MUKAI’S PROGRAM FOR CURVES ON A K3 SURFACE

E. ARBARELLO, A. BRUNO, E. SERNESI

Abstract. Let $C$ be a general element in the locus of curves in $M_g$ lying on some K3 surface, where $g$ is congruent to 3 mod 4 and greater than or equal to 15. Following Mukai’s ideas, we show how to reconstruct the K3 surface as a Fourier-Mukai transform of a Brill-Noether locus of rank two vector bundles on $C$.

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1. Introduction

Let $K_g$ be the moduli stack of pairs $(S,H)$ where $S$ is a K3 surface and $H$ is a very ample line bundle on $S$ such that $H^2 = 2g - 2$. Let $P_g$ be the stack of pairs $(S,C)$ such that $(S,H) \in K_g$ and $C \in |H|$ is smooth and irreducible. Finally let $M_g$ be the moduli stack of smooth curves of genus $g$. The stacks $K_g$, $P_g$, $M_g$ are smooth Deligne-Mumford stacks of dimensions $19, 19+g, 3g-3$ respectively. We have natural morphisms:

\[
\begin{array}{ccc}
M_g & \overset{m_g}{\longrightarrow} & P_g \\
& \searrow & \downarrow \kappa_g \\
& & K_g
\end{array}
\]

where $\kappa_g$ realizes $P_g$ as an open subset of a $\mathbb{P}^9$-fibration. In \cite{4}, Theorem 5, the authors prove that for $g = 11$ and $g \geq 13$ the morphism $m_g$ is a birational map onto its image using properties of the Gauss map for the canonical divisor (also known as Wahl map). Related references are \cite{16}, \cite{17}.

In the last page of \cite{12}, Mukai laid out a beautiful program to actually reconstruct a K3 surface from a curve lying on it, thus giving a rational inverse of $m_g$, whenever the genus $g$ is congruent to 3, mod 4 and greater or equal than 11, and this program was successfully carried out by him, for the case $g = 11$, in \cite{10}.
In our work we take Mukai’s paper \cite{10} as a blueprint and generalize it to all genera which are congruent to 3, mod 4 and greater or equal than 11.

Let \((S, C)\) be a general point in \(\mathbb{P}_g\), with \(g = 2s + 1, \ s \geq 5\) odd. Mukai’s strategy to reconstruct the surface \(S\) from the curve \(C\) is as follows: consider the Brill-Noether locus \(M_C(2, K_C, s)\) which is the moduli space of semistable rank-two vector bundles on \(C\) having canonical determinant and possessing at least \(s + 2\) linearly independent sections. Then \(M_C(2, K_C, s)\) is a K3 surface and the surface \(S\) can be obtained as an appropriate Fourier-Mukai transform of it.

When \(g = 11\) the proof consists of three main steps. One first considers pairs \((S', C')\) where \(S'\) is a K3 surface of a special type, and proves with ad-hoc constructions that \(g\) \(S\) to \(S\) can be obtained as an appropriate Fourier-Mukai transform of it. Finally one shows the existence of an appropriate polarization \(h\) on \(M_C(2, K_C, 5)\) which induces an isomorphism between \(S\) and the Fourier-Mukai transform of \(M_C(2, K_C, 5)\) with respect to \(h\).

The first difficulty in trying to extend this proof is that when \(g = 2s + 1, \ s \geq 6\), the expected dimension of \(M_C(2, K_C, s)\) is zero for \(s = 6\) and negative for \(s \geq 7\), so that it is not even clear that \(M_C(2, K_C, s)\) is non-empty when \(s \geq 7\). However, in her paper \cite{15}, Voisin associates a rank-two vector bundle \(E_L\) to each base-point-free pencil \(|L|\) on \(C\) of degree \(s + 2\). Each of these bundles is exhibited as an extension

\[
0 \rightarrow K_C L^{-1} \rightarrow E_L \rightarrow L \rightarrow 0
\]

and one can prove that Voisin’s bundles \(E_L\) are stable, (see, for instance, Lemma 2.5, Proposition 3.1 and Remark 5.11) and that, as \(L\) varies in \(W_{s+2}^1(C)\), they describe a one-dimensional locus in \(M_C(2, K_C, s)\).

Consider on the K3 surface \(S\) the Mukai vector \(v = (2, |C|, s)\) and denote by \(M_v(S)\) the moduli space of \([C]\)-stable, rank-two vector bundles \(E\) on \(S\) with \(c_1(E) = |C|\) and \(\chi(S, E) = s + 2\). For a general K3 surface \(M_v(S)\) is again a smooth K3 surface. One of our main results is the following:

**Theorem 6.1.** For a general \((C, S) \in \mathbb{P}_g\), \(g = 2s + 1, \ s \geq 5\), there is a unique irreducible component \(V_C(2, K_C, s)\) of \(M_C(2, K_C, s)\) containing the Voisin’s bundles \(E_L\). By sending \(E\) to \(E_C\) one obtains a well defined isomorphism

\[
\sigma : M_v(S) \rightarrow V_C(2, K_C, s)_{\text{red}}
\]

In particular \(V_C(2, K_C, s)_{\text{red}}\) is a smooth K3 surface.

Note that we only assumed \(g\) to be odd in Theorem 6.1. Let now \(M_C(2, K_C)\) be the moduli space of rank two vector bundles on \(C\) with determinant equal to \(K_C\). Write, for simplicity, \(T = V_C(2, K_C, s)_{\text{red}} \subset M_C(2, K_C)\). Following Mukai’s program, let \(U\) be a universal bundle on \(C \times T\), let \(\pi_C\) and \(\pi_T\) be the natural projections from \(C \times T\) to \(C\) and \(T\), respectively, and consider the determinant of the cohomology

\[
h_{\text{det}} = (\det R^1 \pi_{T*} U) \otimes (\det \pi_{T*} U)^{-1}
\]

For \(s\) odd, (i.e. \(g \equiv 3\) mod 4), we prove that \(h_{\text{det}}\) is a genus \(g\) polarization on \(T\) and that \(U\) can be chosen in such a way that the map

\[
C \rightarrow \hat{T} = M_T(T)
\]

\[x \mapsto U_{\{x\} \times T}\]

is an embedding and we have the following theorem (see also the more detailed statement in \cite{17}).
Theorem 7.1. Let \((C, S)\) be a general point of \(\mathcal{P}_g\), where \(g = 2s + 1\), and \(s\) is odd and greater than or equal to 5. Let \(T = V_C(2, K_C, s)_{\text{red}}\). Consider the Mukai vector \(\hat{v} = (2, h_{\text{det}}, s)\). Then any K3 surface containing \(C\) is isomorphic to \(\hat{T} = M_{\hat{v}}(T)\).

In proving both Theorem 6.1 and Theorem 7.1, the basic tool consists in degenerating the surface \(S\) to a rather special K3 surface where both the geometry of the moduli space \(M_v(S)\) and the properties of the morphism \(\sigma : M_v(S) \to V_C(2, K_C, s)_{\text{red}}\) are made transparent by virtue of an explicit isomorphism \(S \cong M_v(S)\) (see Proposition 5.7).

The special K3 surfaces we consider are the direct generalizations of those considered by Mukai in his analysis of the genus 11 case. Namely, we consider a K3 surface \(S\) such that

\[(1.3) \quad \text{Pic}(S) = \mathbb{Z} \cdot [A] \oplus \mathbb{Z} \cdot [B],\]

with

\[(1.4) \quad |C| = |A + B|, \quad \text{and} \quad g(A) = s, \quad g(B) = 1\]

This means that the elliptic pencil \(|B|\) cuts out on \(C\) a \(g_{s+1}^1\) which we call \(\xi\), while the linear system \(|A|\) cuts out on \(C\) the residual series, a \(g_{3s-1}^3\) which we call \(\eta\). An isomorphism

\[(1.5) \quad \rho : S \to M_v(S)\]

is obtained by assigning to each \(x \in S\) the vector bundle \(\mathcal{E}_x\) defined as the unique extension

\[0 \to \mathcal{O}_S(B) \to \mathcal{E}_x \to I_x(A) \to 0\]

The isomorphism \(\rho\) makes \(S\) self-dual, from the Fourier-Mukai point of view. Such self-duality is the key to prove Theorem 6.1 for pairs \((C, S)\) satisfying (1.3) and (1.4). Moreover, in this case \(S \cong M_C(2, K_C, s) = V_C(2, K_C, s)\) (Theorem 5.1). The geometry of the special surface \(S\) is quite different from the \(g = 11\) case, and requires a number of new auxiliary results that are proved in §4. Moreover the negativity of the expected dimension of \(M_C(2, K_C, s)\) is the reason for some lengthening in the proof of Theorem 5.1.

The embedding of \(S\) in \(\mathbb{P}^s\) via the linear series \(|A|\) also plays a fundamental role. Denote by \(T\) and \(\Gamma\) the images of \(S\) and \(C\) respectively, via this embedding. Then, the quadratic hull of \(T\) coincides with the quadratic hull of \(\Gamma\) and, as such, it classifies extensions on \(C\):

\[0 \to \xi \to E \to \eta \to 0\]

The rank-two vector bundles \(E\), or better their stable models, obtained in this way, parametrize \(M_C(2, K_C, s)\) giving, via (1.5), a geometrical interpretation of the isomorphism (1.2).

In the general case Theorem 6.1 is proved by a variational argument similar to Mukai’s, with the added difficulty coming from the negativity of the expected dimension of the Brill-Noether locus. We consider a family of pairs \((S, C)\) with a special fibre satisfying (1.3) and containing a general pair with Picard rank one among its fibres. By applying some deformation theory arguments we are able to control the behaviour of the map (1.5) on the general fibre, overcoming the fact that \(M_C(2, K_C, s)\) has negative expected dimension.

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2. Moduli of sheaves on a K3 surface

Let \((S, C)\) be a pair consisting of a K3 surface \(S\) and a nonsingular curve \(C \subset S\) of genus

\[ g(C) = g = 2s + 1, \]

for some \(s \geq 5\). We let \(M_{v,C}(S)\) be the moduli space of \([C]\)-semistable sheaves with Mukai vector \(v\) on \(S\) and polarization \([C]\). The Mukai vector of a sheaf \(F\) is given by

\[ v(F) = (r(F), c_1(F), \chi(F) - r(F)) \]

where \(r(F)\) denotes the rank of \(F\). From now on we consider the case in which the Mukai vector \(v\) is given by

\[ (2.1) \quad v = (2, [C], s) \]

and we will write

\[ M_{v,C}(S) = M_v(S) \]

As already anticipated in the Introduction, in this paper we will mostly consider the following two cases, to which we give a name.

