Tan Nhat Tran and Akiyoshi Tsuchiya

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Worpitzky-compatible subarrangements of braid arrangements and cocomparability graphs

Tan Nhat Tran\(^*\),\(^a\) and Akiyoshi Tsuchiya\(^b\)

\(^a\)Tan Nhat Tran, Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo 060-0810, Japan
\(^b\)Akiyoshi Tsuchiya, Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

E-mails: trannhattan@math.sci.hokudai.ac.jp, akiyoshi@ms.u-tokyo.ac.jp

Abstract. The class of Worpitzky-compatible subarrangements of a Weyl arrangement together with an associated Eulerian polynomial was recently introduced by Ashraf, Yoshinaga and the first author, which brings the characteristic and Ehrhart quasi-polynomials into one formula. The subarrangements of the braid arrangement, the Weyl arrangement of type \(A\), are known as the graphic arrangements. We prove that the Worpitzky-compatible graphic arrangements are characterized by cocomparability graphs. This can be regarded as a counterpart of the characterization by Stanley and Edelman–Reiner of free and supersolvable graphic arrangements in terms of chordal graphs. Our main result yields new formulas for the chromatic and graphic Eulerian polynomials of cocomparability graphs.

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1. Introduction

Let \(V\) be an \(\ell\)-dimensional Euclidean vector space with the standard inner product \((\cdot,\cdot)\). Let \(\Phi\) be an irreducible (crystallographic) root system in \(V\), with a fixed positive system \(\Phi^+ \subseteq \Phi\) and the associated set of simple roots \(\Delta := \{\alpha_1, \ldots, \alpha_\ell\}\). For \(m \in \mathbb{Z}\) and \(\alpha \in \Phi\), define the affine hyperplane \(H_{\alpha,m}\) by \(H_{\alpha,m} := \{x \in V \mid (\alpha, x) = m\}\). For \(\Psi \subseteq \Phi^+\), the Weyl subarrangement \(\mathcal{A}_\Psi\) is defined by \(\mathcal{A}_\Psi := \{H_{\alpha,0} \mid \alpha \in \Psi\}\). In particular, \(\mathcal{A}_{\Phi^+}\) is called the Weyl arrangement. Define the partial order \(\geq\) on \(\Phi^+\) as follows: \(\beta_1 \geq \beta_2\) if \(\beta_1 - \beta_2 \in \sum_{i=1}^\ell \mathbb{Z}_{\geq 0} \alpha_i\). A subset \(\Psi \subseteq \Phi^+\) is an ideal of
Theorem 4.16. In this paper we prove that, interestingly, the Worpitzky-compatible graphic arrangement is free and supersolvable. Yoshinaga and the first author in the study of characteristic quasi-polynomials of Weyl subarrangements (Worpitzky-)compatible arrangements was introduced by Ashraf, Yoshinaga and the first author in the study of characteristic quasi-polynomials of Weyl subarrangements and Ehrrath theory [2]. It is shown that any ideal subarrangement is free [11]. Various free subarrangements of a Weyl arrangement of type \( B \) were studied, e.g., [6, 13, 23]. Remarkably, a striking result of Abe–Barakat–Cuntz–Hoge–Terao [1] asserts that any ideal subarrangement is free.

Although characterizing free subarrangements of an arbitrary Weyl arrangement is still a challenging problem, in the case of braid arrangement, the free subarrangements can be completely analyzed using the connection to graphs. It follows from the works of Stanley [21] and Edelman–Reiner [6] that free and supersolvable graphic arrangements are synonyms, and they correspond to chordal graphs (every induced cycle in the graph has exactly 3 vertices). Chordal graphs are a superclass of (unit) interval graphs (each vertex can be associated with (a unit) an interval on the real line, and two vertices are adjacent if the associated intervals have a nonempty intersection). More strongly, a graph is an interval graph if and only if it is a cocomparability (Definition 1) and chordal graph [9] (see Figure 1 for an illustration). It should also be noted that the ideal subarrangements of a braid arrangement are parametrized by unit interval graphs (Theorem 16).

Definition 1. A graph is called a comparability graph if its edges can be transitively oriented, i.e., if \( u \to v \) and \( v \to w \), then \( u \to w \). A graph is called a cocomparability graph if its complement is a comparability graph.

