Invariant group orderings and Galois conjugates

Peter A. Linnell*  
Department of Mathematics  
Virginia Tech  
Blacksburg  
VA 24061-0123  
USA

Akbar H. Rhemtulla†  
Department of Mathematics  
University of Alberta  
Edmonton  
AL Canada T6G 2G1

Dale P. O. Rolfsen‡  
Department of Mathematics  
University of British Columbia  
Vancouver  
BC Canada V6T 1Z2

Abstract

This paper investigates conditions under which a given automorphism of a residually torsion-free nilpotent group respects some ordering of the group. For free groups and surface groups, this has relevance to ordering the fundamental groups of three-dimensional manifolds which fibre over the circle.

Key words: ordered group, residually torsion-free nilpotent group, invariant ordering, fibred knot

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1 Introduction

We consider orderings of a residually torsion-free nilpotent group $H$, that is a group $H$ which has a descending sequence of normal subgroups $H_1 \supseteq H_2 \supseteq \cdots$ such that $H/H_i$ is torsion-free nilpotent for all $i$ and $\bigcap H_i = 1$, and their invariance under automorphisms, motivated by some questions in topology. A group $G$ is said to be ordered if there is a strict total ordering $<$ of its elements which is invariant under multiplication on both sides, that is $f < g$ implies
\( hf < hg \) and \( fh < gh \) for all \( f, g, h \in G \); we shall sometimes say that \( G \) is bi-ordered in this situation, to emphasize the two-sidedness of the order. The set \( P \) of all \( g \in G \) greater than the identity is called the positive cone of the ordering, and satisfies:

1. \( P \) is a sub-semigroup, that is, closed under multiplication, and \( 1 \notin P \).
2. If \( 1 \neq g \in G \), then either \( g \in P \) or \( g^{-1} \in P \), but not both.
3. If \( g \in G \) and \( p \in P \), then \( g^{-1}pg \in P \); that is, \( P \) is normal in \( G \).

Conversely, if \( P \) is a subset satisfying (1), (2) and (3), then it defines an ordering by the formula \( f < g \Leftrightarrow f^{-1}g \in P \). If \( P \) satisfies only (1) and (2), then it defines a left-invariant ordering and we say that \( G \) is a left-ordered group. Clearly an ordered group is left ordered, and a left-ordered group is torsion free.

Many groups are orderable, including free groups and torsion-free nilpotent groups, and more generally residually torsion-free nilpotent groups. Orderable groups have unique roots: for \( 0 \neq n \in \mathbb{Z} \), we have \( g^n = h^n \) if and only if \( g = h \) in \( G \). Another pleasant property of orderable groups is that they obey the zero divisor conjecture: if \( G \) is orderable and \( k \) is any integral domain, then the group ring \( kG \) has no nontrivial zero-divisors, and in fact embeds in a skew field.

A subset \( X \) of a left-ordered group \( G \) is convex if whenever \( x, y, z \in G \) satisfy \( x, z \in X \) and \( x < y < z \), then \( y \in X \). The collection of convex subgroups of a left-ordered group is linearly ordered by inclusion. An ordered group is Archimedean if the powers of every non-identity element are cofinal in the ordering. By theorems of Hölder and Conrad [2, Theorems 1.3.4 and 7.2.1], every Archimedean left-ordered group embeds (by a homomorphism preserving the order) into the additive real numbers \( \mathbb{R} \).

We say that an automorphism \( \varphi : G \to G \) respects an ordering \( < \) if \( g < h \Leftrightarrow \varphi(g) < \varphi(h) \). In this setting we also say that \( < \) is a \( \varphi \)-invariant ordering. This is equivalent to the equation \( \varphi(P) = P \).

If \( 1 \to A \to B \to C \to 1 \) is an exact sequence of groups and \( A \) and \( C \) are left-ordered, then \( B \) can also be left-ordered by the lexicographic order; specifically if we view \( C \) as \( B/A \), then for \( g, h \in B \), we define \( g > h \) if and only if \( h^{-1}g > 1 \) in \( A \). In this situation, the positive cone consists of the inverse image of the positive cone of \( C \) in \( B \) together with the positive cone of \( A \). This inheritance under extensions does not hold in general for two-sided orderings; for example the Klein bottle group, \( \langle x, y \mid x^2 = y^3 \rangle \), is an extension of \( \mathbb{Z} \) by \( \mathbb{Z} \) which is left orderable but not orderable. However the above recipe does provide a two-sided ordering, provided the ordering of \( A \) is respected by the automorphisms of \( A \) induced by conjugation by elements of \( B \). In particular, this holds if \( A \) is central in \( B \).

