How likely can a point be in different Cantor sets

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Abstract

Given a positive integer $m \geq 2$, let $\mathcal{K} = \{K_\lambda : \lambda \in (0, 1/m]\}$ be a class of self-similar sets
with each $K_\lambda = \left\{ \sum_{i=1}^{\infty} d_i \lambda^i : d_i \in \{0,1,\ldots,m-1\}, i \geq 1 \right\}$. In this paper we investigate the likelihood of a point in the self-similar sets of $\mathcal{K}$. More precisely, for a given point $x \in (0,1)$ we consider the parameter set $\Lambda(x) = \{\lambda \in (0, 1/m] : x \in K_\lambda\}$, and show that $\Lambda(x)$ is a topological Cantor set having zero Lebesgue measure and full Hausdorff dimension. Furthermore, by constructing a sequence of Cantor subsets of $\Lambda(x)$ with large thickness we show that for any $x,y \in (0,1)$ the intersection $\Lambda(x) \cap \Lambda(y)$ also has full Hausdorff dimension.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Given an integer $m \geq 2$, for $\lambda \in (0, 1/m]$ let $K_\lambda$ be the self-similar set generated by the iterated function system (IFS) $\Psi_\lambda := \{f_i(x) = \lambda(x + i) : i = 0, 1, \ldots, m-1\}$. In other words, $K_\lambda$ is the

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unique non-empty compact set in \( \mathbb{R} \) satisfying \( K_\lambda = \bigcup_{i=0}^{m-1} f_i(K_\lambda) \). So we can rewrite \( K_\lambda \) as

\[
K_\lambda = \left\{ \sum_{i=1}^{\infty} d_i \lambda^i : d_i \in \{0, 1, \ldots, m-1\}, i \geq 1 \right\}.
\] (1.1)

When \( \lambda = 1/m \), it is clear that \( K_{1/m} = [0, 1] \). When \( \lambda \in (0, 1/m) \), it is easy to see that \( K_\lambda \) is a Cantor set with its convex hull \( \text{conv}(K_\lambda) = [0, \frac{m^{1-1/\lambda}}{1-\lambda}] \) and the Hausdorff dimension of \( K_\lambda \) is given by \(-\log m/\log \lambda\) (cf. [8]).

Let \( \mathcal{K} = \{K_\lambda : \lambda \in (0, 1/m)\} \) be the class of all self-similar sets defined as in (1.1). We are interested in the following question: how likely can a point be in different self-similar sets of \( \mathcal{K} \)? In other words, for a point \( x \in [0, 1] \) the above question is to ask how large of the parameter set

\[
\Lambda(x) := \{ \lambda \in (0, 1/m] : x \in K_\lambda \}.
\]

Observe that the point 0 always belongs to \( K_\lambda \) for all \( \lambda \in (0, 1/m] \). Then \( \Lambda(0) = (0, 1/m] \). Furthermore, \( 1 \in K_\lambda \) if and only if \( \lambda = 1/m \). This implies \( \Lambda(1) = \{1/m\} \). So, it is interesting to study \( \Lambda(x) \) for \( x \in (0, 1) \). The main difficulty in the study of \( \Lambda(x) \) is that \( \Lambda(x) \) always involves infinitely many (a continuum of) self-similar sets in \( \mathcal{K} \). We will prove in theorem 1.1 that \( \Lambda(x) \) is a topological Cantor set in \( \mathbb{R} \): a nonempty compact set in \( \mathbb{R} \) having neither interior nor isolated points. And \( \Lambda(x) \) is a Lebesgue null set but has full Hausdorff dimension (see corollary 1.3). Furthermore, by carefully constructing a sequence of Cantor subsets of \( \Lambda(x) \) with large thickness we show in theorem 1.6 that for any two points \( x, y \in (0, 1) \) the intersection \( \Lambda(x) \cap \Lambda(y) \) also has full Hausdorff dimension. As a by-product we show that \( \Lambda(x) \) contains arbitrarily long arithmetic progressions (see remark 4.3).

Our motivation to study \( \Lambda(x) \) comes from unique beta expansions. When \( \lambda \in (1/m, 1) \) the IFS \( \Psi_\lambda = \{f_i\}_{i=0}^{m-1} \) has overlaps, that is, for the attractor \( K_\lambda \) we have \( f_i(K_\lambda) \cap f_{i+1}(K_\lambda) \neq \emptyset \) for all \( 0 \leq i < m-1 \). Denote by \( U_\lambda \) the set of \( x \in K_\lambda \) such that \( x \) has a unique coding, i.e., there exists a unique sequence \( (d_i) \in \{0, 1, \ldots, m-1\}^\infty \) such that \( x = \sum_{i=1}^{\infty} d_i \lambda^i \). The last two authors and their coauthors recently studied in [10] the topological and fractal properties of the set \( U(x) := \{\lambda \in (1/m, 1) : x \in U_\lambda \} \). They showed that \( U(x) \) has zero Lebesgue measure for all \( x > 0 \), and \( \dim_H U(x) = 1 \) for any \( x \in (0, 1) \). Moreover, for typical \( x > 0 \) the set \( U(x) \) contains isolated points. Dajani et al. [5] showed that for any \( x \in (0, 1] \) the algebraic sum \( U(x) + U(x) \) contains interior points. When \( x = 1 \), the set \( U(1) \) was extremely studied. Komornik and Loreti [9] showed that the topological closure of \( U(1) \) is a topological Cantor set. Allaart and the second author studied in [1] the local dimension of \( U(1) \), and showed that \( U(1) \) has more weight close to \( 1/m \). Bonanno et al. [4] found its close connection to the bifurcation set of \( \alpha \)-continued fractions and many other dynamical systems.

Another motivation to study \( \Lambda(x) \) is from the work of Boes et al. [3], where they considered a similar class of Cantor sets \( C_\lambda \) in \([0, 1]\) (they in fact considered a class of fat Moran sets with positive Lebesgue measures). They showed that \( \Lambda'(x) := \{\lambda \in (0, 1/2) : x \in C_\lambda \} \) is of first category for any \( x \in (0, 1) \), and then they concluded that for a second category set of \( \lambda \in (0, 1/2) \) the Cantor set \( C_\lambda \) excluding the two endpoints contains only irrational numbers.

Now we state our main results. First we show that \( \Lambda(x) \) is a topologically small set.

**Theorem 1.1.** For any \( x \in (0, 1) \) the set \( \Lambda(x) \) is a topological Cantor set with min \( \Lambda(x) = \frac{x}{m-1+x} \) and max \( \Lambda(x) = 1/m \).

Theorem 1.1 suggests that \( \Lambda(x) \) can be obtained by successively removing a sequence of open intervals from the closed interval \([0, 1 - x \frac{1}{m-1+x}]\). At the end of section 2 we will introduce a geometrical construction of \( \Lambda(x) \) (see figure 1 for an example).
Figure 1. A geometrical construction of $\Lambda(x)$ with $x = 1/2$ and $m = 2$. See example 2.6 for more explanation.

Observe that the self-similar set $K_\lambda$ is *dimensional homogeneous*, i.e., for any $y \in K_\lambda$ we have $\lim_{\delta \to 0} \dim_H(K_\lambda \cap (y - \delta, y + \delta)) = \dim_H K_\lambda$. This means that $K_\lambda$ is distributed uniformly in the dimensional sense. Our next result states that the local dimension of $\Lambda(x)$ behaves differently.

**Theorem 1.2.** Let $x \in (0, 1)$. Then for any $\lambda \in \Lambda(x)$ we have

$$\lim_{\delta \to 0} \dim_H(\Lambda(x) \cap (\lambda - \delta, \lambda + \delta)) = \dim_H K_\lambda = \frac{\log m}{-\log \lambda}.$$ 

As a direct consequence of theorem 1.2 we obtain that $\Lambda(x)$ has zero Lebesgue measure and full Hausdorff dimension.

**Corollary 1.3.** For any $x \in (0, 1)$ the set $\Lambda(x)$ is a Lebesgue null set, and $\dim_H \Lambda(x) = 1$.

**Remark 1.4.** By corollary 1.3 it follows that $\bigcup_{x \in (0,1) \cap \mathbb{Q}} \Lambda(x)$ has zero Lebesgue measure. This implies that

$$(0, 1/m] \setminus \bigcup_{x \in (0,1) \cap \mathbb{Q}} \Lambda(x) = \{\lambda \in (0, 1/m]: K_\lambda \subset ((0,1) \cap \mathbb{Q})^c\}$$

has full Lebesgue measure $1/m$. In other words, for Lebesgue almost every $\lambda \in (0, 1/m]$ the set $K_\lambda \setminus \{0\}$ contains only irrational numbers.

Another consequence of theorem 1.2 gives the following result for the segment of $\Lambda(x)$.

**Corollary 1.5.** Let $x \in (0, 1)$. Then for any open interval $(a, b)$ with $\Lambda(x) \cap (a, b) \neq \emptyset$ we have

$$\dim_H(\Lambda(x) \cap (a,b)) = \sup_{\lambda \in \Lambda(x) \cap (a,b)} \dim_H K_\lambda.$$ 

Note by theorem 1.1 that $\Lambda(x)$ is a topological Cantor set for any $x \in (0, 1)$. So by corollary 1.5 it follows that the function $h_5 : \lambda \mapsto \dim_H(\Lambda(x) \cap (0, \lambda))$ is a non-decreasing Devil’s staircase for any $x \in (0, 1)$.

Our last result shows that for any two points $x, y \in (0, 1)$ there are infinitely many $\lambda \in (0, 1/m]$ such that $K_\lambda$ contains both points $x$ and $y$. In other words, we show that $\Lambda(x) \cap \Lambda(y)$ contains infinitely many points.
Theorem 1.6. For any \(x, y \in (0, 1)\) we have \(\dim_H(\Lambda(x) \cap \Lambda(y)) = 1\).

The paper is organised in the following way. In section 2 we show that \(\Lambda(x)\) is a topological Cantor set, and prove theorem 1.1. In section 3 we study the local dimension of \(\Lambda(x)\), and prove theorem 1.2, corollaries 1.3 and 1.5. In section 4 we construct a sequence of Cantor subsets \((E_k(x))\) of \(\Lambda(x)\) such that the thickness of \(E_k(x)\) goes to infinity as \(k \to \infty\). Based on this we show that for any \(x, y \in (0, 1)\) the intersection \(\Lambda(x) \cap \Lambda(y)\) has full Hausdorff dimension, and prove theorem 1.6.

2. Topological structure of \(\Lambda(x)\)

Note that for \(\lambda \in (0, 1/m)\) the IFS \(\Psi_\lambda\) satisfies the strong separation condition. Then there exists a natural bijective map from the symbolic space \(\{0, 1, \ldots, m - 1\}^\mathbb{N}\) to the self-similar set \(K_\lambda\) defined by

\[
\pi_\lambda : \{0, 1, \ldots, m - 1\}^\mathbb{N} \to K_\lambda; \quad (d_i) \mapsto \sum_{i=1}^{\infty} d_i \lambda^i. \tag{2.1}
\]

The infinite sequence \((d_i)\) is called a (unique) coding of \(\pi_\lambda((d_i))\) with respect to the IFS \(\Psi_\lambda\). Now we fix \(x \in (0, 1)\), and define a map from \(\Lambda(x) \setminus \{1/m\}\) to \(\{0, 1, \ldots, m - 1\}^\mathbb{N}\) by

\[
\Phi_x : \Lambda(x) \setminus \{1/m\} \to \{0, 1, \ldots, m - 1\}^\mathbb{N}; \quad \lambda \mapsto \pi_\lambda^{-1}(x).
\]

So \(\Phi_x(\lambda)\) is the unique coding of \(x \in K_\lambda\) with respect to the IFS \(\Psi_\lambda\). For convenience, when \(\lambda = 1/m\) we define \(\Phi_x(1/m)\) as the greedy \(m\)-adic expansion of \(x\), i.e., \(\Phi_x(1/m)\) is the lexicographically largest sequence \((d_i) \in \{0, 1, \ldots, m - 1\}^\mathbb{N}\) such that \(x = \sum_{i=1}^{\infty} d_i / m^i\).

Now we recall some terminology from symbolic dynamics (cf [12]). In this paper we fix our alphabet \(\{0, 1, \ldots, m - 1\}\), and let \(\{0, 1, \ldots, m - 1\}^\mathbb{N}\) denote the set of all infinite sequences \((d_i)\) with each element \(d_i \in \{0, 1, \ldots, m - 1\}\). For a word \(w\) we mean a finite string of digits, i.e., \(w = w_1w_2 \ldots w_n\) for some \(n \geq 0\). Here \(n\) is called the length of \(w\). In particular, when \(n = 0\) we call \(w = \epsilon\) the empty word. Let \(\{0, 1, \ldots, m - 1\}^n\) denote the set of all finite words. Then \(\{0, 1, \ldots, m - 1\}^n = \bigcup_{n=0}^{\infty} \{0, 1, \ldots, m - 1\}^n\), where \(\{0, 1, \ldots, m - 1\}^n\) denotes the set of all words of length \(n\). For two words \(u = u_1 \ldots u_n, v = v_1 \ldots v_n\) we write \(uv = u_1 \ldots u_nv_1 \ldots v_n \in \{0, 1, \ldots, m - 1\}^n\). Similarly, for a sequence \(a = a_1a_2 \ldots \in \{0, 1, \ldots, m - 1\}^\mathbb{N}\) we write \(ua = a_1 \ldots ua(a_2) \ldots\). For \(j \in \mathbb{N}\) we denote by \(u^j = u \ldots u\) the \(j\)-fold concatenation of \(u\) with itself, and by \(u^\infty = uu\ldots\) the period sequence with period block \(u\). For a word \(u = u_1 \ldots u_n\), if \(u_k < m - 1\), then we write \(u^\infty = u_1 \ldots u_n (u_k + 1)\). So \(u^\infty\) is also a word. Throughout the paper we will use lexicographical order \(\prec, \preceq, \succ, \succeq\) and \(\asymp\) between sequences in \(\{0, 1, \ldots, m - 1\}^\mathbb{N}\). For example, for two sequences \((a_i), (d_i)\) we say \((a_i) \prec (d_i)\) if there exists \(n \in \mathbb{N}\) such that \(a_1 \ldots a_{n-1} = b_1 \ldots b_{n-1}\) and \(a_n < b_n\). Also, we write \((a_i) \preceq (d_i)\) if \((a_i) < (b_i)\) or \((a_i) = (b_i)\).

