CERTAIN COMPLEX REPRESENTATIONS OF $SL_2(\bar{F}_q)$

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Abstract. We introduce the representation category $\mathcal{C}(G)$ for a connected reductive algebraic group $G$ which is defined over a finite field $\mathbb{F}_q$ of $q$ elements. We show that this category has many good properties for $G = SL_2(\bar{F}_q)$. In particular, it is an abelian category and a highest weight category. Moreover, we classify the simple objects in $\mathcal{C}(G)$ for $G = SL_2(\bar{F}_q)$.

1. Introduction

Let $G$ be a connected reductive algebraic group defined over the finite field $\mathbb{F}_q$ of $q$ elements. Let $k$ be another field and all representations in this paper are over $k$. According to a result of Borel and Tits [1, Theorem 10.3 and Corollary 10.4], we know that except the trivial representation, all other irreducible representations of $kG$ (the group algebra of $G$) are infinite-dimensional if $G$ is semisimple and $k$ is infinite with char $k \neq$ char $\bar{F}_q$. Denote by $G_{q^a}$ the set of $\mathbb{F}_{q^a}$-points of $G$, then we have $G = \bigcup G_{q^a}$. With this basic fact, N.H. Xi studied the abstract representations of $G$ over $k$ by taking the direct limit of the finite-dimensional representations of $G_{q^a}$ and he got many interesting results in [13]. In particular, he showed that the infinite-dimensional Steinberg module is irreducible when char $k = 0$ or char $k = \text{char} \bar{F}_q$. Afterwards, R.T. Yang proved the irreducibility of the Steinberg module for any field $k$ with char $k \neq \text{char} \bar{F}_q$ (see [14]). Later, motivated by Xi’s idea, the structure of the permutation module $k[G/B]$ ($B$ is a fixed Borel subgroup of $G$ ) was studied in [4] for the cross characteristic case and in [5] for the defining characteristic case. We studied the general abstract induced module $M(\theta) = kG \otimes_{kB} k\theta$ in [6] for any field $k$ with char $k \neq \text{char} \bar{F}_q$ or $k = \mathbb{F}_q$, where $T$ is a maximal torus contained in a Borel subgroup $B$ and $\theta$ is a character of $T$ which can also be regarded as a character of $B$ through the homomorphism $B \to T$. The induced module
$\mathbb{M}(\theta)$ has a composition series (of finite length) if $\text{char } k \neq \text{char } \overline{\mathbb{F}}_q$. In the case $k = \overline{\mathbb{F}}_q$ and $\theta$ is a rational character, $\mathbb{M}(\theta)$ has such composition series if and only if $\theta$ is antidominant (see [6] for details). In both cases, the composition factors of $\mathbb{M}(\theta)$ are $E(\theta)_J$ with $J \subset I(\theta)$ (see Section 2 for the explicit setting).

Now we have a large class of irreducible $kG$-modules. Let $kG$-Mod be the $kG$-module category. However, this category is too big and thus in the paper [9], we introduce the principal representation category $\mathcal{O}(G)$ which was supposed to have many good properties. In particular, we conjectured that this category is a highest weight category in the sense of Cline, Parshall and Scott [8]. However, recently X.Y.Chen constructed a counter example (see [3]) to show that this conjecture is not valid in general with the setting given in [9, Section 4]. Thus it deserves to explore other categories besides the category $\mathcal{O}(G)$ in [9]. We hope that there is a category which satisfies certain good properties such as “finite-ness” and “semi-simplicity”, which is also like the BGG category $\mathcal{O}$ in the representations of complex semisimple Lie algebras. In this paper we introduce a full subcategory $\mathcal{C}(G)$ of $kG$-Mod, whose objects are finitely generated by some $T$-eigenvectors (see Section 2 for the definition of $\mathcal{C}(G)$). The main part of this paper is devoted to study the category $\mathcal{C}(G)$ for $G = SL_2(\overline{\mathbb{F}}_q)$.

The rest of this paper is organized as follows: Section 2 contains some preliminary results and we also introduce the category $\mathcal{C}(G)$ in this section. From Section 3 to Section 5, we assume that $k$ is an algebraically closed field of characteristic 0 and study the category $\mathcal{C}(G)$ for $G = SL_2(\overline{\mathbb{F}}_q)$. In Section 3, we classify the simple $kG$-modules with $T$-stable lines. In particular, we get all the simple objects in $\mathcal{C}(G)$. Then we show that $\mathcal{C}(G)$ is an abelian category and has certain good properties in Section 4. In Section 5 we prove that $\mathcal{C}(G)$ is a highest weight category.

2. Background and preliminary results

Let $G$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, the algebraic closure of $\mathbb{F}_q$, e.g. $G = GL_n(\overline{\mathbb{F}}_q)$, $SL_n(\overline{\mathbb{F}}_q)$. Let $B$ be a Borel subgroup, and $T$ be a maximal torus contained in $B$, and $U = R_u(B)$ be the unipotent radical of $B$. We identify $G$ with $G(\overline{\mathbb{F}}_q)$ and do likewise for the various subgroups of $G$ such as $B$, $T$, $U$, etc. We denote by $\Phi = \Phi(G; T)$ the corresponding root system, and by $\Phi^+$ (resp. $\Phi^-$) the set of positive (resp. negative) roots determined by $B$. Let $W = N_G(T)/T$ be the corresponding
Weyl group. We denote by $\Delta = \{\alpha_i \mid i \in I\}$ the set of simple roots and by $S = \{s_i := s_{\alpha_i} \mid i \in I\}$ the corresponding simple reflections in $W$. For each $\alpha \in \Phi$, let $U_\alpha$ be the root subgroup corresponding to $\alpha$ and we fix an isomorphism $\varepsilon_\alpha : \bar{F}_q \rightarrow U_\alpha$ such that $t\varepsilon_\alpha(c)t^{-1} = \varepsilon_\alpha(\alpha(t)c)$ for any $t \in T$ and $c \in \bar{F}_q$. For any $w \in W$, let $U_w$ (resp. $U'_w$) be the subgroup of $U$ generated by all $U_\alpha$ with $w(\alpha) \in \Phi^+$ (resp. $w(\alpha) \in \Phi^-$). One is refereed [2] for more details.

Now let $k$ be an algebraically closed field of characteristic 0 and all the representations in this paper are over $k$. Let $\hat{T}$ be the set of characters of $T$. Each $\theta \in \hat{T}$ can be regarded as a character of $B$ by the homomorphism $B \rightarrow T$. Let $k_{\theta}$ be the corresponding $B$-module. We consider the induced module $M(\theta) = kG \otimes_k B k_{\theta}$. Let $1_{\theta}$ be a fixed nonzero element in $k_{\theta}$. We abbreviate $x 1_{\theta} := x \otimes 1_{\theta} \in M(\theta)$ for $x \in G$. It is not difficult to see that $M(\theta)$ has a basis $\{\hat{x} \mid x \in W, u \in U_{w-1}\}$ by the Bruhat decomposition, where $\hat{x}$ is a fixed representative of $w \in W$.

For each $i \in I$, let $G_i$ be the subgroup of $G$ generated by $U_\alpha$, $U_{-\alpha}$ and set $T_i = T \cap G_i$. For $\theta \in \hat{T}$, define the subset $I(\theta)$ of $I$ by

$$I(\theta) = \{i \in I \mid \theta|_{T_i} \text{ is trivial}\}.$$ 

The Weyl group $W$ acts naturally on $\hat{T}$ by

$$(w \cdot \theta)(t) := \theta^w(t) = \theta(\hat{w}^{-1} t \hat{w})$$

for any $\theta \in \hat{T}$.

Let $J \subset I(\theta)$, and $G_J$ be the subgroup of $G$ generated by $G_i$, $i \in J$. We choose a representative $\hat{w} \in G_J$ for each $w \in W_J$ (the standard parabolic subgroup of $W$). Thus, the element $w 1_{\theta} := \hat{w} 1_{\theta}$ ($w \in W_J$) is well-defined. For $J \subset I(\theta)$, we set

$$\eta(\theta)_J = \sum_{w \in W_J} (-1)^{\ell(w)} w 1_{\theta},$$

where $\ell$ is the length function on $W$. Let $M(\theta)_J = kG \eta(\theta)_J$ the $kG$-module which is generated by $\eta(\theta)_J$.

For $w \in W$, denote by $\mathcal{R}(w) = \{i \in I \mid ws_i < w\}$. For any subset $J \subset I$, we set

$$X_J = \{x \in W \mid x \text{ has minimal length in } xW_J\}.$$ 

We have the following proposition.
Proposition 2.1. [6] Proposition 2.5] For any \( J \subset I(\theta) \), the \( kG \)-module \( M(\theta)_J \) has the form

\[
M(\theta)_J = \sum_{w \in X_J} kUw\theta(\subscript{}w) = \sum_{w \in X_J} kUw_{Jw^{-1}}w\theta(\subscript{}w).
\]

In particular, the set \( \{uw\theta(\subscript{}w) \mid w \in X_J, u \in U_{w_{Jw^{-1}}} \} \) forms a basis of \( M(\theta)_J \).

