Explicit Formulae for Noncommutative Deformations of $\mathbb{CP}^N$ and $\mathbb{CH}^N$

Akifumi Sako, Toshiya Suzuki and Hiroshi Umetsu

Kushiro National College of Technology
Otanoshike-Nishi 2-32-1, Kushiro 084-0916, Japan

MSC 2010: 53D55, 81R60

Abstract

We give explicit expressions of a deformation quantization with separation of variables for $\mathbb{CP}^N$ and $\mathbb{CH}^N$. This quantization method is one of the ways to perform a deformation quantization of Kähler manifolds, which is introduced by Karabegov. Star products are obtained as explicit formulae in all order in the noncommutative parameter. We also give the Fock representations of the noncommutative $\mathbb{CP}^N$ and $\mathbb{CH}^N$.

1 Introduction

Deformation quantizations were introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [1] as a method to quantize spaces. After [1], several ways of deformation quantization were proposed [2,3,4,5]. In particular, deformation quantizations of Kähler manifolds were provided in [6,7,8,9]. In this article, we consider the deformation quantization with separation of variables that are one of the ways to construct the noncommutative Kähler manifolds introduced by Karabegov [11,12,13].

In many cases, deformation quantization of a manifold is given by a star product which is defined in a form of a formal power series of deformation parameter $\hbar$. The power series is obtained as solutions of an infinite system of differential equations, and it is proved that there exists a unique deformation quantization as the solution of the system. The existence of the solution is proved for a wide class of manifolds, however explicit expressions of deformation quantizations are constructed only for few kinds of manifolds. For example, Euclidean spaces are deformed by using the Moyal product, and on manifolds with spherically symmetric metrics explicit star products are given in the context of the Fedosov’s deformation quantization [4].

\footnote{For a recent review, see [10].}
The aim of this article is to give explicit expressions of deformation quantizations with separation of variables for $\mathbb{C}P^N$ and $\mathbb{C}H^N$. To construct star products, we have to solve the infinite system of differential equations. In these cases, as will be shown, differential equation systems are solvable, and expressions of star products are explicitly given in all order of $\hbar$. A noncommutative deformation of $\mathbb{C}P^N$ was investigated by performing the phase space reduction in [19]. We will comment on the connection between their star product and our result.

We also give the Fock representations of noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$. The Fock representations of star products are also used in investigations of field theories on noncommutative spaces. In particular, the Fock representations give useful methods of constructing solitons and instantons in noncommutative field theories. Further, matrix models corresponding to noncommutative field theories can be obtained from the Fock representations, and quantum analyses of the models are actively pursued.

The organization of this article is as follows. In Section 2, we review the deformation quantization with separation of variables proposed by Karabegov. In Section 3, a star product for $\mathbb{C}P^N$ is given explicitly by using the deformation quantization with separation of variables. In Section 4, we give the Fock representation of the star product obtained in Section 3. In Section 5, a star product for $\mathbb{C}H^N$ is constructed explicitly by the similar way to the one in $\mathbb{C}P^N$. Finally, we summarize our results and discuss their several perspectives in Section 6.

2 Review of the deformation quantization with separation of variables

In this section, we review the deformation quantization with separation of variables to construct noncommutative Kähler manifolds.

An $N$-dimensional complex Kähler manifolds is defined by using a Kähler potential. Let $\Phi$ be a Kähler potential and $\omega$ be a Kähler 2-form:

$$\omega := ig_{kl}dz^k \wedge d\bar{z}^l,$$

$$g_{kl} := \frac{\partial^2 \Phi}{\partial z^k \partial \bar{z}^l}. \quad (2.1)$$

In this paper, we use the Einstein summation convention over repeated indices. The $g^{kl}$ is the inverse of the metric $g_{kl}$:

$$g^{kl}g_{lm} = \delta_{km}. \quad (2.2)$$

---

2 Star products on the fuzzy $\mathbb{C}P^N$ are investigated in [14, 15, 16]. A deformation quantization of the hyperbolic plane was provided in [17].
In the following, we denote
\[ \frac{\partial_k}{\partial z^k}, \frac{\partial \bar{k}}{\partial \bar{z}^k}. \]  
(2.3)

Deformation quantization is defined as follows.

**Definition 1** (Deformation quantization (weak sense)). Deformation quantization is defined as follows. \( \mathcal{F} \) is defined as a set of formal power series:
\[
\mathcal{F} := \left\{ f \mid f = \sum_k f_k \hbar^k, \; f_k \in C^\infty \right\}. 
\]  
(2.4)

A star product is defined as
\[
 f \ast g = \sum_k C_k(f, g) \hbar^k 
\]  
(2.5)
such that the product satisfies the following conditions.

1. \( \ast \) is associative product.
2. \( C_k \) is a bidifferential operator.
3. \( C_0 \) and \( C_1 \) are defined as
   \[ C_0(f, g) = fg, \]  
   \[ C_1(f, g) - C_1(g, f) = i\{f, g\}, \]  
   where \( \{f, g\} \) is the Poisson bracket.
4. \( f \ast 1 = 1 \ast f = f \).

Note that this definition of the deformation quantization is weaker than the usual definition of deformation quantization. The difference between them is in (2.7). In the strong sense of deformation quantization the condition \( C_1(f, g) = \frac{i}{2}\{f, g\} \) is required. For example, the Moyal product satisfies this condition. But deformation quantizations with the separation of variables do not satisfy this condition. In the following, “deformation quantization” is used in this weak sense.

**Definition 2** (A star product with separation of variables). \( \ast \) is called a star product with separation of variables when
\[ a \ast f = af \]  
(2.8)
for a holomorphic function \( a \) and
\[ f \ast b = fb \]  
(2.9)
for an anti-holomorphic function \( b \).
We use
\[ D^l = g^{lk} \partial_k = i\{z^l, \cdot\} \]
and
\[ S := \{ A | A = \sum_\alpha a_\alpha D^\alpha, \ a_\alpha \in C^\infty \}, \]
where \( \alpha \) is a multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \).

There are some useful formulae. \( D^l \) satisfies the following equations.
\[ [D^l, D^m] = 0, \ \forall l, m \quad (2.10) \]
\[ [D^l, \partial_m \Phi] = \delta^l_m \quad (2.11) \]
\[ \partial_k = g_{ki} D^i. \quad (2.12) \]

Using them, one can construct a star product as differential operator \( A_f \) such that \( f \ast g = A_f g \).

**Theorem 2.1.** For arbitrary \( \omega \), there exist a star product with separation of variables \( \ast \) and it is constructed as follows. Let \( f \) be an element of \( \mathcal{F} \) and \( A_n \in S \) be a differential operator whose coefficients depend on \( f \) i.e.
\[
A_n = a_{n,\alpha}(f)D^\alpha, \ D^\alpha = \prod_{i=1}^n (D^i)^{\alpha_i}, \ (D^\alpha) = g^{il} \partial_l,
\]
where \( \alpha \) is an multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \). Then,
\[
\tilde{A}_f = \sum_{n=0}^\infty \hbar^n A_n \quad (2.14)
\]
is uniquely determined such that it satisfies the following conditions.
1. When \( R_{\partial_l \Phi} = \partial_l \Phi + \hbar \partial_l \),
\[
[\tilde{A}_f, R_{\partial_l \Phi}] = 0 . \quad (2.15)
\]
2. \[
\begin{align*}
\tilde{A}_f 1 &= f \ast 1 = f, \\
\tilde{A}_fg &= f \ast g, \\
\tilde{A}_h(\tilde{A}_gf) &= h \ast (g \ast f) = (h \ast g) \ast f = \tilde{A}_h g f.
\end{align*}
\]
Recall that each two of $D^i$ commute each other, so if multi index $\alpha$ is fixed then the $A_n$ is uniquely determined. These conditions (2.16)-(2.18) teach us that $\tilde{A}_f g = f * g$ is deformation quantization.

