SCHOTTKY UNIFORMIZATION AND VECTOR BUNDLES OVER RIEMANN SURFACES

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Abstract. We study a natural map from representations of a free group of rank $g$ in $GL(n, \mathbb{C})$, to holomorphic vector bundles of degree 0 over a compact Riemann surface $X$ of genus $g$, associated with a Schottky uniformization of $X$. Maximally unstable flat bundles are shown to arise in this way. We give a necessary and sufficient condition for this map to be a submersion, when restricted to representations producing stable bundles. Using a generalized version of Riemann's bilinear relations, this condition is shown to be true on the subspace of unitary Schottky representations.

1. Introduction

Let $X$ be a compact Riemann surface of genus $g$ and $\mathcal{G}$ be the sheaf of germs of holomorphic functions from $X$ to $GL(n, \mathbb{C})$. The inclusion $GL(n, \mathbb{C}) \hookrightarrow \mathcal{G}$ (where $GL(n, \mathbb{C})$ is identified with its constant sheaf on $X$), defines a map:

$$\mathcal{V} : H^1(X, GL(n, \mathbb{C})) \to H^1(X, \mathcal{G}),$$

that sends a flat $GL(n, \mathbb{C})$-bundle into the corresponding holomorphic vector bundle of rank $n$ over $X$. Two such flat bundles are said to be analytically equivalent if they have the same image under $\mathcal{V}$. By a slight abuse of terminology, we will also say that a holomorphic vector bundle $E$ is flat if it lies in the image of $\mathcal{V}$; this happens if and only if every indecomposable component of $E$ has degree 0, by a classical theorem of Weil [W].

There is a well know bijection between the space of flat $GL(n, \mathbb{C})$-bundles over $X$ and the space of representations of the fundamental group of $X$, $\pi_1 = \pi_1(X)$, modulo overall conjugation $G_n := \text{Hom}(\pi_1, GL(n, \mathbb{C}))/GL(n, \mathbb{C})$; it is given explicitly by:

$$E : G_n \to H^1(X, GL(n, \mathbb{C})), \quad \rho \mapsto E_\rho := \tilde{X} \times_{\rho} \mathbb{C}^n,$$

where $\pi_1$ acts diagonally on the trivial rank $n$ vector bundle over the universal cover $\tilde{X}$ of $X$.

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The subset $\mathcal{M}_n^{st} \subset H^1(X, \mathcal{G})$ of stable holomorphic bundles of rank $n$ and degree $0$ is a non-singular quasi-projective algebraic variety of dimension $n^2(g - 1) + 1$, as shown by Mumford [Mu]. Therefore, if we restrict attention to the subset $G_n^{st} := \{\rho : E_\rho$ is stable} $\subset G_n$ which is a (smooth) complex manifold of twice the dimension of $\mathcal{M}_n^{st}$ (see [Gu]), we get a map:

\[
V : G_n^{st} \to \mathcal{M}_n^{st}, \quad \rho \mapsto V(\rho) := V(E_\rho)
\]

which is surjective, by Weil’s theorem (a stable bundle is indecomposable) and is holomorphic (see §3, below) because of the universal property of the (coarse) moduli space $\mathcal{M}_n^{st}$ ([NS1]).

The well known theorem of Narasimhan and Seshadri [NS2], implies that the restriction of $V$ to the subset $U_n^{st} = \{\rho \in \text{Hom}(\pi_1, U(n))/U(n) : \rho$ is irreducible} is a diffeomorphism.

Now let us fix a canonical basis of $\pi_1 = \pi_1(X)$: elements $a_1, ..., a_g, b_1, ..., b_g$ that generate $\pi_1$, subject to the single relation $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1$. Let $F_g$ be a free group on $g$ generators $B_1, ..., B_g$, and $p : \pi_1 \to F_g$ be the homomorphism given by $p(a_i) = 1, p(b_i) = B_i, i = 1, ..., g$. Then we can form the exact sequence of groups:

\[
1 \to N \to \pi_1 \xrightarrow{p} F_g \to 1,
\]

where $N$ is the smallest normal subgroup of $\pi_1$ containing $a_1, ..., a_g$. A Schottky uniformization, representing $X$ as a quotient of a domain of the Riemann sphere $\mathbb{P}^1$ by a free subgroup of $PSL(2, \mathbb{C})$, gives us a preferred representation $\sigma_X \in \text{Hom}(F_g, PSL(2, \mathbb{C}))$ which is unique up to conjugation (see Thm. [4]). By analogy, we will say that $\rho \in \text{Hom}(\pi_1, GL(n, \mathbb{C}))$ is a Schottky representation if it lies in the image of the inclusion $i := p^* : \text{Hom}(F_g, GL(n, \mathbb{C})) \hookrightarrow \text{Hom}(\pi_1, GL(n, \mathbb{C}))$, induced from sequence [4]). Similarly, a holomorphic vector bundle $E$ is called a Schottky bundle (of rank $n$) if it is in the image of the composition

\[
S_n := \text{Hom}(F_g, GL(n, \mathbb{C}))/GL(n, \mathbb{C}) \xrightarrow{i} G_n \xrightarrow{\rho \mapsto E} H^1(X, \mathcal{G}).
\]

In particular, a Schottky vector bundle is always flat, and therefore isomorphic to a direct sum of indecomposable vector bundles of degree 0. It is easy to see that all line bundles of degree 0 are Schottky and that $S_n^{st} := \{\rho : E_\rho$ is stable} $\subset S_n$ is a complex manifold of the same dimension as $\mathcal{M}_n^{st}$ (Prop. [4]).

In section 2, we give the first non-trivial examples of Schottky vector bundles. Let us say that an indecomposable vector bundle $E$ of rank 2 and degree 0 is maximally unstable, if it is given by a nontrivial extension $0 \to L \to E \to L^{-1} \to 0$, where $L$ is a square root of the canonical line bundle $K$. We will show that:
Theorem 1. Every maximally unstable indecomposable vector bundle of rank 2 and degree 0 is Schottky.

In section 3 we study the spaces of representations $G_{n}^{st}$ and $S_{n}^{st}$, and consider the restriction

$$W := V|_{S_{n}^{st}} : S_{n}^{st} \rightarrow M_{n}^{st}, \quad \rho \mapsto W(\rho) := V(i(\rho))$$

of $V$ to $S_{n}^{st}$ which is again holomorphic (Prop. 3). One can pose the following problem:

Problem 1. Is the above map $W$ surjective, or at least onto a dense open subset of $M_{n}^{st}$?

The answer seems to be unknown at present, to the best of our knowledge, except for the simplest cases: rank 1 or genus one\(^1\) where it is positive (see §6, Appendix). This problem, which can be called the Schottky uniformization for vector bundles, may be of interest for the theory of generalized theta functions, because we can describe them as holomorphic functions on $S_{n}^{st}$, by pulling back the determinant line bundle on the moduli space of semistable bundles using $W$ (see Beauville [B]). It can also be useful in describing the Kähler metric on the moduli space of stable bundles (see Takhtajan and Zograf [T1], [Z1]) by analogy with the fact that the Fuchsian and Schottky uniformizations of a compact Riemann surface are related through a potential for the Weil-Peterson Kähler metric on Teichmüller space ([Z1]).

In section 4, we consider the following period map:

$$P_{Ad_\rho} : H^0(X, \text{End}E_\rho \otimes K) \rightarrow H^1(\pi_1, Ad_\rho), \quad P_{Ad_\rho}(\phi) := [\gamma \mapsto \int_\gamma \phi]$$

where $\rho \in G_{n}$, $Ad_\rho$ denotes the $\pi_1$-module of $n \times n$ matrices $M$ with the action $\gamma \cdot M = \rho(\gamma)M\rho(\gamma)^{-1}$, $H^1(\pi_1, Ad_\rho)$ the first cohomology of $\pi_1$ with values in this module, and $K$ is the canonical line bundle on $X$. In terms of this period map, we get a necessary and sufficient condition for $W$ to be locally invertible.

Theorem 2. Let $X$ have genus $g \geq 2$. Then the differential of $W$ at a representation $\rho$, $dW_\rho : T_{\rho}S_{n}^{st} \rightarrow T_{E_\rho}M_{n}^{st}$ is invertible if and only if $H^1(F_g, Ad_\rho) \cap \text{Im}(P_{Ad_\rho}) = 0$.

In §5, we consider the case of representations that are both Schottky and unitary. This space is also important, since it is one of the Bohr-Sommerfeld orbits of the real polarization of the pre-quantum system of flat unitary connections on $X$ (see Tyurin [Ty]). We use a generalization of Riemann’s bilinear relations to prove:

\(^1\)in a slightly modified version, to take into account semistable bundles.
Theorem 3. If \( \rho \in S_n^{st} \cap U_n^{st} \) then \( dW_\rho \) is surjective. In particular, The image \( W(S_n^{st}) \) contains a nonempty open set \( U \subset M_n^{st} \) (in the complex topology) such that \( W(S_n^{st} \cap U_n^{st}) \subset U \).

In section 6, we study Schottky bundles in the easy case \( g = 1 \), and in the appendix, we consider the case of line bundles, which have been considered before from another viewpoint (compare [K], Ch. VI, §4).

2. Schottky uniformization and unstable vector bundles.

We begin by recalling the classical Schottky uniformization of compact Riemann surfaces. For details we refer to [C, AS, MS, BS1, BS2]. Schottky groups are an important class of Kleinian groups: groups of Möbius transformations acting properly discontinuously on some domain of the Riemann sphere \( \mathbb{P}^1 \). A marked Schottky group of genus \( g \) is a strictly loxodromic (this includes hyperbolic) finitely generated free Kleinian group \( \Sigma \) of rank \( g \) [MS], together with a choice of \( g \) generators \( T_1, \ldots, T_g \in PSL(2, \mathbb{C}) \) of \( \Sigma \). Two marked Schottky groups \( (\Sigma, T_1, \ldots, T_g) \) and \( (\Sigma', T'_1, \ldots, T'_g) \) are said to be equivalent if there exists a Möbius transformation \( M \) such that \( T'_i = MT_iM^{-1} \) for all \( i = 1, \ldots, g \). Thus, the set of equivalence classes of marked Schottky groups of genus \( g \) is a subset of \( \text{Hom}(F_g, PSL(2, \mathbb{C}))/PSL(2, \mathbb{C}) \) (where \( F_g \) is a free group on \( g \) generators, and \( PSL(2, \mathbb{C}) \) acts by overall conjugation), called Schottky space of genus \( g \). It is an open set in \( \mathbb{C}^{3g-3} \). Let us denote by \( \Omega_\Sigma \subset \mathbb{P}^1 \) the domain of discontinuity of \( \Sigma \) (for \( g > 1 \), its complement is a Cantor set [C, AS]).

A Schottky group \( \Sigma \) gives rise to a compact Riemann surface \( X := \Omega_\Sigma/\Sigma \). Every marked Schottky group \( (\Sigma, T_1, \ldots, T_g) \) has a (non unique) standard fundamental domain (see [C]), which is a region \( D \subset \mathbb{P}^1 \) bounded by smooth closed curves \( C_1, \ldots, C_g, C'_1, \ldots, C'_g \), each lying on the outside of all the others, such that \( T_i(C_i) = C'_i \). If we orient each \( C_i \) clockwise and each \( C'_i \) counterclockwise, the canonical holomorphic map \( \Omega_\Sigma \to X \), sends the boundary curves of \( D \) onto smooth non-intersecting simple oriented closed curves \( \alpha_1, \ldots, \alpha_g \) on \( X \).

In this way, we see that a marked Schottky group plus the choice of a standard fundamental domain determines a Riemann surface with a distinguished set of curves \( \{\alpha_1, \ldots, \alpha_g\} \). Conversely, the classical retrosection theorem of Koebe (see [BS1, AS]), states that every compact Riemann surface arises this way: \textit{For every compact Riemann surface of genus } \( g \) \textit{with a choice of } \( g \) \textit{smooth simple non-intersecting, homologically independent, oriented closed curves } \( \alpha_1, \ldots, \alpha_g \), \textit{there exists a marked}
Schottky group of genus $g$, $(\Sigma, T_1, ..., T_g)$ and a fundamental domain for $\Sigma$ with $2g$ boundary curves $C_1, ..., C_g, C'_1, ..., C'_g$, such that $X = \Omega_{\Sigma}/\Sigma$ and the map $\Omega_{\Sigma} \rightarrow X$ sends both $C_i$ and $C'_i$ to $\alpha_i$, preserving orientations as above. The marked Schottky group $(\Sigma, T_1, ..., T_g)$ satisfying these conditions is uniquely determined by $(X, \alpha_i)$ up to equivalence.

Recalling that a choice of canonical basis for $\pi_1$ induces a sequence:

\[ 1 \rightarrow N \rightarrow \pi_1 \xrightarrow{\rho} F_g \rightarrow 1, \]

this classical theorem can be restated in the following way. To avoid heavier notation, we will denote representations and their equivalence classes under conjugation by the same symbols; hopefully, this should cause no confusion.

**Theorem 4.** Let $X$ be a Riemann surface of genus $g \geq 1$ with a canonical basis for $\pi_1$. Then there is a unique (up to conjugation) representation $\sigma_X \in \text{Hom}(F_g, \text{PSL}(2, \mathbb{C}))$ such that $X = \Omega_{\Sigma}/\Sigma$ where $\Sigma = \text{Im}(\sigma_X)$.

**Definition 1.** A representation $\rho$ of $\pi_1$ in a group $G$ will be called a Schottky representation, if it lies in the image of the inclusion $i : \text{Hom}(F_g, G) \hookrightarrow \text{Hom}(\pi_1, G)$. A holomorphic vector bundle $E$ over $X$ is called a Schottky vector bundle over $X$ if $E = \mathcal{V} \circ E_\rho$, for some Schottky representation $\rho \in \text{Hom}(\pi_1, \text{GL}(n, \mathbb{C}))$ (recall the maps (1, 2)).

It is clear that $\rho$ is Schottky if and only if $\rho(a_i) = 1$, for all $i = 1, ..., g$. Another easy consequence of the definition is that the tensor product and direct sum of Schottky vector bundles is also Schottky. Later, we will see that every line bundle of degree 0 is Schottky (see §A), so the property of being Schottky is preserved under tensoring by a line bundle of degree 0.

We now give some interesting examples\footnote{I thank I. Biswas for suggesting to consider these examples.} of Schottky vector bundles of rank 2, which are not semistable for $g \geq 2$. Let $E$ be an extension $0 \rightarrow L \rightarrow E \rightarrow L^{-1} \rightarrow 0$, where $L$ is a line bundle of degree $g - 1$. For $E$ to be indecomposable, the space that classifies these extensions $H^1(X, L^2) = H^0(X, KL^{-2})^*$, has to be non zero, so $L$ is one of the $2^{2g}$ square roots of the canonical bundle $K$. In this case $E$ is unique, and called a maximally unstable indecomposable vector bundle of rank 2 and degree 0, and denoted here by $E_L$. We now prove that these bundles are Schottky.

**Proof of theorem 4.** Let $L$ be a square root of $K$ (i.e., $L^2 = K$). First, note that $E_{L \otimes L_0} = E_L \otimes L_0$, for any line bundle $L_0$ such that $L_0^2 = \mathcal{O}$; moreover if
$E_L$ is Schottky, so is $E_L \otimes L_0$, by the remarks after definition \[. Therefore, if the result is true for some square root of $K$, then it is also true for any other square root. If $g = 1$, and $\mathbb{I}$ is the trivial line bundle, $E_{\mathbb{I}}$ is Atiyah’s bundle $F_2$ (see section \[), so the result follows from lemma \[ below. So, assuming $g \geq 2$, consider the following diagram, induced by the quotient homomorphism $\nu : SL(2, \mathbb{C}) \to PSL(2, \mathbb{C})$, whose commutativity is easily established, and whose vertical sequences are principal fibrations:

\[
\begin{array}{ccc}
\text{Hom}(F_g, \mathbb{Z}_2) & \xrightarrow{i} & \text{Hom}(\pi_1, \mathbb{Z}_2) \\
\downarrow & & \downarrow \\
\text{Hom}(F_g, SL(2, \mathbb{C})) & \xrightarrow{i} & \text{Hom}(\pi_1, SL(2, \mathbb{C})) \\
\downarrow \nu & & \downarrow \nu \\
\text{Hom}(F_g, PSL(2, \mathbb{C})) & \xrightarrow{i} & \text{Hom}(\pi_1, PSL(2, \mathbb{C}))^+ \\
& & \downarrow \mu \\
& & H^1(X, PSL(2, \mathbb{C}))^+
\end{array}
\]

Here $\text{Hom}(\pi_1, PSL(2, \mathbb{C}))^+$ denotes the connected component which contains the image of the inclusion $i : \text{Hom}(F_g, PSL(2, \mathbb{C})) \hookrightarrow \text{Hom}(\pi_1, PSL(2, \mathbb{C}))$, and similarly for $H^1(X, PSL(2, \mathbb{C}))^+$. Using the techniques of Gunning (\[), we see that the marked Schottky group $\sigma_X \in \text{Hom}(F_g, PSL(2, \mathbb{C}))$, of theorem \[ gives rise to a flat projective Schottky bundle $P_{i(\sigma_X)} \in H^1(X, PSL(2, \mathbb{C}))$, which by construction is in the image of the natural map from the space of all projective structures on $X$ to $H^1(X, PSL(2, \mathbb{C}))$. Then, By \[, th. 2, there is a vector bundle $E_1$ such that $\mu(E_1) = P_{i(\sigma_X)}$ and $E_1$ has divisor order $g - 1$, so that it can be given as an extension $0 \to L_1 \to E_1 \to L_1^{-1} \to 0$, with $\text{deg}(L_1) = g - 1$. $E_1$ is indecomposable because of Weil’s theorem, thus $L_1^2 = K$, and by uniqueness, $E_1 = E_{L_1}$. On the other hand, if $\rho \in \text{Hom}(F_g, SL(2, \mathbb{C}))$ is such that $\nu(\rho) = \sigma_X$, then $E_{i(\rho)}$ is Schottky by definition, and the commutativity of the diagram implies $\mu(E_{i(\rho)}) = P_{i(\sigma_X)} = \mu(E_1)$. Thus, by exactness, $E_{L_1}$ and $E_{i(\rho)}$, differ by tensoring with a $\mathbb{Z}_2$-bundle $L_2 \in H^1(X, \mathbb{Z}_2)$ ($L_2^2 = 0$), so $E_{L_1}$ is Schottky as required.

Considering vector bundles of the form $E_L \oplus \mathbb{I}^n$, we get Schottky vector bundles which are clearly not semistable, for any $g \geq 2$ and any rank $\geq 2$.

3. Spaces of Schottky representations.

To study the maps $V, W$ of (\[, \[, we start by studying the spaces of representations $G_n$ and $S_n$. Let $\text{Hom}(\pi_1, GL(n, \mathbb{C}))^o$ and $\text{Hom}(F_g, GL(n, \mathbb{C}))^o$ denote the subsets consisting of representations $\rho$ having only scalar commutants (i.e, if a matrix $M$ commutes with all matrices in the image of $\rho$, then $M$ is a scalar) and let

\[
G_n^o := \text{Hom}(\pi_1, GL(n, \mathbb{C}))^o / GL(n, \mathbb{C}) \quad S_n^o := \text{Hom}(F_g, GL(n, \mathbb{C}))^o / GL(n, \mathbb{C})
\]
be their quotients under simultaneous conjugation. We remark that, in terms of 
the initial map $\mathcal{V}$, this constitutes no restriction, since by [Gu, Gu2] every flat 
holomorphic bundle admits a representation with only scalar commutants. Their 
tangent spaces can be given in terms of cohomology of groups with coefficients in 
group modules. Denote by $H^k(\Gamma, M)$ the $k$-th cohomology group of $\Gamma$ with coefficients in 
the $\Gamma$-module $M$. The adjoint representation, together with a representa-
$\mathcal{V}$ion $\rho$ of $\pi_1$ (resp. $F_g$) in $GL(n, \mathbb{C})$, endows the space of all $n \times n$ matrices, 

with a $\pi_1$- (resp. $F_g$-) module structure $(\gamma \cdot M := \rho(\gamma)M\rho(\gamma)^{-1}$ for a matrix $M$), 
denoted by $Ad\rho$. When $\rho$ has only scalar commutants, it is not difficult to see that 
dimension $H^1(F_g, Ad\rho) = n^2(g-1)+1$, which is also the dimension of $M_n^{st}$. Note that the 
inclusion $Z^1(F_g, Ad\rho) \hookrightarrow Z^1(\pi_1, Ad\rho) : (B_1, ..., B_g) \mapsto (0, ..., 0, B_1, ..., B_g)$ induces an 
inclusion $H^1(F_g, Ad\rho) \hookrightarrow H^1(\pi_1, Ad\rho)$, which is complex linear.

In [Gu, Gu2] it is shown that $G_n^o$ has the structure of a complex analytic manifold 
of dimension $2(n^2(g-1)+1)$ such that the natural projection $Hom(\pi_1, GL(n, \mathbb{C}))^o \to \ G_n^o$ is a complex analytic principal $PGL(n, \mathbb{C})$-bundle, and whose tangent space at the 
equivalence class of $\rho$ is $H^1(\pi_1, Ad\rho)$. $G_n^o$ has also the structure of a complex 
symplectic manifold $\mathfrak{g}$ whose symplectic form $\omega$ is defined by cup product, 
an invariant bilinear pairing $B$ on the Lie algebra of $GL(n, \mathbb{C})$, and evaluation on the 
fundamental homology class $c \in H_2(\pi_1, \mathbb{C})$, $\omega : H^1(\pi_1, Ad\rho) \times H^1(\pi_1, Ad\rho) \to H^2(\pi_1, Ad\rho \otimes Ad\rho) \xrightarrow{B} H^2(\pi_1, \mathbb{C}) \to \mathbb{C}$ (see [Gd]). For the case of $G_n^o$ we find:

**Proposition 1.** $G_n^o$ is a complex analytic manifold of dimension $n^2(g-1)+1$ whose 
tangent space at the equivalence class of the representation $\rho$ is $H^1(F_g, Ad\rho)$. Moreover, $G_n^o$ is a Lagrangian submanifold of $G_n^o$.

**Proof.** Since $\rho \in Hom(F_g, GL(n, \mathbb{C}))$ has only scalar commutants, the action 
of $PGL(n, \mathbb{C})$ by conjugation is free and given $\rho, \rho' \in Hom(F_g, GL(n, \mathbb{C}))^o$ in the 
same orbit, there is a unique $T \in PGL(n, \mathbb{C})$ such that $\rho' = T\rho T^{-1}$. The same 
arguments used in [Gu], §9, prove that $S_n^o = Hom(F_g, GL(n, \mathbb{C}))^o/PGL(n, \mathbb{C})$ (the center acts trivially) is a complex manifold, and that the tangent spaces agree with the 
required cohomology groups. Moreover, $S_n^o$ is an analytic submanifold of $G_n^o$, since the inclusion 
$H^1(F_g, Ad\rho) \hookrightarrow H^1(\pi_1, Ad\rho)$ is complex linear when $\rho$ is Schottky. 
Since $F_g$ is a free group, $H^2(F_g, M) = 0$, for any $F_g$-module $M$. Therefore, the 
symplectic form vanishes on any two tangent vectors to $S_n^o$. 

Observe that $U_n^o = Hom(\pi_1, U(n))^o/U(n)$ is also a subset of $G_n^o$, diffeomorphic 
to $M_n^{st}$ by the theorem of Narasimhan-Seshadri (since a unitary representation with 

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$^3G_n^o$ has also the structure of a hyperKähler manifold but we will not need this extra structure.
only scalar commutants is irreducible). When \( \rho \) is a unitary representation of \( \pi_1 \), the Lie algebra of \( U(n) \) is again a \( \pi_1 \)-module (denoted \( \text{Ad}_\rho^R \)) via the adjoint representation composed with \( \rho \). The tangent space of \( U_n^o \) at the equivalence class of the representation \( \rho \) can be identified with \( H^1(\pi_1, \text{Ad}_\rho^R) \), but the inclusion \( H^1(\pi_1, \text{Ad}_\rho^R) \hookrightarrow H^1(\pi_1, \text{Ad}_\rho) \) is not complex linear. Thus, \( U_n^o \) sits inside \( G_n^o \) as a real analytic submanifold but not complex analytic, in contrast to \( S_n^o \).

**Lemma 1.** For every \( n \geq 1 \), \( g \geq 2 \), there are unitary, Schottky irreducible representations of \( \pi_1(X) \) in \( GL(n, \mathbb{C}) \). Therefore, there are stable Schottky vector bundles of any given rank.

**Proof.** We may suppose that \( n \geq 2 \) (see the appendix). Let \( \lambda_1, ..., \lambda_n \) be \( n \) distinct complex numbers of modulus 1. Let \( B_1 = \text{diag}(\lambda_1, ..., \lambda_n) \), and \( B_2 \) be the permutation matrix \( e_1 \mapsto e_2, ..., e_n \mapsto e_1 \), for a canonical basis \( e_1, ..., e_n \in \mathbb{C}^n \). It is easy to see that \( B_1, B_2 \) form an irreducible set of unitary matrices (i.e, no subspace of \( \mathbb{C}^n \) is preserved by both). Hence, the representation of \( \pi_1 \) given by \( \rho(a_i) = 1 \), \( i = 1, ..., g \); \( \rho(b_i) = B_1 \), \( i = 1, 2 \); \( \rho(b_i) = 1 \), \( i = 3, ..., g \), is unitary, Schottky and irreducible. Stability is clear from Narasimhan - Seshadri’s theorem.

Consider now the subsets \( S_n^{st} \) and \( G_n^{st} \) of \( S_n^o \) and \( G_n^o \), respectively, consisting of conjugacy classes of representations \( \rho \) such that \( E_\rho \) is stable (since a stable bundle is simple, \( \rho \) has only scalar commutants), and let \( \mathcal{E} \) be the holomorphic family of vector bundles over \( X \) parametrized by the complex manifold \( G_n^o \), whose fiber over \( X \times \{ \rho \} \) is \( E_\rho \). It is given by the following \( \pi_1 \)-action on \( G_n^o \times \tilde{X} \times \mathbb{C}^n \):

\[
\gamma \cdot (\rho, z, v) = (\rho, \gamma \cdot z, \rho(\gamma)v) \quad \forall \gamma \in \pi_1, (z, v) \in \tilde{X} \times \mathbb{C}^n.
\]

**Proposition 2.** Let \( X \) be a compact Riemann surface of genus \( g \geq 2 \), and let \( n \geq 1 \). \( S_n^{st} \) and \( G_n^{st} \) are dense open complex submanifolds of \( S_n^o \) and \( G_n^o \), respectively.

**Proof.** In any holomorphic parameter space, stable bundles form the complement of an analytic subset [NS2], Thm. 2(B). Since \( G_n^{st} \) is non-empty (irreducible unitary representations give stable bundles, for \( g \geq 2 \)), the existence of the family \( \mathcal{E} \) implies that \( G_n^{st} \) is open and dense in \( G_n^o \). The same applies to \( S_n^{st} \), since stable Schottky bundles exist, by lemma [I].

**Proposition 3.** \( V : G_n^{st} \to M_n^{st} \) and \( W : S_n^{st} \to M_n^{st} \) are holomorphic maps.

**Proof.** This follows from [NS1], §2, where it is shown that the space of isomorphism classes of simple vector bundles with degree 0 and rank \( n \), \( M_n^{sim} \), verifies the universal property of a coarse moduli space in the holomorphic category.
In other words, for every holomorphic family $\mathcal{A}$ of simple vector bundles over $X$, parametrized by a complex manifold $S$, the “universal map” $S \to \mathcal{M}_n$ sending $s \in S$ to the equivalence class of $\mathcal{A}_s$ is holomorphic. Since our maps $V$ and $W$ are actually the “universal maps” for the holomorphic families $\mathcal{E}|_{\mathcal{G}_n}$ and $\mathcal{E}|_{\mathcal{S}_n}$ constructed above (consisting of stable, hence simple, bundles), and $\mathcal{M}_n$ is an open subset of $\mathcal{M}_n^{\text{sim}}$, we have the coarse moduli space universal property for $\mathcal{M}_n^{\text{st}}$ as well. \hfill \Box

We end this section by observing that $\mathcal{S}_n^0$ can be viewed as a natural generalization of Schottky space considered in [C]. Recall the canonical map $\nu : \text{Hom}(F_2, \text{SL}(2, \mathbb{C})) \to \text{Hom}(F_2, \text{PSL}(2, \mathbb{C}))$.

**Proposition 4.** Schottky space of genus $g \geq 2$ is contained in $\nu(\mathcal{S}_n^0)$. More concretely, if the image of a representation $\sigma : F_n \to \text{PSL}(2, \mathbb{C})$ is a Schottky group $\Sigma'$ of genus $g$, then $\nu^{-1}(\sigma)$ has only scalar commutants.

**Proof.** By definition of Schottky group, every $T \in \Sigma' = \text{Im}(\sigma)$ is loxodromic, and any two generators of $\Sigma'$ have distinct fixed points. Since two non trivial Möbius transformations commute if and only if they have the same fixed points, only the identity commutes with all elements in $\Sigma'$. Therefore, every representation $\rho \in \nu^{-1}(\sigma)$ has only scalar commutants. \hfill \Box

4. The period map.

To compute the differential of $W$, we now consider a certain period map, and prove theorem [A]. Given a representation $\sigma : \pi_1 \to \text{GL}(n, \mathbb{C})$, a global holomorphic section of $E_\sigma \otimes K$ will be called an $E_\sigma$-differential. In terms of a local coordinate $z \in \hat{X}$, an $E_\sigma$-differential can be viewed as a closed, holomorphic differential 1-form $\phi = \phi(z)dz$, satisfying $\phi(\gamma z)\gamma'(z) = \sigma(\gamma)\phi(z)$, for all $\gamma \in \pi_1$. Let $V_\sigma$ denote the (left) $\pi_1$-module defined on $\mathbb{C}^n$ by the representation $\sigma$. For a fixed basepoint $z_0 \in \hat{X}$ and fixed $\phi$, simple computations show that the map $\Phi_{z_0} : \pi_1 \to V_\sigma$ defined by $\gamma \mapsto \int_{z_0}^{\gamma z_0} \phi$, is a cocycle in $Z^1(\pi_1, V_\sigma)$, (since $\Phi_{z_0}(\gamma_1 \gamma_2) = \Phi_{z_0}(\gamma_1) + \gamma_1 \cdot \Phi_{z_0}(\gamma_2)$) whose equivalence class $[\Phi_{z_0}] \in H^1(\pi_1, V_\sigma)$ does not depend on the basepoint $z_0$ (because $\Phi_{z_1}(\gamma) = \int_{z_1}^{\gamma z_1} \phi = (\sigma(\gamma) - 1)\int_{z_0}^{\gamma z_0} \phi + \Phi_{z_0}(\gamma)$).

**Definition 2.** Given a representation $\sigma$, the map:

$$P_\sigma : H^0(X, E_\sigma \otimes K) \to H^1(\pi_1, V_\sigma)$$

defined by $P_\sigma(\phi) = [\Phi_{z_0}]$ is called the period map associated to $\sigma$.

If $\rho_1, \rho_2 \in \text{Hom}(\pi_1, \text{GL}(n, \mathbb{C}))$, one can ask when are $E_{\rho_1}$ and $E_{\rho_2}$ analytically equivalent. Let $z_0 \in \hat{X}$, $p : \hat{X} \to X$ be the universal cover, and denote by $f_\phi$ the
unique solution of the differential equation $f^{-1}df = p^*\phi$, $f(z_0) = I$, for a given $\phi \in H^0(X, \text{End} E_{\rho_1} \otimes K)$.

**Lemma 2.** The following are equivalent: (1) $E_{\rho_1} \cong E_{\rho_2}$, (2) there is a holomorphic function $f : \tilde{X} \to GL(n, \mathbb{C})$ such that $f(\gamma z) = \rho_2(\gamma)f(z)\rho_1(\gamma)^{-1}$ for all $\gamma \in \pi_1, z \in \tilde{X}$, and (3) There exists $\omega \in H^0(X, \text{End} E_{\rho_1} \otimes K)$ and $C \in GL(n, \mathbb{C})$ such that $\rho_2(\gamma) = Cf_\omega(\gamma z)\rho_1(\gamma)f_\omega(z)^{-1}C^{-1}$, for all $\gamma \in \pi_1, z \in \tilde{X}$.

**Proof.** To prove (1) $\iff$ (2), note that an isomorphism between $E_{\rho_1}$ and $E_{\rho_2}$ is a holomorphic global section of $E_{\rho_1}^* \otimes E_{\rho_2} = E_{\rho_1}^{-1} \otimes E_{\rho_2}$, consisting of invertible matrices. So it corresponds to a holomorphic $f : \tilde{X} \to GL(n, \mathbb{C})$ such that $f(\gamma \cdot z) = (\rho_1^{-1} \otimes \rho_2)(\gamma)f(z) = \rho_2(\gamma)f(z)\rho_1(\gamma)^{-1}$.

To prove (2) $\iff$ (3), let $f$ be as in (2) and put $h(z) := C^{-1}f(z)$ where $C := f(z_0)^{-1}$. We obtain $h(z_0) = I$, $\rho_2(\gamma) = Ch(\gamma z)\rho_1(\gamma)h(z)^{-1}C^{-1}$ and $\omega := h^{-1}dh$ belongs to $H^0(X, \text{End} E_{\rho_1} \otimes K)$, since $d(C^{-1}\rho_2(\gamma)C) = 0$ is equivalent to $((h^{-1}dh) \circ \gamma)' = \rho_1(\gamma)(h^{-1}dh)\rho_1(\gamma)^{-1}$. Conversely, if we have (3), then clearly $f = Ch$ verifies (2).

\[\square\]

For a fixed $\rho \in \text{Hom}(\pi_1, GL(n, \mathbb{C}))$ one can define the following map

$$Q_\rho : H^0(X, \text{End} E_{\rho} \otimes K) \to G_n,$$

$$\omega \mapsto [f_\omega(\gamma z_0)\rho(\gamma)]$$

which, because of the previous lemma, does not depend on $z_0$, or on the choice of representative $\rho$ in its conjugacy class. In fact, using this map, lemma 2 is easily seen to be equivalent to

**Lemma 3.** Two bundles $E_{\rho_1}$ and $E_{\rho_2}$ are analytically equivalent if and only if there is $\omega \in H^0(\text{End} E_{\rho_1} \otimes K)$ such that $Q_{\rho_1}(\omega) = \rho_2$.

If we now restrict to representations $\rho$ with only scalar commutants, then $Q_\rho$ becomes a holomorphic map, due to the analytic dependence of $f_\phi$ on $\phi$.

**Lemma 4.** (a) The differential of $Q_\rho$ at the origin, $d(Q_\rho)_0$, coincides with $P_{\text{Ad}_\rho}$.

(b) For any $\rho \in G_n^{st}$, $Ker \text{d}V_\rho = \text{Im} \text{d}(Q_\rho)_0$.

**Proof.** (a) First note that both maps $d(Q_\rho)_0$ and $P_{\text{Ad}_\rho}$ are defined from $H^0(X, \text{End} E_{\rho} \otimes K)$ to $H^1(\pi_1, \text{Ad}_\rho)$. Let $\eta \in H^0(\text{End} E_{\rho} \otimes K)$ and $t \in \mathbb{C}$. For small $t$, we can expand $f_t\eta$ as: $f_t\eta(z) = I + t \int_{z_0}^z \eta + O(t^2)$. Let $\rho_t$ denote $Q_\rho(t\eta)$, for brevity. Discarding second order terms, we find:

$$\rho_t(\gamma) = f_{t\eta}(\gamma z_0)\rho(\gamma) = \rho(\gamma) + t\int_{z_0}^\gamma \eta(\gamma + t\eta(\gamma)) + O(t^2).$$
The derivative of the curve of representations \( \rho_t \) is given by \( \dot{\rho}_t \rho_t^{-1} \), so the differential at \( \phi = 0 \) in the \( \eta \) direction is finally given by:

\[
[d(Q_\rho)_0(\eta)](\gamma) = \dot{\rho}_t \rho_t^{-1} |_{t=0} (\gamma) = \lim_{t \to 0} \frac{Q_\rho(t\eta)(\gamma) - \rho(\gamma)}{t} \rho(\gamma)^{-1} = \int_{\gamma^z} \eta
\]

(b) For any \( \eta \in H^0(X, \text{End} E_\rho \otimes K) \) and \( t \in \mathbb{C} \), we have \( V(Q_\rho(t\eta)) \cong V(Q_\rho(0)) \) (by lemma \( \exists \)). Letting \( t \to 0 \), we see that the differential of \( V \circ Q_\rho \), at the origin is zero:

\[
d(V \circ Q_\rho)_0(\eta) = dV \circ d(Q_\rho)_0(\eta) = 0.
\]

Hence \( \text{Im}(Q_\rho)_0 \subset \text{Ker} dV_\rho \). Conversely, if \( \phi \in \text{Ker} dV_\rho \subset H^1(\pi_1, \text{Ad}_\rho) \), then \( \phi \) is tangent to the fiber of the map \( V \) at \( \rho \), which means tangent to the image of \( Q_\rho \) at 0, so that there is an \( \eta \in H^0(X, \text{End} E_\rho \otimes K) \), such that \( \phi = d(Q_\rho)_0(\eta) \).

\[\square\]

Proof of theorem \( \exists \): Since \( S^*_n \) and \( M^*_n \) have the same dimension, \( n^2(g-1)+1 \) (prop. \( \emptyset \)), we just need to show that \( dW_\rho \) has trivial kernel. Since \( W \) is the composition \( S^*_n \xrightarrow{i} G^*_n \xrightarrow{V} M^*_n \), the differential at \( \rho \) is the composition: \( H^1(F_g, \text{Ad}_\rho) \xrightarrow{d_1} H^1(\pi_1, \text{Ad}_\rho) \xrightarrow{dV_\rho} T_{E_\rho} M^*_n \), and so, (by lemma \( \exists \)):

\[\text{Ker} dW_\rho = H^1(F_g, \text{Ad}_\rho) \cap \text{Ker} dV_\rho = H^1(F_g, \text{Ad}_\rho) \cap \text{Im}(P_{Ad_\rho}).\]

\[\square\]

5. Unitary Schottky vector bundles.

In this section we compute, for unitary Schottky bundles, the period map using a generalized version of Riemann’s bilinear relations, and prove theorem \( \exists \). If \( \sigma \) is a unitary representation of \( \pi_1 \) in a vector space, there is a hermitian inner product \( \langle , \rangle \), which is invariant under the \( \pi_1 \) action: \( \langle \gamma \cdot v_1, \gamma \cdot v_2 \rangle = \langle v_1, v_2 \rangle \) for all \( \gamma \in \pi_1 \) and vectors \( v_1, v_2 \) and so, we can define a “global” hermitian inner product on the space of \( E_\sigma \)- differentials, as follows:

\[
(\phi_1, \phi_2) := i \int_D \langle h_1(z), h_2(z) \rangle \ dz \wedge \overline{dz},
\]

where \( D \in \mathcal{X} \) is a fundamental domain for \( X = \mathcal{X}/\pi_1 \), and \( \phi_i = h_i(z) dz \) for \( z \in \mathcal{X} \). (positive definiteness follows from \( \frac{i}{2} dz \wedge \overline{dz} = dx \wedge dy \)). From the Hermitian pairing \( \langle , \rangle \), we can also form a pairing in \( H^1(\pi_1, \mathcal{V}_\sigma) \), as follows. First allow our cocycles \( \Phi \in H^1(\pi_1, \mathcal{V}_\sigma) \) to be defined by linearity on the group ring \( \mathbb{Z}[\pi_1] \). Let \( R_k = \prod_{j=1}^k a_j b_j a_j^{-1} b_j^{-1} \) \((k = 1, ..., g)\), and put \( R = R_g \). Using the notation of the Fox calculus, we set:

\[
\frac{\partial R}{\partial a_k} = R_{k-1} - R_k b_k, \quad \frac{\partial R}{\partial b_k} = R_{k-1} a_k - R_k.
\]

There is a natural involution \( \# \) in \( \mathbb{Z}[\pi_1] \) given by \( \# : \sum n_j \gamma_j \mapsto \sum n_j \gamma_j^{-1} \), so that, for example

\[
\# \frac{\partial R}{\partial a_k} = R_{k-1}^{-1} - b_k^{-1} R_k^{-1}, \quad \# \frac{\partial R}{\partial b_k} = a_k^{-1} R_{k-1}^{-1} - R_k^{-1}.
\]
In this notation, the fundamental 2-cycle \( c \in H_2(\pi_1, \mathbb{Z}) \), corresponding to \([X]\), under the isomorphism \( H_2(X, \mathbb{Z}) \cong H_2(\pi_1, \mathbb{Z}) \), is given by \( c := \sum_{k=1}^{g} \left[ \frac{\partial \nu}{\partial a_k} \right] \), where the bar notation \([a/b] \ (a, b \in \pi_1) \) denotes the equivalence class of the 2-cycle \((1, a, ab)\) in group homology (see [Br], ch. II). Finally, for \( \Phi, \Psi \in Z^1(\pi_1, \mathbb{V}_{\sigma}) \) define:

\[
\Phi \cup \Psi := \sum_{k=1}^{g} \left( -\langle \Phi(\# \frac{\partial \nu}{\partial a_k}), \Psi(a_k) \rangle + \langle \Phi(\# \frac{\partial \nu}{\partial b_k}), \Psi(b_k) \rangle \right)
\]

This pairing is well defined in cohomology (it depends only on the cohomology classes \([\Phi], [\Psi] \in H^1(\pi_1, \mathbb{V}_{\sigma})\)) and, by an easy calculation, it is the composition of the cup product, followed by contraction in \( \mathbb{V}_{\sigma} \), using \( \langle \, , \rangle \), and by evaluation on the fundamental 2-cycle \( c \in H_2(\pi_1, \mathbb{Z}) \) (compare [Ga], [St]):

\[
\Psi : H^1(\pi_1, \mathbb{V}_{\sigma}) \times H^1(\pi_1, \mathbb{V}_{\sigma}) \xrightarrow{\langle \, , \rangle} H^2(\pi_1, \mathbb{V}_{\sigma} \otimes \mathbb{V}_{\sigma}) \xrightarrow{\langle \, , \rangle} H^2(\pi_1, \mathbb{C}) \xrightarrow{\cdot} \mathbb{C}.
\]

**Proposition 5.** (Bilinear relations for \( E_{\sigma} \)-differentials).

Let \( (\mathbb{V}_{\sigma}, \langle \, , \rangle) \) be a unitary representation. Then, for all \( \phi, \psi \in H^0(X, E_{\sigma} \otimes K) \), we have:

\[
\langle \phi, \psi \rangle = i \{ P_{\sigma}(\phi) \cup P_{\sigma}(\psi) \}.
\]

**Proof.** Let \( \Phi(\gamma) = \int_{z_0}^{z_0} \phi \) be a cocycle representative of \( P_{\sigma}(\phi) \) and similarly for \( \psi \). Let \( f(z) := \int_{z_0}^{z} \phi \), so that \( \phi = df \) and \( f(\gamma z) = \gamma \cdot f(z) + \Phi(\gamma) \), for all \( \gamma \in \pi_1 \). By Stokes’ theorem, we have:

\[
\langle \phi, \psi \rangle = i \int_{\partial D} \langle f(z, \psi(z))dz = i \left( \sum_{k=1}^{4g} \int_{\gamma_k} \langle f(z), \psi(z) \rangle dz \right),
\]

where the curves \( \gamma_k \) are the 4g sides of the boundary of the polygon \( D \subset \hat{X} \), whose vertices can be ordered as \( \{ z_0, a_1z_0, a_1b_1z_0, a_1b_1a_1^{-1}z_0 \equiv R_1b_1z_0, R_1z_0, \ldots, R_gz_0 \equiv z_0 \} \).

Half of the 4g sides give (using the notation \( f^\gamma = f \circ \gamma \)):

\[
\int_{R_k^{-1}a_kz_0}^{R_kb_kz_0} \langle f, \psi \rangle dz + \int_{R_k^{-1}a_kz_0}^{R_kb_kz_0} \langle f, \psi \rangle dz =
\]

\[
= \int_{z_0}^{a_kz_0} \langle f^{R_k^{-1}}, \psi^{R_k^{-1}} \rangle \frac{R_kb_k'(z)}{R_kb_k(z)}dz - \int_{z_0}^{a_kz_0} \langle f^{R_kb_k}, \psi^{R_kb_k} \rangle \frac{(R_kb_k)'(z)}{R_kb_k(z)}dz =
\]

\[
= \int_{z_0}^{a_kz_0} [\langle R_k^{-1} \cdot f + \Phi(R_k^{-1}), R_k^{-1} \cdot \psi \rangle - \langle R_kb_k \cdot f + \Phi(R_kb_k), R_kb_k \cdot \psi \rangle] dz =
\]

\[
= \langle \Phi(R_k^{-1}), R_k^{-1} \cdot \Psi(a_k) \rangle - \langle \Phi(R_kb_k), R_kb_k \cdot \Psi(a_k) \rangle =
\]

\[
= -\langle \Phi(R_k^{-1}), \Psi(a_k) \rangle + \langle \Phi(b_k^{-1}R_k^{-1}), \Psi(a_k) \rangle = -\langle \Phi(\# \frac{\partial R}{\partial a_k}), \Psi(a_k) \rangle.
\]
A similar computation for the remaining $2g$ sides gives the desired formula. 

We observe that, in the special case $V_{σ} = C$ with the trivial action of $π_1$, and the usual inner product $⟨z_1, z_2⟩ = z_1 \overline{z_2}$, $z_1, z_2 ∈ C$, this proposition reduces to the classical Riemann bilinear relations, because it says that for any $φ, ψ ∈ H^0(X, K)$, we have (here all $R_k$ are trivial):

$$\int_X \bar{φ}ψ \, dz ∧ \overline{dz} = -i⟨φ, ψ⟩ = Φ \lrcorner Ψ = \sum_{k=1}^g (Φ(a_k)\overline{Ψ(b_k)} - Φ(b_k)\overline{Ψ(a_k)}).$$

Let us now consider the important case of a representation which is both Schottky and unitary. Recall that $P_{Adρ} : H^0(X, EndE_ρ \otimes K) → H^1(π_1, Adρ)$.

**Proposition 6.** If $ρ ∈ S_n \cap U_n$ then $H^1(F_g, Adρ) \cap Im(P_{Adρ}) = 0$.

**Proof.** Since $F_g$ is a free group, $H^2(F_g, M) = 0$, for any $F_g$-module $M$. Therefore, if $P_{Adρ}(φ) ∈ H^1(F_g, Adρ)$, then $P_{Adρ}(φ) \lrcorner P_{Adρ}(φ) = 0$ and by proposition $\boxempty$, $(φ, φ) = 0$. Therefore, $φ = 0$. 

Applying this proposition to representations producing stable bundles, we can now prove theorem $\boxempty$.

**Proof of theorem $\boxempty$**. When $ρ ∈ S_n^{st} \cap U_n^{st}$, prop. $\boxempty$ and Thm. $\boxempty$, toghether imply that $dWρ$ is invertible. The set $R := \{ρ ∈ S_n^{st} : det(dWρ) = 0\}$ is a closed analytic subset of $S_n^{st}$ which is not the whole set. Since $W$ is a holomorphic map between the complex manifolds $S_n^{st}$ and $M_n^{st}$, on the complement $S_n^{st} \setminus R$, $W$ is a local diffeomorphism, and therefore, it is an open map. So, we can take $U = W(S_n^{st} \setminus R)$. 

6. **Schottky bundles over an elliptic curve.**

Let $X$ be a compact Riemann surface of genus 1, and $I$ the trivial line bundle over $X$. Atiyah $[A]$ proved the following:

**Theorem 5.** (a) For any $n ≥ 1$, there is a unique indecomposable vector bundle of rank $n$ and degree 0 over $X$ denoted $F_n$, such that $dim H^0(X, F_n) = 1$. Moreover, (for $n > 1$) $F_n$ is the unique nontrivial extension $0 → I → F_n → F_{n-1} → 0$. (b) Every indecomposable vector bundle $E$, of rank $n$ and degree 0 over $X$, is isomorphic to $F_n ⊗ det E$.

**Lemma 5.** For every $n ≥ 1$, the bundle $F_n$ is Schottky.

**Proof.** Write $X = C/⟨a, b | ab = ba⟩$, where $a, b ∈ π_1(X)$ act by $a·z = z+1$, $b·z = z + τ$ and $Imτ > 0$. Consider the Schottky representation $ρ_n ∈ Hom(F_g, GL(n, C))$.
given by assigning to $b$ the $n \times n$ matrix $N$ whose entries are all zero except for ones on the principal diagonal and on the diagonal above it ($N_{i,i} = N_{i,i+1} = 1$ are the nonzero entries). We claim that $E_{\rho_n} = \mathbb{F}_n$. Clearly $\mathbb{F}_1 = E_{\rho_1} = \mathbb{I}$, so assume that $E_{\rho_{n-1}} = \mathbb{F}_{n-1}$. By construction, $E_{\rho_n}$ is an extension of the form $0 \rightarrow \mathbb{I} \rightarrow E_{\rho_n} \rightarrow E_{\rho_{n-1}} = \mathbb{F}_{n-1} \rightarrow 0$. and so, by Thm. 5(a), the lemma can be proved by showing that $\dim H^0(X, E_{\rho_n}) = 1$. Sections of $E_{\rho_n}$ over $X$ correspond to holomorphic functions $s = (s_1, \ldots, s_n) : \mathbb{C} \rightarrow \mathbb{C}^n$ satisfying $s(\gamma z) = \rho_n(\gamma)s(z)$ $\forall \gamma \in \pi_1$. This means that $s_n$ is a constant (being an entire doubly periodic function), and that the other components of $s$ have 1 as period, and verify $s_i(z + \tau) = s_i(z) + s_{i+1}(z)$ $\forall i = 1, \ldots, n-1$. Since an abelian differential with zero “$a$-periods” (in this case $ds_{n-1}$) has to be zero, we get $s_n = 0$, and $s_{n-1}$ constant. Repeating this argument we get $s_i = 0$, for all $i = 2, \ldots, n$, and $s_1$ is a constant and so, $\dim H^0(X, E_{\rho_n}) = 1$. \hfill \Box