An application of invariant ordering is to the fundamental groups of manifolds which fibre over the circle. If \( M^n \) fibres over \( S^1 \), with fibre \( F^{n-1} \), their groups fit into an exact sequence

\[
1 \to \pi_1(F) \to \pi_1(M) \to \pi_1(S^1) = \mathbb{Z} \to 1.
\]
One may consider $M$ as a mapping torus of a monodromy map $f : F \to F$, that is

$$M \cong F \times [0,1]/(x,1) \sim (f(x),0), \quad x \in F.$$  

The fundamental group $\pi_1(M)$ is an HNN extension of $\pi_1(F)$ defined by the automorphism $f_* : \pi_1(F) \to \pi_1(F)$, and so it is easy to see the following.

**Proposition 1.1.** If $M$ is a manifold which fibres over the circle with fibre $F$, then $\pi_1(M)$ is left orderable if and only if $\pi_1(F)$ is left orderable. Moreover $\pi_1(M)$ is orderable if and only if $\pi_1(F)$ has a (two-sided) ordering invariant under the $\pi_1$-monodromy $f_* : \pi_1(F) \to \pi_1(F)$.

We will now consider lower dimensions. By a surface, we understand a metric space each of whose points has a neighborhood homeomorphic with the Euclidean plane or half-plane. It is known that almost all surface groups are orderable; more precisely [11, Theorem 3] yields

**Proposition 1.2.** If $F$ is any connected surface (compact or not), then $\pi_1(F)$ is orderable unless $F$ is a projective plane or Klein bottle. The Klein bottle group is left orderable but not orderable.

In particular, all orientable surface groups are orderable. The nonorientable closed surfaces have orderable groups if and only if the surface has negative Euler characteristic, or equivalently, the surface is the connected sum of at least three projective planes. The projective plane’s fundamental group, being finite, is certainly not left orderable.

The following theorem is [8, Theorem 1.1] in the case that $\pi_1(F)$ is free, as in fibred knot complements $M = S^3 \setminus N(K)$. The more general case in which $F$ may be a closed orientable surface was proved in [9, Corollary 2.3].

**Theorem 1.3.** Suppose that $M^3$ is an orientable 3-manifold which fibres over $S^1$, with compact orientable fibre $F^2$ and monodromy $f : F \to F$. Then $\pi_1(M)$ is orderable if all the eigenvalues of the homology monodromy $f_* : H_1(F) \to H_1(F)$ are real and positive. In particular, a fibred knot in $S^3$ or a homology sphere, has orderable knot group if all the roots of its Alexander polynomial are real and positive.

One of the main points of the present paper is to investigate the extent to which the condition on the eigenvalues is necessary in Theorem 1.3 above. Our main result for producing examples is Proposition 3.4.

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## 2 Abelian orderings and Galois conjugates

Let $\theta$ be an endomorphism of the finite rank torsion-free abelian group $A$. The eigenvalues of $\theta$ will mean the complex eigenvalues of the $\mathbb{C}$-linear transformation
$\theta \otimes 1$ induced by $\theta$ on the finite dimensional $\mathbb{C}$-vector space $A \otimes \mathbb{Z} \mathbb{C}$. If two algebraic complex numbers have the same minimal polynomial over $\mathbb{Q}$, they are said to be Galois conjugates.

**Lemma 2.1.** If the finite rank abelian group $A \neq 1$ is a subgroup of the additive group of real numbers $\mathbb{R}$ and the endomorphism $\theta: A \to A$ is multiplication by the real number $\alpha$, then the eigenvalues of $\theta$ are the Galois conjugates of $\alpha$.

**Proof.** Let $f$ denote the minimal polynomial of $\alpha$ over $\mathbb{Q}$. Then $f(\alpha) = 0$ and $f$ is irreducible in $\mathbb{Q}[X]$, more or less by definition. Observe that if $g \in \mathbb{Z}[X]$, then $g(\theta)$ is given by multiplication by $g(\alpha)$. It follows that $f(\theta) = 0$. Therefore the eigenvalues of $\theta$ satisfy $f$. Since $f$ is irreducible, it follows that the eigenvalues of $\theta$ are precisely the roots of $f$, in other words the eigenvalues of $\theta$ are the Galois conjugates of $\alpha$. \qed

**Proposition 2.2.** Let $A$ be a torsion-free abelian group of finite rank and let $\theta$ be an automorphism of $A$. Then $\theta$ preserves an order if and only if for each eigenvalue of $\theta$, at least one of its Galois conjugates is a positive real number.