The following proposition is used in Section 3 and Section 5.

**Proposition 2.2.** We denote a left operation for a generic function $f$ as $L_f := \tilde{A}_f$ i.e. $L_f g = f * g$. The right operation for $f$ is defined similarly by $R_f g := g * f$. $L_f$ ($R_f$) is obtained by using $L_{z^i}$ ($R_{z^i}$) where $L_{z^i}$ ($R_{z^i}$) is defined by $L_{z^i} g = \tilde{z}^i * g$ ($R_{z^i} g = g * z^i$):

\[
L_f = \sum_{\alpha} \frac{1}{\alpha!} \left( \frac{\partial}{\partial z} \right)^\alpha f \left( L_z - \tilde{z} \right)^\alpha, \tag{2.19}
\]

\[
R_f = \sum_{\alpha} \frac{1}{\alpha!} \left( \frac{\partial}{\partial z} \right)^\alpha f \left( R_z - z \right)^\alpha. \tag{2.20}
\]

### 3 Star product with separation of variables on $\mathbb{CP}^N$

In the inhomogeneous coordinates $z^i$ ($i = 1, 2, \cdots, N$), the Kähler potential of $\mathbb{CP}^N$ is given by

\[
\Phi = \ln \left( 1 + |z|^2 \right), \tag{3.1}
\]

where $|z|^2 = \sum_{k=1}^{N} z^k \bar{z}^k$. The metric $(g_{ij})$ is

\[
ds^2 = 2g_{\bar{i}j} dz^i dz^{\bar{j}}, \tag{3.2}
\]

\[
g_{ij} = \partial_i \partial_j \Phi = \frac{(1 + |z|^2) \delta_{ij} - z^i \bar{z}^j}{(1 + |z|^2)^2}, \tag{3.3}
\]

and the inverse of the metric $(g^{\bar{i}j})$ is

\[
g^{\bar{i}j} = (1 + |z|^2) \left( \delta_{ij} + z^i \bar{z}^j \right). \tag{3.4}
\]

We here summarize useful relations in the following calculations;

\[
\partial_{i_1} \partial_{i_2} \cdots \partial_{i_n} \Phi = (-1)^{n-1}(n-1)! \partial_{i_1} \Phi \partial_{i_2} \Phi \cdots \partial_{i_n} \Phi, \tag{3.5}
\]

\[
\left[ \partial_{i_1}, D^{\bar{j}} \right] = \partial_{i_1} D^{\bar{j}} + \delta_{i_1} \Phi D^{\bar{k}}, \tag{3.6}
\]

\[
\left[ \partial_{i_1}, c_{j_1j_2} \cdots \bar{j}_n \right] D^{\bar{j}_1} D^{\bar{j}_2} \cdots D^{\bar{j}_n} = \partial_{i_1} c_{j_1j_2} \cdots \bar{j}_n D^{\bar{j}_1} D^{\bar{j}_2} \cdots D^{\bar{j}_n} + n c_{j_1j_2} \cdots \bar{j}_n \partial_{i_1} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_n} + n c_{j_1j_2} \cdots \bar{j}_{n-1} \partial_{i_1} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{n-1}}, \tag{3.7}
\]

\[
\text{where } c_{j_1j_2} \cdots j_n = -\frac{1}{N!} \delta^{\bar{j}_1}_{j_1} \delta^{\bar{j}_2}_{j_2} \cdots \delta^{\bar{j}_n}_{j_n}.
\]
where the coefficients \( c_{j_1j_2...j_n} \) are totally symmetric under the permutations of the indices.

We construct the operator \( L_{zl} \), which is corresponding to the left star product by \( \bar{z}^l \). \( L_{zl} \) is defined as a power series of \( \hbar \),

\[
L_{zl} = z^l + \hbar D^l + \sum_{n=2}^{\infty} \hbar^n A_n, \tag{3.8}
\]

where \( A_n (n \geq 2) \) is a formal series of the differential operators \( D^k \). We assume that \( A_n \) has the following form,

\[
A_n = \sum_{m=2}^{n} a_m^{(n)} \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l, \tag{3.9}
\]

where the coefficients \( a_m^{(n)} \) do not depend on \( z^l \) and \( \bar{z}^l \).

From the requirement of \( [L_{zl}, \partial_{l} \Phi + \hbar \partial_{l}] = 0 \), the operators \( A_n \) are recursively determined by the equations

\[
[A_n, \partial_{l} \Phi] = [\partial_{l}, A_{n-1}], \quad (n \geq 2) \tag{3.10}
\]

where \( A_1 = D^l \), \( A_2 = \partial_{l} \Phi D^{j_1} D^l \) is easily obtained from the above equation. Using the expression (3.9), the left hand side of the recursion relation (3.10) becomes

\[
[A_n, \partial_{l} \Phi] = \sum_{m=2}^{n} a_m^{(n)} \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi \left[ D^{j_1} \cdots D^{j_{m-1}} D^l, \partial_{l} \Phi \right]
\]

\[
= \sum_{m=2}^{n-1} a_m^{(n)} \left\{ m \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi \partial_{l} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l + \delta_{il} \partial_{j_1} \Phi \partial_{j_2} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l \right\}
\]

\[
+ a_2^{(n)} \left( \partial_{l} \Phi D^l + \delta_{jl} \partial_{l} \Phi D^j \right). \tag{3.11}
\]

On the other hand, the right hand side of (3.10) is calculated as

\[
[\partial_{l}, A_{n-1}] = \sum_{m=2}^{n-1} a_m^{(n-1)} \left[ \partial_{l}, \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l \right]
\]

\[
= \sum_{m=2}^{n-1} \left( a_m^{(n-1)} + ma_m^{(n-1)} \right)
\]

\[
\times \left( m \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi \partial_{l} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l + \delta_{il} \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l \right)
\]

\[
+ a_2^{(n-1)} \left( \partial_{l} \Phi D^l + \delta_{jl} \partial_{l} \Phi D^j \right). \tag{3.12}
\]
Equating (3.11) with (3.12), we find
\[ a_2^{(n)} = a_2^{(n-1)} = \cdots = a_2^{(2)} = 1, \]  
(3.13)
and the following recursion relation
\[ a_m^{(n)} = a_{m-1}^{(n-1)} + (m-1)a_m^{(n-1)}. \]  
(3.14)
To solve this equation, we introduce a generating function
\[ \alpha_m(t) \equiv \sum_{n=m}^{\infty} t^n a_m^{(n)}, \]  
(3.15)
for \( m \geq 2 \). Then the relation (3.14) is written as
\[ \alpha_m(t) = t[\alpha_{m-1}(t) + (m-1)\alpha_m(t)]. \]  
(3.16)
This is solved as
\[ \alpha_m(t) = \frac{t}{1 - (m-1)t}\alpha_{m-1}(t) \\
= t^{m-2} \prod_{n=2}^{m-1} \frac{1}{1-nt} \alpha_2(t). \]  
(3.17)
Since \( \alpha_2(t) \) is easily calculated from (3.13) as
\[ \alpha_2(t) = \sum_{n=2}^{\infty} t^n a_2^{(n)} = \sum_{n=2}^{\infty} t^n = \frac{t^2}{1-t}, \]  
(3.18)
\( \alpha_m(t) \) is determined as
\[ \alpha_m(t) = t^m \prod_{n=1}^{m-1} \frac{1}{1-nt} = \frac{\Gamma(1 - m + \frac{1}{t})}{\Gamma(1 + \frac{1}{t})}, \]  
(3.19)
The function \( \alpha_m(t) \) actually coincides with the generating function for the Stirling numbers of the second kind \( S(n,k) \), and \( a_m^{(n)} \) is related to \( S(n,k) \) as
\[ a_m^{(n)} = S(n-1,m-1). \]  
(3.20)

