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GENERAL BOUNDARY VALUE PROBLEM
FOR AN INTEGRODIFFERENTIAL SYSTEM AND ITS ADJOINT

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(Continuation)**)

4. WEAKLY NONLINEAR BOUNDARY VALUE PROBLEM

Notation. Given a B-space \( B \) with the norm \( \| \cdot \|_B \), \( u_0 \in B \) and \( Q > 0 \), the set \( \{ u \in B : \| u - u_0 \|_B \leq Q \} \) is denoted by \( B(u_0, Q; B) \).

Definition 4.1. Let \( B_1, B_2 \) be B-spaces and let \( \varepsilon_0 > 0 \). An operator \( F : u \in B_1, \varepsilon \in [0, \varepsilon_0] \rightarrow F(\varepsilon)(u) \in B_2 \) is said to be locally lipschitzian in \( u \) near \( \varepsilon = 0 \) if, given an arbitrary \( u_0 \in B_1 \), there exist \( \alpha(u_0) > 0 \), \( Q(u_0) > 0 \) and \( \varepsilon(u_0) > 0 \) such that

\[
\| F(\varepsilon)(u_2) - F(\varepsilon)(u_1) \|_{B_2} \leq \alpha(u_0) \| u_2 - u_1 \|_{B_1},
\]

for all \( u_1, u_2 \in B(u_0, Q(u_0); B_1) \) and \( \varepsilon \in [0, \varepsilon(u_0)] \).

Hereafter we suppose

(\( \mathcal{A} \)) \hspace{1cm} A \in L^1_{n,n}, \hspace{1cm} G \in L^2[\mathbb{R}^n], \hspace{1cm} L \in \mathcal{B}V_{n,n} \ (m = n).

The mappings

\[
\Phi : x \in \mathcal{A}C, \hspace{1cm} \varepsilon \in [0, \varepsilon_0] \rightarrow \Phi(\varepsilon)(x) \in L^1,
\]

\[
\Lambda : x \in \mathcal{A}C, \hspace{1cm} \varepsilon \in [0, \varepsilon_0] \rightarrow \Lambda(x) \in \mathcal{B}_n
\]

are locally lipschitzian in \( x \) near \( \varepsilon = 0 \) and continuous in \( \varepsilon \in [0, \varepsilon_0] \) for any \( x \in \mathcal{A}C \) fixed, \( \varepsilon_0 > 0 \).

*) The last paragraph (§ 5) was added.
**) The first part was published in this Cas. pest. mat. 97 (1972), 399—419.
Let us consider the weakly nonlinear boundary value problem \((\mathcal{P}_\varepsilon)\)

\[
\begin{align*}
\dot{x} &= A(t) x + \int_a^b [d_s G(t, s)] x(s) + \varepsilon \Phi(\varepsilon) (x)(t), \\
\int_a^b [dL(s)] x(s) + \varepsilon \Lambda(\varepsilon)(x) &= 0,
\end{align*}
\]

where \(\varepsilon \geq 0\) is a small parameter.

We proceed formally as in § 3 and write the problem \((\mathcal{P}_\varepsilon)\) in the equivalent form as the system of equations for \(x \in \mathcal{A}^e, h \in \mathcal{L}^2\) and \(c \in \mathcal{R}_a\)

\[
\begin{align*}
-x(t) + X(t) c + \int_a^t X(t) X^{-1}(s) h(s) ds + \varepsilon P_0(\varepsilon)(x)(t) &= 0, \\
-h(t) + H_1(t) c + \int_a^b K(t, s) h(s) ds + \varepsilon P_1(\varepsilon)(x)(t) &= 0, \\
Cc + \int_a^b H_2(s) h(s) ds + \varepsilon P_2(\varepsilon)(x) &= 0,
\end{align*}
\]

where \(X(t)\) has the same meaning as before \(((3,3))\) and

\[
\begin{align*}
H_1(t) &= \int_a^b [d_s G(t, s)] X(s), \\
H_2(t) &= \left(\int_t^b [dL(s)] X(s)\right) X^{-1}(t) = A(\varepsilon)(x) + \int_a^b [dL(s)] X(s) \\
K(t, s) &= \left(\int_s^b [d_s G(t, \sigma)] X(\sigma)\right) X^{-1}(s), \\
C &= \int_a^b [dL(s)] X(s), \\
P_0(\varepsilon)(x)(t) &= X(t) \int_a^t X^{-1}(s) \Phi(\varepsilon)(x)(s) ds, \\
P_1(\varepsilon)(x)(t) &= \int_a^b [d_s G(t, s)] \left(X(s) \int_s^a X^{-1}(\sigma) \Phi(\varepsilon)(x)(\sigma) d\sigma\right) = \\
&= \int_a^b \left(\int_s^a [d_s G(t, \sigma)] X(\sigma)\right) X^{-1}(s) \Phi(\varepsilon)(x)(s) ds = \int_a^b K(t, s) \Phi(\varepsilon)(x)(s) ds, \\
P_2(\varepsilon)(x) &= \Lambda(\varepsilon)(x) + \int_a^b [dL(s)] \left(X(s) \int_s^a X^{-1}(\sigma) \Phi(\varepsilon)(x)(\sigma) d\sigma\right) = \\
&= \Lambda(\varepsilon)(x) + \int_a^b [dL(s)] X(s) X^{-1}(s) \Phi(\varepsilon)(x)(s) ds = \\
&= \Lambda(\varepsilon)(x) + \int_a^b H_2(s) \Phi(\varepsilon)(x)(s) ds.
\end{align*}
\]
By assumptions of this paragraph $K \in \mathcal{L}_2$, $H_1 \in \mathcal{L}^2_{n,a}$ and $P_0$, $P_1$ and $P_2$ are mappings of $\mathcal{A} \times [0, \varepsilon_0]$ into $\mathcal{A}$, $\mathcal{L}^2$ and $\mathcal{R}_n$, respectively, locally lipschitzian in $x$ near $\varepsilon = 0$ and continuous in $\varepsilon \in [0, \varepsilon_0]$ for any $x \in \mathcal{A}$ fixed. For example, in the case of $P_1$ we have for $x_1, x_2 \in \mathcal{A}$, $t \in J$ and $\varepsilon_1, \varepsilon_2 \in [0, \varepsilon_0]$

$$\|P_1(\varepsilon_2)(x_2)(t) - P_1(\varepsilon_1)(x_1)(t)\| \leq \beta \varrho^b G(t, \cdot) \|\Phi(\varepsilon_2)(x_2) - \Phi(\varepsilon_1)(x_1)\|_1,$$

where $\beta = \sup_{t, \varepsilon \in J}\|X(t)X^{-1}(s)\|$. Hence

$$\|P_1(\varepsilon_2)(x_2) - P_1(\varepsilon_1)(x_1)\|_2 \leq \alpha \|\Phi(\varepsilon_2)(x_2) - \Phi(\varepsilon_1)(x_1)\|_1,$$

