Operator Product Expansion and Calculation of the Two-Loop Gell-Mann-Low Function

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Abstract

A simple method is developed that makes it possible to determine the $k$-loop coefficient of the $\beta$-function if the operator product expansion for certain polarization operators in the $(k-1)$ loop is known. The calculation of the two-loop coefficient of the Gell-Mann-Low function becomes trivial – it reduces to a few algebraic operations on already known expressions. As examples, spinor, scalar, and supersymmetric electrodynamics are considered. Although the respective results for $\beta^{(2)}$ are known in the literature, both the method of calculation and certain points pertaining to the construction of the operator product expansion are new.
1 Introduction

The present paper is devoted to the calculation of the charge renormalization in gauge theories. Although the history of the problem extends over decades, we should like to propose a method for the determination of $\beta$-functions which, in our view, is of interest both for its simplicity and for certain other merits. The main idea is as follows. By separating out the integration over one of the virtual lines in the graphs for the effective Lagrangian in the external field method, we reduce the problem of the $k$-loop $\beta$-function to the construction of an operator product expansion in the $(k - 1)$ loop. More precisely, since we are studying the effective charge, in the operator product expansion we need only one term, proportional to the square of the field strength tensor of the external field. The calculation of $\beta^{(2)}$ (the second coefficient in the Gell-Mann-Low function) becomes extremely simple. If we make use of certain already known results, the determination of $\beta^{(2)}$ reduces to a few purely algebraic operations that, in essence, do not require even a single integration.

The resulting expression for the effective action has from the outset a one-logarithm form. Therefore, accuracy in the ultraviolet-regularization procedure is not required.

At present, dimensional regularization (dimensional reduction in supersymmetric theories) is most often used for this purpose. The proposed method makes it possible to work directly in four-dimensional space.

As examples we find $\beta^{(2)}$ in spinor, scalar, and supersymmetric electrodynamics. The treatment of the latter two cases is also of interest in that new points in such a well studied problem as the construction of the operator product expansion (OPE) are rather unexpectedly revealed.

We start from the operator product expansion for the polarization operator

$$\Pi_{\mu\nu}(k) = i \int e^{ikx} d^4x \langle T\{J_\mu(x)J_\nu(0)\}\rangle,$$

where $J_\mu(x)$ is the electromagnetic current of the scalar particles. It is found that in the one-loop approximation the coefficient of $F_{\mu\nu}^2$ is determined not only by the contribution of virtual momenta $p \sim k$ but also by the region of momenta $p$ of the order of the momenta of the external field $F_{\mu\nu}$. Nevertheless the result obtained in this way is the correct result for the OPE coefficient, since a change in the momentum of the external field begins to affect $C_n$ only on the scale $\sim k$. 


2 General elements

First of all we shall formulate certain points that apply in equal measure to all the problems that will be discussed below. We introduce in this section all the necessary notation and explain the general strategy. The initial Lagrangian has the form

\[ \mathcal{L} = -\frac{1}{4e_0^2} F_{\mu\nu} F^{\mu\nu} + \text{matter} \]  

(1)

where \( e_0 \) is the bare coupling constant (charge), \( F_{\mu\nu} \) is the photon-field strength tensor,

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

and "matter" in (1) correspond to the usual kinetic terms of the matter fields (in the supersymmetric model a Yukawa coupling is added too; see below). The fields in (1) are bare (unrenormalized) fields, i.e., fields normalized at the ultraviolet cutoff mass \( M_0 \).

Next, starting from the Lagrangian (1), we calculate the effective Lagrangian that takes account of virtual fluctuations with momenta \( p \),

\[ \mu \leq p \leq M_0 \]

(2)

where \( \mu \) is a running parameter – the so-called normalization point:

\[ \mathcal{L}_{\text{eff}} = -\frac{1}{4e^2(\mu)} F_{\mu\nu} F^{\mu\nu} + \text{other structures} \]

(3)

The tensor \( F_{\mu\nu} \) in (3) must be regarded as the external field, and the coefficient multiplying \( F_{\mu\nu}^2 \) contains the effective (i.e., normalized at the point \( \mu \)) charge. The Lagrangian \( \mathcal{L}_{\text{eff}} \) is a fully appropriate Lagrangian in respect of fluctuations with frequencies smaller than \( \mu \). After this definition of the charge, the reader familiar with the background-field (external-field) method [2] will understand immediately that it is in the framework of this method that we intend to work.

