Quantizing Dirac and Nambu Brackets

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Abstract. We relate classical and quantum Dirac and Nambu brackets. At the classical level, we use the relations between the two brackets to gain some insight into the Jacobi identity for Dirac brackets, among other things. At the quantum level, we suggest that the Nambu bracket is the preferred method for introducing constraints, although at the expense of some unorthodox behavior, which we describe in detail.

1 Introduction

We have recently devoted some time to understand the quantization of Nambu brackets (NBs) using non-Abelian methods. Indirectly, our results are also a quantization of Dirac brackets (DBs) because the two types of brackets are related, as we will explain. Since it is the centenary of Dirac’s birth, and since Dirac did have an affiliation with the University of Miami, this Conference series, and Behram Kursunoglu, it seemed appropriate to contribute something on this subject to these Proceedings at this time.

2 Classical Theory

Poisson and Dirac brackets The ordinary Poisson bracket (PB) is the antisymmetric bilinear formed from any pair of phase-space functions \( A(x, p), B(x, p) \):

\[
\{ A, B \}_{PB} = \sum_i \frac{\partial (A, B)}{\partial (x_i, p_i)} = - \{ B, A \}_{PB} .
\] (1)

For a particle whose motion is constrained, however, this is not the most appropriate tool to take apart and understand the dynamics. The Dirac bracket is the preferred bilinear in the constrained case. It is defined for any even number of constraints. First compute the antisymmetric matrix formed from PBs between pairs of phase-space constraints \( C_i(x, p) \),

\[
A_{ij} = \{ C_i, C_j \}_{PB} .
\] (2)

Then compute its inverse, \( A_{ij}^{-1} \), assuming the latter exists.

The Dirac bracket is then defined in terms of PBs as

\[
\{ f, g \}_{DB} = \{ f, g \}_{PB} - \sum_{ij} \{ f, C_i \}_{PB} A_{ij}^{-1} \{ C_j, g \}_{PB} = - \{ g, f \}_{DB} .
\] (3)

1Talk given by the first author at the Coral Gables Conference 11-14 December 2002.

2Our list of references here will be abbreviated. A more complete set of references is given in the correlated talk by C Zachos, also contributed to these Proceedings [2].
By construction, the DB of any individual constraint, among those incorporated into the DB definition, with any function \( f \) on the phase-space, will vanish identically.

\[
(C_k, f)_{DB} = (C_k, f)_{PB} - \sum_{i,j} (C_k, C_i)_{PB} A^{-1}_{ij} (C_j, f)_{PB} = (C_k, f)_{PB} - (C_k, f)_{PB} = 0.
\]

The Jacobi Identity  A nontrivial property is the Jacobi identity (JI) for the Dirac bracket:

\[
0 = \{f, \{g, h\}_{DB}\}_{DB} + \{g, \{h, f\}_{DB}\}_{DB} + \{h, \{f, g\}_{DB}\}_{DB}.
\]

This is not so easily demonstrated by direct calculation. It is surprisingly difficult (see p 42, L on QM, [2]) to establish the JI for the DB without relating it to something else. We will need

\[
\delta A^{-1} = -A^{-1} (\delta A) A^{-1},
\]

as is valid for any derivation \( \delta \), as well as the JI for the Poisson bracket. The Jacobi identity is itself a property very similar in form to the derivation property, only it applies to the action of one bracket functional on another. So, with

\[
\delta f g \equiv \{f, g\}_{PB},
\]

not only do we have

\[
\delta f (gh) \equiv \{f, gh\}_{PB} = (\delta f g) h + g (\delta f h),
\]

but we also have

\[
\delta f (\{g, h\}_{PB}) \equiv \{f, \{g, h\}_{PB}\}_{PB} = \{(\delta f g), h\}_{PB} + \{g, (\delta f h)\}_{PB}.
\]

That is to say

\[
0 = \{f, \{g, h\}_{PB}\}_{PB} + \{g, \{h, f\}_{PB}\}_{PB} + \{h, \{f, g\}_{PB}\}_{PB}.
\]

Now as a consequence of the PB being a derivation, it is trivially true, for commuting quantities, that

\[
\{f, gh\}_{DB} = \{f, gh\}_{PB} - \sum_{i,j} \{f, C_i\}_{PB} A^{-1}_{ij} \{C_j, gh\}_{PB} = \{f, g\}_{DB} h + g \{f, h\}_{DB},
\]

so the DB is also a derivation. But it is not so obviously true that the DB acts in an analogous fashion on other DBs:

\[
\{f, \{g, h\}_{DB}\}_{DB} = \{\{f, g\}_{DB}, h\}_{DB} + \{g, \{f, h\}_{DB}\}_{DB}.
\]

This is exactly what we want to establish to prove the JI for DBs. The terms that get in the way involve the DB acting on the constraint functionals: \( \{\cdot, C\}_{PB} \) and \( A^{-1} \). Fortunately, not only are the constraints themselves invariant under the action of the DB, but so are these functionals, at least to a sufficient degree (as explained below). Thus the Jacobi identity ultimately holds.

However, just for extra fun, let us use only the derivation property of the Poisson bracket, and not perform any re-ordering of terms (in anticipation of quantization, so that all products could be non-commutative \( \star \) products [2]). By direct calculation, we then obtain a slew of terms (arranged in cyclic triples):

\[
\{f, \{g, h\}_{DB}\}_{DB} + \{g, \{h, f\}_{DB}\}_{DB} + \{h, \{f, g\}_{DB}\}_{DB} =
\]
\[
\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}
\]

(13a)

\[
+ \{C_i, f\} A_{im}^{-1} \{A_{mn}, h\} A_{nj}^{-1} \{C_j, g\} - \frac{1}{2} \{C_i, f\} A_{im}^{-1} \{C_j, g\} A_{nj}^{-1} \{A_{mn}, h\}
\]

(13b)

\[
- \frac{1}{2} \{C_j, h\} A_{nj}^{-1} \{A_{mn}, f\} A_{im}^{-1} \{C_i, g\}
\]

(13c)

\[
+ \{C_i, h\} A_{im}^{-1} \{A_{mn}, g\} A_{nj}^{-1} \{C_j, f\} - \frac{1}{2} \{C_i, h\} A_{im}^{-1} \{C_j, f\} A_{nj}^{-1} \{A_{mn}, g\}
\]

(13d)

\[
+ \{C_j, g\} A_{nj}^{-1} S_{mn}^{-1} (h) - \{C_j, g\} A_{nj}^{-1} S_{mn}^{-1} (h) A_{im}^{-1} \{C_i, f\}
\]

(13e)

\[
+ \{C_j, h\} A_{nj}^{-1} S_{mn}^{-1} (f) - \{C_j, h\} A_{nj}^{-1} S_{mn}^{-1} (f) A_{im}^{-1} \{C_i, g\}
\]

(13f)

\[
+ \{C_i, h\} A_{im}^{-1} \{C_j, f\} A_{jn}^{-1} S_{mn}^{-1} (g) - \{C_i, h\} A_{im}^{-1} \{C_j, f\} A_{jn}^{-1} S_{mn}^{-1} (g) A_{im}^{-1} \{C_i, h\}
\]

