SMALL RESOLUTIONS OF TORIC VARIETIES ASSOCIATED TO STRING POLYTOPES OF SMALL INDICES

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Abstract. Let $G$ be a semisimple algebraic group over $\mathbb{C}$. For a reduced word $i$ of the longest element in the Weyl group of $G$ and a dominant integral weight $\lambda$, one can construct the string polytope $\Delta_i(\lambda)$, whose lattice points encode the character of the irreducible representation $V_\lambda$. The string polytope $\Delta_i(\lambda)$ is singular in general and combinatorics of string polytopes heavily depend on the choice of $i$. In this paper, we study combinatorics of string polytopes when $G = SL_{n+1}(\mathbb{C})$. When $i$ has small indices and $\lambda$ is regular, we explicitly construct a small desingularization of the toric variety $X_{\Delta_i(\lambda)}$ using a Bott manifold. As a byproduct, we show that if $i$ has small indices then $\Delta_i(\lambda)$ is integral for any dominant integral weight $\lambda$, which in particular implies that the anticanonical limit toric variety $X_{\Delta_i(\lambda)}$ of a partial flag manifold $G/P$ is Gorenstein Fano. Furthermore, we apply our result to symplectic topology of the full flag manifold $G/B$ and obtain a formula of the disk potential of the Lagrangian torus fibration on $G/B$ obtained from a flat toric degeneration of $G/B$ to the toric variety $X_{\Delta_i(\lambda)}$.

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1. Introduction

Let $X$ be a smooth projective variety over $\mathbb{C}$. Lazarsfeld–Mustaţă [LM09] and Kaveh–Khovanskii [KK12] independently provided a systematic way of producing a semigroup $\Gamma$ and a convex body $\Delta$ (called a Newton–Okounkov body) from the choice of a polarization $(X, L)$ and a valuation $\nu$ on the ring of sections of $L$. When $\Gamma$ is finitely generated, Anderson [And13] showed that $\Delta$ is a rational convex polytope and constructed a toric degeneration of $X$ whose central fiber is the toric variety $X_\Delta$ associated to $\Delta$. This construction generalizes the previous works of [GL96, Cal02, KM05] for toric degenerations of Schubert varieties.

One of interesting examples of Newton–Okounkov bodies is a string polytope introduced by Littelmann [Lit98] (also see [BZ03, BZ06]). For a semisimple algebraic group $G$ over $\mathbb{C}$ of rank $n$, fix a reduced word $i$ of the longest element of the Weyl group of $G$ and a dominant integral weight $\lambda$. Then one can obtain a string polytope $\Delta_i(\lambda)$, a rational convex polytope in $\mathbb{R}^n$ where $n$ is the complex dimension of $G/B$, in which the lattice points parametrize elements of the dual crystal basis of the irreducible representation $V_\lambda$ of $G$ with highest weight $\lambda$ via the string parametrization. Kaveh [Kav15] proved that the string parametrization associated to $i$ induces a valuation $\nu_i$ on the function field of $G/B$ such that the string polytope $\Delta_i(\lambda)$ coincides.
with the Newton–Okounkov body for \((G/B, \mathcal{L}_\lambda, \nu_1)\) where \(\mathcal{L}_\lambda\) is a line bundle over \(G/B\) determined by the weight \(\lambda\).

The theory of Newton–Okounkov bodies plays a role of bridge between algebraic theory and symplectic geometry. Under the finite generateness of \(\Gamma\), Harada and Kaveh [HK15] produced a completely integrable system \(\Phi\) on \(X\) making the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & X_0 \\
\downarrow{\Phi} & & \downarrow{\Phi_0} \\
\Delta & \xrightarrow{} & \Delta_0
\end{array}
\]

where

- \(X_0(=X_{\Delta})\) is a projective toric variety associated to the Newton–Okounkov polytope \(\Delta\) with a moment map \(\Phi_0\),
- \(\phi\) is a continuous map (or a degeneration map) which is a symplectomorphism outside the singular locus of \(X_0\).

The system \(\Phi\) leads to a Lagrangian torus fibration on \(X\) over \(\Delta\). Harada and Kaveh provide the existence of the system \(\Phi\) for the string polytope \(\Delta_i(\lambda)\), which is the Newton–Okounkov body of \((G/B, \mathcal{L}_\lambda, \nu_1)\) in [HK15 Corollary 3.36].

A key step toward understanding Floer theory and deriving a local Landau–Ginzburg mirror complex chart of \(\Phi\): \(X \to \Delta\) is to compute the (Floer) disk potential of \(\Phi\) introduced by Fukaya–Oh–Ohta–Ono. The disk potential arises from counting invariants of holomorphic disks bounded by its fiber, see [CO06, Corollary 3.36]. Our motivated observation is that if \(\Delta_i(\lambda)\) is an integral\(^1\) toric polytope associated to \(\Delta\) has a small resolution. In this regard, it is a meaningful question asking whether the toric variety associated to \(\Delta\) has a small resolution.

In this manuscript, we focus on the case where \(G = \text{SL}_{n+1}(\mathbb{C})\) and study the integrality of the string polytope \(\Delta_i(\lambda)\) for a dominant integral weight \(\lambda\). Our motivated observation is that if \(\Delta_i(\lambda_P)\) is integral where \(\lambda_P\) is the weight corresponding to the anticanonical bundle of \(G/P\), the associated semigroup \(\Gamma\) is finitely generated (and hence it yields the diagram (1.1)) and the toric variety associated to \(\Delta_i(\lambda_P)\) is Gorenstein Fano, see [And13] and [Rus08, Ste19]. In fact, we will see that the integrality of \(\Delta_i(\lambda)\) holds when \(\Delta_i(\lambda)\) admits a small resolution (see Proposition 2.8). In this regard, we address the following question.

**Question 1.1.** When does the toric variety associated to \(\Delta_i(\lambda)\) admit a small resolution? Can we construct the small resolution explicitly?

The latter question is initiated from the observation of Batyrev, Ciocan-Fontanine, Kim, and van Straten [BCKvS00, Proposition 3.1.2]. They proved that the toric variety \(X_0\) associated to the Gelfand–Cetlin polytope admits a small resolution by constructing an explicit small resolution \(\psi: B \to X_0\) where \(B\) is a Bott manifold\(^2\). Note that the Gelfand–Cetlin polytope \(GC(\lambda)\) is unimodularly equivalent\(^3\) to the string polytope \(\Delta_i(\lambda)\) for the standard reduced expression

\[i_0 := (1, 2, 1, 3, 2, 1, \ldots, n, n-1, \ldots, 1).\]

In order to state our main result, we need to introduce some notions. For any reduced word \(i\), consider the reduced words obtained by interchanging two consecutive numbers \(i\) and \(j\) satisfying \(|i - j| > 1\). We call such an operation a 2-move. We say that two reduced words \(i\) and \(i'\) are equivalent if one can be obtained from the other by applying a sequence of 2-moves. Each equivalent class is called a commutation class. One

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1 A resolution of a variety is called small if the exceptional loci have codimension greater than one.

2 A Bott manifold is an iterated \(\mathbb{C}P^1\)-bundle over \(\mathbb{C}P^1\) such that each \(\mathbb{C}P^1\)-fibration is given by the sum of two line bundles (see [GK94]).

3 Two polytopes \(P\) and \(Q\) in \(\mathbb{R}^n\) are unimodularly equivalent if there exists an affine transformation \(T: x \to Ax + v\) such that \(A \in \text{GL}_n(\mathbb{Z})\), \(v \in \mathbb{Z}^n\), and \(T(P) = Q\).
important property of a reduced word is that for any given \( i \), by applying 2-moves to \( i \) repeatedly, we obtain a new reduced word \( i_2 \) having the consecutive descending subsequence \( D_n \) where

\[
D_n := (n, n-1, \ldots, 1).
\]

We can similarly produce a new reduced word \( i_3 \) equivalent to \( i \) and having the consecutive ascending subsequence \( A_n = (1, 2, \ldots, n) \) (see Proposition 4.2 or [CKLP19 Proposition 3.2]).

Using the above properties, for each sequence \( \delta \in \{A, D\}^n \) of letters consisting of ‘\( A \)’ and ‘\( D \)’, where each letters stand for ‘ascending’ and ‘descending’ respectively, one can associate a non-negative integer vector \( \text{ind}_\delta(i) \in \mathbb{Z}^n \) called the \( \delta \)-index of \( i \) (see Definition 4.7). The present authors proved in [CKLP19] that a string polytope \( \Delta_i(\lambda) \) is unimodularly equivalent to the Gelfand–Cetlin polytope \( GC(\lambda) \) if and only if the \( \delta \)-index of \( i \) is the zero vector for some \( \delta \in \{A, D\}^n \). We say that \( i \) has small indices if \( \text{ind}_\delta(i) = (0, \ldots, 0, k) \) for some \( \delta \in \{A, D\}^n \) and \( k \leq 2 \) (see Definition 6.5). Now we are ready to state our main theorem.

**Theorem 1.2** (Theorem 6.10). Let \( i \) be a reduced word of the longest element in the Weyl group of \( SL_{n+1}(\mathbb{C}) \) and \( \lambda \) a regular dominant integral weight. If \( i \) has small indices, then the toric variety \( X_{\Delta_i(\lambda)} \) associated to the string polytope \( \Delta_i(\lambda) \) admits a small desingularization \( X_{\tilde{\Sigma}_i} \). Moreover, the smooth projective toric variety \( X_{\tilde{\Sigma}_i} \) is isomorphic to a blow-up of a Bott manifold.

At first glance, the condition of small indices seems to be restricted, yet it includes more cases due to the possibilities of a choice of \( \delta \in \{A, D\}^n \). Every reduced word of the longest element has small indices for \( n \leq 3 \). When \( G = SL_3(\mathbb{C}) \), there are 20 commutation classes (out of 62) having small indices, see Appendix B. It is worthwhile to mention that Theorems 1.2 and 1.3 are proved for wider classes of reduced words (see Assumption 6.3). We note that the integrality of string polytopes has been conjectured in [AB04, Conjecture 5.8], but it recently turned out that there exist non-integral string polytopes by [Ste19, Example 7.5]. As every string polytope unimodularly equivalent to the Gelfand–Cetlin polytope has small indices, Theorem 1.2 generalizes [BCFKvS00 Proposition 3.1.2].

**Theorem 1.3** (Corollaries 6.13, 6.14 and 6.15). Let \( i \) be a reduced word of the longest element in the Weyl group of \( SL_{n+1}(\mathbb{C}) \). Suppose that \( i \) has small indices. Then we have the following.

1. For any dominant integral weight \( \lambda \), the string polytope \( \Delta_i(\lambda) \) is integral.
2. For a parabolic subgroup \( P \), the toric variety \( X_{\Delta_i(\lambda_P)} \) is Gorenstein Fano.
3. For a Borel subgroup \( B \) and a regular dominant integral weight \( \lambda \), let \( X = \{X_t \mid t \in \mathbb{C}\} \) be the corresponding toric degeneration, i.e., \( X_t \cong G/B \) for \( t \in \mathbb{C}\setminus\{0\} \) and \( X_0 \cong X_{\Delta_i(\lambda)} \), and let \( \Phi : G/B \rightarrow \Delta_i(\lambda) \) be the completely integrable system. Then, for the Lagrangian submanifold \( L(u) := \Phi^{-1}(u) \subset G/B \), the Floer theoretical disk potential of \( L(u) \) can be computed by the combinatorics of \( \Delta_i(\lambda) \), where \( u \) is an interior point in \( \Delta_i(\lambda) \).

It is worthwhile to mention that Theorems 1.2 and 1.3 are proved for wider classes of reduced words (see Assumption 6.3). We note that the integrality of string polytopes has been conjectured in [AB04, Conjecture 5.8], but it recently turned out that there exist non-integral string polytopes by [Ste19, Example 7.5]. Our main theorem presents a sufficient condition on the integrality of string polytopes.

This paper is organized as follows. In Section 2, we introduce notations and some well-known facts on toric varieties and resolutions of toric varieties. We also introduce the notion of Bott manifolds and toric varieties and resolutions of toric varieties. In Section 3, we describe string polytopes in terms of explicit defining inequalities using Gleižer–Postnikov description. In Section 4, we explain certain operations on the set of reduced words, called the extension and contraction, introduced in the previous work [CKLP19] and illustrate how a string polytope changes when applying each operations. In Section 5, we study the combinatorics of string polytopes having \( \delta \)-indices of the form \((0, \ldots, 0, k)\) and describe the defining inequalities in terms of rigorous paths. In Section 6, we give a construction of the small desingularization of the toric variety \( X_{\Delta_i(\lambda)} \) associated to the string polytope \( \Delta_i(\lambda) \) when \( i \) has small indices and present the main theorem. We also list some corollaries of our main theorem. Finally in Section 7, the proof of the main theorem will be provided.

We have two appendices. The relation between Dynkin diagram automorphisms and combinatorics of string polytopes is explained in Appendix A. In Appendix B, we give the classification of reduced words of the longest element in \( G_7 \) having small indices.
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2. Resolutions of singular toric varieties, and Bott manifolds

In this section, we recall some well-known facts on toric varieties and resolutions of singular toric varieties from [CLS11]. And then, we study certain smooth projective toric varieties, called Bott manifolds, which will be used to construct small resolutions of toric varieties associated to string polytopes. (See [GK94] and [BPT5 §7.8] for more details of Bott manifolds.)

Let $n$ be a positive integer. Let $M$ be the character lattice of a torus $T \cong (\mathbb{C}^*)^n$ and $N$ the lattice of one-parameter subgroups of $T$. We denote

$$ M_R := M \otimes \mathbb{Z} \mathbb{R} \quad \text{and} \quad N_R := N \otimes \mathbb{Z} \mathbb{R} $$

so that $M_R \cong \mathbb{R}^n$ and $N_R \cong \mathbb{R}^n$. Let $X_\Sigma$ be the toric variety of a fan $\Sigma$ in $N_R$ and $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ a torus-invariant Cartier divisor on $X_\Sigma$ where $\Sigma(k)$ is the set of $k$-dimensional cones in $\Sigma$ and $D_\rho$ is the torus-invariant prime divisor corresponding to a ray $\rho \in \Sigma(1)$. The divisor $D$ is called basepoint free if $O_{X_\Sigma}(D)$ is generated by global sections. There are several ways to determine whether $D$ is basepoint free or not. To present them, we recall the corresponding polyhedron $P_D$, Cartier data $\{m_\sigma\}_{\sigma \in \Sigma(n)}$, and the support function $\varphi_D$.

For a Cartier divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, the polyhedron $P_D \subset M_R$ is defined by

$$ P_D = \{ m \in M_R \mid \langle m, u_{\rho} \rangle \geq -a_\rho \quad \text{for all} \ \rho \in \Sigma(1) \}. $$

Here, $u_\rho$ is the integral vector generating a ray $\rho$. Note that when the fan $\Sigma$ is complete, then $P_D$ is bounded, i.e., $P_D$ is a polytope. Since the divisor $D$ is Cartier, there exist $m_\sigma \in M$ for each $\sigma \in \Sigma$ such that

$$ \langle m_\sigma, u_\rho \rangle = -a_\rho \quad \text{for all} \ \rho \in \sigma. $$

We call $\{m_\sigma\}_{\sigma \in \Sigma(n)}$ the Cartier data.

For a Cartier divisor $D$ on a toric variety $X_\Sigma$, its support function $\varphi_D : |\Sigma| \to \mathbb{R}$ is determined by the following properties:

- $\varphi_D$ is linear on each cone $\sigma \in \Sigma$, and
- $\varphi_D(u_\rho) = -a_\rho$ for all $\rho \in \Sigma(1)$.

Indeed the explicit formula of $\varphi_D|_\sigma$ for each $\sigma \in \Sigma(n)$ is determined by the Cartier data $\{m_\sigma\}_{\sigma \in \Sigma(n)}$:

$$ \varphi_D(u) = \langle m_\sigma, u \rangle \quad \text{for all} \ \sigma \in \Sigma. $$

We call a subset $P \subset \{u_\rho \mid \rho \in \Sigma(1)\}$ a primitive collection if $\text{Cone}(P) \notin \Sigma$ but $\text{Cone}(P \setminus \{x\}) \in \Sigma$ for every $x \in P$. We denote by $\text{PC}(\Sigma)$ the set of primitive collections of $\Sigma$. From the definition, we observe the following.

Lemma 2.1. Let $\Sigma$ be a smooth fan and $S \subset \{u_\rho \mid \rho \in \Sigma(1)\}$. Then $\text{Cone}(S) \in \Sigma$ if and only if $P \notin S \quad \text{for any} \ P \in \text{PC}(\Sigma)$

Proof. Suppose that $\text{Cone}(S) \notin \Sigma$. Then we may find a primitive collection $\mathcal{P} \subset S$ in an inductive way. This proves the "if" part. Conversely, assume that $S$ contains some $P \in \text{PC}(\Sigma)$. If $\text{Cone}(S) \in \Sigma$, then $\text{Cone}(P)$ is also in the fan $\Sigma$ since the cone $\text{Cone}(S)$ is simplicial. But this contradicts to the assumption $P \in \text{PC}(\Sigma)$. This completes the proof.

The following theorem presents equivalent conditions on the basepoint freeness of $D$ using primitive collections.

Theorem 2.2 ([CLS11 Proposition 6.1.1, Theorems 6.3.12 and 6.4.9]). Let $X_\Sigma$ be a projective simplicial toric variety and let $D$ be a Cartier divisor. The following are equivalent:

1. $D$ is basepoint free.
Example 2.3. Let \( \{ \} \) \( \Sigma \) be given by \( PC(\Sigma) = \ldots \). These figures and Theorem 2.2 imply that \( D \) \( \text{fan} \) \( \Sigma \) generated by four rays whose generators are one can check the following:

\[
\phi_D \left( \sum_{x \in P} x \right) \geq \sum_{x \in P} \phi_D(x)
\]

for all \( P \in PC(\Sigma) \).

Example 2.5. Let \( X_\Sigma \) be the Hirzebruch surface \( H_2 := \mathbb{C}P(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(2)) \) associated with the complete fan \( \Sigma \) generated by four rays whose generators are \( \{ u_1, u_3 \} \), \( \{ u_2, u_4 \} \). Since the support function is linear on each cone \( \sigma \in \Sigma \), one can check the following:

\[
\phi_D(u_1 + u_3) = \phi_D(2u_4) = 2 \cdot 0 \geq \phi_D(u_1) + \phi_D(u_3) = 0 + 0 = 0,
\]

\[
\phi_D(u_2 + u_4) = \phi_D((0, 0)) = 0 \geq \phi_D(u_2) + \phi_D(u_4) = -1 + 0 = -1.
\]

This computation shows that \( D \) is basepoint free again by Theorem 2.2. In a similar manner, we can check that \( D' \) is not basepoint free as

\[
\phi_{D'}(u_1 + u_3) = \phi_{D'}(2u_4) = 2 \cdot 0 \not\geq \phi_{D'}(u_1) + \phi_{D'}(u_3) = 0 + 1 = 1,
\]

\[
\phi_{D'}(u_2 + u_4) = \phi_{D'}((0, 0)) = 0 \geq \phi_{D'}(u_2) + \phi_{D'}(u_4) = -1 + 0 = -1.
\]

The normal fan \( \Sigma_{P_D} \) of the polytope \( P_D \) for a basepoint free divisor \( D \) has the following property:

**Proposition 2.4** ([CLSIH Proposition 6.2.5]). Assume that \( |\Sigma| \) is complete of full dimension \( n \). Let \( D = \sum_{\rho} a_{\rho} D_{\rho} \) be a basepoint free Cartier divisor on \( X_\Sigma \) with the polytope \( P_D \). Then, if \( v \in P_D \) is a vertex, then the corresponding cone \( \sigma_v \) in the normal fan \( \Sigma_{P_D} \) is the union

\[
\sigma_v = \bigcup_{\substack{\sigma \in \Sigma(n) \\sigma_{mr} = v}} \sigma.
\]

Indeed, the fan \( \Sigma \) is a refinement of \( \Sigma_{P_D} \).

