Non-Standard Asymptotics in High Dimensions: Manski’s Maximum Score Estimator Revisited

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Abstract: Manski’s celebrated maximum score estimator for the binary choice model has been the focus of much investigation in both the econometrics and statistics literatures, but its behavior under growing dimension scenarios still largely remains unknown. This paper seeks to address that gap. Two different cases are considered: $p$ grows with $n$ but at a slow rate, i.e. $p/n \to 0$; and $p \gg n$ (fast growth). By relating Manski’s score estimation to empirical risk minimization in a classification problem, we show that under a soft margin condition \cite{11}, \cite{23} involving a smoothness parameter $\alpha > 0$, the rate of the score estimator in the slow regime is essentially $((p/n) \log n)^{\alpha+2}$, while, in the fast regime, the $l_0$ penalized score estimator essentially attains the rate $(s_0 \log p \log n)^{1/3}$ in the slow and fast growth scenarios respectively, which can be viewed as high-dimensional analogues of cube-root asymptotics\cite{9}: indeed, this work is possibly the first study of a non-regular statistical problem in a high-dimensional framework. We also establish upper and lower bounds for the minimax $L_2$ error in the Manski’s model that differ by a logarithmic factor, and construct a minimax-optimal estimator in the setting $\alpha = 1$. Finally, we provide computational recipes for the maximum score estimator in growing dimensions that show promising results.

1. Introduction

The maximum score estimator was first introduced by Charles Manski in his seminal paper\cite{12} in connection with the stochastic utility model of choice, and has been extensively studied in both the econometrics and the statistics literatures. The binary choice model can be considered as a linear regression model with missing data. More specifically, let

$$Y_i^* = X_i' \beta^0 + \epsilon_i$$

where $\{X_i, \epsilon_i\}$ are i.i.d pairs, the distribution of $\epsilon_i$ is allowed to depend on $X_i$ and $\text{med}(Y_i^*|X_i) = X_i' \beta^0$ (i.e. $\text{med}(\epsilon_i|X_i) = 0$), but instead of observing the full data, we only see $\{Y_i, X_i\}$ where $Y_i = \text{sgn}(Y_i^*)$. The regression parameter $\beta^0$ is of interest. The population score function is defined as:

$$S(\beta) = \mathbb{E}(Y \text{sgn}(X' \beta)) = \mathbb{E}(\text{sgn}(Y^* \cdot X' \beta))$$

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and the corresponding sample score function is:

\[ S_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} Y_i \text{sgn}(X_i^T \beta) = \frac{1}{n} \sum_{i=1}^{n} \text{sgn}(Y_i^*(X_i^T \beta)). \]

The maximum score estimator is defined as any value of \( \beta \) that maximizes the sample/empirical score function:

\[ \hat{\beta}_n = \arg\max_{\beta, ||\beta||=1} S_n(\beta). \]

Note that some norm restriction on \( \beta \) is important both for identifiability of \( \beta \) in this model, as well as for meaningful optimization. As \( \beta^0 \) is only identifiable and estimable up to direction, in what follows, we take \( ||\beta_0|| = 1 \). We also note that the choice of the maximizer is not important; in fact there is no unique maximizer. In follow-up work [13], Manski proved the consistency of \( \hat{\beta}_n \) to the true \( \beta^0 \) and some large deviation results under mild assumptions. The asymptotic distribution properties of the maximum score estimator were established by Kim and Pollard [9] who proved that under additional assumptions

\[ (\beta^0 - \hat{\beta}_n) = O_p(n^{-\frac{1}{2}}), \]

and that the normalized difference converges in distribution to the maximizer of a quadratically drifted Gaussian process which has a non-Gaussian distribution. Shortly thereafter, Horowitz [7] established, under smoothness conditions beyond those in [9] that the estimator obtained by maximizing a kernel smoothed version of the score function can improve the rate of the smoothed estimator to nearly \( O_p(n^{-\frac{1}{2}}) \). One advantage of Horowitz’s estimator over the original maximum score estimator, from a practical viewpoint, is that the limit distribution in his setting is Gaussian and therefore more amenable to inference, while the quantiles of the non-Gaussian limit are hard to determine. More recently, Seo and Otsu [22][21] have extended the asymptotic results on the score estimator to dependent data scenarios. Alternatively, resampling techniques can also be used for inference. Manski and Thompson [14] suggested that the usual bootstrap yields a good approximation of the distribution of the maximum score estimator, but it turns out that the bootstrap is actually inconsistent, as shown in Abrevaya and Huang [2] (but see also [20]). More recently, a model–based smoothed bootstrap approach was proposed by Patra et.al. [16]. Generic \((m \text{ out of } n)\) subsampling techniques [17] can of course be used in principle, but typically suffer from imprecise coverage unless the subsample size \( m \) is well-chosen, which is typically a difficult problem.

**Connections to empirical risk minimization:** The maximum score estimator is naturally connected to a classification problem with two classes. In Manski’s problem, we have observations \( \{X_1, Y_1\}, \ldots, \{X_n, Y_n\} \), where \( X_i \in \mathbb{R}^p \) and \( Y_i \in \{-1, 1\} \), these being the labels of the two classes. The conditional class probabilities are specified by

\[ \eta(x) = \mathbb{P}(Y = 1|X = x) = 1 - F_{e|X=x}(-x^T \beta^0). \]

For classifying the \( Y_i \)’s using an arbitrary classifier \( h \) under 0-1 loss, the population risk is given by \( L(h) = \mathbb{P}(Y \neq h(X)) \). Consider the set of classifiers corresponding to all possible hyperplanes, i.e.

\[ G = \{g_\beta : g_\beta(x) = \text{sgn}(x^T \beta), ||\beta|| = 1\}. \]
The population risk under 0-1 loss for this family is then given by:

\[ L(\beta) = L(g_\beta) = P(Y \neq \text{sgn}(X^T\beta)), \]

and is consistently estimated by the empirical risk \( L_n(\beta) = \mathbb{P}_n(Y \neq \text{sgn}(X^T\beta)). \) From the structure of the model, it is easy to see that the Bayes’ classifier, i.e. the classifier which minimizes the population risk in this model (over all possible classifiers) is precisely \( g_{\beta_0} \) and therefore minimizes \( L(\beta): \)

\[
\text{sgn}(\eta(x) - 0.5) = \text{sgn}(1 - F_\epsilon|x=x(-x^T\beta_0) - 0.5) \\
= \text{sgn}(0.5 - F_\epsilon|x=x(-x^T\beta_0)) \\
= \text{sgn}(x^T\beta_0) = g_{\beta_0} \quad [\because \text{med}(\epsilon|X) = 0].
\]

Thus \( \hat{\beta}_n := \arg \min_{\beta} L_n(\beta) \) empirically estimates the Bayes classifier. By simple algebra \( S(\beta) = 1 - 2L(\beta) \) and \( S_n(\beta) = 1 - 2L_n(\beta). \) Since the former is maximized at \( \beta_0 \) and the latter at \( \hat{\beta}_n, \) it follows that \( \hat{\beta}_n \) is one particular choice for \( \hat{\beta}_n. \) Thus, the maximum score estimator is the minimizer of the empirical risk in this classification problem. The rate of estimation of \( \beta_0 \) depends on two crucial factors: (1) The manner in which \( P(Y = 1|X) \) changes across the hyperplane and (2) The distribution of \( X_i \)’s near the hyperplane. If the conditional probability shifts from \( 1/2 \) rather slowly as we move away from the hyperplane, we have a ‘fuzzier’ classification problem and estimation becomes more challenging. On the other hand, the distribution of the \( X_i \)’s governs the density of observed points around the hyperplane, with higher concentration of points being conducive to improved inference. As far as our knowledge goes, there is no work on the high-dimensional aspect of this model, so this paper bridges a gap in the literature.

2. Our contributions:

We study the behavior of the maximum score estimator in growing dimensions, both when \( p = o(n) \) and \( p \gg n \) and investigate minimax bounds in the underlying problem. We also provide a recipe for computing the maximum score estimator in both regimes, as well as a model selection algorithm when the dimension exceeds the sample size. We articulate our contributions below:

**Theoretical study in growing dimensions:** Section 3.1 deal with the moderate growth setting i.e. \( p = o(n) \), while Section 3.2 investigate the fast growth regime: \( p \gg n. \) In the moderate growth setting, we find the rate of convergence of the maximum score estimator in the \( \ell_2 \) norm to be \( O_p\left(\left((p \log n)/n\right)^{\alpha+2}\right) \) under Tsybakov’s low noise margin assumption \([11],[23]\) with smoothing parameter \( \alpha \) (see Assumption (A1) below). We also establish that under \( \alpha = 1, \) which is, statistically, the most interesting setting\(^1\), the minimax lower bound for the estimation problem is \( (p/n)^{\frac{3}{4}} \), and derive an upper bound to within a logarithmic factor of this lower bound. Furthermore, under the condition \( \alpha = 1, \) we are able to construct

\(^1\)See Section 6.
an estimator which attains the convergence rate $O_p \left( (p/n)^{1/3} \right)$ and is, therefore, minimax optimal. In the $p \gg n$ regime, we demonstrate that under a sparsity constraint, the best subset selection method provides a super-set of the active covariates with exponentially high probability. We also show that the rate of $\ell_2$ convergence of the maximum score estimator under the $\ell_0$ penalty and the margin condition is $O_p \left( \left( (s_0 \log p \log n)/n \right)^{\alpha/2} \right)$. Last but not least, we also derive minimax upper and lower bounds when $p \gg n$: the upper bound is the same as the rate of convergence, whilst the lower bound for $\alpha = 1$ is better by a log factor, namely $\left( (s_0 \log p)/n \right)^{1/3}$. We remark that the $l_2$ metric is a natural measure of distance in this problem since the angle between two unit-norm vectors, which measures how far apart in direction they are, is a function of the $l_2$ norm. All proofs are collected in the appendices.

**Computational Recipes:** In Section 4, we propose computational procedures for the maximum score estimator in growing dimensions. As our model is non-convex and non-differentiable, we cannot use techniques like gradient descent due to the presence of multiple local maxima. Our idea is to obtain a good initial estimator, to which end we propose an algorithm, namely, linear smoothing via adaboost. Next, starting from this initial estimator, we perform gradient descent using a sigmoid loss function, a smoothed version of $0-1$ loss, to arrive at the final estimate. When $p \gg n$, we propose another method called Penalised Adaboost Method to select the active variables. The idea behind this method is to penalize a base classifier if it uses some covariate that has not been selected previously. This produces an estimated set of active variables as well as initial values of the corresponding coefficients via linear smoothing. Now, using only the selected variables and their initial values, we apply gradient descent on the smoothed sigmoid function to obtain the final estimate. As will be seen, our computational procedures give reasonably satisfactory results.

### 3. Asymptotic properties and minimax bounds

We start with some assumptions on the distribution on $X$ and the behavior of $P(Y = 1 \mid X = x)$ near the true hyperplane which play a central role in the subsequent development. To control the behavior of $P(Y = 1 \mid X = x)$, we introduce the margin condition, also known as Tsybakov’s low noise assumption [11], [23], which is a way of quantifying how the above conditional probability deviates from 1/2 near the true hyperplane in terms of a smoothness parameter $\alpha > 0$.

**Assumption (A1): [Soft margin Assumption]** Let $P$ denote the joint distribution of $(X, \epsilon)$ in dimension $p \equiv p_n$. Then, with $\eta(x) := P(Y = 1 \mid X = x)$,

$$
P \left( \left| \eta(X) - \frac{1}{2} \right| \leq t \right) \leq C t^\alpha \quad \forall \ 0 \leq t \leq t^*$$

for some smoothness parameter $\alpha \geq 1$ and for some $0 < t^* \leq \frac{1}{2}$.

Our next assumption regarding the marginal distribution of $X$ is that the probability of the wedge shaped region between the true hyperplane and any other hyperplane under the distribution of $X$ is related to the angle between the corresponding normal vectors.
Assumption (A2): [Distribution assumption on covariates] The distribution of $X$ satisfies the following condition:

$$c' \| \beta - \beta^0 \|_2 \leq \mathbb{P}_X (\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \beta^0))$$

for all $\beta \in S^{p-1}$, where the constant $c' > 0$, does not depend on $n, p$.

Discussion: The above assumption plays a critical role in this paper, relating the underlying geometry in the problem to the probability distribution of the covariates. It is used, for example, in the below proposition, to relate the curvature of the population score function around its maximizer $\beta^0$ to the angle between $\beta^0$ and a generic unit vector $\beta$. The magnitude of the curvature plays a pivotal role in deriving the rate of convergence of Manski’s estimator in both the slow and fast growth regimes (Theorems 3.2 and 3.7 respectively), where upper tail probabilities for $\| \hat{\beta} - \beta^0 \|$ are related to upper tail probabilities for $S(\beta^0) - S(\hat{\beta})$ (which is also the difference in the population risks at these two vectors) via Assumption (A2). In that respect, this assumption can be viewed as an analogue of the compatibility or restricted eigenvalue condition in the classical high-dimensional linear regression problem, which helps convert bounds on the prediction error of the Lasso estimator to rates of convergence. Examples of families of distributions that satisfy (A2) are available in Section 5.

Proposition 3.1. Under Assumptions (A1) and (A2), the curvature of the population score function around the truth satisfies:

$$S(\beta^0) - S(\beta) \geq u_- \| \beta - \beta^0 \|^2$$

for all $\beta \in S^{p-1}$ where $\kappa = (1 + \alpha)/\alpha$ and $u_- = 4\alpha C^{-\frac{1}{\alpha}}(\alpha + 1)^{-\kappa} (c')^{\kappa}$.

The proof of this proposition relies on relating $S(\beta) - S(\beta^0)$ to $\mathbb{P}_X (\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \beta^0))$, and the latter to $\| \beta - \beta^0 \|_2$, via Assumption (A2). Note that larger values of $\alpha$ produce smaller values of $\kappa$ which translate to a larger curvature of $S$ around its maximizer. Larger curvatures correspond to greater sensitivity of $S$ to $\beta$ and translate to better convergence rates, as we see in the subsequent theorems.

3.1. Rate of convergence when: $p/n \to 0$

We first establish a rate of convergence for $\hat{\beta}$.

Theorem 3.2. Under Assumptions A1 and A2, we have

$$\| \hat{\beta} - \beta^0 \|_2 = O_p ((p/n) \log (n/p))^{\frac{\alpha}{\alpha+2}}$$

Using an exponential tail probability bound on $\| \hat{\beta} - \beta^0 \|_2$ we can further establish that:

$$\sup_{\mathcal{P}} \mathbb{E} \left( \| \hat{\beta}_n - \beta \|_2 \right) \leq K_U \left( \frac{p \log (n/p)}{n} \right)^{\frac{\alpha}{\alpha+2}},$$

where $\mathcal{P}$ is the set of all joint distributions $\{X,Y\}$ satisfy assumptions (A1) and (A2) with $\text{med}(Y^*|X) = X^T \beta$ for some $\beta \in S^{p-1}$, and $K_U > 0$ is some constant.
Our proof relies on a concentration inequality (Theorem 2 from [15]) to obtain a bound on the excess risk \( S(\hat{\beta}) - S(\beta^0) \), which, along with Assumption (A2), yields a concentration bound on \( \| \hat{\beta} - \beta^0 \|_2 \). A natural question that arises here is whether the logarithm in the above rate, which arises from the effect of growing dimension on the shattering numbers of the linear classifiers involved, can be dispensed with. While it is unclear whether the exact \((p/n)^{\alpha/2}\) rate is achievable, we demonstrate, in what follows, that for \( \alpha = 1 \), it is possible to construct an estimator whose rate of convergence is \((p/n)^{1/3}\) under the following additional assumption.

**Assumption (A2:upper):**

(i) The distribution of \( X \) satisfies
\[
\mathbb{P}_X(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \beta^0)) \leq C' \| \beta - \beta^0 \|_2
\]
for all \( \{ \beta : \| \beta - \beta_0 \|_2 \leq 1 \} \), where the constant \( C' > 0 \) does not depend on \( n,p \).

(ii) For some small \( u_0 > 0 \),
\[
S(\beta_0) - S(\beta) \leq u_+ \| \beta - \beta^0 \|_2
\]
for all \( \{ \beta : \| \beta - \beta_0 \|_2 \leq u_0 \} \), where the constant \( u_+ > 0 \) does not depend on \( n,p \).

The construction of this estimator can be briefly described as follows: Generate (enough) points randomly from the surface of the sphere, such that with high probability some of the generated points are in a sufficiently small neighborhood of \( \beta^0 \). Then, maximize the empirical score function on the generated points. We show in the following theorem that this empirical maximizer converges to the truth at rate \((p/n)^{1/3}\):

**Theorem 3.3.** Suppose the margin condition (Assumption A1) is satisfied for \( \alpha = 1 \) and that Assumptions (A2) and (A2:upper) hold. Then, there exists an estimator \( \hat{\beta} \), which can be constructed by the above technique, such that
\[
\| \hat{\beta} - \beta^0 \|_2 = O_p \left( (p/n)^{1/3} \right).
\]

**Remark:** Assumption (A2:upper (ii)) as well as the construction of the grid estimator take into account the the fact that \( \alpha = 1 \). In that sense, the new estimator is not adaptive, whereas the maximum score estimator is agnostic to the value of \( \alpha \) and it is possible that the log factor in its convergence rate is the price that one pays for adaptivity. For more insight into the (A2:upper) assumption, see Section 5.

Finally, we show that the minimax lower bound for this estimation problem under \( \alpha = 1 \) is \((p/n)^{1/3}\) i.e. we cannot estimate the linear discriminator at a better rate without more assumptions:

**Theorem 3.4 (Minimax Lower bound).** Let \( \mathcal{P}(C,c',\alpha) \) denote the class of distributions of \( (X,\epsilon) \) in dimension \( p_n + 1 \equiv p + 1 \) that satisfy Assumptions (A1) and (A2) and define \( \mathcal{P} = \bigcup_{c',C,1} \mathcal{P}(c',C,1) \). If \( n = o(e^p) \), we have the following lower bound on the minimax risk:
\[
\inf_{\hat{\beta}_n} \sup_{\mathcal{P}} \mathbb{E} \left( \| \hat{\beta}_n - \beta^0 \|_2^2 \right) \geq K_L \left( \frac{p}{n} \right)^{\frac{2}{3}},
\]
for some constant \( K_L \).
**Remark:** The proof of this result follows the general pattern of constructing an appropriate null hypothesis along with a sequence of local alternatives that approach the null at rate \((p/n)^{1/3}\). The core challenge here is in constructing the local alternatives cleverly. As we are in a growing dimension regime, we resort to Assouad’s lemma to tackle the problem. The same minimax rate is true for the smaller class of distributions formed by intersecting \(P\) with the class of distributions satisfying (A2:upper) for some positive constants \(C', u_0, U_+\), since the local alternatives constructed in the proof satisfy this condition as well. Therefore, the grid estimator continues to be minimax rate optimal for this smaller class of distributions.