For \( J \subset I(\theta) \), define

\[
E(\theta)_J = M(\theta)_J / M(\theta)'_J,
\]

where \( M(\theta)'_J \) is the sum of all \( M(\theta)_K \) with \( J \subset I(\theta) \). We denote by \( C(\theta)_J \) the image of \( \theta(\subscript{}\theta) \) in \( E(\theta)_J \). We also set

\[
Z_J = \{ w \in X_J \mid \mathcal{R}(ww_J) \subset J \cup (I \setminus I(\theta)) \}.
\]

The following proposition gives a basis of \( E(\theta)_J \).

Proposition 2.2. [6] Proposition 2.7] For \( J \subset I(\theta) \), we have

\[
E(\theta)_J = \sum_{w \in Z_J} kw_{Jw^{-1}}wC(\theta)_J.
\]

In particular, the set \( \{uwC(\theta)_J \mid w \in Z_J, u \in U_{w_{Jw^{-1}}} \} \) forms a basis of \( E(\theta)_J \).

The \( kG \)-modules \( E(\theta)_J \) are irreducible and thus we get all the composition factors of \( M(\theta) \) (see [6] Theorem 3.1]). According to [6] Proposition 2.8], one has that \( E(\theta)_1 K_1 \) is isomorphic to \( E(\theta)_2 K_2 \) as \( kG \)-modules if and only if \( \theta_1 = \theta_2 \) and \( K_1 = K_2 \).

For a \( kG \)-module \( M \), an element \( \xi \in M \) is called a \( T \)-eigenvector if \( t\xi = \lambda(t)\xi \) for some \( \lambda \in \widehat{T} \). Set \( M_\lambda = \{ \xi \in M \mid t\xi = \lambda(t)\xi \} \), which is called the weight space corresponding to \( \lambda \in \widehat{T} \). A character \( \lambda \in \widehat{T} \) is called a weight of \( M \) if \( M_\lambda \neq 0 \) and then we denote the weight set of \( M \) by \( Wt(M) \). Let \( M_T = \bigoplus_{\lambda \in \widehat{T}} M_\lambda \) and set \( \mathcal{W}(M) = \dim M_T \). With previous discussion and the form of \( M(\theta)_J \) and \( E(\theta)_J \) (see (2.1) and (2.2)), we are interested in the \( kG \)-module \( M \) which satisfies the following condition:

\( \ast \) \( n = \mathcal{W}(M) < +\infty \) and there exist \( T \)-eigenvectors \( \xi_1, \xi_2, \ldots, \xi_n \) such that \( M \cong \bigoplus_{i=1}^n kU\xi_i \) as \( kB \)-modules.

Let \( \mathcal{C}(G) \) be the full subcategory of \( kG \)-Mod, which consists of the \( kG \)-modules satisfying the condition \( \ast \). From the definition of \( \mathcal{C}(G) \), it seems
very difficult to judge whether it is an abelian category. Naturally, we have the following fundamental questions: (1) Is the category \( \mathcal{C}(G) \) an abelian category? (2) Is this category noetherian or artinian? (3) Classify all the simple objects in \( \mathcal{C}(G) \). We will solve these problems for \( G = SL_2(\overline{F}_q) \) in the following discussion.

3. Simple modules with \( T \)-stable lines.

From now on, let \( G = SL_2(\overline{F}_q) \), \( T \) be the diagonal matrices and \( U \) be the strictly upper unitriangular matrices in \( SL_2(\overline{F}_q) \). Let \( B \) be the Borel subgroup generated by \( T \) and \( U \), which is the upper triangular matrices in \( SL_2(\overline{F}_q) \). As before, let \( N \) be the normalizer of \( T \) in \( G \) and \( W = N/T \) be the Weyl group. We set \( \hat{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), which is the simple reflection of \( W \).

There are two natural isomorphisms 
\[ h : \overline{F}_q^* \to T, \quad h(c) = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}; \quad \varepsilon : \overline{F}_q \to U, \quad \varepsilon(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \]

which satisfies \( h(c)\varepsilon(x)h(c)^{-1} = \varepsilon(c^2x) \). The simple root \( \alpha : T \to \overline{F}_q^* \) is given by \( \alpha(h(c)) = c^2 \). Moreover, one has that

\[ \hat{s}\varepsilon(x)\hat{s} = \varepsilon(-x^{-1})\hat{sh}(-x)\varepsilon(-x^{-1}). \]  

(3.1)

In the following we often denote \( \theta(x) := \theta(h(x)) \) simply. For any finite subset \( X \) of \( G \), let \( X := \sum_{x \in X} x \in kG \). This notation will be frequently used later.

In order to understand the irreducible \( kG \)-modules with \( T \)-stable lines, it is enough to study the simple quotients of the induced module \( \text{Ind}^G_T k_{\theta} \) for each \( \theta \in \widehat{T} \). It is not difficult to see that \( \text{Ind}^G_T k_{\theta} \cong \text{Ind}^G_T k_{\theta^*} \) as \( kG \)-modules. We will consider the following two cases: (1) \( \theta \) is trivial; (2) \( \theta \) is nontrivial. When \( \theta \) is trivial, it is easy to see that

\[ \text{Ind}^G_T k_{\text{tr}} \cong \text{Ind}^G_N k_+ \oplus \text{Ind}^G_N k_- , \]

where \( k_+ \) is the trivial representation of \( N \) and \( k_- \) is the sign representation of \( W \), which also can be regarded as a representation of \( N \).

In our case \( G = SL_2(\overline{F}_q) \), according to [6, Theorem 3.1], \( \mathcal{M}(\text{tr}) = \text{Ind}^G_T k_{\text{tr}} \) has a unique submodule \( \text{St} \) (the Steinberg module) and the corresponding quotient module is trivial. The Steinberg module \( \text{St} = kU\eta_s \), where \( \eta_s = (1-s)1_{\text{tr}} \). Then we have \( \hat{s}\eta_s = -\eta_s \) and \( \hat{s}\varepsilon(x)\eta_s = (\varepsilon(-x^{-1})-1)\eta_s \) for \( x \not= 0 \)
by (3.1). For any nontrivial character \( \theta \in \hat{T} \), \( M(\theta) = \text{Ind}_B^G \kappa_\theta \) is a simple \( kG \)-module. The main theorem of this section is as following.

**Theorem 3.1.** (1) The trivial \( kG \)-module \( k_{tr} \) is the unique simple quotient module of \( \text{Ind}_N^G k_+ \) and the Steinberg module \( \text{St} \) is the unique simple quotient module of \( \text{Ind}_N^G k_- \). (2) When \( \theta \) is nontrivial, \( \text{Ind}_T^G k_\theta \) has just two simple quotient modules \( M(\theta) \) and \( M(\theta^*) \).

**Proof of Theorem 3.1(1).** For convenience, we abbreviate \( x_1^+ := x \otimes 1_+ \in \text{Ind}_N^G k_+ \) and \( x_1^- := x \otimes 1_- \in \text{Ind}_N^G k_- \) for \( x \in G \), where \( 1_+ \) (resp. \( 1_- \)) is a fixed nonzero element in \( k_+ \) (resp. \( k_- \)). Firstly we consider the simple quotient of \( \text{Ind}_N^G k_+ \). We construct a submodule of \( \text{Ind}_N^G k_+ \) as following 

\[
M_+ = \{ \sum_{g \in G} a_g g_1^+ \mid \sum_{g \in G} a_g = 0 \}.
\]

Then it is easy to see that \( \text{Ind}_N^G k_+/M_+ \) is the trivial \( kG \)-module. Now let \( \xi \notin M_+ \) which has the following expression

\[
\xi = \sum_{x, y \in \overline{\mathbb{F}}_q} a_{x,y} \varepsilon(x) \hat{s} \varepsilon(y) 1_+,
\]

where \( \sum_{x, y \in \overline{\mathbb{F}}_q} a_{x,y} \neq 0 \). Firstly there exists an integer \( m \in \mathbb{N} \) such that \( x, y \in \mathbb{F}_{q^m} \) when \( a_{x,y} \neq 0 \). Now let \( n > m \) with \( m \mid n \) and \( u \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^m} \). We consider the element \( \eta = \hat{s} \varepsilon(\alpha) \xi \), which has the form

\[
\eta = \sum_{x, y \in \mathbb{F}_q} a_{x,y} \varepsilon(-(u + x)^{-1}) \hat{s} \varepsilon((u + x) - (u + x)^2 y) 1_+.
\]

Choose \( \mathcal{D}_{q^n} \subset T \) such that \( \alpha : \mathcal{D}_{q^n} \to \mathbb{F}_{q^n}^* \) is a bijection. Thus it is easy to check the element

\[
\overline{\mathcal{D}}_{q^n} \hat{U}_{q^n} \eta = \left( \sum_{x, y \in \mathbb{F}_q} a_{x,y} \hat{U}_{q^n} \hat{s} \hat{U}_{q^n} 1_+ \right) \in \mathbb{G} 1_+
\]

which implies that \( \hat{U}_{q^n} \hat{s} \hat{U}_{q^n} 1_+ \in \kappa G \xi \).

On the other hand, we consider the element

\[
\overline{\mathcal{G}}_{q^n} \xi = \left( \sum_{x, y \in \mathbb{F}_q} a_{x,y} \hat{G}_{q^n} 1_+ \right) \in \kappa G \xi.
\]

Noting that

\[
\overline{\mathcal{G}}_{q^n} 1_+ = (q^n - 1)(2\hat{U}_{q^n} 1_+ + \hat{U}_{q^n} \hat{s} \hat{U}_{q^n} 1_+) \in \kappa G \xi,
\]
therefore we have $\mathcal{U}_q^+ \in kG\xi$ and hence $1_+ \in kG\xi$ by [10] Lemma 2.6. Thus for any element $\xi \notin M_+$, we have $kG\xi = \text{Ind}_N^G k_+$. So, the trivial module is the unique simple quotient module of $\text{Ind}_N^G k_+$.

