

\[(2.48) \quad T_0^{-1}(f_0) = f_3 - \frac{\alpha_0}{f_0} + \frac{\alpha_0 + \alpha_1 + \alpha_2}{f_2 + \frac{\alpha_0}{f_1 + \frac{\alpha_0}{f_0}}}, \]

\[(2.49) \quad T_0^{-1}(f_1) = f_0 - \frac{\alpha_0 + \alpha_1}{f_1 + \frac{\alpha_0}{f_0}}, \]

\[(2.50) \quad T_0^{-1}(f_3) = f_2 + \frac{\alpha_0 + \alpha_1}{f_1 + \frac{\alpha_0}{f_0}}. \]

Using (2.50) in (2.48) along with the constraint to eliminate \(f_3\) we can rewrite the latter equation as

\[(2.51) \quad f_1[n] + \frac{\alpha_0 + n}{f_0[n]} = \sqrt{t} - f_0[n+1] + \frac{n+1 - \alpha_3}{\sqrt{t} - f_1[n+1]]. \]

This identity can then be employed in (2.49) to arrive at

\[(2.52) \quad f_0[n]f_1[n] + f_0[n-1]f_1[n-1] = \alpha_1 + \sqrt{t}f_1[n] + \frac{n - \alpha_3}{\sqrt{t} - f_1[n]}f_1[n], \]

which is the first of the coupled equations (2.46). The second equation of the set (2.46) follows immediately from (2.48) rewritten to read

\[(2.53) \quad \frac{1}{f_1[n]f_1[n+1]} = \frac{1}{f_0[n]f_1[n]} + \frac{f_0[n]f_1[n] + \alpha_0 + n}{f_0[n]f_1[n] - \alpha_1}. \]

The simplest way to derive the difference equation for \(tH[n]\) is to recast
the Schlesinger transformations in terms of canonical variables \(q,p\),

\[
(2.54) \quad tq[n+1] = t + p[n] - \frac{\alpha_0 + \alpha_1 + n}{q[n] + \frac{\alpha_0 + n}{t + p[n]}},
\]

\[
(2.55) \quad p[n+1] = -tq[n] - \frac{(\alpha_0 + n)t}{t + p[n]} + \frac{(n+1-\alpha_3)t}{-p[n] + \frac{\alpha_0 + \alpha_1 + n}{q[n] + \frac{\alpha_0 + n}{t + p[n]}}},
\]

\[
(2.56) \quad tq[n-1] = p[n] - \frac{n-\alpha_3}{1-q[n]} + \frac{1-\alpha_0 - n}{q[n] + \frac{\alpha_2 + \alpha_3 - n}{p[n] + \frac{\alpha_3 - n}{1-q[n]}},}
\]

\[
(2.57) \quad t + p[n-1] = tq[n] + \frac{(\alpha_2 + \alpha_3 - n)t}{p[n] + \frac{\alpha_3 - n}{1-q[n]}}.
\]

Considering the first two equations (2.54, 2.55) we find that

\[
(2.58) \quad qp[n+1] + \alpha_2 = \frac{[\alpha_0 + n + q[n](t + p[n])][q_0 + n + q[p[n]](t + p[n]) - \alpha_1 p[n] + \alpha_2 t]}{p[n][\alpha_1 - q[n](t + p[n])] + (\alpha_0 + \alpha_1 + n)t}
\]

and expressing this in terms of the Hamiltonian yields

\[
(2.59) \quad qp[n+1] + \alpha_2 = \frac{[tq[n] + \alpha_0 + n + q[p[n]]][tH[n] + (\alpha_3 - n)qp[n] - \alpha_2 t]}{tH[n] + (\alpha_3 - n)qp[n] + (\alpha_3 - n - 1)tq[n] + (\alpha_0 + \alpha_1 + n)t}
\]

In an analogous way we find for the down-shifted product

\[
(2.60) \quad qp[n-1] - \alpha_1 = \frac{[p[n] + \alpha_3 - n - q[p[n]]][tH[n] + (1-\alpha_0 - n)qp[n] + \alpha_1 t]}{tH[n] + (1-\alpha_0 - n)qp[n] + (1-\alpha_0 - n)p[n] - (\alpha_2 + \alpha_3 - n)t}.
\]

Now the up-shifted equation can be solved for \(q\) and the down-shifted one for \(p\) allowing their product \(qp[n]\) to be expressed in terms of just the Hamiltonian and other members of the \(qp\) sequence. The final result is (2.47).
3 Application to the finite LUE

3.1 The case $N = 1$

Comparing the case $N = 1$ of the integrals (1.7) and (1.8) with those in (2.41) and (2.42) we see that

\[
\tilde{E}_1((0, s); a, \mu) = C s^{a+\mu+1} \tau^s[1](s) \bigg|_{v_1-v_2=\mu}^{v_3-v_1=a+1} \\
\tilde{E}_1((s, \infty); a, \mu) = C s^{a+\mu+1} \tau^a[1](s) \bigg|_{v_1-v_2=\mu}^{v_3-v_1=a+1}
\]

Recalling (2.15), we thus have that

\[
t \frac{d}{dt} \log \left( t^{-\mu} \tilde{E}_1((0, t); a, \mu) \right) = \sigma^{s(1)}(t) \\
\frac{d}{dt} \log \left( t^{-\mu} \tilde{E}_1((t, \infty); a, \mu) \right) = \sigma^{a(1)}(t),
\]

where both $\sigma^{s(1)}(t)$ and $\sigma^{a(1)}(t)$ satisfy the Jimbo-Miwa-Okamoto $\sigma$ form of $P_V$ (1.37) with

\[
\nu_0 = 0, \quad \nu_1 = -\mu, \quad \nu_2 = a + 1, \quad \nu_3 = 1.
\]

Note that in the case $\mu = 0$ this is consistent with (1.38).

3.2 The general $N$ case

Although it is not at all immediately obvious, the $n \times n$ determinant formed by substituting (2.41) and (2.42) in (2.43) can be identified with the general $n$ cases of the integrals (1.7) and (1.8) respectively. Consider first (2.43) with initial value (2.41).

**Proposition 3.1** Let $\bar{\tau}[n] \bar{\tau}[n]$ be specified by the determinant formula (2.43) with

\[
(3.1) \quad \bar{\tau}[1](t) = t^{1/2} e^{\frac{b}{2}} F_1(v_3-v_1, v_3-v_2+1; -t).
\]

Then we have

\[
(3.2) \quad \bar{\tau}[n] \propto t^{n^2/2} \prod_{l=1}^{n} \int_0^1 du_l e^{tu_l} u_l^{v_1-v_2} (1-u_l)^{v_3-v_1-n} \prod_{1 \leq j < k \leq n} (u_k - u_j)^2.
\]
It follows from this that

\[ \tilde{E}_N((t, \infty); a, \mu) = Ct^{(a+\mu)N+N^2}\tau[N](t) \]  

and

\[ t \frac{d}{dt} \log \left( t^{-\mu} \tilde{E}_N((t, \infty); a, \mu) \right) = U_N(t; a, \mu) \]  

where \( U_N(t; a, \mu) \) satisfies the Jimbo-Miwa-Okamoto \( \sigma \) form of the \( P_V \) differential equation (1.37) with

\[ \nu_0 = 0, \quad \nu_1 = -\mu, \quad \nu_2 = N + a, \quad \nu_3 = N, \]  

subject to the boundary condition

\[ U_N(t; a, \mu) \sim aN + N^2 - N \frac{a+N}{a+\mu+2N} t. \]  

Equivalently \( U_N(t; a, \mu) \) is equal to the auxiliary Hamiltonian (2.12) with \( j = 1 \) and parameters (3.5).

**Proof:** First observe that if \( \{\tilde{\tau}[n]\} \) satisfies the Toda equation (2.32) with \( \tilde{\tau}[0] = 1 \), then \( \{t^{\nu_c}\tilde{\tau}[n]\} \) is also a solution which is given by the determinant formula (2.43) with \( \tilde{\tau}[1](t) \mapsto t^{\nu_c}\tilde{\tau}[1](t) \). Choosing \( c = -1/2 \) and substituting (3.1) for \( \tilde{\tau}[1](t) \) we thus have

\[ t^{-n/2}\tilde{\tau}[n] = \det \left[ \delta^{j+k}_{1F1} \left( v_1-v_2+1, v_3-v_2+1; t \right) \right]_{j,k=0,...,n-1} \]  

where use has been made of the Kummer relation

\[ _1F1(a, c; -t) = e^{-t}_1F1(c-a, c; t). \]  

Our first task is to use elementary row and column operations to eliminate the operator \( \delta^{j+k} \) in (3.7). Starting with elementary column operations, in column \( k \) (\( k = n-1, n-2, \ldots, 1 \) in this order) make use of the identity

\[ \delta_1F1(a, c; t) = a\left( _1F1(a+1, c; t) - _1F1(a, c; t) \right) \]
and add \((v_1 - v_2 + 1)\) times column \(k - 1\). This gives

\[
t^{-n/2} \tau[n] \propto \det \left[ \delta_1 F_1(v_1 - v_2 + 1, v_3 - v_2 + 1; t) \right. \\
\left. \delta_1^{j+k-1} F_1(v_1 - v_2 + 2, v_3 - v_2 + 1; t) \right]_{j=0, \ldots, n-1, k=1, \ldots, n-1}.
\]

