Safe Coverage of Moving Domains for Vehicles With Second-Order Dynamics

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Abstract—Autonomous coverage of a specified area by robots operating in close proximity with each other has many potential applications such as real-time monitoring of rapidly changing environments, and search and rescue; however, coordination and safety are two fundamental challenges. For coordination, we propose a distributed controller for covering moving, compact domains which consists in a double integrator with bounded input forces. This control policy is based on artificial potentials and alignment forces designed to promote desired vehicle-domain and intervehicle separations and relative velocities. We prove that certain coverage configurations are locally asymptotically stable. For safety, we establish energy conditions for collision-free motion and utilize Hamilton–Jacobi (HJ) reachability theory for last-resort pairwise collision avoidance. We derive an analytical solution to the associated HJ partial differential equation corresponding to the collision avoidance problem between two double integrator vehicles. We demonstrate our approach in several numerical simulations involving vehicles covering convex and nonconvex moving domains.

Index Terms—Artificial potentials, autonomous robots, coverage control, decentralized control, Hamilton–Jacobi (HJ) reachability, swarm intelligence.

I. INTRODUCTION

Two major challenges—coordination and safety—arise when autonomous systems cooperate in close proximity with each other. In this article, we consider the problem of controlling multiple autonomous systems to safely cover a desired, possibly moving area in a decentralized manner. Multiagent coverage of a moving domain is important when a large area of interest is dynamically changing or when the target of interest needs to be identified quickly [1], [2]. Applications include real-time surveillance of dynamic environments, efficient search and rescue, and multiagent aerobatics.

The goal of coverage control problems is to deploy agents to a possibly moving domain of interest to achieve an optimal sensing. A common solution is through minimizing a coverage functional involving a Voronoi tessellation and the locations of vehicles within the tessellation [3]–[6]. This is a high-dimensional optimization problem which needs to be solved in real time. Instead of following this procedure, we take a novel approach and achieve coverage through swarming by artificial potentials [7]–[9]. The coverage configurations of interest in this article are states in which agents spread evenly over the domain. Such configurations are also referred to in the literature as balanced or anticonsensus states [10]. In this aim, we set into the control law a certain desired distance between agents and between agents and the domain’s boundary.

Hamilton–Jacobi (HJ) reachability [11], [12] has seen success in collision avoidance [13], air traffic management [14], and emergency landing [15]. HJ reachability analysis involves solving an HJ partial differential equation (PDE) to compute a backward reachable set (BRS) representing states from which danger is inevitable. By using the derived optimal controller on the BRS boundary, safety can be guaranteed despite worst-case actions of another agent.

Contributions: In this article, we develop a new approach to self-collective coordination of autonomous agents that aim to reach and cover a moving target domain. We aim to enable 1) reaching and spreading over a target domain without having set a priori the coverage configuration, 2) use of a distributed control policy from which self-organization emerges at the group level, and 3) guarantee of collision-avoidance throughout the coordination process. In this aim, we consider a control policy based on double integrator dynamics, which includes both a coverage and a safety controller. The coverage controller brings vehicles inside a target domain, spreads them over the target domain, and aligns vehicle and domain velocities with each other. On the other hand, the safety controller guarantees collision avoidance of vehicles.

We emphasize that the proposed coverage controller is done through agent swarming; there is no leader and no order among the agents. This means that the controller does not rely on the well functioning of each individual agent. Such self-collective and cooperative behavior is present in systems of interacting agents in the physics and biology literature [16]–[20]. Swarming by interaction potentials has also been used to contain follower agents within the convex hull of leaders [21], [22]. An agent search and target-locating algorithm based on a swarming model was studied in [23].

Previously, coverage of moving domains using agents with first-order dynamics via Voronoi tessellations was studied in [2]. In [24] and [25], controllers were developed for agents with second-order dynamics to respectively track multiple targets and fence a moving domain. To the best of our knowledge, we present, for the first time, a controller for coverage of moving domains using agents with second-order dynamics and provide theoretical guarantees on stability. Second-order dynamics account for effect of acceleration, which is essential in robotics applications, especially where safety is a concern.

The safety controller is derived from HJ reachability analysis, which provides pairwise safety guarantees for agents with known nonlinear dynamics, experiencing disturbances, and with bounded control. Instead of numerically solving an associated Hamilton–Jacobi–Isaacs (HJI) PDE as is typically done [12], we derive the analytical solution to the PDE to eliminate numerical errors and the need to specify computation bounds. This least-restrictive approach to collision avoidance in coverage problems is novel, being fundamentally different from methods used in the literature on coverage by Voronoi tessellations [5]. More
generally, one could also incorporate other least-restrictive collision avoidance controllers such as [13], [26], and [27], for various scalability and safety guarantees.

The article is organized as follows. Section II presents the background. In Section III, we formulate the coverage problem, and in Section IV, we study the safe coverage problem for moving domains with double integrator dynamics. In Section V, we demonstrate our approach with numerical simulations.

