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Determining the Number of Communities in Degree-corrected Stochastic Block Models

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Abstract

We propose to estimate the number of communities in degree-corrected stochastic block models based on a pseudo likelihood ratio. For estimation, we consider a spectral clustering together with binary segmentation method. This approach guarantees an upper bound for the pseudo likelihood ratio statistic when the model is over-fitted. We also derive its limiting distribution when the model is under-fitted. Based on these properties, we establish the consistency of our estimator for the true number of communities. Developing these theoretical properties require a mild condition on the average degree – growing at a rate faster than log(n), where n is the number of nodes. Our proposed method is further illustrated by simulation studies and analysis of real-world networks. The numerical results show that our approach has satisfactory performance when the network is sparse and/or has unbalanced communities.

Keywords: Clustering, community detection, degree-corrected stochastic block model, K-means, regularization.

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1 Introduction

Advances in modern technology have facilitated the collection of network data which emerge in many fields including biology, bioinformatics, physics, economics, sociology and so forth. Therefore, developing effective analytic tools for network data has become a focal area in statistics research over the past decade. Network data often have natural communities which are groups of interacting objects (i.e., nodes); pairs of nodes in the same group tend to interact more than pairs belonging to different groups. For example, in social networks, communities can be groups of people who belong to the same club, be of the same profession, or attend the same school; in protein-protein interaction networks, communities are regulatory modules of interacting proteins. In many cases, however, the underlying structure of network data is not directly observable. In such cases, we need to infer the latent community structure of nodes from knowledge of their interaction patterns.

The stochastic block model (SBM) proposed by Holland, Laskey, and Leinhardt (1983) is a random graph model tailored for clustering nodes, and it is commonly used for recovering the community structure in network data. SBM has one limitation: it assumes that all nodes in the same community are stochastically equivalent (i.e., they have the same expected degrees). To overcome this limitation, Karrer and Newman (2011) proposed the degree-corrected stochastic block model (DCSBM) which allows for degree heterogeneity within communities. In the literature, various methods have been proposed for estimation of SBM and DCSBM. They include but are not limited to modularity maximization (Newman and Girvan, 2004), likelihood-based methods (Amini, Chen, Bickel, and Levina, 2013; Bickel and Chen, 2009; Choi, Wolfe, and Airoldi, 2012; Zhao, Levina, and Zhu, 2012), the method of moments (Bickel, Chen, and Levina, 2011), spectral clustering (Jin, 2015; Joseph and Yu, 2016; Lei and Rinaldi, 2015; Qin and Rohe, 2013; Rohe, Chatterjee, and Yu, 2011; Sarkar and Bickel, 2015; Su, Wang, and Zhang, 2017), and spectral embedding (Lyzinski, Sussman, Tang, Athreya, and Priebe, 2014; Sussman, Tang, Fishkind, and Priebe, 2012). In most, if not all, works, theoretical properties such as consistency and asymptotic distributions are built based on the assumption that the true number of communities $K_0$ is known.

In practice, prior information of the number of communities is often unavailable. Ac-
curately estimating $K_0$ from the network data is of crucial importance, as the following community detection procedure relies upon it. Determining the number of communities can be regarded as a model selection problem. Naturally, people would like to consider the popular model selection methods such as cross-validation (CV) or likelihood-based methods. However, tailoring those methods for SBMs or DCSBMs and establishing the theoretical support are challenging, as network data are complex in nature.

There are a few methods developed by pioneer works for estimating $K_0$. Among them, the eigenvalue-based methods have been widely applied; see Bordenave, Lelarge, and Massoulié (2015), Bickel and Sarkar (2016), Le and Levina (2015) and Lei (2016) for the hypothesis testing methods on eigenvalues. These methods can be computationally fast, but they only use partial information from the data – the eigenvalues. Empirically, the good behavior of eigenvalues often requires a very large sample size. In order to make use of all the information from the data, we need to estimate the graph model (SBM or DCSBM). To this end, spectral clustering is considered as a quick and effective way, and it has been proven to have reliable theoretical basis (Jin, 2015; Joseph and Yu, 2016; Lei and Rinaldo, 2015; Qin and Rohe, 2013; Rohe et al., 2011; Sarkar and Bickel, 2015; Su et al., 2017). Based on the spectral clustering method for estimating the graph model, Chen and Lei (2018) and Li, Levina, and Zhu (2016) proposed network cross-validation (NCV) and edge cross-validation (ECV), respectively, for selecting the number of communities. In particular, Chen and Lei (2018) showed that NCV underestimates the number of communities with probability approaching zero. Nevertheless, it does not rule out the overselection of the number of communities. In practice, CV methods can be computationally intensive when the number of folds is large; it can lead to unstable results when the number of folds or the number of random sample splittings (or repetitions in the ECV case) is small. Another appealing method for model selection is the likelihood-based approach considered in Wang and Bickel (2016). It does not need iterations or random sample splittings, and with the help of a BIC-type penalty, it can select the correct number of communities in standard SBMs consistently. However, for either SBMs or DCSBMs, optimizing the likelihood function which involves summing over all possible community memberships is computationally intractable for even moderate sample sizes. As a result, Wang and Bickel (2016) use a variational EM algorithm to approximate...
the likelihood.

In this article, we propose a new method by taking advantage of both spectral clustering and likelihood principle. The method is devised for DCSBM, but can be naturally applied to SBM as it is a special case of DCSBM. For determining the number of communities $K$, we propose a pseudo conditional likelihood ratio to compare the goodness-of-fit of two DCSBMs estimated by using $K$ and $K + 1$, respectively, as the number of communities. For estimation, directly using spectral clustering can be an appealing choice as it is computationally fast. However, when $K > K_0$, it remains unclear about theoretical properties for the resulting estimators of the DCSBM obtained through the standard spectral clustering approach. This hinders the use of goodness-of-fit methods for model selection by using spectral clustering for estimation. To overcome the difficulty, we estimate the DCSBM with $K$ communities by using spectral clustering; based on this estimate, we propose a binary segmentation method for estimating the DCSBM with $K + 1$ communities. This approach guarantees consistency of the estimator for the model with $K + 1$ communities when the estimator for the model with $K$ communities is consistent. The binary segmentation technique has been used in the seminal work Vostrikova (1981) for change-point detection and in recent work Wang and Su (2017, proof of Theorem 3.2) for latent group recovery. Our idea of adapting this method to estimate DCSBM has not been considered by others. Based on the proposed estimation approach, we show that the pseudo likelihood ratio has a sound theoretical basis, and the resulting estimator of the number of communities is consistent.

It is worth noting that for establishing the consistency of estimating $K_0$, we only require the average degree to grow with the number of nodes $n$ faster than $\log(n)$, whereas Wang and Bickel (2016) needs it to be faster than $n^{1/2}\log(n)$, i.e., the approach considered in Wang and Bickel (2016) needs a much denser network than our method for good finite sample performance. As pointed out by Wang and Bickel (2016, Section 2.5), their approach needs a very stringent condition on the average degree, because the slow convergence rate of the estimate of the node degree variation passes on to the likelihood ratio. On the contrary, it is not carried on to our pseudo likelihood ratio because of the mutual cancellation of the slow-convergence parts. As a result, this allows us to relax the strong restriction on the average degree in theory. Moreover, we develop thorough theoretical results for the estimators in
DCSBMs, whereas Chen and Lei (2018), Li et al. (2016) and Wang and Bickel (2016) focus on the SBMs.

The rest of the paper is organized as follows. We establish the consistency of our estimators for the number of communities under DCSBMs in Section 2. Section 4 compares the performance of our method with various existing methods in different simulated networks. Section 5 illustrates the proposed method using several real data examples. Section 6 concludes. The proofs of all results are relegated to the mathematical appendix.

Notation. Throughout the paper, we write \( [M]_{ij} \) as the \((i, j)\)-th entry of matrix \( M \). Without confusion, we sometimes simplify \( [M]_{ij} \) as \( M_{ij} \). In addition, we write \( [M]_{i} \) as the \(i\)-th row of \( M \). \( \|M\| \) and \( \|M\|_F \) denote the spectral norm and Frobenius norm of \( M \), respectively. Note that \( \|M\| = \|M\|_F \) when \( M \) is a vector. We use \( 1 \{ \cdot \} \) to denote the indicator function which takes value 1 when \( \cdot \) holds and 0 otherwise. For a vector \( \mathbf{a} = (a_1, ..., a_n)^\top \), let \( \text{diag} (\mathbf{a}) \) be the diagonal matrix whose diagonal is \( \mathbf{a} \), and let \( \|\mathbf{a}\| = (\sum a_i^2)^{1/2} \) be its L_2 norm. Let \( \iota_n \), \( \#S \), and \( [n] \) be the \( n \)-dimensional vector of ones, the cardinality of set \( S \), and the integer sequence \( \{1, 2, \cdots, n\} \), respectively. \( C, c, \) and \( c' \) denote arbitrary positive constants that are independent of \( n \), but may not be the same in different contexts.

2 Methodology

2.1 Degree-corrected SBM

Let \( A \in \{0, 1\}^{n \times n} \) be the adjacency matrix. By convention, we do not allow self-connection, i.e., \( A_{ii} = 0 \). Let \( \hat{d}_i = \sum_{j=1}^n A_{ij} \) denote the degree of node \( i \), \( D = \text{diag}(\hat{d}_1, \ldots, \hat{d}_n) \). We regularize the degree for each node as \( \hat{d}_i^\tau = \hat{d}_i + \tau \) where \( \tau \) is a regularization parameter. Let \( D_{\tau} = \text{diag}(\hat{d}_1 + \tau, \ldots, \hat{d}_n + \tau) \). The regularized sample graph Laplacian is

\[
L_{\tau} = D_{\tau}^{-1/2} AD_{\tau}^{-1/2}.
\]

The network is generated by a degree-corrected stochastic block model with \( K_0 \) true communities. The communities, which represent a partition of the \( n \) nodes, are assumed to be fixed beforehand. Denote \( Z_{K_0} = \{[Z_{K_0}]_{ik}\} \) as the \( n \times K_0 \) binary matrix providing the
true cluster memberships of each node, i.e., \( [Z_{K_0}]_{ik} = 1 \) if node \( i \) is in \( C_{k,K_0} \) and \( [Z_{K_0}]_{ik} = 0 \) otherwise, where \( C_{1,K_0}, \ldots, C_{K_0,K_0} \) are denoted as the communities identified by \( Z_{K_0} \). For \( k = 1, \ldots, K_0 \), let \( n_k(Z_{K_0}) = \#C_{k,K_0} \), the number of nodes in \( C_{k,K_0} \). Given the \( K_0 \) communities, the edges between nodes \( i \) and \( j \) are chosen independently with probability depending on the communities that nodes \( i \) and \( j \) belong to. In particular, for nodes \( i \) and \( j \) belonging to clusters \( C_{k,K_0} \) and \( C_{l,K_0} \), respectively, the probability of edge between \( i \) and \( j \) is given by

\[
P_{ij} = \theta_i \theta_j B_{kl}(Z_{K_0}),
\]

where the block probability matrix \( B(Z_{K_0}) = \{B_{kl}(Z_{K_0})\}, k, l = 1, \ldots, K_0, \) is a symmetric matrix with each entry between \((0, 1]\). The \( n \times n \) edge probability matrix \( P = \{P_{ij}\} \) represents the population counterpart of the adjacency matrix \( A \). Let \( \Theta = \text{diag}(\theta_1, \ldots, \theta_n) \). Then we have

\[
P = \Theta Z_{K_0} B(Z_{K_0}) Z_{K_0}^T \Theta^T.
\]

Note that \( \Theta \) and \( B(Z_{K_0}) \) are only identifiable up to scale. Following the lead of Su et al. (2017, Theorem 3.3), we adopt the following normalization rule:

\[
\sum_{i \in C_{k,K_0}} \theta_i = n_k(Z_{K_0}), \quad k = 1, \ldots, K_0. \tag{2.1}
\]

Apparently, the DCSBM becomes the standard SBM when \( \theta_i = 1 \) for each \( i = 1, \ldots, n \).

### 2.2 Estimation of the number of communities

Due to Wilson, Stevens, and Woodall (2016), for a given number of communities \( K \) and a generic estimator \( \hat{Z}_K \) of the community memberships with corresponding estimated communities \( \{\hat{C}_{k,K}\}_{k=1}^K \), the maximum likelihood estimator (MLE) for \( \theta_i \) in DCSBM is \( \hat{\theta}_i = \frac{\hat{d}_i n_k(\hat{Z}_K)}{\sum_{i' \in \hat{C}_{k,K}} \hat{d}_{i'}} \) for \( i \in \hat{C}_{k,K} \), where \( n_k(\hat{Z}_K) = \sum_{i=1}^n 1\{[\hat{Z}_K]_{ik} = 1\} \), and the MLE for \( B_{kl}(\hat{Z}_K) \) is \( \hat{B}_{kl}(\hat{Z}_K) = \frac{O_{k,l}(\hat{Z}_K)}{n_{k,l}(\hat{Z}_K)} \) for \( k, l = 1, \ldots, K \), where

\[
O_{k,l}(\hat{Z}_K) = \sum_{i=1}^n \sum_{j \neq i} 1\{[\hat{Z}_K]_{ik} = 1, [\hat{Z}_K]_{jl} = 1\} A_{ij};
\]
Therefore, for given \( \hat{Z}_K \), for \( i \in \hat{C}_{k,K} \) and \( j \in \hat{C}_{l,K} \) with \( k \neq l \), the MLE of \( P_{ij} \) is

\[
\hat{P}_{ij}(\hat{Z}_K) = \frac{O_{k,l}(\hat{Z}_K)\hat{d}_i\hat{d}_j}{(\sum_{i' \in \hat{C}_{k,K}}\hat{d}_{i'}) (\sum_{j' \in \hat{C}_{l,K}}\hat{d}_{j'})} = \frac{O_{k,l}(\hat{Z}_K)\hat{d}_i\hat{d}_j}{(\sum_{i'=1}^{K} O_{k,i'}(\hat{Z}_K))(\sum_{j'=1}^{K} O_{l,j'}(\hat{Z}_K))} := \hat{M}_{k,l}(\hat{Z}_K)\hat{d}_i\hat{d}_j,
\]

and for \( i, j \in \hat{C}_{k,K} \), it is

\[
\hat{P}_{ij}(\hat{Z}_K) = \frac{O_{k,k}(\hat{Z}_K)\hat{d}_i\hat{d}_j}{(\sum_{i',j' \in \hat{C}_{k,K}}\hat{d}_{i'}\hat{d}_{j'})} := \hat{M}_{k,k}(\hat{Z}_K)\hat{d}_i\hat{d}_j.
\]

Our procedure of estimating \( K_0 \) requires to obtain two classifications \((\hat{Z}_K, \hat{Z}_{K+1}^b)\) based on \( K \) and \( K+1 \) communities, respectively. To this end, we estimate \( \hat{Z}_K \) via spectral clustering and \( \hat{Z}_{K+1}^b \) via a binary segmentation technique with the algorithm given in Section 2.3. We compute \( \hat{P}_{ij}(\hat{Z}_{K+1}^b) \) and \( \hat{M}_{k,l}(\hat{Z}_{K+1}^b) \) in the same way as \( \hat{P}_{ij}(\hat{Z}_K) \) and \( \hat{M}_{k,l}(\hat{Z}_K) \), and propose the following pseudo likelihood ratio (pseudo-LR) to measure the deviance of goodness-of-fit of DCSBM estimated with \( K \) and \( K+1 \) communities:

\[
L_n(\hat{Z}_{K+1}^b, \hat{Z}_K) = \frac{1}{2} \sum_{i \neq j} \left( \frac{\hat{P}_{ij}(\hat{Z}_{K+1}^b)}{\hat{P}_{ij}(\hat{Z}_K)} - 1 \right)^2. \tag{2.2}
\]

Let

\[
R(K) = \begin{cases} 
\frac{L_n(\hat{Z}_{K+1}^b, \hat{Z}_K)}{\eta_n} & K = 1 \\
\frac{L_n(\hat{Z}_{K+1}^b, \hat{Z}_K)}{L_n(\hat{Z}_K, \hat{Z}_{K-1})} & K \geq 2
\end{cases}
\]

where \( \eta_n = c_n n^2 \) and \( c_n \) is a positive constant. We estimate the true number of communities \( K_0 \) by \( \hat{K}_j, j = 1, 2, \) where

\[
\hat{K}_1 = \arg\min_{1 \leq K \leq K_{\max}} R(K),
\]

and

\[
\hat{K}_2 = \min(\hat{K}_1, \hat{K}_2),
\]

\[1\text{The superscript } b \text{ in } \hat{Z}_{K+1}^b \text{ denotes that it is estimated by a binary segmentation from } \hat{Z}_K.\]
where \( \tilde{K}_2 = \min\{K \in \{1, \cdots, K_{\text{max}}\}, R(K) \leq h_n\} \) if \( \min_{1 \leq K \leq K_{\text{max}}} R(K) \leq h_n \) and \( \tilde{K}_2 = K_{\text{max}} \) otherwise, in which \( h_n \) is a tuning parameter to be specified later.

To understand the above estimators of \( K_0 \), we focus on the case where \( K_0 \geq 2 \). If one is sure that \( K_0 \geq 2 \), there is no need to define \( R(1) \) and one can redefine \( \hat{K}_1 = \arg \min_{2 \leq K \leq K_{\text{max}}} R(K) \). By Theorems 3.3 and 3.4 in Section 3.3, we have

\[
L_n(\hat{Z}_b^b K, \hat{Z}_b K-1) \asymp n^2 \quad \text{for} \quad 2 \leq K \leq K_0 \quad \text{and} \quad L_n(\hat{Z}_b^b K_0+1, \hat{Z}_b K_0) \leq o_{\text{a.s.}}(n^{\rho_n-1}),
\]

where \( a_n \asymp b_n \) means that \( P(c \leq a_n/b_n \leq C) \to 1 \) as \( n \to \infty \) for some positive constants \( c \) and \( C \), \( a.s. \) denotes almost surely, and the parameter \( \rho_n \) characterizes the sparsity of the network such that \( n \rho_n/\log(n) \to \infty \) (see Assumption 4 in Section 3.2). This result implies that

\[
R(K) \asymp 1 \quad \text{for} \quad 2 \leq K < K_0 \quad \text{and} \quad R(K_0) = o_p(1).
\]

As a result, the minimizer of \( R(K) \) satisfies \( \hat{K}_1 > K_0 \) with probability approaching 1 (w.p.a.1) as \( n \to \infty \). Such a result is similar to that in Chen and Lei (2018) who showed that NCV can not underestimate the number of communities asymptotically. The introduction of \( \tilde{K}_2 \) along with some side conditions on the tuning parameter \( h_n \) (i.e., \( h_n \to 0 \) and \( n \rho_n h_n \to \infty \)) ensures that \( P(\hat{K}_2 = K_0) \to 1 \) as \( n \to \infty \). Consequently, \( \tilde{K}_2 \) consistently estimates the number of communities in large samples.