Summarizing the above calculations, \( L_{z^I} \) becomes
\[ L_{z^I} = z^l + hD^l + \sum_{n=2}^{\infty} h^n \sum_{m=2}^{n} a_m^{(n)} \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D_{j_1} \cdots D_{j_{m-1}} D^l \\
= z^l + hD^l + \sum_{m=2}^{\infty} \left( \sum_{n=m}^{\infty} h^n a_m^{(n)} \right) \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D_{j_1} \cdots D_{j_{m-1}} D^l \\
= z^l + \sum_{m=1}^{\infty} \alpha_m(h) \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D_{j_1} \cdots D_{j_{m-1}} D^l. \]  
(3.21)
Here we defined $\alpha_1(t) = t$. Similarly, it can be shown that the right star product by $z^i$, $R_z f = f \ast z^i$ is expressed as

\[ R_z f = z^i + \sum_{m=1}^{\infty} \alpha_m(h) \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^i, \]

(3.22)

where $D^i = g^{\bar{j}i} \partial_{\bar{j}}$.

From the theorem 2.1 proposition 2.2 (3.21) and (3.22), we obtain the following theorem.

**Theorem 3.1.** A star product with separation of variables for $\mathbb{C}P^N$ with the Kähler potential $\Phi = \ln \left(1 + |z|^2\right)$ is given by

\[ f \ast g = L_f g = R_g f. \]

(3.23)

Here differential operators $L_f$ and $R_g$ are determined by the differential operators $L_{\bar{z}}$ and $R_{\bar{z}}$ whose expressions are given in (3.21) and (3.22) through the relation (2.19) and (2.20), respectively.

We can now calculate the star products among $z^i$ and $\bar{z}^i$,

\[ z^i \ast z^j = z^i z^j, \]
\[ z^i \ast \bar{z}^j = z^i \bar{z}^j, \]
\[ \bar{z}^i \ast z^j = \bar{z}^i z^j, \]
\[ \bar{z}^i \ast \bar{z}^j = \bar{z}^i \bar{z}^j + \frac{\hbar}{1 - \hbar} \bar{z}^i \bar{z}^j \left(1 + |z|^2\right)_{2F1} \left(1, 1; 1 - 1/\hbar; -|z|^2\right), \]

(3.24) (3.25) (3.26) (3.27)

where $2F1$ is the Gauss hypergeometric function. Here we used the following equation

\[ D^{j_1} \cdots D^{j_m} z^i = (m - 1)! (1 + |z|^2)^m \]
\[ \times \left[ \sum_{k=1}^{m} \delta_{i j_k} \bar{z}^{j_1} \cdots \bar{z}^{j_{k-1}} z^{j_{k+1}} \cdots z^{j_m} + m z^i \bar{z}^{j_1} \cdots \bar{z}^{j_m} \right], \]

(3.28)

where the hat over a term means that it is to be omitted from the product.

There are several ways of having deformation quantization by a reduction from higher dimensional manifolds. Within the framework of Karabegov’s method for
Kähler manifolds, a general reduction procedure of deformation quantizations with separation of variables was considered in [18]. When the standard star product with separation of variables corresponding to a Kähler potential \( \rho = \psi(z, \bar{z})u\bar{u} \), where \( z \) and \( u \) are holomorphic coordinates, is given, the reduction procedure eliminates the variables \( u, \bar{u} \) and produces the standard star product with separation of variables corresponding to the Kähler potential \( \ln|\psi| \). This reduction procedure can be applied to the case of the reduction from \( \mathbb{C}^{N+1}\setminus\{0\} \) to \( \mathbb{C}P^N \) and gives the same star product as the one used in this article. On the other hand, in [19], a star product on \( \mathbb{C}P^N \) was constructed by performing the phase space reduction from \( \mathbb{C}^{N+1}\setminus\{0\} \). The expression of their star product, denoted as \( *_B \), for functions \( f \) and \( g \) on \( \mathbb{C}P^N \) is given

\[
 f *_B g = f g + \sum_{m=1}^{\infty} \hbar^m \sum_{s=1}^{m} \sum_{k=1}^{s} \frac{k^{m-1}(-1)^{m-k}}{s!(s-k)!(k-1)!} \left( |\zeta|^2 \right)^s \frac{\partial^s f}{\partial \zeta^{A_1} \cdots \partial \zeta^{A_s}} \frac{\partial^s g}{\partial \bar{\zeta}^{\bar{A}_1} \cdots \partial \bar{\zeta}^{\bar{A}_s}}, \tag{3.29}
\]

where \( \zeta^{A_i}, \bar{\zeta}^{\bar{A}_j} \) are the homogeneous coordinates. This is also the star product with separation of variables, and thus (3.24)-(3.26) hold trivially under \( *_B \) product. \( \bar{z}^i *_B z^j \) is calculated as

\[
 \bar{z}^i *_B z^j = \bar{z}^i z^j + \hbar \delta_{ij}(1 + |z|^2) \tilde{F}_1(\bar{|z}|^2) + \hbar \bar{z}^i z^j(1 + |z|^2) \tilde{F}_2(|z|^2), \tag{3.30}
\]

where \( z^i = \zeta^i/\zeta^0, \bar{z}^i = \bar{\zeta}^i/\bar{\zeta}^0, \) and

\[
 \tilde{F}_1(\bar{|z}|^2) \equiv \sum_{m=0}^{\infty} \sum_{s=0}^{m} \sum_{k=1}^{s+1} \frac{\hbar^m s! k^m (-1)^{m+1-k}}{(s+1-k)!(k-1)!} (1 + |\zeta|^2)^s, \tag{3.31}
\]

\[
 \tilde{F}_2(|z|^2) \equiv \sum_{m=0}^{\infty} \sum_{s=0}^{m} \sum_{k=1}^{s+1} \frac{\hbar^m (s+1)! k^m (-1)^{m+1-k}}{(s+1-k)!(k-1)!} (1 + |\zeta|^2)^s. \tag{3.32}
\]

We can show that \( \tilde{F}_1(\bar{|z}|^2) \) satisfies the hypergeometric equation and the boundary conditions for \( 2F_1(1,1;1-1/\hbar;\bar{|z}|^2) \), and thus \( \tilde{F}_1(\bar{|z}|^2) = 2F_1(1,1;1-1/\hbar;\bar{|z}|^2) \). Similarly, \( \tilde{F}_2(|z|^2) = 2F_1(1,2;2-1/\hbar;|z|^2)/(1-\hbar) \) can be also shown. Therefore it turns out \( \bar{z}^i *_B z^j = \bar{z}^i *_B z^j \). These facts lead to \( f * g = f *_B g \). Namely, this calculation shows that the star product constructed by Karabegov’s method coincides with the star product \( *_B \) in [19]. As far as we know, the origin of this coincidence of the star products obtained by these different methods is not apparent at this time.
4 Fock representation

The left star product by $\partial_i \Phi$ and the right star product by $\partial^* \Phi$ are respectively written as

$$L_{\partial_i \Phi} = \hbar \partial_i + \partial_i \Phi = \hbar e^{-\Phi/\hbar} \partial_i e^{\Phi/\hbar},$$  \hspace{0.1cm} (4.1)

$$R_{\partial^* \Phi} = \hbar \partial^* + \partial^* \Phi = \hbar e^{-\Phi/\hbar} \partial^* e^{\Phi/\hbar}.$$  \hspace{0.1cm} (4.2)