II. BACKGROUND

A. Signed Distance

An oriented measure of how far a point $x$ is from a given domain $\Omega$ is the signed distance, defined by

$$b(x) = \begin{cases} \min_{x_p \in \partial \Omega} \|x - x_p\|, & \text{if } x \notin \Omega \\ -\min_{x_p \in \partial \Omega} \|x - x_p\|, & \text{if } x \in \Omega. \end{cases}$$

If $\nabla b(x)$ exists, then there exists a unique $P_{\partial \Omega}(x) \in \partial \Omega$, called the projection of $x$ on $\partial \Omega$, such that

$$b(x) = \begin{cases} \|P_{\partial \Omega}(x) - x\|, & \text{if } x \notin \Omega \\ -\|P_{\partial \Omega}(x) - x\|, & \text{if } x \in \Omega \end{cases}$$

and $\nabla b(x) = \frac{x - P_{\partial \Omega}(x)}{b(x)}$.

B. Hamilton–Jacobi Reachability

In this article, our safety controller is based on HJ reachability, which guarantees pairwise collision avoidance between agents. HJ reachability is a game-theoretic method, in which one agent (Player 1) takes on the role of the “pursuer” or equivalently the “control,” and the other agent (Player 2) takes on the role of the “evader” or “disturbance.” Formally, consider the two-player differential game described by

$$\dot{z} = f(z(t), u(t), d(t)) \quad z(0) = x$$

where $z \in \mathbb{R}^n$ is the joint state of the players, $u \in U$ is the control input of Player 1, and $d \in D$ is the control input of Player 2. We assume $f : \mathbb{R}^n \times U \times D \to \mathbb{R}^n$ is uniformly continuous, bounded, and Lipschitz continuous in $z$ for fixed $u$ and $d$, and $u(\cdot), d(\cdot)$ are measurable functions. By the Picard–Lindelöf Theorem, (3) has a unique local in time solution.

In this differential game, the goal of Player 2 (the disturbance) is to drive the system into some target set using only nonanticipative strategies [11], [28], while Player 1 (the control) aims to drive the system away from it. This differential game setup is crucial for making safety guarantees for agents that have nonlinear dynamics with bounded control and experiencing disturbances.

We introduce the time-to-reach problem as follows.

(Time-to-reach) Find the time to reach a target set $\Gamma_D$ while avoiding the obstacle $\Gamma_S$ from any initial state $x$, in a scenario where Player 1 maximizes the time, while Player 2 minimizes the time. Player 2 is restricted to using nonanticipative strategies, with the knowledge of Player 1’s current and past decisions. Such a time is denoted by $\phi(x)$.

Following [11], given $u(\cdot)$ and $d(\cdot)$, the time to reach a closed target set $\Gamma_D$ with compact boundary, while avoiding the obstacle $\Gamma_S$, is defined as

$$T_x[u,d] = \min \{t \in \Gamma_D \text{ and } z(s) \notin \Gamma_S, \forall s \in [0,t]\}.$$

Note that “target set” refers to a set of interest [12] and, in our case, the set of states to be avoided. Now, the time-to-reach problem reduces to finding

$$\phi(x) = \min_{u,d \in D} T_x[u,d]$$

where $\Theta$ represents the set of nonanticipative strategies, which intuitively states that agents cannot react to future information; this notion is defined formally in [11] and [28]. The collection of all the states that are reachable in a finite time is the capturability set $\mathbb{R}^n = \{x \in \mathbb{R}^n : \phi(x) < +\infty\}$.

Applying the dynamic programming principle [29], one can obtain $\phi$ as the viscosity solution of the stationary HJ PDE

$$\min_{u,d \in D} \{ -\nabla \phi(z) \cdot f(z,u,d) - 1 \} = 0 \text{ in } \mathbb{R}^n \setminus (\Gamma_D \cup \Gamma_S),$$

$$\phi(0) = 0 \text{ on } \Gamma_D, \quad \phi(z) = +\infty \text{ on } \Gamma_S.$$ (4)

This PDE can be solved using finite difference methods such as the Lax–Friedrichs method [11]. Also, from the solution $\phi(x)$, one can obtain the control input for optimal avoidance

$$u^*(z) = \arg \min_{u,d \in D} \{ -\nabla \phi(z) \cdot f(z,u,d) - 1 \}.$$ (5)

Note that the control input bounds of both players are explicit in (4) and (5), allowing us to guarantee collision avoidance while accounting for the control input bounds. In the second part of Section IV-C, we will apply the theory presented above to our problem, with a specific form of (3), which leads to corresponding specific forms of (4) and (5).

III. PROBLEM FORMULATION

We consider a group of $N$ vehicles, denoted by $Q_i, i = 1, \ldots, N$, with the double integrator dynamics given by

$$p_i = v_i, \quad v_i = u_i.$$ (6)

Here, $p_i = (p_{i,x}, p_{i,y})$ and $v_i = (v_{i,x}, v_{i,y})$ are the position and velocity of $Q_i$, respectively, and $u_i = (u_{i,x}, u_{i,y})$ is the control force applied to this vehicle. For practical reasons, we set thresholds on velocities and control inputs

$$\|v_i\| \leq v_{\text{max}}, \quad \|u_i\| \leq u_{\text{max}}.$$ (7)

Given a predefined collision radius $c_r$, a vehicle is considered safe if there are no other vehicles within distance $c_r$

$$\|p_i - p_j\| > c_r, \quad \text{for any } j \neq i.$$ (8)

We are interested in certain configurations of the agents in the target domain, defined as follows.

