HARMONIC MORPHISMS FROM THE CLASSICAL COMPACT SEMISIMPLE LIE GROUPS
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ABSTRACT. In this paper we introduce a new method for manufacturing harmonic morphisms from semi-Riemannian manifolds. This is employed to yield a variety of new examples from the compact Lie groups $\text{SO}(n)$, $\text{SU}(n)$ and $\text{Sp}(n)$ equipped with their standard Riemannian metrics. We develop a duality principle and show how this can be used to construct the first known examples of harmonic morphisms from the non-compact Lie groups $\text{SL}_n(\mathbb{R})$, $\text{SU}^*(2n)$, $\text{Sp}(n, \mathbb{R})$, $\text{SO}^*(2n)$, $\text{SO}(p, q)$, $\text{SU}(p, q)$ and $\text{Sp}(p, q)$ equipped with their standard dual semi-Riemannian metrics.

1. INTRODUCTION

The notion of a minimal submanifold of a given ambient space is of great importance in differential geometry. Harmonic morphisms $\phi : (M, g) \to (N, h)$ between semi-Riemannian manifolds are useful tools for the construction of such objects. They are solutions to over-determined non-linear systems of partial differential equations determined by the geometric data of the manifolds involved. For this reason harmonic morphisms are difficult to find and have no general existence theory, not even locally.

For the existence of harmonic morphisms $\phi : (M, g) \to (N, h)$ it is an advantage that the target manifold $N$ is a surface i.e. of dimension 2. In this case the problem is invariant under conformal changes of the metric on $N^2$. Therefore, at least for local studies, the codomain can be taken to be the complex plane with its standard flat metric. For the general theory of harmonic morphisms between semi-Riemannian manifolds we refer to the excellent book [2] and the regularly updated on-line bibliography [6].

The Riemannian manifold Sol is one of the eight famous 3-dimensional homogeneous geometries and has the structure of a Lie group compatible with its Riemannian metric. Baird and Wood have shown in [3] that Sol does not allow any local harmonic morphisms with values in a surface. This fact has been the main motivation for the research leading to this paper.

We introduce the notion of an eigenfamily of complex valued functions on a given semi-Riemannian manifold $(M, g)$. We show how such a family can be used to construct a variety of harmonic morphisms on open and dense subsets of $M$.

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We focus our attention on the classical semi-simple Lie groups and construct eigenfamilies in the compact Riemannian cases of
\[ \text{SO}(n), \text{SU}(n) \text{ and } \text{Sp}(n) \]
inducing a variety of new harmonic morphisms on these important spaces. The examples constructed by our new method are the first harmonic morphisms on \( \text{SO}(n), \text{SU}(n) \) and \( \text{Sp}(n) \) which are not invariant under the action of the subgroups
\[ \text{SO}(p) \times \text{SO}(q), \text{SU}(p) \times \text{U}(q) \text{ and } \text{Sp}(p) \times \text{Sp}(q), \]
respectively, with \( n = p + q \). For the known invariant solutions leading to harmonic morphisms on the Grassmannians, see [8].

In Theorem 7.1 we prove a useful duality principle and show how this can be used to obtain eigenfamilies on the non-compact semi-Riemannian Lie groups
\[ \text{SL}_n(\mathbb{R}), \text{SU}^*(2n), \text{Sp}(n,\mathbb{R}), \text{SO}^*(2n), \text{SO}(p,q), \text{SU}(p,q) \text{ and } \text{Sp}(p,q). \]
This leads to the construction of the first known examples of harmonic morphisms in all these cases. It should be noted that the non-compact semi-simple Lie groups
\[ \text{SO}(n,\mathbb{C}), \text{SL}_n(\mathbb{C}) \text{ and } \text{Sp}(n,\mathbb{C}) \]
are complex manifolds and hence their coordinate functions form orthogonal harmonic families, see Definition 2.4. This means that in these cases the problem is more or less trivial.

Throughout this article we assume, when not stating otherwise, that all our objects such as manifolds, maps etc. are smooth, i.e. in the \( C^\infty \)-category. For our notation concerning Lie groups we refer to the wonderful book [10].

2. Harmonic Morphisms

Let \( M \) and \( N \) be two manifolds of dimensions \( m \) and \( n \), respectively. Then a semi-Riemannian metric \( g \) on \( M \) gives rise to the notion of a Laplacian on \( (M,g) \) and real-valued harmonic functions \( f : (M,g) \to \mathbb{R} \). This can be generalized to the concept of a harmonic map \( \phi : (M,g) \to (N,h) \) between semi-Riemannian manifolds being a solution to a semi-linear system of partial differential equations, see [2].

**Definition 2.1.** A map \( \phi : (M,g) \to (N,h) \) between semi-Riemannian manifolds is called a harmonic morphism if, for any harmonic function \( f : U \to \mathbb{R} \) defined on an open subset \( U \) of \( N \) with \( \phi^{-1}(U) \) non-empty, the composition \( f \circ \phi : \phi^{-1}(U) \to \mathbb{R} \) is a harmonic function.