Example 2.5. Let \( \Sigma \) be the fan of \( H_2 \) as in Example 2.3. We consider divisors

\[
D = D_2 \quad \text{and} \quad D'' = D_1.
\]

The polytopes \( P_D, P_{D''} \) and the Cartier data are given in Figures 2(1) and 2(3). By Theorem 2.2 both Cartier divisors \( D \) and \( D'' \) are basepoint free. The normal fans \( \Sigma_{P_D} \) and \( \Sigma_{P_{D''}} \) of the polytopes are given in Figures 2(4) and 2(5) respectively. One can see that \( \Sigma \) is a refinement of both of \( \Sigma_{P_D} \) and \( \Sigma_{P_{D''}} \).
Solution of the linear equations: called a small resolution of the system in (2.2). Hence the system in (2.2) can be reduced to the system

\[ \hat{\Sigma} \text{ admits a small desingularization} \]

that is not a simple vertex, i.e., it could correspond to a non-simplicial maximal cone of the fan \( \Sigma \). When a singular toric variety admits a small resolution, the corresponding polytopes are integral:

As a direct corollary of Proposition 2.4, we have the following:

**Corollary 2.7.** Let \( \Sigma \) be a smooth complete polytopal fan in \( N_\mathbb{R} \), and let \( D \) be a basepoint free Cartier divisor on \( X_\Sigma \). If \( \Sigma_{PD} \) is singular and \( \Sigma(1) = \Sigma_{PD}(1) \), then \( X_\Sigma \) is a small desingularization of \( X_{\Sigma_{PD}} \).

When a singular toric variety admits a small resolution, the corresponding polytopes are integral:

**Proposition 2.8.** Let \( X_\Sigma \) be a singular projective toric variety of dimension \( n \). If \( X_\Sigma \) admits a small desingularization \( \hat{X}_\Sigma \), then the polytope \( P_D \) is integral for every Cartier divisor \( D \). That is, each vertex of \( P_D \) is contained in the lattice \( M \cong \mathbb{Z}^n \).

**Proof.** Let \( D = \sum_{k=1}^m a_k D_k \), where \( m = |\Sigma(1)| \). Choose a vertex \( v \) of \( P_D \). Then the coordinate of \( v \) is the solution of the linear equations:

\[
\{ (m, u_k) = -a_k \mid k \in J \}
\]

where \( J \) is a subset of \([m] := \{1, 2, \ldots, m\}\) satisfying that \( \sigma_v = \text{Cone}(u_k \mid k \in J) \). Here \( |J| \geq n \) since \( v \) may not be a simple vertex, i.e., it could correspond to a non-simplicial maximal cone of the fan \( \Sigma_{PD} \). Since the fan \( \Sigma \) admits a small desingularization \( \hat{\Sigma} \), every non-simplicial maximal cone admits a smooth subdivision by the definition. Hence the system in (2.2) can be reduced to the system \( \{ (m, u_k) = -a_k \mid k \in J' \} \) such that \( |J'| = n \) and the set \( \{ u_k \mid k \in J' \} \) forms a \( \mathbb{Z} \)-basis of the lattice \( N \). Therefore the solution of the system (2.2) is contained in \( M \cong \mathbb{Z}^n \), and the result follows. \( \square \)

Now we introduce Bott manifolds which are smooth projective toric varieties.

**Definition 2.9.** A Bott tower \( B^*_n \) of height \( n \) is a tower of fiber bundles

\[ B_n \overset{p_n}{\longrightarrow} B_{n-1} \overset{p_{n-1}}{\longrightarrow} \cdots \overset{p_2}{\longrightarrow} B_1 \longrightarrow pt, \]

of smooth projective toric varieties, where \( B_1 = \mathbb{C}P^1 \) and \( B_j = \mathbb{C}P(O_{B_{j-1}}(\xi_{j-1})) \) for \( 2 \leq j \leq n \). Here, \( \xi_{j-1} \) is a complex line bundle over \( B_{j-1} \). We refer to \( B_j \) as a \( j \)-stage Bott manifold (or just a Bott manifold).
For instance, the complex projective line $\mathbb{C}P^1$, the Hirzebruch surface $H_k = \mathbb{C}P(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(k))$, and the product $\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ are examples of Bott manifolds. By [Har77 Exercise II.7.9], the Picard group of $B_{j-1}$ is isomorphic to the free abelian group of rank $j - 1$. There is a canonical way of constructing an isomorphism from $\mathbb{Z}^{j-1}$ to $\text{Pic}(B_{j-1})$ as follows. Let $\eta_{j, j-1}$ be the dual of the tautological line bundle over $B_{j-1}$ and define $\eta_{j, i}$ for $1 \leq i \leq j - 2$ to be the pullback bundle $\eta_{j, i} := p_{i+1}^* \circ \cdots \circ p_1^* (\eta_{k+1, i})$. Then the map

$$\mathbb{Z}^{j-1} \to \text{Pic}(B_{j-1}), \quad (a_1, \ldots, a_{j-1}) \mapsto (\eta_{j,1})^{\otimes a_1} \otimes \cdots \otimes (\eta_{j,j-1})^{\otimes a_j}$$

is an isomorphism. Hence for each line bundle $\xi_{j-1}$, there exist integers $a_j, \ldots, a_{j,j-1}$ such that

$$\xi_{j-1} \cong (\eta_{j,1})^{\otimes a_1} \otimes \cdots \otimes (\eta_{j,j-1})^{\otimes a_j}. \quad (2.3)$$

It has been known from [GK94 §2.3] that an $n$-stage Bott manifold is determined by the set of integers $\{a_{j,i} \mid 1 \leq i < j \leq n\}$. Moreover, the fan of a Bott manifold can be described as follows:

**Theorem 2.10 (cf. [CLSI1] §7.3 and [BPI5 Theorem 7.8.6]).** Suppose that $B_\ast$ is a Bott tower determined by $\{a_{j,i} \mid 1 \leq i < j \leq n\}$. Then $B_n$ is a smooth projective toric variety corresponding to the complete fan $\Sigma$ with $2n$ ray vectors

$$v_j = -e_j + a_{j+1, j} e_{j+1} + \cdots + a_{n,j} e_n, \quad w_j = e_j \quad \text{for } 1 \leq j \leq n,$$

and $2^n$ maximal cones generated by the sets $\{v_j \mid j \in S\} \cup \{w_j \mid j \notin S\}$ for all subsets $S \subset [n]$. Here, $\{e_1, \ldots, e_n\}$ is the standard basis of $N_\mathbb{R} \cong \mathbb{R}^n$.

**Example 2.11.** The fan of Hirzebruch surface $H_k = \mathbb{C}P(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(k))$ has four ray vectors

$$[v_1 \ v_2 \ w_1 \ w_2] = \begin{bmatrix} -1 & 0 & 1 & 0 \\ k & -1 & 0 & 1 \end{bmatrix},$$

and has four maximal cones parameterized by $S \subset [2] = \{1, 2\}$:

| $S$ | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{1, 2\}$ |
|-----|-------------|----------|----------|-----------|
| Cone | Cone$(w_1, w_2)$ | Cone$(v_1, w_2)$ | Cone$(v_2, w_1)$ | Cone$(v_1, v_2)$ |

See Example 2.3 and Figure 1 for the fan of $H_2$.

By the description of the maximal cones of the fan of a Bott manifold in Theorem 2.10 we are able to list all primitive collections of $B_n$.

**Proposition 2.12.** Let $\Sigma$ be the fan of an $n$-stage Bott manifold with ray vectors $\{v_j, w_j \mid 1 \leq j \leq n\}$. Then there are exactly $n$ primitive collections of $\Sigma$.

$$\text{PC}(\Sigma) = \{\{v_j, w_j\} \mid 1 \leq j \leq n\}.$$ 

Suppose that $A$ and $B$ are lower triangular matrices in $\text{GL}_n(\mathbb{Z})$ such that the diagonal entries of $A$ are all $-1$ and that of $B$ are all $1$. Then the pair $(A, B)$ defines an $n$-stage Bott manifold where the column vectors of the matrix $B^{-1} A$ correspond to $v_j$’s in (2.4) together with $\{e_j \mid 1 \leq j \leq n\}$. Hence, without loss of generality, we may consider such matrices $A$ and $B$ when we define a Bott manifold.

**Example 2.13.** Suppose that we have the following vectors:

$$[v_1 \ v_2 \ v_3 \ w_1 \ w_2 \ w_3] = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$ 

Since both matrices $[v_1 \ v_2 \ v_3]$ and $[w_1 \ w_2 \ w_3]$ are lower triangular matrices in $\text{GL}_3(\mathbb{Z})$ where the former one has diagonal entries $-1$ and the latter one has diagonal entries $1$, they define a 3-stage Bott manifold.

We recall a star subdivision of fans and blow-ups of a toric variety. See [Ewa96 V.6] and [CLSI1 §3.3].

**Definition 2.14 ([CLSI1 Definition 3.3.17]).** Let $\Sigma$ be a fan in $N_\mathbb{R} \cong \mathbb{R}^n$ and assume $\tau \in \Sigma$ has the property that all cones of $\Sigma$ containing $\tau$ are smooth. Let $u_\tau = \sum_{p \in \tau(1)} u_p$ and for each cone $\sigma \in \Sigma$ containing $\tau$, set $\Sigma_\sigma^\tau(\tau) = \{\text{Cone}(A) \mid A \subset \{u_\tau\} \cup \sigma(1), \ \tau(1) \not\subset A\}$. 

Then the star subdivision of $\Sigma$ relative to $\tau$ is the fan
\[ \Sigma^*(\tau) = \{ \sigma \in \Sigma \mid \sigma \nsubseteq \tau \} \cup \bigcup_{\tau \subseteq \sigma} \Sigma^*_n(\tau). \]

The fan $\Sigma^*(\tau)$ is a refinement of $\Sigma$ and induces a toric morphism $\psi: X_{\Sigma_*(\tau)} \to X_\Sigma$. Under the map $\psi$, $X_{\Sigma^*(\tau)}$ becomes the blowup of $X_\Sigma$ along the orbit closure $V(\tau)$. Moreover, if the fan $\Sigma$ is polytopal, i.e., $\Sigma$ is the normal fan of a certain polytope $P$, then so is the fan $\Sigma^*(\tau)$. If $\Sigma$ is a smooth fan in addition, then the star subdivision $\Sigma^*(\tau)$ is also smooth (see [BZ01], Theorem V.6.2).

We finish up this section with the following proposition which will be used in Section 7.

**Proposition 2.15** ([Sat00, Theorem 4.3]). Let $\Sigma$ be a finite complete simplicial fan in $\mathbb{R}^n$ and $\tau \in \Sigma$. Then the primitive collections of the star subdivision $\Sigma^*(\tau)$ are

- $G(\tau) := \{ u_\rho \mid \rho \in \tau(1) \}$,
- $P \in PC(\Sigma)$ such that $G(\tau) \nsubseteq P$, and
- the minimal elements in the set $\{ (P \setminus G(\tau)) \cup \{ u_\tau \} \mid P \in PC(\Sigma), P \cap G(\tau) \neq \emptyset \}$.

**Example 2.16.** Let $\Sigma$ be the fan of the 3-stage Bott manifold defined in Example 2.13. Let $\tau = \text{Cone}(v_1, w_2)$. Then, by Proposition 2.15 we have that
\[ PC(\Sigma^*(\tau)) = \{ \{ v_1, w_2 \}, \{ v_1, w_1 \}, \{ v_2, w_2 \}, \{ v_3, w_3 \}, \{ v_2, v_1 + w_2 \}, \{ w_1, v_1 + w_2 \} \}. \]

3. **Gleizer–Postnikov’s description of string polytopes**

Let $G$ be a connected semisimple algebraic group of rank $n$ over $\mathbb{C}$ and $\mathfrak{g}$ its Lie algebra. Fixing a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and an enumeration of the simple roots $\alpha_1, \ldots, \alpha_n$, we have the Chevalley generators $\{ e_i, f_i, \alpha_i^\vee \mid 1 \leq i \leq n \}$ and the Weyl group $W$ generated by reflections $s_i$ through the hyperplanes orthogonal to the simple roots $\alpha_i$. Here $\alpha_i^\vee$ is the coroot of $\alpha_i$. The weight lattice $\Lambda$ is the set of all $\lambda \in \mathfrak{t}^*$ such that $\lambda(\alpha_i^\vee) \in \mathbb{Z}$ and $\Lambda$ has a $\mathbb{Z}$-basis consisting of the fundamental weights $\varpi_1, \ldots, \varpi_n$, which are determined by the relation $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$. We call a weight $\lambda = \lambda_1 \varpi_1 + \cdots + \lambda_n \varpi_n$ dominant if $\lambda_i \geq 0$ for all $i = 1, \ldots, n$, and regular dominant if $\lambda_i > 0$ for all $i = 1, \ldots, n$. Let $\Lambda_+$ denote the set of dominant integral weights.

In this section, we briefly review the combinatorial theory of string polytopes of type $\mathbb{A}$ as well as several notations introduced in the earlier work [CKLP19]. From now on, we assume that $G = SL_{n+1}(\mathbb{C})$. The Weyl group of $G$ can be naturally identified with the symmetric group $S_{n+1}$, where the reflections associated to the simple roots $\alpha_1, \ldots, \alpha_n$ in $\mathfrak{t}^*$ correspond to the simple transpositions $s_1, \ldots, s_n$ in $S_{n+1}$ ($s_i = (i, i+1)$) respectively.

Let $w_0^{(n+1)}$ be the longest element of $S_{n+1}$ and denote by $R(w_0^{(n+1)})$ the set of reduced words representing $w_0^{(n+1)}$, i.e.,
\[ R(w_0^{(n+1)}) = \{ i = (i_1, \ldots, i_n) \mid [i] = [n]^{\bar{n}} | s_{i_1} s_{i_2} \cdots s_{i_n} = w_0^{(n+1)} \} \]
where $\bar{n}$ is the length of the longest element $w_0^{(n+1)}$ which can be computed as
\[ \bar{n} = \frac{n(n+1)}{2}. \]

For a reduced word $i \in R(w_0^{(n+1)})$ and a dominant integral weight $\lambda = \lambda_1 \varpi_1 + \cdots + \lambda_n \varpi_n$, the string polytope $\Delta_i(\lambda)$ is a convex polytope defined in the Euclidean space $\mathbb{R}^n$ such that lattice points in $\Delta_i(\lambda)$ parameterize the dual canonical basis elements of the irreducible $G$-module $V_\lambda$ with highest weight $\lambda$. In Appendix A, we present the original definition of string polytopes following from Littelmann’s paper [Lit08].

The string polytope $\Delta_i(\lambda)$ can be obtained as the intersection of two convex rational polyhedral cones, the string cone $C_i$ and the $\lambda$-cone $C^\lambda_i$. There are several ways of describing the string cone $C_i$, see [Lit08], [BZ01], and [GP00] for instance. Throughout this paper, we follow Gleizer–Postnikov’s description [GP00] of the string cone $C_i$ and Rusinko’s description [Rus08] of the $\lambda$-cone $C^\lambda_i$ where both descriptions use so-called a wiring diagram. We also refer to [CLKP19] Sections 2 and 3 for more details.

**Definition 3.1.** For a reduced word $i = (i_1, \ldots, i_{\bar{n}}) \in R(w_0^{(n+1)})$, the wiring diagram $G(i)$ is a pseudoline arrangement consisting of a family of $(n+1)$-vertical piecewise straight lines such that

- each pair of wires must intersect exactly once, and
the \( j \)th crossing of wires (from the top) is located in the \( i_j \)th column (from the left) of \( G(i) \) for each \( j = 1, 2, \ldots, \bar{n} \).

We call each crossings nodes and label them as \( t_1, t_2, \ldots, t_\bar{n} \) from the top to the bottom.

In Figure 3, we present wiring diagrams for reduced words \((1, 2, 1, 3, 2, 1)\) and \((1, 3, 2, 1, 3, 2)\) in \( R(w_0^{(4)}) \).

We label the pseudolines \( \ell_1, \ell_2, \ldots, \ell_{\bar{n} + 1} \) and call \( \ell_k \) the \( k \)th wire. Also the upper end and lower end of each wire \( \ell_k \) are labeled by \( U_k \) and \( L_k \), respectively.

**Definition 3.2** ([GP00, Section 5.1]). For a given \( i \in R(w_0^{(n+1)}) \) and \( k \in [n] \), define \( G(i,k) \) to be the oriented wiring diagram \( G(i) \) where the orientations on each wire is given such that

- the first \( k \) wires \( \ell_1, \ldots, \ell_k \) are oriented upward, and
- the other wires \( \ell_{k+1}, \ldots, \ell_{\bar{n}+1} \) are oriented downward.

(1) A rigorous path is an oriented path on \( G(i,k) \) for some \( k \in [n] \) satisfying
- it starts at \( L_k \) and ends at \( L_{k+1} \),
- it respects the orientation of \( G(i,k) \),
- it passes through each node at most once, and
- it does not include forbidden fragments given in Figure 4.

**Figure 4.** Forbidden fragments.

We denote the set of rigorous paths by \( \mathcal{GP}(i) \).

(2) A node \( t \) is called a peak of a rigorous path \( \gamma \in \mathcal{GP}(i) \) if \( t \) is a local maximal node of the path \( \gamma \) with respect to the height of the diagram \( G(i) \).

(3) Among peaks of a rigorous path \( \gamma \in \mathcal{GP}(i) \), we call the global maximal node of the path \( \gamma \) the maximal peak.

(4) Each node \( t_j \) assigns the chamber defined to be the region \( C_j \) enclosed by wires such that
- \( t_j \) is the unique peak of the boundary of \( C_j \), and
- any wire does not intersect the interior of \( C_j \).

(5) For each chamber \( C_j \), define a new variable

\[
m_j := \sum_{i=1}^{\bar{n}} a_i t_i, \quad a_i = \begin{cases} 1 & \text{if } t_i \in C_j \text{ is in the same column as } t_j, \\ -1 & \text{if } t_i \in C_j \text{ is in one column to the right or left of } t_j, \\ 0 & \text{otherwise}. \end{cases}
\]

We call \( m_j \)'s chamber variables. See also [CKLP19, Definition 4.1].

