Some Results on
the Boundary Control of
Systems of Conservation Laws

FABIO ANCONA ∗
ALBERTO BRESSAN †
GIUSEPPE MARIA COCLITE ‡

Abstract

This note is concerned with the study of the initial boundary value problem for systems of conservation laws from the point of view of control theory, where the initial data is fixed and the boundary data are regarded as control functions. We first consider the problem of controllability at a fixed time for genuinely nonlinear Temple class systems, and present a description of the set of attainable configurations of the corresponding solutions in terms of suitable Oleinik-type estimates. We next present a result concerning the asymptotic stabilization near a constant state for general \( n \times n \) systems. Finally we show with an example that in general one cannot achieve exact controllability to a constant state in finite time.

1991 Mathematical Subject Classification: 35L65, 35B37

Key Words: hyperbolic systems, conservation laws, Temple class systems, boundary control, attainable set.

---

∗Dipartimento di Matematica and C.I.R.A.M., P.zza Porta S. Donato, n. 5, 40123 - Bologna, Italy; E-mail: ancona@ciram3.ing.unibo.it.

†SISSA-ISAS, via Beirut 2-4, 34014 - Trieste, Italy; E-mail: bressan@sissa.it.

‡SISSA-ISAS, via Beirut 2-4, 34014 - Trieste, Italy; E-mail: coclite@sissa.it.
1 Introduction

Consider an $n \times n$ system of conservation laws on a bounded interval

\[ u_t + f(u)_x = 0, \quad t \geq 0, \quad x \in ]a, b[, \]  

with the initial condition

\[ u(0, x) = \varphi(x), \quad a \leq x \leq b, \]  

and a weak form of the Dirichlet boundary conditions

\[ u_i(t, a) = \alpha_i(t), \quad u_i(t, b) = \beta_i(t), \quad t > 0 \]  

(see [13, 14, 19] and reference therein for several weak formulations of (1.3)).

We want to study the effect of boundary conditions on the solution of (1.1) from the point of view of control theory. Namely, following the same approach adopted in [5] for scalar conservation laws, we take the initial data $\varphi$ fixed, and, regarding the measurable maps $\alpha_i, \beta_i$ in (1.3) as control functions, we want to investigate the property of the attainable set for (1.1)-(1.2), i.e. of the set

\[ \mathcal{A}(T) \doteq \left\{ u(T, \cdot) : \ u \text{ is a sol. to } (1.1) - (1.3) \right\}, \]  

which consists of all profiles that can be attained at a fixed time $T > 0$, by entropy weak solutions of (1.1)-(1.3). For the definitions and the basic properties of weak solutions we refer to [8]. See also [17, 18, 1, 2] for results concerning the existence and $L^1$ stability of entropy weak solutions of the mixed problem taking values in the space $BV$ of functions with bounded variation, and [10, 4] for the case of $L^\infty$ solutions (with possibly unbounded variation) of Temple class systems.

Throughout, we shall assume:

$(H_1)$ the map $f : \Omega \to \mathbb{R}^n$ is smooth and $\Omega \subset \mathbb{R}^n$ is open;

$(H_2)$ the system (1.1) is strictly hyperbolic, i.e. the Jacobian matrix $Df(u)$ has $n$ real and distinct eigenvalues $\lambda_1(u) < ... < \lambda_n(u)$, $u \in \Omega$ (with a corresponding basis of eigenvectors $\{r_1(u), \ldots, r_n(u)\}$);

$(H_3)$ each characteristic field $r_i$ is linearly degenerate or genuinely nonlinear in the sense of Lax [15];

$(H_4)$ there exist $p \in \{1, ..., n\}$ and $c_0 > 0$ such that

\[ \lambda_1(u) < ... < \lambda_p(u) \leq -c_0 < 0 < c_0 \leq \lambda_{p+1}(u) < ... < \lambda_n(u), \quad u \in \Omega. \]  

(1.5)
By \((H_1)\), for a solution defined on the strip \(t \geq 0, \ x \in [a,b]\), there will be \(n - p\) characteristics entering at the boundary point \(x = a\), and \(p\) characteristics entering at \(x = b\). The initial-boundary value problem is thus well posed if we prescribe \(n - p\) scalar conditions at \(x = a\), and \(p\) scalar conditions at \(x = b\) (see [16]).