(13g)

\[
+ \{C_i, f\} A_{ij}^{-1} \{C_j, \{g, h\}\} + \{C_i, f\} A_{ij}^{-1} \{h, \{C_j, g\}\} + \{g, \{C_i, h\}\} A_{ij}^{-1} \{C_j, f\}
\]

(13h)

\[
+ \{C_i, g\} A_{ij}^{-1} \{C_j, \{h, f\}\} + \{C_i, g\} A_{ij}^{-1} \{f, \{C_j, h\}\} + \{h, \{C_i, f\}\} A_{ij}^{-1} \{C_j, g\}
\]

(13i)

\[
+ \{C_i, h\} A_{ij}^{-1} \{C_j, \{f, g\}\} + \{C_i, h\} A_{ij}^{-1} \{g, \{C_j, f\}\} + \{f, \{C_i, g\}\} A_{ij}^{-1} \{C_j, h\}
\]

(13j)

\[
+ \{C_i, f\} A_{ij}^{-1} \{C_k, g\} \{C_j, A_{kl}^{-1}\} \{C_l, h\} + \{C_i, g\} A_{ij}^{-1} \{C_k, h\} \{C_j, A_{kl}^{-1}\} \{C_l, f\}
\]

(13k)

\[
+ \{C_i, h\} A_{ij}^{-1} \{C_k, f\} \{C_j, A_{kl}^{-1}\} \{C_l, g\}
\]

For purposes of the classical discussion, all the unlabeled brackets on the RHS of (13) are PBs. Here we have also defined

\[
S_{jk} (f) = S_{kj} (f) = \frac{1}{2} \{C_j, \{C_k, f\}\} + \frac{1}{2} \{C_k, \{C_j, f\}\}
\]

(14)

and similarly for \(S_{jk} (g)\) and \(S_{jk} (h)\).

Now, if all quantities commute, as is the case for classical PBs, then each individually numbered line on the RHS of (13) vanishes separately. The only line presenting any issue is perhaps the last one. However, for commuting quantities it is just

\[
\{C_i, f\} \{C_k, g\} \{C_l, h\} (A_{ij}^{-1} \{C_j, A_{kl}^{-1}\} + A_{kj}^{-1} \{C_j, A_{li}^{-1}\} + A_{ij}^{-1} \{C_j, A_{ik}^{-1}\})
\]

\[
= - \{C_i, f\} \{C_k, g\} \{C_l, h\} A_{ij}^{-1} A_{km}^{-1} A_{nl}^{-1} \times
\]

\[
\times \left( \{C_j, A_{mn}\} + \{C_m, A_{nj}\} + \{C_n, A_{jm}\} \right)
\]

(15)

where in the last expression, we used (6) written as \(\{C_j, A_{kl}^{-1}\} = -A_{km}^{-1} \{C_j, A_{mn}\} A_{nl}^{-1}\), etc., and the antisymmetry of \(A^{-1}\). But now, from the definition (2),

\[
\{C_j, A_{mn}\}_{PB} + \{C_m, A_{nj}\}_{PB} + \{C_n, A_{jm}\}_{PB} = 0
\]

(16)

is just the JI for PBs. Thus when all quantities commute, the DB JI is established. But when all quantities do not commute, the individual lines on the RHS of (13) do not vanish, in general, nor does their sum.
Now, clearly, what we have done is *not quite correct* if things do not commute. Our definition of the Dirac bracket \( \{ f, g \}_{DB} \) is not manifestly antisymmetric in \( f \) and \( g \) if the products involved in \( \{ f, C_i \} A^{-1}_{ij} \{ C_j, g \} \) do not commute. We should have at least defined
\[
\{ f, g \}_{DB} = \{ f, g \} - \frac{1}{2} \{ f, C_i \} A^{-1}_{ij} \{ C_j, g \} + \frac{1}{2} \{ g, C_i \} A^{-1}_{ij} \{ C_j, f \}.
\]
Perhaps there is even a better definition. In any case, our previous answer can be corrected just by antisymmetrizing the constraint terms with respect to such interchanges. But this really does not make it any easier to see how the terms on the RHS of (13) can cancel, in general, when non-commuting products are involved. To better understand how all this machinery functions, it would be technically sweet to have another route to quantization.

**Classical Nambu brackets** There is a multi-linear, totally antisymmetric bracket, introduced by Nambu [4]. On the full phase-space for a particle with \( N \) degrees of freedom the highest rank, or *maximal*, Nambu bracket (NB) is essentially just the Jacobian
\[
\{ A_1, A_2, \ldots, A_{2N} \}_{NB} = \frac{\partial (A_1, A_2, \ldots, A_{2N})}{\partial (x_1, p_1, x_2, p_2, \ldots, x_N, p_N)}.
\]
Physically, this has a geometrical interpretation in terms of phase-space gradients.

The surfaces illustrated in the Figure are isoclines for two different phase-space functions, respectively \( A_1 \) and \( A_2 \). A particular phase-space tangent \( \mathbf{v} \) lies at the intersection of these two surfaces. That local phase-space tangent at the point depicted is actually given by the joint cross-product of all but one (e.g. \( A_{2N} \)) of the local phase-space gradients of the \( A \)s contained within the NB. The complete, maximal NB provides a projection onto \( \mathbf{v} \) of the remaining phase-space gradient (e.g. \( \nabla A_{2N} \)). Other possible intersections along the \( A_1 \) surface are also shown as contours representing other values for \( A_2 \), but the corresponding constant \( A_2 \) surfaces are not shown.
The Fundamental Identity  The classical NB obeys a simple combinatorial identity known as the “fundamental identity”, or simply “FI”. If you attempt to totally antisymmetrize $2N + 1$ commuting entries chosen from among $2N$ possibilities, you obviously obtain zero. As a consequence,

$$\{\{A_1, \cdots, A_{2N}\}_NB, B_2, \cdots, B_{2N}\}_NB = \{\{A_1, B_2, \cdots, B_{2N}\}_NB, A_2, \cdots, A_{2N}\}_NB$$

$$+ \{A_1, \{A_2, B_2, \cdots, B_{2N}\}_NB, \cdots, A_{2N}\}_NB + \cdots + \{A_1, \cdots, \{A_{2N}, B_2, \cdots, B_{2N}\}_NB\}_NB.$$  (19)

This has the form needed to say that the action of one maximal NB functional on another is similar to a derivation. That is, in an obvious notation,

$$\delta_{NB} \{A_1, A_2, \cdots, A_{2N}\}_NB = \delta_{NB} A_1, \cdots, A_{2N}\}_NB + \cdots + \{A_1, \cdots, \delta_{NB} A_{2N}\}_NB.$$  (20)

It is a bit misleading to call this particular identity “fundamental”, since the same relation holds whenever any NB is applied to $\{A_1, A_2, \cdots, A_{2N}\}_NB$. In other words, classically there are a set of such identities. We will discuss them all below, after defining NBs with fewer entries. More importantly, however, it is a less fortunate misnomer to call (19) “fundamental” from the standpoint of the quantum theory discussed below.

