The cosmological constant problem or how the quantum vacuum drives the slow accelerating expansion of the Universe

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Abstract

I argue that a solution to the cosmological constant problem is to assume that the expectation value of the quantum vacuum stress-energy tensor is proportional to the metric tensor with a negative energy density and positive pressure. This assumption is confirmed by an explicit calculation of the vacuum expectation for the free electromagnetic and Dirac fields of quantum electrodynamics. As a consequence the metric of the universe might correspond to a FLRW with accelerating expansion only after averaging over large scales, but at small scales it gives rise to an extremely rapid fluctuation between expansion and contraction in every small region, with different phases in different points. The vacuum stress-energy tensor has fluctuations that lead to short periods of expansion. A calculation with plausible approximations leads to an estimate of the accelerating expansion that fits in the observed value.

1 Introduction

The Cosmological Constant Problem[1] arises because, according to standard wisdom, the quantum vacuum should give rise to an effective cosmological term, in Einstein’s equation of general relativity, about $10^{123}$ times too big
(see Wang et al.\textsuperscript{2} and references therein). After the discovery that the Universe is in accelerating expansion\textsuperscript{3} the problem has turned out more dramatic. Indeed the accelerating expansion is attributed to a so-called dark energy, but there is increasing evidence that dark energy is equivalent to a cosmological constant.

Actually the problem arises as a consequence of the following three hypotheses (units $c = \hbar = 1$ are used throughout):

H1- The stress-energy tensor of the quantum vacuum is proportional to the metric tensor, therefore equivalent to a cosmological term. This means that the vacuum pressure, $P^{\text{vac}}$, is isotropic and opposite of the vacuum energy density, $\rho^{\text{vac}}$, if both are measured in a local Minkowski space. That is

$$P^{\text{vac}} = -\rho^{\text{vac}}.$$ 

(1)

The reason for this hypothesis is the assumed Lorentz invariance of the vacuum.

H2. The vacuum energy density is positive, that is

$$\rho^{\text{vac}} > 0.$$ 

(2)

H3. The absolute value of the energy density is of order the Planck density, that is

$$|\rho^{\text{vac}}| \sim \rho_{\text{Planck}} = \frac{1}{\hbar G^2} \sim 10^{125} \rho_B,$$

(3)

where $\rho_B$ is the mean baryonic density of the universe. The reason for this hypothesis is that a straightforward calculation of the energy of the quantum vacuum fields gives a divergent result and there is no natural cutoff except at the Planck scale.

If we ignore, for simplicity, the (baryonic and dark) matter content of the universe then hypothesis H1 and H2 lead to a metric that may be written

$$ds^2 = -dt^2 + a(t)(dx^2 + dy^2 + dz^2),$$

(4)

where the parameter $a(t)$ fulfills

$$\frac{d^2a}{dt^2} = -\frac{4\pi G}{3} (\rho^{\text{vac}} + 3P^{\text{vac}}) a = \frac{8\pi G}{3} \rho^{\text{vac}} a,$$

(5)

that implies

$$a(t) = a(0)e^{Ht}, H \equiv \pm \sqrt{\frac{8\pi G}{3} \rho^{\text{vac}}},$$

(6)
where the choice of an initial positive value for the (Hubble parameter) $H$ leads to an accelerating expansion of the universe. Finally if we accept hypothesis $H3$ the acceleration is huge, many orders greater than observed. The conclusion changes but slightly if we take matter (baryonic and dark) into account. This is the cosmological constant problem.

It is obvious that in order to escape from the problem one at least of the three above hypotheses should be wrong. For some people the solution is to reject $H3$ by the simple expedience of assuming that the huge vacuum energy density is an artifact of the quantization, but the vacuum does not contribute any energy or pressure. Indeed this was implicitly assumed by Einstein when he introduced in 1917 a cosmological constant in his equations, in order to allow for a static universe, that he later rejected when the Hubble expansion was discovered. However there are good reasons to assume that the quantum vacuum is not devoid of energy, for instance phenomena like the Lamb shift or the Casimir effect. Also if the accelerating expansion of the universe is not a quantum vacuum effect, then a cosmological constant should be introduced, say by hand, without any fundamental reason. A weaker assumption than putting $\rho^{\text{vac}} = 0$ is to include a cutoff at energies much smaller than Planck’s, eq.(3). A proposal was made by Zeldovich in 1976[4] substituting the following for eq.(3)

$$\rho^{\text{vac}} = \frac{Gm^6}{\hbar^4}. \quad (7)$$

This fits in the needed dark energy if $m$ is of order the pion mass, but no good reason is known for that.