Definition III.1 ($r$-Subcover): A group of agents is an $r$-subcover for a compact domain $\Omega \subseteq \mathbb{R}^2$ if we have the following.

1) The distance between any two vehicles is at least $r$.
2) The signed distance from any vehicle to $\Omega$ is less than or equal to $-\frac{r}{2}$.

Definition III.2 ($r$-Cover): An $r$-subcover for $\Omega$ is an $r$-cover for $\Omega$ if its size is maximal (i.e., no larger number of agents can be an $r$-subcover for $\Omega$).

The $r$-cover definition is closely related to the packing problem for circular objects of radius $\frac{r}{2}$ in a container with shape $\Omega$ [30]. Having an $r$-cover implies the container is full and there is no room for more of such objects.

We consider a target domain that moves with prescribed constant velocity $v_d$ (in particular, for a static domain, $v_d = 0$). Specifically, let $\Omega \subseteq \mathbb{R}^2$ be a compact domain and define $\Omega_t = \Omega + t v_d$, representing the moving domain at time $t$. Alternatively, if one sets an arbitrary marker point $p_d$ (e.g., the center of mass) in $\Omega$, its motion is given by $p_d(t) = p_d + tv_d$. 


We are interested in covering the moving domain $\Omega_t$ (see Definitions III.1 and III.2). For this reason, we want the vehicles to reach asymptotically, as $t \to \infty$, the velocity of the target domain, while maintaining a cohesive group through dynamics. This is expressed by the concept of flocking [19], [31], [32]. We adapt below the definition of flocking from [33] to the problem of moving target.

Definition III.3 (Flocking with a moving target): A group of vehicles has a time-asymptotic flocking with a target domain moving with constant velocity $v_d$ if its positions and velocities $\{p_i, v_i\}, i = 1, \ldots, N$ satisfy the following two conditions.

1) The relative positions with respect to the marker point in the domain are uniformly bounded in time (cohesiveness)

$$\exists R > 0 \text{ such that } \sum_{i=1}^{N} ||p_i(t) - p_d(t)||^2 \leq R, \quad \forall t \geq 0.$$  

2) The relative velocities with respect to the moving domain go to zero asymptotically in time (velocity alignment)

$$\lim_{t \to +\infty} \sum_{i=1}^{N} ||v_i(t) - v_d||^2 = 0.$$  

The safe domain coverage problem that we investigate is the following. Consider a compact domain $\Omega_t$ that moves with constant velocity $v_d$ in the plane, and $N$ vehicles with dynamics described by (6) and (7), starting from safe initial conditions. Find $r > 0$ and a control policy that leads to an $r$-cover for $\Omega_t$ that flocks with the moving target, while satisfying the safety condition (8) at any time.

IV. CONTROLLERS: DESIGN AND ASYMPTOTIC BEHAVIOR

The controller we propose has two components: coverage and safety. We will present them separately.

A. Coverage Controller

Static domain: The static case ($v_d = 0$) involves a simpler coverage controller; so we present it first [34]. Define $p_{ij} := p_i - p_j$ and denote by $P_{O\Omega}(p_i)$ the closest point of $\partial \Omega$ to $p_i$ (the projection of $p_i$ on $\partial \Omega$). Also, define $h_i := p_i - P_{O\Omega}(p_i)$ and denote by $[[h_i]]$ the signed distance of $p_i$ from $\partial \Omega$—see (1) and Fig. 1. The control force proposed in [34] is given by

$$u_i = \sum_{j \neq i} f_j (||p_{ij}||) \frac{P_{ij}}{||P_{ij}||} - f_h (||[h_i]||) \frac{h_i}{||[h_i]||} - a_v v_i$$

where the three terms in the right-hand side represent intervehicular, vehicle-domain, and braking forces, respectively. Also, $a_v$ is a fixed positive constant.

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forces. The proposed control force is given by

\[
    u_i = -\sum_{j \neq i}^N f_i(||\tilde{p}_{ij}||) \frac{\tilde{p}_{ij}}{||\tilde{p}_{ij}||} - \tilde{f}_h(||\tilde{h}_i||) \frac{\tilde{h}_i}{||\tilde{h}_i||}
\]

\[-7pt\]

\[
    \sum_{j \neq i} f_{al}(||\tilde{p}_{ij}||) (v_{ij} - a_v \tilde{v}_{ij}) \]

(12)

Here, \( p_{ij} := p_i - p_j \), \( v_{ij} := v_i - v_j \), and \( P_{\Omega i} (p_i) \) denote the projection of \( p_i \) on \( \partial \Omega \). Also, \( h_i := p_i - P_{\Omega i} (p_i) \), and \( ||h_i|| \) denotes the signed distance of \( p_i \) from \( \partial \Omega \). \( f_{al} \) is a nonnegative communication function and \( a_v \) is a positive constant. Indeed, change variables to \( p_i - \tilde{p}_i \) as \( P \cdot \dot{v} \) between the interacting vehicles. For \( \dot{v} \) (13) with \( \Phi \), using \( \tilde{h}_i \) parallel \( -\Omega \) \( p_i \) \( v_i \) \( v_j \) \( \tilde{v}_{ij} \) \( v_{ij} \) \( \tilde{v}_i \) \( v_i \) \( v_j \) \( \tilde{v}_{ij} \) \( v_{ij} \).

