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$p$-adic analytic interpolation

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Abstract. Let $K$ be a complete ultrametric algebraically closed field. We study the Kernel of infinite van der Monde Matrices and show close connections with the zeroes of analytic functions. We study when such a matrix is invertible. Finally we use these results to obtain interpolation processes for analytic functions. They are more accurate if $K$ is spherically complete.

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1. NOTATIONS, DEFINITIONS AND THEOREMS

$K$ denotes an algebraically closed field complete for an ultrametric absolute value. Given $a \in K$, $r > 0$, we denote by $d(a, r)$ (resp. $d(a, r^-)$) the disk $\{x \in K : |x - a| \leq r\}$ (resp. $\{x \in K : |x - a| < r\}$). Given $r > 0$ we denote by $C(0, r)$ the circle $d(0, r) \setminus d(0, r^-)$.

Given $r_1, r_2 \in \mathbb{R}_+$ such that $0 < r_1 < r_2$, we denote by $\Gamma(0, r_1, r_2)$ the set $d(0, r_2^-) \setminus d(0, r_1)$.

Given $r > 0$, we denote by $A(d(0, r^-))$ the algebra of the power series $\sum_{n=0}^{\infty} b_n x^n$ converging for $|x| < r$.

Given $K$-vector spaces $E$, $F$, $\mathcal{L}(E, F)$ will denote the space of the $K$-linear mappings from $E$ into $F$.

$\mathcal{E}$ will denote the $K$-vector space of the sequences in $K$, and $\mathcal{E}_0$ will denote the subspace of the bounded sequences. The identically zero sequence will be denoted by $(0)$.

$\mathcal{E}_1$ will denote the set of the sequences $(a_n)$ such that $\limsup_{n \to \infty} \sqrt[n]{|a_n|} \leq 1$. So $\mathcal{E}_1$ is seen to be a subspace of $\mathcal{E}$ isomorphic to the space $A(d(0, 1^-))$, and obviously contains $\mathcal{E}_0$.

Let $M_\infty$ be the set of the infinite matrices $(\lambda_{i,j})$ with coefficients in $K$.

$\delta_{i,j}$ will denote the Kronecker symbol. $I_\infty$ will denote the infinite identical matrix defined as $\lambda_{i,j} = \delta_{i,j}$.

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In this paper, \((a_n)\) will denote an injective sequence in \(d(0,1^-)\) such that \(a_n \neq 0\) for every \(n > 0\). and we will denote by \(M(a_n)\) the infinite matrix \(M = (\lambda_{i,j})\) defined as 
\[
\lambda_{i,j} = (a_i)^j, \quad (i,j) \in \mathbb{N} \times \mathbb{N}.
\]
A matrix \(M = (\lambda_{i,j}) \in M_\infty\) will be said to be \textit{bounded} if there exists \(A \in \mathbb{R}_+\) such that \(|\lambda_{i,j}| \leq A\) whenever \((i,j) \in \mathbb{N} \times \mathbb{N}\).

\(M\) will be said to be \textit{line-vanishing} if for each \(i \in \mathbb{N}\), we have \(\lim_{j \to \infty} \lambda_{i,j} = 0\).

A line-vanishing matrix \(M\) is seen to define a \(K\)-linear mapping \(\psi_M\) from \(E_0\) into \(E\).

So the matrix \(M = M(a_n)\) clearly defines a \(K\)-linear mapping \(\psi_M\) from \(E_1\) into \(E\), because given a sequence \((b_n) \in E_1\), the series \(\sum_{n=0}^{\infty} b_n(a_j)^n\) is obviously convergent.

Lemmas 1 and 2 are immediate:

**Lemma 1:** Let \(M \in M_\infty\) be line vanishing.

The three following statements are equivalent:

- \(\psi_M\) is continuous
- \(\psi_M\) is an endomorphism of \(E_0\)
- \(M\) is bounded.

In particular, Lemma 1 applies to matrices of the form \(M(a_n)\).

**Lemma 2:** Let \(M = M(a_n)\) and let \((b_n) \in E_1\). Then \((b_n)\) belongs to \(\text{Ker} \phi_M\) if and only if the analytic function \(f(t) = \sum_{n=0}^{\infty} b_n t^n\) admits each point \(a_j\) for zero.

**Theorem 1:** Let \(M = M(a_n)\). Then \(\text{Ker} \phi_M \neq \{(0)\}\) if and only if \(\lim_{n \to \infty} |a_n| = 1\).

Besides \(\text{Ker} \psi_M \neq \{(0)\}\) if and only if \(\prod_{n=0}^{\infty} |a_n| > 0\).

**Theorem 2:** Let \(b = (b_n) \in E_0\). There exists an injective sequence \((a_n)\) in \(d(0,1^-)\) such that \(b \in \text{Ker} \psi_{M(a_n)}\) if and only if \(b\) satisfies \(|b_j| < \sup_{n \in \mathbb{N}} |b_n|\) for all \(j \in \mathbb{N}\).

**Definitions and notations:** An injective sequence \((a_n)\) in \(d(0,1^-)\) will be called a \textit{regular sequence} if \(\inf_{n \neq m} |a_n - a_m| > 0\) and \(\lim_{n \to \infty} |a_n| = 1\).