In a recent paper by Q. Wang, Z. Zhu and W. G. Unruh[2] the authors reject hypothesis $H1$. In the following I summarize the arguments of that paper that are relevant for us. They recognize that for quantum vacuum the expectation

$$\langle P^{\text{vac}} + \rho^{\text{vac}} \rangle = 0 \quad (8)$$

is the standard lore but claim that the stress-energy tensor has fluctuations which are of the same order as the expectation value, and thus semiclassical gravity which uses the expectation value of the stress-energy tensor as the source for the gravitational field is a bad approximation. Thus also the idea that the metric is sufficiently smooth at short scales that one can apply Lorentz symmetry arguments to the stress-energy tensor is highly suspect. On much longer scales than the cutoff these fluctuations average out and the
effective spacetime becomes approximately Lorentz invariant, that is Lorentz invariance is an emergent phenomenon. Then they substitute for eq. (1) the following

\[ \rho_{\text{vac}} > -3P_{\text{vac}}, \]  

see their eq.(42). Taking hypothesis \( H_2 \) into account eq. (9) leads to an approximate metric similar to eq.(4), but the parameter \( a \) fulfils

\[ \frac{d^2a}{dt^2} = -\Omega^2 a, \quad \Omega^2 = \frac{4\pi G}{3} (\rho_{\text{vac}} + 3P_{\text{vac}}) > 0, \]

where now the parameter \( a \) should depend on space coordinates, that is \( a = a(x, y, z, t) \). Thus the quantum vacuum would not produce expansion of the universe, but rapid oscillation between expansion and contraction in every small region. On the latter part of the article\cite{2} the authors claim to prove that in the said oscillations the expansion dominates slightly over the contraction thus driving the slow accelerating expansion of the universe. I shall not comment here on that part of the paper.

I propose that the hypothesis to be rejected is \( H_2 \). The positivity of vacuum energy density is a cherished assumption but there is a fundamental reason to reject it, namely that an actual calculation of the stress-energy tensor of the quantum vacuum leads to eq.(8) with \( \langle \rho_{\text{vac}} \rangle < 0 \), at least for the fields of quantum electrodynamics, as is shown in the next section. In this case eq.(10) holds true, that is the quantum vacuum stress-energy produces rapid fluctuations in the metric, rather than accelerating expansion. I shall show that the fluctuations of the vacuum stress-energy tensor give rise to an effective slow expansion in addition to the metric fluctuations. Thus my conclusion is similar to the one of Wang et al. \cite{2}, but from different assumptions.

In order to study the consequences of rejecting hypothesis \( H_2 \) I will consider a metric of FLRW type with spatial dependence, that is

\[ ds^2 = -dt^2 + a^2(x, y, z, t) \left[ dx^2 + dy^2 + dz^2 \right]. \]  

Hence finding the components of the Einstein equation is straightforward (see e.g. \cite{2}), but only two are needed for our purposes here. One of them is the following, that turns out to be independent of the space coordinates (compare with eq.(5)),

\[ \frac{d^2a}{dt^2} \equiv \ddot{a} = -\frac{4\pi G}{3} (\rho + 3P) a, \]
and it holds true for any diagonal stress tensor, not necessarily fulfilling hypothesis $H_1$. The second equation is the following

$$4\pi G (\rho + P) = 0 = -2\ddot a + 2 \left( \frac{\dot a}{a} \right)^2 + \frac{1}{3a^2} \left( \nabla a \right)^2 - \frac{2}{3a^2} \left( \nabla^2 a \right),$$

where I do assume hypothesis $H_1$. The interesting and well known result is that when $\rho + 3P < 0$ there is a solution of both eqs. (12) and (13) with the parameter $a$ independent of space coordinates, that is given by eqs. (11) to (10), meaning an exponential homogeneous expansion of space. It may be shown that this solution fulfils also the remaining Einstein equations for the metric eq. (11). In sharp contrast, if $\rho + 3P > 0$ the solution of eq. (12) is of the form of eq. (10), but no solution of both eqs. (12) and (13) exists with $a$ independent of space coordinates. This would mean that space fluctuates locally between expansion and contraction with different phases in different points.

### 2 The stress-energy of the vacuum in quantum electrodynamics

In this section I shall prove that hypothesis $H_2$ is indeed falsified by a straightforward quantum calculation of the stress tensor of the vacuum. I shall do that for the fields of quantum electrodynamics (QED) but it may be extended to other quantum fields. Indeed for free Bose fields both the vacuum density and the pressure are positive, but for free Fermi fields the density is negative and the pressure positive, as shown in the following. In the particular case of QED here studied this makes possible that eq. (11) holds true, approximately, for the vacuum expectations.

