CONSTRUCTION OF HYPERBOLOIDAL INITIAL DATA

LARS ANDERSSON

1. Introduction

Let \((M, g, \Omega)\) be the conformal rescaling of a 3+1 dimensional vacuum spacetime \((\tilde{M}, \tilde{g})\), with \(g = \Omega^2 \tilde{g}\), where \(\Omega\) is a smooth function on \(M\) vanishing on the boundary \(\partial M\). If \(M\) has smooth null boundary \(I\), with connected components of topology \(S^2 \times \mathbb{R}\), it follows that the Weyl tensor of \((M, g)\) vanishes on \(I\). From this follows peeling properties for \((\tilde{M}, \tilde{g})\). This picture of isolated systems in general relativity, developed by Penrose, see [16, 9], is useful for studying the mass and angular momentum of spacetimes, as well as gravitational radiation.

Friedrich has found a first order symmetric–hyperbolic version of the Einstein evolution equations, called the conformally regular field equations, which may be extended through \(I\). The Cauchy data for this system on a future Cauchy surface \(M\), asymptotic to \(I\) so that \(\bar{M} \cap \partial M = \partial M\), include \(g_{ab}, K_{ab}, \Omega\) as well as components of the rescaled Weyl tensor \(\Omega^{-1} C^{a}_{\beta\gamma\delta}\). In order to get a regular evolution at \(I\), these data must be regular up to \(\partial M\). It was shown by the author, Chrusciel and Friedrich [6, 4, 3, 5] that under certain conditions on the boundary geometry of \(M\), the Cauchy data for the conformally regular field equations, are smooth up to \(\partial M\).

Using the conformally regular field equations, Friedrich has shown that the maximal vacuum development of data (called hyperboloidal data) on a future Cauchy surface intersecting \(I\), regular up to \(\partial M\) is a spacetime which has a “smooth piece of \(I\)”. Further, for small data the maximal vacuum development has a null boundary with future complete null generators and a regular timelike infinity.

A programme has been initiated to numerically evolve the Einstein equations using the conformally regular field equations [14, 15, 13, 12]. This approach may have advantages over the “traditional” approach which treats the asymptotic flatness condition by introducing boundary conditions far away from the isolated system under study, and attempts to observe for example gravitational wave signatures on this boundary.

In this note we will discuss the conformal procedure for constructing solutions \((g_{ab}, K_{ab}, \Omega)\) to the constraint equations and the geometric conditions for regularity at \(I\). I will also briefly discuss the Cauchy problem for the Einstein evolution equations at \(I\), directly from the point of view of the Einstein equations in \((M, g)\).
2. Preliminaries

The following index conventions will be used. Greek indices $\alpha, \beta, \ldots$ run over 0, \ldots, 3, lower case Latin indices $a, b, c, \ldots$ run over 1, 2, 3, and upper case Latin indices $A, B, C, \ldots$ run over 2, 3.

Let $(M, g)$ be a 3+1 dimensional globally hyperbolic space–time with covariant derivative $D$. Introduce coordinates $(x^a) = (t, x^a)$ on $M$, and consider the foliation $M_t$ consisting of level sets of $t$. We will often drop the subscript $t$ on $M_t$ and associated fields. $M$ will be a compact manifold with boundary $\partial M$, and closure $\bar{M} = M \cup \partial M$. Let $T$ be the normal of $M$, assumed to be timelike, and let $g$ be the induced positive definite metric on $M$, with covariant derivative $\nabla$. We will use an index $T$ to denote contraction with $T$, for example $t_T = T^a t_a$. Let $\Sigma = D_T \Omega$ and assume $|\Sigma| > 0$. The lapse function $N$ and shift vectorfield $X$ are defined by $\partial_t = NT + X$. The second fundamental form $K$ is defined by $K_{ab} = (D_a e_b, T)$. Then we have $K_{ab} = -\frac{1}{2} \mathcal{L}_T g_{ab} = -\frac{1}{2} N^{-1} (\partial_t - X) g_{ab}$ and $D_a e_b = (\nabla_a e_b - T K_{ab})$.