Let \((a_n)\) be a regular sequence and let \(\rho = \inf_{n \neq m} |a_n - a_m|\). For every \(r \in ]0,1[,\) we will denote by \(\Omega((a_n), r)\) the set \(d(0,1^-) \setminus \left( \bigcup_{n \in \mathbb{N}} d(a_n, r^-) \right)\), and by \(\Omega(a_n)\) the set \(d(0,1^-) \setminus \left( \bigcup_{n \in \mathbb{N}} d(a_n, \rho^-) \right)\).
Let \( a = (a_n) \) and \( b = (b_n) \) be two sequences in \( K \). We will denote by \( a \ast b \) the convolution product \( (c_n) \) defined as \( c_n = \sum_{j=0}^{n} a_j b_{n-j} \).

**Theorem 3:** Let \( (\alpha_n) \) be a regular sequence of \( d(0,1^-) \) such that there exists \( g \in A(d(0,1^-)) \) satisfying

(i) \( \alpha_n \) is a zero of order 1 of \( g \) for all \( n \in \mathbb{N} \).
(ii) \( g(x) \neq 0 \) whenever \( x \in d(0,1^-) \) \( \backslash \{ \alpha_n : n \in \mathbb{N} \} \).
(iii) \( \lim_{x \to 0, x \in \partial d(\alpha_n)} |g(x)| = +\infty \).

Let \( M = M(\alpha_n) \). Then \( \psi_M \) is injective but its image does not contain \( \mathcal{E}_0 \). Also there exists \( P = (\lambda_{i,j}) \in \mathcal{M}_\infty \) (not unique) satisfying

(1) \( P \) is line-vanishing.
(2) \( \lim_{n \to \infty} \lambda_{n,j} \alpha_h^n = 0 \) for all \( (j, h) \in \mathbb{N} \times \mathbb{N} \).
(3) \( \sum_{n=0}^{\infty} \lambda_{n,j} \alpha_h^n = \delta_{j,h} \) for all \( (j, h) \in \mathbb{N} \times \mathbb{N} \).
(4) \( MP = PM = I_\infty \).
(5) \( P(b) \in \mathcal{E}_1 \) for all \( b \in \mathcal{E}_0 \).
(6) \( MP(b) = b \) for all \( b \in \mathcal{E}_0 \).
(7) \( \psi_P \) is injective.

Let \( (\nu_n) \) be a sequence in \( K \) such that \( |\nu_0| \geq |\nu_n| \) for every \( n > 0 \). For every \( j \in \mathbb{N} \), let

\( (\mu_{n,j})_{n \in \mathbb{N}} \) denote the sequence \( \left( \frac{1}{\sum_{m=0}^{\infty} \nu_m \alpha_j^m} \right) ((\lambda_{n,j} \ast (\nu_n)) \). Then the matrix \( Q = (\mu_{i,j}) \) also satisfies properties (1) – (7) and is not equal to \( P \) for infinitely many sequences \( (\nu_n) \).

**Remarks.** 1. Mainly, the proof of Theorem 3 takes inspiration from that of Lemma 3 in [7]. However, in this lemma, the considered matrix, roughly, was \( P \). Here the matrix we consider is a van der Monde matrix \( M \) and we look for \( P \).

2. Given \( M \), the matrix \( P \) depends on \( g \) and therefore is not unique satisfying (1) – (7). Indeed \( \mathcal{M}_\infty \) is not a ring because the multiplication of matrices is not always defined and even when it is defined, is not always associative. As a consequence, if \( P, P' \) satisfy \( MP = MP' = PM = P'M = I_\infty \), we cannot conclude \( P' = P \).

Actually we can consider \( \phi_M \circ \psi_P \in \mathcal{L}(\mathcal{E}_0, \mathcal{E}) \) and then this is the identity in \( \mathcal{E}_0 \). Next we can consider \( \psi_P \circ \psi_M \in \mathcal{L}(\mathcal{E}_0, \mathcal{E}_1) \) and this is the identity in \( \mathcal{E}_0 \). But we cannot consider \( \psi_P \circ (\phi_M \circ \psi_P) \) because \( \psi_P \) is not defined in \( \mathcal{E}_1 \). In the same way, we cannot consider \( (\psi_P \circ \psi_M) \circ \psi_P \) because \( \psi_P \circ \psi_M \) is only defined in \( \mathcal{E}_0 \).

We consider the matrix \( P \) and look for "inverses" \( M \) such that \( MP = PM = I_\infty \). Suppose that there exists a bounded matrix \( M' \neq M \) such that \( PM' = M'P = I_\infty \). Now we can consider \( \phi_{M'} \circ (\psi_P \circ \psi_M) \in \mathcal{L}(\mathcal{E}_0, \mathcal{E}) \). Since \( \psi_P \circ \psi_M \) is the identity in \( \mathcal{E}_0 \), then \( \phi_{M'} \circ (\psi_P \circ \psi_M) \) is equal to \( \psi_{M'} \). Next we can consider \( (\phi_{M'} \circ \psi_P) \circ \psi_M \in \mathcal{L}(\mathcal{E}_0, \mathcal{E}) \). Since
\( \phi_{M'} \circ \psi_P \) is the identity on \( \mathcal{E}_0 \), we have \( (\phi_{M'} \circ \psi_P) \circ \psi_M = \psi_M \) and therefore \( \psi_M = \psi_{M'} \), hence \( M = M' \).

3. Let \( P, Q \in \mathcal{M}_\infty \) satisfy (1) – (7). Let \( \mathcal{E}' = \psi_P(\mathcal{E}_0) \), let \( \mathcal{E}'' = \psi_Q(\mathcal{E}_0) \). Then the restriction of \( \phi_M \) to \( \mathcal{E}' \) (resp. \( \mathcal{E}'' \)) is just the reciprocal of \( \psi_P \) (resp. \( \psi_Q \)).