### 3.2. Rate of convergence when \(p \gg n\)

We now turn to the case where the dimension of the covariate vector is larger than \(n\). The natural structural assumption on \(\beta^0\) here is that of sparsity and our estimation procedure needs to incorporate this constraint. Since our loss function is non-convex (indeed, non-differentiable!), adding an \(\ell_1\) penalty is not effective for variable selection. One possible route is to consider an \(\ell_1\) penalized version of the smoothed score estimator, namely:

\[
M_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} \left( Y_i - \frac{1}{2} \right) \frac{\exp(\xi_n \beta' X_i)}{1 + \exp(\xi_n \beta' X_i)} - \lambda_1 \|eta\|_1
\]

where we have used the logistic function for smoothing (though other choices are possible). Here \(\xi_n\) controls the rate at which the smoothed criterion function approaches the indicator and \(\lambda_1\) controls the degree of penalization. One may hope for consistency with this penalized approach if the tuning parameter \(\xi_n \uparrow \infty\), but as we are in non-convex territory, the problem is likely theoretically formidable.

In this paper, we study the empirical risk minimization problem under \(l_0\) penalization on \(\beta^0\); i.e. we use a bound (assumed known) on the number of non-zero co-ordinates of \(\beta^0\). Inference on high-dimensional parameters using the \(l_0\) penalty is a well-studied topic in statistics, dating back, at least, to the work of best subset selection by Hocking et.al. [6]. The consistency of the population risk of the best linear predictor under an \(\ell_0\) constraint in a general regression set-up was theoretically studied in Greenshtein and Ritov [5], while minimax rates for \(\ell_q\) constrained estimation \([q \in [0, 1]]\) in high dimensional linear regression (with a known level of sparsity) were more recently calculated by Raskutti et.al. [19].

A key feature of high dimensional inference is model selection. Under the sparsity constraint, most variables are inactive and a good model selection algorithm needs to include the active set with high probability but relatively few inactive variables. Though model selection/ feature selection in the high-dimensional linear regression model has been studied extensively over the past two decades (e.g. see [28], [8], [25], [26], [27] and references therein), the problem remains relatively unaddressed in the classification set-up. Our results are based on a combinatorial model search, i.e. we search all possible models with a prespecified sparsity level and select the one that maximizes the score function. One caveat here is that, if the minimum non-zero element of the true parameter is too small (relative to the noise in the model), it will not be detectable. We therefore need an assumption on
how fast $\beta^0_{\min} \to 0$, where $\beta^0_{\min}$ is the minimum absolute value of the non-zero coefficients of the true $\beta^0$. We formally state our assumptions on the model below.

**Assumption (A3): [Sparsity Assumption]** There exists known $s_0$ with $\|\beta^0\|_0 \leq s_0$, where $s_0$ depends on $n, p$ in such a way that $\frac{s_0 \log p \log n}{n} \to 0$ as $n \to \infty$.

**Assumption (A4): [Beta-min Assumption]** If $\beta^0_{\min}$ denotes the minimum absolute value of the non-zero coefficients of $\beta^0$, then $\beta^0_{\min} \geq \left( \frac{30 \sqrt{\alpha}}{u_-} \right)^{\frac{1}{\alpha}} \left( \frac{2 A s_0 \log (ep/s_0) \log n}{n} \right)^{\alpha + \frac{2}{\alpha}}$, where $A = \log_2 e$ and $u_-$ is the constant we obtained from Proposition 3.1.

To understand the subsequent theorems, we need to introduce some notation. Define

$$\mathcal{M} \equiv \mathcal{M}_{s_0} = \{ m : m \subset \{1, 2, \ldots, p\}, |m| \leq s_0 \},$$

Here $m \in \mathcal{M}$ denotes a model with sparsity $\leq s_0$. Let $\mathcal{H} = \{ h_\beta : \beta \in S^{p-1} \}$, where $h_\beta : \mathbb{R}^p \times \{-1, 1\} \to \{-1, 1\}$ is defined as $h_\beta(x, y) = y \text{sgn}(x^T \beta)$. Also, for $m \in \mathcal{M}$ define:

1. $\mathcal{H}_m = \{ h_\beta \in \mathcal{H} : \beta_i \neq 0 \iff i \in m \}$
2. $\hat{\beta}_m = \text{argmax}_{\beta : h_\beta \in \mathcal{H}_m} S_n(\beta)$, and $\hat{h}_m = h_{\hat{\beta}_m}$
3. $\tilde{\beta}_m = \text{argmax}_{\beta : h_\beta \in \mathcal{H}_m} S(\beta)$, and $\tilde{h}_m = h_{\tilde{\beta}_m}$

**Theorem 3.5.** Suppose $m^* = \text{supp}(\beta^0)$. Consider any two models $m_1, m_2$ such that $m_1 \not\subset m^*$ and $m_2 \supseteq m^*$, with $|m_1| = |m_2| = s_0$. Then under Assumptions (A1 - A4):

$$P \left( L_n(\hat{\beta}_{m_1}) > L_n(\hat{\beta}_{m_2}) \right) \geq 1 - 2e^{-2s_0 \log (ep/s_0)}$$

where $L_n = (1 - S_n)/2$ is the empirical misclassification risk.

The proof depends on excess risk concentration results from the ERM (empirical risk minimization) literature along with the beta-min condition stated in Assumption (A4). The above theorem quickly leads to the observation that the empirical loss function will be minimized at some super-set of the true model $m^*$ with exponentially high probability, as stated in the following corollary:

**Corollary 3.6.** Denote the empirically selected model by $\hat{m}$, i.e. $\hat{m} = \text{argmin}_{m \in \mathcal{M}} L_n(\hat{\beta}_m)$. Then, under Assumptions (A1 - A4), we have

$$P(\hat{m} \not\subset m^*) \leq 2e^{-s_0 \log (ep/s_0)}$$

Now, consider our estimator for $\beta^0$ given by

$$\hat{\beta}_m \Delta \hat{\beta} = \text{argmax}_{\beta : \|\beta\|_1, \|\beta\|_0 \leq s_0} S_n(\beta).$$

The following theorem provides concentration rate of $\hat{\beta}$:

**Theorem 3.7.** Under Assumptions (A1 - A3), we have

$$\|\hat{\beta} - \beta^0\|_2 = O_p \left( \frac{s \log (p/s_0) \log n}{n} \right)^{\alpha/\alpha + 2}. $$
Using an exponential tail probability bound on $\|\hat{\beta} - \beta^0\|_2$, we can further establish that

$$
\sup_{\mathcal{P}} \mathbb{E}\left( \|\hat{\beta} - \beta\|_2 \right) \leq \tilde{K}_U \left( \frac{s_0 \log (p/s_0) \log n}{n} \right)^{\frac{\alpha}{\alpha + 2}}
$$

for some constant $\tilde{K}_U$, where $\mathcal{P}$ is the collection of all distributions of $\{X, Y\}$ such that $\text{med}(Y^*|X) = X^T\beta$ for some $\beta \in S^{p-1}$ and the assumptions are satisfied.

Our next result provides a lower bound on the minimax error rate. We see that, as in the moderate growth regime, the lower bound is better by a logarithmic factor, which implies that the maximum score estimator is almost minimax optimal.

**Theorem 3.8.** Let $\mathcal{P}(C, c', \alpha, s_0)$ denote the class of distributions of $(X, \epsilon)$ in dimension $p_n + 1 \equiv p_1 + 1$ that satisfy Assumptions (A1), (A2) and (A3), and define $\mathcal{P} = \bigcup_{(c', C, 1, s_0)} \mathcal{P}(c', C, 1, s_0)$. Then, we have the following lower bound on the minimax risk for estimating $\beta^0$:

$$
\inf_{\hat{\beta}} \sup_{\mathcal{P}} \mathbb{E}\left( \|\hat{\beta} - \beta^0\|_2 \right) \geq \tilde{K}_L \left( \frac{s_0 \log (p/s_0)}{n} \right)^{\frac{3}{2}}
$$

for some constant $\tilde{K}_L > 0$.

Just like the minimax lower bound proof in the moderate growth case, the proof of this theorem also relies on the construction of a sequence of alternatives, along with the use of Fano’s inequality, to arrive at the final result.

### 4. Computational Aspects

In this section we introduce some new methods to compute the maximum score estimator, both in the moderate and fast growth regimes. In addition, we also provide a model selection technique which works well in terms of variable selection on simulated data.

#### 4.1. $p = o(n)$

The criterion to be maximized for computing the score estimator is non-differentiable, hence traditional optimization methods like gradient descent or Newton scoring do not apply. One idea is to use a smooth surrogate for the non-differentiable score function, replacing it by

$$
SM_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} \left( Y_i - \frac{1}{2} \right) \frac{\exp(\xi \beta' X_i)}{1 + \exp(\xi \beta' X_i)}.
$$

As $\xi$ increases, $SM_n(\beta)$ converges to $S_n(\beta)$. As $SM_n(\beta)$ is non-convex, the efficiency of iterative algorithms like gradient descent will rely heavily on the starting point. We propose computing a workable initial estimator using nonparametric classification techniques, e.g. adaboost and then employing gradient descent starting with this initial estimator.
**Linear Smoothing Via Adaboost:** We now describe our algorithm to obtain the initial estimator. We divide the available data into three sets: a training set, a development set and a test set. We build our model using the training and development data, and then use the test data to get an unbiased estimate of the misclassification error (as reported in the tables). The steps of the algorithm can be described as follows:

1. Run the adaboost algorithm using stumps as base classifiers, where stumps are the data points \( \{X_{i,j}\} \). The total number of iterations, say \( T \), is a tuning parameter of this algorithm.

2. **Early Stopping:** After each iteration \( t \) (where \( t \in \{1, 2, \ldots, T\} \)), validate the model on the development data by computing the empirical misclassification error. Let \( T' \) be the first time after which the error starts increasing. Then, consider the adaboost classifier up-to \( T' \)th iteration, i.e.:

\[
F(x) = \sum_{t=1}^{T'} \alpha_t S_t(x)
\]

where \( S_t \) is the stump selected in \( t^{th} \) iteration and \( \alpha_t = 0.5 \log (1 - \epsilon_t)/\epsilon_t \), \( \epsilon_t \) being the error of \( S_t \). If \( T' > T \) i.e. the error keeps on decreasing, we terminate at the \( T^{th} \) step. We use \( \text{sgn}(F) \) for prediction.

3. Define \( A_j \) to be the collection of all \( t \)'s such that \( S_t \) is the stump on the \( j^{th} \) covariate. (\( A_j = \phi \), if that covariate is not selected at all). We can write \( F(x) = \sum_{j: A_j \neq \phi} F_j(x) \), where \( F_j(x) = \sum_{t \in A_j} \alpha_t S_t(x) \). Thus, \( F_j \) extracts the contribution of the \( j^{th} \) covariate to the final Adaboost classifier. We get an initial estimate of \( \beta_j^0 \) via simple linear regression using the \( j^{th} \) covariate of the training data and its Adaboost prediction, i.e. using the data \( \{(X_{1,j}, F_j(X_1)), (X_{2,j}, F_j(X_2)), \ldots, (X_{N,j}, F_j(X_N))\} \), where \( N \) is the number of training samples. The estimate is:

\[
\hat{\beta}_j = \frac{\sum_{i=1}^{N}(X_{i,j} - \bar{X}_j)F_j(X_i)}{\sum_{i=1}^{N}(X_{i,j} - \bar{X}_j)^2}
\]

If \( A_j = \phi \), set \( \hat{\beta}_j = 0 \).

Upon obtaining an initial estimator of \( \beta^0 \) using the above procedure, we use projected gradient descent (As our parameter space is sphere) on the smoothed version of the maximum score model to get the second stage (final) estimate of \( \beta^0 \). To summarize our algorithm:

**Algorithm 1 Gradient Descent using Linearly Smoothed Adaboost**
1: Divide the data into three parts: Training, development and test data.

2: Apply Linear smoothing via adaboost algorithm on training data and development data to get an initial estimate.

3: Starting from the initial estimator obtained in previous step, perform the projected gradient descent using the smoothed version of maximum score estimator \( (SM_n(\beta)) \) to get the final estimator \( \hat{\beta} \) of \( \beta^0 \).

**Simulation results:** We now present some simulation results. We denote the number of
training samples by $n$, and the number of development and test samples by $n_1$ and $n_2$ respectively.

a) The covariates were generated as $X_1, \ldots, X_n \sim \mathcal{N}(0, \Sigma_p)$, where $\Sigma_p$ is an AR(1) matrix with $\rho = 0.3$. Define $X_{n,p}$ to be $n \times p$ matrix of observations.

b) The true parameter was taken as $\beta_0 / \| \beta_0 \|_2$ where $\beta_0^i = 1/(20 + i)$ for $i \in \{1, 2, 3, \ldots, p\}$.

c) The error vector components $\epsilon_{n,i} \sim \mathcal{N}(0, 1 + |X_{n,p}c_0|_i)$ for $i \in \{1, 2, \ldots, n\}$, where $c_0$ was taken to be a vector of length $p$ generated from $\mathcal{N}(0, 1)$. (The vector $c_0$ was generated once and remained the same over all simulations to avoid additional randomness).

d) The mean vector $\mu_n^0$ is defined as $X_{n,p}\beta_0^0$.

e) Finally, the response $Y_n^* = \mu_n^0 + 0.5 \sigma(\mu_n^0) \epsilon_n$ and the observations are $Y_n = \text{sgn}(Y_n^*)$.

(Here $\sigma(a_n)$ means standard deviation of the vector $a_n$.)

For the initial classifier using Adaboost, we have used $T = 400$ iterations. The value of $T'$, of course, varies from simulation from simulation. For each value of $p$, we have taken $n = p^2$ and $n_1 = n_2 = n/2$. For each fixed value of $p$ we have done 30 simulations and take the average of parameters of interest. For other tuning parameters like $\xi, \alpha$ (where $\xi$ comes from the function $SM_n(\beta)$ and $\alpha$ is the step/ learning rate in gradient descent), we have used cross-validation. Below is the table of output of different values of $n$ and $p$:

| Dim (P) | Bayes Error | Initial Train Error | Initial Test Error | Initial Norm-diff | Final Train Error | Final Test Error | Final Norm-diff |
|---------|-------------|---------------------|-------------------|------------------|------------------|-----------------|----------------|
| 50      | 0.1261      | 0.1284              | 0.1413            | 0.1574           | 0.1061           | 0.1458          | 0.1981         |
| 100     | 0.1235      | 0.1269              | 0.1322            | 0.1205           | 0.1191           | 0.1303          | 0.1098         |
| 200     | 0.1219      | 0.1288              | 0.1313            | 0.1306           | 0.1173           | 0.1268          | 0.0903         |
| 500     | 0.1209      | 0.1468              | 0.1476            | 0.2368           | 0.1196           | 0.1227          | 0.0513         |
| 1000    | 0.12        | 0.1540              | 0.1452            | 0.2731           | 0.1193           | 0.1163          | 0.0353         |

Here **Bayes Error** is an estimate of the misclassification error of true classifier based on simulated data. **Initial Train Error** indicates the training error of the initial classifier obtained using **Linear smoothing via adaboost** algorithm. **Initial Test error** is the error of the same classifier on test sample. **Initial Norm diff** indicates $\| \beta^0 - \hat{\beta}_1 \|_2$ where $\hat{\beta}_1$ is initial classifier. **Final Train Error** denoted the training error of the final classifier $\hat{\beta}$ obtained upon applying the gradient descent algorithm starting from the $\hat{\beta}_1$. **Final Test Error** is the test error of this final classifier. **Final Norm-Diff** is norm-difference between $\hat{\beta}$ and $\beta^0$ after applying gradient descent.

As can be seen in the simulations **Linear smoothing via adaboost** gives a good initial estimator of $\beta^0$, and the final estimator improves the test error for higher values of $p$. 
4.2. Model selection and estimation when $p \gg n$

As mentioned in the Introduction, our problem can be viewed as binary classification problem with linear Bayes’ classifier. Under the sparsity assumption, a few good covariates contribute to the classification. To identify these covariates, we introduce a penalized classification approach for variable selection.

**Penalised Adaboost:** Our simulation results from the previous section show that adaboost provides satisfactory initial estimates both in terms of miss classification error and norm difference. Motivated by this observation, we propose a penalized version of adaboost for variable selection. The idea is to apply linear smoothing via adaboost with an additional penalty term. At each iteration, we penalize those base classifiers (stumps) that involve covariates not selected in previous iterations by increasing the missclassification error by a positive number $\rho$. More precisely:

**Algorithm 2 Penalised adaboost**

1: Start will decision stumps as weak learners and set $T \leftarrow$ Total number of Iterations.
2: Initial $t = 1$, the iteration counter.
3: Initiate null vector $N$ and $P$ to store row and column number of the selected Stump.
4: Initial $F_0 \equiv 0$ function
5: Initialize Weights $W = \{1/n\}$ for all training samples.
6: while $t < T'$ do
7: Calculate weighted error of each stumps.
8: For stumps whose column index $\notin P$, add penalty ($\rho$) to its error.
9: Choose the best stump $S$ from the error matrix.
10: Store its index to $N$ and $P$.
11: Define $\alpha_t = \frac{1}{2} \log \left( \frac{1-\epsilon_t}{\epsilon_t} \right)$ where $\epsilon_t$ is the minimum error.
12: Update weights according to adaboost update.
13: Update $F_t = F_{t-1} + \alpha_t S$.
14: Update $t \leftarrow t + 1$
15: return $F_t$

The penalized adaboost algorithm returns a classifier that uses a small fraction of the possible covariates. We run Linear Smoothing via Adaboost with these selected covariates to get an initial estimate, $\hat{\beta}_1$, of $\beta_0$, and then perform a projected gradient descent algorithm on $SM_n(\beta)$ using the selected covariates again with $\hat{\beta}_1$ as the starting value. In summary:

**Algorithm 3 Gradient Descent using Penalised Adaboost**

1: Divide the data into three parts: training, development and test.
2: Apply Penalized Adaboost on the training data to select the active co-variates.
3: Apply Linear smoothing via adaboost on selected covariates to get initial estimate $\hat{\beta}_1$ and set 0 to unselected co-ordinates.
4: Extract those columns of the data-matrix which co-variates have been selected by penalized adaboost.
5: Starting from $\hat{\beta}_1$, perform projected gradient descent on $SM_n(\beta)$ using the excerpted data to get the final estimator $\hat{\beta}$ of maximum score model.
Simulation results: We are using same setup as before, but with \( p > n : p = n^{1.1} \) and \( s_0 = \sqrt{n} \), where \( s_0 \) is the number of true active co-efficient. The active co-variates are being chosen randomly from the set \( \{1, 2, \ldots, p\} \) and the \( i^{th} \) chosen variable is assigned the value \( \frac{1}{20 + i} \) for \( i \in \{1, \ldots, s_0\} \). We then normalize the coefficient vector to have norm 1. As before, we do 30 iterations for each simulation setting and take the average result.

