The Darboux transformation of the derivative nonlinear Schrödinger equation

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Abstract

The $n$-fold Darboux transformation (DT) is a $2 \times 2$ matrix for the Kaup–Newell (KN) system. In this paper, each element of this matrix is expressed by a ratio of the $(n+1) \times (n+1)$ determinant and $n \times n$ determinant of eigenfunctions. Using these formulae, the expressions of the $q^{[n]}$ and $r^{[n]}$ in the KN system are generated by the $n$-fold DT. Further, under the reduction condition, the rogue wave, rational traveling solution, dark soliton, bright soliton, breather solution and periodic solution of the derivative nonlinear Schrödinger equation are given explicitly by different seed solutions. In particular, the rogue wave and rational traveling solution are two kinds of new solutions. The complete classification of these solutions generated by one-fold DT is given.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The derivative nonlinear Schrödinger equation,

$$i q_t - q_{xx} + i(q^2 q^*)_x = 0,$$

one of the most important integrable systems in the mathematics and physics, is usually called DNLS (or DNLSI) equation. Here ‘$*$’ denotes the complex conjugation, and the subscript $x$ (or $t$) denotes the partial derivative with respect to $x$ (or $t$). This equation is originated from two fields of applied physics. The first is plasma physics in which the DNLS governs the evolution of small but finite-amplitude Alfvén waves that propagate quasi-parallel to the magnetic field [1, 2]. Recently, this equation is also used to describe large-amplitude magnetohydrodynamic (MHD) waves in plasmas [3, 4]. Further, it is natural to improve the DNLS equation in more
practical plasmas. For example, DNLS truncation model [5] and DNLS with nonlinear Landau damping [6]. In the second area, nonlinear optics, the sub-picosecond or femtosecond pulses in single-mode optical fibers are modeled by the DNLS [7–9].

However, the crucial feature of the DNLS is that the integrability such as the dynamical evolution of the associated physical system can be given analytically by using its exact solution. Under the vanishing boundary condition (VBC), Kaup and Newell (KN) [10] firstly proposed an inverse scattering transform (IST) with a revision in their pioneer works, and obtained a one-soliton solution. Later, Kawata [11] further solved DNLS under VBC and non-vanishing boundary condition (NVBC) to obtain a two-soliton solution, and introduced ‘paired soliton’ which is now regarded as one kind of breather solution. The N-soliton formula [12] of the DNLS with VBC is expressed by determinants with the help of pole-expansion. Further, the IST of the DNLS with VBC is re-considered by Huang’s group [13–16] and then the explicit form of the N-soliton is obtained by some algebraic techniques. Now we turn to the DNLS under NVBC, and some special solutions are obtained and the existence of the algebraic soliton is also given [17]. This is followed by the paired soliton of the DNLS from the IST [18]. Wadati et al [19] have given the stationary solutions of the DNLS under the plane wave boundary and the contributions of the derivative term in the DNLS equation. Recently, to avoid the multi-value problem, Chen and Lam [20] revised the IST for the DNLS under NVBC by introducing an affine parameter, and then obtain a single breather solution, which can be reduced to the dark soliton and bright soliton. Further applications on this method can be found in [21]. Cai and Huang [22] found the action-angle variables of the DNLS explicitly by constructing its Hamiltonian formalism.

Similar to many usual soliton equations, the DNLS is also solved by the Hirota method [23] and Darboux transformation (DT) [24, 25] besides IST. By comparing with the corresponding results [26–28] of the nonlinear Schrödinger (NLS) equation, the DT [24, 25] of the DNLS has the following essential distinctness:

- the kernel of one-fold DT is one dimensional, and it can be defined by one eigenfunction of linear system defined by spectral problem;
- the DNLS will be invariant under one-fold DT associated with a pure imaginary eigenvalue (see the last paragraph of the section 2).

Some solutions [24] including multi-soliton and quasi-periodic solutions are obtained by this DT from a trivial seed: zero solution (or vacuum). Steudel [25] has obtained a general formula of the solutions $q^{(N)}$ and $r^{(N)}$ of the KN system in terms of Vandermonde-like determinants by $n$-fold DTs, and then given $n$-soliton and $N$-phase solutions from zero seed, $n$-breather solutions from non-zero seed: monochromatic wave. Unlike the usual DT, Steudel used solutions of Riccati equations, which are transformed from the linear partial differential equations of the spectral problem for the DNLS, to construct the solutions of the DNLS. So the first difficulty in his method is to solve nonlinear Riccati, which is not solvable in general. To overcome this difficulty, Steudel has made an Ansatz (see equation (51) in [25]) and introduced his favorite Seahorse functions. Moreover, the classification of the solutions (see figure 1 in [25]) generated by the DT is very interesting and useful. But the conditions of parameters to generate the dark soliton and bright soliton of the DNLS are not clear. Therefore, it is natural to question whether the difficult Riccati equations are indeed unavoidable for the DT from non-zero seeds and whether the classification of solutions generated by one-fold DT can be fixed thoroughly or not.
It is interesting that the Ablowitz–Kaup–Newell–Segur (AKNS) system \cite{29} can be mapped to the KN system by a gauge transformation \cite{30}. Moreover, there exist other two kinds of derivative NLS equations, i.e. the DNLSII \cite{31}

\begin{equation}
 iq_t + q_{xx} + iqq^*q_x = 0,
 \end{equation}

and the DNLSIII \cite{32}

\begin{equation}
 iq_t + q_{xx} - iq^2q_x^* + \frac{1}{2}q^3q^{*2} = 0,
 \end{equation}

and a chain of gauge transformations between them: DNLSII $\overset{(a)}{\Rightarrow}$ DNLSI $\overset{(b)}{\Rightarrow}$ DNLSIII. Here (a) denotes equation (2.12) in \cite{30}, and (b) denotes: equation (4) $\rightarrow$ equation (3) $\rightarrow$ equation (6) with $\gamma = 0$ in \cite{23}. But these transformations cannot preserve the reduction conditions in a spectral problem of the KN system and involve complicated integrations. So each of them deserves investigating separately.

There are two aims of this paper. The first is to present a detailed derivation of the DT for the DNLS and its determinant representation. Using this representation, the solutions of the DNLS can be expressed by the solutions (eigenfunctions) of the linear partial differential equations of the spectral problem of the KN system instead of the solutions of the nonlinear Riccati equations, which shows that the nonlinear Riccati equation and Seahorse functions are indeed avoidable for the DT from non-zero seeds. The second is to present a complete classification of the solutions generated by one-fold DT from zero seed and non-zero seeds: constant solution and periodic solution with a constant amplitude.

The organization of this paper is as follows. In section 2, a relatively simple approach to DT for the KN system is provided, and then the determinant representation of the $n$-fold DT and formulae of $q^{[n]}$ and $r^{[n]}$ expressed by the eigenfunctions of the spectral problem are given. The reduction of the DT of the KN system to the DNLS equation is also discussed by choosing paired eigenvalues and eigenfunctions. In section 3, under specific reduction conditions, several types of particular solutions are given from zero seed and non-zero seeds: constant solution and periodic solution with a constant amplitude. The complete classification of the dark soliton, bright soliton and periodic solution are given in a table for one-fold DT of the DNLS equation. In particular, two kinds of new solutions: rational traveling solution and rogue wave are given. The conclusion is given in section 4.

2. Darboux transformation

Let us start from the first non-trivial flow of the KN system \cite{10}:

\begin{align}
 r_t - ir_{xx} - (r^2q)_x &= 0, \\
 q_t + iq_{xx} - (rq^2)_x &= 0;
\end{align}

these are exactly reduced to the DNLS equation \cite{1} for $r = -q^*$ while the choice $r = q^*$ would lead to equation \cite{1} with the sign of the nonlinear term changed. The Lax pairs corresponding to coupled DNLS equations \cite{4} and \cite{5} can be given by the KN spectral problem \cite{10}

\begin{align}
 \partial_t \psi &= (J\lambda^2 + Q\lambda)\psi = U\psi, \\
 \partial_t \psi &= (2J\lambda^4 + V_3\lambda^3 + V_2\lambda^2 + V_1\lambda)\psi = V\psi,
\end{align}

with

\begin{align*}
 \psi &= \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, & J &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & Q &= \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix},
\end{align*}
Here $\lambda$, an arbitrary complex number, is called the eigenvalue (or spectral parameter) and $\psi$ is called the eigenfunction associated with $\lambda$ of the KN system. Equations (4) and (5) are equivalent to the integrability condition $U_t - V_x + [U, V] = 0$ of (6) and (7).

The main task in this section is to present a detailed derivation of the DT of the DNLS and the determinant representation of the $n$-fold transformation. Based on the DT for the NLS [26–28] and the DNLS [24, 25], the main steps are as follows: (1) to find a $2 \times 2$ matrix $T$ so that the KN spectral problem, equations (6) and (7), is covariant, and then to obtain a new solution $(q^{[1]}, r^{[1]})$ expressed by the elements of $T$ and the seed solution $(q, r)$; (2) to find the expressions of the elements of $T$ in terms of the eigenfunctions of the KN spectral problem corresponding to the seed solution $(q, r)$; (3) to obtain the determinant representation of the $n$-fold DT $T_n$ and new solutions $(q^{[n]}, r^{[n]})$ by $n$-time iteration of the DT; (4) to consider the reduction condition $q^{[n]} = -(r^{[n]})^*$ by choosing a special eigenvalue $\lambda_k$ and its eigenfunction $\psi_k$, and then to obtain $q^{[n]}$ of the DNLS equation expressed by its seed solution $q$ and its associated eigenfunctions $\{\psi_k, k = 1, 2, \ldots, n\}$. However, we shall use the kernel of the $n$-fold DT ($T_n$) to fix it in the third step instead of iteration.

