Positivity of Legendrian Thom polynomials

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Abstract

We study Legendrian singularities arising in complex contact geometry. We define a one-parameter family of bases in the ring of Legendrian characteristic classes such that any Legendrian Thom polynomial has nonnegative coefficients when expanded in these bases. The method uses a suitable Lagrange Grassmann bundle on the product of projective spaces. This is an extension of a nonnegativity result for Lagrangian Thom polynomials obtained by the authors previously. For a fixed specialization, other specializations of the parameter lead to upper bounds for the coefficients of the given basis. One gets also upper bounds of the coefficients from the positivity of classical Thom polynomials (for mappings), obtained previously by the last two authors.

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1 Introduction

The aim of the present paper (which is a continuation of [5], [17] and [14]) is to study the positivity of Legendrian Thom polynomials. Such positivity properties are nowadays widely investigated in algebraic geometry, see the monograph of Lazarsfeld [13].

Our main goal is to define a certain one-parameter family of bases in the ring of Legendrian characteristic classes. Any Legendrian Thom polynomial has nonnegative coefficients when expanded in any member of this family (Theorem 10 and 11).

The main difference, comparing with the previous papers (in particular, with [5] and [14, Remark 14]) is the definition of Legendre singularity classes to which these Thom polynomials are attached. They are introduced as closed algebraic subsets in the space parametrizing pairs of Legendrian submanifolds in the contact space (see Proposition 3). As in [5], [17] and [14], when we study Thom polynomials of commonly bounded degree, it is enough to deal with finite jets of germs of submanifolds through the origin.

The principal technique involved in the proof of these theorems is transversality with respect to some stratification of a Lagrange Grassmann bundle. This is a subject of Section 4. The key technical result is Theorem 6. We show that the intersection (in the jet bundle) of a Legendre singularity class with the preimage of the closure of a stratum of the stratification, is represented by a nonnegative cycle. In fact, for our purposes we need a certain Lagrange Grassmann bundle over the product of two projective $n$-spaces, see Theorem 10.

In Section 6, we examine the parameter of the constructed family of bases, and give a precise proof of a result announced in [14], describing algebraically the basis corresponding to the value of the parameter equal to 1 (see Theorem 12).

A concrete form of the resulting one-parameter family of bases follows from a degeneracy locus formula from [9] (based on [12] and [16]). (Another formula was announced in [7].) We discuss in Section 7 simplifications in these formulas occuring thanks to special vector bundles in our geometric constructions.

Let us fix the value of the parameter. It turns out that the nonnegativity of coefficients of the bases for some other values of the parameter can imply upper bounds of the coefficients in the basis for the given parameter. This is a subject of Section 8.

In Section 9, we show (Proposition 17) that for nonempty stable singularity classes the corresponding Legendrian (and Lagrangian) Thom polynomials are nonzero. This is an amelioration of the main result of [14]. The proof uses the fact [17] that the Thom polynomials for functions $\mathbb{C}^n \to \mathbb{C}$ are nonzero for nonempty singularity classes. We also show how this last result gives some upper bounds on the coefficients of Legendrian Thom poly-
nomials in the basis from Theorem 12.

In Section 10, we list some examples of Legendrian Thom polynomials expanded in different bases from the family.

In the Appendix, we prove some new positivity result for the intersection coefficients in the Lagrange Grassmann bundle. This result concerns the stratification transverse to the one studied in Section 4. It relies on Anderson’s “large group” action [1].

2 Some Legendrian geometry

Fix \( n \in \mathbb{N} \). Suppose for the moment that \( W \) is a vector space of dimension \( n \), and \( \xi \) is a vector space of dimension one. Set

\[
V := W \oplus (W^* \otimes \xi).
\]

We study Legendrian submanifolds in the standard contact space

\[
V \oplus \xi = W \oplus (W^* \otimes \xi) \oplus \xi,
\]
equipped with the standard contact form

\[
\alpha := dx - \sum_{i=1}^{n} p_i \, dq_i,
\] (1)

where \( x \) is a coordinate of \( \xi \) and \( q_1, q_2, \ldots, q_n \) are the coordinates of \( W \). The dual coordinates \( p_1, p_2, \ldots, p_n \) of \( W^* \otimes \xi \) are determined by the choice of \( x \) and \( q_i \)'s. Different choices of \( x \) give proportional contact forms. The contact form \( \alpha \) does not depend on the choice of the coordinates of \( W \).

Legendrian submanifolds of \( V \oplus \xi \) are maximal integral submanifolds of \( \alpha \), i.e. the manifolds of dimension \( n \) with tangent spaces contained in \( \text{Ker}(\alpha) \).

The space \( V \) is equipped with the symplectic form

\[
\omega := \sum_{i=1}^{n} dp_i \wedge dq_i,
\] (2)

which again depends on the coordinate of \( \xi \). It is well defined as an element of \( \bigwedge^2 V^* \otimes \xi \). A submanifold of \( V \) is Lagrangian if \( \omega \) restricted to its tangent spaces vanishes. The choice of the parameter \( x \) causes in the sequel the necessity of using vector bundles.

For simplicity, by Legendrian (resp. Lagrangian) submanifolds we shall mean the germs of such submanifolds through the origin in \( V \oplus \xi \) and \( V \), respectively.

We will use the following crucial fact ([2, Proposition p.313]):
Lemma 1 The projection of a Legendrian submanifold from $V \oplus \xi$ to $V$ is a Lagrangian submanifold. All the Lagrangian submanifolds of $V$ are obtainable in this way. Moreover, a Legendrian submanifold of $V \oplus \xi$ is uniquely determined by its Lagrangian image.

Therefore we easily translate our constructions in the contact space to the parallel constructions in the symplectic space.

In this section, we describe a space which parameterizes pairs of Legendrian submanifolds $L_1, L_2$. We say that two pairs of Legendrian submanifolds in $V \oplus \xi$ are contact equivalent if they differ by a holomorphic contactomorphism of $V \oplus \xi$.

We first note the following property of Lagrangian submanifolds.

Lemma 2 Any pair of Lagrangian submanifolds is symplectic equivalent to a pair $(L_1, L_2)$ such that $L_1$ is a linear Lagrangian submanifold and the tangent space $T_0 L_2$ is equal to $W$.

Proof. The lemma follows essentially from the Darboux Theorem (see [2], Theorem, p.287). Indeed, it follows from this theorem that any Lagrangian submanifold is symplectomorphic to a linear one (given by vanishing of the $p$-coordinates in the notation of the theorem). We apply this to the first Lagrangian submanifold, getting a linear $L'_1$. Applying then an appropriate rotation, we get the tangency condition for $L_2$, equal to the rotated second Lagrangian submanifold. The image $L_1$ of $L'_1$ under this rotation is linear. The pair $(L_1, L_2)$ satisfies the assertion of the lemma. □

Proposition 3 Any pair of Legendrian submanifolds is contact equivalent to a pair $(L_1, L_2)$ such that the projection of $L_1$ to $V$ is a linear subspace and the tangent space $T_0 L_2$ is equal to $W$.

Proof. We use the bijection between the Legendrian submanifolds of $V \oplus \xi$ and Lagrangian submanifolds of the symplectic space $V$, and the assertion follows from Lemma 2. □

Since a vector space $\xi$ has no distinguished coordinate, we have to perform the above construction assuming that $\xi$ is a line bundle over some base space. We could have used a universal base for bundles but it is more convenient to consider fiber bundles defined over various base spaces. Also, it is convenient to assume that $W$ is a (possibly nontrivial) vector bundle.

