GLOBAL NONLINEAR STABILITY OF RAREFACTION WAVES
FOR COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH
TEMPERATURE AND DENSITY DEPENDENT
TRANSPORT COEFFICIENTS

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ABSTRACT. We study the nonlinear stability of rarefaction waves to the Cauchy
problem of one-dimensional compressible Navier-Stokes equations for a viscous
and heat conducting ideal polytropic gas when the transport coefficients depend
on both temperature and density. When the strength of the rarefaction waves
is small or the rarefaction waves of different families are separated far enough
initially, we show that rarefaction waves are nonlinear stable provided that
\((\gamma - 1) \cdot H^3(\mathbb{R})\)-norm of the initial perturbation is suitably small with \(\gamma > 1\)
being the adiabatic gas constant.

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initial data.

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1. Introduction and main result. This paper is concerned with the precise description of the large time behavior of global smooth large-amplitude solutions to the Cauchy problem of the following one-dimensional compressible Navier-Stokes equations in the Lagrangian coordinates

\[
\begin{cases}
v_t - u_x = 0, \\
    u_t + p(v, \theta)_x = \left(\frac{\mu(v, \theta) u_x}{v}\right)_x, \\
    \left(e + \frac{u^2}{2}\right)_t + (up(v, \theta))_x = \left(\frac{\mu(v, \theta) u u_x}{v}\right)_x + \left(\frac{\kappa(v, \theta) \theta_x}{v}\right)_x,
\end{cases}
\]

with prescribed initial data

\[
(v(t, x), u(t, x), \theta(t, x))|_{t=0} = (v_0(x), u_0(x), \theta_0(x)),
\]

\[
\lim_{x \to \pm \infty} (v_0(x), u_0(x), \theta_0(x)) = (v_{\pm}, u_{\pm}, \theta_{\pm}).
\]

Here \( v_{\pm}, u_{\pm}, \theta_{\pm} > 0 \) are constants to be specified later, the unknowns \( v, u, \theta > 0, p > 0, e \) represent, respectively, the specific volume, the velocity, the absolute temperature, the pressure, and the specific internal energy of the gas. The transport coefficients \( \mu(v, \theta) \) (viscosity) and \( \kappa(v, \theta) \) (heat-conductivity) are prescribed through constitutive relations as functions of \( v \) and \( \theta \) which are assumed to satisfy

\[
\mu(v, \theta) > 0, \quad \kappa(v, \theta) > 0
\]

for all \( v > 0 \) and \( \theta > 0 \).

Throughout this paper, we focus on the gases for which the constitutive relations between the five thermodynamic variables \( v, p, e, \theta, \) and \( s \) satisfy the relations provided by the kinetic theory of gases (cf. [2]), and we consider only ideal, polytropic gases:

\[
p(v, \theta) = \frac{R \theta}{v} = Av^{-\gamma} \exp\left(\frac{\gamma - 1}{R} s\right), \quad e = C_v \theta = \frac{R \theta}{\gamma - 1},
\]

where \( s \) denotes the specific entropy of the gas and the specific gas constants \( A, R, \) and the specific heat at constant volume \( C_v \) are positive constants and \( \gamma > 1 \) is the adiabatic constant. To simplify the presentation, we can assume without loss of generality that \( A = R = 1 \) in the rest of this paper.

Our study of the compressible Navier-Stokes equations (1) for a viscous and heat conducting ideal polytropic gas satisfying the constitutive relations (4) with temperature and density dependent transport coefficients \( \mu(v, \theta) \) and \( \kappa(v, \theta) \) verifying (3) are motivated by the following considerations:

- If one derives the compressible Navier-Stokes equations (1) from the Boltzmann equation with slab symmetry by employing the celebrated Chapman-Enskog expansion, cf. [2, 5, 15, 24, 43, 46], the corresponding five thermodynamic variables \( v, p, e, \theta, \) and \( s \) satisfy the constitutive relations (4) for ideal, polytropic gases and the transport coefficients \( \mu \) and \( \kappa \) depend on temperature. In particular, if the inter-molecular potential is proportional to \( r^{-\alpha} \) with \( \alpha > 1 \), where \( r \) represents the intermolecular distance, then \( \mu \) and \( \kappa \) satisfy

\[
\mu, \quad \kappa \propto \theta^{\frac{\alpha-1}{2\alpha}} = \begin{cases} 
\theta, \text{ for the Maxwellian molecule (} \alpha = 4\text{),} \\
\sqrt{\theta}, \text{ for the elastic spheres (} \alpha \to +\infty\text{)}
\end{cases}
\]

- For certain class of solid-like materials, cf. [3, 6, 13, 20], the viscosity \( \mu \) depends on density and the heat-conductivity \( \kappa \) depends on density and temperature;
Experimental results for gases at very high temperatures show that both the viscosity $\mu$ and the heat-conductivity $\kappa$ may depend on density and temperature, cf. \cite{20, 48}.

There have been many results on the global solvability and the precise descriptions of the large time behaviors of the corresponding global solutions to the initial value problem and the initial-boundary value problems of the one-dimensional compressible Navier-Stokes equations (1)-(3)-(4). For the purpose of this introduction we consider only a small selection of the very extensive literature on the corresponding results concerning the Cauchy problem (1)-(3)-(4), (2) as in the following (For the corresponding results on the initial-boundary value problems of (1)-(3)-(4), those interested are referred to the survey paper \cite{31} and the references cited therein):

For the case with small initial perturbation, the results is quite complete for general transport coefficients satisfying (3). In fact, by employing the energy method developed by Matsumura and Nishida in \cite{32}, if the initial data is small perturbation of certain profiles such as the constant states and some elementary waves like diffusion waves, rarefaction waves, viscous shock waves, viscous contact discontinuities, and some of their superpositions, the corresponding results on the global solvability and the nonlinear stability of these elementary waves to the Cauchy problem (1)-(3)-(4), (2) are well-understood provided that both the initial data and these profiles are assumed to be without vacuum, mass and temperature concentrations, or vanishing temperatures and both the initial perturbation and the strengths of these elementary waves are sufficiently small, cf. \cite{7, 8, 9, 11, 17, 26, 27, 28, 29, 30, 33, 34} and the references cited therein. Note that although all these results are concerned with the case when the transport coefficients $\mu$ and $\kappa$ are positive constants, the generalizations of these results to the case of temperature and density dependent viscosity $\mu$ and heat-conductivity $\kappa$ satisfying (3) are straightforward for small initial perturbation.

For the corresponding results for large data, the story is quite different and the forms of the dependence of transport coefficients $\mu$ and $\kappa$ on the density $\rho \equiv \nu^{-1}$ and/or the temperature $\theta$ have strong influence on the analysis and the corresponding results obtained up to now can be classified as follows:

- For positive constant transport coefficients, a general global existence result is obtained in \cite{1, 14, 21, 22, 49}, but since the upper and lower bounds on $\nu$ and $\theta$ obtained there are not uniform with respect to the time variable, the corresponding results on the precise description of the large time behaviors of the global solutions constructed there are unknown. For such a problem, for the isentropic case, the global nonlinear stability of rarefaction waves were well-established in \cite{35, 36} and for the nonlinear stability of viscous shock waves with large initial perturbation, only partial progress was made in \cite{47}. As for the full compressible Navier-Stokes equations (1), some Nishida-Smoller type results are obtained in \cite{4, 18, 19, 38, 39} (For the corresponding original Nishida-Smoller type global existence result for one-dimensional ideal polytropic isentropic compressible Euler system, please refer to \cite{37}, whereas the non-ientropic case was analyzed by Liu \cite{25} and also Temple \cite{42}), while for the case of general adiabatic exponent $\gamma > 1$, the nonlinear stability of the non-vacuum constant states was solved only recently in \cite{23} and the nonlinear stability of the superposition of rarefaction waves and viscous contact waves was obtained in \cite{10}, cf. also \cite{44, 45} for the corresponding results for the
outflow problem. A key ingredient in all of these analysis is the explicit expression of the specific volume \( v(t,x) \) given in [22] and its decent localized version obtained in [12], which leads to the desired uniform pointwise a priori estimates on the specific volume \( v(t,x) \) which guarantee that no vacuum nor concentration of mass occur;

- Much effort has been invested in generalizing the above approach to other cases and in particular to models satisfying (5). However, as pointed out in [15], this has proved to be challenging and temperature dependence of the viscosity \( \mu \) has turned out to be especially problematic, but one has been able to incorporate various forms of density dependence in \( \mu \) and also temperature dependence in \( \kappa \), cf. [3, 15, 20, 40, 41] and the references cited therein. Even so, it is worth to pointing out that since the upper and lower bounds on \( v(t,x) \) and \( \theta(t,x) \) obtained in [3, 15, 20, 40, 41] depends on time, the problem on the large time behaviors of the global solutions constructed there is unknown;

- For the case when the transport coefficients depend on both density and temperature and satisfy (3), the only results available now are obtained in [24, 46], where the nonlinear stability of non-vacuum constant states with large data are obtained.

Thus a natural question is: For the Cauchy problem (1)-(4), (2) with density and temperature dependent transport coefficients satisfying (3), can we obtain the global nonlinear stability of some elementary waves such as rarefaction waves, viscous shock waves, viscous contact waves, and some of their superpositions? Here “global nonlinear stability” means “the corresponding nonlinear stability result with large initial perturbation”. And our main purpose here is to deduce the global nonlinear stability of rarefaction waves with large initial perturbation.

Since our interest is to show the nonlinear stability of the expansion waves for (1), it is convenient to work with the equations for the entropy \( s \) and the absolute temperature \( \theta \). To this end, we get from (1) and (4) that

\[
s_t = \left( \frac{\kappa(v, \theta)\theta_x}{v\theta} \right)_x + \frac{\kappa(v, \theta)\theta^2_x}{v\theta^2} + \frac{\mu(v, \theta)u_x^2}{v\theta} \tag{6}
\]

and

\[
\theta_t + \frac{\theta}{\gamma - 1} u_x = \frac{\mu(v, \theta)u_x^2}{v} + \left( \frac{\kappa(v, \theta)\theta_x}{v} \right)_x. \tag{7}
\]

Recall that the gas constants \( A \) and \( R \) are normalized to be unity.

In fact, for smooth solutions, equations (1)_{1,-(1)_{2,-(1)_{3}}} are equivalent to equations (1)_{1,-(1)_{2,-(6)}} or (1)_{1,-(1)_{2,-(7)}}. In what follows, we will consider (1)_{1,-(1)_{2,-(6)}} with the initial data

\[
(v(t,x), u(t,x), s(t,x)) |_{t=0} = (v_0(x), u_0(x), s_0(x)), \lim_{x \to \pm \infty} (v_0(x), u_0(x), s_0(x)) = (v_\pm, u_\pm, s_\pm). \tag{8}
\]

Here \( s_0(x) = \frac{1}{\gamma - 1} \ln \left( \frac{\theta_0(x)v_0^{\gamma-1}(x)}{\gamma} \right) \) and \( s_\pm = \frac{1}{\gamma - 1} \ln \left( \frac{\theta_\pm v_\pm^{\gamma-1}}{\gamma} \right) \). Since we will focus on the expansion waves to (1), we assume that \( s_+ = s_- = \bar{s} \) in the rest of this paper.

For expansion waves, the right hand side of (1) decays faster than each individual term on the left hand side. Therefore, the compressible Navier-Stokes equations (1) may be approximated, time-asymptotically, by the Riemann problem of the
compressible Euler equations:

\[
\begin{cases}
  v_t - u_x = 0, \\
  u_t + (v^{-\gamma}e^{(\gamma-1)s})_x = 0, \\
  s_t = 0
\end{cases}
\]  \tag{9}

with Riemann data

\[ (v(t, x), u(t, x), s(t, x))|_{t=0} = (v^R_0(x), u^R_0(x), s^R_0(x)) \]

\[ = \begin{cases}
  (v_-, u_-, s_-), & x < 0, \\
  (v_+, u_+, s_+), & x > 0.
\end{cases} \tag{10}
\]

We consider the case when the Riemann problem (9), (10) admits a unique global weak (rarefaction wave) solution \((V^R(x), U^R(x), S^R(x))\) which consists of a rarefaction wave of the first family, denoted by \((V^R_1(x), U^R_1(x), \bar{s})\), and another of the third family, denoted by \((V^R_3(x), U^R_3(x), \bar{s})\). That is, there exists a unique constant state \((v_m, u_m) \in \mathbb{R}^2 (v_m > 0)\) such that \((v_m, u_m) \in \mathbb{R}_1(v_-, u_-)\) and \((v_+, u_+) \in \mathbb{R}_3(v_m, u_m)\). Here

\[
\mathbb{R}_1(v_-, u_-, \bar{s}) = \left\{ (v, u, s) \mid u = u_- + \int_{v_-}^{v} \sqrt{\gamma z^{-\gamma-1} \exp((\gamma - 1)s)} dz, u \geq u_-, s = \bar{s} \right\},
\]

\[
\mathbb{R}_3(v_m, u_m, \bar{s}) = \left\{ (v, u, s) \mid u = u_m - \int_{v_m}^{v} \sqrt{\gamma z^{-\gamma-1} \exp((\gamma - 1)s)} dz, u \geq u_m, s = \bar{s} \right\}.
\]

In other words, the unique weak solution \((V^R(x), U^R(x), S^R(x))\) to the Riemann problem (9), (10) is given by

\[
\left( V^R_1(x), U^R_1(x), S^R_1(x) \right) = \left( V^R_1(x) + V^R_3(x) - v_m, U^R_1(x) + U^R_3(x) - u_m, \bar{s} \right),
\] \tag{11}

where \((V^R_i(x), U^R_i(x), S^R_i(x))(i = 1, 3)\) are determined by the following equations:

\[
\begin{align*}
  S^R(x) &= \bar{s}, \\
  U^R_1(x) - \int_1^{V^R_1(x)} \sqrt{\frac{\gamma \exp((\gamma - 1)s)}{z^{\gamma+1}}} dz &= u_- - \int_1^{V^R_1(x)} \sqrt{\frac{\gamma \exp((\gamma - 1)s)}{z^{\gamma+1}}} dz, \\
  \lambda_1 (V^R_1(x), \bar{s}) &= 0, \\
  \lambda_1 (v, s) &= -\sqrt{\frac{\gamma \exp((\gamma - 1)s)}{z^{\gamma+1}}}, \tag{12}
\end{align*}
\]

\[
\begin{align*}
  U^R_3(x) + \int_1^{V^R_3(x)} \sqrt{\frac{\gamma \exp((\gamma - 1)s)}{z^{\gamma+1}}} dz &= u_m + \int_1^{V^R_3(x)} \sqrt{\frac{\gamma \exp((\gamma - 1)s)}{z^{\gamma+1}}} dz, \\
  \lambda_3 (V^R_3(x), \bar{s}) &= 0, \\
  \lambda_3 (v, s) &= \sqrt{\frac{\gamma \exp((\gamma - 1)s)}{z^{\gamma+1}}}.
\end{align*}
\]

To study the nonlinear stability of the expansion waves, as in [34], we first construct a smooth approximation to the above Riemann solution (11). For this purpose, let \(\omega_i(t, x) (i = 1, 3)\) be the unique global smooth solution to the following
Cauchy problem:
\[
\begin{aligned}
\left\{ \begin{array}{l}
\omega_{it} + \omega_i \omega_{ix} = 0, \\
\omega_i(t, x)|_{t=0} = \omega_{i0}(x) = \frac{w_{i+} + w_{i-}}{2} + \frac{w_{i+} - w_{i-}}{2} K_q \int_0^x (1 + y^2)^{-\sigma} dy,
\end{array} \right.
\end{aligned}
\] (13)
where \( q > \frac{3}{2}, K_q = (\int_0^{+\infty} (1 + y^2)^{-\sigma} dy)^{-1}, \sigma > 0 \) is a positive constant which will be specified later, and
\[
\begin{aligned}
\omega_{1-} = \lambda_1(v_-, \bar{s}) = -\gamma \sqrt{v_{-}^{-\gamma-1}} \exp ((\gamma - 1)\bar{s}), \\
\omega_{1+} = \lambda_1(v_m, \bar{s}) = -\gamma \sqrt{v_m^{-\gamma-1}} \exp ((\gamma - 1)\bar{s}), \\
\omega_{3-} = \lambda_3(v_m, \bar{s}) = \gamma \sqrt{v_m^{-\gamma-1}} \exp ((\gamma - 1)\bar{s}), \\
\omega_{3+} = \lambda_3(v_+, \bar{s}) = \gamma \sqrt{v_+^{-\gamma-1}} \exp ((\gamma - 1)\bar{s}).
\end{aligned}
\]

Then, by setting \( \sigma = \delta = |v_- - v_+| + |u_- - u_+| \), the smooth approximation of the rarefaction wave profile \((V(t, x), U(t, x), S(t, x))\) is constructed as follows:
\[
(V(t, x), U(t, x), S(t, x)) = (V_1(t + t_0, x) + V_3(t + t_0, x) - v_m, U_1(t + t_0, x) + U_3(t + t_0, x) - u_m, \bar{s}),
\] (14)
where \( t_0 > 0 \) is a sufficiently large but fixed positive constant which will be determined later, \((V_i(t, x), U_i(t, x))(i = 1, 3)\) are defined by the following equations:
\[
\left\{ \begin{array}{l} \\
\lambda_4(V_i(t, x), \bar{s}) = \omega_i(t, x), i = 1, 3, \\
\lambda_1(v, s) = -\gamma \sqrt{v^{-\gamma-1}} \exp ((\gamma - 1)s), \\
\lambda_3(v, s) = \gamma \sqrt{v^{-\gamma-1}} \exp ((\gamma - 1)s), \\
U_1(t, x) = u_- + \int_{v_-}^{V_1(t,x)} \sqrt{v z^{-\gamma-1}} \exp ((\gamma - 1)z)dz, \\
U_3(t, x) = u_m - \int_{v_m}^{V_3(t,x)} \sqrt{v z^{-\gamma-1}} \exp ((\gamma - 1)z)dz,
\end{array} \right.
\] (15)
and \( \Theta(t, x) \) is defined by
\[
\Theta(t, x) = V(t, x)^{1-\gamma} \exp ((\gamma - 1)\bar{s}).
\]

We now turn to state our main results in this paper, to this end, if we denote
\[
l = \frac{t_0}{\delta},
\]
where \( \delta = |v_- - v_+| + |u_- - u_+| \) is the strength of the rarefaction waves, then our main result can be summarized as follows:

**Theorem 1.1** (Main result). Suppose that
- \( N_{0i} = \| (v_0(x) - V(0, x), u_0(x) - U(0, x), \frac{\theta_0(x) - \Theta(0, x)}{\sqrt{\gamma - 1}}) \|_{H^i(\mathbb{R})} \) \((i = 1, 2, 3)\) is bounded by some positive constant independent of \( \gamma - 1 \);
- There exist \((\gamma - 1)\)-independent positive constants \( 0 < V < 1, V > 1, 0 < \Theta < 1, \Theta > 1 \) such that \( 2V \leq v_0(x), V(t, x) \leq \frac{1}{2}V, 2\Theta \leq \theta_0(x), \Theta(t, x) \leq \frac{1}{2}\Theta, \forall(t, x) \in \mathbb{R}_+ \times \mathbb{R}; \)
- The constants \( v_\pm, u_\pm, \) and \( \theta_\pm \) do not depend on \( \gamma - 1; \)
The transport coefficients \( \mu(v, \theta) \) and \( \kappa(v, \theta) \) satisfy (3) and there exist constants \( a \leq 0, b \geq -\frac{1}{2} \) such that

\[
\mu(v) \equiv \min_{\theta \in [\underline{\theta}, \bar{\theta}]} \{ \mu(v, \theta) \} \sim \begin{cases} v^a, & v \to 0_+, \\ v^b, & v \to \infty, \end{cases}
\]

(16)

There exists a sufficiently small positive constant \( \varepsilon_0 \) which depends only on \( N_0, V, V_0, \underline{\theta}, \) and \( \bar{\theta} \) such that

\[
1 < \gamma \leq 1 + \varepsilon_0, \quad l \geq \frac{1}{\varepsilon_0}
\]

Then the Cauchy problem (1)–(6), (8) admits a unique global solution \((v(t, x), u(t, x), s(t, x))\) which satisfies

\[
(v - V, u - U, \theta - \Theta, s - \bar{s}) (t, x) \in C ([0, +\infty), H^3(\mathbb{R})) ,
\]

\[
\left( \frac{\partial}{\partial x} (u - U), \frac{\partial}{\partial x} (\theta - \Theta) \right) (t, x) \in L^2 ([0, +\infty), H^3(\mathbb{R})) ,
\]

and

\[
\underline{\theta} \leq \theta(t, x) \leq \bar{\theta},
\]

\[
C^{-1} \leq v(t, x) \leq C
\]

hold for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\) and some positive constant \( C \geq 1 \) which depends only on \( N_0, V, V_0, \underline{\theta}, \) and \( \bar{\theta} \).