where

$$\alpha = \beta \varrho^b G(t, \cdot)\|_2.$$

Let $K_0 \in \mathcal{L}_2$, $K_1 \in \mathcal{L}^2_{n,a}$ and $K_2 \in \mathcal{L}^2_{n,a}$ be again such that $K(t, s) = K_0(t, s) + K_1(t)K_2(s)$, $\|\|K_0\|| < 1$. Let $\Gamma$ be the resolvent kernel of $K_0$ and let $\hat{H}_1$ and $\hat{K}_1$ be again defined by (3,10). ($\Gamma \in \mathcal{L}_2$, $\hat{H}_1 \in \mathcal{L}^2_{n,a}$ and $\hat{K}_1 \in \mathcal{L}^2_{n,a}$, of course.) Then the system (4,3) becomes

$$-x(t) + U(t) b + \varepsilon R_0(\varepsilon)(x)(t) = 0,$$

$$Bb + \varepsilon R(\varepsilon)(x) = 0,$$

where $B$ is given by (4,4), (3,9), (3,10) and (3,12),

$$U(t) = \left(X(t) \left[ I + \int^{t_a} X^{-1}(s) \hat{H}_1(s) ds \right], \ \ X(t) \int_{a}^{t} X^{-1}(s) \hat{K}_1(s) ds \right),$$

$$R_0(\varepsilon)(x)(t) = P_0(\varepsilon)(x)(t) + X(t) \int_{a}^{t} X^{-1}(s) P_1(\varepsilon)(x)(s) ds,$$

$$R(\varepsilon)(x) = \left( \int^{b_a} \hat{K}_2(s) P_1(\varepsilon)(x)(s) ds \right),$$

$$\hat{H}_2(t) = H_2(t) + \int_{a}^{b} H_2(s) \Gamma(s, t) ds, \ \ \hat{K}_2(t) = K_2(t) + \int_{a}^{b} K_2(s) \Gamma(s, t) ds,$$

$$h(t) = \hat{H}_1(t) c + \hat{K}_1(t) d + \varepsilon \left[ P_1(\varepsilon)(x)(t) + \int_{a}^{b} \Gamma(t, s) P_1(\varepsilon)(x)(s) ds \right],$$

$$d = \int_{a}^{b} K_2(s) h(s) ds, \ \ b = (c', d')'.$$

Clearly, $U(t)$ is absolutely continuous on $J$, $\hat{H}_2 \in \mathcal{L}^2_{n,a}$, $\hat{K}_2 \in \mathcal{L}^2_{n,a}$, $R_0$ and $R$ are mappings of $\mathcal{A} \times [0, \varepsilon_0]$ into $\mathcal{A}$ and $\mathcal{R}_{n+a}$, respectively, locally lipschitzian in $x$ near $\varepsilon = 0$ and continuous in $\varepsilon \in [0, \varepsilon_0]$ for any $x \in \mathcal{A}$ fixed.
The further investigation of our problem rather depends on whether $\det B \neq 0$ or $\det B = 0$. In the former simple (so called noncritical) case the following theorem holds.

**Theorem 4.1.** Let the boundary value problem $(\mathcal{P}_\epsilon)$ be given and let the assumptions $(\mathcal{A})$ be fulfilled. Let the limit problem $(\mathcal{P}_0)$ have only the trivial solution. Then there exists $\epsilon^* > 0$ such that for any $\epsilon \in [0, \epsilon^*]$ there exists a unique solution $x_\epsilon$ of $(\mathcal{P}_\epsilon)$, while $\|x_\epsilon\|_{\mathcal{A}^c} \to 0$ for $\epsilon \to 0^+$.

**Proof.** Let $(\mathcal{P}_0)$ have only the trivial solution. Then by Corollary 1 of Theorem 3.1

$$x(t) = \varepsilon [R_0(\varepsilon)\, (x) (t) - U(t)\, B^{-1} \, R(\varepsilon) \, (x)] = \varepsilon T(\varepsilon) \, (x) \, (t).$$

It follows immediately from the above argument that the operator $T : \mathcal{A}^c \times \times [0, \varepsilon_0] \to \mathcal{A}^c$ is locally lipschitzian in $x$ near $\varepsilon = 0$ and continuous in $\varepsilon \in [0, \varepsilon_0]$ for any $x \in \mathcal{A}^c$ fixed. Hence the fixed point theorem for contractive operators ([8]) can be applied.

**Remark 4.1.** The given boundary value problem $(\mathcal{P}_\epsilon)$ is certainly noncritical e.g. if in (4.3)

a) $\det C \neq 0$ and $1$ is not an eigenvalue of $K(t, s) - H_1(t) \, C^{-1} \, H_2(s)$,

b) $1$ is not an eigenvalue of $K$ and

$$\det \left(C + \int_a^b H_2(s) \left[H_1(s) + \int_a^b Q(s, \sigma) \, H_1(\sigma) \, d\sigma \right] \, ds\right) \neq 0,$$

where $Q$ is the resolvent kernel of $K$.

In the critical case $(\det B = 0)$ some further notations are needed.

**Notation.** $\mathcal{N}_0$ denotes the naturally ordered set $\{1, 2, \ldots, n + n'\}$. If $\mathcal{S}$ is a naturally ordered subset of $\mathcal{N}_0$, then $\mathcal{S}^*$ denotes the naturally ordered complement of $\mathcal{S}$ with respect to $\mathcal{N}_0$. The number of elements of a set $\mathcal{S} \subset \mathcal{N}_0$ is denoted by $\gamma(\mathcal{S})$.

Let $C = (c_{i,j})_{i,j \in \mathcal{N}_0}$ be an $(n + n') \times (n + n')$-matrix and let $\mathcal{S} \subset \mathcal{N}_0$, $\mathcal{V} \subset \mathcal{N}_0$, then $C_{\mathcal{S},\mathcal{V}}$ denotes the matrix $(c_{i,j})_{i \in \mathcal{S}, \, j \in \mathcal{V}}$. Similarly if $b$ is an $(n + n')$-vector ($b = (b_j)_{j \in \mathcal{N}_0}$) and $\mathcal{S} \subset \mathcal{N}_0$, then $b_{\mathcal{S}} = (b_j)_{j \in \mathcal{S}}$. (Analogously for matrix or vector functions and operators.) $\mathcal{N}$ denotes the naturally ordered set $\{1, 2, \ldots, n\}$. The sign $+$ is defined by $b = b_{\mathcal{S}} + b_{\mathcal{S}^*}$.

