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Monoids, their boundaries, fractals and $C^*$-algebras

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Abstract: In this note we establish some connections between the theory of self-similar fractals in the sense of John E. Hutchinson (cf. [3]), and the theory of boundary quotients of $C^*$-algebras associated to monoids. Although we must leave several important questions open, we show that the existence of self-similar $M$-fractals for a given monoid $M$, gives rise to examples of $C^*$-algebras generalizing the boundary quotients $\partial C^*_\lambda(M)$ discussed by X. Li in [4, §7, p. 71]. The starting point for our investigations is the observation that the universal boundary of a finitely one-generated monoid carries naturally two topologies. The cone topology plays a prominent role in the construction of these boundary quotients, while the cone topology can be used to define canonical measures on the attractor of an $M$-fractal for a finitely one-generated monoid $M$.

Keywords: Monoids, boundaries, fractals, $C^*$-algebras

MSC: 20M30, 47D03, 28A80

1 Introduction

On a monoid $\mathcal{M}$ (=semigroup with unit $1_\mathcal{M}$) there is naturally defined a reflexive and transitive relation “$\preceq$”, i.e., for $\omega, \tau \in \mathcal{M}$ one defines $\omega \preceq \tau$ if, and only if, there exists $\sigma \in \mathcal{M}$ satisfying $\omega = \tau \cdot \sigma$. In particular, one may consider $(\mathcal{M}, \preceq)$ as a partially ordered set. Moreover, if $\mathcal{M}$ is $\mathbb{N}_0$-graded, then $(\mathcal{M}, \preceq)$ is a (noetherian) partially ordered set (see Corollary 3.7). Such a poset has a poset completion $i_\mathcal{M} : \mathcal{M} \rightarrow \bar{\mathcal{M}}$ (see § 2.3), and one defines the universal boundary $\partial \mathcal{M}$ of $\mathcal{M}$ by

$$\partial \mathcal{M} = (\bar{\mathcal{M}} \setminus \text{im}(i_\mathcal{M})) / \approx,$$  \hspace{1cm} (1.1)

where $\approx$ is the equivalence relation induced by “$\preceq$” on $\bar{\mathcal{M}} \setminus \text{im}(i_\mathcal{M})$ (see § 2.3). For several reasons (cf. Theorem A, Theorem B, Theorem C) one may consider $\partial \mathcal{M}$ as the natural boundary associated to the monoid $\mathcal{M}$. However, it is less clear what topology one should consider. Apart from the cone topology $\mathcal{T}_c(\bar{\mathcal{M}})$ there is another potentially finer topology $\mathcal{T}_f(\bar{\mathcal{M}})$ which will be called the fine topology on $\partial \mathcal{M}$ (cf. § 2.6), i.e., the identity

$$\text{id}_{\partial \mathcal{M}} : (\partial \mathcal{M}, \mathcal{T}_f(\bar{\mathcal{M}})) \rightarrow (\partial \mathcal{M}, \mathcal{T}_c(\bar{\mathcal{M}}))$$  \hspace{1cm} (1.2)

is a continuous map. The monoid $\mathcal{M}$ will be said to be $\mathcal{T}$-regular, if (1.2) is a homeomorphism. E.g., finitely generated free monoids are $\mathcal{T}$-regular (cf. Proposition 3.11, § 4.1). The universal boundary $\partial \mathcal{M} = (\partial \mathcal{M}, \mathcal{T}_f(\bar{\mathcal{M}}))$ with the fine topology can be identified with the Laca boundary $\widehat{E}(\mathcal{M})$ of the monoid $\mathcal{M}$. This topological space plays an essential role for defining boundary quotients of $C^*$-algebras associated to monoids (cf. [4, § 7], [5]). Indeed one has the following (cf. Theorem 3.10).

**Theorem A.** The map $\overline{\chi} : (\partial \mathcal{M}, \mathcal{T}_f(\bar{\mathcal{M}})) \rightarrow \widehat{E}(\mathcal{M})$ defined by (3.21) is a homeomorphism.

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Remark 1.1. (a) By Theorem A, the topological space $(\partial \mathcal{M}, \mathcal{T}(\mathcal{M}))$ is totally-disconnected and compact and thus it has the nicest topological regularity property that one can wish for. On the contrary, in general one can only show that $(\partial \mathcal{M}, \mathcal{T}(\mathcal{M}))$ is a $T_0$-space, which is a low level regularity property. Indeed, if $(\partial \mathcal{M}, \mathcal{T}(\mathcal{M}))$ happens to be Hausdorff, then (1.2) is necessarily a homeomorphism and $\mathcal{M}$ is $\mathcal{T}$-regular (cf. Proposition 3.11).

We will now work with the cone topology $\mathcal{T}(\mathcal{M})$.

(b) If $\phi: \mathcal{O} \rightarrow \mathcal{M}$ is a surjective graded homomorphism of finitely 1-generated monoids, then, by construction, $\phi$ induces a surjective, continuous and open map

$$\partial \phi: (\partial \mathcal{O}, \mathcal{T}(\mathcal{O})) \rightarrow (\partial \mathcal{M}, \mathcal{T}(\mathcal{M}))$$

(cf. Proposition 3.2). This property can be used to establish the following.

Theorem B. Let $\mathcal{M}$ be a finitely 1-generated $\mathbb{N}_0$-graded monoid. Then $\partial \mathcal{M}$ carries naturally a Borel probability measure

$$\mu_\mathcal{M}: \text{Bor}(\partial \mathcal{M}) \rightarrow \mathbb{R}^+ \cup \{\infty\}$$

induced by the canonical homomorphism of monoids $\phi_\mathcal{M}: \mathcal{T}(\mathcal{M}) \rightarrow \mathcal{M}$ (cf. (3.4)).

On the other hand the induced mapping $\phi_\mathcal{E}$ is given by a map

$$\phi_\mathcal{E}: \mathcal{E}(\mathcal{M}), \rightarrow \mathcal{E}(\mathcal{O}),$$

(cf. Proposition 3.12). Hence for the purpose of constructing Borel measures the fine topology seems to be inappropriate.

Theorem B can be used to define the $\mathcal{C}^*$-algebra

$$\mathcal{C}(\mathcal{M}, \mu_\mathcal{M}) = \langle \beta_\omega, \beta_\omega^* \mid \omega \in \mathcal{M} \rangle \subseteq \mathcal{B}(L^2(\partial \mathcal{M}, C, \mu_\mathcal{M}))$$

for every finitely 1-generated $\mathbb{N}_0$-graded monoid $\mathcal{M}$, where $\beta(\omega)$ is the mapping induced by left multiplication with $\omega$ (cf. § 4.5). We will show by explicit calculation that for the monoid $\mathcal{F}_n$, freely generated by a set of cardinality $n$, the $\mathcal{C}^*$-algebra $\mathcal{C}(\mathcal{F}_n, \mu_{\mathcal{F}_n})$ coincides with the Cuntz algebra $\mathcal{O}_n$ (cf. Proposition 4.4), while for the right-angled Artin monoid $\mathcal{M}^\Gamma$ associated to the finite graph $\Gamma$, $\mathcal{C}(\mathcal{M}^\Gamma, \mu_{\mathcal{M}^\Gamma})$ coincides with the boundary quotients introduced by Crisp and Laca in [2] (cf. § 4.6). Nevertheless the following more general question remains unanswered.

Question 1. Let $\mathcal{M}$ be a finitely 1-generated $\mathbb{N}_0$-graded monoid with the left cancellation property. Does $\mathcal{C}(\mathcal{M}, \mu_\mathcal{M})$ coincide with the boundary quotient $\partial \mathcal{C}(\mathcal{M})$ defined by X. Li in [4, Definition 7.9]?

From now on we will assume that the $\mathbb{N}_0$-graded monoid $\mathcal{M} = \bigcup_{k \in \mathbb{N}_0} \mathcal{M}_k$ is finitely 1-generated. In the context of self-similar fractals in the sense of John E. Hutchinson (cf. [3]) it will turn out to be convenient to endow $\partial \mathcal{M}$ with the cone topology $\mathcal{T}(\mathcal{M})$. Let $(X, d)$ be a complete metric space with a left $\mathcal{M}$-action $\alpha: \mathcal{M} \rightarrow \text{C}(X, X)$ by continuous maps. Such a presentation will be said to be contracting, if there exists a positive real number $\delta < 1$ such that

$$d(\alpha(s)(x), \alpha(s)(y)) \leq \delta \cdot d(x, y),$$

for all $x, y \in X, s \in \mathcal{M}$ (cf. [3, § 2.2]). For such a metric space $(X, d)$ there exists a unique compact subset $K \subseteq X$ such that

1. $K = \bigcup_{s \in \mathcal{M}} \alpha(s)(K),$
2. $K = \text{cl}\{ \text{Fix}(\alpha(t)) \mid t \in \mathcal{M} \} \subseteq X.$

Obviously, by definition every map $\alpha(t) \in \text{C}(X, X)$ is contracting, and thus has a unique fixed point $x_t \in X$. For short we call $K = K(\alpha) \subset X$ the attractor of the representation $\alpha$. One has the following (cf. Proposition 5.4).

Theorem C. Let $\mathcal{M} = \bigcup_{k \in \mathbb{N}_0} \mathcal{M}_k$ be a finitely 1-generated $\mathbb{N}_0$-graded monoid, let $(X, d)$ be a compact metric space and let $\alpha: \mathcal{M} \rightarrow \text{C}(X, X)$ be a contracting representation of $\mathcal{M}$. Then for any point $x \in X$, $a$ induces a continuous map

$$\kappa_x: \partial \mathcal{M} \rightarrow K(\alpha).$$
Moreover, if \( \mathcal{M} \) is \( \mathcal{T} \)-regular, then \( \kappa_\tau \) is surjective.

Under the general hypothesis of Theorem C we do not know whether the topological space \((\partial \mathcal{M}, \mathcal{C}_0(\mathcal{M}))\) is necessarily compact (see Question 3). However, in case that it is compact, we call \((\partial \mathcal{M}, \mathcal{C}_0(\mathcal{M}))\) universal attractor of the finitely 1-generated \( \mathbb{N}_0 \)-graded \( \mathcal{T} \)-regular monoid \( \mathcal{M} \).