Next, in column \(k\) \((k = n - 1, n - 2, \ldots, 2\) in this order) make use of the identity (3.8) again and add \((v_1 - v_2 + 2)\) times column \(k - 1\) to obtain (3.9)

\[
t^{-n/2} \tau[n] \propto \det \left[ \delta_1 F_1(v_1 - v_2 + 1, v_3 - v_2 + 1; t) \right. \\
\left. \delta_1^{j+k-1} F_1(v_1 - v_2 + 2, v_3 - v_2 + 1; t) \right]_{j=0, \ldots, n-1, k=2, \ldots, n-1}.
\]

Further use of (3.8) in an analogous fashion gives (3.10)

\[
t^{-n/2} \tau[n] \propto \det \left[ \delta_1 F_1(v_1 - v_2 + 1 + k, v_3 - v_2 + 1; t) \right]_{j, k=0, \ldots, n-1}.
\]

At this stage we use elementary row operations to eliminate the operation \(\delta^j\) in (3.10). In row \(j\) \((j = n - 1, n - 2, \ldots, 1\) in this order) make use of the identity

\[
\frac{d}{dt} F_1(a, c; t) = \frac{a - c}{c} F_1(a, c + 1; t) + F_1(a, c; t)
\]

and subtract row \(j - 1\) to get

\[
t^{-n/2} \tau[n] \propto \\
\det \left[ F_1(v_1 - v_2 + 1 + k, v_3 - v_2 + 1; t) \\
(v_3 - v_1 - k) \delta^{-1} t_1 F_1(v_1 - v_2 + 1 + k, v_3 - v_2 + 2; t) \right]_{j=1, \ldots, n-1, k=0, \ldots, n-1}.
\]

Next, in row \(j\) \((j = n - 1, n - 2, \ldots, 2\) in this order) make use of the identity (3.11) again and subtract 2 times row \(j - 1\) to obtain

\[
t^{-n/2} \tau[n] \propto \\
t \det \left[ F_1(v_1 - v_2 + 1 + k, v_3 - v_2 + 1; t) \\
(v_3 - v_1 - k) t_1 F_1(v_1 - v_2 + 1 + k, v_3 - v_2 + 2; t) \\
(v_3 - v_1 - k) 2 \delta^{-2} t_2 F_1(v_1 - v_2 + 1 + k, v_3 - v_2 + 3; t) \right]_{j=2, \ldots, n-1, k=0, \ldots, n-1}.
\]
Further use of (3.11) in an analogous fashion gives
\[
\begin{align*}
& t^{-n/2} \tau[n] \propto \\
& t^{n(n-1)/2} \det \left[ (v_3 - v_1 - k)_{j,k=0,\ldots,n-1} \right] \\
& \left( v_3 - v_1 - k \right)_{j,k=0,\ldots,n-1} \propto t^{n(n-1)/2} \det \left[ (v_3 - v_1 - k)_{j,k=0,\ldots,n-1} \right]
\end{align*}
\]
thus eliminating entirely the operation \( \delta \).

We now substitute the integral representation for \( _1F_1 \) deducible from (2.41) to obtain
\[
\begin{align*}
& t^{-n/2} \tau[n] \propto t^{n(n-1)/2} \det \left[ \int_0^1 e^{tu} u^{k+v_1-v_2} (1-u)^{v_3-v_1-1+j-k} du \right]_{j,k=0,\ldots,n-1} \\
& = t^{n(n-1)/2} \prod_{l=1}^n \int_0^1 du_l e^{tu_l} u_l^{v_1-v_2} (1-u_l)^{v_3-v_1-n} \\
& \cdot \det \left[ u_{j+1}^k (1-u_{j+1})^{n+j-k-1} \right]_{j,k=0,\ldots,n-1}.
\end{align*}
\]
(3.12)

Regarding the determinant in the last line of this expression, we have
\[
\begin{align*}
& \det \left[ u_{j+1}^k (1-u_{j+1})^{n+j-k-1} \right]_{j,k=0,\ldots,n-1} \\
& = \prod_{j=0}^{n-1} (1-u_{j+1})^j \det \left[ u_{j+1}^k \right]_{j,k=0,\ldots,n-1},
\end{align*}
\]
where the equality follows by adding the \( k+1 \)-th column to the \( k \)-th \((k = n-1, \ldots, 1)\) in a sequence of sweeps \( l = 1, \ldots, n-1 \). All factors other than the determinant in the multidimensional integral are symmetric in \( \{u_j\} \) so we can symmetrise this term without changing the value of the integral (apart from a constant factor \( n! \) which is not relevant to the present discussion). Noting that
\[
\begin{align*}
& \text{Sym} \left( \prod_{j=0}^{n-1} (1-u_{j+1})^j \det \left[ u_{j+1}^k \right]_{j,k=0,\ldots,n-1} \right) = \frac{\pm}{n!} \prod_{j<k} (u_j - u_k)^2,
\end{align*}
\]
(3.13)
for some sign \( \pm \), and substituting in (3.12) gives (3.2).

\[
\begin{align*}
& t^{-n/2} \tau[n] \propto t^{n(n-1)/2} \prod_{l=1}^n \int_0^1 du_l e^{tu_l} u_l^{v_1-v_2} (1-u_l)^{v_3-v_1-n} \prod_{j<k} (u_j - u_k)^2.
\end{align*}
\]
Recalling (2.44), changing variables $u_j \mapsto 1 - u_j$ and comparing with the final integral in (1.8) we thus have that

$$
\tau[N](t) \propto t^{-(a+\mu)N-N^2} \tilde{E}_N((t, \infty); a, \mu)
$$

provided the parameters are given by (3.5). The result (3.4) now follows by substituting this result in (2.15) with $j = 1$. For the boundary condition (3.6), we see from (3.4) and (1.8) that

$$
U_N(t; a, \mu) \sim t^{-aN + N^2 - \tilde{E}_N((0, t); a, \mu)}
$$

where $J_N(a, \mu)[\sum_{j=1}^N \lambda_j]$ denotes the integral (1.20) with an additional factor of $\sum_{j=1}^N \lambda_j$ in the integrand. Noting that $\sum_{j=1}^N \lambda_j$ can be written as a ratio of alternants allows the integral to computed using the method of orthogonal polynomials and leads to the result (3.6).

Next we consider (2.43) with initial value (2.42).

**Proposition 3.2** Let $\bar{\tau}[n]$ be specified by the determinant formula (2.43) with

$$
\bar{\tau}[1](t) = t^{1/2} \psi(v_1-v_2+1, v_3-v_2+1; t).
$$

Then we have

$$
\bar{\tau}[n] \propto t^{n^2/2} \prod_{l=1}^n \int_0^\infty du_l e^{-tu_l} u_l^{v_1-v_2} (1 + u_l)^{v_3-v_1-n} \prod_{1 \leq j < k \leq n} (u_k - u_j)^2.
$$

It follows from this that

$$
\tilde{E}_N((0, t); a, \mu) = C t^{(a+\mu)N+N^2} \tau[N](t)
$$

and

$$
t \frac{d}{dt} \log \left( t^{-N\mu} \tilde{E}_N((0, t); a, \mu) \right) = V_N(t; a, \mu)
$$

where $V_N(t; a, \mu)$ is equal to the auxiliary Hamiltonian (2.12) with $j = 1$ and parameters (3.5), and so satisfies the Jimbo-Miwa-Okamoto form of
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$P_V(1.37)$ with parameters $(3.3)$. The latter is to be solved subject to the boundary condition

\[(3.18) \quad V_N(t; a, \mu) \sim -Nt + N(a - \mu) - \frac{N(N + \mu)a}{t} + O(1/t^2).\]

**Proof:** Analogous to $(3.7)$ we have

\[(3.19) \quad t^{-n/2}\bar{\tau}[n] = \det \left[ \delta^{j+k} \psi(v_1 - v_2 + 1, v_3 - v_2 + 1; t) \right]_{j,k=0,...,n-1} \]

We now adopt the identical strategy as used in the proof of Proposition 3.1, with the identities $(3.8)$ and $(3.11)$ replaced by

\[
\delta\psi(a, c; t) = a \left( (a - c + 1)\psi(a + 1, c; t) - \psi(a, c; t) \right)
\]

and

\[
\frac{d}{dt}\psi(a, c; t) = \psi(a, c; t) - \psi(a, c + 1; t)
\]

respectively. The results $(3.15)$ and $(3.17)$ are then obtained by repeating the working which led to $(3.2)$ and $(3.4)$.

For the boundary condition, we see from $(3.17)$ and $(1.7)$ that

\[V_N(t; a, \mu) \sim -Nt + N(a - \mu) - \frac{aI_N(\mu)\sum_{j=1}^{N} \lambda_j}{tI_N(\mu)},\]

where $I_N(\mu)\sum_{j=1}^{N} \lambda_j$ denotes the first integral in $(1.20)$ with an extra factor of $\sum_{j=1}^{N} \lambda_j$ in the integrand. This integral can be computed by changing variables $\lambda_j \mapsto \epsilon \lambda_j$ in the definition of $I_N(\mu)$, differentiating with respect to $\epsilon$, and setting $\epsilon = 1$.