Although it is well known, for later convenience I recall the derivation of the energy density and pressure of the free electromagnetic field. The classical Lagrangian of the free field may be written in terms of the four-vector potential $A_\mu$ in the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu}.$$  

(14)

Hence it is straightforward to get the stress-energy tensor of the free field. After quantization the operators of the energy density (or Hamiltonian density), $\hat{\rho}_{EM}$, and the pressure, $\hat{P}_{EM}$, may be written in terms of the electric, $\hat{E}$,
and the magnetic, $\hat{B}$, field operators as follows

$$\hat{\rho}_{EM} \equiv \frac{1}{2} \left( \hat{E}^2 + \hat{B}^2 \right) = 3 \hat{P}_{EM}. \tag{15}$$

Expanding in plane waves we get (ignoring the time dependence)

$$\hat{\rho}_{EM} = \hat{\rho}_{EM0} + \hat{\rho}_{EM1} + \hat{\rho}_{EM2}, \quad \hat{\rho}_{EM0} = \frac{1}{2V} \sum_{k, \varepsilon} k,$$

$$\hat{\rho}_{EM1} = \frac{1}{2V} \sum_{k, \varepsilon} \sum_{k', \varepsilon'} \sqrt{k k'} \varepsilon \cdot \varepsilon' \hat{\alpha}_{k', \varepsilon}^\dagger \hat{\alpha}_{k, \varepsilon} \exp \left[ i (k - k') \cdot \mathbf{r} \right],$$

$$\hat{\rho}_{EM2} = \frac{1}{4V} \sum_{k, \varepsilon} \sum_{k', \varepsilon'} \sqrt{k k'} \varepsilon \cdot \varepsilon' \hat{\alpha}_{k, \varepsilon} \hat{\alpha}_{k', \varepsilon'} \exp \left[ i (k + k') \cdot \mathbf{r} \right] + h.c., \tag{16}$$

where $k \equiv |k|, k' \equiv |k'|, \varepsilon$ is the polarization vector that fulfils $\varepsilon \cdot k = 0$, and similar for $\varepsilon'$, h.c. means Hermitean conjugate and for notational simplicity I have labelled

$$\varepsilon \cdot \varepsilon' \equiv \varepsilon \cdot \varepsilon' + \frac{1}{kk'} (k \times \varepsilon) \cdot (k' \times \varepsilon'). \tag{17}$$

In $\hat{\rho}_{EM1}$ I have written the operators in normal order, taking the commutation relations into account.

The Hamiltonian is obtained by performing a space integral of the energy density eq.(15), that is

$$\hat{H}_{EM} = \lim_{V \to \infty} \int_V \hat{\rho}_{EM} (\mathbf{r}) d^3 \mathbf{r} = \lim_{V \to \infty} \int_V [\hat{\rho}_{EM1} (\mathbf{r}) + \hat{\rho}_{EM2} (\mathbf{r})] d^3 \mathbf{r}$$

$$= \sum_{k, \varepsilon} k (\hat{\alpha}_{k, \varepsilon}^\dagger \hat{\alpha}_{k, \varepsilon} + \frac{1}{2}). \tag{18}$$

The term $\hat{\rho}_{EM2}$ does not contribute to the Hamiltonian, and therefore to the quantum expectation of the vacuum energy, because the space integral of the latter eq.(15) leads to $k' = -k$, whence

$$\int_V \hat{\rho}_{EM2} (\mathbf{r}) d^3 \mathbf{r} = \frac{1}{4V} \sum_{k, \varepsilon \varepsilon'} k \hat{\alpha}_{k, \varepsilon}^\dagger \hat{\alpha}_{-k, \varepsilon} \varepsilon \cdot \varepsilon' = 0, \tag{19}$$

as may be easily proved taking eq.(17) into account with the choice $\varepsilon = \varepsilon'$. 

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For the free electromagnetic field the vacuum state, \( |0\rangle \), may be defined as the state with the minimal energy amongst the eigenvectors of the operator eq.(18). It is a state with zero photons and it has the properties

\[
\alpha_{k,\varepsilon} |0\rangle = 0, \quad \langle0| \alpha_{k,\varepsilon}^\dagger = 0,
\]

whence the vacuum expectation of the energy density and the pressure are

\[
\rho_{EM} = \lim_{V \to \infty} \frac{1}{V} \sum_k k \to (2\pi)^{-3} \int k d^3 k, \\
P_{EM} = \frac{\Lambda^4}{24\pi^2},
\]

where in the continuous limit an integral has been substituted for the sum in \( k \), and a cutoff, \( \Lambda \), has been introduced in the photon energies.

The stress-energy tensor of the Dirac electron-positron field is

\[
T^{\mu\nu} = \frac{i}{2} (\bar{\psi} \gamma^\mu \frac{\partial}{\partial x^\mu} \psi - \frac{\partial}{\partial x^\mu} \bar{\psi} \gamma^\mu \psi), \quad \bar{\psi} \equiv \psi^\dagger \beta, \quad \gamma^0 \equiv \beta, \quad \gamma^k \equiv \alpha^k,
\]

where \( \alpha^k \) and \( \beta \) are Dirac’s matrices and \( \psi \) is the quantized field. Therefore the energy (or Hamiltonian) density operator \( \hat{\rho}_D \) is

\[
\hat{\rho}_D = T_0^0 = \frac{i}{2} \left( \psi^\dagger \frac{\partial}{\partial x^0} \psi - \frac{\partial \psi^\dagger}{\partial x^0} \psi \right).
\]