Moreover, it holds that

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left\{ \left| (v(t, x) - V^R(t, x), u(t, x) - U^R(t, x), s(t, x) - \bar{s}) \right| \right\} = 0.
\]

(17)

Remark 1. Some remarks concerning Theorem 1.1 are listed below:

- The introduction of the parameter \( l \) is motivated by [4] and the main purpose is to control the possible growth of the solutions to the Cauchy problem (1)–(6), (8) generated by both the interactions of rarefaction waves from different families and the interactions between the rarefaction waves and the solution itself. It is worth to pointing out that when one considers the nonlinear stability of single rarefaction waves, it is unnecessary to introduce such a parameter and in such a case, the difficulty induced by the interactions of rarefaction waves from different families will not occur and due to the structure of the equations (1)–(6) and noticing that the nice properties of the smooth approximation \((V(t, x), U(t, x), S(t, x))\) of the rarefaction waves, especially the fact that \( \| (V_{xx}(t, x), U_{xx}(t, x)) \|_{L^p(\mathbb{R})} \in L^1(\mathbb{R}_+) \) for each \( p > 1 \) which is due to the way to construct the smooth approximation of the rarefaction waves, cf. (13), introduced first in [35], the difficulty involving the interactions between the rarefaction waves and the solution itself can be controlled suitably even for large rarefaction waves. Thus one can deduce a similar Nishida-Smoller type nonlinear stability result for single strong rarefaction waves.

- From the proof of Theorem 1.1, the precise dependence of the positive constant \( \varepsilon_0 \) on \( N_0, V, V_0, \underline{\theta}, \) and \( \bar{\theta} \) can be given explicitly, cf. (71), (72), and (73).

- Due to

\[
\theta_0(x) - \Theta(0, x) = v_0(x)^{1 - \gamma} e^{(\gamma - 1)\theta_0(x)} - V(0, x)^{1 - \gamma} e^{(\gamma - 1)\bar{s}} ,
\]

\[
\bar{s} = \frac{1}{\gamma - 1} \ln \left( \theta_+ v_\gamma^{\gamma - 1} \right) ,
\]
and noticing that the assumptions we imposed in Theorem 1.1 tell us that

- Both \( \hat{v}_0(x) \) and \( \hat{V}(0, x) \) are bounded by some positive constants independent of \( \gamma - 1 \) from below and above,
- \((\gamma - 1)\hat{s}\) is bounded by some constant independent of \( \gamma - 1 \),
- it is easy to see that there exists some positive constant \( C \) depending only on
  \[ \| (\hat{v}_0(x), \hat{V}(0, x)) \|_{L^\infty(\mathbb{R})}, \hat{v}_\pm, \hat{\theta}_\pm \]
  such that

\[
\frac{\| \hat{\theta}_0(x) - \Theta(0, x) \|_{H^1(\mathbb{R})}}{\sqrt{\gamma - 1}} \leq C \sqrt{\gamma - 1} \left( 1 + \| e^{(\gamma - 1)s_0(x)} \|_{L^\infty(\mathbb{R})} \right) \times \| (\hat{v}_0(x) - \hat{V}(0, x), s_0(x) - \hat{s}) \|_{H^1(\mathbb{R})}
\]

hold for \( i = 1, 2, 3 \). Thus a sufficient condition to guarantee that \( N_0(i = 1, 2, 3) \) is bounded by some positive constant independent of \( \gamma - 1 \) is

\[
\sqrt{\gamma - 1} \left| e^{(\gamma - 1)s_0(x)} \right|_{L^\infty(\mathbb{R})} \| s_0(x) - \hat{s} \|_{H^1(\mathbb{R})}
\]

is bounded by some positive constant independent of \( \gamma - 1 \).

Before concluding this section, we outline the main idea used to yield the above result. As is usual in constructing global solutions with large-amplitude to nonlinear partial differential equations, the main difficulty lies in how to control the possible growth of the solutions induced by the nonlinearity of the equations under considerations. For our problem, in addition to the above difficulty, we need also to control the possible growth of the solutions to the Cauchy problem \((1)\) generated by the interactions of rarefaction waves from different families and the interaction between the solutions and the rarefaction waves. The key point is to get the desired uniform positive lower and upper bounds on the specific volume \( v(t, x) \) and the absolute temperature \( \theta(t, x) \). Our main observations are the following:

- Firstly, the constitutive relations \((4)\) together with the equation \((7)\) suggest that one can perform the desired energy type estimates based on the a priori assumption \( \sup_{t \in [0, T]} \| \theta(t, x) - \Theta(t, x) \|_{H^1(\mathbb{R})} \leq \epsilon \) for some sufficiently small \( \epsilon > 0 \) and some \( T > 0 \) which implies that \( \Theta \leq \theta(t, x) \leq \Theta \) for all \( (t, x) \in [0, T] \times \mathbb{R} \);
- Secondly, with the above estimate on \( \theta(t, x) \) in hand, we then perform some energy type estimates by using the smallness of both such an \( \epsilon \) and \( \gamma - 1 \) to control the possible growth of the solutions of the Cauchy problem \((1)\) \((1)_1- (1)_{2}, (6), (8)\) induced by the nonlinearity of the equations \((1)_{1}, (1)_{2}, (6)\) and by exploiting the largeness of the parameter \( l \) to control the possible growth generated by both the interactions of rarefaction waves from different families and the interactions between the rarefaction waves and the solution itself. These estimates together with the argument developed by Kanel’ in \([16]\) can lead to an estimate on the uniform lower and upper bounds on \( v(t, x) \) in terms of \( N_{03}, \bar{V}, \bar{\Theta}, \) and \( \bar{\Theta} \);
- Thirdly, a further energy type estimate can yield the following estimate on

\[
\| \theta(t, x) - \Theta(t, x) \|_{H^1(\mathbb{R})} \leq \sqrt{\gamma - 1} C (N_{03}, \bar{V}, \bar{\Theta}, \bar{\Theta})
\]

for some positive constant \( C \) \((N_{03}, \bar{V}, \bar{\Theta}, \bar{\Theta})\) depending only on \( N_{03}, \bar{V}, \bar{\Theta}, \) \( \bar{\Theta} \). Such an estimate implies that \( \epsilon \) is of the order \( \sqrt{\gamma - 1} \) and then a carefully designed continuation argument can close the whole analysis so that
the local solution can be extended step by step to a global one provided that both \( \gamma - 1 \) and \( l^{-1} \) are assumed to be sufficiently small.

This paper is arranged as follows: We will give some properties of the smooth approximation of the rarefaction wave solutions in Section 2. In Section 3, by using the largeness of \( l \) and the smallness of both \( \| \theta(t, x) - \Theta(t, x) \|_{H^3(\mathbb{R})} \) and \( \gamma - 1 > 0 \) to control the possible growth of the solution \( (v(t, x), u(t, x), s(t, x)) \) to the Cauchy problem (1.1) under our consideration and by both the interactions of rarefaction waves from different families and the interactions between the rarefaction waves and the solution itself, we deduce the desired energy type estimates. Finally, we extend the local solution step by step to a global one by combining the a priori estimates obtained in Section 3 with the continuation argument and give the proof of Theorem 1.1 in Section 4.

**Notations.** Throughout the rest of this paper, \( C \) or \( O(1) \) will be used to denote a generic positive constant which is independent of \( t, t_0, \delta, \gamma - 1, \) and \( x \) but may depend on \( N_{03}, v_\pm, u_\pm, \theta_\pm, \nabla, \nabla, \Theta, \) and \( \Theta \) and \( C_i(\cdot, \cdot)(i \in \mathbb{Z}_+) \) stands for some generic constants depending only on the quantities listed in the parentheses. Note that all these constants may vary from line to line.

For two functions \( f(x) \) and \( g(x) \), \( f(x) \sim g(x) \) as \( x \to a \) means that there exists a generic positive constant \( C > 0 \) which is independent of \( t, t_0, \delta, \gamma - 1, \) and \( x \) but may depend on \( N_{03}, v_\pm, u_\pm, \theta_\pm, \nabla, \nabla, \Theta, \) and \( \Theta \) such that \( C^{-1} f(x) \leq g(x) \leq C f(x) \) in a neighborhood of \( a \). \( B \lesssim B' \) means that there is a generic positive constant \( C > 0 \) independent of \( t, t_0, \delta, \gamma - 1, \) and \( x \) such that \( B \leq C B' \), while \( B \sim B' \) means that \( B \lesssim B' \) and \( B' \lesssim B \). \( H^s(\mathbb{R})(l \geq 0) \) denotes the usual Sobolev space with standard norm \( \| \cdot \|_l \) and \( \| \cdot \|_0 = \| \cdot \| \) will be used to denote the usual \( L^2 \)-norm.

For \( 1 \leq p \leq +\infty, f(x) \in L^p(\mathbb{R}), \| f \|_{L^p} = \left( \int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}} \). It is easy to see that \( \| f \|_{L^2} = \| \cdot \| \). Finally, \( \| \cdot \|_{L^\infty_x} \) and \( \| \cdot \|_{L^\infty_x} \) are used to denoted \( \| \cdot \|_{L^\infty(\mathbb{R})} \) and \( \| \cdot \|_{L^\infty([0, t] \times \mathbb{R})} \) respectively.

**2. Preliminaries.** In this section, we give some basic estimates and identities which will be used later. First, we list some properties of the global smooth functions \( (V(t, x), U(t, x), S(t, x), \Theta(t, x)) \) constructed in (13) and (14).

According to (1), (12), and (14), we know that \( (V(t, x), U(t, x), S(t, x)) \) solves the following problem:

\[
\begin{align*}
V_t - U_x & = 0, \\
U_t + \left( \frac{\Theta}{V} \right)_x & = g(V, \Theta)_x, \\
\left( \frac{\Theta}{V^\gamma} + \frac{U^2}{2} \right)_t + \left( \frac{\Theta U}{V} \right)_x & = q(V, \Theta), \\
\Theta_t + (\gamma - 1) \frac{\Theta U}{V} & = r(V, \Theta), \\
S_t & = 0,
\end{align*}
\]

where

\[
\begin{align*}
g(V, \Theta) & = \frac{\Theta}{V} - \frac{\Theta_1}{V_1} - \frac{\Theta_3}{V_3} - \frac{\Theta_m}{V_m}, \\
q(V, \Theta) & = \left( \frac{\Theta}{V^\gamma} - \frac{\Theta_1}{V_1^\gamma} - \frac{\Theta_3}{V_3^\gamma} \right) + \left( \frac{U_1^2}{2} - \frac{U_1^2}{2} - \frac{U_3^2}{2} \right)_t, \\
r(V, \Theta) & = (\gamma - 1) \left( \frac{\Theta U}{V} - \frac{\Theta_1 U_1}{V_1} - \frac{\Theta_3 U_3}{V_3} \right),
\end{align*}
\]
and \( \theta_m = v_m^{1-\gamma} e^{(\gamma-1)\bar{s}} \) with \( \bar{s} = \frac{1}{\gamma - 1} \ln \left( \theta_m v_m^{\gamma-1} \right) \). Notice that \( g(V, \Theta), q(V, \Theta), \) and \( r(V, \Theta) \) measure the the interactions of the rarefaction waves from different families.

To present some estimates on \((V(t, x), U(t, x), S(t, x))\), we first state some results on the Cauchy problem \((13)\) in the following lemma

**Lemma 2.1.** For each \( i \in \{1, 3\} \), the Cauchy problem \((13)\) admits a unique global smooth solution \( \omega_i(t, x) \) which satisfies the following properties:

(i): \( \omega_i < \omega_i(t, x) < \omega_+ \), \( \omega_i(x, t) > 0 \) for each \( (t, x) \in \mathbb{R}_+ \times \mathbb{R} \);

(ii): For any \( p \) with \( 1 \leq p \leq \infty \), there exists a constant \( C_{p, q} \), depending only on \( p, q \), such that

\[
\begin{align*}
\|\omega_{1x}(t)\|_{L^p}^p &\leq C_{p, q} \min \left\{ \sigma^{p-1} \omega_i^p, \omega_i t^{-p+1} \right\}, \\
\|\omega_{1xx}(t)\|_{L^p}^p &\leq C_{p, q} \min \left\{ \sigma^{p-1} \omega_i^p, \sigma(p-1)(1 - \frac{1}{q}) \omega_i^{q-2} t^{-p-\frac{2}{q}+2} \right\}, \\
\|\omega_{1xxx}(t)\|_{L^p}^p &\leq C_{p, q} \min \left\{ \sigma^{3p-1} \omega_i^p, \sigma(2p-1)(1 - \frac{1}{q}) \omega_i^{q-2} t^{-p-\frac{2}{q}+2} \right\}, \\
\|\omega_{1xxxx}(t)\|_{L^p}^p &\leq C_{p, q} \min \left\{ \sigma^{4p-1} \omega_i^p, \sigma(3p-1)(1 - \frac{1}{q}) \omega_i^{q-2} t^{-p-\frac{2}{q}+2} \right\};
\end{align*}
\]

(iii): If \( 0 < \omega_{1-} \left( < \omega_{1+} \right) \) and \( q \) is suitably large, then

\[
\begin{align*}
|\omega_i(t, x) - \omega_{1-}| &\leq C\omega_i \left( 1 + (\sigma x)^2 \right)^{-\frac{q}{2}} \left( 1 + (\sigma \omega_{1-} t)^2 \right)^{-\frac{q}{2}}, x \leq 0, \\
|\omega_i(t, x)| &\leq c\sigma \omega_i \left( 1 + (\sigma x)^2 \right)^{-\frac{q}{2}} \left( 1 + (\sigma \omega_{1+} t)^2 \right)^{-\frac{q}{2}}, x \leq 0;
\end{align*}
\]

(iv): If \( \omega_{1-} < \omega_{1+} \leq 0 \) and \( q \) is suitably large, then

\[
\begin{align*}
|\omega_i(t, x) - \omega_{1+}| &\leq C\omega_i \left( 1 + (\sigma x)^2 \right)^{-\frac{q}{2}} \left( 1 + (\sigma \omega_{1-} t)^2 \right)^{-\frac{q}{2}}, x \leq 0, \\
|\omega_i(t, x)| &\leq c\sigma \omega_i \left( 1 + (\sigma x)^2 \right)^{-\frac{q}{2}} \left( 1 + (\sigma \omega_{1+} t)^2 \right)^{-\frac{q}{2}}, x \leq 0;
\end{align*}
\]

(v): \( \lim_{t \to +\infty} \sup_{x \in \mathbb{R}} |\omega_i(t, x) - \omega_i^R \left( \frac{z}{t} \right) | = 0 \);

Here \( \omega_i = \omega_{1i} - \omega_{1-} > 0 \) and \( \omega_i^R \left( \frac{z}{t} \right) \) is the unique rarefaction wave solution of the corresponding Riemann problem \((1.19)_1\), i.e.,

\[
\omega_i^R(z) = \begin{cases} 
\omega_{1-}, & z \leq \omega_{1-}, \\
\omega_{1-} \leq z \leq \omega_{1+}, & \\
\omega_{1+}, & z \geq \omega_{1+}.
\end{cases}
\]

Based on the results obtained in Lemma 2.1 and from \((14)\) and \((15)\), we can deduce that

**Lemma 2.2.** Letting \( \sigma = \delta, q = 2 \), the smooth approximations \((V(t, x), U(t, x), \Theta(t, x))\) constructed in \((14)\) and \((15)\) have the following properties:

(i): \( V_1(t, x) = U_1(t, x) > 0 \) for each \( (t, x) \in \mathbb{R}_+ \times \mathbb{R} \);

(ii): For any \( p \) with \( 1 \leq p \leq \infty \) there exists a constant \( C_p \), depending only on \( p \), such that

\[
\begin{align*}
\|V_{xx}(t)\|_{L^p}^p &\leq C_p \min \left\{ \delta^{2p-1}, \delta(t + t_0)^{p-1} \right\}, \\
\|V_{xxx}(t)\|_{L^p}^p &\leq C_p \min \left\{ \delta^{3p-1}, \delta^{\frac{2p-1}{2}}(t + t_0)^{-\frac{2p-1}{2}} \right\}, \\
\|V_{xxxx}(t)\|_{L^p}^p &\leq C_p \min \left\{ \delta^{4p-1}, \delta^{\frac{2p-1}{3}}(t + t_0)^{-\frac{2p-1}{3}} \right\}, \\
\|V_{xxxxx}(t)\|_{L^p}^p &\leq C_p \min \left\{ \delta^{5p-1}, \delta^{\frac{2p-1}{4}}(t + t_0)^{-\frac{2p-1}{4}} \right\}.
\end{align*}
\]
It is obvious that \( \|V_x(t)\|_{L^2}^2 \) is not integrable with respect to \( t \). However we can get for any \( r > 0 \) and \( p > 1 \) that
\[
\int_0^\infty \| (V_x, U_x, \Theta_x) (t) \|_{L_{2+r}^r(t)}^2 \, dt \leq C(r) \delta t_0^{-r},
\]
\[
\int_0^\infty \| (V_{xx}, U_{xx}, \Theta_{xx}) (t) \|_{L_p^p} \, dt \leq O(1) \left( \frac{t_0}{\delta} \right)^{-\frac{1}{2}} (1+\frac{3}{5})^{-\frac{1}{2}}(2-\frac{3}{5})^{-\frac{1}{2}},
\]
\[
\int_0^\infty \| (V_{xxx}, U_{xxx}, \Theta_{xxx}) (t) \|_{L_p^p} \, dt \leq O(1) \left( \frac{t_0}{\delta} \right)^{-\frac{1}{2}} (3-\frac{3}{5})^{-\frac{1}{2}};
\]
(iii): For each \( p \geq 1 \),
\[
\left\| \left( g(V, \Theta)_x, \frac{1}{\gamma - 1} r(V, \Theta), q(V, \Theta) \right) (t) \right\|_{L_p^p} \leq C(p) \delta \frac{3}{2} (t + t_0)^{-\frac{3}{2}}.
\]
Especially,
\[
\int_0^\infty \left\| \left( g(V, \Theta)_x, \frac{1}{\gamma - 1} r(V, \Theta), q(V, \Theta) \right) (t) \right\|_{L_p^p} \, dt \leq C(p) \left( \frac{t_0}{\delta} \right)^{-\frac{3}{2}};
\]
(iv): \( \lim_{t \to +\infty} \sup_{x \in \mathbb{R}} |(V(t, x), U(t, x), \Theta(t, x)) - (V^R \left( \frac{x}{v} \right), U^R \left( \frac{x}{v} \right), \Theta^R \left( \frac{x}{v} \right))| = 0; \)
(v): \( |(V(t, x), U(t, x), \Theta(t, x))| \leq O(1) |(V_x(t, x), U_x(t, x), \Theta_x(t, x))| \).

**Remark 2.** Recall that the quantities \( g(V, \Theta) \), \( r(V, \Theta) \), and \( q(V, \Theta) \) represent the interaction of waves from different families. From the estimates obtained in (iii) of Lemma 2.2, it is easy to see that one can control \( g(V, \Theta) \), \( r(V, \Theta) \), and \( q(V, \Theta) \) suitably if the parameter \( l = \frac{t_0}{\delta} \) introduced in (14) in the construction of a smooth approximation of the rarefaction wave profile is chosen sufficiently large.