DBs as NBs  There are some elementary but important relationships between Dirac brackets and Nambu brackets [1, 3].

For a particle with $N$ degrees of freedom but subject to $2n$ constraints, consider a Nambu bracket involving any two functions $A, B$ of the dynamical variables on the full $2N$ dimensional phase-space. Along with $A, B$ insert into the bracket all of the constraints as well as $k = N - n - 1$ factors of $x_i, p_i$ and sum from 1 to $N$ all $k$ of the $i$‘s, i.e. take $k$ symplectic traces. This reduces the Nambu bracket to a Dirac bracket. Thus

$$\{A, B\}_DB \propto \sum_{i_1, \cdots, i_k=1}^N \frac{\partial (A, B, C_1, C_2, \cdots, C_{2n}, x_{i_1}, p_{i_1}, \cdots, x_{i_k}, p_{i_k})}{\partial (x_1, p_1, \cdots, x_N, p_N)}.$$  (21)

with the proportionality to be determined (see [26] and [34] below). Hence the Dirac bracket always follows directly from the Nambu bracket.

Any maximal even rank CNB can also be resolved into products of Poisson brackets. For example, for systems with two degrees of freedom, $\{A, B\}_PB = \frac{\partial (A, B)}{\partial (x_1, p_1)} + \frac{\partial (A, B)}{\partial (x_2, p_2)}$, and the 4-bracket $\{A, B, C, D\}_NB \equiv \frac{\partial (A, B, C, D)}{\partial (x_1, p_1, x_2, p_2)}$ resolves as

$$\{A, B, C, D\}_NB = \{A, B\}_PB \{C, D\}_PB - \{A, C\}_PB \{B, D\}_PB - \{A, D\}_PB \{C, B\}_PB,$$  (22)

in comportance with full antisymmetry under permutations of $A, B, C,$ and $D$. The general result for maximal rank $2N$ brackets for systems with a $2N$-dimensional phase-space is

$$\{A_1, \cdots, A_{2N}\}_NB = \sum_{\text{all } (2N)! \text{ perms} \{\sigma_1, \sigma_2, \cdots, \sigma_{2N}\}} \frac{\text{sgn} (\sigma)}{2^N N!} \{A_{\sigma_1}, A_{\sigma_2} \cdots A_{\sigma_{2N-1}}, A_{\sigma_{2N}}\}_PB,$$  (23)

where sgn $(\sigma) = (-1)^{\pi (\sigma)}$ with $\pi (\sigma)$ the parity of the permutation $\{\sigma_1, \sigma_2, \cdots, \sigma_{2N}\}$. The sum only gives $(2N - 1)! = (2N)! / (2^N N!)$ distinct products of PBs on the RHS, not $(2N)!$. Each such distinct product appears with net coefficient $\pm 1$. 

5
The proof of the relation (23) is remarkably elementary. Both left- and right-hand sides of the expression are sums of $2N$-th degree monomials linear in the $2N$ first-order partial derivatives of each of the $A$s. Both sides are totally antisymmetric under permutations of the $A$s. Hence, both sides are also totally antisymmetric under interchanges of partial derivatives. Thus, the two sides must be proportional. The only issue left is the constant of proportionality. This is easily determined to be 1 just by comparing the coefficients of any given term appearing on both sides of the equation, e.g. $\partial_{x_1} A_1 \partial_{p_1} A_2 \cdots \partial_{x_N} A_{2N-1} \partial_{p_N} A_{2N}$.

This is essentially a special case of Laplace’s theorem on the general minor expansions of determinants, although it must be said that we have never seen it written, let alone used, in exactly this form, either in treatises on determinants or in textbooks on classical mechanics.

For similar relations to hold for sub-maximal even rank Nambu brackets, these must first be defined. It is easiest to just define sub-maximal even rank CNBs by their Poisson bracket resolutions:

$$\{A_1, \ldots, A_{2n}\}_{NB} = \sum_{\text{perms } \sigma} \frac{\text{sgn}(\sigma)}{2^n n!} \{A_{\sigma_1}, A_{\sigma_2}\}_{PB} \cdots \{A_{\sigma_{2n-1}}, A_{\sigma_{2n}}\}_{PB},$$

(24)

only here we allow $n < N$. So defined, these sub-maximal CNBs enter in further recursive expressions. For example, for systems with three or more degrees of freedom, $\{A, B\}_{PB} = \frac{\partial(A,B)}{\partial(x_1,p_1)} + \frac{\partial(A,B)}{\partial(x_2,p_2)} + \frac{\partial(A,B)}{\partial(x_3,p_3)} + \cdots$, and a general 6-bracket expression resolves as

$$\{A_1, A_2, A_3, A_4, A_5, A_6\}_{NB} = \{A_1, A_2\}_{PB} \{A_3, A_4, A_5, A_6\}_{NB} - \{A_1, A_3\}_{PB} \{A_2, A_4, A_5, A_6\}_{NB}$$
$$+ \{A_1, A_4\}_{PB} \{A_2, A_3, A_5, A_6\}_{NB} - \{A_1, A_5\}_{PB} \{A_2, A_3, A_4, A_6\}_{NB}$$
$$+ \{A_1, A_6\}_{PB} \{A_2, A_3, A_4, A_5\}_{NB},$$

(25)

with the 4-brackets resolvable into PBs as above. Sub-maximal brackets defined in this way are the same as those obtained by taking symplectic traces of maximal brackets.

$$\{A_1, \ldots, A_{2n}\}_{NB} = \frac{1}{(N-n)!} \sum_{i_1, \ldots, i_{N-n}=1}^{N} \frac{\partial (A_1, \ldots, A_{2n}; x_{i_1}, p_{i_1}, \ldots, x_{i_{N-n}}, p_{i_{N-n}})}{\partial (x_1, p_1, \ldots, x_N, p_N)}.$$

(26)

This permits the building-up of higher even rank brackets proceeding from initial PBs involving all degrees of freedom. The general recursion relation with this $2n = 2 + (2n - 2)$ form is

$$\{A_1, \ldots, A_{2n}\}_{NB} = \{A_1, A_2\}_{PB} \{A_3, \ldots, A_{2n}\}_{NB}$$
$$+ \sum_{j=3}^{2n-1} (-1)^j \{A_1, A_j\}_{PB} \{A_2, \ldots, A_{j-1}, A_{j+1}, \ldots, A_{2n}\}_{NB}$$
$$+ \{A_1, A_{2n}\}_{PB} \{A_2, \ldots, A_{2n-1}\}_{NB},$$

(27)

and features $2n - 1$ terms on the RHS. (Of course, one can permute the subscripts to get equivalent forms for the RHS.) This recursive result is equivalent to taking the PB resolution as a definition for $2n < 2N$ elements, as can be seen by substituting the PB resolutions of the $(2n - 2)$-brackets on the RHS. Similar relations obtain when the $2n$ elements in the CNB are partitioned into sets of $(2n - 2k)$ and $2k$ elements, with suitable antisymmetrization with respect to exchanges between the two sets.

