ON GAUGE-EQUIVALENT FORMULATIONS OF $N=4$ SKdV HIERARCHY

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Abstract

We point out that the $N = 4$ supersymmetric KdV hierarchy, when written through the prepotentials of the bosonic chiral and antichiral $N = 2$ supercurrents, exhibits a freedom related to the possibility to choose different gauges for the prepotentials. In particular, this implies that the Lax operator for the $N = 4$ SKdV system and the associated realization of $N = 4$ supersymmetry obtained in solv-int/9802003 are reduced to the previously known ones. We give the prepotential form of the ‘small’ $N = 4$ superconformal algebra, the second hamiltonian structure algebra of the $N = 4$ SKdV hierarchy, for two choices of gauge.
1. $N = 4$ supersymmetric KdV (SKdV) system with the ‘small’ $N = 4$ superconformal algebra (SCA) as the second hamiltonian structure was firstly constructed in [1] in terms of harmonic $N = 4$ superfields. Later on, it was reformulated in terms of $N = 2$ superfields [2] that allowed to prove its integrability (the existence of $N = 4$ SKdV hierarchy) by constructing the relevant superfield Lax representations. Two different $N = 2$ superfield Lax operators for $N = 4$ SKdV were proposed [3, 4]. The crucial property of one of them is the manifest preservation of $N = 2$ chirality [3], while another one [4] is characterized by the commutativity with one of $N = 2$ covariant spinor derivatives [5].

In a recent preprint [7] it was claimed that there exists one more Lax operator for $N = 4$ SKdV hierarchy, possessing the same commutativity property as that of [4]. One of the aims of the present note is to demonstrate that these Lax operators are in fact identical to one another. The seeming difference between them is due to different, though gauge-equivalent, representations of two of the basic superfields of $N = 4$ SKdV system, namely the spin 1 chiral and antichiral superfields, through their spin 1/2 spinor prepotentials. In ref. [4] the prepotentials were chosen to be chiral and antichiral $N = 2$ superfields, while in [7] there was used another gauge in which the prepotentials coincide and are given by a general $N = 2$ superfield. We explicitly give the gauge transformation relating these two gauges and show that the realization of $N = 4$ supersymmetry declared in [7] to be new is gauge-equivalent to that given in [4]. We also present the realization of classical ‘small’ $N = 4$ SCA on the spin 1/2 prepotentials in both gauges. It proves to be non-local as distinct from the well-known realization in terms of the spin 1 supercurrents.

2. We start by recalling the salient features of the Lax representation for $N = 2$ SKdV hierarchy given in [4].

We use the standard notation for the covariant derivatives of $N = 2$ superspace $Z \equiv \{z, \theta, \theta\}$

\[
D = \frac{\partial}{\partial \theta} - \frac{1}{2} \theta \frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial z},
\]

\[
\{D, \bar{D}\} = -\frac{\partial}{\partial z}, \quad D^2 = \bar{D}^2 = 0 .
\] (1)

The $N = 4$ SKdV Lax operator of ref. [4] and the $N = 4$ SKdV hierarchy equations, in the form adapted for our purpose, read

\[
L = \partial - J - \bar{D}\partial^{-1}(DJ) - F\bar{D}\partial^{-1}(DF) - \bar{D}\partial^{-1}(FD\bar{F}) ,
\] (2)

\[
\frac{\partial}{\partial t_k} L = - [ (L^k)_{\geq 1}, L] ,
\] (3)

where the subscript $\geq 1$ marks the strictly differential part of pseudo-differential operator. In eq. (2), the parentheses mean that the differential operator inside them acts only on the expression inside; otherwise, the operator is assumed to act freely to the right [4]. The general spin 1 $N = 2$ superfield $J(Z)$ and the chiral spin 1/2 superfields $F(Z), \bar{F}(Z)$,

\[
DF = \bar{D} \bar{F} = 0 ,
\] (4)

\[\text{1. The relation between these two operators is discussed in [4].}\]

\[\text{2. This convention applies only to the Lax operators and to the operators in the r.h.s. of the Poisson brackets defining the second hamiltonian structure algebra; in all other cases (e.g., in the evolution equations) any operator (including } \partial^{-1} ) \text{ is assumed to act only on the function standing to its right.}\]
are the basic objects of $N=4$ SKdV hierarchy in this $N=2$ superfield formulation. The spin 1 chiral supercurrents $\Phi(Z), \overline{\Phi}(Z)$ which were used in \cite{1,2} and which, together with $J(Z)$, constitute the standard basis of the ‘small’ $N=4$ SCA are related to $F, \overline{F}$ by

$$\Phi = \overline{D} F, \quad \overline{\Phi} = D \overline{F}$$

(5)

$$\overline{D} \Phi = 0, \quad D \overline{\Phi} = 0.$$  

(6)

It is straightforward to check that $[D, L] = 0$.

