A Heat Kernel Lower Bound for Integral Ricci Curvature

Xianzhe Dai† Guofang Wei‡

Abstract
In this note we give a heat kernel lower bound in term of integral Ricci curvature, extending Cheeger-Yau’s estimate.

1 Introduction

Heat kernel is one of the most fundamental quantities in geometry. It can be estimated both from above and below in terms of Ricci curvature (see [CY, LY, CGT]). The heat kernel upper bound has been extended to integral Ricci curvature by Gallot in [Ga]. Here we extend Cheeger-Yau’s lower bound [CY] to integral Ricci curvature.

Our notation for the integral curvature bounds on a Riemannian manifold \((M, g)\) is as follows. For each \(x \in M\) let \(r(x)\) denote the smallest eigenvalue for the Ricci tensor \(\text{Ric} : T_x M \to T_x M\), and for any fixed number \(\lambda\) define

\[\rho(x) = |\min \{0, r(x) - (n - 1)\lambda\}|.\]

Then set

\[k(p, \lambda, R) = \sup_{x \in M} \left( \int_{B(x, R)} \rho^p \right)^{\frac{1}{p}},\]

\[\bar{k}(p, \lambda, R) = \sup_{x \in M} \left( \frac{1}{\text{vol} B(x, R)} \cdot \int_{B(x, R)} \rho^p \right)^{\frac{1}{p}}.\]

These curvature quantities evidently measure how much Ricci curvature lies below \((n - 1)\lambda\) in the (normalized) integral sense. And \(\bar{k}(p, \lambda, R) = 0\) iff \(\text{Ric} \geq (n - 1)\lambda\).

Let \(E(x, y, t)\) denote the heat kernel of the Laplace-Beltrami operator on a closed manifold \((M, g)\). For any real number \(\lambda\) we denote \(E_\lambda(x, y, t)\) the heat kernel on the model space of constant curvature \(\lambda\). Our main result is

*1991 Mathematics Subject Classification. Primary 53C20.
†Partially supported by NSF Grant # DMS-9704296
‡Partially supported by NSF Grant # DMS-9626419.
Theorem 1.1 Let $n > 0$ be an integer, $p > (n+1)/2$, $\lambda \leq 0$ real numbers and $D > 0$. Then there exists an explicitly computable $\epsilon_0 = \epsilon(n, p, \lambda, D)$ such that for any $(M, g)$ with $\text{diam} M \leq D$ and $\bar{k}(p, \lambda, D) \leq \epsilon_0$,

$$E(x, y, t) \geq E_\lambda(x, y, t) - k(p, \lambda, D)C(n, p, \lambda, D)(t^{\frac{n+1}{2}} + 1),$$

for any $x, y \in M$ and $t > 0$.

The basic strategy is the same as in Cheeger-Yau, namely, one transplants the heat kernel on the model space to $M$ and compare using Duhamel's principle. The new difficulty lies in controlling an error term which would be zero in the presence of the pointwise Ricci curvature bound. This is overcome by employing 1) the mean curvature estimate in [PW]; 2) a comparison of volume element (integrated over the directional sphere); 3) Gallot’s upper bound estimate [Ga] of the heat kernel, together with a remarkable result of Grigor’y an [Gr] which furnishes us with a Gaussian upper bound for the heat kernel.

2 Basic Facts on Heat Kernel

Here we fix our notation and collect basic facts on the heat kernel which will be used in our proof.

As in [CY] we can define the Laplace-Beltrami operator for generalized Dirichlet and Neumann boundary conditions on a general Riemannian manifold (possibly incomplete) by choosing appropriate domains. The two coincide for a complete manifold. The corresponding heat kernel can simply be defined using spectral theorem. The heat kernel thus defined is always positive ([CY, Lemma 1.1]), which will be essential for our discussion.

The models as used in [CY] need only to have the right mean curvature on the distance spheres. Here we restrict our models to the standard models, namely, simply connected spaces of constant sectional curvature. The following result [CY, Lemma 2.3] is critical for Cheeger-Yau’s theorem as well as in here.