**Definition 1.1** Given \(\varphi \in L^1([a,b]), \ v \in \Omega, \) and \(T > 0, \) we say that the problem (1.1)-(1.2) is exact controllable at time \(T\) to the state \(v\) if and only if there exist measurable maps \(\alpha_i, \beta_i\) such that the solution of (1.1)-(1.3) satisfies

\[
    u(T, \cdot) \equiv v, \quad \text{a.e. in } [a,b].
\]

**Definition 1.2** Given \(\varphi \in L^1([a,b]), \ v \in \Omega,\) we say that the problem (1.1)-(1.2) is asymptotic stabilizable near the state \(v\) if and only if there exist measurable maps \(\alpha_i, \beta_i\) such that the solution of (1.1)-(1.3) satisfies

\[
    u(t, \cdot) \longrightarrow v, \quad \text{in } L^1([a,b]) \quad \text{as } t \longrightarrow +\infty.
\]

In this note we present some recent results obtained by the authors [3, 9] concerning both the problem of exact controllability and of asymptotic stabilization near a constant state. We will first consider the case of Temple systems [20], for which it was obtained a characterization of the corresponding attainable set (1.4) in terms of suitable Oleinik-type estimates, which is a natural extension of the results in [5, 6] concerning scalar conservation laws. For general nonlinear systems, one cannot expect such an exact controllability result. Indeed, even if all wave-fronts in the initial data exit from the interval \([a,b]\) within finite time, they can generate new waves by interacting among themselves. In turn (figure 3), further interactions can produce a sequence of wave-fronts remaining within the interval \([a,b]\) for all times \(t > 0\). Therefore, the effect of the initial data on the solution \(u(T, \cdot)\) may never be completely erased, no matter how large we choose the terminal time \(T\). Hence, we will present a result concerning the asymptotic stabilization of a general system of conservation laws near a constant state. Finally, we discuss a counterexample to the exact controllability concerning a class of \(2 \times 2\) systems for which, in general, a constant state \(u^*\) cannot be attained, in a finite time \(T\).

An outline of these results established in [3, 9] is given in the following sections.

## 2 The Attainable Set for Temple Class Systems

Our first result is concerned with the problem of exact controllability of Temple class systems [10, 20], which are systems that satisfy the following additional assumption.
(H_5) There exists a complete set of Riemann coordinates \( w = (w_1, \ldots, w_n) \) such that each level set \( \{ u; \ w_i(u) = \text{constant} \} \) is an hyperplane.

As a consequence, all integral curves of the eigenvectors are straight lines and coincide with the Hugoniot curves. We shall also assume

(H_6) as \( w \) ranges within the product set \( \Gamma = [w_1^-, w_1^+] \times \cdots \times [w_n^-, w_n^+] \), the corresponding state \( u \) remains inside the domain \( \Omega \) and each characteristic field is genuinely nonlinear.

In the case of systems of this type, the well-posedness theory for the mixed problem was established in [4] within domains of \( L^\infty \) functions (with possibly unbounded variation). Here, the boundary condition is formulated in terms of the strong \( L^1 \) trace of the solution \( u \) at the boundary and, in the same spirit of [13], is based on the definition of a time-dependent set of admissible boundary data, that is related to the notion of Riemann problem. Moreover, for sake of uniqueness, it was introduced in [3, 4] a definition of entropy admissible weak solution to the mixed problem that includes an entropy admissibility condition of Oleinik type.

Notice that, for such systems, wave interactions can only change the speed of wave fronts, without modifying their amplitudes. Therefore, the only restriction to boundary controllability is the decay due to genuine nonlinearity. We thus consider a set of maps, defined in terms of the above Riemann coordinates, that satisfy certain Oleinik-type conditions on the decay of positive waves.

\[
K^\rho := \left\{ \psi \in L^\infty([a, b], \Gamma) \ ; \ \begin{array}{l}
\frac{w_i(\psi(y)) - w_i(\psi(x))}{y - x} \leq \frac{\rho}{x - a} \\
\quad \text{for a.e. } a < x < y < b, \\
\quad \text{if } i \in \{p + 1, \ldots, n\} \\
\frac{w_i(\psi(y)) - w_i(\psi(x))}{y - x} \leq \frac{\rho}{b - y} \\
\quad \text{for a.e. } a < x < y < b, \\
\quad \text{if } i \in \{1, \ldots, p\}
\end{array} \right\},
\]

(2.6)

The inequalities in (2.6) reflect the fact that positive waves entering through the boundary at \( a \) or at \( b \) decay in time. Therefore, their density is inversely proportional to their distance from their entrance point on the boundary.

We can now state our first main result (see [3]).

