A FAMLY OF ETA QUOTIENTS AND AN EXTENSION OF THE RAMANUJAN-MORDELL THEOREM

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ABSTRACT. Let \( k \geq 2 \) be an integer and \( j \) an integer satisfying \( 1 \leq j \leq 4k - 5 \). We define a family \( \{ C_{j,k}(z) \}_{1 \leq j \leq 4k-5} \) of eta quotients, and prove that this family constitute a basis for the space \( S_{2k}(\Gamma_0(12)) \) of cusp forms of weight \( 2k \) and level 12. We then use this basis together with certain properties of modular forms at their cusps to prove an extension of the Ramanujan-Mordell formula.

Key words and phrases: Ramanujan-Mordell formula, Dedekind eta function, eta quotients, eta products, theta functions, Eisenstein series, Eisenstein forms, modular forms, cusp forms, Fourier coefficients, Fourier series.

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1. INTRODUCTION

Let \( \mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{C} \) denote the sets of positive integers, non-negative integers, integers, rational numbers and complex numbers, respectively. Let \( N \in \mathbb{N} \). Let \( \Gamma_0(N) \) be the modular subgroup defined by

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad c \equiv 0 \pmod{N} \right\}.
\]

Let \( k \in \mathbb{Z} \). We write \( M_k(\Gamma_0(N)) \) to denote the space of modular forms of weight \( k \) for \( \Gamma_0(N) \), and \( E_k(\Gamma_0(N)) \) and \( S_k(\Gamma_0(N)) \) to denote the subspaces of Eisenstein forms and cusp forms of \( M_k(\Gamma_0(N)) \), respectively. It is known that

\[
M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N)).
\]

The Dedekind eta function \( \eta(z) \) is the holomorphic function defined on the upper half plane \( \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \) by the product formula

\[
\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).
\]

An eta quotient is defined to be a finite product of the form

\[
f(z) = \prod_{\delta} \eta^{n_\delta}(\delta z),
\]
where $\delta$ runs through a finite set of positive integers and the exponents $r_\delta$ are non-zero integers. By taking $N$ to be the least common multiple of the $\delta$’s we can write the eta quotient (1.2) as

$$f(z) = \prod_{1 \leq \delta | N} \eta^{r_\delta}(\delta z),$$

where some of the exponents $r_\delta$ may be 0. When all the exponents $r_\delta$ are nonnegative, $f(z)$ is said to be an eta product.

As in [11] throughout the paper we use the notation $q = e(z) := e^{2\pi i z}$ with $z \in \mathbb{H}$, and so $|q| < 1$ and $q^{1/24} = e(z/24)$. Ramanujan’s theta function $\varphi(z)$ is defined by

$$\varphi(z) = \sum_{n=0}^{\infty} q^{n^2}.$$ 

It is known that $\varphi(z)$ can be expressed as an eta quotient as

$$\varphi(z) = \frac{\eta^3(2z)}{\eta^2(z)\eta^2(4z)}.$$ 

For $a_j \in \mathbb{N}$, $1 \leq j \leq 4k$, we define

$$N(a_1, \ldots, a_{4k}; n) := \text{card}\{(x_1, \ldots, x_{4k}) \in \mathbb{Z}^{4k} | n = a_1x_1^2 + \cdots + a_{4k}x_{4k}\}.$$ 

Then we have

$$\varphi(a_1z) \cdots \varphi(a_{4k}z) = \sum_{n=0}^{\infty} N(a_1, \ldots, a_{4k}; n)q^n.$$ 

The value of $N(a_1, \ldots, a_{4k}; n)$ is independent of the order of the $a_j$’s.

Let $k \geq 2$ be an integer, and let $a_j \in \{1, 3\}$, $1 \leq j \leq 4k$, with an even number of $a_j$’s equal to 3. Then we write

$$N(a_1, \ldots, a_{4k}; n) = N(1^{4k-2i}, 3^{2i}; n),$$

where $i$ is an integer with $0 \leq i \leq 2k$. Ramanujan [17] stated a formula for $N(1^{2k}, 3^{0}; n)$, which was proved by Mordell in [14], see also [7, 3].

In this paper we define a family $\{C_{j,k}(z)\}_{1 \leq j \leq 4k-5}$ of eta quotients, and prove that this family constitute a basis for the space $S_{2k}(\Gamma_0(12))$ of cusp forms of weight $2k$ and level 12. We then use this basis together with certain properties of modular forms at their cusps to prove an extension of the Ramanujan-Mordell formula, that is, we give a formula for $N(1^{4k-2i}, 3^{2i}; n)$.

