Topology optimization of permanent magnets for stellarators

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Abstract
We introduce a topology optimization method to design permanent magnets for advanced stellarators. Recent research shows that permanent magnets have great potential to simplify stellarator coils. We adopt state-of-the-art numerical techniques to determine the presence of magnets in the entire design space. The FAMUS code is developed and it can design engineering-feasible permanent magnets for general stellarators satisfying the constraints of the maximum material magnetization and explicit forbidden regions. FAMUS has been successfully verified against the previously proposed linear method. Three different permanent magnet designs together with planar TF coils for a half-Tesla NCSX configuration have been obtained for demonstrations. The designs have good accuracy in generating the desired equilibrium and offer considerably large plasma access on the outboard side. The results show that FAMUS is a flexible, advanced numerical tool for future permanent magnet stellarator designs.

Keywords: stellarator, permanent magnet, topology optimization, numerical optimization, simple coil, stellarator design

(Some figures may appear in colour only in the online journal)

1. Introduction
Stellarators have attractive properties, like low recirculating power, steady-state operation and free of disruptions, but they also suffer the engineering complexities, especially complicated non-planar coils. Recently, the concept of using permanent magnets to simplify stellarator coils was introduced. Helander et al [1] proposed a mathematical solution of a curl-free, ‘one-sided’ magnetization distribution such that the effective surface current can be determined by the NESCOIL [2] method. Landreman [3] developed a least-squares minimization method to optimize the magnetization in space together with a Picard iteration method to constrain the maximum magnetization. In a previous paper [4], we introduced a linear method to design perpendicular magnets which employs the idea that current potential is the magnetic dipole moment per unit area. An iterative scheme was also developed to incrementally stack multi-layer magnetizations using the linear method. However, to obtain engineering feasible designs, we need more sophisticated methods.

Stellarator optimization is normally divided into two steps:

1. Configuration optimization using fixed-boundary MHD equilibrium codes to obtain desirable physics properties [5, 6].
2. Coil optimization as a ‘reverse-engineering’ step to design coils for supporting the target equilibrium [7, 8].

The two steps usually will have to be iterated several times until a good balanced solution is obtained. Afterwards, a detailed engineering design will be carried out. The design of permanent magnets is an alternative approach to the coil optimization step. Therefore, in this problem, we are assuming there exists a target configuration and only some engineering constraints will be considered. After an applicable design was obtained, more comprehensive engineering analyses would be performed.

We identify the problem of designing permanent magnets as a topology optimization problem. Topology optimization [9] is defined as a mathematical method to deter-
The magnetic field produced by a given magnetization. In textbooks [16], there are three ways, magnetic charges, effective currents and discretized magnetic dipoles. To numerically compute the magnetic field, various methods [17] have been developed using the finite-element method, the finite-difference method and others. The problem that we are going to discuss in this paper is distinct from other existing applications. First, the computation domain in this problem is much larger. A stellarator usually has a major radius bigger than 1 meter (some even larger than 10 meters). This indicates that a large amount of magnetic material is required. For example, a half-Tesla three-period quasi-axisymmetric stellarator might need 3 tons of magnets [4]. Second, the geometry is more complicated. For modern stellarators, the configurations are optimized to have desirable properties, like good particle confinement and MHD stability. Therefore, the plasma shape is generally complicated and so is the vessel. The complicated geometries create difficulty when generating meshes in the design space. Third, the distance between the plasma and permanent magnets is normally sizable. Permanent magnets are sensitive to temperature and will be demagnetized under neutron fluxes [18]. Thus, magnets need to be placed outside the vessel (and outside the blanket for fusion reactors). In this paper, we will optimize the magnet arrangements iteratively, so the calculation of magnetic field from magnetization should be computationally cheap. Herein, the dipole representation is used. Elements that are sufficiently small will be approximated by one magnetic dipole. The detailed distribution inside each element will not be important during the stage of optimization. In addition, the present magnets are usually homogeneously magnetized which could be well approximated by a dipole when evaluating the far field. The dipole representation is concise and easy, while the accuracy increases with distance. According to [19], the relative error of the dipole approximation for a cubical homogeneous permanent magnet is about 0.01% at the distance of 7 times of the magnet’s dimension.

In this paper, we will introduce a novel topology optimization method used to design permanent magnets for stellarators. Ampere’s law implies that permanent magnets cannot produce the entire magnetic field needed for stellarators [1], so external coils will always be needed. It is expected that permanent magnets will be used to generate only part of the magnetic field while the main magnetic field is produced by coils. Previous results show that advanced stellarator configurations can be realized by permanent magnets together with planar coils [1, 4], which are exceptionally attractive. Herein, we will use planar coils to generate the toroidal field and permanent magnets to produce the others. Permanent magnets also have limitations. For example, they cannot be turned off and the produced magnetic field is relatively small. The amount of required permanent magnets will rely on how much field they have to produce.

This paper is organized as follows. The methodology of designing permanent magnets using the topology optimization method is described in section 2. Section 3 shows a benchmark with the previous linear method for validations. In section 4, three designs of permanent magnets for a half-Tesla three-period quasi-axisymmetric stellarator are demonstrated. It includes a perpendicular-only solution, an orientation-optimized solution and a solution combined with simplified geometry. We then summarize in section 5.