Let $X$ be a topological space, $W$ a complex rank $n$ vector bundle over $X$, and $\xi$ a complex line bundle over $X$. The fibers of $W$, $\xi$, $V$, etc. over a point $x \in X$ are denoted by $W_x$, $\xi_x$, $V_x$. Let

$$\tau : \text{Leg}(W, \xi) \to X$$
denote the Legendre Grassmann bundle parametrizing Legendrian submanifolds in $V_x \oplus \xi_x$, $x \in X$ whose projections to $V_x$ are linear spaces. We shall often identify $\text{Leg}(W, \xi)$ with the Lagrange Grassmann bundle 

$$\tau : \text{LG}(V, \omega) \to \text{X}$$

since any Legendre submanifold in $V_x \oplus \xi_x$ is determined by its projection to $V_x$.

The pull-backs to $\text{Leg}(W, \xi)$ of the bundles $W$, $V$ and $\xi$ will be denoted by the same letters, if no confusion occurs. We have the tautological bundle over $\text{Leg}(W, \xi)$. It is denoted by $R_{W, \xi}$, or by $R$ for short. The symplectic form $\omega$ gives an isomorphism

$$V \cong V^* \otimes \xi.$$  \hfill (3)

We have the tautological sequence on $\text{Leg}(W, \xi)$:

$$0 \to R \to V \to R^* \otimes \xi \to 0.$$  \hfill (4)

Fix $k \in \mathbb{N}$. We identify two Legendrian submanifolds if the degree of their tangency at 0 is greater than or equal to $k$. The equivalence class will be called “a $k$-jet of a submanifold”.

Influenced by Proposition 3, we define $^1C^k(W, \xi)$ to be the set of pairs of $k$-jets of Legendrian submanifolds $(x \in X)$

$$(L_1, L_2) \subset V_x \oplus \xi_x$$

such that the projection of $L_1$ to $V_x$ is a linear space and $T_0L_2 = W_x$. Let

$$\pi : C^k(W, \xi) \to \text{Leg}(W, \xi)$$

denote the projection such that $\pi(L_1, L_2) = L_1$.

Every $k$-jet of a Legendrian submanifold $L$ in $V_x \oplus \xi_x$ is the graph of a 1-form

$$\alpha : W_x \to (W_x^* \otimes \xi_x) \oplus \xi_x.$$  

The condition that $L$ is Legendrian is equivalent to $d\alpha = 0$. Therefore there exists a polynomial function

$$f : W_x \to \xi_x$$

of degree $k + 1$ such that $\alpha = df$. The assumption $0 \in L$ is equivalent to $df(0) = 0$. In other words, $L$ is the graph of the 1-jet of $f$. If, in addition, the tangency condition

$$T_0L = W_x$$

$^1$We follow here a suggestion of Kazarian (see [7, Sect. 4.1]).
holds, then the second jet of $f$ vanish at 0.

By the above, we can identify $\pi^{-1}(W_x)$ with

$$\bigoplus_{i=3}^{k+1} \text{Sym}^i(W_x^*) \otimes \xi_x.$$  

In fact, we obtain the following Cartesian square:

$$
\begin{array}{ccc}
\mathcal{C}^k(W, \xi) & \xrightarrow{\pi} & \text{Leg}(W, \xi) \\
\downarrow & & \downarrow \tau \\
\bigoplus_{i=3}^{k+1} \text{Sym}^i(W^*) \otimes \xi & \rightarrow & X.
\end{array}
$$

In other words, we identify $\mathcal{C}^k(W, \xi)$ with (the space of) the pull back of the bundle

$$\bigoplus_{i=3}^{k+1} \text{Sym}^i(W^*) \otimes \xi$$

to $\text{Leg}(W, \xi)$.

A pair $(L, f)$, consisting of a $k$-jet of a Legendrian submanifold

$L \subset V_x \oplus \xi_x$

and $\xi_x$-valued polynomial

$$f \in \bigoplus_{i=3}^{k+1} \text{Sym}^i(W_x^*) \otimes \xi_x,$$

corresponds to the pair of $k$-jets

$$(L, \text{graph}(j^1(f))) \in \mathcal{C}^k(W_x, \xi_x).$$

Note that the projection $\pi$ and the inverse inclusion (zero section of the bundle) are homotopy equivalences.

By a Legendre singularity class we mean a closed algebraic subset

$$\Sigma \subset \mathcal{C}^k(C^n, \mathbb{C}),$$

invariant with respect to holomorphic contactomorphisms of $\mathbb{C}^{2n+1}$. (It is a union of contact equivalence classes.) Additionally, we assume that the singularity class $\Sigma$ is stable with respect to enlarging the dimension of $W$. Since any changes of coordinates of $W$ and $\xi$ induce holomorphic contactomorphisms of $V \oplus \xi$, any Legendre singularity class $\Sigma$ defines a cycle

$$\Sigma(W, \xi) \subset \mathcal{C}^k(W, \xi).$$
If \( X = \mathbb{P}^n \), \( \xi = \mathcal{O}(1) \) and \( W = \mathbb{I}^n \) is the trivial bundle of rank \( n \), then the cohomology
\[
H^\ast(\text{Leg}(W, \xi), \mathbb{Z}) \cong H^\ast(\mathcal{C}^k(W, \xi), \mathbb{Z})
\]
is isomorphic to the ring of Legendrian characteristic classes for degrees smaller than or equal to \( n \). The element \( PD[\Sigma(W, \xi)] \) of \( H^\ast(\mathcal{C}^k(W, \xi), \mathbb{Z}) \), which is the Poincaré dual of \( [\Sigma(W, \xi)] \) is called the Legendrian Thom polynomial of \( \Sigma \) (cf. [5]).

We have the following presentation of the Legendrian Thom polynomial in terms of Chern classes. Consider the virtual bundle
\[
A := W^* \otimes \xi - R_{W, \xi}.
\]
Using the sequence (4), we get the relation
\[
A + A^* \otimes \xi = 0.
\]
For a Legendre singularity class \( \Sigma \), the Poincaré dual of the class \( [\Sigma(W, \xi)] \) is given by the Thom polynomial
\[
T^\Sigma = T^\Sigma(a_1, a_2, \ldots; s)
\]
evaluated at the Chern classes
\[
a_i = c_i(A) = c_i(-A^* \otimes \xi) \quad \text{and} \quad s = c_1(\xi).
\]

**Remark 4** The ring of Legendrian characteristic classes is the quotient of the polynomial ring
\[
\mathbb{Z}[a_1, a_2, a_3, \ldots; u]
\]
by the relations coming from the identity (6). After inverting 2 and applying the twist by \( \xi^{-2} \) we obtain the ring of Lagrangian classes extended by one additional free variable \( t \), that is
\[
\mathbb{Z}[\frac{1}{2}][a'_1, a'_2, a'_3, \ldots; t]/\mathcal{I},
\]
where \( \mathcal{I} \) is generated by the polynomials
\[
(a'_i)^2 + 2 \sum_{k=1}^{i} (-1)^k a'_{i+k} a'_{i-k}, \quad i > 0.
\]
The even Chern classes \( a'_{2i} \) are expressed by odd ones and this ring is just the polynomial ring
\[
\mathbb{Z}[\frac{1}{2}][a'_1, a'_3, a'_5, \ldots; t].
\]
A similar procedure can be applied to the untwisted variables \( a_i \).