### 2.3 Estimation of the memberships

The proposed pseudo-LR given in (2.2) depends on \((\hat{Z}_K, \hat{Z}_K^b+1)\) which are obtained through the following algorithm. Denote the spectral decomposition of \( L_\tau \) as

\[
L_\tau = \hat{U}_n \hat{\Sigma}_n \hat{U}_n^T,
\]

where \( \hat{\Sigma}_n = \text{diag}(\hat{\sigma}_1, \ldots, \hat{\sigma}_n) \) with \( |\hat{\sigma}_1| \geq |\hat{\sigma}_2| \geq \cdots \geq |\hat{\sigma}_n| \geq 0 \), and \( \hat{U}_n \) is the corresponding eigenvectors such that \( \hat{U}_n^T \hat{U}_n = I_n \). For each \( K = 1, \cdots, K_{\text{max}} \), let

\[
\hat{\nu}_{iK} = \frac{\hat{u}_i^T(1 : K)}{||\hat{u}_i^T(1 : K)||},
\]

where \( \hat{u}_i^T \) is the i-th row of \( \hat{U}_n \) and \( \hat{u}_i^T(1 : K) \) collects the first \( K \) elements of \( \hat{u}_i^T \). We estimate the pair of community memberships \((\hat{Z}_K, \hat{Z}_K^b+1)\) by the following algorithm.
1. Divide \( \{\hat{\nu}_{iK}\}_{i=1}^{n} \) into \( K \) groups by the k-means algorithm with \( K \) centroids. Denote the membership matrix as \( \hat{Z}_K \) with the corresponding communities \( \{\hat{C}_{k,K}\}_{k=1}^{K} \).

2. Divide each \( \hat{C}_{k,K} \) into two subgroups by applying the k-means algorithm on \( \{\hat{\nu}_{iK+1}\}_{i=\hat{C}_{k,K}} \). Denote the two subgroups as \( \hat{C}_{k,K}(1) \) and \( \hat{C}_{k,K}(2) \).

3. For each \( k = 1, \cdots, K \), compute
   \[
   \hat{Q}_K(k) = \frac{\hat{\Phi}(\hat{C}_{k,K}) - \hat{\Phi}(\hat{C}_{k,K}(1)) - \hat{\Phi}(\hat{C}_{k,K}(2))}{\#\hat{C}_{k,K}},
   \]
   where for an arbitrary index set \( C \), \( \hat{\Phi}(C) = \sum_{i \in C} \|\hat{\nu}_{iK+1} - \frac{\sum_{i \in C} \hat{\nu}_{iK+1}}{\#C}\|^2 \).

4. Choose \( \hat{k} = \arg \max_{1 \leq k \leq K} \hat{Q}_K(k) \). Denote \( \{\hat{C}_{k,K+1}\}_{k=1}^{K+1} = \{\hat{C}_{k,K}\}_{k=\hat{k}}^{\hat{k}}, \hat{C}_{k,K}(1), \{\hat{C}_{k,K}\}_{k>\hat{k}}, \hat{C}_{k,K}(2) \) as the new groups for \( K + 1 \). The corresponding membership matrix is denoted as \( \hat{Z}_{K+1}^b \).

In the above algorithm, step 1 applies the standard spectral clustering approach for obtaining \( \hat{Z}_K \), and step 2-4 is a binary segmentation method for obtaining \( \hat{Z}_{K+1}^b \). This procedure is computationally fast. Moreover, the algorithm leads to \( \hat{C}_{k,K+1}^b = \hat{C}_{k,K} \) for \( k \neq \hat{k} \) and \( \hat{C}_{k,K+1}^b \cup \hat{C}_{K+1,K+1}^b = \hat{C}_{k,K} \), so that it ensures consistency of the parameter estimators in the DCSBM with \( K + 1 \) communities when the estimators are consistent in the model with \( K \) communities.

### 3 Theory

#### 3.1 Identification

Recall that the regularized graph Laplacian is
\[
L_\tau = D_\tau^{-1/2} AD_\tau^{-1/2}.
\]

Its population counterpart is
\[
\mathcal{L}_\tau = \mathcal{D}_\tau^{-1/2} P \mathcal{D}_\tau^{-1/2},
\]
where \( \mathcal{D}_\tau = \mathcal{D} + \tau I_n \) and \( \mathcal{D} = \text{diag}(d_1, \ldots, d_n) \) with \( d_i = \sum_{j=1}^{n} P_{ij} \). Let \( \pi_{kn} = n_k(Z_{K_0})/n \) and \( \Pi_n = \text{diag}(\pi_{1n}, \cdots, \pi_{Kn}) \).
Assumption 1 Let \( H(Z_{K_0}) = \rho_n^{-1}B(Z_{K_0}) \) for some \( \rho_n > 0 \), \( W_k = \sum_{i=1}^{K_0} H_{k,i}(Z_{K_0})\pi_{in} \), \( D_H = \text{diag}(W_1, \ldots, W_{K_0}) \), and \( H_0(Z_{K_0}) = D_H^{-1/2}H(Z_{K_0})D_H^{-1/2} \). Then, (1) \( H(Z_{K_0}) \) is not varying with \( n \), (2) as \( n \to \infty \), \( H_0(Z_{K_0}) \to H_0^*(Z_{K_0}) \) where \( H_0^*(Z_{K_0}) \) has full \( K_0 \) rank, (3) all elements of \( H_0^*(Z_{K_0}) \) are positive, and (4) there exist two constants \( \underline{\theta} \) and \( \overline{\theta} \) such that \( 0 < \underline{\theta} \leq \inf_i \theta_i \leq \sup_i \theta_i \leq \overline{\theta} \).

Several remarks are in order. First, Assumption 1 implies that the degree is of order of magnitude \( n\rho_n \). The network can be semi-dense if \( \rho_n \to 0 \). Second, Assumption 1(1) is just for notational simplicity. All our results still hold if \( H(Z_{K_0}) \) depends on \( n \) and converges to some limit. Third, \( K_0 \) in Assumption 1(2) is the true number of communities. Last, for simplicity, we restrict \( \theta_i \) to be bounded and bounded away from zero. This assumption can be relaxed at the cost of more complicated notation.

Next, let \( \Theta_\tau = \text{diag}(\theta_1^\tau, \ldots, \theta_n^\tau) \), where \( \theta_i^\tau = \theta_id_i/(d_i + \tau) \) for \( i = 1, \ldots, n \), \( n_i^\tau(Z_{K_0}) = \sum_{i \in C_k, \pi_{ki}^{K_0}} \theta_i^\tau \), and \( \Pi_n^\tau = \text{diag}(n_1^\tau(Z_{K_0})/n, \ldots, n_{K_0}^\tau(Z_{K_0})/n) \).

Assumption 2 Suppose

\[
\Pi_n \to \Pi_\infty = \text{diag}(\pi_{1\infty}, \ldots, \pi_{K_0\infty}), \quad \Pi_n^\tau \to \Pi_\infty^\tau = \text{diag}(\pi_{1\infty}^\tau, \ldots, \pi_{K_0\infty}^\tau),
\]

and that both \( \{\pi_{k\infty}\}_{k=1}^{K_0} \) and \( \{\pi_{k\infty}^\tau\}_{k=1}^{K_0} \) are bounded and bounded away from zero.

The first convergence in Assumption 2 essentially requires the average degrees of nodes are of the same order of magnitude across the true \( K_0 \) communities. The second convergence in Assumption 2 can be easily satisfied, say, by choosing \( \tau \) to be the average degree \( \bar{d} \) in the network.

Consider the spectral decomposition of \( \mathcal{L}_\tau \),

\[
\mathcal{L}_\tau = U_{1n} \Sigma_{1n} U_{1n}^T,
\]

where \( \Sigma_{1n} = \text{diag}(\sigma_{1n}, \ldots, \sigma_{Kn}) \) is a \( K_0 \times K_0 \) matrix that contains the eigenvalues of \( \mathcal{L}_\tau \) such that \( |\sigma_{1n}| \geq |\sigma_{2n}| \geq \cdots \geq |\sigma_{Kn}| > 0 \) and \( U_{1n}^T U_{1n} = I_{K_0} \).
Theorem 3.1 Suppose Assumptions 1 and 2 hold. Let $u_i^T$ be the $i$-th row of $U_{1n}$. Then (1) there exists a $K_0 \times K_0$ matrix $S_n^r$ such that $(S_n^r)^T S_n^r = I_{K_0}$ and $U_{1n} = \Theta_0^{1/2} Z_{K_0} (\Theta_0 Z_{K_0})^{-1/2} S_n^r$. (2) If $[Z_{K_0}]_{i,:} = [Z_{K_0}]_{j,:}$, then $\frac{u_i}{\|u_i\|} - \frac{u_j}{\|u_j\|} = 0$; if $[Z_{K_0}]_{i,:} \neq [Z_{K_0}]_{j,:}$, then $\frac{u_i^T}{\|u_i^T\|} - \frac{u_j^T}{\|u_j^T\|} = \sqrt{2}$. (3) Let $[S_n^r](1 : K)$ and $[S_n^r]_{k,(1 : K)}$ denote the first $K$ columns of $S_n^r$ and its $k$-th row, respectively. There exist some $K \times K$ orthonormal matrix $O_s$, a $K_0 \times K_0$ matrix $S_{\infty}$ and a positive constant $c$ such that for any $K \leq K_0$, $[S_n^r]_{k,(1 : K)} O_s \rightarrow [S_{\infty}]_{k,(1 : K)}, [S_{\infty}]_{k,(1 : K)}$ has rank $K$, and for any $k = 1, \ldots, K_0$ and $K = 1, \ldots, K_0$, 

$$\lim \inf_n \| [S_n^r]_{k,(1 : K)} \| \geq c.$$ 

(4) For any $K = 1, \ldots, K_0$, the nodes can be divided into $L_K$ groups, denoted by $\{G_{l,K}\}_{l=1}^{L_K}$ such that for any $l = 1, \ldots, L_K$, 

$$\lim \sup_n \sup_{i,j \in G_{l,K}} \left\| \frac{u_i^T (1 : K)}{\|u_i^T (1 : K)\|} - \frac{u_j^T (1 : K)}{\|u_j^T (1 : K)\|} \right\| = 0$$

and for any $l \neq l'$ and some constant $c > 0$ independent of $n$, 

$$\lim \inf_n \inf_{i \in G_{l,K}, j \in G_{l',K}} \left\| \frac{u_i^T (1 : K)}{\|u_i^T (1 : K)\|} - \frac{u_j^T (1 : K)}{\|u_j^T (1 : K)\|} \right\| \geq c.$$ 

In addition, $K \leq L_K \leq K_0$.

Several remarks are in order. First, Theorem 3.1 (1) and (2) have already been established in the literature. See Qin and Rohe (2013) and Su et al. (2017). Second, Theorem 3.1 (3) implies that, for $i \in C_{k,K_0}$, the norm of $(\theta_i)^{-1/2} (n_k (Z_{K_0}))^{1/2} u_i^T (1 : K)$ for any $K = 1, \ldots, K_0$ is bounded away from zero uniformly over $i$, which ensures that the fraction $\frac{u_i^T (1 : K)}{\|u_i^T (1 : K)\|}$ is well defined. This result is similar to Jin (2015, Lemma 2.5). Third, Theorem 3.1 (4) implies that the first $K$ columns of eigenvectors after row normalization still contain information for at least $K$ communities, when $K \leq K_0$. In particular, when $K = K_0$, $L_{K_0} = K_0$ and Theorem 3.1 (2) implies that Theorem 3.1 (4) holds with the true communities, i.e., $\{G_{l,L_{K_0}}\}_{l=1}^{L_{K_0}} = \{C_{k,K_0}\}_{k=1}^{K_0}$. Therefore, $\{G_{l,K}\}_{l=1}^{L_K}$ can be viewed as the true communities identified by the first $K$ columns of eigenvectors. Fourth, by the definition of $G_{l,K}$, for any $k = 1, \ldots, K_0$ and any $K = 1, \ldots, K_0$, the true community $C_{k,K_0}$ belongs to one of $\{G_{l,K}\}_{l=1}^{L_K}$.
3.2 Properties of the estimated memberships

In the following, we aim to show that, under certain conditions, when $K \leq K_0$, $\hat{Z}_K = Z_K$ and $\hat{Z}^b_K = Z^b_K$ almost surely (a.s.) for some deterministic membership matrices $Z_K$ and $Z^b_K$. We denote the communities identified by $Z_K$ and $Z^b_K$ as $\{C_{k,K}\}_{k=1}^K$ and $\{(C^b_{k,K})\}_{k=1}^{K+1}$, respectively. Note that $L_K$ is not necessarily equal to $K$. This implies that neither $\{C_{k,K}\}_{k=1}^K$ nor $\{(C^b_{k,K})\}_{k=1}^{K+1}$ is necessarily equal to the true communities $\{G_{l,K}\}_{l=1}^{L_K}$. We can view $Z_K$ and $Z^b_K$ as the pseudo true values of our estimation procedure described in Section 2.2.

We slightly abuse the notation by calling $Z_K$ evaluated at $K = K_0$ as the pseudo true membership matrix when $K = K_0$ while $Z^b_K$ as the true membership matrix. Theorem 3.2 below shows that when $K = K_0$, the pseudo true values $Z_K$ and $Z^b_K$ are equal to the true membership matrix $Z_{K_0}$. Therefore, the notation is still consistent and we can just write $Z_{K_0}$ as the (pseudo) true membership matrix for $K = K_0$.

We first define $Z_K$ and $Z^b_K$ for $2 \leq K \leq K_0$.

For $i \in C_{k,K}$ and $k = 1, \ldots, K_0$, let

$$\nu_{iK} = \overline{\nu}_{kK} := \frac{[S_{\infty}(1 : K)]_{i, k}}{||[S_{\infty}(1 : K)]_{i, k}||}.$$ 

Then $Z_K$ is defined by (conceptually) applying k-means algorithm to $\{\nu_{iK}\}_{i=1}^n$ with $K$ centroids. Let $g_{iK}$ denote the membership for node $i$ obtained this way, i.e.,

$$g_{iK} = \arg \min_{1 \leq k \leq K} ||\nu_{iK} - \alpha^*_k|| \quad \text{and} \quad \{\alpha^*_k\}_{k=1}^K = \arg \min_{\alpha_1, \ldots, \alpha_K} \sum_{l=1}^{K_0} \pi_{lK} \min_{1 \leq k \leq K} ||\nu_{lK} - \alpha_k||^2. \quad (3.1)$$

Then $[Z_K]_{ik} = 1$ if $g_{iK} = k$, $[Z_K]_{ik} = 0$ otherwise, and $C_{k,K} = \{i : g_{iK} = k\}$. We define $Z^b_{K+1}$ for $K = 1, \ldots, K_0 - 1$ as follows.

1. Given $\{C_{k,K}\}_{k=1}^K$, let $\tilde{C}_{k,K} = C_{k,K} \cap G_{l,K+1}$, for $l = 1, \ldots, L_K$. For $i \in \tilde{C}_{k,K}$,

$$\nu^b_{iK+1} = \frac{\sum_{j \in \tilde{C}_{k,K}} u^l_j(1:K+1)}{||u^l_i(1:K+1)||} \frac{u^l_i(1:K+1)}{\#\tilde{C}_{k,K}}.$$ 

We divide each $C_{k,K}$ into two subgroups by applying the k-means algorithm to $\{\nu^b_{iK+1}\}_{i \in C_{k,K}}$ with two centroids. Denote the two subgroups as $C_{k,K}(1)$ and $C_{k,K}(2)$.

2When $K = 1$, we can trivially define $Z_1 = Z^b_1 = [n] = \{1, 2, \ldots, n\}$. 

2. For each $k = 1, \ldots, K$, compute

$$Q_K(k) = \frac{\Phi(C_{k,K}) - \Phi(C_{k,K}(1)) - \Phi(C_{k,K}(2))}{\#C_{k,K}},$$

where for an arbitrary index set $C$, $\Phi(C) = \sum_{i \in C} \|\nu_{iK+1} - \frac{\sum_{i \in C} \nu_{iK+1}}{\#C} \|^2$.

3. Choose $k^* = \arg \max_{1 \leq k \leq K} Q_K(k)$. Denote $\{ C_{k,K} \}_{k=1}^{K+1}$ as the new groups in step $Z_K^{b+1}$.

We make the following assumption.

**Assumption 3**

1. Suppose $Z_K$ and $Z_K^b$ obtained via (3.1) and the above procedure, respectively, are uniquely defined for $K = 1, \ldots, K_0$. (2) There exists a positive constant $c$ independent of $n$ such that $Q_K(k^*) - \max_{k \neq k^*} Q_K(k) \geq c$ for $K = 2, \ldots, K_0 - 1$.

Several remarks are in order. First, the uniqueness requirement is mild. If $L_K = K$, then obviously $\{ C_{k,K} \}_{k=1}^K = \{ G_{l,K} \}_{l=1}^{L_K}$, which implies $Z_K$ is uniquely defined. Second, we have $L_{K_0} = K_0$. Therefore, by definition, $\{ C_{k,K_0} \}_{k=1}^{K_0}$ defined by $Z_{K_0}$ equal $\{ G_{l,K_0} \}_{l=1}^{K_0}$ which are true community memberships. Third, when $L_K = K$ and $L_{K+1} = K + 1$ for $K \leq K_0 - 1$, by the pigeonhole principle, there only exists one $k \in \{1, \ldots, K\}$, denoted as $k^\dagger$ such that $C_{k^\dagger,K} = G_{k^\dagger,K}$ contains two of $\{ \tilde{C}_{k^\dagger,K+1,i} \}_{i=1}^{K+1}$. Then by Theorem 3.1(4), there exists some constant $c > 0$ such that $Q_K(k^\dagger) \geq c$ and $Q_K(k) \to 0$ for $k \neq k^\dagger$. In this case, $k^* = k^\dagger$ and Assumption 3(2) holds. Fourth, Assumption 3 is in spirit close to Wang and Bickel (2016, Assumption 2.1, Theorem 5.3). It is more of a notational convenience than necessity. Under Assumption 3, we show later that the pseudo likelihood ratio after re-centering is asymptotically normal. If Assumption 3 fails and $(Z_K, Z_K^b)$ are not unique, it can be anticipated that the pseudo likelihood ratio after re-centering will be asymptotically mixture normal with weights depend on the probability of choosing one classification among all possibilities. Last, although Assumption 3 is used to characterize the limiting distribution of the re-centered pseudo likelihood ratio, it does not affect the rate of bias term in the under-fitting case. Because the bias term will dominate the centered term, we actually only need the rate of bias to show the validity of our selection procedure. Therefore, even if Assumption 3 fails, it is reasonable to expect that our procedure can still consistently select the true number of communities as established in Section 3.3.
Assumption 4 Assume $\rho_n n / \log(n) \to \infty$ and $\tau = O(n \rho_n)$.

Recall that the degree of the network is of order of magnitude $n \rho_n$. Assumption 4 requires the degree to diverge faster than $\log(n)$, which is weaker than what is assumed in Wang and Bickel (2016). For DCSBM, Wang and Bickel (2016) requires that $n^{1/2} \rho_n / \log(n) \to \infty$, or equivalently, the degree diverges to infinity faster than $n^{1/2} \log(n)$, which implies a denser network. The key reason for our weaker requirement is that we use pseudo instead of true likelihood ratios. In DCSBM, the rate of convergence for the estimator $\hat{\theta}_i$ of $\theta_i$ is much slower than that for the estimator of block probability matrix. By using the ratio $\frac{\hat{P}_{ij}(\hat{Z}^b_{K+1})}{\hat{P}_{ij}(Z_K)}$ in the definition of pseudo-likelihood ratio, the components of $\hat{\theta}_i$’s that cause the slower convergence rate in both the numerator and the denominator cancel each other, so that the convergence rate of $\frac{\hat{P}_{ij}(\hat{Z}^b_{K+1})}{\hat{P}_{ij}(Z_K)}$ will not be affected by the slower convergence rate of $\hat{\theta}_i$’s.

