MINIMAL FAITHFUL PERMUTATION REPRESENTATIONS FOR LINEAR GROUPS

NEELIMA BORADE AND Ramin TAKLOO-BIGHASH

Abstract. In this paper we study the minimal faithful permutation representations of $\text{SL}_n(\mathbb{F}_q)$ and $\text{GL}_n(\mathbb{F}_q)$.

1. Introduction

By a classical theorem of Cayley any finite group can be realized as a subgroup of a finite permutation group. In fact, for a finite group $G$ of size $|G|$, Cayley’s construction identifies $G$ with a subgroup of $S_{|G|}$, and the embedding is given by the regular action of $G$ on itself. One can often find a smaller permutation group $S_n$ containing a copy of $G$. Given a finite group $G$ we define $p(G)$ to be the smallest natural number $n$ such that $S_n$ has a subgroup isomorphic to $G$. We call the corresponding permutation representation on $n$ letters the minimal faithful representation of $G$. By the proof of Cayley’s theorem, $p(G) \leq |G|$. Despite its innocent sounding definition, computing $p(G)$ in general is an unsolved problem. Johnson’s paper [14] seems to be the first reference which addresses the problem of finding $p(G)$ as such and obtains various results, among which is the classification of finite groups $G$ such that $p(G) = |G|$, [14, Theorem 1]. Johnson [14] also gives the value of $p(G)$ for Abelian groups. Various other classes of groups, including $p$-groups, some easy semi-direct products, and some solvable groups, are studied in [9, 10], though no explicit general formulae are obtained.

The study of $p(G)$ for classical simple groups goes back to Galois who proved if $q > 11$ is a prime number, then $p(\text{PSL}_2(\mathbb{F}_q)) = q + 1$. Summarizing results by Cooperstein, Grechkoseeva, Mazurov, and Vasil’ev [6,12,13,18,21], Table 4 of [13] contains values of $p(G)$ for classical simple groups and exceptional simple groups of Lie type. The Atlas [5] contains the value of $p(G)$ for all finite sporadic simple groups. For a simple group $G$ computing $p(G)$ and determining the corresponding minimal faithful representation is tantamount to finding a subgroup of...
G of minimal index. In his thesis, Patton \cite{16} determined subgroups of minimal index in $\text{SL}_n(F_q)$ and $\text{SP}_{2m}(F_q)$ for $q$ an odd prime power. Cooperstein \cite{6} computed the minimal index of a subgroup for the remaining classical groups over finite fields using a generalization of Patton’s method. Given $G$, if $H$ is the subgroup of minimal index in $G$, we obtain a permutation representation of $G$ on $G/H$. Cooperstein calls the size of the set $|G/H|$ the \textit{minimal degree} of $G$. In this paper, we denote the minimal degree of a group $G$ by $\rho(G)$. An important point to note is that this permutation representation is, in general, \textit{not} faithful, and for that reason, unless $G$ is simple, the minimal degree $\rho(G)$ of $G$ is not necessarily equal to $p(G)$. In fact, Table 1 of \cite{6} has a column listing the size of the kernel of the corresponding permutation representation for each case.

In general, computing $p(G)$ for a non-simple group $G$, and determining the structure of the corresponding faithful permutation representation, is far more complicated than computing the minimal degree $\rho(G)$ of $G$. For comparison, if $G$ is a group of order $q^n$ for a prime number $q$, then by Sylow’s theorem, $G$ has a subgroup of order $q^{n-1}$, and as a result, $\rho(G)$ is always equal to $q^n/q^{n-1} = q$. On the other hand, by \cite{9,10}, the quantity $p(G)$ is related to the fine structure theory of $G$.

There are very few results in literature about the computation of $p(G)$ for a non-simple finite group $G$ of Lie type. In fact, to the best of our knowledge, save for the few examples worked out in the Atlas \cite{5}, the only known family of example is $G = \text{SL}_2(F_q)$ for $q$ a prime power. Theorem 3.7 of \cite{7} claims to have computed $p(\text{GL}_2(F_q))$ but as we will see, this result is wrong; the correct result for this case is stated as Theorem 5.1 below where we bootstrap the result on $\text{SL}_2$ to build minimal faithful representations of $\text{GL}_2$. The passage from $\text{SL}_2$ to $\text{GL}_2$ is not trivial. Observe that $\text{GL}_2(F_q) = D_1 \cdot \text{SL}_2(F_q)$, where $D_1$ is the group of diagonal matrices with any $a \in F_q^\times$ as the top left element on its diagonal and every other element equal to 1. This might suggest that the degree of the minimal faithful permutation representation for $\text{GL}_2(F_q)$ can be easily computed from that of $\text{SL}_2(F_q)$ by utilizing the semi-direct product decomposition. In fact, Lemma 2.4 in \cite{8} claims that the degree of the minimal faithful permutation representation of the semi-direct product $G \rtimes H$ is the same as that of $G$. However, there is a mistake in line six of the proof. The authors’ claim that if

\footnote{Even some of the experts reading an early draft of this manuscript, including at least one referee, have been confused about the distinction between the minimal degree $\rho(G)$ a la Cooperstein and $p(G)$.}
$B_1, \ldots, B_k$ gives a minimal faithful representation of $G$, then the core in $G H$ of $B_1 H \cap \cdots \cap B_k H$ is the core in $G H$ of $B_1 \cap \cdots \cap B_k$ times the core in $G H$ of $H$. This is not necessarily true, and in fact false in our case. In general the subgroup structure of semi-direct product is very complicated, and by Lemma 1.3, maximal subgroups of the direct product of groups depend on both groups and are typically quite intricate. Even for a familiar group like the dihedral group the minimal faithful permutation representation is surprisingly small, [9].

This is a summary of what we accomplish in this paper:

(i) We determine $p({\text{SL}_3}({F_q}))$ for all odd prime powers $q \geq 3$, c.f. Theorem 2.1.
(ii) When $n \geq 4$, we compute $p({\text{SL}_n}({F_q}))$ for infinitely many pairs $(n, q)$, including the cases where $\gcd(n, q-1) = 1$, c.f. Theorem 2.3.
(iii) We formulate a conjecture for $p({\text{SL}_n}({F_q}))$ for all large enough odd prime powers $q$ and all $n$, c.f. Conjecture 2.5.
(iv) We classify all minimal faithful permutation representations of ${\text{GL}_2}({F_q})$ for $q$ an odd prime power $\geq 3$ and compute $p({\text{GL}_2}({F_q}))$, c.f. Theorem 5.1.
(v) We classify all minimal faithful permutation representations of ${\text{GL}_3}({F_q})$ for $q$ a prime power $\geq 3$ and compute $p({\text{GL}_3}({F_q}))$, c.f. Theorem 6.1.
(vi) For all $n, q$, we construct a faithful permutation representation of ${\text{GL}_n}({F_q})$ and we conjecture that it is minimal, c.f. Proposition 3.4 and Conjecture 3.6. Also, see Corollary 3.5.

As in the case of ${\text{SL}_2}({F_q})$ treated in [2], our computation of $p({\text{SL}_3}({F_q}))$ uses the classification of the maximal subgroups, Tables 8.3 and 8.4 of [3]. The fact that our proof of this fact relies on the classification of subgroups is an impediment to computing $p({\text{SL}_n}({F_q}))$ in general. For $n \geq 4$, starting with Patton’s subgroup [16], we construct in the proof of Theorem 2.3 a core-free collection of subgroups of ${\text{SL}_n}({F_q})$. In general we cannot prove that this collection gives a minimal faithful permutation representation, but we can do so if $n, q$ satisfy a technical condition, very divisibility in Definition 2.2. In cases where there is a classification of maximal subgroups, e.g. as in the tables in [3], we have verified that these core-free collections are in fact minimal. An illustrative example is explained in Example 2.4. Inspired by these examples we conjecture that the core-free collections constructed in the proof of Theorem 2.3 are in general minimal, Conjecture 2.5.
To construct faithful collections of $\text{GL}_n(\mathbb{F}_q)$ we use our knowledge of faithful collections of $\text{SL}_n(\mathbb{F}_q)$. By using the faithful collection constructed for $\text{SL}_n(\mathbb{F}_q)$ we construct a faithful collection for $\text{GL}_n(\mathbb{F}_q)$ in §3. In §5 and §6 we prove that this faithful collection is in fact minimal for $\text{GL}_2(\mathbb{F}_q)$ and $\text{GL}_3(\mathbb{F}_q)$, respectively. Here too the proof relies on the classification of maximal subgroups of $\text{SL}_2(\mathbb{F}_q)$ and $\text{SL}_3(\mathbb{F}_q)$. The idea is to start with an arbitrary minimal faithful collection of $\text{GL}_2(\mathbb{F}_q)$, respectively $\text{GL}_3(\mathbb{F}_q)$, and try to beat the bound given by the faithful collection of §3. The surprising thing is that for $\text{GL}_2(\mathbb{F}_q)$ and $\text{GL}_3(\mathbb{F}_q)$ the minimal faithful collections are very rigid and one can completely classify them, Theorem 5.12 and Theorem 6.1. What facilitates translating the minimality problem from $\text{GL}_n(\mathbb{F}_q)$ to $\text{SL}_n(\mathbb{F}_q)$ is a clever idea from [7] which we formulate as Lemma 4.1 and Lemma 4.2. In this case too we can eliminate our reliance on the subgroup classification if we assume $n, q$ satisfy the technical condition of Definition 2.2. These considerations, plus other numerical examples, motivate Conjecture 3.6.

It is clear from our presentation that much of what we do here can be extended to other finite classical groups and their similitude counterparts. It would be interesting to investigate quasi-permutation representations of linear groups [22]. Behravesh and his collaborators [2, 7] have started this investigation, but there is much to be done. Understanding minimal faithful permutation representations and their cousins, minimal faithful quasi-permutation representations, is not only of clear inherent interest, but also of importance in computations. For example, [4] highlights some applications of minimal faithful permutation representations in computational group theory. For another class of applications, see [11].

This paper is organized as follows. Section 2 contains the results on $\text{SL}_n(\mathbb{F}_q)$. In Section 3 we construct a faithful permutation representation of $\text{GL}_n(\mathbb{F}_q)$. Section 4 collects a number of lemmas that are used in the next two sections. We study minimal faithful permutation representations of $\text{GL}_2(\mathbb{F}_q)$ in §5 and $\text{GL}_3(\mathbb{F}_q)$ in §6 by trying to beat the faithful permutation representation construction in §4. We end the paper with some general comments and future directions.

The second author is partially supported by a Collaboration Grant (Award number 523444) from the Simons Foundation. This paper owes a great deal of intellectual debt to the papers [2, 7]. We thank Lior Silberman and Ben Elias for useful conversations. We wish to thank...
Roman Bezrukavnikov and Annette Pilkington who independently simplified our first step of the proof of Lemma 5.6. We wish to thank the anonymous referee for their careful reading of the paper. We used SageMath to carry out some of the numerical calculations and experiments used in the paper.

**Notation.** In this paper $GL_n$ stands for the algebraic group of $n \times n$ matrices with non-zero determinant, and $SL_n$ the subgroup of $GL_n$ with elements of determinant equal to 1. The finite field with $q$ elements is denoted by $\mathbb{F}_q$. The integer $q$ is the power of a fixed prime number $p$. We fix $p, q$ throughout the paper. Write $g := \gcd(n, q - 1)$ and let the prime factorization of $g$ be as follows, $g = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$. Given a natural number $m$, we can write $m = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_t^{b_t} \cdots q_l^{b_l}$, where the $p_i$'s and $q_j$'s are distinct. We set $m_{n,q} := p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ and $T_{n,q}(m) = \sum_{r=1}^r q_r^{b_r}$. For example, if $n = 3$, then $g := \gcd(3, q - 1) = 1$ or 3. If $3 \mid m$ and $g = 3$, then $m_{3,q} := \text{the highest power of 3 dividing } m$ and $T_{3,q}(m)$ is the sum of all the primes in the prime factorization of $m$ except for 3, along with their appropriate powers. If $g = 1$, then $m_{3,q} := 1$ and $T_{3,q}(m)$ is the sum of all the primes in the prime factorization of $m$.

