A BERNSTEIN TYPE RESULT FOR GRAPHICAL SELF-SHRINKERS IN $\mathbb{R}^4$  

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ABSTRACT. Self-shrinkers are important geometric objects in the study of mean curvature flows, while the Bernstein Theorem is one of the most profound results in minimal surface theory. We prove a Bernstein type result for graphical self-shrinker surfaces with co-dimension two in $\mathbb{R}^4$. Namely, under certain natural conditions on the Jacobian of any smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$, we show that the self-shrinker which is the graph of this map must be affine linear. The proof relies on the derivation of structure equations of graphical self-shrinkers in term of the parallel form, and the existence of some positive functions on self-shrinkers related to these Jacobian conditions.

1. Introduction

A smooth submanifold $\Sigma^n$ in $\mathbb{R}^{n+k}$ is a self-shrinker, if the equation

(1.1) $\vec{H} + \frac{1}{2} \vec{F}^\perp = 0$

holds for any point $\vec{F}$ on $\Sigma^n$. Here $\vec{H}$ is the mean curvature vector of $\Sigma^n$ and $\perp$ is the projection of $\vec{F}$ into the normal bundle of $\Sigma^n$.

Self-shrinkers are important in the study of mean curvature flows for at least two reasons. First, if $\Sigma$ is a self-shrinker, it is easily checked that

$$\Sigma_t = \sqrt{-t} \Sigma,$$

for $-\infty < t < 0$, is a solution to the mean curvature flow. Hence self-shrinkers are self-similar solutions to the mean curvature flow. On the other hand, by Huisken ([?]), the blow-ups around a type I singularity converge weakly to nontrivial self-shrinkers after rescaling and choosing subsequences. Since there are no closed minimal submanifolds in Euclidean space, the finite time singularity of mean curvature flows for initial compact data is unavoidable. Therefore it is desirable to classify self-shrinkers under various geometric conditions.

1.1. Motivation. The rigidity of graphical minimal submanifolds in Euclidean spaces is summarized as the Bernstein theorem. In this subsection, we always assume that $f$ is a smooth map from $\mathbb{R}^n$ into $\mathbb{R}^k$, and $\Sigma = (x, f(x))$
is the graph of $f$. The Bernstein theorem states that if $\Sigma$ is minimal, then $\Sigma$ is totally geodesic under the following conditions:

1. $n \leq 7$ and $k = 1$ by Simon ([?]);
2. any $n$ and $k = 1$ with $|Df| = o(\sqrt{|x|^2 + |f|^2})$ as $|x| \to \infty$ by Ecker-Huisken ([?]);
3. $n = 2$ and any $k$ with $|Df| \leq C$ by Chern-Osserman [?];
4. $n = 3$ and any $k$ with $|Df| \leq C$ by Fischer-Colbrie([?]);
5. any $n \geq 2$ and $k \geq 2$ with more restrictive conditions (See Remark 2.4) by Wang ([?] also see Jost-Xin and Yang ([?], [?]�).

On the other hand, a self-shrinker is minimal in $(\mathbb{R}^{n+k}, e^{-\frac{|x|^2}{2n}} dx^2)$ where $dx^2$ is the standard Euclidean metric ([?]). The interests in Bernstein type results for graphical self-shrinkers are revived due to the works of Ecker-Huisken ([?]) and Wang ([?]). They showed that a graphical self-shrinker $\Sigma$ with co-dimension $k = 1$ is a hyperplane through 0 without any restriction on dimension $n$. This implies that the graphical self-shrinker may have the similar rigidity as the graphical minimal submanifold for co-dimension $k \geq 2$.

From the historical results above, naturally we are first interested in the rigidity of graphical self-shrinker surfaces with co-dimension $k \geq 2$.

There are two main difficulties to study graphical self-shrinker surfaces with higher codimension. First, the techniques for minimal submanifolds in Euclidean space are generally not available. In the case of self-shrinker surfaces, there are no corresponding harmonic functions ([?]) and monotonicity formula of the tangent cone at infinity in the minimal surface theory ([?], [?]), also §17 in [?]). Second, the contrast between the hypersurface and higher co-dimensional submanifolds are another obstacle to study self-shrinkers with higher co-dimension. In the hypersurface case, the normal bundle is trivial, the mean curvature and second fundamental form are scale functions. In the higher co-dimension case, the normal bundle can be highly non-trivial. In general the computations related to mean curvature and second fundamental form in this situation are very involved except few cases.

However, recent progresses on self-shrinkers and graphical mean curvature flows offer new tools to overcome these obstacles under some conditions. By Cheng-Zhou [?] and Ding-Xin [?] there is a polynomial volume growth property for completely immersed, proper self-shrinkers (Definition 3.6). With this property, the integration technique gives good estimates if there are well-behaved structure equations satisfied by self-shrinkers (Lemma 3.10).

On the other hand, graphical self-shrinkers have a lot of structure equations in terms of parallel form (Theorem 2.7 and 3.5). This approach is inspired by the works of Wang ([?], [?]) and Tsui-Wang ([?]) to investigate graphical mean curvature flows with arbitrary codimension in product manifolds. They obtained evolution equations of the Hodge star of parallel forms along mean curvature flows (see Remark 2.3).

The contribution in this paper is to apply the parallel form’s theory into
the study of graphical self-shrinkers. In $\mathbb{R}^4$ both of codimension and di-
mension of a graphical self-shrinker surface are two. Then $\mathbb{R}^4$ provides four
parallel 2-forms to reflect various properties of a graphical self-shrinker sur-
face, which is explained in the next subsection.

