THE IRREDUCIBLE REPRESENTATIONS OF 3-DIMENSIONAL SKLYANIN ALGEBRAS

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Abstract. In this article a complete description is given of the simple representations of a 3-dimensional Sklyanin algebra associated to a torsion point. In order to determine these irreducible representations, a review is given of classical results regarding representation theory of graded rings with excellent homological and algebraic properties.

1. Introduction

The 3-dimensional Sklyanin algebras form the most important class of Artin-Schelter regular algebras of global dimension 3. These algebras are parametrized by the following geometric data: an elliptic curve $E$ and a point $\tau \in E$. In addition, they form a class of graded algebras with excellent ringtheoretical and homological properties, which was shown in the original papers by Artin, Tate and Van den Bergh, [5] and [3].

Recently, the representation theory of these algebras have gained attention, see for example [12], [8], [21] and [20]. In [18] it was shown that a Sklyanin algebra is a finite module over its center if and only if $\tau$ is a torsion point of $E$, in which case the Sklyanin algebra is a maximal order over its center. Consequently, in the case of a torsion point, it is useful to describe the simple representations.

The point of this article is to explain the connection between the study of fat point modules, $\text{PGL}_n \times \mathbb{C}^*$-stabilizers of simple representations and $\mathbb{C}^*$-families of (non-trivial) simple representations of a graded algebra $A$ with good properties. This connection is known by the experts, but it is not well-known by the generic mathematician working in noncommutative algebraic geometry. As a consequence, we verify [21, Conjecture 1.5] using these methods.

Conjecture 1. [21, Conjecture 1.5] The simple representations of a 3-dimensional Sklyanin algebra with $\tau$ a torsion point of order $n$ with $(n, 3) = 3$ are of dimension $n$, $n/3$ or 1. The simple representations of dimension $n/3$ form a $3:1$-cover over three lines intersecting in a unique point.

1.1. Statement. The author hereby acknowledges that his own contribution is mainly combining previous results by Artin, Tate, Van den Bergh, Le Bruyn, Smith and others.

1.2. Conventions. In this article, the following notations and conventions are used.

- We will work over $\mathbb{C}$.
- The element $\omega \in \mathbb{C}^*$ will be a primitive third root of unity.
- The element $\rho \in \mathbb{C}^*$ will be a primitive $n$th root of unity for a specified $n \in \mathbb{N}$.
• For a graded algebra $A$, $\text{Proj}(A) = \qgr(A)$ is the category of finitely generated, graded $A$-modules, modulo torsion modules. Recall that there is an automorphism on $\text{Proj}(A)$ by the shift functor: if $M = \bigoplus_{i \in \mathbb{N}} M_i \in \text{Proj}(A)$, then $M[1]$ is the graded module with degree $k$-part isomorphic to $M_{k+1}$.

• For any algebra, the set $\text{irrep}A$ is the set of simple representations of $A$ up to equivalence. With $\text{irrep}_kA$ for $k \in \mathbb{N}$ the subset of $k$-dimensional simple representations up to equivalence are denoted.

• If $A$ is a sheaf of coherent algebras over $X$ with $X$ a projective scheme, then $\text{irrep}A$ denotes the sheaf of sets of simple representations up to equivalence over $X$, that is,

$$(\text{irrep}A)(U) = \text{irrep}A|_U.$$

With $\text{irrep}_kA$ we denote the subsheaf of sets of simple $k$-dimensional representations up to equivalence over $X$.

2. Noncommutative projective geometry

2.1. Cayley-Hamilton algebras. In this section a review of the results of [10], [9] and [11] is discussed. We will assume that

- $A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \ldots$
  is a graded algebra, finitely generated in degree one with quadratic relations,
- $A$ is a domain,
- $A$ has global dimension $d$ and $H_A(t) = (1 - t)^{-d}$ for some $d \in \mathbb{N}$, and
- $A$ is a finite module over its center $R = Z(A)$, which is a normal domain.

The algebras of current interest, the Sklyanin algebras at torsion points, have all these properties by [19, Corollary 1.3]. Let $n \in \mathbb{N}$ be equal to

$$n = \max\{m : \exists \phi : A \rightarrow \mathbb{C}_m(\mathbb{C}) \text{ algebra morphism}\}.$$

Under these conditions, $A$ is a Cayley-Hamilton of degree $n$, that is, there exists an $R$-linear map $\text{tr} : A \rightarrow R$ such that

- $\text{tr}(1) = n$,
- $\text{tr}(ab) = \text{tr}(ba)$ for each $a, b \in A$, and
- $\chi_{n,a}(a) = 0$ for each $a \in A$.

The element $\chi_{n,a}(a)$ is the $n$th Cayley-Hamilton polynomial of degree $n$, expressed in the sums of powers of $a$, as explained in [14, Section 2.3].