- **Rank-1 case:** \(\text{Pic}(S) = \mathbb{Z} \cdot [C]\).
- **Rank-2 case:** \(\text{Pic}(S) = \mathbb{Z} \cdot [A] \oplus \mathbb{Z} \cdot [B]\), with \([C] = [A + B]\), \(A\) and \(B\) nonsingular connected and \(g(A) = s, g(B) = 1\). We then write

\[ \mathcal{O}(B)_{|C} = \xi, \quad \mathcal{O}(A)_{|C} = \eta \]

so that \(\xi\) is a \(g^1_{2s+1}\) and \(\eta\), the residual of \(\xi\), is a \(g^1_{3s-1}\). For the existence of K3 surfaces of this type we refer to Theorem 1.1 in \([8]\).

**Lemma 2.1.** In both the Rank-1 and the Rank-2 cases the Mukai vector \((2.1)\) is primitive and there are no walls relative to \(v\).

**Proof.** In the Rank-1 case the lemma is obvious. Let us assume that we are in the Rank-2 case. The curve \(C\) is clearly primitive and therefore \(v\) is primitive as well. Let us show that there are no walls relative to \(v\) in the ample cone of \(S\) (see \([7]\) Definition 4.C.1). Following the notation of Theorem 4.C.3 in \([7]\), it suffices to prove that there is no element

\[ \lambda = 2c_1(F') - c_1(F) \]

where \(F\) is a \(\mu_C\)-semistable sheaf in \(M_{v,C}(S)\) and \(F' \subset F\) is a rank-1 subsheaf with \(\mu_C(F') = \mu_C(F)\), such that

\[ (2.2) \quad -\Delta \leq \lambda^2 \leq 0 \]

where \(\Delta = 4c_2(F) - c_1^2(F) = 4(s+2) - 4s = 8\). We may write \(c_1(F') = hA + kB\). Now (2.2) reads

\[ -8 \leq (2h - 1)^2(2s - 2) + 2(2k - 1)(2h - 1)(s + 1) \leq 0 \]

The equality \(\mu_C(F') = \mu_C(F)\), gives \(2h(s - 1) + k(s + 1) = 2s\) and the above inequalities can be written as

\[ -4 \leq -s(2h - 1)^2 \leq 0 \]

and this has no solutions for \(s \geq 5\). So there are no walls. \(\square\)

**Lemma 2.2.** Assume that the Mukai vector \((2.1)\) is primitive and there are no walls relative to \(v\). Then the moduli space \(M_v(S)\), if not empty, is a smooth K3 surface and all of its points represent locally free sheaves. In particular this happens if we are in the Rank-1 or in the Rank-2 case.
Proof. Applying Theorem 4.C.3 and Lemma 1.2.13 in [7] we deduce that all sheaves in \( M_v(S) \) are [C]-stable. Since \( v \) is isotropic it follows that \( M_v(S) \) is a smooth and irreducible projective surface ([7], Theorem 6.1.8) which is a K3 by [13]. Since in this case [C]-stability is equivalent to \( \mu \)-stability from [7] Remark 6.1.9 p. 145 it follows that all the points of \( M_v(S) \) represent locally free sheaves. The last assertion is a consequence of Lemma 2.1. □

From the lemma it follows that, under its assumptions, \( [C]-(\text{semi})\)stability is computed in terms of the \( C \)-slope which is defined by

\[
\mu_C(F) = \frac{c_1(F) \cdot C}{r(F)}
\]

In particular this happens if we are in the the Rank-1 or in the Rank-2 case. Let us recall the definition of Lazarsfeld-Mukai bundle.

**Definition 2.3.** Let \( L \) be a globally generated pencil on \( C \subset S \). The Lazarsfeld-Mukai bundle \( \tilde{E}_L \) is the dual of the rank-2 vector bundle \( \tilde{F}_L \) defined by the exact sequence

\[
0 \to \tilde{F}_L \to H^0(L) \otimes O_S \xrightarrow{ev} L \to 0
\]

**Remark 2.4.** Often, in the literature, the bundle \( \tilde{F}_L \) is denoted by the symbol \( F_{C,L} \) and its dual bundle \( \tilde{E}_L \) is denoted by the symbol \( E_{C,L} \).

For these bundles one easily computes the basic invariants:

\[
\begin{align*}
  r(\tilde{E}_L) &= 2, \quad c_1(\tilde{E}_L) = [C], \quad c_2(\tilde{E}_L) = \deg L, \\
  h^0(\tilde{F}_L) &= h^0(\tilde{E}_L(-C)) = h^2(\tilde{E}_L) = 0, \\
  h^1(\tilde{F}_L) &= h^1(\tilde{E}_L(-C)) = h^1(\tilde{E}_L) = 0, \\
  h^0(\tilde{E}_L) &= h^0(L) + h^1(L)
\end{align*}
\]

As far as the \( C \)-slope is concerned, we have

\[
\mu_C(\tilde{E}_L) = 2s
\]

We will need the following Lemma (see also Remark 5.11).

**Lemma 2.5.** Assume that we are in the Rank-1 case. Let \( |L| \) be a \( g^1_{s+2} \) on \( C \). Then \( \tilde{E}_L \) is stable. In particular \( M_v(S) \neq \emptyset \).

**Proof.** Observe that the \( g^1_{s+2} \) is automatically base-point-free because, by Lazarsfeld’s proof of Petri conjecture, the curve \( C \) is Petri and thus has no \( g^1_{s+1} \). Since \( \text{Pic}(S) = \mathbb{Z} \cdot [C] \) and \( c_1(\tilde{E}_L) = [C] \), we may assume that a destabilizing sequence has the form

\[
0 \to O(nC) \to \tilde{E}_L \to I_X((1-n)C) \to 0
\]

where \( n \geq 1 \) and \( X \subset S \) is a zero-dimensional subscheme. But this is absurd since \( h^0(S, O(nC)) > h^0(S, \tilde{E}_L) = s + 2 \). The last assertion is obvious. □

**Definition 2.6.** Let \( L \) be a globally generated pencil on \( C \). The restriction to \( C \) of the Lazarsfeld-Mukai bundle \( \tilde{E}_L \) is called the Voisin bundle of \( L \) and it is denoted by the symbol \( E_L \).
Remark 2.7. It is a very remarkable fact that the bundle $E_L$ only depends on $C$, $L$ and the first infinitesimal neighborhood $C_2$ of $C$ in $S$. In [15] Voisin proves that, even more generally, one may construct a bundle having all the properties of $E_L$ starting from $C$, $L$ and an embedded ribbon $C_2 \subset \mathbb{P}^g$ having $C$ as hyperplane section. She also observes that the datum of an embedded ribbon $C_2 \subset \mathbb{P}^g$ having $C$ as hyperplane section, is equivalent to the datum of an element $u$ in the kernel of the dual of the Gaussian map

$$H^1(C, T^2_C) \rightarrow H^1(C, T_{\mathbb{P}^g} \otimes T_C)$$

Moreover she proves that if $R_L \in H^0(C, K_C + 2L)$ is the ramification divisor of the map determined by $|L|$, the class $uR_L \in H^1(C, T_C + 2L)$ determines an extension

$$(2.5) \quad 0 \rightarrow L \rightarrow E_L \rightarrow KL^{-1} \rightarrow 0$$

which splits at the level of cohomology so that

$$(2.6) \quad h^0(E_L) = h^0(L) + h^1(L) = h^1(E_L)$$

From (2.6) it follows that if $L$ a general element of $W^1_{s+2}$ then $E_L$ is a rank-two vector bundle on $C$ with determinant equal to $K_C$ and with $s+2$ linearly independent sections. From the Brill-Noether point of view this is most unusual. Certainly a general curve of genus $2s+1$ admits no such a vector bundle. The next section is devoted to the analysis of those Brill-Noether loci that are relevant in our study of curves lying on K3 surfaces.

3. Brill-Noether loci for moduli of vector bundles on $C$

Let $(S, C)$ be as in the previous section. We denote by $M_C(2, K_C)$ the moduli space of rank two, semistable vector bundles on $C$ with determinant equal to $K_C$. We consider the Brill-Noether locus

$$M_C(2, K_C, s) = \{ [F] \in M_C(2, K_C) \mid h^0(C, F) \geq s + 2 \}$$

A point $[F] \in M_C(2, K_C)$ corresponding to a stable bundle is smooth for $M_C(2, K_C)$ and

$$T_{[F]}(M_C(2, K_C)) = H^0(S^2F)^{\vee} \cong \mathbb{C}^{3g-3},$$

It is well known that the Zariski tangent space to the Brill-Noether locus $M_C(2, K_C, s)$ at a point $[F]$ can be expressed in terms of the “Petri” map

$$(3.1) \quad \mu : S^2H^0(F) \rightarrow H^0(S^2F)$$

Indeed

$$(3.2) \quad T_{[F]}(M_C(2, K_C, s)) = \text{Im}(\mu)^\perp$$

In particular $M_C(2, K_C, s)$ has expected dimension $3g-3 - \binom{s+3}{2}$. Also notice that, if $F$ is a rank two vector bundle on $C$ with determinant equal to $K_C$, since $\chi(S^2F) = \chi(F \otimes F^\vee) - \chi(K_C) = 3g-3$, we get:

$$(3.3) \quad F \text{ stable } \Rightarrow \ h^0(S^2F) = 3g-3 \Rightarrow \ h^1(S^2F) = 0.$$ 

Proposition 3.1. Assume that we are in the Rank-1 case. Let $v$ be the Mukai vector (2.1). Then there is a well defined morphism

$$(3.4) \quad \sigma : M_v(S) \rightarrow M_C(2, K_C, s)$$

$$(\mathcal{E}) \mapsto [\mathcal{E}|_C]$$
Proof. We first show that for every \([E] \in M_v(S)\) the vector bundle \(E|_C\) is stable. Suppose this is not the case. Then there is an exact sequence

\[
0 \to \alpha \to E|_C \to K\alpha^{-1} \to 0
\]

with \(d = \deg \alpha \geq g - 1 = 2s\). From (3.5) we get a diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
E(-C) \\
\downarrow \\
V \\
\downarrow \\
\alpha \\
\downarrow \\
E|_C \\
\downarrow \\
K\alpha^{-1} \\
\downarrow \\
0
\end{array}
\]

The rank two locally free sheaf \(V\) satisfies

\[
c_1(V) = 0, \quad c_2(V) = c_2(E) - c_1(E) \cdot |C| + c_1(K\alpha^{-1}) = s + 2 - d \leq 2 - s < 0
\]

In particular \(V \cong V^\vee\), but \(V\) is not the trivial bundle. We have

\[
\chi(V) = \chi(E(-C)) + \chi(\alpha) = s + 2 + d - g + 1 \geq s + 2.
\]