Thus, it is natural to ask which class of Weyl subarrangements generalizes the cocomparability graphs. Recently, the notion of (Worpitzky-)compatible arrangements was introduced by Ashraf, Yoshinaga and the first author in the study of characteristic quasi-polynomials of Weyl subarrangements and Ehrrath theory [2]. It is shown that any ideal subarrangement is compatible [2, Theorem 4.16]. In this paper we prove that, interestingly, the Worpitzky-compatible graphic arrangements are characterized by cocomparability graphs, which gives an answer to the aforementioned question.

To state the result formally, we first recall the concept of compatibility. A connected component of \( V \setminus \bigcup_{\alpha \in \Phi^+, m \in \mathbb{Z}} H_{\alpha, m} \) is called an alcove. Let \( A \) be an alcove. A wall of \( A \) is a hyperplane that supports a facet of \( A \). The ceilings of \( A \) are the walls which do not pass through the origin and have the origin on the same side as \( A \). The upper closure \( A^\Diamond \) of \( A \) is the union of \( A \) and its facets.
supported by the ceilings of \( A \). Let \( P^\Diamond := \{ x \in V \mid 0 < (\alpha_i, x) \leq 1 \ (1 \leq i \leq \ell) \} \) be the fundamental parallelepiped (of the coweight lattice) of \( \Phi \). Thus,

\[
P^\Diamond = \bigcup_{A: \text{alcove}, A \subseteq P^\Diamond} A^\Diamond,
\]

which is known as the Worpitzky partition, e.g., [26, Proposition 2.5], [10, Exercise 4.3].

**Definition 2 (cf. [2, Definition 4.8]).** A subset \( \Psi \subseteq \Phi^+ \) is said to be Worpitzky-compatible (or compatible for short) if every nonempty intersection of the upper closure \( A^\Diamond \) of an alcove \( A \subseteq P^\Diamond \) and an affine hyperplane w.r.t. a root in \( \Psi \) can be lifted to a facet intersection. That is, \( A^\Diamond \cap H_{\alpha, m_\alpha} \) for \( \alpha \in \Psi, m_\alpha \in \mathbb{Z} \) either is empty, or is contained in a ceiling \( H_{\beta, m_\beta} \) of \( A \) with \( \beta \in \Psi, m_\beta \in \mathbb{Z} \). If \( \Psi \) is compatible, the Weyl subarrangement \( \mathcal{A}_\Psi \) is said to be compatible as well.

Our main result is the following.

**Theorem 3.** Let \( G = (V, E) \) be a graph with \( |V| = \ell \). Then \( G \) has a labeling using elements from \( \{1, 2, \ldots, \ell\} \) so that \( \mathcal{A}(G) \) is a compatible graphic arrangement if and only if \( G \) is a cocomparability graph.

The cocomparability graph is a combinatorial object while Worpitzky-compatibility is a geometric property. Our main result connects the geometric property of a graphic arrangement with the combinatorial property of the underlying graph. In particular, our main result yields a correspondence between interval graphs and graphic arrangements that are compatible and free. We record the results in Table 1.

**Table 1.** Parallel concepts in type \( A \).

| Graph class      | Weyl subarrangement class       | Location          |
|------------------|---------------------------------|-------------------|
| cocomparability  | compatible (= strongly compatible)| Theorems 3, 9     |
| chordal          | free (= supersolvable)          | [6, 21]           |
| interval         | compatible \( \cap \) free      | Corollary 15      |
| unit interval    | ideal                           | Theorem 16        |

**Figure 1.** Relationship between graph classes: \( U \subseteq I = \text{Ch} \cap \text{Co} \). Co: cocomparability, Ch: chordal, I: interval, U: unit interval.
2. Proof of the main result

First, we introduce a new subclass of Worpitzky-compatible sets, which will play a key role in the proof of our main result.

Definition 4. A subset $\Psi \subseteq \Phi^+$ is said to be strongly (Worpitzky-)compatible if for any $\alpha \in \Psi$ and for every choice of positive roots $\beta_1, \ldots, \beta_m \in \Phi^+$ such that $\alpha \in \sum_{i=1}^m \mathbb{Z}_{>0} \beta_i$, there exists $k$ with $1 \leq k \leq m$ such that $\beta_k \in \Psi$.

The strong Worpitzky-compatibility is a combinatorial property of the root system. This concept was made by inspiration of the following lemma, which is an important result in [2].

Lemma 5. Let $A \subseteq P^\vee$ be an alcove. If there exist $\alpha \in \Phi^+$, $r_\alpha \in \mathbb{Z}$ so that $A^\vee \cap H_{\alpha,r_\alpha}$ is a face of $A^\vee$, where $H_{\beta_j,r_{\beta_j}}$ are the ceilings of $A$, then $\alpha \in \sum_{j=1}^m \mathbb{Z}_{\geq0} \beta_j$.