For $n, k \in \mathbb{N}$ we define the sum of divisors function $\sigma_k(n)$ by

$$\sigma_k(n) = \sum_{1 \leq m | n} m^k.$$
If \( n \notin \mathbb{N} \) we set \( \sigma_k(n) = 0 \). We define the Eisenstein series \( E_{2k}(z) \) by

\[
E_{2k}(z) := -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,
\]

where \( B_{2k} \) are Bernoulli numbers defined by the generating function

\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.
\]

The cusps of \( \Gamma_0(N) \) can be represented by rational numbers \( a/c \), where \( a \in \mathbb{Z}, \ c \in \mathbb{N}, \ c|N \) and \( \gcd(a,c) = 1 \), see [15, p. 320] and [8, p. 103]. We can choose the representatives of cusps of \( \Gamma_0(12) \) as

\[
1, 1/2, 1/3, 1/4, 1/6, \infty.
\]

Throughout the paper we use \( \infty \) and \( 1/12 \) interchangeably as they are equivalent cusps for \( \Gamma_0(12) \).

Let \( f(z) \) be an eta quotient given by (1.3). A formula for the order \( v_{a/c}(f) \) of \( f(z) \) at the cusp \( a/c \) (see [15, p. 320] and [12, Proposition 3.2.8]) is given by

\[
v_{a/c}(f) = \frac{N}{24 \gcd(c^2, N)} \sum_{1 \leq \delta | N} \gcd(\delta, c)^2 \cdot r_\delta.
\]

We use the following theorem to determine if a given eta quotient is in \( M_k(\Gamma_0(N)) \). See [13, Proposition 1, p. 284], [11, Corollary 2.3, p. 37], [9, p. 174], [12] and [10].

**Theorem 1.1. (Ligozat)** Let \( f(z) \) be an eta quotient given by (1.3) which satisfies the following conditions:

\begin{align*}
\text{(L1)} & \quad \sum_{1 \leq \delta | N} \delta \cdot r_\delta \equiv 0 \pmod{24}, \\
\text{(L2)} & \quad \sum_{1 \leq \delta | N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}, \\
\text{(L3)} & \quad \text{For each } d | N, \sum_{1 \leq \delta | N} \frac{\gcd(d, \delta)^2 \cdot r_\delta}{\delta} \geq 0, \\
\text{(L4)} & \quad \sqrt{\prod_{1 \leq \delta | N} \delta^{r_\delta}} \in \mathbb{Q}, \\
\text{(L5)} & \quad k = \frac{1}{2} \sum_{1 \leq \delta | N} r_\delta \text{ an even integer}.
\end{align*}

Then \( f(z) \in M_k(\Gamma_0(N)) \). Furthermore if all inequalities in (L3) are strict then \( f(z) \in S_k(\Gamma_0(N)) \).
For $N = 12$ in (L4) of Theorem 1.1, we have

\[
\sqrt{\prod_{1 \leq \delta \leq 12} \delta} = 2^{r_4 + r_{12}} \sqrt{2^{r_2 + r_6} 3^{r_3 + r_6 + r_{12}}},
\]

Thus the expression in (1.9) is a rational number if and only if

\[
r_2 + r_6 \equiv 0 \pmod{2} \text{ and } r_3 + r_6 + r_{12} \equiv 0 \pmod{2}.
\]

2. Statements of main results

For $j, k \in \mathbb{Z}$ we define an eta quotient $C_{j,k}(z)$ by

\[
C_{j,k}(z) := \left( \frac{\eta^{10}(2z) \eta^5(3z) \eta(4z) \eta^2(6z)}{\eta^{15}(z) \eta^3(12z)} \right) \left( \frac{\eta^2(2z) \eta(3z) \eta^3(12z)}{\eta^2(z) \eta(4z) \eta^2(6z)} \right)^j \left( \frac{\eta^6(z) \eta(6z)}{\eta^3(2z) \eta^2(3z)} \right)^k
\]

(2.1) \quad \equiv q^j + \sum_{n=j+1}^{\infty} c_{j,k}(n)q^n.

In the following theorem we give a basis for $M_{2k}(\Gamma_0(12))$ when $k \geq 2$.

**Theorem 2.1.** Let $k \geq 2$ be an integer.

(a) The family \( \{ C_{j,2k}(z) \}_{1 \leq j \leq 4k-5} \) constitute a basis for $S_{2k}(\Gamma_0(12))$.

(b) The set of Eisenstein series

\[
\{ E_{2k}(z), E_{2k}(2z), E_{2k}(3z), E_{2k}(4z), E_{2k}(6z), E_{2k}(12z) \}
\]

constitute a basis for $E_{2k}(\Gamma_0(12))$.