It is easy to see that the spectral problems (6) and (7) are transformed to

$$
\psi^{[1]}_x = U^{[1]} \psi^{[1]}, \quad U^{[1]} = (T_x + T U)T^{-1}.
$$

(8)

$$
\psi^{[1]}_t = V^{[1]} \psi^{[1]}, \quad V^{[1]} = (T_t + T V)T^{-1}.
$$

(9)

under a gauge transformation

$$
\psi^{[1]} = T \psi.
$$

(10)

By cross-differentiating (8) and (9), we obtain

$$
U^{[1]}_t - V^{[1]}_x + [U^{[1]}, V^{[1]}] = T(U_t - V_x + [U, V])T^{-1}.
$$

(11)

This implies that, in order to make equations (4) and (5) invariant under the transformation (10), it is crucial to search a matrix $T$ so that $U^{[1]}, V^{[1]}$ have the same forms as $U, V$. At the same time, the old potentials (or seed solution) $(q, r)$ in spectral matrixes $U, V$ are mapped into new potentials (or new solution) $(q^{[1]}, r^{[1]})$ in transformed spectral matrixes $U^{[1]}, V^{[1]}$.

2.1. One-fold DT of the KN system

Considering the universality of the DT, suppose that the trial Darboux matrix $T$ in equation (10) is of the form

$$
T = T(\lambda) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \lambda + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix},
$$

(12)

where $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1$ are the functions of $x, t$ need to be determined. From

$$
T_x + T U = U^{[1]} T,
$$

(13)

comparing the coefficients of $\lambda^j$, $j = 3, 2, 1, 0$, it yields

$$
\begin{align*}
\lambda^3 : b_1 &= 0, \quad c_1 = 0, \\
\lambda^2 : q a_1 - 2i b_0 - q^{[1]} d_1 &= 0, \quad -r^{[1]} a_1 + r d_1 + 2ic_0 = 0, \\
\lambda^1 : a_1 x + r b_0 - q^{[1]} c_0 &= 0, \quad d_1 x + q c_0 - r^{[1]} b_0 = 0, \quad q a_0 - q^{[1]} d_0 = 0, \\
\lambda^0 : r^{[1]} a_0 + r d_0 &= 0, \\
\end{align*}
$$

(14)

\[ \text{J. Phys. A: Math. Theor. 44 (2011) 305203 S Xu et al} \]
The last equation shows that \( a_0, b_0, c_0, d_0 \) are the functions of \( t \) only. Similarly, from

\[
T_t + TV = V^{[1]}_t T, 
\]

(15)

comparing the coefficients of \( \lambda^j, j = 4, 3, 2, 1, 0 \), it implies

\[
\begin{align*}
\lambda^2 : & \quad -2ib_0 - q^{[1]}d_1 + qa_1 = 0, \quad 2ic_0 - 2r^{[1]}a_1 + 2rd_1 = 0, \\
\lambda^3 : & \quad -r^{[1]}q^{[1]}a_1 - 2q^{[1]}c_0 + a_1rq + 2rb_0 = 0, \quad qa_0 - q^{[1]}d_0 = 0, \\
r.d. : & \quad -d_1rq + q^{[1]}r^{[1]}d_1 + 2qc_0 - 2r^{[1]}b_0 = 0, \\
\lambda^4 : & \quad a_0rq - a_0r^{[1]}q^{[1]} = 0, \quad a_1rq^2 - r^{[1]}q^{[1]}d_1 - b_0rq + q^{[1]}d_1i - a_1q + i - r^{[1]}q^{[1]}b_0 = 0, \\
c_0r + r^{[1]}q^{[1]}a_1 + d_1r^2q + r^{[1]}q^{[1]}c_0 = d_1r + i - r^{[1]}q^{[1]}d_1 = 0, \quad r^{[1]}q^{[1]}d_0 - r_0d_0 = 0, \\
\lambda^5 : & \quad a_1 + q^{[1]}c_0i + b_0rq^2 - r^{[1]}q^{[1]}c_0 = b_0r^2 = 0, \\
-r^{[1]}q^{[1]}d_0 + a_0rq^2 + q^{[1]}x_0 = r_0 = 0, \\
d_0r, i + d_0r^2q - r^{[1]}q^{[1]}a_0 - r^{[1]}q^{[1]}a_0 = 0, \quad d_1 - c_0q + i + c_0rq^2 - r^{[1]}q^{[1]}b_0 - r^{[1]}q^{[1]}b_0 = 0, \\
\lambda^0 : & \quad a_0 = b_0 = c_0 = d_0 = 0. 
\end{align*}
\]

(16)

The last equation shows that \( a_0, b_0, c_0, d_0 \) are the functions of \( x \) only. So \( a_0, b_0, c_0, d_0 \) are constants.

In order to obtain the non-trivial solutions, we present a DT under the condition \( a_0 = 0, d_0 = 0 \). Based on equations (14) and (16) and without losing any generality, let the Darboux matrix \( T \) be of the form

\[
T = T_1(\lambda; \lambda_1) = \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix} \lambda + \begin{pmatrix} 0 & b_0 \\ c_0 & 0 \end{pmatrix}.
\]

(17)

Here \( a_1 \) and \( d_1 \) are the undetermined functions of \((x, t)\), which will be expressed by the eigenfunction associated with \( \lambda_1 \) in the KN spectral problem. First of all, we introduce \( n \) eigenfunctions \( \psi_j \) as

\[
\psi_j = \begin{pmatrix} \phi_j \\ \varphi_j \end{pmatrix}, \quad j = 1, 2, \ldots, n, \quad \phi_j = \phi_j(x, t, \lambda_j), \quad \varphi_j = \varphi_j(x, t, \lambda_j).
\]

(18)

**Theorem 1.** The elements of one-fold DT are parameterized by the eigenfunction \( \psi_1 \) associated with \( \lambda_1 \) as

\[
d_1 = \frac{1}{a_1}, \quad a_1 = -\frac{\phi_1}{\psi_1}, \quad b_0 = c_0 = \lambda_1,
\]

(19)

\[
\Leftrightarrow T_1(\lambda; \lambda_1) = \begin{pmatrix} -\frac{\lambda_1 \phi_1}{\psi_1} & \lambda_1 \\ \lambda_1 & -\frac{\lambda_1 \varphi_1}{\psi_1} \end{pmatrix}.
\]

(20)

and then the new solutions \( q^{[1]} \) and \( r^{[1]} \) are given by

\[
q^{[1]} = \left( \begin{array}{c} \phi_1 \\ \varphi_1 \end{array} \right)^2 q + 2i \psi_1 \lambda_1, \quad r^{[1]} = \left( \begin{array}{c} \phi_1 \\ \varphi_1 \end{array} \right)^2 r - 2i \frac{\phi_1}{\psi_1} \lambda_1,
\]

(21)

and the new eigenfunction \( \psi^{[1]}_j \) corresponding to \( \lambda_j \) is

\[
\psi^{[1]}_j = \begin{pmatrix} \frac{1}{\psi_1} - \lambda_j \phi_j & \varphi_j \\ -\phi_j & \frac{1}{\varphi_j} \end{pmatrix}.
\]

(22)
Proof. Note that \((a_1 d_1)x = 0\) is derived from equation (14), and then take \(a_1 = \frac{1}{a_1}\) in the following. By transformation equations (17) and (14), new solutions are given by

\[
q^{[1]} = \frac{a_1}{d_1}q - 2\frac{b_0}{d_1}, \quad r^{[1]} = \frac{d_1}{a_1}q + 2\frac{c_0}{a_1}.
\]  
(23)

By using a general fact of the DT, i.e. \(T^*_1(\lambda; \lambda_1)|_{\lambda = \lambda_1} \psi_1 = 0\), then equation (19) is obtained. Next, substituting \((a_1, d_1, a_0, b_0)\) given in equation (19) back into equation (23), new solutions are given as in equation (21). Further, by using the explicit matrix representation equation (20) of \(T_1\), \(\psi_j^{[1]}\) is given by

\[
\psi_j^{[1]} = T_j(\lambda; \lambda_1)|_{\lambda = \lambda_1} \psi_j = \begin{pmatrix}
-\frac{\lambda_j}{\phi_1} & \lambda_1 \\
\lambda_1 & -\frac{\lambda_j}{\phi_1}
\end{pmatrix}_{\lambda_1 = \lambda_1} \begin{pmatrix}
\phi_j \\
\psi_j
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\phi_1} & -\lambda_j & \phi_j & \psi_j \\
-\lambda_1 & \phi_1 & \psi_1
\end{pmatrix}
\].
(24)

Last, a tedious calculation shows that \(T^*_1\) in equation (20) and new solutions indeed satisfy equation (15) (or equivalently equation (16)). So the KN spectral problem is covariant under the transformation \(T^*_1\) in equations (20) and (21), and thus it is the DT of equations (4) and (5).

It is easy to find that \(T_j\) is equivalent to the Imai’s result (see equation (7) of [24]) and to the Steudel’s result (see equation (21) of [25]). Our derivation is more transparent, and new solutions \(q^{[1]}\) and \(r^{[1]}\) can be constructed by the eigenfunction \(\psi_j\), which is a solution of linear partial differential equations (4) and (5). This is simpler than Steudel’s method to solve nonlinear Riccati equations. The remaining problem is how to guarantee the validity of the reduction condition, i.e. \(q^{[1]} = - (r^{[1]})^*\). We shall solve it at the end of this section by choosing special eigenfunctions and eigenvalues.

