Evaluation of binary classifiers for asymptotically dependent and independent extremes

Juliette Legrand\textsuperscript{a}, Philippe Naveau\textsuperscript{a} and Marco Oesting\textsuperscript{b} \textsuperscript{*}

\textsuperscript{a}Laboratoire des sciences du climat et de l’environnement (EstimR), Université Paris-Saclay, CNRS, CEA, UVSQ, 91191 Gif-sur-Yvette, France
\textsuperscript{b}Stuttgart Center of Simulation Science \& Institute for Stochastics and Applications, University of Stuttgart, 70569 Stuttgart, Germany

Abstract

Machine learning classification methods usually assume that all possible classes are sufficiently present within the training set. Due to their inherent rarities, extreme events are always under-represented and classifiers tailored for predicting extremes need to be carefully designed to handle this under-representation. In this paper, we address the question of how to assess and compare classifiers with respect to their capacity to capture extreme occurrences. This is also related to the topic of scoring rules used in forecasting literature. In this context, we propose and study a risk function adapted to extremal classifiers. The inferential properties of our empirical risk estimator are derived under the framework of multivariate regular variation and hidden regular variation. A simulation study compares different classifiers and indicates their performance with respect to our risk function. To conclude, we apply our framework to the analysis of extreme river discharges in the Danube river basin. The application compares different predictive algorithms and test their capacity at forecasting river discharges from other river stations.

Keywords: Multivariate Extreme Value Theory; Binary Classification; Multivariate Regular Variation; Hidden Regular Variation; River Discharges

\textsuperscript{*}Part of this work was supported by the DAMOCLES-COST-ACTION on compound events, the French national program (FRAISE-LEFE/INSU and 80 PRIME CNRS-INSU), and the European H2020 XAIDA (Grant agreement ID: 101003469) and funded by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy - EXC 2075 - 390740016. The authors also acknowledge the support of the French Agence Nationale de la Recherche (ANR) under reference ANR-20-CE40-0025-01 (T-REX project), the ANR-Melody and the Stuttgart Center for Simulation Science (SimTech).
1 Introduction

In statistics, extreme events are generally defined as occurrences with a very low probability (see, e.g. [Resnick, 2003; Asadi et al., 2015]). For many applications in finance, environmental sciences and risk analysis, such rare events can be viewed as the binary response of a complex system driven by a set of explanatory variables (see, e.g. Mhalla et al., 2020). In this context, this setup can be expressed as a binary imbalanced classifiers problem, i.e. one class label (extreme events) has a very low number of observations and the other category (non extreme events) is over represented (see, e.g. He and Ma, 2013; Jalalzai et al., 2018). To illustrate this, one can look at the question of flood risk. For example, the website of the European Copernicus Emergency Management Service provides, for all major European rivers, maps of static flood protection levels for a few important return periods like the 100 year return level. Given a return level, it is of great societal and economical interest for risk managers to assess how statistical models can predict occurrences of such large events, i.e., to compare binary classifiers that aim at predicting a sequence of \{-1, +1\} where +1 corresponds to a rare event. In this work, our goal is not to construct a new binary classifier to predict, say 100 year occurrences, but rather to propose new risk functions that compare binary classifiers that have been built from historical binary samples corresponding to these rare event occurrences. Although the set of explanatory covariates given to build classifiers can be assumed to be identical for all classifiers, it is not evident that a classifier tailored for modeling 100 year occurrences will be optimal for different return periods. Hence, we authorize the data scientist to change his/her classifier according to the chosen flood protection level under study, say \( u \). From a practical point of view, we will always assume in this work that a data set like Table 1 is given to us.

Table 1: Ingredients to compare classifiers for extremes in this study (numbers in bold corresponds to misses; either false positives or false negatives). Extreme binary events change with the threshold choice. Here the threshold \( u \) is greater than \( v \).

| Label | Observed events above threshold | Forecasted classifier outputs for threshold |
|-------|---------------------------------|------------------------------------------|
|       | \( v \) \( u \)               | \( v \) \( u \)                           |
| 1     | -1    -1                         | +1    -1                                  |
| 2     | +1    -1                         | +1    -1                                  |
| 3     | +1    +1                         | -1    -1                                  |
|       |       .                           |       .                                  |
|       |       .                           |       .                                  |
| n-2   | -1    -1                         | -1    -1                                  |
| n-1   | +1    +1                         | +1    -1                                  |
| n     | -1    -1                         | -1    -1                                  |

In this illustrative example, the two scalars \( u \) and \( v \) can be set by an agency like, for example, the aforementioned Copernicus Emergency Management Service, and they

\( ^1 \)Two thresholds are needed in Table 1 if we are in the asymptotic independence case, but only one threshold \( u \) will be necessary for the asymptotic dependence case, see Section 2 for details.
can correspond to the 100-year and 150-year return levels, respectively. In this context, one main question is determine which risk functions can insure coherent rankings between classifiers, and also to identify under which theoretical umbrella probability results and inferential guarantees can be given. The later point can be addressed by leveraging multivariate extreme value theory (MEVT) that provides mathematically sound dependence structures (see, e.g. Coles et al., 1999).

Concerning binary classifiers, it is important to understand why basic loss functions are inefficient at comparing classifiers for extremes. To see this, we can revisit the classical expected loss that simply counts the number of times that a given classifier provides a wrong answer. For example, the expected loss applied to the “always optimistic” classifier that never forecasts +1 appears to improve as \( u \) increases (fewer and fewer extremes are observed), and it even goes to zero for very large values of \( u \). So, never predicting an extreme appears to be a optimal strategy with respect to the classical expected loss.

To avoid this undesirable feature, one natural idea is to re-scale the expected loss by the probability of observing an extreme. In this case, the rescaled associated risk now goes towards one as \( u \) gets large. But, this does not fix the imbalance nature in assessing extreme classifiers. To explain this, one can introduce another trivial case: the “crying wolf” classifier who always predicts an extreme, see also the forecaster’s dilemma (e.g. Lerch et al., 2017). In this case, one can show that the previous rescaling trick does not work as the rescaled loss function goes towards infinity as \( u \) gets large. This wrongly indicates that the “crying wolf” classifier is much worse than the overly optimistic one. Both of them are unreasonable in practice and there is no reason to strongly favour one over the other one. For this reason, we propose and study a new class of loss functions, see Equation (4), that has the advantage of handling such imbalanced cases and be in compliance with MEVT. We will show that our new loss function has strong connections with existing scores and can be viewed as an extension of some classical loss functions. In particular, the critical success index, also called the threat score in the weather literature (e.g. Schaefer, 1990) is closely linked to this risk function. This index computes the total number of correct event forecasts (hits) divided by the total number of forecasts plus the number of misses (hits + false alarms + misses). In the context of rare events forecasts, Stephenson et al. (2008) highlighted some advantages and drawbacks of various risk functions, including the critical success index. These authors linked forecast scoring rules with two dependence indices used in EVT (see Coles et al., 1999). The first one is the tail coefficient, called \( \chi \) in EVT, that provides the probability of true positive event given a extreme event whenever the forecast can be assumed to be calibrated, i.e. observations and forecasts follow the same marginal distributions. In MEVT, a positive \( \chi \) indicates asymptotic dependence. The second index, called \( \overline{\chi} \), captures some second order extremal dependence information, called asymptotic independence in MEVT. Stephenson et al. (2008) advocated the use of \( \overline{\chi} \) and coined the term: the extreme dependency score. Later on, Ferro and Stephenson (2011) proposed two different scores and studied their properties. But their link with the concept of asymptotic independence was not clear and the convergence results of their estimators were not fully developed. Hence, there is also a mathematical interest to study loss functions for the asymptotic independent case.

In the machine learning literature, there is a vast body of work about imbalanced data (see, e.g. Hilario et al., 2018). For example, the so-called functions of precision and recall are two metrics well-used to score binary classifiers in learning research, see, e.g., He and
Ma (2013) and the $F$-score. Still, the review paper by Haixiang et al. (2017) did not mention any imbalanced method based on EVT, even though some studies have proposed binary regression models for imbalanced data based on univariate EVT distributions (e.g. Wang and Dey, 2010). Nonetheless, to our knowledge, very few theoretical links have been made to bridge imbalanced learning and MEVT. One noticeable exception is Jalalzai et al. (2018) who worked on binary classifiers for extremes under the regular variation hypothesis (see, e.g. Resnick, 2003). But they differ from our work on two aspects. First, their object of interest was extremes defined as rare events coming from binary classifiers. In our case, we consider extremes as rare events from either the responses or the binary classifier outputs. The second difference with our work is that, although they considered in details many mathematical aspects under the asymptotic dependence case, they did not explore the asymptotic independence situation. In this study, one part of our results is based on the concept of hidden regular variation (see, e.g. Ledford and Tawn, 1996; Heffernan and Resnick, 2005; Ferro, 2007). In particular, we take advantage of the model of Ramos and Ledford (2009) to derive the asymptotic properties of our estimators. Table 5 in Appendix A summarizes the main features of some risk measures used to compare binary classifiers for extremes, see also Table I in Stephenson et al. (2008) for a score comparison.

Our paper is organized as follows. In Section 2, we propose and study a risk function that can handle both the asymptotic dependence and asymptotic independence cases. Estimators are also constructed and their asymptotic properties derived. Section 3 focuses on a simulation example that highlights the difficulty to compare common classifiers in the case of asymptotic independence. In Section 4, we revisit the well studied example of the Danube river application and see how the choice of the metric can change the ranking of classifiers. Note that, besides the proofs of all propositions, the appendix addresses the questions of how to optimize the linear classifier for extremes and how to choose the relevant features, see Appendix B.

## 2 Risk functions for extremal binary classifiers

As already mentioned in the introduction, we consider binary response variables $Y^{(u)} \in \{-1, +1\}$ depending on some index $u$ which we interpret as level of rarity of events. Thus, an increasing level $u$ means that the events considered to be extreme are becoming more and more rare. For example, two events $\{Y^{(u)} = +1\}$ and $\{Y^{(v)} = +1\}$ in Table 1 are nested in the sense of the ranking between the columns named $u$ and $v$. Concerning binary classifiers, the same level-dependent property appears as a natural requirement to compare extremal classifiers. To mathematically formalize this condition, we denote $g^{(u)}(X)$ the threshold-dependent classifier based on the covariate set $X$. The level-dependent condition can be given then.

**Definition 1** The random variable $Y$ is said to a level-dependent binary response variable whenever

$$\{Y^{(u)} = +1\} \subseteq \{Y^{(v)} = +1\} \text{ for } u \geq v.$$
Similarly, a level-dependent binary classifier \( g = \{g^{(u)}\}_{u \geq \infty} \) is a set of measurable functions \( g^{(u)} : \mathbb{R}^d \to \{-1, +1\} \) that are ordered in the sense that
\[
\{g^{(u)}(X) = +1\} \subseteq \{g^{(v)}(X) = +1\} \quad \text{for} \ u \geq v.
\]
Moreover, we set the range of the level \( u \) to \([0, \infty)\) with \( \mathbb{P}(Y^{(0)} = +1) = 1 \), \( i.e. \) the response is always considered as extreme at the minimal level \( u = 0 \), while \( \mathbb{P}(Y^{(u)} = +1) \to 0 \) as \( u \to \infty \). Note that fixing the threshold range to \([0, \infty)\) is arbitrary and it could be replaced if the data range was different. This convention simplifies notations (avoid integral transform) and eases the interpretation of extremes.

The fourth and fifth columns of Table 1 illustrate the nested ordering of \( g \). Without loss of generality, we have assumed in this definition that all the classifiers to be considered depend on the same vector of input variables \( X \in \mathbb{R}^d \).

### 2.1 Overview over risk functions and extremal dependence

As outlined in the introduction, a reasonable risk function should be based on the classical classification loss \( 1\{g^{(u)}(X) \neq Y^{(u)}\} \) and should, at least asymptotically, penalize the two trivial estimators “always optimistic” and “crying wolf” equally. A possible choice, satisfying these two conditions is the weighted classification loss function
\[
\ell_u(g; (x, y)) = \frac{1}{\mathbb{P}(Y^{(u)} = +1 \text{ or } g^{(u)}(X) = +1)} 1\{g^{(u)}(x) \neq y\}
\]
and the associated expected risk
\[
R^{(u)}(g) = \mathbb{E}(\ell_u(g; (X, Y^{(u)})) = \frac{\mathbb{P}(g^{(u)}(X) \neq Y^{(u)})}{\mathbb{P}(Y^{(u)} = +1 \text{ or } g^{(u)}(X) = +1)}.
\] (1)

By construction, the event \( \{g^{(u)}(X) \neq Y^{(u)}\} \) already implies that (either) \( Y^{(u)} = +1 \) or \( g^{(u)}(X) = +1 \) and therefore, necessarily, \( R^{(u)}(g) \in [0, 1] \), with \( R^{(u)}(g) = 0 \) indicating a perfect classifier and \( R^{(u)}(g) = 1 \) a bad one. In particular, the “always optimistic” classifier \( g^{(u)} \equiv -1 \) possesses unit risk with \( R^{(u)}(g) = 1 \) at each level \( u > 0 \). Similarly, the risk of the “crying wolf” classifier \( g^{(u)} \equiv +1 \) is then equal to \( R^{(u)}(g) = \mathbb{P}(Y^{(u)} = -1) \). It therefore converges to one as \( u \to \infty \). These two naive classifiers are, hence, equally disqualified by \( R^{(u)}(g) \) for large \( u \). This upper unit value of \( R^{(u)}(g) \) provides a clear benchmark to assess any other level-dependent binary classifiers \( g \) satisfying the existence of the limit
\[
R(g) = \lim_{u \to \infty} R^{(u)}(g).
\]
We call such classifiers extremal and \( R(g) \) the extremal risk, which has to satisfy \( R(g) \in [0, 1] \) by definition. Then, \( 1 - R^{(u)}(g) \) can be understood as a critical success index for extremes, (e.g. Schaefer, 1990).