From the definition of the star product given in the previous section, we easily find

$$\partial_i \Phi \ast z^j - z^j \ast \partial_i \Phi = \hbar \delta_{ij}, \quad z^i \ast z^i - z^i \ast z^i = 0, \quad \partial_i \Phi \ast \partial_j \Phi - \partial_j \Phi \ast \partial_i \Phi = 0,$$  \hspace{0.1cm} (4.3)

$$\bar{z}^i \ast \partial_j \Phi - \partial_j \Phi \ast \bar{z}^i = \hbar \delta_{ij}, \quad \bar{z}^i \ast \bar{z}^i - \bar{z}^i \ast \bar{z}^i = 0, \quad \partial_i \Phi \ast \partial_j \Phi - \partial_j \Phi \ast \partial_i \Phi = 0.$$  \hspace{0.1cm} (4.4)

Hence, $\{z^i, \partial_j \Phi \mid i, j = 1, 2, \cdots, N\}$ and $\{\bar{z}^i, \partial_j \Phi \mid i, j = 1, 2, \cdots, N\}$ constitute $2N$ sets of the creation-annihilation operators under the star product. But, it is noted that operators in $\{z^i, \partial_j \Phi\}$ does not commute with ones in $\{\bar{z}^i, \partial_j \Phi\}$, e.g., $z^i \ast \bar{z}^j - \bar{z}^j \ast z^i \neq 0$.

Here, we would like to construct the Fock representation of the star product. First we show that $e^{-\Phi/\hbar} = (1 + |z|^2)^{-1/\hbar}$ is the vacuum projection. $e^{-\Phi/\hbar}$ is annihilated by the left star product of $\partial_i \Phi$ and $z^i$,

$$\partial_i \Phi \ast e^{-\Phi/\hbar} = L_{\partial_i \Phi} e^{-\Phi/\hbar} = \hbar e^{-\Phi/\hbar} \partial_i e^{\Phi/\hbar} e^{-\Phi/\hbar} = 0,$$  \hspace{0.1cm} (4.5)

$$\bar{z}^i \ast e^{-\Phi/\hbar} = L_{\bar{z}^i} e^{-\Phi/\hbar} = \left( \bar{z}^i + \sum_{m=1}^{\infty} \alpha_m(h) \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{\bar{j}_1} \cdots D^{\bar{j}_{m-1}} \right) e^{-\Phi/\hbar} = 0.$$  \hspace{0.1cm} (4.6)

Here the following equation is used

$$D^{\bar{j}_1} \cdots D^{\bar{j}_m} e^{-\Phi/\hbar} = D^{\bar{j}_1} \cdots D^{\bar{j}_m} (1 + |z|^2)^{-1/\hbar}$$

$$= (-1)^m \frac{\Gamma(1 + 1/\hbar)}{\Gamma(1 - m + 1/\hbar)} z^{j_1} \cdots z^{j_{m-1}} (1 + |z|^2)^{-m-1/\hbar}.$$  \hspace{0.1cm} (4.7)

Similarly, it is shown that $e^{-\Phi/\hbar}$ is annihilated by the right star product of the $\partial_i \Phi$ and $z^i$,

$$e^{-\Phi/\hbar} \ast \partial_i \Phi = e^{-\Phi/\hbar} \ast z^i = 0.$$  \hspace{0.1cm} (4.8)

Next, we show that $e^{-\Phi/\hbar}$ satisfies the relation

$$e^{-\Phi/\hbar} \ast f(z, \bar{z}) = e^{-\Phi/\hbar} f(0, \bar{z})$$  \hspace{0.1cm} (4.9)
for a function $f(z, \bar{z})$ such that $f(z, \bar{w})$ can be expanded as Taylor series with respect to $z^i$ and $\bar{w}^j$, respectively. To show the relation, we note that the differential operator $R_{z^i}$ corresponding to the right product of $z^i$ contains only partial derivatives by $\bar{z}^j$, and thus commutes with $z^k$. Moreover, $R_{z^i}e^{-\Phi/h} = e^{-\Phi/h} \ast z^i = 0$ as mentioned above. From these, the relation (4.9) is shown as

$$e^{-\Phi/h} \ast f(z, \bar{z}) = R_f e^{-\Phi/h}$$

\[= \sum_{k_1, \ldots, k_N=0}^{\infty} \frac{1}{k_1! \cdots k_N!} \partial_{k_1} \cdots \partial_{k_N} f(z, \bar{z}) \prod_{m=1}^{N} (R_{z^m} - z^m)^{\ast k_m} e^{-\Phi/h} \]

\[= \sum_{k_1, \ldots, k_N=0}^{\infty} \frac{1}{k_1! \cdots k_N!} \partial_{k_1} \cdots \partial_{k_N} f(z, \bar{z}) \prod_{m=1}^{N} (-z^m)^{\ast k_m} e^{-\Phi/h} \]

\[= e^{-\Phi/h} f(0, \bar{z}). \quad (4.10)\]

Similarly, the following equation holds

$$f(z, \bar{z}) \ast e^{-\Phi/h} = f(z, 0) e^{-\Phi/h}. \quad (4.11)$$

As a specific case of the equation (4.9), the idempotency of $e^{-\Phi/h}$ is obtained,

$$e^{-\Phi(z,\bar{z})/h} \ast e^{-\Phi(z,\bar{z})/h} = e^{-\Phi(z,\bar{z})/h} e^{-\Phi(0,\bar{z})/h} = e^{-\Phi(z,\bar{z})/h}, \quad (4.12)$$

where $\Phi(0, \bar{z}) = 0$ is used.

By using the relations (4.9) and (4.11), it is possible to calculate explicitly star products containing $e^{-\Phi/h}$ as follows,

$$e^{-\Phi/h} \ast (\partial_{i_1} \Phi(z, \bar{z}) \cdots \partial_{i_n} \Phi(z, \bar{z})) = e^{-\Phi/h} (\partial_{i_1} \Phi(0, \bar{z}) \cdots \partial_{i_n} \Phi(0, \bar{z}))$$

\[= \bar{z}^{i_1} \cdots \bar{z}^{i_n} e^{-\Phi/h} \]

\[= e^{-\Phi/h} \ast z^{i_1} \cdots z^{i_n}, \quad (4.13)\]

$$\left(\partial_{i_1} \Phi(z, \bar{z}) \cdots \partial_{i_n} \Phi(z, \bar{z})\right) \ast e^{-\Phi/h} = z^{i_1} \cdots z^{i_n} e^{-\Phi/h}$$

\[= z^{i_1} \cdots z^{i_n} \ast e^{-\Phi/h}. \quad (4.14)\]

We then consider a class of functions

$$M_{i_1 \cdots i_m; j_1 \cdots j_n} = \frac{z^{i_1} \cdots z^{i_m} \bar{z}^{j_1} \cdots \bar{z}^{j_n}}{\sqrt{m! n! \alpha_m(h) \alpha_n(h)}} e^{-\Phi/h}, \quad (4.15)$$
where \( \alpha_n(h) \) is defined in (3.19). \( M_{i_1 \cdots i_m; j_1 \cdots j_n} \) is totally symmetric under permutations of \( i \)'s and \( j \)'s, respectively. It is also useful to represent this function as

\[
M_{i_1 \cdots i_m; j_1 \cdots j_n} = \frac{1}{\sqrt{m!n!\alpha_m(h)\alpha_n(h)}} z^{i_1} \cdots z^{i_m} e^{-\Phi/h} \ast (\partial_{j_1} \Phi \cdots \partial_{j_n} \Phi)
\]

where \( \delta \) by the star product in Theorem 3.1, and its algebra is given by (4.18).