The vehicle-domain alignment force drives the velocity of the vehicles to align with the velocity of the domain. This is a reasonable assumption in applications such as drone surveillance or drone search and rescue. In these cases, the domain may be specified by the user and the drones typically have access to localization systems such as GPS.

The intervehicle alignment force, which controls the alignment of vehicle \( i \)'s velocity with the velocities of the rest of the vehicles, depends on the relative distance \( ||\tilde{p}_{ij}|| \) between the interacting vehicles. For a communication function \( f_{al} \) that is nonincreasing (this is a typical assumption in the literature [32], [33]), vehicles align more strongly with their neighbors and less with vehicles further apart. The results presented in this article correspond to a communication function in the form

\[
f_{al}(||\tilde{p}_{ij}||) = C_{al} e^{-\frac{||\tilde{p}_{ij}||}{\tau_{al}}}
\]

where \( C_{al} \) and \( \tau_{al} \) are constants associated with the alignment strength and alignment range, respectively. This function was considered in [20] in the context of honeybee swarms.

The vehicle-domain alignment force controls the velocity of the vehicles to the domain’s velocity \( v_d \). Also, the braking force in the static domain model (9) can be interpreted as an alignment force that brings the vehicles to a stop.

By changing to relative coordinates with respect to the frame of the moving domain, one can recover the case of a stationary domain \( (v_d = 0) \). Indeed, change variables to \( \tilde{p}_i := p_i - tv_d \), \( \tilde{v}_i := v_i - v_d \)

(13)

and note that the intervehicle positions and velocities are invariant to this change of coordinates, i.e.,

\[
    \tilde{p}_{ij} := \tilde{p}_i - \tilde{p}_j = p_{ij}, \quad \tilde{v}_{ij} := \tilde{v}_i - \tilde{v}_j = v_{ij}.
\]

Also, by translation, the distance to the target domain satisfies

\[
    h_i = (p_i - tv_d) - P_{\Omega_2} (p_i - tv_d) = \tilde{p}_i - P_{\Omega_2} (\tilde{p}_i).
\]

Hence, in the new variables, the signed distances \( ||\tilde{h}_i|| \), where

\[
    \tilde{h}_i := \tilde{p}_i - P_{\Omega_2} (\tilde{p}_i)
\]

are with respect to the initial (fixed) domain \( \Omega \).

Therefore, (12) can be written in the new variables as

\[
    \tilde{u}_i = -\sum_{j \neq i}^N f_i(||\tilde{p}_{ij}||) \frac{\tilde{p}_{ij}}{||\tilde{p}_{ij}||} - f_{al}(||\tilde{h}_i||) \frac{\tilde{h}_i}{||\tilde{h}_i||}
\]

\[-7pt\]

\[
    \sum_{j \neq i} f_{al}(||\tilde{p}_{ij}||) \tilde{v}_{ij} - a_v \tilde{v}_i.
\]

Note that this corresponds to the dynamics in the original variables for a stationary domain.

B. Asymptotic Behavior

We first investigate the dynamics with control (12) for a stationary target \( (v_d = 0) \), using \( f_i \) and \( f_{al} \) described above. Consider the following candidate for a Lyapunov function, consisting of kinetic plus (artificial) potential energy

\[
    \Phi = \frac{1}{2} \sum_{i=1}^N \left( \tilde{p}_i \cdot \tilde{p}_i + \sum_{j \neq i}^N V_f ||\tilde{p}_{ij}|| + 2V_h (p_i) \right).
\]

As each term in \( \Phi \) is nonnegative, the global minimum of \( \Phi \) is 0. At the global minimum \( \Phi = 0 \), the vehicles are at rest \( (\tilde{p}_i = 0, \forall i) \), and the configuration is an \( r \)-cover of \( \Omega \). Indeed, using (10) and (11), \( V_f (p_i) = 0 \) and \( V_h (p_i) = 0 \) imply \( ||p_i - p_j|| \geq r_d \) and \( ||p_i - P_{\Omega_2} (p_i)|| \leq \frac{r_d}{2} \), for all \( i, j \).

The time derivative of \( \Phi \) can be calculated as follows:

\[
    \dot{\Phi} = \sum_{i=1}^N \left( -\sum_{j \neq i} f_{al}(||\tilde{p}_{ij}||) (v_i - v_j) - a_v v_i \right).
\]

(15)

For the intervehicle alignment term, write

\[
    \sum_{i=1}^N v_i \cdot \sum_{j \neq i} f_{al}(||\tilde{p}_{ij}||) (v_i - v_j)
\]

\[
    \frac{1}{2} \sum_{i=1}^N v_i \cdot \sum_{j \neq i} f_{al}(||\tilde{p}_{ij}||) (v_i - v_j) + \frac{1}{2} \sum_{j=1}^N v_j \cdot \sum_{i \neq j} f_{al}(||\tilde{p}_{ij}||) (v_j - v_i)
\]

where, in the second term in the right-hand side, we renamed \( i \leftrightarrow j \) as indices of summation. As \( ||\tilde{p}_{ij}|| = ||p_i|| \), we then get

\[
    \sum_{j \neq i} f_{al}(||\tilde{p}_{ij}||) (v_i - v_j) = \frac{1}{2} \sum_{i=1}^N \sum_{j \neq i} f_{al}(||\tilde{p}_{ij}||) ||v_i - v_j||^2.
\]

