(Super-) String in Two Dimensional Black Hole and Target Space Dualities

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ABSTRACT

We review the recently proposed string theory in two dimensional black hole background. Especially, the structure of the duality in the target space is discussed. The duality is analogous to “$R \rightarrow 1/R$” symmetry of a compactified boson. We consider the duality in more general target space manifolds which have Killing symmetries and we give an explicit formula which connects two different manifolds which are dual to each other. Superstring theory in two dimensional black hole background is also discussed based on supersymmetric $SU(1,1)/U(1)$ gauged Wess-Zumino-Witten model.

⋆ Talk at YITP workshop, “Developments in Strings and Field Theories”, Kyoto, JAPAN
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1. Introduction

Recently it was shown that the $SU(1,1)/U(1)$ gauged Wess-Zumino-Witten (GWZW) model describes strings in a two dimensional black hole. This model gives a simple toy model of black hole, i.e., an example of space-time which has singularities. The string propagation and Hawking radiation in this black hole were discussed in Ref. 2. Since this model is a model of string, the target space has a structure of duality, which is analogous to "$R \to 1/R$" symmetry of a compactified boson. Furthermore, this model can be regarded as the two dimensional gravity coupled with $c = 1$ conformal matter (free boson) when the level $k$ of $SU(1,1)$ current algebra equals to $9/4$ ($c = 26$). We expect that this model could be one of toy models which provide a clue to solve the dynamics of more "realistic" string models.

The supersymmetric extension of this model appeared\cite{6} as an exact solution of ten-dimensional superstring theory corresponding to black fivebranes. In Ref. 8, the supersymmetric extension based on $SU(1,1)/U(1)$ supersymmetric GWZW (SGWZW) model was discussed. It has been shown\cite{9} that supersymmetric $SU(1,1)/U(1)$ coset model has $N = 2$ supersymmetry due to Kazama-Suzuki\cite{10} mechanism and this model is equivalent to $\mathcal{N} = 2$ superconformal models proposed by Dixon, Lykken and Peskin.\cite{11} The central charge $c$ of this system is given by,

$$c = \frac{3k}{k-2}. \quad (1.1)$$

Here $k$ is the level of $SU(1,1)$ current algebra. When $k = \frac{5}{2}$, the central charge $c$ equals to 15 and this conformal field theory describes a critical Neveu-Schwarz-Ramond string theory. The $\mathcal{N} = 1$ supergravity coupled with $c = \frac{3}{2}$ ($\hat{c} = 1$) superconformal matter would be described by this critical theory. Furthermore the $\mathcal{N} = 2$ superconformal symmetry of this model suggests that pure $\mathcal{N} = 2$ supergravity would be also described by this model when $c = 6$ ($k = 4$). Due to $\mathcal{N} = 2$ superconformal symmetry, superstring theories in the two dimensional black
hole background can be constructed by imposing GSO projection. We expect that this superstring theory would be a continuum theory corresponding to Marinari and Parisi's superstring theory in one dimension\cite{12}.

In the next section, we review the relation between $SU(1,1)/U(1)$ GWZW model and string theories in two dimensional black hole based on Ref.1. In section 3, the duality structure of the target space is discussed. We also consider the duality of more general manifolds which have Killing symmetries. We derive a formula which appeared in Ref.13. This formula connects two different manifolds which are dual to each other. In section 4, we briefly review the relation between $SU(1,1)/U(1)$ GWZW model and two dimensional gravity. In section 5, the supersymmetric extension of this model is discussed in the basis of $SU(1,1)/U(1)$ GWZW model. And the last section is devoted to summary and discussion.

2. $SU(1,1)/U(1)$ gauged Wess-Zumino-Witten model and string theories in two dimensional black hole

The action $I_{WZW}^{(g)}$ of Wess-Zumino-Witten (WZW) model is given by,

$$I_{WZW}^{(g)} = \frac{k}{8\pi} \int d^2 x \text{tr} g^{-1} \partial_\mu gg^{-1} \partial^\mu g - \frac{ik}{12\pi} \int d^3 x \epsilon^{\mu \nu \rho} \text{tr} g^{-1} \partial_\mu gg^{-1} \partial_\nu gg^{-1} \partial_\rho g .$$

(2.1)

Here a matrix field $g$ is an element of a group $G$. The Lagrangian theory corresponding to the $G/H$ coset model ($H \subset G$) is given by gauging a vector symmetry, $g \to h^{-1} gh (h \in H)$, in the action (2.1). The action $I_{V}^{(g, h_{L,R})}$ of the GWZW model takes the following form:

$$I_{V}^{(g, h_{L,R})} = I_{WZW}^{(h_{L}^{-1} gh_{R})} - I_{WZW}^{(h_{L}^{-1} h_{R})} .$$

(2.2)

Here matrix fields $h_{L}$ and $h_{R}$ are elements of $H$. If $H$ is an abelian subgroup, we can gauge an axial symmetry, $g \to hgh$, instead of the vector symmetry. The
corresponding action \( I^A(g, h_{L,R}) \) is given by

\[
I^A(g, h_{L,R}) = I^{WZW}(h_L gh_R) - I^{WZW}(h_L^{-1} h_R).
\]  

(2.3)

Since we now consider \( SU(1,1)/U(1) \) GWZW model, the gauge symmetry can be vector (2.2) or axial (2.3). The action in the vectorially-gauged WZW model (2.2) has the form different from that of the action in the axially-gauged WZW model (2.3) but the corresponding conformal field theories are equivalent. As we find in the next section, these two models are related to each other by the duality.

There are compact and non-compact abelian subgroups in \( SU(1,1) \). When we gauge the compact subgroup, the resulting model describes the string propagation in a two dimensional Euclidean manifold. On the other hand, the target space which has Minkowski signature is obtained by gauging the non-compact subgroup.