Thus \(2h^0(V) \geq s + 2\) so that \(h^0(V) \geq 4\). Therefore \(V\) cannot be stable, otherwise \(\dim \operatorname{End}(V) = h^0(V \otimes V^\vee) = h^0(V \otimes V) = 1\). Suppose \(V\) is strictly semistable, then by Jordan-Hölder we have an exact sequence

\[
0 \to \mathcal{L} \to V \to \mathcal{L}^{-1} \otimes I_X \to 0
\]

where \(X \subset S\) is a 0-dimensional subscheme and \(c_1(\mathcal{L}) \cdot C = 0\). We also have

\[
\chi(V) = \chi(\mathcal{L}) + \chi(\mathcal{L}^{-1} \otimes I_X)
\]

so that

\[
0 < -c_2(V) = c_1(\mathcal{L})^2 - h^0(\mathcal{O}_X) \leq c_1(\mathcal{L})^2
\]

Since \(c_1(\mathcal{L}) \cdot C = 0\) this contradicts the fact that \(c_1(\mathcal{L}) = hC\) for some integer \(h\). It follows that \(V\) contains a destabilizing line subbundle \(\mathcal{L}\). Therefore \(\mathcal{L}\) satisfies

\[
c_1(\mathcal{L}) \cdot C > 0, \quad c_1(\mathcal{L})^2 \geq -c_2(V) \geq s - 2
\]

Then we must have \(c_1(\mathcal{L}) = hC\) for some positive integer \(h\) and this contradicts the stability of \(E\). We may then conclude that sending \(E \mapsto E|_C\) gives a well defined morphism \(\sigma' : M_v(S) \to M_C(2, K_C)\).

By Lemma 2.5, for every pencil \(L\) of degree \(s + 2\) we have

\[
[\tilde{E}_L] \in M_v(S)
\]

Since, by 2.3, we have \(h^0(S, \tilde{E}_L(-C)) = h^1(S, \tilde{E}_L) = h^2(S, \tilde{E}_L) = 0\), the same relations must be true for a general point \([E] \in M_v(S)\). Looking then at the exact sequence

\[
0 \to E(-C) \to E \to E|_C \to 0
\]
we easily see that for a general point \([E] \in M_v(S)\) the point \([E|_C]\) belongs to \(M_C(2,K_C,s)\) i.e. \(h^0(C,E|_C) \geq s + 2\). But \(M_v(S)\) is a smooth K3 surface, in particular it is irreducible. Thus the image of \(\sigma'\) must be contained in \(M_C(2,K_C,s)\).

**Remark 3.2.** The proof of Proposition 3.1 can be extended to the Rank-2 case using the Hodge Index Theorem but for the last part one needs the existence of a base point free pencil \(L\) of degree \(s + 2\) and the stability of \(\tilde{E}_L\). The existence of \(L\) will be proved in Proposition 4.5 and the stability in Remark 5.11. In Section 5 we will take a different approach to the study of \(\tilde{E}_L\) and \(E_L\) cannot be deduced from the above mentioned papers.

Finally, we want to study the differential of the morphism \(\sigma : M_v(S) \rightarrow M_C(2,K_C,s)\), whenever it is well defined.

**Proposition 3.3.** Assume that the morphism
\[
\sigma : M_v(S) \rightarrow M_C(2,K_C,s)
\]
\[
\mathcal{E} \mapsto \mathcal{E}|_C
\]
is well defined. Look at the composition:
\[
\sigma' : M_v(S) \xrightarrow{\sigma} M_C(2,K_C,s) \xrightarrow{j} M_C(2,K_C)
\]
where \(j\) is the inclusion. Let \(\mathcal{E} \in M_v(S)\) and consider the following conditions:

(i) \(H^1(S,S^2\mathcal{E}) = 0\).
(ii) \(H^0(S,S^2\mathcal{E}(-C)) = 0\).
(iii) \(S^2H^0(S,\mathcal{E}) \rightarrow H^0(S,S^2\mathcal{E})\) is surjective.

Then:

1. If (i) is satisfied then \(d\sigma'\) is injective at \([\mathcal{E}]\).
2. If (i),(ii) and (iii) are satisfied then \(M_C(2,K_C,s)\) is nonsingular of dimension 2 at \(\mathcal{E}|_C\) and \(\sigma\) is étale at \(\mathcal{E}\).

**Proof.**
(1) - Write \(E = \mathcal{E}|_C\). We use the isomorphism
\[
(3.8) \quad \mathcal{E}^\vee \otimes \mathcal{E} = \mathcal{E}^{\vee} \otimes \mathcal{E}^{\vee}(C) \cong (S^2\mathcal{E}^\vee + \wedge^2\mathcal{E}^\vee)(C) \cong S^2\mathcal{E}^\vee(C) \oplus \mathcal{O}_S
\]
We have
\[
T_{[\mathcal{E}]}(M_v(S)) = H^1(S,\mathcal{E} \otimes \mathcal{E}^\vee) = H^1(S,S^2\mathcal{E}^\vee(C))
\]
\[
T_{[E]}(M_C(2,K_C)) = H^0(C,S^2E)^\vee = H^1(C,S^2E^\vee(K_C))
\]
Thus \(d\sigma'\) is the restriction homomorphism
\[
d\sigma' : H^1(S,S^2\mathcal{E}^\vee(C)) \rightarrow H^1(C,S^2\mathcal{E}^\vee(K_C))
\]
Hence
\[
\ker \sigma' = H^1(S,S^2\mathcal{E}^\vee) = H^1(S,S^2\mathcal{E})
\]
(2) - Consider the following commutative diagram of maps:

\[
\begin{array}{ccc}
S^2H^0(E) & \xrightarrow{a} & H^0(S^2E) \\
\downarrow{b} & & \downarrow{d} \\
S^2H^0(E) & \xrightarrow{c} & H^0(S^2E)
\end{array}
\]

Condition (ii) implies that \(d\) is injective and (iii) implies that (c) is surjective. Therefore

\[\text{corank}(a) \leq \text{corank}(d) \leq h^1(S^2E(-C))\]

Now consider the decomposition (3.3). Since \(E\) is stable we have \(h^0(E^\vee \otimes E) = 1\) and \(h^1(E^\vee \otimes E) = 2\) and from the decomposition we deduce that \(h^1(S^2E(-C)) = 2\) as well, using hypothesis (ii). Therefore \(\text{corank}(a) \leq 2\); since \(coker(a)^{\perp} = T[E]MC(2, K_C, s)\), we conclude that \(\dim[T[E]MC(2, K_C, s)] \leq 2\). But from (i) and (1) it follows that \(MC(2, K_C, s)\) has dimension \(\geq 2\) at \([E]\), and this proves (2).

\[\square\]

4. Geometry of \((S, C)\) in the Rank-2 case

As in the previous two sections we denote by \((S, C)\) a pair consisting of a K3 surface \(S\) and a smooth curve \(C \subset S\) of genus

\[g(C) = 2s + 1, \quad s \geq 5\]

In this section we assume that we are in the Rank-2 case and we prove a few technical results.

Lemma 4.1. a) \(h^0(S, \mathcal{O}(nA + mB)) = 0\), whenever \(n < 0\).

b) \(h^i(S, \mathcal{O}(A - B)) = h^i(S, \mathcal{O}(B - A)) = 0\), for all \(i\).

c) Every element in \(|A|\) is integral and has Clifford index \(\geq 2\). In particular \(|A|\) is very ample.

Proof. a) is immediate by restricting to \(B\).

Let us prove b). We have \((A - B)^2 = -4\) and, by point a), we also have \(h^2(S, \mathcal{O}(A - B)) = 0\). Hence \(h^1(S, \mathcal{O}(A - B)) = h^0(S, \mathcal{O}(A - B))\). Suppose \(h^0(S, \mathcal{O}(A - B)) \neq 0\). Let \(D\) be an effective divisor linearly equivalent to \(A - B\). If \(D\) is connected, then by Lemma 2.2 in [14], we get \(h^1(S, \mathcal{O}(D)) = 0\) and we are done. Otherwise

\[D = D_1 + D_2, \quad D_1 \cdot D_2 = 0, \quad h^0(S, \mathcal{O}(D_i)) \geq 1, \quad i = 1, 2\]

\[D_1 = nA + mB, \quad D_2 = hA + kB, \quad n + h = 1, \quad m + k = -1\]

Since \(D_1\) and \(D_2\) are effective, by a) we must have \(n \geq 0, h \geq 0\). Thus either \(n = 1, h = 0\), or \(n = 0, h = 1\). In any event, we would get \(D_1 \cdot D_2 \neq 0\) which is absurd.

Let us prove the first part of c). Write \(A = \Gamma + \Delta\), with both \(\Gamma\) and \(\Delta\) effective. Then \(\Gamma = nA + mB\). Since \(A \cdot B \geq \Gamma \cdot B = nA \cdot B\), we get \(n \leq 1\). Since \(h^0(S, \mathcal{O}(\Gamma)) \neq 0\), by a) we must have \(n = 0, 1\). If \(n = 0\) then \(mB\) must be a subcurve of \(A\) against point b). If \(n = 1\), then \(\Gamma\) is a sub curve of \(A\), we must have \(m < 0\) which again violates b).

Let us now prove the second part of point c). By the main theorem of [6], and by its refinement in [3], all curves \(A' \in |A|\) have the same Clifford index and \(\text{Cliff}(A')\) is computed by the restriction to \(A'\) of an invertible sheaf \(L\) on \(S\). So, in order to complete the proof of c) we must exclude the existence of \(L \in \text{Pic}(S)\) with either of the following properties:

(i) \(L^2 = 0\) and \(L \cdot A' = 2\) (i.e. \(A'\) is hyperelliptic).
(ii) \(L^2 = 0\) and \(L \cdot A' = 3\) (i.e. \(A'\) is trigonal).
(iii) \( s = 6 \) and \( L^2 = 2 \) and \( L \cdot A = 5 \) (i.e. \( A' \) is isomorphic to a nonsingular plane quintic).

Let us consider (ii). We must have \( L = nA + mB \) and the two conditions translate into

\[
(nA + mB)^2 = 2n^2(s - 1) + 2nm(s + 1) = 0, \quad 2n(s - 1) + m(s + 1) = 3 \quad (s \geq 5)
\]

which are clearly incompatible. The hyperelliptic case (i) is similar.

In case (iii) we must have:

\[
(nA + mB)^2 = 2n(5n + 7m) = 2, \quad 10n + 7m = 5
\]

implying the impossible identity \( 5n(n + 1) = 2 \).

\[ \square \]

**Remark 4.2.**

A) From point c) of Lemma 4.1 it follows that, via the linear system \(|A|\), the surface \( S \) is embedded in \( \mathbb{P}^s \) as a projectively normal surface whose ideal is generated by quadrics.