Lemma 2.1 Let $E_\lambda(r, t)$ denote the heat kernel on the model space of constant curvature $\lambda$, where $r = d(x, y)$. Then, for all $r, t > 0$,

$$\frac{\partial}{\partial r} E_\lambda(r, t) < 0.$$

As we mentioned before, we also need uniform upper bounds on heat kernel. This is established in [Ga] for integral Ricci curvature.
Theorem 2.2 Given any real number \( \lambda \leq 0 \), \( p > \frac{n}{2} \) and \( D > 0 \), there exists an explicitly computable \( \epsilon_0 = \epsilon(n, p, \lambda, D) \) such that for any \((M, g)\) with \( \text{diam} M \leq D \) and \( k(p, \lambda, D) \leq \epsilon_0 \) one has

\[
E(x, y, t) \leq C(n, p, \lambda, D)(t^{-p} + 1),
\]

for any \( x, y \in M \) and \( t > 0 \).

However this estimate is not sufficient for our purpose. Fortunately one has the following recent amazing result of [Gr, Theorem 1.1], which translate Gallot’s estimate into a Gaussian upper bound. We let \( f(t) \), \( g(t) \) denote regular functions in sense of [Gr] (which includes all piecewise power functions with nonnegative exponents).

Theorem 2.3 Let \( x, y \) be two points on an arbitrary smooth connected Riemannian manifold \( M \), for which one has

\[
E(x, x, t) \leq \frac{1}{f(t)}, \quad E(y, y, t) \leq \frac{1}{g(t)},
\]

for all \( 0 < t < T \leq \infty \). Then for any \( C > 4 \) and some \( \delta = \delta(C) > 0 \),

\[
E(x, y, t) \leq \frac{4A}{\sqrt{f(\delta t)g(\delta t)}} e^{-\frac{d(x, y)^2}{Ct}},
\]

where \( A \) is a constant coming from \( f \) and \( g \).

Corollary 2.4 With the assumption of Theorem 2.2, we have

\[
E(x, y, t) \leq C(n, p, \lambda, D)(t^{-p} + 1)e^{-\frac{d(x, y)^2}{Ct}}.
\]

The final piece of information we need is a similar Gaussian type estimate on the derivative of the heat kernel on the model space.

Proposition 2.5 For the model space we have

\[
|\frac{\partial}{\partial r} E_{\lambda}(r, t)| < C(n, \lambda)(t^{-\frac{n+1}{2}} + 1)e^{-\frac{d(x, y)^2}{Ct}}.
\]

Proof. The key point here is that the space derivative deteriorates the bound only by a factor of \( t^{-\frac{1}{2}} \), whereas the time derivative deteriorates the bound by a factor of \( t \). This follows easily from, say, the gradient estimate (Harnack inequality) of Li-Yau [LY], which asserts that a positive solution of the heat equation satisfies, for all \( \alpha > 1 \),

\[
\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n}{\sqrt{2}\alpha - 1}H + \frac{n}{2} \frac{\alpha^2}{t},
\]

where \( H \) denotes the lower bound on the Ricci curvature. \( \blacksquare \)
3 Comparison of Volume Element

In [PW] a mean curvature comparison estimate is given in terms of $k(p, \lambda, R)$ and therefore one obtains the relative volume comparison for integral Ricci curvature. Here we need a comparison of integral of the volume element just over the directional spheres (instead of the balls).

Let $M^n$ be a complete Riemannian manifold and $x \in M$. Around $x$ use exponential polar coordinates and write the volume element as $d\text{vol} = \omega d\theta_{n-1} \wedge dt$, where $d\theta_{n-1}$ is the standard volume element on the unit sphere $S^{n-1}(1)$. As $t$ increases $\omega$ becomes undefined but we can simply define it to be zero at those $t$’s. We have the important equation $\omega' = m\omega$, where the prime indicates differentiation along the radial direction and $m$ is the mean curvature of the distance spheres around $x$.

In the space form $M^n_\lambda$ of constant sectional curvature $\lambda$, we can similarly write the volume element as $d\text{vol} = \omega_\lambda d\theta_{n-1} \wedge dt$ and $\omega_\lambda' = m_\lambda \omega_\lambda$.

