How many quantum phase transitions exist inside the superconducting dome of the iron pnictides?

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Recent experiments on two iron-pnictide families suggest the existence of a single quantum phase transition (QPT) inside the superconducting dome despite the fact that two separate transition lines - magnetic and nematic - cross the superconducting dome at Tc. Here we argue that these two observations are actually consistent. We show, using a microscopic model, that each order coexists with superconductivity for a wide range of parameters, and both transition lines continue into the superconducting dome below Tc. However, at some Tmerge < Tc, the two transitions merge and continue down to T = 0 as a single simultaneous first-order nematic/magnetic transition. We show that superconductivity has a profound effect on the character of this first-order transition, rendering it weakly first-order and allowing strong fluctuations to exist near the QPT.

Introduction. A common theme across different phase diagrams of unconventional superconductors (SC) is the idea of one or more continuous quantum phase transitions (QPT’s) under the SC dome [1]. Examples include heavy fermion materials [2,3], cuprates [4], and iron pnictides [5]. Such QPT is generally associated with a non-superconducting (SC) order which penetrates into the SC dome [4,6-11]. Direct experimental access to this putative QPT requires killing the SC order, which can be challenging in high-temperature superconductors due to high value of the critical magnetic field needed to destroy SC [1]. Besides, one may worry that the applied magnetic field induces orders not present in the zero-field phase diagram. An alternative is to search for the QPT directly inside the SC dome. However, there is no guarantee that the non-SC continuous phase transition persists down to T = 0, as it may become first-order if the SC and non-SC orders do not coexist microscopically [12,14,16].

In the iron pnictides, measurements of the T = 0 SC penetration depth across the phase diagram of BaFe2(As1−xPx)2 found a pronounced peak at x ≈ 0.3 inside the SC dome, consistent with the existence of a single continuous or “almost” continuous QPT [15]. Because in the phase diagram of this and other iron pnictides, e.g. Ba(Fe1−xCox)2As2, a spin-density wave (SDW) transition line meets the SC dome near the highest Tc [13,14], it is natural to identify the observed peak with a magnetic quantum critical point, like in heavy fermions and other materials. However, unlike other quantum materials, in the iron pnictides there is not only one, but two separate phase transition lines that cross the SC dome [20,21]. Besides the SDW transition at Tm, there is also a nematic/structural transition at Tn > Tm, below which the tetragonal C4v symmetry of the system is spontaneously broken down to an orthorhombic C2v [22,23]. This peculiar feature raises the issue of how many QPTs - if any - exist inside the SC dome.

In this paper we address this issue by using a microscopic electronic model which describes simultaneously the magnetic, nematic, and superconducting phases. We show that, for a wide range of parameters, SDW and nematic orders coexist with superconductivity, such that both the Tm and Tn lines penetrate into the SC dome. However, at T = 0 our results show that there is a single QPT at which nematic and magnetic orders appear simultaneously via a first order transition. We argue that this first-order transition is intrinsically weak due to the presence of SC order, which helps to explain the observed strong enhancement of the penetration depth at the QPT.

Figure 1: Schematic phase diagram of the iron pnictides, summarizing our main results. The spin-density wave (SDW, blue curve) and nematic (Nem, red curve) transition lines separately cross the superconducting dome (SC, green curve), and coexist with superconductivity immediately below Tc. As temperature is lowered, one of the transitions becomes first-order (in this example, the SDW one), and at a smaller T the two transition lines merge onto a single simultaneous weakly first-order transition (dashed line) which persists down to T = 0 and gives rise to a single QPT inside the dome (yellow dot). The gray dots denote tricritical points. The back-bending of the lines below Tc may or may not take place (see Refs. [4,15,16].
The schematic phase diagram is shown in Fig. 1. Our results reconcile the existence of two phase transitions above $T_c$, with the penetration depth measurements deep inside the SC dome, which point to a single phase transition at $T = 0$.

**Microscopic model.** We consider a minimal three-pocket low-energy model consisting of one circular hole pocket at the center of the 1-Fe Brillouin zone and two elliptical electron pockets centered at momenta $Q_X = (\pi, 0)$ and $Q_Y = (0, \pi)$. The band dispersions are parametrized in terms of the momentum $k$ and angle $\theta$ as $\varepsilon_{\Gamma,k} = -\varepsilon_k = -\frac{\omega}{2m} + \mu$, $\varepsilon_{X,k+Q_1} = \varepsilon_k + 2\delta_0 + 2\delta_2 \cos 2\theta$, and $\varepsilon_{Y,k+Q_2} = \varepsilon_k + 2\delta_0 - 2\delta_2 \cos 2\theta$, where $\delta_0$ is proportional to the chemical potential, and $\delta_2$, to the ellipticity of the electron pockets [22, 24]. The Hamiltonian of the model is $H = H_2 + H_4$, where $H_2 = \sum_{k,a} \varepsilon_{k,a} a_{k,a}^\dagger a_{k,a}$, with spin index $\sigma$ and band index $a$, and $H_4$ contains eight different 4-fermion interactions [22, 28]. Accordingly, we introduce two fluctuating fields: the SDW one, $M_j = \sum_k \varepsilon_{k,i} \sigma \beta \delta_{k,j} \beta$, where $j = X, Y$, and the SC one, $\Delta_i = \sum_k \varepsilon_{k,i} \sigma \beta \delta_{k,i} \beta$, where $i = X, Y, \Gamma$. We use them to decouple the 4-fermion interaction terms in $H_4$, integrate over the fermions, and obtain the effective action $S_{\text{eff}}[\Delta_i, M_j]$. To avoid lengthy formulas, we assume that all inter-band pairing interactions have the same magnitude, in which case $\Delta_\sigma = -\sqrt{2}\Delta_{X,Y} = \Delta$. Expanding $S_{\text{eff}}[\Delta, M_{\downarrow}]$ in $\Delta$ and $M_{\downarrow}$, we obtain

$$S_{\text{eff}} = a_m \left( M_{X}^2 + M_{Y}^2 \right) + a_s \Delta^2 + \frac{u_m}{2} \left( M_{X}^2 + M_{Y}^2 \right)^2 - \frac{g_m}{2} \left( M_{X}^2 - M_{Y}^2 \right)^2 + \frac{u_s}{2} \Delta^4 + \lambda \Delta^2 \left( M_{X}^2 + M_{Y}^2 \right)$$

where all coefficients are expressed in terms of the 4-fermion interactions in the SDW/SC channels and the band parameters $(\delta_0, \delta_2)$ (see Supplementary Material (SM) for details). The coefficients $a_m$ and $a_s$ vanish at the mean-field SDW and SC transition temperatures $T_{m,0}$ and $T_{c,0}$, while the coefficients $u_m > g_m, u_s$, and $\lambda$ are all positive at not very low $T$.

**Mean-field analysis.** To set the stage, we first analyze this action at the mean-field level, when the $M$ and $\Delta$ fields do not fluctuate. In this case, there is no preemptive nematic order. Still, a straightforward analysis of Eq. 1 shows that the tetragonal symmetry is broken below $T_{m,0}$, because the SDW is a stripe order with either $M_X = 0, M_Y \neq 0$ or $M_X \neq 0, M_Y = 0$. The competing SC and SDW orders coexist microscopically as long as the quartic coefficients satisfy the condition $\lambda < \sqrt{u_s(u_m - g_m)}$ (Refs. [12, 13, 16]). This happens in the region of the $(\delta_0, \delta_2)$ space denoted by the light-blue area of Fig. 2 [29]. For parameters in this range, the continuous SDW transition line penetrates into the SC state, albeit with a different slope [4]. Whether it survives down to $T = 0$ depends on the interplay between the reduction of $u_m$ with decreasing $T$ in the absence of superconductivity and the increase of $u_m$ promoted by the corrections due to superconductivity. We present the analysis of this interplay in the SM. The key result is that for a wide range of parameters the mean-field SDW transition remains second-order down to $T = 0$ and ends up at a magnetic quantum critical point under the SC dome.

