THE MOMENT OF AN OPERATOR IN THE FREE GROUP FACTOR $L(F_N)$

ILWOO CHO

Abstract. In this paper, we will define an operator $X_n$ by the total sum of all word with their length $n$ such that $X_n = \sum_{|w|=n} w$ in the free group factor $L(F_N)$, where $F_N$ is the free group with $N$-generators. We will construct the recurrence relation of the operator product $x^k X_n$, where $x = X_1$ is the generating operator of $L(F_N)$, for $k, n \in \mathbb{N}$. By this recurrence relation, we can compute the moment $\tau(x^k X_n)$ of $x^k X_n$, for the cases when $k = n$ and $k < n$, where $\tau : L(F_N) \to \mathbb{C}$ is the canonical trace on the free group factor $L(F_N)$.

From mid 1980's, Free Probability Theory has been developed. Here, the classical concept of Independence in Probability theory is replaced by a noncommutative analogue called Freeness (See [9]). There are two approaches to study Free Probability Theory. One of them is the original analytic approach of Voiculescu and the other one is the combinatorial approach of Speicher and Nica (See [1], [2] and [3]). Let $A$ be a von Neumann algebra and let $\varphi : A \to \mathbb{C}$ be a linear functional satisfying that $\varphi(a^*) = \overline{\varphi(a)}$, for all $a \in A$. Then the algebraic pair $(A, \varphi)$ is called the $W^*$-probability space. All elements in $(A, \varphi)$ are called random variables. The basic free probabilistic information of the fixed random variable $a \in (A, \varphi)$ is the (free) moments $\varphi(a^n)$, for $n \in \mathbb{N}$, of the random variable $a$. Throughout this paper, let

$$F_N = \langle g_1, g_2, ..., g_N \rangle$$

be the free group with $N$-generators. Then we can construct the free group factor $L(F_N)$, i.e.,

$$L(F_N) = \mathbb{C}[F_N]^{\text{op}}.$$  

This von Neumann algebra is indeed a factor, because the group $F_N$ is icc. (Recall that the group von Neumann algebra $L(G)$ is a factor if and only if the group $G$ is an icc group.) Let $a$ be an operator in $L(F_N)$. Then there exists the Fourier expansion of $x$,

$$a = \sum_{g \in F_N} \alpha_g u_g, \text{ with } \alpha_g \in \mathbb{C}, \text{ for all } g \in F_N.$$  

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We can regard all $g \in F_N$ as unitaries $u_g$ in $L(F_N)$. For the convenience, we will denote these unitaries $u_g$ just by $g$. With this notation, it is easy to check that

$$g^* = u_g^* = u_g^{-1} = u_{g^{-1}} = g^{-1} \text{ in } L(F_N),$$

where $g^{-1}$ is the group inverse of $g$ in $F_N$. We can define the canonical trace $\tau : L(F_N) \to \mathbb{C}$ by

$$\tau \left( \sum_{g \in F_2} \alpha g \right) = \alpha e.$$

Then the algebraic pair $(L(F_N), \tau)$ is a $W^*$-probability space.

In [15], we re-computed the moments of the generating operator

$$x = \sum_{j=1}^{2N} (g_j + g_j^{-1})$$

of $L(F_N)$, by using the following recurrence diagram,

where

$$\nearrow : (2N - 1) + [\text{former term}]$$

$$\searrow : (2N - 1) \cdot [\text{former term}]$$

and

$$\swarrow : (2N) \cdot [\text{former term}].$$

The numbers in the recurrence diagram are came from the well-known relations.
\[ X_1 X_1 = X_2 + 2Ne \]

and

\[ X_1 X_n = X_{n+1} + (N - 1)X_{n-1}, \]

for all \( N, n \in \mathbb{N} \setminus \{1\} \) (See [16]). For example, since \( x = X_1 \) and \( x^3 = X_1 X_1 X_1 \), we can have

\[ x^3 = X_1 (X_2 + 2N e) = X_1 X_2 + 2N \cdot X_1 = X_3 + ((N - 1) + 2N)X_1 = X_3 + q_1^3 X_1. \]