\footnote{For a smooth variety \( Y \), Poincaré duality \( PD : H^B_k(Y, \mathbb{Z}) \to H^{2 \dim Y-k}(Y, \mathbb{Z}) \) is the inverse of \( \cap[Y] : H^{2 \dim Y-k}(Y, \mathbb{Z}) \to H^B_k(Y, \mathbb{Z}) \), where \( BM \) (Borel-Moore) indicates that we deal with closed, possibly noncompact cycles. Apart from that, for a given basis of homology \( x_i \in H_k(Y, \mathbb{Z}) \), \( i = 1, \ldots, \text{rk} H_k(Y, \mathbb{Z}) \) we will consider the dual basis \( x^*_i \in (H_k(Y, \mathbb{Z}))^* = H^k(Y, \mathbb{Z}) \) in the sense of linear algebra.}
3 Legendre Grassmann bundles

We describe two “transverse” cell decompositions of the Legendre Grassmannians.

To begin with, let $\xi, \alpha_1, \alpha_2, \ldots, \alpha_n$ be vector spaces of dimensions equal to one and let

$$W := \bigoplus_{i=1}^{n} \alpha_i, \quad V := W \oplus (W^* \otimes \xi).$$

We have a twisted symplectic form $\omega$ defined on $V$ with values in $\xi$. The Lagrangian Grassmannian $LG(V, \omega)$ is a homogeneous space with respect to the group action of the symplectic group $Sp(V, \omega) \subset \text{End}(V)$. Fix two “opposite” standard isotropic flags in $V$:

$$F^+ := \bigoplus_{i=1}^{h} \alpha_i, \quad F^- := \bigoplus_{i=1}^{h} \alpha^*_{n-i+1} \otimes \xi$$

for $h = 1, 2, \ldots, n$ and consider two subgroups $B^\pm \subset Sp(V, \omega)$ which are the Borel groups preserving the flags $F^\pm$. The orbits of $B^\pm$ in $LG(V, \omega)$ form two “opposite” cell decompositions $\{\Omega_I(F^\pm, \xi)\}$ of $LG(V, \omega)$ (Bruhat decompositions), indexed by strict partitions

$I \subset \rho := (n, n-1, \ldots, 1)$

(see, e.g., [15]). The cells of the $\Omega^+$-decomposition are transverse to the cells of the $\Omega^-$-decomposition.

We pass now to the relative version of the above decompositions.

The description just presented is functorial with respect to the automorphisms of the lines $\xi$ and $\alpha_i$’s, (they form a torus $(\mathbb{C}^*)^{n+1}$). Thus the construction of the cell decompositions can be repeated for bundles $\xi$ and $\{\alpha_i\}_{i=1}^{n}$ over any base $X$. We obtain a Lagrange Grassmann bundle

$$\tau : LG(V, \omega) = Leg(W, \xi) \to X$$

and a group bundle (group scheme over $X$)

$$Sp(V, \omega) \to X$$

together with two subgroup bundles $B^\pm \to X$. Moreover, $Leg(W, \xi)$ admits two (relative) stratifications

$$\{\Omega_I(F^\pm, \xi) \to X\}_{\text{strict } I \subset \rho}.$$

Assume that $X = G/P$ is a compact manifold, homogeneous with respect to an action of a linear group $G$. Then $X$ admits an algebraic cell decomposition $\{\sigma_\lambda\}$ (it is again a Bruhat decomposition). The subsets

$$Z^-_{I\lambda} := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F^-, \xi)$$
form an algebraic cell decomposition of \( \text{Leg}(W, \xi) \), called \textit{distinguished} in the following. The classes of their closures give a basis of homology. Note that each \( Z_{\lambda}^- \) is transverse to each stratum \( \Omega_f(F^+_\lambda, \xi) \), where \( J \subset \rho \) is a strict partition.

Similarly, we define the subsets
\[
Z_{\lambda}^+ := \tau^{-1}(\sigma_\lambda) \cap \Omega_f(F^+_\lambda, \xi).
\]

**Example 5** If \( X = \mathbb{P}^1 \), \( W = 1 \), \( \xi = \mathcal{O}(d) \) (for \( d > 0 \)), then \( \text{Leg}(W, \xi) \) is the Hirzebruch surface \( \Sigma_d \) which can be presented as the sum of the space of the line bundle \( \xi \) and the section at the infinity,
\[
\Sigma_d = \xi \cup \mathbb{P}^1_\infty.
\]

Then \( \mathbb{P}^1_0 \), the zero section of the bundle \( \xi \), is a stratum of the \( \Omega^+ \)-decomposition and the section at infinity \( \mathbb{P}^1_\infty \) is a stratum of the \( \Omega^- \)-decomposition. For the cell decomposition of \( X = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \), we obtain two cell decompositions of \( \Sigma_d \). The closures of the one-dimensional cells are the following:
\[
\left\{ Z_{\lambda}^+ \right\} = \left\{ \tau^{-1}(\infty), \mathbb{P}^1_0 \right\}, \quad \left\{ Z_{\lambda}^- \right\} = \left\{ \tau^{-1}(\infty), \mathbb{P}^1_\infty \right\}.
\]

Two resulting bases of homology are mutually dual with respect to the intersection product. In other words, the basis of
\[
H^2(\Sigma_d, \mathbb{Z}) = (H_2(\Sigma_d, \mathbb{Z}))^*,
\]
consisting of Poincaré duals of the \( Z^+ \)-basis, coincides with the Kronecker dual basis to the \( Z^- \)-basis.

The cycles of \( Z^+ \)-decomposition have the following property: any effective cycle has a nonnegative intersection number with them. This is not true for the elements of \( Z^- \) decomposition: for example the self-intersection of \( \mathbb{P}^1_\infty \) is equal to \(-d\).

We shall generalize Example 5 in the Appendix.

### 4 Positivity in a jet bundle on a Legendrian Grassmannian

We pass now to a nonnegativity result on the Legendrian Thom polynomials and the stratification \( Z^- \). Consider the following vector bundle on \( X \):
\[
F := \bigoplus_{i=3}^{k+1} \text{Sym}^i(W^*) \otimes \xi.
\]

Pulling back \( F \) to \( \text{Leg}(W, \xi) \), we get the following jet bundle:
\[
\pi : E := \tau_* F \rightarrow \text{Leg}(W, \xi).
\]
(With the previous notation, the space of the bundle $E$ is equal to $C^k(W, \xi).$)

This vector bundle is equipped with a $B^+$-action, i.e. an appropriate map

$$B^+ \times_X E \to E$$

over $X$.

A singularity class $\Sigma$ defines the subset $\Sigma(W, \xi) \subset E$, which is fibered over $X$. Moreover, $\Sigma(W, \xi)$ is preserved by the action of $B^+$ since this group consists of holomorphic symplectomorphisms preserving $W$:

$$\begin{array}{ccc}
\Sigma(W, \xi) & \subset & E \\
\circ & \to & LG(V, \omega) \\
B^+ & \to & X
\end{array}$$

We now state the main technical result of the present paper.

**Theorem 6** Fix $I \subset \rho$ and $\lambda$. Suppose that the vector bundle $E$ is globally generated. Then, in $E$, the intersection of $\Sigma(W, \xi)$ with the closure of any $\pi^{-1}(Z^\lambda_I)$ is represented by a nonnegative cycle.

