The relativistic dynamics of oppositely charged two fermions interacting with external uniform magnetic field

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\textbf{ABSTRACT}

We investigated the relativistic dynamics of oppositely charged two fermions interacting with an external uniform magnetic field. We chose the interaction of each fermion with the external magnetic field in the symmetric gauge, and obtained a precise solution of the corresponding fully-covariant two-body Dirac equation that derived from quantum electrodynamics via action principle. The dynamic symmetry of the system we deal with allowed us to determine the relativistic Landau levels of such a neutral system, without using any group theoretical method. As a result, we determined the eigenfunctions and eigenvalues of the corresponding two-body Dirac Hamiltonian.

1. Introduction

Landau quantization is the quantization of cyclotron orbits of the motion of a charged particle moving in a magnetic field [1]. In the non-relativistic regime, Landau quantization was discussed for many different cases [2, 3, 4]. In the relativistic regime, Landau quantization firstly was discussed by Jackiw [5] (see also [6]). Afterwards, many experimental and theoretical studies were conducted on this subject [7, 8, 9, 10, 11, 12, 13]. Today, it is though that the magnetic fields exist in all over the spacetime background and they magnetize the universe [14, 15]. We do not know yet exactly the origins of intra-cluster, galactic and cosmological magnetic fields, but it is predicted that dynamo effects in turbulent fluids can exponentially amplify the seed fields [16]. Although it is not easy to fully explain the galactic and cluster magnetic fields with dynamo theory [17], it is suggested that these fields that permeate the universe may have arisen as a result of the compression of a primordial field [16, 18]. Therefore, the determination of the magnetic field effects at all scales from planets to stars, from galaxies to galaxy clusters and even in the inner galactic environment is still one of the important and active research fields [19, 20]. Also, it has been studied for many years that the structure of the vacuum is modified in the presence of electromagnetic fields [21, 22]. The modification of the vacuum can cause some novel phenomena such as photon decays into electron-positron electron pair [23], the birefringence of a photon [22] and splitting of a photon [24]. It is natural to expect that these effects can occur, since the vacuum (in QED) is regarded as filled with electrons. Hence, they can react like an ordinary medium in the presence of external fields. We think that magnetic fields exist at almost every point in the universe [25] and they may be responsible for many interesting physical effects. The effect illustrated in Nambu-Jona-Lasinio model is that a constant magnetic field is a strong catalyst of a dynamical flavour symmetry breaking [26]. It is though that this point may be important in the phenomena like Hall effect in the condensed matter systems [26]. The presence of magnetic fields can deeply affect the dynamics of relativistic and semi-relativistic systems and hence the determination of the dynamics of systems in magnetic field is a subject of great interest in many areas of physics [27]. The origin of the magnetic field differs from one system to another, but in some cases we expose the systems to the effect of external magnetic fields in order to better understand the underlying physics of systems or to determine the effect of the magnetic field on the systems. Of course, the production of magnetic fields can be a natural feature of the systems. In this present manuscript, we will investigate the relativistic dynamics of oppositely charged two fermions interacting with external magnetic field, without detaily discussing the origin of the external magnetic field.

It is worth to mention that, in general, in the relativistic regime phenomenologically established one-time equations are used to describe the relativistic dynamics of interacting particles. These phenomenologically established equations include free Hamiltonians for each particle plus interparticle interaction potentials [28, 29]. It is crucial that there is also two time problem in the determination of relativistic dynamics of two particles. In the literature, the accepted first relativistic two-body equation was introduced by Breit [30]. The history of relativistic two-body equations goes long way back. More details about them can be found in [31, 32]. It is important to note that the equation introduced by Breit does not hold in every situation, due to the retardation effects. Bethe and Salpeter introduced another formalism [33] to overcome this problem. The Bethe-Salpeter formalism provided a different approach to one-electron atom problems, but this formalism was not fully-acceptable for bound-states. In later years, a complete fully-covariant two-body Dirac equation [34] was derived from quantum electrodynamics with the help of action principle. Moreover, this equation is very similar to the former two-body equation introduced by Kemmer, Fermi and Yang [28, 29]. In 3 + 1 dimensions, the fully-covariant two-body equation gives a 16×16 dimensional matrix equation including the most gen-
eral electric and magnetic potentials. But, in 4-dimensions, the solution of this equation requires group theoretical methods [35] and the well-known energy spectrum for Hydrogen-like atoms can be obtained only via a perturbative solution of this equation [35, 36, 37]. Nevertheless, it is showed that the fully-covariant two-body Dirac equation can be exactly solvable for low dimensional systems [38]. In the literature, we can state that there are very few studies based on two-body problem in magnetic field and there is no a non-perturbative energy spectrum in closed form obtained via an exact solution of a well-established fully-covariant two-body Hamiltonian. Previously, the separation of center of mass motion coordinates of a two-body system (neutral) in homogeneous magnetic fields was introduced [39] and then the theoretical foundations of two-body systems were studied in the presence of both homogeneous [40] and inhomogeneous [41] magnetic fields.