Having defined CNBs with fewer than $2N$ entries, we present the sub-maximal extensions of the so-called fundamental identity. These new identities can be proven either by using the PB resolutions, or by taking symplectic traces of (19). For arbitrary strings of elements, $A = A_1, \ldots, A_n$
and \( B = B_1, \ldots, B_k \), with \( n \) even and \( k \) odd, and for any additional phase-space “weight” \( V \), it follows that

\[
\{B, V \{A\}_{NB}\}_{NB} - \{V \{B, A_1\}_{NB}, A_2, \ldots, A_n\}_{NB} - \cdots - \{A_1, A_2, \ldots, V \{B, A_n\}_{NB}\}_{NB} = \{B_1, V \{B_2, B_3, \ldots, B_k\}_{NB}, A\}_{NB} - \cdots + \{B_k, V \{B_1, B_2, \ldots, B_{k-1}\}_{NB}, A\}_{NB},
\]

in an obvious notation: \( \{B, V \{A\}_{NB}\}_{NB} = \{B_1, \ldots, B_k, V \{A_1, \ldots, A_n\}_{NB}\}_{NB}, \) etc. When the \( A \) string is maximal, i.e. \( n = 2N \), the RHS of (28) vanishes identically, for any odd value of \( k \). (The RHS also vanishes identically when \( k = 1 \), for any even \( n \), if \( V \) is a numerical constant.) We may use (28) to prove the DB II, but first we need to express DBs precisely as NBs.

The PB resolutions lead to a precise, useful relation between Nambu and Dirac brackets, valid for any even number of constraints. Take \( A_1 = f \), \( A_2 = g \), \( A_3 = C_1, \ldots \), then for a \( 2n \)-bracket containing \( f, g \), and \( (2n - 2) \) constraints, we first extract \( f \) into a PB in all possible ways.

\[
\{f, g, C_1, C_2, \cdots\}_{NB} = \{f, g\}_{PB} \{C_1, C_2, \cdots\}_{NB} + \sum_{j=1}^{2n-2} (-1)^j \{f, C_j\}_{PB} \{g, C_1, \cdots, C_{j-1}, C_{j+1}, \cdots\}_{NB}.
\]

Likewise extract \( g \) as a second sum, taking care to note that \( C_j \) has already been removed from the remaining Nambu bracket. Thus

\[
{f, g, C_1, C_2, \cdots}_{NB} = \{f, g\}_{PB} \{C_1, C_2, \cdots\}_{NB} + \sum_{j=2}^{2n-2} \{f, C_{j-1}\}_{PB} \{C_1, \cdots, C_{j-2}, C_{j+1}, \cdots, C_{2n-2}\}_{NB} \{C_j, g\}_{PB}
\]

\[
- \sum_{j=2}^{2n-2} \{f, C_j\}_{PB} \{C_1, \cdots, C_{j-2}, C_{j+1}, \cdots, C_{2n-2}\}_{NB} \{C_{j-1}, g\}_{PB}
\]

\[
+ \sum_{j=1}^{2n-2} \sum_{k \leq j-2} (-1)^{j+k} \{f, C_j\}_{PB} \{C_1, \cdots, C_{k-1}, C_{k+1}, \cdots, C_{j-1}, C_{j+1}, \cdots, C_{2n-2}\}_{NB} \{C_k, g\}_{PB}
\]

\[
- \sum_{j=1}^{2n-2} \sum_{k \geq j+2} (-1)^{j+k} \{f, C_j\}_{PB} \{C_1, \cdots, C_{j-1}, C_{j+1}, \cdots, C_{k-1}, C_{k+1}, \cdots, C_{2n-2}\}_{NB} \{C_k, g\}_{PB}.
\]

All the remaining NBs contain only constraints. Divide by \( \{C_1, C_2, \cdots\}_{NB} \) to find

\[
\frac{1}{\{C_1, C_2, \cdots\}_{NB}} \{f, g, C_1, C_2, \cdots\}_{NB} = \{f, g\}_{PB} - \sum_{j,k} \{f, C_j\}_{PB} A_{jk}^{-1} \{C_k, g\}_{PB}.
\]

Here we have noted \( \sqrt{\det A} = \{C_1, C_2, \cdots, C_{2n-2}\}_{NB} \), and we have used the expression for the inverse of \( \{C_j, C_k\} = A_{jk} \) in terms of constraint minors, written as NBs. That is,

\[
A_{j-1 j}^{-1} = -A_{j-1 j}^{-1} = -\{C_1, \cdots, C_{j-2}, C_{j+1}, \cdots, C_{2n-2}\}_{NB} / \sqrt{\det A},
\]

\[
A_{j k \leq j-2}^{-1} = -(-1)^{j+k} \{C_1, \cdots, C_{k-1}, C_{k+1}, \cdots, C_{j-1}, C_{j+1}, \cdots, C_{2n-2}\}_{NB} / \sqrt{\det A},
\]

\[
A_{j k \geq j+2}^{-1} = (-1)^{j+k} \{C_1, \cdots, C_{j-1}, C_{j+1}, \cdots, C_{k-1}, C_{k+1}, \cdots, C_{2n-2}\}_{NB} / \sqrt{\det A}.
\]
(That this is correct as an expression for $A^{-1}_{jk}$ follows from (27) or its permutations.) The RHS of (31) is precisely the classical Dirac bracket. So we conclude

$$\{f, g\}_{DB} = \frac{1}{\{C_1, C_2, \ldots\}_{NB}} \{f, g, C_1, C_2, \ldots\}_{NB}.$$ 

(34)

No muss, and no fuss! Note that both left- and right-hand sides are homogeneous of degree zero in each of the constraints.

This last result may be used to provide an alternate proof that the classical DB obeys the JI. Since the classical NB is trivially a derivation, we have

$$\{\{C_1, C_2, \ldots\}_{NB}\}^3 \{\{f, g\}_{DB}, h\}_{DB} = \{\{C_1, C_2, \ldots\}_{NB}\}^2 \{\{f, g, C_1, C_2, \ldots\}_{NB}, h, C_1, C_2, \ldots\}_{NB} \{\{f, g, C_1, C_2, \ldots\}_{NB}, h, C_1, C_2, \ldots\}_{NB}.$$ 

(35)

Upon adding the cycled terms and using the generalization of the fundamental identity to arbitrary rank CNBs, (28), it is now straightforward to show

$$\{\{C_1, C_2, \ldots\}_{NB}\}^3 \{\{f, g\}_{DB}, h\}_{DB} + \{\{g, h\}_{DB}, f\}_{DB} + \{\{h, f\}_{DB}, g\}_{DB} = 0.$$ 

(36)

Hence the result (12).

This derivation of the Jacobi Identity for DBs should be compared with that for PBs. In particular, it is well-known that the simplest conceptual way to prove the PB JI is as a classical limit ($\hbar \to 0$) of the JI for quantum commutators, the latter commutator JI being nothing but an encoding of associativity of trilinear operator products on Hilbert space. It might be asked whether (12) also follows as a classical limit of a quantum construction encoding nothing but operator associativity. The answer would be in the affirmative, as the proof we have just sketched using NBs does indeed follow as a classical limit of an operator statement. The identity (28) actually results from the $\hbar \to 0$ limit of an encoding of nothing but multilinear operator associativity. To fully appreciate this, we must consider the quantization of NBs.