Proof. See [2, Proof of Theorem 4.16]. □

Let $\mathcal{I}$ be the set of all ideals of $\Phi^+$. In addition, let $\mathcal{C}$ (resp., $\mathcal{SC}$) be the set of all compatible (resp., strongly compatible) sets of $\Phi^+$. We exhibit a relation between these sets (see also Figure 2 for an illustration).

Theorem 6. If $\Phi$ is an irreducible root system, then

$$\mathcal{I} \subseteq \mathcal{SC} \subseteq \mathcal{C}.$$  

Proof. The first inclusion is clear. The second inclusion follows from Lemma 5. □

Remark 7. Example 27 illustrates a strongly compatible set but not an ideal (w.r.t. any positive system of $\Phi$). There exists a compatible set that is not strongly compatible when $\Phi$ is of type $G_2$ (or $B_2$) [2, Example 4.18(d)].

For any alcove $A$ and $\gamma \in \Phi^+$, there exists a unique integer $r$ with $r - 1 < (x, \gamma) < r$ for all $x \in A$. We denote this integer by $r(A, \gamma)$. The function $r(A, -) : \Phi^+ \to \mathbb{Z}$ determines the position of $A$, sometimes, it is called the address of $A$.

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1 We thank Christian Stump and Takuro Abe for pointing out that the notion of strongly compatible subsets is essentially equivalent to that of coclosed of the root system. Recall that a subset $\Psi \subseteq \Phi^+$ is said to be coclosed if for any $\alpha \in \Psi$ and for $\beta_1, \beta_2 \in \Phi^+$ such that $\alpha = \beta_1 + \beta_2$, either $\beta_1 \in \Psi$ or $\beta_2 \in \Psi$. 

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Lemma 8. Suppose that for each $\gamma \in \Phi^+$ we are given a positive integer $r_\gamma$. There is an alcove $A$ with $r(A, \gamma) = r_\gamma$ for all $\gamma \in \Phi^+$ (in other words, the function $r : \Phi^+ \to \mathbb{Z}$ with $r(\gamma) = r_\gamma$ is the address of an alcove $A$) if and only if $r_\gamma + r_{\gamma'} - 1 \leq r_{\gamma + \gamma'} \leq r_\gamma + r_{\gamma'}$ whenever $\gamma, \gamma' \in \Phi^+$.

Proof. This was first proved by Shi in terms of coroots [20, Theorem 5.2]. The statement here is formulated in terms of roots, which can be found in, e.g., [3, Lemma 2.4].

Surprisingly, there is no difference between compatible and strongly compatible sets in the case of type $A$.

Theorem 9. If $\Phi$ is of type $A_\ell$, then every compatible set is strongly compatible, i.e.,

$$\mathcal{F}^c = \mathcal{C}.$$

Proof. Let $\Psi \subseteq \Phi^+$ be a compatible set. Let $\alpha \in \Psi$, and suppose that there are $\beta_1, \ldots, \beta_m \in \Phi^+$ such that $\alpha = \sum_{i=1}^m \beta_i$. Thus, $\alpha = \sum_{i=1}^m \beta_i = \alpha_1 + \alpha_{s+1} + \ldots + \alpha_t$ for some $1 \leq s \leq t \leq \ell$ with $\alpha_j \in \Delta$ for all $s \leq j \leq t$. We want to show that there exists $k$ with $1 \leq k \leq m$ such that $\beta_k \in \Psi$.

For $\beta = \sum_{i=1}^\ell d_i \alpha_i \in \Phi^+$, denote $\text{supp}(\beta) := \{\alpha_i \mid d_i > 0\}$. The above expression of $\alpha$ induces a partition of $\Delta$ as follows:

$$\Delta = S_0 \sqcup S_1 \cup \cdots \cup S_m \sqcup S_{m+1},$$

where $S_0 := \{\alpha_1, \ldots, \alpha_{s-1}\}$, $S_{m+1} := \{\alpha_{t+1}, \ldots, \alpha_\ell\}$, $S_i := \text{supp}(\beta_i)$ for $1 \leq i \leq m$. For each $\gamma \in \Phi^+$, define

$$r_\gamma := \#\{0 \leq i \leq m+1 \mid \text{supp}(\gamma) \cap S_i \neq \emptyset\}.$$

For any $\gamma, \gamma' \in \Phi^+$ with $\gamma + \gamma' \in \Phi^+$, it is not hard to show the following facts:

(a) $r_{\gamma + \gamma'} = r_\gamma + r_{\gamma'}$ if there is no $i$ with $0 \leq i \leq m + 1$ such that $\text{supp}(\gamma) \cap S_i \neq \emptyset$ and $\text{supp}(\gamma') \cap S_i \neq \emptyset$.

(b) $r_{\gamma + \gamma'} = r_\gamma + r_{\gamma'} - 1$, otherwise.

By Lemma 8, there is an alcove $A$ with $r(A, \gamma) = r_\gamma$ for all $\gamma \in \Phi^+$. In particular, $r(A, \alpha) = m$ and $r(A, \beta_i) = 1$ for $1 \leq i \leq m$. Clearly, $A \subseteq P^\circ$.

Fix $\beta_i$ with $1 \leq i \leq m$. We will show that $H_{\beta_i,1}$ is a ceiling of $A$. The method used in [3, Theorem 3.11] or [24, Theorem 3.1] applies here as well. First, the above formulas imply that

(a) $r_{\beta_i + \gamma} = r_\gamma + r_{\beta_i}$ if $\gamma, \beta_i + \gamma \in \Phi^+$ and

(b) $r_{\beta_i} = r_\gamma + r_{\gamma'} - 1$ if $\beta_i = \gamma + \gamma'$ for $\gamma, \gamma' \in \Phi^+$.

Next, we again apply Lemma 8 to find another alcove, called $B$ with $r(B, \beta_i) = r_{\beta_i} + 1 = 2$, $r(B, \gamma) = r_\gamma$ for all $\gamma \in \Phi^+ \setminus \{\beta_i\}$. Comparing the addresses of alcoves $A$ and $B$ implies that $H_{\beta_i,1}$ is a wall, hence a ceiling of $A$. Thus, the intersection $P := \bigcap_{i=1}^m H_{\beta_i,1} \cap A^\circ$ is a non-empty face, moreover, contained in the face $Q := H_{\alpha,m} \cap A^\circ$ of $A^\circ$. Since any proper face of a polytope is the intersection of all facets containing it, if $Q = \bigcap_{\delta \in D} H_{\delta, n_\delta} \cap A^\circ$ where each $H_{\delta, n_\delta}$ is a wall of $A^\circ$ and $D$ is a set of roots, then $D \subseteq \{\beta_1, \ldots, \beta_m\}$ and $n_\delta = 1$ for all $\delta \in D$. Applying Lemma 5 to the face $Q = H_{\alpha,m} \cap A^\circ = \bigcap_{\delta \in D} H_{\delta, n_\delta} \cap A^\circ$ implies that $\alpha = \sum_{\delta \in D'} \delta$ with $D' \subseteq D$. We must have $D' = \{\beta_1, \ldots, \beta_m\}$ and $P = Q$. Since $\Psi$ is a compatible set, by Definition 2, there exists $k$ with $1 \leq k \leq m$ such that $\beta_k \in \Psi$. This completes the proof.

Thanks to Theorem 9, when $\Phi$ is of type $A$, the concept of compatibility originally defined by geometry of alcoves and affine hyperplanes can now be rephrased in terms of combinatorial property of graphs.

Corollary 10. Let $G = ([\ell], \mathcal{E})$ be a graph. The following are equivalent.

(i) $\mathcal{A}(G)$ is a compatible graphic arrangement, i.e., $\Psi(G) \subseteq \Phi^+ (A_{\ell-1})$ is a compatible set.

(ii) $G$ has the following property: if $i < j$, $[i, j] \in \mathcal{E}$ and $(p_1, \ldots, p_m)$ is any sequence such that $i = p_1 < p_2 < \cdots < p_m = j$, then there exists $p_a$ with $1 \leq a < m$ such that $(p_a, p_{a+1}) \in \mathcal{E}$. 

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Proof. The implication (i) ⇒ (ii) is clear. Now we prove (ii) ⇒ (i). Let \( \alpha = \epsilon_i - \epsilon_j \in \Psi(G) \). If there are \( \beta_1, \ldots, \beta_m \in \Phi^+ \) such that \( \alpha = \sum_{i=1}^m \beta_i \), then \( \alpha = \sum_{\beta \in M} \beta \) where \( M = \{ \beta_1, \ldots, \beta_m \} \). Therefore, \( \epsilon_i - \epsilon_j = \sum_{p \in H} \epsilon_p - \sum_{q \in T} \epsilon_q \) for \( H, T \subseteq [\ell] \). Note that \( \epsilon_i - \epsilon_j \geq \epsilon_p - \epsilon_q \) if and only if \( i \leq p < q \leq j \). The independence of \( \{\epsilon_1, \ldots, \epsilon_\ell\} \) implies \( H \setminus T = \{i\} \) and \( T \setminus H = \{j\} \). Thus, there exist \( p_1, \ldots, p_{m+1} \in \mathbb{Z}_{\geq 0} \) with \( i = p_1 < p_2 < \cdots < p_{m+1} = j \) such that \( M = \{ \epsilon_{p_a} - \epsilon_{p_{a+1}} \mid 1 \leq a \leq m \} \). Therefore, there is \( p_a \) such that \( (p_a, p_{a+1}) \in \mathcal{E} \) hence \( \epsilon_{p_a} - \epsilon_{p_{a+1}} \in \Psi(G) \cap M \). This completes the proof. \( \square \)

Definition 11. An ordering \( v_1 < v_2 < \cdots < v_\ell \) of the vertices of a graph \( G \) is an umbrella-free ordering (or a cocomparability ordering) if \( i < k < j \) and \( \{v_i, v_j\} \in \mathcal{E} \), then either \( \{v_i, v_k\} \in \mathcal{E} \) or \( \{v_k, v_j\} \in \mathcal{E} \) or both.

Lemma 12. Let \( G = (V, \mathcal{E}) \) be a graph with \( |V| = \ell \). An ordering \( v_1 < v_2 < \cdots < v_\ell \) of the vertices of \( G \) is an umbrella-free ordering if and only if “<” satisfies the following condition: if \( i < j \), \( \{v_i, v_j\} \in \mathcal{E} \) and \( (p_1, \ldots, p_m) \) is any sequence such that \( i = p_1 < p_2 < \cdots < p_m = j \), then there exists \( p_a \) with \( 1 \leq a < m \) such that \( \{v_{p_a}, v_{p_{a+1}}\} \in \mathcal{E} \).

Proof. The implication (⇐) is obvious. To prove (⇒), suppose to the contrary that \( \{v_{p_a}, v_{p_{a+1}}\} \notin \mathcal{E} \) for all \( 1 \leq a < m \). Since \( p_1 < p_{m-1} < p_m \), we must have \( \{v_{p_1}, v_{p_{m-1}}\} \notin \mathcal{E} \). Repeating yields \( \{v_{p_1}, v_{p_3}\} \notin \mathcal{E} \) with \( p_1 < p_2 < p_3 \). \{v_{p_1}, v_{p_2}\} \notin \mathcal{E} \), \{v_{p_2}, v_{p_3}\} \notin \mathcal{E} \). This is a contradiction. \( \square \)

We are now in position to prove our main result.

Proof of Theorem 3. It is known that a graph \( G \) is a cocomparability graph if and only if \( G \) has an umbrella-free ordering, e.g., [14, Section 2]. By Lemma 12, this is equivalent to saying that if we label vertex \( v_i \) of \( G \) by \( i \in [\ell] \), then \( G \) has the property in Corollary 10. The rest follows from Corollary 10. \( \square \)

Remark 13. By [6, Theorem 3.3], a graph \( G \) is chordal if and only if \( \mathcal{A}(G) \) is free (= supersolvable) for any labeling of \( G \) using \( [\ell] \) (since if \( G \) and \( G' \) are isomorphic, then \( \mathcal{A}(G) \) and \( \mathcal{A}(G') \) have isomorphic intersection lattices). Given a cocomparability graph \( G \), Theorem 3, however, can only tell the existence of a labeling that makes \( \mathcal{A}(G) \) compatible (see Examples 25 and 26). We will see in Remark 24 that such a labeling is sufficient for computing some polynomial invariants of the graph.

Remark 14. Comparability and cocomparability graphs also have forbidden induced subgraph characterizations, see, e.g., [8,25].

From the main result Theorem 3, we obtain a characterization of graphic arrangements that are compatible and free.

Corollary 15. Let \( G = (V, \mathcal{E}) \) be a graph with \( |V| = \ell \). The following are equivalent.

(i) \( G \) has a labeling using \( [\ell] \) so that \( \mathcal{A}(G) \) is a compatible and free graphic arrangement.

(ii) \( G \) has an ordering \( v_1 < \cdots < v_\ell \) of its vertices such that if \( i < k < j \) and \( \{v_i, v_j\} \in \mathcal{E} \), then \( \{v_i, v_k\} \in \mathcal{E} \).

(iii) \( G \) is an interval graph.

Proof. The equivalence (ii) ⇔ (iii) is well known, e.g., [17, Theorem 4]. The equivalence (i) ⇔ (iii) follows from Theorem 3, Remark 13, and the fact that a graph is an interval graph if and only if it is a cocomparability and chordal graph [9, Theorem 2]. \( \square \)

We complete Table 1 by giving a graphic characterization of ideal subarrangements of braid arrangements, which was suggested to us by Shuhei Tsujiie.
Theorem 16. Let \( G = (V, \mathcal{E}) \) be a graph with \( |V| = \ell \). The following are equivalent.

(i) \( G \) has a labeling using \( \ell \) so that \( \mathcal{A}(G) \) is an ideal-graphic arrangement, i.e., \( \Psi(G) \subseteq \Phi^+(\mathbb{A}_{\ell-1}) \) is an ideal.

(ii) \( G \) has an ordering \( v_1 < \cdots < v_\ell \) of its vertices such that if \( i < k < j \) and \( \{v_i, v_j\} \in \mathcal{E} \), then \( \{v_i, v_k\} \in \mathcal{E} \) and \( \{v_k, v_j\} \in \mathcal{E} \).

(iii) \( G \) is a unit interval graph.

Proof. The equivalence (ii) \( \Leftrightarrow \) (iii) is well known, e.g., [15, Theorem 1]. The equivalence (i) \( \Leftrightarrow \) (ii) is not hard. The key observation is \( \varepsilon_i - \varepsilon_j \geq \varepsilon_p - \varepsilon_q \) if and only if \( i \leq p < q \leq j \).

3. Application to graph polynomials of Cocomparability graphs

Owing to Theorem 3, we will be able to give new formulas for the chromatic polynomial and the (reduced) graphic Eulerian polynomial of cocomparability graphs.

Let \( G = ([\ell], \mathcal{E}) \) be a simple graph, i.e., no loops and no multiple edges. Let \( c_G(t) \) be the chromatic polynomial of \( G \). The graphic Eulerian polynomial \( W_G(t) \) of \( G \) is the polynomial defined by

\[
\sum_{q \geq 0} c_G(q) t^q = \frac{W_G(t)}{(1-t)^{\ell+1}}.
\]

The coefficients of \( W_G(t) \) are proved to be nonnegative integers, and have various combinatorial interpretations, e.g., [4, 5, 22]. Let us recall one of its interpretations following the last two references. Denote by \( \mathfrak{S}_{\ell} \) the symmetric group on \( [\ell] \).

Definition 17. Let \( G = ([\ell], \mathcal{E}) \) be a graph. For \( \pi = \pi_1 \cdots \pi_\ell \in \mathfrak{S}_{\ell} \), the rank \( \rho(\pi) \) of \( \pi \) (\( 1 \leq i \leq \ell \) is defined to be the largest integer \( r \) so that there are values \( 1 \leq i_1 < i_2 < \cdots < i_r = i \) with \( \{\pi_{i_j}, \pi_{i_{j+1}}\} \in \mathcal{E} \) for all \( 1 \leq j \leq r - 1 \). We say that \( \pi \in \mathfrak{S}_{\ell} \) has a graphic descent (w.r.t. \( G \)) at \( i \in [\ell - 1] \) if either

1. \( \rho(\pi_i) > \rho(\pi_{i+1}) \), or
2. \( \rho(\pi_i) = \rho(\pi_{i+1}) \) and \( \pi_i > \pi_{i+1} \).

The graphic Eulerian numbers \( w_k(G) \) (\( 1 \leq k \leq \ell \)) are defined by

\[
w_k(G) = \#\{ \pi \in \mathfrak{S}_{\ell} \mid \text{\( \pi \) has exactly \( \ell - k \) graphic descents} \}.
\]

Theorem 18. We have \( W_G(t) = \sum_{k=1}^{\ell} w_k(G) t^k \). Equivalently,

\[
c_G(t) = \sum_{k=1}^{\ell} w_k(G) \binom{t + \ell - k}{\ell}.
\]

Proof. See, e.g., [5, Theorem 2], [22, Theorem 6]. The equivalence of the formulas holds true in a more general setting, e.g., [4, Theorem 2.1].

Remark 19. If \( G \) is the empty graph, then the graphic descent is the ordinary descent, i.e., the index \( i \in [\ell - 1] \) such that \( \pi_i > \pi_{i+1} \). In this case, \( W_G(t) \) is known as the classical \( \ell \)-th Eulerian polynomial, which first appeared in a work of Euler [7].

It is a standard fact that \( c_G(t) \) is divisible by \( t \). The reduced graphic Eulerian polynomial of \( G \) is the polynomial \( Y_G(t) \) defined by

\[
\sum_{q \geq 1} \frac{c_G(q)}{q} t^q = \frac{Y_G(t)}{(1-t)^{\ell}}.
\]

The polynomial \( Y_G(t) \) also has nonnegative integer coefficients and can be interpreted in terms of the \( h \)-polynomial of a certain relative complex, e.g., [12, Theorem 3.5]. However, unlike \( W_G(t) \), less seems to be known about combinatorial interpretations of the coefficients of \( Y_G(t) \). We will give a combinatorial interpretation for \( Y_G(t) \) when \( G \) is a cocomparability graph (Theorem 23). One can readily compute \( Y_G(t) \) from \( W_G(t) \) and vice versa as we will see below. Write \( Y_G(t) = \sum_{k=1}^{\ell} Y_k(G) t^k \) (note that \( Y_G(0) = 0 \)).
Proposition 20. The polynomials \( W_G(t) \) and \( Y_G(t) \) satisfy the Eulerian recurrence, i.e.,
\[
W_G(t) = t(1-t) \frac{d}{dt} Y_G(t) + t \ell Y_G(t).
\]
Equivalently, for every \( 1 \leq k \leq \ell \),
\[
w_k(G) = ky_k(G) + \ell k - 1) y_{k-1}(G).
\]
Conversely, if \( F(t) \) is a polynomial such that \( F(0) = 0 \), and \( W_G(t) \) and \( F(t) \) satisfy the Eulerian recurrence, then \( F(t) = Y_G(t) \).

Proof. Straightforward. \( \square \)

Now to link the Eulerian polynomials above with the cocomparability graphs, we need to recall the notion of \( \mathcal{A} \)-Eulerian polynomial following [2, Section 4]. We will focus only on type A. Let \( G^c = (V, \mathcal{E}(G^c)) \) denote the complement graph of a graph \( G \).

Definition 21. Let \( G = ([\ell], \mathcal{E}) \) be a graph. We say that \( \pi = \pi_1 \ldots \pi_{\ell} \in \mathcal{S}_\ell \) has an \( \mathcal{A} \)-descent (w.r.t. \( G \)) at \( i \in [\ell] \) if \( \pi_i > \pi_{i+1} \) and \( (\pi_1, \pi_{i+1}) \in \mathcal{E}(G^c) \) \( (\pi_{i+1}, \pi_1) \in \mathcal{E}) \). For \( 0 \leq k \leq \ell \), define
\[
f_k(G) = \frac{1}{\ell} \# \{ \pi \in \mathcal{S}_\ell \mid \pi \text{ has exactly } \ell - k \text{ \( \mathcal{A} \) -descents} \}.
\]
It is easily seen that \( f_0(G) = 0 \). Set \( F_G(t) := \sum_{k=1}^\ell f_k(G) t^k \).

Recall the notation \( \Psi(G) = \{e_i - e_j \mid \{i, j\} \in \mathcal{E} \} \subseteq \Phi^+(A_{\ell-1}) \) in Section 1. It is not hard to check that \( F_G(t) \) equals the \( \mathcal{A} \)-Eulerian polynomial [2, Definition 4.2] of \( \Psi(G) \). If \( G \) is the empty graph, then \( F_G(t) \) is the classical \( (\ell - 1) \)-th Eulerian polynomial. Unlike \( W_G(t) \) or \( Y_G(t) \), the polynomial \( F_G(t) \) is in general not a graph invariant in the sense that it depends on the labeling of \( G \) (see Examples 25 and 26). The proposition below says computing \( F_G(t) \) requires only \( (\ell - 1)! \) permutations.

Proposition 22. If \( 1 \leq k \leq \ell \), then
\[
f_k(G) = \# \{ \pi \in \mathcal{S}_\ell \mid \pi_1 = \ell \text{ and } \pi \text{ has exactly } \ell - k \text{ \( \mathcal{A} \) -descents} \}.
\]

Proof. When \( G \) is the empty graph on \( [\ell] \), it is already proved in [19, Exercise 1.11]. The same argument applies to the general case. Let \( \tau = 23 \ldots \ell 1 \in \mathcal{S}_\ell \), then \( \tau \) cyclically shifts the numbers 1 to \( \ell \). The action of \( \tau \) on \( \mathcal{S}_\ell \) by right multiplication partitions \( \mathcal{S}_\ell \) into \( (\ell - 1)! \) orbits of size \( \ell \). The number of \( \mathcal{A} \)-descents is constant on each orbit. Also, we can choose the permutation \( \pi \) with \( \pi_1 = \ell \) to be a representative of each orbit. \( \square \)

The following is the main result of this section.

Theorem 23. Let \( G = ([\ell], \mathcal{E}) \) be a graph. The following are equivalent.

(i) \( G \) is a cocomparability graph and the ordering \( 1 < 2 < \cdots < \ell \) is an umbrella-free ordering.
(ii) The ordering \( 1 < 2 < \cdots < \ell \) is an umbrella-free ordering.
(iii) \( \Psi(G) \subseteq \Phi^+(A_{\ell-1}) \) is compatible (= strongly compatible).
(iv) The chromatic polynomial of \( G \) is given by
\[
c_G(t) = t \sum_{k=1}^n f_k(G) \binom{t + \ell - 1 - k}{\ell - 1}.
\]
(v) \( F_G(t) \) equals the reduced graphic Eulerian polynomial \( G \), i.e.,
\[
F_G(t) = Y_G(t) = (1 - t)^\ell \sum_{q \geq 1} \frac{c_G(q)}{q} t^q.
\]
Equivalently, \( F_G(t) = Y_K(t) \), where \( K \) is any graph isomorphic to \( G \).
(vi) The polynomials $W_G(t)$ and $F_G(t)$ satisfy the Eulerian recurrence, i.e.,

$$W_G(t) = t(1-t) \frac{d}{dt} F_G(t) + t\ell F_G(t)$$

Equivalently, for every $1 \leq k \leq \ell$,

$$w_k(G) = k f_k(G) + (\ell - k + 1) f_{k-1}(G).$$

**Proof.** The equivalences (i) $\iff$ (ii) $\iff$ (iii) follow from Corollary 10 and Lemma 12. The statements (iii), (iv) and (v) are simply those in [2, Theorems 4.11 and 4.24] when restricting the root system to type $A$, hence they are equivalent. The equivalence (v) $\iff$ (vi) follows from Proposition 20 and the fact that $F_G(0) = 0$ (Definition 21). \qed

**Remark 24.** Thus, to compute $c_G(t)$, $Y_G(t)$ and $W_G(t)$ for a given cocomparability graph $G$, in principle, we can do as follows: find an umbrella-free ordering $v_1 < v_2 < \cdots < v_\ell$ of its vertices (which can be done in linear time [16]), label each vertex $v_i$ by $i \in [\ell]$, and apply Theorem 23. In addition, Theorem 23 says that $F_G(t)$ is an invariant of cocomparability graphs if and only if the ordering $1 < 2 < \cdots < \ell$ is an umbrella-free ordering.

**Example 25.** Let $G$ be a graph given in Figure 3. Thus, $\Psi(G) = \{\alpha_1, \alpha_2\} \subseteq \Phi^+(A_3)$, which is an ideal, hence compatible (Theorem 6). The computation of $F_G(t)$ and $W_G(t)$ is illustrated in Table 2. Let $\tau = 2341$. There are 8 permutations having 2 $A$-descents and 16 having 1 $A$-descent. Thus, $F_G(t) = 4t^3 + 2t^2$. There are 4 permutations having 2 graphic descents (cells in green), 4 having 0 graphic descent (cells in red), and the remaining 16 have 1 graphic descent. Thus, $W_G(t) = 4t^4 + 16t^3 + 2t^2$. The calculation is consistent with Theorem 23.

**Example 26.** Let $G'$ be a graph given in Figure 3. Note that $G$ and $G'$ have isomorphic underlying unlabeled graphs. But $\Psi(G')$ is not compatible, which can be checked either by Theorem 23(v) ($Y_G'(t) \neq F_G'(t)$), or by Theorem 23(ii) ($1 < 2 < 4, \{1, 4\} \in \mathcal{E}$ but $\{1, 2\} \notin \mathcal{E}$, $\{2, 4\} \notin \mathcal{E}$).

**Example 27.** Let $G$ be the claw $K_{1,3}$ given in Figure 3. By Theorem 23(ii), $\Psi(K_{1,3})$ is compatible. By Theorem 16, $\Psi(K_{1,3})$ is not an ideal w.r.t. any positive system since $K_{1,3}$ is not an interval graph.

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Figure 3. Three cocomparability graphs with 4 vertices.

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