Assume that $M$ has null boundary $\partial M$ and let $\bar{M} = M \cup \partial M$. Let $\Omega$ be a positive function on $M$ with $\Omega = 0$, $d\Omega \neq 0$ on $\partial M$ and denote by $\bar{g}$ the conformally related metric $\bar{g} = \Omega^{-2} g$. We will call $g$ the unphysical metric and $\bar{g}$ the physical metric. We will use $\langle \cdot, \cdot \rangle$ for the inner product induced on $TM$ by $g$. Geometric quantities associated with $\bar{g}$ will be decorated with a tilde $\sim$, for example the covariant derivative $\bar{D}$ and the Ricci tensor $\bar{R}_{\alpha\beta}$. We will consider only the case when $(\bar{M}, \bar{g})$ satisfies the vacuum Einstein equations $\bar{R}_{\alpha\beta} = 0$. Let $\omega = \Omega |_M$. Assume that $\partial M = M \cap \partial M$, has positive definite induced metric and that $\omega$ is a defining function for $\partial M$, i.e. $d\omega |_{\partial M} \neq 0$. We will refer to $\partial M$ as $\mathcal{I}$ and the surface $\partial M \subset \partial M$ as a cross section of $\mathcal{I}$.

There is a gauge ambiguity in the conformal rescaling. Let $\bar{\Omega} = \Omega \Theta^{-1}$, $\bar{g} = \Omega^{-2} g$ for some positive function $\Theta$ which is bounded and bounded away from zero on $M$. Then $\bar{g} = \Omega^{-2} g$ is another conformal rescaling of $g$. A conformal gauge change can be used to control the mean curvature of $\partial M$, as well as the mean curvature of $M$. In particular, the conformal gauge freedom can be used so that the level sets of $\omega$ are the leaves of the gauss foliation w.r.t. $\partial M$, i.e. there are coordinates $(x^a) = (x, y^A)$, so that in a neighborhood of $\partial M$, we have $\omega = x$, and the metric takes the form

\[ g_{ab} dx^a dx^b = dx^2 + h_{AB} dy^A dy^B. \]

This possible without loss of generality, cf. [4, Lemma 2.1]. By construction we have $x(p) = d(p, \partial M)$ and the unit normal $\partial M$ is $\eta = \partial_x$. We use the index 1 for contraction with $\eta$, for example $t_1 = \nabla^a x t_{ab} = \eta^a t_{ab}$. Let $\nabla A$ and $\bar{R}_{AB}$ denote the covariant derivative with respect to $h_{AB}$ and the Ricci tensor of $h_{AB}$. The second fundamental form of $\partial M$ with respect to $\eta$ is defined by $\lambda_{AB} = \langle \nabla A e_B, \eta \rangle$. Then we have $\nabla A e_B = \bar{\nabla} A e_B + \eta \lambda_{AB}$ and $\lambda_{AB} = -\frac{1}{2} \mathcal{L}_\eta h_{AB}$.

The quantities associated to $(g_{ij}, K_{ij}, N, T)$ in the physical space–time are

\[
\begin{align*}
\tilde{g}_{ij} &= \omega^{-2} g_{ij}, & \tilde{K}_{ij} &= \omega^{-1} K_{ij} + \Sigma \tilde{g}_{ij}, \\
\tilde{N} &= \omega^{-1} N, & \tilde{T} &= \omega T.
\end{align*}
\]

The shift vectorfield $X$ does not scale, so $\tilde{X} = X$. The physical Weyl tensor satisfies $\tilde{C}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$ since $\tilde{R}_{\alpha\beta} = 0$. Further, $\tilde{C}^{\alpha}_{\beta\gamma\delta} = C^{\alpha}_{\beta\gamma\delta}$. 

A symmetric tensor field \( A_{ab} \) on \( M \) is said to satisfy the shear free condition if
\[
(A_{AB} - \frac{1}{2} h^{CD} A_{CD} h_{AB})\big|_{\partial M} = \Sigma \left( \Sigma \lambda_{AB} - \frac{1}{2} h^{CD} \lambda_{CD} h_{AB} \right)\big|_{\partial M}.
\]
In the following we will consider only hypersurfaces intersecting \( \mathcal{I}^+ \) which in view of our conventions force \( \Sigma < 0 \), so that \( \Sigma/|\Sigma| = -1 \). The spacetime \((M, g)\) is said to be shear free at \( \partial M \) if \( K_{ab} \) satisfies the shear free condition. In order for \( g \) to be in \( C^2(\bar{M}) \), it is necessary that \( K_{ab} \) satisfy the shear free condition, cf. [4, Prop. 3.1]. In the following we only consider initial data which are shear free. Note that a change of conformal gauge can be used to get \( h^{AB} \lambda_{AB}\big|_{\partial M} = 0 \).

3. CONFORMAL RESCALINGS OF MINKOWSKI SPACE

In order to understand the Einstein equations near \( \mathcal{I} \) it is important to choose an appropriate conformal compactification and a suitable foliation near \( \mathcal{I} \). In this section we will display a few examples in the simplest case, Minkowski space.