A grading on an algebra $A$ is equivalent to an action of the torus group $\mathbb{C}^*$, by the rule

$$a \in A_n \Leftrightarrow t \cdot a = t^k a \text{ for each } k \in \mathbb{N}, t \in \mathbb{C}^*, a \in A.$$

As such, $\text{tr}$ is also a degree-preserving map, that is, $\text{tr}(A_n) \subset A_n$.

**Example 1.** Let $n \in \mathbb{N}$ be an integer strictly larger than 1 and take the algebra

$$A_p = \mathbb{C}_p[x,y] = \mathbb{C}(x,y)/(xy - pyx),$$

classically graded by $\deg(x) = \deg(y) = 1$. Then $Z(A_p) = \mathbb{C}[x^n, y^n]$. The natural trace map on $A_p$ is the $\mathbb{C}$-linear extension of:

$$\text{tr}(x^k y^l) = \begin{cases} nx^k y^l, & \text{if } (k,l) \in (n\mathbb{N})^2, \\ 0, & \text{otherwise} \end{cases}$$
The couple \((A, \text{tr})\) determines a Cayley-Hamilton algebra of degree \(n\).

To \(A\) is associated the affine variety of trace-preserving \(n\)-dimensional representations, that is,

\[
X_A = \text{rep}_k A = \{ \phi : A \rightarrow \mathfrak{m}_n(\mathbb{C}) \text{ algebra morphism} : \text{tr} \circ \phi = \phi \circ \text{tr} \},
\]

with the trace on \(\mathfrak{m}_n(\mathbb{C})\) being the usual trace map. The projective special linear group of degree \(n\), \(\text{PGL}_n(\mathbb{C})\) acts on this variety by conjugating matrices, such that \(\text{PGL}_n(\mathbb{C})\)-orbits correspond to isomorphism classes of \(n\)-dimensional representations. By a result of Artin \([1, \text{Section 12.6}]\), one has

\[
X_A/\text{PGL}_n(\mathbb{C}) \cong \text{Spec}(R).
\]

In addition, as \(A\) is graded, \(\mathbb{C}^*\)-acts on \(X_A\). The only closed orbit, as in the commutative case, is the trivial representation \((A/\mathbb{A})^\oplus_n\), which is also the only orbit with a non-trivial \(\mathbb{C}^*\)-stabilizer. From the fact that the actions of \(\mathbb{C}^*\) and \(\text{PGL}_n(\mathbb{C})\) on \(X_A\) commute, it follows that \(X_A\) is a \(\text{PGL}_n(\mathbb{C}) \times \mathbb{C}^*\)-variety.

Let \(M\) be a simple \(n\)-dimensional \(A\)-representation, then its \(\text{PGL}_n(\mathbb{C})\)-stabilizer is trivial by Schur’s lemma. As mentioned before, the \(\mathbb{C}^*\)-stabilizer of \(M\) is also trivial, but the \(\text{PGL}_n(\mathbb{C}) \times \mathbb{C}^*\)-stabilizer doesn’t have to be trivial. From \([6, \text{lemma 4, theorem 2}]\), it follows that this stabilizer is always finite and is conjugated to the group generated by \((g_\zeta, \zeta) \in \text{PGL}_n(\mathbb{C}) \times \mathbb{C}^*\), with \(\zeta = \rho^{2e}\) for some \(e \in \mathbb{N}\) and

\[
g_\zeta = \text{diag}(1, \ldots, 1, \zeta, \ldots, \zeta, \zeta^{-1}, \ldots, \zeta^{-1})
\]

If \(M\) is a \(k\)-dimensional simple \(A\)-representation with \(k < n\), one can look at \(\text{rep}_k A\) and calculate the \(\text{PGL}_k(\mathbb{C}) \times \mathbb{C}^*\)-stabilizer.

2.2. **Graded matrix rings.** Let \(M\) be a simple \(A\)-module of dimension \(k\) with \(k \leq n\) and let \(m = \text{Ann}_A(M)\). Then we have a map

\[
A_m \xrightarrow{\phi} \mathfrak{m}_k(\mathbb{C})
\]

Let

\[
m^g = \max\{ I \subset m : I \text{ homogeneous ideal} \}.
\]

