BREAKING OF TOPOLOGICAL SYMMETRY

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ABSTRACT

The coupling of topological matter to topological Yang-Mills theory in four dimensions is considered and a model is presented. It is shown that, contrary to the two-dimensional case, this coupling may lead to a breaking of the topological symmetry. This means that the vacuum expectation values of the observables of the theory loose their invariance under small deformations of the metric while the action of the model possesses all the symmetries corresponding to the case with no coupling.

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Matter couplings to topological Yang-Mills and topological gravity in two dimensions can be constructed without losing the topological features of the theory [1,2,3]. In other words, these couplings can be constructed while maintaining the $Q$-symmetry of the models, and it turns out that the resulting theory possesses an energy-momentum tensor which is $Q$-exact. The aim of this letter is to point out that in four dimensions the picture that emerges seems to be different. We will present a simple model of topological matter in four dimensions and its coupling to topological Yang-Mills theory (or Donaldson-Witten theory) [4]. The resulting theory has a $Q$-symmetry but its energy-momentum tensor is not $Q$-exact. This implies that the observables leading to Donaldson invariants in topological Yang-Mills theory may get non-topological corrections due to the presence of matter couplings. We will show that it is also possible to add mass terms while preserving $Q$-invariance, leading to a further breaking of the topological character of the theory.

Let us begin constructing topological matter in four dimensions. Our starting point is a twisted version of the hypermultiplet of $N = 2$ supersymmetry [5,6]. The resulting models are different than the ones obtained after twisting $N = 4$ supersymmetry, or $N = 2$ conformal supergravity as constructed in [7] and [8], respectively. In four dimensions, the Lorentz and internal generators of $N = 2$ supersymmetry can be grouped as the ones of $SU(2)_L \times SU(2)_R \times SU(2)_I \times U(1)$. The hypermultiplet is made out of four fields which transform as $(0, 0, 1/2)^0, (1/2, 0, 0)^1, (0, 1/2, 0)^{-1}$ and $(0, 0, 1/2)^2$ respect to those generators. The superindex denotes the $U(1)$ eigenvalue. The field transforming as $(0, 0, 1/2)^2$ is auxiliary. The twisting consists of the replacement of $SU(2)_R \times SU(2)_I$ by $SU(2)'_R$, being this the diagonal sum of $SU(2)_R$ and $SU(2)_I$. Under the twisted algebra $SU(2)_L \times SU(2)'_R \times U(1)$ the component fields of the twisted hypermultiplet transform as $(0, 1/2)^0, (0, 1/2)^1, (1/2, 0)^{-1}$ and $(0, 1/2)^2$. We will denote these fields as $H_\alpha, u_\alpha, v_\dot{\alpha}$ and $K_\alpha$ respectively. Clearly, $H_\alpha$ and $K_\alpha$ are commuting fields while $u_\alpha$ and $v_\dot{\alpha}$ are anticommuting. The $U(1)$ quantum numbers of the $N = 2$ fields now play the role of ghost
numbers. The resulting $Q$-transformations are,

$$\begin{align*}
\delta H_\alpha &= \epsilon u_\alpha, \\
\delta u_\alpha &= -\epsilon K_\alpha, \\
\delta v_\dot{\alpha} &= i\epsilon \partial_{\alpha\dot{\alpha}} H^\alpha, \\
\delta K_\alpha &= i\epsilon \partial_{\alpha\dot{\alpha}} v^{\dot{\alpha}},
\end{align*}$$

(1)

where $\epsilon$ is a constant anticommuting parameter. These transformations indicate that $Q^2 \neq 0$. This is not surprising since the $N = 2$ hypermultiplet possesses central charges. Other models of topological quantum field theories where central charges are present have been previously studied in two dimensions [9,3]. The commuting central charge transformations can be easily found from (1) and the fact that $2Q^2 = Z$, where $Z$ is the central charge generator. They turn out to be,

$$\begin{align*}
\delta_z H_\alpha &= -z K_\alpha, \\
\delta_z u_\alpha &= -iz \partial_{\alpha\dot{\alpha}} v^{\dot{\alpha}}, \\
\delta_z v_\dot{\alpha} &= iz \partial_{\alpha\dot{\alpha}} u^\alpha, \\
\delta_z K_\alpha &= z \Box H_\alpha.
\end{align*}$$

(2)

where $z$ is a commuting constant parameter. It is simple to verify that indeed,

$$[Q, Z] = 0.$$  

(3)

The presence of central charges breaks the $U(1)$ symmetry, which in the twisted theory is just the ghost number symmetry, into $Z_4$. It can be verified explicitly that, indeed, the ghost number is preserved in the transformations (1) and (2) modulo 4.