(c) The set

\[
\{ E_{2k}(\delta z) \mid \delta = 1, 2, 3, 4, 6, 12 \} \cup \{ C_{j,2k}(z) \mid 1 \leq j \leq 4k-5 \}
\]

constitute a basis for $M_{2k}(\Gamma_0(12))$.

For convenience we set

\[
\alpha_k = \frac{-4k}{(2^{2k} - 1)(3^{2k} - 1)B_{2k}},
\]

where $B_{2k}$ are Bernoulli numbers given in (1.7). Also we write \([j]f(z) := a_j \) for \( f(z) = \sum_{n=0}^{\infty} a_n q^n \). We now give an extension of the Ramanujan-Mordell Theorem.

**Theorem 2.2.** Let $k \geq 2$ be an integer and $i$ an integer satisfying $0 \leq i \leq 2k$. Let $\alpha_k$ be as in (2.2). Then

\[
\varphi^{4k-2i}(z) \varphi^{2i}(3z) = \sum_{r \mid 12} b_{(r,i,k)} E_{2k}(rz) + \sum_{1 \leq j \leq 4k-5} a_{(j,i,k)} C_{j,2k}(z),
\]

where

\[
b_{(1,i,k)} = (-1)^k(3^{2k-i} + (-1)^{i+1}) \cdot \alpha_k,
\]

(2.3)
Let \( a \) follows from (2.1) that 
\[
\{ \}
\]
respectively. 
(2.3)–(2.8) 
Thus the set 
\[
\{ \}
\] 
constitute a basis for \( E \) 

The Fourier series expansions of \( \eta \) given by the Fourier series expansions of \( \eta \) are given in (2.9), (2.1), and (2.3)–(2.8) respectively. 

3. PROOF OF THEOREM 2.1 

(a) By Theorem 1.1 we see that \( C_{j,2k}(z) \in S_{2k}(\Gamma_0(12)) \) for \( 1 \leq j \leq 4k - 5 \). It follows from (2.4) that \( \{ C_{j,2k}(z) \}_{1 \leq j \leq 4k - 5} \) is a linearly independent set. We deduce from the formulae in [18, Section 6.3, p. 98] that 
\[
\dim(S_{2k}(\Gamma_0(12))) = 4k - 5.
\]
Thus the set \( \{ C_{j,2k}(z) \}_{1 \leq j \leq 4k - 5} \) is a basis for \( S_{2k}(\Gamma_0(12)) \).

(b) It follows from [18, Theorem 5.9] that 
\[
\{ E_{2k}(z), E_{2k}(2z), E_{2k}(3z), E_{2k}(4z), E_{2k}(6z), E_{2k}(12z) \}
\]
constitute a basis for \( E_{2k}(\Gamma_0(12)) \).

(c) Appealing to [11], the assertion follows from (a) and (b).

4. FOURIER SERIES EXPANSIONS OF \( \eta(rz) \) AND \( E_{2k}(rz) \) AT CERTAINCUSPS 

Let \( k \geq 2 \) be an integer. For convenience we set \( \eta_r(z) = \eta(rz) \) for \( r \in \mathbb{N} \). We also set 
\[
(4.1) \quad A_c = \begin{bmatrix} -1 & 0 \\ c & -1 \end{bmatrix} \in SL_2(\mathbb{Z}).
\]
The Fourier series expansions of \( \eta_r(z) \) for \( r = 1, 2, 3, 4, 6, 12 \) at the cusp \( 1/c \) are given by the Fourier series expansions of \( \eta_r(A_c^{-1}z) \) at the cusp \( \infty \).
In [11, Theorem 1.7 and Proposition 2.1] we take $L = \begin{bmatrix} x & y \\ u & v \end{bmatrix} = L_r$ as

$$L_1 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix},$$

$$L_4 = \begin{bmatrix} -4 & 1 \\ -1 & 0 \end{bmatrix}, \quad L_6 = \begin{bmatrix} -6 & 1 \\ -1 & 0 \end{bmatrix}, \quad L_{12} = \begin{bmatrix} -12 & 1 \\ -1 & 0 \end{bmatrix},$$

and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A_1$, where $A_1$ is given by (4.1). We obtain the Fourier series expansions of $\eta_r(z)$ for $r = 1, 2, 3, 4, 6, 12$ at the cusp 1 as

$$\eta_1(A_1^{-1}z) = e^{\pi i/3}(-z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left(\frac{n^2}{24}(z + 1)\right),$$

$$\eta_2(A_1^{-1}z) = \frac{e^{5\pi i/12}}{21/2}(-z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left(\frac{n^2}{48}(z + 1)\right),$$

$$\eta_3(A_1^{-1}z) = \frac{e^{\pi i/2}}{31/2}(-z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left(\frac{n^2}{72}(z + 1)\right),$$

$$\eta_4(A_1^{-1}z) = \frac{e^{7\pi i/12}}{2}(-z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left(\frac{n^2}{96}(z + 1)\right),$$

$$\eta_6(A_1^{-1}z) = \frac{e^{3\pi i/4}}{61/2}(-z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left(\frac{n^2}{144}(z + 1)\right),$$

$$\eta_{12}(A_1^{-1}z) = \frac{e^{5\pi i/4}}{121/2}(-z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left(\frac{n^2}{288}(z + 1)\right).$$