Minkowski space is \( \mathbb{R}^4 \) with the flat line element
\[
d s^2 = -dt^2 + dr^2 + r^2 d\sigma^2,
\]
in radial coordinates \((t, r, \theta, \phi)\), where \( d\sigma^2 = d\theta^2 + \sin^2(\theta) d\phi^2 \) is the line element on \( S^2 \). We will consider two conformal rescalings of Minkowski space. First, define new coordinates \( \tau, \rho \) by
\[
t + r = \tan \left( \frac{\tau + \rho}{2} \right), \quad t - r = \tan \left( \frac{\tau - \rho}{2} \right),
\]
on
\[
U = \{\tau, \rho : -\pi < \tau < \pi, \quad 0 < \rho < \pi, \quad |\tau + \rho| < \pi, \quad |\tau - \rho| < \pi \}.
\]
Rescaling with the conformal factor
\[
\Omega(\tau, \rho) = 2 \cos \left( \frac{\tau + \rho}{2} \right) \cos \left( \frac{\tau - \rho}{2} \right),
\]
which is positive on \( U \), gives [10, p. 450-451]
\[
d s^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -dr^2 + d\rho^2 + \sin^2(\rho) d\sigma^2,
\]
which is the line element of the Einstein universe \( \mathbb{R} \times S^3 \). We have \( N \equiv 1, X \equiv 0 \).

The null part of the boundary of \( U \) has two components with \( \tau > 0 \) or \( \tau < 0 \), called future and past null infinity, denoted \( \mathcal{I}^\pm \). The point \((\tau, \rho) = (0, \pi)\) is called spatial infinity and denoted \( i_0 \). The points \( i_\pm \) with \((\tau, \rho) = (\pm \pi, 0)\) are called future and past timelike infinity. All inextendible future directed null curves start at \( \mathcal{I}^- \) and end at \( \mathcal{I}^+ \).

The hyperboloids \( t^2 - r^2 = u^2 \) in Minkowski space correspond to hypersurfaces in \( U \) intersecting \( \partial U \) at \( \tau = \pi/2 \) at different angles depending on \( u \), cf. figure [1]. In particular, the hyperboloid with sectional curvature \(-1\) corresponds to \( \tau = \pi/2 \). On the other hand, the time translated hyperboloids given by \( t = \sqrt{1 + r^2} + u \) intersect \( \mathcal{I} \) at \( \tau = \pi/2 + \arctan(u) \), see figure [2]. In this coordinate system, the future translated hyperboloids approach timelike infinity \( i_+ \) as \( u \) increases. The Lapse \( N \) is given at \( \mathcal{I} \) by
\[
\lim_{r \to \infty} N = \frac{1}{\sqrt{1 + u^2}} = \cos(\tau - \pi/2)
\]
which tends to zero as \( \tau \) increases to \( \pi \).
Figure 1. Hyperboloids $t^2 - r^2 = u^2$ in conformally compactified Minkowski space.

Figure 2. Time translated hyperboloids $t = \sqrt{1 + r^2} + u$ in conformally compactified Minkowski space.

For the second conformal rescaling, introduce new coordinates $\psi, \tau$ by

$$
t = \coth(\psi) + \tau,
$$

$$
r = 1 / \sinh(\psi).
$$

This is a singular coordinate transformation, but this does not cause a problem as we are only interested in the behavior near infinity. In terms of the new coordinates, the Minkowski line element takes the form

$$
\tilde{ds}^2 = -d\tau^2 + 2 \sinh^{-2}(\psi) d\psi d\tau + \sinh^{-2}(\psi)(d\psi^2 + d\sigma^2).
$$

After a conformal rescaling with

$$
\Omega = \sinh(\psi),
$$

we get the line element $ds^2 = \Omega^2 \tilde{ds}^2$ given by

$$
ds^2 = -\sinh^2(\psi) d\tau^2 + 2 d\psi d\tau + (d\psi^2 + d\sigma^2).
$$

The hypersurfaces $\tau =$constant correspond to the time translated hyperboloids $t = \sqrt{1 + r^2} + \tau$ in Minkowski space. The conformally rescaled spacetime $\mathbf{M}$ has
null boundary $\mathcal{I} = \{ \psi = 0 \}$, and so the boundary of this conformal rescaling does not include $\mathcal{I}_-$ or $i_0, i_\pm$. Further, the closure of $\mathcal{M}$ is not compact.