Lemma 2.1. Let \(x \in (0, 1)\). Then \(\Phi_x\) is strictly decreasing in \(\Lambda(x)\) with respect to the lexicographical order in \(\{0, 1, \ldots, m - 1\}^\mathbb{N}\).

Proof. Let \(\lambda_1, \lambda_2 \in \Lambda(x)\) with \(\lambda_1 < \lambda_2\). Write \((a_i) = \Phi_x(\lambda_1)\) and \((b_i) = \Phi_x(\lambda_2)\). Suppose on the contrary that \((a_i) \preceq (b_i)\). Then

\[
x = \sum_{i=1}^{\infty} a_i \lambda_1^i \leq \sum_{i=1}^{\infty} b_i \lambda_1^i < \sum_{i=1}^{\infty} b_i \lambda_2^i = x,
\]
leading to a contradiction.

As a direct consequence of lemma 2.1 we can determine the extreme values of $\Lambda(x)$.

**Lemma 2.2.** For any $x \in (0, 1)$ we have $\min \Lambda(x) = m^{-1}$ and $\max \Lambda(x) = 1/m$.

**Proof.** Note that $K_{1/m} = [0, 1]$. So by the definition of $\Lambda(x)$ it is clear that $\max \Lambda(x) = 1/m$.

On the other hand, by lemma 2.1 it follows that the smallest element $\lambda = \min \Lambda(x)$ satisfies $\Phi_\lambda(\lambda) = (m - 1)^\infty$. This gives that $x = (m - 1)/m$, and hence, $\min \Lambda(x) = \lambda = m^{-1}$.

Recall that $K_\lambda$ is the self-similar set generated by the IFS $\Psi_\lambda = \{f_i(x) = \lambda(x + i)\}_{i=0}^{m-1}$. Geometrically, $K_\lambda$ can be constructed via a decreasing sequence of nonempty compact sets. More precisely,

$$K_\lambda = \bigcap_{n=1}^\infty K_\lambda(n) \quad \text{with} \quad K_\lambda(n) = \bigcup_{d_1, \ldots, d_n \in \{0, 1, \ldots, m-1\}^n} f_{d_1} \circ \ldots \circ f_{d_n} \left(0, \frac{(m-1)\lambda}{1 - \lambda}\right).$$

When $\lambda \in (0, 1/m)$, for each $n \in \mathbb{N}$ the set $K_\lambda(n)$ is the union of $m^n$ pairwise disjoint subintervals of equal length $\lambda^{m(m-1)}$. Furthermore, $K_\lambda(n+1) \subset K_\lambda(n)$ for any $n \geq 1$.

Denote by $d_H$ the Hausdorff metric in the space $\mathcal{C}(\mathbb{R})$ consisting of all non-empty compact sets in $\mathbb{R}$ (cf [6]). In the following lemma we show that the set sequence $\{K_\lambda(n)\}_{n=1}^\infty$ converges to $K_\lambda$ uniformly.

**Lemma 2.3.** $d_H(K_\lambda(n), K_\lambda) \to 0$ uniformly as $n \to \infty$ for all $\lambda \in (0, 1/m]$.

**Proof.** Note that $K_\lambda \subset K_\lambda(n)$ for any $\lambda \in (0, 1/m]$ and $n \in \mathbb{N}$, and $K_\lambda(n)$ is the union of $m^n$ pairwise disjoint subintervals of equal length $\lambda^{m(m-1)}$. This implies that

$$d_H(K_\lambda(n), K_\lambda) \leq \lambda^{m(m-1)} \leq \frac{1}{m^n} \to 0 \quad \text{as} \quad n \to \infty$$

for any $\lambda \in (0, 1/m]$.

**Lemma 2.4.** For any $x \in (0, 1)$ the set $\Lambda(x)$ is closed.

**Proof.** Suppose $(\lambda_j)$ is a sequence in $\Lambda(x)$ with $\lim_{j \to \infty} \lambda_j = \lambda_0$. We will show that $\lambda_0 \in \Lambda(x)$, i.e., $x \in K_{\lambda_0}$. By lemma 2.3 it follows that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $\lambda \in (0, 1/m]$ and any $n > N$ we have

$$d_H(K_\lambda(n), K_{\lambda_0}) < \frac{\varepsilon}{3} \quad (2.2)$$

Take $n > N$. Since $\lambda_j \to \lambda_0$ as $j \to \infty$, there exists $J \in \mathbb{N}$ such that for any $j > J$ we have

$$d_H(K_{\lambda_j}(n), K_{\lambda_0}(n)) < \frac{\varepsilon}{3} \quad (2.3)$$

Then by (2.2) and (2.3) it follows that for any $j > J$ and $n > N$,

$$d_H(K_{\lambda_j}(n), K_{\lambda_0}(n)) < d_H(K_{\lambda_j}(n), K_{\lambda_j}(n)) + d_H(K_{\lambda_j}(n), K_{\lambda_0}(n)) + d_H(K_{\lambda_0}(n), K_{\lambda_0}(n)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies that

$$d_H(K_{\lambda_j}(n), K_{\lambda_0}) \to 0 \quad \text{as} \quad j \to \infty.$$
Now suppose on the contrary that \( x \notin K_{\lambda_0} \). Note that \( K_{\lambda_0} \) is compact. Then \( \text{dist}(x, K_{\lambda_0}) = \inf \{ |x - y| : y \in K_{\lambda_0} \} > 0 \), and thus by (2.4) it follows that \( \text{dist}(x, K_{\lambda_j}) > 0 \) for all \( j \in \mathbb{N} \) sufficiently large. This leads to a contradiction with \( x \in K_{\lambda_j} \) for all \( j \geq 1 \). Hence, \( x \in K_{\lambda_0} \), and thus \( \Lambda(x) \) is closed. \( \square \)

To prove theorem 1.1 we still need the following lemma.

**Lemma 2.5.** Let \( x \in (0, 1) \) and \( q \in (\frac{1}{m+1}, 1/m) \). Then there exists a constant \( C > 0 \) such that for any two \( \lambda_1, \lambda_2 \in \Lambda(x) \cap (0, q] \) we have

\[
|\pi_q(\Phi_x(\lambda_1)) - \pi_q(\Phi_x(\lambda_2))| > C|\lambda_1 - \lambda_2|.
\]

**Proof.** Note by lemma 2.2 that \( \min \Lambda(x) = \frac{1}{m+1} \). So \( \Lambda(x) \cap (0, q] \neq \emptyset \) for any \( q > \frac{1}{m+1} \). Take \( \lambda_1, \lambda_2 \in \Lambda(x) \cap (0, q] \) with \( \lambda_1 < \lambda_2 \). By lemma 2.1 we have

\[
(a_i) := \Phi_x(\lambda_1) > \Phi_x(\lambda_2) =: (b_i).
\]

Then there exists \( n \in \mathbb{N} \) such that \( a_1 \ldots a_{n-1} = b_1 \ldots b_{n-1} \) and \( a_n > b_n \). By using \( \lambda_1, \lambda_2 \in \Lambda(x) \) we obtain that

\[
\sum_{i=1}^{n-1} b_i\lambda_2^{-i} \leq \sum_{i=1}^{n} b_i\lambda_2^{-i} = x = \sum_{i=1}^{n-1} a_i\lambda_1^{-i} + \sum_{i=n}^{\infty} (m-1)\lambda_1^{-i},
\]

where for \( n = 1 \) we set the summation over an empty index set as zero. This implies that

\[
\frac{x}{\lambda_1\lambda_2}(\lambda_2 - \lambda_1) = \frac{x}{\lambda_1} - \frac{x}{\lambda_2} \leq \sum_{i=1}^{n-1} a_i\lambda_1^{-i} - \sum_{i=1}^{n-1} b_i\lambda_2^{-i} + \sum_{i=n}^{\infty} (m-1)\lambda_1^{-i-1}
\]

\[
= \sum_{i=1}^{n-1} a_i\lambda_1^{-i} - \sum_{i=1}^{n-1} a_i\lambda_2^{-i} + \frac{m-1}{1-\lambda_1}\lambda_1^{-n-1}
\]

\[
\leq \frac{m-1}{1-\lambda_1}\lambda_1^{-n-1}.
\] (2.5)

Observe that \((a_i)\) and \((b_i)\) are distinct at the first place \( n \). Then

\[
|\pi_q((a_i)) - \pi_q((b_i))| \geq q^n - \frac{m-1}{1-q}q^{n+1} = \frac{1-mq}{1-q}q^n.
\] (2.6)

Hence, by (2.5) and (2.6) we conclude that

\[
|\pi_q(\Phi_x(\lambda_1)) - \pi_q(\Phi_x(\lambda_2))| = |\pi_q((a_i)) - \pi_q((b_i))| \\
\geq \frac{1-mq}{1-q}q^n \geq \frac{1-mq}{1-q}\lambda_1^n \\
\geq \frac{1-mq}{1-q} \times \frac{(1-\lambda_1)x}{(m-1)\lambda_2}|\lambda_1 - \lambda_2| \\
\geq \frac{(1-mq)x}{(m-1)q}|\lambda_1 - \lambda_2|
\]

as desired. \( \square \)
Proof of theorem 1.1. By lemmas 2.2 and 2.4 we only need to prove that \( \Lambda(x) \) has neither isolated nor interior points. First we prove that \( \Lambda(x) \) has no isolated points. Let \( \lambda \in \Lambda(x) \). We split the proof into the following two cases.

Case I. \( \lambda \in (0, 1/m) \). Then \( \Phi_s(\lambda) = (d_i) \) is the unique coding of \( x \) with respect to the IFS \( \Psi_s \). For \( n \geq 1 \) let \( p_n \) be the unique root in \((0, 1)\) of the equation

\[
x = \sum_{i=1}^{n-1} d_i p_n^i + d_n p_n^n,
\]

where \( d_i = d_i + 1 \) (mod \( m \)). Then for all sufficiently large \( n \in \mathbb{N} \) we have \( p_n \in (0, 1/m) \) and \( \Phi_s(p_n) = d_1 \ldots d_{n-1}0^\infty \). So \( p_n \in \Lambda(x) \) for all large \( n \in \mathbb{N} \). Furthermore, by lemma 2.5 it follows that \( p_n \to \lambda \) as \( n \to \infty \). Thus, \( \lambda \) is not isolated in \( \Lambda(x) \).

Case II. \( \lambda = 1/m \). Then \( \Phi_s(1/m) = (d_i) \) is the greedy \( m \)-adic expansion of \( x \). Since \( x \in (0, 1) \), there exist infinitely many \( n \in \mathbb{N} \) such that \( d_n < m - 1 \). For all such \( n \) we define \( p_n \in (0, 1) \) so that \( x = \sum_{i=1}^{n-1} d_i p_n^i + (m - 1)p_n^n \). Then \( p_n \in (0, 1/m) \) and \( \Phi_s(p_n) = d_1 \ldots d_{n-1}(m - 1)0^\infty \). So \( p_n \in \Lambda(x) \) for all \( n \in \mathbb{N} \) satisfying \( d_n < m - 1 \). Again, by lemma 2.5 it follows that \( p_n \) converges to \( 1/m \) along a suitable subsequence, and thus \( 1/m \) is not isolated in \( \Lambda(x) \).

By cases I and II we conclude that \( \Lambda(x) \) has no isolated points. Next we prove that \( \Lambda(x) \) has no interior points. It suffices to prove that for any two points \( \lambda_1, \lambda_2 \in \Lambda(x) \setminus \{1/m\} \) there must exist \( \lambda \) between \( \lambda_1 \) and \( \lambda_2 \) but not in \( \Lambda(x) \).

Take \( \lambda_1, \lambda_2 \in \Lambda(x) \setminus \{1/m\} \) with \( \lambda_1 < \lambda_2 \). Then by lemma 2.1 we have

\[
(a_i) = \Phi_s(\lambda_1) \succ \Phi_s(\lambda_2) = (b_i).
\]

So there exists \( n \in \mathbb{N} \) such that \( a_1 \ldots a_{n-1} = b_1 \ldots b_{n-1} \) and \( a_n > b_n \). We define two sequences

\[
\xi = b_1 \ldots b_n(m - 1)\infty, \quad \zeta = b_1 \ldots b_{n-1}(b_n + 1)0^\infty.
\]

Then \( (b_i) \prec \xi \prec \zeta \prec (a_i) \). So by using \( 0 < \lambda_1 < \lambda_2 < 1/m \) it follows that

\[
\pi_{\lambda_1}(\xi) < \pi_{\lambda_1}(\zeta) \leq \pi_{\lambda_1}(a_i) = x = \pi_{\lambda_2}(b_i) \leq \pi_{\lambda_2}(\xi) < \pi_{\lambda_2}(\zeta).
\]

(2.7)

Let \( I_\lambda := (\pi_{\lambda}(\xi), \pi_{\lambda}(\zeta)) \). Then (2.7) implies that \( x \) is between two disjoint open intervals \( I_{\lambda_1} \) and \( I_{\lambda_2} \). Observe that the map \( \lambda \mapsto [\pi_{\lambda}(\xi), \pi_{\lambda}(\zeta)] \) is continuous with respect to the Hausdorff metric \( d_H \). So there must exist \( \lambda \in (\lambda_1, \lambda_2) \) such that \( x \in I_\lambda \). Since \( I_\lambda \cap K_\lambda = \emptyset \), it follows that \( x \notin K_\lambda \), i.e., \( \lambda \in (\lambda_1, \lambda_2) \setminus \Lambda(x) \). This completes the proof.