**Preemptive nematic order.** We next go beyond mean-field and include fluctuations of the SDW fields $M_X$ and $M_Y$. We do not include fluctuations of $\Delta$ as we are interested in what happens deep inside the SC dome, where $\Delta$ is close to its saturation value. To account for fluctuations of $M_\uparrow$ we replace $a_m$ in Eq. 1 by the SDW susceptibility $\chi_0^{-1}(Q_1, q, \omega_n) = a_m + q^2 + f(\omega_n)$, where $\omega_n = 2\pi n T$ is the Matsubara frequency and $f(\omega_n)$ is proportional to $|\omega_n|$ in the normal state and $\omega_n^2$ deep inside the SC dome. We then introduce two Hubbard-Stratonovich fields $\psi = u_m \left( M_X^2 + M_Y^2 \right)$ and $\varphi = g_m \left( M_X^2 - M_Y^2 \right)$, integrate the partition function over $M(\mathbf{q}, \omega_n)$, and obtain the effective action $S_{\text{eff}}[\Delta, \psi, \varphi]$. Fluctuations of $M_\uparrow$ and of $\psi$ and $\varphi$ are conjugated - if $M_\uparrow$ fluctuates weakly, fluctuations of $\varphi$ and $\psi$ are strong, but if $M_\uparrow$ fluctuates strongly, as we now assume, fluctuations of $\psi$ and $\varphi$ are weak, and the effective action $S_{\text{eff}}[\Delta, \psi, \varphi]$ can in turn be analyzed in the saddle-point approximation (see SM). The field $\langle \psi \rangle$ is non-zero at any temperature and its primary effect is to shift the “pure” SDW transition temperature from $T_{m,0}$ down to $T_{m,0}$. Strictly in $d = 2$, $T_{m,0} = 0$, but in quasi-2D systems, like iron-pnictides, $T_{m,0}$ is finite even for small inter-layer coupling. Our analysis, for which we used the expansion to order $M^4$ in Eq. 1 is valid when $(T_n - T_{m,0})/T_n \ll 1$. A non-zero $\langle \varphi \rangle$ appears only below a certain $T_n$ and breaks the tetragonal $C_4$ symmetry down to $C_2$, inducing an orthorhombic distortion and orbital order [22]. If $\langle \varphi \rangle$ becomes non-zero at $T_n > T_{m,0}$, there exists a temperature range in which the system displays nematic order $\langle \varphi \rangle \neq 0$ but no long-range magnetic order ($M_\uparrow = 0$).

Consider first the normal state. Integrating over the $M_j$ fields we obtain the effective action $S_{\text{eff}}[0, \psi, \varphi]$

$$S_{\text{eff}}[0, \psi, \varphi] = \frac{\varphi^2}{2g_m} - \frac{\psi^2}{2u_m} + \frac{3}{2} \int_q \log \left[ (\chi_0^{-1} + \psi)^2 - \varphi^2 \right]$$

where $\int_q = T \sum_{\omega_n} \int d^dq/(2\pi)^d$. This action has been analyzed before in several contexts [22, 23, 25]. For quasi-2D systems, the behavior depends on the ratio $\alpha = u_m/g_m \geq 1$. For $\alpha$ relevant to near-optimally doped BaFe$_2$(As$_{1-x}$P$_x$)$_2$ and Ba(Fe$_{1-x}$Co$_x$)$_2$As$_2$, the nematic transition is second order and occurs at $T_n > T_{m,0}$, i.e. before the “pure” SDW transition. A non-zero $\langle \varphi \rangle$ shifts the SDW transition upwards from $T_{m,0}$ to $T_n$, but still $T_{m,0} < T_n$. In this situation, there are two split second-order transition lines, $T_n$ and $T_{m,0}$, which separately cross the $T_c$ line. Our goal now is to find the fate of these transitions inside the SC dome.
the vicinity of the point where the nematic transition line $T_n$ hits $T_c$, and analyze whether nematic or order can coexist with SC. We assume that $T_c$ is large enough, set $d = 2$ - which is safe for the nematic order - and neglect the dynamic part of $\chi_0$. Keeping it will not change our results qualitatively (see SM). We obtain $\psi$ from the saddle-point equation $\delta S_{\text{eff}} / \delta \psi = 0$, substitute the result back into the effective action and obtain $S_{\text{eff}}(\Delta, \varphi)$.

Expanding the action we obtain

$$S_{\text{eff}}(\Delta, \varphi) = a_n\varphi^2 + \frac{u_n}{2} \varphi^4 + \tilde{a}_\Delta \Delta^2 + \frac{\tilde{u}_\Delta}{2} \Delta^4 + \lambda \varphi^2 \Delta^2$$  \hspace{1cm} (3)

where the coefficients are given by

$$\lambda = \frac{\lambda}{2(u_m + g_m)}, \quad \tilde{u}_\Delta = \frac{\lambda^2}{u_m + g_m}, \quad u_n = \frac{1}{6} \frac{u_m - 2g_m}{u_m + g_m}$$ \hspace{1cm} (4)

Notice that all coefficients originate from the SDW/SC action [11], i.e. the coupling between the nematic and SC order parameters is a consequence of the coupling between the SDW and SC fields ($\lambda \propto \Lambda$) [20]. In the absence of SC, the nematic transition is second-order when $u_n > 0$, i.e. $\alpha = u_m / g_m > 2$, which we assume to hold.

It follows from Eq. (3) that nematic and SC orders coexist when $\lambda < \sqrt{\tilde{u}_\Delta u_n}$, which in terms of the original Ginzburg-Landau coefficients gives $\lambda < \sqrt{u_s(u_m - 2g_m)}$. Although this is a more restrictive condition than $\lambda < \sqrt{u_s(u_m - g_m)}$ for the coexistence between mean-field SDW and SC, it is still satisfied in a rather wide range of parameters ($\delta_0, \delta_2$), including the region of small $\delta_0$ and $\delta_2$ (the red region in Fig. 2b).

In this parameter range, the second-order $T_n$ line continues below $T_c$, albeit with a different slope. Because the condition for SDW-SC coexistence is the same both in mean-field and in the presence of Gaussian fluctuations [14], in the same red region of Fig. 2b, the SDW $T_m$ line also continues as a second-order transition line into the SC dome.

**Nematic and SDW transitions at $T = 0$.** At $T = 0$, Matsubara frequencies form a continuous set and the dynamics of $\chi_0(q, \omega_n)$ cannot be neglected. Deep in the SC state, the spin dynamics is propagating $\chi^{-1}_0(q, q, \omega_n) = a_m + \epsilon^2 + \omega_n^2$, i.e. the quantum system behaves like the classical system in an effective dimension $d_{\text{eff}} = d + 1 = 3$.