It is obvious that \( e^2(\mu) \) depends on \( e_0 \) and \( M_0/\mu \). The Gell- Mann-Low function \( \beta(\alpha) \) is defined as

\[ \beta(\alpha) = \frac{\partial \alpha(\mu)}{\partial \log \mu} \Big|_{\alpha_0, M_0 \text{ fixed}} \]

(4)

where \( \alpha = e^2/4\pi \).

In the two-loop approximation

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\[
\frac{1}{\alpha} = \frac{1}{\alpha_0} + \beta^{(1)} \log \frac{M_0}{\mu} + \beta^{(2)} \alpha_0 \log \frac{M_0}{\mu} + \text{terms not containing } M_0, \tag{5}
\]
and, consequently,
\[
\beta(\alpha) = \alpha^2 \left( \beta^{(1)} + \alpha \beta^{(2)} + \ldots \right) \tag{6}
\]
The coefficient $\beta^{(1)}$ is determined by the simplest, one-loop graph and, of course, does not require any commentary (see, e.g., Ref. 1; the results are collected in the Table). Our problem is to find $\beta^{(2)}$ in the simplest possible way.

| Table 1: Coefficients of the Gell-Mann–Low function (defined in (5) and (6)). |
|---------------------------------------------------------------|
| Spinor electrodyn. | Scalar electrodyn. | Supersymmetric electrodyn. |
| $\beta^{(1)}$ | $\frac{2}{3\pi}$ | $\frac{1}{6\pi}$ | $\frac{1}{\pi}$ |
| $\beta^{(2)}$ | $\frac{1}{2\pi^2}$ | $\frac{1}{2\pi^2}$ | $\frac{1}{\pi^2}$ |

### 3 Spinor electrodynamics

The model includes the photon and electron and is described by the Lagrangian
\[
\mathcal{L} = -\frac{1}{4e_0^2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \gamma^\mu D_\mu \psi, \tag{7}
\]
where $\psi$ is the Dirac spinor,
\[
i D_\mu = i \partial_\mu + A_\mu, \tag{8}
\]
and the mass term has been omitted. The effective Lagrangian \[3\] in the two-loop approximation is determined by the graph in Fig. 1, where the solid line denotes the electron propagator in the external photon field. In fact, in the expansion of the propagator in the photon field one needs to keep only terms $O(F_{\mu\nu})$ and $O(F_{\mu\nu}^2)$, since no other terms lead to an $F_{\mu\nu}^2$ structure in $\mathcal{L}_{\text{eff}}$.

Furthermore, it is obvious that the photon Green function in Fig. 1 corresponds to the propagation of a free photon:
\[
D_{\mu\nu}(x) = g_{\mu\nu} \frac{ie_0^2}{4\pi^2} \frac{1}{x^2}. \tag{9}
\]
The corresponding expression for the two-loop $\mathcal{L}_{\text{eff}}$ can be written in the form

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{2} \int d^4x \left(-iD^{\mu\nu}(x)\right) \Pi_{\mu\nu}^{(F^2)}(x),$$

where $\Pi_{\mu\nu}(x)$ is the polarization operator:

$$\Pi_{\mu\nu} = i\langle T\{j_{\mu}(x)j_{\nu}(0)\}\rangle, \quad j_{\mu} = \bar{\psi}\gamma^{\mu}\psi. \quad (11)$$

The superscript $(F^2)$ in $(10)$ means that in the operator expansion for $\Pi_{\mu\nu}$ we are interested only in the single operator $F_{\mu\nu}(0)F^{\mu\nu}(0)$:

$$\Pi_{\mu\nu}(x) = ... + C_{\mu\nu}(x)F^2(0) + ... \quad (12)$$

If we consider $\mathcal{L}_{\text{eff}}$ in two loops, the coefficient $C_{\mu\nu}(x)$ must be calculated in the one-loop approximation. The expression for $C_{\mu\nu}(x)$ in this approximation is well defined – it requires neither infrared nor ultraviolet regularization. In fact, $C_{\mu\nu}(x)$ is obtained by simple multiplication of the two electron propagators $S(0,x)$ and $S(x,0)$. The quantity $\Pi_{\mu\nu}$ appearing in $(10)$ has the form const $\cdot F^2(0)$ $x^{-2}$, and the factor $\log(M_0/\mu)$ (see (5)) arises from the integration over $d^4x$ that is performed at the very end. In fact, according to $(10)$ we have

$$\mathcal{L}_{\text{eff}}^{(2)}(0) \sim \left(\int d^4x \frac{1}{x^4}\right) F^2(0) \sim \left(\log \frac{|x_{\text{min}}|}{|x_{\text{max}}|}\right) F^2(0).$$

The construction of the operator product expansion for $\Pi_{\mu\nu}(x)$ has been discussed repeatedly in the literature in connection with the QCD sum rules [1]. This problem – the determination of the $F^2$ operator in $\Pi_{\mu\nu}$ – has been discussed in full detail in the review Ref. 3 (p. 609). It is instructive, however, to go through this exercise again so that we can then stress those aspects that distinguish spinor electrodynamics from the scalar model and supersymmetric model.