1.2. **Statement of the main result.** In view of results mentioned above, it is interesting to investigate the graphical self-shrinker surface in $\mathbb{R}^4$. We obtain the following Bernstein type result by studying the Jacobian of a smooth map.

**Theorem 1.1.** Let $f = (f_1(x_1, x_2), f_2(x_1, x_2))$ be a smooth map from $\mathbb{R}^2$ into $\mathbb{R}^2$ with its Jacobian $J_f = (\frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1})$ satisfying one of the following conditions:

1. $J_f + 1 > 0$ for all $x \in \mathbb{R}^2$;
2. $1 - J_f > 0$ for all $x \in \mathbb{R}^2$.

If $\Sigma = (x, f(x))$ is a self-shrinker surface in $\mathbb{R}^4$, then $\Sigma$ is a plane through 0.

**Remark 1.2.** Notice that in the main result, the co-dimension of $\Sigma$ is two.

In particular, this implies that if $f$ is a diffeomorphism on $\mathbb{R}^2$ the graphical self-shrinker of $f$ in $\mathbb{R}^4$ is totally geodesic. One can compare this theorem with the results about the Jacobian of graphical minimal surfaces in $\mathbb{R}^4$ ([?], ??).

Let us explain conditions (1) and (2) in more details. Let $\Sigma$ be the graphical self-shrinker in Theorem 1.1. We take $(x_1, x_2, x_3, x_4)$ as the coordinate of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. Let $\eta_1 = dx_1 \wedge dx_2$, $\eta_2 = dx_3 \wedge dx_4$, $\eta' = \eta_1 + \eta_2$ and $\eta'' = \eta_1 - \eta_2$. First we choose a proper orientation on $\Sigma$ such that $*\eta_1 > 0$ (Def. 2.2). The direct computation shows that $*\eta' = (1 + J_f) * \eta_1$ and $*\eta'' = (1 - J_f) * \eta_1$ (Lemma 3.1). Then conditions (1) and (2) correspond to $*\eta' > 0$ and $*\eta'' > 0$, respectively. When both conditions (1) and (2) are satisfied, then we have $|J_f| < 1$, which means the map $f$ is area-decreasing. In [?], ??], assuming $f$ is area-decreasing, together with additional curvature conditions, they showed the mean curvature flow of the graph of some smooth map stays graphical and exists for all time.

Note that the usual maximum principle does not apply to our non-compact submanifold. Our main technical tool to treat this problem is Lemma 3.10, where we use a cutoff function and apply the Divergence Theorem: a technique also used in [?] for the case of hypersurface. The crucial condition is the polynomial volume growth property of graphical self-shrinkers. It can be of independent interest as we state it in a more general formulation (Lemma 3.10).

1.3. **Plan of the paper.** In §2, we discuss the parallel form and the geometry of graphical self-shrinkers. The structure equation of self-shrinkers in terms of parallel forms is summarized in Theorem 2.7. In §3, we apply Theorem 2.7 to the cases of $*\eta'$, $*\eta''$. For example, in Theorem 3.5 we derive...
that \( \ast \eta' = \eta'(e_1, e_2) \) satisfies the equation

\[
\Delta(\ast \eta') + \ast \eta'( \left( h_{1k}^3 - h_{2k}^4 \right)^2 + \left( h_{4k}^1 + h_{2k}^3 \right)^2 ) - \frac{1}{2} \langle \vec{F}, \nabla(\ast \eta') \rangle = 0,
\]

where \( h_{ij}^0 \) are the second fundamental form and \( \Delta (\nabla) \) is the Laplacian (covariant derivative) of \( \Sigma \). With the polynomial volume growth property, Lemma 3.10 implies that

\[
\left( h_{1k}^3 - h_{2k}^4 \right)^2 + \left( h_{4k}^1 + h_{2k}^3 \right)^2 \equiv 0,
\]

if \( \ast \eta' \) is a positive function. This implies that the graphical self-shrinker is minimal. Similar conclusion can be achieved for \( \ast \eta'' \). We then show that it is actually totally geodesic.

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2. **Parallel Forms**

The parallel forms in Euclidean space play the fundamental role in this paper. We will record the structure equation of self-shrinkers in terms of parallel forms. This equation can be quite general and we will present these results for submanifolds of arbitrary (co-)dimensions in general Riemannian manifolds in §2.1, and then restrict to self-shrinkers in Euclidean spaces in §2.2.

2.1. **Parallel forms and their Hodge star.** We will adapt the notations in [?] and [?]. Assume that \( N^n \) is a smooth n-dimensional submanifold in a Riemannian manifold \( M^{n+k} \) of dimension \( n+k \). We denote by \( \{e_i\}_{i=1}^n \) the orthonormal basis of the tangent bundle of \( N \) and \( \{e_\alpha\}_{\alpha=n+1}^{n+k} \) the orthonormal basis of the normal bundle of \( N \). The Riemann curvature tensor of \( M \) is defined by

\[
R(X,Y)Z = -\bar{\nabla}_X \bar{\nabla}_Y Z + \bar{\nabla}_Y \bar{\nabla}_X Z + \bar{\nabla}_{[X,Y]} Z,
\]

for smooth vector fields \( X, Y \) and \( Z \). The second fundamental form \( A \) and the mean curvature vector \( \vec{H} \) are defined as

\[
A(e_i, e_j) = (\bar{\nabla}_{e_i} e_j) = h_{ij}^\alpha e_\alpha
\]

(2.1)

\[
\vec{H} = (\bar{\nabla}_{e_i} e_i) = h_{ii}^\alpha e_\alpha = h^\alpha e_\alpha.
\]

(2.2)

Here we used Einstein notation and \( h^\alpha = h_{ii}^\alpha \).