2.2. \(n\)-fold DT for the KN system

The key task in this subsection is to establish the determinant representation of the \(n\)-fold DT for the KN system. To this purpose, set

\[
D = \begin{pmatrix}
a & 0 \\
0 & d
\end{pmatrix}, \quad A = \begin{pmatrix}
0 & b \\
c & 0
\end{pmatrix},
\]

as in [24].

According to the form of \(T_1\) in equation (17), the \(n\)-fold DT should be of the form [24]

\[
T_n = T_n(\lambda; \lambda_1, \lambda_2, \ldots, \lambda_n) = \prod_{l=0}^{n-1} P_l \lambda^l,
\]  
(25)

with

\[
P_n = \begin{pmatrix}
a_n & 0 \\
0 & d_n
\end{pmatrix} \in D, \quad P_{n-1} = \begin{pmatrix}
0 & b_{n-1} \\
c_{n-1} & 0
\end{pmatrix} \in A, \quad P_l \in D (\text{if } l - n \text{ is even}), \quad P_l \in A (\text{if } l - n \text{ is odd}).
\]  
(26)

Here \(P_0\) is a constant matrix, and \(P_l (1 \leq i \leq n)\) is the function of \(x\) and \(t\). In particular, \(P_0 \in D\) if \(n\) is even and \(P_0 \in A\) if \(n\) is odd, which leads to the separate discussion on the
determinant representation of $T_n$ in the following by means of its kernel. Specifically, from algebraic equations,

$$\psi_k^{[n]} = T_n(\lambda; \lambda_1, \ldots, \lambda_n)_{\lambda = \lambda_k} \psi_k = \sum_{i=0}^{n} P_i \lambda_i \psi_k = 0, \quad k = 1, 2, \ldots, n,$$

(27)

the coefficients $P_i$ are solved by Cramer’s rule. Thus we obtain the determinant representation of the $T_n$.

Theorem 2.

1. For $n = 2k$ ($k = 1, 2, 3, \ldots$), the $n$-fold DT of the KN system can be expressed by

$$T_n = T_n(\lambda; \lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{pmatrix} \frac{(T_n)_{11}}{W_n} & \frac{(T_n)_{12}}{W_n} \\ \frac{(T_n)_{21}}{W_n} & \frac{(T_n)_{22}}{W_n} \end{pmatrix}, \quad (28)$$

with

$$W_n = \begin{vmatrix} \lambda_n^3 \phi_1 & \lambda_{n-2}^2 \phi_1 & \lambda_{n-1}^1 \phi_1 & \lambda_{n-3}^1 \phi_1 & \ldots & \lambda_{n-1}^1 \phi_1 & \lambda_{n-3}^1 \phi_1 & \ldots & \lambda_{n-3}^1 \phi_1 \\ \lambda_n^{3-1} \phi_2 & \lambda_{n-2}^2 \phi_2 & \lambda_{n-1}^1 \phi_2 & \lambda_{n-3}^1 \phi_2 & \ldots & \lambda_{n-1}^1 \phi_2 & \lambda_{n-3}^1 \phi_2 & \ldots & \lambda_{n-3}^1 \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n^1 \phi_n & \lambda_{n-1}^2 \phi_n & \lambda_{n-2}^1 \phi_n & \lambda_{n-3}^1 \phi_n & \ldots & \lambda_{n-1}^2 \phi_n & \lambda_{n-2}^1 \phi_n & \ldots & \lambda_{n-3}^1 \phi_n \\ \end{vmatrix}$$

$$(T_n)_{11} = \begin{pmatrix} \lambda_n^0 & \lambda_{n-2}^2 & 0 & \ldots & \lambda_{n-3}^1 & \ldots & 0 & \lambda_{n-1}^{2} & \lambda_1 \lambda_2 \ldots \lambda_n \\ \lambda_n^1 \phi_1 & \lambda_{n-2}^2 \phi_1 & \lambda_{n-1}^1 \phi_1 & \lambda_{n-3}^1 \phi_1 & \ldots & \lambda_{n-1}^1 \phi_1 & \lambda_{n-3}^1 \phi_1 & \ldots & \lambda_{n-3}^1 \phi_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n^1 \phi_n & \lambda_{n-2}^2 \phi_n & \lambda_{n-1}^1 \phi_n & \lambda_{n-3}^1 \phi_n & \ldots & \lambda_{n-1}^2 \phi_n & \lambda_{n-2}^1 \phi_n & \ldots & \lambda_{n-3}^1 \phi_n \\ \end{pmatrix}$$

$$(T_n)_{12} = \begin{pmatrix} 0 & 0 & \lambda_{n-3}^2 & 0 & \ldots & 0 & \lambda_n^1 & \lambda_1 \lambda_2 \ldots \lambda_n \\ \lambda_n^1 \phi_1 & \lambda_{n-2}^2 \phi_1 & \lambda_{n-1}^1 \phi_1 & \lambda_{n-3}^1 \phi_1 & \ldots & \lambda_{n-1}^1 \phi_1 & \lambda_{n-3}^1 \phi_1 & \ldots & \lambda_{n-3}^1 \phi_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n^1 \phi_n & \lambda_{n-2}^2 \phi_n & \lambda_{n-1}^1 \phi_n & \lambda_{n-3}^1 \phi_n & \ldots & \lambda_{n-1}^2 \phi_n & \lambda_{n-2}^1 \phi_n & \ldots & \lambda_{n-3}^1 \phi_n \\ \end{pmatrix}$$

$$\tilde{W}_n = \begin{vmatrix} \lambda_n^1 \phi_1 & \lambda_{n-1}^2 \phi_1 & \lambda_{n-2}^2 \phi_1 & \lambda_{n-3}^1 \phi_1 & \ldots & \lambda_{n-3}^1 \phi_1 & \lambda_n^1 \phi_1 \\ \lambda_n^2 \phi_2 & \lambda_{n-2}^2 \phi_2 & \lambda_{n-1}^2 \phi_2 & \lambda_{n-3}^2 \phi_2 & \ldots & \lambda_{n-3}^2 \phi_2 & \lambda_n^2 \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_n^1 \phi_n & \lambda_{n-1}^2 \phi_n & \lambda_{n-2}^2 \phi_n & \lambda_{n-3}^1 \phi_n & \ldots & \lambda_{n-3}^1 \phi_n & \lambda_n^1 \phi_n \\ \end{vmatrix}$$

$$(T_n)_{21} = \begin{pmatrix} 0 & 0 & \lambda_{n-3}^2 & 0 & \ldots & 0 & \lambda_n^1 & \lambda_1 \lambda_2 \ldots \lambda_n \\ \lambda_n^1 \phi_1 & \lambda_{n-2}^2 \phi_1 & \lambda_{n-1}^2 \phi_1 & \lambda_{n-3}^2 \phi_1 & \ldots & \lambda_{n-1}^2 \phi_1 & \lambda_{n-3}^2 \phi_1 & \ldots & \lambda_{n-3}^2 \phi_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n^2 \phi_n & \lambda_{n-2}^2 \phi_n & \lambda_{n-1}^2 \phi_n & \lambda_{n-3}^2 \phi_n & \ldots & \lambda_{n-1}^2 \phi_n & \lambda_{n-3}^2 \phi_n & \ldots & \lambda_{n-3}^2 \phi_n \\ \end{pmatrix}$$

$$\tilde{(T_n)_{21}} = \begin{pmatrix} 0 & 0 & \lambda_{n-3}^2 & 0 & \ldots & 0 & \lambda_n^1 & \lambda_1 \lambda_2 \ldots \lambda_n \\ \lambda_n^1 \phi_1 & \lambda_{n-2}^2 \phi_1 & \lambda_{n-1}^2 \phi_1 & \lambda_{n-3}^2 \phi_1 & \ldots & \lambda_{n-1}^2 \phi_1 & \lambda_{n-3}^2 \phi_1 & \ldots & \lambda_{n-3}^2 \phi_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n^2 \phi_n & \lambda_{n-2}^2 \phi_n & \lambda_{n-1}^2 \phi_n & \lambda_{n-3}^2 \phi_n & \ldots & \lambda_{n-1}^2 \phi_n & \lambda_{n-3}^2 \phi_n & \ldots & \lambda_{n-3}^2 \phi_n \\ \end{pmatrix}$$

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\[(T_n)_{22} = \begin{bmatrix}
    \lambda^n & 0 & \lambda^{n-2} & 0 & \cdots & \lambda^2 & 0 & \lambda_1 \lambda_2 \cdots \lambda_n \\
    \lambda_1 \phi_1 & \lambda_1^{-1} \phi_1 & \lambda_1^{-3} \phi_1 & \cdots & \lambda_1^3 \phi_1 & \lambda_1 \phi_1 & \lambda_1 \lambda_2 \cdots \lambda_n \phi_1 \\
    \lambda_2 \phi_2 & \lambda_2^{-1} \phi_2 & \lambda_2^{-3} \phi_2 & \cdots & \lambda_2^3 \phi_2 & \lambda_2 \phi_2 & \lambda_2 \lambda_3 \cdots \lambda_n \phi_2 \\
    \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
    \lambda_n \phi_n & \lambda_n^{-1} \phi_n & \lambda_n^{-3} \phi_n & \cdots & \lambda_n^3 \phi_n & \lambda_n \phi_n & \lambda_n \lambda_{n+1} \cdots \lambda_n \phi_1 \\
\end{bmatrix}
\]