### 3 Quantum Theory

**Definition of QNBs** Define the quantum Nambu bracket (QNB) as a fully antisymmetrized multilinear sum of operator products in an associative enveloping algebra,

$$[A_1, A_2, \ldots, A_k] \equiv \sum_{\sigma \in \text{perms} \{1,2,\ldots,k\}} \text{sgn} (\sigma) A_{\sigma_1} A_{\sigma_2} \cdots A_{\sigma_k},$$ 

(37)

where sgn($\sigma$) = $(-1)^{\pi(\sigma)}$ with $\pi(\sigma)$ the parity of the permutation $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$. This definition is also due to Nambu [4], although such structures have independently appeared in the mathematical literature [7].
Recursion relations. There are various ways to obtain QNBs recursively, from products involving fewer operators. For example, a QNB involving \( k \) operators has both left- and right-sided resolutions of single operators multiplying QNBs of \( k - 1 \) operators.

\[
\{A_1, A_2, \cdots, A_k\} = \sum_{k! \text{ perms } \sigma} \frac{\text{sgn} (\sigma)}{(k-1)!} A_{\sigma_1} \{A_{\sigma_2}, \cdots, A_{\sigma_k}\} \\
= \sum_{k! \text{ perms } \sigma} \frac{\text{sgn} (\sigma)}{(k-1)!} \{A_{\sigma_1}, \cdots, A_{\sigma_{k-1}}\} A_{\sigma_k} .
\]  

(38)

On the RHS there are actually only \( k \) distinct products of single elements with \((k-1)\)-brackets, each such product having a net coefficient \( \pm 1 \). The denominator compensates for replication of these products in the sum over permutations. (We leave it as an elementary exercise for the reader to prove this result.)

For example, the 2-bracket is obviously just the commutator \([A, B] = AB - BA\), while the 3-bracket may be written in either of two convenient ways,

\[
[A, B, C] = A[B, C] + B[C, A] + C[A, B] \\
= [A, B] C + [B, C] A + [C, A] B .
\]  

(39)

Summing these two RHS lines gives anticommutators containing commutators on the RHS.

\[
2 \times [A, B, C] = \{A, [B, C]\} + \{B, [C, A]\} + \{C, [A, B]\} .
\]  

(40)

The last expression is to be contrasted to the Jacobi identity obtained by taking the difference of the two RHS lines.

\[
0 = [A, [B, C]] + [B, [C, A]] + [C, [A, B]] .
\]  

(41)

Similarly for the 4-bracket,

\[
[A, B, C, D] = A[B, C, D] - B[C, D, A] + C[D, A, B] - D[A, B, C] \\
= -[B, C, D] A + [C, D, A] B - [D, A, B] C + [A, B, C] D .
\]  

(42)

Summing these two lines gives

\[
2 \times [A, B, C, D] = [A, [B, C, D]] - [B, [C, D, A]] + [C, [D, A, B]] - [D, [A, B, C]] ,
\]  

(43)

while taking the difference gives

\[
0 = \{A, [B, C, D]\} - \{B, [C, D, A]\} + \{C, [D, A, B]\} - \{D, [A, B, C]\} .
\]  

(44)

There may be some temptation to think of the last of these as something like a generalization of the Jacobi identity, and in principle, it is, but in a crucially limited way, so that temptation should be checked. The more appropriate and complete generalization of the Jacobi identity is given systematically below.

**Jordan products** Define a fully symmetrized, generalized Jordan operator product, or GJP,

\[
\{A_1, A_2, \cdots, A_k\} \equiv \sum_{\text{all } k! \text{ perms } \{\sigma_1, \sigma_2, \cdots, \sigma_k\}} A_{\sigma_1} A_{\sigma_2} \cdots A_{\sigma_k} ,
\]  

(45)
as introduced, in the bilinear form at least, by Pascual Jordan to render non-abelian algebras into abelian algebras at the expense of non-associativity. The generalization to multi-linears was suggested by Kurosh [7], but the idea was not used in any previous physical application, as far as we know.

A GJP also has left- and right-sided recursions,

\[
\{A_1, A_2, \cdots, A_k\} = \sum_{k! \text{ perms } \sigma} \frac{1}{(k-1)!} A_{\sigma_1} \{A_{\sigma_2}, A_{\sigma_3}, \cdots, A_{\sigma_k}\}
\]

On the RHS there are again only \(k\) distinct products of single elements with \((k-1)\)-GJPs, each such product having a net coefficient +1. The denominator again compensates for replication of these products in the sum over permutations. (We leave it as another elementary exercise for the reader to prove this result.)

For example, a Jordan 2-product is obviously just an anticommutator \(\{A, B\} = AB + BA\), while a 3-product is given by

\[
\{A, B, C\} = \{A, B\} C + \{A, C\} B + \{B, C\} A
= A \{B, C\} + B \{A, C\} + C \{A, B\}.
\]

Equivalently, taking sums and differences, we obtain

\[
2 \times \{A, B, C\} = \{A, \{B, C\}\} + \{B, \{A, C\}\} + \{C, \{A, B\}\},
\]

as well as the companion of the Jacobi identity often encountered in super-algebras,

\[
0 = [A, \{B, C\}] + [B, \{A, C\}] + [C, \{A, B\}].
\]

Similarly for the 4-product,

\[
\{A, B, C, D\} = A \{B, C, D\} + B \{C, D, A\} + C \{D, A, B\} + D \{A, B, C\}
= \{A, B, C\} D + \{B, C, D\} A + \{C, D, A\} B + \{D, A, B\} C.
\]

Summing gives

\[
2 \times \{A, B, C, D\} = \{A, \{B, C, D\}\} + \{B, \{C, D, A\}\} + \{C, \{D, A, B\}\} + \{D, \{A, B, C\}\},
\]

while subtracting gives

\[
0 = [A, \{B, C, D\}] + [B, \{C, D, A\}] + [C, \{D, A, B\}] + [D, \{A, B, C\}].
\]

Again the reader is warned off the temptation to think of the last of these as a bona fide generalization of the super-Jacobi identity. It is a valid identity of course, following from nothing but associativity, but there is a superior and complete set of identities to be given later.
(Anti)Commutator resolutions  As in the classical case, it is always possible to resolve even rank brackets into sums of commutator products, very usefully. For example,

\[
[A, B, C, D] = [A, B][C, D] - [A, C][B, D] - [A, D][C, B] \\
+ [C, D][A, B] - [B, D][A, C] - [C, B][A, D].
\]  

(53)

An arbitrary even bracket of rank \(2n\) breaks up into \((2^n)!/(2^n) = n!(2n-1)!!\) such products. Another way to say this is that even QNBs can be written in terms of GJPs of commutators. The general result is

\[
[A_1, \ldots, A_{2n}] = \sum_{(2n)! \text{ perms } \sigma} \frac{\text{sgn} (\sigma)}{2^n n!} \{[A_{\sigma_1}, A_{\sigma_2}], \ldots, [A_{\sigma_{2n-1}}, A_{\sigma_{2n}}]\}.  
\]  

