A heuristic wave equation parameterizing BEC dark matter halos with a quantum core and an isothermal atmosphere

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Received 15 September 2021 / Accepted 10 February 2022 / Published online 14 March 2022
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Abstract. The Gross–Pitaevskii–Poisson equations that govern the evolution of self-gravitating Bose–Einstein condensates, possibly representing dark matter halos, experience a process of gravitational cooling and violent relaxation. We propose a heuristic parametrization of this complicated process in the spirit of Lynden-Bell’s theory of violent relaxation for collisionless stellar systems. We derive a generalized wave equation that was introduced phenomenologically in Chavanis (Eur Phys J Plus 132:248, 2017) involving a logarithmic nonlinearity associated with an effective temperature $T_{\text{eff}}$ and a damping term associated with a friction $\xi$. These terms can be obtained from a maximum entropy production principle and are linked by a form of Einstein relation expressing the fluctuation-dissipation theorem. The wave equation satisfies an $H$-theorem for the Lynden-Bell entropy and relaxes towards a stable equilibrium state which is a maximum of entropy at fixed mass and energy. This equilibrium state represents the most probable state of a Bose–Einstein condensate dark matter halo. It generically has a core-halo structure. The quantum core prevents gravitational collapse and may solve the core-cusp problem. The isothermal halo leads to flat rotation curves in agreement with the observations. These results are consistent with the phenomenology of dark matter halos. Furthermore, as shown in a previous paper (Chavanis in Phys Rev D 100:123506, 2019), the maximization of entropy with respect to the core mass at fixed total mass and total energy determines a core mass–halo mass relation which agrees with the relation obtained in direct numerical simulations. We stress the importance of using a microcanonical description instead of a canonical one. We also explain how our formalism can be applied to the case of fermionic dark matter halos.

1 Introduction

The nature of dark matter (DM) is still unknown and remains one of the greatest mysteries of modern cosmology even after almost 100 years of research.

The suggestion that DM may constitute a large part of the universe was made by Zwicky [1]. Using the virial theorem to infer the average mass of galaxies within the Coma cluster, he obtained a much higher value than the mass of luminous material. He realized therefore that some mass was missing to account for the observations (he called this unseen mass dunkle Materie). This virial mass discrepancy in galaxy clusters was confirmed later by more accurate measurements of velocity dispersion showing that DM should represent about 90% of the mass of the cluster [2].

Another evidence of the missing mass problem and of the existence of DM comes from the study of the rotation curves of disk galaxies [3–6]. The rotational velocity of hydrogen clouds in spiral galaxies measured from the Doppler effect is found first to increase near the galaxy center in agreement with Newtonian theory but then to saturate to an approximately constant value $v_\infty \sim 200$ km/s, even at large distances where very little baryonic matter can be detected, instead of decreasing according to the Keplerian law (like for the rotation of the planets around the Sun). This suggests that galaxies are surrounded by an extended halo of DM, whose mass $M(r) \sim v_\infty^2 r / G$ increases linearly with radius. This can be conveniently modeled by a classical isothermal gas whose density decreases as $r^{-2}$ [7].

On the cosmological side, the observations of distant type Ia supernovae [8–11] and the recent Planck satellite measurements of the Cosmic Microwave Background (CMB) radiation [12,13] have revealed that the content of the universe is made of about 70% dark energy, 25% DM, and 5% baryonic (visible) matter. DM is also required to interpret the data of gravitational lensing [14–16] and the observations of the Bullet Cluster, resulting from the collision of two clusters of galaxies, in which the baryonic and the DM components are clearly separated [17].
There have been some attempts to interpret the observations without assuming the existence of DM. For example Milgrom [18] proposed a modification of Newton’s law (MOND theory) on galactic scales to explain the rotation curves of spiral galaxies without invoking DM. Other theories of modified gravity have been introduced as alternatives to DM [19] but the DM hypothesis is favored by most astrophysicists. It is likely that DM is made of a new type of particles, interacting only gravitationally with ordinary matter, not yet included in the standard model of particle physics. In the standard cold dark matter (CDM) model, DM is assumed to be made of (still hypothetical) weakly interacting massive particles (WIMPs) with a mass in the GeV-TeV range. They may correspond to supersymmetric (SUSY) particles [20]. These particles freeze out from thermal equilibrium in the early universe and, as a consequence of this decoupling, cool off rapidly as the universe expands. In the warm dark matter (WDM) model [21,22], DM is thought to be made of fermions, such as massive neutrinos, with a mass in the keV range. Other [21,22], DM is thought to be made of fermions, such as massive neutrinos, with a mass in the keV range. Other theories of modified gravity have been introduced as alternatives to DM [19] but the DM hypothesis is favored by most astrophysicists. It is likely that DM is made of a new type of particles, interacting only gravitationally with ordinary matter, not yet included in the standard model of particle physics. In the standard cold dark matter (CDM) model, DM is assumed to be made of (still hypothetical) weakly interacting massive particles (WIMPs) with a mass in the GeV-TeV range. They may correspond to supersymmetric (SUSY) particles [20]. These particles freeze out from thermal equilibrium in the early universe and, as a consequence of this decoupling, cool off rapidly as the universe expands. In the warm dark matter (WDM) model [21,22], DM is thought to be made of fermions, such as massive neutrinos, with a mass in the keV range. Other popular DM candidates are bosons like the axion. The QCD axion [23] with a mass $m \sim 10^{-4} \text{eV}/c^2$ was initially proposed as a solution of the charge parity (CP) problem of quantum chromodynamics (QCD) [24] but ultra light axions (ULA) arising from string theory with a mass possibly as small as $m \sim 10^{-22} \text{eV}/c^2$ are also actively considered at present [25].

In the standard CDM model, DM is represented as a classical pressureless gas at zero temperature ($T = 0$) described by the Euler–Poisson equations, or as a collisionless system of classical particles described by the Vlasov–Poisson equations [26]. The CDM model works remarkably well at large (cosmological) scales and is consistent with ever improving measurements of the CMB from WMAP and Planck missions [12,13]. However, it experiences serious problems at small (galactic) scales. In particular, classical collisionless $N$-body numerical simulations predict that DM halos should be cuspy [27], with a density diverging as $r^{-1}$ for $r \to 0$, while observations reveal that they have a constant density core [28]. On the other hand, the CDM model predicts an over-abundance of small-scale structures (subhalos/satellites), much more than what is observed around the Milky Way [29–31]. Finally, dissipationless CDM simulations predict that the majority of the most massive subhaloes of the Milky Way are too dense to host any of its bright satellites. These problems are referred to as the “core-cusp” problem [32], the “missing satellites” problem [29–31], and the “too big to fail” problem [33]. The expression “small-scale crisis of CDM” has been coined [34]. The small-scale problems of the CDM model are somehow related to the assumption that DM is pressureless.

These problems may be relieved if we assume that DM is warm or if the DM particles are self-interacting. In the WDM model the dispersion of the particles is responsible for a pressure force that can halt gravitational collapse and prevent the formation of cusps [22]. Similarly, in the self-interacting dark matter (SIDM) model with a large scattering cross section but negligible annihilation or dissipation [38], collisions can cause the relaxation of the particles in the regions of high density and establish an isothermal (Maxwellian) distribution function (DF). As a result, the system presents an isothermal core (instead of a cusp) and a NFW halo. When this model is confronted to observations it is found that the cross section $\sigma/m$ of the DM particles depends on their velocity dispersion. This may be explained by a dark photon model where self-interactions are described by a Yukawa potential [39], by a short-range interaction model with a large scattering length [40], or by a dark fusion model [41]. For dwarf and low surface brightness ( LSB) galaxies, the cross-section is approximately constant with a value $\sigma/m \sim 1 \text{cm}^2/\text{g}$. Observations of the Bullet Cluster lead to an upper limit $\sigma/m < 1.25 \text{cm}^2/\text{g}$.

Another possibility to solve the small-scale crisis of CDM is to take into account the quantum nature of the DM particle. In this paper, we shall assume that the DM particle is a boson like an axion [25]. At very low temperatures, bosons form self-gravitating Bose–Einstein condensates (BECs). In that case, DM halos

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2 The possibility to solve the CDM crisis without changing the basic assumptions of the CDM model invokes the feedback of baryons that can transform cusps into cores [35–37].

3 Some authors have considered the case where the DM particle is a fermion like a massive neutrino (see the Introduction of Ref. [42] for a short review and an exhaustive list of references). In this model, gravitational collapse is prevented by the quantum pressure arising from the Pauli exclusion principle.

4 The condensation of integer spin particles was theoretically predicted by Bose [43] and Einstein [44,45] in 1924 and observed for the first time in laboratory experiments of dilute alkali gases in 1995 [46–48]. The condensation occurs when the particles in the gas become correlated quantum mechanically, i.e., when the de Broglie thermal wavelength of a particle $\lambda_{\text{dB}} = \sqrt{2\pi\hbar^2/mk_B T}$ turns out to be greater than the mean interparticle distance $l = n^{-1/3}$ (i.e. $n\lambda_{\text{dB}}^3 > 1$). The exact condensation temperature is given by

$$T_c = \frac{2\pi\hbar^2 n^{2/3}}{mk_B (3/2)^{2/3}}$$

with $\zeta(3/2) = 2.612 \ldots$ If we apply these results to ULAs with a mass $m = 2.92 \times 10^{-22} \text{eV}/c^2$ and consider a typical DM halo density $\rho = 7.02 \times 10^{-3} M_{\odot}/\text{pc}^3$ (medium spiral), we get $T_c = 4.82 \times 10^{16} \text{K}$. On the other hand, the typical temperature of the halo (obtained from the virial relation $k_B T_m/m \sim v_h^2 = GM_\odot/r_h$) is $T_m \sim 4.41 \times 10^{-25} \text{K}$ where we have taken $(k_B T_m/m)^{1/2} \approx 108 \text{km/s}$. Therefore, we find that $T_m \ll T_c$. This inequality shows that $T_m$ is necessarily an out-of-equilibrium (effective)
can be viewed as gigantic bosonic atoms at $T = 0$ where the bosonic particles are condensed in a single macroscopic quantum state. They are described by a scalar field (SF) that can be interpreted as the wavefunction $\psi(r,t)$ of the condensate. The evolution of the wave function of the condensate is governed by the Schrödinger–Poisson equations when the bosons are noninteracting or by the Gross–Pitaevskii–Poisson (GPP) equations when the bosons are self-interacting. By using the Madelung [50] transformation, these wave equations may be written in the form of hydrodynamic equations including a quantum potential. The wave properties of the SF are negligible at large (cosmological) scales where the SF behaves as CDM, but they become relevant at small (galactic) scales and can prevent gravitational collapse. This model has been given several names such as wave DM, fuzzy dark matter (FDM), BECDM, $\psi$DM, or SFDM. We shall refer to this model as BECDM. In the BECDM model, gravitational collapse is prevented by the quantum pressure arising from the Heisenberg uncertainty principle or from the scattering of the bosons (when the self-interaction is repulsive). This leads to DM halos presenting a central core instead of a cusp. Since the quantum Jeans length is nonzero [52,60–69], the formation of small-scale structures is suppressed even at $T = 0$. Therefore, quantum mechanics may be a way to solve the small-scale problems of the CDM model such as the core-cusp problem and the missing satellite problem.

The GPP equations have a very complicated dynamics. A self-gravitating BEC at $T = 0$ that is not initially in a steady state undergoes Jeans instability, gravitational collapse (free fall), displays damped oscillations, and finally settles down on a quasistationary state (virialization) by radiating part of the scalar field [70–72]. This is the process of gravitational cooling initially introduced by Seidel and Suen [70] in the context of boson stars.

As a result of gravitational cooling, the system reaches an equilibrium configuration with a core-halo structure. The condensed core (soliton/BEC) is stabilized by quantum mechanics and has a smooth (finite) density. This is a stable stationary solution of the GPP equations at $T = 0$ (ground state). Gravitational collapse is prevented by the quantum potential arising from the Heisenberg principle or by the pressure $P = 2\pi a_s \hbar^2 \rho^2 / m^3$ arising from the self-interaction of the bosons which is measured by their scattering length $a_s$. This quantum core (ground state) is surrounded by a halo of scalar radiation corresponding to the quantum interferences of excited states. As shown by Schive et al. [73,74], and further discussed in [75–78], these interferences produce time-dependent small-scale density granules – or quasiparticles – of the size of the solitonic core $\lambda_{DB} \sim h / mv \sim 1$ kpc (de Broglie wavelength) and with an effective mass $m_{\text{eff}} \sim \rho \lambda_{DB}^3 \sim 10^5 M_\odot \gg m$ that counter self-gravity and create an effective thermal pressure. These noninteracting excited states are analogous to collisionless particles in classical mechanics. As a result, the halo behaves essentially as CDM and is approximately isothermal with an equation of state $P = \rho T_{\text{eff}}$ involving an effective temperature $T_{\text{eff}}$ resulting from a process of collisionless violent relaxation (see below). The quantum core (soliton) may solve the core-cusp problem of the CDM model and the isothermal halo where the density decreases as $r^{-2}$ yields flat rotation curves in agreement with the observations. In Ref. [81] we have constructed a core-halo solution of BECDM halos in the Thomas-Fermi (TF) approximation based on an equation of state of the form $P = 2\pi a_s \hbar^2 \rho^2 / m^3 + \rho T_{\text{eff}}$ (see Fig. 1). At a general level, the core-halo structure of DM halos (and the presence of granules) has been clearly evidenced in numerical simulations of the Schrödinger–Poisson equations [73,74,82–88] and in the comparison with observations (see, e.g., [89,90]). For noninteracting bosons, the core mass-halo mass relation $M_c \propto M_h^{1/3}$ has been obtained numerically in Ref. [74] and explained heuristically from a nonlocal uncertainty principle.

Gravitational cooling is a dissipationless relaxation mechanism similar in some respect to the concept of violent relaxation introduced by Lynden-Bell [91] in the context of collisionless stellar systems described by the Vlasov–Poisson equations. A collisionless stellar system that is not initially in a dynamically stable steady state undergoes Jeans instability, gravitational collapse (free

\footnote{A repulsive self-interaction ($a_s > 0$) stabilizes the quantum core. By contrast, an attractive self-interaction destabilizes the quantum core above a maximum mass $M_{\text{max}} = 1.012 h / \sqrt{Gm_s a_s}$ first identified in [52].

The halo cannot be exactly isothermal otherwise it would have an infinite mass [7]. In reality, the density in the halo decreases as $r^{-3}$, similar to the NFW [27] and Burkert [28] profiles, or even as $r^{-4}$ (see Appendix D of [79] and Appendix I of [80]), instead of $r^{-2}$ corresponding to the isothermal sphere [7]. This extra-confinement may be due to incomplete relaxation, tidal effects, and stochastic perturbations as discussed in Appendix B of [81]. We stress that the halo is in a dynamical equilibrium (virialized) state but in an out-of-equilibrium thermodynamical equilibrium state (see footnote 4). As discussed in [75–78], the quasiparticles are responsible for a slow (secular) collisional evolution of the halo towards thermodynamical equilibrium on a timescale of the order of the Hubble time. By this process part of the halo condense (since $T < T_c$) and feeds the soliton. We shall not consider this collisional regime in the present paper.}
fall), displays damped oscillations, and finally settles down on a quasistationary (virialized) state by sending some of the particles at large distances. This process, which takes place on a dynamical timescale, is related to phase mixing and nonlinear Landau damping.

Lynden-Bell [91] developed the statistical mechanics of violent relaxation and determined the coarse-grained DF at statistical equilibrium (most probable state) from a maximum entropy principle (MEP) taking into account all the constraints of the Vlasov–Poisson equations. The quasistationary state reached by the system as a result of violent relaxation is expected to be in this most probable state. The Lynden-Bell DF is similar to the Fermi–Dirac distribution but with, of course, a completely different interpretation. It takes into account a sort of exclusion principle implied by the Vlasov equation, similar to the Pauli exclusion principle for fermions, but of nonquantum origin. In the theory of Lynden-Bell, further developed by Chavanis and Sommeria [92], the quasistationary state generally has a core-halo structure with a completely degenerate core at $T = 0$ (effective fermion ball) and an isothermal atmosphere with an effective temperature $T_{\text{eff}}$. The core has a polytropic equation of state $P = (1/5)(3/(4\pi\eta))^{2/3}\rho^{5/3}$ and the halo has an isothermal equation of state $P = \rho T_{\text{eff}}$. The degenerate core (in the sense of Lynden-Bell) may solve the core-cusp problem of the CDM model. On the other hand, the density decreases as $r^{-2}$ in the isothermal halo, yielding flat rotation curves in agreement with the observations [3–6]. This core-halo structure has been studied in detail in models of self-gravitating fermions and in relation to Lynden-Bell’s theory of violent relaxation (see Sec. V.A of [81] for an exhaustive list of references). In the analogy between the gravitational cooling of self-gravitating BECs and the violent relaxation of collisionless self-gravitating systems, the bosonic core (BEC/soliton) corresponds to the effective fermion ball and the halo made of scalar radiation corresponds to the isothermal halo predicted by Lynden-Bell. Actually, since a collisionless system of bosons is described by the Vlasov–Poisson equations at large scales where quantum effects are negligible (see [84] for the Schrödinger–Poisson–Vlasov–Poisson correspondence) it is very likely that both processes – gravitational cooling and violent relaxation – occur in self-gravitating BECs and may even correspond to the same phenomenon. As a result, self-gravitating BECs should have a core that is partly bosonic (soliton) and partly fermionic (in the sense of Lynden-Bell), surrounded by an effective isothermal halo. Gravitational cooling and violent relaxation may be at work during hierarchical clustering, a process by which small DM halos merge and form larger halos in a bottom-up structure formation scenario. It is believed that DM halos acquire an approximately isothermal profile, or more realistically a NFW or Burkert profile (see footnote 7), as a result of successive mergings. Gravitational cooling and violent relaxation explain how collisionless self-gravitating systems can rapidly thermalize and acquire a large effective temperature $T_{\text{eff}}$ even if $T = 0$ fundamentally.

In the context of the violent relaxation of collisionless stellar systems, we have derived a relaxation equation for the coarse-grained DF $\mathcal{F}(r, v, t)$ by using a maximum entropy production principle (MEPP) [94, 96, 97]. This coarse-grained Vlasov equation has the form of a fermionic Kramers equation involving a diffusion term and a friction term. The diffusion term accounts for fluctuations and effective thermal effects while the friction term accounts for collisionless dissipation (nonlinear Landau damping). The diffusion and friction coefficients are linked by a form of Einstein relation expressing the fluctuation-dissipation theorem. The

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**Fig. 1** Density profile of a BECDM halo (in the TF approximation) of mass $M_b = 10^{12} M_\odot$ surrounding the Milky Way (we have taken $a_s/m^3 = 3.28 \times 10^3$ fm/(eV/c^2)^3) [81]. It has a core-halo structure with a quantum core of mass $M_c = 6.39 \times 10^{10} M_\odot$ and radius $R_c = 1$ kpc. The dashed line corresponds to a purely isothermal DM halo without quantum core.

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9 Lynden-Bell [91], who was concerned with the study of elliptical galaxies, argued that these objects are described by the nondegenerate limit of his theory where his DF is similar to the Boltzmann distribution. However, his theory may also apply to fermionic DM halos where degeneracy effects may be important as suggested in [79, 80, 92]. The theory of violent relaxation explains how a collisionless self-gravitating system may reach an isothermal distribution on a very short timescale (of the order of the dynamical time $t_D$) without recourse to collisions or gravitational encounters that operate on a relaxation timescale $t_R \sim (N/\ln N)t_D$ much larger than the age of the universe [7]. The theory of violent relaxation thus solves a notoriously important timescale problem in astrophysics [79, 80, 91, 92].

9 The connection between self-gravitating BECs and the Lynden-Bell theory of violent relaxation was mentioned in [81, 93] and in the Appendix of [84]. There also exist interesting analogies between the violent relaxation of stellar systems and the formation of two-dimensional large-scale vortices like Jupiter’s Great Red Spot [94, 95].

10 Alternatively, we can describe the effective collision term by a fermionic Landau operator [96–101].
The paper is organized as follows. In Sect. 2, we recall the wave function approach of BECDM halos based on the Schrödinger–Poisson equations. We discuss the hydrodynamic representation of these equations based on the Madelung transformation and introduce the concepts of gravitational cooling and violent relaxation. In Sect. 3, we recall the DF approach of BECDM halos based on the Wigner–Poisson equations. We propose a heuristic coarse-graining of these equations based on a MEPP and derive the corresponding hydrodynamic equations. In Sect. 4, we obtain a generalized wave equation which is equivalent to the hydrodynamic equations associated with the coarse-grained Wigner–Poisson equations. This generalized wave equation provides a heuristic parametrization of the complex dynamics of BECDM halos. In Sect. 5, we study the equilibrium states of this equation and show that their structure agrees with the phenomenology of BECDM halos. In Sect. 6, we stress the importance of developing a microcanonical description of the process of violent relaxation and show how it can be implemented in our parametrization. In Sect. 7, we explain how our formalism can be applied to the case of fermionic DM halos. The Appendices provide additional results that are needed in our study. In Appendix A, we derive the basic properties of the Wigner distribution. In Appendix B, we recall the basic properties of the generalized GPP equations introduced in [93]. We mention that, in addition to providing a parametrization of the processes of gravitational cooling and violent relaxation, these equations can serve as a numerical algorithm to construct stable stationary solutions of the GPP equations. In Appendix C, we discuss certain aspects of the dynamical evolution of classical collisionless self-gravitating systems. We point out in particular the limitation of the single-speed solution and the need to introduce more sophisticated parametrizations. In Appendix D, we discuss the violent relaxation of classical collisionless self-gravitating systems towards a quasistationary state in terms of statistical mechanics. In Appendix E, we discuss the classical limit $\hbar \to 0$ of the quantum equations derived in this paper. In Appendix F, we briefly consider the Vlasov–Bohm equation instead of the Wigner equation. In Appendix G, we explain how our results can be extended to multistate systems like in the case of fermions.

2 Wave function approach

2.1 Schrödinger–Poisson equations

Let us consider a system of $N$ bosons of individual mass $m$ interacting via a binary potential $u(|r - r'|)$. Their Hamiltonian is given, in the second quantization, by
\[ \hat{H} = \int dr \hat{\psi}^\dagger(r) \left( -\frac{\hbar^2}{2m} \Delta \right) \hat{\psi}(r) + \frac{1}{2} \int dr dr' \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(r') \mu(|r - r'|) \hat{\psi}(r) \hat{\psi}(r'), \] (2)

where \( \hat{\psi}(r) \) and \( \hat{\psi}^\dagger(r) \) are the bosonic field operators that annihilate and create a particle at the position \( r \), respectively. We write the field operator in the Heisenberg representation as

\[ \hat{\psi}(r, t) = \psi(r, t) + \tilde{\psi}(r, t), \] (3)

where the expectation value of the field operator \( \psi(r, t) = \langle \hat{\psi}(r, t) \rangle \) is the condensate wavefunction and \( \tilde{\psi}(r, t) \) is the noncondensate field operator whose average is zero by construction: \( \langle \tilde{\psi}(r, t) \rangle = 0 \). In this manner, the BEC contribution has been separated from the total bosonic field operator. The condensate wavefunction \( \psi(r, t) \) is a classical field and its squared modulus determines the condensate mass density \( \rho(r, t) = |\psi(r, t)|^2 \). It is normalized such that \( M = \int \rho(r, t) dr \) represents the total mass of the condensate.

The Heisenberg equation of motion for the quantum field operator \( \hat{\psi}(r, t) \) corresponding to the many-body Hamiltonian given by Eq. (2) is

\[ i\hbar \frac{\partial}{\partial t} \hat{\psi}(r, t) = [\hat{\psi}, \hat{H}] = \left[ -\frac{\hbar^2}{2m} \Delta + m \int dr' \hat{\psi}^\dagger(r) \mu(|r - r'|) \hat{\psi}(r') \right] \hat{\psi}(r, t). \] (4)

This equation is exact and describes the dynamics of a system of interacting bosons at arbitrary temperature \( T \). If we take the average of this equation, we obtain an exact equation for the condensate wavefunction \( \psi(r, t) \). This equation contains correlation terms which describe the interaction between the condensate and the thermal cloud of noncondensed bosons. This equation must be supplemented by a kinetic equation for the distribution function \( f(r, p, t) \) of the noncondensed bosons which itself depends on the interaction with the condensate. This yields a system of two coupled differential equations for the condensate and the thermal cloud. These equations describe the collisional evolution of the system of interacting bosons. The corresponding kinetic theory is studied in detail in Ref. [105]. However, this is not the regime that we consider in the present paper.

In the following, we assume that \( T \ll T_c \) and \( N \gg 1 \), where \( T_c \) is the condensation temperature and \( N \) is the number of bosons. In that case, most of the bosons lie in the same single-particle quantum state and we can make the mean field approximation which consists in replacing the field operator \( \hat{\psi}(r, t) \) by the condensate wavefunction \( \psi(r, t) \). This yields the mean field equation

\[ i\hbar \frac{\partial}{\partial t} \psi(r, t) = \left[ -\frac{\hbar^2}{2m} \Delta + m \int dr' \mu(|r - r'|) |\psi(r', t)|^2 \right] \psi(r, t). \] (5)

This equation gives a very good description of the condensate when \( T \ll T_c \) and \( N \gg 1 \). It can be written as a Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi_{\text{tot}} \psi \] (6)

with a mean field potential

\[ \Phi_{\text{tot}}(r, t) = \int u(|r - r'|) \rho(r', t) dr'. \] (7)

produced by the particles self-consistently. We now assume that the potential of interaction \( u(|r - r'|) \) can be written as the sum of a long-range potential \( u_{\text{LR}}(|r - r'|) \) and a short-range potential \( u_{\text{SR}}(|r - r'|) \) so that \( u = u_{\text{LR}} + u_{\text{SR}} \). The long-range potential corresponds to the gravitational interaction and is given by \( u_{\text{LR}} = -G/|r - r'| \). The short-range potential takes into account binary collisions. In a dilute cold gas they can be modeled by a zero-range pseudo-potential \( u_{\text{SR}} = g\delta(|r - r'|) \) of strength \( g = 4\pi a_s \hbar^2 m^3 \), where \( a_s \) is the s-wave scattering length of the bosons. The scattering length \( a_s \) can be positive (corresponding to a repulsive self-interaction) or negative (corresponding to an attractive self-interaction). Under these conditions, the total mean field potential can be written as \( \Phi_{\text{tot}}(r, t) = \Phi(r, t) + h(\rho) \), where \( \Phi(r, t) = -G \int \rho(r', t) |r - r'|^{-1} dr' \) is the gravitational potential and \( h(\rho) = g\rho \) is an effective potential modeling short-range interactions. When this form of potential is substituted into Eq. (6), we obtain the GPP equations

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + \frac{4\pi a_s \hbar^2}{m^2} |\psi|^2 \psi, \] (8)

\[ \Delta \Phi = 4\pi G |\psi|^2. \] (9)

When the self-interaction of the bosons can be neglected they reduce to the Schrödinger–Poisson equations

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi, \] (10)

\[ \Delta \Phi = 4\pi G |\psi|^2. \] (11)

The Schrödinger–Poisson equations can be combined into a single equation of the form

\[ i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \Delta - \frac{G m}{|r - r'|} |\psi|^2(r', t) \right] \psi, \] (12)
classical field of quasiparticles is we write the wave function as the form of hydrodynamic equations. To that purpose, globular clusters containing long, of the order of the Hubble time, like in the case of \( \psi \). However, in the limit of very large occupation numbers \( \text{operators with appropriate commutation rules [see Eq. (4)].} \)

In principle “second-quantize” the Schrödinger equation, which then becomes an equation for the evolution of field in the limit of very large occupation numbers. See, e.g., Ref. \[7\]. In the case of bosonic DM halos, the equation equation is adequate.