(2) For \( n = 2k + 1 \) \((k = 1, 2, 3, \ldots)\), then

\[ T_n = T_n(\lambda; \lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{pmatrix}
    (T_n)_{11} & (T_n)_{12} \\
    (T_n)_{21} & (T_n)_{22} \\
\end{pmatrix} \]

with

\[ Q_n = \begin{bmatrix}
    \lambda^n & 0 & \lambda^{n-2} & 0 & \cdots & \lambda^2 & 0 & \lambda \\
    \lambda_1 \phi_1 & \lambda_1^{-1} \phi_1 & \lambda_1^{-3} \phi_1 & \cdots & \lambda_1^3 \phi_1 & \lambda_1 \phi_1 & -\lambda_1 \lambda_2 \cdots \lambda_n \phi_1 \\
    \lambda_2 \phi_2 & \lambda_2^{-1} \phi_2 & \lambda_2^{-3} \phi_2 & \cdots & \lambda_2^3 \phi_2 & \lambda_2 \phi_2 & -\lambda_1 \lambda_2 \cdots \lambda_n \phi_2 \\
    \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
    \lambda_n \phi_n & \lambda_n^{-1} \phi_n & \lambda_n^{-3} \phi_n & \cdots & \lambda_n^3 \phi_n & \lambda_n \phi_n & -\lambda_1 \lambda_2 \cdots \lambda_n \phi_n \\
\end{bmatrix}
\]
Next, we consider the transformed new solutions \(q^{[n]}, r^{[n]}\) of the KN system corresponding to the \(n\)-fold DT. Under a covariant requirement of the spectral problem of the KN system, the transformed form should be

\[
\partial_v \psi^{[n]} = (J \phi^2 + Q^{[n]} \lambda) \psi = U^{[n]} \psi,
\]

with

\[
\psi = \left( \begin{array}{c} \phi \\ \varphi \end{array} \right), \quad J = \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right), \quad Q^{[n]} = \left( \begin{array}{cc} 0 & q^{[n]} \\ r^{[n]} & 0 \end{array} \right),
\]

and then

\[
T_n + T_n U = U^{[n]} T_n.
\]

Substituting \(T_n\) given by equation (25) into equation (33), and then comparing the coefficients of \(\lambda^{n+1}\), it yields

\[
q^{[n]} = \frac{a_n}{d_n} q - 2 \frac{b_{n-1}}{d_n}, \quad r^{[n]} = \frac{d_n}{a_n} r + 2 i \frac{c_{n-1}}{a_n}.
\]

Furthermore, substituting \(a_n, b_{n-1}, c_{n-1}\) which are obtained from equation (28) for \(n = 2k\) and from equation (29) for \(n = 2k + 1\), into (34), the new solutions \((q^{[n]}, r^{[n]})\) are given by

\[
q^{[n]} = \frac{\Omega_{11}}{\Omega_{21}} q + 2 i \frac{\Omega_{11} \Omega_{22}}{\Omega_{21}}, \quad r^{[n]} = \frac{\Omega_{21}}{\Omega_{11}} r - 2 i \frac{\Omega_{21} \Omega_{22}}{\Omega_{11}}.
\]

Here, \((1)\) for \(n = 2k\),

\[
\Omega_{11} = \left| \begin{array}{cccc} \lambda_1^{n-1} \varphi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-3} \varphi_1 & \ldots & \lambda_1 \varphi_1 \\ \lambda_2^{n-1} \varphi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-3} \varphi_2 & \ldots & \lambda_2 \varphi_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n^{n-1} \varphi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-3} \varphi_n & \ldots & \lambda_n \varphi_n \end{array} \right|,
\]

\[
\Omega_{12} = \left| \begin{array}{cccc} \lambda_1^{n-1} \varphi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-3} \varphi_1 & \ldots & \lambda_1 \varphi_1 \\ \lambda_2^{n-1} \varphi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-3} \varphi_2 & \ldots & \lambda_2 \varphi_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n^{n-1} \varphi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-3} \varphi_n & \ldots & \lambda_n \varphi_n \end{array} \right|,
\]

\[
\Omega_{21} = \left| \begin{array}{cccc} \lambda_1^{n-1} \varphi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-3} \varphi_1 & \ldots & \lambda_1 \varphi_1 \\ \lambda_2^{n-1} \varphi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-3} \varphi_2 & \ldots & \lambda_2 \varphi_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n^{n-1} \varphi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-3} \varphi_n & \ldots & \lambda_n \varphi_n \end{array} \right|,
\]

\[
\Omega_{22} = \left| \begin{array}{cccc} \lambda_1^{n-1} \varphi_1 & \lambda_1^{n-2} \varphi_1 & \lambda_1^{n-3} \varphi_1 & \ldots & \lambda_1 \varphi_1 \\ \lambda_2^{n-1} \varphi_2 & \lambda_2^{n-2} \varphi_2 & \lambda_2^{n-3} \varphi_2 & \ldots & \lambda_2 \varphi_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n^{n-1} \varphi_n & \lambda_n^{n-2} \varphi_n & \lambda_n^{n-3} \varphi_n & \ldots & \lambda_n \varphi_n \end{array} \right|.
\]
(2) for $n = 2k + 1$,

\[
\Omega_{i1} = \begin{vmatrix}
\lambda_1^{-n} \varphi_1 & \lambda_1^{-n-1} \varphi_1 & \lambda_1^{-n-2} \varphi_1 & \ldots & \lambda_1 \varphi_1 \\
\lambda_2^{-n} \varphi_2 & \lambda_2^{-n-1} \varphi_2 & \lambda_2^{-n-2} \varphi_2 & \ldots & \lambda_2 \varphi_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_n^{-n} \varphi_n & \lambda_n^{-n-1} \varphi_n & \lambda_n^{-n-2} \varphi_n & \ldots & \lambda_n \varphi_n
\end{vmatrix},
\]

\[
\Omega_{i2} = \begin{vmatrix}
\lambda_1^{-n} \phi_1 & \lambda_1^{-n-1} \phi_1 & \lambda_1^{-n-2} \phi_1 & \ldots & \lambda_1 \phi_1 \\
\lambda_2^{-n} \phi_2 & \lambda_2^{-n-1} \phi_2 & \lambda_2^{-n-2} \phi_2 & \ldots & \lambda_2 \phi_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_n^{-n} \phi_n & \lambda_n^{-n-1} \phi_n & \lambda_n^{-n-2} \phi_n & \ldots & \lambda_n \phi_n
\end{vmatrix},
\]

\[
\Omega_{j1} = \begin{vmatrix}
\lambda_1^{-n} \varphi_1 & \lambda_1^{-n-1} \varphi_1 & \lambda_1^{-n-2} \varphi_1 & \ldots & \lambda_1 \varphi_1 \\
\lambda_2^{-n} \varphi_2 & \lambda_2^{-n-1} \varphi_2 & \lambda_2^{-n-2} \varphi_2 & \ldots & \lambda_2 \varphi_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_n^{-n} \varphi_n & \lambda_n^{-n-1} \varphi_n & \lambda_n^{-n-2} \varphi_n & \ldots & \lambda_n \varphi_n
\end{vmatrix},
\]

\[
\Omega_{j2} = \begin{vmatrix}
\lambda_1^{-n} \phi_1 & \lambda_1^{-n-1} \phi_1 & \lambda_1^{-n-2} \phi_1 & \ldots & \lambda_1 \phi_1 \\
\lambda_2^{-n} \phi_2 & \lambda_2^{-n-1} \phi_2 & \lambda_2^{-n-2} \phi_2 & \ldots & \lambda_2 \phi_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_n^{-n} \phi_n & \lambda_n^{-n-1} \phi_n & \lambda_n^{-n-2} \phi_n & \ldots & \lambda_n \phi_n
\end{vmatrix},
\]

We are now in a position to consider the reduction of the DT of the KN system so that $q^{[n]} = -(r^{[n]})^*$; then the DT of the DNLS is given. Under the reduction condition $q = -r^*$, the eigenfunction $\psi_k = \left(\begin{array}{c} \phi_k \\
\varphi_k \end{array}\right)$ associated with the eigenvalue $\lambda_k$ has the following properties [24]:

1. $\phi_k^* = \varphi_k$, $\lambda_k = -\lambda_k^*$;
2. $\phi_k^* = \varphi_k$, $\varphi_k^* = \phi_k$, $\lambda_k^* = -\lambda_k$, where $k \neq l$.

Note that the denominator $W_n$ of $q^{[n]}$ is a modulus of a non-zero complex function under reduction condition, so the new solution $q^{[n]}$ is non-singular. For the one-fold DT $T_1$, set

$\lambda_1 = i\beta_1$ (a pure imaginary constant), and its eigenfunction $\psi_1 = \left(\begin{array}{c} \phi_1 \\
\varphi_1 \end{array}\right)$; (38)

then $T_1$ in theorem 1 is the DT of the DNLS. We note that $q^{(1)} = -(r^{(1)})^*$ holds with the help of equation (21), $q = -r^*$ and this special choice of $\psi_1$. This is an essential distinctness of DT between DNLS and NLS, because the one-fold transformation of AKNS cannot preserve the reduction condition to the NLS. Furthermore, for the two-fold DT, according to the above property (ii), set

$\lambda_2 = -\lambda_1^*$ and its eigenfunction $\psi_2 = \left(\begin{array}{c} \varphi_1 \\
\phi_1 \end{array}\right)$;

then $q^{[2]} = -(r^{[2]})^*$ can be verified from equation (35) and $T_2$ given by equation (28) is the DT of the DNLS. Of course, in order to obtain $q^{[2]} = -(r^{[2]})^*$ so that $T_2$ also becomes the DT
of the DNLS, we can also set
\[ \lambda_l = i\beta_l \text{(pure imaginary) and its eigenfunction } \psi_l = \begin{pmatrix} \phi_l \\ \phi_l^* \end{pmatrix}, \quad l = 1, 2. \] (40)

There are many choices to guarantee \( q^{[n]} = -(r^{[n]})^* \) for the \( n \)-fold DTs when \( n > 2 \). For example, setting \( n = 2k \) and \( l = 1, 3, \ldots, 2k - 1 \), and then choosing the following \( k \) distinct eigenvalues and eigenfunctions in \( n \)-fold DTs:
\[ \lambda_l \leftrightarrow \psi_l = \begin{pmatrix} \phi_l \\ \phi_l^* \end{pmatrix}, \quad \text{and} \quad \lambda_{2l} = -\lambda_{2l-1}^*, \leftrightarrow \psi_{2l} = \begin{pmatrix} \psi_{2l-1}^* \\ \phi_{2l-1}^* \end{pmatrix} \] (41)
so that \( q^{[2k]} = -(r^{[2k]})^* \) in equation (35). Then \( T_{2k} \) with these paired eigenvalue \( \lambda_l \) and paired eigenfunctions \( \psi_l(i = 1, 3, \ldots, 2k - 1) \) is reduced to the \( (2k) \)-fold DT of the DNLS. Similarly, \( T_{2k+1} \) in equation (29) can also be reduced to the \((2k+1)\)-fold DT of the DNLS by choosing one pure imaginary \( \lambda_{2k+1} = i\beta_{2k+1} \) (pure imaginary) and \( k \) paired eigenvalues \( \lambda_{2l} = -\lambda_{2l-1}^*(l = 1, 2, \ldots, k) \) with corresponding eigenfunctions according to properties (i) and (ii).

3. Particular solutions
3.1. DTs applied to zero seed
For \( q = r = 0 \), equations (6) and (7) are solved by
\[ \psi_k = \begin{pmatrix} \phi_k \\ \phi_k^* \end{pmatrix}, \quad \phi_k = \exp \left( i(\lambda_k^2x + 2\lambda_k^4t) \right), \quad \phi_k^* = \exp \left( -i(\lambda_k^2x + 2\lambda_k^4t) \right). \] (42)

Case 1 (\( N = 1 \)). Under the choice equation (38), substituting \( \psi_l \) in equation (42) back into equation (21) with \( \lambda_1 = i\beta_1 \), then one solution of the DNLS is
\[ q^{[1]} = -2\beta_1 \exp(-2i(\beta_1^2x + 2\beta_1^4t)), \] (43)
which is not a soliton but a periodic solution with a constant amplitude.

Case 2 (\( N = 2 \)). Considering the choice in equation (40) with \( \lambda_1 = i(l + m) \), \( \lambda_2 = i(l - m) \), and substituting the eigenfunctions in equation (42) back into \( T_2 \), the result of the DT of the DNLS is then simply found from (35):
\[ q^{[2]} = -4\beta_1 \exp \left( \begin{pmatrix} m \cos(2G) - il \sin(2G) \end{pmatrix}^3 \right) \frac{\exp(2iF)}{(m^2 - l^2) \cos(2G)^2 + l^2)^2}, \] (44)
which is a quasi-periodic solution, and here \( F = -l^2x + 2l^4t + 12l^2m^2t - m^2x + 2m^4t \), and \( G = 8l^3mt - 2lmx + 8lm^3t \). Furthermore, considering the choice in equation (39) with \( \lambda_1 = \alpha_1 + i\beta_1 \), \( \lambda_2 = -\alpha_1 + i\beta_1 \), and using the eigenfunctions in equation (42), then the solution of the DNLS generated by the two-fold DT is simply found from (35):
\[ q^{[2]} = 4i\alpha_1 \exp \left( \begin{pmatrix} -i\alpha_1 \cosh(2\Gamma) + \beta_1 \sinh(2\Gamma) \end{pmatrix} \right)^3 \frac{\exp(2ih)}{((-\alpha_1^2 - \beta_1^2)^2 \cosh(2\Gamma)^2 + \beta_1^2)^2}, \] (45)
with \( h = -\beta_1^2x + 2\beta_1^4t - 12\alpha_1^2\beta_1^2t + \alpha_1^2x + 2\alpha_1^4t \) and \( \Gamma = -8\alpha_1 \beta_1^2t + 2\alpha_1 \beta_1x + 8\alpha_1 \beta_1^3t \).

By letting \( \alpha_1 \to 0 \) in (45), it becomes a rational solution
\[ q^{[2]} = 4\beta_1 \exp \left( 2i\beta_1^2(-x + 2\beta_1^2t) \right) \frac{(4i\beta_1^2(4\beta_1^2t - x) - 1)^3}{(16\beta_1^4(4\beta_1^2t - x)^2 + 1)^2}. \] (46)
with an arbitrary real constant $\beta_1$. Obviously, the rational solution is a linear soliton, and its trajectory is defined explicitly by

$$x = 4\beta_1^2 t,$$

(47)
on the $(x - t)$ plane. The solutions $q^{[1]}$ and $q^{[2]}$ of the DNLS equation are consistent with the results of [24, 25] except the rational solution. So the rational solution $q^{[2]}$ in equation (46) of the DNLS equation is first found in this paper, which is plotted in figure 1.

3.2. DTs applied to non-zero seeds: constant solution and periodic solution

Set $a$ and $c$ as the two complex constants, and take $c > 0$ without loss of generality; then $q = c \exp(i(ax + (-c^2 + a)at))$ is a periodic solution of the DNLS equation, which will be used as a seed solution of the DT. Substituting $q = c \exp(i(ax + (-c^2 + a)at))$ into the spectral problem, equations (6) and (7), and using the method of separation of variables and the superposition principle, the eigenfunction $\psi_k$ associated with $\lambda_k$ is given by

$$\begin{pmatrix}
\varphi_k(x, t, \lambda_k)[1, k] \\
\varphi_k(x, t, \lambda_k)[2, k]
\end{pmatrix} = \begin{pmatrix}
\varphi 1(x, t, \lambda_k)[1, k] + \varphi 2(x, t, \lambda_k)[1, k] + \varphi 1^*(x, t, -\lambda_k^*)[2, k] \\
\varphi 1(x, t, \lambda_k)[2, k] + \varphi 2(x, t, \lambda_k)[2, k] + \varphi 1^*(x, t, -\lambda_k^*)[1, k] \\
\varphi 2^*(x, t, -\lambda_k^*)[1, k] \\
\varphi 2^*(x, t, -\lambda_k^*)[2, k]
\end{pmatrix}. \quad (48)$$

Here

$$\begin{pmatrix}
\varphi 1(x, t, \lambda_k)[1, k] \\
\varphi 1(x, t, \lambda_k)[2, k]
\end{pmatrix} = \begin{pmatrix}
\exp\left(\frac{\sqrt{s}(x+2\lambda_k^2t+(c^2-\nu\lambda_k^2))}{2} - \frac{1}{2}(i(ax + (-c^2 + a)at))\right) \\
\exp\left(-\frac{\sqrt{s}(x+2\lambda_k^2t+(c^2-\nu\lambda_k^2))}{2} + \frac{1}{2}(i(ax + (-c^2 + a)at))\right)
\end{pmatrix},$$

(49)

$$\begin{pmatrix}
\varphi 2(x, t, \lambda_k)[1, k] \\
\varphi 2(x, t, \lambda_k)[2, k]
\end{pmatrix} = \begin{pmatrix}
\exp\left(\frac{\sqrt{s}(x+2\lambda_k^2t+(c^2-\nu\lambda_k^2))}{2} - \frac{1}{2}(i(ax + (-c^2 + a)at))\right) \\
\exp\left(-\frac{\sqrt{s}(x+2\lambda_k^2t+(c^2-\nu\lambda_k^2))}{2} + \frac{1}{2}(i(ax + (-c^2 + a)at))\right)
\end{pmatrix}. \quad (50)$$

Figure 1. Rational solution $|q^{[2]}|^2$ of the DNLS with $\beta_1 = 0.5$. 
\[ \sigma 1(x, t, \lambda_k) = \left( \frac{\sigma 1(x, t, \lambda_k)[1, k]}{\sigma 1(x, t, \lambda_k)[2, k]} \right), \quad \sigma 2(x, t, \lambda_k) = \left( \frac{\sigma 2(x, t, \lambda_k)[1, k]}{\sigma 2(x, t, \lambda_k)[2, k]} \right). \]

\[ s = -a^2 - 4\lambda^3 - 4\lambda^2(c^2 - a). \]

Note that \( \sigma 1(x, t, \lambda_k) \) and \( \sigma 2(x, t, \lambda_k) \) are two different solutions of the spectral problem, equations (6) and (7), but we can only obtain the trivial solutions through the DT of the DNLS by setting the eigenfunction \( \psi_0 \) as one of them.

What is more, we can get richer solutions by using (48).

**Case 3** \((N = 1)\). Under the choice in equation (38) with \( \psi_1 \) given by equation (48) and \( \lambda_1 = i\beta_1 \), the one-fold DT of the DNLS generates

\[ |q^{(1)}|^2 = c^2 - 2a + \frac{2(2\beta_1^2 + a^2 - 8c^2\beta_1^2)}{a + 2\beta_1^2 + 2c\beta_1 \cosh(K(x - 2\beta_1^2 t + at - c^2 t))}, \]

with \( K = \sqrt{4c^2\beta_1^2 - (2\beta_1^2 + a)^2} \), according to equation (21). By letting \( x \to \infty, t \to \infty \), so \( |q^{(1)}|^2 \to c^2 - 2a \). The trajectory is defined implicitly by

\[ x = 2\beta_1^2 t + at - c^2 t = 0. \]

\[ q^{(1)} \] in equation (49) gives a soliton solution if \( 4c^2\beta_1^2 - (2\beta_1^2 + a)^2 > 0 \), and gives a periodic solution if \( 4c^2\beta_1^2 - (2\beta_1^2 + a)^2 < 0 \). This classification is consistent with that of Steudel (see figure 1 of [25]). Further, we find that \( q^{(1)} \) in equation (49) can generate a dark soliton if \( c^2 - 2a > (c - \beta_1)^2 \) and a bright solition if \( c^2 - 2a < (c - \beta_1)^2 \). Here

\[ |q^{(1)}|^2_{\text{extreme}} = (c^2 - 2a) + \frac{2(2\beta_1^2 + a^2 - 4c^2\beta_1^2)}{a + 2\beta_1 c + \beta_1^2} = (2\beta_1 - c)^2. \]

Note that, \( \delta = K^2 \) has four roots of \( \beta_1 \) and \( \delta_0 = (2\beta_1 - c)^2 - (c^2 - 2a) \) has two roots of \( \beta_1 \) in general. Combining the conditions of the bright/dark soliton and periodic solutions, a complete classification of the different solutions generated by one-fold DT is obtained, as given in table 1. The depth of the dark soliton is \( 2(-a + 2\beta_1 c - 2\beta_1^2) \) and the height of the bright soliton is \( 2(a - 2\beta_1 c + 2\beta_1^2) \). Particularly, for \( a = 0 \), the seed solution \( q = c \) is a positive constant, and then the one-fold DT of the DNLS generates a dark soliton under the condition \( 0 < \beta_1 < c \), the bright soliton under \( -c < \beta_1 < 0 \), and a periodic solution under \( \beta_1 < -c \) and \( \beta_1 > c \). To illustrate the table, figure 2 is plotted for the case of \( c > 0 \) and \( a < 0 \). Set \( y_1 = (c - 2\beta_1)^2, y_2 = 4c^2\beta_1^2 - (2\beta_1^2 + a)^2 = \delta \) and \( \lambda_3 = c^2 - 2a \) with specific parameters \( a = -1.5 \) and \( c = 0.8 \). There are four roots of \( y_2 \), which are \((\beta_1)_1 > (\beta_1)_2 > (\beta_1)_3 > (\beta_1)_4 \). Note that the \((\beta_1)_1 \) and \((\beta_1)_3 \) are also the roots of \( y_1 = y_3 = \delta_0 \). We can see from figure 2 that \( q^{(1)} \) in equation (49) gives the bright soliton when \( \beta_1 \in ((\beta_1)_1, (\beta_1)_3) \) because \( y_2 > 0 \) and \( y_1 > y_3 \), the dark soliton when \( \beta_1 \in ((\beta_1)_2, (\beta_1)_1) \) because \( y_2 < 0 \) and \( y_1 < y_3 \), and the periodic solutions for other three cases of \( \beta_1 \) because \( y_2 < 0 \).

**Case 4** \((N = 2)\). Under the choice in equation (40) with \( \lambda_1 = i\beta_1, \lambda_2 = i\beta_2 \) and \( \beta_1 \neq \beta_2 \), the solution of the DNLS equation is generated by two-fold DT from (35) as

\[ q^{(2)} = \frac{(\beta_1 \phi_1^* \phi_2 - \beta_2 \phi_1 \phi_2^*)^2}{(\beta_1 \phi_1^* \phi_2 - \beta_2 \phi_1^* \phi_2^*)^2} = 2 \frac{(\beta_1^2 - \beta_2^2) \phi_1 \phi_2 (\beta_1^2 \phi_2^2 - \beta_2 \phi_1 \phi_2)}{(\beta_1 \phi_1^* \phi_2^2 - \beta_2 \phi_1^* \phi_2^2)^2}, \]

where \( \phi_1 \) and \( \phi_2 \) are given by equation (48). Similarly, under the choice in equation (39) with one paired eigenvalue \( \lambda_1 = \alpha_1 + i\beta_1 \) and \( \lambda_2 = -\alpha_1 + i\beta_1 \), the two-fold DT equation (35) of the DNLS equation implies a solution

\[ q^{(2)} = \frac{(\lambda_1 \phi_1 \phi_2^* - \lambda_2 \phi_1 \phi_2)^2}{(-\lambda_2 \phi_1 \phi_2^* + \lambda_1 \phi_1 \phi_2^*)^2} q + 2i \frac{(\lambda_1^2 - \lambda_2^2) \phi_1 \phi_2^* (\lambda_1 \phi_1 \phi_2^* - \lambda_2 \phi_1 \phi_2^*)}{(-\lambda_2 \phi_1 \phi_2^* + \lambda_1 \phi_1 \phi_2^*)^2}, \]
Figure 2. Intervals of $\beta_1$ in the one-fold DT generate different solutions $q^{[1]}$ (dark soliton, bright soliton and periodic solution) under specific parameters $a = -1.5$ and $c = 0.8$. Here $y_1 = (c - 2\beta_1)^2$, $y_2 = 4c^2\beta_1^2 - (2\beta_1^2 + a)^2$ and $y_3 = c^2 - 2a$. There are five intervals of $\beta_1$ divided by the four roots of $y_2$. From the left to the right, the second interval and the fourth interval correspond to the bright soliton and the dark soliton respectively. The other three intervals correspond to the periodic solutions.

Table 1. Classification of the solutions $q^{[1]}$ generated by the one-fold DT in case 3 according to the intervals of the eigenvalue $\lambda_1 = i\beta_1$.

| Classification of the solutions generated by one-fold DT | $c = 0$ | $\forall \beta_1 \in \mathbb{R}$ | Periodic solutions |
|----------------------------------------------------------|--------|-------------------------------|------------------|
| Zero seed                                                | $c = 0$| $\forall \beta_1 \in \mathbb{R}$| Periodic solutions|
| Constant seed                                            | $a = 0, c > 0$ | $0 < \beta_1 < c$ | Dark solitons |
|                                                          | $-c < \beta_1 < 0$ | Bright solitons |
|                                                          | $\beta_1$ belongs to other two intervals | Periodic solutions |
| Periodic seed                                            | $a > 0, c > 0$ | $\frac{1}{2}c - \frac{1}{2}\sqrt{c^2 - 2a} < \beta_1 < \frac{1}{2}c + \frac{1}{2}\sqrt{c^2 - 2a}$ | Dark solitons |
|                                                          | $-\frac{1}{2}c - \frac{1}{2}\sqrt{c^2 - 2a} < \beta_1 < -\frac{1}{2}c + \frac{1}{2}\sqrt{c^2 - 2a}$ | Bright solitons |
|                                                          | $\beta_1$ belongs to other three intervals | Periodic solutions |
| Periodic seed                                            | $a < 0, c > 0$ | $-\frac{1}{2}c + \frac{1}{2}\sqrt{c^2 - 2a} < \beta_1 < -\frac{1}{2}c - \frac{1}{2}\sqrt{c^2 - 2a}$ | Dark solitons |
|                                                          | $-\frac{1}{2}c - \frac{1}{2}\sqrt{c^2 - 2a} < \beta_1 < -\frac{1}{2}c + \frac{1}{2}\sqrt{c^2 - 2a}$ | Bright solitons |
|                                                          | $\beta_1$ belongs to other three intervals | Periodic solutions |

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with \( \varphi_1 \) and \( \varphi_1 \) given by equation (48). Two concrete examples of equation (52) are given below.

1. For simplicity, let \( a = 2\alpha_1^2 - 2\beta_1^2 + c^2 \) so that \( \text{Im}(-a^2 - 4\lambda_1^4 - 4\lambda_1^2(c^2 - a)) = 0 \); then

\[
|q^{[2]}|^2 = -16\alpha_1\beta_1 \times \\
\frac{w_1 \cos(f_1) \cos(f_2) + w_2 \sin(f_1) \sin(f_2) + w_3}{w_4 \cos(f_1) \cos(f_2) + w_5 \sin(f_1) \sin(f_2) + w_6 \cos(2f_2) + w_7 \cosh(2f_1) + w_8 + c^2},
\]

(53)

\[
w_1 = c\alpha_1(c^2 - 4\beta_1^2)(c^2 + 4\alpha_1^2),
w_2 = -c\beta_1(c^2 + 4\alpha_1^2)(c^2 - 4\beta_1^2),
w_3 = 2\alpha_1\beta_1(c^2 - 4\beta_1^2)(4\alpha_1^2 + c^2),
w_4 = 8c\alpha_1^2\beta_1^2(c^2 + 4\alpha_1^2),
w_5 = -8c\alpha_1\beta_1^2(c^2 - 4\beta_1^2),
w_6 = c^2\alpha_1^2(c^2 + 4\alpha_1^2) + c\beta_1^2(c^2 - 4\beta_1^2),
w_7 = 16\alpha_1^2\beta_1^2(\alpha_1^2 + \beta_1^2),
w_8 = c^4(\alpha_1^2 - \beta_1^2) + 16\alpha_1^2\beta_1^2(\alpha_1^2 - \beta_1^2) + 4c^2(\alpha_1^2 + \beta_1^2)^2,
\]

\[
f_1 = K1(4\alpha_1^2 t - 4\beta_1^2 t + x),
f_2 = 4K1\alpha_1\beta_1 t,
\]

\[
K1 = \sqrt{16\alpha_1^2\beta_1^2 - 4c^2\alpha_1^2 + 4c^2\beta_1^2 - c^4}.
\]

By letting \( x \to \infty, t \to \infty \), so that \( |q^{[2]}|^2 \to c^2 \), the trajectory of this solution is defined explicitly by

\[
x = -4\alpha_1^2 t + 4\beta_1^2 t,
\]

(54)

from \( f_1 = 0 \) if \( K1^2 > 0 \), and by

\[
t = 0
\]

(55)

from \( f_2 = 0 \) if \( (K1)^2 < 0 \). According to equation (53), we can obtain the Ma breather [33] (time-periodic breather solution) and the Akhmediev breather [34] (space-periodic breather solution) solutions. In general, the solution in equation (53) evolves periodically along the straight line with a certain angle of \( x \)-axis and \( t \)-axis. The dynamical evolution of \( |q^{[2]}|^2 \) in equation (53) for different parameters is plotted in figures 3–5, which give a visual verification of the three cases of trajectories. Inspired by the extensive research of rogue wave [4, 34] for the NLS equation, a limit procedure [34] is used to construct the rogue wave of the DNLS equation in the following. By letting \( c \to -2\beta_1 \) in (52) with \( \text{Im}(-a^2 - 4\lambda_1^4 - 4\lambda_1^2(c^2 - a)) = 0 \), it becomes a rogue wave,

\[
h_{\text{rogue wave}}^{[2]} = \frac{r1r2r3}{r4r5}
\]

(56)

\[
r1 = 2\exp(2i(\alpha_1^2 + \beta_1^2)(2t\alpha_1^2 + x - 2t\beta_1^2))
\]

\[
r2 = \beta_1(16\beta_1^2\alpha_1^2(4\alpha_1^2 + x)^2 + 16\beta_1^4(4\beta_1^2 - x)^2 + 8i\beta_1^2(x + 4t\alpha_1^2 - 8t\beta_1^2) + 1)
\]

\[
r3 = 2(16\beta_1^2\alpha_1^2(4\alpha_1^2 + x)^2 + 16\beta_1^4(4\beta_1^2 - x)^2 - 8\alpha_1\beta_1(x + 4t\alpha_1^2 - 8t\beta_1^2) + 1)
\]

\[
\times (-\alpha_1 + 16\beta_1(\beta_1^2 - \alpha_1^2)t - 4\beta_1(\alpha_1^2 + \beta_1^2)x + 16i\alpha_1\beta_1^2(\alpha_1^2 + \beta_1^2)t - i\beta_1)
\]
Figure 3. The dynamical evolution of $|q^{[2]}|^2$ (time-periodic breather) in equation (53) on the $(x-t)$ plane with specific parameters $\alpha_1 = \beta_1$, $\beta_1 = 0.5$ and $c = 0.8$. The trajectory is a line $x = 0$.

Figure 4. The dynamical evolution of $|q^{[2]}|^2$ (space-periodic breather) in equation (53) on the $(x-t)$ plane with specific parameters $\alpha_1 = \beta_1$, $\beta_1 = 0.5$ and $c = 1.5$. The trajectory is a line $t = 0$.

\[-16 \beta_1^2 \alpha_1^2 (4t \alpha_1^2 + x)^2 + 16 \beta_1^4 (4t \beta_1^2 - x)^2 + 8i \beta_1^2 (x + 4t \alpha_1^2 - 8t \beta_1^2) + 1) \times (\alpha_1 + 16 \beta_1 (\beta_1^4 - \alpha_1^4) t - 4 \beta_1 (\alpha_1^2 + \beta_1^2) x + 16i \alpha_1 \beta_1^2 (\alpha_1^2 + \beta_1^2) t + \beta_1 i) \]

$r_4 = \alpha_1 + 16 \beta_1 (\beta_1^4 - \alpha_1^4) t - 4 \beta_1 (\alpha_1^2 + \beta_1^2) x + 16i \alpha_1 \beta_1^2 (\alpha_1^2 + \beta_1^2) t + \beta_1 i$

$r_5 = (-16 \beta_1^2 \alpha_1^2 (4t \alpha_1^2 + x)^2 - 16 \beta_1^4 (4t \beta_1^2 - x)^2 + 8 i \beta_1^2 (x + 4t \alpha_1^2 - 8t \beta_1^2) - 1)^2.$

By letting $x \to \infty$, $t \to \infty$, so that $|q^{[2]}_{\text{rogue wave}}|^2 \to 4 \beta_1^2$, the maximum amplitude of $|q^{[2]}_{\text{rogue wave}}|^2$ occurs at $t = 0$ and $x = 0$ and is equal to $36 \beta_1^2$, and the minimum amplitude
Figure 5. The dynamical evolution of the solution $|q^{(2)}|^2$ in equation (53) for case 4(a). It evolves periodically along a straight line with certain angle of x-axis and t-axis under specific parameters $\alpha_1 = 0.65, \beta_1 = 0.5$ and $c = 0.95$.

Figure 6. The dynamical evolution of $|q^{(2)}|^2$ given by equation (56) on the $(x-t)$ plane with specific parameters $\alpha_1 = 1/2$ and $\beta_1 = 1/2$. By letting $x \to \infty, t \to \infty$, so that $|q^{(2)}|^2 \to 1$, the maximum amplitude of $|q^{(2)}|^2$ occurs at $t = 0$ and $x = 0$ and is equal to 9, and the minimum amplitude of $|q^{(2)}|^2$ occurs at $t = \pm \sqrt{15/10}$ and $x = \pm \frac{9\sqrt{3}}{10} \cdot \frac{\sqrt{3}}{\sqrt{3}}(4\alpha_1^2 + \beta_1^2)$, and is equal to 0. Through Figures 6 and 7 of $|q^{(2)}|^2$, the main features (such as large amplitude and local property on the $(x-t)$ plane) of the rogue wave are shown. We have found that $|q^{(2)}|^2$ in equation (53) gives the same result of $|q^{(2)}|^2$ by taking the limit of $c \to -2\beta_1$.

(2) When $a = \frac{\sqrt{2}}{2}$, from equation (48), it is not difficult to find that there are two sets of collinear eigenfunctions,

$$
\begin{align*}
(\sigma^1(x, t, \lambda_k)[1, k]) & \quad \text{and} \quad (\sigma^{2*}(x, t, -\lambda_k^*)[2, k]), \\
(\sigma^1(x, t, \lambda_k)[2, k]) & \quad \text{and} \quad (\sigma^{2*}(x, t, -\lambda_k^*)[1, k]).
\end{align*}
$$

(57)
Figure 7. Contour plot of the wave amplitudes of $|q_2^2|$ in the $(x-t)$ plane is given by equation (56) for $\alpha_1 = \frac{1}{2}$ and $\beta_1 = \frac{1}{2}$.

Figure 8. The dynamical evolution of $|q_2^2|$ in case 4(b) on the $(x-t)$ plane with specific parameters $\alpha_1 = 0.5$, $\beta_1 = 0.35$ and $c = 0.85$. It evolves periodically along a straight line on the $(x-t)$ plane.

Therefore, the eigenfunction $\psi_k$ associated with $\lambda_k$ for this case is given by

$$
\begin{pmatrix}
\phi_k(x, t, \lambda_k) \\
\psi_k(x, t, \lambda_k)
\end{pmatrix} = \begin{pmatrix}
\sigma_1(x, t, \lambda_k)[1, k] + \sigma_1^*(x, t, -\lambda_k^*)[2, k] \\
\sigma_1(x, t, \lambda_k)[2, k] + \sigma_1^*(x, t, -\lambda_k^*)[1, k]
\end{pmatrix}.
$$

(59)

Here

$$
\begin{pmatrix}
\sigma_1(x, t, \lambda_k)[1, k] \\
\sigma_1(x, t, \lambda_k)[2, k]
\end{pmatrix} = \frac{1}{\sqrt{\sigma_0}} \exp(i(\lambda_k^2 x + 2\lambda_k^4 t + \frac{1}{2}c^2 x - \frac{1}{4}c^4 t)).
$$

Under the choice in equation (39) with $\lambda_1 = \alpha_1 + i\beta_1$, $\lambda_2 = -\alpha_1 + i\beta_1$, and the $\psi_1$ given by equation (59), the solution $q_2^2$ is given simply by equation (35). Figure 8 is plotted for $|q_2^2|^2$, which shows the periodical evolution along a straight line on the $(x-t)$ plane.

Case 5 ($N = 4$). According to the choice in equation (41) with two distinct eigenvalues $\lambda_1 = \alpha_1 + i\beta_1$, $\lambda_3 = \alpha_3 + i\beta_3$, substituting $\psi_1$ and $\psi_3$ defined by equation (48) into equation (35), the new solution $q_4^2$ generated by the four-fold DT is given. Its analytical expression is
omitted because it is very complicated, but $|q^{[4]}|^2$ are plotted in figures 9 and 10 to show the dynamical evolution on the $(x-t)$ plane: (a) let $a = 2a_i^2 - 2\beta_i^2 + c^2$, $i = 1, 3$, so that $\text{Im}(-a^2 - 4\lambda_i^4 - 4\lambda_i^2(c^2 - a)) = 0$; then figure 9 shows intuitively that two breathers may have parallel trajectories; (b) two breathers have an elastic collision so that they can preserve their profiles after interaction, which is verified in figure 10.