(54)

An even GJP also resolves into symmetrized products of anticommutators.

\[
\{A_1, \ldots, A_{2n}\} = \sum_{(2n)! \text{ perms } \sigma} \frac{1}{2^n n!} \{\{A_{\sigma_1}, A_{\sigma_2}\}, \ldots, \{A_{\sigma_{2n-1}}, A_{\sigma_{2n}}\}\}.  
\]  

(55)

As in the classical bracket formalism, the proofs of these relations are elementary. Both left- and right-hand sides of the expressions are sums of \(2n\)-th degree monomials linear in each of the \(A_s\). Both sides are either totally antisymmetric, in the case of QNBs, or totally symmetric, in the case of GJPs, under permutations of the \(A_s\). Thus the two sides must be proportional. The only issue left is the constant of proportionality. This is easily determined to be 1 just by comparing the coefficients of any given term appearing on both sides of the equation, e.g. \(A_1A_2\cdots A_{2n-1}A_{2n}\).

The classical limit  Since Poisson brackets are straightforward classical limits of commutators,

\[
\lim_{\hbar \to 0} \left( \frac{1}{i\hbar} \right)^2 [A, B] = \{A, B\}_{PB},
\]  

(56)

it follows that the commutator resolution of all even QNBs directly specifies their classical limit. (For a detailed approach to the classical limit, including sub-dominant terms of higher order in \(\hbar\), see, e.g., the Moyal Bracket discussion in [2].)

For example, from

\[
[A, B, C, D] = \{[A, B], [C, D]\} - \{[A, C], [B, D]\} - \{[A, D], [C, B]\},
\]  

(57)

with due attention to factors of 2, the classical limit emerges as

\[
\frac{1}{2} \times \lim_{\hbar \to 0} \left( \frac{1}{i\hbar} \right)^2 [A, B, C, D] \\
= \{A, B\}_{PB}\{C, D\}_{PB} - \{A, C\}_{PB}\{B, D\}_{PB} - \{A, D\}_{PB}\{C, B\}_{PB} \\
= \{A, B, C, D\}_{NB}.
\]  

(58)

And so it goes with all other even rank Nambu brackets. For a \(2n\)-bracket, one sees that

\[
\frac{1}{n!} \times \lim_{\hbar \to 0} \left( \frac{1}{i\hbar} \right)^n [A_1, \ldots, A_{2n}] \\
= \sum_{(2n)! \text{ perms } \sigma} \frac{\text{sgn} (\sigma)}{2^n n!} \{A_{\sigma_1}, A_{\sigma_2}\}_{PB}\{A_{\sigma_3}, A_{\sigma_4}\}_{PB} \cdots \{A_{\sigma_{2n-1}}, A_{\sigma_{2n}}\}_{PB} \\
= \{A_1, \ldots, A_{2n}\}_{NB}.  
\]  

(59)
This is another way to establish that there are indeed \((2n - 1)!!\) independent products of \(n\) Poisson brackets summing up to give the PB resolution of the classical Nambu \(2n\)-bracket. Once again due attention must be given to a critical additional factor of \(n!\) (as in the denominator on the LHS) since the GJPs on the RHS of the commutator resolution will, in the classical limit, always replicate the same classical product \(n!\) times.

**The Leibniz rule failure and derivators** Define the derivator to measure the failure of the simplest Leibniz rule for QNBs,

\[
\Delta_B (A, \mathcal{A}) \equiv (A, \mathcal{A} | B_1, \cdots, B_k) \\
\equiv [AA, B_1, \cdots, B_k] - A[B_1, \cdots, B_k] - [A, B_1, \cdots, B_k]\mathcal{A}.
\]

The first term on the RHS is a \((k + 1)\)-bracket acting on just the product of \(A\) and \(\mathcal{A}\), the order of the bracket being evident in the pre-superscript of the \(\Delta_B\) notation. This reads in an obvious way. For instance, \(\Delta_B\) is a “4-delta of \(B\)s”. That notation also emphasizes that the \(B\)s act on the pair of \(A\)s. The second notation makes explicit all the \(B\)s and is useful for computer code.

Any \(\Delta_B\) acts on all pairs of elements in the enveloping algebra \(\mathfrak{a}\) to produce another element in \(\mathfrak{a}\).

\[
\Delta_B : \mathfrak{a} \times \mathfrak{a} \mapsto \mathfrak{a}.
\]

When \(\Delta_B\) does not vanish the corresponding bracket with the \(B\)s does not define a derivation on \(\mathfrak{a}\). The derivator \(\Delta_B (A, \mathcal{A})\) is linear in both \(A\) and \(\mathcal{A}\), as well as linear in each of the \(B\)s.

Less trivially, from explicit calculations, we find inhomogeneous recursion relations for these derivators.

\[
(A, \mathcal{A} | B_1, \cdots, B_k) = \sum_{k! \text{ perms } \sigma} \frac{1}{2} \text{sgn} (\sigma) \left( (A, \mathcal{A} | B_{\sigma_1}, \cdots, B_{\sigma_{k-1}}) B_{\sigma_k} + (-1)^k B_{\sigma_k} (A, \mathcal{A} | B_{\sigma_1}, \cdots, B_{\sigma_{k-1}}) \right) \\
+ \sum_{k! \text{ perms } \sigma} \frac{1}{2} \text{sgn} (\sigma) \left( [A, B_{\sigma_k}] [B_{\sigma_1}, \cdots, B_{\sigma_{k-1}}, \mathcal{A}] \right) - [A, B_{\sigma_1}, \cdots, B_{\sigma_{k-1}}] [B_{\sigma_k}, \mathcal{A}] \\
+ \frac{(-1)^{k+1} - 1}{2} A [B_1, \cdots, B_k] \mathcal{A}.
\]

Alternatively, we may write this so as to emphasize the number of distinct terms on the RHS and distinguish between the even and odd bracket cases. The first two terms under the sum on the RHS give a commutator/anticommutator for \(k\) odd/even, and the last term is absent for \(k\) odd, while for \(k\) even it may be viewed as a type of obstruction in the recursion relation for the odd quantum bracket.

The obstruction is clarified if we specialize to \(n = 1\), i.e. the 3-bracket case. Since commutators are always derivations, one has \(\Delta_B (A, \mathcal{A}) = 0\), and the first RHS line vanishes for the \(\Delta_B (A, \mathcal{A})\) case above. So we have just

\[
(A, \mathcal{A} | B_1, B_2) = [A, B_2] [B_1, \mathcal{A}] - [A, B_1] [B_2, \mathcal{A}] - A [B_1, B_2] \mathcal{A}.
\]

The first two terms on the RHS are \(O (\hbar^2)\) while the last is \(O (\hbar)\). It is precisely this last term which was responsible for some of Nambu's misgivings concerning his quantum 3-bracket. In particular, even in the extreme case when both \(A\) and \(\mathcal{A}\) commute with the \(B\)s, \(\Delta_B (A, \mathcal{A})\) does not vanish.
By contrast, for the even \((2n+2)\)-bracket, all terms on the RHS are generically of the same order, \(O(\hbar^{n+1})\). In terms of combinatorics, this seems to be the only feature for the simple, possibly failed, Leibniz rule that distinguishes between even and odd brackets.

