MULTI-STATE CANALYZING FUNCTIONS OVER FINITE FIELDS

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Abstract. In this paper, we extend the definition of Boolean canalyzing functions to the canalizing functions over finite field \( \mathbb{F}_q \), where \( q \) is a power of a prime. We obtain the characterization of all the eight classes of such functions as well as their cardinality. When \( q = 2 \), we obtain a combinatorial identity by equating our result to the formula in [2]. Finally, for a better understanding to the magnitude, we obtain the asymptotes for all the eight cardinalities as either \( n \to \infty \) or \( q \to \infty \).

1. Introduction

In 1993, canalyzing Boolean rules were introduced by S. Kauffman [6] as biologically appropriate rules in Boolean network models of gene regulatory networks. When comparing the class of canalizing functions to other classes of functions with respect to their evolutionary plausibility as emergent control rules in genetic regulatory system, it is informative to know the number of canalizing functions with a given number of input variables [2]. However, the Boolean network modeling paradigm is rather restrictive, with its limit to two possible functional levels, ON and OFF, for genes, proteins, etc. Many discrete models of biological networks therefore allow variables to take on multiple states. Commonly used discrete multi-state model types are the so-called logical models [19], Petri nets [18], and agent-based models [17]. It was shown in [20] and [21] that many of these models can be translated into the rich and general mathematical framework of polynomial dynamical systems over a finite field \( \mathbb{F}_q \). (Software to carry out this translation is available at [http://dvd.vbi.vt.edu/cgi-bin/git/adam.pl](http://dvd.vbi.vt.edu/cgi-bin/git/adam.pl)).

In this paper, we generalize the concept of Boolean canalizing rules to the multi-state case, that is, to functions over any finite fields \( \mathbb{F}_q \), thus generalizing the results in [2]. We provide formulas for the cardinalities of all the eight classes canalizing functions. We also obtain the asymptotes of these cardinalities as either \( n \to \infty \) or \( q \to \infty \).

2. Preliminaries

In this section we introduce the definition of a canalizing function. Let \( \mathbb{F} = \mathbb{F}_q \) be a finite field with \( q \) elements, where \( q \) is a power of a prime. If \( f \) is a \( n \) variable function from \( \mathbb{F}^n \) to \( \mathbb{F} \), it is well known [11] that \( f \) can be expressed as a polynomial, called the algebraic normal form (ANF):

\[
f(x_1, x_2, \ldots, x_n) = \sum_{k_1=0}^{q-1} \sum_{k_2=0}^{q-1} \cdots \sum_{k_n=0}^{q-1} a_{k_1 k_2 \ldots k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}
\]

where each coefficient \( a_{k_1 k_2 \ldots k_n} \in \mathbb{F} \) is a constant. The number \( k_1 + k_2 + \cdots + k_n \) is the multivariate degree of the term \( a_{k_1 k_2 \ldots k_n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \) with nonzero coefficient \( a_{k_1 k_2 \ldots k_n} \). The greatest degree of all the terms of \( f \) is called the algebraic degree, denoted by \( \deg(f) \). The greatest degree of each individual variable \( x_i \) will be denoted by \( \deg(f)_i \). Let \( [n] = \{1, 2, \ldots, n\} \).

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It is shown in [20] that it is no restriction of generality to consider models in which the set of states of the model variables have the algebraic structure of a finite field. The above fact that any function \( F^n \rightarrow F \) can be represented as a polynomial makes the results of this paper valid in the most general setting of models that are given as dynamical systems generated by iteration of set functions.

We now define a notion of canalyzing function in multi-state setting, which is a straightforward generalization of the Boolean case.

**Definition 2.1.** A function \( f(x_1, x_2, \ldots, x_n) \) is canalyzing in the \( i \)-th variable with canalyzing input value \( a \in F \) and canalyzed output value \( b \in F \) if \( f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = b \), for any \( (x_1, x_2, \ldots, x_n) \).

In other words, a function is canalyzing is there exists one variable \( x_i \) such that, if \( x_i \) receives certain inputs, this by itself determines the value of the function. For the purpose of the proofs below we will need to use families of canalyzing functions for which part of the specification is fixed, such as the variable \( x_i \) or \( a \) or \( b \) or some combination. For ease of notation, we will refer to a canalyzing function just as canalyzing if no additional information is specified. A function that is canalyzing in variable \( i \) with canalyzing input value \( a \in F \) and canalyzed output value \( b \in F \) will be referred to as \( < i : a : b > \) canalyzing.

We introduce an additional concept.

**Definition 2.2.** \( f(x_1, x_2, \ldots, x_n) \) is essential in variable \( x_i \) if there exist \( r, s \in F \) such that \( f(x_1, \ldots, x_{i-1}, r, x_{i+1}, \ldots, x_n) \neq f(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n) \).

**Example 2.3.** Let \( q = 5, n = 3 \). \( f(x_1, x_2, x_3) = 2(x_1 - 3)^3(x_1 - 2)x_2 + 1 \). Then this function is essential on \( x_1 \) and \( x_2 \) but not essential on \( x_3 \). It has algebraic degree 5 with \( \deg(f)_1 = 4 \) and \( \deg(f)_2 = 1 \). Note that \( f \) is canalyzing in \( x_1 \) with canalyzing input value 3 and canalyzed output value 1, i.e. \( f \) is \( < 1 : 3 : 1 > \) canalyzing. Note that \( f \) is also \( < 1 : 2 : 1 > \) and \( < 2 : 0 : 1 > \) canalyzing. Since \( f \) is not essential in \( x_3 \) it cannot be \( < 3 : a : b > \) canalyzing for any \( a, b \in F_q \).