First of all, for the external photon field we use the Fock-Schwinger gauge [2,4,5]

$$x^\mu A_{\mu}(x) = 0. \quad (13)$$

(A review of this technique is given in Ref. 3.) In this gauge the four-potential $A_{\mu}(x)$ is expressed in terms of the fields strength tensor $F_{\mu\nu}$:

$$A_{\mu}(x) = \frac{1}{2 \cdot 0!} x^\rho F_{\rho\mu}(0) + \frac{1}{3 \cdot 1!} x^\alpha x^\rho (\partial_\alpha F_{\rho\mu}(0)) + ... \quad (14)$$
In fact, since we are interested only in the $F^2$ structure in the effective Lagrangian, it is possible (and necessary) to confine ourselves to just the first term of the expansion in the right-hand side of (14). This corresponds to a constant external electromagnetic field. Henceforth, terms with derivatives of $F$ will be consistently omitted.

If we neglect the mass, the electron propagator will have the form

$$S(x,0) = \langle x \mid \frac{1}{\slashed{D}} \mid 0 \rangle = -i\langle T\{\bar{T}\psi(x)\bar{\psi}(0)\} \rangle = \frac{1}{2\pi^2} \frac{\hat{x}^\alpha}{x^4} - \frac{1}{8\pi^2} \frac{x^\alpha}{x^2} \tilde{F}_{\alpha\beta} \gamma^\beta \gamma^5 + ... \quad (15)$$

where $\hat{x} = \gamma^\mu x_\mu$ and the ellipses denote operators that cannot give $F^2$ in $\mathcal{L}_{\text{eff}}$ (e.g., $\partial_\alpha F_{\beta\gamma}$ or $F_{\mu\alpha} F^{\nu\alpha} - \frac{1}{4} \delta^{\nu}_{\mu} F_{\alpha\beta} F^{\alpha\beta}$).

The absence in (15) of a term of the form $(\hat{x}(\log x^2)F_{\alpha\beta}F^{\alpha\beta}$ has an independent theoretical explanation. It follows from the general theorem [5] which states that $S(x,0)$ has no logarithmic singularity in a self-dual external field. Dimensionally, generally speaking, a non-singular term $\hat{x}F_{\alpha\beta}F^{\alpha\beta}$ could arise. For the following analysis it is important that there should not in fact be such a term. We note that the method described, e.g., in Ref. 6 for constructing propagators in an external field fixes only the singular terms. Therefore, in order to elucidate whether or not a term $\hat{x}F^2$ is present in $S(x,0)$ it is convenient to make direct use of the equation for the propagator $S(x,0)$:

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figures.png}
\caption{Fig. 1: Too-loop graph for $\mathcal{L}_{\text{eff}}$ in spinor electrodynamics, Sec. 3. The solid line is the electron propagator in the background field; Fig. 2: Too-loop graph for $\mathcal{L}_{\text{eff}}$ in scalar electrodynamics (Sec. 4). The double solid line is the $\phi$ field propagator in the background field; Fig. 3: Additional contribution due to the photino exchange in supersymmetric electrodynamics (Sec. 5). The single and double solid lines denote the propagators of the spinor and scalar fields, respectively, in the external field.}
\end{figure}
\[ i \gamma^\mu D_\mu S(x, 0) = \delta^{(4)}(x). \] (16)

If we substitute \( A_\mu(x) \) in the form
\[ A_\mu = \frac{1}{2} x^\rho F^\rho_\mu \] (17)

it is easily verified that there is no term \( \hat{x} F^2 \) in \( S(x, 0) \). However, a nonsingular term \( \sim F^2 \) appears and plays an important role in the propagator of a scalar particle. We shall postpone the relevant discussion to Sec. 4.