We note that a forbidden fragment can appear only when \( \ell_i \) crosses over \( \ell_j \) such that
- \( i > j \) where the orientation of both wires is downward, or...
In Figure 5, one can find some oriented wiring diagrams and rigorous paths for Example 3.3. Any rigorous path \( \gamma \) can be expressed by

\[
(3.2) \quad \ell_{r_1} \rightarrow \cdots \rightarrow \ell_{r_p}, \quad r_1 = k, \quad r_{p+1} = k + 1
\]

which records wires in order which appear in the travel along the path. The expression (3.2) is called a wire-expression\(^4\) of \( \gamma \). We denote by

\[
\text{node}(\gamma) = \{ \ell_{r_i} \cap \ell_{r_{i+1}} \mid i = 1, \ldots, p \}
\]

the set of nodes on \( \gamma \) appearing at the intersections of consecutive wires in (3.2).

**Example 3.3.** In Figure 5, one can find some oriented wiring diagrams and rigorous paths for \( i = (1, 2, 1, 3, 2, 1) \) and \( i' = (1, 3, 2, 1, 3, 2) \). Also, one can find chambers for the word \( (1, 3, 2, 1, 3, 2) \) in Figure 5.

![Figure 5. Oriented wiring diagrams for \( i = (1, 2, 1, 3, 2, 1) \) and \( i' = (1, 3, 2, 1, 3, 2) \), and chambers.](image)

Now we are ready to define \( C_i \), \( C_i^\lambda \), and \( \Delta_i(\lambda) \) each of which is a convex object in \( \mathbb{R}^n \). We will use the coordinate system \( (t_1, \ldots, t_n) \) by abuse of notation.

**Definition 3.4 (GP00, Rus08).** Let \( i = (i_1, \ldots, i_n) \in R(w_i^{(n+1)}) \) and \( \lambda = \lambda_1 \varpi_1 + \cdots + \lambda_n \varpi_n \in \Lambda^+ \).

1. Let \( \gamma \) be a rigorous path in \( G(i, k) \) for some \( k \in [n] \). The string inequality for \( \gamma \) is defined by

\[
\sum_{j=1}^{\hat{n}} a_j t_j \geq 0, \quad \text{where } a_j := \begin{cases} 1 & \text{if } \gamma \text{ travels from } \ell_r \text{ to } \ell_s \text{ at } t_j \text{ and } r < s, \\ -1 & \text{if } \gamma \text{ travels from } \ell_r \text{ to } \ell_s \text{ at } t_j \text{ and } r > s, \\ 0 & \text{otherwise}. \end{cases}
\]

The string cone \( C_i \) is the set of points in \( \mathbb{R}^n \) satisfying all string inequalities.

2. For each node \( t_j \) in \( G(i) \), the \( \lambda \)-inequality for \( t_j \) is defined by

\[
t_j \leq \lambda_j + \sum_{k > j} b_k t_k, \quad \text{where } b_k := \begin{cases} 1 & \text{if the node } t_k \text{ is in one column to the right or left of } t_j, \\ -2 & \text{if the node } t_k \text{ is in the same column as } t_j, \\ 0 & \text{otherwise}. \end{cases}
\]

The \( \lambda \)-cone \( C_i^\lambda \) is the set of points in \( \mathbb{R}^n \) satisfying all \( \lambda \)-inequalities.

**Remark 3.5 (CKLP19, Section 4.1).** In terms of chamber variables, the description of the string polytope becomes much simpler. Under the change of coordinates \( (t_1, \ldots, t_n) \rightarrow (m_1, \ldots, m_n) \), we may describe the string cone \( C_i \) as follows.

\[
C_i = \left\{ (m_1, \ldots, m_n) \in \mathbb{R}^n \left| \sum_{\gamma \subset \text{region enclosed by } \gamma} m_j \geq 0, \quad \gamma \in \mathcal{GP}(i) \right. \right\}.
\]

\(^4\)There is another type of expression, called a node-expression of a rigorous path. See CKLP19 (2.2) in Section 2.1.
On the other hand, we have the vectors string polytope can be written by
\[
\lambda
\]
is defined as the intersection of the string cone and the \(\lambda\)-cone. Therefore, for \(\lambda\)

\[
\lambda
\]

Note that in [BF19], they also used a similar description of string polytopes (see, for instance, [BF19, Figure 6]).

**Definition 3.6.** Let \(i = (i_1, \ldots, i_n) \in R(w_0^{n+1})\) and \(\lambda = \lambda_1 \varpi_1 + \cdots + \lambda_n \varpi_n \in \Lambda^+\). The *string polytope* \(\Delta_4(\lambda)\) is defined as the intersection of the string cone and the \(\lambda\)-cone. In terms of the chamber variables \(m_j\)'s, the string polytope can be written by

\[
\Delta_4(\lambda) := C_4 \cap C_\lambda = \bigcap_{\gamma \in \mathcal{GP}(i)} \{ m \in \mathbb{R}^n \mid (w_\gamma, m) \geq 0 \} \cap \bigcap_{j=1}^n \{ m \in \mathbb{R}^n \mid (v_j, m) + \lambda_j \geq 0 \},
\]

where \(w_\gamma\) and \(v_j\) denote the coefficient vectors of the string inequality for \(\gamma\) and the \(\lambda\)-inequality for \(m_j\) as in Remark 3.5 respectively. Indeed,

\[
(3.3) \quad w_\gamma = \sum_{v_j \in \text{region enclosed by } \gamma} e_j, \quad v_j = - \sum_{k \geq j, i_k = i_j} e_k
\]

where \(\{e_1, \ldots, e_n\}\) is the standard basis of \(\mathbb{R}^n\).

**Example 3.7.** Let \(i = (1,3,2,1,3,2) \in R(w_0^{(4)})\). Then there are seven rigorous paths, and each path \(\gamma\) defines the following vector \(w_\gamma\): (See Figure 5[3])

\[
\begin{align*}
& w_{\ell_1 \to \ell_2} = (1,0,1,0,1,0), \quad w_{\ell_1 \to \ell_4 \to \ell_2} = (0,0,1,0,1,0), \quad w_{\ell_1 \to \ell_3 \to \ell_2} = (0,0,0,0,1,0), \\
& w_{\ell_2 \to \ell_3} = (0,0,0,0,0,1), \quad w_{\ell_3 \to \ell_4} = (0,1,1,0,0), \quad w_{\ell_3 \to \ell_4 \to \ell_2} = (0,0,1,1,0,0), \quad w_{\ell_3 \to \ell_4} = (0,0,0,1,0,0).
\end{align*}
\]

On the other hand, we have the vectors \(v_j\) for \(1 \leq j \leq 6\):

\[
\begin{align*}
& v_1 = (-1,0,0,-1,0,0), \quad v_2 = (0,-1,0,0,-1,0), \quad v_3 = (0,0,-1,0,0), \\
& v_4 = (0,0,0,-1,0,0), \quad v_5 = (0,0,0,0,-1,0), \quad v_6 = (0,0,0,0,0,-1).
\end{align*}
\]

Therefore, for \(\lambda = \lambda_1 \varpi_1 + \lambda_2 \varpi_2 + \lambda_3 \varpi_3\), the string polytope \(\Delta_4(\lambda)\) is expressed as follows.

\[
\Delta_4(\lambda) = \left\{ (m_1, \ldots, m_6) \in \mathbb{R}^6 \mid \begin{array}{c}
m_1 + m_3 + m_5 \geq 0, \quad m_3 + m_5 \geq 0, \quad m_5 \geq 0, \quad m_6 \geq 0, \\
m_2 + m_3 + m_4 \geq 0, \quad m_3 + m_4 \geq 0, \quad m_4 \geq 0, \\
-m_1 - m_4 + \lambda_1 \geq 0, \quad -m_2 - m_5 + \lambda_3 \geq 0, \quad -m_3 - m_6 + \lambda_2 \geq 0, \\
-m_4 + \lambda_1 \geq 0, \quad -m_5 - \lambda_3 \geq 0, \quad -m_6 + \lambda_2 \geq 0.
\end{array} \right\}
\]

In the remaining of this section, we observe some combinatorial properties of rigorous paths which will be used later.

**Proposition 3.8** ([CKLP19 Proposition 4.6]). Let \(\lambda\) be a regular dominant weight and \(i \in R(w_0^{(n+1)})\). Then the expression in Definition 3.6 is non-redundant in the string polytope \(\Delta_4(\lambda)\). Indeed, when we consider the normal fan \(\Sigma_{\Delta_4(\lambda)}\) of the string polytope, the set of ray generators of the fan \(\Sigma_{\Delta_4(\lambda)}\) is the same as

\[
\{ w_\gamma \mid \gamma \in \mathcal{GP}(i) \} \cup \{ v_j \mid j = 1, \ldots, n \}.
\]

Let \(R_i\) be the closed region enclosed by the path \(\ell_i \to \ell_{i+1}\) and \(R_i^\circ\) its interior for each \(i \in [n]\). See Figure 6 for the regions \(R_i\) of the word \((4,3,4,2,3,4,1,2,3,4,5,4,6,5,4,3,2,1,4,3,2) \in R(w_0^{(7)})\).
Lemma 3.9. Let \( i \in R(w_0^{(n+1)}) \) and \( \gamma = (\ell_1 \rightarrow \ell_{r_1} \rightarrow \cdots \rightarrow \ell_{r_s} \rightarrow \ell_{i+1}) \in \mathcal{GP}(i) \). Then \( \gamma \subset R_i \).

4. Extensions and contractions on reduced words

In this section, we introduce two operations called a contraction and an extension that produce a new reduced word in \( R(w_0^{(n)}) \) and \( R(w_0^{(n+2)}) \), respectively, for each reduced word \( i \in R(w_0^{(n+1)}) \). See also [CKLP19 Section 3.3] for more details.

Recall that for a given \( i \in R(w_0^{(n+1)}) \), one can produce a new reduced word in \( R(w_0^{(n+1)}) \) by a braid move. There are two types of braid moves, called 2-move and 3-move, where

- (2-move) exchanging \((i, j)\) with \((j, i)\) for \(|i - j| > 1\), i.e., \( s_is_j = s_js_i \).
- (3-move) exchanging \((i, i + 1, i)\) with \((i + 1, i, i + 1)\), i.e., \( s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \).

According to Tits’ Theorem [Tit69], every pair of reduced words in \( R(w_0^{(n+1)}) \) is connected by a sequence of braid moves. Define an equivalence relation \( \sim \) on \( R(w_0^{(n+1)}) \) such that

\[
i \sim i' \iff \text{i and } i' \text{ are related by a sequence of 2-moves.}
\]

From the definition of string cones, it immediately follows that if two reduced words \( i \) and \( i' \) differ by a sequence 2-moves, then the corresponding string cones differ by the change of coordinates. Hence, using the equivalence relation \( \sim \), we can state the following.

Lemma 4.1 ([CKLP19 Lemma 3.1]). If \( i \sim i' \), then the string polytopes \( \Delta_i(\lambda) \) and \( \Delta_{i'}(\lambda) \) are the same up to coordinate changes for any dominant integral weight \( \lambda \in \Lambda_+ \).

The following proposition observed in [CKLP19] suggests two canonical representatives for each equivalence class in \( R(w_0^{(n+1)})/\sim \). See also [CKLP19 Example 3.3].

Proposition 4.2 ([CKLP19 Proposition 3.2]). For any \( i = (i_1, \ldots, i_n) \in R(w_0^{(n+1)}) \), we may rearrange \( i \) using 2-moves so that

\[
i \sim (i'_1, \ldots, i'_n, n, n-1, \ldots, 2, 1, i'_{n+1}, \ldots, i'_d) =: i''_D =: i''_D.
\]
for some integer $u \geq 0$ and $i'_j \in [n]$. Similarly, there exist an integer $v \geq 0$ and $i''_j$'s in $[n]$ such that
\[ i \sim (i''_1, \ldots, i''_v, 1, 2, \ldots, n - 1, n, i''_{v+n+1}, \ldots, i''_n) =: i''_A =: A_n =: i''_D \]
where ‘$D$’ and ‘$A$’ stand for ‘descending’ and ‘ascending’, respectively. Here, the descending chain $D_n$ and ascending chain $A_n$ are given by
\[ D_n := (n, n - 1, \ldots, 2, 1) \quad \text{and} \quad A_n := (1, 2, \ldots, n - 1, n). \]

Using Proposition 4.2, we can define an index of $i$ as follows.

**Definition 4.3** ([CKLP19 Definition 3.4]). For each $i \in R(w_0^{(n+1)})$ with
\[ i \sim i''_D D_n i''_D \sim i''_A A_n i''_A, \]
define
\[ \text{ind}_D(i) := |i''_D| \quad \text{and} \quad \text{ind}_A(i) := |i''_A| \]
and call them the $D$-index of $i$ and the $A$-index of $i$, respectively.

**Remark 4.4.** Note that $\text{ind}_D(i)$ and $\text{ind}_A(i)$ count the number of nodes in $G(i)$ below $\ell_{n+1}$ and $\ell_1$, respectively, as explained in the proof of [CKLP19 Proposition 3.5].

We introduce some notations as follows. For a word $i = (i_1, \ldots, i_k)$, we denote by
\[ i + 1 := (i_1 + 1, \ldots, i_k + 1) \quad \text{and} \quad i - 1 := (i_1 - 1, \ldots, i_k - 1). \]
We also denote by $[a, b] := \{a, a+1, \ldots, b\}$ for $a, b \in \mathbb{Z}$.

**Definition 4.5** ([CKLP19 Definition 3.6]). For any $i \in R(w_0^{(n+1)})$ and $s \in \{0, 1, \ldots, \bar{n}\}$, assume that
\[ i = (i_1, \ldots, \bar{i}_{n-s}, \bar{i}_{n-s+1}, \ldots, i_\bar{n}) =: i''(s) \sim i''_D D_n i''_D \sim i''_A A_n i''_A \]
where both words $i''_D D_n i''_D$ and $i''_A A_n i''_A$ are minimal in the sense that 2-moves are used as little as possible to obtain them from $i$, respectively.

1. The $D$-contraction of $i$, denoted by $C_D(i)$, is the reduced word $i''_D D_n i''_D - 1 \in R(w_0^{(n)})$.
2. The $A$-contraction of $i$, denoted by $C_A(i)$, is the reduced word $(i''_A - 1) i''_A \in R(w_0^{(n)})$.
3. For $s \in [0, \bar{n}]$, the $D$-extension of $i$ at $s$, denoted by $E_D(s)(i)$, is the reduced word
\[ i''(s) D_{n+1} (i''(s) + 1) \in R(w_0^{(n+2)}). \]
4. For $s \in [0, \bar{n}]$, the $A$-extension of $i$ at $s$, denoted by $E_A(s)(i)$, is the reduced word
\[ (i''(s) + 1) A_{n+1} i''(s) \in R(w_0^{(n+2)}). \]

**Remark 4.6.** The extension map is surjective up to 2-move (see [CKLP19 Section 3.3]). Indeed, the following composition is surjective:
\[ R(w_0^{(n+1)}) \times [0, \bar{n}] \xrightarrow{E_s} R(w_0^{(n+2)}) \xrightarrow{\pi} R(w_0^{(n+2)})/\sim \xrightarrow{[E_s]} \]

Note that, starting with the empty set, we may apply extensions $\ell$ times repeatedly so that one can get a reduced word (depending on the choice of $A$ and $D$ in each steps) in $R(w_0^{(n+1)})$. Similarly for each $i \in R(w_0^{(n+1)})$ and the choice of $A$ or $D$ in each steps, one can apply contractions repeatedly and finally get the empty set.

**Definition 4.7.** (1) For any sequence $\delta = (\delta_1, \ldots, \delta_n) \in \{A, D\}^n$ and an integer vector $I = (I_1, \ldots, I_n)$ with $I_i \in [0, \bar{i} - 1]$, we obtain a reduced word
\[ i_\delta(I) := (E_{\delta_n}(I_n) \circ \cdots \circ E_{\delta_1}(I_2) \circ E_{\delta_1}(I_1))() \in R(w_0^{(n+1)}). \]
Here, $\bar{i} - 1 = i(i - 1)/2$ is given in (3.1).
Lemma 4.9. (CKLP19, Lemma 5.1)

It is not hard to check that $\ell$ node lying on that $\in GP$ this follows. Note that if $i, i'$ is not in $\text{Im} \Psi$ does not imply that two words are the same up to 2-moves.

Now, we investigate how the set of rigorous paths enlarges by extensions or contractions.

Example 4.8. Let $\delta = (D, D, D)$ and $I = (0, 0, 2)$. Then, we have that

$$\Psi(i, i') : GP(i) \hookrightarrow GP(E(s)(i))$$

Moreover,

$$\text{Im} \Psi(i, s) = \{ \gamma \in GP(E(s)(i)) \mid \text{node}(\gamma) \text{ does not contain a node lying on } \ell \}$$

where $\ell_D := \ell_{n+2}$ and $\ell_A := \ell_1$.

We call a rigorous path $\gamma \in GP(E(s)(i))$ $\bullet$-new if it is not in $\text{Im} \Psi(i, s)$. More generally, we can define $\bullet$-new paths for general reduced words as follows. Note that if $i \sim i'$, then there is a natural identification between $GP(i)$ and $GP(i')$. Since the extension is surjective up to 2-moves (see Remark 4.6), for any $i \in R(w_0^{(n+1)})$, there exist $i' \in R(w_0^{(n)})$, $\bullet \in \{ A, D \}$, and $s \in [0, n-1]$ such that $E(s)(i') \sim i$. Thus we say that $\gamma \in GP(i)$ $\bullet$-new if $\gamma$ is $\bullet$-new in $GP(E(s)(i'))$. Equivalently, $\gamma \in GP(i)$ is $\bullet$-new if node(\gamma) contains a node lying on $\ell$. See [CKLP19, Section 5] for more details.

For any reduced word $i \in R(w_0^{(n+1)})$, the authors provide in [CKLP19, Propositions 5.6 and 5.7] an explicit way of finding $n$ $\bullet$-new paths (called canonical) in $GP(i)$ for each $\bullet = A$ and $D$.

Proposition 4.11 ([CKLP19, Proposition 5.6]). Let $i \in R(w_0^{(n+1)})$ and $k \in [n]$. Let $t_{jk}$ be the node at which $\ell_k$ and $\ell_{n+1}$ intersect. Then there exists a rigorous path $\gamma_D(i, k) \in GP(i)$ such that

- it has a unique peak $t_{jk}$,
- it travels from $t_k$ to $\ell_{n+1}$ at $t_{jk}$,
- it is below $\ell_{n+1}$,
- with respect to the wire-expression of $\gamma_D(i, k)$:
  $$\ell_{r_p} \to \cdots \to \ell_{r_1} \to \ell_k \to \ell_{n+1} \to \ell_{u_q} \to \cdots \to \ell_{u_1} (= \ell_{r_{k+1}}),$$
- the sequences $r_1, \ldots, r_p$ and $u_1, \ldots, u_q$ are increasing, and
- $\gamma_D(i, k) \subset R_{a_k}$, where $a_k = \max \{ a \mid t_{jk} \in R_a \}$.