To construct matter actions we will introduce a second multiplet which can be thought as the complex conjugate of the one just described. We will denote the component fields of this multiplet by $\overline{H}_\alpha$, $\overline{u}_\alpha$, $\overline{v}_{\dot{\alpha}}$ and $\overline{K}_\alpha$. Clearly, under $Q$ and $Z$
they have the same transformation properties as in (1) and (2). For example, the corresponding $Q$ transformations are,

\[
\begin{align*}
\delta H_\alpha &= \epsilon \overline{\sigma}_\alpha, \\
\delta \overline{\sigma}_\alpha &= -\epsilon K_\alpha, \\
\delta \overline{\sigma}_\dot{\alpha} &= i \epsilon \partial_{\dot{\alpha}} \overline{H}^\alpha, \\
\delta K_\alpha &= i \epsilon \partial_{\dot{\alpha}} \overline{v}^\dot{\alpha},
\end{align*}
\]

(4)

The ghost number assignment for these fields is the same as the one used for their counterparts with no overlines. The matter action which is invariant under $Q$ and $Z$ takes the form:

\[
L_f = L_{f0} + m L_f^m = \int d^4x \left[ H_\alpha \Box H_\alpha + i \overline{u}^\alpha \partial_{\dot{\alpha}} \dot{v}^\dot{\alpha} - i \overline{v}^\dot{\alpha} \partial_{\dot{\alpha}} u^\alpha + \overline{K}^\alpha K_\alpha \\
+ m \left( \overline{K}^\alpha H_\alpha - \overline{H}^\alpha K_\alpha \right) + m \left( \overline{u}^\alpha u_\alpha + \overline{v}^\dot{\alpha} v_\dot{\alpha} \right) \right],
\]

(5)

where $m$ is the bare mass associated to the twisted hypermultiplet. Notice that $K_\alpha$ and $\overline{K}_\alpha$ play the role of auxiliary fields. For $m \neq 0$ it is convenient to redefine these fields so that they appear quadratically in the action:

\[
\begin{align*}
\overline{K}_\alpha &= \overline{K'}_\alpha + m \overline{H}_\alpha, \\
K_\alpha &= K'_\alpha - m H_\alpha.
\end{align*}
\]

(6)

The mass terms in (5) become,

\[
m L_f^m = \int d^4x \left[ m^2 \overline{H}^\alpha H_\alpha + m \left( \overline{u}^\alpha u_\alpha + \overline{v}^\dot{\alpha} v_\dot{\alpha} \right) \right].
\]

(7)

In order to study if this model leads to a topological quantum field theory we must formulate it for an arbitrary four-manifold. Since the model contains spinors this manifold must admit at least one spin structure, i.e., it must be a spin manifold. Actually, we will endow our manifold also with an $SU(N)$ gauge
connection $A_\mu$. Vector indices will be denoted by greek letters from the middle of
the alphabet. The ones from the beginning of the alphabet are reserved for spinor
indices. Let us denote by $M$ an arbitrary four-dimensional spin manifold without
boundary. Introducing a vierbein $e_{a\mu}$ on such a manifold, one finds that the action

$$L^{e,A} = L^{e,A}_0 + mL^{e,A}_m$$

$$= \int_M d^4x e \left[ \bar{H}^\alpha (\Box + \frac{1}{4} R) H_\alpha + \frac{i}{2} \bar{H}^\alpha F^+_{\alpha\beta} H_\beta + i\bar{u}^\alpha \nabla_{\alpha\hat{\alpha}} v^{\hat{\alpha}} - i\bar{\nu}^{\hat{\alpha}} \nabla_{\alpha\hat{\alpha}} u^\alpha \right. \right. $$

$$+ \bar{K}^\alpha K'_{\alpha} + m^2 \bar{H}^\alpha H_\alpha + m(\bar{u}^\alpha u_\alpha + \bar{\nu}^{\hat{\alpha}} v^{\hat{\alpha}}) \left. \right] , \tag{8}$$

is invariant under the following $Q$ and $Z$ transformations,

$$\delta H_\alpha = \epsilon u_\alpha, \quad \delta \bar{z} H_\alpha = -z K_\alpha,$$

