Entire solutions of multivalued nonlinear Schrödinger equations in Sobolev spaces with variable exponent

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Abstract. We establish the existence of an entire solution for a class of stationary Schrödinger equations with subcritical discontinuous nonlinearity and lower bounded potential that blows-up at infinity. The abstract framework is related to Lebesgue–Sobolev spaces with variable exponent. The proof is based on the critical point theory in the sense of Clarke and we apply Chang’s version of the Mountain Pass Lemma without the Palais–Smale condition for locally Lipschitz functionals. Our result generalizes in a nonsmooth framework a result of Rabinowitz [35] on the existence of ground-state solutions of the nonlinear Schrödinger equation.

Key words: Schrödinger equation, entire solution, Lipschitz functional, Clarke generalized gradient, critical point.

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1 Introduction and auxiliary results

The Schrödinger equation plays the role of Newton’s laws and conservation of energy in classical mechanics, that is, it predicts the future behaviour of a dynamic system. The linear form of Schrödinger’s equation is

$$\Delta \psi + \frac{8\pi^2 m}{\hbar^2} (E(x) - V(x)) \psi = 0,$$

where $\psi$ is the Schrödinger wave function, $m$ is the mass, $\hbar$ denotes Planck’s constant, $E$ is the energy, and $V$ stands for the potential energy. The structure of the nonlinear Schrödinger equation is much more complicated. This equation is a prototypical dispersive nonlinear partial differential equation that has been central for almost four decades now to a variety of areas in Mathematical Physics. The relevant fields of application may vary from optics and propagation of the electric field in optical fibers (Hasegawa and Kodama [20], Malomed [27]), to the self-focusing and collapse of Langmuir waves in plasma physics (Zakharov [40]) and the behaviour of deep water waves and freak waves (the so-called rogue waves) in the ocean (Benjamin and Feir [5] and Onorato, Osborne, Serio and Bertone [32]). The nonlinear Schrödinger equation also describes various phenomena arising in: self-channelling of a high-power ultra-short laser in matter, in the theory of Heisenberg ferromagnets and magnons, in dissipative quantum mechanics, in condensed matter theory, in plasma physics (e.g., the Kurihara superfluid film equation). We refer to Ablowitz, Prinari and Trubatch [1], Grosse and Martin [19] and Sulem [38] for a modern overview, including applications.
Consider the model problem
\[ i\hbar \psi_t = -\frac{\hbar^2}{2m} \Delta \psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \quad \text{in } \mathbb{R}^N \ (N \geq 2), \tag{1} \]
where \( p < 2N/(N-2) \) if \( N \geq 3 \) and \( p < +\infty \) if \( N = 2 \). In physical problems, a cubic nonlinearity corresponding to \( p = 3 \) is common; in this case \( (1) \) is called the Gross-Pitaevskii equation. In the study of Eq. \( (1) \), Oh [31] supposed that the potential \( V \) is bounded and possesses a non-degenerate critical point at \( x = 0 \). More precisely, it is assumed that \( V \) belongs to the class \( (V_a) \) (for some \( a \)) introduced in Kato [22]. Taking \( \gamma > 0 \) and \( \hbar > 0 \) sufficiently small and using a Lyapunov-Schmidt type reduction, Oh [31] proved the existence of a standing wave solution of Problem \( (1) \), that is, a solution of the form
\[ \psi(x, t) = e^{-iEt/\hbar}u(x). \tag{2} \]
Note that substituting the ansatz \( (2) \) into \( (1) \) leads to
\[ -\frac{\hbar^2}{2} \Delta u + (V(x) - E)u = |u|^{p-1}u. \]
The change of variable \( y = h^{-1}x \) (and replacing \( y \) by \( x \)) yields
\[ -\Delta u + 2(V_h(x) - E)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{3} \]
where \( V_h(x) = V(hx) \).