Hence, from (15), we find

\[
    \dot{\Phi} = \frac{1}{2} \sum_{i=1}^N \left( \tilde{p}_i \cdot \tilde{p}_i + \sum_{j \neq i} V_f (\tilde{p}_{ij}) + 2V_h (\tilde{p}_i) \right).
\]

(16)

Then, by the calculations for the stationary target above

\[
    \dot{\Phi} = \frac{1}{2} \sum_{i=1}^N \left( \tilde{p}_i \cdot \tilde{p}_i + \sum_{j \neq i} V_f (\tilde{p}_{ij}) + 2V_h (\tilde{p}_i) \right).\]

(17)

Note that \( \dot{\Phi} \) is negative semidefinite and equal to zero if and only if \( \tilde{v}_i = 0 \) (or, equivalently, \( v_i = v_d \)) for all \( i \), i.e., when vehicles’ velocities are aligned with the velocity of the domain.
Theorem IV.2 (Flocking with the moving target): Consider a target domain \( \Omega_t \) that moves with constant velocity \( v_d \), and a group of \( N \) vehicles with smooth dynamics governed by (6), with the control law given by (12). Then, the group of agents has a time-asymptotic flocking with the moving target \( \Omega_t \).

Proof: We have to check the conditions in Definition III.3. For group cohesiveness (condition 1), we will use relative coordinates. The key idea is that the vehicle-domain potential \( V_i \) is confining the vehicles and keeps them as a group [31]. Note that in relative coordinates, the distances to the target are with respect to the fixed domain \( \Omega \) [see (14)].

Using that the kinetic energy and the potential \( V_i \) are nonnegative and \( \Phi \) given by (16) is nonincreasing, we have

\[
\sum_{i=1}^{N} V_i (\tilde{p}_i (t)) \leq \Phi (t) \leq \Phi (0).
\]

To show the boundedness of \( \tilde{p}_i \), we only need to consider the case \( \tilde{p}_i \notin \Omega \) as, otherwise, the relative positions are inside the compact set \( \Omega \). Using the expression (11) for \( V_i \), we then find

\[
\frac{a_h}{2} \sum_{i=1}^{N} \left( \| \tilde{p}_i (t) - P_{\Omega} \tilde{p}_i (t) \| + \frac{r_d}{2} \right)^2 \leq \Phi (0).
\]

This shows that the distances from \( \tilde{p}_i (t) \) to the domain \( \Omega \) remain bounded by \( \sqrt{2 \Phi (0)/a_h} \) when \( \tilde{p}_i \notin \Omega \). Restoring the original variables, we can then conclude that there exists \( R > 0 \) such that \( \| p_i (t) - p_d (t) \| \leq R \), for all \( i \) and \( t \geq 0 \).

To show velocity alignment (condition 2), we first note that the velocities are also uniformly bounded in time. Indeed, since the potentials \( V_i \) and \( V_{\Omega} \) are nonnegative and \( \Phi \) is nonincreasing, we have

\[
\sum_{i=1}^{N} \| \tilde{v}_i (t) \|^2 \leq 2 \Phi (t) \leq 2 \Phi (0).
\]

Hence, the solutions \( (\tilde{p}_i (t), \tilde{v}_i (t)) \) of the relative system are confined within a compact set through dynamics. By LaSalle invariance principle, we conclude that the solutions approach asymptotically the largest invariant set in \( \{ \Phi = 0 \} \). Consequently, we infer by (17) that as \( t \to \infty \), the vehicles’ velocities approach the velocity of the target domain.

Remark IV.3: The asymptotic states are critical points of \( \Phi \) that satisfy \( \tilde{v}_i = 0 \) for all \( i \). Alternatively, these equilibria are critical points of the artificial potential energy \( \sum_{i=1}^{N} \sum_{j \neq i} V_i (\tilde{p}_{ij}) + 2V_\Omega (\tilde{p}_i) \). We expect that almost every solution of the relative system will approach asymptotically a local minimum of this potential energy.

Most relevant to our study are the \( r_d \)-covers defined in Section III. In some certain simple geometries, for example, a square number of vehicles in a square domain or a triangular number of vehicles in a triangular domain, the \( r_d \)-covers are isolated equilibria. In such cases, we have the following result.

Proposition IV.4: Consider a target domain \( \Omega_t \) that moves with constant velocity \( v_d \), and a group of \( N \) vehicles with dynamics defined by (6) and (12). Let the relative equilibrium of interest be of the form \( \tilde{p}_i = 0 \), \( \| \tilde{b}_i \| \geq r_d \) and \( \| \tilde{h}_i \| \leq \frac{a_h}{2} \) for \( i, j = 1, \ldots, N \) (see Definitions III.1 and III.2) and assume that this equilibrium configuration is isolated. Also assume that there is a neighborhood about the equilibrium in which the control law remains smooth. Then, the relative equilibrium is a global minimum of the sum of all the artificial potentials and is locally asymptotically stable.