$$\delta u_\alpha = -\epsilon K_\alpha, \quad \delta \bar{z} u_\alpha = -iz \nabla_{\alpha\hat{\alpha}} v^{\hat{\alpha}},$$

$$\delta \bar{v}^{\hat{\alpha}} = i\epsilon \nabla_{\alpha\hat{\alpha}} H^\alpha, \quad \delta \bar{z} \bar{v}^{\hat{\alpha}} = iz \nabla_{\alpha\hat{\alpha}} u^\alpha,$$

$$\delta K_\alpha = i\epsilon \nabla_{\alpha\hat{\alpha}} v^{\hat{\alpha}}, \quad \delta \bar{z} K_\alpha = z \left[ (\Box + \frac{1}{4} R) H_\alpha + \frac{i}{2} F^+_{\alpha\beta} H_\beta \right]. \tag{9}$$

In (8) and (9) $\nabla_\mu$ denotes a covariant derivative respect to the vierbein $e_{a\mu}$ and the
gauge connection $A_\mu$, $R$ is the curvature scalar, and $F^+_{\alpha\beta}$ is the anti-self-dual part
of the gauge field strength, $F^+_{\alpha\beta} = C^{\hat{\alpha}\hat{\beta}} F_{\alpha\hat{\alpha},\beta\dot{\beta}}$. Of course, $\Box$ denotes the covariant
Laplacian. The matter fields with no overline transform under a given representa-
tion of $SU(N)$ while the ones with overlines transform under the complex conjuga te
representation. Notice that in this generalized setting the transformations (9) get
terms involving the Riemann curvature and the gauge field strength. A similar
set of transformations as the ones in (9) holds for the fields with overline. The
generalized transformations (9) verify the algebra (3).

Certainly, the action $L^{e,A}$ in (8) is not topological because of the mass terms.
However, if $m = 0$ the action is topological. To verify this we must analyze whether
or not the energy-momentum tensor is $Q$-exact. First, notice that the action is $Q$-
exact. It turns out that,
\[ \mathcal{L}^{e,A}_0 = \{ Q, \Lambda^{e,A} \}, \]  
(10)
where
\[ \Lambda^{e,A} = \frac{1}{2} \int_M d^4 x e \left[ i \mathcal{H}^\alpha \nabla_{\alpha \dot{\alpha}} v^{\dot{\alpha}} + iv^{\dot{\alpha}} \nabla_{\alpha \dot{\alpha}} H^\alpha - K^\alpha u_\alpha - \bar{\pi}^\alpha K_\alpha \right]. \]  
(11)
The invariance under \( Q \) and \( Z \) of \( \mathcal{L}^{e,A}_0 \) follows simply from (3) and the fact that,
\[ [Z, \Lambda] = 0. \]  
(12)
The form of (10) does not imply in general that the theory is topological. Only when \( Q \) and \( \delta \delta^e_{\alpha \mu} \) commute this implication holds. This is not the case, however, in this model as can be concluded from the transformations (9). Indeed, it turns out that the energy-momentum tensor is not \( Q \)-exact. One finds:
\[ T^{e,A}_{\mu \nu} = \{ Q, \Lambda^{e,A}_{\mu \nu} \} + \frac{1}{2} g_{\mu \nu} T^{e,A}_0, \]  
(13)
where,
\[ \Lambda^{e,A}_{\mu \nu} = \frac{i}{4} \left[ \mathcal{H}^\alpha (\sigma_{(\mu)}_{\alpha \beta} \nabla_{\nu}) v^{\beta} \right. \]
\[ \left. - \nabla_{(\mu} \mathcal{H}^\alpha (\sigma_{\nu)}_{\alpha \beta} v^{\beta} + \bar{\pi}^{(\beta}(\sigma_{\mu)}_{\alpha} \nabla_{\nu) \alpha} H^\alpha \right] \]  
(14)
and,
\[ T^{e,A}_0 = H^\alpha \nabla_{\gamma} \gamma \dot{\alpha} ^{\gamma} \nabla_{\alpha \beta} H^\alpha + i\mathcal{H}^\alpha \nabla_{\alpha \beta} \nabla_{\gamma} \gamma ^{\beta} H^\gamma + i\bar{\pi}^{\gamma} \nabla_{\alpha \beta} u^{\alpha} - iv^{\dot{\alpha}} \nabla_{\alpha \beta} \pi^{\dot{\alpha}} \]  
\[ \left. - i\bar{\pi}^{\gamma} \nabla_{\alpha \beta} v^{\dot{\alpha}} + iu^{\alpha} \nabla_{\alpha \beta} \bar{\pi}^{\dot{\alpha}} - 2K^\alpha K_\alpha. \right] \]  
(15)