Expansion of the quantum field \( \psi \) in plane waves gives, after some algebra,

\[
\hat{\rho}_D = \frac{1}{V} \sum_{p,s} \sqrt{m^2 + p^2} \left( b_{p,s}^\dagger b_{p,s} - d_{p,s} d_{p,s}^\dagger \right) = \frac{1}{V} \sum_{p,s} \sqrt{m^2 + p^2} \left( b_{p,s}^\dagger b_{p,s} + d_{p,s}^\dagger d_{p,s} - 1 \right),
\]

where \( V \) is the normalization volume, \( b_{p,s}, (d_{p,s}) \) is the electron (positron) destruction operator and \( b_{p,s}^\dagger, (d_{p,s}^\dagger) \) the creation operator, \( m \) being the electron (or positron) mass, \( p \) its momentum and \( s \) its spin. It is remarkable the quantity \(-1\) in the latter form of eq.(24), which is a consequence of the anticommutation rules of Fermi field operators and gives rise to a negative contribution to the vacuum energy. The vacuum state, \( |0\rangle \), consists of zero...
electrons and zero positrons. The expectation value of the density operators eq.24 gives, after performing the (trivial) sum of polarizations and passing to the continuous limit $V \to \infty$,

$$
\rho_D = \langle 0 | \hat{\rho}_D | 0 \rangle = -\pi^{-2} \int_0^{P_{\text{max}}} \sqrt{m^2 + p^2} p^2 dp = -\pi^{-2} \int_{m}^{\Lambda} \sqrt{E^2 - m^2} E^2 dE
$$

$$
= -\frac{1}{4\pi^2} \left[ \Lambda (\Lambda^2 - \frac{1}{2}m^2) \sqrt{\Lambda^2 - m^2} - \frac{1}{2}m^4 \cosh^{-1} \left( \frac{\Lambda}{m} \right) \right]
$$

$$
= -\frac{1}{4\pi^2} \left[ \Lambda^4 - \Lambda^2 m^2 - \frac{1}{2}m^4 \ln \left( \frac{\Lambda}{m} \right) \right] + O \left( \frac{m^6}{\Lambda^2} \right), \quad (25)
$$

where $E = \sqrt{m^2 + p^2}$ and $\Lambda >> m$ is a cutoff energy for the electron and positron.

In order to get the contribution of the free Dirac field to the vacuum pressure it is convenient to calculate the expectation value of the trace of the tensor $T^{\mu\nu}$ eq.(22). For the quantized tensor we get

$$
\rho_D = \langle 0 \left| \hat{\rho}_D + 3 \hat{P}_D \right| 0 \rangle = \rho_D + 3P_D = m \langle 0 | \bar{\psi} \psi | 0 \rangle = -\frac{1}{V} \sum_{\mathbf{p},s} \frac{m^2}{E} \left( b_{\mathbf{p},s}^\dagger b_{\mathbf{p},s} + d_{\mathbf{p},s}^\dagger d_{\mathbf{p},s} - 1 \right) \quad (26)
$$

where we introduce the differential operators $\hat{H} \equiv i \partial / \partial t$, $\hat{\mathbf{p}} \equiv -i \nabla$, the third equality takes Dirac’s equation into account and the latter follows after a plane waves expansion. The vacuum expectation value is

$$
\langle 0 | T^{\mu\nu} | 0 \rangle = \langle 0 | \hat{\rho}_D + 3\hat{P}_D | 0 \rangle = \rho_D + 3P_D = m \langle 0 | \bar{\psi} \psi | 0 \rangle = -\frac{1}{V} \sum_{\mathbf{p},s} \frac{m^2}{E}
$$

$$
\rightarrow \frac{m^2}{2\pi^2} \left[ -\Lambda^2 + m^2 + m^2 \ln \left( \frac{\Lambda}{m} \right) \right] + O \left( \frac{m^6}{\Lambda^2} \right), \quad (27)
$$

whence

$$
P_D = -\frac{1}{3}\rho_D + \frac{m^2}{6\pi^2} \left[ -\Lambda^2 + m^2 + m^2 \ln \left( \frac{\Lambda}{m} \right) \right] + O \left( \frac{m^6}{\Lambda^2} \right). \quad (28)
$$

It is interesting that for the electron-positron field the vacuum energy is negative but the vacuum pressure is positive and from the calculation it may
be realized that this is a general fact for spin-1/2 (Dirac) fields. In contrast for the electromagnetic field both energy and pressure are positive and we may conjecture that this is a general fact for Bose fields, although I omit the proof. The difference derives from the commutation rules (for Bose fields) vs. anticommutation (for Fermi fields).