As an example, we explicitly give the second flow equations

$$\frac{\partial J}{\partial t} = \left[ D, \overline{D} \right] J' + 2JJ' - 2(DF \overline{DF})' = \left[ D, \overline{D} \right] J' + 2JJ' - 2(\Phi \overline{\Phi})',$$  

(7)

$$\frac{\partial F}{\partial t} = -F'' - 2D(J \overline{DF}), \quad \frac{\partial \overline{F}}{\partial t} = \overline{F''} - 2\overline{D}(J \overline{DF}).$$

(8)

Acting on the last two equations by $\overline{D}, D$ and making use of the relation (5), we can equivalently rewrite this system in terms of $\Phi, \overline{\Phi}$. The $\Phi, \overline{\Phi}$ form of eqs. (8) is as follows

$$\frac{\partial \Phi}{\partial t} = -\Phi'' - 2D(J \Phi), \quad \frac{\partial \overline{\Phi}}{\partial t} = \overline{\Phi''} - 2\overline{D}(J \overline{\Phi}).$$

(9)

Similar equivalent forms can be given for other flows. This equivalence between the $F$ and $\Phi$ representations is based on the invertibility of the relation (5): due to the chirality of $F, \overline{F}$ one can express the latter, up to an unessential constant Grassmann zero-mode, in terms of $\Phi, \overline{\Phi}$

$$F = -\partial^{-1} D \Phi, \quad \overline{F} = -\partial^{-1} \overline{D} \overline{\Phi}.$$  

(10)

The $N=4$ SKdV hierarchy equations reveal covariance under an extra $N=2$ supersymmetry

$$\delta J = \epsilon D \Phi + \overline{\epsilon} \overline{D} \overline{\Phi}, \quad \delta \Phi = -\overline{\epsilon} \overline{D} J, \quad \delta \overline{\Phi} = -\epsilon D J.$$  

(11)

Here, $\epsilon, \overline{\epsilon}$ are complex Grassmann parameters (they are not mutually conjugated in general). Together with the explicit $N=2$ supersymmetry these transformations constitute $N=4$ supersymmetry in one dimension. An equivalent realization in terms of the superfields $F, \overline{F}$ is non-local

$$\delta J = -\epsilon F' - \overline{\epsilon} \overline{F'}, \quad \delta F = -\overline{\epsilon} D \partial^{-1} \overline{D} J, \quad \delta \overline{F} = -\epsilon \overline{D} D \partial^{-1} J.$$  

(12)

It is easy to directly check the covariance of the second flow equations (7) - (9) under the transformations (11) or (12).

Note that here and in what follows we do not need to impose any reality conditions on the involved superfields.

3. It has been noticed in \cite{4} that the superfields $F, \overline{F}$ can be considered as prepotentials of $\Phi, \overline{\Phi}$ in some special gauge. Let us elaborate on this interpretation in more detail and show that the description of $N=4$ SKdV hierarchy in terms of $J(Z)$ and a general single spinor superfield $g(Z)$ proposed in \cite{7} merely amounts to another choice of gauge for the prepotential.

A general solution of the chirality conditions (3) is as follows

$$\Phi = \overline{D} v, \quad \overline{\Phi} = D \overline{v},$$

(13)
with $v, \overline{v}$ being general complex spin $1/2$ fermionic prepotentials (they, like $\Phi$ and $\overline{\Phi}$, are not obliged to be mutually conjugated). The prepotentials are defined up to gauge transformation

$$v \Rightarrow v + \overline{D}\lambda, \quad \overline{v} \Rightarrow \overline{v} - D\overline{\lambda}, \quad (14)$$

where $\lambda, \overline{\lambda}$ are arbitrary complex bosonic superfield parameters. In turn, the latter are defined up to the freedom

$$\lambda \Rightarrow \lambda + \overline{D}\omega, \quad \overline{\lambda} \Rightarrow \overline{\lambda} + D\overline{\omega} \quad (15)$$

$\omega, \overline{\omega}$ being complex fermionic superfield functions.