**Theorem 2.1** Let (H) be a system of Temple class, and assume that (H_1)-(H_6) are verified. Then, letting \( \mathcal{A}(T) \) be the attainable set defined in (1.4) (in which the solution is understood as an “entropy admissible weak solution”), the following hold:

(i) for every fixed \( \tau > 0 \), there exists \( \rho = \rho(\tau) > 0 \) such that

\[
\mathcal{A}(\tau) \subseteq K^\rho, \quad \tau \geq \tau;
\]

(2.7)
(ii) there exist $T > 0$ and $\rho' < \rho(T)$, such that

$$K^{\rho'} \subseteq A(\tau), \quad \tau > T; \quad (2.8)$$

(iii) $A(T)$ is a compact subset of $L^1([a,b])$ for each $T > 0$.

The first property $(i)$ is an immediate consequence of the definition of entropy admissible solution that satisfies suitable Oleinik-type estimates.

The proof of $(ii)$ is established in two steps.

1) Backward Construction of Front Tracking Solutions. We take

$$\tau > T \equiv \frac{4}{\lambda_{\min}} b - a, \quad \lambda_{\min} \equiv \min_i |\lambda_i|, \quad (2.9)$$

and, for a given function $\psi \in K^{\rho'}$, we construct a sequence of approximate solutions $u^\nu$ on the strip $[0, \tau] \times [a, b]$ such that

$$u^\nu(\tau, \cdot) = \psi^\nu \quad \psi^\nu \xrightarrow{L^1} \psi,$$

$$u^\nu(0, \cdot) = \varphi^\nu \quad \varphi^\nu \xrightarrow{L^1} \varphi, \quad (2.10)$$

with the following procedure. We partition the strip $[0, \tau] \times [a, b]$ in three regions. On the rectangle $[(3/4)T, \tau] \times [a, b]$, starting from $t = \tau$, we construct backward in time the front tracking solution $u^\nu$ relying on the fact that the Oleinik estimates of the definition (2.6) of $K^{\rho'}$ guarantee that two rarefaction fronts of the same family never cross in $\Omega$ (see figure 1a). The total number of wave-fronts in $u^\nu(t, \cdot)$ decreases as $t \downarrow (3/4)T$ whenever a (backward) front crosses the boundary points $x = a, x = b$. Therefore, since the maximum time taken by fronts of $u^\nu$ to cross the interval $[a, b]$ is $(b - a)/\lambda_{\min}$, the definition (2.9) of $T$ guarantees that all the (backward) fronts of $u^\nu$ will hit the boundaries $x = a, x = b$ within some time $\tau' \in [(3/4)T, \tau]$. Hence, there will be some constant state $\omega \in \Omega$ such that $u^\nu((3/4)T, \cdot) \equiv \omega$. We next define $u^\nu$ on the rectangle $[0, T/4] \times [a, b]$ as the restriction to $[0, T/4] \times [a, b]$ of the front tracking solution to the Cauchy problem for $\Omega$; with initial data

$$\varphi(x) = \begin{cases} 
\varphi^\nu(a+) & \text{if } x < a, \\
\varphi^\nu(x) & \text{if } a \leq x \leq b, \\
\varphi^\nu(b-) & \text{if } x > b.
\end{cases}$$

Since $u^\nu$ contains only fronts originated at the points of the segment $\{(0, x) ; x \in [a, b]\}$, because of (2.9), these wave-fronts cross the whole interval $[a, b]$ and exit from the boundaries $x = a, x = b$ before time $T/4$. Hence, there will be some state $\omega^\nu \in \Omega$ such that $u^\nu(T/4, \cdot) \equiv \omega^\nu$. Finally, we define $u^\nu(t, \cdot)$ for $t \in [T/4, (3/4)T]$ so that $u^\nu(T/4, \cdot) \equiv \omega^\nu, u^\nu((3/4)T, \cdot) \equiv \omega$. 

F. Ancona and A. Bressan and G. M. Coclite
2) Convergence of Front Tracking Solutions. Since the sequence of approximate solutions constructed by our algorithm satisfy the Oleinik-type estimates on the decay of positive waves, they have a uniformly bounded variation on each interval of the type \([a+\varepsilon, b-\varepsilon]\), \(\varepsilon > 0\). Thus, applying Helly’s Theorem, and by a diagonal procedure, we can extract a subsequence that converges in \(L^1\) to a weak solution \(u\) of (1.1). We then extend a regularity property established by A.Bressan and P.G. LeFloch [11] for solutions with small total variation of general genuinely nonlinear systems, to the case of solutions with arbitrarily large variation of genuinely nonlinear Temple class systems. This property guarantees that, for Temple systems, solutions of the mixed problem (1.1)-(1.3) are continuous outside a countable number of Lipschitz curves. As an immediate consequence we deduce that the solution \(u\) admits a strong \(L^1\) trace at the boundaries \(x = a, x = b\) and satisfies the corresponding boundary conditions.