Let \( \nabla \) be the covariant derivative of \( \Sigma \) with respect to the induced metric. Then \( \nabla^\perp \) \( A \) can be written as follows:

\[
\nabla^\perp_{e_k} A(e_i, e_j) = h_{ij,k}^\alpha e_\alpha.
\]

(2.3)
Note that $h_{ij,k}^a$ is not equal to $e_k(h_{ij}^a)$ unless $\Sigma$ is a hypersurface. In fact, we have:

**Lemma 2.1.** $h_{ij,k}^a$ takes the following form:

$$h_{ij,k}^a = e_k(h_{ij}^a) + h_{ij}^\beta(e_\alpha, \nabla_{e_\alpha} e_\beta) - C_{ki}^l h_{lj}^a - C_{kj}^l h_{li}^a,$$

where $\nabla_{e_i} e_j = C_{ij}^k e_k$.

**Proof.** By its definition, $h_{ij,k}^a = \langle \nabla_{e_k}^\perp A(e_i, e_j), e_\alpha \rangle$.

The conclusion follows from expanding $\nabla_{e_k}^\perp A(e_i, e_j)$:

$$h_{ij,k}^a = (\nabla_{e_k}^\perp A(e_i, e_j), e_\alpha) - \langle A(\nabla_{e_k} e_i, e_j), e_\alpha \rangle - \langle A(e_i, \nabla_{e_k} e_j), e_\alpha \rangle$$

$$= \langle \nabla_{e_k} (h_{ij}^\beta), e_\alpha \rangle - C_{ki}^l h_{lj}^a - C_{kj}^l h_{li}^a$$

$$= e_k(h_{ij}^a) + h_{ij}^\beta(e_\alpha, \nabla_{e_\alpha} e_\beta) - C_{ki}^l h_{lj}^a - C_{kj}^l h_{li}^a.$$

\[\square\]

For later calculation, we recall that the Codazzi equation is

$$R_{aijk} = h_{ij,k}^a - h_{ik,j}^a,$$

where $R_{aijk} = R(e_\alpha, e_i, e_k, e_j)$.

**Definition 2.2.** An $n$-form $\Omega$ is called parallel if $\nabla \Omega = 0$, where $\nabla$ is the covariant derivative of $M$.

The Hodge star $*\Omega$ on $N$ is defined by

$$*\Omega = \frac{\Omega(X_1, \ldots, X_n)}{\sqrt{\det(g_{ij})}}$$

where \{X_1, \ldots, X_n\} is a local frame on $N$ and $g_{ij} = \langle X_i, X_j \rangle$.

**Remark 2.3.** We denote by $M$ the product manifold $N_1 \times N_2$, $\Omega$ the volume form of $N_1$. Then $\Omega$ is a parallel form in $M$. If $N$ is a graphical manifold over $N_1$. Then $*\Omega > 0$ on $N$ for an appropriate orientation. For example, the graphical self-shrinker $\Sigma$ in §1.2 satisfies that $*\Omega > 0$ on $\Sigma$ where $\Omega$ is $dx_1 \wedge \cdots \wedge dx_n$.

$*\Omega$ is independent of the frame \{X_1, \ldots, X_n\}, up to a fixed orientation. This fact greatly simplifies our calculation. When \{X_1, \ldots, X_n\} is the orthonormal frame \{e_1, \ldots, e_n\}, $*\Omega = \Omega(e_1, \ldots, e_n)$.

The evolution equation of $*\Omega$ along mean curvature flows is the key ingredient in (2.5).

**Remark 2.4.** In [?], the author proved if $\Sigma = \langle x, f(x) \rangle$ is minimal where $f : \mathbb{R}^n \to \mathbb{R}^k$, there exists a $0 < \delta < 1$ and $K > 0$ such that $|\lambda_i \lambda_j| \leq 1 - \delta$ and $*\Omega > K$, then $\Sigma$ is affine linear.

Here $\{\lambda_i\}_{i=1}^n$ is the eigenvalue of $df$, $\Omega$ is $dx_1 \wedge \cdots \wedge dx_n$. 


The following equation (2.7) first appeared as equation (3.4) in [?] in the proof of the evolution equation of $\ast \Omega$ along the mean curvature flow. We provide a proof for the sake of completeness.

**Proposition 2.5.** Let $N^n$ be a smooth submanifold of $M^{n+k}$. Suppose $\Omega$ is a parallel $n$-form and $R$ is the Riemann curvature tensor of $M$. Then $\ast \Omega = \Omega(e_1, \cdots, e_2)$ satisfies the following equation:

$$
\Delta(\ast \Omega) = -\sum_{i,k} (h_{ik}^\alpha)^2 \ast \Omega + \sum_i (h_i^\alpha \sum_k R_{akik}) \Omega_{ia} + 2 \sum_{i<j,k} h_{ik}^\alpha h_{jk}^\beta \Omega_{ia,j\beta}.
$$

Here $\Delta$ denotes the Laplacian on $N$ with respect to the induced metric, and $h_{ik}^\alpha = h_{ii,k}^\alpha$. In the second group of terms, $\Omega_{ia} = \Omega(\hat{e}_1, \cdots, \hat{e}_n)$ with $\hat{e}_s = e_s$ for $s \neq i$ and $\hat{e}_s = e_\alpha$ for $s = i$. In the last group of terms, $\Omega_{ia,j\beta} = \Omega(\hat{e}_1, \cdots, \hat{e}_n)$ with $\hat{e}_s = e_s$ for $s \neq i,j$, $\hat{e}_s = e_\alpha$ for $s = i$ and $\hat{e}_s = e_\beta$ for $s = j$.