From (1.4) and (4.2)–(4.7) we obtain the Fourier series expansions of $\varphi(z)$ and $\varphi(3z)$ at the cusp 1 as

$$\varphi(A_1^{-1}z) = \frac{\eta_5^2(A_1^{-1}z)}{\eta_1^2(A_1^{-1}z)\eta_3^2(A_1^{-1}z)}$$

$$= \frac{e^{\pi i/4}}{21/2}(-z - 1)^{1/2} \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left(\frac{n^2}{48}(z + 1)\right) \right)^5$$

$$= \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left(\frac{n^2}{24}(z + 1)\right) \right)^2 \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left(\frac{n^2}{96}(z + 1)\right) \right)^2,$$

$$\varphi(3A_1^{-1}z) = \frac{\eta_5^2(A_1^{-1}z)}{\eta_3^2(A_1^{-1}z)\eta_1^2(A_1^{-1}z)}.$$
we obtain the Fourier series expansions of
\[
\frac{e^{\pi i/4}}{6^{1/2}} (z - 1)^{1/2} \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{144} (z + 1) \right) \right)^5.
\]

(4.9) 
\[
\left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{72} (z + 1) \right) \right)^2 \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{288} (z + 1) \right) \right)^2.
\]

Similarly, by taking \( A = A_3 \) in [11, Theorem 1.7 and Proposition 2.1] and \( L \) as
\[
L_1 := \begin{bmatrix} -1 & -5 \\ -3 & -16 \end{bmatrix}, \quad L_2 := \begin{bmatrix} -2 & -5 \\ -3 & -8 \end{bmatrix}, \quad L_3 := \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix},
\]
\[
L_4 := \begin{bmatrix} -4 & -5 \\ -3 & -4 \end{bmatrix}, \quad L_6 := \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}, \quad L_{12} := \begin{bmatrix} -4 & 1 \\ -1 & 0 \end{bmatrix}
\]
we obtain the Fourier series expansions of \( \eta_r(z) \) for \( r = 1, 2, 3, 4, 6, 12 \) at the cusp 1/3 as
\[
\eta_1(A_3^{-1}z) = e^{5\pi i/3} (-3z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{24} (z - 5) \right),
\]

(4.10) 
\[
\eta_2(A_3^{-1}z) = \frac{e^{7\pi i/12}}{2^{1/2}} (-3z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{48} (z - 5) \right),
\]

(4.11) 
\[
\eta_3(A_3^{-1}z) = e^{\pi i/3} (-3z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{24} (3z + 1) \right),
\]

(4.12) 
\[
\eta_4(A_3^{-1}z) = \frac{e^{17\pi i/12}}{2} (-3z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{96} (z - 5) \right),
\]

(4.13) 
\[
\eta_6(A_3^{-1}z) = \frac{e^{5\pi i/12}}{2^{1/2}} (-3z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{48} (3z + 1) \right),
\]

(4.14) 
\[
\eta_{12}(A_3^{-1}z) = \frac{e^{\pi i/12}}{2} (-3z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{96} (3z + 1) \right).
\]

(4.15) 

From (1.4) and (4.10)–(4.15) we obtain the Fourier series expansions of \( \varphi(z) \) and \( \varphi(3z) \) at the cusp 1/3 as

\[
\varphi(A_3^{-1}z) = \frac{\eta_5^5(A_3^{-1}z)}{\eta_2^2(A_3^{-1}z) \eta_4^2(A_3^{-1}z)} \\
\frac{e^{3\pi i/4}}{2^{1/2}} (-3z - 1)^{1/2} \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{48} (z - 5) \right) \right)^5
\]

(4.16) 
\[
\left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{24} (z - 5) \right) \right)^2 \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e \left( \frac{n^2}{96} (z - 5) \right) \right)^2.
\]
\[
\varphi(3A_4^{-1}z) = \frac{\eta_5^5(A_3^{-1}z)}{\eta_5^2(A_3^{-1}z)\eta_{12}^2(A_3^{-1}z)} \frac{e^{\pi i/4}}{2^{1/2}} (-3z - 1)^{1/2} \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{48}(3z + 1) \right) \right)^5 \\
= \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{24}(3z + 1) \right) \right)^2 \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{96}(3z + 1) \right) \right)^2.
\]