At last, the following theorem is obtained.

**Theorem 4.1.** Let \( \mathcal{M} = \left\{ 1, \sum_{i,j} a_{ij} M_{i;j} \right\} \) be a set of linear combinations of \( M_{i_1 \cdots i_m; j_1 \cdots j_n} \) defined by (4.17) in \( \mathbb{C}P^N \), where \( i, j \) are multi-index of \( i = (i_1, \cdots, i_m) \) and \( j = (j_1, \cdots, j_n) \) and \( a_{ij} \in \mathbb{C} \). Then \( \mathcal{M} \) is a ring whose multiplication is defined by the star product in Theorem 3.1 and its algebra is given by (4.18).
Further, the star products between $M_{i_1 \cdots i_m; j_1 \cdots j_n}$ and one of $z^k, \partial_k \Phi, \bar{z}^k$ and $\bar{\partial}_k \Phi$ are calculated as follows,

$$z^k * M_{i_1 \cdots i_m; j_1 \cdots j_n} = \sqrt{\frac{m + 1}{-m + 1 + 1/\hbar}} M_{ki_1 \cdots k_i m; j_1 \cdots j_n},$$

$$\partial_k \Phi * M_{i_1 \cdots i_m; j_1 \cdots j_n} = \hbar \sqrt{-m + 1 + 1/\hbar} \sum_{l=1}^{m} \delta_{kl} M_{i_1 \cdots i_m; j_1 \cdots j_n},$$

$$\bar{z}^k * M_{i_1 \cdots i_m; j_1 \cdots j_n} = \frac{1}{\sqrt{m(-m + 1 + 1/\hbar)}} \sum_{l=1}^{m} \delta_{kl} M_{i_1 \cdots i_m; j_1 \cdots j_n},$$

$$\bar{\partial}_k \Phi * M_{i_1 \cdots i_m; j_1 \cdots j_n} = \hbar \sqrt{(m + 1)(-m + 1 + 1/\hbar)} M_{ki_1 \cdots k_i m; j_1 \cdots j_n},$$

$$M_{i_1 \cdots i_m; j_1 \cdots j_n} * z^k = \sqrt{\frac{n + 1}{-n + 1 + 1/\hbar}} \sum_{l=1}^{n} \delta_{kl} M_{i_1 \cdots i_m; j_1 \cdots j_n},$$

$$M_{i_1 \cdots i_m; j_1 \cdots j_n} * \partial_k \Phi = \hbar \sqrt{(n + 1)(-n + 1 + 1/\hbar)} M_{i_1 \cdots i_m; j_1 \cdots j_n},$$

$$M_{i_1 \cdots i_m; j_1 \cdots j_n} * \bar{z}^k = \sqrt{\frac{n + 1}{-n + 1 + 1/\hbar}} M_{i_1 \cdots i_m; j_1 \cdots j_n},$$

$$M_{i_1 \cdots i_m; j_1 \cdots j_n} * \bar{\partial}_k \Phi = \hbar \sqrt{-n + 1 + 1/\hbar} \sum_{l=1}^{n} \delta_{kl} M_{i_1 \cdots i_m; j_1 \cdots j_n}.$$

## 5 The case of $\mathbb{C}H^N$

The Kähler potential of $\mathbb{C}H^N$ is given by

$$\Phi = -\ln \left(1 - |z|^2 \right).$$

The metric $g_{i\bar{j}}$ and the inverse metric $g^{i\bar{j}}$ are defined by

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{(1 - |z|^2) \delta_{i\bar{j}} + \bar{z}^i z^j}{(1 - |z|^2)^2},$$

$$g^{i\bar{j}} = (1 - |z|^2) \left( \delta_{i\bar{j}} - \bar{z}^i z^j \right).$$
Then we find the following relations similar to (3.5)–(3.7):

\[ \partial_{i_1} \partial_{i_2} \cdots \partial_{i_n} \Phi = (n - 1)! \partial_{i_1} \Phi \partial_{i_2} \Phi \cdots \partial_{i_n} \Phi, \]

\[ [\partial_i, D^j] = - \partial_i \Phi D^j - \delta_{ij} \partial_k \Phi D^k, \]

\[ [\partial_i, c_{j_1 j_2 \cdots j_n} \bar{D}^{j_1} \bar{D}^{j_2} \cdots \bar{D}^{j_n}] = \partial_i c_{j_1 j_2 \cdots j_n} \bar{D}^{j_1} \bar{D}^{j_2} \cdots \bar{D}^{j_n} - nc_{j_1 \cdots j_n} \partial_i \Phi \bar{D}^{j_1} \cdots \bar{D}^{j_n} - n(n - 1)c_{j_1 \cdots j_{n-1}} \partial_i \Phi D^{j_1} \cdots \bar{D}^{j_{n-1}}. \]

The operator \( L_{z^l} \) is expanded as a power series of the noncommutative parameter \( h \),

\[ L_{z^l} = z^l + hD^l + \sum_{n=2}^{\infty} h^n B_n. \]

We assume that \( B_n \) has the following form,

\[ B_n = \sum_{m=2}^{n} (-1)^{n-1} b_m^{(n)} \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l. \]

The factor \((-1)^{n-1}\) in the front of the coefficient \( b_m^{(n)} \) is introduced for convenience.

Requiring \([L_{z^l}, \partial_i \Phi + h \partial_l] = 0\), it is found that \( b_m^{(n)} \) should satisfy similar relations to (3.13) and (3.14),

\[ b_2^{(n)} = b_2^{(n-1)} = \cdots = b_2^{(2)} = 1, \]

\[ b_m^{(n)} = b_{m-1}^{(n-1)} + (m - 1)b_m^{(n-1)}. \]

Hence \( b_m^{(n)} \) coincides with \( a_m^{(n)} \), and we obtain the explicit representation of the star product with separation of variables on \( \mathbb{C}H^N \),

\[ L_{z^l} = z^l + hD^l + \sum_{n=2}^{\infty} h^n \sum_{m=2}^{n} (-1)^{n-1} b_m^{(n)} \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l \]

\[ = z^l + \sum_{m=1}^{\infty} (-1)^{m-1} \beta_m(h) \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l, \]

\[ R_{z^l} = z^l + hD^l + \sum_{n=2}^{\infty} h^n \sum_{m=2}^{n} (-1)^{n-1} b_m^{(n)} \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l \]

\[ = z^l + \sum_{m=1}^{\infty} (-1)^{m-1} \beta_m(h) \partial_{j_1} \Phi \cdots \partial_{j_{m-1}} \Phi D^{j_1} \cdots D^{j_{m-1}} D^l, \]
with
\[ \beta_n(t) = (-1)^n \alpha_n(-t) = \frac{\Gamma(1/t)}{\Gamma(n + 1/t)}. \] (5.12)

From the theorem 2.1 and proposition 2.2, we obtain the following theorem.

**Theorem 5.1.** A star product with separation of variables for \( \mathbb{C}H^N \) with the Kähler potential \( \Phi = -\ln(1 - |z|^2) \) is given by
\[ f * g = L_f g = R_g f. \] (5.13)

Here differential operators \( L_f \) and \( R_g \) are determined through the relation (2.19) and (2.20), respectively, by the differential operators \( L_{\bar{z}} \) and \( R_z \) whose expressions are given in (5.10) and (5.11).