**Proof.** Recall that $\nabla$ and $\hat{\nabla}$ are the covariant derivatives of $N$ and $M$, respectively. Fixing a point $p$ on $\Sigma$, we assume that $\{e_1, \cdots, e_n\}$ is normal at $p$ with respect to $\nabla$. Lemma 2.1 implies that

$$
\nabla_{e_k} e_j(p) = 0, \quad h_{ij,k}^\alpha(p) = e_k(h_{ij}^\alpha(p)) + h_{ij}^\alpha(e_\alpha, \hat{\nabla}_{e_k} e_\beta)(p).
$$

Since $\nabla \Omega = 0$, we have

$$
\nabla_{e_k}(\ast \Omega) = \Omega(\nabla_{e_k} e_1, \cdots, e_n) + \cdots + \Omega(e_1, \cdots, \nabla_{e_k} e_n)
$$

$$
= \sum_i h_{ik}^\alpha \Omega_{ia}.
$$

For $\nabla_{e_k} \nabla_{e_k}(\ast \Omega)$, we get

$$
\nabla_{e_k} \nabla_{e_k}(\ast \Omega) = \sum_i e_k(h_{ik}^\alpha) \Omega_{ia} + \sum_i h_{ik}^\alpha e_k(\Omega_{ia}).
$$

The second term in (2.10) can be computed as

$$
\sum_i h_{ik}^\alpha e_k(\Omega_{ia}) = \sum_i h_{ik}^\alpha \Omega(e_1, \cdots, \nabla_{e_k} e_\alpha, \cdots, e_n) + 2 \sum_{i<j} h_{ik}^\alpha h_{jk}^\beta \Omega_{ia,j\beta}
$$

$$
= \sum_{i,\alpha} -(h_{ik}^\alpha)^2 \ast \Omega + h_{ik}^\alpha h_{jk}^\beta \Omega_{ia,j\beta} + 2 \sum_{i<j} h_{ik}^\alpha h_{jk}^\beta \Omega_{ia,j\beta}.
$$

Plugging this into (2.10) yields that

$$
\nabla_{e_k} \nabla_{e_k}(\ast \Omega) = -\sum_{i,\alpha} (h_{ik}^\alpha)^2 \ast \Omega + 2 \sum_{i<j} h_{ik}^\alpha h_{jk}^\beta \Omega_{ia,j\beta} + \sum_i h_{ik}^\alpha \Omega_{ia}
$$

$$
= -\sum_{i,\alpha} (h_{ik}^\alpha)^2 \ast \Omega + 2 \sum_{i<j} h_{ik}^\alpha h_{jk}^\beta \Omega_{ia,j\beta} + \sum_i (h_{kk,i} + R_{akik}) \Omega_{ia}.
$$
in view of (2.8) and (2.5), we can finally conclude that
\[
\Delta(\ast \Omega(p)) = \nabla_{e_k} \nabla_{e_k}(\ast \Omega)(p) - \nabla_{e_k} e_k(\ast \Omega)(p)
\]
\[
= - \sum_{i,k,\alpha} (h_{ik}^\alpha)^2 \ast \Omega + 2 \sum_{i<j,k} h_{ik}^\alpha h_{jk}^\beta \Omega_{i\alpha,j\beta} + \sum_i h_{i}^\alpha + \sum_k R_{ki}^{\alpha} \Omega_{i\alpha}.
\]
The conclusion follows.

The conclusion follows. \(\square\)

2.2. Self-shrinkers in Euclidean space. In this subsection, we only consider the case when \(M^{n+k}\) is the Euclidean space and \(N^n\) is a self-shrinker.

**Lemma 2.6.** Let \(\Omega\) be a parallel \(n\)-form in \(\mathbb{R}^{n+k}\). Suppose \(N^n\) is an \(n\)-dimensional self-shrinker in \(\mathbb{R}^{n+k}\). Using the notation in Proposition 2.5, we have:

\[
(2.11) \sum_i \Omega_{i\alpha} h_{i}^\alpha = \frac{1}{2} \langle \vec{F}, \nabla(\ast \Omega) \rangle
\]

where \(\vec{F}\) is any point on \(N\).

**Proof.** As in (2.8), we assume that \(\{e_1, \ldots, e_n\}\) is normal at \(p\). From (2.9), we compute \(\nabla(\ast \Omega)\) as follows:

\[
\nabla(\ast \Omega) = \nabla_{e_k}(\ast \Omega)e_k = (\sum_i h_{ik}^\alpha \Omega_{i\alpha})e_k.
\]

This leads to

\[
(2.12) \frac{1}{2} \langle \vec{F}, \nabla(\ast \Omega) \rangle = \frac{1}{2} \langle \vec{F}, e_k \rangle (\sum_i h_{ki}^\alpha \Omega_{i\alpha}).
\]