The effective action in terms of $\psi$ and $\varphi$ has the same form as in the absence of SC, Eq. (2), but with renormalized coefficients and in $d_{\text{eff}} = 3$. Anticipating that nematic transition may trigger an instantaneous magnetic transition, we introduce an SDW order parameter $m$ and write $S_{\text{eff}}$ in terms of $\psi, \varphi,$ and $m$. We again use $\delta S_{\text{eff}} / \delta \psi = 0$ to eliminate $\psi$ and obtain the action in terms of $\varphi$ and $m$:

$$S_{\text{eff}}[\varphi, m] = \frac{\varphi^2}{2 g_m} - \frac{r(r - 2\tilde{a}_m)}{2\pi m^2} (r - |\varphi|)$$

$$- \frac{(r + \varphi)^{3/2} + (r - \varphi)^{3/2}}{4\pi}$$ \hspace{1cm} (5)

where $r = \tilde{a}_m - (3u_m / (8\pi)) (\sqrt{r + \varphi} + \sqrt{r - \varphi}) + (3u_m / (2\pi))^2 m^2$ is a function of $\varphi$ and $m$, and $\tilde{a}_m = a_m + (3\Lambda u_m) / (2\pi)^2$ is the renormalized distance to the $T = 0$ SDW transition in the absence of nematicity. Note that $\tilde{a}_m$ is the only term that depends on the upper cutoff $\Lambda$ of the momentum-frequency integration and that $S_{\text{eff}}$ is an even function of $\varphi$. The magnetic order parameter $m$ satisfies the equation of state $m(r - |\varphi|) = 0$. It vanishes if the nematic order parameter either emerges continuously or jumps to a value $|\varphi| < r$, but can become non-zero if $\varphi$ jumps at the nematic transition to $|\varphi| = r$. That $m$ can become non-zero right at the nematic transition can also be understood by looking at the SDW susceptibility $\chi(Q) \propto 1/r$. For $\varphi = 0$, the SDW susceptibility diverges when $r = 0$, which happens at $\tilde{a}_m = 0$. If the nematic transition occurs at $\tilde{a}_m > 0$, preempting the magnetic transition, the static SDW susceptibility splits into $\chi(Q) \propto 1/(r \pm \varphi)$ (plus sign for $i = X$ and minus sign for $i = Y$). If $\varphi$ jumps to $|\varphi| = r$ at the nematic transition, one of the $\chi(Q_i)$ diverges, which implies that $m$ may also jump to a non-zero value.

We analyzed $S_{\text{eff}}[\varphi, m]$ by reducing $\tilde{a}_m$ from some initially large positive value down to zero. In the range, $|\varphi| < \varphi_0$, where $\varphi_0 = \frac{3u}{2\pi} \left( \frac{3u}{2\pi} + \frac{32\tilde{a}_m - \frac{3u}{2\pi}}{32\tilde{a}_m - \frac{3u}{2\pi}} \right)^2$, we have $|\varphi| \leq r$ and hence $m = 0$. On the other hand, for $|\varphi| > \varphi_0$, we have $r = |\varphi|$ and then $m = m(\varphi, \tilde{a}_m)$ is non-zero and is determined from the equation on $r (= |\varphi|)$. Our results are shown in Fig. 3 where we plotted $S_{\text{eff}}[\varphi]$ in both regions at various $\tilde{a}_m$. For large $\tilde{a}_m$, $S_{\text{eff}}[\varphi]$ has a minimum at $\varphi = 0$ and monotonically increases with $|\varphi|$. When $\tilde{a}_m$ becomes smaller than $\tilde{a}_m_{c1} = (3u/4\pi)^2 / (2(\alpha - 1))$, $S_{\text{eff}}[\varphi]$ develops inflection points at $|\varphi| > \varphi_0$. Upon decreasing $\tilde{a}_m$ further, these inflection points split in two pairs of local maximum and minimum $\varphi = \pm \varphi_{\text{max}}$ and $\varphi = \pm \varphi_{\text{min}}$. As $\tilde{a}_m$ is lowered, $S_{\text{eff}}[\varphi = \pm \varphi_{\text{min}}]$ eventually becomes lower.
The effective action $S_{\text{eff}}[\varphi]$ at $T = 0$ and $d_{\text{eff}} = 3$, as a function of the nematic order parameter $\varphi$ for various $\bar{a}_m$, which measure the distance to the pure SDW $T = 0$ transition. From top to bottom, $\bar{a}_m \left(\frac{\varphi^2}{m_{\text{cr}}^2}\right) = 0.0345, 0.0320, 0.0310, 0.03034,$ and 0.0295. The dashed line $\varphi = \varphi_0$ separates the region where nematic order does not induce magnetic order ($|\varphi| < \varphi_0$) from the regions where magnetic order is simultaneously induced ($|\varphi| > \varphi_0$). We set $u_m/g_m = 5$. 

Figure 3: The effective action $S_{\text{eff}}[\varphi]$ at $T = 0$ and $d_{\text{eff}} = 3$, as a function of the nematic order parameter $\varphi$ for various $\bar{a}_m$, which measure the distance to the pure SDW $T = 0$ transition. From top to bottom, $\bar{a}_m \left(\frac{\varphi^2}{m_{\text{cr}}^2}\right) = 0.0345, 0.0320, 0.0310, 0.03034,$ and 0.0295. The dashed line $\varphi = \varphi_0$ separates the region where nematic order does not induce magnetic order ($|\varphi| < \varphi_0$) from the regions where magnetic order is simultaneously induced ($|\varphi| > \varphi_0$). We set $u_m/g_m = 5$. 

than $S_{\text{eff}}[\varphi = 0]$, i.e. the system undergoes a first-order nematic transition in which the nematic order parameter jumps from $\varphi = 0$ to $\varphi = \pm \varphi_{\text{min}}$. This happens at $m_{\text{cr}} \approx f(\alpha) a_{m_{\text{cr}}}$, where $f(\alpha \approx 1) = 0.657$ and $f(\alpha \gg 1) \to 1$. Because $\varphi_{\text{min}} > \varphi_0$, the jump in the nematic order parameter is strong enough to induce a simultaneous first-order magnetic transition. Since at the transition both $\varphi$ and $m$ jump simultaneously to finite values, there is only one first-order QPT under the SC dome (see Fig. 1).

We verified that at $\bar{a}_m = \bar{a}_{m_{\text{cr}}}$ the coefficient of the $\varphi^2$ term in $S_{\text{eff}}[\varphi]$ remains positive for all $\bar{a}_m \equiv u_m/g_m$, i.e. the first-order nematic transition preempts not only the SDW transition but also the potential second-order nematic transition. This result is in contrast with earlier works (Ref. [31]) which suggested that the nematic and magnetic transitions remain separate and second-order down to $T = 0$ at $d_{\text{eff}} = 3$.

An important issue is the strength of this first-order transition. For $d_{\text{eff}} = 3$, the jump in the nematic order parameter $\delta \varphi \propto (\alpha - 1)^{-2}$ decreases when $\alpha$ increases and is of order $\alpha^{-2}$ at large $\alpha$. This is a general consequence of the fact that $d_{\text{eff}} = 3$ is the borderline between the regimes of simultaneous and split nematic/SDW transitions, since for $d_{\text{eff}} = 3 - \epsilon$ the two transition become split and second-order for $\alpha > \frac{3}{2}$. Interestingly, we found that, in a wide region of $(\delta_0, \delta_2)$, $\alpha$ becomes large in the SC state (see Fig. 2(b)), i.e. the presence of superconductivity makes the first-order transition weaker. Note in this regard that the effective dimension $d_{\text{eff}} = 3$ is also a direct consequence of the presence of SC, which changes the spin dynamics by propagating. Without SC, the spin dynamics would be diffusive with $d_{\text{eff}} = 2 + z = 4$, and the first-order transition would be much stronger [37].

The weak character of the first-order QPT inside the SC dome is also manifested in the temperature range $0 < T < T_c$. By combining the present results at $T = 0$ and near $T_c$ with the earlier analysis of the classical phase diagram in quasi-2D systems [22], we find that the nematic and magnetic transition lines merge at some nonzero temperature $T_{\text{merge}} < T_c$, below which the two orders develop simultaneously via a first-order transition (see Fig. 1). The details of the phase diagram near $T_{\text{merge}}$ depend on the degree of the 3D anisotropy, with either the nematic or the magnetic transition line becoming second-order immediately above $T_{\text{merge}}$ and the other transition line remaining first-order up to a somewhat larger $T$. Most importantly, $T_{\text{merge}}$ also scales as $1/\alpha^2$ and is small at large $\alpha = u_m/g_m$. As a result, the system behaves almost like the nematic and SDW second-order transition lines would merge right at $T = 0$. In this special case, stripe and non-stripe magnetic states are degenerate, and the SDW order parameter manifold is enhanced to $O(6)$ (see Eq. 1), leading to enhanced quantum fluctuations near the QPT.