Then \(m^g\) is a maximal homogeneous ideal. By a graded version of Artin-Wedderburn \([13, \text{theorem 1.5.8}]\), there is an integer \(e \in \mathbb{N} \setminus \{0\}\) and a \(k\)-tuple \((a_1, \ldots, a_k) \in \mathbb{N}^k\) with \(0 \leq a_1 \leq a_2 \leq \ldots \leq a_k < e\) such that

\[
A_m^e/m^g \cong \mathfrak{m}_k(\mathbb{C}[t, t^{-1}])((a_1, a_2, \ldots, a_k))
\]

with \(\text{deg}(t) = e\). If \(\text{deg}(t) = 1\) then we will suppress the integers \(a_1, \ldots, a_k\). This isomorphism is a graded isomorphism. Recall that for a \(\mathbb{Z}\)-graded ring \(S\) and a \(k\)-tuple \((a_1, \ldots, a_k) \in \mathbb{Z}^k\) the matrix ring \(\mathfrak{m}_k(S)(a_1, a_2, \ldots, a_k)\) has degree \(m\)-part for \(m \in \mathbb{Z}\)

\[
\mathfrak{m}_k(S)(a_1, a_2, \ldots, a_k)_m = \begin{bmatrix}
S_m & S_{m-a_1+a_2} & \cdots & S_{m-a_1+a_k} \\
S_{m-a_2+a_1} & S_m & \cdots & S_{m-a_2+a_k} \\
\vdots & \vdots & \ddots & \vdots \\
S_{m-a_k+a_1} & S_{m-a_k+a_2} & \cdots & S_m
\end{bmatrix}.
\]
By construction, there is a commutative diagram

\[
\begin{array}{ccc}
A_m & \xrightarrow{\phi} & M_k(\mathbb{C}) \\
\downarrow{\psi} & & \downarrow{t \mapsto 1} \\
A_{m^\rho} & \xrightarrow{} & M_k(\mathbb{C}[t, t^{-1}]) \langle a_1, a_2, \ldots, a_k \rangle
\end{array}
\]

By Theorem 1, \(M_k(\mathbb{C}[t, t^{-1}])\langle a_1, a_2, \ldots, a_k \rangle\) is generated in degree one over its center if and only if

\[(a_1, a_2, \ldots, a_k) = (0, \ldots, 0, 1, \ldots, 1, e-1, \ldots, e-1)\]

with \(m_j > 0\) and \(\sum_{j=0}^{k-1} m_j = k\), which we will assume throughout this paper. By assumption, this means that if \(x \in A_1\), then

\[
\psi(x) = \begin{bmatrix}
0 & 0 & \ldots & M(x)_{0t} \\
M(x)_1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & M(x)_{e-1} & 0
\end{bmatrix}
\]

with \(M(x)_{0t} \in M_{m_1 \times m_{l-1}}(\mathbb{C})\), indices taken in \(\mathbb{Z}_e\). But then, by construction, \(\phi\) has as \(\text{PGL}_k(\mathbb{C}) \times \mathbb{C}^*\)-stabilizer the cyclic group \(\langle (g_\zeta, \zeta) \rangle\).

**Example 2.** Continuing example 1 with \(\rho = -1\), then the center of \(A = A_{-1}\) is \(\mathbb{C}[x^2, y^2]\). Over \(m = (x^2 - 1, y^2 - 1)\) lies a unique two-dimensional simple representation. Then \(m^\rho = (x^2 - y^2)\). As \(Z(A_{m^\rho}) \subset \oplus_{k \in \mathbb{Z}} (A_{m^\rho})_k\) and \(A_{m^\rho}\) is generated over its center by degree one elements, it follows that

\[A_{m^\rho}/m^\rho \cong \mathbb{M}_2(\mathbb{C}[t, t^{-1}])\langle 0, 1 \rangle, \deg(t) = 2.\]

By construction, a \(\mathbb{C}^*\)-family of simple two-dimensional representations of \(A\) is determined by

\[
\begin{bmatrix}
0 & t \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
0 & -t \\
1 & 0
\end{bmatrix} \in \mathbb{M}_2(\mathbb{C}[t, t^{-1}])\langle 0, 1 \rangle.
\]

For each \(t \neq 0\) and associated algebra morphism \(\phi_t\) obtained by specialising these two matrices, the \(\text{PGL}_2(\mathbb{C}) \times \mathbb{C}^*\)-stabilizer is determined by

\[
\left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, -1 \right\rangle.
\]

**2.3. Periodic fat point modules.** Associated to

\[
A_{m^\rho} \xrightarrow{\psi} M_k(\mathbb{C}[t, t^{-1}])\langle 0, \ldots, 0, 1, \ldots, 1 - e, \ldots, e - 1 \rangle
\]

are \(e\)-\(e\)-periodic fat point modules ([10], Section 6). Recall that a fat point module is a simple object of \(\text{Proj}(A)\). For \(j \in \mathbb{N}\), let \(V_j = \mathbb{C}^m\) if \(j \equiv l \mod \mathbb{Z}_e\). Let \(M(\psi) = \oplus_{j \in \mathbb{N}} V_j\) and define an action of \(A\) by extending