If a function has exactly one essential variable, say \( x_i \), then its ANF is

\[
f = a_{q-1}x_i^{q-1} + \ldots + a_1x_i + a_0
\]

there exist a \( j \geq 1 \) such that \( a_j \neq 0 \). There are \( q^q - q \) many such functions since all the constants should be excluded. \( \frac{1}{q}(q^q - q) = q^{q-1} - 1 \) many of them have fixed canalyzed value \( b \) for any \( b \in F \) since each number is equal. In total, there are \( n(q^{q-1} - 1) \) many one essential variable canalyzing function with fixed canalyzed value \( b \) for any \( b \) since there are \( n \) variables.

There is only one constant function with fixed canalyzed value which is itself.

**Notation 2.4.** For \( i \in 0, 1, \ldots, n \) and \( a, b \in F_q \) we will use the following notation,

- \( C^{i}_{a,b} \): The set of all functions that are canalyzing in the \( i \)-th variable with canalyzing input value \( a \) and canalyzed output value \( b \).
- \( C^{i \ast}_{a,b} \): The set of all functions that are canalyzing in the \( i \)-th variable with some canalyzing input value in \( F_q \) and canalyzed output value \( b \).
- \( C^{i}_{a,s} \): The set of all functions that are canalyzing in the \( i \)-th variable with canalyzing input value \( a \) and some canalyzed output value in \( F_q \).
- \( C^{i}_{a,b} \): The set of all functions that are canalyzing on some variable with canalyzing input value \( a \) and canalyzed output value \( b \).
- \( C^{i \ast}_{a,s} \): The set of all functions that are canalyzing on some variable with some canalyzing input value in \( F_q \) and some canalyzed output value in \( F_q \).
- \( C^{i \ast}_{a,b} \): The set of all functions that are canalyzing on some variable with canalyzing input value \( a \) and some canalyzed output value in \( F_q \).
$C_{a,b}^*$: The set of all functions that are canalyzing on some variable with some canalyzing input value in $\mathbb{F}_q$ and canalyzing output value $b$.

$C_{a,*}^*$: The set of all functions that are canalyzing on some variable with some canalyzing input value in $\mathbb{F}_q$ and some canalyzing output value in $\mathbb{F}_q$, i.e., this set consists of all the canalyzing functions.

We have the following propositions.

**Proposition 2.5.** $C_{a,b_1}^i \cap C_{a,b_2}^i = \emptyset$ whenever $b_1 \neq b_2$.

**Proposition 2.6.** $C_{a,*}^{i_1} \cap C_{a,*}^{i_2} = \emptyset$ whenever $b_1 \neq b_2$ and $i_1 \neq i_2$.

**Proof.** Let $f \in C_{a,b_1}^i \cap C_{a,b_2}^i$, then there exist $a_1$ such that the value of $f$ should be always $b_1$ if we let $x_i = a_1$. Similarly, there exist $a_2$ such that the value of $f$ should be $b_2$ if we let $x_i = a_2$. But $b_1 \neq b_2$, a contradiction. \[\square\]

With the above notations, we have

$$C_{a,*}^i = \bigcup_{b \in \mathbb{F}} C_{a,b}^i = \bigcup_{a \in \mathbb{F}} C_{a,*}^i = \bigcup_{i \in [n]} C_{i,*}^i,$$

$$C_{a,b}^i = \bigcup_{i \in [n]} C_{i,b}^i = \bigcup_{a \in \mathbb{F}} C_{a,*}^i,$$

$$C_{a,*}^i = \bigcup_{a \in \mathbb{F}} C_{a,b}^i,$$

$$C_{i,b}^i = \bigcup_{i \in [n]} C_{i,b}^i,$$

$$C_{a,b}^i = \bigcup_{i \in [n]} C_{i,b}^i.$$

For any set $S$, we use $|S|$ to stand for its cardinality. We use $C(n,k) = \frac{n!}{k!(n-k)!}$ to stand for the binomial coefficients.

Obviously, for the above notations, the cardinality are same for different values of $i$, $a$ and $b$. In other words, we have $|C_{a_1,b_1}^i| = |C_{a_2,b_2}^i|$, $|C_{a,b}^i| = |C_{i,c}^i|$, $|C_{a,b}^i| = |C_{a,d}^i|$ and etc.

### 3. Characterization and Enumeration of Canalyzing Functions over $\mathbb{F}$

Similar to [1] we have

**Lemma 3.1.** $f(x_1,x_2,...x_n)$ is $< i : a : b >$ canalyzing iff

$$f(X) = f(x_1,x_2,...,x_n) = (x_i - a)Q(x_1,x_2,...,x_n) + b$$

where $\deg(Q)_i \leq q - 2$.

**Proof.** From the algebraic normal form of $f$, we rewrite it as $f = x_i^{q-1}g_1(X_i) + x_i^{q-2}g_2(X_i) + ... + x_ig_1(X_i) + g_0(X_i)$, where $X_i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. Using long division we get $f(X) = f(x_1,x_2,...,x_n) = (x_i - a)Q(x_1,x_2,...,x_n) + r(X_i)$. Since $f(X)$ is $< i : a : b >$ canalyzing, we get $f(X) = f(x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_n) = b$ for any $(x_1, x_2, ..., x_n)$, i.e., $r(X_i) = b$ for any $X_i$. So $r(X_i)$ must be the constant $b$. We finished the necessity. The sufficiency is obvious. \[\square\]

The above lemma means $f(X)$ is $< i : a : b >$ iff $(x_i - a)|(f(X) - b)$.

Now we get our first formula.

**Lemma 3.2.** For any $i \in [n]$, $a$, $b \in \mathbb{F}$, there are $q^{n-1}$ many $< i : a : b >$ canalyzing functions. In other words, $|C_{a,b}^i| = q^{n-1}$.

