Some existence problems regarding partial Latin squares

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Abstract
Latin squares are interesting combinatorial objects with many applications. When working with Latin squares, one is sometimes led to deal with partial Latin squares, a generalization of Latin squares. One of the problems regarding partial Latin square and with applications to Latin squares is whether a partial Latin square with a given set of conditions exists. The goal of this article is to introduce some problems of this kind and answer some existence questions regarding partial Latin squares.

1 Introduction

A partial Latin square (or PLS for short) $P$ is a finite nonempty subset of $\mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ for which the restriction maps $Pr_{ij} : P \to \mathbb{N}^2$ are injective for $1 \leq i < j \leq 3$. Here $Pr_{ij} : \mathbb{N}^3 \to \mathbb{N}^2$ is the projection map on the $(i,j)$th factor. A partial Latin square can be represented by using an array in the following way. Consider an array whose rows and columns are indexed by natural numbers. To the $(i,j)$th cell of the array, assign $k$ if $(i, j, k) \in P$, and let it remain empty if no such $k$ exists. The resulting array, denoted by $A(P)$, has the following properties: it has only a finitely many nonempty cells and every natural number appears at most once in each row and each column of $A(P)$. It is easy to see that $P \mapsto A(P)$ gives a 1-1 correspondence between the set of partial Latin squares and the set of arrays having the mentioned properties. Similarly $P$ can also be represented on finite arrays. In this representation, the entries of the cells are usually called the symbols of $P$.

Given a partial Latin square $P$, we can associate some parameters to it. The first parameter is the number of elements of $P$ which is called the volume of $P$ and denoted by $v(P)$. Put $R(P) = Pr_1(P)$, $C(P) = Pr_2(P)$ and $S(P) = Pr_3(P)$ where $Pr_i : \mathbb{N}^3 \to \mathbb{N}$ is the projection map on the $i$th factor. The number $r(P) = |R(P)|$ is called the number of rows of $P$ where $|X|$ stands for the cardinality of a set $X$. Similarly $c(P) = |C(P)|$ is called the number of columns of $P$ and $s(P) = |S(P)|$ is called the number of symbols of $P$. To get more parameters for $P$, let $R(P)$ consist of natural numbers $i_1 < i_2 < ... < i_{r(P)}$.

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Then we obtain natural numbers \(|P_{r_i}^{-1}(i) \cap P| \) for \( i = i_1, \ldots, i_r(p) \). These natural numbers are called the row-parameters of \( P \). In a similar way, the column-parameters and symbol-parameters of \( P \) are defined.

The question handled in this paper is the following.

**Question 1.1.** Suppose that natural numbers \( m_1, \ldots, m_r, n_1, \ldots, n_c \) and \( p_1, \ldots, p_s \) are given. How can one decide if there is a partial Latin square \( P \) having row-parameters \( m_1, \ldots, m_r \), column-parameters \( n_1, \ldots, n_c \) and symbol-parameters \( p_1, \ldots, p_s \) ?

A remark about this question is in order. One can easily derive some necessary conditions on \( m_1, \ldots, m_r, n_1, \ldots, n_c, p_1, \ldots, p_s \) for the existence of such a PLS. The author is unaware if a "reasonable" set of necessary and sufficient conditions exists in the literature. In any case, this question is partly answered in this paper.

## 2 Existence of partial Latin squares

Before tackling Question 1.1, we need the following lemma from Graph Theory, see \([1]\) for the relevant material in Matching Theory.

**Lemma 2.1.** Suppose that \( G = (X, Y) \) is a bipartite graph such that the degree of each vertex in \( G \) is less than or equal to a given natural number \( n \). Suppose that \( X_1 \subset X \) and \( Y_1 \subset Y \) are two sets of vertices such that \( d_G(z) = n \) for all \( z \in X_1 \cup Y_1 \). Then \( G \) has a matching covering all the vertices in \( X_1 \cup Y_1 \).

**Proof.** First we show that \( G \) has a matching \( M \) covering all the vertices in \( X_1 \). In fact for every subset \( Z \subset X_1 \), we have \( n|Z| = \sum_{z \in N_G(Z)} d(z) \leq n|N_G(Z)| \), i.e. \( |Z| \leq |N_G(Z)| \) where \( N_G(Z) \) is the set of vertices in \( G \) which are adjacent to some vertex in \( Z \). By Hall’s theorem, \( G \) has a matching \( M \) which covers \( X_1 \). Similarly \( G \) has a matching \( N \) which covers \( Y_1 \). By deleting some edges if necessary, we can furthermore assume that \( M \) has \(|X_1|\) edges and \( N \) has \(|Y_1|\) edges. Let \( M \Delta N \) be the symmetric difference of \( M \) and \( N \). It is known (and in fact easy to see) that \( M \Delta N \) is a vertex-disjoint union of cycles and paths. We construct a matching \( K \subset M \cup N \) covering all the vertices in \( X_1 \cup Y_1 \) in some steps.