Let $\chi = \text{rank} \,(B) < n + n'$, while

$$\det B_{\mathcal{S}^*,\mathcal{V}^*} \neq 0 \quad \text{and} \quad B_{\mathcal{S},\mathcal{V}_0} - WB_{\mathcal{S}^*,\mathcal{V}_0} = 0,$$

(4.7)
\( v(\mathcal{S}^*) = v(\mathcal{S}^*) = \chi \) and \( W \) is an \((n + n' - \chi) \times \chi\)-matrix. Let us put \( v = n + n' - \chi \), \( B_1 = B_{\mathcal{S}^*, \mathcal{S}^*} B_2 = B_{\mathcal{S}^*, \mathcal{S}^*} \), \( \gamma = b_{\mathcal{S}^*} \) and \( \delta = b_{\mathcal{S}^*} \). Then (4.5) yields

\[
(4.8) \quad \gamma = -B_1^{-1} B_2 \delta - \varepsilon B_1^{-1} R_{\mathcal{S}^*}(\varepsilon)(x) .
\]

Inserting (4.8) and \( b = \gamma + \delta \) into (4.5) we obtain that (4.5) is equivalent to the system of equations for \( x \in \mathcal{A} \) and \( \delta \in \mathcal{A}_v \),

\[
(4.9) \quad -x(t) + V(t) \delta + \varepsilon S(\varepsilon)(x)(t) = 0 ,
T(\varepsilon)(x) = 0 ,
\]

where

\[
(4.10) \quad V(t) = U_{\mathcal{S}^*, \mathcal{S}^*} (t) - U_{\mathcal{S}^*, \mathcal{S}^*} (t) B_1^{-1} B_2 ,
\]

\( S : x \in \mathcal{A} \), \( \varepsilon \in [0, \varepsilon_0] \rightarrow S(\varepsilon)(x) = R_0(\varepsilon)(x) - U_{\mathcal{S}^*, \mathcal{S}^*}(\cdot) B_1^{-1} R_{\mathcal{S}^*}(\varepsilon)(x) \in \mathcal{A} \),

\( T : x \in \mathcal{A} \), \( \varepsilon \in [0, \varepsilon_0] \rightarrow T(\varepsilon)(x) = R_{\mathcal{S}^*}(\varepsilon)(x) - W R_{\mathcal{S}^*}(\varepsilon)(x) \in \mathcal{A}_v \).

\( V(t) \) is absolutely continuous on \( J \) and it is easy to verify that the operators \( S \) and \( T \) have the same smoothness properties as \( \Phi \), \( \Lambda \), \( P_0 \), \( P_1 \) etc.

Let \( \varepsilon > 0 \), then \( x \in \mathcal{A} \) is a solution to the boundary value problem \((\mathcal{P}_\varepsilon)\) iff \((x, \delta)\), where

\[
\delta = b_{\mathcal{S}^*} \quad \text{and} \quad b = \left( \int_a^b K_2(t) \left( \int_a^b [d_4 G(t, s)] x(s) \right) ds \right) = \left( \int_a^b \left[ \int_a^b K_2(s) G(s, t) ds \right] x(t) \right) ,
\]

is a solution to (4.9). (All solutions \( x_0 \) of the limit problem \((\mathcal{P}_0)\) are given by \( x_0(t) = V(t) \delta \), where \( \delta \) is an arbitrary \( v \)-vector.) To investigate further the existence of a solution (and its dependence on \( \varepsilon \)) to \((\mathcal{P}_\varepsilon)\) various principles in accordance with the smoothness of the operators \( \Phi \) and \( \Lambda \) may be used. Below we state two existence theorems which can serve as models. The first one is obtained by the use of the Newton method for equations in \( \mathcal{B} \)-spaces.

**Proposition 1.** Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be \( \mathcal{B} \)-spaces and let \( \varepsilon_0 > 0 \). Let \( \mathcal{U} \subset \mathcal{A}_1 \) and let \( F \) be an operator: \((u, \varepsilon) \in \mathcal{U} \times [0, \varepsilon_0] \rightarrow F(\varepsilon)(u) \in \mathcal{A}_2 \). Let us assume that

(i) the equation \( F(0)(u) = 0 \) possesses a solution \( u_0 \in \mathcal{U} \);

(ii) there exists \( \varepsilon_0 > 0 \) such that \( F \) is continuous in \((u, \varepsilon) \in \mathcal{U}_0 \times [0, \varepsilon_0] = \mathcal{U}(u_0, \varepsilon_0; \mathcal{A}_1) \times [0, \varepsilon_0] \) and for all \((u, \varepsilon) \in \mathcal{U}_0 \times [0, \varepsilon_0] \) possesses a \( \mathcal{G} \)-derivative \( F'_u(\varepsilon)(u) \) with respect to \( u \) which is continuous in \((u, \varepsilon) \in \mathcal{U}_0 \times [0, \varepsilon_0] \);

(iii) \( F'_u(0)(u_0) \) possesses a bounded inverse \([F'_u(0)(u_0)]^{-1}\).
Then there exist \( \varepsilon^* > 0 \) and \( Q^* > 0 \) such that for any \( \varepsilon \in [0, \varepsilon^*] \) the equation \( F(\varepsilon)(u) = 0 \) possesses one and only one solution \( u^*(\varepsilon) \) in \( \mathcal{U}(u_0, Q^*; \mathcal{B}_1) \). The mapping \( \varepsilon \in [0, \varepsilon^*] \to u^*(\varepsilon) \in \mathcal{B}_1 \) is continuous and \( u^*(\varepsilon) \to u_0 \) in \( \mathcal{B}_1 \) if \( \varepsilon \to 0^+ \).

(For the proof see [19], p. 355. Similar theorems are proved also in [8] or [16].)

**Remark 4.1.** Let us notice that the assertion of Proposition 1 can be equivalently reformulated as follows.

There exists \( \varepsilon^* > 0 \) such that for all \( \varepsilon \in [0, \varepsilon^*] \) there exists a unique solution \( u^* = u^*(\varepsilon) \in \mathcal{U}_0 \) of the equation \( F(\varepsilon)(u) = 0 \) continuous in \( \varepsilon \in [0, \varepsilon^*] \) and such that \( u^*(0) = u_0 \).

To be able to apply Proposition 1 to the boundary value problem \((\mathcal{P}_e)\) we have to add some further assumptions concerning the differentiability of \( \Phi \) and \( \Lambda \) to those used until now. It is easy to verify that if \( \mathcal{U} \subset \mathcal{A} \) and \( \Phi \) and \( \Lambda \) are continuous in \( (x, \varepsilon) \in \mathcal{U} \times [0, \varepsilon_0] \) and for all \( (x, \varepsilon) \in \mathcal{U} \times [0, \varepsilon_0] \) possess a G-derivative with respect to \( x \) which is continuous in \( (x, \varepsilon) \in \mathcal{U} \times [0, \varepsilon_0] \), then the same holds also for the operators \( S \) and \( T \).

**Theorem 4.2.** Let the boundary value problem \((\mathcal{P}_e)\) fulfilling the assumptions \((\mathcal{A})\) be given. Let the limit problem \((\mathcal{P}_0)\) admit a nonzero solution (i.e. \( \det B \neq 0 \)). Let the matrix function \( V \) and the operators \( T \) and \( T_0 \) be defined by \((4,7)\), \((4,10)\) and \((4,11)\)

\[
T_0 : \delta \in \mathcal{R}_v \to T_0(\delta) = T(0)(V(\cdot) \delta) \in \mathcal{R}_v.
\]

Suppose

(I) the limit problem \((\mathcal{P}_0)\) possesses a solution \( x_0 \) such that \( T_0(\delta_0) = 0 \) for \( \delta_0 = (b_0)^T \), where

\[
b_0 = \begin{pmatrix} x_0(a) \\
\int_a^b \int_s^b K_2(s, t) G(s, t) \, ds \, x_0(t) \end{pmatrix};
\]

(II) there exists \( Q_0 > 0 \) such that \( \Phi \) and \( \Lambda \) are continuous in \( (x, \varepsilon) \in \mathcal{U}_0 \times [0, \varepsilon_0] = \mathcal{U}(x_0, Q_0; \mathcal{A} \mathcal{E}) \times [0, \varepsilon_0] \) and for all \( (x, \varepsilon) \in \mathcal{U}_0 \times [0, \varepsilon_0] \) possess a G-derivative with respect to \( x \) continuous in \( (x, \varepsilon) \in \mathcal{U}_0 \times [0, \varepsilon_0] \);

(III) the Jacobian

\[
\det \left( \frac{D T_0}{D \delta} (\delta_0) \right)
\]

is nonzero.