The following characterization of harmonic morphisms between semi-Riemannian manifolds is due to Fuglede and generalizes the corresponding
well-known result of [4, 9] in the Riemannian case. For the definition of horizontal conformality we refer to [2].

**Theorem 2.2.** [5] A map \( \phi : (M, g) \to (N, h) \) between semi-Riemannian manifolds is a harmonic morphism if and only if it is a horizontally (weakly) conformal harmonic map.

The next result generalizes the corresponding well-known theorem of Baird and Eells in the Riemannian case, see [1]. It gives the theory of harmonic morphisms a strong geometric flavour and shows that the case when \( n = 2 \) is particularly interesting. In that case the conditions characterizing harmonic morphisms are independent of conformal changes of the metric on the surface \( N^2 \). For the definition of horizontal homothety we refer to [2].

**Theorem 2.3.** [7] Let \( \phi : (M, g) \to (N^n, h) \) be a horizontally conformal submersion from a semi-Riemannian manifold \((M, g)\) to a Riemannian manifold \((N, h)\). If

1. \( n = 2 \) then \( \phi \) is harmonic if and only if \( \phi \) has minimal fibres,
2. \( n \geq 3 \) then two of the following conditions imply the other:
   (a) \( \phi \) is a harmonic map,
   (b) \( \phi \) has minimal fibres,
   (c) \( \phi \) is horizontally homothetic.

In what follows we are mainly interested in complex valued functions \( \phi, \psi : (M, g) \to \mathbb{C} \) from semi-Riemannian manifolds. In this situation the metric \( g \) induces the complex-valued Laplacian \( \tau(\phi) \) and the gradient \( \text{grad}(\phi) \) with values in the complexified tangent bundle \( T^C M \) of \( M \). We extend the metric \( g \) to be complex bilinear on \( T^C M \) and define the symmetric bilinear operator \( \kappa \) by

\[
\kappa(\phi, \psi) = g(\text{grad}(\phi), \text{grad}(\psi)).
\]

Two maps \( \phi, \psi : M \to \mathbb{C} \) are said to be orthogonal if

\[
\kappa(\phi, \psi) = 0.
\]

The harmonicity and horizontal conformality of \( \phi : (M, g) \to \mathbb{C} \) are given by the following relations

\[
\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 0.
\]

**Definition 2.4.** Let \((M, g)\) be a semi-Riemannian manifold. Then a set

\[
\mathcal{E} = \{ \phi_i : M \to \mathbb{C} \mid i \in I \}
\]

of complex valued functions is said to be an eigenfamily on \( M \) if there exist complex numbers \( \lambda, \mu \in \mathbb{C} \) such that

\[
\tau(\phi) = \lambda \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \phi \psi
\]

for all \( \phi, \psi \in \mathcal{E} \). A set

\[
\Omega = \{ \phi_i : M \to \mathbb{C} \mid i \in I \}
\]
is said to be an orthogonal harmonic family on $M$ if for all $\phi, \psi \in \Omega$

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \psi) = 0.$$  

The next result shows that an eigenfamily on a semi-Riemannian manifold can be used to produce a variety of local harmonic morphisms.

**Theorem 2.5.** Let $(M, g)$ be a semi-Riemannian manifold and

$$\mathcal{E} = \{\phi_1, \ldots, \phi_n\}$$

be a finite eigenfamily of complex valued functions on $M$. If $P, Q : \mathbb{C}^n \to \mathbb{C}$ are linearly independent homogeneous polynomials of the same positive degree then the quotient

$$\frac{P(\phi_1, \ldots, \phi_n)}{Q(\phi_1, \ldots, \phi_n)}$$

is a non-constant harmonic morphism on the open and dense subset

$$\{p \in M \mid Q(\phi_1(p), \ldots, \phi_n(p)) \neq 0\}.$$  

A proof of Theorem 2.5 can be found in Appendix A. For orthogonal harmonic families we have the following useful result.

**Theorem 2.6.** Let $(M, g)$ be a semi-Riemannian manifold and

$$\{\phi_k : M \to \mathbb{C} \mid k = 1, \ldots, n\}$$

be a finite orthogonal harmonic family on $(M, g)$. Let $\Phi : M \to \mathbb{C}^n$ be the map given by $\Phi = (\phi_1, \ldots, \phi_n)$ and $U$ be an open subset of $\mathbb{C}^n$ containing the image $\Phi(M)$ of $\Phi$. If

$$\mathcal{H} = \{h_i : U \to \mathbb{C} \mid i \in I\}$$

is a family of holomorphic functions then

$$\{\psi : M \to \mathbb{C} \mid \psi = h(\phi_1, \ldots, \phi_n), \ h \in \mathcal{H}\}$$

is an orthogonal harmonic family on $M$.