Definition 3.1 Suppose there are two membership matrices $Z_1$ and $Z_2$ with corresponding communities $\{C_{k,j}\}^{K_j}_{k=1}$, $j = 1, 2$, respectively. Then we say $Z_1$ is finer than $Z_2$ if for any $k_1 = 1, \cdots, K_1$, there exists $k_2 = 1, \cdots, K_2$ such that

$$C_{k_1,1} \subset C_{k_2,2}.$$ 

In this case, we write $Z_1 \succeq Z_2$.

Theorem 3.2 If Assumptions 1-4 hold, then (1) for $K = 1, \cdots, K_0$,

$$\hat{Z}_K = Z_K \quad \text{a.s. and } Z_{K_0} \succeq Z_K,$$

(2) for $K = 1, \cdots, K_0 - 1$,

$$\hat{Z}^b_{K+1} = Z^b_{K+1} \quad \text{a.s. and } Z_{K_0} \succeq Z^b_{K+1},$$

and (3) after relabeling, we have $\hat{C}^b_{k,K+1} = C_{k,K}$ for $k = 1, \cdots, K - 1$ and $C_{K,K} = \hat{C}^b_{K,K+1} \cup \hat{C}^b_{K+1,K+1}$, for $K = 1, \cdots, K_0$, a.s.

Theorem 3.2 (1) and (2) show that $\hat{Z}_K$ and $\hat{Z}^b_K$ will equal to their pseudo true counterparts almost surely. This is the oracle property of estimating the community membership when we
either under- or just-fit the model, i.e., $K \leq K_0$. On the other hand, it is very difficult, if not completely impossible, to show the similar oracle property in the over-fitting case, i.e., $K > K_0$. In particular, we are unable to uniquely define $Z_{K_{0}+1}^b$ and show that $\hat{Z}_{K_{0}+1}^b = Z_{K_{0}+1}^b$.

As pointed out by Wang and Bickel (2016), even in the population level (i.e., the probability matrix is observed), “embedding a $K$-block model in a larger model can be achieved by appropriately splitting the labels $Z$ and there are an exponential number of possible splits.” However, Theorem 3.2(3) with $K = K_0$ shows that, for any $k = 1, \cdots, K_0 + 1$, there exists some $k'$ such that $\hat{C}_{k,K_0+1}^b \subset \hat{C}_{k',K_0}$, which should be one of the true communities based on the oracle property. We can use this feature to handle the over-fitting case.

### 3.3 Properties of the pseudo-LR and the estimated number of communities

Without loss of generality, we assume that $\hat{Z}_K^b$ is obtained by splitting the last group in $\hat{Z}_{K-1}$ into the $(K-1)$-th and $K$-th groups in $\hat{Z}_K^b$. Further denote, for $k, l = 1, \cdots, K$ and $k \leq l$,

\[
\Gamma_{k,l}^0 (Z_K^b) = \sum_{s \in I(C_k^b), t \in I(C_l^b)} H_{st}(Z_{K_0}) \pi_{s\infty} \pi_{t\infty}
\]

and

\[
\Gamma_{l}^0 (Z_K^b) = \sum_{s \in I(C_{K_0}^b), t = 1, \cdots, K_0} H_{st}(Z_{K_0}) \pi_{s\infty} \pi_{t\infty},
\]

where $I(C_{k,K}^b)$ denotes a subset of $[K_0]$ such that if $m \in I(C_{k,K}^b)$, then $C_{m,K_0} \subset C_{k,K}^b$.

**Assumption 5** If $K_0 \geq 3$, then for any $K = 3, \cdots, K_0$, at least one of the following terms is not exactly equal to one:

\[
\frac{\Gamma_{k,l}^0 (Z_K^b)[\Gamma_{K-1}^0 (Z_K^b) + \Gamma_{K,K}^0 (Z_K^b)]}{\Gamma_l^0 (Z_K^b)[\Gamma_{k,K-1}^0 (Z_K^b) + \Gamma_{k,K}^0 (Z_K^b)]}, \quad k = 1, \cdots, K-2, \quad l = K-1, K,
\]

\[
\frac{\Gamma_{K-1,K-1}^0 (Z_K^b)[\Gamma_{K-1}^0 (Z_K^b) + \Gamma_{K,K}^0 (Z_K^b)]^2}{[\Gamma_{K-1}^0 (Z_K^b)]^2[\Gamma_{K-1,K-1}^0 (Z_K^b) + 2\Gamma_{K-1,K}^0 (Z_K^b) + \Gamma_{K,K}^0 (Z_K^b)]},
\]

\[
\frac{\Gamma_{K-1,K}^0 (Z_K^b)[\Gamma_{K-1}^0 (Z_K^b) + \Gamma_{K,K}^0 (Z_K^b)]^2}{[\Gamma_{K-1}^0 (Z_K^b)][\Gamma_{K-1,K-1}^0 (Z_K^b) + 2\Gamma_{K-1,K}^0 (Z_K^b) + \Gamma_{K,K}^0 (Z_K^b)]},
\]

\[
(3.3)
\]

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\[
\frac{\Gamma^0_{K,K}(Z^b_K)[\Gamma^0_{K-1}(Z^b_K) + \Gamma^0_{K}(Z^b_K)]^2}{(\Gamma^0_{K}(Z^b_K))^2[\Gamma^0_{K-1,K-1}(Z^b_K) + 2\Gamma^0_{K-1,K}(Z^b_K) + \Gamma^0_{K,K}(Z^b_K)]}.
\]

(3.4)

If \( K = K_0 = 2 \), then at least one of three terms (3.2)–(3.4) is not exactly one.

If the last two columns of \([\Gamma^0_{k,l}(Z^b_K)]\) are exactly the same, then all terms in Assumption 5 are equal to one. Assumption 5 rules out this case when \( K \leq K_0 \).

**Theorem 3.3** If Assumptions 1–4 hold, then, for \( 2 \leq K \leq K_0 \), there exists \( \tilde{B}_{K,n} \) such that

\[
\tilde{\omega}_{K,n}^{-1} n^{-1} \rho_n^{1/2} [L_n(\hat{Z}^b_K, \hat{Z}_{K-1}) - \tilde{B}_{K,n}] \sim N(0,1)
\]

where \( \tilde{\omega}_{K,n} = (\tilde{\omega}^2_{K,n})^{1/2} \), and the asymptotic bias \( \tilde{B}_{K,n} \) and variance \( \tilde{\omega}^2_{K,n} \) are defined in (7.10) and (7.23), respectively, in the appendix. If, in addition, Assumption 6 holds, then there exist two positive constants \( (c_{K1}, c_{K2}) \) potentially dependent on \( K \) such that

\[
c_{K2}n^2 \geq \tilde{B}_{K,n} \geq c_{K1}n^2.
\]

Theorem 3.3 shows that in the under-fitting case, the asymptotic bias term that is of order \( n^2 \) will dominate the centered pseudo likelihood ratio that is of order \( n\rho_n^{-1/2} \). However, when we over-fit the model, i.e., \( K > K_0 \), the asymptotic bias term will be zero. The sudden change in the orders of magnitude of the pseudo-LR \( L_n(\hat{Z}^b_K, \hat{Z}_{K-1}) \) provides useful information on the true number of communities.

**Assumption 6** There exists some sufficiently small constant \( \varepsilon \) such that

\[
\inf_{1 \leq k \leq K_0+1} n_k(\hat{Z}^b_{K_0+1})/n \geq \varepsilon.
\]

Assumption 6 always holds in our simulation. By Theorem 3.2, \( \hat{Z}_{K_0} = Z_{K_0} \) a.s. Suppose we obtain \( \hat{Z}^b_{K_0+1} \) by splitting the last community (i.e., the \( C_{K_0,K_0} \)) into two groups by binary segmentation. In simulation, we observe that the two new groups \( \hat{C}^b_{K_0,K_0+1} \) and \( \hat{C}^b_{K_0+1,K_0+1} \) have close to even sizes. In addition, we can modify our estimation procedure of \( \hat{Z}^b_{K_0+1} \) to ensure that Assumption 6 holds automatically. In particular, suppose \( n_{K_0}(\hat{Z}^b_{K_0+1}) \leq n\varepsilon \), then let

\[
\hat{C}^b_{K_0,K_0+1} = \hat{C}^b_{K_0,K_0+1} \cup \hat{C}^b_{K_0+1,K_0+1}(1) \quad \text{and} \quad \hat{C}^b_{K_0+1,K_0+1} = \hat{C}^b_{K_0+1,K_0+1}(2),
\]

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where \( \hat{C}_{b, K_0+1}^{K_0+1}(1) \) and \( \hat{C}_{b, K_0+1}^{K_0+1}(2) \) are obtained by randomly and evenly dividing \( \hat{C}_{b, K_0+1}^{K_0+1} \). Then \( \hat{C}_{b, K_0+1}^{b, new} \) and \( \hat{C}_{b, K_0+1}^{b, new} \) satisfy Assumption 6. Although we do not know \( K_0 \) a priori, we can apply this modification for any \( K = 1, \ldots, K_{\max} \). When \( K < K_0 \), Theorem 3.2(2) shows that, for some sufficiently small \( \varepsilon \),

\[
n_k(\hat{Z}_{K+1}^b) = n_k(Z_{K+1}^b) \geq \inf_k n_k(Z_{K_0}) \geq n\varepsilon \quad a.s.
\]

Therefore, the modification will never take action when \( K < K_0 \), which implies that all our results still hold under this modification.

**Theorem 3.4** Suppose that Assumptions 1–6 hold. Then

\[
0 \leq L_n(\hat{Z}_{K_0+1}^b, \hat{Z}_{K_0}) \leq O_{a.s.}(n\rho^{-1}).
\]

In addition, if \( h_n \to 0 \) and \( n\rho_nh_n \to \infty \), then

\[
P(\hat{K}_1 \geq K_0) \to 1,
\]

and

\[
P(\hat{K}_2 = K_0) \to 1.
\]

Several remarks are in order. First, Theorem 3.4 establishes the upper bound for the pseudo likelihood ratio in the over-fitting case. Like Wang and Bickel (2016), we are unable to obtain its exact limiting distribution because we do not have the oracle property for \( \hat{Z}_{K_0+1}^b \). The more profound reason for the lack of oracle property is that we have limited knowledge on the asymptotic behavior of the \( (K_0+1) \)-th column of the eigenvector matrix \( \hat{U}_n \). Fortunately, the upper bound is sufficient for the consistent estimation of \( K_0 \) with the help of the tuning parameter \( h_n \). Second, the tuning parameter \( \eta_n = c_\eta n^2 \) is only needed to deal with the case \( K = 1 \) in which the pseudo likelihood ratio cannot be defined. As remarked before, this tuning parameter is not needed if we are sure that \( K_0 \geq 2 \) so that we can obtain the estimate \( \hat{K}_1 \) by searching over \( K \in [2, K_{\max}] \). Alternatively, one can separately test \( K_0 = 1 \) using other methods, e.g., the eigenvalue-based test proposed by Bickel and Sarkar (2016), and then use our methods to select \( K \) for \( K \geq 2 \). In this case, one can also avoid the use of \( \eta_n \). Third, we show that \( \hat{K}_1 \) cannot under-estimate the number of communities in large
samples. This result is similar to that in Chen and Lei (2018) who showed that NCV does not underestimate the number of communities with w.p.a.1. Fourth, to obtain a consistent estimate of $K_0$, we can employ the estimator $\hat{K}_2$ which demands the specification of the tuning parameter $h_n$. This parameter plays the role of the penalty term in Wang and Bickel (2016)'s BIC-type information criterion. We propose to use $h_n = c_h d^{-1/2}$ where $d$ is the average degree of the network. We will investigate the sensitivity of the performance of $\hat{K}_2$ in the constant $c_h$ in Section 4.

As mentioned in the introduction, our pseudo-likelihood-based method has some computational advantages in comparison with the existing methods. In particular, it is well known that the likelihood-based method of Wang and Bickel (2016) is computationally intensive even after one uses a variational EM algorithm to approximate the true likelihood. The NCV method of Chen and Lei (2018) and the ECV method of Li et al. (2016) can also be computationally intensive when the number of folds is large.

4 Numerical Examples on Simulated Networks

4.1 Background and methods

In this section, we conduct simulations to evaluate the performance of our proposed method. We call our pseudo-LR estimators $\hat{K}_1$ and $\hat{K}_2$ as PLR1 and PLR2, respectively. Moreover, we compare our proposed method with four other approaches, including LRBIC (Wang and Bickel, 2016), NCV (Chen and Lei, 2018), ECV (Li et al., 2016) and BHMC (Le and Levina, 2015). LRBIC considers a likelihood-based approach for estimating the latent node labels and selecting models. LRBIC is only designed for the standard SBMs. It requires one to set the maximum number of communities ($K_{\text{max}}$) and to choose a tuning parameter to control the order of the BIC-type penalty. NCV applies cross-validation (CV) from spectral clustering, while ECV uses CV with edge sampling for choosing between SBM and DCSBM and selecting the number of communities simultaneously. NCV requires one to set $K_{\text{max}}$ and to choose two tuning parameters, viz, the number of folds for the CV and the number of repetitions to reduce the randomness of the estimator due to random sample splitting.
ECV requires one to set $K_{\text{max}}$ and to choose two tuning parameters, viz, the probability for an edge to be drawn and the number of replications. BHMC is developed by using the network Bethe-Hessian matrix with moment correction. It requires the selection of a scalar parameter to define the Bethe Hessian matrix and another one for fine-tuning. Like our method, BHMC can be generally applied to both SBM and DCSBM. We use the R package “randnet” to implement these four methods, and set $K_{\text{max}} = 10$ for all methods that require a maximal value when searching over $K$’s.

### 4.2 Data generation mechanisms and settings

We consider the following mechanisms to generate the connectivity matrix $B = \{B_{k\ell}\}_{1 \leq k, \ell \leq K_0}$.

**Setting 1 (S1).** Let $B_{k\ell} = 0.5\rho n^{-1/2}\{1 + I(k = \ell)\}$ for $1 \leq k, \ell \leq K_0$, and for some $\rho > 0$.

**Setting 2 (S2).** We first simulate $W = (W_1, \ldots, W_{M_0})^\top$ from Unif$(0, 0.3)^{M_0}$, where Unif($a, b)^{M_0}$ denotes an $M_0$-dimensional uniform distribution on $[a, b]$ and $M_0 = (K_0 + 1)K_0/2$. Let the main diagonal of $B$ be the $K_0$ largest elements in $W$ and the upper triangular part of $B$ contain the rest elements in $W$. Let $B_{k\ell} = B_{\ell k}$ for all $1 \leq k, \ell \leq K_0$. We use the generated $B$ with the smallest singular value no smaller than 0.1.

All simulation results are based on 200 realizations. S1 considers different sparsity levels for different values of $\rho$, and S2 allows all entries in $B$ to be different. The membership vector is generated by sampling each entry independently from $\{1, \ldots, K_0\}$ with probabilities $\{0.4, 0.6\}, \{0.3, 0.3, 0.4\}$ and $\{0.25, 0.25, 0.25, 0.25\}$ for $K_0 = 2, 3$ and 4, respectively. We consider both SBMs and DCSBMs. For the DCSBMs, we generate the degree parameters $\theta_i$ from Unif$(0.2, 1)$ and further normalize them to satisfy the condition (2.1).

### 4.3 Results

For our method, we let $\tau = \bar{d}$ and $c_\eta = 0.05$. Note that for computing the PLR2 estimator $\hat{K}_2$, we need a tuning parameter $h_n$. We set $h_n = c_h\bar{d}^{-1/2}$. We first would like to examine the performance of the PLR2 estimator when $c_h$ takes different values. Consider $c_h = 0.5, 1.0, 1.5, 2.0$. Let $\rho = 3, 4, 5$ for design S1. Tables [1] and [3] report the mean of $\hat{K}_2$ by the PLR2 method and the proportion (prop) of correctly estimating $K_0$ among 200 simulated
datasets when data are generated from SBMs and DCSBMs, respectively, for $n = 500, 1000$ and $K_0 = 1, 2, 3, 4$. For comparison, Tables 2 and 4 report those statistics for the PLR1 estimate $\hat{K}_1$. It is worth noting that when $c_h = 0$, the two estimates $\hat{K}_1$ and $\hat{K}_2$ are exactly the same. Comparing Tables 1 and 3 to Tables 2 and 4, we see that for smaller values of $c_h$, the behavior of $\hat{K}_2$ is more similar to that of $\hat{K}_1$. Moreover, Tables 1 and 3 show that the PLR2 estimator has similar performance at $c_h = 0.5, 1.0, 1.5, 2.0$ for design S1, and its performance improves when the value of $\rho$ or the sample size $n$ increases. However, for design S2, PLR2 behaves better at $c_h = 0.5, 1.0$. Overall, both PLR1 and PLR2 at $c_h = 0.5, 1.0$ have good performance, and PLR2 with $c_h = 1.0$ slightly outperforms PLR1 and PLR2 with $c_h = 0.5$.