2. Methodology

2.1. Magnetic field calculation

The magnetic field generated by a magnetic dipole is calculated by the Biot-Savart law, \( \mathbf{B} = \frac{\mu_0}{4\pi} \sum_{i=1}^{D} \left( \frac{3m_i \mathbf{r}_i}{|\mathbf{r}_i|^3} - \frac{1}{|\mathbf{r}_i|^5} m_i \right) \), where \( m_i \) is the magnetic moment. The entire computational domain can be discretized into numerous elements and each element can be approximated by one dipole at the center when the element size is sufficiently smaller than the distance between the magnet and the plasma. Suppose the total number of discrete dipoles is \( D \), then the total magnetic field is a summation over the dipoles,

\[
B_M = \frac{\mu_0}{4\pi} \sum_{i=1}^{D} \left( \frac{3m_i \mathbf{r}_i}{|\mathbf{r}_i|^3} - \frac{1}{|\mathbf{r}_i|^5} m_i \right),
\]

where \( m_i = \sum_{i=x,y,z} m_i^e \mathbf{e}_i \) is the magnetic moment of \( i \)th magnetic dipole, \( \mathbf{r}_i = \sum_{i=x,y,z} \mathbf{r}_i^e \mathbf{e}_i \) the position vector from the \( i \)th dipole to the evaluation point.

2.2. Constraint on the maximum magnetization

One of the explicit constraints that we have to follow is that the magnetization cannot exceed the remanence of materials \( (B_r) \). The remanence varies with materials and material grades. For example, neodymium-iron-boron(NdFeB) magnet, which is one of the most common magnets, normally has a remanence of 1.0 – 1.5 T, while FeCoNi magnets may have \( B_r = 2.9 \) T [20]. In this paper, we will choose \( B_r = 1.4 \) T for all the calculations, as this is one of the most widely commercially available magnets on the market. However, it does not mean the remanence is fixed. One could change the value of \( B_r \) when using different materials.

The magnetic moment of each element is the volume integral of the magnetization, \( m = \iiint \mathbf{M} \mathrm{d}V \). If we consider homogeneous magnetization and take the relative permeability as
\( \mu = 1 \), each dipole will have a maximum allowable magnetic moment, which is then calculated as \( m_{0i} = B_i / \mu_0 V_i \) \((V_i \) is the volume for the \( i \)th element and \( \mu_0 \) the vacuum permeability). To better represent the constraint of the maximum magnetization, we shall split the magnetic moment of each dipole into magnitude and orientation. A local spherical coordinates is used to represent the (unit) orientation,

\[
m_i(\theta_i, \phi_i) = m_{0i} \begin{pmatrix} \sin \theta_i \cos \phi_i \\ \sin \theta_i \sin \phi_i \\ \cos \theta_i \end{pmatrix},
\]

\( \theta_i \in [-\pi, \pi] \) is the inclination angle and \( \phi_i \in [-\pi, \pi] \) the azimuth angle. Instead of using three components in Cartesian coordinates, we reduce the number of variables to two \((\theta, \phi)\) and explicitly enforce the constraint on the maximum allowable magnetic moment \( (m_{0i}) \).

### 2.3. The density method

The density method in topology optimization, which is also called solid isotropic microstructure with penalization \([21]\), represents the presence of material in each element with a normalized density varying from 0 to 1,

\[
\rho = \begin{cases} 
0 & \text{no material;} \\
1 & \text{material.} 
\end{cases}
\]

In numerical optimizations, it is generally more complicated to do discrete optimization than optimizing continuous parameters. Therefore, the density is allowed to be continuous. For the consideration of simplifying assembly and reducing purchase cost, one would expect the magnets have uniform strength (and size). In other words, the intermediate values of the normalized density \( (0 < \rho < 1) \) should be avoided. A special technique is applied to penalize the intermediate values. The normalized density is written in an exponential form, \( \rho = \rho_0 q^p \), where \( p \in [0, 1] \) is the free parameter and \( q \) the penalize coefficient. The penalize coefficient is specified by users and the larger \( q \) is set, the more stiff the distribution is. By using a relatively large value of \( q \), like \( q = 7 \), one could achieve a polarized distribution for the normalized density. Now, the final representation for the magnetic moment of each dipole becomes

\[
m_i(p_i, \theta_i, \phi_i) = p_i q^p m_{0i} \begin{pmatrix} \sin \theta_i \cos \phi_i \\ \sin \theta_i \sin \phi_i \\ \cos \theta_i \end{pmatrix},
\]

where the free variables for the \( i \)th dipole are \( p_i, \theta_i, \) and \( \phi_i \), \( p_i \) controls the density and \( \{\theta_i, \phi_i\} \) determines the orientation. \( m_{0i} \) comes from the product of the maximum allowable magnetization and the element volume.

### 2.4. Objective functions

To produce the required magnetic field, the primary objective function in coil optimization codes is usually the Neumann boundary condition, which is the normal magnetic field on a prescribed toroidal surface \( S \). Likewise, to optimize the permanent magnets, we can employ an objective function for the magnetic field in the following form,

\[
F_B(p, \theta, \phi) = \int_S (B_M \cdot n - B_{n_{\text{tgt}}})^2 \, da,
\]

where \( B_{n_{\text{tgt}}} \) is the target normal field that permanent magnets have to cancel off and \( n = N / |N| \) the unit surface normal vector. Once the boundary condition is satisfied, permanent magnets will generate the desired magnetic field to help support the MHD equilibrium. Normally, the last-closed-flux-surface (LCFS) is used as the plasma boundary. The magnetic field is tangential on the flux surface, i.e., \( (B_M + B_{\text{coils}} + B_{\text{plasma}}) \cdot n = 0 \) on \( S \). So we have \( B_{n_{\text{tgt}}} = -(B_{\text{coils}} + B_{\text{plasma}}) \cdot n \), where \( B_{\text{coils}} \) comes from external coils and \( B_{\text{plasma}} \) from plasma currents. The normal field at an arbitrary point on the plasma boundary generated by the dipoles can be calculated as,