As will be described in Section 2.2, \( R^{(u)}(g) \) can be calculated from the three probabilities \( \mathbb{P}(Y^{(u)} = +1) \), \( \mathbb{P}(g^{(u)}(X) = +1) \) and \( \mathbb{P}(Y^{(u)} = +1, g^{(u)}(X) = +1) \). Perceiving these three expressions as functions of \( u \in [0, \infty) \), they are all non-increasing and going from 1 to 0 and therefore have similar properties as tail functions of random variables. This allows us to define terms that are well-known from extreme value theory when considering tails of
continuous random variables. For instance, two random variables are called tail equivalent if both corresponding tail functions, \( i.e. \) the probabilities of being extreme, decay at the same rate. Analogously, we could consider the two probabilities of observing an extreme event and classifying an event as extreme as the level of extremity tends to infinity. Similarly, the terms of asymptotic dependence and asymptotic independence focus on the probability of two random variables being jointly extreme, while we could take the probability of events being both observed and classified as an extreme.

**Definition 2** Let \( Y \) be a level-dependent binary response variable and \( g \) be a level-dependent binary classifier, see Definition 1.

1. \( Y \) and \( g \) are called tail comparable if the limit
   \[
   c(g) := \lim_{u \to \infty} \frac{\mathbb{P}(g(u)(X) = +1)}{\mathbb{P}(Y(u) = +1)} \in [0, \infty]
   \]
   exists. They are called tail equivalent if \( c(g) \in (0, \infty) \). Analogously, we say that \( g \) has a lighter tail than \( Y \) if \( c(g) = 0 \), while \( g \) is said to possess a heavier tail than \( Y \) if \( c(g) = \infty \).

2. Assume that \( Y \) and \( g \) are tail equivalent. Then, we call them asymptotically dependent if
   \[
   \chi^*(g) := \lim_{u \to \infty} \mathbb{P}(g(u)(X) = +1 \mid Y(u) = +1) > 0,
   \]
   while we call them asymptotically independent if
   \[
   \lim_{u \to \infty} \mathbb{P}(g(u)(X) = +1 \mid Y(u) = +1) = \lim_{u \to \infty} \mathbb{P}(Y(u) = +1 \mid g(u)(X) = +1) = 0.
   \]

Note that, tail equivalence of \( Y \) and \( g \) implies that the limits of two conditional probabilities \( \mathbb{P}(g(u)(X) = +1 \mid Y(u) = +1) \) and \( \mathbb{P}(Y(u) = +1 \mid g(u)(X) = +1) \) as \( u \to \infty \) just differ by the factor \( c(g) \). In particular, \( Y \) and \( g \) are asymptotically independent if and only if \( \chi^*(g) = 0 \). If \( c(g) = 1 \), \( \chi^* \) is very closely related to the tail dependence coefficient \( \chi \) of two random variables (see Coles et al., 1999) that is usually used to define asymptotic independence.

In Section 2.2, we will demonstrate that both tail equivalence and asymptotic dependence of \( Y \) and \( g \) are necessary conditions to obtain an extremal risk \( R(g) < 1 \), \( i.e. \) a better performance than the one of the naive classifiers. While a classifier \( g \) can often be modified to become tail equivalent by some appropriate transformation \( \phi \) of the level \( u \) to be considered, \( i.e. \), we compare \( Y(u) \) with \( g^{(\phi(u))} \) instead of \( g(u) \), there are situations where no asymptotic dependent classifier exists, but all tail equivalent classifiers are asymptotically independent, e.g., if the covariate vector \( X \) does not comprise the main drivers for extremes. Even in this case, however, it is still of interest to compare different asymptotically independent classifiers as the covariate vector \( X \) might contain at least some information on the extremes. Thus, we need to modify the risk function under consideration.

To this end, we note that, in the case of asymptotic independence, we typically face the following situation where the probability mass \( B_u = \mathbb{P}(g(u)(X) = Y(u) = +1) \) becomes negligible compared to the sum of probability masses \( A_u + C_u = \mathbb{P}(g(u)(X) = +1) + \mathbb{P}(Y(u) = +1) \), as \( u \) gets large. The top graph in Figure 1 schematically displays the
measurable sets over which the probabilities $A_u$, $B_u$ and $C_u$ are computed. When $B_u = o(A_u + C_u)$, the classifier $g^{(u)}$ needs to squeeze some second order information from $X$, i.e. to zoom into a smaller set over which $A_u + C_u$ is estimated by reducing its size. This is possible by introducing a moderate threshold $v$ smaller than $u$. The bottom right panel of Figure 1 displays the new sets, and their corresponding probabilities, that will be used to compute our new risk function $R^{(u,v)}$. In other words, we restrict our attention to the event $\{g^{(v)}(X) = +1, Y^{(v)} = +1\}$ and evaluate the risk $R^{(u)}$ on this smaller set. This leads to the conditional risk function

$$R^{(u,v)}(g) = \frac{\mathbb{P}(g^{(u)}(X) \neq Y^{(u)} | g^{(v)}(X) = Y^{(v)} = +1)}{\mathbb{P}(g^{(u)}(X) = +1 \text{ or } Y^{(u)} = +1 | g^{(v)}(X) = Y^{(v)} = +1)}, \quad u \geq v \geq 0. \quad (2)$$

As we assumed that $\mathbb{P}(g^{(0)}(X) = +1) = \mathbb{P}(Y^{(0)} = +1) = 1$, we can see that the definition of the conditional risk function provides a generalization of the risk function $R^{(u)}$ defined in [1] with $R^{(u)}(g) = R^{(u,0)}(g)$. Similar to the extremal risk, we can also consider some extremal limit of the conditional risk by letting $u \to \infty$ and $v \to \infty$ at the same rate. More precisely, for every $\varepsilon \in (0, 1)$, we define the extremal conditional risk $R_\varepsilon$ by

$$R_\varepsilon(g) = \lim_{u,v \to \infty} R^{(u,v)}(g). \quad (3)$$

In the following sections, we will first investigate the original risk function $R^{(u)}(g) = R^{(u,0)}(g)$ defined in [1] and study its properties for asymptotically dependent and asymptotically independent classifiers, before turning to more general conditional risk $R^{(u,v)}(g)$ from [2]. The following lemma collects several results that will form the basis for our further analysis.

**Lemma 3** Let $g$ be a level-dependent binary classifier with $g^{(0)}(\cdot) \equiv +1$ and let $u \geq v \geq 0$. Then, the conditional risk $R^{(u,v)}(g)$ can be written as

$$R^{(u,v)}(g) = 1 - \left[ \frac{1}{\mathbb{P}(Y^{(u)} = +1 | g^{(u)}(X) = +1, Y^{(v)} = +1)} + \frac{1}{\mathbb{P}(g^{(u)}(X) = +1 | g^{(v)}(X) = +1, Y^{(u)} = +1)} - 1 \right]^{-1}, \quad (4)$$

In addition, we have the three following properties for $R$:

(a) For all $u \geq 0$, the conditional risk $R^{(u,v)}(g)$ is non-increasing in $v$ with $R^{(u,u)}(g) = 0$.

(b) Let $h$ be another level-dependent binary classifier. If $g^{(u)} \equiv h^{(u)}$ and $g^{(v)} \leq h^{(v)}$ for some $v < u$, then

$$R^{(u,v)}(g) \leq R^{(u,v)}(h).$$

(c) Let $a, b$ be some positive constants and $c_{v,v'}$ be some positive function, such that, for any $v, v' \in [0, u]$,

$$\mathbb{P}(g^{(v)}(X) = +1, Y^{(v')} = +1) = c_{v,v'} \left( \mathbb{P}(g^{(v)}(X) = +1) \right)^a \left( \mathbb{P}(Y^{(v')} = +1) \right)^b \quad (5)$$
Figure 1: Schematic representations of the ratio $R^{(u)}$ defined by (1) and $R^{(u,v)}$ defined by (2). The asymptotic dependent (resp. independent) case occurs when the probability mass $B_u$ is comparable (resp. negligible) to $A_u + C_u$, as $u$ gets large. By removing mass below $v$, the probability mass $B_{u,v}$ can remain comparable to $A_{u,v} + C_{u,v}$ in the asymptotic independent case.
Then

\[
R^{(u,v)}(g) = 1 - \left[ \frac{c_{v,u}}{c_{u,u}} \left( \frac{1}{\mathbb{P}(g^{(u)}(X) = +1 \mid g^{(v)}(X) = +1)} + \frac{1}{\mathbb{P}(Y^{(u)} = +1 \mid Y^{(v)} = +1)} - 1 \right) \right]^{-1}. \tag{6}
\]

We deduce from Equation (4) that \( R^{(u,v)}(g) = 0 \) if and only if

\[
\mathbb{P}(Y^{(u)} = +1 \mid g^{(u)}(X) = +1) = \mathbb{P}(g^{(u)}(X) = +1 \mid g^{(v)}(X) = Y^{(u)} = +1) = 1.
\]

The second property of this lemma indicates that, for a given pair of thresholds \( u \geq v \geq 0 \), the risk function becomes smaller if the function \( g^{(v)} \) becomes as small as possible.

Equation (5) can be viewed as a mixing condition that leads to a simple expression of \( R^{(u,v)}(g) \) based on probabilities of disjoint events.

### 2.2 The extremal risk \( R \)

In this section, we will further investigate the original risk function \( R^{(u)}(g) = R^{(u,0)}(g) \) defined in (1). Applying Lemma 3 with \( v = 0 \) and using the fact that \( \mathbb{P}(Y^{(0)} = +1) = \mathbb{P}(g^{(0)}(X) = +1) = 1 \), we obtain

\[
R^{(u)}(g) = 1 - \left[ \frac{1}{\mathbb{P}(Y^{(u)} = +1 \mid g^{(u)}(X) = +1)} + \frac{1}{\mathbb{P}(g^{(u)}(X) = +1 \mid Y^{(u)} = +1)} - 1 \right]^{-1}.
\]

This leads to the expression

\[
R^{(u)}(g) = 1 - \frac{\mathbb{P}(g^{(u)}(X) = +1 \mid Y^{(u)} = +1)}{1 - \mathbb{P}(g^{(u)}(X) = +1 \mid Y^{(u)} = +1) + \mathbb{P}(g^{(u)}(X) = +1) / \mathbb{P}(Y^{(u)} = +1)}. \tag{7}
\]

We can use this representation to analyze the effect of tail equivalence of \( Y \) and \( g \) as defined in Definition 2.

- If \( g \) has a (partially) lighter tail than \( Y \) in the sense that \( \liminf_{u \to \infty} \mathbb{P}(g^{(u)}(X) = +1) / \mathbb{P}(Y^{(u)} = +1) = 0 \), we also have \( \liminf_{u \to \infty} \mathbb{P}(g^{(u)}(X) = +1 \mid Y^{(u)} = +1) = 0 \), and consequently, by Equation (7),

  \[ \limsup_{u \to \infty} R^{(u)}(g) = 1. \]

- In the case where \( g \) possesses a (partially) heavier tail than \( Y \) in the sense that \( \limsup_{u \to \infty} \mathbb{P}(g^{(u)}(X) = +1) / \mathbb{P}(Y^{(u)} = +1) = \infty \), we also see by Equation (7) that

  \[ \limsup_{u \to \infty} R^{(u)}(g) = 1. \]

This indicates that, whenever \( g \) and \( Y \) are tail comparable, but not tail equivalent, the classifier \( g \) cannot outperform naive classifiers with respect to the risk \( R^{(u)}(g) \) for large \( u \).

While tail equivalence can be related to marginal properties of the classifier and the observations, the extremal risk also depends on the dependence between those two which is measured in terms of \( \chi^*(g) \), as the following lemma shows.
Lemma 4 Assume that the two limits $c(g) \in (0, \infty)$ and $\chi^*(g) \in [0, 1]$ defined in Definition 2 exist. Then, $\chi^*(g) \leq c(g)$ and the extremal risk has the expression

$$R(g) = 1 - \frac{\chi^*(g)}{1 + c(g) - \chi^*(g)}.$$  \hspace{1cm} (8)

In particular,

$$R(g) = 0 \text{ if and only if } c(g) = \chi^*(g) = 1.$$  

This lemma indicates that $c(g) = 1$ is a necessary condition to have $R(g) = 0$. Note that this condition implies that the constant $\chi^*(g)$ simply corresponds to the “classical” tail dependence coefficient $\chi(g)$ of the events $\{g^{(u)}(X) = +1\}$ and $\{Y^{(u)} = +1\}$ as $u \to \infty$, i.e.

$$\chi(g) := \lim_{u \to \infty} \mathbb{P}(\text{if } f(X) + N > u \text{ then } f(X) + N > u, \text{ otherwise } g^{(u)}(X) = 1),$$

and so, the case $c(g) = 1$ simplifies the expression of the risk

$$R(g) = 1 - \frac{\chi(g)}{2 - \chi(g)}.$$  \hspace{1cm} (9)

This equality tells us that any asymptotically calibrated classifier with $\chi(g) = 0$ always produces a risk function $R(g) = 1$. Consequently, any asymptotically independent classifier is as uninformative as the two naive classifiers. A reasonable strategy will be to dismiss all asymptotically independent classifiers and find/construct new asymptotically dependent classifiers with positive $\chi(g)$. But, finding asymptotically dependent classifiers can be complex in practice, and in addition, in some not so exotic setups, this is not always possible. To see this, we consider the simple non-linear regression model in the following lemma.