### 3. The Riemannian Lie Group $\text{GL}_n(\mathbb{C})$

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ of left-invariant vector fields on $G$. Then a Euclidean scalar product $g$ on the algebra $\mathfrak{g}$ induces a left-invariant Riemannian metric on the group $G$ and turns it into a Riemannian manifold. If $Z$ is a left-invariant vector field on $G$ and $\phi, \psi : U \to \mathbb{C}$ are two complex valued functions defined locally on $G$ then the first and second order derivatives satisfy

$$Z(\phi)(p) = \frac{d}{ds}[\phi(p \cdot \exp(sZ))]|_{s=0},$$

$$Z^2(\phi)(p) = \frac{d^2}{ds^2}[\phi(p \cdot \exp(sZ))]|_{s=0}.$$

The tension field $\tau(\phi)$ and the $\kappa$-operator $\kappa(\phi, \psi)$ are given by

$$
\tau(\phi) = \sum_{Z \in \mathcal{B}} Z^2(\phi) \quad \text{and} \quad \kappa(\phi, \psi) = \sum_{Z \in \mathcal{B}} Z(\phi)Z(\psi)
$$

where $\mathcal{B}$ is any orthonormal basis of the Lie algebra $\mathfrak{g}$.

Let $\mathbf{GL}_n(\mathbb{C})$ be the complex general linear group equipped with its standard Riemannian metric induced by the Euclidean scalar product on the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ given by

$$
g(Z, W) = \Re \text{tr} ZW^*.
$$

For $1 \leq i, j \leq n$ we shall by $E_{ij}$ denote the element of $\mathfrak{gl}_n(\mathbb{R})$ satisfying

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}
$$

and by $D_t$ the diagonal matrices

$$D_t = E_{tt}.
$$

For $1 \leq r < s \leq n$ let $X_{rs}$ and $Y_{rs}$ be the matrices satisfying

$$X_{rs} = \frac{1}{\sqrt{2}}(E_{rs} + E_{sr}), \quad Y_{rs} = \frac{1}{\sqrt{2}}(E_{rs} - E_{sr}).
$$

With the above notation we have the following easily verified matrix identities

$$
\sum_{r<s} X^2_{rs} = \frac{(n-1)}{2} I_n, \quad \sum_{r<s} Y^2_{rs} = -\frac{(n-1)}{2} I_n, \quad \sum_{t=1}^n D_t^2 = I_n,
$$

$$
\sum_{r<s} X_{rs}E_{ij}X^t_{rs} = \frac{1}{2}(E_{ij} + \delta_{ij}(I_n - 2E_{ij})),
$$

$$
\sum_{r<s} Y_{rs}E_{ij}Y^t_{rs} = -\frac{1}{2}(E_{ij} - \delta_{ij}I_n),
$$

$$
\sum_{t=1}^n D_tE_{ij}D^t_t = \delta_{ji}E_{ij}.
$$

4. The Riemannian Lie group $\mathbf{SO}(n)$

In this section we construct eigenfamilies of complex valued functions on the special orthogonal group

$$
\mathbf{SO}(n) = \{ x \in \mathbf{GL}_n(\mathbb{R}) \mid x \cdot x^t = I_n, \det x = 1 \}.
$$

The Lie algebra $\mathfrak{so}(n)$ of $\mathbf{SO}(n)$ is the set of skew-symmetric matrices

$$
\mathfrak{so}(n) = \{ X \in \mathfrak{gl}_n(\mathbb{R}) \mid X + X^t = 0 \}
$$

and for this we have the canonical orthonormal basis

$$\{ Y_{rs} \mid 1 \leq r < s \leq n \}.$$
Lemma 4.1. For $1 \leq i, j \leq n$ let $x_{ij} : \text{SO}(n) \to \mathbb{R}$ be the real valued coordinate functions given by

$$x_{ij} : x \mapsto e_i \cdot x \cdot e_j^t$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis for $\mathbb{R}^n$. Then the following relations hold

$$\tau(x_{ij}) = -\frac{(n-1)}{2} x_{ij},$$

$$\kappa(x_{ij}, x_{kl}) = -\frac{1}{2} (x_{il}x_{kj} - \delta_{jl} \sum_{t=1}^{n} x_{it}x_{kt}).$$

Proof. It follows directly from the definition of the functions $x_{ij}$ that if $X$ is an element of the Lie algebra $\text{so}(n)$ then the first and second order derivatives satisfy

$$X(x_{ij}) : x \mapsto e_i \cdot x \cdot X \cdot e_j^t$$

and

$$X^2(x_{ij}) : x \mapsto e_i \cdot x \cdot X^2 \cdot e_j^t.$$

Employing the above mentioned matrix identities we then yield

$$\tau(x_{ij}) = \sum_{r<s} Y^2_{rs}(x_{ij}) = \sum_{r<s} e_i \cdot x \cdot Y^2_{rs} \cdot e_j^t = -\frac{(n-1)}{2} x_{ij},$$

$$\kappa(x_{ij}, x_{kl}) = \sum_{r<s} e_i \cdot x \cdot Y_{rs} \cdot e_j^t \cdot e_l \cdot Y^t_{rs} \cdot x^t \cdot e_k^t = e_i \cdot x \cdot \left( \sum_{r<s} Y_{rs} \cdot E_{jl} \cdot Y^t_{rs} \right) \cdot x^t \cdot e_k^t = -\frac{1}{2} (x_{il}x_{kj} - \delta_{jl} \sum_{t=1}^{n} x_{it}x_{kt}).$$

\[\square\]

