Dimensional cross-over in self-organised super-radiant phases of ultra-cold atoms inside a cavity

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Abstract
We consider a condensate of ultra-cold bosonic atoms in a linear optical cavity illuminated by a two-pump configuration where each pump makes different angles with the direction of the cavity axis. We show that such a configuration allows a smooth transition from a one-dimensional quantum optical lattice configuration to a two-dimensional quantum optical lattice configuration induced by the cavity–atom interaction. Using a Holstein–Primakoff transformation, we find the atomic density profile of such a self-organised ground state in the super-radiant phase as a function of the angular orientations of the pumps in such a dynamical quantum optical lattice, and also provide an analysis of their structures in coordinate and momentum space. In the later part of the paper, we show how the corresponding results can also be qualitatively understood in terms of an extended Bose–Hubbard model in such a quantum optical lattice potential.

Keywords: ultra-cold atomic condensate, super-radiance, optical cavity, lattice-supersolid, self-organisation, extended Bose–Hubbard model

(Some figures may appear in colour only in the online journal)

1. Introduction
The pioneering work of Dicke [1] that predicted the excitation of a super-radiant phase [2–4] by a radiation pulse obtained a convincing experimental demonstration by the Esslinger group in a system of ultra-cold atoms inside a cavity [5]: a normal to super-radiant phase transition was observed in an ultra-cold atomic Bose–Einstein (BH) condensate through self-organisation [6]. This observation was preceded by the experimental observation of super-radiant Rayleigh scattering from ultra-cold atoms in free space [7] and ring cavities [8]. These experiments opened up a new direction in the study of exotic quantum many-body phases [9–12] with an ultra-cold atomic condensate trapped in optical lattice potentials.

Initial studies of such quantum many-body phases of ultra-cold atomic systems involved free-space optical lattice potentials that are not affected by the atomic density distribution [13, 14]; hence the optical lattice potential acts as a classical external potential on the ultra-cold atoms [15, 16] and does not have their dynamics. In comparison, atomic condensates trapped in a cavity-generated dynamical quantum optical lattice potential [17–20] have a significant impact on the structure and strength of the lattice potential [11]. In particular, when these trapped atoms are directly illuminated by a transverse pump beam the excited atoms scatter the pump photons, which finally populate the cavity mode. This position-dependent atom–photon coupling gives rise to a position-dependent scattered-field amplitude. It can generate novel self-organised quantum many-body phases of the atoms through cavity-mediated long-range interactions [21], such as a lattice super-solid phase through a Dicke type transition [5] or a more ideal super-solid phase [22–25] where continuous gauge symmetry and continuous translational symmetry are spontaneously
broken [26] leading to the simultaneous existence of an off-diagonal long-range order (a property of superfluids) and a diagonal long-range order (a property of solids) [27–45].

The existence of competing short-range and cavity-induced long-range interaction in the bosonic lattice model provides a host of novel quantum phases, such as superfluid, super-solid, Mott insulator and charge density wave [46, 47], their novel collective excitations [48], metastability and avalanche dynamics in Mott insulator and density-wave phases [49]. The transition from a coherent superfluid phase to an incoherent Mott insulator, both lying in a super-radiant regime, was also studied by combining the BH model and the Dicke model [50] and also by using the multi-configuration time-dependent Hartree method for indistinguishable particles [51]. The quantum properties of light also get significantly modified due to the interplay of cavity-mediated long-range interactions and the short-range processes of the atom [52]. They can be designed and optimised to create new types of quantum simulators [53] for both single- and multi-mode cavities. Other significant works in this direction explored the following aspects: quantum magnetism by simulating a quantum spin Hamiltonian with multi-component ultra-cold atoms in a linear cavity pumped by external lasers [54], p-band induced self-organisation and dynamics in an optical cavity [55], spin-entanglement and magnetic competition in spinor quantum optical lattices in a cavity [56], creation of various topologically non-trivial phases in a cavity–atom system [57, 58], parametric instabilities in a driven-dissipative Bose–Einstein condensate (BEC) in a cavity [59], a dissipation-engineered family of dark states in cavity–atom systems [60], the role of the atomic correlations in the dynamical instability generated in a cavity–BEC system [61], the possibility of the existence of intertwined and vestigial order in a crossed-cavity–multimode BEC system [62], super-radiant scattering and dynamical instability in a system of a linear cavity illuminated by a single pump [63], the recent observation of a time crystal stabilised by dissipation in a driven open cavity–BEC system [64] and so on. Cavity-like periodic patterns in the atomic density have also been observed in free-space systems under certain conditions [65] and Dicke super-radiance was also studied in fermionic gases [66].

Most of the works mentioned above considered a single-pump laser in a linear cavity [5, 6, 9, 17–21, 46–61, 63–66], ring cavity [8] or crossed-cavity [22–24, 62] set-up. This does not change the dimensionality of the self-organised super-radiant phases and their corresponding quantum optical lattice potential for a given set-up. It may be noted that in the simulation of various quantum many-body phases that have a wide-ranging spread from hard condensed matter problems to the systems studied in high-energy physics (for a review see, e.g., [67]) with table-top ultra-cold atomic systems in a classical optical lattice, the dimensionality of the classical optical lattice plays a significant role [68] in determining the nature of the simulated quantum system. The variety of quantum systems that can be simulated by ultra-cold atoms can be significantly enhanced if the dynamical quantum optical lattice potential created inside an optical cavity can also be made into different dimensions, and in an interchangeable way. In the current work, by considering an ultra-cold BEC placed inside a linear cavity illuminated by two pump beams making angles of $\theta_1$ and $\theta_2$ with the cavity axis we propose a simple way of achieving this by varying the relative angle between these classical pumps.

The existence of two tuning parameters $\theta_1$ and $\theta_2$ offers the possibility of realisation of a large number of self-organised phases. We demonstrate that by changing these angles in the super-radiant regime, one can continuously transform from a one-dimensional (1D) self-organised (SO) lattice supersolid phase to a two-dimensional (2D) SO lattice supersolid phase. The dimensionality is clearly identified by identifying the atomic density maxima and minima in coordinate space and analysing them in momentum space. These findings form one of the main results of our paper. Using a Holstein–Primakoff (HP) transformation [69, 70], we determine the dynamically generated quantum optical lattice potential in these SO phases and the corresponding atomic density distribution that shows the change in dimensionality as the angles made by two pumps with the cavity axis are varied in the range $(\theta_1, \theta_2) \in [0, \pi/2]$. We additionally show how such SO phases evolve as a function of the increasing intra-cavity photon number. Our proposed set-up enjoys the possibility of experimental realisation since it is intermediate between the already realised experimental set-up of a single cavity illuminated by a single transverse pump [5] and a crossed-cavity [23, 24] illuminated by a single pump [22, 25]. It can also be generalised to other cavity set-ups in addition to the linear one we have considered here.

To consolidate our theoretical analysis further, in the later part of this work, using the tight-binding approximation in the dynamical quantum optical lattice potential, we derive an extended Bose–Hubbard model (EBHM) Hamiltonian for our system (for a detailed review of various Hubbard models in ultra-cold atomic systems see [71]) in certain ranges of the cavity parameters in terms of the atomic field operators only from the microscopic many-body Hamiltonian for such an atom–photon system. The EBHM written in this form makes it easier to capture the cavity-mediated long-range interaction, which is responsible for the various self-organised lattice supersolid phases that we observed. We point out when such an EBHM is relatively more useful compared with the HP approximation for describing the super-radiant phase inside a cavity. Consequently, this allows us to compare our approach studying such SO phases with the EBHM derived for the classical optical lattice potential of ultra-cold atomic systems [72–75] that also predicts a transition from a density wave to a supersolid phase. We also evaluate the quantum optical lattice potential using this EBHM and compare it with the same obtained under HP transformation and the balanced pump condition (to be defined later).

Accordingly, we organise the rest of the paper in the following way. In section 2 we introduce the model microscopic Hamiltonian for our system and discuss the scattering states of the atoms. In section 3 we introduce the HP approximation to calculate the properties of the super-radiant phases and show the method of calculation within this approach in detail. In section 4 we discuss the main results of this paper under the HP approximation. In section 4.1 we provide the results for the
2. Model system and the Hamiltonian

We consider a linear cavity with a single cavity mode characterised by frequency $\omega_c$ and wave vector $k$, illuminated by two pump beams at frequency $\omega_p$ making angles $\theta_1$ and $\theta_2$ with the cavity axis. The cavity is loaded with a BEC with $N = 1.05 \times 10^5$ Rb atoms in the $|F, m_F\rangle = |1, -1\rangle$ state, where $F$ and $m_F$ are the total angular momentum and the corresponding magnetic quantum number. The cavity is detuned from the pump laser frequency by $\Delta_c = \omega_p - \omega_c$. $a(\hat{a}^\dagger)$ is the annihilation (creation) operator which annihilates (creates) a photon in the cavity mode with wave vector $k$. We have taken pump 1 to be along the $y$-direction and this choice gives

$$k = k\sin(\theta_1)x + k\cos(\theta_1)y,$$

$$k_1 = k\hat{y}, k_2 = -k\sin(\theta_2 - \theta_1)x + k\cos(\theta_2 - \theta_1)y,$$

where $k = \frac{2\pi}{\lambda_p}$ with $\lambda_p$ being the pump wavelength.