We call the path $\gamma_D(i, k)$ a canonical D-new path.
Example 4.12. Let \( i = (4, 3, 4, 2, 3, 4, 1, 2, 3, 4, 5, 4, 6, 5, 4, 3, 2, 1, 4, 3, 2) \) ∈ \( R(w^{(7)}_0) \). See Figure \( \text{[6]} \) for the wiring diagram of \( i \). Then,

\[
i = E_D(3)(4, 3, 4, 2, 3, 4, 1, 2, 3, 4, 5, 4, 6, 5, 4, 3, 2, 1, 4, 3, 2, 1).
\]

For \( k = 3 \), there are two \( D \)-new rigorous paths which satisfy the first four conditions on Proposition \[4.11\]

\[
\gamma_1 := (\ell_1 \to \ell_3 \to \ell_4) \subset R_3, \quad \gamma_2 := (\ell_6 \to \ell_3 \to \ell_7) \subset R_6.
\]

Since \( 6 > 3 \), the path \( \gamma_2 \) is the canonical \( D \)-new path \( \gamma_D(i, 3) \), and \( \gamma_1 \) is a \( D \)-new path but not canonical. By the similar observations, one can find the following canonical \( D \)-new paths.

\[
\begin{align*}
\gamma_D(i, 1) &= (\ell_1 \to \ell_7 \to \ell_2), \quad \gamma_D(i, 2) = (\ell_2 \to \ell_7 \to \ell_6 \to \ell_3), \quad \gamma_D(i, 3) = (\ell_6 \to \ell_3 \to \ell_7), \\
\gamma_D(i, 4) &= (\ell_6 \to \ell_4 \to \ell_7), \quad \gamma_D(i, 5) = (\ell_6 \to \ell_5 \to \ell_7), \quad \gamma_D(i, 6) = (\ell_6 \to \ell_7).
\end{align*}
\]

5. Combinatorics of string polytopes of index \((0, \ldots, 0, k)\)

In this section, we compare \( \mathcal{GP}(C_D(i)) \) with \( \mathcal{GP}(i) \) which are sets of rigorous paths. Throughout this section, we assume that \( i \sim i_\delta(I) \in R(w_0^{(n+1)}) \) with \( I = (0, \ldots, 0, k) \) for some \( k \leq n-1 \) and \( \delta = (\delta_1, \ldots, \delta_n) \in \{A, D\}^n \). Without loss of generality, we may assume that

\[
\delta_n = D
\]

by Proposition \[A.7\]. Moreover, by Lemma \[4.1\] we may assume that

\[
i = i_\delta(I) (= i_{\delta_1, \ldots, \delta_n}(0, \ldots, 0, k)).
\]

Then the \((\delta_1, \ldots, \delta_{n-1})\)-index of the contraction \( C_D(i) \in R(w_0^{(n)}) \) becomes the zero vector, i.e.,

\[
\text{ind}(\delta_1, \ldots, \delta_{n-1})(C_D(i)) = (0, \ldots, 0) \in \mathbb{Z}^{n-1}.
\]

Therefore, \( C_D(i) \) defines the string polytope which is unimodularly equivalent to the Gelfand–Cetlin polytope by [CKLP19, Theorem 6.7]. In particular, we have

\[
|\mathcal{GP}(C_D(i))| = \frac{n(n-1)}{2} = \bar{n} - n.
\]

Now we consider the canonical inclusion map defined in Lemma \[4.10\],

\[
\Psi := \Psi_D(C_D(i), k): \mathcal{GP}(C_D(i)) \hookrightarrow \mathcal{GP}(i)
\]

which sends a rigorous path \( \gamma = (\ell_i \to \ell_{r_1} \to \cdots \to \ell_{r_p} \to \ell_{i+1}) \) in \( \mathcal{GP}(C_D(i)) \) to the path

\[
(5.1) \quad \Psi(\gamma) = (\ell_i \to \ell_{r_1} \to \cdots \to \ell_{r_p} \to \ell_{i+1}) \in \mathcal{GP}(i).
\]

Theorem 5.1. Let \( \delta = (\delta_1, \ldots, \delta_n) \in \{A, D\}^n \) and \( I = (0, \ldots, 0, k) \) with \( k \leq n-1 \). Assume that \( n \geq 2 \) and \( \delta_n = D \). Then,

\[
|\mathcal{GP}(i_\delta(I))| = \begin{cases} 
\bar{n} + k - 1 & \text{if } \delta_{n-1} = D \text{ and } k = n - 1, \\
\bar{n} + k & \text{if } \delta_{n-1} = D \text{ and } k < n - 1, \\
\bar{n} & \text{if } k = 0; \text{ or } \delta_{n-1} = A \text{ and } k = n - 1, \\
\bar{n} + 1 & \text{if } \delta_{n-1} = A \text{ and } 0 < k < n - 1.
\end{cases}
\]

Indeed, \( |\mathcal{GP}(i_\delta(I))| \) depends only on \( k, \delta_{n-1} \) and \( \delta_n \).

To prove Theorem 5.1 we need the following lemma.

Lemma 5.2. Let \( i \in R(w_0^{(n+1)}) \) and \( \gamma = (\ell_i \to \ell_{r_1} \to \cdots \to \ell_{r_p} \to \ell_{i+1}) \in \mathcal{GP}(i) \). Then \( i+1 \notin \{r_1, \ldots, r_p\} \). In particular, with respect to the upward orientation of \( \ell_i \), if \( \ell_i \) meets \( \ell_{r_j} \) just before intersecting \( \ell_{i+1} \), then \( \gamma \) should be \( \ell_i \to \ell_{r_j} \to \ell_{i+1} \).

Proof. Assume on the contrary that \( r_j = i + 1 \) for some \( j \in \{1, \ldots, p - 1\} \). Then there are four possible configurations of the sub-path \( \ell_{r_j} \to \ell_{r_{j+1}} \) as follows (where each red broken arrow describes a part of the path \( \gamma \)).
By Lemma 5.2, the path $\gamma$ should be contained in the region $R_i$. Since the region $R_i$ is enclosed by paths $\ell_{i-1}$ and $\ell_i$, the path $\gamma$ should travel the shaded parts in Figure 7. But, for Cases 2 and 4, the red paths are not contained in $R_i$, and hence those cases are excluded.

For Case 1, since the wire $\ell_{r_{j+1}}$ goes downward, $r_{j+1} > i + 1$. Thus the wire $\ell_{r_{j+1}}$ is on the left hand side of the wire $\ell_{i-1}$ on the bottom. But it is impossible since each pair of wires meets only once. Similarly, Case 3 is also impossible since $r_{j+1} < i + 1$ and so the orientation of $\ell_{r_{j+1}}$ should be downward. Therefore the result follows.

**5.1 Case 1**: $(\delta_{n-1}, \delta_n) = (D, D)$.

**Proposition 5.3.** Let $\delta = (\delta_1, \ldots, \delta_n) \in \{A, D\}^n$ and $i \in R(w_0^{(n+1)})$. Suppose that $\delta_{n-1} = \delta_n = D$. If $i \sim i_k(I)$ with $I = (0, \ldots, 0, k)$ for some $k \leq n - 1$, then

$$|GP(i)| = \begin{cases} \bar{n} + k - 1 & \text{if } k = n - 1, \\ \bar{n} + k & \text{if } k < n - 1. \end{cases}$$

**Figure 8.** Rigorous paths $\ell_{n-k-1} \rightarrow \ell_n \rightarrow \ell_{n+1} \rightarrow \ell_{n-k}$ (green), $\gamma_0 = (\ell_{n-k-1} \rightarrow \ell_n \rightarrow \ell_{n-k})$ (dotted red) and $\gamma_1 = (\ell_{n-i} \rightarrow \ell_{n+1} \rightarrow \ell_{n-i+1})$ (blue) when $(\delta_{n-1}, \delta_n) = (D, D)$.

**Proof.** By Lemma 5.2, the only possible region $R_i$ which contains two different rigorous paths having the same maximal peak is $R_{n-k-1}$. This is because $R_{n-k-1}$ is the only region which contains a node in its interior. One can see in Figure 8 that wires $\ell_{n+1}$ and $\ell_n$ intersect at the node $\bar{t}_{n-2k}$ in the interior of $R_{n-k-1}$. This produces a D-new path of Case II-2 in [CKLP'T9] Proposition 5.10 (see the green path in Figure 8):

$$\ell_{n-k-1} \rightarrow \ell_n \rightarrow \ell_{n+1} \rightarrow \ell_{n-k}, \quad (5.3)$$
which is not canonical since its peak, \( t_{n-(n+k)} \), does not lie on the wire \( \ell_{n+1} \) (this violates the first condition in Proposition 4.11). Note that there is another rigorous path \( \bar{\gamma}_0 := (\ell_{n-k-1} \rightarrow \ell_n \rightarrow \ell_{n-k}) \) having the same maximal peak \( t_{n-(n+k)} \). See the dotted red path in Figure 8.

On the other hand, one can see that for \( 1 \leq j_1 < j_2 \leq n \),
\[
R_j^1 \cap R_j^2 \neq \emptyset \iff n-k-1 \leq j_1 \leq n-2 \text{ and } j_2 = n
\]
(c.f. Figure 6). These intersections produce \( k-1 \) D-new paths of Case I-2 in [CKLP19 Proposition 5.10]:
\[
\gamma_i := (\ell_{n-i} \rightarrow \ell_{n-i+1} \cdots ) \quad \text{for } 2 \leq i \leq k
\]
which are not canonical. See the blue path in Figure 8. Therefore, there are exactly \( k \) non-canonical D-new paths when \( n-k-1 > 0 \), i.e., \( k < n-1 \).

If \( k = n-1 \), then no red dotted line appears in Figure 8 and so there are exactly \( k-1 \) non-canonical D-new paths (blue paths). This completes the proof.

**Example 5.4.** Let \( \delta = (D, D, A, A, D, D) \) and \( I = (0, 0, 0, 0, 0, 3) \). Let
\[
i := i_3(I) = (4, 3, 4, 2, 3, 4, 1, 2, 3, 4, 5, 4, 6, 5, 4, 3, 2, 1, 4, 3, 2) \in R(w_0^{(7)}).
\]
The regions \( R_i \) are presented in Figure 6. There are three D-new paths which are not canonical (see red paths in Figure 9):
\[
\ell_2 \rightarrow \ell_6 \rightarrow \ell_7 \rightarrow \ell_3, \quad \ell_4 \rightarrow \ell_7 \rightarrow \ell_4 \rightarrow \ell_7 \rightarrow \ell_5.
\]
Note that the canonical D-new paths \( \gamma_D(i, k) \) for \( k = 3, 4 \) are \( \ell_6 \rightarrow \ell_3 \rightarrow \ell_7 \) and \( \ell_6 \rightarrow \ell_4 \rightarrow \ell_7 \), respectively (see Example 4.12 and blue paths in Figures 9(3) and 9(2)).

**Figure 9. D-new paths in Example 5.4**

### 5.2. Case 2: \((\delta_{n-1}, \delta_n) = (A, D)\).

**Lemma 5.5.** Let \( k \) be a positive integer satisfying \( k \leq n-1 \). Consider sequences \( \delta = (\delta_1, \ldots, \delta_{n-2}, A, D) \) and \( \delta' = (\delta_1, \ldots, \delta_{n-2}, D, A) \) in \( \{A, D\}^n \). Then, we have the equivalence
\[
i_{\delta} \left( \begin{array}{c} 0, \ldots, 0, k \\ n-1 \end{array} \right) \sim i_{\delta'} \left( \begin{array}{c} 0, \ldots, 0, n-k-1 \\ n-1 \end{array} \right).
\]

**Proof.** Let \( I = (0, \ldots, 0, k) \). We set \( i := i_3(I) \). Since \( \text{ind}_D(i) = k \) and \( \text{ind}_A(C_D(i)) = 0 \), the last part of the sequence \( i \) has the following form:
\[
i = (i', 1^{n-k-1} 2^{n-k-1} \ldots n-k-1 \underbrace{n, n-1, \ldots, n-k+1}_{D_n}, n-k-1, \ldots, 2, 1, n-k+1, \ldots, n).
\]
Here, \( i' = i_{(\delta_1, \ldots, \delta_{n-2})}(0, \ldots, 0) + 1 = C_D(C_A(i)) + 1 \). Then, the boxed numbers in the above equation form the sequence \( A_n \). Thus we have that

\[
(5.4) \quad i \sim (i', n, n-1, \ldots, n-k+1, \underbrace{1 \ldots n-k-1}_A, n-k, \underbrace{n-k+1 \ldots n}_A, n-k-1, \ldots, 2, 1).
\]

On the other hand, we have that

\[
i(0, \ldots, 0, n-k-1) = (E_A(n-k-1) \circ E_D(0))(i_{(\delta_1, \ldots, \delta_{n-2})}(0, \ldots, 0))
= (E_A(n-k-1) \circ E_D(0))(i' - 1)
= (E_A(n-k-1))(i', n-1, n-2, \ldots, 1)
= (i', n, n-1, \ldots, n-k+1, 1, 2, \ldots, n, n-k-1, \ldots, 2, 1).
\]

Hence the equivalence \((5.4)\) proves the lemma. \(\square\)

**Example 5.6.** Let \( i = (2, 3, 2, 1, 2, 3) \in R(u_0^{(4)}). \) Then we have that

\[
i = i_{(A, A, D)}(0, 0, 2) = i_{(A, D, A)}(0, 0, 0).
\]

**Proposition 5.7.** Let \( \delta = (\delta_1, \ldots, \delta_n) \in \{A, D\}^n \) and \( I = (0, \ldots, 0, k) \) with \( 0 < k < n - 1 \). Suppose that \( \delta_{n-1} = A \) and \( \delta_n = D \). If \( i \sim i(I) \), then \( |GP(i)| = \bar{n} + 1 \).

**Proof.** By Lemma 5.2, the only possible enclosed region \( R \) which contains two different rigorous paths which have the same maximal peak is \( R_{k+1} \) (see Figure 10). Note that in the interior of \( R_{k+1} \), wires \( \ell_{n+1} \) and \( \ell_1 \) meet at \( \ell_{n-k+1} \). This produces that a \( D \)-new path of Case I-1 in [CKLP19, Proposition 5.10]:

\[
(5.5) \quad \ell_{k+1} \rightarrow \ell_{n+1} \rightarrow \ell_1 \rightarrow \ell_{k+2}.
\]

which is not canonical as its maximal peak \( \tilde{n}_{-(n+k)} \) does not lie on the wire \( \ell_{n+1} \) (this violates the first condition in Proposition 4.11). Note that there is another rigorous path \( \tilde{\gamma}_0 := (\ell_{k+1} \rightarrow \ell_1 \rightarrow \ell_{k+2}) \) having the same maximal peak \( \tilde{n}_{-(n+k)} \). On the other hand, in this case, we have \( R_j^{(k)} \cap R_j^{(k)} = \emptyset \) for all \( 1 \leq j_1 < j_2 \leq n \). Consequently, there is exactly one non-canonical \( D \)-new path, and therefore the result follows. \(\square\)

![Figure 10](image-url)
Example 5.8. Let \( i := i_{(D,A,A,A,D)}(0,0,0,2) = (4,3,4,2,3,4,1,2,5,4,3,2,1,4,5) \). In this case, the regions \( R_i \) can be expressed by
\[
R_1 = \mathcal{C}_{15}, \quad R_2 = \mathcal{C}_6 \cup \mathcal{C}_9 \cup \mathcal{C}_{14}, \quad R_3 = \mathcal{C}_3 \cup \mathcal{C}_5 \cup \mathcal{C}_8 \cup \mathcal{C}_{10} \cup \mathcal{C}_{11}, \\
R_4 = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_7 \cup \mathcal{C}_{12}, \quad R_5 = \mathcal{C}_{13}.
\]
One can easily check that \( R_{j_1}^o \cap R_{j_2}^o = \emptyset \) for any \( 1 \leq j_1 < j_2 \leq 5 \), and there is one \( D \)-new path which is not canonical (see the dotted red path in Figure 11(2)):
\[
\ell_3 \to \ell_6 \to \ell_1 \to \ell_4.
\]
In this case, we have five canonical \( D \)-new paths:
\[
\gamma_D(i,1) = (\ell_3 \to \ell_6 \to \ell_4), \quad \gamma_D(i,2) = (\ell_2 \to \ell_6 \to \ell_3), \quad \gamma_D(i,3) = (\ell_3 \to \ell_6 \to \ell_4), \\
\gamma_D(i,4) = (\ell_4 \to \ell_6 \to \ell_5), \quad \gamma_D(i,5) = (\ell_5 \to \ell_6).
\]
For example, the canonical \( D \)-new path \( \gamma_D(i,3) \) is the blue path in Figure 11(2).

By Propositions 5.3 and 5.7, we prove Theorem 5.1.

Proof of Theorem 5.1. In case of \( (\delta_{n-1},\delta_n) = (D,D) \), the result follows from Proposition 5.3. For the case of \( (\delta_{n-1},\delta_n) = (A,D) \), the result follows
- from [CKLP19] Theorem 6.7 when \( k = 0 \),
- from Lemma 5.5 and [CKLP19] Theorem 6.7] when \( k = n - 1 \),
- from Proposition 5.7 when \( 0 < k < n - 1 \).

On the other hand, from the proofs of Propositions 5.3 and 5.7 we obtain the following consequence. Let \( \delta = (\delta_1, \ldots, \delta_n) \in \{A,D\}^n \), \( I = (0, \ldots, 0, k) \) with \( k \leq n - 1 \), and \( i = i_k(I) \). Assume that \( n \geq 2 \) and \( \delta_n = D \). For each \( j \in [\bar{n}] \), the number of rigorous paths having maximal peak at \( t_j \) is 1 or 2. Moreover, if two rigorous paths \( \gamma \) and \( \gamma' \) have the same maximal peak \( t_j \), then there are two possibilities:
1. either the region enclosed by one of the paths is contained in that of the other, or
2. the node \( t_j \) is on the wire \( \ell_{n+1} \) and only one of them is a canonical \( D \)-new path. (The other one is automatically a non-canonical \( D \)-new path.)

For the purpose of later use, we define the following paths.

Definition 5.9. For each \( j \in [\bar{n}] \), define \( \gamma_j \) to be
- \( \gamma \) if there exists only one rigorous path \( \gamma \) having the maximal peak \( t_j \),
- the path enclosing the larger region for Case (1),
- the canonical \( D \)-new path for Case (2).

For the remaining rigorous paths, we label them as follows (cf. Figure 8 and 10).
Example 5.10. The following examples exhibit $\gamma_i$’s as well as $\tilde{\gamma}_j$’s defined in Definition 5.9.

1. Let $i := i_{(A,A,D)} (0,0,1) = (2,1,3,2,1,3) \in R(w_0^{(4)})$. Then, for $s \neq 2$, there exists only one rigorous path whose maximal peak is $t_s$. For $s = 2$, there are two paths

$$\gamma := (\ell_2 \to \ell_1 \to \ell_3) \quad \text{and} \quad \gamma' := (\ell_2 \to \ell_4 \to \ell_1 \to \ell_3)$$

whose maximal peak is $t_2$. Then the region enclosed by $\gamma$ is $C_2 \cup C_4$ and the region enclosed by $\gamma'$ is $C_2 \cup C_3 \cup C_4$ (see Figure 12(1)). Thus we have that

$$\gamma_2 = \gamma', \quad \tilde{\gamma}_0 = \gamma.$$

2. Let $i := i_{(D,D,D)} (0,0,1) = (1,2,3,2,1,2)$. Then, for $s \neq 2$, there exists only one rigorous path whose maximal peak is $t_s$. For $s = 2$, there are two paths

$$\gamma := (\ell_1 \to \ell_3 \to \ell_2) \quad \text{and} \quad \gamma' := (\ell_1 \to \ell_4 \to \ell_2)$$

whose maximal peak is $t_2$. Since the region enclosed by $\gamma$ is $C_2 \cup C_3$ and that of $\gamma'$ is $C_2 \cup C_3 \cup C_4$ (see Figure 12(2)), we obtain

$$\gamma_2 = \gamma', \quad \tilde{\gamma}_0 = \gamma.$$

3. Let $i := i_{(D,D,D)} (0,0,2) = (1,3,2,1,3,2) \in R(w_0^{(4)})$. Then, for $s = 3$, there are two paths

$$\gamma := (\ell_1 \to \ell_4 \to \ell_2) \quad \text{and} \quad \gamma' := (\ell_3 \to \ell_1 \to \ell_4)$$

whose maximal peak is $t_3$ (see Figure 5(2)). In this case, the regions enclosed by these paths are not contained in each other. Since $\gamma' = \gamma_D(1,1)$ is canonical (while $\gamma$ is not canonical), it follows that

$$\gamma_3 = \gamma', \quad \tilde{\gamma}_2 = \gamma.$$

In this case, $\tilde{\gamma}_0$ is not defined. (Indeed, $(k,n) = (2,3)$ and $n-k-1 = 0$ so that $|GP(i)| = n+k-1 = 7$. See Proposition 5.3.)