The foliation $\mathcal{M}_\tau$ of level sets of $\tau$ has Lapse $N = \cosh(\psi)$, and Shift $X = \partial_\psi$ so that the time like normal $T$ is $T = \cosh^{-1}(\psi)(\partial_\tau - \partial_\psi)$. The foliation $\mathcal{M}_\tau$ is static so that $K_{ab} = 0$. The induced geometry on $\mathcal{M}_\tau$ is the cylinder $d\psi^2 + d\sigma^2$, with totally geodesic boundary $\{ \psi = 0 \}$. The future sheets of the hyperboloids $t^2 - r^2 = u^2$ of mean curvature $-3/u$ intersect $\mathcal{I}$ at $\tau = 0$, see figure 3.

4. CONFORMAL CONSTRAINT EQUATIONS

The Einstein vacuum equations $\hat{\mathbf{R}}_{\alpha\beta} = 0$ imply the constraint equations

\begin{align}
(6a) & \quad \hat{R} + (\hat{K}_a^a)^2 - \hat{K}_{ab}\hat{K}^{ab} = 0, \\
(6b) & \quad \hat{\nabla}^a\hat{K}_{ab} - \hat{\nabla}_b\hat{K}^b = 0.
\end{align}

We will consider the constraint equations, in the special case of constant mean curvature $\nabla_a \hat{K}^a = 0$, as a system of equations for the unphysical data and rewrite them in a form which exhibits the freely specifiable data.

In the form of the constraint equations that we will present, the freely specifiable data are $(g_{ij}, A_{ij})$, where $g_{ij}$ is a metric on $\mathcal{M}$, and $A_{ij}$ is a trace free, symmetric tensor field on $\mathcal{M}$ satisfying the shear free condition. We will assume for simplicity $A_{1a}|_{\partial \mathcal{M}} = 0$, which does not restrict the degrees of freedom.

For a solution $\rho$ be a defining function for $\partial \mathcal{M}$, $\rho = x$ near $\partial \mathcal{M}$. Assume $\Sigma |_{\partial \mathcal{M}}$ is a nonzero constant. Set $\hat{g}_{ab} = \rho^{-2}g_{ab}$, define $\hat{\nabla}, \hat{\mathbf{R}}$ etc. w.r.t. $\hat{g}$ and raise and lower indices on hatted fields with $\hat{g}$. Given data $g_{ij}, A_{ij}$, a solution $(g_{ij}, \hat{K}_{ij}, \omega)$ to the constraint equations is constructed by solving the system $\hat{\mathbf{R}}$ below. We will refer to the system $\hat{\mathbf{R}}$ as the conformal constraint equations. Existence, uniqueness and regularity of solutions to the conformal constraint equations has been analyzed in the papers $\hat{\mathbf{R}}$.

For a vector field $Y$, let

\begin{align}
(7a) & \quad L(Y)_{ij} = \nabla_i Y_j + \nabla_j Y_i - \frac{2}{3}\nabla_k Y^k g_{ij}, \\
(7b) & \quad \hat{\nabla}^i \hat{L}(Y)_{ij} = \hat{\nabla}^i (\rho^{-1} A_{ij}),
\end{align}

For a solution $Y$ to the system

\begin{align}
(7a) & \quad L(Y)_{ij} = \nabla_i Y_j + \nabla_j Y_i - \frac{2}{3}\nabla_k Y^k g_{ij}, \\
(7b) & \quad \hat{\nabla}^i \hat{L}(Y)_{ij} = \hat{\nabla}^i (\rho^{-1} A_{ij}),
\end{align}

Figure 3. Hyperboloids $t^2 - r^2 = u^2$ in the conformal rescaling $\hat{\mathbf{R}}$. 
I satisfying a natural causal condition (domain of dependence of compact sets is compact) 
(7). There it was proved that in an asymptotically Schwarzschild spacetime, constant mean 
curvature hypersurfaces in asymptotically flat spacetimes was studied 
∇ the constant mean curvature condition 
4.1. **Constant mean curvature hypersurfaces.** The existence of complete constant 
mean curvature hypersurfaces in asymptotically flat spacetimes was studied 
g solve the physical constraint equations (6) with mean curvature \( \tilde{g} \) corresponding to a solution \( g, K, \omega \) of the conformal constraint equations (7) solve the physical constraint equations (6) with mean curvature \( \tilde{g} K = 3 \Sigma \), under 
the constant mean curvature condition \( \nabla_a \Sigma = 0 \).

4.1. **Constant mean curvature hypersurfaces.** The existence of complete constant 
mean curvature hypersurfaces in asymptotically flat spacetimes was studied in [7]. There it was proved that in an asymptotically Schwarzschild spacetime satisfying a natural causal condition (domain of dependence of compact sets is compact), given a cross section of \( \mathcal{J}_+ \), and a number \( \Sigma < 0 \), there is a unique CMC hypersurface intersecting \( \mathcal{J}_+ \) at the given cross section. The proof uses a barrier construction, together with the causality condition, to prove that the solutions to a sequence of boundary value problems converge to a complete CMC hypersurface.