Define $\psi = \psi(t, \theta) = \max\{0, m(t, \theta) - m_\lambda(t, \theta)\}$ and $0$ whenever it becomes undefined. The following mean curvature comparison estimate is established in [PW].

**Theorem 3.1** For $p > n/2, \lambda \leq 0$,

$$
\left( \int_{B(x,r)} \psi^{2p} d\text{vol} \right)^{\frac{1}{2p}} \leq C(n, p) \left( k(p, \lambda, r) \right)^{\frac{1}{2}}. 
$$  (3.1)

With the help of the mean curvature comparison estimate we deduce a comparison estimate for the volume element.

**Lemma 3.2** There is a constant $C(n, p, \lambda, R)$ such that for any $p > \frac{n+1}{2}$, $\lambda \leq 0$, $r \leq R$, if $k(p, \lambda, R) \leq 1$, then we have

$$
\frac{\int_{S^{n-1}} \omega(r, \theta) d\theta_{n-1}}{\int_{S^{n-1}} \omega_\lambda(r, \theta) d\theta_{n-1}} \leq 1 + C(n, p, \lambda, R) \left( k(p, \lambda, R) \right)^{\frac{1}{2}} \tag{3.2}
$$

**Remark.** The assumption $k(p, \lambda, R) \leq 1$ is only for the simplicity of the statement.

**Proof.** We will prove a more general relative version. Define $u(r) = \frac{\int_{S^{n-1}} \omega(r, \theta) d\theta_{n-1}}{\int_{S^{n-1}} \omega_\lambda(r, \theta) d\theta_{n-1}}$.

From the beginning of the proof of Lemma 2.1 in [PW], we have for $0 \leq r_1 < r_2 \leq R$,

$$
u(r_2) - u(r_1) \leq \frac{1}{\text{vol}S^{n-1}} \int_{r_1}^{r_2} \int_{S^{n-1}} \frac{\omega}{\omega_\lambda} d\theta_{n-1} \wedge dt.
$$

Using Hölder’s inequality, we have

$$
\int_{r_1}^{r_2} \int_{S^{n-1}} \psi \frac{\omega}{\omega_\lambda} d\theta_{n-1} \wedge dt
\leq \left( \int_{0}^{R} \int_{S^{n-1}} \psi^{2p} \omega d\theta_{n-1} \wedge dt \right)^{1/2p} \cdot \left( \int_{r_1}^{r_2} \left( \frac{1}{p} \int_{S^{n-1}} \omega d\theta_{n-1} \right)^{1/p} dt \right)^{1 - \frac{1}{2p}}
\leq C(n, p) \left( k(p, \lambda, R) \right)^{1/2} \left( \int_{0}^{R} \omega_\lambda^{1+\frac{1}{p}} dt \right)^{\frac{1}{1+\frac{1}{p}}} \cdot \left( \int_{r_1}^{r_2} \left( \frac{1}{p} \int_{S^{n-1}} \omega_\lambda d\theta_{n-1} \right)^{1+\frac{1}{p}} dt \right)^{\frac{1}{1+\frac{1}{p}}},
$$
where $\alpha > 0$ is chosen so that $p > \frac{(1+\alpha)n+1}{2}$, therefore $\int_0^R \omega_\lambda \frac{1}{2p-1} dt$ is integrable. Thus $u(r_2)$ satisfies the integral inequality

$$u(r_2) - u(r_1) \leq C(n, p, \lambda, R) (k(p, \lambda, R))^{\frac{1}{2}} \left( \int_{r_1}^{r_2} (u(t))^{1+\frac{1}{\alpha}} dt \right)^{\frac{1}{\alpha+1}} \left(1 - \frac{1}{2p}\right).$$

This implies

$$(u(r_2) - u(r_1))_+ \leq C k^{\frac{1}{2}} \left( \int_{r_1}^{r_2} (u(t))^{1+\frac{1}{\alpha}} dt \right)^{\frac{1}{\alpha+1}} \left(1 - \frac{1}{2p}\right).$$