**Proof.** We may assume that $A \neq 1$. First suppose $\theta$ preserves an order on $A$. If $1 = A_0 < A_1 < \cdots < A_n = A$ is a series of convex subgroups in $A$, then each $A_{i+1}/A_i$ is a torsion-free abelian group with rank at least 1, consequently $n$ is at most the rank of $A$. We therefore have a finite series of convex subgroups $1 = A_0 < A_1 < \cdots < A_n = A$ of $A$, where $A_{i-1} \neq A_i$ and there are no convex subgroups strictly between $A_{i-1}$ and $A_i$ for all $i$. Since $\theta$ maps convex subgroups to convex subgroups, it follows that $\theta A_i = A_i$ for all $i$. Thus $\theta$ induces an order preserving automorphism on the ordered group $A_i/A_{i-1}$. By H"older’s theorem [2 Theorem 1.3.4] we may consider $A_i/A_{i-1}$ as a subgroup of the additive group of $\mathbb{R}$ (with its natural order), and then a theorem of Hion [2 Theorem 1.5.1] tells us that this automorphism induced by $\theta$ is multiplication by a positive real number. It follows from Lemma 2.1 that for each eigenvalue of $\theta$, at least one of its Galois conjugates is a positive real number. Conversely suppose all the eigenvalues of $\theta$ have a Galois conjugate which is a positive real number. If $1 = A_0 < A_1 < \cdots < A_n = A$ is a sequence of subgroups of $A$ with each $A_{i+1}/A_i$ torsion-free, then $n$ is at most the rank of $A$. Therefore we may choose a finite series of $\theta$-invariant subgroups $1 = A_0 < A_1 < \cdots < A_n = A$ where $A_i/A_{i-1}$ is torsion free and if $B$ is a $\theta$-invariant subgroup such that $A_{i-1} < B \leq A_i$, then $A_i/B$ is a torsion group.

We now fix $i$ and view $A_i/A_{i-1}$ as a $\mathbb{Z}[X]$-module, where the action of $X$ is induced by $\theta$. Then $A_i/A_{i-1} \otimes \mathbb{Z} \mathbb{Q}$ is an irreducible $\mathbb{Q}[X]$-module. By the structure theorem for modules over a principal ideal domain applied to the ring $\mathbb{Q}[X]$ and the $\mathbb{Q}[X]$-module $A_i/A_{i-1} \otimes \mathbb{Z} \mathbb{Q}$, we may consider $A_i/A_{i-1}$ as a subgroup of $\mathbb{R}$, and $\theta$ as multiplication by a positive real number, because each eigenvalue of $\theta$ has at least one of its Galois conjugates a positive real number. Thus each $A_i/A_{i-1}$ has a $\theta$-invariant order. We can now use the lexicographic ordering to obtain a $\theta$-invariant order of $A$. \qed
3 Orderings of Residually Torsion-free Nilpotent Groups

Suppose $F$ is a finitely generated nonabelian free group and $\theta: F \to F$ is an automorphism. It was shown in [8, Theorem 2.6] that if (as in Theorem 1.3) all the eigenvalues of the corresponding map on the abelianization $H_1(F, \mathbb{Z})$ are real and positive, then one can construct a $\theta$-invariant order on $F$. In this section we show that the condition that all the eigenvalues be positive is not necessary. We will also consider generalizations to residually torsion-free nilpotent groups.