This result indicates that a wide class of spacetimes with regular conformal compactification are foliated by CMC hypersurfaces. The regularity of the height functions of these CMC hypersurfaces at \( \mathcal{J} \) has not been studied in detail, nor has the existence proof been carried out in the conformally compactified setting. This is a natural problem which should be studied further.

The conditions for regularity of CMC solutions of the conformal constraint equations constructed using the procedure discussed above is well understood, see the discussion in section 4.3 below. However, it is not at all well understood how these results relate to the regularity of CMC foliations in the physical or unphysical spacetime, evolved from these data, nor what spatial gauge conditions are suitable in this situation, see section 3 below.

4.2. **Degenerate elliptic equations.** The system (7) of conformal constraint equations is a degenerate elliptic system for \( (u, Y) \). We will briefly describe the regularity properties of solutions to systems of this type. As an example consider the Lichnerowicz equation (7d). The principal part of the right hand side of (7d) is essentially of the form

\[
L = L_x + B,
\]

where \( L_x \) is the ordinary differential operator \( L_x = x^2 \partial_x^2 + ax \partial_x + b \), and \( B \) is of lower order in \( \partial_x \). In case \((1 - a)^2 > 4b\), the equation for \( L_x x^\alpha = 0 \) has real distinct characteristic roots \( \alpha_\pm \) and the equation \( L_x u = f \) has a solution \( u = o(x^{\alpha_-}) \) for sufficiently regular \( f = o(x^{\alpha_-}) \).

The operator appearing in the Lichnerowicz equation has critical exponents \( \alpha_- = -1, \alpha_+ = 3 \). Let \( L \) be this operator and consider the equation \( Lu = f \), for \( f \in C^\infty(M) \). Suppose \( f \) is of the form \( f = f_1 x + f_2 x^2 + f_3 x^3 \). Then a solution \( u = o(x^{-1}) \) is of the form

\[
u = 1 + u_1 x + u_2 x^2 + u_3 x^3 \ln x + u_4 x^3 \text{ higher order terms,}
\]

where \( u_{3,1} = cf_3 \) for some explicit constant \( c \). This example reflects the fact, cf. 4-3, 4-4, 4-6 that solutions to degenerate elliptic systems are in general nonsmooth.
at \( \partial M \), instead the general form of the solution has a polylogarithmic expansion of the form
\[ u = \sum u_{i,j} x^i \ln^j x. \]
In case the RHS and the coefficients are smooth, the logarithm terms in \( u \) appear first at the critical exponent \( \alpha_+ \).

The terms in the expansion of \( u \) up to and including the first logarithm term are computable in terms of the data, which implies that explicit geometric conditions for the regularity of solutions of the conformal constraint equations \( \mathcal{E} \), and hence of the physical spacetime constructed from these data, can be found.

4.3. Regularity of solutions to the conformal constraint equations. We will now discuss the detailed conditions for regularity of the conformal data \((g_{ab}, K_{ab}, \Omega)\) at \( \partial M \). We will assume that free data \((g_{ij}, A_{ij}) \in C^\infty(\bar{M})\) are given and consider the conditions needed for the solution \((g_{ij}, K_{ij}, \omega)\) of the conformal constraint equations, produced by solving the system \( \mathcal{E} \), to be in \( C^\infty(\bar{M}) \).

In the case when \( M \) is a moment of time symmetry in the unphysical space–time, i.e. \( A_{ij} = K_{ij} = 0 \), there is a simple criterion for regularity of the solution to the constraint equations. Let
\[ \lambda^\ast_{AB} = \lambda_{AB} - \frac{1}{2} h^{CD} \lambda_{CD} \lambda_{AB} \]
denote the trace free part of \( \lambda_{AB} \). Then the solution to the conformal constraint equation is in \( C^\infty(\bar{M}) \) if and only if the conformal density \( \mathcal{C} \) on \( \partial M \), defined by
\[ \mathcal{C} = \nabla^A \nabla^B \lambda_{AB} + \lambda^*_{AB} R_{AB} - \frac{1}{2} h^{CD} \lambda_{CD} \lambda_{AB} \]
vanishes. In particular, this holds if \( \partial M \) is totally umbilic, i.e. if
\[ \lambda_{AB} - \frac{1}{2} h^{CD} \lambda_{CD} \lambda_{AB} \big|_{\partial M} = 0. \]
In general if this condition does not hold, then the solution has an expansion in powers of \( x \) and \( \ln x \) near \( \partial M \) and is thus of finite regularity at \( \partial M \).