Let $v = \max \{u - u(r_1), 0\} = (u - u(r_1))_+$. Then $u \leq v + u(r_1)$ and we have

$$v \leq C k^{\frac{1}{2}} \left( \int_{r_1}^{r_2} (v(t) + u(r_1))^{1+\frac{1}{\alpha}} dt \right)^{\frac{1}{\alpha+1}} \left(1 - \frac{1}{2p}\right).$$

Or

$$v^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \leq (C k^{\frac{1}{2}})^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \int_{r_1}^{r_2} (v(t) + u(r_1))^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} dt.$$  \(3.3\)

Write

$$(v(t) + u(r_1))^{\frac{\alpha+1}{\alpha} - \left(\frac{1}{2p-1}\right)} = \left( (v(t) + u(r_1))^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \right)^{\frac{1}{2p}}.$$  \(3.3\)

Now we use the inequality

$$(a + b)^q \leq 2^{q-1}(a^q + b^q), \quad a, b \geq 0, \quad q \geq 1$$

to obtain

$$(v(t) + u(r_1))^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \leq \left[ 2^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \left( v(t)^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} + u(r_1)^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \right) \right]^{\frac{1}{1-\frac{1}{2p}}}.$$

And letting $w = v^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}}$, \(3.3\) becomes

$$w \leq (C k^{\frac{1}{2}})^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \int_{r_1}^{r_2} 2^{\frac{1}{\alpha} + \frac{1}{2p}} \left( w(t) + u(r_1)^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \right)^{\frac{1}{2p-1}} dt.$$  \(3.3\)

Let $\bar{w}$ be the solution of

$$\begin{cases}
\bar{w}' & = 2^{\frac{1}{\alpha} + \frac{1}{2p}} \left( C k^{\frac{1}{2}} \right)^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \left( \bar{w}(t) + u(r_1)^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \right)^{\frac{1}{2p-1}} \\
\bar{w}(r_1) & = 0
\end{cases}$$

Then

$$\bar{w}(r_2) = \left[ \left( u(r_1)^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} \right)^{\frac{1}{2p}} + \frac{1}{2p} 2^{\frac{1}{\alpha} + \frac{1}{2p}} \left( C k^{\frac{1}{2}} \right)^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}} (r_2 - r_1) \right]^{2p} - u(r_1)^{\frac{\alpha+1}{\alpha} - \frac{1}{2p-1}}.$$
By Gronwall inequality, we have

\[ w \leq \bar{w}, \]

which means

\[
(u(r_2) - u(r_1))_+ \\
\leq \left[ (u(r_1) \frac{\alpha + 1}{\alpha} \frac{1}{2p-1} + \frac{1}{2p} \frac{\alpha + 1}{\alpha} \frac{2p}{2p-1} \left( Ck^{\frac{1}{2}} \right)^{\frac{\alpha + 1}{\alpha} \frac{2p}{2p-1}} (r_2 - r_1) \right)^{2p} - u(r_1) \frac{\alpha + 1}{\alpha} \frac{2p}{2p-1}] \right]^{\frac{1}{2p-1}} (1 - \frac{1}{2p})^{\frac{1}{2p-1}} \]

Using the inequality

\[(x + a)^q - a^q \leq qx(x + a)^{q-1}, \quad q \geq 1,\]

we get

\[ (u(r_2) - u(r_1))_+ \]

\[ \leq \left( (2^{\frac{1}{\alpha} + \frac{1}{2p}} (r_2 - r_1))^{\frac{1}{\alpha + 1} (1 - \frac{1}{2p})} \right) \left( u(r_1) \frac{\alpha + 1}{\alpha} \frac{1}{2p-1} + \frac{1}{2p} \frac{\alpha + 1}{\alpha} \frac{2p}{2p-1} \left( Ck^{\frac{1}{2}} \right)^{\frac{\alpha + 1}{\alpha} \frac{2p}{2p-1}} (r_2 - r_1) \right)^{q}, \]

where \( q = (2p - 1) \frac{\alpha}{\alpha + 1} \left( 1 - \frac{1}{2p} \right). \)

In particular, when \( r_1 = 0 \) and \( k(p, \lambda, R) \leq 1 \), we get

\[ u(r_2) - 1 \leq C(n, p, \lambda, R) (k(p, \lambda, R))^{\frac{1}{2}}. \]

\[ \square \]

4 Proof of Theorem

We follow the same basic strategy as in Cheeger-Yau, starting with the Duhamel’s Principle which needs to be justified because of the singularity of the distance function at the cut locus.