A relevant result is that the stress-energy tensor of the QED vacuum fulfills eq. (28) modulo terms logarithmically divergent with the cutoff, therefore of order $m^4$ if we assume that a cutoff $\Lambda$ exists, even if it is of order the Planck energy. Indeed we have

$$\rho_{QED} + P_{QED} = \rho_{EM} + P_{EM} + \rho_D + P_D = \frac{m^4}{12\pi^2} \left[ 2 + 3 \ln \left( \frac{\Lambda}{m} \right) \right]. \quad (29)$$

This is a remarkable cancelation taking into account the huge values of the density and the pressure, that are of order $\Lambda^4$, about $10^{90} m^4$ if $\Lambda$ is the Planck mass. Nevertheless the sum eq. (29) is still high, about $10^{35}$ times the dark energy density.

A quantity of interest is

$$\rho_{QED} + 3P_{QED} = \rho_{EM} + 3P_{EM} + \rho_D + 3P_D$$

$$= \frac{\Lambda^4}{4\pi^2} - \frac{\Lambda^2 m^2}{2\pi^2} + \frac{m^4}{2\pi^2} \left[ \ln \left( \frac{\Lambda}{m} \right) + 1 \right], \quad (30)$$

where eqs. (21) and (28) have been taken into account.

I point out that the total energy density operator of QED should contain also a term for the interaction, that may be written

$$\hat{\rho}_{\text{int}} (r,t) = -e \hat{\psi}^\dagger \alpha \hat{\psi} \cdot \hat{A}. \quad (31)$$

The operators $\hat{\psi}, \hat{\psi}^\dagger$ and $\hat{A}$ contain two terms each when expanded in plane waves, every term corresponding to an infinite sum. One of these terms has creation operators and the other one annihilation operators. This gives rise to 8 terms for $\hat{\rho}_{\text{int}}$, eq. (31). I will write only the two terms that will survive in the Hamiltonian. We get (ignoring the time dependence)

$$\hat{\rho}_{\text{int}} (r) = \sum_{p,q,k,s,s',\varepsilon} \left[ \zeta_n \hat{a}_{k,\varepsilon} \hat{b}_{p,s} \hat{d}_{q,s'} \exp \left[ i (p + q + k) \cdot r \right] + \text{h.c} \right]$$

$$\zeta_n = -e \frac{m}{V^{1/2} \sqrt{2kEE'}} \nu_s^\dagger (p) \alpha \cdot \varepsilon v_{s'} (q), \quad (32)$$
where h.c. means Hermitean conjugate, $u^\dagger_{s'}$ and $v_{s'}$ are spinors, and $n$ stands for $\{p, q, k, s, s', \varepsilon\}$. The interaction Hamiltonian $\hat{H}_{int}$ is the space integral of $\hat{\rho}_{int}(r)$ within the volume $V$. One of the terms of the Hamiltonian may create triples electron-positron-photon and the other term may annihilate triples. In this paper I will neglect the interaction energy and pressure. They are typically smaller than the other terms considered in this section by about $\alpha = 1/137$.

### 3 Fluctuations in the QED vacuum

The study of fluctuations requires calculating the stress-energy tensor over finite regions. For instance we may calculate the vacuum energy density in a region of volume $v$ using an energy density operator of the form

$$\hat{\rho}^v \equiv v^{-1} \int_v d^3 r \hat{\rho}(r) \rightarrow \rho^v = \langle \text{vac} | \hat{\rho}^v | \text{vac} \rangle,$$

where $\hat{\rho}(r)$ is the quantum operator for the energy density (see e.g. eq.(16)). At a difference with section 2 where a similar calculation involved the limit $V \rightarrow \infty$, see eq.(21), here the volume $v$ remains finite. The consequence is that the density fluctuates in the sense that

$$\langle \text{vac} | \left( v^{-1} \int_v d^3 r \hat{\rho} \right)^2 | \text{vac} \rangle > \langle \text{vac} | v^{-1} \int_v d^3 r \hat{\rho} | \text{vac} \rangle^2,$$

that would become an equality in the limit $v \rightarrow \infty$. Similar results hold for the pressure.

The most interesting quantity for us is the sum $\rho_{QED} + 3P_{QED}$ of components of the vacuum stress-energy tensor, see eq.(12). In order to study the fluctuations we define the following operator

$$\hat{Q} \equiv v^{-1} \int_v d^3 r [\hat{\rho}_{EM} + 3\hat{P}_{EM} + \hat{\rho}_{D} + 3\hat{P}_{D}],$$

(33)

where eqs.(15), (16) and (27) have been taken into account. The probability distribution predicted by quantum mechanics for the quantity $Q$, associated to the operator $\hat{Q}$, is

$$F(Q) = \frac{1}{2\pi} \int d\lambda \exp(-i\lambda Q) \langle \text{vac} | \exp(i\lambda \hat{Q}) | \text{vac} \rangle,$$