In this present paper we hoped to fill this gap in the literature and we investigated the relativistic dynamics of oppositely charged two fermions interacting with an external uniform magnetic field. At the beginning, we wrote the fully-covariant two body Dirac equation in 3-dimensional Minkowski spacetime [38] and we chose the interaction of each fermion with the external magnetic field in the symmetric gauge, since this problem has 2 + 1-dimensional dynamical symmetry [42]. The dynamic symmetry of the system allowed us to determine the eigenfunctions and eigenvalues (in closed-form) of the corresponding two-body Dirac Hamiltonian, without using any group theoretical method.

2. Fully-covariant two-body Dirac equation in 2+1 dimensions

In a general three dimensional spacetime background, the relativistic dynamics of charged two fermions interacting with external uniform magnetic field can be investigated via the following fully-covariant two-body Dirac equation,

\[
\left\{ \left[ i\sigma_a^\mu \gamma_\mu^0 + i b_1 I_2 \right] \gamma_\mu^0 \right\} \Psi (x_1, x_2) = 0, \\
\gamma_\mu^0 = \left( \gamma_0^2 + \gamma_0^1 \right), \\
b_1 = \frac{m_1 c}{\hbar}, \quad b_2 = \frac{m_2 c}{\hbar}, \quad (\eta = 0, 1, 2),
\]

in which the superscripts (1) and (2) refer to the first fermion with the mass \( m_1 \) and second fermion with mass \( m_2 \), respectively. In Eq. (1), \( I_2 \) is \( 2 \times 2 \) dimensional unit matrix, \( \Psi (x_1, x_2) \) is the bi-local Dirac field (massive) that is constructed by a direct product (\( \otimes \)) of arbitrary massive two Dirac fields as follows,

\[
\Psi (x_1, x_2) = Y (x_1) \cdot \chi (x_2).
\]

e_1 and e_2 are charges of these particles, \( c \) is usual Planck constant, the letter \( c \) represents to the light speed, \( A_\eta \) and \( \Gamma_\eta \) correspond to the vector potentials and spinorial affine connections, respectively. Even though Eq. (1) does not seem to be manifestly covariant at first look, the \( \gamma_\mu^0 \) means \( \gamma_\mu^0 I_\eta \) in this equation (\( I_\eta \) is a timelike vector \( I_\eta = (100) \)). Eq. (1) includes spin algebra spanned by direct productions of the Dirac matrices (for more details about its structure see [34]). As we mentioned in above, the problem we deal with can be studied in 2 + 1 dimensional flat Minkowski space-time background that can be represented via the following line element,

\[
ds^2 = c^2 dt^2 - dx^2 - dy^2. \tag{3}
\]

It is clear that the relativistic dynamics of the system we deal with does not changed by spinorial affine connections in Eq. (1), since they vanish [43]. In the presence of the external uniform magnetic field, the vector potentials in Eq. (1) can be chosen in the symmetric gauge as follows,

\[
A_0^{(1)} = 0, \quad A_1^{(1)} = -\frac{B_0 y_1}{2}, \quad A_2^{(1)} = \frac{B_0}{2}, \\
A_0^{(2)} = 0, \quad A_1^{(2)} = -\frac{B_0 y_2}{2}, \quad A_2^{(2)} = \frac{B_0}{2},
\]

in which \( B_0 \) relates with the strength of external uniform magnetic field and \( x_\mu, y_\mu (\mu = 1, 2) \) pairs correspond to the spatial coordinates of the particles in the spacetime background represented by Eq. (3). For the line element given in Eq. (3), the Dirac matrices can be chosen as in the following [38],

\[
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where \( \sigma^x, \sigma^y \) and \( \sigma^z \) are the Pauli spin matrices.