![Figure 12. Wiring diagrams and chambers.](image-url)
6. Small resolutions on string polytopes of small indices

In this section, we associate a certain Bott manifold $B_i$, an iterated $\mathbb{C}P^1$-bundle, to each reduced word $i$. Moreover, we prove that a small desingularization of the toric variety $X_{\Delta_i(\lambda)}$ associated to the string polytope $\Delta_i$ can be obtained by blowing up the Bott manifold $B_i$ under the assumption that $i$ has small indices (see Theorem 6.10).

Suppose that $\delta = (\delta_1, \ldots, \delta_n) \in \{A, D\}^n$ with $\delta_n = D$ and let $i = i_\delta(I)$ with $I = (\ldots, 0, k)$ where $0 \leq k \leq n - 1$. Recall from (3.1) that the length of $i$ is $n = n(n + 1)/2$, which is the same as the dimension of the string polytope $\Delta_i(\lambda)$ when $\lambda$ is regular dominant. Among the elements in $\mathcal{G}(i)$, we have $\tilde{n}$ number of rigorous paths $\{\gamma_j \mid j \in [\tilde{n}]\}$ as in Definition 5.9 and the paths assign integral vectors $w_j := w_{\gamma_j}$ for $j \in [\tilde{n}]$

where $w_\cdot$ is the coefficient vector (with respect to chamber variables) of the string inequality for the path $\gamma$ (see [3.3]). Note that the matrix $[w_1 \cdots w_\tilde{n}]$ is a lower triangular matrix whose diagonal entries are all 1. On the other hand, we can associate $\tilde{n}$ integral vectors $v_1, \ldots, v_{\tilde{n}}$ to each $\lambda$-inequalities such that the matrix $[v_1 \cdots v_{\tilde{n}}]$ is a lower triangular matrix whose diagonal entries are all $-1$. See also (3.3) for the definition of $v_j$. Using these vectors, one obtain a Bott manifold (see Theorem 2.10).

Definition 6.1. Let $i = i_\delta(I)$ with $I = (\ldots, 0, k)$ and $0 \leq k \leq n - 1$. Define $B_i$ to be the Bott manifold determined by the vectors $\{v_j, w_j \mid 1 \leq j \leq \tilde{n}\}$ in the sense of Theorem 2.10. We denote by $\Sigma_i$ the fan of $B_i$.

Remark 6.2. For $i \in R(w_0^{(n+1)})$ satisfying $i \sim i_\delta(0, \ldots, 0)$ for some $\delta \in \{A, D\}^n$, the string polytope $\Delta_i(\lambda)$ is unimodularly equivalent to the Gelfand–Cetlin polytope $GC(\lambda)$ by [CKLP19, Theorem 6.7]. We call such a reduced word $i$ a Gelfand–Cetlin type. Moreover, it has been proved in [BCFKvS00, Proposition 3.1.2] that the toric variety $X_{GC(\lambda)}$ associated to the Gelfand–Cetlin polytope $GC(\lambda)$ admits a small desingularization for any dominant integral weight $\lambda$. Indeed, for $i_0 := (1, 2, 1, 3, 2, 1, \ldots, n, n-1, \ldots, 1)$, considering the affine transformation sending the string polytope $\Delta_{i_0}(\lambda)$ to $GC(\lambda)$ constructed in [CKLP19, Section 6], one can see that the fan of Bott manifold $B_{i_0}$ is same as that of the small desingularization of $X_{GC(\lambda)}$ constructed in [BCFKvS00, Proposition 3.1.2] for a regular dominant integral weight $\lambda$.

From now on, we suppose that a reduced word $i \in R(w_0^{(n+1)})$ satisfies the following assumption:

Assumption 6.3. (1) $i \sim i_\delta(I)$ with $I = (\ldots, 0, k)$ for some $0 \leq k \leq n - 1$ and (2) assuming $\delta_n = D$,

$$\begin{cases} 
0 \leq k \leq 2 & \text{if } (\delta_{n-1}, \delta_n) = (D, D), \\
0 \leq k \leq n - 1 & \text{if } (\delta_{n-1}, \delta_n) = (A, D).
\end{cases}$$

We construct a smooth projective toric variety associated to $i$. Recall from Theorem 5.1 that

$$\mathcal{G}(i) = \begin{cases} 
\{\gamma_j \mid j \in [\tilde{n}]\} & \text{if } k = 0; \text{ or } \delta_{n-1} = A \text{ and } k = n - 1, \\
\{\gamma_j \mid j \in [\tilde{n}]\} \cup \gamma_0 & \text{if } \delta_{n-1} = A \text{ and } 0 < k < n - 1; \text{ or } \delta_{n-1} = D \text{ and } k = 1, \\
\{\gamma_j \mid j \in [\tilde{n}]\} \cup \gamma_0 \cup \gamma_2 & \text{if } \delta_{n-1} = D \text{ and } (k, n) = (2, 3), \\
\{\gamma_j \mid j \in [\tilde{n}]\} \cup \gamma_0 \cup \gamma_2 & \text{if } \delta_{n-1} = D \text{ and } k = 2, n > 3,
\end{cases}$$

in which the number of rigorous paths is at most $\tilde{n} + 2$.

Proposition 6.4. Each vector $w_{\gamma_i}$ for $i = 0, 2$ can be expressed as a linear combination of $w_j$'s and $v_j$'s as follows.

(1) When $\delta_{n-1} = D$ and $k = 1, 2$:

$$w_{\gamma_0} = w_{\tilde{n}-(n+k)} + v_{\tilde{n}-2k} + w_{\tilde{n}-k+1},$$

$$w_{\gamma_2} = w_{\tilde{n}-3} + v_{\tilde{n}-2} + w_{\tilde{n}-1}.$$ 

(2) When $\delta_{n-1} = A$ and $0 < k < n - 1$:

$$w_{\gamma_0} = w_{\tilde{n}-(n+k)} + v_{\tilde{n}-n} + w_{\tilde{n}-k+1}.$$
Proof. First we consider the case $\delta_{n-1} = D$. Then using Figure 8, one can see that
\[
\begin{align*}
\mathbf{w}_{\gamma_0} &= \mathbf{e}_{n-(n+k)} + \mathbf{e}_{n-2k-1} \\
&= (\mathbf{e}_{n-(n+k)} + \mathbf{e}_{n-2k-1} + \mathbf{e}_{n-2k}) + (-\mathbf{e}_{n-2k} - \mathbf{e}_{n-k+1}) + \mathbf{e}_{n-k+1} \\
&= \mathbf{w}_{\gamma_0} + \mathbf{v}_{n-2k} + \mathbf{w}_{n-k+1}.
\end{align*}
\]
For $\gamma_2$, the path $\gamma_2$ exists only when $k = 2$ and we have
\[
\mathbf{w}_{\gamma_2} = \mathbf{e}_{n-3} + \mathbf{v}_{n-1} = \mathbf{w}_{n-3} + \mathbf{v}_{n-2} + \mathbf{w}_{n-1}.
\]
This proves the claim $\gamma_2$. Now assume $\delta_{n-1} = A$. From Figure 10, we can easily see that
\[
\begin{align*}
\mathbf{w}_{\gamma_0} &= \mathbf{e}_{n-(n+k)} + \mathbf{e}_{n-n+1} \\
&= (\mathbf{e}_{n-(n+k)} + \mathbf{e}_{n-n+1} + \mathbf{e}_{n-n}) + (-\mathbf{e}_{n-n} - \mathbf{e}_{n-k+1}) + \mathbf{e}_{n-k+1} \\
&= \mathbf{w}_{\gamma_0} + \mathbf{v}_{n-n} + \mathbf{w}_{n-k+1}.
\end{align*}
\]
This completes the proof.

Example 6.5. In the following examples, we compute $\gamma_i$'s for $i = 0$ or 2 as well as $\mathbf{w}_j$'s and $\mathbf{v}_j$'s explicitly, and also confirm Proposition 6.4 in each cases.

1. Let $\mathbf{i} := \mathbf{i}_{(A,A,D)} (0,0,1) = (2,1,3,2,1,3) \in R(\mathbf{v}_0^{(4)})$. From Example 5.10(1) and Figure 12(1) we obtain
\[
\begin{bmatrix}
\mathbf{v}_1 & \cdots & \mathbf{v}_6 \\
\mathbf{w}_1 & \cdots & \mathbf{w}_6
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1
\end{bmatrix}.
\]
Note that $|\mathcal{GP}(\mathbf{i})| = 7$ and the path $\gamma_0 = \ell_2 \rightarrow \ell_1 \rightarrow \ell_3$ defines a vector $(0,1,0,1,0,0)$. Moreover, we have that
\[
(0,1,0,1,0,0) = \mathbf{w}_2 + \mathbf{v}_3 + \mathbf{w}_6.
\]

2. Let $\mathbf{i} := \mathbf{i}_{(D,D,D)} (0,0,1) = (1,2,3,2,1,3)$. Following observations in Example 5.10(1) and Figure 12(2) we have
\[
\begin{bmatrix}
\mathbf{v}_1 & \cdots & \mathbf{v}_6 \\
\mathbf{w}_1 & \cdots & \mathbf{w}_6
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0
\end{bmatrix}.
\]
In this case, the path $\gamma_0 = \ell_1 \rightarrow \ell_3 \rightarrow \ell_2$ defines a vector $(0,1,0,1,0,0)$ expressed by
\[
(0,1,0,1,0,0) = \mathbf{w}_2 + \mathbf{v}_3 + \mathbf{w}_6.
\]

3. Let $\mathbf{i} := \mathbf{i}_{(D,D,D)} (0,0,2) = (1,3,2,1,3,2)$. Again by Example 5.10(3) (also, see Example 3.7) and Figure 5(2) we may check that
\[
\begin{bmatrix}
\mathbf{v}_1 & \cdots & \mathbf{v}_6 \\
\mathbf{w}_1 & \cdots & \mathbf{w}_6
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1
\end{bmatrix}.
\]
In this case, we have $|\mathcal{GP}(\mathbf{i})| = 7$ and the path $\gamma_2 = \ell_1 \rightarrow \ell_4 \rightarrow \ell_2$ defines a vector $(0,0,1,0,1,0)$. Moreover, we have that
\[
(0,0,1,0,1,0) = \mathbf{w}_3 + \mathbf{v}_4 + \mathbf{w}_5.
\]
In the rest of this section, we will show that for each $i$ satisfying Assumption 6.3, there exists a smooth complete fan whose ray generators are $\{w_\gamma \mid \gamma \in \mathcal{GP}(i)\} \cup \{v_j \mid j \in [\bar{n}]\}$, and this fan can be obtained as a refinement of the fan associated to the Bott manifold $B_i$.

For each $\overline{\gamma}_i \in \mathcal{GP}(i)$ in Definition 5.9, define two subsets $A_i$ and $B_i$ of $[\bar{n}]$ such that

$$w_{\overline{\gamma}_i} = \sum_{a \in A_i} w_a + \sum_{b \in B_i} v_b.$$  

From Proposition 6.4 we see that $A_i \cap B_i = \emptyset$. In particular, we have

$$(6.2) \quad \{w_a \mid a \in A_i\} \cup \{v_b \mid b \in B_i\} \nsubseteq \{w_j, v_j\} \quad \text{for any } j \in [\bar{n}].$$

This implies that the cone $\mathrm{Cone}(\{w_a \mid a \in A_i\} \cup \{v_b \mid b \in B_i\})$ is contained in the fan $\Sigma_i$ of the Bott manifold $B_i$. (See Theorem 2.10) For example, if $|\mathcal{GP}(i)| = \bar{n} + 2$, i.e., $\delta_{n-1} = D$ with $n > 3$ and $k = 2$, the paths $\overline{\gamma}_0$ and $\overline{\gamma}_2$ respectively correspond to the set $\{w_{\bar{n}-(n+2)}, w_{\bar{n}-4}, w_{\bar{n}-1}\}$ and the set $\{w_{\bar{n}-3}, w_{\bar{n}-2}, w_{\bar{n}-1}\}$ by Proposition 6.4. Therefore

$$\mathrm{Cone}(w_{\bar{n}-(n+2)}, w_{\bar{n}-4}, w_{\bar{n}-1}) \in \Sigma_i \quad \text{and} \quad \mathrm{Cone}(w_{\bar{n}-3}, w_{\bar{n}-2}, w_{\bar{n}-1}) \in \Sigma_i$$

by Lemma 2.1 and (6.2).

Let $\tau \in \Sigma_i$ such that

$$\tau := \begin{cases} 
\mathrm{Cone}(w_{\bar{n}-(n+k)}, w_{\bar{n}-n}, w_{\bar{n}-k+1}) & \text{if } \delta_{n-1} = A, \\
\mathrm{Cone}(w_3, v_4, w_5) & \text{if } \delta_{n-1} = D \text{ and } (k, n) = (2, 3), \\
\mathrm{Cone}(w_{\bar{n}-(n+k)}, v_{\bar{n}-2k}, w_{\bar{n}-k+1}) & \text{if } \delta_{n-1} = D \text{ and } (k, n) \neq (2, 3).
\end{cases}$$

Since $\tau \in \Sigma_i$ for each cases, we can think of the star subdivision $\Sigma^*_i(\tau)$ of the fan $\Sigma_i$ of $B_i$ along the cone $\tau$.

On the other hand, when $(\delta_{n-1}, \delta_n) = (D, D)$ with $k = 2$ and $n > 3$, the primitive collection $\mathrm{PC}(\Sigma^*_i(\tau))$ consists of

$$\{w_{\bar{n}-(n+2)}, v_{\bar{n}-4}, w_{\bar{n}-1}\}, \{w_{\bar{n}-3}, v_{\bar{n}-2}, w_{\bar{n}-1}\}, \{w_{\bar{n}-5}, w_{\bar{n}-4}, w_{\bar{n}-1}\}, \{w_{\bar{n}-5}, v_{\bar{n}-4}, w_{\bar{n}-1}\}, \{w_j, v_j\}_{j \in [\bar{n}]}$$

by Propositions 2.12 and 2.15. Moreover, Lemma 2.1 implies that the cone

$$(6.4) \quad \tau_2 := \mathrm{Cone}(w_{\bar{n}-3}, v_{\bar{n}-2}, v_{\bar{n}-1})$$

is contained in the fan $\Sigma^*_i(\tau)$ since each element of $\mathrm{PC}(\Sigma^*_i(\tau))$ is not contained in $\{w_{\bar{n}-3}, v_{\bar{n}-2}, w_{\bar{n}-1}\}$.

**Definition 6.6.** Suppose that $i$ satisfies Assumption 6.3. Let $\Sigma_i$ be the fan of the Bott manifold $B_i$ defined in Definition 6.1. We define the fan $\widehat{\Sigma}_i$ by

$$\widehat{\Sigma}_i := \begin{cases} 
\Sigma_i & \text{if } |\mathcal{GP}(i)| = \bar{n}, \\
\Sigma^*_i(\tau) & \text{if } |\mathcal{GP}(i)| = \bar{n} + 1, \\
(\Sigma^*_i(\tau))^*(\tau_2) & \text{if } |\mathcal{GP}(i)| = \bar{n} + 2.
\end{cases}$$

By the property of the star subdivision (see the paragraph below Definition 2.14 and Proposition 3.8), we get the following.

**Proposition 6.7.** Suppose that $i$ satisfies Assumption 6.3. Then the fan $\widehat{\Sigma}_i$ is a smooth polytopal fan. Moreover, for any regular dominant integral weight $\lambda$, we have

$$\{u_\rho \mid \rho \in \widehat{\Sigma}_i(1)\} = \{w_\gamma \mid \gamma \in \mathcal{GP}(i)\} \cup \{v_j \mid 1 \leq j \leq \bar{n}\} = \{u_\rho \mid \rho \in \Sigma_{\Delta_i(\lambda)}(1)\}$$

where $\Sigma_{\Delta_i(\lambda)}$ is the normal fan of the string polytope $\Delta_i(\lambda)$.

Before to state our main theorem, we introduce the following notion.

**Definition 6.8.** We say that $i$ has small indices if $\mathrm{ind}_i(\delta) = (0, \ldots, 0, k)$ for some $\delta \in \{A, D\}^n$ and $k \leq 2$.

We will see later that every reduced word in $R(w_0^{(4)})$ has small indices.

**Proposition 6.9.** Suppose that $i \in R(w_0^{(\bar{n}+1)})$ has small indices. Then $i$ satisfies Assumption 6.3.
Now we observe the last part of the word \( \iota' \):

1. \((\delta_{n-2}, \delta_{n-1}) = (D, D): \iota' = (\ldots, n-2, n-3, 2, 1, n-1, \ldots, n-2, 3, 2, 1) \).
2. \((\delta_{n-2}, \delta_{n-1}) = (A, D): \iota' = (\ldots, 1, 2, \ldots, n-2, n-1, n-2, 3, 2, 1) \).
3. \((\delta_{n-2}, \delta_{n-1}) = (D, A): \iota' = (\ldots, n-1, n-2, 2, 1, 2, \ldots, n-2, n-1) \).
4. \((\delta_{n-2}, \delta_{n-1}) = (A, A): \iota' = (\ldots, 2, 3, n-1, 1, 2, \ldots, n-3, n-2, n-1) \).

One can see that, in any case, the last two words (2, 1) or \((n-2, n-1)\) of \( \iota' \) do not change when applying 2-move on \( \iota' \). This implies that, by setting \( \iota'' = (i_1', \ldots, i_{n-2}'') \) and \( \iota'' = (i_1', \ldots, i_{n-2}'') \), we have

\[
(i_1', \ldots, i_{n-2}'') \sim (i_1', \ldots, i_{n-2}'') \quad \text{and} \quad (i_1'', \ldots, i_{n-2}'') = (i_1''', \ldots, i_{n-2}'''').
\]

Therefore, we get

\[
i = E_{\delta_n}(k)(\iota'') \sim E_{\delta_n}(k)(\iota'') = i\delta(0, \ldots, 0, k)
\]

since \( k \leq 2 \). Hence we prove the proposition. \( \square \)

Now we are ready to state our main theorem which states that the smooth projective toric variety \( X_{\Sigma_i} \) provides a small desingularization of \( X_{\Delta_i(\lambda)} \) when \( i \) satisfies Assumption 6.3. (The proof will be given in Section 7.)