(4.17)

Again by taking \( A = A_4 \) in [11, Theorem 1.7 and Proposition 2.1] and \( L \) as

\[
L_1 := \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}, \quad L_2 := \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}, \quad L_3 := \begin{bmatrix} -3 & 1 \\ -4 & 1 \end{bmatrix}, \\
L_4 := \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}, \quad L_6 := \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}, \quad L_{12} := \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}
\]

we obtain the Fourier series expansions of \( \eta_r(z) \) for \( r = 1, 2, 3, 4, 6, 12 \) at the cusp 1/4 as

\[
\eta_1(A_4^{-1}z) = e^{13\pi i/12}(-4z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{24}(z + 1) \right),
\]

(4.18)

\[
\eta_2(A_4^{-1}z) = e^{\pi i/6}(-4z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{12}(z + 1) \right),
\]

(4.19)

\[
\eta_3(A_4^{-1}z) = \frac{e^{5\pi i/12}}{3^{1/2}}(-4z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{72}(z + 1) \right),
\]

(4.20)

\[
\eta_4(A_4^{-1}z) = e^{\pi i/12}(-4z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{6}(z + 1) \right),
\]

(4.21)

\[
\eta_6(A_4^{-1}z) = \frac{e^{\pi i/3}}{3^{1/2}}(-4z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{36}(z + 1) \right),
\]

(4.22)

\[
\eta_{12}(A_4^{-1}z) = \frac{e^{5\pi i/12}}{3^{1/2}}(-4z - 1)^{1/2} \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{18}(z + 1) \right).
\]

(4.23)

From (4.14) and (4.18)–(4.23) we obtain the Fourier series expansions of \( \varphi(z) \) and \( \varphi(3z) \) at the cusp 1/4 as

\[
\varphi(A_4^{-1}z) = \frac{\eta_5^5(A_4^{-1}z)}{\eta_1^5(A_4^{-1}z)\eta_{12}^2(A_4^{-1}z)}
\]
\[ e^{\pi i/2}(-4z - 1)^{1/2} \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{12} (z + 1) \right) \right)^5 
\]

\[ \frac{\left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{24} (z + 1) \right) \right)^2 \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{6} (z + 1) \right) \right)^2}{\left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{72} (z + 1) \right) \right)^2 \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{18} (z + 1) \right) \right)^2} \]

\[ \varphi(3A_4^{-1}z) = \frac{\eta_0^6(A_4^{-1}z)}{\eta_3^6(A_4^{-1}z) \eta_{12}^6(A_4^{-1}z)} \]

\[ \frac{1}{3}(-4z - 1)^{1/2} \left( \sum_{n \geq 1} \left( \frac{12}{n} \right) e\left( \frac{n^2}{36} (z + 1) \right) \right)^5 
\]

Table 4.1: First terms of \( \varphi^{4k-2i}(z) \varphi^{2i}(3z) \) at certain cusps

| Cusp | \( \infty \) | 1 | 1/2 | 1/3 | 1/4 | 1/6 |
|------|-------------|---|-----|-----|-----|-----|
| Term | \( (-z - 1)^{2k} \frac{(-1)^k}{2^{2k} 3^i} \) | \( (-3z - 1)^{2k} \frac{(-1)^{i+k}}{2^{2k} 3^i} \) | \( (-4z - 1)^{2k} \frac{(-1)^i}{3^i} \) | \( 0 \) |

By (1.8), we have

\[ v_{1/2}(\varphi^{4k-2i}(z) \varphi^{2i}(3z)) = 3k - i > 0, \]

\[ v_{1/6}(\varphi^{4k-2i}(z) \varphi^{2i}(3z)) = k + i > 0 \]

for all \( 1 \leq i \leq 2k \), that is, the first terms of the Fourier series expansions of \( \varphi^{4k-2i}(z) \varphi^{2i}(3z) \) at cusps 1/2 and 1/6 are 0. This completes Table 4.1.

The following theorem is an analogue of [11, Proposition 2.1] for the Eisenstein series \( E_{2k}(tz) \). For convenience we set \( E_{(2k,t)}(z) = E_{2k}(tz) \) for \( t \in \mathbb{N} \).
Theorem 4.1. Let $k \geq 2$ be an integer and $t \in \mathbb{N}$. The Fourier series expansion of $E_{(2k,t)}(z)$ at the cusp $1/c \in \mathbb{Q}$ is

$$E_{(2k,t)}(A_c^{-1}z) = \left(\frac{g}{t}\right)^{2k} (-cz - 1)^{2k} E_{2k}\left(\frac{g^2}{t} z + \frac{yg}{t}\right),$$

where $g = \gcd(t,c)$, $y$ is some integer, and $A_c$ is given in (4.1). 