B) Let \( I_S \) and \( I_C \) be the ideal sheaves of \( S \), respectively \( C \), in \( \mathbb{P}^s \). Recall that \( 2A - C \sim A - B \) as divisors on \( S \). From point A) and point b) of Lemma 4.1 we deduce that \( H^0(S, \mathcal{O}(2A)) \cong H^0(C, 2\eta) \) and in particular we get a surjection

\[
m : S^2H^0(C, \eta) \to H^0(C, 2\eta) \to 0
\]

and an equality

\[
H^0(S, I_S(2)) = H^0(C, I_C(2)) = \ker(m)
\]

**Lemma 4.3.** Let \( D \subset S \) be a finite closed subscheme of length \( d \geq 1 \). Assume that

\[
h^0(S, I_D(A)) \geq \max \left\{ 3, s - \frac{d - 1}{2} \right\}
\]

Then \( d = 1 \).

**Proof.** We view \( S \) embedded in \( \mathbb{P}^s = \mathbb{P}H^0(S, \mathcal{O}(A))^\vee \). Consider a hyperplane \( H \) passing through \( D \), i.e defining a non-zero element of \( H^0(S, I_D(A)) \). We set \( A = H \cap S \). We may view \( D \) as a subscheme of the integral curve \( A \). As such it defines a rank-one torsion free sheaf on \( A \) which we still denote by \( D \). From (4.3) we get

\[
h^0(A, \omega_A(-D)) \geq 2
\]

Thus, by Riemann-Roch on \( A \):

\[
h^0(A, \mathcal{O}_A(D)) = h^0(A, \omega_A(-D)) + d - s + 1 \geq \frac{d + 1}{2}
\]

Therefore either \( h^0(A, \mathcal{O}_A(D)) = 1 \) and \( d + 1 \leq 2 \), implying that \( d \leq 1 \), which is precisely what we aim at, or \( h^0(A, \mathcal{O}_A(D)) \geq 2 \), which, together with (4.4) tells us that \( D \) contributes to the Clifford index of \( A \). Let us see that this case can not occur. By (4.5) we get

\[
\text{Cliff } D = d - 2h^0(A, \mathcal{O}_A(D)) + 2 \\
\leq d - 2 \left( \frac{d + 1}{2} \right) + 2 \leq 1
\]

and this implies that \( \text{Cliff}(A) \leq 1 \), contradicting Lemma [1.1c).
Remark 4.4. In a sense, the technical lemma we just proved is our substitute for Mukai’s Lemma 7 which is ubiquitous in [10]. It suggests the possibility of introducing the notion of Clifford index of a 0-dimensional closed subscheme $D$ on a polarized K3 surface $(S, H)$ by letting
\[ \text{Cliff}(D) = 2g - d - 2h^0(S, I_D(H)) + 2 \]
where $d = \text{length}(D)$ and $g = \frac{1}{2}H^2 + 1$ is the genus of $H$. One says that $D$ contributes to the Clifford index of $H$ if both $h^0(S, I_D(H)) \geq 3$ and $h^0(S, I_D(H)) + d - g + 1 \geq 3$. A straightforward generalization of the proof of Lemma 4.3 gives that, if all $H' \in |H|$ are integral, then
\[ \text{Cliff}(H) = \min \{ \text{Cliff}(D) : D \subset S \text{ contributes to } \text{Cliff}(H) \} \]
Our next aim is to prove the following Proposition regarding linear series of degree $s + 2$ on $C$ in the Rank-2 case.

**Proposition 4.5.** a) Let $|L|$ be a degree-$(s + 2)$, base-point-free pencil on $C \subset S$. Then $L$ is Petri.  
b) $h^0(C, \eta\xi^{-1}) = 1$ and thus $\xi$ is a smooth isolated point of $W^1_{s+1}$.  
c) There exists on $C$ a base-point-free $g^1_{s+2}$.

We are going to use the following Lemma due to Green-Lazarsfeld and Donagi-Morrison. We take its statement from [5], Lemma 2.1.

**Lemma 4.6.** (Green-Lazarsfeld and Donagi-Morrison) Let $S$ be a K3 surface. Let $|L|$ be a base-point-free pencil on a smooth curve $C$ lying on $S$. If the Lazarsfeld-Mukai bundle $\mathcal{E}_L$ is not simple (i.e. it has non-trivial automorphisms), then there exists line bundles $M$ and $N$ on $S$ and a zero-dimensional subscheme $X \subset S$ such that
i) $h^0(S, M) \geq 2$, $h^0(S, N) \geq 2$.  
ii) $N$ is base-point-free.  
iii) There is an exact sequence
\[ 0 \to M \to \mathcal{E}_L \to N \otimes I_X \to 0 \]
Moreover if $h^0(S, M - N) = 0$ then $\text{Supp}(X) = \emptyset$ and the above sequence splits.

**Proof.** (of Proposition 4.5) As far as point a) is concerned we proceed exactly as in Lazarsfeld’s proof of Petri’s conjecture. It is then enough to prove that $\mathcal{E}_L$ is simple. Suppose it is not so. By the preceding Lemma there is an exact sequence (4.7). Since $\text{Pic}(S) = \mathbb{Z} \cdot [A] \oplus \mathbb{Z} \cdot [B]$, $C = A + B$ and $c_1(\mathcal{E}_L) = C$ we must have
\[ M = mA + nB, \quad N = hA + kB, \quad (m + h - 1)A + (n + k - 1)B \sim 0 \]
from condition i) of the Lemma and from Lemma 4.1 we get that the pair $\{M, N\}$ coincides with the pair $\{A, B\}$. On the other hand, by Lemma 4.1 we have $H^0(A - B) = 0$. Using Lemma 4.6 again, the sequence (4.7) splits and $\mathcal{E}_L = \mathcal{O}(A) \oplus \mathcal{O}(B)$. This is absurd since $h^0(S, \mathcal{E}_L) = s + 2$.

Regarding item b), looking at the Petri map $H^0(C, \xi) \otimes H^0(C, \eta) \to H^0(C, \omega_C)$, and using the b.p.f.p.t we see that $h^0(C, \eta\xi^{-1}) \geq 1$. On the other hand, looking at the exact sequence
\[ 0 \to \mathcal{O}_S(-2B) \to \mathcal{O}_S(A - B) \to \eta\xi^{-1} \to 0 \]
we see that $h^0(C, \eta\xi^{-1}) \leq 1$. 

(11)
As far as point c) is concerned, consider the smooth locus
\[ V = \{ \xi(p) \mid p \in C \} \subset W_{s+2}^{1} \]
To analyze \( W_{s+2}^{1} \) along \( V \) we look at the Petri map.
\[ \mu_{0,\xi(p)} : H^{0}(C,\xi(p)) \otimes H^{0}(C,\eta(-p)) \to H^{0}(C,K_{C}) \]
By hypothesis, \( H^{0}(C,\xi(p)) = H^{0}(C,\xi) \) and
\[ \ker \mu_{0,\xi(p)} = H^{0}(C,\eta\xi^{-1}(-p)) \]
By b), we know that \( h^{0}(C,\eta\xi^{-1}) = 1 \). Let \( D \neq 0 \) be the divisor of a non-zero section of \( \eta\xi^{-1} \). By Brill-Noether theory it follows that \( V \) is a one-dimensional component of \( W_{s+2}^{1} \) and that \( \xi(p) \) is a singular point of \( W_{s+2}^{1} \) if and only if \( p \in \text{Supp} \ D \). Moreover, in this case
\[ \dim T_{\xi(p)}W_{s+2}^{1} = 2 \]
A priori it could be that, for \( p \in \text{Supp} \ D \), the dimension of \( W_{s+2}^{1} \) at \( \xi(p) \) is equal to 1 and \( \xi(p) \) is an embedded point. However, for a determinantal variety of the correct dimension this cannot be the case. Thus there must be a one-dimensional component \( V' \) of \( W_{s+2}^{1} \), distinct from \( V \), and meeting \( V \) at \( \xi(p) \).

Now suppose that there is no base-point-free \( g_{s+2}^{1} \) in \( V' \). Then
\[ V' = \{ \xi'(p') \mid p' \in C \} \]
for some fixed \( \xi' \in W_{s+1}^{1} \), with \( \xi' \neq \xi \). Therefore there is a point \( p' \in C \), such that
\[ \xi(p) = \xi'(p') \]
But then \( \xi = \xi' \), contrary to the assumption.

Proposition 4.5, together with the results in [2], gives the following proposition.

**Proposition 4.7.** Set \( g = 2s+1 \). Let \( M_{g,s+1}^{1} \subset M_{g} \) be the irreducible divisor of curves possessing a \( g_{s+1}^{1} \). Let \( [C] \) be a general point in \( M_{g,s+1}^{1} \). Then there is a unique \( g_{s+1}^{1} \) on \( C \) and every degree-\((s+2)\) base-point-free pencil on \( C \) satisfies Petri’s condition.

5. Brill-Noether loci in the Rank-2 case

In this section we assume that we are in the Rank-2 case. Using Lemma 4.1 b) and proceeding as in Section 3 of [10], to each point \( x \in S \) we associate a rank-2, pure sheaf \( E_{x} \) defined as the unique extension

\[ 0 \to \mathcal{O}_{S}(B) \to E_{x} \to I_{x}(A) \to 0. \]

We define
\[ E_{x} = E_{x}|_{C} \]
The main theorem we want to prove in this section is the following generalization of Theorem 3 of [10].

**Theorem 5.1.** By associating to \( x \in S \) the bundle \( E_{x} \) on \( C \) we obtain an isomorphism between \( S \) and \( M_{C}(2,K_{C},s) \).

The proof of the theorem will be obtained from the following chain of facts to be proved:
• Both $E_x$ and $\mathcal{E}_x$ are stable. Moreover $[\mathcal{E}_x] \in M_v(S)$ and $[E_x] \in M_C(2, K_C, s)$.

• The map

$$\rho : S \to M_v(S)$$

$$x \mapsto [\mathcal{E}_x]$$

is an isomorphism of K3 surfaces.

• From these two facts it follows that the morphism

$$\sigma : M_v(S) \to M_C(2, K_C, s)$$

$$[\mathcal{E}] \mapsto [\mathcal{E}|_C]$$

is well defined.

• $\sigma$ is bijective.

• For every $x \in S$

$$\dim T_{[\mathcal{E}_x]}(M_C(2, K_C, s)) = 2$$

and the differential of $\sigma$ is an isomorphism at $[\mathcal{E}_x]$, for every $x \in S$.