The microscopic many-body Hamiltonian for the system is written as

$$\hat{H}_{MB} = (-\hbar\Delta_c\hat{a}^\dagger\hat{a}) + \int \int \hat{\Psi}^\dagger(x,y) \left( \frac{\hbar^2}{2M_a} \nabla^2 + V(r) \right) \hat{\Psi}(x,y) + \frac{\hbar g_0}{2} |\hat{\Psi}(x,y)|^2 - \mu_0 \hat{\Psi}(x,y)dxdy,$$

where

$$V(r) = \hbar g_1 \cos(k_1 \cdot r) \cos(k_2 \cdot r) (\hat{a}^\dagger + \hat{a}) + \hbar g_2 \cos(k_2 \cdot r) \cos(k_1 \cdot r) (\hat{a}^\dagger + \hat{a}) + \hbar U_0 \cos^2(k_1 \cdot r) \hat{a}^\dagger \hat{a} + \hbar U_0 \cos^2(k_2 \cdot r) + \hbar U_0 \cos(k_1 \cdot r) \cos(k_2 \cdot r),$$

and $\hat{\Psi}(x,y)$ ($\hat{\Psi}^\dagger(x,y)$) is the atomic field operator which annihilates (creates) a particle at position $(x,y)$. The number operator is $N = \int d\hat{r} \hat{\Psi}^\dagger(\hat{r}) \hat{\Psi}(\hat{r})$. $U_p = \Omega_p^2/\Delta_a$ is the potential depth of the transverse pump potentials formed by the two pumps where $\Omega_p$ is the maximum pump Rabi frequency. $g_0$ is the maximum atom–photon coupling strength and $U_0 = g_0^2/\Delta_a$ is the depth of the potential formed by the cavity field. $U_0$ denotes the maximum shift in the resonance frequency for a single intracavity photon. $\eta = \Omega_0 g_0/\Delta_a$ is the two-photon Rabi frequency for the cavity and signifies the strength of the interaction between the pumps and the cavity field. $g_2 = 4\pi a_0 \hbar^2 N/m_a V$ is the strength of the short-range $s$-wave collisions with scattering length $a_s$.

In subsequent discussion, using a HP transformation, we shall see the effect of these angular changes on the self-organised phases inside the cavity.

3. Holstein–Primakoff approach

We expand the atomic field operator in the momentum modes $\hat{\Psi}(x,y)$ as shown in equation (4). These processes couple the BEC zero-momentum state (in grey at coordinate $|0, 0\rangle$) to the eight momentum modes $|\pm k_1, \pm k_2\rangle$, where $k_1$ and $k_2$ are defined in equations (1) and (2). Explicit tabulation of these momentum states is given in the caption to figure 1.

To gain more insight into the effect of orientation of the two pumps, we additionally provide the momentum diagrams for a few angles in figure 2. The red arrow in the central circle in figure 2 shows the angle of pump 1 ($\theta_1$) and the blue arrow shows the angle of pump 2 ($\theta_2$) with the cavity axis. We keep $\theta_1$ constant and vary $\theta_2$. The details of the figure are explained in the captions. We start with case (1), $\theta_1 = \pi/6, \theta_2 = \pi/4$, where we show the eight separate momentum states along with the zero-momentum state (in grey at coordinate $|k_1, k_2\rangle = |0, 0\rangle$) as shown in figure 1(b). Then as $\theta_2$ increases to $\pi/2$ in case (2), we see that the blue momentum states rotate by the same amount as there is a change in the value of $\theta_2$, therefore transferring the rotation in real space to that in reciprocal space. In case (3), $\theta_1 = \pi/6, \theta_2 = \pi/2$, two blue momentum states coincide with the zero-momentum state. We define this coincidence as the degeneracy in momentum space. As expected these momentum states only depend on the angle between the two pumps, namely $(0 < \theta_2 - \theta_1 < \pi/2)$.
Figure 1. (a) Schematic for the single cavity–two pump system. The atomic condensate is trapped at the intersection of the two pump beams. Pump 1 makes an angle $\theta_1$ and pump 2 makes an angle $\theta_2$ with the cavity axis. Wave vectors $k_1$, $k_2$, $k_c$ and $\Omega_p$ are explained in the text. (b) The momentum diagram shows the nine momentum states in terms of $\theta_1$ and $\theta_2$ $|0, 0\rangle$, $A_1 = | -\hbar k \sin \theta_1, \hbar k (1 - \cos \theta_1)\rangle$, $A_2 = | \hbar k \sin \theta_1, \hbar k (1 + \cos \theta_1)\rangle$, $A_3 = | -\hbar k \sin \theta_1, \hbar k (-1 - \cos \theta_1)\rangle$, $A_4 = | -\hbar k \sin \theta_1, \hbar k (-1 + \cos \theta_1)\rangle$, $B_1 = | \hbar k (\sin \theta_1 - \sin(\theta_2 - \theta_1))\rangle$, $B_2 = | \hbar k (\sin \theta_1 - \sin(\theta_2 - \theta_1))\rangle$, $B_3 = | \hbar k (\cos \theta_1 + \cos(\theta_2 - \theta_1))\rangle$, $B_4 = | \hbar k (\cos \theta_1 - \cos(\theta_2 - \theta_1))\rangle$, $B_5 = | \hbar k (\cos \theta_1 - \cos(\theta_2 - \theta_1))\rangle$, $B_6 = | \hbar k (\cos \theta_1 + \cos(\theta_2 - \theta_1))\rangle$. The annotations of the colour, dotted and dashed lines are explained in the text in section 2.

Figure 2. The momentum diagrams for the two pump–cavity system for a few specific values of the two pump angles $\theta_1$ and $\theta_2$ which are indicated in each figure. In (3), three momentum states with same $|k_x, k_y\rangle$ coincide at the origin whereas in (4) all the four blue and red momentum states coincide. The annotations of the colour, dotted and dashed lines are same as in figure 1(b).
\[ \Psi(x, y) = \psi_0 \hat{c}_0 + \psi_1 \hat{c}_1 + \psi_1 \hat{c}^+_1 + \psi_2 \hat{c}_2 + \psi_2 \hat{c}^+_2, \]  

(5)

where \( \psi_0 = \sqrt{1/\Lambda} \) represents the BEC zero-momentum mode, \( \psi_{1\pm} = \sqrt{2/\Lambda} \cos [(k_z \pm k_\parallel) \cdot \mathbf{r}] \) and \( \psi_{2\pm} = \sqrt{2/\Lambda} \cos [(k_z \pm k_\parallel) \cdot \mathbf{r}] \) represent the atomic modes with momenta \( k_z \pm k_\parallel \) and \( k_z \pm k_\parallel \), respectively. \( \hat{c}_0, \hat{c}_1^\pm \) and \( \hat{c}_2^\pm \) respectively create an atom at \( \mathbf{r} \) and an excitation with energy \( \hbar \omega_0 \pm \hbar \lambda \) and \( \hbar \omega_0 \pm 2 \hbar \lambda \). We substitute the expansion of the atomic field operator in equation (3) to obtain the following effective many-body Hamiltonian:

\[
H_{MB} = -\hbar \Delta_x \hat{a}^\dagger \hat{a} + \sum_{i=\pm 1, \pm 2} \hbar \omega_i \hat{c}_i^\dagger \hat{c}_i + \frac{\hbar \lambda}{\sqrt{N}} (\hat{a}^\dagger + \hat{a}) \sum_{j=\pm 1, \pm 2} \left( \hat{c}_j^\dagger \hat{c}_0 + \hat{c}_0^\dagger \hat{c}_j \right),
\]

(6)

where \( \omega_{rec} \) is the frequency associated with the recoil energy, \( E_{rec} = \hbar \omega_{rec} \). \( \omega_{rec} = \hbar k^2 / 2 M_0, \omega_0 = 2(1 + \cos \theta_1) \omega_{rec}, \omega_{-1} = 2(1 - \cos \theta_1) \omega_{rec}, \omega_{2} = 2(1 + \cos \theta_2) \omega_{rec}, \Delta_c = \omega_0 - \omega_c = \omega_{-1} N \), \( \lambda = \frac{\sqrt{\Delta_c}}{\sqrt{N}} \) is the coupling parameter between the atom and the two pump modes. The atomic momentum states in figures 1(b) and 2 describe the scattering of a photon with momentum \( \hbar k \) from pump 1 (red) and pump 2 (blue) into the cavity mode. There are two ways to reach the excited momentum states \( \pm \hbar k \pm k_\parallel \) from the BEC zero-momentum state \( \mathbf{0} \). The dotted red (blue) lines show the absorption of the pump 1 (pump 2) photon accompanied by emission of a photon into the cavity, identified by the operator \( \hat{a}^\dagger \hat{c}_1^\dagger \hat{c}_0 (\hat{a}^\dagger \hat{c}_1^\dagger \hat{c}_0) \). The solid red (blue) lines show the absorption of a cavity photon accompanied by emission of the photon into pump 1 (pump 2), identified by the operator \( \hat{a}^\dagger \hat{c}_1 \hat{c}_0 (\hat{a}^\dagger \hat{c}_1 \hat{c}_0) \). The reverse processes are not shown in the diagram. These processes correspond to the operators \( \hat{a}^\dagger \hat{c}_1^\dagger \hat{c}_1 + (\hat{a}^\dagger \hat{c}_0^\dagger \hat{c}_0) \) and \( \hat{a}^\dagger \hat{c}_0^\dagger \hat{c}_1 + (\hat{a}^\dagger \hat{c}_1^\dagger \hat{c}_0) \).

To underscore the similarity of the Hamiltonian (6) with the prototype Dicke model, we use the generalized HP transformation [70]

\[
\begin{align*}
\hat{c}_p^\dagger \hat{c}_0^\dagger &= b_p^\dagger b_q^\dagger, \\
\hat{c}_p^\dagger \hat{c}_0 &= b_p^\dagger \Theta_0(N) b_q, \\
\hat{c}_p^\dagger \hat{c}_0^\dagger &= \Theta_0(N)' b_q^\dagger b_p, \\
\end{align*}
\]

(7)

where

\[
\Theta_0(N) = \sqrt{\frac{N - \sum_{p \neq 0} b_p^\dagger b_p}{N}}. 
\]

Here 0 is the reference state and \( p, q = \pm 1, \pm 2 \) are the non-zero momentum states of the system. \( b_p \) are the new HP bosonic operators and satisfy the commutation relation—\( [b_p, b_q^\dagger] = \delta_{p,q} \). The expectation value for these bosonic modes is \( \langle b_p^\dagger b_q^\dagger \rangle = N \). However, all of them are macroscopic in the limit \( N \to \infty \) in order to make the HP approximation valid. Substituting these expressions into the Hamiltonian (6), we get

\[
H_{MB} = -\hbar \Delta_x \hat{a}^\dagger \hat{a} + \sum_{p=\pm 1, \pm 2} \hbar \omega_p b_p^\dagger b_p \\
+ \frac{\hbar \lambda}{\sqrt{N}} (\hat{a}^\dagger + \hat{a}) \sum_{p=\pm 1, \pm 2} \left( b_p^\dagger \Theta_0(N) + \Theta_0(N)' b_p \right). 
\]

(9)

It may be noted that if the number of excited states \( p \) is 1 we can directly use a pseudospin-\( \frac{1}{2} \) representation of these bosonic operators and the last term of the Hamiltonian indicates the coupling of a single bosonic mode with a large single spin 5. That is the prototype Dicke Hamiltonian [2, 3] which has been experimentally verified to show a super-radiant phase transition above a critical cavity–pump detuning [5]. Here, we present a generalisation of the HP approximation because of the existence of more than one excited state. In the subsequent discussion, we shall directly use the Hamiltonian (6).