Based on the above results, we let $c_h = 1.0$ for the PLR2 estimator. For evaluating the performance of the six methods at different sparsity levels, we let $\rho = 0.5, 1, 2, 3, 4, 5, 6$ for design S1, so that the average expected degree ranges from 7.0 to 83.9, for instance, at $K_0 = 4$ and $n = 500$ for the DCSBMs. Figure 1 shows the proportions of correctly estimating $K_0$ among 200 simulated datasets versus the values of $\rho$ for the six methods: PLR1 (solid lines), PLR2 (dash-dot lines), LRBIC (dashed lines), NCV (dotted lines), ECV (thin dash-dot lines) and BHMC (thin dotted lines), when data are simulated from design S1 with $K_0 = 2, 3, 4$ and $n = 500$. The results for the SBMs and DCSBMs are shown in the left and right panels, respectively. We observe that our proposed methods PLR1 and PLR2 have similar performance with PLR2 moderately better when $K_0 = 2$. Moreover, PLR1 and PLR2 have larger proportions of correctly estimating $K_0$ than the other four methods at small values of $\rho$. This indicates that PLR1 and PLR2 outperform other methods for sparse designs. The BHMC method performs better than LRBIC, NCV and ECV at $K_0 = 2, 3$, but its performance becomes inferior to that of the other three methods when $K_0 = 4$. It is worth noting that for larger $K_0$, it correspondingly requires a larger $\rho$ in order to successfully estimate $K_0$. When $\rho$ is sufficiently large, eventually all methods can successfully estimate $K_0$. Compared to the other four methods, PLR1 and PLR2 require less constraints on the sparsity level $\rho$ in order to correctly estimate $K_0$. For example, for the DCSBMs with $K_0 = 4$, the proportions of correctly estimating $K_0$ are 0.38 for PLR1 and PLR2, whereas the proportions are close to zero for other methods at $\rho = 3$. For the DCSBMs with $K_0 = 2$,
Table 1: The mean of $\hat{K}_2$ and the proportion (prop) of correctly estimating $K_0$ among 200 simulated datasets when data are generated from SBMs.

| $\rho$ | $c_h$ | $K_0 = 1$ | $K_0 = 2$ | $K_0 = 3$ | $K_0 = 4$ |
|--------|-------|-----------|-----------|-----------|-----------|
|        |       | 0.5 1.0 1.5 2.0 | 0.5 1.0 1.5 2.0 | 0.5 1.0 1.5 2.0 | 0.5 1.0 1.5 2.0 |
|        |       | $n = 500$ | $n = 1000$ | $n = 1000$ | $n = 1000$ |
| S1     | 3 mean | 1.035 1.000 1.000 1.000 | 2.025 2.000 2.000 2.000 | 3.060 3.060 3.000 3.000 | 3.465 3.465 3.430 3.355 |
|        | prop   | 0.995 1.000 1.000 1.000 | 0.995 1.000 1.000 1.000 | 0.990 0.990 1.000 1.000 | 0.355 0.355 0.350 0.330 |
|        | 4 mean | 1.000 1.000 1.000 1.000 | 2.030 2.000 2.000 2.000 | 3.115 3.015 3.000 3.000 | 4.085 4.085 4.085 4.005 |
|        | prop   | 1.000 1.000 1.000 1.000 | 0.995 1.000 1.000 1.000 | 0.975 0.995 1.000 1.000 | 0.925 0.925 0.925 0.925 |
|        | 5 mean | 1.000 1.000 1.000 1.000 | 2.000 2.000 2.000 2.000 | 3.000 3.000 3.000 3.000 | 4.060 4.060 4.060 4.000 |
|        | prop   | 1.000 1.000 1.000 1.000 | 1.000 1.000 1.000 1.000 | 1.000 1.000 1.000 1.000 | 0.980 0.980 0.980 1.000 |
| S2     | mean   | 1.000 1.000 1.000 1.000 | 2.000 2.000 2.000 2.000 | 3.000 3.000 2.035 2.000 | 4.000 3.995 3.820 3.620 |
|        | prop   | 1.000 1.000 1.000 1.000 | 1.000 1.000 1.000 1.000 | 1.000 1.000 0.035 0.000 | 1.000 0.995 0.895 0.795 |
Table 2: The mean of $\hat{K}_1$ and the proportion (prop) of correctly estimating $K_0$ among 200 simulated datasets when data are generated from SBMs.

|       | $n = 500$ |                     | $n = 1000$ |                     |
|-------|-----------|---------------------|------------|---------------------|
|       | $K_0 = 1$ | $K_0 = 2$ | $K_0 = 3$ | $K_0 = 4$ | $K_0 = 1$ | $K_0 = 2$ | $K_0 = 3$ | $K_0 = 4$ |
| $\rho$ |           |           |           |           |           |           |
| S1    | 3         | mean     | 1.035     | 2.095     | 3.115     | 3.465     | 1.000     | 2.055     | 3.040     | 4.080     |
|       |           | prop     | 0.995     | 0.980     | 0.975     | 0.355     | 1.000     | 0.990     | 0.985     | 0.980     |
|       | 4         | mean     | 1.000     | 2.045     | 3.060     | 4.085     | 1.000     | 2.000     | 3.015     | 4.020     |
|       |           | prop     | 1.000     | 0.990     | 0.990     | 0.925     | 1.000     | 1.000     | 0.995     | 0.995     |
|       | 5         | mean     | 1.000     | 2.020     | 3.015     | 4.060     | 1.000     | 2.000     | 3.045     | 4.030     |
| S2    |           | prop     | 1.000     | 0.995     | 0.995     | 0.980     | 1.000     | 1.000     | 0.990     | 0.990     |
|       |           | mean     | 1.000     | 2.000     | 3.110     | 4.000     | 1.000     | 2.000     | 3.000     | 4.000     |
|       |           | prop     | 1.000     | 1.000     | 0.980     | 1.000     | 1.000     | 1.000     | 1.000     | 1.000     |

the proportions are 0.71 and 0.89 for PLR1 and PLR2, respectively, and they are less than 0.1 for other methods at $\rho = 0.5$.

For further demonstration, Tables 5-7 report the mean of the estimated number of communities and the proportion (prop) of correctly estimating $K_0$ for designs S1 and S2 with $n = 500$. For S1, we observe the same pattern as shown in Figure 1. For S2 in which all entries of $B$ are different, the six methods have comparable performance.

## 5 Real Data Examples

In this section, we evaluate the performance of our method on several real-world networks.

### 5.1 Jazz musicians network

We apply the methods to analyze the collaboration network of Jazz musicians. The data was obtained from The Red Hot Jazz Archive digital database (www.redhotjazz.com). In our analysis, we include 198 bands that performed between 1912 and 1940. We study the
Table 3: The mean of $\hat{K}_2$ and the proportion (prop) of correctly estimating $K$ among 200 simulated datasets when data are generated from DCSBMs.

|   | $K_0 = 1$          | $K_0 = 2$          | $K_0 = 3$          | $K_0 = 4$          |
|---|-------------------|-------------------|-------------------|-------------------|
|   | $\rho$ $c_h$ 0.5 1.0 1.5 2.0 | $\rho$ $c_h$ 0.5 1.0 1.5 2.0 | $\rho$ $c_h$ 0.5 1.0 1.5 2.0 | $\rho$ $c_h$ 0.5 1.0 1.5 2.0 |
| S1 | mean 1.000 1.000 1.000 1.000 | mean 2.095 2.000 2.000 2.000 | mean 3.070 3.070 3.000 3.000 | mean 3.675 3.675 3.615 3.380 |
|   | prop 1.000 1.000 1.000 1.000 | prop 0.980 1.000 1.000 1.000 | prop 0.980 0.980 1.000 1.000 | prop 0.380 0.380 0.390 0.370 |
| S2 | mean 1.000 1.000 1.000 1.000 | mean 2.035 2.000 2.000 2.000 | mean 3.025 3.000 3.000 3.000 | mean 4.175 4.150 4.100 4.050 |
|   | prop 1.000 1.000 1.000 1.000 | prop 0.990 1.000 1.000 1.000 | prop 0.990 1.000 1.000 1.000 | prop 0.915 0.920 0.935 0.940 |
| S2 | mean 1.000 1.000 1.000 1.000 | mean 2.000 2.000 2.000 2.000 | mean 3.020 3.000 3.000 3.000 | mean 4.045 4.015 4.000 4.000 |
|   | prop 1.000 1.000 1.000 1.000 | prop 1.000 1.000 1.000 1.000 | prop 0.995 1.000 1.000 1.000 | prop 0.985 0.995 1.000 1.000 |
|   | $n = 500$         | $n = 1000$        | $n = 500$         | $n = 1000$        |

|   | $\rho$ $c_h$ 0.5 1.0 1.5 2.0 | $\rho$ $c_h$ 0.5 1.0 1.5 2.0 | $\rho$ $c_h$ 0.5 1.0 1.5 2.0 | $\rho$ $c_h$ 0.5 1.0 1.5 2.0 |
|---|-------------------|-------------------|-------------------|-------------------|
| S1 | mean 1.000 1.000 1.000 1.000 | mean 2.050 2.000 2.000 2.000 | mean 3.000 3.000 3.000 3.000 | mean 4.060 4.045 4.025 4.025 |
|   | prop 1.000 1.000 1.000 1.000 | prop 0.990 1.000 1.000 1.000 | prop 1.000 1.000 1.000 1.000 | prop 0.980 0.985 0.990 0.995 |
| S2 | mean 1.000 1.000 1.000 1.000 | mean 2.000 2.000 2.000 2.000 | mean 3.000 3.000 3.000 3.000 | mean 4.020 4.000 4.000 4.000 |
|   | prop 1.000 1.000 1.000 1.000 | prop 1.000 1.000 1.000 1.000 | prop 1.000 1.000 1.000 1.000 | prop 0.995 1.000 1.000 1.000 |
| S2 | mean 1.000 1.000 1.000 1.000 | mean 2.000 2.000 2.000 2.000 | mean 3.000 3.000 3.000 2.030 | mean 4.000 4.000 4.000 3.210 |
|   | prop 1.000 1.000 1.000 1.000 | prop 1.000 1.000 1.000 1.000 | prop 1.000 1.000 1.000 0.03 | prop 1.000 1.000 1.000 0.605 |
Figure 1: The proportions of correctly estimating $K_0$ versus the values of $\rho$ for the six methods, when data are simulated from design S1 with $K_0 = 2, 3, 4$ and $n = 500$. 
Table 4: The mean of $\hat{K}_1$ and the proportion (prop) of correctly estimating $K_0$ among 200 simulated datasets when data are generated from DCSBM.

|   | $n = 500$ | $n = 1000$ |
|---|-----------|------------|
|   | $K_0 = 1$ | $K_0 = 2$ | $K_0 = 3$ | $K_0 = 4$ | $K_0 = 1$ | $K_0 = 2$ | $K_0 = 3$ | $K_0 = 4$ |
|   |           |           |           |           |           |           |           |           |
| S1 | 3 mean    | 1.000     | 2.095     | 3.070     | 3.675     | 1.000     | 2.050     | 3.000     | 4.060     |
|    | prop      | 1.000     | 0.980     | 0.980     | 0.380     | 1.000     | 0.990     | 1.000     | 0.980     |
|    | 4 mean    | 1.000     | 2.090     | 3.025     | 4.175     | 1.000     | 2.000     | 3.000     | 4.020     |
|    | prop      | 1.000     | 0.980     | 0.990     | 0.915     | 1.000     | 1.000     | 1.000     | 0.995     |
|    | 5 mean    | 1.000     | 2.035     | 3.030     | 4.045     | 1.000     | 2.000     | 3.000     | 4.045     |
|    | prop      | 1.000     | 0.990     | 0.995     | 0.985     | 1.000     | 1.000     | 1.000     | 0.985     |
| S2 | mean      | 1.000     | 2.000     | 3.035     | 4.005     | 1.000     | 2.000     | 3.000     | 4.000     |
|    | prop      | 1.000     | 1.000     | 0.995     | 0.995     | 1.000     | 1.000     | 1.000     | 1.000     |

The community structure of the band network in which there are 198 nodes representing bands and 2742 unweighted edges indicating at least one common musician between two bands. The left panel of Figure 2 shows the degree distribution for the jazz band network. The minimal, average and maximum degrees of this network are 1.0, 27.7 and 100.0, respectively. Moreover, the distribution of degrees spreads over the range from 1 to 62 with four degree values outside this range. This indicates that the node degrees are highly varying for this network.

Let $K_{\text{max}} = 10$ for all methods. We apply our proposed PLR1 and PLR2 methods to estimate the number of communities and obtain that $\hat{K}_1 = 3$ and $\hat{K}_2 = 3$, so that three communities are identified by both methods. For further illustration, the right panel of Figure 2 depicts the band network with 198 nodes divided into three communities. The results confirm the community structure mentioned in [Gleiser and Danon, 2003] that the band network is divided into two large communities based on geographical locations where the bands recorded, and the largest community also splits into two communities due to a racial segregation. Moreover, we obtain the estimated edge probabilities within communities.
Table 5: The mean of $\hat{K}$ by the six methods and the proportion (prop) of correctly estimating $K_0$ among 200 simulated datasets for $K_0 = 2$ and $n = 500$.

|       | $\rho$ |     | S1  |     |     |     |     | S2  |     |     |     |     |
|-------|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|       |        |     | 0.5 | 1   | 2   | 3   | 4   | 5   | 6   |     |     |     |
| SBM   |        |     |     |     |     |     |     |     |     |     |     |     |
| PLR1  | mean   | 2.865 | 2.380 | 2.235 | 2.095 | 2.045 | 2.020 | 2.000 | 2.000 |    |     |     |
|       | prop   | 0.765 | 0.880 | 0.960 | 0.980 | 0.990 | 0.995 | 1.000 | 1.000 |    |     |     |
| PLR2  | mean   | 2.290 | 2.285 | 2.025 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 |    |     |     |
|       | prop   | 0.875 | 0.900 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |    |     |     |
| LRBIC | mean   | 1.000 | 1.000 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 |    |     |     |
|       | prop   | 0.000 | 0.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |    |     |     |
| NCV   | mean   | 1.055 | 1.105 | 2.205 | 2.005 | 2.010 | 2.020 | 2.000 | 2.005 |    |     |     |
|       | prop   | 0.045 | 0.095 | 0.815 | 0.995 | 0.990 | 0.995 | 1.000 | 0.995 |    |     |     |
| ECV   | mean   | 1.000 | 1.000 | 2.005 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 |    |     |     |
|       | prop   | 0.000 | 0.000 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |    |     |     |
| BHMC  | mean   | 1.065 | 1.865 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 |    |     |     |
|       | prop   | 0.065 | 0.845 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |    |     |     |
| DCSBM |        |     |     |     |     |     |     |     |     |     |     |     |
| PLR1  | mean   | 3.015 | 2.425 | 2.120 | 2.095 | 2.090 | 2.035 | 2.025 | 2.000 |    |     |     |
|       | prop   | 0.710 | 0.905 | 0.980 | 0.980 | 0.980 | 0.990 | 0.995 | 1.000 |    |     |     |
| PLR2  | mean   | 2.275 | 2.205 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 |    |     |     |
|       | prop   | 0.890 | 0.950 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |    |     |     |
| LRBIC | mean   | 1.000 | 1.000 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 |    |     |     |
|       | prop   | 0.000 | 0.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |    |     |     |
| NCV   | mean   | 1.150 | 1.170 | 2.040 | 1.970 | 1.995 | 2.000 | 2.000 | 2.005 |    |     |     |
|       | prop   | 0.090 | 0.130 | 0.790 | 0.960 | 0.975 | 1.000 | 1.000 | 0.995 |    |     |     |
| ECV   | mean   | 1.000 | 1.010 | 2.000 | 2.005 | 2.000 | 2.000 | 2.000 | 2.000 |    |     |     |
|       | prop   | 0.000 | 0.010 | 0.990 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 |    |     |     |
| BHMC  | mean   | 1.080 | 1.880 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 | 2.000 |    |     |     |
|       | prop   | 0.080 | 0.880 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |    |     |     |
Table 6: The mean of $\hat{K}$ by the six methods and the proportion (prop) of correctly estimating $K_0$ among 200 simulated datasets for $K_0 = 3$ and $n = 500$.

|                | $\rho$ | S1       | S2       | SBM          |
|----------------|--------|----------|----------|--------------|
|                | 0.5    | 1        | 2        | 3            | 4    | 5    | 6    |
| PLR1           | mean   | 3.035    | 2.715    | 2.975        | 3.115 | 3.060 | 3.015 | 3.000 | 3.110 |
|                | prop   | 0.080    | 0.085    | 0.535        | 0.975 | 0.990 | 0.995 | 1.000 | 0.980 |
| PLR2           | mean   | 2.125    | 2.595    | 2.975        | 3.060 | 3.015 | 3.000 | 3.000 | 3.000 |
|                | prop   | 0.045    | 0.075    | 0.535        | 0.990 | 0.995 | 1.000 | 1.000 | 1.000 |
| LRBIC          | mean   | 1.000    | 1.000    | 1.005        | 2.960 | 3.000 | 3.000 | 3.000 | 3.000 |
|                | prop   | 0.000    | 0.000    | 0.000        | 0.960 | 1.000 | 1.000 | 1.000 | 1.000 |
| NCV            | mean   | 1.045    | 1.050    | 1.495        | 2.830 | 3.015 | 3.015 | 3.000 | 3.030 |
|                | prop   | 0.000    | 0.000    | 0.070        | 0.710 | 0.985 | 0.995 | 1.000 | 0.970 |
| ECV            | mean   | 1.000    | 1.000    | 1.400        | 2.905 | 3.005 | 3.000 | 3.000 | 3.005 |
|                | prop   | 0.000    | 0.000    | 0.045        | 0.905 | 0.995 | 1.000 | 1.000 | 0.995 |
| BHMC           | mean   | 1.055    | 1.160    | 2.335        | 3.000 | 3.000 | 3.000 | 3.000 | 3.000 |
|                | prop   | 0.000    | 0.000    | 0.335        | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

|                | $\rho$ | S1       | S2       | DCSBM        |
|----------------|--------|----------|----------|--------------|
|                | 0.5    | 1        | 2        | 3            | 4    | 5    | 6    |
| PLR1           | mean   | 2.925    | 2.930    | 3.180        | 3.070 | 3.025 | 3.030 | 3.025 | 3.035 |
|                | prop   | 0.070    | 0.149    | 0.530        | 0.980 | 0.990 | 0.995 | 0.995 | 0.995 |
| PLR2           | mean   | 2.125    | 2.830    | 3.150        | 3.070 | 3.000 | 3.000 | 3.000 | 3.000 |
|                | prop   | 0.075    | 0.100    | 0.535        | 0.980 | 0.990 | 1.000 | 1.000 | 1.000 |
| LRBIC          | mean   | 1.000    | 1.000    | 1.025        | 2.955 | 3.000 | 3.000 | 3.000 | 3.000 |
|                | prop   | 0.000    | 0.000    | 0.000        | 0.955 | 1.000 | 1.000 | 1.000 | 1.000 |
| NCV            | mean   | 1.040    | 1.065    | 1.595        | 2.955 | 3.000 | 3.005 | 3.000 | 3.010 |
|                | prop   | 0.005    | 0.000    | 0.085        | 0.820 | 0.990 | 0.995 | 1.000 | 0.990 |
| ECV            | mean   | 1.000    | 1.000    | 1.350        | 2.940 | 3.005 | 3.000 | 3.000 | 3.000 |
|                | prop   | 0.000    | 0.000    | 0.030        | 0.930 | 0.995 | 1.000 | 1.000 | 1.000 |
| BHMC           | mean   | 1.055    | 1.145    | 2.415        | 2.995 | 3.000 | 3.000 | 3.000 | 3.000 |
|                | prop   | 0.000    | 0.000    | 0.415        | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 |
Table 7: The mean of $\hat{K}$ by the six methods and the proportion (prop) of correctly estimating $K_0$ among 200 simulated datasets for $K_0 = 4$ and $n = 500$.

| Method   | $\rho$ | S1 | S2 |
|----------|--------|----|----|
|          |        | 0.5| 1  |
|          |        | 2  | 3  |
|          |        | 4  | 5  |
|          |        | 6  |    |
|          |  SBM   |    |    |
| PLR1     | mean   | 2.665 | 2.850 | 3.200 | 3.465 | 4.085 | 4.060 | 4.000 | 4.000 |
|          | prop   | 0.015 | 0.025 | 0.035 | 0.355 | 0.925 | 0.980 | 1.000 | 1.000 |
| PLR2     | mean   | 2.300 | 2.850 | 2.665 | 3.465 | 4.085 | 4.060 | 4.000 | 3.995 |
|          | prop   | 0.015 | 0.025 | 0.025 | 0.355 | 0.925 | 0.980 | 1.000 | 0.995 |
| LRBIC    | mean   | 1.000 | 1.000 | 1.000 | 1.005 | 3.840 | 4.000 | 4.000 | 4.000 |
|          | prop   | 0.000 | 0.000 | 0.000 | 0.000 | 0.920 | 1.000 | 1.000 | 1.000 |
| NCV      | mean   | 1.015 | 1.020 | 1.004 | 1.500 | 4.030 | 4.005 | 4.000 | 4.060 |
|          | prop   | 0.000 | 0.000 | 0.000 | 0.070 | 0.740 | 0.965 | 1.000 | 0.940 |
| ECV      | mean   | 1.000 | 1.000 | 1.000 | 1.370 | 3.905 | 4.000 | 4.000 | 4.000 |
|          | prop   | 0.000 | 0.000 | 0.000 | 0.035 | 0.845 | 1.000 | 1.000 | 1.000 |
| BHMC     | mean   | 1.035 | 1.020 | 1.200 | 2.330 | 3.610 | 3.985 | 4.000 | 4.000 |
|          | prop   | 0.000 | 0.000 | 0.000 | 0.015 | 0.630 | 0.985 | 1.000 | 1.000 |
|          | DCSBM  |    |    |    |    |    |    |    |    |
| PLR1     | mean   | 2.750 | 2.780 | 2.765 | 3.675 | 4.175 | 4.045 | 4.010 | 4.005 |
|          | prop   | 0.030 | 0.040 | 0.040 | 0.380 | 0.915 | 0.985 | 0.995 | 0.995 |
| PLR2     | mean   | 2.105 | 2.655 | 2.745 | 3.675 | 4.150 | 4.015 | 4.000 | 4.005 |
|          | prop   | 0.000 | 0.015 | 0.040 | 0.380 | 0.920 | 0.995 | 1.000 | 0.995 |
| LRBIC    | mean   | 1.000 | 1.000 | 1.000 | 1.005 | 3.845 | 4.000 | 4.000 | 4.000 |
|          | prop   | 0.000 | 0.000 | 0.000 | 0.000 | 0.920 | 1.000 | 1.000 | 1.000 |
| NCV      | mean   | 1.050 | 1.003 | 1.045 | 1.805 | 4.005 | 4.015 | 4.020 | 4.060 |
|          | prop   | 0.000 | 0.000 | 0.000 | 0.100 | 0.700 | 0.980 | 0.980 | 0.940 |
| ECV      | mean   | 1.000 | 1.000 | 1.000 | 1.435 | 3.895 | 4.000 | 4.005 | 4.005 |
|          | prop   | 0.000 | 0.000 | 0.000 | 0.040 | 0.840 | 1.000 | 0.995 | 0.995 |
| BHMC     | mean   | 1.075 | 1.015 | 1.285 | 2.360 | 3.575 | 3.985 | 4.000 | 4.000 |
|          | prop   | 0.000 | 0.000 | 0.000 | 0.050 | 0.600 | 0.985 | 1.000 | 1.000 |
Figure 2: Left panel shows the degree distribution; right panel depicts the jazz band network with three communities.