In the case where \( K_{ij} \neq 0 \) and consequently \( A_{ij} \neq 0 \), we make the following simplifying assumptions. We consider shear free data \((g_{ij}, A_{ij})\), such that
\[
\begin{align*}
(8a) & \quad h^{AB} \lambda_{AB} \big|_{\partial M} = 0, \\
(8b) & \quad A_{ij} \big|_{\partial M} = 0.
\end{align*}
\]
Equation \((8a)\) should be viewed as a conformal gauge condition, which can always be satisfied if the data \((g_{ij}, A_{ij})\) is smooth up to boundary. Similarly, equation \((8b)\) may be viewed as a gauge type condition as it is possible to achieve this by subtracting a term \( \rho^{-1} L_{ij} \) from \( A_{ij} \). If \((8)\) holds, then the fields \( \omega, Y^i \) satisfy
\[
\begin{align*}
(9a) & \quad \omega = |\Sigma| x + O(x^3), \\
(9b) & \quad Y^i = O(x^3),
\end{align*}
\]
and \( K_{ab} \) is determined up to first order by \( A_{ab} \). If \((9)\) holds, then the fields \( \omega, K_{ij}, \omega^{-1} C_{\alpha\beta\gamma\delta} \) are in \( C^\infty(\bar{M}) \) if and only if the equations
\[
\begin{align*}
(10a) & \quad \partial_x K_{AB} - \frac{1}{2} h^{CD} \partial_x K_{CD} h_{AB} = 0, \\
(10b) & \quad \nabla^A \nabla^B \lambda_{AB} + R_{AB} \lambda^{AB} = 0, \\
(10c) & \quad \nabla^B (\partial_x K_{AB} + \frac{3}{2} \lambda_{CD} \lambda^{CD} h_{AB}) = 0,
\end{align*}
\]
8 LARS ANDERSSON

cf. [1, Eq. (4.21)–(4.23), (4.33)], hold on \( \partial M \). Note that our sign conventions for \( \epsilon \) and \( K_{ab} \) are opposite those of [1]. The particular form we present here holds assuming (8). Equation (10a), which states that \( \partial_x K_{AB} \) is proportional to \( h_{AB} \) on \( \partial M \), is a consequence of the vanishing of the Weyl tensor on \( \partial M \), while equations (10b) and (10c) follow from smoothness of \( K_{ab} \) up to \( \partial M \). In particular, it can be shown that if \( \partial M \approx S^2 \), then

\[
(11) \quad \left( \partial_x K_{AB} + \frac{3}{2} \lambda^{CD} \lambda_{CD} h_{AB} \right) \bigg|_{\partial M} = 0.
\]

5. The initial value problem

In this section we will consider the Einstein evolution equations directly from the point of view of the unphysical Cauchy data. Since we know from the work of Friedrich [10] that the maximal vacuum extension of conformally regular data on \( M \) has regular conformal boundary near \( \partial M \), it follows that solutions to a well–posed formulation of the Einstein evolution equations in the unphysical space–time will have this property. Working directly in terms of the unphysical Cauchy data allows one to relate this problem to the extensive literature on the numerical solution of the Einstein evolution equations. In particular, it is interesting to consider gauge choices and hyperbolic reformulations.

Consider a foliation \( M_t \) of \((M, g)\) with \( \partial_t = N T + X \). The structure equations of the foliation imply the Einstein evolution equations

\[
(12a) \quad \partial_t g_{ij} = -2N K_{ij} + \mathcal{L}_X g_{ij},
\]
\[
(12b) \quad \partial_t K_{ij} = -\nabla_i \nabla_j N + N (R_{ij} + tr K_{ij} - 2K_{im} K^m_j - R_{ij}) + \mathcal{L}_X K_{ij},
\]

and the constraint equations

\[
(13a) \quad R - |K|^2 + (tr K)^2 = 2R_{TT},
\]
\[
(13b) \quad \nabla_i tr K - \nabla^j K_{ij} = R_{Ti}.
\]

The unphysical Ricci tensor \( R_{\alpha\beta} \) is given by

\[
R_{\alpha\beta} = \tilde{R}_{\alpha\beta} - \Omega^{-1} [2 D_\alpha D_\beta \Omega + D_\alpha D_\gamma \Omega g_{\gamma\beta}] + 3\Omega^{-2} D_\alpha \Omega D_\gamma \Omega g_{\gamma\beta},
\]

the unphysical Ricci scalar is \( R = g^{\alpha\beta} R_{\alpha\beta} \), and \( |K|^2 = K_{ij} K^{ij} \). The condition \( D_T \Omega = \Sigma \) may be viewed as an evolution equation for \( \Omega \),

\[
(13c) \quad \partial_t \Omega = N \Sigma + X \Omega.
\]