Lemma 5 Assume that the observations $Y^{(u)}$ are generated by a non-linear regression model: for all $u \geq 0$,

$$Y^{(u)} = \begin{cases} +1, & \text{if } f(X) + N > u \\ -1, & \text{otherwise} \end{cases},$$

where $N$ corresponds to a regularly varying random noise and $X$ corresponds to the explanatory variables, independent of $N$. If

$$\mathbb{P}(f(X) > u) = o(\mathbb{P}(N > u)) \ (u \to \infty),$$

then for any level-dependent classifier $g$, we always have

$$R(g) = 1$$

provided that $R(g)$ exists. Hence, no classifier can outperform naive classifiers for this regression model.

\footnote{Within the framework of forecast verification, $c(g) = 1$ means that $g$ and $Y$ are asymptotically calibrated.}
Note that even if the forecaster knows exactly the function \( f(\cdot) \) and has drawn from the explanatory \( X \), the “ideal” classifier

\[
g^{(u)}(X) = \begin{cases} +1, & \text{if } f(X) > u \\ -1, & \text{otherwise} \end{cases}
\]

will perform badly, \textit{i.e.} \( R(g) = 1 \). In addition, the classical trick of using ranks to avoid the problem of marginals discrepancy cannot be applied here. For example, suppose that \( f(X) + N \) is unit Fréchet distributed, then transforming the marginals of \( X \) into unit Fréchet random variables, say into \( \tilde{X} \), does not remove the issue as the unobserved noise \( N \) has still have heavier tails than \( \tilde{f}(\tilde{X}) = f(X) \) for some function \( \tilde{f}(\cdot) \). An example of a simulation of this framework is given in Section 3 (see also Figure 2). So, a finer risk measure is needed that is able to distinguish different classifiers in case of asymptotic independence, as defined in Definition 2 between the classifiers and the observations \( Y \).

### 2.3 The extremal conditional risk \( R_{\xi} \)

The choice of the conditions in Lemma 3 brings new possibilities to construct finer risk measures for extremes than \( R^{(u)} \). In particular, the conditions allow us to exclude events that are not at least “a little extreme” (w.r.t. some smaller level \( v \)) both in terms of the observation \( Y^{(u)} \) and the classifier \( g^{(v)} \). Such a modeling strategy is at the core of hidden regular variation and asymptotic independence models. More precisely, we first need to fix marginal features. We assume that both \( u \mapsto P(g^{(u)}(X) = +1) \) and \( u \mapsto P(Y^{(u)} = +1) \) are regularly varying with indices \( \alpha_g > 0 \) and \( \alpha_Y > 0 \), respectively. This means that for any \( \varepsilon \in (0, 1) \),

\[
\lim_{u \to \infty} P(g^{(u)}(X) = +1 | g^{(\varepsilon u)}(X) = +1) = \varepsilon^{\alpha_g} \quad \text{and} \quad \lim_{u \to \infty} P(Y^{(u)} = +1 | Y^{(\varepsilon u)} = +1) = \varepsilon^{\alpha_Y}.
\]

These limits have to be understood in terms of Equation (6), \textit{i.e.} in terms of \( P(g^{(u)}(X) = +1 | g^{(v)}(X) = +1) \) and \( P(Y^{(u)} = +1 | Y^{(v)} = +1) \) with \( v = \varepsilon u \). To apply (6), the mixing condition (5) needs to be satisfied. To do so, we opt for an extended version of the framework of Ramos and Ledford (2009), \textit{i.e.}

\[
P[g^{(u)}(X) = +1, Y^{(v)} = +1] = L(u, v)(u^{-\alpha_g} v^{-\alpha_Y})^{1/2\eta}, \tag{10}
\]

where \( \eta \in (0, 1] \) indicates the rate of decay of the joint probability as a function of \( u \) and \( v \) and \( L(\cdot, \cdot) \) is bivariate slowly varying function, \textit{i.e.} there exists a limit function \( \ell^s : (0, \infty) \times (0, \infty) \to (0, \infty) \) defined as

\[
\ell^s(s, t) = \lim_{u \to \infty} \frac{L(us, ut)}{L(u, u)}, \quad s, t > 0
\]

and satisfying \( \ell^s(cs, ct) = \ell^s(s, t) \) for all \( c, s, t > 0 \). The parameter \( \eta \) measures the extremal dependence strength. The case \( \eta = 1 \) (and \( \lim_{u \to \infty} L(u, u) > 0 \)) corresponds to the asymptotic dependence of \( Y \) and \( g \) while \( \eta < 1 \) (or \( \eta = 1 \) and \( \lim_{u \to \infty} L(u, u) = 0 \)) corresponds to the asymptotic independence case. In particular, if \( \eta = .5 \), then independence appears in the extremes. If \( .5 < \eta < 1 \) (\( 0 < \eta < .5 \)) the extremes are said to be positively (negatively) associated.

Now, noticing that the joint probability in Equation (10) corresponds to the mixing condition (5), we can apply Lemma 3 see Appendix D for a proof.
Proposition 6 Under the Ramos and Ledford model defined by (10), the extremal conditional risk in (3) can be expressed as

\[ R_\varepsilon(g) = 1 - \frac{1}{\ell^*(\varepsilon, 1)\varepsilon^{-\alpha_g/2\eta} + \ell^*(1, \varepsilon)\varepsilon^{-\alpha_g/2\eta} - 1} \]

for any \( \varepsilon \in (0, 1) \)

such that \( \ell^* \) is continuous at \((\varepsilon, 1)\) and \((1, \varepsilon)\).

For fixed \( \varepsilon \in [0, 1) \), the risk function \( R_\varepsilon(g) \) decreases with increasing \( \eta \). So, given all parameters are fixed but \( \eta \), the forecaster should aim at maximizing \( \eta \). In practice, two forecasters, say \( g_1 \) and \( g_2 \), may produce different \( \ell^*(\cdot, \cdot) \) and \( \alpha_g \). Consequently, the minimization of \( R_\varepsilon(g) \) can also depend, besides \( \eta \), on other parameters.

Table 2: Overview of the risk measures used in this study.

| Notation       | Usage          | Associated limiting risk          |
|----------------|----------------|---------------------------------|
| \( R^{(u)}(g) \) [1] | Asymp. dependence | \( R_\varepsilon \) = \lim_{u \to \infty} R^{(u)}(g) \) Extremal risk |
| \( R^{(u,v)}(g) \) [2] | Asymp. independence | \( R_\varepsilon(g) = \lim_{u,v \to \infty} R^{(u,v)}(g) \) Extremal conditional risk |

2.4 Risk function inference

Concerning the estimation of \( R_\varepsilon(g) \) defined by (3), the empirical estimator can be easily computed from the sample \((g^{(u)}(X_i), Y_i^{(u)}), g^{(v)}(X_i), Y_i^{(v)}(X_i))_{i=1,\ldots,n}\) based on two thresholds \( u, v \) such that \( v/u \approx \varepsilon \). More precisely, such an estimator is given by

\[
\hat{R}^{(u,v)}_n(g) = \frac{\sum_{i=1}^n 1\{g^{(u)}(X_i) \neq Y_i^{(u)}, Y_i^{(v)} = 1, g^{(v)}(X_i) = 1\}}{\sum_{i=1}^n 1\{\max\{g^{(u)}(X_i), Y_i^{(u)}\} = 1, Y_i^{(v)} = 1, g^{(v)}(X_i) = 1\}}. \tag{11}
\]

The following proposition describes the asymptotic property of such an estimator.

Proposition 7 Assume that the risk function \( R_\varepsilon(g) \) defined by (3) exists. Consider two sequences \( u_n, v_n \to \infty \) such that \( v_n/u_n \to \varepsilon \) and \( np_g(u_n, v_n) \to \infty \) as \( n \to \infty \) where

\[
p_g(u_n, v_n) := \mathbb{P}(\max\{g^{(u)}(X), Y^{(u)}\} = 1, g^{(v)}(X) = Y^{(v)} = 1).
\]

If

\[
\lim_{n \to \infty} \sqrt{np_g(u_n, v_n)} \left( \frac{\mathbb{P}(g^{(u)}(X) \neq Y^{(u)}), g^{(v)}(X) = Y^{(v)} = 1}{p_g(u_n, v_n)} - R_\varepsilon(g) \right) = 0,
\]

then the empirical estimator \( \hat{R}^{(u,v)}_n(g) \) is asymptotically normal with negligible bias, i.e.,

\[
\sqrt{np_g(u_n, v_n)} \left( \hat{R}^{(u,v)}_n(g) - R_\varepsilon(g) \right) \xrightarrow{d} \mathcal{N}(0, R_\varepsilon(g)(1 - R_\varepsilon(g))).
\]
Remark 8 From the proof of Proposition \textsuperscript{7} it follows that
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{\max\{g^{(u)}(X_i), Y_i^{(u)}\} = 1, Y_i^{(v)} = 1, g^{(v)}(X_i) = 1\} \rightarrow \mathbb{P}_g(u, v)
\]
in probability as \( n \to \infty \), i.e., the numerator is a consistent estimator of \( p_g(u, v) \). As \( p_g(u, v) \) occurs in the rate of convergence in Proposition \textsuperscript{7} this estimator can be used to assess the variance of \( \hat{R}_n^{(u,v)}(g) \).

While Proposition \textsuperscript{7} provides asymptotic normality of an estimator for the extremal conditional risk \( R(g) \) for a single classifier \( g \), the comparison of the extremal conditional risks of two classifiers \( g_1 \) and \( g_2 \) requires joint asymptotic normality of the estimators \( \hat{R}_n^{(u,v)}(g_1) \) and \( \hat{R}_n^{(u,v)}(g_2) \). In order to obtain such a result, some more assumptions on the joint behaviour of \( g_1^{(u)}(X) \) and \( g_2^{(u)}(X) \) are needed. Note that the rates \( (np_{g_1}(u, v))^{1/2} \) and \( (np_{g_2}(u, v))^{1/2} \) might be different. Thus, joint convergence holds at a potentially slower rate \( [n(p_{g_1} \land p_{g_2})(u, v)]^{1/2} \). In this case, it might happen that one of the asymptotic variances is equal to zero.

**Proposition 9** Let \( g_1 \) and \( g_2 \) be two extremal classifiers and \( \varepsilon \in [0, 1] \). Assume that the both risk functions \( R_\varepsilon(g_1) \) and \( R_\varepsilon(g_2) \) exist as well as the two limits
\[
c_\varepsilon^{(1)} := \lim_{u,v \to \varepsilon} \frac{p_{g_1}(u,v) \land p_{g_2}(u,v)}{p_{g_1}(u,v)} \in [0, 1] \quad \text{and} \quad c_\varepsilon^{(2)} := \lim_{u,v \to \varepsilon} \frac{p_{g_1}(u,v) \land p_{g_2}(u,v)}{p_{g_2}(u,v)} \in [0, 1].
\]
Furthermore, we assume that the limits \( r_\varepsilon \in [0, 1], q_\varepsilon^{(12)} \in [0, 1], q_\varepsilon^{(21)} \in [0, 1] \) and \( p_\varepsilon \in [0, 1] \) given in the appendix exist.

Consider two sequences \( u_n, v_n \to \infty \) such that \( v_n/u_n \to \varepsilon \) and \( n(p_{g_1} \land p_{g_2})(u_n, v_n) \to \infty \) as \( n \to \infty \). If
\[
\lim_{n \to \infty} \sqrt{n(p_{g_1} \land p_{g_2})(u_n, v_n)} \left( \frac{\mathbb{P}(g_1^{(u_n)}(X) \neq Y^{(u_n)}, g_1^{(v_n)}(X) = Y^{(v_n)} = 1)}{p_{g_1}(u_n, v_n)} - R_\varepsilon(g_1) \right) = 0
\]
and
\[
\lim_{n \to \infty} \sqrt{n(p_{g_1} \land p_{g_2})(u_n, v_n)} \left( \frac{\mathbb{P}(g_2^{(u_n)}(X) \neq Y^{(u_n)}, g_2^{(v_n)}(X) = Y^{(v_n)} = 1)}{p_{g_2}(u_n, v_n)} - R_\varepsilon(g_2) \right) = 0
\]
then the two empirical estimators \( \hat{R}_n^{(u,v)}(g_1) \) and \( \hat{R}_n^{(u,v)}(g_2) \) are jointly asymptotically normal with negligible bias, i.e.,

\[
\sqrt{n(p_{g_1} \land p_{g_2})(u_n, v_n)} \left( \hat{R}_n^{(u,v)}(g_1) - R_\varepsilon(g_1), \hat{R}_n^{(u,v)}(g_2) - R_\varepsilon(g_2) \right) \xrightarrow{d} \mathcal{N} \left( \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], \left[ \begin{array}{cc} c_\varepsilon^{(1)} R_\varepsilon(g_1)(1 - R_\varepsilon(g_1)) & \sigma_{12} \\ \sigma_{12} & c_\varepsilon^{(2)} R_\varepsilon(g_2)(1 - R_\varepsilon(g_2)) \end{array} \right] \right),
\]
where
\[
\sigma_{12} = \sqrt{c_\varepsilon^{(1)} c_\varepsilon^{(2)}} \left( r_\varepsilon - q_\varepsilon^{(21)} R_\varepsilon(g_1) - q_\varepsilon^{(12)} R_\varepsilon(g_2) + p_\varepsilon R_\varepsilon(g_1) R_\varepsilon(g_2) \right)
\]
3 Simulations

3.1 A simple linear setup

In the following, the necessary ingredients as in Table 1 are computed using continuous responses that we truncate. This approach is aimed at obtaining the necessary input variables and should not be considered as the reference framework for the application of the two developed extremal risk measures.