Remark 1.2. Let \( \mathcal{M} \) be a finitely 1-generated monoid. Then \( \partial \mathcal{M} \) carries canonically a probability measure \( \mu_\mathcal{M} \) (cf. §4.5). Thus, by Theorem C, the attractor of the \( \mathcal{M} \)-fractal \((X, d), (\alpha)\) carries the contact probability measure \( \mu_x = \mu^\alpha_x \) for every point \( x \in X \), which is given by

\[
\mu_x(B) = \mu_\mathcal{M}(\kappa_\tau^{-1}(B)), \quad B \in \text{Bor}(K).
\]  

(1.8)

By (1), the monoid \( \mathcal{M} \) is acting on \( K \), and thus also on \( L^2(K, C, \mu_\mathcal{M}) \) by bounded linear operators \( \gamma(\omega) \), \( \omega \in \mathcal{M} \) (cf. §5.2). This defines a \( C^* \)-algebra (cf. §5.2)

\[
C^*(\mathcal{M}, X, d, \mu_\mathcal{M}) = \langle \gamma(\omega), \gamma(\omega)^* \mid \omega \in \mathcal{M} \rangle \subseteq \mathcal{B}(L^2(K, C, \mu_\mathcal{M})).
\]  

(1.9)

In case that the equivalence relation \( \sim \) generated by \( \preceq \) on \( \partial \mathcal{M} \) is different from \( = \) (cf. (1.1)) the canonical map \( \jmath : \partial \mathcal{M} \to \partial \mathcal{M} / \sim \) is not the identity.

Question 2. Does there exist a finitely 1-generated \( \tau \)-regular monoid \( \mathcal{M} \) for which the map \( \jmath \) is not the identity, and an \( \mathcal{M} \)-fractal \((X, d), (\alpha)\) such that \( C^*(\mathcal{M}, X, \partial \mu_\mathcal{M}) \) is not isomorphic to \( \partial C^*_\lambda(\mathcal{M}) \)?

2 Posets and their boundaries

A poset (or partially ordered set) is a set \( X \) together with a reflexive and transitive relation \( \preceq : X \times X \to \{t, f\} \) with the property that for all \( x, y \in X \) satisfying \( x \preceq y \) and \( y \preceq x \) follows that \( x = y \). By \( \mathbb{N} = \{1, 2, \ldots\} \) we denote the set of positive integers, and by \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) we denote the set of non-negative integers, i.e., \( \mathbb{N}_0 \) is a commutative monoid.

2.1 Cones, cocones and intervalls

For a poset \((X, \preceq)\) and \( \tau, \omega \in X \) the set

\[
C_\omega = \{ x \in X \mid x \preceq \omega \}
\]

(2.1)

will be called the cone defined by \( \omega \), and

\[
\mathcal{C}_\tau = \{ y \in X \mid y \succeq \tau \}
\]

(2.2)

the cocone defined by \( \tau \). For \( \tau \preceq \omega \) the set

\[
[\tau, \omega] = \mathcal{C}_\tau \cap C_\omega = \{ x \in X \mid \tau \preceq x \preceq \omega \}
\]

(2.3)

is called the closed intervall from \( \tau \) to \( \omega \), i.e., \([\omega, \omega] = \{\omega\}\). The poset \((X, \preceq)\) is said to be noetherian, if \( \text{card}(\mathcal{C}_\tau) < \infty \) for all \( \tau \in X \).

2.2 Complete posets

For a poset \((X, \preceq)\) let

\[
\mathcal{D}(\mathbb{N}, X, \preceq) = \{ f \in \mathcal{F}(\mathbb{N}, X) \mid \forall n, m \in \mathbb{N} : n \leq m \implies f(n) \preceq f(m) \}
\]

(2.4)
denote the set of decreasing functions which we will - if necessary - identify with the set of decreasing sequences. A poset \((X, \preceq)\) is said to be complete, if for all \(f \in \mathscr{P}(N, X, \preceq)\) there exists an element \(z \in X\) such that

\[ (CP_1) \quad f(n) \succeq z \text{ for all } n \in N, \quad \text{and} \]

\[ (CP_2) \quad \text{if } y \in X \text{ satisfies } f(n) \succeq y \text{ for all } n \in N, \text{ then } z \succeq y. \]

Note that - if it exists - \(z \in X\) is the unique element satisfying (i) and (ii) for \(f \in \mathscr{P}(N, X, \preceq)\). As usually, \(z = \min(f)\) is called the minimum of \(f \in \mathscr{P}(N, X, \preceq)\).

### 2.3 The poset completion of a poset

Let \((X, \preceq)\) be a poset. For \(u, v \in \mathscr{P}(N, X, \preceq)\) put

\[ u \preceq v \iff \forall n \in N \exists k \in N: \quad u(kn) \preceq v(n), \tag{2.5} \]

and put

\[ u \sim v \iff (u \preceq v \land v \preceq u) \lor (v \preceq u \land u = c_m, m = \min(u)), \tag{2.6} \]

where \(c_z \in \mathscr{P}(N, X, \preceq), z \in X\), is given by \(c_z(n) = z\) for all \(n \in N\).

Let \(\sim\) be the equivalence relation generated by \(\sim\) and put \(\overline{X} = \mathscr{P}(N, X, \preceq)/\sim\). Then the following properties hold for \((\overline{X}, \preceq)\).

**Proposition 2.1.** Let \((X, \preceq)\) be a poset.

(a) The relation \(\preceq\) defined by (2.5) is reflexive and transitive.

(b) For any strictly increasing function \(\alpha: N \to N\) and \(u \in \mathscr{P}(N, X, \preceq)\) one has \(u = u \circ \alpha\).

(c) Define for \([u], [v] \in \overline{X}\) that \([u] \preceq [v]\) if, and only if, \(u \preceq v\). Then \((\overline{X}, \preceq)\) is a poset.

(d) \((\overline{X}, \preceq)\) is complete.

**Proof.** (a) The relation \(\preceq\) is obviously reflexive. Let \(u, v, w \in \mathscr{P}(N, X, \preceq)\), \(u \preceq v, v \preceq w\). Then for all \(n \in N\) there exists \(h_n, k_n \in N\) such that \(u(h_n) \preceq v(k_n) \preceq w(n)\). Thus, \(u \preceq w\).

(b) Let \(u \in \mathscr{P}(N, X, \preceq)\) and let \(\alpha: N \to N\) be a strictly increasing function. Let \(m < n, m, n \in N\). Since \(\alpha\) is strictly increasing, \(a(m) < a(n)\). Then there exist \(m_0, n_0 \in N\) such that \(m_0 < a(m) < a(n) \leq n_0\). Then one has \(u(m_0) \succeq u(a(m)) \succeq u(a(n)) \succeq u(n_0)\). Thus \(u \preceq u \circ \alpha\) and \(u \circ \alpha \preceq u\), proving that \(u = u \circ \alpha\).

(c) Let \([u], [v] \in \overline{X}\), \([u] \preceq [v]\) and \([v] \preceq [u]\). Then, by definition, \(u \preceq v\) and \(v \preceq u\), and thus \(u = v\), i.e., \([u] = [v]\).

(d) Define \(\{u_k\}_{k \in N} \in \mathscr{P}(N, X, \preceq)\), i.e., \(u_k \in X\) for all \(k \in N\). Then one has \(u_1 \succeq u_2 \succeq \ldots\) by definition. Since each \(u_k \in \mathscr{P}(N, X, \preceq)\), one has \(u_k(n) \succeq u_k(m)\) for all \(k \in \mathbb{N}, n, m \in \mathbb{N}\). We define \(v \in \mathscr{P}(N, X, \preceq)\) by \(v(n) = u_n(n), n \in \mathbb{N}\). Then \([v] \in \overline{X}\) is the minimum of \(\{u_k\}_{k \in N}\). This yields the claim.

Assigning every element \(x \in X\) the equivalence class containing the constant function \(c_x \in \mathscr{P}(N, X, \preceq)\) yields a strictly decreasing mapping of posets \(i_X: X \to \overline{X}\). From now on \((X, \preceq)\) will be considered as a sub-poset of \((\overline{X}, \preceq)\). The poset \((\overline{X}, \preceq)\) will be called the poset completion of \((X, \preceq)\). The following fact is straightforward.

**Fact 2.2.** The map \(i_X\) is a bijection if, and only if, \((X, \preceq)\) is complete.

**Example 2.1.** Let \(X = \mathbb{N} \cup \{\infty\}\) and define \(n \preceq m\) if, and only if, \(n \geq m\), where \(\preceq\) denotes the natural order relation. Then the poset \((X, \preceq)\) is complete and \(\overline{X} = X\).

### 2.4 The universal boundary of a poset

For a poset \((X, \preceq)\) the poset \(\partial X = \overline{X} \setminus \text{im}(i_X)\) will be called the universal boundary of the poset \((X, \preceq)\). From now on we use the notation \(x \triangleright y\) as a short form for \(x \succeq y\) and \(x \neq y\). A function \(f: \mathbb{N} \to X\) will be said to be strictly decreasing, if \(f(n + 1) < f(n)\) for all \(n \in \mathbb{N}\). The following fact will turn out to be useful.
Fact 2.3. Let \( f \in \mathcal{S}(\mathbb{N}, X, \preceq) \) be a decreasing function such that \([f] \in \partial X\). Then there exists a strictly decreasing function \( h \in \mathcal{S}(\mathbb{N}, X, \preceq) \) such that \( f = h \), i.e., \([f] = [h]\).

Proof. By hypothesis, \( f = \text{im}(f) \) is an infinite set. In particular, the set \( \Omega = \{ \min(f^{-1}(j)) \mid j \in J \} \) is an infinite and unbounded subset of \( \mathbb{N} \). Let \( e : \mathbb{N} \to \Omega \) be the enumeration function of \( \Omega \), i.e., \( e(1) = \min(\Omega) \), and recursively one has \( e(k+1) = \min(\Omega \setminus \{e(1), \ldots, e(k)\}) \). Then, by construction, \( h = f \circ e \) is strictly decreasing, and, by Proposition 2.1(b), one has \( f = h \), and hence the claim.

Fact 2.4. Let \( (X, \preceq) \) be a noetherian poset, and let \((\overline{X}, \preceq)\) be its completion. Then for all \( \tau \in X \) one has \( \mathcal{O}_\tau(\overline{X}) \subseteq X \). In particular, \( \mathcal{O}_\tau(\overline{X}) = \mathcal{O}_\tau(X) \), where the cocones are taken in the respective posets.