Remark We note that the quantum force in Eq. (19) can be written as

\[
-\frac{1}{m} \nabla Q = -\frac{1}{\rho} \partial_j P^Q_{ij},
\]

where \( P^Q_{ij} \) is an anisotropic quantum pressure tensor. Therefore, the quantum Euler equation (19) may be rewritten as

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \partial_j P^Q_{ij} - \nabla \Phi.
\]

Remark The mean field approximation is justified for two reasons. First, the temperature of our system is much smaller than the condensation temperature \( T_c \). Therefore, we are in a situation where most of the bosons are condensed in the same quantum state. As a result, we can treat the wave function as a classical field (the condition \( T \ll T_c \) is equivalent to a large occupation number \( N = n_{\text{eff}}^3 \sim 10^{94} \gg 1 \) \[78\]). In that case, the Schrödinger equation can be interpreted as a classical wave equation. On the other hand, in the case of systems with long-range interactions such as self-gravitating systems, the collisional relaxation time scales like \( (N/\ln N) t_D \) and is generally extremely large (see, e.g., Ref. \[7\]). In a first regime, the evolution of the system is essentially collisionless and a mean field approximation can be implemented. In the analogy with the kinetic theory of classical self-gravitating systems (see, e.g., \[106\]), Eq. (4) is the counterpart of the Klimontovich equation which is an exact equation equivalent to the Liouville equation or to the Hamilton equations of motion, and Eq. (5) is the counterpart of the Vlasov equation which is a mean field equation valid in the collisionless regime. It describes the evolution of a smooth wave function \( \psi(r,t) \).

2.2 Madelung transformation

We can use the Madelung \[50\] transformation to write the Schrödinger–Poisson equations (10) and (11) under the form of hydrodynamic equations. To that purpose, we write the wave function as

\[
\psi(r,t) = \sqrt{\rho(r,t)} e^{iS(r,t)/\hbar},
\]

where \( \rho(r,t) \) is the mass density and \( S(r,t) \) is the action given by

\[
\rho = |\psi|^2 \quad \text{and} \quad S = -\frac{i}{2} \ln \left( \frac{\psi}{\psi^*} \right).
\]

Following Madelung \[50\], we introduce the velocity field

\[
\mathbf{u} = \frac{\nabla S}{m}.
\]

Since the velocity is potential, the flow is irrotational: \( \nabla \times \mathbf{u} = 0 \). Substituting Eq. (14) into Eqs. (10) and (11) and separating the real and the imaginary parts, we find that the Schrödinger–Poisson equations are equivalent to hydrodynamic equations of the form

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\frac{\partial S}{\partial t} + \frac{1}{2 m} (\nabla S)^2 + m \Phi + Q &= 0, \\
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\frac{1}{m} \nabla Q - \nabla \Phi, \\
\Delta \Phi &= 4\pi G \rho, \\
Q &= -\frac{\hbar^2}{2 m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{4 m} \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 \right]
\end{align*}
\]

11 To describe a system of bosons in interaction we should in principle “second-quantiz” the Schrödinger equation, which then becomes an equation for the evolution of field operators with appropriate commutation rules [see Eq. (4)]. However, in the limit of very large occupation numbers \( N \gg 1 \), the classical description of the Schrödinger wave equation is adequate.

12 In the case of spiral or elliptical galaxies containing \( N \sim 10^{12} \) stars, the relaxation time is much larger than the age of the universe \[7\]. In the case of bosonic DM halos, the relaxation time is reduced by Bose stimulation (the number of quasiparticles is \( N_{\text{eff}} \sim 10^9 \) \[78\]) but it remains relatively long, of the order of the Hubble time, like in the case of globular clusters containing \( N \sim 10^5 \) stars \[7\].

13 We can also base the collisional kinetic theory of self-gravitating bosons on an equation similar to Eq. (5) for a classical field \( \psi_c(r,t) \) provided that we take fluctuations into account (i.e., \( \psi \) can be decomposed into \( \psi_c = \psi + \delta \psi \) where \( \psi \) is a smooth component and \( \delta \psi \) is a stochastic component arising from two-body collisions).
2.3 Gravitational cooling and violent relaxation

The Schrödinger–Poisson equations (10) and (11) conserve the mass
\[ M = \int |\psi|^2 \, dr \] (25)
and the energy
\[ E_{\text{tot}} = \frac{\hbar^2}{2m^2} \int |\nabla \psi|^2 \, dr + \frac{1}{2} \int |\psi|^2 \Phi \, dr, \] (26)
which is the sum of the kinetic energy \( \Theta \) and the gravitational energy \( W \).

Equivalently, the quantum Euler–Poisson equations (17)–(20) conserve the mass
\[ M = \int \rho \, dr \] (27)
and the energy
\[ E_{\text{tot}} = \Theta_c + \Theta_Q + W, \] (28)
where the first term is the classical kinetic energy
\[ \Theta_c = \int \rho \frac{u^2}{2} \, dr, \] (29)
the second term is the quantum kinetic energy
\[ \Theta_Q = \frac{1}{m} \int \rho Q \, dr, \] (30)
and the third term is the gravitational energy
\[ W = \frac{1}{2} \int \rho \Phi \, dr. \] (31)

The decomposition \( \Theta = \Theta_c + \Theta_Q \) arises naturally from the Madelung transformation [52].

Since the Schrödinger–Poisson equations (and the corresponding hydrodynamic equations) are reversible, they do not satisfy an \( H \)-theorem. As a result, their relaxation towards a quasistationary state (if the system is not initially in such a state) is not trivial.\(^{14}\) Numerical simulations show that the Schrödinger–Poisson equations experience a process of gravitational cooling [70–72] which is similar to the process of violent relaxation of collisionless stellar systems described by the Vlasov–Poisson equations [91]. This process takes place on a very short timescale, of the order of the dynamical or free fall time. As a result of gravitational cooling and violent relaxation, the system displays damped oscillations and achieves a quasistationary state with a core-halo structure which is in virial equilibrium. The virial theorem reads\(^{15}\)
\[ 2\langle \Theta \rangle + \langle W \rangle = 0, \] (32)
where \( E_{\text{tot}} = \Theta + W \) is constant. This core-halo structure has been evidenced in direct numerical simulations of the Schrödinger–Poisson equations [73,74,82–88].

The quantum core corresponds to the ground state of the Schrödinger–Poisson equations. It is described by a stationary wavefunction of the form
\[ \psi(r, t) = \phi(r)e^{-iEt/\hbar}, \] (33)
where \( \phi(r) \) and \( E \) are real. These quantities are determined by the Eigenvalue problem
\[ -\frac{\hbar^2}{2m^2} \Delta \phi + m \Phi \phi = E \phi, \] (34)
\[ \Delta \Phi = 4\pi G\phi^2. \] (35)
The ground state is the solution of these equations with the lowest Eigenenergy \( E \). This solution is spherically symmetric and has no node so that the density profile decreases monotonically (see, e.g., [52,108]).

In the hydrodynamic representation, the core is determined by the condition of quantum hydrostatic equilibrium
\[ \frac{\rho}{m} \nabla Q + \rho \nabla \Phi = 0 \] (36)
coupled to the Poisson equation (20). Equations (34) and (36) are equivalent. Indeed, since \( \phi = \sqrt{\rho} \), Eq. (34) can be rewritten as
\[ Q + m \Phi = E. \] (37)
Taking the gradient of this equation and multiplying by \( \rho/m \), we obtain Eq. (36).

The quantum core (ground state) can also be obtained by minimizing the energy at fixed mass [52]:
\[ \min \{ E_{\text{tot}} \mid M \text{ fixed} \}. \] (38)
First, one can show that an extremum of energy at fixed mass is a steady state of the Schrödinger–Poisson equations. Indeed, writing the variational principle as
\[ \delta E_{\text{tot}} - \frac{\mu}{m} \delta M = 0, \] (39)
\(^{14}\) This is a notoriously difficult mathematical problem even at the classical level, i.e., for the Vlasov–Poisson equations. The relaxation of the Vlasov–Poisson equations towards a quasistationary state on the coarse-grained scale is related to the concepts of violent relaxation, phase mixing and non-linear Landau damping [107].

\(^{15}\) See Appendix G of [93] for a general derivation of the quantum virial theorem applying to self-gravitating BECs from the hydrodynamic representation of the GPP equations.
where $\mu/m$ is a Lagrange multiplier (global chemical potential) taking into account the mass constraint, we get \[ Q + m\Phi = \mu, \] (40)
which is equivalent to Eq. (37) with $E = \mu$. This establishes that the Eigenenergy is equal to the global chemical potential. Furthermore, one can show that an equilibrium state of the Schrödinger–Poisson equations is dynamically stable if, and only if, it is a minimum of energy at fixed mass ($\delta^2 E_{\text{tot}} > 0$ for all perturbations that conserve mass).\(^{16}\) These results are very general for dynamical systems \[109\]. They are basically due to the fact that $E_{\text{tot}}$ and $M$ are individually conserved by the Schrödinger–Poisson equations.

The equations for the ground state can be solved numerically \[71–75,108,110–112\] leading to what is generally called a soliton. In general, the soliton (quantum core) has a mass $M_c$ and an energy $E_c$ that differ from the initial mass $M$ and initial energy $E$ of the system. Therefore, the excess mass and the excess energy must be redistributed in a halo made of scalar waves (radiation) resulting from the interference of excited states. Numerical simulations of the Schrödinger–Poisson equations \[73,74,82–88\] show that this halo has a density profile that is consistent with the NFW profile obtained in CDM simulations \[27\].\(^{17}\)

The halo profile is also relatively close to an isothermal profile with an effective temperature $T_{\text{eff}}$ (see, e.g., the Appendix of \[84\]). The analogy between the process of gravitational cooling and the process of violent relaxation will help us determining the structure of the halo. To that purpose, we have to introduce a DF approach as done in Sect. 3.

### 2.4 Interferences

To better understand the structure of the halo, it is useful to follow Ref. \[113\] and decompose the wavefunction under the form
\[
\psi(r, t) = \sum_n c_n \psi_n(r) e^{-iE_n t/\hbar}, \tag{41}
\]
where $\psi_n(r)$ are the Eigenfunctions of the time-independent Schrödinger equation constructed with the locally averaged gravitational potential $\Phi(r)$ and $c_n$ is a random complex amplitude for mode $n$ (the gravitational potential is obtained self-consistently from the Poisson equation $\Delta \Phi = 4\pi G \sum_n |c_n|^2 |\psi_n(r)|^2$). Instead of many particles, one deals here with a single wave that has many noninteracting Eigenstates. The exact density profile $\rho = |\psi|^2$ is given by
\[
\rho(r, t) = \sum_n |c_n|^2 |\psi_n(r)|^2 + \sum_{n \neq m} c_n^* c_m \psi_n(r) \psi_m(r) e^{i(E_m - E_n)t/\hbar}. \tag{42}
\]

The first term in Eq. (42) determines the smooth, stationary, profile of the DM halo (the one that can be compared to the NFW profile for example). Here, $p_n = |c_n|^2$ is a weighting factor which is proportional to the probability of the $n$-th state. The fundamental mode $n = 0$ (ground state) corresponds to the condensate (soliton) forming the quantum core and the excited states $n > 0$ give rise to the halo. The second term in Eq. (42), which is time-dependent, represents the interferences of the different Eigenstates. It gives rise to density granules – or quasiparticles – of size $\lambda_{\text{df}}$ that provide pressure support against self-gravity and that can cause a slow (secular) collisional evolution of the halo \[75–78\]. Developing this approach, Lin et al. \[113\] derived the classical particle DF $f(r, v)$ of the halo using analytical and numerical techniques and found that it is well-fitted by the fermionic King model \[80,96\].\(^{18}\) By contrast, the ground state (soliton) is a highly nonlinear object that cannot be described by this DF. In the following sections, we shall develop a complementary description of the core-halo structure of DM halos based on a kinetic approach.

**Remark** We note that the Schrödinger–Poisson equations \[10\] and \[11\], or the corresponding quantum Euler–Poisson equations \[17\]–\[20\] do not relax towards an equilibrium state. Indeed, the density $\rho(r, t)$ from Eq. (42) is always time-dependent due to small-scale interferences. However, if we locally average over these fluctuations, we get a smooth density that tends to an equilibrium state [first term in Eq. (42)]. This is very similar to the process of violent relaxation based on the Vlasov–Poisson equations \[C1\] and \[C2\] \[91\]. In that case, the fine-grained DF $f(r, v, t)$ never reaches an equilibrium state but develops filaments at smaller and smaller scales. However, if we locally average over this filamentation, we obtain a coarse-grained DF $\bar{f}(\mathbf{r}, \mathbf{v}, t)$ that rapidly relaxes towards a quasistationary state $\bar{f}_{\text{QSS}}(\mathbf{r}, \mathbf{v})$ (see Appendix D).

\(^{16}\) See Appendix B of \[104\] for a detailed proof of this result.

\(^{17}\) This is true in a "smoothed-out" sense where the density profile is averaged on a scale larger than the de Broglie length. Indeed, the small-scale interferences produce time-dependent granules (quasiparticles) of size $\lambda_{\text{df}}$ that can induce a slow collisional evolution of the halo \[75–78\]. We shall not consider this collisional evolution here and remain at a purely collisionless level.

\(^{18}\) In principle, for a system of bosons, we would expect that $f(r, v)$ is given by the Bose–Einstein or Rayleigh-Jeans DF. That would be the case if the statistical equilibrium state resulted from a collisional relaxation \[78\]. However, in the present context, the evolution of the system is collisionless and the DF is given by the Lynden-Bell DF which is similar to the Fermi–Dirac DF (see below). This is why Lin et al. \[113\] find that the DF of the halo is well-described by the Fermi–Dirac DF \[91,92\] or by the fermionic King model \[80,96\]. In the nondegenerate limit, these DFs reduce to the Boltzmann DF and to the classical King model.
3 Distribution function approach

3.1 Wigner equation

From the wavefunction $\psi(r, t)$ we can define the Wigner DF $f(r, v, t)$ by [114]

$$f(r, v, t) = \frac{m^3}{(2\pi \hbar)^3} \int dy \ e^{i m v \cdot y / \hbar} \times \psi^* \left( r + \frac{y}{2}, t \right) \psi \left( r - \frac{y}{2}, t \right). \quad (43)$$

The term in the second line represents the density matrix. One can check that $\int f dv = |\psi|^2 = \rho$ (see Appendix A). On the other hand, using the Schrödinger equation (10), one can show that $f(r, v, t)$ satisfies an equation of the form

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \frac{im^4}{(2\pi \hbar)^3 \hbar} \int e^{i m (v' - v) \cdot r / \hbar} \times \left[ \Phi \left( r + \frac{y}{2}, t \right) - \Phi \left( r - \frac{y}{2}, t \right) \right] f(r, v', t) dy dv' = 0, \quad (44)$$

which is called the Wigner (or Wigner-Moyal) equation [114,115]. It can be viewed as the quantum generalization of the Vlasov equation to which it reduces in the classical limit $\hbar \to 0$ (see Appendix A). For self-gravitating BECs, the Wigner equation must be coupled to the Poisson equation

$$\Delta \Phi = 4\pi G \int f dv. \quad (45)$$

Equations (44) and (45) form the Wigner–Poisson equations. The Wigner–Poisson equations conserve the energy $E = (1/2) \int f v^2 dr dv + (1/2) \rho \Phi dr$ and the mass $M = \int f dv dr$ (see Appendix A).

**Remark** The Wigner DF $f(r, v, t)$ is real but not necessarily positive. Therefore, it does not have the status of a true DF. One way to overcome this difficulty is to use the Husimi [116] representation which is essentially a smoothed version of the Wigner quasiprobability distribution.

3.2 Coarse-grained Wigner equation

As for the classical Vlasov–Poisson equations (see Refs. [94,96–101] and Appendix D), it is relevant to operate a coarse-graining on the Wigner–Poisson equations. This is especially justified at sufficiently large scales (i.e. in the halo) where quantum effects are weak and the system essentially behaves as a classical collisionless self-gravitating gas described by the Vlasov–Poisson equations. Therefore, it is expected to experience phase mixing, nonlinear Landau damping, and violent relaxation [7]. This coarse-grained description will allow us to take into account the processes of violent relaxation and gravitational cooling mentioned in Sect. 2.3.

As in the case of the classical Vlasov equation, the coarse-grained DF $\mathcal{F}(r, v, t)$ does not satisfy the Wigner equation because of phase-space correlations (see Appendix D2). These phase-space correlations introduce an effective “collision” term in the right hand side of the coarse-grained Wigner equation. For simplicity, we shall neglect quantum corrections in this “collision” term and use the same parametrization as for the classical process of violent relaxation.\(^{19}\) This parametrization is based on a heuristic MEPP (see Ref. [94] and Appendix D2). Therefore, we propose a coarse-grained Wigner equation of the form

$$\frac{\partial \mathcal{F}}{\partial t} + v \cdot \frac{\partial \mathcal{F}}{\partial r} - \frac{im^4}{(2\pi \hbar)^3 \hbar} \int e^{i m (v' - v) \cdot r / \hbar} \times \left[ \Phi \left( r + \frac{y}{2}, t \right) - \Phi \left( r - \frac{y}{2}, t \right) \right] \mathcal{F}(r, v', t) dy dv'$$

$$= \frac{\partial}{\partial v} \cdot \left[D \left( \frac{\partial \mathcal{F}}{\partial v} + \beta \mathcal{F}(1 - \mathcal{F}/\eta_0) v \right) \right], \quad (46)$$

coupled to the Poisson equation

$$\Delta \Phi = 4\pi G \int \mathcal{F} dv. \quad (47)$$

The left hand side of Eq. (46) is the usual Wigner term (we can write $\Phi$ instead of $\Phi$ since it is a smooth field produced by $f$ averaged over $v$). The right hand side of Eq. (46) can be interpreted as a fermionic Kramers (or Fokker–Planck) “collision” term. As a result, the coarse-grained Wigner equation (46) is similar to the fermionic Wigner–Kramers equation. The fermionic nature of the collision term arises from the Lynden-Bell exclusion principle ($\mathcal{F} \leq \eta_0$) introduced in the context of collisionless stellar systems [91]. Here, $\eta_0$ denotes the maximum value of the initial DF.\(^{20}\) The Lynden-Bell exclusion principle is similar to the

\(^{19}\) As we shall see, the effective collision term accounts for the formation of the halo. In the halo, quantum effects are erased on the coarse-grained scale (they manifest themselves only on the fine-grained scale through the presence of granules [75–78]). By contrast, we keep quantum effects in the left hand side of the coarse-grained Wigner equation (advection term). This term accounts for the formation of the soliton in which quantum effects are dominant.

\(^{20}\) If DM is made of fermions, we have $\eta_0 \sim m^4/\hbar^3$ (see footnote 34 in [80]). Therefore, degeneracy effects in the sense of Lynden-Bell are important. They lead to a quantum core in the form of a fermion ball. By contrast, the Heisenberg uncertainty principle (quantum potential) is negligible, or small, for fermions. For condensed bosons, this is the opposite. Indeed, $\eta_0$ is in general very large ($\eta_0 \gg m^4/\hbar^3$) so that degeneracy effects in the sense of Lynden-Bell are usually negligible ($\mathcal{F} \ll \eta_0$). By contrast, the Heisenberg uncertainty principle (quantum potential) is important for bosons. It leads to a quantum core in the form of a soliton. Here, for the sake of generality, we shall take into account all these effects although some of them may be negligible depending on the situation.
Pauli exclusion principle in quantum mechanics but with, of course, a completely different interpretation (see Appendix D1).

The first term in the fermionic Kramers collision operator is a diffusion and the second term is a friction. The friction may be interpreted as a form of nonlinear Landau damping since it is associated with a collisionless relaxation. The friction coefficient and the diffusion coefficient are linked by the Einstein relation

$$\xi = \beta D,$$

where $$\beta = 1/T_{\text{eff}}$$ is an inverse effective temperature (see Appendix D2). The Einstein relation (48) expresses the fluctuation-dissipation theorem. The diffusion coefficient $$D$$ is not given by the MEPP but it can be calculated by developing a quasilinear theory of the process of violent collisionless relaxation like in [96–101]. This theory is valid in a regime of “gentle” relaxation. In the present context, it leads to the Wigner–Landau equation

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \frac{im^2}{(2\pi \hbar)^3} \int e^{im(v-v')y/\hbar} \prod_{i=1}^{m} \left( \Phi \left( r + \frac{y_i}{2}, t \right) - \Phi \left( r - \frac{y_i}{2}, t \right) \right) f(r,v',t) dv' = \frac{\partial}{\partial v_j} K_{ij} \left\{ \left( 1 - \frac{T}{T_0} \right) \frac{\partial f}{\partial v_j} - \frac{1}{2} \frac{\partial^2 f}{\partial v_j^2} \right\},$$

where $$u = v' - v$$ is the relative velocity and $$\ln \Lambda = \ln(R/\epsilon_r)$$ the Coulomb logarithm. The Wigner–Kramers equation (46) is recovered from the Wigner–Landau equation (49) in a thermal bath approximation, and the diffusion coefficient can be explicitly calculated like in Ref. [96]. It is then explicitly demonstrated that the process of collisionless violent relaxation takes place on a few dynamical times.\(^{21}\)

\(^{21}\) The fact that the coarse-grained DF $$\bar{f}(r,v,t)$$ relaxes towards the Lynden-Bell DF on a few dynamical times is interpreted by Kadmenson and Pogutse [98] in terms of “collisions” between macroparticles with a large effective mass $$m_{\text{eff}} \approx \tau_0 c_\epsilon^2 c_s^2$$ (these macroparticles are fundamentally different from the quasiparticles introduced by [75] in relation to the collisional evolution of FDM halos on a secular timescale). In the foregoing equations, $$\epsilon_r(t)$$ and $$\epsilon_v(t)$$ denote the position and velocity correlation scales of the fluctuations $$\delta f$$ (see Appendix D2). These correlation scales are expected to decrease with time. If this decay is sufficiently rapid, it can slow down and even stop the collisionless relaxation before the Lynden-Bell DF is reached. This kinetic blocking can account for incomplete relaxation and solve the infinite mass problem of the Lynden-Bell DF (see [94,96,97,101] and more specifically [117] for additional discussion about the concept of incomplete relaxation). This may explain why DM halos have a NFW or Burkert profile instead of a Lynden-Bell profile (see footnote 7).

For classical particles ($$\hbar = 0$$), the fermionic Kramers collision term accounts for the relaxation of the coarse-grained DF towards the Lynden-Bell distribution

$$\bar{f}_{LB}(r,v) = \frac{\eta_0}{1 + e^{\epsilon(v^2/2 + \Phi(r)) - \alpha}},$$

where $$\epsilon = v^2/2 + \Phi(r)$$ is the individual energy of the particles by unit of mass. The Lynden-Bell distribution, which is similar to the Fermi–Dirac distribution, is a particular steady state of the Vlasov–Poisson equations (C1) and (C2). It represents the most probable state of the system at statistical equilibrium in the two-level case (see Appendix D1). It is obtained by maximizing the Lynden-Bell entropy at fixed mass and energy. This is also the only equilibrium state of the coarse-grained Vlasov equation (D33). It cancels individually the advection (Vlasov) term and the collision (fermionic Kramers) term. Now, for quantum systems ($$\hbar \neq 0$$), the Lynden-Bell distribution (51) is not an exact steady state of the Wigner equation (44) or of the coarse-grained Wigner equation (46) because it does not cancel the advection (Wigner) term exactly.\(^{22}\) However, in the present work, we shall disregard this difficulty. Indeed, we expect that the Lynden-Bell distribution provides a reasonable description of the halo where quantum effects are weak. In that case, it cancels the advection (Wigner) term approximately. Quantum effects are important in the core (soliton) and they are taken into account in the advection (Wigner) term whose cancelation is equivalent to the condition of quantum hydrostatic equilibrium from Eq. (36). By contrast, in the core, the Lynden-Bell distribution is usually subdominant. The distinction between a quantum core and an approximately isothermal halo (in the sense of Lynden-Bell) will become clear in the hydrodynamic representation of the coarse-grained Wigner equation developed below. The importance of violent relaxation in establishing an isothermal-like halo in the BECDM model was stressed in [81,93].

3.3 Truncated Lynden-Bell distribution

As is well-known, the Lynden-Bell distribution (51) coupled to the Poisson equation (47) generates configurations with an infinite mass.\(^{23}\) Therefore, strictly speaking, the coarse-grained distribution $$\bar{f}(r,v,t)$$ does not relax towards the Lynden-Bell distribution (51) since the mass is necessarily finite. This is related to the problem of incomplete violent relaxation [91,94] and to

\(^{22}\) Quantum effects (Heisenberg uncertainty principle) lead to a small modification of the Lynden-Bell DF of the order $$O(\hbar^2)$$ in the same manner that they lead to a small modification of the Boltzmann distribution for a system at thermal equilibrium [114].

\(^{23}\) Mathematically, this is because the Lynden-Bell distribution reduces to the Boltzmann distribution (D19) at large distances (where the system is dilute) so the density $$\rho$$ decreases as $$r^{-2}$$ like for the self-gravitating isothermal sphere [7].
the fact that the Lynden-Bell distribution does not take into account the escape of high energy particles. Nevertheless, the Lynden-Bell distribution is expected to be approximately valid for tightly bound particles with sufficiently negative energies ($\epsilon < 0$).