**Theorem 6.10.** Let \( i \) be a reduced word of the longest element in the Weyl group of \( SL_{n+1}(\mathbb{C}) \) and \( \lambda \) a regular dominant integral weight. If \( i \) satisfies Assumption 6.3, then the toric variety \( X_{\Sigma_i} \) is a small desingularization of the toric variety \( X_{\Delta_i(\lambda)} \). In particular if \( i \) has small indices, then the toric variety associated to a string polytope admits a small resolution.

Assumption 6.3 will play a crucial role in the proof. We will consider the case when \( i \) does not satisfy Assumption 6.3 in Example 7.1. In this case, our smooth projective toric variety does not give a small resolution of \( X_{\Delta_i(\lambda)} \).

**Remark 6.11.** One may ask whether there exists a small resolution of \( X_{\Delta_i(\lambda)} \) when \( i \) does not necessarily satisfy Assumption 6.3. Unfortunately, the answer is negative. By Proposition 2.8 if the toric variety \( X_{\Delta_i(\lambda)} \) admits a small resolution for a regular dominant weight \( \lambda \), then the string polytope \( \Delta_i(\mu) \) becomes an integral polytope for any dominant integral weight \( \mu \). However Steinert [Ste19] recently provides an example of a non-integral string polytope. More precisely, for \( i = (1, 3, 2, 1, 3, 2, 4, 3, 2, 1, 5, 4, 3, 2, 1) \), the string polytope \( \Delta_i(\pi_3) \) is not integral (see [Ste19] Example 7.5). Consequently, the toric variety \( X_{\Delta_i(\lambda)} \) cannot admit a small resolution for a regular dominant integral weight \( \lambda \). One can easily check that the word \( i \) does not satisfy Assumption 6.3.

Now we consider the string polytopes for \( n \leq 3 \). When \( n = 1 \) or 2, then all string polytopes are Gelfand–Cetlin type, and hence they admit small resolutions (see Remark 6.2. When \( n = 3 \), there are 16 reduced words of the longest element in \( S_4 \) by the hook length formula (see [BB03] Corollary 7.4.8))

\[
|R(w^{(n+1)}_0)| = \frac{(n+1)!}{n!3^{n+1}(n-1)\cdots(2n-1)}.
\]

Under the equivalence relation given by 2-moves and the involution \( \iota: [3] \rightarrow [3] \) on the Dynkin diagram as in Example 6.6, one can classify the reduced words into four types as we see below. In other words, there are at most four types of string polytopes up to unimodular equivalence.

Type 1. \((1, 2, 1, 3, 2, 1), (1, 2, 3, 1, 2, 1), (2, 1, 2, 3, 2, 1), (2, 3, 2, 1, 2, 3), (3, 2, 3, 1, 2, 3), (3, 2, 1, 3, 2, 3)\).

Type 2. \((1, 2, 3, 2, 1, 2), (3, 2, 1, 2, 3, 2)\).

Type 3. \((1, 3, 2, 3, 1, 2), (1, 3, 2, 1, 3, 2), (3, 1, 2, 3, 1, 2), (3, 1, 2, 1, 3, 2)\).

Type 4. \((2, 1, 3, 2, 3, 1), (2, 3, 1, 2, 3, 1), (2, 1, 3, 2, 1, 3), (2, 3, 1, 2, 1, 3)\).
For each type, we can compute \( \delta \)-indices (for a particularly chosen \( \delta \) as below) and see that all the reduced words in \( R(w_0^{(i)}) \) have small indices:

\[
\begin{align*}
(1, 2, 1, 3, 2, 1) &= i_{(D,D,D)}(0, 0, 0), \\
(1, 2, 3, 2, 1, 2) &= i_{(D,D,D)}(0, 0, 1), \\
(1, 3, 2, 1, 3, 2) &= i_{(D,D,D)}(0, 0, 2), \\
(2, 1, 3, 2, 1, 3) &= i_{(D,A,D)}(0, 0, 1).
\end{align*}
\]

Therefore we obtain the following.

**Corollary 6.12.** For any \( i \in R(w_0^{(i)}) \) and a regular dominant weight \( \lambda \), the toric variety \( X_{\Delta_i(\lambda)} \) associated to a string polytope \( \Delta_i(\lambda) \) admits a small resolution.

For \( n = 4 \), there exist reduced words of the longest element which do not satisfy Assumption 6.3. In Appendix B we calculate all \( \delta \)-indices for every reduced word in \( R(w_0^{(5)}) \). See Table 1.

Now we illustrate some corollaries of Theorem 6.10. The following, the integrality of string polytopes, is an immediate consequence of Proposition 2.8.

**Corollary 6.13.** Suppose that \( i \in R(w_0^{(n+1)}) \) satisfies Assumption 6.3. Then the string polytope \( \Delta_i(\lambda) \) is integral for any dominant integral weight \( \lambda \).

Next, let \( P \subset G \) be a parabolic subgroup and \( \lambda_P \) the weight corresponding to the anticanonical bundle of the partial flag variety \( G/P \). (For example, we have \( \lambda_B = 2\bar{\omega}_1 + \cdots + 2\bar{\omega}_n \) for a Borel subgroup \( B \subset G \).

The following corollary guarantees the reflexivity of string polytopes for the weight \( \lambda_P \).

**Corollary 6.14.** Let \( P \subset G \) be a parabolic subgroup and \( \lambda_P \) the weight corresponding to the anticanonical bundle of a partial flag variety \( G/P \). Suppose that \( i \in R(w_0^{(n+1)}) \) satisfies Assumption 6.3. Then the string polytope \( \Delta_i(\lambda_P) \) is reflexive, and therefore the toric variety associated to the string polytope \( \Delta_i(\lambda_P) \) is a Gorenstein Fano variety.

**Proof.** Note that the string polytope \( \Delta_i(\lambda_P) \) is integral by Corollary 6.13. Moreover, it was proved in [Rus08, Theorem 7] that the dual \( (\Delta_i(\lambda_P) - m)^\circ \) of the translation of the string polytope is integral for a certain vector \( m \in M_\mathbb{R} \cong \mathbb{R}^n \). Thus \( \Delta_i(\lambda_P) \) is reflexive and therefore the projective toric variety associated to \( \Delta_i(\lambda_P) \) is a Gorenstein Fano variety by Theorem 8.3.4 in [CLS11].

Finally we introduce some applications of Theorem 6.10 to the symplectic geometry of flag varieties. For a given symplectic manifold \( (M, \omega) \) and a Lagrangian submanifold \( L \subset M \) with a suitable condition (weakly unobstructed for example), one can define a function \( \Psi \) called a Landau–Ginzburg superpotential or a disc potential. The function \( \Psi \) is defined on some set called a Maurer–Cartan space. Roughly speaking, the potential function \( \Psi \) is a Laurent series and it encodes the number of holomorphic discs (of Maslov index two) bounded by \( L \).

Now let \( \lambda \) be a regular dominant integral weight. Our goal is to compute the potential function \( \Psi \) of some Lagrangian tori in the flag variety \( G/B \) obtained from a toric degeneration \( G/B \) by using the work of Nishinou–Nohara–Ueda [NNU10].

By Caldero [Cal02], there exists a toric degeneration

\[
\mathcal{X} = \{ X_t \mid t \in \mathbb{C} \} \quad \text{such that} \quad X_t \cong G/B \quad \text{for} \quad t \in \mathbb{C} \setminus \{ 0 \} \quad \text{and} \quad X_0 \cong X_{\Delta_i(\lambda)}.
\]

From \( \mathcal{X} \), one can obtain a completely integrable system

\[
\Phi_{t,\lambda}: G/B \to \mathbb{R}^n
\]

in the sense of Harada–Kaveh such that the image \( \Phi_{t,\lambda}(G/B) \) is the string polytope \( \Delta_i(\lambda) \) (see [HK15, Corollary 3.36]). In particular, every fiber \( L(u) := \Phi_{t,\lambda}^{-1}(u) \) becomes a Lagrangian torus with respect to a Kostant–Kirillov–Souriau symplectic form on \( G/B \). The following corollary, obtained directly from Theorem 6.10 [NNU10, Theorem 10.1], and [NNU12, Theorem 1], states that the potential function for each \( L(u) \) can be computed in terms of \( \Delta_i(\lambda) \) as follows.

**Corollary 6.15.** Let \( B \) be a Borel subgroup of \( G = SL_{n+1}(\mathbb{C}) \). Let denote a regular dominant integral weight by \( \lambda = \lambda_1\bar{\omega}_1 + \cdots + \lambda_n\bar{\omega}_n \) and suppose that \( i \in R(w_0^{(n+1)}) \) satisfies Assumption 6.3. Then the disk potential of \( L(u) \) is

\[
(6.7) \quad \Psi \Omega_i(x) = \sum_{\gamma \in \mathcal{P}(i)} e^{(w_\gamma, z)} T(w_\gamma, u) + \sum_{j=1}^n e^{(v_j, z)} T(v_j, u) + \lambda_j.
\]
for $x \in H^1(L(u), \Lambda_0)$. Here, $\Lambda_0$ is the Novikov ring $\{ \sum_{i=0}^{\infty} a_i T^{\mu_i} \mid a_i \in \mathbb{C}, \mu_i \in \mathbb{R}_{\geq 0}, \lim_{i \to \infty} \mu_i = \infty \}$.

**Remark 6.16.** In Corollary 6.15 note that the potential function for $L(u)$ is defined on some set (called the Maurer Cartan space) denoted by $\mathcal{M}(L(u))$. It is proved by Nishino-Nohara-Ueda [NU12 Theorem 1] that $H^1(L(u), \Lambda_0) \subset \mathcal{M}(L(u))$ when the toric variety $X_{\Delta(\lambda)}$ associated to $\Delta(\lambda)$ admits a small resolution. The formula of $\mathfrak{PO}_1$ in Corollary 6.15 is indeed for the restriction of $\mathfrak{PO}_1$ to $H^1(L(u), \Lambda_0)$.

**Example 6.17.** Let $\lambda = (1, 3, 2, 1, 3, 2) \in R(w^{(4)}_0)$ and $\lambda = \lambda_1 \varpi_1 + \lambda_2 \varpi_2 + \lambda_3 \varpi_3$. (See Example 3.7 for the inequalities for the corresponding string polytope.) Then the potential function for the Lagrangian torus fiber determined by the string polytope $\Delta_i(\lambda)$ is given by

$$
\mathfrak{PO}_1 = e^{x_1 + x_3 + x_5} T^{u_1 + u_3 + u_5} + e^{x_2 + x_4} T^{u_2 + u_4} + e^{x_3 + x_5} T^{u_3 + u_5} + e^{x_4} T^{u_4} + e^{x_5} T^{u_5}
$$

This potential function can be written as

$$
\mathfrak{PO}_1 = y_1 y_3 y_5 + y_3 y_5 + y_5 + y_6 + y_2 y_3 y_4 + y_3 y_4 + y_4 + q_1 y_1 y_4 + q_2 y_2 y_5 + q_3 y_3 y_6 + y_4 + q_4 y_5 + q_5 y_6
$$

by setting $q_i = T^{\lambda_i}$ for $1 \leq i \leq 3$ and $y_j = e^{x_j} T^{u_j}$ for $1 \leq j \leq 6$.

**Remark 6.18.** Suppose that reduced words $i$ and $i'$ in $R(w^{(n)}_0)$ are related by a 3-move. Namely,

$$
i = (i_1, i, i + 1, i, i_2) \leftrightarrow (i_1, i + 1, i, i + 1, i_2) = i'.
$$

Berenstein and Zelevinsky proved that there is a piecewise-linear automorphism $i^* T_1: \mathbb{R}^n \to \mathbb{R}^n$ preserving the lattice and such that

$$
i^* T_1(\Delta_i(\lambda) \cap \mathbb{Z}^n) = \Delta_i(\lambda) \cap \mathbb{Z}^n
$$

for a dominant integral weight $\lambda$ (see [BZ93 Theorem 2.7]). More precisely, suppose that the 3-move occupies positions $k, k + 1, k + 2$. Then, the map $i^* T_1$ leaves all the components of a coordinate $t = (t_1, \ldots, t_n)$ except $t_k, t_{k+1}, t_{k+2}$ and changes $(t_k, t_{k+1}, t_{k+2})$ to

$$(6.8) \quad (\max(t_{k+2}, t_{k+1} - t_k), t_k + t_{k+2}, \min(t_k, t_{k+1} - t_{k+2})).$$

For any two words $i$ and $i'$ which are related by a 3-move and have small indices, two potentials $\mathfrak{PO}_1$ and $\mathfrak{PO}_{i'}$ are related by the coordinate change whose tropical lift is (6.8).

7. **Proof of Theorem 6.10**

In this section, we will give a proof of Theorem 6.10. Assume that $i$ satisfies Assumption 6.3. Since the normal fan of the string polytope $\Sigma_{\Delta_i(\lambda)}$ is independent of the choice of a regular dominant weight $\lambda$, we may assume that

$$\lambda = 2 \sum_{i=1}^{n} \varpi_i.$$

Let $\hat{\Sigma}_i$ be the fan, the refinement of the fan $\Sigma_i$ associated to the Bott manifold $B_i$, defined in Definition 6.6 and $D_j$ the torus-invariant prime divisor corresponding to the ray generator $v_j$ for each $j \in [\hat{n}]$. Let $D$ be a Cartier\footnote{Any Weil divisor of a smooth toric variety is Cartier, see [LS11] Proposition 4.2.6.} divisor given by

$$D = \sum_{j=1}^{\hat{n}} 2D_j.$$

Then the polyhedron associated to the divisor $D$ is precisely

$$P_D = \Delta_i(\lambda)$$

by the definition of the polytope $P_D$ (see (2.1)) and the string polytope (see Definition 3.6). Therefore, to prove Theorem 6.10 it is enough to show that $D$ is a basepoint free divisor on $X_{\Sigma_i}$ by Corollary 2.7.
To check whether the divisor $D$ is basepoint free, we apply Theorem 2.2 to primitive collections in $\text{PC}(\hat{\Sigma}_i)$. That is, it is enough to show that the support function $\varphi_D$ satisfies

$$\varphi_D \left( \sum_{x \in \mathcal{P}} x \right) \geq \sum_{x \in \mathcal{P}} \varphi_D(x)$$

for every primitive collection $\mathcal{P} \in \text{PC}(\hat{\Sigma}_i)$. We divide into four cases and prove our claim by case-by-case analysis as follows.

**Case 1:** $k = 0$. In this case, we have $|\mathcal{GP}(i)| = \bar{n}$ and so each region $R_i$ does not contain any node in its interior. In addition, there exists a unique rigorous path $\gamma_j$ having a peak $t_j$ for each $j \in [\bar{n}]$ and every rigorous path travels along at most 3 wires by Lemma 5.2. Namely, each path is of the form:

\[
\begin{cases}
\ell_i \rightarrow \ell_{i+1} \\
\ell_i \rightarrow \ell_p \rightarrow \ell_{i+1}
\end{cases}
\]

for some $i, p \in [\bar{n}]$.

We first claim that it satisfies either

$$w_j + v_j = 0,$$

or

there exist $j_1 \neq j_2$ such that $w_j + v_j = w_{j_1} + v_{j_2}$.

Figures 13 and 14 present all possible local pictures of the path $\gamma_j$ around the maximal peak $t_j$ in case where $t_j$ is not on $\ell_{n+1}$ (and therefore some wire $\ell_q$ passes through the interior of the region $R_i$). Note that $w_j + v_j = 0$ only when $t_j$ is lying on $\ell_{n+1}$. Moreover, the orientation of the wire $\ell_q$ in each case is uniquely determined as the forbidden pattern in Figure 4 should be avoided. One can easily see that the path $\ell_i \rightarrow \ell_q \rightarrow \ell_{i+1}$ is a rigorous path corresponding to the vector $w_{j_1}$. Let $t_{j_2}$ be the node described in the figures. Then the nodes $t_j$ and $t_{j_2}$ are on the same column while $t_{j_1}$ is located on the next column of $t_j$, which implies that $j_1 \neq j_2$. Thus the claim follows from the definitions of $w_j$'s and $v_j$'s. Note that it is straightforward from Figures 13 and 14 that the region assigned by $w_j + v_j$ is equal to that of $w_{j_1} + v_{j_2}$ since the regions associated to $w_j$ and $w_{j_1}$ (respectively $v_j$ and $v_{j_2}$) are differ by $C_j$ (respectively $-C_j$).
It remains to show that $D$ is basepoint free. Since $j_1 \neq j_2$, the cone $\text{Cone}(w_{j_1}, v_{j_2})$ is contained in $\hat{\Sigma}_i$. Moreover, the sum $w_{j_1} + v_{j_2}$ is contained in that cone. Using the fact that the support function $\varphi_D$ is linear on each cone, we get

\begin{equation}
\varphi_D(w_{j_1} + v_{j_2}) = 0 \geq \varphi_D(w_{j_1}) + \varphi_D(v_{j_2}) = 0 + (-2), \text{ or}
\end{equation}

\begin{equation}
\varphi_D(w_{j_1} + v_{j_2}) = \varphi_D(w_{j_1} + v_{j_2}) = \varphi_D(w_{j_1}) + \varphi_D(v_{j_2}) = 0 + (-2) \geq \varphi_D(w_{j_1}) + \varphi_D(v_{j_2}) = 0 + (-2).
\end{equation}

Therefore the inequalities (7.1) hold for primitive collections $\{w_j, v_j\} \mid 1 \leq j \leq \bar{n}$, and the result follows.

**Case 2:** $\delta_{n-1} = D$ and $k = 1$. In this case, the wiring diagram can is described in Figure 15 and there is a unique non-canonical path $\gamma_0$ (blue path).

![Figure 15](image_url)

**Figure 15.** Rigorous paths $\gamma_{n-(n+1)} = (\ell_{n-2} \to \ell_n \to \ell_{n+1} \to \ell_{n-1})$ (dotted red) and $\gamma_0 = (\ell_{n-2} \to \ell_n \to \ell_{n-1})$ (blue) for $\delta_{n-1} = D$ and $k = 1$.

Recall from (6.3) that $\tau$ is the cone generated by the set $\{w_{n-(n+1)}, v_{n-2}, w_n\}$. To check the basepoint freeness of $D$, we consider the set of primitive collections of $\hat{\Sigma}_i = \Sigma_i^*$

\[ \text{PC}(\hat{\Sigma}_i) = \{w_j, v_j\} \mid 1 \leq j \leq \bar{n} \]

\[ \cup \{w_{n-(n+1)}, v_{n-2}, w_n\} \cup \{w_{\gamma_0}, v_{n-(n+1)}\} \cup \{w_{\gamma_0}, w_{n-2}\} \cup \{w_{\gamma_0}, v_n\} \]

obtained by Proposition 2.15 where $w_{\gamma_0}$ is the vector generating the ray in the fan $\hat{\Sigma}_i$ which corresponds to the path $\gamma_0$.

Let $t_x$ be the node at the second from the bottom among nodes lying on the first column (see Figure 15). Similarly to **Case 1**, we see that (7.2) holds since

- for each node $t_j \neq t_x, t_{n-(n+1)}$, the local shape around the node coincides with one of the pictures in Figures 13 and 14 and so $(j_1, j_2)$ is uniquely determined,
- for $t_x$, we have $w_x + v_x = w_{n-(n+1)} + v_{n-1}$

and therefore we take $(j_1, j_2) = (n - (n + 1), n - 1)$,

- for $t_{n-(n+1)}$, we have $w_{n-(n+1)} + v_{n-(n+1)} = w_{n-3} + v_n$.

and so we take $(j_1, j_2) = (n - 3, n)$.