3. Radial equations

As is usual with two-body problems, the center of mass motion coordinates and relative motion coordinates can be separated (covariantly) via help of the following expressions [38],

\[
R_\eta = \frac{1}{M} \left( m_1 x_1^{(1)} + m_2 x_2^{(2)} \right), \quad r_\eta = x_1^{(1)} - x_2^{(2)}, \\
x_\eta^{(1)} = \frac{m_1}{M} r_\eta + R_\eta, \quad x_\eta^{(2)} = -\frac{m_2}{M} r_\eta + R_\eta, \\
\delta x_\eta^{(1)} = \delta_{r_\eta} + \frac{m_1}{M} \delta R_\eta, \quad \delta x_\eta^{(2)} = -\delta_{r_\eta} + \frac{m_2}{M} \delta R_\eta, \\
\delta x_\eta^{(1)} + \delta x_\eta^{(2)} = \delta R_\eta.
\]

It is important to underline that the total energy of the system is determined according to the proper time \( \tau_0 \) of the system, since there is no relative time differences between the first and second fermions. Now, we can assume that the center of mass is located at the origin of the spacetime background \( (R_1 = R_2 = 0) \). At that rate, by substituting Eq. (4), Eq. (5) and Eq. (6) into Eq. (1) one can acquire the following matrix equation,