We need to consider the rational lower central series $G^r_n$ (for $n$ a positive integer) of the group $G$; here the superscript $r$ indicates “rational”. Recall that the lower central series of a group $G$ is defined inductively by $G_1 = G$ and $G_{n+1} = [G, G_n]$ for $n \geq 1$. Then $G^r_n = \{ g \in G \mid g^m \in G_n \text{ for some positive integer } m \}$. Since $G/G_n$ is a nilpotent group, $G^r_n$ is a characteristic subgroup of $G$ for all positive integers $n$. Furthermore, each $G^r_n/G^r_{n+1}$ is a torsion-free group which lies in the center of $G/G_n$. It is easy to see that $\bigcap_{n=1}^{\infty} G^r_n = 1$ if and only if $G$ is residually torsion-free nilpotent. If $\theta$ is an automorphism of $G$, then $\theta$ induces automorphisms on $G_n/G_{n+1}$ and $G^r_n/G^r_{n+1}$ for all $n \geq 1$. Thus if $A$ is the infinite cyclic group, we can make each $G_n/G_{n+1}$ into a $\mathbb{Z}A$-module by making the generator of $A$ act by $\theta$. Then [10, 5.2.5] tells us that we have a $\mathbb{Z}A$-epimorphism from $(G/G_2)^{\otimes n}$, the $n$-fold tensor product with diagonal $A$-action, onto $G_n/G_{n+1}$. We may also view each $G^r_n/G^r_{n+1}$ as a $\mathbb{Z}A$-module. It follows that we have a $\mathbb{Q}A$-epimorphism from $(G/G_2^{\otimes 2} \otimes \mathbb{Q})^{\otimes n}$ onto $(G^r_n/G^r_{n+1}) \otimes \mathbb{Q}$. Suppose now that $G/G_2$ has finite rank and let $\alpha_1, \ldots, \alpha_p$ denote the eigenvalues of the automorphism induced by $\theta$ on $G/G_2$. Then we see that $G^r_n/G^r_{n+1}$ is a finite rank torsion-free abelian group for all $n$, and that the eigenvalues of the automorphism induced by $\theta$ on each $G^r_n/G^r_{n+1}$ are just products of $\alpha_1, \ldots, \alpha_p$.

To apply Proposition 2.2, we want a condition that will ensure that at least one of the Galois conjugates of such a product is a positive real number. For convenience, we make the following definition.

**Definition 3.1.** Let $f \in \mathbb{Q}[X]$ be a monic polynomial and let $f = f_1 \cdots f_n$ be its factorization into monic irreducible polynomials (so $n$ is a nonnegative integer and each $f_i$ is irreducible). Then we say that $f$ is a **special** polynomial if for each $i$, at least one of the following two conditions is satisfied.

(i) $f_i$ has odd prime power degree, negative constant term, and all its roots are real.

(ii) All the roots of $f_i$ are real and positive.

In an earlier version of this paper, we defined $f$ to be special if it always satisfied condition [3.11] (a stronger condition). With this definition, Wangshan Lu in his M.Sc. thesis [5] classified all the special Alexander polynomials of fibred knots with degree less than 10.

We can now state...
Lemma 3.2. Let $f \in \mathbb{Q}[X]$ be a special polynomial. If $\alpha$ is a product of the roots of $f$, then at least one of the Galois conjugates of $\alpha$ is a positive real number.

Proof. Let $f = f_1 \cdots f_n$ be the factorization of $f$ into monic irreducible polynomials. We prove the result by induction on $n$, the case $n = 0$ being clear because $\alpha = 1$ (empty product) in this case. Let $\alpha_1, \ldots, \alpha_p$ be the roots of $f_1$, let $\beta_1, \ldots, \beta_q$ be the roots of $f_2 \cdots f_n$, and let $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$. Then

$$\alpha = \alpha_1^{r_1} \cdots \alpha_p^{r_p} \beta_1^{s_1} \cdots \beta_q^{s_q}$$

where the $r_i, s_i$ are nonnegative integers. Clearly all the Galois conjugates of $\alpha$ are real numbers. Set $\beta = \beta_1^{s_1} \cdots \beta_q^{s_q}$. Suppose all the Galois conjugates of $\alpha$ are negative. First consider the case when all the roots of $f_1$ are positive. By induction on $n$, some Galois conjugate of $\beta$ is positive and it follows that some Galois conjugate of $\alpha$ is also positive.

Therefore we may assume that $f_1$ satisfies condition 3.3, and here we may assume that $p$ is a power of the odd prime $p'$. Let $c$ denote the negative of the constant term of $f_1$, a positive real number. By considering a Sylow $p'$-subgroup of the Galois group of $K$ over $\mathbb{Q}$, there is a $p'$-subgroup $P$ of field automorphisms of $K$ which acts transitively on $\{\alpha_1, \ldots, \alpha_p\}$. Set $\bar{P} = |P|/p$, so $\bar{P}$ is the order of the stabilizer in $P$ of $\alpha_1$. Then we have

$$\prod_{\theta \in P} \theta(\alpha) = (\alpha_1 \cdots \alpha_p)^{\bar{P}(r_1 + \cdots + r_p)} \prod_{\theta \in P} \theta(\beta) = c^{\bar{P}(r_1 + \cdots + r_p)} \prod_{\theta \in P} \theta(\beta).$$