(This procedure is necessary since there are two rigorous paths having the maximal peak $t_{n-(n+1)}$.) Since every pair $\{w_{j_1}, v_{j_2}\}$ does not contain any primitive collection in $\text{PC}(\hat{\Sigma}_i)$, Lemma 2.1 implies that

\[ \text{Cone}(w_{j_1}, v_{j_2}) \in \hat{\Sigma}_i. \]
For the remaining four primitive collections in PC(\(\tilde{\Sigma}_4\)), we get the following relations:

\[
\begin{align*}
\mathbf{w}_{\bar{n}-(n+1)} + \mathbf{v}_{\bar{n}-2} + \mathbf{w}_{\bar{n}} &= \mathbf{w}_{\bar{n}0}, & \mathbf{w}_{\bar{n}0} + \mathbf{v}_{\bar{n}-(n+1)} &= \mathbf{w}_{\bar{n}-3} + \mathbf{v}_{\bar{n}-2}, \\
\mathbf{w}_{\bar{n}0} + \mathbf{v}_{\bar{n}-2} &= \mathbf{w}_{\bar{n}-(n+1)} + \mathbf{w}_{\bar{n}-1}, & \mathbf{w}_{\bar{n}0} + \mathbf{v}_{\bar{n}} &= \mathbf{w}_{\bar{n}-(n+1)} + \mathbf{v}_{\bar{n}-2}.
\end{align*}
\]

Since the map \(\varphi_D\) is linear on each cone in \(\tilde{\Sigma}_4\), we have that

\[
\begin{align*}
\varphi_D(\mathbf{w}_j + \mathbf{v}_j) &= \varphi_D(\mathbf{w}_j) + \varphi_D(\mathbf{v}_j) = 0 + (-2) = -2, \\
\varphi_D(\mathbf{w}_{\bar{n}-(n+1)} + \mathbf{v}_{\bar{n}-2} + \mathbf{w}_{\bar{n}}) &= \varphi_D(\mathbf{w}_{\bar{n}0}) = 0
\end{align*}
\]

Therefore the inequality (7.1) holds for every primitive collection and this proves the theorem for Case 2.

**Case 3:** \(\delta_{n-1} = D\) and \(k = 2\). In this case, we have \(n \geq 3\) as \(0 \leq k \leq n - 1\). We divide into two cases: \(n = 3\) (\(|GP(1)| = \bar{n} + 1\)) and \(n > 3\) (\(|GP(1)| = \bar{n} + 2\). See (6.1).

For the first 6 primitive collections, we can directly read off the following relations from Figure 5(2):

\[
\begin{align*}
\mathbf{w}_1 + \mathbf{v}_1 &= \mathbf{w}_{\bar{n}2} + \mathbf{v}_4, & \mathbf{w}_2 + \mathbf{v}_2 &= \mathbf{w}_3 + \mathbf{v}_5, & \mathbf{w}_3 + \mathbf{v}_3 &= \mathbf{w}_4 + \mathbf{v}_6, \\
\mathbf{w}_j + \mathbf{v}_j &= 0 & \text{for } j = 4, 5, 6.
\end{align*}
\]

(For instance for the first equality, both \(\mathbf{w}_1 + \mathbf{v}_1\) and \(\mathbf{w}_{\bar{n}2} + \mathbf{v}_4\) correspond to the formal sum \(\mathbf{E}_3 - \mathbf{E}_4 + \mathbf{E}_5\) in Figure 5(2)). Observe that none of the pairs of summands on the right hand side of each relations in (7.5) contain any of primitive collections, which implies that each pair generates a cone in \(\tilde{\Sigma}_4\) by Lemma 2.1.

For the other primitive collections, similarly we obtain the followings:

\[
\begin{align*}
\mathbf{w}_3 + \mathbf{v}_4 + \mathbf{w}_5 &= \mathbf{w}_{\bar{n}2}, & \mathbf{w}_{\bar{n}2} + \mathbf{v}_3 &= \mathbf{w}_5 + \mathbf{v}_6, & \mathbf{w}_{\bar{n}2} + \mathbf{v}_4 &= \mathbf{w}_3 + \mathbf{w}_5, & \mathbf{w}_{\bar{n}2} + \mathbf{v}_5 &= \mathbf{w}_3 + \mathbf{v}_4.
\end{align*}
\]

Since the support function \(\varphi_D\) is linear on each cone in \(\tilde{\Sigma}_4\) and

\[
\varphi_D(\mathbf{v}_j) = -2, \quad \varphi_D(\mathbf{w}_j) = \varphi_D(\mathbf{w}_{\bar{n}2}) = 0,
\]

the relations in (7.5) and (7.6) imply that the support function \(\varphi_D\) satisfies the desired inequalities (7.1).

(This conclusion is straightforward since for each relations, the number of \(\mathbf{v}_j\)'s on the left is greater then equal to that on the right, cf. (7.4).)

Now let us consider the case of \(k = 2\) with \(n > 3\). The main difference from the case \((k, n) = (2, 3)\) is that one more non-canonical path \(\gamma_0\) (the blue path in Figure 16) appears. Therefore, as in (6.1), there are two non-canonical paths \(\gamma_0\) and \(\gamma_2\) with

\[
\begin{align*}
\mathbf{w}_{\bar{n}0} &= \mathbf{w}_{\bar{n}-(n+2)} + \mathbf{v}_{\bar{n}-4} + \mathbf{w}_{\bar{n}-1}, & \mathbf{w}_{\bar{n}2} &= \mathbf{w}_{\bar{n}-3} + \mathbf{w}_{\bar{n}-1} + \mathbf{v}_{\bar{n}-2}.
\end{align*}
\]
See Proposition 6.4. For the fan \( \hat{\Sigma}_1 = (\Sigma^*_1(\tau))^{*}(\tau_2) \) defined in (6.4), where \( \tau = \text{Cone}(w_{\tilde{n}-(n+2)}, v_{n-4}, w_{\tilde{n}-1}) \) and \( \tau_2 = \text{Cone}(w_{\tilde{n}-3}, w_{\tilde{n}-1}, v_{\tilde{n}-2}) \) in (6.3) and (6.4), the set of primitive collections of \( \hat{\Sigma}_1 \) is

\[
\text{PC}(\hat{\Sigma}_1) = \{ \{w_j, v_j\} \mid 1 \leq j \leq \tilde{n} \}
\]

\[\cup \{w_{\tilde{n}-(n+2)}, v_{n-4}, w_{\tilde{n}-1}\} \cup \{w_{\tilde{n}0}, v_{n-(n+2)}\} \cup \{w_{\tilde{n}0}, w_{\tilde{n}-4}\} \cup \{w_{\tilde{n}0}, v_{\tilde{n}-1}\}
\]

\[\cup \{w_{\tilde{n}-3}, w_{\tilde{n}-1}, v_{\tilde{n}-2}\} \cup \{w_{\tilde{n}2}, v_{n-3}\} \cup \{w_{\tilde{n}2}, v_{n-1}\} \cup \{w_{\tilde{n}2}, w_{\tilde{n}-2}\} \cup \{w_{\tilde{n}2}, w_{\tilde{n}-(n+2)}, v_{n-4}\}
\]

which can be obtained by Proposition 2.15.

Let \( t_x \) and \( t_y \) be the nodes at the second from the bottom among nodes lying on the first column and at the third from the bottom among nodes lying on the second column, respectively. See Figure 16 for nodes \( t_x \) and \( t_y \). For simplicity, we only draw the case \( x > y \), but in general \( x \) does not need to be greater than \( y \). Similarly to (7.2), we have

\[w_j + v_j = 0; \text{ or}
\]

there exist \( j_1 \neq j_2 \) such that

\[
w_j + v_j = \begin{cases} w_{j_1} + v_{j_2} & \text{if } j \neq x \\ w_{\tilde{n}0} + v_{j_2} & \text{if } j = x \end{cases}
\]

because of the following observation:

- for each node \( t_j \neq t_x, t_y, t_{n-(n+1)} \), the local shape around the node coincides with one of the pictures in Figures 13 and 14 and so \( (j_1, j_2) \) is uniquely determined in each case.
- for \( t_x \), we have

\[w_x + v_x = w_{\tilde{n}0} + v_{\tilde{n}-2},
\]

- for \( t_y \), we have

\[w_y + v_y = w_{n-(n+2)} + v_{n-3}
\]

and therefore we take \( (j_1, j_2) = (\tilde{n} - (n + 2), \tilde{n} - 3) \),

- for \( t_{n-(n+2)} \), we have

\[w_{\tilde{n}-(n+2)} + v_{\tilde{n}-(n+2)} = w_{\tilde{n}-5} + v_{\tilde{n}-1}
\]

and so we take \( (j_1, j_2) = (\tilde{n} - 5, \tilde{n} - 1) \).

In particular, we have \( j_1 \neq j_2 \).

For showing that \( D \) is basepoint free, we apply the same procedure as in the previous cases as follows. For the first \( \tilde{n} \) primitive collections, we can similarly show that each pair \( \{w_{j_1}, v_{j_2}\} \) as well as \( \{w_{\tilde{n}0}, v_{j_2}\} \) does not contain any primitive collection of \( \hat{\Sigma}_1 \) listed in (7.7) and so it generates a cone in \( \hat{\Sigma}_1 \) using Lemma 2.1. More precisely, it is rather straightforward that \( \{w_{j_1}, v_{j_2}\} \) does not contain any of (7.7) since \( j_1 \neq j_2 \).

For \( \{w_{\tilde{n}0}, v_{j_2}\} \), we need to show that

\[j_2 \neq \tilde{n} - (n + 2), \tilde{n} - 1
\]

which follows from our observations above that \( j_2 = \tilde{n} - 2 \).

For remaining nine primitive collections in (7.7), we obtain the following relations:

\[
\begin{align*}
&\{w_{\tilde{n}-(n+2)} + v_{\tilde{n}-4} + w_{\tilde{n}-1} = w_{\tilde{n}0},
&\{w_{\tilde{n}0} + v_{\tilde{n}-(n+2)} = w_{\tilde{n}-5} + v_{\tilde{n}-4},
&\{w_{\tilde{n}0} + v_{\tilde{n}-1} = w_{\tilde{n}-2},
&\{w_{\tilde{n}2} = w_{\tilde{n}-3} + v_{\tilde{n}-2},
&\{w_{\tilde{n}2} + w_{\tilde{n}-(n+2)} + v_{\tilde{n}-4} = w_{\tilde{n}0} + w_{\tilde{n}-3} + v_{\tilde{n}-2}.
\end{align*}
\]

See Figure 16 [Note: The figure is not provided here.] We can check that the right hand side of each relation in (7.11) generates a cone in \( \hat{\Sigma}_1 \) in a similar fashion. Combining all relations (7.8), (7.9), (7.10), and (7.11), and the linearity of the support function \( \varphi_D \) each cone together with the informations

\[
\varphi_D(v_j) = -2, \quad \varphi_D(w_j) = \varphi_D(w_{\tilde{n}0} = \varphi_D(w_{\tilde{n}2} = 0,
\]

we see that \( \varphi_D \) satisfies the desired inequalities (7.1). (As mentioned at the end of the first part of Case 3, the conclusion immediately follows from that the number of \( v_j \)'s on the left is greater than equal to that on the right for each relations.) This completes the proof for Case 3.
By Proposition 2.4, we have that $\delta_{n-1} = D$ and $k = 2$. In particular we have (7.2) because

$$w_{n-k+1} + w_{n-k} = w_{n-k+1}$$

Case 4: $\delta_{n-1} = A$ and $0 < k < n - 1$. In this case, there is one non-canonical path $\gamma_0$ with $w_{\gamma_0} = w_{\gamma_0} + v_{\gamma_0} + w_{\gamma_0}$ and the fan $\Sigma_4$ is given by $\Sigma_4(\tau)$ where $\tau = \text{Cone}(w_{\gamma_0} + v_{\gamma_0} + w_{\gamma_0})$. (The picture for this case is described in Figure 10.) By Proposition 2.4 we have that

$$PC(\Sigma_4) = \{ \{w_{j}, v_{j}\} \mid 1 \leq j \leq \tilde{n}\}$$

$$+ \{ w_{\tilde{n}-n+k} + v_{\tilde{n}-n} + w_{\tilde{n}-k+1} \} \cup \{ w_{\gamma_0}, v_{\tilde{n}-n} \} \cup \{ w_{\gamma_0}, v_{\tilde{n}-n-k+1} \}.$$

Figure 16. When $\delta_{n-1} = D$ and $k = 2$, rigorous paths $\gamma_2 = (\ell_{n-2} \rightarrow \ell_{n+1} \rightarrow \ell_{n-1})$ (dotted red) and $\gamma_0 = (\ell_{n-3} \rightarrow \ell_{n} \rightarrow \ell_{n-2})$ (blue).

Let $t_x$ be the node at the second from the bottom among nodes lying on the $(n-k)$th column (in the painted region in Figure 10). We can similarly prove that (7.2) holds because

- for each node $t_j \neq t_x$, the local shape around the node coincides with one of the pictures in Figures 13 and 14, so $(j_1, j_2)$ is uniquely determined,
- for $t_x$, we have

$$w_{x} + v_{x} = w_{\tilde{n}-(n-k)} + v_{\tilde{n}-n-1}.$$

In particular we have $j_1 \neq j_2$. Moreover, one can easily see that $\{w_{j_1}, v_{j_2}\}$ does not contain any of the primitive collections of $\Sigma_4$ and so it generates a cone in $\Sigma_1$ for every $\{w_{j_1}, v_{j_2}\}$. Furthermore, the inequalities in (7.1) hold for each $\{w_{j_1}, v_{j_2}\}$. (Indeed, the inequalities in (7.1) are equalities since the left and right hand side of the relations contains the same number (one) of $v_j$'s.)

For the rest four primitive collections in (7.12), we have

$$w_{\tilde{n}-(n-k)} + v_{\tilde{n}-n} + w_{\tilde{n}-k+1} = w_{\gamma_0}, \quad w_{\gamma_0} + v_{\tilde{n}-(n-k)} = w_{\tilde{n}-n+1} + v_{\tilde{n}-n+2},$$

$$w_{\gamma_0} + w_{\tilde{n}-n} = w_{\tilde{n}-(n-k)} + w_{\tilde{n}-k+1}, \quad w_{\gamma_0} + v_{\tilde{n}-(n-k)} + w_{\gamma_0} + w_{\tilde{n}-k+1} = w_{\tilde{n}-(n-k)} + v_{\tilde{n}-n}.$$

Since the support function $\varphi_D$ is linear on each cone and

$$\varphi_D(v_j) = -2, \quad \varphi_D(w_{j_1}) = \varphi_D(w_{\gamma_0}) = 0,$$

the relations in (7.13) imply that the support function $\varphi_D$ satisfies the desired inequalities (7.1).

We finalize this section by presenting an example of a smooth projective toric variety $X_{\Sigma_1}$ such that $\{u_\rho | \rho \in \Sigma_1(1)\} = \{u_\rho | \rho \in \Sigma_{\Delta_1}(1)\}$ but $X_{\Sigma_1}$ is not a small desingularization of $X_{\Delta_1}$.

Example 7.1. Suppose that

$$i = i_{D,A,A,A,D,D}(0,0,0,0,0,3) = (4,3,4,2,3,4,1,2,3,4,5,4,6,5,4,3,2,1,4,3,2) \in R(w_0^{(7)}).$$
Since \( \text{ind}_A(i) = 9 \) and \( \text{ind}_D(C_A(i)) = 3 \neq 0 \), the word \( i \) does not satisfy Assumption 6.3. By Theorem 5.1 (also, see Example 5.14 and Figure 6.9), the number of rigorous paths is \( 21 + 3 = 24 \). Following Definition 6.1, we may find \( \{v_j, w_j \mid 1 \leq j \leq 21\} \). For remaining three non-canonical paths \( \bar{\gamma}_0, \bar{\gamma}_2, \bar{\gamma}_3 \) (see Definition 5.9), we have the following relations:

\[
\bar{\gamma}_0 = w_{12} + v_{15} + w_{19}, \quad \bar{\gamma}_2 = w_{17} + v_{18} + w_{20}, \quad \bar{\gamma}_3 = w_{16} + v_{17} + v_{18} + w_{19} + w_{21}.
\]

By setting

\[
\tau := \text{Cone}(w_{12}, v_{15}, w_{19}), \quad \tau_2 := \text{Cone}(w_{17}, v_{18}, w_{20}), \quad \tau_3 := \text{Cone}(w_{16}, v_{17}, v_{18}, w_{19}, w_{21}),
\]

we may define the fan \( \bar{\Sigma}_4 \) by

\[
\bar{\Sigma}_4 = \left( (\Sigma^* (\tau))^* (\tau_2) \right)^* (\tau_3)
\]

Then the set of primitive collections are given by

\[
\text{PC}(\bar{\Sigma}_4) = \{ \{w_j, v_j\} \mid 1 \leq j \leq 21\}
\]

\[
\cup \{w_{12}, v_{15}, w_{19}\} \cup \{w_{16}, v_{17}, w_{18}\} \cup \{w_{17}, v_{18}, w_{20}\}
\]

\[
\cup \{w_{16}, v_{17}, v_{18}, w_{21}\} \cup \{w_{17}, v_{18}, w_{20}\} \cup \{w_{17}, v_{18}\} \cup \{w_{18}, w_{20}\}
\]

\[
\cup \{w_{12}, v_{15}, w_{19}\} \cup \{w_{16}, v_{17}, v_{18}, w_{21}\} \cup \{w_{17}, v_{18}\} \cup \{w_{18}, w_{20}\}
\]

\[
\cup \{w_{12}, v_{15}, w_{19}\} \cup \{w_{16}, v_{17}, v_{18}, w_{21}\} \cup \{w_{17}, v_{18}\} \cup \{w_{18}, w_{20}\}.
\]

Considering the collection \( \mathcal{P} = \{w_{16}, v_{17}, v_{18}, w_{21}\} \), we have the relation

\[
w_{21} = v_{19} = e_1 = w_{16} + v_{17} + v_{18} + w_{21},
\]

where \( \{e_1, \ldots, e_{21}\} \) is the set of standard basis vectors in \( \mathbb{R}^{21} \). Since the set \( \{w_{16}, v_{17}, v_{18}, w_{21}\} \) does not contain any primitive collection in \( \text{PC}(\bar{\Sigma}_4) \), the summation \( w_{16} + v_{17} + v_{18} + w_{21} \) is contained in \( \text{Cone}(w_{16}, v_{17}, v_{18}, w_{21}) \). For the divisor \( D = \sum_{j=1}^{21} 2D_j \) as in the proof of Theorem 6.10 and its support function \( \varphi_D \), we have that

\[
\varphi_D(w_{21} + v_{19}) = 0 + (-2) + (-2) + 0 \geq \varphi_D(w_{21}) + \varphi_D(v_{19}) = 0 + (-2).
\]

Therefore the collection \( \mathcal{P} \) does not satisfy the inequality, so that \( D \) is not a basepoint free divisor on \( \bar{\Sigma}_4 \). Hence one cannot say that the toric variety \( X_{\bar{\Sigma}_4} \) is a small desingularization of the toric variety \( X_{\Delta(i)} \) even though \( \bar{\Sigma}_4 \) is a smooth polytopal fan such that

\[
\{u_{\rho} \mid \rho \in \bar{\Sigma}_4(1)\} = \{u_{\rho} \mid \rho \in \Sigma_{\Delta(i)}(1)\}.
\]

Note that we can choose other vectors to construct a Bott manifold (and there exist finitely many choices in this case). But one can check that none of them defines a small desingularization of the toric variety \( X_{\Delta(i)} \).