In eq. (14) \( \sigma_{\mu} = (1, \sigma_i) \), where \( \sigma_i, i = 1, 2, 3 \), are the Pauli matrices, and \( \sigma_{\mu \nu} \) (\( \bar{\sigma}_{\mu \nu} \)) are the generators of Lorentz transformations on undotted (dotted) spinors.
Although $T^{c,A}$ is not $Q$-exact it has the property that it vanishes on-shell. In theories where the action is $Q$-exact, as it is the case here, this lack of $Q$-exactness of the energy momentum-tensor does not break the topological symmetry. Standard arguments [4] show that in this case the classical limit is exact and therefore terms which vanish on-shell are harmless.

The vacuum expectation value of any product of operators which are invariant under $Q$ leads to topological invariants. Actually, the same arguments show that the resulting vacuum expectation values are in this case also invariant under deformations of the gauge connection $A_{\mu}$. The gauge current turns out to be $Q$-exact:

$$J_{\alpha\dot{\alpha}} = \{Q, i\overline{\theta}^{\alpha} v^{\dot{\beta}} + i\overline{\sigma}^{\dot{\alpha}} H^{\alpha}\}. \quad (16)$$

Unfortunately, the form of the $Q$-transformations of the fields in (9) indicates that there are not operators invariant under $Q$ in this theory. One could make, however, the observation that according to (9) the field $u_{\alpha}$ $Q$-transforms into the auxiliary field $K_{\alpha}$. A similar situation to this occurs in type B sigma models in two dimensions [10,3,11,12]. When an operator is not anihilated by $Q$ but it is proportional to auxiliary fields it may lead to topological invariants. Let us analyze the situation in this case.

Let us consider, for example, the following operator,

$$\phi(P)^n = (\overline{\sigma}^{\alpha}(P)u_{\alpha}(P))^n, \quad (17)$$

where $P \in M$. From (9) it follows that,

$$[Q, \phi^n] = -n(\overline{u}^\gamma u_\gamma)^{n-1}(\overline{K}^{\alpha} u_{\alpha} - \overline{u}^{\dot{\alpha}} K_{\alpha}). \quad (18)$$

The vacuum expectation value of the operator (17) has the following dependence
on \( e^{a\mu} \) and \( A^\mu \),

\[
\begin{align*}
\delta e^{a\mu}_{\alpha}(P') \langle \phi(P)^n \rangle &= \int [DX] \phi(P)^n T_{\mu}^{e\alpha}(P') \exp(-\mathcal{L}^e_A), \\
\delta A^{\alpha\bar{\alpha}}_{\alpha}(P') \langle \phi(P)^n \rangle &= \int [DX] \phi(P)^n J_{\alpha\bar{\alpha}}(P') \exp(-\mathcal{L}^e_A),
\end{align*}
\]

(19)

where \([DX]\) denotes the full functional integral measure. Using (13), (16), the fact that the action \( eQ \)-exact, and assuming that \([DX]\) is invariant under \( Q \), it turns out that if \( P \neq P' \):

\[
\delta e^{a\mu}_{\alpha}(P') \langle \phi(P)^n \rangle = 0, \quad \delta A^{\alpha\bar{\alpha}}_{\alpha}(P') \langle \phi(P)^n \rangle = 0.
\]

(20)

To get (20) one just has to realize that the quadratic terms in the auxiliary fields multiplying the exponential of the action do not occur at coincident points for \( P \neq P' \). On the other hand, it also holds that the vacuum expectation value \( \langle \phi(P)^n \rangle \) is independent of the point \( P \). This follows from the fact that \( d\phi^n \) is \( Q \)-exact up to terms linear in the auxiliary fields:

\[
d\phi^n = n\{Q, (\bar{H}^\alpha du_\alpha - d\bar{u}^\alpha H_\alpha)(\bar{u}^\gamma u_\gamma)^{n-1}\} + (\text{linear } K\text{-terms}).
\]

(21)

Thus, the operators \( \phi^n \) lead to quantities which are invariant under small deformations of the vierbein and the gauge connection.