(Of course, it is sufficient to assume that $F$ on $X$ is globally generated.) This result is, in fact, true for any effective $B^+$-invariant cycle on $E$, fibered over $X$, taken instead of $\Sigma(W, \xi)$. Its proof is modelled on the techniques of [8].

Before proving the theorem, we shall establish some preliminary result.

For a subset $Y \subset E$ and a global section $s \in H^0(E)$, we denote by $s + Y$ the fiberwise translate of $Y$ by $s$.

Fix a stratum $\Omega = \Omega_J(F^+_\bullet, \xi)$, and set

$$\Xi := \Sigma(W, \xi) \cup \pi^{-1}\Omega.$$ 

We need the following lemma which is a variant of [8, Lemma 1].

**Lemma 7** Let $Y \subset E_{\mid \Omega}$ be a subvariety. Then there exists an open, dense subset $U \subset H^0(E)$ such that for any section $s \in U$, the translate $s + \Xi$ has proper intersection with $Y$.

**Proof.** Let

$$H^0(E) \times \Xi \to E_{\mid \Omega}$$

be the fiberwise translate.

We claim that $q$ is flat (in fact, it is a fibration). The question is local. We find an open set $X' \subset X$ such that the bundles $\alpha_{\mid X'}$ and $\xi_{\mid X'}$ are trivial. Over $X'$ the set $\Xi$ is the product of $X'$ and some variety. Therefore,

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3This means that it is a Zariski locally trivial fibration over $X$. 

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it is enough to assume that $X$ is a point. Further, note that $\Xi \to \Omega$ is a fibration since $\Omega$ is homogeneous with respect to $B^+$ and $\Xi$ is a $B^+$-invariant subset of $E$. Thus the question reduces to a single fiber of $E$. The fiber is homogeneous with respect to the action of $H^0(E)$, and for $x \in X$ the map

$$H^0(E) \times \Xi_x \to H^0(E) \to E_x,$$

is clearly a fibration.

Applying [8, Lemma 1], the assertion of the lemma follows. □

**Proof of Theorem 6**

For a strict partition $J \subset \rho$, we set

$$E_J := \pi^{-1}(\Omega_J(P^+_\ast, \xi)).$$

Applying Lemma 7, we get an open dense subset $U_J \subset H^0(E)$ such that for $s \in U_J$ the intersection

$$(s + (\Sigma(W, \xi) \cap E_J)) \cap (\pi^{-1}(\overline{Z_{I\lambda}}) \cap E_J)$$

is proper inside $E_J$. We now pick

$$s \in \bigcap_{\text{strict } J \subset \rho} U_J,$$

and set

$$\Sigma(W, \xi)' := s + \Sigma(W, \xi).$$

Since the cycle $\pi^{-1}(\overline{Z_{I\lambda}})$ is transverse to the stratification $\{E_J\}_{\text{strict } J \subset \rho}$ of $E$, an easy dimension count shows that $\pi^{-1}(\overline{Z_{I\lambda}})$ intersects properly $\Sigma(W, \xi)'$ in $E$.

Theorem 6 now follows since by [4, §8.2] the intersection

$$[\Sigma(W, \xi)] \cdot [\pi^{-1}(\overline{Z_{I\lambda}})] = [\Sigma(W, \xi)'] \cdot [\pi^{-1}(\overline{Z_{I\lambda}})]$$

is represented by a nonnegative cycle. □

We record the following corollary. Let

$$\iota : Leg(W, \xi) \to \mathcal{C}^k(W, \xi)$$

be the zero section, and

$$\iota^* : H^*(\mathcal{C}^k(W, \xi), \mathbb{Z}) \to H^*(Leg(W, \xi), \mathbb{Z})$$

be the induced map on cohomology. We write

$$\iota^* PD[\Sigma(W, \xi)] := \sum_{I, \lambda} \gamma_{I\lambda}[\overline{Z_{I\lambda}}]^*,$$

where $\gamma_{I\lambda}$ are integers.
Corollary 8 If $F$ is globally generated, then $\gamma_{I\lambda}$ are nonnegative.

Proof. We have

$$\gamma_{I\lambda} = \langle \iota^* PD[\Sigma(W, \xi)], [Z_{I\lambda}] \rangle = [\Sigma(W, \xi)] \cdot [Z_{I\lambda}] \geq 0$$

by Theorem 6. □

5 A family of bases in which any Legendrian Thom polynomial has positive expansion

We shall apply Theorem 6 in the situation when all $\alpha_i$ are equal to the same line bundle $\alpha$ (i.e. $W = \alpha^\oplus n$) and $\alpha^{-m} \otimes \xi$ is globally generated for $m \geq 3$.

Example 9 We shall consider the following three cases: the base is always $X = \mathbb{P}^n$ and

1. $\xi_1 = \mathcal{O}(-2), \quad \alpha_1 = \mathcal{O}(-1)$,

2. $\xi_2 = \mathcal{O}(1), \quad \alpha_2 = 1$,

3. $\xi_3 = \mathcal{O}(-3), \quad \alpha_3 = \mathcal{O}(-1)$,

These cases were for the authors the key examples supporting their working conjecture that the assertion of Theorem 10 holds true.

Case 1 was the subject of [14, Remark 14], where the basis related to the distinguished cell decomposition of $\text{Leg}(W_1, \xi_1)$ was investigated. In this case, for degrees $\leq n$, the cohomology $H^*(\text{Leg}(W_1, \xi_1), \mathbb{Z}[\frac{1}{2}])$ is isomorphic to the ring of Legendrian characteristic classes tensored by $\mathbb{Z}[\frac{1}{2}]$.

In Case 2, the integral cohomology $H^*(\text{Leg}(W_2, \xi_2), \mathbb{Z})$ is isomorphic to the ring of Legendrian characteristic classes up to degree $n$. The distinguished cell decomposition of $\text{Leg}(W_2, \xi_2)$ gives us another basis of cohomology.

In Case 3, the cohomology of $\text{Leg}(W_3, \xi_3)$ is isomorphic, up to degree $n$, to the ring of Legendrian characteristic classes, provided we invert the number three this time.

The positivity property in Case 1 was known to us (see [14, Remark 14]), whereas in Cases 2 and 3, it was Kazarian who suggested the positivity. His conjecture was supported by computation of all the Thom polynomials up to degree seven.

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In general, $H^*(\text{Leg}(W, \xi), \mathbb{Q})$ is isomorphic to the ring of Legendrian characteristic classes up to degree $\dim W$ if $\xi$ is nontrivial. The case of $W = W^{(p,q)}$ and $\xi = \xi^{(p,q)}$, where
\[
\xi^{(p,q)} = \xi_2^{\otimes p} \otimes \xi_3^{\otimes q} \quad \text{and} \quad a = a^{(p,q)} = a_2^{\otimes p} \otimes a_3^{\otimes q} = a_3^{\otimes q}
\]
($p, q$ are integers), will be used in Section 6.

To overlap all these three cases we consider the product
\[
X := \mathbb{P}^n \times \mathbb{P}^n
\]
and set
\[
W := p_1^* \mathcal{O}(-1)^{\oplus n}, \quad \xi := p_1^* \mathcal{O}(-3) \otimes p_2^* \mathcal{O}(1),
\]
where $p_i : X \to \mathbb{P}^n$, $i = 1, 2$, are the projections. Restricting the bundles $W$ and $\xi$ to the diagonal, or to the factors we obtain the three cases considered above.