In relation to $(1.15)$ we note from the fact that $(2.37)$ is linear, with linearly independent solutions $(2.41)$ and $(2.42)$, that

\[(3.20) \quad \tau^{\xi}[1](t) = \frac{e^{-t}}{\Gamma(1 + v_1 - v_2)} \left( \int_{-1}^{\infty} -\xi \int_{-1}^{0} e^{-tu}u^{v_1 - v_2}(1 + u)^{v_3 - v_1 - 1} du \right)\]

is proportional to the most general solution of $(2.37)$. Substituting $(3.20)$ in $(2.33)$ with $n = 1$ and $v_4 = v_1$, forming $(2.43)$, and simplifying by noting that the considerations of the proof of Proposition 3.2 (excluding the discussion of the boundary condition) remain valid independent of the value of $\xi$, we obtain the following result.
Proposition 3.3 Let \( \hat{\tau}[n] \) be specified by the determinant formula \((2.43)\) with \( \hat{\tau}[1] \) proportional to \( t^{1/2} \) times \((3.21)\). Then

\[
\tilde{E}_N((0,s); a, \mu; \xi) = Ct^{(a+\mu)(N+2)} \tau[N](t)
\]

and

\[
t \frac{d}{dt} \left( t^{-N} \tilde{E}_N((0,0); a, \mu; \xi) \right) = W_N(t; a, \mu)
\]

where \( W_N(t; a, \mu) \), like \( U_N(t; a, \mu) \) from Proposition \((3.4)\) and \( V_N(t; a, \mu) \) from Proposition \((3.2)\), is equal to the auxiliary Hamiltonian \((2.14)\) with \( j = 1 \) and parameters \((3.5)\), and so satisfies the Jimbo-Miwa-Okamoto form of PV \((1.37)\) with parameters \((3.5)\). In the case \( \mu = 0 \), with \( \rho(t) \) denoting the eigenvalue density of the LUE, the latter is to be solved subject to the boundary condition

\[
\lim_{t \to 0} W_N(t; a, 0) ~ -\xi \rho(t) ~ -\xi \frac{\Gamma(N+a+1)}{\Gamma(N)\Gamma(a+1)\Gamma(a+2)} t^a.
\]

3.3 A relationship between transcendents

The evaluations \((1.49)\) for \( p_{\min}(s; a) \) and \((1.40)\) for \( \tilde{E}_N((0,s); a, 0) \) substituted into \((1.11)\) imply that

\[
V_N(t; a, 2) = -(a + 1 + 2N) + t + V_{N+1}(t; a, 0) + t V'_{N+1}(t; a, 0).
\]

In our previous study \([17]\) we encountered an analogous identity relating some \( P_{IV} \) transcendents. Like \((3.24)\), the \( P_{IV} \) identity in \([17]\) was discovered using a relation of the type \((1.11)\). A subsequent independent derivation using Bäcklund transformations was also found and presented. Likewise the identity \((3.24)\) can be derived from the formulas \((2.26)\) and Table \((2.1)\). An identical formula relates \( U_N(t; a, 2) \) and \( U_{N+1}(t; a, 0) \) which can be understood from the evaluation of \( p_{\max}(s; a) \) and \( \hat{E}_N((s, \infty); a, 0) \).

Proposition 3.4 Introduce the shift operators \( T_2 \) and \( T_3 \), which have the action on the parameters \( v \) specified by

\[
T_2 \cdot v = (v_1 + \frac{3}{4}, v_2 - \frac{1}{4}, v_3 - \frac{1}{4}, v_4 - \frac{1}{4}),
\]

\[
T_3 \cdot v = (v_1 - \frac{1}{4}, v_2 - \frac{1}{4}, v_3 + \frac{3}{4}, v_4 - \frac{1}{4}).
\]
and consequently from (2.28) and Table 2.4 the representation

(3.26) \[ T_2 = s_1 \pi s_3 s_2, \quad T_3 = s_2 s_1 \pi s_3. \]

Then we have

(3.27) \[ T_3 T_2^{-1} V_{N+1}(t; a, 0) = V_N(t; a, 2) \]

and use of (2.20) and Table 2.1 on the LHS reduces this to (3.24).

**Proof:** We take advantage of the arbitrariness in (2.13) which with \( \{ \nu_i \} \) given by (1.38) with \( N \mapsto N + 1 \) allows us to write

(3.28) \[ V_{N+1}(t; a, 0) = \sigma^{(1)} \bigg|_{\begin{array}{c} v_2 - v_1 = a + N + 1, \\ v_3 - v_1 = 0, \\ v_4 - v_1 = N + 1 \end{array}}. \]

The actions (3.25) then imply

(3.29) \[ T_3 T_2^{-1} V_{N+1}(t; a, 0) = \sigma^{(1)} \bigg|_{\begin{array}{c} v_2 - v_1 = a + N, \\ v_3 - v_1 = -2, \\ v_4 - v_1 = N \end{array}}. \]

which recalling (1.41) and (2.13) is the equation (3.27).

To evaluate the action of \( T_3 T_2^{-1} \) on the LHS of (3.27), first note from (2.12) and the final equation in (2.19) that for general parameters

\[ \sigma^{(1)} = tH + (v_3 - v_1)(v_2 - v_1). \]

We remark that the simplifying feature of arranging the parameters so that \( \alpha_2 = v_3 - v_1 = 0 \) in (3.28) is that (2.18) takes the reduced form

(3.29) \[ tH \bigg|_{\alpha_2=0} = \frac{1}{2} \left[ q(q-1)(p+t) - (\alpha_1 + \alpha_3)q + \alpha_1 \right]. \]

The explicit formula (3.29) will become important later on. For now see the action of \( T_3 T_2^{-1} \) on \( \sigma^{(1)} \) for general values of the parameters.

First we note that the generators of the extended type \( A_3^{(1)} \) affine Weyl group possess the algebraic properties (2.20) which can be used in the formula for \( T_3 T_2^{-1} \) implied by (3.26) to show that

\[ T_3 T_2^{-1} = s_2 s_3 s_0 s_1 s_3. \]
Now, it follows from (2.26) and Table 2.1 that
\[ s_0 s_1 s_3(tH) = tH + \alpha_0 \frac{t}{p + t} + \alpha_0(\alpha_2 - 1) + (\alpha_1 + \alpha_0)t. \]

Let us now consider separately the second term in (3.30). Use of Table 2.1 shows
\[ s_3 s_1(\alpha_0 \frac{t}{p + t}) = \frac{(1 - \alpha_2)tq(q - 1)}{q(q - 1)(p + t) - \alpha_1(q - 1) - \alpha_3q}. \]
Furthermore, we see from Table 2.1 that in the special circumstance \( \alpha_2 = 0 \) (i.e. \( v_3 - v_1 = 0 \)), \( s_2 \) acts like the identity on \( p \) and \( q \) so we have
\[ s_2 s_3 s_1(\alpha_0 \frac{t}{p + t}) \bigg|_{\alpha_2=0} = \frac{tq(q - 1)}{q(q - 1)(p + t) - \alpha_1(q - 1) - \alpha_3q} \]
\[ = t \frac{(tH)'}{tH} \bigg|_{\alpha_2=0} \]
where the final equality follows from (3.29) and (2.18). For the remaining terms in (3.30), we see from (2.26) and Table 2.1 that
\[ s_2 s_3 s_1 \left[ tH + \alpha_0(\alpha_2 - 1) + (\alpha_1 + \alpha_0)t \right] \bigg|_{\alpha_2=0} = tH + t - \alpha_0. \]
The above results together with the simple formula
\[ T_3 T_2^{-1}(\alpha_1 \alpha_2) \bigg|_{\alpha_2=0} = 2(1 - \alpha_1) = 2(1 - v_2 + v_1) \]
imply
\[ (3.32) \quad T_3 T_2^{-1}\sigma^{(1)} \bigg|_{v_3 - v_1 = 0} = tH \bigg|_{v_3 - v_1 = 0} + t + 1 - (v_2 - v_1) - (v_4 - v_1) + t \frac{(tH)'}{tH} \bigg|_{v_3 - v_1 = 0}. \]
Substituting the values of \( v_2 - v_1 \) and \( v_4 - v_1 \) from (3.28) gives (3.24).

### 3.4 Difference Equations

We know the logarithmic derivatives of the \( \tau \)-functions \( U_N(t; a, \mu) \) and \( V_N(t; a, \mu) \) satisfy the second order second degree differential equation (1.37). Here we will utilise the Schlesinger transformation theory to show that they also satisfy third order difference equations in both \( a \) and \( \mu \) variables.
Proposition 3.5 The logarithmic derivative $U_N(t; a, \mu)$ satisfies a third order difference equation in the variable $a$

\begin{equation}
- t(U - U) = \left\{ \left( \overline{U} - U + a + \mu + 1 \right) \left[ N(\mu + t) + (a + 1)U - aU \right] \\
+ (a + 1)t \left[ \overline{U} - U - N \right] \right\} \\
\div \left\{ N(a + \mu + 1 + t) + \overline{U} - (a + 1)(\overline{U} - U) \right\} \\
\times \left\{ \left( \overline{U} - \overline{U} + a + \mu \right) \left[ N\mu + (N + a + \mu + 1)U - (N + a + \mu)\overline{U} \right] \\
+ t \left[ \mu N - \mu \overline{U} + (N + a + \mu)U - (N + a)\overline{U} \right] \right\} \\
\div \left\{ - \mu(a + \mu + t) + U - (N + a + \mu)(\overline{U} - U) \right\}
\end{equation}