\[
\left( \gamma^0 \gamma^0 \right) \Psi + \frac{m_1 c}{\hbar} (I_2 \gamma^0) \Psi + \frac{m_2 c}{\hbar} (\gamma^0 I_2) \Psi \\
+ \left( \gamma^1 \gamma^0 \right) \left( \sigma_y + \frac{m_1}{M} \sigma_y - iB_0 \frac{m_2}{M} \gamma^1 \right) \Psi
\]
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\[
+ (\gamma^2 - 1) \left( \partial_t + \frac{m_1}{M} \partial_y - i \frac{B_1 m_1}{M} x \right) \Psi
\]
\[
+ (\gamma^2 - 1) \left( -\partial_x + \frac{m_2}{M} \partial_X + i \frac{B_2 m_2}{M} Y \right) \Psi
\]
\[
+ (\gamma^2 - 1) \left( -\partial_y + \frac{m_1}{M} \partial_Y - i \frac{B_1 m_1}{M} X \right) \Psi = 0,
\]
\[
B_1 = \frac{e_1 B_0}{2hc}, \quad B_2 = \frac{e_2 B_0}{2hc}, \quad \Omega = \begin{pmatrix} 0 & \mu_1 B_{r-} & \mu_1 i K_- & 0 \\ -\mu_1 B_{r+} & 0 & 0 & \mu_1 i K_+ \\ 0 & -\mu_1 i K_- & 0 & -\mu_1 B_{r+} \\ 0 & 0 & \mu_2 B_{r-} & -i K_- \end{pmatrix}, \quad \text{in which } x, y, X, Y \text{ are the spatial coordinates of the relative motion and center of mass motion, respectively.}
\]
\[
\Psi(t, \vec{r}, \vec{R}) = e^{-i\omega t} e^{i \vec{K} \cdot \vec{r} \Omega (\vec{r})},
\]
\[
\Omega (\vec{r}) = \begin{pmatrix} \psi_1 (\vec{r}) \\ \psi_2 (\vec{r}) \\ \psi_3 (\vec{r}) \\ \psi_4 (\vec{r}) \end{pmatrix}, \quad \text{in which \( \omega \) is the total frequency determined according to the proper time of the system and \( \vec{K} \) relates with the spatial momentum of the center of mass motion (\( h \vec{K} \)). For the ansatz Eq. (8) defined for a moving \( \vec{K} \neq 0 \) system formed by oppositely charged (\( e_1 = e, e_2 = -e \)) and arbitrary massive two fermions, by multiplying the Eq. (7) with \( \gamma^0 \gamma^0 \) from left, one can obtain the following matrix equation,}
\[
\begin{pmatrix} \varepsilon - M & \tilde{d}_- & -\tilde{d}_- & 0 \\ -\tilde{d}_+ & \varepsilon - \Delta m & 0 & -\tilde{d}_- \\ \tilde{d}_+ & 0 & \varepsilon + \Delta m & \tilde{d}_- \\ 0 & -\tilde{d}_+ & \varepsilon + M & \tilde{d}_- \end{pmatrix} \Omega (\vec{r}) = 0,
\]
\[
M = \frac{(m_1 + m_2)c}{\hbar}, \quad \Delta m = \frac{(m_1 - m_2)c}{\hbar}, \quad B = \frac{eB_0}{2hc}, \quad K_\pm = K_x \pm i K_y, \quad \mu_q = m_q/(m_1 + m_2), \quad (q = 1, 2, \ldots),
\]
\[
\varepsilon = \frac{\alpha}{c}, \quad \tilde{d}_- = d_x + i d_y, \quad r_+ = x + iy, \quad K_+ = n \pm i K_y, \quad \mu_q = m_q/(m_1 + m_2), \quad (q = 1, 2, \ldots),
\]
\[
\begin{pmatrix} \varepsilon - M & \tilde{d}_- & -\tilde{d}_- & 0 \\ -\tilde{d}_+ & \varepsilon - \Delta m & 0 & -\tilde{d}_- \\ \tilde{d}_+ & 0 & \varepsilon + \Delta m & \tilde{d}_- \\ 0 & -\tilde{d}_+ & \varepsilon + M & \tilde{d}_- \end{pmatrix} \Omega (\vec{r}) = 0,
\]
\[
\begin{pmatrix} \psi_1 (\vec{r}) \\ \psi_2 (\vec{r}) \\ \psi_3 (\vec{r}) \\ \psi_4 (\vec{r}) \end{pmatrix} \mapsto \begin{pmatrix} \psi_1 (r) e^{i(\pm - 1)\phi} \\ \psi_2 (r) e^{i\phi} \\ \psi_3 (r) e^{i\phi} \\ \psi_4 (r) e^{i(\pm + 1)\phi} \end{pmatrix},
\]
\[
\begin{align*}
\varepsilon \psi_1 (r) - M \psi_2 (r) + & \left( 2 \partial_r + i \frac{\Delta m}{M} \right) \psi_3 (r) - B \psi_4 (r) = 0, \\
\varepsilon \psi_2 (r) - M \psi_1 (r) + & \left( \frac{\partial r}{r} - B \frac{\Delta m}{M} \right) \psi_3 (r) - i K \psi_4 (r) = 0, \\
\varepsilon \psi_3 (r) - \Delta m \psi_4 (r) + & \left( \frac{\partial r}{r} - B \frac{\Delta m}{M} \right) \psi_2 (r) = 0, \\
\varepsilon \psi_4 (r) - \Delta m \psi_3 (r) - B \psi_1 (r) + i K \psi_2 (r) = 0,
\end{align*}
\]
\[
\text{in which,}
\begin{align*}
\phi_1 (r) &= \psi_1 (r) + \psi_4 (r), \quad \phi_2 (r) = \psi_1 (r) - \psi_4 (r), \\
\phi_3 (r) &= \psi_2 (r) - \psi_3 (r), \quad \phi_4 (r) = \psi_2 (r) + \psi_3 (r),
\end{align*}
\]
\[
\begin{align*}
\varepsilon \phi_1 (z) - M \phi_2 (z) + & \sqrt{\frac{\gamma}{B}} \partial_z \phi_3 (z) - \sqrt{\frac{\gamma}{B}} B \phi_4 (z) = 0, \\
\varepsilon \phi_2 (z) - M \phi_1 (z) = 0, \\
\varepsilon \phi_3 (z) - \frac{\gamma}{B} \phi_1 (z) - 2 \sqrt{\frac{\gamma}{B} \partial_z \phi_1 (z) = 0,} \\
\varepsilon \phi_4 (z) - \frac{\gamma}{B} B \phi_1 (z) = 0.
\end{align*}
\]