4. Conclusions

In this paper, a detailed derivation of the DT from the KN system and then the determinant representation of the $n$-fold case are given in theorems 1 and 2. Each element of $n$-fold DT matrix $T_n$ is expressed by the determinant of the eigenfunctions of the spectral problem in equations (6) and (7). The determinant representations of the new solution $q^{[n]}$ and $r^{[n]}$
of the KN system are also given in equation (35). Furthermore, by the special choice of the eigenvalue $\lambda_k$ and its eigenfunction $\psi_k$ to construct $T_n$ so that $q^{[n]} = -\psi_k^{[n]}$, $T_n$ is also reduced to the $n$-fold DT of the DNLS equation and $q^{[n]}$ is a solution of the DNLS. To illustrate our method, solutions of five specific cases are discussed by analytical formulae and figures. In particular, a complete classification of the solutions of the DNLS equation generated by the one-fold DT is given in table 1.

By comparing with the known results [24, 25] of the DT for the DNLS equation, our results provide the following improvements.

- A detailed derivation of the DT and the determinant representation of $T_n$. This representation is useful to compute the soliton surfaces of the DNLS equation in the future as we have done for the NLS equation [28]. The rogue wave and the rational traveling wave are firstly given about the DNLS equation. The rational solution has been used by us in a separate preprint to construct the rogue wave of the variable coefficient DNLS equation [35].
- A complete and thorough classification of the solution generated by the one-fold DT. The bright soliton and dark soliton are also classified, which is not published before. At the same time, our results show the nonlinear and difficult Riccati equations in [25], which are transformed from the linear equations of the spectral problem, and Seahorse functions are indeed avoidable. Of course, these do not disaffirm the merits of the method in [25].
- The general solution (48) of the linear partial differential equations in a spectral problem is crucial to obtain the non-trivial solution of the DNLS equation.
- The solution in equation (53) is a relatively general form of the breather solution of the DNLS, which can evolve periodically along any straight line on the $(x - t)$ plane by choosing different values of the parameters $\alpha_1, \beta_1, c$. It has two well-known reductions: the Ma breather going periodically along the $t$-axis, and the Akhmediev breather going periodically along the $x$-axis.

Finally, we would like to mention the DT [36] of the DNLSIII. Unlike the DNLS equation, Fan’s results show that the kernel of the one-fold DT of the DNLSIII is two dimensional, and then support again the necessity of the separate study of the three kinds of derivative nonlinear Schrödinger equations. So we shall consider the determinant representation of the DT for DNLSII and DNLSIII in the near future. Moreover, we are also interested in the periodic solutions with a variable amplitude of the DNLS equation.

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**References**

[1] Mjølhus 1976 On the modulational instability of hydromagnetic waves parallel to the magnetic field *J. Plasma Phys.* 16 321–34

[2] Spangler S P 1997 *Nonlinear Waves and Chaos in Space Plasmas* ed T Hada and H Matsumoto (Tokyo: Terrapub) p 171

[3] Ruderman M S 2002 DNLS equation for large-amplitude solitons propagating in an arbitrary direction in a high-$\beta$ Hall plasma *J. Plasma Phys.* 67 271–6
[4] Fedun V, Ruderman M S and Erdélyi R 2008 Generation of short-lived large-amplitude magnetohydrodynamic pulses by dispersive focusing Phys. Lett. A 372 6107–10
Ruderman M S 2010 Freak waves in laboratory and space plasmas. Freak waves in plasmas Eur. Phys. J. 185 57–66
[5] Sánchez-Arriaga G, Sammartin J R and Elsakar S A 2007 Damping models in the truncated derivative nonlinear Schrödinger equation Phys. Plasmas 14 082108
Sánchez-Arriaga G, Hada T and Nariyuki Y 2009 The truncation model of the derivative nonlinear Schrödinger equation Phys. Plasmas 16 042302
[6] Sanchez-Arriaga G, Sanmartin J R and Elsakar S A 2007 Damping models in the truncated derivative nonlinear Schrödinger equation Phys. Plasmas 14 082108
Sanchez-Arriaga G, Hada T and Nariyuki Y 2009 The truncation model of the derivative nonlinear Schrödinger equation Phys. Plasmas 16 042302
[7] Tzoor N and Jain M 1981 Self-phase modulation in long-radius optical waveguides Phys. Rev. A 23 1266–70
[8] Anderson D and Lisak M 1983 Nonlinear asymmetric self-phase modulation and self-steepening of pulses in long optical waveguides Phys. Rev. A 27 1393–8
[9] Govind P A 2001 Nonlinear Fibers Optics 3rd edn (New York: Academic)
[10] Kaup D J and Newell A C 1978 An exact solution for a derivative nonlinear Schrödinger equation J. Math. Phys. 19 798–801
[11] Kawata T, Kobayashi N and Inoue H 1979 Soliton solution of the derivative nonlinear Schrödinger equation J. Phys. Soc. Japan 46 1008–15
[12] Huang N-N and Chen Z-Y 1990 Alfven solitons J. Phys. A: Math. Gen. 23 439–53
[13] Zhou G-Q and Huang N-N 2007 An N-soliton solution to the DNLS equation based on revised inverse scattering transform J. Phys. A: Math. Theor. 40 13607–23
[14] Huang N N 2007 Marchenko equation for the derivative nonlinear Schrödinger equation Chin. Phys. Lett. 24 894–7
[15] He J-C and Chen Z-Y 2008 Comment on revision of Kaup Newell’s works on IST for DNLS equation Commun. Theor. Phys. 50 1369–74
[16] Yang C-N, Yu J-L, Cai H and Huang N-N 2008 Inverse scattering transform for the derivative nonlinear Schrödinger equation Chin. Phys. Lett. 25 421–4
[17] Ichikawa Y H and Watanabe S 1977 Solitons and envelope solitons in collisionless plasmas J. Phys. (Paris) C6–15
[18] Kawata T and Inoue H 1978 Exact solutions of the derivative nonlinear Schrödinger equation under the nonvanishing conditions J. Phys. Soc. Japan 44 1968–76
[19] Ichikawa Y H, Konno K, Wadati M and Sanuki H 1980 Spiky soliton in circular polarized Alfvén wave J. Phys. Soc. Japan 48 279–86
[20] Chen X-J and Lam W K 2004 Inverse scattering transform for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions Phys. Rev. E 69 066604
[21] Lashkin V M 2007 N-soliton solutions and perturbation theory for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions J. Phys. A: Math. Theor. 40 6119–32
[22] Cai H and Huang N-N 2006 The Hamiltonian formalism of the DNLS equation with a nonvanished boundary value J. Phys. A: Math. Gen. 39 5007–14
[23] Kakei S, Sasa N and Satsuma 1995 Bilinearization of a generalized derivative nonlinear Schrödinger equation J. Phys. Soc. Japan 64 1519–26
[24] Imai K 1999 Generalization of Kaup–Newell inverse scattering formulation and Darboux transformation J. Phys. Soc. Japan 68 355–9
[25] Steudel H 2003 The hierarchy of multi-soliton solutions of the derivative nonlinear Schrödinger equation J. Phys. A: Math. Gen. 36 1931–46
[26] Neugebauer G and Meinel R 1984 General N-soliton solution of the AKNS class on arbitrary background Phys. Lett. A 100 467–70
[27] Matveev V B and Salle M A 1991 Darboux Transformations and Solitons (Berlin: Springer)
[28] He J S, Zhang L, Cheng Y and Li Y S 2006 Determinant representation of Darboux transformation for the AKNS system Sci. China A 12 1867–78
[29] Ablowitz M J, Kaup D J, Newell A C and Segur H 1973 Nonlinear evolution equations of physical significance Phys. Rev. Lett. 31 125–7
Ablowitz M J and Clarkson P A 1991 Solitons, nonlinear evolution equations and inverse scattering (Cambridge: Cambridge University Press)
[30] Wadati M and Sogo K 1983 Gauge transformations in soliton theory J. Phys. Soc. Japan 52 394–38
[31] Chen H H, Lee Y C and Liu C S 1979 Integrability of nonlinear Hamiltonian systems by inverse scattering method Phys. Scr. 20 490–2
[32] Gerdjikov V S and Ivanov I 1983 A quadratic pencil of general type and nonlinear evolution equations: II. Hierarchies of Hamiltonian structures J. Phys. B: Phys. 10 130–43
[33] Ma Y-C 1979 The perturbed plane-wave solutions of the cubic Schrödinger equation Stud. Appl. Math. 60 43–58
[34] Akhmediev N N and Korneev V I 1986 Modulation instability and periodic solutions of the nonlinear Schrödinger equation Theor. Math. Phys. 69 1080–93
Akhmediev N, Soto-Crespo J M and Ankiewicz A 2009 Extreme waves that appear from nowhere: on the nature of rogue waves Phys. Lett. A 373 2137–45
[35] He J, Xu S and Wang L 2011 The Rogue wave of the variable coefficient derivative nonlinear Schrödinger equation (in preparation)
[36] Fan E 2000 Darboux transformation and soliton-like solutions for the Gerdjikov–Ivanov equation J. Phys. A: Math. Gen. 33 6925–33