Our main simulated example is therefore based on a simple linear regression model but with the feature that the components of the input vector $X$ do not have the same tail behavior and the noise is regularly varying, see Lemma 5. More precisely, the multivariate vector $X$ is defined as follows

$$
\begin{align*}
X_1 &\sim \text{Pareto}(3) \\
X_2 &\sim \text{Pareto}(2) \\
X_3 &\sim \text{Exp}(1) \\
X_4 &\sim \text{Exp}(2)
\end{align*}
$$

where all $X_i$ are independent with $X_1$ and $X_2$ Pareto distributed with respective tail index 2 and 3, and $X_3$ and $X_4$ exponentially distributed with respective scale parameters 1 and 2. For the simulations, we use the same framework as in Lemma 5, i.e. assuming that the events of interest are given by

$$
Y(u) := \begin{cases} 
+1, & \text{if } X_1 + N > u \\
-1, & \text{otherwise}
\end{cases},
$$

(12)

for $u \geq 0$, where $N \sim \text{Pareto}(2)$ represents an independent noise with heavier tail than $X_1$. So, given a sample $(\{X_{j,i}\}_{1 \leq j \leq 4}, N_i)_{i=1,\ldots,n}$ with $n = 10000$, our goal is to compare different classifiers in terms of predicting extreme occurrences, here defined as the event $\{X_1 + N > u\}$ with $u$ equal to the 97th percentile of $X_1 + N$. In this simulation setup, it is clear from (12) that all variables but $X_1$ are useless to explain the extreme occurrences of $Y(u)$. In addition, Lemma 5 tells us that the relevant information contained in the variable $X_1$ is hidden by the heavier noise $N$, i.e. we are in the case of asymptotic independence.

An example of such simulation is given in Figure 2. The left panel displays a scatter plot between the response variable $f(X) + N$ (left axis) and the explanatory variable $f(X)$ (right axis). In the present simulation framework, $f(X) = X_1$. As expected, no sign of asymptotic dependence can be found in the upper corner. In the right panel, we remove the mass along the axis (gray points) by conditioning on the joint event $\{X_1 > v\} \cap \{Y(v) = +1\}$ with $v = \varepsilon u$ and $\varepsilon = .7$, i.e. we consider the “ideal” classifier $\{g(v)(X) = +1\} := \{X_1 > v\}$. The right panel zooms on these specific points and highlights a clear dependence between $X_1 + N$ and $X_1$ that was hidden by the heavy tailed noise $N$ in $X_1 + N$.

In practice, we do not know the ideal choice of classifier $g$, so we need to introduce different classifiers and compare them.

3.2 Classifiers descriptions

Table 4 provides the list of classifiers that we compare with our metrics (11). This list contains some of the most standard classifiers found in the literature (see, e.g. Hastie
et al., 2009): logistic regression \((Logistic)\), decision tree \((Tree)\), random forest \((RF)\) and support vector machine \((SVM)\). We apply them with their built-in cost function that is not necessarily fine-tuned to forecast extremes. This is not an issue because our main goal is to compare existing forecasters, and not to create new ones (see, e.g. Jalalzai et al., 2018; Wang and Dey, 2010, for such developments). An example of binary outputs from the decision tree classifier is given in Table 3. The difference between the left and right panels corresponds to the training set based on two different thresholds \(u\) and \(v\), \(i.e.\) the ingredients needed in our study.

**Table 3:** Contingency table depicting an example of predicted binary output for the decision tree classifier with two different thresholds \(v\) and \(u > v\), with \(v = 0.4u\). The results shown are computed upon a testing set whose sample size is equal to 3000 (30% of the data)

| Threshold \(v\) | \(Y^{(v)} = +1\) | \(Y^{(v)} = -1\) |
|-----------------|-----------------|-----------------|
| \(g^{(v)}(X) = +1\) | 646             | 213             |
| \(g^{(v)}(X) = -1\) | 662             | 1479            |

| Threshold \(u > v\) | \(Y^{(u)} = +1\) | \(Y^{(u)} = -1\) |
|---------------------|-----------------|-----------------|
| \(g^{(u)}(X) = +1\) | 12              | 0               |
| \(g^{(u)}(X) = -1\) | 66              | 2922            |

In order to fix a baseline in terms of performance, we compare these four classic classifiers with a linear classifier defined as follows

\[
g_{\theta}^{(u)}(X) = \begin{cases} +1, & \theta^\top X > u, \\ -1, & \theta^\top X \leq u, \end{cases} \quad \text{for some } \theta \in [0, \infty)^d.
\]

This specific classifier is discussed in more details in Appendix B. In short, Proposition 10 shows that under mild conditions, one can find some optimal \(\theta^*\) such that the associated extremal risk \(R(g_{\theta^*})\) is minimal. In such context, and given the simulation framework (12), we expect the optimal linear classifier \(g_{\theta^*}^{(u)}\) to be the best among the different classifiers considered.
Table 4: Summary and key features of the different classifiers studied. See for example Hastie et al. (2009) for a comprehensive review of the last four classification methods.

| Method                | Main features                                                                                                                                 |
|-----------------------|---------------------------------------------------------------------------------------------------------------------------------------------|
| Linear classifier     | Simple binary classifier, parameters estimation based on the minimization of the risk function over the set of contributing variables, theoretical value of $R(g_\theta)$ inferred from spectral decomposition. |
| Logistic regression   | Parametric linear model with a lasso penalty, coefficients of less contributing variables are set to zero.                                       |
| Decision trees        | Easy to interpret, gives relative importance of each variables, learns simple decision rules inferred from the input.                             |
| Random forests        | Builds multiple decision trees combined by majority vote, better predictive power than decision trees.                                          |
| Support vector machines| Finds the best hyperplane to separate two overlapping classes, generally performs better than the other classifiers.                          |

3.3 Implementation and results

We split our simulated data set in two: 70% for a training part, over which we train our different classifiers to get good predictive power; 30% for a testing part, which we use to estimate different risks $R^{(u,v)}$. Since we consider here simulated data, we assume that we can artificially generate several data sets as in Table 1 for a range of thresholds $v$. Note that each algorithm has the same inputs, in particular the same binary sequence describing the events $\{Y^{(u)} = +1\}$ with $u$ set to be equal to the 97th percentile of $X_1 + N$, and $\{Y^{(v)} = +1\}$ with $v = \varepsilon u$ for different values of $\varepsilon$. This cross-validation procedure has been repeated 50 times. The sample used to compute our risk functions is based on the bivariate binary vectors $(g^{(u)}(\{X_{j,i}\}_{1 \leq j \leq 4}), \{Y_{i}^{(u)} = 1\})_{i=1,...,n}$. In addition, the binary outputs of the classifiers $g$ are obtained under the threshold $u$ and some thresholds $v < u$, so the training part has to be performed twice (once for each threshold). Then, the empirical risk estimator defined by (11) can be computed. Figure 3 shows the sensitivity of the classifier ranking with respect to the value of $v$.

According to Lemma 5, our five classifiers cannot asymptotically outperform the naive classifiers. And as $u$ gets to infinity, the estimated risks should converge to 1, the worst possible value. The top-left panel in Figure 3, corresponding to the case $v = 0$, shows that each classifier has a risk near to one at high, but finite, level $u$. So, to better discriminate classifiers, we need to remove the masses along the axes by setting a positive value for $v$. As we increase $v$, the size of sets needed to compute (11) becomes smaller; see the values of $n_{vk}$ in the legend of each panel. Hence, the blue box plots become wider as $v$ increases: a classical bias-variance trade-off.

As we know from (12) that the true generative process is linear, $v = 0.4u$ appears as a reasonable value to balance the bias-variance trade-off. In this case, our linear classifier,
Figure 3: Estimation of $R^{(u,v_k)}(g)$ for different classifiers $g$ (cross-validation with 50 repetitions) and $k \in \{0, 1, 2, 3\}$. In red (top-left) are the estimates for $v_0 = 0$ and in blue for different values of $v_k = \varepsilon_k u$, $k > 0$ and $\varepsilon_k \in \{0.4, 0.6, 0.8\}$. At the top of each plot, the value of $\varepsilon_k$ and the number of points $n_{v_k}$ such that $\{Y^{(v_k)} = +1\}$ from the testing set are given.
tailored to handle linear asymptotic independence cases, outperforms the other classifiers. And among these four other classifiers, the decision tree appears to be the best, but it is still far from the optimal linear solution. More importantly, for this specific empirical study the overall ranking seems to be insensitive to the values of $v > 0$.

Other simulations concerning the regular variation case are available upon request.

4 Danube river discharges

![River map of the upper Danube basin](image)

Figure 4: River map of the upper Danube basin, showing sites of the 31 gauging stations along the Danube and its tributaries. Water flows toward gauging station 1. The stations represented by a green triangle shaped dot are the three stations of interest as described in Section 4.

We now apply our assessment approach to summer daily river discharges (measured in $m^3/s$) at 31 stations spread over the upper Danube basin, see Figure 4, and recorded over the time period 1960-2010 in June, July and August. These observations have been studied by the EVT community (see, e.g. Asadi et al., 2015; Mhalla et al., 2020; Gnecco et al., 2021). This dataset was made available by the Bavarian Environmental Agency (http://www.gkd.bayern.de).

To remove temporal clustering in extreme river discharges, Mhalla et al. (2020) in their Section 5 implemented a declustering step, we use here these declustered data. Each station then contains $n = 428$ observations that we will consider temporally independent. In order to reduce the large discrepancy in terms of discharges magnitude among stations, we force the starting value of all 31 time series to equal zero by subtracting to each station its minimum. Then, we re-normalize each time series by its range (i.e. the difference between the maximum and the minimum of each time series).

These post processing treatments are useful to display and interpret the data at hand and do not impact the classifiers performance.

Although all 31 station recordings are available, we can artificially remove one station and try to predict its extreme occurrences from a given subset of other weather stations. In this section, we remove station 1 (downstream) and try to predict its extreme occurrences from only stations 23 and 24, which are indirect tributaries to the main river flow. So, this
Figure 5: Estimation of $R^{(u,v)}(g)$ for five different classifiers (cross-validation with 50 repetitions, 70% train and 30% test), threshold $u$ is the 0.85 quantile of $X_1$. In red (left) are the estimates when $v = 0$ and in blue (right) for $v = 0.6u < u$. The length of the testing set is equal to 129, this leads to around 60 points such that $\{Y^{(v)} = +1\}$ and nearly 20 points such that $\{Y^{(u)} = +1\}$.

setup is complex\(^4\) for two reasons. First, station 1, as a downstream point that accumulates all discharges, has a much heavier tail than the two tributaries. Second, empirical investigations (not reported here) show that it is difficult to determine if we are in the asymptotically dependent or independent case.

As in Table 1, we need to have at our disposal the occurrences of extreme river discharges at station 1 for two different thresholds $v$ and $u$ (with $v < u$). Note that we could assume that we do not observe the whole time series of river discharges at this station, only the extreme occurrences for two different levels. Furthermore, we could also assume that we only have outputs from different classifiers, as in Table 1 previously built using the observations of stations 23 and 24. However, in the following, we assume that we can observe the whole time series of river discharges at stations 23 and 24, that we use to built different classifiers and then compare their performances.

Figure 5 summarizes our findings. Considering a conditioning set with the lower threshold $v$ implies that only 190 points remain from the original length of 428 data points per station. This can explain why, looking at Figure 5, the uncertainty in the risk estimate increases when considering $R^{(u,v)}$ (blue boxes, right panel) instead of $R^{(u,0)} = R^{(u)}$ (red boxes, left panel).

First, unlike the simulation example in Section 3.1, we do not want to assume here that we can observe extreme occurrences at station 1 at any level $v$. Furthermore, and as already mentioned, it is not clear to assess whether our river discharges analysis using the three selected weather stations belongs to the framework of asymptotic independence or not. Still, it is reassuring that the ranking of the classifiers in Figure 5 appears to be insensitive

\(^4\)Section C treats a simpler case where the extreme occurrences at station 1 are predicted from the whole set of remaining stations. In this case, strong dependencies among station 1 and other stations can be observed. So, the main issue is to select these stations, a problem discussed in Section B.
to the values of \( R^{(u)} \) or \( R^{(u,v)} \), i.e. whether the data are asymptotically dependent or not. This hints that, among all the classifiers, the logistic regression with lasso penalty seems to perform better than the four other classification methods. This ranking of classifiers is specific to this particular example. No general conclusions about lasso techniques for extremes should be drawn.

Besides this river example, we advocate practitioners to compute risk functions that can both handle the asymptotic dependence and asymptotic independence cases. This also complements the recent tools used to discriminate between the two cases (see, e.g. Ahmed et al., 2022). In addition, the linear classifier could provide a simple benchmark with well understood properties with respect to \( R^{(u)} \), see Proposition 10.

5 Discussion

This work focused on the elaboration of risk measures tailored for binary classification of extreme events. The main hypothesis of this study is that the ingredients we are working with are in the form of Table 1, i.e. we have at our disposal binary outputs of extreme events based on different thresholds.

Building on the multivariate regular variation and the hidden regular variation frameworks, two extremal risk measures (see Table 2) have been developed, which can encompass different type of asymptotic dependence between the observed binary events and the binary forecasts.

To illustrate the theoretical results of the paper and their applicability, a simulation study and an application to a river network dataset have been proposed.

Finally, we would like to point out that even if we assume that the thresholds are imposed by a given application, the subject of optimizing an \( \varepsilon \) could be addressed in a future work.

6 Supplementary Materials

A R package is available on GitHub that implements the empirical estimation of the risk function developed in this paper and can be used either to reproduce the results of the conducted classifier comparisons or to perform new comparisons using other binary classifiers. The data used in the application are available in the R package graphicalExtremes (Engelke et al., 2019).

References

Ahmed, M., Maume-Deschamps, V., and Ribereau, P. (2022). Recognizing a spatial extreme dependence structure: A deep learning approach. *Environmetrics*, 33(4):e2714.