Setting
\[
(\varphi(t, x), \psi(t, x), \phi(t, x), \xi(t, x)) = (v(t, x) - V(t, x), u(t, x) - U(t, x), \theta(t, x) - \Theta(t, x), s(t, x) - \bar{s}),
\]
we can deduce that \( (\varphi(t, x), \psi(t, x), \phi(t, x), \xi(t, x)) \) solves
\[
\begin{cases}
\varphi_t - \psi_x = 0, \\
\psi_t + \left[ \frac{\theta}{v} \Theta \right]_x = \left( \mu(v, \theta) \frac{\omega_x}{v} \right)_x - g(V, \Theta)_x, \\
\left( \frac{\psi_t}{\gamma - 1} + \frac{\theta}{v} \psi_x + \left( \frac{\theta}{v} - \Theta \right) U_x \right)_x = \left( \frac{\kappa(v, \theta) \theta_x}{v^2} \right)_x + \frac{\mu(v, \theta) \omega_x^2}{v^2} - r(V, \Theta)_x, \\
\xi_t = \left( \frac{\kappa(v, \theta) \theta_x}{v^2} \right)_x + \frac{\kappa(v, \theta) \theta_x^2}{v^2} + \frac{\mu(v, \theta) \omega_x^2}{v^2}
\end{cases}
\]
with initial data
\[
(\varphi(t, x), \psi(t, x), \phi(t, x), \xi(t, x))|_{t=0} = (\varphi_0(x), \psi_0(x), \phi_0(x), \xi_0(x)) = (v_0(x) - V(0, x), u_0(x) - U(0, x), \theta_0(x) - \Theta(0, x), s_0(x) - \bar{s}).
\]

On the other hand, it is well-known that
\[
\eta(v, u, \theta; V, U, \Theta) = \Theta \Phi \left( \frac{v}{V} \right) + \frac{1}{2} (u - U)^2 + \frac{\Theta}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right), \quad \Phi(s) = s - \ln s - 1
\]
is a convex entropy to the compressible Navier-Stokes equations (1) around the smooth rarefaction wave profile \((V(t, x), U(t, x), \Theta(t, x), S(t, x))\) which satisfies the following identity

\[
\eta_t(v, u, \theta; V, U, \Theta) + \left( \left( \frac{\partial}{\partial x} \Theta \right) V \right)_x + \left\{ \frac{\mu(v, \theta)\Theta^2}{v} \frac{\psi^2}{v^2} + \frac{\kappa(v, \theta)\Theta^2}{v\theta^2} \right\} \\
\left. + \left( v - \gamma e^{(\gamma - 1)s} - \gamma V e^{(\gamma - 1)s} + (\gamma - 1) \varphi (\gamma - 1) e^{(\gamma - 1)s} \right) U \right.x \\
= \left( \frac{\mu(v, \theta)\psi \psi_x}{v} + \frac{\kappa(v, \theta)\phi_x \psi}{v\theta} \right)_x + \frac{\mu(u, v, \theta)U_x \phi_x \psi}{v} + \frac{\kappa(v, \theta)\Theta_x \phi_x \psi}{v\theta} \\
- \frac{\mu(v, \theta)U_x \phi_x \psi}{v} + \frac{2\mu(v, \theta)U_x \phi_x \psi}{v\theta} + \frac{\kappa(v, \theta)\Theta_x \phi_x \psi}{v\theta^2} + \frac{\kappa(v, \theta)\Theta_x \phi_x \psi}{v} \\
- \frac{\mu(v, \theta)U_x \phi_x \psi}{v} + \frac{\mu(v, \theta)U_x \phi_x \psi}{v\theta} + \frac{\mu(v, \theta)U_x \phi_x \psi}{v\theta^2} + \frac{\mu(v, \theta)\Theta_x \phi_x \psi}{v} \\
+ \frac{\kappa(v, \theta)\Theta_x \phi_x \psi}{v} + \frac{\kappa(v, \theta)\Theta_x \phi_x \psi}{v\theta} \\
- q(V, \Theta) + g(V, \Theta) \right. x - r(V, \Theta) \xi. \tag{21}
\]

Moreover, \((19)_1\) together with \((19)_2\) imply

\[
\left( \frac{\mu(v, \theta)\phi_x}{v} \right)_t = \psi_t + \left( \frac{\partial}{\partial x} \Theta \right) V_x + \frac{\mu(u, v, \theta)U_x \phi_x \psi}{v} + \frac{\mu(u, v, \theta)U_x \phi_x \psi}{v\theta} + \frac{\mu(u, v, \theta)U_x \phi_x \psi}{v\theta^2} + \frac{\mu(v, \theta)\Theta_x \phi_x \psi}{v} \\
+ \frac{\kappa(v, \theta)\Theta_x \phi_x \psi}{v} + \frac{\kappa(v, \theta)\Theta_x \phi_x \psi}{v\theta}.
\tag{22}
\]

The identity (22) plays an essential role in our analysis and we’d like to use several sentences here to explain our main idea to deduce our main result. Those terms appeared in \(J\), especially the last two terms, represent the terms induced by the nonlinearity of the compressible Navier-Stokes equations (1), while the term in \(K\) measures the interaction of rarefaction waves from different families, and those terms in \(L\) reflect the interaction between the rarefaction waves and the solution itself. In the following sections, we will show how to control the corresponding terms related to \(J\) by the smallness of \(\gamma - 1\) and the terms related to \(K\) and \(L\) by the largeness of the parameter \(l\) introduced in (14).

3. Energy estimates. To prove Theorem 1.1, we first define the following set of functions for which the solutions to the Cauchy problem (19), (20) will be sought:

\[
X^k(0, T; A_0, A_1; B_0, B_1) = \left\{ (\varphi, \psi, \phi, \xi)(t, x) \in C([0, T]; H^k(\mathbb{R})) \right. \\
(\psi_x, \phi_x)(t, x) \in L^2([0, T]; H^k(\mathbb{R})) \\
A_0 \leq V(t, x) + \varphi(t, x) \leq A_1, \\
B_0 \leq \Theta(t, x) + \phi(t, x) \leq B_1 \right\}.
\]

For the local solvability of the Cauchy problem (19), (20) in the above set of functions, one has
Lemma 3.1 (Local existence). Under the assumptions listed in Theorem 1.1, there exists a sufficiently small positive constant $t_1$, which depends only on $\mathbf{V}, \mathbf{V}, \Theta, \overline{\Theta}$, and $\left\| (\varphi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}}) \right\|_3$ such that the Cauchy problem (19)-(20) admits a unique smooth solution $(\varphi(t,x), \psi(t,x), \phi(t,x), \xi(t,x)) \in X^3(0, t_1; \mathbf{V}, \mathbf{V}, \Theta, \overline{\Theta})$ which satisfies

$$
\begin{align*}
0 < \mathbf{V} \leq \varphi(t,x) + \mathbf{V}(t,x) \leq \mathbf{V}, \\
0 < \Theta \leq \phi(t,x) + \Theta(t,x) \leq \overline{\Theta}
\end{align*}
$$

for all $(t,x) \in [0, t_1] \times \mathbb{R}$ and

$$
\max_{t \in [0, t_1]} \left\{ \left\| \left( \varphi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right)(t) \right\| \right\} \leq 2 \left\| \left( \varphi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_3.
$$

Suppose that such a local solution $(\varphi(t,x), \psi(t,x), \phi(t,x), \xi(t,x))$ constructed in Lemma 3.1 has been extended to the time step $t = T > t_1$ and satisfies the a priori assumption

$$
\| \phi(t) \|_3 \leq \epsilon, \quad 0 < M_1^{-1} \leq v(t,x) \leq M_1, \quad \| (\varphi(t), \psi(t)) \|_3 \leq N_1
$$

for all $x \in \mathbb{R}, 0 \leq t \leq T$, some sufficiently small positive constant $\epsilon > 0$, and some generic positive constants $M_1, N_1$ (without loss of generality, we may assume in the rest of this manuscript that $M_1 \geq 1, N_1 \geq 1$), what we want to do next is to deduce some energy type estimates in terms of $\| (\varphi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}}) \|_3, \mathbf{V}, \mathbf{V}, \Theta, \overline{\Theta}$, but are independent of $\epsilon, M_1, N_1$. Before going to the details of the analysis, we’d like to mention again that our main idea here is to use the largeness of $\mathbf{l}$ and the smallness of both $\epsilon$ and $\gamma - 1$ to control the possible growth of the solution $(\varphi(t,x), \psi(t,x), \phi(t,x), \xi(t,x))$ constructed in Lemma 3.1 induced by the nonlinearity of the equations (19), the interaction of rarefaction waves from different families, and the interaction between the rarefaction waves and the solution $(\varphi(t,x), \psi(t,x), \phi(t,x), \xi(t,x))$ itself.

Now we turn to perform the desired energy type estimates based on the a priori assumption (23). Before doing so, we first point out that although the precise expressions of the positive constants $C(\mathbf{V}, \mathbf{V}, \Theta, \overline{\Theta})$ and $C(M_1)$ appeared in the right hand side of the energy type estimates throughout the rest of this paper can indeed be given explicitly, to simplify the presentation, we will simply denote them by $C(\mathbf{V}, \mathbf{V}, \Theta, \overline{\Theta})$ and $C(M_1)$ even though they may vary from line to line.

Before performing the energy type estimates on the solution $(\varphi(t,x), \psi(t,x), \phi(t,x), \xi(t,x))$ defined on the strip $\Pi_T = [0, T] \times \mathbb{R}$, we first deduce from the assumption $0 < \epsilon < \min \left\{ \frac{\overline{\Theta}}{2}, \Theta \right\}$ that

$$
\sup_{t \in [0,T]} \left\{ \| \phi(t) \|_{L^\infty_x} \right\} \leq \sup_{t \in [0,T]} \left\{ \| \phi(t) \|_1^\frac{1}{2} \| \phi_x(t) \|_1^\frac{1}{2} \right\} \leq \| \phi(t) \|_3.
$$

Consequently, we have for each $(t,x) \in [0,T] \times \mathbb{R}$ that

$$
\theta(t,x) = \Theta(t,x) + \phi(t,x) \leq \frac{\overline{\Theta}}{2} + \| \phi(t) \|_3 \leq \overline{\Theta}
$$

and

$$
\theta(t,x) = \Theta(t,x) + \phi(t,x) \geq 2\overline{\Theta} - \| \phi(t) \|_3 \geq \Theta.
$$
With the above estimate in hand, we have by integrating the entropy identity (21) with respect to $t$ and $x$ over $[0,t] \times \mathbb{R}$ that

$$\int_{\mathbb{R}} \left\{ \Theta \Phi \left( \frac{v}{\sqrt{v}} \right) + \frac{1}{2} \nabla^{2} + \frac{\Theta}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) \right\} (t, x) dx$$

$$+ \int_{0}^{t} \int_{\mathbb{R}} \left\{ \mu(v, \theta) \Theta^{2} \psi_{x}^{2} + \kappa(v, \theta) \Theta^{2} \psi_{x}^{2} \right\} (\tau, x) dx d\tau$$

$$+ \int_{0}^{t} \int_{\mathbb{R}} \left\{ (\gamma - 1) e^{(\gamma - 1) s} - V - (\gamma - 1) e^{(\gamma - 1) s} + (\gamma - 1) e^{(\gamma - 1) s} \psi \right.$$ 

$$\left. \left(- (\gamma - 1) V - (\gamma - 1) e^{(\gamma - 1) s} \right) U_{x} \right\} (\tau, x) dx d\tau$$

$$= \int_{\mathbb{R}} \left\{ \Theta \Phi \left( \frac{v}{\sqrt{v}} \right) + \frac{1}{2} \nabla^{2} + \frac{\Theta}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) \right\} (0, x) dx + \sum_{j=1}^{5} R_{j}.$$ 

Here

$$R_{1} = \int_{0}^{t} \int_{\mathbb{R}} \mu(v, \theta) U_{x} \psi_{x} + \frac{\kappa(v, \theta)}{v\theta} \mu(v, \theta) U_{x} \psi_{x}$$

$$+ 2 \frac{\mu(v, \theta)}{v\theta} U_{x} \psi_{x} + \frac{\kappa(v, \theta)}{v^{2}\theta} \mu(v, \theta) U_{x} \psi_{x}$$

$$+ \frac{\kappa(v, \theta)}{v^{2}} \mu(v, \theta) U_{x} \psi_{x} \left(0, x\right) dx d\tau,$$

$$R_{2} = \int_{0}^{t} \int_{\mathbb{R}} \mu(v, \theta) \psi_{x} + \frac{\kappa(v, \theta)}{v\theta} \mu(v, \theta) \psi_{x}$$

$$+ 2 \frac{\mu(v, \theta)}{v\theta} \psi_{x} + \frac{\kappa(v, \theta)}{v^{2}\theta} \mu(v, \theta) \psi_{x}$$

$$+ \frac{\kappa(v, \theta)}{v^{2}} \mu(v, \theta) \psi_{x}, \left(0, x\right) dx d\tau,$$

$$R_{3} = \int_{0}^{t} \int_{\mathbb{R}} \mu(v, \theta) V_{x} U_{x} \psi_{x} + \frac{\kappa(v, \theta)}{v\theta} \mu(v, \theta) V_{x} U_{x} \psi_{x}$$

$$+ 2 \frac{\mu(v, \theta)}{v\theta} V_{x} U_{x} \psi_{x} + \frac{\kappa(v, \theta)}{v^{2}\theta} \mu(v, \theta) V_{x} U_{x} \psi_{x}$$

$$+ \frac{\kappa(v, \theta)}{v^{2}} \mu(v, \theta) V_{x} U_{x} \psi_{x}, \left(\tau, x\right) dx d\tau,$$

$$R_{4} = \int_{0}^{t} \int_{\mathbb{R}} \mu_{0}(v, \theta) U_{x} \psi_{x} + \frac{\kappa_{0}(v, \theta)}{v\theta} \mu_{0}(v, \theta) U_{x} \psi_{x}$$

$$+ 2 \frac{\mu_{0}(v, \theta)}{v\theta} U_{x} \psi_{x} + \frac{\kappa_{0}(v, \theta)}{v^{2}\theta} \mu_{0}(v, \theta) U_{x} \psi_{x}$$

$$+ \frac{\kappa_{0}(v, \theta)}{v^{2}} \mu_{0}(v, \theta) U_{x} \psi_{x}, \left(\tau, x\right) dx d\tau,$$

$$R_{5} = \int_{0}^{t} \int_{\mathbb{R}} \left\{ - q(V, \Theta) + g(V, \Theta) U - g(V, \Theta) V_{x} \psi - r(V, \Theta) \xi \right\} \left(\tau, x\right) dx d\tau.$$

From the a priori assumption (23) and its consequence (25), (24), we have

$$\int_{\mathbb{R}} \left\{ \Phi \left( \frac{v}{\sqrt{v}} \right) + \frac{\phi^{2}}{v} + \frac{\phi^{2}}{\gamma - 1} \right\} (t, x) dx$$

$$+ \int_{0}^{t} \int_{\mathbb{R}} \left\{ \frac{\mu(v, \theta) \phi^{2}}{v} + \frac{\kappa(v, \theta) \phi^{2}}{v} \right\} (\tau, x) dx d\tau$$

$$\leq C \left( V, \bar{V}, \Theta, \bar{\Theta} \right) \left\| \left( \varphi_{0}, \psi_{0}, \frac{\phi_{0}}{\sqrt{\gamma - 1}} \right) \right\|^{2} + \sum_{j=1}^{5} R_{j}'.$$

Here $R_{j}' = C(V, \bar{V}, \Theta, \bar{\Theta}) |R_{j}| (j = 1, 2, 3, 4, 5)$ and note that the positive constant $C(V, \bar{V}, \Theta, \bar{\Theta})$ in (26) is independent of $M_{1}$ and $N_{1}$.

Now we estimate $R_{j}' (j = 1, 2, 3, 4, 5)$ term by term. In fact, from the a priori assumption (23), Cauchy-Schwarz’s inequality, and Lemma 2.2, we have

$$R_{1}' \leq \int_{0}^{t} \int_{\mathbb{R}} \left( - \frac{1}{2} \frac{\mu(v, \theta) \phi^{2}}{v^{3}} + \frac{1}{2} \frac{\mu(v, \theta) \phi^{2}}{v} + \frac{1}{3} \frac{\kappa(v, \theta) \phi^{2}}{v} \right) (\tau, x) dx d\tau$$

$$+ C(M_{1}) t \int_{0}^{t} \|U_{x}(\tau)\|_{L_{\infty}}^{2} \left( \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (\tau) \int_{0}^{t} d\tau$$
\[
R'_{2} \leq O(1)M_1 \int_{0}^{t} (\|U_{xx}(\tau)\| \|\psi(\tau)\| + \|\Theta_{xx}\| \|\phi(\tau)\|) \, d\tau \\
\leq \int_{0}^{t} \|\Psi_{xx}(\tau)\| \, d\tau \\
+ C(M_1) \int_{0}^{t} (\|L_{xx}(\tau)\| \|\psi(\tau)\|^2 + \|\Theta_{xx}\| \|\phi(\tau)\|^2) \, d\tau \\
\leq O(1)l^{-\frac{1}{4}} + C(M_1)l^{-\frac{1}{4}} \int_{0}^{t} (1 + \tau)^{-\frac{1}{4}} \left\| \left( \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (\tau) \right\|^2 \, d\tau,
\]

\[
R'_{3} \leq O(1) \int_{0}^{t} \int_{\mathbb{R}} \left( |V_1||U_{xx}|^2 + |U_{xx}|^2 + |V_2|\|\Theta_{xx}\|^2 \right) (\tau, x) \, dxd\tau \\
+ C(M_1) \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{1}{v^4} + \frac{1}{v^2} \right) \psi^2 + |U_{xx}|^2 \frac{\phi^2}{v^2} + |\Theta_{xx}|^2 \frac{\phi^2}{v^2} \right) (\tau, x) \, dxd\tau \\
\leq O(1)l^{-\frac{1}{4}} + C(M_1)l^{-\frac{1}{4}} \int_{0}^{t} (1 + \tau)^{-\frac{1}{4}} \left\| \left( \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (\tau) \right\|^2 \, d\tau,
\]

\[
R'_{4} \leq \frac{1}{3} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta)\phi^2}{\psi} \right) (\tau, x) \, dxd\tau \\
+ C(M_1) \int_{0}^{t} \|U_{x}(\tau)\|^2 \left\| \left( \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (\tau) \right\|^2 \, d\tau \\
\leq \frac{1}{3} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta)\phi^2}{\psi} \right) (\tau, x) \, dxd\tau \\
+ C(M_1)l^{-\frac{1}{4}} \int_{0}^{t} (1 + \tau)^{-\frac{1}{4}} \left\| \left( \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (\tau) \right\|^2 \, d\tau,
\]

and

\[
R'_{5} \leq O(1) \int_{0}^{t} \|(q(V, \Theta), g(V, \Theta))_x(\tau)\|_{L^1} \, d\tau \\
+ O(1) \int_{0}^{t} \|(g(V, \Theta)_x, r(V, \Theta)) (\tau)\| \|\psi, \xi\| (\tau) \, d\tau \\
\leq O(1) \int_{0}^{t} \|(q(V, \Theta), g(V, \Theta)_x) (\tau)\|_{L^1} \, d\tau \\
+ O(1) \int_{0}^{t} \|(g(V, \Theta)_x, r(V, \Theta)) (\tau)\| \, d\tau \\
+ O(1) \int_{0}^{t} \|(g(V, \Theta)_x, r(V, \Theta)) (\tau)\| \|\psi, \xi\| (\tau) \|^2 \, d\tau.
\]
\[\leq O(1)l^{-\frac{1}{2}} + O(1)l^{-\frac{1}{2}} \int_0^t (1 + \tau)^{-\frac{3}{2}} \left\| \left( \varphi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (\tau) \right\|^2 d\tau.\]