where $U := U_N(t; a, \mu), \overline{U} := U_N(t; a - 1, \mu), \overline{U} := U_N(t; a + 1, \mu)$, etc and the boundary conditions are expressed by $U_N(t; a, \mu)$ at three consecutive $a$-values for all $N, t, \mu$. In addition $U_N(t; a, \mu)$ satisfies a third order difference equation in $\mu$

\begin{equation}
- t(U - \overline{U}) = \\
\left\{ \left( \overline{U} - U + 2N + a + \mu + 1 \right) \left[ -N(N + a) + (N + \mu + 1)U - (N + \mu)\overline{U} \right] \\
- t[-N(N + a) + (\mu + 1)\overline{U} - (N + \mu + 1)\overline{U} + NU] \right\} \\
\div \left\{ -N(2N + a + \mu + 1 - t) + \overline{U} - (N + \mu + 1)(\overline{U} - U) \right\} \\
\times \left\{ \left( \overline{U} - \overline{U} + 2N + a + \mu \right) \left[ -N(N + a) + (N + a + \mu + 1)U - (N + a + \mu)\overline{U} \right] \\
- t[-N(N + a) - \mu \overline{U} + (N + a + \mu)U - (N + a)\overline{U}] \right\} \\
\div \left\{ -(N + a)(2N + a + \mu - t) + U - (N + a + \mu)(\overline{U} - \overline{U}) \right\}
\end{equation}

where now $\overline{U} := U_N(t; a, \mu - 1), \overline{U} := U_N(t; a, \mu + 1)$.

Proof: The difference equation generated by $T_0^{-1} (2.47)$ with the parameter identification $v_2 - v_1 = -\mu, v_3 - v_1 = N + a, v_4 - v_1 = N$ is not directly useful as this leads to a difference equation in both $N$ and $a$ (there
are no difference equations in the parameter $N$ alone - only combined ones with $a$ or $\mu$). However difference equations generated by the other shift operators can be simply found using this one with a permutation of the parameter identification. Thus for the difference equation in $a$ one can use $v_2 - v_1 = -\mu, v_3 - v_1 = N, v_4 - v_1 = N + a$ and the relation $tH = U_N(t; a, \mu) + \mu N$ which gives the result is (3.33). The second result (3.34) follows from the parameter identification $v_2 - v_1 = N, v_3 - v_1 = N + a, v_4 - v_1 = -\mu$ and $tH = U_N(t; a, \mu) - N(N+a)$.

Both of these difference equations are of the third order and linear in the highest order difference and it may be possible to integrate these once and reduce them to second order equations, however we do not pursue this question here. Although we have stated the difference equations for $U_N(t; a, \mu)$, it is clear that $V_N((0, s); a, \mu)$ satisfies these as well although subject to different boundary conditions.

### 3.5 \( \tilde{E}_N((0, s); a, \mu) \) for \( a \in \mathbb{Z}_+ \)

In a previous study [16] the quantities $\tilde{E}_N((0, s); a, 0)$ and $\tilde{E}_N((0, s); a, 2)$ for $a \in \mathbb{Z}_+$ were expressed in terms of $a \times a$ determinants. This was done using the method of orthogonal polynomials to simplify the corresponding multiple integrals. In fact we can easily express $\tilde{E}_N((0, s); a, \mu)$ for $a \in \mathbb{Z}_+$, with $\mu$ general, as an $a \times a$ determinant using the methods of the present study.

**Proposition 3.6** For $a \in \mathbb{Z}_+$ the function

\[
\sigma^{(3)}(t) = \frac{t}{d} \log \left(t^{-N\mu - \frac{1}{2} a(a-1)} e^{-t(N+a)} \det \left[ \delta^{j+k}(e^L N(-t)) \right]_{j,k=0,\ldots,a-1} \right) 
\]

(3.35)

\[
= \frac{t}{d} \log \left(t^{-N\mu} e^{-Nt} \det \left[ \frac{d^j}{dt^j} L_{N+k}^\mu(-t) \right]_{j,k=0,\ldots,a-1} \right)
\]

Consequently, for $a \in \mathbb{Z}_{\geq 0}$,

\[
\tilde{E}_N((0, s); a, \mu) \propto e^{-Ns} \det \left[ \frac{d^j}{ds^j} L_{N+k}^\mu(-s) \right]_{j,k=0,\ldots,a-1}.
\]

(3.36)
Proof: Choose

\[ v_3 - v_1 = -N, \quad v_3 - v_2 = \mu \]

in (3.1). Then according to (2.43), for \( a \in \mathbb{Z}_{\geq 1} \)

\[ t^{-a/2} \tau[a] = \det \left[ \delta^{i+k} e^t F_1(-N, \mu + 1; -t) \right]_{j,k=0,\ldots,a-1}. \]

Using the fact that

\[ \delta \left[ L_N^\mu (-t) e^t \right] \]

where \( L_N^\mu \) denotes the Laguerre polynomial, substituting (3.38) in (2.44) and then substituting the resulting expression in (2.15) gives the first formula for \( \sigma^{(3)} \) in (3.35). In specifying the parameters in (2.15) we have made use of (3.37), the fact that \( v_4 - v_1 = a \), as well as (2.2). The theory noted in the sentence containing (2.13) tells us that \( \sigma^{(3)} \) satisfies (1.37)

\[ \nu_i = \begin{cases} \nu_0 = 0, & \nu_1 = v_2 - v_3 = -\mu, & \nu_2 = v_1 - v_3 = N, & \nu_3 = v_4 - v_3 = a + N. \end{cases} \]

To obtain the second equality in (3.35) we make use of the identity

\[ \delta \left[ L_N^\mu (-t) e^t \right] = \left[ (N + 1)L_{N+1}^\mu (-t) - (N + \mu + 1)L_N^\mu (-t) \right] e^t \]

in conjunction with elementary column operations, proceeding in an analogous fashion to the derivation of (3.10) using the identity (3.8). This shows

\[ \det \left[ \delta^{i+k} e^t L_N^\mu (-t) \right]_{j,k=0,\ldots,a-1} \propto \det \left[ \delta^{i+k} (e^t L_{N+k}^\mu (-t)) \right]_{j,k=0,\ldots,a-1}. \]

The second equality in (3.35) now follows by applying to this the general identities

\[ \det \left[ \delta^{i+k} (u(t) f_k(t)) \right]_{j,k=0,\ldots,a-1} = (u(t))^a \det \left[ \delta^{i+k} f_k(t) \right]_{j,k=0,\ldots,a-1} \]

\[ \det \left[ \delta^{i+k} f_k(t) \right]_{j,k=0,\ldots,a-1} = t^{a(a-1)/2} \det \left[ \frac{d^j}{dt^j} f_k(t) \right]_{j,k=0,\ldots,a-1}. \]
The function in the logarithm of the second equality in (3.35) is of the form $t^{-N \mu} e^{-N t}$ times a polynomial in $t$. But for $a \in \mathbb{Z}_{\geq 0}$, we know from the second integral formula in (1.7) that $t^{-N \mu} \tilde{E}((0, t); a, \mu)$ has the same structure. Since $\sigma^{(3)}$ also satisfies the same differential equation as $V_N(t; a, \mu)$ in (3.17), the formula (3.36) follows. \[ \Box \]

The determinant formula (3.38) can be written as an $a$-dimensional integral by using (3.2). Because (3.37) implies an exponent $-(N + a)$ for the factors $(1 - u_j)$ in the integrand, we must first modify the interval of integration. For this purpose it is convenient to first change variables $u_j \mapsto 1 - u_j$. Then instead of the interval of integration $[0, 1]$ we choose a simple, closed contour which starts at $u_j = 1$ and encircles the origin. In particular, choosing this contour as the unit circle in the complex $u_j$ plane gives

$$
t^{-a/2} \pi[a] \propto t^{a(a-1)/2} e^{at} \left( \prod_{j=1}^{a} (1 + e^{-2\pi i x_j})^N (1 + e^{2\pi i x_j})^\mu e^{se^{2\pi i x_j}} \right)_{\text{CUE}_a}.
$$

Consequently we have the following generalization of (1.39).

**Proposition 3.7** For $a \in \mathbb{Z}_{\geq 0}$,

(3.39) \hspace{1cm} \tilde{E}_N((0, s); a, \mu)

$$
= e^{-Ns} \frac{M_a(0, 0)}{M_a(\mu, N)} \left( \prod_{j=1}^{a} (1 + e^{-2\pi i x_j})^N (1 + e^{2\pi i x_j})^\mu e^{se^{2\pi i x_j}} \right)_{\text{CUE}_a}.
$$

We remark that the identity (3.39) is itself a special case of a known more general integral identity \[11\]. The latter identity involves the PDFs

$$
\frac{1}{C} \prod_{l=1}^{N} \lambda_l^\alpha e^{-\beta \lambda_l/2} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta \quad (\lambda_l > 0)
$$

and

$$
\frac{1}{C} \prod_{1 \leq j < k \leq N} |e^{2\pi i x_k} - e^{2\pi i x_j}|^\beta \quad (-1/2 < x_l < 1/2)
$$
defining what we will term the ensembles $L_\beta E$ and $C_\beta E$ respectively. For $\beta = 2$, these PDFs were introduced in (1.4) and (1.32) as the LUE and CUE. For $\beta = 1$ and 4 these PDFs also have a random matrix interpretation. They correspond to the case of an orthogonal and symplectic symmetry respectively, and give rise to the matrix ensembles LOE, COE and LSE, CSE. Now define

$$\tilde{E}^{(\beta)}_N((0, s); a, \mu) = \left\langle \prod_{l=1}^{N} \chi_{(s, \infty)}^{(l)}(\lambda_l - s)\mu \right\rangle_{L_\beta E}$$

(note that this reduces to (1.7) for $\beta = 2$). Then it follows from results in [11], derived using the theory of certain multi-variable hypergeometric functions based on Jack polynomials, that for $a \in \mathbb{Z}_{\geq 0}$

$$(3.40) \quad \tilde{E}^{(\beta)}_N((0, s); a, \mu) = e^{-Ns} \frac{M^{(\beta)}_a(0,0)}{M^{(\beta)}_a(2(\mu+1)/\beta-1, N)} \times \left\langle \prod_{j=1}^{a} (1 + e^{2\pi i x_j})^{2(\mu+1)/\beta-1}(1 + e^{-2\pi i x_j})^N e^{se^{2\pi i x_j}} \right\rangle_{C(4/\beta)E_a},$$

where $M^{(\beta)}(a, b)$ denotes the integral (1.30) with exponent 2 in the product of differences replaced by $\beta$.