\[x \in A_1, v \in M(\psi)_j = V_j : xv = M(x)_{j+1}v\] as an element of \(M(\psi)_{j+1} = V_{j+1}\).
As a \( \mathbb{C}[t]\)-module, this is just \( \bigoplus_{i=0}^{e-1}(\mathbb{C}[t])[i]^{\oplus m_i} \). Of course, this makes sense as \( t \) acts faithfully on this module and so
\[
\text{END}\left( \bigoplus_{i=0}^{e-1}(\mathbb{C}[t,t^{-1}])[i]^{\oplus m_i} \right) = \mathbb{M}_k(\mathbb{C}[t,t^{-1}])/(0,0,\ldots,0,1,\ldots,1,\ldots,e-1,\ldots,e-1).
\]

By surjectivity of \( \psi \), \( M(\psi) \) is a fat point module of \( A \). However, if \( e > 1 \), then \( M(\psi)[k] \not\cong M(\psi) \) for \( 1 \leq k \leq e-1 \), but \( M(\psi)[e] \cong M(\psi) \) in \( \text{Proj}(A) \). By definition, \( M(\psi) \) is an \( e \)-periodic fat point module of multiplicity \( (m_0,m_1,\ldots,m_{e-1}) \).

The other way round, suppose that \( M \) is an \( e \)-periodic fat point module of multiplicity \( (m_0,m_1,\ldots,m_{e-1}) \). Then to \( M \) is associated a \( \sum_{i=0}^{e-1} m_i = k \)-dimensional representation \( S \) by the following rule:
\[
x \in A_1, v \in M_j : \text{ if } xv = M(x)_{j+1}v \text{ as an element of } M_{j+1} = V_{j+1}
\]
then one associates the algebra map
\[
\psi_M(x) = \begin{bmatrix}
0 & 0 & \ldots & M(x)_0 \\
M(x)_1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & M(x)_{e-1} & 0
\end{bmatrix}.
\]
This algebra map corresponds to a simple representation of \( A \), for if this was not simple, then this would imply that \( M \) is not a fat point module.

In short, we have

**Theorem 2.** [14] Proposition 3] There is a one-to-one correspondence between

- simple \( k \)-dimensional representations with \( \text{PGL}_k(\mathbb{C})\times \mathbb{C}^* \)-stabilizer generated by \( (g_c,\zeta) \) with \( g_c = \text{diag}(1,\ldots,1,\zeta,\ldots,\zeta,\ldots,\zeta,\ldots,\zeta) \) and \( m_0, m_1, \ldots, m_{e-1} \)

- shift-equivalence classes of fat point modules of period \( e \) and multiplicity \( (m_0,m_1,\ldots,m_{e-1}) \).

If \( m_0 = m_1 = \ldots = m_{e-1} = 1 \), then \( M \) is called a point module. In fact, one has

**Proposition 1.** If \( A \) has finite global dimension and has Hilbert series \( (1-t)^{-d} \), then the Hilbert series of a fat point \( M \) is \( 1 \)-periodic, that is, \( H_M(t) = m(1-t)^{-1} \) for some \( m \in \mathbb{N} \).

**Proof.** Let \( M \) be an \( e \)-periodic fat point. As \( M \) has a finite projective resolution, it follows that
\[
\frac{f(t)}{(1-t)^l} = \frac{p(t)}{1-t^e} \text{ with } (1-t,f(t)) = 1, \deg(p(t)) = e-1.
\]
But then it follows that \( (1-t)^l \) divides \( 1-t^e \), which can only happen if \( l = 1 \). In this case,
\[
f(t)(1+t+\ldots+t^{e-1}) = p(t),
\]
which can only happen if \( f(t) = m \) is constant. But then the Hilbert series of \( M \) is \( m(1-t)^{-1} \).

The integer \( m \) of the proposition is called the multiplicity of \( M \).
Example 3. Continuing example 2, one sees that the algebra morphism
\[ \frac{A}{m^g} \rightarrow M_2(\mathbb{C}[t, t^{-1}])(0, 1), (x, y) \mapsto \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -t \\ 1 & 0 \end{pmatrix} \]
determines two point modules, the point module \( P \) with associated infinite quiver
\[
\begin{array}{c}
1 \\
\downarrow 1 \\
1 \\
\downarrow 1 \\
\downarrow 1 \\
\downarrow 1 \\
\cdots
\end{array}
\]
and its shift \( P[1] \).