We also introduce notation for some subgroups of $GL_n(\mathbb{F}_q)$ that will be used throughout the paper. Let

$$
D_t = \left\{ \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \mid a \in A_t \right\}
$$

and

$$
Z_t = \left\{ \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a \in A_t \right\},
$$

where recall that $A_t$ is the unique subgroup of $\mathbb{F}_q^\times$ of size $(q - 1)/t$. We usually denote $Z_1$ by $Z$. Note that any subgroup of $Z$ is of the form $Z_t$ for some $t \mid q - 1$. In fact, $Z_t$ is the unique subgroup of $Z$ of order $(q - 1)/t$. For $s, t$ divisors of $q - 1$ we have

$$
Z_s \cap Z_t = Z_{\text{lcm}(s,t)}.
$$

(1.1)

Let

$$
GL_n(\mathbb{F}_q)^t = \{ g \in GL_n(\mathbb{F}_q) \mid \det g \in A_t \}. 
$$
Then it is clear that $D_t$, $Z_t$, and $GL_n(F_q)^t$ are subgroups of $GL_n(F_q)$, and that $GL_n(F_q)^t = D_t \cdot SL_n(F_q)$. We note that for $t \mid q - 1$, (1.2) $[GL_n(F_q) : GL_n(F_q)^t] = t$.

We define $P$ to be the set of all matrices

$$\begin{pmatrix} B & x \\ 0 & b \end{pmatrix}$$

where $B \in GL_{n-1}(F_q)$, $b^{-1} = \det B \in F_q^\times$, $x \in F_q^{n-1}$ and $Q$ to be the set of all matrices

$$\begin{pmatrix} C & x \\ 0 & c \end{pmatrix}$$

where, $c^{-1} = \det(C) \in A_{(q-1)n,q}$, $x \in F_q^{n-1}$, $C \in GL_{n-1}(F_q)$ and $B, C$, and $x$ arbitrary. By Patton [16], $P$ is the subgroup of $SL_n(F_q)$ of minimal index.

The standard reference for minimal faithful permutation representations of finite groups is Johnson’s classical paper [14]. In order to construct a faithful permutation representation of a group $G$ we need to construct a collection of subgroups $\{H_1, \ldots, H_\ell\}$ such that $\text{core}_G(H_1 \cap \cdots \cap H_\ell) = \{e\}$. Recall that for a subgroup $H$ of $G$, $\text{core}_G(H)$ is the largest normal subgroup of $G$ contained in $H$, i.e.,

$$\text{core}_G(H) = \bigcap_{x \in G} xHx^{-1}.$$ 

We call a collection $\{H_1, \ldots, H_\ell\}$ of subgroups of $G$ faithful if $\text{core}_G(H_1 \cap \cdots \cap H_\ell) = \{e\}$. In this case the left action of $G$ on the disjoint union

$$A = G/H_1 \cup \cdots \cup G/H_\ell$$

is faithful. Note that $|A| = \sum_i |G/H_i|$. A collection $\{H_1, \ldots, H_\ell\}$ is called minimal faithful if

(1) $\text{core}_G(H_1 \cap \cdots \cap H_\ell) = \{e\}$,
(2) $\sum_i |G/H_i|$ is minimal among all collections of subsets satisfying (1).

In this case, $\sum_i |G/H_i|$ is denoted by $p(G)$. The papers [10][14] and the thesis [9] contain many examples of explicit computations of $p(G)$ for various groups $G$.

2. Minimal faithful permutation representations of $SL_n(F_q)$

In this section we study the case of $SL_n(F_q)$ for $q$ odd and $n \geq 2$. 
2.1. The case of $\text{SL}_2(\mathbb{F}_q)$. We recall the construction for $\text{SL}_2(\mathbb{F}_q)$ for odd $q$ from [2]. Write $q - 1 = 2^r \cdot m$ with $m$ odd. Let $\varpi$ be a generator of the cyclic group $\mathbb{F}_q^\times$. For $t | q - 1$, set $A_t = \langle \varpi^t \rangle$. The set $A_t$ is the unique subgroup of $\mathbb{F}_q^\times$ of size $(q - 1)/t$, as introduced in §1. Note that if $s, t$ are divisors of $q - 1$, then $A_s \cap A_t = A_{\text{lcm}(s,t)}$. Set
\begin{equation}
H_{\text{odd}} = \{ \begin{pmatrix} a & x \\ a^{-1} & 1 \end{pmatrix} \mid a \in A_{2^r}, x \in \mathbb{F}_q \}.
\end{equation}
Then Theorem 3.6 of [2] says that $H_{\text{odd}}$ is a core free subgroup of $\text{SL}_2(\mathbb{F}_q)$ of minimal index, i.e., the action of $\text{SL}_2(\mathbb{F}_q)$ on $\text{SL}_2(\mathbb{F}_q)/H_{\text{odd}}$ is a minimal faithful representation. Also, it is easy to see that $[\text{SL}_2(\mathbb{F}_q) : H_{\text{odd}}] = (q - 1)^2(q + 1)$.

2.2. The case of $\text{SL}_3(\mathbb{F}_q)$. In this section, we compute the minimal faithful permutation representation of $\text{SL}_3(\mathbb{F}_q)$ for $q$ an odd prime power. In [1] Aschbacher classified maximal subgroups of finite classical groups. He shows that each such maximal subgroup lies in one of the 8 classes of subgroups $\mathcal{C}_1, \ldots, \mathcal{C}_8$ or its socle is a non-abelian simple group and it belongs to a special class $\mathcal{S}$. Following this classification the maximal subgroups of $\text{SL}_3(\mathbb{F}_q)$ have been compiled in Tables 8.3 and 8.4 of [3]. Using this we compute the order of the maximal subgroups of $\text{SL}_3(\mathbb{F}_q)$ and conclude the result below.

**Theorem 2.1.** The minimal faithful permutation representation of $\text{SL}_3(\mathbb{F}_q)$ for $q$ an odd prime power depends on $g = \gcd(q - 1, 3)$. There are two cases.

1. If $g = 1$, then the subgroup $M = E_q^2 : \text{GL}_2(\mathbb{F}_q)$ of class $\mathcal{C}_1$ gives the minimal faithful permutation representation of $\text{SL}_3(\mathbb{F}_q)$ and $p(\text{SL}_3(\mathbb{F}_q)) = \frac{[\text{SL}_3(\mathbb{F}_q)]}{[E_q^2 : \text{GL}_2(\mathbb{F}_q)]} = \frac{q^3 - 1}{q - 1}$.
2. If $g = 3$, then the maximal subgroup $G_3$ of $M$ with trivial intersection with the order 3 subgroup of the center and whose order is not coprime to 3 gives the minimal faithful permutation representation of $\text{SL}_3(\mathbb{F}_q)$ and $p(\text{SL}_3(\mathbb{F}_q)) = \frac{[\text{SL}_3(\mathbb{F}_q)]}{|G_3|} = \frac{(q^3 - 1)(q - 1)}{q - 1}$.

**Proof.** After case by case analysis, we deduce that the subgroup of $\text{SL}_3$ with maximal order is $E_q^2 : \text{GL}_2(\mathbb{F}_q)$.

Case 1: When $g = 1$ this maximal subgroup has trivial core and gives the minimal faithful permutation representation.

Case 2: When $g = 3$ i.e. $q \equiv 1 \mod 3$, then each maximal subgroup of $\text{SL}_3(\mathbb{F}_q)$ has order not coprime to 3 and the maximal subgroup $M = E_q^2 : \text{GL}_2(\mathbb{F}_q)$ has non-trivial core. Elements of $M$ have the form:
\[
\begin{pmatrix}
a & b & \ast \\
c & d & \ast \\
0 & 0 & \text{det } \gamma^{-1}
\end{pmatrix}
\]
where \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). By the argument Theorem 2.3, the minimal faithful permutation representation in this case is offered by the maximal subgroup \( G_3 \) of \( M \) with trivial intersection with the order 3 subgroup of the center and whose order is not coprime to 3, as long as \( n = 3 \) is very divisible in the sense of Definition 2.2 below.

We claim that whenever \( g = 3 \) the subgroup \( G_3 \) gives the minimal faithful permutation representation of \( \text{SL}_3(\mathbb{F}_q) \) for all values of \( q \) i.e. even when 3 is not very divisible. We reason as follows: the order of \( G_3 \) is given by \( \frac{(q^2-1)(q^2-q)q^2}{(q-1)^3} \). The maximal core-free subgroup we want is this maximal core free subgroup \( G_3 \) of \( M \), as the order of each maximal subgroup of \( \text{SL}_3(\mathbb{F}_q) \) belonging to classes outside of \( C_1 \) is verified to be smaller than that of \( G_3 \). The result follows immediately.

2.3. The case where \( n \geq 4 \). In order to state our theorem we need a definition.

**Definition 2.2.** Suppose \( g = p_1^{r_1} \cdots p_s^{r_s} \) and for each \( j \) between 1 to \( s \) let \( p_j^{a_j} \) be the highest power of \( p_j \) dividing \( q - 1 \). If \( n \) is such that \( p_j^{a_j} \) divides \( n \) for \( 1 \leq j \leq s \), then we call \( n \) very divisible relative to \( q \). We usually drop relative to \( q \). For example, \( n \) is very divisible whenever \( q - 1 \) divides \( n \).

**Theorem 2.3.** For very divisible \( n \geq 4 \) the minimal faithful representation of \( \text{SL}_n(\mathbb{F}_q) \) is computed as follows:

If we have that \( g = 1 \), then the maximal subgroup \( P \) of class \( C_1 \) gives the minimal faithful permutation representation of \( \text{SL}_n(\mathbb{F}_q) \) and \( p(\text{SL}_n(\mathbb{F}_q)) = \frac{|\text{SL}_n(\mathbb{F}_q)|}{|P|} = q^{n-1} \frac{q-1}{q-1} \) by [16]. When \( g > 1 \) we have that the minimal faithful permutation representation of \( \text{SL}_n(\mathbb{F}_q) \) is given by subgroups \( H_1, \ldots, H_s \) and has size \( p(\text{SL}_n(\mathbb{F}_q)) = \frac{q^{n-1}}{q-1}(p_1^{a_1} + \cdots + p_s^{a_s}) \). Further, each \( H_j \) is the biggest subgroup of \( P \) with trivial intersection with the order \( p_j \) central subgroup.

**Proof.** If we have that \( g = 1 \), then the maximal subgroup \( P \) of \( \text{SL}_n(\mathbb{F}_q) \) is core free and gives the minimal faithful permutation representation by [16]. In this case, we have \( p(\text{SL}_n(\mathbb{F}_q)) = \frac{|\text{SL}_n(\mathbb{F}_q)|}{|P|} = q^{n-1} \frac{q-1}{q-1} \). Hence, for the remainder of the proof we will assume \( g > 1 \). Suppose, \( \{H_1, \ldots, H_t\} \) was a minimal faithful representation of \( \text{SL}_n(\mathbb{F}_q) \). Then, \( \text{core}_{C_1}(H_1 \cap \cdots \cap H_t) = \{e\} \). We consider two cases based on order considerations of the groups of our minimal faithful collection.

**Case 1)**: No subgroup \( H_i \) has order coprime to \( g = p_1^{r_1} \cdots p_s^{r_s} \), so each
$H_i$ has non trivial intersection with the order $p_k$ subgroup of the center for some $k \in \{1, \ldots, s\}$. However, $\text{core}_G(H_1 \cap \cdots \cap H_s) = \{e\}$ implies that for each prime factor $p_j$ dividing $g$ there exists a subgroup in our collection $H_{ij}$ with trivial intersection with the order $p_j$ subgroup of the center. If this was not true, then $\text{core}_G(H_1 \cap \cdots \cap H_j)$ would contain an element of order $p_j$. Write $G_j$ as the maximal subgroup of $P$ with trivial intersection with the order $p_j$ subgroup of the center and whose order is not coprime to $g$. We will show $H_{ij} = G_j$. Observe that $G_j$ being a subgroup of $P$ is the set of all matrices