Now we consider the simple quotient modules of $\text{Ind}_N^G k_-$. For convenience, we denote by

$$\Lambda(z) = (\dot{s}\varepsilon(z) + 1 - \varepsilon(-z^{-1}))1_-$$

for each $z \in \bar{F}_q^*$. Then we have $h(c)\Lambda(z) = \Lambda(c^{-2}z)$ for any $h(c) \in T$. Moreover, it is easy to check that

$$(3.2) \quad \begin{align*}
\varepsilon(z)\Lambda(z^{-1}) &= -\Lambda(-z^{-1}), \\
\dot{s}\varepsilon(x)\Lambda(y) &= \varepsilon(-x^{-1})\Lambda(x(xy - 1)) + \Lambda(x) - \Lambda(y^{-1}(xy - 1)),
\end{align*}$$

where $xy \neq 1$. Let $M_-$ be the submodule of $\text{Ind}_N^G k_-$ which is generated by $\Lambda(z), z \in \bar{F}_q^*$. Since we have the equation

$$\dot{s}\varepsilon(z)\eta_s = (\varepsilon(-z^{-1}) - 1)\eta_s$$

in the Steinberg module $St = kU\eta_s$. Thus it is not difficult to see that $\text{Ind}_N^G k_-/M_-$ is isomorphic to the Steinberg module.

Let $\zeta \in \text{Ind}_N^G k_-$ which is not in $M_-$. According to (3.2), then $\zeta$ has the following expression

$$\zeta = \sum_{xy \neq 1} a_{x,y}\varepsilon(x)\Lambda(y)1_- + \sum_{z \in \bar{F}_q^*} b_z\Lambda(z)1_- + \sum_{u \in \bar{F}_q} c_u\varepsilon(u)1_-,$$

where $c_u \neq 0$ for some $u$. There exists an integer $m \in N$ such that $x, y, z, u \in \bar{F}_q^m$ when $a_{x,y} \neq 0, b_z \neq 0$ and $c_u \neq 0$. Without lost of generality, we can assume that $c_0 \neq 0$. Moreover, we can assume that $\sum_{u \in \bar{F}_q} c_u \neq 0$. Otherwise, we can consider $\dot{s}\zeta$ instead of $\zeta$. Indeed, if we write

$$\dot{s}\zeta = \sum_{xy \neq 1} a'_{x,y}\varepsilon(x)\Lambda(y)1_- + \sum_{z \in \bar{F}_q^*} b'_z\Lambda(z)1_- + \sum_{u \in \bar{F}_q} c'_u\varepsilon(u)1_-,$$

it is easy to see that $\sum_{u \in \bar{F}_q} c'_u = -c_0$ which is nonzero. For the convenience of later discussion, we denote by

$$A = \sum_{xy \neq 1} a_{x,y}, \quad B = \sum_{z \in \bar{F}_q^*} b_z, \quad C = \sum_{u \in \bar{F}_q} c_u.$$

Note that $C$ is nonzero by our assumption.
Now let $n > m$ with $m | n$ and $v \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^m}$. We consider the element

$$\dot{s}\varepsilon(v)\zeta := \sum_{xy \neq 1} f_{x,y} \varepsilon(x)\Lambda(y) 1_- + \sum_{z \in \bar{\mathbb{F}}_q^*} g_z \Lambda(z) 1_- + \sum_{u \in \bar{\mathbb{F}}_q^*} h_u \varepsilon(u) 1_-.$$  

Then by (3.2), we get

(3.3)  

$$\sum_{xy \neq 1} f_{x,y} = A + B, \quad \sum_{z \in \bar{\mathbb{F}}_q^*} g_z = C, \quad \sum_{u \in \bar{\mathbb{F}}_q^*} h_u = 0.$$  

Choose $\mathcal{D}_{q^n} \subset T$ such that $\alpha : \mathcal{D}_{q^n} \to \bar{\mathbb{F}}_{q^n}$ is a bijection. Combining (3.3), it is not difficult to get

(3.4)  

$$\mathcal{D}_{q^n} \underline{U}_{q^n} \dot{s}\varepsilon(v)\zeta = (A + B + C) \underline{U}_{q^n} \underline{s} \underline{U}_{q^n} 1_- \in kG\zeta.$$  

On the other hand, we also have

(3.5)  

$$\mathcal{D}_{q^n} \underline{U}_{q^n} \zeta = (A + B) \underline{U}_{q^n} \underline{s} \underline{U}_{q^n} 1_- + (q^n - 1) C \underline{U}_{q^n} 1_- \in kG\zeta.$$  

If $A + B + C \neq 0$ and noting that $C \neq 0$, then we get $\underline{U}_{q^n} 1_- \in kG\zeta$ using (3.4) and (3.5). Now assume that $A + B + C = 0$, then by (3.5), we have

$$\zeta' := \underline{U}_{q^n} \underline{s} \underline{U}_{q^n} 1_- - (q^n - 1) \underline{U}_{q^n} 1_- \in kG\zeta.$$  

Noting that $\zeta'$ can be written as

$$\zeta' = \sum_{z \in \bar{\mathbb{F}}_q^*} \underline{U}_{q^n} \Lambda(z) 1_- - (q^n - 1) \underline{U}_{q^n} 1_-$$

using (3.2), thus it is not difficult to see that the sum of the coefficients in $\zeta'$ is $-2(q^n - 1)$. So we can discuss $\zeta'$ instead of $\zeta$ form the beginning and also have $\underline{U}_{q^n} 1_- \in kG\zeta$.

In conclusion, we have $\underline{U}_{q^n} 1_- \in kG\zeta$ and thus we get $1_- \in kG\zeta$ by [10, Lemma 2.6]. Then for any element $\zeta \notin \mathbb{M}_-$, we see that $kG\zeta = \text{Ind}_N^G k_\zeta$. So, the Steinberg module is the unique simple quotient module of $\text{Ind}_N^G k_\zeta$. Hence Theorem 3.1(1) is proved.  

Before the proof of Theorem 3.1(2), we need to introduce some properties of the nontrivial group homomorphisms from $\bar{\mathbb{F}}_q^*$ to $k^*$, where $k$ is an algebraically closed field of characteristic 0 as before.

**Proposition 3.2.** Let $\lambda : \bar{\mathbb{F}}_q^* \to k^*$ be a nontrivial group homomorphism and $u_1, u_2, \ldots, u_n \in \bar{\mathbb{F}}_q^*$, which are different from each other. Let $x_1, x_2, \ldots, x_n$ be $n$ variables with values in $\bar{\mathbb{F}}_q^*$. Denote by $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and by
\[ x = (x_1, x_2, \ldots, x_n). \] Let \( A_n^\lambda(x) \) be the matrix whose entry in row \( i \) and column \( j \) is \( \lambda(x_i + u_j) \). Thus there exist infinitely many \( x \in (\mathbb{F}_q^*)^n \) such that \( \det A_n^\lambda(x) \neq 0 \).

**Proof.** The lemma is obvious when \( n = 1 \). Assume that this proposition holds when \( n \leq m \). Now we consider the case for \( n = m + 1 \). We set

\[ u^k = (u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_{m+1}), \quad x' = (x_1, x_2, \ldots, x_m). \]

\[ \det A_n^\lambda(x) = \sum_{k=1}^{m+1} (-1)^{m+1+k} \lambda(x_{m+1} + u_k) \det A_n^\lambda(x'). \]

For each fixed integer \( k = 1, 2, \ldots, m+1 \), we have infinitely many \( x' \in (\mathbb{F}_q^*)^m \) such that \( \det A_n^\lambda(x') \neq 0 \) by the inductive hypothesis. Hence it is enough to show that for any element \( a_1, a_2, \ldots, a_{m+1} \in k^* \), there exists an element \( x_0 \in \mathbb{F}_q^* \) such that

\[ a_1 \lambda(x_0 + u_1) + a_2 \lambda(x_0 + u_2) + \cdots + a_{m+1} \lambda(x_0 + u_{m+1}) \neq 0. \]

For an integer \( s \), we let \( \Gamma_s = \lambda(\mathbb{F}_q^*) \), which is a finite cyclic group in \( k^* \). Set \( b_k = -a_k/a_{m+1} \) and we deal with the following equation

\[ b_1 y_1 + b_2 y_2 + \cdots + b_m y_m = 1 \]

with solutions \( y = (y_1, y_2, \ldots, y_m) \in (\Gamma_s)^m \). Denote the solution set of this equation by \( S(b, \Gamma_s) \). Then by [21, Theorem 1.1], for any \( s \in \mathbb{N} \), we have \( |S(b, \Gamma_s)| \leq C(m) \) for some integer \( C(m) \) only depends on \( m \). Since \( \lambda : \mathbb{F}_q^* \rightarrow k^* \) is nontrivial and then we can choose the integer \( s \) large enough such that

\[ |\Gamma_s/\ker \lambda| > C(m) + m. \]

Thus there exists \( x_0 \in \mathbb{F}_q^* \) such that \( x_0 + u_j \neq 0 \) for any \( j = 1, 2, \ldots, m+1 \) and (3.6) holds. The proposition is proved. \( \square \)

According to the above proof of Proposition 3.2 we get the following corollary immediately.