After these preliminary comments, it will not seem surprising to the reader that the subsequent calculation of \( \mathcal{L}_{\text{eff}} \) amounts to two or three simple algebraic operations. First of all,
\[ \Pi^{(F^2)}_{\mu\nu}(x) = i \text{Tr} \{ \gamma_\mu S^{(F)}(x, 0) \gamma_\nu S^{(F)}(0, x) \} = -i \frac{1}{192 \pi^4} F^2(0) \frac{2 x_\mu x_\nu + x^2 g_{\mu\nu}}{x^4}, \] (18)

where we have taken into account that
\[ S(x, 0) = \gamma^0 S^\dagger(0, -x) \gamma^0; \]
\[ \tilde{F}_{\alpha\beta}(0) \tilde{F}_{\gamma\delta}(0) \rightarrow -\frac{1}{12} F^2(0) \left( g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \right). \]

The result for \( \Pi^{(F^2)}_{\mu\nu} \) turns out to be automatically transverse, as it should be for a \( T \)-product of conserved currents.

It remains to take the last step. Substituting (18) and (9) into (10) and going over to Euclidean space \( (x_0 \rightarrow ix_4) \), we obtain
\[ \mathcal{L}_{\text{eff}}^2 = -\int_{|x|_{\text{min}}}^{|x|_{\text{max}}} d^4 x \frac{e_0^2}{x^4 256 \pi^6} F^\mu_{\nu}(0) F^{\mu\nu}(0) = -\frac{e_0^2}{2^7 \pi^4} \left( \log \frac{M_0}{\mu} \right) F^2(0), \] (19)

where \( |x|_{\text{min}} = M_0^{-1} \) and \( |x|_{\text{max}} = \mu^{-1} \). In terms of \( \beta^{(2)} \) (see (5)) the result (19) obviously reduces to
\[ \beta^{(2)} = \frac{1}{2 \pi^2} \quad \text{(spinor electrodynamics)}, \] (20)

which coincides with the well known expression for the second coefficient of the Gell-Mann–Low function (see, e.g., Ref. 1).
4 Scalar electrodynamics

The matter Lagrangian in the scalar case is \( \Delta L = (D_\mu \phi)^\dagger (D_\mu \phi) \), where the mass term has been omitted. The effective Lagrangian in the two-loop approximation is determined by the graph in Fig. 2.

Now, when we have formulated the main stages of the procedure for the example of spinor electrodynamics, the calculation of \( \beta^{(2)} \) in scalar electrodynamics proceeds considerably faster. We shall not dwell on those points that are the same in the two models.

First of all, the general expressions (10), (11) determining \( L^{(2)}_{\text{eff}} \) remain valid\(^1\); the only difference being that the particle current now has the form

\[
J_\mu = i\phi^\dagger \overset{\leftrightarrow}{D}_\mu \phi = i[\phi^\dagger D_\mu \phi - (D_\mu \phi)^\dagger \phi]. \tag{21}
\]

Next, since a derivative appears in the definition of the current, to calculate \( \Pi_{\mu\nu} \) it is necessary to know the function describing the propagation of a scalar particle from the point \( y \) to the point \( x \) and only after the differentiation can we set \( y = 0 \)\(^2\).

The Green function of the massless scalar field has the form

\[
G(x, y) = \left\langle x \left| \frac{1}{p^2} \right| y \right\rangle = -i\langle T\{\phi(x)\phi^\dagger(y)\}\rangle
\]

\[
= \frac{i}{4\pi^2} \frac{1}{(x-y)^2} + \frac{1}{8\pi^2} \frac{x^\mu y^\rho}{(x-y)^2} F_{\mu\rho}(0)
\]

\[
= \frac{i}{512\pi^2} (x-y)^2 F^2(0) - \frac{i}{384\pi^2} \frac{x^2 y^2 - (xy)^2}{(x-y)^2} F^2(0) + \ldots. \tag{22}
\]

Here ... denotes terms with derivatives of \( F_{\mu\nu} \) and terms of the type

\[
F_{\mu\alpha} F^{\nu\alpha} - \frac{1}{4} \delta_\mu^\nu (F_{\alpha\beta} F^{\alpha\beta}).
\]

\(^1\)We note that the tadpole-type graphs generated by the contact term \( A_\mu A^\mu \phi^\dagger \phi \) does not give a contribution to the \( \beta \) function and for this reason is not considered.