Substituting the above estimates into (26), we can get that
\[
\left\| \left( \sqrt{\Phi} \left( \frac{v}{V} \right), \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (t) \right\|^2 + \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_x^2}{v} + \frac{\kappa(v, \theta)\phi_x^2}{v} \right) (\tau, x) dx d\tau
\leq C \left( \frac{\nu}{\sqrt{\gamma - 1}} \right) \left\{ \left\| \left( \varphi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|^2 + l^{-\frac{1}{2}} \right.
\]
\[\left. + l^{-\frac{1}{2}} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_x^2}{v^3} \right) (\tau, x) dx d\tau \right. \]
\[\left. + C(M_1)l^{-\frac{1}{4}} \int_0^t \int_{\mathbb{R}} (1 + \tau)^{-\frac{3}{2}} \left\| \left( \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (\tau) \right\|^2 d\tau \right. \]
\[\left. + l^{-\frac{1}{2}} \int_0^t (1 + \tau)^{-\frac{3}{2}} \left\| \left( \varphi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (\tau) \right\|^2 d\tau \right\}. \tag{27}
\]

If we choose \( l \geq l_1 > 1 \) sufficiently large such that
\[
C \left( \frac{\nu}{\sqrt{\gamma - 1}} \right) l_1^{-\frac{1}{2}} \leq 1,
\]
\[
C \left( \frac{\nu}{\sqrt{\gamma - 1}} \right) C(M_1) l_1^{-\frac{1}{4}} \leq 1, \tag{28}
\]
then we can deduce by exploiting Gronwall’s inequality in (27) that
\[
\left\| \left( \sqrt{\Phi} \left( \frac{v}{V} \right), \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (t) \right\|^2 + \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_x^2}{v} + \frac{\kappa(v, \theta)\phi_x^2}{v} \right) (\tau, x) dx d\tau
\leq C \left( \frac{\nu}{\sqrt{\gamma - 1}} \right) \left\{ \left\| \left( \varphi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|^2 \right.
\]
\[\left. + 1 + l^{-\frac{1}{2}} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_x^2}{v^3} \right) (\tau, x) dx d\tau \right\}. \tag{29}
\]

Now we turn to control the term \( \int_0^t \int_{\mathbb{R}} \frac{\mu(v, \theta)\psi_x^2}{v^3} (\tau, x) dx d\tau \) appeared in the right hand side of (29). To this end, we get from (22) that
\[
\left( \frac{1}{2} \left\{ \left( \frac{\mu(v, \theta)\phi_x}{v} \right)^2 \right\} + \frac{\mu(v, \theta)\theta\phi_x^2}{v^3} \right)
\]
\[= \left( \frac{\mu(v, \theta)\phi_x}{v} \right)_t - \left( \frac{\mu(v, \theta)\phi_x}{v} \right)_x + \frac{\mu(v, \theta)\phi_x^2}{v} \]
\[- \mu(v, \theta)U_x\psi_x + \mu(v, \theta)\psi_x U_x + \mu(v, \theta)U_x\psi - \mu(v, \theta)U_x^2 \rho \psi \]
\[+ \left( \frac{V_x}{v} - \frac{\theta}{v^2} \right) + \left( \frac{\Theta_x}{v^2} \right) \left( \frac{1}{v} - \frac{1}{V^2} \right) + \frac{\phi_x}{v} \mu(v, \theta)\phi_x \]
\[+ \left( \frac{\mu(v, \theta)\phi_x}{v} - \psi \right) \left( \frac{\mu(v, \theta)V_x u_x}{v^2} - \mu(v, \theta)V_x u - \mu(v, \theta)U_{xx} \right). \]
\[-\frac{\mu_\theta(v, \theta)\theta_x u_x}{v} + \frac{\mu_\theta(v, \theta)\varphi_x \theta_t}{v}\).\\n
Integrating this identity with respect to \(t\) and \(x\) over \([0, t] \times \mathbb{R}\), we obtain

\[
\int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x}{v} \right)^2 (t, x)dx \leq C (V, \overline{V}, \Theta, \overline{\Theta}) \left\{ \| (\varphi_0, \psi_0) \|^2 + \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\psi_x^2}{v} \right) (\tau, x)dx d\tau \right\}
\]

\[
+ \sum_{j=6}^{10} |R_j|.
\]

Here

\[
\begin{align*}
R_6 &= C(V, \overline{V}, \Theta, \overline{\Theta}) \int_\mathbb{R} \left( \frac{\mu(v, \theta)\psi_x}{v} \right) (\tau, x)dx, \\
R_7 &= C(V, \overline{V}, \Theta, \overline{\Theta}) \int_0^t \int_\mathbb{R} \left[ \left( \frac{\mu(v, \theta)U_x v_x}{v^2} - \frac{\mu_\psi(v, \theta)U_x \theta_x}{v} \right) + \Theta \left( \frac{1}{v^2} - \frac{1}{\overline{v}^2} \right) \right] (\tau, x)dx d\tau, \\
R_8 &= C(V, \overline{V}, \Theta, \overline{\Theta}) \int_0^t \int_\mathbb{R} \left[ g(V, \Theta) \frac{\mu(v, \theta)\varphi_x}{v} \right] (\tau, x)dx d\tau, \\
R_{10} &= C(V, \overline{V}, \Theta, \overline{\Theta}) \int_0^t \int_\mathbb{R} \left[ \left( \frac{\mu(v, \theta)\varphi_x}{v} - \psi \right) \left( \frac{\mu(v, \theta)V_x u_x}{v^2} - \frac{\mu_\psi(v, \theta)U_x \theta_x}{v} \right) \right] (\tau, x)dx d\tau.
\end{align*}
\]

Now we deal with \(R_6, R_7, R_8, R_9,\) and \(R_{10}\) term by term. For this purpose, by applying the Cauchy-Schwarz inequality, we can get from Lemma 2.2 and the a priori assumption (23) that

\[
|R_6| \leq \frac{1}{2} \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x}{v} \right)^2 (t, x)dx + O(1) \| \psi \|^2,
\]

\[
|R_7| \leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x)dx d\tau + \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta)\varphi_x}{v} \right) (\tau, x)dx d\tau + C(M_1) \int_0^t \| U_{xx} (\tau) \| d\tau \\
+ C(M_1) \int_0^t \int_\mathbb{R} \left[ M_1 U^2_x (\tau, x) + (M_1^2 + M_1^3) V_x^2 (\tau, x) \right] \psi^2 (\tau, x)dx d\tau + C(M_1) \int_0^t \left( \frac{\mu(v, \theta)\varphi_x}{v} \right) (\tau, x)dx d\tau
\]

\[
\leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x)dx d\tau + \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta)\varphi_x}{v} \right) (\tau, x)dx d\tau + C(M_1) \int_0^t \left( 1 + \tau \right)^{-\frac{3}{2}} \| \psi(\tau) \|^2 d\tau + C(M_1) l^{-\frac{1}{4}}.
\]
\[ |R_8| \leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta)\theta_x^2}{v} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \int_\mathbb{R} \left( M_1 V_x^2 \varphi^2 + M_1^2 V_x^2 \varphi^3 + M_1^3 V_x^2 \varphi^3 \right) (\tau, x) \, dx \, d\tau \\
\leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta)\theta_x^2}{v} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) l^{-\frac{1}{2}} \int_0^t (1 + \tau)^{-\frac{3}{2}} \left\| \left( \varphi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (\tau) \right\|^2 \, d\tau,
\]

and

\[ |R_9| \leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \int_\mathbb{R} g(V(\tau, x), \Theta(\tau, x))^2 \, dx \, d\tau \\
\leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x) \, dx \, d\tau + C(M_1) l^{-\frac{1}{2}}.
\]

As to \( R_{10} \), noticing that

\[
R_{10} = C(\nabla, \nabla, \Theta, \Theta) \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x}{v} - \psi \right) (\tau, x) \\
\times \left( \frac{\mu(v, \theta)V_x u_x}{v^2} - \frac{\mu(v, \theta)V_x u_x}{v} \right) (\tau, x) \, dx \, d\tau \\
- \frac{\mu(v, \theta)U_{xx}}{v} - \frac{\mu(v, \theta)\varphi_x \theta_x}{v} + \frac{\mu(v, \theta)\varphi_x \theta_x}{v} \right) (\tau, x) \, dx \, d\tau \\
= C(\nabla, \nabla, \Theta, \Theta) \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x}{v} \right) (\tau, x) \left( \frac{\mu(v, \theta)V_x u_x}{v^2} \right) (\tau, x) \, dx \, d\tau \\
- \frac{\mu(v, \theta)V_x u_x}{v} - \frac{\mu(v, \theta)U_{xx}}{v} - \frac{\mu(v, \theta)\varphi_x \theta_x}{v} + \frac{\mu(v, \theta)\varphi_x \theta_x}{v} \right) (\tau, x) \, dx \, d\tau \\
+ C(\nabla, \nabla, \Theta, \Theta) \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_x \theta_x}{v^2} \right) (\tau, x) \, dx \, d\tau \\
+ C(\nabla, \nabla, \Theta, \Theta) \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\psi \varphi_x \theta_x}{v} \right) (\tau, x) \, dx \, d\tau \\
+ C(\nabla, \nabla, \Theta, \Theta) \int_0^t \int_\mathbb{R} \left( -\frac{\mu(v, \theta)\psi \varphi_x \theta_x}{v} \right) (\tau, x) \, dx \, d\tau \\
:= \sum_{j=1}^{4} K_j,
\]
where \( K_j (j = 1, 2, 3, 4) \) denote the corresponding terms in the right hand side of the above identity.

To bound these \( K_j (j = 1, 2, 3, 4) \), by exploiting Lemma 2.2 and the a priori assumption (23) again, we can get that \( K_1 \) and \( K_2 \) can be estimated as follows

\[
|K_1| \leq \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \varphi_x^2}{v^3} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \int_{\mathbb{R}} \left\{ \phi_x^2 \varphi_x^2 + \psi_x^2 \Theta_x^2 + \phi_x^2 U_x^2 + \Theta_x^4 + U_{xx}^2 + \varphi_x^2 V_x^2 + V_x^4 \right\} \, dx \, d\tau
\]

\[
\leq \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \varphi_x^2}{v^3} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) \left( e^2 + M_x^2 l^{-\frac{1}{2}} \right) \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \varphi_x^2}{v} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) l^{-\frac{1}{2}} \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta) \varphi_x^2}{v} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \int_{\mathbb{R}} \left( \Theta_x^4 + U_{xx}^2 \right) (\tau, x) \, dx \, d\tau
\]  

(31)

and

\[
|K_2| \leq C(M_1) (\gamma - 1) \left( \| \phi_x \|^2_{L^\infty_x} + \| \Theta_x \|^2_{L^\infty_x} + \| \phi_{xx} \|_{L^\infty_x} + \| \Theta_{xx} \|_{L^\infty_x} \right) \\
+ \| (\psi_x, V_x, \psi_x, U_x) \|^2_{L^1_x} + \| \psi_x \|_{L^\infty_x} + 1 \right) \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \varphi_x^2}{v^3} \right) (\tau, x) \, dx \, d\tau
\]

\[
\leq C(M_1) (\gamma - 1) (N_1 + 1) \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \varphi_x^2}{v^3} \right) (\tau, x) \, dx \, d\tau.
\]  

(32)

For \( K_3 \), since

\[
\frac{\theta_t}{\gamma - 1} = \frac{\kappa_0(v, \theta) \varphi_x^2 + \kappa(v, \theta) \varphi_{xx} + \kappa_0(v, \theta) \psi_x \varphi_x + \mu(v, \theta) u_x^2 - \theta u_x - \kappa(v, \theta) \varphi_x}{v^2},
\]

we have

\[
K_3 = C \left( \mathbf{V}, \mathbf{V}, \Theta, \Theta \right) \int_0^t \int_{\mathbb{R}} \psi(\tau, \varphi) \left( \frac{\mu(v, \theta) V_x u_x}{v^2} + \mu(v, \theta) V_x u_x \right) \\
+ \frac{\mu(v, \theta) U_{xx}}{v} + \frac{\mu_0(v, \theta) \varphi_{xx} u_x}{v} \right) (\tau, x) \, dx \, d\tau
\]

\[
= C(\mathbf{V}, \mathbf{V}, \Theta, \Theta) \int_0^t \int_{\mathbb{R}} \psi(\tau, \varphi) \left[ \frac{\mu_0(v, \theta) \varphi_x \psi_x}{v} + \frac{\mu(v, \theta) (\varphi_x U_x + \psi_x \Theta_x)}{v} \right] \\
+ \frac{\mu(v, \theta) \psi_x V_x}{v} - \frac{\mu(v, \theta) \psi_x V_x}{v^2}
\]

(33)
\[
\begin{align*}
&+ \left( \frac{\mu(v, \theta) U_x V_x}{v} + \mu(v, \theta) \Theta_x U_x + \frac{\mu(v, \theta) V_x U_x}{v^2} + \frac{\mu(v, \theta) U_x^2}{v} \right) (\tau, x) d\tau \\
= \sum_{j=1}^{3} K_{3,j},
\end{align*}
\]
and \( K_{3,j} (j = 1, 2, 3) \) can be estimated as in the following
\[
|K_{3,1}| \leq O(1) \int_0^t \int_R \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right) (x, \tau) \, dx \, d\tau \\
\leq C(M_1) \int_0^t \|\psi(\tau)\|_{L^\infty} \left\| \left( \frac{\kappa^2 \psi_x}{\sqrt{\theta}} \right)(\tau) \right\| \left\| \left( \frac{\mu^2 \psi_x}{\sqrt{\theta}} \right)(\tau) \right\| \, d\tau \\
\leq C(M_1) \int_0^t \|\psi(\tau)\| \left\| \left( \frac{\kappa^2 \psi_x}{\sqrt{\theta}} \right)(\tau) \right\| \left\| \left( \frac{\mu^2 \psi_x}{\sqrt{\theta}} \right)(\tau) \right\| \, d\tau \\
\leq \frac{1}{10} \int_0^t \int_R \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \left\| \sqrt{v(\tau)} \right\|_{L^\infty} \left\| \frac{1}{\sqrt{v(\tau)}} \right\| \left\| \psi(\tau) \right\| \left\| \left( \frac{\mu^2 \psi_x}{\sqrt{\theta}} \right)(\tau) \right\| \, d\tau \\
	imes \|\phi_x(\tau)\| \left\| \left( \frac{\kappa^2 \phi_x}{\sqrt{\theta}} \right)(\tau) \right\| \, d\tau \\
\leq \frac{1}{10} \int_0^t \int_R \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right) (\tau, x) \, dx \, d\tau \\
+ \epsilon C(M_1) N_1 \int_0^t \int_R \left( \frac{\mu(v, \theta) \psi_x^2 + \kappa(v, \theta) \phi_x^2}{v} \right) (\tau, x) \, dx \, d\tau,
\end{align*}
\]
\[
|K_{3,2}| \leq \frac{1}{10} \int_0^t \int_R \left( \frac{\mu(v, \theta) \psi_x^2 + \kappa(v, \theta) \phi_x^2}{v} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \|U_x(\tau)\|_{L^\infty} \|\psi(\tau)\|^2 \, d\tau \\
\leq \frac{1}{10} \int_0^t \int_R \left( \frac{\mu(v, \theta) \psi_x^2 + \kappa(v, \theta) \phi_x^2}{v} \right) (\tau, x) \, dx \, d\tau \\
+ C(M_1) l^{-\frac{1}{2}} \int_0^t (1 + \tau)^{-\frac{1}{2}} \|\psi(\tau)\|^2 \, d\tau, 
\]
and
\[
|K_{3,3}| \leq \int_0^t \int_R \left( V_x(\tau, x) \right)^2 \, dx \, d\tau \\
+ C(M_1) \int_0^t \int_R \left( V_x(\tau, x) \right)^2 \left( \frac{1}{v^2} + \frac{1}{v^4} \right) \, (\tau, x) \, dx \, d\tau \\
+ \int_0^t \|U_{xx}(\tau)\| \, d\tau + C(M_1) \int_0^t \|U_{xx}(\tau)\| \|\psi(\tau)\|^2 \, d\tau \\
\leq O(1) l^{-\frac{1}{2}} + C(M_1) l^{-\frac{1}{2}} \int_0^t (1 + \tau)^{-\frac{1}{2}} \|\psi(\tau)\|^2 \, d\tau.
\]
Inserting (34), (35), and (36) into (33), one can deduce that $K_3$ can be bounded as in the following

$$|K_3| \leq \frac{1}{5} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_x^2}{v} \right) (\tau, x) dx d\tau + \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta)\phi_x^2}{v} \right) (\tau, x) dx d\tau$$

$$+ O(1)l^{-\frac{1}{2} + \epsilon} C(M_1) N_1 \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_x^2 + \kappa(v, \theta)\phi_x^2}{v} \right) (\tau, x) dx d\tau$$

(37)

$$+ C(M_1) l^{-\frac{1}{2}} \int_0^t (1 + \tau)^{-\frac{3}{2}} \|\psi(\tau)\|^2 d\tau$$

$$+ C(M_1) l^{-\frac{1}{2} + \epsilon} \int_0^t (1 + \tau)^{-\frac{3}{2}} \|\psi(\tau)\|^2 d\tau,$$

As to $K_4$, since

$$K_4 = C(\mathcal{V}, \mathcal{V}, \Theta, \Theta)(\gamma - 1) \int_0^t \int_{\mathbb{R}} \left( -\frac{\mu_\theta(v, \theta)\psi_x}{v} \right) (\tau, x)$$

$$\times \left[ \left( \frac{\kappa_\theta(v, \theta)\theta_x^2 + \kappa(v, \theta)\theta_{xx} + \mu(v, \theta)u_x^2 - \theta_{ux}}{v} \right) + \frac{\kappa(v, \theta)V_x\theta_x}{v^2} - \frac{\kappa_\theta(v, \theta)V_x\theta_x}{v} \right]$$

$$+ \left( \frac{\kappa(v, \theta)\varphi_x\theta_x}{v^2} - \frac{\kappa(v, \theta)\theta_x\varphi_x}{v^2} \right) (\tau, x) dx d\tau$$

(38)

$$= \sum_{j=1}^2 K_{4,j},$$

we have from the smallness of $\epsilon$ that

$$|K_{4,1}| \leq \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x) dx d\tau$$

$$+ (\gamma - 1)^2 C(M_1) \int_0^t \int_{\mathbb{R}} \left( \psi_x^2 (\theta_x^4 + \theta_{xx}^2 + u_x^4 + u_{xx}^2 + V_x^4) \right) (\tau, x) dx d\tau$$

(39)

$$\leq \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x) dx d\tau$$

$$+ C(M_1)(\gamma - 1)^2 N_1^2 \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta)\phi_x^2}{v} \right) (\tau, x) dx d\tau$$

$$+ C(M_1)(\gamma - 1)^2 N_1^4 \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta)\phi_x^2 + \mu(v, \theta)\psi_x^2}{v} \right) (\tau, x) dx d\tau$$

$$+ C(M_1)(\gamma - 1)^2 N_1^2 l^{-\frac{1}{2}} \int_0^t (1 + \tau)^{-\frac{3}{2}} \|\psi(\tau)\|^2 d\tau$$

and

$$|K_{4,2}| \leq C(M_1)(\gamma - 1) \|(\psi, \varphi_x, \Theta_x)\|_{L_\infty^\infty} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x) dx d\tau$$

(40)

$$\leq C(M_1)(\gamma - 1) N_1 \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x) dx d\tau.$$
Lemma 3.2. Under the assumptions listed in Lemma 3.1, and the above conditions (42), (43), we have

\[
\int_{\mathbb{R}} \left( \frac{\mu(v,\theta)\varphi_{x}^{2}}{v} \right) (t,x)dx + \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v,\theta)\varphi_{x}^{2}}{v^{3}} \right) (\tau,x)d\tau d\tau
\lesssim \|\psi(t)\|^{2} + C(M_{1}) \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v,\theta)\varphi_{x}^{2}}{v} + \frac{\kappa(v,\theta)\phi_{xx}^{2}}{\nu} \right) (\tau,x)d\tau d\tau
\]
\[ + 1 + \| (\varphi_{0x}, \psi_0) \|^2 + (\gamma - 1) \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right)(\tau, x) \, dx \, d\tau \\
+ \int_0^t (1 + \tau)^{-\frac{1}{2}} \left\| \left( \sqrt{\frac{v}{\Phi}}, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right)(\tau) \right\|^2 d\tau. \quad (44) \]

Now we turn to deal with the term \( \int_0^t \int_\mathbb{R} \kappa(v, \theta) \phi_{xx}^2 \, dx \, d\tau \). To this end, multiplying \((19)_3\) by \(-\phi_{xx}\) gives

\[
\left( \frac{\phi_x(t)}{2(\gamma - 1)} \right)_t + \frac{k(v, \theta)}{v} \phi_{xx}^2 - \frac{1}{\gamma - 1} (\phi_x \phi_t)_x \\
= \frac{\theta}{v} \psi_x \phi_{xx} + \left\{ \frac{k(v, \theta) \varphi_x \phi_x}{v^2} - \frac{\mu(v, \theta) \psi_x^2}{v} - \frac{\kappa_{\theta}(v, \theta) \phi_x^2}{v} - \frac{\kappa_{\theta}(v, \theta) \varphi_x \phi_x}{v} \right\} \phi_{xx} \\
+ \left\{ \frac{k(v, \theta) V_x \phi_x}{v^2} + \frac{\kappa(v, \theta) \theta \phi_x}{v^2} - 2 \frac{\mu(v, \theta) U_x \psi_x}{v} - 2 \frac{\kappa_{\theta}(v, \theta) \varphi_x \phi_x}{v} \right\} \phi_{xx} \\
- \frac{\kappa_{\theta}(v, \theta) \theta \phi_{xx}}{v} + \frac{1}{\gamma - 1} r(V, \Theta) \phi_{xx}. \]

Integrating this identity with respect to \( t \) and \( x \) over \([0, t] \times \mathbb{R}\), we can get that

\[
\left\| \frac{\phi_x(t)}{\sqrt{\gamma - 1}} \right\|^2 + \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right)(\tau, x) \, dx \, d\tau \leq \| \phi_{0x} \|^2 + \sum_{i=11}^{16} |R_i|.
\]

Here \( R_i (i = 11, \ldots, 16) \) denote those terms corresponding to the terms in the right hand side of the above identity.