At the end of this section we describe the geometrical construction of \( \Lambda(x) \). By theorem 1.1 it follows that \( \Lambda(x) \) is a Cantor set in \( \mathbb{R} \) for any \( x \in (0, 1) \). So, it can be obtained by successively removing a sequence of open intervals from \( \text{conv}(\Lambda(x)) = [\frac{x}{m-1+x}, \frac{1}{m}] \). Let \( (x_i) = \Phi_s(1/m) \) be the greedy \( m \)-adic expansion of \( x \). Since \( x \in (0, 1) \), there exists a smallest \( \ell \in \mathbb{N} \) such that \( x_\ell < m - 1 \). Note that \( \Phi_s(\frac{x}{m-1+x}) = (m - 1)^\infty \). We call a word \( w \in \{0, 1, \ldots, m - 1\}^* \) admissible in \( \Phi_s(\Lambda(x)) \) if

\[
(x_i) \prec w0^\infty \prec w(m - 1)^\infty \prec (m - 1)^\infty.
\]

Since \( x_i = m - 1 \) for all \( i < \ell \), it follows that any admissible word \( w \) has length at least \( \ell \). For each admissible word \( w \) we define the associated basic interval \( \Lambda_w = [p_w, q_w] \) by

\[
\Phi_s(p_w) = w(m - 1)^\infty \quad \text{and} \quad \Phi_s(q_w) = w0^\infty.
\]
Then for two admissible words $u, w$, if $u$ is a prefix of $w$, then $J_w \subset J_u$. Furthermore, for any admissible word $w$ there must exist an admissible word $v$ such that $w$ is a prefix of $v$, i.e., $w$ has an offspring $v$. Denote by $\mathcal{A}(x) = \bigcup_{n=1}^{\infty} A_n(x)$ the set of all admissible words, where $A_n(x)$ consists of all admissible words of length $n$. So the basic intervals $\{J_w : w \in \mathcal{A}(x)\}$ have a tree structure, and the Cantor set $\Lambda(x)$ can be constructed geometrically as

$$\Lambda(x) = \bigcap_{n=1}^{\infty} \bigcup_{w \in A_n(x)} J_w.$$  

**Example 2.6.** Let $m = 2$ and $x = 1/2$. Then $\text{conv}(\Lambda(x)) = [1/3, 1/2]$. Furthermore, $\Phi_x(1/2) = 1^\infty$ and $\Phi_x(1) = 10^\infty$. Then any admissible word has length at least $\ell = 2$. By the definition of admissible words it follows that

$$A_2(x) = \{10, 11\}, \quad A_3(x) = \{100, 101, 110, 111\}, \ldots,$$

and in general, for any $n \in \mathbb{N}$ we have

$$A_{n+1}(x) = \{1u : u \in \{0, 1\}^n\}.$$  

So, in the first step we remove the open interval $H_1 = (0.366025, 0.396608) \sim (110^\infty, 101^\infty)$ from the convex hull $[1/3, 1/2]$; and in the next step we remove two open intervals

$$H_2 = (0.342508, 0.352201) \sim (1101^\infty, 1100^\infty),$$

$$H_3 = (0.423854, 0.435958) \sim (1010^\infty, 1001^\infty).$$

This procedure can be continued, and after finitely many steps we can get a good approximation of $\Lambda(x)$ (see figure 1).

### 3. Fractal properties of $\Lambda(x)$: local dimension

In this section we will investigate the local dimension of $\Lambda(x)$, and prove theorem 1.2. Our proof will be split into the following two cases: (I) local dimension of $\Lambda(x)$ at $\lambda = 1/m$; (II) local dimension of $\Lambda(x)$ at $\lambda \in (0, 1/m)$.

#### 3.1. Local dimension of $\Lambda(x)$ at $\lambda = 1/m$

Observe that $1/m \in \Lambda(x)$ for any $x \in (0, 1)$. In this part we will show that the local dimension of $\Lambda(x)$ at $1/m$ is one.

**Proposition 3.1.** For any $x \in (0, 1)$ we have

$$\lim_{\delta \to 0} \dim_H \left( \Lambda(x) \cap \left( \frac{1}{m} - \delta, \frac{1}{m} + \delta \right) \right) = 1 = \dim_H K_{1/m}.$$  

Our strategy to prove proposition 3.1 is to construct a large subset of $\Lambda(x) \cap (1/m - \delta, 1/m + \delta)$ with its Hausdorff dimension arbitrarily close to one. Let $x \in (0, 1)$. For $k \in \mathbb{N}$ we set

$$\Lambda_k(x) := \{ \lambda \in \Lambda(x) : \exists N \text{ such that } \sigma^N(\Phi_x(\lambda)) \text{ does not contain } k \text{ consecutive zeros} \}.$$
where $\sigma$ is the left-shift map on $\{0, 1, \ldots, m - 1\}^\mathbb{N}$. Note that for any $\lambda \in \Lambda(x) \setminus \bigcup_{k=1}^\infty \Lambda_k(x)$ the coding $\Phi_\lambda(x)$ must end with $0^\infty$. Thus the difference between $\Lambda(x)$ and $\bigcup_{k=1}^\infty \Lambda_k(x)$ is at most countable. So, by the countable stability of Hausdorff dimension it follows that

$$\dim_H \Lambda(x) = \dim_H \bigcup_{k=1}^\infty \Lambda_k(x) = \sup_{k \geq 1} \dim_H \Lambda_k(x).$$

In the following we will give a lower bound for the Hausdorff dimension of $\Lambda_k(x)$. Let $(x_i) = \Phi_\lambda(1/m)$ be the greedy $m$-adic expansion of $x$. Since $x \in (0,1)$, there exist infinitely many digits $x_i < m - 1$. So there exists a subsequence $(n_i) \subset \mathbb{N}$ such that $x_{n_i} < m - 1$ for all $i \geq 1$. For a large integer $j$, let $\gamma_j$ be the unique root in $(0,1/m)$ of the equation

$$x = \sum_{i=1}^{n_j-1} x_i \gamma_{j,i} + (x_{n_j} + 1) \gamma_{j,i}^\infty + \sum_{i=n_j+1}^\infty (m-1) \gamma_{j,i}.$$

Then

$$\Phi_\lambda(\gamma_j) = x_1 \ldots x_{n_j}^\infty (m-1)^\infty, \quad \text{and} \quad \gamma_j \nearrow 1/m \quad \text{as} \quad j \to \infty. \quad (3.1)$$

Define

$$\Gamma_{k,j}(x) := \left\{ x_1 \ldots x_{n_j}^+ d_1 d_2 \ldots \in \{0, 1, \ldots, m - 1\}^\mathbb{N} : (d_i) \text{ does not contain } k \text{ consecutive zeros} \right\}. \quad (3.2)$$

**Lemma 3.2.** Let $x \in (0, 1)$ and $k \in \mathbb{N}$. Then for a large $j \in \mathbb{N}$ we have

$$\Gamma_{k,j}(x) \subseteq \Phi_\lambda(\Lambda_k(x) \cap [\gamma_j, 1/m)).$$

**Proof.** Take a sequence $(c_i) \in \Gamma_{k,j}(x)$. Then the equation $x = \sum_{i=1}^\infty c_i \lambda^i$ determines a unique $\lambda \in (0,1/m)$, i.e., $\Phi_\lambda(x) = (c_i)$. Note that

$$\Phi_\lambda(1/m) = (x_i) \prec \Phi_\lambda(\lambda) = (c_i) \prec x_1 \ldots x_{n_j}^\infty (m-1)^\infty = \Phi_\lambda(\gamma_j).$$

By lemma 2.1 we conclude that $\lambda \in [\gamma_j, 1/m)$. Furthermore, by the definition of $\Gamma_{k,j}(x)$ it follows that $\sigma^0((c_i))$ does not contain $k$ consecutive zeros. So, $\lambda \in \Lambda_k(x)$. This completes the proof. \qed

To give a lower bound of $\dim_H(\Lambda_k(x) \cap [\gamma_j, 1/m))$ we still need the following lemma.

**Lemma 3.3.** Let $x \in (0, 1)$ and $k \in \mathbb{N}$. Then for a large $j \in \mathbb{N}$ there exists $C > 0$ such that for any $\lambda_1, \lambda_2 \in \Phi_\lambda^{-1}(\Gamma_{k,j}(x))$ we have

$$|\pi_{\gamma_j}(\Phi_\lambda(\lambda_1)) - \pi_{\gamma_j}(\Phi_\lambda(\lambda_2))| \leq C |\lambda_1 - \lambda_2|.$$

**Proof.** Let $\lambda_1, \lambda_2 \in \Phi_\lambda^{-1}(\Gamma_{k,j}(x))$ with $\lambda_1 < \lambda_2$. Then by lemma 3.2 we have $\lambda_1, \lambda_2 \in \Lambda_k(x) \cap [\gamma_j, 1/m)$. Furthermore, $(a_i) = \Phi_\lambda(\lambda_1)$ and $(b_i) = \Phi_\lambda(\lambda_2)$ have a common prefix of length at least $n_j$. Since $\lambda_1 < \lambda_2$, by lemma 2.1 we have $(a_i) \succ (b_i)$. So there exists $n > n_j$ such
that \( a_1 \ldots a_n = b_1 \ldots b_{n-1} \) and \( a_n > b_n \). Note that \( \sigma^\mu((a_i)) \) does not contain \( k \) consecutive zeros. Then
\[
\lambda_{n+k}^{n+k} + \sum_{i=1}^{n} a_i \lambda_i' < \sum_{i=1}^{\infty} a_i \lambda_i' = x = \sum_{i=1}^{\infty} b_i \lambda_i' < \sum_{i=1}^{n} a_i \lambda_i'.
\]
Rearranging the above equation gives that
\[
\lambda_{n+k}^{n+k} < \sum_{i=1}^{n} a_i (\lambda_i' - \lambda_i) < \sum_{i=1}^{\infty} (m-1)(\lambda_i' - \lambda_i) = \frac{m-1}{(1-\lambda_1)(1-\lambda_2)}(\lambda_2 - \lambda_1).
\]
Since \( \lambda_1, \lambda_2 \in [\gamma_j, 1/m) \), this implies that
\[
\lambda_n < \frac{m-1}{\lambda_1'(1-\lambda_1)(1-\lambda_2)}(\lambda_2 - \lambda_1) < \frac{m^2}{\gamma_j'(m-1)}(\lambda_2 - \lambda_1). \tag{3.3}
\]
Note that \((a_i)\) and \((b_i)\) have a common prefix of length \( n-1 \). It follows that
\[
|\pi_{\gamma_j}(\lambda_1) - \pi_{\gamma_j}(\lambda_2)| \leq \gamma_j^{n-1} (m-1)\gamma_j = \frac{m-1}{1-\gamma_j} \gamma_j^n.
\]
Therefore, by (3.3) it follows that
\[
|\pi_{\gamma_j}(\Phi(x) \lambda_1) - \pi_{\gamma_j}(\Phi(x) \lambda_2)| = |\pi_{\gamma_j}(\lambda_1) - \pi_{\gamma_j}(\lambda_2)|
\leq \frac{m-1}{1-\gamma_j} \gamma_j^n \leq \frac{m-1}{1-\lambda_1} \lambda_1^n
\leq \frac{m-1}{1-\gamma_j} \times \frac{m^2}{\gamma_j'(m-1)}|\lambda_1 - \lambda_2|
= \frac{m^2}{(1-\gamma_j)\gamma_j'(m-1)}|\lambda_1 - \lambda_2|
\]
as desired. \( \square \)

**Lemma 3.4.** Let \( x \in (0, 1) \) and \( k \in \mathbb{N} \). Then for a large \( j \in \mathbb{N} \) we have
\[
\dim_H(\Lambda_k(x) \cap [\gamma_j, 1/m)) \geq \frac{(k-1) \log m + \log(m-1)}{-k \log \gamma_j}.
\]