Degree distribution of jazz band network

| Degree | Fraction of bands |
|--------|-------------------|
| 0      | 0.00              |
| 1      | 0.01              |
| 2      | 0.02              |
| 3      | 0.03              |

Jazz band network

which are $\hat{B}_{kk} = 0.349, 0.297, 0.358$ for $k = 1, 2, 3$, respectively, and edge probabilities between communities which are $\hat{B}_{12} = 0.029$, $\hat{B}_{13} = 0.087$ and $\hat{B}_{23} = 0.007$. Lastly, we obtain the estimated number of communities as 8, 3, 6 and 7, respectively, by the LRBIC, NCV, ECV and BHMC methods.

5.2 Political books network

We investigate the community structure of a network of US political books (available at www.orgnet.com) by different methods. In this network, there are 105 nodes representing books about US politics published around the 2004 presidential election and sold by the online bookseller Amazon.com, and there are 441 edges representing frequent co-purchasing of books by the same buyers. The left graph of Figure 3 shows the degree distribution for the political books network with the average degree being 8.4. We see that the degree has a right skewed distribution with most values ranging from 2 to 9. Let $K_{\text{max}} = 10$. We identify $\hat{K}_1 = \hat{K}_2 = 3$ communities by both PLR1 and PLR2. This result is consistent with
Figure 3: Left panel shows the degree distribution; right panel depicts the political books network with three communities.

**Degree distribution of political books network**

| Degree | Fraction of bands |
|--------|-------------------|
| 5      | 0.00              |
| 10     | 0.05              |
| 15     | 0.10              |
| 20     | 0.15              |
| 25     | 0.20              |

The ground-truth community structure that these books are actually divided into three categories “liberal”, “neutral” and “conservative” according to their political views (Newman, 2006). For further demonstration, we plot the political books network with three communities in the right panel of Figure 3. Groups 1, 2 and 3 represent the estimated communities of liberal, conservative and neutral books. We also obtain the estimated edge probabilities within communities which are $\hat{B}_{kk} = 0.219, 0.224, 0.164$ for $k = 1, 2, 3$, and the edge probabilities between communities which are $\hat{B}_{12} = 0.001, \hat{B}_{13} = 0.019$ and $\hat{B}_{23} = 0.224$. We see that groups 1 and 2 from two different political affiliations are very weakly connected. We apply the LRBIC, NCV, ECV and BHMC methods, and obtain the estimated number of communities as 3, 6, 8 and 4, respectively, by these four methods.

### 5.3 Facebook friendship network

We apply our methods to a large social network which contains friendship data of Facebook users (available at www.snap.stanford.edu). A node represents a user and an edge represents...
a friendship between two users. The data have 4039 nodes and 88218 edges. We use the nodes with the degree between 10 and 300. As a result, there are 2901 nodes and 80259 edges in our analysis. The left graph of Figure 4 shows the degree distribution for the Facebook friendship network with the average degree being 55.33. The degree distribution is again right skewed. Let $K_{\text{max}} = 20$. By using the proposed PLR1 and PLR2 methods, we identify $\hat{K}_1 = \hat{K}_2 = 11$ communities. The right panel of Figure 4 shows the estimated community structure of the Facebook friendship network with eleven identified communities. We can observe sub-communities of friends who are tightly connected through mutual friendships. Lastly, the LRBIC, NCV, ECV and BHMC methods found 19, 19, 20 and 14 communities, respectively.
6 Conclusion

We propose a new pseudo conditional likelihood ratio method for selecting the number of communities in DCSBMs. The method can be naturally applied to SBMs. For estimating the model, we consider the spectral clustering together with a binary segmentation algorithm. This estimation approach enables us to establish the limiting distribution of the pseudo likelihood ratio when the model is under-fitted, and derive the upper bound for it when the model is over-fitted. Based on these properties, we show the consistency of our estimator for the true number of communities. Our method is computationally fast as the estimation is based on spectral clustering, and it also has appealing theoretical properties for the sparse and degree-corrected designs. Moreover, our numerical results show that the proposed method has good finite sample performance in various simulation designs and real data applications, and it outperforms several other popular methods in sparse networks.

There are several interesting extensions of this work for future research. First, we are interested in extending the method to other block models such as overlapping and bipartite SBMs. Second, we would like to adapt the method to dynamic SBMs. Last, we will investigate the theoretical properties of the approach by allowing the number of communities to grow with the number of nodes. In sum, the aforementioned three possible avenues can be future studies, and they require separate thorough investigation.

7 Appendix

7.1 Proofs of results in Section 3

Proof of Theorem 3.1. The first two results are proved in Su et al. (2017, Theorem 3.3). For part (3), by the proof of Su et al. (2017, Theorem 3.3), $S_n$ is the $K_0 \times K_0$ eigenvector matrix of $(\Pi_n^r)^{1/2} H_0(Z_{K_0})(\Pi_n^r)^{1/2}$ with the corresponding eigenvalues ordered from the biggest to the smallest in absolute values. By Assumptions 1 and 2, we have

$$(\Pi_n^r)^{1/2} H_0(Z_{K_0})(\Pi_n^r)^{1/2} \rightarrow \Pi_\infty^{1/2} H_0(Z_{K_0})\Pi_\infty^{1/2} := S_\infty \Sigma_\infty S_\infty.$$
By Davis-Kahan Theorem in [Yu, Wang, and Samworth (2015)] and the fact that the smallest eigenvalue in absolute value of $\Pi_{\infty}^{1/2}H_0'(Z_{K_0})\Pi_{\infty}^{1/2}$ is nonzero, there exists a $K \times K$ orthogonal matrix $O_s$ such that $S_n^\tau[1 : K|O_s \rightarrow S_{\infty}[1 : K]$ where $S_{\infty}$ is the eigenvector matrix of $\Pi_{\infty}^{1/2}H_0'(Z_{K_0})\Pi_{\infty}^{1/2}$ and is of full rank. In addition, by Assumptions 1(2) and 2, all elements in $\Pi_{\infty}^{1/2}H_0'(Z_{K_0})\Pi_{\infty}^{1/2}$ are positive. By [Horn and Johnson (1990), Lemma 8.2.1], all elements in the first column of $S_{\infty}$ are strictly positive. This implies that, for any $k = 1, \ldots, K_0$,

$$\lim_{n} \inf ||[S_n^\tau]_{k,1 : K}|| = \lim_{n} \inf ||[S_n^\tau]_{k,1 : K}O_s|| = ||[S_{\infty}]_{k,1 : K}|| \geq ||[S_{\infty}]_{k,1}|| > 0.$$  

Recall that if $i \in C_{k,K_0}$, then

$$u_i^T(1 : K) = (\theta_i^\tau)^{1/2}(n_k^T(Z_{K_0}))^{-1/2}S_n^\tau(1 : K).$$

Because $S_n^\tau(1 : K)$ is a $K_0 \times K$ matrix, it is easy to see that $L_K \leq K_0$. In addition, if $i \in C_{k,K_0}$ and $j \in C_{l,K_0}$,

$$||u_i^T(1 : K)|| - ||u_j^T(1 : K)|| \leq ||[S_n^\tau]_{k,1 : K}|| - ||[S_n^\tau]_{l,1 : K}||O_s|| \leq ||[S_{\infty}]_{k,1 : K}|| - ||[S_{\infty}]_{l,1 : K}||.$$  

(7.1)

Because $S_{\infty}$ is of full rank, the first $K$ columns of $S_{\infty}$ should have rank $K$. This implies the $K_0$ number of $K \times 1$ row vectors $\{[S_{\infty}]_{k,1 : K}||[S_{\infty}]_{l,1 : K}\}_{k=1}^{K_0}$ take at least $K$ distinct values. Therefore, $L_K \geq K$. Last, we call nodes $i$ and $j$ are equivalent if

$$\lim_{n \to \infty} ||u_i^T(1 : K)|| - ||u_j^T(1 : K)|| = 0.$$  

Then $G_{l,K}$ can be constructed as the equivalence class of the above equivalence relation. Let

$$I = \left\{(k,l) : ||[S_{\infty}]_{k,1 : K}|| - ||[S_{\infty}]_{l,1 : K}|| \neq 0, k = 1, \ldots, K_0, l = 1, \ldots, K_0\right\}.$$  

In view of the fact that the cardinality of $I$ is finite, we have

$$c^* = \min_{(k,l) \in I} ||[S_{\infty}]_{k,1 : K}|| - ||[S_{\infty}]_{l,1 : K}|| > 0.$$  

Then, by (7.1), if nodes $i$ and $j$ are not equivalent,

$$\lim_{n} \inf ||u_i^T(1 : K)|| - ||u_j^T(1 : K)|| \geq c^* > 0.$$  

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This implies that $\{G_{l,K}\}_{l=1}^{L_K}$ constructed as the equivalence class satisfy the two requirements in Theorem 3.1(4) with $c = c^*$. ■

**Proof of Theorem 3.2.** First, we prove Theorem 3.2(1). Let $\hat{g}_{iK}$ be the membership estimated by the k-means algorithm with $K$ centroids, i.e.,

$$\hat{g}_{iK} = \arg\min_{1 \leq k \leq K} ||\hat{\nu}_{iK} - \hat{\alpha}_k||$$

and

$$\{\hat{\alpha}_k\}_{k=1}^{K} = \arg\min_{\alpha_1, \ldots, \alpha_K} \frac{1}{n} \sum_{i=1}^{n} \min_{1 \leq k \leq K} ||\hat{\nu}_{iK} - \alpha_k||^2.$$  

Because $L_2$-norm is invariant under rotation,

$$\hat{g}_{iK} = \arg\min_{1 \leq k \leq K} ||\hat{\nu}_{iK} \hat{O}_{Kn} - \hat{\alpha}_k||$$

and

$$\{\hat{\alpha}_k\}_{k=1}^{K} = \arg\min_{\alpha_1, \ldots, \alpha_K} \frac{1}{n} \sum_{i=1}^{n} \min_{1 \leq k \leq K} ||\hat{\nu}_{iK} \hat{O}_{Kn} - \alpha_k||^2. \quad (7.2)$$

where $\hat{O}_{Kn}$ is a $K \times K$ orthonormal matrix such that $\hat{O}_{Kn} = \bar{U} \bar{V}^T$, $\bar{U} \bar{\Sigma} \bar{V}^T$ is the singular value decomposition of $\hat{U}_n(1 : K)^T \hat{U}_n(1 : K)$, and $U_n$ is the population analogue of $\hat{U}_n : \mathcal{L}_r = U_n \Sigma_n U_n^T$. Here, $\Sigma_n = \text{diag}(\sigma_{1n}, \ldots, \sigma_{Kn}, 0, \ldots, 0)$ is a $n \times n$ matrix and we suppress the dependence of $\bar{U}$, $\bar{\Sigma}$, and $\bar{V}$ on $K$.

We aim to show

$$\sup_i 1\{\hat{g}_{iK} \neq g_{iK}\} = 0 \ a.s. \quad (7.3)$$

Suppose that

$$\sup_{1 \leq i \leq n} ||\hat{\nu}_{iK}^T \hat{O}_{Kn} - \nu_{iK}^T|| = o_{a.s.}(1), \quad (7.4)$$

which we will prove later. In addition, by (3.1),

$$\{\alpha^*_k\}_{k=1}^{K} = \arg\min_{\alpha_1, \ldots, \alpha_K} \sum_{l=1}^{K_0} \pi_{ln} \min_{1 \leq k \leq K} ||\bar{\nu}_{lK} - \alpha_k||^2.$$  

Then for any $k = 1, \ldots, K$, we have

$$\alpha^*_k = \sum_{l \leq K_0; C_{l,K_0} \subseteq C_{k,K}} \psi_{n,k,l} \bar{\nu}_{lK},$$

or in matrix form,

$$(\alpha^*_1, \ldots, \alpha^*_K) = (\bar{\nu}_{1K}, \ldots, \bar{\nu}_{L_K,K}) \Psi_n',$$

where $\psi_{n,k,l} = \pi_{ln} / (\sum_{l \leq K_0; C_{l,K_0} \subseteq C_{k,K}} \pi_{ln})$ for $k = 1, \ldots, K$ and $l = 1, \ldots, L_K$, and $\Psi_n = [\psi_{n,k,l}]$. Note that $L_K \geq K$. By Assumption 2 $\Psi_n \to \Psi_\infty$, where $[\Psi_\infty]_{k,l} = \frac{\pi_{l\infty}}{\sum_{l \leq K_0; C_{l,K_0} \subseteq C_{k,K}} \pi_{l\infty}}$.
Theorem 3.1(3) shows that, if \( \tau \) for any \( z \) where \( d \), then (7.3) follows Su et al. (2017, Theorem 2.3) with their \( \hat{\theta} \). Because \( Z \) holds with \( c \) because Assumption 2 implies Su et al. (2017, Assumption \( \nu \), this verifies Su et al. (2017, Assumption 10). Hence, by Su et al. (2017, Theorem 3.4),

\[
\Psi_\infty' = (\Psi_{1,\infty}, \Psi_{2,\infty}),
\]

where \( \Psi_{1,\infty} \) is a \( K \times K \) diagonal matrix with strictly positive diagonal elements. Therefore, \( \Psi_\infty \) has rank \( K \). By Theorem 3.1(3), \( \tilde{\psi}_1, \cdots, \tilde{\psi}_{L_K,K} \) also has rank \( K \). This implies, the limit of the \( K \times K \) matrix \( (\alpha_1^*, \cdots, \alpha_K^*) \) is of full rank. Therefore, there exists a constant \( \zeta > 0 \) such that

\[
\liminf_n \min_{k \neq k'} |\alpha_k^* - \alpha_{k'}^*| > \zeta. \tag{7.5}
\]

Then (7.3) follows Su et al. (2017, Theorem 2.3) with their \( \hat{\psi}_{in} = \hat{\psi}_{1K} \hat{O}_{Kn} \) and \( \hat{\beta}_{g_{Kn}} = \nu_{1K}^T \) because Assumption 2 implies Su et al. (2017, Assumption 2) and Su et al. (2017, Assumption 4) holds with \( c_{2n} = o(1) \) and \( c_{1n} = \zeta > 0 \) due to (7.4) and (7.5).

Now we turn to prove (7.4). Note that, by the proof of Su et al. (2017, Theorem 3.3), \( \sigma_{Kn} \) is the smallest in absolute value eigenvalue of \( \mu_n^{1/2} \hat{O}_{Kn} \). Since

\[
\mu_n^{1/2} \hat{O}_{Kn} \rightarrow (\mu_1^{1/2} \hat{O}_{K_0}(\mu_\infty) \) which has full rank, \( \inf_n |\sigma_{Kn}| \geq \inf_n |\sigma_{Kn}| > 0 \) for any \( K \leq K_0 \). Second, Assumption 4 implies Su et al. (2017, Assumption 11). Last, let \( d_i^* = d_i + \tau \). Since \( \tau \leq Mn\rho_n \) for some \( M > 0 \) and \( d_i \approx n\rho_n \), we have

\[
d_i^*/d_i \approx 1.
\]

Therefore, there exist constants \( C > c > 0 \) such that

\[
C \geq \sup_{k,n} n_{ik}^* d_i^*/(n d_i) \geq \inf_{k,n} n_{ik}^* d_i^*/(n d_i) \geq c.
\]

This verifies Su et al. (2017, Assumption 10). Hence, by Su et al. (2017, Theorem 3.4),

\[
\sup_{i} (\mu_i^*)^{1/2} \theta_i^{-1/2} \| \hat{u}_i (1 : K)^T \hat{O}_{Kn} - u_i^T (1 : K) \| \leq \log^{1/2}(n)(n\rho_n)^{-1/2} = o(1) \quad \text{a.s.}, \tag{7.6}
\]

where \( z_i \) denotes the membership index of node \( i \), viz, \( z_i = k \) if \( [Z_{K_0}]_{ik} = 1 \). In addition, Theorem 3.1(3) shows that, if \( i \in C_{k,K_0} \) for any \( k = 1, \cdots, K_0 \), then

\[
\liminf_n (\mu_i^*)^{1/2} \theta_i^{-1/2} \| \mu_i^T u_i (1 : K) \| = \liminf_n \|[S_n]_{ik} (1 : K)\| \geq c.
\]

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Therefore,
\[
\sup_i ||\hat{\nu}_{ik}^T \hat{O}_{Kn} - \nu_{ik}^T||
\leq \sup_i ||\hat{\nu}_{ik}^T \hat{O}_{Kn} - \frac{u_i(1:K)}{||u_i(1:K)||}|| + \sup_i \frac{||u_i(1:K)||}{||u_i(1:K)||} - \nu_{ik}^T||
\leq \sup_{1 \leq i \leq n} \frac{||\hat{\nu}_{ik}^T \hat{O}_{Kn} - u_i(1:K)||}{||u_i(1:K)||} + o(1)
\leq \frac{o_{a.s.}(1)}{c - o_{a.s.}(1)} + o(1) = o_{a.s.}(1),
\] (7.7)
where the second inequality holds because of the definition of $\nu_{ik}$ and Theorem 3.1. This concludes the proof of (7.3). We also note that, by definition, for any $K = 1, \ldots, K_0$ and $k = 1, \ldots, K_0$, there exists $l = 1, \ldots, L_K$ such that $C_{k,K_0} \subset G_{l,K}$. In addition, by (3.1) and Assumption 3(1), for any $l = 1, \ldots, L_K$, there exists $k' = 1, \ldots, K$ such that $G_{l,K} \subset C_{k',K}$. Therefore,

$$
C_{k,K_0} \subset G_{l,K} \subset C_{k',K} \quad \text{and} \quad Z_{K_0} \supseteq Z_K.
$$

**Second, we prove Theorem 3.2(2).** We know from Theorem 3.2(1) that $\hat{Z}_{K-1} = Z_{K-1}$ a.s., i.e., $\hat{C}_{k,K-1} = C_{k,K-1}$ for $k = 1, \ldots, K-1$. We aim to show that $\hat{Z}_K = Z_K$ almost surely for $K = 2, \ldots, K_0$. Recall $\hat{C}_{k,K-1} = C_{k,K-1} \cap G_{l,K}$. We divide $[K-1]$ into two subsets $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $k \in \mathcal{K}_1$ if there exists at least two indexes $l_1$ and $l_2$ such that both $\hat{C}_{k_1,K}$ and $\hat{C}_{k_2,K}$ are nonempty sets and $\mathcal{K}_2 = [K-1] \setminus \mathcal{K}_1$. Note that $L_K \geq K > K-1$. Therefore, by the pigeonhole principle, $\mathcal{K}_1$ is nonempty. We divide the proof into three steps. For a generic $k \in \mathcal{K}_1$, denote $\hat{C}_{k,K-1}(1)$ and $\hat{C}_{k,K-1}(2)$ as two subsets of $C_{k,K-1}$ which are obtained by applying k-means algorithm on $\{\hat{\nu}_m(1:K)\}_{m \in C_{k,K-1}}$ with two centroids. Similarly, let $C_{k,K-1}(1)$ and $C_{k,K-1}(2)$ as two subsets of $C_{k,K-1}$ which are obtained by applying k-means algorithm on $\{\nu_{ik}^b\}_{i \in C_{k,K-1}}$ with two centroids, where $\nu_{ik}^b$ is defined in Section 3.2. In the first step, we aim to show $\hat{k} = k^* \in \mathcal{K}_1$ a.s., where $\hat{k}$ is defined in Step 4 of the procedure in Section 2.2. In the second step, we aim to show that $\hat{C}_{k^*,K-1}(1) = C_{k^*,K-1}(1)$ and $\hat{C}_{k^*,K-1}(2) = C_{k^*,K-1}(2)$ a.s. These two results imply that

$$
C_{k^*,K-1}(1) = \hat{C}_{k,K-1}(1) \quad \text{and} \quad C_{k^*,K-1}(2) = \hat{C}_{k,K-1}(2),
$$
which completes the proof of $\hat{Z}_K = Z_K$ for $k = 1, \ldots, K_0$. Last, in the third step, we show that $Z_{K_0} \supseteq Z_{K+1}$. 