Comparison with experiments Our results can be directly applied to iron pnictides, particularly to BaFe$_2$(As$_{1-x}$P$_x$)$_2$ and Ba(Fe$_{1-x}$Co$_{x})_2$As$_2$, whose SC domes are crossed by two split second-order magnetic and nematic transition lines. Microscopic coexistence between SDW and superconductivity has been established in both cases by NMR [38, 40], and a suppression of the orthorhombic order parameter (proportional to $\varphi$ in our model) has been found inside the SC dome [29, 21]. According to our calculations, there should be a single simultaneous weak first-order nematic/SDW QPT at $T = 0$. This can be verified by measurements of the $T = 0$ SC penetration depth across the SC dome. In BaFe$_2$(As$_{1-x}$P$_x$)$_2$, a single peak in the penetration depth has been observed near optimal doping [17]. Experiments cannot resolve whether it implies a second-order or weakly first-order transition, but the fact is that there is a single transition at $T = 0$, despite two split transitions crossing into the SC dome, in agreement with our theory.

The peak in the penetration depth (but not a divergence) is expected due to $O(3)$ SDW fluctuations [18, 19]. $T_{\text{merge}}$ is small, as it is for large $\alpha$ (see Fig. 2), the emerging $O(6)$ symmetry further enhances the strength of the peak. In Ba(Fe$_{1-x}$Co$_x$)$_2$As$_2$, penetration depth measurements have so far not identified a peak inside the SC dome, yet the penetration depth was found to increase below a certain doping [41]. This increase is expected in the SDW + SC phase due to the competition between SC and SDW orders [12, 13], and in this regard the experimental result is again consistent with the existence of a single transition point at $T = 0$.

To summarize, in this paper we considered the behavior of the SDW and nematic transition lines inside the SC dome. We argued that, for a wide range of parameters, both orders coexist with SC, and the two transition lines separately penetrate into the SC dome as continuous second-order transitions. However, as temperature is lowered, they merge at some small but finite $T_{\text{merge}}$, giving rise to a single $T = 0$ QPT, at which nematic and SDW orders appear simultaneously via a weakly first-
order transition. The first-order character of the transition is ultimately due to the nematic instability, which becomes first-order on its own deep inside the SC dome, and changes the character of the would-be second-order SDW transition. The weak character of the transition is, on the other hand, a direct consequence of the coexistence with the SC order, which makes the spin dynamics propagating and enhances the ratio of the quartic couplings, pushing the system to the borderline between the first-order and second-order regimes.

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For $d_{ij} = 4$, the magnitude of the jump in $\varphi$ is generally of order of the upper cutoff $\Lambda$, and gets smaller only for very large $\alpha$.

We consider a model with one hole pocket at the $\Gamma$ point and two symmetry-related elliptical electron pockets at the $X$ and $Y$ points of the unfolded Brillouin Zone (BZ). Of the eight electronic interactions present in this model, two contribute to the superconducting (SC) and spin-density wave (SDW) channels: the electron-hole density-density interactions ($U_1$) and electron-hole pair hopping interactions ($U_3$). The Hamiltonian is given by $H = H_2 + H_4$, with:

$$H_2 = \sum_{k,i \in \{X,Y,\Gamma\}} \varepsilon_{k,i} c_{k\sigma,i}^\dagger c_{k\sigma,i}$$

$$H_4 = \sum_{k,i \in \{X,Y\}} U_1 c_{k\sigma,i}^\dagger c_{k\sigma,i}^\dagger c_{k\delta,j} c_{k\delta,j} + h.c.\, \delta_{\alpha\beta} \delta_{\gamma\delta}$$

where summation over spin indices is implied. The dispersions $\varepsilon_{k,i}$ are given in the main text as function of $\delta_0$ (which is proportional to the chemical potential) and $\delta_2$ (which is proportional to the ellipticity of the electron pockets). In $H_4$ we retain terms only in the spin and the pairing sector and define the staggered spin operators $S_i = \frac{1}{2} \sum_{k} c_{k\sigma,i}^\dagger \sigma_{\alpha\beta} c_{k\beta,i}$ and the pairing operators $b_i = \sum_{k} c_{k\uparrow,i} c_{-k\downarrow,i}$. We can then rewrite $H_4$ as:

$$H_4 = -(U_1 + U_3) \sum_{i \in \{X,Y\}} S_i \cdot S_i + 2U_3 \sum_{i \in \{X,Y\}} (b_i b_i + h.c.)$$

After introducing the Hubbard-Stratonovich fields $M_{\{X,Y\}}$, $\Delta_h$ for $b_\Gamma$, and $\Delta_c$ for $b_{\{X,Y\}}$, we obtain the action $S$ as function of the fermionic fields as well as the SDW and SC fluctuating fields (assumed to be homogeneous):

$$S[\Psi, M_i, \Delta_i] = \frac{2}{(U_1 + U_3)} \left( M_X^2 + M_Y^2 \right) - \frac{4}{U_3} \Delta_h \Delta_c - \int_k \hat{\Psi}_k^\dagger \left( i\omega_n - \hat{H}_k \right) \hat{\Psi}_k$$

Here, $k = (\omega_n, k)$, with $\omega_n = (2n + 1) \pi T$ denoting the fermionic Matsubara frequency, and $\int_k = T \sum_{\omega_n} \int \frac{d^d k}{(2\pi)^d}$. The 12-dimensional Nambu operator is given by $\hat{\Psi}_k^\dagger = \left( \psi_{k,\Gamma}^\dagger, \psi_{k,X}^\dagger, \psi_{k,Y}^\dagger, \psi_{-k,\Gamma}, \psi_{-k,X}, \psi_{-k,Y} \right)$, with:

$$\psi_{k,i}^\dagger = \left( c_{k\uparrow,i}^\dagger, c_{-k\uparrow,i}^\dagger, c_{-k\downarrow,i}^\dagger, c_{k\downarrow,i}^\dagger \right)$$
and:

\[ \mathcal{H}_k = \begin{pmatrix} \varepsilon_g & -\Delta_h(i\sigma_y) & -M_X \cdot \sigma & 0 & -M_Y \cdot \sigma & 0 \\ \Delta_h(i\sigma_y) & -\varepsilon_g & 0 & M_X \cdot \sigma^* & 0 & M_Y \cdot \sigma^* \\ -M_X \cdot \sigma & 0 & \varepsilon_X & -\Delta_e(i\sigma_y) & 0 & 0 \\ 0 & M_X \cdot \sigma^* & \Delta_e(i\sigma_y) & -\varepsilon_X & 0 & 0 \\ -M_Y \cdot \sigma & 0 & 0 & 0 & \varepsilon_Y & -\Delta_e(i\sigma_y) \\ 0 & M_Y \cdot \sigma^* & 0 & 0 & 0 & \Delta_e(i\sigma_y) \end{pmatrix} \]  

(S5)