The last two items finally give:

• $\sigma$ is in fact an isomorphism of smooth K3 surfaces and therefore $\sigma \rho : S \to M_C(2, K_C, s)$ is an isomorphism.

We need to establish a number of preliminary results. First of all, since $H^1(S, \mathcal{O}(B)) = 0$, from (5.1) we get the exact sequence

$$0 \to H^0(S, \mathcal{O}_S(B)) \to H^0(S, \mathcal{E}_x) \to H^0(S, I_x(A)) \to 0.$$  

$$H^0(S, \mathcal{E}_x) = s + 2, \quad H^i(S, \mathcal{E}_x) = 0, \quad i = 1, 2$$

and $\mathcal{E}_x$ is generated by global sections. Notice that since $\det \mathcal{E}_x = \mathcal{O}(C)$ we have

$$\mathcal{E}_x^\vee \cong \mathcal{E}_x(-C)$$

From the sequence and (5.5) we get

$$0 \to \mathcal{E}_x^\vee \to \mathcal{E}_x \to E_x \to 0$$

we then get an isomorphism

$$H^0(S, \mathcal{E}_x) \cong H^0(C, E_x)$$

and in particular

$$\dim H^0(C, E_x) = s + 2 = h^0(C, \xi) + h^0(C, \eta) - 1.$$  

If $x \notin C$ restricting (5.1) to $C$ gives

$$0 \to \xi \to E_x \to \eta \to 0$$

If $x \in C$ then, factoring out the torsion from $I_x(A) \otimes \mathcal{O}_C$, we get

$$0 \to \xi(x) \to E_x \to \eta(-x) \to 0$$

We are now going to prove two results that are key elements in the proof of the stability of $E_x$ (Proposition 5.5). 

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Lemma 5.2. The extension (5.11) is non-split.

Proof. To prove this we proceed as in the proof of Mukai’s Proposition 3 in [10]. Let \( E_x \to \xi(x) \) be a splitting. Consider the diagram

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & & \downarrow & & \\
\mathcal{E}_x(-C) & \cong & \mathcal{E}_x(-C) & & \\
\downarrow & & \downarrow & & \\
0 & \to & \mathcal{F} & \to & \mathcal{E}_x & \to & \xi(x) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \eta(-x) & \to & \mathcal{E}_x & \to & \xi(x) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & 0 & & & & & & \\
\end{array}
\]

Then \( H^0(S, \mathcal{F}) \) is an \( s \)-dimensional subspace of \( H^0(S, \mathcal{E}_x) \) mapping onto \( H^0(S, I_xA) \). We have \( c_1(\mathcal{F}) = 0 \) so that \( \wedge^2 \mathcal{F} = \mathcal{O}_S \). Look at the evaluation map \( H^0(S, \mathcal{F}) \otimes \mathcal{O}_S \to \mathcal{F} \subset \mathcal{E}_x \) and take its second wedge product

\[
\wedge^2(H^0(S, \mathcal{F}) \otimes \mathcal{O}_S) \xrightarrow{\alpha} \wedge^2 \mathcal{F} = \mathcal{O}_S \subset \wedge^2 \mathcal{E}_x = \mathcal{O}_S(C)
\]

where \( \alpha = (c_1, \ldots, c_n) \), with \( c_i \in \mathbb{C}, \, i = 1, \ldots, n \). Restricting \( \alpha \) to \( C \) we see that it vanishes on \( x \) and therefore vanishes identically. Thus, the image of the evaluation map \( H^0(S, \mathcal{F}) \otimes \mathcal{O}_S \to \mathcal{F} \subset \mathcal{E}_x \) is of rank one and is isomorphic to \( I_x(A) \), which is a contradiction, since \( \mathcal{E}_x \) is non split.

\[ \square \]

Lemma 5.3. Let \( E \) be a rank two vector bundle (not necessarily semi-stable) with canonical determinant on \( C \). Suppose that \( h^0(E) = s + 2 \) and that \( E \) contains a line sub bundle isomorphic to \( \xi \) or \( \xi(p) \) for a point \( p \in C \). Then either

a) \( E \) is stable, or

b) there is an exact sequence \( 0 \to \xi \to E \to \eta \to 0 \), and \( \eta(-p) \) is a destabilizing subsheaf of \( E \).

c) \( E = \xi(p) \oplus \eta(-p) \)

Proof. Assume that \( E \) is not stable; then we must have an exact sequence

\[
0 \to \alpha \to E \to \beta \to 0
\]
where $\alpha$ is a line bundle on $C$ of degree greater or equal to $g - 1$. We have two possible diagrams

\[
\begin{array}{ccccccc}
0 & \rightarrow & \xi & \rightarrow & E & \rightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \xi(p) & \rightarrow & E & \rightarrow & 0
\end{array}
\]

Since $\deg \alpha \geq 2s > s + 1 = \deg \xi$, (resp. $\deg \alpha \geq 2s > s + 2 = \deg \xi(p)$) there can be no injective map from $\alpha$ to $\xi$ (resp. $\xi(p)$). Thus, we must have $\alpha = \eta(-D)$ and $\beta = \xi(D)$ for some positive divisor of degree $d \leq s - 1$ (resp. $d \leq s - 2$). We have

\[
s + 2 = h^0(E) \leq h^0(\xi(D)) + h^0(\eta(-D))
\]

(5.13)

\[
= 2h^0(\eta(-D)) + s + 1 + d - g + 1
\]

\[
= 2h^0(\eta(-D)) - s + 1 + d
\]

Thus

\[
h^0(\eta(-D)) \geq s - \frac{d - 1}{2}
\]

(5.14)

Thus $h^0(\eta(-D)) \geq s - \frac{d - 1}{2} \geq \frac{s + 2}{2} \geq 3$, (resp. $h^0(\eta(-D)) \geq \frac{s + 3}{2} \geq 3$) since $s \geq 5$. We can then apply Lemma [1.3] and we get $d \leq 1$. Then the only possibility are the ones described in points b) and c).

\[
\square
\]

The next result needed to prove the stability of $E_x$ and $E_x$ is Mukai's Lemma 2 in [10]. We include its statement for the convenience of the reader.

**Lemma 5.4. (Mukai)** Let $L$ be a line bundle on a smooth curve $C$ and consider non-trivial extensions

\[
0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0
\]

with $M = K_C L^{-1}$.

1) The extensions $E$ with $h^0(E) = h^0(L) + h^0(M)$ are parametrized by the projective space $\mathbb{P}^* \text{Coker}\{S^2 H^0(M) \rightarrow H^0(M^2)\}$.

2) Assume that the multiplication map $S^2 H^0(M) \rightarrow H^0(M^2)$ is surjective. Then $h^0(C, E) \leq h^0(C, L) + h^0(C, M) - 1$. Moreover, the non-trivial extensions $E$ such that $h^0(C, E) = h^0(C, L) + h^0(C, M) - 1$ are parametrized by the quadratic hull of the image of $\Phi_{|M|} : C \rightarrow \mathbb{P}^* H^0(C, M)$. More precisely, for every point $x$ of the quadric hull, there is a unique extension $E$ such that the image of the linear map $H^0(C, E) \rightarrow H^0(C, M)$ is the codimension one subspace corresponding to $x$.
Lemma 11). The local to global spectral sequence of ext gives
\[ 0 \to O \to E \to I \to 0 \]
which is a relative version of (5.16). We then have a universal extension
\[ 0 \to O \to E \to I \to 0 \]
Proof. Assume first that \( x \not\in C \). We are then in case (5.10). By (5.9) the sequence cannot split.
By Lemma 5.3, if \( E \) is not stable then it sits in an exact sequence
\[ 0 \to \eta(-p) \to E \to \xi(p) \to 0 \]
for some \( p \in C \) and we have \( H^0(C, E_x) \cong H^0(C, \xi(p)) \oplus H^0(C, \eta(-p)) = H^0(C, \xi) \oplus H^0(C, \eta(-p)) \).
Thus the codimension one subspace of \( H^0(C, \eta) \) is \( H^0(C, \eta(-p)) \). But then \( x = p \in C \) which is a contradiction.
Let then \( x \in C \). Then we are in case (5.11). By Lemma 5.2, this sequence cannot split. The stability of \( E \) now follows immediately from Lemma 5.3.
The stability of \( E \) is now clear. If \( L \) is a destabilizing subsheaf of \( E \) then \( L \cdot C \geq C^2/2 \). But then \( L \mid C \) would destabilize \( E \). The last assertion is a consequence of (5.5) and (5.9).

Remark 5.6. From the preceding arguments we learned that points \( x \) in the quadratic hull of \( C \subset \mathbb{P}^s \), not belonging to \( C \), correspond to extensions of type (5.10) where \( E \) is a stable bundle. On the other hand, points \( x \) belonging to \( C \) correspond to extensions of type
\[ a) \quad 0 \to \xi \to E' \to \eta \to 0 \]
where \( E' \) is destabilized by \( \eta(-x) \). Finally, if \( D \) is the divisor of a section of \( \eta\xi^{-1} \), a point \( x \in \text{Supp}(D) \), corresponds to an extension of type:
\[ b) \quad 0 \to \xi \to E'' \to \eta \to 0 \]
and \( E'' \) is clearly unstable. In both cases a) and b) the "stable limit" replacing \( E' \) (resp. \( E'' \)) is \( E \) as in (5.11).

We next come to:

Proposition 5.7. The map
\[ \rho : S \to M_v(S) \]
\[ x \mapsto [E_x] \]
is an isomorphism of K3 surfaces.