Using integration by part and heat equation, we have

\[
E(x, y, t) = E_\lambda(x, y, t) \\
= - \int_0^t \int_M \frac{d}{ds} (E_\lambda(x, w, t - s)) \cdot E(w, y, s) d \text{vol} \, ds \\
+ \int_0^t \int_M E_\lambda(x, w, t - s) \cdot \frac{d}{ds} (E(w, y, s)) d \text{vol} \, ds \\
= - \int_0^t \int_M \frac{d}{ds} (E_\lambda(x, w, t - s)) \cdot E(w, y, s) d \text{vol} \, ds \\
- \int_0^t \int_M E_\lambda(x, w, t - s) \cdot \Delta E(w, y, s) d \text{vol} \, ds. \quad (4.1)
\]
Now
\[
\frac{d}{ds} E_\lambda = -\Delta E_\lambda = \frac{\partial^2}{\partial r^2} E_\lambda + m_\lambda(r) \frac{\partial}{\partial r} E_\lambda
\]
\[
= -\Delta E_\lambda - (m(r, \theta) - m_\lambda(r)) \frac{\partial}{\partial r} E_\lambda
\]
\[
\leq -\Delta E_\lambda - (m(r, \theta) - m_\lambda(r)) + \frac{\partial}{\partial r} E_\lambda,
\]
since \( \frac{\partial}{\partial r} E_\lambda \leq 0 \) by Lemma 2.1. Hence the righthand side of (4.1) is
\[
\geq \int_0^t \int_M \Delta E_\lambda(x, w, t - s) \cdot E(w, y, s) d\text{vol} ds
\]
\[
- \int_0^t \int_M E_\lambda(x, w, t - s) \cdot \Delta E(w, y, s) d\text{vol} ds
\]
\[
- \int_0^t \int_M (m(r, \theta) - m_\lambda(r)) + |\frac{\partial E_\lambda}{\partial r}(x, w, t - s)| \cdot E(w, y, s) d\text{vol} ds.
\]
The first two terms combined can be shown to be nonnegative using the same argument as in [CY] (using certain convexity property of the distance function at the cut locus). The last term is the extra error term, which is
\[
\geq - \int_0^t \left( \int_M (m(r, \theta) - m_\lambda(r))^q d\text{vol} \right)^{1/q} \left( \int_M \left| \frac{\partial E_\lambda}{\partial r}(x, w, t - s) E(w, y, s) \right|^q d\text{vol} \right)^{1/q} ds,
\]
for some \( q \leq 2p \) to be chosen later. Here \( q' = \frac{q}{q-1} \).

Now the first factor is controlled by Theorem 3.1 and the volume comparison estimate from [PW, Theorem 1.1]. For the second factor, according to Corollary 2.4 and Proposition 2.5,
\[
\left| \frac{\partial E_\lambda}{\partial r}(x, w, t - s) E(w, y, s) \right| \leq C[(t - s)^{-\frac{n+1}{2}} + 1]s^{-p_1 + 1} e^{-\frac{d^2(x,w)}{5s}}.
\]
Here \( p_1 = \frac{n}{2} + \alpha \) will be chosen so that \( \alpha > 0 \) is suitably small. In order to apply Corollary 2.4 we now need that \( \overline{k}(\lambda, p_1, D) \) is small than an explicit constant \( \epsilon_0 \) (as determined by Gallot [Ga]).