\(^2\)The Fock-Schwinger gauge \( x_\mu A^\mu = 0 \) distinguishes the coordinate origin. Therefore, in gauge non-invariant quantities there is no translational invariance and \( G(x, y) \neq G(x - y, 0) \), where \( G \) is the propagator of the scalar field.
The last term in the right-hand side does not give a contribution to $\mathcal{L}_{\text{eff}}^{(2)}$ since the coefficient in it is proportional to $y^2$. The most important term in (22) is the third, the appearance of which was for us a surprise. In fact, it appears with a coefficient that is nonsingular as $x \to y$, and so is not fixed in the framework of the standard procedure of expanding in the momenta (see Ref. 6). Thus, in this aspect the situation differs radically from that which we obtained for the spinor propagator. In the case of the spinor Green function all the terms necessary for the calculation of $\mathcal{L}_{\text{eff}}^{(2)}$ had singular coefficients and were determined in the framework of the standard OPE procedure (i.e., expansion in singularities).

The fact that $(x - y)^2 F^2$ is present in $G(x, y)$ is easily checked using the equation of motion:

$$-D^2 G(x, y) = \delta^{(4)}(x - y).$$

This exercise becomes especially simple if we set $y = 0$. Then, from general arguments,

$$G(x, 0) = \frac{i}{4\pi^2 x^2} + C x^2 F^2(0) + O(F^3).$$

Here $C$ is a certain constant (in $G(x, 0)$ there can be no term linear in $F$). Then from (23)

$$-D^2 G(x, 0) = \left[ -\left( \frac{\partial}{\partial x^\mu} \right)^2 + (A_\mu(x))^2 + 2i A_\mu \partial_\mu \right] G(x, 0) = \delta^{(4)}(x)$$

where we have made use of the fact that, by virtue of (17), $\partial_\mu A_\mu(x) = 0$. Next, $A_\mu \partial^\mu G(x, 0)$ can be omitted because of the absence in $G(x, 0)$ of a term linear in $F$. Finally, the relation

$$\left\{ -\left( \frac{\partial}{\partial x} \right)^2 + (A_\mu(x))^2 \right\} G(x, 0) = 0, \quad \left( A_\mu^2 = \frac{1}{16} x^2 F^2(0) \right)$$

makes it possible to determine the constant $C$:}

$$C = \frac{i}{512\pi^2}.$$

It is usually assumed that terms nonsingular in $x$ (in momentum space they have the form of a $\delta$ function and derivatives, e.g., $x^2 \leftrightarrow [(\partial/\partial q_\mu)^2 \delta^{(4)}(q)]$) are connected not with small but with large distances. In the present case we shall see that this is not so. The term $(x - y)^2 F^2(0)$, like other terms in (22), comes from short distances, and cannot be dropped from (22) without violating the equations of motion.
Indeed, it is clear that this contribution is important for calculating the part singular in $x$ in $\Pi_{\mu\nu}(x)$. The reason is that in $\Pi_{\mu\nu}(x)$ the part that is polynomial in $x$ is multiplied by the singular propagator of the free scalar field (more precisely, by its derivative). As a result, the product is singular. Although, in principle, the procedure for constructing $\Pi^{(F^2)}_{\mu\nu}(x)$ is the same as for the spinor case analyzed above, there is a slight technical complication associated with the presence of the covariant derivative in the definition (21) of the current. If $y \to 0$, then $D_{\mu y}$ can be assumed to coincide with the ordinary derivative $\partial/\partial y^\mu$. In the one-loop approximation $\Pi_{\mu\nu}(x,y)|_{y\to 0}$ has the form

$$\Pi_{\mu\nu}(x,y) = 2i \left\{ D_{\mu}G(x,y) \left( \partial/\partial y^\nu \right) G(y,x) - D_{\mu}G(x,y) \left( \partial /\partial y^\nu \right) G(y,x) \right\}. \quad (24)$$

If we make use of the expression (17) for $A_{\mu}(x)$, then in $\Pi^{(F^2)}_{\mu\nu}(x, y \to 0)$ two types of contribution arise:

$$\Pi^{(F^2)}_{\mu\nu} = \Pi^{(1)}_{\mu\nu} + \Pi^{(2)}_{\mu\nu},$$

where

$$\Pi^{(1)}_{\mu\nu} = x^\rho F_{\rho\mu}(0) \left[ G(x,y) \frac{\partial}{\partial y^\nu} G(y,x) - G(y,x) \frac{\partial}{\partial y^\nu} G(x,y) \right]_{y=0},$$

$$\Pi^{(2)}_{\mu\nu} = 2i \left[ \frac{\partial}{\partial x^\mu} G(x,y) \frac{\partial}{\partial y^\nu} G(y,x) - \left( \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} G(x,y) \right) G(y,x) \right]_{y=0},$$

$$= - \frac{i}{64\pi^4 x^4} (x_\mu x_\nu) F^2(0), \quad (25)$$

where we have used the explicit expression (22) for the propagator. We stress that $\Pi^{(2)}_{\mu\nu}$ owes its origin entirely to the nonsingular term in (22). If we had not taken the latter into account, we would have obtained a non-transversal result for $\Pi_{\mu\nu}$.