Given a cycle \( C \) in \( M \Delta N \), we put \( K_C \) to be the set of edges of \( C \) which belong to \( M \). Clearly \( K_C \) covers all the vertices of \( C \).

Next suppose that \( P = v_1, \ldots, v_m \) is a maximal path in \( M \Delta N \) with edges \( v_1v_2 \in M, v_2v_3 \in N, \ldots \). Since vertex \( v_2 \) is covered by both \( M \) and \( N \), we have \( v_2 \in X_1 \cup Y_1 \). W consider two cases depending on whether \( v_2 \in X_1 \) or \( v_2 \in Y_1 \).

First suppose that \( v_2 \in X_1 \). Then \( v_1 \notin Y_1 \), since otherwise there would exist some vertex \( x \) such that \( xv_1 \in N \), a contradiction to the fact that \( P \) is a maximal path in \( M \Delta N \). It is now easy to see that we must have \( v_3 \in Y_1, v_4 \in X_1, \ldots \). If \( v_m \in X_1 \) (i.e. \( m \) is even ), then set \( K_P \) to be the set of edges of \( P \) used in \( M \). If \( v_m \in Y_1 \) (i.e. \( m \) is odd), then put \( K_P \) to be the set of edges of \( P \) used in \( N \). Either way, it can be seen that \( K_P \) covers all the vertices of \( P \) belonging to
$X_1 \cup Y_1$. Now consider the second case, i.e. $v_2 \in Y_1$. Then we must clearly have $v_1 \in X_1$. In this case put $K_P$ to be the set of edges of $P$ used in $M$. Then $K_P$ covers all the vertices of $P$ belonging to $X_1 \cup Y_1$. To see this, note that either $m$ is odd in which case $v_3 \in X_1, v_4 \in Y_1, \ldots v_{m-1} \in Y_1, v_m \in X \setminus X_1$, or $m$ is even in which case $v_3 \in X_1, v_4 \in Y_1, \ldots v_{m-1} \in Y_1, v_m \in Y \setminus Y_1$.

Similarly we define $K_P$ where $P = v_1, \ldots, v_m$ is a maximal path in $M \Delta N$ with edges $v_1v_2 \in \overline{N}, v_2v_3 \in M, \ldots$.

Now define $K$ to be the following set of edges of $G$, $K = (M \cap N) \cup (\cup Q K_Q)$ where $Q$ ranges over the set of cycles and maximal paths in $M \Delta N$. I claim that $K$ is a matching covering all the vertices in $X_1 \cup Y_1$. First we prove that $K$ is a matching. In the way we have defined $K_Q$’s, it is clear that no vertex is covered by more than one edge in $\cup Q K_Q$. It is also clear that $M \Delta N$ is a matching. Finally, since $M \cap N$ and $\cup Q K_Q$ have no vertex in common, we see that $K$ is in fact a matching. Since every vertex of $X_1 \cup Y_1$ belongs to $M \cap N$ or one of the cycles or paths of $M \Delta N$, we see that every vertex of $X_1 \cup Y_1$ is covered by some edge of $K$, as demonstrated above when defining $K_Q$’s.

$\square$

### 2.1 Special cases of Question 1.1

We start with a useful lemma.

**Lemma 2.2.** Let $B$ be a nonempty set of $v$ cells of an $r \times c$ array. Suppose that $B$ has $n_i > 0$ cells in the $i$th row and $n_j > 0$ cells in the $j$th row for each $i$ and $j$. Then the cells in $B$ can be filled out with natural numbers in such a way that we obtain a PLS, $P$ with $s(P) = \max(n_1, \ldots, n_r, m_1, \ldots, m_c)$.

**Proof.** Proof by induction on $t = \max(n_1, \ldots, n_r, m_1, \ldots, m_c)$. If $t = 1$, then it implies that $n_i = 1$ and $m_j = 1$ for all $i, j$ which means $B$ has exactly one cell in each row and one cell in each column and consequently, we can easily construct the desired PLS, $P$ with just one symbol.