5.1. Gauge condition at \( \partial M \). It is natural to require \( \partial_t \Omega \big|_{\partial M} = 0 \). From (13c) and \( \omega = |\Sigma x + h.o., \partial_t \Omega \big|_{\partial M} = N \Sigma + |\Sigma| \big|_{\partial M} \), so \( \partial_t \Omega \big|_{\partial M} = 0 \) implies

\[
N \big|_{\partial M} = (X, \eta) \big|_{\partial M}.
\]

In view of this, a natural boundary condition for \( N \) and \( X \) is

\[
(14) \quad N \big|_{\partial M} = 1, \quad X \big|_{\partial M} = \eta,
\]

which corresponds to the asymptotic behavior of Lapse and Shift in the second conformal compactification discussed in section 3. In a neighborhood of \( \partial M \), we may decompose \( X \) as \( X = \alpha \eta + \beta \). Then the boundary condition for \( X \) is \( \alpha = 1, \beta = 0 \) on \( \partial M \).
5.2. Evolution at $\partial M$. As discussed in [4, §5], the shear free condition is necessary in order for the development of the data on $M$ to have a regular conformal boundary. It is convenient to introduce the notation $\lambda = h^{CD}\lambda_{CD}$, $\kappa = h^{CD}K_{CD}$. We relax the condition $\lambda|_{\partial M} = 0$ used in section 4.3.

Imposing the boundary gauge condition (14), a calculation shows that $\partial_t h_{AB}|_{\partial M} = -(\kappa + \lambda)h_{AB}|_{\partial M}$. In particular, $\partial_t h_{AB}|_{\partial M}$ is pure trace. For a symmetric tensor $t_{AB}$, let $t^*_{AB} = t_{AB} - \frac{1}{2}h^{CD}t_{CD}h_{AB}$ denote the trace free part. Define the shear tensor $\sigma_{AB}$ by $\sigma_{AB} = \lambda^*_{AB} + K^*_{AB}$, so that the shear free condition can be formulated as $\sigma_{AB}|_{\partial M} = 0$.

From the fact that $\partial_t h_{AB}|_{\partial M}$ is pure trace together with the fact that $M$ is 3–dimensional, it follows that $\partial_t \sigma_{AB}|_{\partial M} = [\partial_t (K_{AB} + \lambda_{AB})]^*$. Thus, if $\partial_t (K_{AB} + \lambda_{AB})|_{\partial M}$ is pure trace, then the shear free condition is conserved by the evolution. The evolution equations and the boundary conditions imply

$$\partial_t \lambda_{AB}|_{\partial M} = \partial_x (\lambda_{AB}) + N_1 K_{AB} - \frac{1}{2} \mathcal{L}_{X_1} g_{AB}|_{\partial M},$$

where $N_1 = \partial_x N|_{\partial M}$, $X_1 = [\partial_x, X]|_{\partial M}$. After a lengthy calculation one finds,

$$\partial_t \sigma_{AB}|_{\partial M} = N_1 K^*_{AB} - \frac{1}{2} [\mathcal{L}_{X_1} g_{AB}]^*. $$

The leading order term $N_1$ is determined by the regularity of $R_{\alpha\beta}$ up to $\partial M$. Therefore the condition for preserving the shear free condition under evolution, $\partial_t \sigma_{AB}|_{\partial M} = 0$ shows up as a gauge condition for $X$. Due to the shear free condition, the equation

$$N_1 K^*_{AB} - \frac{1}{2} [\mathcal{L}_{X_1} g_{AB}]^*|_{\partial M} = 0,$$

can be solved by putting $X_1|_{\partial M} = N_1 \eta$, in other words by imposing the condition that $X$ is parallel to $\eta$ at $\partial M$ to first order, and that $\partial_t$ is null to first order.

6. Discussion

In this note, I have shown how to construct solutions of the conformal constraint equations, and indicated the first steps in analyzing the evolution problem for the Einstein vacuum evolution equations, directly in the unphysical setting. It seems worthwhile to explore gauge choices, hyperbolic reformulations, boundary conditions etc. for the evolution equations directly in this setting. This makes it possible to tie in to the extensive literature on the standard form of the Einstein evolution equations. As an example, it is interesting to look at reformulations of the Einstein equations, derived by adding multiples of the constraints in the evolution equations, which have improved stability of the constraints, see for example [11]. It is natural to ask here if there are such reformulations which are also better behaved at $\mathcal{I}$. The gauge conditions that have been considered so far in the literature on the numerical solution of the Einstein equations are mainly hyperbolic in nature, and
therefore suffer from well known problems such as gauge singularities [2, 1]. It is known that CMC time gauge gives a well posed form of the Einstein evolution equations. Therefore it seems natural to consider this time gauge also for the hyperboloidal IVP.