To estimate \( |R_i|, i = 11, \ldots, 16 \) term by term, we have from the a priori assumption \((23)\), Cauchy-Schwarz’s inequality, and Lemma 2.2 that

\[
|R_{11}| \leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right)(\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right)(\tau, x) \, dx \, d\tau,
\]

\[
|R_{12}| \leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right)(\tau, x) \, dx \, d\tau \\
+ C(M_1) \left( N_1^2 + 1 \right) \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta) \psi_x^2}{v} + \kappa(v, \theta) \phi_x^2 \right)(\tau, x) \, dx \, d\tau,
\]

\[
|R_{13}| \leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right)(\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \left\| U_x(\tau) \right\|_{L^\infty} \left\| \left( \phi_x, \frac{\phi}{\sqrt{\gamma - 1}} \right)(\tau) \right\|^2 d\tau \\
\leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right)(\tau, x) \, dx \, d\tau,
\]

\[
|R_{14}| \leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right)(\tau, x) \, dx \, d\tau \\
+ C(M_1) \int_0^t \left\| U_x(\tau) \right\|_{L^\infty} \left\| \left( \phi_x, \frac{\phi}{\sqrt{\gamma - 1}} \right)(\tau) \right\|^2 d\tau \\
\leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right)(\tau, x) \, dx \, d\tau.
\]
\[ +C(M_1)l^{-\frac{1}{2}} \int_0^t (1 + \tau)^{-\frac{3}{4}} \left\| \begin{pmatrix} \varphi \\ \frac{\phi}{\sqrt{\gamma - 1}} \end{pmatrix} (\tau) \right\|^2 d\tau, \]

\[ |R_{14}| \leq \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \kappa(v, \theta) \phi_{xx}^2 \right) (\tau, x) d\tau dx + C(M_1) \int_0^t \int_{\mathbb{R}} \left( \Theta_x \mu(v, \theta) \varphi_x^2 \left( \frac{1}{v^3} + \frac{1}{v} \right) + V_x^2 \kappa(v, \theta) \phi_x^2 \left( \frac{1}{v^3} + \frac{1}{v} \right) + U_x^2 \mu(v, \theta) \psi_x^2 \right) (\tau, x) d\tau dx \]

\[ \leq \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \kappa(v, \theta) \phi_{xx}^2 \right) (\tau, x) d\tau dx + C(M_1) \int_0^t \int_{\mathbb{R}} \left( \mu(v, \theta) \varphi_x^2 + \kappa(v, \theta) \phi_x^2 \mu(v, \theta) \psi_x^2 \right) (\tau, x) d\tau dx, \]

\[ |R_{15}| \leq \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \kappa(v, \theta) \phi_{xx}^2 \right) (\tau, x) d\tau dx + C(M_1) \int_0^t \int_{\mathbb{R}} V_x^2 (\tau, x) d\tau dx \]

\[ \leq \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \kappa(v, \theta) \phi_{xx}^2 \right) (\tau, x) d\tau dx + C(M_1) l^{-1}, \]

and

\[ |R_{16}| \leq \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \kappa(v, \theta) \phi_{xx}^2 \right) (\tau, x) d\tau dx + C(M_1) \int_0^t \int_{\mathbb{R}} \left( \Theta_x \mu(v, \theta) \varphi_x^2 + \frac{1}{(\gamma - 1)^2} \tau V, \Theta \right) (\tau, x) d\tau dx \]

\[ \leq \frac{1}{10} \int_0^t \int_{\mathbb{R}} \left( \kappa(v, \theta) \phi_{xx}^2 \right) (\tau, x) d\tau dx + C(M_1) l^{-\frac{1}{2}}. \]

Having obtained the above estimates, if we choose \( l \geq l_3 \geq l_2 > 1 \) suitably large such that

\[ C(M_1) l_3^{-\frac{1}{2}} \leq 1, \quad (45) \]

then we can obtain the following result

**Lemma 3.3.** Under the smallness of \( l \) in (45) and assumptions listed in Lemma 3.2, we can get

\[ \left\| \frac{\phi_x(t)}{\sqrt{\gamma - 1}} \right\|^2 + \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right) (\tau, x) d\tau dx \leq C(M_1) (N_1^2 + 1) \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \varphi_x^2}{v} + \frac{\kappa(v, \theta) \phi_x^2}{v} \right) (\tau, x) d\tau dx + \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right) (\tau, x) d\tau dx + \int_0^t (1 + \tau)^{-\frac{1}{2}} \left\| \left( \frac{\phi}{\sqrt{\gamma - 1}}, \varphi \right) (\tau) \right\|^2 d\tau + 1. \]

Now if \( \gamma - 1 \) is further assumed to be small enough such that

\[ (\gamma - 1)C(M_1)(N_1^2 + 1) \leq 1, \quad (47) \]
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we can conclude from (29), (44), and (46) that
\[
\left\| \left( \sqrt{\Phi \left( \frac{v}{\gamma - 1} \right)}, \psi, \frac{\phi}{\sqrt{\gamma - 1}}, \frac{\mu(v, \theta)\varphi_x}{v} \right) (t) \right\|^2 \\
+ \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_x^2}{v} + \frac{\kappa(v, \theta)\phi_x^2}{v} + \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x) dx d\tau 
\]
\[
\lesssim \left\| \left( \psi_0, \varphi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_1 ^2. 
\]

Here and in the rest of this manuscript, without loss of generality, we can assume that \( \left\| \left( \psi_0, \varphi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_1 \geq 1. \)

Having obtained the estimate (48), we now turn to use Kanel’s method [16] to deduce a uniform positive lower and upper bounds for \( v(t, x). \) To do so, we need to deduce the \( L^2 \)-norm estimate on \( \mu(v, \theta)\tilde{v}_x \), where \( \tilde{v} = \frac{v}{\text{v}}. \) For this purpose, due to \( \mu(v, \theta)\tilde{v}_x = \mu(v, \theta)\varphi_x - \mu(v, \theta) \left( \frac{V_x}{v} - \frac{V}{\text{v}} \right), \)
we can get from Lemma 2.2 that
\[
\left\| \left( \mu(v, \theta)\tilde{v}_x \right) (t) \right\|^2 \leq 2 \left\| \left( \mu(v, \theta)\varphi_x \right) (t) \right\|^2 + C (\Theta, \overline{\Theta}) C(M_1) \| V_x(t) \|^2 \\
\leq 2 \left\| \left( \mu(v, \theta)\varphi_x \right) (t) \right\|^2 + C (\Theta, \overline{\Theta}) C(M_1) l^{-1}. 
\]

Thus the estimate (48) tells us that
\[
\left\| \left( \sqrt{\Phi(\tilde{v})}, \psi, \frac{\phi}{\sqrt{\gamma - 1}}, \frac{\mu(v, \theta)\varphi_x}{\tilde{v}} \right) (t) \right\|^2 \\
+ \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_x^2}{v} + \frac{\kappa(v, \theta)\phi_x^2}{v} + \frac{\mu(v, \theta)\varphi_x^2}{v^3} \right) (\tau, x) dx d\tau 
\]
\[
\lesssim \left\| \left( \psi_0, \varphi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_1^2. 
\]

The above analysis yields the following result

**Lemma 3.4.** Under the assumptions (28), (42), (43), (45), (47) and the a priori assumption (23), we can deduce that (49) holds.

Now we use the estimate (49) to yield an estimate on the positive lower and upper bounds on the specific volume \( v(t, x). \) To this end, we first get from the a priori assumption (23) together with the estimates (25) and (24) that
\[
\mu(v(t, x), \theta(t, x)) \geq \mu(v(t, x)) \]
holds for all \( (t, x) \in [0, T] \times \mathbb{R}. \)

Secondly, since the constants \( v_\pm, u_\pm, \) and \( \theta_\pm \) do not depend on \( \gamma - 1 \) and \( \mu(v) \) is assumed to satisfy (16), one can easily deduce that there exist a positive constant \( C(\Theta, \overline{\Theta}, v_\pm, u_\pm, \theta_\pm) \) depending only on \( \Theta, \overline{\Theta}, v_\pm, u_\pm, \) and \( \theta_\pm \) and a function \( \bar{\mu}(v) \) satisfying
\[
\bar{\mu}(v) \sim \begin{cases} 
  v^a, & v \to 0^+, \\
  v^b, & v \to \infty
\end{cases}
\]

(51)
such that
\[ \mu(v(t, x)) \geq C(\Theta, \Sigma, v, u, \Theta, \Sigma) \mu(\bar{v}(t, x)) := C(\Theta, \Sigma, v, u, \Theta, \Sigma) \mu \left( \frac{v(t, x)}{V(t, x)} \right) \]
(52)
holds true for all \((t, x) \in [0, T] \times \mathbb{R} \),
(50), (51), and (52) together with the estimate (49) imply
\[ \left\| \left( \sqrt{\Phi(\bar{v})}, \frac{\mu(\bar{v})}{v} \bar{v}_x \right) (t) \right\|^2 \lesssim \left\| \left( \psi_0, \varphi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_1^2. \]
(53)
With the estimate (53) in hand, if we setting
\[ \Psi(\bar{v}) = \int_1^\infty \frac{\sqrt{\Phi(z)}}{z} \tilde{\mu}(z) dz, \quad \Phi(z) = z - \ln z - 1, \]
we can deduce from the assumption (51) that there exist positive constants \(A_1, A_2\) such that
\[ |\Psi(\bar{v})| \geq A_1 \left( \bar{v}^{b+2} + |\ln \bar{v}|^{2} \right) - A_2. \]
Moreover, one can get from the estimate (53) that
\[ |\Psi(\bar{v})| = \left| \int_{-\infty}^x \Psi(\bar{v}(y)) dy \right| \leq \int_{\mathbb{R}} \left| \left( \sqrt{\Phi(\bar{v})} \frac{\mu(\bar{v})}{v} \bar{v}_x \right) (t, x) \right| dx \leq \int_{\mathbb{R}} \left| \left( \sqrt{\Phi(\bar{v})} \frac{\mu(\bar{v})}{v} \bar{v}_x \right) (t, x) \right| dx \leq \left\| \left( \psi_0, \varphi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_1^2. \]
Then we deduce from the above inequalities that there exists a generic positive constant \(C\) which may depend only on \(v, u, \Theta, \Sigma, v, \Sigma\) such that
\[ \exp \left( -C \left\| \left( \varphi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_1^{1+2b} \right) \leq v(t, x) \leq C \left\| \left( \psi_0, \varphi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_1^{1+2b}. \]
(54)
Furthermore, we can get from the above estimate and the estimate (49) that
\[ \left\| \left( \varphi, \psi, \varphi_x, \frac{\phi}{\sqrt{\gamma - 1}} \right) (t) \right\|^2 \lesssim \exp \left( O(1) \left\| \left( \varphi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_1^{1+2b} \right) \]
(55)
\[ \int_0^t \left\| (\varphi_x(\tau), \psi_x(\tau), \varphi_x(\tau)) \right\|^2 d\tau \lesssim \exp \left( O(1) \left\| \left( \varphi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_1^{1+2b} \right). \]
Now we turn to derive certain energy type estimates on \((\varphi_x(t, x), \phi_x(t, x), \psi_x(t, x))\). To this end, we need to yield an estimate on \(\|\psi_x(t)\|^2\), which is the main content of the following lemma

**Lemma 3.5.** Under the assumptions listed in Lemma 3.4, we can get that
\[ \|\psi_x(t)\|^2 + \int_0^t \int_\mathbb{R} \left( \frac{\psi_{xx}}{v} \right)(\tau, x) d\tau dx \lesssim N_{01}^2 \exp \left( O(1) N_{01}^2 \right). \]
Proof. From (18) and (19), we have

\[
\left( \frac{\psi^2}{2} \right)_t + \frac{\mu(v, \theta) \psi_x^2}{v} - (\psi_t \psi_x)_x = \left( -\frac{\theta}{v^2} \varphi_x + \frac{1}{v} \phi_x \right) \psi_{xx} + \left( \frac{\mu(v, \theta) \varphi_x \psi_x}{v^2} - \frac{\mu_{\theta}(v, \theta) \varphi_x \psi_x}{v} - \frac{\mu_v(v, \theta) \varphi_x \psi_x}{v} \right) \psi_{xx}
\]

\[+ V_x \left( -\frac{\theta}{v^2} + \Theta_x \left( \frac{1}{v} - \frac{1}{V} \right) \psi_{xx} + g(V, \Theta) \psi_{xx} \right)
\]

\[+ \left( \frac{\mu(v, \theta) (V_x \psi_x + U_x \varphi_x)}{v^2} - \frac{\mu_{\theta}(v, \theta) (U_x \varphi_x + \Theta_x \psi_x)}{v} - \frac{\mu_v(v, \theta) (V_x \psi_x)}{v} \right) \psi_{xx}.
\]

Integrating this identity with respect to \( t \) and \( x \) over \([0, t] \times \mathbb{R}\), we obtain

\[
\|\psi_x(t)\|^2 + \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right) (\tau, x) \, dx \, d\tau \quad (56)
\]

\[
\leq C \left( \nabla \cdot V, V, \Theta, \Theta \right) \left\{ \|\psi_{xx}\|^2 + \sum_{i=17}^{23} |R_i| \right\}.
\]

Letting \( R_i' = C(\nabla \cdot V, V, \Theta, \Theta)|R_i|(i = 17, \ldots, 23) \), we now estimate \( R_i'(i = 17, \ldots, 23) \) term by term. For this purpose, we can get from Lemma 2.2, (54), (55), Cauchy-schwarz's inequality, and the fact \( \|\phi_x\|_{L^2([0, T])} \leq \epsilon \leq 1 \) that

\[
R_{17} \leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right) (\tau, x) \, dx \, d\tau + O(1) \int_0^t \int_{\mathbb{R}} \left( \frac{\varphi_x^2}{v^2} + \frac{\phi_x^2}{v} \right) (\tau, x) \, dx \, d\tau
\]

\[
\leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right) (\tau, x) \, dx \, d\tau + O(1) N_{01}^2 \exp (O(1) N_{01}^2),
\]

\[
R_{18} \leq \frac{1}{9} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right) (\tau, x) \, dx \, d\tau
\]

\[
+ O(1) N_{01}^2 \exp (O(1) N_{01}^2) \int_0^t \int_{\mathbb{R}} \left( \varphi_x^2 \psi_x^2 \left( \frac{1}{v^2} + \frac{1}{V} \right) \right) (\tau, x) \, dx \, d\tau
\]

\[
+ O(1) N_{01}^2 \exp (O(1) N_{01}^2) \int_0^t \int_{\mathbb{R}} \left( \frac{\phi_x^2 \psi_x^2}{v} \right) (\tau, x) \, dx \, d\tau
\]

\[
\leq \frac{1}{9} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right) (\tau, x) \, dx \, d\tau
\]

\[
+ O(1) N_{01}^2 \exp (O(1) N_{01}^2) \int_0^t \|\psi_x(\tau)\|_{L^2}^2 \|\varphi_x(\tau)\|_{L^2}^2 \, d\tau
\]

\[
+ O(1) N_{01}^2 \exp (O(1) N_{01}^2) \int_0^t \|\phi_x(\tau)\|_{L^2}^2 \|\psi_x(\tau)\|_{L^2}^2 \, d\tau
\]

\[
\leq \frac{1}{9} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_x^2}{v} \right) (\tau, x) \, dx \, d\tau
\]

+O(1)N^{2}_{01} \exp \left( O(1)N^{2}_{01} \right) \int_{0}^{t} \| \psi_{x}(\tau) \| \| \psi_{xx}(\tau) \| d\tau \\
+O(1)N^{2}_{01} \exp \left( O(1)N^{2}_{01} \right) \int_{0}^{t} \| \psi_{x}(\tau) \|^{2} d\tau \\
\leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau + O(1)N^{2}_{01} \exp \left( O(1)N^{2}_{01} \right),

R'_{19} \leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau \\
+O(1)N^{2}_{01} \exp \left( O(1)N^{2}_{01} \right) \int_{0}^{t} \int_{\mathbb{R}} \left( V_{x}^{2} \left( \frac{\varphi^{2}}{v} + \frac{\varphi^{2} + \phi^{2}}{v^{3}} \right) \right) (\tau, x) d\tau d\tau \\
\leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau + O(1)N^{2}_{01} \exp \left( O(1)N^{2}_{01} \right),

R'_{20} \leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau \\
+O(1)N^{2}_{01} \exp \left( O(1)N^{2}_{01} \right) \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\Theta_{x}^{2} \phi^{2}}{v} \right) (\tau, x) d\tau d\tau \\
\leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau + O(1)N^{2}_{01} \exp \left( O(1)N^{2}_{01} \right),

R'_{21} \leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau \\
+O(1)N^{2}_{01} \exp \left( O(1)N^{2}_{01} \right) \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{V_{x}^{2} \psi_{x}^{2} + U_{x}^{2} \varphi^{2}}{v^{3}} \right) (\tau, x) d\tau d\tau \\
+O(1)N^{2}_{01} \exp \left( O(1)N^{2}_{01} \right) \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{U_{x}^{2} \phi^{2} + \Theta_{x}^{2} \phi_{x}^{2} + U_{x}^{2} \phi^{2}}{v} \right) (\tau, x) d\tau d\tau \\
\leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau + O(1)N^{2}_{01} \exp \left( O(1)N^{2}_{01} \right),

R'_{22} \leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau \\
+O(1) \exp \left( O(1)N^{2}_{01} \right) \int_{0}^{t} \left( \| U_{xx}(\tau) \|_{L^{\infty}} + \| U_{x}(\tau) \|_{L^{\infty}} \right) d\tau \\
\leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau + O(1) \exp \left( O(1)N^{2}_{01} \right),

and

R'_{23} \leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau \\
+O(1) \exp \left( O(1)N^{2}_{01} \right) \int_{0}^{t} \| g(V, \Theta)_{x}(\tau) \|^{2} d\tau \\
\leq \frac{1}{8} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta) \psi_{xx}^{2}}{v} \right) (\tau, x) d\tau d\tau + O(1) \exp \left( O(1)N^{2}_{01} \right).
Then we collect the above estimates in (56) to deduce the desired estimate and this completes the proof of Lemma 3.5. □