4 \tau-function theory of $P_{III}$ and hard edge scaling

In the Laguerre ensemble the eigenvalues are restricted to be positive. For $N$ large, the spacing between the eigenvalues in the neighbourhood of the origin is of order $1/N$. By scaling the coordinates as in (1.54) the eigenvalue spacing is then of order 1 and well defined distributions result. In this limit the $P_V$ system degenerates to the $P_{III}$ system so it is appropriate to revise the $\tau$-function theory of the latter.

4.1 Okamoto $\tau$-function theory of $P_{III}$

Following [31], the Hamiltonian theory of $P_{III}$ (actually the $P_{III}'$ system rather than the $P_{III}$ ) can be formulated in terms of the Hamiltonian

$$(4.1) \quad \tau H = q^2 p^2 - (q^2 + v_1 q - t)p + \frac{1}{2}(v_1 + v_2)q.$$
Thus by substituting (4.1) in the Hamilton equations (2.3) and eliminating \( p \) we find that \( y(s) \) satisfies the PIII differential equation

\[
\frac{d^2 y}{ds^2} = \frac{1}{y} \left( \frac{dy}{ds} \right)^2 - \frac{1}{s \, ds} \frac{dy}{ds} + \frac{1}{s} (\alpha y^2 + \beta) + \gamma y^3 + \delta \frac{1}{y}
\]

with \( q(t) = sy(s), \ t = s^2 \) and

\[
\alpha = -4v_2, \ \beta = 4(v_1 + 1), \ \gamma = 4, \ \delta = -4.
\]

Note that with \( H \) specified by (4.1), the first of the Hamilton equations (2.3) gives

\[
tq' = 2q^2p - (q^2 + v_1q - t).
\]

Thus, by using this equation to eliminate \( p \) in (4.1), we see that \( tH \) can be expressed as an explicit rational function of \( q \) and \( q' \). Analogous to Proposition 2.1, it is straightforward to show that \( tH \) plus a certain linear function in \( t \) satisfies a second order second degree equation.

**Proposition 4.1** [31] With \( H \) specified by (4.1), define the auxiliary Hamiltonian

\[
h = tH + \frac{1}{4} v_1^2 - \frac{1}{2} t.
\]

The auxiliary Hamiltonian \( h \) satisfies the differential equation

\[
(th'')^2 + v_1v_2h' - (4(h')^2 - 1)(h - th') - \frac{1}{4}(v_1^2 + v_2^2) = 0.
\]

**Proof:** Following [31], we note from (4.1) and the Hamiltonian equations (2.3) that

\[
h' = p - \frac{1}{2}
\]

(4.5)

\[
2h' = t(1 - p)pq + v_1p - \frac{1}{2}(v_1 + v_2).
\]

Using the first equation to substitute for \( p \) in the second gives

\[
q = \frac{th'' - v_1h' + \frac{1}{2}v_2}{\frac{1}{2}(1 - 4(h')^2)}, \quad qp = \frac{th'' - v_1h' + \frac{1}{2}v_2}{(1 - 2h')},
\]
while we can check from (4.3), (4.1) and the first equation in (4.5) that
\[ h - th' = (qp - \frac{1}{2}v_1)^2 - q[qp - \frac{1}{2}(v_1 + v_2)]. \]
Substituting for \( qp \) and \( q \), and simplifying, gives (4.4).

Of interest in the random matrix application is the variant of (4.4) satisfied by
\[ \sigma_{III}(t) := -(tH) \Big|_{t\to t/4} - \frac{v_1}{4}(v_1 - v_2) + \frac{t}{4}. \]
A straightforward calculation using the result of Proposition 4.1 shows that \( \sigma_{III} \) satisfies
\[
(4.7) 
(t\sigma''_{III})^2 - v_1v_2(\sigma'_{III})^2 + \sigma'_III(4\sigma'_{III} - 1)(\sigma_{III} - t\sigma'_{III}) - \frac{1}{4^3}(v_1 - v_2)^2 = 0.
\]
Note that with the \( \tau \)-function defined in terms of the Hamiltonian (4.1) by (2.14), we have
\[
(4.8) \quad \sigma_{III}(t) = -i \frac{d}{dt} \log \left( e^{-t/4}e^{v_1(v_1 - v_2)/4} \tau(t/4) \right).
\]

### 4.2 Bäcklund transformations and Toda lattice equation

For the Hamiltonian (4.1), Okamoto [31] has identified two Bäcklund transformations with the property (2.16):
\[
(4.9) \quad T_1 \cdot \mathbf{v} = (v_1 + 1, v_2 + 1), \quad T_2 \cdot \mathbf{v} = (v_1 + 1, v_2 - 1).
\]
The operators \( T_1 \) and \( T_2 \) can be constructed out of more fundamental operators \( s_0, s_1, s_2 \) associated with the underlying \( B_2 \) root lattice, whose action (following [31] and [25]) on \( \mathbf{v}, p, q \) is given in Table 4.2. According to Table 4.2 we have
\[
(4.10) \quad T_1 = s_0s_2s_1s_2, \quad T_2 = s_2s_0s_2s_1.
\]
Table 4.2. Bäcklund transformations relevant to the $P_{III}$ Hamiltonian (4.1).

Analogous to the situation with the Hamiltonian (2.30) in the $P_{V}$ theory, introducing the sequence of Hamiltonians by

$$H[n] := H_{(v_1,v_2)\rightarrow(v_1+n,v_2+n)}$$

where $H$ is given by (4.1), the operator $T_1$ can be used to establish a Toda lattice equation for the corresponding $\tau$-function sequence (2.31).

**Proposition 4.2** [31, 25] The $\tau$-function sequence corresponding to the Hamiltonian sequence (4.11) obeys the Toda lattice equation

$$\delta^2 \log \bar{\tau}[n] = \bar{\tau}[n-1] \bar{\tau}[n+1] - \bar{\tau}^2[n], \quad \delta := t \frac{d}{dt}$$

where

$$\bar{\tau}[n] := t^{n/2} \tau[n](t/4).$$

**Proof:** Analogous to (2.34) and (2.35) we have

$$\delta \log \frac{\bar{\tau}[n-1] \bar{\tau}[n+1]}{\bar{\tau}^2[n]} = q[n](1 - p[n]) - T_1^{-1} q[n](1 - T_1^{-1} p[n]).$$

Making use of (4.10), we can use Table 4.2 to explicitly compute $T_1^{-1} q[n]$ and $T_1^{-1} p[n]$. This shows

$$q[n](1 - p[n]) - T_1^{-1} q[n](1 - T_1^{-1} p[n]) = -\frac{1}{p[n]} \left( 2q[n]p^2[n] - (2q[n] + v_1)p[n] + \frac{1}{2}(v_1 + v_2) \right).$$
But according to (4.1) and the second equation in (4.5), this latter expression is equal to $\delta \log(\frac{dt}{dt}tH)$. Substituting in (4.14) we deduce that

$$\frac{d}{dt}\delta \log \tau[n] = C \frac{\tau[n-1] \tau[n+1]}{\tau^2[n]}$$

Substituting for $\tau[n]$ according to (4.13) and taking $C = 1$ gives (4.12).

4.3 Classical solutions

Okamoto [31] has shown that for the special choice of parameters $v_1 = -v_2$, the PIII system admits a solution with $\tau[0] = 1$, and allows $\tau[1]$ to be evaluated as a Bessel function.

**Proposition 4.3** [31] For the special choice of parameters

(4.15) \[ v_1 = -v_2 \]

in (4.4) it is possible to choose $\tau[0] = 1$. The first member $\tau[1]$ of the $\tau$-function sequence corresponding to (4.11), after the substitution $t \mapsto t/4$, satisfies the equation

(4.16) \[ t(\hat{\tau}[1])'' + (v_1 + 1)(\hat{\tau}[1])' - \frac{1}{4} \hat{\tau}[1] = 0, \quad \hat{\tau}[1] := \tau[1](t/4). \]

For $v_1 \notin \mathbb{Z}$, this equation has the two linearly independent solutions in terms of Bessel functions

(4.17) \[ \hat{\tau}[1] = t^{-v_1/2}I_{\pm v_1}(\sqrt{t}) \]

(for $v_1 \in \mathbb{Z}$, $I_{v_1}$ and $I_{-v_1}$ are proportional and (4.17) only provides one independent solution).