This recurrence diagram represents that

\[ x^{2k} = X_{2k} + p_{2k-2}^{2k} X_{2k-2} + \ldots + p_2^{2k} X_2 + p_0^{2k} e \]

and

\[ x^{2k+1} = X_{2k+1} + q_{2k-1}^{2k+1} X_{2k-1} + \ldots + q_3^{2k} X_3 + q_1^{2k} X_1, \]

for all \( k \in \mathbb{N} \), where \( p_j^{2k} \)’s and \( q_i^{2k+1} \)’s are gotten from the above recurrence diagram and where \( e \) is the identity of \( F_N \) and where \( X_n = \sum_{|w|=n} w \) is the total sum of all words with their length \( n \), as an operator in the free group factor \( L(F_N) \), for all \( n \in \mathbb{N} \). Therefore, we can get that all odd moments of the generating operator \( x \) of \( L(F_N) \) vanish. More precisely, we have that

\[ \tau (x^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ p_0^n & \text{if } n \text{ is even}, \end{cases} \]

where \( p_0^n \)’s are gotten from the above recurrence diagram, for all \( n \in 2\mathbb{N} \).

In this paper, we will consider the operators \( x^k X_n \), in \( L(F_N) \), for \( k, n \in \mathbb{N} \). By regarding them as random variables in the \( W^* \)-probability space \( (L(F_N), \tau) \), we can compute the moment \( \tau(x^k X_n) \). In order to do that, we will construct another recurrence relation to express \( x^k X_n \), in terms of \( X_j \)’s. This recurrence relation is needed because there is no concrete recurrence relations for \( X_m X_n \), where \( m, n \in \mathbb{N} \setminus \{1\} \). One of the main results of this paper is that if \( k = n \), then

\[ \tau (x^n X_n) = \tau \left( r_1^{(n)} X_{2n} + r_2^{(n)} X_{2n-2} + \ldots + r_n^{(n)} X_2 + r_{n+1}^{(n)} e \right) = r_{n+1}^{(n)}, \]
where the sequence \( (r_1^{(n)}, \ldots, r_{n+1}^{(n)}) \) is the coefficient sequence of \( (r + (N - 1))^n \), for all \( n \in \mathbb{N} \). Here, \( r \) is just an indeterminant. Also, it is shown that if \( n > k \), then \( \tau(x^kX_n) = 0 \).

1. **The Operator** \( x^kX_n \) **in** \( L(F_N) \)

Let’s consider the coefficient of \( (r + (N - 1))^n \), for \( n \in \mathbb{N} \), where \( r \) is an arbitrary indeterminant. Then we have that the Pascal’s triangle expressing the coefficients of \( (r + (N - 1))^n \), as follows:

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2(N - 1) & (N - 1)^2 & & & \\
1 & 3(N - 1) & 3(N - 1)^2 & (N - 1)^3 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

\[\rightarrow (r + (N - 1))^0 \quad \rightarrow (r + (N - 1))^1 \quad \rightarrow (r + (N - 1))^2 \quad \rightarrow (r + (N - 1))^3 \]

**Definition 1.1.** We will denote the coefficients of \( (r + (N - 1))^n \) by the sequence \( (r_1^{(n)}, \ldots, r_{n+1}^{(n)}) \), for all \( n \in \mathbb{N} \). The sequence \( (r_1^{(n)}, \ldots, r_{n+1}^{(n)}) \) is called the coefficient sequence of \( (r + (N - 1))^n \), for each \( n \in \mathbb{N} \). Remark that, in all cases, \( r_1^{(n)} = 1 \).

For example, the coefficient sequence of \( (r + (N - 1))^3 \) is

\[ (1, 3(N - 1), 3(N - 1)^2, (N - 1)^3) \].