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**Step 1.** We show that $\hat{k} = k^* \in \mathcal{K}_1$ a.s. For a generic $k \in \mathcal{K}_1$, because the $L_2$-norm is invariant under rotation, we can regard the procedure as applying k-means algorithm to $\hat{\beta}_{in} = \hat{O}_{Kn}^T \hat{v}_{iK}$ for $i \in C_{k,K-1}$. Further denote $\beta_{in} = \nu_{iK}^b$. Then, $\beta_{in} = \beta_{jn}$ if $i, j \in \hat{C}_{k,K-1}^l$ for some $l$, and

$$\sup_{i \in C_{k,K-1}} \|\beta_{in} - \beta_{in}\| \leq \sup_{i \in C_{k,K-1}} \|\hat{v}_{iK}^T \hat{O}_{Kn} - \frac{u_i(1 : K)}{\|u_i(1 : K)\|}\| + \sup_{i \in C_{k,K-1}} \|\frac{u_i(1 : K)}{\|u_i(1 : K)\|} - (\nu_{iK}^b)^T\| \leq \frac{o_{a.s.}(1)}{c - o_{a.s.}(1)} + \sup_{1 \leq l \leq L_k} \sup_{i,j \in C_{k,K-1} \cap G_{l,K}} \|\frac{u_i(1 : K)}{\|u_i(1 : K)\|} - \frac{u_j(1 : K)}{\|u_j(1 : K)\|}\| \leq \frac{o_{a.s.}(1)}{c - o_{a.s.}(1)} + o(1) = o_{a.s.}(1),$$

where the first inequality holds by the triangle inequality, the second inequality holds by similar arguments as used in establishing (7.7) and the definition of $\nu_{iK}^b$, the third inequality holds because of the definition of $\{G_{l,K}\}_{l=1}^{L_K}$. In addition, by the definition of $\{G_{l,K}\}_{l=1}^{L_K}$ again, there exists some positive constant $c$ such that, for $l \neq l'$, $\hat{C}_{k,K}^l \neq \emptyset$, and $\hat{C}_{k,K}^{l'} \neq \emptyset$,

$$\inf_{i \in \hat{C}_{k,K}^l, j \in \hat{C}_{k,K}^{l'}} \|\beta_{in} - \beta_{jn}\| \geq c > 0.$$

Therefore, by Lemma 7.1 in the next subsection, we have, for any $k \in \mathcal{K}_1$, $Q_{K-1}(k) = \hat{Q}_{K-1}(k) + o_{a.s.}(1)$. For $k \in \mathcal{K}_2$, $Q_{K-1}(k) = o(1)$ and $\hat{Q}_{K-1}(k) = o_{a.s.}(1)$. Therefore, $Q_{K-1}(k) = \hat{Q}_{K-1}(k) + o_{a.s.}(1)$ for $k = 1, \ldots, K - 1$. Recall that

$$k^* = \arg\max_{1 \leq k \leq K} Q_K(k).$$

We claim $\hat{k} = k^*$ a.s. Suppose not. Then by Assumption 3,

$$0 \leq \hat{Q}_{K-1}(\hat{k}) - \hat{Q}_{K-1}(k^*) = Q_{K-1}(\hat{k}) - Q_{K-1}(k^*) + o_{a.s.}(1) \leq o_{a.s.}(1) - c.$$

This is a contradiction.

**Step 2.** We show that $\hat{C}_{k^*,K-1}(1) = C_{k^*,K-1}(1)$ and $\hat{C}_{k^*,K-1}(2) = C_{k^*,K-1}(2)$ a.s. Because $Z_{K-1}$ and $Z_K^b$ are unique, Lemma 7.1 implies, up to some relabeling,

$$C_{k^*,K-1}(1) = \hat{C}_{k^*,K-1}(1) \quad \text{and} \quad C_{k^*,K-1}(2) = \hat{C}_{k^*,K-1}(2). \quad (7.8)$$
Therefore, $\hat{Z}_k^b = Z_k^b$ for $k = 1, \cdots, K_0$.

**Step 3.** We show that $Z_{K_0} \succeq Z_{K+1}^b$. For any $k = 1, \cdots, K_0$ and any $K = 2, \cdots, K_0$, Theorem 3.2(1) shows that there exists $k' = 1, \cdots, K - 1$ such that $C_{k,K_0} \subset C_{k',K-1}$. If $k' \neq k^*$, then $C_{k,K_0} \subset C_{k',K-1} = C_{k'',K}$ for some $k'' = 1, \cdots, K$. If $k' = k^*$, we know that $C_{k,K_0} \subset G_{l,K}$ for some $l = 1, \cdots, L_K$. Therefore,

$$C_{k,K_0} \subset C_{k^*,K-1} \cap G_{l,K} = \tilde{C}_{k^*,K-1}^l.$$

Last, by Lemma 7.1 we know that

$$\tilde{C}_{k^*,K-1}^l \subset \text{ either } C_{k^*,K-1}(1) \text{ or } C_{k^*,K-1}(2).$$

Therefore, there exists $k'' = 1, \cdots, K$ such that

$$C_{k,K_0} \subset \tilde{C}_{k^*,K-1}^l \subset C_{k'',K}.$$

This completes the proof of Theorem 3.2(2).

For Theorem 3.2(3), the result holds by the construction of $\hat{Z}_{K+1}^b$ for $K = 1, \cdots, K_0$ and the fact that $\hat{Z}_K = Z_K$ for $K = 1, \cdots, K_0$. ■

**Proof of Theorem 3.3.** By Theorem 3.2(3), without loss of generality, we assume that $\hat{Z}_K^b$ is obtained by splitting the last group in $\hat{Z}_{K-1}$ into the $(K-1)$-th and $K$-th groups in $\hat{Z}_K$, i.e.,

$$\hat{C}_{k,K-1} = \hat{C}_{k,K}, \text{ for } k = 1, \cdots, K - 2 \text{ and } \hat{C}_{K-1,K-1} = \hat{C}_{K-1,K} \cup \hat{C}_{K,K}.$$

Therefore, for any $k,l \leq K - 2$, if $i \in \hat{C}_{k,K} = \hat{C}_{k,K-1}$ and $j \in \hat{C}_{l,K} = \hat{C}_{l,K-1}$, we have

$$O_{k,l}(\hat{Z}_K^b) = O_{k,l}(\hat{Z}_{K-1}), \quad \sum_{i' \in \hat{C}_{k,K}} \hat{d}_{i'} = \sum_{i' \in \hat{C}_{k,K-1}} \hat{d}_{i'}, \quad \text{and thus, } \quad \hat{P}_{ij}(\hat{Z}_K^b) = \hat{P}_{ij}(\hat{Z}_{K-1}).$$

By Theorem 3.2 we have $\hat{Z}_K^b = Z_K^b$ a.s. for $K \leq K_0$. Then by (2.2) and the definition
of $\hat{P}_{ij}(\cdot)$,

$$L_n(\hat{Z}_K^b, \hat{Z}_{K-1}^b)$$

$$= 2 \sum_{k=1}^{K-2} \left\{ \sum_{l=K-1}^{K} 0.5n_{k,l}(Z_K^b) \left( \frac{\hat{M}_{k,l}(Z_K^b)}{\hat{M}_{k,k-1}(Z_{K-1})} - 1 \right)^2 \right\} + \left\{ 0.5 \left[ n_{K-1,K-1}(Z_K^b) \left( \frac{\hat{M}_{K-1,K-1}(Z_K^b)}{\hat{M}_{K-1,K-1}(Z_{K-1})} - 1 \right)^2 \right] \right\}$$

$$+ 2n_{K-1,K}(Z_K^b) \left( \frac{\hat{M}_{K-1,K}(Z_K^b)}{\hat{M}_{K-1,K-1}(Z_{K-1})} - 1 \right)^2 + n_{K,K}(Z_K^b) \left( \frac{\hat{M}_{K,K}(Z_K^b)}{\hat{M}_{K-1,K-1}(Z_{K-1})} - 1 \right)^2 \right\}$$

$$= 2 \sum_{k=1}^{K-2} I_{kn} + II_n.$$ 

For $i \in C_{k,K}^b$ and $j \in C_{l,K}^b$, $k, l = 1, \cdots, K$, the population counterpart of $\hat{P}_{ij}(\hat{Z}_K^b)$ is

$$P_{ij}(Z_K^b) = \frac{E[O_{k,l}(Z_K^b)]d_i d_j}{\sum_{i, j' \in C_{l,K}^b \ni j' \neq j} d_i d_j} = M_{k,l}(Z_K^b) d_i d_j. \quad (7.9)$$

Let

$$\tilde{B}_{K,n} = 2 \sum_{k=1}^{K-2} I_{kn} + II_n, \quad (7.10)$$

where

$$I_{kn} = \sum_{l=K-1}^{K} 0.5n_{k,l}(Z_K^b) \left( \frac{M_{k,l}(Z_K^b)}{M_{k,k-1}(Z_{K-1})} - 1 \right)^2, \quad (7.11)$$

$$II_n = 0.5n_{K-1,K-1}(Z_K^b) \left( \frac{M_{K-1,K-1}(Z_K^b)}{M_{K-1,K-1}(Z_{K-1})} - 1 \right)^2$$

$$+ n_{K-1,K}(Z_K^b) \left( \frac{M_{K-1,K}(Z_K^b)}{M_{K-1,K-1}(Z_{K-1})} - 1 \right)^2 + 0.5n_{K,K}(Z_K^b) \left( \frac{M_{K,K}(Z_K^b)}{M_{K-1,K-1}(Z_{K-1})} - 1 \right)^2. \quad (7.12)$$

Note that $O_{k,l}(Z_K^b)$ is independent across $1 \leq k, l \leq K$. Let

$$V_{k,l}(Z_K^b) = \frac{\sum_{s \in I(C_{k,k}^b), t \in I(C_{l,k}^b)} [n_{\theta}^{(1)}(s,t)H_{st}(Z_{K_0}) - n_{\theta}^{(2)}(s,t)H_{st}(Z_{K_0})B_{st}(Z_{K_0})]}{n^2},$$

where $n_{\theta}^{(m)}(k) = \sum_{i \in C_{k,K_0}} \theta_{i}^{m}$ for $m = 1, \cdots, 4$,

$$n_{\theta}^{(1)}(s,t) = n_{\theta}^{(1)}(s)n_{\theta}^{(1)}(t) - n_{\theta}^{(2)}(s)1 \{s = t\},$$

and

$$n_{\theta}^{(2)}(s,t) = n_{\theta}^{(2)}(s)n_{\theta}^{(2)}(t) - n_{\theta}^{(4)}(s)1 \{s = t\}.$$
Similarly,
\[ n^{-1} \rho_n^{-1/2} \{ O_k,l(Z_K^b) - E[O_k,l(Z_K^b)] \} - N_K(k, l) = o_p(1), \quad k \neq l, \quad (7.13) \]
where \( N_K(k, l) \) is normally distributed with expectation zero and variance \( V_{k,l}(Z_K^b) \),
\[ n^{-1} \rho_n^{-1/2} \{ O_{k,k}(Z_K^b) - E[O_{k,k}(Z_K^b)] \} - N_K(k, k) = o_p(1), \quad k = K - 1, K, \]
where \( N_K(k, k) \) is normally distributed with zero expectation and variance \( 2V_{k,k}(Z_K^b) \), and
\[ \{ \{ N_K(k, l) \}_{k=1, \ldots, K-2, l=K-1, K}, N_K(K - 1, K), N_K(K - 1, K - 1), N_K(K, K) \} \]
are mutually independent.

Next, we consider the linear expansions for \( \hat{I}_{kn} - I_{kn} \) and \( \hat{II}_n - II_n \) separately in Steps 1 and 2 below.

**Step 1. We consider the linear expansion of \( \hat{I}_{kn} - I_{kn} \).

In this step, we focus on the case in which \( k = 1, \ldots, K - 2 \) and \( l = K - 1, K \). Note that
\[
\frac{\hat{M}_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} = \frac{O_{k,l}(Z_K^b)/\{ \sum_{k'=1}^{K} O_{l,l'}(Z_K^b) \}}{O_{k,K-1}(Z_{K-1})/\{ \sum_{k'=1}^{K} O_{K-1,l'}(Z_{K-1}) \}} = \frac{O_{k,l}(Z_K^b)/\{ \sum_{k'=1}^{K} O_{l,l'}(Z_K^b) \}}{\sum_{k=K-1}^{K} O_{k,l}(Z_K^b)/\{ \sum_{k=K-1}^{K} \sum_{k'=1}^{K} O_{l,l'}(Z_K^b) \}}.
\]

Similarly,
\[
\frac{M_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} = \frac{E[O_{k,l}(Z_K^b)]/\{ \sum_{k=K-1}^{K} E[O_{l,l'}(Z_K^b)] \}}{\{ \sum_{k=K-1}^{K} E[O_{k,l}(Z_K^b)] \}/\{ \sum_{k=K-1}^{K} \sum_{k'=1}^{K} E[O_{l,l'}(Z_K^b)] \}}. \quad (7.14)
\]

Then, by the delta method and some tedious calculation, we have
\[
n^{1/2} \rho_n^{1/2} [\hat{M}_{k,l}(Z_K^b) - M_{k,l}(Z_K^b)] = \frac{N_K(k, l)}{\Gamma_{1,l}(Z_K^b)} - \frac{\Gamma_{k,l}(Z_K^b) [\sum_{k'=1}^{K} N_K(l, k')]}{\Gamma_{1,l}^2(Z_K^b)} + o_p(1),
\]
where \( N_K(K - 1, K) = N_K(K, K - 1) \),
\[
\Gamma_{k,l}(Z_K^b) = n^{-2} \rho_n^{-1} E[O_{k,l}] = \Gamma_{k,l}^b(Z_K^b) + o(1), \quad (7.15)
\]
and
\[
\Gamma_{1,l}(Z_K^b) = n^{-2} \rho_n^{-1} \sum_{k'=1}^{K} E[O_{l,k'}(Z_K^b)] = \Gamma_{1,l}^b(Z_K^b) + o(1).
\]

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Similarly,

\[
n_{\rho_n}^{1/2} \left[ M_{k,K-1}(Z_{K-1}) - M_{k,K-1}(Z_{K-1}) \right] = \frac{N_K(k, K - 1) + N_K(k, K)}{\Gamma_{K-1}(Z_K^b) + \Gamma_K(Z_K^b)} \left[ \frac{\Gamma_{K-1}(Z_K^b) + \Gamma_K(Z_K^b)}{\Gamma_{K-1}(Z_K^b) + \Gamma_K(Z_K^b)} \right] + o_p(1).
\]

By Taylor expansion, we have

\[
n_{\rho_n}^{1/2} \left( \frac{\dot{M}_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} - \frac{M_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} \right) = \frac{1}{M_{k,K-1}(Z_{K-1})} \left[ \frac{N_K(k, l)}{\Gamma_l(Z_K^b)} - \frac{\Gamma_l(Z_K^b)(\sum_{l'=1}^K N_K(l, l'))}{\Gamma_l(Z_K^b)} \right] - \frac{M_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} \cdot \frac{N_K(k, K - 1) + N_K(k, K)}{\Gamma_{K-1}(Z_K^b) + \Gamma_K(Z_K^b)} \left( \frac{\Gamma_{K-1}(Z_K^b) + \Gamma_K(Z_K^b)}{\Gamma_{K-1}(Z_K^b) + \Gamma_K(Z_K^b)} \right) + o_p(1).
\]

This, in conjunction with the fact that \(a^2 - b^2 = (a - b)^2 + 2(a - b)b\), implies that

\[
n_{\rho_n}^{1/2} \left( \dot{I}_kn - I_kn \right) = \sum_{l=K-1}^K 0.5n_{\rho_n}^{1/2} n_{k,l}(Z_K^b) \left( \frac{\dot{M}_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} - \frac{M_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} \right)^2 + \sum_{l=K-1}^K n_{\rho_n}^{1/2} n_{k,l}(Z_K^b) \left( \frac{\dot{M}_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} - \frac{M_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} \right) \left( \frac{M_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} - 1 \right) + \sum_{l=K-1}^K \pi_k(Z_K^b)\pi_l(Z_K^b) \left( \frac{M_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} - 1 \right) n_{\rho_n}^{1/2} \left( \frac{\dot{M}_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} - \frac{M_{k,l}(Z_K^b)}{M_{k,K-1}(Z_{K-1})} \right) + o_p(1),
\]

where the second equality follows from the fact that

\[
\frac{n_k(Z_K^b)}{n} \to \pi_k(Z_K^b) = \sum_{m \in I(C_{k,K}^b)} \pi_{m_{\infty}},
\]

with \(\pi_{m_{\infty}}\) defined in Assumption 2 and that \(n_{\rho_n}^{1/2} \to \infty\) as \(n \to \infty\) under Assumption 4. 