\[
B_M \cdot n(\theta_i, \zeta_i) = \frac{1}{|N|} \sum_{i=1}^{D} \left( g_i \cdot m_i \right),
\]

where we parameterize the plasma boundary using the poloidal angle \( \theta \) and toroidal angle \( \zeta \). \( N \) is the normal vector of the plasma boundary and \( g \) is the 'inductance' matrix. Since the geometric quantities do not change during the optimizations, we can compute \( g \) once and then store it in the memory,

\[
g_{ij}(\theta_i, \zeta_i) = \frac{\mu_0}{4\pi} \left( \frac{3r_i \cdot N_i}{|r_i|^3} - \frac{1}{|r_i|^3} N_i \right).
\]

The first-order derivatives of \( F_B \) can be analytically computed,

\[
\frac{\partial F_B}{\partial p_i} = 2 \int_{\theta_i} \int_{\zeta_i} \left( \frac{1}{|N_i|} \sum_{j=1}^{D} g_{ij} \cdot m_j - B_{n_{\text{tgt}}}^{\text{coils}} \right) q_p q^{p-1} m_{0i} \times (g_i^{\text{coils}} \sin \theta_i \cos \phi_i + g_i^{\text{coils}} \sin \phi_i \sin \theta_i + g_i^{\text{coils}} \cos \theta_i) d\theta_i d\zeta_i \ ,
\]

\[
\frac{\partial F_B}{\partial \theta_i} = 2 \int_{\theta_i} \int_{\zeta_i} \left( \frac{1}{|N_i|} \sum_{j=1}^{D} g_{ij} \cdot m_j - B_{n_{\text{tgt}}}^{\text{coils}} \right) p_i q^p m_{0i} \times (g_i^{\text{coils}} \cos \theta_i \cos \phi_i + g_i^{\text{coils}} \cos \phi_i \sin \theta_i - g_i^{\text{coils}} \sin \theta_i) d\theta_i d\zeta_i \ ,
\]

\[
\frac{\partial F_B}{\partial \phi_i} = 2 \int_{\theta_i} \int_{\zeta_i} \left( \frac{1}{|N_i|} \sum_{j=1}^{D} g_{ij} \cdot m_j - B_{n_{\text{tgt}}}^{\text{coils}} \right) p_i q^p m_{0i} \times (-g_i^{\text{coils}} \sin \theta_i \sin \phi_i + g_i^{\text{coils}} \sin \theta_i \cos \phi_i) d\theta_i d\zeta_i \ .
\]

The inverse problem of the Biot–Savart law is ill-posed and coil optimization codes usually have other penalty functions on coil geometry to obtain engineering favorable solutions.
For the permanent magnet problem, we adopt a regularization/penalization term on the total number of magnetic moments, which is calculated as,
\[
F_M(p) = \sum_{i=1}^{D} |m_i|^2.
\] (11)

\(F_M\) is determined by the density parameter only. The derivative of \(F_M\) with respect to \(p_i\) is
\[
\frac{\partial F_M}{\partial p_i} = 2q_i \rho_i^{2q-1} (m_{0i})^2,
\] (12)

while \(\partial F_M / \partial \theta_i\) and \(\partial F_M / \partial \phi_i\) are zero. There is another constraint that we have to consider for a practical design. The access to the plasma is essential for heating and diagnostic systems, which implies the magnets cannot cover the entire space. Thus, there are forbidden regions where the magnets cannot be present. Instead of employing new objective functions to preserve the forbidden regions, we have a more straightforward way. For the elements inside the forbidden regions, the associated parameters \((p_i, \theta_i, \phi_i)\) will not be varied and \(p_i\) is zero during the optimization. By doing so, we can exclude the magnets inside the forbidden regions.

The final objective function is then the weighted summation of \(F_B\) and \(F_M\),
\[
F(p, \theta, \phi) = F_B + \lambda F_M,
\] (13)

where \(\lambda\) is a user-specified weight for the penalization.

### 2.5. Numerical implementation

The FAMUS (Flexible Advanced Magnets Used for Stellarators) code\(^1\) has been developed, on the foundation of the FOCUS [8] code. The problem of finding optimal magnet layout can be expressed in mathematical languages as,
\[
\min F(p, \theta, \phi) = \int \left[ |B_M(p, \theta, \phi) \cdot n - B^\text{tgt}_{\theta, \phi} |^2 \right] \text{d}a + \lambda \sum_{i=1}^{D} |m_i(p, \theta, \phi)|^2
\] (14)

s. t. \(p_i \in [0, 1], \theta_i \in [-\pi, \pi], \phi_i \in [-\pi, \pi], i = 1, \ldots, D\). (15)

The variables are the density and orientation of each magnetic dipole, and they have explicit bounds. For simplification, we assume the magnetization does not change when the background field changes. In other words, we will omit the interactions between each dipole and thus the variables are separable. It is beneficial to use message passing interface (MPI) for parallelization and each CPU can handle a sub-group of dipoles. To minimize the target function, we use a limited-memory quasi-Newton algorithm (the L-BFGS-B library) [22]

with defined bounds. The gradient is calculated analytically using the formulas in equation (8), (9), (10) and (12). To reduce the number of degrees of freedom, FAMUS can enforce toroidal periodicity and/or stellarator symmetry [23] if necessary.

