Sharp Entropy Bounds for Plane Curves and Dynamics of the Curve Shortening Flow

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Abstract

We prove that a closed immersed plane curve with total curvature \(2\pi m\) has entropy at least \(m\) times the entropy of the embedded circle, as long as it generates a type I singularity under the curve shortening flow (CSF). We construct closed immersed plane curves of total curvature \(2\pi m\) whose entropy is less than \(m\) times the entropy of the embedded circle. As an application, we extend Colding-Minicozzi’s notion of a generic mean curvature flow to closed immersed plane curves by constructing a piecewise CSF whose only singularities are embedded circles and type II singularities.

1 Introduction

Huisken [18] conjectured that for mean curvature flow (MCF) from generic initial embedded hypersurfaces, all singularities are spheres or cylinders. In a recent fundamental paper [11], Colding and Minicozzi made an important step towards establishing Huisken’s genericity conjecture. In that paper, they define the entropy of an immersed hypersurface \(\Sigma \subset \mathbb{R}^{n+1}\) to be
\[
\lambda(\Sigma) = \sup_{x_0, t_0} \left(4\pi t_0\right)^{-n/2} \int_{\Sigma} e^{-\frac{|x-x_0|^2}{4t_0}} \, d\mu,
\]
where the supremum is taken over all translations \(x_0 \in \mathbb{R}^{n+1}\) and rescalings \(t_0 > 0\) of \(\Sigma\). Entropy is nonincreasing along MCF and is constant along the flow precisely for shrinkers, the contracting self-similar solutions. Entropy is used in the following important way: if \(\Sigma\) is a shrinker arising as the limit of a blowup sequence of a singular point of the MCF starting at the initial hypersurface \(\Sigma'\) then \(\lambda(\Sigma) \leq \lambda(\Sigma')\). Entropy thus provides a vital tool for ruling out certain singularities for a MCF.

The first main theorem of this paper gives entropy lower bounds for plane curves in terms of a topological quantity, the turning number, which is the total curvature divided by \(2\pi\). We denote by \(\Gamma_m \subset \mathbb{R}^2\) the \(m\)-covered circle with radius \(\sqrt{2}\), i.e. \(\Gamma_m\) is an immersion \(S^1 \to \mathbb{R}^2\) whose turning number is \(m\) and whose image is the circle of radius \(\sqrt{2}\). It is easy to see that \(\Gamma_m\) is a shrinker for all \(m \geq 1\).

**Theorem A.** Let \(\Gamma \subset \mathbb{R}^2\) be a closed immersed curve with turning number \(m\).

**A.1** If the CSF starting at \(\Gamma\) generates a type I singularity, then \(\lambda(\Gamma) \geq \lambda(\Gamma_m)\).

**A.2** There exist closed immersed curves \(\Gamma' \subset \mathbb{R}^2\) with turning number \(m\), but \(\lambda(\Gamma') < \lambda(\Gamma_m)\). Any such curve generates only type II singularities under the CSF.
Straightforward calculations show that
\[
\lambda(\Gamma_m) = m \cdot \lambda(\Gamma_1) = m \sqrt{2\pi/e}.
\]
In other words, the entropy of the \(m\)-covered circle is \(m\) times the entropy of the embedded circle.

An important special case of Theorem A.1 is when \(\Gamma \subset \mathbb{R}^2\) is a closed shrinker of turning number \(m\). Closed shrinkers for the CSF are called Abresch-Langer curves [1]. For any pair of relatively prime integers \(m, n \in \mathbb{N}\) satisfying
\[
\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2},
\]
there exists a closed shrinker \(\Gamma_{m,n}\) with turning number \(m\) and whose curvature has \(2n\) critical points. Moreover, every closed shrinker is of this form (other than the round circles). Theorem A.1 implies that
\[
\lambda(\Gamma_{m,n}) \geq \lambda(\Gamma_p)
\]
for every Abresch-Langer curve \(\Gamma_{m,n}\) and every \(p\)-covered circle \(\Gamma_p\) with \(m \geq p\). The normalization constant \((4\pi)^{-n/2}\) in the definition of entropy (1.1) is chosen so that hyperplanes have entropy 1. It follows that
\[
\lambda(\Sigma) = \lambda(\Sigma \times \mathbb{R})
\]
for all hypersurfaces \(\Sigma \subset \mathbb{R}^{n+1}\). Using this fact, Theorem A.1 immediately implies entropy bounds on higher dimensional shrinkers \(\Sigma \subset \mathbb{R}^{n+1}\) of the form \(\Sigma = \Gamma \times \mathbb{R}^{n-1}\), where \(\Gamma\) is a closed plane shrinker.

The key tool for proving Theorem A.2 is the notion of entropy instability. A shrinker \(\Sigma \subset \mathbb{R}^{n+1}\) is entropy unstable if there exists a variation \((\Sigma_\epsilon)_{\epsilon \in (-\delta, \delta)}\) of \(\Sigma\) which decreases entropy. Entropy stability of embedded shrinkers was first introduced in [11]. Our method to construct closed curves with turning number \(m\) but entropy less than that of the \(m\)-covered circle \(\Gamma_m\) is to show that \(\Gamma_m\) is entropy unstable for all \(m \geq 2\). We not only show that multiply-covered circles are entropy unstable, but also that all closed shrinkers \(\Gamma \subset \mathbb{R}^2\) other than the embedded circle are entropy unstable. Furthermore, we calculate their entropy index, which is the number of linearly independent variations which reduce the entropy of the shrinker (see Liu [21]).

Colding and Minicozzi use entropy stability to define a piecewise \(MCF\) starting at a closed hypersurface \(\Sigma \subset \mathbb{R}^{n+1}\) as a finite collection of MCFs \(\Sigma^i\) on time intervals \([t_i, t_{i+1}]\) so that each \(\Sigma^i_{t_{i+1}}\) is a graph over \(\Sigma^i_{t_i+1}\) of a function \(u_i\) and
\[
\lambda(\Sigma^i_{t_{i+1}}) \leq \lambda(\Sigma^i_{t_i+1}).
\]
Piecewise MCF provides an ad hoc notion of generic MCF that is important in many applications: it provides a method of continuing a flow through unstable singularities. If a MCF reaches an entropy unstable singularity, the flow is slightly perturbed to decrease entropy so that the singularity can never reoccur along the flow. Colding and Minicozzi construct a piecewise MCF for closed embedded surfaces in \(\mathbb{R}^3\) which becomes extinct in a round point [11]. We obtain a similar result by constructing a piecewise CSF for closed immersed curves in \(\mathbb{R}^2\).

**Theorem B.** Let \(\Gamma \subset \mathbb{R}^2\) be a closed immersed curve. Then there exists a piecewise CSF starting at \(\Gamma\) and defined up to time \(T\) at which the flow either becomes extinct in an embedded circle, or has type II singularities. Moreover, if \(\Gamma\) has turning number greater than 1, the latter case holds.
It is worth noting that the break-points of the piecewise CSF constructed in Theorem B can be made arbitrarily small in the \( C^\infty \) norm. Furthermore, a result of Gage-Hamilton [13] and Grayson [14] states that the CSF starting at a closed embedded plane curve becomes convex and eventually becomes extinct in a “round point”. See [2] for an elegant direct proof. It follows that the piecewise CSF of Theorem B starting at a closed embedded plane curve reduces to the usual CSF whose only singularity is an embedded circle.

A singularity of a MCF defined up to time \( T \) is of type I if the curvature blows up no faster than \((T - t)^{-1/2}\); otherwise, the singularity is of type II. Type I singularities are important in the singularity analysis of MCF because Huisken [16] shows that a rescaling of a type I singularity is a smooth shrinker. More generally, every limit of rescalings of a MCF around a fixed point in spacetime is modeled on a possibly singular shrinker [16],[17]. The study of shrinkers is therefore central in the analysis of singularity formation of the MCF. Shrinkers can equivalently be defined as immersed hypersurfaces \( \Sigma \subset \mathbb{R}^n \) satisfying the equation

\[
H = \frac{1}{2} \langle x, n \rangle.
\]

Straightforward calculations show that hyperplanes passing through the origin, round spheres of radius \( \sqrt{2}\pi \) centered at the origin, and cylinders of the form \( S^k \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1} \) are all shrinkers.

Much work has concerned the construction and classification of shrinkers [3],[7],[19], and for the case \( n = 1 \), a complete classification is known [1].