Example 2.2. Let \( X = A \sqcup B \), where \( A, B = \mathbb{Z} \) and define
\[
n \preceq m \iff \left( (n, m \in A \lor n, m \in B) \land n \preceq m \right) \lor (n \in A \land m \in B),
\]
where “\( \preceq \)” denotes the natural order relation on \( \mathbb{Z} \). Then \((X, \preceq)\) is a poset and its completion is given by \( \overline{X} = \mathbb{Z} \sqcup \{ -\infty \} \sqcup \mathbb{Z} \sqcup \{ -\infty \} \). For \( n \in A \), one has \( \mathcal{O}_n(X) \neq \mathcal{O}_n(\overline{X}) \), since \( -\infty \in B \) is in \( \mathcal{O}_n(\overline{X}) \), but not in \( \mathcal{O}_n(X) \).

2.5 The cone topology

Let \((X, \preceq)\) be a poset, and let \((\overline{X}, \preceq)\) denote its completion. For \( \tau, \omega \in X \) let
\[
S(\tau, \omega) = \{ x \in X \mid x \preceq \tau \land x \preceq \omega \}.
\]
By transitivity,
\[
C_\tau(\overline{X}) \cap C_\omega(\overline{X}) = \bigcup_{\omega \in S(\tau, \omega)} C_\omega(\overline{X}).
\]
In particular,
\[
\mathcal{B}_\tau(\overline{X}) = \{ \{ x \} \mid x \in X \} \cup \{ C_\omega(\overline{X}) \mid \omega \in X \}
\]
is a base of a topology \( \mathcal{B}_\tau(\overline{X}) \) - the cone topology - on \( \overline{X} \). By construction, the subspace \( X \) is discrete and open, and the subspace \( \partial X \) is closed.

For \( \omega \in \overline{X} \) let \( \mathcal{N}_\omega(\omega) \) denote the set of all open neighborhoods of \( \omega \) with respect to the cone-topology, and put \( \mathcal{S}(\omega) = \bigcap_{U \in \mathcal{N}_\omega(\omega)} U \). Then, by construction, one has \( \mathcal{S}(\omega) = \{ \omega \} \) for \( \omega \in X \), and \( \mathcal{S}(\omega) = C_\omega(\overline{X}) \) for \( \omega \in \partial X \). This implies the following.

Proposition 2.5. Let \((X, \preceq)\) be a poset, and let \((\overline{X}, \preceq)\) denote its completion. Then \((\overline{X}, \mathcal{B}_\tau(\overline{X}))\) is a \( T_0 \)-space (or Kolmogorov space).

Proof. Let \( \tau, \omega \in \overline{X}, \tau \neq \omega \). If either \( \tau \in X \) or \( \omega \in X \), then either \{\( \tau \)\} or \{\( \omega \)\} is an open set. So we may assume that \( \tau, \omega \in \partial X \). As \( \mathcal{S}(\omega) = C_\omega(\overline{X}) \), either there exists \( U \in \mathcal{N}_\omega(\omega) \), \( \tau \notin U \), or \( \tau \preceq \omega \). By changing the role of \( \omega \) and \( \tau \), either there exists \( V \in \mathcal{N}_\tau(\tau) \), \( \omega \notin V \), or \( \omega \preceq \tau \). Since \( \tau \preceq \omega \) and \( \omega \preceq \tau \) is impossible, this yields the claim.

2.6 The fine topology

For a partially ordered set \((X, \preceq)\) let
\[
\mathcal{F} = \{ \{ \tau \}, C_\tau(\overline{X}), C_\tau(\overline{X})^C \mid \tau \in X \}
\]
denote the set of all subsets of \( X \) of cardinality 1, all cones and their complements in \( X \). Then \( \mathcal{F} \) is a subbasis of a topology \( \mathcal{F}(\overline{X}) \) on \( \overline{X} \) which we will call the fine topology on \( \overline{X} \). In particular, the set \( \Omega = \{ X = \bigcap_{i \in \sigma} X_i \mid X_1, \ldots, X_\tau \in \mathcal{F} \} \) is a base of the topology \( \mathcal{F}(\overline{X}) \). By definition, this topology has the following properties.
Fact 2.6. Let \((X, \preceq)\) be a partially ordered set. Then
(a) \((X, \mathcal{T}(X))\) is a \(T_2\)-space (or Hausdorff space).
(b) \(\mathcal{T}(X) \subseteq \mathcal{T}(\tilde{X})\).

2.7 The \(\sim\)-boundary

There is another type of boundary for a poset, the \(\sim\)-boundary, which seems to be relevant for the study of fractals (see (5.6)). Let \((X, \preceq)\) be a noetherian poset, and let \((\tilde{X}, \preceq)\) denote its completion. Put

\[
\Omega = \Delta(X) \cup \{ (\varepsilon, \eta) \in \partial X \times \partial X \mid \varepsilon \preceq \eta \},
\]

where \(\Delta(X) = \{ (x, x) \mid x \in X \}\), and let \(\sim\) denote the equivalence relation on \(\tilde{X}\) generated by the relation \(\Omega\). Then one has a canonical map

\[
\pi: \tilde{X} \to X,
\]

where \(\tilde{X} = X/\sim\). By construction, \(\pi|_X\) is injective. The set \(\tilde{\partial}X = \tilde{X} \setminus \pi(X)\) will be called the \(\sim\)-boundary of the poset \((X, \preceq)\). We put

\[
I(\sim) = \{ (\omega, \tau) \in \tilde{X} \times \tilde{X} \mid \omega \sim \tau \} \subseteq \tilde{X} \times \tilde{X}
\]

The set \(\tilde{X}\) carries the quotient topology \(\mathcal{T}_0(\tilde{X})\) with respect to the mapping \(\pi\) and the topological space \((\tilde{X}, \mathcal{T}(\tilde{X}))\). In particular, the subspace \(\pi(X) \subseteq \tilde{X}\) is discrete and open, and \(\partial X \subseteq \tilde{X}\) is closed. For \(\omega \in \tilde{X}\) we put \(\tilde{C}_\omega = \pi(C_\omega(X))\). The space \(\tilde{X}\) will be considered merely as topological space. It has the following property.

Proposition 2.7. The topological space \((\tilde{X}, \mathcal{T}_0(\tilde{X}))\) is a \(T_1\)-space.

Proof. For \(\omega \in \tilde{X}\) one has

\[
\mathcal{A}(\pi(\omega)) = \pi\left( \bigcap_{\tau \sim \omega} \mathcal{A}(\tau) \right) = \pi\left( \bigcap_{\tau \sim \omega} C_\tau(\tilde{X}) \right) = \{ \pi(\omega) \}. \tag{2.15}
\]

This yields the claim. \(\square\)

3 Monoids and their boundaries

A monoid (or semigroup with unit) \(\mathcal{M}\) is a set with an associative multiplication \(\cdot: \mathcal{M} \times \mathcal{M} \to \mathcal{M}\) and a distinguished element \(1 \in \mathcal{M}\) satisfying \(1 \cdot x = x \cdot 1 = x\) for all \(x \in \mathcal{M}\). For a monoid \(\mathcal{M}\) we denote by

\[
\mathcal{M}^* = \{ x \in \mathcal{M} \mid \exists y \in \mathcal{M}: x \cdot y = y \cdot x = 1 \} \tag{3.1}
\]

the maximal subgroup contained in \(\mathcal{M}\).

3.1 \(\mathbb{N}_0\)-graded monoids

The set of non-negative integers \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\) together with addition is a monoid. A monoid \(\mathcal{M}\) together with a homomorphism of monoids \(\sim:\mathcal{M} \to \mathbb{N}_0\) is called an \(\mathbb{N}_0\)-graded monoid. For \(k \in \mathbb{N}_0\) one defines \(\mathcal{M}_k = \{ x \in \mathcal{M} \mid |x| = k \}\). The \(\mathbb{N}_0\)-graded monoid \(\mathcal{M}\) is said to be connected, if \(\mathcal{M}_0 = \{1\}\). One has the following straightforward fact.

Fact 3.1. For a connected \(\mathbb{N}_0\)-graded monoid \(\mathcal{M}\) one has \(\mathcal{M}^* = \{1\}\).

If \(\mathcal{Q}\) and \(\mathcal{M}\) are \(\mathbb{N}_0\)-graded monoids, a homomorphism \(\phi: \mathcal{Q} \to \mathcal{M}\) is a homomorphism of \(\mathbb{N}_0\)-graded monoids, if \(\phi(\mathcal{Q}_k) \subseteq \mathcal{M}_k\) for all \(k \in \mathbb{N}_0\). The following property is straightforward.
Proposition 3.2. Let $\phi : \mathcal{D} \rightarrow \mathcal{M}$ be a homomorphism of $\mathbb{N}_0$-graded monoids. Then $\phi$ is monotone, i.e., $x, y \in \mathcal{D}$, $x \leq y$ implies $\phi(x) \leq \phi(y)$, and thus induces a monotone map

$$\mathcal{D}\phi : \mathcal{D}(\mathbb{N}, \mathcal{D}, \leq) \rightarrow \mathcal{D}(\mathbb{N}, \mathcal{M}, \leq).$$

(3.2)

Let $\hat{\phi} : \mathcal{D} \rightarrow \mathcal{M}$ denote the induced map. Let $\partial \phi : \partial \mathcal{D} \rightarrow \partial \mathcal{M}$ be the map induced by $\phi$. Then $\partial \phi$ is continuous with respect to the cone topology.