Recalling that \(\vec{H} = h^\alpha e_\alpha\), we have \(h^\alpha = -\frac{1}{2} \langle \vec{F}, e_\alpha \rangle\) since \(\vec{H} + \frac{1}{2} \vec{F}^\perp = 0\). Taking the derivative of \(h^\alpha\) with respect to \(e_i\), we get

\[
e_i(h^\alpha) = \frac{1}{2} h_{ik}^\alpha \langle \vec{F}, e_k \rangle - \frac{1}{2} \langle \vec{F}, e_\beta \rangle \langle \nabla_{e_i} e_\alpha, e_\beta \rangle
\]

\[
(2.13) = \frac{1}{2} h_{ik}^\alpha \langle \vec{F}, e_k \rangle - h_\beta \langle \nabla_{e_i} e_\beta, e_\alpha \rangle.
\]

Since we assume that \(\nabla_{e_i} e_j(p) = 0\), (2.4) yields that \(h_{kk,i}^\alpha(p) = e_i(h_{kk}^\alpha)(p) + h_\beta \langle \nabla_{e_i} e_\beta, e_\alpha \rangle(p)\). Then we conclude that

\[
h_{i}^\alpha(p) = e_i(h^\alpha)(p) + h_\beta \langle \nabla_{e_i} e_\beta, e_\alpha \rangle(p).
\]

Comparing above with (2.13), we get \(h_{i}^\alpha(p) = \frac{1}{2} h_{ik}^\alpha \langle \vec{F}, e_k \rangle(p)\). The lemma follows from combining this with (2.12).

Using Proposition 2.5 and Lemma 2.6, we obtain the following structure equation of self-shrinkers in terms of the parallel form.
Theorem 2.7. (Structure Equation) In $\mathbb{R}^{n+k}$, suppose $\Sigma$ is an $n$-dimensional self-shrinker. Let $\Omega$ be a parallel $n$-form, then $*\Omega = \Omega(e_1, \cdots, e_n)$ satisfies that

\begin{equation}
\Delta(*\Omega) + (h_{ik}^\alpha)^2 * \Omega - 2 \sum_{i<j} \Omega_{i\alpha,j\beta} h_{ik}^\alpha h_{jk}^\beta - \frac{1}{2} \langle \vec{F}, \nabla(*\Omega) \rangle = 0,
\end{equation}

where $\vec{F}$ is the coordinate of the point on $\Sigma$ and $\Omega_{i\alpha,j\beta} = \Omega(\hat{e}_1, \cdots, \hat{e}_n)$ with $\hat{e}_s = e_s$ for $s \neq i, j$, $\hat{e}_s = e_\alpha$ for $s = i$ and $\hat{e}_s = e_\beta$ for $s = j$.

This theorem enables us to obtain various information of self-shrinkers for different parallel forms. We will apply this idea to our particular situation in next section.

3. Graphical self-shrinkers in $\mathbb{R}^4$

From this section on, we will focus on the graphical self-shrinkers in Euclidean space. The structure equations of graphical self-shrinkers will be derived in Proposition 3.5. The polynomial volume growth property plays an essential role in Lemma 3.10, which is our main technical tool.

3.1. Structure equations for graphical self-shrinkers in $\mathbb{R}^4$. We consider the following four different parallel 2-forms in $\mathbb{R}^4$:

\begin{align*}
\eta_1 &= dx_1 \wedge dx_2, & \eta' &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \\
\eta_2 &= dx_3 \wedge dx_4 & \eta'' &= dx_1 \wedge dx_2 - dx_3 \wedge dx_4
\end{align*}

Recall that for a smooth map $f = (f_1(x_1, x_2), f_2(x_1, x_2))$, its Jacobian $J_f$ is

\begin{equation}
J_f = \left( \frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \frac{\partial f_2}{\partial x_2} \right);
\end{equation}

Lemma 3.1. Suppose $\Sigma = (x, f(x))$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth map. Then

\[ *\eta_2 = J_f * \eta_1; \]

Proof. Notice that $*\eta_1$ and $*\eta_2$ are independent of the choice of the local frame. Denote by $e_1 = \frac{\partial}{\partial x_1} + \frac{\partial f_1}{\partial x_1} \frac{\partial}{\partial x_3} + \frac{\partial f_2}{\partial x_1} \frac{\partial}{\partial x_4}$, $e_2 = \frac{\partial}{\partial x_2} + \frac{\partial f_1}{\partial x_2} \frac{\partial}{\partial x_3} + \frac{\partial f_2}{\partial x_2} \frac{\partial}{\partial x_4}$ and $g_{ij} = \langle e_i, e_j \rangle$. Then

\[ *\eta_2 = \frac{dx_3 \wedge dx_4(e_1, e_2)}{\sqrt{\det(g_{ij})}} = J_f \frac{\sqrt{\det(g_{ij})}} = J_f * \eta_1; \]

The above lemma is not enough to explore structure equations in Theorem 2.7. We need further information about the microstructure of a point on $\Sigma$. 


Lemma 3.2. Assume $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a smooth map. Denote by $df$ the differential of $f$. Then for any point $x$,

1. There exist oriented orthonormal bases $\{a_1, a_2\}$ and $\{a_3, a_4\}$ in $T_x \mathbb{R}^2$ and $T_{f(x)} \mathbb{R}^2$ respectively: such that

$$
\begin{align*}
\text{df}(a_1) &= \lambda_1 a_3, \\
\text{df}(a_2) &= \lambda_2 a_4;
\end{align*}
$$

Here ‘oriented’ means $dx_i \wedge dx_{i+1}(a_i, a_{i+1}) = 1$ for $i = 1, 3$.