In this section, we will find the recurrence relation for \( x^kX_n \), where \( k, n \in \mathbb{N} \). Observe that, since \( x = X_1 \), in our case, we have that

\[
x^kX_n = x^{k-1}xX_n = x^{k-1}X_1X_n \\
= x^{k-1}(X_{n+1} + (N - 1)X_{n-1}) \\
= x^{k-2}(X_1(X_{n+1} + (N - 1)X_{n-1})) \\
= x^{k-2}(X_1X_{n+1} + (N - 1)X_1X_{n-1}) \\
= x^{k-2}((X_{n+2} + (N - 1)X_n) + (N - 1)(X_n + (N - 1)X_{n-2}))
\]
\[
\begin{align*}
&= x^{k-2} \left( X_{n+2} + 2(N-1)X_n + (N-1)^2X_{n-2} \right) \\
&= x^{k-3} \left( X_1X_{n+2} + 2(N-1)X_1X_n + (N-1)^2X_1X_{n-2} \right) \\
&= x^{k-3} \left( X_{n+3} + 3(N-1)X_{n+1} + 3(N-1)^2X_{n-1} + (N-1)^3X_{n-3} \right) \\
&= \ldots.
\end{align*}
\]

Thus we can easily verify that:

**Theorem 1.1.** Let \( k \) and \( p \) be numbers in \( \mathbb{N} \) satisfying that \( p \leq k \). Then \( x^k X_n \) is

\[
(1.1) \quad x^k = x^{p} \left( r_1^{(p)} X_{n+p} + r_2^{(p)} X_{n+p-2} + \ldots + r_p^{(p)} X_{n-p+2} + r_{p+1}^{(p)} X_{n-p} \right),
\]

where \( (r_1^{(p)}, \ldots, r_{p+1}^{(p)}) \) is the coefficient sequence of \( (r + (N-1))^p \). \( \Box \)

The above theorem is proved by the induction on \( p \), after taking the sufficiently big \( k \). By the previous theorem, we have that:

**Corollary 1.2.** (1) If \( k < n \) in \( \mathbb{N} \), then we have that

\[
x^k X_n = r_1^{(k)} X_{n+k} + r_2^{(k)} X_{n+k-2} + \ldots + r_k^{(k)} X_{n-k+2} + r_{k+1}^{(k)} X_{n-k}.
\]

(2) If \( k > n \) in \( \mathbb{N} \), then we have that

\[
x^k X_n = x^{k-n} \left( r_1^{(n)} X_{2n} + r_2^{(n)} X_{2n-2} + \ldots + r_n^{(n)} X_2 + r_{n+1}^{(n)} \right).
\]

(3) If \( k = n \) in \( \mathbb{N} \), then we have that

\[
x^n X_n = r_1^{(n)} X_{2n} + r_2^{(n)} X_{2n-2} + \ldots + r_n^{(n)} X_2 + r_{n+1}^{(n)}.
\]

**Proof.** It is easy to prove (3), by (1.1). Now, assume that \( k = n + k' \), for some \( k' \in \mathbb{N} \). Then, by (3), we can verify the result of (2). Similarly, we can get (1). \( \blacksquare \)

**Example 1.1.** Let \( k = 3 \) and \( n = 3 \). Then, by the previous corollary, we have that

\[
x^3 X_3 = r_1^{(3)} X_6 + r_2^{(3)} X_4 + r_3^{(3)} X_2 + r_4^{(3)} e,
\]

where \( (r_1^{(3)}, r_2^{(3)}, r_3^{(3)}, r_4^{(3)}) \) is the coefficient sequence of \( (r + (N-1))^3 \). Now, take \( k = 2 \) and \( n = 3 \). Then
\[ x^2 X_3 = x (X_1 X_3) = X_1 (X_4 + (N - 1)X_2) \]
\[ = X_5 + 2(N - 1)X_3 + (N - 1)^2 X_1. \]

Now, we will take \( k = 5 \) and \( n = 3 \). Then
\[ x^5 X_3 = x^2 (x^3 X_3) \]
\[ = x^2 \left( r_1^{(3)} X_6 + r_2^{(3)} X_4 + r_3^{(3)} X_2 + r_4^{(3)} e \right). \]