Proof. The Fourier series expansion of $E_{(2k,t)}(z)$ at the cusp $1/c$ is given by the Fourier series expansion of $E_{(2k,t)}(A_c^{-1}z)$ at the cusp $\infty$. We have

$$E_{(2k,t)}(A_c^{-1}z) = E_{(2k,t)}\left(\frac{-z}{-cz - 1}\right) = E_{2k}\left(\frac{-tz}{-cz - 1}\right) = E_{2k}(\gamma z),$$

where $\gamma = \begin{bmatrix} -t & 0 \\ -c & -1 \end{bmatrix}$. As $\gcd(t/g, c/g) = 1$, there exist $y, v \in \mathbb{Z}$ such that

$$\frac{t}{g}(-v) + \frac{c}{g} y = 1. \text{ Thus } L := \begin{bmatrix} -t/g & y \\ -c/g & v \end{bmatrix} \in SL_2(\mathbb{Z}). \text{ Then for } k \geq 2, \text{ we have}$$

$$E_{(2k,t)}(A_c^{-1}z) = E_{2k}(LL^{-1}\gamma z) = (-c)\left(\frac{-vt + cy}{t}z + y\right)^{2k} E_{2k}\left(\frac{-vt + cy}{t}z + y\right)
\quad = \left(\frac{vt - cy}{g} z + \frac{vt - cy}{g}\right)^{2k} E_{2k}\left(\frac{g^2}{t} z + \frac{yg}{t}\right)$$

$$= (g/t)^{2k}(-cz - 1)^{2k} E_{2k}\left(\frac{g^2}{t} z + \frac{yg}{t}\right),$$

which completes the proof. \qed

It follows from Theorem 4.1 and (1.6) that the first term of the Fourier series expansion of $E_{2k}(tz)$ at the cusp $1/c$ is

$$(4.27) \left(\frac{g}{t}\right)^{2k} (-cz - 1)^{2k} \left(-\frac{B_{2k}}{4k}\right).$$

5. Proofs of Theorems 2.2 and 2.3

Let $k \geq 2$ be an integer and $i$ an integer with $0 \leq i \leq 2k$. By (1.4) we have

$$\varphi^{4k-2i}(z)\varphi^{2i}(3z) = \frac{\eta^{20k-10i}(2z)}{\eta^{8k-4i}(z)\eta^{8k-4i}(4z)} \cdot \frac{\eta^{10i}(6z)}{\eta^{4i}(3z)\eta^{4i}(12z)}.$$ 

By Theorem 1.1, we have $\varphi^{4k-2i}(z)\varphi^{2i}(3z) \in M_{2k}(\Gamma_0(12))$. By Theorem 2.1(c), we have

$$(5.1) \varphi^{4k-2i}(z)\varphi^{2i}(3z) = \sum_{r|12} b_{r,i,k} E_{2k}(rz) + \sum_{1 \leq j \leq 4k-5} a_{j,i,k} C_{j,2k}(z)$$

for some constants $b_{r,i,k}, b_{2,i,k}, b_{3,i,k}, b_{4,i,k}, b_{6,i,k}, b_{12,i,k}, a_{1,i,k}, \ldots, a_{(4k-5,i,k)}$. Since $C_{1,k}(z), \ldots, C_{4k-5,k}(z)$ are cusp forms, the first terms of their Fourier series expansions at all cusps are 0.
By appealing to (1.27) and Table 4.1 we equate the first terms of the Fourier series expansions of (5.1) in both sides at cusps $\infty$, 1, 1/2, 1/3, 1/4, 1/6 to obtain the system of linear equations