$$
\begin{pmatrix}
B_j & x \\
0 & b_j
\end{pmatrix}
$$

where, $b_j^{-1} = \det(B_j) \in \mathbb{F}_q^\times$, $x \in \mathbb{F}_q^{n-1}$, $B_j$ is a subgroup of $\text{GL}_{n-1}(\mathbb{F}_q)$ and $a, x$ arbitrary. Moreover, since $G_j$ is the biggest subgroup of $P$ with trivial intersection with the order $p_j$ central subgroup, $B_j$ must be the largest subgroup of $\text{GL}_{n-1}(\mathbb{F}_q)$, which has trivial intersection with the order $p_j$ subgroup of the center. Since $p_j$ divides $g = \gcd(q-1,n)$, $\text{SL}_{n-1}(\mathbb{F}_q)$ will not contain a central subgroup of order $p_j$. As a central element \( \begin{pmatrix} a & 0 & \ldots & 0 \\ 0 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & a \end{pmatrix} \) in $\text{SL}_{n-1}(\mathbb{F}_q)$ will satisfy $a^{q-1} = a^{n-1} = 1$, then $|a| | \gcd(q-1,n-1)$ and $\gcd(q-1,n)$, then $|a|!1$ and consequently $a = 1$. Hence, we may assume that $B_j$ contains $\text{SL}_{n-1}(\mathbb{F}_q)$. By lemma 3.1, $B_j$ has the form $\text{GL}_{n-1}(\mathbb{F}_q)^t = \{g \in \text{GL}_{n-1}(\mathbb{F}_q) \mid \det g \in A_t\}$. Also, by Lemma 3.2, $Z \cap \text{GL}_{n-1}(\mathbb{F}_q)^t = Z_{\frac{q-1}{\gcd(n-1,f)}}$. Recalling order of $Z_t$ is $\frac{q-1}{t}$, we require $p_j \nmid \frac{\gcd(n-1,f)(q-1)}{t}$. Also $p_j | n$ implies $p_j \nmid n-1$, so $p_j \nmid \gcd(n-1,t)$. Thus, $p_j \nmid \frac{q-1}{t}$. We want $B_j$ to be the largest possible size, which forces $t$ to be the smallest such that $p_j \nmid \frac{q-1}{t}$. Hence, $t = (q-1)p_j = \frac{p_j}{p_j}$. That is the highest power of $p_j$ dividing $q-1$. Hence, $B_j = \text{GL}_{n-1}(\mathbb{F}_q)^{p_j} = D_{p_j}^{\infty} \cdot \text{SL}_{n-1}(\mathbb{F}_q)$ and $\text{GL}_{n-1}(\mathbb{F}_q) = D_1 \cdot \text{SL}_{n-1}(\mathbb{F}_q)$. Further, $D_1 = D_{\frac{q-1}{t}}^{\infty} \cdot D_1$, gives $\text{GL}_{n-1}(\mathbb{F}_q) = D_{\frac{q-1}{p_j}}^{\infty} \cdot B_j$. Writing $C_j = D_{\frac{q-1}{p_j}}^{\infty}$ we have $\text{GL}_{n-1}(\mathbb{F}_q) = C_j \cdot B_j$. Let $P_j$ be the set of all matrices

$$
\begin{pmatrix}
C_j & 0 \\
0 & C_j
\end{pmatrix}
$$
where, $c_j^{-1} = \det(C_j) \in \mathbb{F}_q^\times$, $x \in V_{n-1}$, $C_j$ as defined above, $a, x$ arbitrary. Hence, $P = G_j.P_j$. Observe that $|P_j| = |C_j| = |D_{\frac{q-1}{p_j}}| = p_j^{a_j} = |Z_{\frac{q-1}{p_j}}|$. Thus, $R.Z_{\frac{q-1}{p_j}} = Z_{\frac{q-1}{p_j}}.R$ is a subgroup of $GL_n(\mathbb{F}_q)$. If $p_j^{a_j} | n$ for all $j$, then $Z_{\frac{q-1}{p_j}}$ is a subgroup of $SL_n(\mathbb{F}_q)$ and so is $R.Z_{\frac{q-1}{p_j}}$. But, $P$ is the largest subgroup of $SL_n(\mathbb{F}_q)$, hence index of $P$ in $SL_n(\mathbb{F}_q) = \frac{|SL_n(\mathbb{F}_q)|}{|G_j||P_j|} \leq \frac{|SL_n(\mathbb{F}_q)|}{|R||Z_{\frac{q-1}{p_j}}|}$. Using that $|Z_{\frac{q-1}{p_j}}| = |P_j|$ and simplifying we obtain, $|R| \leq |G_j|$ as claimed. Hence, $H_{ij} = G_j$ and for the collection to be minimal it must consist of $H_{i_1}, \ldots, H_{i_\ell}$ with degree $[G : G_i] + \cdots + [G : G_s] = |G : P| \cdot (|P_1| + \cdots + |P_s|) = \frac{q-1}{q-1} \cdot (p_1^{a_1} + \cdots + p_s^{a_s})$.

Case 2): There is some $i$, such that $1 \leq i \leq \ell$ and $\gcd(|H_i|, g) = 1$. We recall from Lemma 3.2 of [4], that the only proper normal subgroups of $SL_n(\mathbb{F}_q)$ are central subgroups. Using this it follows that any such subgroup $H_i$ with $\gcd(|H_i|, g) = 1$ has trivial core, because any element of the center has order dividing $g$. The intersection of any such subgroups $H_i$ will consequently have trivial core as well. Labeling the groups in our minimal faithful collection that have order coprime to $g$ with a $'$ mark, that is $H_i'$ and labeling the other subgroups $H_i$ in our collection without the $'$ mark we obtain the inequality $\Sigma_i[G : H_i'] + \Sigma_j[G : H_j] \geq \Sigma_i[G : H_i']$. Hence, the collection with only the subgroups $\{H_i'\}$ for some indexing set $i \in I$ not only has trivial core thus forming a faithful collection, but also it has lower degree than the minimal faithful collection comprising all the subgroups $H_i$. This is a contradiction, unless our minimal faithful collection has all subgroups of the form $H_i'$, that is each subgroup in our collection has order coprime to $g$. However, in this case we recall again that any such subgroup has trivial intersection with the center $Z(SL_n(\mathbb{F}_q))$ and thus it has trivial core. Thus, we could replace the collection of subgroups $H_i'$ with a single subgroup $H$ of maximal order coprime to $g$. This would give a faithful collection with order lesser than the minimal one, forcing our minimal faithful collection to comprise of a single subgroup $H$ satisfying $(|H|, g) = 1$. Next, following the proof of Lemma 3.5 in [2] we write $S$ to denote $Z(SL_n(\mathbb{F}_q))$ to observe that $S \cap H = \{1\}$ and $SH/S \subset G/S$. This gives us that $SH/S \cong H/S \cap H \cong H \subset G/S \cong PSL_n(\mathbb{F}_q)$. In other words subgroups $H$ of $G$ of order coprime to $g$ can be identified with subgroups of $PSL_n(\mathbb{F}_q)$ of order coprime to $g$. However, we know from Patton’s result in [16] that $\rho(PSL_n(\mathbb{F}_q))$ is $\frac{q^n - 1}{q - 1}$. Combining all this information with the fact that $PSL_n(\mathbb{F}_q) = SL_n(\mathbb{F}_q)/Z(SL_n(\mathbb{F}_q))$
and $|\text{PSL}_n(\mathbb{F}_q)| = |\text{SL}_n(\mathbb{F}_q)|/g$ we have that $\rho(\text{SL}_n(\mathbb{F}_q)) = |G|/|H| = |\text{PSL}_n(\mathbb{F}_q)|g/|H|$ is at least $\rho(\text{PSL}_n(\mathbb{F}_q))g = (q^n - 1)/(q - 1) \times g$. However, recalling the prime factorization of $g$ as $\prod_j p_j^{a_j}$ we obtain $\rho(\text{PSL}_n(\mathbb{F}_q))g \geq (q^n - 1)/(q - 1) \times (\prod_j p_j^{a_j}) > (q^n - 1)/(q - 1) \times (\sum_j p_j^{a_j})$ by Lemma 4.3, which is the index in the previous case. Since the size of the faithful collection in case 1 is smaller than the one provided by the subgroup $H$ in this case, we conclude that the minimal faithful permutation representation is always provided by the subgroups in case 1.

**Example 2.4.** For example, if $n = 4$ and $q = 41$, then $g = \gcd(4, 40) = 4$ and $n$ is not a very divisible natural number. Utilizing the classification in Table 8.8 of [3] we deduce that the maximal subgroup of $\text{SL}_4(\mathbb{F}_{41})$ is given by $E_{41}^3 : \text{GL}_3(41)$ i.e. the subgroup whose elements have the form:

$$
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
0 & 0 & 0 & \det \gamma^{-1}
\end{pmatrix}
$$

where $\gamma$ is the matrix

$$
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix}
$$

This subgroup has index $\frac{q^4 - 1}{q - 1}$. This subgroup has trivial core when $g = 1$, but in our case $g$ is 4. Theorem 2.3 implies that the minimal faithful permutation representation of $\text{SL}_4(\mathbb{F}_{41})$ is provided by the maximal subgroup $G_4$ of $E_{41}^3 : \text{GL}_3(41)$ with trivial core, as long as $n$ is very divisible. In our case $n$ is not very divisible, as 40 is divisible by $2^5$, while $n = 4$ is not. We still claim that the minimal faithful permutation representation of $\text{SL}_4(\mathbb{F}_{41})$ is provided by the maximal subgroup $G_4$ of $E_{41}^3 : \text{GL}_3(41)$ with order coprime to 4 in this case. By Theorem 2.3, $G_4$ has index $\frac{(q^4 - 1)(q - 1)^2}{(q - 1)^2}$ and its order can be verified to be larger than the order of each maximal subgroup of $\text{SL}_4(\mathbb{F}_{41})$ and thus it is the required subgroup.

This examples and others like it support the following conjecture.

**Conjecture 2.5.** For any $n$, Theorem 2.3 holds true for all $\text{SL}_n(\mathbb{F}_q)$ for $q$ an odd prime power.

### 3. A faithful collection for $\text{GL}_n(\mathbb{F}_q)$

In this section we construct a faithful collection for $\text{GL}_n(\mathbb{F}_q)$.

The following lemma is a consequence of the Lattice Isomorphism Theorem:
Lemma 3.1. If $H$ is a subgroup of $GL_n(F_q)$ which contains $SL_n(F_q)$, then there is $t | q - 1$ such that $H = GL_n(F_q)^t$.

The following lemma is important:

Lemma 3.2. $Z \cap GL_n(F_q)^t = Z_{\frac{t}{\gcd(n,t)}}$.

Proof. $Z \cap GL_n(F_q)^t$ is a subgroup of $Z$, so it must be of the form $Z_s$ for some $s \mid (q - 1)$. We need to determine the diagonal elements of the form

\[
\begin{pmatrix}
a & 0 & \ldots & 0 \\
0 & a & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a
\end{pmatrix}
\]

that can be written in the form

\[
\begin{pmatrix}
\varpi^kt & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
a_{t_1} & 0 & \ldots & 0 \\
0 & a_{t_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a_{t_{n-1}}
\end{pmatrix}
\]

for some integer $0 \leq k < (q - 1)/t$, such that $a_1, a_2, \ldots, a_n \in F_q^{\times}$ and $a_1a_2 \cdots a_n = 1$. Observe that, $\varpi^kt = a_2 = \cdots = a_n = a$ gives us $a^n(\varpi^kt)^{-1} = 1$, which means $\varpi^kt = a^n$. Letting $a = \varpi^t$ we obtain $\varpi^kt = \varpi^{nt}$. This is true if and only if $kt \equiv nt \mod q - 1$. There will be a solution for $\ell$ if and only if $\gcd(n, q - 1)$ divides $kt$ if and only if $\frac{\gcd(n,q-1)}{\gcd(n,q-1,t)}$ divides $\frac{t}{\gcd(n,q-1,t)}k$. But, $\frac{\gcd(n,q-1)}{\gcd(n,q-1,t)}$ and $\frac{t}{\gcd(n,q-1,t)}$ are coprime. Hence, $\gcd(n,q-1)$ divides $kt$ if and only if $\frac{t}{\gcd(n,q-1,t)}$ divides $k$. But, $\ell | q - 1$ implies $\gcd(n,q-1,t) = \gcd(n,t)$. Combining everything $\varpi^kt = \varpi^{nt}$ has a solution for $\ell$ if and only if $\frac{\gcd(n,q-1)}{\gcd(n,t)}$ divides $k$. Observe $0 \leq k < \frac{q-1}{t}$. Hence, the number of possibilities for $k$ are

\[
\frac{q-1}{t} - \frac{1}{t} = \frac{(q-1)\gcd(n,t)}{t\gcd(n,q-1)}.
\]

Dividing $kt \equiv nt \mod q - 1$ throughout by $\gcd(n,q-1)$ we obtain

\[
\ell \equiv \left(\frac{n}{\gcd(n,q-1)}\right)^{-1} \frac{kt}{\gcd(n,q-1)} \mod \frac{q-1}{\gcd(n,q-1)}
\]

Hence, for each value of $k$ there will be $\gcd(n,q-1)$ corresponding values for $\ell$. Thus, the total number of possibilities for $a$ i.e. for $k\ell$ are $\frac{(q-1)\gcd(n,t)}{t\gcd(n,q-1)} \cdot \gcd(n,q-1) = \frac{(q-1)\gcd(n,t)}{t}$. Thus, we conclude that $Z \cap GL_n(F_q)^t = Z_{\frac{t}{\gcd(n,t)}}$ as claimed. \qed
To state our results we need a couple of pieces of notation. Let \( P, Q, m_{n,q}, \) and \( T_{n,q} \) be defined in the same manner as in §1.