For a while, we consider the case that the axial symmetry generated by the compact abelian subgroup of \( SU(1,1) \) is gauged.

By parametrizing \( g \) by,

\[
g = \exp\left(\frac{i}{2} \phi_L \sigma_2\right) \exp(\frac{1}{2} r \sigma_1) \exp\left(\frac{i}{2} \phi_R \sigma_2\right),
\]

(2.4)

with \( \sigma_i \) the Pauli matrices, we obtain the action \( I^{SU(1,1)/U(1)} \) of the \( SU(1,1)/U(1) \) GWZW model

\[
I^{SU(1,1)/U(1)} = \frac{k}{4\pi} \int d^2 z \left[-\partial \phi_L \bar{\partial} \phi_L - \partial \phi_R \bar{\partial} \phi_R - 2 \cosh r \partial \phi_L \bar{\partial} \phi_R + \partial r \bar{\partial} r \right]
\]

\[
+ \frac{k}{2\pi} \int d^2 z \left[-(1 + \cosh r) A \bar{A} + A(\bar{\partial} \phi_R + \cosh r \bar{\partial} \phi_L) + (\partial \phi_L + \cosh r \partial \phi_R) \bar{A} \right].
\]

(2.5)

* By using the Polyakov-Wiegmann formula, the action is given by the integration of Lagrangian density in two dimensions since the Wess-Zumino term is a total derivative in the parametrization (2.4).
Here gauge fields $A$ and $\bar{A}$ are defined by,

$$A \equiv h^{-1}_L \partial h_L, \quad \bar{A} \equiv h^{-1}_R \bar{\partial} h_R.$$ \hfill (2.6)

The action $I^{SU(1,1)/U(1)}$ is invariant under the following axial gauge transformation:

$$\delta \phi_{L,R} = \alpha, \quad \delta A = -\frac{i}{2} \partial \alpha, \quad \delta \bar{A} = -\frac{i}{2} \bar{\partial} \alpha.$$ \hfill (2.7)

We fix the gauge symmetry (2.7) by choosing the following gauge condition:

$$\phi_L = -\phi_R = \phi.$$ \hfill (2.8)

By integrating gauge fields $A$ and $\bar{A}$ and rescaling $r \to \frac{r}{2}$, we obtain,

$$I = \frac{k}{\pi} \int d^2 z (\partial r \bar{\partial} r + \tanh^2 r \partial \phi \bar{\partial} \phi) - \frac{1}{\pi} \int d^2 z \sqrt{h} R^{(2)} [2 \ln(\cosh r + 1) + c].$$ \hfill (2.9)

Here $h = \det h_{\mu \nu}$ ($h_{\mu \nu}$ is a metric on the string world sheet), $R^{(2)}$ is a scalar curvature on the string world sheet and $c$ is an undetermined constant. The second term comes from the integration of gauge fields $A$ and $\bar{A}$; we have evaluated $\det^{-1}(\cosh r + 1)$ by using heat kernel method. The obtained action (2.9) can be regarded as the action of string $\sigma$ model whose target space metric $g_{\mu \nu}$ and dilaton field $\Phi$ are given by,

$$g_{rr} = 1, \quad g_{\phi \phi} = \tanh^2 r, \quad g_{r \phi} = g_{\phi r} = 0,$$ \hfill (2.10)

$$\Phi = 2 \ln(\cosh r + 1) + c.$$ \hfill (2.11)

When $r$ goes to infinity, the action (2.9) takes the form:

$$I = \frac{k}{\pi} \int d^2 z \partial r \bar{\partial} r - \frac{2}{\pi} \int d^2 z \sqrt{h} R^{(2)} r + \frac{k}{\pi} \int d^2 z \partial \phi \bar{\partial} \phi.$$ \hfill (2.12)

The system described by the action (2.12) is a direct product of Coulomb gas $r$ and free boson $\phi$. On the other hand, when $r$ vanishes, $g_{\phi \phi}$ in Eq.(2.10) vanishes.
and the target space metric (2.10) becomes singular. However the target space curvature $R$, which is given by

$$R = \frac{4}{\cosh^4 r},$$

(2.13)
is not singular ($R \to 4$ when $r \to 0$). Therefore we find that the target space metric has the form of semi-infinite cigar.

The theory corresponding to the case that we gauge the axial symmetry generated by non-compact abelian subgroup of $SU(1, 1)$ is given by replacing $\sigma_2$ by $i\sigma_3$ in the parametrization (2.4) or simply analytic continuing $\phi \to it$. The corresponding target space metric is given by,

$$g_{rr} = 1, \quad g_{tt} = -\tanh^2 r, \quad g_{rt} = g_{tr} = 0,$$

(2.14)

If we define new coordinates $u, v$ by imitating the Kruskal coordinates of the Schwarzschild black hole in four dimensions

$$u = -\sinh r e^{-t}, \quad v = \sinh r e^t,$$

(2.15)

the target space metric can be rewritten by

$$g_{uu} = g_{vv} = 0, \quad g_{uv} = g_{vu} = -\frac{1}{1 - uv}.$$

(2.16)

This metric (2.16) tells that the surface which is defined by $uv = 0$ ($r = 0$) is a horizon, which is a null surface* without curvature singularities. Therefore the target space can be regarded as a two-dimensional black hole. On the other hand, the surface which satisfies $uv = 1$ is a really singular surface where the scalar curvature diverges. The target space is divided into six regions:

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* Null surfaces in general $D$ dimensional space are defined as follows: Let the equation $f(x_i) = 0$ define $D - 1$ dimensional hypersurface embedded in $D$ dimensional space. Here $x_i$ ($i = 1, \cdots, D$) are coordinates in $D$ dimensional space. Then a normal vector $\mathbf{F}$ is given by $\mathbf{F} = (\partial_1 f, \partial_2 f, \cdots, \partial_D f)$. The null surface is a surface where the normal vector $\mathbf{F}$ has the vanishing length, $\mathbf{F} \cdot \mathbf{F} = g^{ij}\partial_i f \partial_j f = 0$. In the case that we are now considering, the function $f$ is given by $f = uv$ and we find $\mathbf{F} \cdot \mathbf{F} = -4(1 - uv)uv$. Since $\mathbf{F} \cdot \mathbf{F}$ vanishes when $uv = 0$, the surface defined by the equation $uv = 0$ is a null surface.
I, IV : \( uv < 0 \) \((u > 0, v < 0 \) in the region I and \( u > 0, v < 0 \) in the region IV) \\
II, III : 0 < uv < 1 \((u, v > 0 \) in the region II and \( u, v < 0 \) in the region III) \\
V, VI : uv > 1 \((u, v > 0 \) in the region V and \( u, v < 0 \) in the region VI) \\

The meanings of the regions V and VI, which lie beyond the singularity, will be clarified in the next section. The regions V and VI are related to the regions I and IV by the duality.

Finally we consider the case that the vector symmetry generated by compact or non-compact abelian subgroup of \( SU(1, 1) \) is gauged. By using the parametrization (2.4) and the following gauge fixing condition, instead of (2.8),

\[ \phi_L = \phi_R = \phi , \]  

we obtain the following gauge fixed action,

\[ I = \frac{k}{\pi} \int d^2 z (\partial r \bar{\partial} r \pm \coth^2 r \partial \phi \bar{\partial} \phi) + \text{(dilaton term)} . \]  

Here the sign + (−) corresponds to the case that the compact (non-compact) subgroup is gauged. The obtained metric

\[ g_{rr} = 1, \quad g_{\phi\phi} = \pm \coth^2 r, \quad g_{r\phi} = g_{\phi r} = 0 , \]  

is singular, i.e., \( g_{\phi\phi} \) diverges at \( r = 0 \). The target space has the form like a trumpet or funnel. The vector transformation \( g \rightarrow h^{-1}gh \) has a fixed point \( r = 0 \) in the parametrization (2.4), \( g \rightarrow h^{-1}gh = g \). Therefore the singularity appears in a way similar to orbifolds. On the other hand, the axial transformation \( g \rightarrow hgh \) has no fixed point and we obtain a smooth manifold.
3. The duality of the target space manifold

In this section we discuss the duality structure of the target space. We also consider the more general manifolds which have Killing symmetries. More detailed discussion was given in Ref.13. (See also Refs. 14, 15 and references therein.)

As shown in the last section, the target space when we gauge the axial symmetry generated by non-compact abelian subgroup is divided by six regions. If we define new coordinates \((t, s)\) in regions V and VI where \(uv > 1\),

\[
\begin{align*}
  u &= \cosh s e^{-t}, \\
  v &= \cosh s e^{t},
\end{align*}
\]

the target space metric can be rewritten by

\[
\begin{align*}
  g_{ss} &= 1, & g_{tt} &= -\coth^2 s, & g_{st} = g_{ts} &= 0.
\end{align*}
\]

The obtained metric (3.2) is identical with the metric (2.19) which appears when the vector symmetry generated by non-compact abelian subgroup is gauged. This imply the regions V and VI are equivalent to the region I and IV since the corresponding conformal field theories are equivalent. Kiritsis has proved the equivalence as path integrals of GWZW model. The equivalence is a generalization of the invariance which appears in case of a boson compactified on the circle. When we change the radius \(R\) of the circle into \(1/R\), the spectrum of the boson is invariant.

Before we gauge the abelian symmetry, the WZW action (2.1) has axial \(\otimes\) vector abelian symmetries. After we gauge the axial (vector) symmetry, there remains a global vector (axial) symmetry. The global symmetry becomes a Killing symmetry of the target space metric. As we show later, Killing symmetries connect the metrics between two different target spaces. We define Killing symmetries as follows : Let’s consider the metric \(g_{\mu\nu} = g_{\mu\nu}(x_0, x_1, \cdots, x_{D-1})\) in \(D\) dimensional space. Here \(x_i\)'s are coordinates in the space. If the metric \(g_{\mu\nu}\) does not depend
on one of coordinates, say $x_i$, when we choose a special coordinates system,

$$\frac{\partial g_{\mu\nu}}{\partial x_i} = 0 \ ,$$

the metric $g_{\mu\nu}$ has an abelian symmetry,

$$g_{\mu\nu}(x_0, x_1, \ldots, x_i, \ldots, x_{D-1}) \rightarrow g_{\mu\nu}(x_0, x_1, \ldots, x_i + a_i, \ldots, x_{D-1})
= g_{\mu\nu}(x_0, x_1, \ldots, x_i, \ldots, x_{D-1}) \ ,$$

i.e., the metric $g_{\mu\nu}$ is invariant under the transformation $x_i \rightarrow x_i + a_i$. Here $a_i$ is a parameter of the transformation. We call this symmetry as a Killing symmetry. In case of $SU(1,1)/U(1)$ coset model, the $SU(1,1)$ WZW model (2.1) has global axial symmetry $\delta \phi_L = \delta \phi_R = \alpha$ and global vector symmetry $\delta \phi_L = -\delta \phi_R = \beta$ in the parametrization (2.4). When we gauge the axial symmetry and fix the gauge symmetry by the gauge condition (2.8), the system has a global vector symmetry $\delta \phi = \beta$, which is a Killing symmetry of the target space metric (2.10). Note that the metric in Eq.(2.10) does not depend on $\phi$.