Proof. We follow again Mukai’s line of reasoning [10] (p. 194, before Lemma 8, and p. 195 after Lemma 11). The local to global spectral sequence of ext gives
\[ H^j(S, \mathcal{E}xt^j_{\mathcal{O}_S}(I_x(A), \mathcal{O}_S(B))) \to \text{Ext}^{j+j}_{\mathcal{O}_S}(I_x(A), \mathcal{O}_S(B)) \]
By Lemma 4.1 the natural map
\[ \text{Ext}^1_{\mathcal{O}_S}(I_x(A), \mathcal{O}_S(B)) \to H^0(S, \mathcal{E}xt^1_{\mathcal{O}_S}(I_x(A), \mathcal{O}_S(B))) \cong \mathbb{C} \]
is an isomorphism so that the extension (5.11) is the unique non trivial extension of \( I_x(A) \) by \( \mathcal{O}_S(B) \). Now one can perform a relative version of this construction. We let \( T \) be a copy of \( S \) and \( \Delta \) be the diagonal of \( S \times T \). We have an isomorphism
\[ \mathcal{E}xt^1_{\mathcal{O}_{S \times T}}(I_{\Delta}(p^*A), \mathcal{O}_{S \times T}(q^*B)) \to q_*\mathcal{E}xt^1_{\mathcal{O}_{S \times T}}(I_{\Delta}(p^*A), \mathcal{O}_{S \times T}(q^*B)) \cong \mathcal{O}_T(B - A) \]
which is a relative version of (5.16). We then have a universal extension
\[ 0 \to \mathcal{O}_{S \times S}(p^*B) \to F \to I_{\Delta}(p^*A + q^*(B - A)) \to 0 \]
whose restriction to \( S \times \{x\} \) is (5.1). This gives a well defined morphism

\[
\rho : S \to M_v(S)
\]

\( x \mapsto [\mathcal{E}_x] \)

As \( S \) and \( M_v(S) \) are smooth K3 surfaces, to prove that \( \rho \) is an isomorphism it suffices to show that it is injective and for this it suffices to show that

\[
\dim \text{Hom}(O(B), \mathcal{E}_x) = 1, \quad \text{i.e.} \quad h^0(S, \mathcal{E}_x(-B)) = 1
\]

But this follows readily from the exact sequence

\[
0 \to O_S \to \mathcal{E}_x(-B) \to I_x(A - B) \to 0
\]

\[\square\]

**Corollary 5.8.** \( \sigma : M_v(S) \to M_C(2, K_C, s) \) is well defined.

**Proof.** The corollary is an immediate consequence of Propositions 5.5 and 5.7. \(\square\)

**Proposition 5.9.** \( \sigma \) is bijective.

**Proof.** - \( \sigma \) is injective.

Clearly what we have to prove is that \( \dim \text{Hom}(\xi, E_x) = 1 \), or in other words that \( h^0(C, E_x \xi^{-1}) = 1 \).

From the exact sequence

\[
0 \to O_S(-B - C) \to \mathcal{E}_x(-B) \to E_x \xi^{-1} \to 0
\]

we get

\[
H^0(S, \mathcal{E}_x(-B)) \cong H^0(C, E_x \xi^{-1})
\]

From the sequence

\[
0 \to O_S \to \mathcal{E}_x(-B) \to I_x(A - B) \to 0
\]

we get \( H^0(S, \mathcal{E}_x(-B)) \cong \mathbb{C} \).

- \( \sigma \) is surjective.

Let \([E] \in M_C(2, K_C, s)\). Let us recall Mukai’s Lemma 1 in [10]. Again, we include its statement for the convenience of the reader.

**Lemma 5.10.** (Mukai) Let \( E \) be a rank two vector bundle of canonical determinant \( \zeta \) a line bundle on \( C \). If \( \zeta \) is generated by global sections, then we have

\[
\dim \text{Hom}_{\mathcal{O}_C}(\zeta, E) \geq h^0(E) - \deg \zeta
\]

Since \( h^0(E) \geq s + 2 \), by the preceding lemma, there must be an exact sequence

\[
0 \to \xi(D) \to E \to \eta(-D) \to 0
\]

for some effective divisor \( D \) of degree \( d \) on \( C \). Since \( E \) is stable we must have

\[
\deg(\xi(D)) = s + 1 + d \leq \deg(E)/2 = 2s,
\]

i.e. \( d \leq s - 1 \). But then, as in the proof of Lemma 5.3, we deduce that \( d \leq 1 \). Two cases can occur. Either:

\[
0 \to \xi(p) \to E \to \eta(-p) \to 0
\]

or

\[
0 \to \xi \to E \to \eta \to 0
\]

\[\square\]
Then one concludes exactly as in Mukai’s paper \cite{Mukai} (pp. 195-196) by using Lemma \ref{lem5.10} as follows. In the first case \( E \cong E_p \) because the extension does not split and is unique. In the second case the coboundary

\[ H^0(C, \eta) \to H^1(C, \xi) \]

has rank one. We then apply point 2) in Lemma \ref{lem5.4} together with the fact that, by \ref{thm5.2}, the quadratic hull of \( \Phi_{|\eta|}(C) \) is exactly \( S \). We thus find a point \( x \in S \) such that \( H^0(S, I_x A) = \text{Im}[H^0(S, E) \to H^0(C, \eta) = H^0(S, A)] \). By the uniqueness again we have \( E = E_x = E_x|_C \). \hfill \Box

**Remark 5.11.** The two vector bundles \( \tilde{E}_L \) and \( E_L \) are stable also in the Rank-2 case, for every choice of a base-point-free pencil \( |L| \) of degree \( s + 2 \).

The proof of this fact runs as follows. By Theorem \ref{thm5.1} it is enough to prove that \( \tilde{E}_L \) is stable. Suppose not, and let \( N \) be a subsheaf of \( \tilde{E}_L \) with slope greater or equal than \( 2s = \mu_C(\tilde{E}_L) \). Then \( \alpha = N|_C \) destabilizes \( E_L \). On the other hand, by Mukai’s Lemma \ref{lem5.10} \( \text{Hom}(\xi, E_L) \neq 0 \) and we have an exact sequence

\[ 0 \to \xi(D) \to E_L \to \eta(-D) \to 0 \]

for some positive divisor \( D \). We may write \ref{eq5.13} with \( E \) replaced by \( E_L \) and conclude, in exactly the same way, that \( \deg(D) \leq 1 \). We can then proceed as in the proof of Lemma \ref{lem5.3} and prove that either \( \alpha = \eta \) or \( \alpha = \eta(-p) \). On the other hand, we have an exact sequence

\[ 0 \to L \to E_L \to K_CL^{-1} \to 0 \]

We must then have either \( h^0(C, L\alpha^{-1}) \neq 0 \) or \( h^0(C, K_CL^{-1}\alpha^{-1}) \neq 0 \). For degree reasons, the only possibility is that \( \alpha = \eta(-p) \) and \( h^0(C, K_CL^{-1}\alpha^{-1}) \neq 0 \). This implies that \( L = \xi(p) \) but then \( L \) can not be base-point-free. This contradiction proves our claims.

Next, we prepare the ground for the proof of the last step. From the exact sequence \ref{lem5.4} we deduce the following exact sequences

\begin{align}
(5.18) & \quad 0 \to U \to S^2E_x \to O_S(I_x^2A^2) \to 0 \\
(5.19) & \quad 0 \to O_S(2B) \to U \to O_S(I_x(A + B)) \to 0 \\
\end{align}

In particular

\[ U \cong E_x(B) \]

We also have

\begin{align}
(5.20) & \quad 0 \to U \to S^2H^0(E_x) \to S^2H^0(I_xA) \to 0 \\
(5.21) & \quad 0 \to S^2H^0(S, B) \to U \to H^0(B) \otimes H^0(I_xA) \to 0 \\
\end{align}

**Lemma 5.12.** \( H^0(S, S^2E_x(-C)) = 0 \)

**Proof.** We have an exact sequence

\begin{align}
(5.22) & \quad 0 \to (S^2E_x)(-C) \to S^2E_x \to S^2E_x \to 0 \\
\end{align}
On the other hand we have an exact sequence
\[(5.23) \quad 0 \to \mathcal{E}_x(-A) \to S^2\mathcal{E}_x(-C) \to \mathcal{O}_S(I^2_x(A - B)) \to 0\]
By Lemma (4.1) we have \(H^0(I^2_x(A - B)) = 0\) and we see that
\[(5.24) \quad H^0(\mathcal{E}_x(-A)) = 0\]
by looking at the exact sequence
\[(5.25) \quad 0 \to \mathcal{O}_S(B - A) \to \mathcal{E}_x(-A) \to I_x \to 0\]

**Lemma 5.13.**
a) \(S^2H^0(S, \mathcal{O}_S(B)) \to H^0(S, \mathcal{O}_S(2B))\) is an isomorphism
b) \(H^0(S, \mathcal{O}_S(B) \otimes H^0(S, I_x(A)) \to H^0(S, I_x(A + B))\) is injective
c) \(F : S^2H^0(S, I_x(A)) \to H^0(S, I^2_x(2A))\) is surjective

**Proof.**
a) Follows from the base-point-free-pencil trick.
b) Follows again from the base-point-free-pencil trick.
c) Let \(x \in S\). Let \(A\) be a generic hyperplane section of \(S\) given by the equation \(s_A = 0\) and assume \(x \notin A\). Consider the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & \ker(F) & \to & S^2H^0(S, I_xA) & \overset{F}{\to} & H^0(S, I^2_xA^2) & \cong & \mathbb{C}^{4s-5} \\
\downarrow & & \downarrow h & & \equiv & & \downarrow r & & \\
0 & \to & \ker(f) & \to & S^2H^0(A, \omega_A) & \overset{f}{\to} & H^0(A, \omega_A^2) & \cong & \mathbb{C}^{3s-3} \\
\downarrow & & \downarrow coker(h) & & \downarrow & & \downarrow & & \\
0 & & & & & & 0 & & \end{array}
\]

Consider an element \(s_A \cdot t\) with \(t \in H^0(S, I^2_xA)\). We can choose coordinates \(\{x_0, \ldots, x_s\}\) so that
\[x = [1, 0, \ldots, 0], \quad s_A = x_0, \quad t = x_1, \quad T_x(S) = \{x_1 = \cdots = x_{s-2} = 0\}\]
To prove that \(s_A \cdot t\) lies in \(\text{Im}(F)\) one must find \(\widetilde{Q} \in S^2H^0(S, I_xA)\), i.e. a quadric which is singular in \(x\), such that \(\widetilde{Q}|_S = (x_0x_1)|_S\). In other words we must find a quadric \(Q \in I_S(2)\) such that
\[\widetilde{Q} = \lambda Q + \mu x_0x_1, \quad \mu \neq 0\]
is singular in \(x\). We must have
\[0 = \left(\frac{\partial \widetilde{Q}}{\partial x_j}\right)_x = \lambda \left(\frac{\partial Q}{\partial x_j}\right)_x, \quad j \neq 1, \quad 0 = \left(\frac{\partial \widetilde{Q}}{\partial x_1}\right)_x = \lambda \left(\frac{\partial Q}{\partial x_1}\right)_x + \mu\]
Since the ideal of $S$ is generated by quadrics (Remark 4.2) we may choose $Q$ in $I_S(2)$ such that $T_\alpha(Q) = \{x_1 = 0\}$, so that

$$Q = x_0x_1 + \sum_{i\neq 0, j\neq 0} b_{ij}x_i x_j$$

We may then set $\tilde{Q} = Q - x_0x_1$.

We are now ready to prove:

**Proposition 5.14.** For every $x \in S$

$$\dim T_{[E_x]}(M_C(2, K_C, s)) = 2$$

and the differential of $\sigma : M_v(S) \to M_C(2, K_C, s)$ is an isomorphism at $[E_x]$, for every $x \in S$.

