Remark on norm compactness in $L^p(\mu, X)$.

Y. Askoura

October 29, 2020

Lemma, Université Paris II, 4 rue Blaise Desgoffe, 75006 Paris. youcef.askoura@u-paris2.fr

Abstract

We prove a compactness criterion in $L^p(\mu, X)$: a subset of $L^p(\mu, X)$ is relatively norm compact iff the set of integrals of its functions over any measurable set is relatively norm compact, it satisfies the Fréchet oscillation restriction condition and it is $p$-uniformly integrable. The proof is elementary.

1 Introduction

Relative norm compactness in spaces of Lebesgue-Bochner integrable functions is characterized by tightness together with a concentration condition (reduction of oscillations) in [14, 1, 9] for functions defined on an interval $[0, T]$. Rossi and Savaré [12] provides some generalization of these results steel for functions defined on intervals. Note that in [14] the tightness is expressed by assuming that some integrals must belong to some compact set. The general results we know, proved by Diaz and Mayoral [3] (see van Neerven [10, 11] for a different proof and [2] for the particular case $L^1(\mu, X)$), characterize norm compactness through tightness, scalar relative compactness and the $p$-uniform integrability. The scalar relative compactness is obtained by Bocce criterion: a reduction of oscillations.

Thereafter, we adopt an “integral" tightness. Instead of assuming tightness, that is: except on an arbitrarily small measurable set, the values of the considered functions belong uniformly to a compact set, we assume that their integrals over any measurable set belong to a compact set. We prove a compactness criterion as described in the abstract. Comparatively to Diaz and Mayoral [3] results, the Bocce criterion is replaced by the Fréchet one, by modifying slightly the tightness notion used in Diaz and Mayoral [5].

*This is a preliminary version.
2 Main results

Let \((\Omega, \mathcal{A}, \mu)\) be a finite measure space, where \(\mathcal{A}\) is a \(\sigma\)-algebra and \(\mu\) a countably additive positive and finite measure. The set \(\mathcal{A}^+\) refers to the set of measurable subsets \(E \in \mathcal{A}\) with \(\mu(E) > 0\). We do not make a difference between two measurable sets with a \(\mu\)-null symmetric measure.

Consider a Banach space \(X\) normed by \(\| \cdot \|\). Denote by \(L^p(\mu, X)\), \(1 \leq p < +\infty\), the Lebesgue-Bochner space consisting of all equivalence classes of (strongly) \(\mu\)-measurable \(\mu\)-a.e. equal functions \(f\) defined from \(\Omega\) to \(X\) such that \(\|f\|^p\) is \(\mu\)-integrable. The usual norm in \(L^p(\mu, X)\) is denoted by \(\| \cdot \|_p\). The set \(\Pi\) consisting of all finite partitions \(\pi \subset \mathcal{A}^+\) of \(\Omega\) is directed by refinement. Given \(\pi \in \Pi\), define on \(L^1(\mu, X)\) the conditional expectation operator

\[
\mathbb{E}_\pi(f)(\cdot) = \sum_{K \in \pi} \frac{1}{\mu(K)} \int_K fd\mu \chi_K(\cdot),
\]

where \(\chi_K(\cdot)\) stands for the characteristic function of \(K\).

Recall that a subset \(\mathcal{H} \subset L^p(\mu, X)\), \(1 \leq p < +\infty\), is said to be \(p\)-uniformly integrable iff, the set \(\{\|f\|^p : f \in \mathcal{H}\}\) is uniformly integrable. That is,

\[
\lim_{M \to +\infty} \int_{\|f\|^p > M} \|f\|^p d\mu = 0, \text{ uniformly in } f \in \mathcal{H}.
\]

Equivalently, \(\mathcal{H}\) is bounded in \(L^p(\mu, X)\) and \(\lim_{\mu(E) \to 0} \int_E \|f\|^p d\mu = 0\), uniformly in \(f \in \mathcal{H}\). For \(p > 1\) and a finite \(\mu\), it results straightforwardly from Hölder’s inequality that a \(p\)-uniformly integrable set is \((1-)\)-uniformly integrable.

**Definition 1.** A subset \(\mathcal{H} \subset L^p(\mu, X)\) is said to be integral tight iff, for every \(E \in \mathcal{A}\), \(\{\int_E f d\mu : f \in \mathcal{H}\}\) is relatively norm compact.