**Corollary 3.3.** Let \( \lambda : \mathbb{F}_q^* \rightarrow k^* \) be a nontrivial group homomorphism and \( u_1, u_2, \ldots, u_n \in \mathbb{F}_q^* \), which are different from each other. Given \( a_1, a_2, \ldots, a_n \in k^* \), then there exists infinitely many elements \( x \in \mathbb{F}_q^* \) such that

\[ a_1 \lambda(x + u_1) + a_2 \lambda(x + u_2) + \cdots + a_n \lambda(x + u_n) \neq 0. \]
Remark 3.4. Proposition 3.2 and Corollary 3.3 do not hold when \( k = \mathbb{F}_q \).

Indeed, we can choose \( q \) elements \( u_1, u_2, \ldots, u_q \in \mathbb{F}_q^* \), which are different from each other and satisfy \( u_1^q + u_2^q + \cdots + u_q^q = 0 \). Let \( \lambda : \mathbb{F}_q^* \to \mathbb{F}_q^* \) be a group homomorphism such that \( \lambda(x) = x^q \). Then it is easy to see that

\[
\lambda(x + u_1) + \lambda(x + u_2) + \cdots + \lambda(x + u_q) = 0
\]

for any \( x \in \mathbb{F}_q^* \) with \( x + u_i \neq 0 \), which is a counter example to Proposition 3.2 and Corollary 3.3. However I guess that Proposition 3.2 and Corollary 3.3 still hold when \( \text{char } k \neq \text{char } \mathbb{F}_q \). Other methods need to be developed.

With the above preparations, now we prove Theorem 3.1(2).

Proof of Theorem 3.1(2). Let \( \varphi_e : \text{Ind}_T^G k_\theta \to \mathcal{M}(\theta) \) be the natural morphism such that \( \varphi_e(1_\theta) = \hat{1}_\theta \), where \( \hat{1}_\theta \) is a fixed nonzero element in \( \mathcal{M}(\theta)_\theta \). Let \( \varphi_s : \text{Ind}_T^G k_\theta \to \mathcal{M}(\theta^s) \) be the morphism such that \( \varphi_s(1_\theta) = s\hat{1}_\theta^s \). Both \( \mathcal{M}(\theta) \) and \( \mathcal{M}(\theta^s) \) are the simple quotient modules of \( \text{Ind}_T^G k_\theta \). Now let \( \xi \) be a nonzero element in \( \text{Ind}_T^G k_\theta \). We will show that \( \mathbb{G}^G \xi = \text{Ind}_T^G k_\theta \) if \( \xi \notin \ker \varphi_e \) and \( \xi \notin \ker \varphi_s \). Thus \( \text{Ind}_T^G k_\theta \) has no other simple quotient modules except \( \mathcal{M}(\theta) \) and \( \mathcal{M}(\theta^s) \).

Claim (1): One has that \( \bar{U}_q sU_q^* \hat{1}_\theta \in \mathbb{G}^{G} \xi \) when \( \xi \notin \ker \varphi_e \).

Proof of Claim (1): Now we write \( \xi \) as following

\[
\xi = \sum_{x \in \mathbb{F}_q} a_x \varepsilon(x) \hat{1}_\theta + \sum_{y, z \in \mathbb{F}_q} b_{y, z} \varepsilon(y) s\varepsilon(z) \hat{1}_\theta.
\]

Noting that \( \xi \notin \ker \varphi_e \), the following equation

\[
\varphi_e(\xi) = \sum_{x \in \mathbb{F}_q} a_x \hat{1}_\theta + \sum_{y, z \in \mathbb{F}_q} b_{y, z} \varepsilon(y) s\varepsilon(z) \hat{1}_\theta \neq 0
\]

tells us that \( \sum_{x \in \mathbb{F}_q} a_x \neq 0 \) or \( \sum_{z \in \mathbb{F}_q} b_{y, z} \neq 0 \) for some \( y \). Without lost of generality, we can assume that \( \sum_{x \in \mathbb{F}_q} a_x \neq 0 \). Otherwise, it is enough to consider the element \( s^{-1} \varepsilon(-y_0) \xi \) instead of \( \xi \). Moreover, we can also assume that \( \sum_{x \in \mathbb{F}_q} a_x \neq 0 \). Otherwise, we can consider the element \( t\xi \) for some \( t \in T \) instead of \( \xi \). Indeed, it is easy to see that the sum of the coefficients in \( t\xi \) is

\[
(3.7) \quad \theta(t) \sum_{x \in \mathbb{F}_q} a_x + \theta^s(t) \sum_{y, z \in \mathbb{F}_q} b_{y, z}.
\]
Noting that \( \sum_{x \in \mathbb{F}_q} a_x \) and \( \sum_{y,z \in \mathbb{F}_q} b_{y,z} \) are nonzero, we can choose one \( t \in T \) such that (3.7) is nonzero since \( \theta \) is nontrivial.

With the assumption that
\[
\sum_{x \in \mathbb{F}_q} a_x \neq 0 \quad \text{and} \quad \sum_{x \in \mathbb{F}_q} a_x + \sum_{y,z \in \mathbb{F}_q} b_{y,z} \neq 0,
\]
we show that \( U_{q^n} s U_{q^n}^* \in \mathbb{k}G \xi \) for some \( n \in \mathbb{N} \). Firstly there exists \( m \in \mathbb{N} \) such that \( x, y, z \in \mathbb{F}_{q^m} \) when \( a_x \neq 0 \) and \( b_{y,z} \neq 0 \). Let \( n > m \) and \( m|n \).

Given an element \( u \in \mathbb{F}_{q^n} \setminus \mathbb{F}_{q^m} \), then the element \( \eta := s \varepsilon(u) \xi \) has the following form
\[
\eta = \sum_{x \in \mathbb{F}_q} a_x s \varepsilon(u + x) \mathbf{1}_{\theta} + \sum_{y,z \in \mathbb{F}_q} b'_{y,z} \varepsilon(-(u + y)^{-1}) s \varepsilon((u + y)^2 z - (u + y)) \mathbf{1}_{\theta},
\]
where \( b'_{y,z} = b_{y,z} \theta(h(-(u + y))) \). Thus if we denote by \( \eta \) as following
\[
\eta = \sum_{x,y \in \mathbb{F}_q} f_{x,y} \varepsilon(x) s \varepsilon(y) \mathbf{1}_{\theta},
\]
then we get \( f_x := \sum_{y \in \mathbb{F}_q} f_{x,y} \neq 0 \) for some \( x \). Let \( v \in \mathbb{F}_q^* \) such that \( v + x \neq 0 \) when \( f_{x,y} \neq 0 \) and we consider the element
\[
\zeta := s \varepsilon(v) \eta = \sum_{x,y \in \mathbb{F}_q} f_{x,y} \theta(h(-(v + x))) \varepsilon(-(v + x)^{-1}) s \varepsilon((v + x)^2 y - (v + x)) \mathbf{1}_{\theta}.
\]
By Lemma 3.3 we can choose some element \( v \) such that the elements \( (v + x)^2 y - (v + x) \) are nonzero when \( f_{x,y} \neq 0 \) and this element \( v \) also makes
\[
f := \sum_{x \in \mathbb{F}_q} f_x \theta(h(-(v + x))) \neq 0.
\]
As before we choose \( \mathcal{C}_{q^n} \subset T \) such that \( \alpha : \mathcal{C}_{q^n} \to \mathbb{F}_{q^n}^* \) is a bijection. Therefore we get
\[
\sum_{t \in \mathcal{C}_{q^n}} \alpha^*(t)^{-1} t U_{q^n} s U_{q^n}^* \mathbf{1}_{\theta} = f U_{q^n} s U_{q^n}^* \mathbf{1}_{\theta}
\]
which implies that \( U_{q^n} s U_{q^n}^* \mathbf{1}_{\theta} \in \mathbb{k}G \xi \). Claim (♣) is proved.

Claim (♠): When \( \xi \notin \ker \varphi_{\alpha} \), one has an element
\[
\xi' = \sum_{x \in \mathbb{F}_q} a_x \varepsilon(x) \mathbf{1}_{\theta} + \sum_{y \in \mathbb{F}_q} b_y \varepsilon(y) s \mathbf{1}_{\theta} + \sum_{u,v \in \mathbb{F}_q, v \in \mathbb{F}_q^*} c_{u,v} \varepsilon(u) s \varepsilon(v) \mathbf{1}_{\theta} \in \mathbb{k}G \xi
\]
such that $\sum_{y \in \overline{F}_q} b_y \neq 0$.