It may be pointed out that an optical cavity is characterised by the Purcell factor \( \beta \) and the intra-cavity field decay rate \( \kappa \). For \( \beta > 1 \), scattering into modes not supported by the cavity is practically eliminated. The typical experimental situation that motivates our theoretical proposal (such as [5]) is carried out in the regime \( \beta > 1 \) and \( \kappa \gg \omega_{rec} \). This criterion allows kinetic energy transfer for the backscattering of two photons into different momentum modes that are supported by the cavity. As a result, in this case the HP approach works. An alternative regime where \( \kappa < \omega_{rec} \) was also explored in experiments [76].

The Dicke Hamiltonian can be exactly diagonalized in the thermodynamic limit (\( N \to \infty \)) using the HP approximation [69, 70]. It shows a continuous transition from a normal to a super-radiant phase in this limit as a function of the critical value of the atom–pump coupling \( \lambda \) defined in (18). Given the fact that our system contains a finite but large number of particles, application of this HP approximation provides a reasonable value at which such transition occurs as established by recent experiments [5, 22]. The HP approximation, however, breaks down when there is a superfluid to insulator type of transition and one needs a different method to study such a system [50]. We shall discuss this issue in more detail in section 6.

We now expand the atomic and cavity field operators using the HP transformation [69, 70]

\[
\hat{a} = \sqrt{N} \alpha + \delta \hat{a},
\]

(10a)

\[
\hat{c}_1^\pm = \sqrt{N} \Psi_1^\pm + \delta \hat{c}_1^\pm,
\]

(10b)

\[
\hat{c}_2^\pm = \sqrt{N} \Psi_2^\pm + \delta \hat{c}_2^\pm,
\]

(10c)

\[
\hat{c}_0 = \sqrt{N} - \hat{c}_1 - \hat{c}_1^+ - \hat{c}_2 - \hat{c}_2^+ + \delta \hat{c}_0,
\]

(10d)

where the first term in each expansion represents the ground state expectation value and the second term is the fluctuation. It may be pointed out that \( \psi_{0,1,2,3} \) are the wave functions...
for the momentum modes $|0, k, \pm k_{1,2}\rangle$ whereas $\Psi_{0,1,2,3}$ are the mean field values of $\tilde{c}_{0,1,2,3}$. Inserting expressions (10a)–(10d) into equation (6), the many-body Hamiltonian can be split into three parts and is written as

$$\hat{H}_{MB} = \hat{H}_{m=0}^{(0)} + \sqrt{\hat{H}_{m=0}^{(1)}} + \hat{H}_{m=0}^{(2)},$$

with each part scaling as $N^{(2-n)/2}$. In expression (11)

$$\hat{H}_{m=0}^{(0)} = -\hbar \Delta_{\alpha} \alpha^2 + h \omega_{1+} \Psi_{1+}^2 + h \omega_{2+} \Psi_{2+}^2 + h \omega_{1-} \Psi_{1-}^2 + h \omega_{2-} \Psi_{2-}^2 + 4 \hbar \omega_{1+} \Psi_{1+} \Psi_{1-} + 4 \hbar \omega_{2+} \Psi_{2+} \Psi_{2-}$$

and $\hat{H}_{m=0}^{(1)}$ and $\hat{H}_{m=0}^{(2)}$ are respectively linear and quadratic in fluctuations.

3.1 Ground state properties

The ground state energy is obtained from $\frac{\partial \hat{H}_{m=0}^{(0)}}{\partial \Psi_{1+}} = 0$, $\frac{\partial \hat{H}_{m=0}^{(0)}}{\partial \Psi_{1-}} = 0$ and $\frac{\partial \hat{H}_{m=0}^{(0)}}{\partial \Psi_{2+}} = 0$. For $\alpha$ we obtain:

$$\alpha = \frac{2 \lambda}{\Delta_{\alpha}} \Psi_{1+} \Psi_{1-} \Psi_{2+} \Psi_{2-}.$$ (13)

Substituting the value of $\alpha$ in $\hat{H}_{m=0}^{(0)}$ gives

$$\hat{H}_{m=0}^{(0)} = h \omega_{1+} \Psi_{1+}^2 + h \omega_{1-} \Psi_{1-}^2 + h \omega_{2+} \Psi_{2+}^2 + h \omega_{2-} \Psi_{2-}^2 + 4 \hbar \omega_{1+} \Psi_{1+} \Psi_{1-} + 4 \hbar \omega_{2+} \Psi_{2+} \Psi_{2-} + 8 \hbar \omega_{1+} \Psi_{1+} \Psi_{1-} \Psi_{2+} \Psi_{2-}.$$ (14)

Expression (14) is now extremized with respect to $\Psi_{1+}, \Psi_{1-}, \Psi_{2+}$ and $\Psi_{2-}$, which yields four equations (11a)–(11d) (for details see appendix A). These are solved simultaneously subject to the conditions

$$\Psi_{1+}^2 + \Psi_{1-}^2 + \Psi_{2+}^2 + \Psi_{2-}^2 = 1;$$ (15)

$$0 < \Psi_{1\pm} < 1; \quad 0 < \Psi_{2\pm} < 1;$$ (16)

to obtain the solutions for $\Psi_{1+}$, $\Psi_{1-}$, $\Psi_{2+}$ and $\Psi_{2-}$. Then these values are substituted in equation (13) to obtain $\alpha$ as a function of $\Delta_{\alpha}$. The critical detuning is given as (for a detailed derivation see appendix B)

$$\Delta_{\alpha} = -\frac{4 \lambda^2}{\omega_{1+}} - \frac{4 \lambda^2}{\omega_{1-}} - \frac{4 \lambda^2}{\omega_{2+}} - \frac{4 \lambda^2}{\omega_{2-}},$$ (17)

where $\omega_{1+} = \omega_{1-} = \omega_{2+} = \omega_{2-}$. In the presence of atom–atom interactions and considering the cavity decay rate $\kappa$, the critical detuning is modified as

$$\Delta_{\kappa} = -\frac{2 \lambda^2}{\omega_{10}} - \sqrt{-\frac{4 \lambda^4}{\omega_{10}^2} - \kappa^2} - \frac{2 \lambda^2}{\omega_{20}} - \sqrt{-\frac{4 \lambda^4}{\omega_{20}^2} - \kappa^2},$$ (18)

where $\omega_{10} = \sqrt{\frac{\hbar \lambda |\Psi_{1+}|^2}{2m} + \omega_{1}}$ and $\omega_{20} = \sqrt{\frac{\hbar \lambda |\Psi_{2+}|^2}{2m} + \omega_{2}}$.

In figure 3(b), we provide a plot of $\alpha$ values as a function of $\Delta_{\alpha}$ for different values of $\theta_{1}$ and $\theta_{2}$. As can be seen from figure 3(a), for $\theta_{1} = \theta_{2}$ the critical detuning, $\Delta_{\alpha}$, is maximum and then it decreases symmetrically from the maximum value as $|\theta_{1} - \theta_{2}|$ increases. For a fixed value of $\theta_{1} (\theta_{2})$, the critical detuning increases up to $\theta_{2}(\theta_{1}) = \pi/2$ and then decreases symmetrically up to $\theta_{2}(\theta_{1}) = \pi$. This happens because as $\theta_{2}$ increases, $\omega_{2}$ decreases, resulting in a decrease in $|\Delta_{\alpha}|$ (see equations (17) and (18)). Below $\Delta_{\alpha}$, the system is in the normal phase and $\Psi_{1+} = \Psi_{2+} = \Psi_{1-} = \Psi_{2-} = 0$; therefore, $\Psi_{0} = 1$ and $\alpha = 0$, which represents uniform atomic density. At $\Delta_{c} = \Delta_{\alpha}$, the system enters a self-organized super-solid phase, and $\Psi_{1+} = \Psi_{2+} = \Psi_{1-} = \Psi_{2-} \neq 0$, which results in $\Psi_{0} \neq 1$ and $\alpha \neq 0$. As $|\Delta_{\alpha}|$ decreases, $\alpha$ increases. From equations (11a)–(11d), we can see that $\Psi_{1\pm}$ and $\Psi_{2\pm}$ depend...
on $\omega_{1,\pm}$ and $\omega_{2,\pm}$, respectively. We observe that $\omega_{2,\pm}(\pi \pm \theta_2) = \omega_{2,\mp}(\theta_2)$. This results in $\Psi_{2,\pm} \leftrightarrow \Psi_{2,\mp}$. But this interchange in $\Psi_{2,\pm}$ and $\Psi_{2,\mp}$ does not affect the value of $\alpha$ in equation (13) because it depends on the total sum $\Psi_{1,\pm} + \Psi_{1,\mp} + \Psi_{2,\pm} + \Psi_{2,\mp}$. Therefore, for angles $\pi \pm \theta_2$, we obtain same values of $\alpha$ as pointed out in figure 3(b).

Following equation (13), at resonance $\alpha$ diverges as $\Delta_\alpha \rightarrow 0$. Accordingly, all the plots in figure 3(b) for different $\theta_{1,2}$ asymptotically approach the same curves when $\Delta_\alpha$ approaches 0. To ascertain the behaviour of the self-organised phases close to this resonance value, we have chosen arbitrarily a value of $\Delta_\alpha$ close to zero where corresponding values of $\alpha$ for various $\theta_{1,2}$ are practically same within the numerical precision of our computation. We call this $\Delta_\alpha$. $\Delta_\alpha$ is an intermediate value of the detuning that lies between $\Delta_\alpha$ for various $\theta_{1,2}$ and $\Delta_\alpha$. In the next section we shall discuss the self-organised atomic phases for these different values of the detuning $\Delta_\alpha$.