From (44), (46), (55), and (54), we deduce the following result

**Lemma 3.6.** Under the assumptions listed in Lemma 3.4, we have that

\[
\|\phi_x(t)\|^2 + \int_0^t \int_\mathbb{R} \left( \frac{\phi_x^2}{v} \right) (\tau, x) \, dx \, d\tau \leq O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right)
\]

and

\[
\|\varphi_x(t)\|^2 + \int_0^t \int_\mathbb{R} \left( \frac{\varphi_x^2}{v^3} \right) (\tau, x) \, dx \, d\tau \leq O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right).
\]

The above analysis implies that under a priori assumption (23), the smallness assumption of \(\gamma - 1, \epsilon > 0\) and the largeness of \(l\), there exists some positive constant \(C\) which may depend only on \(v, u, \theta, \sqrt{\gamma - 1}, \sqrt{\gamma - 1}, \), there exists some positive constant \(C\) such that

\[
\left\| \left( \varphi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (t) \right\|_1^2 + \int_0^t \left( \|\varphi_x(\tau)\|^2 + \|\psi_x(\tau)\|^2 \right) d\tau \leq C \left( V, \nabla, \Theta, \Theta, \Theta \right) N_{01}^2 \exp \left( O(1)N_{01}^2 \right).
\]

Now we derive the second order energy estimates on \((\varphi(t, x), \psi(t, x), \phi(t, x))\). To do so, for the corresponding estimates on \(\|\psi_{xx}(t)\|^2\), we differentiate (19) with respect to \(x\) once and multiply the final result by \(-\psi_{xxx}(t, x)\) to get that

\[
\left( \frac{\psi_{xx}^2}{2} \right)_t + \frac{\mu(v, \theta)\psi_{xx}^2}{v} - (\psi_{xx})_x
\]

\[
= g(V, \Theta)_{xx} \psi_{xx} + \left( \frac{\theta}{v} - \frac{\Theta}{V} \right)_{xx} \psi_{xx}
\]

\[
+ \left( \frac{\mu(v, \theta)\psi_{xx}}{v} - \left( \frac{\mu(v, \theta)u_x}{v} \right)_{xx} \right) \psi_{xx}.
\]

Integrating this identity with respect to \(t\) and \(x\) over \([0, t] \times \mathbb{R}\), we have

\[
\|\psi_{xx}(t)\|^2 + \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\psi_{xx}^2}{v} \right) (\tau, x) \, dx \, d\tau \leq C \left( V, \nabla, \Theta, \Theta, \Theta \right) \left\{ \|\psi_{xx}\|^2 + \sum_{i=1}^3 |I_i| \right\}.
\]

Letting \(I'_1 = C(V, \nabla, \Theta, \Theta, \Theta) |I_i| (i = 1, 2, 3)\) and by using Lemma 2.2, (57), and Cauchy-schwarz’s inequality, we can bound \(I'_i (i = 1, 2, 3)\) term by term as follows

\[
I'_1 \leq \frac{1}{4} \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\psi_{xx}^2}{v} \right) (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right) \int_0^t \int_\mathbb{R} g(V(\tau, x), \Theta(\tau, x))^2 \, dx \, d\tau
\]

\[
\leq \frac{1}{4} \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\psi_{xx}^2}{v} \right) (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right).
\]

\[
I'_2 = C(V, \nabla, \Theta, \Theta) \int_0^t \int_\mathbb{R} \left( \frac{\phi_{xx} + \Theta_{xx}}{v} - \frac{2(\phi_x + \Theta_x)v_x}{v^2} - \frac{\theta(\phi_{xx} + V_{xx})}{v^2} \right)
\]

\[
\leq C(V, \nabla, \Theta, \Theta) \left\{ \|\phi_{xx} + \Theta_{xx}\|_1 + \|2(\phi_x + \Theta_x)v_x\|_1 + \|\theta(\phi_{xx} + V_{xx})\|_1 \right\}.
\]
\[\begin{align*}
&+ \frac{2\theta(\varphi_x + V_x)^2}{v^3} - \frac{\Theta_{xx} V_x}{V^2} + \frac{2\Theta_x V_x}{V^2} + \frac{\Theta V_{xx}}{V^2} - \frac{2\Theta V_x^2}{V^3} \bigg|_{\tau=0}^{\tau=\tau} (\tau, x) \psi_{xxx}(\tau, x) d\tau d\tau \\
&\leq \frac{1}{4} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_{xxx}^2}{v} \right)(\tau, x) d\tau d\tau \\
&+ O(1) \exp \left( \frac{N_{01}^2}{N_{01}} \right) \int_0^t \int_{\mathbb{R}} \left( \phi_x^2 + \Theta_x^2 + \varphi_x^2 + \varphi_{xx}^2 + V_x^2 \right)^2 (\tau, x) d\tau d\tau \\
&\leq \frac{1}{4} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_{xxx}^2}{v} \right)(\tau, x) d\tau d\tau \\
&+ O(1) \exp \left( \frac{N_{01}^2}{N_{01}} \right) \left( \int_0^t \|\phi_{xx}(\tau)\|^2 d\tau + \int_0^t \|\Theta_{xx}(\tau)\|^2 d\tau \\
&\quad + \int_0^t \|\phi_x(\tau)\|^2_{L^\infty} \|\varphi_x(\tau)\|^2 d\tau + \int_0^t \|V_x(\tau)\|^2_{L^\infty} \|\phi_x(\tau)\|^2 d\tau \\
&\quad + \int_0^t \|\Theta_x(\tau)\|^2_{L^\infty} \|\varphi_x(\tau)\|^2 d\tau + \int_0^t \|\Theta_{xx}(\tau)\|^2 d\tau \\
&\quad + \int_0^t \|\varphi_x(\tau)\|^2_{L^\infty} \|\varphi_{xx}(\tau)\|^2 d\tau + \int_0^t \|\varphi_{xx}(\tau)\|^2 d\tau \right) \\
&\leq \frac{1}{4} \int_0^t \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)\psi_{xxx}^2}{v} \right)(\tau, x) d\tau d\tau + O(1) N_{01}^2 \exp \left( \frac{N_{01}^2}{N_{01}} \right) \\
&+ O(1) \exp \left( \frac{N_{01}^2}{N_{01}} \right) \int_0^t \|\varphi_{xx}(\tau)\|^2 d\tau,
\end{align*}\]
+\psi_x^2 \varphi_{xx} + \varphi_{xx}^2 \right) (\tau, x)dx d\tau + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \\
\leq \frac{1}{4} \int_0^t \int_\mathbb{R} \left( \frac{\mu (v, \theta) \psi_{xx}^2}{v} \right) (\tau, x) dx d\tau + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \|\varphi_{xx}(\tau)\|^2 d\tau \\
+ O(1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \int_\mathbb{R} \left( |(\psi_{xx}, \varphi_{xx})|^2 |(\varphi_x, \psi_x, \varphi_x)|^2 \right) (\tau, x) dx d\tau \\
+ \psi_x^2 \varphi_{xx}^2 + \varphi_{xx}^2 \right) (\tau, x) dx d\tau + O(1) N_{01}^2 \exp(O(1) N_{01}^2).

Inserting the estimates of \(I_i(1 \leq i \leq 3)\) into (58), we can deduce from (54), (25), and (24) that

\[
\|\psi_{xx}(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2 d\tau \\
\leq O(1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \int_\mathbb{R} \left( |(\psi_{xx}, \varphi_{xx})|^2 |(\varphi_x, \psi_x, \varphi_x)|^2 \right) (\tau, x) dx d\tau \\
+ O(1) N_{02}^2 \exp(O(1) N_{02}^2).
\]

To deduce an estimate on \(\|\varphi_{xx}(t)\|^2\), we have by differentiating (19) with respect to \(x\) once and by multiplying the result by \(-\varphi_{xx}(t, x)\) to get that that

\[
\frac{\left( \frac{\varphi_{xx}^2}{2(\gamma - 1)} \right)}{v} + \frac{\kappa(v, \theta) \varphi_{xx}^2}{v} - \frac{1}{\gamma - 1} \left( \varphi_{xx} \varphi_{xx, x} \right) x \\
= \frac{\theta}{v} \psi_x \varphi_{xxx} + \frac{\theta \psi_{xx} \varphi_{xxx}}{v} + \left( \frac{\theta}{v} - \Theta \right) U_x \varphi_{xxx} \\
+ \frac{\theta}{v - \Theta} U_{xxx} \varphi_{xxx} + \frac{1}{\gamma - 1} r(V, \Theta) x \varphi_{xxx} \\
+ \left[ \frac{\kappa(v, \theta) \varphi_{xxx}}{v} - \left( \frac{\kappa(v, \theta) \varphi_x}{v} \right)_{xxx} \right] \varphi_{xxx} \\
- \left( \frac{\mu(v, \theta) u_x^2}{v} \right) \varphi_{xxx}.
\]

Integrating the above identity with respect to \(t\) and \(x\) over \([0, t] \times \mathbb{R}\), we have

\[
\left\| \frac{\varphi_{xx}(t)}{\sqrt{\gamma - 1}} \right\|^2 + \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \varphi_{xx}^2}{v} \right) (\tau, x) dx d\tau \\
\leq C \left( \mathcal{Y}, \mathcal{V}, \Theta, \mathcal{B} \right) \left\| \frac{\varphi_{0xx}}{\sqrt{\gamma - 1}} \right\|^2 + \sum_{i=4}^{11} I_i
\]

and \(I_i (i = 4, \ldots, 11)\) can be controlled term by term as follows:

\[
I_4 = C(\mathcal{Y}, \mathcal{V}, \Theta, \mathcal{B}) \int_0^t \int_\mathbb{R} \left( \frac{\theta}{v} \right) \psi_x \varphi_{xxx} (\tau, x) dx d\tau \\
= C(\mathcal{Y}, \mathcal{V}, \Theta, \mathcal{B}) \int_0^t \int_\mathbb{R} \left[ \left( \frac{\varphi_x + \Theta_x}{v} - \frac{\theta (\varphi_x + V_x)}{v^2} \right) \psi_x \varphi_{xxx} \right] (\tau, x) dx d\tau \\
\leq \frac{1}{9} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \varphi_{xx}^2}{v} \right) (\tau, x) dx d\tau
\]

+ \frac{O(1) \exp(O(1) N_{01}^2)}{\int_0^t \int_\mathbb{R} \left( \frac{\varphi_x^2 + \Theta_x^2 + \varphi_x^2 + V_x^2}{v_x} \right) (\tau, x) \psi^2(\tau, x) dx d\tau}.
\[ I_5 = C(\mathcal{V}, \Theta, \Xi, \Xi) \int_0^t \int_\mathbb{R} \left[ \left( \frac{\theta}{v} \psi_{xx} \phi_{xxx} \right) (\tau, x) \right] \, dx \, d\tau + O(1)N_{01}^2 \exp\left( O(1)N_{01}^2 \right), \]

\[
\leq \frac{1}{9} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xxx}^2}{v} \right) \left( \kappa(v, \theta) \phi_{xxx}^2 \right) \left( \frac{\theta}{v} \psi_{xx} \phi_{xxx} \right) (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp\left( O(1)N_{01}^2 \right), \]

\[
I_6 = C(\mathcal{V}, \Theta, \Xi, \Xi) \int_0^t \int_\mathbb{R} \left[ \left( \frac{\Theta_x}{V} + \Theta V_x \frac{1}{V^2} \right) U_x \phi_{xxx} \right] (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp\left( O(1)N_{01}^2 \right), \]

\[
\leq \frac{1}{9} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xxx}^2}{v} \right) \left( \frac{\Theta_x}{V} + \Theta V_x \frac{1}{V^2} \right) U_x \phi_{xxx} \left( \frac{\theta}{v} \psi_{xx} \phi_{xxx} \right) (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp\left( O(1)N_{01}^2 \right), \]

\[
I_7 = C(\mathcal{V}, \Theta, \Xi, \Xi) \int_0^t \int_\mathbb{R} \left[ \left( \frac{\theta}{v} \psi_{xx} \phi_{xxx} \right) (\tau, x) \right] \, dx \, d\tau + O(1)N_{01}^2 \exp\left( O(1)N_{01}^2 \right), \]

\[
\leq \frac{1}{9} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xxx}^2}{v} \right) \left( \frac{\theta}{v} \psi_{xx} \phi_{xxx} \right) (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp\left( O(1)N_{01}^2 \right), \]

\[
I_8 = C(\mathcal{V}, \Theta, \Xi, \Xi) \int_0^t \int_\mathbb{R} \left[ \frac{1}{\gamma - 1} r(V, \Theta) \phi_{xxx} \right] (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp\left( O(1)N_{01}^2 \right), \]

\[
\leq \frac{1}{9} \int_0^t \int_\mathbb{R} \left( \frac{\kappa(v, \theta) \phi_{xxx}^2}{v} \right) \left( \frac{\theta}{v} \psi_{xx} \phi_{xxx} \right) (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp\left( O(1)N_{01}^2 \right), \]

\[
I_9 = C(\mathcal{V}, \Theta, \Xi, \Xi) \int_0^t \int_\mathbb{R} \left[ \left( \frac{\kappa(v, \theta) \phi_{xxx}}{v} \right) \phi_{xxx} \right] (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp\left( O(1)N_{01}^2 \right), \]
\[
\leq \frac{1}{9} \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right) (\tau, x) \, dx \, d\tau
\]

\[
+ O(1) \exp \left( O(1) N_{01}^3 \right) \int_0^t \int_{\mathbb{R}} \left( |\phi_x + \Theta_x| + |\phi_x + \Theta_x|^2 |\varphi_x + V_x|^2 \right)
\]

\[
+ |\phi_x + \Theta_x|^2 |\varphi_x + V_x|^2 + |\phi_x + \Theta_x|^4 |\varphi_x + V_x|^2
\]

\[
+ |\varphi_x + V_x, \phi_x + \Theta_x| (\tau, x) \, dx \, d\tau
\]

\[
\leq \frac{1}{9} \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right) (\tau, x) \, dx \, d\tau
\]

\[
+ O(1) \exp \left( O(1) N_{01}^3 \right) \left( 1 + \int_0^t \| \varphi_{xx}(\tau) \|^2 \, d\tau \right)
\]

\[
+ O(1) N_{01}^3 \int_0^t \int_{\mathbb{R}} \left( \phi_{xx}^2 + \phi_x^2 + \phi_{xx}^2 \right) (\tau, x) \, dx \, d\tau,
\]

where we have used the following estimate

\[
\int_0^t \int_{\mathbb{R}} \left( \phi_x^4 + \varphi_x^2 \right) (\tau, x) \, dx \, d\tau
\]

\[
\leq \int_0^t \left( \| \phi_x (\tau) \|^2 \| \phi_{xx}(\tau) \|^2 \left( \| \phi_x(\tau) \|^2 + \| \varphi_x(\tau) \|^2 \right) \right)
\]

\[
+ \| \phi_x(\tau) \|^2 \| \phi_x(\tau) \|^2 \| \varphi_{xx}(\tau) \|^2 \, d\tau
\]

\[
\leq O(1) N_{01}^3 \exp \left( O(1) N_{01}^3 \right) \left( 1 + \int_0^t \| \varphi_{xx}(\tau) \|^2 \, d\tau \right),
\]

\[
I_{10} = C(V, \nabla \cdot \Theta, \Theta) \int_0^t \int_{\mathbb{R}} \left[ \left( \frac{\kappa(v, \theta) \phi_{xxx}}{v} \right) (\tau, x) \right] \, dx \, d\tau
\]

\[
\leq \frac{1}{9} \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right) (\tau, x) \, dx \, d\tau
\]

\[
+ O(1) N_{01}^3 \exp \left( O(1) N_{01}^3 \right),
\]

and

\[
I_{11}
\]

\[
= C(V, \nabla \cdot \Theta, \Theta) \int_0^t \int_{\mathbb{R}} \left[ \left( \frac{\mu(v, \theta) u_x^2}{v} \right) \right] (\tau, x) \, dx \, d\tau
\]

\[
\leq \frac{1}{9} \int_0^t \int_{\mathbb{R}} \left( \frac{\kappa(v, \theta) \phi_{xx}^2}{v} \right) (\tau, x) \, dx \, d\tau
\]

\[
+ O(1) N_{01}^3 \exp \left( O(1) N_{01}^3 \right) \int_0^t \int_{\mathbb{R}} \left( \psi_x \psi_{xx} + \psi_x^3 \right) (\tau, x) \, dx \, d\tau
\]

\[
+ \psi_{xx}^2 + \psi_x^2 + \psi_x \psi_{xx} (\tau, x) \, dx \, d\tau.
\]
\[
\leq \frac{1}{9} \int_0^t \int_R \left( \frac{\kappa(v, \theta) \phi_{xxx}^2}{v} \right) (\tau, x) dx d\tau \\
+ O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right) + O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right) \int_0^t \int_R (\psi_x^2 \psi_{xx}^2) (\tau, x) dx d\tau \\
+ O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right) \int_0^t \int_R \|\psi_x(\tau)\|^2 \|\psi_{xx}(\tau)\|^2 \left( \|\phi_x(\tau)\|^2 + \|\phi_{xx}(\tau)\|^2 \right) d\tau \\
\leq \frac{1}{9} \int_0^t \int_R \left( \frac{\kappa(v, \theta) \phi_{xxx}^2}{v} \right) (\tau, x) dx d\tau + O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right) \\
+ O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right) \int_0^t \int_R \psi_x^2 (\tau, x) \psi_{xx}^2 (\tau, x) dx d\tau.
\]

Plugging the above estimates on \( I_i \) \((i = 4, \ldots, 11)\) into (60), one has from (54), (25), and (24) that

\[
\left\| \frac{\phi_{xx}(t)}{\sqrt{\gamma - 1}} \right\|^2 + \int_0^t \int_R \|\phi_{xx}(\tau)\|^2 d\tau \leq O(1)N_{02}^2 \exp \left( O(1)N_{01}^2 \right) \\
+ O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right) \int_0^t \int_R \psi_x^2 (\tau, x) \psi_{xx}^2 (\tau, x) dx d\tau \tag{61}
\]

To get an estimate on \( \|\varphi_{xx}(t)\|^2 \), we first differentiate (22) with respect to \( x \) once, multiply the result by \( (\mu(v, \theta) \frac{\varphi_x}{v})_x \), and then integrate the final identity with respect to \( t \) and \( x \) over \([0, t] \times R\) to deduce that

\[
\frac{1}{2} \left\| \frac{\mu(v, \theta) \varphi_x}{v^2} - \frac{\theta}{v} \frac{\phi_x}{v} \right\|^2 + \int_0^t \int_R \psi_x \left( \frac{\mu(v, \theta) \varphi_x}{v} \right)_x dx d\tau \\
+ \int_0^t \int_R \frac{\theta}{v} \frac{\phi_x}{v} \left( \frac{\mu(v, \theta) \varphi_x}{v} \right)_x dx d\tau \\
+ \int_0^t \int_R g(V, \Theta) \left( \frac{\mu(v, \theta) \varphi_x}{v} \right)_x dx d\tau + I_{15}.
\]