Using the representations of the star product, we can calculate the star products among \( z^i \) and \( \bar{z}^i \),
\[ z^i * z^j = z^i z^j, \] (5.14)
\[ z^i * \bar{z}^j = \bar{z}^i z^j, \] (5.15)
\[ \bar{z}^i * \bar{z}^j = \bar{z}^i \bar{z}^j, \] (5.16)
\[ \bar{z}^i * z^j = \bar{z}^i z^j + \hbar \delta_{ij}(1 - |z|^2)_2 F_1 \left(1, 1; 1 + 1/\hbar; |z|^2\right) \]
\[ - \frac{\hbar}{1 + \hbar} \bar{z}^i \bar{z}^j (1 - |z|^2)_2 F_1 \left(1, 2; 2 + 1/\hbar; |z|^2\right). \] (5.17)

Here the following equation similar to (3.28) is used,
\[ D_{\bar{z}^j} \ldots D_{\bar{z}^m} z^i = (-)^{m-1}(m-1)! (1 - |z|^2)^m \]
\[ \times \left[ \sum_{k=1}^{m} \delta_{ij_k} \bar{z}^j_k \ldots \bar{z}^j_m - m \bar{z}^i \bar{z}^j \ldots \bar{z}^{jm} \right]. \] (5.18)

As in the case of \( \mathbb{C}P^N \), \( \{ z^i, \partial_j \Phi \} \) and \( \{ \bar{z}^i, \partial_j \Phi \} \) satisfy the commutation relations for the creation-annihilation operators. Also \( e^{-\Phi/\hbar} \) is the vacuum projection operator,
\[ \partial_i \Phi * e^{-\Phi/\hbar} = 0, \] (5.19)
\[ \bar{z}^i * e^{-\Phi/\hbar} = 0, \] (5.20)
\[ e^{-\Phi/\hbar} * \partial_i \Phi = 0, \] (5.21)
\[ e^{-\Phi/\hbar} * z^i = 0. \] (5.22)
\[ e^{-\Phi/h} \ast e^{-\Phi/h} = e^{-\Phi/h}. \] (5.23)

Here we used the following relation which is corresponding to (4.7) in the case of \( \mathbb{C}P^N \),
\[ D^{\bar{j}_1} \ldots D^{\bar{j}_m} e^{-\Phi/h} = D^{\bar{j}_1} \ldots D^{\bar{j}_m} (1 - |z|^2)^{1/h} \]
\[ = (-1)^m \frac{\Gamma(1/h + m)}{\Gamma(1/h)} z^{\bar{j}_1} \ldots z^{\bar{j}_m} (1 - |z|^2)^{1/h + m}. \] (5.24)

As in the case of \( \mathbb{C}P^N \), we consider a class of functions
\[ N_{i_1 \ldots i_m;j_1 \ldots j_n} = \frac{z^{i_1} \ldots z^{i_m} z^{\bar{j}_1} \ldots z^{\bar{j}_n}}{\sqrt{m!n!\beta_m(h)\beta_n(h)}} e^{-\Phi/h} \]
\[ = \frac{1}{\sqrt{m!n!\beta_m(h)\beta_n(h)}} z^{i_1} \ast \ldots \ast z^{i_m} \ast e^{-\Phi/h} \ast (\partial_{j_1} \Phi \ldots \partial_{j_n} \Phi) \]
\[ = \frac{1}{\sqrt{m!n!\beta_m(h)\beta_n(h)}} \left( (\partial_{i_1} \Phi \ldots \partial_{i_m} \Phi) \ast e^{-\Phi/h} \ast z^{\bar{j}_1} \ast \ldots \ast z^{\bar{j}_n} \right) \]
\[ = \frac{1}{\sqrt{m!n!\beta_m(h)\beta_n(h)}} (\partial_{i_1} \Phi \ast \ldots \ast \partial_{i_m} \Phi) \ast e^{-\Phi/h} \ast z^{\bar{j}_1} \ast \ldots \ast z^{\bar{j}_n}. \] (5.25)

\( N_{i_1 \ldots i_m;j_1 \ldots j_n} \) is totally symmetric under permutations of \( i \)'s and \( j \)'s, respectively. Then we can show that these functions form a closed algebra
\[ N_{i_1 \ldots i_m;j_1 \ldots j_n} \ast N_{k_1 \ldots k_r;l_1 \ldots l_s} = \delta_{m}^{k_1 \ldots k_r} \delta_{n}^{l_1 \ldots l_s} N_{i_1 \ldots i_m;j_1 \ldots j_n}. \] (5.26)

Moreover, the star products between \( N_{i_1 \ldots i_m;j_1 \ldots j_n} \) and one of \( z^k, \partial_k \Phi, \bar{z}^k \) and \( \partial_k \Phi \)
are calculated as follows,

\[ z^k \ast N_{i_1 \cdots i_m;j_1 \cdots j_n} = \sqrt{\frac{m+1}{m+1/h}} N_{ki_1 \cdots i_m;j_1 \cdots j_n}, \quad (5.27) \]

\[ \partial_k \Phi \ast N_{i_1 \cdots i_m;j_1 \cdots j_n} = \hbar \sqrt{\frac{m-1+1/h}{m}} \sum_{l=1}^{m} \delta_{kl} N_{i_1 \cdots i_l \cdots i_m;j_1 \cdots j_n}, \quad (5.28) \]

\[ z^k \ast N_{i_1 \cdots i_m;j_1 \cdots j_n} = \frac{1}{\sqrt{m(m-1+1/h)}} \sum_{l=1}^{m} \delta_{kl} N_{i_1 \cdots i_l \cdots i_m;j_1 \cdots j_n}, \quad (5.29) \]

\[ \partial_k \Phi \ast N_{i_1 \cdots i_m;j_1 \cdots j_n} = \hbar \sqrt{(m+1)(m+1/h)} N_{ki_1 \cdots i_m;j_1 \cdots j_n}, \quad (5.30) \]

\[ N_{i_1 \cdots i_m;j_1 \cdots j_n} \ast z^k = \frac{1}{n(n-1+1/h)} \sum_{l=1}^{n} \delta_{kl} N_{i_1 \cdots i_l \cdots i_m;j_1 \cdots j_n}, \quad (5.31) \]

\[ N_{i_1 \cdots i_m;j_1 \cdots j_n} \ast \partial_k \Phi = \hbar \sqrt{(n+1)(n+1/h)} N_{i_1 \cdots i_m;j_1 \cdots j_n k}, \quad (5.32) \]

\[ N_{i_1 \cdots i_m;j_1 \cdots j_n} \ast z^k = \sqrt{\frac{n+1}{n+1/h}} N_{i_1 \cdots i_m;j_1 \cdots j_n k}, \quad (5.33) \]

\[ N_{i_1 \cdots i_m;j_1 \cdots j_n} \ast \partial_k \Phi = \hbar \sqrt{\frac{n-1+1/h}{n}} \sum_{l=1}^{n} \delta_{kl} N_{i_1 \cdots i_l \cdots i_m;j_1 \cdots j_n}. \quad (5.34) \]

\[ 6 \text{ Summary and discussion} \]

In this paper, we obtained explicit expressions of star products in $\mathbb{C}P^N$ and $\mathbb{C}H^N$ by using the deformation quantization with separation of variables proposed by Karabegov. In this quantization method, a star product by a function is represented by a formal series of differential operators, which is obtained as the solution of an infinite system of differential equations. We gave the explicit solutions of the equations in the case of $\mathbb{C}P^N$ and $\mathbb{C}H^N$. The operators corresponding to the left (right) star multiplications of functions are determined as the power series of $\hbar$ in which each term contains the Stirling numbers of the second kind, the Kähler potentials of the manifolds, and the differential operators.