Then there exists \( \varepsilon^* > 0 \) such that for all \( \varepsilon \in [0, \varepsilon^*] \) there exists a unique solution \( x^*(\varepsilon) \) to \((\mathcal{P}_e)\) continuous in \( \varepsilon \in [0, \varepsilon^*] \) as a mapping \([0, \varepsilon^*] \to \mathcal{A} \mathcal{E}\) and such that \( x^*(0) = x_0 \).
Proof. Let us denote $\mathcal{B} = \mathcal{A} \times \mathcal{R}$ and

$$F : (x, \delta) \in \mathcal{B}, \quad \varepsilon \in [0, \varepsilon_0] \rightarrow \begin{pmatrix} -x + V(.) \delta + \varepsilon S(\varepsilon)(x) \\ T(\varepsilon)(V(.) \delta + \varepsilon S(\varepsilon)(x)) \end{pmatrix} \in \mathcal{B}.$$ 

($\mathcal{B}$ is a B-space with the norm $\| (x, \delta) \|_\mathcal{B} = \| x \|_\mathcal{A} + \| \delta \|_\mathcal{R}$)

We shall verify that the operator $F$ fulfills all the assumptions of Proposition 1.

(i) For $(x, \delta) \in \mathcal{B}$ we have

$$F(0) (x, \delta) = \begin{pmatrix} -x + V(.) \delta \\ T(0)(V(.) \delta) \end{pmatrix} = \begin{pmatrix} -x + V(.) \delta \\ T_0(\delta) \end{pmatrix}.$$ 

Let $x_0$ be a solution to $(\mathcal{B}_0)$ such that $T_0(\delta_0) = 0$ for $\delta_0 = (b_0, \gamma)$, where

$$b_0 = \left( \int_a^b d_1 \int_a^b K_2(s) G(s, t) \, ds \right) x_0(t).$$ 

Then $x_0 = V(.) \delta_0$ and hence $F(0) (x_0, \delta_0) = 0$.

(ii) Since the operators $S$ and $T$ have the same smoothness properties as $\Phi$ and $\Lambda$, there exist $\varepsilon_1 > 0$ and $\vartheta_1 > 0$ such that $F$ fulfills the assumption (ii) of Proposition 1 on $\mathcal{U}_1 \times [0, \varepsilon_1] = \mathcal{U}(x_0, \delta_0), \quad \mathcal{Q}_1; \mathcal{B} \times [0, \varepsilon_1]$ while for $(x, \delta, \varepsilon) \in \mathcal{U}_1 \times [0, \varepsilon_1]$ and $(\bar{x}, \bar{y}) \in \mathcal{B},$

$$[F(x, \delta)(\varepsilon, x, \delta)](\bar{x}, \bar{y}) = \begin{pmatrix} -\bar{x} + V(.) \bar{\delta} + \varepsilon[S_\varepsilon'(\varepsilon)(x)] \bar{x} \\ [-T(x)(V(.) \delta + \varepsilon S(\varepsilon)(x))](V(.) \bar{\delta}) + \varepsilon[T(\varepsilon)(V(.) \delta + \varepsilon S(\varepsilon)(x))][S_\varepsilon'(\varepsilon)(x)] \bar{x} \end{pmatrix}.$$ 

In particular

$$J_0(\bar{x}, \bar{y}) = [F(x, \delta)(0)(x_0, \delta_0)](\bar{x}, \bar{y}) = \begin{pmatrix} -\bar{x} + V(.) \bar{\delta} \\ [-T(0)(V(.) \delta)(V(.) \bar{\delta})] \bar{x} \end{pmatrix} = \begin{pmatrix} -\bar{x} + V(.) \bar{\delta} \\ \left[ DT_0(\delta_0) \right] \bar{\delta} \end{pmatrix}.$$

(iii) Given an arbitrary couple $(x, \delta) \in \mathcal{B},$

$$J_0(\bar{x}, \bar{y}) = \begin{pmatrix} x \\ \delta \end{pmatrix}$$

iff

$$\bar{\delta} = \left[ \frac{DT_0}{D\delta}(\delta_0) \right]^{-1} \delta \quad \text{and} \quad \bar{x} = V(.) \bar{\delta} + x.$$
Thus the operator $J_0$ possesses an inverse

$$J_0^{-1} : (x, \delta) \in \mathcal{B} \rightarrow \left( x + V(.) \left[ \frac{D^T}{D\delta} (\delta_0) \right]^{-1} \delta \right) \in \mathcal{B},$$

the boundedness of $J_0^{-1}$ being obvious.

Applying Proposition 1 we complete the proof.

The system (4.9) can be simplified by means of the following

**Proposition 2.** There exists $\epsilon_1 > 0$ such that for every $\epsilon \in [0, \epsilon_1]$ and $\delta \in \mathcal{R}$, there exists a unique solution $x = \Xi(\delta) \in \mathcal{C}$ of the equation

$$(4.9)_2 - x + V(.) \delta + \epsilon S(\delta)(x) = 0,$$

the operator $\Xi : \mathcal{R} \times [0, \epsilon_1] \rightarrow \mathcal{C}$ being continuous in $(\delta, \epsilon)$ and locally lipschitzian in $\delta$ near $\epsilon = 0$.

**Proof.** The existence and uniqueness of the desired solution $x = \Xi(\delta) \in \mathcal{C}$ for all $\delta \in \mathcal{R}$ and $\epsilon \in [0, \epsilon_2]$ with some $\epsilon_2 > 0$ and the continuity of $\Xi$ in $(\delta, \epsilon) \in \mathcal{R} \times [0, \epsilon_2]$ are evident. Given an arbitrary $\delta_0 \in \mathcal{R}$, let us denote

$$x_0 = V(.) \delta_0 = \Xi(0)(\delta_0).$$

Let $\beta = \beta(\delta_0) > 0$, $\epsilon_3 = \epsilon(\delta_0) > 0$ and $\eta = \eta(\delta_0) > 0$ be such that

$$\|S(\epsilon)(x_1) - S(\epsilon)(x_2)\|_{\mathcal{C}} \leq \beta \|x_2 - x_1\|_{\mathcal{C}}$$

for all $x_1, x_2 \in \mathcal{U}(x_0, \eta; \mathcal{C})$ and $\epsilon \in [0, \epsilon_3]$. In virtue of the continuity of $\Xi$ in $(\delta, \epsilon)$ there exist $\sigma = \sigma(\delta_0) > 0$ and $\epsilon_4 = \epsilon_4(\delta_0) > 0$ ($\epsilon_4 \leq \epsilon_3$) such that $\Xi(\delta) \in \mathcal{U}(x_0, \sigma; \mathcal{C})$ for all $\delta \in \mathcal{U}(\delta_0, \sigma; \mathcal{R})$ and $\epsilon \in [0, \epsilon_4]$. Hence for $\delta_1, \delta_2 \in \mathcal{U}(\delta_0, \sigma; \mathcal{R})$ and $\epsilon \in [0, \epsilon_4]

$$\|\Xi(\delta_2) - \Xi(\delta_1)\|_{\mathcal{C}} \leq \|V\|_{\mathcal{C}} \|\delta_2 - \delta_1\| + \epsilon \beta \|\Xi(\delta_2) - \Xi(\delta_1)\|_{\mathcal{C}}.$$

Wherefrom, putting $\epsilon_1 = \epsilon_1(\delta_0) = \min(\epsilon_4, (2\beta)^{-1})$ our assertion follows.