**Proof.** Let $\tau \in \mathcal{M}$. Then the monotony of $\mathcal{D}\phi$ implies that

$$\hat{\phi}^{-1}(C_r(\mathcal{M})) = \bigcup_{y \in \mathcal{Y}} C_y(\mathcal{D}),$$

(3.3)

where $\mathcal{Y} = \{ q \in \mathcal{D} \mid \hat{\phi}(q) \in C_r(\mathcal{M}) \}$. Thus $\hat{\phi}$ and $\partial \phi$ are continuous. \qed

### 3.2 1-generated monoids

For any set $Y$ there exists a free monoid $\mathcal{F}(Y)$ which is naturally $\mathbb{N}_0$-graded. Moreover, $\mathcal{F}(Y)$ is connected and $\mathcal{F}(Y)_1 = Y$. For an $\mathbb{N}_0$-graded monoid $\mathcal{M}$ there exists a canonical homomorphism of $\mathbb{N}_0$-graded monoids

$$\phi_{\mathcal{M}} : \mathcal{F}(\mathcal{M}_1) \rightarrow \mathcal{M}$$

satisfying $\phi_{\mathcal{M}_1} = \text{id}_{\mathcal{M}_1}$. The $\mathbb{N}_0$-graded monoid $\mathcal{M}$ is said to be 1-generated, if $\phi_{\mathcal{M}}$ is surjective. In particular, such a monoid is connected. By definition, free monoids are 1-generated. Moreover, $\mathcal{M}$ is said to be finitely 1-generated, if it is 1-generated and $\mathcal{M}_1$ is a finite set. The following important question remains unanswered in this paper.

**Question 3.** Does there exist a finitely 1-generated monoid $\mathcal{M}$ satisfying $\mathcal{F}(\mathcal{M}) \neq \mathcal{F}(\mathcal{M})$?

### 3.3 Monoids as posets

Let $\mathcal{M}$ be a monoid. For $x \in \mathcal{M}$, put

$$\mathcal{M}x = \{ yx \mid y \in \mathcal{M} \};$$

(3.5)

$$x\mathcal{M} = \{ xy \mid y \in \mathcal{M} \}.$$  

(3.6)

For $x, y \in \mathcal{M}$ we define

$$x \preceq y \iff x\mathcal{M} \subseteq y\mathcal{M},$$

(3.7)

i.e., $x \preceq y$ if, and only if, there exists $z \in \mathcal{M}$ such that $x = yz$.

### 3.4 Left cancellative monoids

A monoid $\mathcal{M}$ is said to be left cancellative if $xy = xz$ implies $y = z$ for all $x, y, z \in \mathcal{M}$; and right cancellative if $yx = zx$ implies $y = z$ for all $x, y, z \in \mathcal{M}$.

**Proposition 3.3.** Let $\mathcal{M}$ be a left-cancellative monoid. For $x, y \in \mathcal{M}$ one has $\mathcal{M}x = \mathcal{M}y$ if, and only if, there exists $z \in \mathcal{M}^\ast$ such that $y = xz$.

**Proof.** For $z \in \mathcal{M}^\ast$ one has $z\mathcal{M} = \mathcal{M}$. Thus for $x \in \mathcal{M}$ and $y = xz$, multiplying by $x$ from the left yields $y\mathcal{M} = x\mathcal{M}$. Viceversa, suppose $x\mathcal{M} = y\mathcal{M}$ for $x, y$ in $\mathcal{M}$. Then there exist $z, w \in \mathcal{M}$ such that $y = xz$ and $x = yw$. Hence $y = ywz$ and $x = xzw$. Thus left cancellation implies $zw = 1 = wz$, showing that $z, w \in \mathcal{M}^\ast$. \qed
Corollary 3.4. Let \( \mathcal{M} \) be a left-cancellative monoid. Then \((\mathcal{M} \setminus \mathcal{M}^*, \preceq)\) is a poset.

Remark 3.5. If left cancellation is replaced by right cancellation, then one has \( x\mathcal{M} = y\mathcal{M} \) if, and only if, there exists \( z \in \mathcal{M}^* \) such that \( y = zx \).

### 3.5 1-generated monoids as posets

Proposition 3.6. Let \( \mathcal{M} \) be a connected \( \mathbb{N}_0 \)-graded monoid. For \( x, y \in \mathcal{M} \) one has \( x\mathcal{M} = y\mathcal{M} \) if, and only if, \( x = y \).

Proof. Suppose \( x\mathcal{M} = y\mathcal{M} \), for \( x, y \in \mathcal{M} \). Then there exist \( z, w \in \mathcal{M} \) such that \( x = yz \) and \( y = xw \), so \( |x| = |y| + |z| \) and \( |y| = |x| + |w| \). Thus \( |z| = 0 = |w| \). Since \( \mathcal{M} \) is connected, this implies \( z = 1 = w \). \( \square \)

As a consequence one obtains the following.

Corollary 3.7. Let \( \mathcal{M} \) be a 1-generated \( \mathbb{N}_0 \)-graded monoid. Then \((\mathcal{M}, \preceq)\) is a poset. If \( \mathcal{M} \) be finitely 1-generated, then \((\mathcal{M}, \preceq)\) is a noetherian poset.

Remark 3.8. The following example shows that the universal boundary \( \partial \mathcal{M} \) is in general different from the \( \sim \)-boundary \( \bar{\partial} \mathcal{M} \). Let \( \mathcal{M} = (x, y, z \mid xz = zx) \). Consider

\[
\begin{align*}
 f_1 &: \mathbb{N} \to M, \quad f_1(n) = (xz)^n, \\
 f_2 &: \mathbb{N} \to M, \quad f_2(n) = x^n, \\
 f_3 &: \mathbb{N} \to M, \quad f_3(n) = z^n.
\end{align*}
\]

(3.8)

Then \( f_2 \triangleright f_1 \preceq f_3 \). Hence \( \pi(f_1) = \pi(f_2) = \pi(f_3) \in \bar{\partial} \mathcal{M} \), and \( \pi: \partial M \to \bar{\partial} \mathcal{M} \) is not injective.

### 3.6 Abelian semigroups generated by idempotents

Let \( E \) be an abelian semigroup being generated by a set of elements \( \Sigma \subseteq E \) satisfying \( \sigma^2 = \sigma \) for all \( \sigma \in \Sigma \), i.e., all elements of \( \Sigma \) are idempotents. Then every element \( u \in E \) is an idempotent, and one may define a partial order \( \preceq \) on \( E \) by

\[
u \preceq v \iff u \cdot v = v,
\]

(3.9)

for \( u, v \in E \). Let \( \mathcal{R} = \{ (u, v) \in \Sigma \times \Sigma \mid u \preceq v \} \). By definition, one has

\( E = \{ u = \sigma_1 \cdots \sigma_r \mid \sigma_1 \in \Sigma \} \).

(3.10)

Hence

\[
E \simeq \mathcal{F}^{ab}(\Sigma)/R,
\]

(3.11)

where \( \mathcal{F}^{ab}(\Sigma) \) is the free abelian semigroup over the set \( \Sigma \), and \( R \) is the relation

\[
R = \{ (u \nu, v) \mid (u, v) \in \mathcal{R} \} \subseteq \mathcal{F}^{ab}(\Sigma) \times \mathcal{F}^{ab}(\Sigma),
\]

(3.12)

i.e., \( E = \mathcal{F}^{ab}(\Sigma)/R^\sim \), where \( R^\sim \) is the equivalence relation on \( \mathcal{F}^{ab}(\Sigma) \) generated by the set \( R \). Let

\[
\tilde{E} = \{ \chi: E \to \{0, 1\} \mid \chi \text{ a semigroup homomorphism}, \chi(0) = 0, \chi \not\equiv 0 \}
\]

(3.13)

Then \( \tilde{E} \) coincides with the set of characters of the \( \mathcal{C} \)-algebra \( \mathcal{C}(E) \) generated by \( E \) (satisfying \( e^* = e \) for all \( e \in E \)), and hence carries naturally the structure of a compact topological space. By construction, \( \tilde{E} \) can be identified with a subset of \( \mathcal{F}(\Sigma, \{0, 1\}) \) - the set of functions from \( \Sigma \) to \( \{0, 1\} \). In more detail,

\[
\tilde{E} = \{ \phi \in \mathcal{F}(\Sigma, \{0, 1\}) \mid \forall (u, v) \in \mathcal{R} : \phi(v) = \phi(u) \cdot \phi(v) \}
\]

(3.14)

Thus identifying \( \mathcal{F}(\Sigma, \{0, 1\}) \) with \( \{0, 1\}^\Sigma \), one obtains that

\[
\tilde{E} = \{ (\eta_\sigma)_{\sigma \in \Sigma} \in \{0, 1\}^\Sigma \mid \forall (u, v) \in \mathcal{R} : \sigma_v = \sigma_u \cdot \sigma_v \}.
\]

(3.15)
3.7 The semigroup of idempotents generated by a set of subsets of a set

Let $X$ be a set, and let $S \subseteq \mathcal{P}(X)$ be a set of subsets of $X$. Then $S$ generates an algebra of sets $\mathcal{A}(S) \subseteq \mathcal{P}(X)$, i.e., the sets of $\mathcal{A}(S)$ consist of the finite intersections of sets in $S$. Then

$$E(S) = \langle I_A \mid A \in \mathcal{A}(S) \rangle \subseteq \mathcal{P}(X, \{0, 1\})$$  \hspace{1cm} (3.16)

is an abelian semigroup being generated by the set of idempotents

$$\Sigma = \{ I_Y \mid Y \in S \}.$$  \hspace{1cm} (3.17)

Moreover, by (3.14), one has

$$\widehat{E}(S) = \{ \phi \in \mathcal{F}(S, \{0, 1\}) \mid \forall U, V \in S, V \subseteq U : \phi(V) = \phi(U) \cdot \phi(V) \}$$  \hspace{1cm} (3.18)

3.8 The Laca-boundary of a monoid

Let $\mathcal{M}$ be a 1-generated monoid. Then one chooses

$$S = \{ \omega \cdot \mathcal{M} \mid \omega \in \mathcal{M} \}$$  \hspace{1cm} (3.19)

to consist of all principal right ideals. For short we call the compact set $\partial \mathcal{M} = \widehat{E}(S)$ for $S$ as in (3.19) the Laca boundary of $\mathcal{M}$. For an infinite word $\omega = (\omega_k) \in \mathcal{D}(\mathbb{N}, \mathcal{M}, \geq)$ and for $\tau \in \mathcal{M}$ one defines the element $\chi_\omega \in \widehat{E}(S)$ by $\chi_\omega(r, \mathcal{M}) = 1$ if, and only if, there exists $k \in \mathbb{N}$ such that $\omega_k \in r, \mathcal{M}$, i.e., $\tau \geq \omega_k$, and thus $\tau \geq \omega$. This yields a map

$$\chi : \mathcal{D}(\mathbb{N}, \mathcal{M}, \leq) \longrightarrow \widehat{E}(S)$$  \hspace{1cm} (3.20)