4. Results and discussion

Using the solutions of equations (A1a)–(A1d) along with equation (13) and equation (5), we can now evaluate the dynamical quantum optical lattice and the corresponding atomic density in the self-organised super-radiant phases for different values of $\theta_1$ and $\theta_2$ and a set of $\Delta_\alpha$.

4.1. Quantum optical lattice potential in super-radiant phases

To calculate the dynamical optical lattice potential, we replace $\hat{a}(\hat{a}^\dagger)$ in equation (4) by $\alpha(\alpha^*)$ and plot $V(r)$ as a function of $x$ and $y$. The resulting expression becomes

$$V(r) = \hbar \eta \cos(k_x \cdot r) \cos(k_y \cdot r)(\alpha + \alpha^*) + hU_0 \cos^2(k_x \cdot r)|\alpha|^2 + hU_p \cos^2(k_x \cdot r) + hU_p \cos(k_x \cdot r) \cos(k_y \cdot r) \cos(k_y \cdot r).$$

The optical lattice potentials are thus determined by the mean-field value of the photon creation and annihilation operators, which are determined from the coupled atom–photon dynamics inside the cavity. We define $\Delta_{\alpha,\pm}$ as a value just above $\Delta_{\alpha}$.

Under the HP approximation $\alpha = \alpha^*$, which we determine numerically from equation (13) and substitute in the expression (19) at this detuning $\Delta_{\alpha,\pm}$, and provide a few representative plots for fixed $\theta_1 = \pi/\theta_2$ and varying $\theta_2$ in figure 4. In figure 4(a) $\theta_2 = \pi/\theta_2$, and we obtain stripes of maxima (yellow) and minima (blue) forming a 1D potential along the $y$-direction. It may be noted that for $\theta_1 = \theta_2 = \pi/\theta_2$, we get $\Psi_{1,\pm} = \Psi_{2,\pm}$ and $\Psi_{1,\pm} = \Psi_{2,\mp}$, which is the solution of an effective single-pump-single-cavity arrangement which has an intensity $2U_p$ [77].

As we change $\theta_2 = \pi/\theta_2$ in figure 4(b), the positions of the minima start to change and they gradually start to form a 2D structure. We can understand this by following the top row of the dotted ellipse in these figures. For $\theta_2 = \pi/\theta_2$, the sites of the minima of the top row shift towards the left and form a rectangular potential with lattice vectors $\vec{P}$ and $\vec{Q}$. They are given as

$$\vec{P} = -\lambda_p \frac{\sin(\theta_2 - \theta_1)}{1 + \cos(\theta_2 - \theta_1)} \hat{x} + \frac{\lambda_p}{2} \hat{y},$$

$$\vec{Q} = \lambda_p \frac{\sin(\theta_2 - \theta_1)}{1 + \cos(\theta_2 - \theta_1)} \hat{x} + \frac{\lambda_p}{2} \hat{y}.$$ 

As we increase $\theta_2$ from $\pi/\theta_2$ to $2\pi/\theta_2$ in figure 4(e), the sites of the minima form a rhombic lattice structure. For $\theta_2 = \pi/\theta_2$ and at $\theta_2 = \pi$ in figures 4(c) and (f) one gets parallelogramatic lattices tilted in mutually opposite orientations. Exactly same lattice is obtained when $\theta_2$ is increased in multiples of $\pi$, which shows that there is a symmetry in the structure of the potential about the pump 1 ($y$)-axis. In the case of photon numbers, this symmetry exists about the axis perpendicular to the cavity axis. This is because for any change $\pm \Delta \theta$ from this axis we obtain the same cavity field amplitude.

In figure 5, we plot the optical lattice potential that we actually use in the calculation of atomic density in the self-organised phases under the HP approximation. The corresponding expression is just the $\alpha$-dependent part in expression (19) and hence gives us the dynamic part of the potential

$$V(r) = \hbar \eta \cos(k_x \cdot r) \cos(k_y \cdot r)(\alpha + \alpha^*) + hU \cos^2(k_x \cdot r)|\alpha|^2 + hU_p \cos(k_x \cdot r) \cos(k_y \cdot r).$$

The dotted lines in figures 5(a)–(i) show the minima of the potential and will correspond to the maxima of the atomic density. For each $\theta_2$ we plot the optical lattice potentials from top to bottom in increasing order of $\Delta_\alpha$. As expected, the potential is deepest (see the colour bar) in the lowest row, namely in figures 5(g)–(i). From left to right in each row with increasing $\theta_2$, the quantum optical lattice shows a transition from a 1D form to a 2D form. The corresponding density patterns will be discussed in more detail in section 4.2.

4.2. Self-organised atomic density

Compared with the normal phase (the condensate) where the atoms only populate the zero-momentum state in the super-radiant phase, other momentum states depicted in figure 2 become populated at different values of the pump angles $\theta_{1,2}$ leading to a phase transition. To plot the atomic densities in these new phases, we substitute the numerical solutions of equations (A1a)–(A1d), in equation (5). Then $A|\Psi|^2$ is plotted as a function of $x$ and $y$ and $\bar{\lambda}$ for fixed $\theta_1 = \pi/\theta_2$ and variable $\theta_2$ in figures 6(a)–(i). It may be noted that for $\Delta_\alpha < \Delta_{\alpha,r} = \alpha = 0$. For $\theta_1 = \pi/\theta_2$, $\alpha$ becomes non-zero for $\Delta_\alpha < \Delta_{\alpha,r}$, which results in the localisation of atoms (figures 6(a), (d) and (g)) at the minima of the optical lattice potential, showing a 1D variation given in figures 5(a), (d) and (g). For $\Delta_\alpha = \Delta_{\alpha,m}$ and $\Delta_\alpha = \Delta_{\alpha,r}$, this 1D localisation gets stronger due to increase in $\alpha$. As we increase $\theta_2$, the arrangement of the minima starts to deviate from this perfect one-dimensionality. Consequently the atoms start relocating them according to the new pattern of potential minima and form a periodic pattern that
is intermediate between a 1D and a 2D pattern. This can be seen in (figures 6(b), (c) and (h)). For \( \theta_2 = \frac{2\pi}{3} \), increasingly 2D arrangements of potential minima are available for occupation (see figures 5(c), (f) and (i)), and this gives rise to a prominent 2D variation of the atomic density in figures 6(c), (f) and (i) where the unit cell is identified inside each figure. Vertically downward in this column the atomic density increases as \( \alpha \) increases with increasing \( \Delta_c \), making the 2D structure more prominent. The appearance of such self-organised periodic modulation of the superfluid density above the critical detuning \( \Delta_c \) in a finite system is a hallmark of the lattice superfluid phase [22]. These figures thus show a clear dimensional cross-over in self-organized lattice superfluid phases in the super-radiant regime and represent the central result of this work. Because of the presence of several momentum components in expression (5), there are also secondary atomic density minima, some of which are marked with a red circle (see figure 6(i)). This self-organisation is an outcome of the cavity-mediated long-range interaction between the atoms. The explicit form of this long-range interaction appears clearly in the EBM derived under the self-consistent tight-binding approximation in this quantum optical lattice potential. In the next section, we will derive the EBM for this system and relate the BH parameters with the obtained \( \alpha \) values and the dynamical optical lattice potential obtained under HP transformation.

5. Classification of the super-radiant phases

The solutions of equations (A1a)–(A1d) for \( \Delta_s < \Delta_{cr} \), yield \( \psi_1 = \psi_2 = 0 \), which gives \( \psi_0 = 1 \) and the atomic density \( |\psi(x,y)|^2 = 1/4 \). This characterises a homogeneous superfluid phase (HSF) in the normal region. In this phase, the cavity photon number is zero, as shown in figure 3(b). For \( \Delta_s > \Delta_{cr} \),

\[
\psi_{1+} = \psi_{2+} = \psi_{1-} = \psi_{2-} \neq 0, 
\]

which gives \( \psi_0 \neq 1 \) and the system enters a super-radiant phase with the appearance of output cavity photons making \( \alpha \neq 0 \).

The atomic density can be obtained by substituting the mean field part of each operator from equations (10a)–(10d) in the expression of the atomic operator (5), namely:

\[
|\psi|^2 = |\psi_0|^2 + |\psi_1+|^2 + |\psi_1-|^2 + |\psi_2+|^2 + |\psi_2-|^2 = |\psi_0|^2 + |\psi_1+|^2 + |\psi_1-|^2 + |\psi_2+|^2 + |\psi_2-|^2 + 2\psi_0^*\psi_1^- + 2\psi_0\psi_1^+ + 2\psi_1^*\psi_2^- + 2\psi_1\psi_2^+ + 2\psi_2^*\psi_3^- + 2\psi_2\psi_3^+ + 2\psi_3^*\psi_4^- + 2\psi_3\psi_4^+ + 2\psi_4^*\psi_5^- + 2\psi_4\psi_5^+ + 2\psi_5^*\psi_6^- + 2\psi_5\psi_6^+ + 2\psi_6^*\psi_7^- + 2\psi_6\psi_7^+ + 2\psi_7^*\psi_8^- + 2\psi_7\psi_8^+ + 2\psi_8^*\psi_9^- + 2\psi_8\psi_9^+ + 2\psi_9^*\psi_{10}^- + 2\psi_9\psi_{10}^+ + 2\psi_{10}^*\psi_{11}^- + 2\psi_{10}\psi_{11}^+. 
\]

The first five terms of the resulting expression are proportional to the single-mode density \( |\psi_{ij}|^2 \) with \( i = 0, 1, 2 \). We refer to them as the self-terms. The other terms contain the overlap of such single-mode superfluid order parameters at a specific \( k \), and are proportional to \( \psi_{ij} \psi_{ij}^* \) with \( i \neq j \) and \( i,j = 0, 1, 2 \). We refer to such terms as cross-terms. Our calculation shows that each type of term significantly contributes towards the formation of a self-organised lattice structure in the super-radiant regime. We provide explicitly the contribution of the...
Figure 5. (a)–(i) Dynamical optical lattice potential for a fixed value of $\theta_1$ and varying $\theta_2$ (indicated on the top of each figure) for different values of $\Delta_c$ (indicated on the left) according to expression (20). As $\Delta_c$ increases, the depth of the potential increases but the structure of the potential remains the same. The dashed black line indicates the locations of potential minima. See details in section 4.1.