**Remark 4.2.** It could be shown that if $\delta_0 \in \mathcal{R}$, $x_0 = V(.) \delta_0$ and $S$ possesses for all $(x, \epsilon) \in \mathcal{U}(x_0, \eta_1; \mathcal{C}) \times [0, \epsilon_1]$ ($\eta_1 > 0$) a G-derivative with respect to $x$ continuous in $(x, \epsilon) \in \mathcal{U}(x_0, \eta_1; \mathcal{C}) \times [0, \epsilon_1]$, then there exist $\epsilon_2 > 0$ and $\eta_2 > 0$
such that for all \((\delta, \varepsilon) \in U(\delta_0, Q_2; R_v) \times [0, \varepsilon_2]\) \(\Xi\) possesses a G-derivative with respect to \(\delta\) continuous in \((\delta, \varepsilon) \in U(\delta_0, Q_2; R_v) \times [0, \varepsilon_2]\). (For \(\delta \in R_v\)

\[
\left[\Xi'_\varepsilon(\varepsilon)(\delta)\right] \bar{\delta} = (i - \varepsilon[S'_\varepsilon(\varepsilon)(\Xi(\varepsilon)(\delta))]^{-1} (V(. \bar{\delta}))
\]

where \(i\) denotes the identity operator in \(\mathcal{C}\).

Inserting \(x = \Xi(\varepsilon)(\delta)\) into (4.9) we get

\[(4,12) \Theta(\varepsilon)(\delta) = T(\varepsilon)(\Xi(\varepsilon)(\delta)) = 0 .
\]

The second existence theorem for the critical case is based on the notion of the Brouwer topological degree and does not require any assumptions of the differentiability of \(\Phi\) and \(\Lambda\). It follows from the following proposition. (For the definition of the Brouwer topological degree see J. CRONIN [4].)

**Proposition 3.** Let \(\mathcal{G}\) be a bounded open set in \(R_v\) and let \(f\) be a continuous mapping of the closure \(\bar{\mathcal{G}}\) of \(\mathcal{G}\) in \(R_v\) into \(R_v\). Let \(f(\delta) \neq 0\) on the frontier \(\partial \mathcal{G}\) of \(\mathcal{G}\) in \(R_v\) and let the degree \(d(f, \mathcal{G}, 0)\) of \(f\) with respect to \(0 \in R_v\) and \(\mathcal{G}\) be nonzero. Then the equation \(f(\delta) = 0\) has at least one solution in \(\mathcal{G}\) and there exists \(\eta > 0\) such that for every continuous mapping \(g : \mathcal{G} \rightarrow R_v\) with \(\sup \|f(\delta) - g(\delta)\| < \eta\) there exists in \(\mathcal{G}\) at least one solution of the equation \(g(\delta) = 0\).

**Proof.** The mapping

\[
h : \delta \in \mathcal{G}, \ t \in [0, 1] \rightarrow h(\delta, t) = f(\delta) + (1 - t)(g(\delta) - f(\delta))
\]

is a continuous mapping of \(\mathcal{G} \times [0, 1]\) into \(R_v\) with \(h(\delta, 0) = g(\delta)\) and \(h(\delta, 1) = f(\delta)\). If

\[
\|f(\delta)\| \geq 2\eta > 0 \text{ and } \|f(\delta) - g(\delta)\| < \eta \text{ on } \partial \mathcal{G},
\]

then for all \(\delta \in \partial \mathcal{G}\) and \(t \in [0, 1]\)

\[
\|h(\delta, t)\| \geq \|f(\delta)\| - \|f(\delta) - g(\delta)\| > \eta > 0 .
\]

Proposition 2 is now an immediate consequence of Existence Theorem ([4], p. 32) and of Theorem of Invariance under Homotopy ([4], p. 31).

**Theorem 4.3.** Let the boundary value problem \((P_\varepsilon)\) fulfilling the assumptions \((\mathcal{A})\) be given. Let the limit problem \((P_0)\) admit a nonzero solution (i.e. \(\det B = 0\)). Let the matrix function \(V\) and the operators \(T\) and \(T_0\) be given by (4.7), (4.10) and (4.11). Suppose

(1) the limit problem \((P_0)\) possesses a solution \(x_0\) such that \(T_0(\delta_0) = 0\) for \(\delta_0 = (b_0)s\), where

\[
b_0 = \left(\int_a^b d, \int_a^b K_2(s) G(s, t) ds\right) x_0(t)
\]


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(II) there exists a bounded open subset $G$ of $\mathcal{R}$ such that $T_0(\delta) \neq 0$ for $\delta \in \partial G$ and $d(T_0, G, 0) < 0$.

Then there exists $\varepsilon^* > 0$ such that for every $\varepsilon \in [0, \varepsilon^*]$ there exists at least one solution to $(P_\varepsilon)$.

Proof. It is easy to verify that the operator $T_0 : \mathcal{R} \times [0, \varepsilon_0] \to \mathcal{R}$ is locally lipschitzian in $\delta \in \mathcal{R}$ near $\varepsilon = 0$ and continuous in $\varepsilon \in [0, \eta_1]$ with some $\eta_1 > 0$ small enough for any $\delta \in \mathcal{R}$ fixed. By Heine-Borel Covering Theorem we may assume that there exists $\eta_2 > 0$ such that for every $\varepsilon \in [0, \eta_2]$ there exists at least one solution to $(P_\varepsilon)$.