Concerning (iii), the compactness of \(\mathcal{A}(T)\) is achieved with the same type of arguments used to establish the convergence of the approximate solutions in the proof of (ii).

3 Asymptotic Stabilization

We now consider a general \(n \times n\) system and show that, starting with an initial data with
small oscillations, the system can be asymptotically steered to any constant state (see [Theorem 1]).

**Theorem 3.1** Let \( K \) be a compact, connected subset of the open domain \( \Omega \subset \mathbb{R}^n \) and assume that \([L,A]\) satisfies \((H_1)-(H_4)\). Then there exist constants \( C_0, \delta, \kappa > 0 \) such that the following holds. For every constant state \( u^* \in K \) and every initial data \( \varphi : [a,b] \to K \) with \( \text{Tot.Var.}\{\varphi\} < \delta \), there exists an entropy weak solution \( u = u(t,x) \) of \([L,A]\) such that, for all \( t > 0 \),

\[
\text{Tot.Var.}\{u(t,\cdot)\} \leq C_0 e^{-2\tau t}, \quad \|u(t,\cdot) - u^*\|_{L^\infty} \leq C_0 e^{-2\tau t}.
\]

(3.11)

The idea of the proof is as follows. Call \( \lambda_* > 0 \) a lower bound for the absolute value of all wave speeds and set \( \tau = (b-a)/\lambda_* \). In this way, all waves present in the solution at a given time \( t \) will exit through one the boundaries within time \( t+\tau \). On the first interval \([0,\tau]\) we let all waves exit, arranging the boundary values at \( x = a \) and at \( x = b \) so that no reflected waves ever enter the domain \([a,b]\).

Therefore, the only waves present in the solution at time \( \tau \) are those generated by interactions, in the interior of the interval \( ]a,b[ \). They can be estimated as

\[
\text{Tot.Var.}\{u(\tau,\cdot)\} \leq C_0 \text{Tot.Var.}\{u(0,\cdot)\}^2.
\]

The above estimate shows that the solution \( u(\tau,\cdot) \) remains very close to some constant state, say \( u^\dagger \). If \( u^\dagger \neq u^* \), we suitably change the boundary conditions, producing new incoming waves at \((\tau,b), (2\tau,a)\), achieving the bounds

\[
\text{Tot.Var.}\{u(3\tau,\cdot)\} \leq C_0 \text{Tot.Var.}\{u(0,\cdot)\}^2, \quad \|u(3\tau,\cdot) - u^*\|_{L^\infty} \leq C_0 \|u(0,\cdot) - u^*\|_{L^\infty}^2.
\]

Repeating inductively the same strategy in the time intervals \([3\tau,6\tau], [6\tau,9\tau], \ldots\) , we obtain the result.
An interesting question is whether one can reach exactly a constant state \( u^* \) within a finite time \( T \). By Theorem 2.1, this is certainly the case for Temple class systems. In the final part of this paper, we show that this exact controllability cannot be attained in finite time, in general.

Our counterexample is concerned with a class of \( 2 \times 2 \) strictly hyperbolic, genuinely nonlinear systems with the property that the interaction of two shocks of the same family generates a shock in the other family (see [9, Theorem 2]). This is the case for the system (see [12]):

\[
\begin{align*}
\rho_t + (u \rho)_x &= 0, \\
u_t + \left( \frac{u^2}{2} + \frac{K^2}{\gamma - 1} \rho^{\gamma - 1} \right)_x &= 0.
\end{align*}
\]  

(4.12)

with \( 1 < \gamma < 3 \). Here \( \rho > 0 \) and \( u \) denote the density and the velocity of a gas, respectively.

**Theorem 4.1** Consider a \( 2 \times 2 \) system of conservation laws satisfying \((H_1)\) and the following \((H_7)\) there exist \( 0 < \lambda_* < \lambda^* \) such that

\[-\lambda^* < \lambda_1(u) < -\lambda_* < 0 < \lambda_* < \lambda_2(u) < \lambda^*,\]

\[D\lambda_1 \cdot r_1 > 0, \quad D\lambda_2 \cdot r_2 > 0,\]
\[ r_1 \wedge r_2 < 0, \quad r_1 \wedge (Dr_1 \cdot r_1) < 0, \quad r_2 \wedge (Dr_2 \cdot r_2) < 0, \]

where \( r_1(u), r_2(u) \) are the right eigenvectors of \( Df(u) \).

Let \( \varphi \in BV([a,b];\mathbb{R}^2) \) with small total variation and a dense set of shocks. Every entropic solution of (1.1)-(1.2) has a dense set of shocks in \( u(t,\cdot) \), for each \( t \geq 0 \). In particular, \( u(t,\cdot) \) cannot be a constant.