Proof of Claim (♠): Firstly we study the form of the elements in $\ker \varphi_s$. Let

$$\varpi_1 = \sum_{x \in \overline{F}_q} f_x \varepsilon(x)(e + \sum_{u \in \overline{F}_q} f_{x,u} \theta^s(-u) \varepsilon(u) \hat{s} \varepsilon(u^{-1})) \mathbf{1}_\theta,$$

where $\sum_{u \in \overline{F}_q} f_{x,u} + 1 = 0$ for any $x \in \overline{F}_q$ and set

$$\varpi_2 = \sum_{y \in \overline{F}_q} \varepsilon(y) \sum_{v \in \overline{F}_q} g_{y,v} \theta^s(-v) \varepsilon(v) \hat{s} \varepsilon(v^{-1}) \mathbf{1}_\theta,$$

where $\sum_{v \in \overline{F}_q} g_{y,v} = 0$ for any $y \in \overline{F}_q$. Moreover we let

$$\varpi_3 = \sum_{z \in \overline{F}_q} h_z \varepsilon(z) \mathbf{1}_\theta, \quad \text{where} \sum_{z \in \overline{F}_q} h_z = 0.$$

It is easy to check that $\varpi_1, \varpi_2, \varpi_3 \in \ker \varphi_s$. Denote by $\Omega_i$ the set of the elements with the form of $\varpi_i$ for $i = 1, 2, 3$. Then it is not difficult to see that each element of $\ker \varphi_s$ is a linear combination of the elements in $\Omega_1, \Omega_2$ and $\Omega_3$.

Now let $\xi$ be the following

$$\xi = \sum_{x \in \overline{F}_q} \alpha_x \varepsilon(x) \mathbf{1}_\theta + \sum_{y \in \overline{F}_q} \beta_y \varepsilon(y) \mathbf{1}_\theta + \sum_{u \in \overline{F}_q, v \in \overline{F}_q^*} \gamma_{u,v} \varepsilon(u) \hat{s} \varepsilon(v) \mathbf{1}_\theta$$

If $\sum_{y \in \overline{F}_q} \beta_y \neq 0$, then $\xi$ has already satisfied our requirement. Otherwise, we can write

$$\xi = \varpi + \sum_{u \in \overline{F}_q, v \in \overline{F}_q^*} \tau_{u,v} \varepsilon(u) \hat{s} \varepsilon(v) \mathbf{1}_\theta, \quad \tau_{u,v} \in k$$

such that $\varpi \in \ker \varphi_s$. Since $\xi \notin \ker \varphi_s$, the element

$$\eta := \sum_{u \in \overline{F}_q, v \in \overline{F}_q^*} \tau_{u,v} \varepsilon(u) \hat{s} \varepsilon(v) \mathbf{1}_\theta$$

is not in $\ker \varphi_s$ and in particular, it is not in $\Omega_2$. Now for $x \in \overline{F}_q$, set

$$\Xi_x = \{(u, v) \mid \tau_{u,v} \neq 0 \text{ and } (u - x)v = 1\}.$$

Since $\eta \notin \ker \varphi_s$, there exist an element $x_0$ such that

$$\sum_{(u, v) \in \Xi_{x_0}} \tau_{u,v} \theta^s(-v) \neq 0$$
by some easy computation. Now we consider the following element

\[ \xi' = s \varepsilon(-x_0) \xi = s \varepsilon(-x_0)(\varpi + \eta). \]

Firstly, we have \( s \varepsilon(-x_0) \varpi \in \ker \varphi_s \), which is a linear combination of elements in \( \Omega_1, \Omega_2 \) and \( \Omega_3 \). For the second part \( s \varepsilon(-x_0) \eta \), it is easy to check that when \( (u, v) \notin \Xi_{x_0} \), the element

\[ s \varepsilon(-x_0) \varepsilon(u) s \varepsilon(v) 1_\theta = \varepsilon(u') s \varepsilon(v') 1_\theta \]

for some \( u' \in \bar{\mathbb{F}}_q \) and \( v' \in \bar{\mathbb{F}}^*_q \). On the other hand, for \( (u, v) \in \Xi_{x_0} \), we have

\[ s \varepsilon(-x_0) \sum_{(u, v) \in \Xi_{x_0}} \tau_{u,v} \varepsilon(u) s \varepsilon(v) 1_\theta = \sum_{(u, v) \in \Xi_{x_0}} \tau_{u,v} \theta(-v^{-1}) \varepsilon(-v) s 1_\theta. \]

Noting that \( \theta(-v^{-1}) = \theta^s(-v) \) and using (3.8), it is easy to check that \( \xi = s \varepsilon(-x_0) \xi \in kG \xi \) satisfies our requirement.

With Claim (♠) and Claim (♣), now we give the proof of Theorem 3.1(2). Let \( \xi \) be an element such that \( \xi \notin \ker \varphi_e \) and \( \xi \notin \ker \varphi_s \). By Claim (♠), there exists an element

\[ \xi' = \sum_{x \in \mathbb{F}_q} a_x \varepsilon(x) 1_\theta + \sum_{y \in \mathbb{F}_q} b_y \varepsilon(y) s 1_\theta + \sum_{u \in \mathbb{F}_q, v \in \mathbb{F}_q} c_{u,v} \varepsilon(u) \varepsilon(v) 1_\theta \in kG \xi \]

such that \( \sum_{y \in \mathbb{F}_q} b_y \neq 0 \). We denote by

\[ A = \sum_{x \in \mathbb{F}_q} a_x, \quad B := \sum_{y \in \mathbb{F}_q} b_y, \quad C = \sum_{u \in \mathbb{F}_q, v \in \mathbb{F}_q} c_{u,v} \]

for simple. Let \( n \in \mathbb{N} \) such that \( x, y, u, v \in \mathbb{F}_{q^n} \) when \( a_x \neq 0, b_y \neq 0 \) and \( c_{u,v} \neq 0 \). As before we choose \( \mathcal{D}_{q^n} \subset T \) such that \( \alpha : \mathcal{D}_{q^n} \rightarrow \mathbb{F}_{q^n}^* \) is a bijection. Hence it is not difficult to see that

\[ \sum_{t \in \mathcal{D}_{q^n}} \theta^s(t)^{-1} t U_{q^n}^\alpha \xi' = A \sum_{t \in \mathcal{D}_{q^n}} \theta(t^2) 1_\theta + (q^n - 1) B \overline{U_{q^n}}^s s 1_\theta + C \overline{U_{q^n}}^s \overline{U_{q^n}^s} 1_\theta, \]

which is in \( kG \xi \). When \( n \) is big enough and \( \theta \) is nontrivial on \( \mathbb{F}_{q^n}^* \), we have

\[ \sum_{t \in \mathcal{D}_{q^n}} \theta(t^2) = 0. \]

By Claim (♣), we have \( U_{q^n}^s \overline{U_{q^n}^s} 1_\theta \in kG \xi \) and then we get \( \overline{U_{q^n}^s} 1_\theta \in kG \xi \) since \( B \) is nonzero. So using [10] Lemma 2.6, we have \( 1_\theta \in kG \xi \) and thus \( kG \xi = \text{Ind}_T^G k_\theta \). Therefore the induced module \( \text{Ind}_T^G k_\theta \) has only two simple quotient modules \( M(\theta) \) and \( M(\theta^s) \). The theorem is proved.
4. Abelian category $\mathcal{C}(G)$

In this section, we show that the category $\mathcal{C}(G)$ is an abelian category for $G = SL_2(\mathbb{F}_q)$. For convenience, we denote

$$\text{Irr}(G) = \{k_{\text{tr}}, St, M(\theta) \mid \theta \in \hat{T} \text{ is nontrivial}\}.$$  

This is the set of the simple objects in $\mathcal{C}(G)$. By [7, Theorem 2.5], the induced module $\text{Ind}_{T}^{B} k_{\theta}$ has a unique simple $k_{B}$-submodule

$$S_{\theta} = \{\sum_{x \in \mathbb{F}_q^*} a_x \epsilon(x)1_{\theta} \mid \sum_{x \in \mathbb{F}_q^*} a_x = 0\}$$

and the corresponding quotient module is $k_{\theta}$.

**Lemma 4.1.** Let $S = S_{\text{tr}}$ be the unique simple $k_{B}$-submodule of $\text{Ind}_{T}^{B} k_{\theta}$. Then $S^n$ (the direct sum of $n$-copies of $S$) can not assign a $k_{G}$-module structure for any $n \in \mathbb{N}$.

**Proof.** Firstly, each element $\xi \in S^n$ has the following form

$$\xi = \sum_{j,\mu} f_{j,\mu}(e - \epsilon(x_{j,\mu}))\lambda_j,$$

where each $\lambda_j$ denotes the trivial character for $j = 1, 2, \ldots, n$. Suppose $S^n$ has a $k_{G}$-module structure and for each $i = 1, 2, \ldots, n$, we set

$$\dot{s}(e - \epsilon(1))\lambda_i = \sum_{j,\mu} g_{j,\mu}^i (e - \epsilon(x_{j,\mu}^i))\lambda_j,$$

where $x_{j,\mu}^i \in \bar{\mathbb{F}}_q^*$, $g_{j,\mu}^i \in k$.