**Conjecture.** Under the hypothesis of Theorem 1, every matrix satisfying properties (1) – (7) is of the form

\[
\mu_{n,j} = \left( \frac{1}{\sum_{m=0}^{\infty} \nu_m \alpha_{n,m}^j} \right) ((\lambda_n, \nu_n))
\]

**Theorem 4:** Let \( K \) be spherically complete, and let \( (\alpha_n) \) be a sequence in \( d(0,1^-) \) satisfying \( |\alpha_n - \alpha_m| \geq \min(|\alpha_n|,|\alpha_m|) \) whenever \( n \neq m \), \( \lim_{n \to -\infty} |\alpha_n| = 1 \), and \( \prod_{n=0}^{\infty} |\alpha_n| = 0 \).

Then \( \mathcal{M}(\alpha_n) \) admits inverses \( P \) such that, for every bounded sequence \( b := (b_n) \) in \( K \), the sequence \( a := (a_n) = P(b) \) defines a function \( f(x) = \sum_{n=0}^{\infty} a_n x^n \in A(d(0,1^-)) \) satisfying \( f(\alpha_n) = b_n \).

**Theorem 5:** Let \( (\alpha_n) \) be a regular sequence in \( d(0,1^-) \). There exists a regular sequence \( (\gamma_n) \in d(0,1^-) \) such that \( (\alpha_n) \) is a subsequence of \( (\gamma_n) \) satisfying: for every inverse matrix \( P \) of \( \mathcal{M}(\gamma_n) \) and for every bounded sequence \( b = (b_n) \) of \( K \), the sequence \( a = P(b) := (a_n) \) defines an analytic function \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) such that \( f(\gamma_j) = b_j \) whenever \( j \in \mathbb{N} \).

2. **PROVING THEOREMS 1 AND 2.**

For each set \( D \) in \( K \), we denote by \( H(D) \) the set of the analytic elements in \( D \) (i.e., the completion of the set of the rational functions with no pole in \( D \)).

Given \( f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-)) \), one defines the valuation function \( v(f,\mu) \) in the interval \( ]0, +\infty[ \) as \( v(f,\mu) = \inf_{n \in \mathbb{N}} (v(b_n) + n\mu) \).

**Lemma 3** Let \( f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-)) \). For every \( \mu > 0 \), \( f \) satisfies

\[
v(f,\mu) = \lim_{v(t) \to \mu, v(t) \neq \mu} v(f(t)). \text{ For every } x \in d(0,1^-), \text{ } f \text{ satisfies } v(f(x)) \geq v(f, v(x)).
\]

**Lemma 4** For every \( r \in ]0,1[ \), \( f \) satisfies \( -\log \|f\|_{d(0,r)} = v(f, -\log r) \).

Besides \( f \) is bounded in \( d(0,1^-) \) if and only if the sequence \( (b_n) \) belongs to \( \mathcal{E}_0 \). If \( f \) is bounded in \( d(0,1^-) \), then \( \|f\|_{d(0,1^-)} = \sup_{n \in \mathbb{N}} |b_n| \) and \( -\log \|f\|_{d(0,1^-)} = \lim_{\mu \to 0} v(f,\mu) \).
Lemma 4: Let $f(t) \in A(d(0,1^-))$ and let $r_1, r_2 \in (0,1)$ satisfy $r_1 < r_2$. If $f$ admits $q$ zeros in $d(0,r_1)$ (taking multiplicities into account) and $t$ distinct zeros $\alpha_1, \ldots, \alpha_t$, of multiplicity order $\zeta_j$ $(1 \leq j \leq t)$ respectively in $\Gamma(0, r_1, r_2)$, then $f$ satisfies

$$v(f, -\log r_2) - v(f, -\log r_1) = -\sum_{j=1}^{t} \zeta_j (v(a_j) + \log r_2) - q(\log r_2 - \log r_1).$$

Proof of Theorem 1. Let $b = (b_n) \in \mathcal{E}_1 \backslash \{(0)\}$ and let $f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-))$.

First we suppose $\ker \phi_M \neq \{(0)\}$ and therefore we can assume $b \in \ker \phi_M$. Then, by Lemma 2, $f$ satisfies $f(\alpha_j) = 0$ for every $j \in \mathbb{N}$. But for every $r \in ]0,1[\), we know that $f$ belongs to $H(d(0,r))$ and has finitely many zeros in $d(0,r)$. Hence we have $\lim_{n \to \infty} |a_n| = 1$.

Reciprocally, let the sequence $(a_n)$ satisfy $\lim_{n \to \infty} |a_n| = 1$. By Proposition 5 in [4], we know that there exists a not identically zero analytic function $f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-))$ which admits each $\alpha_j$ as a zero. Hence we have $\sum_{n=0}^{\infty} b_n a_j^n = 0$, and of course the sequence $(b_n)$ belongs to $\mathcal{E}_1$, hence to $\ker \phi_M$.

Now we suppose that $\ker \psi_M \neq (0)$ and we assume that the sequence $(b_n)$ belongs to $\ker \psi_M$. In particular $\ker \phi_M \neq (0)$ and therefore $\lim_{n \to \infty} |a_n| = 1$. Without loss of generality we may clearly assume $|a_n| \leq |a_{n+1}|$ for all $n \in \mathbb{N}$. Besides, by definition we have $|a_1| > 0$.