(cf. [5, § 2.2]). By definition, it has the following property:

**Proposition 3.9.** For $\omega = (\omega_k) \in \mathcal{D}(\mathbb{N}, \mathcal{M}, \leq), \tau \in \mathcal{M}$, one has $\chi_\omega(\tau, \mathcal{M}) = 1$ if, and only if, $\tau \geq \omega$. In particular, one has $\chi_\eta = \chi_\omega$ if, and only if, $\eta = \omega$, and hence $\chi$ induces an injective map

$$\overline{\chi} : \partial \mathcal{M} \longrightarrow \partial \mathcal{M}.$$  \hspace{1cm} (3.21)

**Proof.** The first part has already been established before. Let $\eta = (\eta_k)$. Then by the first part, $\omega \geq \eta$ implies that for all $\tau \in \mathcal{M}$ one has

$$\chi_\omega(\tau, \mathcal{M}) = 1 \Rightarrow \chi_\eta(\tau, \mathcal{M}) = 1.$$  \hspace{1cm} (3.22)

Thus as $\text{im}(\chi_\omega) \subseteq \{0, 1\}$ one concludes that $\omega \geq \eta$ and $\omega \geq \eta$ implies that $\chi_\omega = \chi_\eta$. On the other hand $\chi_\omega = \chi_\eta$ implies that $1 = \chi_\omega(\eta_k, \mathcal{M}) = \chi_\eta(\eta_k, \mathcal{M})$ for all $k \in \mathbb{N}$. In particular, $\eta \geq \omega$. Interchanging the roles of $\eta$ and $\omega$ yields $\omega \geq \eta$, and thus $\eta = \omega$ (cf. section 2.3). The last part is a direct consequence of the definition of $\partial \mathcal{M}$. \qed

The following theorem shows that for a 1-generated $\mathbb{N}_0$-graded monoid $\mathcal{M}$ its universal boundary $\partial \mathcal{M}$ with the fine topology is a totally-disconnected compact space.

**Theorem 3.10.** The map $\overline{\chi} : (\partial \mathcal{M}, \mathcal{T}_f(\mathcal{M})) \longrightarrow \partial \mathcal{M}$ is a homeomorphism.

**Proof.** It is well known that $\chi$ is surjective (cf. [5, Lemma 2.3]), and thus $\overline{\chi}$ is surjective. By Proposition 3.9, $\overline{\chi}$ is injective, and hence $\overline{\chi}$ is a bijection. The sets

$$U^\tau_\varepsilon = \{ \eta \in \widehat{E}(\mathcal{M}) \mid \eta(\tau, \mathcal{M}) = \varepsilon \}, \quad \tau \in \mathcal{M}, \quad \varepsilon \in \{0, 1\}$$  \hspace{1cm} (3.23)

form a subbasis of the topology of $\widehat{E}(\mathcal{M})$, and

$$\chi^{-1}(U^\tau_0) = C_\varepsilon(\mathcal{M}) \cap \partial \mathcal{M}$$  \hspace{1cm} (3.24)

by (3.22). Hence $\chi^{-1}(U^\tau_0) = C_\varepsilon(\mathcal{M}) \cap \partial \mathcal{M}$ and this yields the claim. \qed
The proof of Theorem 3.10 has also shown that
\[ \chi^{-1} : \partial_M \rightarrow (\partial \mathcal{M}, \mathcal{T}(\mathcal{M})) \]  
(3.25)
is a bijective and continuous map. Thus, if \((\partial \mathcal{M}, \mathcal{T}(\mathcal{M}))\) is Hausdorff, then \(\chi^{-1}\) is a homeomorphism (cf. [1, § 9.4, Corollary 2]). This has the following consequence.

**Proposition 3.11.** Let \(\mathcal{M}\) be a 1-generated \(\mathbb{N}_0\)-graded monoid such that \((\partial \mathcal{M}, \mathcal{T}(\mathcal{M}))\) is Hausdorff. Then \(\mathcal{M}\) is \(\mathcal{T}\)-regular.

In contrast to Proposition 3.2 one has the following property for the Laca boundary of monoids.

**Proposition 3.12.** Let \(\phi : \mathcal{D} \rightarrow \mathcal{M}\) be a surjective homomorphism of connected \(\mathbb{N}_0\)-graded monoids. Then \(\phi\) induces an injective continuous map \(\phi_\partial : \partial \mathcal{M} \rightarrow \partial \mathcal{D}\).

*Proof.* By Proposition 3.6, \(\phi\) induces a map \(\phi_\Sigma : \Sigma(\mathcal{D}) \rightarrow \Sigma(\mathcal{M})\) given by
\[ \phi_\Sigma(\omega \mathcal{D}) = \phi(\omega) \mathcal{M}. \]  
(3.26)
Moreover, for \(x, y \in \mathcal{D}\) one has \(x \preceq y\), if and only if, \(x \mathcal{D} \subseteq y \mathcal{D}\), if and only if there exists \(z \in \mathcal{D}\) such that \(x = y \cdot z\). From the last statement one concludes that \(\phi_\Sigma(x \mathcal{D}) \subseteq \phi_\Sigma(y \mathcal{D})\). Thus, by (3.11), \(\phi_\Sigma\) induces a homomorphism of semigroups
\[ \phi_E : E(\mathcal{D}) \rightarrow E(\mathcal{M}), \]  
(3.27)
and thus a map
\[ \phi_\Sigma^0 : \widehat{E}(\mathcal{M}) \cup \{0\} \rightarrow \widehat{E}(\mathcal{D}) \cup \{0\}. \]  
(3.28)
If \(\phi\) is surjective, then \(\phi_E\) is surjective, and \(\phi_\Sigma^0\) restricts to a map
\[ \phi_\Sigma^0 : \widehat{E}(\mathcal{M}) \rightarrow \widehat{E}(\mathcal{D}). \]  
(3.29)
It is straightforward to verify that \(\phi_E^0\) is continuous and injective. \(\square\)

## 4 Free monoids and trees

Let \(\mathcal{F}_n = \mathcal{F}(x_1, \ldots, x_n)\) be the free monoid on \(n\) generators. Let \(S = \{x_1, \ldots, x_n\}\) be the set of generators, and let \(|\_| : \mathcal{F}_n \rightarrow \mathbb{N}_0\) be the grading morphisms, i.e., \(|y| = 1\) if and only if \(y \in S\). The Cayley graph \(\Gamma(\mathcal{F}_n, S)\) of \(\mathcal{F}_n\) with respect to \(S\) is the graph defined by
\[ V = \{ x | x \in \mathcal{F}_n \} \]  
(4.1)
\[ E = \{ (x, xx_i) \in V \times V | x \in \mathcal{F}_n, x_i \in S \}. \]  
(4.2)
The *origin* and *terminus* maps \(o, t : E \rightarrow V\) are given by the projection onto the first and second coordinate, respectively. Then \(\Gamma(\mathcal{F}_n, S)\) is an \(n\)-regular tree with root 1 and all edges pointing away from 1. The graph \(\Gamma(\mathcal{F}_n, S)\) coincides with an orientation of the \(n\)-regular tree \(T_n\).

### 4.1 The boundary of the \(n\)-regular tree

The boundary \(\partial T_n\) of \(T_n\) is the set of equivalence classes of infinite paths without backtracking under the relation \(\sim\) defined by the shift, i.e.,
\[ v_0 v_1 v_2 \cdots \sim v_1 v_2 v_3 \cdots \]  
(4.3)
We denote by \([v, w]\) the unique path starting at \(v\) in the class \(\omega\) and define

\[ I_v = \{ \omega \in \partial T_n \mid v \in [1, \omega) \} \]  

the interval of \(\partial T_n\) starting at \(v\). Then \(\partial T_n\) is compact with respect to the topology \(\mathcal{T}\) generated by \(\{I_v\}_{v \in V}\).

For any \([\rho] \in \partial T_n\) there exists a unique ray \(\rho = (e_k)_{k \in \mathbb{N}}, o(\rho) = o(e_1) = 1\). One can assign to \(\rho\) the decreasing function \(\omega_\rho \in \mathcal{P}(\mathbb{N}, \mathcal{T}_n, \leq)\) given by \(\omega_\rho(k) = t(e_k)\). The map \(\varphi: \partial T_n \to \partial \mathcal{T}_n\) given by

\[ \varphi([\rho]) = [\omega_\rho] \]  

is a bijection. Hence one can identify \(\partial T_n\) with \(\partial \mathcal{T}_n\).

### 4.2 The space \((\mathcal{F}_n, \mathcal{T}_c(\mathcal{F}_n))\)

Every cone \(C_r(\mathcal{F}_n)\) defines a rooted subtree \(T_r\) of \(T_n\) satisfying \(\partial T_r = \partial \mathcal{T}_n \cap C_r(\mathcal{F}_n)\). Thus every covering \(\bigcup_{r \in U} C_r(\mathcal{F}_n) \cap \partial \mathcal{T}_n\) of the boundary of \(\partial \mathcal{T}_n\) by cones defines a forest \(F = \bigcup_{r \in U} T_r\). Let \(F = \bigcup_{i \in I} F_i\) be the decomposition of \(F\) in connected components. Then \(\partial T_n = \partial F = \bigcup_{i \in I} \partial F_i\), where \(\cup\) denotes disjoint union. Hence the compactness of \(\partial T_n\) implies \(|I| < \infty\).

As \(\partial F_i \subseteq \partial T_n\) is closed, and hence compact, a similar argument shows that there exist finitely many cones \(C_{r_{ij}}, 1 \leq j \leq r_i\), such that \(F_i = \bigcup_{1 \leq j \leq r_i} T_{r_{ij}}\). Thus, if \(\bigcup V\) is an open covering of \(\mathcal{F}_n\) by open sets, it can be refined to a covering \(\bigcup U\), where \(U\) consists either of a cone \(C_{r(\mathcal{F}_n)}\) or of a singleton set \(\{\omega\}, \omega \in \mathcal{F}_n\). Let \(A \subseteq T_n\) be the subtree being generated by the vertices \(r_{i,j}\). Then \(A\) is a finite subtree, and the only vertices of \(T_n\) not being covered by \(\bigcup_{i,j} C_{r_{i,j}}(\mathcal{F}_n)\) are contained in \(V(A)\). This shows that \((\mathcal{F}_n, \mathcal{T}_c(\mathcal{F}_n))\) is a compact space.