Let $P, Q : \text{SO}(n) \to \mathbb{C}$ be homogeneous polynomials of the coordinate functions $x_{ij} : \text{SO}(n) \to \mathbb{C}$ of degree one i.e. of the form

$$P = \text{trace}(A \cdot x^t) = \sum_{i,j=1}^{n} a_{ij}x_{ij} \quad \text{and} \quad Q = \text{trace}(B \cdot x^t) = \sum_{k,l=1}^{n} b_{kl}x_{kl}$$

for some $A, B \in \mathbb{C}^{n \times n}$. As a direct consequence of Lemma 4.1 we then yield

$$PQ + 2 \kappa(P, Q) = \sum_{i,j,k,l=1}^{n} a_{ij}b_{kl}x_{ij}x_{kl} + 2 \sum_{i,j,k,l=1}^{n} a_{ij}b_{kl} \kappa(x_{ij}, x_{kl})$$

$$= \sum_{i,j,k,l=1}^{n} a_{ij}b_{kl}x_{ij}x_{kl} - \sum_{i,j,k,l=1}^{n} a_{ij}b_{kl}x_{kj}x_{il} + \sum_{i,j,k,l=1}^{n} a_{ij}b_{kj}x_{it}x_{kl}$$

$$= \sum_{i,j,k,l=1}^{n} (a_{ij}b_{kl} - a_{kj}b_{ij})x_{ij}x_{kl} + \text{trace}(AB^t xx^t).$$
Comparing coefficients we see that $PQ + 2\kappa(P, Q) = 0$ if $AB^t = 0$ and 
\[
\det \begin{pmatrix} a_{ij} & b_{il} \\ a_{kj} & b_{kl} \end{pmatrix} = (a_{ij}b_{kl} - a_{kj}b_{il}) = 0
\]
for all $1 \leq i, j, k, l \leq n$.

**Theorem 4.2.** Let $V$ be a maximal isotropic subspace of $\mathbb{C}^n$ and $p \in \mathbb{C}^n$ be a non-zero element. Then the set 
\[
\mathcal{E}_V(p) = \{ \phi_a : SO(n) \to \mathbb{C} \mid \phi_a(x) = \text{trace}(p'ax^t), \ a \in V \}
\]
of complex valued functions is an eigenfamily on $SO(n)$.

**Proof.** Assume that $a, b \in V$ and define $A = p^t a$ and $B = p^t b$. By construction any two columns of the matrices $A$ and $B$ are linearly dependent. This means that for all $1 \leq i, j, k, l \leq n$
\[
\det \begin{pmatrix} a_{ij} & b_{il} \\ a_{kj} & b_{kl} \end{pmatrix} = (a_{ij}b_{kl} - a_{kj}b_{il}) = 0.
\]
Furthermore we have $AB^t = 0$. Hence $P^2 + 2\kappa(P, P) = 0$, $PQ + 2\kappa(P, Q) = 0$, $Q^2 + 2\kappa(Q, Q) = 0$ and the statement follows directly from Lemma 4.1 and the calculations above. □

Applying the fact that $xx^t = I$ for each $x \in SO(n)$ we get a simplified formula for the $\kappa$ operator
\[
\kappa(x_{ij}, x_{kl}) = \frac{1}{2}(\delta_{ik}\delta_{jl} - x_{il}x_{kj}).
\]
With a similar analysis to that above one yields the result of Theorem 4.3. It should be noted that having employed the special property $xx^t = I$ the eigenfamily $\mathcal{E}(p)$ can not be used directly in the duality of Theorem 7.1.

**Theorem 4.3.** Let $p$ be a non-zero isotropic element of $\mathbb{C}^n$ i.e. such that $(p, p) = 0$. Then the set 
\[
\mathcal{E}(p) = \{ \phi_a : SO(n) \to \mathbb{C} \mid \phi_a(x) = \text{trace}(p'ax^t), \ a \in \mathbb{C}^n \}
\]
of complex valued functions is an eigenfamily on $SO(n)$.

**Example 4.4.** For $z, w \in \mathbb{C}$ let $p$ be the isotropic element of the 4-dimensional complex vector space $\mathbb{C}^4$ given by 
\[
p(z, w) = (1 + zw, i(1 - zw), i(z + w), z - w).
\]
This gives us the complex 2-dimensional deformation of eigenfamilies $\mathcal{E}_p$ each consisting of complex valued functions
\[
\phi_a : SO(4) \to \mathbb{C}
\]
of the form
\[
\phi_a(x) = (1 + zw)(a_1x_{11} + a_2x_{21} + a_3x_{31} + a_4x_{41}) \\
+i(1 - zw)(a_1x_{12} + a_2x_{22} + a_3x_{32} + a_4x_{42}) \\
+i(z + w)(a_1x_{13} + a_2x_{23} + a_3x_{33} + a_4x_{43})
\]
\[(z - w)(a_1x_{14} + a_2x_{24} + a_3x_{34} + a_4x_{44}).\]

5. The Riemannian Lie group \(SU(n)\)

In this section we construct eigenfamilies of complex valued functions on the unitary group \(U(n)\). They can be used to construct local harmonic morphisms on the special unitary group \(SU(n)\). The unitary group \(U(n)\) is the compact subgroup of \(GL_n(\mathbb{C})\) given by
\[U(n) = \{ z \in GL_n(\mathbb{C})| \ z \cdot z^* = I_n \} \]
The circle group \(S^1 = \{ w \in \mathbb{C} | |w| = 1 \}\) acts on the unitary group \(U(n)\) by multiplication \((e^{i\theta}, z) \mapsto e^{i\theta}z\) and the orbit space of this action is the special unitary group \(SU(n) = \{ z \in U(n) | \det z = 1 \}\).