Of course, the \( K = 0 \) case is a relatively simple case, but any pairing effect becomes important in this case.
4. Energy spectrum

One can solve the de-coupled equation system in Eq. (11) for \( \varphi_1(z) \) and arrive at the following 2\(^{nd} \) order wave equation,

\[
\frac{\partial^2 \varphi_1(z)}{\partial z^2} + \frac{1}{z} \frac{\partial \varphi_1(z)}{\partial z} - \frac{1}{4} \left( \frac{z^2 + 1}{z^2} - \frac{\epsilon^2 - M^2}{2Bz} \right) \varphi_1(z) = 0,
\]

This equation can be reduced into the well-known shape of the Whittaker differential equation via defining the ansatz that reads \( \varphi_1(z) = \frac{1}{\sqrt{z}} \chi(z) \).

\[
\frac{\partial^2 \chi(z)}{\partial z^2} + \left( \frac{4}{z} + \frac{1}{z^2} - \frac{1}{4} \right) \chi(z) = 0,
\]

and the solution function of this wave equation is given in follows [44, 45],

\[
\chi(z) = QW_{\mu,v}(z)
\]

in which the \( Q \) is the normalization constant. The condition of the solution function to be polynomial is given as follows [44, 45],

\[
\frac{1}{2} + v - \mu = -n, \quad (n \geq 0),
\]

in which the \( n \) is principal quantum number (non-negative integer). The expression in Eq. (13) leads to the quantization condition for the formation of such a system. With the help of Eq. (12) and Eq. (13) one can acquire the following non-perturbative spectrum in energy domain by assuming \( \epsilon < 0 \),

\[
E = \pm 2mc^2 \sqrt{1 - \frac{\omega_c h}{mc^2}(n + 1), \quad \omega_c = \frac{|e|B_0}{mc},
\]

in which \( \omega_c \) is the cyclotron frequency for each fermion. Eq. (14) clearly gives the relativistic Landau levels of a neutral system (spinless) formed by oppositely charged two fermions (non-interacting). Dependence of total energy \( (E) \) on the strength of the external homogeneous magnetic field can be seen in Figure 1. Also, one can obtain all of the spinor components in Eq. (11) as follows,

\[
\begin{pmatrix}
\varphi_1(z) \\
\varphi_2(z) \\
\varphi_3(z) \\
\varphi_4(z)
\end{pmatrix} = Q \begin{pmatrix}
\frac{W_{\mu\nu}(z)}{2} \\
\frac{W_{\mu\nu}(z)}{2} \\
\frac{W_{\mu\nu}(z)}{2} \\
\frac{W_{\mu\nu}(z)}{2}
\end{pmatrix} \begin{pmatrix}
\frac{M}{\sqrt{\epsilon z}} \\
\frac{M}{\sqrt{\epsilon z}} \\
\frac{M}{\sqrt{\epsilon z}} \\
\frac{M}{\sqrt{\epsilon z}}
\end{pmatrix}.
\]

5. Results and Discussion

In this study, we investigated the relativistic dynamics of oppositely charged two fermions interacting with an external homogeneous magnetic field. To obtain a non-perturbative energy spectrum of such a system, we solved the corresponding fully-covariant two-body Dirac equation in 2 + 1 dimensional flat Minkowski spacetime background, since this problem has 2 + 1-dimensional dynamical symmetry. For a spinless and static system formed by oppositely charged two fermions (non-interacting), the dynamic symmetry of the problem we deal with allowed us to obtain the eigenfunctions and eigenvalues (in closed-form) of the corresponding fully-covariant two-body Dirac Hamiltonian, without using any group theoretical method. As it expected, the obtained energy spectrum (Eq. (14)) shows that the total energy \( (E) \) of the system closes to total rest mass energy \( (2mc^2) \) of the system when the external magnetic field is very weak \( (\omega_c h \ll mc^2) \). It is clear that in Eq. (14), the term associated with the external magnetic field does not vanish even in the ground state. Also, the total energy value closes to the zero when \( \omega_c h \approx mc^2 \) and \( n = 0 \). The non-perturbative energy spectrum in Eq. (14) also indicates that this system can decay when \( \omega_c h > mc^2 \) in any physically possible quantum state.

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