By Lemma 3 we know that $\inf_{n \in \mathbb{N}} v(b_n) = \lim_{\mu \to 0^+} v(f,\mu) = \lim_{|z| \to 0^-} f(z) = -\log \|f\|_{d(0,1^-)}$. Now for each $\mu > 0$, let $q(\mu)$ be the unique integer such that $v(a_n) \geq \mu$ for every $n \leq q(\mu)$ and $v(a_n) < \mu$ for every $n > q(\mu)$. By Lemma 4, we check

$$v(f, \mu) - v(f, v(a_1)) \leq \sum_{j=2}^{q(\mu)} \mu - v(a_j) + 2(\mu - v(a_1)).$$

Since $v(f,\mu)$ is bounded when $\mu$ approaches 0, by (1) it is seen that $\sum_{j=1}^{\infty} v(a_j)$ must be bounded and therefore we have $\prod_{n=1}^{\infty} |a_n| > 0$.

Reciprocally we suppose $\prod_{n=1}^{\infty} |a_n| > 0$. We can easily check that $\lim_{n \to \infty} |a_n| = 1$, and then we can assume $|a_n| \leq |a_{n+1}|$ for all $n \in \mathbb{N}$ without loss of generality. For each
$j \in \mathbb{N}$ we put $P_j(x) = \prod_{m=1}^j (1 - x/a_m)$. By Theorem 1 in [2], we can check that there exists $f \in A(d(0,1^-))$ ($f$ not identically zero) satisfying

1. $f(a_m) = 0$ for all $m \in \mathbb{N}$, and
2. $v(f, \mu) \geq v(P_{q(\mu)}, \mu) - 1$ for all $\mu > 0$.

Now we notice that if $\mu_1 > \mu_2 > 0$ then we have $v(P_{q(\mu_1)}, \mu_1) = v(P_{q(\mu_2)}, \mu_1)$ and then we see that $\lim_{\mu \to 0^+} v(P_{q(\mu)}, \mu) = \sum_{j=1}^\infty v(a_j)$. But by (2) we have $\sum_{j=1}^\infty v(a_j) < +\infty$ and therefore by (4), $v(f, \mu)$ is bounded in $]0, +\infty[$. Let $f(t) = \sum_{n=0}^\infty b_n t^n$. By Lemma 3 the sequence $(b_n)$ is bounded and by (3) it clearly belongs to Ker.$\psi_M$. This finishes the proof of Theorem 1.

Lemma 5: Let $f(t) = \sum_{n=0}^\infty b_n t^n \in A(d(0,1^-))$ and let $r \in (0,1)$. Then $f$ admits at least one zero in $C(0,r)$ if and only if there exist $k,l \in \mathbb{N}$ ($k < l$) such that $|b_k| r^k = |b_l| r^l$.

Proof of Theorem 2. As a consequence of Lemma 5, a function $f(t) = \sum_{n=0}^\infty b_n t^n \in A(d(0,1^-))$ admits infinitely many zeros in $d(0,1^-)$ if and only if $|b_j| < \sup_{n \in \mathbb{N}} |b_n|$ for every $j \in \mathbb{N}$. Then the conclusion comes from Lemma 2.

3. PROVING THEOREM 3.

As an application of Corollary (of Theorem 5) in [8], we have this lemma.

Lemma 6: Let $f \in A(d(0,1^-))$ have a regular sequence of zeros $(b_n)$ and satisfy

$$\lim_{\substack{|x| \to 1^- \\ x \in \Omega(b_n)}} |f(x)| = +\infty.$$ Then $1/f$ belongs to $H(\Omega(b_n))$.

Proof of Theorem 3. We may obviously assume $|\alpha_n| \leq |\alpha_{n+1}|$ and therefore $\alpha_n \neq 0$ whenever $n > 0$. Since $g$ is not bounded in $d(0,1^-)$, by Lemma 3 we have $\lim_{\mu \to 0^+} v(g, \mu) = -\infty$, and by Lemma 4 the sequence of the zeros $(\alpha_n)$ satisfies $\prod_{n=1}^\infty |\alpha_n| = 0$, hence $\psi_M$ is injective.

Now we look for $P$. Since $g$ admits each $\alpha_j$ as a simple zero, it factorizes in $A(d(0,1^-))$ in the form $\psi_j(x)(1 - x/\alpha_j)$ and we have $\psi_j(\alpha_j) \neq 0$. We put $g_j(x) = \frac{\psi_j(x)}{\psi_j(\alpha_j)}$. Then $g_j$ belongs to $A(d(0,1^-))$ and may be written as $\sum_{n=0}^\infty \lambda_{n,j} x^n$. We denote by $P$ the matrix
and we will show that satisfies Properties (1) - (7).

For convenience, we put \( D = \Omega(\alpha_n) \). Since \( \lim_{|x|^{-1}} |g(x)| = +\infty \), by Lemma 6, we know that \( 1/g \) belongs to \( H(D) \). For each \( n \in \mathbb{N} \), we put \( u_n = x^n/g \). Then in \( H(D) \), \( u_n \) has a Mittag-Leffler series ([3], [5]) of the form \( \sum_{j=0}^{\infty} \frac{\beta_{j,n}}{1 - x/\alpha_j} \). Now we put \( \theta_j = \psi_j(\alpha_j) \) and we have \( g(x) = \theta_j g_j(x)(1-x/\alpha_j) \). We will compute the \( \beta_{j,n} \). Let \( \nu_{j,n} = (1-x/\alpha_j)u_n \). Then we have \( \nu_{j,n}(\alpha_i) = \frac{\alpha_i^n}{g_j(\alpha_j) \theta_j} \). But since \( g_j(\alpha_j) = 1 \) whenever \( j \in \mathbb{N} \), we see that \( \beta_{j,n} = \alpha_j^n/\theta_j \), hence \( x^n g(x) = \sum_{j=0}^{\infty} \frac{\alpha_j^n}{\theta_j (1 - x/\alpha_j)} \). We notice that \( \| \frac{\alpha_j^n}{1 - x/\alpha_j} \|_D = \frac{|\alpha_j|^n+1}{\rho} \) and then we have \( \lim_{j \to \infty} |\theta_j| = +\infty \), because the sequence of the terms \( x^n/g(x) \) must tend to 0. Now we have \( x^n = \sum_{j=0}^{\infty} \frac{\alpha_j^n g(x)}{\theta_j (1 - x/\alpha_j)} \), while \( g_j(x) = \frac{g(x)}{\theta_j (1 - x/\alpha_j)} \). Since \( g_j(x) = \sum_{n=0}^{\infty} \lambda_{n,j} x^n \), we obtain