The space $\text{Leg}(W, \xi)$ has a distinguished cell decomposition $Z^{-I}_I \lambda$ where $I$ runs over strict partitions contained in $\rho$, and $\lambda = (a, b)$ with $a$ and $b$ natural numbers smaller than or equal to $n$. The classes of closures of the cells of this decomposition give a basis of the homology of $\text{Leg}(W, \xi)$. The dual basis of cohomology is denoted by
\[
e_{I,a,b} = [Z^{-I}_I(a,b)]^\ast.
\]
From geometric reasons it is clear that the basis $\{e_{I,a,b}\}$ consists of Poincaré duals $\text{PD}[Z^+_{I(a,b)}]$ of the $Z^+_{I(a,b)}$'s.

Let $v_1$ and $v_2$ be the first Chern classes of $p_1^*(\mathcal{O}(1))$ and $p_2^*(\mathcal{O}(1))$.

By the definition of $Z^+_{I(a,b)}$ in $H^*(\text{Leg}(W, \xi), \mathbb{Z})$, we have
\[
e_{I,a,b} = e_{I,0,0} v_1^a v_2^b.
\]
Moreover, we have
\[
e_{I,0,0} = \text{PD}[\Omega_I(F^+, \xi)].
\]

With $X$, $W$ and $\xi$ as in (7) and (8), we have:

**Theorem 10** Let $\Sigma$ be a Legendre singularity class. Then $\text{PD}[\Sigma(W, \xi)]$ has nonnegative coefficients in the basis $\{e_{I,a,b}\}$.

**Proof.** We apply Corollary 8. In its notation, the vector bundle $F$ on $X = \mathbb{P}^n \times \mathbb{P}^n$ is equal to
\[
F = \bigoplus_{j=3}^{k+1} \text{Sym}^j(W^*) \otimes \xi = \bigoplus_{j=3}^{k+1} \text{Sym}^j(1^n) \otimes p_1^* \mathcal{O}(j-3) \otimes p_2^* \mathcal{O}(1).
\]
Since $F$ is globally generated, the assertion of the theorem follows from Corollary 8, where $\lambda$ runs over all pairs $(a, b)$ of natural numbers less than or equal to $n$. □
6 The parameter $p/q$ and the basis for $p = q = 1$

In algebraic expressions of the present paper, we shall use $\tilde{Q}$-functions of [16] and their geometric interpretation from [15]. The reader can find in [14, Sect.3] a summary of their properties in the notation which will be also used here.

Fix a Legendre singularity class $\Sigma$. By Theorem 10, we know that the Thom polynomial of $\Sigma$, evaluated at the Chern classes of $A = W^* \otimes \xi - R$ and $c_1(\xi) = v_2 - 3v_1$, is a nonnegative $\mathbb{Z}$-linear combination of the following form:

$$T^\Sigma = \sum_{I,a,b} \gamma_{I,a,b} e_{I,a,b} = \sum_{I,a,b} \gamma_{I,a,b} PD[\Omega_I(F^+, \xi)]v_1^a v_2^b.$$

We want to find an additive basis of the ring of Legendrian characteristic classes with the property that any Legendrian Thom polynomial is a nonnegative combination of basis elements. To this end, we take a geometric model of the classifying space: $\text{Leg}(W^{(p,q)}, \xi^{(p,q)})$ (see the previous section), and the desired basis is given by the cohomology basis Kronecker dual to the $\mathbb{Z}^-$-basis. More precisely, dividing the cohomology ring $H^*(\text{Leg}(W, \xi), \mathbb{Q})$ by the relation

$$q \cdot v_1 = p \cdot v_2,$$

that is specializing the parameters to $v_1 = p \cdot t$, $v_2 = q \cdot t$, we obtain the ring $H^*(\text{Leg}(W^{(p,q)}, \xi^{(p,q)}), \mathbb{Q})$ isomorphic to the ring of Legendrian characteristic classes in degrees up to $n$ (provided that $c_1(\xi) = v_2 - 3v_1$ is not specialized to 0 and $(p, q) \neq (0, 0)$.)

From Theorem 10, we obtain:

**Theorem 11** If $p$ and $q$ are nonnegative, $q - 3p \neq 0$ and $(p, q) \neq (0, 0)$ then the Thom polynomial is a nonnegative combination of the $PD[\Omega_I(F^+, \xi)]^t$'s.

The family $PD[\Omega_I(F^+, \xi)]^t$ is a one-parameter family of bases depending on the parameter $p/q$.

Though the main theme of the present paper is the existence of one-parameter family of bases in which every Legendrian Thom polynomial has positive expansion, we shall give also some results on algebraic form of the $PD[\Omega_I(F^+, \xi)]^t$'s.

First, we come back to Case 1 from Example 9. This corresponds to fixing the parameter to be 1, i.e. $p = 1$ and $q = 1$. This corresponds to setting $v_1 = v_2 = t$. Geometrically, this means that we study the restriction of the bundles $W$ and $\xi$ to the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$, or, we study $W_1 = W_{p+1}$.
$O(-1)^{\oplus n}$ and $\xi = \xi_1 = O(-2)$. Set $\zeta := O(-1)$. Then $\xi^{\frac{1}{2}} = \zeta$. From (6), we have

$$A^* \otimes \xi^{\frac{1}{2}} + A \otimes \xi^{-\frac{1}{2}} = 0. \quad (11)$$

In general, when the bundle $\xi$ admits a square root $\zeta$ it is convenient to give another description of the space $\text{Leg}(W, \xi)$. Let us define $W' = W \otimes \zeta^{-1}$ and $V' = V \otimes \zeta^{-1}$.

Then $V'$ is equipped with a symplectic form with constant coefficients, and

$$V' = W' \oplus W'^*.$$

In our case,

$$X = \mathbb{P}^n, \quad W = O(-1)^{\oplus n}, \quad \text{and} \quad \xi = O(-2).$$

Then $W'$ and $V'$ become trivial bundles:

$$W' = 1^n, \quad V' = 1^{2n}.$$

We have

$$\text{Leg}(W, \xi) = LG(\mathbb{C}^{2n}) \times \mathbb{P}^n, \quad (12)$$

where $LG(\mathbb{C}^{2n}) = LG(\mathbb{C}^{2n}, \omega)$ and $\omega$ is the standard nondegenerate symplectic form on $\mathbb{C}^{2n}$.

Let $R'$ denote the tautological bundle on $LG(\mathbb{C}^{2n})$. Under the identification (12), $R'$ pulled back from $LG(\mathbb{C}^{2n})$ to $\text{Leg}(W, \xi)$ is equal to $R \otimes \zeta^{-1}$.

We thus have

$$A \otimes \zeta^{-1} = W^* \otimes \zeta - R \otimes \zeta^{-1} = W'^* - R' = 1^n - R' = R'^* - 1^n. \quad (13)$$

The distinguished cell decomposition of $\text{Leg}(W, \xi) = LG(\mathbb{C}^{2n}) \times \mathbb{P}^n$ is of the product form. By the cohomological properties of $\tilde{Q}$-functions ([15]), the basis of cohomology consists of the following functions

$$\tilde{Q}_I(R'^* \cdot t^j) = \tilde{Q}_I(R^* \otimes \xi^{\frac{1}{2}}) \cdot c_1(\xi^{-\frac{1}{2}})^j,$$

where $I$ runs over strict partitions in $\rho$.