One can make use of the gauge freedom (14) to choose different gauges for the prepotentials $v, \overline{v}$. Let us first demonstrate that the choice

$$v_1 = F, \quad \overline{v}_1 = \overline{F}, \quad DF = \overline{D}\overline{F} = 0 \quad (16)$$

is just one of such possible gauge-fixings. We start from a general $v$ and should show the existence of a gauge function $\lambda_1$ such that

$$F = v + \overline{D}\lambda_1. \quad (17)$$

Using the algebra of covariant derivatives (1), it is easy to check that this condition fixes $\overline{D}\lambda_1$ up to an unessential complex Grassmann constant (it can be absorbed into $F$)

$$\overline{D}\lambda_1 = \partial^{-1}\overline{D}Dv \quad \Rightarrow \quad \lambda_1 = \partial^{-1}Dv + \overline{D}\omega, \quad (18)$$

whence

$$F = v + \partial^{-1}\overline{D}Dv. \quad (19)$$

For $\overline{F}$ one obtains an analogous expression in terms of $\overline{v}$.

Another possible gauge choice is to identify $v$ and $\overline{v}$

$$v_2 = \overline{v}_2 = g. \quad (20)$$

Let us show the existence of gauge transformation relating the gauges (14) and (20)

$$F = g + \overline{D}\lambda_2, \quad \overline{F} = g - D\overline{\lambda}_2. \quad (21)$$

Once again, the use of the algebra (1) allows one to determine $\overline{D}\lambda_2, D\overline{\lambda}_2$ up to unessential Grassmann constants in terms of either $g$ or $F, \overline{F}$. As the result we get the following simple invertible relations between $F, \overline{F}$ and $g$

$$F = g + \partial^{-1}\overline{D}Dg, \quad \overline{F} = g + \partial^{-1}D\overline{D}g \quad (22)$$

$$g = F - \partial^{-1}\overline{D}D\overline{F} = F + \overline{F}. \quad (23)$$

It should be stressed once more that the chiral supercurrents $\Phi, \overline{\Phi}$ are gauge-invariant quantities, i.e. they do not depend on the choice of gauge for $v, \overline{v}$. In other words, from the point of view of $N = 4$ SKdV hierarchy it does not matter which kind of the prepotential representation has been chosen for these objects

$$\Phi = \overline{D}F = \overline{D}g, \quad \overline{\Phi} = D\overline{F} = Dg. \quad (24)$$
The \( F, \overline{F} \) representation was used in \([4]\). It is just the \( g \) representation that was used in \([7]\). Obviously, even more choices are possible.

A common feature of the above gauges is that they fix the prepotentials up to a constant, i.e. allow no non-trivial residual gauge freedom. Respectively, the field component sets of the prepotentials and the supercurrents \( \Phi, \overline{\Phi} \) precisely match each other in these gauges. Just due to this property one can establish invertible relations between the gauge-fixed prepotentials and \( \Phi, \overline{\Phi} \). Using the explicit relation \((23)\), it is easy, e.g., to find the analog of the relation \((10)\) for the gauge \((20)\)

\[
g = -\partial^{-1} \left( D \Phi + \overline{D} \overline{\Phi} \right) . \tag{25}
\]

With all these explicit relations at hand, it is straightforward to be convinced that all the equations of \( N = 4 \) SKdV hierarchy in the \( g \)-description \([7]\) directly stem from those in the \( F, \overline{F} \) description \([4]\). E.g., the second flow equations \((8)\) entail, through the relation \((23)\), the following evolution equation for \( g \)

\[
\frac{\partial g}{\partial t_2} = [D, \overline{D}] g' - 2D (J \overline{D} g) - 2\overline{D} (J D g) \tag{26}
\]

(it again implies \((3)\) through the correspondence \((24)\)). The same is true for the \( N = 4 \) supersymmetry transformations: eqs. \((12)\) have the following gauge-equivalent form in terms of the superfields \( J, g \) (one should use the relations \((22)\) and \((23)\))

\[
\delta J = \epsilon D \overline{D} g + \epsilon \overline{D} D g , \quad \delta g = -\epsilon \partial^{-1} \overline{D} D J - \epsilon \overline{D} \partial^{-1} D \overline{D} J . \tag{27}
\]

It is interesting to see how the chiral subvariety of the \( N = 4 \) SKdV equations look in terms of \( v, \overline{v} \) before any gauge fixing. For the second flow one gets from \((9), (13)\) (we explicitly write only the equation for \( v \))

\[
\frac{\partial v}{\partial t_2} = -v'' - 2D(J \Phi) + \overline{D} X . \tag{28}
\]

Here the superfield \( X(Z) \) describes an arbitrariness related to the gauge freedom \((14)\). To different gauges for \( v, \overline{v} \) there correspond special choices of \( X \) (and \( X \)). A similar arbitrariness persists in the higher-dimension flows written through \( v, \overline{v} \). Let us now apply to the Lax operator \((2)\). Keeping in mind the invertible relations between \( \Phi, \overline{\Phi} \) on the one hand and \( \overline{F}, F \) or \( g \) on the other, it is natural to expect that it admits a unique gauge-invariant representation in terms of the supercurrents \( \Phi, \overline{\Phi} \). Then its \( F, \overline{F} \) form \((2)\) or a gauge-equivalent \( g \) form should follow from this ‘master’ representation upon substituting different expressions for \( \Phi, \overline{\Phi} \) through the prepotentials \( v, \overline{v} \). This is indeed so. The \( \Phi, \overline{\Phi} \) representation of \((2)\) is as follows

\[
L = \partial - J - \overline{D} \partial^{-1} (DJ) - \partial^{-1} \Phi \overline{\Phi} + \partial^{-1} (D \Phi) \partial^{-1} \overline{D} \overline{\Phi} . \tag{29}
\]

Substituting for \( \Phi, \overline{\Phi} \) their expressions through \( \overline{F}, F \) (eqs. \((3)\)), using the \( D, \overline{D} \) algebra \((1)\) and the identities like

\[
(F') = \partial F - F \partial ,
\]

it is straightforward to show that \((29)\) is identical to \((2)\). Of course, one can arrive at \((29)\) by substituting \((11)\) in \((2)\). The \( g \) form of the Lax operator can be obtained by substituting the relevant expressions for \( \Phi, \overline{\Phi} \) from eqs. \((24)\).
Comparing the operator (29) with the Lax operator proposed in [7], one observes them to be identical to each other up to the following redefinitions
\[ \Phi = \overline{G}, \quad \overline{\Phi} = G, \] (30)

\[ G, \overline{G} \Rightarrow iG, i\overline{G} \quad (F, \overline{F} \Rightarrow iF, i\overline{F}), \quad g \Rightarrow ig \]. (31)

Let us notice that the Lax operator (29) can be brought into the following suggestive form
\[ L = \partial - J - \Phi \partial^{-1} \overline{\Phi} - D \partial^{-1} \left[ D, J + \Phi \partial^{-1} \overline{\Phi} \right] , \] (32)
from which it is easy, e.g., to reveal that on the subspace of the chiral wave functions \( \Psi(Z) \), \( D\Psi(Z) = 0 \), this operator is reduced to that proposed in [3]
\[ L \Rightarrow -D\overline{\Phi} - D\overline{\Phi} \partial^{-1} \left\{ J + \Phi \overline{\partial}^{-1} \overline{\Phi} \right\} D\overline{\Phi} \partial^{-1} . \] (33)

The existence of such a correspondence for more general case was mentioned in [6].

All the basic features of the prepotential formulation of \( N = 4 \) SkdV hierarchy remain valid as well for another known \( N = 2 \) hierarchy with the ‘small’ \( N = 4 \) SCA as the second hamiltonian structure, the so called ‘quasi’ \( N = 4 \) SkdV system [8] (its characteristic feature is lacking of \( N = 4 \) supersymmetry). In particular, it admits gauge-equivalent formulations in terms of the superfields \( J, F, \overline{F} \) or \( J, g \).