2.4. Sheaf of algebras \( A \). In this subsection we follow [9]. Let \( g \) be a central
element of \( A \) of degree strictly larger than 0. Localizing \( A \) at \( g \) one gets the graded
ring \( \Lambda = A[\frac{1}{g}] \) with center \( \Sigma = R[\frac{1}{g}] \) and taking the degree 0 part \( \Lambda_0 \), one gets an
algebra over \( \Sigma_0 \) which is finitely generated as a \( \Sigma_0 \)-module. Gluing these algebras,
one gets a sheaf of orders \( A \), finite over \( R = \text{Proj}(R) \). We have \( \Lambda_0 \subset Q^0(A)_0 \),
which is a division algebra, say of PI-degree \( s \). However, it is not always so that
\( Z(\Lambda_0) = \Sigma_0 \).

Example 4. For any quantum algebra \( A_\rho \) from example 1, localizing at for example
\( x \), one finds that \( \Lambda_0 = \mathbb{C}[\frac{y}{x}] \) and \( \Sigma_0 = \mathbb{C}[\frac{y}{x}]^n \).

Consider now a fat point module \( M \) which is not annihilated by \( g \). Then \( g \)
acts faithfully on \( M \) and \( M[\frac{1}{g}] \) is a graded \( A[\frac{1}{g}] \)-module. The module \( M[\frac{1}{g}] \)
becomes a gr-simple module (that is, it has no non-trivial graded submodules). By
the equivalence of categories of Dade’s Theorem [13, Theorem 3.1.1], \( N = M[\frac{1}{g}]_0 \)
is a simple \( \Lambda_0 \)-module.

Let \( Z = \text{Spec}(Z(A)) \) be the central proj of \( A \). By a result of Artin [1, Section
12.6], \( Z \) parametrizes isomorphism classes of semi-simple trace-preserving \( s \)-
dimensional representations of \( A \). There is a surjective map
\[
Z \xrightarrow{\Phi} R.
\]
This map will be finite and therefore generically \( e \)-to-one for some \( e \in \mathbb{N} \).

Proposition 2. The integer \( e \) is equal to the least common multiple of the degrees
of homogeneous elements of \( R \).

Proof. Localize \( A \) at a finite number of generators of \( R \), say \( B = A[(x_1 \cdots x_t)^{-1}] \)
with \( R = \mathbb{C}[x_1, \ldots, x_t]/(I) \) for \( I \) some set of relations. Then any simple representation
of dimension \( n \) of \( B \) comes from an algebra morphism
\[
B_{m^g} \rightarrow M_n(\mathbb{C}[t, t^{-1}])(0, \ldots, 0, 1, \ldots, 1, \ldots, e - 1, \ldots, e - 1),
\]
for some maximal graded ideal \( m^g \) of \( B \), which we know from the previous discussions
corresponds to a sum of \( e \) distinct but shift-equivalent fat point modules of
multiplicity \( \frac{1}{e} \). As a fat point corresponds to a simple representation of \( B_0 \), we find
that the PI-degree of \( A \) is \( s = \frac{1}{e} \). As there are \( e \) fat points lying over the graded
prime ideal \( m^g \cap R \), it follows that the map
\[
Z \xrightarrow{\Phi} R
\]
is generically \( e \)-to-one. \( \square \)
The shift functor on $\text{Proj}(A)$ induces an automorphism on the fat points of a fixed multiplicity. As we now know that fat points correspond to simple representations of $A$, this shift functor also induces an automorphism $\alpha$ on $Z$. As $Z = A/\text{PGL}_n(\mathbb{C})$ and $\text{Spec}(R) = A/\text{PGL}_n(\mathbb{C})$, it follows that $\Phi$ is the quotient map by the group $G = \langle \alpha \rangle$.

**Theorem 3.** The following sets are in one-to-one correspondence:

- $e$-periodic multiplicity $m$-fat points of $A$,
- $e$ $m$-dimensional simple representations of $A$ such that the corresponding points in $Z$ form an orbit under $\alpha$,
- graded maximal ideals $m^g$ of $A$ such that $A/m^g \cong \mathbb{M}_{em}(\mathbb{C}[t,t^{-1}])(0,\ldots,0,1,\ldots,1,\ldots,e-1,\ldots,e-1), \deg(t) = e$,
- simple $me$-dimensional representations with $\text{PGL}_{me}(\mathbb{C}) \times \mathbb{C}^*$-stabilizer generated by a subgroup conjugated to $\langle (g_\zeta, \zeta) \rangle$ with $g_\zeta = \text{diag}(1,\ldots,1,\zeta,\ldots,\zeta^{-1},\ldots,\zeta^{-1})$.