As in the scalar case we have (see [9, Section 3]).

**Lemma 4.1 (Oleinik type estimate)** Let \( u = u(t,x) \) be a solution of (1.1) with \( n = 2 \) and satisfying \((H_1)\) and \((H_7)\). There exist \( k, \delta > 0 \) such that, if

\[
\text{Tot.Var.}(u(t,\cdot)) < \delta,
\]

then

\[
\omega_i(t,y) - \omega_i(t,x) < \frac{k}{t^r} (y - x), \quad x < y, \quad t > 0, \quad i = 1, 2,
\]

where \( \omega_1, \omega_2 \) are the Riemann coordinates associated to (1.1).
By the previous properties, a shock can never be completely canceled by interactions with rarefaction waves of the same family. Hence, it can only disappear by exiting from one of the boundaries $x = a$ or $x = b$. However, if the set of shocks at time $t = 0$ is everywhere dense, these shocks will interact among each other on a dense set of points in the domain $[a, b] \times [0, \infty[$, and give rise to a dense set of new shocks. One can arrange so that the total strength of these shocks quickly approaches zero, according to Theorem 3.1, but cannot become exactly zero within finite time. For all details we refer to [4], Section 3.

**Remark 4.1** The previous analysis breaks down in the case of the $p$–system, because in this case the interaction of two shocks of the same family produces a centered rarefaction wave of the other family. In particular, an Oleinik type estimate cannot hold.

**References**

[1] D. Amadori, Initial-boundary value problems for nonlinear systems of conservation laws, *Nonlin. Diff. Equat. Appl.* **4** (1997), 1-42.

[2] D. Amadori and R. M. Colombo, Continuous dependence for $2 \times 2$ conservation laws with boundary, *J. Differential Equations* **138** (1997), 229-266.

[3] F. Ancona and G. M. Coclite, On the Attainable set for Temple Class Systems with Boundary Controls, Ref. S.I.S.S.A. 10/2002/M.

[4] F. Ancona and P. Goatin, Uniqueness and Stability of $L^\infty$ Solutions for Temple Class Systems with Boundary and Properties of the Attainable Sets, *SIAM J. Math. Anal.*, to appear.

[5] F. Ancona and A. Marson, On the attainable set for scalar non-linear conservation laws with boundary control, *SIAM J. on Control and Optimization* **36** (1998), 290-312.

[6] F. Ancona and A. Marson, Scalar non-linear conservation laws with integrable boundary data, *Nonlinear Anal.* **35** (1999), pp. 687-710.

[7] P. Baiti and A. Bressan, The semigroup generated by a Temple class system with large data, *Differ. Integ. Equat.* **10** (1997), 401-418.

[8] A. Bressan, Hyperbolic Systems of Conservation Laws. The one dimensional Cauchy problem, Oxford Univ. Press, 2000.

[9] A. Bressan and G. M. Coclite, On the Boundary Control of System of Conservation Laws, *SIAM J. Control Optim.*, to appear.
[10] A. Bressan and P. Goatin, Stability of $L^\infty$ solutions of Temple class systems, *Differ. Integ. Equat.*, 13, (2000), 1503-1528.

[11] A. Bressan and P.G. LeFloch, Structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws, *Indiana Univ. Math. J.*, 48, (1999), 43-84.

[12] R. DiPerna, Global solutions to a class of nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* 26 (1973), 1-28.

[13] F. Dubois and P.G. LeFloch, Boundary conditions for non-linear hyperbolic systems of conservation laws, *J. Differential Equations* 71 (1988), 93-122.

[14] K.T. Joseph and P.G. LeFloch, Boundary layers in weak solutions of hyperbolic conservation laws, *Arch. Rational Mech. Anal.* 147 (1999), 47-88.

[15] P. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* 10 (1957), 537-566.

[16] T. Li and W. Yu, Boundary Value Problems for Quasilinear Hyperbolic Systems, Duke University Mathematics Series V, 1985.

[17] T.P. Liu, Initial-boundary value problem for gas dynamics, *Arch. Rational Mech. Anal.* 64 (1977), 137-168.

[18] M. Sablé-Tougeron, Méthode de Glimm et probléme mixte, *Ann. Inst. Henri Poincaré* 10 (1993), 423-443.

[19] D. Serre, Systems of Conservation Laws I, II, Cambridge University Press, 2000.

[20] B. Temple, Systems of conservation laws with invariant submanifolds, *Trans. Amer. Math. Soc.* 280 (1983), 781-795.