We use the maxima and minima of these atomic density plots to identify the resulting self-organised lattice structures in the super-radiant regime. To that end we first set $F(x, y) = |\Psi(x, y)|^2$. The extrema of the atomic density can be obtained from the conditions

$$F_x = \frac{\partial F}{\partial x} = 0, \quad F_y = \frac{\partial F}{\partial y} = 0.$$ 

This gives us two equations with two unknown variables, $x$ and $y$, and their solution will give us the extrema of the atomic density. Maxima and minima of $F(x,y)$ can be determined from

$$F_{xx} = \frac{\partial^2 |\Psi(x,y)|^2}{\partial x^2}, \quad F_{yy} = \frac{\partial^2 |\Psi(x,y)|^2}{\partial y^2}, \quad F_{xy} = \frac{\partial^2 |\Psi(x,y)|^2}{\partial x\partial y}.$$ 

For $F_{xx}F_{yy} - F_{xy} > 0$, the solution $F(x,y)$ can be a maximum or a minimum point. Then, if $F_{xx} < 0$ and $F_{yy} < 0$, $(x,y)$ is a maximum point, and if $F_{xx} > 0$ and $F_{yy} > 0$, $(x,y)$ is a minimum point. If $F_{xx}F_{yy} - F_{xy} < 0$, $(x,y)$ is a saddle point. In general such points have to be obtained through numerical computation, but for some specific combinations of angles we can determine the maxima and minima from an analytically solvable equations. For example for $\theta_1 = \theta_2$, where the two-pump condition degenerates into a single-pump case the $x$ and $y$ coordinates for the maxima and minima are

$$x_{\text{max}} = \frac{\lambda_p((n + m) - (n - m)\cos \theta_1)}{2\sin \theta_1},$$

$$x_{\text{min}} = \frac{(n - m)\lambda_p}{2\sin \theta_1},$$

$$y_{\text{max}} = \frac{(n - m)\lambda_p}{2},$$

$$y_{\text{min}} = \frac{\lambda_p((n + m + 1) - (n - m)\cos \theta_1)}{2\sin \theta_1}.$$ 

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Details of these calculations are provided in appendix C.

5.1. Momentum diagram from the atomic density

Self-organisation in super-radiant regime is manifested by the sudden build-up of the cavity field accompanied by the
formation of the momentum peaks in the absorption image of
the atomic cloud after its sudden release [5]. An idea about
these momentum peaks can be obtained by taking the Four-
ier transform (FT) of the atomic density that was analysed
in section 5. We take the FT of equation (21)

\[ \mathcal{F}(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i k \cdot r} |\Psi(x, y)|^2 \, dr. \]  

(22)

For the self-terms in the expansion (21) it gives

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i k \cdot r} |\psi_j \pm \Psi_j \pm |^2 \, dr = \delta(k_x - K_{jx \pm}) \delta(k_y - K_{jy \pm}), \]  

(23)

where \( j = 0, \pm 1, \pm 2 \), \( K_{jx \pm} \) is the \( x \)-component of \( k_x \pm \bar{k}_y \) and \( K_{jy \pm} \) is the \( y \)-component of \( \bar{k}_x \pm k_y \). Similar expressions can also be obtained with the cross-terms.

In figure 8, we show the FTs of the atomic densities in the
\( k_x \) and \( k_y \) planes for representative values of \( \theta_1 \) and \( \theta_2 \) along with the corresponding momentum scattering diagram and the real space lattice structure of the super-radiant phases. The top row, namely figures 8(a)–(c), plots the momentum-scattering diagram, whereas the bottom row, figures 8(g)–(i), shows the FT of the atomic density and their peaks. The middle row, that is figures 8(d)–(f), depicts the maxima in the real space density by which one can identify the self-organised lattice structure. The central peak in the momentum distribution at \( k_x = k_y = 0 \) is scaled by a factor of 30 to show it alongside the rest of the momentum peaks which appear due to the interference of other momenta values and are not captured in figures 8(a)–(c).

The peak locations in the \( k_x, k_y \) plane corresponding to the self-terms for all three combinations of \( \theta_1, \theta_2 \) are presented in a tabular form in table 1. These momentum peaks are linked with lattice spacing in the 1D and 2D lattices obtained in figures 6 and 7 by the formula \( k = 2\pi/\lambda_p \). To show that, we

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**Figure 6.** (a)–(i) The atomic density (plotted along the colour axis) using the HP approximation as a function of \( x \) and \( y \) for \( \Delta_c, \Delta_m \) and \( \Delta_s \). The sequence of these figures as well as other details are same as in figure 5. The red circles in (i) indicate the location of secondary density maxima in the 2D lattice–supersolid structure and due to the presence of more than one non-zero component of momentum \( k \) in the superfluid density.
find out the lattice spacing of the 1D lattice in figure 6(a). The lattice spacing for this case is \( \frac{\lambda}{2} = \frac{\lambda_{\text{xy}}}{2} \), which corresponds to a momentum wave vector \( \approx 2k \). It can now be checked that the corresponding spacing between the momentum peaks in table 1 (for \( \theta_1 = \pi / 6, \theta_2 = \pi / 6 \)) is \( 2k \). Similarly, the spacing between the peaks at \( \theta_1, \theta_2 = \pi / 2, \pi / 3 \) and \( \theta_1, \theta_2 = 2\pi / 3, \pi / 2 \) is also \( 2k \). This demonstrates the relation between the real space lattice and the momentum space diagram. However, in general, the presence of multiple momentum peaks leads to a more complex lattice structure whose shape has to be obtained numerically.

As can be seen in figure 8(d) for \( \theta_1 = \pi / 6, \theta_2 = \pi / 6 \), the density maxima having the same height are very closely spaced along the y-axis whereas they are well separated along the x-axis. We call such parallel tube-like high-density regions a 1D lattice supersolid. In figure 8(e), for \( \theta_1 = \pi / 6, \theta_2 = \pi / 3 \), some of the maxima shift from the 1D structure and show the intermediate stage of a 1D to 2D transition in the structure of the lattice supersolids. In figure 8(f), for \( \theta_1 = \pi / 6, \theta_2 = 2\pi / 3 \), the atomic density maxima are separated almost equally along the x- and y-axes, giving a 2D supersolid. As can be seen from the lower panel, the corresponding momentum peaks also change.

6. Bose–Hubbard Model for a two-pump system

Since a dynamical optical lattice is formed inside the cavity, following standard procedure we expand the atomic field operator using the site-localized Wannier functions [13, 78, 79] as

\[
\tilde{\Psi}(x,y) = \sum_{p,q} \tilde{b}_{p,q} w_{p,q}(x,y),
\]

where \( \tilde{b}_{p,q} (\tilde{b}_{p,q}^\dagger) \) annihilates (creates) an atom at site \( (p,q) \) of the cavity and \( w_{p,q}(x,y) \) is the corresponding maximally localized wave function. It may be noted that these Wannier functions themselves are dynamic since they depend on \( \alpha \) [80–83].

Using the properties of these Wannier functions following standard procedure an effective BH Hamiltonian for the system can be derived as

\[
\hat{H}_{\text{BH}} = \left[ E_1 \hat{B}_x + E_2 \hat{B}_y - h (\Delta_1 - U_0 J_{00} \hat{N} - U_0 \delta \hat{a}^\dagger \hat{a} + \eta (\hat{a}^\dagger + \hat{a}) \right] \times \left( (\hat{J}_{11} + \hat{J}_{12}) \hat{B}_x + (\hat{J}_{11} + \hat{J}_{12}) \hat{B}_y + (\hat{J}_{01} + \hat{J}_{02}) \hat{N} \right) + \frac{U_0}{2} \sum_{p,q} \tilde{b}_{p,q} (\tilde{b}_{p,q}^\dagger - 1) - \mu_0 \hat{N}.
\]

It may be noted that the hopping amplitudes along the x- and y-directions \( E_1, E_2 \) and the on-site energies \( E_0 \) whose expressions were given in expression (21). The atomic density (plotted along the colour axis) is calculated using the HP approximation as a function of \( x \) and \( y \). The red dots show the maxima and the white dots show the minima of the atomic density. The pump angles corresponding to each figure is shown on the top of each figure.

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**Figure 7.** The self-interference and cross-interference terms of the atomic density for different cavity–pump detunings \( \Delta_x, \Delta_y \) and \( \Delta_z \), as given in expression (21). The atomic density (plotted along the colour axis) is calculated using the HP approximation as a function of \( x \) and \( y \). The red dots show the maxima and the white dots show the minima of the atomic density. The pump angles corresponding to each figure is shown on the top of each figure.
Figure 8. (a)–(c) The momentum scattering diagrams for three \((\theta_1, \theta_2)\) combinations. (d)–(f) The waterfall plot of the atomic density, which clearly shows the transition of a 1D lattice supersolid at \(\theta_1 = \pi/6, \theta_2 = \pi/6\) to a 2D lattice supersolid at \(\theta_1 = \pi/6, \theta_2 = 2\pi/3\). The points of maximum density (shown by purple dots) are well separated along the \(x\)-direction. The separation between the successive secondary maxima along the \(y\)-direction is negligible compared with the separation of the local minima along the \(y\)-direction. This forms 1D tubes of atoms extending along the \(y\)-direction. As we change the value of pump angle \(\theta_2\) to \(\pi/3\), we can see a separation of the maxima along the \(y\)-direction as well. The peak value of the maximum atomic density also shows a variation compared with the 1D case where all the maxima have the same peak value. This indicates the onset of the transition of the 1D lattice supersolid to a 2D supersolid. For \(\theta_2 = 2\pi/3\), the points of maximum atomic density have equal heights and a comparable separation along the \(x\)- and \(y\)-directions. Therefore, this arrangement can be identified as a 2D lattice supersolid. (g)–(i) Plots of the Fourier transform of the atomic density for \(\theta_1, \theta_2\). The additional peaks in these plots are due to the presence of secondary density maxima (shown in figure 6) which are a result of the cross-interference terms shown in figure 7.