We have to deal with the singularity caused by the heat kernel at \( t = 0 \). Divide \( s \) to \( s^{1/2} + s^{t/2} \). If \( t > 1 \), then we divide further so \( s = s^{t/2} + s^{(t-1)/2} + s^{(t-2)/2} \). In the latter case the estimate for the middle term is straightforward. For \( 0 \leq s \leq t/2 \), we have (and we may assume that \( s \leq \frac{1}{2} \) by the above discussion)
\[
(t - s)^{-\frac{n+1}{2}} \leq (t/2)^{-\frac{n+1}{2}}, \quad e^{-\frac{d^2(x,w)}{5s}} \leq 1,
\]
which implies
\[
\int_0^{t/2} \left( \int_M \left| \frac{\partial E_\lambda}{\partial r} (x, w, t - s) E(w, y, s) \right|^q \, d\text{vol} \right)^{\frac{1}{q}} \, ds \leq C \left( t^{-\frac{n+1}{2}} + 1 \right) \int_0^{t/2} \left( \int_M e^{-\frac{q'd^2(w, y)}{8s}} \, d\text{vol} \right)^{\frac{1}{q'}} \, ds.
\]

Now, writing out the integral over the space using the exponential polar coordinate around \(y\)
\[
\int_M e^{-\frac{q'd^2(w, y)}{8s}} \, d\text{vol} = \int_0^D e^{-\frac{q'r^2}{8s}} \left( \int_{S^{n-1}} \omega(r, \theta) d\theta \right) dr.
\]
Here it is used that the integral is over the whole manifold. With the curvature assumption on \(k(p, \lambda, D)\), we can apply the comparison estimate for the volume element Lemma 3.2 and get
\[
\int_M e^{-\frac{q'd^2(w, y)}{8s}} \, d\text{vol} \leq C \int_0^D e^{-\frac{q'r^2}{8s}} \omega_\lambda (r) \, dr.
\]
We then make a change of coordinate \(r_1 = \frac{r}{\sqrt{s}}\), deducing
\[
\int_M e^{-\frac{q'd^2(w, y)}{8s}} \, d\text{vol} \leq C s^{\frac{n}{2}} \int_0^\infty e^{-q'r_1^2} \frac{\omega_\lambda (r_1 s^{\frac{1}{2}})}{s^{\frac{n-1}{2}}} \, dr_1.
\]
Making use of the inequality \(\frac{\omega_\lambda (r_1 s^{\frac{1}{2}})}{s^{\frac{n-1}{2}}} \leq r_1^{n-1} e^{-e(n-1)\sqrt{\lambda} r_1} \) (which can be easily verified) and noticing that since \(s \leq 1\) this term is dominated by \(e^{-q'r_1^2}\), we finally arrive at the following estimate for the \(\int_0^{t/2}\) part of the error term
\[
C \left( t^{-\frac{n+1}{2}} - p_1 + \frac{1}{2q'} + 1 \right),
\]
where \(p_1\) and \(q\) need to be chosen to satisfy the inequality
\[
-p_1 + \frac{n}{2q'} + 1 > 0.
\]

Similarly one has (this time using the exponential polar coordinate around \(x\))
\[
\int_{t/2}^t \left( \int_M \left| \frac{\partial E_\lambda}{\partial r} (x, w, t - s) E(w, y, s) \right|^q \, d\text{vol} \right)^{\frac{1}{q}} \, ds \leq C \left( t^{-\frac{n+1}{2}} - p_1 + \frac{1}{2q'} + 1 \right).
\]
Finally, we note that suitable choice for \(p_1\) and \(q\) can be easily made. For example, \(q = n + 1\) and \(p_1 = \frac{n+1}{2}\) will do.
References

[CGT] J. Cheeger, M. Gromov, M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Diff. Geom. 17 (1982) 15-53.

[CY] J. Cheeger, S.T. Yau, *A lower bound for the heat kernel*, Comm. Pure Appl. Math. 34 (1981) 465-480.

[Ga] S. Gallot, *Isoperimetric inequalities based on integral norms of Ricci curvature*, Astérisque, 157-158, (1988), 191-216.

[Gr] A. Grogor’yan, *Gaussian upper bounds for the heat kernel on arbitrary manifolds*, J. Diff. Geom. 45 (1997) 33-52.

[LY] P. Li, S.T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. 156 (1986), no. 3-4, 153–201.

[PW] P. Petersen, G. Wei, *Relative volume comparison with integral curvature bounds*, GAFA 7 (1997) 1031-1045.

Department of Mathematics, University of California, Santa Barbara, CA 93106
dai@math.ucsb.edu
wei@math.ucsb.edu