(34)
It is easy to see that $\hat{\rho}_{EM} + 3\hat{P}_{EM}$ commutes with $\hat{\rho}_D + 3\hat{P}_D$, the former involving electromagnetic field operators and the latter electron and positron operators. Furthermore it may be realized that the expectation eq. (27) is actually an eigenvalue of the operator $\hat{\rho}_D + 3\hat{P}_D$ with eigenvector $|0\rangle$. Therefore we may rewrite eq. (34) in the form

$$F(Q) = \frac{1}{2\pi} \int d\lambda \exp \left\{ i\lambda \left[ v^{-1} \int R d^3r (\hat{\rho}_D + 3\hat{P}_D) - Q \right] \right\} S$$

$$S \equiv \langle 0 | \exp \left\{ i\lambda \left[ v^{-1} \int R d^3r (\hat{\rho}_{EM} + 3\hat{P}_{EM}) \right] \right\} | 0 \rangle,$$

where I approximate $|vac\rangle$ by $|0\rangle$. The vacuum ground state $|vac\rangle$, including the interaction eq. (32), is different from the ground state $|0\rangle$ of the free fields alone, but our approximation in eq. (35) is consistent with neglecting the interaction, as was made in section 2. Taking into account eq. (16) we may write

$$\hat{\rho}_{EM} + 3\hat{P}_{EM} = 2\hat{\rho}_{EM} = \frac{1}{2V} \sum_{k,\varepsilon} \sum_{k',\varepsilon'} \sqrt{kk'} \varepsilon \ast \varepsilon'$$

$$\times \left[ \hat{a}_{k,\varepsilon} \exp (ik \cdot r) + \hat{a}_{k,\varepsilon}^\dagger \exp (-ik \cdot r) \right]$$

$$\times \left[ \hat{a}_{k',\varepsilon'} \exp (ik' \cdot r) + \hat{a}_{k',\varepsilon'}^\dagger \exp (-ik' \cdot r) \right].$$

Calculating accurately the probability density $F(Q)$ via eq. (35) would be cumbersome. In fact we should expand $S$ in powers of $\lambda$, that is redefine

$$S \equiv 1 + \sum_{n=1}^{\infty} \frac{(i\lambda)^n}{n!} \langle 0 | \left( v^{-1} \int_R 2\hat{\rho}_{EM} d^3r \right)^n | 0 \rangle,$$

then calculate the general term and finally sum the series. In addition the interaction energy operator, $\hat{\rho}_{int}$, should be included in the vacuum expectation. In this paper I will not make an accurate calculation but an estimate of $F(Q)$ using some approximations as explained in the following.

I will assume that the probability density $F(Q)$ is a Gaussian that is

$$F(Q) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{(Q - Q_0)^2}{2\sigma^2} \right],$$
and get the parameters $Q_0$ and $\sigma$ via identifying

$$\langle Q \rangle \equiv \int Q F(Q) dQ = \langle 0 | Q | 0 \rangle, \quad \langle Q^2 \rangle = \langle 0 | \hat{Q}^2 | 0 \rangle,$$

whence taking eq. (30) into account we get

$$Q_0 = \langle 0 | v^{-1} \int_R d^3 r [2\hat{\rho}_{EM} + \hat{\rho}_D + 3\hat{P}_D] | 0 \rangle = \rho_D + 3P_D + 2\rho_{EM},$$

$$= \frac{1}{4\pi^2} \left[ \Lambda^4 - 2\Lambda^2 m^2 + 2m^4 \left[ \ln \left( \frac{\Lambda}{m} \right) + 1 \right] \right]. \quad (39)$$

The former equality may be checked easily from the expressions of the density and pressure operators, eqs. (22) to (24).

In order to get the variance of the distribution, $\sigma^2$, it is convenient to use eq. (16), that is

$$\sigma^2 = v^{-2} \langle 0 | (\int_R 2\hat{\rho}_{EM} d^3 r)^2 | 0 \rangle - (2\rho_{EM})^2$$

$$= v^{-2} \langle 0 | (\int_R 2\hat{\rho}_{EM} d^3 r)^2 | 0 \rangle = \frac{v^{-2}}{8V^2} \sum_{k, \varepsilon} \sum_{k', \varepsilon'} kk' (\varepsilon * \varepsilon')^2$$

$$\times (\hat{\alpha}_{k, \varepsilon} \hat{\alpha}_{k', \varepsilon'} \hat{\alpha}_{k, \varepsilon} \hat{\alpha}_{k', \varepsilon'}) \int_v d^3 r \int_v d^3 r' \exp \left[ i (k + k') \cdot (r - r') \right],$$

$$= \frac{v^{-2}}{4V^2} \sum_k \sum_{k'} kk' \left( \frac{k-k'}{kk'} \right)^2 \left| \int_v d^3 r \exp \left[ i (k + k') \cdot r \right] \right|^2, \quad (40)$$

where a sum of polarizations has been performed in the latter equality. I point out that $v$ is a small volume, say with size the Planck length, whilst $V$ is an integration volume introduced for calculational convenience. In fact it allows treating the fields via an expansion in plane waves, putting $V \rightarrow \infty$ at some stage of the calculation in order to substitute integrals for the more involved sums.