Remark 4.3. The methods of this paragraph can be also applied if $L \in \mathcal{B} \mathcal{V}_{m,n}$ and $\Lambda : \mathcal{A} \mathcal{C} \to \mathcal{R}_m$, where generally $m \neq n$. Of course, the situation is no more predetermined so largely by the fact whether the limit problem $(P_0)$ admits a nonzero solution or not. Let the $(m + n') \times (n + n')$-matrix $B$ be defined by (4.4), (3.9), (3.10) and (3.12). Let the $n \times (n + n')$-matrix function $U$ and the operators $R_0 : \mathcal{A} \mathcal{C} \times \times [0, \varepsilon_0] \to \mathcal{A} \mathcal{C}$ and $R : \mathcal{A} \mathcal{C} \times [0, \varepsilon_0] \to \mathcal{R}_{n+n}$, be given by (4.4) and (4.6). Then again an $n$-vector function $x \in \mathcal{A} \mathcal{C}$ is a solution to the boundary value problem $(P_\varepsilon)$ if a couple $(x, b)$, where

\[ b = \left( \int_a^b \left[ \int_a^t G(s, t) \right] x(t) \right), \]

is a solution to the system of operator equations ((4.5))

\[ -x + U(.) b + \varepsilon R_0(\varepsilon) (x) = 0, \]
\[ Bb + \varepsilon R(\varepsilon) (x) = 0. \]

Let $m < n$ and rank $(B) = m + n'$. Let us denote $\mathcal{M} = \{1, 2, \ldots, m + n'\}$ and let $\mathcal{V} \subset \mathcal{N}_0$ be such that $\mathcal{V}(\mathcal{V}) = n - m$ and det $B_{\mathcal{M}, \mathcal{V}} = 0$. Putting $\gamma = b_\mathcal{V}$, $\delta = b_\mathcal{V}$, $B_1 = B_{\mathcal{M}, \mathcal{V}}$ and $B_2 = B_{\mathcal{M}, \mathcal{V}}$, (4.5) becomes

\[ (4.13) \]
\[ -x + V(.) \delta + \varepsilon S(\varepsilon) (x) = 0, \]

where the $n \times (n - m)$-matrix function $V$ and the operator $S$ are given by (4.10). Given an arbitrary $\delta_0 \in \mathcal{R}_{n-m}$, the function $x_0 = V(.) \delta_0$ is a solution to the limit problem $(P_0)$ and by Proposition 2 there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in [0, \varepsilon^*]$ there exists a unique solution $x^*(\varepsilon)$ to $(P_\varepsilon)$ continuous in $\varepsilon \in [0, \varepsilon^*]$ as a mapping $[0, \varepsilon^*] \to \mathcal{A} \mathcal{C}$ and such that $x^*(0) = x_0$. The given boundary value problem $(P_\varepsilon)$ can be treated similarly as the noncritical case for $m = n$, although the limit problem $(P_0)$ possesses a nonzero solution. On the other hand, if $\varepsilon > 0$, $m > n$ and rank $(B) = n + n'$, then (4.5) is equivalent to the system

\[ (4.14) \]
\[ -x + \varepsilon S(\varepsilon) (x) = 0, \quad T(\varepsilon) (x) = 0 \]

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with $S$ and $T$ defined analogously as in (4,10). Now the function $x$ is uniquely determined by $(4,14)_1$ and to be a solution to the given problem $(\mathcal{P}_*)$ with $\varepsilon > 0$ it has to satisfy $(4,14)_2$. Hence the boundary value problem $(\mathcal{P}_*)$ has generally no solution, though the limit problem $(\mathcal{P}_0)$ has only the trivial solution (cf. Corollary 1 of Theorem 3,1). In the other cases we meet analogous situations.

5. LINEAR BOUNDARY VALUE PROBLEM — FUNCTIONAL ANALYSIS APPROACH

Let us turn back to the linear boundary value problem $(\mathcal{P})$ given by

\begin{align}
\dot{x} - A(t) x &= \int_a^b [dG(t, s)] x(s) = f(t), \\
\int_a^b [dL(s)] x(s) &= l,
\end{align}

where $A \in L^1_{n,m}, f \in L^1, G \in L^2[\mathcal{BV}], L \in \mathcal{BV}_{m,n}$ and $l \in \mathcal{B}_{m}$. Without any loss of generality we may assume that for all $t \in J G(t, .)$ and $L$ are continuous from the right on the open interval $(a, b)$.

In [20] D. Wexler derived the true adjoint (in the sense of functional analysis) to the boundary value problem 

\[ \dot{x} - A(t) x = f(t), \quad Lx = l, \]

where $A \in L^1_{n,m}, f \in L^1, L$ is a continuous linear mapping of $\mathcal{F}$ into some B-space $\Lambda$ and $l \in \Lambda$. In this paragraph we apply his ideas to the boundary value problem $(\mathcal{P})$.

The special form of the operator $L$ and the different choice of a dual space to the space $\mathcal{F}$ of continuous functions on $J$ (measures are replaced by functions of bounded variation) enables us to prove that the problem $(\mathcal{P}^*)$ derived in § 3 ((3,16), (3,17)) is equivalent to the true adjoint of $(\mathcal{P})$.

First, we have to introduce some new notations.

$L^\infty$ denotes the B-space of all row $n$-vector functions measurable and essentially bounded on $J$. It is well-known that $L^\infty$ is a dual B-space to the B-space $L^1 = L^1_{n,1}$ of column $n$-vector functions $L$-integrable on $J$. The value of a functional $y^* \in L^\infty$ on $x \in L^1$ is given by

\[ \langle x, y^* \rangle = \int_a^b y^*(s) x(s) \, ds \]

and the norm of $y^*$ is $\|y^*\|_{\infty} = \sup_{t \in J} \|y^*(t)\|$. Functions from $L^\infty$ which coincide a.e. on $J$ are identified with one another.

$BV^+$ is the B-space of all row $n$-vector functions of bounded variation on $J$ and continuous from the right on $(a, b)$ ($BV^+ \subset BV^+_{1,n}$). $C^*$ denotes the dual B-space
to the space $\mathcal{C}$ of column $n$-vector functions continuous on $J$, i.e. $\mathcal{C}^*$ is formed by all functions from $\mathcal{B}Y^*$ which vanish at $a$. Given an arbitrary functional $y' \in \mathcal{C}^*$, its value on $x \in \mathcal{C}$ is given by

$$\langle x, y' \rangle_{\mathcal{C}^*} = \int_a^b \left[ dy'(t) \right] x(t)$$

and $\|y'\|_{\mathcal{C}^*} = \text{var}_a^b y'$. The zero element of $\mathcal{C}^*$ is the function vanishing everywhere on $J$.

$\mathcal{A}C^*$ denotes the dual B-space to the B-space $\mathcal{AC}$ of column $n$-vector functions absolutely continuous on $J$. The value of a functional $y' \in \mathcal{AC}^*$ on $x \in \mathcal{AC}$ is denoted by $\langle x, y' \rangle_{\mathcal{AC}^*}$. Let us notice that we can consider $([20], 2,1) \mathcal{C}^* \subset \mathcal{AC}^*$ and $\langle x, y' \rangle_{\mathcal{AC}^*} = \langle x, y' \rangle_{\mathcal{AC}^*}$ for $x \in \mathcal{AC}$ and $y' \in \mathcal{C}^*$. Moreover, since the topology of $\mathcal{AC}$ is stronger than that induced by $\mathcal{C}(\|x\|_x = \sup_j \|x(t)\|)$ and $\mathcal{AC}$ is dense in $\mathcal{C}$, the zero elements of $\mathcal{AC}^*$ and $\mathcal{AC}^*$ coincide.