**Proof:** Substituting (4.15) into the definition (4.1) of $H$ we see that it is possible to choose

(4.18) \[ p = 0, \quad H = 0, \]
the latter allowing us to take \( \tau[0] = 1 \). To calculate \( q \) we use the Hamilton equation

\[
(4.19) \quad tq' = \frac{\partial tH}{\partial p} \bigg|_{v_1 = -v_2} = -(q^2 + v_1q - t).
\]

Now, it follows from (2.27) with \( H \) given by (4.1) and \( T_0^{-1} \) by \( T_1 \) that

\[
q = t \frac{d}{dt} \log \tau[1].
\]

Substituting this in (4.19) and changing variables \( t \mapsto t/4 \) gives (4.16).

For definiteness take the + sign in (4.17) and let \( v_1 = \nu \). This substituted in (4.13) with \( n = 1 \), and the corresponding formula for \( \bar{\tau}[1] \) substituted in (2.43) shows that

\[
(4.20) \quad t^{n(n-1)/2} \bar{\tau}[n] = \det [\delta^{j+k} I_\nu(\sqrt{t})]_{j,k=0,\ldots,n-1},
\]

where use has also been made of the theory in the first sentence of the proof of Proposition 3.1. The determinant in (4.20) can be written in a form which is independent of the operator \( \delta \).

**PROPOSITION 4.4** We have

\[
(4.21) \quad \det [\delta^{j+k} I_\nu(\sqrt{t})]_{j,k=0,\ldots,n-1} = (t/4)^{n(n-1)/2} \det [I_{j-k+\nu}(\sqrt{t})]_{j,k=0,\ldots,n-1}.
\]

**PROOF:** Let \( t = s^2 \) and note that

\[
\delta := t \frac{d}{dt} = \frac{1}{2} s \frac{d}{ds} =: \frac{1}{2} \delta_s
\]

to conclude

\[
(4.22) \quad \det [\delta^{j+k} I_\nu(\sqrt{t})]_{j,k=0,\ldots,n-1} = 2^{-n(n-1)} \det [\delta^{j+k} I_\nu(s)]_{j,k=0,\ldots,n-1}.
\]

We now adopt a very similar strategy to that used in the proofs of Propositions 3.1 and 3.2. Briefly, we first make use of the identity

\[
\delta_s I_\nu(s) = s I_{\nu+1}(s) + \nu I_\nu(s)
\]
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together with elementary row operations to eliminate the operator $\delta^j$ in (4.22), obtaining

$$\det[\delta^j I_{\nu}(s)]_{j,k=0,\ldots,n-1} = \det[\delta^k s^j I_{\nu+j}(s)]_{j,k=0,\ldots,n-1}. \quad (4.23)$$

Next, use is made of the identity

$$\delta s^j I_{\nu+j}(s) = s^{j+1} I_{\nu+j-1}(s) - \nu s^j I_{\nu+j}(s) \quad (4.24)$$

to perform elementary column operations thus reducing the RHS of (4.23) to the form

$$\det[s^j I_{\nu+j}(s) \ldots \delta s^{k-1} s^{j+1} I_{\nu+j-1}(s) \ldots]_{j=0,\ldots,n-1, k=1,\ldots,n-1}. \quad (4.25)$$

Next we apply elementary column operations, using the identity (4.24) with $j \mapsto j+1$, $\nu \mapsto \nu-2$, to show that (4.25) is equal to

$$\det[s^j I_{\nu+j}(s) \ldots \delta s^{k-2} s^{j+2} I_{\nu+j-2}(s) \ldots]_{j=0,\ldots,n-1, k=2,\ldots,n-1}.$$

Continuing in this fashion, using the identity (4.24) with appropriate substitutions, reduces the RHS of (4.23) to

$$\det[s^j I_{\nu+j}(s)]_{j,k=0,\ldots,n-1}.$$

Substituting in (4.22) gives (4.21).

COROLLARY 4.5  The function

$$\sigma_{III}(t) = -t \frac{d}{dt} \log \left( e^{-t/4} t^{\nu^2/2} \det[I_{j-k^{+\nu}(\sqrt{t})}]_{j,k=0,\ldots,n-1} \right) \quad (4.26)$$

satisfies the equation (4.47) with parameters

$$\left( v_1, v_2 \right) = \left( \nu + n, -\nu + n \right) \quad (4.27)$$

and boundary condition

$$\sigma_{III}(t) \sim \frac{t}{4} - \frac{nt^{1/2}}{2} - \left( \frac{\nu^2}{2} - \frac{n^2}{4} \right) + \cdots \quad (4.28)$$
Proof: It follows from Proposition 4.4, (4.20) and (1.13) that

\[ \tau[n](t/4) \propto t^{-n\nu/2} \det[I_{j-k+\nu}(\sqrt{t})]_{j,k=0,\ldots,n-1} \]

The equation satisfied by \( \sigma_{III}(t) \) follows by substituting this, and the parameter values (4.27), in (4.8). To obtain the boundary condition, we make use of the well known Toeplitz determinant formula

\[ \det[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-i(j-k)\theta} d\theta]_{j,k=0,\ldots,n-1} = \frac{1}{n!} \frac{1}{(2\pi)^n} \prod_{l=1}^{n} \int_{-\pi}^{\pi} d\theta_l \prod_{1 \leq j < k \leq n} |e^{i\theta_k} - e^{i\theta_j}|^2 \]

together with the integral representation

\[ I_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta + z \cos \theta} d\theta, \quad n \in \mathbb{Z} \]

to rewrite the determinant in (4.26) as a multidimensional integral,

(4.29) \[ \det[I_{j-k+\nu}(\sqrt{t})]_{j,k=0,\ldots,n-1} = \frac{1}{n!} \frac{1}{(2\pi)^n} \prod_{l=1}^{n} \int_{-\pi}^{\pi} d\theta_l \frac{e^{\sqrt{t}\cos \theta_l - i\nu \theta_l}}{\prod_{1 \leq j < k \leq n} |e^{i\theta_k} - e^{i\theta_j}|^2} \]

valid for \( \nu \in \mathbb{Z} \). For large \( t \) the dominant contribution to the above integral comes from the neighbourhood of \( \theta_j = 0 \) (\( j = 1, \ldots, n \)). Expanding to leading order about these points and changing variables shows

\[ \det[I_{j-k+\nu}(\sqrt{t})]_{j,k=0,\ldots,n-1} \sim t_{-n^2/4}. \]

Substituting this in (4.26) gives (4.28).

The result of Corollary 4.3 is relevant to the hard edge scaling of \( \tilde{E}_N((0,s);a,\mu) \), which gives the quantity \( \tilde{E}^{\text{hard}}(t;a,\mu) \) defined by (1.69). Now, in an earlier study [16], it was shown that for \( a \in \mathbb{Z}_{\geq 0} \)

\[ \tilde{E}^{\text{hard}}(t;a,0) = e^{-t/4} \det[I_{j-k}(\sqrt{t})]_{j,k=0,\ldots,a-1} \]

(4.30) \[ \tilde{E}^{\text{hard}}(t;a,2) \propto e^{-t/4} t^{-a} \det[I_{j-k+2}(\sqrt{t})]_{j,k=0,\ldots,a-1} \]
Thus, as already noted in \[16\] in the case of $\tilde{E}_{\text{hard}}(t; a, 0)$ (deduced from knowledge of (1.70)), it follows from Corollary 4.5 that $\tilde{E}_{\text{hard}}(t; a, \mu)$ for $\mu = 0$ and $\mu = 2$ can be characterised as the solution of the equation (1.7) with parameters $(v_1, v_2) = (a + \mu, a - \mu)$.

The results (4.30) were obtained by computing the limit of the RHS of (3.36) using the asymptotic formula

$$e^{-x/2} x^{\alpha/2} L_N^\alpha(-x) \sim N^{\alpha/2} I_a(2(Nx)^{1/2}).$$

The same approach allows $\tilde{E}_{\text{hard}}(t; a, \mu)$ to be computed for general $\mu$ and $a \in \mathbb{Z}_\geq 0$, giving the result

$$\tilde{E}_{\text{hard}}(t; a, \mu) \propto e^{-t/4} t^{-\mu a/2} \det[I_{j-k+\mu}(\sqrt{t})]_{j,k=0,...,a-1}.$$

(4.31)

It then follows from Corollary 4.5 that the result (1.73) holds for general $\mu$ and $a \in \mathbb{Z}_\geq 0$. For general $a > -1$ the result (1.73) can be deduced from the first formula in (1.40). This task will be undertaken in subsection 4.5.