We can improve the Lynden-Bell distribution by taking into account the escape of unbound particles with positive energy ($\epsilon > 0$) or by taking into account tidal effects from neighboring systems. In particular, from the fermionic Vlasov-Kramers equation (D33), one can derive a truncated Lynden-Bell distribution of the form [96]

$$
\mathcal{f} = A \frac{e^{-\beta(\epsilon-\epsilon_m)} - 1}{\epsilon - \epsilon_m} \quad (\epsilon \leq \epsilon_m),
$$

$$
\mathcal{f} = 0 \quad (\epsilon \geq \epsilon_m),
$$

This distribution vanishes above a certain escape energy $\epsilon_m$ which is equal to zero for isolated systems and which is strictly negative for tidally truncated systems. The truncated Lynden-Bell distribution (52) is similar to the fermionic King model [80,96,118]. In the nondegenerate (dilute) limit, it becomes similar to the classical King model

$$
\mathcal{f} = A \left[ e^{-\beta(\epsilon-\epsilon_m)} - 1 \right] \quad (\epsilon \leq \epsilon_m),
$$

$$
\mathcal{f} = 0 \quad (\epsilon \geq \epsilon_m),
$$

which was introduced in relation to globular clusters evolving under the effect of two-body encounters [119]. In the present context, the thermalization of the system is due to Lynden-Bell’s type of violent collisionless relaxation and the fermionic nature of the DF is related to Lynden-Bell’s exclusion principle arising from the Vlasov equation.

The truncated Lynden-Bell distribution (or fermionic King model) has a finite mass and a maximum density in phase space which prevents gravitational collapse. It has been studied in detail in [80]. The corresponding configurations have a core-halo structure with a degenerate core similar to a “fermion ball” (a polytrope of index $n = 3/2$) and an isothermal halo. Because of the truncation, the isothermal halo does not extend to infinity. The density drops to zero at a finite radius identified with the tidal radius.

**Remark** In the context of BECDM halos, Lin et al. [113], by developing the model from Eqs. (41) and (42) and using results from numerical simulations, observed that the virialized state produced by gravitational cooling, in addition of containing a solitonic core (arising from the bosonic nature of the particles), has a DF consistent with the truncated Lynden-Bell distribution (52) and (53). This corroborates our previous qualitative arguments [93] according to which the (truncated) Lynden-Bell distribution provides a good description of the “atmosphere” of DM halos surrounding the solitonic core.

### 3.4 Quantum Jeans equations

Taking the hydrodynamic moments of the coarse-grained Wigner equation (46), we obtain the following equations:

$$
\frac{\partial f}{\partial t} + \nabla \cdot (f \mathbf{u}) = 0,
$$

$$
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{\rho} \partial_j P_{ij} - \nabla \Phi.
$$

where we have introduced the local density

$$
\rho = \int f \, d\mathbf{v},
$$

the local velocity

$$
\mathbf{u} = \frac{1}{\rho} \int f \mathbf{v} \, d\mathbf{v},
$$

and the pressure tensor

$$
P_{ij} = \int f (\mathbf{v} - \mathbf{u})_i (\mathbf{v} - \mathbf{u})_j \, d\mathbf{v}.
$$

Equations (56) and (57) are called the quantum damped Jeans equations. They coincide with the classical damped Jeans equations derived from the coarse-grained Vlasov equation (see Ref. [94] and the Appendix of [84]). Indeed, $\hbar$ does not explicitly appear in these equations. Explicit factors of $\hbar$ enter only in the higher moment equations of the hierarchy. We note that these hydrodynamic equations are not closed since the pressure tensor (60) depends on the coarse-grained DF $\mathcal{f}(\mathbf{v}, \mathbf{u}, t)$ which is not explicitly known in general. In the following, we propose a heuristic manner to close these equations by combining the results obtained in two extreme limits of our formalism corresponding to $D = 0$ and $\hbar = 0$ respectively.

#### 3.4.1 Hydrodynamic representation of the fine-grained Wigner equation

If we consider the fine-grained Wigner equation (44), there is no collision term ($D = \xi = 0$) and we obtain the quantum Jeans equations

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,
$$

$$
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{\rho} \partial_j P_{ij} - \nabla \Phi.
$$

24 For simplicity, we assume here that $D$ is constant.
25 More generally, we can build up an infinite hierarchy of such equations by taking the successive moments of the DF.
They coincide with the classical Jeans equations derived from the fine-grained Vlasov equation (see Ref. [7] and Appendix C2). Indeed, \( h \) does not explicitly appear in these equations. In the classical case \( (h = 0) \), these equations are not closed [7]. However, in the quantum case \( (h \neq 0) \), they can be closed! Indeed, since the Wigner equation \( (44) \) is equivalent to the Schrödinger equation \( (10) \), the quantum Jeans equations \( (61) \) and \( (62) \) obtained from the Wigner equation must coincide with the quantum Euler equations \( (17) \) and \( (19) \) obtained from the Schrödinger equation. This implies that the pressure tensor \( P_{ij} \) in Eq. \( (62) \) is exactly given by Eq. \( (23) \), i.e.,

\[
P_{ij} = P_{ij}^Q.
\]

This result can also be obtained by a direct calculation, substituting Eq. \( (43) \) into Eq. \( (60) \) and using Eqs. \( (14) \)–\( (16) \). In the same manner, one can show that the density defined by Eq. \( (58) \) and the velocity defined by Eq. \( (59) \) are equivalent to Eqs. \( (14) \)–\( (16) \). These calculations are detailed in Appendix A.

3.4.2 Hydrodynamic representation of the coarse-grained Vlasov equation

If we consider the process of violent relaxation but ignore quantum effects \( (h = 0) \), we are led back to the situation studied in Ref. [94] (see also the Appendix of Ref. [84]) based on the coarse-grained Vlasov equation. This leads to the classical damped Jeans equations that are equivalent to Eqs. \( (56) \) and \( (57) \) (see the comment made after Eq. \( (60) \)). These equations are not closed. In Ref. [94], it was proposed to compute the pressure tensor \( (60) \) by making a LTE approximation for the DF based on the Lynden-Bell statistics

\[
\bar{f}_{\text{LTE}}(\mathbf{r}, \mathbf{v}, t) = \frac{\eta}{1 + e^{(\mathbf{v} - \mathbf{u}(\mathbf{r}, t))^{2}/2 - \alpha(\mathbf{r}, t)}}, \tag{64}
\]

where \( \mathbf{u}(\mathbf{r}, t) \) is the local velocity \( (59) \) and \( \alpha(\mathbf{r}, t) \) is a local chemical potential which can be related to the local density \( \rho(\mathbf{r}, t) \) by substituting Eq. \( (64) \) into Eq. \( (58) \). If we now compute the pressure tensor \( (60) \) with the DF \( (64) \), we obtain

\[
P_{ij} = P_{\text{LB}}(\rho)\delta_{ij}, \tag{65}
\]

where \( P_{\text{LB}}(\rho) \) is the Lynden-Bell equation of state which is similar to the Fermi–Dirac equation of state (see Appendix D1). On the other hand, in the last term of Eq. \( (57) \), we shall make for simplicity the approximation

\[
\int \bar{f}(1 - \bar{f}/\gamma_0)\mathbf{v} \, d\mathbf{v} \simeq \rho \mathbf{u}, \tag{66}
\]

which amounts to neglecting degeneracy effects in the friction term. This is a relevant approximation in the dilute halo which is nondegenerate or weakly degenerate (in the sense of Lynden-Bell). With the closure from Eq. \( (65) \) and the approximation from Eq. \( (66) \), the damped Jeans equation \( (57) \) becomes

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P_{\text{LB}} - \nabla \varphi - \xi \mathbf{u}, \tag{67}
\]

where \( P_{\text{LB}}(\rho) \) is the Lynden-Bell pressure and \( \xi \) is the friction coefficient given by the Einstein relation \( (48) \).

Eq. \( (67) \) is called the damped Euler equation. The Lynden-Bell equation of state is given in parametric form by Eqs. \( (D9) \) and \( (D10) \). It has the same form as the Fermi–Dirac equation of state in quantum mechanics except that \( g \eta^{3}/\hbar^{3} \) (where \( g \) is the multiplicity of the quantum states) is replaced by \( \gamma_0 \), the maximum value of the DF. The Lynden-Bell equation of state is essentially an isothermal equation of state with a modification at high densities taking into account the Lynden-Bell exclusion principle which is similar to the Pauli exclusion principle in quantum mechanics. In the nondegenerate limit, valid at low densities, it reduces to the classical isothermal equation of state

\[
P_{\text{th}} = \rho T_{\text{eff}} \tag{68}
\]

with an effective temperature \( T_{\text{eff}} \). In the completely degenerate limit, valid at high densities, we get a pressure of zero-point energy

\[
P_{\text{LB}}^{(0)} = \frac{1}{5} \left( \frac{3}{4\pi \gamma_0} \right)^{2/3} \rho^{5/3}, \tag{69}
\]

corresponding to a polytrope of index \( \gamma = 5/3 \) and polytropic constant \( K = (1/5)(3/4\pi \gamma_0)^{2/3} \) like in the theory of nonrelativistic white dwarf stars [120]. The complete Lynden-Bell equation of state can be conveniently approximated by

\[
P_{\text{LB}} = P_{\text{th}} + P_{\text{LB}}^{(0)} = \rho T_{\text{eff}} + \frac{1}{5} \left( \frac{3}{4\pi \gamma_0} \right)^{2/3} \rho^{5/3}, \tag{70}
\]

where the first term accounts for the effective temperature and the second term accounts for the Lynden-Bell exclusion principle. When coupled to gravity the Lynden-Bell equation of state leads to configurations with a core-halo structure. They are made of a “fermionic” core (in the sense of Lynden-Bell) and an isothermal halo. These core-halo structures have been computed in [92] in relation to the Lynden-Bell theory of violent relaxation but they also appear in numerous works on fermionic DM halos in which the DM particle

\[\footnote{The Lynden-Bell pressure and the friction term are in some sense related to the correlation function \( \delta \rho \delta \Phi \) that emerges from the coarse-graining of the quantum Euler–Poisson equations \( (17) \)–\( (20) \), where \( \delta \rho \) and \( \delta \Phi \) denote the fluctuations about the coarse-grained (smooth) fields. However, the proper description of these correlations requires the analysis of Appendix D2 in phase space.}
is a fermion like a massive neutrino (see the Introduction of Ref. [42] and Sec. V.A of [81] for an exhaustive list of references).

3.4.3 Hydrodynamic representation of the coarse-grained Wigner equation

We now consider the coarse-grained Wigner equation (46) and propose to close the quantum damped Jeans equations (56) and (57) by simply superposing the results obtained previously. Therefore, we propose to approximate the pressure tensor of Eq. (60) by

\[ P_{ij} = P_{ij}^Q + P_{LB}(\rho)\delta_{ij}, \]

where \( P_{ij}^Q \) is the quantum pressure tensor (see Sect. 3.4.1) and \( P_{LB}(\rho) \) is the Lynden-Bell pressure (see Sect. 3.4.2).

Remark In principle, there is a correction of order \( O(h^2) \) in the Lynden-Bell pressure due to the effect of the Heisenberg uncertainty principle (see footnote 22) and the presence of the granules in the halo. However, we shall neglect this small correction.

4 Heuristic equations parameterizing the complex dynamics of BECDM halos

4.1 Hydrodynamic equations

Combining the previous results, we find that the hydrodynamic equations parameterizing the complex dynamics of BECDM halos in our model are

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \]
\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P_{LB} - \frac{1}{\rho} \nabla P_{\text{int}} - \frac{1}{m} \nabla Q - \nabla \Phi - \xi \mathbf{u}, \]
\[ \Delta \Phi = 4\pi G \rho \].

(72) (73) (74)

For the sake of generality, we have considered the case of self-interacting BECs and we have added the pressure

\[ P_{\text{int}}(\rho) = \frac{2\pi a_s h^2}{m^3} \rho^2 \]

due to their self-interaction (see, e.g., Ref. [52]).\(^{27}\) This is a polytropic equation of state of index \( \gamma = 2 \) and polytropic constant \( K = 2\pi a_s h^2/m^3 \). When \( P_{LB} = 0 \) and \( \xi = 0 \), we recover the hydrodynamic equations associated with the standard GPP equations (8) and (9).

\(^{27}\) According to Eq. (8), this amounts to making the replacement \( \Phi \rightarrow \Phi + 4\pi a_s h^2\rho/m^3 \) in Eq. (67).

However, the process of violent relaxation generates an additional pressure \( P_{LB} \) and a friction \( \xi \).

If we introduce the total pressure \( P = P_{LB} + P_{\text{int}} \), the quantum damped Euler equation (73) can be rewritten as

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P_{LB} - \frac{1}{\rho} \nabla P_{\text{int}} - \frac{1}{\rho} \nabla P_{\text{th}}^Q - \frac{1}{m} \nabla Q - \nabla \Phi - \xi \mathbf{u}. \]

(76)

On the other hand, if we use the approximate expression of the Lynden-Bell equation of state from Eq. (70), we obtain

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P_{\text{poly}} - \frac{1}{\rho} \nabla P_{\text{th}} - \frac{1}{m} \nabla Q - \nabla \Phi - \xi \mathbf{u}, \]

(77)

involving a general polytropic equation of state

\[ P_{\text{poly}} = K \rho^\gamma \]

(79)

and an isothermal equation of state

\[ P_{\text{th}} = \rho T_{\text{eff}}. \]

(80)

We can then incorporate several polytropic equations of state in this model by simply summing the pressures.

Remark In the strong friction limit \( \xi \rightarrow +\infty \), we can neglect the inertial term (l.h.s.) in the damped quantum Euler equation (73) and substitute the resulting equation into the continuity equation (72), thereby obtaining the quantum Smoluchowski–Poisson equations [121]

\[ \frac{\xi \partial \rho}{\partial t} = \nabla \cdot \left( \nabla P_{\text{int}} + \nabla P_{LB} + \frac{\rho}{m} \nabla Q + \rho \nabla \Phi \right), \]
\[ \Delta \Phi = 4\pi G \rho. \]

(81) (82)

If we neglect the quantum potential, making the TF approximation, we recover the classical Smoluchowski–Poisson equations

\[ \frac{\xi \partial \rho}{\partial t} = \nabla \cdot \left( \nabla P_{\text{int}} + \nabla P_{LB} + \rho \nabla \Phi \right), \]
\[ \Delta \Phi = 4\pi G \rho. \]

(83) (84)

that have been exhaustively studied in [122] and references therein. These equations were introduced in relation to a rather academic model of self-gravitating
Brownian particles [123]. The present study suggests that they may have some applications in the context of DM (see Ref. [124] for an illustration of the formation of a DM halo with a core-halo structure in the framework of these equations). We can also obtain an equation intermediate between the quantum Euler equation and the quantum Smoluchowski equation. It has a form [93]

\[
\frac{\partial^2 \rho}{\partial t^2} + \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \nabla P_{\text{int}} + \nabla P_{\text{LB}} + \frac{\rho}{m} \nabla Q + \rho \nabla \Phi \right)
\]  

similar to the telegrapher’s equation.

4.2 Wave equation

We can now use the Madelung transformation of Sect. 2.2 backwards to derive the generalized wave equation corresponding to the generalized hydrodynamic equations (72) and (73). If we first neglect the terms corresponding to violent relaxation (\(\xi = P_{\text{LB}} = 0\)), the hydrodynamic equations (72) and (73) reduce to

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (86)
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P_{\text{int}} - \frac{1}{m} \nabla Q - \nabla \Phi. \quad (87)
\]

They correspond to a GP equation of the form [52]

\[
\imath \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \left[ \Phi + h_{\text{int}}(|\psi|^2) \right] \psi, \quad (88)
\]

where the effective potential \(h_{\text{int}}(|\psi|^2)\), which can be interpreted as an enthalpy, takes into account the self-interaction of the bosons. For a general barotropic equation of state \(P_{\text{int}}(\rho)\), it is determined by the relation [52,93]

\[
h'_{\text{LB}}(\rho) = \frac{P'_{\text{int}}(\rho)}{\rho}. \quad (89)
\]

For a general polytropic equation of state of the form of Eq. (79) we have

\[
h_{\text{poly}}(\rho) = \frac{K \gamma}{\gamma - 1} |\psi|^2(\gamma - 1), \quad (90)
\]

For the standard BEC with the polytropic equation of state (75), we obtain

\[
h_{\text{int}}(|\psi|^2) = \frac{4\pi a_s \hbar^2}{m^3} |\psi|^2, \quad (91)
\]

leading to the standard GP equation (8).

If we now account for violent relaxation, we obtain the generalized GPP equations

\[
\imath \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \left[ \Phi + h_{\text{int}}(|\psi|^2) + h_{\text{LB}}(|\psi|^2) \right] \psi
\]

\[\left(-\frac{\hbar^2}{2m} + \frac{\xi}{2} \int \frac{\psi^2}{\psi^*} - \int \frac{\psi}{\psi^*} \right) \psi, \quad (92)\]

\[
\Delta \Phi = 4\pi G |\psi|^2. \quad (93)
\]

Equation (92) is the wave equation associated with Eqs. (72) and (73). As compared to Eq. (88) there are two new terms. The Lynden-Bell enthalpy \(h_{\text{LB}}(|\psi|^2)\) and the friction term \(\xi\). The Lynden-Bell enthalpy is determined by the Lynden-Bell equation of state \(P_{\text{LB}}(\rho)\) through the relation

\[
h'_{\text{LB}}(\rho) = \frac{P'_{\text{LB}}(\rho)}{\rho}. \quad (94)
\]

Apparently, it is not possible to give a simple explicit expression of \(h_{\text{LB}}\). If we introduce the total enthalpy \(h = h_{\text{LB}} + h_{\text{int}}\), the generalized GP equation (92) can be written as

\[
\imath \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \left[ \Phi + h(|\psi|^2) \right] \psi
\]

\[\left(-\frac{\hbar^2}{2m} + \frac{\xi}{2} \int \frac{\psi^2}{\psi^*} - \int \frac{\psi}{\psi^*} \right) \psi, \quad (95)\]

This is the wave equation associated with Eqs. (72) and (76). It is of the form of the generalized GPP equations introduced and studied in [93] (see also Appendix B). If we use the approximate expression of the Lynden-Bell equation of state from Eq. (70), we get

\[
h_{\text{LB}}(|\psi|^2) = h^{(0)}_{\text{LB}}(|\psi|^2) + h^{\text{th}}_{\text{LB}}(|\psi|^2) \quad (96)
\]

with

\[
h^{(0)}_{\text{LB}}(|\psi|^2) = \frac{1}{2} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} |\psi|^{4/3} \quad (97)
\]

and

\[
h^{\text{th}}_{\text{LB}}(|\psi|^2) = T_{\text{eff}} \ln(|\psi|^2). \quad (98)
\]
In that case, the generalized GP equation (92) can be written as

\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta \psi + mT_{\text{eff}} \ln(|\psi|^2)\psi + m\left[\Phi + h_{\text{int}}(|\psi|^2) + h_{\text{LB}}^{(0)}(|\psi|^2)\right] \psi
\]

\[
-\frac{\hbar}{2} \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi,
\]

where the effective thermal term has been made explicit. This is the wave equation associated with Eq. (77). Equation (99) may be viewed as a coarse-grained GP equation parameterizing the processes of violent relaxation and gravitational cooling. For the standard BEC, we get

\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta \psi + mT_{\text{eff}} \ln(|\psi|^2)\psi + 4\pi\alpha_s \hbar^2 \rho + \frac{3}{8\pi\eta_0} m^2 \rho^{4/3} \psi + m\Phi \psi
\]

\[
-\frac{\hbar}{2} \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi,
\]

where all the terms have been made explicit. The structural wave equation associated with Eqs. (72) and (78) is

\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \Delta \psi + mT_{\text{eff}} \ln(|\psi|^2)\psi + K\gamma m \rho^{2(\gamma-1)} \psi + m\Phi \psi
\]

\[
-\frac{\hbar}{2} \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi.
\]

Remark It is interesting to note that, at a formal level, the generalized GP equation (95) allows us to make a connection between the Schrödinger (or GP) equation of quantum mechanics ($\xi = 0$) and the (generalized) Smoluchowski equation of Brownian theory ($\xi \to +\infty$) [93]. We also note that the generalized GP equation (99) including a temperature term and a friction term can be obtained from a “unified” formalism based on a generalization of the theory of scale relativity to the case of dissipative systems [125,126]. In this unified formalism, the temperature and the friction appear as two manifestations of the same concept.

5 Equilibrium states

The generalized GPP equations (92) and (93), or equivalently the hydrodynamic equations (72)–(74), satisfy an $H$-theorem for a generalized free energy $F = \Theta_{\text{E}} + \Theta_{Q} + U_{\text{int}} + U_{\text{LB}} + W$ (see [93] and Appendix B for the definition of the different functionals) and relax towards a stable equilibrium state which minimizes $F$ at fixed mass $M$. We can determine this equilibrium state in different manners.

The equation of quantum hydrostatic equilibrium, which corresponds to the steady state of the quantum Euler equation (73), reads

\[
\nabla P_{\text{LB}} + \nabla P_{\text{int}} + \frac{\rho}{m} \nabla Q + \rho \nabla \Phi = 0.
\]

This equation describes the balance between the Lynden-Bell pressure, the pressure due to the self-interaction of the bosons, the quantum potential, and the gravitational force. If we introduce the total pressure $P = P_{\text{LB}} + P_{\text{int}}$, we can rewrite Eq. (102) as [93]

\[
\frac{\rho}{m} \nabla Q + \nabla P + \rho \nabla \Phi = 0.
\]

This is the equilibrium state of Eq. (76). Combined with the Poisson equation (74) we obtain the fundamental equation of quantum hydrostatic equilibrium determining the structure of DM halos [93]

\[
\frac{\hbar^2}{2m^2} \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \nabla \cdot \left( \frac{\nabla P}{\rho} \right) = 4\pi G \rho.
\]

The foregoing equations can also be obtained from the generalized GPP equations (92) and (93). Writing $\psi(r,t) = \phi(r)e^{-iEt/\hbar}$, where $\phi(r) = \sqrt{\rho(r)}$ and $E$ are real, the stationary solutions of the generalized GPP equations are determined by the Eigenvalue problem [93]

\[
-\frac{\hbar^2}{2m^2} \Delta \phi + m \left[ \Phi + h_{\text{int}}(\rho) + h_{\text{LB}}(\rho) \right] \phi = E\phi,
\]

\[
\Delta \Phi = 4\pi G \phi^2,
\]

where $E$ is the Eigenenergy. Dividing Eq. (105) by $\phi$, we get

\[
Q + m\Phi + mh_{\text{int}}(\rho) + mh_{\text{LB}}(\rho) = E.
\]

Taking the gradient of this equation and using Eqs. (89) and (94), we recover the condition of quantum hydrostatic equilibrium from Eq. (102).

29 The $H$-theorem and the relaxation towards an equilibrium state are due to the friction term $\xi > 0$ which provides a source of dissipation and implies the irreversibility of the generalized GPP equations (92) and (93). By contrast, the usual GP equations (8) and (9) conserve the energy and are reversible. Their relaxation towards a quasistationary state is due to gravitational cooling and violent relaxation and can be understood only at a coarse-grained scale (see Sects. 2.3 and 2.4). It is in this sense that the generalized GPP equations (92) and (93) provide a parametrization of the ordinary GPP equations (8) and (9) taking into account the processes of gravitational cooling and violent relaxation.
Finally, an equilibrium state of the GPP equations can be obtained by extremizing the free energy $F$ at fixed mass $M$. Writing the variational principle as

$$
\delta F - \frac{\mu}{m} \delta M = 0, \quad (108)
$$

where $\mu/m$ is a Lagrange multiplier (global chemical potential) taking into account the mass constraint, we get

$$
Q + m\Phi + mh_{\text{int}}(\rho) + mh_{\text{LB}}(\rho) = \mu, \quad (109)
$$

which is equivalent to Eq. (107) with $E = \mu$. This establishes that the Eigenenergy is equal to the global chemical potential. Furthermore, one can show that an equilibrium state of the generalized GPP equations is stable if, and only if, it is a minimum of free energy at fixed mass ($\delta^2 F > 0$ for all perturbations that conserve mass). These results are a consequence of the H-Theorem [93].