Following Ref. 2, we integrate out the fermions and expand for small M, \Delta, obtaining the effective action

\[ S_{\text{eff}} [M_i, \Delta_i] = \frac{2}{(U_1 + U_3)} (M_X^2 + M_Y^2) - \frac{4}{U_3} \Delta_h \Delta_e + \frac{1}{2} \int \text{Tr} (\hat{G}_0 \hat{V})^2 + \frac{1}{2} \int \text{Tr} (\hat{G}_0 \hat{V})^4 + O(V^6) \]

where \( \hat{G}_0 = \text{diag}(G_T, \hat{G}_T, G_X, \hat{G}_X, G_Y, \hat{G}_Y) \) and \( \hat{V} \) is the same as \( \mathcal{H} \) but with the diagonal entries set to zero. Here, we introduced the non-interacting Green’s functions. \( G_i^{-1} = i \omega - \varepsilon_i \) and \( \hat{G}_i^{-1} \equiv -\hat{G}_{i,-}^{-1} = i \omega + \varepsilon_i \). To simplify our analysis, we consider the \( s^+ \) SC gap structure given by the solution of the linearized gap equations, which gives \( \Delta_e/\Delta_h = -1/\sqrt{2} \). We obtain:

\[ S_{\text{eff}} [M_i, \Delta_i] = a_m \left( M_X^2 + M_Y^2 \right) + a_s \Delta^2 + \frac{u_m}{2} \Delta^4 + \frac{u_m^{(1)} + u_m^{(2)}}{4} \left( M_X^2 + M_Y^2 \right)^2 - \frac{u_m^{(2)} - u_m^{(1)}}{4} \left( M_X^2 - M_Y^2 \right)^2 + \lambda \left( M_X^2 + M_Y^2 \right) \Delta^2 \]  

(S6)

with \( \Delta_h \equiv \Delta \) and Ginzburg-Landau coefficients:

\[ a_m = \frac{2}{(U_1 + U_3)} + 2 \int_k G_T G_X \]
\[ a_s = \frac{4}{\sqrt{2} U_3} + 2 \int_k \left( G_T \hat{G}_T + G_X \hat{G}_X \right) \]
\[ u_m^{(1)} = \int_k G_T^2 G_X^2 \]
\[ u_m^{(2)} = \int_k G_T^2 G_X G_Y \]
\[ \lambda = 2 \int_k \left( G_T^2 \hat{G}_T G_X + \frac{1}{2} G_X^2 \hat{G}_X G_T - \frac{1}{\sqrt{2}} G_T \hat{G}_T G_X \hat{G}_X \right) \]
\[ u_s = 2 \int_k \left( G_T^2 \hat{G}_T^2 + \frac{1}{2} G_X^2 \hat{G}_X^2 \right) \]  

(S7)

Evaluating the momentum integrals above give:

\[ \int_k G_T G_X = -2\pi \rho_T \sum_{n>0} \left( \frac{\omega_n}{\omega_n^2 + \tilde{\mu}_X^2} \right) \]
\[ \int_k \left( G_T \hat{G}_T + G_X \hat{G}_X \right) = -4\pi \rho_T \sum_{n>0} \frac{1}{\omega_n} \]
\[ \int_k G_T^2 G_X^2 = \pi \rho_T \sum_{n>0} \left( \frac{\omega_n (\omega_n^2 - 3\tilde{\mu}_X^2)}{(\omega_n^2 + \tilde{\mu}_X^2)^3} \right) \]
\[ \int_k G_T^2 G_X G_Y = \pi \rho_T \sum_{n>0} \left( \frac{\omega_n [(\omega_n^2 - \tilde{\mu}_X \tilde{\mu}_Y)^2 - \tilde{\mu}_X \tilde{\mu}_Y (\tilde{\mu}_X + \tilde{\mu}_Y)^2]}{(\omega_n^2 + \tilde{\mu}_X^2)^2 (\omega_n^2 + \tilde{\mu}_Y^2)^2} \right) \]
\[ \int_k G_k^2 \tilde{G}_k^2 = \int_k G_k^2 \tilde{G}_k^2 = \pi \rho_F T \sum_{n>0} \frac{1}{\omega_n^3} \]

\[ \int_k G_k^2 \tilde{G}_k \tilde{G}_k = \int_k G_k^2 \tilde{G}_k \tilde{G}_k = \pi \rho_F T \sum_{n>0} \left( \frac{\omega_n}{(\omega_n^2 + \tilde{\mu}_X^2)^2} \right) \]

\[ \int_k G_k \tilde{G}_k \tilde{G}_k \tilde{G}_k = \pi \rho_F T \sum_{n>0} \left( \frac{1}{\omega_n (\omega_n^2 + \tilde{\mu}_X^2)} \right) \]

where \( \rho_F \) is the density of states at the Fermi level, \( \langle \rangle \) refers to angular averaging over the Fermi surface, and \( \tilde{\mu}(x,y) = \delta_0 + \delta_2 \cos \theta \).

### B. Nematicity and superconductivity

From Eq (S6) we can follow the steps in Ref. [2] explained in the main text and introduce the new Hubbard-Stratonovic fields \( \varphi \) and \( \psi \) corresponding to \( M_X^2 + M_Y^2 \) (thermal fluctuations) and \( M_X^2 - M_Y^2 \) (nematic order parameter). After integrating out the Gaussian magnetic fluctuations in the paramagnetic phase, the new effective action is

\[ S_{\text{eff}} = \frac{\varphi^2}{2g_m} + \frac{\psi^2}{2u_m} + a_s \Delta^2 + \frac{u_s}{2} \Delta^4 + \frac{N}{2} \int \ln \left( \psi + \lambda \Delta^2 + \chi_0^{-1} - \varphi^2 \right) \]

where \( u_m \equiv \frac{u^{(1)} + u^{(2)}}{2}, g_m \equiv \frac{u^{(2)} - u^{(1)}}{2}, \) and \( N \) is the number of components of the magnetic order parameter. Here, \( \chi_0^{-1}(Q_i, q, \omega_n) = a_m + q^2 + f(\Omega_n) \), where \( \Omega_n = 2\pi n T \) is the Matsubara frequency and \( f(\Omega_n) \) is proportional to \( [\Omega_n] \) in the normal state and \( \Omega_n^2 \) deep inside the SC state. At the temperature where the nematic transition meets the SC transition line, we can restrict our analysis to \( \Omega_n = 0 \). Furthermore, since the pnictides are layered materials, we focus here in the case \( d = 2 \), which gives:

\[ \int \frac{d^2 q}{(2\pi)^2} \ln \left((r + q^2)^2 - \varphi^2 \right) = \frac{1}{4\pi} \left[ 2r(1 + \ln \Lambda^2) - (r + \varphi) \ln (r + \varphi) - (r - \varphi) \ln (r - \varphi) \right] \]

where \( \Lambda \) is the upper momentum cutoff and we defined \( r \equiv \psi + \lambda \Delta^2 + a_m \). To proceed with the saddle point approximation, we rescale \( \Delta^2 \rightarrow n \Delta^2 \) as well as the quartic coefficients

\[ (u_m, u_s, g_m, \lambda) \rightarrow \frac{(u_m, u_s, g_m, \lambda)}{n} \]  

(S8)

where \( n = \frac{N}{\pi \tau} \). Then, \( S_{\text{eff}} \) acquires an overall factor of \( n \), rendering the saddle-point approximation exact in the limit \( N \rightarrow \infty \). It follows that:

\[ \frac{\tilde{S}_{\text{eff}}}{2g_m n} = \varphi^2 - \frac{\psi^2}{2u_m} + a_s \Delta^2 + \frac{u_s}{2} \Delta^4 + r \ln \left( \frac{\Lambda^4}{r^2 - \varphi^2} \right) + 2r - \varphi \ln \left( \frac{r + \varphi}{r - \varphi} \right) \]

where, for convenience, we performed one additional rescaling:

\[ (u_m, u_s, \lambda, a_m, a_s, \varphi, \psi, r, \Lambda^2) \rightarrow 2g_m (u_m, u_s, \lambda, a_m, a_s, \varphi, \psi, r, \Lambda^2) \]  

(S9)

Using the saddle-point equation \( \frac{\partial \tilde{S}_{\text{eff}}}{\partial \psi} = 0 \), we can eliminate \( r \), which is given implicitly as a function of \( \varphi \) and \( \Delta \):  

\[ r = \tilde{a}_m + \lambda \Delta^2 - u_m \ln \left( r^2 - \varphi^2 \right) \]  

(S10)