To retain the precision of modeling, FAMUS tends to use a large number of dipoles, which means the dimensionality is (extremely) large. It is not an issue for the computation speed since the efficiency of MPI is considerably high and the usage of analytically calculated derivatives makes sure the problem can be easily scaled-up. However, the large number of variables makes it easy to get trapped in local minima. An example of FAMUS finding different solutions with different initial guesses is shown in appendix A. A good initial guess is thus important for successful optimizations. In [4], we introduced a linear method to design perpendicular magnets using current potential. It can provide a good initial guess for FAMUS. Besides, we can linearly solve the magnitude and/or orientation of each magnetic dipole if we do not enforce the constraint over the maximum magnetization, as shown in appendix B. We might not be able to incorporate the constraint of the maximum magnetization without further modifications, but the linear solved solution can be truncated manually and should provide a relative good starting point for FAMUS non-linear optimizations. In addition, other approaches, like the one proposed by Helander et al [1] and REGCOIL_PM [3], can also be used for initialization.

During the optimization, we represent small cells as magnetic dipoles. Once the arrangement of magnetic dipoles is determined by FAMUS, the original shapes of each magnet will be recovered. The magnets with negligible magnitude \((\rho \sim 0)\) can be directly removed. The topology of the required permanent magnets is then obtained. The next step is to cut magnets and design supporting structures. The size of magnets can be adjusted if necessary, i.e. big magnets can be split into smaller pieces and adjacent magnets that have the same orientation can be replaced by a larger magnet. The technique of cutting magnets is important and will have to consider various factors. It is beyond the scope of this paper and the details will be discussed in future work.

### 3. Benchmark with the surface magnetization method

To validate the FAMUS code, we can compare the solution optimized by FAMUS with the one obtained by the surface magnetization method [4], which shall be called as ‘the linear method’ hereafter. For this benchmark, a two-period rotating elliptical stellarator was used. Details of the numerical setup for the linear method can be found in section 3.1 of the reference [4]. By using the linear method, a single-layer of magnetic dipoles \((128 \times 128 \text{ per period})\) that are perpendicular to the winding surface was obtained to support the vacuum configuration together with a toroidal field.

To prepare FAMUS runs, the positions of the magnetic dipoles were input to FAMUS and their orientations were fixed to be perpendicular to the winding surface (pointing outwards). The allowable moment \(m_{0i}\) for each magnetic dipole

\(^1\) https://princetonuniversity.github.io/FOCUS/famus.html
was set to the maximum value of the magnetic moment in the linear method solution. Therefore, the normalized density in the linear method solution is in the range of $[-1, 1]$ and the distribution is shown in figure 1. In FAMUS runs, only the normalized density of each dipole was allowed to vary in the same range of $p_i \in [-1, 1]$. The number of free parameters was 16384. The initial density of each dipole was set to a small, non-zero number, $1.0 \times 10^{-3}$ and the exponential factor was $q = 1$. $F_B$ was reduced from $8.73 \times 10^{-3}$ to $5.20 \times 10^{-13}$ Tm$^2$ in 100 iterations. The elapsed wall time was 389.28 s using 32 CPUs (Intel Xeon E5-2670@2.60GHz) with MPI parallel. Figure 1 shows the comparison between the distribution of the density optimized by FAMUS and the target solution from the linear method. The difference between the two solutions are negligible and the average difference in the normalized density is $9.94 \times 10^{-4}$ (maximum difference $8.91 \times 10^{-3}$). The results indicate that FAMUS can find exactly the same solution (to some numerical tolerance) as the linear method if we restrict the magnets to be perpendicular only and relax the constraint on the maximum allowable magnetization.

4. Applications on NCSX

Now we can apply FAMUS to practical configurations. For demonstration, we choose the quasi-axisymmetric stellarator, NCSX [24], as the target configuration. NCSX was partly built at the Princeton Plasma Physics Laboratory and unfortunately canceled due to cost overrun. It was originally designed with 18 modular coils to generate the main magnetic field. In our study, we are going to only use the 18 planar TF coils to provide the essential toroidal field. One of the designed equilibria, C09R00, was scaled to have a volume-averaged magnetic field of $\langle B \rangle = 0.5$ T, which is the maximum field produced by the TF coils. Some other parameters of C09R00 are $N_{fp} = 3$, $R_0 = 1.44$ m, $a = 0.32$ m, $V_{\text{plasma}} = 2.96$ m$^3$, and $\langle \beta \rangle = 4.09\%$.

To further embrace engineering constraints, we will re-use the NCSX vacuum vessel [25], which has been built as well. Ports have been opened for diagnostics and heating systems. These ports are the explicit forbidden regions wherein permanent magnets can not be placed. In this paper, we are going to consider the three largest ports with extensions, the vertical ports (upper and lower), the neutral-beam port and the port 4, while other ports can also be reserved if necessary.

In the following, we will present three applications to the NCSX configuration. The first one fixes the magnet orientations and only the normalized density is varied. The second one has arbitrary orientations to improve the magnetic field efficiency and uses less magnets. The third one is incorporated with simplified geometries for the magnets. We will use the finite-thickness solution obtained by the linear method [4] as a reference. As one of the earliest investigations, the linear method solution is restricted to perpendicular magnets and cannot explicitly exclude ports. Furthermore, it was obtained by incrementally stacking magnet layers, so it is not an overall optimization.

4.1. Permanent magnets in perpendicular direction

The magnetic dipoles from the linear method are lying on a sequence of surfaces offset from the vessel. The innermost surface is 2 cm away from the vessel and the outermost is 22 cm away, while the distance between two adjacent surfaces is 1 cm. The resolution of magnetic dipoles is 20 (radial) $\times$ 128 (poloidal) $\times$ 64 (toroidal) per half period. The orientation of each dipole is aligned to be perpendicular to the supporting
The distribution of the normalized density for the multi-layer perpendicular magnetization from the linear method.