**Lemma 3.3.** We have

\[
\text{core}_{\text{GL}_n(\mathbb{F}_q)}(Q \cdot D_1) = Z_{(q-1),n,q} \quad \text{and} \quad \text{core}_{\text{GL}_n(\mathbb{F}_q)}(P \cdot D_1) = Z.
\]

**Proof.** Observe that \( \text{SL}_n(\mathbb{F}_q) \) is not contained in \( Q \cdot D_1 \). So, the core must be the intersection of \( Q \cdot D_1 \) with the center \( Z \) of \( \text{GL}_n(\mathbb{F}_q) \). For an element of \( Q \cdot D_1 \) to be in the center we require the element \( D = \begin{pmatrix} C & x \\ 0 & c \end{pmatrix} \) in \( Q \) to be diagonal. In particular, if \( D = \text{diag}(c_1, \ldots, c_{n-1}) \), then multiplying this by an element \( \text{diag}(b, 1, \ldots, 1) \) of \( D_1 \) should give us a diagonal matrix. That is, if \( c_1 b = c_2 = \cdots = c_{n-1} = c \) and using \( c^{-1} = \det C \) we have the equation \( c_1 c^{n-2} = c^{-1} \) i.e. \( c^{n-1} b^{-1} = c^{-1} \) or \( c^n = b \). Note that \( b \) is any element in \( \mathbb{F}_q^* \), so as long as \( c \neq 0 \) pick \( b = c^n \) and get the element \( \text{diag}(c, c, \ldots, c) \) in \( Q \cdot D_1 \cap Z \). So, the core of \( Q \cdot D_1 \) is \( Z_{(q-1),n,q} \). By similar considerations we obtain \( \text{core}_{\text{GL}_n(\mathbb{F}_q)}(P \cdot D_1) = Z \). \( \square \)

We can now give a construction of a faithful permutation representation which we will later prove to be minimal for \( n = 2 \) and 3.

**Proposition 3.4.** Write \( q - 1 = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_1^{b_1} \cdots q_t^{b_t} \), with \( g = \gcd(n, q - 1) = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}, \ r_1, \ldots, r_s \geq 1 \) and the \( p_i \)'s and \( q_i \)'s distinct. Then the collection of subgroups

\[
\{ Q \cdot D_1, \ \text{GL}_n(\mathbb{F}_q)^{q_1^{b_1} \cdots q_t^{b_t}} \}
\]

is a faithful collection. The corresponding faithful permutation representation has size

\[
\frac{q^n - 1}{q - 1} \cdot (q - 1)_{n,q} + T_{n,q}(q - 1).
\]

If \( g = 1 \) then the collection of subgroups

\[
\{ P \cdot D_1, \ \text{GL}_n(\mathbb{F}_q)^{q_1^{b_1} \cdots q_t^{b_t}} \}
\]

is a faithful collection. The corresponding faithful permutation representation has size

\[
\frac{q^n - 1}{q - 1} + T_{n,q}(q - 1).
\]

**Proof.** We need to show that the subgroup \( U = Q \cdot D_1 \cap \text{GL}_n(\mathbb{F}_q)^{q_1^{b_1} \cap \cdots \cap \text{GL}_n(\mathbb{F}_q)^{q_t^{b_t}}} \) is core-free. By Lemma 3.3, \( \text{core}_{\text{GL}_n(\mathbb{F}_q)}(Q \cdot D_1) \) is a subset of \( Z \), so the core of \( U \) is a subgroup of \( Z \). Since any subgroup of \( Z \) is normal, we just need to compute the intersection \( U \cap Z \). Lemmas
Lemmas 3.2 and 3.3 implies that this intersection is $Z_{(q-1)_{n,q}} \cap Z_{q_1} \cap \ldots \cap Z_{q_t}$. As $\text{lcm}((q-1)_{n,q}, q_1^{b_1}, \ldots, q_t^{b_t}) = q-1$, Equation (1.1) says

$$Z_{(q-1)_{n,q}} \cap Z_{q_1} \cap \ldots \cap Z_{q_t} = Z_{q-1} = \{e\}.$$ 

This means that the collection is faithful. The size of the corresponding permutation representation is equal to

$$[\text{GL}_n(\mathbb{F}_q) : Q \cdot D_1] + [\text{GL}_n(\mathbb{F}_q) : \text{GL}_n(\mathbb{F}_q)^{q_1^{b_1}}] + \cdots + [\text{GL}_n(\mathbb{F}_q) : \text{GL}_n(\mathbb{F}_q)^{q_t^{b_t}}]$$

from [16] and (1.2).

By symmetry, when $g = 1$, we need to show that the subgroup $U' = P \cdot D_1 \cap \text{GL}_n(\mathbb{F}_q)^{q_1^{b_1}} \cap \cdots \cap \text{GL}_n(\mathbb{F}_q)^{q_t^{b_t}}$ is core-free. By Lemma 3.3, $\text{core}_{\text{GL}_n(\mathbb{F}_q)}(P \cdot D_1)$ is $Z$, so the core of $U'$ is a subgroup of $Z$. Since any subgroup of $Z$ is normal, we just need to compute the intersection $U' \cap Z$.

Lemmas 3.2 and 3.3 implies that this intersection is $Z \cap Z_{q_1} \cap \ldots \cap Z_{q_t}$. As $\text{lcm}(1, q_1^{b_1}, \ldots, q_t^{b_t}) = q-1$, Equation (1.1) says

$$Z \cap Z_{q_1} \cap \ldots \cap Z_{q_t} = Z_{q-1} = \{e\}.$$ 

This means that the collection is faithful. The size of the corresponding permutation representation is equal to

$$[\text{GL}_n(\mathbb{F}_q) : P \cdot D_1] + [\text{GL}_n(\mathbb{F}_q) : \text{GL}_n(\mathbb{F}_q)^{q_1^{b_1}}] + \cdots + [\text{GL}_n(\mathbb{F}_q) : \text{GL}_n(\mathbb{F}_q)^{q_t^{b_t}}]$$

$$= \frac{q^n - 1}{q - 1} + q_1^{b_1} + \cdots + q_t^{b_t}$$

after using Equations [16] and (1.2). This finishes the proof of the proposition. \qed

**Corollary 3.5.** We have

$$p(\text{GL}_n(\mathbb{F}_q)) \leq \frac{q^n - 1}{q - 1} \cdot (q-1)_{n,q} + T_{n,q}(q-1).$$

We make the following conjecture:

**Conjecture 3.6.** Let $q \geq 5$ be an odd prime power. Then if $g = 1$,

$$p(\text{GL}_n(\mathbb{F}_q)) = \frac{q^n - 1}{q - 1} + T_{n,q}(q-1),$$

whereas if $g > 1$, then

$$p(\text{GL}_n(\mathbb{F}_q)) = p(\text{SL}_n(\mathbb{F}_q)) + T_{n,q}(q-1).$$
We will verify this conjecture for \( n = 2 \) and \( n = 3 \) in §5 and §6, respectively. That the conjecture is true for very divisible \( n \) is a consequence of Lemma 4.5 below.

4. Minimal faithful collections

In this section we will prove some general lemmas about minimal faithful collections that will help us determine these collections for \( n = 2 \) and 3 in the next two sections.

**Lemma 4.1.** Let \( H \) be a subgroup of \( \text{GL}_n(\mathbb{F}_q) \). Then

\[
[H : H \cap \text{SL}_n(\mathbb{F}_q)] = \frac{|H \cdot \text{SL}_n(\mathbb{F}_q)|}{|\text{GL}_n(\mathbb{F}_q)|} \cdot (q - 1).
\]

**Proof.** We observe that \( H \cdot \text{SL}_n(\mathbb{F}_q) \) is a subgroup of \( \text{GL}_n(\mathbb{F}_q) \). We have

\[
|H \cdot \text{SL}_n(\mathbb{F}_q)| = \frac{|H| \cdot |\text{SL}_n(\mathbb{F}_q)|}{|H \cap \text{SL}_n(\mathbb{F}_q)|}.
\]

Hence,

\[
[H : H \cap \text{SL}_n(\mathbb{F}_q)] = \frac{|H|}{|H \cap \text{SL}_n(\mathbb{F}_q)|} = \frac{|\text{GL}_n(\mathbb{F}_q)|}{(\text{GL}_n(\mathbb{F}_q) : H \cdot \text{SL}_n(\mathbb{F}_q))} \cdot \frac{1}{|\text{SL}_n(\mathbb{F}_q)|} = \frac{q - 1}{[\text{GL}_n(\mathbb{F}_q) : H \cdot \text{SL}_n(\mathbb{F}_q)]},
\]

as claimed. \( \square \)

**Lemma 4.2.** Let \( H \) be a subgroup of \( \text{GL}_n(\mathbb{F}_q) \) such that \( H \cap \text{SL}_n(\mathbb{F}_q) \) is a core free subgroup of \( \text{SL}_n(\mathbb{F}_q) \). Then,

\[
[\text{GL}_n(\mathbb{F}_q) : H] \geq p(\text{SL}_n(\mathbb{F}_q)) \cdot [\text{GL}_n(\mathbb{F}_q) : H \cdot \text{SL}_n(\mathbb{F}_q)].
\]

**Proof.** Since \( H \cap \text{SL}_n(\mathbb{F}_q) \) is a core free subgroup of \( \text{SL}_n(\mathbb{F}_q) \) we have the inequality

\[
[\text{SL}_n(\mathbb{F}_q) : H \cap \text{SL}_n(\mathbb{F}_q)] \geq p(\text{SL}_n(\mathbb{F}_q)).
\]

Next,

\[
[\text{GL}_n(\mathbb{F}_q) : H] = \frac{[\text{GL}_n(\mathbb{F}_q) : H \cap \text{SL}_n(\mathbb{F}_q)]}{[H : H \cap \text{SL}_n(\mathbb{F}_q)]}.
\]

By Lemma 4.1 this expression is equal to

\[
= \frac{[\text{GL}_n(\mathbb{F}_q) : H \cap \text{SL}_n(\mathbb{F}_q)]}{q - 1} \cdot [\text{GL}_n(\mathbb{F}_q) : H \cdot \text{SL}_n(\mathbb{F}_q)]
\]

\[
= [\text{SL}_n(\mathbb{F}_q) : H \cap \text{SL}_n(\mathbb{F}_q)] \cdot [\text{GL}_n(\mathbb{F}_q) : H \cdot \text{SL}_n(\mathbb{F}_q)]
\]

as claimed.
\[ \geq p(\text{SL}_n(\mathbb{F}_q)) \cdot [\text{GL}_n(\mathbb{F}_q) : H \cdot \text{SL}_n(\mathbb{F}_q)], \]

by Equation (4.1). \hfill \Box

**Lemma 4.3.** For natural numbers \(a_1, \ldots, a_k \geq 2\), at least one of which is strictly larger than 2, we have

\[ \sum_i a_i < \prod_i a_i. \]

**Proof.** Proof is by induction, without loss of generality assume \(a_1 \geq 3\). We have

\[ (a_1 - 1) \cdot (a_2 - 1) \geq 2. \]

Simplifying gives \(a_1a_2 \geq a_1 + a_2 + 1\). The rest is clear. \hfill \Box

**Corollary 4.4.** We have

\[ T_{n,q}(q - 1) < p(\text{SL}_n(\mathbb{F}_q)). \]

when \(n = 2\) or \(3\) or \(n > 3\) is a very divisible integer.

**Proof.** If \(n = 2\), then by lemma 4.3

\[ T_{2,q}(q - 1) \leq \frac{q - 1}{(q - 1)_2,q} < q - 1 < q + 1 < (q - 1)_{2,q}(q + 1) = p(\text{SL}_2(\mathbb{F}_q)) \]

upon using the statements in §2.1.

If \(n = 3\), we have two cases depending on whether or not 3 divides \(q - 1\).