The operators

$$D : x \in \mathcal{AC} \to \dot{x} \in \mathbb{L}^1,$$

$$A : x \in \mathcal{AC} \to A(t) x(t) \in \mathbb{L}^1,$$

$$G : x \in \mathcal{AC} \to \int_a^b \left[ d_s G(t, s) \right] x(s) \in \mathbb{L}^1,$$

$$\mathcal{B}_1 : x \in \mathcal{AC} \to Dx - Ax - Gx \in \mathbb{L}^1$$

and

$$\mathcal{B}_2 : x \in \mathcal{AC} \to \int_a^b \left[ dL(s) \right] x(s) \in \mathbb{R}^m$$

are linear and continuous. Hence the operator

$$(5,3) \hspace{1cm} \mathcal{B} : x \in \mathcal{AC} \to \begin{pmatrix} \mathcal{B}_1 x \\ \mathcal{B}_2 x \end{pmatrix} \in \mathbb{L}^1 \times \mathbb{R}^m$$

is linear and continuous, too. Its adjoint $\mathcal{B}^*$ is a linear continuous operator $\mathbb{L}^\infty \times \mathbb{R}^m \to \mathcal{AC}^*$ defined on $(y', \lambda') \in \mathbb{L}^\infty \times \mathbb{R}^m$ by

$$\langle \mathcal{B}^* x, y' \rangle_{\mathcal{AC}^*} + \lambda' (\mathcal{B}_2 x) = \langle x, \mathcal{B}^* (y', \lambda') \rangle_{\mathcal{AC}^*} \text{ for all } x \in \mathcal{AC}.$$ 

The boundary value problem $(P)$ can be now written in the form

$$(5,4) \hspace{1cm} \mathcal{B} x = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$ 

Let us derive an explicit form for $\mathcal{B}^*$. For $x \in \mathcal{AC}$ and $(y', \lambda') \in \mathbb{L}^\infty \times \mathbb{R}^m$ we have

$$\langle x, \mathcal{B}^* (y', \lambda') \rangle_{\mathcal{AC}^*} = \langle \mathcal{B}_1 x, y' \rangle_{\mathcal{AC}^*} + \lambda' (\mathcal{B}_2 x) = \langle Dx, y' \rangle_{\mathcal{AC}^*} - \langle Ax, y' \rangle_{\mathcal{AC}^*} -$$

$$- \langle Gx, y' \rangle_{\mathcal{AC}^*} + \lambda' (\mathcal{B}_2 x) = \langle x, D^* y' - A^* y' - G^* y' + \mathcal{B}^* \lambda' \rangle_{\mathcal{AC}^*}$$

and

$$\mathcal{B}^* (y', \lambda') = D^* y' - A^* y' - G^* y' + \mathcal{B}^* \lambda' ,$$

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where $D^*$, $A^*$, $G^*$ and $B_2^*$ are adjoint operators to $D$, $A$, $G$ and $B_2$, respectively. Thus the adjoint equation to (5.4) is

$$(5.5) \quad D^*y' - A^*y' - G^*y' + B_2^* \lambda' = 0$$

(where 0 means the zero element of $\mathcal{H}^*$, of course).

Given an arbitrary $x \in \mathcal{H}$ and $y' \in L^\infty$, it holds by Lemma 2,7

$$\int_a^b y'(t) \left( \int_a^b [d_s G(t, s)] x(s) \right) dt = \int_a^b \left[ \int_a^b y'(s) (G(s, t) - G(s, a)) ds \right] x(t).$$

As a consequence, since $\int_a^b y'(s) (G(s, t) - G(s, a)) ds \in \mathcal{H}^*$, we have

$$\langle x, G^*y' \rangle_{\mathcal{H}^*} = \langle Gx, y' \rangle_{\mathcal{H}} = \int_a^b x y'(s) (G(s, t) - G(s, a)) ds$$

and

$$(5.6) \quad G^* : y' \in L^\infty \rightarrow \int_a^b y'(s) (G(s, t) - G(s, a)) ds \in \mathcal{H}^*.$$ 

By a similar argument the operators $A^*$ and $B_2^*$ are defined by

$$(5.7) \quad A^* : y' \in L^\infty \rightarrow \int_a^t y'(s) A(s) ds \in \mathcal{H}^*$$

and

$$(5.8) \quad B_2^* : \lambda' \in B_2^* \rightarrow \lambda' (L(t) - L(a)) \in \mathcal{H}^*.$$

Furthermore,

$$(5.9) \quad D^* : y' \in \mathcal{H}^* \rightarrow -y'(t) + R(y')(t) \in \mathcal{H}^*,$$

where

$$(5.10) \quad R(y')(t) = \begin{cases} y'(a) & \text{for } t = a, \\ 0 & \text{for } a < t < b, \\ y'(b) & \text{for } t = b. \end{cases}$$

The operator $Dx - Ax$ maps $\mathcal{H}$ onto $L^1$. Hence $y' \in L^\infty$ being an arbitrary solution to $D^*y' - A^*y' = 0$, $y'(t) = 0$ a.e. on $J$. Moreover, given an arbitrary $g^* \in \mathcal{H}^*$, the equation

$$(5.11) \quad D^*y' - A^*y' = g'$$

has a solution in $L^\infty$ iff

$$(5.12) \quad \int_a^b [dg^*(s)] X(s) = 0,$$

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where $X$ denotes again the fundamental matrix solution of $Dx - Ax = 0$ (cf. (3,3)). Suppose $g \in \mathcal{C}^*$ and (5.11) has a solution in $L^\infty$. Then this solution is unique in $L^\infty$. Let us put for $t \in J$

$$ z'(t) = -\left(\int_a^t [dg(s)]X(s)\right)X^{-1}(t). $$

Since $z' \in \mathcal{C}^*$ and $R(z')(t) \equiv 0$ by (5.10) and (5.12), we have by (5.7), (5.9), Lemma 1.1 and (3.3)

$$ D*z' - A*z' = -z'(t) + \int_a^t \left(\int_a^s [dg(\sigma)]X(\sigma)\right)X^{-1}(\sigma)A(\sigma)\, d\sigma = $$

$$ = -z'(t) + \int_a^t [dg(s)] \left(X(s) \int_s^t X^{-1}(\sigma)A(\sigma)\, d\sigma\right) = g'(t). $$

It follows that $z'$ is the unique solution of (5.11) in $L^\infty$. Applying this to (5.5) and taking into account (5.6)–(5.8), we obtain that to any solution $(y', \lambda') \in L^\infty \times \mathcal{B}^*$ of (5.5) there exists a solution $(\eta', \lambda')$ of (5.5) such that $\eta' \in \mathcal{B}^+$, $\lambda'$ is continuous at $a$ from the right and at $b$ from the left and $y'(t) = \eta'(t)$ a.e. on $J$ ($y' = \eta'$ in $L^\infty$).