If for some \( \xi \in \mathbb{R}^N \setminus \{0\} \), \( V(x + s\xi) = V(x) \) for all \( s \in \mathbb{R} \), equation \( (1) \) is invariant under the Galilean transformation
\[ \psi(x, t) \mapsto \psi(x - \xi t, t) \exp \left( i\xi \cdot \frac{x}{h} - \frac{1}{2}i|\xi|^2t/\hbar \right) \psi(x - \xi t, t). \]
Thus, in this case, standing waves reproduce solitary waves travelling in the direction of \( \xi \). In a celebrated paper, Rabinowitz [35] proved that Equation \( (3) \) has a ground-state solution (mountain-pass solution) for \( \hbar > 0 \) small, under the assumption that \( \inf_{x \in \mathbb{R}^N} V(x) > E \). After making a standing wave ansatz, Rabinowitz reduces the problem to that of studying the semilinear elliptic equation
\[ -\Delta u + a(x)u = f(x, u) \quad \text{in } \mathbb{R}^N, \tag{4} \]
under suitable conditions on \( a \) and assuming that \( f \) is smooth, superlinear and subcritical.

Our purpose in this paper is to study the multivalued version of Equation \( (3) \). For a more general class of differential operators, the so-called \( p(x) \)-Laplace operators. This degenerate quasilinear operator is defined by \( \Delta_{p(x)}u := \text{div}(|\nabla u|^{p(x)-2}\nabla u) \) (where \( p(x) \) is a certain function whose properties will be stated in what follows) and it generalizes the celebrated \( p \)-Laplace operator \( \Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u) \), where \( p > 1 \) is a constant. The \( p(x) \)-Laplace operator possesses more complicated nonlinearity than the \( p \)-Laplacian, for example, it is inhomogeneous. We only recall that \( \Delta_p \) describes a variety of phenomena in the nature. For instance, the equation governing the motion of a fluid involves the \( p \)-Laplace operator. More exactly, the shear stress \( \tau \) and the velocity gradient \( \nabla u \) of the fluid are related in the manner that \( \tau(x) = r(x)|\nabla u|^{p-2}\nabla u \), where \( p = 2 \) (resp., \( p < 2 \) or \( p > 2 \)) if the fluid is Newtonian (resp., pseudoplastic or dilatant). Other applications of the \( p \)-Laplacian also appear in the study of flow through porous media \( (p = 3/2) \), Nonlinear Elasticity \((p \geq 2)\), or Glaciology \((1 < p \leq 4/3)\).

The analysis we develop in this paper is carried out in terms of Clarke’s critical point theory for locally Lipschitz functionals and in generalized Sobolev spaces. That is why we recall in this section
some basic facts related to Clarke’s generalized gradient (see Clarke \[9, 10\] for more details) and Lebesgue-Sobolev spaces with variable exponent.

Let $E$ be a real Banach space and assume that $I : E \to \mathbb{R}$ is a locally Lipschitz functional. Then the Clarke generalized gradient is defined by

$$\partial I(u) = \{ \xi \in E^* ; I^0(u, v) \geq \langle \xi, v \rangle, \text{ for all } v \in E \},$$

where $I^0(u, v)$ stands for the directional derivative of $I$ at $u$ in the direction $v$, that is,

$$I^0(u, v) = \limsup_{\lambda \to 0^+} \frac{I(w + \lambda v) - I(w)}{\lambda}.$$

Variable exponent Lebesgue spaces appeared in the literature for the first time already in a 1931 article by W. Orlicz \[33\]. In the years 1950 this study was carried on by Nakano \[30\] who made the first systematic study of spaces with variable exponent. Later, the Polish mathematicians investigated the modular function spaces (see, e.g., the basic monograph Musielak \[29\]). Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context we refer to the works of Sharapudinov \[37\], Tsenov \[39\] and Zhikov \[41\, 42\]. For deep results in weighted Sobolev spaces with applications to partial differential equations and nonlinear analysis we refer to the excellent monographs by Drabek, Kufner and Nicolosi \[13\], Hyers, Isac and Rassias \[21\], Kufner and Persson \[25\], and Precup \[34\]. We also refer to the recent works by Diening \[11\], Ruzicka \[36\] and Chen, Levine and Rao \[8\] for applications of Sobolev spaces with variable exponent in the study of electrorheological fluids or in image restoration.

We recall in what follows some definitions and basic properties of the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an arbitrary domain with smooth boundary.