Now suppose that a natural number $p$ is given and the lemma holds for all natural numbers $t < p$. We need to prove the lemma for $t = p$. Without loss of generality, we can assume that $n_r \leq n_{r-1} \leq \ldots \leq n_1$ with $n_1 = \cdots = n_r = p$ but $n_{r+1} < p$. Similarly we can assume that $m_c \leq m_{c-1} \leq \ldots \leq m_1$ with $m_1 = \cdots = m_{c_1} = p$ but $m_{c_1+1} < p$.

Now consider the following bipartite graph $G$. The set of vertices of $G$ is the union of $X = \{1, \ldots, r\}$ and $Y = \{1, \ldots, c\}$. The vertex $x \in X$ is adjacent to $y \in Y$ if cell $(x, y)$ of the array belongs to $B$. Setting $X_1 = \{1, \ldots, r_1\}$ and $Y_1 = \{1, \ldots, c_1\}$, we can apply Lemma 2.1 to obtain a matching $K$ of $G$ covering all the vertices in $X_1 \cup Y_1$. Let the edges of the matching correspond to cells $(i_1, j_1), \ldots, (i_k, j_k)$. Set $B' = B \setminus \{(i_1, j_1), \ldots, (i_k, j_k)\}$.

It is now easy to see that no row or column of the array can have more than $p - 1$ cells belonging to $B'$. However note that the first row or the first column has $p - 1$ cells belonging to $B'$. So, by induction, we can construct a PLS on $B'$ with symbols $1, \ldots, p - 1$. Now if we fill out the remaining cells of $B$ with $p$, then it can easily be seen that we have a PLS on $B$ with exactly $p$ symbols.
The most general form of Question [1.1] answered in this paper, is the following.

**Theorem 2.3.** Suppose that natural numbers \( n_1, \ldots, n_r, m_1, \ldots, m_c \) and \( s \) are given. Then there is a PLS, \( P \) having row-parameters \( n_1, \ldots, n_r \) and column-parameters \( m_1, \ldots, m_c \) such that \( s(P) = s \) if and only if the following hold: (1) \( n_1 + \cdots + n_r = m_1 + \cdots + m_c = v \). (2) For subsets \( I \subseteq \{1, \ldots, r\} \) and \( J \subseteq \{1, \ldots, c\} \) we have \( \sum_{i \in I} n_i + \sum_{j \in J} m_j \leq v + |J||I| \). (3) \( \max(n_1, \ldots, n_r, m_1, \ldots, m_c) \leq s \leq v \).

**Proof.** First suppose that such a PLS, \( P \) exists. Then it is clear that the first condition holds where \( v \) is just the volume of \( P \). To see the second condition, consider an \( r \times s \) matrix \( E \) where \( E_{ij} = 1 \) if cell \((i, j)\) belongs to \( P \) and \( E_{ij} = 0 \) otherwise. The well-known criteria of the Gale-Ryser theorem, see [2] for example, gives the condition (2). Finally, we see that \( v \), the volume of \( P \), is at least the number of the symbols of \( P \) and the number of symbols \( s \) cannot be less than the number of cells of \( P \) in some row or column. Therefore (3) must hold.

Conversely, suppose that conditions (1), (2) and (3) hold. According to the Gale-Ryser theorem, the first two conditions imply that there is a \((0,1)\)-matrix \( E \) whose row-sum vector is \((n_1, \ldots, n_r)\) and whose column-sum vector is \((m_1, \ldots, m_c)\). Consider the following set \( B \) of cells of an \( r \times c \) array. Cell \((i, j)\) belongs to \( B \) if and only if \( E_{ij} = 1 \). It is immediate that \( B \) has \( n_i \) cells in row \( i \) and \( m_j \) cells in column \( j \) for every \( i, j \). By Lemma 2.2 there is a PLS, \( Q \) on \( B \) with exactly \( s_0 = \max(n_1, \ldots, n_r, m_1, \ldots, m_c) \) symbols. Let the symbols be \( 1, \ldots, s_0 \). Choose \( s - s_0 \) arbitrary cells of \( Q \) and change their symbols to \( s_0 + 1, \ldots, s \) in an arbitrary order such that each symbol \( s_0 + 1, \ldots, s \) is used exactly once. This is possible since \( s_0 \leq s \leq v \). The result is now a PLS having the desired conditions.

Another special case of Question [1.1] is given below.

**Proposition 2.4.** Suppose that natural numbers \( n_1, \ldots, n_r, c \) and \( s \) are given. Then there is a PLS, \( P \) having row-parameters \( n_1, \ldots, n_r \) such that \( c(P) = c \) and \( s(P) = s \), if and only if \( \max(c, s) \leq n_1 + \cdots + n_r \leq cs \) and \( n_i \leq \min(c, s) \) for every \( i = 1, \ldots, r \).