To conclude this subsection we make two remarks. The first is that substituting (4.29) in (4.31) shows that for $\mu \in \mathbb{Z}$ and $a \in \mathbb{Z}_\geq 0$

$$\tilde{E}_{\text{hard}}(t; a, \mu) \propto e^{-t/4} t^{-\mu a/2} \prod_{l=1}^{\alpha} \int_{-\pi}^\pi d\theta_l e^{\sqrt{t} \cos \theta_l - i \mu \theta_l} \prod_{1 \leq j < k \leq a} |e^{i \theta_k} - e^{i \theta_j}|^2$$

(4.32)

This identity, relating an average in the infinite LUE scaled at the hard edge to an average in the CUE of dimension $a$, has previously been derived \[11\] as a scaled limit of (3.39). With $c := 2(\mu + 1)/\beta$ and $\mu$ such that $c \in \mathbb{Z}_\geq 0$, it was also shown that the identity (3.40) in the hard edge limit reduces to

$$\tilde{E}^{(\beta)}_{\text{hard}}(t; a, \mu) = C e^{-t/4} \left(\frac{1}{t}\right)^{(c-1)a} \left(\frac{1}{2\pi}\right)^a \times \prod_{l=1}^{\alpha} \int_{-\pi}^\pi d\theta_l e^{\sqrt{t} \cos \theta_l - i(c-1)\theta_l} \prod_{1 \leq j < k \leq a} |e^{i \theta_k} - e^{i \theta_j}|^{4/\beta}$$

(4.33)

where

$$C = \prod_{j=1}^{a} \frac{\Gamma(1 + 2/j)}{\Gamma(1 + 2j/\beta)} \frac{\Gamma(c + 2(j - 1)/\beta)}{\Gamma(1 + 2j/\beta)}$$
and $\tilde{E}^{(\beta)\text{hard}}$, like $\tilde{E}^{\text{hard}}$ specified by (1.69), is normalised so that it equals unity for $t = 0$. As a second remark, we note that a feature of the differential equation (4.4) is that it is unchanged by the mapping $t \mapsto -t$, $v_2 \mapsto -v_2$. Using this, it follows from (4.3) and (4.6) that (4.7) is also satisfied by \[ \sigma_{\text{III}}(t) \mid_{v_2 \rightarrow -v_2} + \frac{t}{2} + v_1 v_2/2 = -t \frac{d}{dt} \log \left( e^{-t/4} \nu_1(v_1 - v_2)/4 \nu(-t/4) \right) \mid_{v_2 \rightarrow -v_2} \] where the equality follows from (4.8). But by an appropriate modification of Propositions 4.3 and 4.4, a determinant formula for $\tau(-t/4) \mid_{v_2 \rightarrow -v_2}$ can be given for certain $(v_1, v_2)$. Performing these modifications and substituting the resulting formula in (4.34) leads us to the conclusion that \[ -t \frac{d}{dt} \log \left( e^{-t/4} \nu_2/2 \det[J_{j-k+\nu}(\sqrt{t})]_{j,k=0,...,n-1} \right) \] satisfies (4.7) with parameters \[ (v_1, v_2) = (\nu + n, \nu - n). \]

In the $n = \mu = 2$ and $\nu = a$ case, substituting (4.35) for $\sigma(-t)$ in (1.78) gives the well known [12] expression for $\rho^{\text{hard}}(s)$ in terms of Bessel functions.

### 4.4 Schlesinger Transformations

Integrable difference equations are found to arise from the Schlesinger transformations generated by $T_1, T_2$ [34, 33, 26] acting on $q$ and $p$ and we consider these here, along with the difference equations for the Hamiltonians.

**Proposition 4.6** [33] *The Schlesinger transformations of the P_{III} system for the operators $T_1, T_2$ generating the parameter sequences $(v_1+n, v_2+n)$ (respectively $(v_1+n, v_2-n)$) acting on the transcendent $q$ correspond to second order difference equations of the alternate discrete Painlevé II, d-P_{II},*
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equation (respectively its dual)

\begin{align}
\frac{1}{2} v_1 + v_2 + 2 + 2n &+ \frac{1}{2} \frac{q[n]q[n+1] + t}{q[n]-1} = q^{-1} - \frac{q}{t} + \frac{v_2 + n}{t}, \\
\frac{1}{2} v_2 - v_1 - 2 - 2n &+ \frac{1}{2} \frac{q[n]q[n+1] - t}{q[n]-1} = q^{-1} + \frac{q}{t} + \frac{v_1 + n}{t}.
\end{align}

Further, under the action of $T_1$, $tH[n+1] - tH[n] = q[n](1 - p[n])$ and this satisfies the second order difference equation

\begin{align}
q(1-p)[n] \left\{ q(1-p)[n] + \frac{1}{2}(v_1 - v_2) \right\} \\
= t \left\{ q(1-p)[n + 1] + q(1-p)[n] + \frac{1}{2}(v_1 - v_2) \right\} \\
\div \left\{ q(1-p)[n + 1] + q(1-p)[n] + v_1 + 1 + n \right\} \\
\times \left\{ q(1-p)[n] + q(1-p)[n - 1] + \frac{1}{2}(v_1 - v_2) \right\} \\
\div \left\{ q(1-p)[n] + q(1-p)[n - 1] + v_1 + n \right\},
\end{align}

while under the action of $T_2$, $tH[n+1] - tH[n] = q[n]p[n]$ and this satisfies the second order difference equation

\begin{align}
qp[n] \left\{ qp[n] - \frac{1}{2}(v_1 + v_2) \right\} &= -t \left\{ qp[n + 1] + qp[n] - \frac{1}{2}(v_1 + v_2) \right\} \\
\div \left\{ qp[n + 1] + qp[n] - v_1 - 1 - n \right\} \\
\times \left\{ qp[n] + qp[n - 1] - \frac{1}{2}(v_1 + v_2) \right\} \\
\div \left\{ qp[n] + qp[n - 1] - v_1 - n \right\}.
\end{align}

PROOF: In the case of the $T_1$ transformation the action on the canon-
ical variables in the forward and reverse directions are

\begin{align*}
(4.40) \quad q[n + 1] &= -\frac{t}{q} + \frac{\frac{1}{2}(v_1 + v_2 + 2 + 2n)t}{q[q(p - 1) - \frac{1}{2}(v_1 - v_2)] + t} \\
(4.41) \quad p[n + 1] &= \frac{q}{t} [q(p - 1) - \frac{1}{2}(v_1 - v_2)] + 1 \\
(4.42) \quad q[n - 1] &= \frac{t}{\frac{1}{2}(v_1 + v_2 + 2n) - q} \\
(4.43) \quad p[n - 1] &= 1 - t^{-1} \left[ \frac{\frac{1}{2}(v_1 + v_2 + 2n)}{p} - q \right] \\
& \quad \times \left\{ \left[ \frac{\frac{1}{2}(v_1 + v_2 + 2n)}{p} - q \right] (1 - p) - \frac{1}{2}(v_1 - v_2) \right\}
\end{align*}

By combining (4.40) and (4.41) into the form

\begin{equation}
(4.44) \quad q[n + 1] + \frac{t}{q[n]} = \frac{\frac{1}{2}(v_1 + v_2 + 2 + 2n)}{p[n + 1]}
\end{equation}

and eliminating \( p \) between this equation and (4.42) we have the result (4.36). The difference equation for the Hamiltonian can be found by solving

\begin{equation}
(4.45) \quad q(1-p)[n + 1] + q(1-p)[n] + \frac{1}{2}(v_1 - v_2)
\end{equation}

\begin{equation*}
= -\frac{1}{2}(v_1 + v_2 + 2 + 2n) \frac{q(1-p)[n] + \frac{1}{2}(v_1 - v_2)}{q(1-p)[n] + \frac{1}{2}(v_1 - v_2) - t/q}
\end{equation*}

for \( q \) and

\begin{equation}
(4.46) \quad q(1-p)[n - 1] + q(1-p)[n] + \frac{1}{2}(v_1 - v_2) = \frac{1}{2}(v_1 + v_2 + 2n) \frac{1 - p[n]}{p[n]}
\end{equation}

for \( p \) and then reforming \( q(1-p) \). The result is (4.38). The corresponding results for the \( T_2 \) sequence can be found in a similar manner.
4.5 Hard edge scaling $E_{\text{hard}}^{\mu}(t; a, \mu)$

According to (1.66) and (1.40) we have for general $a > -1$

\[(4.47) \quad E_{\text{hard}}^{\mu}(s; a, \mu) = \exp \int_{0}^{s} v(t; a, \mu) \frac{dt}{t}\]

where

\[(4.48) \quad v(t; a, \mu) = \lim_{N \to \infty} \left(V_N(t/4N; a, \mu) + \mu N\right).\]

In this limit the differential equation (1.37) characterizing the $P_V$ auxiliary Hamiltonian (2.12) and specifying $V_N$ degenerates to the differential equation (4.4) characterizing the $P_{\text{III}}$ auxiliary Hamiltonian (4.3) thus identifying $v$ with this quantity.