Asadi, P., Davison, A. C., and Engelke, S. (2015). Extremes on river networks. *The Annals of Applied Statistics*, 9(4):2023 – 2050.

Coles, S., Heffernan, J., and Tawn, J. (1999). Dependence measures for extreme value analyses. *Extremes*, 2:339–365.
Engelke, S., Hitz, A. S., and Gnecco, N. (2019). *graphicalExtremes: Statistical Methodology for Graphical Extreme Value Models*. R package version 0.1.0.9000.

Engelke, S. and Ivanovs, J. (2021). Sparse structures for multivariate extremes. *Annual Review of Statistics and Its Application*, 8(1):241–270.

Ferro, C. A. T. (2007). A probability model for verifying deterministic forecasts of extreme events. *Weather and Forecasting*, 22(5):1089 – 1100.

Ferro, C. A. T. and Stephenson, D. B. (2011). Extremal dependence indices: Improved verification measures for deterministic forecasts of rare binary events. *Weather and Forecasting*, 26(5):699 – 713.

Gnecco, N., Meinshausen, N., Peters, J., and Engelke, S. (2021). Causal discovery in heavy-tailed models. *The Annals of Statistics*, 49(3):1755 – 1778.

Haixiang, G., Yijing, L., Shang, J., Mingyun, G., Yuanyue, H., and Bing, G. (2017). Learning from class-imbalanced data: Review of methods and applications. *Expert Systems with Applications*, 73:220–239.

Hastie, T., Tibshirani, R., and Friedman, J. (2009). *The Elements of Statistical Learning: Data Mining, Inference, and Prediction* (2nd edition). Springer series in statistics. Springer, New York.

He, H. and Ma, Y. (2013). *Imbalanced Learning: Foundations, Algorithms, and Applications*. Wiley-IEEE Press, 1st edition.

Heffernan, J. and Resnick, S. (2005). Hidden regular variation and the rank transform. *Advances in Applied Probability*, 37(2):393–414.

Hilario, A. F., Lopez, S. G., Galar, M., Prati, R. C., Krawczyk, B., and Herrera, F. (2018). *Learning from Imbalanced Data Sets*. Springer International Publishing.

Jalalzai, H., Clémençon, S., and Sabourin, A. (2018). On Binary Classification in Extreme Regions. In *Advances in Neural Information Processing Systems*, pages 3092–3100.

Ledford, A. W. and Tawn, J. (1996). Statistics for near independence in multivariate extreme values. *Biometrika*, 83(1):169–187.

Lerch, S., Thorarinsdottir, T. L., Ravazzolo, F., and Gneiting, T. (2017). Forecaster’s Dilemma: Extreme Events and Forecast Evaluation. *Statistical Science*, 32(1):106 – 127.

Mhalla, L., Chavez-Demoulin, V., and Dupuis, D. J. (2020). Causal mechanism of extreme river discharges in the upper danube basin network. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 69(4):741–764.

Ramos, A. and Ledford, A. (2009). A new class of models for bivariate joint tails. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 71(1):219–241.

Resnick, S. (2003). *A Probability Path*. Modern Birkhäuser Classics. Birkhäuser Boston.
Schaefer, J. T. (1990). The critical success index as an indicator of warning skill. *Weather and Forecasting*, 5(4):570 – 575.

Stephenson, D. B., Casati, B., Ferro, C. A. T., and Wilson, C. A. (2008). The extreme dependency score: a non-vanishing measure for forecasts of rare events. *Meteorological Applications*, 15(1):41–50.

Van der Vaart, A. W. and Wellner, J. A. (1996). *Weak Convergence and Empirical Processes*. Springer Series in Statistics. Springer, New York.

Wang, X. and Dey, D. K. (2010). Generalized extreme value regression for binary response data: An application to B2B electronic payments system adoption. *The Annals of Applied Statistics*, 4(4):2000 – 2023.
## A Overview of different risk measures for imbalanced binary data

Table 5: Key features of some risk measures for extremes. The integer $n$ denotes the sample size, $h$ the number of “hits” (true positive), $f$ the number of “false alarm”, $m$ the number of misses (false negative) and $r$ the number of correct rejection (see Table 6). The third column refers to the type of dependence when $u$ is large, see Definition 2. The last column focuses on the asymptotic properties of the proposed risk measure estimator.

| Name                                | Definition                                                   | Asymptotic extremal dependence type | Estimators Properties                  |
|-------------------------------------|--------------------------------------------------------------|------------------------------------|----------------------------------------|
| Critical success index              | $h/(h + f + m)$                                              | Dependent case                     | To be covered in this work             |
| Asymptotic risk in the extremes     | $\mathbb{P}(f + m \mid \|X\| > u)$ where $\|X\| > u$ defined extremes in the covariate space $X$ | Dependence case                    | Derived in (Jalalzai et al., 2018)     |
| Extreme dependency score            | $\frac{2 \log ((h + m)/n)}{\log (h/n)} - 1$                | Independence case                  | NA in (Stephenson et al., 2008)        |
| Extremal dependence index           | $\frac{\log \frac{f}{f + r} - \log \frac{h}{h + m}}{\log \frac{f}{f + r} + \log \frac{h}{h + m}}$ | Independence case                  | NA in (Ferro and Stephenson, 2011)     |
| Extension of the critical success index | see [2]                                         | Independent case                    | To be covered in this work             |

Table 6: Contingency table representing the different outputs for binary classification.

| $Y^{(u)} = +1$ | $Y^{(u)} = -1$ |
|----------------|----------------|
| $g^{(u)}(X) = +1$ | $h$           | $f$ |
| $g^{(u)}(X) = -1$ | $m$           | $r$ |
B Linear Classifiers

B.1 Definition, Basic Properties and Inference

In this section, we consider a specific type of classifiers which in this paper is referred to as linear classifiers, i.e. classifiers of the form

\[ g_\theta(X) = \begin{cases} +1, & \theta^\top X > u, \\ -1, & \theta^\top X \leq u, \end{cases} \]

for some \( \theta \in [0, \infty)^d \),

where, for notational convenience, we assume that \( X \) is nonnegative, i.e. \( X \in [0, \infty)^d \).

Intuitively, these classifiers assume that each covariate – provided that it has an effect at all — has an amplification effect on the extremity of the event to be classified and model the strength of these effects by the weight vector \( \theta \). To obtain an optimal linear classifier of \( g_\theta(X) \), i.e. some weight vector \( \theta^* \) such that the extremal risk \( R(g_{\theta^*}) \) gets minimal, we need to impose some joint extremal dependence structure on \( g \) and \( Y \) which allow us to calculate \( R \) from (8).

Even though some of the results can also be obtained in a similar manner in a more general framework for the extremal conditional risk \( R_\varepsilon \), henceforth, we will focus on the asymptotically dependent case where we might find some optimal classifier with unconditional extremal risk \( R(g_{\theta^*}) < 1 \). As discussed before, in this case, \( g_{\theta^*} \) has to be tail equivalent to \( Y \), i.e.

\[ \frac{\mathbb{P}((\theta^*)^\top X > u)}{\mathbb{P}(Y(u) = +1)} \to c(g) \in (0, \infty) \quad \text{as} \quad u \to \infty. \]

As the tail behavior of a linear combination is mainly driven by the component with the most heavy tail, it is natural to assume the tail behavior of \( \|X\|_\infty \) jointly with the asymptotic behavior of \( u \mapsto \mathbb{P}(Y(u) = +1) \). In order to obtain asymptotic dependence, we need an additional assumption similar to joint regular variation in classical extreme value theory.

To establish such a condition, let us have a closer look at the function \( u \mapsto \mathbb{P}(Y(u) = +1) \) and assume that it is eventually continuous. Then, we can define the random variable

\[ H = \sup\{u \geq 0 : Y(u) = +1\}. \]

By definition of \( N \) and the fact that the sets \( \{Y(u) = +1\} \) are nested for different levels \( u \), we note that \( H > u \) implies that \( Y(u) = +1 \) while \( H < u \) implies that \( Y(u) = -1 \). To cover the case \( H = u \), we note that

\[ \begin{align*} \mathbb{P}(H = u) &= \mathbb{P}(Y(v) = +1 \text{ for all } v < u \text{ and } Y(v) = -1 \text{ for all } v > u) \\
&= \mathbb{P}(Y(v) = +1 \text{ for all } v < u) - \mathbb{P}(Y(v) = +1 \text{ for some } v > u) \\
&= \lim_{v \uparrow u} \mathbb{P}(Y(v) = +1) - \lim_{v \downarrow u} \mathbb{P}(Y(v) = +1) = 0 \end{align*} \]

for sufficiently large \( u \) as the function \( v \mapsto \mathbb{P}(Y(v) = +1) \) as assumed to be eventually continuous. Thus, for sufficiently large \( u \) and with probability one,

\[ Y(u) = \begin{cases} +1, & \text{if } H > u \\
-1, & \text{if } H \leq u \end{cases} \]
Thus, Equation (13) means that \((\theta^*)^\top X\) and \(H\) are tail equivalence, while asymptotic dependence between \(g\) and \(Y\) is equivalence to asymptotic dependence of \((\theta^*)^\top X\) and \(H\) in the classical notion of extreme value theory.

A natural assumption is therefore that \((X, H)\) is jointly regularly varying on \([0, \infty)\) with index \(\alpha > 0\), i.e. there exist an \(\alpha\)-Pareto random variable \(P\) and, independently of \(P\), a random vector \((\Gamma, \Omega) \in [0, \infty)^d \times [0, \infty)\), the so-called spectral tail vector, on the unit sphere \(\{x \in [0, \infty)^{d+1} : \|x\|_\infty = 1\}\) such that

\[
\left(\left(\frac{\|X, H\|_\infty}{u}, \frac{\|X, H\|_\infty}{\|X, H\|_\infty}\right) \bigg| \|X, H\|_\infty > u\right) \overset{d}{\to} (P, (\Gamma, \Omega)), \text{ as } u \to \infty.
\]

Here, we additionally assume that \(P(\|\Gamma\|_\infty > 0) > 0\) and \(P(\Omega > 0) > 0\). In this setup, \(\|X\|_\infty\) and \(H\) are tail equivalent in the sense that

\[
\lim_{u \to \infty} \frac{P(\|X\|_\infty > u)}{P(H > u)} = \lim_{u \to \infty} \frac{P(\|X\|_\infty > u \mid \|X, H\|_\infty > u)}{P(\|X, H\|_\infty > u)} = \frac{P(P \cdot \|\Gamma\|_\infty > 1)}{P(P \cdot \Omega > 1)} = \frac{\mathbb{E}(\|\Gamma\|_\infty^\alpha)}{\mathbb{E}(\Omega^\alpha)} \in (0, \infty),
\]

This is the minimal requirement on the link between the covariates \(X\) and the unobserved extremes of \(H\) essentially saying that at least one component of \(X\) is tail equivalent to \(H\). It is important to highlight that we do not exclude the case that \(\Gamma_i = 0\) a.s. for some \(i \in \{1, \ldots, d\}\) which means that \(X_i\) possesses a lighter tail than \(H\). This property can be read off from the quantity

\[
c_i = \lim_{u \to \infty} \frac{P(X_i > u)}{P(H > u)} = \frac{\mathbb{E}(\Gamma_i^\alpha)}{\mathbb{E}(\Omega^\alpha)} \in [0, \infty).
\]

Thus, \(\Gamma_i = 0\) a.s. if and only if \(c_i = 0\). By including this case, we therefore admit that most of the components of \(X\) may not contribute to the extremes of the vector \(H\).

Under these conditions, we obtain that, for all \(\theta \in [0, \infty)^d\), the classifier \(g_\theta\) is an extremal classifier as

\[
R(g_\theta) = \lim_{u \to \infty} \frac{\mathbb{P}[\max\{\theta^\top X, H\} > u] - \mathbb{P}[\min\{\theta^\top X, H\} > u]}{\mathbb{P}(H > u \text{ or } \theta^\top X > u)}
= \frac{\mathbb{E}(\max\{\theta^\top \Gamma, \Omega\}^\alpha) - \mathbb{E}(\min\{\theta^\top \Gamma, \Omega\}^\alpha)}{\mathbb{E}(\max\{\theta^\top \Gamma, \Omega\}^\alpha)} = 1 - \frac{\mathbb{E}(\min\{\theta^\top \Gamma, \Omega\}^\alpha)}{\mathbb{E}(\max\{\theta^\top \Gamma, \Omega\}^\alpha)} \in [0, 1]. \tag{14}
\]

Equation (14) implies that the function \(\theta \mapsto R(g_\theta)\) is well-defined and continuous on \([0, \infty)^d\). Its value does not depend on those components \(\theta_i\) for which \(\Gamma_i = 0\) a.s., which is equivalent to \(c_i = 0\) as discussed above. Thus, in the following, we will consider this function only on the parameter set

\[
C = \{\theta \in [0, \infty)^d : \theta_i = 0 \text{ for all } i \text{ s.t. } c_i = 0\},
\]

containing all the relevant information – here, note that, in practice, identifying the components \(i \in \{1, \ldots, d\}\) such that \(c_i = 0\) a.s., from a given data set is a necessary step for the correct specification of the set \(C\).
From the consideration in the Introduction, it can be easily seen that $R(g_0) = 1$; the case $\theta = 0$ corresponding to the trivial always optimistic classifier. Furthermore, denoting the set of indices $j$ with $c_j > 0$ by $J$, we can see that

$$R(g_\theta) \geq 1 - \frac{\mathbb{E}(\Omega^\alpha)}{\|\theta\|_\infty \cdot \min_{j \in J} \mathbb{E}(\Gamma_j^\alpha)} \to 1$$

(15)

as $\|\theta\|_\infty \to \infty$. By the continuity of $\theta \mapsto R(g_\theta)$, we obtain that the function attains a global minimum on the domain $C$.