**Theorem 6.** Every flat vector bundle over a Riemann surface of genus one is Schottky.

**Proof.** By Weil’s theorem, we may assume that $E$ is indecomposable of degree 0. Then by Thm. 5(b), $E = \mathbb{F}_n \otimes \det E$. Since $\mathbb{F}_n$ is Schottky and $\det E$ is a line bundle, we conclude that $E$ is also Schottky (see the remarks after definition [4]). \hfill \Box

Over a Riemann surface of genus one, there are no stable vector bundles of degree 0 and rank $n > 1$ (see Tu, [42]), so the map $V$ does not make sense in this case, but we can construct an analog, considering the moduli space of semistable vector bundles of degree 0, which is isomorphic to the $n$-th symmetric product of the Jacobian: $\mathcal{M}^ss_n = \text{Sym}^n(Jac(X))$ ([42]). Similarly, we can consider semisimple representations:

$G^ss_n = \{ \rho \in \text{Hom}(\pi_1, GL(n, \mathbb{C})) : \rho(\gamma) \text{ is diagonalizable for all } \gamma \in \pi_1 \}/GL(n, \mathbb{C})$,

Since $\pi_1$ is commutative, all matrices $\rho(\gamma)$ can be simultaneously diagonalized, and so, $G^ss_n$ is the $n$-th symmetric product of one dimensional representations, $G^ss_n = \text{Sym}^n(\text{Hom}(\pi_1, \mathbb{C}^*))$. It is then easy to verify that $V_n : G^ss_n \rightarrow \mathcal{M}^ss_n$ (sending $\rho$ to $E_\rho$) is the $n$-th symmetric power of $V_1 : \text{Hom}(\pi_1, \mathbb{C}^*) \rightarrow \mathcal{M}_1$ ([42]). The space of Schottky representations in $G^ss_n$, will be $S^ss_n = \text{Sym}^n(\text{Hom}(F_g, \mathbb{C}^*))$, and we can describe the map $W_1 : S^ss_n \rightarrow \mathcal{M}_n^ss$ using the lattice $\Lambda := \{2\pi i\omega \}$ $\subset H^0(X, K)$ (where $\omega$ is a normalized differential), as follows:

**Proposition 7.** For a Riemann surface of genus 1, the map $W : S^ss_n \rightarrow \mathcal{M}^ss_n$ is the $n$-th symmetric power of the $\Lambda$-bundle $W_1 : \text{Hom}(F_g, \mathbb{C}^*) \rightarrow \mathcal{M}_1$, where $\Lambda$ acts on $\text{Hom}(F_g, \mathbb{C}^*)$ as in Prop. 5.
It is known that every line bundle of degree 0 is Schottky (see the discussion in [K], Ch. VI, §4). In our setting, this can be easily obtained as follows. The moduli space of degree 0 holomorphic line bundles \( \mathcal{M}_1 \) is the Jacobian variety of \( X \), \( \text{Jac}(X) \), which is a group under tensor product. \( H^1(X; \mathbb{C}^*) = \text{Hom}(\pi_1, \mathbb{C}^*) \) is also an abelian group (under tensor product of representations) isomorphic to \((\mathbb{C}^*)^{2g}\) upon the choice of generators of \( \pi_1 \); and the map \( V = V_1 : G_1 = \text{Hom}(\pi_1, \mathbb{C}^*) \to \text{Jac}(X) \) becomes a (non-algebraic) surjective homomorphism (a degree 0 line bundle is flat). The holomorphic map \( Q_\rho \) gives now an explicit action of \( H^0(X, K) \) on \( \text{Hom}(\pi_1, \mathbb{C}^*) \):

\[
\omega \cdot \rho := Q_\rho(\omega)(\gamma) = e^{f_1}, \omega(\gamma),
\]

which represents the Jacobian as \( \text{Hom}(\pi_1, \mathbb{C}^*)/H^0(X, K) \). Let \( \rho_1 \in \text{Hom}(\pi_1, \mathbb{C}^*) \) represent a line bundle \( L \); finding a Schottky representation \( \rho_2 \) of the same \( L \) amounts, by lemma 3 to finding a holomorphic differential \( \omega \) with \( \rho_2(a_j) = e^{f_1}/\omega \rho_1(a_j) = 1 \), for all \( j \). Since this equation is always solved with \( \omega = -\sum_{j=1}^{g} \log(\rho_1(a_j)) \omega_j \) (for any choice of branches of log, and where \( \omega_1, \ldots, \omega_g \) is a normalized basis of \( H^0(X, K) \), i.e, \( \int a_i \omega_j = \delta_{ij} \) and \( \int a_i \omega_j = \Pi_{ij} \) \((i, j = 1, \ldots, g)\); \( \Pi_{ij} \) is the period matrix, symmetric with positive definite imaginary part), we see that every degree 0 line bundle \( L \), admits a Schottky representation.

Finally, to describe all such representations, consider the lattice \( \Lambda := \{2\pi i(n_1\omega_1 + \ldots + n_g\omega_g) : n_1, \ldots, n_g \in \mathbb{Z} \} \) inside \( H^0(X, K) \). Then, two Schottky representations \( \rho_1 \) and \( \rho_2 \) produce the same holomorphic line bundle if and only if there exists \( \omega \in \Lambda \) such that \( Q_{\rho_1}(\omega) = \rho_2 \). To see this, let \( E_{\rho_1} = E_{\rho_2} \) and \( \omega \in H^0(X, K) \) be the form such that \( Q_{\rho_1}(\omega) = \rho_2 \); then \( e^{f_1} \omega = 1 \) for all \( j \). Writing \( \omega = \sum c_i \omega_i \), this implies \( c_j \in 2\pi i\mathbb{Z} \), which means \( \omega \in \Lambda \). The converse is immediate, and we conclude that:

**Proposition 8.** The map \( W = W_1 : S_1 = \text{Hom}(F_g, \mathbb{C}^*) \to \mathcal{M}_1 = \text{Jac}(X) \) is a holomorphic principal \( \Lambda \)-bundle, under the action \([\Lambda]\) of \( \Lambda \) on \( S_1 \).

Let \( 1 \) denote the trivial representation. In contrast to the case of stable bundles, Schottky bundles do not determine a unique representation:

**Corollary 1.** If \( E \) is a Schottky vector bundle, then there are infinitely many non-conjugate Schottky representations that give rise to \( E \).

**Proof.** If \( E = E_\rho \) for some \( \rho \in \text{Hom}(F_g, \text{GL}(n, \mathbb{C})) \), then \( \rho \otimes (\omega \cdot 1) \) is a non-conjugate Schottky representation, for all \( \omega \in \Lambda \setminus \{0\} \). Moreover \( E_{\rho \otimes (\omega \cdot 1)} = E_\rho \otimes E_{\omega \cdot 1} = E_\rho \otimes 1 = E \), by Prop. 8. \( \square \)
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