The Lie algebra \(u(n)\) of the unitary group \(U(n)\) satisfies
\[u(n) = \{ Z \in \mathbb{C}^{n \times n} | Z + Z^* = 0 \}\]
and for this we have the canonical orthonormal basis \(\{Y_{rs}, iX_{rs} | 1 \leq r < s \leq n \} \cup \{iD_t | t = 1, \ldots, n \}\).

Lemma 5.1. For \(1 \leq i, j \leq n\) let \(z_{ij} : U(n) \to \mathbb{C}\) be the complex valued coordinate functions given by \(z_{ij} : z \mapsto e_{i} \cdot z \cdot e_{j}^t\) where \(\{e_1, \ldots, e_n\}\) is the canonical basis for \(\mathbb{C}^n\). Then the following relations hold
\[\tau(z_{ij}) = -nz_{ij} \text{ and } \kappa(z_{ij}, z_{kl}) = -z_{il}z_{kj}.\]

Proof. The proof is similar to that of Lemma 4.1. \(\square\)

Let \(P, Q : U(n) \to \mathbb{C}\) be homogeneous polynomials of the coordinate functions \(z_{ij} : U(n) \to \mathbb{C}\) of degree one i.e. of the form
\[P = \text{trace}(A \cdot z^t) = \sum_{i,j=1}^{n} a_{ij}z_{ij}\]
and \(Q = \text{trace}(B \cdot z^t) = \sum_{k,l=1}^{n} b_{kl}z_{kl}\) for some \(A, B \in \mathbb{C}^{n \times n}\). As a direct consequence of Lemma 5.1 we then yield
\[PQ + \kappa(P, Q) = \sum_{i,j,k,l=1}^{n} (a_{ij}b_{kl} - a_{kj}b_{il})z_{ij}z_{kl}.\]
Comparing coefficients we see that \(\kappa(P, Q) + PQ = 0\) if for all \(1 \leq i, j, k, l \leq n\)
\[\det\begin{pmatrix} a_{ij} & b_{il} \\ a_{kj} & b_{kl} \end{pmatrix} = (a_{ij}b_{kl} - a_{kj}b_{il}) = 0.\]

Theorem 5.2. Let \(p\) be a non-zero element of \(\mathbb{C}^n\). Then the set
\[\mathcal{E}(p) = \{ \phi_a : U(n) \to \mathbb{C} | \phi_a(z) = \text{trace}(p^taz^t), \ a \in \mathbb{C}^n \}\]
of complex valued functions is an eigenfamily on \(U(n)\).
Proof. Assume that $a, b \in \mathbb{C}^n$ and define $A = p^t a$ and $B = p^t b$. By construction any two columns of the matrices $A$ and $B$ are linearly dependent. This means that for all $1 \leq i, j, k, l \leq n$
\[ \det \begin{pmatrix} a_{ij} & b_{il} \\ a_{kj} & b_{kl} \end{pmatrix} = (a_{ij}b_{kl} - a_{kj}b_{il}) = 0 \]
so $P^2 + \kappa(P, P) = 0$, $PQ + \kappa(P, Q) = 0$ and $Q^2 + \kappa(Q, Q) = 0$. The statement now follows directly from Lemma 5.1. □

It should be noted that the local harmonic morphisms on the unitary group $U(n)$ that we obtain by employing Theorem 2.5 are invariant under the circle action and hence induce local harmonic morphisms on the special unitary group $SU(n)$.

**Example 5.3.** The 3-dimensional sphere $S^3$ is diffeomorphic to the special unitary group $SU(2)$ given by
\[ SU(2) = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \mid |z|^2 + |w|^2 = 1 \right\}. \]
For $p = (1, 0) \in \mathbb{C}^2$ we get the eigenfamily
\[ \mathcal{E}(p) = \{ \phi_a : U(n) \to \mathbb{C} \mid \phi_a(z) = a_1 z + a_2 w, \ a = (a_1, a_2) \in \mathbb{C}^n \}. \]
By choosing $a = (1, 0)$ and $b = (0, 1)$ and applying Theorem 2.5 we obtain the well known globally defined harmonic morphism
\[ \phi = \frac{\phi_a}{\phi_b} : SU(2) \to S^2 \]
called the Hopf map satisfying
\[ \phi\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = \frac{z}{w}. \]

6. THE RIEMANNIAN LIE GROUP $Sp(n)$

In this section we construct eigenfamilies of complex valued functions from the quaternionic unitary group $Sp(n)$ being the intersection of the unitary group $U(2n)$ and the standard representation of the quaternionic general linear group $GL_n(\mathbb{H})$ in $\mathbb{C}^{2n \times 2n}$ given by
\[ (z + jw) \mapsto q = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}. \]
The Lie algebra $sp(n)$ of $Sp(n)$ satisfies
\[ sp(n) = \left\{ \begin{pmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{pmatrix} \in \mathbb{C}^{2n \times 2n} \mid Z^* + Z = 0, \ W^t - W = 0 \right\}. \]
and for this we have the standard orthonormal basis which is the union of the following three sets
\[ \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mid 1 \leq r < s \leq n \right\}. \]
\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & iX_{rs} \\
n & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix}
\frac{1}{\sqrt{2}} & -X_{rs} & 0 \\
n & 0 & 0 \end{bmatrix} \mid 1 \leq r < s \leq n,
\]
\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & iD_t \\\nn & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix}
\frac{1}{\sqrt{2}} & iD_t & 0 \\
n & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & D_t \\\nn & 0 & 0 \end{bmatrix} \mid 1 \leq t \leq n.
\]

**Lemma 6.1.** For \(1 \leq i, j \leq n\) let \(z_{ij}, w_{ij} : \text{Sp}(n) \to \mathbb{C}\) be the complex valued coordinate functions given by
\[
z_{ij} : g \mapsto e^{i \cdot g \cdot e^t j}, \quad w_{ij} : g \mapsto e^{i \cdot g \cdot e^t n+j},
\]
where \(\{e_1, \ldots, e_{2n}\}\) is the canonical basis for \(\mathbb{C}^{2n}\). Then the following relations hold
\[
\tau(z_{ij}) = -\frac{2n+1}{2} \cdot z_{ij}, \quad \tau(w_{ij}) = -\frac{2n+1}{2} \cdot w_{ij},
\]
\[
\kappa(z_{ij}, z_{kl}) = -\frac{1}{2} \cdot z_{il}z_{kj}, \quad \kappa(w_{ij}, w_{kl}) = -\frac{1}{2} \cdot w_{il}w_{kj},
\]
\[
\kappa(z_{ij}, w_{kl}) = -\frac{1}{2} \left[ w_{il}z_{kj} - \delta_{jl} \cdot \sum_{t=1}^{n} (z_{it}w_{kt} - w_{it}z_{kt}) \right].
\]

**Proof.** The proof is similar to that of Lemma 4.1 but more involved. \(\square\)

**Theorem 6.2.** Let \(p\) be a non-zero element of \(\mathbb{C}^n\). Then the set
\[
\mathcal{E}(p) = \{\phi_{ab} : \text{Sp}(n) \to \mathbb{C} \mid \phi_{ab}(g) = \text{trace}(p^t az^t + p^t bw^t), \ a, b \in \mathbb{C}^n\}
\]
of complex valued functions is an eigenfamily on \(\text{Sp}(n)\).

**Proof.** Let \(a, b, c, d\) be arbitrary elements of \(\mathbb{C}^n\) and define the complex valued functions \(P, Q : \text{Sp}(n) \to \mathbb{C}\) by
\[
P = \text{trace}(p^t az^t + p^t bw^t) \quad \text{and} \quad Q = \text{trace}(p^t cz^t + p^t dw^t).
\]
Then a simple calculation shows that
\[
PQ + 2\kappa(P, Q) = [(a, d) - (b, c)] \sum_{i,k,t=1}^{n} (z_{it}w_{kt} - w_{it}z_{kt}) = 0.
\]
Automatically we also get \(P^2 + 2\kappa(P, P) = 0\) and \(Q^2 + 2\kappa(Q, Q) = 0\). \(\square\)

7. **The Duality**

In this section we show how a real analytic eigenfamily \(\mathcal{E}\) on a semi-Riemannian non-compact semi-simple Lie group \(G\) gives rise to a real-analytic eigenfamily \(\mathcal{E}^*\) on its Riemannian compact dual \(U\) and vice versa. The method of proof is borrowed from a related duality principle for harmonic morphisms from Riemannian symmetric spaces, see [8].

Let \(W\) be an open subset of \(G\) and \(\phi : W \to \mathbb{C}\) be a real analytic map. Let \(G^\mathbb{C}\) denote the complexification of the Lie group \(G\). Then \(\phi\) extends uniquely to a holomorphic map \(\phi^\mathbb{C} : W^\mathbb{C} \to \mathbb{C}\) from some open subset \(W^\mathbb{C}\).
of $G^\mathbb{C}$. By restricting this map to $U \cap W^\mathbb{C}$ we obtain a real analytic map $\phi^*: W^* \to \mathbb{C}$ from some open subset $W^*$ of $U$.

**Theorem 7.1.** Let $\mathcal{E}$ be a family of maps $\phi: W \to \mathbb{C}$ and $\mathcal{E}^*$ be the dual family consisting of the maps $\phi^*: W^* \to \mathbb{C}$ constructed as above. Then $\mathcal{E}^*$ is an eigenfamily if and only if $\mathcal{E}$ is an eigenfamily.

**Proof.** Let $g = k + p$ be a Cartan decomposition of the Lie algebra of $G$ where $k$ is the Lie algebra of a maximal compact subgroup $K$. Furthermore let the left-invariant vector fields $X_1, \ldots, X_n \in p$ form a global orthonormal frame for the distribution generated by $p$ and similarly $Y_1, \ldots, Y_m \in \mathfrak{k}$ form a global orthonormal frame for the distribution generated by $\mathfrak{k}$. We shall now assume that $\phi$ and $\psi$ are elements of the eigenfamily $\mathcal{E}$ on the semi-Riemannian $W$ i.e.

$$\tau(\phi) = -\sum_{k=1}^m Y^2_k(\phi) + \sum_{k=1}^n X^2_k(\phi) = \lambda \cdot \phi,$$

$$\kappa(\phi, \psi) = -\sum_{k=1}^m Y_k(\phi)Y_k(\psi) + \sum_{k=1}^n X_k(\phi)X_k(\psi) = \mu \cdot \phi \cdot \psi.$$

By construction and by the unique continuation property of real analytic functions the extension $\phi^\mathbb{C}$ of $\phi$ satisfies the same equations.

The Lie algebra of $U$ has the decomposition $u = k + i\mathfrak{p}$ and the left-invariant vector fields $iX_1, \ldots, iX_n \in i\mathfrak{p}$ form a global orthonormal frame for the distribution generated by $i\mathfrak{p}$. Then

$$\tau(\phi^*) = \sum_{k=1}^m Y^2_k(\phi^*) + \sum_{k=1}^n (iX_k)^2(\phi^*) = -\lambda \cdot \phi^*$$

$$\kappa(\phi^*, \phi^*) = \sum_{k=1}^m Y_k(\phi^*)Y_k(\psi^*) + \sum_{k=1}^n (iX_k)(\phi^*)(iX_k)(\psi^*) = -\mu \cdot \phi^* \cdot \psi^*.$$

This shows that $\mathcal{E}^*$ is an eigenfamily. The converse is similar. \hfill \Box

8. **The semi-Riemannian Lie group $GL_n(\mathbb{C})$**

Let $h$ be the standard left-invariant semi-Riemannian metric on the general linear group $GL_n(\mathbb{C})$ induced by the semi-Euclidean scalar product on the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ given by

$$h(Z, W) = \Re \text{trace } ZW.$$

Then we have the orthogonal decomposition

$$\mathfrak{gl}_n(\mathbb{C}) = \mathcal{W}_+ \oplus \mathcal{W}_-$$

of Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ where

$$\mathcal{W}_+ = \{ Z \in \mathfrak{gl}_n(\mathbb{C}) | Z - Z^* = 0 \}.$$
is the subspace of Hermitian matrices and
\[ W_+ = \{ Z \in \mathfrak{gl}_n(\mathbb{C}) \mid Z + Z^* = 0 \} \]
is the subspace of skew-Hermitian matrices. The scalar product is positive definite on \( W_+ \) and negative definite on \( W_- \). This means that for two complex valued functions \( \phi, \psi : U \to \mathbb{C} \) locally defined on \( \text{GL}_n(\mathbb{C}) \) the differential operators \( \tau \) and the \( \kappa \) satisfy
\[
\tau(\phi) = \sum_{Z \in B_+} Z^2(\phi) - \sum_{Z \in B_-} Z^2(\phi),
\]
\[
\kappa(\phi, \psi) = \sum_{Z \in B_+} Z(\phi)Z(\psi) - \sum_{Z \in B_-} Z(\phi)Z(\psi)
\]
where \( B_+ \) and \( B_- \) are orthonormal bases for \( W_+ \) and \( W_- \), respectively.

Employing the duality principle of Theorem 7.1 we can now easily construct harmonic morphisms from the non-compact semi-Riemannian Lie groups
\[
\text{SL}_n(\mathbb{R}), \text{SU}^*(2n), \text{Sp}(n, \mathbb{R}), \text{SO}^*(2n), \text{SO}(p, q), \text{SU}(p, q), \text{Sp}(p, q)
\]
via the following classical dualities \( G \cong U \):
\[
\text{SL}_n(\mathbb{R}) = \{ x \in \text{GL}_n(\mathbb{R}) \mid \det x = 1 \} \cong \text{SU}(n),
\]
\[
\text{SU}^*(2n) = \{ g = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \mid g \in \text{SL}_{2n}(\mathbb{C}) \} \cong \text{SU}(2n),
\]
\[
\text{Sp}(n, \mathbb{R}) = \{ g \in \text{SL}_{2n}(\mathbb{R}) \mid g \cdot J_n \cdot g^t = J_n \} \cong \text{Sp}(n),
\]
\[
\text{SO}^*(2n) = \{ z \in \text{SU}(n, n) \mid z \cdot I_{nn} \cdot J_n \cdot z^t = I_{nn} \cdot J_n \} \cong \text{SO}(2n),
\]
\[
\text{SO}(p, q) = \{ x \in \text{SL}_{p+q}(\mathbb{R}) \mid x \cdot I_{pq} \cdot x^t = I_{pq} \} \cong \text{SO}(p + q),
\]
\[
\text{SU}(p, q) = \{ z \in \text{SL}_{p+q}(\mathbb{C}) \mid z \cdot I_{pq} \cdot z^* = I_{pq} \} \cong \text{SU}(p + q),
\]
\[
\text{Sp}(p, q) = \{ g \in \text{GL}_{p+q}(\mathbb{H}) \mid g \cdot I_{pq} \cdot g^* = I_{pq} \} \cong \text{Sp}(p + q).
\]
Here we have used the standard notation
\[
I_{pq} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \quad \text{and} \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.
\]

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Appendix A.