**Proof.** In view of Proposition 3.3 we must verify the following three conditions:

(i) $H^1(S, S^2E_x) = 0$.

(ii) $H^0(S, S^2E_x(-C)) = 0$.

(iii) $S^2H^0(S, E_x) \to H^0(S, S^2E_x)$ is surjective.

We start with $(iii)$. Look at (5.18), (5.20) and (5.21). We get a diagram

\[
\begin{array}{cccccc}
0 & \to & E_x & \to & E_x(B) & \to 0 \\
& & u & & & \\
0 & \to & H^0(S, U) & \to & H^0(S^2E_x) & \to H^1(S, U) \\
& & & & c & & \\
& & & & 0 & & \\
\end{array}
\]

We will show that $u$ is an isomorphism. Let us first show that

$$H^1(S, U) = 0,$$

where $U = E_x(B)$

For this we look at the sequence

$$0 \to E_x \to E_x(B) \to E_x(B)|_B \to 0$$

We get: $H^1(U) = H^1(E_x(B)) = H^1(E_x(B)|_B) = H^1(E_x|_B)$. From the exact sequence

$$0 \to E_x(-B) \to E_x \to E_x|_B \to 0$$

we get

$$H^1(E_x|_B) \cong H^2(E_x(-B))$$

However, $H^2(E_x(-B)) = H^0(E_x(-A)) = 0$ by (5.24). In conclusion: $H^1(U) = 0$.

We now claim that $u$ is an isomorphism. Consider the diagram

\[
\begin{array}{cccccc}
0 & \to & S^2H^0(S, B) & \to & U & \to H^0(S, B) \otimes H^0(S, I_x A) & \to 0 \\
& & u & & & \\
0 & \to & H^0(S, 2B) & \to & H^0(S, U) & \to H^0(S, I_x(A + B)) & \to 0 \\
& & & & & & \\
& & & & & & H^1(S, 2B) \to 0 \\
\end{array}
\]
Since $S^2H^0(B) \to H^0(2B)$ is an isomorphism and
\[ H^0(B) \otimes H^0(I_xA) \to H^0(I_x(A + B)) \]
is injective, the claim follows from a dimension count.

From Lemma 5.13 we know that $F$ is surjective and so is $Fl$ and therefore $m$ and thus $c$, proving $iii$).

Item (ii) is Lemma 5.12.

To prove (i) we look at the exact sequence
\[ 0 \to H^1(U) = 0 \to H^1(S^2E_x) \to H^1(I_x^2A^2) \]
But now $H^1(I_x^2A^2) = 0$ as it follows from the exact sequences
\[ 0 \to I_xA^2 \to A^2 \to A^2_x \to 0, \quad 0 \to I_x^2A^2 \to I_xA^2 \to A^2_x \otimes I_x/I_x^2 \to 0 \]
and from the ampleness of $A$.

The proof of Theorem 5.1 is now complete.

6. Brill-Noether loci in the Rank-1 case

The purpose of this section is to prove the following:

**Theorem 6.1.** Let $(S, C)$ be a general pair belonging to the Rank-1 case. There is a unique, generically smooth, 2-dimensional irreducible component $V_C(2, K_C, s)$ of $M_C(2, K_C, s)$, containing the Voisin bundles $E_L$, with $L \in W_{s+2}^1(C)$, such that $\sigma$ induces an isomorphism of $M_v(S)$ onto $V_C(2, K_C, s)_{\text{red}}$. In particular $V_C(2, K_C, s)_{\text{red}}$ is a K3 surface.

Before going into the proof we need some preliminaries. In the next statement we will refer to the notations introduced in diagram (1.1).

**Lemma 6.2.** Let $(S_0, C_0) \in \mathcal{P}_g$ be a pair belonging to the Rank-2 case. Then there exists a nonsingular affine curve $B$ and a pair $(S, \mathcal{C})$ with the following properties. There is a diagram of smooth families over $B$
\[ \begin{array}{ccc} \mathcal{C} & \to & S \\ \downarrow & & \downarrow \\ B & \to & \mathbb{P}^g \end{array} \]
whose fibre $(S(b_0), C(b_0))$ over $b_0$ is $(S_0, C_0)$, and such that for all $b \neq b_0$ outside a countable subset the fibre $(S(b), C(b)) \in \mathcal{P}_g$ is a pair belonging to the Rank-1 case.

**Proof.** Let $\mathcal{H}_g$ be the open subset of the Hilbert scheme of $\mathbb{P}^g$ parametrizing nonsingular K3 surfaces of degree $4s$ and let $\mathcal{F}_g \to \mathcal{H}_g$ be the open subset of the flag Hilbert scheme parametrizing pairs $C \subset S \subset \mathbb{P}^g$ with $|S| \in \mathcal{H}_g$ and $C \in |\mathcal{O}_S(1)|$. Then $(S_0, C_0)$ corresponds to a point $b_0 \in \mathcal{F}_g$. Let $B \subset \mathcal{F}_g$ be a general nonsingular affine curve through $b_0$. Then the pullback to $B$ of the universal family over $\mathcal{F}_g$ has the required properties.

**Proof of Theorem 6.1.** The family (6.1) defines naturally a varying Mukai vector $v(b)$ such that $v = v(b_0)$; as $b \in B$ varies the moduli spaces $M_{v(b)}(S(b))$ fit into a family $\varphi : M_v(S/B) \to B$.
of projective surfaces. Modulo shrinking $B$ if necessary, we may assume that this is a family of K3 surfaces. Similarly, the moduli spaces $\mathcal{M}_{C(b)}(2, K_{C(b)})$ fit into a smooth proper family $\mathcal{M}_{C/B}(2, \omega_{C/B}) \rightarrow B$ of relative dimension $3g - 3$. By the openness of (semi)stability ([7], Proposition 2.3.1) and the properness of $\varphi$ we may assume that the restriction morphisms

$$\sigma'_b : M_v(b)(S(b)) \rightarrow M_{C(b)}(2, K_{C(b)})$$

are well defined. They define a morphism of relative moduli spaces:

$$\mathcal{M}_v(S/B) \xymatrix{ \ar[r]^\Sigma' & \mathcal{M}_{C/B}(2, \omega_{C/B}) \ar[d]_{\varphi} } B$$

Over $b_0$ we have

$$\Sigma'(b_0) = \sigma'_0 : M_{v_0}(S_0) \rightarrow M_{C_0}(2, K_{C_0})$$

which is an embedding, with image $M_{C_0}(2, K_{C_0}, s) = V_{C_0}(2, K_{C_0}, s)$. Therefore, modulo shrinking $B$ if necessary, we may assume that for all $b \in B$ we have that $\Sigma'(b) = \sigma'_b$ embeds the K3 surface $M_v(b)(S(b))$ into $M_{C(b)}(2, K_{C(b)})$, and the image is contained in $M_{C(b)}(2, K_{C(b)}, s)$.

Modulo performing an étale base change we may further assume that there is a line bundle $L$ on $C$ such that $L(b) \in W_{s+2}^1(C(b))$ and $L_0 := L(b_0)$ is a base point free $g^1_{s+2}$. Consider the corresponding family $E_C$ of Voisin bundles on $S$. The vector bundle $E_C(b_0)$ over $S_0$ satisfies conditions (i),(ii) and (iii) of Proposition 3.3 as all bundles in $M_v(S_0)$ do (see Section 9). Therefore by upper-semicontinuity we may assume that all bundles $E_C(b)$ satisfy at least (i) and (ii) as well. Moreover, by construction, they also satisfy $h^0(E_C(b)) = s + 2$, so that in particular $S^2H^0(E_C(b))$ has constant dimension. Moreover they also satisfy $h^0(S^2E_C(b)) = h^0(S^2E_C(b)_{|C(b)})-2$, as shown by the exact sequence:

$$0 \rightarrow S^2E_C(b)(-C(b)) \rightarrow S^2E_C(b) \rightarrow S^2E_C(b)_{|C(b)} \rightarrow 0$$

because $h^1(S^2E_C(b)(-C(b))) = 2$. Therefore semicontinuity applies and condition (iii) can be also assumed to be satisfied for all $b \in B$.

We now apply Proposition 3.3 and we deduce that $M_{C(b)}(2, K_{C(b)}, s)$ is smooth of dimension 2 at $\sigma_b(E_C(b))$. Therefore $\sigma_b$ embeds $M_v(b)(S(b))$ into an irreducible 2-dimensional generically smooth component $V_C(b)(2, K_{C(b)}, s)$ of $M_{C(b)}(2, K_{C(b)}, s)$ whose reduction is therefore isomorphic to $M_v(b)(S(b))$. This component is uniquely determined by the condition of containing the bundles $E_{C(b)}$.

7. The Fourier-Mukai transform

As usual we consider a pair $(S, C)$ which we assume to be either in the Rank-1 or in the Rank-2 case. If we are in the Rank-1 case we denote by $T$ the K3 surface $V_C(2, K_C, s)_{\text{red}}$ introduced in the previous section. We do the same in the Rank-2 case where, by virtue of Theorem 5.1, we have $V_C(2, K_C, s)_{\text{red}} = M_C(2, K_C, s)$. In both cases we have an isomorphism

$$\sigma : M_v(S) \xrightarrow{\sim} T, \quad \text{where} \quad v = (2, [C], s)$$

We will always view the K3 surface $T$ as a sub variety of $M_C(2, K_C)$. We further assume that $s = 2t + 1$, i.e. $g \equiv 3 \mod 4$

Following Mukai’s program (Remark 10.3 in [12]) and its implementation in genus eleven (Section 4 of [10]) we are going to prove the following theorem.
Theorem 7.1. There exists a Poincaré bundle

\[ \mathcal{U} \]
\[ \downarrow \]
\[ C \times T \]

unique up to isomorphism, having the following properties. Denote by \( \pi_C : C \times T \rightarrow C \) and \( \pi_T : C \times T \rightarrow T \) the two projections, then

\begin{enumerate}[i)]
  \item \( \mathcal{U}_{C \times \{E\}} \cong E, \quad \forall \ [E] \in T \subset M_C(2, K_C) \)
  \item \( \text{det}(\mathcal{U}) \cong K_C \boxtimes h_{\text{det}}, \text{ where } h_{\text{det}} = (\text{det} R^1 \pi_T \mathcal{U}) \otimes (\text{det} \pi_T \mathcal{U})^{-1} \)
\end{enumerate}