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\[
\phi_{k,l}(k) = \pi_k(Z_K^b) \pi_l(Z_K^b) \left( \frac{M_{k,l}(Z_K^b)}{M_{k,k-1}(Z_K^b)} \left( \frac{1}{M_{k,k-1}(Z_K^b)} - \frac{1}{M_{k,k-1}(Z_K^b)} \right) \right) \left[ \frac{1\{l' = k\} - \Gamma_l(Z_K^b)}{\Gamma_l(Z_K^b)} \right] \\
- \sum_{l=K-1}^{K} \pi_k(Z_K^b) \pi_l(Z_K^b) \left( \frac{M_{k,l}(Z_K^b)}{M_{k,k-1}(Z_K^b)} \left( \frac{1}{M_{k,k-1}(Z_K^b)} - \frac{1}{M_{k,k-1}(Z_K^b)} \right) \right) \\
\times \left[ \frac{1\{l' = k\} - \Gamma_{k,k-1}(Z_K^b) + \Gamma_{k,k}(Z_K^b)}{\Gamma_{k,k-1}(Z_K^b) + \Gamma_{k,k}(Z_K^b)} \right], \quad l' = 1, \ldots, K-2, \quad l = K-1, K,
\]

\[
\phi_{K-1,K}(k) = - \pi_k(Z_K^b) \pi_{K-1}(Z_K^b) \left( \frac{M_{k,K-1}(Z_K^b)}{M_{k,k-1}(Z_K^b)} \left( \frac{1}{M_{k,k-1}(Z_K^b)} - \frac{1}{M_{k,k-1}(Z_K^b)} \right) \right) \Gamma_{k,k-1}(Z_K^b) \\
+ \sum_{l=K-1}^{K} \pi_k(Z_K^b) \pi_l(Z_K^b) \left( \frac{M_{k,l}(Z_K^b)}{M_{k,k-1}(Z_K^b)} \left( \frac{1}{M_{k,k-1}(Z_K^b)} - \frac{1}{M_{k,k-1}(Z_K^b)} \right) \right) \frac{\Gamma_{k,k-1}(Z_K^b) + \Gamma_{k,k}(Z_K^b)}{\Gamma_{k,k-1}(Z_K^b) + \Gamma_{k,k}(Z_K^b)} \\
= - \sum_{l=K-1}^{K} \pi_k(Z_K^b) \pi_l(Z_K^b) \left( \frac{M_{k,l}(Z_K^b)}{M_{k,k-1}(Z_K^b)} \left( \frac{1}{M_{k,k-1}(Z_K^b)} - \frac{1}{M_{k,k-1}(Z_K^b)} \right) \right) \Gamma_{k,k-1}(Z_K^b) \\
+ \sum_{l=K-1}^{K} \pi_k(Z_K^b) \pi_l(Z_K^b) \left( \frac{M_{k,l}(Z_K^b)}{M_{k,k-1}(Z_K^b)} \left( \frac{1}{M_{k,k-1}(Z_K^b)} - \frac{1}{M_{k,k-1}(Z_K^b)} \right) \right) \frac{\Gamma_{k,k-1}(Z_K^b) + \Gamma_{k,k}(Z_K^b)}{\Gamma_{k,k-1}(Z_K^b) + \Gamma_{k,k}(Z_K^b)} \\
= \frac{\Gamma_{k,k-1}(Z_K^b) - \Gamma_{k,k}(Z_K^b)}{\Gamma_{k,k-1}(Z_K^b) + \Gamma_{k,k}(Z_K^b)}
\]

Step 2. We consider the linear expansion of \( \tilde{I}_n - I_n \).

Note that

\[
\tilde{M}_{K-1,K-1}(Z_K^b) - M_{K-1,K-1}(Z_K^b) = \frac{O_{K-1,K-1}(Z_K^b) - E[O_{K-1,K-1}(Z_K^b)]}{\sum_{i',j' \in C_{K-1,K-1}^b, i' \neq j'} d_{i'}d_{j'}} - \frac{E[O_{K-1,K-1}(Z_K^b)]}{\sum_{i',j' \in C_{K-1,K-1}^b, i' \neq j'} d_{i'}d_{j'}} \left( \sum_{i',j' \in C_{K-1,K-1}^b, i' \neq j'} d_{i'}d_{j'} \right). \]
By the proof of Su et al. (2017, Lemma 3.1), we have

\[
\sup_i |\hat{d}_i/d_i - 1| = O_{\text{a.s.}}(\log^{1/2}(n)(n\rho_n)^{-1/2}) = o_{\text{a.s.}}(1). \tag{7.18}
\]

Therefore,

\[
n^{-4}\rho_n^{-2} \sum_{i',j' \in C_{K-1,K}^b, i' \neq j'} \hat{d}_{i'} \hat{d}_{j'} = n^{-4}\rho_n^{-2} \left( \sum_{i',j' \in C_{K-1,K}^b, i' \neq j'} d_{i'}d_{j'} \right) [1 + o_{\text{a.s.}}(1)]
\]

\[
= [\Gamma_{K-1}^2(Z_K^b) - n^{-4}\rho_n^{-2} \sum_{i \in C_{K-1,K}^b} d_i^2] [1 + o_{\text{a.s.}}(1)]
\]

\[
= [\Gamma_{K-1}^2(Z_K^b) + O(n^{-1})] [1 + o_{\text{a.s.}}(1)] = \Gamma_{K-1}^2(Z_K^b) + o_{\text{a.s.}}(1),
\]

where the last inequality holds because \(\sup_i d_i \leq n\rho_n\). Also note that, by (7.18),

\[
n^{-3}\rho_n^{-3/2} \sum_{i',j' \in C_{K-1,K}^b, i' \neq j'} (\hat{d}_{i'} \hat{d}_{j'} - d_{i'}d_{j'})
\]

\[
= n^{-3}\rho_n^{-3/2} \left[ (\sum_{i' \in C_{K-1,K}^b} \hat{d}_{i'})^2 - (\sum_{i' \in C_{K-1,K}^b} d_{i'})^2 \right] - n^{-3}\rho_n^{-3/2} \left[ \sum_{i' \in C_{K-1,K}^b} (\hat{d}_{i'}^2 - d_{i'}^2) \right]
\]

\[
= n^{-3}\rho_n^{-3/2} \left[ (\sum_{i' \in C_{K-1,K}^b} \hat{d}_{i'} - d_{i'}) (\sum_{i' \in C_{K-1,K}^b} d_{i'}) (2 + o_{\text{a.s.}}(1)) \right]
\]

\[
- n^{-3}\rho_n^{-3/2} \left[ \sum_{i' \in C_{K-1,K}^b} (\hat{d}_{i'} - d_{i'})d_{i'}(2 + o_{\text{a.s.}}(1)) \right]
\]

\[
= n^{-3}\rho_n^{-3/2} \left[ (\sum_{i' \in C_{K-1,K}^b} \hat{d}_{i'} - d_{i'}) (\sum_{i' \in C_{K-1,K}^b} d_{i'}) (2 + o_{\text{a.s.}}(1)) \right] + O_{\text{a.s.}}(\log^{1/2}(n)n^{-1/2})
\]

\[
= n^{-3}\rho_n^{-3/2} \left( \sum_{l' = 1}^{K} \left( O_{K-1,l'}(Z_K) - E[O_{K-1,l'}(Z_K)] \right) - \sum_{i' \in C_{K-1,K}^b} P_{i'i'}(Z_{K_0}) \right)
\]

\[
\times \left( \sum_{l' = 1}^{K} E[O_{K-1,l'}(Z_K)] + \sum_{i' \in C_{K-1,K}^b} P_{i'i'}(Z_{K_0}) \right) \left( 2 + o_{\text{a.s.}}(1) \right) + o_{\text{a.s.}}(1)
\]

\[
= 2\Gamma_{K-1}(Z_K) \left( \sum_{l' = 1}^{K} N_K(K - 1, l') \right) + o_{\text{a.s.}}(1),
\]

where the second inequality holds due to (7.18), the third equality holds because of (7.18) and the facts that \(\sup_i d_i \leq n\rho_n\) and \(#C_{K-1,K}^b \leq n\), the fourth inequality holds due to the definition of \(d_i\), and the last inequality holds because of (7.13) – (7.16).
Then, by the delta method,
\[ n^3 \rho_n^{3/2} [\hat{M}_{K-1,K-1}(Z^b_K) - M_{K-1,K-1}(Z^b_K)] = \frac{N_K(K-1,K-1)}{\Gamma_{K-1}(Z^b_K)} - \frac{2\Gamma_{K-1,K-1}(Z^b_K)\sum_{l'=1}^K N_K(K-1,l')}{\Gamma_{K}(Z^b_K)} + o_p(1). \] (7.19)

Similarly,
\[ n^3 \rho_n^{3/2} (\hat{M}_{K,K}(Z^b_K) - M_{K,K}(Z^b_K)) = \frac{N_K(K,K)}{\Gamma_{K}(Z^b_K)} - \frac{2\Gamma_{K,K}(Z^b_K)\sum_{l'=1}^K N_K(K,l')}{{\Gamma_{K}(Z^b_K)}^2} + o_p(1). \]

Furthermore, we have
\[ \hat{M}_{K-1,K}(Z^b_K) - M_{K-1,K}(Z^b_K) = \frac{O_{K-1,K}(Z^b_K) - E[O_{K-1,K}(Z^b_K)]}{(\sum_{i'\in C_{K-1,K}^b} \hat{d}_{i'}) (\sum_{j'\in C_{K,K}^b} \hat{d}_{j'})} \]
\[ - \frac{E[O_{K-1,K}(Z^b_K)] [\sum_{i'\in C_{K-1,K}^b} \hat{d}_{i'}] (\sum_{j'\in C_{K,K}^b} \hat{d}_{j'}) - (\sum_{i'\in C_{K-1,K}^b} \hat{d}_{i'}) (\sum_{j'\in C_{K,K}^b} \hat{d}_{j'})}{(\sum_{i'\in C_{K-1,K}^b} \hat{d}_{i'}) (\sum_{j'\in C_{K,K}^b} \hat{d}_{j'}) (\sum_{i'\in C_{K-1,K}^b} \hat{d}_{i'}) (\sum_{j'\in C_{K,K}^b} \hat{d}_{j'})}. \]

Therefore,
\[ n^3 \rho_n^{3/2} [\hat{M}_{K-1,K}(Z^b_K) - M_{K-1,K}(Z^b_K)] = \frac{N_K(K-1,K)}{\Gamma_{K-1}(Z^b_K)\Gamma_{K}(Z^b_K)} - \frac{\Gamma_{K-1,K}(Z^b_K) [\Gamma_{K-1}(Z^b_K) \sum_{l'=1}^K N_K(l',K) + \Gamma_{K,K}(Z^b_K) \sum_{l'=1}^K N_K(l',K-1)]}{{\Gamma_{K}(Z^b_K)}^2} \] + o_p(1). (7.20)

Finally, noting that
\[ \hat{M}_{K-1,K-1}(Z_{K-1}) = \frac{O_{K-1,K-1}(Z_{K-1})}{\sum_{i',j'\in C_{K-1,K-1}} \hat{d}_{i'} \hat{d}_{j'}} \]
\[ = \frac{O_{K-1,K-1}(Z^b_K) + 2O_{K-1,K}(Z^b_K) + O_{K,K}(Z^b_K)}{\sum_{i',j'\in C_{K-1,K-1}} \hat{d}_{i'} \hat{d}_{j'} + \sum_{i'\not\in C_{K-1,K-1}} \hat{d}_{i'} \hat{d}_{j'} + \sum_{i'\not\in C_{K-1,K-1}} \hat{d}_{i'} \hat{d}_{j'}} , \]
we have
\[ n^3 \rho_n^{3/2} (\hat{M}_{K-1,K-1}(Z_{K-1}) - M_{K-1,K-1}(Z_{K-1})) = \frac{N_K(K-1,K) + 2N_K(K-1,K) + N_K(K,K)}{[\Gamma_{K-1}(Z^b_K) + \Gamma_{K,K}(Z^b_K)]^2} \]
\[ - \frac{\Gamma_{K-1,K-1}(Z^b_K) + 2\Gamma_{K-1,K}(Z^b_K) + \Gamma_{K,K}(Z^b_K)}{[\Gamma_{K-1}(Z^b_K) + \Gamma_{K,K}(Z^b_K)]^3} \sum_{l'=1}^K 2[N_K(K-1,l') + N_K(K,l')] + o_p(1). \] (7.21)
For \( s, t = K - 1, \) \( K, \) let \( \hat{m}_{s,t}(Z_K^b) = n^2 \rho_n \hat{M}_{s,t}(Z_K^b) \) and

\[
m_{s,t}(Z_K^b) = n^2 \rho_n M_{s,t}(Z_K^b) = \frac{\Gamma_{s,t}^0(Z_K^b)}{\Gamma_{s,t}^0(Z_K^b)}[1 + o(1)].
\]

Define \( m_{K-1,K-1}(Z_{K-1}) \) and \( \hat{m}_{K-1,K-1}(Z_{K-1}) \) similarly. By the previous calculations, we have

\[
\hat{m}_{s,t}(Z_K^b) = m_{s,t}(Z_K^b)[1 + o_a.s.(1)].
\]

Hence,

\[
n\rho_n^{1/2} \left( \frac{\hat{M}_{K-1,K-1}(Z_K^b)}{M_{K-1,K-1}(Z_{K-1})} - \frac{M_{K-1,K-1}(Z_K^b)}{M_{K-1,K-1}(Z_{K-1})} \right)
= n^3 \rho_n^{3/2} \left[ \hat{M}_{K-1,K-1}(Z_K^b) - M_{K-1,K-1}(Z_K^b) \right]
- m_{K-1,K-1}(Z_K^b) n^3 \rho_n^{3/2} \left[ \hat{M}_{K-1,K-1}(Z_{K-1}) - M_{K-1,K-1}(Z_{K-1}) \right] + o_p(1),
\]

\[
n\rho_n^{1/2} \left( \frac{\hat{M}_{K,K}(Z_K^b)}{M_{K-1,K-1}(Z_{K-1})} - \frac{M_{K,K}(Z_K^b)}{M_{K-1,K-1}(Z_{K-1})} \right)
= n^3 \rho_n^{3/2} \left[ \hat{M}_{K,K}(Z_K^b) - M_{K,K}(Z_K^b) \right]
- m_{K,K}(Z_K^b) n^3 \rho_n^{3/2} \left[ \hat{M}_{K-1,K-1}(Z_{K-1}) - M_{K-1,K-1}(Z_{K-1}) \right] + o_p(1),
\]

and

\[
n\rho_n^{1/2} \left( \frac{\hat{M}_{K-1,K}(Z_K^b)}{M_{K-1,K-1}(Z_{K-1})} - \frac{M_{K-1,K}(Z_K^b)}{M_{K-1,K-1}(Z_{K-1})} \right)
= n^3 \rho_n^{3/2} \left[ \hat{M}_{K-1,K}(Z_K^b) - M_{K-1,K}(Z_K^b) \right]
- m_{K-1,K}(Z_K^b) n^3 \rho_n^{3/2} \left[ \hat{M}_{K-1,K-1}(Z_{K-1}) - M_{K-1,K-1}(Z_{K-1}) \right] + o_p(1).
\]
Then, by (7.19)–(7.21),

\[
\begin{align*}
& n^{-1} \rho_n^{1/2} (\widetilde{III}_n - I_n) \\
& = n \rho_n^{1/2} \left[ \pi_{K-1}^2 (Z_K^b) \left( \frac{\tilde{M}_{K-1,K-1}(Z_K^b)}{M_{K-1,K-1}(Z_K)} - \frac{M_{K-1,K-1}(Z_K^b)}{M_{K-1,K-1}(Z_K)} \right) \\
& \quad + 2 \pi_{K-1}(Z_K^b) \pi_K(Z_K^b) \left( \frac{\tilde{M}_{K-1,K}(Z_K^b)}{M_{K-1,K-1}(Z_K)} - \frac{M_{K-1,K}(Z_K^b)}{M_{K-1,K-1}(Z_K)} \right) \\
& \quad + \pi_K^2(Z_K^b) \left( \frac{\tilde{M}_{K,K}(Z_K^b)}{M_{K-1,K-1}(Z_K)} - \frac{M_{K,K}(Z_K^b)}{M_{K-1,K-1}(Z_K)} \right) \right] + o_p(1) \\
& = n^3 \rho_n^{3/2} \left[ \pi_{K-1}^2 (Z_K^b) \left[ \tilde{M}_{K-1,K-1}(Z_K^b) - M_{K-1,K-1}(Z_K^b) \right] \\
& \quad + \frac{2 \pi_{K-1}(Z_K^b) \pi_K(Z_K^b) \tilde{M}_{K-1,K}(Z_K^b) - M_{K-1,K}(Z_K^b)}{m_{K-1,K-1}(Z_K-1)} \\
& \quad + \frac{\pi_K^2(Z_K^b) \tilde{M}_{K,K}(Z_K^b) - M_{K,K}(Z_K^b)}{m_{K-1,K-1}(Z_K-1)} \right] + o_p(1) \\
& \times n^3 \rho_n^{3/2} \tilde{M}_{K-1,K-1}(Z_K-1) - M_{K-1,K-1}(Z_K-1) + o_p(1) \\
& = \sum_{l=1}^{K-2} \sum_{l'=1}^{K} \phi_{l',l}(K-1) N_K(l', l) + \phi_{K-1,K-1}(K-1) N_K(K-1, K-1) \\
& \quad + \phi_{K-1,K}(K-1) N_K(K-1, K) + \phi_{K,K}(K-1) N_K(K, K) + o_p(1),
\end{align*}
\]

where, by denoting \( \phi = \frac{\pi_{K-1}^2 (Z_K^b) m_{K-1,K-1}(Z_K^b) + 2 \pi_{K-1}(Z_K^b) \pi_K(Z_K^b) m_{K-1,K}(Z_K^b) + \pi_K^2(Z_K^b) m_{K,K}(Z_K^b)}{m_{K-1,K-1}(Z_K-1)} \), we have

\[
\begin{align*}
\phi_{l',K-1}(K-1) & = - \frac{2 \pi_{K-1}^2 (Z_K^b) \Gamma_{K-1,K-1}(Z_K^b)}{\Gamma_{K-1,K-1}(Z_K^b) m_{K-1,K-1}(Z_K-1)} - \frac{2 \pi_{K-1}(Z_K^b) \pi_K(Z_K^b) \Gamma_{K-1,K-1}(Z_K^b)}{\Gamma_{K-1,K-1}(Z_K^b) m_{K-1,K-1}(Z_K-1)} \\
& \quad - 2 \phi \left[ \Gamma_{K-1,K-1}(Z_K^b) + 2 \Gamma_{K-1,K}(Z_K^b) + \Gamma_{K,K}(Z_K^b) \right] \left[ \Gamma_{K-1,K-1}(Z_K^b) + \Gamma_{K,K}(Z_K^b) \right]^{3/2}(Z_K^b) m_{K-1,K-1}(Z_K-1)^3, \\
& \quad l' = 1, \ldots, K-2,
\end{align*}
\]

\[
\begin{align*}
\phi_{l',K}(K-1) & = - \frac{2 \pi_{K-1}^2 (Z_K^b) \Gamma_{K,K}(Z_K^b)}{\Gamma_{K,K}(Z_K^b) m_{K-1,K-1}(Z_K-1)} - \frac{2 \pi_{K-1}(Z_K^b) \pi_K(Z_K^b) \Gamma_{K,K}(Z_K^b)}{\Gamma_{K,K}(Z_K^b) m_{K-1,K-1}(Z_K-1)} \\
& \quad - 2 \phi \left[ \Gamma_{K-1,K-1}(Z_K^b) + 2 \Gamma_{K-1,K}(Z_K^b) + \Gamma_{K,K}(Z_K^b) \right] \left[ \Gamma_{K-1,K-1}(Z_K^b) + \Gamma_{K,K}(Z_K^b) \right]^{3/2}(Z_K^b) m_{K-1,K-1}(Z_K-1)^3, \\
& \quad l' = 1, \ldots, K-2,
\end{align*}
\]
\[
\phi_{K-1,K-1}(K-1) = \frac{\pi^2_{K-1}(Z_{K}^b)}{\Gamma_{K-1}(Z_{K}^b)m_{K-1,K-1}(Z_{K-1})} - \frac{2\pi^2_{K-1}(Z_{K}^b)\Gamma_{K-1,K-1}(Z_{K-1})}{\Gamma^3_{K-1}(Z_{K}^b)m_{K-1,K-1}(Z_{K-1})} - \frac{2\pi_{K-1}(Z_{K}^b)\pi_{K}(Z_{K}^b)\Gamma_{K-1,K}(Z_{K}^b)}{\Gamma_{K-1,K}(Z_{K}^b)m_{K-1,K-1}(Z_{K-1})},
\]

and

\[
\phi_{K-1,K}(K-1) = \frac{\pi^2_{K}(Z_{K}^b)}{\Gamma_{K}(Z_{K}^b)m_{K-1,K-1}(Z_{K-1})} - \frac{2\pi^2_{K}(Z_{K}^b)\Gamma_{K,K}(Z_{K}^b)}{\Gamma^3_{K}(Z_{K}^b)m_{K-1,K-1}(Z_{K-1})} - \frac{2\pi_{K-1}(Z_{K}^b)\pi_{K}(Z_{K}^b)\Gamma_{K-1,K}(Z_{K}^b)}{\Gamma_{K-1,K}(Z_{K}^b)m_{K-1,K-1}(Z_{K-1})}.
\]