In this way, we obtain

**Theorem 12** The Thom polynomial for a Legendre singularity class $\Sigma$ is a combination:

$$T^\Sigma = \sum_{j \geq 0} \sum_I \alpha_{I, j} \tilde{Q}_I(A \otimes \xi^{-\frac{1}{2}}) \cdot t^j, \quad (14)$$

where

$$t = \frac{1}{2} c_1(\xi^*) \in H^2(X, \mathbb{Z}[\frac{1}{2}]),$$

$I$ runs over strict partitions in $\rho$, and $\alpha_{I, j}$ are nonnegative integers.

**Remark 13** We get the result announced in [14, Remark 14], where it should read “$t = \frac{1}{2} c_1(\xi^*)$”, and where we used the notation $L^* - 1^{\dim L}$ for the bundle $A$.  

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7 Formulas for $PD[\Omega I(F^+, \xi)]$

By virtue of Theorem 10, (9) and (10) we are interested in expressions for $PD[\Omega I(F^+, \xi)]$ in the Chern classes of the bundles $\xi, R, W, F^+_i, i = 1, \ldots, n$. When $\xi$ is trivial, such a formula was found in [9, Theorem 1, Corollary 4]. The formula is:

$$PD[\Omega I(F^+, \xi)] = \pm \sum J \tilde{Q}_J(R^*) \sum K \tilde{Q}_{(J \cup K')}(W^*) \det \left( c_{k_p-j'_q}(F^+ p-j'_q) \right).$$

(15)

We refer the reader to [9] for details about the summations and signs. Twisting the bundles $R, W, F^+_i, i = 1, \ldots, n$ by $\xi^{-\frac{1}{2}}$, we get an expression for $PD[\Omega I(F^+, \xi)]$. It is desirable to have it in a close form as a polynomial in the Chern classes of the bundles involved.

Note that in our situation, there is a simplification: $W = \alpha^{\oplus n}$ and the flag $F^+_i$ consists of the partial sums of the line bundle $\alpha$. This should cause simplifications in (15).

Example 14 By [16, Theorem 9.3] (see also [9, Cor. 5]) if $\xi = 1, I = \{h\}$, then

$$PD[\Omega_h(F^+_h, 1)] = c_h(R^* - F^+_{n+1-h}).$$

Hence

$$PD[\Omega_h(F^+_h, \xi)] = c_h(R^* \otimes \xi^{\frac{1}{2}} - F^+_{n+1-h} \otimes \xi^{-\frac{1}{2}})$$

$$= c_h((R^* \otimes \xi - F^+_{n+1-h}) \otimes \xi^{-\frac{1}{2}}).$$

Note that for any virtual bundle $E$ of dimension $h-1$ and for any line bundle $\zeta$ we have $c_h(E \otimes \zeta) = c_h(E)$. Hence

$$PD[\Omega_h(F^+_h, \xi)] = c_h(R^* \otimes \xi - F^+_{n+1-h}).$$

In our situation ($W = \alpha^{\oplus n}$), the above formula can be written in the form

$$PD[\Omega_h(F^+_h, \xi)] = c_h(R^* \otimes \xi - W + \alpha^{\oplus h-1})$$

$$= c_h(W^* \otimes \xi - R + \alpha^{\oplus h-1})$$

$$= c_h(A + \alpha^{\oplus h-1}).$$

Remark 15 In the letter [7] to the last author, Kazarian announces another expression for $PD[\Omega I(F^+, \xi)]$. We quote from [7] (modulo some change of notation):

"Let $u$ be a variable. For a tuple of integers $i_1 > \cdots > i_r > 0$ and a collection of formal series $a^{(k)}_1 = 1 + a^{(k)}_1 + a^{(k)}_2 + \cdots, k = 1, \ldots, r$, we define multivariate twisted Schur $Q$-function as the following formal combination of monomials

$$\tilde{Q}_{i_1, \ldots, i_r}(u; a^{(1)}_1, \ldots, a^{(r)}_r) = \sum_{s_1, \ldots, s_r, k} w_{s_1, \ldots, s_r, k} u^k a^{(1)}_{i_1 + s_1} \cdots a^{(r)}_{i_r + s_r},$$

16"
where the integer coefficients \( w_{s_1, \ldots, s_r, k} \) are determined by the expansion

\[
\sum_{s_1, \ldots, s_r, k} w_{s_1, \ldots, s_r, k} u^k \frac{t_1^{s_1} \cdots t_r^{s_r}}{t_1 + t_2 + \cdots + t_r + u} = \prod_{1 \leq i < j \leq r} \frac{1 - t_i/t_j}{1 + t_i/t_j + u/t_j}.
\]

We present the fraction \( \frac{1 - t_i/t_j}{1 + t_i/t_j + u/t_j} = \frac{t_i - t_j}{t_i + t_j + u} \) in the form that makes its power expansion unambiguous. More rigorously, we take the power expansion of this rational function as \( t_1 \to \infty, t_2/t_1 \to \infty, \ldots, t_n/t_{n-1} \to \infty \).

Then (loc.cit.) the following formula is announced: for \( I = (i_1, \ldots, i_r) \),

\[
\text{PD}[\Omega_I(F^+, \xi)] = \tilde{Q}_{i_1, \ldots, i_r} (c_1(\xi); c_\bullet(V - R - F_{n-i_1+1}^+), \ldots, c_\bullet(V - R - F_{n-i_r+1}^+)).
\]

Note that we have for any \( j \),

\[
V - R - F_j^+ = R^* \otimes \xi - F_j^+.
\]

Hence \( \text{PD}[\Omega_I(F^+, \xi)] \) is equal to

\[
\tilde{Q}_I(c_1(\xi); c_\bullet(R^* \otimes \xi - F_{n-i_1+1}^+), \ldots, c_\bullet(R^* \otimes \xi - F_{n-i_r+1}^+)).
\]

Note that in our situation, we have

\[
R^* \otimes \xi - F_{n-i+1}^+ = R^* \otimes \xi - \alpha^{n-i+1} = R^* \otimes \xi - W + \alpha^{-1} = A + \alpha^{-1}.
\]

and thus \( \text{PD}[\Omega_I(F^+, \xi)] \) is equal to

\[
\tilde{Q}_I(c_1(\xi); c_\bullet(A + \alpha^{-1}), \ldots, c_\bullet(A + \alpha^{r-1})).
\]

**Remark 16** Our interest in the proof of the announced formula (16) is stimulated by the fact that the proof of the parallel formula (15) is based on very tricky and lengthy computations from [12].

### 8 Upper bounds of coefficients

In this section, we shall translate the positivity result in Theorem 10 into restrictions for the coefficients of Thom polynomials.

Let us fix the value of the parameter. It turns out that the nonnegativity of coefficients of the bases for some other values of the parameter can imply upper bounds of the coefficients in the basis for the given parameter. We plan to discuss this more systematically elsewhere. Here we consider the 3 bases in degree two from Cases 1, 2 and 3 in Example 9. We shall call them the first, second and third basis, respectively.