**Proposition 10** Additionally to the assumptions above on the joint distribution of $(X, H)$ with $\alpha > 1$, assume that there exists a function $a(u)$ with $a(u) \to 0$ as $u \to \infty$ such that

$$\mathbb{P}(u^{-1}X \in A \mid \|X, H\| > u) \leq (1 + a(u)) \mathbb{P}(PT \in A)$$

(16)

for all $A \subset [0, \infty)$. Furthermore, let $u_n \to \infty$ and $n \mathbb{P}(H > u_n) \to \infty$ such that, for every compact subset $K \subset C$,

$$\sup_{\theta \in K} \sqrt{n} \mathbb{P}(H > u_n) \left| \frac{\mathbb{P}(g_{\theta}^{(u_n)}(X) \neq Y^{(u_n)}) \mathbb{P}(H > u_n) - \mathbb{E}\left(\max\{\theta^\top \Gamma, \Omega\}^\alpha\right) - \mathbb{E}\left(\min\{\theta^\top \Gamma, \Omega\}^\alpha\right)}{\mathbb{E}(\Omega^\alpha)} \right| = 0$$

(17)

and

$$\sup_{\theta \in K} \sqrt{n} \mathbb{P}(H > u_n) \left| \frac{\mathbb{P}(\max\{g_{\theta}^{(u_n)}(X), Y^{(u_n)}\} = 1) - \mathbb{E}\left(\max\{\theta^\top \Gamma, \Omega\}^\alpha\right)}{\mathbb{E}(\Omega^\alpha)} \right| = 0.$$  

(18)

If the function $\theta \mapsto R(g_\theta)$ has a unique minimizer $\theta^*$ in $C$, then the estimator

$$\hat{\theta}_{n,u_n} = \arg\min_{\theta \in C} \hat{R}_{n,u_n}^{(u_n,0)}(g_\theta).$$

is consistent, i.e. $\hat{\theta}_{n,u_n} \to_p \theta^*$.

Given the set $C$, this result provides a strategy to find the optimal $\theta$, i.e., the best linear classifier. Determining the set $C$ requires the identification of the relevant features, i.e., the index set $J$ such that $c_j > 0$ if and only if $j \in J$. This is discussed in more detail in the following subsection.

**B.2 Feature Selection**

The notion of sparsity quickly comes into play when doing classification. This is all the more true when one is only interested in the extremes. Among the whole data set, only a small proportion will truly contribute to the extremal behavior of the variable of interest. Here, we develop a method to identify the informative signals in terms of extremes among a large data set, assuming that the covariates and the observations $Y^{(u)}$ are tail comparable.

For a comprehensive review of existing methods on sparsity and multivariate extremes we highly recommend the work of [Engelke and Ivanovs (2021)](https://www.statlab.fraunhofer.de/). As we have seen above, for linear classifiers, all the relevant features $X_i$ necessarily satisfy $c_i > 0$. Thus, feature selection can be based on estimation of the $c_i$ which can be done according to the following proposition.
Proposition 11 Assume that
\[ c_i = \lim_{u \to \infty} \frac{\mathbb{P}(X_i > u)}{\mathbb{P}(Y^{(u)} = +1)} \]
exists. If \( u_n \to \infty \) and \( n \mathbb{P}(Y^{(u)} = +1) \to \infty \), then
\[ \frac{\sum_{j=1}^{n} \mathbb{1}\{X_{j,i} > u_n\}}{\sum_{j=1}^{n} \mathbb{1}\{Y_{j}^{(u_n)} = +1\}} \to p_i c_i. \]
If, additionally,
\[ \sqrt{n} \mathbb{P}(Y^{(u_n)} = +1) \left( \frac{\mathbb{P}(X_i > u_n)}{\mathbb{P}(Y^{(u_n)} = +1)} - c_i \right) \to 0 \]
and
\[ \chi_i^* = \lim_{u \to \infty} \mathbb{P}(X_i > u \mid Y^{(u)} = +1) \in [0,1] \]
exists, then, we have
\[ \sqrt{n} \mathbb{P}(H > u_n) \left( \frac{\sum_{j=1}^{n} \mathbb{1}\{X_{j,i} > u_n\}}{\sum_{j=1}^{n} \mathbb{1}\{Y_{j}^{(u_n)} = +1\}} - c_i \right) \to_{d_{n \to \infty}} \mathcal{N}(0, c_i \cdot [1 - 2\chi_i^* + c_i]). \]

C River network

This section deals with a simpler case than the application in Section 4. Here an application could be the following: we want to know which stations should continue to be maintained to prevent extreme floods and maybe some stations are not necessary.

Here we still want to predict the extreme events at station 1. We define an extreme event as an event exceeding the 0.85 quantile of \( X_1 \) and we assume that the whole set of remaining stations is available. In this case, strong dependencies among station 1 and other stations can be observed. So, the main issue is to select these stations following the procedure presented in Appendix B.

As detailed in Appendix B, we identify the stations that may not contribute to the extremes of \( X_1 \) by estimating \( (c_i)_{i \in \{2,...,31\}} \). The estimation of the \( c_i \) on all the data is presented in Table 7. We found that among the 30 stations, only three stations are relevant: stations 2, 13 and 30. Looking at Figure 4, these stations correspond to the stations closest to \( X_1 \). Figure 6 shows the scatter plots between these stations and station 1.

Once the contributing variables have been identified, we compare the performance of several classifiers, on the one hand keeping all the data and on the other hand keeping only the informative stations. Since there is a strong dependence between the data, we assume that it is sufficient here to look at the risk \( R^{(u,0)} \). Comparison results are shown in Figure 7.

By definition of the linear classifier, the estimation is already done by keeping only the informative variables, which is why the estimates are identical for this specific classifier. As for the other classifiers, we see some improvements when keeping only the informative variables: the risk estimates are slightly smaller. This means that even if we remove a lot of information by going from 30 explanatory variables to 3, these 3 remaining stations contain all the information in terms of extremes of station 1.
Table 7: Empirical estimates of $\hat{c}_i$ (as defined in proposition 11) for each station. The values different from zero are highlighted in red.

| id | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ |
|----|-------|-------|-------|-------|-------|-------|-------|-------|-----------|-----------|
| $\hat{c}_i$ | 0.06  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00  | 0.00      | 0.00      |

| id | $X_{12}$ | $X_{13}$ | $X_{14}$ | $X_{15}$ | $X_{16}$ | $X_{17}$ | $X_{18}$ | $X_{19}$ | $X_{20}$ | $X_{21}$ |
|----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $\hat{c}_i$ | 0.00     | 0.30     | 0.00     | 0.00     | 0.00     | 0.00     | 0.00     | 0.00     | 0.00     | 0.00     |

| id | $X_{22}$ | $X_{23}$ | $X_{24}$ | $X_{25}$ | $X_{26}$ | $X_{27}$ | $X_{28}$ | $X_{29}$ | $X_{30}$ | $X_{31}$ |
|----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $\hat{c}_i$ | 0.00     | 0.00     | 0.00     | 0.00     | 0.00     | 0.00     | 0.00     | 0.00     | 0.30     | 0.00     |

Figure 6: Correlation plot between $X_1$ and the stations considered to contribute to the extremes of $X_1$ according to the table 7 (i.e. for which $c_i \neq 0$). The blue dotted lines represent the threshold $u$ defined by the 85th percentile of $X_1$.

Figure 7: Estimation of $R^{(u,0)}(g)$ for different classifiers $g$ (cross-validation - 70% train, 30% test - with 50 repetitions). The red distributions come from the estimation with all the stations, and the green distributions represent the estimations with only the variables having $\hat{c}_i \neq 0$. 
D Proofs

D.1 Proof of Lemma 3:

Using that \( \{g^{(u)}(X) = +1\} \subseteq \{g^{(v)}(X) = +1\} \) and \( \{Y^{(u)} = +1\} \subseteq \{Y^{(v)} = +1\} \) for \( u \geq v \), we can write

\[
\mathbb{P}(g^{(u)}(X) = +1 \text{ or } Y^{(u)} = +1 \mid g^{(v)}(X) = +1, Y^{(v)} = +1) = \frac{\mathbb{P}(g^{(u)}(X) = +1, Y^{(v)} = +1)}{\mathbb{P}(g^{(v)}(X) = +1, Y^{(v)} = +1)},
\]

\[
= \frac{\mathbb{P}(g^{(u)}(X) = +1, Y^{(v)} = +1) + \mathbb{P}(g^{(v)}(X) = +1, Y^{(u)} = +1) - \mathbb{P}(g^{(u)}(X) = +1, Y^{(u)} = +1)}{\mathbb{P}(g^{(v)}(X) = +1, Y^{(v)} = +1)}.
\]

In the same way, we have

\[
\mathbb{P}(g^{(u)}(X) \neq Y^{(u)} \mid g^{(v)}(X) = +1, Y^{(v)} = +1) = \frac{\mathbb{P}(g^{(u)}(X) = +1, Y^{(v)} = +1) + \mathbb{P}(g^{(v)}(X) = +1, Y^{(u)} = +1) - 2 \mathbb{P}(g^{(u)}(X) = +1, Y^{(u)} = +1)}{\mathbb{P}(g^{(v)}(X) = +1, Y^{(v)} = +1)}.
\]

Hence, we deduce that

\[
R^{(u,v)}(g) = \frac{\mathbb{P}(g^{(u)}(X) \neq Y^{(u)} \mid g^{(v)}(X) = +1, Y^{(v)} = +1)}{\mathbb{P}(g^{(u)}(X) = +1 \text{ or } Y^{(u)} = +1 \mid g^{(v)}(X) = +1, Y^{(v)} = +1)} - 1 \geq 0.
\]

The expression given by \([4]\) follows.

Item (a) of the lemma follows from the representation \([4]\) together with the inequality

\[
\mathbb{P}(U \mid V) \geq \mathbb{P}(U \mid W), \text{ if the events } U, V \text{ and } W \text{ satisfy } U \subseteq V \subseteq W.
\]

For item (b), note that

\[
R^{(u,v)}(g) \leq R^{(u,v)}(h)
\]

\[
\iff (1 - R^{(u,v)}(g))^{-1} \leq (1 - R^{(u,v)}(h))^{-1}
\]

\[
\iff \frac{\mathbb{P}(g^{(u)}(X) = +1, Y^{(v)} = +1)}{\mathbb{P}(g^{(u)}(X) = +1, Y^{(u)} = +1)} \leq \frac{\mathbb{P}(h^{(u)}(X) = +1, Y^{(v)} = +1)}{\mathbb{P}(h^{(u)}(X) = +1, Y^{(u)} = +1)}
\]

\[
\iff \mathbb{P}(g^{(u)}(X) = +1, Y^{(v)} = +1) \leq \mathbb{P}(h^{(u)}(X) = +1, Y^{(v)} = +1) + \mathbb{P}(h^{(u)}(X) = +1, Y^{(u)} = +1)
\]

\[
\iff \mathbb{P}(g^{(u)}(X) = +1, Y^{(v)} = +1) \leq \mathbb{P}(h^{(u)}(X) = +1, Y^{(v)} = +1) \leq \mathbb{P}(h^{(u)}(X) = +1, Y^{(v)} = +1) + \mathbb{P}(h^{(v)}(X) = +1, Y^{(u)} = +1)
\]

\[
29
\]
As, by assumption, \( P(g^{(u)}(X) = +1, Y^{(u)} = +1) = P(h^{(u)}(X) = +1, Y^{(u)} = +1) \) and \( P(g^{(v)}(X) = +1, Y^{(v)} = +1) = P(h^{(v)}(X) = +1, Y^{(v)} = +1) \), but also the inequality \( P(g^{(v)}(X) = +1, Y^{(u)} = +1) \leq P(h^{(v)}(X) = +1, Y^{(u)} = +1) \) holds, we obtain

\[
R^{(u,v)}(g) \leq R^{(u,v)}(h),
\]
i.e. the second statement (b).

Item (c) is a direct consequence of [4].

D.2 Proof of Lemma 4:

First, we note that, by definition,

\[
\chi^*(g) = c(g) \cdot \lim_{n \to \infty} P(Y^{(u)} = +1 \mid g^{(u)}(X) = +1) \leq c(g).
\]

By taking the limits separately for every single term in (7), we directly obtain (8).

From that, we can immediately see that

\[
R(g) = 0 \text{ if and only if } 1 + c(g) = 2\chi^*(g).
\]

Now, assume that \( \chi^*(g) < c(g) \). Then, \( R(g) = 0 \) would imply that \( 1 + \chi^*(g) < 2\chi^*(g) \) which is equivalent to \( \chi^*(g) > 1 \) in contradiction to the definition of \( \chi^*(g) \). Thus, \( R(g) = 0 \) implies \( \chi^*(g) = c(g) \). Plugging this into (8), we obtain \( c(g) = \chi^*(g) = 1 \). Conversely, \( c(g) = \chi^*(g) = 1 \) obviously leads to \( R(g) = 0 \). □

D.3 Proof of Lemma 5:

Let \((u_n)_{n \in \mathbb{N}}\) be a sequence of positive thresholds with \( u_n \to \infty \) as \( u \to \infty \). From the discussion at the beginning of Section 2.2, we know that we can only get \( R(g) < 1 \), only if the sequence \( P(g^{(u_n)}(X) = +1)/P(Y^{(u_n)} = +1) \) is bounded away from 0 and \( \infty \), i.e. there exist constants \( 0 < C_1 < C_2 < \infty \) such that

\[
C_1 \leq \frac{P(g^{(u_n)}(X) = +1)}{P(Y^{(u_n)} = +1)} \leq C_2 \quad \text{for all } n \in \mathbb{N}.
\] (19)

Thus, this will be assumed in the following. Let \( w \) such that \( \delta := \lim_{u \to \infty} w(u)/u \in (0,1) \), then we can write that

\[
P(Y^{(u_n)} = +1 \mid g^{(u_n)}(X) = +1) = \frac{P(f(X) + N > u_n, g^{(u_n)}(X) = +1)}{P(g^{(u_n)}(X) = +1)} \leq \frac{P(f(X) > w(u_n), g^{(u_n)}(X) = +1)}{P(g^{(u_n)}(X) = +1)} + \frac{P(N > u_n - w(u_n), g^{(u_n)}(X) = +1)}{P(g^{(u_n)}(X) = +1)}.
\]

Since \( g^{(u_n)}(X) \) and \( N \) are independent, the second term reduces to \( P(N > u_n - w(u_n)) \) which converges to 0 as \( n \to \infty \) since \( u - w(u) \sim (1 - \delta)u \) as \( u \) gets large.