In this appendix we prove the result stated in Theorem 2.5. It shows how the elements of an eigenfamily $E$ of complex valued functions on a semi-Riemannian manifold $(M, g)$ can be used to produce a variety of harmonic morphisms defined on open and dense subsets of $M$. The first result shows how the operators $\tau$ and $\kappa$ behave with respect to products.

**Lemma A.1.** Let $(M, g)$ be a semi-Riemannian manifold and $E_1, E_2$ be two families of complex valued functions on $M$. If there exist complex numbers $\lambda_1, \mu_1, \lambda_2, \mu_2, \mu \in \mathbb{C}$ such that for all $\phi_1, \phi_2 \in E_1$ and $\psi_1, \psi_2 \in E_2$

$\tau(\phi_1) = \lambda_1 \phi_1, \quad \kappa(\phi_1, \phi_2) = \mu_1 \phi_1 \phi_2,$

$\tau(\psi_1) = \lambda_2 \psi_1, \quad \kappa(\psi_1, \psi_2) = \mu_2 \psi_1 \psi_2,$

$\kappa(\phi_1, \psi_1) = \mu \phi_1 \psi_1$

then the following relations hold

$\tau(\phi_1 \psi_1) = (\lambda_1 + 2\mu + \lambda_2)\phi_1 \psi_1,$

$\kappa(\phi_1 \psi_1, \phi_2 \psi_2) = (\mu_1 + 2\mu + \mu_2)\phi_1 \psi_1 \phi_2 \psi_2$

for all $\phi_1, \phi_2 \in E_1$ and $\psi_1, \psi_2 \in E_2$.

**Proof.** The statement is an immediate consequence of the following basic facts concerning first and second order derivatives of products

$X(\phi_1 \psi_1) = X(\phi_1) \psi_1 + \phi_1 X(\psi_1),$

$X^2(\phi_1 \psi_1) = X^2(\phi_1) \psi_1 + 2X(\phi_1)X(\psi_1) + \phi_1 X^2(\psi_1).$

□

The following result shows how the operators $\tau$ and $\kappa$ behave with respect to quotients.

**Lemma A.2.** Let $(M, g)$ be a semi-Riemannian manifold and $P, Q : M \to \mathbb{C}$ be two complex valued functions on $M$. If there exists a complex number $\lambda \in \mathbb{C}$ such that $\tau(P) = \lambda P$ and $\tau(Q) = \lambda Q$

then the quotient $\phi = P/Q$ is a harmonic morphism if and only if

$Q^2 \kappa(P, P) = PQ \kappa(P, Q) = P^2 \kappa(Q, Q).$

**Proof.** For first and second order derivatives of the quotient $P/Q$ we have the following basic facts

$X(\phi) = \frac{X(P)Q - PX(Q)}{Q^2},$

$X^2(\phi) = \frac{Q^2 X^2(P) - 2QX(P)X(Q) + 2PX(Q)X(Q) - PQX^2(Q)}{Q^3}$

leading to the following formulae for $\tau(\phi)$ and $\kappa(\phi, \phi)$

$Q^3 \tau(\phi) = Q^2 \tau(P) - 2Q \kappa(P, Q) + 2P \kappa(Q, Q) - PQ \tau(Q),$
\[ Q^1 \kappa(\phi, \phi) = Q^2 \kappa(P, P) - 2PQ \kappa(P, Q) + P^2 \kappa(Q, Q). \]

The statement is a direct consequence of these relations. \qed

**Proof of Theorem 2.5.** For an eigenfamily \( \mathcal{E} = \{\phi_1, \ldots, \phi_n\} \) on the semi-Riemannian manifold \((M, g)\) we define the infinite sequence

\[ \{\mathcal{E}^k\}_{k=1}^\infty \]

by induction

\[ \mathcal{E}^1 = \mathcal{E} \quad \text{and} \quad \mathcal{E}^{k+1} = \mathcal{E}^1 \cdot \mathcal{E}^k = \{ \phi \cdot \psi | \phi \in \mathcal{E}^1, \psi \in \mathcal{E}^k \}. \]

It then follows from the fact that

\[ \kappa(\phi, \psi) = k \mu_1 \phi \psi \]

for all \( \phi \in \mathcal{E}^1 \) and \( \psi \in \mathcal{E}^k \) and Lemma A.1 that each \( \mathcal{E}^{k+1} \) is an eigenfamily on \( M \). With this at hand the statement of Theorem 2.5 is an immediate consequence of Lemma A.2. \qed

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