These arguments show that in order to build non-trivial observables one must study the cohomology of \( Q \) modulo terms linear in the auxiliary fields \( K_\alpha \) and \( \bar{K}_\alpha \). The form of the transformations (1) indicates that this cohomology is trivial unless one could regard quantities like \( \bar{H}^\alpha H_\alpha \) as a local coordinate on some manifold with non-trivial topology. We will not analyze that possibility in this paper.

Due to the quadratic form of the action \( \mathcal{L}^e_0 \) and the presence of the symmetry \( Q \), the computation of the vacuum expectation value of products of \( Q \)-invariant operators modulo linear \( K \)-terms reduces to an integration over zero modes. In
other words, after expanding the fields entering the functional integral into zero and non-zero modes, the integration of the last ones reduces to a ratio of determinants whose value is 1. The presence of zero modes leads to a ghost number anomaly which as usual implies certain selection rule for the observables of the theory. Let us study the form that this selection rule takes in a situation which will be of interest in the analysis of the full theory. Consider the case in which $M$ is $S^4$ and $A_\mu$ is an $SU(2)$ anti-self-dual connection ($F^+_{\alpha\beta} = 0$) of second Chern class $k$. In this case, since $R > 0$, there are not $H_\alpha$-zero modes in $\mathcal{L}^{e,A}_{0}$. Respect to the spinor fields $u_\alpha$ and $v_\dot{\alpha}$, their structure of zero modes depends on the representation chosen. If they belong to the $SU(2)$ representation of isospin $j/2$, there are $\frac{1}{16}j(j+1)(j+2)k$ $v_\dot{\alpha}$-zero modes, while there are not $u_\alpha$-zero modes [13]. Thus, the selection rule that emerges is that operators which could possibly lead to a non-zero vacuum expectation value would contain ghost number $-\frac{1}{6}j(j+1)(j+2)k$. This selection rule by itself is strong enough to argue that there are not non-trivial observables in the situation considered since the form of the transformations (1) indicates that there are not $Q$-invariant operators of negative ghost number.

The topological matter theory described by the action $\mathcal{L}^{e,A}_{0}$ in (8) does not seem to provide observables leading to topological invariants. However, this theory can be coupled to topological Yang-Mills or topological gravity modifying the values of the observables of those theories. In the rest of this paper we will describe the topological matter coupling to the first of these theories.

Before describing the coupling, let us recall first the structure of Donaldson-Witten theory. The action of this theory is [4],

$$
\mathcal{L}^{DW} = \frac{1}{g^2} \int_M d^4xe^{\text{Tr}}\left[Q, \frac{1}{4}(F^+_{\alpha\gamma} + G_{\alpha\gamma})\chi^\alpha\gamma + i\lambda\nabla_{\alpha\dot{\beta}}\psi^{\dot{\alpha}\beta} + \frac{i}{2}\lambda[\eta,\phi]\right]
$$

$$
- \frac{1}{g^2} \int_M d^4xe^{\text{Tr}}\left[\frac{1}{4}(F^+)^2 - \frac{1}{4}(G)^2 - \chi^\alpha\gamma\nabla_{\alpha\dot{\beta}}\psi^{\dot{\alpha}\beta} + \frac{i}{4}\phi\{\chi_{\alpha\gamma},\chi^\alpha\gamma\}
\right.
$$

$$
+ \eta \nabla_{\alpha\dot{\beta}}\psi^{\dot{\alpha}\beta} - i\lambda\{\psi_{\alpha\dot{\beta}},\psi^{\dot{\alpha}\beta}\} - \lambda\nabla_{\alpha\dot{\beta}}\nabla_{\alpha\dot{\beta}}\phi + \frac{i}{2}\phi\{\eta,\eta\} + \frac{1}{2}[\lambda,\phi]^2\right],
$$

(22)
where $\lambda$ and $\phi$ are commuting scalar fields, and $\eta$, $\psi_{\alpha\dot{\beta}}$ and $\chi_{\alpha\beta}$ are anticommuting scalar, vector and anti-self-dual fields, respectively. $G_{\alpha\beta}$ is an auxiliary commuting and anti-self-dual field [14]. In (22) $g$ denotes the gauge coupling constant. The operator $Q$ in (22) is the corresponding one to the following transformations,

$$
\begin{align*}
\delta A_{\alpha\dot{\beta}} &= \epsilon \psi_{\alpha\dot{\beta}}, \\
\delta \psi_{\alpha\dot{\beta}} &= -\epsilon \nabla_{\alpha\dot{\beta}} \phi, \\
\delta \lambda &= \epsilon \eta, \\
\delta \eta &= i\epsilon [\lambda, \phi],
\end{align*}
$$

Since $Q^2$ is just a gauge transformation the action (22) is manifestly invariant under the transformations (23).