Suppose 3 does not divide \(q - 1\), so by Theorem 2.1 \(p(\text{SL}_3(\mathbb{F}_q)) = \frac{q^2 - 1}{q - 1}\). By Lemma 4.3

\[ T_{3,q}(q - 1) \leq \frac{q - 1}{(q - 1)_3,q} < q - 1 < q + 1 < \frac{q^3 - 1}{q - 1} = p(\text{SL}_3(\mathbb{F}_q)). \]

Next, suppose 3 divides \(q - 1\), so by Theorem 2.1 \(p(\text{SL}_3(\mathbb{F}_q)) = (q^2 + q + 1)(q - 1)_{3,q}\). By Lemma 4.3

\[ T_{3,q}(q - 1) \leq \frac{q - 1}{(q - 1)_3,q} < q - 1 < p(\text{SL}_3(\mathbb{F}_q)). \]

For general very divisible \(n > 3\) by Lemma 4.3 and Theorem 2.3

\[ T_{n,q}(q - 1) \leq \frac{q - 1}{(q - 1)_n,q} \leq q - 1 < (q - 1) \frac{(q^{n-1} + q^{n-2} + \cdots + 1)}{q - 1} = \]

\[ \leq \frac{(q^n - 1)}{q - 1} \leq \frac{(q^n - 1)}{q - 1} (\sum_j p_j) \leq p(\text{SL}_n(\mathbb{F}_q)). \]

\hfill \Box
Now let $\mathcal{C} = \{H_1, \ldots, H_\ell\}$ be a minimal faithful collection of $\text{GL}_n(\mathbb{F}_q)$ for $n$ prime. Recall, that the collection being faithful means that we have core$\mathcal{C}(H_1 \cap \cdots \cap H_\ell) = \{e\}$. In particular, no central subgroup of $\text{GL}_n(\mathbb{F}_q)$ can lie in the intersection of the groups $H_i$, since the center of $\text{GL}_n(\mathbb{F}_q)$ is always a normal subgroup. Further recall that $Z(\text{SL}_n(\mathbb{F}_q)) \subseteq Z(\text{GL}_n(\mathbb{F}_q))$, implies that for each prime factor $p_j$ dividing $g$ there must exist a subgroup $H_{ij}$ in our collection, such that $H_{ij} \cap \text{SL}_n(\mathbb{F}_q)$ has trivial intersection with the order $p_j$ subgroup of $Z(\text{SL}_n(\mathbb{F}_q))$. If this was not true, then core$\mathcal{C}(H_1 \cap \cdots \cap H_\ell)$ would contain the order $p_j$ subgroup of $Z(\text{SL}_n(\mathbb{F}_q))$. Recall, that any central element of $\text{SL}_n(\mathbb{F}_q)$ has order dividing $g$, since it has the form $\text{diag}(a, \ldots, a)$ with $a^g = a^g - 1 = 1$ i.e. $a^g = 1$. Also, since we are assuming $n$ is prime we must have that $g$ equals either 1 or $n$, as $g$ divides $n$. Thus, in the case of $n$ prime, we conclude that there must be some index $i$ for which the subgroup $H_i \cap \text{SL}_n(\mathbb{F}_q)$ of $\text{SL}_n(\mathbb{F}_q)$ does not contain the central element of $\text{SL}_n(\mathbb{F}_q)$ of order $n$, if it exists, thus making it a core free subgroup of $\text{SL}_n(\mathbb{F}_q)$.

**Lemma 4.5.** In the case that $n = 2, 3$, or $n > 3$ is very divisible and prime, there is at most one $i$ such that $H_i \cap \text{SL}_n(\mathbb{F}_q)$ is a core free subgroup of $\text{SL}_n(\mathbb{F}_q)$ and $H_i \cdot \text{SL}_n(\mathbb{F}_q) = \text{GL}_n(\mathbb{F}_q)$.

*Proof.* Suppose $H_i \cap \text{SL}_n(\mathbb{F}_q), H_j \cap \text{SL}_n(\mathbb{F}_q)$ are both core free subgroups of $\text{SL}_n(\mathbb{F}_q)$. Hence, by Lemma 4.2, $[\text{GL}_n(\mathbb{F}_q) : H_i] + [\text{GL}_n(\mathbb{F}_q) : H_j]$ is larger than or equal to

$$([\text{GL}_n(\mathbb{F}_q) : H_i \cdot \text{SL}_n(\mathbb{F}_q)] + [\text{GL}_n(\mathbb{F}_q) : H_j \cdot \text{SL}_n(\mathbb{F}_q)]) \cdot p(\text{SL}_n(\mathbb{F}_q))$$

which is at least $2p(\text{SL}_n(\mathbb{F}_q))$. By Corollary 4.4 we have $2p(\text{SL}_n(\mathbb{F}_q)) > p(\text{SL}_n(\mathbb{F}_q)) + T_{n,q}(q - 1)$, and this latter quantity, by Corollary 3.5 and 2.3 is larger than or equal to $p(\text{GL}_n(\mathbb{F}_q))$. Consequently, if $H_i \neq H_j$ or if $H_i \cdot \text{SL}_n(\mathbb{F}_q) \neq \text{GL}_n(\mathbb{F}_q)$,

$$[\text{GL}_n(\mathbb{F}_q) : H_i] + [\text{GL}_n(\mathbb{F}_q) : H_j] > p(\text{GL}_n(\mathbb{F}_q)).$$

This contradicts the assumption that $\mathcal{C}$ is a minimal faithful collection. Without loss of generality let $i = 1$.

Our goal is to minimize

$$[\text{GL}_n(\mathbb{F}_q) : H_2] + \cdots + [\text{GL}_n(\mathbb{F}_q) : H_\ell].$$

We need a lemma.

**Lemma 4.6.** Suppose $t \mid q - 1$, and $t \neq q - 1$. Let $H(t)$ be the subgroup of $\text{GL}_n(\mathbb{F}_q)$ with minimal $[\text{GL}_n(\mathbb{F}_q) : H]$ among the subgroups that satisfy $Z \cap H = Z_t$. Then

$$H(t) = \text{GL}_n(\mathbb{F}_q)^{dt},$$
where $d$ is any divisor of $g = \gcd(n, q - 1)$. Furthermore, 
$$[\text{GL}_n(\mathbb{F}_q) : H(t)] = dt.$$ 

**Proof.** By Lemma 3.2 there is a subgroup $H$ containing $\text{SL}_n(\mathbb{F}_q)$ which satisfies the conditions of the lemma. In fact we will show the subgroup $H$ in the statement of the lemma must contain $\text{SL}_n(\mathbb{F}_q)$.

Let $m_t$ be the minimum degree $[\text{GL}_n(\mathbb{F}_q) : K]$ over subgroups $K$ of $\text{GL}_n(\mathbb{F}_q)$ containing $\text{SL}_n(\mathbb{F}_q)$ and $Z_t$. Now, the $H$ in the lemma satisfies $[\text{GL}_n(\mathbb{F}_q) : H] = [\text{GL}_n(\mathbb{F}_q) : H \cap \text{SL}_n(\mathbb{F}_q)]/[H : H \cap \text{SL}_n(\mathbb{F}_q)] = [\text{GL}_n(\mathbb{F}_q) : H \cap \text{SL}_n(\mathbb{F}_q)]/\text{gcd}([\text{GL}_n(\mathbb{F}_q) : H], \text{SL}_n(\mathbb{F}_q))$ by Lemma 4.1. Observe, $H.\text{SL}_n(\mathbb{F}_q)$ is a subgroup of $\text{GL}_n(\mathbb{F}_q)$ containing both $Z_t$ (as $H$ contains it) and $\text{SL}_n(\mathbb{F}_q)$. Hence, $[\text{GL}_n(\mathbb{F}_q) : H] \geq [\text{GL}_n(\mathbb{F}_q) : H \cap \text{SL}_2(\mathbb{F}_q)]/\text{gcd}(n,q-1)$.

We require $[\text{GL}_n(\mathbb{F}_q) : H]$ to be minimal among all subgroups containing $Z_t$ by definition. So, we need to minimize $[\text{GL}_n(\mathbb{F}_q), H \cap \text{SL}_n(\mathbb{F}_q)]$. For this, it’s clear that $[H \cap \text{SL}_n(\mathbb{F}_q)]$ must be maximized, hence $H$ contains $\text{SL}_n(\mathbb{F}_q)$. So, by Lemma 3.1 $H = \text{GL}_n(\mathbb{F}_q)^{t'}$ for some $t'|q-1$.

Combining Lemma 3.2 and $H \cap Z = Z_t$, we obtain $t = \frac{t'}{\gcd(n,t')}$. However, $\gcd(n,t')|\gcd(n,q-1) = g \implies t' = dt$ for some divisor $d$ of $g$ and $H = \text{GL}_n(\mathbb{F}_q)^{dt}$ as claimed. The last assertion follows from Equation 1.2.

5. Minimal faithful permutation representations of $\text{GL}_2(\mathbb{F}_q)$

We will determine the size and the structure of the minimal faithful permutation representations of the group $\text{GL}_2(\mathbb{F}_q)$, for an odd prime power $q$. Theorem 3.7 of [7] claims to have determined at least the size of the minimal faithful permutation representation, but there is a typo in the answer, and it appears to us that the proof presented is not correct. The proof we present here is inspired by the results and techniques of [2] and [7].

We will modify the notation slightly noting that $\gcd$ of 2 and $q - 1$ is always 2, since $q$ is odd. Given a natural number $n$, we can write $n = 2^r \prod_{p \text{ odd}} p^{e_p}$. We set $n_{2,q} = 2^r$ and $T_{2,q}(n) = \sum_{p \text{ odd}} p^{e_p}$. Also given a finite group $G$ we let $p(G)$ be the size of the minimal faithful permutation representation of $G$, i.e., the size of the smallest set $A$ on which $G$ has a faithful action. We will be proving the following result:

**Theorem 5.1.** If $q \geq 3$ is an odd prime power, then 
$$p(\text{GL}_2(\mathbb{F}_q)) = p(\text{SL}_2(\mathbb{F}_q)) + T_{2,q}(q - 1) = (q + 1)(q - 1)^2 + T_{2,q}(q - 1).$$
In fact we prove a much stronger theorem, Theorem 5.12, where we identify all minimal faithful sets for $GL_2(\mathbb{F}_q)$. The equality $p(\text{SL}_2(\mathbb{F}_q)) = (q + 1)(q - 1)_{2,q}$ is Theorem 3.6 of [2]. To prove our theorem we first construct a faithful permutation representation of size $p(\text{SL}_2(\mathbb{F}_q)) + T_{2,q}(q - 1)$ and then we proceed to find all minimal faithful permutation representations of $GL_2(\mathbb{F}_q)$ by trying to beat this bound. The proof we present here is elementary but rather subtle.

Recall the construction of the minimal faithful permutation representation of $\text{SL}_2(\mathbb{F}_q)$ from §2.1. Set
\begin{equation}
(5.1) \quad GH_{\text{odd}} = D_1 \cdot H_{\text{odd}}.
\end{equation}
Then $GH_{\text{odd}}$ is a subgroup of $GL_2(\mathbb{F}_q)$, and we have
\begin{equation}
(5.2) \quad [\text{GL}_2(\mathbb{F}_q) : GH_{\text{odd}}] = (q - 1)_{2,q}(q + 1).
\end{equation}

Lemma 5.2. We have
\[ \text{core}_{\text{GL}_2(\mathbb{F}_q)}(GH_{\text{odd}}) = Z_{2^r}. \]

Proof. Any normal subgroup of $\text{GL}_2(\mathbb{F}_q)$ which does not contain $\text{SL}_2(\mathbb{F}_q)$ is a subgroup of $Z$, so we just need to show $Z \cap GH_{\text{odd}} = Z_{2^r}$. For $a \in A_{2^r}$ we have
\[ \begin{pmatrix} a^2 & 1 \\ 1 & a \end{pmatrix} \cdot \begin{pmatrix} a^{-1} \\ a \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}. \]

We can now give a construction of a faithful permutation representation which we will later prove to be minimal.

Proposition 5.3. Write $q - 1 = 2^r \cdot p_1^{e_1} \cdots p_k^{e_k}$ with $p_1, \ldots, p_k$ distinct odd primes, and $e_1, \ldots, e_k \geq 1$. Then the collection of subgroups
\[ \{GH_{\text{odd}}, \text{GL}_2(\mathbb{F}_q)^{p_1^{e_1}}, \ldots, \text{GL}_2(\mathbb{F}_q)^{p_k^{e_k}} \} \]
is a faithful collection. The corresponding faithful permutation representation has size $p(\text{SL}_2(\mathbb{F}_q)) + T_{2,q}(q - 1)$.

Proof. We need to show that the subgroup $U = GH_{\text{odd}} \cap \text{GL}_2(\mathbb{F}_q)^{p_1^{e_1}} \cap \cdots \cap \text{GL}_2(\mathbb{F}_q)^{p_k^{e_k}}$ is core free. By Lemma 5.2, $\text{core}_{\text{GL}_2(\mathbb{F}_q)}(GH_{\text{odd}})$ is a subset of $Z$, so the core of $U$ is a subgroup of $Z$. Since any subgroup of $Z$ is normal, we just need to compute the intersection $U \cap Z$. Lemmas 3.2 and 5.2 imply that this intersection is $Z_{2^r} \cap Z_{p_1^{e_1}} \cap \cdots Z_{p_k^{e_k}}$. As $\text{lcm}(2^r, p_1^{e_1}, \ldots, p_k^{e_k}) = q - 1$, Equation (1.1) says
\[ Z_{2^r} \cap Z_{p_1^{e_1}} \cap \cdots Z_{p_k^{e_k}} = Z_{q - 1} = \{e\}. \]
This means that the collection is faithful. The size of the corresponding permutation representation is equal to

\[
[GL_2(\mathbb{F}_q) : G_{H_{\text{odd}}}]+ [GL_2(\mathbb{F}_q) : GL_2(\mathbb{F}_q)^p_i]+ \cdots + [GL_2(\mathbb{F}_q) : GL_2(\mathbb{F}_q)^p_i]
\]

\[= (q-1)_{2,q}(q+1) + p_i^1 + \cdots + p_i^n\]

after using Equations (5.2) and (1.2). This finishes the proof of the proposition.

Corollary 5.4. We have

\[p(GL_2(\mathbb{F}_q)) \leq (q-1)_{2,q}(q+1) + T_{2,q}(q-1) = p(SL_m(\mathbb{F}_q)) + T_{2,q}(q-1).\]

Without loss of generality let \(i = 1\).

Lemma 5.5. The \(H_1 \cap SL_2(\mathbb{F}_q)\) is a conjugate of \(H_{\text{odd}}\) in \(SL_2(\mathbb{F}_q)\), with \(H_{\text{odd}}\) defined by Equation (2.1).

Proof. Since \((-1, -1) \not\in H_1\), \(H_1 \cap SL_2(\mathbb{F}_q)\) will have odd order. By Lemma 3.5 of [2], up to conjugation, we have the following possibilities for \(H_1 \cap SL_2(\mathbb{F}_q)\):

(A) a cyclic subgroup of odd order dividing \(q + 1\);

(B) a subgroup of odd order of the upper triangular matrices

\[T(2, q) = \{ \begin{pmatrix} a & \cdot \\ \cdot & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ \cdot & 1 \end{pmatrix} \mid a \in \mathbb{F}_q^\times, x \in \mathbb{F}_q \} \}

In case (A), since \(|H_1 \cap SL_2(\mathbb{F}_q)| = (q + 1)/t\) is odd, and \(q + 1\) are even numbers, we must have \(t \geq 2\). By the proof of Lemma 4.2 and the statement of Lemma 4.3

\[|GL_2(\mathbb{F}_q) : H_1| = |SL_2(\mathbb{F}_q) : H_1 \cap SL_2(\mathbb{F}_q)| \cdot |GL_2(\mathbb{F}_q) : H_1 \cdot SL_2(\mathbb{F}_q)|\]

\[= |SL_2(\mathbb{F}_q) : H_1 \cap SL_2(\mathbb{F}_q)| = \frac{|SL_2(\mathbb{F}_q)|}{|H_1 \cap SL_2(\mathbb{F}_q)|} = \frac{q(q+1)(q-1)}{q^1 + 1} \cdot t \geq 2\frac{q(q+1)(q-1)}{q^1 + 1}.

We determine the cases where \(\frac{q(q+1)(q-1)}{q^1 + 1} \geq (q-1)_{2,q} \cdot (q+1)\). We need \(\frac{q(q+1)(q-1)}{q^1 + 1} \geq (q-1)_{2,q}\). We have two cases:

- If the denominator is \(q-1\), then we need \(q \geq (q-1)_{2,q}\), and that’s obviously true.
- If the denominator is \(q+1\), then we want \(q(q-1) \geq (q+1)(q-1)_{2,q}\). For this write \(q-1 = 2^r m\) with \(m\) odd, then we need \((2^r m + 1)2^r m \geq (2^r m + 2)2^r\), or what is the same, \((2^r m + 1)m \geq 2^r m + 2\). If \(m \geq 3\), then this last inequality is definitely satisfied, but if \(m = 1\), it is not true.
This discussion means that unless \( q = 2^r + 1 \), \( [\text{GL}_2(\mathbb{F}_q) : H_1] \geq 2p(\text{SL}_2(\mathbb{F}_q)) \) which by Corollary 4.4 is strictly bigger than \( p(\text{SL}_2(\mathbb{F}_q)) + T_{2,q}(q-1) \). This last statement, by Corollary 5.4, contradicts the minimality of \( C \).

Now we examine the case where \( q = 2^r + 1 \). Note that in this case \( T_{2,q}(q-1) = 0 \) as \( q-1 \) has no odd prime factors. One easily checks that

\[
\frac{2q(q+1)(q-1)}{q+1} > (q-1)_{2,q}(q+1).
\]

This means that \( [\text{GL}_2(\mathbb{F}_q) : H_1] > p(\text{SL}_2(\mathbb{F}_q)) + T_{2,q}(q-1) \) which again, by Corollary 5.4, contradicts the minimality of \( C \).

Now we examine case (B). In this case, if we write \( q-1 = 2^r m \), there is a divisor \( m_0 \) of \( m \) such that

\[
H_1 \cap \text{SL}_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & 1 \\ a^{-1} & 1 \end{pmatrix} \in A_{2^r m_0} \cap \text{SL}_2(\mathbb{F}_q) \right\}.
\]

(Note that this is up to conjugation only, but with a change of basis, we may assume it to be true.) Then

\[
[\text{SL}_2(\mathbb{F}_q) : H_1 \cap \text{SL}_2(\mathbb{F}_q)] = [\text{SL}_2(\mathbb{F}_q) : H_{\text{odd}}] \cdot [H_{\text{odd}} : H_1 \cap \text{SL}_2(\mathbb{F}_q)] = p(\text{SL}_2(\mathbb{F}_q)).
\]

By the proof of Lemma 4.2 we have

\[
[\text{GL}_2(\mathbb{F}_q) : H_1] = [\text{SL}_2(\mathbb{F}_q) : H_1 \cap \text{SL}_2(\mathbb{F}_q)] \cdot [\text{GL}_2(\mathbb{F}_q) : H_1 \cdot \text{SL}_2(\mathbb{F}_q)] = p(\text{SL}_2(\mathbb{F}_q)),
\]

upon using Lemma 4.5. Again as before if \( m_0 > 1 \), we conclude \( [\text{GL}_2(\mathbb{F}_q) : H_1] \geq 2 \), and we get a contradiction. \( \square \)

Without loss of generality we may assume \( H_1 \cap \text{SL}_2(\mathbb{F}_q) = H_{\text{odd}} \).

For \( n, 0 \leq n < q-1 \), define a subgroup \( D(n) \subset \text{GL}_2(\mathbb{F}_q) \) by

\[
D(n) = \left\{ \begin{pmatrix} a^{n+1} & 1 \\ a^{-n} & 1 \end{pmatrix} : a \in \mathbb{F}_q^\times \right\}.
\]

Set

\[
GH(n) = D(n) \cdot H_{\text{odd}}.
\]

The subgroup \( GH(0) \) is what we had called \( GH_{\text{odd}} \) in Equation (5.1).

**Lemma 5.6.** There is an \( n, 0 \leq n < q-1 \), such that \( H_1 = GH(n) \).
Proof. The proof of this lemma is in two steps. In the first step we show that $H_1$ is a subgroup of upper triangular matrices in $\text{GL}_2(\mathbb{F}_q)$, and then we identify it explicitly. The simple argument we give for the first step was suggested independently by Roman Bezrukavnikov and Annette Pilkington. We start with the observation that $H_1 \cap \text{SL}_2(\mathbb{F}_q)$ is normal in $H_1$. By Lemma 5.5, $H_1 \cap \text{SL}_2(\mathbb{F}_q)$ consists of upper triangular matrices and contains all upper triangular unipotent matrices. Suppose \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_1 \), and let \( \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \) be an arbitrary upper triangular unipotent matrix. Then since the matrix
\[
 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ \frac{x^2 - bc}{ad - bc} & * \end{pmatrix}
\]
for all $x$, we must have $c = 0$.

Now we proceed to identify $H_1$ explicitly. By the proof of Lemma 4.2 and the statements of Lemmas 4.5 and 5.5 we have
\[
[\text{GL}_2(\mathbb{F}_q) : H_1] = [\text{SL}_2(\mathbb{F}_q) : H_{odd}].
\]
This means $|H_1| = (q - 1) \cdot |H_{odd}|$. So we need to find $(q - 1)$ representatives for the quotient $H_1/H_{odd}$. By Lemma 5.5, the determinant $\text{det}: H_1 \to \mathbb{F}_q^\times$ is surjective. In particular, if $\varpi$ is a generator of $\mathbb{F}_q^\times$, there is a matrix $X$ in $H_1$, upper triangular by the first part, such that $\text{det} X = \varpi$. Since by Lemma 5.5, $H_1$ contains all upper triangular unipotent matrices, we may assume that $X$ is diagonal. Since $\varpi$ is a generator of $\mathbb{F}_q^\times$, we may write $X = \begin{pmatrix} \varpi^{n+1} \\ \varpi^m \end{pmatrix}$. Since $\varpi = \text{det} X = \varpi^{n+m+1}$, we conclude $n+m \equiv 0 \pmod{(q-1)}$, or $m \equiv -n$. So if we let $X_n = \begin{pmatrix} \varpi^{n+1} \\ \varpi^{-n} \end{pmatrix}$, then $X_n \in H_1$ and $\text{det} X_n = \varpi$. The elements \( \{ X_n^i \mid 0 \leq i < q - 1 \} \) provide the $(q - 1)$ representatives for $H_1/H_{odd}$ that we need.

**Corollary 5.7** (From the proof). We have
\[
[\text{GL}_2(\mathbb{F}_q) : H_1] = p(\text{SL}_2(\mathbb{F}_q)).
\]

**Proof.** This is Equation (5.3). \(\square\)

**Lemma 5.8.** We have
\[
\text{core}_{\text{GL}_2(\mathbb{F}_q)}(H_1) = \mathbb{Z}_{2r}.
\]

**Proof.** By Lemma 5.6, it suffices to prove $\text{core}_{\text{GL}_2(\mathbb{F}_q)}(\text{GH}(n)) = \mathbb{Z}_{2r}$ for each $n$, and that means we need to determine $Z \cap \text{GH}(n)$. Suppose
we have an element of the form
\[ t = \left( a^{n+1} \right. \left. \begin{array}{c} a^n \\ 1 \end{array} \right) \cdot \left( b^{-1} \right. \left. \begin{array}{c} b \\ 1 \end{array} \right), \quad a \in \mathbb{F}_q^*, b \in A_{2^r} \]
and suppose \( t \in Z \). This means \( a^{2n+1} = b^2 \). Write \( a = \varpi^i, b = \varpi^{2^r \cdot j} \).
Then we have
\[ (2n + 1)i \equiv 2^{r+1}j \mod (q - 1). \]
Let \( g = \gcd(2n + 1, q - 1) \). Then \( j = gu \) for some \( u \). Then if \( k \) is a multiplicative inverse of \( (2n + 1)/g \) modulo \((q - 1)/g\), we have
\[ i \equiv k \cdot 2^{r+1} \cdot u \mod \frac{q - 1}{g}, \]
or
\[ i = k \cdot 2^{r+1} \cdot u + \frac{q - 1}{g}s \]
for some \( s \). So if for any \( u, s \) we set \( a = \varpi^i, b = \varpi^{2^r \cdot j} \) with \( i, j \) satisfying
\[ \begin{aligned}
  i &= k \cdot 2^{r+1} \cdot u + \frac{q - 1}{g}s \\
  j &= ug,
\end{aligned} \]
then \( a^{2n+1} = b^2 \). Now we examine the matrix \( t \). We see that \( a^{-n} \cdot b \) is equal to \( \varpi \) raised to the power
\[ -n(k \cdot 2^{r+1} \cdot u + \frac{q - 1}{g}s) + u \cdot g \cdot 2^r \]
\[ = (-2nk + g)2^r \cdot u - n \cdot \frac{q - 1}{g} \cdot s \]
\[ = 2^r \cdot \left( (-2nk + g) \cdot u - n \cdot \frac{q - 1}{2^r \cdot g} \cdot s \right). \]
We will show that
\[ (5.4) \quad \gcd(2nk - g, n \cdot \frac{q - 1}{g}) = 1. \]
Let us first look at \( \gcd(2nk - g, n) \). This is equal to \( \gcd(g, n) \) which is equal to 1, as \( g \mid 2n + 1 \) and \( \gcd(n, 2n + 1) = 1 \). This means
\[ \gcd(2nk - g, n \cdot \frac{q - 1}{g}) = \gcd(2nk - g, \frac{q - 1}{g}) \]
\[ = \gcd((2n + 1)k - g - k, \frac{q - 1}{g}) \]
\[ = \gcd(g \left\{ \frac{2n + 1}{g} \cdot k - 1 \right\} - k, \frac{q - 1}{g}) \]
\[ = \gcd(-k, \frac{q - 1}{g}) \]
In the above computation we have used the fact that \( k \) is multiplicative inverse of \( (2^n + 1)/g \) modulo \( (q - 1)/g \), so \( k \cdot (2^n + 1)/g - 1 \) is divisible by \( (q - 1)/g \). Now that we have established Equation (5.4) we observe that since \(-2nk + g\) is odd we have

\[
gcd(-2nk + g, -n \cdot \frac{q - 1}{2^r \cdot g}) = 1.
\]

This means that there are integers \( s, u \) such that the corresponding \( a, b \) satisfy

\[
a - nb = a^{n+1}b^{-1} = \varpi^{2^s}, \quad \text{and that whenever} \quad a^{-n}b = a^{n+1}b^{-1} \quad \text{for} \quad a \in \mathbb{F}_q^\times, b \in A_{2^r}, \quad \text{then the common value is of the form} \quad \varpi^{f \cdot 2^s} \quad \text{for some integer} \quad f.
\]

This finishes the proof of the lemma.

Now that we have identified the possibilities for \( H_1 \) and its core, we optimize the choices of \( H_2, \ldots, H_{\ell} \). Define natural numbers \( t_2, \ldots, t_{\ell} \) by setting

\[
Z \cap H_i = Z_{t_i}, \quad 2 \leq i \leq \ell.
\]

We can pick each \( t_i \) to be a divisor of \( q - 1 \). By Equation (1.1) and Lemma 6.5, the statement

\[
\text{core}_{\text{GL}_2(\mathbb{F}_q)}(H_1 \cap \cdots \cap H_\ell) = \{e\}
\]

is equivalent to

\[
\text{lcm}(2^r, t_2, \ldots, t_{\ell}) = q - 1.
\]

**Corollary 5.9.** Suppose \( t \mid q - 1 \), and \( t \neq q - 1 \). Let \( H(t) \) be the subgroup of \( \text{GL}_2(\mathbb{F}_q) \) with minimal \([\text{GL}_2(\mathbb{F}_q) : H]\) among the subgroups that satisfy \( Z \cap H = Z_t \). Then

\[
H(t) = \begin{cases} 
\text{GL}_2(\mathbb{F}_q)^t & \text{t odd;} \\
\text{GL}_2(\mathbb{F}_q)^{2t} & \text{t even.}
\end{cases}
\]

Furthermore,

\[
[\text{GL}_2(\mathbb{F}_q) : H(t)] = \begin{cases} 
t & \text{t odd;} \\
2t & \text{t even.}
\end{cases}
\]

To finish the proof of Theorem 5.1 we have to solve the following optimization problem for \( n = q - 1 \).

**Problem 5.10.** Suppose a natural number \( n = 2^r m \) with \( m \) odd is given. For a natural number \( t \), set \( \epsilon(t) = (3 + (-1)^t)/2 \). Find natural numbers \( \ell, t_2, \ldots, t_{\ell} \) such that

- \( \text{lcm}(2^r, t_2, \ldots, t_{\ell}) = n; \)
- \( \sum_{i=2}^{r} \epsilon(t_i) t_i \) is minimal.

We call \((t_2, \ldots, t_{\ell})\) the optimal choice for \( n \).
Lemma 5.11. Write $n = 2^r p_1^{e_1} \cdots p_k^{e_k}$ with $p_i$'s distinct odd primes. Then the optimal choice for $n$ is $(p_1^{e_1}, \ldots, p_k^{e_k})$.

Proof. Suppose $(t_2, \ldots, t_\ell)$ is an optimal choice for $n$. If some $t_i$ is even, say equal to $2^s$, replacing $t_i$ by $s$ does not change the lcm in the statement Problem 5.10, but decreases the value of $\sum \epsilon(t_i)t_i$. Since $(t_2, \ldots, t_\ell)$ is optimal for $n$, this means that all of the $t_i$'s have to be odd. Next, write each $t_i$ as the product of prime powers $\pi_1^{m_1} \cdots \pi_v^{m_v}$. By Lemma 4.3, $\sum_j \pi_j^{m_j} \leq t_i$ with equality only when $v = 1$. Again, since $(t_2, \ldots, t_\ell)$ is optimal, this means each $t_i$ is a prime power. It is also clear that if $i \neq j$, then $(t_i, t_j) = 1$, because otherwise $t_i, t_j$ will be powers of the same prime, and we can throw away the one with smaller exponent. \qed

Putting everything together we have proved the following theorem:

Theorem 5.12. We have

$$p(\text{GL}_2(\mathbb{F}_q)) = p(\text{SL}_2(\mathbb{F}_q)) + T_{2,q}(q-1).$$

For $0 \leq n \leq q-2$, set

$$C_n = \{GH(n), \text{GL}_2(\mathbb{F}_q)^{p_1^{e_1}}, \ldots, \text{GL}_2(\mathbb{F}_q)^{p_k^{e_k}}\}.$$

Up to conjugacy we have $q-1$ classes of minimal faithful collections for $\text{GL}_2(\mathbb{F}_q)$ and they are given by $C_n$, $0 \leq n \leq q-2$.

6. Minimal faithful permutation representations of $\text{GL}_3(\mathbb{F}_q)$

In this section we will be proving

Theorem 6.1. We have

$$p(\text{GL}_3(\mathbb{F}_q)) = p(\text{SL}_3(\mathbb{F}_q)) + T_{3,q}(q-1).$$

For $0 \leq n \leq q-2$ set

$$C_n = \{GH(n), \text{GL}_3(\mathbb{F}_q)^{p_1^{e_1}}, \ldots, \text{GL}_3(\mathbb{F}_q)^{p_k^{e_k}}\}.$$

Then up to conjugacy we have $q-1$ classes of minimal faithful sets for $\text{GL}_3(\mathbb{F}_q)$ and they are given by the sets $C_n$, $0 \leq n \leq q-2$.

We use notation as per Lemma 4.5, so we let $C = \{H_1, \ldots, H_\ell\}$ be a minimal faithful collection of $\text{GL}_3(\mathbb{F}_q)$ and let $H_1$ be such that $H_1 \cap \text{SL}_n(\mathbb{F}_q)$ doesn't contain the element of $Z(\text{SL}_n(\mathbb{F}_q))$ of order 3 if $3 \mid q-1$. Note, such $H_1$ is unique by Lemma 4.5.
Lemma 6.2. The subgroup \( H_1 \cap \text{SL}_3(\mathbb{F}_q) \) is a conjugate of \( G_3 \) in \( \text{SL}_3(\mathbb{F}_q) \), when \( g = 3 \) i.e. \( 3 | q - 1 \).

In the case of \( g = 1 \) we have that \( H_1 \cap \text{SL}_3(\mathbb{F}_q) \) is a conjugate of the subgroup \( E_2^q : \text{GL}_2(\mathbb{F}_q) \) of class \( C_1 \).

Proof.

\[
|H_1 : \text{SL}_3(\mathbb{F}_q)| = \frac{|H_1| \cdot |\text{SL}_3(\mathbb{F}_q)|}{|H_1 \cap \text{SL}_3(\mathbb{F}_q)|} = |\text{GL}_3(\mathbb{F}_q)|.
\]

We want the index of \( H_1 \) in \( \text{GL}_3(\mathbb{F}_q) \) i.e. \( \frac{|\text{GL}_3(\mathbb{F}_q)|}{|H_1|} \) to be minimized i.e. \( \frac{|\text{SL}_3(\mathbb{F}_q)|}{|H_1 \cap \text{SL}_3(\mathbb{F}_q)|} \) to be minimized. \( H_1 \cap \text{SL}_3(\mathbb{F}_q) \) is then the maximal core free subgroup of \( \text{SL}_3(\mathbb{F}_q) \), which we already showed is given by \( G_3 \). Similarly, in the case of \( g = 1 \) we have that \( H_1 \cap \text{SL}_3(\mathbb{F}_q) \) is the maximal core free subgroup of \( \text{SL}_3(\mathbb{F}_q) \), given by \( E_2^q : \text{GL}_2(\mathbb{F}_q) \).

Without loss of generality we may assume \( H_1 \cap \text{SL}_3(\mathbb{F}_q) = G_3 \) for \( g = 3 \) and \( H_1 \cap \text{SL}_3(\mathbb{F}_q) = E_2^q : \text{GL}_2(\mathbb{F}_q) \) for \( g = 1 \).

For \( n, 0 \leq n < q - 1 \), define a subgroup \( D(n) \subset \text{GL}_n(\mathbb{F}_q) \) by

\[
D(n) = \{ \begin{pmatrix} a^{n+1} & a^1 & a^{-n} \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} | a \in \mathbb{F}_q^\times \}.
\]

Set

\[
GH(n) = D(n) \cdot G_3,
\]

when \( g = 3 \) and

\[
GH(n) = D(n) \cdot E_2^q : \text{GL}_2(\mathbb{F}_q)
\]

when \( g = 1 \).

Lemma 6.3. There is an \( n, 0 \leq n < q - 1 \), such that \( H_1 = GH(n) \).

Proof. Observe that \( H_1 \cap \text{SL}_3(\mathbb{F}_q) \) is normal in \( H_1 \). By Lemma 6.2, \( H_1 \cap \text{SL}_3(\mathbb{F}_q) \) consists of matrices with vanishing entries in their last row's first column and second column and contains all upper triangular unipotent matrices. Suppose \( \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in H_1 \), and let \( \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) be an arbitrary upper triangular unipotent matrix. Then since the matrix

\[
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}
\]
for all $x$, we must have $dh = eg$ if either of $g$ or $h$ are non-zero.

By symmetric considerations letting \[
\begin{pmatrix}
1 & x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
be an arbitrary upper triangular unipotent matrix we obtain
\[
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix}^{-1}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & x \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix}
\]
\[
= \begin{pmatrix}
* & * & * \\
* & * & * \\
xg(bg - ah)/* & xh(bg - ah)/* & *
\end{pmatrix}
\]
for all $x$. Hence, $bg = ah$ if either of $d$ or $e$ are non-zero. In totality we have $dh = eg$ and $bg = ah$ if either of $g$ or $h$ are non-zero. However, in this scenario assuming $g \neq 0$ without loss of generality, $e = \frac{dg}{g}$ and $b = \frac{ah}{g}$, so $ea - bd = \frac{dha}{g} - \frac{dha}{g} = 0$. This implies that the determinant of the original matrix
\[
\begin{pmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{pmatrix}
\]
vanishes. Contradiction! Hence, both $g$ and $h$ equal 0. Now we proceed to identify $H_1$ explicitly. By the proof of Lemma 4.2 and the statements of Lemmas 4.5 and 6.2 we have
\[
[GL_3(\mathbb{F}_q) : H_1] = [SL_3(\mathbb{F}_q) : G_3].
\]
This means $|H_1| = (q-1) \cdot |G_3|$. So we need to find $(q-1)$ representatives for the quotient $H_1/G_3$. By Lemma 4.5, the determinant $det : H_1 \to \mathbb{F}_q^\times$ is surjective. In particular, if $\varpi$ is a generator of $\mathbb{F}_q^\times$, there is a matrix $X$ in $H_1$, with vanishing entries in its first column’s second row and third row by the first part, such that $det X = \varpi$. Since by Lemma 6.2, $H_1$ contains all upper triangular unipotent matrices, we may assume that $X$ is diagonal. Since $\varpi$ is a generator of $\mathbb{F}_q^\times$, we may write $X = \begin{pmatrix} \varpi^{n+1} & \varpi^m & \varpi^k \\ \varpi^{k+1} & \varpi^{-n-1} & \varpi^1 \end{pmatrix}$. Since $\varpi = det X = \varpi^{n+m+k+1}$, we conclude $n + m + k \equiv 0 \mod (q-1)$, or $m + k \equiv -n$. So if we let $X_n = \begin{pmatrix} \varpi^{n+1} & \varpi^{-n-1} \\ \varpi^1 & \varpi^1 \end{pmatrix}$, then $X_n \in H_1$ and $det X_n = \varpi$. The elements $\{X_n^i \mid 0 \leq i < q-1\}$ provide the $(q-1)$ representatives for
$H_1/G_3$ that we need.
Note, when $g = 1$ replacing $G_3$ by $E_q^2 : GL_2(\mathbb{F}_q)$ and applying the same reasoning as above gives us the desired conclusion. □

**Corollary 6.4** (From the proof). We have

$$[GL_3(\mathbb{F}_q) : H_1] = (q - 1) \cdot p(SL_3(\mathbb{F}_q)).$$

**Lemma 6.5.**

$$\text{core}_{GL_3(\mathbb{F}_q)}(H_1) = Z^{3^r}.$$  
where $r$ is the highest power of 3 dividing $q - 1$.

**Proof.** By Lemma 6.3 it suffices to prove $\text{core}_{GL_n(\mathbb{F}_q)}(GH(n)) = Z^{3^r}$ for each $n$, and that means we need to determine $Z \cap GH(n)$. Suppose we have an element of the form

$$t = \begin{pmatrix} a^{n+1} & a^{-n-1} \\ a & b^{-2} \\ 1 & b \end{pmatrix}, \quad a \in \mathbb{F}_q^\times, b \in A_3.$$  
and suppose $t \in Z$. This means $a^{2n+2} = b^3$ and $a^n = b^3$. Write $a = \varpi^i, b = \varpi^{3^r \cdot j}$. Then we have

$$(2n + 2) \cdot i \equiv n \cdot i \equiv 3^{r+1}j \mod (q - 1).$$

Let $g = \gcd(2n + 2, q - 1)$. Then $j = gu$ for some $u$. Then if $k$ is a multiplicative inverse of $(2n + 2)/g$ modulo $(q - 1)/g$, we have

$$i \equiv k \cdot 3^{r+1} \cdot u \mod \frac{q - 1}{g},$$

or

$$i = k \cdot 3^{r+1} \cdot u + \frac{q - 1}{g} s$$

for some $s$. So if for any $u, s$ we set $a = \varpi^i, b = \varpi^{3^r \cdot j}$ with $i, j$ satisfying

$$\begin{align*}
    i &= k \cdot 3^{r+1} \cdot u + \frac{q - 1}{g} s, \\
    j &= ug,
\end{align*}$$

and $q - 1 | (n + 2) \cdot i$ then $a^{2n+2} = a^n = b^3$. Now we examine the matrix $t$. We see that $a^{-n-1} \cdot b$ is equal to $\varpi$ raised to the power

$$(-n - 1)(k \cdot 3^{r+1} \cdot u + \frac{q - 1}{g} s) + u \cdot g \cdot 3^r$$

$$= (3 \cdot (-n - 1) \cdot k \cdot u + u \cdot g)3^r + \frac{q - 1}{g} \cdot s$$

$$= 3^r \cdot \left\{ (3 \cdot (-n - 1) \cdot k + g) \cdot u + \frac{q - 1}{3^r \cdot g} \cdot s \right\}. $$
We will show that

\[(6.2) \quad \gcd(3 \cdot (-n - 1) \cdot k + g, \frac{q - 1}{g}) = 1.\]

We have

\[
\gcd(3 \cdot (-n - 1) \cdot k + g, \frac{q - 1}{g}) = \gcd(2 \cdot (-n - 1) \cdot k + g + (-n - 1) \cdot k, \frac{q - 1}{g})
\]

\[
= \gcd(g \left\{ \frac{2 \cdot (-n - 1) \cdot k}{g} + 1 \right\} + (-n - 1) \cdot k, \frac{q - 1}{g})
\]

\[
= \gcd((-n - 1) \cdot k, \frac{q - 1}{g}) = 1.
\]

In the above computation we have used the fact that \(k\) is multiplicative inverse of \((2n + 2)/g\) modulo \((q - 1)/g\), so \(2 \cdot (-n - 1) \cdot k/g + 1\) is divisible by \((q - 1)/g\). Now that we have established Equation (5.4) we observe that

\[
\gcd(2 \cdot (-n - 1) \cdot k + g, q - 1) = 1.
\]

This means that there are integers \(s, u\) such that the corresponding \(a, b\) satisfy

\[
a^{n+1}b^{-2} = a^{-n-1}b^1 = ab = \rho^{3r},
\]

and that whenever \(a^{n+1}b^{-2} = a^{-n-1}b^1 = ab\) for \(a \in \mathbb{F}_q^\times, b \in A_{3^r}\), then the common value is of the form \(\rho^{f \cdot 3^r}\) for some integer \(f\). This finishes the proof of the lemma. \(\square\)

Our goal is to minimize

\[
[GL_3(\mathbb{F}_q) : H_2] + \cdots + [GL_3(\mathbb{F}_q) : H_\ell].
\]

We use the following corollary to Lemma 4.6

**Corollary 6.6.** Suppose \(t | q - 1\), and \(t \neq q - 1\), then in the case of \(n = 3\), we have \(g = 1\) or 3. So, the subgroup \(H(t)\) of \(GL_3(\mathbb{F}_q)\) with minimal \([GL_3(\mathbb{F}_q) : H]\) among the subgroups that satisfy \(Z \cap H = Z_t\) is given by

\[
H(t) = \begin{cases} GL_3(\mathbb{F}_q)^t & 3 \nmid t; \\ GL_3(\mathbb{F}_q)^{3t} & 3 | t. \end{cases}
\]

Furthermore,

\[
[GL_3(\mathbb{F}_q) : H(t)] = \begin{cases} t & 3 \nmid t; \\ 3t & 3 | t. \end{cases}
\]

To finish the proof of Theorem 5.12 we have to solve the following optimization problem for \(n = q - 1\).
Problem 6.7. Suppose a natural number \( n = 3^r m \) with \( m \) coprime to 3 is given. For a natural number \( t \), set \( \epsilon(t) = (4 + 2(-1)^t)/2 \). Find natural numbers \( \ell, t_2, \ldots, t_\ell \) such that

- \( \text{lcm}(3^r, t_2, \ldots, t_\ell) = n; \)
- \( \sum_{i=2}^{\ell} \epsilon(t_i) t_i \) is minimal.

We call \((t_2, \ldots, t_\ell)\) the optimal choice for \( n \).

Lemma 6.8. Write \( n = 3^r p_1^{e_1} \cdots p_k^{e_k} \) with \( p_i \)'s distinct odd primes. Then the optimal choice for \( n \) is \((p_1^{e_1}, \ldots, p_k^{e_k})\).

Proof. Suppose \((t_2, \ldots, t_\ell)\) is an optimal choice for \( n \). If some \( t_i \) is even, say equal to \( 3s \), replacing \( t_i \) by \( s \) does not change the lcm in the statement Problem 6.7, but decreases the value of \( \sum_i \epsilon(t_i) t_i \). Since \((t_2, \ldots, t_\ell)\) is optimal for \( n \), this means that all of the \( t_i \)'s have to be coprime to 3. Next, write each \( t_i \) as the product of prime powers \( \pi_1^{m_1} \cdots \pi_v^{m_v} \). By Lemma 4.3, \( \sum_j \pi_j^{m_j} \leq t_i \) with equality only when \( v = 1 \). Again, since \((t_2, \ldots, t_\ell)\) is optimal, this means each \( t_i \) is a prime power. It is also clear that if \( i \neq j \), then \( (t_i, t_j) = 1 \), because otherwise \( t_i, t_j \) will be powers of the same prime, and we can throw away the one with smaller exponent. \( \square \)

Putting everything together finishes the proof of Theorem 6.1.

7. General remarks/Future Work

Although we do not have a way to tackle the conjecture for general \( n \) at least for a very divisible prime \( n \) the algorithm described in \S 5 and \S 6 extends to other GL\(_n\)(\( \mathbb{F}_q \)) for small \( n \). We observe that for \( n \) prime we can always isolate a subgroup, in our minimal faithful collection, whose order is coprime to \( g = \text{gcd}(n, q - 1) \). Call this subgroup \( H_1 \). Then, we may analyze \( H_1 \cap \text{SL}_n(\mathbb{F}_q) \) and observe that it’s the largest core free subgroup of \( \text{SL}_n \). To compute it explicitly we use Patton’s result in [16]. Namely, that the maximal subgroup \( P \) of \( \text{SL}_n(\mathbb{F}_q) \) has the form

\[
\begin{pmatrix}
B & x \\
0 & b
\end{pmatrix}
\]

where, \( b^{-1} = \det(B) \in \mathbb{F}_q^\times \), \( x \in \mathbb{F}_q^{n-1} \), \( B \in \text{GL}_{n-1}(\mathbb{F}_q) \) and \( b, x \) arbitrary. Then, at least heuristically, the maximal core free subgroup \( H_1 \) must be a subgroup of \( P \). This forces \( A \) to be the maximal subgroup of \( \text{GL}_{n-1}(\mathbb{F}_q) \), with order coprime to \( g \). We can express \( A \) as \( D \cdot A' \), where \( A' \) is the maximal subgroup of \( \text{SL}_{n-1}(\mathbb{F}_q) \) with order coprime to \( g \). Now we apply Patton’s result again and iterate the procedure. We keep doing this until we obtain a \( 2 \times 2 \) matrix, which we can handle.
by computing the maximal subgroup of $\text{SL}_2(\mathbb{F}_q)$ of order coprime to $g$. The case of general $n$ will require new techniques other than those described in this paper.

References

[1] M. Aschbacher, *On the maximal subgroups of the finite classical groups*, Inv. Math. **76**, 469–514, 1984.

[2] H. Behravesh, *Quasi-permutation representations of $\text{SL}_2(\mathbb{F}_q)$ and $\text{SL}_2(\mathbb{F}_q)$*, Glasgow Mathematical Journal **41**, 393–408, 1999.

[3] Bray, John N., et al., *The Maximal Subgroups of the Low-Dimensional Finite Classical Groups*, (London Mathematical Society Lecture Note Series), Cambridge University Press, 2013.

[4] John J. Cannon, Derek F. Holt, and William R. Unger, *The use of permutation representations in structural computations in large finite matrix groups*, Journal of Symbolic Computation **95**, 26–38, 2019.

[5] Conway, John Horton, *Atlas of finite groups: maximal subgroups and ordinary characters for simple groups*, Oxford University Press, 1985.

[6] Bruce Cooperstein, *Minimal degree for a permutation representation of a classical group*, Israel J. Math. **30**, 215–235, 1978.

[7] M. R. Darafsheh, M. Ghorbany, A. Daneshkhah, and H. Behravesh, *Quasi-permutation representations of the group $\text{GL}_2(\mathbb{F}_q)$*, Journal of Algebra **243**, 142–167, 2001.

[8] D. Easdown, Michael Hendriksen, *Minimal permutation representations of semidirect products of groups*, Journal of Group Theory **19**, 1017–1048, 2016.

[9] B. Elias, *Minimally faithful group actions and $p$-groups*, Princeton University Senior Thesis, 2005.

[10] B. Elias, L. Silberman, and R. Takloo-Bighash, *Minimal permutation representations of nilpotent groups*, Experiment. Math. **19**, no. 1, 121–12, 2010.

[11] Graves, Christina; Graves, Stephen J.; Lauderdale, L.-K. *Vertex-minimal graphs with dihedral symmetry I*. Discrete Math. 340 (2017), no. 10, 2573–2588.

[12] M. A. Grechkoseeva, *On minimal permutation representations of classical simple groups*. Siberian Math. J. **44** (2003), no. 3, 443–462.

[13] S. Guest, J. Morris, C. E. Praeger, and P. Spiga, *On the maximum orders of elements of finite almost simple groups and primitive permutation groups*, Trans. Amer. Math. Soc. **367**, 7665–7694, 2015.

[14] D. L. Johnson, *Minimal permutation representations of finite groups*, Amer. J. Math. **93**, 857–866, 1971.

[15] V. D. Mazurov, *Minimal permutation representations of finite simple classical groups*. Special linear, symplectic, and unitary groups, Algebra and Logic **32** (1993), no. 3, 142–153 (1994).

[16] W. H. Patton, *The Minimum Index For Subgroups In Some Classical Groups: A Generalization Of A Theorem Of Galois*, University of Illinois at Chicago P.h.D. Thesis, 1972.

[17] Jacques Thévenaz, *Maximal Subgroups of Direct Products*, Journal of Algebra **198**, 352 – 361, 1997.
[18] A. V. Vasil’ev, *Minimal permutation representations of finite simple exceptional groups of types $G_2$ and $F_4* (Russian, with Russian summary), Algebra i Logika 35 (1996), no. 6, 663–684, 752, DOI 10.1007/BF02366397; English transl., Algebra and Logic 35 (1996), no. 6, 371–383. MR1454682 (98d:20055)

[19] A. V. Vasil’ev, *Minimal permutation representations of finite simple exceptional groups of types $E_6$, $E_7$ and $E_8* (Russian, with Russian summary), Algebra i Logika 36 (1997), no. 5, 518–530, 599–600, DOI 10.1007/BF02671607; English transl., Algebra and Logic 36 (1997), no. 5, 302–310. MR1601169 (98m:20055)

[20] A. V. Vasil’ev, *Minimal permutation representations of finite simple exceptional groups of twisted type* (Russian, with Russian summary), Algebra i Logika 37 (1998), no. 1, 17–35, 122, DOI 10.1007/BF02684081; English transl., Algebra and Logic 37 (1998), no. 1, 9–20. MR1672901 (99m:20115)

[21] A. V. Vasil’ev and V. D. Mazurov, *Minimal permutation representations of finite simple orthogonal groups* (Russian, with Russian summary), Algebra i Logika 33 (1994), no. 6, 603–627, 716, DOI 10.1007/BF00756348; English transl., Algebra and Logic 33 (1994), no. 6, 337–350 (1995). MR1347262 (96i:20063)

[22] W. J. Wong, *Linear groups analogous to permutation groups*, J. Austral. Math. Soc. Ser. A 3 (1963), 180–184.

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 S. Morgan Str, Chicago, IL 60607, USA

Email address: nb4296@princeton.edu

Email address: rtakloo@uic.edu