Proof: Note that \( \Phi \) vanishes at such equilibria, as \( V_i (\tilde{p}_{ij}) = 0 \) and \( V_{\Omega} (\tilde{p}_i) = 0 \), for all \( i, j \). Hence, these relative equilibria are global minimizers of the potential energy. Their local asymptotic stability follows from LaSalle invariance principle, using the fact that the equilibria are isolated.

Remark IV.5: All considerations in this subsection apply to the case of zero interindividial alignment forces (\( f_{ij} = 0 \)). In such a case, by working in the moving frame of the domain, the problem reduces, in fact, to the one studied in [34].

Remark IV.6: The previous theoretical results apply to the unconstrained dynamics, i.e., constraints (7) are not enforced. However, close to the desired operation point, the coverage forces are small enough and do not need to be thresholded, in which case the theoretical results are indeed valid.

Choosing an adequate \( r_d \) when solving the safe-domain-coverage problem leads to a nonlinear optimization problem [see (37)], which, in general, can be quite difficult. We set the value of this parameter based on the assumption that any vehicle is covering roughly the same square area, i.e.,

\[
r_d = \sqrt{\frac{\text{Area} (\Omega)}{N}}.
\]

C. Collision Avoidance

For small initial energies, collision avoidance can be shown directly. For general cases, we introduce a safety controller based on HJ reachability analysis.

Small initial energy: The following result holds for initial data with small energy \( \Phi \).

Proposition IV.7: Consider a target domain \( \Omega \) and a group of \( N \) vehicles with dynamics defined by (6) and (12). Assume the energy \( \Phi (0) \) of the initial configuration satisfies

\[
\Phi (0) < \int_{r_d}^{r_c} f_i (s) \, ds = \frac{a_h}{2} (c_r - r_d)^2.
\]

Then, no vehicle collision can occur for all \( t \geq 0 \).

Proof: Suppose by contradiction that there is a time \( t_\ast \), when vehicles \( k \) and \( l \) are at collision radius from each other, i.e., \( \| p_k (t_\ast) - p_l (t_\ast) \| = c_r \). Given that \( V_i \) is nonnegative, the intervehicle potential energy at \( t_\ast \) can be bounded below as

\[
\frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} V_i (p_{ij} (t_\ast)) \geq V_k (p_k (t_\ast)) - p_l (t_\ast)) = \int_{r_d}^{r_c} f_i (s) \, ds.
\]

On the other hand, using that the kinetic energy and the potential \( V_{\Omega} \) are nonnegative, we have

\[
\frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} V_i (p_{ij} (t)) \leq \Phi (t) \leq \Phi (0).
\]

Now combine the two sets of inequalities above to find \( \Phi (0) \geq \int_{r_d}^{r_c} f_i (s) \, ds \), which contradicts the assumption on \( \Phi (0) \).

Collision avoidance via Hamilton–Jacobi theory: For general configurations, we use HJ reachability analysis.

Consider the dynamics between two vehicles \( Q_i, Q_j \), defined in terms of their relative states

\[
pr_{i,x} = p_{i,x} - p_{j,x}, \quad vr_{i,x} = v_{i,x} - v_{j,x},
\]

\[
pr_{i,y} = p_{i,y} - p_{j,y}, \quad vr_{i,y} = v_{i,y} - v_{j,y}
\]

where the vehicle \( Q_i \) is the evader, located at the origin, and \( Q_j \) is the pursuer, the latter being considered as the model disturbance. The
relative dynamical system can be written as

\[
\begin{align*}
\dot{p}_{r,x} &= v_{r,x}, & \dot{v}_{r,x} &= u_{i,x} - u_{j,x}, \\
\dot{p}_{r,y} &= v_{r,y}, & \dot{v}_{r,y} &= u_{i,y} - u_{j,y}.
\end{align*}
\]

(19)

with \( \|u_i\|, \|u_j\| \leq u_{\text{max}} \), where \( u_i = (u_{i,x}, u_{i,y}) \) and \( u_j = (u_{j,x}, u_{j,y}) \) are the control inputs of the agents \( Q_i \) and \( Q_j \), respectively. From the perspective of agent \( Q_i \), the control inputs of \( Q_j \) are treated as worst-case disturbance.

System (19) can be put in the general form (3) from Section II, with \( z = (p_{r,x}, p_{r,y}, v_{r,x}, v_{r,y}) \), \( u = (u_{i,x}, u_{i,y}) \), \( d = (d_x, d_y) := (u_{j,x}, u_{j,y}) \), and \( f(z, u, d) \) being the right-hand side of (19).

According to (8), the set of unsafe states to be avoided are described by the target set \( \Gamma_D = \{ z : p_{r,x}^2 + p_{r,y}^2 \leq c_p^2 \} \). The obstacle set \( \Gamma_S \) is the empty set as it is not needed. Let \( \psi(z) \) be the time it takes for the solution of (19), with starting point \( z \in \mathbb{R}^\ast \setminus \Gamma_D \), to reach \( \Gamma_D \) when the disturbance and control inputs are optimal. As the two vehicles have the same capabilities, we make the educated guess that the optimal nonanticipative strategy for the pursuer is to copy the evader accelerations, having so a zero relative acceleration. This implies that the relative velocity \( v_r \) will remain constant through time.