Moreover:

\begin{enumerate}[iii)]
  \item \( h_{\text{det}} \) is a polarization of genus \( g \) on \( T \).
  \item For each \( x \in C \), the vector bundle \( \mathcal{U}_x = \mathcal{U}_{\{x\} \times T} \) is stable and \( \mathcal{U}_x \in M_{\tilde{\nu}}(T) \), where \( \tilde{\nu} = (2, h_{\text{det}}, s) \)
  \item The morphism \( C \rightarrow \tilde{T} = M_{\tilde{\nu}}(T) \) defined by \( x \mapsto \mathcal{U}_x \) is an embedding.
  \item The Fourier-Mukai transform \( (\tilde{T}, \hat{h}_{\text{det}}) \) of \( (T, h_{\text{det}}) \) is isomorphic to \( (S, h) \), where \( h = [C] \).
\end{enumerate}

Proof. Write

\[ M_C(2, K_C) = R/PGL(\nu), \quad R \subset \text{Quot} \]

consider the quotient map

\[ p : R \rightarrow M_C(2, K_C) \]

and set

\[ R' = p^{-1}(T) \]

Let \( \tilde{\mathcal{U}} \) be the restriction to \( C \times R' \) of the universal bundle over \( C \times \text{Quot} \). Consider the projection

\[ \pi_{R'} : C \times R' \rightarrow R' \]

The sheaf \( \pi_{R'} \tilde{\mathcal{U}} \) is a vector bundle of rank \( s + 2 \), while \( \pi_{R'}(\tilde{\mathcal{U}} \boxtimes K_C) \) is a vector bundle of rank \( 8s \); indeed \( h^0(E \otimes K_C) = 3 \text{deg} K_C + 2(1 - g) = 8s \). Since \( s = 2t + 1 \), the two integers \( 8s \) and \( s + 2 \) are relatively prime and we can find integers \( x \) and \( y \) such that \( 1 + x(s + 2) + y8s = 0 \).

Consider then the vector bundle on \( C \times R' \):

\[ \mathcal{V} = \tilde{\mathcal{U}} \boxtimes \pi_{R'}^* (\text{det}(\pi_{R'} \tilde{\mathcal{U}})^x \otimes (\text{det}(\pi_{R'} \tilde{\mathcal{U}} \boxtimes K_C))^y) \]

The action of a central element \( c \in \mathbb{C}^* \subset GL(\nu) \) on the three factors are : \( 1, e^{x(s + 2)} \) and \( e^{8ys} \). Thus the vector bundle \( \mathcal{V} \) is acted on by \( PGL(\nu) \) and descends to a Poincaré bundle \( \mathcal{V} \) on \( C \times T \). Since \( T \) is regular there exists a line bundle \( L \) on \( T \) such that

\[ \text{det} \mathcal{V} = K_C \boxtimes L \]

Thus

\[ \mathcal{V}^\vee \boxtimes \pi_C^* K_C \cong \mathcal{V} \boxtimes \pi_T^* L^{-1} \]

As a consequence by Serre duality:

\[ (R^1 \pi_T^* \mathcal{V})^\vee \cong \pi_T^* (\mathcal{V}^\vee \boxtimes \pi_C^* K_C) \cong \pi_T^* \mathcal{V} \otimes L^{-1} \]

Hence

\[ h_{\text{det}} = (\text{det} R^1 \pi_T^* \mathcal{V}) \otimes (\text{det} \pi_T^* \mathcal{V})^{-1} \cong L^{s + 2} \otimes (\text{det} \pi_T^* \mathcal{V})^{-2} \]

Now the universal bundle

\[ \mathcal{U} = \mathcal{V} \boxtimes L^{t + 1} \boxtimes (\text{det} \pi_T^* \mathcal{V})^{-1} \]

satisfies both \( i \) and \( ii \). The unicity follows from the fact that \( \text{Pic}(T) \) is torsion free.
Since properties \( iii), \ iv) \) are invariant under small deformations, we may limit ourselves to the rank-two case. In this case we have the universal extension
\[
0 \to \mathcal{O}_{S \times T}(\rho^* B) \to \mathcal{F} \to \mathcal{I}_\Delta(\rho^* A + \tau^*(B - A)) \to 0
\]
where \( \rho \) and \( \tau \) are the projections \( S \times T \to S \) and \( S \times T \to T \). Moreover we identify \( S \) and \( T \) via the isomorphism
\[
\begin{align*}
S & \longrightarrow M_\circ(S) \xrightarrow{\sigma} T \\
x & \mapsto \mathcal{E}_x \xmapsto{} \mathcal{E}_x = \mathcal{E}_{x|C}
\end{align*}
\]
(c.f. Theorem \ref{thm:iso} and Proposition \ref{prop:emb}). We have
\[
\det(\mathcal{F}|_{C \times T}) = K_C \otimes \mathcal{O}(B - A)
\]
Consider
\[
0 \to \tau_* \mathcal{O}_{S \times T}(\rho^* B) \to \tau_* \mathcal{F} \to \tau_* \mathcal{I}_\Delta(\rho^* A + \tau^*(B - A)) \to R^1 \tau_* \mathcal{O}_{S \times T}(\rho^* B) \to 0
\]
which gives
\[
0 \to H^0(\xi) \otimes \mathcal{O}_T \to \tau_* (\mathcal{F}|_{C \times T}) \to H^0(\eta) \otimes \mathcal{O}_T (B - A) \to \mathcal{O}_T (B) \to 0
\]
We know that
\[
h_{\det} = L^{s+2} \otimes \det \tau_*(\mathcal{F}|_{C \times T})^{-2}
\]
We then have
\[
L = \mathcal{O}(B - A), \quad \det \tau_*(\mathcal{F}|_{C \times T}) = \mathcal{O}(sB - (s + 1)A), \quad h_{\det} = \mathcal{O}(sA - (s - 2)B)
\]
Thus \( h_{\det} \) is a positive polarization and its genus is given by
\[
g(h_{\det}) = \frac{1}{2}(sA - (s - 2)B)^2 + 1 = 2s + 1
\]
proving \( iii \).

In the rank-two case the normalized Poincaré bundle is given by
\[
\mathcal{U} = \mathcal{F}|_{C \times T} \otimes \tau^* \mathcal{O}((t + 1)A - tB)
\]
Let \( C' \) be a smooth element in \( |sA - (s - 2)B| \). Set \( B' = (t + 1)A - tB \) and \( A' = tA - (t - 1)B \). Under the identification given by \( \ref{eq:iso} \) we consider \( C', A' \) and \( B' \) as divisors in \( T \). We may then consider the rank-two case given by the decomposition
\[
\text{Pic}(T) = \mathbb{Z} \cdot A' \oplus \mathbb{Z} \cdot B', \quad |C'| = |A' + B'|
\]
For this case the universal extension can be given by tensoring \( \ref{eq:iso} \) by \( \tau^* \mathcal{O}(B') \). For each \( x \in S \) setting \( \mathcal{U}_x = \mathcal{U}_{\{x\} \times T} \), we get
\[
0 \to \mathcal{O}_T (B') \to \mathcal{U}_x \to I_x (A') \to 0
\]
We also get an isomorphism (Theorem \ref{thm:iso})
\[
\begin{align*}
T & \longrightarrow M_\circ(T) \\
x & \mapsto \mathcal{U}_x
\end{align*}
\]
and a fortiori an embedding
\[
\begin{align*}
C & \longrightarrow M_\circ(T) \\
x & \mapsto \mathcal{U}_x
\end{align*}
\]
Finally we want to show that \( (\hat{T}, \hat{h}_{\det}) = (M_\circ(T), \hat{h}_{\det}) \) may be identified with \( (S, h) \). Since we have the isomorphism \( \sigma : \hat{S} = M_\circ(S) \to T \), we have \( (\hat{T}, \hat{h}_{\det}) = (\hat{S}, h') = (S, h') \) for some polarization \( h' \) of genus \( g \). In the rank one case we necessarily have \( h' = h \). Let us show that also in the rank two
case we have $\hat{h}_{\text{det}} = h$. To simplify notation we will prove the equivalent statement that $\hat{h} = h_{\text{det}}$, which means

(7.5) \[ [\hat{C}] = [C'] = [sA - (s - 2)B] \]

From [11], we recall the procedure one has to follow to construct $\hat{h}$, starting from $h$.

We let $\hat{S} = M_v(S)$, where $v = (2, h, s)$

Then $\hat{S}$ is again a K3 surface and there is a universal family $\mathcal{F}$ on $\hat{S} \times S$. Let

\[ c_1(\mathcal{F}) = h + \phi \in H^2(S) \oplus H^2(\hat{S}), \quad \text{and} \quad c_2^{\text{mid}}(\mathcal{F}) \in H^2(S) \oplus H^2(\hat{S}) \]

be the first Chern class of $\mathcal{E}$ and the middle Künneth component of the second Chern class respectively. Define a class $\psi \in H^2(\hat{S})$ by

\[ h \cup c_2^{\text{mid}}(\mathcal{F}) = p \otimes \psi \in H^4(S) \oplus H^2(\hat{S}) \]

where $p$ is the fundamental class of $S$. Both $\phi$ and $\psi$ are algebraic by Lefschetz theorem. Then the class $\hat{h}$ is given by:

\[ \hat{h} = \psi - 2s\phi \]

We now consider the rank two case in which $\text{Pic}(S) \cong \mathbb{Z} \cdot A \oplus \mathbb{Z} \cdot B$ and we look at the exact sequence (7.1)

(7.6) \[ 0 \to \mathcal{O}_{\hat{S} \times S}(p^*B) \to \mathcal{F} \to I_\Delta(p^*A + q^*(B - A)) \to 0 \]

We get

\[ c_1(\mathcal{F}) = c_1(p^*(A + B)) + c_1(q^*(B - A)) \]

Thus

\[ h = c_1(p^*(A + B)), \quad \phi = c_1(q^*(B - A)) \]

On the other hand

\[ c_2(\mathcal{F}) = c_1(p^*(B)) \cup (c_1(p^*(A) + c_1(q^*(B - A))) + \Delta \]

so that

\[ c_2(\mathcal{F})^{\text{mid}} = c_1(p^*(B)) \cup (c_1(q^*(B - A))) + \Delta \]

Therefore, as a class in $H^2(\hat{S})$,

\[ \psi = ((A + B \cdot B)[B - A] + [A + B] = (s + 2)B - sA \]

As a conclusion

\[ \hat{h} = sA - (s - 2)B. \]
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E-mail address: ea@mat.uniroma1.it, bruno@mat.uniroma3.it, sernesi@mat.uniroma3.it

Dipartimento di Matematica, Sapienza, Università di Roma, Roma.

Dipartimento di Matematica e Fisica, Università Roma Tre, Roma.

Dipartimento di Matematica e Fisica, Università Roma Tre, Roma.