Application and Implementation of Steinmetz Solid

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Abstract. This article will start from how to calculate the volume of Steinmetz Solid by double integration, and expand to formation, cross section and unfolding of it by Matlab in a brand new way, in order to explore the internal structure and application. Further, we expand to polygonal Steinmetz Solid by Matlab to acquire more comprehensive understanding and mastery of it.

1. Introduction
The earliest idea of Steinmetz Solid was derived from the calculation of the volume of the sphere, and was first discovered by the ancient Chinese mathematician Liu Hui⁴. He created such unique 3D geometry, each of which was square in cross-sections and was circumscribed to the circle of the sphere at the same height. Since the area ratio of a circle to its circumscribed square is $\pi : 4$, the ratio of the volume of Steinmetz Solid to the volume of the inscribed sphere is $4 : \pi$⁵. Therefore, as long as the volume of Steinmetz Solid can be solved, the volume of the corresponding sphere can be calculated. In recent years, Steinmetz Solid also has relevant research in medicine⁶. This article will start from the question of how to generate the Steinmetz Solid, and its cross-sections and how to unfold the Steinmetz Solid by Matlab to acquire more comprehensive understanding and mastery of it.

2. Double Integration
Steinmetz Solid $V$ is shown as below. According to the symmetry of the coordinate axis, it can be divided into 8 parts on the basis of the volume of the first limit portion $V_1$, that is $V = 8V_1$, then the volume on the first limit portion $V_1$ can be calculated using double integration⁷,⁸.

According to the projection on the $xoy$ plane, the volume of Steinmetz Solid can be calculated:
\[ V_1 = \int_0^R \int_0^{\sqrt{R^2 - x^2}} \int_0^{\sqrt{R^2 - y^2}} \int_0^{\sqrt{R^2 - z^2}} \sqrt{R^2 - x^2} \, dy \, dx = \frac{2}{3} R^3, \quad V = 8V_1 = \frac{16}{3} R^3. \]

3. Generation of Steinmetz Solid

3.1. Intersection equation of Steinmetz Solid

From another point of view, the generation of Steinmetz Solid can theoretically be regarded as a 3D figure formed by the intersection of two cylinders. The problem with the intersection of two cylinders can be determined by the three views of Steinmetz Solid.

![Different Views of Steinmetz Solid](image1)

Since the left view and the top view of Steinmetz Solid are both circular, the coordinate equation of Steinmetz Solid on the \( xoy \) plane and the \( xoz \) plane can be determined, and both should satisfy the parameter equation of the circle. Since the two intersection lines of the main view are straight and at an angle of 45 degrees to the horizontal plane, Steinmetz Solid has the following on the \( yoz \) plane:

\[ yz = 0 \quad \text{or} \quad yz = -6. \]

Therefore, the parametric equation of the intersection of Steinmetz Solid is:

\[
\begin{cases} 
  x = R \cos \theta \\
  y = R \sin \theta \\
  z = R \sin \theta 
\end{cases} \quad \theta \in (0, \pi), \]

\[
\begin{cases} 
  x = R \cos \theta \\
  y = R \sin \theta \\
  z = -R \sin \theta 
\end{cases} \quad \theta \in (0, 2\pi). \]

Where \( R \) is the radius of the circle and \( \theta \) is the central angle. Matlab code about generation of the intersection point is as follows:

```matlab
ntL=15; R=2; tL12=linspace(0,pi,ntL); tL34=linspace(pi,2*pi,ntL);
xL1=R*cos(tL12);yL1=R*sin(tL12);zL1=R*sin(tL12);
xL2=R*cos(tL12);yL2=R*sin(tL12);zL2=-R*sin(tL12);
xL3=R*cos(tL34);yL3=R*sin(tL34);zL3=R*sin(tL34);
xL4=R*cos(tL34);yL4=R*sin(tL34);zL4=-R*sin(tL34);
```

3.2. Cylindrical equation of Steinmetz Solid

Lateral cylinder equation:

\[
\begin{cases} 
  x = R \cos \theta \\
  y = h \\
  z = R \sin \theta 
\end{cases} \quad \theta \in (0, 2\pi), \]

in the process of generating the cylinder, we use the method of starting from the intersection line and gradually drawing points to the sides.

As for the lateral cylinder parallel to the \( xoy \) plane, start from a certain intersection line, in the same \( xoy \) plane, the coordinate values of the lateral cylinder in \( x \) and \( z \) axis on both sides are unchanged. For the coordinate value in \( y \) axis, plot points between each interval, and the coordinate value of the point at both ends of a given cylinder in \( y \) axis is \([-6, 6]\), that is \( h = 12 \), and finally the cylinder corresponding to Steinmetz Solid is generated.

![Steinmetz Solid Is Generated by Cylinders](image2)
Matlab code about generation of the cylinder point is as follows:

\[
\begin{align*}
X_{d1a} &= [1; 1] \cdot x_{L1}; \\
Y_{d1a} &= y_{L1}; 3R \cdot \text{ones(size(y_{L1}))}; \\
Z_{d1a} &= [1; 1] \cdot z_{L1}; \\
X_{d2a} &= [1; 1] \cdot x_{L2}; \\
Y_{d2a} &= y_{L2}; 3R \cdot \text{ones(size(y_{L2}))}; \\
Z_{d2a} &= [1; 1] \cdot z_{L2}; \\
X_{d3a} &= [1; 1] \cdot x_{L3}; \\
Y_{d3a} &= y_{L3}; -3R \cdot \text{ones(size(y_{L3}))}; \\
Z_{d3a} &= [1; 1] \cdot z_{L3}; \\
X_{d4a} &= [1; 1] \cdot x_{L4}; \\
Y_{d4a} &= y_{L4}; -3R \cdot \text{ones(size(y_{L4}))}; \\
Z_{d4a} &= [1; 1] \cdot z_{L4};
\end{align*}
\]

3.3. Surface equation of Steinmetz Solid
Since the intersection coordinate equation of Steinmetz Solid has been obtained, in the process of obtaining the surface equation, if the corresponding points on the intersection line are filled, the surface of Steinmetz Solid can be obtained.

\[
\begin{align*}
X_{p1a} &= [1; 1] \cdot x_{L1}; \\
Y_{p1a} &= [1; -1] \cdot y_{L1}; \\
Z_{p1a} &= [1; 1] \cdot z_{L1}; \\
X_{p1b} &= [1; 1] \cdot x_{L1}; \\
Y_{p1b} &= [1; 1] \cdot y_{L1}; \\
Z_{p1b} &= [1; -1] \cdot z_{L1}; \\
X_{p2a} &= [1; 1] \cdot x_{L4}; \\
Y_{p2a} &= [1; -1] \cdot y_{L4}; \\
Z_{p2a} &= [1; 1] \cdot z_{L4}; \\
X_{p2b} &= [1; 1] \cdot x_{L4}; \\
Y_{p2b} &= [1; 1] \cdot y_{L4}; \\
Z_{p2b} &= [1; -1] \cdot z_{L4};
\end{align*}
\]

3.4. Flow chart of Steinmetz Solid
Finally, according to the above process, the flow chart of the animation of Steinmetz Solid can be obtained as follows:

4. Application of Steinmetz Solid

4.1. Cross-section area of Steinmetz Solid
In order to obtain the cross-section area of Steinmetz Solid, you need to pay attention to the three views. Since the shape of the main view is a square, the shape of the side view and the top view is a circle, so the cross section of Steinmetz Solid should be related to both.

As seen from the above figure, the cross section from one end to the other is constantly changing. Since the inside can fill a sphere, its cross section is a square concluding an inscribed circle.

When the cross section is changing, the maximum radius of the inscribed circle is taken as the radius of the inscribed sphere. At this time, the inner square and the outer square coincide. Then, after passing through the section at the maximum radius, the section continues to shrink until a point.

The animation demo of the cross-section area of Steinmetz Solid is as follows:
Figure 4.2 The Animation Demo of Cross-section Area of Steinmetz Solid
Given the radius of the inscribed sphere is $R$, the intersection equation of $x$ and $y$ axis is $x = R\cos \theta$, $y = R\sin \theta$. Further analysis, the intersection equation of $x$ axis is consistent with the section circle of it, and the radius of section circle is consistent with the intersection equation of $y$ axis. Therefore, the cross section equation is $x = x_i$, $y = y_i \cos \phi$, $z = y_i \sin \phi$.

According to the above calculation process, the flow chart of the cross-section area generation animation of Steinmetz Solid can be obtained as follows:

![Flow Chart of Cross-section Area of Steinmetz Solid](image)

Matlab code about generation of cross section is as follows:
```matlab
ntLd=31; R=2; tLd=linspace(0,pi,ntLd); xLd=R*cos(tLd); yLd=R*sin(tLd);
for i=1:ntLd
    ntr=51; tr=linspace(0,2*pi,ntr);
    xr=xLd(i); ry=yLd(i); yr=ry*cos(tr); zr=ry*sin(tr);
end
```

Based on theory and double integration, the area of Steinmetz Solid can be determined.

![The Structure of the Area of Steinmetz Solid](image)

From the structure of Steinmetz Solid, the area is equal to 16 times of the right figure, that is $S = 16A$.

About $A$, the equation is $z = \sqrt{R^2 - x^2}$, then $z_i' = \frac{-x}{\sqrt{R^2 - x^2}}$, $z_i'' = 0$, $\sqrt{1 + z_i'^2 + z_i''^2} = \frac{R}{\sqrt{R^2 - x^2}}$.

Thus $A = \iint_{D} \frac{R}{\sqrt{R^2 - x^2}} \, dx \, dy = \int_{0}^{R} \int_{0}^{\sqrt{R^2 - x^2}} \frac{R}{\sqrt{R^2 - x^2}} \, dy \, dx = \int_{0}^{R} R \, dx = R^2$, $S = 16A = 16R^2$.

4.2. Polygonal Steinmetz Solid
We take the example of a pentagon and use the section accumulation method in 4.1 to explore how to generate a polygonal Steinmetz Solid, and display and analyze the variation of the section and area. Comparing the similarities and differences between pentagon and square, the method of generating sections is similar. The difference lies in the coordinates of the section polygon. As for pentagon, given the radius of inscribed sphere $R$, the intersection equation of $x$ and $y$ axis is also $x_i = R\cos \theta$,
\[y_i = R \sin \theta,\] and the cross section equation is \[x_i = x_i, \quad y = y_i \cos \phi, \quad z = y_i \sin \phi.\] Known that the radius of a pentagon inscribed circle (section circle) is \(y_i\), so the radius of the pentagon is \(R_a = \frac{y_i}{\cos \left(\frac{\alpha}{2}\right)}\), and the cross section equation is \[x_i = x_i, \quad y = R_a \cos \gamma, \quad z = R_a \sin \gamma.\]

**Figure 4.5** Flow Chart of Polygonal Steinmetz Solid

**Figure 4.6** Schematic Diagram of Polygonal Steinmetz Solid

Matlab code about generation of cross section and pentagon is as follows:

```matlab
for i=1:ntLd % Section coordinate
    rx=xLd(i); ry=yLd(i); yr=ry*cos(tr); zr=ry*sin(tr); % Radius and circle coordinates of the circle
    Ra=ry/cos(dsit/2); xps1=rx*ones(1,n+1); yps1=Ra*cos(sits); % Vertex sequence
end
```

4.3. **Unfolding of Steinmetz Solid**

To unfold the pentagon Steinmetz Solid, it is expanded by a spherical surface. You need to pay attention to the two parts: one is the angle \(\beta\) that is rotated longitudinally from \(\pi\) to 0, and the other is the angle \(\omega\) that gradually expands laterally. For the middle line of each two intersection lines, by splitting in the vertical and horizontal directions, the coordinate value in \(x\) axis is \(x = R \cos(\pi - \beta + \omega)\), since the longitudinal angle \(\pi - \beta\) is gradually reduced, and the lateral radius is incremented based on the upper layer. Therefore, the radius of the circle parallel to the \(yoz\) plane is \(r = R(\pi - \beta) + R \sin(\pi - \beta + \omega)\).

Since the expanded contour of vertices remains pentagon, according to the calculation of vertices of polygon, the \(x\) and \(y\) coordinates of the middle line of each two intersection lines can be obtained.

Polygon radius is \(R_a = \frac{r}{\cos \left(\frac{\alpha}{2}\right)}\), where \(\alpha\) is the angle between every two intersections after expansion.

The middle line coordinate of each two intersection lines when expanding is \(x = a \cos(\pi - \beta + \omega),\) \(y = R_a \cos \gamma,\) \(z = R_a \sin \gamma.\)
The following discussion is how to draw each part after expanding. According to the triangle relationship, the side length of polygon on the $yoz$ plane is $dL = 2R_s \tan \left( \frac{\alpha}{2} \right)$, so the coordinate points of each intersection line after expansion are: $x_p = x$, $y_p = y + \frac{dL}{2} \cos \left( \gamma \pm \frac{\pi}{2} \right)$, $y_p = y + \frac{dL}{2} \cos \left( \gamma \pm \frac{\pi}{2} \right)$.

Figure 4.7 Flow Chart of Unfolding of Steinmetz Solid

Figure 4.8 Schematic Diagram of Unfolding of Steinmetz Solid

Matlab code about unfolding of Steinmetz Solid is as follows:

```matlab
ntLd=31; n=5; dsitT=pi/n; % Central angle of the polygon
sits=sits+dsitT; tLd=linspace(pi,0,ntLd);
for i=1:ntLd % Calculate the coordinates of the middle line of each intersection line when expanding
ti=tLd(i);
for j=i:ntLd
tj=tLd(j);
    XpL(:,j)=R*cos(pi-ti+tj); ryz=(pi-ti)*R+R*sin(pi-ti+tj); % Expanding coordinates and radius
    Ra=ryz/cos(dsit/2); yps1=Ra*cos(sits); zps1=Ra*sin(sits); YpL(:,j)=yps1'; ZpL(:,j)=zps1';
end
for j=1:n
    XpLa=XpL(j,:);YpLa=YpL(j,:)+dL/2*cos(sits(j)+pi/2); ZpLa=ZpL(j,:)+dL/2*sin(sits(j)+pi/2);
    XpLb=XpL(j,:);YpLb=YpL(j,:)+dL/2*cos(sits(j)-pi/2); ZpLb=ZpL(j,:)+dL/2*sin(sits(j)-pi/2);
end
```

5. Conclusion
Starting from double integration, this paper deduces the solution to the volume of Steinmetz Solid. By Matlab programming, the generation and cross section of it are realized in a new method. Extending to polygon, we use Matlab to realize the generation of polygonal Steinmetz Solid and cross section. At the same time, Mapping 3D to 2D, the expansion of Steinmetz Solid is realized by Matlab, therefore, a more complete understanding of Steinmetz Solid can be acquired.
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References
[1] Wei Zhang. The Origin and Proof of Zu Geng Principle [J]. Journal of Chongqing Second Normal University, 2010, 23(3):113-115.
[2] Qiuhai Wang, Shaoquan Chen. A Comparative Study of Volume Calculation Methods between China and the West [J]. Philosophy of Science and Technology, 1996(1):24-28.
[3] Mao Yujiang, Song Jie, Wei Jie, Wang Manyi. Prevention of unrecognized joint penetration during internal fixation of hip fractures: a geometric model based on Steinmetz Solid [J]. Hip International: the journal of clinical and experimental research on hip pathology and therapy, 2010, 20(4).
[4] Donghai Zhang. Parametric Equations of Steinmetz Solid and Its Application in GeoGebra [J]. Middle School Mathematics Monthly, 2015(06):48-50.
[5] Zhiling Yuan, Shuhuan Lu. Inquiry Teaching Design Based on Mathematical Culture—Zu Geng Principle and Sphere Volume [J]. Mathematics Teaching Research, 2007(10):12-15.