We introduce two new variables in the simulation table to measure the variable selection performance of Penalized Adaboost, namely:

1. True Disc. True Discoveries, i.e. the proportion of the true active variables selected on an average over 30 iterations.
2. False Disc. False Discoveries, i.e. the proportion of null covariates selected as active variables on average over 30 iterations.

| Dim (P) | True Disc. | False Disc. | Bayes Error | Initial Train Error | Initial Test Error | Initial Norm-diff | Final Train Error | Final Test Error | Final Norm-diff |
|---------|------------|-------------|-------------|---------------------|-------------------|-----------------|------------------|---------------|---------------|
| 10000   | 0.8426     | 0.0019      | 0.1252      | 0.1509              | 0.1826            | 0.3907          | 0.1186           | 0.17           | 0.3324        |
| 5000    | 0.8056     | 0.0062      | 0.1259      | 0.1557              | 0.1993            | 0.4752          | 0.1156           | 0.1876         | 0.4284        |
| 2000    | 0.7677     | 0.0216      | 0.1251      | 0.1463              | 0.2423            | 0.6209          | 0.0985           | 0.2286         | 0.5854        |
| 1000    | 0.7014     | 0.0316      | 0.1286      | 0.1463              | 0.2729            | 0.7622          | 0.0913           | 0.2688         | 0.7261        |
| 700     | 0.6683     | 0.0351      | 0.1328      | 0.1525              | 0.2907            | 0.8022          | 0.0969           | 0.2824         | 0.7753        |

From the table, it is evident that, penalised adaboost performs well for variable selection in high dimension scenario. As \( p \) increases, the true discovery rate move towards 1 and 0 respectively, indicating that asymptotically, the model selection aspect continues to improving.

5. Concluding Discussion

We close with a discussion of various aspects of the high-dimensional binary choice model and our approach to the problem.

5.1. Why is \( \alpha = 1 \) interesting?

We argue why the case \( \alpha = 1 \) is more relevant than \( \alpha > 1 \) by showing that a rich family of distributions satisfy the margin condition for \( \alpha = 1 \) under some natural assumptions. For any \( \alpha < \alpha' \), the family with margin condition \( \alpha' \) is a subfamily of the margin condition \( \alpha \), because \( \Pr_X (\eta(X) - 0.5) \leq t) \leq C t^{\alpha'} \leq C t^\alpha \), for all \( 0 \leq t \leq t^* < 1 \). Hence, \( \alpha = 1 \) represents the most general class of distributions.

Assume that (a) \( X \perp \perp \epsilon \) and the density of \( \epsilon \), say \( f \), does not depend on \( p \); (b) \( f(x) \geq c_\delta > 0 \) on \( \{-\delta, \delta\} \) for some \( \delta > 0 \); (c) the density of \( X^T \beta_0 \) is bounded by a positive number...
\( \leq k \) on \((-\delta', \delta')\) for some \( \delta' > 0 \), with \( k, \delta' \) not depending on \( p \). Then, for \( 0 \leq t \leq t^* \), where \( \delta \wedge \delta' > (F^{-1}_\epsilon(0.5 + t^*) \vee -F^{-1}_\epsilon(0.5 - t^*)) \):

\[
P_X (|\eta(X) - 0.5| \leq t) = P_X \left(|F_\epsilon(-X^T\beta^0) - F(0)| \leq t\right)
= P_X \left(F^{-1}_\epsilon(0.5 - t) \leq -X^T\beta^0 \leq F^{-1}_\epsilon(0.5 + t)\right)
\leq P_X \left(|X^T\beta^0| \leq (F^{-1}_\epsilon(0.5 + t) \vee -F^{-1}_\epsilon(0.5 - t))\right)
\leq 2k(F^{-1}_\epsilon(0.5 + t) \vee -F^{-1}_\epsilon(0.5 - t))
\leq \frac{2kt}{c_\delta}
\]

which is the condition corresponding \( \alpha = 1 \).

Using an inverse-Lipschitz type condition, one can also let \( \epsilon \) depend on \( X \). Suppose that the conditional distribution of \( \epsilon \) given \( X \) satisfies:

\[
|F_{\epsilon|X=x}(x^T\beta^0) - 0.5| = |F_{\epsilon|X=x}(x^T\beta^0) - F_{\epsilon|X=x}(0)| \geq C(|x^T\beta^0| \wedge \xi) \text{ a.e. } X
\]

for some \( C, \xi > 0 \) independent of \( p \), for almost surely \( X \sim P_X \). This holds, for example, if almost surely, for all \( x \), the conditional density \( f_{\epsilon|X=x}(\xi) \geq c > 0 \) on a fixed neighborhood \((-\delta', \delta')\) around 0, with \((c, \delta)\) not depending on \( p \). The margin condition is now satisfied for \( \alpha = 1 \) under the same condition on the density of \( X^T\beta^0 \) as before. An example of the dependence requirement of \( \epsilon \) on \( X \) is \( \epsilon|X = x \sim \mathcal{N}(0, 1 + (||x||_2 \wedge 1)) \).

On the other hand, there are two natural scenarios where \( \alpha > 1 \) is satisfied. The first is when the distribution of \( X^T\beta^0 \) satisfies \( P(|X^T\beta^0| \leq t) \leq C t^{\alpha_0} \) for \( \alpha_0 > 1 \). This, along with the Inverse-Lipschitz type assumption on the conditional distribution of \( \epsilon|X \), yields models where the margin condition is satisfied for \( \alpha_0 \). Note, however, that the upper bound on the distribution function of \( |X^T\beta^0| \), forces the density of \( X^T\beta^0 \) to be 0 at the point 0, if one assumes that the density is continuous. This is quite restrictive.

The second natural scenario arises when \( P(Y = 1|X = x) \) changes rapidly near the hyperplane. Consider the following setup:

\[
F_{\epsilon|X=x}(t) = \frac{1}{2} + \frac{\text{sgn}(t) |t|^{1/\alpha_0}}{2} \quad \forall \quad -1 \leq t \leq 1
\]

for \( \alpha_0 > 1 \). Then, the assumption that the density of \( X^T\beta^0 \) is uniformly bounded in \( p \) in a neighborhood of 0 that is also free of \( p \), leads to the margin condition with parameter \( \alpha_0 \). Here, it is the rather special structure of \( \eta(x) \) that gives the margin condition.

### 5.2. Exploring and relaxing our assumptions

It is of interest to investigate sufficient conditions under which assumptions (A2) and (A2:upper) hold. We show in Lemma A.2 in the appendix that (A2) and (A2:upper(i))
hold simultaneously when $X$ arises from an elliptically symmetric distribution centered at 0, under some restrictions on the minimum and maximum eigenvalues of its orientation matrix. Assumption (A2:upper(ii)) also holds for elliptically symmetric distributions centered at 0 but under some further mild conditions, as demonstrated in Lemma A.3.

To allow for scenarios when $X$ can have non-zero mean (more generally, when $X$ is not centered at 0), we would also like to consider $X$ of the form $(1, X)$ where $X$ is a random vector with mean 0; i.e. we include an intercept term in the model. While (A2) in its current form is not a sensible assumption when $X$ includes an intercept term (since for large values of the co-efficient of the intercept the wedge appearing in Assumption (A2) could have negligible to zero probability), a restricted version of (A2) [in the sense that the inequality in (A2) is fulfilled for all $\beta$ sufficiently close to $\beta_0$, where the parameter $\beta_0$ is itself subject to certain constraints] is, indeed, verifiable for families of distributions including elliptically symmetric $X$ centered at the origin. The main results of this manuscript still continue to hold, but to accommodate the restricted version of (A2), the proofs presented in the appendix need to be slightly modified. We skip the details.

Recall that (A2) says that $c'\|\beta - \beta^0\|_2 \leq P(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\beta^0))$ for all $\beta \in S^{p-1}$ for some constant $c' > 0$. If (A2) is assumed only locally, in the sense that the inequality in the preceding sentence holds only on the set $\{\beta : \|\beta - \beta^0\|_2 \leq \delta\}$, for some fixed $\delta > 0$, Proposition 3.1 will be true only on this set. Consequently, proving the rate of convergence of Manski’s maximum score estimator will require establishing consistency of $\hat{\beta}$ for $\beta^0$ in the $\ell_2$ norm, in order to guarantee that $\hat{\beta}$ will eventually be in a $\delta$ neighborhood of $\beta^0$.

**Theorem 5.1.** Suppose that Assumption A1 holds and that the inequality in Assumption A2 is only satisfied for $\{\beta : \|\beta - \beta^0\|_2 \leq \delta\}$. Then, in the regime that $p/n \to 0$, $\|\hat{\beta} - \beta^0\|_2 \overset{P}{\to} 0$ as $n \to \infty$. The same conclusion holds in the regime $p > n$ under the assumption that $s \log p/n \to 0$.

Under this localized version of (A2), the minimax upper bound results for $\hat{\beta}$ as presented in Theorems 3.2 and 3.7 are no longer guaranteed (this is easily seen from an inspection of the proof of the minimax upper bound), and whether the grid estimator is $(p/n)^{1/3}$ consistent in the slow regime is also unclear. However, all other results presented in the paper do go through.

### 5.3. Asymptotic distribution

In their seminal paper, Kim and Pollard [9] proved that for fixed $p$, $n^{1/3}(\hat{\beta} - \beta^0)$ converges in distribution to the maximizer of a Gaussian process with quadratic drift. Our treatment of the binary choice model should be contrasted with their approach: while they assumed the continuous differentiability of both the density of $X$ and $\eta(x) = P(Y = 1 | X = x)$ and a compact support for $X$, we have made no such assumptions. We have tackled those aspects of this problem from the classification point of view, with assumptions on the growth of $P(Y = 1 | X = x)$ near the Bayes hyperplane and in addition, conditions on the distribution of $X$ to ensure that sufficiently many observations are available around the Bayes hyperplane. As far as the asymptotic distribution of the score estimator in growing dimensions (or
functionals thereof) is concerned, this is, in itself, a mathematically formidable problem, well outside the scope of this paper. Based on what we know in the fixed $p$ setting, the forms of such distributions are likely to be extremely complicated. The question remains whether tractable asymptotic distributions for making inference on components of $\beta_0$ in the growing $p$ setting could be obtained for smoothed versions of the score estimator, in the spirit of Horowitz’s paper [7]. This is likely to be an interesting but challenging avenue for future research on this subject.

Appendix A: Proofs of Theorems

A.1. Proof of Proposition 3.1

Fix $\beta : \|\beta\|_2 = 1$. Denote $X_\beta = \{x : \text{sgn}(x^T \beta) \neq \text{sgn}(x^T \beta^0)\}$. At-first we will establish a relation between $S(\beta^0) - S(\beta)$ and $P_X(X_\beta)$. Towards that end, we have the following lemma:

Lemma A.1. Under Assumption A1, $S(\beta^0) - S(\beta) \geq c_1(P_X(X_\beta))^\kappa$ where $c_1 = \frac{2\alpha}{C_1^{(\alpha+1)\kappa}}$.

Proof. 

\[
S(\beta^0) - S(\beta) = 2 \int_{X_\beta} |E(Y|X)|f(x) \, dx \\
= 4 \int_{X_\beta} |1 - F_{\lambda X}(-x^T \beta^0) - 0.5|f(x) \, dx \\
= 4 \int_{X_\beta} |\eta(x) - 0.5|f(x) \, dx \\
\geq 4t P_X(|\eta(x) - 0.5| \geq t, X_\beta) \quad \text{[for some } 0 \leq t \leq t^*] \\
\geq 4t (P_X(X_\beta) - P_X(|\eta(x) - 0.5| \leq t)) \\
\geq 4t (P_X(X_\beta) - Ct^{\alpha})
\]

If we maximize RHS with respect to $t$, the maximum is attained at $\hat{t} = \left(\frac{P_X(X_\beta)}{\alpha+1}C\right)^\frac{1}{\alpha}$. This is a feasible value of $t$, i.e. $0 \leq \hat{t} \leq t^*$ if $P_X(X_\beta) \leq C(\alpha + 1)(t^*)^\alpha$. If the Assumption (A1) is valid for some $C$, then it true for all $C_1 > C$. Hence, without loss of generality we can assume $C \geq (\alpha + 1)(t^*)^\alpha$, i.e. $0 \leq \hat{t} \leq t^*$ is true always. Under this condition, putting the value of $\hat{t}$ we have for all $\beta \in S^{p-1}$:

\[
S(\beta^0) - S(\beta) \geq \frac{4\alpha}{C^{\frac{1}{\alpha}}(\alpha+1)^{\kappa}}(P_X(X_\beta))^\kappa
\]

Using the relation between $P_X(X_\beta)$ and $\|\beta - \beta^0\|_2$ via Assumption (A2), we conclude the proof of the Proposition.

\[\Box\]
A.2. Assumptions (A2) and (A2:upper) revisited

Lemma A.2. Suppose that $X_{p \times 1}$ follows an elliptically symmetric distribution centered at 0, with density $f_X(x) = |\Sigma|^{-1/2}g(x^T\Sigma^{-1}x)$, where $g$ is a non-negative function. Assume that:

$$\inf_P \frac{\lambda_{\min}(\Sigma)}{\lambda_{\max}(\Sigma)} \geq c_3 > 0.$$ 

Then $X$ satisfies Assumption A2.

Proof. First, we prove that for $X \sim \mathcal{N}(0, \Sigma_p)$ with the above displayed condition holding. Observe that $\mathbb{P}(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\beta^0))$ depends on the two-dimensional geometry of $X$, i.e. only on the distribution of $(X^T\beta, X^T\beta^0)$. To make the calculations easier, we transform $X$ into $Y$ where the first two-coordinates of $Y$ corresponds to $(X^T\beta, X^T\beta^0)$. Consider the following orthogonal matrix:

$$P = \begin{bmatrix}
\beta^0' \\
\beta'^-(\beta^0, \beta)^0' \\
\sqrt{1-(\beta^0, \beta)^2} \\
v_3 \\
\vdots \\
v_p
\end{bmatrix}$$

where $\beta^0', \beta'^-(\beta^0, \beta)^0', v_3, \ldots, v_p$ forms an orthonormal basis of $\mathbb{R}^p$ ($v_3, \ldots, v_p$ can be constructed using the Gram-Schmidt algorithm). If we define $Y = PX$, then $Y_1 = X^T\beta^0$ and $X^T\beta = a_1Y_1 + a_2Y_2$ where $a_1 = (\beta^0, \beta), a_2 = \sqrt{1-(\beta, \beta)^2}$. Then the probability of the wedge shaped region becomes:

$$\mathbb{P}(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\beta^0))$$

$$= \int_{\beta^0 x \geq 0} f_X(x) \, dx + \int_{\beta^0 x < 0} f_X(x) \, dx$$

$$= \int_{y_1 \geq 0, a_1 y_1 + a_2 y_2 < 0} f_X(P^{-1}y) \, dy + \int_{y_1 < 0, a_1 y_1 + a_2 y_2 \geq 0} f_X(P^{-1}y) \, dy$$

$$= \int_{y_1 \geq 0, a_1 y_1 + a_2 y_2 < 0} \frac{1}{\sqrt{2\pi(P\Sigma P^T)}} e^{-\frac{1}{2}y'(P\Sigma P^T)^{-1}y} \, dy + \int_{y_1 < 0, a_1 y_1 + a_2 y_2 \geq 0} \frac{1}{\sqrt{2\pi(P\Sigma P^T)}} e^{-\frac{1}{2}y'(P\Sigma P^T)^{-1}y} \, dy$$

$$= \int_{y_1 \geq 0, a_1 y_1 + a_2 y_2 < 0} \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{1}{2}y'(\Sigma)^{-1}y} \, dy_1 \, dy_2 + \int_{y_1 < 0, a_1 y_1 + a_2 y_2 \geq 0} \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{1}{2}y'(\Sigma)^{-1}y} \, dy_1 \, dy_2$$

[Marginalise over $y_3, \ldots, y_p$, with $\Sigma$ being the leading 2 $\times$ 2 block of $(P\Sigma P^T)$]

$$\geq \frac{1}{2\pi \sqrt{\lambda_1 \lambda_2}} \left[ \int_{y_1 \geq 0, a_1 y_1 + a_2 y_2 < 0} e^{-\frac{1}{2\lambda_1}(y_1^2 + y_2^2)} \, dy_1 \, dy_2 + \int_{y_1 < 0, a_1 y_1 + a_2 y_2 \geq 0} e^{-\frac{1}{2\lambda_2}(y_1^2 + y_2^2)} \, dy_1 \, dy_2 \right]$$

[where $\lambda_1, \lambda_2$, are two eigenvalues of $\Sigma$]

$$\geq \frac{1}{2\pi \sqrt{\lambda_1 \lambda_2}} \left[ \int_{y_1 \geq 0, a_1 y_1 + a_2 y_2 < 0} \int_{\tan^{-1}(-a_1/a_2) + 2\pi}^{\tan^{-1}(-a_1/a_2) + 2\pi} r \, dr \, d\theta + \int_{y_1 < 0, a_1 y_1 + a_2 y_2 \geq 0} \int_{\tan^{-1}(-a_1/a_2) + \pi}^{\tan^{-1}(-a_1/a_2) + \pi} r \, dr \, d\theta \right]$$

[Polar]
= \frac{1}{\pi} \sqrt{\frac{\lambda_2}{\lambda_1}} \left[ \tan^{-1}(a_1/a_2) + \frac{\pi}{2} \right] \tag{A.1}

Now, for \( ||\beta - \beta^0|| = \delta, a_1 = \langle \beta, \beta^0 \rangle = 1 - \frac{\delta^2}{2} \). Hence we get,

\[
\frac{\lambda_1}{\lambda_2} = \frac{a_1}{\sqrt{1 - a_1^2}} = \frac{1 - \frac{\delta^2}{2}}{\sqrt{1 - \left(1 - \frac{\delta^2}{2}\right)^2}} = \frac{1 - \frac{\delta^2}{2}}{\delta \sqrt{1 - \frac{\delta^2}{4}}}
\]

Using this we obtain,

\[
\left( \tan^{-1}(a_1/a_2) + \frac{\pi}{2} \right) = \left( \tan^{-1} \left[ -\frac{1 - \frac{\delta^2}{2}}{\delta \sqrt{1 - \frac{\delta^2}{4}}} \right] + \frac{\pi}{2} \right)
\]

It can be easily seen (i.e. by differentiating) that the function \( \frac{\tan^{-1} \left[ -\frac{1 - \frac{\delta^2}{2}}{\delta \sqrt{1 - \frac{\delta^2}{4}}} \right] + \frac{\pi}{2}}{\delta} \) is an increasing function of \( \delta \) for \( 0 \leq \delta \leq 2 \). More precisely, observing that

\[
\lim_{\delta \downarrow 0} \frac{\tan^{-1} \left[ -\frac{1 - \frac{\delta^2}{2}}{\delta \sqrt{1 - \frac{\delta^2}{4}}} \right] + \frac{\pi}{2}}{\delta} = 1
\]

we conclude, for \( 0 \leq \delta \leq 2 \):

\[
1 \leq \frac{\tan^{-1} \left[ -\frac{1 - \frac{\delta^2}{2}}{\delta \sqrt{1 - \frac{\delta^2}{4}}} \right] + \frac{\pi}{2}}{\delta} \leq \frac{\pi}{2}.
\]

In conjunction with A.1, this gives:

\[
\mathbb{P}_X(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \beta^0)) \geq \frac{1}{\pi} \sqrt{\frac{\lambda_2}{\lambda_1}} \|\beta - \beta^0\|_2.
\]

Finally using the fact that

\[
\lambda_{\min}(P\Sigma P^T) = \lambda_{\min}(\Sigma) \leq \lambda_2 \leq \lambda_1 \leq \lambda_{\max}(P\Sigma P^T) = \lambda_{\max}(\Sigma)
\]

we have \( \sqrt{\frac{\lambda_2}{\lambda_1}} \geq \sqrt{c_\lambda} \). Combining these, we conclude:

\[
\mathbb{P}_X(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \beta^0)) \geq \sqrt{c_\lambda} \|\beta - \beta^0\|_2.
\]

Now, on to general \( X \). By our assumption on \( X \) in the statement of the lemma, \( X \sim \mathcal{E}(0, \Sigma_p) \) i.e. \( X = \Sigma_p^{1/2} Y \) for some spherically symmetric random variable \( Y \). We know

\[
Y \overset{d}{=} \frac{Z}{\|Z\|} g(\|Z\|)
\]
for some $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $Z \sim \mathcal{N}(0, I_p)$. Using the relation we have:

$$
P_X(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \beta^0))
= P_X \left( \text{sgn} \left( \frac{\Sigma^{1/2} Z}{\|Z\|} g(\|Z\|) \right) \neq \text{sgn} \left( \frac{(\Sigma_p^{1/2} Z)^T \beta^0}{\|Z\|} g(\|Z\|) \right) \right)
= P_X \left( \text{sgn} \left( \Sigma^{1/2} Z \right) \neq \text{sgn} \left( (\Sigma_p^{1/2} Z)^T \beta^0 \right) \right)
$$

which again falls back to $\mathcal{N}(0, \Sigma_p)$ situation. The upper bound can be established via a similar calculation, where we need a finite upper bound on $\sup_p \lambda_{\min}(\Sigma_p)$: this is given by $\frac{1}{c_X}$.