Figure 2.

The distribution of the normalized density for the FAMUS-optimized solution with fixed orientations.

Figure 3.

The linear method solution is directly read by FAMUS and used as an initial guess, while the magnetic dipoles in the forbidden regions are filtered out (fixed to be zero during the optimization). To penalize the intermediate values in the density function, we chose the penalization coefficient \( q = 7 \). Free variables were the normalized density of each magnetic dipole, \( p_i \in [-1, 1] \). The orientation was fixed during the optimizations. The total number of variables was 154843 while there were 8997 elements inside the forbidden regions. FAMUS can further minimize the normal field, optimize the layout, reduce the magnet volume. However, we found that initializing with \( p_i = 1.0 \) yielded a better solution with larger access on the outboard side. Figure 3 shows the distribution of the normalized density optimized from the initial guess of \( p_i = 1.0 \) with \( q = 7 \). In 200 iterations (26 minutes of wall time with 256 CPUs using MPI), \( F_B \) was reduced to \( 2.91 \times 10^{-5} \, \text{T}^2 \, \text{m}^2 \) and the total used magnetic moment was \( 5.49 \times 10^5 \, \text{A} \cdot \text{m}^2 \) (equivalent to 0.50 m³ NeFeB magnet).

Significant difference in the distribution of the normalized density can be observed when comparing figures 2 and 3. Since we are using structured grids for the meshes of dipoles, we can compute the radial accumulation of the normalized density. As shown in figure 4, the original solution from the linear method covers almost the entire space. The FAMUS optimized solution can explicitly incorporate ports and has large area of zero-density regions where no magnets are needed. The magnets are mostly localized on the inboard side. It can be more clearly observed when viewing from the outboard side, as illustrated in figure 5, in which we removed the magnets with \( |\rho| < 0.1 \). The almost-open outboard side can provide considerably large access to the plasma, which can be used for heating systems, diagnostics and maintenance. The large access to the plasma would be exceptionally attractive to future fusion reactors which rely on remote maintenance.
Figure 5. Outboard view of the perpendicular permanent magnets together with TF coils for the half-Tesla NCSX configuration. Magnets with normalized density $|\rho| < 0.1$ are removed for clarity.

Figure 6. Histograms of the normalized densities for the linear method solution and FAMUS-optimized solution. The intervals are of equal size (0.1) in the range of [0.0, 1.0] for both cases. The rectangles are not plotted with full width to clarify the distinction.

We can check if the density method is working as expected. In figure 6, we plot out the histograms of the normalized density for the linear method solution and FAMUS-optimized solution. The FAMUS-optimized solution has a more polarized distribution. More than half of the magnets (52.49%) have a density smaller than 0.1 and 30.18% of the magnets have a density larger than 0.9. The intermediate values are indeed penalized by adopting the exponential function of the density. The accuracy will be mostly retained when the magnets are retained only at the places of $\rho > 0.9$.

4.2. Permanent magnets with arbitrary orientations

We can now relax the restriction of orientation to improve the magnetic efficiency. The ‘Halbach’ array [26] can improve the magnetic efficiency by a factor of two in simple 1D and 2D cases. Thus, we can try to reduce the thickness of the design space from 20 to 10 cm. The resolution of the structured mesh is now 10 (radial) × 128 (poloidal) × 64 (toroidal) per half period. Again, stellarator symmetry is enforced. To avoid being trapped in local minima, we initialized the dipole orientation to be perpendicular to the vessel and split the optimization into three steps. First, the density of each dipole was fixed to be one (including those in the forbidden regions) and the orientations were optimized to only reduce the normal field error, $F_B$. $F_B$ was decreased from $7.07 \times 10^{-2}$ to $7.71 \times 10^{-6}$ T$^2$m$^2$ in 100 iterations. Subsequently, the density was allowed to be varied and both $F_B$ and $F_M$ were minimized. Also, $q = 8$ was used to penalize the intermediate values for the density. Since the orientation is varying, we are not using negative values for $\rho$ to indicate the opposite direction. In 200 iterations, $F_B$ was further reduced to $2.81 \times 10^{-6}$ T$^2$m$^2$ while the total amount of the used magnetic moment was reduced...
by 31.1%. In the third step, the magnetic dipoles in the forbidden regions were filtered out. The orientations and normalized densities of the other dipoles were further optimized. The final solution is shown in figure 7, while \( F_B = 6.57 \times 10^{-6} \, T^2/m^2 \) and the total used magnetic moment is \( 4.75 \times 10^5 \, A \cdot m^2 \) (equivalent to 0.43 m\(^3\) NeFeB magnet).

We can check the orientation of each magnetic dipole. In figure 8, we plot the value of \( \frac{\textbf{m} \cdot \textbf{n}'}{|\textbf{m}|} \) where \( \textbf{n}' \) is the vessel normal. This quantity represents the angle between the dipole orientation and the vessel normal. The dipoles with |\( \rho | < 0.1 \) have been clipped and most of the dipoles are located on the inboard side as well. On the inboard side, tilt stripes are observed. At the bean-shaped cross-section, the dipoles are pointing outwards at the upper half, and inwards at the lower regions. Between the two regions, the dipoles are mostly tangential to the vessel normal. This is somewhat similar to a Hablach array with the pattern of \( \uparrow \rightarrow \downarrow \leftarrow \uparrow \). Of course, the geometries are fully 3D and the actual distribution is much more complicated than ideal Hablach arrays.