Using $t \in T$ to act on both sides, then it is not difficult to see that for $y \in \mathbb{F}_q^*$, we have

$$\dot{s}(e - \epsilon(y))\lambda_i = \sum_{j,\mu} g_{j,\mu}^i (e - \epsilon(y^{-1}x_{j,\mu}^i))\lambda_j.$$  

Therefore we have

$$\dot{s}(\epsilon(z))\dot{s}(e - \epsilon(1))\lambda_i = \sum_{j,\mu} g_{j,\mu}^i \dot{s}(\epsilon(z) - \epsilon(x_{j,\mu}^i + z))\lambda_j$$

$$= \sum_{j,\mu} g_{j,\mu}^i \sum_{k,\nu} g_{k,\nu}^j (\epsilon(z^{-1}x_{k,\nu}^j) - \epsilon((x_{j,\mu}^i + z)^{-1}x_{k,\nu}^j))\lambda_k$$

(4.1)
for any element $z \in \bar{\mathbb{F}}_q^*$ such that $x_{j,\mu}^i + z \neq 0$. On the other hand, since $s \in \mathbb{Z} \setminus z = \varepsilon(-z^{-1})s\theta(-z)\varepsilon(-z^{-1})$, we get

$$s \varepsilon(z)s = \varepsilon(-z^{-1})s\theta(-z)\varepsilon(-z^{-1}),$$

and

$$\lambda_i = \varepsilon(-z^{-1})s\varepsilon(-z) - \varepsilon(z^2 - z))\lambda_i = \sum_{i,\nu} g_{i,\nu}(\varepsilon(-z^{-1}(1 + x_{j,\nu})) - \varepsilon(z^{-1}(z - 1)^{-1}(x_{j,\nu} - z + 1))\lambda_k.$$

Combining the above two equations (1.1) and (1.2), we get the following

$$\sum_{j,\mu} g_{j,\mu} g_{i,\nu}(\varepsilon(z^{-1}x_{j,\nu}) - \varepsilon((x_{j,\mu} + z)^{-1}x_{k,\nu}))\lambda_k = \sum_{k,\nu} g_{k,\nu}(\varepsilon(-z^{-1}(1 + x_{j,\nu}^i)) - \varepsilon(z^{-1}(z - 1)^{-1}(x_{j,\nu}^i - z + 1)))\lambda_k$$

for any $z \in \bar{\mathbb{F}}_q^*$ such that $x_{j,\mu}^i + z \neq 0$ and $i = 1, 2, \ldots, n$. However it is not difficult to see that (4.3) is not an identity. Indeed, for any fixed $i = 1, 2, \ldots, n$, if there exists $x_{i,\nu}^j \neq -1$, then we can choose one $z_0 \in \bar{\mathbb{F}}_q^*$ such that $z_0^{-1}(z_0 - 1)^{-1}(x_{k,\nu}^i + z_0 + 1)$ is different from the following set

$$\{z_0^{-1}x_{k,\nu}^i, (x_{j,\mu} + z_0)^{-1}x_{k,\nu}^i, -z_0^{-1}(1 + x_{k,\nu}^i) \mid g_{k,\nu}^i \neq 0, g_{j,\mu}^i \neq 0, g_{k,\nu}^i \neq 0\}.$$

If there is only one $x_{k,\nu}^i = -1$ such that $g_{k,\nu}^i \neq 0$, then (4.3) does not hold obviously. Therefore $S^n$ cannot assign a $\mathbb{G}$-module structure. The lemma is proved.

In general, using the same discussion as in Lemma 4.1, we have the following proposition.

**Proposition 4.2.** Let $S_\lambda$ be the unique simple $\mathbb{G}$-submodule of $\text{Ind}_{\mathbb{F}}^{\mathbb{G}} k_\lambda$ for each $\lambda \in \hat{T}$. One has that $\bigoplus_{\lambda \in \hat{T}} S_\lambda^{n_\lambda}$ can not assign a $\mathbb{G}$-module structure, where $n_\lambda \in \mathbb{N}$ and only finitely many $n_\lambda$ are nonzero for $\lambda \in \hat{T}$.

**Theorem 4.3.** The category $\mathcal{C}(G)$ is an abelian category.

**Proof.** Recall the definition of $\mathcal{C}(G)$ in Section 1, it is enough to show that $\mathcal{C}(G)$ is closed under taking subquotients. Let $M$ be a $\mathbb{G}$-module with $\mathbb{T}$-eigenvectors $\xi_1, \xi_2, \ldots, \xi_n$ such that $M \cong \bigoplus_{i=1}^n \mathbb{U}_i \xi_i$ as $\mathbb{G}$-modules. Following [7, Theorem 2.5], when $\xi \in M_\theta$, each $\mathbb{U}_\xi$ is isomorphic to $\text{Ind}_{\mathbb{N}}^{\mathbb{G}} k_\theta$ or $k_\theta$. Thus $M$ is a $\mathbb{G}$-module of finite length. Let $N$ be a simple quotient of $M$, then $N$ has a $\mathbb{T}$-eigenvector which implies $N \in \text{Irr}(G)$. The Steinberg module $S_t$ is isomorphic to $\text{Ind}_{\mathbb{T}}^{\mathbb{G}} k_t$ as $\mathbb{G}$-modules and $M(\theta)$ is isomorphic
to $k_\theta \oplus \text{Ind}_B^T k_\theta$ as $kB$-modules. However since $\bigoplus_{\lambda \in T} S_{\lambda}$ can not assign a $kG$-module structure by Proposition 4.2, it is easy to see that the subquotients of $M$ are also in $\mathcal{C}(G)$. The theorem is proved.

\begin{corollary}
The category $\mathcal{C}(G)$ is noetherian and artinian.
\end{corollary}

A $kG$-module can be regarded as a $kT$-module (resp. $kB$-module) naturally. We denote by $\mathcal{C}(G)_T$ (resp. $\mathcal{C}(G)_B$) the full subcategory of $kT$-Mod (resp. $kB$-Mod), which consists of the objects in $\mathcal{C}(G)$.

\begin{corollary}
One has that $\text{Hom}_T(k_\theta, -) : \mathcal{C}(G)_T \to \text{Vect}$ is an exact functor for any $\theta \in \hat{T}$. Thus given a short exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

in the category $\mathcal{C}(G)$, one has that $M^T \cong M_1^T \oplus M_2^T$ as $T$-modules. In particular, we have $\text{Wt}(M) = \text{Wt}(M_1) \cup \text{Wt}(M_2)$.

\textit{Proof.} Using the setting in \cite[Section 2]{12}, when $k$ is an algebraically closed field of characteristic zero, $kB$ is a locally Wedderburn algebra. Thus by \cite[Lemma 3]{12}, $k_\theta$ is an injective $kB$-module which implies the exactness of $\text{Hom}_B(-, k_\theta)$. For $M \in \mathcal{C}(G)$, we have $M \cong \bigoplus_{i=1}^n kU\xi_i$ as $kB$-modules, where $\xi_i \in M_{\lambda_i}$. By \cite[Theorem 2.5]{7}, the induced module $\text{Ind}_T^B k_\theta$ has a unique simple $kB$-submodule $S_{\theta}$ and the corresponding quotient module is $k_\theta$. Thus each $kU\xi_i$ is isomorphic to $\text{Ind}_T^B k_{\lambda_i}$ or $k_{\lambda_i}$ as $kB$-modules. Therefore it is easy to see that

$$\text{Hom}_T(k_\theta, M) \cong \text{Hom}_B(M, k_\theta),$$

which implies the exactness of $\text{Hom}_T(k_\theta, -)$. The rest part is obvious. \hfill \Box

\begin{corollary}
Given a short exact sequence in $\mathcal{C}(G)$, then it is split regarded as a short exact sequence in $\mathcal{C}(G)_B$.
\end{corollary}

\textit{Proof.} By the definition of $\mathcal{C}(G)$ and Proposition 4.2 noting that the set of simple objects in the category $\mathcal{C}(G)$ is

$$\text{Irr}(G) = \{k_{\text{tr}}, \text{St}, M(\theta) \mid \theta \in \hat{T} \text{ is nontrivial}\},$$

the corollary is proved. \hfill \Box
5. Highest weight category

In this section, we show that \(\mathcal{C}(G)\) is a highest weight category for \(G = SL_2(\bar{F}_q)\). Firstly we recall the definition of highest weight category (see [8]).

**Definition 5.1.** Let \(\mathcal{C}\) be a locally artinian, abelian, \(k\)-linear category with enough injective objects that satisfies Grothendieck’s condition. Then we call \(\mathcal{C}\) a highest weight category if there exists a locally finite poset \(\Lambda\) (the “weights” of \(\mathcal{C}\)), such that:

(a) There is a complete collection \(\{S(\lambda)_{\lambda \in \Lambda}\}\) of non-isomorphic simple objects of \(\mathcal{C}\) indexed by the set \(\Lambda\).

(b) There is a collection \(\{A(\lambda)_{\lambda \in \Lambda}\}\) of objects of \(\mathcal{C}\) and, for each \(\lambda\), an embedding \(S(\lambda) \subset A(\lambda)\) such that all composition factors \(S(\mu)/S(\lambda)\) satisfy \(\mu < \lambda\). For \(\lambda, \mu \in \Lambda\), we have that \(\dim_k \text{Hom}_\mathcal{C}(A(\lambda), A(\mu))\) and \([A(\lambda) : S(\mu)]\) are finite.

(c) Each simple object \(S(\lambda)\) has an injective envelope \(I(\lambda)\) in \(\mathcal{C}\). Also, \(I(\lambda)\) has a good filtration \(0 = F_0(\lambda) \subset F_1(\lambda) \subset \ldots\) such that:

(i) \(F_1(\lambda) \cong A(\lambda)\);

(ii) for \(n > 1\), \(F_n(\lambda)/F_{n-1}(\lambda) \cong A(\mu)\) for some \(\mu = \mu(n) > \lambda\);

(iii) for a given \(\mu \in \Lambda\), \(\mu = \mu(n)\) for only finitely many \(n\);

(iv) \(\bigcup F_i(\lambda) = I(\lambda)\).