### 5.1 Core-halo structure

We now show that the equilibrium states of the GPP equations (92) and (93) have a core-halo structure. If we use the approximate expression of the Lynden-Bell equation of state from Eq. (70), we can rewrite Eq. (102) as

$$
\frac{\rho}{m} \nabla Q + \nabla P^{(0)}_{\text{LB}} + \nabla P_{\text{int}} + \nabla P^{\text{th}}_{\text{LB}} + \rho \nabla \Phi = 0. \quad (110)
$$

It corresponds to a structural equation of the form

$$
\frac{\rho}{m} \nabla Q + \nabla P_{\text{poly}} + \nabla P_{\text{th}} + \rho \nabla \Phi = 0, \quad (111)
$$

which is the equilibrium state of Eq. (78). It involves a polytropic equation of state and an isothermal equation of state. The total pressure is [81,125]

$$
P = K\rho^\gamma + \rho T_{\text{eff}}. \quad (112)
$$

Combining Eq. (111) with the Poisson equation (74) we obtain the fundamental differential equation [81,125]

$$
\frac{\hbar^2}{2m^2} \Delta \left( \frac{\nabla \rho}{\sqrt{\rho}} \right) - \frac{K \gamma}{\gamma - 1} \Delta \rho^{\gamma - 1} - T_{\text{eff}} \Delta \ln \rho = 4\pi G\rho. \quad (113)
$$

For noninteracting BECs it reduces to

$$
\frac{\hbar^2}{2m^2} \Delta \left( \frac{\nabla \rho}{\sqrt{\rho}} \right) - T_{\text{eff}} \Delta \ln \rho = 4\pi G\rho. \quad (114)
$$

In the TF approximation, we get

$$
- \frac{K \gamma}{\gamma - 1} \Delta \rho^{\gamma - 1} - T_{\text{eff}} \Delta \ln \rho = 4\pi G\rho. \quad (115)
$$

If we define

$$
\rho = \rho_0 e^{-\psi}, \quad \xi = \left( \frac{4\pi G\rho_0}{T_{\text{eff}}} \right)^{1/2} r, \quad (116)
$$

$$
\chi = \frac{K\gamma \rho_0^{\gamma - 1}}{T_{\text{eff}}}, \quad \epsilon = \frac{2\pi G\rho_0 \hbar^2}{m^2 T_{\text{eff}}^2}, \quad (117)
$$

where $\rho_0$ is the central density, we find that Eq. (113) takes the form of a generalized Emden equation [81,125]

$$
\epsilon \Delta \left( e^{\psi/2} \Delta e^{-\psi/2} \right) + \Delta \psi + \chi \nabla \cdot \left[ e^{-(\gamma - 1)\psi} \nabla \psi \right] = e^{-\psi}. \quad (118)
$$

For noninteracting BECs it reduces to

$$
\epsilon \Delta \left( e^{\psi/2} \Delta e^{-\psi/2} \right) + \Delta \psi = e^{-\psi}. \quad (119)
$$

In the TF approximation, we get

$$
\Delta \psi + \chi \nabla \cdot \left[ e^{-(\gamma - 1)\psi} \nabla \psi \right] = e^{-\psi}. \quad (120)
$$

Alternatively, if we define

$$
\rho = \rho_0 \theta^n, \quad \xi = \left( \frac{4\pi G\rho_0}{T_{\text{eff}}} \right)^{1/2} \frac{n}{(n + 1)\rho_0^{1/n - 1}} r, \quad (121)
$$

$$
\sigma = \frac{\epsilon}{\chi^2} = \frac{2\pi G \hbar^2}{K^2 \gamma^2 m^2 \rho_0^{2\gamma - 3}}, \quad (122)
$$

we find that Eq. (113) takes the form of a generalized Lane-Emden equation [81,125]

$$
- \frac{\sigma}{n^2} \Delta \left( \frac{\Delta \theta^{n/2}}{\theta^{n/2}} \right) + \frac{1}{\chi} \Delta \ln \theta + \Delta \theta = -\theta^n. \quad (123)
$$

In the TF approximation, we get

$$
\frac{1}{\chi} \Delta \ln \theta + \Delta \theta = -\theta^n. \quad (124)
$$

The above equations describe the balance between the quantum potential taking into account the Heisenberg uncertainty principle, the pressure due to the self-interaction of the bosons, the pressure due to the Lynden-Bell exclusion principle, the pressure due to effective thermal effects, and the self-gravity. The solutions have a core-halo structure with a quantum core and an isothermal halo (see Ref. [81] for explicit calculations of DM halos in the case of a standard self-gravitating BEC with repulsive self-interaction corresponding to a polytropic index $\gamma = 2$ in the TF approximation). The quantum core has a bosonic nature due to the Heisenberg uncertainty principle (soliton) and to the self-interaction of the particles leading to a polytropic core of index $n = 1$ (boson ball). It also has a fermionic nature in the sense of Lynden-Bell leading to a polytropic core of index $n = 3/2$ (fermion...
Remark Recalling the expressions of $P_{\text{LB}}^{(0)}$ and $P_{\text{int}}$ [see Eqs. (69) and (75)], we find that the total pressure is explicitly given by

$$P = \frac{1}{5} \left( \frac{3}{4 \pi \eta_0} \right)^{2/3} \rho^{5/3} + \frac{2 \pi a_s \hbar^2}{m^3} \rho^2 + \rho T_{\text{eff}}. \quad (125)$$

Neglecting the Lynden-Bell pressure and making the TF approximation, we obtain a differential equation of the form

$$-\frac{4 \pi a_s \hbar^2}{m^2} \Delta \rho - T_{\text{eff}} \Delta \ln \rho = 4 \pi G \rho. \quad (126)$$

The equation of state (125) and the differential equation (126) have been studied specifically in [81] (see Fig. 1). On the other hand, in Appendix E of [102] we have shown that the soliton resulting from the equilibrium between the gravitational attraction and the quantum repulsion (Heisenberg uncertainty principle) is similar to a polytrope of index $\gamma = 3/2$ (i.e. $n = 2$) with an equation of state

$$P = \left( \frac{2 \pi G \hbar^2}{9 m^2} \right)^{1/2} \rho^{3/2}. \quad (127)$$

depending on the gravitational constant $G$. Therefore, to compute the structure of a soliton surrounded by an isothermal halo, instead of solving Eq. (114) with the quantum term, we can solve Eq. (115) without the quantum term but with the pressure from Eq. (127). This leads to a differential equation of the form

$$-\left( \frac{2 \pi G \hbar^2}{m^2} \right)^{1/2} \Delta \sqrt{\rho} - T_{\text{eff}} \Delta \ln \rho = 4 \pi G \rho, \quad (128)$$

corresponding to a total pressure

$$P = \left( \frac{2 \pi G \hbar^2}{9 m^2} \right)^{1/2} \rho^{3/2} + \rho T_{\text{eff}}. \quad (129)$$

5.2 Quantum core

In the core, we can neglect effective thermal effects and take $P_{\text{LB}}^{\text{th}} = 0$. In that case, Eq. (110) reduces to

$$\frac{\rho}{m} \nabla Q + \nabla P_{\text{LB}}^{(0)} + \nabla P_{\text{int}} + \rho \nabla \Phi = 0. \quad (130)$$

It corresponds to a structural equation of the form

$$\frac{\rho}{m} \nabla Q + \nabla P_{\text{poly}} + \rho \nabla \Phi = 0. \quad (131)$$

Combined with the Poisson equation (74), we get

$$\frac{\hbar^2}{2m^2} \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) - \frac{K \gamma}{\gamma - 1} \Delta \rho^{\gamma - 1} = 4 \pi G \rho. \quad (132)$$

For noninteracting BECs it reduces to

$$\frac{\hbar^2}{2m^2} \Delta \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = 4 \pi G \rho. \quad (133)$$

In the TF approximation, we get

$$-\frac{K \gamma}{\gamma - 1} \Delta \rho^{\gamma - 1} = 4 \pi G \rho. \quad (134)$$

With the change of variables from Eq. (121), Eq. (132) takes the form of a quantum Lane–Emden equation

$$-\frac{\sigma}{m^2} \Delta \left( \frac{\Delta \theta^{n/2}}{\theta^{n/2}} \right) + \Delta \theta = -\theta^n. \quad (135)$$

In the TF approximation, we recover the ordinary Lane–Emden equation [120]

$$\Delta \theta = -\theta^n. \quad (136)$$

The equilibrium of the core is due to the balance between the quantum potential, the self-interaction of the bosons, the pressure due to the Lynden-Bell exclusion principle, and the gravitational attraction. Equation (132) with $\gamma = 2$ corresponding to a standard BEC has been solved analytically (using a Gaussian ansatz) in [52] and numerically in [108] for an arbitrary (repulsive, attractive or vanishing) self-interaction. It describes a compact quantum core. Because of quantum effects (or because of the Lynden-Bell exclusion principle), the central density is finite instead of diverging as in the CDM model. Therefore, quantum mechanics is able to solve the core-cusp problem.

5.3 Isothermal halo

In the halo, we can neglect quantum effects and take $Q = P_{\text{LB}}^{(0)} = 0$ and $P_{\text{int}} = 0$. In that case, Eq. (110) reduces to

$$\nabla P_{\text{LB}}^{\text{th}} + \rho \nabla \Phi = 0. \quad (137)$$
Combined with the Poisson equation (74), we get
\[ -T_{\text{eff}} \Delta \ln \rho = 4\pi G \rho, \]  
which is equivalent to the ordinary Emden equation [120]
\[ \Delta \psi = e^{-\psi}. \]  
The equilibrium of the halo is due to the balance between the effective thermal pressure and the gravitational attraction. The Boltzmann-Poisson equation (138), or the Emden equation (139), has no simple analytical solution and must be solved numerically (see, e.g., [81]). However, its asymptotic behavior is known analytically [120]. The density of a self-gravitating isothermal halo decreases as \( \rho(r) \sim T_{\text{eff}}/(2\pi G r^2) \) for \( r \to +\infty \), corresponding to an accumulated mass \( M(r) \sim 2T_{\text{eff}} r/G \) increasing linearly with \( r \). This leads to the rotation curves
\[ v^2(r) = \frac{G M(r)}{r} \to v_{\infty}^2 = 2T_{\text{eff}}, \]  
in agreement with the observations [7].

5.4 Summary

In conclusion, the physical meaning of the generalized GPP equations (92) and (93) is clear. The damping term forces the system to relax towards a stable equilibrium state with a core-halo structure. The friction term and the thermal term provide a parametrization of gravitational cooling and violent relaxation. The quantum core is able to solve the core-cusp problem and the isothermal halo accounts for the flat rotation curves of the galaxies. This core-halo structure is in agreement with the phenomenology of BECDM halos.

6 Canonical and microcanonical ensembles

In the previous sections, we have developed a canonical ensemble description in which the effective temperature \( T_{\text{eff}} \) is fixed. The corresponding thermodynamic potential is the free energy (see Appendix B)
\[ F = \Theta_c + \Theta_Q + U_{\text{int}} + U_{\text{LB}}^{(0)} + W + U_{\text{LB}}^{th}, \]  
where
\[ \Theta_c = \int \rho \frac{u^2}{2} \, dr, \]  
\[ \Theta_Q = \frac{1}{m} \int \rho Q \, dr, \]  
\[ U_{\text{int}} = \frac{2\pi a_s h^2}{m^3} \int \rho^2 \, dr, \]  
\[ U_{\text{LB}}^{(0)} = \frac{3}{10} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \int \rho^{5/3} \, dr, \]  
\[ W = \frac{1}{2} \int \rho \Phi \, dr, \]  
\[ U_{\text{LB}}^{th} = T_{\text{eff}} \int \rho (\ln \rho - 1) \, dr, \]
are the classical kinetic energy, the quantum kinetic energy, the internal energy of self-interaction, the internal energy associated with the Lynden-Bell pressure of zero-point energy, the gravitational energy, and the internal energy associated with the Lynden-Bell thermal pressure. The generalized GPP equations (93) and (99) satisfy an \( H \)-theorem for the free energy (141) and relax towards an equilibrium state which minimizes \( F \) at fixed mass \( M \) (see Appendix B3). This equilibrium state is canonically stable. However, for self-gravitating systems, the statistical ensembles are inequivalent [128–131]. One can show that an equilibrium state that is canonically stable is necessarily microcanonically stable but the reciprocal is wrong [132]. There exist equilibrium states that are microcanonically stable while being canonically unstable. Therefore, we may miss important equilibrium states by using a canonical description instead of a microcanonical one. For example, it was shown in Ref. [102] that the core-halo configurations of BECDM halos with a quantum core and an isothermal halo, similar to those observed in numerical simulations, are canonically unstable while being generically microcanonically stable. These core-halo configurations have a negative specific heat. We know that systems with negative specific heats are unstable in the canonical ensemble while they may be stable in the microcanonical ensemble. Therefore, the generalized GPP equations (93) and (99) with a constant temperature \( T_{\text{eff}} \) do not relax towards the important core-halo configurations observed in direct numerical simulations since they are unstable in the canonical ensemble. This is a drawback of this model.

In addition to these considerations of thermodynamic stability, there is no reason why the effective temperature \( T_{\text{eff}} \) of the halo should be constant. On the contrary, it should adapt itself so as to conserve the total energy. Indeed, according to the discussion of Sect. 2.3, the mass \( M - M_c \) and the energy \( E_{\text{tot}} - E_c \) that are not contained in the core should be stored in the halo.

It is therefore important to develop a microcanonical model of BECDM halos where the total energy is fixed. This can be simply achieved by letting the effective temperature \( T_{\text{eff}}(t) \) depend on time so as to conserve the total energy
\[ E_{\text{tot}} = \Theta_c + \Theta_Q + U_{\text{int}} + U_{\text{LB}}^{(0)} + W + \Theta_{\text{LB}}^{th}, \]  
where
\[ \Theta_{\text{LB}}^{th} = \frac{3}{2} M T_{\text{eff}}. \]

30 This is also the Lynden-Bell entropy multiplied by \(-T_{\text{eff}}\) (see Appendix B3). As a result, we can write \( F = E_{\text{tot}} - T_{\text{eff}} S \) up to constant additive terms.

31 See [81,102] for a detailed discussion.
is the Lynden-Bell thermal energy (see Appendix B2). Therefore, \( T_{\text{eff}}(t) = 2(E_{\text{tot}} - \Theta_c(t) - \Theta_Q(t) - U_{\text{int}}(t) -\frac{U_{\text{LB}}(t)}{M}) \). In line with the discussion of Sect. 5, the total energy \( E_{\text{tot}} = E_c + E_h \) can be decomposed into the energy of the core \( E_c \simeq (\Theta_c)_c + \Theta_Q + U_{\text{int}} + U_{\text{LB}}(t) + W_c \) dominated by quantum effects and the energy of the halo \( E_h \simeq (\Theta_c)_h + \Theta_{\text{LB}}^h + W_h \) dominated by thermal effects. One can show that the generalized GPP equations (93) and (99) with the time-dependent temperature \( T_{\text{eff}}(t) \) satisfy an \( H \)-theorem for the Lynden-Bell entropy

\[
S = - \int \rho (\ln \rho - 1) \, d\mathbf{r} + \frac{3}{2} M \ln T_{\text{eff}}, \tag{150}
\]

and relax towards an equilibrium state which maximizes \( S \) at fixed mass \( M \) and energy \( E_{\text{tot}} \) (see Appendix B2). This equilibrium state is microcanonically stable. As a result, the generalized GPP equations (93) and (99) with a time-dependent temperature \( T_{\text{eff}}(t) \) determined by Eqs. (148) and (149) with \( E_{\text{tot}} \) fixed, relax towards a core-halo structure, similar to the one observed in direct numerical simulations of BECDM, since it is stable in the microcanonical ensemble. Furthermore, we showed in Ref. [102] that the core mass-halo mass relation \( M_c(M_h) \) obtained by maximizing the entropy at fixed mass and energy reproduces the relation observed in direct numerical simulations. This confirms that our effective thermodynamical description is relevant to describe BECDM halos. At equilibrium, the virial theorem reads (see Appendix B2)

\[
2(\Theta + \Theta_{\text{LB}}) + W = 0, \tag{151}
\]

where \( E_{\text{tot}} = \Theta + \Theta_{\text{LB}} + W \) is constant (we consider noninteracting systems \( U_{\text{int}} = 0 \) for simplicity). This relation is in agreement with Eq. (32) with \( (W') \simeq W \) and \( (\Theta) = \Theta + \Theta_{\text{LB}} \), where \( \Theta_{\text{LB}} = \Theta_{\text{LB}}^0 + \Theta_{\text{LB}} \) is the Lynden-Bell energy taking into account fine-grained correlations.

**Remark** For simplicity, we have assumed that the temperature \( T_{\text{eff}}(t) \) is uniform. We can develop a more general model where the temperature \( T(r,t) \) is spatially inhomogeneous (see Ref. [94] and Appendix D4). Its evolution equation can be obtained from the second moment of the Wigner–Kramers equation. This leads to a system of three hydrodynamic equations of the form of Eqs. (D65)–(D67) or their generalization given in [94].

### 7 Fermionic DM

Although the previous formalism has been developed for self-gravitating bosons in the form of BECs, it can also be applied to self-gravitating fermions with only minor modifications. In the case of fermions, one can ignore the quantum potential in a first approximation.\(^{32}\) A system of collisionless self-gravitating fermions is then described by the classical Vlasov–Poisson equations (C1) and (C2). These equations exhibit a process of violent relaxation leading to the Lynden-Bell DF (D5). As we have seen, the Lynden-Bell DF is similar to the Fermi–Dirac DF. Actually, in the case of fermions, the maximum phase space density \( \eta_0 \) fixed by the Lynden-Bell exclusion principle is of the same order as the quantum bound \( m^3/h^3 \) fixed by the Pauli exclusion principle (see footnote 20). Therefore, in that case, the Lynden-Bell DF truly coincides with the Fermi–Dirac DF. As a result, the statistical theory of violent relaxation is able to justify the establishment of the Fermi–Dirac DF in a fermionic DM halo on a very short timescale, of the order of a few dynamical times, without the need of collisions which require much more time to develop (see footnote 8) [80]. The resulting fermionic DM halos have a core-halo structure made of a quantum core (fermion ball) surrounded by an isothermal halo. Such configurations have been studied by many authors (see Sec. V.A of [81] for an exhaustive list of references). However, they have an infinite mass. To solve this problem, a kinetic theory of collisionless violent relaxation on the coarse-grained scale has been developed in Refs. [94,96–101]. It leads to a fermionic Vlasov–Kramers or fermionic Vlasov–Landau equation. This kinetic equation relaxes towards the truncated Lynden-Bell DF (52) and (53) [96]. This DF has a finite mass (see also footnote 21) and is stabilized against gravitational collapse by the Pauli or Lynden-Bell exclusion principle. This leads to the fermionic King model of DM halos studied in [80,133].

In a more general approach, we can take the quantum potential into account. In that case, the starting point of the analysis is the Hartree–Fock equations (G3) and (G4) which can be viewed as multistate Schrödinger–Poisson equations. These equations take the Heisenberg uncertainty principle and the Pauli exclusion principle into account. They are equivalent to the quantum Euler–Poisson equations (G30)–(G32) which are similar to Eqs. (17)–(21) except that they include an additional pressure term [see Eq. (G27)] arising from the Pauli exclusion principle (this term comes from multistate fluctuations). These equations are also equivalent to the Wigner equation (44). We can then introduce a coarse-graining and proceed as in Sects. 3–6. The only difference is that we have to take into account the Pauli exclusion principle that prevents two fermions with equal spin to occupy the same quantum state. In the statistical approach, the Pauli exclusion principle creates a Fermi–Dirac pressure \( P_{\text{FD}} \).\(^{33}\) This pressure can be decomposed into a thermal pressure \( P_{\text{th}} \) and a

\(^{32}\) This is because, in DM models, fermions have a much larger mass than bosons (see, e.g., [102]).

\(^{33}\) As discussed above, in the case of fermions, the Fermi–Dirac pressure \( P_{\text{FD}} \) is equivalent to the Lynden-Bell pressure \( P_{\text{LB}} \). However, for the sake of clarity (and generality) we shall treat these two terms separately.
pressure of zero point energy \( P_{FD}^{(0)} \) corresponding to a completely degenerate Fermi gas at \( T = 0 \). The Fermi pressure plays a role similar to the pressure \( P_{int} \) arising from the self-interaction of the bosons. We can also take into account the Slater correction which is equivalent to a pressure \( P_{slater} \) (see Appendix G).\(^{34}\) Since the thermal cloud is taken into account in \( P_{th} \), we just have to add the Fermi pressure \( P_{FD}^{(0)} \) and the Slater pressure \( P_{slater} \) in the formalism of Sects. 3–6. Below, we briefly write the basic equations that result from this formalism.

The equation of state of the Fermi gas at \( T = 0 \) is [120]

\[
P_{FD}^{(0)}(\rho) = \frac{1}{20} \left( \frac{3}{\pi} \right)^{2/3} \frac{\hbar^2}{m^{5/3}} \rho^{5/3}. \tag{152}
\]

Using the results of Appendix B, we obtain the enthalpy

\[
h_{FD}^{(0)}(\rho) = \frac{1}{8} \left( \frac{3}{\pi} \right)^{2/3} \frac{\hbar^2}{m^{5/3}} \rho^{2/3} \tag{153}
\]

and the potential

\[
V_{FD}^{(0)}(\rho) = \frac{3}{40} \left( \frac{3}{\pi} \right)^{2/3} \frac{\hbar^2}{m^{5/3}} \rho^{5/3}. \tag{154}
\]

The Slater equation of state is (see Appendix G)

\[
P_{slater}(\rho) = \frac{C_s}{4m} \rho^{4/3}. \tag{155}
\]

Using the results of Appendix B, we obtain the enthalpy

\[
h_{slater}(\rho) = \frac{C_s}{m} \rho^{1/3} \tag{156}
\]

and the potential

\[
V_{slater}(\rho) = \frac{3C_s}{4m} \rho^{4/3}. \tag{157}
\]

Collecting these results, we obtain a generalized wave equation of the form

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m T_{eff} \ln(|\psi|^2) \psi + \frac{1}{8} \left( \frac{3}{\pi} \right)^{2/3} \frac{\hbar^2}{m^{5/3}} |\psi|^{4/3} \psi + C_s |\psi|^{2/3} \psi + \frac{1}{2} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \hbar m \psi^{4/3} \psi + m \Phi \psi - i \frac{\hbar}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi \tag{158}
\]

\( |\psi|^{2/3} \) nonlinearity corresponds to the Slater exchange energy term and the \(|\psi|^{4/3} \) nonlinearity accounts for the Pauli exclusion principle at \( T = 0 \) like in the TF theory of the atoms [134,135]. The von Weizsäcker correction to the TF theory is accounted for in the kinetic term \(- (\hbar^2/2m) \Delta \psi \). The wave equation (158) with \( \xi = T_{eff} = 1/\eta_0 = 0 \) is expected to display a process of violent relaxation leading to a fermion ball at \( T = 0 \) surrounded by a halo of scalar radiation. This is similar to the process of gravitational cooling experienced by the GPP equations for bosons (the fermion ball is the counterpart of the bosonic condensate). The coarse-grained equation (158), with the friction term and the Lynden-Bell terms (exclusion principle and effective temperature) retained, parameterizes this process of violent relaxation. Note that this equation also takes into account the Heisenberg uncertainty principle through the kinetic term. This provides an additional source of small-scale regularization, in addition to the Pauli (or Lynden-Bell) exclusion principle, preventing gravitational collapse.

Remark For the sake of completeness, we briefly consider the case of a relativistic Fermi gas. In the ultrarelativistic limit, the equation of state of the Fermi gas at \( T = 0 \) is [120]

\[
P(\rho) = \frac{1}{8} \left( \frac{3}{\pi} \right)^{1/3} \frac{\hbar c}{m^{4/3}} \rho^{4/3}. \tag{159}
\]

Using the results of Appendix B, we obtain the enthalpy

\[
h(\rho) = \frac{1}{2} \left( \frac{3}{\pi} \right)^{1/3} \frac{\hbar c}{m^{4/3}} \rho^{1/3} \tag{160}
\]

and the potential

\[
V(\rho) = \frac{3}{8} \left( \frac{3}{\pi} \right)^{1/3} \frac{\hbar c}{m^{4/3}} \rho^{4/3}. \tag{161}
\]

This leads to a generalized wave equation of the form

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m T_{eff} \ln(|\psi|^2) \psi + \frac{1}{2} \left( \frac{3}{3\pi} \right)^{1/3} \frac{\hbar c}{m^{1/3}} |\psi|^{2/3} \psi + m \Phi \psi - i \frac{\hbar}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi \tag{162}
\]

8 Conclusion

In this paper, we have tried to justify with more precise arguments the generalized GPP equations (92) and (93) introduced heuristically in [81,93]. These equations aim at providing a parametrization of the complicated
processes of gravitational cooling and violent relaxation experienced by a self-gravitating BEC described by the ordinary GPP equations (8) and (9). First, we have recalled that the Schrödinger equation (10) for the wave function is equivalent to the Wigner equation (44) for the DF. Then, we have introduced a coarse-grained Wigner equation (46) by analogy with the coarse-grained Vlasov equation introduced in connection to Lynden-Bell’s theory of violent relaxation (see [84] for the Schrödinger–Poisson–Vlasov–Poisson correspondence). This equation is consistent with a MEP. From the coarse-grained Wigner equation, proceeding as in [94], we have derived a set of quantum hydrodynamic equations (72)–(74) which include a quantum potential, a Lynden-Bell pressure, and a friction force. The Lynden-Bell pressure can be decomposed into a polytropic equation of state and an isothermal equation of state [see Eq. (70)]. The isothermal equation of state is valid at low or mid densities. It takes into account effective thermal effects which describe scalar radiation. The polytropic equation of state is valid at high densities. It takes into account degeneracy effects in the sense of Lynden-Bell. Finally, we have used the inverse Madelung transformation to derive the corresponding wave equations (92) and (93). These generalized GPP equations include an effective potential associated with the Lynden-Bell pressure and a damping term. The Lynden-Bell potential can be written as the sum of a power-law potential describing degeneracy effects and a logarithmic potential describing effective thermal effects [see Eq. (96)]. The self-interaction of the bosons can be taken into account as usual by introducing an additional potential in the GPP equations [see Eq. (91)]. This leads to Eq. (100). We have explained the importance of developing a microcanonical description of the process of violent relaxation instead of a canonical one. This can be achieved in our formalism by letting the effective temperature depend on time (and possibly position) so as to conserve the total energy [see Eqs. (148) and (149) or Eqs. (D65)–(D68)].

The generalized GPP equations satisfy an $H$-theorem for the Lynden-Bell entropy and relax towards a stable equilibrium state (viralized state) which is a maximum entropy state at fixed mass and energy. This quasistationary state, which can be viewed as the most probable state of the system, has a core-halo structure. It is made of a quantum core surrounded by an isothermal halo. The core results from the balance between the gravitational attraction, the repulsion due to the Heisenberg uncertainty principle, the self-interaction of the bosons, and the repulsion due to the Lynden-Bell exclusion principle. The halo results from the balance between the gravitational attraction and the Lynden-Bell effective thermal pressure. The quantum core solves the core-cusp problem. The isothermal halo accounts for the flat rotation curves of the galaxies. Since this core-halo structure is consistent with a MEP, the core mass can be obtained by maximizing the entropy $S(M_c)$ at fixed halo mass $M_h$ and halo energy $E_h$. Interestingly, we have shown in [102] that this maximization problem is equivalent to the “velocity dispersion tracing” relation and returns the core mass-halo mass relation $M_c(M_h)$ observed in direct numerical simulations of the GPP equations (we have derived a general $M_c(M_h)$ relation valid for bosons with a repulsive or an attractive self-interaction and for fermions [102,104]). Therefore, our effective thermodynamic approach is consistent with the structure of BECDM halos.

We have proposed that a similar generalized wave equation may be relevant for self-gravitating fermions [see Eqs. (158) and (162)]. In that case, the pressure due to the Pauli exclusion principle replaces the pressure due to the self-interaction of the bosons. The corresponding wave equation relaxes towards a stable equilibrium state with a core-halo structure made of a quantum core and an isothermal halo. This is similar to the equilibrium state of the generalized GPP equations except that the bosonic condensate is replaced by a fermion ball stabilized by the Pauli exclusion principle. Explicit density profiles with a core-halo structure have been computed from our model in the case of BECs with a repulsive self-interaction in the TF approximation [81]. They are in good agreement with the density profiles of DM halos obtained from observations or from direct numerical simulations. Similar results will be reported for noninteracting bosons and fermions in forthcoming contributions (see [127,136]).

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.]

Appendix A: Properties of the Wigner distribution

In this Appendix, we check that the density $\rho$, velocity $u$ and pressure tensor $P_{ij}$ obtained from the Wigner DF (43) coincide with the expressions obtained from the Schrödinger equation (10).

To that purpose, let us first establish useful identities. Using Eqs. (15) and (16), we get

$$\rho = |\psi|^2 = \psi\psi^* \quad (A1)$$

and

$$u = \frac{\nabla S}{m} = \frac{\hbar}{2m\rho}(\psi^*\nabla\psi - \psi\nabla\psi^*). \quad (A2)$$

We also note that

$$\int dv \, e^{imv\cdot y/\hbar} = (2\pi)^3 \delta \left( \frac{my}{\hbar} \right) = \left( \frac{2\pi\hbar^3}{m^3} \right) \delta(y). \quad (A3)$$

35 The Lynden-Bell exclusion principle is usually negligible for self-gravitating bosons (for which $\eta_0 \gg m^4/\hbar^3$) but it may be relevant for self-gravitating fermions (for which $\eta_0 \sim gm^4/\hbar^3$).