For the first term, we rewrite the ratio as follows

\[
\frac{P(f(X) > w(u_n), g^{(u_n)}(X) = +1)}{P(g^{(u_n)}(X) = +1)} \leq \frac{P(f(X) > w(u_n))}{P(g^{(u_n)}(X) = +1)} = \frac{P(f(X) > w(u_n))P(N > w(u_n))}{P(N > w(u_n))P(g^{(u_n)}(X) = +1)}.
\]
Since \( w(u_n) \to \infty \) and \( \mathbb{P}(f(X) > u) = o(\mathbb{P}(N > u)) \), the ratio \( \frac{\mathbb{P}(f(X)>w(u_n))}{\mathbb{P}(N>w(u_n))} \) goes to 0 as \( n \to \infty \). From the assumption \( w(u) \sim \delta u \) and regular variation of \( N \), \( \frac{\mathbb{P}(N>w(u_n))}{\mathbb{P}(f(Y)>w(u_n))} \) behaves as a constant when \( n \to \infty \). The only remaining term is \( \frac{\mathbb{P}(g(u_n)(X) = +1)}{\mathbb{P}(N > u_n)} \) which is bounded due to Equation (19). So,

\[
\lim_{n \to \infty} \mathbb{P}(g(u_n)(X) = +1 | Y^{(u_n)} = +1) = 0.
\]

By Equations (7) and (19), this implies

\[
\lim_{n \to \infty} R(u_n)(g) = 1.
\]

As this holds true for any sequence \((u_n)_{n \in \mathbb{N}}\), we obtain \( R(g) = 1 \).

D.4 Proof of Proposition 6:

The Ramos and Ledford model corresponds to the special case of (5): For all \( 0 \leq v_1, v_2 \leq v \), we have

\[
\mathbb{P}[g(v_1)(X) = +1, Y^{(v_2)} = +1] = L(v_1, v_2)(v_1^{-\alpha_0}v_2^{-\alpha_Y})^{1/2}\eta
= c_{v_1, v_2}(\mathbb{P}(g(v_1)(X) = +1))^{a}(\mathbb{P}(Y^{(v_2)} = +1))^{b}
\]

with

\[
\mathbb{P}(g(v_1)(X) = +1) = L_g(v_1)(v_1/u)^{-\alpha_0}u^{-\alpha_0},
\]

\[
\mathbb{P}(Y^{(v_2)} = +1) = L_Y(v_2)(v_2/u)^{-\alpha_Y}u^{-\alpha_Y}
\]

for slowly varying functions \( L_g \) and \( L_Y \) and

\[
c_{v_1, v_2} = \frac{L(v_1, v_2)}{(L_g(v_1)L_Y(v_2))^{1/2}\eta}, \quad a = b = \frac{1}{2\eta}.
\]

Then, from Equation (6),

\[
R(u,v)(g) = 1 - \left[ \frac{c_{u,v}}{c_{u,u}}(\mathbb{P}(g(u)(X) = +1 | g(v)(X) = +1))^{-a} + \frac{c_{u,v}}{c_{u,u}}(\mathbb{P}(Y^{(u)} = +1 | Y^{(v)} = +1))^{-b} - 1 \right]^{-1}.
\]

Letting \( u \) get large and noting that, by the continuity of \( \ell^* \),

\[
\lim_{u \to \infty} \frac{c_{v,u}}{c_{u,u}} = \ell^*(\varepsilon, 1) \quad \text{and} \quad \lim_{u \to \infty} \frac{c_{u,v}}{c_{u,u}} = \ell^*(1, \varepsilon)
\]

provide the required result.
D.5 Proof of Propositions 7 and 9:

First of all, we note that Proposition 7 is a special case of Proposition 9 with \( g_1 = g_2 = g \) which implies \( c_s^{(1)} = c_s^{(2)} = 1 \). Thus, it suffices to show Proposition 9. To this end, we define the constants

\[ r_\epsilon := \lim_{v, u \to \epsilon} \frac{\mathbb{P}(g_1^{(u)}(X) \neq Y^{(u)}, g_2^{(u)}(X) \neq Y^{(v)}, g_1^{(v)}(X) = g_2^{(v)}(X) = Y^{(v)} = 1)}{(p_{g_1} \land p_{g_2})(u_n, v_n)}, \]

\[ q_\epsilon^{(12)} := \lim_{v, u \to \epsilon} \frac{\mathbb{P}(g_1^{(u)}(X) \neq Y^{(u)}, \max\{g_2^{(u)}(X), Y^{(u)}\} = 1, g_1^{(v)}(X) = g_2^{(v)}(X) = Y^{(v)} = 1)}{(p_{g_1} \land p_{g_2})(u_n, v_n)}, \]

\[ q_\epsilon^{(21)} := \lim_{v, u \to \epsilon} \frac{\mathbb{P}(\max\{g_1^{(u)}(X), Y^{(u)}\} = 1, \max\{g_2^{(u)}(X), Y^{(u)}\} = 1, g_1^{(v)}(X) = g_2^{(v)}(X) = Y^{(v)} = 1)}{(p_{g_1} \land p_{g_2})(u_n, v_n)}, \]

\[ p_\epsilon := \lim_{v, u \to \epsilon} \frac{\mathbb{P}(\max\{g_1^{(u)}(X), Y^{(u)}\} = 1, \max\{g_2^{(u)}(X), Y^{(u)}\} = 1, g_1^{(v)}(X) = g_2^{(v)}(X) = Y^{(v)} = 1)}{(p_{g_1} \land p_{g_2})(u_n, v_n)}, \]

all of which exist in \([0, 1]\) by the assumptions of the proposition.

For \( n \in \mathbb{N} \) and \( i \in \{1, \ldots, n\} \), let

\[
Z_{i,n} = \begin{pmatrix}
\frac{\sqrt{c_s^{(1)}}}{\sqrt{n p_{g_1}(u_n, v_n)}} \left[ \mathbb{1}\{g_1^{(u_n)}(X_i) \neq Y_i^{(u_n)}\}, g_1^{(v_n)}(X_i) = Y_i^{(v_n)} = 1 \right] \\
- \mathbb{P}(g_1^{(u_n)}(X) \neq Y^{(u_n)}, g_1^{(v)}(X) = Y^{(v)} = 1) \\
\frac{\sqrt{c_s^{(1)}}}{\sqrt{n p_{g_1}(u_n, v_n)}} \left[ \mathbb{1}\{\max\{g_1^{(u_n)}(X_i), Y_i^{(u_n)}\} = 1, g_1^{(v_n)}(X_i) = Y_i^{(v_n)} = 1 \right] \\
- \mathbb{P}(\max\{g_1^{(u_n)}(X), Y^{(u)}\} = 1, g_1^{(v)}(X) = Y^{(v)} = 1) \\
\frac{\sqrt{c_s^{(2)}}}{\sqrt{n p_{g_2}(u_n, v_n)}} \left[ \mathbb{1}\{g_2^{(u_n)}(X_i) \neq Y_i^{(u_n)}\}, g_2^{(v_n)}(X_i) = Y_i^{(v_n)} = 1 \right] \\
- \mathbb{P}(g_2^{(u_n)}(X) \neq Y^{(u_n)}, g_2^{(v)}(X) = Y^{(v)} = 1) \\
\frac{\sqrt{c_s^{(2)}}}{\sqrt{n p_{g_2}(u_n, v_n)}} \left[ \mathbb{1}\{\max\{g_2^{(u_n)}(X_i), Y_i^{(u_n)}\} = 1, g_2^{(v_n)}(X_i) = Y_i^{(v_n)} = 1 \right] \\
- \mathbb{P}(\max\{g_2^{(u_n)}(X), Y^{(u)}\} = 1, g_2^{(v)}(X) = Y^{(v)} = 1)
\end{pmatrix}.
\]

First, we will show that

\[
\sum_{i=1}^{n} Z_{i,n} \xrightarrow{d_{n \to \infty}} N(0, \Sigma),
\]

where

\[
\Sigma = \begin{pmatrix}
c_s^{(1)} R_\epsilon(g_1) & c_s^{(1)} R_\epsilon(g_1) & \sqrt{c_s^{(1)} c_s^{(2)}} R_\epsilon & \sqrt{c_s^{(1)} c_s^{(2)}} p_\epsilon \\
\sqrt{c_s^{(1)} c_s^{(2)}} R_\epsilon & c_s^{(1)} R_\epsilon(g_1) & \sqrt{c_s^{(1)} c_s^{(2)}} p_\epsilon & \sqrt{c_s^{(1)} c_s^{(2)}} q_\epsilon^{(2)} \\
\sqrt{c_s^{(1)} c_s^{(2)}} p_\epsilon & \sqrt{c_s^{(1)} c_s^{(2)}} p_\epsilon & c_s^{(2)} R_\epsilon(g_2) & c_s^{(2)} R_\epsilon(g_2) \\
\sqrt{c_s^{(1)} c_s^{(2)}} q_\epsilon & \sqrt{c_s^{(1)} c_s^{(2)}} q_\epsilon^{(2)} & c_s^{(2)} R_\epsilon(g_2) & c_s^{(2)} R_\epsilon(g_2)
\end{pmatrix}.
\]
which, by the Cramér-Wold device is equivalent to
\[ \sum_{i=1}^{n} a^\top Z_{i,n} \xrightarrow{\text{d}}_{n\to\infty} \mathcal{N}(0, a^\top \Sigma a) \quad \forall a \in \mathbb{R}^4, \]

To this end, we note that, by a straightforward calculation
\[ \lim_{n\to\infty} \sum_{i=1}^{n} \text{Var}(a^\top Z_{i,n}) = a^\top \Sigma a. \]
Furthermore, we have that
\[ |a^\top Z_{i,n}| \leq \|a\|_1 \|Z_{n,i}\|_\infty \leq \|a\|_1 \cdot \left( \frac{\sqrt{C_1(1)}}{\sqrt{n p_{g_1}(u_n, v_n)}} + \frac{\sqrt{C_2(2)}}{\sqrt{n p_{g_2}(u_n, v_n)}} \right) \xrightarrow{n\to\infty} 0. \]
Consequently, for each \( \eta > 0 \), there exists \( n_0 > 0 \) such that \( |a^\top Z_{i,n}| < \eta \) for all \( n > n_0 \) and, therefore,
\[ \sum_{i=1}^{n} \mathbb{E}[(a^\top Z_{i,n})^2 1\{a^\top Z_{i,n} > \eta\}] = 0, \]
i.e. the triangular array \((a^\top Z_{i,n})_{i=1,\ldots,n,n\in\mathbb{N}}\) satisfies the Lindeberg condition. Thus, by Lindeberg’s Central Limit Theorem
\[ \sum_{i=1}^{n} a^\top Z_{i,n} \xrightarrow{n\to\infty} \mathcal{N}(0, a^\top \Sigma a) \quad \forall a \in \mathbb{R}^4. \]
This implies that
\[ \sqrt{n(p_{g_1} \wedge p_{g_2})(u_n, v_n)} \left( \begin{array}{c}
\frac{\sum_{i=1}^{n} 1\{g_1^{(u_n)}(X_i) \neq Y^{(u_n)}(v_n) = g_1^{(v_n)}(X_i) = 1\} - \mathbb{P}(g_1^{(u_n)}(X) \neq Y^{(u_n)}(v_n) = g_1^{(v_n)}(X) = 1) }{p_{g_1}(u_n, v_n)} - 1 \\
\frac{\sum_{i=1}^{n} 1\{\max\{g_1^{(u_n)}(X_i), Y^{(u_n)}(v_n) = g_1^{(v_n)}(X_i) = 1\} > 1\} - \mathbb{P}(g_1^{(u_n)}(X) \neq Y^{(u_n)}(v_n) = g_1^{(v_n)}(X) = 1) }{p_{g_1}(u_n, v_n)} - 1 \\
\frac{\sum_{i=1}^{n} 1\{g_2^{(u_n)}(X_i) \neq Y^{(u_n)}(v_n) = g_2^{(v_n)}(X_i) = 1\} - \mathbb{P}(g_2^{(u_n)}(X) \neq Y^{(u_n)}(v_n) = g_2^{(v_n)}(X) = 1) }{p_{g_2}(u_n, v_n)} - 1 \\
\frac{\sum_{i=1}^{n} 1\{\max\{g_2^{(u_n)}(X_i), Y^{(u_n)}(v_n) = g_2^{(v_n)}(X_i) = 1\} > 1\} - \mathbb{P}(g_2^{(u_n)}(X) \neq Y^{(u_n)}(v_n) = g_2^{(v_n)}(X) = 1) }{p_{g_2}(u_n, v_n)} - 1 
\end{array} \right) \]
\[ = (1 + o(1)) \sum_{i=1}^{n} Z_{i,n} \xrightarrow{n\to\infty} \mathcal{N}(0, \Sigma), \]
Applying the multivariate Delta method to the function
\[ (f_1, g_1, f_2, g_2) \mapsto \left( \frac{f_1}{g_1}, \frac{f_2}{g_2} \right) \]
yields the result. \(\square\)
E Proofs of the appendix

E.1 Proof of Proposition 10

The proof is based on the following lemma which is proven in Subsection E.3

Lemma 12 Under the assumptions from Proposition 10, for every compact subset $K \subset C$, the sequences of processes $\{A_n(\theta), \theta \in K\}$ and $\{B_n(\theta), \theta \in K\}$ defined by

$$
A_n(\theta) = \sqrt{\frac{n}{\mathbb{P}(H > u_n)}} \left( \frac{1}{n} \sum_{i=1}^{n} 1 \{\{\theta^\top X_i > u_n\} \cap \{H_i > u_n\}\} - \mathbb{P}(g_{\theta}^{(u_n)}(X) \neq Y^{(u_n)}) \right)
$$

$$
B_n(\theta) = \sqrt{\frac{n}{\mathbb{P}(H > u_n)}} \left( \frac{1}{n} \sum_{i=1}^{n} 1 \{\{\theta^\top X_i > u_n\} \cup \{H_i > u_n\}\} - \mathbb{P}(\max\{g_{\theta}^{(u_n)}(X), Y^{(u_n)}\} = 1) \right)
$$

jointly converge to a centered bivariate Gaussian process $\{(A(\theta), B(\theta)), \theta \in K\}$ weakly in $\ell^2(K)$.