**4.3. Permanent magnets with simplified geometry**

In the above cases, we used structured grids that are discretized along the curvilinear coordinates to fit the vacuum vessel. One can also provide different meshes to meet specific requirements for the magnets. For example, magnet designs with simplified geometry have been studied \cite{27} with a view toward reducing the complexity of construction. One of these implementations, which shall be called ‘curved bricks’, is shown in figure 9. The magnets are cut along the grids of cylindrical coordinates \((R, \phi, Z)\). Each brick has dimensions of 4.9 cm \( \times \) 0.041 rad \( \times \) 4.9 cm; hence, all magnets at a given major radius have the same shape. The toroidally curved magnets can be effectively approximated by cuboidal magnets. In figure 9, each magnet is replaced by four cuboidal magnets. As in previous cases, magnets that collide with selected ports are excluded. Gaps of 0.1 cm \( \times \) 0.0025 rad \( \times \) 0.1 cm are reserved between each magnet for mounting infrastructure. The total thickness of the magnet layer is about 20 cm (not uniform). Details of the design can be found in reference \cite{27}. Now, the magnet information is imported to FAMUS and each magnet is approximated by one magnetic dipole located at the center of mass. The orientations and magnitudes of the dipoles are optimized by FAMUS to reduce the normal field error and the number of used magnetic moments. The final distribution of the normalized density for each dipole is illustrated in figure 9.

The realized normal field error for the radial bricks is \( 1.15 \times 10^{-6} \, T^2/m^2 \) and the equivalent volume for magnets is 0.64 m\(^3\). The normal field error is lower than other cases, which is not surprising as the curved-brick case allows the dipole orientation to be varied and uses more magnetic moment than the previous orientation-optimized case. Like the other two solutions, most of the non-zero magnets in the curved bricks solution are located at the inboard side; therefore, it may be possible to refine the configuration by removing many of the outboard magnets altogether.

**4.4. Free-boundary equilibria using permanent magnets**

In the above, we have demonstrated three different designs using permanent magnets. They are designed following different goals and have various performance in \( F_B \) and \( F_M \). In table 1, we list the normal field error evaluated in the form of the mean value of |\( B \cdot n \)|/|\( B \)| on the plasma boundary and the equivalent volume of NdFeB magnets per half period (calculated by \( \sum_{i=1}^{n} |n_i|/M_0 \), where \( M_0 = B_x / \mu_0 \) and \( B_x = 1.4 \, T \)). The linear method solution shown in figure 2 is also included as a reference. The curved bricks case has the lowest field error while the orientation-optimized case uses the minimum amount of magnets. In both cases, orientation is allowed to be varied which demonstrates that relaxing the orientation can improve the magnetic efficiency. The linear method solution has the worst performance as it was not fully optimized and only perpendicular magnets were used.

The value of normal field error on the LCFS of the fixed-boundary equilibrium is one criteria to assess the accuracy of produced magnetic field. We have also reconstructed the free-boundary equilibria using the VMEC code \cite{28}. In the free-boundary VMEC calculations, we used the same pressure profile and toroidal current density profile as the fixed-boundary equilibrium. An MGRID file that contains the external magnetic field in a prescribed volume is necessary for free-boundary runs. In this problem, the external magnetic field is generated by TF coils and permanent magnets. For each design of permanent magnets, we have generated an MGRID file together with the TF coils and all the MGRID files have the same numerical resolution \((101 \times 101 \times 72)\). In all the VMEC runs, we used Mpol = 11, Ntor = 6, Ns = 256. As shown in
Figure 8. Orientations of the magnetic dipoles. The colors stand for the cosine value of the angle between the dipole orientation and the vessel normal. The dipoles with \( |\rho| < 0.1 \) have been removed.

Table 1. Comparisons of the normal field error and the magnet volume for different NCSX permanent magnet designs.

| Solutions                  | Linear method | Perpendicular-only | Orientation-optimized | Curved bricks |
|----------------------------|---------------|--------------------|-----------------------|---------------|
| Ave. \( \frac{|B \cdot n|}{|B|} \) | \( 1.10 \times 10^{-2} \) | \( 2.46 \times 10^{-3} \) | \( 1.62 \times 10^{-3} \) | \( 7.26 \times 10^{-4} \) |
| Magnets vol. (m³)          | 0.68          | 0.50               | 0.43                  | 0.64          |

Figure 9. The distribution of the normalized density for the curved bricks case with arbitrary orientation. The magnets are all cuboidal and will be mounted on independent toroidal ribs.

Figure 10. LCFS of the free-boundary equilibria from different permanent magnet designs and the fixed-boundary equilibrium. The bean-shaped surfaces locate at \( \zeta = 0 \) and bullet-shaped surfaces at \( \zeta = \pi/3 \).

Figure 11. All the designs have considerably high accuracy in reconstructing the MHD equilibrium compared to the fixed-boundary calculation. This is also revealed in the comparison of the rotational transform (\( \iota \)) profiles in figure 11. The linear
method solution has the largest deviations as a reflection of the worst normal field error. Other solutions have considerably high accuracy. Compared to the fixed-boundary calculations, the discrepancy in the boundary shapes and $\iota$ profiles is almost indistinguishable.