Set

$$C_+ (\Omega) = \{ h; h \in C(\Omega), h(x) \geq 2 \text{ for all } x \in \Omega \}.$$

For any $h \in C_+ (\Omega)$ we define

$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{ and } \quad h^- = \inf_{x \in \Omega} h(x).$$

For any $p(x) \in C_+ (\Omega)$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{ u; \text{ } u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)} \, dx < \infty \}.$$

On this space we define the Luxemburg norm by the formula

$$|u|_{p(x)} = \inf \left\{ \mu > 0; \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} \, dx \leq 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces \[23\, \text{Theorem 2.5}\], the Hölder inequality holds \[23\, \text{Theorem 2.1}\], they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ \[23\, \text{Corollary 2.7}\] and continuous functions are dense if $p^+ < \infty$ \[23\, \text{Theorem 2.11}\]. The inclusion between Lebesgue spaces also generalizes naturally \[23\, \text{Theorem 2.8}\]: if $0 < |\Omega| < \infty$ and $r_1, r_2$ are variable exponents so that $r_1(x) \leq r_2(x)$ almost everywhere in $\Omega$ then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$. 

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We denote by $L^{p'}(x)(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'}(x)(\Omega)$ the H"older type inequality

$$\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p} + \frac{1}{p'} \right) |u|_{p(x)}^{p(x)} |v|_{p'(x)}^{p'(x)}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} \, dx.$$  

If $(u_n), u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$$

$$|u_n - u|_{p(x)} \to 0 \iff \rho_{p(x)}(u_n - u) \to 0.$$  

Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [14].

Next, we define $W^{1,p(x)}_0(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$  

The space $(W^{1,p(x)}_0(\Omega), \| \cdot \|_{p(x)})$ is a separable and reflexive Banach space. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding $W^{1,p(x)}_0(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact (if $\Omega$ is bounded) and continuous (for arbitrary $\Omega$), where $p^*(x) = \frac{np(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \geq N$. We refer to Edmunds and Rákosník [15] [16], Fan, Shen and Zhao [17], Fan and Zhao [18], and Kováčik and Rákosník [23] for further properties of variable exponent Lebesgue-Sobolev spaces.

2 The main result

For any function $h(x, \cdot) \in L_\text{loc}^\infty(\mathbb{R})$ we denote by $\underline{h}$ (resp., $\overline{h}$) the lower (resp., upper) limit of $h$ in its second variable, that is,

$$\underline{h}(x,t) = \lim_{\varepsilon \searrow 0} \text{essinf}_{|t-s| < \varepsilon} \{h(x,s); \, |t-s| < \varepsilon\}; \quad \overline{h}(x,t) = \lim_{\varepsilon \searrow 0} \text{esssup}_{|t-s| < \varepsilon} \{h(x,s); \, |t-s| < \varepsilon\}.$$  

Let $a \in L_\text{loc}^\infty(\mathbb{R}^N)$ be a variable potential such that, for some $a_0 > 0$,

$$a(x) \geq a_0 \quad \text{a.e. } x \in \mathbb{R}^N \quad \text{and} \quad \text{ess lim}_{|x| \to \infty} a(x) = +\infty.$$  

We assume throughout this paper that $p \in C_+(\mathbb{R}^N)$ ($N \geq 2$) such that $p^+$ is finite.
Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that, for some $C > 0$, $q \in \mathbb{R}$ with $p^+ < q + 1 \leq Np^-/(N - p^-)$ if $p^- < N$ and $p^+ < q + 1 < +\infty$ if $p^- \geq N$, and $\mu > p^+$, we have

$$|f(x,t)| \leq C(|t| + |t|^q) \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

\begin{equation}
\lim_{\varepsilon \to 0} \text{esssup} \left\{ \frac{|f(x,t)|}{|t|^{p^- - 1}} \left| (x,t) \in \mathbb{R}^N \times (-\varepsilon, \varepsilon) \right. \right\} = 0;
\end{equation}

\begin{equation}
0 \leq \mu F(x,t) \leq tf(x,t) \quad \text{a.e. } (x,t) \in \mathbb{R}^N \times [0, +\infty).
\end{equation}

Our hypothesis $q \leq Np^-/(N - p^-)$ enables us to allow an almost critical behaviour on $f$. We also point out that we do not assume that the nonlinearity $f$ is continuous. An example of nonlinearity that fulfills assumptions (10)–(12) is any positive discontinuous function $f(x,t)$ with subcritical growth that obeys like $t^a$ ($a > p^+$) in a neighborhood of the origin; for instance, take $N = 3$, $p^- = 2$, $p^+ = 5$ (say, $p(x) = (7 + 3 \sin |x|)/2$), and

$$f(x,t) = \begin{cases} 
t^5 + t^6 & \text{if } (x,t) \in \mathbb{R}^N \times [0,1) \\
et^t & \text{if } (x,t) \in \mathbb{R}^N \times [1,10] \\
\rho(x) + t^5 & \text{if } (x,t) \in \mathbb{R}^N \times (10, \infty) \\
-f(x,t) & \text{if } (x,t) \in \mathbb{R}^N \times (-\infty,0),
\end{cases}$$

where $\rho(x) = +1$ if $|x| \in \mathbb{Q}$ and $\rho(x) = 0$, otherwise.

Let $E$ denote the set of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $[a(x)]^{1/p(x)}u \in L^{p(x)}(\mathbb{R}^N)$ and $\nabla u \in L^{p(x)}(\mathbb{R}^N)$. Then $E$ is a Banach space if it is endowed with the norm

$$\|u\|_E := \left| [a(x)]^{1/p(x)} u \right|_{p(x)} + |\nabla u|_{p(x)}.$$

We remark that $E$ is continuously embedded in $W^{1,p(x)}(\mathbb{R}^N)$. In the case $p(x) \equiv 2$ and if the potential $a(x)$ fulfills more general hypotheses than [4], then the embedding $E \subset L^{q+1}(\mathbb{R}^N)$ is compact, whenever $2 \leq q < (N + 2)/(N - 2)$ (see, e.g., Bartsch, Liu and Weth [3] and Bartsch, Pankov and Wang [4]).

We do not know if this compact embedding still holds true in our “variable exponent” framework and under assumption [4].

Throughout this paper we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $E^*$ and $E$.

Set $F(x,t) := \int_0^t f(s)ds$ and

$$\Psi(u) := \int_{\mathbb{R}^N} F(x,u(x))dx.$$ 

We observe that $\Psi$ is locally Lipschitz on $E$. This follows by [10], Hölder’s inequality and the continuous embedding $E \subset L^{q+1}(\mathbb{R}^N)$. Indeed, for all $u, v \in E$,

$$|\Psi(u) - \Psi(v)| \leq C\|u - v\|_E,$$

where $C = C(\|u\|_E, \|v\|_E) > 0$ depends only on $\max\{\|u\|_E, \|v\|_E\}$. 

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In this paper we are concerned with the problem

\[
\begin{cases}
-\text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) + a(x)|u|^{p(x)-2}u \in [\underline{f}(x, u), \overline{f}(x, u)] & \text{in } \mathbb{R}^N \\
u \geq 0, \, u \not\equiv 0 & \text{in } \mathbb{R}^N.
\end{cases}
\]  

(13)

We notice that the semilinear anisotropic case corresponding to \( p(x) \equiv 2 \) has been analyzed in Gazzola and Rădulescu [12].

We refer to Bertone–do Ó [6] and Kristály [24] for the study (by means of other methods) of certain classes of Schrödinger type equations which involve discontinuous nonlinearities.

**Definition 1.** We say that \( u \in E \) is a solution of Problem (13) if \( u \geq 0, \, u \not\equiv 0 \), and \( 0 \in \partial I(u) \), where

\[
I(u) := \int_{\mathbb{R}^N} \frac{1}{p(x)} \left( |\nabla u|^{p(x)} + a(x)|u|^{p(x)} \right) dx - \int_{\mathbb{R}^N} F(x, u^+) dx, \quad \text{for all } u \in E.
\]

The mapping \( I : E \to \mathbb{R} \) is called the energy functional associated to Problem (13). Our previous remarks show that \( I \) is locally Lipschitz on the Banach space \( E \).

The above definition may be reformulated, equivalently, in terms of hemivariational inequalities. More precisely, \( u \in E \) is a solution of (13) if \( u \geq 0, \, u \not\equiv 0 \) in \( \mathbb{R}^N \), and

\[
\int_{\mathbb{R}^N} \left( |\nabla u|^{p(x)-2} \nabla u \nabla v + a(x)|u|^{p(x)-2}uv \right) dx + \int_{\mathbb{R}^N} (-F)^0(x, u; v) dx \geq 0, \quad \text{for all } v \in E.
\]

Our main result is the following

**Theorem 1.** Assume that hypotheses (9)–(12) are fulfilled. Then Problem (13) has at least one solution.

**3 Proof of Theorem 1**

We first claim that there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
f(x, t) \geq C_1 t^{\mu-1} - C_2 \quad \text{a.e. } (x, t) \in \mathbb{R}^N \times [0, +\infty).
\]  

(14)

Indeed, by the definition of \( \underline{f} \) we deduce that

\[
\underline{f}(x, t) \leq f(x, t) \quad \text{a.e. } (x, t) \in \mathbb{R}^N \times [0, +\infty).
\]  

(15)

Set \( F(x, t) := \int_0^t \underline{f}(x, s) ds \). Thus, by our assumption (12),

\[
0 \leq \mu F(x, t) \leq t \underline{f}(x, t) \quad \text{a.e. } (x, t) \in \mathbb{R}^N \times [0, +\infty).
\]  

(16)

Next, by (16), there exist positive constants \( R \) and \( K_1 \) such that

\[
\underline{F}(x, t) \geq K_1 t^{\mu} \quad \text{a.e. } (x, t) \in \mathbb{R}^N \times [R, +\infty).
\]  

(17)

Our claim (14) follows now directly by relations (15), (16) and (17).

Next, we observe that

\[
\partial I(u) = -\text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) + a(x)|u|^{p(x)-2}u - \partial \Psi(u^+) \quad \text{in } E^*.
\]
So, by [7, Theorem 2.2] and [28, Theorem 3], we have
\[ \partial \Psi(u) \subset [f(x, u(x)), \overline{f}(x, u(x))] \quad \text{a.e. } x \in \mathbb{R}^N, \]
in the sense that if \( w \in \partial \Psi(u) \) then
\[ f(x, u(x)) \leq w(x) \leq \overline{f}(x, u(x)) \quad \text{a.e. } x \in \mathbb{R}^N. \] (18)

This means that if \( u_0 \) is a critical point of \( I \), then there exists \( w \in \partial \Psi(u_0) \) such that
\[ -\text{div} ([|\nabla u_0|^{p(x)} - 2\nabla u_0] + a(x)|u_0|^{p(x)} - 2u_0 = w \quad \text{in } E^*. \]

This argument shows that, for proving Theorem [1] it is enough to show that the energy functional \( I \) has at least a nontrivial critical point \( u_0 \in E, u_0 \geq 0 \). We prove the existence of a solution of Problem [13] by arguing that the hypotheses of Chang’s version of the Mountain Pass Lemma for locally Lipschitz functionals (see Chang [7]) are fulfilled. More precisely, we check the following geometric assumptions:

\[ I(0) = 0 \text{ and there exists } v \in E \text{ such that } I(v) \leq 0; \] (19)

there exist \( \beta, \rho > 0 \) such that \( I \geq \beta \) on \( \{u \in E; \|u\|_E = \rho\} \). (20)

**Verification of (19).** Fix \( w \in C_c^\infty(\mathbb{R}^N) \setminus \{0\} \) such that \( w \geq 0 \) in \( \mathbb{R}^N \). In particular, we have
\[ \int_{\mathbb{R}^N} \left(|\nabla w|^{p(x)} + a(x)w^{p(x)}\right) dx < +\infty. \]

So, by (14) and choosing \( t > 1 \),
\[ I(tw) = \int_{\mathbb{R}^N} \frac{t^{p(x)}}{p(x)} \left(|\nabla w|^{p(x)} + a(x)w^{p(x)}\right) dx - \Psi(tw) \]
\[ \leq \frac{t^{p^+}}{p_-} \int_{\mathbb{R}^N} \left(|\nabla w|^{p(x)} + a(x)w^{p(x)}\right) dx + C_2 t \int_{\mathbb{R}^N} w dx - C_1 t^\mu \int_{\mathbb{R}^N} w^\mu dx. \]

Since, by hypothesis, \( 1 < p^+ < \mu \), we deduce that \( I(tw) < 0 \) for \( t > 1 \) large enough.

**Verification of (20).** Our hypotheses (10) and (11) imply that, for any \( \epsilon > 0 \), there exists some \( C_\epsilon > 0 \) such that
\[ |f(x, t)| \leq \epsilon |t| + C_\epsilon |t|^{q+1} \quad \text{a.e. } (x, t) \in \mathbb{R}^N \times \mathbb{R}. \] (21)

By (21) and Sobolev embeddings in variable exponent spaces we have, for any \( u \in E \),
\[ \Psi(u) \leq \epsilon \int_{\mathbb{R}^N} \frac{1}{p(x)} |u|^{p(x)} dx + \frac{A_\epsilon}{q+1} \int_{\mathbb{R}^N} |u|^{q+1} dx \leq \epsilon \int_{\mathbb{R}^N} \frac{1}{p(x)} |u|^{p(x)} dx + C_4 \|u\|_{L^{q+1}(\mathbb{R}^N)}^{q+1}, \]
where \( \epsilon \) is arbitrary and \( C_4 = C_4(\epsilon) \). Thus, by our hypotheses,
\[ I(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\right) dx - \Psi(u^+) \]
\[ \geq \frac{1}{p^+} \int_{\mathbb{R}^N} \left[|\nabla u|^{p(x)} + (a_0 - \epsilon)|u|^{p(x)}\right] dx - C_4 \|u\|_{L^{q+1}(\mathbb{R}^N)}^{q+1} \geq \beta > 0, \]

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for $\|u\|_E = \rho$, with $\rho$, $\varepsilon$ and $\beta$ are small enough positive constants.

Denote
\[
P := \{\gamma \in C([0,1], E); \gamma(0) = 0, \gamma(1) \neq 0 \text{ and } I(\gamma(1)) \leq 0\}
\]
and
\[
c := \inf_{\gamma \in P} \max_{t \in [0,1]} I(\gamma(t)).
\]
Set
\[
\lambda_I(u) := \min_{\zeta \in \partial I(u)} \|\zeta\|_{E^*}.
\]
We are now in position to apply Chang’s version of the Mountain Pass Lemma for locally Lipschitz functionals (see Chang [7]). So, there exists a sequence \(\{u_n\} \subset E\) such that
\[
I(u_n) \to c \quad \text{and} \quad \lambda_I(u_n) \to 0.
\]
Moreover, since $I(|u|) \leq I(u)$ for all $u \in E$, we can assume without loss of generality that $u_n \geq 0$ for every $n \geq 1$. So, for all positive integer $n$, there exists \(\{w_n\} \subset \partial \Psi(u_n) \subset E^*\) such that, for any $v \in E$,
\[
\int_{\mathbb{R}^N} \left( |\nabla u_n|^p(x) - 2 \nabla u_n \nabla v + a(x) u_n^{p(x)-1} v \right) dx - \langle w_n, v \rangle \to 0 \quad \text{as} \quad n \to \infty.
\]
Note that for all $u \in E$, $u \geq 0$, the definition of $\Psi$ and our hypotheses yield
\[
\Psi(u) \leq \frac{1}{\mu} \int_{\mathbb{R}^N} u(x) f(x, u(x)) dx.
\]
Therefore, by (18), for every $u \in E$, $u \geq 0$, and for any $w \in \partial \Psi(u),$
\[
\Psi(u) \leq \frac{1}{\mu} \int_{\mathbb{R}^N} u(x) w(x) dx.
\]
Hence
\[
I(u_n) \geq \frac{\mu - p^+}{\mu p^+} \int_{\mathbb{R}^N} \left( |\nabla u_n|^p(x) + a(x) u_n^{p(x)} \right) dx + \frac{1}{\mu} \int_{\mathbb{R}^N} \left( |\nabla u_n|^p(x) + a(x) u_n^{p(x)} - w_n u_n \right) dx + \frac{1}{\mu} \int_{\mathbb{R}^N} w_n u_n dx - \Psi(u_n)
\]
\[
\geq \frac{\mu - p^+}{\mu p^+} \int_{\mathbb{R}^N} \left( |\nabla u_n|^p(x) + a(x) u_n^{p(x)} \right) dx + \frac{1}{\mu} \int_{\mathbb{R}^N} \left( |\nabla u_n|^p(x) + a(x) u_n^{p(x)} - w_n u_n \right) dx
\]
\[
= \frac{\mu - p^+}{\mu p^+} \int_{\mathbb{R}^N} \left( |\nabla u_n|^p(x) + a(x) u_n^{p(x)} \right) dx + \frac{1}{\mu} \int_{\mathbb{R}^N} (-\Delta_p(x) u_n + a u_n - w_n, u_n)
\]
\[
= \frac{\mu - p^+}{\mu p^+} \int_{\mathbb{R}^N} \left( |\nabla u_n|^p(x) + a(x) u_n^{p(x)} \right) dx + o(1) \|u_n\|_E.
\]
This relation and (22) show that the Palais-Smale sequence \(\{u_n\}\) is bounded in $E$. It follows that \(\{u_n\}\) converges weakly (up to a subsequence) in $E$ and strongly in $L^p(x)(\mathbb{R}^N)$ to some $u_0 \geq 0$. Taking into account that $w_n \in \partial \Psi(u_n)$ for all $n$, that $u_n \to u_0$ in $E$ and that there exists $w_0 \in E^*$ such that $w_n \to w_0$ in $E^*$ (up to a subsequence), we infer that $w_0 \in \partial \Psi(u_0)$. This follows from the fact that the map $u \mapsto F(x, u)$ is compact from $E$ into $L^1$. Moreover, if we take $\varphi \in C^\infty_c(\mathbb{R}^N)$ and let $\omega := \text{supp} \varphi$, then by (23) we get
\[
\int_{\omega} \left( |\nabla u_0|^p(x) - 2 \nabla u_0 \nabla \varphi + a(x) u_0^{p(x)-1} \varphi - w_0 \varphi \right) dx = 0.
\]
So, by relation (4) p.104 in Chang [7] and by the definition of \((-F)^0\), we deduce that
\[
\int_\omega \left( |\nabla u_0|^{p(x)-2} \nabla u_0 \nabla \varphi + a(x) u_0^{p(x)-1}(\varphi) \right) dx + \int_\omega (-F)^0(x, u_0; \varphi) dx \geq 0.
\]
By density, this hemivariational inequality holds for all \(\varphi \in E\) and this means that \(u_0\) solves Problem [13].