\[
(8) \quad x^n = \sum_{j=0}^{\infty} \alpha_j^n (\sum_{h=0}^{\infty} \lambda_{h,j} x^h).
\]

In particular, (8) holds in every disk \( d(0,r) \) with \( r \in ]0,1[ \). But then we know that \( \| g_j \|_{d(0,r)} = \sup_{\lambda \in \mathbb{N}} |\lambda_{j,h}| r^h \leq \frac{\| \psi_j \|_{d(0,r)}}{|\theta_j|} \). Now, we have \( \| \phi_j \|_{d(0,r)} \leq \| g \|_{d(0,r)} \) as soon as \( |\alpha_i| > r \) because then \( \| 1/(1 - x/\alpha_j) \|_{d(0,r)} = 1 \) and therefore the sequence \( \| \phi_j \|_{d(0,r)} \) is bounded. Then the family \( (\lambda_{h,j} x^h)_{j,h \in \mathbb{N}} \) tends to zero when \( j \) tends to +\( \infty \), uniformly with respect to \( h \). In particular, \( P \) is line-vanishing. For each \( h \in \mathbb{N} \), we put \( s_h = \sup_{h \in \mathbb{N}} |\lambda_{h,j}| \). We will show

\[
(9) \quad \lim_{h \to +\infty} \sup_{h \in \mathbb{N}} s_h^{1/h} \leq 1.
\]

Indeed this is equivalent to show that for every \( r \in ]0,1[ \), we have

\[
(10) \quad \lim_{h \to +\infty} s_h r^h = 0.
\]
Let \( r \in ]0, 1[ \) and let \( \epsilon > 0 \). Since the family \( \{|\lambda_{h,j}| r^h\}_{j,h \in \mathbb{N}} \) tends to zero uniformly with respect to \( h \) when \( j \) tends to \( +\infty \), there clearly exists \( N \) such that \( |\lambda_{h,j}| r^h < \epsilon \) whenever \( j > N \), whenever \( h \in \mathbb{N} \), hence for every \( h \in \mathbb{N} \), we have \( s_h r^h \leq \max_{1 \leq j \leq N} |\lambda_{h,j}| r^h \). But for each fixed \( i \in \mathbb{N} \), we know that \( \lim_{h \to \infty} |\lambda_{h,j}| r^h = 0 \), hence \( \lim_{h \to \infty} \max_{1 \leq j \leq N} |\lambda_{h,j}| r^h = 0 \). This finishes showing (10). Therefore (9) is proven and so is (2).

Now, we can apply the limits inversion theorem and, then, by (8), we have

\[
(11) \quad x^n = \sum_{h=0}^{\infty} \left( \sum_{j=0}^{\infty} \alpha_j^n \lambda_{h,j} \right) x^h,
\]

whenever \( x \in d(0, r) \). Actually this is true for all \( r \in ]0, 1[ \) and therefore (11) holds for all \( x \in d(0, 1^-) \). Hence we have \( \sum_{j=0}^{\infty} \alpha_j^n \lambda_{h,j} = 0 \) whenever \( n \neq h \) and \( \sum_{j=0}^{\infty} \alpha_j^n \lambda_{n,j} = 1 \). So (3) is satisfied.

Thus we have proven that \( PM = I_\infty \). Now we check that \( MP = I_\infty \). For every \( h \neq j \), we have \( g_j(\alpha_h) = g(\alpha_h) = 0 \), hence \( \sum_{h=0}^{\infty} \alpha_h^n \lambda_{h,j} = 0 \). Besides, it is seen that \( g_j(\alpha_j) = 1 \), hence \( \sum_{n=0}^{\infty} \alpha_j^n \lambda_{n,j} = 1 \). So we conclude that \( MP = I_\infty \) and this finishing proving (4).

Now, we will check that \( P(b) \in E_1 \) for all \( b \in E_0 \). Let \( b := (b_n) \in E_0 \), let \( a := (a_n) = P(b) \) and let \( f(t) = \sum_{n=0}^{\infty} a_n t^n \). For each \( j \in \mathbb{N} \) we put \( f_j(t) = \sum_{m=0}^{j} b_m g_m(t) \). Then \( f_j \) belongs to \( A(d(0, 1^-)) \) for all \( j \in \mathbb{N} \). Let \( r \in ]0, 1[ \). Like the family \( \{|\lambda_{n,j}| r^n\} \), the family \( \{|\lambda_{n,j}| r^n\} \) tends to zero uniformly with respect to \( n \) when \( j \) tends to \( +\infty \). That way, in \( H(d(0, r)) \) we have \( \lim_{j \to \infty} \|f - f_j\|_{d(0, r)} = 0 \) and therefore \( f \) belongs to \( H(d(0, r)) \). This is true for all \( r \in ]0, 1[ \) and therefore \( f \) belongs to \( A(d(0, 1^-)) \). Hence \( P(b) \in E_1 \). This shows (5).