In the following, we show that Killing symmetries connect the metrics between the two different target spaces. We consider the following $\sigma$ model action which has $D$ dimensional target space.

$$S = -\frac{1}{2} \int d^2z \tilde{g}_{\mu\nu} \partial x^\mu \partial x^\nu + \int d^2z \sqrt{h} R^{(2)}(2) \Phi + \cdots \quad \mu, \nu = 0, 1, \ldots, D - 1 \quad (3.5)$$

Here $\tilde{g}_{\mu\nu}$ is a sum of $D$ dimensional metric $g_{\mu\nu}$ and anti-symmetric tensor $a_{\mu\nu}$:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + a_{\mu\nu} \ .$$

We now assume that $\tilde{g}_{\mu\nu}$, $\Phi$ and etc. do not depend on one of coordinates, say $x_0$. 

The partition function is given by

\[ Z = \int [dx^\mu] e^S \]

\[ = \int [dx^\mu] e^{-\frac{1}{2} \int d^2z (\tilde{g}_{00} \partial x^0 \bar{\partial} x^0 + \tilde{g}_{0i} \partial x^i \bar{\partial} x^i + \tilde{g}_{ij} \partial x^i \bar{\partial} x^j) + \int d^2z \sqrt{\tilde{h}} R^{(2)} \Phi + ...} \quad (3.7) \]

\[ i, j = 1, 2, \cdots, D - 1 \].

By using the following formula

\[ e^{-ab^2} = \frac{1}{2\sqrt{\pi a}} \int_{-\infty}^{\infty} dy e^{-\frac{y^2}{4a} + iby}, \quad (3.8) \]

we can rewrite the partition function (3.7)

\[ Z = \int [dx^\mu] [dy][\bar{dy}] \det^{-1} \tilde{g}_{00} \]

\[ e^{-\frac{1}{2} \int d^2z \{ \frac{\partial}{\partial y} + \tilde{g}_{0i} \partial x^i \} (2iy + \tilde{g}_{0i} \partial x^i) + \frac{\bar{\partial}}{\bar{\partial} \bar{y}} (2i\bar{y} + \tilde{g}_{i0} \bar{\partial} x^i) \bar{\partial} x^0 + \tilde{g}_{ij} \partial x^i \bar{\partial} x^j \} + \int d^2z \sqrt{\tilde{h}} R^{(2)} \Phi + ...} \quad (3.9) \]

The integration of \( x_0 \) gives a constraint

\[ \partial (iy + \tilde{g}_{0i} \partial x^i) + \bar{\partial} (i\bar{y} + \tilde{g}_{i0} \partial x^i) = 0, \quad (3.10) \]

which is solved by

\[ iy + \tilde{g}_{0i} \partial x^i = \bar{\partial} \phi, \quad i\bar{y} + \tilde{g}_{i0} \partial x^i = -\partial \phi. \quad (3.11) \]

Here \( \phi \) is a real bosonic field.\(^*\) Note that the Jacobian is c-number when we change the variables \( y \) and \( \bar{y} \) into \( \phi \). Furthermore, we can evaluate \( \det^{-1} \tilde{g}_{00} \) by using heat

\(^*\) This reality assignment is correct when we Wick-rotate into Minkowski path integral where \( iy \) and \( i\bar{y} \) are real.
kernel method. Then the partition function $Z$ in Eq.(3.7) or (3.9) is rewritten by

$$Z = \int [dx^\mu] [d\phi] \times e^{-\frac{1}{2} \int d^2 z \left( \frac{1}{g_{00}} \partial^2 \phi - \frac{\tilde{g}_{00}}{g_{00}} \partial \phi \partial x' + 2 \frac{\tilde{g}_{00}}{g_{00}} \partial x' \partial \phi + (\tilde{g}_{ij} - \frac{\tilde{g}_{00} \tilde{g}_{0j}}{g_{00}}) \partial x' \partial x' \right)} + \int d^2 z \sqrt{hR} (\Phi + \ln g_{00}) + \cdots \quad .$$

(3.12)

If we regard $\phi$ as a new coordinate $x'_0$ of the target space, $x'_0 = \phi$, we find new metric $\tilde{g}'_{\mu\nu}$ and new dilaton field $\Phi'$ are given by

$$\begin{align*}
\tilde{g}'_{00} &= \frac{1}{g_{00}}, & \tilde{g}'_{0i} &= -\frac{g_{0i}}{g_{00}}, & \tilde{g}'_{0'} &= \frac{g_{00}}{g_{00}}, & \tilde{g}'_{ij} &= \tilde{g}_{ij} - \frac{g_{00} g_{0j}}{g_{00}}, \\
\Phi' &= \Phi + \ln g_{00} \quad .
\end{align*}$$

(3.13)

(3.14)

Equations (3.13) and (3.14) appeared in Ref.13. These equations give a generalization of “$R \to 1/R$” symmetry of a compactified boson.

4. $SU(1,1)/U(1)$ coset model and two dimensional gravity

In this section, we explain the relation between $SU(1,1)/U(1)$ GWZW model and two dimensional gravity.