Actually, to show that \(\mathcal{C}(G)\) is a highest weight category, the main difficulty is to prove that the category \(\mathcal{C}(G)\) has enough injective objects.

**Proposition 5.2.** For any \(M, N\) in \(\mathcal{C}(G)\) such that \(\text{Wt}(M) \cap \text{Wt}(N) = \emptyset\), we have \(\text{Ext}^1_{\mathcal{C}(G)}(M, N) = 0\).

**Proof.** Since each object in \(\mathcal{C}(G)\) is of finite length, it is enough to show that \(\text{Ext}^1_{\mathcal{C}(G)}(M, N) = 0\) for any simple object \(M, N \in \mathcal{C}(G)\), where \(\text{Wt}(M) \cap \text{Wt}(N) = \emptyset\). Recall that \(\text{Irr}(G) = \{k_{tr}, St, M(\theta) | \theta \in \hat{T} \text{ is nontrivial}\}\) is the set of simple objects in \(\mathcal{C}(G)\), we will show that

\[
\text{Ext}^1_{\mathcal{C}(G)}(k_{tr}, M(\lambda)) = 0, \quad \text{Ext}^1_{\mathcal{C}(G)}(St, M(\mu)) = 0 \quad \text{and} \quad \text{Ext}^1_{\mathcal{C}(G)}(M(\theta), S) = 0,
\]

where \(S\) is a simple object whose weights are different with \(\theta\) and \(\theta^a\).

(1) \(\text{Ext}^1_{\mathcal{C}(G)}(k_{tr}, M(\lambda)) = 0\). Suppose we have a short exact sequence

\[
0 \rightarrow M(\lambda) \rightarrow M \rightarrow k_{tr} \rightarrow 0 \quad (5.1)
\]

in \(\mathcal{C}(G)\). Then using Corollary [4.6] there exists an element \(\xi \in M\) such that \(z\xi = \xi\) for any \(z \in B\). Noting that \(M_{tr} = k\xi\), thus \(s^a\xi = a\xi\) for some
\( a \in \mathbb{k}^\ast \). However the following equation
\[
\dot{s}\varepsilon(x)\dot{s}\xi = \varepsilon(-x^{-1})\dot{s}h(-x)\varepsilon(-x^{-1})\xi
\]
shows that \( a \) must be 1. Therefore the short exact sequence (5.1) is split which implies that \( \text{Ext}_{\mathcal{C}(G)}^1(k_{tr}, M(\lambda)) = 0 \).

(2) \( \text{Ext}_{\mathcal{C}(G)}^1(St, M(\mu)) = 0 \). Suppose we have a short exact sequence
\[
0 \rightarrow M(\mu) \rightarrow N \rightarrow St \rightarrow 0
\]
in \( \mathcal{C}(G) \). By Corollary 4.6, there exists an element \( \eta \in N_{tr} \) such that \( \dot{s}\eta = b\eta \) for some \( b \in \mathbb{k}^\ast \) and \( \dot{s}\varepsilon(1)\eta = (\varepsilon(-1) - e)\eta + \varpi \) for some \( \varpi \in M(\mu) \). Using \( \dot{s} \) to act on both sides, we get \( b = -1 \). Moreover we have
\[
\dot{s}\varepsilon(x)\eta = (\varepsilon(1))\dot{s}\varepsilon(x)\eta + \varepsilon(-x^{-1}) \dot{s}\varepsilon(x) \varepsilon(1) \eta.
\]
Now we consider the element \( \dot{s}\varepsilon(z)\dot{s}\varepsilon(1)\eta \). Firstly we have
\[
\dot{s}\varepsilon(z)\dot{s}\varepsilon(1)\eta = \varepsilon(-z^{-1})\dot{s}h(-z)\varepsilon(-z^{-1})\dot{s}\varepsilon(1)\eta = \varepsilon(-z^{-1})\dot{s}\varepsilon(z^2 - z)\eta.
\]
On the other hand, we get
\[
\dot{s}\varepsilon(z)\dot{s}\varepsilon(1)\eta = \dot{s}(\varepsilon(z) - \varepsilon(z))\eta + \dot{s}\varepsilon(z)\varpi.
\]
Compare the above two equations (5.3) and (5.4), it is not difficult to get
\[
\dot{s}\varepsilon(z)\varpi = \varepsilon(-z^{-1})h((z^2 - z)^{-\frac{1}{2}}))\varpi + (h(z^{-\frac{1}{2}}) - h((z - 1)^{-\frac{1}{2}}))\varpi
\]
for any \( z \in \overline{F}_q \). Denote by
\[
\varpi = f1_\mu + \sum_{x \in \overline{F}_q} g_x \varepsilon(x)1_\mu, \quad \text{where } f, g_x \in \overline{F}_q.
\]
Then it is easy to see that \( f = 0 \). Moreover if \( g_x \neq 0 \) when \( x \neq 0, -1 \), then (5.5) can not hold for any \( z \in \overline{F}_q \). Substitute \( \varpi = a\dot{s}1_\mu + b\varepsilon(1)\dot{s}1_\mu \) into (5.5) and we get \( \varpi = 0 \). Thus the short exact sequence (5.2) is split which implies that \( \text{Ext}_{\mathcal{C}(G)}^1(St, M(\mu)) = 0 \).

(3) \( \text{Ext}_{\mathcal{C}(G)}^1(M(\theta), S) = 0 \) for some simple object \( S \) whose weights are different with \( \theta \) and \( \theta^s \). Suppose we have a short exact sequence
\[
0 \rightarrow S \rightarrow L \rightarrow M(\theta) \rightarrow 0
\]
in \( \mathcal{C}(G) \). Then by Corollary 4.6, there exists \( \zeta \in L_\theta \) such that \( u\zeta = \zeta \) and thus \( \dot{s}\zeta \in L_{\theta^s} \) since the weights of \( S \) are different with \( \theta \) and \( \theta^s \). Thus the short exact sequence (5.6) is split and we have \( \text{Ext}_{\mathcal{C}(G)}^1(M(\theta), S) = 0 \). □
**Proposition 5.3.** One has that $k_{tr}$ and $M(\theta)$ are injective objects in the category $\mathcal{C}(G)$ for any $\theta \in T$.

**Proof.** By [12] Theorem 1, we see that $k_{tr}$ is an injective $kG$-module and thus it is injective in $\mathcal{C}(G)$. According to Proposition 5.2, it is enough to verify that $\text{Ext}^1(\mathcal{C})$ is injective and we have $\text{Ext}^1(\mathcal{C}, \mathcal{C})$. Noting that $\text{Ext}^1(\mathcal{C}_G(\text{tr}), M(tr)) = 0$, $\text{Ext}^1(\mathcal{C}_G(\text{tr}, St), M(tr)) = 0$, and $\text{Ext}^1(\mathcal{C}_G(\text{tr}, k_{tr}), M(tr)) = 0$, we get a long exact sequence

$$0 \rightarrow \text{Hom}(M(tr), k_{tr}) \rightarrow \text{Hom}(M(tr), St) \rightarrow \text{Hom}(M(tr), k_{tr}) \rightarrow \text{Ext}(\mathcal{C}_G(\text{tr}, St), M(tr)) \rightarrow \text{Ext}(\mathcal{C}_G(\text{tr}, k_{tr}), M(tr)) \rightarrow \text{Ext}(\mathcal{C}_G(\text{tr}, k_{tr}), k_{tr}) \rightarrow \text{Ext}(\mathcal{C}_G(\text{tr}, k_{tr}), k_{tr}) \rightarrow \ldots$$

It is easy to check that $\text{Hom}(M(tr), k_{tr}) = 0$ and $\text{Hom}(M(tr), St) = 0$ because $k_{tr}$ is injective in $\mathcal{C}(G)$, it is enough to verify that $\text{Ext}(\mathcal{C}_G(\text{tr}, St), St) \cong k$. Given a short exact sequence

$$0 \rightarrow St \rightarrow M \rightarrow k_{tr} \rightarrow 0$$

in $\mathcal{C}(G)$, then by Corollary 4.6 we get

$$\text{Hom}(M(tr), M) \cong \text{Hom}(k_{tr}, M) \cong k$$

using Frobenius reciprocity. Thus $M \cong M(tr)$ or $M \cong St \oplus k_{tr}$. Therefore $\text{Ext}(\mathcal{C}_G(\text{tr}, St), M(tr)) \cong k$ and hence $\text{Ext}(\mathcal{C}_G(\text{tr}, M(tr)) = 0$.