Let us show (6). Let \( b := (b_0, \ldots, b_n, \ldots) \) be a bounded sequence. Let \( a = P b \), and let \( a = (a_0, \ldots, a_n, \ldots) \). We will show

\[
(12) \quad \limsup_{n \to \infty} |a_n|^{1/n} \leq 1.
\]

Without loss of generality, we may assume \( |b_j| \leq 1 \), whenever \( j \in \mathbb{N} \). Then we have \( |a_n| \leq \sup_{j \in \mathbb{N}} |\lambda_{n,j}| = s_n \), therefore \( \limsup_{n \to \infty} |a_n|^{1/n} \leq \limsup_{n \to \infty} s_n^{1/n} \leq 1 \). Now, by (12), it is seen that for all \( j \in \mathbb{N} \), the series \( \sum_{n=0}^{\infty} a_n \alpha_j^n \) is convergent and therefore we may consider
$Ma = M(Pb)$. By definition, for each $i \in \mathbb{N}$, we have $a_i = \sum_{j=0}^{\infty} \lambda_{i,j} b_j$. Let $Ma = (x_h)_{h \in \mathbb{N}}$.

For each $h \in \mathbb{N}$ we have $x_h = \sum_{m=0}^{\infty} \alpha_h^m a_m = \sum_{m=0}^{\infty} \alpha_h^m (\sum_{j=0}^{\infty} \lambda_{m,j} b_j)$. Let $r = |\alpha_h|$. As we saw, the family $|\lambda_{m,j} b_j| r^m$ tends to 0 when $m$ tends to $+\infty$, uniformly with respect to $j$. Hence by the Limits Inversion Theorem, we have

$$\sum_{m=0}^{\infty} \alpha_h^m (\sum_{j=0}^{\infty} \lambda_{m,j} b_j) = \sum_{j=0}^{\infty} b_j \sum_{m=0}^{\infty} \lambda_{m,j} \alpha_h^m.$$ 

Hence by (3), we see that $x_j = b_j$ and this finishes proving (6). Then by (6) $\psi_P$ is clearly injective.

Finally we will prove the last statement of the theorem. Let $\phi(x) = \sum_{n=0}^{\infty} \nu_n x^n$. The function $\phi$ belongs to $A(d(0,1^-))$ and is invertible in $A(d(0,1^-))$ thanks to the inequality $|\nu_0| > |\nu_n|$ whenever $n > 0$. Hence the function $G(x) = g(x)\phi(x)$ is easily seen to satisfy i), ii), iii), iv) like $g$. Then $G$ factorizes in $A(d(0,1^-))$ and can be written as $\phi_j(x)(1-x/\alpha_j)$

with $\phi_j(x) = \psi_j(x)\phi(x)$. Hence we put $G_j(x) = \frac{\phi_j(x)}{\phi_j(\alpha_j)} = \frac{g_j(x)\phi(x)}{\phi(\alpha_j)}$. Now it is clearly seen that the power series of $G_j$ is $\sum_{n=0}^{\infty} \mu_{n,j} x^n$. By definition, the matrix $Q$ satisfies the same properties as $P$. But when $\phi$ is not a constant function, for each fixed $j \in \mathbb{N}$, we do not have $\mu_{n,j} = \lambda_{n,j}$ for all $n \in \mathbb{N}$. Hence $Q$ is different from $P$. As a consequence we see that $\psi_M$ is not surjective, it would be an automorphism of $E_0$ and therefore $\psi_P$ would also be an automorphism of $E_0$ and it would be unique. This ends the proof of Theorem 3.

4. PROVING THEOREMS 4 AND 5

Notation. For each integer $q \in \mathbb{N}^*$, we will denote by $G(q)$ the group of the $q$-roots of 1.

Lemma 7 : Let $(a_n)$ be a sequence in $d(0,1^-)$ such that $\lim_{n \to \infty} |a_n| = 1$. For each $s \in \mathbb{N}$, there exists a prime integer $q > p$ and $\zeta \in G(q)$ such that $|\zeta^h a_s - a_j| = \max(|a_s|, |a_j|)$ for every $j \in \mathbb{N}$, for every $h = 1, \ldots, q - 1$.

Proof. Let $r = |a_s|$. Since $\lim_{n \to \infty} |a_n| = 1$, the circle $C(0, r)$ contains finitely many terms of the sequence $(a_n)$. Without loss of generality we may assume $|a_n| < r$ whenever $n < l$, $|a_n| > r$ whenever $n > t$ and $|a_n| = r$, whenever $n = l, \ldots, t$ (with obviously $l \leq s \leq t$). Whatever $q \in \mathbb{N}$, $\zeta \in G(q)$ are, it is seen that we have $|\zeta^h a_s - a_j| = |a_s|$ for all $j < l$ and $|\zeta^h a_s - a_j| = |a_j|$ for all $j > t$. In the residue class field $k$ of $K$, for every $j = l, \ldots, t$ let $\gamma_j$ be the class of $a_j/a_s$. There does exist a prime integer $q > p$ such that the polynomial $p(x) = x^q - 1$ admits none of the $\gamma_j$ ($l \leq j \leq t$) as a zero. Hence, for
every $q$-root $\zeta$ of 1 in $K$, we have $\zeta^h \neq \zeta_j$ whenever $j = 1, \ldots, t$, whenever $h = 1, \ldots, q - 1$. Now let $\zeta$ be a $q$-th root of 1 in $K$. Then by classical properties of the polynomials, we have $|\zeta^h - a_j| = 1$, hence $|\zeta^h a_s - a_j| = |a_s| = r$ whenever $h = 1, \ldots, q - 1$, whenever $j = 1, \ldots, t$. This completes the proof of Lemma 7.