4. As the last topic, we will give how the ‘small’ \( N = 4 \) SCA, the second hamiltonian structure algebra of \( N = 4 \) SkdV hierarchy, is realized on the \( N = 2 \) SCA supercurrent \( J(Z) \) and the prepotentials \( v(Z), \overline{v}(Z) \) in the gauges (19), (24).

The classical non-vanishing Poisson brackets (PB) defining this algebra on the supercurrents \( J, \Phi, \overline{\Phi} \) can be written as follows (see, e.g. [4])
\[ \{ J(1), J(2) \} = \left( \partial \left[ D, \overline{D} \right] + \partial J + J \partial + D J \overline{D} + \overline{D} J D \right) \delta(1,2) , \] (34)
\[ \{ J(1), \Phi(2) \} = - \left( \overline{D} D \Phi + D \Phi D \right) \delta(1,2) , \] (35)
\[ \{ J(1), \overline{\Phi}(2) \} = - \left( D \overline{D} \overline{\Phi} + D \overline{\Phi} D \right) \delta(1,2) , \] (36)
\[ \{ \Phi(1), \overline{\Phi}(2) \} = \left( \partial D \overline{D} + D J \overline{D} \right) \delta(1,2) , \] (37)

where \( \delta(1,2) = (\theta_1 - \theta_2)(\overline{\theta}_1 - \overline{\theta}_2)\delta(z_1 - z_2) \) and the differential operators in the r.h.s. are evaluated at the second point of \( N = 2 \) superspace. Using the inverse relations (10), it is straightforward to rewrite PBs (34) - (37) in terms of \( F, \overline{F} \)
\[ \{ J(1), F(2) \} = \left( F D \overline{D} - D \overline{D} D F + \partial^{-1} D \overline{D} D F D \overline{D} \right) \delta(1,2) , \] (38)
\[ \{ J(1), \overline{F}(2) \} = \left( \overline{F} D D - D D \overline{F} + \partial^{-1} D D \overline{F} D D \right) \delta(1,2) , \] (39)
\[ \{ F(1), \overline{F}(2) \} = \left( D D - \partial^{-1} D D J \overline{D} \partial^{-1} \right) \delta(1,2) . \] (40)

Quite analogously, using the relations (23) or (25), one easily restores from eqs. (34) - (37) or (38) - (40) the \( J, g \) form of the ‘small’ \( N = 4 \) SCA. It is given by the PB (34) and the following
\[ ^3 \text{The precise correspondence with the PBs of ref. [4] is achieved via substitutions } \Phi \leftrightarrow \overline{\Phi} , \{} \rightarrow -\{} . \]
PBs involving $g(Z)$

$$\{J(1), g(2)\} = \left( \partial g + \frac{1}{2} g \partial - D g \overline{D} - \overline{D} g D - \frac{1}{2} \partial^{-1} \left[ D, \overline{D} \right] g \left[ D, \overline{D} \right] \right) \delta(1, 2) , \quad (41)$$

$$\{g(1), g(2)\} = - \left( \left[ D, \overline{D} \right] + \frac{1}{2} J + \frac{1}{2} \partial^{-1} \left[ D, \overline{D} \right] J \partial^{-1} \left[ D, \overline{D} \right] \right) \delta(1, 2) . \quad (42)$$

An interesting peculiarity of the ‘small’ $N = 4$ SCA written in this way is the unavoidable presence of non-local terms in the r.h.s. of the defining PBs. In particular, the relations (38), (39), (41) imply that the fermionic superfields $F, \overline{F}$ and $g$ are not primary in the standard sense with respect to $N = 2$ SCA (the notion of conformal spin is still meaningful). Note that the set of PBs (34), (41), (42) differs from the PBs of $N = 3$ SCA in the $N = 2$ superfield notation [7] just by such non-local terms.

As a final remark, we note that it would be desirable to find an analog of the above PBs for the arbitrary prepotentials $v, \overline{v}$ before imposing any gauge condition on them. Such an algebra (if existing) is expected to be more general than the ‘small’ $N = 4$ SCA. The latter would follow from this general algebra as some its reduction upon imposing the appropriate gauge conditions and applying the Dirac procedure. Knowing the general algebra could also fix the above-mentioned uncertainty in the evolution equations for $v, \overline{v}$.

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