Combining (7.17) and (7.22), we have

\[
n^{-1}p^1_{n}^{1/2}[L_n(\hat{Z}_K, \hat{\zeta}_K) - \bar{B}_K] = \sum_{l=1}^{K-2} \sum_{l'=1}^{K} \phi_{l',l}N_{K}(l',l) + \phi_{K-1,K-1}N_{K}(K-1, K-1) + \phi_{K-1,K}N_{K}(K-1, K) + o_p(1),
\]

where

\[
\phi_{l',l} = \sum_{k=1}^{K-2} 2\phi_{l',l}(k) + \phi_{l',l}(K-1), \quad l' = 1, \ldots, l, \quad l = K-1, K.
\]

Letting

\[
\tilde{\alpha}_{K,n}^2 = \sum_{l'=1}^{K-2} \sum_{l=K-1,K; l' \leq l} \phi_{l',l}^2 V_{l',l}(Z_{K}) + \phi_{K-1,K-1}^2 2V_{K-1,K-1}(Z_{K}) + \phi_{K,K}^2 2V_{K,K}(Z_{K}) + \phi_{K-1,K}^2 V_{K-1,K}(Z_{K}), \tag{7.23}
\]

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we have
\[
\tilde{\omega}_{k_2,n}^{-1} n^{-1} \rho_n^{-1/2} \left[ L_n(\hat{Z}_{K_2}, \hat{Z}_{K_2}) - \hat{B}_{K,n} \right] \rightsquigarrow N(0, 1).
\]

**Step 3.** We now prove the second result in the theorem.

By (7.14), (7.14), (7.15) and (7.16), for \( k = 1, \cdots, K - 2 \), we have
\[
n^{-2} I_{kn} \rightarrow \sum_{l=K-1}^{K} 0.5 \pi_k(Z^K_l) \pi_l(Z^K_l) \left( \frac{\Gamma^0_{k,l}(Z^K_l)[\Gamma^0_{K-1,K-1}(Z^K_l) + \Gamma^0_{K,K}(Z^K_l)] - 1}{\Gamma^0_{l,l}(Z^K_l)[\Gamma^0_{k,k-1}(Z^K_l) + \Gamma^0_{k,k}(Z^K_l)]} \right)^2.
\]

Similarly, by (7.12), (7.14), (7.15) and (7.16), we have
\[
n^{-2} II_n \rightarrow 0.5 \pi^2_{K-1}(Z^K_K) \left( \frac{\Gamma^0_{K-1,K-1}(Z^K_K)[\Gamma^0_{K-1,K-1}(Z^K_K) + \Gamma^0_{K,K}(Z^K_K)] - 1}{\Gamma^0_{K-1,K-1}(Z^K_K)} \frac{\Gamma^0_{K-1,K-1}(Z^K_K) + \Gamma^0_{K,K}(Z^K_K) + 1}{\Gamma^0_{K,K}(Z^K_K)} \right)^2 + 0.5 \pi^2_{K-1}(Z^K_K) \left( \frac{\Gamma^0_{k,k-1}(Z^K_K)[\Gamma^0_{k,k-1}(Z^K_K) + \Gamma^0_{k,k}(Z^K_K)] - 1}{\Gamma^0_{k,k-1}(Z^K_K)} \frac{\Gamma^0_{k,k-1}(Z^K_K) + \Gamma^0_{k,k}(Z^K_K) + 1}{\Gamma^0_{k,k}(Z^K_K)} \right)^2.
\]

Clearly, there exits \( c_{K_2} < \infty \) such that
\[
n^{-2} \hat{B}_{K,n} = \sum_{k=1}^{K-2} n^{-2} I_{kn} + n^{-2} II_n \leq c_{K_2}.
\]

In addition, Assumption 5 implies that at least one of the squares is nonzero. Therefore, there exists a constant \( c_{K_1} > 0 \) such that
\[
n^{-2} \hat{B}_{K,n} = \sum_{k=1}^{K-2} n^{-2} I_{kn} + n^{-2} II_n \geq c_{K_1}.
\]

**Proof of Theorem 3.4.** We consider the bound of \( L_n(\hat{Z}_{K_0+1}^b, \hat{Z}_{K_0}) \). We say \( z \) is a \( n \times (K_0 + 1) \) membership matrix for \( n \) nodes and \( K_0 + 1 \) groups if there is only one element in each row of \( z \) that takes value 1, and the rest of the entries are zero. Say \( Z_{ik} = 1 \), then we say that the \( i \)-th node is identified in group \( k \). Let
\[
\mathcal{V}_{K_0+1} = \left\{ z \text{ is a } n \times (K_0 + 1) \text{ membership matrix s.t.}
\begin{align*}
&z \text{ is a } n \times (K_0 + 1) \text{ membership matrix s.t.} \\
&\text{every group identified by } Z \text{ is a subset of one of the true communities and } \\
&\inf_{1 \leq k \leq K} n_k(Z)/n \geq \varepsilon
\end{align*}
\right\}.
\]

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Without loss of generality, we assume that \( \hat{Z}_{K_0+1}^b \) is obtained by splitting the last group in \( \hat{Z}_{K_0} \) into the \( K_0 \)-th and \( (K_0+1) \)-th groups in \( \hat{Z}_{K_0+1}^b \). Theorem 3.2 and Assumption 6 we have \( \hat{Z}_{K_0+1}^b \in \mathcal{V}_{K_0+1} \) a.s. Let \( z_{K_0+1} \) be an arbitrary realization of \( \hat{Z}_{K_0+1}^b \) such that \( z_{K_0+1} \in \mathcal{V}_{K_0+1} \) and \( h(\cdot|z_{K_0+1}) \) be a surjective mapping: \( [K_0 + 1] \mapsto [K_0] \) that maps the community index identified by \( z_{K_0+1} \) into the true community index in \( [K_0] \) for any \( z_{K_0+1} \in \mathcal{V}_{K_0+1} \). Then, we have

\[
h(k|z_{K_0+1}) = k, \quad k = 1, \ldots, K_0 - 1
\]

and

\[
h(K_0|z_{K_0+1}) = h(K_0 + 1|z_{K_0+1}) = K_0.
\]

By (7.9), for any \( z_{K_0+1} \in \mathcal{V}_{K_0+1} \), \( i \in C_{k,K_0+1} \) and \( j \in C_{l,K_0+1} \), \( k = 1, \ldots, K_0 - 1, \ l = K_0, K_0 + 1 \),

\[
P_{ij}(z_{K_0+1}) = B_h(k|z_{K_0+1})h(l|z_{K_0+1})\theta_i\theta_j = B_{k,K_0}\theta_i\theta_j = P_{ij}(Z_{K_0}).
\]

Therefore,

\[
\frac{M_{k,l}(z_{K_0+1})}{M_{k,K_0}(Z_{K_0})} = \frac{P_{ij}(z_{K_0+1})}{P_{ij}(Z_{K_0})} = 1, \quad k = 1, \ldots, K_0 - 1, \ l = K_0, K_0 + 1. \tag{7.24}
\]

Similarly,

\[
\frac{M_{K_0,K_0}(z_{K_0+1})}{M_{K_0,K_0}(Z_{K_0})} = \frac{M_{K_0,K_0+1}(z_{K_0+1})}{M_{K_0,K_0}(Z_{K_0})} = \frac{M_{K_0+1,K_0+1}(z_{K_0+1})}{M_{K_0,K_0}(Z_{K_0})} = 1. \tag{7.25}
\]

By Theorem 3.2, \( \tilde{Z}_{K_0} = Z_{K_0} \) and \( \hat{Z}_{K_0+1}^b \in \mathcal{V}_{K_0+1} \) a.s. Therefore, (7.24) and (7.25) still hold when \( z_{K_0+1} \) and \( Z_{K_0} \) are replaced by \( \hat{Z}_{K_0+1}^b \) and \( \tilde{Z}_{K_0} \). Then,

\[
L_n(\hat{Z}_{K_0+1}^b, \tilde{Z}_{K_0})
\]

\[
= 2 \sum_{k=1}^{K_0-1} \sum_{l=K_0}^{K_0+1} 0.5 n_{k,l}(\hat{Z}_{K_0+1}^b) \left( \frac{\hat{M}_{k,l}(\hat{Z}_{K_0+1}^b)}{\hat{M}_{k,K_0}(\tilde{Z}_{K_0})} - 1 \right)^2
\]

\[
+ 0.5 \left[ n_{K_0,K_0}(\hat{Z}_{K_0+1}^b) \left( \frac{\hat{M}_{K_0,K_0}(\hat{Z}_{K_0+1}^b)}{\hat{M}_{K_0,K_0}(\tilde{Z}_{K_0})} - 1 \right)^2 + 2 n_{K_0,K_0+1}(\hat{Z}_{K_0+1}^b) \left( \frac{\hat{M}_{K_0,K_0+1}(\hat{Z}_{K_0+1}^b)}{\hat{M}_{K_0,K_0+1}(\tilde{Z}_{K_0})} - 1 \right)^2 \right]. \tag{7.26}
\]

For the first term in (7.26),

\[
0.5 n_{k,l}(\hat{Z}_{K_0+1}^b) \left( \frac{\hat{M}_{k,l}(\hat{Z}_{K_0+1}^b)}{\hat{M}_{k,K_0}(\tilde{Z}_{K_0})} - 1 \right)^2 \lesssim n^2 \sup_{z_{K_0+1} \in \mathcal{V}_{K_0+1}} \left( \frac{\hat{M}_{k,l}(z_{K_0+1})}{\hat{M}_{k,K_0}(\tilde{Z}_{K_0})} - \frac{M_{k,l}(z_{K_0+1})}{M_{k,K_0}(\tilde{Z}_{K_0})} \right)^2.
\]
The rate of the RHS of the above display depends on that of
\[ \sup_{z_{K_0+1} \in V_{K_0+1}} |O_{k,l}(z_{K_0+1}) - E[O_{k,l}(z_{K_0+1})]|. \]

By Bernstein inequality,
\[
P\left( \sup_{z_{K_0+1} \in V_{K_0+1}} |O_{k,l}(z_{K_0+1}) - E[O_{k,l}(z_{K_0+1})]| \geq C n^{3/2} \rho_n^{1/2} \right)
\leq 2^n \exp\left( -\frac{C^2 n^3 \rho_n}{\theta n^2 + C n^{3/2} \rho_n^{1/2}/3} \right) \leq \exp(-C'n)
\]
for some constant \( C' > 0 \). Therefore,
\[
\sup_{z_{K_0+1} \in V_{K_0+1}} |O_{k,l}(z_{K_0+1}) - E[O_{k,l}(z_{K_0+1})]| = O_{a.s.}(n^{3/2} \rho_n^{1/2}).
\]
It also implies the uniform consistency that
\[
\sup_{z_{K_0+1} \in V_{K_0+1}} |n^{-2} \rho_n^{-1} O_{k,l}(z_{K_0+1}) - \Gamma_{k,l}(z_{K_0+1})| = O_{a.s.}(n \rho_n^{-1/2}) = o_{a.s.}(1).
\]

Following the same Taylor expansion detailed in Step 1 of the proof of Theorem 3.3, we have
\[
\sup_{z_{K_0+1} \in V_{K_0+1}} \left| \frac{\dot{M}_{k,l}(z_{K_0+1})}{\dot{M}_{k,K_0}(Z_{K_0})} - \frac{M_{k,l}(z_{K_0+1})}{M_{k,K_0}(Z_{K_0})} \right| = O_{a.s.}(n \rho_n^{-1/2}).
\]

Therefore,
\[
0.5 n_{k,l} (\hat{Z}_{K_0+1}^b) \left( \frac{\dot{M}_{k,l}(\hat{Z}_{K_0+1}^b)}{\dot{M}_{k,K_0}(\hat{Z}_{K_0})} - 1 \right)^2 = O_{a.s.}(n \rho_n^{-1}).
\]
The rest of the terms in (7.26) can be bounded similarly. Thus, we conclude that
\[
L_n(\hat{Z}_{K_0+1}^b, \hat{Z}_{K_0}) = O_{a.s.}(n \rho_n^{-1}). \tag{7.27}
\]

Next, we study the asymptotic property of \( \hat{K}_1 \). If \( K_0 = 1 \), \( P(\hat{K}_1 \geq 1) = 1 \) holds trivially. If \( K_0 \geq 2 \),
\[
R(1) \asymp \frac{n^2}{\eta_n} \asymp 1.
\]
When \( 2 \leq K < K_0 \), by Theorem 3.3
\[
R(K) \asymp \frac{\hat{B}_{K-1} + O_p(n \rho_n^{-1/2})}{\hat{B}_K + O_p(n \rho_n^{-1/2})} \asymp 1.
\]

When $K = K_0$, by Theorem 3.3 and (7.27),

$$R(K_0) \lesssim \frac{n \rho^{-1}_n}{c_{K_1}n^2 + O_p(n \rho^{-1/2}_n)} \to 0.$$  

Since $n^2/(n \rho^{-1}_n) = n \rho_n \to \infty$ under Assumption 4,

$$P( \hat{K}_1 \geq K_0) \leq P \left( R(K_0) < \max_{K < K_0} R(K) \right) \to 1.$$  

Now, we study the asymptotic property of $\tilde{K}_2$. If $K_0 = 1$,

$$R(1) \lesssim \frac{1}{n \rho_n} \to 0.$$  

Therefore, $P(\tilde{K}_2 = 1) = P(R(1) \leq h_n) \to 1$ because $n \rho_n h_n \to \infty$ as $n \to \infty$. If $K_0 \geq 2$, by Theorem 3.3 and (7.27),

$$\begin{cases} R(K) \asymp \frac{n^2}{n \rho_n} \to \infty, & \text{if } K = 1, \\ R(K) \asymp 1, & \text{if } 2 \leq K < K_0, \\ R(K) \lesssim \frac{n \rho^{-1}_n}{n^2} \asymp \frac{1}{n \rho_n} \to 0, & \text{if } K = K_0. \end{cases}$$  

This, in conjunction with the conditions that $n \rho_n h_n \to \infty$ and $h_n \to 0$ as $n \to \infty$ implies that

$$P(\tilde{K}_2 = K_0) = P \left( \min_{1 \leq K < K_0} R(K) > h_n, R(K_0) \leq h_n \right) \to 1.$$  

It follows that $P(\hat{K}_2 = K_0) \geq P(\tilde{K}_1 \geq K_0, \tilde{K}_2 = K_0) \to 1$. ■

7.2 Technical lemmas

The following lemma is based on Wang and Su (2017, proof of Theorem 3.2). We state it here for completeness and clarity of presentation.

**Lemma 7.1** Let $C$ be a set of nodes and $\{\hat{\beta}_i\}_{i \in C}$ be a sequence of $K \times 1$ vectors such that $\sup_{i \in C} ||\hat{\beta}_i - \beta_i|| = o_{a.s.}(1)$ and $\sup_{i \in C} ||\beta_i|| = O(1)$. In addition, suppose $\{\beta_i\}_{i \in C}$ has $L$ distinct vectors and we group index $i$ into $L$ mutually exclusive groups $\{C_l\}_{l=1}^L$ such that if $i, j \in C_l$, $\beta_i = \beta_j$ and for any $i \in C_l$, $j \in C_{l'}$, $l \neq l'$, $\inf_{i,j,n} ||\beta_i - \beta_j|| > c > 0$. We apply
the binary segmentation algorithm on $\{\beta_i\}_{i=1}^n$ and $\{\hat{\beta}_i\}_{i=1}^n$ and obtain two sets of mutually exclusive groups $(C(1), C(2))$ and $(\hat{C}(1), \hat{C}(2))$, respectively. Then, for any $l = 1, \cdots, L,$

$$C_l \subset \text{either } C(1) \text{ or } C(2)$$

and

$$\frac{\hat{\Phi}(C) - \hat{\Phi}(\hat{C}(1)) - \hat{\Phi}(\hat{C}(2))}{\#C} = \frac{\Phi(C) - \Phi(C(1)) - \Phi(C(2))}{\#C} + o_{a.s.}(1),$$

where for a generic index set $C$,

$$\hat{\Phi}(C) = \sum_{i \in C} ||\hat{\beta}_i - \frac{\sum_{i \in C} \hat{\beta}_i}{\#C}||^2$$

and

$$\Phi(C) = \sum_{i \in C} ||\beta_i - \frac{\sum_{i \in C} \beta_i}{\#C}||^2.$$ 

If we further assume $(C(1), C(2))$ is uniquely defined, then after relabeling, $\hat{C}(1) = C(1)$ and $\hat{C}(2) = C(2)$ a.s.

**Proof of Lemma 7.1.** The proof is same as that of Wang and Su (2017, Theorem 3.2), and thus, is omitted. Wang and Su (2017) focus on the case $L = 3.$ The proof is further divided into three sub-cases. By Assumption 3, $(C(1), C(2))$ is uniquely defined. Therefore, the last sub-case in Wang and Su (2017, the proof of Theorem 3.2) is ruled out. ■

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