The observables of Donaldson-Witten theory, which lead to topological invariants, are arbitrary products of operators [4],

$$
O^{(\gamma)} = \int_{\gamma} W_{k_{\gamma}},
$$

where $\gamma$ is a homology cycle of $M$ of dimension $k_{\gamma}$, and $W_{k_{\gamma}}$ is one of the differential forms,

$$
\begin{align*}
W_0 &= \frac{1}{2} \text{Tr} \phi^2, \\
W_1 &= \text{Tr}(\phi \wedge \psi), \\
W_2 &= \text{Tr}(\frac{1}{2} \psi \wedge \psi + i\phi \wedge F), \\
W_3 &= i\text{Tr}(\psi \wedge F), \\
W_4 &= -\frac{1}{2} \text{Tr}(F \wedge F).
\end{align*}
$$

The coupling of the topological matter described in (8) to the full topological multiplet of Donaldson-Witten theory can be obtained by twisting its corresponding $N = 2$ counterpart. It turns out that the resulting theory can be truncated making it simpler. In what follows we will describe the truncated theory. Details
of the truncation will be presented elsewhere. The full action takes the following form,

$$\mathcal{L} = \mathcal{L}^{\text{DW}} + \mathcal{L}_0 + m\mathcal{L}_m,$$  \hspace{1cm} (26)

where,

$$\mathcal{L}_0 = \int d^4x e\left[ \bar{H}^\alpha (\Box + \frac{1}{4} R) H_\alpha + i\bar{H}^\alpha F_{\alpha\beta}^+ H^\beta + i\bar{H}^\alpha \nabla_{\alpha\dot{\alpha}}\bar{v}^\dot{\alpha} - i\bar{v}^\dot{\alpha} \nabla_{\alpha\dot{\alpha}} u^\alpha \\
+ \bar{K}'^\alpha K'_\alpha + \bar{H}^\alpha \psi_{\alpha\beta}\bar{v}^\beta - \bar{v}^\beta \psi_{\alpha\beta} H^\alpha + i\bar{v}^\dot{\alpha}\phi v_\dot{\alpha} \right],$$  \hspace{1cm} (27)

$$m\mathcal{L}_m = \int d^4x e\left[ m^2 \bar{H}^\alpha H_\alpha + m(\bar{v}^\beta u_\alpha + \bar{v}^\dot{\alpha} v_\dot{\alpha}) - im\bar{H}^\alpha \phi H_\alpha \right].$$

The coupling modifies the $Q$ and $Z$-transformations (9). They take now the following form:

$$\delta H_\alpha = \epsilon u_\alpha,$$
$$\delta u_\alpha = -\epsilon(K_\alpha + i\phi H_\alpha),$$
$$\delta v_\dot{\alpha} = i\epsilon \nabla_{\alpha\dot{\alpha}} H^\alpha,$$
$$\delta K_\alpha = i\epsilon \nabla_{\alpha\dot{\alpha}} v^\dot{\alpha},$$
$$\delta z H_\alpha = -z(K_\alpha + i\phi H_\alpha),$$
$$\delta z u_\alpha = -i z(\nabla_{\alpha\dot{\alpha}}\bar{v}^\dot{\alpha} + \phi u_\alpha),$$
$$\delta z v_\dot{\alpha} = z(i\nabla_{\alpha\dot{\alpha}} u^\alpha + \psi_{\alpha\beta} H^\beta),$$
$$\delta z K_\alpha = z[(\Box + \frac{1}{4} R) H_\alpha + \frac{i}{2} F_{\alpha\beta}^+ H^\beta + \psi_{\alpha\beta} v^\dot{\alpha}].$$  \hspace{1cm} (28)

Of course, one has a similar set of transformations for the fields with overlines. In the coupled theory the algebra $2Q^2 = Z$ holds. The $z$-transformations of the fields in the topological Yang-Mills multiplet are just gauge transformations with gauge parameter $z\phi$.