Lemma 8: Let $(a_n)$ be a regular sequence and let $\rho = \inf_{n \neq m} |a_n - a_m|$. There exists a sequence $(b_n)$ in $d(0,1^-)$ satisfying:

1. $\lim_{n \to \infty} |b_n| = 1$.
2. $|b_n - b_m| \geq \rho$ whenever $n \neq m$.
3. $(a_n)$ is a subsequence of $(b_n)$.
4. There exists a sequence $(q_n)$ of prime integers different from $p$ satisfying $\lim_{n \to \infty} q_n = +\infty$, such that for every $m \in \mathbb{N}$, $\zeta \in G(q_n)$, $\zeta b_n$ is another term of the sequence $(b_n)$.
5. There exists $f \in A(d(0,1^-))$ admitting each $b_n$ as a simple zero and having no other zero in $d(0,1^-)$, satisfying $\lim_{|x| \to 1^-} |f(x)| = +\infty$.

Proof. First we will construct a sequence $(b'_n)$ satisfying (1), (2), (3), (4). Let $(q_j)$ be a strictly increasing sequence of prime integers strictly bigger than $p$ and, for each $j \in \mathbb{N}$, let $S_j = \{0, q_j, \ldots, q_j - 1\}$. We will show that a good choice of the sequence $(q_j)$ enables us to obtain

$$(6) \quad |b'_n - b'_m| = \max(|b'_n|, |b'_m|)$$

for every couple $(n, m)$ satisfying $n \neq m$ and $(n, m) \neq (s_i, s_j)$ whenever $(i, j) \in \mathbb{N} \times \mathbb{N}$. In other words $|b'_n - b'_m| = \max(|b'_n|, |b'_m|)$ must be true all time except when $n = m$ and when $(b'_n, b'_m)$ is equal to some couple $(a_{s_i}, a_{s_j})$. For each $t \in \mathbb{N}$, let $F_t = \{s_0, s_1, \ldots, s_t\}$ and let $E_t$ be $\{0, 1, \ldots, s_t - 1\} \setminus F_t$. Assume that $q_0, q_1, \ldots, q_{t-1}$ have been chosen to satisfy the following properties $(\alpha_t)$ and $(\beta_t)$

$$(\alpha_t) \quad |b'_n - a_{s_j}| = \max(|b'_n|, |a_{s_j}|) \text{ for all } j \in \mathbb{N}, \text{ for all } n \in E_t.$$

$$(\beta_t) \quad |b'_n - b'_m| = \max(|b'_n|, |b'_m|) \text{ for all } (n, m) \in E_t \times E_t \text{ such that } n \neq m.$$ We will choose $q_t$ such that both $(\alpha_{t+1})$, $(\beta_{t+1})$ are satisfied. Indeed, by Lemma 7 we can take a prime integer $u$ such that, given $\zeta_t \in G(u)$, we have $|\zeta_t a_t - a_j| = \max(|a_t|, |a_j|)$ for all $j \in \mathbb{N}$, for all $h = 1, \ldots, u - 1$, $|\zeta_t a_t - b'_n| = \max(|a_t|, |b'_n|)$ for all $n < s_t$, for all $h = 1, \ldots, u - 1$. Thus we can take $q_t = u$ and we see that both $(\alpha_{t+1})$, $(\beta_{t+1})$ are satisfied. Hence we can construct the sequence $(q_t)$ by induction and, therefore, the sequence $(b'_n)$ satisfying (6) is now constructed. Then it is easily checked that the sequence $(b'_n)$ so obtained satisfies (1), (2), (3), (4).
Now let \( \{r_0, \ldots, r_n, \ldots\} = \{|a_j| : j \in \mathbb{N}\} \) and let \( D = \Omega(b_n) \). The infinite product 
\[
g(x) = \prod_{j=0}^{\infty} (1 - (x/a_j)^{q_j})
\]
converges in \( A(d(0, 1^-)) \) and has no zero in \( d(0, r) \cap D \) because, by construction of the sequence \((b'_n)\), each zero of \( g \) is one of the points \( b'_m \) for some \( m \in \mathbb{N} \). Hence it is seen that we have \(|g(x)| \geq 1\) for every \( x \in d(0, 1^-) \setminus \bigcup C(0, r_n) \). For each \( n \in \mathbb{N} \), let \( \Sigma_n = D \cap C(0, r_n) \), let \( \tau_n = \inf_{x \in \Sigma_n} |g(x)| \), let \( \sigma_n \in (r_n, r_{n+1}) \cap |K| \), let \( c_n \in C(0, \sigma_n) \), and let \( u_n > \min(p, n) \) be a prime integer such that \( \tau_n(c_n)^{u_n} > n + 1 \). Since 
\[
\lim_{n \to \infty} u_n = +\infty,
\]
it is seen that the infinite product \( h(x) = \prod_{n=0}^{\infty} (1 - (x/c_n)^{u_n}) \) converges in \( A(d(0, 1^-)) \). Let \( D' = \Omega((c_n, \rho) \) and let \( D'' = D' \cap D \). Let \( h(x) = \sum_{n=0}^{\infty} \lambda_n x^n \) and, for each \( r \in (0, 1) \), let \( M(r) = \sup_{n \in \mathbb{N}} |\lambda_n| r^n \). Each pole of \( h \) is simple and is of the form \( \zeta c_n \) with \( \zeta \in G(u_n) \). Hence it is seen that \( h \) satisfies \(|h(x)| \geq M |x| / \rho \) for all \( x \in D' \). Hence if \( x \in D'' \setminus \bigcup \Sigma_n \), then we have 
\[
|g(x)h(x)| = M(r_n)\tau_n \geq (r_n/r_{n-1})^{u_{n-1}} \tau_n > n + 1
\]
and finally we have
\[
(7) \quad \lim_{|x| \to 1, x \in D''} |g(x)h(x)| = +\infty.
\]
Now let \((b''_n)\) be the sequence of the zeros of \( g \). Clearly \((b''_n)\) satisfies (1) and (4) and also satisfies \(|b''_n - b'_m| = \max(|b''_n|, |b'_m|)\) whenever \( n, m \in \mathbb{N} \) and \(|b''_n - b'_m| = \max(|b''_n|, |b'_m|)\) whenever \( n \neq m \). Now we put \( b_{2n} = b'_n \) and \( b_{2n+1} = b''_n \). The sequence \((b_n)\) clearly satisfies (1), (2), (3), (4) and also satisfies (5) because the zeros of \( h \) are the \( b''_n \) while those of \( g \) are the \( b'_n \). Thus the zeros of \( f \) are just the \( b_n \), and then, by (7), we have 
\[
\lim_{|x| \to 1, x \in \Omega(b_n)} |f(x)| = +\infty.
\]
This ends the proof of Lemma 8.