Write $\gamma = \prod_{\theta \in P} \theta(\beta)$, so $\prod_{\theta \in P} \theta(\alpha) = c^{\bar{P}(r_1 + \cdots + r_p)} \gamma$. Since $\gamma$ is a product of the roots of $f_2 \cdots f_n$, by induction on $n$ there is a field automorphism $\phi$ of $K$ such that $\phi(\gamma)$ is a positive real number. Then

$$\prod_{\theta \in P} \phi\theta(\alpha) = c^{\bar{P}(r_1 + \cdots + r_p)} \phi(\gamma). \tag{3.3}$$

The left hand side of (3.3) is a product of $|P|$ negative numbers and hence is negative, whereas the right hand side is positive. This contradiction completes the proof. \[\square\]

Proposition 3.4. Let $f \in \mathbb{Q}[X]$ be a special polynomial (see Definition 3.1), let $G$ be a residually torsion-free nilpotent group, let $\theta$ be an automorphism of $G$, and let $\phi: G/G_2' \to G/G_2'$ be the automorphism induced by $\theta$. Assume that $G/G_2'$ has finite rank and that the eigenvalues of $\phi$ are roots of $f$. Then $G$ has a bi-ordering invariant under $\theta$.

Proof. Recall that if $Z$ is a central subgroup of the group $G$, and $Z$ and $G/Z$ are orderable, then one can use the lexicographic ordering to bi-order $G$. To show that $G$ has a bi-order which is invariant under $\theta$, it will be sufficient to show that each $G/G''_n$ has a bi-order which is invariant under $\theta$, by [2] Theorem 1.3.2(a)]. Thus it will be sufficient to show that each $G''_n/G''_{n+1}$ has an order which is invariant under $\theta$. Lemma 3.2 tells us that each eigenvalue of the
induced action of \( \theta \) on \( G^n_r/G^n_{r+1} \) has at least one of its Galois conjugates a positive real number. We now see from Proposition 3.4 that \( G^n_r/G^n_{r+1} \) has an ordering invariant under \( \theta \) and the result follows.

Remark. To apply Proposition 3.4 we need examples of residually torsion-free nilpotent groups \( G \) such that \( G/G^r \) has finite rank. The latter condition is easily satisfied; it will certainly be the case if \( G \) is finitely generated. The former is certainly satisfied if \( G \) is torsion-free nilpotent or free.

Recall that the group \( G \) is fully residually free means that given \( g_1, \ldots, g_n \in G \), then there exists a homomorphism \( G \rightarrow F \) where \( F \) is a free group such that \( \theta(g_i) \neq 1 \) for all \( i \). Certainly a fully residually free group is residually torsion-free nilpotent. Benjamin Baumslag [1] top of p. 414 proved that if \( 4 \leq n \in \mathbb{Z} \) and \( 0 \neq w_1, \ldots, w_n \in \mathbb{Z} \), then \( \langle a_1, \ldots, a_n \mid a_1^{w_1} \cdots a_n^{w_n} = 1 \rangle \) is fully residually free. Thus in particular all surface groups, orientable or not, are residually torsion-free nilpotent, except the projective plane, Klein bottle, and the group \( \langle a, b, c \mid a^2b^2c^2 = 1 \rangle \), the fundamental group of the connected sum of exactly three projective planes (the torus group is fully residually free, though this particular case does not follow from Baumslag’s paper). The group \( \langle a, b, c \mid a^2b^2c^2 = 1 \rangle \) is not fully residually free [6]; we do not know whether or not it is residually torsion-free nilpotent.

Another situation where Proposition 3.4 can be applied is as follows. In [4] p. 17, Labute defines an element \( x \) of a free group \( F \) to be primitive if \( x \neq 1 \), and if \( x \in F_n \setminus F_{n+1} \) (where \( F_n \) denotes the lower central series of \( F \) ), then \( x \) is not a \( d \)th power modulo \( F_{n+1} \) for any integer \( d \geq 2 \). Then [4] Theorem on p. 17] shows that a one-relator group, where the relator is a primitive element of the ambient central series, has a lower central series with torsion-free factors. Thus in particular if the one-relator group is also residually nilpotent, then the group is residually torsion-free nilpotent.