When evaluating the configuration performance, what really matters are the plasma properties. For example, NCSX was optimized to be quasi-axisymmetric to reduce the neo-classical transport. Other than comparing the LCFS and $\iota$ profiles, we can also check the neoclassical transport of each equilibrium. Here, we use the NEO code [29] to calculate the effective ripple, $\epsilon_{\text{eff}}$. The value of $\epsilon_{\text{eff}}^{3/2}$ of each equilibrium is shown in figure 12. They all have equivalent levels of $\epsilon_{\text{eff}}^{3/2}$. The curved bricks case, which has the lowest normal field error, achieves the closest performance to the fixed-boundary equilibrium. But the linear method case, which has the highest normal field error, is not the most deviated one. The optimized perpendicular-only case attains the lowest $\epsilon_{\text{eff}}^{3/2}$ at $s = 0.0$–0.35 and the highest at $s = 0.5$–0.9 ($s$ denotes the normalized toroidal flux). After comparing the $B_{\text{man}}$ spectra, we found that the bumpy components ($m = 0, n \neq 0$), especially the $m = 0, n = 2$ component, in the perpendicular-only and the linear method case were significantly smaller ($\sim 50\%$) than other two, as shown in figure C3. The lower $\epsilon_{\text{eff}}$ near the axis for the perpendicular-only and linear method case is probably due to the reduction of the field bumpiness, as the bumpy field could affect the neoclassical transport near the axis [30]. It is interesting to see that perpendicular magnets can effectively reduce the field bumpiness, even smaller than the fixed-boundary target. Meanwhile, the slight deterioration of $\epsilon_{\text{eff}}$ at $s = 0.5$–0.9 for the perpendicular-only case is possibly caused by the $m = 4, n = 1$ component, as shown in figure C4. The differences in the effective ripple indicate that the normal field error is not completely correlated to the effective ripple. Generally, smaller normal field error means higher fidelity in generating the target magnetic field. However, deviations in the magnetic field sometimes might lead to nearby configurations with an improvement in some physics properties. A new opportunity for designing coils and permanent magnets is to directly optimize physics properties.

5. Summary

In this paper we have introduced a topology optimization method to design permanent magnets for stellarators. The allowable design space for magnets is divided into numerous elements and we use magnetic dipoles to approximate the magnets in each element. The density method with a penalization technique is used to determine if the magnet should be present in each element. The FAMUS code is then developed and it can design engineering-feasible permanent magnets for general stellarators satisfying the constraints of the maximum material magnetization and explicit forbidden regions. We have successfully validated FAMUS with the previously proposed linear method. Three different permanent magnet designs for a half-Tesla quasi-axisymmetric configuration have been demonstrated. FAMUS optimized solutions are found to have the following advantages:

1. For the perpendicular-only case, the FAMUS solution has lower normal field error and uses less magnets than the linear method.
2. FAMUS can optimize the magnitude and/or orientation of all the magnets. By relaxing the orientation of permanent magnets, the so-called ‘Halbach’ arrays are formed and the magnetic efficiency is enhanced.
3. FAMUS can effectively determine the magnet layout and reserve space for ports. The solutions tend to have completely open outboard access.

Free-boundary reconstructions using permanent magnets and planar TF coils are carried out. All the three designs are...
proven to have good accuracy in supporting the target equilibrium.

The three permanent magnet designs for NCSX show attractively large access on the outboard side. This is extremely favorable for future fusion reactors which requires easy access for remote maintenance. Possible reasons for the outboard clearance are as follows. The outboard side usually has lower field strength and tends to require less magnets. Particularly, NCSX has a bean-shaped cross-section. The concave regions, which requires extra shaping magnetic field, are on the inboard side. In addition, the penalization on the total amount of magnetic moment tends to remove the magnets on the outboard side as they are further from the plasma and have larger magnetic moment (because the element size is larger). Although we have only shown the quasi-axisymmetric stellarator, the new tool we have implemented can be applied to any configurations.

In this paper, we are concentrating on the introduction of the new topology optimization method for design permanent magnets. For future work, we can use the FAMUS code to investigate the difference in the permanent magnet layouts for various stellarators and to study the availability of permanent magnets in reactor-size experiments. Although the magnetic field from permanent magnets with existing materials is considerably lower than the magnetic field appeared in proposed fusion reactor designs, permanent magnets are still attractive to future fusion reactors, because only part of the magnetic field will be produced by permanent magnets. In this paper, we are using permanent magnets to produce all the poloidal field and the amount of permanent magnets required by a half-Tesla experiment is sizeable. It is expected that less magnets will be needed if the external coils can produce some rotational transform. We would envision stellarator reactors that use permanent magnets at some critical locations to simplify the coils. We can also adopt more advanced optimization algorithms, like the stochastic gradient descent [31], to better handle the local minimum. More engineering constraints, like the calculation of magnetic forces and the assembly tolerance, could also be implemented in the future. FAMUS requires non-trivial efforts on generating meshes, which is particularly challenging when dealing with the complicated geometries of stellarators. Powerful meshing tools can be used in the future to help obtain better solutions.

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Appendix A. Evidence of local minima

For the benchmark case, we initialized the FAMUS run with a uniform normalized density, ρ = 0.001. However, if the initial guess was ρ = 1.0, we would get a different solution, as shown in figure A1. The normal field error is $F_B = 3.82 \times 10^{-11}$ T²m² after 500 iterations. Although the normal field error is considerably low, the distributions of the normalized density for the two solutions are distinct. The differences in the two solutions initialized from different guesses indicate that they are trapped in different local minima. This is not surprising because there are 16384 variables in the optimization problem and we are not using any global optimization algorithms.

The clear discrepancy in FAMUS solutions from different initial guesses can also be observed in figure A2. Two different perpendicular-only solutions, as introduced in section 4.1, have been obtained by using FAMUS with the same settings. The only difference is that one was initialized with $ρ = 0.001$ and the other one with $ρ = 1.0$.