The covariantly gauge fixed theory of two dimensional gravity coupled with conformal matter has conformal symmetry. The action was considered to be given by a sum of the actions of Faddev-Popov ghost, the conformal matter and Coulomb gas which corresponds to Liouville mode $\varphi$ of the metric.\footnote{16,17} The total central charge of this system vanishes. When the conformal anomaly $c$ of the conformal matter is less than one, the action successfully describes the two dimensional gravity. When $c = 1$, however, the analysis based on matrix models has clarified that there exist logarithmic corrections for the scaling.\footnote{18,19,20} The corrections can be understood by the cut off for the Liouville mode $\varphi : \varphi \geq 0$\footnote{21,22} This cut off is necessary in order to obtain finite theories.
The conformal field theory which describes two dimensional gravity coupled with \( c = 1 \) conformal matter, \( i.e. \), free boson (\( d = 1 \) non-critical string) will satisfy the following two requirement:

1. In some asymptotic region, the theory describes the system of Coulomb gas coupled with a free boson.

2. There is a natural cut off for the mode of the Coulomb gas.

The \( SU(1,1)/U(1) \) GWZW model (2.9) corresponding to the case that the axial symmetry generated by compact abelian subgroup of \( SU(1,1) \) is gauged satisfies these two requirement if we identify \( r \) with the Liouville mode \( \varphi \) and \( \phi \) with the free boson. The one loop analysis of the \( \beta \) function in two dimensional target space \( \sigma \) model\(^{[23]}\) suggests that the black hole solution (2.9) is a unique solution whose \( \beta \) functions vanish when there is no tachyon background (cosmological term).\(^*\) Several authors\(^{[25,26,27]}\) investigated the relation between \( SU(1,1)/U(1) \) coset model and \( d = 1 \) non-critical string. Especially, the spectrum of \( SU(1,1)/U(1) \) coset model was analyzed in detail in Ref.27. It was found that the spectrum contains some of the states found in the studies of \( d = 1 \) non-critical string theory but, in addition, there appears new states not previously found. The actual spectrum could be smaller if there is any truncation consistent with modular invariance.

5. Superstring in Two Dimensional Black Hole

In this section, superstring theory in two dimensional black hole background is discussed based on supersymmetric \( SU(1,1)/U(1) \) GWZW model.\(^8\)

The action \( S^{WZW}(G) \) of \( N = 1 \) supersymmetric WZW model is given by:\(^{28}\)

\[
S^{WZW}(G) = \frac{k}{2\pi} \int d^2 z d^2 \theta \text{tr} G^{-1} DGG^{-1} DG
- \frac{k}{2\pi} \int dt d^2 zd^2 \theta [\text{tr} G^{-1} DGG^{-1} DG^{-1} \partial_t G + (D \leftrightarrow \bar{D} \text{term})] .
\]

\(^*\) Two loop analysis was given in Ref.4. The effects of the tachyon background was considered in Ref.24.
Here we define covariant derivatives $D$ and $\bar{D}$ by using holomorphic and anti-holomorphic Grassmann coordinates $\theta$ and $\bar{\theta}$

$$D \equiv \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z}, \quad \bar{D} \equiv \frac{\partial}{\partial \bar{\theta}} - \bar{\theta} \frac{\partial}{\partial \bar{z}}.$$  \hspace{1cm} (5.2)

The matrix superfield $G$, which is an element of a group $\mathcal{G}$, is given by

$$G = \exp(i \sum_a T a \Phi^a).$$ \hspace{1cm} (5.3)

Here $\Phi^a$ is a superfield $\Phi^a = \phi^a + \theta \psi^a + \bar{\theta} \bar{\psi}^a + \theta \bar{\theta} f^a$ and $T^a$ is a generator of the algebra corresponding to $\mathcal{G}$. The action (5.1) satisfies Polyakov-Wiegmann type formula:

$$S_{\text{WZW}}(GH) = S_{\text{WZW}}(G) + S_{\text{WZW}}(H) + \frac{k}{2 \pi} \int d^2 z d^2 \theta \text{tr} G^{-1} D G \bar{D} H H^{-1}. \hspace{1cm} (5.4)$$

Here $H$ is also an element of $\mathcal{G}$. This formula guarantees that the system described by the action (5.1) has super Kac-Moody symmetry and $N = 1$ superconformal symmetry.

If $\mathcal{F}$ is an abelian subgroup of $\mathcal{G}$, we can gauge the following global axial symmetry in the action (5.1), which is given by an element $F$ of $\mathcal{F}$,

$$G \rightarrow F G F.$$ \hspace{1cm} (5.5)

The action of $\mathcal{G}/\mathcal{F}$ SGWZW is given by,

$$S_{\mathcal{G}/\mathcal{F}}^\mathcal{G}(G, A) = S_{\text{WZW}}^\mathcal{G}(F_L G F_R) - S_{\text{WZW}}^\mathcal{G}(F_L^{-1} F_R)$$

$$= S_{\text{WZW}}^\mathcal{G}(G) + \frac{k}{2 \pi} \int d^2 z d^2 \theta \text{tr}(A \bar{A} + AG \bar{G}^{-1} + G^{-1} DG \bar{A} + A \bar{D} GG^{-1}).$$ \hspace{1cm} (5.6)
Here $F_{L,R} \in \mathcal{F}$ and gauge fields $A, \tilde{A}$ are defined by,

$$A = F^{-1}_L DF_L, \quad \tilde{A} = F^{-1}_R \bar{D}F_R.$$  \hspace{1cm} (5.7)

The action (5.6) is invariant under the following $U(1)$ gauge transformation

$$G \to FGF, \quad A \to A + F^{-1}DF, \quad \tilde{A} \to \tilde{A} + F^{-1} \bar{D}F.$$ \hspace{1cm} (5.8)

A supersymmetric extension of string theory in a two dimensional black hole background is given by setting $G = SU(1,1)$ in the action (5.6). We start with considering $SU(1,1)$ SWZW model. By parametrizing $G$ by,

$$G = \exp(\frac{i}{2} \Phi_L \sigma_2) \exp(\frac{1}{2} R \sigma_1) \exp(\frac{i}{2} \Phi_R \sigma_2),$$ \hspace{1cm} (5.9)

with $\sigma_i$ the Pauli matrices, we obtain the action $S^{SU(1,1)}$ of $SU(1,1)$ SWZW model

$$S^{SU(1,1)} = \frac{k}{2\pi} \int d^2zd^2\theta [-\frac{1}{2} D\Phi_L \bar{D}\Phi_L - \frac{1}{2} D\Phi_R \bar{D}\Phi_R$$
$$\quad - \cosh R D\Phi_L \bar{D}\Phi_R + \frac{1}{2} DR DR].$$ \hspace{1cm} (5.10)