Let \( s = c_1(\xi) \) and let us list the classes of degree two. The first basis of the Legendrian characteristic classes consists of:

\[
c_2(A + \alpha_1) = a_2 + \frac{1}{2} s a_1, \quad \frac{1}{2} c_1(\xi)c_1(A) = -\frac{1}{2} s a_1, \quad \left( -\frac{1}{2} c_1(\xi) \right)^2 = \frac{1}{4} s^2,
\]

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by Example 14, since \( c_1(\xi_1) = -2c_1(\mathcal{O}(1)) \) and \( c_1(\alpha_1) = -c_1(\mathcal{O}(1)) \). The second basis is

\[
c_2(A) = a_2, \quad s c_1(A) = s a_1, \quad s^2
\]
since here \( \alpha_2 = 1 \). The third basis is

\[
c_2(A + \alpha_3) = a_2 + \frac{1}{3} s a_1, \quad -\frac{1}{3} c_1(\xi_3) c_1(A) = -\frac{1}{3} s a_1, \quad \left( -\frac{1}{3} c_1(\xi_3) \right)^2 = \frac{1}{9} s^2,
\]
since \( c_1(\xi_3) = -3c_1(\mathcal{O}(1)) \) and \( c_1(\alpha_3) = -c_1(\mathcal{O}(1)) \).

The Thom polynomial of the singularity \( A_3 \) is of the form

\[
3(a_2 + \frac{1}{2} s a_1) - \kappa \frac{1}{2} s a_1
\]
with \( \kappa \geq 0 \), by the positivity in the first basis. The positivity in the second basis gives the condition

\[
\frac{3}{2} - \frac{1}{2} \kappa \geq 0
\]
that is \( \kappa \leq 3 \). When we write the Thom polynomial in the third basis we see that the coefficient of \(-\frac{1}{3} s a_1\) is equal to \( \kappa - 1 \). It follows that \( \kappa \geq 1 \).

Recall that the only Legendrian Thom polynomial of degree 2 is the one of the singularity \( A_3 \), displayed in the first basis as:

\[
T^{A_3} = 3\tilde{Q}_2 + t\tilde{Q}_1,
\]
i.e. with \( \kappa = 1 \).

Another upper bound for the coefficients of the expansions of Legendrian Thom polynomials can be obtained by the method of Example 21.

9 Legendrian vs. classical Thom polynomials

In this section, we shall use the basis from Case 1 in Example 9, i.e., we put \( t = v_1 = v_2 \).

**Proposition 17** For a nonempty stable Legendre singularity class \( \Sigma \) the Lagrangian Thom polynomial (i.e. \( T^{\Sigma} \) evaluated at \( t = 0 \)) is nonzero.

**Proof.** For a Legendre singularity class \( \Sigma \) consider the associated singularity class of maps \( f : M \to C \) from \( n \)-dimensional manifolds to curves (see [5, p.729] and [6, p.123]). We denote the related Thom polynomial by \( T^{\Sigma} \).

According to [5, pp. 708–709], we have

\[
T^{\Sigma} = T^{\Sigma} \cdot c_n(T^*M \otimes f^*TC).
\]
(17)

We know by [17, Theorem 4] that the Thom polynomial \( T^{\Sigma} \) is nonzero. Moreover, it follows from the proof (loc.cit.) that \( T^{\Sigma} \), specialized with \( f^*TC = 1 \) i.e. \( t = 0 \), is also nonzero. The assertion follows from the equation (17). \( \Box \)

Consequently, we get an improvement of Theorem 10.
Corollary 18 For a nonempty stable Legendre singularity class \( \Sigma \) the (Legendrian) Thom polynomial \( T^\Sigma \) is nonzero.

Remark 19 Suppose that we are in the situation of (17). We shall use the expansions of Legendrian Thom polynomials in the basis for \( v_1 = v_2 = t \), studied in the previous section. Set \( \xi := f^*TC \). If \( A = T^*M \otimes \xi - TM \), having the Thom polynomial presented as in (14), we want to compute the \( \tilde{Q} \)-functions of

\[
A \otimes \xi^{-\frac{1}{2}} = (T^*M \otimes \xi - TM)^* \otimes \xi^{-\frac{1}{2}} = E^* - E,
\]

where \( E = TM \otimes \xi^{-\frac{1}{2}} \). Since for every strict partition \( I \),

\[
\tilde{Q}_I(E^* - E) = Q_I(E^*),
\]

where \( Q_I \) denotes the classical Schur \( Q \)-function [18], we get the desired expression by changing any \( \tilde{Q}_I(A \otimes \xi^{-\frac{1}{2}}) \) to \( Q_I(E^*) \).

The following procedure mimics the passing from the LHS to RHS in Eq. (17), where \( T^\Sigma \) is given as a \( \mathbb{Z}[t] \)-combination of the \( \tilde{Q}_I(A \otimes \xi^{-\frac{1}{2}}) \)'s, and \( T^\Sigma \) is to be written as a \( \mathbb{Z} \)-combination of the Schur functions \( S_J(T^*M - \xi^*) \) (for Schur functions in virtual bundles we refer to [11], and for Schur \( Q \)-functions to [15]).

Procedure 20 We start from a \( \mathbb{Z}[t] \)-combination of the polynomials discussed in Remark 19: \( \tilde{Q}_I(A \otimes \xi^{-\frac{1}{2}}) = Q_I(T^*M \otimes \xi^{\frac{1}{2}}) \).

- We write \( Q_I(T^*M \otimes \xi^{\frac{1}{2}}) \) as a combination of the \( S_J(T^*M \otimes \xi^{\frac{1}{2}}) \)'s (here we use a combinatorial rule from [19] decomposing Schur \( Q \)-functions into \( S \)-functions);
- we expand any \( S_J(T^*M \otimes \xi^{\frac{1}{2}}) \) as a combination of the \( S_K(T^*M)t^i \)'s (here we use a formula from [10] for decomposition of the Schur polynomials of twisted bundles);
- we multiply the obtained combination by \( c_n(T^*M \otimes \xi) \) (here we use the factorization formula for Schur functions from [3]); we eventually get a combination of the \( S_L(T^*M - \xi^*) \)'s with the coefficients being polynomials in \( n \).

Example 21 We shall examine now how positivity of Schur function expansions of Thom polynomials for mappings \( \mathbb{C}^n \rightarrow \mathbb{C} \), proved in [17], implies some upper bounds on the coefficients of a Legendrian Thom polynomial in the expansion (14).

Let us consider a degree 2 cohomology class already considered in Section 8 of the form

\[
3\tilde{Q}_2 + \kappa t\tilde{Q}_1,
\]

(18)
where \( \kappa \) is an integer.

We fix \( n \geq 2 \). We apply Procedure 20 to (18):

\[
(3\tilde{Q}_2 + \kappa t\tilde{Q}_1)(T^*M \otimes \xi^1) \cdot c_n(T^*M \otimes \xi) =
(6(S_2 + S_1^2) + 2\kappa tS_1)(T^*M \otimes \xi^1) \cdot c_n(T^*M \otimes \xi) =
(6(S_2 + S_1^2) + 2t(\kappa - n)S_1 + t^2n(n - 2\kappa))(T^*M) \cdot c_n(T^*M \otimes \xi)
\]

By the factorization formula, the last expression is equal to

\[
6(S_1^{n-1} + S_1^{n-2}) + (6n - \kappa)S_1n^2 + \left(\frac{3}{2}n^2 - \kappa \frac{n}{2}\right)S_1n^2 + 2
\]

evaluated at \( T^*M - \xi^* \).

Suppose that for each \( n \geq 2 \) we have in (19) a nonnegative combination of Schur functions, i.e.

\[
6n - \kappa \geq 0 \quad \text{and} \quad \frac{3}{2}n^2 - \kappa \frac{n}{2} \geq 0 .
\]

This implies that \( \kappa \leq 6 \).