**Lemma A.3.** If the function $\eta(x)$ satisfies that

$$|\eta(x) - 1/2| = |F_{1[X=x]}(-x^T \beta^0) - F_{1[X=x]}(0)| \leq k |x^T \beta^0|,$$

for some constant $k$ a.e. with respect to the measure of $X$ and the distribution of $X$ follows a consistent family of elliptical distribution with $f_X(x) = |\Sigma_p|^{-1/2} g_p(x^T \Sigma_p^{-1} x)$. Assume that $g_2$ (the density component corresponding to the two dimensional marginal of $X$) is a decreasing function on $R$ and the eigenvalues of orientation matrix $\Sigma_p$ satisfies:

$$0 < \lambda_- \leq \lambda_{\min}(\Sigma_p) \leq \lambda_{\max}(\Sigma_p) \leq \lambda_+ < \infty$$

for all $p$. Then, under the first condition of Assumption (A2:upper) we have

$$S(\beta^0) - S(\beta) \leq u_+ \|\beta - \beta^0\|^2_2$$

for all $\beta \in S^{p-1}$ where $u_+ = 4\pi k k_1 \frac{\lambda_+}{\sqrt{\lambda_-}}$.

**Proof.** As in the proof of proposition 3.1 we have (with the same notation):

\[
\begin{align*}
S(\beta^0) - S(\beta) &= 4 \int_{X_\beta} |\eta(x) - 1/2| f(x) \, dx \\
&\leq 4k \int_{X_\beta} |x^T \beta^0| f(x) \, dx \\
&= 4k \left[ \int_{y_1 \geq 0} \left| y_1 \right| f_{Y_1,Y_2}(y_1, y_2) \, dy_1 dy_2 + \int_{y_1 < 0} \left| y_1 \right| f_{Y_1,Y_2}(y_1, y_2) \, dy_1 dy_2 \right] \\
&= 4k |\tilde{\Sigma}|^{-1/2} \left[ \int_{y_1 \geq 0} \left| y_1 \right| g_2(y^T \tilde{\Sigma}^{-1} y) \, dy_1 dy_2 + \int_{y_1 < 0} \left| y_1 \right| g_2(y^T \tilde{\Sigma}^{-1} y) \, dy_1 dy_2 \right] \\
&= 4k |\tilde{\Sigma}|^{-1/2} \left[ \int_{\frac{\pi}{2}}^{\frac{\pi}{2}(a_2/a_1)+\pi} \int_0^\infty r^2 |\cos(\theta)| g_2(r^2/\lambda_2) \, dr \, d\theta + \int_0^{\tan^{-1}(a_2/a_1)+\pi} \int_0^\infty r^2 |\cos(\theta)| g_2(r^2/\lambda_2) \, dr \, d\theta \right] \quad [\lambda_1 \leq \lambda_2 are the two eigenvalues of \tilde{\Sigma}]
\end{align*}
\]
\[
\leq 8kk_1 \frac{\lambda_2}{\sqrt{\lambda_1}} \cos (\tan^{-1}(a_1/a_2))(\tan^{-1}(a_1/a_2) + \pi/2)
\]
\[
\leq 4\pi kk_1 \frac{\lambda_2}{\sqrt{\lambda_1}} \| \beta - \beta^0 \|_2 \leq 4\pi kk_1 \frac{\lambda_+}{\sqrt{\lambda_-}} \| \beta - \beta^0 \|_2^2
\]

\[\square\]

**Remark A.1.** The Lipschitz type condition i.e. \(|\eta(x) - 1/2| \leq k|x^T \beta^0|\) controls how the function varies around the true hyperplane. This condition is easily satisfied if we assume that the conditional density of \(\epsilon\) given \(X\) has an uniform upper bound over all \(x\) and dimension. Note that the two conditions in the above lemma and Assumption (A2:upper(i)) are readily satisfied, for example, for a broad class of elliptically symmetric densities centered at 0.

### A.3. Proof of Theorem 3.2

We use a concentration inequality for excess risk due to (Theorem 2 of [15]) to establish the rate of convergence of the maximum score estimator. We provide a restatement of that theorem here for convenience.

**Theorem A.4.** Let \(\{Z_i = (X_i, Y_i)\}_{i=1}^n\) be i.i.d. observations taking values in the sample space \(Z : X \times Y\) and let \(F\) be a class of real-valued functions defined on \(X\). Let \(\gamma : F \times Z \rightarrow [0,1]\) be a loss function, and suppose that \(f^* \in F\) uniquely minimizes the expected loss function \(P(\gamma(f, .))\) over \(F\). Define the empirical risk as \(\gamma_n(f) = (1/n) \sum_{i=1}^n \gamma(f, Z_i)\), and \(\bar{\gamma}_n(f) = \gamma_n(f) - P(\gamma(f, .))\). Let \(l(f^*, f) = P(\gamma(f, .)) - P(\gamma(f^*, .))\) be the excess risk. Consider a pseudo-distance \(d\) on \(F \times F\) satisfying \(\text{Var}_P[\gamma(f,.) - \gamma(g,.)] \leq d^2(f,g)\). Finally, let \(C_1 := \{h : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \text{ non-decreasing, continuous, } h(x)/x \text{ non-increasing on } [0, \infty) \text{ and } h(1) \geq 1\}\). Assume that:

1. There exists \(F \subseteq F\) and a countable subset \(F' \subseteq F\), such that for each \(f \in F\), there is a sequence \(\{f_k\}\) of elements of \(F'\) satisfying \(\gamma(f_k, z) \rightarrow \gamma(f, z)\) as \(k \rightarrow \infty\), for every \(z \in Z\).
2. \(d(f, f^*) \leq \omega \left( \sqrt{l(f^*, f)} \right) \forall f \in F\), where \(\omega \in C_1\)
3. For every \(f \in F'\)

\[
\sqrt{n} \mathbb{E} \left[ \sup_{g \in F' : d(f, g) \leq \sigma} \left| \gamma_n(f) - \bar{\gamma}_n(g) \right| \right] \leq \phi(\sigma)
\]

for every \(\sigma > 0\) such that \(\phi(\sigma) \leq \sqrt{n}\sigma\), where \(\phi \in C_1\).

Let \(\epsilon_*\) be the unique positive solution of \(\sqrt{n}\epsilon_*^2 = \phi(\omega(\epsilon_*))\). Let \(\hat{f}\) be the (empirical) minimizer of \(\gamma_n\) over \(F\) and \(l(f^*, F) = \inf_{f \in F} l(f^*, f)\). Then, there exists an absolute constant \(K\) such that for all \(y \geq 1\), the following inequality holds:

\[
\mathbb{P} \left( l(f^*, \hat{f}) > 2l(f^*, F) + Ky\epsilon_*^2 \right) \leq e^{-y}
\]
In our problem, the set of classifiers
\[ \mathcal{F} = \mathcal{F}_p = \{ f_\beta : \mathbb{R}^p \to \{-1, 1\}, \ f_\beta(X) = \text{sgn}(X^T \beta), \beta \in S^{p-1} \} , \]
and \( \mathcal{Z} = \mathbb{R}^p \times \{-1, 1\} \). Define \( \gamma(f_\beta, (X, Y)) = (1 - Y \text{sgn}(X^T \beta))/2 \), so that \( \gamma_n(f_\beta) = (1 - S_n(\beta))/2 \) and \( P(\gamma(f_\beta, .)) = (1 - S(\beta))/2 \), and consequently, \( \tilde{\gamma}_n(f_\beta) = -(S_n(\beta) - S(\beta))/2 \). Also, note that \( f^* \equiv f_{\beta_0} \) and the excess risk is \( l(f_{\beta_0}, f_\beta) = (S(\beta_0) - S(\beta))/2 \). To verify the first assumption, take \( \mathcal{F} = F \) and take \( F' = \{ f_\beta \in F : \beta \in S_1 \} \) where \( S_1 \) is a countable dense subset of \( S^{p-1} \). It is easy to check that the convergence criterion in condition (1) of the above theorem is satisfied on the set \( X_0 \times \{-1, 1\} \) where \( X_0 \) is the set of all \( x \) such that \( \beta^T x \neq 0 \) for all \( \beta \in S_1 \). Since \( X_1 \) is continuous and \( S_1 \) is countable, \( X_0 \) has probability 1, and this is sufficient for the conclusions of the theorem to hold.

As shown in Proposition 3.1, for margin parameter \( \alpha \) we have:
\[ l(f_{\beta_0}, f_\beta) = (S(\beta_0) - S(\beta))/2 \geq h^n \mathbb{P}^\kappa(\text{sgn}(X^T_\beta) \neq \text{sgn}(X^T_{\beta_0})) \]
for some constant \( h \), where \( \kappa = (1 + \alpha)/\alpha \). Define the pseudo-distance
\[ d(f_{\beta_1}, f_{\beta_2}) = P^{1/2}(\text{sgn}(X^T_{\beta_1}) \neq \text{sgn}(X^T_{\beta_2})) \]
This is a valid distance because,
\[
\text{Var}_P(\gamma(f_{\beta_1}, .) - \gamma(f_{\beta_2}, .)) = \frac{1}{4} \text{Var}_P(Y \text{sgn}(X^T \beta_1) - Y \text{sgn}(X^T \beta_2)) \\
\leq \frac{1}{4} \mathbb{E}(\text{sgn}(X^T \beta_1) - \text{sgn}(X^T \beta_2))^2 \\
= \mathbb{E}(\mathbb{I}(\text{sgn}(X^T \beta_1) \neq \text{sgn}(X^T \beta_2))) \\
= \mathbb{P}(\text{sgn}(X^T \beta_1) \neq \text{sgn}(X^T \beta_2)) \\
= d^2(f_{\beta_1}, f_{\beta_2})
\]
Hence we have \( l(f_{\beta_0}, f(\beta)) \geq h^n d^2(\beta, f_{\beta_0}) \). If we take \( \omega(x) = h^{-1/2} x^{1/\kappa} \), then assumption (2) of the previous theorem is satisfied.

Let \( \mathcal{H}(\mathcal{F}) := \log | \{ \{ x : \beta^T x > 0 \} \cap \{ X_1, X_2, \ldots, X_n \} : \| \beta \| = 1 \} | \) denote the random combinatorial entropy of \( \mathcal{F} \). From the arguments in Section 2.4 of [15], we can take
\[ \phi(\sigma) = L \sigma \sqrt{(1 + \mathbb{E}(\mathcal{H}(\mathcal{F})))/n} \]
and
\[ \epsilon^2 \leq \left( \frac{L^2 (1 + \mathbb{E}(\mathcal{H}(\mathcal{F})))}{n} \right)^{\frac{\kappa}{\alpha - 1}} \sqrt{\frac{L^2 (1 + \mathbb{E}(\mathcal{H}(\mathcal{F})))}{n}} ,
\]
for some constants \( L > 0 \). Since the collection \( \mathcal{F} \) has VC dimension \( p \), using Sauer’s Lemma (see e.g. [10]), we know that \( \mathcal{H}(\mathcal{F}) \) is uniformly bounded by \( p(1 + \log (n/p)) \), and therefore:
\[ \mathbb{E}(\mathcal{H}(\mathcal{F})) \leq p \left( 1 + \log \left( \frac{n}{p} \right) \right) \leq 2p \log \frac{n}{p} \quad [\text{As } p/n \to 0] .
\]
Using the upper bound on the expectation, it follows that
\[ \epsilon^2 \leq \left( \frac{L^2 p \log (n/p)}{n} \right)^{\frac{\kappa}{\alpha - 1}} , \]
when $\kappa > 1$ (or $\alpha < \infty$). We conclude, using Theorem A.4, that:

$$
P \left( S(\beta^0) - S(\hat{\beta}) \geq Ky \left( \frac{L^2 p \log n / p}{n} \right)^{\frac{\kappa}{2\kappa - 1}} \right) \leq e^{-y}
$$

for all $y \geq 1$. Using Proposition 3.1, we get the following concentration bound:

$$
P \left( \|\hat{\beta} - \beta^0\|_2 \geq t \left( \frac{p \log n / p}{n} \right)^{\frac{1}{2\kappa - 1}} \right) \leq P \left( S(\beta^0) - S(\hat{\beta}) \geq t^\kappa u_\left( \frac{p \log n / p}{n} \right)^{\frac{\kappa}{2\kappa - 1}} \right) \leq e^{-t^\kappa \kappa \kappa^{-1} \frac{u_{-}}{u -}} \ \text{[For } t \geq (K_1K^{2\kappa/\kappa - 1}/u_\left)^{1/\kappa}]}
$$

which tends to 0 as $t \to \infty$. Hence we can conclude that

$$
\|\hat{\beta} - \beta^0\|_2 = O_P \left( \left( \frac{p \log n / p}{n} \right)^{\frac{1}{2\kappa - 1}} \right) = O_P \left( \left( \frac{p \log n / p}{n} \right)^{\frac{\alpha \alpha}{2\kappa - 1}} \right).
$$

The upper bound on the expectation follows from this exponential tail bound on $\|\hat{\beta} - \beta^0\|_2$.

Define $r_n = \left( \frac{p \log n / p}{n} \right)^{\frac{\alpha}{2\kappa - 2}}$. Then:

$$
\mathbb{E} \left( \frac{1}{r_n} \|\hat{\beta} - \beta^0\|_2 \right) = \int_0^{\infty} \mathbb{P}(\|\hat{\beta} - \beta^0\|_2 \geq Tr_n) dT
$$

$$
= \int_0^{(K_1K^{2\kappa/\kappa - 1}/u_\left)^{1/\kappa}} \mathbb{P}(\|\hat{\beta} - \beta^0\|_2 \geq Tr_n) dT + \int_{(K_1K^{2\kappa/\kappa - 1}/u_\left)^{1/\kappa}}^{\infty} \mathbb{P}(\|\hat{\beta} - \beta^0\|_2 \geq Tr_n) dT
$$

$$
\leq (K_1K^{2\kappa/\kappa - 1}/u_\left)^{1/\kappa} + \int_{(K_1K^{2\kappa/\kappa - 1}/u_\left)^{1/\kappa}}^{\infty} e^{-\frac{Tr_n}{K_1}} dT
$$

$$
\leq (K_1K^{2\kappa/\kappa - 1}/u_\left)^{1/\kappa} + f(\kappa) < \infty
$$

which completes the proof of minimax upper bound.

### A.4. Proof of Theorem 3.3

We generate $a_n(8n/p)^{\frac{p - 1}{4}}$ points uniformly from the surface of the sphere (where $a_n \uparrow \infty$ will be chosen later), maximize the empirical score function $S_n(\beta)$ over these selected points and show that the maximizer achieves the desired rate. Define $T_n = a_n(8n/p)^{(p-1)/3}$ and $E_n$ to be the collection of $T_n$ points generated uniformly. We start with the following technical lemma that plays a key role in the proof.

**Lemma A.5.** Suppose $D(x, r)$ denotes a spherical cap around $x$ of radius $r$, i.e.

$$
D(x, r) = \{y \in S^{p-1} : \|x - y\|_2 \leq r\}
$$
Then we have
\[
\frac{1}{2}(r/2)^{p-1} \leq \sigma(D(x, r)) \leq \frac{1}{2\sqrt{2}} r^{p-1}
\]
for \(0 \leq r \leq 1\) and \(p \geq 8\), where \(\sigma\) is the uniform measure on the sphere, i.e. the proportion of the surface of the spherical cap to the surface area of the sphere.

For a brief discussion on this Lemma, see Appendix B.

The next lemma shows that we can find at least one point in our collection which is within a distance of \((p/n)^{1/3}\) of \(\beta_0\) with probability \(\uparrow 1\).

**Lemma A.6.** Let \(\Omega_{-1,n}\) denote the event that there exists at least one \(\beta' \in E_n\) such that \(||\beta' - \beta^0||_2 \leq (p/n)^{1/3}\). Then \(P(\Omega_{-1,n}) \rightarrow 1\).

**Proof.** Using Lemma A.5 we have the following bound:
\[
\mathbb{P}(\exists \beta' \in E_n \text{ such that } ||\beta' - \beta^0||_2 \leq (p/n)^{1/3}) \geq 1 - \left(1 - \frac{1}{2}(p/8n)^{p+1}\right) a_n(8n/p)^{p-1} \rightarrow 1 \text{ as } n \rightarrow \infty. \]
\[\square\]

Let \(\tilde{\beta}\) denote the point closest to \(\beta^0\). On \(\Omega_{-1,n}\), \(||\tilde{\beta} - \beta^0|| \leq (p/n)^{1/3}\). To establish the convergence rate, we will use a specific version of the shelling argument. Fix \(T > 0\), sufficiently large. (In fact, as we work our way through the proof we will keep enhancing the value of \(T\) as and when necessary, but as this will be done finitely many times, it won’t have a bearing on the rate of convergence.) Consider shells \(C_i\) around the true parameter \(\beta^0\), where
\[
C_i = \{\beta \in S^{p-1} : ||\beta - \beta^0||_2 \leq T(p/n)^{1/3}2^i\} = D(\beta^0, r_i)
\]
with \(r_i = T(p/n)^{1/3}2^i\), for \(i = 0, 1, \ldots, A_n\) and \(A_n \xrightarrow{A} \frac{1}{3}\log_2 (n/p) - \log_2 T\). We will compute an upper bound on the number of elements of \(B_i = E_n \cap C_i\) for all \(i \in \{0, 1, \cdots, A_n\}\).