Consequently, to find all solutions of (5.5) in $L^\infty \times \mathcal{B}^*$, it is sufficient to consider instead of $\mathcal{B}^*$ its restriction $\mathcal{B}^*_0$ on $\mathcal{V} \times \mathbb{R}^*_m$, where $\mathcal{V}$ is formed by all functions from $\mathcal{B}^+$ which are continuous at $a$ from the right and at $b$ from the left. By (5.6)–(5.9)

$$ \mathcal{B}^*_0(y', \lambda') = -y'(t) + R(y')(t) - \int_a^t y'(s)A(s)\, ds + \lambda'(L(t) - L(a)) - $$

$$ - \int_a^b y'(s) (G(s, t) - G(s, a))\, ds \in \mathcal{C}^*. $$

In other words, the equation (5.5) for $(y', \lambda') \in L^\infty \times \mathcal{B}^*$ is equivalent to the equation

$$ -y'(t) + R(y')(t) - \int_a^t y'(s)A(s)\, ds + \lambda'(L(t) - L(a)) - $$

$$ - \int_a^b y'(s) (G(s, t) - G(s, a))\, ds = 0 \quad \text{on} \quad J $$

for $(y', \lambda') \in \mathcal{V} \times \mathcal{B}^*_m$. In particular, (5.13) yields

$$ y'(a) - y'(a) = 0 \quad \text{for} \quad t = a, $$

(5.14) $$ y'(t) = -\int_a^t y'(s)A(s)\, ds + \lambda'(L(t) - L(a)) - \int_a^b y'(s) (G(s, t) - G(s, a))\, ds $$

for $t \in (a, b), $
and
\begin{equation}
0 = - \int_{a}^{b} y'(s) A(s) \, ds + \lambda \cdot (L(b) - L(a)) - \int_{a}^{b} y'(s) (G(s, b) - G(s, a)) \, ds
\end{equation}
for \( t = b \).

Furthermore, from (5,14) we have
\begin{equation}
y'(a) = y'(a+) = \lambda \cdot (L(a+) - L(a)) - \int_{a}^{b} y'(s) (G(s, a+) - G(s, a)) \, ds
\end{equation}
and consequently (5,14) becomes
\begin{equation}
y'(t) = y'(a) - \int_{a}^{t} y'(s) A(s) \, ds + \lambda \cdot (L(t) - L(a+)) - \int_{a}^{t} y'(s) (G(s, t) - G(s, a+)) \, ds \quad \text{for} \quad t \in (a, b).
\end{equation}

Making use of (5,15), (5,14) can be modified as follows
\begin{equation}
y'(t) = \int_{t}^{b} y'(s) A(s) \, ds - \lambda \cdot (L(b) - L(t)) + \int_{a}^{b} y'(s) (G(s, b) - G(s, t)) \, ds \quad \text{for} \quad t \in (a, b).
\end{equation}

Thus
\begin{equation}
y'(b) = y'(b-) = -\lambda \cdot (L(b) - L(b-)) + \int_{a}^{b} y'(s) (G(s, b) - G(s, b-)) \, ds
\end{equation}
and
\begin{equation}
y'(t) = y'(b) + \int_{t}^{b} y'(s) A(s) \, ds + \lambda \cdot (L(t) - L(b-)) - \int_{a}^{b} y'(s) (G(s, t) - G(s, b-)) \, ds \quad \text{for} \quad t \in (a, b).
\end{equation}

Let us define
\begin{align*}
G(t, s) &= \begin{cases} 
G(t, a+) & \text{for} \ t \in J \text{ and } s = a , \\
G(t, s) & \text{for} \ t \in J \text{ and } a < s < b , \quad L^0(s) = \begin{cases} 
L(a+) & \text{for} \ s = a , \\
L(s) & \text{for} \ a < s < b , \\
L(b-) & \text{for} \ s = b , 
\end{cases}
\end{cases} \\
C(t) &= G(t, a+) - G(t, a) \quad \text{and} \quad D(t) = G(t, b) - G(t, b-) \quad \text{for} \ t \in J \text{ and}
\end{align*}

\begin{align*}
M &= L(a+) - L(a) , \quad N = L(b) - L(b-) .
\end{align*}

Then from (5,16), (5,17), (5,19) and (5,20) we can conclude that the equation (5,13) (and hence also (5,5)) is equivalent to the system of equations for \((y', \gamma') \in L^\infty \times \mathcal{P}^\infty_m (\gamma' = -\lambda')\)
\[ y'(t) = y'(a) - \int_a^t y'(s) A(s) \, ds - \gamma'(L_0(t) - L_0(a)) - \int_a^b y'(s) (G_0(s, t) - G_0(s, a)) \, ds \quad \text{on } J , \]

\[ y'(a) = -\gamma'M - \int_a^b y'(s) C(s) \, ds , \quad y'(b) = \gamma'N + \int_a^b y'(s) D(s) \, ds . \]

In the introduced notation, the original boundary value problem \( (\mathcal{P}) \) assumes the form
\[
\dot{x} = A(t) \, x + C(t) \, x(a) + D(t) \, x(b) + \int_a^b [d_s G_0(t, s)] \, x(s) + f(t) ,
\]
\[
M \, x(a) + N \, x(b) + \int_a^b [dL_0(s)] \, x(s) = l
\]
and (5.21), (5.22) is exactly its adjoint \( (\mathcal{P}^*) \) derived in § 3 ((3,16), (3,17)).

As a consequence we have that the adjoint \( (\mathcal{P}^*) \) of \( (\mathcal{P}) \) from § 3 and the true adjoint (5,5) of \( (\mathcal{P}) \) are equivalent.

From the fundamental "alternative" theorem concerning linear equations in B-spaces ([5] VI, §6) and from Theorem 3,1 it follows that the operator \( \mathcal{B} \) of the boundary value problem \( (\mathcal{P}) \) defined by (5,3) has a closed range in \( \mathcal{L}^1 \times \mathbb{R}_n \).

Remark. The closedness of the range \( \mathcal{B}(\mathcal{A}C) \) of the operator \( \mathcal{B} \) can be also shown directly in a similar way as D. Wexler did in [20] § 3 for the operator
\[
\mathbf{x} \in \mathcal{A}C \rightarrow \left( \begin{array}{c} \dot{x} - A(t) \, x \\ L \mathbf{x} \end{array} \right) \in \mathcal{L}^1 \times \mathbb{R}_m ,
\]
where \( L \) is a continuous linear mapping of \( \mathcal{A}C \) into some B-space \( \Lambda \). In fact, let the matrix \( B \) and the operator
\[
\Psi : \left( \begin{array}{c} f \\ l \end{array} \right) \in \mathcal{L}^1 \times \mathbb{R}_m \rightarrow \Psi(f, l) = w \in \mathbb{R}_{m+n} .
\]
be defined by (4,4), (3,9), (3,10) and (3,12). Let us put
\[
\Theta : b \in \mathbb{R}_{m+n} \rightarrow Bb \in \mathbb{R}_{m+n} .
\]
Given \( f \in \mathcal{L}^1 \) and \( l \in \mathbb{R}_m \), the corresponding boundary value problem \( (\mathcal{P}) \) possesses a solution (i.e. \( (f', l') \in \mathcal{B}(\mathcal{A}C) \)) iff \( \Psi(f, l) \in \Theta(\mathbb{R}_{m+n}) \). Hence
\[
\mathcal{B}(\mathcal{A}C) = \Psi^{-1}(\Theta(\mathbb{R}_{m+n})) .
\]
Since \( \Psi \) and \( \Theta \) are continuous linear operators and \( \dim \Theta(\mathbb{R}_{m+n}) < \infty \), the set \( \Psi^{-1}(\Theta(\mathbb{R}_{m+n})) \) is certainly closed.
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