Example 3.5. Let \( p \) be a power of an odd prime, let \( F \) denote the free group of rank \( p \) with free generators \( x_1, \ldots, x_p \), and let \( f(X) = X^p + f_{p-1}X^{p-1} + \cdots + f_1X - 1 \in \mathbb{Z}[X] \) be a polynomial which is irreducible in \( \mathbb{Q}[X] \) and has all roots real; note that \( f \) is a special polynomial (Definition 3.1). For example \( p = 3 \) and \( f(X) = x^3 - 3x - 1 \), which has two negative roots. Then we can define an automorphism \( \theta \) of \( F \) by the formula

\[
\theta x_1 = x_2, \quad \theta x_2 = x_3, \quad \ldots, \quad \theta x_{p-1} = x_p, \quad \theta x_p = x_1x_2^{-f_1}\ldots x_p^{-f_{p-1}}.
\]

Then the eigenvalues of the automorphism of \( F \) induced by \( \theta \) are roots of \( f \). Since \( F/F^2 \) has finite rank, namely \( p \), and \( F \) is residually torsion-free nilpotent, it follows from Proposition 2.2 that \( F \) has an ordering which is invariant under \( \theta \).

Example 3.6. Let \( n \) be a positive integer, let \( F = \langle x, y \rangle \) be the free group of rank 2 with generators \( x, y \), and let \( G = \langle g \mid g^n = 1 \rangle \) be the cyclic group of order \( n \). By mapping \( F \) onto \( G \) by the map \( x \mapsto g, \ y \mapsto 1 \), we may write \( G = F/R \) where \( R \) is the kernel of this map. Then \( R \) is a finitely generated free group.
and conjugation by $F$ on $R$ gives $R/R'$ (where $R'$ indicates the commutator subgroup of $R$) the structure of a $\mathbb{Z}G$-module. Furthermore, [3] Proposition 5.10 shows that $R/R' \cong \mathbb{Z} \oplus \mathbb{Z}G$. Let $\theta$ indicate the automorphism $r \mapsto xr^{-1}$ of $R$ and let $\phi$ be the automorphism of $R/R'$ induced by $\theta$. Since $\phi$ has order $n$, the eigenvalues of $\phi$ are precisely $n$th roots of 1. Also by restricting a bi-order on $F$ to $R$, we see that $\theta$ preserves a bi-order on $R$.

This contrasts with the example of an automorphism of a free group which is periodic of period $n$, where $2 \leq n \in \mathbb{Z}$; that is $\theta^n$ is the identity, but $\theta$ is not the identity. Its eigenvalues are also $n$th roots of 1. However no ordering of the free group can be $\theta$-invariant. For if (say) $w \in F$ and $w < \theta(w)$, then $\theta(w) < \theta^2(w)$, etc. and we obtain the contradiction $w < \theta^n(w) = w$.

This shows in particular, that one cannot determine whether an automorphism of a finitely generated free group preserves a bi-order by looking at its action on the abelianization of the free group.

Finally we give an explicit example of a fibred knot whose group is ordered, but is not covered by the criterion of [3] Theorem 1.1, that is a fibred knot whose Alexander polynomial doesn’t have all roots real and positive. Consider the polynomial $f(x) := x^6 + 3x^5 - x^4 - 7x^3 - x^2 + 3x + 1$. This factors as $(x^3 + x^2 - 2x - 1)(x^3 + 2x^2 - x - 1)$. It is easily checked that both factors are irreducible and have all roots real, hence $f(x)$ is a special polynomial (Definition 3.1). Furthermore $f$ has negative roots. If we multiply $f(x)$ by $-x^{-3}$, we obtain $-x^3 - 3x^2 + x + 7 + x^{-1} - 3x^{-2} - x^{-3}$. This is symmetric in $x$ and $x^{-1}$, the sum of its coefficients is 1, and its leading coefficient is $-1$, hence it is the Alexander polynomial of a fibred knot. The corresponding Conway polynomial is $\nabla(z) := 1 - 20z^2 - 9z^4 - z^6$, because $\nabla(x^{-1/2} + x^{1/2}) = -x^3 - 3x^2 + x + 7 + x^{-1} - 3x^{-2} - x^{-3}$. An explicit example of a fibred knot with Conway polynomial $\nabla$ is given below.

![Diagram of fibred knot](image-url)

This follows from [7] Theorem 6; in that theorem the Conway polynomial is called the Kauffman polynomial. The numbers 20 and 9 indicate the number of full twists in the direction indicated (a total of 40 and 18 crossovers, including the ones drawn).
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