36 One can also account for the self-interaction of the fermions and introduce a self-interaction pressure in addition to the pressure due to the Pauli exclusion principle.
We are now ready to compute the first moments of the Wigner DF

\[
f(r, v, t) = \frac{m^3}{(2\pi\hbar)^3} \int dy \psi^{\ast}(r + \frac{y}{2}, t) \psi(r - \frac{y}{2}, t).
\]  

(A4)

Integrating Eq. (A4) over \(v\) and using Eq. (A3), we obtain

\[
\rho \equiv \int f \, dv = |\psi|^2, 
\]

(A5)

which coincide with Eq. (A1).

Multiplying Eq. (A4) by \(v\) and integrating over \(v\), we get

\[
\int f \, dv = \frac{m^3}{(2\pi\hbar)^3} \int dv dy \psi^{\ast}(r + \frac{y}{2}, t) \psi(r - \frac{y}{2}, t) \]

\[
= \frac{m^3}{(2\pi\hbar)^3} \int dv dy \frac{\partial}{\partial y} \left( e^{i m v y / \hbar} \right) \frac{\hbar}{i m}
\]

\[
\times \psi^{\ast}(r + \frac{y}{2}, t) \psi(r - \frac{y}{2}, t)
\]

\[
= - \frac{m^3}{(2\pi\hbar)^3} \int dv dy e^{i m v y / \hbar} \frac{\hbar}{i m}
\]

\[
\times \frac{\partial}{\partial y} \left[ \psi^{\ast}(r + \frac{y}{2}, t) \psi(r - \frac{y}{2}, t) \right],
\]

(A6)

where we have used an integration by parts to obtain the last equality. Calculating the derivative of the term in brackets, and using Eq. (A3), we obtain

\[
\rho \mathbf{u} \equiv \int f \, dv = \frac{\hbar}{2im} (\psi^{\ast} \nabla \psi - \psi \nabla \psi^{\ast}),
\]

(A7)

which coincide with Eq. (A2).

Multiplying Eq. (A4) by \(v_i v_j\), integrating over \(v\), and proceeding as in Eq. (A6), we obtain

\[
\int f v_i v_j \, dv = \frac{m^3}{(2\pi\hbar)^3} \int dv dy e^{i m v y / \hbar}
\]

\[
\times \psi^{\ast}(r + \frac{y}{2}, t) \psi(r - \frac{y}{2}, t)
\]

\[
= \frac{m^3}{(2\pi\hbar)^3} \int dv dy \frac{\partial^2}{\partial y_i \partial y_j} \left( e^{i m v y / \hbar} \right) \frac{\hbar}{i m}
\]

\[
\times \psi^{\ast}(r + \frac{y}{2}, t) \psi(r - \frac{y}{2}, t)
\]

\[
= \frac{m^3}{(2\pi\hbar)^3} \int dv dy e^{i m v y / \hbar} \frac{\hbar}{i m}
\]

\[
\times \frac{\partial^2}{\partial y_i \partial y_j} \left[ \psi^{\ast}(r + \frac{y}{2}, t) \psi(r - \frac{y}{2}, t) \right].
\]

(A8)

Calculating the second derivative of the term in brackets, and using Eq. (A3), we get

\[
\int f v_i v_j \, dv = - \frac{\hbar^2}{4m^2} \left( \psi^{\ast} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi^{\ast} \frac{\partial^2 \psi}{\partial x_j \partial x_i} \right)
\]

\[
- \psi \frac{\partial \psi^{\ast}}{\partial x_i} \frac{\partial \psi}{\partial x_j} + \psi \frac{\partial^2 \psi}{\partial x_i \partial x_j}.
\]

(A9)

Using Eqs. (A7) and (A9), we obtain after simplification

\[
P_{i j} = \frac{\hbar^2}{4m^2} \left( \psi^{\ast} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} + \psi \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right)
\]

\[
- \psi^{\ast} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi \frac{\partial^2 \psi^{\ast}}{\partial x_i \partial x_j}.
\]

(A11)

The quantum pressure tensor obtained from the Schrödinger equation (10) can be written as [see Eq. (23)]

\[
P_{i j}^Q = \frac{\hbar^2}{4m^2} \left( \rho \partial_{x_i} \rho \rho - \partial_{x_i} \rho \right).
\]

(A12)

Using Eq. (A1), we get after simplification

\[
P_{i j}^Q = \frac{\hbar^2}{4m^2} \left( \psi^{\ast} \frac{\partial \psi}{\partial x_i} \partial \psi + \psi \frac{\partial \psi^{\ast}}{\partial x_i} \partial \psi^{\ast}
\]

\[
- \psi^{\ast} \frac{\partial^2 \psi^{\ast}}{\partial x_i \partial x_j} - \psi \frac{\partial^2 \psi}{\partial x_i \partial x_j}.
\]

(A13)

Comparing Eqs. (A11) and (A13) we see that

\[
P_{i j} = P_{i j}^Q.
\]

(A14)

Let us now check that the expression of the energy in the Wigner representation is the same as in the Schrödinger representation. In terms of the Wigner DF, the total energy is given by

\[
E = \int \frac{u^2}{2} \, dv + \frac{1}{2} \int \rho \Phi \, dr.
\]

(A15)

According to Eqs. (A10) and (A14), it can be rewritten as

\[
E = \int \rho \frac{u^2}{2} \, dr + \frac{1}{2} \int P_{i i}^Q \, dr + \frac{1}{2} \int \rho \Phi \, dr.
\]

(A16)

Using Eq. (G.22) of [93] establishing that \(\int P_{i i}^Q \, dr = 2\Theta_Q\), we find that

\[
E = \int \rho \frac{u^2}{2} \, dr + \int \rho \frac{Q}{m} \, dr + \frac{1}{2} \int \rho \Phi \, dr,
\]

(A17)

which is in agreement with Eq. (28) obtained from the Schrödinger equation. From this equivalence, we can directly conclude that the Wigner–Poisson equations conserve the energy.

Finally, we derive the Wigner equation (44). Taking the time derivative of the Wigner DF (A4), we get

\[
\frac{\partial f}{\partial t} = \frac{m^3}{(2\pi\hbar)^3} \int dv \psi^{\ast}(r + \frac{y}{2}, t) \frac{\partial \psi^{\ast}}{\partial t} \left( r - \frac{y}{2}, t \right)
\]

\[
+ \frac{\partial \psi}{\partial t} \left( r + \frac{y}{2}, t \right) \psi \left( r - \frac{y}{2}, t \right).
\]

(A18)

The next step is to substitute the Schrödinger equation (10) into the term in brackets of Eq. (A18). The contribution of the potential term in the Schrödinger equation is
\[
\left( \frac{\partial f}{\partial t} \right)_{\text{pot}} = \frac{im^4}{(2\pi\hbar)^4} \int dy e^{im\mathbf{v} \cdot \mathbf{y}/\hbar} \\
\times \left[ \Phi(\mathbf{r} + \mathbf{y}/2, t) - \Phi(\mathbf{r} - \mathbf{y}/2, t) \right] \\
\times \phi^\ast(\mathbf{r} + \mathbf{y}/2, t) \psi(\mathbf{r} - \mathbf{y}/2, t) \\
= \frac{im^4}{(2\pi\hbar)^4} \int dyd\mathbf{v}' e^{im(\mathbf{v} - \mathbf{v}')} y/\hbar \\
\times \left[ \Phi(\mathbf{r} + \mathbf{y}/2, t) - \Phi(\mathbf{r} - \mathbf{y}/2, t) \right] f(\mathbf{r}, \mathbf{v}', t).
\]
\tag{A19}
\]

The second equality can be checked directly by substituting Eq. (A4), integrating over \( \mathbf{v}' \), and using Eq. (A3). The contribution of the kinetic term in the Schrödinger equation is

\[
\left( \frac{\partial f}{\partial t} \right)_{\text{kin}} = \frac{im^2}{2(2\pi\hbar)^2} \int dy e^{im\mathbf{v} \cdot \mathbf{y}/\hbar} \\
\times \phi^\ast(\mathbf{r} + \mathbf{y}/2, t) \Delta_\mathbf{r} \psi(\mathbf{r} - \mathbf{y}/2, t) \\
= -\frac{im^3}{(2\pi\hbar)^3} \mathbf{v} \cdot \int dy e^{im\mathbf{v} \cdot \mathbf{y}/\hbar} \\
\times \psi^\ast(\mathbf{r} + \mathbf{y}/2, t) \nabla_y \cdot \nabla_\mathbf{r} \psi(\mathbf{r} - \mathbf{y}/2, t). 
\tag{A20}
\]

Integrating by parts, we get

\[
\left( \frac{\partial f}{\partial t} \right)_{\text{kin}} = \frac{im^2}{2(2\pi\hbar)^2} \int dy e^{im\mathbf{v} \cdot \mathbf{y}/\hbar} \\
\times \nabla_y \psi^\ast(\mathbf{r} + \mathbf{y}/2, t) \nabla_y \psi(\mathbf{r} - \mathbf{y}/2, t) \\
-\frac{m^3}{(2\pi\hbar)^3} \mathbf{v} \cdot \int dy e^{im\mathbf{v} \cdot \mathbf{y}/\hbar} \\
\times \psi^\ast(\mathbf{r} + \mathbf{y}/2, t) \nabla_y \psi(\mathbf{r} - \mathbf{y}/2, t). 
\tag{A21}
\]

Making the same operations with the kinetic term of the complex conjugate Schrödinger equation and adding the two results, we obtain

\[
\left( \frac{\partial f}{\partial t} \right)_{\text{kin}} = -\frac{m^3}{(2\pi\hbar)^3} \mathbf{v} \cdot \int dy e^{im\mathbf{v} \cdot \mathbf{y}/\hbar} \\
\times \psi^\ast(\mathbf{r} + \mathbf{y}/2, t) \nabla_y \psi(\mathbf{r} - \mathbf{y}/2, t) \\
-\frac{m^3}{(2\pi\hbar)^3} \mathbf{v} \cdot \int dy e^{im\mathbf{v} \cdot \mathbf{y}/\hbar} \\
\times \psi(\mathbf{r} - \mathbf{y}/2, t) \nabla_y^\ast \psi^\ast(\mathbf{r} + \mathbf{y}/2, t) = -\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}}. 
\tag{A22}
\]

The second equality can be checked directly by substituting Eq. (A4). Summing Eqs. (A19) and (A22), we obtain the Wigner equation (44).

Remark When \( \hbar \to 0 \), we can make the approximation \( \Phi(\mathbf{r} \pm \mathbf{y}/2, t) \simeq \Phi(\mathbf{r}, t) \pm \nabla \Phi(\mathbf{r}, t) \cdot \mathbf{y}/2 \) in the integral of the Wigner equation (44) and we obtain

\[
\frac{df}{dt} + \mathbf{v} \cdot \frac{df}{dr} \\
= \frac{im^4}{(2\pi\hbar)^4} \nabla \Phi \cdot \int e^{im(\mathbf{v} - \mathbf{v}') \cdot \mathbf{y}/\hbar} y f(\mathbf{r}, \mathbf{v}', t) dyd\mathbf{v}' \\
= \frac{m^3}{(2\pi\hbar)^3} \nabla \Phi \cdot \frac{\partial}{\partial t} \int e^{im(\mathbf{v} - \mathbf{v}') \cdot \mathbf{y}/\hbar} f(\mathbf{r}, \mathbf{v}', t) dyd\mathbf{v}' \\
= \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}}. 
\]
\[ \Delta \Phi = 4\pi G\rho^2. \]  

(B7)

Dividing Eq. (B6) by \( \phi \), we get

\[ Q + m\phi + mh(\rho) + mT_{\text{eff}} \ln \rho = E, \]  

(B8)

where \( Q \) is the quantum potential defined by Eq. (21).

Using the Madelung [50] transformation (see Sect. 2.2), the hydrodynamic equations corresponding to the generalized GPP equations (B1) and (B2) are

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \]  

(B9)

\[ \frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + m[\Phi + T_{\text{eff}} \ln \rho + h(\rho)] 
+ Q + \xi(S - \langle S \rangle) = 0, \]  

(B10)

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{m} \nabla Q - \frac{1}{\rho} \nabla P - \frac{1}{\rho} \nabla P_{\text{th}} - \xi u, \]  

(B11)

\[ \Delta \Phi = 4\pi G\rho, \]  

(B12)

where the barotropic equation of state \( P(\rho) \) is determined by the relation (see Appendix B5)

\[ h'(\rho) = \frac{P'(\rho)}{\rho}. \]  

(B13)

For a general polytropic equation of state of the form

\[ P_{\text{poly}} = K\rho^\gamma, \]  

(B14)

the enthalpy and the potential are given by

\[ h_{\text{poly}}(\rho) = \frac{K\gamma}{\gamma - 1} \rho^{\gamma - 1}, \quad V_{\text{poly}}(\rho) = \frac{K}{\gamma - 1} \rho^\gamma. \]  

(B15)

In particular, for the standard self-interacting BEC described by the polytropic equation of state

\[ P_{\text{int}}(\rho) = \frac{2\pi a_s h^2}{m^3} \rho^2, \]  

(B16)

we have

\[ h_{\text{int}}(\rho) = \frac{4\pi a_s h^2}{m^3} \rho, \quad V_{\text{int}}(\rho) = \frac{2\pi a_s h^2}{m^3} \rho^2. \]  

(B17)

For the Lynden-Bell pressure of zero-point energy

\[ P_{\text{LB}}^{(0)} = \frac{1}{5} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \rho^{5/3}, \]  

(B18)

we have

\[ h_{\text{LB}}^{(0)}(\rho) = \frac{1}{2} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \rho^{2/3}, \]  

(B19)

\[ V_{\text{LB}}^{(0)}(\rho) = \frac{3}{10} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \rho^{5/3}. \]  

(B20)

For the Lynden-Bell isothermal equation of state

\[ P_{\text{th}} = \rho T_{\text{eff}}, \]  

(B21)

the enthalpy and the potential are given by

\[ h_{\text{th}}(\rho) = T_{\text{eff}} \ln \rho, \quad V_{\text{th}}(\rho) = T_{\text{eff}} \rho \ln \rho - 1. \]  

(B22)

An equilibrium state of the quantum damped Euler equations (B9)–(B12) satisfies the condition of quantum hydrostatic equilibrium

\[ \frac{\rho}{m} \nabla Q + \nabla P + \rho \nabla \Phi + \nabla P_{\text{th}} = 0. \]  

(B23)

This equation is equivalent to Eq. (B8). Indeed, dividing Eq. (B23) by \( \rho \), using Eq. (B13) and integrating, we obtain Eq. (B8), where \( E \) appears as a constant of integration.

### 2. Microcanonical model

We introduce the mass

\[ M = \int \rho \, d\mathbf{r}, \]  

(B24)

and the energy

\[ E_{\text{tot}} = \Theta_c + \Theta_Q + U + W + \Theta_{\text{th}}, \]  

(B25)

which includes the classical kinetic energy

\[ \Theta_c = \int \rho \frac{\mathbf{u}^2}{2} \, d\mathbf{r}, \]  

(B26)

the quantum kinetic energy

\[ \Theta_Q = \frac{1}{m} \int \rho Q \, d\mathbf{r}, \]  

(B27)

the internal energy

\[ U = \int V(\rho) \, d\mathbf{r}, \]  

(B28)

the gravitational energy

\[ W = \frac{1}{2} \int \rho \Phi \, d\mathbf{r}, \]  

(B29)

and the thermal energy

\[ \Theta_{\text{th}} = \frac{3}{2} MT_{\text{eff}}. \]  

(B30)

We also introduce the entropy (up to an additive constant)

\[ S = - \int \rho (\ln \rho - 1) \, d\mathbf{r} + \frac{3}{2} M \ln T_{\text{eff}}. \]  

(B31)

The functionals (B26)–(B29) are justified in [93] and the functionals (B30) and (B31) are justified in Appendix D3.

In the microcanonical situation, the total energy \( E_{\text{tot}} \) is fixed and the temperature \( T_{\text{eff}}(t) \) evolves in time so as to satisfy the constraint from Eq. (B25) yielding

\[ \frac{3}{2} MT_{\text{eff}}(t) = E_{\text{tot}} - \Theta_c(t) - \Theta_Q(t) - U(t) - W(t). \]  

(B32)

The generalized GPP equations (B1) and (B2) with the time-dependent temperature \( T_{\text{eff}}(t) \) given by Eq. (B32)\(^{37}\)

\[ \frac{3}{2} MT_{\text{eff}}(t) = E_{\text{tot}} - \frac{\rho^2}{2m^2} \int |\nabla \psi|^2 \, d\mathbf{r} \]

\[ - \int V(|\psi|^2) \, d\mathbf{r} - \frac{1}{2} \int |\psi|^2 \Phi \, d\mathbf{r}. \]  

(B33)

\(^{37}\) In terms of the wavefunction, the effective temperature is given by
conservethe energy (by construction) and satisfy an $H$- 
theorem for the entropy (B31). This can be proven as fol-
lows. From Eq. (B31), we have
\[
\dot{S} = - \int \ln \rho \frac{\partial \rho}{\partial t} \, dr + \frac{3}{2} M \frac{\dot{T}_{\text{eff}}}{T_{\text{eff}}}. \tag{B34}
\]
Using the equation of continuity (B9) and integrating by
parts, we get
\[
\dot{S} = - \int \mathbf{u} \cdot \nabla \rho \, dr + \frac{3}{2} M \frac{\dot{T}_{\text{eff}}}{T_{\text{eff}}}. \tag{B35}
\]
On the other hand, taking the time derivative of Eq. (B32),
and using Eqs. (B26)–(B29), we obtain
\[
\frac{3}{2} M \dot{T}_{\text{eff}} = - \int \left( \frac{\mathbf{u}^2}{2} + \frac{Q}{m} + h + \Phi \right) \frac{\partial \rho}{\partial t} \, dr - \int \rho \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} \, dr,
\]
where we have used the identities of Appendix C of [93].
Using the equation of continuity (B9), the damped quantum
Euler equation (B11), and recalling the identity of vector
analysis ($\mathbf{u} \cdot \nabla \mathbf{u} = \nabla (\mathbf{u}^2/2) - \mathbf{u} \times (\nabla \times \mathbf{u})$) and the fact that
the flow is irrotational ($\nabla \times \mathbf{u} = 0$) since $\mathbf{u} = \nabla S/m$, we
get after simplification (using straightforward integrations
by parts)
\[
\frac{3}{2} M \dot{T}_{\text{eff}} = \int \mathbf{u} \cdot \nabla P_{\text{th}} \, dr + \xi \int \rho \mathbf{u}^2 \, dr. \tag{B37}
\]
Combining Eqs. (B35) and (B37), and using Eq. (B21), we
finally obtain
\[
\dot{S} = \frac{\xi}{T_{\text{eff}}} \int \rho \mathbf{u}^2 \, dr \geq 0. \tag{B38}
\]
Therefore, the entropy increases monotonically. At equili-
brum (where $\dot{S} = 0$), we get $\mathbf{u} = 0$ leading to the quantum
equation of hydrostatic equilibrium (B23).
We can easily show that the generalized GPP equations in
the microcanonical ensemble relax towards an equilibrium
state that maximizes the entropy at fixed mass and energy.
First of all, an extremum of entropy at fixed mass and energy
is determined by the variational principle
\[
\delta S + \alpha \delta M = 0, \tag{B39}
\]
where $\alpha = \mu/mT_{\text{eff}}$ is a Lagrange multiplier taking into
account the conservation of mass [the conservation of energy
is taken into account in Eq. (B32)]. Using
\[
\delta S = - \int \ln \rho \delta \rho \, dr + \frac{3}{2} M \dot{T}_{\text{eff}} \delta T_{\text{eff}} \tag{B40}
\]
and
\[
\frac{3}{2} M \delta T_{\text{eff}} = - \int \left( \frac{Q}{m} + h + \Phi \right) \delta \rho \, dr, \tag{B41}
\]
we obtain
\[
mT_{\text{eff}} \ln \rho + Q + mh + m\Phi = \mu, \tag{B42}
\]
which is equivalent to Eq. (B38) with $\mu = E$. This shows
that the Eigenenergy $E$ coincides with the chemical poten-
tial $\mu$. Taking the gradient of Eq. (B42) and using Eq. (B13),
we recover the equation of quantum hydrostatic equilibrium
(B23). Therefore, an equilibrium state of the generalized
GPP equations is an extremum of entropy at fixed mass
and energy. On the other hand, by proceeding as explained
at the end of Appendix D2, we can show that only entropy
maxima (not minima or saddle points) at fixed mass and
energy are dynamically stable [127]. From these results, we
conclude that the generalized GPP equations with a time-
dependent effective temperature relax towards an equilib-
rium state which maximizes the entropy at fixed mass and
energy:
\[
\max \{ S \mid M, E_{\text{tot}} \text{ fixed} \}. \tag{B43}
\]
Proceeding as in Appendix G of [93], we obtain the damped quantum virial theorem
\[
\frac{1}{2} \frac{\ddot{I}}{I} + \frac{1}{2} \xi \dot{I} = 2(\Theta_c + \Theta_{\text{th}} + \Theta_Q) + 3 \int P \, dr + W, \tag{B44}
\]
where $I = \int r^2 \rho \, dr$ is the moment of inertia. According to
Eq. (B25), we have $\Theta_c + \Theta_{\text{th}} + \Theta_Q = E_{\text{tot}} - U - W$ so we can rewrite Eq. (B44) as
\[
\frac{1}{2} \frac{\ddot{I}}{I} + \frac{1}{2} \xi \dot{I} = 2E_{\text{tot}} - 2U - W + 3 \int P \, dr. \tag{B45}
\]
For a polytropic equation of state, the internal energy sat-
sifies the identity [see Eqs. (B14) and (B15)]
\[
U_{\text{poly}} = \frac{1}{\gamma - 1} \int P_{\text{poly}} \, dr, \tag{B46}
\]
and the damped quantum virial theorem becomes
\[
\frac{1}{2} \frac{\ddot{I}}{I} + \frac{1}{2} \xi \dot{I} = 2E_{\text{tot}} + (3\gamma - 5)U_{\text{poly}} - W. \tag{B47}
\]
We note that the term involving the internal energy vanishes
for the particular index $\gamma = 5/3$ which corresponds to the
Lynden-Bell pressure of zero-point energy (B18).

Remark In the strong friction limit $\xi \rightarrow +\infty$, we can neglect the inertial term (l.h.s.) in the damped quantum Euler equation (B11) and get
\[
\mathbf{u} = -\frac{1}{\xi \rho} \left( T_{\text{eff}} \nabla \rho + \nabla P + \frac{\rho}{m} \nabla Q + \rho \nabla \Phi \right). \tag{B48}
\]
Substituting this relation into the continuity equation (B9)
we obtain the quantum Smoluchowski–Poisson equations
\[
\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( T_{\text{eff}} \nabla \rho + \nabla P + \frac{\rho}{m} \nabla Q + \rho \nabla \Phi \right), \tag{B49}
\]
\[
\Delta \Phi = 4\pi G \rho. \tag{B50}
\]
Since $|\mathbf{u}| = O(1/\xi)$ in the strong friction limit, we can neglect the classical kinetic energy $\Theta_c$ in Eq. (B32). There-
fore, the evolution of the effective temperature is given by
\[
\frac{3}{2} M \dot{T}_{\text{eff}}(t) = E_{\text{tot}} - \Theta_Q(t) - U(t) - W(t). \tag{B51}
\]
\[\text{38}\] The structure of this equation shows that a system described by the generalized GPP equations undergoes damped oscillations towards a virialized state. These damped oscillations are characteristic of the process of gravitational cooling and violent relaxation (see Sect. 2.3). In this sense, the generalized GPP equations provide a parametrization of the ordinary GPP equations.
The $H$-theorem takes the form
\[
\dot{S} = \frac{1}{\xi T_{\text{eff}}} \int \frac{1}{\rho} \left( T_{\text{eff}} \nabla \rho + \nabla P + \frac{\rho}{m} \nabla Q + \rho \nabla \Phi \right)^2 \, dr \geq 0,
\]
and we have the same general properties as those discussed after Eq. (B38). On the other hand, the overdamped quantum virial theorem is given by
\[
\frac{1}{2} \dot{\xi} I = 2(\Theta_{\text{th}} + \Theta_Q) + 3 \int P \, dr + W. \tag{B53}
\]
Using $\Theta_{\text{th}} + \Theta_Q = E_{\text{tot}} - U - W$, we can rewrite Eq. (B53) as
\[
\frac{1}{2} \xi I = 2E_{\text{tot}} - 2U - W + 3 \int P \, dr. \tag{B54}
\]
For a polytropic equation of state, using Eq. (B46), the overdamped quantum virial theorem becomes
\[
\frac{1}{2} \xi I = 2E_{\text{tot}} + (3\gamma - 5)U_{\text{poly}} - W. \tag{B55}
\]

3. Canonical model

In the canonical ensemble, the temperature $T_{\text{eff}}$ is fixed and the appropriate thermodynamic potential is the free energy
\[
F = E_{\text{tot}} - T_{\text{eff}} S. \tag{B56}
\]
Using Eqs. (B25) and (B31), we obtain (up to an additive constant)
\[
F = \Theta_c + \Theta_Q + U + W + U_{\text{th}}, \tag{B57}
\]
where
\[
U_{\text{th}} = T_{\text{eff}} \int \rho \ln(\rho - 1) \, dr \tag{B58}
\]
is the internal energy associated with the thermal pressure. This returns the free energy introduced in [93].

The generalized GPP equations (B1) and (B2) with a fixed temperature $T_{\text{eff}}$ satisfy an $H$-theorem for the free energy (B57). Indeed, using calculations similar to the previous ones (see also Appendix D of [93]) we obtain
\[
\dot{F} = -\xi \int \rho u^2 \, dr \leq 0. \tag{B59}
\]
Therefore, the free energy decreases monotonically. At equilibrium (where $\dot{F} = 0$), we get $u = 0$ leading to the equation of quantum hydrostatic equilibrium (B23).

We can easily show that the generalized GPP equations in the canonical ensemble relax towards an equilibrium state that minimizes the free energy at fixed mass. First of all, an extremum of free energy at fixed mass is determined by the variational principle
\[
\delta F - \frac{\mu}{m} \delta M = 0, \tag{B60}
\]
where $\mu/m$ is a Lagrange multiplier taking into account the conservation of mass. Using calculations similar to the previous ones (see also section 3.4 of [93]) we obtain Eq. (B42) which is equivalent to Eq. (B8) with $\mu = E$ or to the quantum equation of hydrostatic equilibrium (B23). Therefore, an equilibrium state of the generalized GPP equations is an extremum of free energy at fixed mass. On the other hand, by proceeding as explained at the end of Appendix D2 we can show that only minima (not maxima or saddle points) of free energy at fixed mass are dynamically stable [127]. From these results, we conclude that the generalized GPP equations with a constant effective temperature relax towards an equilibrium state which minimizes the free energy at fixed mass:
\[
\min \{ F \mid M \text{ fixed} \}. \tag{B61}
\]
The damped quantum virial theorem is given by Eq. (B44) or, equivalently, by
\[
\frac{1}{2} \dot{\xi} I + \frac{1}{2} \xi I = 2(\Theta_c + \Theta_Q) + 3MT_{\text{eff}} + 3 \int P \, dr + W, \tag{B62}
\]
where $T_{\text{eff}}$ is constant.