Appendix B. Least-squares minimization to calculate the magnetic moment

Equation 1 implies that the magnetic field is linear to magnetic moment, if we fix the position of each dipole. We want to minimize the normal field error on the plasma boundary, $F_B$, which is quadratic of $B_n$. It can be converted to a least-squares minimization problem and solved linearly. The derivative of $F_B$ with respect to an arbitrary parameter, for instance $m_i^k$ (ith component of the magnetic moment of the k-th dipole), is

$$\frac{\partial F_B}{\partial m_i^k} = \int_S 2(B_M \cdot n - B_n^g) \left(\frac{\partial B_M}{\partial m_i^k} \cdot n\right) \, da,$$

$$= \int_S 2 \left[ \frac{\mu_0}{4\pi} \sum_{j=1}^D \left( \frac{3m_j \cdot r_j - 1}{|r_j|^3} m_j \right) \cdot n - B_n^g \right] \times \left[ \frac{\mu_0}{4\pi} \left( \frac{3r_i \cdot n}{|r_i|^3} r_k - \frac{1}{|r_k|^3} N \right) \right] \, da,$$

$$= 2\Delta \theta \Delta \zeta \sum_{i_n} \sum_{i_\zeta} \left[ \frac{\mu_0}{4\pi} \frac{1}{N} \sum_{j=1}^D \left( \frac{3r_j \cdot N}{|r_j|^3} r_j - \frac{1}{|r_j|^3} N \right) \cdot m_j - B_n^g \right] \left[ \frac{\mu_0}{4\pi} \left( \frac{3r_i \cdot N}{|r_i|^3} r_k - \frac{1}{|r_k|^3} N \right) \cdot e_i \right],$$

$$= 2\Delta \theta \Delta \zeta \sum_{i_n} \sum_{i_\zeta} \left[ \frac{1}{N} \sum_{j=1}^D g_j \cdot m_j - B_n^g \right] (g_k \cdot e_i),$$

where $g(\theta, \zeta)$ is the ‘inductance’ matrix as calculated by equation 7. To minimize $F_B$, we can solve $\frac{\partial F_B}{\partial m_i^k} = 0$, and we obtain a linear system,

$$\sum_j \sum_{i_n} A_{i,j,k} m_j^i = b_k^i,$$

where the matrix element is

$$A_{i,j,k} = \Delta \theta \Delta \zeta \sum_{i_n} \sum_{i_\zeta} \left( \frac{1}{N} g_j^i g_k^i \right).$$
**Figure A1.** Distributions of two FAMUS runs for the rotating ellipse case initialized with different densities ($\rho = 1.0$ and $\rho = 0.001$). The $\rho = 0.001$ case (right) is the one shown in figure 1.

**Figure A2.** The distributions of the normalized density for two perpendicular-only permanent magnet designs for the half-Tesla NCSX. The two solutions were optimized by FAMUS with the same settings, while the left one (a) was initialized with an uniform guess of $\rho = 0.001$ and the right one (b) was $\rho = 1.0$.

and the right-hand side

$$ b'_k = \Delta \theta \Delta \zeta \sum_{i_u} \sum_{i_c} (B^u_{i_u} g'_c) \ . $$  \hspace{1cm} \text{(B4)}

We can also add a regularization term, $F_M$, which is the overall magnetic moment (equivalent to the total amount of PM material). The combined objective function, $F$, is now expressed as equation 13. Likewise, if we want to minimize $F_M$ only, we have

$$ \frac{\partial F_M}{\partial m'_k} = 2m'_k = 0 \ . $$ \hspace{1cm} \text{(B5)}

After packing all the parameters (magnetic moments) into one column vector with 3D entries, $x = [m'_1, m'_2, m'_3, m'_4, \ldots, m'_D]^T$, we can solve the linear equations,

$$ Ax = b \ , $$ \hspace{1cm} \text{(B6)}
If we separate the orientation and magnitude of each dipole, the magnetic moment can be represented as $m_j = I_j \hat{m}_j$. Similarly, we can write down the linear equations for solving the magnitude,

$$\sum_j A_{jk} I_j = b_k,$$  \hspace{1cm} \text{(B7)}

where

$$A_{jk} = \Delta \theta \Delta \zeta \sum_{i_0} \sum_{i_\zeta} \left( \frac{1}{N} h_{i_\theta i_\zeta} \right),$$ \hspace{1cm} \text{(B8)}

$$b_k = \Delta \theta \Delta \zeta \sum_{i_0} \sum_{i_\zeta} (B_{\text{uni}}^{i_\theta i_\zeta} h_k),$$ \hspace{1cm} \text{(B9)}

Right now, the ‘inductance’ equation is

$$h_k(\theta, \zeta) = \frac{\mu_0}{4\pi} \left( \frac{3r_k \cdot N \cdot \hat{m}_k}{|r_k|^2} - \frac{1}{|r_k|^2} N_m \cdot \hat{m}_k \right).$$ \hspace{1cm} \text{(B10)}

Both the two least-squares minimization methods cannot explicitly address the constraint of the maximum allowable magnetization. When the number of dipoles is large, inverting a large matrix using the normal QR decomposition or singular value decomposition might be considerably slow.

### Appendix C. Comparison of magnetic field spectra

The following is the comparison of the bumpy components ($m = 0, n \neq 0$) and the $m = 4, n = 1$ component in each equilibrium, as discussed in section 4.4. $B_{\text{uni}}$ is calculated in Boozer coordinates using the code BOOZ_XFORM and the components are normalized by the Frobenius norm of the axisymmetric terms ($n = 0$).

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