**Appendix A. Dynkin diagram automorphisms and string polytopes**

Let \( G \) be a connected semisimple algebraic group of rank \( n \) over \( \mathbb{C} \) and \( g \) its Lie algebra. Fixing a Cartan subalgebra \( t \) of \( g \) and an enumeration of the simple roots \( \alpha_1, \ldots, \alpha_n \), we have the Chevalley generators \( \{e_i, f_i, \alpha_i^\vee \mid 1 \leq i \leq n\} \) and the Weyl group \( W \) generated by reflections \( s_i \) through the hyperplanes orthogonal to the simple roots \( \alpha_i \). Here \( \alpha_i^\vee \) is the coroot of \( \alpha_i \). The weight lattice \( \Lambda \) is the set of all \( \lambda \in t^* \) such that \( \lambda(\alpha_i^\vee) \in \mathbb{Z} \) and \( \Lambda \) has a \( \mathbb{Z} \)-basis consisting of the fundamental weights \( \varpi_1, \ldots, \varpi_n \), which are determined by the relation \( \langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j} \). We call a weight \( \lambda = \lambda_1 \varpi_1 + \cdots + \lambda_n \varpi_n \) dominant if \( \lambda_i \geq 0 \) for all \( i = 1, \ldots, n \). Let \( \Lambda_+ \) denote the set of dominant integral weights.

For a dominant weight \( \lambda \), we denote a (finite-dimensional) irreducible representation of \( G \) with highest weight \( \lambda \) by \( V_\lambda \). Then \( V_\Lambda \) has a remarkable basis \( B_\Lambda \) consisting of the nonzero vectors \( b_\Lambda \), where \( b \) lies in the specialization at \( q = 1 \) of the Lusztig canonical basis for the quantized enveloping algebra \( U_q(g) \) of \( g \) over \( \mathbb{C}(q) \) (for details, see [Kas90] and [Kav13, Section 3]). Denote by \( \hat{e}_i, \hat{f}_i : B_\Lambda \rightarrow B_\Lambda \cup \{0\} \) the raising and lowering Kashiwara operators for \( V_\Lambda^\star \). This depends on a reduced word \( i = (i_1, \ldots, i_{\tilde{n}}) \in [n]^{\tilde{n}} \) for the longest element \( w_0 \in W \), \( w_0 = s_{i_1} \cdots s_{i_{\tilde{n}}} \), where \( \tilde{n} = \ell(w_0) \). The set of reduced words for \( w \in W \) will be denoted by \( R(w) \).
Definition A.1. For a reduced word \( i = (i_1, \ldots, i_n) \in R(w_0) \), we define a map \( \Phi_i : B^*_\lambda \to \mathbb{Z}_{\geq 0}^n \) by \( \Phi_i(b) = (t_1, \ldots, t_n) \), where
\[
\begin{align*}
t_1 &= \max \{ a \mid f_{i_1}^a(b) \neq 0 \}, \\
t_2 &= \max \{ a \mid f_{i_2}^a f_{i_1}^a(b) \neq 0 \}, \\
&\vdots \\
t_n &= \max \{ a \mid f_{i_n}^a \cdots f_{i_2}^a f_{i_1}^a(b) \neq 0 \}.
\end{align*}
\]
This map is called \textit{string parametrization} of \( B^*_\lambda \) with respect to \( i \).

Proposition A.2 \([\text{Lit98}]\) Proposition 1.5 and \([\text{BZ01}]\) Proposition 3.5). \textit{There exists a (unique) rational polyhedral convex cone} \( C_i \subset \Lambda \times \mathbb{R}^n \) such that the union \( \bigcup_{\lambda \in \Lambda_+} \{ (\lambda, \Phi_i(b)) \mid b \in B^*_\lambda \} \) is the intersection of \( C_i \) with the lattice \( \Lambda \times \mathbb{Z}^n \).

The projection of \( C_i \) to the second factor \( \mathbb{R}^n \) is also a rational polyhedral convex cone and we call it \textit{string cone} \( C_i \) associated to \( i \in R(w_0) \). Since the highest weight \( G \)-module of the weight 0 is trivial, \( C_i \) intersects with \( \{0\} \times \mathbb{R}^n \) only at the origin. Thus the slice of the cone \( C_i \) at a fixed \( \lambda \in \Lambda_+ \) is a rational polyhedral polytope in \( \mathbb{R}^n \).

Definition A.3. For a dominant weight \( \lambda \in \Lambda_+ \) and a reduced word \( i = (i_1, \ldots, i_n) \in R(w_0) \), the slice of \( C_i \) at \( \lambda \) is the \textit{string polytope} \( \Delta_i(\lambda) = \{ t \mid (\lambda, t) \in C_i \} \subset \mathbb{R}^n \).

It follows from \([\text{Lit98}]\) that the string polytope \( \Delta_i(\lambda) \) can be obtained by intersecting the string cone \( C_i \) with the \( \lambda \)-cone:
\[
\Delta_i(\lambda) = C_i \cap \{ t \in \mathbb{R}^n_{\geq 0} \mid l_j(t) \leq \langle \lambda, \alpha_i^\vee \rangle \text{ for } 1 \leq j \leq n \},
\]
where \( l_1, \ldots, l_n \) are linear functions defined by
\[
l_j(t) := t_j + \langle t_{j+1} \alpha_i^{j+1} + \cdots + t_n \alpha_i^n, \alpha_i^\vee \rangle \text{ for } 1 \leq j \leq n.
\]

In this case where \( G = \text{SL}_{n+1}(\mathbb{C}) \), the inequalities defining the string cone \( C_i \) are described explicitly in \([\text{BZ01}]\) Proposition 3.14] and can be written from the Gleizer–Postnikov’s paths in the wiring diagram in \([\text{GP00}]\). Also, the inequalities defining the \( \lambda \)-cone can be read off from the wiring diagram in \([\text{Rus08}]\). See Section \[3\]

Example A.4. Let \( G = \text{SL}_3(\mathbb{C}) \), and \( \lambda = 2\varpi_1 + 2\varpi_2 \). Let \( i = (1, 2, 1) \). Then the linear functions \( l_1, l_2, l_3 \) are given by
\[
l_1(t) = t_1 - t_2 + 2t_3, \quad l_2(t) = t_2 - t_3, \quad l_3(t) = t_3.
\]
The string cone \( C_i \) is the set of points \( (t_1, t_2, t_3) \in \mathbb{R}^3 \) satisfying
\[
t_1 \geq 0, \quad t_2 \geq t_3 \geq 0.
\]
The string polytope \( \Delta_i(\lambda) \) is given by the set of points \( t = (t_1, t_2, t_3) \in \mathbb{R}^3 \) satisfying:
\[
0 \leq t_1 \leq t_2 - 2t_3 + 2, \\
t_3 \leq t_2 \leq t_3 + 2, \\
0 \leq t_3 \leq 2,
\]
which is described in Figure \[17\].
Definition A.5. Let $\mathfrak{g}$ be a simple Lie algebra with Cartan matrix $C = (c_{i,j})_{1 \leq i,j \leq n}$. A bijection $\theta: [n] \to [n]$ satisfying $c_{\theta(i), \theta(j)} = c_{i,j}$ for all $i,j$ is called a Dynkin diagram automorphism.

Example A.6. 
1. For $\mathfrak{g} = \mathfrak{s}l_{n+1}(\mathbb{C})$, the involution $\iota: [n] \to [n]$ defined by $\iota(i) = n+1-i$ for $1 \leq i \leq n$ is a Dynkin diagram automorphism (see Figure 18(1)).
2. A non-trivial Dynkin diagram automorphism exists only when $\mathfrak{g}$ is a Lie algebra of type $A_n$ ($n \geq 2$), $D_n$, or $E_6$ (see Figures 18(2) and 18(3)). All these algebras except $D_4$ have a unique non-trivial Dynkin diagram automorphism of order 2. Since $D_4$ also has a Dynkin diagram automorphism of order 3, the group of its Dynkin diagram automorphisms is isomorphic to the symmetric group $S_3$ (see Figure 18(4)).

Note that the group of Dynkin diagram automorphisms of a semisimple Lie algebra $\mathfrak{g}$ is isomorphic to the group of outer automorphisms of $\mathfrak{g}$ [OV90, Section 4 of Chapter 4]).

Proposition A.7. Let $\theta$ be a Dynkin diagram automorphism of $\mathfrak{g}$. If we denote $\theta(i) = (\theta(i_1), \ldots, \theta(i_n))$ for a reduced word $i = (i_1, \ldots, i_n) \in R(w_0)$, then the rational polyhedral convex cone $C_\theta(i) \subset \Lambda_R \times \mathbb{R}^n$ is equal to $C_1$ defined in Proposition A.2. Consequently, we have the same string polytopes $\Delta_{\theta(i)}(\lambda) = \Delta_i(\lambda)$ for any dominant weight $\lambda \in \Lambda^+$. 

Proof. A Dynkin diagram automorphism naturally induces a Lie algebra automorphism $\tilde{\theta}: \mathfrak{g} \to \mathfrak{g}$ such that $\tilde{\theta}(e_i) = e_{\theta(i)}$, $\tilde{\theta}(f_i) = f_{\theta(i)}$, $\tilde{\theta}(\alpha_i^\vee) = \alpha_{\theta(i)}^\vee$ for all $i$ (we will the same notation for simplicity). Then it also induces a $\mathbb{C}(q)$-algebra automorphism $\tilde{\theta}: U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ preserving the $\mathbb{C}(q)$-subalgebra $U_q^-(\mathfrak{g})$ generated by the Chevalley generators $\{ f_i \mid i = 1, \ldots, n \}$ corresponding to negative roots. Therefore, we have a $\mathbb{C}(q)$-linear automorphism $\tilde{\theta}: V_\lambda \to V_\lambda$ induced from $\tilde{\theta}: U_q^{-}(\mathfrak{g}) \to U_q^{-}(\mathfrak{g})$.

Because $\tilde{\theta} \circ \tilde{e}_i = \tilde{e}_{\theta(i)} \circ \tilde{\theta}$ and $\tilde{\theta} \circ \tilde{f}_i = \tilde{f}_{\theta(i)} \circ \tilde{\theta}$ on $V_\lambda$ by [NS03, Lemma 2.3.2], the crystal structure of the crystal basis $B_\lambda$ of $V_\lambda$ is stable under $\tilde{\theta}$. Hence, we obtain $\Phi_{\theta(i)}(\tilde{\theta}(\tilde{b})) = \Phi_i(b)$ for any $i \in R(w_0)$ so that $C_{\theta(i)} = C_1$. Since the string polytope $\Delta_i(\lambda) \subset \mathbb{R}^n$ is the slice of $C_i$ at $\lambda$, we conclude that $\Delta_{\theta(i)}(\lambda) = \Delta_i(\lambda)$. □
APPENDIX B. REDUCED WORDS OF THE LONGEST ELEMENT IN $S_5$ HAVING SMALL INDICES.

In this section, we observe reduced words in $R(w_0^{(5)})$, and present elements in $R(w_0^{(5)})$ which have small indices. Using the hook length formula \[\binom{n}{k}\], there are 768 many reduced words of the longest element in $S_5$. By the result \[\text{[B99, 33]},\] there are 62 reduced words up to 2-moves. Furthermore, considering the involution in Example \[\text{A.6(1)}\] and Proposition \[\text{A.7},\] it is enough to consider 31 elements in $R(w_0^{(5)})$ to study combinatorics of the string polytopes $\Delta_{i}(\lambda)$. In Table \[\text{[1]}\] we consider these 31 elements and check whether they have small indices or not. The number on the first column is the index given in \[\text{[B99, Table 1]}\].

It has been known from \[\text{[AB04, Example 5.7]}\] that the string polytopes $\Delta_{i}(\lambda)$ are integral for $n \leq 4$ and $\lambda = \sum_{i=1}^{n} 2\omega_i$. Moreover, one can check that $\Delta_{i}(\omega_i)$ are also integral for all $1 \leq i \leq n$ using the computer program SAGE. Indeed, for $n \leq 3$, we already proved that $X_{\Delta_{i}(\lambda)}$ admits a small desingularization (in Theorem \[\text{6.10}\]) so that $\Delta_{i}(\lambda)$ is integral (in Corollary \[\text{6.13}\]). We may address the following question.

**Question B.1.** *When does the toric variety $X_{\Delta_{i}(\lambda)}$ admit a small desingularization for $i \in R(w_0^{(5)})$ and a regular dominant integral weight $\lambda$?*

| # | reduced word $i$ | $i(I)$ | small indices | $|GP(i)|$ |
|---|---|---|---|---|
| 1 | $(1,2,1,3,2,1,4,3,2,1)$ | $i = i(D,D,D,D)$ | $(0,0,0,0)$ | $\circ$ | 10 |
| 2 | $(2,1,2,3,2,1,4,3,2,1)$ | $i = i(D,A,D,D)$ | $(0,0,0,0)$ | $\circ$ | 10 |
| 3 | $(1,2,3,2,1,2,4,3,2,1)$ | $i \sim i(D,A,D,A)$ | $(0,0,0,0,0,0,0,0)$ | $x$ | 11 |
| 4 | $(1,2,1,3,2,4,3,2,1,2)$ | $i \sim i(D,D,D,D)$ | $(0,0,0,0)$ | $\circ$ | 11 |
| 5 | $(2,1,3,2,3,1,4,3,2,1)$ | $i \sim i(D,D,D,A)$ | $(0,0,0,0,0,0,0,0)$ | $x$ | 11 |
| 6 | $(2,1,2,3,2,4,3,2,1,2)$ | $i = i(D,A,D,D)$ | $(0,0,0,0)$ | $\circ$ | 11 |
| 7 | $(1,3,2,3,1,2,4,3,2,1)$ | $i \sim i(D,A,D,A)$ | $(0,0,0,0,0,0,0,0)$ | $x$ | 11 |
| 8 | $(1,2,3,2,1,4,3,2,3,1)$ | $i \sim i(D,D,D,D)$ | $(0,0,0,0,0,0)$ | $\times$ | 11 |
| 9 | $(1,2,1,3,4,3,2,3,1,2)$ | $i \sim i(D,A,D,D)$ | $(0,0,0,0,0,0,0,0)$ | $x$ | 11 |
| 10 | $(2,1,3,2,1,4,3,4,2,1)$ | $i \sim i(D,D,A,D)$ | $(0,0,0,0,0,0,0,0)$ | $x$ | 11 |
| 11 | $(2,3,2,1,2,3,4,3,2,1)$ | $i = i(D,D,D,D)$ | $(0,0,0,0)$ | $\circ$ | 10 |
| 12 | $(2,1,3,2,3,4,3,2,1,2)$ | $i \sim i(D,D,D,A)$ | $(0,0,0,0,0,0,0,0)$ | $x$ | 13 |
| 13 | $(2,1,2,3,4,3,2,3,1,2)$ | $i \sim i(D,D,A,D)$ | $(0,0,0,0,0,0,0,0)$ | $x$ | 13 |
| 14 | $(1,3,2,3,1,4,3,2,3,1)$ | $i \sim i(D,D,D,D)$ | $(0,0,0,0,0,0,0,0)$ | $\circ$ | 13 |
| 15 | $(3,2,1,2,3,4,3,2,3,1)$ | $i \sim i(D,A,D,A)$ | $(0,0,0,0,0,0,0,0)$ | $x$ | 11 |
| 16 | $(1,2,3,2,4,3,2,1,2,3)$ | $i \sim i(D,D,A,A)$ | $(0,0,0,0,0,0,0,0)$ | $x$ | 14 |
| 17 | $(1,2,1,4,3,4,3,2,3,1,2)$ | $i \sim i(D,D,D,D)$ | $(0,0,0,0,0,0,0,0)$ | $\times$ | 12 |
| 18 | $(1,2,3,4,3,2,3,2,3,2,3)$ | $i = i(D,D,D,A)$ | $(0,0,0,0,0,0,0,0,0,0)\times$ | $\circ$ | 13 |
| 19 | $(2,3,2,1,2,4,3,4,2,1)$ | $i \sim i(D,D,A,D)$ | $(0,0,0,0,0,0,0,0,0,0)$ | $\times$ | 13 |
| 20 | $(2,1,3,2,4,3,4,2,1,2)$ | $i \sim i(D,A,D,D)$ | $(0,0,0,0,0,0,0,0,0,0)$ | $\circ$ | 13 |
| 21 | $(3,2,3,1,2,3,4,3,2,1,2)$ | $i = i(D,A,A,D)$ | $(0,0,0,0,0,0,0,0,0,0)$ | $\times$ | 13 |
| 22 | $(2,1,2,3,4,3,2,3,1,2)$ | $i \sim i(D,D,D,D)$ | $(0,0,0,0,0,0,0,0,0,0)$ | $\circ$ | 13 |
| 23 | $(1,3,2,1,4,3,4,2,3,1)$ | $i \sim i(D,D,D,D)$ | $(0,0,0,0,0,0,0,0,0,0)$ | $\times$ | 15 |
| 24 | $(3,2,1,2,3,4,3,2,3,1)$ | $i \sim i(D,D,D,D)$ | $(0,0,0,0,0,0,0,0,0,0)$ | $\times$ | 14 |
| 25 | $(1,3,2,3,4,3,2,1,2,3,2)$ | $i \sim i(D,A,A,D)$ | $(0,0,0,0,0,0,0,0,0,0,0)$ | $\circ$ | 13 |
| 26 | $(1,2,3,4,3,2,3,1,2,3,2)$ | $i \sim i(D,D,D,D)$ | $(0,0,0,0,0,0,0,0,0,0,0)$ | $\times$ | 14 |
| 27 | $(1,2,4,3,4,2,1,2,3,2)$ | $i \sim i(D,D,D,D)$ | $(0,0,0,0,0,0,0,0,0,0,0)$ | $\times$ | 13 |
| 28 | $(3,2,3,1,2,4,3,4,2,1)$ | $i \sim i(D,A,A,D)$ | $(0,0,0,0,0,0,0,0,0,0,0)$ | $\circ$ | 13 |
| 29 | $(2,1,3,4,3,2,3,4,1,2)$ | $i \sim i(D,A,D,D)$ | $(0,0,0,0,0,0,0,0,0,0,0)$ | $\times$ | 17 |
| 30 | $(2,1,4,3,2,3,4,3,1,2)$ | $i \sim i(D,D,A,D)$ | $(0,0,0,0,0,0,0,0,0,0,0)$ | $\times$ | 15 |
| 31 | $(1,2,4,3,2,3,1,2,3)$ | $i \sim i(D,A,D,D)$ | $(0,0,0,0,0,0,0,0,0,0,0)$ | $\times$ | 14 |

**Table 1.** Reduced words in $R(w_0^{(5)})$. 
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