Collecting $\Pi^{(1)}_{\mu\nu}$ and $\Pi^{(2)}_{\mu\nu}$, we arrive at

$$\Pi^{(F^2)}_{\mu\nu} = - \frac{i}{192\pi^4 x^4} (x^2 g_{\mu\nu} + 2x_\mu x_\nu) F^2(0), \quad (26)$$

which satisfies the transversality condition $(\partial/\partial x^\mu) \Pi_{\mu\nu} = 0$ and coincides numerically with the result (18). Therefore, without repeating the subsequent calculations, we conclude that
\[ \beta^{(2)} = \frac{1}{2\pi^2} \]  
(scalar electrodynamics), \hspace{1cm} (27)

in complete agreement with the literature (see, e.g., Ref. 1).

5 Supersymmetric electrodynamics

In superfield notation the action has the form

\[ S_{SUSY\text{QED}} = \frac{1}{4e_0^2} \int d^2\theta d^4 x W^2 + \frac{1}{4} \int d^2\theta d^2\bar{\theta} d^4 x (Te^V T + S e^{-V} S), \]  
(28)

where \( T \) and \( S \) are two left-handed chiral superfields with opposite charges, \( V \) is a real superfield, incorporating the field of the photon and photino, and \( W \) is the stress superfield. In components, the SQED Lagrangian is

\[
\mathcal{L} = -\frac{1}{4e_0^2} F_{\mu\nu}^2 + \frac{i}{e_0} \bar{\lambda} \sigma^\mu \partial_\mu \lambda + \sum_{q=\pm} \left[ (D_\mu \phi_q)^\dagger (D^\mu \phi_q) + \bar{\psi}_q i \sigma^\mu D_\mu \psi_q \right. \\
\left. - i\sqrt{2} \phi_q^\dagger (\lambda \psi_q) + i\sqrt{2} (\bar{\psi}_q \bar{\lambda}) \phi_q \right] - e_0^2 (\phi_+^\dagger \phi_+ - \phi_-^\dagger \phi_-)^2. \]  
(29)

In the language of components the matter sector includes two Weyl spinors \( \psi_\alpha^q, (\alpha = 1, 2) \) with charges \( q = \pm \) which can be combined into a single Dirac spinor \( \Psi \) describing the electron. To each Weyl spinor corresponds its complex scalar field \( \phi_q \). Both the electron and the scalar fields have the usual gauge coupling with the photon \( (D_\mu = \partial_\mu \mp i qA_\mu) \).

The gauge sector contains not only the photon but also the photino (charge 0), described by the Weyl spinor \( \lambda^\alpha \) where \( \alpha = 1, 2 \). The self-interaction of the scalar field, (the square of the \( D \) term) has no effect in the calculation of \( \mathcal{L}_{\text{eff}}^{(2)} \) and appears only in higher orders.

The calculation is conveniently performed by going over to the Dirac spinor

\[
\Psi = \begin{pmatrix} \psi_+ \\ \bar{\psi}_- \end{pmatrix}.
\]

In this notation the vertex of the interaction of \( \lambda \) with \( \Psi \) takes the form

\[ 324 \]
\[ L_{\text{int}} = -\sqrt{2} i \left[ \phi^\dagger_+ \left( \tilde{\lambda} \frac{1 + \gamma^5}{2} \Psi \right) + \phi^\dagger_- \left( \tilde{\lambda} \frac{1 - \gamma^5}{2} \Psi \right) \right] + \text{H.c.}. \] (30)

Thus, in the calculation of \( L^{(2)}_{\text{eff}} \) in supersymmetric electrodynamics it is necessary to take into account all three diagrams depicted in Figs. 1, 2, and 3. (In the diagrams of Figs. 2 and 3 scalar particles of two types \( q = \pm \) are propagating.) We have found the diagrams of Figs. 1 and 2 in the preceding sections.

The only additional contribution is connected with the graph in Fig. 3.

As in the case of photon exchange, we take into account the fact that the photino does not interact directly with the external field. Then \( L^{(2)}_{\text{eff}} \) (Fig. 3) is represented in the form

\[ L^{(2)}_{\text{eff}} \text{(Fig. 3)} = -2i \int_{x_{\text{min}}}^{x_{\text{max}}} d^4 x \left\langle T \left\{ \lambda_\beta(0) \bar{\lambda}(x) \right\} \right\rangle \Pi^{(F^2)}_{\alpha\beta}(x) \] (31)

where the polarization operator \( \Pi^{(F^2)}_{\alpha\beta} \) is defined as

\[ \Pi_{\alpha\beta}(x) = \left\langle T \left\{ \phi(x) \Psi_\alpha(x), \phi(0) \bar{\Psi}_\beta(0) \right\} \right\rangle = i^2 G(0, x) S_{\alpha\beta}(x, 0). \] (32)

In this expression for \( \Pi_{\alpha\beta} \) we have already taken into account both types scalar. We recall that the photino propagator is free and has the form

\[ \left\langle T \left\{ \lambda_\beta(0) \bar{\lambda}(x) \right\} \right\rangle = -\frac{i e_0^2}{2\pi^2} \frac{\hat{x}_{\alpha\beta}}{x^4}. \] (33)

Since the expressions for the propagators \( S(x, 0) \) and \( G(0, x) \) are known (see (115) and (22)), the calculation of \( \Pi^{(F^2)}_{\alpha\beta}(x) \) proceeds trivially:

\[ \Pi^{(F^2)}_{\alpha\beta} = -\frac{i}{1024\pi^4} \frac{\hat{x}_{\alpha\beta}}{x^2} F^2(0). \] (34)

We note that only the \( F^2 \) part in \( G(0, x) \) and the free term in \( S(x, 0) \) have cooperated here. Substituting (34) and (33) into (31) we obtain

\[ L^{(2)}_{\text{eff}} \text{(Fig. 3)} = \frac{e_0^2}{2^7\pi^4} \left( \log \frac{M_0}{\mu} \right) F^2(0). \] (35)

In terms of \( \beta^{(2)} \) (see (5)) the result (35) obviously reduces to

\[ \beta^{(2)} \text{(Fig. 3)} = -\frac{1}{2\pi^2}. \] (36)

Adding now the electron loop and the two loops with the scalars (Fig. 2), we arrive...
at the conclusion that in supersymmetric electrodynamics

$$\beta^{(2)} = \frac{1}{\pi^2} \quad \text{(SUSY QED)}.$$  \hfill (37)

6 Conclusion

The question of the calculation of the second coefficient of the Gell-Mann-Low function arose in connection with the fact that the ratio $\beta^{(2)}/\beta^{(1)}$ in supersymmetric electrodynamics has been obtained recently by a completely different method [7] and it was desirable to compare the prediction of Ref. 7 with direct calculations. Since, unfortunately, we did not succeed in finding standard calculations of $(\beta^{(2)}/\beta^{(1)})_{\text{SUSY QED}}$ in the literature, it was necessary to devise a method that would make it possible to calculate the graphs for $\beta^{(2)}$ within a reasonable interval of time. In this way, we have obtained

$$\beta(\alpha)_{\text{SUSY QED}} = \frac{\alpha^2}{\pi} \left(1 + \frac{\alpha}{\pi} + \ldots\right),$$ \hfill (38)

which is in agreement with the prediction of Ref. 7, according to which

$$\beta(\alpha)_{\text{SUSY QED}} = \frac{\alpha^2}{\pi} (1 + \gamma_m) = \frac{\alpha^2}{\pi} \left(1 + \frac{\alpha}{\pi} + \ldots\right),$$

where $\gamma_m$ is the anomalous mass dimension. The above calculation of $\beta^{(2)}$, in our opinion, convincingly demonstrates the effectiveness of the method of calculating the coefficients $\beta^{(i)}$ by means of the operator-product-expansion method. The two-loop calculation is maximally simplified, since it uses prepared blocks, i.e., a simple multiplication of known propagators takes place. In addition, it is not necessary to display particularly high accuracy in the determination of the regularization procedure.

A new aspect for us was the necessity of taking account of terms nonsingular in $x$ in the propagator of the scalar field. Formally, this corresponds to the region of virtual momenta of the order of the momenta of the external field, although the result for the polarization operator corresponds to the normal OPE. Further discussion of this question – the relationship between the Wilson operator product expansion and the analysis performed in scalar and supersymmetric electrodynamics – will be given in a separate publication [8].