2. Moreover, we have $\lambda_1 \lambda_2 = J_f$.

Proof. Fix a point $x$. First, we prove the existence of (1). By the Singular Value Decomposition Theorem (P291, [?]) there exist two $2 \times 2$ orthogonal matrices $Q_1, Q_2$ such that

$$
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix} = Q_1 \begin{pmatrix}
\lambda_1' & 0 \\
0 & \lambda_2'
\end{pmatrix} Q_2
$$

with $\lambda_1', \lambda_2' \geq 0$.

Letting $\lambda_1 = \det(Q_1)\lambda_1'\det(Q_2)$, $\lambda_2 = \lambda_2'$, $A = \det(Q_1)Q_1$, $B = \det(Q_2)Q_2$, we find that $\det(A) = \det(B) = 1$. And we have

$$
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix} = A \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} B
$$

We consider the new basis $(a_1, a_2)^T = A^T(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T$, $(a_3, a_4)^T = B(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})^T$, then $dx_1 \wedge dx_2(a_1, a_2) = 1$ and $dx_3 \wedge dx_4(a_3, a_4) = 1$ ($A^T$ is the transpose of $A$). Moreover, (3.4) implies that

$$
\text{df}(a_1, a_2)^T = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} (a_3, a_4)^T
$$

Now we obtain (1).

Next we prove (2). Since $dx_1 \wedge dx_2(a_1, a_2) = 1$ and $dx_3 \wedge dx_4(a_3, a_4) = 1$, there exist two $2 \times 2$ orthogonal matrices $C, D$ with $\det(C) = \det(D) = 1$ such that $(a_1, a_2)^T = C(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^T$, $(a_3, a_4)^T = (\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})^T$. Then

$$
\text{df}(a_1, a_2)^T = C \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2}
\end{pmatrix}^T;
$$

$$
= C \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix} (a_3, a_4)^T;
$$

$$
= \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} (a_3, a_4)^T
$$

therefore $\lambda_1 \lambda_2 = \det(C)J_f\det(D) = J_f$. We obtain (2). The proof is complete. □

Remark 3.3. The conclusion (2) is independent of the special choice of $\{a_i\}_{i=1}^4$ satisfying (1).
With these bases, we construct the following local frame for later use.

**Definition 3.4.** Fixing a point \( p = (x, f(x)) \) on \( \Sigma \), we construct a special orthonormal basis \( \{e_1, e_2\} \) of the tangent bundle \( T\Sigma \) and \( \{e_3, e_4\} \) of the normal bundle \( N\Sigma \) such that at the point \( p \) we have for \( i = 1, 2 \):

\[
(3.5) \quad e_i = \frac{1}{\sqrt{1 + \lambda_i^2}}(a_i + \lambda_i a_{2+i}); \quad e_{2+i} = \frac{1}{\sqrt{1 + \lambda_i^2}}(a_{2+i} - \lambda_i a_i);
\]

where \( \{a_1, a_2, a_3, a_4\} \) are from (3.3).

For a parallel 2-form \( \Omega \), we have \( *\Omega = \Omega(e_1, e_2) \). Applying (3.5) and \( \lambda_1 \lambda_2 = Jf \), direct computations show that \( *\eta_1, *\eta_2, *\eta' \) and \( *\eta'' \) take the following form:

\[
(3.6) \quad *\eta_1 = \frac{1}{\sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)}} > 0,
\]

\[
(3.7) \quad *\eta_2 = \frac{\lambda_1 \lambda_2}{\sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)}},
\]

\[
(3.8) \quad *\eta' = (1 + Jf)(*\eta_1),
\]

\[
(3.9) \quad *\eta'' = (1 - Jf)(*\eta_1).
\]

There is a symmetric relation between \( *\eta_1 \) and \( *\eta_2 \). More precisely, we have the structure equations for graphical self-shrinkers in \( \mathbb{R}^4 \) as follows:

**Theorem 3.5.** Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is a smooth map, and \( \Sigma = (x, f(x)) \) is a graphical self-shrinker in \( \mathbb{R}^4 \). Using the notations in Definition 3.4, we have

\[
(3.10) \quad \Delta(*\eta_1) + *\eta_1(h_1^{3k}h_2^{k2} - h_1^{4k}h_2^{k3}) - \frac{1}{2} \langle \vec{F}, \nabla(*\eta_1) \rangle = 0;
\]

\[
(3.11) \quad \Delta(*\eta_2) + *\eta_2(h_1^{3k}h_2^{k2} - h_1^{4k}h_2^{k3}) - \frac{1}{2} \langle \vec{F}, \nabla(*\eta_2) \rangle = 0;
\]

\[
(3.12) \quad \Delta(*\eta') + *\eta'(h_1^{3k}h_2^{k2} + h_1^{4k}h_2^{k3}) - \frac{1}{2} \langle \vec{F}, \nabla(*\eta') \rangle = 0;
\]

\[
(3.13) \quad \Delta(*\eta'') + *\eta''(h_1^{3k}h_2^{k2} + h_1^{4k}h_2^{k3}) - \frac{1}{2} \langle \vec{F}, \nabla(*\eta'') \rangle = 0,
\]

where \( h_{ij} = \langle \nabla e_i, e_j \rangle \) are the second fundamental form of \( \Sigma \), \( \Delta \) and \( \nabla \) are the Laplacian and the covariant derivative of \( \Sigma \) respectively.

**Proof.** It suffices to show the first two equations (3.10) and (3.11), since the other two (3.12) and (3.13) follow from combining (3.10) and (3.11) together.