\[
\begin{align*}
    b_{(1,i,k)} + b_{(2,i,k)} + b_{(3,i,k)} + b_{(4,i,k)} + b_{(6,i,k)} + b_{(12,i,k)} &= -\frac{4k}{B_{2k}}, \\
    b_{(1,i,k)} + \frac{b_{(2,i,k)}}{2^{2k}} + \frac{b_{(3,i,k)}}{3^{2k}} + \frac{b_{(4,i,k)}}{4^{2k}} + \frac{b_{(6,i,k)}}{6^{2k}} + \frac{b_{(12,i,k)}}{12^{2k}} &= \frac{(-1)^k}{2^{2k}3^{i}} \cdot \frac{-4k}{B_{2k}}, \\
    b_{(1,i,k)} + b_{(2,i,k)} + \frac{b_{(3,i,k)}}{2^{2k}} + \frac{b_{(4,i,k)}}{4^{2k}} + \frac{b_{(6,i,k)}}{6^{2k}} + \frac{b_{(12,i,k)}}{12^{2k}} &= 0, \\
    b_{(1,i,k)} + b_{(2,i,k)} + \frac{b_{(3,i,k)}}{3^{2k}} + \frac{b_{(4,i,k)}}{4^{2k}} + \frac{b_{(6,i,k)}}{6^{2k}} + \frac{b_{(12,i,k)}}{12^{2k}} &= \frac{(-1)^{i+k}}{2^{2k}} \cdot \frac{-4k}{B_{2k}}, \\
    b_{(1,i,k)} + b_{(2,i,k)} + \frac{b_{(3,i,k)}}{2^{2k}} + \frac{b_{(4,i,k)}}{4^{2k}} + \frac{b_{(6,i,k)}}{6^{2k}} + \frac{b_{(12,i,k)}}{12^{2k}} &= 0.
\end{align*}
\]

Solving the above system of linear equations, we obtain the asserted expressions for $b_{(r,i,k)}$ for $r = 1, 2, 3, 4, 6, 12$ in (2.3)–(2.8). By (2.1), we have $[j]C_{j,k}(z) = 1$ for each $j$ with $1 \leq j \leq 4k - 5$. Equating the coefficients of $q^j$ in both sides of (5.1) we obtain

\[
N(1^{4k-2i}j^2; j) = \sum_{r|12} b_{(r,i,k)}\sigma_{2k-1}(j/r) + \sum_{1\leq i \leq j-1} a_{(i,i,k)}c_{i,2k}(z) + a_{(j,i,k)}.
\]

We isolate $a_{(j,i,k)}$ to complete the proof of Theorem 2.2. Finally, Theorem 2.3 follows from (1.5), (1.6), (2.1) and Theorem 2.2.

6. Examples and Remarks

We now illustrate Theorems 2.2 and 2.3 by some examples.

**Example 6.1.** We determine $N(1^6, 3^2; n)$ for all $n \in \mathbb{N}$. We take $k = 2$ and $i = 1$ in Theorems 2.2 and 2.3. By (2.3)–(2.8) we have

\[
(6.1) \quad \begin{cases} 
    b_{(1,1,2)} = 28/5, & b_{(2,1,2)} = 0, & b_{(3,1,2)} = -108/5, \\
    b_{(4,1,2)} = -448/5, & b_{(6,1,2)} = 0, & b_{(12,1,2)} = 1728/5.
\end{cases}
\]

We compute $N(1^6, 3^2; n)$ for $n = 1, 2, 3$ as $4k - 5 = 3$, and obtain

\[
(6.2) \quad N(1^6, 3^2; 1) = 12, \quad N(1^6, 3^2; 2) = 60, \quad N(1^6, 3^2; 3) = 164.
\]

By (2.9), (6.1) and (6.2), we obtain

\[
a_{(1,1,2)} = 32/5, \quad a_{(2,1,2)} = 48, \quad a_{(3,1,2)} = 576/5.
\]

Then, by Theorem 2.3, for all $n \in \mathbb{N}$, we have

\[
N(1^6, 3^2; n) = \frac{28}{5}\sigma_3(n) - \frac{108}{5}\sigma_3(n/3) - \frac{448}{5}\sigma_3(n/4) + \frac{1728}{5}\sigma_3(n/12)
\]

which agrees with the known results, see for example [4]. We note that the last two coefficients in the above expression are different from the ones in [4] since we used a different basis for the space of cusp forms.

**Example 6.2.** We determine \( N(1^4, 3^8; n) \) for all \( n \in \mathbb{N} \). We take \( k = 3 \) and \( i = 4 \) in Theorems 2.2 and 2.3. By (2.3)–(2.8) we have

\[
(6.3) \quad \begin{cases}
  b_{(1,4,3)} = 8/91, & b_{(2,4,3)} = 0, & b_{(3,4,3)} = 720/91, \\
  b_{(4,4,3)} = -512/91, & b_{(6,4,3)} = 0, & b_{(12,4,3)} = -4608/91.
\end{cases}
\]

We compute \( N(1^4, 3^8; n) \) for \( n = 1, 2, 3, 4, 5, 6, 7 \) as \( 4k - 5 = 7 \), and obtain

\[
(6.4) \quad \begin{cases}
  N(1^4, 3^8; 1) = 8, & N(1^4, 3^8; 2) = 24, & N(1^4, 3^8; 3) = 48, \\
  N(1^4, 3^8; 4) = 152, & N(1^4, 3^8; 5) = 432, \\
  N(1^4, 3^8; 6) = 720, & N(1^4, 3^8; 7) = 1344.
\end{cases}
\]