**Example 5.** Let

$$A = \mathbb{C}[x,y,z]/(yz - \omega z y, zx - \omega x z, xy - \omega y x)$$

be an Artin-Schelter regular algebra of global dimension 3. Then the center of $A$ is isomorphic to $R \cong \mathbb{C}[u, v, w, d]/(uvw - d^3)$. However, the central proj of $A$ is isomorphic to $\mathbb{P}^2$, and in fact $\text{Proj}(A) \cong \text{coh}(\mathbb{P}^2)$. In fact, $\text{Proj}(R) = \mathbb{P}^2/\mathbb{Z}_3$, with $\mathbb{Z}_3$ acting on $\mathbb{P}^2 = \mathbb{P}(\mathbb{C} \mathbb{Z}_3)$. The last fact can be proved using invariant theory: if $\mathbb{P}^2 = \text{Proj}(\mathbb{C}[x,y,z])$ with $\mathbb{Z}_3$-weights 0, 1, 2, then

$$\mathbb{C}[x,y,z]^{\mathbb{Z}_3} = \mathbb{C}[x,y^3,z^3,zy].$$

As $\text{Proj}(C) = \text{Proj}(C^{(k)})$ for any affine, commutative ring $C$,

$$\text{Proj}(\mathbb{C}[x,y^3,z^3,zy]) \cong \text{Proj}(\mathbb{C}[x,y^3,z^3,zy]^{(3)})$$

$$= \text{Proj}(\mathbb{C}[x^3,y^3,z^3,xzy])$$

$$\cong \text{Proj}(\mathbb{C}[u,v,w]/(uvw - d^3)).$$

Consequently, the map

$$\mathbb{P}^2 \xrightarrow{\Phi} \mathbb{P}^2/\mathbb{Z}_3$$

is indeed just the quotient map by the automorphism $\alpha = \text{diag}(1,\omega,\omega^2)$.

Armed with this knowledge, we can now tackle the problem of describing all the simple representations of 3-dimensional Sklyanin algebras.

3. 3-dimensional Sklyanin algebras

In this section, set $p = 3$. Let $E$ be an elliptic curve embedded in $\mathbb{P}^2$ and a torsion point $\tau \in E$ of order $n > 1$. The associated 3-dimensional Sklyanin algebra $A = A(E, \tau)$ is a finite module over its center by [19, Theorem 1.2]. In this section we will give a complete description of the simple representations of $A$.

The first thing we have to notice is: if $g$ is the unique central element of degree 3, then all fat points of $A/(g)$ are point modules, which are parametrized by $E$. This
follows from the fact that $A/(g) = \mathcal{O}_\tau(E)$ is the twisted coordinate of $E$ associated to $\tau$ \cite[Theorem 6.8]{1}. In addition, the shift functor \cite{1} on $E \subset \operatorname{Proj}(\mathcal{O}_\tau(E))$ is addition with $\tau$, so that each point module is $n$-periodic. From this we deduce using Theorem 3.

**Proposition 3.** Each non-trivial simple representation of $\mathcal{O}_\tau(E)$ is $n$-dimensional with $\operatorname{PGL}_n(\mathbb{C}) \times \mathbb{C}^*$-stabilizer isomorphic to $\mathbb{Z}_n$.

This fact is independent of the conditions $(n, 3) = 1$ or $(n, 3) = 3$.

3.1. $(n, 3) = 1$. The first thing one has to determine is the center of $A$.

**Theorem 4.** \cite[Theorem 4.8]{1} The center of $A$ is isomorphic to

$$R = \mathbb{C}[u, v, w, g]/(\Psi(u, v, w) - g^n),$$

with $\deg(u) = \deg(v) = \deg(w) = n, \deg(g) = 3$ and $\Psi(u, v, w)$ a homogeneous polynomial of degree $3n$ which embeds the elliptic curve $E' = E/\langle \tau \rangle$ in $\mathbb{P}^2_{[u:v:w]}$. Consequently, the central proj is $\mathbb{P}^2 \cong \operatorname{Proj}(R)$.

For the sake of completeness, a review of the discussion in \cite{8} will be given. By \cite[Theorem 3.4]{2}, all the fat points that are not annihilated by $g$ are of multiplicity $n$, which also follows from \cite[Theorem 7.3]{4} and the above discussion regarding simple representations of the sheaf $A$ of dimension $n$ and multiplicity $n$ fat point modules.

**Theorem 5.** \cite[Lemma 1, Theorem 4]{8} The simple representations of $A$ come in three types, let $S$ be a simple $A$-module with annihilator $m = \operatorname{Ann}(S)$.