**Proof of Theorem 4.** Without loss of generality we may obviously assume \(|\alpha_n| \leq |\alpha_{n+1}|\) whenever \( n \in \mathbb{N} \). Let \( \rho = |\alpha_0| \). Hence by hypothesis each disk \( d(\alpha_q, \rho^-) \) contains no point \( \alpha_n \) for each \( n \neq q \). Let \( D = \Omega((\alpha_n), \rho^-) \).

For each \( n \in \mathbb{N} \), let \( T_n \) be the hole \( d(\alpha_n, \rho^-) \) of \( D \). Since \(|\alpha_n| = 0\), it is shortly checked that the sequence \((T_n, 1)\) is a \( T \)-sequence of \( D \) ([8]). Then, since \( K \) is spherically complete, by [4], Theorem 4, there exists \( g \in A(d(0, 1^-)) \) admitting each \( \alpha_n \) as a simple zero and having no zero else in \( d(0, 1^-) \). Therefore, as \( \prod_{n=0}^{\infty} |\alpha_n| = 0 \), is is seen that \( g \)
satisfies \( \lim_{|x| \to 1} |g(x)| = +\infty \). Now we can apply Theorem 3, which shows that the matrix \( M = M(a_n) \) admits inverses \( P \). Then the sequence \( (a_n) \) satisfies \( \sum_{n=0}^{\infty} a_n a_j^n = b_j \) for every \( j \in \mathbb{N} \) and this clearly ends the proof of Theorem 4.

**Proof of Theorem 5.** By Lemma 8, there exists a regular sequence \( (\gamma_n) \) of \( d(0,1^-) \) such that \( (a_n) \) is a subsequence of \( (\gamma_n) \) together with an analytic function \( g \in A(d(0,1^-)) \) admitting each \( \gamma_m \) as a simple zero and having no other zero in \( d(0,1^-) \), satisfying \( \lim_{|x| \to 1^-} |g(x)| = +\infty \) with \( \rho = \inf_{n \neq m} |\gamma_n - \gamma_m| \). Then, by Theorem 3, the matrix \( M = M(\gamma_n) \) admits line-vanishing inverses \( M' \) satisfying \( M(M'(b)) = b \) for all bounded sequence \( b = (b_n) \). Let \( a := (a_n) = M'(b) \). Thus we have \( M(a) = b \) and therefore \( \sum_{n=0}^{\infty} a_n \gamma_j^n = b_j \) whenever \( j \in \mathbb{N} \). This ends the proof of Theorem 5.

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**REFERENCES**

[1] AMICE, Yvette, *Les nombres p-adiques*, P.U.F. 1975.

[2] FRESNEL, Jean, DE MATHAN, Bernard, *L'image de la transformation de Fourier p-adique*, C.R.A.S. Paris, Série A, 278 (1974), 653-656.

[3] KRASNER, Marc, *Prolongement analytique uniforme et multiforme dans les corps valués complets. Les tendances géométriques en algèbre et théorie de nombres*. Clermont-Ferrand 1964, pp 97-141. Centre Nationale de la Recherche Scientifique (1966) (Colloques internationaux du C.N.R.S., Paris, 143).

[4] LAZARD, Michel, *Les zéros d'une fonction analytique sur un corps valué complet*, Publications Mathématiques, 14 (1962), 47-75, IHES (PUF).

[5] ROBBA, Philippe, *Fonctions analytiques sur les corps valués ultramétriques complets. Prolongement analytique et algèbres de Banach ultramétriques*, Astérisque, 10 (1973), 109-220.

[6] SARMANT, Marie-Claude, *Produits méromorphes*, Bulletin des Sciences Mathématiques: 109 (1985), 155-178.

[7] SARMANT, Marie-Claude, ESCASSUT, Alain, *Prolongement analytique à travers un T-filtre*, Studia Scientiarum Mathematicarum Hungarica, 22 (1987), 407-444.

[8] SARMANT, Marie-Claude, ESCASSUT, Alain, *Fonctions analytiques et produits croulants*, Collectanea Mathematica, 36 (1985), 199-218.
[9] SERRE, Jean Pierre, *Endomorphismes complètement continus d'espaces de Banach p-adiques*, Publications Mathématiques n 12, IHES (1962), 69-85.

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