If \( p_r \) and \( v_r \) are such that a collision can occur, there exists a collision point \( c_p \); see Fig. 3. This will be one of the intersection points of the line through \( p_r \) in the direction \( v_r \) and the circle of radius \( c_p \) centered at the origin. To find the collision time, we replace the coordinates of \( c_p = (p_{r,x} + \psi(z)v_{r,x}, p_{r,y} + \psi(z)v_{r,y}) \) into the canonical equation of the circle. Hence, the collision time is the minimum of the two solutions of the quadratic equation

\[
(v_{r,x}^2 + v_{r,y}^2)\psi^2(z) + 2(p_{r,x}v_{r,x} + p_{r,y}v_{r,y})\psi(z) + (p_{r,x}^2 + p_{r,y}^2 - c_p^2) = 0.
\]

(20)

By implicit differentiation of (20), we find

\[
\frac{\partial \psi}{\partial v_{r,x}} = \frac{-v_{r,x}\psi^2(z) - p_{r,x}\psi(z)}{(u_{r,x}^2 + v_{r,y}^2)\psi(z) + (p_{r,x}v_{r,x} + p_{r,y}v_{r,y})}
\]

(22a)

and

\[
\frac{\partial \psi}{\partial v_{r,y}} = \frac{-v_{r,y}\psi^2(z) - p_{r,y}\psi(z)}{(u_{r,x}^2 + v_{r,y}^2)\psi(z) + (p_{r,x}v_{r,x} + p_{r,y}v_{r,y})}
\]

(22b)

and, hence, from (21), we can derive a closed expression for the optimal avoidance controller.

The static HJI PDE (4) is typically approximated by finite difference methods such as the one presented in [11]. Our approach, using an analytic solution, leads to two main advantages. First, we do not have to deal with large amounts of memory and long computational times involved in refinements of the numerical resolution. Second, while numerical methods can only compute the solution in a bounded domain, an analytical solution allows us to have the best possible resolution in unbounded domains. This allows us to predict and react to possible collisions arbitrarily far into the future.

D. Overall Control Logic

In this subsection, we describe how to switch between the two controllers presented above. We consider that vehicle \( Q_i \) is in potential conflict with vehicle \( Q_j \) if the time to collision \( \psi(z_i) \) (here, \( z_i \) denotes the relative current state of the two vehicles) is less than or equal to a specified duration \( t_{\text{safety}} \). In such a case, \( Q_i \) must use the safety controller; otherwise, the coverage controller is used. \( t_{\text{safety}} \) represents the time to collision assuming another agent’s worst-case controls, which would not be applied “willingly” but could occur by chance when the coverage control happens to be worst case. In principle, \( t_{\text{safety}} \) may be chosen based on the maximum duration that worst-case control may be applied by chance. This can be hard to estimate in practice, but the theoretical interpretation of \( t_{\text{safety}} \) is still informative.

If a vehicle detects more than one conflict, it will apply the control policy of the first conflict detected at that particular time. Algorithm 1 describes the overall control logic for a generic vehicle \( Q_i \). In Algorithm 1, lines 6 and 7 can be obtained from (21), (22a), and (22b) (also note the normalization step in line 14), while line 12 comes from the explicit coverage control (9).

V. NUMERICAL SIMULATIONS

We show three numerical simulation scenarios for vehicles using the coverage controller (12). While the first two scenarios are covered by the theory, the last one illustrates how our strategy still leads to appropriate final configurations even when the domain follows noninertial trajectories.

The safety issues are addressed in Section IV-C. We note that safety issues may also arise when a vehicle needs to avoid two or more vehicles at the same time. Our safety controller does not guarantee collision avoidance in such cases. Guaranteed collision avoidance for more than two vehicles is an unsolved problem, as explored for example in [13].

Triangular domain: Consider an equilateral triangular domain moving with constant velocity, \( \dot{c}_T = \left( \frac{v}{2}, \frac{v}{2}, \frac{v}{2} \right) \) being covered by a triangular number of vehicles, i.e., \( N = \frac{n(n+1)}{2}, \; n \in \mathbb{N} \). At the start of the simulation, the vehicles lie on a line outside the domain [see Fig. 4(a)]. The evolution for a group of \( N = 10 \) agents is illustrated in Fig. 4(b)-(d). The tails represent the 1-s history of the vehicles’ positions.

Some effects of strong alignments, that is, large \( a_i \) or \( C_{\text{col}} \) values, include vehicles spreading slowly inside the domains or, in some cases, not reaching the target formation (see also [38]). On the other
Fig. 4. Vehicles covering and following a moving triangular domain, when $N = 10$, $c_v = 2$ (m/s), $v_{max} = 10$ (m/s), $u_{max} = 3$ (m/s$^2$), $t_{safe} = 5$ (s), $a_1 = 1$ (m/s$^2$), $a_h = 2$ (m/s$^2$), $a_v = 0.2$ (m/s$^2$), $C_{al} = 0.2$ (m/s$^2$), $r_d = 7.79$ (m), $v_d = (\sqrt{3}/2, \sqrt{3}/2)$ (m/s), $A = 292.28$ (m$^2$), and $r_d = \sqrt{A/N} = 5.4$ (m). Vehicles start in linear formation. (a) $t = 0$(s); (b) $t = 9$(s); (c) $t = 24$(s); (d) $t = 60$(s).