Here

\[
I_{15} = \int_0^t \int_R \left( \frac{\mu(v, \theta) V_x u_x}{v^2} - \frac{\mu(v, \theta) V_x u_x}{v} - \frac{\mu(v, \theta) U_{xx}}{v} - \frac{\mu(v, \theta) \theta x u_x}{v} \\
+ \frac{\mu(v, \theta) \phi_x}{v} \right)_x \left( \frac{\mu(v, \theta) \varphi_x}{v} \right)_x dx d\tau
\]
To deduce an estimate on $I_{12}$, due to

$$
\int_0^t \int_\mathbb{R} |\theta(t, x) \varphi_x(t, x)| |\psi_{xx}(t, x)| \, dx \, dt \\
\leq O(1)(\gamma - 1) N_{01}^2 \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} (|\psi_{xx} \varphi_x|)(t, x) \left(|\psi_x + U_x|^2 + |\varphi_x + \Theta_x|^2 \right) (t, x) \, dx \, dt \\
+ O(1)(\gamma - 1) N_{01}^2 \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} (\varphi_x^2 \phi_x^2 + \varphi_x^2 \phi_x^2 + \varphi_x^2 \phi_x^2 + \varphi_x^2 \varphi_x^2)(t, x) \, dx \, dt \\
+ O(1)(\gamma - 1) N_{01}^2 \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} (\varphi_x(\varphi_x - \varphi_x - \varphi_x - \varphi_x)(t, x) \, dx \, dt \\
+ O(1)(\gamma - 1) N_{01}^2 \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} \varphi_x^2(\varphi_x)(t, x) \, dx \, dt \\
+ O(1)(\gamma - 1) N_{01}^2 \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} \varphi_x^2(\varphi_x)(t, x) \, dx \, dt \\
+ O(1)(\gamma - 1) N_{01}^2 \exp(O(1)N_{01}^2),
$$

$$
\int_0^t \int_\mathbb{R} |\varphi_x(\varphi_x)(t, x)| \psi_{xx}(t, x) \, dx \, dt \\
\leq O(1)N_{01}^2 \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} |\varphi_x(\varphi_x)(t, x)| \, dx \, dt \\
+ O(1)N_{01}^2 \exp(O(1)N_{01}^2),
$$

and noticing

$$
I_{12} = \int_0^t \int_\mathbb{R} \left( \psi_x \left( \frac{\mu(\nu, \theta) \varphi_x}{\nu} \right) \right) \, dx \, dt \quad \text{d}x \, \text{d}t \\
= \int_\mathbb{R} \left( \psi_x \left( \frac{\mu(\nu, \theta) \varphi_x}{\nu} \right) \right) \, dx - \int_\mathbb{R} \left( \psi_x \left( \frac{\mu(\nu, \theta) \varphi_0}{\nu} \right) \right) \, dx \\
+ \int_0^t \int_\mathbb{R} \left( \psi_x \left( \frac{\mu(\nu, \theta) \varphi_x}{\nu} \right) \right) \, dx \, dt \\
= \int_\mathbb{R} \left( \psi_x \left( \frac{\mu(\nu, \theta) \varphi_x}{\nu} \right) \right) \, dx - \int_\mathbb{R} \left( \psi_x \left( \frac{\mu(\nu, \theta) \varphi_0}{\nu} \right) \right) \, dx.
\[ I_{12} \leq \frac{1}{4} \left\| \left( \frac{\mu(v, \theta)}{v} \right) \varphi_x(t) \right\|^2 + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \]
\[ + O(1)(\gamma - 1) N_{01}^2 \exp(O(1) N_{01}^2) \left\{ \int_0^t \int_R \varphi_x^2(\tau, x) \phi_{xx}^2(\tau, x) dx d\tau \right. \]
\[ + \left. \int_0^t \| \varphi_{xx}(\tau) \|^2 d\tau \right\}. \]

Similarly, \( I_{13} \) and \( I_{14} \) can be estimated as in the following:

\[ I_{13} = \int_0^t \int_R \left( \frac{\mu(v, \theta)}{v} \varphi_x + \frac{\mu(v, \theta)}{v} \varphi_x + \mu(v, \theta) \varphi_x + \mu(v, \theta) \varphi_x \right) (\tau, x) \]
\[ \times \left( \frac{\theta_{xx} + 2\theta_{xx}}{v} + \frac{2\theta_{xx}}{v^3} + \frac{2\theta_{xx}}{v^3} - \frac{2\theta_{xx}}{v^3} \right) (\tau, x) d\tau \]
\[ \leq -\frac{6}{7} \int_0^t \int_R \left( \frac{\mu(v, \theta)}{v} \phi_{xx}^2 \right) (\tau, x) d\tau d\tau + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \]
\[ + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \left\{ \int_0^t \int_R \left\{ \left| \phi_{xx} \phi_{xx} \right| + \left| \phi_{xx} \right| + \left| \phi_{xx} \right|^2 \right\} \right\} \]
\[ \leq -\frac{4}{5} \int_0^t \int_R \left( \frac{\mu(v, \theta)}{v} \phi_{xx}^2 \right) (\tau, x) d\tau d\tau + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \]
\[ + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \int_R \left\{ \phi_{xx}^2 + \phi_{xx}^2 + \phi_{xx}^2 \right\} (\tau, x) d\tau \]
\[ \leq -\frac{2}{3} \int_0^t \int_R \left( \frac{\mu(v, \theta)}{v} \phi_{xx}^2 \right) (\tau, x) d\tau d\tau + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \]
\[ + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \left( \| \phi_{xx}(\tau) \|^2 + \| \phi_{xx}(\tau) \|^2 \| \phi_{xx}(\tau) \|^2 + \| \phi_{xx}(\tau) \|^2 \right) \right\} \]
\[ \leq -\frac{2}{3} \int_0^t \int_R \left( \frac{\mu(v, \theta)}{v} \phi_{xx}^2 \right) (\tau, x) d\tau d\tau + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \],

\[ I_{14} = \int_0^t \int_R \left( \frac{\mu(v, \theta)}{v} \varphi_x + \frac{\mu(v, \theta)}{v} \varphi_x + \mu(v, \theta) \varphi_x \right) \]
\[ - \frac{\mu(v, \theta)}{v^2} \right) (\tau, x) g(V(\tau, x), \Theta(\tau, x))_x x d\tau \]
\[ \leq \frac{1}{20} \int_0^t \int_R \left( \frac{\mu(v, \theta)}{v} \phi_{xx}^2 \right) (\tau, x) d\tau d\tau + O(1) N_{01}^2 \exp(O(1) N_{01}^2) \].
As to $I_{15}$, it is easy to see that

$$I_{15} \leq \frac{1}{20} \int_0^t \int_\mathbb{R} \left( \frac{\mu(u, \theta) \theta \phi_x^2}{\nu^3} \right) (\tau, x) \, dx \, d\tau + O(1) N_{01}^2 \exp(O(1) N_{01}^2)$$

$$+ O(1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \int_\mathbb{R} \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x)$$

$$\times \left\{ \left[ \left| \psi_x \phi_x^2 + |\phi_x x \phi_x \psi_x| + |\phi_x x x \phi_x \psi_x| + |\phi_x x \psi_x| + |\phi_x x x \psi_x| \right] + |\phi_x \phi_x \phi_x \psi_x| + |\phi_x x \phi_x \phi_x \psi_x| + |\phi_x \phi_x \phi_x \psi_x| \right\} (\tau, x) \, dx \, d\tau$$

where

$$D_1 = O(1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \int_\mathbb{R} \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x)$$

$$\times \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x) \, dx \, d\tau$$

$$D_2 = O(1) (\gamma - 1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \int_\mathbb{R} \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x)$$

$$\times \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x) \, dx \, d\tau$$

$$D_3 = O(1) (\gamma - 1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \int_\mathbb{R} \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x)$$

$$\times \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x) \, dx \, d\tau$$

$$D_4 = O(1) (\gamma - 1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \int_\mathbb{R} \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x)$$

$$\times \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x) \, dx \, d\tau$$

$$D_5 = O(1) (\gamma - 1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \int_\mathbb{R} \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x)$$

$$\times \left( \phi_x^2 + |\phi_x \phi_x| + |\phi_x x| + |\phi_x x x| \right) (\tau, x) \, dx \, d\tau.$$

Since

$$\int_0^t \int_\mathbb{R} \left( \phi_x^2 + |\phi_x x| + |\phi_x x x| \right) (\tau, x) \, dx \, d\tau$$

$$\leq \int_0^t \left( \|\phi_x(\tau)\|_{L^\infty}^2 + \|\psi_x(\tau)\|_{L^\infty}^2 + \|\phi_x(\tau)\|_{L^\infty}^2 \right) \|\phi_x, \psi_x, \phi_x\| (\tau) \, d\tau$$

$$\leq O(1) N_{01}^2 \exp(O(1) N_{01}^2) \int_0^t \left( \|\phi_x(\tau)\| \|\phi_x x(\tau)\| \right) \, dx \, d\tau.$$
\[
+ \| \varphi_x (\tau) \| \| \varphi_{xx} (\tau) \| + \| \psi_x (\tau) \| \| \psi_{xx} (\tau) \| ) \, d\tau \\
\leq \frac{1}{20} \int_0^t \int_\mathbb{R} \left( \frac{\mu (v, \theta) \varphi_{xx}^2}{\nu^3} \right) (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp(O(1)N_{01}^2),
\]

\( D_1 \) can be estimated as

\[
D_1 \leq \frac{1}{20} \int_0^t \int_\mathbb{R} \left( \frac{\mu (v, \theta) \varphi_{xx}^2}{\nu^3} \right) (\tau, x) \, dx \, d\tau \\
+ O(1)N_{01}^2 \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} \left\{ \varphi_x^2 \varphi_{xx}^2 + \varphi_x^2 \varphi_{xx}^2 + \varphi_x^2 \varphi_{xx}^2 + \varphi_x^2 \varphi_{xx}^2 + \varphi_x^2 \varphi_{xx}^2 \right\} (\tau, x) \, dx \, d\tau \\
+ O(1)N_{01}^2 \exp(O(1)N_{01}^2) \leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\mu (v, \theta) \varphi_{xx}^2}{\nu^3} \right) (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp(O(1)N_{01}^2)
\]

For \( D_i (i = 2, 3, 4, 5) \), under the assumption that \( \epsilon \) and \( \gamma - 1 \) are chosen sufficiently small, we can obtain

\[
D_2 \leq \frac{1}{20} \int_0^t \int_\mathbb{R} \left( \frac{\mu (v, \theta) \varphi_{xx}^2}{\nu^3} \right) (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp(O(1)N_{01}^2)
\]

\[
+ O(1)(\gamma - 1)^2 N_{01}^4 \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} \left( \varphi_x^2 + \varphi_{xx}^2 + \varphi_{xx}^2 \right) (\tau, x) \, dx \, d\tau \\
\leq \frac{1}{10} \int_0^t \int_\mathbb{R} \left( \frac{\mu (v, \theta) \varphi_{xx}^2}{\nu^3} \right) (\tau, x) \, dx \, d\tau + O(1)N_{01}^2 \exp(O(1)N_{01}^2),
\]

\[
D_3 \leq (\gamma - 1)N_{01}^2 \int_0^t \int_\mathbb{R} \left( (|\varphi_{xx}| + |\varphi_{xx}| + |\psi_{xx}|) \right) (\tau, x) \, dx \, d\tau \\
+ (\gamma - 1)N_{01}^2 \int_0^t \int_\mathbb{R} \left( \frac{\mu (v, \theta) \varphi_{xx}^2}{\nu^3} \right) (\tau, x) \, dx \, d\tau \\
+ O(1)(\gamma - 1)N_{01}^2 \exp(O(1)N_{01}^2) \times \int_0^t \int_\mathbb{R} \left( \varphi_x^2 + \varphi_{xx}^2 + \varphi_{xx}^2 \right) (|\psi_{xx}| + |\varphi_{xx}| + |\psi_{xx}|) (\tau, x) \, dx \, d\tau \\
+ O(1)(\gamma - 1)N_{01}^2 \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} \left( \varphi_x^2 + \varphi_{xx}^2 + \varphi_{xx}^2 \right) (\tau, x) \, dx \, d\tau \\
+ O(1)N_{01}^2 \exp(O(1)N_{01}^2) \leq O(1)(\gamma - 1)N_{01}^2 N_{01} \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} \left( \frac{\mu (v, \theta) \varphi_{xx}^2}{\nu^3} \right) (\tau, x) \, dx \, d\tau \\
+ O(1)N_{01}^2 \exp(O(1)N_{01}^2),
\]

\[
D_4 \leq O(1)(\gamma - 1)N_{01}^2 \exp(O(1)N_{01}^2) \int_0^t \int_\mathbb{R} \left( \frac{\mu (v, \theta) \varphi_{xx}^2}{\nu^3} \right) (\tau, x) \, dx \, d\tau
\]
\[ I_{15} \leq \frac{9}{20} \int_0^t \int_\mathbb{R} \left( \frac{\mu(v, \theta)\varphi_{xx}}{v^3} \right) (\tau, x) dx d\tau + O(1)N_{01}^2 e^{O(1)N_{01}^2} \]

\[ + O(1)N_{01}^2 e^{O(1)N_{01}^2} \int_0^t \int_\mathbb{R} [\varphi_x^2 (\varphi_{xx}^2 + \psi_{xx}^2) + \psi_x^2 \phi_{xx}^2 + \varphi_x^2 \psi_{xx}^2] (\tau, x) d\tau d\tau \\
 + O(1)(\gamma - 1)^{\frac{1}{2}} N_{01}^2 e^{O(1)N_{01}^2} \int_0^t \|\phi_{xx}(\tau)\|^2 d\tau. \]

Based on the above estimates, we finally get that

\[ \|\varphi_{xx}(t)\|^2 + \int_0^t \|\varphi_{xx}(\tau)\|^2 d\tau \]

\[ \leq O(1)N_{02}^2 e^{O(1)N_{01}^2} \]

\[ + O(1)N_{01}^2 e^{O(1)N_{01}^2} \int_0^t \int_\mathbb{R} [\psi_{xx}^2 + \phi_{xx}^2 + \psi_{xx}^2 \phi_{xx}^2 + \phi_{xx}^2 \psi_{xx}^2] (\tau, x) d\tau d\tau \\
 + O(1)(\gamma - 1)^{\frac{1}{2}} N_{01}^2 e^{O(1)N_{01}^2} \int_0^t \|\phi_{xxx}(\tau)\|^2 d\tau, \]
where we used the following result
\[
\left\| \left( \frac{\mu(v, \theta) \varphi_x}{v} \right)_x(t) \right\|^2 \geq O(1)N_{01}^2 \exp \left( -O(1)N_{01}^2 \right) \| \varphi_{xx}(t) \|^2
\]
\[\geq -O(1)N_{01}^2 \exp \left( O(1)N_{01}^2 \right) .\]

A suitable linear combination of (59), (61), and (63) yields the following result

**Lemma 3.7.** Under the same conditions listed in Lemma 3.4, if we further assume that \( \gamma + 1 \) is sufficiently small such that the assumption (62) holds, then we have
\[
\left\| \left( \varphi_{xx}, \psi_{xx}, \frac{\phi_{xx}}{\sqrt{\gamma - 1}} \right) \right\|^2 + \int_0^t \| (\varphi_{xx}, \psi_{xxx}, \phi_{xxx}) (\tau) \| d\tau \\
\lesssim N_{02}^2 \exp \left( O(1)N_{01}^2 \right) .
\]

**Proof.** In fact, by using the smallness of \( \gamma - 1 \), we can get by multiplying (63) by a sufficiently large positive number \( \lambda \) and by adding the resulting inequality with (59) and (61) that
\[
\left\| \left( \varphi_{xx}, \psi_{xx}, \frac{\phi_{xx}}{\sqrt{\gamma - 1}} \right) \right\|^2 + \int_0^t \| (\varphi_{xx}, \psi_{xxx}, \phi_{xxx}) (\tau) \| d\tau \\
\leq O(1)N_{01}^2 e^{O(1)N_{01}^2} \int_0^t \int_{\mathbb{R}} \left( \phi^2_x + \psi^2_x + \frac{\phi^2_{xx}}{\gamma - 1} \right) \left( \varphi^2_{xx} + \psi^2_{xx} \right) (\tau, x)d\tau d\tau
\]
\[+ O(1)N_{02}^2 e^{O(1)N_{01}^2} + O(1)N_{02}^2 e^{O(1)N_{01}^2} \int_0^t \int_{\mathbb{R}} \left[ \left( \psi^2_x + \frac{\phi^2_x}{\sqrt{\gamma - 1}} \right) \varphi^2_{xx} \right] (\tau, x)d\tau d\tau .\]

Since
\[
\int_0^t \int_{\mathbb{R}} \left( \psi^2_x + \frac{\phi^2_x}{\sqrt{\gamma - 1}} \varphi^2_{xx} \right) (\tau, x)d\tau d\tau
\]
\[\leq \int_0^t \| \varphi_{xx}(\tau) \|^2 \left( \| \psi_x(\tau) \| \| \psi_{xx}(\tau) \| + \| \phi_x(\tau) \| \| \phi_{xx}(\tau) \| \right) d\tau \]
and
\[
\int_0^t \int_{\mathbb{R}} \left[ \left( \phi^2_x + \psi^2_x + \frac{\phi^2_x}{\sqrt{\gamma - 1}} \right) \left( \psi^2_{xx} + \frac{\phi^2_{xx}}{\gamma - 1} \right) \right] (\tau, x)d\tau d\tau
\]
\[\leq \int_0^t \left( \| \psi_{xx}(\tau) \|^2 + \| \phi_{xx}(\tau) \|^2 \right) d\tau \times \left( \| \psi_x(\tau) \|^2 + \| \psi_{xx}(\tau) \|^2 + \| \phi_x(\tau) \|^2 + \| \phi_{xx}(\tau) \|^2 \right) d\tau
\]
\[+ \int_0^t \| \varphi_x(\tau) \|^2 \left( \| \psi_{xx}(\tau) \| \| \psi_{xxx}(\tau) \| + \| \phi_{xx}(\tau) \| \| \phi_{xxx}(\tau) \| \right) d\tau ,
\]
the estimate stated in Lemma 3.7 follows immediately from the estimate (55) and by inserting (65) and (66) into (64), and by employing Gronwall’s inequality. This completes the proof of Lemma 3.7. \( \square \)

Similarly, we can deduce that

**Lemma 3.8.** Under the same conditions listed in Lemma 3.4, if we further assume \( \gamma - 1 \) small enough such that
\[K_2(\gamma - 1)^2 N_{02}^2 \exp(N_{01}^2 \exp(K_2 N_{01}^2)) \leq 1\]
holds for some sufficiently large constant $K_2$ depending only on $v_\pm$, $u_\pm$, $\theta_\pm$, $\mathcal{V}$, $\mathcal{V}$, $\mathcal{O}$, and $\mathcal{O}$, then we have
\[
\left\| \left( \varphi_{xxx}, \psi_{xxx}, \frac{\phi_{xxx}}{\sqrt{\gamma - 1}} \right) (t) \right\|^2 + \int_0^t \left\| (\varphi_{xxx}, \psi_{xxx}, \phi_{xxx}) (\tau) \right\|^2 d\tau \\
\leq N_{03}^2 \exp(\exp(O(1) N_{02}^2)).
\]

As a direct consequence of the results obtained in Lemmas 3.2-3.8, we have the following corollary.

**Corollary 3.1.** Let $$(\varphi(t, x), \psi(t, x), \phi(t, x))$$ be the local solution constructed in Lemma 3.1 which has been extended to the time step $t = T \geq t_1$ and assume that $$(\varphi(t, x), \psi(t, x), \phi(t, x))$$ satisfies the a priori assumption in (23), then if $l$ is chosen sufficiently large and $\epsilon > 0$ and $\gamma - 1 > 0$ are chosen sufficiently small such that
\[
\begin{aligned}
&C(V, V, \Theta, \Theta) l^{-\frac{1}{2}} \leq 1, \\
&C(V, V, \Theta, \Theta) C(M_1) l^{-\frac{1}{2}} \leq 1, \\
&C(M_1) l^{-\frac{1}{2}} + C(M_1) N_{01}^2 l^{-\frac{1}{2}} \leq 1, \\
&C(M_1) l^{-\frac{1}{2}} \leq 1
\end{aligned}
\]
hold for some positive constants $C(V, V, \Theta, \Theta)$ and $C(M_1)$ which depend only on $V, V, \Theta, \Theta$ and $M_1$ respectively,
\[
\begin{aligned}
&C(M_1) N_1 \epsilon \leq 1, \\
&C(M_1)(\gamma - 1)(N_1 + 1) \leq \frac{1}{M}, \\
&C(M_1)(\gamma - 1) N_1^2 \leq 1, \\
&C(M_1)(\gamma - 1) N_1 \leq \frac{1}{M}, \\
&(\gamma - 1) C(M_1) (N_1^2 + 1) \leq 1, \\
&(\gamma - 1) N_1^2 \leq 1, \\
&K_1 (\gamma - 1) N_1^2 N_{01}^2 e^{K_1 N_{01}^2} \leq \frac{1}{M}, \\
&K_2 (\gamma - 1)^{\frac{1}{2}} N_{01}^2 e^{N_{01}^2 \exp(K_2 N_{02}^2)} \leq 1
\end{aligned}
\]
hold true for some positive constants $C(M_1)$ depending only on $M_1$ and $K_1, K_2$ which are sufficiently large but depend only on $v_\pm$, $u_\pm$, $\theta_\pm$, $\mathcal{V}$, $\mathcal{V}$, $\mathcal{O}$, and $\mathcal{O}$, one can deduce that there exists a generic positive constant $K_3$ which depends only on $V, V, \Theta, \Theta$, and $M_3$ such that
\[
\left\| \left( \varphi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (t) \right\|^2 + \int_0^t \left( \|\varphi_x(\tau)\|^2_2 + \|\psi_x(\varphi_x)(\tau)\|^2_2 \right) d\tau \\
\leq K_3 N_{03}^2 \exp(\exp(K_3 N_{02}^2))
\]
holds for any $0 \leq t \leq T$.