We also constructed the Fock representations of the star products by using the fact that $\{z^i, \partial_j \Phi\}$ and $\{z^i, \partial_j \Phi\}$ constitute $2N$ sets of the creation-annihilation operators under the star product. We first identified the function $e^{-\Phi/h}$ corresponding to the vacuum projection. Then we considered the functions which are derived by multiplying polynomials of $z^i$ and $\bar{z}^i$ on $e^{-\Phi/h}$, and showed that these functions form the closed algebra under the star product.

Now, we have three comments. Firstly, the operator $L_f$ of the left star multiplication by a function $f$ which is given in Section 3 for $\mathbb{C}P^N$ and in Section 5 for
\( \mathbb{C}H^N \), respectively, can be represented by using the covariant derivatives. To this end, we show that \( L_f \) on these manifolds has the following form,

\[
L_f = \sum_{n=0}^{\infty} c_n(h) g_{j_1 k_1} \cdots g_{j_n k_n} (D^{j_1} \cdots D^{j_n} f) D^{\bar{k}_1} \cdots D^{\bar{k}_n}. \tag{6.1}
\]

The coefficient \( c_n(h) \) is determined by the condition \( [L_f, h \partial_t + \partial_t \Phi] = 0 \). For the case of \( \mathbb{C}P^N \), this condition becomes

\[
[L_f, h \partial_t + \partial_t \Phi] = \sum_{n=1}^{\infty} [n(1 - h(n - 1)) c_n(h) - h c_{n-1}(h)] \times g_{j_1 k_1} \cdots g_{j_{n-1} k_{n-1}} (D^{j_1} D^{j_{n-1}} f) D^{\bar{k}_1} \cdots D^{\bar{k}_{n-1}} = 0. \tag{6.2}
\]

By solving the recursion relation, \( n(1 - h(n - 1)) c_n(h) - h c_{n-1}(h) = 0 \), under the initial condition \( c_0 = 1 \), \( c_n(h) \) is obtained as

\[
c_n(h) = \frac{\Gamma(1 - n + 1/h)}{n! \Gamma(1 + 1/h)} = \frac{\alpha_n(h)}{n!}, \tag{6.3}
\]

where \( \alpha_n(h) \) in given in \( (3.19) \). The first two terms in the power series of \( h \) of \( L_f \) are calculated,

\[
L_f \equiv f + h g_{j k} (D^j f) D^k \ (\mod h^2). \tag{6.4}
\]

Similarly, the operator \( L_f \) on \( \mathbb{C}H^N \) can be represented in the form of \( (6.1) \) with \( c_n(h) = \beta_n(h)/n! \) where \( \beta_n(h) \) is defined in \( (5.12) \).

The expression of \( L_f \) \( (6.1) \) can be rewritten by the use of the covariant derivatives on the manifolds. Non-vanishing components of the Christoffel symbols on a Kähler manifolds are only \( \Gamma^{i}_{jk} \) and \( \Gamma^{\bar{i}}_{j\bar{k}} \). Hence, for scalars \( f \) and \( g \)

\[
g^{j_1 k_1} \cdots g^{j_n k_n} \nabla_{k_1} \cdots \nabla_{k_n} f = g^{j_1 k_1} \nabla_{k_1} \left( g^{j_2 k_2} \cdots g^{j_n k_n} \nabla_{k_2} \cdots \nabla_{k_n} f \right)
= g^{j_1 k_1} \partial_{k_1} \left( g^{j_2 k_2} \cdots g^{j_n k_n} \nabla_{k_2} \cdots \nabla_{k_n} f \right)
= D^{j_1} \left( g^{j_2 k_2} \cdots g^{j_n k_n} \nabla_{k_2} \cdots \nabla_{k_n} f \right)
= D^{j_1} \cdots D^{j_n} f,
\]

\[
g^{\bar{j}_1 k_1} \cdots g^{\bar{j}_n k_n} \nabla_{k_1} \cdots \nabla_{k_n} g = D^{\bar{j}_1} \cdots D^{\bar{j}_n} g. \tag{6.5}
\]

Using these relations, \( L_f g \) becomes

\[
L_f g = f \ast g = \sum_{n=0}^{\infty} c_n(h) g^{j_1 k_1} \cdots g^{j_n k_n} (\nabla_{j_1} \cdots \nabla_{j_n} f) (\nabla_{k_1} \cdots \nabla_{k_n} g). \tag{6.7}
\]
In this article, we treat $h$ as a formal parameter. Now we consider the specific case of $h = 1/L$ ($L \in \mathbb{N}$) and the star product in a function space $\mathcal{M}_L$ spanned by

$$z^{i_1} \cdots z^{i_m} \bar{z}^{j_1} \cdots \bar{z}^{j_n} \over (1 + |z|^2)^L, \quad (m, n \leq L).$$

In this case, the series in (6.1) terminates at $n = L$, because

$$D^{j_1} \cdots D^{j_{L+1}} f = 0, \quad D^{k_1} \cdots D^{k_{L+1}} g = 0,$$

where $f, g \in \mathcal{M}_L$. Then, the expression of the star product coincides with the one in [14].

Secondly, let us try to extend the covariant expression of $L_f$ (6.7) to locally symmetric Kähler manifolds, $\nabla \mu R_{\nu \rho \sigma} \lambda = 0$. We assume the following form of $L_f$,

$$L_f g = \sum_{n=0}^{\infty} T_{n}^{j_1 \cdots j_n, k_1 \cdots k_n} (\nabla_{j_1} \cdots \nabla_{j_n} f) (\nabla_{k_1} \cdots \nabla_{k_n} g). \quad (6.8)$$

Here $g$ is a scalar function and $T_{n}^{j_1 \cdots j_n, k_1 \cdots k_n}$ is a covariantly constant tensor, $\nabla T_n = 0$, and completely symmetric under permutations of $\bar{j}$'s and $k$'s, respectively. Requiring $[L_f, \partial_{\bar{i}} \Phi + h \partial_{\bar{i}}] = 0$, recursion relations for $T_{n}^{j_1 \cdots j_n, k_1 \cdots k_n}$ are derived,

$$[nT_{n}^{j_1 \cdots j_n, k_1 \cdots k_n} g_{k_{n}i} - hT_{n-1}^{j_1 \cdots j_{n-1}, k_1 \cdots k_{n-1}} \delta_{i}^{j_n}]$$

$$- h \frac{n(n-1)}{2} T_{n}^{j_1 \cdots j_{n-1}, k_1 \cdots k_{n-1}} R_{ipq}^{k_{n}k_{n-1}} (\nabla_{j_1} \cdots \nabla_{j_{n-1}} f) (\nabla_{k_1} \cdots \nabla_{k_{n-1}} g) = 0. \quad (6.9)$$

Since the recursion relations include only the metric and the Riemann tensor, $T_{n}^{j_1 \cdots j_n, k_1 \cdots k_n}$ is determined as a function of these quantities and satisfies $\nabla T_n = 0$. Further, the recursion relations are simplified in the case of $\mathbb{C}P^N$. Because $R_{ijkl} = -g_{ij} g_{kl} - g_{ik} g_{lj}$ on $\mathbb{C}P^N$ with the metric (3.3), it can be shown that $L_f$ has the covariant form (6.7).

Thirdly, we consider relations between star products on different patches. Let $\bigcup U_i \quad (U_i = \{ |\zeta^0 : \zeta^1 : \cdots : \zeta^N |(\zeta^i \neq 0) \})$ be an open covering of $\mathbb{C}P^N$, where $\zeta^k$ is a homogeneous coordinate. We define an inhomogeneous coordinate $z^k = k_{\zeta}^k$ in $U_0$ and $w_1 = \zeta^0, w^k = \zeta^k \quad (k \geq 2)$ in $U_1$. Consider the mapping from $U_0$ to $U_1$. The transformation between these inhomogeneous coordinates is given by

$$w^1 = \frac{1}{z^1}, \quad w^k = \frac{z^k}{z^1} \quad (k \geq 2). \quad (6.10)$$
Under this transformation, the Kähler potential is changed as
\[ \Phi(z) = \ln(1 + |z|^2) = \ln(1 + |w|^2) - \ln w^1 - \ln w^1. \] (6.11)

From the lemma 3 in [12], it is found that the star product does not change under the transformation. Similarly, the form of the star product is invariant under mappings between other patches.