### 10 Examples of Legendrian Thom polynomials

The Thom polynomials expanded in the basis \( \{ e_{I,a,b} \} \) (see Section 5) are (the summands in bold represent the Lagrangian Thom polynomials):

- **A\(_2\)**: \( \tilde{Q}_1 \)
- **A\(_3\)**: \( 3\tilde{Q}_2 + v_2\tilde{Q}_1 \)
- **A\(_4\)**: \( 12\tilde{Q}_3 + 3\tilde{Q}_{21} + (3v_1 + 7v_2)\tilde{Q}_2 + (v_1v_2 + v_2^2)\tilde{Q}_1 \)
- **D\(_4\)**: \( \tilde{Q}_{21} \).

The last equation means that the Thom polynomial of the singularity \( D_4 \) written in all bases from the family is equal to \( \tilde{Q}_{21} \).

Similarly the Thom polynomial of the singularity \( P_8 \) in all bases from the family is equal to \( \tilde{Q}_{321} \). Next we have

- **A\(_5\)**: \( 60\tilde{Q}_4 + 27\tilde{Q}_{31} + (6v_1 + 16v_2)\tilde{Q}_{21} + (39v_1 + 47v_2)\tilde{Q}_3 + (6v_1^2 + 22v_1v_2 + 12v_2^2)\tilde{Q}_2 + (2v_1^2v_2 + 3v_1v_2^2 + v_2^3)\tilde{Q}_1 \)
- **D\(_5\)**: \( 6\tilde{Q}_{31} + 4v_2\tilde{Q}_{21} \),

and analogously to \( D_5 \),

- **P\(_9\)**: \( 12\tilde{Q}_{421} + 12v_2\tilde{Q}_{321} \).

Let us specialize \( v_1 = v_2 = t \). The Thom polynomials for singularities of codimensions lower or equal to six are listed in \([14]\). Here are the Legendrian Thom polynomials for the consecutive singularities \( A_8, D_8, E_8, X_9, P_9 \) displayed in the notation from (14). Again, the summands in bold represent the Lagrangian Thom polynomials. The formulas are due to Kazarian, \([7]\).
The lower codimensional classes are displayed in [14], where the expression for $A_7$ should read:

$$A_7 : \quad 135\tilde{Q}_{321} + 1275\tilde{Q}_{42} + 2004\tilde{Q}_{51} + 2520\tilde{Q}_{6} + t(7092\tilde{Q}_{5} + 4439\tilde{Q}_{41} + 1713\tilde{Q}_{32}) + t^2(3545\tilde{Q}_{31} + 7868\tilde{Q}_{4}) + t^3(1106\tilde{Q}_{21} + 4292\tilde{Q}_{3}) + t^41148\tilde{Q}_{2} + t^5120\tilde{Q}_{1}.$$
Inspired by the paper [1] of Anderson, we consider some “large group” action. Assume that 
\[ \alpha_i^* \otimes \alpha_j \quad \text{for} \quad i < j \quad \text{and} \quad \alpha_i^* \otimes \alpha_j^* \otimes \xi \quad \text{for all} \quad i, j \]
are globally generated. Consider the group \( \Gamma B^- \) of global sections of the bundle \( B^- \to X \).

**Lemma 22** For each point \( x \in X \), the restriction map from \( \Gamma B^- \) to the fiber \( B^-_x \) is surjective.

**Proof.** The group \( B^-_x \) is generated by two subgroups:

- \( B^-_W \): the automorphisms of \( W \) inducing automorphisms of \( W^* \otimes \xi \) which preserve the flag \( F^-_{\bullet} \),

- \( N^- \): the maps \( W \to W^* \otimes \xi \) which belong to 
  \[ \text{Sym}^2(W^*) \otimes \xi \subset W^* \otimes W^* \otimes \xi = Hom(W, W^* \otimes \xi). \]

We identify the elements of \( B^-_W \) with matrix \( \{ b_{ij} \} \), whose entries over \( x \in X \) belong to the fiber 
\[ Hom(\alpha_i, \alpha_j)_x = (\alpha_i^* \otimes \alpha_j)_x \]
for \( i \leq j \), or are zero for \( i > j \). The group bundle \( B^-_W \) is generated by the global sections. Similarly, the group bundle \( N^- \) is a quotient bundle of 
\[ Hom(W, W^* \otimes \xi) \cong W^* \otimes W^* \otimes \xi \]
isomorphic to \( \text{Sym}^2(W^*) \otimes \xi \); therefore it is globally generated. This proves the lemma. \( \square \)

For a strict partition \( J \subset \rho \), let us denote by \( \Omega^-_J \) the space of the stratum \( \Omega_J(F^-_{\bullet}, \xi) \to X \).

From the lemma, it follows

**Corollary 23** The group \( \Gamma B^- \) acts on \( LG(V, \omega) \), preserving fibers, and in each fiber its orbits coincide with the strata of the stratification \( \{ \Omega^-_J \} \).

Assume that \( X \) is homogeneous with respect to a linear group \( G \) and the transformation group acts on the line bundles \( \xi \) and \( \alpha_i \). For instance, \( X \) is a product of projective spaces, and each line bundle involved is a tensor product of the \( O(j) \)'s.

We define \( H \) to be the subgroup of \( \text{Aut}(LG(V, \omega)) \) generated by \( \Gamma B^- \) and \( G \) (it is the semidirect product of these groups). The variety \( H \) is irreducible.

From the above, it follows
Lemma 24 The group $H$ acts transitively on each stratum $\Omega_J$: $G$ transports any fiber to any other fiber, and $\Gamma B^-$ acts transitively inside the fibers.

We now state the following positivity result.

Theorem 25 The intersection of any nonnegative cycle on $LG(V, \omega)$ with any $\overline{Z_{I\lambda}^+}$ is represented by a nonnegative cycle.

Proof. Let $Y$ be a nonegative cycle on $LG(V, \omega)$. We shall find a translate $h \cdot Y$ by an element $h \in H$ which is transverse to any $Z_{I\lambda}^+$.

By Lemma 24, we can use the Bertini-Kleiman transversality theorem [8] for $H$ acting on $\Omega_J$. By this theorem, there exists an open, dense subset $U_{J\lambda} \subset H$ with the following property: if $h \in U_{J\lambda}$ then $h \cdot (Y \cap \Omega_J)$ is transverse to $Z_{I\lambda}^+ \cap \Omega_J$. Set

$$U_J := \bigcap_{I, \lambda} U_{J\lambda}.$$  

We get an open, dense subset $U_J \subset H$ with the following property: if $h \in U_J$, then $h \cdot (Y \cap \Omega_J)$ is transverse to any $Z_{I\lambda}^+ \cap \Omega_J$ (transversality in $\Omega_J$). Since $\Omega_J$ is transverse to all strata $Z_{I\lambda}^+$ of $LG(V, \omega)$, this transversality holds also in the whole ambient space. Set

$$U := \bigcap_{\text{strict } J \subset \rho} U_J.$$  

Pick $h \in U$. Then $Y' = h \cdot Y$ is transverse to all the $Z_{I\lambda}^+$'s.

The theorem now follows since by [4, Sect. 8.2] the intersection

$$Y' \cdot \overline{Z_{I\lambda}^+}$$

is represented by a nonnegative cycle.  \(\square\)

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