Spacetime and the Holographic Renormalization Group

Vijay Balasubramanian\textsuperscript{1,2} and Per Kraus\textsuperscript{3}\textsuperscript{†}

\textsuperscript{1}Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA
\textsuperscript{2}Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA
\textsuperscript{3}Enrico Fermi Institute, University of Chicago, Chicago, IL 60637, USA

Abstract

Anti-de Sitter (AdS) space can be foliated by a family of nested surfaces homeomorphic to the boundary of the space. We propose a holographic correspondence between theories living on each surface in the foliation and quantum gravity in the enclosed volume. The flow of observables between our “interior” theories is described by a renormalization group equation. The dependence of these flows on the foliation of space encodes bulk geometry.

1 Introduction

The holographic principle \cite{1} states that quantum gravity on a manifold can be described by a theory defined on the boundary of that manifold. The simplest realization of this principle has been in AdS space, which, in certain cases, can be described by a local conformal field theory (CFT) defined on the AdS boundary \cite{2}. The correlation functions of the CFT describe the experiments of an observer who prepares field configurations at infinity and measures their amplitudes.

A strong version of the holographic principle would assert that quantum gravity on any volume contained within a manifold can be described by a theory defined on the

\textsuperscript{*}vijayb@pauli.harvard.edu
\textsuperscript{†}pkraus@theory.uchicago.edu
boundary of that volume. The holographic dual would then describe experiments of an observer who prepares field configurations on the interior boundary and measures their amplitudes.

However, such an interior holographic dual within AdS cannot be a local theory. To see this, assume that the boundary theory is local, and that bulk objects near the boundary correspond to local excitations in the dual. Then, imagine sending a light ray radially through AdS$_{d+1}$, which has a metric:

$$ds^2 = -(1 + r^2/\ell^2)\, dt^2 + (1 + r^2/\ell^2)^{-1} \, dr^2 + r^2 \, d\Omega_{d-1}^2.$$  (1)

The time taken for a light ray to propagate from a point on the sphere at fixed $r$ to the antipodal point is:

$$t_{\text{bulk}} = 2 \int_0^r \frac{dr}{1 + r^2/\ell^2} = 2\ell \tan^{-1}(r/\ell) = \pi\ell - \frac{2\ell^2}{r} + O(r^{-2})$$  (2)

$$t_{\text{bndy}} = \frac{\pi r}{\sqrt{1 + r^2/\ell^2}} = \pi\ell - \frac{\pi \ell^3}{2r^2} + O(r^{-3}).$$  (3)

As noted in [5], in the large $r$ limit the bulk and boundary propagation times are equal, indicating the potential consistency of a local holographic description. But when $r$ is finite, $t_{\text{bndy}} > t_{\text{bulk}}$, so that nonlocal boundary dynamics will be necessary to yield the same arrival times. This also tells us that a holographic description of flat space should be nonlocal since taking $\ell \to \infty$ at fixed $r$ yields $t_{\text{bndy}} = \frac{\pi}{2}t_{\text{bulk}}$.

In this note, we address the issue of interior holographic duals for AdS by adopting a Wilsonian renormalization group (RG) perspective. To describe a subset of a system we “integrate out” the excluded degrees of freedom. In general this will induce an infinite set of interactions in the remaining theory, making it nonlocal. In the AdS context, we foliate spacetime by surfaces $\partial M_\rho$ of constant radial coordinate $\rho$, with enclosed volume $M_\rho$. We fix the values of the fields $\Phi_\rho$ on $\partial M_\rho$ and perform the bulk path integral over the excluded volume. The result is a nonlocal functional of $\Phi_\rho$ which we treat as a boundary contribution to the bulk action describing $M_\rho$. Responses of the resulting interior path integral to variations of $\Phi_\rho$ describe experiments carried out by observers placed on $\partial M_\rho$. We identify these responses with the correlation functions of a holographic dual defined on the interior boundary. Related work has appeared recently in [6]. For some other discussions of RG equations in the AdS/CFT context, see [7].

The observer at $\partial M_\rho$ naturally probes the interior volume with pointlike variations of the fields $\Phi_\rho$. In the semiclassical limit, the bulk equations of motion tell us that these variations turn into extended variations of the fields at infinity (see, e.g., [8] and references therein). This spreading of the fields increases as $\partial M_\rho$ is moved into the interior. In the CFT dual, these boundary values of bulk fields map onto sources smeared over a characteristic scale specified by the the position of the inner
boundary. It is then appropriate to integrate out CFT degrees of freedom at lengths shorter than this scale. This suggests that the interior holographic theories described above are related to the CFT duals of AdS spaces by coarsening transformations. We will demonstrate that this is the case and show that, for any nested family of foliating surfaces for AdS, there is an RG equation describing the flow of observables in the corresponding series of interior holographic duals. Spacetime diffeomorphisms relate foliating families and are realized as relations between different flows.

2 Defining The Inner Correspondence

We will consider Euclidean AdS, which is topologically a ball. Foliate AdS by a family of topologically spherical surfaces indexed by a parameter \(\rho\) approaching 0 at the boundary and \(\infty\) at the center. Let \(\partial M_{\rho}\) be any element of this foliating family, with \(M_{\rho}\) being the enclosed volume. The AdS/CFT correspondence for the boundary at \(\rho = 0\) is written as \([3, 4]\):

\[
e^{-Z_0[\Phi_0]} = \int_{M_0} D\Phi e^{-S_0[\Phi]} = \left< e^{-\int_{\partial M_0} \Phi_0 O} \right> = e^{-S_{\text{CFT}}(\Phi_0)}
\] (4)

The two terms on the left represent the string theory path integral on AdS evaluated as a functional of the boundary data \(\Phi_0\). On the right hand side is the effective action for the dual conformal field theory defined on the boundary manifold \(\partial M_0\) in the presence of sources \(\Phi_0\). The spacetime action \(S_0\) contains both bulk and boundary contributions:

\[
S_0[\Phi] = \int_{M_0} \mathcal{L}[\Phi] + \int_{\partial M_0} B_0[\Phi_0].
\] (5)

where the boundary terms are chosen to cancel divergences arising from the bulk integral (see, e.g., \([4]\)). Upon performing the bulk path integral, \(Z_0[\Phi_0]\) becomes a functional of \(\Phi_0\) defined on \(\partial M_0\). Since the conformal factor on the boundary of AdS actually diverges, it is convenient to cut off the space at some small \(\rho = \epsilon\), which can be understood as a kind of ultraviolet regulator for the CFT \([10]\). (We will always take \(\epsilon \to 0\) in the end). We write:

\[
Z_\epsilon[\Phi_\epsilon] = \sum_{n=1}^{\infty} \int_{\partial M_\epsilon} \left[ \prod_{j=1}^{n} d\mathbf{b}_j \sqrt{\gamma_\epsilon(\mathbf{b}_j)} \Phi_\epsilon(\mathbf{b}_j) \right] c_n(\epsilon; \mathbf{b}_1 \cdots \mathbf{b}_n)
\] (6)

Here \(\mathbf{b}\) are boundary coordinates and \(\gamma_\epsilon\) is the determinant of the induced metric on \(\partial M_\epsilon\). The boundary term \(B_0\) in (3), when restricted to the surface \(\partial M_\epsilon\), eliminates various contact terms that would otherwise make the expansion singular as \(\epsilon \to 0\) \([1]\). The correlation functions of the dual CFT are precisely the coefficients \(c_n\) in the \(\epsilon \to 0\) limit.

We are interested in defining a suitable *inner correspondence* between quantum gravity on \(M_\rho\) and some theory defined on the boundary \(\partial M_\rho\). In the field theory
limit we would like an equation analogous to (4):

\[ e^{-Z_\rho[\Phi_\rho]} = \int_{M_\rho} \mathcal{D}\Phi e^{-S_\rho[\Phi]} = e^{-S_{\text{CFT}}(\Phi_\rho)}. \] (7)

Consider an observer stationed on \( \partial M_\rho \). Such an observer can probe physics in the region \( M_\rho \) by measuring the amplitudes for various field configurations \( \Phi_\rho \) to occur. The amplitudes are given by the path integral in the full AdS spacetime subject to the boundary condition that \( \Phi = \Phi_\rho \) on \( \partial M_\rho \). It is convenient to perform the path integral in two steps. First, integrate over fields in the excluded volume \( M_0 - M_\rho \) to get a nonlocal functional of \( \Phi_\rho \):

\[ e^{-Z_\rho[\Phi_\rho]} = \int_{M_\rho} \mathcal{D}\Phi \int_{M_0 - M_\rho} \mathcal{D}\Phi_0 e^{-S_0[\Phi]} = \int_{M_\rho} \mathcal{D}\Phi e^{-S_\rho[\Phi]} \] (8)

where

\[ S_\rho[\Phi] = \int_{M_\rho} \mathcal{L}[\Phi] + \int_{\partial M_\rho} B_\rho[\Phi]. \] (9)

\( S_\rho[\Phi] \) encapsulates the physics in \( M_\rho \).

The virtue of first integrating over the bulk fields in the excluded volume is that we can envision doing the analogous procedure in the gauge theory. Roughly speaking, fields \( \Phi_\rho \) correspond to smeared fields \( \Phi_0 \) at the outer boundary, and hence to smeared sources in the gauge theory. In the CFT it is then natural to form an effective action by integrating over field modes with wavelengths shorter than the smearing length. Matters will be made more concrete shortly when we consider the semiclassical limit.

To compute bulk correlation functions on \( \partial M_\rho \) we perform the remaining path integral over \( M_\rho \) to obtain a functional of \( \Phi_\rho \),

\[ Z_\rho[\Phi_\rho] = \sum_{n=1}^{\infty} \int_{\partial M_\rho} \left[ \prod_{j=1}^{n} db_j \sqrt{\gamma_\rho(b_j)} \Phi_\rho(b_j) \right] c_n(\rho; b_1 \cdots b_n). \] (10)

We have obtained a one parameter set of correlation functions \( c_n(\rho; b_1 \cdots b_n) \) indexed by \( \rho \) which, by construction, reduce to those in (6) as \( \rho \to \epsilon \). The dependence on \( \rho \) is naturally interpreted as the renormalization group evolution of the correlation functions.

### 2.1 Semiclassical correspondence

In the semiclassical, small curvature, limit the bulk path integral for the “outer correspondence” (4) is dominated by its saddlepoints. So, in the corresponding limit of the dual CFT, (4) becomes

\[ e^{-S_{\text{cl}}(\Phi_0)} = e^{-S_{\text{CFT}}(\Phi_0)}. \] (11)

The left hand side is now simply the AdS classical action (5) evaluated as a functional of boundary data. We might have expected multiple saddlepoints to contribute, but,
in Euclidean signature, and at least in the nearly free limit, demanding regularity for supergravity fields uniquely specifies classical solutions given the boundary data. This even applies to the metric when the boundary data specifies a conformal structure sufficiently close to the round one (see discussion and references in [4]). If we admit black hole spacetimes the topology at infinity is no longer a sphere — there is also a circle whose periodicity must be chosen to achieve regularity of the solution at the origin. Again, the boundary conditions uniquely select the classical solution. Multiple solutions can exist for fixed boundary conditions if we admit different bulk topologies. For the present we will neglect the matter of summing over these solutions since they do not arise in the analysis of pure AdS.

To define the “inner correspondence” in the field theory limit we simply integrated over the fields in the excluded volume $M_0 - M_\rho$. In the semiclassical limit this amounts to evaluating the action for a classical solution in the excluded volume with fields taking values $\Phi_\rho$ at the inner boundary. Again, there is a unique solution in the bulk with the prescribed boundary conditions.

We can compute the full bulk action associated with classical solutions and express it in terms of either the fields $\Phi_\epsilon$ or $\Phi_\rho$:

\[
S_{cl} = \sum_{n=1}^{\infty} \int_{\partial M_\epsilon} \prod_{j=1}^{n} db_j \sqrt{\gamma_\epsilon(b_j)} \Phi_\epsilon(b_j) c_n(\epsilon; b_1 \cdots b_n) \tag{12}
\]

\[
= \sum_{n=1}^{\infty} \int_{\partial M_\rho} \prod_{j=1}^{n} db_j \sqrt{\gamma_\rho(b_j)} \Phi_\rho(b_j) c_n(\rho; b_1 \cdots b_n).
\]

To derive an RG equation we must relate the correlation functions at $\rho$ to those at $\epsilon$. Such a relation is found by noting, as above, that the classical fields $\Phi_\epsilon$ are uniquely specified by $\Phi_\rho$. We display this relation in terms of a propagator:

\[
\Phi_\epsilon(b) = \int_{\partial M_\rho} \sqrt{\gamma_\rho(b')} G_{\epsilon\rho}(b, b') \Phi_\rho(b'). \tag{13}
\]

### 2.2 The meaning of our construction

The meaning of our construction is most easily grasped by considering two point functions in the inner and outer theories. In the semiclassical limit, the outer CFT two point function for widely separated operators is computed from a classical bulk geodesic between two boundary points. Our procedure for computing inner two point functions amounts to extending the geodesic between two interior points until they reach the outer boundary, and adding in the action for the excluded part of the trajectory. Since the geodesics spread on the way from the interior boundary to the exterior, interior correlators at one separation are given by exterior correlators at a larger separation. More concretely, consider AdS in Poincaré coordinates:

\[
ds^2 = \frac{\ell^2}{\rho^2} (d\rho^2 + db^2) \tag{14}
\]
Consider a scalar field in AdS in a representation of the conformal group with weight $\Delta$. Disturbances of this field on the AdS boundary ($\rho = 0$) propagate to the surface at fixed $\rho$ via the kernel

$$G_{bb} \sim \frac{\rho^\Delta}{(\rho^2 + |b - b'|^2)^\Delta}$$

So a point disturbance at $\partial M_0$ grows to a coordinate size $\rho$ at $\partial M_\rho$. Conversely, a given point on $\partial M_\rho$ is affected by fields within a patch of coordinate size $\rho$ on the outer boundary. Now imagine a observer on $\partial M_\rho$ who probes the system with local sources. In terms of the original CFT, such an observer only has access to sources which are smeared over coordinate size $\rho$. So her experiments can be reproduced by an effective action in which degrees of freedom smaller than $\rho$ have been integrated out — short distance information has been lost. Integrating out degrees of freedom induces an infinite series of higher derivative terms, multiplied by powers of the dimensionful scale $\ell$. If one tries to pass to the flat space limit by sending $\ell \to \infty$, the coefficients of the higher derivative terms diverge, signalling an increasingly nonlocal description.

Our procedure realizes the argument of Susskind and Witten that the degrees of freedom in an interior holographic dual should scale like the boundary area. Consider the surfaces $\partial M_\rho$ and $\partial M_\rho'$ in Poincaré coordinates. We have just argued that they are related to coarsenings of the CFT at infinity at scales $\rho$ and $\rho'$. Let us assume, following Susskind and Witten, that there is a fixed number of degrees of freedom per coarsened cell in the CFT at infinity. Then the ratio of degrees of freedom in the theories on $\partial M_\rho$ and $\partial M_\rho'$ is the ratio of areas, as desired.

An alternative procedure for defining the holographic dual of an interior volume is to simply cut off the interior path integral at some finite boundary. This is unappealing because there are physical processes in which particles emerge from the interior region, propagate in the exterior, and then reenter the interior. These processes are analogous to the virtual effect of massive degrees of freedom in a Wilsonian effective action. In both cases, simply cutting off the theory throws out relevant physics. In our approach, the effect of virtual processes is encoded in the nonlocal boundary terms.

### 3 RG Flow of Observables

We will now show that the flow of observables between our “inner” theories is described by a renormalization group equation. As before, foliate Euclidean AdS by a family of surfaces homeomorphic to the boundary, and let $n^\mu$ be the outward pointing
normal to this family of surfaces. Then, if the spacetime metric is $g_{\mu\nu}$, the induced metric on a given foliating surface is $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. In an adapted coordinate system, with $\rho$ being the radial direction, the metric admits an ADM-like decomposition:

$$
\begin{align*}
  g_{\mu\nu} &= g_{\rho\rho} \, d\rho^2 + \gamma_{ij} \, (db^j + V^j \, d\rho)(db^i + V^i \, d\rho) \\
  n^\mu &= \delta^{\mu\rho} / \sqrt{g_{\rho\rho}}
\end{align*}
$$

Then, from (12) and (13) we find that

$$
c_n(\rho; b_1 \cdots b_n) = \int_{\partial M} \left[ \prod_{j=1}^n \int_{\partial \mathcal{M}_\epsilon} d\rho' \, \sqrt{\gamma_{\epsilon}(b_j')} \, G_{\epsilon\rho}(b_j', b_j) \right] c_n(\epsilon; b'_1 \cdots b'_n) \quad (17)
$$

We have just learned that the observables of the “inner” theory are precisely the “outer” CFT correlators convolved against the kernel $G_{\epsilon\rho}$. To make progress, consider situations where we can undo the convolution by an integral transform. For example, if the metric on $\partial M_\epsilon$ is proportional to the identity, the Fourier transform converts the convolution into a product. We will therefore refer to $c_n$ in the deconvolved basis as the “momentum space” correlator $\tilde{c}_n$,

$$
c_n(\rho; b_1 \cdots b_n) = \tilde{G}_{\epsilon\rho}(b_1, k_1) \cdots \tilde{G}_{\epsilon\rho}(b_n, k_n) \tilde{c}_n(\epsilon; k_1 \cdots k_n) \quad (18)
$$

Here the variables $k$ parametrize the deconvolution basis. The correlator $\tilde{c}_n(\epsilon; \cdots)$ is independent of the index $\rho$ of the interior surface. So the $\rho$ dependence of the inner observables is summarized by:

$$
n^\mu \nabla_\mu c_n(\rho; b_1 \cdots b_n) + \left[ \sum_j n^\mu \nabla_\mu \ln \tilde{G}_{\epsilon\rho}(b_j, k_j) \right] c_n(\rho; b_1 \cdots b_n) = 0 \quad (19)
$$

where $n^\mu$ is the normal vector to $\partial \mathcal{M}_\rho$. (13) is an RG equation describing Wilsonian flow of correlators in the gauge theory, in correspondence with the observations of spacetime observers stationed on the fixed surfaces $\partial \mathcal{M}_\rho$. The gradient operator acts on the coordinates $b$ as well as on $\rho$. This equation looks unfamiliar because it has been written for a general foliation of AdS. We will see below that in Poincaré coordinates it displays all the expected features of Wilsonian renormalization group flow.

3.1 Example: Poincaré coordinates

In Poincaré coordinates the metric of AdS is:

$$
ds^2 = \frac{\ell^2}{\rho^2} (d\rho^2 + db^2)
$$

7
and we are interested in surfaces of fixed $\rho$. We will work out the relation between inner and outer observables for massive scalars. In AdS$_{d+1}$ the operator dual to such a scalar has dimension $\Delta$:

$$\Delta = \frac{d}{2} + \nu, \quad \nu = \frac{1}{2}\sqrt{d^2 + 4m^2}. \quad (21)$$

To Fourier transform both sides of (17) it is convenient to define the inner and outer correlators in momentum space:

$$\tilde{c}_n(\rho; k_1 \cdots k_n) = \int_{\partial M_\rho} \left[ \prod_{j=1}^n \gamma^j \sqrt{\gamma^j} \right] c_n(\rho; b_1 \cdots b_n) \quad (22)$$

Next, since the propagator $G_{\epsilon \rho}$ approaches $\delta(b - b')/\sqrt{\gamma(\rho)}$ as $\epsilon \to \rho$, the Fourier transform with respect to $b'$ gives $\tilde{G}_{\epsilon \rho}(b, k) = e^{i k \cdot b}$. It is easy to construct a massive scalar mode solution that approaches such a plane wave on $\partial M_\rho$ from the complete bases provided in, e.g., [11]. The propagator is then a Bessel function:

$$\tilde{G}_{\epsilon \rho}(b, k) = \left( \frac{\epsilon}{\rho} \right)^{d/2} \left( \frac{K_\nu(\epsilon)}{K_\nu(\rho \epsilon)} \right) e^{i k \cdot b} \quad (23)$$

with $q^2 = k \cdot k$. This gives:

$$\tilde{c}_n(\rho; k_1 \cdots k_n) = \left( \frac{\rho}{\epsilon} \right)^{-n d/2} \left( \prod_j K_\nu(q_j \epsilon) \right) \tilde{c}_n(\epsilon; k_1 \cdots k_n) \quad (24)$$

To gain further insight we need the power series expansion of the Bessel function:

$$K_\nu(z) \propto z^{-\nu}[1 + F(z^2)] \quad (25)$$

$$F(z^2) = \sum_{n=0}^{\infty} e_{-\nu}(n) z^{2n - \nu} \sum_{n=0}^{\infty} e_{\nu}(n) z^{2n}$$

(For purposes of argument we have have taken $\nu$ to be generic – when $\nu$ is integral the expansion also involves logarithmic terms.) Using this in (24) we find:

$$\tilde{c}_n(\rho; k_1 \cdots k_n) = \left( \frac{\rho}{\epsilon} \right)^{n(\Delta - d)} \left[ \prod_j \frac{1 + F(q_j^2 \epsilon^2)}{1 + F(q_j^2 \rho^2)} \right] \tilde{c}_n(\epsilon; k_1 \cdots k_n) \quad (26)$$

We implicitly understood all along that $\epsilon \to 0$. Since the theory is conformally invariant, this limit yields the scaling behaviour

$$\tilde{c}_n(\epsilon; k_1 \cdots k_n) = \epsilon^{n(\Delta - d)} \tilde{c}_n(k_1 \cdots k_n) \quad (27)$$

\footnote{The precise coefficients $e_\nu$ are not important for us.}
where \( \tilde{c}_n \) is finite. Rearranging terms, the inner correlator becomes

\[
\rho^{-n(\Delta-d)} \left[ \prod_j (1 + F(q_j^2 \rho^2)) \right] \tilde{c}_n(\rho; k_1 \cdots k_n) = \tilde{c}_n(k_1 \cdots k_n) \tag{28}
\]

First consider \( (q_j \rho) \ll 1 \) for all \( j \). Then the interior correlators at \( \rho \) and \( \rho' \) are related by a rescaling \( (\rho/\rho')^n(\Delta-d) \). This is exactly the behaviour expected for low energy correlation functions in a Wilsonian effective treatment. We argued in Sec. 2 that the observables on the surface \( \partial M_\rho \) in Poincaré coordinates were smeared at a scale \( \rho \). A Wilsonian treatment requires a rescaling of coordinates to keep the numerical size of the cutoff fixed. Precisely this effect is achieved by the Weyl factor in the metric on \( \partial M_\rho \) which keeps the proper size of the smearing fixed. This in turn results in scaling of the correlators as we flow inwards (to the infrared).

More generally, since \( \tilde{c} \) on the right hand side of (28) is independent of \( \rho \), we have an RG equation:

\[
\left[ \rho \frac{\partial}{\partial \rho} - n(\Delta-d) \right] \tilde{c}_n(\rho; k_1 \cdots k_n) + \left[ \sum_j \rho \frac{\partial}{\partial \rho} \ln[1 + F(q_j^2 \rho^2)] \right] \tilde{c}_n(\rho; k_1 \cdots k_n) = 0
\]

When all the momenta \( q \) are small, the second term vanishes and, as expected, we have the RG equation for pure scaling of infrared Wilsonian correlators. Violations of scaling appear in the second term and are suppressed at low momenta. Transforming back into position space gives:

\[
\left[ \rho \frac{\partial}{\partial \rho} - n(\Delta-d) \right] c_n + \left[ \sum_j \rho \frac{\partial}{\partial \rho} \ln[1 + F(\rho^2 \nabla_i^2)] \right] c_n = 0 \tag{30}
\]

### 3.2 Bulk Field Equations from CFT?

In the semiclassical limit, the interior effective theories that we have constructed are related to the exterior CFT by a renormalization group transformation, suggesting a direct relation between the bulk field equations and the RG equations in the CFT. This is at first surprising since the bulk field equations are second order while the RG equations are first order. However, there is no real conflict because demanding regularity of the bulk solutions in Euclidean space eliminates one solution, making the equations effectively first order. Related observations have been made in [1].

The connection can be made more explicit by recalling the correspondence between boundary behavior of the bulk fields in \( AdS_{d+1} \) and sources and operators in the gauge theory [3, 4, 11, 12, 13]. Up to a \( \rho \) dependent scaling, sources correspond to the boundary values of bulk fields while operators correspond to their radial derivatives. Schematically:

\[
J \sim \Phi, \quad O \sim \rho \partial_\rho \Phi. \tag{31}
\]
In the CFT, $J$ appears as a coupling to the gauge invariant operator $\mathcal{O}$ of the form $\int J(b)\mathcal{O}(b)$. In (31), $\mathcal{O}$ is understood as the expectation value of the operator. Now consider the structure of the bulk equation for a free scalar field of mass $m$:

$$\nabla^2 - m^2 \Phi(\rho) e^{i\vec{k} \cdot \vec{x}} = 0 \Rightarrow \left[ \rho^2 \partial_\rho^2 + (1 - d) \rho \partial_\rho - \vec{k}^2 \rho^2 - m^2 \right] \Phi(\rho) = 0. \quad (32)$$

If we use the relations (31), the field equation takes the form

$$\left[ \rho \frac{\partial}{\partial \rho} + d_0 \right] \mathcal{O} - \left[ d_1 + d_2 \vec{k}^2 \rho^2 \right] J = 0. \quad (33)$$

Again, we are being schematic — $d_0$, $d_1$, and $d_2$ are constants. The source $J$ is not an independent variable since it determines the expectation value for $\mathcal{O}$. In momentum space, $J$ can be expressed as $\mathcal{O}$ times a function of $\vec{k}^2$. Using this, we find that (33) has the same form as (29) with $n = 1$.

To make this connection precise, various issues such as the scheme dependence of the RG equations must be confronted. Nevertheless, there is reason to hope that the field equations of supergravity can be derived from the CFT via the renormalization group. Work in this direction is in progress.

## 4 Discussion: Geometry and RG Flows

We have argued that there is a natural way to define an “interior” holographic correspondence between physics inside finite volumes $\mathcal{M}_\rho$ and a theory on the boundary $\partial \mathcal{M}_\rho$. The correlation functions of the interior theory are related to the exterior observables by a coarsening transformation. A given family of foliating surfaces then leads to a particular flow of smeared observables summarized by a renormalization group equation. Changing the foliation leads to a different flow. In fact, we are learning that spacetime geometry arises in a holographic context as the geometry of the space of RG flows.

Consider a CFT defined on a plane and a family of theories derived from it by coarsening transformations. Concretely, let $\phi(b)$ be a field in the CFT, and define coarsened fields $\phi(\rho; b)$ by convolving $\phi$ against a kernel $K_\rho$ which has a characteristic scale $\rho$. As $\rho$ increases from 0 to $\infty$, we arrive at a family of smeared theories. In some natural sense there should be a geometry on this “stack” of theories. First of all, a coarsening transformation should be accompanied by a rescaling of lengths, and that is implemented by rescaling the metric of the smeared theories. In addition, we would like a notion of distance or separation between the original CFT and its cousins that depends on the coarsening parameter $\rho$. For the class of kernels inspired by AdS/CFT, we have learned that there is a natural distance, and it is given by the geodesic length between the fixed $\rho$ Poincaré surfaces. In this sense, anti-de Sitter spaces induce a geometry on a certain class of RG flows of the dual CFTs.
In general there is no requirement that a field should be coarsened uniformly. Indeed, it is often convenient in lattice field theory to use meshes of different sizes in different regions. Within the AdS/CFT correspondence this freedom is realized in our ability to pick a general family of foliating surfaces for AdS space. Diffeomorphisms, or transformations between different families of foliating surfaces, are then manifestly realized as transformations between different RG flows. The explicit action of diffeomorphisms on our flows is easy to derive by acting with the generators on the kernel $G_{\epsilon \rho}$.

Several questions arise. First, why is the class of smearing kernels selected by the AdS/CFT correspondence distinguished? After all, there are many more ways to smear field variables than implied by solutions to AdS wave equations. Second, is there a natural geometry on the space of RG flows that can be derived intrinsically from gauge theory considerations? Given two theories with the same set of fields, we can detect the difference between them by computing and comparing correlation functions. It is possible that there is a natural measure of distance between theories that can be derived in this way. The answers to these questions are likely to be intimately related. We are looking for a statement that some classes of coarsening transformations lead to RG flows on which there is a natural geometry. That geometry, in a holographic context, is spacetime.

Acknowledgments:
We thank Steve Giddings, David Gross, Gary Horowitz, Finn Larsen, Miao Li, Emil Martinec, George Minic, Nikita Nekrasov, Joe Polchinski, and Ruud Siebelink for helpful discussions. V.B. is supported by the Harvard Society of Fellows and NSF grants NSF-PHY-9802709 and NSF-PHY-9407194. P.K. is supported by NSF grant PHY-9600697.

References
[1] G. 't Hooft, “Dimensional reduction in quantum gravity”, Salamfest 1993:0284-296, gr-qc/9310026; L. Susskind, “The world as a hologram”, J.Math.Phys.36:6377-6396,1995, hep-th/9409089.

[2] J. Maldacena, “The large N limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys.2:231-252 (1998), hep-th/9711200.

[3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, “Gauge theory correlators from noncritical string theory”, Phys. Lett. B428:105 (1998), hep-th/9802109.

[4] E. Witten, “Anti-de Sitter space and holography”, Adv. Theor. Math. Phys.2:253 (1998), hep-th/9802150.
[5] G. Horowitz, N. Itzhaki, “Black holes, shock waves, and causality in the AdS/CFT correspondence”, hep-th/9901012;
L. Susskind, “Holography in the flat space limit”, hep-th/9901079;
D. Kabat and G. Lifschytz, “Gauge theory origins of supergravity causal structure”, hep-th/9902073;
D. Bak and S.J. Rey, “Holographic view of causality and locality via branes in AdS/CFT correspondence”, hep-th/9902101.

[6] M. Porrati and A. Starinets, “RG fixed points in supergravity duals of 4-d field theory and asymptotically AdS spaces”, hep-th/9903083

[7] E. Alvarez and C. Gomez, “Geometric holography, the renormalization group and the c theorem”, Nucl. Phys. B541 (1999) 441, hep-th/9807220;
E.T. Akhmedov, “A remark on the AdS/CFT correspondence and the renormalization group flow”, Phys. Lett. B442 (1998) 152, hep-th/9806217.

[8] A. Peet and J. Polchinski, “UV/IR relations in AdS dynamics”, Phys. Rev. D18 (1978) 3565, hep-th/9809022

[9] G. Chalmers and K. Schalm, “Holographic normal ordering and multiparticle states in the AdS/CFT correspondence”, hep-th/9901143;
V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity”, hep-th/9902121.

[10] L. Susskind and E. Witten, “The holographic bound in anti-de Sitter space”, hep-th/9805114.

[11] V. Balasubramanian, P. Kraus and A. Lawrence, “Bulk vs. boundary dynamics in anti-de Sitter spacetime”, Phys. Rev. D59, (1999), 046003, hep-th/9805171.

[12] T. Banks, M. Douglas, G. Horowitz, and E. Martinec, “AdS dynamics from conformal field theory”, hep-th/9808016

[13] V. Balasubramanian, P. Kraus, A. Lawrence and S.P. Trivedi, “Holographic probes of anti-de Sitter spacetimes”, hep-th/9808017, to appear in Phys. Rev. D.