By (2.9), (6.3) and (6.4), we obtain

\[
a_{(1,4,3)} = 720/91, \quad a_{(2,4,3)} = 14880/91, \quad a_{(3,4,3)} = 123376/91, \quad a_{(4,4,3)} = 40640/7, \\
a_{(5,4,3)} = 1248448/91, \quad a_{(6,4,3)} = 1551360/91, \quad a_{(7,4,3)} = 792576/91.
\]

Then, by Theorem 2.3, for all \( n \in \mathbb{N} \), we have

\[
N(1^4, 3^8; n) = \frac{8}{91} \sigma_5(n) + \frac{720}{91} \sigma_5(n/3) - \frac{512}{91} \sigma_5(n/4) - \frac{46080}{91} \sigma_5(n/12) \\
+ \frac{720}{91} c_{1,6}(n) + \frac{14880}{91} c_{2,6}(n) + \frac{123376}{91} c_{3,6}(n) + \frac{40640}{7} c_{4,6}(n) \\
+ \frac{1248448}{91} c_{5,6}(n) + \frac{1551360}{91} c_{6,6}(n) + \frac{792576}{91} c_{7,6}(n),
\]

which agrees with the known results, see for example [1].

**Example 6.3.** Ramanujan [17] stated a formula for \( N(1^{2k}, 3^0; n) \), which was proved by Mordell in [14], see also [7,3]. By taking \( i = 0 \) in Theorems 2.2 and 2.3, we obtain

\[
N(1^{2k}, 3^0; n) = \frac{4k}{(2^k - 1)B_{2k}} \left( (-1)^{k+1} \sigma_{2k-1}(n) + (1 + (-1)^k) \sigma_{2k-1}(n/2) \\
-2^{2k} \sigma_{2k-1}(n/4) \right) + \sum_{1 \leq j \leq 4k-5} a_{(j,0,k)} c_{j,2k}(n),
\]

where \( a_{(j,0,k)} \) and \( c_{j,2k}(n) \) are given by (2.9) and (2.1), respectively. The coefficients of \( \sigma \)-functions in the above formula agree with those in [7, theorem 1.1] and [3, theorem 4.1]. Different coefficients in the cusp part are due to the choice of our basis for the space \( S_{2k}(\Gamma_0(12)) \) of cusp forms.
Remark 6.1. Throughout the paper we assumed that $k \geq 2$. For $k = 1$ we have $\dim(S_2(\Gamma_0(12))) = 0$. A basis for $M_2(\Gamma_0(12)) = E_2(\Gamma_0(12))$ is given in [2], see also [19, 5, 6].

Remark 6.2. Let $k \geq 2$ be an integer. Let $N \in \mathbb{N}$ and $\chi$ a Dirichlet character of modulus dividing $N$. We write $M_k(\Gamma_0(N), \chi)$ to denote the space of modular forms of weight $k$ with multiplier system $\chi$ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ to denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N), \chi)$, respectively. We deduce from the formulae in [18, Section 6.3, p. 98] that
\[
\dim(S_{2k-1}(\Gamma_0(12), \chi_1)) = 4k - 7, \\
\dim(S_{2k-1}(\Gamma_0(12), \chi_2)) = 4k - 6, \\
\dim(S_{2k}(\Gamma_0(12), \chi_3)) = 4k - 4,
\]
are Legendre-Jacobi-Kronecker symbols. By appealing to a more general version of Theorem 1.1 (Ligozat), see for example [16, Theorem 1.64] and [11, Corollary 2.3, p. 37], we deduce that the families of eta quotients
\[
\left\{ C_{j, 2k-1}(z) \right\}_{1 \leq j \leq 4k-7}, \\
\left\{ \frac{\eta^4(z)\eta(4z)\eta(12z)}{\eta^4(2z)\eta^2(6z)} C_{j, 2k-1}(z) \right\}_{1 \leq j \leq 4k-6}, \\
\left\{ \frac{\eta^4(z)\eta(4z)\eta(12z)}{\eta^4(2z)\eta^2(6z)} C_{j, 2k}(z) \right\}_{1 \leq j \leq 4k-4}
\]
constitute a basis for $S_{2k-1}(\Gamma_0(12), \chi_1)$, $S_{2k-1}(\Gamma_0(12), \chi_2)$, $S_{2k}(\Gamma_0(12), \chi_3)$, respectively, where $\chi_1, \chi_2, \chi_3$ are given in (6.5).

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