Table 1. The position of the prominent peaks (other than the central peak at \(k_x = k_y = 0\)), which appear due to the self-terms in the momentum diagram for three combinations of angles \(\theta_1, \theta_2\) as given in figure 7 (a subset of the cases presented in figure 6).

| \(\theta_1\) | \(\theta_2\) | \((k_x, k_y)_{1+}\) | \((k_x, k_y)_{1-}\) | \((k_x, k_y)_{2+}\) | \((k_x, k_y)_{2-}\) |
|-------------|-------------|----------------|----------------|----------------|----------------|
| \(\pi/6\)   | \(\pi/6\)   | \(k/2, (1 + \sqrt{3}/2)k\) | \(k/2, (1 - \sqrt{3}/2)k\) | \(k/2, (1 + \sqrt{3}/2)k\) | \(k/2, (1 - \sqrt{3}/2)k\) |
| \(\pi/3\)   | \(\pi/3\)   | \(k/2, (1 + \sqrt{3}/2)k\) | \(k/2, (1 - \sqrt{3}/2)k\) | \(k/2, (1 + \sqrt{3}/2)k\) | \(k/2, (1 - \sqrt{3}/2)k\) |
| \(2\pi/3\)  | \(2\pi/3\)  | \(k/2, (1 + \sqrt{3}/2)k\) | \(k/2, (1 - \sqrt{3}/2)k\) | \(-k/2, \sqrt{3}k/2\) | \(3k/2, \sqrt{3}k/2\) |

are given in appendix E, are directly due to the transverse pumping. The other set of hopping and on-site interactions \((J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8, J_9, J_{10})\) are due to the photons scattered by the atoms and are respectively given in appendix E and \(U_i = g_{2D} \int dx dy |w_{p,q}(x, y)|^4\) is the on-site interaction strength between the atoms. \(B_i = \sum_{p,q} (b_{p,q} b_{p+1,q} + \bar{b}_{p,q} \bar{b}_{p+1,q})\) \(B_i = \sum_{p,q} (b_{p,q} b_{p+1,q} + \bar{b}_{p,q} \bar{b}_{p+1,q})\) represents long-range hopping along the \(x\)-and \(y\)-directions. Here \((p+1,q)\) refers to a site along the \(x\)-direction and \((p,q+1)\) refers to a site along the \(y\)-direction.

\[
\delta = J_1 \hat{B}_1 + J_2 \hat{B}_2, \\
\hat{n}_{p,q} = \bar{b}_{p,q} \bar{b}_{p,q}, \quad \hat{N} = \sum_{p,q} \hat{n}_{p,q}, \\
\hat{\rho}_0 = \rho_0 - E_0.
\]

For the parameters considered in this work, \(\kappa \gg \omega_{rec}\), which refers to the bad cavity limit. In this limit, the cavity decay rate, \(\kappa\), is the fastest time scale. \(\kappa = 1/\tau\), where \(\tau\) is the photon storage time or the total time the photon spends in the cavity. For large values of \(\kappa\), \(\tau\) is small and, as a result,
the photons do not stay inside the cavity for a long time [84]. Therefore, the cavity field reaches a steady state well before the atoms. This allows us to adiabatically eliminate the cavity field dynamics by setting $\hbar \dd 
abla = 0$ and obtain the extended BH Hamiltonian in terms of the atomic operators only. The corresponding expression for $\hat{a}$ is [21]

$$\hat{a} = \frac{\eta (J_{11} + J_{12}) \hat{B}_x + \eta (J_{11} + \bar{J}_{12}) \hat{B}_y + \eta (\bar{J}_{01} + \bar{J}_{02}) \hat{N}}{\Delta_c - U_0 \hat{J}_0 \hat{N} - U_0 \delta + i \kappa}.$$  

(26)

For a fixed number of atoms, $N = \langle \hat{N} \rangle$, we expand $\hat{a}$ in tunneling matrix elements, $J_x$ and $J_y$, as follows:

$$\hat{a} = \frac{\eta (J_{11} + \bar{J}_{12}) \hat{B}_x + \eta (J_{11} + \bar{J}_{12}) \hat{B}_y + \eta (\bar{J}_{01} + \bar{J}_{02}) N}{\Delta_c - U_0 \hat{J}_0 \hat{N} + i \kappa} \times \left( 1 + \frac{U_0 (J_{11} \hat{B}_x + J_{12} \hat{B}_y)}{\Delta_c + i \kappa} + \ldots \right).$$  

(27)

and retain only up to the first-order terms in the expansion. This truncated steady-state solution of equation (27) is substituted in equation (25) and we retain terms of second order in $\hat{B}_x$ and $\hat{B}_y$ to get the effective EBHM Hamiltonian as

$$\hat{H}_{BH} = \frac{U_2}{2} \sum_{p,q} \hat{a}_{p,q} (\hat{n}_{p,q} - 1) - \mu_0 \hat{N} + \hat{E}_r \hat{B}_x + \hat{E}_s \hat{B}_y + U_0 \hat{B}_x^2 + U_0 \hat{B}_y^2 + \hat{O}(3).$$  

(28)

We provide a comparison of the above-obtained EBHM with other EBHM models studied in [21, 46, 48, 80] in appendix D. The detailed expressions for the parameters that appear in the EBHM Hamiltonian in (28) are given in appendix E. It is possible to determine the quantum phases (ground state) associated with such an effective EBHM Hamiltonian using sophisticated numerical techniques such as dynamical mean-field theory, and for a single-pump system such work has been done to obtain strongly correlated lattice super-solid phases in such systems [83]. Extending such an exercise for the current EBHM Hamiltonian (28) in our two-pump model is computationally demanding and is outside the scope of the current work. Hopefully this can be explored in future investigations. Before discussing the EBHM further, in the following paragraphs we shall provide a brief discussion about when and why such a BH approximation is useful to describe such a cavity-based ultra-cold system compared with the HP approximation described in section 3. It is well known that a BH Hamiltonian was successfully used to describe the superfluid to Mott insulator transition in ultra-cold atomic systems [13] in a classical optical lattice. In such systems, the phase fluctuations, $(\Delta \phi)$, and the number of fluctuations, $\Delta N$, follow the uncertainty relation [85, 86]

$$\Delta N \Delta \phi = 1.$$  

(29)

The tunnelling amplitude $J$ and the on-site interaction strength $U_{int}$ can be given by the following well-known analytical formula valid for a classical optical lattice potential with depth $V_0$ much greater than the recoil energy $E_{rec}$ ($V_0 \gg E_{rec}$) [87]:

$$J = \frac{4}{\sqrt{\pi}} \frac{V_0}{E_{rec}} \exp \left( -2 \sqrt{\frac{V_0}{E_{rec}}} \right),$$

$$U_{int} = \sqrt{\frac{8}{\pi}} k a_{rec} \left( \frac{V_0}{E_{rec}} \right)^{3/4}.$$  

(30)

With the increase in $V_0$, the tunnelling of atoms between the minima of the optical lattice decreases. This in turn decreases the particle number fluctuations, resulting in an increase in the phase fluctuations as seen in the uncertainty relation (29). This leads to the loss of phase coherence and the eventual emergence of a phase-incoherent Mott insulator phase in place of a phase-coherent superfluid phase.

In the system under consideration, instead of a classical optical lattice, a dynamic quantum optical lattice is formed inside a cavity. Nevertheless, in figure 9(a) we have used the expressions defined in (30) to evaluate the analogue quantities of BH parameters for such a dynamic quantum optical lattice potential. To that end, we consider $V_0$ to be the depth of the optical lattice potential in equation (19), and have used the later expressions $\alpha (\alpha')$ obtained from (31). For the system under consideration in this work $E_{rec}$ is defined in section 3 below equation (6). Using this value, we see a crossing of $J$ and $U_{int}$ at $\Delta_p = -3.35$ MHz. Beyond this cavity-pump detuning, the analogue of on-site interactions in the prototype BH model, calculated with the help of quantities defined for the current cavity–atom system, starts to dominate the corresponding tunnelling between adjacent wells.

These results, plotted in figure 9(a), are now compared with the similar quantities that appear in the EBHM Hamiltonian (28) derived for the current system under consideration. These quantities are plotted in figure 9(b). In the system under consideration, the dynamical quantum optical lattice potential gets deeper with increase in the output photon number $|\alpha|^2$. For the EBHM obtained in equation (28), the equivalent of $J$ defined for prototype BH model (30) is the tunnelling strengths $(E_r, E_s)$ which are defined in equations (E1) and (E2) in appendix E. They become comparable to the corresponding $U_j$ near $\Delta_c \sim \Delta_{p1,p2} = -3.37(-3.25)$ MHz, and beyond this point $U_j$ starts to dominate the tunnelling strengths, $(E_r, E_s)$, and eventually tunnelling will be completely prohibited near $\Delta_c = \Delta_p$.

It may be pointed out that the relation (30) is not rigorously valid for the EBHM defined in equation (28). Nevertheless, the good agreement between the values of the detuning parameters $\Delta_p$ and $\Delta_{p1,p2}$ evaluated in these two different ways acts as a consistency check on the EBHM parameters derived for the two-pump cavity–atom system under consideration. We can therefore conclude from the above discussion that the phase coherence between two neighbouring wells in the dynamical
quantum optical lattice potential is lost when the on-site interaction strength becomes comparable to the tunnelling strength \([85, 86]\), at approximately \(\Delta_x \approx 3.37(3.25)\) MHz, and the calculation from the HP approximation is less reliable beyond this point. The EBHM model on the other hand can describe such a phase-incoherent Mott insulator phase in the system, even though we are not demonstrating the same in the current work.