**Proof.** In Lemma 3.1 $Q$ can be any polynomial with $\deg(Q)_i \leq q - 2$. Its ANF is $\sum_{k_0=0}^{q-1} \sum_{k_2=0}^{q-2} \sum_{k_0=0}^{q-1} a_{k_0,k_2}x_1k_0x_2k_2...x_nk_n$. Since each coefficient has $q$ many choices and there are $(q - 1)q^{n-1} = q^n - q^{n-1}$ monomials, we get what we want. \[\square\]

Because $C_{a,*}^i = \bigcup_{b \in \mathbb{F}} C_{a,b}^i$, by Proposition 2.5 we get
Corollary 3.3. \(|C_{a,s}^i| = q(q^n-q^{n-1}) = q^n - q^{n-1} + 1\).

Lemma 3.4. For any \(\{a_1, a_2, ..., a_k\} \subset \mathbb{F}\), \(f(X) \in \bigcap_{j=1}^k C_{a_j,b}^i\) iff
\[
f(X) = f(x_1, x_2, ..., x_n) = (\prod_{j=1}^k (x_i - a_j))Q(x_1, x_2, ..., x_n) + b, \text{ where } \deg(Q)_i \leq q - k - 1.
\]

Proof. From Lemma 3.2, we know \((x_i - a_j)(f(X) - b)\) for \(j = 1, 2, ..., k\). So their product since they are pairwise coprime. □

Lemma 3.5. \(|\bigcap_{j=1}^k C_{a_j,b}^i| = q^{n-kq^{n-1}}\) for any \(\{a_1, a_2, ..., a_k\} \subset \mathbb{F}\).

Proof. This is similar to the proof of Lemma 3.2 by Lemma 3.4 □

Note: If \(k = q\), the above number is 1. This is because it means \((x_i - a_j)(f(X) - b)\) for all the \(a_j, j = 1, 2, ..., q\), i.e., \((x_i^q - x_i)(f(X) - b)\), where \(x_i^q - x_i = \prod_{a \in \mathbb{F}}(x_i - a)\). So \(f(X) - b = 0\) which means \(f(x) = b\).

Theorem 3.6. For any \(i \in [n], b \in \mathbb{F}\), \(|C_{a,b}^i| = |\bigcup_{a \in \mathbb{F}} C_{a,b}^i| = q^n - (q^{n-1} - 1)^q\).

Proof. By Inclusion and Exclusion Principle, we have \(|C_{a,b}^i| = |\bigcup_{a \in \mathbb{F}} C_{a,b}^i| = \sum_{a \in \mathbb{F}} |C_{a,b}^i| - \sum_{a_1, a_2 \in \mathbb{F}} |C_{a_1,b}^i \cap C_{a_2,b}^i| + \cdots + (-1)^{k-1} \sum_{a_1, a_2, ..., a_k \in \mathbb{F}} |\bigcap_{j=1}^k C_{a_j,b}^i| + \cdots + (-1)^{q-1} = C(q, 1)q^{n-q^{n-1}} - C(q, 2)q^{n-2q^{n-1}} + (-1)^{q-1}C(q, q)q^{n-kq^{n-1}} + \cdots + 1 = \sum_{k=1}^{q} (-1)^{k-1}C(q, k)q^{n-kq^{n-1}} = q^n \sum_{k=1}^{q} (C(q, k)(-q^{-q^{n-1}}))^k = q^n (1 - (1 - q^{-q^{n-1}})^q) = q^n - (q^n - 1)^q\). □

Similarly,

Lemma 3.7. For any \(\{i_1, i_2, ..., i_k\} \subset [n], f(X) \in \bigcap_{j=1}^k C_{a,b}^{i_j}\) iff
\[
f(X) = f(x_1, x_2, ..., x_n) = (\prod_{j=1}^k (x_{i_j} - a))Q(x_1, x_2, ..., x_n) + b, \text{ where } \deg(Q)_{i_j} \leq q - 1, j = 1, 2, ..., k.
\]

Lemma 3.8. \(|\bigcap_{j=1}^k C_{a,b}^{i_j}| = q^{(q-1)^k q^{n-k}}\) for any \(\{i_1, i_2, ..., i_k\} \subset [n]\).

Theorem 3.9. \(|C_{a,c}^*| = \sum_{1 \leq k \leq n} (-1)^{k-1}C(n, k)q^{(q-1)^k q^{n-k}}\).

Proof. \(|C_{a,c}^*| = |\bigcup_{i \in [n]} C_{a,b}^i| = \sum_{1 \leq i \leq n} |C_{a,c}^i| - \sum_{1 \leq i < j \leq n} |C_{a,c}^i \cap C_{a,c}^j| + \cdots + (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |\bigcap_{j=1}^k C_{a,c}^{i_j}| + \cdots + (-1)^{n-1} |\bigcap_{j=1}^n C_{a,c}^{i_j}| = C(n, 1)q^{(q-1)^n} - C(n, 2)q^{(q-1)^2 q^{n-2}} + \cdots + (-1)^{n-1}C(n, k)q^{(q-1)^k q^{n-k}} + \cdots + (-1)^{n-1} q^{(q-1)^n} = \sum_{1 \leq k \leq n} (-1)^{k-1}C(n, k)q^{(q-1)^k q^{n-k}}\). □

Corollary 3.10. \(|C_{a,c}^*| = q \sum_{1 \leq k \leq n} (-1)^{k-1}C(n, k)q^{(q-1)^k q^{n-k}}\).

Proof. \(C_{a,c}^* = \bigcup_{b \in \mathbb{F}} C_{a,c}^b\), we need to show \(C_{a,c}^{b_1} \cap C_{a,c}^{b_2} = \emptyset\) if \(b_1 \neq b_2\). Suppose \(f \in C_{a,c}^{b_1} \cap C_{a,c}^{b_2}\), then there exist \(i_1\) and \(i_2\) such that \(f \in C_{a,c}^{i_1} \cap C_{a,c}^{i_2}\) since \(C_{a,c}^{i} = \bigcup_{i \in [n]} C_{a,c}^{i}\). If \(i_1 \neq i_2\), we get a contradiction by Proposition 2.3.5. If \(i_1 = i_2\), we get a contradiction by Proposition 2.6 since \(C_{a,c}^{i_1} \subset C_{a,c}^{i_1, b_1}\) and \(C_{a,c}^{i_2} \subset C_{a,c}^{i_2, b_2}\). □

Now, we are going to find the formula for the number of all the analyzing functions with given analyzable value \(b\). In other words, the formula of \(|C_{a,b}^*|\).

Let \(S_b = \{C_{a,b}^i | i \in [n], a \in \mathbb{F}\}\) for any \(b \in \mathbb{F}\). By Inclusion and Exclusion Principle, we have \(|C_{a,b}^*| = |\bigcup_{i \in [n]} C_{a,b}^i| = |\bigcup_{i \in [n]} \bigcup_{a \in \mathbb{F}} C_{a,b}^i| = \sum_{b=1}^{q^l} (-1)^{k-1} N_k\), where \(N_k = \sum_{S \subset S_b, |S| = k} |\bigcap_{T \in S} T|\).

In order to evaluate \(N_k\), we write all the elements in \(S_b\) as the following \(n \times q\) matrix:
\[
C_{a_1,b}^1, C_{a_2,b}^1, ..., C_{a_q,b}^1
\]
For the necessity, we use induction principle.

Lemma 3.11. Suppose \( a, b \) are any two elements in \( F \) and let \( a, b \neq 0 \). Then \( f = Q_k(X) \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 \), where \( A_0 = b, A_t \in F, A_t \) is determined by \( a_1, \ldots, a_{t+1}, b_1, \ldots, b_{t+1}, t = 1, 2, \ldots, k-1 \). \( \text{deg}(Q_k)_i \leq q - k - 1 \).

Proof. For the necessity, we use induction principle.

For \( k = 1 \), it is true by the definition and Lemma 3.11. If \( f \in \bigcap_{j=1}^{k} C_{a_j, b_j} \) implies \( f \in \bigcap_{j=1}^{k-1} C_{a_j, b_j} \), by the assumption, we have \( f = Q_{k-1}(X) \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 \). Since \( f \in C_{a_k, b_k} \), we get

\[
 f(x_1, \ldots, x_{i-1}, a_k, x_{i+1}, \ldots, x_n) = Q_{k-1} \prod_{j=1}^{k-1}(a_k - a_j) + A_{k-2} \prod_{j=1}^{k-2}(a_k - a_j) + \ldots + A_1(a_k - a_1) + A_0
\]

for any \( X = \{x_1, \ldots, x_{i-1}, a_k, x_{i+1}, \ldots, x_n\} \). In other words, \( Q_{k-1}(X) \prod_{j=1}^{k-1}(a_k - a_j) + A_{k-1} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \)

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]

\[
 (x_i - a_k)Q_k + A_{k-1} \prod_{j=1}^{k-1}(x_i - a_j) + A_{k-2} \prod_{j=1}^{k-2}(x_i - a_j) + \ldots + A_1(x_i - a_1) + A_0 = \]
= Q_k(X) \prod_{j=1}^{k}(x_i-a_j) + A_{k-1} \prod_{j=1}^{k-1}(x_i-a_j) + ... + A_1(x_i-a_1) + A_0.
We finish the proof of necessity.
When \( x_i = a_1 \), we have \( f = A_0 = b_1 \), so \( f \in C_{a_1,b_1}^i \).
When \( x_i = a_2 \), we set \( f = A_1(a_2-a_1) + A_0 = b_2 \), we get a unique solution for \( A_1 \) such that \( f \in C_{a_2,b_2}^i \).
When \( x_i = a_k \), we get a unique solution for \( A_{k-1} \) such that \( f \in C_{a_k,b_k}^i \). In summary, \( f \in \bigcap_{j=1}^{k} C_{a_j,b_j}^i \).
\[ \square \]

From the above Lemma, we immediately obtain

**Lemma 3.15.** \( |\bigcap_{j=1}^{k} C_{a_j,b_j}^i| = q^{(q-k)q^{n-1}} \) given \( \{a_1, a_2, ..., a_k\} \subset \mathbb{F} \) and \( \{b_1, b_2, ..., b_k\} \subset \mathbb{F} \)

In order to evaluate \( |C_{s,*}^i| \), we need to generalize Lemma 3.14.
To save space, we just focus on the cardinality in the following lemma.

**Lemma 3.16.** \( a_{11}, a_{12}, ... a_{1k_1}; a_{21}, a_{22}, ... a_{2k_2}; ...; a_{r1}, a_{r2}, ... a_{rk_r} \) are \( k_1 + k_2 + ... + k_r \) distinct elements of \( \mathbb{F} \), \( \{b_1, b_2, ..., b_k\} \subset \mathbb{F} \). Then
\[ |\bigcap_{j=1}^{k_1} C_{a_{1j},b_1}^i \bigcap_{j=1}^{k_2} C_{a_{2j},b_2}^i \bigcap ... \bigcap_{j=1}^{k_r} C_{a_{rj},b_r}^i| = q^{(q-k_1-k_2-...-k_r)q^{n-1}} \]

**Proof.** By Lemma 3.14, \( f \in \bigcap_{j=1}^{k_1} C_{a_{1j},b_1}^i \) iff \( f = Q(X) \prod_{j=1}^{k_1} (x_i-a_{1j}) + b_1, deg(Q) \leq q - k_1 - 1 \), i.e., we have
\[ \bigcap_{j=1}^{k_1} C_{a_{1j},b_1}^i = \{ Q(X) \prod_{j=1}^{k_1} (x_i-a_{1j}) + b_1 | Q, deg(Q) \leq q - k_1 - 1 \} \]
Let \( f \in \bigcap_{j=1}^{k_1} C_{a_{1j},b_1}^i \), then \( f = Q(X) \prod_{j=1}^{k_1} (x_i-a_{1j}) + b_1 \). If we also have \( f \in C_{a_{2j},b_2}^i \), let \( x_i = a_{21} \), we get
\[ f(x_1, x_1-a_{21}, x_{i+1}, ..., x_{n}) = Q(x_1, x_1-a_{21}, x_{i+1}, ..., x_{n}) \prod_{j=1}^{k_1} (a_{2j}-a_{1j}) + b_1 = b_2 \]
for any \( (x_1, x_1-a_{21}, x_{i+1}, ..., x_{n}) \). The coefficient \( \prod_{j=1}^{k_1} (a_{2j}-a_{1j}) \) is nonzero, so we can solve for \( Q \) and get \( Q \in C_{a_{2j},b_2}^i \) for some \( A_1 \in \mathbb{F} \). Hence we can write
\[ Q = (x_i-a_{21})Q_1 + A_1 \], i.e., \( f = (x_i-a_{21})Q_1 \prod_{j=1}^{k_1} (x_i-a_{1j}) + O(x_i) \), where \( O(x_i) \) is a one variable polynomial whose coefficients are completely determined by \( a_{ij}, b_i, deg(Q_1) = deg(O(x_i)) \leq k_1 \).

Obviously, we can repeat the above process, to get
\[ f \in \bigcap_{j=1}^{k_1} C_{a_{1j},b_1}^i \bigcap ... \bigcap_{j=1}^{k_r} C_{a_{rj},b_r}^i \] iff
\[ f = Q(\prod_{j=1}^{k_1} (x_i-a_{1j}))(\prod_{j=1}^{k_2} (x_i-a_{2j}))...(\prod_{j=1}^{k_r} (x_i-a_{rj})) + O(x_i) \]. Where \( deg(Q) \leq q - k_1 - k_2 - ... - k_r - 1 \) and \( O(x_i) \) is a uniquely determined polynomial of \( x_i \) and \( deg(O(x_i)) \leq k_1 + ... + k_r - 1 \). Hence, we know the cardinality is \( q^{(q-k_1-k_2-...-k_r)q^{n-1}} \).
\[ \square \]

Now, we are ready to find the cardinality of \( C_{s,*}^i \). We have

**Theorem 3.17.**
\[ |C_{s,*}^i| = q! \frac{\sum_{k=1}^{q} (-1)^{k-1} q(q-k)q^{n-1}}{(q-k)!} \sum_{k_1+...+k_q=k,0 \leq k_i \leq q} \frac{1}{k_1!k_2!...k_q!} \]

**Proof.** \( C_{s,*}^i = \bigcup_{b \in \mathbb{F}} C_{s,b}^i = \bigcup_{b \in \mathbb{F}} \bigcap_{a \in \mathbb{F}} C_{a,b}^i \).
Let \( S_i = \{ C_{a,b}^i | a, b \in \mathbb{F} \} \), we get \( |C_{s,*}^i| = \sum_{a=1}^{q} (-1)^{k-1} N_k \). Where \( N_k = \sum_{s \in S_i, |s|=k} |\bigcap_{T \in s} T| \).
In order to evaluate \( N_k \), we write all the elements in \( S_i \) as the following \( q \times q \) matrix.

\[ C_{a_1,b_1}^i, C_{a_2,b_1}^i, ..., C_{a_q,b_1}^i \]
\[ C_{a_1,b_2}^i, C_{a_2,b_2}^i, ..., C_{a_q,b_2}^i \]

...............
\[ C_{a_1,b_1} C_{a_2,b_2} \ldots C_{a_q,b_q} \]

For any \( s \in S_1 \) with \( |s| = k \), we will chose \( k \) elements from the above matrix to form \( s \).

Suppose \( k_1 \) of it elements are from the first row (there are \( C(q,k_1) \) many ways to do so). Let these \( k_1 \) elements be \( C_{a_{11},b_1} C_{a_{12},b_1} \ldots C_{a_{k_1},b_1} \).

Suppose \( k_2 \) of its elements are from the second row, otherwise the intersection will be \( \phi \) by Proposition 2.5 (there are \( C(q-k_1,k_2) \) many ways to do so). Let these \( k_2 \) elements be \( C_{a_{21},b_2} C_{a_{22},b_2} \ldots C_{a_{2k_2},b_2} \).

Suppose \( k_q \) of its elements are from the last row (there are \( C(q-k_1-k_2-\ldots-k_{q-1},k_q) \) many ways to do so). Let these \( k_q \) elements be \( C_{a_{q1},b_q} C_{a_{q2},b_q} \ldots C_{a_{qk_q},b_q} \).

\[ k_1 + k_2 + \ldots + k_q = k, \quad 0 \leq k_i \leq q, \quad i = 1, 2, \ldots, q. \]

\[ N_k = \sum_{s \subseteq S_1, |s| = k} |\cap_{T \in s} T| = \sum_{k_1 + \ldots + k_q = k, 0 \leq k_i \leq q} C(q,k_1) C(q-k_1,k_2) \ldots C(q-k_1-\ldots-k_{q-1},k_q) I_{k_1,k_2,\ldots,k_q}, \]

where \( I_{k_1,k_2,\ldots,k_q} = |(C_{a_{11},b_1} \cap C_{a_{21},b_2} \cap \ldots \cap C_{a_{q1},b_q})| \).

By Lemma 3.16 we know \( I_{k_1,k_2,\ldots,k_q} = q^{(q-k_1-k_2-\ldots-k_{q-1})q^{n-1}} = q^{(q-k)q^{n-1}} \), this number is zero if \( k > q \).

A straightforward computation shows that

\[ C(q,k_1) C(q-k_1,k_2) \ldots C(q-k_1-\ldots-k_{q-1},k_q) = \frac{q!}{k_1! k_2! \ldots k_q!} q^{(q-k)q^{n-1}}. \]

Hence, we get

\[ |C_{*,*}| = \sum_{k=1}^{q} (-1)^{k-1} N_k = \sum_{k=1}^{q} (-1)^{k-1} \frac{q!}{k_1! k_2! \ldots k_q!} q^{(q-k)q^{n-1}} \]

\[ \frac{q!}{(q-k)!} \sum_{k_1 + \ldots + k_q = k, 0 \leq k_i \leq q} \frac{1}{k_1! k_2! \ldots k_q!}. \]

Now we begin to evaluate \( |C_{*,*}| \). We have

**Theorem 3.18.**

\[ |C_{*,*}| = \sum_{k=1}^{q} (-1)^{k-1} U_k + \sum_{k=1}^{nq} (-1)^{k-1} V_k, \]

where

\[ U_k = n \sum_{t_1 + t_2 + \ldots + t_q = k, 0 \leq t_i \leq q} \frac{q!}{t_1! t_2! \ldots t_q!} q^{(q-k)q^{n-1}} = \frac{nq!}{(q-k)!} q^{(q-k)q^{n-1}} \sum_{t_1 + t_2 + \ldots + t_q = k, 0 \leq t_i \leq q} \frac{1}{t_1! t_2! \ldots t_q!}. \]

\[ V_k = q \sum_{k_1 + \ldots + k_n = k, 0 \leq k_i \leq k-1, 0 \leq k_i \leq q} (\prod_{j=1}^{n} C(q,k_j)) q^{\prod_{j=1}^{n} (q-k_j)}. \]

**Proof.** First \( C_{*,*} = \bigcup_{i \in [n]} \bigcup_{a \in F} \bigcup_{b \in F} C_{a,b}^i \). Let \( S = \{ C_{a,b}^i | i \in [n], a, b \in F \} \), then

\[ |C_{*,*}| = nq^2 - \sum_{k=1}^{q} (-1)^{k-1} N_k, \]

where \( N_k = \sum_{s \subseteq S, |s| = k} |\cap_{T \in s} T| \).

We write all the \( nq^2 \) elements of \( S \) as the following \( n \) many \( q \times q \) matrices.

\[ C_{a_1,b_1} C_{a_1,b_2} \ldots C_{a_1,b_q} C_{a_2,b_1} C_{a_2,b_2} \ldots C_{a_2,b_q} \]

\[ C_{a_2,b_1} C_{a_2,b_2} \ldots C_{a_2,b_q} \]
where the one in Lemma 3.16 and \( k = 1 \)
i some fixed \( 3.11. \) But these two cases are not disjoint.
\( M \) \( k \) non empty intersection, we know all these \( M \).

If \( 0 \leq i \leq n \), there exist \( k \) \( k \).
Hence, a typical intersection either looks like the one in Lemma 3.16 or the one in Lemma 3.11.

We call this matrix \( M_1 \).

We combine all the above \( M \), \( k \) \( C \) \( M \).

We call this matrix \( M_2 \).

We call this matrix \( M_n \).

We combine all the above \( M_i \), \( i = 1, 2, \ldots, n. \) to form a \( nq \times q \) matrix \( M \) whose first \( q \) rows are \( M_1 \), the second \( q \) rows are \( M_2, \ldots, \) the last \( q \) rows are \( M_n \).

We are going to chose \( k \) elements from \( M \) to form the intersection. In order to get a possible non empty intersection, we know all these \( k \) elements must come from either the same \( M_i \) (for some fixed \( i \)) or all of them from the same column of \( M \) by Proposition 2.6. Inside the fixed \( M_i \), each elements must come from different rows by Proposition 2.3.

Hence, a typical intersection either looks like the one in Lemma 3.16 or the one in Lemma 3.11. But these two cases are disjoint.

Suppose we chose \( k_i \) elements from \( M_i \), \( i = 1, 2, \ldots, n, \) \( k_1 + k_2 + \ldots + k_n = k, \) \( 0 \leq k_i \leq k, \) \( i = 1, 2, \ldots, n. \)

If there exist \( i \) such that \( k_i = k \), then \( k_j = 0, \forall j \neq i. \) This implies the intersection looks like the one in Lemma 3.16 and \( k \).

If \( 0 \leq k_i \leq k - 1, \forall i \in [n] \), then the intersection looks like the one in Lemma 3.11 and \( k \leq nq. \)

The above two cases are disjoint now. By Lemma 3.16 and Lemma 3.12 we get
\[
N_k = \sum_{s \subseteq S, |s| = k} |\bigcap_{t \in s} T| = \sum_{k_1 + \ldots + k_n = k, 0 \leq k_i \leq k} = \sum_{\exists i, k_i = k} + \sum_{k_i \leq k - 1, i = 1, \ldots, n} = U_k + V_k
\]
where
\[
U_k = n \sum_{t_1 + t_2 + \ldots + t_q = k, 0 \leq t_i \leq q} \frac{q!}{t_1!t_2!\ldots t_q!(q-k)!q^{(q-k)q-1}} = \frac{nq!}{(q-k)!q^{(q-k)q-1}} \sum_{t_1 + t_2 + \ldots + t_q = k, 0 \leq t_i \leq q} \frac{1}{t_1!t_2!\ldots t_q!}
\]
\[
V_k = q \sum_{k_1 + \ldots + k_n = k, 0 \leq k_i \leq k - 1, 0 \leq k_i \leq q} \prod_{j=1}^{n} C(q, k_j)q^{\sum_{j=1}^{n} (q-k_j)}
\]

Hence,
\[
|C^k_{s,x}| = \sum_{k=1}^{nq^2} (-1)^{k-1} N_k = \sum_{k=1}^{nq^2} (-1)^{k-1} (U_k + V_k) = \sum_{k=1}^{q} (-1)^{k-1} U_k + \sum_{k=1}^{nq} (-1)^{k-1} V_k,
\]
\( \square \)
In the following, we will reduce the formula $|C_{*,*}|$ when $q = 2$ and compare it with the one in [2].

$$|C_{*,*}| = \sum_{k=1}^{2n} (-1)^{k-1}U_k + \sum_{k=1}^{2n} (-1)^{k-1}V_k,$$

where

$$U_k = n \sum_{t_1+t_2=k,0 \leq t_1 \leq 2} \frac{2!}{t_1!t_2!(2-k)!} 2^{(2-k)2^{n-1}}$$

$$V_k = 2 \sum_{k_1+\ldots+k_n=k,0 \leq k \leq 1,0 \leq k_1 \leq 2} \sum_{j=1}^{n} (\prod_{j=1}^{n} C(2,k_j)) 2\Pi_{j=1}^{n}(2-k_j)$$

A simple calculation shows that $U_1 = 4n2^{n-1} = C(n,1)2^{2}2^{2n-1}$ and $U_2 = 4n$.

$V_1 = 0$ since the condition of the sum is not satisfied.

$$V_2 = 2 \sum_{k_1+\ldots+k_n=2,0 \leq k \leq 1} \sum_{j=1}^{n} (\prod_{j=1}^{n} C(2,k_j)) 2\Pi_{j=1}^{n}(2-k_j) = C(n,2)2^32^{2n-2}$$

When $3 \leq k \leq 2n$

$$V_k = 2 \sum_{k_1+\ldots+k_n=k,0 \leq k \leq 2} \sum_{j=1}^{n} (\prod_{j=1}^{n} C(2,k_j)) 2\Pi_{j=1}^{n}(2-k_j)$$

$$= C(n,k)2^{k+1}2^{2n-k} + \sum_{1 \leq t \leq \frac{k}{2}} C(n,t)C(n-t,k-2t)2^{k-2t+1}$$

Hence, when $q = 2$,

$$|C_{*,*}| = -4n + \sum_{1 \leq k \leq n} (-1)^{k+1} C(n,k)2^{k+1}2^{2n-k}$$

$$+ \sum_{3 \leq k \leq 2n} (-1)^{k-1} \sum_{1 \leq t \leq \frac{k}{2}} C(n,t)C(n-t,k-2t)2^{k-2t+1}$$

When $n=1$, 2, 3, 4, one can obtain (without calculator) the sequence 4, 14, 120, 3514. These results are consistent with those in [2]. By [2], the cardinality of $C_{*,a}$ should be

$$|C_{*,*}| = 2((-1)^n - n) + \sum_{1 \leq k \leq n} (-1)^{k+1} C(n,k)2^{k+1}2^{2n-k}.$$  

So, we obtain the following combinatorial identity (for any positive integer $n$).

$$\sum_{3 \leq k \leq 2n} (-1)^{k-1} \sum_{1 \leq t \leq \frac{k}{2}} C(n,t)C(n-t,k-2t)2^{k-2t+1} = 2((-1)^n + n)$$

The left sum should be explained as 0 if $n = 1$. As usual, $C(n,k)$ is 0 if $k > n$.

For general $q$, from Lemma 3.2 we know $|C_{i,b}| = q^{n^2-q^{n-1}}$, since $C_{*,*} = \bigcup_{i \in [n]} \bigcup_{a \in \mathbb{F}} \bigcup_{b \in \mathbb{F}} C_{i,a,b}$, we obtain $|C_{*,*}| \leq nq^2q^{(q-1)q^{n-1}}$.

In order to get an intuitive idea about the magnitude of all the cardinality numbers, We will find their asymptote as $n \to \infty$ or $q \to \infty$.

We have the following notation
Definition 3.19. $f(x) \cong g(x)$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

Now, we can list all the cardinalities asymptotically, we have

Theorem 3.20.

$$|C_{a,b}^i| = q(q-1)q^{n-1};$$

$$|C_{a,*}^i| = q(q-1)q^{n-1};$$

$$|C_{*,b}^i| \cong nq(q-1)q^{n-1}, |C_{*,b}^i| \cong q(q-1)q^{n-1};$$

$$|C_{a,b}^*| \cong nq(q-1)q^{n-1}, |C_{a,b}^*| \cong n(q-1)q^{n-1};$$

$$|C_{a,*}^*| \cong nqq(q-1)q^{n-1}, |C_{a,*}^*| \cong nqq(q-1)q^{n-1};$$

$$|C_{*,b}^*| \cong nqq(q-1)q^{n-1}, |C_{*,b}^*| \cong nqq(q-1)q^{n-1};$$

Proof. The first two rows are previous lemma and corollary.

We will give a proof for the last row, the others are similar and easier.

$$|C_{*,*}^*| = \sum_{k=1}^{q} (-1)^{k-1} U_k + \sum_{k=1}^{nq} (-1)^{k-1} V_k.$$

$$U_k = \frac{nq!}{(q-1)!} q^{(q-1)q^{n-1}} \sum_{t_1+t_2+\ldots+t_q=1, 0 \leq t_i \leq q} \frac{1}{t_1!t_2!\ldots t_q!} = nqq(q-1)q^{n-1}.$$

When $2 \leq k \leq q$, we have

$$U_k = \frac{nq!}{(q-k)!} q^{(q-k)q^{n-1}} \sum_{t_1+t_2+\ldots+t_q=k} \frac{1}{t_1!t_2!\ldots t_q!} \leq nqq(q-2)q^{n-1} \sum_{0 \leq t_i \leq q, i=1,2,\ldots,q} 1 = nqq(q-2)q^{n-1}(q+1)^q.$$

So, $\lim_{n \to \infty} \frac{V_k}{U_k} = 0$ for $2 \leq k \leq q$.

$V_1 = 0$ since the condition of the sum is not satisfied.