First we consider the equation (3.10). Using the frame in (3.5), the third term in Theorem 2.7 becomes:

\[
2(\eta_1)_{\alpha \beta \gamma} h_{ik}^\alpha h_{jk}^\beta = 2dx_1 \wedge dx_2(e_3, e_4)(h_1^{3k}h_2^{k2} - h_1^{4k}h_2^{k3})
\]

\[
= 2 \frac{\lambda_1 \lambda_2}{\sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)}}(h_1^{3k}h_2^{k2} - h_1^{4k}h_2^{k3})
\]

\[
= 2 * \eta_2(h_1^{3k}h_2^{k2} - h_1^{4k}h_2^{k3}).
\]
Here in the second line we used the fact that $dx_1 \wedge dx_2(a_1, a_2) = 1$. Plugging this into (2.14), we obtain (3.10).

Similarly we obtain that

$$2(\eta_2)_{i_0,j_3} h_{ik}^3 h_{jk}^2 = 2 dx_3 \wedge dx_4(e_3, e_4)(h_{1k}^3 h_{2k}^4 - h_{2k}^3 h_{1k}^4)
= \frac{1}{\sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)}} (h_{1k}^4 h_{2k}^3 - h_{1k}^3 h_{2k}^4)
= 2 * \eta_1 (h_{1k}^3 h_{2k}^4 - h_{1k}^4 h_{2k}^3).$$

(3.14)

Here in the second line we used the fact that $dx_3 \wedge dx_4(a_3, a_4) = 1$. Then (3.11) follows from plugging (3.14) into (2.14). □

3.2. Volume growth for self-shrinkers. We will state our main analytic tool in a more general setting since it may be of independent interest. In this subsection, we will consider graphical self-shrinkers of $n$-dimensional in $\mathbb{R}^{n+k}$.

**Definition 3.6.** Let $N^n$ be a complete, immersed $n$-dimensional submanifold in $\mathbb{R}^{n+k}$, we say $N$ has the polynomial volume growth property, if for any $r \geq 1$,

$$\int_{N \cap B_r(0)} dvol \leq Cr^n,$$

where $B_r(0)$ is the ball in $\mathbb{R}^{n+k}$ centered at 0 with radius $r$.

Recently Cheng-Zhou [?] and Ding-Xin [?] showed the polynomial volume growth property is automatic under the following condition, but without the restriction of dimension and codimension.

**Theorem 3.7.** ([?, ?]) If $N^n$ is a $n$-dimensional complete, immersed, proper self-shrinker in $\mathbb{R}^{n+k}$, then it satisfies the polynomial volume growth property.

**Remark 3.8.** The properness assumption can not be removed. See for example Remark 4.1 in [?].

Note that any graphical self-shrinker in Euclidean space is embedded, complete and proper. Therefore we state:

**Corollary 3.9.** Let $\Sigma = (x, f(x))$ be a smooth graphical self-shrinker in $\mathbb{R}^4$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth map. Then $\Sigma$ has the polynomial volume growth property.

The following lemma is crucial for our argument:

**Lemma 3.10.** Let $N^n \subset \mathbb{R}^{n+k}$ be a complete, immersed smooth $n$-dimensional submanifold with at most polynomial volume growth. Assume there are a positive function $g$ and a nonnegative function $K$ satisfying the following inequality:

$$0 \geq \Delta g - \frac{1}{2} \langle \tilde{F}^i, \nabla g \rangle + Kg,$$

(3.15)
Here $\Delta (\nabla)$ is the Laplacian (covariant derivative) of $N^n$, and $\vec{F}$ is the position vector of $N^n$. Then $g$ is a (positive) constant and $K \equiv 0$.

**Proof.** Fixing $r \geq 1$, we denote by $\phi$ a compactly supported smooth function in $\mathbb{R}^{n+k}$ such that $\phi \equiv 1$ on $B_r(0)$ and $\phi \equiv 0$ outside of $B_{r+1}(0)$ with $|\nabla \phi| \leq |D\phi| \leq 2$. Here $D\phi$ and $\nabla \phi$ are the gradient of $\phi$ in $\mathbb{R}^{n+k}$ and $N^n$ respectively.

Since $g$ is positive, let $u = \log g$. Then the inequality (3.15) becomes

$$0 \geq \Delta u - \frac{1}{2} \langle \vec{F}, \nabla u \rangle + (K + (\nabla u)^2).$$

Multiplying $\phi e^{-|\vec{F}|^2/4}$ to the right-hand side of the above equation and integrating on $N^n$, we get

$$0 \geq \int_N \phi^2 \text{div}_N (e^{-|\vec{F}|^2/4} \nabla u) + \int_N \phi^2 e^{-|\vec{F}|^2/4} (K + |\nabla u|^2)$$

$$= - \int_N 2\phi \langle \nabla \phi, \nabla u \rangle e^{-|\vec{F}|^2/4} + \int_N \phi^2 e^{-|\vec{F}|^2/4} (K + |\nabla u|^2)$$

$$\geq - \int_N 2|\nabla \phi|^2 e^{-|\vec{F}|^2/4} + \int_N \phi^2 e^{-|\vec{F}|^2/4} (K + |\nabla u|^2/2).$$

(3.16)

In (3.16), we used the inequality

$$|2\phi \langle \nabla \phi, \nabla u \rangle| \leq \frac{\phi^2 |\nabla u|^2}{2} + 2|\nabla \phi|^2.$$

Now we estimate, using the construction that $|\nabla \phi| \leq |D\phi| \leq 2$,

$$\int_{N \cap B_r(0)} e^{-|\vec{F}|^2/4} (K + |\nabla u|^2/2) \leq \int_N \phi^2 e^{-|\vec{F}|^2/4} (K + |\nabla u|^2/2)$$

$$\leq \int_N 2|\nabla \phi|^2 e^{-|\vec{F}|^2/4} \quad \text{by (3.16)}$$

$$\leq 8 \int_{N \cap (B_{r+1}(0) \setminus B_r(0))} e^{-|\vec{F}|^2/4}$$

$$\leq 8 C(r + 1)^n e^{-\frac{r^2}{4}}.$$

In the last line we use the fact that the submanifold $N^n$ has the polynomial volume growth property.