Furthermore, the cutoff \( \Lambda \) has been absorbed into a redefinition of the quadratic term, \( \tilde{a}_m = a_m + 2u_m \ln \Lambda^2 \). The action then can be written as:

\[ S \equiv \frac{\tilde{S}_{\text{eff}}}{2g_m n} = \varphi^2 + \frac{r^2}{2u_m} + 2r - \varphi \ln \left( \frac{r + \varphi}{r - \varphi} \right) + \left( a_s - \frac{\lambda \tilde{a}_m}{u_m} \right) \Delta^2 + \left( \frac{u_s}{2} - \frac{\lambda^2}{2u_m} \right) \Delta^4 \]  

(S11)
Since we are interested in the region of the phase diagram where the nematic transition line crosses the SC dome, we expand the action for small $\varphi$ and $\Delta$. In particular, we substitute in Eq. (S10):

$$r = r_0 + b_1 \varphi^2 + b_2 \varphi^4 + c_1 \Delta^2 + c_2 \Delta^4 + d \varphi^2 \Delta^2$$  \hfill (S12)

where $r_0$ is the solution with $\varphi = 0$, $\Delta = 0$, and find the coefficients $b_i$, $c_i$, and $d$. Substituting this form in $\bar{S}$ and expanding for small $\varphi$ and $\Delta$, we obtain:

$$\bar{S} = a_n \varphi^2 + \frac{u_n}{2} \varphi^4 + \tilde{a}_s \Delta^2 + \frac{\tilde{u}_s}{2} \Delta^4 + \tilde{\lambda} \varphi^2 \Delta^2$$

with the Ginzburg-Landau coefficients:

$$\tilde{a}_s = a_s - \frac{\lambda}{u_m} (a_m - r_0)$$
$$a_n = 1 - \frac{1}{r_0}$$
$$\tilde{\lambda} = \frac{r_0 (r_0 + 2u_m)}{\lambda}$$
$$\frac{\tilde{u}_s}{2} = \frac{u_s - \lambda^2}{r_0 + 2u_m}$$
$$\frac{u_n}{2} = \frac{(-r_0 + u_m)}{6r_0 (r_0 + 2u_m)}$$  \hfill (S13)

Since we consider the vicinity of the nematic transition, where $a_n = 0$, we can set $r_0 = 1$ in the quartic coefficients $\tilde{\lambda}$, $\tilde{u}_s$, and $u_n$. Going back to the original variables via Eqs. (S8) and (S9), we then obtain the results in Eq. (4) of the main text.

II. SPIN-DENSITY WAVE TRANSITION INSIDE THE SC DOME

To obtain the SDW action deep inside the SC state, where the SC gap $\Delta$ is nearly saturated, we go back to the original action (S3) and treat $\Delta$ as a parameter, expanding only in powers of $M_i$:

$$S_{(SC)}^{(SC)} [M_i] = a_m (M_X^2 + M_Y^2) + \frac{u_m^{(1)} + u_m^{(2)}}{4} (M_X^2 + M_Y^2)^2 - \frac{u_m^{(2)} - u_m^{(1)}}{4} (M_X^2 - M_Y^2)^2$$  \hfill (S14)

As a result of this procedure, the Ginzburg-Landau coefficients depend now not only on the modified normal Green’s function $G_{i,k}$, but also on the anomalous Green’s function $F_{i,k}$:

$$G_{i,k} = \frac{-i \omega_n + \varepsilon_{k,i}}{\omega_n^2 + \varepsilon_{k,i}^2 + \Delta_i^2}$$
$$F_{i,k} = \frac{\Delta_i}{\omega_n^2 + \varepsilon_{k,i}^2 + \Delta_i^2}$$  \hfill (S15)

In particular, we obtain:

$$a_m = \frac{1}{4(U_1 + U_3)} + 2 \int_k (F_1 F_X + G_1 G_X)$$
$$u_m^{(1)} = \int_k \left[ 4F_1 F_X G_1 G_X + F_1^2 (F_X^2 + G_X G_Y) + F_1^2 G_1 G_X + G_1^2 G_X^2 \right]$$
$$u_m^{(2)} = \int_k \left[ F_1^2 (F_X F_Y + G_X G_Y) + G_1^2 G_X G_Y + G_1 G_X F_X F_Y + 4F_1 F_X G_1 G_Y \right]$$  \hfill (S16)
where \( \tilde{G}_{i,k} = -G_{i,-k} \). We define the quasi-particle excitation energy \( E_i = \sqrt{\Delta^2 + \varepsilon_i^2} \) and consider the \( T = 0 \) limit, where the Matsubara sum becomes an integral over frequencies. Performing this integration yields:

\[
\begin{align*}
\int \frac{1}{k} (G_X G_X + F_Y F_Y) &= \int \frac{1}{2(E + E_X)} \left[ -1 + \frac{\varepsilon_X \varepsilon_X + \Delta_X^2}{E X} \right] \\
\int \frac{1}{k} G_X G_Y F_Y &= \int \frac{1}{4(E + E_X)} \left[ -\Delta_X^2 \frac{E X}{E} + \frac{\Delta_X \Delta_X \varepsilon_X \varepsilon_X}{(E X)^2} \Xi \right] \\
\int \frac{1}{k} F_Y^2 F_Y &= \int \frac{1}{4(E + E_X)} \left[ \frac{\Delta_X^2}{E X} \right]^2 \\
\int \frac{1}{k} (F_X^2 G_Y + F_Y^2 G_X) &= \int \frac{1}{4(E + E_X)} \left[ -\Delta_X^2 \frac{E X}{E} - \frac{\Delta_X^2 \varepsilon_X \varepsilon_X}{(E X)^2} \Xi \right] \\
\int \frac{1}{k} G_Y G_X F_X &= \int \frac{1}{4(E + E_X)} \left[ -\Delta_X \frac{E X}{E} + \frac{\Delta_X \varepsilon_X \varepsilon_X \varepsilon_Y + A + B}{E X} \right] \\
\int \frac{1}{k} G_Y G_Y &= \int \frac{1}{4(E + E_X)} \left[ -\Delta_X \frac{E X}{E} + \frac{\Delta_X \varepsilon_X \varepsilon_Y + A + B}{E X} \right]
\end{align*}
\]

where \( \Xi = 3 + \frac{E_X^2 + E_Y^2}{E X E Y} \). We also obtain:

\[
\begin{align*}
\int \frac{1}{k} F_Y^2 &= \int \frac{1}{E X} \frac{\Delta_X^2}{A + B} \\
\int \frac{1}{k} F_Y^2 G_Y &= \int \frac{1}{k} -\Delta_X \frac{E X}{E} \frac{\varepsilon_X \varepsilon_Y + A + B}{E X} \\
\int \frac{1}{k} G_Y G_Y &= \int \frac{1}{E X} \frac{\Delta_X \varepsilon_X \varepsilon_Y + A + B}{E X} \\
\int \frac{1}{k} G_Y G_Y &= \int \frac{1}{E X} \frac{\Delta_X \varepsilon_X \varepsilon_Y + A + B}{E X} \\
\int \frac{1}{k} G_Y G_Y &= \int \frac{1}{E X} \frac{\varepsilon_X \varepsilon_Y + A + B}{E X}
\end{align*}
\]

with:

\[
\begin{align*}
A &= E_X (E_X + E_X + E_Y) (2E_T + E_X + E_Y) \\
B &= (E_T + E_X) (E_T + E_Y) (E_X + E_Y) \\
C &= (E_T + E_X + E_Y) (-E_T^2 + E_X E_Y) \\
F &= E_T (E_T + E_X) (E_T + E_Y) \\
D &= 4E_T E_X E_Y (E_T + E_X)^2 (E_T + E_Y)^2 (E_X + E_Y)
\end{align*}
\]