Remark: In the strong friction limit $\xi \to +\infty$, we obtain the quantum Smoluchowski–Poisson equations (B49) and (B50) where $T_{\text{eff}}$ is constant. The $H$-theorem now reads
\[
\dot{F} = -\xi \int \frac{1}{\rho} \left( T_{\text{eff}} \nabla \rho + \nabla P + \frac{\rho}{m} \nabla Q + \rho \nabla \Phi \right)^2 \, dr \leq 0. \tag{B63}
\]
On the other hand, the overdamped quantum virial theorem is given by Eq. (B53) or, equivalently, by
\[
\frac{1}{2} \xi I = 2\Theta_Q + 3MT_{\text{eff}} + 3 \int P \, dr + W, \tag{B64}
\]
where $T_{\text{eff}}$ is constant.

4. Numerical algorithm

A stationary solution of the ordinary GPP equations
\[
\imath \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \left[ \Phi + \frac{dV}{d|\psi|^2} \right] \psi, \tag{B65}
\]
\[
\Delta \Phi = 4\pi G|\psi|^2, \tag{B66}
\]
is of the form $\psi(r, t) = \phi(r) e^{-iEt/\hbar}$, where $\phi(r) = \sqrt{\rho(r)}$ and $E$ (Eigenenergy) are real. These quantities are determined by the Eigenvalue problem
\[
-\frac{\hbar^2}{2m} \Delta \phi + m [\Phi + h(\rho)] \phi = E\phi, \tag{B67}
\]
\[
\Delta \Phi = 4\pi G\phi^2. \tag{B68}
\]
Dividing Eq. (B67) by $\phi$, we get
\[
Q + m\Phi + mh(\rho) = E, \tag{B69}
\]
where $Q$ is the quantum potential defined by Eq. (21).

Using the Madelung transformation, the ordinary GPP equations (B65) and (B66) are equivalent to the hydrodynamic equations
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \tag{B70}
\]
\[
\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + m\Phi + mh(\rho) + Q = 0, \tag{B71}
\]
\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{m} \nabla Q - \frac{1}{\rho} \nabla P - \nabla \Phi, \tag{B72}
\]
\[
\Delta \Phi = 4\pi G\rho. \tag{B73}
\]
A stationary solution of the quantum Euler–Poisson equations (B70)–(B73) satisfies the condition of quantum hydrostatic equilibrium
\[
\frac{\rho}{m} \nabla Q + \nabla P + \rho \nabla \Phi = 0. \tag{B74}
\]
This equation is equivalent to Eq. (B69) as can be seen by taking the gradient of Eq. (B69) and using Eq. (B13).

On the other hand, a stationary solution of the ordinary GPP equations (B65) and (B66) is an extremum of energy

$$E_{\text{tot}} = \Theta_c + \Theta_Q + U + W,$$

(B75)
at fixed mass $M$. Indeed, writing the variational principle as

$$\delta E_{\text{tot}} = \frac{\mu}{m} \delta M = 0,$$

(B76)
where $\mu/m$ is a Lagrange multiplier (global chemical potential) taking into account the mass constraint, we get

$$Q + m\Phi + mh(\rho) = \mu,$$

(B77)
which is equivalent to Eq. (B69) with $E = \mu$, and to Eq. (B74). Furthermore, it can be shown that an equilibrium state is dynamically stable if, and only if, it is a minimum of energy at fixed mass [52]. Therefore, a stable stationary solution of the GPP equations is determined by the minimization problem

$$\min \left\{ E_{\text{tot}} \mid M \text{ fixed} \right\},$$

(B78)
We stress that the ordinary GPP equations do not relax towards the stationary state that minimizes the energy at fixed mass (ground state) since these equations are reversible and the energy is conserved. They rather experience a process of gravitational cooling and violent relaxation (on a coarse-grained scale) towards a quasistationary state with a core-halo structure (see Sect. 2.3). The characterization of this core-halo state was the topic of this paper.

Independently from the physical problem treated in this paper, it is an interesting mathematical problem in itself to be able to construct stationary solutions of the ordinary GPP equations (B65) and (B66). However, it is difficult in practice to numerically solve the nonlinear Eigenvalue problem defined by Eqs. (B67) and (B68) and make sure that the solution is dynamically stable. Interestingly, the generalized GPP equations (B1) and (B2) with $T_{\text{eff}} = 0$ provide a useful numerical algorithm to reach that goal. Indeed, we have shown in Appendix B3 that these equations satisfy an H-theorem and that they relax towards a stationary state that minimizes the free energy $F$ defined by Eq. (B57) at fixed mass $M$. Since $F$ with $U_{th} = 0$ coincides with $E_{\text{tot}}$, this equilibrium state solves the minimization problem (B78). Therefore, it is a dynamically stable steady state of the ordinary GPP equations (B65) and (B66). Consequently, the generalized GPP equations (B1) and (B2) with $T_{\text{eff}} = 0$ provide a useful numerical algorithm to construct stable stationary solutions of the ordinary GPP equations (B65) and (B66). By construction, the equilibrium solution reached by the generalized GPP equations (B1) and (B2) with $T_{\text{eff}} = 0$, which are true relaxation equations, is guaranteed to be a stable stationary solution of the ordinary GPP equations (B65) and (B66).

$$\frac{\mu}{m} \text{ is a Lagrange multiplier (global chemical potential)}$$

Remark Instead of solving the generalized GPP equations (B1) and (B2), we may equivalently solve the quantum damped Euler equations (B9)–(B12) or the simpler (diffusive) quantum Smoluchowski–Poisson equations (B49) and (B50) with $T_{\text{eff}} = 0$ which also relax towards a stationary solution that satisfies the minimization problem (B78).

5. General identities for a cold gas

For a cold ($T = 0$) gas, the first principle of thermodynamics

$$d \left( \frac{u}{\rho} \right) = -Pd \left( \frac{1}{\rho} \right) + Td \left( \frac{u}{\rho} \right),$$

(B79)
where $u$ is the density of internal energy, $s$ the density of entropy, $\rho$ the mass density, and $P$ the pressure reduces to

$$d \left( \frac{u}{\rho} \right) = -Pd \left( \frac{1}{\rho} \right) = \frac{P}{\rho^2} dp.$$

(B80)
Introducing the enthalpy per particle

$$h = \frac{P + u}{\rho},$$

(B81)
we get

$$du = hdp \quad \text{and} \quad dh = \frac{dP}{\rho}.$$  

(B82)
Comparing Eq. (B81) with the Gibbs–Duhem relation at $T = 0$:

$$u = -P + Ts + \frac{\mu}{m} \rho \quad \Rightarrow \quad \frac{\mu}{m} = \frac{P + u}{\rho},$$

(B83)
we see that the enthalpy $h(\rho)$ is equal to the local chemical potential $\mu(\rho)$ by unit of mass: $h(\rho) = \mu(\rho)/m$. On the other hand, for a barotropic gas for which $P = P(\rho)$, the foregoing equations can be written as

$$P(\rho) = \rho^2 \left( \frac{u(\rho)}{\rho} \right)' = \rho u'(\rho) - u(\rho),$$

(B84)
$$P'(\rho) = \rho u''(\rho), \quad h(\rho) = P(\rho) + u(\rho),$$

(B85)
$$h(\rho) = u'(\rho), \quad h'(\rho) = \frac{P'(\rho)}{\rho},$$

(B86)
$$u(\rho) = \rho \int_{0}^{\rho} \frac{P'(\rho)}{\rho^2} d\rho'.$$

(B87)
Comparing Eq. (B86) with Eqs. (B3) and (B13), we see that the potential $V(\rho)$ represents the density of internal energy

$$u(\rho) = V(\rho).$$

(B88)
We then have

$$P(\rho) = \rho^2 \left( \frac{V(\rho)}{\rho} \right)' = \rho V'(\rho) - V(\rho),$$

(B89)
$$P'(\rho) = \rho V''(\rho), \quad h(\rho) = \frac{P(\rho) + V(\rho)}{\rho},$$

(B90)
$$h(\rho) = V'(\rho), \quad h'(\rho) = \frac{P'(\rho)}{\rho},$$

(B91)
$$V(\rho) = \rho \int_{0}^{\rho} \frac{P'(\rho)}{\rho^2} d\rho'.$$

(B92)
The squared speed of sound is

$$c_s^2 = P'(\rho) = \rho V''(\rho).$$

(B93)
Appendix C: Classical collisionless self-gravitating systems

In this Appendix and in the following one, we discuss certain aspects of the dynamical evolution of classical collisionless self-gravitating systems to facilitate the comparison with the dynamical evolution of quantum self-gravitating systems treated in the main text.

1. Vlasov equation

A classical collisionless self-gravitating system (such as a stellar system or such as CDM) is governed by the Vlasov–Poisson equations

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = 0, \quad (C1)
\]

\[
\Delta \Phi = 4\pi G \int f \, dv, \quad (C2)
\]

where \( f = f(r, v, t) \) is the six-dimensional DF. The Vlasov equation is also known as the collisionless Boltzmann equation \([7]\). It states that, in the absence of encounters (“collisions”) between the particles, the density (DF) of a “fluid” particle is conserved when we follow its motion in phase space, i.e., \( Df/Dt = 0 \) where \( Df/DT = \partial f/\partial t + v \cdot \partial f/\partial r - \nabla \Phi \cdot \partial f/\partial v \) is the material derivative (Stokes operator).

As a result, the Vlasov–Poisson equations conserve the energy \( E = (1/2) \int f u^2 \, dr \, dv + (1/2) \int \rho \Phi \, dr \), the mass \( M = \int f \, dv \) and an infinite class of Casimir integrals of the form \( \int h(f) \, dv \) where \( h(f) \) is an arbitrary function of \( f \) \([91, 94]\).

Remark. The Vlasov equation can be viewed as the expression of the Liouville theorem in the individual phase space. Under the circumstance in which stellar encounters can be ignored, each star can be idealized as an independent conservative system described by the Hamiltonian \( H = v^2/2 + \Phi(r, t) \) yielding the equations of motion \( dr/dt = v, \, dv/dt = -\nabla \Phi \). The equation of continuity \( \partial f/\partial t + \nabla \cdot (fu_0) = 0 \), where \( \nabla = (\partial_x, \partial_y) \) is a generalized nabla operator and \( u_0 = (v, -\nabla \Phi) \) a generalized velocity field, and the fact that the flow in phase space in incompressible, \( \nabla \cdot u_0 = 0 \), leads to the Liouville equation \( \partial f/\partial t + u_0 \cdot \nabla f = 0 \), which is the Vlasov equation.

2. Hydrodynamics of the Vlasov equation: Jeans equations

From the Vlasov equation, we can derive a system of hydrodynamic equations called the Jeans equations.\(^{40}\) By integrating the Vlasov equation \((C1)\) over the velocity, we get the continuity equation (expressing the local mass conservation)

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \quad (C3)
\]

where we have introduced the local density

\[
\rho = \int f \, dv \quad (C4)
\]

and the local velocity

\[
u = \langle v \rangle = \frac{1}{\rho} \int f v \, dv. \quad (C5)\]

Then, multiplying the Vlasov equation \((C1)\) by \( v \) and integrating over the velocity, we obtain the momentum equation

\[
\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j} \int f v_i v_j \, dv = -\rho \frac{\partial \Phi}{\partial x_i}. \quad (C6)
\]

Introducing the difference \( w = v - u \) between the velocity \( v \) of a particle and the mean local velocity \( u(r, t) \), and using the fact that \( \langle w \rangle = 0 \), we get

\[
\int f v_i v_j \, dv = \rho u_i u_j + \int f w_i w_j \, dv. \quad (C7)
\]

Therefore, the momentum equation \((C6)\) takes the form

\[
\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_i u_j) = -\partial_i P_j - \rho \frac{\partial \Phi}{\partial x_i}, \quad (C8)
\]

where we have introduced the pressure tensor \( -P_j \) is the stress tensor)

\[
P_{ij} = \rho \langle w_i w_j \rangle = \int f (v - u_i)(v - u_j) \, dv. \quad (C9)
\]

It can be written as

\[
P_{ij} = \rho \langle (v_i - u_i)(v_j - u_j) \rangle. \quad (C10)
\]

Using the continuity equation \((C3)\), we obtain the identity

\[
\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \otimes u) = \rho \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right]. \quad (C11)
\]

As a result, the momentum equation \((C8)\) can be rewritten as

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho} \partial_i P_j - \nabla \Phi. \quad (C12)
\]

These equations are essentially those for a compressible fluid which is supported by pressure in the form of a velocity dispersion. These equations are not closed because the pressure tensor \( P_{ij} \) depends on the DF which is not explicitly known in general. Actually, we can build up an infinite hierarchy of equations by introducing higher and higher moments of the velocity. In general, there is no simple way to close this hierarchy of Jeans equations except in the single speed case (see Appendix C3). In more general cases, some approximations must be introduced.

3. Single-speed solution: pressureless Euler equations

The Vlasov–Poisson equations admit a particular solution of the form

\[
f(r, v, t) = \rho(r, t) \delta(v - u(r, t)). \quad (C13)
\]

This is called the single-speed solution because there is a single velocity attached to any given point \( r \) at time \( t \). It corresponds to the "dust model" where the pressure is zero because there is no thermal motion (at a given location all

\(^{40}\) Actually, the Vlasov equation \([137]\) was introduced by Jeans \([138]\) (see \([139]\)) and the Jeans equations were introduced by Maxwell (see \([7]\)).
the particles have the same velocity). The density $\rho(\mathbf{r}, t)$ and the velocity $\mathbf{u}(\mathbf{r}, t)$ satisfy the pressureless Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (C14)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \Phi, \quad (C15)$$

$$\Delta \Phi = 4\pi G \rho. \quad (C16)$$

These equations are exact. They can be deduced from the Jeans equations (C3) and (C12) by closing the hierarchy with the condition $P_{ij} = 0$ obtained by substituting Eq. (C13) into Eq. (C9). They correspond to very particular initial conditions where the particles at a given location all have the same velocity. However, there is a well-known difficulty with the solution (C13). Even if one starts with a DF of the form of Eq. (C13) then, after a finite time, the solution of the Vlasov–Poisson equations becomes multi-streaming because of particle crossing. This leads to the formation of caustics (singularities) in the density field at shell-crossing. Therefore, Eq. (C13) ceases to be valid. This phenomenon renders the pressureless hydrodynamical description (C14)–(C16) useless beyond the first time of crossing when the fast particles cross the slow ones. Therefore, after shell-crossing, the pressureless Euler equations are not defined anymore and we must come back to the original Vlasov–Poisson equations, or to the Jeans equations, because we need to account for a velocity dispersion. Indeed, the velocity field becomes multi-valued even if, initially, it is single-valued. The pressureless Euler equations are only valid until shell crossing and they fail as soon as orbit crossing (multi-streaming) occurs.

Different attempts to cure this problem have been proposed in order to continue using a hydrodynamical model.

(i) A first heuristic possibility to avoid multi-streaming is to introduce a viscosity term $\nu \Delta \mathbf{u}$ in the momentum equation (C15) yielding the Navier–Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \Phi + \nu \Delta \mathbf{u}. \quad (C17)$$

In order for the diffusion term to have a smoothing effect only in the regions where particle-crossing is about to occur, the viscosity $\nu$ should be small. More precisely, the limit $\nu \to 0$ should be taken, which is different from setting $\nu = 0$.41

(ii) A second heuristic possibility is to introduce a pressure term $(-1/\rho) \nabla P$ in the momentum equation (C15) yielding

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \Phi - \frac{1}{\rho} \nabla P, \quad (C18)$$

where $P$ is the fluid pressure, a local quantity given by a specified equation of state which takes into account velocity dispersion. This amounts to closing the hierarchy of Jeans equations with the isotropy ansatz $P_{ij} = P(\rho) \delta_{ij}$. In this manner, there is no shell-crossing singularities. The velocity dispersion giving rise to the pressure can be a consequence of the multi-streaming or it can be already present in the initial condition. We need $P \to 0$ at large scales to recover the CDM model and $P \neq 0$ at small scales to avoid singularities.42

(iii) A third heuristic possibility, proposed by Widrow and Kaiser [144], is to replace the Vlasov equation by a wave equation having the form of a Schrödinger equation with an effective Planck constant $\hbar_{\text{eff}}$ controlling the spatial resolution.43 Using the Madelung transformation, this prescription leads, instead of Eq. (C15), to a momentum equation of the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \Phi - \frac{1}{\rho} \partial_i P_{ij}^{Q}, \quad (C19)$$

with an effective “quantum” pressure tensor

$$P_{ij}^{Q} = -\frac{\hbar_{\text{eff}}^2}{4m^2} \rho \partial_i \partial_j \ln \rho = \frac{\hbar_{\text{eff}}^2}{4m^2} \left( \frac{1}{\rho} \partial_i \rho \partial_j \rho - \partial_i \partial_j \rho \right). \quad (C20)$$

Interestingly, the effective quantum pressure tensor $P_{ij}^{Q}$ bears some resemblance with the Jeans pressure tensor $P_{ij}$ of Eq. (C10) with the velocity average operators replaced by density gradients (a nonlocal quantity). This amounts to closing the hierarchy of Jeans equations with the condition $P_{ij} = P_{ij}^{Q}$.44 Therefore, although the system is classical, the procedure of Widrow and Kaiser [144] amounts to closing the Jeans equations by introducing an effective quantum potential of order $O(\hbar_{\text{eff}}^2/m^2)$. When $\hbar_{\text{eff}}/m \to 0$ the effective quantum pressure $P_{ij}^{Q}$ tends to zero at large scales while $P_{ij}^{Q} \neq 0$ at small scales (in line with the fact that quantum mechanics is negligible at large scales and important at small scales in BECDM), thereby preventing singularities. This is exactly what is required. The effective quantum pressure tensor acts as a regularizer of caustics and singularities in classical solutions. The quantum pressure also replaces the role of the viscosity in the adhesion model (see footnote 41) [145]. Therefore, the Schrödinger method can handle multiple streams in phase space.

(iv) A fourth possibility is to use the heuristic kinetic theory of violent relaxation developed by Chavanis et al. [94]

42 Sensible expressions of $P(\rho)$ have been considered by Buchert et al. [142]. In particular, they singled out an equation of state of the form $P(\rho) \propto \rho^{\gamma}$ that leads, under certain assumptions, to a viscous Burgers-like equation in cosmology. Interestingly, as discussed in [143], this expression is similar to the pressure created by self-interacting bosons in the BECDM model.

43 The Schrödinger equation is equivalent to the Wigner equation. In the Schrödinger equation, $\psi(\mathbf{r}, t)$ encodes both position and momentum information in a single position-space function. It is argued that, when $\hbar_{\text{eff}} \to 0$, the Vlasov–Poisson equations are recovered and that a finite value of $\hbar_{\text{eff}}$ provides a small-scale regularization of the dynamics. In that case, the Schrödinger–Poisson system has nothing to do with quantum mechanics since it aims at describing the evolution of classical collisionless matter under the influence of gravity.

44 For self-gravitating BECs described by the “true” Schrödinger equation, this identification is exact (see Sect. 3.4.1). Inversely, this identification may suggest an interpretation of the quantum potential $P_{ij}^{Q}$ in terms of a classical kinetic theory.
(see also the Appendix of [84]). In that case, the pressureless Euler equation (C15) is replaced by the damped Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla \Phi - \frac{1}{\rho} \nabla P_{LB} - \xi \mathbf{u}, \quad \text{(C21)}$$

where $P_{LB}$ is the Lynden-Bell pressure (or a generalized pressure) and $\xi$ is a friction coefficient accounting for nonlinear Landau damping.

In these different hydrodynamic models, we can go beyond the first time of crossing. Therefore, these hydrodynamic models are defined for all times. The solution of these hydrodynamic equations is expected to remain close to the solution of the Vlasov–Poisson equations for all times provided that $\nu \to 0$, $P \to 0$, and $h_{\text{coll}}/m \to 0$ in these respective models. We note that the limits $\nu, P, h_{\text{coll}}/m \to 0$ are crucially different from taking $P = \nu = h_{\text{coll}}/m = 0$.

**Remark** The Vlasov–Poisson equations (C1) and (C2) are valid for all times. Similarly, the Jeans equations (C3)–(C12), which are equivalent to the Vlasov equation, are valid for all times but they are not closed (we have to consider the whole hierarchy of equations). By contrast, the pressureless Euler–Poisson equations (C14)–(C16) are valid only until the first time of crossing. Before that time they coincide with the Vlasov–Poisson equations and after that time they break down. The modified Euler–Poisson equations (C17)–(C21) are valid for all times. If $\nu \to 0$, $P \to 0$ and $h_{\text{coll}} \to 0$, they are expected to be close to the Vlasov–Poisson equations. In particular, Eqs. (C19)–(C20) are equivalent to the Schrödinger and Wigner equations, which are themselves equivalent to the Vlasov equation when $h_{\text{coll}} \to 0$.

### Appendix D: Violent relaxation of classical collisionless self-gravitating systems

The Vlasov–Poisson equations (C1) and (C2) describing classical collisionless stellar systems are known to experience a process of violent relaxation [91] caused by the rapid fluctuations of the strongly varying gravitational potential at the early stage of galaxy formation. While the fine-grained DF $f(r, v, t)$ always evolves in time, forming intermingled filaments in phase space at smaller and smaller scales, the coarse-grained DF $\bar{f}(r, v)$, which smooths out this intricate filamentation, rapidly relaxes towards a quasistationary state $\bar{f}_{\text{QSS}}(r, v)$ in this process. This takes place on a very short timescale of the order of the dynamical time $t_D$. For $t > t_D$, the evolution of $f(r, v, t)$ occurs on scales smaller than the coarse-graining mesh. This complicated dynamics is associated with phase mixing, violent relaxation and nonlinear Landau damping. As a result, the coarse-grained DF $\bar{f}(r, v, t)$ does not satisfy the Vlasov equation. Phase-space correlations introduce an effective “collision” term $\mathcal{C}(\bar{f})$ on the right hand side of the coarse-grained Vlasov equation. This collision term drives the relaxation of the coarse-grained DF towards the quasistationary state. The determination of the quasistationary state $\bar{f}_{\text{QSS}}(r, v)$ and of the collision term $\mathcal{C}(\bar{f})$ is a problem of fundamental interest but also of great difficulty because of the very nonlinear nature of the process. Here, we tackle this problem through a thermodynamical approach. We use a MEP to determine the quasistationary state $\bar{f}_{\text{QSS}}(r, v)$ and we use a MEPP to determine the collision term $\mathcal{C}(\bar{f})$. At a general level, the MEP determines the most probable equilibrium state of the system and the MEPP determines the most probable evolution of the system [146].

#### 1. Maximum entropy principle

In a seminal paper, Lynden-Bell [91] argued that the coarse-grained DF $\bar{f}_r(r, v, t)$ violently relaxes towards a quasistationary state $\bar{f}_{\text{QSS}}(r, v)$ which maximizes a suitable mixing entropy while taking into account all the constraints of the Vlasov equation. In the two-level approximation of the theory, where the fine-grained DF $f(r, v, t)$ takes only two values $f = \eta_0$ and $f = 0$, the constraints reduce to the conservation of mass and energy and the Lynden-Bell entropy reads

$$S = -\eta_0 \int \left\{ \frac{7}{\eta_0} \ln \frac{7}{\eta_0} + \left( 1 - \frac{7}{\eta_0} \right) \ln \left( 1 - \frac{7}{\eta_0} \right) \right\} \, d\mathbf{v}.$$  \hspace{1cm} (D1)

It can be obtained from a combinatorial analysis taking into account the specificities of the Vlasov equation. In particular, the coarse-grained DF must satisfy the inequality $\bar{f}(r, v, t) \leq \eta_0$ arising from the incompressibility of the flow in phase space and the conservation of the DF on the fine-grained scale. This constraint is similar to the Pauli exclusion principle in quantum mechanics and this is why the Lynden-Bell entropy (D1) resembles the Fermi–Dirac entropy of quantum mechanics. In this sense, the process of violent relaxation is similar in some respects to the collisional relaxation of self-gravitating fermionic particles. The Lynden-Bell DF corresponds to a fourth type of statistics corresponding to distinguishable particles experiencing an exclusion principle [91].

According to the MEP, the DF of the quasistationary state $\bar{f}_{\text{QSS}}$ maximizes the Lynden-Bell entropy (D1) at fixed mass

$$M = \int \rho \, d\mathbf{r} \quad \text{(D2)}$$

and energy

$$E = \int \frac{1}{2} \rho v^2 \, d\mathbf{v} + \frac{1}{2} \int \rho \Phi \, d\mathbf{r} = K + W,$$  \hspace{1cm} (D3)

where $K$ is the kinetic energy and $W$ the potential (gravitational) energy. Introducing Lagrange multipliers and writing the variational principle under the form

$$\delta S - \beta \delta E + \alpha \delta M = 0,$$  \hspace{1cm} (D4)

we find that the extrema of entropy at fixed mass and energy correspond to the Lynden-Bell DF

$$\bar{f}_{LB}(r, v) = \frac{\eta_0}{1 + \alpha e^{(v^2/2 + \Phi(r)) - \alpha}}, \quad \text{(D5)}$$

45 The Lynden-Bell entropy is proportional to the logarithm of the number of microstates corresponding to a given macrostate. This is a measure of disorder. Therefore, the maximization of $S$ under constraints determines the most probable state of the system, i.e., the macrostate that is the most represented at the “microscopic” level.

46 The general case is treated in [91,94,101].

47 More generally, the coarse-grained DF must always be smaller than the maximum value of the fine-grained (or initial) DF.
where $\epsilon = v^2/2 + \Phi(r)$ is the energy of a particle by unit of mass. The Lagrange multipliers $\beta = 1/T_{\text{eff}}$ and $\alpha = \mu_{\text{eff}}/T_{\text{eff}}$ are the effective inverse temperature and the effective chemical potential (divided by the effective temperature). The Lynden-Bell DF (D5) which maximizes the Lynden-Bell entropy at fixed mass and energy is the most probable, or most mixed, state taking into account all the constraints of the Vlasov equation. The Lynden-Bell DF is similar to the Fermi–Dirac DF in quantum mechanics provided that we make the correspondence

$$\eta_0 = \frac{gm^4}{h^3},$$

(D6)

where $g = 2s + 1$ is the multiplicity of the quantum states. In particular, the Lynden-Bell DF (D5) satisfies the constraint $\tilde{f}_{LB}(r, v) \leq \eta_0$ which is similar to the Pauli exclusion principle in quantum mechanics. We note that the effective temperature $T_{\text{eff}}$ has not the dimension of a temperature. It should rather be interpreted as a velocity dispersion. However, we shall use this notation which is more transparent. We also note that the mass $m$ of the particles does not appear in the Lynden-Bell theory since it is based on the Vlasov equation for collisionless systems which is independent of the mass of the particles. In this sense, we can say that the temperature in Lynden-Bell’s theory is proportional to the mass of the particles [91].