The action (26) represents the coupling of topological matter to Donaldson-Witten theory. Let us analyze the structure of the resulting theory. Certainly, if $m \neq 0$ the topological character of the theory is broken. For $m = 0$ it turns out that contrary to the case of $\mathcal{L}_0^{\text{DW}}$ in (8) the action $\mathcal{L}_0$ in (27) is not $Q$-exact. This can be demonstrated writing the most general local expression of ghost number $-1 \mod 4$ and showing that it does not lead to $\mathcal{L}_0$. The $Z$-symmetry of the theory is very restrictive and somehow the responsible for the non-existence of a reasonable
matter action which is $Q$-exact for the coupled theory. This has very important consequences. As shown below, the energy-momentum tensor of the theory is not $Q$-exact. As in the non-coupled case, the part of the energy-momentum tensor which is not $Q$-exact vanishes on-shell. However, in the coupled case, since the action is not $Q$-exact, one does not possess an argument to disregard the terms in the energy-momentum tensor which are not $Q$-exact and therefore the manifest topological character of the theory is broken. Matter couplings to Donaldson-Witten theory lead in principle to non-topological corrections to Donaldson invariants, i.e., to the vacuum expectation values of the observables (24) computed with $\mathcal{L}^{\text{DW}} + \mathcal{L}_0$.

Let us analyze the computation of the vacuum expectation value of a product of observables of the type (24) of ghost number $r$. We will consider a situation in which $M$ is $S^4$ and the gauge group is $SU(2)$, while the matter fields are valued in a $SU(2)$ representation of dimension $d$. Since $\mathcal{L}^{\text{DW}}$ is $Q$-exact, standard arguments show that the vacuum expectation values are invariant under deformations of the parameter $g$ in (22). This allows to make computations of $Q$-invariant quantities in the limit $g \to 0$, in which the contributions from the functional integral are dominated by the classical configurations of Donaldson-Witten theory.

The classical configurations of the gauge connection are anti-self-dual gauge fields ($F^+_{\alpha\beta} = 0$). For the case in which $M$ is $S^4$ and the gauge group is $SU(2)$ these form a moduli space of dimension $8k - 3$, where $k$ is the second Chern-class. There are in addition $8k - 3 \psi_\mu$-zero modes which are anticommuting. There are not zero modes for the rest of the fields in the gauge multiplet. The contribution from matter fields is computed spanning them around classical configurations. For an anti-self-dual gauge connection of second Chern class $k$, and matter fields in the $SU(2)$ representation of isospin $j/2$, there are $\frac{1}{8} j(j+1)(j+2) k \nu_{\dot{\alpha}}$-zero modes. Since $\nu_{\dot{\alpha}}$ has ghost number $-1$, in order to have a non-vanishing vacuum expectation value the ghost number of the operator entering the functional integral must take the value $r = 8k - 3 - \frac{1}{8} j(j+1)(j+2) k$. The computation of observables can

\begin{itemize}
  \item [\text{*}] We make this choice to be specific but a similar discussion holds in general.
\end{itemize}
be carried out exactly using the invariance under variations of $g$. The resulting expression involves, besides integrations of zero modes, convolutions of these with fermionic and bosonic propagators.

The energy-momentum tensor $T_{\mu\nu}$ corresponding to $\mathcal{L}^{DW} + \mathcal{L}_0$ takes the form:

$$T_{\mu\nu} = \{ Q, \Lambda_{\mu\nu} + \Lambda^{e.A}_{\mu\nu} \} + \frac{1}{2} g_{\mu\nu} T^{e.A}, \quad (29)$$

where $\Lambda_{\mu\nu}$ corresponds to the part of topological Yang-Mills [4], $\Lambda^{e.A}_{\mu\nu}$ is the one given in (14), and,

$$T^{e.A} = H^\alpha \nabla_\gamma \nabla_\alpha \overline{H}^\gamma + \overline{H}^\alpha \nabla_\alpha \nabla_\beta \overline{H}^\beta + i \overline{\psi}^\beta \nabla_\alpha \psi^e - i \overline{\psi}^\alpha \nabla_\alpha \psi^e - i \nabla_\alpha \psi^\alpha \phi^e.$$

(30)

This last quantity vanishes on-shell. However, since the action of the theory is not $Q$-exact one cannot ignore those terms when computing the dependence of vacuum expectation values on the vierbein. For an arbitrary variation of the vierbein one finds,

$$\frac{\delta}{\delta e_{\alpha\mu}} \langle \prod \mathcal{O}^{(\gamma)} \rangle = \frac{1}{2} e e_{\alpha\mu} \langle \prod \mathcal{O}^{(\gamma)} T^{e.A} \rangle \quad (31)$$

where $\prod \mathcal{O}^{(\gamma)}$ denotes an arbitrary product of the operators (24).