**Lemma A.7.** For all \(i \in \{0, 1, \cdots, A_n\}\),
\[
|B_i| = |E_n \cap C_i| \leq 2Tnp_i \leq \frac{1}{2\sqrt{2}} a_n(T2^{i+1})^{p-1} \leq a_n(T2^{i+1})^p
\]
with exponentially high probability where \(p_i = \sigma(D(\beta^0, r_i))\).

**Proof.** Let \(N_i\) denote the number of points in \(E_n \cap B_i\). Then \(N_i \sim \text{Bin}(T_n, p_i)\) where \(p_i = \sigma(D(\beta^0, r_i))\). For \(i = A_n, p_i = 1\). So \(P(N_i > 2Tnp_i) = 0\). Hence we will only confine ourselves to the case \(i \in \{0, 1, \cdots, A_n - 1\}\). In this case, \(r_i \leq 1\) and hence from Lemma A.5 we have \(p_i \leq \frac{1}{2\sqrt{2}} < \frac{1}{2}\). From the Chernoff tail bound for the Binomial distribution (see e.g. need reference) we have, for each \(i\): \(P(N_i > 2Tnp_i) \leq \exp(-T_n D(2p_i||p_i))\), where
\[
D(2p_i||p_i) = 2p_i \log \frac{2p_i}{p_i} + (1 - 2p_i) \log \frac{1 - 2p_i}{1 - p_i} = p_i \log 4 + (1 - 2p_i) \log \frac{1 - 2p_i}{1 - p_i}
\]
is the Kullback-Liebler divergence between Bernoulli($p_i$) and Bernoulli($2p_i$). This can be lower bounded thus:

\[
D(2p_i \mid p_i) = p_i \log 4 + (1 - 2p_i) \log \frac{1 - 2p_i}{1 - p_i}
= p_i \log 4 + (1 - 2p_i) \log \frac{1 - p_i - p_i}{1 - p_i}
= p_i \log 4 + (1 - 2p_i) \log \left(1 - \frac{p_i}{1 - p_i}\right)
\geq p_i \log 4 - (1 - 2p_i) \frac{p_i}{1 - p_i} \quad \text{[\because \log 1 - x \geq \frac{-x}{1 - x}, \ p_i \leq \frac{1}{2}]}
= p_i (\log 4 - 1)
\geq \frac{1}{2} (T2^i)^{p-1} \left(\frac{p}{8n}\right)^{p-1} (\log 4 - 1). \quad \text{[Lemma A.5]}
\]

Using this upper bound we have:

\[
P(N_i > 2T_n p_i) \leq e^{\left(-\frac{T_n}{2}(T2^i)^{p-1} \left(\frac{p}{8n}\right)^{p-1}(\log 4 - 1)\right)}
= e^{\left(-a_n(8n/p)^{p-1} \frac{1}{2}(T2^i)^{p-1} \left(\frac{p}{8n}\right)^{p-1}(\log 4 - 1)\right)}
= e^{\left(-a_n \frac{1}{2}(T2^i)^{p-1}(\log 4 - 1)\right)}
\]

\[\square\]

Define $\Omega_{i,n} = N_i \leq 2T_n p_i$ for $i = \{0, 1, \cdots, A_n - 1\}$ and let $\Omega_n = \cap_{i=-1}^{A_n} \Omega_{i,n}$. The following lemma says that the event $\Omega_n$ happens with high probability:

**Lemma A.8.** For any $T > 1$, $\mathbb{P}(\Omega_n) \to 1$ as $n \to \infty$.

**Proof.** It is enough to show that $\sum_{i=-1}^{A_n} \mathbb{P}(\Omega_{i,n}^c) \to 0$ as $n \to \infty$. We have already established in Lemma A.6 that $\mathbb{P}(\Omega_{-1,n}^c) \to 0$ as $n \to \infty$. Using Lemma A.7:

\[
\sum_{i=0}^{A_n-1} \mathbb{P}(\Omega_{i,n}^c) \leq \sum_{i=0}^{A_n-1} e^{\left(-a_n \frac{1}{2}(T2^i)^{p-1}(\log 4 - 1)\right)}
\]

Now for any fixed $n$, the maximum term obtains when $i = 0$ i.e. $e^{\left(-a_n \frac{1}{2}(T)^{p-1}(\log 4 - 1)\right)}$ which goes to $0$ for $T > 1$. Furthermore, the series under consideration is easily dominated by $\sum_{i=1}^{\infty} e^{-k2^i}$ for some constant $k > 0$, which is clearly finite. Hence the series on the right-side of the above display goes to $0$ with increasing $n$. \[\square\]

The rest of the analysis will be done conditioning on the event $\Omega_n$. Define $\mathbb{P}_n(A) = \mathbb{P}(A \mid \Omega_n)$. Then we have:

\[
\mathbb{P}\left(\|\hat{\beta}\_\text{new} - \beta^0\|_2 > 3T(p/n)^{1/3}\right)
\leq \mathbb{P}\left(\|\hat{\beta}\_\text{new} - \beta^0\|_2 > 3T(p/n)^{1/3} \mid \Omega_n\right) \mathbb{P}(\Omega_n) + \mathbb{P}(\Omega_n^c)
\]
For the rest of the calculations we need to bound $T > 3T(p/n)^{1/3}$ for $\sum_{i=1}^{n} Z_i(\beta) - S_n(\beta)$ and $Z_i(\beta)$ assumes values $\{-2, 0, 2\}$. Also, $E(Z_i(\beta)) = S(\beta) - S(\tilde{\beta})$. Using Proposition 3.1 and Assumption (A2:upper) we have:

$$S(\beta) - S(\tilde{\beta}) = S(\beta) - S(\beta^0) + S(\beta^0) - S(\tilde{\beta})$$

$$\leq -u_-||\beta - \beta^0||_2 + u^+||\tilde{\beta} - \beta^0||_2$$

$$\leq -(u_-/2)||\beta - \beta^0||_2$$

for $T > \sqrt{(2u_+)/u_-}$. This implies $Z_i(\beta)$ has high probability of being negative. We exploit this to prove the concentration. To simplify the calculations, define $\{Y_i(\beta)\}_{i=1}^{n}$ be to be a collection of independent random variables with

$$Y_i(\beta) = \begin{cases} 2, & \text{with prob. } P(Z_i(\beta) = 2 \mid Z_i(\beta) \neq 0) \\ -2, & \text{with prob. } P(Z_i(\beta) = -2 \mid Z_i(\beta) \neq 0) \end{cases}$$

Hence the expectation of $Y_i(\beta)$ is:

$$E(Y_i(\beta)) = E(Z_i(\beta))|Z_i(\beta) \neq 0$$

$$= \frac{E(Z_i(\beta))|Z_i(\beta) \neq 0)P(Z_i(\beta) \neq 0)}{P(Z_i(\beta) \neq 0)}$$

$$= \frac{P(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\tilde{\beta}))}{P(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\tilde{\beta}))}$$

$$\leq \frac{-(u_-/2)||\beta - \beta^0||_2^2}{P(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\tilde{\beta}))}$$

(A.2)

For the rest of the calculations we need to bound $P(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\tilde{\beta}))$. Towards that direction we have the following:

$$P(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\tilde{\beta}))$$

$$= P(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\tilde{\beta}), \text{sgn}(X^T\tilde{\beta}) \neq \text{sgn}(X^T\beta^0))$$

$$+ P(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\tilde{\beta}), \text{sgn}(X^T\tilde{\beta}) = \text{sgn}(X^T\beta^0))$$

$$\leq P(\text{sgn}(X^T\beta) \neq \text{sgn}(X^T\beta^0), \text{sgn}(X^T\tilde{\beta}) = \text{sgn}(X^T\beta^0)) + P(\text{sgn}(X^T\tilde{\beta}) \neq \text{sgn}(X^T\beta^0))$$
\[\leq \mathbb{P}(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \beta^0)) + \mathbb{P}(\text{sgn}(X^T \tilde{\beta}) \neq \text{sgn}(X^T \beta^0))\]
\[\leq 2C' \|\beta - \beta^0\|_2\]

for \( T > 1 \). For the lower bound we have:

\[\mathbb{P}(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \tilde{\beta}))\]
\[\geq \mathbb{P}(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \tilde{\beta}), \text{sgn}(X^T \tilde{\beta}) = \text{sgn}(X^T \beta^0))\]
\[= \mathbb{P}(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \beta^0), \text{sgn}(X^T \tilde{\beta}) = \text{sgn}(X^T \beta^0))\]
\[\geq \mathbb{P}(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \beta^0)) - \mathbb{P}(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \beta^0))\]
\[\geq c' \|\beta - \beta^0\|_2 - C'(p/n)^{1/3}\]
\[\geq (c'/2) \|\beta - \beta^0\|_2\]

when \( T > 2C'/c' \), where we are also using the fact that \( \beta \in B_0 \). Putting the upper bound in equation A.2 we have:

\[E(Y_i(\beta)) \leq -\frac{u}{4C'} \|\beta - \beta^0\|_2\]

So, if \( P(Y_i(\beta) = 2) = P(Z_i(\beta) = 2 \mid Z_i(\beta) \neq 0) = p_2 \) (say) then \( 4p_2 - 2 \leq (-u_-/4C')\|\beta - \beta^0\|_2 \) which implies \( p_2 \leq \frac{1}{2} - (-u_-/16C')\|\beta - \beta^0\|_2 \). Define \( W_i(\beta) = \frac{Y_i(\beta) + 2}{4} \). Then \( W_i(\beta) \sim \text{Ber}(p_2) \). Let \( N \) denote the number of non-zero \( Z_i(\beta) \)'s. Then \( N \sim \text{Bin}(n, p_1) \) where \( p_1 = \mathbb{P}(\text{sgn}(X^T \beta) \neq \text{sgn}(X^T \tilde{\beta})) \).

\[\mathbb{P}_n \left( \sup_{\beta \in B_i \cap B_{i-1}^c} S_n(\beta) - S_n(\tilde{\beta}) \geq 0 \right)\]
\[\leq \sum_{j: \beta_j \in B_i \cap B_{i-1}^c} \mathbb{P}_n \left( S_n(\beta_j) - S_n(\tilde{\beta}) \geq 0 \right)\]
\[\leq \sum_{j: \beta_j \in B_i \cap B_{i-1}^c} \mathbb{P}_n \left( \sum_{i=1}^n Z_i(\beta_j) \geq 0 \right)\]
\[\leq \sum_{j: \beta_j \in B_i \cap B_{i-1}^c} \sum_{m=1}^n \mathbb{P}_n \left( \sum_{i=1}^m Z_i(\beta_j) \geq 0 \mid N = m \right) \mathbb{P}(N = m)\]
\[\leq \sum_{j: \beta_j \in B_i \cap B_{i-1}^c} \sum_{m=1}^n \mathbb{P}_n \left( \sum_{i=1}^m Y_i(\beta_j) \geq 0 \right) \mathbb{P}(N = m)\]
\[\leq \sum_{j: \beta_j \in B_i \cap B_{i-1}^c} \sum_{m=1}^n \mathbb{P}_n \left( \sum_{i=1}^m W_i(\beta_j) \geq \frac{m}{2} \right) \mathbb{P}(N = m)\]
\[\leq \sum_{j: \beta_j \in B_i \cap B_{i-1}^c} \sum_{m=1}^n (4p_2q_2)^m/2 \left( \frac{n}{m} \right) p_1^m (1 - p_1)^{n-m} \quad \text{[Chernoff bound for Binomial tail probability]}\]
\[\leq \sum_{j: \beta_j \in B_i \cap B_{i-1}^c} (1 - p_1 + 2p_1\sqrt{p_2q_2})^n\]
\[\leq \sum_{j: \beta_j \in B_i \cap B_{i-1}^c} e^n \log(1 - p_1(1 - \sqrt{4p_2q_2}))\]
Thus we can take $a_n = p$ to ignore the effect of $(\log a_n / p)$. Putting this back in equation (1) we get:

\[
\sum_{i=1}^{A_n} \mathbb{P} \left( \sup_{\beta \in B_i \cap B_{i-1}^c} S_n(\beta) - S_n(\hat{\beta}) \geq 0 \right) \\
\leq 2 \sum_{i=1}^{A_n} e^p [\log T + i \log 2 - (u^2 \epsilon' / 256C^2) T^{32^3(i-1)}]
\rightarrow 0 \text{ as } p \to \infty ,
\]

for large enough $T$ (by using similar arguments to the one used for handling the earlier series) which proves the theorem.

A.5. Proof of Theorem 3.4

To obtain a lower bound on the minimax error, we use Assouad’s Lemma [3] which we state below for convenience:

**Lemma A.9. [Assouad’s Lemma]** Let $\Omega = \{0, 1\}^m$ (or $\{-1, 1\}^m$) be the set of all binary sequences of length $m$. Let $P_\omega, \omega \in \Omega$ be a set of $2^m$ measures on some space $\{X, A\}$ and let the corresponding expectations be $E_\omega$. Then:

\[\inf_{\hat{\omega}} \sup_{\omega \in \Omega} \mathbb{E}_\omega (d_H(\hat{\omega}, \omega)) \geq \frac{m}{2} (1 - \max_{\omega, \omega'} ||P_n^\omega - P_n^{\omega'}||_{TV})\]

where $\hat{\omega}$ is an estimator based on $n$ i.i.d. observations $z_1, \ldots, z_n \sim P_\omega$, $P_n^\omega$ denotes the $n$-fold product measure of $P_\omega$, $d_H$ is the Hamming distance and $\omega \sim \omega'$ means $d_H(\omega, \omega') = 1$.\(^2\)

To apply this lemma in our model, define for small $\epsilon > 0$:

\[\tilde{\Theta} = \{\beta : \text{ where } \beta = \frac{\gamma}{\|\gamma\|_2}, \gamma_1 = 1, \gamma_j \in \{-\epsilon, \epsilon\}, \forall \ 2 \leq j \leq p\}.\]

\(^2\)For some discussions and applications of this lemma, see [24].
We will motivate the choice of $\epsilon$ in the later part of the proof. Observe that, $\|\gamma\|_2$ is same for all $\gamma \in \Theta$ and equals $\sqrt{1 + (p-1)\epsilon^2}$. For notational simplicity, define $m(\epsilon) = \sqrt{1 + (p-1)\epsilon^2}$. Now, for any $\omega \in \{1,1\}^{p-1}$, define $\gamma_\omega = (1, \epsilon \omega)$ and $\beta_\omega = \gamma_\omega / \|\gamma_\omega\|_2$. This establishes a 1-1 correspondence between $\Omega$ and $\Theta$, with $m = p - 1$. For any $\beta \in \Theta$ define the joint distribution $P_B$ of $(X,Y)$ as:

1. $X \sim \mathcal{N}(0, I_p)$
2. $P_B(Y = 1|X) = \begin{cases} \frac{1}{2} + \beta'X, & \text{if } |\beta'X| \leq \left[ \epsilon \sqrt{p} \vee \frac{|X_1|}{2m(\epsilon)} \right] \wedge 1/4. \\ \frac{1}{2} + \left( \epsilon \sqrt{p} \vee \frac{|X_1|}{2m(\epsilon)} \right) \wedge 1/4 \text{sgn}(\beta'X), & \text{otherwise.} \end{cases}$

The Gaussian distribution of $X$ trivially satisfies Assumption (A2). In the following lemma we show that this construction also satisfies Assumption (A1).

**Lemma A.10.** The above construction of $\eta(x) = P_B(Y = 1|X = x)$ satisfies Assumption (A1) with $\alpha = 1$, when $\epsilon \sqrt{p}$ is sufficiently small.

**Proof.** Fix $t$ such that $0 \leq t < \frac{1}{4}$. Then,

$$
P_X(\|\eta(X) - 0.5\| \leq t) = P_X(\|\eta(X) - 0.5\| \leq t, |\beta'X| \leq \left[ \epsilon \sqrt{p} \vee \frac{|X_1|}{2m(\epsilon)} \right] \wedge 1/4) + P_X(\|\eta(X) - 0.5\| \leq t, |\beta'X| > \left[ \epsilon \sqrt{p} \vee \frac{|X_1|}{2m(\epsilon)} \right] \wedge 1/4) \leq P_X(|\beta^T X| \leq t, |\beta^T X| \leq \left[ \epsilon \sqrt{p} \vee \frac{|X_1|}{2m(\epsilon)} \right] \wedge 1/4) + P_X\left( \left[ \epsilon \sqrt{p} \vee \frac{|X_1|}{2m(\epsilon)} \right] \wedge 1/4 \leq t, |\beta^T X| > \left[ \epsilon \sqrt{p} \vee \frac{|X_1|}{2m(\epsilon)} \right] \wedge 1/4 \right) \leq P_X(|\beta^T X| \leq t) + P_X\left( \left[ \epsilon \sqrt{p} \vee \frac{|X_1|}{2m(\epsilon)} \right] \wedge 1/4 \leq t \right) \leq P_X(|\beta^T X| \leq t) + P_X(|X_1| \leq 2m(\epsilon)t) \leq \sqrt{\frac{2}{\pi}} \left[ t + 2m(\epsilon)t \right] \leq 5 \sqrt{\frac{2}{\pi}}t \]

The last inequality is valid when $m(\epsilon) \leq 2$, which happens for $\epsilon \sqrt{p}$ sufficiently small. \hfill \Box

We use the notation $\beta \sim_j \beta'$ if $\beta$ and $\beta'$ differs only in $j^{th}$ position for $2 \leq j \leq p$. So, in order use Assouad’s lemma, we need an on $\|P_B^n - P_{\beta'}^n\|_{TV}$ when $\beta \sim_j \beta'$ for any $2 \leq j \leq p$. Fix $\beta_1$ and $\beta_2$ and $j \in \{2, \cdots, p\}$ such that $\beta_1 \sim_j \beta_2$. Using the standard relation between the total variation norm and Hellinger distance, we have:

$$
\|P_B^n - P_{\beta_2}^n\|_{TV} \leq \sqrt{H^2(P_{\beta_1}^n, P_{\beta_2}^n)} \leq \sqrt{nH^2(P_{\beta_1}, P_{\beta_2})}
$$

To make the minimax lower bound non-trivial, we will choose $\epsilon = \epsilon(n, p)$ in a way that ensures $H^2(P_{\beta_1}, P_{\beta_2}) \sim n^{-1}$. Towards that, we need the following lemma:
Lemma A.11. If $P_1 = Ber(p_1)$ and $P_2 = Ber(p_2)$ with $p_1, p_2 \in [1/4, 3/4]$, then $H^2(P_1, P_2) \leq \frac{\nu^2}{4\sqrt{3}s(1-s)}$ where $\nu = p_2 - p_1$, $s = (p_1 + p_2)/2$.