From the Lynden-Bell DF (D5), one can determine the density and the pressure through the equations

$$\rho = \int \tilde{f} dv = \int_0^{+\infty} \frac{\eta_0}{1 + e^{\beta[v^2/2 + \Phi(r)] - \alpha}} 4\pi v^2 dv,$$

(D7)

$$P = \frac{1}{3} \int \tilde{f} v^2 dv = \frac{1}{3} \int_0^{+\infty} \frac{\eta_0}{1 + e^{\beta[v^2/2 + \Phi(r)] - \alpha}} 4\pi v^4 dv.$$

(D8)

They can be rewritten as

$$\rho(r) = \frac{4\pi \eta_0 \sqrt{2}}{3^{3/2}} I_{1/2} \left[ e^{\beta \Phi(r) - \alpha} \right],$$

(D9)

$$P(r) = \frac{8\pi \eta_0 \sqrt{2}}{3^{3/2}} I_{3/2} \left[ e^{\beta \Phi(r) - \alpha} \right],$$

(D10)

where

$$I_n(t) = \int_0^{+\infty} \frac{x^n}{1 + e^x} dx$$

(D11)

denote the Fermi–Dirac integrals. Eliminating formally the gravitational potential $\Phi(r)$ between Eqs. (D9) and (D10), we see that the equation of state is barotropic: $P(r) = P[\rho(r)]$. Equations (D9) and (D10) determine the Lynden-Bell equation of state $P_{LB}(\rho)$ in parametric form. The Lynden-Bell equation of state is formally similar to the Fermi–Dirac equation of state. Using Eq. (D8), the kinetic energy can be written as

$$K = \frac{3}{2} \int P_{LB}(\rho) d\rho.$$ 

(D12)

On the other hand, the Lynden-Bell DF (D5), and more generally any DF of the form $f = f(\epsilon)$, implies the condition of hydrostatic equilibrium (see, e.g., [42] and the Remark of Appendix F)

$$\nabla P + \rho \nabla \Phi = 0.$$ 

(D13)

In the completely degenerate limit $\tilde{f}_{LB} \sim \eta_0$, the Lynden-Bell DF reduces to a step function

$$f_{LB}(r, v) = \eta_0 H(\epsilon - \epsilon_{LB}),$$

(D14)

where $H$ is the Heaviside function ($H(x) = 1$ if $x < 1$ and $H(x) = 0$ if $x > 1$) and $\epsilon_{LB}$ is the Lynden-Bell energy, which is the counterpart of the Fermi energy. In that case, Eqs. (D7) and (D8) reduce to

$$\rho = \int_0^{v_{LB}} \eta_0 4\pi v^2 dv = 4\pi \eta_0 \frac{v_{LB}^3}{3},$$

(D15)

$$P = \frac{1}{3} \int_0^{v_{LB}} \eta_0 4\pi v^2 dv = 4\pi \eta_0 \frac{v_{LB}^5}{3},$$

(D16)

where $v_{LB}(r) = \sqrt{2(\epsilon_{LB} - \Phi(r))}$ is the Lynden-Bell velocity. Equations (D15) and (D16) lead to the equation of state

$$P = \frac{1}{5} \left( \frac{3}{4\pi \eta_0} \right)^{2/3} \rho^{5/3}.$$ 

(D17)

This is a polytropic equation of state of index $n = 3/2$ like the one arising in the theory of nonrelativistic white dwarf stars at $T = 0$ corresponding to the ground state of the self-gravitating Fermi gas [120].

In the nondegenerate, or dilute, limit $\tilde{f}_{LB} \ll \eta_0$, the Lynden-Bell entropy becomes similar to the Boltzmann entropy

$$S \approx - \int \tilde{f} \left[ \ln \left( \frac{\tilde{f}_{LB}(r)}{\eta_0} \right) - 1 \right] dv,$$

(D18)

the Lynden-Bell DF becomes similar to the Maxwell-Boltzmann DF

$$\tilde{f}_{LB}(r, v) \approx \eta_0 e^{\alpha - \beta v^2/2 - \Phi(r)},$$

(D19)

and the Lynden-Bell equation of state becomes similar to the equation of state of an isothermal gas

$$P = \rho T_{\text{eff}}.$$ 

(D20)

This equation of state has been studied in relation to isothermal stars [120] and to the statistical mechanics of “collisionless” self-gravitating systems relaxing under the effect of gravitational encounters [128]. Remarkably, the Lynden-Bell theory of violent relaxation explains how a collisionless self-gravitating system can “thermalize” on a very short timescale, much shorter that the collisional relaxation time $t_{\text{relax}} \sim (N/\ln N) t_D$, without the need of “collisions” [91].

So far, we have assumed that the system is isolated so that it conserves the mass and the energy. This corresponds to the microcanonical ensemble. We now consider the canonical ensemble in which the temperature $T_{\text{eff}} = 1/\beta$ is fixed instead of the energy. In that case, the equilibrium state is obtained by minimizing the free energy $F = E - T_{\text{eff}} S$ at fixed mass $M$ or, equivalently, by maximizing the Massieu function $J = S - \beta E$ at fixed mass $M$. The variational principle determining the extrema of free energy at fixed mass reads

$$\delta J + \alpha \delta M = 0.$$ 

(D21)

Since $\beta$ is fixed, this variational principle for the first variations of the thermodynamical potential is equivalent to Eq. (D4) and it returns the Lynden-Bell DF (D5). Therefore, the equilibrium states are the same in the microcanonical
and canonical ensembles (the extrema of entropy at fixed mass and energy coincide with the extrema of free energy at fixed mass). However, their thermodynamical stability (related to the sign of the second variations of the thermodynamical potential) may be different in the microcanonical and canonical ensembles. This is the notion of ensembles inequivalence for systems with long-range interactions [128–131]. An equilibrium state is thermodynamically stable in the microcanonical ensemble if it is a maximum of entropy at fixed mass and energy. This corresponds to

$$\delta^2 J = \delta^2 S - \beta \delta^2 E = -\int \frac{1}{f(1 - f/\eta_0)} \left( \frac{\delta f}{\delta v} \right)^2 \, dv \, dr$$

\[ \frac{1}{2} \beta \int \delta \rho \delta \Phi \, dr \leq 0 \]  

for all perturbations $\delta f$ that conserve mass and energy at first order. An equilibrium state is thermodynamically stable in the canonical ensemble if it is a minimum of free energy at fixed mass. This corresponds to the inequality of Eq. (D22) for all perturbations $\delta f$ that conserve mass. We note that canonical stability implies microcanonical stability but the converse is wrong [132]. For example, it is shown in [102] that the core-halo solution with a negative specific heat is stable in the microcanonical ensemble while it is unstable in the canonical ensemble.

2. Maximum entropy production principle

We now consider the dynamical evolution of the coarse-grained DF $f_\eta(r, \nu, t)$. Writing $f = f_r + \delta f$ and $\Phi = \Phi_r + \delta \Phi$, where $\delta f$ and $\delta \Phi$ denote fluctuations about the coarse-grained fields, and taking the local average of the Vlasov equation (C1), we get

$$\frac{\partial f_r}{\partial t} + \nu \cdot \frac{\partial f_r}{\partial \nu} - \nabla \Phi \cdot \frac{\partial f_r}{\partial \nu} = \frac{\partial}{\partial \nu} \delta f_r \delta \Phi.$$  

(D23)

This equation shows that the correlations of the fluctuations of the gravitational potential and DF create an effective “collision” term $C[f]$ in the r.h.s. of Eq. (D23). In Ref. [94] we obtained an explicit expression of this collision term by using heuristic arguments based on a MEPP. We considered the general case where the fine-grained DF may take an arbitrary number of values. Below, we detail this procedure in the simpler case where the fine-grained DF takes only two values $f = 0$ and $f = \eta_0$ as in the preceding section.

To apply the MEPP, we first write the relaxation equation for the coarse-grained DF under the form

$$\frac{\partial f_r}{\partial t} + \nu \cdot \frac{\partial f_r}{\partial \nu} - \nabla \Phi \cdot \frac{\partial f_r}{\partial \nu} = - \frac{\partial}{\partial \nu} \cdot J,$$  

(D24)

where $J$ is the current to be determined. The form of Eq. (D24) ensures the conservation of mass provided that $J$

decreases sufficiently rapidly for large $|\nu|$. From Eqs. (D1), (D3) and (D24), we get

$$\dot{S} = - \int \frac{1}{f(1 - f/\eta_0)} \, J \cdot \frac{\partial f}{\partial \nu} \, dv \, dr, \quad (D25)$$

$$E = \int J \cdot \nu \, dv \, dr,$$  

(D26)

where we have used straightforward integrations by parts. Following the MEPP, we shall determine the optimal current $J$ which maximizes the rate of entropy production (D25) while satisfying the conservation of energy $\dot{E} = 0$. For this problem to have a solution, we shall also impose a limitation on the current $|J|$ characterized by a bound $C(r, \nu, t)$ which exists but is not explicitly known, so that

$$\frac{J^2}{\eta_0^2} \leq C(r, \nu, t).$$  

(D27)

It can be shown by a convexity argument that reaching the bound (D27) is always favorable for increasing $\dot{S}$, so this constraint can be replaced by an equality. The variational problem can then be solved by introducing at each time $t$ Lagrange multipliers $\beta(t)$ and $1/D_r(r, \nu, t)$ for the two constraints. The condition

$$\delta \dot{S} - \beta(t) \delta \dot{E} - \int \frac{1}{D_r} \delta \left( \frac{J^2}{\eta_0^2} \right) \, dv \, dr = 0 \quad (D28)$$

yields an optimal current of the form

$$J = -D \left[ \frac{\partial f_r}{\partial \nu} + \beta(t) f(1 - f/\eta_0) \nu \right], \quad (D29)$$

where we have set $D = D(1 - f/\eta_0)$ to avoid divergences when $f \rightarrow \eta_0$. The time evolution of the Lagrange multiplier $\beta(t)$ is determined by the conservation of energy $\dot{E} = 0$, introducing Eq. (D29) into the constraint (D26). This yields

$$\beta(t) = - \frac{\int D_r \frac{\partial f_r}{\partial \nu} \cdot \nu \, dv \, dr}{\int D_r \left( 1 - f/\eta_0 \right) \nu^2 \, dv \, dr}. \quad (D30)$$

Note that the optimal current (D29) can be written as

$$J = -D \frac{\partial \alpha}{\partial \nu}, \quad (D31)$$

where

$$\alpha(r, \nu, t) = \ln \left( \frac{f/\eta_0}{1 - f/\eta_0} \right) + \beta(t) \left[ \frac{\nu^2}{2} + \Phi(r, t) \right] \quad (D32)$$

is a time-dependent chemical potential which is uniform at equilibrium according to Eq. (D5). The relation from Eq. (D31) then corresponds to the linear thermodynamics of Onsager [147,148] where the currents are proportional to the gradients of the thermodynamic potentials that are uniform at statistical equilibrium. Therefore, the MEPP can be viewed as a variational formulation of Onsager’s linear thermodynamics [149].

48 Since the process of violent relaxation is very nonlinear, we cannot in principle use perturbation methods (see, however, [96–101] in a regime of “gentle” relaxation). We thus capitalize our ignorance and assume that the system evolves so as to maximize its rate of entropy production at fixed mass and energy. This thermodynamic principle is expected to determine the most probable evolution of the system.

49 We have some freedom on the determination of the diffusion coefficient since it is related to a constraint that is not explicitly known.

50 See Ref. [146] for a discussion of the connection between the MEPP and the minimization of the Onsager-Machlup [150] functional.
Introducing the optimal current (D29) into Eq. (D24), we obtain the relaxation equation

\[
\frac{\partial \mathcal{T}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathcal{T}}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \mathcal{T}}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left[ D \left( \frac{\partial \mathcal{T}}{\partial \mathbf{v}} + \beta(t)(1 - \mathcal{T}/\eta_0)\mathbf{v} \right) \right].
\]

(D33)

Morphologically, this relaxation equation has the form of a nonlinear Fokker–Planck equation or, more precisely, the form of a fermionic Kramers equation [149,151]. The first term is a diffusion term and the second term is a friction term. The fermionic factor \( \mathcal{F}(1 - \mathcal{T}/\eta_0) \) takes into account the Lynden-Bell exclusion principle. The function \( \beta(t) \) can be considered as a time dependent inverse temperature evolving with time so as to conserve the energy (microcanonical formulation).\(^{51}\) The friction coefficient \( \xi \) satisfies a generalized Einstein relation: \( \xi = D \beta \). Note that \( D \) is not determined by the MEP since it is related to the unknown constant \( C(\mathbf{r}, \mathbf{v}, t) \) in Eq. (D27).

It is straightforward to check that Eq. (D33) with the constraint (D30) satisfies an \( H \)-theorem for the Lynden-Bell entropy (D1). From Eq. (D25), we can write

\[
\dot{S} = -\int \frac{1}{\mathcal{F}(1 - \mathcal{T}/\eta_0)} \mathbf{J} \cdot \left[ \frac{\partial \mathcal{T}}{\partial \mathbf{v}} + \beta(t)(1 - \mathcal{T}/\eta_0)\mathbf{v} \right] d\mathbf{r} d\mathbf{v} + \beta(t) \int \mathbf{J} \cdot \mathbf{v} d\mathbf{r} d\mathbf{v}.
\]

(D34)

The last integral vanishes due to the conservation of the energy (see Eq. (D26) with \( E = 0 \)). Using Eq. (D29), we obtain

\[
\dot{S} = \int \frac{\mathbf{J}^2}{D \mathcal{F}(1 - \mathcal{T}/\eta_0)} d\mathbf{r} d\mathbf{v},
\]

(D35)

which is positive (\( \dot{S} \geq 0 \)) provided that \( D > 0 \). This proves the \( H \)-theorem. At equilibrium, we have \( \dot{S} = 0 \) which implies \( \mathbf{J} = 0 \). Then, according to Eq. (D29), we obtain

\[
\frac{\partial \mathcal{T}}{\partial \mathbf{v}} + \beta \mathcal{F}(1 - \mathcal{T}/\eta_0)\mathbf{v} = 0.
\]

(D36)

Integrating this equation with respect to \( \mathbf{v} \), we get

\[
\ln \left( \frac{\mathcal{T}/\eta_0}{1 - \mathcal{T}/\eta_0} \right) + \beta \frac{\mathbf{v}^2}{2} = A(\mathbf{r}),
\]

(D37)

where \( A(\mathbf{r}) \) is a constant of integration that may depend on \( \mathbf{r} \). Taking the gradient of the foregoing equation with respect to \( \mathbf{r} \), we find that

\[
\frac{1}{\mathcal{F}(1 - \mathcal{T}/\eta_0)} \frac{\partial \mathcal{T}}{\partial \mathbf{r}} = \nabla A(\mathbf{r}).
\]

(D38)

On the other hand, since \( \partial \mathcal{F}/\partial \mathbf{r} = 0 \) and \( \mathbf{J} = 0 \), the advection term in Eq. (D24) cancels out:

\[
\mathbf{v} \cdot \frac{\partial \mathcal{T}}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \mathcal{T}}{\partial \mathbf{v}} = 0.
\]

(D39)

Together with Eqs. (D36) and (D38), this implies the relation

\[
\mathbf{v} \cdot (\nabla A + \beta \nabla \Phi) = 0.
\]

(D40)

This relation must be true for all \( \mathbf{v} \) so that

\[
\nabla A + \beta \nabla \Phi = 0,
\]

(D41)

which can be integrated into

\[
A(\mathbf{r}) = -\beta \Phi(\mathbf{r}) + \alpha,
\]

(D42)

connecting the growth rate \( \lambda \) of the perturbation \( \delta \mathbf{f} \sim e^{\lambda t} \) to the second order variations of the free energy (Mussieu function) \( \dot{J} = S - \beta E \) and the second order variations of the rate of entropy production \( \delta^2 \dot{S} \). Since the product \( \lambda \dot{S}J \) is positive because \( \delta^2 \dot{S} \geq 0 \) according to Eq. (D35), we conclude that a stationary solution of Eqs. (D30) and (D33) is linearly dynamically stable (\( \lambda < 0 \)) if, and only if, it is an entropy maximum at fixed mass and energy (\( \delta^2 J < 0 \)). This aesthetic formula shows the equivalence between dynamical and thermodynamical stability for the generalized Fokker–Planck equation (D33) with the constraint from Eq. (D30). Therefore, this equation can only relax towards maxima of \( S \), not towards minima or saddle points.

A relaxation equation of the form of Eq. (D24) appropriate to the canonical ensemble can be obtained by maximizing the rate of free energy dissipation \( J = \dot{S} - \beta \dot{E} \) with the constraint (D27). The corresponding variational principle

\[
\delta J - \int \frac{1}{D} \left( \frac{\mathbf{J}^2}{2f} \right) d\mathbf{r} d\mathbf{v} = 0
\]

(D44)

again yields an optimal current of the form of Eq. (D29) now involving a constant inverse temperature \( \beta \). This equation satisfies an \( H \)-theorem for the Lynden-Bell free energy. From Eqs. (D25) and (D26) we have

\[
\dot{J} = \dot{S} - \beta \dot{E} = -\int \frac{1}{\mathcal{F}(1 - \mathcal{T}/\eta_0)} \mathbf{J} \cdot \frac{\partial \mathcal{T}}{\partial \mathbf{v}} d\mathbf{r} d\mathbf{v} - \beta \int \mathbf{J} \cdot \mathbf{v} d\mathbf{r} d\mathbf{v}.
\]

(D45)

Using Eq. (D29), we obtain

\[
\dot{J} = \int \frac{\mathbf{J}^2}{D \mathcal{F}(1 - \mathcal{T}/\eta_0)} d\mathbf{r} d\mathbf{v},
\]

(D46)

\(^{52}\) This result can also be directly obtained from the \( H \)-theorem by using Lyapunov’s direct method.
which is positive \((\dot{J} \geq 0)\) provided that \(D > 0\). Therefore, the free energy \(F\) decreases monotonically until an equilibrium state of the form (D5) is reached \((H\text{-theorem})\). In the canonical ensemble, we can show that \(2\lambda \delta^2 J = \delta^2 J \geq 0\) \[149\]. Therefore, a stationary solution of the generalized Fokker–Planck equation (D33) with constant temperature is linearly stable if, and only if, it is a \(\{\text{local}\}\) minimum of free energy \(F\) at fixed mass. Therefore, this equation can only relax towards minima of \(F\), not towards maxima or saddle points.

**Remark** By using the MEPP (or the linear thermodynamics of Onsager), we have constructed a kinetic equation for the coarse-grained DF, Eq. (D33), which relaxes towards the Lynden-Bell DF (D5). This kinetic equation has the form of a fermionic Kramers equation. We stress that this thermodynamical approach is phenomenological in nature. It is also possible to derive a kinetic equation for the coarse-grained DF, Eq. (D33), which relaxes towards the free energy \(D > 0\) by making a thermal bath approximation \[96\]. In that case, the diffusion coefficient is uncertain \((\text{it may only describe a} \ \text{“gentle” relaxation})\). How-ever, the fermionic Kramers equation can be obtained from the fermionic Landau equation by making a thermal bath approximation \[96\]. In that case, the diffusion coefficient \((\text{which is not determined by the MEPP})\) can be explicitly calculated. The fermionic Kramers equation with a time-dependent temperature and the fermionic Landau equation both satisfy an \(H\)-theorem for the Lynden-Bell entropy. We note, however, that the fermionic Landau equation conserves the energy locally while the fermionic Kramers equation \(a\) dissipative effects:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{D51}
\]
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P_{\text{ij}} - \nabla \Phi - \xi \mathbf{u}, \tag{D52}
\]

where \(\xi = D \beta\). They are called the damped Jeans equations. On the other hand, the effective collision term in Eq. (D47) provides a source of relaxation which allows us to close the hierarchy of moment equations. Indeed, we can compute the pressure tensor in Eq. (D52) by making a LTE approximation

\[
\mathcal{T}_{\text{LTE}}(\mathbf{r}, \mathbf{v}) = \frac{\rho (\mathbf{r}, t)}{[2\pi T_{\text{eff}}(t)]^{3/2}} \left[\frac{2 \pi m (\mathbf{v})}{2 T_{\text{eff}}(t)}\right]^2. \tag{D53}
\]

In that case, we get

\[
P_{\text{ij}} = P_{\text{th}} \delta_{\text{ij}} \quad \text{with} \quad P_{\text{th}} = \rho T_{\text{eff}}(t). \tag{D54}
\]

This leads to a system of hydrodynamic equations of the form

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{D55}
\]
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P_{\text{th}} - \nabla \Phi - \xi \mathbf{u}, \tag{D56}
\]
\[
\Delta \Phi = 4\pi G \rho, \tag{D57}
\]

called the damped Euler equations. Using the LTE approximation (D53), the total energy (D3) can be written as

\[
E = \int \rho \frac{\mathbf{u}^2}{2} d\mathbf{r} + \frac{3}{2} MT_{\text{eff}} + W. \tag{D58}
\]

3. Hydrodynamic equations

We now take the hydrodynamic moments of the coarse-grained Vlasov equation (D33). To make the results fully explicit, we consider the nondegenerate limit of the theory \((\text{the general case is treated in} \ [94])\). In that case, the coarse-grained Vlasov equation takes the form

\[
\frac{\partial \mathbf{v}}{\partial J} + \mathbf{v} \cdot \nabla \Phi = \frac{2D}{\partial \mathbf{v}} \left[ D \left( \frac{\partial \mathbf{v}}{\partial \mathbf{v}} + \beta(t) \mathbf{v} \right) \right], \tag{D47}
\]

which is similar to the classical Kramers equation. The evolution of the inverse effective temperature \(\beta(t) = 1/T_{\text{eff}}(t)\) is given by

\[
\beta(t) = \frac{1}{\int D f \partial \mathbf{v} \cdot \mathbf{v} d\mathbf{r} d\mathbf{v}}. \tag{D48}
\]

Assuming that \(D\) is constant, and making an integration by parts, we get

\[
\beta(t) = \frac{3M}{2K(t)}, \tag{D49}
\]

where

\[
K(t) = \int f \frac{v^2}{2} d\mathbf{r} d\mathbf{v}. \tag{D50}
\]

is the total kinetic energy. Eq. (D49) can be rewritten as \(K(t) = \frac{3M}{2}T_{\text{eff}}(t)\). This relation shows that the effective temperature \(T_{\text{eff}}(t) = 1/\beta(t)\) can be interpreted as the average kinetic temperature of the system, i.e., \(T_{\text{eff}}(t) = \langle T_{\text{kin}}(r, t) \rangle\) where \(\frac{3}{2} \rho T_{\text{kin}}(r, t) = \int f \frac{v^2}{2} d\mathbf{v}\) is the local kinetic energy. One has \(E = \frac{3M}{2}T_{\text{eff}} + W\).

Taking the hydrodynamic moments of the coarse-grained Vlasov equation (D47), we obtain a system of equations similar to the Jeans equations (see Appendix C2) but including dissipative effects:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{D51}
\]
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P_{\text{ij}} - \nabla \Phi - \xi \mathbf{u}, \tag{D52}
\]

where \(\xi = D \beta\). They are called the damped Jeans equations. On the other hand, the effective collision term in Eq. (D47) provides a source of relaxation which allows us to close the hierarchy of moment equations. Indeed, we can compute the pressure tensor in Eq. (D52) by making a LTE approximation

\[
\mathcal{T}_{\text{LTE}}(\mathbf{r}, \mathbf{v}) = \frac{\rho (\mathbf{r}, t)}{[2\pi T_{\text{eff}}(t)]^{3/2}} \left[\frac{2 \pi m (\mathbf{v})}{2 T_{\text{eff}}(t)}\right]^2. \tag{D53}
\]

In that case, we get

\[
P_{\text{ij}} = P_{\text{th}} \delta_{\text{ij}} \quad \text{with} \quad P_{\text{th}} = \rho T_{\text{eff}}(t). \tag{D54}
\]

This leads to a system of hydrodynamic equations of the form

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{D55}
\]
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P_{\text{th}} - \nabla \Phi - \xi \mathbf{u}, \tag{D56}
\]
\[
\Delta \Phi = 4\pi G \rho, \tag{D57}
\]

called the damped Euler equations. Using the LTE approximation (D53), the total energy (D3) can be written as

\[
E = \int \rho \frac{\mathbf{u}^2}{2} d\mathbf{r} + \frac{3}{2} MT_{\text{eff}} + W. \tag{D58}
\]
In the microcanonical ensemble where the energy $E$ is fixed, this equation determines the evolution of the effective temperature $T_{\text{eff}}(t)$. On the other hand, the Boltzmann-like entropy (D18) can be written (up to an additive constant) as

$$S = - \int \rho (\ln \rho - 1) \, \text{d}r + \frac{3}{2} M \ln T_{\text{eff}}. \quad (D59)$$

This justifies the expressions of $E$ and $S$ used in Appendix B2. One can then show (like in Appendix B2) that the hydrodynamic equations (D55)–(D57) with a time-dependent effective temperature $T_{\text{eff}}(t)$ determined by Eq. (D58) satisfy an $H$-theorem for the entropy $S$ given by Eq. (D59). Analogously, in the canonical ensemble where the effective temperature $T_{\text{eff}}$ is fixed, one can show (like in Appendix B3) that the hydrodynamic equations (D55)–(D57) with a constant effective temperature $T_{\text{eff}}$ satisfy an $H$-theorem for the free energy $F = E - T_{\text{eff}}S$. One can also derive explicit expressions of the virial theorem like in Appendices B2 and B3 (one simply has to take $P = Q = 0$ in the equations of these Appendices).