**Proof.** By lemmas 3.2 and 3.3 it follows that
\[
\dim_H(\Lambda_k(x) \cap [\gamma_j, 1/m)) \geq \dim_H \Phi^{-1}(\Gamma_k(x))\]
\[
\geq \dim_H \pi_{\gamma_j}(\Gamma_{k,j}(x)) = \dim_H \pi_{\gamma_j}(\sigma^\mu(\Gamma_{k,j}(x))),
\]
where the last equality follows by the stability of Hausdorff dimension under similarity transformations. Observe that any sequence \((d_i) \in ([0, 1] \times \{1, \ldots, m-1\})^{k-1} \times \{1, \ldots, m-1\})^\mathbb{N}\) does not contain \( k \) consecutive zeros, and so \((d_i) \in \sigma^\mu(\Gamma_{k,j}(x))\) by (3.2). Therefore,
\[
\dim_H(\Lambda_k(x) \cap [\gamma_j, 1/m)) \geq \dim_H \pi_{\gamma_j}(([0, 1] \times \{1, \ldots, m-1\})^{k-1} \times \{1, \ldots, m-1\})^\mathbb{N})
\leq \frac{(k-1) \log m + \log(m-1)}{-k \log \gamma_j},
\]
completing the proof. \( \square \)
Proof of proposition 3.1. Note that $\bigcup_{k=1}^{\infty} \Lambda_k(x) \subseteq \Lambda(x)$ and the difference set $\Lambda(x) \setminus \bigcup_{k=1}^{\infty} \Lambda_k(x)$ is at most countable. Furthermore, by the definition of $\Lambda_k(x)$ we know that $\Lambda_k(x) \subset \Lambda_{k+1}(x)$ for any $k_1 < k_2$. So by lemma 3.4 it follows that for any large $j \in \mathbb{N}$,

$$\dim_H(\Lambda(x) \cap [\gamma_j, 1/m)) = \dim_H(\bigcup_{k=1}^{\infty} (\Lambda_k(x) \cap [\gamma_j, 1/m)))$$

$$= \lim_{k \to \infty} \dim_H(\Lambda_k(x) \cap [\gamma_j, 1/m))$$

$$\geq \lim_{k \to \infty} \frac{(k-1) \log m + \log (m-1)}{-k \log \gamma_j} = \frac{\log m}{\log \gamma_j}$$

Note by (3.1) that $\gamma_j \to 1/m$ as $j \to \infty$. This implies that

$$\lim_{j \to 0} \frac{\log m}{\log \gamma_j} = 1,$$

completing the proof. \qed

3.2. Local dimension of $\Lambda(x)$ at $\lambda \in \Lambda(x) \setminus \{1/m\}$

In this part we will prove theorem 1.2 for $\lambda \in \Lambda(x) \setminus \{1/m\}$. Fix $x \in (0,1)$ and $\lambda \in \Lambda(x) \setminus \{1/m\}$. Let $(x_i) = \Phi_\lambda(x) \in \{0,1,\ldots,m-1\}^\mathbb{N}$ be the unique coding of $x \in K_\lambda$ with respect to the IFS $\Psi_\lambda$. Let $n \in \mathbb{N}$ large enough such that $x_1 \ldots x_n \neq 0^n$, and then let $\beta_n$ and $\gamma_n$ be the unique roots in $(0,1)$ of the following equations respectively:

$$x = \sum_{i=1}^{n} x_i \beta_n^i + \sum_{i=n+1}^{\infty} (m-1) \beta_n^i \quad \text{and} \quad x = \sum_{i=1}^{n} x_i \gamma_n^i.$$

In fact, by choosing $n \in \mathbb{N}$ sufficiently large we can even require that

$$0 < \beta_n \leq \lambda \leq \gamma_n < 1/m. \quad (3.4)$$

Then $\Phi_\lambda(\beta_n) = x_1 \ldots x_n (m-1)^\infty$ and $\Phi_\lambda(\gamma_n) = x_1 \ldots x_n 0^\infty$. Furthermore, $\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \gamma_n = \lambda$.

Recall from the previous subsection that $\Lambda_k(x)$ consists of all $q \in \Lambda(x)$ such that the tail sequence of $\Phi_\lambda(q)$ does not contain $k$ consecutive zeros. Now for a large $n \in \mathbb{N}$, we also define

$$\Gamma_{\lambda, n}(x) := \left\{ x_1 \ldots x_n d_1 d_2 \ldots \in \{0,1,\ldots,m-1\}^\mathbb{N} : (d_2) \text{ does not contain } k \text{ consecutive zeros} \right\}.$$

Lemma 3.5. Let $x \in (0,1)$, $\lambda \in \Lambda(x) \setminus \{1/m\}$ and $k \in \mathbb{N}$. Then for any $\delta \in (0, \min \{\lambda, 1/m - \lambda\})$ there exists a large $N \in \mathbb{N}$ such that

$$\Gamma_{\lambda, n}(x) \subseteq \Phi_\lambda(\Lambda_k(x) \cap (\lambda - \delta, \lambda + \delta))$$

for any $n \geq N$.

Proof. By (3.4) there exists $N_1 \in \mathbb{N}$ such that for any $n > N_1$ we have $0 < \beta_n \leq \lambda \leq \gamma_n < 1/m$. Take $(c) \in \Gamma_{\lambda, n}(x)$. Then the equation $x = \sum_{i=1}^{\infty} c_i q^i$ determines a unique $q \in (0,1/m)$, i.e., $\Phi_\lambda(q) = (c)$. Observe that

$$\Phi_\lambda(\gamma_n) = x_1 \ldots x_n 0^\infty \leq (c) = \Phi_\lambda(q) \leq x_1 \ldots x_n (m-1)^\infty = \Phi_\lambda(\beta_n).$$
By lemma 2.1 we have $\beta_n \leq q \leq \gamma_n$. Note that $\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \gamma_n = \lambda$. Then for $\delta \in (0, \min \{\lambda, 1/m - \lambda\})$ there exists an integer $N > N_1$ such that for any $n > N$ we have
\[ q \in [\beta_n, \gamma_n] \subset (\lambda - \delta, \lambda + \delta) \subset (0, 1/m). \]

Clearly, by the definition of $\Gamma^1_{x, \delta}(x)$ the tail sequence $\sigma^n((c_i))$ does not contain $k$ consecutive zeros. This proves $q \in \Lambda_\delta(x)$, and hence completes the proof.

Now by lemma 3.5 and the same argument as in the proof of lemma 3.4 we obtain the following lower bound for the dimension of $\Lambda_\delta(x) \cap (\lambda - \delta, \lambda + \delta)$.

**Lemma 3.6.** Let $x \in (0, 1), \lambda \in \Lambda(x) \backslash \{1/m\}$ and $k \in \mathbb{N}$. Then for any $\delta \in (0, \min \{\lambda, 1/m - \lambda\})$ we have
\[
\dim_H(\Lambda_\delta(x) \cap (\lambda - \delta, \lambda + \delta)) \geq \dim_H \pi_{\lambda, -\delta}(\Gamma^1_{x, \delta}(x)) \geq \frac{(k - 1) \log m + \log(m - 1)}{-k \log(\lambda - \delta)}.
\]

**Proof of theorem 1.2.** By proposition 3.1 it suffices to prove the theorem for $\lambda \in \Lambda(x) \backslash \{1/m\}$. By lemma 3.6 it follows that for any $\delta \in (0, \min \{\lambda, 1/m - \lambda\})$ we have
\[
\dim_H(\Lambda(x) \cap (\lambda - \delta, \lambda + \delta)) = \lim_{k \to \infty} \dim_H \bigcup_{n=1}^{\infty} (\Lambda_\delta(x) \cap (\lambda - \delta, \lambda + \delta)) \\
= \lim_{k \to \infty} \lim_{n \to \infty} (\Lambda_\delta(x) \cap (\lambda - \delta, \lambda + \delta)) \\
\geq \lim_{k \to \infty} \frac{(k - 1) \log m + \log(m - 1)}{-k \log(\lambda - \delta)} = \frac{\log m}{-\log(\lambda - \delta)}.
\]

Letting $\delta \to 0$ we obtain that
\[
\lim_{\delta \to 0} \dim_H(\Lambda(x) \cap (\lambda - \delta, \lambda + \delta)) \geq \frac{\log m}{-\log \lambda}. \tag{3.5}
\]

On the other hand, by using lemma 2.5 it follows that
\[
\dim_H(\Lambda(x) \cap (\lambda - \delta, \lambda + \delta)) \leq \dim_H \pi_{\lambda, \beta+\delta} \circ \Phi_\lambda(\Lambda(x) \cap (\lambda - \delta, \lambda + \delta)) \\
\leq \dim_H K_{\lambda+\delta} = \frac{\log m}{-\log(\lambda + \delta)}. \tag{3.6}
\]

Letting $\delta \to 0$ in (3.6), we conclude by (3.5) that
\[
\lim_{\delta \to 0} \dim_H(\Lambda(x) \cap (\lambda - \delta, \lambda + \delta)) = \frac{\log m}{-\log \lambda}. \tag{3.7}
\]

**Proof of corollary 1.3.** By theorem 1.2 it follows that
\[
\dim_H \Lambda(x) \geq \lim_{n \to \infty} \dim_H(\Lambda(x) \cap (1/m - 1/2^n, 1/m + 1/2^n)) = \dim_H K_{1/m} = 1.
\]

So $\Lambda(x)$ has full Hausdorff dimension. In the following it suffices to prove that $\Lambda(x)$ has zero Lebesgue measure for any $x \in (0, 1)$. 

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Theorem 1.2 implies that for any \( \lambda \in \Lambda(x) \setminus \{1/m\} \) there exists \( \delta_\lambda > 0 \) such that \( \text{dim}_H(\Lambda(x) \cap (\lambda - \delta_\lambda, \lambda + \delta_\lambda)) < 1 \), which implies that \( \Lambda(x) \cap (\lambda - \delta_\lambda, \lambda + \delta_\lambda) \) is a Lebesgue null set. Note by theorem 1.1 that for any \( \gamma \in (0, 1/m) \) the segment \( \Lambda(x) \cap (0, \gamma] \) is compact, and then it can be covered by a finite number of open intervals \( \{(\lambda_i - \delta_{\lambda_i}, \lambda_i + \delta_{\lambda_i})\}_{i=1}^N \) for some \( \lambda_i \in \Lambda(x) \) with \( 1 \leq i \leq N \). Since each set \( \Lambda(x) \cap (\lambda_i - \delta_{\lambda_i}, \lambda_i + \delta_{\lambda_i}) \) is a Lebesgue null set, so is \( \Lambda(x) \cap (0, \gamma] \) for any \( \gamma < 1/m \). Let \( (\gamma_n) \) be a sequence in \( (0, 1/m) \) with \( \gamma_n \nearrow 1/m \) as \( n \to \infty \). We conclude that \( \Lambda(x) = \{1/m\} \cup \bigcup_{n=1}^\infty (\Lambda(x) \cap (0, \gamma_n)) \) has zero Lebesgue measure.

**Proof of corollary 1.5.** Note that the function \( D : \lambda \mapsto \text{dim}_H K_\lambda = \frac{\log m}{\log \lambda} \) is continuous and strictly increasing in \((0, 1/m]\). Let \((a, b)\) be an open interval such that \((a, b) \cap \Lambda(x) \neq \emptyset\). We consider the following two cases.

Case (I). There exists a \( \lambda_* \in \Lambda(x) \cap (a, b) \) such that \( D(\lambda_*) = \sup_{\lambda \in \Lambda(x) \cap (a, b)} D(\lambda) \). Then \( \Lambda(x) \cap (\lambda_*, b) = \emptyset \). Take \( \varepsilon > 0 \). By theorem 1.2 it follows that

\[
\text{dim}_H(\Lambda(x) \cap (a, b)) \geq \text{dim}_H(\Lambda(x) \cap (\lambda_* - \delta, \lambda_* + \delta)) > D(\lambda_*) - \varepsilon
\]

(3.7)

for \( \delta > 0 \) sufficiently small. On the other hand, by theorem 1.2 it follows that for any other \( \lambda \in \Lambda(x) \cap [a, b) = \Lambda(x) \cap [a, \lambda_*] \) there must exist a \( \delta_\lambda > 0 \) such that

\[
\text{dim}_H(\Lambda(x) \cap (\lambda - \delta_\lambda, \lambda + \delta_\lambda)) < D(\lambda) + \varepsilon.
\]

(3.8)

Observe that the union of \((\lambda - \delta_\lambda, \lambda + \delta_\lambda)\) with \( \lambda \in \Lambda(x) \cap [a, b) \) is an open cover of the compact set \( \Lambda(x) \cap [a, b) = \Lambda(x) \cap [a, \lambda_*] \). So there exists a finite subcover of \( \Lambda(x) \cap [a, b) \), say \( \{(\lambda_i - \delta_{\lambda_i}, \lambda_i + \delta_{\lambda_i}) : 1 \leq i \leq N\} \), such that \( \Lambda(x) \cap [a, b) \subset \bigcup_{1 \leq i \leq N} \Lambda(x) \cap (\lambda_i - \delta_{\lambda_i}, \lambda_i + \delta_{\lambda_i}) \). Therefore, by (3.8) we conclude that

\[
\text{dim}_H(\Lambda(x) \cap (a, b)) = \text{dim}_H(\Lambda(x) \cap [a, b])
\]

\[
= \max_{1 \leq i \leq N} \text{dim}_H(\Lambda(x) \cap (\lambda_i - \delta_{\lambda_i}, \lambda_i + \delta_{\lambda_i}))
\]

\[
\leq \max_{1 \leq i \leq N} (D(\lambda_i) + \varepsilon) \leq D(\lambda_*) + \varepsilon.
\]

Since \( \varepsilon > 0 \) was taken arbitrarily, this together with (3.7) implies that

\[
\text{dim}_H(\Lambda(x) \cap (a, b)) = \text{dim}_H(\Lambda(x) \cap (a, \lambda_*]) = D(\lambda_*)
\]

\[
= \sup_{\lambda \in \Lambda(x) \cap (a, b)} \text{dim}_H K_\lambda
\]

as desired.

Case (II). There is no \( \lambda_* \in \Lambda(x) \cap (a, b) \) such that \( D(\lambda_*) = \sup_{\lambda \in \Lambda(x) \cap (a, b)} D(\lambda) \). Then there exists a sequence \( (\lambda_n) \subset \Lambda(x) \cap (a, b) \) such that \( \lambda_n \nearrow b \) as \( n \to \infty \), and \( \sup_{n \geq 1} D(\lambda_n) = \sup_{\lambda \in \Lambda(x) \cap (a, b)} D(\lambda) \). By case (I) we know that for any \( n \geq 1 \) we have

\[
\text{dim}_H(\Lambda(x) \cap (a, \lambda_n)) = D(\lambda_n).
\]
By the countable stability of Hausdorff dimension we conclude that
\[ \dim_H(\Lambda(x) \cap (a, b)) = \lim_{n \to \infty} \frac{\lambda_n}{n} \leq \sup_{(a, \lambda_n]} D(\lambda_n) = \sup_{\lambda \in \Lambda(x) \cap (a, b)} D(\lambda). \]

This completes the proof. \(\square\)

4. Large intersection of \(\Lambda(x)\) and \(\Lambda(y)\)

In this section we will show that the intersection of any two parameter sets \(\Lambda(x)\) and \(\Lambda(y)\) contains a Cantor subset of Hausdorff dimension arbitrarily close to one, and then prove theorem 1.6. Our strategy is to show that \(\Lambda(x)\) contains a Cantor subset with its thickness arbitrarily large.

4.1. Construction of Cantor subsets of \(\Lambda(x)\) with large thickness

First we recall from [13] the definition of thickness for a Cantor set in \(\mathbb{R}\). Suppose \(E \subset \mathbb{R}\) is a Cantor set. Then \(E\) can be obtained by successively removing countably many disjoint open intervals \(\{U_i\}_{i=1}^{\infty}\) from the closed interval \(I = \text{conv}(E)\), where \(\text{conv}(E)\) denotes the convex hull of \(E\). In the first step, we remove the open interval \(U_1\) from \(I\), and we obtain two closed subintervals \(L_1\) and \(R_1\). In this case we call \(I\) the father interval of \(U_1\), and call \(L_1, R_1\) the generating intervals of \(U_1\). In the second step, we remove \(U_2\). Without loss of generality we may assume \(U_2 \subset L_1\). Then we obtain two subintervals \(L_2\) and \(R_2\) from \(L_1\). This procedure can be continued. Suppose in the \(n\)th step we remove \(U_n\) from some closed interval \(L_j\), for some \(1 \leq j \leq n - 1\), and we obtain two subintervals \(L_n, R_n\). So \(L_j\) is the father interval of \(U_n\), and \(L_n, R_n\) are the two generating subintervals of \(U_n\). Continuing this procedure indefinitely we obtain the Cantor set \(E\). The thickness of \(E\) introduced by Newhouse [13, definition 3.1] is then defined by

\[ \tau_\Delta(E) := \sup \left\{ \inf_{m \in \mathbb{N}} \left\{ \frac{|L_{\alpha(m)}|}{|U_{\alpha(m)}|}, \frac{|R_{\alpha(m)}|}{|U_{\alpha(m)}|} \right\} : \{U_{\alpha(m)}\}_{m=1}^{\infty} \text{ is a permutation of } \{U_i\}_{i=1}^{\infty} \right\}, \]

(4.1)

where \(L_{\alpha(m)}\) and \(R_{\alpha(m)}\) are the generating intervals of \(U_{\alpha(m)}\) in the procedure, and \(|U|\) denotes the length of a subinterval \(U \subset \mathbb{R}\). It is worth to mention that the supremum in (4.1) can be achieved by ordering the open intervals \(\{U_i\}_{i=1}^{\infty}\) in a decreasing length (cf [2]).

Let \(x \in (0, 1)\), and let \((x_i) = \Phi_x(1/m)\) be the greedy \(m\)-adic expansion of \(x\). Note that the greedy expansion \((x_i)\) can not end with \((m-1)^\infty\). Then there exist infinitely many \(i \geq 1\) such that \(x_i < m - 1\). Denote by \((n_i) \subset \mathbb{N}\) the set of all indices \(i \geq 1\) such that \(x_i < m - 1\). Then for any \(j \geq 1\) we have \(x_{n_j} < m - 1\) and \(x_{n_j+1} \ldots x_{n_j+1} = (m-1)^{j+1-n_j-1}\). For \(j \geq 1\) and \(b \in \{x_{n_j} + 1, x_{n_j} + 2, \ldots, m - 1\}\) let \(p_{jb}, q_{jb} \in (0, 1/m)\) be defined by

\[ \Phi_x(p_{jb}) = x_1 \ldots x_{n_j-1} b (m-1)^\infty \quad \text{and} \quad \Phi_x(q_{jb}) = x_1 \ldots x_{n_j-1} b 0^\infty. \]

(4.2)

Then by lemma 2.1 it follows that
for all \( j \geq 1 \), and \( p_j, q_j \to 1/m \) as \( j \to \infty \). Note that the intervals \( I_{j,b} = [p_j, q_j] \) with \( j \geq 1 \) and \( b \in \{x_n + 1, \ldots, m - 1\} \) are pairwise disjoint. Then we can rename these intervals in an increasing order:

\[
I_1 = [p_1, q_1], \quad I_2 = [p_2, q_2], \ldots, I_k = [p_k, q_k], \quad I_{k+1} = [p_{k+1}, q_{k+1}],
\]

such that \( q_k < p_{k+1} \) for all \( k \geq 1 \). So for each interval \( I_k \) there exist a unique \( j \in \mathbb{N} \) and a unique \( b \in \{x_n + 1, \ldots, m - 1\} \) such that \( I_k = I_{j,b} \). Since we have only finitely many choices for the index \( b \), it follows that \( k \to \infty \) is equivalent to \( j \to \infty \).

Let \( k \geq 1 \). Then \( I_k = I_{j,b} \) for some \( j \geq 1 \) and \( b \in \{x_n + 1, \ldots, m - 1\} \). Now for a word \( w \in \{0, 1, \ldots, m - 1\}^* \) we define the basic interval associated to \( w \) by

\[
I_k(w) = [p_k(w), q_k(w)],
\]

where

\[
\Phi_1(p_k(w)) = x_1 \ldots x_{n_j} b w(m - 1)\infty, \quad \Phi_1(q_k(w)) = x_1 \ldots x_{n_j} b w 0\infty.
\]

Then for the empty word \( \epsilon \) we have \( I_k(\epsilon) = [p_k, q_k] \). Observe that for any two words \( u \) and \( v \), if \( u \) is a prefix of \( v \), then \( I_k(v) \subset I_k(u) \). So, for any \( w \in \{0, 1, \ldots, m - 1\}^* \) we have \( I_k(w) \supseteq \bigcup_{d=0}^{m-1} I_k(wd) \) with the union pairwise disjoint. Furthermore, the left endpoint of \( I_k(w(m - 1)) \) coincides with the left endpoint of \( I_k(w) \), and the right endpoint of \( I_k(w0) \) coincides with the right endpoint of \( I_k(w) \). In other words, these intervals \( I_k(w), w \in \{0, 1, \ldots, m - 1\}^* \) have a tree structure. Therefore,

\[
E_k(x) := \bigcap_{n=0}^{\infty} \bigcup_{w \in \{0, 1, \ldots, m - 1\}^n} I_k(w)
\]

is a Cantor subset of \( \Lambda(x) \). Here the inclusion \( E_k(x) \subset \Lambda(x) \) follows by our construction of \( E_k(x) \) that each \( \lambda \in E_k(x) \) corresponds to a unique coding \( (d_i) \in \{0, 1, \ldots, m - 1\}^\mathbb{N} \) such that \( \Phi_1(\lambda) = (d_i) \), which implies that \( \lambda \in \Lambda(x) \). Since \( E_k(x) \subset \Lambda(x) \) for all \( k \geq 1 \), we then construct a sequence of Cantor subsets of \( \Lambda(x) \). Furthermore, by (4.3) it follows that these Cantor subsets \( E_k(x), k \geq 1 \) are pairwise disjoint, and \( d_H(E_k(x), \{1/m\}) \to 0 \) as \( k \to \infty \), where \( d_H \) is the Hausdorff metric.

Let \( k \geq 1 \). Recall that for a word \( w = w_1 \ldots w_n \) with \( w_n < m - 1 \) we write \( w^+ = w_1 \ldots w_{n-1}(w_n + 1) \). For two neighbouring basic intervals \( I_k(w^+) = [p_k(w^+), q_k(w^+)] \) and \( I_k(w) = [p_k(w), q_k(w)] \) of the same level, we call the open interval \( G_k(w) := (q_k(w^+), p_k(w)) \) the gap between them (see figure 2). Now we write

\[
\tau(E_k(x)) := \inf_{n \geq 1} \min_{w \in \{0, 1, \ldots, m - 1\}^n} \left\{ \frac{|I_k(w^+)|}{|G_k(w)|}, \frac{|I_k(w^+)|}{|G_k(w)|} \right\}
\]

\[
= \inf_{n \geq 1} \min_{w \in \{0, 1, \ldots, m - 1\}^n} \left\{ \frac{q_k(w^+)-p_k(w^+)}{p_k(w)-q_k(w)}, \frac{q_k(w)-p_k(w)}{p_k(w)-q_k(w)} \right\},
\]
\[ \begin{array}{ccc}
I_k(w^+) & G_k(w) & I_k(w) \\
\lambda_1 = p_k(w^+) & \lambda_2 = q_k(w^+) & \lambda_3 = p_k(w) \\
\lambda_4 = q_k(w) \\
\end{array} \]

**Figure 2.** The geometrical structure of the basic intervals \(I_k(w^+) = [p_k(w^+), q_k(w^+)] = [\lambda_1, \lambda_2]\) and \(I_k(w) = [p_k(w), q_k(w)] = [\lambda_3, \lambda_4]\), and the gap \(G_k(w) = (q_k(w^+), p_k(w)) = (\lambda_2, \lambda_3)\).

where for \(w^+ \notin \{0, 1, \ldots, m - 1\}^n\) we set \(|I_k(w^+)| = |I_k(\epsilon)|\).

Note that the quantity \(\tau(E_\epsilon(x))\) is defined directly from our geometric construction of \(E_\epsilon(x)\) in (4.5). Furthermore, comparing with the thickness defined by Newhouse in (4.1) this quantity \(\tau(E_\epsilon(x))\) is much easier to estimate by using their expansions in (4.4). In the next lemma we show that \(\tau_\text{N}(E_\epsilon(x)) \geq \tau(E_\epsilon(x))\).

**Lemma 4.1.** Let \(x \in (0, 1)\). Then for any \(k \geq 1\) we have \(\tau_\text{N}(E_\epsilon(x)) \geq \tau(E_\epsilon(x))\).

**Proof.** Let \(x \in (0, 1)\) and \(k \geq 1\). We remove the open intervals (gaps) \(G_k(w)\) with \(w \in \{0, 1, \ldots, m - 1\}^n\) from \(I_k(\epsilon) = [p_k, q_k]\) in the following way, where \(\epsilon\) is the empty word.

First we remove from \(I_k(\epsilon)\) the open intervals \(G_0(0), G_1(0), \ldots, G_k(m - 2)\) and so on, and in the \((m - 1)\)th step we remove the open interval \(G_k(m - 2)\) (see figure 3). Then by the definitions of thickness defined in (4.1) and (4.6) respectively it follows that

\[
\min_{0 \leq d \leq m - 2} \left\{ \frac{|L_d(d)|}{|G_d(d)|}, \frac{|R_d(d)|}{|G_d(d)|} \right\} \geq \min_{0 \leq d \leq m - 2} \left\{ \frac{|I_d(d + 1)|}{|G_d(d)|}, \frac{|I_d(d)|}{|G_d(d)|} \right\},
\]

where \(L_d(d)\) and \(R_d(d)\) are the generating intervals of \(G_k(d)\) in the construction of \(E_\epsilon(x)\) by Newhouse. After removing the \((m - 1)\) open intervals \(G_0(0), G_1(1), \ldots, G_k(m - 2)\) we obtain \(m\) basic intervals \(I_0(0), I_1(1), \ldots, I_k(m - 1)\).

Next for each basic interval \(I_k(d)\) we remove from \(I_k(d)\) the open intervals \(G_0(0), G_1(0), \ldots, G_k(d(m - 2))\) successively, and get \(m\) basic subintervals \(I_k(0), I_k(1), \ldots, I_k(d(m - 1))\). Proceeding this argument, and suppose we are considering the basic interval \(I_k(w)\) for some \(w\) of length \(n\). Then we successively remove from \(I_k(w)\) the open intervals \(G_0(w)\), \(G_1(w)\), \(\ldots, G_k(w(m - 2))\), and obtain \(m\) basic subintervals \(I_k(0), I_k(1), \ldots, I_k(w(m - 1))\). By the same argument as above one can verify that

\[
\min_{0 \leq d \leq m - 2} \left\{ \frac{|L_d(wd)|}{|G_d(wd)|}, \frac{|R_d(wd)|}{|G_d(wd)|} \right\} \geq \min_{0 \leq d \leq m - 2} \left\{ \frac{|I_d(wd + 1)|}{|G_d(wd)|}, \frac{|I_d(wd)|}{|G_d(wd)|} \right\},
\]

where \(L_d(wd)\) and \(R_d(wd)\) are the generating intervals of \(G_d(wd)\). By induction it follows that

\[
\min_{wd \in \{0, 1, \ldots, m - 1\}^n} \left\{ \frac{|L_d(w)|}{|G_d(w)|}, \frac{|R_d(w)|}{|G_d(w)|} \right\} \geq \min_{wd \in \{0, 1, \ldots, m - 1\}^n} \left\{ \frac{|I_d(w^+)|}{|G_d(w)|}, \frac{|I_d(w)|}{|G_d(w)|} \right\}
\]

for all \(n \geq 1\). Therefore, by (4.1) and (4.6) we conclude that

\[
\tau_{\text{N}}(E_\epsilon(x)) \geq \inf_{n \geq 1} \min_{wd \in \{0, 1, \ldots, m - 1\}^n} \left\{ \frac{|L_d(w)|}{|G_d(w)|}, \frac{|R_d(w)|}{|G_d(w)|} \right\} = \tau(E_\epsilon(x)).
\]

\[\square\]
there exist \( \lambda \).

For simplicity we write \( \ell \). Then there exists a smallest integer \( \pi \lambda \) of \( \ell \).

By the same argument as in the estimate for \( \lambda \) successively. Right: we remove the open intervals \( G_k(0), G_k(1), \ldots, G_k(m-1) \) from \( I_k(\epsilon) \) in one step. Then for any \( d \in \{0, 1, \ldots, m-2\} \) we have \( R_k(d) = I_k(d) \) and \( |I_k(d)| \geq |I_k(d+1)| \).

In the following we show that the thickness of \( E_k(x) \) goes to infinitely as \( k \to \infty \).

**Proposition 4.2.** Let \( x \in (0, 1) \). Then \( \tau(E_k(x)) \to +\infty \) as \( k \to \infty \).

**Proof.** Let \( x \in (0, 1) \), and let \( (x_i) = \Phi(x/m) \) be the greedy \( m \)-adic expansion of \( x \).

Then there exists a smallest integer \( \ell \geq 1 \) such that \( x_\ell > 0 \). Take \( k \) large enough, so there exist \( j > 1, b \in \{x_\ell + 1, \ldots, m-1\} \) such that \( I_k = I_{j+b} \) and \( n_\ell > \ell \). For \( n \geq 1 \) let \( w \in \{0, 1, \ldots, m-1\}^n \) such that \( w^+ \in \{0, 1, \ldots, m-1\}^n \). In view of (4.6) we need to estimate the lower bounds for the two quotients

\[
\frac{q_k(w^+) - p_k(w^+)}{p_k(w) - q_k(w^+)}, \quad \frac{q_k(w) - p_k(w)}{p_k(w) - q_k(w^+)},
\]

For simplicity we write \( \lambda_1 = p_k(w^+), \lambda_2 = q_k(w^+), \lambda_3 = p_k(w) \) and \( \lambda_4 = q_k(w) \). Then \( \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \) (see figure 2). So we need to estimate the lower bounds for \( \lambda_2 - \lambda_1 \) and \( \lambda_4 - \lambda_3 \), and the upper bounds for \( \lambda_3 - \lambda_2 \).

**A lower bound for \( \lambda_2 - \lambda_1 \).** Note by (4.4) that

\[
\pi_{\lambda_1}(x_1 \ldots x_{n_j-1}b^m(m-1)^\infty) = x = \pi_{\lambda_2}(x_1 \ldots x_{n_j-1}b^m0^\infty).
\]

Write \( (c_i) := x_1 \ldots x_{n_j-1}b^w(m-1)^\infty \). Rearranging the above equation and by the definition of \( \pi_{\lambda} \) in (2.1) we obtain that

\[
\frac{m-1}{1 - \lambda_2} \lambda_2^{n_j+1} = \sum_{i=1}^{\infty} c_i \lambda_2^i - \sum_{i=1}^{\infty} c_i \lambda_1^i = \sum_{i=1}^{\infty} c_i (\lambda_2^i - \lambda_1^i) \leq \sum_{i=1}^{\infty} (m-1)(\lambda_2^i - \lambda_1^i) = \frac{m-1}{1 - \lambda_1}(\lambda_2 - \lambda_1).
\]

This implies that

\[
\lambda_2 - \lambda_1 \geq (1 - \lambda_1)\lambda_2^{n_j+1}.
\]

**A lower bound for \( \lambda_4 - \lambda_3 \).** Note by (4.4) that

\[
\pi_{\lambda_4}(x_1 \ldots x_{n_j-1}b^w(m-1)^\infty) = x = \pi_{\lambda_4}(x_1 \ldots x_{n_j-1}b^w0^\infty).
\]

Then by the same argument as in the estimate for \( \lambda_2 - \lambda_1 \) one can verify that

\[
\lambda_4 - \lambda_3 \geq (1 - \lambda_3)\lambda_4^{n_j+1}.
\]
An upper bound for $\lambda_3 - \lambda_2$. Note by (4.4) that
\[ \pi_{\lambda_2}(x_1 \ldots x_{n_j-1}bw^0 m^{-1} 0^\infty) = x = \pi_{\lambda_2}(x_1 \ldots x_{n_j-1}bw(m - 1)^\infty). \]
Write $(d_i) := x_1 \ldots x_{n_j-1}bw0^\infty$. Rearranging the above equation gives that
\[ \lambda_2^{n_j+n} - \frac{m - 1}{1 - \lambda_3} \lambda_3^{n_j+n+1} = \sum_{i=1}^{\infty} d_i \lambda_3^i - \sum_{i=1}^{\infty} d_i \lambda_3^i \geq \lambda_3^{n_j+n+1} - \lambda_2^{n_j+n+1} (\lambda_3 - \lambda_2), \]
where the first inequality follows by the definition of $\ell$ that $\ell < n_j$ and $x_\ell > 0$. So, by using $\lambda_3 > \lambda_2$ it follows that
\[ \lambda_3 - \lambda_2 \leq \lambda_2^{n_j+n+1} - \left( \frac{m - 1}{1 - \lambda_3} \lambda_3^{n_j+n+1} \right) \]
\[ \leq \lambda_2^{n_j+n+1} \left( \frac{m - 1}{1 - \lambda_3} \right) \]
\[ = \lambda_2^{n_j+n+1} \frac{m}{1 - \lambda_3} \left( \frac{1}{m} - \lambda_3 \right). \quad (4.9) \]
Since $\lambda_3 = p_3(w) \nearrow 1/m$ as $k \to \infty$, we still need to estimate $1/m - \lambda_3$. Note that $\Phi, 1/m = (x_i)$. Then
\[ \pi_{\lambda_3}(x_1 \ldots x_{n_j-1}bw(m - 1)^\infty) = x = \pi_{1/m}(x_1x_2 \ldots). \]
This yields that
\[ \pi_{\lambda_3}(0^{n_j-1}(b - x_{n_j})w(m - 1)^\infty) - \pi_{1/m}(0^{n_j-1}x_{n_j+1}x_{n_j+2} \ldots) \]
\[ = \pi_{1/m}(x_1 \ldots x_{n_j}0^\infty) - \pi_{\lambda_3}(x_1 \ldots x_{n_j}0^\infty) \]
\[ \geq (1/m)^\ell - \lambda_3^\ell \geq \lambda_\ell^{n_j-1}(1/m - \lambda_3). \]
So, by using $b \in \{x_{n_j} + 1, \ldots, m - 1\}$ it follows that
\[ 1/m - \lambda_3 \leq \lambda_3^{n_j+n-1} - \left( \pi_{1/m}(0^{n_j-1}(b - x_{n_j})w(m - 1)^\infty) \right) \]
\[ = \lambda_3^{n_j+n-1} - \pi_{\lambda_3}(0^{n_j-1}(b - x_{n_j})w(m - 1)^\infty) \]
\[ < \lambda_3^{n_j+n-1} - \lambda_3^{n_j+n-1}. \quad (4.10) \]
Substituting (4.10) into (4.9) we obtain an upper bound for $\lambda_3 - \lambda_2$:
\[ \lambda_3 - \lambda_2 \leq \frac{m \lambda_2^{n_j+n-1}}{1 - \lambda_3} \lambda_3^{n_j+n+1}. \quad (4.11) \]
Therefore, by (4.7) and (4.11) it follows that
In order to prove theorem 1.6 we recall the following results from Kraft [11, theorem 1.1] and Hunt et al [7, theorem 1]. Two Cantor sets $F_1$ and $F_2$ are called interleaved if

$$F_1 \cap \text{conv}(F_2)^\circ \neq \emptyset \quad \text{and} \quad \text{conv}(F_1)^\circ \cap F_2 \neq \emptyset,$$

where $\text{conv}(F)^\circ$ denotes the interior of the convex hull of $F \subset \mathbb{R}$. In other words, if two Cantor sets are interleaved, then neither set lies in the closure of a gap of the other.

**Lemma 4.4.** Let $x, y \in (0, 1)$. If there exist $i, j \in \mathbb{N}$ such that $E_i(x)$ and $E_j(y)$ are interleaved with $\tau(x, y) := \min\{\tau_N(E_i(x)), \tau_N(E_j(y))\} > 1 + \sqrt{2}$, then $E_i(x) \cap E_j(y)$ contains a Cantor subset whose thickness is of order $\sqrt{\tau(x, y)}$.
By lemma 4.1 and proposition 4.2 it follows that for any \( x \in (0, 1) \) we have \( \tau_{\mathcal{N}}(E_i(x)) \to \infty \) as \( i \to \infty \). So, if we can show that there exist infinitely many pairs \( (i, j) \) such that \( E_i(x) \) and \( E_j(y) \) are interleaved Cantor sets, then theorem 1.6 can be deduced from lemma 4.4 and (4.14). In the following we will show that there exist infinitely many pairs \( (i, j) \) such that \( E_i(x) \) and \( E_j(y) \) are interleaved.

For \( x \in (0, 1) \) we recall from (4.5) that \( E_i(x) \) is a Cantor subset of \( \Lambda(x) \) for any \( k \geq 1 \). For each basic interval \( I_k = \text{conv}(E_i(x)) = [p_k, q_k] \) there exist a unique \( j \geq 1 \) and \( b \in \{x_{n_j} + 1, \ldots, m - 1\} \) such that \( I_k = I_{j,b} \). Then the endpoints \( p_k, q_k \) are determined by

\[
\Phi_x(p_k) = x_1 \ldots x_{n_j-1} b \ (m - 1)^\infty \quad \text{and} \quad \Phi_x(q_k) = x_1 \ldots x_{n_j-1} b \ 0^\infty. \tag{4.15}
\]

First we define the thickness of the interval sequence \( \{I_k : k \geq 1\} \) by

\[
\theta(x) := \liminf_{k \to \infty} \theta_k(x),
\]

where

\[
\theta_k(x) := \min \left\{ \frac{|I_k|}{|G_k|}, \frac{|I_{k+1}|}{|G_{k+1}|} \right\} = \min \left\{ \frac{q_k - p_k}{p_{k+1} - q_k}, \frac{q_{k+1} - p_{k+1}}{p_{k+1} - q_k} \right\}. \tag{4.16}
\]

Here \( G_k = (q_k, p_{k+1}) \) is the gap between \( I_k \) and \( I_{k+1} \) (see figure 4).

In the following lemma we show that the thickness of the interval sequence \( \{I_k : k \geq 1\} \) is infinity for any \( x \in (0, 1) \).

**Lemma 4.5.** For any \( x \in (0, 1) \) we have \( \theta(x) = +\infty \).

**Proof.** The idea to prove this lemma is similar to that for proposition 4.2. By the definition of \( \theta(x) \) we need to estimate the lower bounds \( \theta_k(x) \) for \( k \) large enough. Take \( k \in \mathbb{N} \) sufficiently large. Then there exist a unique \( j \in \mathbb{N} \) and a unique \( b \in \{x_{n_j} + 1, \ldots, m - 1\} \) such that \( I_k = I_{j,b} \). In view of (4.16) we consider the following two cases: (I) \( b \in \{x_{n_j} + 1, \ldots, m - 1\} \); (II) \( b = x_{n_j} + 1 \).

Case (I). \( b \in \{x_{n_j} + 1, \ldots, m - 1\} \). Then \( I_k = I_{j,b} \) and \( I_{k+1} = I_{j,b-1} \). We need to estimate the lower bounds of \( |I_k| = q_k - p_k \) and \( |I_{k+1}| = q_{k+1} - p_{k+1} \), and the upper bounds of \( |G_k| = p_{k+1} - q_k \). Observe by (4.15) that

\[
\pi_{q_k}(x_1 \ldots x_{n_j-1} b 0^\infty) = x = \pi_{p_k}(x_1 \ldots x_{n_j-1} b (m - 1)^\infty).
\]

Write \( (c_j) := x_1 \ldots x_{n_j-1} b (m - 1)^\infty \). Rearranging the above equation and by the definition of \( \pi_{\chi} \) in (2.1) it follows that

\[
\pi_{q_0}(c_j) = x = \pi_{p_0}(c_j) = \pi_{q_0}(x_1 \ldots x_{n_j-1} b (m - 1)^\infty).
\]
\[
\frac{m - 1}{1 - q_k} \sum_{i=1}^{n_j+1} c_i q_k^i - \sum_{i=1}^{\infty} c_i p_k^i = \sum_{i=1}^{\infty} c_i (q_k^i - p_k^i) \\
\leq \sum_{i=1}^{\infty} (m - 1)(q_k^i - p_k^i) = \frac{m - 1}{(1 - p_k)(1 - q_k)}(q_k - p_k).
\]

Whence,
\[
q_k - p_k \geq (1 - p_k)q_k^{-1}. \tag{4.17}
\]

Similarly, one can prove that
\[
q_{k+1} - p_{k+1} \geq (1 - p_{k+1})q_{k+1}^{-1}. \tag{4.18}
\]

Now we turn to the upper bounds of \(p_{k+1} - q_k\). Note by (4.15) that
\[
\pi_{q_k}(x_1 \ldots x_{n_j-1} b0^\infty) = x = \pi_{p_{k+1}}(x_1 \ldots x_{n_j-1} (b - 1)(m - 1)^\infty).
\]

Write \((d_i) = x_1 \ldots x_{n_j-1}(b - 1)0^\infty\). Rearranging the above equation gives that
\[
q_k = \frac{m - 1}{1 - p_{k+1}} p_{k+1}^{-1} = \sum_{i=1}^{\infty} d_i p_{k+1}^i - \sum_{i=1}^{\infty} d_i q_k^i \\
\geq p_{k+1} - q_k \geq q_k^{-1}(p_{k+1} - q_k),
\]

where \(\ell < n_j\) is the smallest integer such that \(x_\ell > 0\). Therefore,
\[
p_{k+1} - q_k \leq q_k^{-1-\ell} \left( q_k^{-1} - \frac{m - 1}{1 - p_{k+1}} p_{k+1}^{-1} \right) \\
\leq q_k^{-\ell+1} \left( 1 - \frac{m - 1}{1 - p_{k+1}} p_{k+1}^{-1} \right) \\
= q_k^{-\ell+1} \left( \frac{m}{1 - p_{k+1}} \right). \tag{4.19}
\]

Since \(p_{k+1} \to 1/m\ as \ k \to \infty\, the upper bound in (4.19) converges to zero much faster than \(q_k^{-\ell+1} \). So we still need to estimate \(1/m - p_{k+1}\). Observe that
\[
\pi_{p_{k+1}}(x_1 \ldots x_{n_j-1} (b - 1)(m - 1)^\infty) = x = \pi_{1/m}(x_1 \ldots x_{n_j-1} x_{n_j+1} \ldots).
\]

Then
\[
(b - 1 - x_{n_j}) p_{k+1}^{-1} \left( \frac{m - 1}{1 - p_{k+1}} p_{k+1}^{-1} \right) = \sum_{i=1}^{n_j} x_i (1/m)^i - \sum_{i=1}^{n_j} x_i p_{k+1}^i \\
\geq \sum_{i=1}^{n_j} x_i \left( 1/m^i - p_{k+1}^i \right) \\
\geq (1/m)^\ell - p_{k+1}^{\ell+1} \geq p_{k+1}^{\ell+1}(1/m - p_{k+1}),
\]
which implies that
\[
1/m - p_{k+1} \leq p_{k+1}^{n_j-1} = p_{k+1}^{n_j-\ell}.
\] (4.20)

Substituting (4.20) into (4.19) we obtain an upper bound for \( p_{k+1} - q_k \):
\[
p_{k+1} - q_k \leq q_k^{n_j-\ell} \left( \frac{m}{1-p_{k+1}} \right)_{p_{k+1}}^{n_j-\ell}.
\] (4.21)

Hence, by (4.17), (4.18) and (4.21) it follows that
\[
\theta_k(x) = \min \left\{ \frac{q_k - p_k}{p_{k+1} - q_k}, \frac{q_{k+1} - p_{k+1}}{p_{k+1} - q_k} \right\} \\
\geq \min \left\{ \frac{(1-p_k)(1-p_{k+1})q_k}{mp_{k+1}^{n_j-\ell}}, \frac{(1-p_{k+1})^2 q_{k+1}}{mp_{k+1}^{n_j-\ell}} \right\} \to +\infty
\]
as \( k \to \infty \) (equivalent to \( j \to \infty \)).

Case (II). \( b = x_n + 1 \). Then \( I_k = I_{b} = [p_k, q_k] \) and \( I_{k+1} = I_{b+1, m-1} = [p_{k+1}, q_{k+1}] \). In this case we have
\[
\Phi_x(p_k) = x_1 \ldots x_n^+ (m-1)^\infty, \quad \Phi_x(q_k) = x_1 \ldots x_n^0 0^\infty;
\]
\[
\Phi_x(p_{k+1}) = x_1 \ldots x_{n_{j+1}} (m-1)^\infty = x_1 \ldots x_{n_j} (m-1)^\infty,
\]
\[
\Phi_x(q_{k+1}) = x_1 \ldots x_{n_{j+1}} (m-1)^0 0^\infty = x_1 \ldots x_{n_j} (m-1)^{j+1-n_j} 0^\infty.
\]

Here we emphasise that by the definition of \( (n_j) \) it follows that \( x_i = m-1 \) for any \( n_j < i < n_{j+1} \). By the same argument as in case I one can prove that
\[
q_k - p_k \geq (1-p_k)q_k^{n_j+1}, \quad q_{k+1} - p_{k+1} \geq (1-p_{k+1})q_{k+1}^{n_j+1},
\] (4.22)

and
\[
p_{k+1} - q_k \leq q_k^{n_j-\ell} \left( \frac{m}{1-p_{k+1}} \right)_{p_{k+1}}^{n_j-\ell}.
\] (4.23)

Therefore, by (4.22) and (4.23) it follows that
\[
\theta_k(x) = \min \left\{ \frac{q_k - p_k}{p_{k+1} - q_k}, \frac{q_{k+1} - p_{k+1}}{p_{k+1} - q_k} \right\} \to +\infty
\]
as \( k \to \infty \).

Hence, by cases (I) and (II) we conclude that \( \theta(x) = +\infty \). \( \Box \)

In order to prove that \( E_i(x) \) and \( E_j(y) \) are interleaved for infinitely many pairs \((i, j)\) we still need the following estimate.

**Lemma 4.6.** Let \( x \in (0, 1) \). Then there exist a constant \( \eta(x) > 1 \) and a large integer \( N \) such that for any \( k \geq N \) we have
\[
\frac{|I_{k+1}|}{|I_k|} = \frac{q_{k+1} - p_{k+1}}{q_k - p_k} < \eta(x).
\]
**Proof.** Suppose \( I_k = I_{j,b} = [p_k, q_k] \) for some \( j \geq 1 \) and \( b \in \{x_{n_j} + 1, \ldots, m - 1\} \). We split the proof into the following two cases: (I) \( b \in \{x_{n_j} + 2, \ldots, m - 1\} \); (II) \( b = x_{n_j} + 1 \).

Case (I). \( b \in \{x_{n_j} + 2, \ldots, m - 1\} \). Then \( I_{k+1} = I_{j,b-1} = [p_{k+1}, q_{k+1}] \). Note by (4.17) that 
\[ q_k - p_k \geq (1 - p_b)q_k^{n_j+1} . \]
In the following it suffices to estimate the upper bounds of \( q_{k+1} - p_{k+1} \). Observe that
\[ \pi_{q_{k+1}}(x_1, \ldots, x_{n_{j-1}}(b - 1)0^\infty) = x = \pi_{p_{k+1}}(x_1, \ldots, x_{n_{j-1}}(b - 1)(m - 1)0^\infty). \]
Write \((c_i) := x_1, \ldots, x_{n_{j-1}}(b - 1)0^\infty\). Rearranging the above equation gives that
\[ \frac{m - 1}{1 - p_{k+1}} p_{k+1}^{n_j+1} = \sum_{i=1}^\infty c_i q_i^{n_j+1} = \sum_{i=1}^\infty c_i p_{k+1}^{n_j+1} \]
\[ \geq q_{k+1} - p_{k+1} = p_{k+1}^{n_j+1}(q_{k+1} - p_{k+1}) , \]
where \( \ell < n_j \) is the smallest integer such that \( x_\ell > 0 \). This implies that
\[ q_{k+1} - p_{k+1} \leq \frac{m - 1}{1 - p_{k+1}} p_{k+1}^{n_j+1} < p_{k+1}^{n_j+1} , \tag{4.24} \]
where the last inequality follows by \( p_{k+1} < 1/m \). Therefore, by (4.17) and (4.24) it follows that
\[ \frac{q_{k+1} - p_{k+1}}{q_k - p_k} < \frac{p_{k+1}^{n_j+1}}{(1 - p_k)q_k^{n_j+1}} = \frac{1}{(1 - p_k)p_{k+1}^{n_j+1}} \left(p_{k+1} - p_k\right)^{n_j+1} . \tag{4.25} \]
Since \( p_k \nearrow 1/m \) as \( k \to \infty \), there exists a large integer \( N_1 \) such that \( p_{k+1} > \frac{1}{2m} \) for any \( k > N_1 \). Then (4.25) implies that
\[ \frac{q_{k+1} - p_{k+1}}{q_k - p_k} < \frac{1}{(1 - 1/m)(1/2m)^\ell} \left(p_{k+1} - p_k\right)^{n_j+1} = \frac{m(2m)^\ell}{m - 1} \left(p_{k+1} - p_k\right)^{n_j+1} \tag{4.26} \]
for any \( k > N_1 \). Note that \( k \to \infty \) is equivalent to \( j \to \infty \). So, to finish the proof it suffices to prove that \( \lim_{k \to \infty} (p_{k+1} - p_k)^{n_j+1} = 1 \). This follows by (4.21) that
\[ \lim_{k \to \infty} \left( \frac{p_{k+1}}{q_k} \right)^{n_j+1} = \lim_{k \to \infty} \left( 1 + \frac{p_{k+1} - q_k}{q_k} \right)^{n_j+1} \]
\[ \leq \lim_{k \to \infty} \left( 1 + q_k^{n_j+1} \frac{m}{1 - p_{k+1}} \right)^{n_j+1} \]
\[ = \exp \left[ \lim_{k \to \infty} \left( (n_j + 1)q_k^{n_j+1} \frac{m}{1 - p_{k+1}} \right) \right] = 1 , \]
and that \( (p_{k+1} - p_k)^{n_j+1} \geq 1 \) for any \( k \geq 1 \). So, by (4.26) there must exist an integer \( N > N_1 \) such that for any \( k > N \),
\[
\frac{q_{k+1} - p_{k+1}}{q_k - p_k} < \frac{(2m)^{\ell+1}}{m - 1}
\]
as desired.

Case (II). \( b = x_{n_j} + 1 \). Then \( I_{k+1} = I_{j+1,m-1} = [p_{k+1}, q_{k+1}] \). Note by (4.22) that \( q_k - p_k \geq (1 - p_k)q_k^{n_j+1} \). In the following it suffices to estimate the upper bounds for \( q_{k+1} - p_{k+1} \). Observe that

\[
\pi_{q_{k+1}}(x_1 \ldots x_{n_j}(m - 1)^{n_j+1-\pi})(\infty) = x = \pi_{p_{k+1}}(x_1 \ldots x_{n_j}(m - 1)^{\infty}).
\]

Then by the same argument as in case (I) one can show that

\[
q_{k+1} - p_{k+1} \leq \frac{n_{j+1}-\ell+1}{m}
\]

This together with (4.22) implies that

\[
\frac{q_{k+1} - p_{k+1}}{q_k - p_k} \leq \frac{n_{j+1}-\ell+1}{(1 - p_k)q_k^{n_j+1}} = \frac{q_k^{n_j+1-n_j}}{(1 - p_k)p_{k+1}} \left( \frac{p_{k+1}}{q_k} \right)^{n_{j+1}+1} \tag{4.27}
\]

Since \( p_k > 1/m \) as \( k \to \infty \), and \( n_{j+1} - n_j \geq 1 \), there exists a large integer \( N_2 \) such that \( p_{k+1} > 1/(2m) \) for any \( k > N_2 \). Then (4.27) implies that

\[
\frac{q_{k+1} - p_{k+1}}{q_k - p_k} \leq \frac{1}{(1 - 1/m)(1/2m)} \left( \frac{p_{k+1}}{q_k} \right)^{n_{j+1}+1} = \frac{m(2m)^{\ell}}{m - 1} \left( \frac{p_{k+1}}{q_k} \right)^{n_{j+1}+1} \tag{4.28}
\]

for any \( k > N_2 \). Note by (4.23) that

\[
\limsup_{k \to \infty} \left( \frac{p_{k+1}}{q_k} \right)^{n_{j+1}+1} = \limsup_{k \to \infty} \left( 1 + \frac{p_{k+1} - q_k}{q_k} \right)^{n_{j+1}+1} \leq \limsup_{k \to \infty} \left( 1 + q_k^{n_j-\ell} \frac{m}{1 - p_k} \right)^{n_{j+1}+1} \left( \frac{p_{k+1}}{q_k} \right)^{n_{j+1}+1} = \exp \left[ \limsup_{k \to \infty} \left( (n_{j+1} + 1) \frac{p_{k+1}}{q_k} \right)^{n_j-\ell} \frac{m}{1 - p_{k+1}} \right] = 1.
\]

So, by (4.28) there must exist an integer \( N > N_2 \) such that for any \( k > N \),

\[
\frac{q_{k+1} - p_{k+1}}{q_k - p_k} < \frac{(2m)^{\ell+1}}{m - 1}.
\]

This completes the proof. \( \square \)

Based on lemmas 4.5 and 4.6 we are ready to show that \( E_i(x) \) and \( E_j(y) \) are interleaved for infinitely many pairs \((i, j) \in \mathbb{N} \times \mathbb{N} \).

**Lemma 4.7.** For any \( x, y \in (0, 1) \) there exists a sequence of pairs \((i_k, j_k) \in \mathbb{N} \times \mathbb{N} \) such that for any \( k \geq 1 \) we have \( i_k < i_{k+1}, j_k < j_{k+1} \), and the two Cantor sets \( E_{i_k}(x), E_{j_k}(y) \) are interleaved.
Proof. Take \(x, y \in (0, 1)\). Clearly the lemma holds true if \(x = y\). So in the following we assume \(x \neq y\). To emphasise the dependence on \(x\) we write \(I_i^x := \text{conv}(E_i(x)) = [p_i^x, q_i^x]\).