- If $g \notin m$ then $S$ is a $n$-dimensional representation of $A$ with trivial $\operatorname{PGL}_n(\mathbb{C}) \times \mathbb{C}^*$-stabilizer. The $\mathbb{C}^*$-orbit of $S$ corresponds to a unique 1-periodic fat point module of multiplicity $n$, which in turn corresponds to a unique $n$-dimensional representation of $A$. Consequently, one has

$$A_m^s/m^g \cong \mathbb{M}_n(\mathbb{C}[t, t^{-1}]) \text{ with } \deg(t) = 1.$$

- If $g \in m$ but $\dim(S) \neq 1$ then $S$ is $n$-dimensional with $\operatorname{PGL}_n(\mathbb{C}) \times \mathbb{C}^*$-stabilizer isomorphic to $\mathbb{Z}_n$. The $\mathbb{C}^*$-orbit of $S$ corresponds to $n$ shift-equivalent point modules, who in turn correspond to $n$ 1-dimensional representations of $A$. Consequently, on has

$$A_m^s/m^g \cong \mathbb{M}_n(\mathbb{C}[t, t^{-1}])/(0, 1, \ldots, n - 1) \text{ with } \deg(t) = n.$$

- If $\dim(S) = 1$ then $S$ is the trivial representation $A/A^+$.

Consequently, in the affine case, we have

$$\begin{align*}
\operatorname{irrep}_n A & \xrightarrow{1:1} \operatorname{irrep} R \setminus \operatorname{V}(u, v, w, g), \\
\operatorname{irrep}_1 A & \xrightarrow{1:1} \operatorname{irrep} V(u, v, w, g)
\end{align*}$$

while in the projective case

$$\begin{align*}
\operatorname{irrep}_n A & \xrightarrow{1:1} \operatorname{Proj}(R) \setminus E' = \mathbb{P}^2 \setminus E', \\
\operatorname{irrep}_1 A & = E \xrightarrow{n:1} E'.
\end{align*}$$
3.2. $(n, 3) = 3$. If $(n, 3) = 3$, then there will be a difference between $\mathcal{R} = \text{Proj}(R)$ and $\mathcal{Z}$. Let $E' = E/(3\tau)$ and $E'' = E'/\langle \tau \rangle$. We will assume that $n \neq 3$.

**Theorem 6.** [14, Theorem 4.8] The center of $A$ is isomorphic to

$$R = \mathbb{C}[u, v, w, g]/(\Psi(u, v, w) + 3z^2g^2 + 3zg^3 + g^n)$$

with $\deg(u) = \deg(v) = \deg(w) = n, \deg(g) = 3$. The polynomial $\Psi(u, v, w)$ of degree $3n$ corresponds to $E''$ embedded in $\mathbb{P}^2_{[u, v, w]}$ and $z$ is a linear polynomial in $u, v, w$ that vanishes on three inflection points of $E''$. By Proposition 2, the map

$$\mathcal{Z} \xrightarrow{\Phi} \mathcal{R}$$

will be a quotient map by an automorphism $\alpha$ of order 3, which is the greatest common divisor of the degrees of the homogeneous elements of $R$. By [18, Theorem 4.7], $\mathcal{Z} \cong \mathbb{P}^2$ and $E' \subset \mathcal{Z}$. Let $s = \frac{2}{3}$, then again by [2, Theorem 3.4], all the fat points not annihilated by $g$ are of multiplicity $s$.

As $E'$ is the ramification locus of $\mathcal{A}$ (that is, the closed subset of $\mathcal{Z}$ where $A$ is not Azumaya), it follows that $\alpha$ has to induce an automorphism of order three on $E'$ without fix points. This can only be if $\alpha$ is conjugated to $\langle \text{diag}(1, \omega, \omega^2) \rangle \subset \text{PGL}_3(\mathbb{C})$. But then, $\alpha$ has three fixed points, which correspond to three fat points of multiplicity $s$ which are 1-periodic. They in turn correspond to three $\mathbb{C}^*$-families of simple representations of dimension $s$, with trivial $\text{PGL}_n(\mathbb{C}) \times \mathbb{C}^*$-stabilizer.

**Theorem 7.** The simple representations of $A$ come in four types. Let $S$ be a simple $A$-module with annihilator $m = \text{Ann}(S)$.

- If $g \notin m$ and $S$ is a quotient of a fat point $F$ that is not fixed by the shift functor, then $S$ is $n$-dimensional with $\text{PGL}_n(\mathbb{C}) \times \mathbb{C}^*$-stabilizer isomorphic to $\mathbb{Z}_3$. The fat point $F$ is 3-periodic and has multiplicity $s$. $F$ and its shifts $F[1]$ and $F[2]$ correspond to three different $s$-dimensional representations of $A$. Consequently, one has

$$A_{ms}/m^9 \cong \mathbb{N}_n(\mathbb{C}[t, t^{-1}])(0, 1, \ldots, 2, 0, 1, \ldots, 2, 0) \; \text{with} \; \deg(t) = 3.$$