The proof of this Lemma can be found in Appendix B. For the rest of the proof, define

$$A_i = \left[ |\beta_i'X| \leq \left\{ \epsilon \sqrt{p} \lor \frac{|X_1|}{2m(\epsilon)} \right\} \land 1/4 \right]$$

for $i = 1, 2$. Now,

$$H^2(P_{\beta_1}, P_{\beta_2}) = E_X \left[ H^2(P_{\beta_1}(Y = 1|X), P_{\beta_2}(Y = 1|X)) \right] = E_X \left[ H^2(P_{\beta_1,X}, P_{\beta_2,X}) \right] \quad \text{[Say]}$$

We next divide the domain of $X$ into two sub-parts and compute the corresponding values of $\nu_X = P_{\beta_1}(Y = 1|X) - P_{\beta_2}(Y = 1|X)$, on these sub-parts.

Case 1: $X \in A_1 \cup A_2$.

Case 2: $X \in A_1^c \cap A_2^c$. Note that, in this case, $|\nu_X| = 0$, if $\text{sign}(\beta_1'X) = \text{sign}(\beta_2'X)$, $|\nu_X| \leq 2 \left( \epsilon \sqrt{p} \lor \frac{|X_1|}{2m(\epsilon)} \right)$ otherwise.

Lemma A.12. Under Case 1, $|\nu_X| = |P_{\beta_1}(Y = 1|X) - P_{\beta_2}(Y = 1|X)| \leq 2\epsilon |X_j|/m(\epsilon)$ where $\beta_1 \sim_j \beta_2$.

Proof. First assume that, $X \in A_1 \cap A_2$. Then,

$$|\nu_X| = |P_{\beta_1}(Y = 1|X) - P_{\beta_2}(Y = 1|X)| = |\beta_1'X - \beta_2'X| = \frac{2\epsilon |X_j|}{m(\epsilon)}$$

Next, consider the case that $X \in A_1 \cap A_2^c$. Then, $|\beta_2'X| > (\epsilon \sqrt{p} \lor |X_1|/2m(\epsilon)) \land 1/4$ but $|\beta_1'X| < (\epsilon \sqrt{p} \lor |X_1|/2m(\epsilon)) \land 1/4$. Hence,

$$|\nu_X| = |P_{\beta_1}(Y = 1|X) - P_{\beta_2}(Y = 1|X)| \leq |\beta_1'X - \beta_2'X| = \frac{2\epsilon |X_j|}{m(\epsilon)}.$$

The third case follows in the exact same manner, by symmetry.

We are now in a position to tackle $H^2(P_{\beta_1}, P_{\beta_2})$ as shown below.

$$H^2(P_{\beta_1}, P_{\beta_2}) \quad \text{(A.3)}$$
\[ \mathbb{E}_X \left[ H^2(P_{\beta_1}, X, P_{\beta_2}, X) \right] \]
\[ \leq \frac{1}{4\sqrt{3}} \mathbb{E} \left[ \frac{\nu_X^2}{s(1-s)} \right] \]
\[ = \frac{1}{4\sqrt{3}} \mathbb{E} \left[ \frac{\nu_X^2}{s(1-s)} \mathbb{1}_{X \in A_1 \cup A_2} + \frac{\nu_X^2}{s(1-s)} \mathbb{1}_{X \in A_1^c \cap A_2^c} \right] \]
\[ \leq \frac{4}{3\sqrt{3}} \mathbb{E} \left[ \nu_X^2 \mathbb{1}_{X \in A_1 \cup A_2} + \nu_X^2 \mathbb{1}_{X \in A_1^c \cap A_2^c} \right] \quad [\because \frac{1}{4} \leq s \leq \frac{3}{4}] \]
\[ \leq \frac{16}{3\sqrt{3}} \mathbb{E} \left[ \frac{\varepsilon^2 X_j^2}{m(\varepsilon)^2} \mathbb{1}_{X \in A_1 \cup A_2} + 4 \left( \varepsilon \sqrt{p} \vee \frac{|X_1|}{2m(\varepsilon)} \right)^2 \mathbb{1}_{X \in A_1^c \cap A_2^c, \text{sign}(\beta_j X) \neq \text{sign}(\beta_j' X)} \right] \]
\[ = \frac{16}{3\sqrt{3}} \mathbb{E}[T_1 + 4T_2] \quad [\text{Say}]. \quad (A.4) \]

We will analyze the expectation of each summand separately. Define \( \tilde{\beta} \Delta = \beta_{[2:p]} \) i.e. \( \tilde{\beta} \) is a vector of dimension \((p - 1)\) which we obtain by removing the first coordinate of \( \beta \), and let \( \tilde{X} \) be defined similarly in terms of \( X \). We have:

\[ \mathbb{E}(T_1) = \frac{1}{m(\varepsilon)^2} \mathbb{E} \left( \varepsilon^2 X_j^2 \mathbb{1}_{X \in A_1 \cup A_2} \right) \]
\[ \leq \frac{1}{m(\varepsilon)^2} \left[ \mathbb{E} \left( \varepsilon^2 X_j^2 \mathbb{1}_{X \in A_1} \right) + \mathbb{E} \left( \varepsilon^2 X_j^2 \mathbb{1}_{X \in A_2} \right) \right] \]
\[ = \frac{2}{m(\varepsilon)^2} \mathbb{E} \left( \varepsilon^2 X_j^2 \mathbb{1}_{X \in A_1} \right) \quad [\because \text{both terms are identically distributed}] \]
\[ \leq \frac{2}{m(\varepsilon)^2} \mathbb{E} \left( \varepsilon^2 X_j^2 \mathbb{1}_{|\beta_j X| \leq \sqrt{p} \vee \frac{|X_1|}{2m(\varepsilon)}} \right) \]
\[ \leq \frac{2}{m(\varepsilon)^2} \left[ \varepsilon^2 \mathbb{E} \left( X_j^2 \mathbb{1}_{|\beta_j X| \leq \frac{|X_1|}{2m(\varepsilon)}} \right) + \varepsilon^2 \mathbb{E} \left( X_j^2 \mathbb{1}_{|X_1| \leq \sqrt{p}} \right) \right] \quad [\because |a + b| \geq |a| - |b|] \]
\[ \leq \frac{2\varepsilon^2}{m(\varepsilon)^2} \left[ \mathbb{E} \left( X_j^2 \mathbb{1}_{|X_1| \leq |\beta_j X|} \right) + \mathbb{E}(X_j^2) P(|X_1| \leq 2m(\varepsilon) \sqrt{p}) \right] \]
\[ \leq \frac{2\varepsilon^2}{m(\varepsilon)^2} \mathbb{E}_X \left( X_j^2 \mathbb{E}_X \left( \mathbb{1}_{|X_1| \leq 2m(\varepsilon) |\beta_j X|} \right) \right) + 4 \sqrt{\frac{2}{\pi m(\varepsilon)} \varepsilon^3 \sqrt{p}} \]
\[ \leq \frac{2\varepsilon^2}{m(\varepsilon)^2} \mathbb{E}_X \left( X_j^2 P \left( |X_1| \leq 2m(\varepsilon) |\beta_j X| | \tilde{X} \right) \right) + 4 \sqrt{\frac{2}{\pi m(\varepsilon)^2} \varepsilon^3 \sqrt{p}} \]
\[ \leq \sqrt{\frac{8}{\pi}} \left[ \frac{2\varepsilon^2}{m(\varepsilon)^2} \mathbb{E}_X \left( X_j^2 |\tilde{\beta} \tilde{X}| \right) \right] \quad [\text{as } X_1 \& \tilde{X} \text{ are independent}] \]
\[ \leq 4 \sqrt{\frac{2}{\pi}} \left[ \frac{\varepsilon^2}{m(\varepsilon)^2} \mathbb{E}(X_j^4) \right] \frac{1}{2} \left( \mathbb{E}(\mathbb{1}_{\tilde{\beta} \tilde{X}}) \right) \frac{1}{2} + \frac{\varepsilon^3 \sqrt{p}}{m(\varepsilon)} \]
\[ \leq 4 \sqrt{\frac{2}{\pi m(\varepsilon)}} \left( \mathbb{E}(\mathbb{1}_{\tilde{\beta} \tilde{X}}) \right) \frac{1}{2} + 4 \sqrt{\frac{2}{\pi m(\varepsilon)} \varepsilon^3 \sqrt{p}} \]
\[ \leq 4 \sqrt{\frac{2}{\pi (1 + \sqrt{3})}} e^3 \sqrt{p} \left[ \frac{1}{m(\varepsilon)^2} + \frac{1}{m(\varepsilon)} \right] \]
\[ \leq 8 \sqrt{\frac{2}{\pi}} (1 + \sqrt{3}) \epsilon^3 \sqrt{p} \quad [\because m(\epsilon) \geq 1] \quad (A.5) \]

For the second part, observe that,
\[
\{ X \in A_1^c \cap A_2^c \text{ and } \text{sign}(\beta_1' X) \neq \text{sign}(\beta_2' X) \} \Rightarrow |\beta_1' X - \beta_2' X| \geq 2\epsilon \sqrt{p} \\
\Rightarrow |X_j| \geq \frac{2\epsilon |X_j|}{m(\epsilon)} \geq 2\epsilon \sqrt{p} \\
\Rightarrow |X_j| \geq m(\epsilon) \sqrt{p}
\]

Using this observation, we get,
\[
E(T_2) \leq \left( E \left( \epsilon \sqrt{p} \frac{|X_j|}{2m(\epsilon)} \right) \right)^{\frac{1}{2}} \left( E \left( 1_{X \in A_1^c \cap A_2^c, \text{sign}(\beta_1' X) \neq \text{sign}(\beta_2' X)} \right) \right)^{\frac{1}{2}} \\
\leq K \left( E \left( 1_{X \in A_1^c \cap A_2^c, \text{sign}(\beta_1' X) \neq \text{sign}(\beta_2' X)} \right) \right)^{\frac{1}{2}} \\
\leq K \left( E \left( \frac{1}{2} |X_j| \geq m(\epsilon) \sqrt{p} \right) \right)^{\frac{1}{2}} \\
\leq Ke^{-\frac{m(\epsilon)^2 p}{4}}, \quad (A.6)
\]

where \( K \) is an absolute constant. Putting together A.4, A.5 and A.6, we get,
\[
H^2(P_{\beta_1}, P_{\beta_2}) = E_X \left( H^2(P_{\beta_1, X}, P_{\beta_2, X}) \right) \leq \frac{16}{3\sqrt{3}} \left[ 8 \sqrt{\frac{2}{\pi}} (1 + \sqrt{3}) \epsilon^3 \sqrt{p} + 4Ke^{-\frac{m(\epsilon)^2 p}{4}} \right] .
\]

Set \( \zeta = \frac{128}{3\sqrt{3}} \sqrt{\frac{2}{\pi}} (1 + \sqrt{3}) \). If we choose \( \epsilon = \left( \frac{1}{2\zeta} \right)^{\frac{1}{2}} n^{-\frac{1}{2}} p^{-\frac{1}{6}} \), then \( E_X \left( H^2(P_{\beta_1}, P_{\beta_2}) \right) \leq \frac{1}{2n} + \frac{64K}{3\sqrt{3}} e^{-\frac{m(\epsilon)^2 p}{4}} \). So we have
\[
\sqrt{n} H^2(P_{\beta_1}, P_{\beta_2}) \leq \sqrt{\frac{1}{2} + \frac{64K}{3\sqrt{3}} ne^{-\frac{m(\epsilon)^2 p}{4}}} \leq \sqrt{\frac{2}{3}}
\]

for all large \( n \), as \( ne^{-\frac{m(\epsilon)^2 p}{4}} \to 0 \), which follows from our assumption \( n = o(e^{fp}) \), for \( f < 1/4 \).

Now we can relate Hamming distance to \( \ell_2 \) distance via
\[
\| \hat{\beta} - \beta_0 \|_2^2 = \frac{\epsilon^2}{m(\epsilon)^2} d_H(\hat{\beta}, \beta_0)
\]

and use Assouad’s lemma to deduce:
\[
\inf_{\beta_n \in \hat{\Theta}} \sup_{P_{\beta_1}, \beta_2 \in \Theta} E_\beta \left( \| \hat{\beta}_n - \beta \|_2^2 \right) \geq \frac{\epsilon^2(p - 1)}{2m(\epsilon)^2} (1 - \max_{\beta \sim \beta_0} \| P_{\beta}^n - P_{\beta_0}^n \|_{TV}) \\
\geq \frac{\epsilon^2(p - 1)}{4} (1 - \sqrt{2/3}) \quad [\because m(\epsilon) \to 1 \text{ as } p \to \infty] \\
\geq K_L \left( \frac{p}{n} \right)^{\frac{1}{2}} \quad [\because \epsilon = \left( \frac{1}{2\zeta} \right)^{\frac{1}{2}} n^{-\frac{1}{2}} p^{-\frac{1}{6}}]
\]
for some constant $\tilde{K}_L$. Finally, let $\hat{\beta}$ be any estimator assuming values in $S^{p-1}$. Define $\tilde{\beta}$ to be the projection of $\hat{\beta}$ on the hypercube, i.e.

$$\tilde{\beta} = \arg\min_{\beta \in \tilde{\theta}} \|\hat{\beta} - \beta\|_2$$

Then for any $\beta \in \tilde{\theta}$ we have:

$$\|\tilde{\beta} - \beta\|_2 \leq \|\tilde{\beta} - \hat{\beta} + \hat{\beta} - \beta\|_2 \leq 2\|\hat{\beta} - \beta\|_2$$

Using this relation we can conclude that:

$$\inf_{\beta_n} \sup_{P_{\beta}} \mathbb{E}_{\beta} \left( \|\hat{\beta}_n - \beta\|_2^2 \right) \geq \frac{1}{4} \inf_{\beta_n \in \tilde{\theta}} \sup_{P_{\beta}; \beta \in \tilde{\theta}} \mathbb{E}_{\beta} \left( \|\hat{\beta}_n - \beta\|_2^2 \right) \geq K_L \left( \frac{P}{n} \right)^{\frac{3}{2}}$$

where $K_L = \tilde{K}_L/4$. This completes the proof.

**A.6. Proof of Theorem 3.5**

The proof depends on following adapted version of Theorem 9.1 of [4]:

**Theorem A.13.** Let $\mathcal{H}$ be a set of classifiers with VC Dimension $d$. Then, for any $t > 0$, with probability at least $1 - e^{-t}$, we can say:

1. $R(\hat{h}_{\mathcal{H}}) - \min_{h \in \mathcal{H}} R(h) \leq 8\sqrt{\frac{2d \log n}{n}} + 2\sqrt{\frac{2t}{n}} = 8\sqrt{2} \sqrt{\frac{d \log n}{n}} + 2\sqrt{2} \sqrt{\frac{t}{n}}$
2. $|R(\hat{h}_{\mathcal{H}}) - R_n(\hat{h}_{\mathcal{H}})| \leq 4\sqrt{\frac{2d \log n}{n}} + 2\sqrt{\frac{t}{2n}} = 4\sqrt{2} \sqrt{\frac{d \log n}{n}} + 2\sqrt{2} \sqrt{\frac{t}{n}}$,

where $\hat{h}_{\mathcal{H}}$ is the best possible empirical classifier in $\mathcal{H}$, the empirical loss $R_n$ is defined by:

$$R_n(h) \equiv R_n(h_{\beta}) = -S_n(\beta) = -\frac{1}{n} \sum_{i=1}^{n} Y_i sgn(X_i^T \beta),$$

and the population loss or risk $R$ as:

$$R(h) \equiv R(h_{\beta}) = -S(\beta) = -E(Y^* sgn(X^T \beta)).$$

For our problem, the set of classifiers under consideration is described by all unit norm vectors $\beta$ in $\mathbb{R}^p$ with number of non-zero co-ordinates bounded above by $s_0$. For a given model $m$, define $\hat{h}_m = \arg\min_{h_{\beta} \in \mathcal{H}_m} R(h_{\beta})$. As $m_2 \supset m^*$, we have $R(\hat{h}_{m_2}) < R(\hat{h}_{m_1})$, since $R(\hat{h}_{m_2})$ is the Bayes’ Risk. Hence:

$$R(\hat{h}_{m_1}) - R(\hat{h}_{m_2}) = -E \left[ Y^* sgn(X^T \beta_{m_1}) - Y^* sgn(X^T \beta_{m_2}) \right]$$
\[ S(\beta_{m_2}) - S(\beta_{m_1}) \geq u_- \|\beta_{m_1} - \beta_{m_2}\|^\kappa \]  
\[ \geq u_- |\beta_{0, \text{min}}|^\kappa \]  
\[ \geq 30\sqrt{2} \frac{2A_s \log (ep/s_0) \log n}{n} \]  
[Assumption(A4)] \hspace{1cm} (A.7)

Now we need an upper bound on the VC dimension of our models. As our collection comprises of all the models with sparsity \( s_0 \), the upper bound can be obtained from the following lemma:

**Lemma A.14.** Let \( C(s_0) = \{\eta(x) = 1_{\beta x \geq 0} : \beta \in \mathbb{R}^p, \|\beta\|_0 \leq s_0\} \) be the set of all \( s_0 \) sparse binary classifiers and \( V(C(s_0)) \) be its VC dimension. Then:

\[ A_s \log \left( \frac{2p}{s_0} \right) \leq V(C(s_0)) \leq 2A_s \log \left( \frac{ep}{s_0} \right) \]

where \( A = \log_2 e \).

**Proof.** See the proof of Lemma 1 in the paper [1] mentioned above. \( \square \)

Define \( V_n = 2A_s \log (ep/s_0) \log n \) where \( 2A_s \log (ep/s_0) \) is an upper bound on the VC dimension of models with number of active elements \( s_0 \) obtained from the above lemma. Take \( t = V_n \). Then, using Theorem A.13 there exists \( \Omega_t \) with probability \( \geq 1 - e^{-V_n} \), such that:

\[ |R_n(\hat{h}_{m_1}) - R(\tilde{h}_{m_1})| \leq |R_n(\hat{h}_{m_1}) - R(\tilde{h}_{m_1}) + R(\tilde{h}_{m_1}) - R(\tilde{h}_{m_1})| \]
\[ \leq |R_n(\hat{h}_{m_1}) - R(\tilde{h}_{m_1})| + |R(\tilde{h}_{m_1}) - R(\tilde{h}_{m_1})| \]
\[ \leq 12\sqrt{2} \sqrt{V(C(s_0)) \log n} + 3\sqrt{2} \frac{t}{n} \]
\[ \leq 15\sqrt{2} \frac{V_n}{n} \]

Similarly there exist \( \Omega'_{t} \) with probability \( \geq 1 - e^{-2V_n} \), such that

\[-15\sqrt{2} \frac{V_n}{n} + R(\tilde{h}_{m_1}) \leq R_n(\hat{h}_{m_2}) \leq R(\tilde{h}_{m_2}) + 15\sqrt{2} \frac{V_n}{n}.\]

Using equation (A.7), we get,

\[ R(\tilde{h}_{m_2}) \leq -30\sqrt{2} \frac{V_n}{n} + R(\tilde{h}_{m_1}). \]

Hence, on the set \( \Omega_t \cap \Omega'_{t} \), we have

\[ R_n(\hat{h}_{m_2}) \leq R(\tilde{h}_{m_2}) + 15\sqrt{2} \frac{V_n}{n} \leq -15\sqrt{2} \frac{V_n}{n} + R(\tilde{h}_{m_1}) \leq R_n(\hat{h}_{m_1}) \]

and

\[ \mathbb{P}(\Omega_t \cap \Omega'_{t}) \geq 1 - 2e^{-V_n} \geq 1 - 2e^{-2s_0 \log (ep/s_0)} \]

as \( 2A_s \log (ep/s_0) \log n \geq 2s_0 \log (ep/s_0). \)
A.7. Proof of Corollary 3.6

Here we will prove that, under $\beta_{p,\min}^0$ condition in Theorem 6, we will select a superset of true active variables with probability exponentially close to 1:

$$P(\hat{m} \not\supset m^*) = P\left( \bigcup_{m \supseteq m^*} \left\{ R_n(\hat{h}_m) \leq \inf_{m' \supseteq m^*} R_n(\hat{h}_{m'}) \right\} \right)$$

$$\leq \sum_{m \supseteq m^*} P\left( R_n(\hat{h}_m) \leq \inf_{m' \supseteq m^*} R_n(\hat{h}_{m'}) \right)$$

$$\leq \sum_{m \supseteq m^*} 2e^{-2V_n}$$

$$\leq 2e^{-2s_0 \log (ep/s_0)} e^{s_0 \log (ep/s_0)}$$

$$= 2e^{-s_0 \log (ep/s_0)}.$$

A.8. Proof of Theorem 3.7

We apply Theorem A.4 again. All notation is as in the proof of Theorem 3.2. Now, define $F = \{ f_\beta : f_\beta(x) = \text{sgn}(\beta'x) : \beta \in S^{p-1}, \|\beta\|_0 \leq s_0\}$. Using the bound obtained in Lemma A.14 on the VC dimension of these sparse classifiers, we obtain:

$$\mathbb{E}(\mathcal{H}(F)) \leq V(F) \log \frac{en}{V(F)}$$

$$\leq 2As_0 \log \left( \frac{ep}{s_0} \right) \log \frac{en}{As_0 \log \left( \frac{2p}{s_0} \right)}$$

$$\leq 3s_0 \log \left( \frac{ep}{s_0} \right) \log n.$$

Hence, as in the proof of Theorem 3.2, using Theorem A.4, we have for margin parameter $\alpha$:

1. $\phi(\sigma) = L \sigma \sqrt{\left( 1 + \mathbb{E}(\mathcal{H}(F)) \right)}$

2. $\epsilon^*_2 \leq \left( \frac{L^2 s_0 \log (ep/s_0) \log n}{n} \right)^{\kappa} n^{-\kappa/2}$ for $\kappa > 1$, i.e. $\alpha < \infty$.

where $\kappa = (1 + \alpha)/\alpha$. This implies, as in the proof of Theorem 3.2 that

$$P\left( S(\beta^0) - S(\hat{\beta}_m) \geq Ky \left( \frac{L^2 s_0 \log (ep/s_0) \log n}{n} \right)^{\kappa} \right) \leq e^{-y}$$

for all $y \geq 1$. Proposition 3.1 then leads to the following concentration bound:

$$P\left( \|\hat{\beta}_m - \beta^0\|_2 \geq t \left( \frac{s_0 \log (ep/s_0) \log n}{n} \right)^{1/2} \right)$$
\[ \begin{align*}
\leq & \mathbb{P} \left( S(\beta^0) - S(\hat{\beta}_m) \geq t^\kappa u_- \left( \frac{s_0 \log(ep/s_0) \log n}{n} \right)^{\frac{1}{2\kappa - 1}} \right) \\
\leq & e^{-\frac{t^\kappa u_-}{KL^{2\kappa}/2\kappa - 1}} \quad \text{[For } t \geq (KL^{2\kappa}/2\kappa - 1/u_-)^{1/\kappa}] 
\end{align*} \]

which tends to 0 as \( t \to \infty \). Hence, we conclude that

\[ \|\hat{\beta} - \beta^0\|_2 = O_P \left( \left( \frac{s_0 \log(ep/s_0) \log n}{n} \right)^{\frac{1}{2\kappa - 1}} \right) = O_P \left( \left( \frac{s_0 \log(ep/s_0) \log n}{n} \right)^{\frac{\alpha}{\alpha + 2}} \right). \]

The upper bound on the expectation follows from this exponential tail bound of \( \|\hat{\beta} - \beta^0\|_2 \).

Define \( r_n = \left( \frac{s_0 \log(ep/s_0) \log n}{n} \right)^{\frac{\alpha}{\alpha + 2}} \). Then:

\[ \mathbb{E} \left( \frac{1}{r_n} \|\hat{\beta} - \beta^0\|_2 \right) \]

\[ \begin{align*}
= & \int_0^\infty \mathbb{P}(\|\hat{\beta} - \beta^0\|_2 \geq Tr_n) \, dT \\
= & \int_0^{(KL^{2\kappa}/2\kappa - 1/u_-)^{1/\kappa}} \mathbb{P}(\|\hat{\beta} - \beta^0\|_2 \geq Tr_n) \, dT + \int_\infty^{(KL^{2\kappa}/2\kappa - 1/u_-)^{1/\kappa}} \mathbb{P}(\|\hat{\beta} - \beta^0\|_2 \geq Tr_n) \, dT \\
\leq & (KL^{2\kappa}/2\kappa - 1/u_-)^{1/\kappa} + \int_\infty^{(KL^{2\kappa}/2\kappa - 1/u_-)^{1/\kappa}} e^{-\frac{p \log n}{KL^{2\kappa}/2\kappa - 1/u_-}} \, dT \\
\leq & (KL^{2\kappa}/2\kappa - 1/u_-)^{1/\kappa} + g(\kappa) < \infty
\end{align*} \]

which completes the proof of the minimax upper bound.

**A.9. Proof of Theorem 3.8**

We use Fano’s inequality along with the Gilbert-Varshamov Lemma. Fano’s inequality (or Local-Fano’s inequality) gives us a lower bound on the minimax risk: If \( \Theta' \subseteq \Theta \) is a finite 2\( \epsilon \) packing set, i.e. for any two \( \theta_i, \theta_j \in \Theta' \), \( \|\theta_i - \theta_j\|_2 \geq 2\epsilon \) with \( |\Theta'| = M \), then, based on \( n \) i.i.d. samples \( z_1, z_2, \ldots, z_n \sim P_\theta \) we have the following minimax lower bound:

\[ \begin{align*}
\inf_{\hat{\theta}} \sup_{\theta \in \Theta} \mathbb{E} \left( \|\hat{\theta} - \theta\|_2 \right) \geq 2^2 \left( 1 - \frac{\frac{p \log n}{KL^2} \sum_{i,j} KL(P_{\theta_i}||P_{\theta_j}) + \log 2}{\log (M - 1)} \right)
\end{align*} \]

The crux of the proof relies on constructing competing models that approach each other at an optimal rate, as \( n \) increases. We start with a preliminary lemma.

**Lemma A.15.** If \( P \sim Ber(p_1) \) and \( Q \sim Ber(q_1) \) and if \( \frac{1}{4} \leq q_1 \leq \frac{3}{4} \), then \( KL(P||Q) \leq \frac{16}{9}(p - q)^2 \)

The proof of the Lemma appears in Appendix B. We next state the Gilbert-Varshamov Lemma for convenience (see [19] and references therein), that guides the construction of \( \Theta' \) in our problem.
Lemma A.16 (Gilbert-Varshamov). Define $d_H$ to be the Hamming distance, i.e. $d_H(x, y) = \sum_{i=1}^{d} 1_{x_i \neq y_i}$ with $d$ being the underlying dimension. Given any $s$ with $1 \leq s \leq \frac{d}{8}$, we can find $w_1, \ldots, w_M \in \{0,1\}^d$ such that:

a) $d_H(w_i, w_j) \geq \frac{s}{2} \forall i \neq j \in \{1,2,\ldots,M\}$.

b) $\log M \geq \frac{s}{8} \log (1 + \frac{d}{2s})$

c) $\|w_j\|_0 = s \forall j \in \{1,2,\ldots,M\}$.

Fix $0 < \delta < 1/4$. To construct a $2\epsilon$ packing set ($\epsilon = \frac{\delta}{4}$) of $S^{p-1}$, consider the following vectors:

$$
\beta_j = \frac{1}{\sqrt{s}} \frac{w_j}{\|w_j\|} \cdot \frac{1}{\sqrt{1+\delta^2}}.
$$

where $w_j \in W$, a subset of $\{0,1\}^{p-1}$ constructed using GV lemma. Let $\Theta' = \{\beta_j : J \in \{1,2,\ldots,M\}\} \subseteq \Theta = S^{p-1}$. For $I \neq J$,

$$
\|\beta_I - \beta_J\|^2 = \frac{\delta^2}{s(1+\delta^2)} d_H(w_I, w_J) \geq \frac{\delta^2}{4}.
$$

For notational simplicity define $m(\delta) = \sqrt{1+\delta^2}$. Denote $P_{\beta_j}(X,Y)$ as the joint distribution of $(X,Y)$ where $X \sim \mathcal{N}(0,I_p)$ and

$$
P_{\beta_j}(Y = 1|X) = \begin{cases} 
\frac{1}{2} + \beta_j^T X, & \text{if } |\beta_j^T X| \leq \left( \frac{1}{4} \vee \frac{\delta \vee |X_j|}{2m(\delta)} \right) \wedge \frac{1}{4} \\
\frac{1}{2} + \left( \delta \vee \frac{|X_j|}{2m(\delta)} \right) \vee \frac{1}{4} \text{sgn}(\beta_j^T X), & \text{if } |\beta_j^T X| > \left( \delta \vee \frac{|X_j|}{2m(\delta)} \right) \vee \frac{1}{4} \end{cases}
$$

for all $J \in \{1,2,\ldots,M\}$. Now, for any $\beta_j \in \Theta'$, we have $\beta_j^T X = \frac{X_j}{m(\delta)} + \tilde{\beta}_j^T \tilde{X}$ where $\tilde{a} = (a_2, a_3, \ldots, a_p)$. As $\|\beta_j\|_0 = s$ by construction, we know $\tilde{\beta}_j^T \tilde{X} = \frac{\delta Z_j}{m(\delta)}$ where $Z_j \sim \mathcal{N}(0,1)$ and independent of $X_1$. Thus, we have $\beta_j^T X = \frac{X_j}{m(\delta)} + \frac{\delta Z_j}{m(\delta)}$. Next we will show that, this construction obeys the margin assumption for $\alpha = 1$ and $t^* = \frac{1}{4}$.

**Lemma A.17.** The above family of distributions satisfy the margin assumption (Assumption A1) for $\alpha = 1$ and $t^* = \frac{1}{4}$.

**Proof.** Fix any $0 \leq t < \frac{1}{4}$.

\[
\mathbb{P}_X(|\eta(X) - 0.5| \leq t) = \mathbb{P}_X \left(|\eta(X) - 0.5| \leq t, |\beta_j^T X| \leq \left[ \delta \vee \frac{|X_j|}{2m(\delta)} \right] \wedge \frac{1}{4} \right) + \mathbb{P}_X \left(|\eta(X) - 0.5| \leq t, |\beta_j^T X| > \left[ \delta \vee \frac{|X_j|}{2m(\delta)} \right] \wedge \frac{1}{4} \right) 
\]

\[
= \mathbb{P}_X \left(|\beta_j^T X| \leq t, |\beta_j^T X| \leq \left[ \delta \vee \frac{|X_j|}{2m(\delta)} \right] \wedge \frac{1}{4} \right) + \mathbb{P}_X \left(|\beta_j^T X| \leq t, |\beta_j^T X| > \left[ \delta \vee \frac{|X_j|}{2m(\delta)} \right] \wedge \frac{1}{4} \right) 
\]

\[
\leq \mathbb{P}_X(|\beta_j^T X| \leq t) + \mathbb{P}_X \left(|\beta_j^T X| > \left[ \delta \vee \frac{|X_j|}{2m(\delta)} \right] \wedge \frac{1}{4} \right) 
\]
Lemma A.19. For any upper-bound the KL divergence:

We analyze each summand separately, starting with $S_\mathcal{A}$
Define the event $A_I = \left\{ |\beta_I^T X| \leq \left( \delta \vee \frac{|X_1|}{2m(\delta)} \right) \right\}$. Then we have the following lemma:

**Lemma A.18.** If $X \in A_I \cup A_J$, then

$$|P_{\beta_I}(Y = 1|X) - P_{\beta_J}(Y = 1|X)| \leq |\beta_I^T X - \beta_J^T X| = |\beta_I^T \tilde{X} - \beta_J^T \tilde{X}|$$

The proof follows the same arguments as that of Lemma A.12 and is skipped. Next, we upper-bound the KL divergence:

**Lemma A.19.** For any $I \neq J \in \{1, 2, \cdots, M\}$, we have

$$KL(P_{\beta_I} || P_{\beta_J}) \leq \frac{128}{3} \sqrt{\frac{2}{\pi}} \left[ \left( \frac{1}{6} + \frac{4\phi(3)}{27} \right) \delta^3 \right],$$

where $\phi$ is the standard normal density.

**Proof.**

$$KL(P_{\beta_I} || P_{\beta_J}) = \mathbb{E}_X (KL(P_{\beta_I}(Y|X) || P_{\beta_J}(Y|X)))$$

$$\leq \frac{16}{3} \mathbb{E}_X (P_{\beta_I}(Y|X) - P_{\beta_J}(Y|X))^2$$

$$\leq \frac{16}{3} \mathbb{E}_X \left( (\beta_I^T X - \beta_J^T X)^2 1_{X \in A_I \cup A_J} \right)$$

$$+ 4 \mathbb{E}_X \left( \left( \delta \vee \frac{|X_1|}{2m(\delta)} \right) \wedge \frac{1}{4} \right)^2 1_{X \in A_I \cap A_J, sgn(\beta_I^T X) \neq sgn(\beta_J^T X)}$$

$$= \frac{16}{3} (S_1 + 4S_2) \quad \text{[say]}$$

We analyze each summand separately, starting with $S_1$.

$$\mathbb{E}_X \left( (\beta_I^T X - \beta_J^T X)^2 1_{X \in A_I \cup A_J} \right)$$

$$\leq 2 \mathbb{E}_X \left( (\beta_I^T X - \beta_J^T X)^2 1_{X \in A_I} \right)$$

$$\leq 2 \mathbb{E}_X \left( (\beta_I^T X - \beta_J^T X)^2 1_{|\beta_I^T X| \leq \left( \delta \vee \frac{|X_1|}{2m(\delta)} \right)} \right)$$

$$\leq 2 \left[ \mathbb{E}_X \left( (\beta_I^T X - \beta_J^T X)^2 1_{|X_1| \leq 2m(\delta) |\beta_I^T X|} \right) + \mathbb{E}_X \left( (\beta_I^T X - \beta_J^T X)^2 1_{|X_1| \leq 2m(\delta) \delta} \right) \right]$$

$$\leq 2 \left[ \mathbb{E}_X \left( (\beta_I^T \tilde{X} - \beta_J^T \tilde{X})^2 P_{X_1 \tilde{X}} (|X_1| \leq 2m(\delta) |\beta_I^T \tilde{X}|) \right) + \mathbb{E}_X \left( (\beta_I^T \tilde{X} - \beta_J^T \tilde{X})^2 P_{X_1} (|X_1| \leq 2m(\delta) \delta) \right) \right]$$
\[
\begin{align*}
&\leq \sqrt{\frac{8}{\pi}} \left[ 2m(\delta) E_X \left( (\beta^T X - \beta^T \tilde{X})^2 | \beta^T \tilde{X} \right) + \frac{8}{m(\delta)} \delta^3 \right] \\
&\leq 4 \sqrt{\frac{2}{\pi}} \left[ m(\delta) \left( E_X (\beta^T X - \beta^T \tilde{X})^2 \right)^{\frac{1}{2}} \left( E_X (|\beta^T X|^2)^{\frac{1}{2}} + \frac{4}{m(\delta)} \delta^3 \right) \right] \\
&\leq 4 \sqrt{\frac{2}{\pi}} \left[ \frac{\sqrt{12}}{m(\delta)^{\frac{3}{2}}} + \frac{4}{m(\delta)} \right] \delta^3 \\
&\leq 8 \sqrt{\frac{2}{\pi}} \left[ 2 + \sqrt{\delta} \right] \delta^3. \tag{A.8}
\end{align*}
\]

Now, on to \(S_2:\)

\[
E_X \left( \left( \delta \vee \frac{|X_1|}{2m(\delta)} \right) \wedge 1/4 \right)^2 1_{X \in A_1^c \cap A_2^c: \text{sgn}(X^T \beta_f) \neq \text{sgn}(X^T \beta_f)}
\]

\[
= 2E_X \left( \left( \left( \delta \vee \frac{|X_1|}{2m(\delta)} \right) \wedge 1/4 \right)^2 1_{X_1 + \delta Z \geq \delta \left( \frac{|X_1|}{2m(\delta)} \right) \wedge 1/4} 1_{X_1 \leq 2m(\delta)} \right)
\]

\[
= 2E_X \left( \delta^2 1_{X_1 + \delta Z \geq \delta \left( \frac{|X_1|}{2m(\delta)} \right) \wedge 1/4} 1_{X_1 \leq 2m(\delta)} \right)
\]

\[
+ 2E_X \left( \frac{|X_1|^2}{4m(\delta)^2} 1_{X_1 + \delta Z \geq \frac{1}{4} X_1} 1_{X_1 \leq \frac{1}{2}} \right)
\]

\[
\leq 2E_X \left( \delta^2 1_{X_1 + \delta Z \geq \delta \left( \frac{|X_1|}{2m(\delta)} \right) \wedge 1/4} 1_{X_1 \leq 2m(\delta)} \right)
\]

\[
+ 2E_X \left( \frac{|X_1|^2}{4m(\delta)^2} 1_{X_1 + \delta Z \geq \frac{1}{4} X_1} 1_{X_1 \leq \frac{1}{2}} \right)
\]

\[
\leq 2 \left[ \sqrt{\frac{2}{\pi}} m(\delta) \delta^3 + 2E_X \left( \frac{|X_1|^2}{4m(\delta)^2} 1_{X_1 \geq -\frac{1}{4} X_1} 1_{X_1 \leq \frac{1}{4}} \right) \right] \tag{A.9}
\]

\[
\leq 2 \left[ \sqrt{\frac{2}{\pi}} m(\delta) \delta^3 + 2E_X \left( \frac{|X_1|^2}{4m(\delta)^2} 1_{X_1 \geq -\frac{1}{4} X_1} 1_{X_1 \leq \frac{1}{4}} \right) \right]
\]

\[
\leq 2 \left[ \sqrt{\frac{2}{\pi}} m(\delta) \delta^3 + 2E_X \left( \frac{|X_1|^2}{4m(\delta)^2} 1_{X_1 \geq -\frac{1}{4} X_1} 1_{X_1 \leq \frac{1}{4}} \right) \right]
\]

\[
\leq 2 \left[ \sqrt{\frac{2}{\pi}} m(\delta) \delta^3 + 2E_X \left( \frac{|X_1|^2}{4m(\delta)^2} 1_{X_1 \leq -\frac{1}{4} X_1} 1_{X_1 \geq \frac{1}{4}} \right) \right]
\]

\[
\leq 2 \left[ \sqrt{\frac{2}{\pi}} m(\delta) \delta^3 + 2E_X \left( \frac{|X_1|^2}{4m(\delta)^2} 1_{X_1 \geq -\frac{1}{4} X_1} 1_{X_1 \leq \frac{1}{4}} \right) \right] \tag{A.10}
\]

\[
\leq 2 \left[ \sqrt{\frac{2}{\pi}} m(\delta) \delta^3 + 2E_X \left( \frac{|X_1|^2}{4m(\delta)^2} 1_{X_1 \leq -\frac{1}{4} X_1} 1_{X_1 \geq \frac{1}{4}} \right) \right]
\]

\[
\leq 2 \left[ \sqrt{\frac{2}{\pi}} m(\delta) \delta^3 + 2E_X \left( \frac{|X_1|^2}{4m(\delta)^2} 1_{X_1 \leq -\frac{1}{4} X_1} 1_{X_1 \geq \frac{1}{4}} \right) \right] + \frac{1}{8} e^{-\frac{9}{32\pi^2}}
\]
\[
\leq 2\left[\frac{2\sqrt{\frac{m(\delta)}{\pi}}}{9m(\delta)^2} E_X \left(Z_J^2 I_{2m(\delta) \leq X_j \leq -\frac{4}{3}m(\delta)} Z_J \leq -3m(\delta)} \right) + \frac{1}{8} e^{-\frac{9}{32\pi^2}} \right]
\]

\[
\leq 4\left[\frac{2\sqrt{\frac{m(\delta)}{\pi}}}{9m(\delta)^2} E_{Z_J} \left(Z_J^2 \Phi \left(\frac{2}{3}Z_J\right) - \Phi (2m(\delta)) I_{Z_J \leq -3m(\delta)} \right) + \frac{1}{16} e^{-\frac{9}{32\pi^2}} \right]
\]

\[
\leq 4\left[\frac{2\sqrt{\frac{m(\delta)}{\pi}}}{27m(\delta)^2} E_{Z_J} \left(Z_J^2 | Z_J + 3m(\delta)| \right) + \frac{\sqrt{\pi}}{16\sqrt{2}} e^{-\frac{9}{32\pi^2}} \right]
\]

\[
\leq 4\left[\frac{2\sqrt{\frac{m(\delta)}{\pi}}}{27m(\delta)^2} \left(\sqrt{3} + 6 \right) + \frac{\sqrt{\pi}}{16\sqrt{2}} \right] \delta^3
\]

\[
\leq 8\left[1 + \frac{\sqrt{3} + 6}{54} + \frac{\sqrt{\pi}}{32\sqrt{2}} \right] \delta^3
\]

Combining equations A.8, A.8 and A.11 we conclude that:

\[
KL(\mathbb{P}_{\beta_j} || \mathbb{P}_{\beta_j}) \leq \frac{128}{3} \sqrt{\frac{2}{\pi}} \left[6 + \sqrt{3} + \frac{2(\sqrt{3} + 6)}{27} + \frac{\sqrt{\pi}}{8\sqrt{2}} \right] \delta^3.
\]

\[\square\]

The final step is a direct application of Fano’s inequality. According to our construction, \(\Theta'\) is a \(2\epsilon\) packing set with \(\epsilon = \frac{\delta}{s}\). For notational simplicity, set

\[
U_c \triangleq \frac{128}{3} \sqrt{\frac{2}{\pi}} \left[6 + \sqrt{3} + \frac{2(\sqrt{3} + 6)}{27} + \frac{\sqrt{\pi}}{8\sqrt{2}} \right].
\]

The upper bound on the KL divergences, in conjunction with Fano’s inequality, gives:

\[
\inf_{\beta} \sup_{\mathbb{P}_\beta} \mathbb{E} \left(\|\hat{\beta} - \beta\|^2\right) \geq \frac{\delta^2}{16} \left(1 - \frac{nU_c \delta^3 + \log \frac{2}{\epsilon}}{\frac{\pi}{2}} \right).
\]

Taking \(\delta = \left(\frac{s \log \frac{p}{s}}{nU_c}\right)^{\frac{1}{3}}\), then we have:

\[
\inf_{\beta} \sup_{\mathbb{P}_\beta} \mathbb{E} \left(\|\hat{\beta} - \beta\|^2\right) \geq \frac{1}{256U_c^2} \left(\frac{s \log \frac{p}{s}}{n}\right)^{\frac{2}{3}} \left(1 - \frac{\delta^4}{\frac{\pi}{2}} \log \frac{p}{s} + \log \frac{2}{\epsilon} \right) \geq \frac{1}{210U_c^2} \left(\frac{s \log \frac{p}{s}}{n}\right)^{\frac{2}{3}},
\]

the last inequality holding true when \(\log 2 \leq \frac{s}{125} \log \frac{p}{s}\), which is true for all large \(s, p\) as \(s \log \frac{p}{s} \to \infty\). \[\square\]
Appendix B: Supplementary Material

B.1. Proof of Theorem 5.1

First we show that as \( p/n \to 0 \),
\[
\sup_{\beta \in \mathbb{S}^{p-1}} \| S_n(\beta) - S(\beta) \|_2 \to 0
\]
i.e. the class of functions \( G = G_p = \{ g_\beta : \mathbb{R}^p \times \{-1,1\} \to \{-1,1\}, g_\beta(x,y) = y \text{sgn}(x^T \beta) \} \) is Glivenko-Cantelli class which is equivalent to showing (for details see [18]):

1. There exists \( G \), an envelope of \( G \) such that \( P^* G \leq \infty \).
2. \( \lim_{n \to \infty} \frac{E^*(\log(N(\epsilon,G_m,L_2(\mathbb{P}_n))))}{n} = 0 \) for all \( M < \infty, \epsilon > 0 \), where \( N(\epsilon,G_m,L_2(\mathbb{P}_n)) \) is the \( \epsilon \) covering number of the set \( G_m = \{ g_\beta 1_{G \leq M} : g_\beta \in G \} \) with respect to \( L_2(\mathbb{P}_n) \) norm.

Clearly \( G \equiv 1 \) is an integrable envelope of \( G \). Now \( G \) is VC class of VC dimension \( v = (p+1) \). Hence, we have:
\[
\sup_{Q} N(\epsilon,G,L_2(Q)) \leq K_v \left( \frac{4\sqrt{e}}{\epsilon} \right)^{2v}
\]
for some universal constant \( K \) and \( 0 \leq \epsilon \leq 1 \). Using this, we have:
\[
\frac{E^*(\log(N(\epsilon,G_m,L_2(\mathbb{P}_n))))}{n} \leq \frac{\log(kv)}{n} + \frac{2v}{n} \log \left( \frac{4\sqrt{e}}{\epsilon} \right) \to 0
\]
if \( v/n \to 0 \iff p/n \to 0 \) which completes the proof.

In the previous step we have established that \( S_n(\beta) \to S(\beta) \) uniformly over \( \beta \). Now we need to prove \( \hat{\beta} = \arg\max_{\beta} S_n(\beta) \) converges to \( \beta^0 = \arg\max_{\beta} S(\beta) \). Towards that we need the following Lemma:

Lemma B.1. Given any \( 0 \leq \epsilon_1 < \epsilon_2 \leq 2 \) and \( \beta_1 \) such that \( \| \beta_1 - \beta^0 \|_2 = \epsilon_2 \), we can find \( \beta_2 \) with \( \| \beta_2 - \beta^0 \|_2 \leq \epsilon_1 \) such that
\[
S(\beta^0) - S(\beta_1) \geq S(\beta^0) - S(\beta_2)
\]

We defer the proof of this lemma to the next subsection. Using Proposition 3.1 we have
\[
S(\beta^0) - S(\beta) \geq u_- \| \beta - \beta^0 \|_2\delta
\]
which is now true for \( \| \beta - \beta^0 \|_2 \leq \delta \) under the assumptions of Theorem 5.1. Suppose \( 0 \leq \epsilon < \delta \), then using Lemma B.1 we have:
\[
\inf_{\beta : \| \beta - \beta^0 \|_2 \geq \epsilon} S(\beta^0) - S(\beta) = \inf_{\epsilon < \| \beta - \beta^0 \|_2 \leq \delta} S(\beta^0) - S(\beta) \geq \epsilon \delta
\]
\[ \mathbb{P}(\|\beta - \beta^0\|_2 > \epsilon) = \mathbb{P}\left( \sup_{\|\beta - \beta^0\|_2 > \epsilon} (S_n(\beta) - S_n(\beta^0)) > 0 \right) \\
= \mathbb{P}\left( \sup_{\|\beta - \beta^0\|_2 > \epsilon} ((S_n - S)(\beta) - (S_n - S)(\beta^0) + S(\beta - \beta^0)) > 0 \right) \\
\leq \mathbb{P}\left( \sup_{\|\beta - \beta^0\|_2 > \epsilon} ((S_n - S)(\beta) - (S_n - S)(\beta^0)) > \inf_{\|\beta - \beta^0\|_2 > \epsilon} (S(\beta^0) - S(\beta)) \right) \\
\leq \mathbb{P}\left( \sup_{\|\beta - \beta^0\|_2 > \epsilon} ((S_n - S)(\beta) - (S_n - S)(\beta^0)) > \inf_{\|\beta - \beta^0\|_2 \leq \delta} (S(\beta^0) - S(\beta)) \right) \\
\leq \mathbb{P}\left( \sup_{\|\beta - \beta^0\|_2 > \epsilon} ((S_n - S)(\beta) - (S_n - S)(\beta^0)) > c\epsilon \right) \\
\rightarrow 0 \ [\because \mathcal{G} \text{ is a GC Class}]
\]

which completes the proof for \( p/n \) going to 0.

The same proof works for \( p \gg n \) under our assumption \((s_0 \log p)/n \rightarrow 0\), because, what is really needed in the above proof is the condition \( V/n \rightarrow 0 \) where \( V \) is the VC dimension of the set of classifiers under consideration. When \( p \gg n \), \( V = O(s_0 \log p) \) under the sparsity assumption, and therefore by our assumption \( V/n \rightarrow 0 \) in this case as well. \(\square\)

\textbf{B.2. Proof of Lemma B.1}

Under the assumption that \( \text{med}(\epsilon|X) = 0 \) in our model, we have for any \( \beta \):

\[ S(\beta) - S(\beta^0) = \int_{X_\beta} |\mathbb{E}(Y|X)| \, dF_X \]

where \( X_\beta = \{x : \text{sgn}(x^T\beta) \neq \text{sgn}(x^T\beta^0)\} \). Now a fix \( \beta_1 \) with \( \|\beta_1 - \beta^0\|_2 = \epsilon_1 \). Define \( \beta_2 = \frac{\alpha\beta_1 + (1 - \alpha)\beta^0}{\alpha\beta_1 + (1 - \alpha)\beta^0} \) for some \( \alpha \in (0, 1/2) \) which will be chosen later. Suppose \( x \in X_{\beta_2} \):

**Case 1:** Suppose \( x^T\beta_2 > 0 > x^T\beta^0 \). Then
\[ \frac{\alpha}{\|\alpha\beta_1 + (1 - \alpha)\beta^0\|_2} x^T\beta_1 = \beta_2^Tx - \frac{1 - \alpha}{\|\alpha\beta_1 + (1 - \alpha)\beta^0\|_2} x^T\beta^0 > 0 \iff x^T\beta_1 > 0 \]

**Case 2:** Suppose \( x^T\beta_2 < 0 < x^T\beta^0 \). Then
\[ \frac{\alpha}{\|\alpha\beta_1 + (1 - \alpha)\beta^0\|_2} x^T\beta_1 = \beta_2^Tx - \frac{1 - \alpha}{\|\alpha\beta_1 + (1 - \alpha)\beta^0\|_2} x^T\beta^0 < 0 \iff x^T\beta_1 < 0 \]

Hence \( X_{\beta_2} \subseteq X_{\beta_1} \). Now \( \|\alpha\beta_1 + (1 - \alpha)\beta^0\|_2 \geq (1 - 2\alpha) \) by triangle inequality and using the fact that \( \|\beta_1\| = \|\beta^0\| = 1 \). Therefore,
\[ \|\beta_2 - \beta^0\|_2 = \left\| \frac{\alpha\beta_1 + (1 - \alpha)\beta^0}{\|\alpha\beta_1 + (1 - \alpha)\beta^0\|_2} - \beta^0 \right\|_2 \]

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\[
\frac{\alpha (\beta_1 - \beta^0) + (1 - \|\alpha \beta_1 + (1 - \alpha)\beta^0\|_2)\beta^0}{\|\alpha \beta_1 + (1 - \alpha)\beta^0\|_2} \leq \frac{\alpha \epsilon_1}{1 - 2 \alpha}
\]

To conclude the proof we choose \(\alpha\) such that \(\alpha \epsilon_1/(1 - 2 \alpha) = \epsilon_2\) i.e. \(\alpha = \epsilon_2/(\epsilon_1 + 2 \epsilon_2)\).

### B.3. Discussion on Lemma A.5

In this sub-section we will discuss Lemma A.5 in little bit details. Consider the following lemma:

**Lemma B.2.** For any fixed \(x \in S^{p-1}\), define \(C(x, \epsilon)\) to be \(\epsilon\)-angular spherical cap around \(x\), i.e.

\[
C(x, \epsilon) = \{y \in S^{p-1} : \langle x, y \rangle \geq \epsilon\}
\]

Then we have

\[
\sigma(C(x, \epsilon)) \leq \frac{1}{2\epsilon \sqrt{p}} (1 - \epsilon^2)^{\frac{p-1}{2}} \leq \frac{1}{2\sqrt{2}} (1 - \epsilon^2)^{\frac{p-1}{2}}
\]

for \(\sqrt{\frac{2}{p}} \leq \epsilon \leq 1\). The last inequality follows from the assumption \(\sqrt{\frac{2}{p}} \leq \epsilon\).

This lemma is a well-known fact in convex geometry. Note that, Lemma A.5 and Lemma B.2 are in different scale as one of them involves the angle and the other one involves the distance. In the following Lemma we bridge this gap:

**Lemma B.3.** For \(0 \leq r \leq 1\) and \(p \geq 8\), we have:

\[
\sigma(D(x, r)) \leq \frac{1}{2\sqrt{2}} r^{p-1}
\]

**Proof.** The proof of this lemma is quite straight forward. Note that \(C(x, \epsilon) = D(x, r)\) where \(\epsilon = (1 - r^2/2)\). If \(r \leq 1\) and \(p \geq 8\) then \(\epsilon \geq \sqrt{\frac{2}{p}}\). Hence we have:

\[
\sigma(D(x, r)) \leq \frac{1}{2\sqrt{2}} \left(1 - \left(1 - \frac{r^2}{2}\right)^2\right)^{\frac{p-1}{2}}
\]

\[
\leq \frac{1}{2\sqrt{2}} r^{p-1} \left(1 - \frac{r^2}{4}\right)^{\frac{p-1}{2}}
\]

\[
\leq \frac{1}{2\sqrt{2}} r^{p-1}
\]

which completes the proof. \(\square\)

Finally using Lemma B.3 we get the upper bound on \(\sigma(D(x, r))\). The lower bound is also found in convex geometry literature. Combining them together, we get Lemma A.5.
B.4. Proof of Lemma A.11

Proof. Define $x - (p_1 - q_1)/2 = \nu/2$. From the definition of Hellinger distance between two Bernoulli Random variables, we get,

$$H^2(P_1, P_2) = 1 - \sqrt{p_1 q_1} - \sqrt{(1 - p_1)(1 - q_1)}$$

$$= 1 - \sqrt{(s + x)(s - x)} - \sqrt{(1 - s + x)(1 - s - x)}$$

$$= 1 - \sqrt{s^2 - x^2} - \sqrt{(1 - s)^2 - x^2}$$

$$= 1 - s \sqrt{\frac{1 - x^2}{s^2} - (1 - s) \sqrt{\frac{1 - x^2}{(1 - s)^2}}}$$

$$= 1 - s \left[ 1 - \frac{x^2}{2s^2} \left( 1 - \frac{x}{s} \right)^{-1/2} \right] - (1 - s) \left[ 1 - \frac{x^2}{2(1 - s)^2} \left( 1 - \frac{x}{(1 - s)} \right)^{-1/2} \right]$$

$$= \frac{x^2}{2s} \left( 1 - \frac{x}{s} \right)^{-1/2} + \frac{x^2}{2(1 - s)} \left( 1 - \frac{x}{(1 - s)} \right)^{-1/2}$$

(B.1)

In the second last line we use mean value theorem:

$$f(x) = \sqrt{1 - x} = 1 - \frac{x}{2} (1 - \tilde{x})^{-1/2}$$

for some $\tilde{x}$ between 0 and $x$. As our parameter space is $[1/4, 3/4]$, we have $p_1 \leq 3q_1$ for any choice of $p_1, q_1$. Hence, $\frac{|p_1|}{s} \leq \frac{1}{2}$ and $\frac{|p_1|}{1 - s} \leq \frac{1}{2}$ which immediately implies $\frac{x^2}{s^2} \leq \frac{1}{4}$ and $\frac{x^2}{(1 - s)^2} \leq \frac{1}{4}$, which, in turn, validates $\left( 1 - \frac{x^2}{s^2} \right)^{-1/2} \leq \frac{2}{\sqrt{3}}$ and $\left( 1 - \frac{x^2}{(1 - s)^2} \right)^{-1/2} \leq \frac{2}{\sqrt{3}}$. Using this in equation B.1 we conclude:

$$\frac{x^2}{2s} \left( 1 - \frac{x}{s} \right)^{-1/2} + \frac{x^2}{2(1 - s)} \left( 1 - \frac{x}{(1 - s)} \right)^{-1/2} \leq \frac{2}{\sqrt{3}} \left[ \frac{x^2}{2s} + \frac{x^2}{2(1 - s)} \right]$$

$$= \frac{(q_1 - p_1)^2}{4\sqrt{3}s(1 - s)}$$

\[\Box\]

B.5. Proof of Lemma A.15

Proof.

$$KL(P || Q) = p_1 \log \frac{p_1}{q_1} + (1 - p_1) \log \frac{1 - p_1}{1 - q_1}$$

$$\leq p_1 (p_1 - q_1) + \frac{1 - p_1}{1 - q_1} (q_1 - p_1) \quad \because \log x \leq x - 1$$

$$= (p_1 - q_1) \left[ \frac{p_1}{q_1} - \frac{1 - p_1}{1 - q_1} \right]$$

$$= \frac{(p_1 - q_1)^2}{q_1(1 - q_1)} \leq \frac{16}{3} (p_1 - q_1)^2 \quad \because \frac{1}{4} \leq q_1 \leq \frac{3}{4}$$

\[\Box\]
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