The following lemma is a straightforward vector valued version of Riesz Theorem. Its proof is a slight modification of the classical one \cite{3} by adding the integral tightness condition.

**Lemma 1.** A subset \(\mathcal{H}\) of \(L^p(\mu, X)\), \(1 \leq p < +\infty\), is relatively norm compact iff,

1) \(\mathcal{H}\) is integral tight and,

2) \(\lim_{\pi} \sup_{f \in \mathcal{H}} \|\mathbb{E}_\pi(f) - f\|_p = 0\).

**Proof.** \(\Rightarrow\) Let \(A \in \mathcal{A}^+\) and \(f \in L^p(\mu, X)\). Remark using Hölder’s inequality that

\[
\left( \int_A \|f\| d\mu \right)^p \leq \mu(A)^{p-1} \int_A \|f\|^p d\mu.
\]

This provides

(a) \(\|\mathbb{E}_\pi(f)\|_p \leq \|f\|_p\), and
(b) the operator

\[ \Lambda_A : L^p(\mu, X) \to X \]

\[ f \mapsto \int_A f \, d\mu \]

is continuous for the norm topologies.

Observe that 1) results immediately from (b). For 2), let \( \varepsilon > 0 \) and consider a finite covering of \( \mathcal{H} \) by open balls \( B(f_i, \varepsilon/4) \) of radius \( \varepsilon/4 \). Since the set of simple functions is dense in \( L^p(\mu, X) \), consider for every \( i \) a simple function \( \varphi_i \) such that \( \|f_i - \varphi_i\|_p \leq \varepsilon/4 \). Let \( \pi \subset \mathcal{A}^+ \) a finite partition of \( \Omega \) such that every \( \varphi_i \) is constant on every element of \( \pi \). Observe that \( \mathbb{E}_\pi(\varphi_i) = \varphi_i \) for all \( i \). Then, for every \( f \in \mathcal{H} \), there is some \( i \) such that

\[ \|\mathbb{E}_\pi(f) - f\|_p \leq \|\mathbb{E}_\pi(f) - \varphi_i\|_p + \|\varphi_i - f_i\|_p + \|f_i - f\|_p \]

\[ \leq \|\mathbb{E}_\pi(f - \varphi_i)\|_p + \varepsilon/2 \]

\[ \leq \|f - \varphi_i\|_p + \varepsilon/2 \]

\[ \leq \varepsilon. \]

It is clear that this formula remains valid for every \( \pi' \geq \pi \). Then 2) is true as well.

\( \Leftarrow \) Conversely, 1) implies that for every \( \pi \in \Pi \), \( \mathbb{E}_\pi(\mathcal{H}) \) is relatively compact as homeomorphic to a relatively compact subset of the product of \( \Lambda_A(\mathcal{H}) \), \( A \in \pi \), consisting in relatively compact sets. From 2), for every \( \varepsilon > 0 \), there is \( \pi \in \Pi \) such that \( \sup_{f \in \mathcal{H}}\|\mathbb{E}_\pi(f) - f\|_p \leq \varepsilon/3 \).

Let \( B(\mathbb{E}_\pi(f_i), \varepsilon/3), i = 1, \ldots, n \) a finite covering of \( \mathbb{E}_\pi(\mathcal{H}) \), by open balls of radius \( \varepsilon/3 \). That is, for every \( f \in \mathcal{H} \), \( \|\mathbb{E}_\pi(f) - \mathbb{E}_\pi(f_i)\| \leq \varepsilon/3 \) for some \( i \in \{1, \ldots, n\} \). Then, \( \mathcal{H} \) is covered by a finite number of open balls \( B(f_i, \varepsilon), i = 1, \ldots, n \) of radius \( \varepsilon \). Hence it is totally bounded.

Condition 2) above is a concentration or an oscillation reduction condition. Recall that the essential oscillation of a function \( f \) on a set \( A \) is:

\[ \text{EssOsc}(f, A) = \inf\{\sup \|f(\omega) - f(\omega')\| : \omega, \omega' \in A \setminus B : \mu(B) = 0\}. \]

**Definition 2 (13, 5).** A subset \( \mathcal{H} \) of \( \mu \)-measurable functions satisfies Fréchet oscillation condition (F) iff, for every \( \varepsilon > 0 \), there is a finite partition \( \pi \in \Pi \), there is a set \( \Omega_f \in \mathcal{A} \) for every \( f \in \mathcal{H} \), such that \( \mu(\Omega \setminus \Omega_f) \leq \varepsilon \), and \( \text{EssOsc}(f, \Omega_f \cap A) \leq \varepsilon \) for all \( A \in \pi \).

Let us denote the mean of a function \( f \in L_1(\mu, X) \) on \( A \in \mathcal{A}^+ \) by \( m_A(f) = \frac{1}{\mu(A)} \int_A f \, d\mu \).

Then, a simple application of the mean value Theorem for Bochner integral \( \text{[4]} \), allows to see that for every \( A \in \mathcal{A}^+ \), and \( f \in L_1(\mu, X), \)

\[ \|f(\omega) - m_A(f)\| \leq \text{EssOsc}(f, A), \mu\text{-a.e. on } A. \]

Indeed, \( m_A(f) = \frac{1}{\mu(A)} \int_A f \, d\mu \in \text{co}f(A \setminus B) \), for all \( B \subset A, B \in \mathcal{A} \) with \( \mu(B) = 0 \). Let \( \varepsilon > 0 \) be fixed. For every \( B \subset A, B \in \mathcal{A}, \mu(B) = 0 \), there is a finite convex combination \( \sum \alpha_k f(w_k') \), \( w_k' \in A \setminus B \), such that

\[ \forall \omega \in \Omega, \|f(\omega) - m_A(f)\| \leq \|f(\omega) - \sum \alpha_k f(w_k')\| + \varepsilon \leq \max_k \|f(\omega) - f(w_k')\| + \varepsilon. \]
Let $B_\varepsilon \subset A$, $B_\varepsilon \in \mathcal{A}$ with $\mu(B_\varepsilon) = 0$, such that

$$\text{EssOsc}(f, A) > \sup \{\|f(\omega) - f(\omega')\| : \omega, \omega' \in A \setminus B_\varepsilon\} - \varepsilon.$$ 

Hence, for all $\omega \in A \setminus B_\varepsilon$, $\|f(\omega) - m_A(f)\| \leq \text{EssOsc}(f, A) + 2\varepsilon$. By considering a sequence $\varepsilon_n$ decreasing to 0, the inequality holds on $A \setminus \cup B_{\varepsilon_n}$.

**Theorem 1.** A set $\mathcal{H} \subset L^p(\mu, X)$ is relatively norm compact if, it satisfies the Fréchet oscillation condition ($\mathcal{F}$), it is integral tight and $p$-uniformly integrable.

**Proof.** In the following we omit the symbol "$d\mu$" under the integral sign and denote simply $\int_A f$ instead of $\int_A f d\mu$. All the integrals are related to the measure $\mu$.

$\implies$ The $p$-uniform integrability of $\mathcal{H}$ is obvious (results for instance from the obvious relative weak compactness of the set $\{\omega \mapsto \|f(\omega)\|^p : f \in \mathcal{H}\}$ in $L^1(\mu)$). The integral tightness follows from Lemma 1. Observe now that we can assume without loss of generality that the functions of $\mathcal{H}$ take their values in a separable complete metric space. Indeed, since $\mathcal{H}$ is relatively norm compact, it is relatively compact for the topology of convergence in measure. Then it is tight [6]. That is, for every $n \geq 1$, there exists a compact subset $D_n$ of $X$ such that $\mu(\{\omega \in \Omega : f(\omega) \notin D_n\}) \leq 1/n$, for every $f \in \mathcal{H}$. Hence, up to modifying the functions of $\mathcal{H}$ on a $\mu$-null set, we can assume that all the functions of $\mathcal{H}$ take their values on the closure of a the $\sigma$-compact set $\bigcup_n D_n$ which is separable and complete. Hence, we can apply the Fréchet Theorem ([13], p. 425) to affirm that $\mathcal{H}$ satisfies ($\mathcal{F}$).

$\impliedby$ We have to prove that Condition 2) of Lemma 1 is satisfied. Then, the conclusion follows from Lemma 1.

Let $\varepsilon > 0$ be fixed. We can assume without loss of generality that $\varepsilon \leq 1$. The $p$-uniform integrability of $\mathcal{H}$ implies its uniform integrability. We know that the uniform integrability of $\mathcal{H}$ implies the uniform integrability of the set of conditional expectations $\{\mathbb{E}_\pi(\|f\|) : f \in \mathcal{H}, \pi \in \Pi\}$. For the reader's convenience, by Markov's inequality

$$\mu(\mathbb{E}_\pi(\|f\|) > M) \leq \frac{1}{M} \int_\Omega \mathbb{E}_\pi(\|f\|) = \frac{1}{M} \int_\Omega \|f\|,$$

and

$$\int_{\mathbb{E}_\pi(\|f\|) > M} \mathbb{E}_\pi(\|f\|) = \int_{\mathbb{E}_\pi(\|f\|) > M} \|f\|.$$

Hence, $\lim_{M \to +\infty} \int_{\mathbb{E}_\pi(\|f\|) > M} \mathbb{E}_\pi(\|f\|) = 0$ uniformly in $f \in \mathcal{H}$, provided that $\mathcal{H}$ is uniformly integrable.

Now, by Jensen's inequality, since $x \mapsto \|x\|^p$ is convex, $\|\mathbb{E}_\pi(f)\|^p \leq \mathbb{E}_\pi(\|f\|^p)$, $\mu$-a.e on $\Omega$. Therefore, the $p$-uniform integrability of $\mathcal{H}$ implies the uniform integrability of

$$\{\|f\|, \|f\|^p, \mathbb{E}_\pi(\|f\|), \mathbb{E}_\pi(\|f\|^p), \|\mathbb{E}_\pi(f)\|^p : f \in \mathcal{H}, \pi \in \Pi\}.$$

It results that, there is $\delta \in [0, \varepsilon]$, such that $\mu(E) \leq \delta$ implies

$$\max \left\{\int_E \|f\|, \int_E \|f\|^p, \int_E \mathbb{E}_\pi(\|f\|), \int_E \mathbb{E}_\pi(\|f\|^p), \int_E \|\mathbb{E}_\pi(f)\|^p\right\} \leq \varepsilon, \forall f \in \mathcal{H}, \forall \pi \in \Pi.$$
Using ($\mathcal{F}$), let $\pi_0 \subset A^+$ be a partition of $\Omega$, $\Omega_f \in A^+, f \in \mathcal{H}$, such that $\mu(\Omega \setminus \Omega_f) \leq \delta \leq \varepsilon$ and $\text{EssOsc}(f, \Omega_f \cap K) \leq \delta \leq \varepsilon$ for all $f \in \mathcal{H}$ and $K \in \pi_0$. Clearly, for every $\pi > \pi_0$, $\text{EssOsc}(f, \Omega_f \cap K) \leq \delta \leq \varepsilon$ for all $f \in \mathcal{H}$ for every $K \in \pi$. Let $\pi > \pi_0$ be fixed and let us denote, for $f \in \mathcal{H}$,

$$\pi_f = \{K \in \pi : \mu(\Omega_f \cap K) > 0\}.$$

Then, for all $f \in \mathcal{H}$, setting $I_\pi = \int_\Omega \|f - \mathbb{E}_\pi(f)\|^p,$

$$I_\pi \leq \sum_{K \in \pi} \int_{\Omega_f \cap K} \|f - m_K(f)\|^p + 2^p \int_{\Omega \setminus \Omega_f} \|f\|^p + \|\mathbb{E}_\pi(f)\|^p,$$

$$\leq 2^p \sum_{K \in \pi_f} \int_{\Omega_f \cap K} \|f - m_{\Omega_f \cap K}(f)\|^p + 2^p \sum_{K \in \pi_f} \mu(\Omega_f \cap K)\|m_{\Omega_f \cap K}(f) - m_K(f)\|^p + 2^{p+1} \varepsilon,$$

$$\leq 2^p \mu(\Omega_f), \varepsilon^p + 2^p \sum_{K \in \pi_f} \mu(\Omega_f \cap K)((\frac{1}{\mu(\Omega_f \cap K)} - \frac{1}{\mu(K)}) \int_{\Omega_f \cap K} f - \frac{1}{\mu(K)} \int_{\Omega_f \cap K} f \|^p + 2^{p+1} \varepsilon,$$

$$\leq 2^p \mu(\Omega_f), \varepsilon^p + 2^{2p} \sum_{K \in \pi_f} \left(\frac{\mu(K \setminus \Omega_f)^p \mu(\Omega_f \cap K)^{1-p}}{\mu(K)^p} \int_{\Omega_f \cap K} f \|^p + \frac{\mu(\Omega_f \cap K)^p}{\mu(K)^p} \int_{\Omega_f \cap K} f \|^p\right) + 2^{p+1} \varepsilon.$$

But, from Hölder’s inequality, for every $K \in \pi$, such that $\mu(\Omega_f \cap K) > 0$ and $\mu(K \setminus \Omega_f) > 0$,

$$\mu(\Omega_f \cap K)^{1-p} \int_{\Omega_f \cap K} f \|^p \leq \int_{\Omega_f \cap K} \|f\|^p$$

and $\int_{K \setminus \Omega_f} \|f\|^p \leq \mu(K \setminus \Omega_f)^{p-1} \int_{K \setminus \Omega_f} \|f\|^p.$

Then,

$$I_\pi \leq 2^p \mu(\Omega_f), \varepsilon^p + 2^{2p} \sum_{K \in \pi_f, \mu(K \setminus \Omega_f) > 0} \left(\frac{\mu(K \setminus \Omega_f)^p \mu(\Omega_f \cap K)^{1-p}}{\mu(K)^p} \int_{\Omega_f \cap K} f \|^p + \frac{\mu(\Omega_f \cap K)^p}{\mu(K)^p} \int_{\Omega_f \cap K} f \|^p\right) + 2^{p+1} \varepsilon,$$

$$\leq 2^p \mu(\Omega_f), \varepsilon^p + 2^{2p} \sum_{K \in \pi_f, \mu(K \setminus \Omega_f) > 0} \left[\left(\frac{\mu(K \setminus \Omega_f)}{\mu(K)}\right)^{p-1} \frac{\mu(\Omega_f \cap K)}{\mu(K)} \int_{\Omega_f \cap K} f \|^p + \frac{\mu(\Omega_f \cap K)^p}{\mu(K)^p} \int_{\Omega_f \cap K} f \|^p\right] + 2^{p+1} \varepsilon,$$

$$\leq 2^p \mu(\Omega_f), \varepsilon^p + 2^{2p} \sum_{K \in \pi_f} \left(\mu(K \setminus \Omega_f) \frac{1}{\mu(K)} \int_{\Omega_f \cap K} f \|^p + \int_{K \setminus \Omega_f} \|f\|^p\right) + 2^{p+1} \varepsilon,$$

$$\leq 2^p \mu(\Omega_f), \varepsilon^p + 2^{2p} \left(\int_{\Omega \setminus \Omega_f} \|f\|^p + \int_{\Omega \setminus \Omega_f} \|f\|^p\right) + 2^{p+1} \varepsilon,$$

$$\leq 2^p \mu(\Omega_f), \varepsilon^p + 2^{2p} \left(\int_{\Omega \setminus \Omega_f} \mathbb{E}_\pi(\|f\|^p) + \int_{\Omega \setminus \Omega_f} \|f\|^p\right) + 2^{p+1} \varepsilon,$$

$$\leq [\mu(\Omega) + 2^{p+1} + 2]2^p \varepsilon = T \varepsilon.$$
Where, we used in the last inequality the fact that \( \varepsilon \in ]0, 1] \) and denoted by \( T \) the constant 
\[ \mu(\Omega) + 2^{p+1} + 2 \] independent from \( f, \pi \) and \( \varepsilon \). Therefore, Condition 2) of Lemma 1
follows straightforwardly.

Remark 1. The proof of the previous Theorem establishes that \( p \)-uniform integrability and Fréchet oscillation restriction condition \((F)\) implies Condition 2) of Lemma 1.

We end with a comment on tightness and integral tightness. Let us start by observing that integral tightness does not imply tightness. Consider the following:

Example 1. Consider the space of absolutely summable sequences \( l^1 \) endowed with its usual norm and set \( \Omega = [0, 1] \) endowed with its Borel \( \sigma \)-algebra \( A \) and set \( \mu \) to be the Lebesque measure. Let \( (r_n)_{n \geq 1} \) be the Rademacher sequence on \([0, 1]\) and \( (e_n)_{n \geq 1} \) the canonical basis of \( l^1 \): \( e_n \) is the element of \( l^1 \), with vanishing terms except the \( n^{th} \) one equals to 1. Set \( H = \{r_n e_n : n \geq 1\} \subset L^1(\mu, l^1) \). For every \( A \in A \),
\[ \| \int_A r_n e_n d\mu \| = | \int_A r_n d\mu \| e_n \| = | \int_A r_n d\mu |. \]

Since, \( (r_n) \) converges weakly in \( L^1(\mu) \) to 0, the sequence of norms \( \| \int_A r_n e_n d\mu \| \) converges to 0 in \( l^1 \). Consequently, the set \( \{\int_A r_n e_n d\mu : n \geq 1\} \) is relatively norm compact in \( l^1 \), for every \( A \in A \). That is \( H \) is integral tight. However, for \( \varepsilon = \frac{1}{2} \), every compact \( K \) in \( l^1 \) cannot contain all the vectors \( e_n, n \geq 1 \). Hence, for every compact \( K \) of \( l^1 \), there is \( e_{n_K} \not\in K \). Then, \( \mu\{\omega \in \Omega : r_{n_K} e_{n_K} \not\in K\} \geq \frac{1}{2} \geq \varepsilon \). This means that \( H \) is not tight.

In contrast with the previous example, it is known that tightness together with \( p \)-uniform integrability imply integral tightness \([4, 8]\). A straightforward application of this result together with the results above allow to establish a vector valued (separable) version of the well known Vitali’s Theorem.

Corollary 1. Assume that \( X \) is separable. A subset \( H \) of \( L^p(\mu, X) \) is relatively norm compact iff it is relatively compact for the convergence in measure topology and \( p \)-uniformly integrable.

Proof. \( \Rightarrow \) is obvious.
\( \Leftarrow \) Since \( H \) is relatively compact in measure, it is tight. Together with its \( p \)-uniform integrability, this implies its integral tightness \([4]\). The Fréchet oscillation \((F)\) results as well from the convergence in measure \([13]\). We conclude the result from Theorem \([4]\).

References

[1] J. P. Aubin, Un théorème de compacité, C. R. Acad. Sci. Paris 256 (1963), 5042-5044.

[2] E. J. Balder, M. Girardi and V. Jalbi, From weak to strong types of \( L^1_E \)-convergence by the Bocce criterion, Studia Mathematica 111(1994), 241-262.
[3] V. I. Bogachev, Measure theory (2 volumes). Springer-Verlag, Berlin, 2007.

[4] C. Castaing, Un résultat de compacité lié à la propriété des ensembles Dunford-Pettis dans $L^1_F(\Omega, A, \mu)$, *Travaux Sém. Anal. Convexe* 9(1979), no. 2, Exp. No. 17, 7 pp.

[5] S. Diaz and F. Mayoral, On compactness in spaces of Bochner integrable functions, *Acta Math. Hungar.*, 83(1999), 231-239.

[6] S. Díaz and F. Mayoral, Compactness in measure and the sequential Bourgain property, *Arch. Math.* 71(1998), 55-62.

[7] J. Diestel and J. J. Uhl, Vector measures. Mathematical Surveys, Number 15, American Mathematical Society, 1977.

[8] V. Jalbi, Contribution aux problèmes de convergence des fonctions vectorielles et des intégrales fonctionnelles, *PhD thesis, University Montpellier II*, 1993.

[9] J. L. Lions, Equations différentielles opérationelles et problèmes aux limites, *Springer, Berlin*, 1961.

[10] J. van Neerven, Compactness in the Lebesgue-Bochner spaces $L^p(\mu, X)$, *Indagationes Mathematicae*, 25 (2014), 389-394.

[11] J. van Neerven, Compactness in vector-valued Banach function spaces, *Positivity* 11 (2007) 461-467.

[12] R. Rossi and G. Savaré, Tightness, Integral Equicontinuity and Compactness for Evolution Problems in Banach Spaces, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 5(2003), 395-431.

[13] M. Saadoune and M. Valadier, Convergence in measure. Local formulation of the Fréchet criterion, *C. R. Acad. Sci. Paris, Série I*, 320(1995), 423-428.

[14] J. Simon, Compact Sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.* 146 (1987), 65-96.