**Proposition 4.7** The function $v$ specified by (4.43) satisfies the differential equation

\[(4.49) \quad (tv'')^2 - (\mu + a)^2(v')^2 - v'(4v + 1)(v - tv') - \frac{\mu}{2}(\mu + a)v' - \frac{\mu^2}{4} = 0.\]

Consequently

\[(4.50) \quad v(t; a, \mu) = -\left(\sigma_{\text{III}}(t) + \mu(\mu + a)/2\right)\]

where $\sigma_{\text{III}}$ is specified by the Jimbo-Miwa-Okamoto $\sigma$-form of the Painlevé III equation (1.71) with parameters (1.74) $v_1 = a + \mu, v_2 = a - \mu$ and is subject to the boundary condition (1.73)

\[(4.51) \quad v(t; a, \mu) \sim -\frac{1}{4}t + \frac{1}{2}at^{1/2} - \frac{1}{4}a(a + 2\mu).\]

**Proof:** Making the replacement $\sigma \to \sigma - N\mu$ in (1.37) with parameters (3.5), changing variables $t \to t/4N$, $\sigma(t/4N) \to v(t)$, and equating terms of order $N^2$ (which is the leading order) on both sides gives (4.49). Substituting (4.50) in (4.49) shows $\sigma_{\text{III}}$ satisfies (1.74) with parameters as specified. The boundary condition is the same as (4.28) with $n = a, \nu = \mu$. \[\blacksquare\]
**Proposition 4.8**  The function \( v(t;a,\mu) \) satisfies a third order difference equation in the variable \( a \) (suppressing the additional dependencies)

\[
\frac{1}{4} a(a+1)t = [v(a+1) - v(a-1) + a + \mu][v(a+2) - v(a) + a + \mu + 1] \\
\times \left[ \frac{1}{4} t - av(a+1) + (a+1)v(a) \right],
\]

and a third order difference equation in \( \mu \)

\[
-\frac{1}{4} t = [v(\mu+1) - v(\mu)][v(\mu+1) - v(\mu) + a] \\
\times \frac{[v(\mu+1) - v(\mu-1) + a + \mu][v(\mu+2) - v(\mu) + a + \mu + 1]}{[v(\mu+1) - v(\mu-1) + a][v(\mu+2) - v(\mu) + a]},
\]

**Proof:**  These two results follow from applying the hard edge scaling form \((4.48)\) to the finite-\(N\) difference equations \((3.33)\) and \((3.34)\). Note that \((4.53)\) is precisely the result that would be inferred from the \(P_{\text{II}}\) difference equation generated by the \(T_2\) shift \((4.39)\) with the correspondence

\[
v(t;a,\mu) = tH \Big|_{t\rightarrow t/4} - t/4.
\]

We have noted in \((1.77)\) that a corollary of Proposition \(4.7\) is the evaluation of \(p_{\text{min}}(s;a)\) in terms of \(v(t;a,2)\). Since, analogous to \((1.11)\),

\[
p_{\text{min}}^\text{hard}(s;a) = -\frac{d}{ds} \tilde{E}_\text{hard}^\text{hard}((0,s);a,0),
\]

the results \((1.77)\) and \((1.70)\) substituted into this formula imply an identity between transcendentals analogous to \((3.24)\). As is the case with \((3.24)\), this identity too can be independently verified.

**Proposition 4.9**  With \(\sigma(t)\) denoting the auxiliary Hamiltonian \((4.6)\) with parameters \((v_1,v_2) = (a+2,a-2)\), and \(\sigma_B(t)\) denoting the same quantity with parameters \((v_1,v_2) = (a,a)\), the identity

\[
\sigma(t) = \sigma_B(t) - 1 - t \frac{\sigma_B'(t)}{\sigma_B(t)},
\]

holds.
Proof: The indirect derivation of this result has been sketched above. A direct derivation can be given by using the properties of the shift operator $T_2$ in (4.9) and (4.10). First, denote by $tH[0]$ the Hamiltonian $tH$ in (4.4) (and similarly define $p[0], q[0]$) so that

\[
\sigma_B(t) = -(tH[0])\bigg|_{t \to t/4} + t/4.
\]

It then follows from the definitions of $T_2$ and $\sigma(s)$ that

\[
\sigma(t) = -(tT_2^2H[0])\bigg|_{t \to t/4} - (a + 2) + t/4.
\]

On the other hand, from the definition (4.1) and the property (2.16) of $T_2$ we have that

\[
tT_2^2H[0] = tH[0] - q[0]p[0] - (T_2q[0])(T_2p[0]).
\]

But from (4.10) and Table 4.2 it follows that

\[
T_2q[0] = \left(\frac{t}{q[0]} - \frac{t}{q[0](q[0]p[0] - a) + t}\right),
\]

\[
T_2p[0] = -q[0] \frac{t}{t}(q[0]p[0] - a)
\]

and thus

\[
tT_2^2H[0] = tH[0] - a - 1 + \frac{\sigma'_B(t)}{\sigma_B(t)}
\]

where to obtain the final term use has been made of (4.1), (4.6) and the analogue of the first equation in (4.5). Substituting in (4.58) and recalling (4.57) gives (4.56).

4.6 The spacing probability $p_2(0; s)$

The evaluation (4.47) has consequence regarding the nearest neighbour spacing distribution $p_2(0; s)$ for the scaled, infinite GUE in the bulk. This quantity has a special place in the Painlevé transcendent evaluation of gap probabilities because it motivated the study of the probability $E_2(0; s)$
of no eigenvalues in an interval of length $s$ in the scaled, infinite GUE through the relation

$$p_2(0; s) = \frac{d^2}{ds^2} E_2(0; s),$$

and $E_2(0; s)$ in turn was the first quantity in random matrix theory to be characterised as the solution of a non-linear equation in the Painlevé theory [22]. More recently, building on the evaluation of $E_2(0; s)$ from [24], it has been shown that

$$p_2(0; s) = -\frac{\tilde{\sigma}(\pi s)}{s} \exp \int_0^{\pi s} \frac{\tilde{\sigma}(t)}{t} dt$$

(a close relative of (1.37) for a particular choice of the parameters) subject to the boundary condition $\tilde{\sigma}(s) \sim -(s^3/3\pi)$. Here we will provide a Painlevé transcendent evaluation of $p_2(0; s)$ distinct from (4.60).

The starting point is the identity

$$\langle \prod_{l=1}^N (\lambda_l^2 - s^2) \chi^{(l)}_{(-\infty, -s) \cup (s, \infty)} \rangle_{\text{GUE}} \propto \tilde{E}_{[N/2]}((0, s^2); 1/2, 2) \tilde{E}_{[(N+1)/2]}((0, s^2); -1/2, 2),$$

which is a variant of a formula given in [14], applicable to general matrix ensembles with a unitary symmetry and an even weight function. Now the LHS of (4.61) multiplied by $s^2$ is proportional to the density function for a spacing between eigenvalues of length $2s$ symmetric about the origin. Recalling that the bulk scaling in the GUE requires

$$\lambda_l \mapsto \frac{\pi \lambda_l}{\sqrt{2N}},$$

it follows by replacing $s$ by $\pi s/\sqrt{2N}$, making use of the definition (1.69), and the property $p_2(0; s) \sim \frac{\pi^2}{3} s^2$ which follows from the boundary condition for $\tilde{\sigma}(s)$ in (4.60) (this fixes the proportionality constant) that

$$p_2(0; 2s) = \frac{(2\pi s)^2}{3} \tilde{E}^\text{hard}((\pi s)^2; 1/2, 2) \tilde{E}^\text{hard}((\pi s)^2; -1/2, 2).$$
According to (4.47) this specifies $p_2(0; s)$ in terms of Painlevé III transcendents whereas (4.60) involves Painlevé V transcendents.

5 Concluding remarks

The results of the paper have already been summarised in Section 1. Here we want to draw attention to a feature of this work (and our previous work [17]) which requires further study. This feature relates to the boundary conditions in the scaled limits, in particular the specification of $\tilde{E}^\text{hard}(s; a, \mu)$ by (1.73). One observes that the boundary condition (1.75) is even in $\mu$, as is the differential equation (1.71) with parameters (1.74). Thus according to this specification the solution itself must be even in $\mu$, but this contradicts the small $t$ behaviour which for the formula (1.73) to be well defined must be

$$\sigma(t) \sim \frac{1}{2} \mu(a + \mu) + O(t^\epsilon), \quad (\epsilon > 0). \quad (5.1)$$

The situation is well illustrated by the case $a = 1$, for which (4.26) shows

$$\sigma(t) = -t \frac{d}{dt} \log \left(e^{-t/4} \mu^2/2 I_\mu(\sqrt{t})\right). \quad (5.2)$$

The small $t$ expansion of $I_\mu(\sqrt{t})$ shows

$$\sigma(t) \sim \frac{1}{2} \mu(\mu + 1) + \frac{\mu}{4(\mu + 1)} t, \quad (5.3)$$

in agreement with (5.1), whereas the large $t$ expansion of $I_\mu(\sqrt{t})$ shows

$$\sigma(t) \sim \frac{t}{4} - \frac{t^{1/2}}{2} + \sum_{l=0}^\infty \frac{a_l}{l^{1/2}}, \quad (5.4)$$

where the $a_l$ are even functions of $\mu$. Thus the asymmetry between $\mu$ and $-\mu$ can only be present in an exponentially small term for $t \to \infty$. This is clear from the exact expression

$$\sigma(t; \mu) - \sigma(t; -\mu) = -\frac{\sin \pi \mu}{\pi} \frac{1}{I_\mu(\sqrt{t})I_{-\mu}(\sqrt{t})}. \quad (5.5)$$
Furthermore, although the boundary condition \((5.1)\) distinguishes the cases \(\pm \mu\), the differential equation \((1.71)\) also requires that the leading (in general) non-analytic term also be specified for the solution to be uniquely specified (for example, with \(\mu = 0\) this term is proportional to \(t^{a+1}\)).

A practical consideration of this discussion is that the boundary condition \((1.75)\) does not uniquely determine the solution of \((1.71)\), so that \(\tilde{E}^{\text{hard}}(s; a, \mu)\) is not uniquely characterised. The same remark applies to the formula \((1.60)\) for \(\tilde{E}^{\text{soft}}(s; \mu)\). Indeed the inadequacy of the boundary condition \((1.62)\) to uniquely determine the solution is already apparent in our discussion of the formula \((1.63)\).

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