We also denote by \(G_i^x := (q_i^x, p_{i+1}^x)\) the gap between the two intervals \(I_i^x\) and \(I_{i+1}^x\). Observe that the closed intervals \(\{I_i^x : i \geq 1\}\) are pairwise disjoint, and for each \(i \geq 1\) the interval \(I_{i+1}^x\) is on the right-hand side of \(I_i^x\) (see figure 4). Furthermore, \(p_i^x \nearrow 1/m\) as \(i \to \infty\). Let \(\eta(x) > 1\) and \(\eta(y) > 1\) be the constants defined as in lemma 4.6, and let \(\eta := \max\{\eta(x), \eta(y)\}\). Then by proposition 4.2, lemmas 4.5 and 4.6 there exist \(N \in \mathbb{N}\) such that for any \(i, j > N\) we have

\[
\tau(E_i(x)) > \eta, \quad \tau(E_j(y)) > \eta; \quad (4.29)
\]

\[
\theta_i(x) = \min \left\{ \frac{|I_i^x|}{|G_i^x|}, \frac{|I_{i+1}^x|}{|G_i^x|} \right\} > \eta, \quad \theta_j(y) = \min \left\{ \frac{|I_j^y|}{|G_j^y|}, \frac{|I_{j+1}^y|}{|G_j^y|} \right\} > \eta, \quad (4.30)
\]

and

\[
\frac{|I_{i+1}^x|}{|I_i^x|} < \eta(x), \quad \frac{|I_{j+1}^y|}{|I_j^y|} < \eta(y). \quad (4.31)
\]

Then (4.30) suggests that for each \(i > N\) the length of the gap \(G_i^x\) is strictly smaller than the length of each of its neighbouring intervals \(I_i^x\) and \(I_{i+1}^x\). Also, for each \(j > N\) the length of the gap \(G_j^y\) is strictly smaller than the length of each of its neighbouring intervals \(I_j^y\) and \(I_{j+1}^y\).

**Claim 1.** There must exist a pair \((i, j) \in \mathbb{N} \times \mathbb{N}\) with \(i, j > N\) such that

\[
\text{int}(I_i^x) \cap \text{int}(I_j^y) \neq \emptyset,
\]

where \(\text{int}(I)\) denotes the interior of an interval \(I\).

**Proof of claim 1.** Take \(\delta > 0\) sufficiently small. Then there exist a smallest integer \(N_1 > N\) such that \(I_i^x \cap (1/m - \delta, 1/m) \neq \emptyset\), and a smallest integer \(N_2 > N\) such that \(I_{i+1}^x \cap (1/m - \delta, 1/m) \neq \emptyset\). So \(I_i^x \subset (1/m - \delta, 1/m)\) for any \(i > N_1\), and \(I_{i+1}^x \subset (1/m - \delta, 1/m)\) for any \(j > N_2\). Therefore, to prove claim 1 it suffices to prove that there exist \(i > N_1\) and \(j > N_2\) such that \(\text{int}(I_i^x) \cap \text{int}(I_j^y) \neq \emptyset\). Suppose this is not true. Then each basic interval \(I_i^x\) with \(i > N_1\) must belong to the closure of a gap of the interval sequence \(\{I_j : j \geq N_2\}\).

In other words, for each \(i > N_1\) there must exist \(j \geq N_2\) such that \(I_i^x \subset \overline{G_j^y} = [q_j^y, p_{j+1}^y]\). Observe that \(p_i^x \nearrow 1/m\) as \(i \to \infty\), and \(p_j^y \nearrow 1/m\) as \(j \to \infty\). So there must exist \(i > N_1\) such that \(I_i^x\) and \(I_{i+1}^x\) belong to the closure of two different gaps of the interval sequence \(\{I_j : j \geq N_2\}\), say \(I_i^x \subset \overline{G_j^x}\) and \(I_{i+1}^x \subset \overline{G_j^y}\) for some \(j' > j \geq N_2\) (see figure 5). Since \(\theta_i(x) > 1\) by (4.30), we have

\[
|I_{i+1}^x| \leq |\overline{G_j^x}| = |G_j^x| \leq \min \{ |I_i^x|, |I_{i+1}^x| \}.
\]

This, together with \(I_{i+1}^x \subset \overline{G_{j'}^y}\), implies that

\[
\theta_i(x) \leq \frac{|I_i^x|}{|\overline{G_j^x}|} = \frac{|I_i^x|}{|G_j^x|} = \frac{|I_i^x|}{|I_{i+1}^x|} < 1,
\]

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which leads to a contradiction with (4.30) that $\theta_j(y) > \eta > 1$ for all $j > N$. Hence, there must exist $i, j > N$ such that $\text{int}(I^i_1) \cap \text{int}(I^j_1) \neq \emptyset$, proving claim 1.

### Claim 2.
There exists a pair $(i_1, j_1) \in \mathbb{N} \times \mathbb{N}$ with $i_1, j_1 > N$ such that the two Cantor sets $E_{i_1}(x)$ and $E_{j_1}(y)$ are interleaved.

#### Proof of claim 2.
By claim 1 we can choose $i, j > N$ such that $\text{int}(I^i_1) \cap \text{int}(I^j_1) \neq \emptyset$. If $I^i_1 \not\subseteq I^j_1$ and $I^j_1 \not\subseteq I^i_1$, then the two Cantor sets $E_{i_1}(x)$ and $E_{j_1}(y)$ are interleaved, and then we are done by setting $i_1 = i$ and $j_1 = j$. Otherwise, without loss of generality we may assume $I^i_1 \subset I^j_1 = \text{conv}(E_{j_1}(y))$. If $I^j_1$ is not contained in the closure of a gap of $E_{j_1}(y)$, then $E_{i_1}(x)$ and $E_{j_1}(y)$ are interleaved Cantor sets, and again we are done by setting $i_1 = i$ and $j_1 = j$. In the following we assume that $I^j_1$ is contained in the closure of a gap of $E_{j_1}(y)$, say $I^j_1 \subset \overline{G^j_1(w)}$ for some word $w_1$ of smallest length. Here $G^j_1(w)$ is the gap between the two basic intervals $I^j_1(w_1)$ and $I^j_1(w_1)$ which are defined as in (4.4), see figure 6. Then the length of the interval $I^j_1(w_1)$ is strictly larger than one. Then the length of the interval $I^j_1(w_1)$ is strictly larger than the length of the gap $G^j_1(w)$. Furthermore, note by (4.30) that the thickness of the interval sequence $\{I^i_1 : i > N\}$ is larger than one, and $p^i_1 \nearrow 1/m$ as $i \to \infty$, we claim that there must exist a smallest integer $s_1 > s_0$ such that

$$\text{int}(I^i_1) \cap \text{int}(I^j_1(w_1)) \neq \emptyset. \quad (4.32)$$

Suppose on the contrary that there is no such $s_1 > s_0$ satisfying $\text{int}(I^i_1) \cap \text{int}(I^j_1(w_1)) \neq \emptyset$. Then there is an $s > s_0$ such that $I^j_1(w_1) < \overline{G^j_1}$. This implies that

$$\theta_j(x) \leq \frac{|I^i_1|}{|G^j_1|} = \frac{|I^i_1|}{|G^j_1|} \leq \frac{|I^i_1|}{|I^j_1(w_1)|} < 1,$$

where the last inequality follows by that $I^j_1 \subset \overline{G^j_1(w_1)}$ and $|G^j_1(w)| < |I^j_1(w_1)|$. However, this leads to a contradiction with (4.30) that $\theta_j(x) > 1$ for all $j > N$. Hence, we prove the existence of $s_1$ such that (4.32) holds.

Note by (4.35) and (4.31) that

$$|I^i_1| \leq \eta |I^i_{j_1-1}| \leq \eta |G^j_1(w_1)| < |I^j_1(w_1)|. \quad (4.33)$$

Since $\text{int}(I^i_1) \cap \text{int}(I^j_1(w_1)) \neq \emptyset$, by (4.33) it follows that if $I^i_1$ is not contained in the closure of a gap of $E_{j_1}(y) \cap I^j_1(w_1)$, then $E_{i_1}(x)$ and $E_{j_1}(y)$ are interleaved, and then we are done by setting $i_1 = s_1$. Otherwise, suppose $I^i_1$ is contained in the closure of a gap of $E_{j_1}(y) \cap I^j_1(w_1)$. Then there exists a word $w_2$ of smallest length such that $I^i_1 \subset \overline{G^j_1(w_1,w_2)}$. Note that the thickness of $E_{j_1}(y)$ and the thickness of $\{I^i_1 : i > N\}$ are both larger than one. Then by the same argument as in (4.32) there exists a smallest integer $s_2 > s_1$ such that

$$\text{int}(I^i_1) \cap \text{int}(I^j_1(w_1,w_2)) \neq \emptyset.$$
Again, by the same reason as in the proof of (4.33) we can prove that $|I_{s_n}^i| < |I_{s_n}^j(w_1, w_2)|$. So, if $I_{s_2}^i$ is not contained in the closure of a gap of $E_j(y) \cap I_{s_2}^j(w_1, w_2)$, then $E_j(x)$ and $E_j(y)$ are interleaved, and then we set $i_1 = s_2$. Repeating the above argument, and we claim that our procedure must stop at some finite time $n$, i.e., there exist an integer $s_n$ and words $w_1, w_2, \ldots, w_n$ such that

$$\text{int}(I_{s_n}) \cap \text{int}(I_{s_n}^j(w_1, w_2 \ldots w_n)) \neq \emptyset, \quad |I_{s_n}^j| < |I_{s_n}^j(w_1, w_2 \ldots w_n)|,$$

and $I_{s_n}^j$ is not contained in the closure of a gap of $E_j(y) \cap I_{s_n}^j(w_1, w_2 \ldots w_n)$. If this claim is true, then we are done by setting $i_1 = s_n$. Suppose it is not true. Then $I_{s_1}^i \subset I_{s_1}^j$ for all large integers $i > s_1$, which implies that $q_i^j \leq q_i^j < 1/m$ for all $i \geq 1$. This leads to a contradiction with $\lim_{i \to \infty} q_i^j = 1/m$. Therefore, we have find a pair $(i_1, j_1) \in \mathbb{N} \times \mathbb{N}$ with $i_1, j_1 > N$ such that the two Cantor sets $E_{i_1}(x)$ and $E_{j_1}(y)$ are interleaved. This proves claim 2.

Now we take $\delta > 0$ small enough such that $1/m - \delta > \max \{q_i^j, q_j^i\}$, and repeating the argument as in claims 1 and 2. Then we can find another pair $(i_2, j_2) \in \mathbb{N} \times \mathbb{N}$ with $i_2 > i_1, j_2 > j_1$ such that $E_{i_2}(x)$ and $E_{j_2}(y)$ are interleaved. We can proceed this argument indefinitely, and then find a sequence of pairs $(i_k, j_k) \in \mathbb{N} \times \mathbb{N}$ such that for any $k \geq 1$ we have $i_k < i_{k+1}, j_k < j_{k+1}$, and the two Cantor sets $E_{i_k}(x)$ and $E_{j_k}(y)$ are interleaved. This completes the proof. \hfill $\square$

**Proof of theorem 1.6.** By proposition 4.2, lemmas 4.4 and 4.7 it follows that $\Lambda(x) \cap \Lambda(y)$ contains a Cantor set of thickness arbitrarily large. More precisely, for the infinite sequence of pairs $(i_k, j_k) \in \mathbb{N} \times \mathbb{N}$ defined as in lemma 4.7 it follows that for any $k \geq 1$,

$$\Lambda(x) \cap \Lambda(y) \supset E_{i_k}(x) \cap E_{j_k}(y). \quad (4.34)$$

Note by lemma 4.1 and proposition 4.2 that

$$\lim_{k \to \infty} \tau_N(E_{i_k}(x)) = \lim_{k \to \infty} \tau_N(E_{j_k}(y)) = +\infty. \quad (4.35)$$

So, by lemma 4.4 and (4.35) it follows that for $k$ sufficiently large the intersection $E_{i_k}(x) \cap E_{j_k}(y)$ contains a Cantor set with its thickness comparable with

$$\sqrt{\min \{\tau_N(E_{i_k}(x)), \tau_N(E_{j_k}(y))\}}.$$

Hence, by (4.14), (4.34) and (4.35) we conclude that

$$\dim_{H}(\Lambda(x) \cap \Lambda(y)) \geq \dim_{H}(E_{i_k}(x) \cap E_{j_k}(y)) \to 1 \quad \text{as } k \to \infty.$$
5. Final remarks

In this paper we consider the class of self-similar sets

\[
K_\lambda = \left\{ \sum_{i=1}^{\infty} d_i \lambda^i : d_i \in \{0, 1, \ldots, m-1\} \quad \forall i \geq 1 \right\}
\]

with one parameter \( \lambda \in (0, 1/m] \). The strategy used in this paper can be applied to study some other one-parameter family of Cantor sets. The monotonicity of the coding map \( \Phi_x \) defined in section 2 is pivotal. Without this monotonicity we will have trouble in constructing Cantor subsets in the parameter space. But we can still modify our method if the map \( \Phi_x \) is piecewise monotonic.

In remark 4.3 (b) we show that the algebraic sum \( \Lambda(x) + \Lambda(y) \) contains an interval for any \( x \in (0, 1) \). It is natural to ask whether it is indeed an interval. If not, can we determine the largest interval \( I(x) \) in \( \Lambda(x) + \Lambda(x) \), and compute the local dimension of \( (\Lambda(x) + \Lambda(x)) \setminus I(x) \)? In general, it is interesting to study the topological structure of \( \Lambda(x) + \Lambda(y) \) for any \( x, y \in (0, 1) \).

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