In figure 3(c) we plotted all the relevant BH parameters as a function of \(\Delta_c\). These parameters naturally become non-zero for \(\Delta_c = \Delta_{s}\). As \(\Delta_c\) increases, \(\alpha\) increases, resulting in an increase in the depth of the dynamical optical lattice potential. We already pointed out that this leads to an increase in on-site energy \(U\) and a simultaneous decrease in the tunneling parameters \(\tilde{E}_{s}\), as occurred in the prototype BH model in a classical optical lattice potential \([13]\). The coefficient of the cavity-mediated long-range interaction term in the EBHM Hamiltonian \((28)\), \(U_{\alpha,\beta,\gamma,\delta}\), that appears before \(B_{\alpha}^2, B_{\beta}^2\) and \(B_{\gamma}B_{\delta}\) respectively increases in absolute magnitude with increasing \(\alpha\), implying stronger cavity-mediated long-range interaction with more scattered photons, but these coefficients carry a negative sign as opposed to the positive sign of the on-site interaction \(U\). The presence of competing interaction terms in the EBHM Hamiltonian \((28)\) with different signs explains the emergence of the lattice super-solid phase in such a system. This EBHM-based description, therefore, allows us to compare and contrast these systems with cavity-mediated long-range interactions with a number of other continuum systems with competing long- and short-range interactions that were simultaneously investigated for such supersolid phases, such as Rydberg atomic condensates \([88, 89]\) and dipolar bosonic quantum gases \([90–92]\).

6.1. Quantum optical lattice potential from the Bose–Hubbard formalism

In the HP approach discussed in section 3, the atomic field operator \(\hat{\Psi}(x, y)\) is expanded in five different modes in equation \((5)\). Whereas in the tight-binding approach that was used to derive the EBHM Hamiltonian \((28)\), the atomic operator is expanded in terms of tightly bound Wannier orbitals \((24)\), each of which is a superposition of Bloch waves of all allowed momentum values. Thus these two approaches use different Hilbert spaces for the expansion of the atomic field operators and a rigorous quantitative comparison between the results obtained from these two approaches is difficult. In some limiting cases we can, however, make some conclusions based on the quantum optical lattice potentials evaluated using both these approaches and comparing them with the one evaluated under the balanced pump condition. We shall do this with the help of figure 10. To evaluate the quantum optical lattice potential from the EBHM we note that since \(\tilde{J}_{x}, \tilde{J}_{y}, \tilde{J}_{s1}, \tilde{J}_{s2}\) and \(\tilde{J}_{y1}\) are small compared with \(\tilde{J}_{01}, \tilde{J}_{02}\) from equation \((27)\) \(\alpha = \langle \hat{a} \rangle\) can be well approximated by

\[
\alpha = \frac{\eta (\tilde{J}_{01} + \tilde{J}_{02}) N}{\Delta_{x} + i\kappa},
\]

\[
\alpha + \alpha^* = \frac{2\Delta_{c} \eta (\tilde{J}_{01} + \tilde{J}_{02}) N}{\Delta_{x}^2 + \kappa^2}. \tag{31}
\]

In the first row, namely figures 10(a)–(c), we provide some representative plots of the optical lattice under the balanced pump condition, where \(\alpha\) is not determined dynamically. Ideally, the balanced pump condition is obtained when \(\alpha \to \infty\). Under such a balanced pump condition, we set...
Figure 10. (a)–(c) Optical lattice potential as a function of $x$ and $y$ for balanced pump conditions defined in section 4.1. (d)–(l) The evaluation of the dynamical optical lattice potential using EBHM and the method detailed in section 6.1. For these figures the detuning and $\alpha$ increases in each column downward. In each row from left to right $\theta_2$ increases. The $x$-axis remains same for all rows. Details are discussed in the text.

$U_0|\alpha|^2 = U_p$ and substitute this in expression (19). In the subsequent plots in figure 10, namely figures 10(d)–(l), we substitute $|\alpha|^2$ and $\alpha + \alpha^*$ calculated with equation (31) in expression (19) to obtain the quantum optical lattice potentials in EBHM.

A comparison of figures 10(a)–(c) with the quantum optical lattice potential of figures 10(j)–(l) corresponding to EBHM shows that the later potentials are approaching the balanced pump condition as $\Delta_c$ is increasing, namely when it approaches the cavity–pump resonance condition. The values of $\alpha$ obtained in the EBHM consider only the leading-order terms and neglects the long-range interactions, but they still show significant agreement with the potential obtained using the balanced pump condition. However, for other values of $\Delta_c$ in the super-radiant regime, the profiles of the dynamical optical lattice potential obtained from HP approximation and plotted in figures 5(a)–(f) is considerably different from that obtained under similar conditions using the BH approximation and plotted in figure 10. It may be pointed out here that in the system considered, the super-radiance phenomenon is defined as the collective emission of a light field by a group of $N$ atoms when they interact with a common light field [1] and emit light with an intensity proportional to $N^2$ [69]. The expression for $\alpha$ in (31) and the presence of the $\alpha^2$ term in the expression for the dynamical optical potential (19) shows rigorously why the phases considered here beyond the critical detuning are called super-radiant phases.

7. Conclusions

We have theoretically demonstrated a dimensional cross-over in the self-organised lattice supersolid phases formed inside a linear cavity as a function of the relative angles between two classical pumps within a HP approach. We provided detailed classification of these self-organised phases by analysing their structures in coordinate and momentum spaces. The corresponding quantum optical potential that is responsible for
such super-radiant phases due to self-organisation is plotted along with the corresponding atomic density and the relation between these plots is explained. In the later part of this work, we derived an effective extended Bose–Hubbard Hamiltonian from the same microscopic Hamiltonian, and with the help of the Bose–Hubbard parameters explain how cavity-mediated long-range interaction is responsible for such super-solid phases. We also evaluated the dynamical optical lattice potential using the EBHM and compared it with the one obtained through the Holstein–Primakoff transformation and the one under the balanced pump condition. Our proposal of observing such dimensional cross-over in a single set-up will hopefully augment further studies in this direction. The collective excitations accompanying such studies are another associated problem that can be looked at in the future.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Equations for mean field values $\Psi_{1\pm,2\pm}$ in section 3

$$\omega_1 \Psi_{1+} - \frac{2\lambda^2}{\Delta_c} \left( \Psi_{1+} (\Psi_{1+} + \Psi_{1-}) - \Psi_0^2 \right) (\Psi_{1+} + \Psi_{1-})$$

$$- \frac{2\lambda^2}{\Delta_c} \Psi_{1+} (\Psi_{1+} + \Psi_{1-})^2 - \frac{2\lambda^2}{\Delta_c} \left( 2\Psi_{1+} (\Psi_{1+} + \Psi_{1-}) - \Psi_0^2 \right)$$

$$\times (\Psi_{1+} + \Psi_{1-}) = 0, \quad (A1a)$$

$$\omega_2 \Psi_{2+} - \frac{2\lambda^2}{\Delta_c} \left( \Psi_{2+} (\Psi_{2+} + \Psi_{2-}) - \Psi_0^2 \right) (\Psi_{2+} + \Psi_{2-})$$

$$- \frac{2\lambda^2}{\Delta_c} \Psi_{2+} (\Psi_{1+} + \Psi_{1-})^2 - \frac{2\lambda^2}{\Delta_c} \left( 2\Psi_{2+} (\Psi_{2+} + \Psi_{2-}) - \Psi_0^2 \right)$$

$$\times (\Psi_{1+} + \Psi_{1-}) = 0, \quad (A1b)$$

$$\omega_1 \Psi_{1-} - \frac{2\lambda^2}{\Delta_c} \left( \Psi_{1-} (\Psi_{1+} + \Psi_{1-}) - \Psi_0^2 \right) (\Psi_{1+} + \Psi_{1-})$$

$$- \frac{2\lambda^2}{\Delta_c} \Psi_{1-} (\Psi_{2+} + \Psi_{2-})^2 - \frac{2\lambda^2}{\Delta_c} \left( 2\Psi_{1-} (\Psi_{1+} + \Psi_{1-}) - \Psi_0^2 \right)$$

$$\times (\Psi_{2+} + \Psi_{2-}) = 0, \quad (A1c)$$

$$\omega_2 \Psi_{2-} - \frac{2\lambda^2}{\Delta_c} \left( \Psi_{2-} (\Psi_{2+} + \Psi_{2-}) - \Psi_0^2 \right) (\Psi_{2+} + \Psi_{2-})$$

$$- \frac{2\lambda^2}{\Delta_c} \Psi_{2-} (\Psi_{1+} + \Psi_{1-})^2 - \frac{2\lambda^2}{\Delta_c} \left( 2\Psi_{2-} (\Psi_{2+} + \Psi_{2-}) - \Psi_0^2 \right)$$

$$\times (\Psi_{1+} + \Psi_{1-}) = 0. \quad (A1d)$$

Appendix B. Derivation for critical detuning in section 3

In this section we provide a derivation for equations (17) and (18). We first find out the Hessian matrix for the Hamiltonian in equation (14) which is given as [70]

$$\begin{bmatrix}
\frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+}^2} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+} \partial \Psi_{1-}} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+} \partial \Psi_{2+}} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+} \partial \Psi_{2-}} \\
\frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+} \partial \Psi_{1-}} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+}^2} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+} \partial \Psi_{2+}} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+} \partial \Psi_{2-}} \\
\frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+} \partial \Psi_{2+}} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+} \partial \Psi_{2-}} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{2+}^2} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{2+} \partial \Psi_{2-}} \\
\frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+} \partial \Psi_{2-}} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{1+} \partial \Psi_{2-}} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{2+} \partial \Psi_{2-}} & \frac{\partial^2 h_{m=0}(0)}{\partial \Psi_{2+}^2}
\end{bmatrix}$$