### 4.3 The space \((\mathcal{M}, \mathcal{T}_c(\mathcal{M}))\)

Let \(\mathcal{M}\) be a \(\mathcal{T}\)-regular finitely 1-generated monoid. Then, by definition, \((\partial \mathcal{M}, \mathcal{T}_c(\mathcal{M}))\) is a Hausdorff space, and hence \((\mathcal{M}, \mathcal{T}_c(\mathcal{M}))\) is a Hausdorff space. By Proposition 3.2, the canonical mapping \(\phi_\mathcal{M}: \mathcal{F} \to \mathcal{M}\) (cf. (3.4)) induces a continuous surjective map \(\phi_\mathcal{M}: \mathcal{F} \to \mathcal{M}\). This shows the following.

**Proposition 4.1.** Let \(\mathcal{M}\) be a finitely 1-generated \(\mathbb{N}_0\)-graded \(\mathcal{T}\)-regular monoid. Then \((\mathcal{M}, \mathcal{T}_c(\mathcal{M}))\) is a compact space.

### 4.4 The canonical probability measure on the boundary of a regular tree

By Carathéodory’s extension theorem the assignment

\[ \mu(I_v) = n^{-|v|} \]  

defines a unique probability measure \(\mu: \text{Bor}(\partial T_n) \to \mathbb{R}_+^\ast\). Hence the corresponding probability measure \(\mu: \text{Bor}(\partial \mathcal{T}_n) \to \mathbb{R}_+^\ast\) satisfies

\[ \mu(\partial \mathcal{T}_n \cap C_r(\mathcal{F}_n)) = n^{-|r|} \text{ for } r \in \mathcal{F}_n. \]  

**Definition.** Let \(\cdot: \mathcal{F}_n \times \partial \mathcal{T}_n \to \partial \mathcal{T}_n\) be the map given by

\[ x \cdot [\omega] = [x\omega], \]  

where \(x\omega: \mathbb{N} \to \mathcal{F}_n\) is given by \((x\omega)(n) = x\omega(n)\)

Note that this action is well defined, since \(\omega \sim \omega'\) implies that \(x\omega \sim x\omega'\).
\textbf{Definition.} Let \( \cdot \cdot \cdot \) \( L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu) \times \mathcal{F}_n \to L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu) \) be the map given by
\[ f.x = \mathcal{F}_n(f) \mu, \quad \text{(4.9)} \]
where
\[ (\mathcal{F}_n(f))(\omega) = f([x\omega]). \quad \text{(4.10)} \]

Note that for \( f \in L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu) \) one has \( \mathcal{F}_n(f) \in L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu) \), since
\[ \| \mathcal{F}_n(f) \|_2^2 = \int_{\partial \mathcal{F}_n} |\mathcal{F}_n(f)(\omega)|^2 \mu(\omega) \leq \int_{\partial \mathcal{F}_n} |f(\omega)|^2 \mu(\omega) \leq \| f \|_2^2, \quad \text{(4.13)} \]
where (4.13) follows since \( x\partial \mathcal{F}_n \subseteq \partial \mathcal{F}_n \).

\textbf{Definition.} For \( z \in \mathcal{F}_n \) we define the map \( T_z : L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu) \to L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu) \) by
\[ T_z(f) = \mathcal{F}_n(f), \quad \text{(4.15)} \]

\textbf{Fact 4.2.} \( \mathcal{F}_n \) acts via \( T \) on \( L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu) \) by bounded linear operators.

\textbf{Proof.} Let \( z \in \mathcal{F}_n \). For \( f, g \in L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu) \), \([\omega] \in \partial \mathcal{F}_n \), one has
\[ (T_z(f + g))(\omega) = (T_z(f))(\omega) + (T_z(g))(\omega) \]
by definition. Thus \( T_z \) is linear. It is also bounded, since
\[ \| T_z \|_\infty = \sup_{\| f \| = 1} \| T_z(f) \|_2 \leq \sup_{\| f \| = 1} \| f \|_2 \leq 1. \quad \text{(4.16)} \]

Hence \( T_z \in \mathcal{B}(L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu)) \) for all \( z \in \mathcal{F}_n \). As \( \mathcal{B}(L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu)) \) is a \( \mathcal{C}^* \)-algebra, \( T_z \) has an adjoint operator \( T_z^* \), which is the bounded operator satisfying
\[ (T_z f, g) = (f, T_z^* g), \quad \text{(4.17)} \]
for all \( f, g \in L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu) \).

\textbf{Fact 4.3.} The bounded operator \( T_z^* \), for \( z \in \mathcal{F}_n \), is given by
\[ (T_z^* f)(\omega) = \begin{cases} 0 & \text{if } [\omega] \notin z\mathcal{F}_n \\ f([\omega']) & \text{if } [\omega] = [\omega']. \end{cases} \quad \text{(4.18)} \]

\textbf{Proof.} Note that \( T_z^* f \in L^2(\partial \mathcal{F}_n, \mathbb{C}, \mu) \), since
\[ \| T_z^* f \|_2^2 = \int_{\partial \mathcal{F}_n} |T_z^* f(\omega)|^2 \mu(\omega) \leq \int_{\partial \mathcal{F}_n} |f(\omega)|^2 \mu(\omega), \quad \text{(4.19)} \]
Let \( f, g \in L^2(\partial F_n, \mathbb{C}, \mu) \). Then one has
\[
\langle f, T^* z g \rangle = \int_{\delta T_n} f(T^* z g) \, d\mu
\]
(4.22)
\[
= \int_{z \delta T_n} f(Tz \overline{g}) \, d\mu
\]
(4.23)
\[
= \int_{z \delta T_n} (Tz f) \overline{g} \, d\mu
\]
(4.24)
\[
\leq \int_{\delta T_n} (Tz f) \overline{g} \, d\mu
\]
(4.25)
\[
= \langle Tz f, g \rangle.
\]
(4.26)
where equality (4.24) holds by
\[
f([z\omega']) T^* z g([z\omega']) = (Tz f)([\omega']) \overline{g}([\omega']).
\]
(4.27)

Proposition 4.4. The following identities hold for all \( x, y \in S \subseteq F_n \)
\[
T^*_x T_y = \delta_{xy};
\]
(4.28)
\[
\sum_{i=1}^n T_{x_i} T^*_n = 1.
\]
(4.29)
In particular, the \( C^* \)-algebra \( C^*(F_n, \mu) \subseteq \mathcal{B}(L^2(\partial F_n, \mathbb{C}, \mu)) \) generated by \( F_n \) is isomorphic to the Cuntz algebra \( \mathcal{O}_n \).

Proof. Let \( x, y \in S \subseteq F_n \) and let \( f \in L^2(\partial F_n, \mathbb{C}, \mu) \). For any \([\omega] \in \partial F_n \) one has
\[
T^*_x T_y f([\omega]) = \delta_{xy} f([\omega])
\]
(4.30)
by Fact 4.3. This proves identity (4.28).

Let \([\omega] \in \partial F_n \). Then there exists \( x_j \in S \) such that \([\omega] \in x_j \partial F_n \). Hence one has
\[
T_{x_j} T^*_x f([\omega]) = \delta_{ij} f([\omega])
\]
(4.31)
for any \( f \in L^2(\partial F_n, \mathbb{C}, \mu) \). This yields the identity (4.29). \( \square \)

4.5 Finitely 1-generated monoids

Let \( \mathcal{M} \) be a finitely 1-generated \( \mathbb{N}_0 \)-graded monoid. Then one has a canonical surjective graded homomorphism \( \phi_{\mathcal{M}} : \mathcal{F} \to \mathcal{M} \), where \( \mathcal{F} \) is a finitely generated free monoid (cf. (3.4)), which induces a continuous map \( \partial \phi : \partial \mathcal{F} \to \partial \mathcal{M} \) (cf. Proposition 3.2). In particular,
\[
\mu_{\mathcal{M}} : \text{Bor}(\partial \mathcal{M}) \to \mathbb{R}_0^+
\]
(4.32)
given by \( \mu_{\mathcal{M}}(A) = \mu(\partial \phi_{\mathcal{M}}^{-1}(A)) \) is a Borel probability measure on \( \partial \mathcal{M} \).

For \( s \in \mathcal{M} \), define the map \( \beta_s : \partial \mathcal{M} \to \partial \mathcal{M} \) by
\[
\beta_s([f]) = [sf], \quad [f] \in \partial \mathcal{M},
\]
(4.33)
where \((sf)(n) = s \cdot f(n)\) for all \(n \in \mathbb{N}, f \in \mathcal{D}(\mathbb{N}, \mathcal{M}, \preceq)\). Then, as \(\beta_s\) is mapping cones to cones, \(\beta_s\) is continuous. Hence one has a representation
\[
\beta: \mathcal{M} \to C(\partial \mathcal{M}, \partial \mathcal{M}).
\]
(4.34)
For \(s \in \mathcal{M}\), let \(\beta_{s,*}: L^2(\partial \mathcal{M}, \mathbb{C}, \mu) \to L^2(\partial \mathcal{M}, \mathbb{C}, \mu)\) be the map defined by
\[
\beta_{s,*}(g)([f]) = g(\beta_s([f])) = g([sf]), \quad g \in L^2(\partial \mathcal{M}, \mu), \ [f] \in \partial \mathcal{M}.
\]
(4.35)
Then one has
\[
\|\beta_{s,*}(g)\|^2 = \int_{\partial \mathcal{M}} |g(\beta_s([f]))|^2 \, d\mu_{\mathcal{M}}
\]
(4.36)
\[
= \int_{\partial \mathcal{M}} |g([sf])|^2 \, d\mu_{\mathcal{M}}
\]
(4.37)
\[
= \int_{\partial \mathcal{M}} |g([f])|^2 \, d\mu_{\mathcal{M}}
\]
(4.38)
\[
\leq \int_{\partial \mathcal{M}} |g([f])|^2 \, d\mu_{\mathcal{M}}
\]
(4.39)
\[
= \|g\|^2,
\]
(4.40)
for all \(g \in L^2(\partial \mathcal{M}, \mathbb{C}, \mu), s \in \mathcal{M}\). Thus,
\[
\|\beta_{s,*}\| = \sup_{\|g\|_2 = 1} \|\beta_{s,*}(g)\|_2 \leq 1
\]
(4.41)
for all \(s \in \mathcal{M}\), i.e., \(\beta_{s,*}\) is a bounded operator on \(L^2(\partial \mathcal{M}, \mathbb{C}, \mu)\). By an argument similar to the one used in the proof of Fact 4.2 one can show that it is also linear. In particular, there exists a representation
\[
\beta_s: \mathcal{M} \to \mathcal{B}(L^2(\partial \mathcal{M}, \mathbb{C}, \mu)).
\]
(4.42)