When $2 \leq k \leq nq$, we have

$$V_k = q \sum_{k_1+\ldots+k_n=k, 0 \leq k_1 \leq q} (\prod_{j=1}^{n} C(q,k_j)) q^{q(q-1)q^{n-1}} \leq q \sum_{0 \leq k_i \leq q, i=1,2,\ldots,n} (nq!) q^{q-1} q^{n-2} = q(q+1)^n nq! q^{q-1} q^{n-2}.$$

Hence,

$$\left| \sum_{k=1}^{nq} (-1)^{k-1} V_k \right| \leq nqq(q+1)^n nq! q^{q-1} q^{n-2}.$$
We obtain
\[
\lim_{n \to \infty} \left| \sum_{k=1}^{nq} (-1)^{k-1}V_k \right| \leq \lim_{n \to \infty} \frac{nqq(q + 1)^n nq!q(q-1)^2q^{n-2}}{nq^2q(q-1)q^{n-1}} = \lim_{n \to \infty} \frac{(q + 1)^n nq!}{q(q-1)q^{n-2}} = 0.
\]

In summary, we obtain
\[
\lim_{n \to \infty} \frac{|C^*_{s,s}|}{nq^2q(q-1)q^{n-1}} = 1
\]

In other words,
\[
|C^*_{s,s}| \underset{n}{=} nq^2q(q-1)q^{n-1}.
\]

From the above proof, it is also clear that we have
\[
\lim_{q \to \infty} \frac{|C^*_{s,s}|}{nq^2q(q-1)q^{n-1}} = 1
\]

In other words,
\[
|C^*_{s,s}| \underset{q}{=} nq^2q(q-1)q^{n-1}.
\]

When \( q = 2 \), the first part of the last row in the above theorem has been obtained in [2].

4. Conclusion

In this paper, we generalized the definition of Boolean canalizing functions to the functions over general finite fields \( \mathbb{F}_q \). We obtain clear characterization for all eight classes of canalizing functions. Using Inclusion and Exclusion Principle, we also obtain eight formulas for the cardinality of these classes. The main idea is from [1] and [2]. Actually, the characterization is motivated from a simple lemma in [1]. The enumeration idea is a natural extension of [2]. By specifying our results to the case \( q = 2 \), we obtain the formula in [2], and derive an interesting combinatorial identity. Finally, for a better understanding to the magnitudes, we provide all the eight asymptotes of these cardinalities as either \( n \to \infty \) or \( q \to \infty \).

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References

[1] A. Jarrah, B. Ropasa and R. Laubenbacher, “Nested Canalizing, Unate Cascade, and Polynomial Functions”, *Physica D* 233 (2007), pp. 167-174.
[2] Winfried Just, Ilya Shmulevich, John Konvalina, “The Number and Probability of Canalizing Functions”, *Physica D* 197 (2004), pp. 211-221.
[3] Yuan Li, “Results on Rotation Symmetric Polynomials Over GF(p)”, *Information Sciences* 178 (2008), pp. 280-286.
[4] David Murrugarra Tomairo, “Nested Canalizing Functions”, (2010) preprint.
[5] Franziska Hinkelmann, Abdul Salam Jarrah, “Inferring Biological Models: Nested Canalizing”, (2010) preprint.
[6] S. A. Kauffman, “The Origins of Order: Self-Organization and Selection in Evolution”, *Oxford University Press, New York, Oxford* (1993).
[7] S. A. Kauffman, C. Peterson, B. samuelesson, C. Troein, “Random Boolean Network Models and the Yeast Transcription Network”, Proc. Natl. Acad. Sci 100 (25) (2003), pp. 14796-14799.
[8] S. Nikolajewa, M. Friedel, T. Wilhelm, “Boolean Network with Biological Relevant Rules Show Ordered Behavior”, BioSystems 90 (2007), pp. 40-47.
[9] E. A. Bender, J. T. Butler, “Asymptotic Approximations for the Number of Fanout-free Functions”, IEEE Trans. Comput 27 (12) (1978), pp. 1180-1183.
[10] T. Sasao, K. Kinoshita, “On the Number of Fanout-free Functions and Unate Cascade Functions”, IEEE Trans. Comput 28 (1) (1979), pp. 66-72.
[11] R. Lidl and H. Niederreiter, “Finite Fields”, Cambridge University Press, New York (1977).
[12] A. Jarrah and R. Laubenbacher, “Discrete Models of Biochemical Networks: The Toric Variety of Nested Canalyzing Functions”, Algebraic Biology In H. Anai, K. Horimoto, and T. Kutsia. editors, number 4545 in LNCS, pages 15-22, springer, (2007).
[13] Charalambos A. Charalambides, “Enumerative Combinatorics”, A CRC Press Company Boca Raton, London, New York, Washington, DC. (2002).
[14] O. Colón-Reyes, A. S. Jarrah, R. Laubenbacher and B. Sturmfels, “Monomial Dynamical System Over Finite Fields”, Complex System 16 (2006), pp. 333-342.
[15] O. Colón-Reyes, R. Laubenbacher and B. Pareigis, “Boolean Monomial Dynamical System”, Annals of Combinatorics (2004) pp. 425-439.
[16] M. Aldana, S. Coppersmith, L. P. Kadanoff, “Boolean Dynamics with Random Couplings,”, in: E. Kaplan, J. E. Marsden, K. R. Sreenivasan(Eds), Perspectives and Problems in Nonlinear Science, Springer-Verlag, New York (2002) pp. 23-89.
[17] M. Pogson et al. (2006) Formal agent-based modelling of intracellular chemical interactions. Biosystems 85:37–45.
[18] L. Steggles et al. (2007) Qualitatively modelling and analysing genetic regulatory networks: a Petri net approach. Bioinformatics 23:336-343.
[19] Thomas R. and D’Ari R. (1989) Biological Feedback. CRC Press, Boca Raton, Florida.
[20] A. Veliz-Cuba, et al. (2010) Polynomial algebra of discrete models in systems biology. Bioinformatics 26:1637-1643.
[21] Hinkelmann, F., Murrugarra, D., Jarrah, A.S., and Laubenbacher, R. (2010) A mathematical framework for agent-based models of complex biological networks. Bull. Math. Biol., in press.

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