*Translated from Russian by P. J. Shepherd*
Addendum to section 4, 1992, from [9]

An important aspect to be elucidated now is the interpretation of the non-singular terms in the operator product expansion discussed in Sec. 4 (see Eq. (23) and below). Can one find a place for such terms within the consistent procedure of separation of short- and large-distance contributions?

To answer this question let us turn to the consideration of $\Pi_{\mu\nu}$ in scalar QED. The puzzling term $x^2 F^2$ in $G(x, 0)$ below Eq. (23) corresponds to the diagram presented in Fig. 4a. Since the propagation function in the lower part of the graph is proportional to $\delta''(p)$, following the ideology of OPE we must actually cut the lower line and then the upper line shrinks to a point (Fig. 4b). Thus, the calculation of Fig. 4a is a two-step process. First, within the standard OPE approach, we calculate the coefficient in front of the operator $(D_\mu \phi)^\dagger(D^\mu \phi)$ (see Fig. 4b) – this coefficient is determined entirely by short distances – and, then, the conversion of $(D_\mu \phi)^\dagger(D^\mu \phi)$ into $F_{\mu\nu}^2$ which can be ascribed to large distances. In other words, the second step is obviously the calculation of a photonic matrix element of $(D_\mu \phi)^\dagger(D^\mu \phi)$.

![Diagram](image_url)

Fig. 4. Figure caption. Operator product expansion associated with the non-singular part in Eq. (22) (see Sec. 3) and the emergence of the condensate (39), (40).

The last term in eq. (22) lead to the “normal” OPE for $\Pi_{\mu\nu}^{(F^2)}$ with the coefficient function determined entirely by short distances.

Formally, the operator $\int d^3 x (D_\mu \phi)^\dagger(D^\mu \phi)$ vanishes because of the equations of motions. One can readily convince oneself, however, that in the external gauge field there is an “anomalous” relation

$$\langle (D_\mu \phi)^\dagger(D^\mu \phi) \rangle$$

stemming from the first term in Eq. (22).
The full Green function \( G(x, y) \) certainly satisfies the equation of motion
\[
-D^2 G(x, y) = \delta^4(x - y) .
\]
However, in calculating \( \Pi^{\mu\nu}_{(F^2)} \) we split it in two pieces: the piece singular in \( x \) is used for determination of the coefficient function while the piece regular in \( x \), \( G^{\text{reg}}(x, y) \), is interpreted as a matrix element. Then \( \langle (D_\mu \phi)^\dagger(D^\mu \phi) \rangle \neq 0 \). More specifically, Eq. (22) implies
\[
\langle (D_\mu \phi)^\dagger(D^\mu \phi) \rangle = \lim_{x \to 0} \left( \frac{-i}{64\pi^2} \frac{F_{\alpha\beta}F^{\alpha\beta}}{x^4} \right) = 1 .
\]
Now we are finally able to explain the difference between our first result for \( \Pi^{\mu\nu}_{\mu\nu} \) quoted in the second line in (25) and the complete answer presented in Eq. (26). The former expression has been obtained by substituting in \( \Pi^{\mu\nu}_{\mu\nu} \) the singular part of \( G(x, y) \). Thus, it corresponds, in the language of OPE, to the genuine contribution of the operator \( F^2 \).

One should not forget, however, another \( \text{dim} = 4 \) operator, \( (D_\mu \phi)^\dagger(D^\mu \phi) \), whose matrix element in the background electromagnetic field reduces to the same structure \( F^2 \) (see Eq. (40)). The coefficient of \( (D_\mu \phi)^\dagger(D^\mu \phi) \) can be trivially extracted from the diagrams in Fig. 5. With no effort we arrive at
\[
\Pi^{\mu\nu}_{\mu\nu}(q) = -2 \left( \frac{g^{\mu\nu}}{q^2} - 2 \frac{q_\mu q_\nu}{q^4} \right) \langle (D_\mu \phi)^\dagger(D^\mu \phi) \rangle .
\] (41)

Invoking Eq. (D.2) in Ref. [3] for the Fourier transformation and Eq. (40) we then find that the contribution of Fig. 4b is equal to
\[
\frac{-i}{64\pi^2} \frac{x_\mu x_\nu}{x^4} F^2(0) ,
\] (42)
precisely the difference between the expression in the second line of (25) and Eq. (26).

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