(2) $\text{Ext}(\mathcal{C}_G(\text{tr}, St), M(tr)) = 0$. Using the short exact sequence

$$0 \rightarrow St \rightarrow M(tr) \rightarrow k_{tr} \rightarrow 0,$$

we get a long exact sequence

$$0 \rightarrow \text{Hom}(M(tr), St) \rightarrow \text{Hom}(M(tr), M(tr)) \rightarrow \text{Hom}(M(tr), k_{tr}) \rightarrow \text{Ext}(\mathcal{C}_G(\text{tr}, St), M(tr)) \rightarrow \text{Ext}(\mathcal{C}_G(\text{tr}, M(tr)), M(tr)) \rightarrow \text{Ext}(\mathcal{C}_G(\text{tr}, k_{tr}), M(tr)) \rightarrow \text{Ext}(\mathcal{C}_G(\text{tr}, k_{tr}), k_{tr}) \rightarrow \ldots$$

Since $k_{tr}$ is injective and we have $\text{Ext}(\mathcal{C}_G(\text{tr}, k_{tr}), M(tr)) = 0$, it is enough to show that $\text{Ext}(\mathcal{C}_G(\text{tr}, St), St) = 0$. Given a short exact sequence

$$(5.7) \quad 0 \rightarrow St \rightarrow N \rightarrow St \rightarrow 0$$

in $\mathcal{C}(G)$, then there exists $\xi_1, \xi_2 \in N_{tr}$ such that $N \cong kU\xi_1 \oplus kU\xi_2$ as $kB$-modules and $kU\xi_1$ is isomorphic to the Steinberg module. Firstly it is
easy to see that \( \dot{s}\xi_2 = -\xi_2 + a\xi_1 \) for some \( a \in k \). Using \( \dot{s} \) to act on both sides, we get \( a = 0 \) and thus \( \dot{s}\xi_2 = -\xi_2 \). On the other hand, we have
\[
\dot{s}\varepsilon(1)\xi_2 = (\varepsilon(-1) - e)\xi_2 + \varpi
\]
for some \( \varpi \in kU\xi_1 \). Denote by \( \varpi = \sum_{x \in \mathbb{F}_q} a_x \varepsilon(x)\xi_1 \) and then we get
\[
\dot{s}\varepsilon(y)\xi_2 = (\varepsilon(-y^{-1}) - e)\xi_2 + \sum_{x \in \mathbb{F}_q} a_x \varepsilon(xy^{-1})\xi_1
\]
for any \( y \in \mathbb{F}_q^* \). Now we consider the element \( \dot{s}\varepsilon(z)\dot{s}\varepsilon(1)\xi_2 \). Firstly we have
\[
(5.8) \quad \dot{s}\varepsilon(z)\dot{s}\varepsilon(1)\xi_2 = \varepsilon(-y^{-1})\dot{s}h(-z)\varepsilon(-z^{-1})\varepsilon(1)\xi_2 = \dot{s}\varepsilon(z^2 - z)\xi_2.
\]
On the other hand, we have
\[
(5.9) \quad \dot{s}\varepsilon(z)\dot{s}\varepsilon(1)\xi_2 = \dot{s}(\varepsilon(z - 1) - \varepsilon(z))\xi_2 + \dot{s}\varepsilon(z)\varpi.
\]
Combining \((5.8)\) and \((5.9)\), it is not difficult to get
\[
(5.10) \quad \dot{s}\varepsilon(z) \sum_{x \in \mathbb{F}_q} a_x \varepsilon(x)\xi_1 = \sum_{x \in \mathbb{F}_q} a_x (\varepsilon(x(z^2 - z))^{-1} - z^{-1}) + \varepsilon(xz^{-1}) - \varepsilon(x(z - 1)^{-1})\xi_1
\]
for any \( z \in \mathbb{F}_q \). If \( a_x \neq 0 \) when \( x \neq 0, -1 \), then the equation cannot hold for any \( z \in \mathbb{F}_q \). However, substitute \( \varpi = a\xi_1 + b\varepsilon(-1)\xi_1 \) into \((5.10)\), we get \( \varpi = 0 \) easily. Thus the short exact sequence \((5.7)\) is split and we have \( \operatorname{Ext}^1_{\mathcal{C}(G)}(\text{St}, \text{St}) = 0 \) which implies that \( \operatorname{Ext}^1_{\mathcal{C}(G)}(\text{St}, M(\text{tr})) = 0 \).

(3) \( \operatorname{Ext}^1_{\mathcal{C}(G)}(M(\lambda), M(\theta)) = 0 \) for \( \lambda = \theta \) or \( \theta^s \). Firstly given a short exact sequence
\[
(5.11) \quad 0 \longrightarrow M(\theta) \longrightarrow K \longrightarrow M(\theta) \longrightarrow 0
\]
in \( \mathcal{C}(G) \), then by Corollary 4.6 we have
\[
K \cong k_\theta \oplus k_\theta \oplus \text{Ind}_F^B k_\theta^s \oplus \text{Ind}_F^B k_\theta^s
\]
as \( kB \)-modules and there exists \( \xi_1, \eta_1 \in K_\theta \) and \( \xi_2, \eta_2 \in K_{\theta^s} \) such that \( k\xi_1 + kU\xi_2 \cong M(\theta) \), which is a \( kG \)-submodule of \( K \). We can assume that \( \xi_2 = \dot{s}\xi_1 \). Now suppose
\[
\dot{s}\eta_1 = a\xi_2 + b\eta_2, \quad \dot{s}\eta_2 = c\xi_1 + d\eta_1
\]
for some \( a, b, c, d \in k \). Using \( \dot{s} \) to act on both sides of the two equations, we get \( c = -\frac{a}{b} \) and \( d = \frac{1}{b} \). In particular, \( b \) is nonzero. Now set \( \tilde{\eta}_2 = a\xi_2 + b\eta_2 \).
Thus $k\eta_1 + kU\tilde{\eta}_2$ is isomorphic to $M(\theta)$. So the short exact sequence (5.11) is split which implies that $\text{Ext}^1_{\mathcal{E}(G)}(M(\theta), M(\theta)) = 0$.

Next we consider the short exact sequence
\begin{equation}
0 \longrightarrow M(\theta) \longrightarrow L \longrightarrow M(\theta^* \lambda) \longrightarrow 0
\end{equation}
in $\mathcal{E}(G)$. There exists $\zeta_1, \varrho_1 \in L_\theta$ and $\zeta_2, \varrho_2 \in L_{\theta^*}$ such that $M(\theta) \cong k\zeta_1 + kU\zeta_2$. We can assume that $\zeta_2 = s\zeta_1$. Now suppose
\[ s\varrho_1 = a'\zeta_2 + b'\varrho_2, \quad s\varrho_2 = c'\zeta_1 + d'\varrho_1, \]
then we also get $c' = -\frac{a'}{b'}$ and $d' = \frac{1}{b'}$. In particular, $b'$ and $d'$ are nonzero. Set $\varrho_1 = c'\zeta_1 + d'\varrho_1$, then we get $M(\theta^*) \cong k\varrho_2 + kU\varrho_1$. Therefore the short exact sequence (5.12) is split and we also get $\text{Ext}^1_{\mathcal{E}(G)}(M(\theta^* \lambda), M(\theta)) = 0$.

The proposition is proved.

\[ \square \]

**Theorem 5.4.** The category $\mathcal{E}(G)$ has enough injective objects.

**Proof.** By Proposition 5.3, the injective envelope for each simple object exists in the category $\mathcal{E}(G)$. Thus a standard argument shows that $\mathcal{E}(G)$ has enough injective objects (see [9, Theorem 2.7]).

\[ \square \]

Now we show that $\mathcal{E}(G)$ is a highest weight category. In Definition 5.1, the set of weights is $\Lambda = \{(\theta, J) \mid \theta \in \hat{T}, J \subset I(\theta)\}$ and we define the order of the weights by
\[(\theta_1, J_1) \leq (\theta_2, J_2), \text{ if } \theta_1 = \theta_2 \text{ and } J_1 \supseteq J_2.\]
Specifically, for $G = SL_2(\tilde{F}_q)$, $\Lambda = \{(\text{tr}, \emptyset), (\text{tr}, \{s\}), (\theta, \emptyset) \mid \theta \text{ is nontrivial}\}$. Let $S(\text{tr}, \{s\}) = A(\text{tr}, \{s\}) = k_{\text{tr}}$, $S(\text{tr}, \emptyset) = A(\text{tr}, \emptyset) = \text{St}$ and $S(\theta, \emptyset) = A(\theta, \emptyset) = M(\theta)$. By Proposition 5.3, we set $I(\text{tr}, \{s\}) = k_{\text{tr}}$, $I(\text{tr}, \emptyset) = M(\text{tr})$ and $I(\theta, \emptyset) = M(\theta)$. It is not difficult to check that $\mathcal{E}(G)$ with this setting satisfies all the condition in Definition 5.1 and thus we have the following theorem.

**Theorem 5.5.** The category $\mathcal{E}(G)$ is a highest weight category.

According to the same discussion as [9, Section 4 and Section 5], all the indecomposable modules in $\mathcal{E}(G)$ are $\text{St}, k_{\text{tr}}$ and $\{M(\theta) \mid \theta \in \hat{T}\}$. Therefore each object in $\mathcal{E}(G)$ is a direct sum of these modules. In particular, the category $\mathcal{E}(G)$ is a Krull–Schmidt category.
Remark 5.6. In [3], X.Y. Chen constructed a complex representation $M$ of $SL_2(\mathbb{F}_q)$, which contains the Steinberg module as a proper submodule and the corresponding quotient module is the trivial module. However $M$ has no $B$-stable line. So, this gives a negative answer to [9, Conjecture 3.7]. Thus the principal representation category $\mathcal{O}(G)$ introduced in [9] is not a highest weight category. Chen’s work shows that $\mathcal{O}(G)$ is very complicated in general. However the representation $M$ he constructed in [3] is not in the category $\mathcal{C}(SL_2(\mathbb{F}_q))$. Therefore, for general reductive algebraic group $G$, the category $\mathcal{C}(G)$ may be a good category for further study.

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