### 4.6 Right-angled Artin monoids

Let \(\Gamma = (V, E)\) be a finite undirected graph, i.e. \(|V| = n < \infty\) and \(E \subseteq \mathcal{P}_2(V)\), where \(\mathcal{P}_2(V)\) denotes the set of subsets of cardinality 2 of \(V\). The right-angled Artin monoid associated to \(\Gamma\) is the monoid \(\mathcal{M}^\Gamma\) defined by
\[
\mathcal{M}^\Gamma = \langle x \in V \mid xy = yx \text{ if } \{x, y\} \in E \rangle^+.
\]
(4.43)
Clearly, \(\mathcal{M}^\Gamma\) is \(\mathbb{N}_0\)-graded and finitely 1-generated. By Luis Paris theorem (cf. [6]), \(\mathcal{M}^\Gamma\) embeds into the right-angled Artin group \(G_\Gamma\). Thus \(\mathcal{M}^\Gamma\) has the left-cancellation property as well as the right-cancellation property. The canonical homomorphism \(\phi_\Gamma: \mathcal{F}(V) \to \mathcal{M}^\Gamma\) is surjective and induces a continuous surjective map
\[
\partial \phi_\Gamma: \partial \mathcal{F}(V) \to \partial \mathcal{M}^\Gamma.
\]
(4.44)
(cf. Proposition 3.2). We denote by \(\mu_\Gamma: \text{Bor}(\partial \mathcal{M}^\Gamma) \to \mathbb{R}_0^+\) the Borel probability measure induced by \(\partial \phi_\Gamma\), i.e., for \(A \in \text{Bor}(\partial \mathcal{M}^\Gamma)\) one has
\[
\mu_\Gamma(A) = \mu(\partial \phi_\Gamma^{-1}(A)),
\]
(4.45)
where \(\mu\) is the measure defined on \(\partial \mathcal{F}(V)\) by (4.7).

**Definition.** Let \(\Gamma = (V, E)\) be a graph, and let \(\Gamma_1 = (V_1, E_1)\) and \(\Gamma_2 = (V_2, E_2)\) be subgraphs of \(\Gamma\). We say that \(\Gamma\) is bipartitely decomposed by \(\Gamma_1\) and \(\Gamma_2\), if \(V = V_1 \sqcup V_2\) and
\[
E = E_1 \sqcup E_2 \sqcup \{ \{v_1, v_2\} \mid v_1 \in V_1, v_2 \in V_2 \}.
\]
(4.46)
In this case we will write \(\Gamma = \Gamma_1 \lor \Gamma_2\). If no such decomposition exists, \(\Gamma\) will be called coconnected.
Any graph $\Gamma$ can be decomposed into connected components $\Gamma_i$, i.e. $\Gamma = \bigcup_{i \in I} \Gamma_i$. In a similar fashion one may define a decomposition in coconnected components.

**Definition.** Let $\Gamma = (V, E)$ be a graph and let $\Gamma^{\text{op}} = \bigsqcup_{i \in I} A_i$ be the decomposition of $\Gamma^{\text{op}}$ in its connected components. We will call

$$\Gamma = \bigvee_{i \in I} A_i^{\text{op}},$$

(4.47)

the decomposition of $\Gamma$ in coconnected components.

One has the following property.

**Fact 4.5.** Let $\Gamma = (V, E)$ be an undirected graph. Then $\Gamma$ is coconnected if, and only if, $\Gamma^{\text{op}}$ is connected. In particular, if $\Gamma^{\text{op}} = \bigsqcup_{i \in I} A_i$ is the decomposition of $\Gamma^{\text{op}}$ in its connected components, then one has

$$\Gamma = \bigvee_{i \in I} A_i^{\text{op}},$$

(4.48)

where $A_i^{\text{op}}$ are coconnected subgraphs of $\Gamma$.

**Proof.** Obviously, the graph $\Gamma = \Gamma_1 \cup \Gamma_2$ is bipartitly decomposed if, and only if, $\Gamma^{\text{op}} = \Gamma_1^{\text{op}} \sqcup \Gamma_2^{\text{op}}$ is not connected. This yields to the claim. ☐

Note that the decomposition in coconnected components implies that any two vertices in different components must be connected by an edge. From this property one concludes the following straightforward fact.

**Fact 4.6.** Let $\Gamma = (V, E)$ be a finite graph with unoriented edges, and let $\Gamma = \bigvee_{i \in I} \Gamma_i$ be its decomposition in coconnected components, $\Gamma_i = (V_i, E_i)$. Then

$$\mathcal{A}^\Gamma = \mathcal{A}^{\Gamma_1} \times \cdots \times \mathcal{A}^{\Gamma_r},$$

(4.49)

where $\mathcal{A}^{\Gamma_i} = \langle v \in V_i \rangle$. In particular, $\partial \mathcal{A}^{\Gamma_i} = x_{1_{\text{st}}} \partial \mathcal{A}^{\Gamma_i}$ and

$$L^2(\partial \mathcal{A}^{\Gamma_i}, C, \mu_{\Gamma_i}) = L^2(\partial \mathcal{A}^{\Gamma_i}, C, \mu_{\Gamma_1}) \otimes \cdots \otimes L^2(\partial \mathcal{A}^{\Gamma_i}, C, \mu_{\Gamma_r}).$$

(4.50)

In [2], J. Crisp and M. Laca has shown the following.

**Theorem 4.7** ([2], Theorem 6.7). Let $\Gamma = (V, E)$ be a finite unoriented graph such that $\Gamma^{\text{op}}$ has no isolated vertices, and let $\Gamma = \bigvee_{i = 1}^r \Gamma_i$ be the decomposition of $\Gamma$ in coconnected components, $\Gamma_i = (V_i, E_i)$. Then the universal $\mathcal{C}^*$-algebra with generators $\{S_x \mid x \in V\}$ subject to the relations

(i) $S_x S_x = 1$ for each $x \in V$;
(ii) $S_x S_y = S_y S_x$, and $S_x^2 S_y = S_y S_x^2$ if $x$ and $y$ are adjacent in $\Gamma$;
(iii) $S_x S_y = 0$ if $x$ and $y$ are distinct and not adjacent in $\Gamma$;
(iv) $\prod_{x \in V}(1 - S_x^2 S_x) = 0$ for each $i \in \{1, \ldots, r\}$;

is canonically isomorphic to the boundary quotient $\partial \mathcal{C}_\mathcal{A}(\mathcal{A}^\Gamma)$ for $\mathcal{A}^\Gamma$ and it is a simple $\mathcal{C}^*$-algebra.

Hence, one has the following proposition.

**Proposition 4.8.** The $\mathcal{C}^*$-algebra $\mathcal{C}^*(\mathcal{A}^\Gamma, \mu_{\Gamma})$ (cf. (1.6)) of a right-angled Artin monoid $\mathcal{A}^\Gamma$ is isomorphic to the boundary quotient $\partial \mathcal{C}_\mathcal{A}(\mathcal{A}^\Gamma)$ of Theorem 4.7.

**Proof.** Let $\Gamma = (V, E)$ be a finite unoriented graph such that $|V| = n$ and let $\Gamma = \bigvee_{i = 1}^r \Gamma_i$ be its decomposition in coconnected components. It is straightforward to verify (i)-(iii) for the set of operators $\{T_x \mid x \in V\}$, where

$$T_x(f(\omega)) = f([x \omega]),$$

(4.51)
and the adjoint operators are given by

\[
(T_x \alpha)(\{u\}) = \begin{cases} 
0 & \text{if } [u] \notin x \mathcal{M}^R \\
\alpha([u']) & \text{if } [u] = x[u'], 
\end{cases}
\]  

(4.52)

where \( f \in L^2(\partial \mathcal{M}^R, \mathbb{C}, \mu_f) \). It remains to prove that it also satisfies (iv). Let

\[
e_j = \prod_{x \in V_i} (1 - T_x T_x^*) \]  

(4.53)

In order to show that \( e_i(f) = 0 \) for all \( f \in L^2(\partial \mathcal{M}^R, \mathbb{C}, \mu_f) \) it suffices to show that \( e_i(f) = 0 \) for \( f = f_1 \otimes \cdots \otimes f_r, \) \( f_1 \in L^2(\partial \mathcal{M}^R, \mathbb{C}, \mu_f) \) (cf. (4.50)). Note that

\[
(1 - T_s T_s^*)(f)(\{u\}) = \begin{cases} 
0 & \text{if } [u] \in x \mathcal{M}^R, \\
f([u]) & \text{otherwise}. 
\end{cases}
\]  

(4.54)

Let \( [u] = [u_1] \cdots [u_l], [u_j] \in \partial \mathcal{M}^R \). Then there exists \( y \in V_i \) such that \( [u_j] \in y \mathcal{M}^R \). Hence, by (4.54)

\[
(1 - T_y T_y^*)(f)(\{u\}) = 0. 
\]  

(4.55)

Hence \( e_i(f) = 0 \) and this yields the claim. \( \square \)

## 5 Fractals

Let \( \mathcal{M} \) be a finitely 1-generated monoid. By an \( \mathcal{M} \)-fractal we will understand a compact metric space \( (X, d) \) with a contracting left \( \mathcal{M} \)-action \( \alpha : \mathcal{M} \rightarrow C(X, X) \), i.e., there exists a real number \( \delta < 1 \) such that for all \( x, y \in X \) and all \( \omega \in \mathcal{M} \setminus \{1\} \) one has

\[
d(\alpha(\omega)(x), \alpha(\omega)(y)) < \delta \cdot d(x, y). 
\]  

(5.1)

The real number \( \delta \) will be called the contraction constant. To the authors knowledge the following important question has not been discussed in the literature yet.