Notice that, in the above formula, we can keep doing our process as follows;
\[ x^2 \left( r_1^{(3)} X_6 + r_2^{(3)} X_4 + r_3^{(3)} X_2 + r_4^{(3)} e \right) \]
\[ = r_1^{(3)} x^2 X_6 + r_2^{(3)} x^2 X_4 + r_3^{(3)} x^2 X_2 + r_4^{(3)} x^2 \]
\[ = r_1^{(3)} \left( r_1^{(2)} X_8 + r_2^{(2)} X_6 + r_3^{(2)} X_4 \right) \]
\[ + r_2^{(3)} \left( r_1^{(2)} X_6 + r_2^{(2)} X_4 + r_3^{(2)} X_2 \right) \]
\[ + r_3^{(3)} \left( r_1^{(2)} X_4 + r_2^{(2)} X_2 + r_3^{(2)} e \right) \]
\[ + r_4^{(3)} \left( X_2 + (2N) e \right) \]
\[ = \left( r_1^{(3)} r_1^{(2)} \right) X_8 + \left( r_1^{(3)} r_2^{(2)} + r_2^{(3)} r_1^{(2)} \right) X_6 \]
\[ + \left( r_1^{(3)} r_3^{(2)} + r_2^{(3)} r_2^{(2)} + r_3^{(3)} r_1^{(2)} \right) X_4 \]
\[ + \left( r_2^{(3)} r_3^{(2)} + r_3^{(3)} r_2^{(2)} + r_4^{(3)} \right) X_2 \]
\[ + \left( r_3^{(3)} r_3^{(2)} + (2N) r_4^{(3)} \right) e. \]

2. The Moment of \( x^k X_n \)
In this chapter, we will compute the moments of the random variable \(x^kX_n\) in our \(W^*\)-probability space \((L(F_N), \tau)\). Remark that to compute the tracial value \(\tau(a)\) of an arbitrary random variable \(a\) in \(L(F_N)\) is to find coefficient of \(e\)-term of \(a\). So, we will try to find the \(e\)-term of operator \(x^kX_n\).

Theorem 2.1. Let \(k, n \in \mathbb{N}\) and let \(x = X_1\) be the generating operator of the free group factor \(L(F_N)\). If \(X_n = \sum w \in L(F_n)\), then

\[
\begin{align*}
(1) & \quad \tau(x^kX_n) = r_{n+1}^{(n)}, \quad \text{whenever } n = k. \\
(2) & \quad \tau(x^kX_n) = 0, \quad \text{whenever } n > k.
\end{align*}
\]

Proof. Assume that \(k = n\). Then, by (3) of the previous corollary, we have that

\[
\tau(x^nX_n) = \tau\left(r_1^{(n)}X_{2n} + r_2^{(n)}X_{2n-2} + \ldots + r_n^{(n)}X_2 + r_{n+1}^{(n)}e\right)
\]

\[
= r_{n+1}^{(n)},
\]

where \(\left(r_1^{(n)}, \ldots, r_{n+1}^{(n)}\right)\) is the coefficient sequence of \((r + (N - 1))^n\). Now, assume that \(n > k\). Then, by (1) of the previous corollary, we can get that

\[
\tau(x^kX_n) = \tau\left(r_1^{(k)}X_{n+k} + r_2^{(k)}X_{n+k-2} + \ldots + r_k^{(k)}X_{n-k} + r_{k+1}^{(k)}X_{n-k}\right)
\]

\[
= 0,
\]

since \(x^kX_n\) does not have the \(e\)-term. □

Now, suppose that \(k > n\) and \(k = n + k'\). Then

\[
(3.1) \quad x^kX_n = x^{k'}x^nX_n
\]

\[
= x^{k'}\left(r_1^{(n)}X_{2n} + r_2^{(n)}X_{2n-2} + \ldots + r_n^{(n)}X_2 + r_{n+1}^{(n)}e\right).
\]

Suppose that \(k' = 1\). Then \(x^{k'} = x = X_1\). So, \(x^{k'}X_j = X_1X_j\) does not contain \(e\)-term, for each \(j = 0, 2, 4, \ldots, 2n\). This shows that

\[
(3.2) \quad \text{if } k' = 1, \text{ then } \tau\left(x^{k'}X_n\right) = 0.
\]

By (2) of the previous theorem, we have that

\[
(3.3)
\]
\[
\tau(x^{k'}x^n X_n) = r_1^{(n)}\tau(x^{k'}X_{2n}) + r_2^{(n)}\tau(x^{k'}X_{2n-2}) + \ldots + r_n^{(n)}\tau(x^{k'}X_2) + r_{n+1}^{(n)}\tau(x^{k'}). \]