**Generalized Jacobi identities and QFIs**

We previously pointed out some elementary identities involving QNBs which are suggestive of generalizations of the Jacobi identity for commutators. Those particular identities, while true, were not designated as “generalized Jacobi identities”, for the simple fact that they do not involve the case where QNBs of a given rank act on QNBs of the same rank. Here we explore QNB identities of the latter type. There are indeed acceptable generalizations of the usual commutators-acting-on-commutators Jacobi identity (i.e. quantum 2-brackets acting on quantum 2-brackets), and these generalizations are indeed valid for all higher rank QNBs (i.e. quantum n-brackets acting on quantum n-brackets). However, there is an essential distinction to be drawn between the even and odd quantum bracket cases.

It is important to note that, historically, there have been some incorrect guesses and false starts in this direction that originated from the so-called fundamental identity obeyed by classical Nambu brackets (see [1] for the irrelevant literature). The correct generalizations of the Jacobi identities which do encode associativity were found independently by groups of mathematicians and physicists [8, 9]. Interestingly, both groups were studying cohomology questions, so perhaps it is not surprising that they arrived at the same result. The acceptable generalization of the Jacobi identity that was found is satisfied by all QNBs, although for odd QNBs there is a significant difference in the form of the final result: It contains an “inhomogeneity”. The correct generalization is obtained just by totally antisymmetrizing the action of one n-bracket on the other. Effectively, this amounts to antisymmetrizing the form of the classical FI over all permutations of the \(A\)s and \(B\)s, including all exchanges of \(A\)s with \(B\)s. To describe some of the results before giving them in detail:

The totally antisymmetrized action of odd \(n\) QNBs on other odd \(n\) QNBs results in \((2n−1)\)-brackets.

The totally antisymmetrized action of even \(n\) QNBs on other even \(n\) QNBs results in zero.

More precisely, the generalized Jacobi identities (GJIs) for arbitrary \(n\)-brackets follow from totally antisymmetrizing the action of any bracket on any other through use of the so-called “shifting bracket argument” [1]. That argument actually leads to a larger set of results which we summarize here, calling them the quantum Jacobi identities, or QJIs. The GJIs are special cases of QJIs for \(k=n−1\).

**QJI for QNBs**

\begin{align}
\sum_{(n+k)! \text{ perms } \sigma} \text{sgn} (\sigma) \left[ [A_{\sigma_1}, \cdots, A_{\sigma_n}, A_{\sigma_{n+1}}, \cdots, A_{\sigma_{n+k}}] \right] \\
= [A_1, \cdots, A_{n+k}] \times n!k! \times \begin{cases} 
(k + 1) & \text{if } n \text{ is odd} \\
\frac{1}{2} \left(1 + (-1)^k \right) & \text{if } n \text{ is even}
\end{cases}
\end{align}

(63)

This result is proven just by computing the coefficient of any selected monomial, e.g. \(A_1 \cdots A_{n+k}\). This is the quantum identity that most closely corresponds to the general classical result (28). While that classical identity holds without requiring antisymmetrization over exchanges of \(A\)s and
Bs, in contrast the quantum identity must be totally antisymmetrized if it is to be a consequence of only the associativity of the underlying algebra of Hilbert space operators. Note that the \( nk! \) on the RHS of (63) may be replaced by just 1 if we sum only over permutations in which the \( A_{i \geq n} \) are interchanged with the \( A_{i < n} \) in \([A_{\sigma_1}, \ldots, A_{\sigma_n}], A_{\sigma_{n+1}}, \ldots, A_{\sigma_{n+k}}\] , and ignore all permutations of the \( A_1, A_2, \ldots, A_n \) among themselves, and of the \( A_{n+1}, \ldots, A_{n+k} \) among themselves.

There is an important specialization of the QJI result: For even \( n \) and odd \( k \),

\[
\sum_{(n+k)! \text{ perms } \sigma} \text{sgn}(\sigma) [A_{\sigma_1}, \ldots, A_{\sigma_n}, A_{\sigma_{n+1}}, \ldots, A_{\sigma_{n+k}}] = 0. \tag{64}
\]

In particular, when \( k = n - 1 \), for \( n \) even, the vanishing RHS obtains. All other \( n \)-not-even and/or \( k \)-not-odd cases of the QJI have the \([A_1, \ldots, A_{n+k}]\) inhomogeneity on the RHS. The classical limit of this last identity is immediate.

In particular, when \( n \) is odd and \( k \) is even, the vanishing RHS obtains. All other \( n \)-not-even and/or \( k \)-not-odd cases of the QJI have the \([A_1, \ldots, A_{n+k}]\) inhomogeneity on the RHS. The classical limit of this last identity is immediate.

\[
\sum_{(n+k)! \text{ perms } \sigma} \text{sgn}(\sigma) \{\{A_{\sigma_1}, \ldots, A_{\sigma_n}\}_NB, A_{\sigma_{n+1}}, \ldots, A_{\sigma_{n+k}}\}_NB = 0. \tag{65}
\]

For example, when the outside bracket is just a PB, this implies for odd \( k \)

\[
\varepsilon_{a_1 \ldots a_k} \{\{B_{a_1}, \ldots, B_{a_{k-1}}\}, B_{a_k}\} = 0, \tag{66}
\]

where \( \varepsilon_{a_1 \ldots a_k} \) is the totally antisymmetric Levi-Civita symbol, and all \( a_i \) are summed from 1 to \( k \). This classical identity is needed to obtain the form of the terms in the last line of (28), and hence to prove the DB JI.

The QJI also permits us to give the correct operator form of the so-called fundamental identities valid for all QNBs. We accordingly call these quantum fundamental identities, or QFIs, and present them in their general form.

**QFI for QNBs**

\[
\sum_{(n+k)! \text{ perms } \sigma} \text{sgn}(\sigma) \left( [A_{\sigma_1}, \ldots, A_{\sigma_n}], A_{\sigma_{n+1}}, \ldots, A_{\sigma_{n+k}} \right)
- \sum_{j=1}^{n} \left[ [A_{\sigma_1}, \ldots, [A_{\sigma_j}, A_{\sigma_{n+1}}, \ldots, A_{\sigma_{n+k}}], \ldots, A_{\sigma_n}] \right]
= [A_1, \ldots, A_{n+k}] \times n!k! \times \begin{cases} 0 & \text{if } k \text{ is odd} \\ (1-n)(k+1) & \text{if } k \text{ is even and } n \text{ is odd} \\ (1-n(k+1)) & \text{if } k \text{ is even and } n \text{ is even} \end{cases}. \tag{67}
\]

Aside from the trivial case of \( n = 1 \), the only way the RHS vanishes without conditions on the full \((n+k)\)-bracket is when \( k \) is odd. All \( n > 1 \), even \( k \) result in the \([A_1, \ldots, A_{n+k}]\) inhomogeneity on the RHS.