The holomorphic (anti-holomorphic) conserved currents $J_i$ ($\tilde{J}_i$) of this system are given by,

$$2kG^{-1}DG = J_1 \sigma_1 + iJ_2 \sigma_2 + J_3 \sigma_3, \quad 2kG^{-1} \bar{D}G = \tilde{J}_1 \sigma_1 + i\tilde{J}_2 \sigma_2 + \tilde{J}_3 \sigma_3,$$ \hspace{1cm} (5.11)

$$J_i = j_i + \theta \tilde{j}_i + \cdots, \quad \tilde{J}_i = \tilde{j}_i + \theta \bar{\tilde{j}}_i + \cdots.$$ \hspace{1cm} (5.12)

Here $\cdots$ express the terms which vanish by using the equations of motion. If we define new currents $\hat{J}_i$ and $\tilde{\hat{J}}_i$ by the following equation

$$\hat{J}_i = \tilde{J}_i - \frac{1}{2k} \epsilon_{ilm} \tilde{j}_l \tilde{j}_m, \quad \tilde{\hat{J}}_i = \tilde{\tilde{J}}_i - \frac{1}{2k} \epsilon_{ilm} \tilde{\tilde{j}}_l \tilde{\tilde{j}}_m,$$ \hspace{1cm} (5.13)

These currents $\hat{J}_i$ and $\tilde{\hat{J}}_i$ do not depend on the fermion currents $j_i$ and $\tilde{j}_i$.  

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By expanding superfields $\Phi_{L,R}$ and $R$ into components,

$$\Phi_{L,R} = \phi_{L,R} + \theta \psi_{L,R} + \bar{\theta} \bar{\psi}_{L,R} + \theta \bar{\theta} f_{L,R},$$

$$\frac{R}{2} = s + \theta \eta + \bar{\theta} \bar{\eta} + \theta \bar{\theta} g,$$

we can rewrite the $SU(1, 1)$ SWZW action $S^{SU(1,1)}$ in Eq.(5.10) by a sum of bosonic $SU(1, 1)$ WZW action $\tilde{S}^{SU(1,1)}$ and free fermion actions:

$$S^{SU(1,1)} = \tilde{S}^{SU(1,1)} + \frac{1}{4k\pi} \int d^2z [j_+ \bar{\partial} j_- - j_2 \bar{\partial} j_2 + \bar{j}_+ \partial \bar{j}_- - \bar{j}_2 \partial \bar{j}_2],$$

$$\tilde{S}^{SU(1,1)} = \frac{k}{2\pi} \int d^2z \left[ -\frac{1}{2}(\partial \phi_L \bar{\partial} \phi_L + \partial \phi_R \bar{\partial} \phi_R) - \cosh(2s)\partial \phi_L \bar{\partial} \phi_R + 2\partial s \bar{\partial} s \right].$$

Here $j_\pm$ and $\bar{j}_\pm$ are defined by

$$j_\pm \equiv j_1 \pm i j_3, \quad \bar{j}_\pm \equiv \bar{j}_1 \pm i \bar{j}_3.$$

The conserved currents corresponding to the non-supersymmetric $SU(1, 1)$ WZW action $\tilde{S}^{SU(1,1)}$ (5.16) are given by $\hat{J}_i$ and $\tilde{J}_i$ in Eq.(5.13).

Fermionic currents $j_\pm$ and $\bar{j}_\pm$ can be written as

$$j_\pm = \frac{k}{2} \exp(\mp i \phi_R)(\eta \pm \frac{i}{2} \sinh(2s) \psi_L),$$

$$\bar{j}_\pm = \frac{k}{2} \exp(\mp i \phi_L)(\bar{\eta} \pm \frac{i}{2} \sinh(2s) \bar{\psi}_R).$$

Note that there appear bosonic factors $\exp(\mp i \phi_R)$ and $\exp(\mp i \phi_L)$. Due to these factors, the boundary conditions of $j_\pm$ and $\bar{j}_\pm$ are twisted although fermions $\eta$, $\bar{\eta}$, $\psi_L$ and $\bar{\psi}_R$, which will be identified later with space-time fermionic coordinates,
should be periodic or anti-periodic. Therefore the eigenvalues of the zero modes of fermion number currents $K$ and $\bar{K}$,

$$K = \frac{1}{4k} (j_+ j_- - j_- j_+) \quad , \quad \bar{K} = \frac{1}{4k} (\bar{j}_+ \bar{j}_- - \bar{j}_- \bar{j}_+) ,$$  \hspace{1cm} (5.19)

which satisfy the following operator product expansions

$$K(z) j_\pm (w) \sim \pm \frac{1}{z - w} j_\pm \quad , \quad \bar{K}(\bar{z}) \bar{j}_\pm (\bar{w}) \sim \pm \frac{1}{\bar{z} - \bar{w}} \bar{j}_\pm ,$$  \hspace{1cm} (5.20)

are not quantized.

We now gauge the abelian symmetry in the action (5.10) by following Eq. (5.6). We consider the case that the abelian symmetry is generated by $\sigma_2$. Since the abelian symmetry is compact, the resulting theory describes the Euclidean black hole. The theory of the Lorentzian black hole can be obtained by replacing $\sigma_2$ by $i\sigma_3$ or simply by analytic continuing $\Phi_{L,R} \rightarrow i \Phi_{L,R}$.