Equation (31) indicates that when coupling topological matter to topological Yang-Mills the manifest topological character of the theory is lost. It is important to study the properties of the vacuum expectation value in the right hand side of (31). For example, it would be interesting to characterize the topologies for which it vanishes. If the right hand side of eq. (31) vanishes this theory leads to a set of invariants which are richer than Donaldson invariants since they are also labeled by the $SU(2)$ representation carried by the matter fields.

Donaldson invariants are polynomials on $H_*(M) \times H_*(M) \times \ldots \times H_*(M)$, i.e., products of the homology groups of $M$. The vacuum expectation values of the observables (24) evaluated in the coupled theory presented in this paper are also
polynomials on $H_\ast(M) \times H_\ast(M) \times \cdots \times H_\ast(M)$. Since the action is $Q$-invariant the quantities $\langle \prod \mathcal{O}^{(\gamma)} \rangle$ are invariant under deformations of the cycles $\gamma$. The argument is the standard one. Since a deformation of $\gamma$ leads to a $Q$-exact deformation of $\mathcal{O}^{(\gamma)}$, the corresponding vacuum expectation value vanishes due to the $Q$-invariance of the action. This argument holds also in the case $m \neq 0$, so even in this case the observables can be regarded as polynomials on $H_\ast(M) \times H_\ast(M) \times \cdots \times H_\ast(M)$.

The observables computed with the action $\mathcal{L}^{\text{DW}} + \mathcal{L}_0$ are polynomials on $H_\ast(M) \times H_\ast(M) \times \cdots \times H_\ast(M)$ which depend on the representation of $SU(2)$ which have been chosen for the matter fields. The resulting quantities may not be topological invariants but still be interesting quantities. Whether or not these quantities may help in the study of four-dimensional spin manifolds is an open question. If the observables were computed with $\mathcal{L}^{\text{DW}} + \mathcal{L}_0 + m\mathcal{L}_m$ a dependence on $m$ would be introduced and the resulting equation (31) would possess additional terms.

The breaking of the topological symmetry which may occur in this theory would imply that the theory possess propagating modes. The breaking is caused by the interaction since in the absence of coupling one is left with two theories which are topological. Many questions should be answered from this point of view. For example, one would like to know if the degeneracy of the vacuum of Donaldson-Witten theory is lifted by the interaction, or if the breaking leads to the presence of physical degrees of freedom.

In a forthcoming paper we will study the theory presented in this letter in full detail. We will analyze its symmetries and its features from both a physical and a mathematical point of view. Also, it would be very interesting to construct different types of topological matter based on other $N = 2$ multiplets like, for example, the relaxed hypermultiplet [15].

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REFERENCES

1. E. Witten, *Comm. Math. Phys.* **118**(1988), 411
2. J.M.F. Labastida, M. Pernici and E. Witten, *Nucl. Phys.* **310**(1988), 611
3. J.M.F. Labastida and P.M. Llatas, *Nucl. Phys.* **379**(1992), 220
4. E. Witten, *Comm. Math. Phys.* **117**(1988), 353
5. P. Fayet, *Nucl. Phys.* **B113**(1976), 135
6. M.F. Sohnius, *Nucl. Phys.* **B138**(1978), 109
7. A. Karlhede and M. Roček, *Phys. Lett.* **B212**(1988), 51
8. J. Yamron, *Phys. Lett.* **B213**(1988), 325
9. J.M.F. Labastida and P.M. Llatas, *Phys. Lett.* **B271**(1991), 101
10. C. Vafa, *Mod. Phys. Lett.* **A6**(1991), 337
11. E. Witten, “Mirror Manifolds and Topological Field Theory”, in *Essays on Mirror Manifolds*, ed. S.-T. Yau (International Press, 1992)
12. C. Vafa, “Topological Mirrors and Quantum Rings”, in *Essays on Mirror Manifolds*, ed. S.-T. Yau (International Press, 1992)
13. R. Jackiw and C. Rebbi, *Phys. Rev.* **D16**(1977), 1052
14. J.M.F. Labastida and M. Pernici, *Phys. Lett.* **B212B**(1988), 56
15. P.S. Howe, K.S. Stelle and P.K. Townsend, *Nucl. Phys.* **B214**(1983), 519