Moreover, if we further assume that $\gamma - 1 > 0$ is sufficiently small such that
\[
\sqrt{\gamma - 1} \sqrt{K_3 N_{03} \exp\left( \frac{1}{2} \exp(K_3 N_{02}^2) \right)} \leq \min \left\{ \frac{\Theta}{2}, \Theta \right\},
\]
then for each $(t, x) \in [0, T] \times \mathbb{R}$, we have
\[
\sup_{t \in [0, T]} \{\|\phi(t)\|_{L^2_x}\} \leq \sup_{t \in [0, T]} \left\{ \|\phi(t)\|^{\frac{1}{2}} \|\phi_x(t)\|^{\frac{1}{2}} \right\} \leq \|\phi(t)\|_3,
\]
and consequently
\[ \theta(t, x) = \Theta(t, x) + \phi(t, x) \leq \frac{\Theta}{2} + \|\phi(t)\|_3 \leq \Theta, \]

and
\[ \theta(t, x) = \Theta(t, x) + \phi(t, x) \geq 2\Theta - \|\phi(t)\|_3 \geq \Theta. \]

Thus
\[ \Theta \leq \theta(t, x) \leq \Theta \]

holds true for all \((t, x) \in [0, T] \times \mathbb{R}\).

4. The proof of our main result. Now we turn to prove Theorem 1.1. Firstly, we can deduce from the conditions listed in Theorem 1.1 and the local existence result stated in Lemma 3.1 that there exists a sufficiently small positive constant \(t_1\), which depends only on \(\Theta, \Theta, V, V\), and \(N_{03} := \left\| \left( \varphi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_3\), such that the Cauchy problem (19), (20) admits a unique smooth solution \((\varphi(t, x), \psi(t, x), \phi(t, x), \xi(t, x)) \in X^3(0, t_1; V, V; \Theta, \Theta)\) which satisfies
\[ V \leq v(t, x) \leq V, \quad \Theta \leq \theta(t, x) \leq \Theta, \quad \text{for all } 0 \leq t \leq t_1, x \in \mathbb{R} \text{ and} \]
\[ \left\| \left( \varphi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) \right\|_3 \leq 2 \left\| \left( \varphi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right) \right\|_3 := 2N_{03} \]

holds for all \(0 \leq t \leq t_1\).

Having obtained (67) and (68), if we set
\[ T = t_1, \quad M_1 = \max \{V, V^{-1}\}, \quad \epsilon = 2\sqrt{\gamma - 1}N_{03}, \quad N_1 = 2N_{03}, \]

and noticing that \(N_{03}, V, V, \Theta, \Theta\), and \(\Theta\) are assumed to be independent of \(\gamma - 1\), we can find two positive constants \(l_4 \geq \max\{l_1, l_2, l_3\} \geq 1\) and some \(\gamma_1 > 1\) such that
\[
\begin{cases}
C (V, V, \Theta, \Theta) l_4^{-\frac{1}{2}} \leq 1, \\
C (V, V, \Theta, \Theta) C (\max \{V, V^{-1}\}) l_4^{-\frac{1}{4}} \leq 1, \\
C (\max \{V, V^{-1}\}) l_4^{-\frac{1}{4}} + 4C (\max \{V, V^{-1}\}) N_{03}^2 l_4^{-\frac{1}{4}} \leq 1, \\
C (\max \{V, V^{-1}\}) l_4^{-\frac{1}{4}} \leq 1,
\end{cases}
\]

\[
\begin{cases}
4\sqrt{\gamma_1} - 1C (\max \{V, V^{-1}\}) N_{03} \leq 1, \\
C (\max \{V, V^{-1}\}) (\gamma_1 - 1) (2N_{03} + 1) \leq \frac{1}{16}, \\
16C (\max \{V, V^{-1}\}) (\gamma_1 - 1)^2 N_{03}^4 \leq 1, \\
2C (\max \{V, V^{-1}\}) (\gamma_1 - 1)N_{03} \leq \frac{1}{16}, \\
(\gamma_1 - 1)C (\max \{V, V^{-1}\}) (4N_{03}^2 + 1) \leq 1, \\
16(\gamma_1 - 1)N_{03}^4 \leq 1, \\
4K_1 (\gamma_1 - 1)N_{03}^2 N_{03}^2 \exp (K_1 N_{03}^2) \leq \frac{1}{20}, \\
K_2 (\gamma_1 - 1)^2 N_{02}^2 \exp (N_{02}^2 \exp (K_2 N_{01}^2)) \leq 1,
\end{cases}
\]
and

\[ 2\sqrt{\gamma_1 - 1}N_{03} \leq \min \left\{ \frac{\Theta}{2}, \Theta \right\} \]

hold.

The above analysis tells us that all the conditions listed in Corollary 3.9 hold with \( T = t_1 \) for all \( l \geq l_4 \) and \( 1 < \gamma \leq \gamma_1 \) and consequently we have from Corollary 3.9 that the local solution \((\varphi(t, x), \psi(t, x), \phi(t, x), \xi(t, x))\) constructed above satisfies

\[ \Theta \leq \theta(t, x) \leq \overline{\Theta}, \quad \exp(-CN_{01}^2) \leq v(t, x) \leq CN_{01}^{1+2b} \]  

(69)

for all \( 0 \leq t \leq t_1, x \in \mathbb{R} \) and

\[
\left\| \left( \varphi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (t) \right\|_3^2 + \int_0^t \left( \|\varphi_x(\tau)\|_3^2 + \|\psi_x, \phi_x(\tau)\|_3^2 \right) d\tau 
\leq K_3N_{03}^2 \exp(\exp(K_3N_{02}^2))
\]  

(70)

holds for all \( 0 \leq t \leq t_1 \).

Now, we take \((\varphi(t_1, x), \psi(t_1, x), \phi(t_1, x), \xi(t_1, x))\) as initial data, we can deduce again from Lemma 3.1 that the local solution \((\varphi(t, x), \psi(t, x), \phi(t, x), \xi(t, x))\) constructed above can be extended to the time step \( t = t_1 + t_2 \) for some suitably small positive constant \( t_2 \) depending only on \( N_{03}, \Theta, \overline{\Theta}, \overline{\Theta} \) and satisfies

\[
\frac{1}{2} \exp(-CN_{01}^2) \leq v(t, x) \leq 2CN_{01}^{1+2b}, \quad \frac{1}{2} \Theta \leq \theta(t, x) \leq 2\overline{\Theta},
\]

and

\[
\left\| \left( \varphi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (t) \right\|_3 \leq 2 \left( \left\| \varphi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right\|_{t_1} \right)_3 
\leq 2\sqrt{K_3N_{03}} \exp(\frac{1}{2} \exp(K_3N_{02}^2))
\]

holds for all \( 0 \leq t \leq t_1 + t_2 \).

If we set

\[
\begin{align*}
T &= t_1 + t_2, \\
\epsilon &= 2\sqrt{\gamma - 1}K_3N_{03} \exp(\frac{1}{2} \exp(K_3N_{02}^2)), \\
M_1 &= 2 \exp(CN_{01}^2), \\
N_1 &= 2\sqrt{K_3N_{03}} \exp(\frac{1}{2} \exp(K_3N_{02}^2)),
\end{align*}
\]

since the positive constants \( C, K_3, \) and \( N_{03} \) are independent of \( \gamma - 1 \), it is easy to see that we can find positive constants \( l_5 \geq l_4 > 1, \ 1 < \gamma_2 \leq \gamma_1 \) such that

\[
\begin{align*}
C \left( V, \overline{V}, \Theta, \overline{\Theta} \right) l_5^{-\frac{1}{2}} &\leq 1, \\
4C \left( V, \overline{V}, \Theta, \overline{\Theta} \right) C \left( 2e^{CN_{01}^2} \right) e^{2CN_{02}^2} l_5^{-\frac{1}{2}} &\leq 1, \\
16C \left( 2e^{CN_{01}^2} \right) e^{4CN_{01}^2} l_5^{-\frac{1}{2}} &\leq 1, \\
8C \left( 2e^{CN_{01}^2} \right) e^{8CN_{01}^2} K_3N_{03}^2 e^{\exp(K_3N_{02}^2)} l_5^{-\frac{1}{2}} &\leq 1, \\
8C \left( 2e^{CN_{01}^2} \right) e^{3CN_{01}^2} l_5^{-\frac{1}{2}} &\leq 1,
\end{align*}
\]  

(71)
And again the above analysis tells us that all the conditions listed in Corollary 3.9 hold with $T = t_1 + t_2$ for all $l \geq l_5$ and $1 < \gamma \leq \gamma_2$ and consequently we have from Corollary 3.9 that the solution $(\varphi(t, x), \psi(t, x), \phi(t, x), \xi(t, x))$ defined on the time interval $[0, t_1 + t_2]$ satisfies (69) and (70) for all $0 \leq t \leq t_1 + t_2$ with the same positive constants $C$ and $K_3$.

Now we take $(\varphi(t_1 + t_2, x), \psi(t_1 + t_2, x), \phi(t_1 + t_2, x), \xi(t_1 + t_2, x))$ as initial data. Noticing that the constants $C$ and $K_3$ in (69) and (70) are independent of the time variable $t$, we can then extend $(\varphi(t, x), \psi(t, x), \phi(t, x), \xi(t, x))$ to the time step $t = t_1 + 2t_2$ by exploiting Lemma 3.1 again. Repeating the above procedure, we can thus extend the solution $(\varphi(t, x), \psi(t, x), \phi(t, x), \xi(t, x))$ step by step to a global one provided that $1 < \gamma \leq \gamma_2$, $l \geq l_5$, as a by-product of the above analysis, we can also deduce that $(\varphi(t, x), \psi(t, x), \phi(t, x), \xi(t, x))$ satisfies

$$
\left\| \left( \varphi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right) (t) \right\|_3^2 + \int_0^t \left( \|\varphi_x(\tau)\|_3^2 + \|\psi_x(\tau)\|_3^2 \right) d\tau
\leq K_3 N_0 e^{\exp(K_3 N_0^2)}
$$

from which the time asymptotic behavior (17) follows easily, and we complete the proof of the Theorem 1.1.

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REFERENCES

[1] S. N. Antontsev, A. V. Kazhikhov and V. N. Monakhov, Boundary Value Problems in Mechanics of Nonhomogeneous Fluids, North-Holland, Amsterdam, 1990.

[2] C. Cercignani, R. Illner and M. Pulvirenti, The Mathematical Theory of Dilute Gases, Appl. Math. Sci. 106, Springer-Verlag, New York, 1994.

[3] C. M. Dafermos and L. Hsiao, Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity, Nonlinear Anal., 6 (1982), 435–454.
[4] R. Duan, H.-X. Liu and H.-J. Zhao, Nonlinear stability of rarefaction waves for the compressible Navier-Stokes equations with large initial perturbation, Trans. Amer. Math. Soc., 361 (2009), 453–493.

[5] H. Grad, Asymptotic theory of the Boltzmann equation II, in rarefied gas dynamics, J.A. Laurmann, ed., Academic Press, New York, 1963, pp. 26–59.

[6] L. Hsiao and S. Jiang, Nonlinear hyperbolic-parabolic coupled systems, In: Handbook of Differential Equations, Vol. 1: Evolutionary Equations. Chapter 4, pp. 287–384. Elsevier, 2004.

[7] F.-M. Huang, J. Li and A. Matsumura, Asymptotic stability of combination of viscous contact wave with rarefaction waves for one-dimensional compressible Navier-Stokes system, Arch. Ration. Mech. Anal., 197 (2010), 89–116.

[8] F.-M. Huang and A. Matsumura, Stability of a composite wave of two viscous shock waves for the full compressible Navier-Stokes equation, Comm. Math. Phys., 289 (2009), 841–861.

[9] F.-M. Huang and Z.-P. Xin, Stability of contact discontinuities for the 1-D compressible Navier-Stokes equations, Arch. Rational Mech. Anal., 179 (2006), 55–77.

[10] F.-M. Huang and T. Wang, Stability of superposition of viscous contact wave and rarefaction waves for compressible Navier-Stokes system, Preprint at arXiv: 1502.00211.

[11] F.-M. Huang, Z.-P. Xin and T. Yang, Contact discontinuity with general perturbations for gas motions, Adv. Math., 219 (2008), 1246–1297.

[12] S. Jiang, Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains, Comm. Math. Phys., 200 (1999), 181–193.

[13] S. Jiang and R. Racke, Evolution Equations in Thermoelasticity, Monogr. Surv. Pure Appl. Math., 112, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[14] S. Jiang and P. Zhang, Global weak solutions to the Navier-Stokes equations for a 1D viscous polytropic ideal gas, Quart. Appl. Math., 61 (2003), 435–449.

[15] H. K. Jenssen and T. K. Karper, One-dimensional compressible flow with temperature dependent transport coefficients, SIAM J. Math. Anal., 42 (2010), 904–930.

[16] Y. Kanel’, On a model system of equations of one-dimensional gas motion, Differ. Uravn., 4 (1968), pp. 374–380.

[17] S. Kawashima and A. Matsumura, Asymptotic stability of travelling wave solutions of systems for one-dimensional gas motion, Commun. Math. Phys., 101 (1985), 97–127.

[18] S. Kawashima, A. Matsumura and K. Nishihara, Asymptotic behaviour of solutions for the equations of a viscous heat-conductive gas, Proc. Japan Acad. Ser. A., 62 (1986), 249–252.

[19] S. Kawashima and T. Nishida, Global solutions to the initial value problem for the equations of one-dimensional motion of viscous polytropic gases, J. Math. Kyoto Univ., 21 (1983), 825–837.

[20] B. Kawohl, Global existence of large solutions to initial-boundary value problems for a viscous, heat-conducting, one-dimensional real gas, J. Differential Equations, 58 (1985), 76–103.

[21] A. V. Kazhikhov, Correctness “in the whole” of the mixed boundary value problems for a model system of equations of a viscous gas (in Russian), Dinamika Sploshn. Sredy, 21 (1975), 18–47.

[22] A. V. Kazhikhov and V. V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, J. Appl. Math. Mech., 41 (1977), 273–282.

[23] J. Li and Z. Liang, Some uniform estimates and large-time behavior for one-dimensional compressible Navier-Stokes system in unbounded domains with large data, Arch. Ration. Mech. Anal., 220 (2016), 1195–1208, arXiv:1404.2214.

[24] H.-X. Liu, T. Yang, H.-J. Zhao and Q.-Y. Zou, One-dimensional compressible Navier-Stokes equations with temperature dependent transport coefficients and large data, SIAM J. Math. Anal., 46 (2014), 2185–2228.

[25] T.-P. Liu, Solutions in the large for the equations of nonsentropic gas dynamics, Indiana Univ. Math. J., 26 (1977), 147–177.

[26] T.-P. Liu, Shock waves for compressible Navier-Stokes equations are stable, Commun. Pure Appl. Math., 39 (1986), 565–594.

[27] T.-P. Liu and Z.-P. Xin, Nonlinear stability of rarefaction waves for compressible Navier-Stokes equations, Comm. Math. Phys., 118 (1988), 451–465.

[28] T.-P. Liu and Z.-P. Xin, Pointwise decay to contact discontinuities for systems of viscous conservation laws, Asian J. Math., 1 (1997), 34–84.
[29] T.-P. Liu and Y.-N. Zeng, Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws, *Mem. Amer. Math. Soc.*, **125** (1997), viii+120 pp.

[30] T.-P. Liu and Y.-N. Zeng, Shock waves in conservation laws with physical viscosity, *Mem. Amer. Math. Soc.*, **234** (2015), vi+168 pp.

[31] A. Matsumura, Inflow and outflow problems in the half space for a one-dimensional isentropic model system of compressible viscous gas. Proceedings of the Third World Congress of Nonlinear Analysts, Part 6 (Catania, 2000), *Nonlinear Anal.*, **47** (2001), 4269–4282.

[32] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.*, **20** (1980), 67–104.

[33] A. Matsumura and K. Nishihara, On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.*, **2** (1985), 17–25.

[34] A. Matsumura and K. Nishihara, Asymptotic toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.*, **3** (1986), 1–13.

[35] A. Matsumura and K. Nishihara, Global stability of the rarefaction waves of a one-dimensional model system for compressible viscous gas, *Comm. Math. Phys.*, **144** (1992), 325–335.

[36] A. Matsumura and K. Nishihara, Global asymptotics toward the rarefaction wave for solutions of viscous p-system with boundary effect, *Quart. Appl. Math.*, **58** (2000), 69–83.

[37] T. Nishida and J. A. Smoller, Solutions in the large for some nonlinear hyperbolic conservation laws, *Comm. Pure Appl. Math.*, **26** (1973), 183–200.

[38] K. Nishihara, T. Yang and H.-J. Zhao, Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations, *SIAM J. Math. Anal.*, **35** (2004), 1561–1597.

[39] M. Okada and S. Kawashima, On the equations of one-dimensional motion of compressible viscous fluids, *J. Math. Kyoto Univ.*, **23** (1983), 55–71.

[40] R.-H. Pan and W.-Z. Zhang, Compressible Navier-Stokes equations with temperature dependent heat conductivity, *Commun. Math. Sci.*, **13** (2015), 401–425.

[41] Z. Tan, T. Yang, H.-J. Zhao and Q.-Y. Zou, Global solutions to the one-dimensional compressible Navier-Stokes-Poisson equations with large data, *SIAM J. Math. Anal.*, **45** (2013), 547–571.

[42] J. B. Temple, Solutions in the large for the nonlinear hyperbolic conservation laws of gas dynamics, *J. Differential Equations*, **41** (1981), 96–161.

[43] W. G. Vincenti and C. H. Kruger, *Introduction to Physical Gas Dynamics*, Cambridge Math. Lib., Cambridge University Press, Cambridge, UK, 1975.

[44] L. Wan, T. Wang and H.-J. Zhao, Asymptotic stability of wave patterns to compressible viscous and heat-conducting gases in the half space, *arXiv:1506.07626*.

[45] L. Wan, T. Wang and Q.-Y. Zou, Stability of stationary solutions to the outflow problem for full compressible Navier-Stokes equations with large initial perturbation, *IOP Publishing Ltd & London Mathematical Society*, **49** (2016), *arXiv:1503.03922*.

[46] T. Wang and H.-J. Zhao, Global large solutions to a viscous heat-conducting one-dimensional gas with temperature-dependent viscosity, Preprint at *arXiv:1505.05252* (2015).

[47] T. Wang, H.-J. Zhao and Q.-Y. Zou, One-dimensional compressible Navier-Stokes equations with large density oscillation, *Kinet. Relat. Models*, **6** (2013), 649–670.

[48] Y. B. Zel’dovich and Y. P. Raizer, *Physics of Shock Waves and High Temperature Hydrodynamic Phenomena, Vol. II*, Academic Press, New York, 1967.

[49] A. A. Zlotnik and A. A. Amosov, On the stability of generalized solutions of equations of one-dimensional motion of a viscous heat-conducting gas, *Siberian Math. J.*, **38** (1997), 663–684.