In the normal phase $\Psi_{1+} = \Psi_{2+} = \Psi_{1-} = \Psi_{2-} = 0$, therefore $\Psi_0 = 1$. The Hessian takes the following form

$$\begin{bmatrix}
\frac{2\omega_1^2 + 8\lambda^2}{\Delta_c} & \frac{8\lambda^2}{\Delta_c} & \frac{8\lambda^2}{\Delta_c} & \frac{8\lambda^2}{\Delta_c} \\
\frac{8\lambda^2}{\Delta_c} & \frac{2\omega_1^2 + 8\lambda^2}{\Delta_c} & \frac{8\lambda^2}{\Delta_c} & \frac{8\lambda^2}{\Delta_c} \\
\frac{8\lambda^2}{\Delta_c} & \frac{8\lambda^2}{\Delta_c} & \frac{2\omega_2^2 + 8\lambda^2}{\Delta_c} & \frac{2\omega_2^2 + 8\lambda^2}{\Delta_c} \\
\frac{8\lambda^2}{\Delta_c} & \frac{8\lambda^2}{\Delta_c} & \frac{2\omega_2^2 + 8\lambda^2}{\Delta_c} & \frac{2\omega_2^2 + 8\lambda^2}{\Delta_c}
\end{bmatrix}$$

At critical detuning, $\Delta_c$, $\Psi_{1+} = \Psi_{2+} = \Psi_{1-} = \Psi_{2-} \neq 0$, therefore $\Psi_0 \neq 1$. Therefore, at this point, the determinant of the Hessian gives us the critical detuning

$$\Delta_c = -\frac{4}{\omega_1^2} \lambda^2 - \frac{4}{\omega_2^2} \lambda^2, \quad (B1)$$

where

$$\omega_1 = \frac{\omega_{1+} \omega_{1-}}{\omega_{1+} + \omega_{1-}}, \quad \omega_2 = \frac{\omega_{2+} \omega_{2-}}{\omega_{2+} + \omega_{2-}}.$$

In presence of atom–atom interactions, $g_{2D}$ and the cavity decay rate, $\kappa$, the critical detuning becomes modified and is given as

$$\Delta_c = -\frac{2\lambda^2}{\omega_0} - \sqrt{-\frac{4\lambda^4}{\omega_0^2} - \kappa^2 - \frac{2\lambda^2}{\omega_0^2} - \sqrt{-\frac{4\lambda^4}{\omega_0^2} - \kappa^2}}.$$
Appendix C. Derivation of the maxima and minima in the atomic density in section 5

In this section we shall describe the analytical technique for obtaining the maxima and minima of the atomic density for a special combination of angles \( \theta_1 \) and \( \theta_2 \). In general such maxima and minima have to be obtained numerically

\[
\frac{\partial |\Psi(x,y)|^2}{\partial x} = 0
\]

\[
\Rightarrow |\Psi_{1+}(x,y)|^2 = 0
\]

\[
|\Psi_{2+}(x,y)|^2 = |\Psi_{1-}(x,y)|^2 = |\Psi_{2-}(x,y)|^2 = 0,
\]

(C1)

\[
\frac{\partial |\Psi(x,y)|^2}{\partial y} = 0
\]

\[
\Rightarrow |\Psi_{1+}(x,y)|^2 = 0
\]

\[
|\Psi_{2+}(x,y)|^2 = |\Psi_{1-}(x,y)|^2 = |\Psi_{2-}(x,y)|^2 = 0,
\]

(C2)

For \( \theta_2 - \theta_1 = 0 \), the corresponding maxima and minima in the atomic density plot can be obtained analytically. To demonstrate that we substitute \( \theta_1 = \theta_2 \) in equations (C1) and (C2) to get

\[
\frac{\partial}{\partial x}|\Psi(x,y)|^2 = \Psi_{1+}(x,y) = \Psi_{2+}(x,y) = \Psi_{1-}(x,y) = \Psi_{2-}(x,y) = 0,
\]

(C3)

\[
\frac{\partial}{\partial y}|\Psi(x,y)|^2 = \Psi_{1+}(x,y) = \Psi_{2+}(x,y) = \Psi_{1-}(x,y) = \Psi_{2-}(x,y) = 0,
\]

(C4)

Since \( \Psi_{1+} \) and \( \Psi_{2+} \) are independent non-zero momentum components of the superfluid order parameter, the solutions of the above equation can be obtained from

\[
kx \sin \theta_1 + ky \cos \theta_1 = m \pi \cos \theta_1, \quad n \pi \cos \theta_1,
\]

whose solution gives us the coordinates of \( x \) and \( y \) where the densities are extrema, namely

\[
x = \frac{\lambda_p}{4}(n + m) - \frac{\lambda_p}{4}(n - m) \cos \theta_1,
\]

\[
y = \frac{\lambda_p}{4}(n - m) \sin \theta_1.
\]

To find out the maxima and minima we need to evaluate \( F_{xx} \) and \( F_{xy} \) at these points and obtain that \( F_{xx} < 0 \) and \( F_{xy} < 0 \) when \( n \) and \( m \) are even integers and \( F_{xx} > 0 \) and \( F_{xy} > 0 \) when \( n \) and \( m \) are odd integers.

Appendix D. Relation between our model and other BH models

A comparison of our BH model in equation (28) with the models considered in [21, 46, 48, 80] also reveals that the long-range interaction terms of our model are proportional to \( \hat{B}_{xy}^2 \) which is different from the models in [21, 46, 48, 80], where the global-range interactions favour particle imbalance between odd and even sites. The difference in the origin of the infinite range interactions in the two models is a consequence of the lattice geometry. The optical lattice potential in [21, 46, 48, 80] has equal depths along the \( x \)- and \( y \)-directions, which gives rise to a square lattice, while in our case we have different lattice depths along the \( x \)- and the \( y \)-directions and we get a distorted square lattice for \( \Delta_x > \Delta_m \).

Appendix E. Expressions for the Bose–Hubbard model in section 6

The hopping amplitudes along the \( x \)- and \( y \)-directions and on-site energy are given by

\[
E_x = \frac{1}{2} \int dx dy \left( w_{p,q}(x,y) \left( -\frac{\hbar^2}{2M_a} \nabla^2 + hU_p \cos^2(k_1 \cdot r) + hU_p \cos^2(k_2 \cdot r) + hU_p \cos(k_1 \cdot r) \cos(k_2 \cdot r) \right) + w_{p+1,q}(x,y) \right),
\]

\[
E_y = \frac{1}{2} \int dx dy \left( w_{p,q}(x,y) \left( -\frac{\hbar^2}{2M_a} \nabla^2 + hU_p \cos^2(k_1 \cdot r) + hU_p \cos^2(k_2 \cdot r) + hU_p \cos(k_1 \cdot r) \cos(k_2 \cdot r) \right) + w_{p,q+1}(x,y) \right),
\]

\[
E_0 = \int dx dy \left( w_{p,q}(x,y) \left( -\frac{\hbar^2}{2M_a} \nabla^2 + hU_p \cos^2(k_1 \cdot r) + hU_p \cos^2(k_2 \cdot r) + hU_p \cos(k_1 \cdot r) \cos(k_2 \cdot r) \right) + w_{p,q}(x,y) \right).
\]
The hopping and on-site interactions due to the photons scattered by the atoms are given as

\[ J_x = \frac{1}{2} \int dx dy w_{p,q}^*(x,y) \cos(k_x \cdot r) w_{p+1,q}(x,y), \]
\[ J_y = \frac{1}{2} \int dx dy w_{p,q}^*(x,y) \cos(k_y \cdot r) w_{p,q+1}(x,y), \]
\[ J_0 = \int dx dy |w_{p,q}(x,y)|^2 \cos(k_x \cdot r), \]
\[ J_{11} = \frac{1}{2} \int dx dy w_{p,q}^*(x,y) \cos(k_x \cdot r) \cos(k_x \cdot r) w_{p+1,q}(x,y), \]
\[ J_{12} = \frac{1}{2} \int dx dy w_{p,q}^*(x,y) \cos(k_x \cdot r) \cos(k_y \cdot r) w_{p,q+1}(x,y), \]
\[ J_{21} = \frac{1}{2} \int dx dy w_{p,q}^*(x,y) \cos(k_y \cdot r) \cos(k_x \cdot r) w_{p+1,q}(x,y), \]
\[ J_{22} = \frac{1}{2} \int dx dy w_{p,q}^*(x,y) \cos(k_y \cdot r) \cos(k_y \cdot r) w_{p,q+1}(x,y). \]

The detailed expressions for the Bose–Hubbard parameters that appear in the Hamiltonian (28) are listed below

\[ E_x = E_x + \frac{2\hbar \Delta \eta^2 (\tilde{J}_{11} + \tilde{J}_{12}) (\tilde{J}_{01} + \tilde{J}_{02}) N}{\Delta^2 + \kappa^2}, \]
\[ + \frac{2\hbar \eta^2 N U_{0,p} (\Delta^2 - \kappa^2) (\tilde{J}_{01} + \tilde{J}_{02})^2}{(\Delta^2 + \kappa^2)^2}, \] 
\[ E_y = E_y + \frac{2\hbar \Delta \eta^2 (\tilde{J}_{11} + \tilde{J}_{12}) (\tilde{J}_{01} + \tilde{J}_{02}) N}{\Delta^2 + \kappa^2}, \]
\[ + \frac{2\hbar \eta^2 N U_{0,p} (\Delta^2 - \kappa^2) (\tilde{J}_{01} + \tilde{J}_{02})^2}{(\Delta^2 + \kappa^2)^2}, \]
\[ U_{in} = \frac{\hbar \Delta \eta^2 (\tilde{J}_{11} + \tilde{J}_{12})^2}{\Delta^2 + \kappa^2}, \]
\[ + \frac{2\hbar \eta^2 N U_{0,p} (\Delta^2 - \kappa^2) (\tilde{J}_{01} + \tilde{J}_{02}) (\tilde{J}_{11} + \tilde{J}_{12})}{(\Delta^2 + \kappa^2)^2}, \]
\[ U_{in} = \frac{\hbar \Delta \eta^2 (\tilde{J}_{11} + \tilde{J}_{12})^2}{\Delta^2 + \kappa^2}, \]
\[ + \frac{2\hbar \eta^2 N U_{0,p} (\Delta^2 - \kappa^2) (\tilde{J}_{01} + \tilde{J}_{02}) (\tilde{J}_{11} + \tilde{J}_{12})}{(\Delta^2 + \kappa^2)^2}. \]

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