By using the parametrization (5.9), the $SU(1,1)/U(1)$ gauged SWZW action takes the form

$$S^{SU(1,1)/U(1)} = S^{SU(1,1)} + \frac{k}{2\pi} \int d^2 z d^2 \theta [4(1 + \cosh R) A \bar{A} + 2iA(\bar{D}\Phi_L + \cosh R \bar{D}\Phi_R) + 2i(D\Phi_L + \cosh R D\Phi_R) \bar{A}] .$$  \hspace{1cm} (5.21)

Here $S^{SU(1,1)}$ is $SU(1,1)$ SWZW action in Eq.(5.10). By the following redefinitions,

$$\Phi \equiv \Phi_L - \Phi_R ,$$

$$A' \equiv A + \frac{i}{2} \cosh R D\Phi_L + D\Phi_R ,$$

$$\bar{A}' \equiv \bar{A} + \frac{i}{2} \cosh R D\Phi_R + \bar{D}\Phi_L ,$$  \hspace{1cm} (5.22)

the action (5.21) can be rewritten as follows,

$$S^{SU(1,1)/U(1)} = \frac{k}{2\pi} \int d^2 z d^2 \theta [\frac{1}{2} \tanh^2 \frac{R}{2} D\Phi \bar{D}\Phi + \frac{1}{2} DR \bar{D}R + 4(1 + \cosh R) A' \bar{A}' ] .$$  \hspace{1cm} (5.23)

The action (5.21) and (5.23) are invariant under the following infinitesimal gauge
transformation corresponding to Eq.(5.8),

\[
\delta \Phi_L = \delta \Phi_R = \Lambda, \quad \delta A = -\frac{i}{2} D\Lambda, \quad \delta \bar{A} = -\frac{i}{2} \bar{D}\Lambda.
\] (5.24)

We fix this gauge symmetry by imposing the following gauge condition

\[
\Phi_L = -\Phi_R = \tilde{\Phi}.
\] (5.25)

By integrating gauge fields \(A\) and \(\bar{A}\) in the action (5.21) or (5.23),* and by integrating auxiliary fields, we obtain the following action,

\[
S^{(1)} = \frac{k}{\pi} \int d^2z \left[ \tanh^2 s \left( \partial \phi \partial \bar{\phi} - \partial \bar{\psi} \bar{\psi} + \psi \bar{\psi} \right) - 2 \frac{\sinh s}{\cosh^3 s} \left( \eta \bar{\psi} \partial \phi + \bar{\eta} \bar{\psi} \partial \phi \right) + 4 \tanh^2 s \eta \bar{\eta} \psi \bar{\psi} + \partial s \partial \bar{s} - \partial \bar{\eta} \bar{\eta} + \eta \bar{\eta} \right].
\] (5.26)

Here we write superfields \(\tilde{\Phi}\) and \(R\) in terms of components:

\[
\tilde{\Phi} = \phi + \theta \psi + \bar{\theta} \bar{\psi} + \theta \bar{\theta} f, \\
\frac{R}{2} = s + \theta \eta + \bar{\theta} \bar{\eta} + \theta \bar{\theta} g.
\] (5.27)

This system has \(N = 1\) supersymmetry since the starting action (5.21) and gauge condition (5.24) are manifestly supersymmetric. In fact, this action is nothing but the action of (1,1) supersymmetric \(\sigma\) model\(^{[30]}\) in two dimensional black hole background.

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* The integration of the gauge fields induces the dilaton term in the action but we now neglect this term. The gauge fixed action which is correct at the quantum level is given later in this paper.
The $N = 1$ supersymmetry in the action (5.26) is extended to $N = 2$ supersymmetry since this action is invariant under the following holomorphic (anti-holomorphic) $U(1)$ symmetry:

$$\delta \psi = -\frac{u(z)}{\tanh s} \eta$$

$$\delta \eta = u(z) \tanh s \psi ,$$

(5.28)

and

$$\delta \bar{\psi} = -\frac{\bar{u}(\bar{z})}{\tanh s} \bar{\eta}$$

$$\delta \bar{\eta} = \bar{u}(\bar{z}) \tanh s \bar{\psi} .$$

(5.29)

Here $u(z)$ ($\bar{u}(\bar{z})$) is a holomorphic (anti-holomorphic) parameter of the transformation. The transformations (5.28) and (5.29) tell that the currents of this $U(1)$ symmetry can be regarded as fermion number currents with respect to space-time fermion coordinates, $\eta, \psi, \bar{\eta}$ and $\bar{\psi}$. By commuting this $U(1)$ symmetry transformation with the original $N = 1$ supersymmetry transformation, we obtain another supersymmetry transformation and we find that the action has $N = 2$ supersymmetry. On the other hand, in case of the Lorentzian black hole, the obtained algebra is not exactly $N = 2$ superconformal algebra.† Usual $N = 2$ superconformal algebra is given by

$$\{G_n, G_m\} = 4L_{n+m} + 2(m-n)J_{n+m} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0} ,$$

$$[J_n, G_m^{\pm}] = \pm G_{n+m} , \quad \text{etc.}$$

(5.30)

and the hermiticities of the operators are assigned by

$$(G_n^{\dagger}) = G_{-n}^{-}, \quad J_n^{\dagger} = J_{-n} .$$

(5.31)

The algebra which appears in the Lorentzian case is identical with Eq.(5.30), but

† Note that any Lorentzian manifold is not Kähler.
the assignment of the hermiticities is different from Eq.(5.31):

\[(G^+)^\dagger_n = G^-_{-n}, \quad (G^-)^\dagger_n = G^-_{-n}, \quad J^\dagger_n = -J_{-n}. \quad (5.32)\]

This is not so surprising since this algebra also appears in flat two dimensional Lorentzian space-time which is a subspace of flat ten dimensional space-time in usual Neveu-Schwarz-Ramond model. Even in the Lorentzian case, we have a $U(1)$ current and superstring theories can be constructed by imposing GSO projection.