Following Ref. [1], we set \( \Delta_T = \Delta \) and \( \Delta_X = -\Delta/\sqrt{2} \). The nature of the mean-field SDW transition inside the SC dome is determined by the quartic coefficients \( u_m \equiv \frac{u_1 + u_2}{2} \), \( g_m \equiv \frac{u_2 - u_1}{2} \). In the absence of SC, \( u_m < 0 \) always at \( T = 0 \), regardless of the band structure parameters \( \delta_0 \) and \( \delta_2 \), implying that at \( T = 0 \) the SDW transition is first-order. This is illustrated in Fig. S1, where we plot the \( T = 0 \) value of \( u_m \) given by Eq. (S7) - i.e. without SC - as function of \( \delta_0 / \delta_F \) (\( \delta_F \) is the fermi energy) for \( \delta_0 = 0 \). The presence of SC can significantly change this result: in Fig. S1b, we plot \( u_m \) at \( T = 0 \) in the presence of SC, as given by Eq. (S16), as function of \( \delta_0 / \delta_2 \) for \( \delta_2 = 0 \). Clearly, for a large enough SC gap \( \Delta \), \( u_m \) changes sign, yielding a second-order SDW transition at \( T = 0 \) as long as \( g_m < u_m \) as well. In Fig. 2 of the main text, we plot the \( T = 0 \) value of the ratio \( \alpha = u_m / g_m \) in the entire \( (\delta_0, \delta_2) \) parameter space.

### III. Nematic Transition Inside the SC Dome

By using the magnetic action inside the SC state, Eq. (S14), with Ginzburg-Landau coefficients renormalized by the SC order, we can go beyond mean-field to study the nematic transition temperature near \( T = 0 \). In the main
text, we showed that at $T = 0$ and $d = 2, z = 1$, there is a simultaneous first-order SDW/nematic transition. We now extend this analysis to finite temperatures. Repeating the same procedure as described in Section II B, we introduce the Hubbard-Stratonovich fields $\varphi$ and $\psi$ and obtain the saddle-point equations:

\[
\frac{\varphi}{\bar{g}_m} = 2T \sum_{\Omega_n} \int q \, dq \left( \frac{1}{\psi + a_m + q^2 + \Omega_n^2 - \varphi} - \frac{1}{\psi + a_m + q^2 + \Omega_n^2 + \varphi} \right)
\]

\[
\frac{\psi}{\bar{u}_m} = 2T \sum_{\Omega_n} \int q \, dq \left( \frac{1}{\psi + a_m + q^2 + \Omega_n^2 - \varphi} + \frac{1}{\psi + a_m + q^2 + \Omega_n^2 + \varphi} \right)
\]

(S28)

where $(\bar{g}_m, \bar{u}_m) = \frac{\Lambda}{\bar{g}_m} (g_m, u_m)$. To perform the Matsubara frequency summation, we use the identity

\[
T \sum_{\Omega_n} f(i\Omega_n) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \coth \left( \frac{x}{2T} \right) \text{Im}[f(x)]
\]

(S29)

valid for any function $f$ of bosonic Matsubara frequencies $\Omega_n = 2n\pi T$. After defining $r = a_m + \psi$ and the rescaled quantities $(\bar{\varphi}, \bar{r}) = (\bar{\varphi}, \bar{r})_{\bar{g}_m}^2, T = \bar{T}\bar{g}_m$, we obtain the saddle-point equations:

\[
\bar{\varphi} = 2\bar{T} \left( \ln \sinh \frac{\sqrt{\bar{r} + \bar{\varphi}}}{2T} - \ln \sinh \frac{\sqrt{\bar{r} - \bar{\varphi}}}{2T} \right)
\]

(S30)

\[
\bar{r} = \bar{a}_m - 2\bar{T} \alpha \left( \ln \sinh \frac{\sqrt{\bar{r} + \bar{\varphi}}}{2T} + \ln \sinh \frac{\sqrt{\bar{r} - \bar{\varphi}}}{2T} \right)
\]

(S31)

where $\bar{a}_m = (a_m + 2\Lambda q \bar{u}_m) / \bar{g}_m^2$, $\Lambda q$ is the momentum integration upper cutoff, and $\alpha = u_m / g_m$. In Fig. S2, we plot the renormalized control parameter $\bar{a}_m$ as function of $\bar{\varphi}$ for different fixed temperatures. We see that for small $\bar{T}$ the first instability (corresponding to the largest $\bar{a}_m$) is at $0 < \bar{\varphi} < \bar{r}$, indicating a first-order nematic transition. As $\bar{T}$ increases, the first instability moves towards $\bar{\varphi} = 0$, indicating a second-order nematic transition. Because here we considered $d = 2$, the magnetic transition only happens at zero temperature, but for an anisotropic quasi-2D system, following Ref. [2], the magnetic transition will remain first-order and simultaneous to the nematic transition up to a temperature $T_{\text{merge}}$, above which one of the two transition become second-order and split from the other.
IV. ON THE PECULIARITIES OF THE HUBBARD-STRATONOVICH TRANSFORMATION

In this section we discuss one subtle issue related to the regularization of the integrals in the Hubbard-Stratonovich transformation. Consider for instance the transformation from an effective action for the SDW fields $\Delta X$ and $\Delta Y$ deep inside the SC state, Eq. (S14). In the notation we introduced after Eq. (S14),

$$S_{\text{eff}}[M_i] = a_m (M_X^2 + M_Y^2) + \frac{u_m}{2} (M_X^2 + M_Y^2)^2 - \frac{g_m}{2} (M_X^2 - M_Y^2)^2$$ (S32)

As we discussed above, in the SC state $u_m$ and $g_m$ are both positive. The partition function, from which we extract the free energy, is given by

$$Z = \int dM_X dM_Y e^{-S_{\text{eff}}[M_i]}.$$ 

The rationale behind the Hubbard-Stratonovich transformation is to rewrite the partition function as an integral over the new fields $\psi$ and $\varphi$ which describe the fluctuations of $M_X^2 + M_Y^2$ and $M_X^2 - M_Y^2$, respectively. This is done by expressing the quartic terms in $e^{-S_{\text{eff}}[M_i]}$ as integrals over the new fields $\psi$ and $\varphi$ of some new effective action $S_{\text{eff}}(M_i, \psi, \varphi)$ which depends only linearly on $M_X^2 + M_Y^2$ and $M_X^2 - M_Y^2$. The integrals over $M_X$ and $M_Y$ in the partition function can then be easily evaluated. Exponentiating the result one expresses the partition function as $Z = \int d\psi d\varphi e^{-S_{\text{eff}}[\psi, \varphi]}$.

For the nematic field $\varphi$, the computational procedure is free from subtleties. We use the mathematical identity which states that $e^{ax^2/2}$ can be expressed, for positive $a$, as

$$e^{ax^2} = \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} dy e^{\frac{y^2}{2a} + xy}$$ (S33)

Indeed,

$$\int_{-\infty}^{\infty} dy e^{\frac{y^2}{2a} + xy} = \int_{-\infty}^{\infty} dy e^{\frac{(y-ax)^2}{2a}} e^{\frac{x^2}{2a}},$$ (S34)

and the integral over $y$ converges and can be trivially evaluated by shifting variables:

$$\int_{-\infty}^{\infty} dy e^{-\frac{(y-ax)^2}{2a}} = 2 \int_{0}^{\infty} dy e^{-y^2/(2a)} = \sqrt{2\pi a}$$ (S35)

Applying this transformation to the $g_m$ term in $e^{-S_{\text{eff}}[M_i]}$, we obtain:

$$e^{g_m (M_X^2 - M_Y^2)^2} = \frac{1}{\sqrt{2\pi g_m}} \int_{-\infty}^{\infty} d\varphi e^{\frac{\varphi^2}{2g_m} + \varphi (M_X^2 - M_Y^2)}$$ (S36)
For the $u_m$ term, however, the Hubbard-Stratonovich transformation is trickier because the corresponding term in the effective action is
\[ e^{-\frac{u_m}{2}(M_x^2 + M_y^2)^2} \] (S37)
where, as we said, $u_m > 0$. We can still formally apply the Hubbard-Stratonovich transformation and obtain
\[ e^{-\frac{u_m}{2}(M_x^2 + M_y^2)^2} = \frac{1}{\Lambda} \int_{-\infty}^{\infty} d\psi e^{\frac{\psi^2}{2u_m}} e^{i\psi(M_x^2 + M_y^2)} \] (S38)
but now the normalization factor
\[ \Lambda = 2 \int_{0}^{\infty} dy e^{\frac{y^2}{2u_m}} \] (S39)
diverges.