If the function $\theta \mapsto R(g_\theta)$ has a unique minimizer $\theta^*$, then, necessarily, $R(g_{\theta^*}) < 1$. Now, similarly to the notation above, let $J$ denote the set of indices $j$ with $c_j > 0$, and let us consider $\theta \in C$ such that $\|\theta\|_\infty > k_0$ for some constant $k_0 > 0$. Then,

$$
\tilde{R}_n^{(u_n,0)}(g_\theta) = 1 - \frac{\sum_{i=1}^{n} 1 \{\min(\theta^\top X_i, H_i) > u_n\}}{\sum_{i=1}^{n} 1 \{\max(\theta^\top X_i, H_i) > u_n\}} \geq 1 - \frac{\sum_{i=1}^{n} 1 \{H_i > u_n\}}{\min_{j \in J} \sum_{i=1}^{n} 1 \{k_0 X_{ij} > u_n\}}
$$

$$
n \to \infty, \ 1 - \max_{j \in J} \frac{\mathbb{E}(\Omega^\alpha)}{k_0 \mathbb{E}(\Gamma_j^\alpha)}
$$

where the right-hand side goes to 1 as $k_0 \to \infty$. Thus, as $\tilde{R}_n^{(u_n,0)}(g_{\theta^*}) \to_p R(g_{\theta^*}) < 1$, we obtain that, for sufficiently large $k_0 \gg \|\theta^*\|$, with probability going to one,

$$
\tilde{R}_n^{(u_n,0)}(g_{\theta^*}) \leq \min_{\theta \in C \setminus \{0, k_0\}^d} \tilde{R}_n^{(u_n,0)}(g_\theta)
$$

and, consequently,

$$
\arg\min_{\theta \in C} \tilde{R}_n^{(u_n,0)}(g_\theta) = \arg\min_{\theta \in C \cap [0,k_0]^d} \tilde{R}_n^{(u_n,0)}(g_\theta).
$$

Now, we note that, by Lemma 12 the bias conditions (17) and (18) and the functional delta method, $\tilde{R}_n^{(u_n,0)}(g_\theta)$ converges in probability to $R(g_\theta)$ uniformly on every compact subset of $C$. In particular,

$$
\sup_{\theta \in C \cap [0,K]^d} \left| \tilde{R}_n^{(u_n,0)}(g_\theta) - R(g_\theta) \right| \to_p 0.
$$

Thus,

$$
\arg\min_{\theta \in C \cap [0,K]^d} \tilde{R}_n^{(u_n,0)}(g_\theta) \to_p \arg\min_{\theta \in C \cap [0,K]^d} R(g_\theta) = \theta^*.
$$

\[\square\]
E.2 Proof of Proposition 11

The proof runs analogously to the proof of Proposition 9: For \( n \in \mathbb{N} \) and \( j \in \{1, \ldots, n\} \), we define

\[
Z_{j,n} = \begin{pmatrix}
\frac{1}{\sqrt{n \mathbb{P}(Y(u_n) = +1)}} \left(1 \{X_{j,i} > u_n\} - \mathbb{P}(X_i = +1)\right) \\
\frac{1}{\sqrt{n \mathbb{P}(Y(u_n) = +1)}} \left(1 \{Y_j(u_n) = +1\} - \mathbb{P}(Y(u_n) = +1)\right)
\end{pmatrix}
\]

in order to first show that

\[
\sum_{i=1}^{n} Z_{i,n} \xrightarrow{n \to \infty} d \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_i \chi_i^* & 1 \\ 1 & c_i \chi_i^* \end{pmatrix} \right)
\]

which, by the Cramér-Wold device is equivalent to

\[
\sum_{i=1}^{n} (a_1 Z_{i,n,1} + a_2 Z_{i,n,2}) \xrightarrow{n \to \infty} d \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, (a_1, a_2) \begin{pmatrix} c_i \chi_i^* & 1 \\ 1 & c_i \chi_i^* \end{pmatrix} (a_1, a_2) \right) \quad \forall a_1, a_2 \in \mathbb{R}, \quad (20)
\]

A straightforward calculation leads to

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \text{Var}(a_1 Z_{i,n,1} + a_2 Z_{i,n,2}) = (a_1, a_2) \begin{pmatrix} c_i \chi_i^* & 1 \\ 1 & c_i \chi_i^* \end{pmatrix} (a_1, a_2).
\]

By the inequality, we can see that

\[
|a_1 Z_{i,n,1} + a_2 Z_{i,n,2}| \leq \frac{|a_1| + |a_2|}{\sqrt{n \mathbb{P}(Y(u_n) = +1)}} \xrightarrow{n \to \infty} 0
\]

for each \( \eta > 0 \), eventually \(|\mathbf{a}^\top Z_{i,n}| < \eta\) with probability one. Hence, we obtain

\[
\sum_{i=1}^{n} \mathbb{E}[|a_1 Z_{i,n,1} + a_2 Z_{i,n,2}|^2 1\{|a_1 Z_{i,n,1} + a_2 Z_{i,n,2}| > \eta\}] = 0,
\]

\textit{i.e.} the triangular array \((\mathbf{a}^\top Z_{i,n})_{i=1, \ldots, n; n \in \mathbb{N}}\) satisfies the Lindeberg condition. Lindeberg’s Central Limit Theorem then yields Equation (20). The desired result follows by applying the Delta method to the function \((f, g) \mapsto f/g\).

E.3 Proof of Lemma 12

We will proof the lemma by applying the Central Limit Theorem 2.11.9 in \textit{Van der Vaart and Wellner (1996)}. To this end, we define the function spaces \(A = \{a_\theta, \theta \in K\}\) and \(B = \{b_\theta, \theta \in K\}\) where

\[
a_\theta : (0, \infty)^d \times (0, \infty) \to \{0, 1\}, \quad a_\theta(x, h) = 1\{\theta^\top x > 1\} \triangle \{h > 1\}
\]

\[
b_\theta : (0, \infty)^d \times (0, \infty) \to \{0, 1\}, \quad b_\theta(x, h) = 1\{\theta^\top x > 1\} \cup \{h > 1\}.
\]

Then, with

\[
Z_{nl}(f) = \frac{1}{\sqrt{n \mathbb{P}(H > u_n)}} f(u_n^{-1}X_l, u_n^{-1}H_l), \quad f \in A \cup B,
\]

35
for \( l = 1, \ldots, n \), we have that
\[
\{ A_n(\theta), \theta \in K \} = \left\{ \sum_{l=1}^{n} (Z_{nl}(f) - \mathbb{E} Z_{nl}(f)), f \in A \right\}
\]
and
\[
\{ B_n(\theta), \theta \in K \} = \left\{ \sum_{l=1}^{n} (Z_{nl}(f) - \mathbb{E} Z_{nl}(f)), f \in B \right\}.
\]

Now, we have that
\[
\max \left\{ \| Z_{nl} \|_A, \| Z_{nl} \|_B \right\} = \sup_{f \in A \cup B} | Z_{nl}(f) | \leq \frac{1}{\sqrt{n \mathbb{P}(H > u_n)}} \text{ a.s.}
\]
for all \( l = 1, \ldots, n \) and \( n \in \mathbb{N} \).

Consequently, we check the Lindeberg condition: For \( k \in \mathbb{N} \), we have
\[
\lim_{n \to \infty} \sum_{l=1}^{n} \mathbb{E} \left( \| Z_{nl} \|_A^k \mathds{1}\{ \| Z_{nl} \|_A > \eta \} \right) \leq \lim_{n \to \infty} \frac{n}{\sqrt{n \mathbb{P}(H > u_n)}} \mathds{1}\{ n \mathbb{P}(H > u_n) < 1/\eta^2 \} = 0
\]
as \( n \mathbb{P}(H > u_n) \to \infty \) by definition. For \( k = 2 \), we obtain a Lindeberg type condition that ensures convergence of \( A_n \) and \( B_n \) to \( A \) and \( B \), respectively, in terms of finite-dimensional distributions. For \( k = 1 \), we obtain the Lindeberg type condition of Theorem 2.11.9 in [Van der Vaart and Wellner (1996)].

It remains to check the equi-continuity condition. In the following, to simplify notation, we assume that \( C = [0, \infty)^d \). Then, for \( \theta^{(1)}, \theta^{(2)} \in [a, b] \subset K \subset C \), we have that
\[
| a_{\theta^{(1)}}(u_n^{-1} X, u_n^{-1}H) - a_{\theta^{(2)}}(u_n^{-1} X, u_n^{-1}H) | \in \{0, 1\}
\]
and
\[
| b_{\theta^{(1)}}(u_n^{-1} X, u_n^{-1}H) - b_{\theta^{(2)}}(u_n^{-1} X, u_n^{-1}H) | \in \{0, 1\}
\]
and the probability that any of those two expressions is equal to one is bounded by the probability
\[
\mathbb{P}(\mathds{1}\{ a^T X > u_n \} \neq \mathds{1}\{ b^T X > u_n \}) = \mathbb{P}(a^T X \leq u_n, b^T X > u_n)
\]
\[
= \mathbb{P} \left( \| (X, H) \| \geq \frac{u_n}{\| K \| d} \right) \mathbb{P} \left( a^T X \leq u_n, b^T X > u_n \mid \| (X, H) \| \geq \frac{u_n}{\| K \| d} \right)
\]
where we use that \( b^T X > u_n \) implies that \( \| X \| > u_n/(\| K \| d) \) with \( \| K \| = \sup_{x \in K} \| x \|_\infty \).

Making use of the fact that \( \mathbb{P}(\| (X, H) \| > u_n/(\| K \| d)) \leq C_0(\| K \| d)^\alpha \mathbb{P}(H > u_n) \) for some constant \( C_0 > 0 \) and the bound given by Equation (10), we obtain that
\[
\mathbb{P}(\mathds{1}\{ a^T X > u_n \} \neq \mathds{1}\{ b^T X > u_n \})
\]
\[
\leq C_0(\| K \| d)^\alpha \mathbb{P}(H > u_n)[1 + a(u_n/(\| K \| d))] \mathbb{P} \left( P a^T \Gamma \leq \| K \| d, P b^T \Gamma > \| K \| d \right)
\]
\[
= C_0(\| K \| d)^\alpha \mathbb{P}(H > u_n)[1 + a(u_n/(\| K \| d))] \mathbb{E}_\Gamma \left( \mathbb{P} \left( P \in \left[ \| K \| d, \| K \| d \right] \mid b^T \Gamma > \frac{a^T \Gamma}{\| K \| d} \right) \right)
\]
\[
\leq C_0 \mathbb{P}(H > u_n)[1 + a(u_n/(\| K \| d))] \mathbb{E} \left( (b^T \Gamma)^\alpha - (a^T \Gamma)^\alpha \right)
\]
\[
\leq 2C_0 \mathbb{P}(H > u_n) \| a - b \|
provided that \( u_n \) is sufficiently large as \( a(u_n/(\|K\|d)) \to 0 \).

Consequently,

\[
\sup_{\|f-g\|<\delta} \sum_{l=1}^{n} E \left[ (Z_{nl}(f) - Z_{nl}(g))^2 \right] = 2C_0\delta,
\]

which tends to 0 as \( \delta \to 0 \). From this inequality, it can also be seen that any partition of \( K \) into hypercubes with length \( \varepsilon^2/(2C_0) \) leads to a valid \( \varepsilon \)-bracketing, \( i.e. \) the number \( N_\varepsilon \propto 1/\varepsilon^{2d} \) grows with a power rate and, so, \( \sqrt{\log(N_\varepsilon)} \) is integrable.

Thus, by Theorem 2.11.9, the bivariate processes \((A_n, B_n)\) converge to a centered bivariate Gaussian process \((A, B)\) weakly in \( \ell^\infty(K) \). The limiting covariance structure can be obtained from

\[
\begin{align*}
\text{Cov}(A(\theta_1), A(\theta_2)) &= \lim_{n \to \infty} \text{Cov}(A_n(\theta_1), A_n(\theta_2)), \\
\text{Cov}(A(\theta_1), B(\theta_2)) &= \lim_{n \to \infty} \text{Cov}(A_n(\theta_1), B_n(\theta_2)), \\
\text{Cov}(B(\theta_1), B(\theta_2)) &= \lim_{n \to \infty} \text{Cov}(B_n(\theta_1), B_n(\theta_2)),
\end{align*}
\]

for \( \theta_1, \theta_2 \in K \).

\[\square\]