For the (small) region with volume $v$ I choose a sphere of radius $R$ so that

$$v = \frac{4}{3} \pi R^3. \quad (41)$$

Then the space integral in eq. (40) is straightforward and we get

$$I \equiv \left| \int_R d^3 r \exp \left[ i (k + k') \cdot r \right] \right|^2 = 4\pi^2 R^6 \left( \sin \frac{x}{x^6} - x \cos \frac{x}{x^6} \right)^2, \quad (42)$$

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where

\[ x \equiv R |k + k'| = R \sqrt{k^2 + k'^2 + 2kk'u}, \quad u \equiv \cos \theta, \quad (43) \]

\( \theta \) being the angle between \( k \) and \( k' \). Thus eq.(40) becomes, in the continuous limit \( V \rightarrow \infty \),

\[ \sigma^2 = \frac{1}{(16\pi^3)^2} \int_0^\Lambda 4\pi k^3 dk \int_0^\Lambda 2\pi k'^3 dk' \int_{-1}^1 du (1 + u)^2 I(x), \quad (44) \]

where \( \Lambda \) is the cutoff defined in section 2. It is convenient to go from the variable \( u \) to \( x \) whence we get

\[ \sigma^2 = \frac{8u^{-2}}{(16\pi^2)^2} \int_0^\Lambda k^3 dk \int_0^\Lambda k'^3 dk' \int_{R|k+k'|}^{R|k-k'|} \frac{xdx}{R^2kk'} I(x) \times \left[ 1 + \frac{x^2 - R^2(k^2 + k'^2)}{2kk'} \right]^2, \]

that is, with the new change of variable \( y = x^2 \),

\[ \sigma^2 = \frac{v^{-2}}{(4\pi)^4 R^6} \int_0^\Lambda dk \int_0^\Lambda dk' \int_{R^2(k+k')^2}^{R^2(k-k')^2} dy I(\sqrt{y}) \left[ y - R^2 (k - k')^2 \right]^2. \quad (45) \]

The integral is involved and I will simplify it approximating \( I \sim (4\pi^2 R^6/9)/(1 + 2y^2) \), that agrees with eq.(12) in both limits \( y << 1 \) and \( y >> 1 \). Thus we get

\[ \sigma^2 = \frac{R^2 v^{-2}}{9 \times 32\pi^2} \int_0^\Lambda dk \int_0^\Lambda k'dk' J(k, k'), \quad (46) \]

where

\[ J \equiv \left[ \frac{y}{2} + \frac{R^2 ((k - k')^2 - 1/2)}{\sqrt{2}} \arctan \left( \frac{\sqrt{2}y}{R^2} \right) - \frac{R^2 (k - k')^2 \ln (1 + 2y^2)}{R^2(k-k')^2} \right]^{R^2(k+k')^2}_{R^2(k-k')^2} \]

\[ = 2R^2kk' - R^2 (k - k')^2 \ln \frac{1 + 2R^4 (k + k')^4}{1 + 2R^4 (k - k')^4} \]

\[ + \frac{2R^2 ((k - k')^2 - 1/2)}{\sqrt{2}} \left[ \arctan \left( \frac{\sqrt{2}R^2 (k + k')^2}{R^2} \right) - \arctan \left( \frac{\sqrt{2}R^2 (k - k')^2}{R^2} \right) \right]. \]

The integrals are lengthy and finding an accurate result is not worth taking into account the other approximations involved. I will report only an upper
bound obtained retaining the positive terms of \( J \) alone. This provides a rough approximation taking into account that \( J > 0 \) as it follows from eq.(40). Thus from the inequality
\[
\frac{\pi}{2} > \arctan \left( \sqrt{2} R^2 (k + k')^2 \right) - \arctan \left( \sqrt{2} R^2 (k - k')^2 \right) > 0,
\]
we estimate
\[
\sigma^2 \lesssim \frac{R^4 v^{-2}}{9 \times 32\pi^2} \int_0^\Lambda k dk \int_0^\Lambda k' dk' \left[ 2kk' + \frac{\pi}{2\sqrt{2}} (k - k')^2 \right] \approx \frac{\Lambda^6 R^4}{3200v^2}. \quad (47)
\]
Taking eq.(41) into account we get finally
\[
\sigma^2 \lesssim \frac{\Lambda^6 R^4}{3200} \left( \frac{3}{4\pi R^3} \right)^2. \quad (48)
\]