It remains to prove that \(u_0 \not\equiv 0\). If \(w_n\) is as in [23], then by (18) (recall that \(u_n \geq 0\)) and (22) (for large \(m\)) we deduce that
\[
\frac{c}{2} \leq I(u_n) - \frac{1}{p} \langle -\Delta p(x)u_n + au_n - w_n, u_n \rangle \\
= \frac{1}{p} \langle w_n, u_n \rangle - \int_{\mathbb{R}^N} F(x, u_n) dx \leq \frac{1}{p} \int_{\mathbb{R}^N} u_n \overline{f}(x, u_n) dx.
\]
(24)

Now, taking into account its definition, one deduces that \(\overline{f}\) verifies (21), too. So, by (24), we obtain
\[
0 < \frac{c}{2} \leq \frac{1}{p} \int_{\mathbb{R}^N} (\varepsilon u_n^2 + A_\varepsilon u_n^{q+1}) dx = \frac{\varepsilon}{p} \|u_n\|^2_{L^2(\mathbb{R}^N)} + \frac{A_\varepsilon}{p} \|u_n\|^{q+1}_{L^{q+1}(\mathbb{R}^N)}.
\]
In particular, this shows that \(\{u_n\}\) does not converge strongly to 0 in \(L^{q+1}(\mathbb{R}^N)\). It remains to argue that \(u_0 \not\equiv 0\). Since both \(\|u_n\|_{L^{q+1}(\mathbb{R}^N)}\) and \(\|\nabla u_n\|_{L^p(\mathbb{R}^N)}\) are bounded, it follows by Lemma I.1 in Lions [26] that the sequence \(\{u_n\}\) “does not vanish” in \(L^p(\mathbb{R}^N)\). Thus, there exists a sequence \(\{z_n\} \subset \mathbb{R}^N\) and \(C > 0\) such that, for some \(R > 0\),
\[
\int_{z_n + BR} u_n^{q+1} dx \geq C.
\]
(25)

We claim that the sequence \(\{z_n\}\) is bounded in \(\mathbb{R}^N\). Indeed, if not, up to a subsequence, it follows by (4) that
\[
\int_{\mathbb{R}^N} a(x) u_n^{q+1} dx \to +\infty \quad \text{as } n \to \infty,
\]
which contradicts our assumption \(I(u_n) = c + o(1)\). Therefore, by (25), there exists an open bounded set \(D \subset \mathbb{R}^N\) such that
\[
\int_D u_n^{q+1} dx \geq C > 0.
\]

In particular, this relation implies that \(u_0 \not\equiv 0\) and our proof is concluded.

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