Partial antisymmetrizations of the individual terms in the general QFI may also be entertained. The result is to find more complicated inhomogeneities, and does not seem to be very informative. At best these partial antisymmetrizations show in a supplemental way how the fully antisymmetrized results are obtained. In certain isolated, special cases (cf. the \( \text{su}(2) \) example of the next section, for which \( k = 3 \)), the bracket effects of select Bs can act as a derivation (essentially because the \( k \)-bracket is equivalent, in its effects, to a commutator). If that is the case, then the quantum version of the classical FI (19) trivially holds. It is also possible in principle for that simple identity
to hold, again in very special situations, if the quantum bracket is not a derivation, through various cancellations among terms. As an aid to finding such peculiar situations, it is useful to resolve the quantum correspondents of the terms in the classical relation into the derivators introduced previously. From the definition of \([A_1, \cdots, A_n]\), and some straightforward manipulations, we find

\[
[[A_1, \cdots, A_n], B] = \sum_{j=1}^{n} [A_1, \cdots, [A_j, B], \cdots, A_n]
\]

\[
= \sum_{n! \text{ perms } \sigma} \text{sgn}(\sigma) \left( \frac{1}{(n-1)!} [A_{\sigma_1}, [A_{\sigma_2}, \cdots, A_{\sigma_n}] | B] + \frac{1}{(n-2)!} A_{\sigma_1} (A_{\sigma_2}, [A_{\sigma_3}, \cdots, A_{\sigma_n}] | B) 
+ \frac{1}{2(n-3)!} [A_{\sigma_1}, A_{\sigma_2}] (A_{\sigma_3}, [A_{\sigma_4}, \cdots, A_{\sigma_n}] | B) 
+ \cdots + \frac{1}{(n-1)!} [A_{\sigma_1}, A_{\sigma_2}, \cdots, A_{\sigma_{n-2}}] (A_{\sigma_{n-1}}, A_{\sigma_n} | B) \right),
\]

(68)

with the abbreviation \(B = B_1, \cdots, B_k\). The terms on the RHS are a sum over \(j = 1, \cdots, n-1\) of derivators between solitary \(A\)s (i.e. 1-brackets) and various \((n-j)\)-brackets, left-multiplied by complementary rank \((j-1)\)-brackets. (There is a similar identity that involves right-multiplication by the complementary brackets.) For example, suppose \(n = 2\). Then we have for any number of \(B\)s

\[
[[A_1, A_2], B] - [[A_1, B], A_2] - [A_1, [A_2, B]] = (A_1, A_2 | B) - (A_2, A_1 | B).
\]

(69)

In principle, this can vanish, even when the action of the \(B\)s is not a derivation, if the \(k\)-derivator is symmetric in the first two arguments. That is, if \((A_1, A_2 | B_1, \cdots, B_k) = \frac{1}{k} (A_1, A_2 | B_1, \cdots, B_k) + \frac{1}{k} (A_2, A_1 | B_1, \cdots, B_k)\). However, we have not found an interesting (nontrivial) physical example where this is the case.

### 4 Quantum entwinement of Dirac Brackets

From the commutator resolution of the QNB,

\[
[f, g, C_1, C_2, \cdots] = \{[f, g], [C_1, C_2], \cdots\} \pm \text{ permutations of the Cs}
\]

\[
- \{[f, C_1], [g, C_2], \cdots\} + \{[g, C_1], [f, C_2], \cdots\} \pm \text{ permutations of the Cs}.
\]

Terms in the first line, where the commutator \([f, g]\) is intact, are the quantum analogues of the first line in the classical relation

\[
\{f, g, C_1, C_2, \cdots\}_{NB} = \{f, g\}_{PB} \{C_1, C_2, \cdots\}_{NB}
+ \sum_{j, k} (-1)^{j+k} \{f, C_j\}_{PB} \{C_1, \cdots, C_{k-1}, C_{k+1}, \cdots, C_{j-1}, C_{j+1}, \cdots\}_{NB} \{C_k, g\}_{PB},
\]

(70)

whereas terms in the commutator resolution where \(f\) and \(g\) appear in different commutators, as in the second line above, are the quantum analogues of the second line in the classical relation. Unfortunately, the coefficient of \([f, g]\) cannot be trivially divided out of the commutator resolution as it could in the classical relation. That is, generalized special Jordan algebras are not division rings (see the Appendix, [1], for an exemplary discussion).

However, if the Hilbert space is suitably partitioned into invariant sectors using projectors, with

\[
[f, g] = \sum_{j,k} \mathbb{P}_j ([f, g])_{jk} \mathbb{P}_k, \quad \mathbb{P}_j^2 = \mathbb{P}_j, \quad \mathbb{P}_j \mathbb{P}_k = 0, \quad j \neq k.
\]

(71)
then the cumulative effect of the \([C_i, C_j]\) terms can possibly be diagonalized,

\[
\sigma \times ([f, g])_\sigma = \{( [f, g])_\sigma, [C_1, C_2], \cdots \} \pm \text{permutations},
\]

(72)

and hence for non-vanishing \(\sigma\) the effects of entwining \([f, g]\) within the constraint commutators can be divided out, sector by sector. On each such invariant sector, an effective Dirac bracket can then be computed.

The result is not a derivation, on the full Hilbert space, but acts like a derivation on each invariant sector. This is the best we can do to define QDBs in terms of QNBs, in general. For further discussion of the use of such projections as well as the physics coded in QNBs for particular examples, see [1], and also [2] in these proceedings. Another recent proposal to implement constraints using projection operators, which came to our attention while writing up this contribution, is to be found in [10].

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References

[1] T Curtright and C Zachos, “Classical and Quantum Nambu Mechanics” [hep-th/0212267] to appear in Phys Rev D.

[2] C K Zachos and T L Curtright, “Deformation Quantization of Nambu Mechanics” [quant-ph/0302106], these proceedings.

[3] P A M Dirac, Can J Math 3 (1951) 1; Lectures on Quantum Mechanics, Yeshiva University 1964 (reprinted by Dover, 2001).

[4] Y Nambu, Phys Rev D7 (1973) 2405.

[5] C Gonera and Y Nutku, Phys Lett A285 (2001) 301.

[6] T Curtright and C Zachos, N J Phys 4 (2002) 83.

[7] A G Kurosh, Russian Math Surveys 24 (1969) 1-13.

[8] P Hanlon and M Wachs, Adv Math 113 (1995) 206-236.

[9] J A de Azcárraga, et al., J Phys A29 (1996) 7993-8010 [hep-th/9605067]; Commun Math Phys 184 (1997) 669-681 [hep-th/9605213]; J Phys A30 (1997) L607-L616 [hep-th/9703019]; Rev R Acad Cien Serie A Mat 95 (2001) 225-248 [physics/9803046].

[10] I Batalin, S Lyakhovich, and R Marnelius, Phys Lett B534 (2002) 201-208 [hep-th/0112175].