**Proof.** First suppose that such a PLS, \( P \) exists. Since \( n_1 + \cdots + n_r \) is the volume of \( P \) and each column has at least one cell in \( P \), we see that \( c \leq n_1 + \cdots + n_r \). Similarly, we have \( s \leq n_1 + \cdots + n_r \). Since \( P \) is a PLS with \( c(P) = c \) and \( s(P) = s \), its volume \( n_1 + \cdots + n_r \) is at most \( st \). It is clear that a row of the array cannot have more than \( c \) cells in \( P \) and it cannot have more than \( s \) cells of the array. In other words, we have \( n_i \leq \min(c, s) \) for all \( i \).

Conversely, suppose that the conditions hold. Without loss of generality we assume that \( c \leq s \). Choose a set \( B \) of cells in an \( r \times c \) array where \( B \) has exactly \( n_i \) cells of the array in row \( i \) for every \( i \). This is possible since \( n_i \leq c \) for every
i = 1, ..., r. For every j = 1, ..., c, let p_j be the number of cells of B in the jth column. Suppose that one of numbers \(p_1, ..., p_c\), say \(p_1\), is greater than \(s\). Since \(p_1 + \cdots + p_c = n_1 + \cdots + n_r \leq cs\), we see that there is some \(p_j \neq s\). Now, since \(p_j < p_1\), there must exist \(1 \leq i \leq r\) such that \((i, 1) \in B\) but \((i, j) \notin B\). Set \(B_1 = (B \setminus \{(i, 1)\}) \cup \{(i, j)\}\). It is easy to see that \(B_1\) has exactly \(n_k\) cells in each row \(k\) for every \(k = 1, ..., r\) and has exactly \(p_1 - 1, p_2, ..., p_j - 1, p_j + 1, p_{j+1}, ..., p_c\) cells in columns \(1, ..., c\) respectively. Continuing this process, we obtain a subset \(B'\) of cells of the array with \(n_i\) cells in row \(i\) and \(m_j \leq s\) cells in column \(j\) for each \(i\) and \(j\). Now it is clear that \(n_1, ..., n_r\) and \(m_1, ..., m_c\) and \(s\) satisfy the conditions in Theorem 2.3 and therefore there is a PLS, \(P\) having row-parameters \(n_1, ..., n_r\), column parameters \(m_1, ..., m_c\) such that \(s(P) = s\). It implies that \(P\) has row-parameters \(n_1, ..., n_r\) and we have \(c(P) = c\), \(s(P) = s\).

The following case of Question 1.1. is the last case treated in this paper.

**Corollary 2.5.** Suppose that natural numbers \(r, c, s\) and \(v\) are given. Then there is a PLS, \(P\) with \(r(P) = r\), \(c(P) = c\), \(s(P) = s\) and \(v(P) = v\) if and only if \(\max(r, c, s) \leq v \leq \min(rc, cs, rs)\).

**Proof.** First suppose that such a PLS, \(P\) exists. Then \(P\) has one cell in each row which means \(r \leq v\). Similarly one can show that \(c \leq v\) and \(s \leq v\). Since \(P\) can be represented on an \(r \times c\) array and \(v\) is the number of cells of the array occupied by \(P\), it is immediate that \(v \leq rc\). Similarly we have \(v \leq cs\) and \(v \leq rs\).

Conversely, suppose that the inequalities hold. Choose a set \(B\) of cells of an \(r \times c\) array such that \(|B| = v\). This is possible since \(v \leq rc\). Following the same argument as in the proof of Proposition 2.4 by starting from \(B\) and using the condition \(v \leq rs\), we can construct a set \(B'\) of cells in the array such that \(B'\) has \(n_i \leq s\) cells in the \(i\)th row for every \(i = 1, ..., r\). It is obvious that \(n_i \leq c\) cells in the \(i\)th row for every \(i = 1, ..., r\). Now natural numbers \(n_1, ..., n_r, c\) and \(s\) satisfy the conditions in Proposition 2.4. Therefore there is a PLS, \(P\) having row-parameters \(n_1, ..., n_r\) such that \(c(P) = c\) and \(s(P) = s\). It is clear that \(P\) is the desired PLS and therefore the proof is complete.

**References**

[1] Bondy, J. A. and Murty, U. S. R. Graph Theory with Applications. New York: North Holland, 1976.

[2] Brualdi, R. and Ryser, H. J. Combinatorial Matrix Theory. New York: Cambridge University Press, 1991.