Choosing a time gauge condition such as the CMC condition, leaves open the choice of spatial gauge. It seems likely that it is advantageous to work with a “fully gauge fixed formulation” for numerical work. The combination of CMC time gauge and harmonic spatial gauge gives a well posed elliptic–hyperbolic system in case of compact Cauchy surface [3]. Does this gauge condition give a well posed system also for the hyperboloidal IVP, with boundary conditions given by a foliation of $\mathcal{I}$? The defining equation for the Shift vector field given in [3] is derived from the evolution equation for the metric and therefore does not depend on the matter content of the spacetime. This makes it possible to apply this formulation both in the physical and unphysical spacetime. It is an interesting problem to explore the consequences of this gauge choice for the hyperboloidal IVP, both from the point of view of the physical and unphysical formulation of the IVP and gauge conditions.

Acknowledgements. This paper is based on a talk given at the Workshop on the Conformal Structure of Space-Times held at Tübingen, April 2–4, 2001. I am grateful to the organizers, Jörg Frauendiener and Helmut Friedrich, for their hospitality and support. This work is supported in part by the Swedish Natural Sciences Research Council (SNSRC), contract no. R-RA 4873-307 and NSF, contract no. DMS 0104402.

References

[1] Miguel Alcubierre, Appearance of coordinate shocks in hyperbolic formalisms of general relativity, Phys. Rev. D (3) 55 (1997), no. 10, 5981–5991.
[2] Miguel Alcubierre and Joan Massó, Pathologies of hyperbolic gauges in general relativity and other field theories, Phys. Rev. D (3) 57 (1998), no. 8, R4511–R4515.
[3] Lars Andersson and Piotr T. Chruściel, Hyperboloidal Cauchy data for vacuum Einstein equations and obstructions to smoothness of null infinity, Phys. Rev. Lett. 70 (1993), no. 19, 2829–2832.
[4] ______, On “hyperboloidal” Cauchy data for vacuum Einstein equations and obstructions to smoothness of scri, Comm. Math. Phys. 161 (1994), no. 3, 533–568.
[5] ______, Solutions of the constraint equations in general relativity satisfying “hyperboloidal boundary conditions”, Dissertationes Math. (Rozprawy Mat.) 355 (1996), 100.
[6] Lars Andersson, Piotr T. Chruściel, and Helmut Friedrich, On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations, Comm. Math. Phys. 149 (1992), no. 3, 587–612.
[7] Lars Andersson and Mirta S. İriondo, Existence of constant mean curvature hypersurfaces in asymptotically flat spacetimes, Ann. Global Anal. Geom. 17 (1999), no. 6, 503–538.
[8] Lars Andersson and Vincent Moncrief, Elliptic–hyperbolic systems and the Einstein equations, gr-qc/0110111, 2001.
[9] Jörg Frauendiener, Conformal infinity, Living Rev. Relativ. 3 (2000), 2000–4, 92 pp. (electronic).
[10] Helmut Friedrich, Cauchy problems for the conformal vacuum field equations in general relativity, Comm. Math. Phys. 91 (1983), no. 4, 445–472.
[11] Simonetta Frittelli and Oscar A. Reula, Well-posed forms of the 3+1 conformally-decomposed Einstein equations, J. Math. Phys. 40 (1999), no. 10, 5143–5156.
[12] Peter Hübner, How to avoid artificial boundaries in the numerical calculation of black hole spacetimes, Classical Quantum Gravity 16 (1999), no. 7, 2145–2164.
[13] ______, A scheme to numerically evolve data for the conformal Einstein equation, Classical Quantum Gravity 16 (1999), no. 9, 2823–2843.
[14] __________, *From now to timelike infinity on a finite grid*, Classical Quantum Gravity **18** (2001), no. 10, 1871–1884.

[15] __________, *Numerical calculation of conformally smooth hyperboloidal data*, Classical Quantum Gravity **18** (2001), no. 8, 1421–1440.

[16] Roger Penrose and Wolfgang Rindler, *Spinors and space-time. Vol. 2*, second ed., Cambridge University Press, Cambridge, 1988, Spinor and twistor methods in space-time geometry.

Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA