Non-Abelian $k$-Vortex Dynamics in $\mathcal{N} = 1^*$ theory and its Gravity Dual

Roberto Auzzi and S. Prem Kumar
Department of Physics, Swansea University, Singleton Park, Swansea
SA2 8PP, U.K.
E-mail: r.auzzi@swansea.ac.uk, s.p.kumar@swansea.ac.uk

Abstract: We study magnetic flux tubes in the Higgs vacuum of the $\mathcal{N} = 1^*$ mass deformation of $SU(N_c)$, $\mathcal{N} = 4$ SYM and its large $N_c$ string dual, the Polchinski-Strassler geometry. Choosing equal masses for the three adjoint chiral multiplets, for all $N_c$ we identify a “colour-flavour locked” symmetry, $SO(3)_C + F$ which leaves the Higgs vacuum invariant. At weak coupling, we find explicit non-Abelian $k$-vortex solutions carrying a $\mathbb{Z}_{N_c}$-valued magnetic flux, with winding, $0 < k < N_c$. These $k$-strings spontaneously break $SO(3)_{C+F}$ to $U(1)_{C+F}$ resulting in an $S^2$ moduli space of solutions. The world-sheet sigma model is a nonsupersymmetric $\mathbb{C}P^1$ model with a theta angle $\theta_{1+1} = k(N_c - k)\theta_{3+1}$ where $\theta_{3+1}$ is the Yang-Mills vacuum angle. We find numerically that $k$-vortex tensions follow the Casimir scaling law $T_k \propto k(N_c - k)$ for large $N_c$. In the large $N_c$ IIB string dual, the $SO(3)_{C+F}$ symmetry is manifest in the geometry interpolating between $AdS_5 \times S^5$ and the interior metric due to a single D5-brane carrying D3-brane charge. We identify candidate $k$-vortices as expanded probe D3-branes formed from a collection of $k$ D-strings. The resulting $k$-vortex tension exhibits precise Casimir scaling, and the effective world-sheet theta angle matches the semiclassical result. S-duality maps the Higgs to the confining phase so that confining string tensions at strong 't Hooft coupling also exhibit Casimir scaling in $\mathcal{N} = 1^*$ theory in the large $N_c$ limit.
1. Introduction

Supersymmetric gauge theories, following the works of [1], have provided a large class of examples where condensation of monopoles is the mechanism for confinement of electric charges. Softly broken $\mathcal{N} = 2$ supersymmetric gauge theories confine via a magnetic version of the Abelian Higgs mechanism. In these theories the confined, heavy, coloured sources are held together by Abelian strings (Abrikosov-Nielsen-Olesen solitons [2, 3]). In contrast, in pure Yang-Mills theory with $SU(N_c)$ gauge group for instance, heavy external charges are expected to be confined by chromoelectric flux tubes which annihilate in groups of $N_c$, which we refer to as $\mathbb{Z}_{N_c}$-strings. One example where the dynamics of $\mathbb{Z}_{N_c}$ strings may be accessed at weak coupling, is presented by the so-called $\mathcal{N} = 1^*$ theory [4, 5, 6, 7, 8, 9] which is a mass deformation of $\mathcal{N} = 4$ theory preserving $\mathcal{N} = 1$ supersymmetry (SUSY). What makes this theory particularly interesting is that it also has a known large $N_c$ string dual [8] which brings with it the possibility of exploring flux tube dynamics in the large $N_c$ limit.

The $\mathcal{N} = 1^*$ theory has extremely rich infrared dynamics and beautiful phase structure, made possible in part by the Olive-Montonen electric-magnetic duality [10] (enlarged to $SL(2, \mathbb{Z})$) which it inherits from its parent $\mathcal{N} = 4$ theory. For example, the theory with $SU(N_c)$ gauge group has a large number of vacua with a mass gap, each of which is realized in a distinct phase. The action of $SL(2, \mathbb{Z})$ exchanges and permutes these vacua. The vacuum in the Higgs phase, where the gauge group is broken completely, is mapped by S-duality (inversion of the gauge coupling) onto the confining vacuum where the gauge group is classically unbroken and the theory confines in the infrared (IR). The $\mathbb{Z}_{N_c}$ chromoelectric flux tubes in the confined phase at strong gauge coupling, get directly mapped to magnetic flux tubes in the Higgs vacuum at weak coupling. At weak coupling, the Higgs vacuum is semiclassical and hence the physics of the associated flux tubes is accessible. The study of these for general $N_c$, and particularly their large $N_c$ gravity duals, will be the subject of this paper.

In recent years, certain special flux tubes at weak coupling have been encountered in gauge theories (with and without SUSY) with $U(N_c)$ gauge group and $N_f$ flavours in the fundamental representation [11, 12] and extensively studied therein [13, 14, 15]. The crucial feature of all these strings at weak coupling is the presence of orientational moduli associated with rotations within a colour-flavour locked symmetry. We will refer to these as “non-Abelian” flux tubes. The interested reader can find reviews in [16, 17, 18, 19]. In the context of $\mathcal{N} = 1^*$ theory with $SU(2)$ gauge group, non-Abelian vortices in the Higgs vacuum were constructed and studied first in [20].
The basic example of the non-Abelian strings is in the context of $\mathcal{N} = 2$, $U(N_c)$ gauge theory with $N_f$ flavours and $N_f = N_c = N$ and a Fayet-Iliopolous term. In this case there is an $SU(N)_{C+F}$ symmetry which is left unbroken by the vacuum; the vortex soliton breaks this symmetry to $(SU(N-1) \times U(1))_{C+F}$. The vortex internal space is then parameterized by

$$\mathcal{M}^{N=2} = \frac{SU(N)_{C+F}}{(SU(N-1) \times U(1))_{C+F}} = \mathbb{CP}^{N-1}.$$ 

In this paper we will study a similar “colour-flavour locked” symmetry that appears in the Higgs vacuum of $\mathcal{N} = 1^*$ theory with $SU(N_c)$ gauge group. When the masses of the three adjoint $\mathcal{N} = 1$ chiral multiplets in the theory are chosen to be equal, an $SO(3)$ subgroup of the original global $SO(6)_R$ symmetry of $\mathcal{N} = 4$ theory is left unbroken. The VEVs of the scalar fields in this phase are proportional to $N_c$ dimensional $SU(2)$ generators. This fact allows to find a specific combination of the global $SO(3)$ and colour generators, that are left unbroken by the VEVs of the adjoint scalars. We denote this combined colour-flavour symmetry as $SO(3)_{C+F}$.

Since all fields in the theory are in the adjoint representation of the gauge group, the topologically stable flux tubes are classified by a $\mathbb{Z}_{N_c}$ quantum number $k = 1, 2, \ldots, N_c - 1$.

In the first part of this paper we find a general ansatz for the $k$-vortex solution, generalizing the $N_c = 2$, $k = 1$ case studied in [20]. The ansatz is given in explicit form for $N_c = 3, 4$ and a natural algorithm for higher rank gauge groups presents itself. Since the equations of motion are not analytically tractable, a numerical solution of the vortex profile functions is necessary. We were able to perform the numerical computations for $k$-vortices in theories with $2 \leq N_c \leq 6$.

The $k$-vortex solution breaks the $SO(3)_{C+F}$ symmetry to $U(1)_{C+F}$, so that the vortex internal moduli space (for every $k$) is parameterized by

$$\mathcal{M}^{N=1^*} = \frac{SO(3)_{C+F}}{U(1)_{C+F}} = \mathbb{CP}^{1}.$$ 

Acting on a given $k$-string solution with the broken symmetry generators rotates the orientation of the non-Abelian magnetic flux within the colour space. A crucial difference between the vortices in $\mathcal{N} = 1^*$ theory and those in theories with $\mathcal{N} = 2$ SUSY, is that the latter are BPS solutions. With $SO(3)$ symmetric masses, the $\mathcal{N} = 1^*$ vortices are far from BPS and have no fermionic “super-orientational” zero modes.

Footnotes:

1Flux tubes at weak coupling with $\mathbb{Z}_{N_c}$ quantum numbers were also studied in numerous papers (see [21], [22] for an incomplete list).

2There is a limiting regime of mass parameters (two masses equal, and the third being relatively small) where the $\mathcal{N} = 1^*$ theory can be viewed as softly broken $\mathcal{N} = 2^*$ theory, but we will not be particularly interested in this limit. Abelian vortices in softly broken $\mathcal{N} = 2^*$ theory are BPS.
The low-energy effective theory for the fluctuations of the light modes on the $k$-string is determined by performing an adiabatic, world-sheet dependent colour-flavour locked rotation. This excites the internal, orientational zero mode degrees of freedom localised on the vortex. The resulting action is that of a sigma model with $S^2$ as target space and the following Lagrangian,

$$S_{1+1} = \int dz \, dt \left( B_{N_c,k}(\partial_s \vec{n})^2 - \frac{g_{1+1}^{N_c,k}}{8\pi} \epsilon_{abc} n^a \partial_s n^b \partial_r n^c \right),$$  \hspace{1cm} (1.1)$$

where $\vec{n}$ is a position vector on the unit sphere. This is an effective theory with a UV cutoff determined by the vortex thickness. Importantly, the effective theory is an asymptotically free quantum theory and its IR dynamics depends strongly on the vacuum theta angle \([14, 23, 24, 25, 26]\). Therefore, while the four dimensional gauge theory is semiclassical, the vortex theory is highly quantum and becomes strongly interacting. The classical value of the sigma model coupling $B_{N_c,k}$ can be determined in terms of the Yang-Mills coupling $g_{YM}^2$, for all $k$ and $N_c$, by a non-trivial numerical calculation involving the vortex profile functions. The classical sigma model coupling constant turns out to be weak for weak gauge coupling. On the other hand, the effective 2-dimensional $\theta$ angle can be computed analytically in general, to yield a simple, but very interesting result,

$$\theta_{1+1}^{N_c,k} = k(N_c - k)\theta_{3+1},$$  \hspace{1cm} (1.2)$$

where $\theta_{3+1}$ is the vacuum angle of the four dimensional gauge theory. This relation is significant for two reasons. First, it satisfies the basic requirement that the Higgs vacuum should be invariant under shifts of $\theta_{3+1}$ by multiples of $2\pi$. Second, whenever $\theta_{1+1} = \pi$, the world-sheet theory is integrable and flows to a conformal fixed point with massless $SO(3)$ doublets as the only excitations. For all other values of $\theta_{1+1}$ the two dimensional theory develops a mass gap and its only excitations are triplets of $SO(3)$ which may be viewed as confined meson-like states made up of doublets. This in turn implies that there exist various special values for $\theta_{3+1}$, determined by (1.2) for every $k$, at which different $k$-vortex theories flow to an interacting conformal fixed point with central charge $c = 1$.

Since we find the explicit $k$-vortex solutions, albeit numerically, we are in a position to ask how their tensions scale with $N_c$. This is a question that has attracted considerable interest in recent years, from various perspectives \([1, 27, 28, 29, 30]\) for gauge theories with a $\mathbb{Z}_{N_c}$ symmetry. We perform a numerical analysis of the semiclassical $k$-string tensions and their ratios for $N_c = 4, 5, 6$. We find that as $N_c$ is increased, the results are extremely well approximated by a Casimir scaling law with an accuracy better than 0.1%. Although we do not yet have an understanding of the physics behind this result, we are able to confirm that Casimir scaling of the tensions
becomes precise in the large $N_c$ gravity dual. At this point it is worth emphasizing that S-duality maps these Higgs phase results at $g_{YM}^2 \ll 1$ to the confining vacuum at $g_{YM}^2 \gg 1$.

The second part of our paper is devoted to a study of $k$-strings in the Higgs vacuum in the large $N_c$, Type IIB string dual obtained by Polchinski and Strassler [8]. The supergravity background which is dual to the Higgs vacuum, becomes applicable when $N_c \to \infty$ and $N_c/g_{YM}^2 \gg 1$. Since this also includes the regime of weak gauge coupling, we cannot expect supergravity to be valid in the entire geometry as a weakly coupled regime would correspond to large curvatures in the string dual. This also occurs in the Higgs vacuum, where the dual background interpolates between $AdS_5 \times S^5$ asymptotics and a deep interior portion generated by a D5-brane wrapped on a flux supported two-sphere. The D5-brane which carries $N_c$ units of D3-brane charge, makes an appearance due to the Myers effect [31] resulting from the $\mathcal{N} = 1^*$ deformation. In the crossover region, near the D5-brane, the geometry becomes strongly curved and we expect large string corrections. Despite this we can certainly look for candidate probe brane configurations that are expected to be dual to the $k$-vortices of the Higgs vacuum in the large $N_c$ limit. By S-duality, the picture in the Higgs vacuum is exchanged with the confining vacuum at strong ’t Hooft coupling: $N_c/g_{YM}^2 \to g_{YM}^2 N_c \gg 1$ which is the usual condition for the validity of supergravity.

With all the above caveats in mind, we look for our candidate probe branes in the dual geometry. The $SO(3)_{C+F}$ is obvious in the geometry as the sphere wrapped by the D5-brane has an $SO(3)$ isometry in the limit of equal masses for the adjoint chiral multiplets. The $k = 1$ vortex is naturally a probe D1-brane in the Higgs vacuum. In thebrane picture, the D1-brane binds to the D5-brane which has a world-volume $B$-field endowing the 5-brane with D3 charge. This bound state corresponds to a magnetic flux tube. In the gravity picture, the probe D1-brane sits at a radial position near the D5-brane. Despite the possibility of stringy corrections to the background, we use the probe Dirac-Born-Infeld action and the Chern-Simons terms to obtain the effective Lagragian in Eq. (1.1), with

\[ B_{N_c,1} = \frac{\pi N_c}{g_{YM}^2}, \quad \theta_{1+1}^{N_c,1} = N_c \theta_{3+1}. \]  

The value of $\theta_{1+1}^{N_c,1}$ which is found in the dual is consistent with Eq. (1.2) for large $N_c$. The tension of this configuration can also be computed (as originally done in [8]) and yields $T_{k=1} = 2\pi m^2 N_c/g_{YM}^2$.

In order to model the $k$-string with $k \sim \mathcal{O}(N_c)$, motivated by the Myers dielectric effect on a collection of $k$ D-strings in the Higgs vacuum, we use a D3-brane with the topology of $\mathbb{R}^{1,1} \times S^2$, with $k$ units of flux in the compact directions. Crucially,
the primary contribution to the tension of this D3-brane is a disc stretching inside the D5-sphere, a picture that we find to be consistent with the baryon vertex in $\mathcal{N} = 1^*$ theory. From the D3-brane picture we find that the vortex tension follows the Casimir scaling law

$$T_k = 2\pi \frac{m^2}{g_{YM}^2} k(N_c - k).$$

reproducing precisely the semiclassical field theory result which was determined numerically. Most remarkably the Chern-Simons terms of the probe brane also compute the theta angle on the $k$-vortex worldsheet, exactly matching Eq. (1.2), the weak coupling gauge theory result.

The agreement between the gravity dual and semiclassical gauge theory physics is surprising and clearly needs an explanation. An important aspect of the probe brane results from the string dual is that the physical quantities that agree with the gauge theory - the $k$-string tension and the worldsheet theta angle - do not appear to receive significant contributions from the strongly curved parts of the geometry. The D3-brane $k$-string tension arises mainly from a disc-like portion that effectively sees a flat geometry inside the D5-sphere, while the theta angle originates in the Chern-Simons term which is insensitive to the metric. Therefore, we believe that the above picture and results are robust.

The paper is organized as follows. In Sect. 2 we review certain aspects of the $\mathcal{N} = 1^*$ field theory. In Sect. 3 we present solitonic solutions for the $k$-strings and present the results of numerical analysis. In Sect. 4 we derive the vortex worldsheet effective action from a direct calculation. Sect. 5 deals with the probe brane calculation in the Polchinski-Strassler background for the Higgs vacuum. Sect. 6 briefly summarizes our conclusions. Some details of the interior geometry in the Polchinski-Strassler background are presented in an Appendix.

2. The Field Theory setting

In this section we cover some of the basic facts regarding the $\mathcal{N} = 1^*$ field theory with $SU(N_c)$ gauge group. We pay particular attention to the theory in the Higgs phase, and for a specific choice of the mass deformation parameters. The physics in this vacuum is related directly via S-duality to the confining phase of the theory.
2.1 The $\mathcal{N} = 1^*$ deformation of $\mathcal{N} = 4$ SYM

We begin by reviewing the field content and the microscopic Lagrangian of the $\mathcal{N} = 1^*$ theory. In the language of $\mathcal{N} = 1$ supersymmetry, the $\mathcal{N} = 1^*$ theory contains an $\mathcal{N} = 1$ vector multiplet $W_\alpha$ and three chiral multiplets $(\Phi_1, \Phi_2, \Phi_3)$, transforming in the adjoint representation of the gauge group which we take to be $SU(N_c)$. The theory is obtained by a relevant, mass deformation of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The superpotential of $\mathcal{N} = 4$ SYM reads,

$$W = \frac{1}{g_{YM}^2} \text{Tr}([\Phi_1, \Phi_2]\Phi_3).$$

(2.1)

The superpotential can be deformed by adding $\mathcal{N} = 1$ SUSY preserving mass terms for the adjoint matter fields,

$$\Delta W = \frac{1}{g_{YM}^2} \sum_{i=1}^{3} \frac{1}{2} m_i \text{Tr}(\Phi_i^2).$$

(2.2)

This is a relevant deformation and the resulting theory exhibits nontrivial dynamics in the infrared, resulting in a rich phase structure. In the UV however, the theory flows to $\mathcal{N} = 4$ SYM, with the gauge coupling remaining a freely adjustable parameter. Thus the $\mathcal{N} = 1^*$ theory has, in addition to three complex mass parameters, a dimensionless, tunable complexified gauge coupling

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta_{3+1}}{2\pi}.$$  

(2.3)

In Euclidean space, the bosonic part of the action is,

$$S_b^E = \int d^4x \left[ \frac{1}{g_{YM}^2} \left( \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \sum_{j=1}^{3} \text{Tr}[D_\mu \Phi_j]^2 + V_F + V_D \right) + \frac{i}{32\pi^2} F_{\mu\nu}^a F^{a\mu\nu} \right],$$

(2.4)

where we have used the same symbol $\Phi_j$, for the chiral superfields as for their lowest (scalar) components. We define the $SU(N_c)$ generators $T^a$, $(a = 1, 2, \ldots N_c^2 - 1)$, (with $F_{\mu\nu} = T_a F_{a\mu\nu}$), with the usual normalization $\text{Tr}(T_a T_b) = \frac{1}{2} \delta_{ab}$, while the gauge covariant derivative is

$$D_\mu \Phi_k = \partial_\mu \Phi_k - i[A_\mu, \Phi_k].$$

(2.5)

The scalar potential is the sum of $V_F$ and $V_D$, the F and D-term contributions respectively:

$$V_F = \text{Tr}\left(w_1 w_1^\dagger + w_2 w_2^\dagger + w_3 w_3^\dagger\right), \quad w_i = \epsilon_{ijk} \Phi_j \Phi_k + m_i \Phi_i,$$

(2.6)
and
\[
V_D = -\frac{1}{4} \text{Tr} \left( [\Phi_1^\dagger, \Phi_1] + [\Phi_2^\dagger, \Phi_2] + [\Phi_3^\dagger, \Phi_3] \right)^2. \tag{2.7}
\]

In this paper we will be mainly interested in the case where the masses of the three adjoint chiral multiplets are equal:
\[
m_1 = m_2 = m_3 = m. \tag{2.8}
\]

With this choice, the superpotential term \( \Delta W \) breaks the \( SO(6)_R \) global symmetry of the \( \mathcal{N} = 4 \) theory to an \( SO(3) \) subgroup under which the complex chiral multiplets \( (\Phi_1, \Phi_2, \Phi_3) \), transform as a triplet. In the \( \mathcal{N} = 1^* \) theory, this \( SO(3) \) acts as an ordinary global symmetry, and not as an R-symmetry.

### 2.2 Higgs and Confining Vacua

The mass deformation above results in a large set of vacuum configurations determined by the F-flatness conditions (modulo complex gauge transformations),
\[
\Phi_i = -\frac{1}{m} \epsilon_{ijk} \Phi_j \Phi_k. \tag{2.9}
\]

As is well-known \[3, 9\], the solutions to these equations may be classified in terms of all \( N_c \)-dimensional representations the \( SU(2) \) algebra. Each such classical ground state then splits into a certain number quantum vacua depending on the non-Abelian gauge symmetry subgroup left unbroken by the classical solution. The quantum ground states are in one to one correspondence with all possible phases of \( SU(N_c) \) gauge theory with adjoint matter, in four dimensions.

Of particular interest are the Higgs and confining vacua which correspond to the \( N_c \) dimensional irreducible representation and the trivial representation, respectively. The VEVs of the adjoint scalars in the Higgs vacuum are proportional to the generators of the irreducible \( SU(2) \) representation with dimension \( N_c \),
\[
\Phi_l = i m J_l, \quad (l = 1, 2, 3). \tag{2.10}
\]

For generic \( N_c \), the \( SU(2) \) representation is labelled by \( j = \frac{N_c-1}{2} \) with \( J_3 \) chosen to be the usual diagonal matrix
\[
J_3 = \text{diag}(j, j-1, \ldots, -j); \quad j = \frac{N_c-1}{2}. \tag{2.11}
\]

The only non-zero elements of the matrices \( J_1, J_2 \) are off-diagonal, given by
\[
(J_1)_{a, a+1} = (J_1)_{a+1, a} = \frac{\sqrt{a(N_c-a)}}{2}, \quad a = 1, 2, \ldots N_c-1 \tag{2.12}
\]
\[
(J_2)_{a, a+1} = -(J_2)_{a+1, a} = -i \frac{\sqrt{a(N_c-a)}}{2}.
\]
The usual relation between the generators of the $SU(2)$ algebra and the quadratic Casimir then follows,

$$J_1^2 + J_2^2 + J_3^2 = j(j+1)1 = \frac{(N_c^2 - 1)}{4}1.$$

This relation leads to a natural association of the Higgs vacuum of $\mathcal{N} = 1^*$ theory with fuzzy sphere configurations of D3 branes in the string theory dual [8].

The results we deduce below for magnetic flux tubes in the Higgs vacuum, will have a direct bearing on the tension of the chromoelectric flux tubes in the confining vacuum. This is because the $SL(2,\mathbb{Z})$ electric-magnetic duality of the parent $\mathcal{N} = 4$ theory permutes different IR phases of the $\mathcal{N} = 1^*$ theory [4, 5, 6]. In particular, the Higgs and confining vacua are exchanged under S-duality: $\tau \rightarrow -1/\tau$.

### 2.3 Colour-Flavour locking

The VEVs of the adjoint scalars in the Higgs vacuum break the $SO(3)$ global symmetry and the $SU(N_c)$ gauge symmetry. However, it is always possible to find a combined global colour-flavour rotation which is unbroken [20]. This combined $SO(3)_{C+F}$ global symmetry subgroup can be understood as follows. Any global $SO(3)$ rotation of the triplet $(\Phi_1, \Phi_2, \Phi_3)$ can be undone by a global colour transformation whose generators are chosen to be proportional to the VEVs of the adjoint scalars i.e., the $N_c$ dimensional $SU(2)$ generators. More explicitly, we first rotate the triplet $(\Phi_1, \Phi_2, \Phi_3)$ with the flavour matrix $U_F = \exp(T_j a_j)$, where $T_j$ are the following $SO(3)$ generators,

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.14)$$

This transformation acts in the flavour space as

$$\Phi \rightarrow U_F \Phi.$$ 

Then let us introduce the global colour matrix $W_C = \exp(i J_l a_l)$, acting as:

$$\Phi_i \rightarrow W_C \Phi_i W_C^\dagger,$$

where $J_l$ are the $N_c$ dimensional representations of $SU(2)$ generators.

A combination of the above flavour and colour rotations are unbroken by the scalar VEVs. The existence of this $SO(3)_{C+F}$ symmetry allows the determination of the worldsheet sigma model of vortices (magnetic flux tubes) in the Higgs vacuum, as we see below.
2.4 Higgs Vacuum Spectrum

The perturbative spectrum in the Higgs vacuum for $N_c = 2$ is given by an $SO(3)_{C+F}$ triplet of massive vector $\mathcal{N} = 1$ multiplets with mass $\sqrt{2}m$, one chiral multiplet with mass $m$ and 5 chiral multiplets with mass $2m$. For general $N_c$ the perturbative spectrum has been computed in [34]. The result is the following: there are always 3 massive vectors multiplets with mass $\sqrt{2}m$ and one chiral multiplet with mass $m$. In addition, for every $k = 2, \ldots, N_c - 1$ there are $4k$ massive chiral multiplets with mass $km$, $2k + 1$ massive vectors with mass $\sqrt{k(k+1)m}$ and lastly, a set of $2N_c + 1$ chiral multiplets with mass $N_cm$. For every $N_c$ all these particles fit in representations of $SO(3)_{C+F}$.

A beautiful feature of the Higgs vacuum of the $\mathcal{N} = 1^*$ theory is that in the large $N_c$ limit it provides a deconstruction of a six dimensional theory compactified on a sphere. In particular, as discussed in [34], the perturbative spectrum of the $U(N_c)$, $\mathcal{N} = 1^*$ theory, is identical to the spectrum of the Maldacena-Nunez twisted compactification of the $\mathcal{N} = (1,1)$ six dimensional $U(1)$ gauge theory on a two-dimensional sphere. This interpretation is a direct consequence of the association of the Higgs vacuum with a fuzzy sphere configuration [35] as described above.

3. The $\mathbb{Z}_{N_c}$ vortex as a soliton

3.1 General discussion

Since the $\mathcal{N} = 1^*$ theory has only fields transforming in the adjoint representation of the gauge group, the Lagrangian is invariant under transformations in the center $\mathbb{Z}_{N_c}$ of $SU(N_c)$. In the Higgs vacuum, magnetic charges valued in $\mathbb{Z}_{N_c} = \pi_1 [SU(N_c)/\mathbb{Z}_{N_c}]$ are confined by magnetic flux tubes, also carrying a $\mathbb{Z}_{N_c}$ charge. Since the fluxes are defined modulo $N_c$, they annihilate in groups of $N_c$. At weak coupling $g_{YM} \ll 1$, the physics in the Higgs vacuum is semiclassical and the magnetic flux tubes should be understood as ordinary non-Abelian vortex string solutions of the classical equations of motion.

In this section we introduce an ansatz for solitonic $k$-strings in the Higgs vacuum of the $\mathcal{N} = 1^*$ theory with gauge group $SU(N_c)$ in the semiclassical limit $g_{YM} \ll 1$.

\footnote{In [34], the spectrum of the $U(N_c)$ gauge theory was determined, which differs slightly from the $SU(N_c)$ theory discussed here. In particular, for $U(N_c)$, there are three additional chiral multiplets with mass $m$ and one massless vector multiplet.}
We will write the ansatz explicitly for $N_c = 2, 3, 4$. The resulting vortices carry magnetic flux $k = 1, 2, \ldots, N_c - 1$, defined modulo $N_c$. Since they annihilate in groups of $N_c$, $N_c = 4$ is the minimal gauge group for which a non-trivial $k = 2$ string appears. There is also a $k = 2$ vortex for $N_c = 3$, but is essentially equivalent to the $k = 1$ vortex under the transformation $k \to N_c - k$.

The generalization of our ansatz to general $k$ and $N_c$ is straightforward, but explicit calculations with these ansätze get quite complicated. For generic $N_c$ and $k$, in order to solve the equations of motion of the gauge theory, we need to introduce an ansatz which depends on $3(N_c - 1)$ independent profile functions for the vortex. We have performed explicit numerical computations for $2 \leq N_c \leq 6$ and generic $k$.

The classical equations of motion for the bosonic fields read
\begin{align}
\partial_\mu F^{\mu\nu} - i [A_\mu, F^{\mu\nu}] &= \frac{i}{2} \sum_{i=1}^{3} \left( [D^\nu \Phi_i, \Phi_i^\dagger] + [D^\nu \Phi_i^\dagger, \Phi_i] \right), \\
D^\mu D_\mu \Phi_i &= (m w_i^\dagger - \epsilon_{ijl}[w_j^\dagger, \Phi_i]) + \frac{\partial V_D}{\partial \Phi_i}.
\end{align}

Below we list explicit vortex solutions which satisfy
\begin{equation}
\Phi_i = -\Phi_i^\dagger, \quad i = 1, 2, 3.
\end{equation}
so that the D-term contribution to the potential is identically zero when evaluated on the solution, and the resulting equations of motion are somewhat simpler. For general $N_c$, there are $N_c - 1$ distinct topological sectors labelled by an integer $k$ with $0 < k < N_c$.

The vortex configurations have the adjoint scalars $\Phi_i$ approaching, at infinity, a gauge transform of their VEVs in the Higgs vacuum. In particular, certain matrix elements of the adjoint scalars undergo a $2\pi$ phase rotation upon winding once around the vortex. This phase rotation corresponds to a gauge transformation (at infinity) which is single-valued in $SU(N_c)/\mathbb{Z}_{N_c}$. In our ansätze below, the solutions with winding number $k = 1$ will have the scalars winding at infinity, effectively generated by
\begin{equation}
Y_1 = \frac{1}{N_c} \text{Diag}(1, \cdots, 1, -(N_c - 1)),
\end{equation}
resulting in a chromomagnetic flux proportional to $Y_1$. Thus the flux picks out a specific direction in colour-flavour space and the associated string is truly non-Abelian.

In these solutions, $\Phi_3$ is chosen to have no azimuthal variation, whilst both $\Phi_1$ and $\Phi_2$ have nontrivial angular dependence, away from the vortex core. In particular,
as the azimuthal angle \( \varphi \) varies from 0 to \( 2\pi \), the components \((\Phi_{1,2})_{N_c-1,N_c}\) wind with a phase \( e^{i\varphi} \), while \((\Phi_{1,2})_{N_c,N_c-1}\) wind with the opposite phase \( e^{-i\varphi} \).

For generic winding \( 1 < k < N_c \) the flux carried by the corresponding \( k \)-vortex is proportional to

\[
Y_k = \text{Diag} \left( \begin{array}{cccc}
\frac{k}{N_c}, \ldots, \frac{k}{N_c}, & -\frac{N_c - k}{N_c}, \ldots, -\frac{N_c - k}{N_c} \\
\end{array} \right),
\]

and away from the vortex core, the adjoint scalars behave as

\[
\Phi_{1,2}(r, \varphi) = e^{iY_k \varphi} \Phi_{1,2}(r, \varphi = 0) e^{-iY_k \varphi}
\]

while \( \Phi_3 \) has only a radial dependence. Under the effect of this rotation, the components \((\Phi_{1,2})_{N_c-k,N_c-k+1}\) and \((\Phi_{1,2})_{N_c-k+1,N_c-k}\) wind around the vortex with phase \( e^{i\varphi} \) and \( e^{-i\varphi} \) respectively. We believe that these are the solutions of lowest tension in each topological sector \( k \), as each field winds at infinity exactly once. The solutions also display an obvious vortex/anti-vortex symmetry, which is evident under the replacement \( k \to N_c - k \).

The explicit vortex solutions will break the \( SO(3)_{C+F} \) global symmetry. However, they are invariant under the action of a \( U(1) \) subgroup corresponding to global rotations acting on \((\Phi_1, \Phi_2)\). The action of the broken global symmetry generators then leads to a \( SU(2)/U(1) \simeq \mathbb{CP}^1 \) moduli space of solutions for generic \((N_c, k)\).

### 3.2 \( N_c = 2 \)

For \( SU(2) \) gauge group, the vortex solutions were first found in [20]. Here we rederive their result for completeness,

\[
\Phi_1 = \frac{im}{2} \psi_1(r) \begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}, \quad \Phi_2 = \frac{im}{2} \psi_1(r) \begin{pmatrix} 0 & -ie^{i\varphi} \\ ie^{-i\varphi} & 0 \end{pmatrix}, \quad \Phi_3 = \frac{im}{2} \kappa_1(r) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where \( \psi_1 \) and \( \kappa_1 \) are profile functions to be determined by the equations of motion. Both approach unity as \( r \to \infty \), in order to match up with the Higgs VEVs. Near the origin \( \psi_1 \) vanishes so that the solution is smooth at \( r = 0 \). It is obvious that this configuration will be invariant under a combination of an \( SO(2) \) flavour rotation.
acting on the pair \((\Phi_1, \Phi_2)\) and a global colour rotation generated by \(\sigma_3\). Hence the full \(SO(3)_{C+F}\) is broken to \(U(1)_{C+F}\).

The gauge field is solved by a typical vortex form

\[
A_x = \frac{-y}{r^2} (1 - f(r)) Y, \quad A_y = \frac{x}{r^2} (1 - f(r)) Y, \quad Y = \frac{1}{2} \sigma_3. \quad (3.8)
\]

This picks out a direction in the colour space and results in a magnetic flux also proportional to \(Y\),

\[
F_{xy} = -\frac{f'(r)}{r} Y. \quad (3.9)
\]

![Figure 1: The vortex profile functions for \(N_c = 2\): \(\kappa_1\) (solid), \(\psi_1\) (long dashes), \(f\) (short dashes).](image)

The ansatz which is axisymmetric under rotations about the \(z\)-axis, can be used to evaluate the action functional per unit length. This yields the vortex tension functional

\[
T = 2\pi \int r \, dr \left( \frac{f'^2}{2r^2} + \frac{m^2\kappa_1^2}{2} + m^2\psi_1^2 + \frac{m^2\psi_1^2 f^2}{r^2} + \frac{m^4}{2} \left( (\kappa_1 - \psi_1^2)^2 + 2\psi_1^2(\kappa_1 - 1)^2 \right) \right). \quad (3.10)
\]

It is easily checked that the equations of motion for the profile functions that follow from varying this tension functional are the same as those following from (3.1) and (3.2). The profile functions are thus determined by solving,

\[
f'' - f' r - f\psi_1^2 2m^2 = 0, \quad (3.11)
\]

\[
\psi_1'' + \frac{\psi_1'}{r} - \frac{\psi_1 f^2}{r^2} = m^2 \psi_1 (\psi_1^2 + \kappa_1^2 - 3\kappa_1 + 1), \quad (3.12)
\]

\[
\kappa_1'' + \frac{\kappa_1'}{r} = m^2 (2\psi_1^2\kappa_1 - 3\psi_1^2 + \kappa_1). \quad (3.13)
\]
The above equations can be solved numerically and the results are plotted in Fig. 1. We learn from the solution that $\psi_1$ grows from zero at the core of the vortex to unity at infinity. At the same time, $\kappa_1$ remains non-zero at the string core whilst approaching 1 asymptotically. Hence, in the core $\Phi_3 \neq 0$, $\Phi_1 = \Phi_2 = 0$, so that there is a Coulomb-like phase, shielded by a crossover region which eventually merges with the Higgs phase vacuum at infinity. The profile function $f(r)$ shows that the magnetic field $\sim f'(r)/r$ is non-zero in the Coulomb-like phase, concentrated in a neighbourhood of the origin, while vanishing in the asymptotic Higgs vacuum.

3.3 $N_c = 3$

Having understood the structure of the $\mathbb{Z}_2$ string for $SU(2)$, we can now apply our general non-Abelian string ansatz described in Section 3.1, to higher rank gauge groups. For $SU(3)$, and for $k = 1$ our general ansatz takes the form,

$$
\Phi_1 = \frac{im}{\sqrt{2}} \begin{pmatrix} 0 & \psi_1 & 0 \\ \psi_1 & 0 & \psi_2 e^{i\varphi} \\ 0 & \psi_2 e^{-i\varphi} & 0 \end{pmatrix}, \quad \Phi_2 = \frac{im}{\sqrt{2}} \begin{pmatrix} 0 & -i\psi_1 & 0 \\ i\psi_1 & 0 & -i\psi_2 e^{i\varphi} \\ 0 & i\psi_2 e^{-i\varphi} & 0 \end{pmatrix}
$$

(3.14)

$$
\Phi_3 = im \begin{pmatrix} \kappa_1 - \kappa_2/2 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & -\kappa_1 - \kappa_2/2 \end{pmatrix}.
$$

The forms of $\Phi_1$ and $\Phi_2$ are both motivated by their Higgs phase VEVs, $imJ_1$ and $imJ_2$ respectively. At infinity, the profile functions $\psi_1$ and $\psi_2$ approach unity, while $\psi_2$ vanishes at the origin so that the solution remains smooth. As we go around the origin, $\Phi_1$ and $\Phi_2$ undergo a phase rotation generated by

$$
Y_1 = \frac{1}{3} \text{Diag}(1, 1, -2).
$$

(3.15)

The magnetic flux for the solution turns out to be proportional to $Y_1$, which satisfies

$$
\exp(2\pi i Y_1) = \text{Diag}(e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3}).
$$

(3.16)

The gauge field is modified slightly from the $SU(2)$ case,

$$
A_x = -\frac{y}{r^2} ((1 - f(r))Y_1 + g(r)\lambda), \quad A_y = \frac{x}{r^2} ((1 - f(r))Y_1 + g(r)\lambda),
$$

(3.17)

where $\lambda = \frac{1}{3} \text{Diag}(1, -1, 0)$ and $g(r)$ is a new profile function. The non-Abelian magnetic field then is

$$
F_{xy} = -\frac{f'(r)}{r} Y_1 + \frac{g'(r)}{r} \lambda.
$$

(3.18)
As in the $SU(2)$ example, we can evaluate the energy per unit length for the ansatz to obtain the tension functional which can be varied to yield the equations of motion,

$$T = 2\pi \int r \, dr \left( \frac{2f'^2}{3r^2} + \frac{g'^2}{2r^2} + \frac{1}{2} m^2(4\kappa_1^2 + 3\kappa_2^2) + 2m^2(\psi_1'^2 + \psi_2'^2) + \frac{1}{2} m^2 \left( \frac{2(2\kappa_1 - \kappa_2 - 2\psi_1^2)^2}{2\kappa_1 + \kappa_2 - 2\psi_2^2} 
+ \frac{1}{2} m^2 \left( 2(\kappa_2 + \psi_1^2 - \psi_2^2)^2 + (2 - 2\kappa_1 + 3\kappa_2)^2 \psi_1^2 + (2 - 2\kappa_1 - 3\kappa_2)^2 \right) \right) \right).$$

(3.19)

Once again the equations of motion following from this functional are consistent with the equations of motion of the full theory, Eqs. (3.1) and (3.2).

For $SU(3)$ gauge group there is also a $k = 2$ vortex. However this is follows from a $k \to N_c - k$ replacement in our $k = 1$ solution. In other words, the $k = 2$ solution will be the identical to the above, with the opposite flux (winding).

### 3.4 $N_c = 4$

The $\mathbb{Z}_{N_c}$ string solution for $SU(4)$ gauge group is particularly interesting, as this is the first instance where we encounter a non-trivial multi-vortex solution, i.e. with winding $k > 1$. We only need to consider the cases with $k = 1, 2$ (the $k = 3, 4$ vortices are identical to $k = 1, 2$ respectively with negative winding).

$k = 1$ solution: The ansatz follows the general pattern described earlier,

$$\Phi_1 = \frac{mi}{2} \begin{pmatrix} 0 & \sqrt{3}\psi_1 & 0 & 0 \\ \sqrt{3}\psi_1 & 0 & 2\psi_2 & 0 \\ 0 & 2\psi_2 & 0 & \sqrt{3}\psi_3 e^{i\varphi} \\ 0 & 0 & \sqrt{3}\psi_3 e^{-i\varphi} & 0 \end{pmatrix},$$

(3.20)

$$\Phi_2 = \frac{mi}{2} \begin{pmatrix} 0 & -i\sqrt{3}\psi_1 & 0 & 0 \\ i\sqrt{3}\psi_1 & 0 & -i2\psi_2 & 0 \\ 0 & i2\psi_2 & 0 & -i\sqrt{3}\psi_3 e^{i\varphi} \\ 0 & 0 & i\sqrt{3}\psi_3 e^{-i\varphi} & 0 \end{pmatrix},$$

$$\Phi_3 = \frac{mi}{2} \begin{pmatrix} 3\kappa_1 - 2\kappa_3 & 0 & 0 & 0 \\ 0 & \kappa_2 + 2\kappa_3 & 0 & 0 \\ 0 & 0 & -\kappa_2 + 2\kappa_3 & 0 \\ 0 & 0 & 0 & -3\kappa_1 - 2\kappa_3 \end{pmatrix}.$$
The following expression is used for the gauge field:

\[
A_x = \frac{-y}{r^2} \left( (1 - f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell \right), \\
A_y = \frac{x}{r^2} \left( (1 - f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell \right),
\]

where

\[
Y_1 = \frac{1}{4} \text{Diag}(1, 1, 1, -3).
\]

which yields \(\exp(2\pi i Y_1) = \text{Diag}(e^{\pi i/2}, e^{\pi i/2}, e^{\pi i/2}, e^{\pi i/2})\). The non-Abelian flux carried by the vortex is proportional to \(Y_1\). The \(g_\ell\)'s are functions of \(r\) vanishing both at \(r = 0\) and at \(r \to \infty\), and \(\lambda_\ell\) are a basis of diagonal matrices with satisfying,

\[
\text{Tr}Y_1 \lambda_\ell = 0, \\
\text{Tr}\lambda_i \lambda_\ell = \frac{1}{2} \delta_{i\ell}.
\]

We choose

\[
\lambda_1 = \frac{1}{\sqrt{12}} \text{Diag}(1, 1, -2, 0), \\
\lambda_2 = \frac{1}{2} \text{Diag}(1, -1, 0, 0).
\]

The string profile can then can be found by the minimization of the energy functional. We do not write the explicit form as it is quite lengthy. The numerical solutions to the resulting equations of motion are shown in Figure 2.

**Figure 2:** The vortex profile for \(N_c = 4\). Left: \(\psi_1\) (solid), \(\psi_2\) (long dashes), \(\psi_3\) (short dashes). Center: \(\kappa_1\) (solid), \(\kappa_2\) (long dashes), \(\kappa_3\) (short dashes). Right: \(f\) (solid), \(g_1\) (long dashes), \(g_2\) (short dashes).

Since \(\psi_3\) vanishes at the origin and all the diagonal elements of \(\Phi_3\) remain non-zero at \(r = 0\), we infer that at the core of the vortex solution, a \(U(1)\) subgroup of the gauge symmetry is unbroken and theory is in a Coulomb phase in that region.

\[k = 2\] solution: We now turn to the \(k = 2\) vortex solution. The relevant configuration for the scalars is now obtained by applying an \(SU(4)\) rotation to the Higgs vacuum VEVs, generated by

\[
Y_2 = \frac{1}{2} \text{Diag}(1, 1, -1, -1)
\]
with $\exp(2\pi i Y_2) = \text{Diag}(-1, -1, -1, -1)$. The chromomagnetic flux is also proportional to $Y_2$. The explicit ansatz is then,

$$
\Phi_1 = \frac{mi}{2} \begin{pmatrix} 0 & \sqrt{3}\psi_1 & 0 & 0 \\ \sqrt{3}\psi_1 & 2\psi_2 e^{i\varphi} & 0 & 0 \\ 0 & 2\psi_2 e^{-i\varphi} & 0 & \sqrt{3}\psi_3 \\ 0 & 0 & \sqrt{3}\psi_3 & 0 \end{pmatrix},
$$

$$
\Phi_2 = \frac{mi}{2} \begin{pmatrix} 0 & -i\sqrt{3}\psi_1 & 0 & 0 \\ i\sqrt{3}\psi_1 & 0 & -i2\psi_2 e^{i\varphi} & 0 \\ 0 & i2\psi_2 e^{-i\varphi} & 0 & 0 \\ 0 & 0 & i\sqrt{3}\psi_3 & 0 \end{pmatrix},
$$

$$
\Phi_3 = \frac{mi}{2} \begin{pmatrix} 3\kappa_1 - 2\kappa_3 & 0 & 0 & 0 \\ 0 & \kappa_2 + 2\kappa_3 & 0 & 0 \\ 0 & 0 & -\kappa_2 + 2\kappa_3 & 0 \\ 0 & 0 & 0 & -3\kappa_1 - 2\kappa_3 \end{pmatrix}.
$$

The gauge fields are still given by Eq. 3.21, with $Y_1$ replaced by $Y_2$ and

$$
\lambda_1 = \frac{1}{2} \text{Diag}(1, -1, 0, 0), \quad \lambda_2 = \frac{1}{2} \text{Diag}(0, 0, 1, -1).
$$

As before the vortex profiles can be found numerically and the results are shown in Fig. 3.

**Figure 3:** Profiles for $N_c = 4$ and $k = 2$. Left: $\psi_1 = \psi_3$ (solid), $\psi_2$ (long dashes). Center: $\kappa_1$ (solid), $\kappa_2$ (long dashes), $\kappa_3 = 0$. Left: $f$ (solid), $g_1 = g_2$ (long dashes).

This solution provides a confirmation of the general picture of these non-Abelian vortices in the Higgs vacuum of $\mathcal{N} = 1^*$ theory. They all have an unbroken $U(1)$ gauge group at their core, while approaching a totally Higgsed phase in the exterior. This is, of course, consistent with the premise that the Higgs vacuum and its excitations should have a semiclassical description. Another general feature is that the non-Abelian strings break the $SO(3)_{C+F}$ symmetry group to a $U(1)$ subgroup. The moduli space of $\mathbb{Z}_{N_c}$ string solutions is therefore isomorphic to $\mathbb{C}P^1 \simeq SU(2)_{C+F}/U(1)$ for all $k$ and $N_c$. We will also confirm this feature of the theory in its large $N_c$ string dual.
The generalization of the vortex ansatz to arbitrary $N_c$ proceeds in a straightforward fashion and requires introducing $3(N_c - 1)$ profile functions $(\psi_i, \kappa_i, f, g_i)$. In the absence of any obvious analytical simplifications, we will not pursue this direction further in this paper.

### 3.5 $k$-string tensions

The study of non-Abelian $k$-string tensions is a topic of great interest and is particularly so in the present context. The non-Abelian vortices of the Higgs vacuum at weak coupling $g_{YM} \ll 1$ are mapped by S-duality of $\mathcal{N} = 1^*$ theory to confining strings at strong coupling $g_{YM} \gg 1$. With the explicit ansätze at hand for general $N_c$ and $k$, we can compute their tensions, albeit numerically. We will then compare these results with the known tensions in a different parametric regime for $\mathcal{N} = 1^*$ theory wherein the vortex strings are almost BPS. It should be pointed out that when $m_1 = m_2 = m_3 = m$, the strings are far from BPS. Nevertheless we will see that the numerical values of the tensions approach the BPS values as $N_c$ is increased.

The tension of semiclassical non-Abelian strings in the Higgs vacuum of $\mathcal{N} = 1^*$ theory has been discussed in [20, 22], in the limit

\[
m_1 = m_2 = m, \quad \text{and} \quad m_3 \ll m. \tag{3.28}\n\]

In this limit the vortex becomes an almost BPS object, due to the fact that in the limit $m_3/m \to 0$, $\mathcal{N} = 2$ supersymmetry is restored. The theory may then be viewed as softly broken $\mathcal{N} = 2^*$ theory. The $\mathcal{N} = 2^*$ theory, with $m_1 = m_2 = m$ and $m_3 = 0$, is realized in the Coulomb phase due to $\Phi_3$ obtaining a VEV. Adding a mass $m_3$ for $\Phi_3$ at the appropriate point on the Coulomb branch moduli space results in complete Higgsing of the theory due to electric degrees of freedom becoming light and condensing. The profile functions of the vortices in this limit are simpler, because it is consistent to take the profiles $\kappa_j(r)$ (equivalently, $\Phi_3$) as constant. For a BPS vortex the tension is exactly proportional to the field condensates:

\[
T_{N_c,k}^{BPS} = 2\pi \frac{mm_3}{g_{YM}^2} k(N_c - k). \tag{3.29}\n\]

This behaviour is the so-called ‘Casimir scaling’ of $k$-string tensions.

The case we have focussed attention on this paper is far from the BPS limit with,

\[
m_1 = m_2 = m_3 = m. \tag{3.30}\n\]

Using our ansatz above we have numerically evaluated the vortex tension functional $T_{N_c,k}$ for $2 \leq N_c \leq 6$ and the results are in Table 1. In this table we have presented
Table 1: Values of $T_{N_c,k}/T_{N_c,k}^{\text{BPS}}$ for $2 \leq N_c \leq 6$ and different $k$.

| $N_c$ | 2     | 3   | 4   | 5   | 6   |
|-------|-------|-----|-----|-----|-----|
| $k = 1$ | 0.894 | 0.926 | 0.943 | 0.954 | 0.961 |
| $k = 2$ |       | 0.944 | 0.954 | 0.962 |
| $k = 3$ |       |       |       | 0.962 |

The main conclusion that we can draw from this numerical data is that for large $N_c$ the $k$-string tension $T_{N_c,k}$ in the theory with $m_3 = m$ quickly approaches the BPS tension formula given by Eq. (3.29). There does not appear to be an obvious explanation for this result. We also note also that for fixed $N_c$ the ratios in the table are, to a very good approximation, independent of $k$.

The numerical results for string tension ratios $T_{N_c,k+1}/T_{N_c,k}$ are also rather striking. For $N_c = 4$ we find the following numerical result,

$$
\frac{T_{N_c=4,k=2}}{T_{N_c=4,k=1}} = 1.334
$$

(3.31)

while the prediction from Casimir scaling is $4/3$. For $N_c = 5$ we find

$$
\frac{T_{N_c=5,k=2}}{T_{N_c=5,k=1}} = 1.501
$$

(3.32)

while the Casimir scaling prediction is $3/2$. Finally, for $N_c = 6$:

$$
\frac{T_{N_c=6,k=2}}{T_{N_c=6,k=1}} = 1.6008, \quad \frac{T_{N_c=6,k=3}}{T_{N_c=6,k=1}} = 1.801
$$

(3.33)

while the Casimir scaling values are $8/5$ and $9/5$.

The numerical results above are striking in that the tension is not a BPS protected quantity, so the accuracy of the Casimir scaling law is better than what we could expect. The Casimir scaling law is only exact in the limit $m_3 << m$, but evidently it is still an extremely good approximation also for $m = m_3$ for the cases $N_c = 4, 5, 6$ which have been studied numerically. This suggests that in the large $N_c$ theory, the $k$-string tensions likely obey a Casimir scaling law in the $\mathcal{N} = 1^*$ theory. This can be best understood by investigating the known large-$N_c$ string dual of the $\mathcal{N} = 1^*$ theory [8], which we will do in Section 5.
4. Effective world-sheet theory

In this section we round off our field theoretic analysis with the construction of the (classical) world-sheet theory of $k$-strings in the Higgs vacuum. For $SU(2)$ gauge group this was already done in [20]. Below we will extend this to $SU(N_c)$ gauge group and general $k$. We will also present a new ingredient, namely the effect on the worldsheet sigma model, of a non-zero $\theta_{3+1}$ angle in the $\mathcal{N} = 1^*$ Yang-Mills theory.

The general class of vortex solutions presented above have the property that they are invariant only under a $U(1)$ subgroup of the colour-flavour locked $SO(3)_{C+F}$ transformations. This unbroken $U(1)$ is a rotation acting on the pair $(\Phi_1, \Phi_2)$ which can be undone by a gauge transformation. The moduli space of inequivalent solutions is thus $\mathbb{C}P^1 \simeq SU(2)_{C+F}/U(1)$. The associated moduli correspond to the orientational modes of the magnetic flux in the string solution.

The low-lying excitations of the worldsheet theory of the vortex will involve, apart from translational zero modes for the center of mass, the adiabatic dynamics of the orientational zero modes. For all $N_c$ and $k$ we see that this is a nonsupersymmetric sigma model (as in the examples discussed in Refs. [20, 32]) with target space $\mathbb{C}P^1$, along with a theta angle that is related in a special way to the four dimensional Yang-Mills theta angle. The absence of supersymmetry makes the present situation different from BPS non-Abelian vortex strings in $\mathcal{N} = 2$ SQCD [11, 15, 12, 14] and also different from the Heterotic vortex string discussed in Refs. [36].

4.1 Kinetic term

Let us consider a vortex oriented along the $z$ axis. In order to obtain the effective world-sheet theory of the orientational zero modes, we introduce an adiabatic $SO(3)_{C+F}$ rotation which depends on the world-sheet coordinates $(z, t)$ of the vortex string. Doing so will turn these moduli (the global rotation parameters) into world-sheet dependent fields. It is best to perform these steps in singular gauge, i.e. where the scalar fields have no winding at infinity and the flux is concentrated near the origin as in [12, 14].

Upon a (worldsheet dependent) colour-flavour locked rotation, the triplet of scalar fields transform in the following way:

$$\vec{\Phi} \rightarrow U_F(z, t) \cdot \left( W_C(z, t) \vec{\Phi} W_C^\dagger(z, t) \right),$$

(4.1)

where the matrix $U_F$ acts in the three dimensional flavour space (2.19) and $W_C$ is a colour transformation (2.16) generated by the $N_c$ dimensional representation of...
$SU(2)$ generators. The gauge fields transform as:

\[ A_{x,y} \rightarrow W_C A_{x,y} W_C^\dagger. \]

(4.2)

A transformation dependent on $z$ and $t$, will of course also generate components of $A_\mu$ along the world-sheet coordinates, as evident from the ordinary gauge transformation $A_s \rightarrow W_C A_s W_C^\dagger + i W_C \partial_s W_C^\dagger$. To be consistent therefore, the full vortex solution will need to be modified and the radial dependence of the new components of the gauge field have to be solved for. We will, using a natural axisymmetric ansatz for all $A_\mu$, obtain the worldsheet effective theory. This can be done along the lines of [32]: $A_s$ are chosen in gauge space in such a way that they are perpendicular to the $A_{x,y}$ and to the derivative $\partial_s (A_{x,y})$. This is

Let us discuss, for simplicity, the $N_c = 2$ $k = 1$ case (already studied in [20]) and subsequently generalize. In singular gauge

\[ \Phi_1 = \frac{im}{2} \psi_1(r) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Phi_2 = \frac{im}{2} \psi_1(r) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \]

\[ \Phi_3 = \frac{im}{2} \kappa_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_x = \frac{y}{r^2} f(r) \sigma_3, \quad A_y = \frac{-x}{r^2} f(r) \sigma_3. \]

(4.3)

(4.4)

Since the solution is symmetric under $U(1)_{C+F}$ rotations around the $\Phi_3$ axis in flavour space, to generate a new solution we need to act on it with one of the broken generators. If this action is chosen to be global, we simply obtain a new solution with the the same tension as the old one, but with the non-Abelian flux pointing in a different direction in colour+flavour space. Let us therefore consider a $(z,t)$ dependent rotation around the $\Phi_2$ axis without loss of generality. We will use the following worldsheet dependent colour and flavour transformations:

\[ W_C \sigma_3 W_C^\dagger = \vec{n}(z,t) \cdot \vec{\sigma}. \]

(4.5)

Then the following ansätze can be used for the gauge field components $A_s$ with $s = z,t$:

\[ A_s = - (\vec{n} \times \partial_s \vec{n})^a \frac{\sigma_a}{2} \rho(r). \]

(4.6)

The gauge orientations for $A_s$ are dictated by the requirement that they be orthogonal in colour space, to both $A_{x,y}$ and $\partial_s A_{x,y}$ (after the world-sheet dependent transformation). We substitute the expression for the gauge fields (4.6) along with the world-sheet dependent transformation (4.2) of the solution (4.4), into the action and obtain

\[ S_{1+1} = \int dz \, dt B_{2,1}(\partial_s \vec{n})^2, \]

(4.7)
where
\[
B_{2,1} = \frac{1}{g_{YM}^2} \int_0^\infty dr \frac{2\pi r}{2} \left( \frac{\rho'^2}{2} + \frac{(\rho - 1)^2 f^2}{2r^2} + \frac{m^2}{2} \left( 2(\kappa_1 - \psi_1)^2(1 - \rho) + (\kappa_1^2 + \psi_1^2)\rho^2 \right) \right).
\]

It is worth noting that this kinetic term for the world-sheet moduli is generated only by gauge kinetic terms depending on \( F_{\mu\nu} \) and scalar kinetic terms \( \sim |D_s \Phi_{1,2}|^2 \) of the four dimensional gauge theory. All other terms including the scalar potential of the gauge theory are invariant under the combined colour-flavour rotations. In order for the sigma model coupling to be finite, we need to impose the boundary conditions
\[
\rho(0) = 1, \quad \rho(r \to \infty) = 0.
\]

The Euler-Lagrange equations for \( \rho(r) \) and other profile functions can be solved numerically to yield the kinetic term for the world-sheet moduli fields. The result, shown in Table 2 for \( SU(2) \): \( B_{2,1} = 0.39 \left( 2\pi/g_{YM}^2 \right) \) matches with that of [20].

The generalization of the above arguments to the \( SU(N_c) \) case, for each of the stable \( k \)-vortices, is actually straightforward. Now the relevant colour transformations are generated by the \( N_c \) dimensional representation of \( SU(2) \) generators. To get the normalization of the kinetic term of the resulting \( \mathbb{CP}^1 \) model, we again consider just a rotation around the \( \Phi_2 \) axis so that,
\[
W_C J_3 W_C^\dagger = \vec{n}(z,t) \cdot \vec{J}.
\]

Then the following ansatz can be used for the gauge fields along the worldsheet
\[
A_s = -(\vec{n} \times \partial_s \vec{n})^a J^a \rho(r).
\]

If we insert these expressions into the gauge theory action, the following term in the vortex effective theory can be found,
\[
S_{1+1} = \int dz \, dt \left( B_{N_c,k}(\partial_s \vec{n})^2 \right).
\]

The general formula for \( B_{N_c,k} \) is complicated and we can only evaluate it on a case by case basis, numerically.

For example, for \( N_c = 3, \ k = 1 \) the explicit expression is,
\[
B_{3,1} = \frac{1}{g_{YM}^2} \int dr \frac{2\pi r}{2} \left( 2\rho'^2 + \frac{(4f^2 - 4gf + 5g^2)(\rho - 1)^2}{4r^2} + \frac{m^2}{2} \left( 4(\rho^2 - 2\rho + 2)\kappa_1^2 + + 8(\rho - 1)(\psi_1 + \psi_2) \kappa_1 + 3(3\rho^2 - 6\rho + 4)\kappa_2^2 - 12(\rho - 1)\kappa_2(\psi_1 - \psi_2) + 2(-4\psi_1 \psi_2(\rho - 1)^2 + (3\rho^2 - 6\rho + 4)\psi_1^2 + (3\rho^2 - 6\rho + 4)\psi_2^2) \right) \right).
\]
The coefficient is then determined numerically by variation of the action functional with the boundary conditions \( \rho(r = 0) = 1 \) and \( \rho(r \to \infty) = 0 \). In Table 2 are shown the numerical values for \( B_{N_c,k} \) for \( 2 \leq N_c \leq 6 \).

| \( N_c \) | 2   | 3   | 4   | 5   | 6   |
|----------|-----|-----|-----|-----|-----|
| \( \frac{g_T M}{2\pi} B_{N,1} \) | 0.390 | 1.181 | 2.343 | 3.867 | 5.847 |
| \( \frac{g_T M}{2\pi} B_{N,2} \) | 2.696 | 4.344 | 6.710 |       |     |
| \( \frac{g_T M}{2\pi} B_{N,3} \) |       |       |       | 6.888 |     |

**Table 2:** Some numerical results for the classical kinetic term for the \( k \)-vortex. For \( N_c = 2 \) there is agreement with the value computed in Ref. [20].

### 4.2 World-sheet Theta Angle

Whenever the Yang-Mills theta angle \( \theta_{3+1} \) is non-vanishing, a new ingredient appears in the vortex world-sheet theory. The Yang-Mills theta angle feeds into the world-sheet theory as a topological term for the \( \mathbb{CP}^1 \) sigma model. The coefficient of this topological term, the world-sheet theta angle denoted as \( \theta_{1+1} \), plays a crucial role in the ensuing world-sheet dynamics. In particular, the IR dynamics is strongly theta-dependent [32, 24, 25, 26].

We begin by demonstrating the mechanism of generation of the world-sheet theta angle for \( SU(2) \) gauge group. In this case the steps involved and the result are rather similar to [32]. The relevant terms can be obtained by a colour-flavour transformation that depends on both \( z \) and \( t \). It will be sufficient to consider the following \((z,t)\) dependent colour-flavour rotation of the vortex fields,

\[
U_F = \exp(J_2 \alpha(z)). \exp(J_1 \beta(t)) , \quad W_C = \exp \left( i \frac{\sigma_2}{2} \alpha(z) \right), \exp \left( i \frac{\sigma_1}{2} \beta(t) \right). \quad (4.14)
\]

The time and space dependent rotation will generate gauge field components \( A_z \) and \( A_t \). These will have to be chosen normal, in colour space, to \( A_{x,y} \) and their derivatives. Using the appropriate ansatz, \( A_s = -\langle \vec{n} \times \partial_s \vec{n} \rangle \langle \vec{\sigma}/2 \rangle \rho(r) \), we obtain

\[
A_z = \left( \frac{\sigma_1}{4} \sin \alpha \sin 2\beta + \frac{\sigma_2}{2} \cos^2 \beta - \frac{\sigma_3}{4} \cos \alpha \sin 2\beta \right) \rho(r) \alpha'(z) , \quad (4.15)
\]

\[
A_t = \left( \frac{\sigma_1}{2} \cos \alpha + \frac{\sigma_3}{2} \sin \alpha \right) \rho(r) \beta(t) .
\]

Introducing the new world-sheet variations into the space-time action action, the theta dependent topological term in the Yang-Mills action then gives rise to a topological term on the world-sheet of the vortex

\[
S_{1+1}^\theta = \frac{\theta_{3+1}}{32\pi^2} \int d^4 x \ F^{\alpha}_{\mu \nu} \tilde{F}^{\alpha}_{\mu \nu} = \frac{\theta_{3+1}}{16\pi^2} \int dz \ dt \ C \alpha'(z) \beta(t) \cos \beta , \quad (4.16)
\]
where
\[ C = \int_0^\infty 2\pi dr \, r \frac{1}{2r} \frac{d}{dr} (\rho^2 - 2\rho)f = \pi. \] (4.17)

Note that \( C \) is obtained by integrating a total derivative and only depends on the values of the profile functions at zero and infinity, namely \( \rho(0) = f(0) = 1 \) and \( \rho(\infty) = f(\infty) = 0 \). Written more covariantly, this leads to the following interaction in the vortex effective action,
\[ L_{1+1}^\theta = -\frac{\theta_{3+1}}{8\pi} \epsilon^{sr} \epsilon^{abc} n^a \partial_s n^b \partial_r n^c \quad s, r = (t, z). \] (4.18)

This is very similar to the case discussed in Ref. [32] and relates the theta angle of the \( \mathbb{CP}^1 \) model to the Yang-Mills theta angle as
\[ \theta_{1+1} = \theta_{3+1} \quad \text{for} \quad SU(2). \] (4.19)

The general result for arbitrary \( N_c \) and \( k \) is more illuminating than the \( SU(2) \) theory. In particular, in the general case the world-sheet theta angle is not equal to the space-time theta angle; the two are related and this relation depends both on \( N_c \) and \( k \).

We may consider the most general colour-flavour rotated gauge field configurations in Eqs. (4.10) and (4.11) and evaluate the topological term on these to produce
\[ L_{1+1}^\theta = -C_{N_c,k} \frac{\theta_{3+1}}{8\pi} \epsilon^{sr} \epsilon^{abc} n^a \partial_s n^b \partial_r n^c. \] (4.20)

The proportionality constant \( C_{N_c,k} \) is again given by the integral of a total derivative,
\[ C_{(N_c,k)} = \int_0^\infty 2\pi dr \, \frac{d}{dr} \left\{ (\rho^2 - 2\rho) (f(r) \text{Tr}(Y_{N_c,k}J_3)) \right\} = \pi k(N_c - k). \] (4.21)

This means that for the \( k \)-vortex, the theta angle of the world-sheet sigma model is determined by \( \theta_{3+1} \) as
\[ \theta_{1+1} = k(N_c - k) \theta_{3+1}. \] (4.22)

So the long-wavelength fluctuations of the world-sheet theory of the \( \mathbb{Z}_{N_c} \) flux tube carrying \( k \) units of magnetic flux, are governed by the effective action
\[ S_{1+1} = \int dz \, dt \left( B_{N_c,k}(\partial_s \tilde{n})^2 - k(N_c - k) \frac{\theta_{3+1}}{8\pi} \epsilon^{sr} \epsilon^{abc} n^a \partial_s n^b \partial_r n^c \right), \] (4.23)

The effective theta angle is an integer multiple of the four dimensional one and is thus guaranteed to respect the invariance of the Higgs vacuum under \( \theta_{3+1} \rightarrow \theta_{3+1} + 2\pi \).

### 4.2.1 Dynamics on the vortex world-sheet

We have seen that the effective long-wavelength dynamics of the \( k \)-vortices in the Higgs vacuum of \( \mathcal{N} = 1^* \) theory with \( SU(N_c) \) gauge group (and with three equal
masses), is given by a $\mathbb{CP}_1$ model for all $N_c$ and $k$. The four dimensional theory being $\mathcal{N} = 1$ supersymmetric, the vortices are non-BPS and the effective world-sheet theory is non-supersymmetric. Thus there are no fermionic super-orientational zero modes. The resulting world-sheet dynamics is different from that of BPS vortex strings in $\mathcal{N} = 2$ SQCD \cite{11, 12} for example.

It is well-known that the value of the theta angle has a strong effect on the IR dynamics of the $\mathbb{CP}_1$ model \cite{23, 24, 25, 26, 38, 32}. First of all the $\mathbb{CP}_1$ model is asymptotically free and so is a strongly coupled theory. This is interesting: the four dimensional field theory is weakly coupled, but the dynamics on the vortex is highly quantum. When $\theta_{1+1} = 0$ and $\theta_{1+1} = \pi$, the model is exactly solvable. Specifically, the spectrum at $\theta_{1+1} = 0$ is known to consist of a single massive $SO(3)$ triplet with an exact S-matrix \cite{24} and the theory has a mass gap. This picture continues to be valid for generic non-zero values of $\theta_{1+1}$. When $\theta_{1+1}$ hits $\pi$, however, something drastic happens. The theory has massless excitations and flows to a $c = 1$ conformal fixed point \cite{37, 34, 25} described by the $SU(2)$ Wess-Zumino-Witten model at level $k = 1$. The spectrum now consists of massless $SU(2)$ doublets. The picture, therefore, is that at generic $\theta_{1+1}$, the doublets are confined and bound into meson-like excitations, transforming as a triplet of $SO(3)$. The singlet state, not having a conserved quantum number, is unstable. It is possible to analyze the spectrum in the vicinity of $\theta_{1+1} = \pi$ \cite{32, 38} and can be interpreted as consisting of “kink-anti-kink” bound states. The string tension between these kinks and anti-kinks (the $SU(2)$ doublets) vanishes as the vacuum angle approaches $\pi$.

The existence of the non-trivial dynamics near $\theta_{1+1} = \pi$ begs the question: how is this reflected in the physics of the four dimensional gauge theory? The situation is particularly intriguing, since nothing obviously drastic happens in the gauge theory when $\theta_{3+1} = \pi/k(N_c - k)$. This merits deeper study, but one obvious possibility is that this concerns the spectrum of confined monopole-dyon states in the Higgs phase. The doublets (kinks) are likely to be the bound states of monopoles with the vortex. These monopole-dyon states exist as massive ’t Hooft-Polyakov monopoles in the $\mathcal{N} = 2^*$ theory in the Coulomb phase. As $\theta_{3+1}$ is dialled, the spectrum of these massive states undergoes a rearrangement and can lead to level crossing between certain mutually non-local states (e.g. the (0,1) monopole and the (1,1) dyon for $SU(2)$). It is possible that the special values of $\theta_{3+1}$ may be the points at which such massive, mutually non-local states become degenerate. If both these states happen to get confined upon breaking the supersymmetry to $\mathcal{N} = 1^*$, then they can appear bound to the magnetic flux tubes. The appearance of such mutually non-local states simultaneously on the world-sheet, may drive the sigma model to an interacting fixed point. We are merely speculating at this stage, but clearly the issue deserves deeper study.
5. The Vortex in the String Dual

The string theory dual of $\mathcal{N} = 1^*$ theory was constructed by Polchinski and Strassler [8] by considering an appropriate deformation of Type IIB string theory on $AdS_5 \times S^5$ background. The undeformed $AdS_5 \times S^5$ background with $N_c$ units of Ramond-Ramond five form flux is dual to the large $N_c$ limit of $SU(N_c)$, $\mathcal{N} = 4$ SYM. The relation between gauge theory and string theory parameters is as follows. The string coupling $g_s$ and the radius of curvature of AdS space are related to the gauge coupling and the 't Hooft coupling respectively as

$$4\pi g_s = g_{YM}^2, \quad \frac{R_{AdS}}{\sqrt{\alpha'}} = (4\pi g_s N_c)^{1/4} \gg 1, \quad C_0 = \frac{\theta_{3+1}}{2\pi} \quad (5.1)$$

where $C_0$ is the Type IIB RR scalar.

The $\mathcal{N} = 1^*$ mass deformation of the $\mathcal{N} = 4$ theory is achieved by switching on a non-normalizable mode for the three-form flux $G_3 = F_3 - \tau H_3$, with $\tau = i/g_s + C_0/2\pi$, the unperturbed Type IIB coupling. The Polchinski-Strassler dual geometry was obtained by treating the $G_3$ flux as a perturbation and solving the Type IIB supergravity equations of motion to linear order in this perturbation. The rich infrared physics of $\mathcal{N} = 1^*$ theory was captured in the string dual using two central ingredients: the Myers dielectric effect [31] and the action of $SL(2, \mathbb{Z})$ duality on the vacua of the theory.

The classical description of the $\mathcal{N} = 1^*$ vacua [8], shows that the scalars get noncommuting expectation values describing fuzzy sphere configurations [35]. In the language of D-branes, this means that that $N_c$ D3-branes on which the parent $\mathcal{N} = 4$ theory lives, acquire non-commuting positions, transverse to their worldvolume. These transverse positions trace out fuzzy $S^2$’s and the configuration can be reinterpreted as 5-branes wrapped on concentric flux supported two-cycles carrying $N_c$ units of D3-brane charge. The full large $N_c$, IIB string dual background interpolates between the “near-shell” geometry generated by the multiple 5-branes and the asymptotically AdS solution towards the boundary of the space. Different $\mathcal{N} = 1^*$ vacua, with the theory realized in different phases, are obtained by the action of the IIB $SL(2, \mathbb{Z})$ transformations on a given fivebrane configuration.

5.1 Polchinski-Strassler Higgs Vacuum

The Polchinski-Strassler description of each $\mathcal{N} = 1^*$ vacuum consists of an asymptotically AdS geometry with a $G_3$ flux turned on, matching onto an interior geometry
generated by \((c, d)\) 5-branes. For instance, classical vacua preserving an \(SU(p)\) gauge symmetry, where \(p\) is a divisor of \(N_c\) are described by \(p\) coincident D5 branes carrying a net D3 charge, provided

\[
\frac{q}{p g_s} \gg 1, \quad q = \frac{N_c}{p}.
\] (5.2)

When \(p\) and \(q\) are such that \(p g_s/q \gg 1\), the vacuum with \(SU(p)\) gauge symmetry is described by \(q\) NS5 branes.

The Higgs vacuum is thus described by a single D5 brane carrying net D3 charge, when

\[
\frac{N_c}{g_s} \gg 1
\] (5.3)

while the confining vacuum is described by a single NS5 brane when

\[
N_c g_s \gg 1.
\] (5.4)

The former is an extremely weak condition in the large \(N_c\) limit, while the latter is the usual condition for the gauge theory to be strongly coupled. The two vacua and these two conditions for the validity of the Polchinski-Strassler supergravity description in the “far-from-shell” region, are exchanged under the S-duality, \(g_s \leftrightarrow 1/g_s\).

In the Higgs vacuum, with \(m_1 = m_2 = m_3 = m\), the metric in the interior matches onto the geometry generated by a D5-brane wrapped on an \(S^2\) carrying \(N_c\) units of D3-charge. The D3-brane worldvolume coordinates are \(x^\mu\), \((\mu = 0, 1, 2, 3)\) wherein the field theory lives. The six transverse directions are denoted as

\[
w^i = x^{7,8,9}, \quad \text{and} \quad y^i = x^{4,5,6}.
\] (5.5)

The D3 branes spread out along the \(w^i\) directions with \(y^i = 0\) and the resulting D5-brane wraps a round sphere of radius

\[
r_0 = \pi \alpha' m N_c.
\] (5.6)

The string frame metric of the Polchinski-Strassler solution corresponding to the Higgs vacuum (D5-brane) with equal masses for the adjoint chiral multiplets and with \(\theta_{3+1} = 0\), is

\[
d s_{\text{string}}^2 = Z_x^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + Z_y^{1/2} (dy^2 + y^2 d\Omega_y^2 + dw^2) + Z_{\Omega}^{1/2} w^2 d\Omega_w^2,
\] (5.7)

where

\[
Z_x = Z_y = \frac{R_{\text{AdS}}^4}{\rho_+^2 \rho_-^2}, \quad Z_{\Omega} = \frac{R_{\text{AdS}}^4 \rho^2}{\rho_+^2 (\rho_+^2 + \rho_-^2)^2}, \quad \rho_\pm = \sqrt{y^2 + (w \pm r_0)^2}.
\] (5.8)
The parameters in the metric are
\[ R_{\text{AdS}}^4 = 4\pi g_s N_c \alpha'^2, \quad \rho_c = (\alpha' m) \sqrt{g_s N_c \pi} . \] (5.9)
and \( r_0 \) is as defined in (5.6). The dilaton is non-constant, approaching its asymptotic value \( g_s \) as \( w^i, y^i \to \infty \), but vanishing close to the D5/D3 brane,
\[ e^{2\Phi} = g^2 \frac{\rho^2}{\rho_-^2 + \rho_c^2}, \quad C_0 = \theta_{3+1} = 0. \] (5.10)
The Polchinski-Strassler (including the R-R and NS-NS potentials) background has a manifest \( SO(3) \) isometry acting on the \( S^2 \) in the \( w \)-plane. This naturally gets identified with the colour-flavour locked \( SO(3)_{C+F} \) symmetry of the Higgs phase of \( \mathcal{N} = 1^* \) theory.

The full solution above is approximate, and is constructed by matching the \( r \gg r_0 \) limit with the “near-shell” solution for \( r \approx r_0 \). The “far-from-shell” region is well approximated by the background generated by D3-brane charge density spread out on a spherical shell and is close to a Coulomb branch configuration. In this regime the D3-brane charge density dominates over the D5 charge density. The “near-shell” regime is well described by the exact solution for a flat D5-brane with D3-brane charge \([41, 42]\). Here the effect of the D5-brane dominates. The flat D5 solution and the matching used by Polchinski and Strassler are reviewed in the Appendix.

An important feature of the Higgs vacuum geometry is that near the D5-brane, supergravity ceases to be applicable. This is due to large transverse curvatures near the D5-brane. We will, however, adopt a pragmatic approach and use the metric for our subsequent analysis, with the aim of identifying certain aspects of the gauge theory vortex dynamics that are robustly captured by the string dual. The questions that we are interested in, will, perhaps surprisingly, turn out to be insensitive to the strongly curved parts of the geometry.

The main ingredient we will need from the region near the spherical D5 shell, is the NS-NS two-form potential (see the Appendix for further details),
\[ B_2 = -\frac{\alpha' \pi N_c}{1 + \rho_-^2 / \rho_c^2} \sin \theta_w \, d\theta_w \wedge d\phi_w . \] (5.11)
Near the shell there are also the non-vanishing R-R potentials, \( C_2 \) and \( C_4 \), which can be extracted from the flat D5 background discussed in Appendix. The form of the R-R potentials at \( \theta_{3+1} = 0 \) will not be relevant for the probe branes that we will study in this section.

The Higgs vacuum solution at any non-zero theta angle for the gauge theory follows upon acting on the solution with \( \theta_{3+1} = 0 \) with an \( SL(2, \mathbb{R}) \) transformation.
in IIB supergravity. Under such a transformation, $\tau \rightarrow \tau + \frac{\theta_{3+1}}{2\pi}$, the 5-brane remains a D5-brane, but the two R-R potentials $C_2$ and $C_0$ are shifted as,

$$C_0 \rightarrow C_0 + \frac{\theta_{3+1}}{2\pi}, \quad C_2 \rightarrow C_2 + \frac{\theta_{3+1}}{2\pi} B_2.$$  \hspace{1cm} (5.12)

Thus at non zero $\theta_{3+1}$, the R-R two form $C_2$ acquires components along the $d\theta_w \wedge d\phi_w$ direction which will be relevant for our probes.

### 5.2 The Vortex as a D1-brane

Magnetic flux tubes in the Higgs vacuum have a natural brane interpretation as bound states of D1-branes with the D5-brane. Such D1-D5/D3 bound states involving the single D5-brane in the Higgs vacuum are possible due to the non-vanishing $B_2$ potential on the 5-brane, responsible for the D3-charge. In this situation, the bound state is a semiclassical instanton of the non-commutative field theory [40] on the 5-brane.

We first review the computation of the D-string tension, dual to a $k = 1$ vortex, in the Higgs vacuum geometry, first done in [8]. The two new ingredients in our analysis will be a derivation of the world-sheet sigma model of the vortex and the effect of the Yang-Mills theta angle.

Following [8], we model a magnetic vortex as a D1-brane probe in this geometry. The DBI action for the probe D1 brane in the geometry reads,

$$S_{\text{DBI}} = \frac{1}{2\pi \alpha'} \int d^2 \xi \left\{ e^{-\Phi} \sqrt{-\text{det}(G_{ab} + B_{ab} + 2\pi \alpha' F_{ab})} \right\}. \hspace{1cm} (5.13)$$

Let us consider a D1-brane oriented in the $x_0, x_1$ directions and the embedding $(\xi_0, \xi_1) = (x_0, x_1)$, so that the pullback of the metric onto the world-sheet is

$$G_{00} = -Z^{-1/2}_x + \ldots, \quad G_{11} = Z^{-1/2}_x + \ldots.$$  \hspace{1cm} (5.14)

The dots correspond to terms involving fluctuations of the string in the transverse $\vec{y}$ and $\vec{w}$ directions, proportional to the derivatives $\partial_0(\vec{y}, \vec{w})$, $\partial_1(\vec{y}, \vec{w})$. While the fluctuations along $\vec{y}$ coordinates cannot be studied in supergravity due to large curvatures, angular fluctuations in the $\vec{w}$ directions will appear to be accessible. The part of the DBI action which does not depend on these derivatives and yields the effective tension of the D1-brane is,

$$S_{\text{DBI}} \approx \frac{1}{2\pi \alpha'} \int d^2 x (Z^{-1/2}_x e^{-\Phi}) \hspace{1cm} (5.15)$$

$$= \frac{1}{2\pi \alpha'} \int d^2 x \sqrt{y^2 + (w + r_0)^2} \sqrt{y^2 + (w - r_0)^2} + \rho_c^2 \frac{g_s R_{\text{AdS}}^2}{c^2}. $$
This is minimized when,
\[ w = r_0 + \sqrt{r_0^2 - \frac{2\rho_c^2}{r_0^2}} \approx r_0 - \frac{\rho_c^2}{2r_0}, \quad y = 0. \tag{5.16} \]
Since \( \rho_c \ll r_0 \), the probe D1-brane sits at a relatively small distance \( \delta w \) from the D5 shell,
\[ \delta w = \frac{\rho_c^2}{2r_0} \approx \frac{g_s \alpha' m}{2}. \tag{5.17} \]
The dilaton and the functions that determine the metric at this point evaluate to
\[ e^{-\Phi} \approx 2\sqrt{\frac{\pi N_c}{g_s^2}}, \quad Z_x^{-1/2} \approx \frac{\sqrt{g_s N_c \pi \alpha' m^2}}{2}, \quad \frac{Z_{\Omega}^{1/2}}{2} = \frac{1}{2} \sqrt{\frac{g_s}{(\pi N_c)^3 \alpha' m^2}}. \tag{5.18} \]
The curvature of the space transverse to the D5-brane, i.e., in the radial \( w \) and \( \vec{y} \) directions can be seen to be substringy at the value of \( w \) giving the location of the D1-brane. However, as we see below, this value for \( w \) corresponds to a large radius sphere in the \( w^{1,2,3} \) space.

Continuing to use the metric (5.7), the tension of the D1-brane at this location is
\[ T_{D1} \approx \frac{2r_0 \rho_c}{g_s R_{AdS}^2} \frac{1}{2 \pi \alpha'} = \frac{N_c m^2}{2 g_s} = \frac{2 \pi N_c m^2}{g_s^2 Y_M}. \tag{5.19} \]
Remarkably, this formula matches the BPS formula (3.29) for \( k = 1 \), in the Higgs vacuum which is only expected to work for softly broken \( \mathcal{N} = 2^* \) theory. This is suggestive that the vortices at large \( N_c \) (and \( N_c/g_s \gg 1 \)) in the Higgs vacuum become BPS objects. Equivalently, the confining strings at large \( N_c \) and \( g_s N_c \gg 1 \) (S-dualizing the Higgs vacuum) obey the BPS tension formula. More evidence in support of this possibility was offered in \[8\]. This is also in agreement with our semiclassical results for large \( N_c \).

At its equilibrium position the D1-brane is transverse to a two-sphere at \( y = 0 \) and \( w = r_0 - g_s \alpha' m/2 \), which is concentric with the dielectric 5-brane sphere. At this location the radius of the transverse two-sphere is
\[ Z_{\Omega}^{1/4} w \approx \frac{R_{AdS}}{2} \approx \frac{1}{2} (4 \pi g_s N_c)^{1/4} \sqrt{\alpha'}. \tag{5.20} \]
Clearly, this is large in string units in the supergravity limit. The D-string is pointlike on the transverse two-sphere, resulting in a \( \mathbb{CP}^1 \) moduli space of vortex solutions. The sphere is large in string units and we can allow for a slow, adiabatic variation of the D1-brane position on the sphere, as a function of the world-sheet coordinates. The polar coordinate of the D-string \( \vec{n}_w \equiv (\theta_w, \phi_w) \), corresponds to the vortex colour-flavour zero mode.
Let us consider an arbitrary dependence of $\vec{n}_w$ on the world-sheet coordinates $(x_0, x_1)$ and introduce this into the DBI action. Taking into account only the contribution of the pullback of the spacetime metric, the following world-volume action results,

$$ S_{\text{DBI}} = \int d^2 x \frac{e^{-\Phi}}{2\pi\alpha'} \sqrt{\text{Det} \left( -Z_x^{-1/2} + (\partial_0 \vec{n}_w)^2 w^2 Z^{1/2}_\Omega (\partial_0 \vec{n}_w) \cdot (\partial_1 \vec{n}_w) w^2 Z^{1/2}_\Omega \right)} \left( (\partial_0 \vec{n}_w) \cdot (\partial_1 \vec{n}_w) w^2 Z^{1/2}_\Omega Z^{-1/2}_x + (\partial_1 \vec{n}_w)^2 w^2 Z^{1/2}_\Omega \right) $$

In this formula we have actually omitted the pullback of the $B_2$ field, which only contributes to a four-derivative term in the vortex world-volume action that we neglect. At the two-derivative level, we find

$$ S_{\text{DBI}} \approx T_{D1} \int d^2 x \left( 1 + (\partial_s \vec{n}_w)^2 w^2 Z^{1/2}_\Omega Z^{1/2}_x \right), $$

where $T_{D1}$ is the tension of the $k = 1$ vortex in Eq.(5.19). From this we get the coefficient of the kinetic term of the $\mathbb{CP}^1$ sigma model,

$$ L_{\text{kin}} = \frac{N_c}{4g_s} (\partial_s \vec{n}_w)^2 = \frac{\pi N_c}{g_{YM}^2} (\partial_s \vec{n}_w)^2 . $$

While it is interesting to perform the above formal manipulations, it is not clear that the classical coupling constant of the sigma model is significant since the $\mathbb{CP}^1$ model is asymptotically free and the coupling constant will run when the sigma model is quantized. Secondly, the coupling can also get large corrections due to stringy effects from the highly curved transverse parts of the geometry. The coefficient of the topological term in the $\mathbb{CP}^1$ model, on the other hand, has special significance.

We now turn to evaluating this from the D1-brane action.

**5.2.1 The Theta term**

To complete the supergravity analysis of the $k = 1$ vortex, we will now see how the Yang-Mills theta terms feeds into the world-sheet sigma model. This feeding-in occurs through the Chern-Simons terms of the D1-brane world-volume theory,

$$ S_{\text{CS}} = \frac{1}{2\pi\alpha'} \int \left[ \exp(2\pi\alpha' F_2 + B_2) \wedge \sum_q C_q \right] = \frac{1}{2\pi\alpha'} \int [C_2 + C_0 B_2] $$

We expect this term to have universal, robust features for two reasons. First, the effect of a non-zero $\theta_{3+1}$ has to be such that physics is periodic under shifts of the
theta angle by $2\pi$. This is particularly true in the Higgs vacuum which is actually invariant under such shifts, while the confining vacuum gets mapped to an oblique-confining phase under the same operation. Furthermore, the Wess-Zumino term in the D-brane action is insensitive to the background metric and may well capture the correct physics even in the supergravity approximation.

Since the probe D1-brane is located relatively very close to the D5-brane shell, we need to use the expressions for $B_2$, $C_2$ and $C_0$ given in (5.12), (5.11) and (A.6) for the “near-shell” region. There is a subtlety surrounding the Wess-Zumino couplings of D-branes involving the pullback of the $B_2$ field (such as $C_0 \wedge B_2$ in Eq. (5.25)), which has been discussed in the references [43]. The upshot of this is that the contribution to Wess-Zumino terms from the pullback of $B_2$ have to be omitted. The term $C_0 \wedge B_2$ is an effective D(-1) ‘charge’ for our probe D1, arising from the pullback of the $B$ field. In our case the contribution corresponds to a D1 world-sheet wrapping an $S^2$ with $|\vec{w}| \approx r_0 - \rho^2/2r_0$, which is a homotopically trivial cycle. From the results of [43], this term is cancelled by bulk contributions, so we have to drop it from the present calculation.

Finally then, the only relevant term is the pullback of the components of $C_2$ along the sphere at constant $w$ and $y = 0$,

$$S_{CS} = \frac{1}{2\pi \alpha'} \int C_2 = \frac{1}{2\pi \alpha'} \frac{\theta_{3+1}}{2\pi} \int B_2|_{\theta_{1+1}=0}.$$  \hspace{1cm} (5.25)

In the large $N_c$ supergravity limit, the magnitude of the near shell $B$ field (5.11) at the the radial position of the D1-brane, is equal to $\alpha' \pi N_c$. The result is a two-derivative theta term for the effective $\mathbb{C}P^1$ sigma model,

$$\mathcal{L}_\theta = \frac{\theta_{1+1}}{8\pi} \epsilon^{sr} \vec{n}_w \cdot (\partial_s \vec{n}_w \times \partial_r \vec{n}_w) , \quad \theta_{1+1} = N_c \theta_{3+1} \quad (s, r) = x^{0,1}. \hspace{1cm} (5.26)$$

This is consistent (at large $N_c$) with our semiclassical field theory calculation done in the previous section. Note that if we were to keep the term proportional to $C_0 \wedge B_2$, we would find $\theta_{1+1} = 2N_c \theta_{3+1}$, which would be inconsistent with our semiclassical expectation.

5.3 The $k$-vortex as a D3 brane

Although we were able to reproduce the sigma model of the $k = 1$ vortex using the dual geometry, the tensions and the theta terms for $k$-strings have a nontrivial dependence on $k$. It is not a priori clear that the Polchinski-Strassler geometry should be able to reproduce these since the IR geometry for the Higgs vacuum will
receive large stringy corrections. However, as before, we will attempt to identify the appropriate D-brane configuration dual to a magnetic $k$-string and investigate whether this can compute the tensions and world-sheet parameters reliably.

It is now well understood in a variety of different confining backgrounds [28] that $k$-string tensions with $k$ of order $N_c$ in large $N_c$ theories, are computed by expanded brane configurations. A collection of multiple probe F/D-strings can blow up into higher dimensional D-branes by a version of the Myers effect, wrapping topologically trivial cycles. A similar, very closely related phenomenon also occurs for Wilson loops in general tensor representations involving sources of varying $N$-ality in large $N_c$ gauge theories [44, 46, 45]. In all these cases, the expanded brane configuration carries a net $k$-string charge by virtue of world-volume electric or magnetic fields.

![Figure 4: The minimal energy configuration for the probe D3 brane in the $\vec{w}$ space is given by the red disc and the blue “polar cap” shown in the figure. The “polar cap” part is located very close to the D5 sphere.](image)

In our large $N_c$ dual to the Higgs vacuum, we look for candidate branes that correspond to $k$-vortices with

$$k \to \infty, N_c \to \infty \quad \text{and} \quad \frac{k}{N_c} \text{ fixed.} \quad (5.27)$$

The most natural object is a D3-brane with topology $\mathbb{R}^{1,1} \times S^2$, and a nonvanishing (magnetic) $F_2$ flux along its compact directions located near the D5 shell. Our candidate probe D3-brane is sketched in Figure 4. It is located at $\vec{y} = 0$ and has the topology of an $S^2$ (but not the shape) in the $\vec{w}$ space. We believe this to be the correct configuration for large enough $k$. For finite or small $k$, the $S^2$ of the probe D3-brane will be a small, smooth, squashed sphere located at some polar angle along the D5-sphere. As $k$ is increased, the squashed sphere grows in size with $k$. Most of this probe D3 sphere will want to stay near the D5 shell where the $B_2$ field is concentrated, in order to minimize its tension. The brane will, however, remain blown up due to the magnetic field on its worldvolume which provides it with the
The requisite D-string charge. The candidate probe brane also breaks the $SO(3)$ isometry to $U(1)$ rotations around the $w_3$ axis, as we expect for the $k$-vortices.

### 5.3.1 A warm-up

As a warm-up exercise, let us compute the DBI action for a spherically symmetric configuration at constant $|\vec{w}|$, with $k$ units of uniform flux on top of it. This will have higher tension than the D-brane in Figure 4. We choose a constant world-volume magnetic field $F_2$ along the $(\theta_w, \phi_w)$ directions and proportional to the volume form of $S^2$,

$$F_2 = \frac{k}{2} \sin \theta_w d\theta_w \wedge d\phi_w. \quad (5.28)$$

The magnetic field induces a D-string charge $k$ for the spherical D3-brane, through its Chern-Simons coupling

$$S_{CS} = \frac{1}{(2\pi)^3\alpha'^2} \int_{S^2 \times \mathbb{R}^{1,1}} 2\pi\alpha' F_2 \wedge C_2 = \frac{1}{2\pi\alpha'} k \int_{\mathbb{R}^{1,1}} C_2. \quad (5.29)$$

The tension for the spherical D-brane will be obtained by minimizing the DBI action with respect to $w$.

$$S_{DBI} = 4\pi \int d^2x \left( \frac{Z_{x}^{-1/2} e^{-\Phi}}{(2\pi)^3\alpha'^2} \sqrt{Z_\Omega w^4 + 4\pi^2 \left( \frac{k\alpha'}{2} - \frac{N\alpha'}{2} \frac{1}{1 + \frac{(w-r_0)^2}{r_0^2}} \right)^2} \right). \quad (5.30)$$

The tension is minimized at $w \approx r_0$ (in the large $N_c$ limit) and we obtain for the $SO(3)$ symmetric setup

$$T_{D3} \approx 2\pi \frac{m^2}{g_{YM}^2} N_c (N_c - k). \quad (5.31)$$

We will see that the tension of the configuration in Figure 4 will be lower than the above and so the $SO(3)$ symmetric $k$-string cannot be stable.

### 5.3.2 The $k$-vortex

Now let us compute the energy of the configuration in Fig. 4. It consists of two parts: i) one which is a piece of a sphere subtending the solid angle parametrized by

$$0 \leq \theta_w \leq \bar{\eta}_k, \quad \text{and} \quad 0 \leq \phi_w \leq 2\pi,$$

that we refer to as the polar cap, and ii) a disc glued to the bottom of the cap. The two parts are distinguished by the fact that polar cap lies close to the D5 shell at $w \approx r_0$ where the $B_2$ field reaches its maximum. Since $B_2$ is non-zero only within a

---
thin region \(\mathbf{5.11}\) of width \(\rho_c \ll r_0\), the disc portion of the expanded brane only sees a vanishing antisymmetric tensor potential. In fact the geometry seen by the disc in the interior of the D5-sphere is basically flat. As a consequence of this, the polar cap can minimize its tension by having a magnetic field switched on that completely cancels the pullback of \(B_2\). This means that the polar cap is close to the D5-sphere and is practically tensionless (suggesting that it is possibly dissolved in the D5). In fact the entire tension of the configuration arises from the disc. The disc itself cannot shrink since its boundary must match on to the boundary of the polar cap, and the size of the latter is fixed by the net D-string charge. Also, in the absence of any \(B_2\) field in the interior of the D5-sphere, the disc has no world-volume magnetic field.

Let us first discuss the polar cap. The D3 world-volume is parameterized by the coordinates \((x_0, x_1, \phi_w, \theta_w)\). With this portion of the brane at constant \(|\vec{w}| \approx r_0\) and \(\vec{y} = 0\), we take the \(F_2\) field to be proportional to the volume form of the two-sphere

\[
F_2 = \frac{k}{(\cos \bar{\eta}_k - 1)} \sin \theta_w (d\theta_w \wedge d\phi_w)
\]  

(5.33)

where \(0 \leq \theta_w \leq \bar{\eta}_k\) and \(0 \leq \phi_w \leq 2\pi\). The normalization is chosen so that this yields a D-string charge \(k\) for the blown up D3-brane. The D1-charge density is concentrated entirely in the polar cap portion.

The DBI action for the probe D3-brane reads

\[
S_{\text{cap}} = \frac{1}{(2\pi)^3 \alpha'^2} \int d^4 \xi \left\{ e^{-\Phi} \sqrt{\left( - \det (G_{ab} + B_{ab} + 2\pi \alpha' F_{ab}) \right)} \right\}.
\]  

(5.34)

where

\[
G_{ab} + B_{ab} + 2\pi \alpha' F_{ab} =
\begin{pmatrix}
-Z_x^{-1/2} & 0 & 0 & 0 \\
0 & Z_x^{-1/2} & 0 & 0 \\
0 & 0 & Z_{\Omega}^{1/2} w^2 & (2\pi \alpha' F_{\theta \phi} + B_{\theta \phi}) \\
0 & 0 & -(2\pi \alpha' F_{\theta \phi} + B_{\theta \phi}) & Z_{\Omega}^{1/2} w^2 \sin^2 \theta_w
\end{pmatrix}
\]  

(5.35)

From this we find the tension of the cap to be

\[
S_{\text{cap}} =
\int d^2 x \, d\phi_w \, d\theta_w \sin \theta_w \left( \frac{Z_x^{-1/2} e^{-\Phi}}{(2\pi)^3 \alpha'^2} \sqrt{Z_{\Omega} w^4 + 4\pi^2 \left( \frac{k\alpha'}{1 - \cos \bar{\eta}_k} - \frac{N\alpha'/2}{1 + (w-r_0)^2 / \rho_c^2} \right)^2} \right).
\]  

(5.36)

The action for the polar cap needs to be extremized with respect to both \(w\) and \(\bar{\eta}_k\). We find that there is a minimum at which the tension vanishes exactly for \(|\vec{w}| = r_0\) and

\[
(1 - \cos \bar{\eta}_k) = \frac{2k}{N_c}.
\]  

(5.37)
Since the action is positive definite and it vanishes exactly at the above values of \( w \) and \( \bar{\eta}_k \), we have found the global minimum of this contribution to the tension. The pullback of \( B_2 \) is exactly cancelled by the magnetic field and the volume of the cycle goes to zero near \( w = r_0 \) due to vanishing \( Z_\Omega \) and the brane becomes tensionless. At this point we expect significant stringy corrections and the DBI approach is invalid. These will likely change the tension for the polar cap, but it is not clear whether the result will become comparable to the much larger contribution to the tension from the disc at the bottom of the polar cap.

The disc lies for most of its extension at \( |\vec{w}| - r_0 \gg \rho_c \). In this limit the relevant metric (at \( \vec{y} = 0 \)) is

\[
ds^2|_{w-r_0 \gg \rho_c} = \frac{w^2 - r_0^2}{R_{\text{AdS}}^2} dx_\mu dx_\nu \eta^{\mu \nu} + \frac{R_{\text{AdS}}^2}{w^2 - r_0^2} (dw^2 + w^2 (d\theta_w^2 + \sin^2 \theta_w d\phi_w^2))
\]

(5.38)

and the dilaton is simply \( e^\Phi = g_s \). The \( B_2 \) field is also small in this region. Hence, the disc portion of the probe D3 has two space-time directions \((x_0, x_1)\), and two directions in the \( \vec{w} \) space, without fluxes. The warp factors from the two different subspaces cancel out and then the resulting DBI action is equivalent to the one for a membrane in flat space with fixed perimeter. So the tension of the disc is given by its area in flat space,

\[
T_{D3} \approx \frac{1}{(2\pi)^3 \alpha'^2 g_s} \frac{1}{(\pi r_0^2 \sin^2 \bar{\eta})} \left( \frac{m^2}{2 g_s} k(N_c - k) \right).
\]

(5.39)

which is indeed less than the tension of the \( SO(3) \) symmetric ansatz in Eq. (5.31). This gives the tension of the \( k \)-vortex and remarkably, matches our weak coupling semiclassical results and the softly broken \( \mathcal{N} = 2^* \) formula. In this analysis we neglected the small region at \( w - r_0 \lesssim \rho_c \); a more careful DBI analysis yields

\[
T_{D3} = \frac{1}{g_s (2\pi)^2 \alpha'^2} \int_0^{r_0 \sin \bar{\eta}} \frac{1}{\sqrt{(r_0 \cos \bar{\eta})^2 + s^2 + (r_0 \cos \bar{\eta})^2}} \left( \frac{(w - r_0)^2}{(w - r_0)^2 + \rho_c^2} \right) sds.
\]

(5.40)

The formula is obtained following a non-trivial cancellation between the dilaton and the metric warp factors. Clearly for \( |w - r_0| \gg \rho_c \), the result is given by the area of the flat disc. It is a good approximation to ignore \( \rho_c \) relative to the disc radius, since

\[
\frac{\rho_c}{r_0} = \sqrt{\frac{g_s}{N_c \pi}} \ll 1.
\]

(5.41)

It is straightforward to check that a more careful extremization does not change the result at the leading order in \( g_s \).

A rather interesting cross-check of the picture above results when one determines the tension of the same kind of configuration (a polar cap with a disc glued) with
generic values of \( \bar{\eta}_k \), i.e. where \( \bar{\eta}_k \) is allowed to be a free parameter instead of being determined by the \( k \)-string charge. The resulting tension formula is then

\[
T_{D3}(\bar{\eta}_k, k) = \frac{m^2}{2g_s} \left( \frac{N_c^2 \sin^2 \bar{\eta}_k}{4} + N_c \left| \frac{N_c}{2} - k \right| \right),
\]

(5.42)

which, indeed always has a minimum at the value of \( \bar{\eta}_k \) given by Eq. (5.37), as shown in Figure 5.

\[\text{Figure 5:} \text{ Energy as a function of } \bar{\eta}_k \text{ for } N = 10, k = 3. \text{ The blue line is proportional to the energy of the “polar cap”, the red one to the energy of the flat disc and the black line is the sum of the two contributions. The minimum is given by Eq. (5.37).}\]

### 5.3.3 Theta term in world-sheet sigma model

Since our expanded D3-brane configuration breaks the \( SO(3) \) isometry to \( U(1) \), it follows that small fluctuations of the orientations of the configuration along the D5-sphere, will lead to a \( 1 + 1 \) dimensional sigma model with target space \( \mathbb{C}P^1 \). As usual, all potentially interesting physics lies in the theta angle of this sigma model. From this picture it is straightforward to find also the \( \theta_{1+1} \) of the effective \( S^2 \) sigma model. This comes from the following Chern-Simons coupling in the D3-brane theory,

\[
S_{\text{CS}}^{\text{cap}} = \frac{1}{(2\pi)^3 \alpha'^2} \int_{\text{cap}} C_2 \wedge (2\pi \alpha' F_2).
\]

(5.43)

For \( \theta_{3+1} \neq 0 \), the components of \( C_2 \) tangential to the polar cap, can be read off from (5.11) and (5.12). Note that the only non-zero contribution comes from the polar cap, since it has a magnetic field,

\[
F_2 = \frac{N_c}{2} \sin \theta_w (d\theta_w \wedge d\phi_w).
\]

(5.44)

Let us denote with \( \vec{n}_w \), the position of the North Pole at the center of the polar cap and for simplicity, let us orient \( \vec{n}_w \) in the \( w^3 \) direction. We also denote the position vector of any point on the polar cap as

\[
\vec{p} = (\sin \theta_w \cos \phi_w, \sin \theta_w \sin \phi_w, \cos \theta_w).
\]

(5.45)
Now, we want to consider the effect of an infinitesimal displacement of the entire polar cap following from the action of a rotation generator in $SO(3)/U(1)$. We allow this displacement to have an adiabatic dependence on the non-compact coordinates $x^0$ and $x^1$. Under the infinitesimal change $\vec{n}_w$ transforms as

$$\vec{n}_w \rightarrow \vec{n}_w + \partial_0 \vec{n}_w \, dx^0 + \partial_1 \vec{n}_w \, dx^1. \quad (5.46)$$

The corresponding displacement for a generic point $\vec{p}$ on the polar cap is,

$$\vec{p} \rightarrow \vec{p} + \partial_0 \vec{p} \, dx^0 + \partial_1 \vec{p} \, dx^1. \quad (5.47)$$

It is easily seen that the variations in the position vector $\vec{p}$ are related to the change in $\vec{n}_w$ as

$$\partial_s \vec{p} = (\vec{n}_w \times \partial_s \vec{n}_w) \times \vec{p}. \quad (5.48)$$

Armed with these relations between the variations in $\vec{p}$ and those in $\vec{n}_w$, we turn to the Chern-Simons term (5.43) in the D-brane action. First of all the pullback of $C_2$ can be re-expressed in terms of the unit vectors corresponding to the location of each point on the polar cap

$$P[C_2] = -\frac{\theta_{3+1} \alpha' N_c}{2} (\partial_0 \vec{p} \times \partial_1 \vec{p}) \cdot \vec{p} \, dx^0 \wedge dx^1, \quad (5.49)$$

which can be written in terms of the unit polar vector $\vec{n}_w$ as

$$P[C_2] = -\frac{\theta_{3+1} \alpha' N_c}{2} \cos \theta_w (\partial_0 \vec{n}_w \times \partial_1 \vec{n}_w) \cdot \vec{n}_w \, dx^0 \wedge dx^1. \quad (5.50)$$

We substitute this in Eq. (5.43) and integrate over the polar angles of the cap to obtain the topological term in the world-sheet sigma model of the $k$-vortex

$$\mathcal{L}_\theta = -\frac{k(N_c - k) \theta_{3+1}}{8\pi} \epsilon^{\sigma \tau} \vec{n}_w \cdot (\partial_\sigma \vec{n}_w \times \partial_\tau \vec{n}_w). \quad (5.51)$$

Once again, despite potential issues with regard to the high curvatures in the vicinity of the D5-sphere, the answer is in agreement with the physics at $g^2_{YM} \ll 1$ and is invariant under shifts of the Yang-Mills theta angle by multiples of $2\pi$. As before, one likely reason for the robust nature of the result is that it originates from the Chern-Simons couplings of the D3-brane.

The coefficient of the kinetic term $\tilde{B}_{N_c,k}$ is a trickier issue in the probe D3 approach. We have already remarked that this will flow in the quantized sigma model. Nevertheless it is an object that we can formally estimate. The contribution to this quantity from the cap is zero; there is a non-zero contribution from the disc that we will estimate here. Let us parameterize the disc in the $\vec{w}$ space with the coordinate $\vec{q}$:

$$\vec{q} = (s \cos \phi, s \sin \phi, r_0 \cos \vec{\eta}_k). \quad (5.52)$$
The infinitesimal variation of $\vec{q}$ is

$$\frac{\partial \vec{q}}{\partial x^s} = \left( \vec{n}_w \times \frac{\partial \vec{n}_w}{\partial x^s} \right) \times \vec{q},$$

which upon inserting in the DBI action, yields

$$L_{\text{kin}} = \tilde{B}_{N_c,k} \left( \frac{\partial \vec{n}_w}{\partial x^s} \right)^2,$$

where

$$\tilde{B}_{N_c,k} = \int ds d\phi \left( \frac{R_{\text{AdS}^5} \left( s^2 \cos^2(\phi) + \cos^2(\bar{\eta}_k) r_0^2 \right) \sqrt{\frac{(r_0 - w)^2 + \rho_c^2}{(s^2 + \cos^2(\bar{\eta}_k) r_0^2) (r_0 - w)^2 + \rho_c^2}}}{16 g_s \pi^3 \alpha'^2 (w + r_0)^2 (r_0 - w)^2 + \rho_c^2} \right),$$

which scales as $1/\sqrt{g_s}$ and not as $1/g_s$ as we would expect from the field theory and also from the D1 probe calculation. The contribution from the interior of the disc is $O(g_s^0)$. The leading contribution of order $O(1/\sqrt{g_s})$ comes from the boundary of the disc at $w - r_0 \approx \rho_c$. This is the region nearby the intersection between the cap and the disc and indeed there we can not trust our guess for the shape of the D5-brane probe.

### 5.4 Relation to the Baryon Vertex

In the context of gravity duals of gauge theories, there exists a close relationship between baryons and flux tubes. Flux tubes are made of the same material as baryons, first seen in the context of the baryon vertex in $\mathcal{N} = 4$ theory [47, 48] and subsequently for gravity duals of confining gauge theories [49, 50, 51]. In all these examples, a baryon vertex with $N_c$ strings attached, represented by a wrapped D5-brane for example, can be deformed and pulled apart into groups of constituent quarks connected by a flux tube. The portion of the D5-brane that looks like a flux tube in the gauge theory is obtained by the 5-brane wrapping an $S^4 \subset S^5$.

Polchinski and Strassler [8] argued that a D5-brane baryon vertex of the UV $\mathcal{N} = 4$ theory, when taken to the IR by moving towards the interior of the $\mathcal{N} = 1^*$ geometry, eventually meets the dielectric 5-brane spheres in the interior. In the confining vacuum when the baryon vertex is moved past the NS5-sphere, by the Hanany-Witten process of brane creation [52], the D5-brane baryon vertex turns into a D3-brane ball filling the space inside the NS5-sphere. Following a similar logic applied to a D5-brane wrapping an $S^4$ inside the $S^5$ in the far UV geometry, in the IR we would expect the $\mathbb{Z}_{N_c}$ flux tube to be a D3-brane with world-volume...
\[ \mathbb{R}^{1,1} \times D_2 \] where the \( D_2 \) is a disc stretching inside and ending on the dielectric sphere. The magnetic flux tube would essentially be the same type of object, obtained by S-duality on the confining vacuum. It is encouraging to see that the crucial portion of our candidate \( k \)-vortex, the expanded D3-brane, is precisely such a D3-brane disc. The tension of the magnetic flux tube arises entirely from this disc. Nevertheless, the polar cap was crucial for providing the boundary condition that stabilized the disc, and for providing the magnetic field responsible for \( k \) units of D-string charge. It would be interesting to understand better, the precise connection between the two slightly differing pictures.

### 5.5 Confining Vacuum

We conclude our discussion on flux tubes in the Polchinski-Strassler dual, with a brief analysis of \( k \)-strings in the confining vacuum. First of all we note that the tension of the \( k \)-string in the confining phase at strong coupling \( g_{YM}^2 N_c \gg 1 \) is simply the S-dual of the magnetic \( k \)-string tension in the Higgs vacuum at weak coupling or \( N_c / g_{YM}^2 \gg 1 \). Hence we learn that at least at large \( N_c \), and large ’t Hooft coupling, the confining \( k \)-strings must obey a Casimir scaling law,

\[ T_{N_c,k}^{\text{confining}} = m^2 \frac{g_{YM}^2}{8\pi} k(N_c - k). \]  

A direct confirmation of this from the corresponding expanded D3-brane in the confining vacuum geometry would be useful, but we leave this for future study. The Casimir scaling for confining string tensions is in contrast to previously encountered sine laws and approximate sine laws in other confining theories [29, 30, 28]. The confining vacuum is manifestly \( SO(3) \) invariant in the absence of any VEVs for the adjoint scalars.

Below we outline the calculation of the tension for a \( k = 1 \) flux tube in the confining vacuum (first done in [8]) to see how it is consistent with the action of S-duality. The configuration corresponding to the confining vacuum is an NS5-brane wrapped on a sphere. The corresponding supergravity background can be found via S-duality from the Higgs vacuum with the fields and the parameters transforming as

\[ g_s \to \tilde{g}_s = \frac{1}{g_s}, \quad \alpha' \to \tilde{\alpha}' = g_s \alpha', \quad \exp(\Phi) \to \exp(\tilde{\Phi}) = \exp(-\Phi), \]  

\[ ds^2 \to d\tilde{s}^2 = g_s \exp(-\Phi) ds^2, \quad (B_2, C_2) \to (\tilde{B}_2, \tilde{C}_2) = (-C_2, B_2). \]

The profile functions for the new background metric are then

\[ Z_x = Z_{\Omega} = \frac{R_{\text{AdS}}^4}{\rho_+^2 (\rho_-^2 + \rho_+^2)}, \quad Z_y = \frac{R_{\text{AdS}}^4 (\rho_-^2 + \rho_+^2)}{\rho_+^4 \rho_-^4}, \]  

\[ (5.58) \]
with the metric having the same form as (5.7). The parameters in the metric and the dilaton are,

\[ r_0 = (\alpha' m) \pi g_s N_c, \quad \rho_c = (\alpha' m) \sqrt{\pi g_s N_c}, \quad e^{2\Phi} = g_s^2 \rho_c^2 + \rho_c^2. \] (5.59)

This is the background for \( \theta_{3+1} = 0 \). The confining string of the gauge theory is identified with the F-string (dissolved in the NS5 sphere), which couples directly to the string metric. The action for the probe fundamental string is

\[ S_{F1} = \frac{1}{2\pi \alpha'} \int d^2x \left\{ \sqrt{-\det(G_{ab})} + B_{ab} \right\}, \] (5.60)

with the string oriented in the \( x_0, x_1 \) directions. Upon evaluating this action we find

\[ S_{F1} = \frac{1}{2\pi \alpha'} \int d^2x \sqrt{Z_{x}^{-1/2}} \left[ \sqrt{y^2 + (w - r_0)^2} \sqrt{y^2 + (w - r_0)^2 + \rho_c^2} \right]. \] (5.61)

The location of the minimum is the same as in the D5 case and the resulting string tension

\[ T_{F1} = \frac{m^2 g_s N_c}{2}, \] (5.62)

is exactly the S-dual of the vortex tension. At the radial position of the string

\[ |w - r_0| \approx \rho_c^2/2r_0 = \alpha' m/2 \ll r_0, \]

the radius of the sphere with constant \( w \) is \( \sqrt{\alpha' \pi g_s N_c} \) which is large in string units. It is interesting that the flux tube in the strongly coupled confining vacuum appears to break the global \( SO(3) \) invariance, as it is point-like on the sphere. This is counter-intuitive, since the \( SO(3) \) is an exact global symmetry of the confining vacuum and not a colour-flavour locked transformation as in the Higgs vacuum. So we do not expect the confining strings to have any orientational zero modes. This should become manifest upon quantizing the associated sigma model, whereby the quantum wavefunction spreads over the entire classical moduli space and the classical zero modes are removed. To obtain the classical sigma model for the flux tube, we can allow the string to fluctuate in the directions tangential to the sphere and these would give the action for the “classical orientational zero modes” of the confining string flux tube. From the Nambu-Goto action for the string we get

\[ S_{F1} = \int d^2x \frac{1}{2\pi \alpha'} \left[ \text{Det} \left( \begin{array}{cc} -Z_{x}^{-1/2} + (\partial_0 \vec{n}_w)^2 & w^2 Z_{\Omega}^{-1/2} \\
(\partial_0 \vec{n}_w)(\partial_1 \vec{n}_w)w^2 Z_{\Omega}^{-1/2} & Z_{x}^{-1/2} + (\partial_1 \vec{n}_w)^2 \end{array} \right) \right]. \] (5.63)

The effective action at the two-derivative level is

\[ S_{F1} = \int d^2x \left( g_s N_c \frac{\alpha'}{4} (\partial_0 \vec{n}_w)^2 \right). \] (5.64)
It is nice to see that the classical coupling is exactly S-dual to the one for the D-string. The theta dependence of the sigma model action could provide us with further clues about the worldsheet dynamics of these flux tubes. However, the confining background above is for $\theta_{3+1} = 0$. Shifting the Yang-Mills theta angle by multiples of $2\pi$ changes the vacuum to one in an oblique-confining phase and the NS5 brane to a $(1, n)$ 5-brane. If we dial $\theta_{3+1}$ as was done in the Higgs vacuum using an $SL(2, \mathbb{R})$ shift, this does not alter the $B_2$ field, although it does change the R-R potential $C_2$. However, the former does not have components along the NS5 sphere, whilst only the RR two form does and we do not know how this couples to the F1 world-sheet in a simple way.

6. Summary and further questions

In this paper we have first studied solitonic $k$-vortices in the Higgs vacuum of the $SU(N_c) \mathcal{N} = 1^*$ theory with equal adjoint masses, transforming under an $SO(3)_{C+T}$ symmetry group. We have found that for every $k$ and $N_c$ the vortex world-sheet theory is a non-supersymmetric $S^2$ sigma model. Perhaps the most interesting feature of the two dimensional world-sheet theory is the relation between its theta angle and the four dimensional one, $\theta_{1+1} = k(N_c - k)\theta_{3+1}$. This has very specific implications for the IR dynamics of the sigma model which is asymptotically free and for general values of $\theta_{1+1}$, has a mass gap with the spectrum consisting of a triplet of $SO(3)$. When $\theta_{1+1} = \pi$ however, the theory is integrable and the spectrum consists of massless doublets of $SO(3)$ and the theory flows to a $c = 1$ conformal fixed point. The doublets of $SO(3)$ which are confined into meson-like triplet states become deconfined and massless at $\theta_{1+1} = \pi$. However this value of the world-sheet theta angle corresponds to a seemingly non-special value of the spacetime theta angle. We have speculated on the possibility that the values:

$$\theta_{3+1} = \frac{\pi}{k(N_c - k)} ,$$

may correspond to a level crossing in the semiclassical spectrum of massive, mutually non-local monopole-dyon states of the parent $\mathcal{N} = 2^*$ theory in the Coulomb phase. These states would be confined in the Higgs vacuum and appear as $SO(3)$ doublets bound to a $k$-vortex. We have not presented any evidence for this, but clearly, further study of the relation between the vortex world-sheet spectrum and the spectrum of the four dimensional theory will reveal interesting physics.

The relation between the theta angles on the worldsheet and spacetime also implies that instantons of charge one in the $k$-string $\mathbb{C}P^1$ sigma model correspond to multi-instantons in the four dimensional theory.
Yet another feature of our non-Abelian string solutions is that their tensions, evaluated numerically, are extremely well approximated by the Casimir scaling law,

\[ T_{N, k} = \frac{2\pi m^2 k(N_c - k)}{g_{YM}^2}, \]

for large \( N_c \). This is remarkable in that the Casimir scaling formula is known to be valid for vortices in the softly broken \( \mathcal{N} = 2^* \) theory because of “almost \( \mathcal{N} = 2 \) SUSY” and the resulting string solutions are BPS. Our solutions are far from BPS and the agreement with the BPS tension formula, at large \( N_c \) is surprising and needs further explanation. The semiclassical Higgs vacuum is mapped by S-duality to the confining vacuum at strong 't Hooft coupling and so confining string tensions in this regime will also obey Casimir scaling.

We have identified the supergravity duals of the vortex strings in the large \( N_c \) limit of \( \mathcal{N} = 1^* \) theory. The dual IIB string background, due to Polchinski and Strassler, has a parametric regime of validity \( N_c/g_s \gg 1 \), which includes the semiclassical regime of weak gauge coupling. For this reason, and more explicitly from the form of the metric itself which is sourced by a D5-brane in the IR, it is expected that the IR physics of the Higgs vacuum lies outside the regime of supergravity due to large curvatures. We find it surprising therefore, that we were not only able to identify the candidate objects dual to \( k \)-vortices in the large \( N_c \) limit, as expanded D3-branes, but also able to compute their tensions and find an exact Casimir scaling in agreement with the semiclassical results. As a bonus we were able to reproduce the semiclassical relation between the Yang-Mills theta angle and the worldsheet theta angle from the candidate wrapped, D-brane configurations. It would definitely be useful to understand better the reason for this agreement and connect to some kind of large \( N_c \) BPS property.

The Higgs vacuum of \( \mathcal{N} = 1^* \) theory, in the large \( N_c \) limit, has been argued to provide a deconstruction of six dimensional supersymmetric gauge theory compactified on a fuzzy sphere [34]. In this picture, the vortices may be reinterpreted as noncommutative instantons [40] of the six-dimensional theory. This presents a potentially fruitful arena for systematically investigating the Higgs vacuum vortices, at least in the large \( N_c \) limit [53], and may explain some of the surprising features above at large \( N_c \). Finally, a closely related situation arises in the beta deformation of \( \mathcal{N} = 4 \) theory at special values of \( \beta \), wherein the resulting theory in its Higgs phase has been argued to deconstruct Little String Theory [33].

Acknowledgements: The authors would like to thank A. Armoni, C. Hoyos-Badajoz, K. Konishi, B. Lucini, C. Nuñez and A. Yung for discussions.
Appendix A: “Near Shell” Background: Flat D5 and NS5 with D3 charge

Below we quote the form of the “near-shell” background in the Higgs (D5) and confining (NS5) solutions of Polchinski and Strassler. In the Higgs (confining) phase, the “near-shell” solutions are directly read off from the metric in string frame for \( p \) flat D5-branes (NS5 branes) with D3-brane charge bound to them \([41, 42]\):

**D5/D3 background:**

\[
ds^2 = \frac{dx_\mu dx_\nu \eta^{\mu\nu}}{\sqrt{1 + \frac{S^2}{\alpha'^2 u^2}}} + \sqrt{1 + \frac{S^2}{\alpha'^2 u^2}} \cos^2 \varphi + 1 \left(1 + \frac{S^2}{\alpha'^2 u^2} (du^2 + u^2 d\Omega_3^2)\right) \tag{A.1}
\]

where

\[
S = \sqrt{\frac{g_s p a'}{\cos \varphi}}, \tag{A.2}
\]

and \( \tan \varphi \) is proportional to the density of D3 brane charge dissolved on the D5-branes. The dilaton and the \( B \) fields are given by:

\[
e^{2\Phi} = g_s^2 \frac{\alpha'^2 u^2}{S^2 \cos^2 \varphi + \alpha'^2 u^2}, \quad B_{45} = -\frac{S^2 \sin \varphi \cos \varphi}{S^2 \cos^2 \varphi + \alpha'^2 u^2}. \tag{A.3}
\]

The RR potentials are:

\[
C_2 = \pm 2 S^2 \cos \varphi \frac{\sin^2 \theta \cos \phi_1 d\theta \wedge d\phi_2}{g_s}, \tag{A.4}
\]
\[
C_4 = \mp \frac{2 S^2 \sin \varphi}{g_s} \frac{r^2 + S^2/2 \cos^2 \varphi}{r^2 + S^2 \cos^2 \varphi} \sin^2 \theta \cos \phi_1 dx_5 \wedge dx_4 \wedge d\theta \wedge d\phi_2 \tag{A.5}
\]
\[
\pm \frac{\sin \varphi}{g_s} \frac{r^2}{r^2 + S^2} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,
\]

where \((\theta, \phi_1, \phi_2)\) are the standard coordinates in \( S^3 \).

In the decoupling limit \( \alpha' \to 0 \), \( \tan \varphi \to \infty \) with \( \alpha' \tan \varphi = b \) held constant, after the rescaling \( \hat{x}_{4,5} = \tan \varphi x_{4,5} \), the solution reads:

\[
ds^2 = \alpha' \left( \frac{u}{g_s p a} \left(\frac{d\hat{x}_4^2 + d\hat{x}_5^2}{1 + a^2 u^2} + dx_\mu dx_\nu \eta^{\mu\nu}\right) + \frac{g_s p a}{u} (du^2 + u^2 d\Omega_3^2)\right), \tag{A.6}
\]
\[
e^{2\Phi} = g_s^2 \frac{\alpha'^2 u^2}{1 + a^2 u^2}, \quad B_{45} = -\frac{\alpha'}{g_s p a} \frac{1}{1 + a^2 u^2},
\]

where we have defined

\[
a = \sqrt{\frac{\alpha' \tan \varphi}{g_s p}}. \tag{A.7}
\]
The identification used in the Polchinski-Strassler solution to interpolate between the near-shell and the asymptotic metric is,

\[ u = \frac{\rho}{\alpha'}, \quad a = \frac{1}{pm\sqrt{g_s N_c \pi}} \quad \bar{x}^{4,5} = \frac{1}{p \alpha' m^2 \pi N_c} w^{1,2}. \]  

(A.8)

**NS5/D3 background:** The metric in string frame for \( p \) flat NS5-branes with D3-brane charge (in the appropriate decoupling limit) is,

\[ ds^2 = \frac{\alpha' g_s^2}{pa^2} \left( \frac{d\bar{x}_4^2 + d\bar{x}_5^2}{\sqrt{1 + a^2 u^2}} + \sqrt{1 + a^2 u^2} dx_\mu dx_\nu \eta^{\mu\nu} \right) + \alpha' p \frac{\sqrt{1 + a^2 u^2}}{u^2} (du^2 + u^2 d\Omega_3^2). \]  

(A.9)

The dilaton and the \( C_2 \) fields are:

\[ e^{2\Phi} = g_s^2 \frac{1 + a^2 u^2}{a^2 u^2}, \quad C_2 = -\frac{g_s^2 \alpha'}{a^2 p} \frac{1}{1 + a^2 u^2} d\bar{x}_4 \wedge d\bar{x}_5. \]  

(A.10)

In the confining vacuum, the identification used in the PS solution between near-shell and the asymptotic metric is,

\[ u = \frac{\rho}{\alpha' g_s}, \quad a = \frac{\sqrt{g_s}}{p m \sqrt{N_c \pi}} \quad \bar{x}^{4,5} = \frac{1}{p g_s \alpha' m^2 \pi N_c} w^{1,2}. \]  

(A.11)

**References**

[1] N. Seiberg and E. Witten, Nucl. Phys. B 426, 19 (1994) [Erratum-ibid. B 430, 485 (1994)] [arXiv:hep-th/9407087].

N. Seiberg and E. Witten, Nucl. Phys. B 431, 484 (1994) [arXiv:hep-th/9408099].

[2] A. Abrikosov, Sov. Phys. JETP 32, 1442 (19yy).

[3] H. B. Nielsen and P. Olesen, Nucl. Phys. B 61, 45 (1973).

[4] C. Vafa and E. Witten, Nucl. Phys. B 431, 3 (1994) [arXiv:hep-th/9408074].

[5] R. Donagi and E. Witten, Nucl. Phys. B 460, 299 (1996) [arXiv:hep-th/9510101].

[6] N. Dorey, JHEP 9907 (1999) 021 [arXiv:hep-th/9906011].

[7] N. Dorey and S. P. Kumar, JHEP 0002 (2000) 006 [arXiv:hep-th/0001103].

[8] J. Polchinski and M. J. Strassler, “The string dual of a confining four-dimensional gauge theory,” arXiv:hep-th/0003136.
[9] M. J. Strassler, “Messages for QCD from the superworld,” Prog. Theor. Phys. Suppl. 131, 439 (1998) [arXiv:hep-lat/9803009].

M. J. Strassler, “Millennial messages for QCD from the superworld and from the string,” arXiv:hep-th/0309140.

[10] C. Montonen and D. I. Olive, Phys. Lett. B 72, 117 (1977).

[11] A. Hanany and D. Tong, JHEP 0307 (2003) 037 [arXiv:hep-th/0306150].

[12] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B 673 (2003) 187 [arXiv:hep-th/0307287].

[13] D. Tong, Phys. Rev. D 69, 065003 (2004) [arXiv:hep-th/0307302].

[14] M. Shifman and A. Yung, Phys. Rev. D 70, 045004 (2004) [arXiv:hep-th/0403149].

[15] A. Hanany and D. Tong, JHEP 0404, 066 (2004) [arXiv:hep-th/0403158].

[16] M. Shifman and A. Yung, Rev. Mod. Phys. 79 (2007) 1139 [arXiv:hep-th/0703267].

[17] D. Tong, “TASI lectures on solitons,” arXiv:hep-th/0509216.

[18] D. Tong, arXiv:0809.5060 [hep-th].

[19] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A 39 (2006) R315 [arXiv:hep-th/0602170].

[20] V. Markov, A. Marshakov and A. Yung, Nucl. Phys. B 709 (2005) 267 [arXiv:hep-th/0408235].

[21] H. J. de Vega and F. A. Schaposnik, Phys. Rev. Lett. 56 (1986) 2564; H. J. de Vega and F. A. Schaposnik, Phys. Rev. D 34 (1986) 3206; J. Heo and T. Vachaspati, Phys. Rev. D 58 (1998) 065011 [arXiv:hep-ph/9801455]; F. A. Schaposnik and P. Suranyi, Phys. Rev. D 62 (2000) 125002 [arXiv:hep-th/0005109]; K. Konishi and L. Spanu, Int. J. Mod. Phys. A 18 (2003) 249 [arXiv:hep-th/0106175].

[22] M. A. C. Kneipp, Phys. Rev. D 76 (2007) 125010 [arXiv:0707.3791 [hep-th]]; M. A. C. Kneipp, Phys. Rev. D 69 (2004) 045007 [arXiv:hep-th/0308086].

[23] E. Witten, Nucl. Phys. B 149, 285 (1979).

[24] A. B. Zamolodchikov and A. B. Zamolodchikov, Annals Phys. 120, 253 (1979).

[25] A. B. Zamolodchikov and A. B. Zamolodchikov, Nucl. Phys. B 379, 602 (1992).

[26] V. A. Fateev, E. Onofri and A. B. Zamolodchikov, Nucl. Phys. B 406 (1993) 521.

[27] B. Lucini and M. Teper, Phys. Rev. D 64, 105019 (2001) [arXiv:hep-lat/0107007].

B. Lucini, M. Teper and U. Wenger, JHEP 0406, 012 (2004) [arXiv:hep-lat/0404008].
[28] C. P. Herzog and I. R. Klebanov, Phys. Lett. B 526 (2002) 388 [arXiv:hep-th/0111078].

[29] M. R. Douglas and S. H. Shenker, Nucl. Phys. B 447 (1995) 271 [arXiv:hep-th/9503163].

[30] A. Hanany, M. J. Strassler and A. Zaffaroni, Nucl. Phys. B 513 (1998) 87 [arXiv:hep-th/9707244].

[31] R. C. Myers, JHEP 9912 (1999) 022 [arXiv:hep-th/9910053].

[32] A. Gorsky, M. Shifman and A. Young, Phys. Rev. D 71 (2005) 045010 [arXiv:hep-th/0412082].

[33] N. Dorey, JHEP 0408 (2004) 043 [arXiv:hep-th/0310117]; N. Dorey, JHEP 0407 (2004) 016 [arXiv:hep-th/0406104].

[34] R. P. Andrews and N. Dorey, Phys. Lett. B 631 (2005) 74 [arXiv:hep-th/0505107];
R. P. Andrews and N. Dorey, Nucl. Phys. B 751 (2006) 304 [arXiv:hep-th/0601098].

[35] J. Madore, Class. Quant. Grav. 9 (1992) 69.

[36] M. Edalati and D. Tong, JHEP 0705 (2007) 005 [arXiv:hep-th/0703045]; D. Tong, JHEP 0709 (2007) 022 [arXiv:hep-th/0703235]; M. Shifman and A. Young, Phys. Rev. D 77 (2008) 125016 [arXiv:0803.0158 [hep-th]]; M. Shifman and A. Young, Phys. Rev. D 77 (2008) 125017 [arXiv:0803.0698 [hep-th]].

[37] I. Affleck, Nucl. Phys. B 265 (1986) 409; I. Affleck and F. D. M. Haldane, Phys. Rev. B 36 (1987) 5291; I. Affleck, Phys. Rev. Lett. 66 (1991) 2429.

[38] D. Controzzi and G. Mussardo, Phys. Rev. Lett. 92 (2004) 021601 [arXiv:hep-th/0307143].

[39] R. Shankar and N. Read, Nucl. Phys. B 336 (1990) 457.

[40] N. Nekrasov and A. S. Schwarz, Commun. Math. Phys. 198 (1998) 689 [arXiv:hep-th/9802068].

[41] M. Alishahiha, Y. Oz and M. M. Sheikh-Jabbari, JHEP 9911 (1999) 007 [arXiv:hep-th/9909215].

[42] J. C. Breckenridge, G. Michaud and R. C. Myers, Phys. Rev. D 55 (1997) 6438 [arXiv:hep-th/9611174].

[43] C. Bachas, M. R. Douglas and C. Schweigert, JHEP 0005 (2000) 048 [arXiv:hep-th/0003037]; W. Taylor, JHEP 0007 (2000) 039 [arXiv:hep-th/0004141];
A. Alekseev, A. Mironov and A. Morozov, Phys. Lett. B 532 (2002) 350 [arXiv:hep-th/0005244]; J. G. Zhou, Nucl. Phys. B 607 (2001) 237 [arXiv:hep-th/0102178].
[44] N. Drukker and B. Fiol, JHEP 0502, 010 (2005) [arXiv:hep-th/0501109].

[45] S. Yamaguchi, JHEP 0605, 037 (2006) [arXiv:hep-th/0603208].

[46] S. A. Hartnoll and S. Prem Kumar, Phys. Rev. D 74, 026001 (2006) [arXiv:hep-th/0603190].

[47] C. G. Callan and J. M. Maldacena, Nucl. Phys. B 513, 198 (1998) [arXiv:hep-th/9708147].

[48] E. Witten, JHEP 9807, 006 (1998) [arXiv:hep-th/9805112].

[49] C. G., Callan, A. Guijosa and K. G. Savvidy, Nucl. Phys. B 547, 127 (1999) [arXiv:hep-th/9810092].

[50] C. G., Callan, A. Guijosa, K. G. Savvidy and O. Tafjord, Nucl. Phys. B 555, 183 (1999) [arXiv:hep-th/9902197].

[51] S. A. Hartnoll and R. Portugues, Phys. Rev. D 70, 066007 (2004) [arXiv:hep-th/0405214].

[52] A. Hanany and E. Witten, Nucl. Phys. B 492, 152 (1997) [arXiv:hep-th/9611230].

[53] R. Auzzi and S. P. Kumar, Work in progress.