In [23], Stone computes the entropy of embedded spheres (and thus also of shrinking cylinders), implying that

\[
2 > \lambda(S^1) > \lambda(S^2) > \cdots \to \sqrt{2}.
\]

In [6], Brakke proves that hyperplanes have least entropy among all shrinkers, and that there is a gap between the entropy of a plane and the entropy of the next lowest shrinker. Colding-Ilmanen-Minicozzi-White [9] prove that the sphere is the shrinker with second lowest entropy, and that a sphere has least entropy of among all closed embedded shrinkers. This result has been generalized by Bernstein-Wang [5], Zhu [25], and Ketover-Zhou [20] to all closed embedded hypersurfaces, not only shrinkers. Very recently, Hershkovits and White [15] proved a lower bound and rigidity theorem for shrinkers: a closed embedded shrinker \( \Sigma \in \mathbb{R}^{n+1} \) with non-trivial \( k \)-th homology has entropy greater than or equal to the entropy of \( S^k \), and equality holds if and only if \( \Sigma = S^k \times \mathbb{R}^{n-k} \). Our results can be viewed as an extension of these results to the case of closed immersed curves.

Stability of Abresch-Langer curves has been studied in a different context by Epstein-Weinstein [12]. Our results show that the Jacobi operator of an Abresch-Langer curve has the same number of negative eigenvalues as the linear stability operator studied by Epstein-Weinstein. Their analysis does not incorporate the action of the group generated by rigid motions and dilations. In particular, for a curve to be in Epstein-Weinstein’s stable manifold, then under the rescaled flow it has to limit into the given shrinker. In other words, for a curve to be in their stable manifold, it is not enough that it limit into a rotation, translation, or dilation of the given shrinker. However, in a generic MCF [11], a curve which limits into a rotation, translation, or dilation of a given shrinker, is considered as part of the stable manifold.

**Outline of the Proofs.** We now outline the proof of Theorem A.1. The main technical challenge is showing that the entropy functional is continuous under small perturbations near a closed shrinker...
Note that the entropy functional not continuous in general: a sequence of rescalings at a point on a sphere converges to a hyperplane in the limit, however, a hyperplane has entropy 1, while a sphere has entropy at least \( \sqrt{2} \). Our method for showing that the entropy functional is continuous near a closed shrinker \( \Gamma \subset \mathbb{R}^2 \) consists of two steps. Firstly, Proposition 2.1 is a general result giving a sufficient condition for the \( F \)-functional of a closed immersed hypersurface \( \Sigma \subset \mathbb{R}^{n+1} \) to be continuous at \( \Sigma \), namely, when \((x_0, t_0)\) is restricted to a compact subset \( K \subset \mathbb{R}^{n+1} \times (0, \infty) \) and variations are restricted to a subset \( B \subset C^{2,\alpha}(\Sigma) \) of variations with a uniform \( C^{2,\alpha} \) bound.

The second step is to provide a quantitative version of [11, Lemma 7.7] to show that we may in fact apply Proposition 2.1 to closed shrinkers \( \Gamma \subset \mathbb{R}^2 \). To that end, Theorem 2.3 shows that the entropy of the perturbed curve \( \Gamma + \epsilon n \) with \( f \in B \) is achieved in \( B_R(0) \times [T_0, T_1] \) for some \( R > 0 \) and some \( 0 < T_1 < T_2 < \infty \). The main technical challenge is to show that \( T_1 > 0 \). The key result we use to conclude the proof is due to Au [4]. Au shows that if \( \Gamma_{m,n} \) is an Abresch-Langer curve, then when \( \epsilon > 0 \) is small enough, the perturbed curve \( \Gamma_{m,n} + \epsilon n \) converges to the \( m \)-covered circle \( \Gamma_m \) under the rescaled CSF. Using Au’s result and the monotonicity property of entropy we conclude the proof.

Next, we give an outline of the proof of Theorem A.2. Our method to construct closed curves with turning number \( m \) but entropy less than that of the \( m \)-covered circle \( \Gamma_m \) is to show that \( \Gamma_m \) is entropy unstable for all \( m \geq 2 \). An important tool for analyzing the entropy stability of a closed hypersurface \( \Sigma \subset \mathbb{R}^{n+1} \) is its \( F \)-functional

\[
F_{x_0, t_0}(\Sigma) = (4\pi t_0)^{-n/2} \int_{\Sigma} e^{-\frac{|x-x_0|^2}{4t_0}} d\mu.
\]

The entropy \( \lambda(\Sigma) \) is then the supremum over all translations \( x_0 \in \mathbb{R}^{n+1} \) and dilations \( t_0 > 0 \) of the \( F \)-functional. It follows from Huisken’s monotonicity formula [16] that shrinkers are precisely the critical points of the \( F_{0,1} \)-functional, the Gaussian area. Therefore, shrinkers are precisely the minimal hypersurfaces of \( \mathbb{R}^{n+1} \) equipped with the conformally changed metric \( g_{ij} = e^{-\frac{|x|^2}{4}} \delta_{ij} \).

We show that the entropy index of a shrinker is the difference between the Morse index \( 1 \) of the shrinker (considered as a minimal surface of \( (\mathbb{R}^{n+1}, e^{-\frac{|x|^2}{4}} \delta_{ij}) \)) and the dimension of the space spanned by the mean curvature function and the component functions of the normal vector field of the shrinker. Consequently, we reduce the calculation of the entropy index of a shrinker to the calculation of the Morse index. For 1-dimensional shrinkers in \( \mathbb{R}^2 \), the Jacobi operator reduces to a 1-dimensional Sturm-Liouville operator. Using the well-developed theory of spectra of Sturm-Liouville operators, we are able to determine exactly how many negative eigenvalues the Jacobi operator of a 1-dimensional closed shrinker has, and thus also the entropy index.

Lastly, we outline the proof of Theorem B. As mentioned above, the possible singularities of the CSF starting at \( \Gamma \) are classified into two categories: type I and type II. Huisken [16, Theorem 3.5] shows that any rescaling of a type I singularity is a shrinker. By Abresch and Langer’s classification of shrinkers for the CSF [1], any rescaling of a type I singularity must therefore be an Abresch-Langer curve, or a (multiply-covered) circle. Corollary 5.8 states that the only entropy stable closed singularity for CSF is the embedded circle. Let \( \Gamma \) be any closed immersed plane curve.

If the singularity is of type I, any rescaling gives an Abresch-Langer curve or a circle, say \( \tilde{\Gamma}_\infty \). If \( \tilde{\Gamma}_\infty \) is not an embedded circle, Lemma 6.1 shows that the flow can be slightly perturbed to a curve \( \Gamma' \) with \( \lambda(\Gamma') < \lambda(\tilde{\Gamma}_{\infty}) \). Consequently, \( \tilde{\Gamma}_\infty \) can never appear as a singularity for the flow starting from

\[\text{Recall that the Morse index of a minimal surface } \Sigma \subset (M, g) \text{ is defined as the number of negative eigenvalues of the Jacobi operator determined by } (M, g) \text{ and } \Sigma \text{ (see [10, p. 41]).} \]
The above process is then repeated with \( \Gamma' \) instead of \( \Gamma \). Since the perturbations can be made with arbitrarily small \( C^\infty \)-norm, the perturbations preserve turning number. Therefore piecewise CSF preserves turning number, and since there are only finitely many closed shrinkers of a given turning number, the piecewise flow terminates after finitely many perturbations.

**Organization of the Paper.** In Section 2 we prove some basic properties of the entropy functional for immersed hypersurfaces and shrinkers. In particular, here we prove the continuity Theorem 2.3 for entropy, which is central to our argument.

In Section 3, we prove Theorem A.1.

In Section 4, we introduce and study some basic properties of index of the shrinkers. The results of this section hold for shrinkers of any dimension.

In Section 5, we compute the precise entropy index of closed immersed shrinkers for the CSF. As a corollary, we obtain Theorem A.2.

In Section 6 we combine our results to prove Theorem B.

**Notation.** Throughout, all hypersurfaces \( \Sigma \subset \mathbb{R}^{n+1} \) are assumed to be immersed and orientable, so that there exists a globally defined unit normal vector field \( \mathbf{n} : \Sigma \to S^n \). We denote by \( x : \Sigma \to \mathbb{R}^{n+1} \) the given immersion and by \( \Sigma + f \mathbf{n} \) the normal variation of \( \Sigma \) by a function \( f : \Sigma \to \mathbb{R} \). The mean curvature of \( \Sigma \) is denoted by \( H : \Sigma \to \mathbb{R} \). It is easy to show (see for example [22]) that MCF is a geometric flow which is invariant under tangential reparameterization, so without ambiguity we may also use the image \( (\Sigma_t)_{t \in [0,T]} \) to denote the flow. In the case of curves, we use the notation \( (\Gamma_t)_{t \in [0,T]} \). Finally, angle brackets \( \langle \cdot, \cdot \rangle \) denote the standard Euclidean inner product.

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## 2 Properties of Entropy

In this section, we will provide conditions under which the entropy functional is continuous at a closed plane shrinker. Note that in general, the entropy functional is not a continuous function on the space of immersed hypersurfaces (with the \( C^\infty \) topology, say). The standard example is given by blowing up a sphere at a point, giving a hyperplane in the limit. Each element of the blowup sequence is a sphere, thus having entropy greater than \( \sqrt{2} \). However, a hyperplane has entropy 1. The results of this section will be used in the Section 3 to prove Theorem A.1.

The following proposition shows that, when a closed hypersurface is slightly perturbed, the \( F \)-functional cannot change too much under the perturbation, as long as \( (x_0, t_0) \) are confined to a compact subset of \( \mathbb{R}^{n+1} \times (0, \infty) \).

**Proposition 2.1.** Let \( \Sigma \subset \mathbb{R}^{n+1} \) be a closed hypersurface and \( K \subset \mathbb{R}^{n+1} \times (0, \infty) \) a compact subset. Then there exists an \( \epsilon_0 > 0 \) and a constant \( C \), both depending only on \( \Sigma \) and \( K \), such that, for all \( f \in C^{2,\alpha}(\Sigma) \) with \( \|f\|_{C^{2,\alpha}} \leq 1 \), all \( \epsilon \geq 0 \) with \( \epsilon < \epsilon_0 \), and all \( (x_0, t_0) \in K \), the following inequality holds:

\[
|F_{x_0, t_0}(\Sigma_\epsilon) - F_{x_0, t_0}(\Sigma)| \leq C\epsilon,
\]

(2.2)
where $\Sigma_\epsilon = \Sigma + \epsilon f n$.

\textbf{Proof.} Let $x: \Sigma \to \mathbb{R}^{n+1}$ be the given immersion and let $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$. In local coordinates around a point $p \in \Sigma$, we can write

$$\bar{g}_{ij} = \left\langle \partial_i (x + \epsilon f n), \partial_j (x + \epsilon f n) \right\rangle = g_{ij} + 2\epsilon f h_{ij} + \epsilon^2 f^2 h_{ik} h_{kj} + \epsilon^2 (\partial_i f) (\partial_j f),$$

where $g_{ij}$ is the metric induced on $\Sigma$ as a hypersurface in $\mathbb{R}^{n+1}$. Since $\Sigma$ is closed, it follows that there exists an $\epsilon_0 > 0$, such that, for any $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$ and any $\epsilon \in \mathbb{R}$ satisfying $|\epsilon| < \epsilon_0$, the tensor $\bar{g}$ defines a metric on $\Sigma_\epsilon$ and the volume measure $\sqrt{\bar{g}}$ and the derivative $\partial \sqrt{\bar{g}}/\partial \epsilon$

are uniformly bounded on compact sets. Define

$$G_{x_0,t_0}(\beta, x) = e^{-\frac{|x-x_0+\beta t_0|^2}{4\epsilon t_0}} \sqrt{\bar{g}}.$$

Then Jacobi’s formula gives

$$\frac{\partial G_{x_0,t_0}(\beta, x)}{\partial \beta} = \frac{1}{2} \left( \text{Tr} \left( \bar{g}^{-1} \frac{\partial \bar{g}}{\partial \beta} \right) - \frac{f(x-x_0, n) + \beta f^2}{t_0} \right) e^{-\frac{|x-x_0+\beta t_0|^2}{4\epsilon t_0}} \sqrt{\bar{g}},$$

which is uniformly bounded on $K$ by a constant depending only on $\epsilon_0$ and $K$. In particular, the constant is independent of the choice of $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$. Therefore

$$|F_{x_0,t_0}(\Sigma_\epsilon) - F_{x_0,t_0}(\Sigma)| = \left| \frac{1}{\sqrt{4\pi t_0}} \int_{\Sigma} (G_{x_0,t_0}(\epsilon, x) - G_{x_0,t_0}(0, x)) dx \right|
\leq \frac{1}{\sqrt{4\pi t_0}} \int_{\Sigma} |G_{x_0,t_0}(\epsilon, x) - G_{x_0,t_0}(0, x)| dx
= \frac{\epsilon}{\sqrt{4\pi t_0}} \int_{\Sigma} \int_0^1 \left| \frac{\partial G_{x_0,t_0}(\epsilon u, x)}{\partial \beta} \right| du dx
\leq \frac{\epsilon}{\sqrt{4\pi t_0}} \int_{\Sigma} \int_0^1 \left| \frac{\partial G_{x_0,t_0}(\epsilon u, x)}{\partial \beta} \right| du dx
\leq \epsilon C$$

as desired. \hfill \Box

The main result of this section, below, shows there exists a compact subset of $\mathbb{R}^2 \times (0, \infty)$ such that the entropy of any curve obtained by perturbing a given closed plane shrinker is attained in this compact set. As a consequence, we may apply the previous proposition to conclude that the entropy functional is continuous under small perturbations of a closed plane shrinker. Theorem 2.3 may be viewed as a quantitative version of \cite[Lemma 7.7]{11}.

\textbf{Theorem 2.3.} Let $\Gamma \subset \mathbb{R}^2$ be a closed shrinker. Then there exists an $\epsilon_0 > 0$ and a compact subset $K \subset \mathbb{R}^2 \times (0, \infty)$, both depending only on $\Gamma$, such that, for any variation $f \in C^{2,\alpha}(\Sigma)$ with $\|f\|_{C^{2,\alpha}} \leq 1$ and any $\epsilon \in \mathbb{R}$ with $|\epsilon| < \epsilon_0$, the following holds:

$$\lambda(\Gamma_\epsilon) = \sup_{(x_0,t_0) \in K} F_{x_0,t_0}(\Gamma_\epsilon),$$

where $\Gamma_\epsilon = \Gamma + \epsilon f n$. 

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Proof. Throughout, if \( f \in C^{2,\alpha}(\Gamma) \), then we will denote by \( \Gamma_\epsilon' \) the perturbed curve \( \Gamma_\epsilon' = \Gamma + \epsilon f \mathbf{n} \).

As stated in the proof of the previous proposition, there exists an \( \epsilon_0 > 0 \) such that \( \Gamma_\epsilon' \) is a closed immersed curve for any \( \epsilon \in \mathbb{R} \). Let \( B \subset C^{2,\alpha}(\Gamma) \times (0, \infty) \) consist of those \( \Gamma_\epsilon' \) with \( \|f\|_{C^{2,\alpha}} \leq 1 \) and \( \epsilon \leq \epsilon_0 \).

Throughout, if \( f \) and \( \epsilon \) such that \( \Gamma_\epsilon' \) is contained in the closed ball \( B_R(0) \subset \mathbb{R}^2 \) of radius \( R \) centered at the origin, which is compact. Colding and Minicozzi [11] show that for any closed hypersurface \( M \subset \mathbb{R}^{n+1} \) with \( \epsilon > 0 \) and each fixed \( t_0 > 0 \), sup \( F_{x_0,t_0}(M) \) is achieved inside the convex hull of \( M \), which is compact, since \( M \) is closed.

Step (1): Bounds on \( |x_0| \). By the compactness of \( \Gamma \) and the uniform \( C^{2,\alpha} \) bound the the variation functions, there exists \( R > 0 \) (independent of \( f \) and \( \epsilon \)) such that \( \Gamma_\epsilon' \) is contained in the closed ball \( B_R(0) \subset \mathbb{R}^2 \) of radius \( R \) centered at the origin, which is compact. Colding and Minicozzi [11]

show that for any closed hypersurface \( M \subset \mathbb{R}^{n+1} \) and each fixed \( t_0 > 0 \), sup \( x_0 \in \mathbb{R}^{n+1} \) \( F_{x_0,t_0}(M) \) is attained in \( B_R(0) \subset \mathbb{R}^2 \).

Step (2): Upper bound on \( t_0 \). Let \( T_1 = (4\pi)^{-n/2} \text{Vol}(B_R) \). Then for any \( t_0 > T_1 \), any \( x_0 \in B_R(0) \), and any hypersurface \( \Gamma_\epsilon' \in \mathcal{B} \), we have

\[
F_{x_0,t_0}(\Gamma_\epsilon') \leq (4\pi t_0)^{-n/2} \text{Length}(\Gamma_\epsilon') \\
\leq (4\pi t_0)^{-n/2} \text{Vol}(B_R) \\
< (4\pi T_1)^{-n/2} \text{Vol}(B_R) \\
= 1.
\]

However, we know that \( \lambda(\Gamma_\epsilon') > 1 \). This can be seen as follows: a closed embedded curve has entropy at least that of the round circle, while a closed curve with self-intersection has entropy at least 2. We conclude that

\[
sup_{(x_0,t_0) \in B_R(0) \times (0,\infty)} F_{x_0,t_0}(\Gamma_\epsilon') = \sup_{(x_0,t_0) \in B_R(0) \times (0,T_1)} F_{x_0,t_0}(\Gamma_\epsilon'),
\]

for all \( \Gamma_\epsilon' \in \mathcal{B} \).

Step (3): Lower bound on \( t_0 \). We observe that for \( \epsilon_0 \) small enough, \( \Gamma_\epsilon' \) has a uniform curvature bound. This follows from the expression of curvature in terms of the first and second derivatives of the immersion. Therefore, we may assume that \( |k| \leq k_0 \) for all \( \epsilon \leq \epsilon_0 \). Let \( R_0 > 0 \) be a constant whose value will be determined. Now for any fixed \( x_0 \) and any \( t_0 \leq R_0^{-3} \), we have

\[
F_{x_0,t_0}(\Gamma_\epsilon') = (4\pi t_0)^{-1/2} \int_{\{|x-x_0| \geq R_0^{-1}\cap \Gamma_\epsilon'}} e^{-\frac{|x-x_0|^2}{4t_0}} ds \\
+ (4\pi t_0)^{-1/2} \int_{\{|x-x_0| \leq R_0^{-1}\cap \Gamma_\epsilon'}} e^{-\frac{|x-x_0|^2}{t_0}} ds.
\]

The first term on the right hand side is bounded by

\[
(4\pi t_0)^{-1/2} e^{-\frac{R_0^{-2}}{4t_0}} \text{Length}(\Gamma_\epsilon') \leq CR_0^{-1/2}
\]

for a universal constant \( C \).

Now we estimate the second term on the right hand side. By change of variables, this term equals

\[
\int_{\{|x-x_0| \leq R_0^{-1}\cap \Gamma_\epsilon'}} e^{-\frac{|x-x_0|^2}{4}} ds,
\]
where \( t_0^{-1} \Gamma_i^f \) is the curve \( \Gamma_i^f \) rescaled by \( t_0^{-1} \) with center at \( x_0 \). Suppose \( \{ |x - x_0| \leq R_0^{-1} \} \cap (t_0^{-1} \Gamma_i^f) \) consists of the connected components \( \gamma_1, \ldots, \gamma_m \). Then each \( \gamma_i \) is a curve with curvature uniformly bounded by \( k_0 t_0 \). In particular, for a given \( \delta > 0 \), when \( t_0 \) is large enough depending on \( k_0 \), the curvature of \( \gamma_i \) is small enough and therefore
\[
\int_{\gamma_i} e^{-|x-x_0|^2} \, ds \leq 1 + \delta.
\]

So we only need to bound the number of such curves to bound the second term.

Let us consider the immersed tubular neighborhood \( N(\Gamma, \alpha) \) of \( \Gamma \) of radius \( \alpha \), which is an open subset of \( \mathbb{R}^2 \) with multiplicity. Here \( \alpha \) only depends on \( \Gamma \) and can be chosen small enough such that the multiplicity of the tubular neighborhood is at most the multiplicity of the curve \( \Gamma \). Let us choose \( t_0 \) small such that the \( \epsilon_0 < \alpha/2 \), and choose \( R_0 \) large such that \( R_0^{-1} < \alpha/2 \), then if the ball \( \{ |x - x_0| \leq R_0^{-1} \} \) intersects \( \Gamma_i^f \) nontrivially, it completely lies in \( N(\Gamma, \alpha) \). Moreover, each \( \gamma_i \) counts the multiplicity of the tubular neighborhood \( N(\Gamma, \alpha) \) once. So the number \( m \) of \( \gamma_i \)'s is no more than the multiplicity of \( N(\Gamma, \alpha) \), hence no more than the multiplicity of \( \Gamma \).

We conclude that
\[
F_{x_0, t_0}(\Gamma_i^f) \leq C R_0^{-1/2} + m(1 + \delta) = m + \delta m + C R_0^{-1/2}.
\]

Here \( \Gamma \) is a shrinking symmetric Fermi--Walker normal, a result of Colding-Minicozzi \cite[Lemma 7.7]{Colding2001} gives that its entropy is achieved at \( F_{x_0, t_0} \). Thus the entropy of \( \Gamma \) is greater than the multiplicity of \( \Gamma \), i.e. \( \lambda(\Gamma) > m + \beta \) for some \( \beta > 0 \). So when \( \epsilon_0 \) is small, Proposition \ref{prop:2.1} gives that \( F_{x_0, t_0}(\Gamma_i^f) > m + \beta/2 \). If we pick \( \delta \) small enough and \( R_0 \) large enough, for \( t_0 \leq R_0^{-3} =: T_0 \) we have
\[
F_{x_0, t_0}(\Gamma_i^f) \leq m + \beta/4 < m + \beta/2 < \lambda(\Gamma).
\]

Combining all of the steps above, the proposition follows, with \( K := B_R(0) \times [T_0, T_1] \).

**Corollary 2.4.** Let \( \Gamma \subset \mathbb{R}^2 \) be a closed shrinker and let \((\Gamma_i)_{i \in \mathbb{N}}\) be a sequence of plane curves which \( C^\infty \) converges to \( \Gamma \). Then
\[
\lim_{i \to \infty} \lambda(\Gamma_i) = \lambda(\Gamma).
\]

**Proof.** It follows from the convergence assumption that there exists an \( N' \in \mathbb{N} \), such that, for \( i > N' \), \( \Gamma_i \) can be written as a graph
\[
\Gamma_i = \Gamma + \epsilon_i f_i \mathbf{n}_i
\]
for some \( f_i \in C^\infty(\Gamma) \) normalized to \( \| f_i \|_{C^\infty} = 1 \) and some \( \epsilon_i > 0 \) with \( \epsilon_i \to 0 \) as \( i \to \infty \). Applying the previous theorem implies that there exists an \( \lambda'' \geq 1 \) and a compact subset \( K \subset \mathbb{R}^2 \times (0, \infty) \) such that, for all \( i > N'' \), the entropy \( \lambda(\Gamma_i) \) is achieved at some \( (x_i, t_i) \in K \). Then by Proposition \ref{prop:2.1}, there exists an \( N \geq N'' \) and a constant \( C \) depending only on \( K \) such that, for any \( (x_0, t_0) \in K \) and any \( i > N \), we have
\[
F_{x_0, t_0}(\Gamma_i) - C \epsilon_i \leq F_{x_0, t_0}(\Gamma) \leq F_{x_0, t_0}(\Gamma_i) + C \epsilon_i.
\]

Consequently,
\[
\sup_K F_{x_0, t_0}(\Gamma_i) - C \epsilon_i \leq \sup_K F_{x_0, t_0}(\Gamma) \leq \sup_K F_{x_0, t_0}(\Gamma_i) + C \epsilon_i.
\]

Since the entropy \( \lambda(\Gamma_i) \) is achieved in \( K \) for all \( i > N \), it follows immediately that
\[
\lambda(\Gamma_i) - C \epsilon_i \leq \lambda(\Gamma) \leq \lambda(\Gamma_i) + C \epsilon_i.
\]

Now taking the limit \( i \to \infty \) gives the desired result. \( \Box \)
The following corollary will be central to our construction of a generic CSF. The previous corollary shows that the entropy is continuous at a closed plane shrinker. However, for the construction of the piecewise CSF in Theorem B, we also need the entropy to be continuous near a closed plane shrinker. The following corollary ensures that the latter condition is satisfied.

**Corollary 2.5.** Let \( \Gamma \subset \mathbb{R}^2 \) be a closed shrinker, \( f \in C^{2,\alpha}(\Gamma) \) a variation, and \( (g_i)_{i \in \mathbb{N}} \subset C^\infty(\Gamma) \) a sequence of functions which \( C^\infty \)-converges to 0. Then there exists an \( \epsilon_0 > 0 \) such that, for all \( \epsilon \in \mathbb{R} \) with \( |\epsilon| < \epsilon_0 \), we have

\[
\lim_{i \to \infty} \lambda(\Gamma_\epsilon + g_i n) = \lambda(\Gamma_\epsilon),
\]

where \( \Gamma_\epsilon = \Gamma + \epsilon f n \).

**Proof.** By the \( C^\infty \) convergence assumption, there exists \( N > 0 \) such that \( \|g_i\|_{C^{2,\alpha}} \leq 1/2 \) for all \( i > N \). Since \( \Gamma \) is compact and \( f \in C^{2,\alpha}(\Gamma) \), there exists \( \epsilon_1 > 0 \) such that \( \|ef\|_{C^{2,\alpha}} \leq 1/2 \) for all \( \epsilon \in \mathbb{R} \) with \( |\epsilon| < \epsilon_1 \). Consequently, \( \|ef + g_i\|_{C^{2,\alpha}} \leq 1 \) for all \( i > N \) and all \( \epsilon \in \mathbb{R} \) with \( |\epsilon| < \epsilon_1 \). By Theorem 2.3, there exists an \( \epsilon_2 > 0 \) with \( \epsilon_2 \leq \epsilon_1 \), an \( N' \geq N \), and a compact subset \( K \subset \mathbb{R}^2 \times (0, \infty) \) such that, for all \( \epsilon \in \mathbb{R} \) with \( |\epsilon| < \epsilon_2 \) and all \( i > N' \), the entropy of the curves \( \Gamma_\epsilon \) and \( \Gamma_\epsilon + g_i n \) is attained in \( K \). By Proposition 2.1, for all \( \epsilon \in \mathbb{R} \) with \( |\epsilon| < \epsilon_0 \) and all \( (x_0, t_0) \in K \), we have

\[
\lim_{i \to \infty} F_{x_0, t_0}(\Gamma_\epsilon + g_i n) = F_{x_0, t_0}(\Gamma_\epsilon).
\]

Since the curves \( \Gamma_\epsilon \) and \( \Gamma_\epsilon + g_i n \) attain their entropies in \( K \) for all \( \epsilon \in \mathbb{R} \) with \( |\epsilon| < \epsilon_0 \) and all \( i > N' \), arguing as in the proof of the previous corollary concludes the proof. \( \square \)

### 3 Entropy and Turning Number

In this section we will prove Theorem A.1, which gives lower bounds on the entropy of closed plane curves which generate type I singularities under the CSF. We will first prove a particular case, in which we only consider the closed plane shrinkers themselves.

**Theorem 3.1.** Let \( \Gamma \subset \mathbb{R}^2 \) be a closed shrinker with turning number \( m \). Then \( \lambda(\Gamma) \geq \lambda(\Gamma_m) \).

**Proof.** We assume \( \Gamma \neq \Gamma_m \), since otherwise the result holds trivially. Let \( x : S^1 \to \mathbb{R}^2 \) be the given immersion of \( \Gamma \). There are two cases to consider.

- **Case (I):** Suppose that the degree of the map \( x : S^1 \to x(S^1) \) is 1 (in other words, \( \Gamma \) is not multiply-covered). Define the constant normal variation function \( f = 1 \) along \( \Gamma \). By Corollary 2.4, if \( \Gamma_\epsilon = \Gamma + \epsilon n \), then

\[
\lim_{\epsilon \to 0} \lambda(\Gamma_\epsilon) = \lambda(\Gamma).
\]

In particular, there exists a constant \( C > 0 \) such that

\[
\lambda(\Gamma) - C\epsilon \leq \lambda(\Gamma_\epsilon) \leq \lambda(\Gamma) + C\epsilon,
\]

for all \( \epsilon > 0 \) small enough. Au [4] shows that the rescaled CSF starting at \( \Gamma_\epsilon \) converges to the the \( m \)-covered circle. By the monotonicity property of entropy, we therefore have

\[
\lambda(\Gamma_m) \leq \lambda(\Gamma_\epsilon) \leq \lambda(\Gamma) + C\epsilon.
\]

Taking the limit \( \epsilon \to 0 \) shows that \( \lambda(\Gamma) \geq \lambda(\Gamma_m) \).
Case (2): Suppose that the degree of the map \( x : S^1 \to x(S^1) \) is greater than 1 (in other words, \( \Gamma \) is a multiply-covered Abresch-Langer curve). Then \( \Gamma \) is a \( k \)-covered Abresch-Langer curve (with \( k \geq 2 \)) \( \Gamma' \), where \( \Gamma' \) has turning number \( p \) satisfying \( kp = m \). By the previous case and (1.2),

\[
\lambda(\Gamma) = k \cdot \lambda(\Gamma') \geq k \cdot \lambda(\Gamma_p) = kp \cdot \lambda(\Gamma_1) = m \cdot \lambda(\Gamma_1) = \lambda(\Gamma_m).
\]

Lemma 3.2. The turning number of a closed immersed curve \( \Gamma \subset \mathbb{R}^2 \) is preserved under the CSF and rescaled CSF.

Proof. CSF and rescaled CSF preserve immersedness and are therefore regular homotopies. Regular homotopy classes of immersions \( S^1 \to \mathbb{R}^2 \) are classified by their turning number, by the Whitney-Graustein theorem \([24, \text{Theorem } 1]\). In other words, two closed immersed plane curves are regularly homotopic if and only if they have the same turning number.

Proof of Theorem A.1. We assume that \( \Gamma \) is not a shrinker, since otherwise the theorem follows immediately from Theorem 3.1. Let \((\Gamma_t)_{t \in [0,T)} \) be the CSF starting at \( \Gamma \). Since all type I singularities are closed shrinkers, they are compact, so there exists a unique singular point \( x_0 \in \mathbb{R}^2 \). Without loss of generality, we may assume \( x_0 \) is the origin. Let \((\tilde{\Gamma}_\tau)_{\tau \in [\log T^{-1/2}, \infty)} \) be a rescaled CSF \([16, \text{Section 2}]\) around \( 0 \in \mathbb{R}^2 \) starting at \( \Gamma \). By \([16, \text{Theorem } 3.5]\), for any sequence \( \tau_i \to \infty \), there exists a subsequence also denoted \( \tau_i \) such that the rescaled curves \( \tilde{\Gamma}_{\tau_i} \) will smoothly converge to a shrinker \( \tilde{\Gamma}_\infty \) as \( \tau_i \to \infty \). By the classification of 1-dimensional shrinkers \([1]\), \( \tilde{\Gamma}_\infty \) must be a multiply-covered circle or a multiply-covered Abresch-Langer curve.

Let \( n \) be the turning number of \( \tilde{\Gamma}_\infty \). By the lemma, the turning number of \( \tilde{\Gamma}_\tau \) is \( m \) for all \( \tau \in [\log T^{-1/2}, \infty) \). By the \( C^\infty \)-convergence \( \tilde{\Gamma}_{\tau_i} \to \tilde{\Gamma}_\infty \), the curvature of \( \tilde{\Gamma}_{\tau_i} \) smoothly converges to the curvature of \( \tilde{\Gamma}_\infty \). Therefore the total curvatures satisfy

\[
n = \lim_{i \to \infty} \frac{1}{2\pi} \int_{\tilde{\Gamma}_{\tau_i}} \tilde{k}_i = m.
\]

This shows that \( \tilde{\Gamma}_\infty \) is a shrinker of turning number \( m \). By the monotonicity of entropy,

\[
\lambda(\tilde{\Gamma}_\tau) \leq \lambda(\Gamma)
\]

for all \( \tau \in [\log T^{-1/2}, \infty) \). By the \( C^\infty \)-convergence \( \tilde{\Gamma}_{\tau_i} \to \tilde{\Gamma}_\infty \), Corollary 2.4 implies that

\[
\lim_{i \to \infty} \lambda(\tilde{\Gamma}_{\tau_i}) = \lambda(\tilde{\Gamma}_\infty),
\]

and since this holds for any blowup sequence, it follows thus that

\[
\lambda(\Gamma) \geq \lambda(\tilde{\Gamma}_\infty).
\]

Since we assume that \( \Gamma \) is not a shrinker and since entropy is constant along the CSF only for shrinkers, the inequality is in fact strict. The theorem now follows from Theorem 3.1.
4 Entropy Index and Morse Index

Recall from the introduction that the shrinkers \( \Sigma \subset \mathbb{R}^{n+1} \) are precisely the minimal surfaces of \( \mathbb{R}^{n+1} \) equipped with the conformally changed metric \( g_{ij} = e^{-\frac{|x|^2}{4}} \delta_{ij} \). In this section, we will provide an important connection between the entropy stability of a shrinker and the stability of the shrinker considered as a minimal surface in \( (\mathbb{R}^{n+1}, e^{-\frac{|x|^2}{4}} \delta_{ij}) \). Namely, Theorem 4.7, shows that a closed shrinker \( \Sigma \subset \mathbb{R}^{n+1} \) is entropy unstable if and only if the Morse index of \( \Sigma \) is greater than \( n+2 \). Consequently, to prove Theorem A.2 we will only need to show that the \( m \)-covered circle has Morse index greater than 3. The results of this section apply to shrinkers \( \Sigma^n \subset \mathbb{R}^{n+1} \) of any dimension.

Definition 4.1 (Entropy stability). Let \( \Sigma^n \subset \mathbb{R}^{n+1} \) be a closed shrinker.

(i) Let \( \Sigma_\varepsilon \) be a variation of \( \Sigma \) in the direction \( f \in C^{2,\alpha}(\Sigma) \). The variation \( \Sigma_\varepsilon \) is entropy stable if there exist some variations \( x_\varepsilon \) of 0 and \( t_\varepsilon \) of 1 such that

\[
\frac{\partial^2}{\partial \varepsilon^2} \bigg|_{\varepsilon = 0} F_{x_\varepsilon, t_\varepsilon}(\Sigma_\varepsilon) \geq 0;
\]

otherwise, the variation is entropy unstable.

(ii) The shrinker \( \Sigma \) is entropy stable if every variation of \( \Sigma \) is entropy stable. Otherwise, \( \Sigma \) is entropy unstable.

(iii) The entropy index of \( \Sigma \) is the number of linearly independent unstable variations.

When there is no possibility for confusion, we simply say a variation or a shrinker is stable (unstable), instead of entropy stable (entropy unstable). In [11, Theorem 0.15], Colding-Minicozzi show that for an entropy unstable shrinker, there exists a variation which will decrease entropy, thus providing the link between entropy and \( F \)-functional. Their theorem is stated for embedded shrinkers, but their proof also works for closed immersed shrinkers.

Theorem 4.2 ([11]). Let \( \Sigma \subset \mathbb{R}^{n+1} \) be a closed shrinker whose entropy index is nonzero. Then \( \Sigma \) is entropy unstable, i.e. there exists a variation \( f \in C^{\infty}(\Sigma) \) and an \( \varepsilon_0 > 0 \) such that \( \lambda(\Sigma_\varepsilon) < \lambda(\Sigma) \) for all \( \varepsilon \in \mathbb{R} \) with \( |\varepsilon| < \varepsilon_0 \) and \( \varepsilon \neq 0 \), where \( \Sigma_\varepsilon = \Sigma + \varepsilon f n \).

We will use two lemmas in the proof of Theorem 4.7:

Lemma 4.3. Let \( \Sigma \subset \mathbb{R}^{n+1} \) be a closed shrinker and let \( f \in C^{2,\alpha}(\Sigma) \) be contained in the subspace of variations spanned by \( H \) and the components \( n_1, \ldots, n_{n+1} \) of the normal vector field \( n : \Sigma \to \mathbb{R} \). Then \( f \) is an entropy stable variation.

Proof. We refer the reader to [11, Section 4] for properties of the second variation of the \( F \)-functional. Let \( f = aH + (Y, n) \) for some \( a \in \mathbb{R} \) and some \( Y \in \mathbb{R}^{n+1} \). Choose any \( y \in \mathbb{R}^{n+1} \) and any \( h \in \mathbb{R} \) and set \( x_\varepsilon = ey, t_\varepsilon = 1 + \varepsilon h \) and \( \Sigma_\varepsilon = \Sigma + \varepsilon f n \). Then with \( F'' = \partial_{\varepsilon\varepsilon} \bigg|_{\varepsilon = 0}(F_{x_\varepsilon, t_\varepsilon}(\Sigma_\varepsilon)) \).
we have
\[
F'' = \int_{\Sigma} \left( -f Lf + 2hf H - h^2 H^2 + f \langle y, n \rangle - \frac{1}{2} \langle y, n \rangle^2 \right) e^{-\frac{|y|^2}{4}} d\mu
\]
\[
= \int_{\Sigma} \left( -a^2 H^2 - \frac{1}{2} \langle Y, n \rangle^2 + 2ahH^2 - h^2 H^2 + \langle Y, n \rangle \langle y, n \rangle - \frac{1}{2} \langle y, n \rangle^2 \right) e^{-\frac{|y|^2}{4}} d\mu
\]
\[
= -\int_{\Sigma} \left( (a-h)^2 H^2 + \frac{1}{2} (\langle Y, n \rangle - \langle y, n \rangle)^2 \right) e^{-\frac{|y|^2}{4}} d\mu.
\]
Thus, if we choose \(h = a\) and \(y = Y\), then \(F'' = 0\), so \(f\) is an entropy stable variation. \(\square\)

Lemma 4.4. Let \(\Sigma \subset \mathbb{R}^{n+1}\) be a closed, orientable hypersurface. Then the component functions \(n_i : \Sigma \to \mathbb{R}\) of the unit normal vector field \(n\) are linearly independent.

Proof. We claim that if the Gauss map \(G : \Sigma \to S^n\) is surjective, then the result holds. Suppose that the Gauss map is surjective, and suppose for contradiction that the \(n_i\) are linearly dependent, i.e. there exists \(y \in \mathbb{R}^{n+1}\) such that \(\langle y, n(p) \rangle = 0\) for all \(p \in \Sigma\). Then since the Gauss map is surjective, \(y\) is orthogonal to every vector in \(\mathbb{R}^{n+1}\), which is true if and only if \(y\) is the zero vector.

Now we show that Gauss map of any closed orientable hypersurface \(x : M^n \to \mathbb{R}^{n+1}\) is surjective. Pick any vector \(v \in S^n\). We wish to show that there exists some \(p \in M\) such that \(n(p) = v\). By the compactness of \(M\), the smooth function \(\langle v, x \rangle : M \to \mathbb{R}\) attains a maximum at some point \(p \in M\). Let \(\{e_i\}\) be a basis of \(T_pM\). Then
\[
0 = \nabla_{e_i} \langle v, x(p) \rangle = \langle v, e_i \rangle
\]
for all \(i = 1, \ldots, n\). Consequently, \(v\) is orthogonal to \(M\) at \(p\), so \(v = \pm n(p)\). Since we have chosen \(p\) to be the maximum, and since \(\langle v, x \rangle\) is not constant (\(M\) is closed), we must in fact have \(v = n(p)\), as desired. \(\square\)

Corollary 4.5. Let \(\Sigma \subset \mathbb{R}^{n+1}\) be a closed shrinker. Then
\[
\dim \text{span}\{H, n_1, \ldots, n_{n+1}\} = n + 2,
\]
where \(n_i\) is the \(i^{th}\) component function of the unit normal vector field \(n\).

Proof. Note that \(H\) and \(n_i\) are linearly independent for all \(i\), since \(LH = H\) and \(Ln_i = \frac{1}{2} n_i\). The proposition now follows from Lemma 4.4. \(\square\)

We are now able to prove the main result of this section, which will be used in the next section to calculate the entropy index of Abresch-Langer curves and \(m\)-covered circles.

Theorem 4.7. Let \(\Sigma \subset \mathbb{R}^{n+1}\) be a closed shrinker with Morse index \(m\). Then \(\Sigma\) has entropy index
\[
\Lambda(\Sigma) = m - n - 2.
\]

Proof. By standard elliptic theory, the Morse index of a compact minimal surface is finite. In particular, \(m\) is finite.

First we will show that the entropy index is at most \(m - n - 2\). By the previous proposition, this will follow immediately if we can show that the entropy index is bounded above by the Morse index. Suppose \(f\) is an entropy unstable variation of \(\Sigma\). Then by definition, for any variations \(x_\epsilon\)
of $0 \in \mathbb{R}^{n+1}$ and $t_\epsilon$ of 1, we have $(F_{x_\epsilon,t_\epsilon}(\Sigma_\epsilon))''|_{\epsilon=0} < 0$. In particular, this holds for the trivial variations $x_\epsilon = 0$ and $t_\epsilon = 1$:

$$(F_{0,1}(\Sigma_\epsilon))''|_{\epsilon=0} = -\int_{\Sigma} fLfe^{-\frac{|x|^2}{4}} d\mu < 0.$$  

This shows that $f$ is an unstable variation of $\Sigma$ when $\Sigma$ is considered as a minimal surface in the conformally changed metric. By definition, the dimension of the space of unstable variations of $\Sigma$ as a minimal surface is the Morse index. This shows that the entropy index is bounded above by the Morse index. Combining this with Lemma 4.3 and Corollary 4.5 gives the desired bound.

Now we will show that the entropy index is at least $m - n - 2$. By definition of the Morse index, there exist $m$ linearly independent eigenfunctions $u_1, \ldots, u_m \in C^{2,\alpha}(\Sigma)$ corresponding to the eigenvalues $\mu_1 < \mu_2 \leq \cdots \leq \mu_m < 0$. Colding and Minicozzi [11, Corollary 5.15] have shown that there exists an orthonormal basis of eigenfunctions of $L$ for the weighted $L^2$ space. Therefore, without loss of generality, we may assume that $u_1, \ldots, u_m$ are orthonormal in the weighted $L^2$ space. The number of functions in $\{u_1, \ldots, u_m\}$ which are orthogonal to $H$ and the components $n_1, \ldots, n_{n+1}$ of $n$ is $m - n - 2$. Pick any such function $u_i$ (if none exists, the entropy index is zero) and let $\Sigma_\epsilon$ be a variation of $\Sigma$ by $u_i n$. Choose any $y \in \mathbb{R}^{n+1}$ and any $h \in \mathbb{R}$ and set $x_\epsilon = ey$ and $t_\epsilon = 1 + \epsilon h$. Then

$$(F_{x_\epsilon,t_\epsilon}(\Sigma_\epsilon))''|_{\epsilon=0} = \int_{\Sigma} \left(-u_i Lu_i - h^2 H^2 - \frac{1}{2}(y, n)^2\right) e^{-\frac{|x|^2}{4}} d\mu$$

$$< \mu_i \int_{\Sigma} u_i^2 e^{-\frac{|x|^2}{4}} d\mu$$

by orthogonality and the assumption that $\mu_i < 0$. Since we had $m - n - 2$ possible linearly independent choices for such a $u_i$, the theorem follows. 

5 Entropy Instability of CSF Singularities

Having established in the previous section the relationship between the entropy index and Morse index of a shrinker, we will now calculate the Morse index of $m$-covered circles $\Gamma_m$ and Abresch-Langer curves $\Gamma_{m,n}$. For 1-dimensional closed shrinkers, the Jacobi operator is Sturm-Liouville.

Proposition 5.1. Let $\Gamma \subset \mathbb{R}^2$ be a shrinker. Then the Jacobi operator $L$ of $\Gamma$ is a Sturm-Liouville operator. In particular, for all $j \geq 1$, the eigenfunctions $u_{2j-1}$ and $u_{2j}$ of $L$ have exactly $2j$ zeros.

Proof. We refer the reader to [11, Section 5] for properties of the Jacobi operator of a shrinker. Differentiate the relation $k = \frac{1}{2}(x, n)$ to obtain $2k_s/k = (x, t)$ and note that $\nabla(\cdot) = \partial_s(\cdot)t$. 

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Therefore

\[ L = \Delta - \frac{1}{2} \langle x, \nabla (\cdot) \rangle + k^2 + \frac{1}{2} \]
\[ = \partial_{ss} - \frac{1}{2} \langle x, \partial_s (\cdot) t \rangle + k^2 + \frac{1}{2} \]
\[ = \partial_{ss} - \frac{k_s}{k} \partial_s + k^2 + \frac{1}{2} \]
\[ = k \left( \partial_s \left( \frac{1}{k} \partial_s \right) + \frac{1}{k} (k^2 + \frac{1}{2}) \right). \]

The latter part of the theorem follows from general Sturm-Liouville theory [8, Theorem 8.3.1].

For \( m \)-covered circles, the Jacobi operator reduces even further, as the curvature is constant. In this case, we are able to calculate the spectrum and the entropy unstable variations exactly (see Remark 5.3).

**Theorem 5.2.** Let \( \Gamma_m \) be an \( m \)-covered circle of radius \( \sqrt{2} \). Then the entropy index of \( \Gamma_m \) is

\[ \Lambda(\Gamma_m) = 2 \lceil \sqrt{2}m \rceil - 4, \]

where \( \lceil \cdot \rceil \) is the ceiling function. In particular, \( \Gamma_1 \) is the only entropy stable circle.

**Proof.** Since \( \Gamma_m \) has constant curvature, \( k_s = 0 \). Thus the Jacobi operator reduces to

\[ L = \partial_{ss} + 1, \]

and the spectrum can be calculated explicitly:

\[ \mu_j = \frac{j^2}{2m^2} - 1 \]

for \( j \in \mathbb{Z} \). Straightforward calculation using the above formula gives that the number of negative eigenvalues is \( 2 \lceil \sqrt{2}m \rceil - 1 \). The result now follows from Theorem 4.7. Note that the variation function corresponding to \( j = 0 \) is constant and hence proportional to \( k \). Furthermore, the variation functions corresponding to \( |j| = m \) are in the span of the component functions of the normal \( n \), since \( \mu_j = -\frac{1}{2} \) in these cases.

**Remark 5.3.** The unstable variation functions of the \( m \)-covered circle can be determined explicitly. A unit speed parameterization \( x : [0, 2\sqrt{2\pi}m] \to \mathbb{R}^2 \) of the \( m \)-covered circle \( \Gamma_m \) of radius \( \sqrt{2} \) is given by

\[ x(\theta) = \sqrt{2} (\cos(\theta/\sqrt{2}), \sin(\theta/\sqrt{2})). \]

Straightforward calculation shows that the directions of unstable variations of \( \Gamma_m \) are given by the collection of functions \( f_j, g_j : [0, 2\sqrt{2\pi}m] \to \mathbb{R} \) defined by

\[ f_j(\theta) = \sin \left( \frac{j}{\sqrt{2m}} \theta \right) \quad \text{and} \quad g_j = \cos \left( \frac{j}{\sqrt{2m}} \theta \right), \]

for \( j \in \mathbb{N} \) such that \( 1 \leq j < \sqrt{2}m \) and \( j \neq m \). By Theorem 5.2, these are all of the entropy unstable variations. Geometrically, varying \( \Gamma_m \) by these functions corresponds to enlarging some circles of \( \Gamma_m \), while contracting the others. The CSF will amplify this perturbation.
The proof of Theorem A.2 now follows easily using Theorem 5.2.

Proof of Theorem A.2. Theorem 5.2 shows that an $m$-covered circle is entropy unstable for $m \geq 2$. By Theorem 4.2, the $m$-covered circle $\Gamma_m$ can be perturbed to a curve $\Gamma'$ with lower entropy. Moreover, since the perturbation can be chosen to be arbitrarily $C^\infty$-small, the curvature of $\Gamma'$ can be chosen close enough to the curvature of $\Gamma_m$ so that the perturbed curve also has turning number $m$. The second part of Theorem A.2 follows immediately from Theorem A.1.

Entropy instability means that we can perturb a shrinker slightly to decrease entropy, and thus that the singularity corresponding to the given shrinker will never occur along the flow starting from the perturbed shrinker. In Section 6, our construction of a piecewise CSF for closed immersed curves, whose only singularities are embedded circles and type II singularities, rests upon the fact that the embedded circle is the only entropy stable closed plane shrinker. The latter fact will follow from Theorem 5.5.

An Abresch-Langer curve $\Gamma_{m,n}$ has entropy index

\begin{equation}
\Lambda(\Gamma_{m,n}) = 2n - 5.
\end{equation}

Proof. Let $L$ be the Jacobi operator of $\Gamma := \Gamma_{m,n}$. We claim that $k_s/k$ is an eigenfunction of $L$ with eigenvalue 0, i.e. that $L(k_s/k) = 0$. Indeed, differentiating the shrinker equation (1.7) gives

\begin{align*}
2L(k_s/k) &= \partial_{ss}(x, t) - k_s k_s(x, t) + (k^2 + 1/2)(x, t) \\
&= \partial_s(1 - k_s(x, n)) - \frac{1}{2}(x, t)(1 - k_s(x, n)) + (k^2 + 1/2)(x, t) \\
&= -k_s(x, n) - k^2(x, t) + \frac{1}{2}k_s(x, t)(x, n) + k^2(x, t) \\
&= -k_s(x, n) + k_s(x, n) \\
&= 0.
\end{align*}

Since $k$ is strictly positive and $k_s$ has exactly $2n$ zeros on $[0, 2\pi m]$, $k_s/k$ is either the $(2n - 1)^{st}$ or $2n^{th}$ eigenfunction of $L$. It remains to show that $k_s/k$ is the $(2n - 1)^{st}$ eigenfunction and that zero is a simple eigenvalue. To do so, we reparameterize $\Gamma$ using the variable

$$
\theta = -\cos^{-1}(e_1, n),
$$

where $e_1 \in \mathbb{R}^2$ is a constant unit vector. This is possible since $\Gamma$ is convex (however, $\theta$ is only locally continuous as a discontinuity appears after one round). It follows from the shrinker equation that if we define the operator $\tilde{L}$ by

$$
\tilde{L} = k^2 \partial_{\theta\theta} + (k^2 + 1/2),
$$

then $\tilde{L}k_\theta = k_s/k$ and $\tilde{L}f = Lf$. The theorem now follows from the linear analysis in [12, Proposition 2.1], however, we provide a proof here for the convenience of the reader.

Consequently, $k_\theta = k_s/k$ and

$$
\tilde{L}k_\theta = \tilde{L}(k_s/k) = L(k_s/k) = 0.
$$
In other words, $k_0$ is an eigenfunction of $\tilde{L}$ with eigenvalue 0. We denote by $\{\mu_j\}$ and $\{\nu_j\}$, respectively, the Dirichlet and Neumann eigenvalues of $\tilde{L}$ on $[0, \pi m/n]$. Let $f_j$ solve the Neumann problem $\tilde{L}f_j = -\nu_j f_j$ on $[0, \pi m/n]$. Since $k$ is an even function, $f_j$ may be extended by odd reflection and then periodically to an eigenfunction of $\tilde{L}$ on $[0, 2\pi m]$ with the same eigenvalue $\nu_j$. The eigenfunction $f_2$ will have exactly $2n$ zeros on $[0, 2\pi m)$ and $k_0$ is the lowest Dirichlet eigenvalue of $\tilde{L}$ on $[0, \pi m/n]$, so $\mu_1 = 0$. From the standard fact that $\mu_1 \geq \nu_2$, it follows that $k_0$ and $f_2$ are, respectively, the $(2n - 1)^{st}$ and $2n^{th}$ eigenfunctions of $\tilde{L}$ on $[0, 2\pi m]$. Moreover, since $\tilde{L}$ and $L$ have the same eigenfunctions, $k_s/k = k_0$ is the $(2n - 1)^{st}$ eigenfunction of $L$.

Next we show that zero is a simple eigenvalue. Note that $\tilde{L}$ is a linear second order differential operator so the equation $\tilde{L}f = 0$ can have at most two linearly independent solutions. If zero were not a simple eigenvalue, then there would exist a nontrivial $2\pi m$ periodic solution $w$ of $\tilde{L}w = 0$. Consequently, any solution of $\tilde{L}f = 0$, being a linear combination of $k_0$ and $w$, which are both $2\pi m$ periodic, would have to be $2\pi m$ periodic. As a result, to show that zero is a simple eigenvalue, it is sufficient to produce a solution to $\tilde{L}f = 0$ which is not $2\pi m$ periodic.

We will now produce such a solution. It follows from the shrinker equation that $k$ solves the ODE

$$k_{\theta \theta} + k - \frac{1}{2k} = 0.$$  

This equation has first integral

$$E = k_0^2 + k^2 - \frac{1}{4} \log k$$

and the general solution can be expressed as $k(\theta + a, E)$. Let

$$u = \frac{\partial k}{\partial E} \bigg|_{E=E_{m,n}},$$

where $E_{m,n}$ is the constant corresponding to $\Gamma_{m,n}$. Straightforward calculation using (5.7) shows that $\tilde{L}u = 0$. Abresch-Langer [1, Proposition 3.2] show that $u$ is not $2\pi m$ periodic. As stated above, it follows that zero is a simple eigenvalue of $\tilde{L}$ and thus also of $L$.

Summarizing, we have shown that $L(k_s/k) = 0$, that $k_s/k$ is the $(2n - 1)^{st}$, and that zero is a simple eigenvalue of $L$. Consequently, $L$ has $2n - 2$ negative eigenvalues. By Theorem 4.7, the Abresch-Langer curve $\Gamma_{m,n}$ has entropy index $2n - 5$.

**Corollary 5.8.** The embedded circle is the only entropy stable closed singularity of the CSF for closed curves.

### 6 Generic CSF

In this section, we prove Theorem B. The following lemma shows that, as a consequence of Corollary 5.8, we can perturb an unstable closed plane shrinker in such a way as to satisfy the entropy conditions (1.6) for a piecewise CSF. For the piecewise flow in Theorem B, we will actually require the entropy inequality (1.6) to be a strict inequality so that we can exclude a shrinker from appearing as a singularity at later times of the flow.
\textbf{Lemma 6.1.} Let $\Gamma \subset \mathbb{R}^2$ be a closed immersed curve such that the CSF $(\Gamma_t)_{t \in [0,T]}$ starting at $\Gamma$ has a type I singularity other than the embedded circle and let $\tilde{\Gamma}_\infty \subset \mathbb{R}^2$ be some blowup sequence limit. Then there exists a $T_0 > 0$ and a function $u \in C^\infty(\Gamma)$ such that the graph $\tilde{\Gamma} = \Gamma_{T_0} + un$ satisfies

\begin{equation}
\lambda(\tilde{\Gamma}) < \lambda(\tilde{\Gamma}_\infty).
\end{equation}

\textbf{Proof.} If the CSF $(\Gamma_t)_{t \in [0,T]}$ has a type I singularity at time $T$, any limit of a rescaling sequence is a closed shrinker and thus is compact. Therefore, there is only one singular point.

Without loss of generality, we may assume that $0 \in \mathbb{R}^2$ is a singular point. By the classification of 1-dimensional shrinkers [1], $\tilde{\Gamma}_\infty$ must be a multiply-covered circle or a multiply-covered Abresch-Langer curve. Since we assume that the shrinker $\tilde{\Gamma}_\infty$ is not an embedded circle, Corollary 5.8 implies that $\tilde{\Gamma}_\infty$ is entropy unstable. By Theorem 2.3, there exists a variation $f \in C^\infty(\tilde{\Gamma}_\infty)$ and an $\epsilon_0 > 0$ such that

$$\lambda(\tilde{\Gamma}_\infty + \epsilon f n) < \lambda(\tilde{\Gamma}_\infty),$$

for all $\epsilon \neq 0$ with $|\epsilon| < \epsilon_0$. By assumption, there exists a sequence $\tau_i \to \infty$ of rescaled times such that the rescaled curves $(\tilde{\Gamma}_{\tau_i})_{i \in \mathbb{N}}$ smoothly converge to $\tilde{\Gamma}_\infty$. Thus, there exists $N' \in \mathbb{N}$ and a sequence $(g_i)_{i \in \mathbb{N}} \subset C^\infty(\tilde{\Gamma}_\infty)$ with $\|g_i\|_{C^\infty} \to 0$ such that, for $i > N'$, the rescaled curve $\tilde{\Gamma}_{\tau_i}$ can be written as a graph

$$\tilde{\Gamma}_{\tau_i} = \tilde{\Gamma}_\infty + g_i n.$$

By the monotonicity of entropy and Corollary 2.5, there exists $\epsilon_1 \leq \epsilon_0$ and $N \geq N'$ such that, for all $\epsilon \neq 0$ with $|\epsilon| < \epsilon_1$ and all $i > N$, we have

$$\lambda(\tilde{\Gamma}_\infty + (\epsilon f + g_i) n) < \lambda(\tilde{\Gamma}_\infty) \leq \lambda(\tilde{\Gamma} + g_i n).$$

Since entropy is invariant under dilations, when the curve $\tilde{\Gamma}_\infty + (\epsilon f + g_i) n$ is rescaled back to original spacetime, the entropy conditions can be satisfied when $|\epsilon| \neq 0$ is small enough, $i$ is large enough. The time $T_0$ then corresponds to the rescaled time $\tau_i$ and the function $u$ is a multiple of the function $\epsilon f$. \hfill \Box

Using the classification of singularities for the CSF of closed curves and Lemma 6.1, we now prove Theorem B.

\textbf{Proof of Theorem B.} We will construct a piecewise CSF with a finite number of discontinuities that eventually becomes extinct in as an embedded circle, or has a type II singularity. We perform a smooth jump just before a (entropy unstable) singular time, where we replace a time slice of the flow by a graph over it, and the crucial point is to show that the entropy decreases below the entropy of the unstable singularity. We repeat this until we get a singular point where every rescaled singularity is either an embedded circle or a type II singularity.

Let $(\Gamma_t)_{t \in [0,T_1]}$ be the CSF starting at $\Gamma$. The CSF $(\Gamma_t)_{t \in [0,T_1]}$ has either a type I or type II singularity at time $T_1$. We consider both cases:

\textbf{Case (I):} If the CSF $(\Gamma_t)_{t \in [0,T_1]}$ has a type I singularity at time $T_1$, any limit of a rescaling sequence is either an entropy stable or entropy unstable closed shrinker. In particular, any limit of a rescaling sequence is compact, and therefore, there is only one singular point $x_1 \in \mathbb{R}^2$ at time $T_1$. 

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Case (I.a): If there is some rescaling sequence limit which is an entropy unstable closed shrinker $\Gamma_{1,\infty}^1$, Lemma 6.1 gives the existence of a $t_2 < T_1$ and a curve $\Gamma_{t_2}^2$ such that $\Gamma_{t_2}^2$ is a graph over $\Gamma_{t_2}^1 := \Gamma_{t_2}$ of a function $u_2 \in C^{2,\alpha}(\Gamma_{t_2}^1)$ satisfying

$$\lambda(\Gamma_{t_2}^2) < \lambda(\Gamma_{1,\infty}^1).$$

Consequently, there exists a CSF $(\Gamma_{t_2}^1)_{t \in [0, t_2]}$ starting at $\Gamma_{t_2}^2$ such that the concatenation of $(\Gamma_{t_2}^1)_{t \in [0, t_2]}$ with $(\Gamma_{t_2}^2)_{t \in [t_2, T_2]}$ is a piecewise CSF starting at $\Gamma$.

Case (I.b): If all rescaling sequences converge to an entropy stable closed shrinker, it must be an embedded circle, so the theorem holds.

Case (II): If the CSF $(\Gamma_t)_{t \in [0, T_1]}$ has a type II singularity at time $T_1$, the theorem holds.

If Case (I.a) applies, we return to cases (I) and (II), this time applied to $(\Gamma_{t_2}^2)_{t \in [t_2, T_2]}$, and repeat the process. Since piecewise CSF preserves turning number and since there are only finitely many closed plane shrinkers of any given turning number, it follows immediately from the fact that

$$\lambda(\Gamma_{t_2}^i) < \lambda(\Gamma_{1,\infty}^{i-1})$$

at each step and from the monotonicity of entropy under piecewise CSF that Case (I.a) can apply only finitely many times. This proves the first part of the theorem.

It was shown in the proof of Theorem A.1 that any blowup sequence of a type I singularity of the CSF for closed curves preserves turning number. It follows that a closed curve with turning number greater than 1 cannot have an embedded circle as the limit of a rescaling sequence. This proves the second part of Theorem B.

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