Letting $r$ go to infinity, we obtain that

$$\int_N e^{-|\vec{F}|^2/4} (K + |\nabla u|^2/2) \leq 0.$$

Since $K$ is nonnegative, we have $K \equiv \nabla u \equiv 0$. Therefore $g$ is a positive constant. □
The proof of Theorem 1.1.} Adapting to our case of graphical self-shrinker surfaces in $\mathbb{R}^4$, we are ready to prove Theorem 1.1.

**Proof.** (of Theorem 1.1) We claim that $\Sigma$ is minimal under the assumptions. We prove this case by case.

**Assuming Condition (1):** the equations (3.6) and (3.8) imply that the parallel form $*\eta'$ has the same sign as $1 + J_f$, hence $*\eta'$ is a positive function. Moreover from (3.12) $*\eta'$ satisfies

$$\Delta(*\eta') + (*\eta')(h_{1k}^3 - h_{2k}^4)^2 + (h_{1k}^4 + h_{2k}^3)^2 - \frac{1}{2} \langle \vec{F}, \nabla(*\eta') \rangle = 0.$$ 

Here $\vec{F} = (x, f(x))$. Since $\Sigma$ has the polynomial volume growth property, using Lemma 3.10, we conclude that

$$(h_{1k}^3 - h_{2k}^4)^2 + (h_{1k}^4 + h_{2k}^3)^2 \equiv 0.$$ 

We then obtain

$$h_{11}^3 = h_{21}^4, \quad h_{22}^3 = -h_{12}^4,$$

$$h_{11}^4 = -h_{21}^3, \quad h_{22}^4 = h_{12}^3.$$ 

Then $\vec{H} = (h_{11}^3 + h_{22}^4)e_3 + (h_{11}^4 + h_{22}^3)e_4 \equiv 0$. So $\Sigma$ is a minimal surface.

**Assuming Condition (2):** this is similar to the above case. (3.6) and (3.9) imply that $*\eta''$ has the same sign as $1 - J_f$, hence $*\eta''$ is positive. From (3.13), it also satisfies

$$\Delta(*\eta'') + (*\eta'')(h_{1k}^3 + h_{2k}^4)^2 + (h_{1k}^4 - h_{2k}^3)^2 - \frac{1}{2} \langle \vec{F}, \nabla(*\eta'') \rangle = 0.$$ 

Again we apply Lemma 3.10 to find that

$$(h_{1k}^3 + h_{2k}^4)^2 + (h_{1k}^4 - h_{2k}^3)^2 \equiv 0.$$ 

Then we have

$$h_{11}^3 = -h_{21}^4, \quad h_{22}^3 = h_{12}^4,$$

$$h_{11}^4 = h_{21}^3, \quad h_{22}^4 = -h_{12}^3.$$ 

Therefore we arrive at:

$$\vec{H} = (h_{11}^3 + h_{22}^4)e_3 + (h_{11}^4 + h_{22}^3)e_4 \equiv 0,$$

which means $\Sigma$ is minimal.

Now $\Sigma$ is a graphical self-shrinker and minimal. From (1.1), we have $\vec{F} \perp \equiv 0$ for any point $\vec{F}$ on $\Sigma$. For any normal unit vector $e_\alpha$ in the normal bundle of $\Sigma$, we have

$$\langle \vec{F}, e_\alpha \rangle \equiv 0.$$ 

Taking derivative with respect to $e_i$ for $i = 1, 2$ from (3.21), we get

$$\langle \vec{F}, e_1 \rangle h_{11}^\alpha + \langle \vec{F}, e_2 \rangle h_{12}^\alpha = 0,$$

$$\langle \vec{F}, e_1 \rangle h_{21}^\alpha + \langle \vec{F}, e_2 \rangle h_{22}^\alpha = 0,$$
Now assume $\vec{F} \neq 0$. Since $\vec{F}^\perp = 0$, $((\vec{F}, e_1), (\vec{F}, e_2)) \neq (0, 0)$. According to the basic linear algebra, we conclude that

(3.22) \[ h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2 = 0 \]

The minimality implies $h_{11}^\alpha = -h_{22}^\alpha$. Hence (3.22) becomes $-(h_{11}^\alpha)^2 = (h_{12}^\alpha)^2$. We find that $h_{ij}^\alpha = 0$ for $i, j = 1, 2$. Therefore $\Sigma$ is totally geodesic except $\vec{F} = 0$. Since $\Sigma$ is a graph, there is at most one point on $\Sigma$ such that $\vec{F} = 0$. By the continuity of the second fundamental form, $\Sigma$ is totally geodesic everywhere.

Now $\Sigma$ is a plane. If $0$ is not on the plane, then we can find a point $\vec{F}_0$ in this plane which is nearest to $0$. It is easy to see that $\vec{F}_0 = \vec{F}_0^\perp \neq 0$. This is contradict that $\vec{F}^\perp = -\vec{H} \equiv 0$.

We complete the proof. \hfill \Box