In order to consider the spectrum of this theory, we choose the following gauge condition instead of Eq. (5.24),

\[\bar{D}A - D\bar{A} = 0. \quad (5.33)\]

This gauge condition allows us to parametrize the gauge fields $A$ and $\bar{A}$ as

\[A = D\Pi, \quad \bar{A} = -\bar{D}\Pi. \quad (5.34)\]

By shifting the fields $\Phi_{L,R}$,

\[\Phi_L \to \Phi_L + 2i\Pi, \quad \Phi_R \to \Phi_R - 2i\Pi, \quad (5.35)\]

the gauge fixed action $S^{(2)}$ is given by a sum of $SU(1,1)$ SWZW action $S^{SU(1,1)}$ in Eq. (5.10), free field action $S^{\Pi}$ and (free) ghost action $S^{\text{FP}}$.

\[S^{(2)} = S^{SU(1,1)} + S^{\Pi} + S^{\text{FP}}, \quad (5.36)\]

\[S^{\Pi} = -\frac{4k}{\pi} \int d^2z d^2\theta D\Pi \bar{D}\Pi, \]

\[S^{\text{FP}} = \frac{k}{2\pi} \int d^2z d^2\theta B\bar{D}\bar{C}.\]

Here $B$ and $C$ are anti-ghost and ghost superfields.
The BRS charge $Q_B$ which defines the physical states is given by

$$Q_B = \oint dz C (D\Pi - \frac{i}{4k} J_2) + \oint d\bar{z} C (\bar{D}\Pi + \frac{i}{4k} \bar{J}_2) .$$  \hspace{1cm} (5.37)

This BRS charge gives constraints on the physical states,

$$D\Pi - \frac{i}{2} J_2 = \bar{D}\Pi + \frac{i}{2} \bar{J}_2 = 0 ,$$  \hspace{1cm} (5.38)

which tell that $B$, $C$, $\Pi$ and $J_2$ (or $\bar{J}_2$) make so-called “quartet” structure similar to the structure which appeared in the quantization of Neveu-Schwarz-Ramond model based on BRS symmetry. \cite{31,32}

The action which describes superstring theory in the two dimensional black hole is simply given by a sum of $SU(1,1)$ WZW action (5.16), free fermion actions (5.15) and the actions of free superfield and free ghost and anti-ghost superfields (5.36). Furthermore the constraints (5.38) imposed by the BRS charge (5.37) can be easily solved with respect to free superfields $\Pi$. Therefore if we can find the spectrum of the bosonic string in the two dimensional black hole, \cite{1,2} we can also find the spectrum of this string theory.

The $U(1)$ current, which corresponds to the transformations (5.28) and (5.29) are given by,

$$J = \frac{-2i}{k-2} \hat{J}_2 + \frac{k}{k-2} K , \hspace{1cm} \bar{J} = \frac{-2i}{k-2} \bar{\hat{J}}_2 + \frac{k}{k-2} \bar{K} . \hspace{1cm} (5.39)$$

Here $\hat{J}_2$ and $\bar{\hat{J}}_2$ are defined by Eq.(5.13) and fermion number currents $K$ and $\bar{K}$ are defined by Eq.(5.19). These $U(1)$ currents commute with the BRS charge (5.37) and we can impose GSO projection consistently. Note that GSO projection does not give any constraint on the representations of $SU(1,1)$ current algebra since the eigenvalues of the zero modes in the currents $K$ and $\bar{K}$ are not quantized although those in $J$ and $\bar{J}$ are quantized.
6. Summary and Discussion

We have reviewed the recently proposed string theory in two dimensional black hole background. Especially, the structure of the duality in the target space was discussed. Furthermore we analyzed the duality structure of more general target space manifolds which have Killing symmetries. The duality is a generalization of “$R \rightarrow 1/R$” symmetry of a compactified boson. We have derived a formula which appeared in Ref.13. This formula connects two different manifolds which are dual to each other.

We have also discussed supersymmetric $SU(1,1)/U(1)$ gauged Wess-Zumino-Witten model. Due to Kazama-Suzuki\cite{10} mechanism, this model has $N = 2$ superconformal symmetry. When the central charge $c = 15$, this conformal field theory describes a critical Neveu-Schwarz-Ramond string theory. The $N = 1$ supergravity coupled with $c = \frac{3}{2} (\hat{c} = 1)$ superconformal matter would be described by this critical theory. The $N = 2$ superconformal symmetry of this model suggests that pure $N = 2$ supergravity would be also described by this model when $c = 6$ ($k = 4$). Due to $N = 2$ superconformal symmetry, superstring theories in the two dimensional black hole background can be constructed by imposing GSO projection. We expect that this superstring theory would be equivalent to the matrix models which have space-time supersymmetry\cite{12,33} and topological superstring theories based on $N = 2$ superconformal topological field theories.\cite{34}

Recently string models based on $SU(1,1)\times U(1)$ coset model were considered in Refs.35 and 36. These models describe the strings in two\cite{35} or three\cite{36} dimensional charged black holes. By adjusting the radius of the $U(1)$ boson, we will obtain $N = 2$ superconformal theory with $c > 3^{[11]}$ in the same way as $N = 2$ minimal model was constructed from $SU(2)\times U(1)\over U(1)$. The obtained model should be equivalent to the model discussed here.

Acknowledgements: I would like to acknowledge discussions with N. Ishibashi, M. Li, J. Lykken and A. Strominger. I am also indebted to M. Kato, E. Kiritsis, A. Sugamoto, T. Uchino and S.-K. Yang for the discussion at the early stage. I wish
to thank the theory groups of SLAC, UC Santa Barbara and Fermilab, where a part of this work was done. I am grateful to M. Rocek for drawing my attention to Ref.13. This work is supported by Soryuushi Shougakkai.
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