**Question 4.** For which finitely 1-generated monoids \( \mathcal{M} \) does there exist an \( \mathcal{M} \)-fractal \( (X, d, \alpha) \)?

**Example 5.1.** Let \( s_1, s_2 : I \rightarrow I, I = [0, 1], \) be defined by \( s_1(x) = \frac{1}{2} x, \ s_2(x) = \frac{1}{2} + s_1(x) \). Then \( \langle s_1, s_2 \rangle \subseteq C(I, I) \) is isomorphic to the free monoid \( \mathcal{F}_2 \) on 2 generators. The \( \mathcal{F}_2 \)-fractal \( (I, d, \alpha) \), where \( d \) is the standard metric and \( \alpha \) is the action described above, has as attractor the Cantor set (see [3], Ex. 3.3).

### 5.1 The action of the universal boundary on an \( \mathcal{M} \)-fractal

Let \( \mathcal{M} \) be a finitely 1-generated monoid with grading \( \langle - ; \mathcal{M} \rangle : \mathcal{M} \rightarrow N_0 \). For a strictly decreasing sequence \( f \in \mathcal{D}(\mathbb{N}, \mathcal{M}, \prec) \) and for \( n, m \in \mathbb{N}_0, m > n \), there exists \( \tau_{m,n} \in \mathcal{M} \setminus \{1\} \) such that \( f(m) = f(n) \cdot \tau_{m,n} \). By induction, one concludes that \( |f(n)| \geq n \). If \( [f] \in \partial \mathcal{M} \), then \( f \) can be represented by a strictly decreasing sequence (cf. Fact 2.3).

As \( \alpha \) is contracting, one concludes that \( (\alpha(f(n)))(x) \) is a Cauchy sequence for every strictly decreasing sequence \( f \in \mathcal{D}(\mathbb{N}, \mathcal{M}, \prec) \) and thus has a limit point \( \alpha(f)(x) = \lim_{n \rightarrow \infty} (\alpha(f(n)))(x) \). In more detail, if \( \alpha \) has contracting constant \( \delta < 1 \), one has for \( n, m \in \mathbb{N}, m > n \), that

\[
d(\alpha(f(m))(x), \alpha(f(n))(x)) < \delta^{j[n]} \cdot d(\alpha(\tau_{m,n})(x), x) \leq \delta^{j[n]} \cdot \text{diam}(X), 
\]  

(5.2)

where \( \text{diam}(X) = \max\{ d(y, z) \mid y, z \in X \} \). Thus one has a map

\[
\langle - ; \mathcal{M} \rangle : \mathcal{D}(\mathbb{N}, \mathcal{M}, \prec) \times X \rightarrow X
\]  

(5.3)

given by \( [f] \cdot x = \alpha(f)(x) \). This map has the following property.
Remark 5.1. (a) Let \((X, d)\) be a compact metric space. For \(A, B \subseteq X\) the Hausdorff metric \(\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_0^+\), where \(\mathcal{P}(X)\) denotes the set of subsets of \(X\), is given by
\[
\delta(A, B) = \sup \{ d(a, B), d(b, A) \mid a \in A, b \in B \},
\]
where \(d(a, B) = \inf \{ d(a, b) \mid b \in B \}\) (cf. [3, (2.4)]).

(b) Let \(\mathcal{M}\) be a finitely 1-generated monoid, and let \((X, d), x\) be an \(\mathcal{M}\)-fractal with attractor \(K \subseteq X\). For \(\mathcal{P} : \mathcal{P}(X) \to \mathcal{P}(X)\) given by
\[
\mathcal{P}(A) = \bigcup_{\sigma \in \mathcal{M}_1} \sigma(A),
\]
it is well known that \((\mathcal{P}^k(A))_{k \in \mathbb{N}},\) where \(\mathcal{P}^k(A) = \mathcal{P}(\mathcal{P}^{k-1}(A))\), converges to \(K\) in the Hausdorff metric (cf. [3, Statement (1)]).

Proposition 5.2. Let \(\mathcal{M}\) be a finitely 1-generated monoid, and let \((X, d), a\) be an \(\mathcal{M}\)-fractal with attractor \(K \subseteq X\). Then the map \((5.3)\) is continuous and \([f] \cdot x \in K\) for all \(f \in \mathcal{D}(\mathcal{N}, \mathcal{M}, \prec)\) and \(x \in X\).

Proof. Let \(f \in \mathcal{D}(\mathcal{N}, \mathcal{M}, \prec)\) be a strictly decreasing function. For \(A = \{x\}\), and \(\mathcal{P}\) as above, the sequence \((\mathcal{P}^k(A))_{k \in \mathbb{N}}\) converges to \(K\) in the Hausdorff metric. Thus for all \(\varepsilon > 0\) there exists \(N(\varepsilon) \in \mathbb{N}\) such that for all \(n > N(\varepsilon)\) one has \(\delta(\mathcal{P}^n(A), K) < \varepsilon\). Hence \(d(\mathcal{P}(n))(x), K) < \varepsilon\) for all \(n > N(\varepsilon)\), and \(\mathcal{P}(n)(x)\) is a clusterpoint of \(K\). As \(K\) is closed this implies \(f(\mathcal{P}(n))(x) \in K\).

The map \((5.3)\) is obviously continuous in the second argument. Moreover, let \(f, h \in \mathcal{D}(\mathcal{N}, \mathcal{M}, \prec), f, h \prec \tau_1, \tau \in \mathcal{M}_.\) Then
\[
d(a(f(x)), a(h)(x)) \leq 2 \cdot \delta|\tau| \cdot \text{diam}(X). \tag{5.4}
\]
Thus \((5.3)\) is continuous.

Proposition 5.3. Let \(f, h \in \mathcal{D}(\mathcal{N}, \mathcal{M}, \prec)\) satisfying \(f \preceq h\). Then, \(\mathcal{P}(n)(x) = a(h)(x)\).

Proof. We may assume that \(f(n) \preceq h(n)\) for all \(n \in \mathbb{N}\), i.e., there exists \(y_n \in \mathcal{M}\) such that \(f(n) = h(n) \cdot y_n\). Then, by the same argument which was used for \((5.2)\), one concludes that
\[
d(\mathcal{P}(n)(x), h(n)(x)) \leq \delta^{h(n)} \cdot \text{diam}(X) \leq \delta^n \text{diam}(X). \tag{5.5}
\]
This yields the claim.

From Proposition 5.3 one concludes that the map \((5.3)\) induces a map
\[
\alpha \cdot \tau : \hat{\mathcal{M}} \times X \to X \tag{5.6}
\]
given by \(\alpha(f) \cdot x = a(f)(x)\) (cf. (2.13)), and thus an action of \(\hat{\mathcal{M}}\) on \(X\).

The following property suggest to think of \((\hat{\mathcal{M}}, \mathcal{P}_c)\) as the universal attractor of an \(\mathcal{M}\)-fractal.

Proposition 5.4. Let \(x \in X\), and let \(K \subset X\) be the attractor of the \(\mathcal{M}\)-fractal \((X, d), a\). Then the induced map
\[
\kappa_x : \hat{\mathcal{M}} \to K \tag{5.7}
\]
given by \(\kappa_x([f]) = a(f)(x)\) is surjective.

Proof. Let \(z \in K\), and \(A = \{x\}\). By (cf. [3, (2.4)]), for all \(\varepsilon > 0\) there exists \(N(\varepsilon) \in \mathbb{N}\) such that for all \(n > N(\varepsilon)\) one has \(\delta(\mathcal{P}^n(A), z) < \varepsilon\), i.e., there exists a sequence \((f_n)_{n \in \mathbb{N}}, f_n \in \mathcal{M}, f_{n+1} \in \bigcup_{\mathcal{M}_1} \{\sigma \cdot f_n\}\), such that \(d(\mathcal{P}(n)(x), z) < \varepsilon\).

If \(\mathcal{M}\) is \(\mathcal{F}\)-regular, then \((\hat{\mathcal{M}}, \mathcal{P}(\mathcal{M}))\) is compact (cf. Proposition 4.1). Hence \((f_n)_{n \in \mathbb{N}}\) has a cluster point \(f \in \hat{\mathcal{M}}\). As \(|f_n| = n\), one has \(f \notin \mathcal{M}\) and thus \(f \in \hat{\mathcal{M}}\). It is straightforward to verify that \([f] \cdot x = z\), showing that \(\kappa_x\) is surjective.
5.2 The C*-algebra associated to an M-fractals for a finitely 1-generated monoid M

Let M be a finitely 1-generated monoid, and let ((X, d), α) be an M-fractal with attractor K. For x ∈ X there exists a continuous mapping κx : ∂M → K (cf. Theorem C). Let µx : Bor(K) → ℝ≥0 be the probability measure given by (1.8). Then M acts on K, and thus also on L²(K, C, µx).

For t ∈ M let γt : L²(K, C, µx) → L²(K, C, µx) be given by

\[ γ_t(g)(x) = g(α_t(x)) \]  

(5.8)

where g ∈ L²(K, C, µx). Hence the monoid M acts on the Hilbert space L²(K, C, µx) by bounded linear operators.

\[ \|γ_t(g)\|² = \int_K |γ_t(g(z))|^² dµ_x = \int_K |g(α_t(z))|^² dµ_x \leq \|g\|² \]  

(cf. § 4.1). One defines the C*-algebra generated by the M-fractal ((X, d), α) by

\[ C^*(M, X, d, µ_x) = \langle γ_t, γ_t^* | t ∈ M \rangle \subseteq B(L²(K, C, µ_x)). \]  

(5.9)

References

[1] N. Bourbaki, General topology. Chapters 1–4, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation. MR 1726779
[2] J. Crisp and M. Laca, Boundary quotients and ideals of Toeplitz C*-algebras of Artin groups, J. Funct. Anal. 242 (2007), no. 1, 127–156. MR 2274018
[3] J. E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981), no. 5, 713–747. MR 625600
[4] X. Li, Semigroup C*-algebras, arXiv:1707.05940v1 (2017).
[5] X. Li, T. Omland, and J. Spielberg, C*-algebras of right LCM one-relator monoids and Artin-Tits monoids of finite type, preprint, July 2018.
[6] L. Paris, Artin monoids inject in their groups, Comment. Math. Helv. 77 (2002), no. 3, 609–637. MR 1933791