Acknowledgement
A.S. is supported by KAKENHI No.23540117 (Grant-in-Aid for Scientific Research (C)). H.U. is supported by KAKENHI No.21740197 (Grant-in-Aid for Young Scientists (B)). We should like to thank an anonymous referee for his helpful comments.

Appendix
Star products on \( \mathbb{C}P^1 \) have been well studied. In the appendix, we summarize the results in the case of \( \mathbb{C}P^1 \) for convenience.

In the case of \( \mathbb{C}P^1 \), \( L_\bar{z} \) and \( R_z \) are given by
\[
L_\bar{z} = \bar{z} + \sum_{m=1}^{\infty} \alpha_m(h) (\bar{\partial} \Phi)^{m-1} \bar{D}^m, \tag{A.12}
\]
\[
R_z = z + \sum_{m=1}^{\infty} \alpha_m(h) (\partial \Phi)^{m-1} D^m, \tag{A.13}
\]
where \( \Phi = \ln(1 + z\bar{z}) \) is the Kähler potential of \( \mathbb{C}P^1 \), \( \bar{D} = g^{\bar{z}z} \bar{\partial} = (1 + z\bar{z}) \partial, \) and \( D = g^{z\bar{z}} \partial = (1 + z\bar{z}) \bar{\partial}. \) \( \alpha_m(h) \) is defined in (3.19). The star products among \( z \) and \( \bar{z} \) become
\[
z \ast z = z^2, \tag{A.14}
\]
\[
z \ast \bar{z} = |z|^2, \tag{A.15}
\]
\[
\bar{z} \ast \bar{z} = \bar{z}^2, \tag{A.16}
\]
\[
\bar{z} \ast z = |z|^2 + h(1 + |z|^2)^2 \, _2F_1(1, 2; 1 - 1/h; -|z|^2) \tag{A.17}
\]

Under the star product, \( \partial \Phi \) and \( z \) (\( \bar{z} \) and \( \bar{\partial} \Phi \)) satisfy the commutation relations of the creation-annihilation operators, respectively,
\[
\partial \Phi \ast z - z \ast \partial \Phi = h, \tag{A.18}
\]
\[
\bar{z} \ast \bar{\partial} \Phi - \bar{\partial} \Phi \ast \bar{z} = h. \tag{A.19}
\]
The function $e^{-\Phi/h} = (1 + |z|^2)^{-1/h}$ is corresponding to the vacuum projection,

\begin{equation}
\partial \Phi \ast e^{-\Phi/h} = \bar{z} \ast e^{-\Phi/h} = e^{-\Phi/h} \ast \partial \Phi = e^{-\Phi/h} \ast z = 0, \tag{A.20}
\end{equation}

\begin{equation}
e^{-\Phi/h} \ast e^{-\Phi/h} = e^{-\Phi/h}. \tag{A.21}
\end{equation}

The following functions

\begin{equation}
M_{mn} = \frac{z^m \bar{z}^n}{\sqrt{m!n!\alpha_m(h)\alpha_n(h)}} e^{-\Phi/h} \tag{A.22}
\end{equation}

form a closed algebra,

\begin{equation}
M_{mn} \ast M_{kl} = \delta_{nk}M_{ml}. \tag{A.23}
\end{equation}

These formulae coincide with the ones of the fuzzy sphere [20, 21] when $h = 1/L$ ($L \in \mathbb{N}$), as mentioned in Section 6.

\section*{References}

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, “Deformation Theory And Quantization. 1. Deformations Of Symplectic Structures,” Annals Phys. 111 (1978) 61.

F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, “Deformation Theory And Quantization. 2. Physical Applications,” Annals Phys. 111 (1978) 111.

[2] M. De Wilde, P. B. A. Lecomte, “Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds,” Lett. Math. Phys. 7, 487 (1983).

[3] H. Omori, Y. Maeda, and A. Yoshioka, “Weyl manifolds and deformation quantization,” Adv. in Math. 85, 224 (1991).

[4] B. Fedosov, “A simple geometrical construction of deformation quantization,” J. Differential Geom. 40, 213 (1994).

[5] M. Kontsevich, “Deformation quantization of Poisson manifolds, I,” Lett. Math. Phys. 66, 157 (2003) [arXiv:q-alg/9709040].

[6] C. Moreno, “ *-products on some Kähler manifolds”, Lett. Math. Phys. 11, 361 (1986).

[7] C. Moreno, “ Invariant star products and representations of compact semisimple Lie groups,” Lett. Math. Phys. 12, 217 (1986).

21
[8] M. Cahen, S. Gutt, J. Rawnsley, “Quantization of Kahler manifolds, II,” Am. Math. Soc. Transl. 337, 73 (1993).

[9] M. Cahen, S. Gutt, J. Rawnsley, “Quantization of Kahler manifolds, IV,” Lett. Math. Phys 34, 159 (1995).

[10] M. Schlichenmaier, “Berezin-Toeplitz quantization for compact Kahler manifolds: A Review of Results,” Adv. Math. Phys. 2010, 927280 (2010) [arXiv:1003.2523 [math.QA]].

[11] A. V. Karabegov, “On deformation quantization, on a Kahler manifold, associated to Berezin’s quantization,” Funct. Anal. Appl. 30, 142 (1996).

[12] A. V. Karabegov, “Deformation quantizations with separation of variables on a Kahler manifold,” Commun. Math. Phys. 180, 745 (1996) [arXiv:hep-th/9508013].

[13] A. V. Karabegov, “An explicit formula for a star product with separation of variables,” [arXiv:1106.4112 [math.QA]].

[14] A. P. Balachandran, B. P. Dolan, J.-H. Lee, X. Martin and D. O’Connor, “Fuzzy complex projective spaces and their star products,” J. Geom. Phys. 43, 184 (2002) [hep-th/0107099].

[15] Y. Kitazawa, “Matrix models in homogeneous spaces,” Nucl. Phys. B 642, 210 (2002) [hep-th/0207115].

[16] D. Karabali, V. P. Nair and S. Randjbar-Daemi, “Fuzzy spaces, the M(atrix) model and the quantum Hall effect,” In *Shifman, M. (ed.) et al.: From fields to strings, vol. 1* 831-875 [hep-th/0407007].

[17] P. Bieliavsky, S. Detournay and P. Spindel, “The Deformation quantizations of the hyperbolic plane,” Commun. Math. Phys. 289, 529 (2009) [arXiv:0806.4741 [math-ph]].

[18] A. Karabegov, “Deformation quantization of a Kähler-Poisson structure vanishing on a Levi nondegenerate hypersurface,” Contemporary Math. 450 (2008), 163.

[19] M. Bordemann, M. Brischle, C. Emmrich, S. Waldmann, “Phase Space Reduction for Star-Products: An Explicit Construction for CP^n,” Lett. Math. Phys. 36 (1996), 357.

[20] J. Madore, “The Fuzzy sphere,” Class. Quant. Grav. 9, 69 (1992).

[21] G. Alexanian, A. Pinzul and A. Stern, “Generalized coherent state approach to star products and applications to the fuzzy sphere,” Nucl. Phys. B 600, 531 (2001) [hep-th/0010187].