4 The accelerating expansion of the universe

From eq.(12) we see that when \( \rho + 3P > 0 \) the space oscillates rapidly between expansion and contraction, but when \( \rho + 3P < 0 \) it expand exponentially, that is
\[
a = a_0 \exp \left( Ht \right), \quad H = \sqrt{\frac{4\pi}{3} G (\rho + 3P)} \quad \text{if} \quad \rho + 3P < 0. \quad (49)
\]
As is well known no satisfactory quantum gravity theory is available and the popular method to treat problems involving simultaneously quantum theory and gravity is the semiclassical approximation. It consists of using as stress-energy tensor, in the right side of Einstein’s equation, the expectation of the quantized tensor in the appropriate quantum state, here the vacuum state \( |0\rangle \). In order to take account of the quantum fluctuations it is necessary to extend the semiclassical approximation as follows. I propose to treat \( F(Q) \), eq.(34) as a probability distribution of the trace, \( \rho_{QED} + 3P_{QED} \), of the stress-energy tensor. This amounts at applying the semiclassical approximation to all expectations \( \langle 0 | (v^{-1} \int_\mathcal{R} 2\rho_{EM} d^3 \mathbf{r})^n | 0 \rangle \), eq.(37). After that Einstein equation provides, in principle, a probability distribution of metrics. Of course making a calculation of this probability distribution would be a formidable task, but here our more modest task will be to estimate the probability that the metric is of the form of eq.(49) in a small region where we assume that stress-energy tensor is homogeneous and isotropic. In fact I
propose to estimate the Hubble parameter $H$ by calculating the expectation (in the sense of classical probability theory) that $Q = \rho + 3P < 0$ from the probability distribution eq.(38). That is I assume that the effective $Q$ for the accelerating expansion is given by the integral

$$Q_{\text{eff}} = \int_{-\infty}^{0} QF(Q) dQ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{Q_0}{\sigma}} \left(x + \frac{Q_0}{\sigma}\right) \exp \left(-\frac{x^2}{2}\right) dx \quad (50)$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left[- \exp \left(-\frac{Q_0^2}{2\sigma^2}\right) + \frac{Q_0}{\sigma} \text{erfc} \left(-\frac{Q_0}{\sqrt{2}\sigma}\right)\right]$$

$$\simeq \frac{\sigma}{\sqrt{2\pi}} \times \frac{Q_0}{\sigma} \times \exp \left(-\frac{Q_0^2}{2\sigma^2}\right) \simeq \frac{Q_0}{\sqrt{2\pi}} \exp \left(-\frac{Q_0^2}{2\sigma^2}\right).$$

Approximating $Q_0$ by the leading term in eq.(39), taking eq.(47) into account, we get

$$Q_0 \simeq \frac{\Lambda^4}{4\pi^2}, \quad \frac{Q_0^2}{2\sigma^2} > \frac{3200\Lambda^2 R^2}{2 \times (3\pi)^2} = 18\Lambda^2 R^2,$$

whence

$$Q_{\text{eff}} < \frac{\Lambda^4}{(2\pi)^{5/2}} \exp \left(-18\Lambda^2 R^2\right).$$

This would fit in the observed value of the dark energy needed to explain the accelerated expansion of the universe if we choose $R \sim 4\Lambda^{-1}$. In fact in this case taking eq.(50) into account we would get

$$\frac{Q_0^2}{2\sigma^2} = 288 \Rightarrow Q_{\text{eff}} \lesssim Q_0 \exp (-284) = 10^{-123}\Lambda^4.$$ 

That is, assuming that $\Lambda^4$ is the Planck density, then $R$ needs to be about 4 times the Planck length.

5 Conclusions

I argue that a solution to the cosmological constant problem is to assume that the expectation value of the quantum vacuum stress-energy tensor is proportional to the metric tensor with a negative energy density and positive pressure. Indeed this fits with the results of a straightforward calculation of the said expectation for free Bose and Fermi fields, at least for the fields of quantum electrodynamics (electromagnetism and electron-positron Dirac
field). As a consequence the metric of the universe might corresponds to a FLRW with accelerated expansion only after averaging on large scales. However at small scales it gives rise to an extremely rapid fluctuation between expansion and contraction in every small region, with different phases in different points. Of course both our assumption and the result of the calculation in QED contradict the standard wisdom that the vacuum energy density should be positive. This puts a difficulty that should be further studied.

The vacuum stress-energy tensor has huge fluctuations of the same order of its average value. This fact implies that some specially big fluctuations may give rise to a tensor with positive energy and negative pressure, leading to short periods of expansion in every small region. The very small probability of such extremely large fluctuations implies that the overall expansion of the universe is very slow. An explicit calculation with plausible approximations leads to an estimate that fits in the observed value of the accelerated expansion.

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References

[1] Steven Weinberg, The cosmological constant problem. *Rev. Mod. Phys.*, 61, 1-23 (1989).

[2] Qingdi Wang, Zhen Zhu, and William G. Unruh, *Phys. Rev. D*, 95, 103504 (2017); [arXiv:1703.00543](https://arxiv.org/abs/1703.00543).

[3] P. J. E. Peebles and Bharat Ratra, *Rev. Mod. Phys.*, 78, 559-606 (2003).

[4] B. Y. Zeldovich, *Sov. Phys. Usp.* 24, 216 (1981).

[5] S. S. Schweber, *An introduction to relativistic quantum field theory*, Harper and Row, New York, 1962.