Let \( j \in \{2, 4, \ldots, 2n-2, 2n\} \) and assume that \( k' < j \). Then the summands in (3.3) satisfy that

\[
(3.4) \quad \tau(x^{k'}X_j) = \tau(x^{k'}X_{j+2}) = \ldots = \tau(x^{k'}X_{2n}) = 0,
\]

by (2) of the previous theorem. So, we can conclude that:

**Proposition 2.2.** Let \( k, k', n \in \mathbb{N} \) and \( k = n + k' \) and let \( k' < 2n \). Assume that \( j \) is the minimal number satisfying that \( k' < j \), where \( j \in \{2, 4, \ldots, 2n-2, 2n\} \). Then

\[
\tau(x^k X_n) = r_{n_j}^{(n)}\tau(x^{k'}X_j) + \ldots + r_n^{(n)}\tau(x^{k'}X_2) + r_{n+1}^{(n)}\tau(x^{k'}). \]

**Proof.** Since \( k > n \), we have that

\[
\tau(x^k X_n) = \tau(x^{k'}X^n X_n)
\]
\[
= \tau(x^{k'}(r_1^{(n)}X_{2n} + r_2^{(n)}X_{2n-2} + \ldots + r_n^{(n)}X_2 + r_{n+1}^{(n)}))
\]
\[
= r_1^{(n)}\tau(x^{k'}X_{2n}) + r_2^{(n)}\tau(x^{k'}X_{2n-2}) + \ldots + r_n^{(n)}\tau(x^{k'}X_2) + r_{n+1}^{(n)}\tau(x^{k'}). \]

Assume that \( j \) is the minimal number satisfying \( k' < j \), where \( j \in \{2, 4, \ldots, 2n\} \). Then, by (3.4), we can get that

\[
(3.5) \quad \tau(x^k X_n) = r_{n_j}^{(n)}\tau(x^{k'}X_j) + \ldots + r_n^{(n)}\tau(x^{k'}X_2) + r_{n+1}^{(n)}\tau(x^{k'}). \]

Note that in the previous proposition, \( \tau(x^{k'}) \) can be computed by the recurrence diagram introduced at the beginning of this paper. The case when \( k' > 2n \) is very
hard to find the concrete formula. However, we can verify that we might have the recursive algorithm for the computation. i.e.,

\[ k = n + k' = n + (2n + k''). \]

So, like the observation for \( k' \), we can do the similar process for \( k'' \). Also, if \( k'' > 4n \), then

\[ k = n + (2n + (4n + k''')) \].

So, we do the similar job for \( k''' \).

Notice that, for \( k, n \in \mathbb{N} \) we have that

\[ x^k X_n = X_n x^k. \]

Recall that \( X_1 X_n = X_n X_1 \), for all \( n \in \mathbb{N} \) in \( L(F_N) \). By the definition of the generating operator \( x, x = X_1 \). So, we have \( xX_n = X_n x \). Hence,

\[ x^k X_n = x^{k=1} X_n = x^{k-1} X_n x = ... = x X_n x^{k-1} = X_n x^k. \]

Therefore, we can get that:

**Proposition 2.3.** \( \tau (x^k X_n) = \tau (X_n x^k) \), for all \( k, n \in \mathbb{N} \).  \( \square \)

**Corollary 2.4.** \( \tau (x^{k_1} X_{n_1} x^{k_2}) = \tau (x^{k_1+k_2} X_n) \), for all \( k_1, k_2, n \in \mathbb{N} \).  \( \square \)

**Corollary 2.5.** \( \tau (x^{k_1} X_{n_1} x^{k_2} X_{n_2} ... x^{k_m} X_{n_m}) = \tau (x^{\sum_{i=1}^m k_i} \cdot \Pi_{j=1}^m X_{n_j}) \).  \( \square \)

Actually the above computation would be very complicated because we do not know the concrete expression for \( X_m X_n \), for all \( m, n \in \mathbb{N} \setminus \{1\} \).

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DEP. OF MATH, UNIV. OF IOWA, IOWA CITY, IA, U. S. A
E-mail address: ilcho@math.uiowa.edu