- If $g \notin m$ and $S$ is a quotient of a fat point $F$ that is fixed by the shift functor, then $S$ is $s$-dimensional with trivial $\text{PGL}_n(\mathbb{C}) \times \mathbb{C}^*$-stabilizer. The multiplicity of $F$ is $s$ and $F$ corresponds to a unique $s$-dimensional representation of $A$. Consequently, one has

$$A_{ms}/m^9 \cong \mathbb{N}_s(\mathbb{C}[t, t^{-1}]) \; \text{with} \; \deg(t) = 1.$$

- If $g \in m$ and $S$ is not one-dimensional, then $S$ is $n$-dimensional with $\text{PGL}_n(\mathbb{C}) \times \mathbb{C}^*$-stabilizer isomorphic to $\mathbb{Z}_n$. The $\mathbb{C}^*$-orbit of $S$ corresponds to $n$ shift-equivalent point modules, who in turn correspond to $n$ 1-dimensional representations of $A$. Consequently, one has

$$A_{ms}/m^9 \cong \mathbb{N}_n(\mathbb{C}[t, t^{-1}])(0, 1, \ldots, n - 1) \; \text{with} \; \deg(t) = n.$$

- If $\dim(S) = 1$ then $S$ is the trivial representation $A/A^+$.

Regarding the center $R$ of $A$, we can now give a different description.

**Proposition 4.** The center $R$ is isomorphic to $\mathbb{C}[u, v, w, d, g]/(uvw - d^3, g^4 - 1)$ with $l$ a linear form in $u, v, w$ and $d$ such that $R/(g)$ is the homogeneous coordinate ring of $E''$, generated in degree $n$. 


Proof. By \cite[Theorem 3.7, Proposition 4.2]{18}, there are three normal elements \(u_0, v_0, w_0\) of degree \(s\) such that \(\text{Proj}(\mathbb{C}[u_0, v_0, w_0]) = \mathbb{P}^2 = \mathbb{Z}\). By the fact that the shift functor corresponds to \(\text{diag}(1, \omega, \omega^2)\) for an open subset of fat points, it follows that each irreducible representation of dimension \(n\) restricted to \(\mathbb{C}[u_0, v_0, w_0]\) is determined by the matrices for \((\lambda, \mu, \eta) \in \mathbb{A}^3\).

\[
\begin{pmatrix}
0_s & 0_s & \lambda 1_s \\
\omega 1_s & 0_s & \omega \mu 1_s \\
\omega^2 1_s & 0_s & \omega^2 \mu 1_s
\end{pmatrix},
\begin{pmatrix}
0_s & 0_s & \mu 1_s \\
\omega \mu 1_s & 0_s & 0_s \\
0_s & \omega^2 \mu 1_s & 0_s
\end{pmatrix},
\begin{pmatrix}
0_s & 0_s & \eta 1_s \\
\omega \eta 1_s & 0_s & 0_s \\
0_s & \omega^2 \eta 1_s & 0_s
\end{pmatrix}
\]

for an open subset of \(\mathbb{A}^3\). But then, \(u = u_0^3, v = v_0^3, w = w_0^3\) and \(d = u_0v_0w_0\) are sent to scalar matrices for an open subset of \(\text{Spec}(R)\), so they are central in \(A\). In addition, they are linearly independent, so they generate \(R^{(n)}\) by Theorem \cite[18]. But then \(g^s = l\) for some linear form in \(u, v, w\) and \(d\). The fact that

\[
\mathbb{V}(uvw - d^3, g^s - l) \cap \mathbb{V}(g) = E''
\]

follows from the fact that the center of \(A/(g)\) is \(\mathcal{O}(E/(\tau)) = \mathcal{O}(E'')\).

In the affine case, we now find

\[
\text{irrep}_n A \overset{1:1}{\longrightarrow} \text{irrep} R \setminus \mathbb{V}(uv, vw, uw),
\]

\[
\text{irrep}_s A \overset{3:1}{\longrightarrow} \mathbb{V}(uv, uw, uw) \setminus \mathbb{V}(u, v, w, g),
\]

\[
\text{irrep}_1 A \overset{1:1}{\longrightarrow} \mathbb{V}(u, v, w, g),
\]

which verifies \cite[Conjecture 1.5]{21}, while in the projective case we find

\[
\text{irrep}_n A \overset{1:1}{\longrightarrow} \text{irrep} Z \setminus E',
\]

\[
\text{irrep}_s A = E \overset{s:1}{\longrightarrow} E',
\]

\[
(\text{irrep} Z = \mathbb{P}^2) \setminus \mathbb{V}(u_0v_0, v_0w_0, u_0w_0) \overset{3:1}{\longrightarrow} \mathcal{R} \setminus \mathbb{V}(uv, vw, wu, d),
\]

\[
\mathbb{V}(u_0v_0, v_0w_0, u_0w_0) \overset{1:1}{\longrightarrow} \mathbb{V}(uv, vw, wu, d).
\]

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