One can avoid the divergence by introducing the imaginary field $\psi \rightarrow i\tilde{\psi}$ instead of the real one, i.e., by writing the exact and well-defined relation
\[ e^{-\frac{u_m}{2}(M_x^2 - M_y^2)^2} = \frac{1}{\sqrt{2\pi u_m}} \int_{-\infty}^{\infty} d\psi e^{\frac{\psi^2}{2u_m}} e^{i\tilde{\psi}(M_x^2 + M_y^2)} \] (S40)

However, later in the calculation the effective action is analyzed as a function of $\tilde{\psi}$ in the complex plane[5] and is taken at the position of the extreme along the imaginary axis of $\tilde{\psi}$, i.e., one actually returns to real $\psi$, for which the normalization factor in Eq. (S38) is the issue.

We argue that the more appropriate way to proceed is to consider $\Lambda$ in Eq. (S39) as the limit
\[ \Lambda = \lim_{\delta \rightarrow \pi} \Lambda_\delta \] (S41)
where
\[ \Lambda_\delta = 2 \int_{0}^{\infty} dy e^{\frac{y^2 e^{i\delta}}{2u_m}} \] (S42)
The well-behaved integral in (S35) corresponds to $\delta = 0$.

Figure S3: Integration contour for the evaluation of the integral $\Lambda_\delta$ from Eq. (S42). The radius $R$ has to be set to infinity at the end of the calculation.

To evaluate $\Lambda_\delta$, consider the integral $J_\delta = \oint dz e^{\frac{z^2 e^{i\delta}}{2}}$ over a complex variable $z$, taken over the contour shown in Fig. S3. The contour consists of two lines - one along the positive real axis and one along a line in the upper
half-plane of \( z \), directed at an angle \( \delta/2 \) with respect to the real axis - and the arc with radius \( R \) which will be set to infinity at the end of the calculation. We label the corresponding contributions as \( J_a \), \( J_b \), and \( \frac{J_c}{2} \), respectively:

\[
J_a = \int_0^R e^{-y^2/(2u_m)} dy
\]

\[
J_b = -e^{i\delta/2} \int_0^R dy e^{-y^2 e^{i\delta}/(2u_m)}
\]

\[
\frac{J_c}{2} = iR \int_0^{\delta/2} e^{i\theta} e^{-R^2 e^{i\theta}/(2u_m)} d\theta
\]  

(S43)

The integral \( J_a \) is elementary and in the limit of \( R \to \infty \) yields \( \sqrt{\pi u_m/2} \). The second integral at \( R \to \infty \) becomes \( J_b \to -(1/2)e^{i\delta/2} \Lambda_\delta \), and the integral \( J_c \) over the arc needs to be carefully analyzed.

Because the function \( e^{-z^2} \) is analytic for any finite \( z \), we have \( J = J_a + J_b + J_c = 0 \), hence

\[
\Lambda_\delta = e^{-i\delta/2} (\sqrt{2\pi u_m} + J_c)
\]  

(S44)

The issue then is to calculate \( J_c \). It is a complex function \( J_c = \text{Re} J_c + i \text{Im} J_c \), whose real and imaginary parts are given by

\[
\text{Re} J_c = -2R \int_0^{\delta/2} e^{-(R^2 \cos 2\theta)/(2u_m)} \sin(\theta - R^2 \sin 2\theta) d\theta
\]

\[
\text{Im} J_c = 2R \int_0^{\delta/2} e^{-(R^2 \cos 2\theta)/(2u_m)} \cos(\theta - R^2 \sin 2\theta) d\theta
\]  

(S45)
At $\delta$ exactly equal to $\pi$, both terms can be readily calculated numerically. We show the results in Fig.~\ref{fig:supp}. While the real part $\text{Re}J_c$ rapidly approaches $-\sqrt{2\pi u_m}$, the imaginary part $\text{Im}J_c$ rapidly approaches $(2u_m/R)e^{R^2/(2u_m)}$. As a result, $\Lambda_\pi \approx (2u_m/R)e^{R^2/(2u_m)}$, which is nothing but $2\int_0^R dy e^{y^2/(2u_m)}$. Obviously, $\Lambda_\pi$ diverges when $R \to \infty$.

We found that the behavior of $\text{Re}J_c$ and $\text{Im}J_c$ changes qualitatively at large $R$ once $\delta$ becomes different than $\pi$ (see Fig.~\ref{fig:supp}). In particular, starting from some critical $R_0$, both $\text{Re}J_c$ and $\text{Im}J_c$ become oscillating functions of $R$. As a result, there is an infinite set of $R_i$ at which $\text{Re}J_c = 0$ and an infinite set of $R_j$ at which $\text{Im}J_c = 0$. We show this behavior in Fig.~\ref{fig:supp} for $\delta = \pi/2$. The value $R_0$ at which oscillations begin is infinite at $\delta = \pi$, but is finite at any $\delta < \pi$ and its value decreases as $\pi - \delta$ increases.

Because both $\text{Re}J_c$ and $\text{Im}J_c$ oscillate, the limit of each of these functions at $R \to \infty$ depends on how one approaches $R \to \infty$. In particular, one can use the fact that there is an infinite set of $R$’s for which $\text{Re}J_c = 0$, $\{R_i\}$, and another one for which $\text{Im}J_c = 0$, $\{R_j\}$. By approaching the limit $R \to \infty$ via the sets $\{R_i\}$ and $\{R_j\}$, we obtain $\lim_{R \to \infty} J_c = 0$.

We emphasize that this is only possible for $\delta < \pi$, when both $\text{Re}J_c$ and $\text{Im}J_c$ oscillate with $R$. Substituting $J_c = 0$ into (S44) we find

$$\Lambda_\delta = e^{-i\delta/2}\sqrt{2\pi u_m} \quad (S46)$$

Hence, $\Lambda_\pi = \lim_{\delta \to \pi} \Lambda_\delta = -i\sqrt{2\pi u_m}$, i.e. it is finite. Note that Eq. (S46) gives the same result for $\Lambda_\delta$ which we would obtain by formally substituting $u_m \to u_m e^{-i\delta}$ into Eq. (S35). For other ways to regularize such an integral see Refs. 1, 2.

The same computational scheme can be applied to the evaluation of the action near the extremum of $S_{\text{eff}}[\psi, \varphi]$. Taken as a function of $\psi$, the extremum is a maximum rather than a minimum. Expanding the action near the maximum at $\psi = \psi_0$, we get $S_{\text{eff}}[\psi, \varphi] = S_{\text{eff}}[\psi_0, \varphi] - A(\psi - \psi_0)^2$ with $A > 0$. For the partition function, we then obtain

$$Z = Z_0 \int d\psi e^{A(\psi - \psi_0)^2} \quad (S47)$$

Formally, the integral diverges and makes the expansion near $\psi_0$ problematic. However, once we define the integral over $\psi$ in the same way as above, the integral becomes finite and one can apply a conventional reasoning (e.g., large $N$ expansion [3]) to justify the approximation in which $S_{\text{eff}}[\psi, \varphi]$, viewed as a function of $\psi$, is taken at the maximum.

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