Algorithm 1: Overall Control Logic for a Generic Vehicle $Q_i$.

IN: State $x_i$ of a vehicle $Q_i$; states $\{x_j\}_{j \neq i}$ of other vehicles $\{Q_j\}_{j \neq i}$; a domain $\Omega$ to cover.

PARAMETER: A time horizon for safety check $t_{safe}$;

OUT: A control $u_i$ for $Q_i$.

1: safe $\leftarrow$ True;
2: for $j \neq i$ do
3: $z \leftarrow x_i - x_j$;
4: if $\nabla(z) \leq t_{safe}$ then
5: safe $\leftarrow$ False;
6: $U_{ix} = -\frac{v_{r,x}v_{r,y}v(x) + v_{r,x}v_{r,y}v(x)}{v_{r,x}v_{r,y}v(x) + v_{r,x}v_{r,y}v(x)}$;
7: $U_{iy} = -\frac{v_{r,x}v_{r,y}v(x) + v_{r,x}v_{r,y}v(x)}{v_{r,x}v_{r,y}v(x) + v_{r,x}v_{r,y}v(x)}$;
8: break for;
9: end if
10: end for
11: if safe then
12: $(U_{ix}, U_{iy}) = \sum_{j \neq i}^N f_i(||p_{ij}||) \frac{b_{ij}}{||p_{ij}||} - f_i(||h_i||) \frac{b_i}{||p_{ij}||} - a_v v_i$;
13: end if
14: $u_i = \max_{U_{ix}, U_{iy}} \frac{||U_{ix}, U_{iy}||}{||U_{ix}, U_{iy}||}$;

RETURN: $u_i$

hand, weak alignments, i.e., small $a_v$ and $C_{al}$ values, cause undesired overshoots, and slower asymptotic flocking. Therefore, it is important to maintain a good balance between the strength of the alignment and coverage forces.

Nonconvex domain: We now study the scenario in which vehicles cover and follow a moving nonconvex domain in the shape of an arrowhead. While the domain preserves its shape, it moves with a constant velocity $v_d = (\sqrt{2}/2, \sqrt{2}/2)$. Different time instants of the simulation are shown in Fig. 5, where the tails represent the vehicles’ positions during the last 20 s of the simulation. Initially, all the nine vehicles lie on a line perpendicular to the movement direction of the target domain, as shown by the tails of the vehicles in Fig. 5(a).

We distinguish two main behaviors: During a first phase of the simulation [see Fig. 5(a)], the vehicles cover the domain approximately evenly, adopting the arrow shape, while in a second phase [see Fig. 5(b)], a clearer domain-following behavior is observed. The oscillations of the two vehicles that are lagging behind are the effect of their proximity to the corners. Indeed, as one of the line segments of the boundary wedge gets closer to the vehicle near the corner, it pushes it toward the other segment of the wedge, a back-and-forth motion that causes the zigzagging. These oscillations can be reduced by reinforcing the velocity alignment.

Unlike the convex case, in nonconvex domains, the projection on the boundary for points outside of the domain may not be unique; this is the case, for instance, of the green vehicle in the middle of the initial setup—see start of the tails in Fig. 5(a). Although the chance for a vehicle to lie in one of these states is extremely unlikely (the set of points where this happens has zero measure), this fact may yield ambiguity in the definition of the domain-vehicle force. We mitigate this issue by considering the contribution from only one of the multiple projection points; consequently, the numerical time evolution may depend on the chosen projection method.

Domain moving in a circle: Finally, we include the case of an accelerating target domain—an equilateral triangular domain moving on a circular path. The triangular domain moves so that its center of mass describes a circular motion of radius 30 with constant angular velocity $\frac{2\pi}{30}$, while aligning its heading to be tangent to this circle (see Fig. 6). Note that this nonminimal path is not covered by our previous theoretical results (Theorem IV.2 and Proposition IV.4).

The vehicles’ time evolution is illustrated in Fig. 6, where the tails represent the vehicles’ 20-s position history. At the beginning, the $N = 6$ vehicles are in the line formation as shown by the beginning of the tails in Fig. 6(a). As in previous simulations, the vehicles try to reach the moving domain, this time rotating around the domain’s circular path [see Fig. 6(a) and (b)]. Then, the vehicles reach coverage of the domain [see Fig. 6(c)] which is maintained by each vehicle by remaining in a circular movement of constant radius [see Fig. 6(d)].

When a vehicle describes a uniform circular movement with angular velocity $\omega$ and radius $r$, its speed remains constant over time and is given by $r \omega$. As the vehicles move asymptotically along circles with different radii, they have different velocities, and, hence, this type of “flock” does not satisfy Definition III.3. In contrast to the case when the domain is
moving along inertial paths, in this case, each vehicle’s control force magnitude does not go asymptotically to zero, but it approaches its centripetal acceleration $r_wa_l^2$ instead.

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