**Remark** We note that, in Eq. (D58), the effective temperature $T_{\text{eff}}$ represents the thermal kinetic energy $\frac{3}{2} MT_{\text{eff}}(t) = \int \frac{w^2}{2} \, \text{d}r \text{d}v$ where $w = v - u(r, t)$ is the fluctuating velocity while the exact relation (D49) indicates that it should represent the total kinetic energy $\frac{1}{2} MT_{\text{eff}}(t) = \int w^2 \, \text{d}r \text{d}v$ including the mean kinetic energy. This is an artefact of the LTE approximation (D53) which involves a uniform temperature $T_{\text{eff}}(t)$ instead of a space-dependent temperature $T(r, t)$. We can solve this problem by replacing $T_{\text{eff}}(t)$ by $T(r, t)$ in Eq. (D53) and introducing a hydrodynamic equation for the local temperature $T(r, t)$ (second moment) as in Ref. [94] (see Appendix D4). However, the LTE approximation (D53) becomes exact close to equilibrium or in the strong friction limit $\xi \to +\infty$ where $|u| = O(1/\xi)$. Indeed, in the strong friction limit, we have

$$u = - \frac{1}{\xi} \rho \left( T_{\text{eff}} \nabla \rho + \rho \nabla \Phi \right). \quad (D60)$$

Substituting this relation into the continuity equation (D55) we obtain the Smoluchowski–Poisson equations

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot (T_{\text{eff}} \nabla \rho + \rho \nabla \Phi), \quad (D61)$$

$$\Delta \Phi = 4 \pi G \rho. \quad (D62)$$

Since $|u| = O(1/\xi)$, we can neglect the kinetic energy $\Theta_k$ in Eq. (D58). Therefore, the evolution of the effective temperature $T_{\text{eff}}(t)$ is given by the energy constraint

$$E = \frac{3}{2} MT_{\text{eff}} + \frac{1}{2} \int \rho \Phi \, \text{d}r. \quad (D63)$$

In the canonical ensemble, Eqs. (D61) and (D62) are valid with fixed $T_{\text{eff}}$. One can then derive the $H$-theorems and the virial theorems like in the Remarks at the end of Appendices B2 and B3 (one simply has to take $P = Q = 0$ in the equations of these Appendices). We refer to the series of papers initiated in [123] for a detailed study of these equations.

### 4. Inhomogeneous temperature

In the previous section, we have assumed for simplicity that the temperature (velocity dispersion) is uniform. In a more elaborate model (see Ref. [94]), we can take into account an inhomogeneous temperature by considering the second moment of the coarse-grained Vlasov equation and closing the hierarchy of hydrodynamic equations with the LTE approximation (D53) with $T_{\text{eff}}(t)$ replaced by $T(r, t)$. This yields (see Refs. [94, 153] for details and generalizations)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \quad (D65)$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = - \frac{1}{\rho} \nabla (\rho T) - \nabla \Phi - \xi u, \quad (D66)$$

$$\frac{3}{2} \left( \frac{\partial T}{\partial t} + u \cdot \nabla T \right) + T \nabla \cdot u = -3 \xi (T - T_{\text{eff}}(t)), \quad (D67)$$

where $T_{\text{eff}}(t) = 1/\beta(t)$ evolves according to Eq. (D49) so as to conserve the total energy

$$E = \int \rho \frac{u^2}{2} \, \text{d}r + \frac{3}{2} \int \rho T \, \text{d}r + W. \quad (D68)$$

The pressure is $P = \rho T$ and the thermal energy is $\Theta_k = \int w^2 \, \text{d}r \text{d}v = \frac{3}{2} \int \rho T \, \text{d}r$. These equations conserve the mass and the energy and satisfy an $H$-theorem for the Boltzmann-like entropy

$$S = - \int \rho (\ln \rho - 1) \, \text{d}r + \frac{3}{2} \int \rho \ln T \, \text{d}r. \quad (D69)$$

As a result, they relax towards the equilibrium Boltzmann-like distribution with a uniform temperature $T = T_{\text{eff}}$. The equation for the entropy density $\sigma = -\ln(\rho T^{-3/2})$ is

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot \sigma \nabla \Phi = 3 \xi \frac{T_{\text{eff}}(t) - T}{T}. \quad (D70)$$

In the absence of dissipation ($\xi = 0$) we recover the usual Euler equations [154] which conserve the entropy. In that case, $\rho T^{-3/2}$ is constant along the fluid (Lagrangian) trajectories, corresponding to the adiabatic law $P \rho^{-5/3} = c$. The damped virial theorem is given by (see, e.g., Appendix G of [93])

$$\frac{1}{2} \dot{I} + \frac{1}{2} \xi I = 2 \Theta_k + 3 \int \rho T \, \text{d}r + W. \quad (D71)$$

Using the energy conservation equation (D68), it can be rewritten as

$$\frac{1}{2} \dot{I} + \frac{1}{2} \xi I = 2E - W. \quad (D72)$$

In the canonical ensemble, $T_{\text{eff}}$ is constant and the hydrodynamic equations satisfy an $H$-theorem for the free energy $F = E - T_{\text{eff}}S$.

### Appendix E: The classical limit $\hbar \to 0$

In this Appendix, we discuss how the quantum equations studied in the present paper pass to the limit $\hbar \to 0$.\footnote{We note that the Schrödinger–Poisson equations (10) and (11) depend only on $\hbar$ through the ratio $\hbar/m$. Therefore, the classical limit corresponds to $\hbar/m \to 0$ (i.e. $\hbar \to 0$ or $m \to +\infty$).} Our discussion is essentially heuristic. A mathematically rigorous treatment of the classical limit $\hbar \to 0$ is difficult but would certainly be very valuable.
1. Classical versus quantum descriptions

In the collisionless regime, classical self-gravitating systems are described by the Vlasov–Bohm equations (C1) and (C2). The Vlasov equation is equivalent to the infinite hierarchy of Jeans equations (see Appendix C2). This is true for all times and for arbitrary initial conditions. Let us now assume that we start from a single-speed initial condition. Then, as long as the DF remains single-speed, the DF is given by Eq. (C13), the pressure tensor vanishes ($P_{ij} = 0$), and the Jeans equations reduce to the pressureless Euler equations (C14) and (C15). This is true until shell crossing, after which the pressureless Euler equations develop singularities and the DF becomes multi-streamed. Before shell crossing the Vlasov equation is equivalent to the pressureless Euler equations but after shell crossing the pressureless Euler equations are not valid anymore. In that case $P_{ij} \neq 0$. Unfortunately, it is not possible to calculate the pressure tensor exactly so the Jeans equations are not closed. One has to come back to the Vlasov–Poisson equations. Some heuristic procedures to compute $P_{ij}$ approximately to continue using a hydrodynamical approach are described in Appendix C3.

In the collisionless regime, quantum self-gravitating systems made of condensed bosons are described by the Schrödinger–Poisson equations (10) and (11). The Schrödinger equation is equivalent to the quantum Euler equations (17)–(19). These equations are also equivalent to the Wigner equation (44), whose exact solution is given by Eq. (43), and to the quantum Jeans equations (see Sect. 3.4.1). For condensed bosons, $P_{ij} = P_{ij}^Q$, where $P_{ij}^Q$ is given by Eq. (23), and the quantum Jeans equations are closed (they reduce to the quantum Euler equations (17)–(19)). This is true for all times and for arbitrary initial conditions.

2. Comparison between the Wigner equation and the Vlasov equation

When $h \to 0$, the Wigner equation (44) reduces to the Vlasov equation (C1) which corresponds to $h = 0$ (see Appendix A). On the other hand, the Wigner equation is equivalent to the Schrödinger equation (10) and to the quantum Euler equations (17)–(19). Therefore, the solution of the Vlasov equation ($h = 0$) is expected to be well-approximated by the solution of the Wigner, Schrödinger and quantum Euler equations with $h \to 0$. In particular, $f(r, v, t) \simeq f_W(r, v, t)$ where $f_W$ is given by Eq. (43). This equivalence is valid for all times. Therefore, for what concerns the Wigner equation, the limit $h \to 0$ is equivalent to $h = 0$.

3. Comparison between the quantum Euler equation and the pressureless Euler equations

When $h \to 0$, the quantum Euler equations (17)–(19) seem to reduce to the pressureless Euler equations (C14) and (C15) which correspond to $P = h = 0$. However, this equivalence is only valid before shell crossing ($t < t_\ast$). After shell crossing ($t > t_\ast$), the quantum Euler equations are still valid while the pressureless Euler equations are not valid anymore (see Appendix C3). Therefore, for what concerns the quantum Euler equations (17)–(19), the limit $h \to 0$ is different from $h = 0$. A small but finite value of $h$ allows us to extend the solutions of the hydrodynamic equations past $t_\ast$ for all times.

4. Conclusion

The Vlasov equation is valid for all times. By contrast, the pressureless Euler equations are valid only for a single speed solution until the first time of crossing. Before that time, they are equivalent to the Vlasov equation but after that time they break down. The Schrödinger equation, the quantum Euler equations, and the Wigner equation are equivalent and they are valid for all times. When $h \to 0$ their solutions are expected to be close to the solution of the Vlasov equation.

In conclusion, the Vlasov equation and the Schrödinger, quantum Euler, and Wigner equations with $h \to 0$ are superior to the classical pressureless Euler equations. They take into account velocity dispersion whereas the classical pressureless fluid description does not. They can be used to describe multistreaming and caustics in the nonlinear regime, whereas the classical pressureless fluid equations break down in that regime. Indeed, the pressureless Euler equations develop shocks so they are not well-defined after the first shock. The velocity dispersion, or the quantum pressure, allow to regularize the dynamics and solve the problems of the classical pressureless hydrodynamic description.

Appendix F: The Vlasov–Bohm equation

Instead of using the rather complicated Wigner equation (44), one could consider the simpler Vlasov–Bohm equation

$$ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial v} - \nabla \Phi \cdot \frac{\partial f}{\partial v} - \frac{1}{m} \nabla Q \cdot \frac{\partial f}{\partial v} = 0. \quad (F1) $$

This equation is similar to the classical Vlasov equation (C1) except that it includes the quantum potential $Q$ from Eq. (21). We stress, however, that this equation is heuristic and that it is not expected to give an exact description of the dynamics. The Vlasov–Bohm–Poisson equations conserve the energy $E = (1/2) \int f v^2 dv + (1/2) \int \rho \Phi dv + (1/m) \int \rho Q dv$ and an infinite class of Casimir integrals of the form $\int f(h) dv$ where $h(f)$ is an arbitrary function of $f$. Coarse-graining the Vlasov–Bohm equation like in Sect. 3.2, we obtain the fermionic Vlasov–Bohm–Kramers equation

$$ \frac{\partial T}{\partial t} + v \cdot \frac{\partial T}{\partial v} - \nabla \Phi \cdot \frac{\partial T}{\partial v} - \frac{1}{m} \nabla Q \cdot \frac{\partial T}{\partial v} = \nabla \left[ D \left( \frac{\partial T}{\partial v} + \beta \frac{1}{T} \right) \right] \quad (F2) $$

instead of Eq. (46) [one can also obtain a fermionic Vlasov–Bohm–Landau equation instead of Eq. (49)]. This equation relaxes towards an equilibrium DF of the form

$$ T_{LB}(r, v) = \frac{\eta_0}{1 + e^{(v^2/2+\Phi(r)Q/m)^{-\alpha}}}. \quad (F3) $$

which can be viewed as a Lynden-Bell DF incorporating the quantum potential. Since $Q$ depends on the density itself
this equation is a complicated integral equation. It implies the condition of quantum hydrostatic equilibrium from Eq. (103) (see the Remark below).

We can then take the hydrodynamic moments of the coarse-grained Vlasov–Bohm–Kramers equation (F2) as in Sect. 3.4. Instead of Eq. (57), we get

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \partial_j P_{ij} - \nabla \Phi - \frac{1}{m} \nabla Q - \frac{1}{\rho} D \beta \int \mathcal{F}(1 - \mathcal{F}/\eta) v \, dv, \quad (F4)$$

in which the quantum potential appears explicitly. Closing the hierarchy of quantum Jeans equations by making the LTE approximation from Eq. (64) to compute the pressure tensor, we obtain

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla P_{\text{LH}} - \frac{1}{m} \nabla Q - \nabla \Phi - \xi u, \quad (F5)$$

which coincides with Eq. (73). Therefore, at the level of the hydrodynamic equations, and within the assumptions made in our approach, the Vlasov–Bohm equation yields the same results as the Wigner equation. The reason is that our approach neglects collective effects (encapsulated in the dielectric function) where differences between the Wigner equation and the Vlasov–Bohm equation occur [127]. This approximation may be justified during the very nonlinear process of violent relaxation.

Remark For spherically symmetric systems, the stationary solutions of the Vlasov–Bohm equation (F1) are of the form $f = f(\epsilon_Q)$ with $\epsilon_Q = v^2/2 + \Phi + Q/m$. We can easily show that this relation implies the condition of quantum hydrostatic equilibrium from Eq. (103). Indeed, defining the density and the pressure by $\rho = \int f \, dv$ and $P = \frac{1}{3} \int f v^2 \, dv$, we get

$$\nabla P = \frac{1}{3} \int v^2 \frac{\partial f}{\partial \epsilon_Q} \, dv = \frac{1}{3} \left( \nabla \Phi + \frac{1}{m} \nabla Q \right) \int v^2 f(\epsilon_Q) \, dv = \frac{1}{3} \left( \nabla \Phi + \frac{1}{m} \nabla Q \right) \int \frac{\partial f}{\partial \epsilon_Q} \cdot v \, dv = - \left( \nabla \Phi + \frac{1}{m} \nabla Q \right) \int f \, dv \quad (F6)$$

yielding

$$\nabla P + \rho \nabla \Phi + \frac{\rho}{m} \nabla Q = 0. \quad (F7)$$

The effective collision term on the right hand side of Eq. (F2) selects the form of the equilibrium DF $f = f(\epsilon_Q)$ among all the possible stationary solutions of the Vlasov–Bohm equation. In the present case, the fermionic Kramers operator selects the Lynden-Bell DF from Eq. (F3).

### Appendix G: Multistate systems

#### 1. Hartree equations

We consider a system of $N$ quantum particles in gravitational interaction. These particles may be fermions or bosons. Fundamentally, they are described by an $N$-body wave function $\Psi(r_1, ..., r_N, t)$ satisfying the exact $N$-body Schrödinger equation [155]

$$ih \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi - \sum_{\alpha < \beta} \frac{G m^2}{|r_{\alpha} - r_{\beta}|} \Psi. \quad (G1)$$

When $N$ is large, this equation is completely untractable. The aim of statistical mechanics and kinetic theory is to obtain simpler models described by one-body equations. Following Hartree [156, 157] we shall make a mean field approximation and ignore correlations among the particles. We thus assume that the $N$-body wave function $\Psi(r_1, ..., r_N, t)$ can be written as a product of $N$ one-body wave functions $\psi_\alpha(r, t)$ so that

$$\Psi(r_1, ..., r_N, t) = \prod_{\alpha=1}^{N} \psi_\alpha(r, t). \quad (G2)$$

The mean field approximation is known to become exact for systems with long-range interactions (like self-gravitating systems) when $N \to +\infty$ in a proper thermodynamic limit [131]. In that case, the evolution of the quantum system is described by $N$ coupled mean field Schrödinger equations of the form

$$ih \frac{\partial \psi_\alpha}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi_\alpha + m \Phi \psi_\alpha, \quad (G3)$$

$$\Delta \Phi = 4\pi G \sum_{\alpha=1}^{N} p_\alpha |\psi_\alpha|^2, \quad (G4)$$

where $p_\alpha$ is the occupation probability of state $\alpha$ such that the normalization condition $\sum_\alpha p_\alpha = 1$ is fulfilled. We shall assume that $p_\alpha$ is constant or that it varies slowly with $r$ and $t$. The total density is $\rho = \sum_\alpha p_\alpha |\psi_\alpha|^2$.

In the mean field approximation, each particle evolves in a self-consistent gravitational potential $\Phi$ that is produced by the particles themselves through the Poisson equation (G4). Equations (G3) and (G4) are called the Hartree (mean field) equations.

From the wavefunctions $\psi_\alpha(r, t)$ of the multistate system, we can define the Wigner DF by

$$f(r, v, t) = \sum_{\alpha=1}^{N} \frac{m^3}{(2\pi \hbar)^3} p_\alpha \int dy e^{i m v \cdot y/\hbar}$$

$$\times \psi_\alpha^*(r - \frac{y}{2}, t) \psi_\alpha(r - \frac{y}{2}, t). \quad (G5)$$

It satisfies the Wigner equation (44). The Wigner DF (G5) involves the density matrix $\rho(r, r', t) = \sum_{\alpha=1}^{N} p_\alpha \psi_\alpha^*(r, t) \psi_\alpha(r, t)$ which appears in the von Neumann equation. The Hartree, Wigner and von Neumann equations are equivalent.

In the case of fermions, the wave function $\Psi$ is antisymmetric with respect to the exchange of any two variables $r_\alpha$ and $r_\beta$. The antisymmetry condition stems from the Pauli exclusion principle which establishes that two fermions cannot share the same position so that the probability density $|\Psi|^2$ vanishes as $r_\alpha = r_\beta$ (for a rigorous treatment, the spin of the fermions must be considered). A system of fermions is necessarily in a mixed quantum state to respect the Pauli exclusion principle. In the Hartree–Fock [158, 159] theory, the $N$-body wave function is expressed as

$$\Psi(r_1, ..., r_N, t) = \frac{1}{\sqrt{N!}} \det[\psi_\alpha(r_\beta, t)]. \quad (G6)$$
In other words, the wave function for many fermions is taken as a Slater determinant which is zero for two particles at the same position. This expression guarantees that the system obeys the Pauli exclusion principle. This leads to the Hartree–Fock equations corresponding to the Hartree Eqs. (G3) and (G4) with an additional exchange energy term introduced by Fock [158]. This exchange energy term can be simplified by using the Slater [160] approximation (see also Dirac [161] and Kohn–Sham [162]) which amounts to introducing a term $C_S p^{1/3} \psi_\alpha$ in the right hand side of Eq. (G3), where $C_S$ is a positive constant in the gravitational case (it is negative in the electrostatic case). The Hartree–Fock equations neglect correlations. More general equations taking into account correlations (assuming that the fermions interact by microscopic forces in addition to the gravitational force) can be obtained through the density functional theory [155] initiated by Kohn and Sham [162].

A gas of bosons at $T = 0$ forms a BEC in which all the bosons are in the same quantum state described by a single wave function $\psi(r,t)$. In the mean field approximation, the $N$-body wave function is expressed as

$$\Psi(r_1, \ldots, r_N, t) = \prod_{\alpha=1}^{N} \psi(r_{\alpha}, t).$$

(G7)

In that case, the evolution of the self-gravitating BEC is described by the Schrödinger–Poisson equations (10) and (11). This amounts to taking $\alpha = 1$ and $p_{\alpha} = 1$ in Eqs. (G3) and (G4). A system of bosons at $T = 0$ is in a pure quantum state.

Remark. Actually, even in the case of BECs, we can have a quantum superposition of modes. Indeed, the wave function $\psi$ can be decomposed into Eigenfunctions of the Schrödinger–Poisson equations involving the fundamental state ($n = 0$) and the excited states ($n > 0$) as in Eq. (41). As a result, the multistate equations (G3) and (G4) are also relevant for bosons with this interpretation (in that case, $p_{\alpha}$ is the probability of mode $\alpha$ and $N$ is the number of modes). The resulting equations can be called the Hartree (mean field) equations for bosons. In the case of fermions, the Hartree equations (G3) and (G4) are basically valid with $p_{\alpha} = 1/N$ if $\alpha = 1, \ldots, N$ and $p_{\alpha} = 0$ otherwise. However, as explained above, they can also be viewed as an expansion over the modes $\alpha$ of the system where $p_{\alpha}$ represents the probability of mode $\alpha$ and $N$ represents the number of modes. Below, we generalize the hydrodynamic representation of the SP equations to the case of a multistate system and show how a pressure term arises in the quantum Euler equations. This pressure term is similar to the one obtained from the approach of Sect. 3.

2. Madelung transformation

We can use the Madelung transformation to rewrite the Hartree–Fock equations (G3) and (G4) under the form of hydrodynamic equations for each state $\alpha$ (we shall use the Slater approximation $C_S p^{1/3} \psi_\alpha$ to evaluate the exchange term for fermions). Let us write the wave function as

$$\psi_\alpha(r,t) = \sqrt{p_\alpha(r,t)} e^{iS_\alpha(r,t)/\hbar},$$

(G8)

where $\rho_\alpha(r,t)$ is the density and $S_\alpha(r,t)$ is the action given by

$$\rho_\alpha = |\psi_\alpha|^2 \quad \text{and} \quad S_\alpha = -\frac{\hbar}{2} \ln \left( \frac{\psi_\alpha}{\psi_\alpha^*} \right).$$

(G9)

Following Madelung, we introduce the velocity field

$$u_\alpha = \frac{\nabla S_\alpha}{m}.$$  

(G10)

Substituting Eq. (G8) into Eqs. (G3) and (G4) and separating the real and the imaginary parts, we find that the Hartree–Fock equations are equivalent to hydrodynamic equations of the form

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot ((\rho_\alpha u_\alpha)) = 0,$$

(G11)

$$\frac{\partial S_\alpha}{\partial t} + \frac{1}{2m} (\nabla S_\alpha)^2 + m\Phi + C_S \rho^{1/3} + Q_\alpha = 0,$$

(G12)

$$\frac{\partial u_\alpha}{\partial t} + (u_\alpha \cdot \nabla)u_\alpha = -\frac{1}{m} \nabla Q_\alpha - \nabla \Phi - \frac{1}{\rho} \nabla P_{\text{Slater}},$$

(G13)

$$\Delta \Phi = 4\pi G \sum_{\alpha=1}^{N} p_\alpha \rho_\alpha,$$

(G14)

where

$$Q_\alpha = -\frac{\hbar^2}{2m} \sqrt{\frac{\rho_\alpha}{p_\alpha}} = -\frac{\hbar^2}{4m} \left[ \frac{\Delta \rho_\alpha}{\rho_\alpha} - \frac{1}{2} \frac{(\nabla \rho_\alpha)^2}{\rho_\alpha^2} \right]$$

(G15)

is the quantum potential and

$$P_{\text{Slater}} = \frac{C_S}{4m} \rho^{4/3}$$

(G16)

is the Slater pressure. It corresponds to a polytrope of index $n = 3$ with a polytropic constant $K_S = C_S/4m$. The Slater pressure is positive ($P_{\text{Slater}} > 0$) meaning that the exchange interaction is effectively repulsive in the gravitational case (it is attractive in the electrostatic case). Using the continuity equation (G11), we obtain the identity

$$p_\alpha \left[ \frac{\partial u_\alpha}{\partial t} + (u_\alpha \cdot \nabla)u_\alpha \right] = \frac{\partial }{\partial t}(p_\alpha u_\alpha) + \nabla (p_\alpha \nabla \rho_\alpha \cdot u_\alpha).$$

(G17)

The quantum Euler equation (G13) can then be rewritten as

$$\frac{\partial }{\partial t}(p_\alpha u_\alpha) + \nabla (p_\alpha u_\alpha \otimes u_\alpha) = -\rho_\alpha \nabla P_{\text{Slater}} - \rho_\alpha \nabla \Phi - \frac{p_\alpha}{m} \nabla Q_\alpha.$$  

(G18)

The foregoing equations are valid for each state $\alpha$. We now introduce the total density

$$\rho = \sum_{\alpha=1}^{N} p_\alpha \rho_\alpha.$$  

(G19)
and the total velocity
\[ u = \frac{1}{\rho} \sum_{\alpha=1}^{N} p_{\alpha} \rho_{\alpha} u_{\alpha}. \] (G20)

From Eqs. (G11), (G19) and (G20), we obtain the continuity equation
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \] (G21)

From Eqs. (G18), (G19) and (G20), we get
\[ \frac{\partial}{\partial t} (\rho u) + \nabla (\rho u \otimes u) = -\nabla P_{\text{Slater}} - \rho \nabla \Phi - \sum_{\alpha=1}^{N} \rho_{\alpha} \frac{\partial}{\partial \alpha} \nabla Q_{\alpha}, \] (G22)

where \( w_{\alpha} = u_{\alpha} - u \). Expanding the advection term in Eq. (G22) and using the identity
\[ \sum_{\alpha=1}^{N} p_{\alpha} \rho_{\alpha} w_{\alpha} = \sum_{\alpha=1}^{N} p_{\alpha} \rho_{\alpha} (u_{\alpha} - u) = \rho u - \rho u = 0, \] (G23)

we are left with
\[ \frac{\partial}{\partial t} (\rho u) + \nabla (\rho u \otimes u) = -\sum_{\alpha=1}^{N} \rho_{\alpha} \nabla (\rho_{\alpha} w_{\alpha} \otimes w_{\alpha}) - \nabla P_{\text{Slater}} - \rho \nabla \Phi - \sum_{\alpha=1}^{N} \rho_{\alpha} \frac{\partial}{\partial \alpha} \nabla Q_{\alpha}. \] (G24)

Using the continuity equation (G21), we obtain the identity
\[ \rho \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right] = \frac{\partial}{\partial t} (\rho u) + \nabla (\rho u \otimes u). \] (G25)

The Euler equation (G24) can then be rewritten as
\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \partial_{\rho} P_{ij} - \frac{1}{\rho} \nabla P_{\text{Slater}} - \frac{1}{\rho m} \sum_{\alpha=1}^{N} \rho_{\alpha} \nabla Q_{\alpha} - \nabla \Phi, \] (G26)

where
\[ P_{ij} = \sum_{\alpha=1}^{N} p_{\alpha} \rho_{\alpha} w_{\alpha} \otimes w_{\alpha}. \] (G27)

is the pressure tensor arising from the difference between the multistate velocities \( u_{\alpha} \) and the total velocity \( u \). It can also be written as
\[ P_{ij} = \sum_{\alpha=1}^{N} p_{\alpha} \rho_{\alpha} u_{\alpha} \otimes u_{\alpha} - \rho u \otimes u. \] (G28)

If we make the approximation
\[ -\frac{1}{\rho m} \sum_{\alpha=1}^{N} p_{\alpha} \rho_{\alpha} \nabla Q_{\alpha} \simeq -\frac{1}{m} \nabla Q, \] (G29)

where \( Q \) is the quantum potential defined by Eq. (21), we obtain the hydrodynamic equations
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \] (G30)
\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \partial_{\rho} P_{ij} - \frac{1}{\rho} \nabla P_{\text{Slater}} - \frac{1}{m} \nabla Q - \nabla \Phi, \] (G31)
\[ \Delta \Phi = 4\pi G \rho. \] (G32)

These equations are not closed since the pressure tensor \( P_{ij} \) depends on the multistate variables. We note that the pressure tensor defined by Eq. (G27) is similar to the pressure tensor defined by Eq. (60) if we identify \( p_{\alpha} \rho_{\alpha} \) with the coarse-grained DF \( \tilde{f}(r, v) \). We shall consider successively the case of bosons and fermions.

3. Bosons

In the case of bosons at \( T = 0 \) forming a BEC, we just have one pure state \( \alpha = p_{\alpha} = 1 \), and we trivially find that the pressure tensor defined by Eq. (G27) vanishes
\[ P_{ij} = 0. \] (G33)

We also have \( P_{\text{Slater}} = 0 \) in that case \( (C_{S} = 0) \). As a result, the hydrodynamic equations (G30)–(G32) reduce to Eqs. (17)–(20) as it should. However, if we write \( \psi \) as a superposition of modes [see Eq. (41)], and use Eqs. (G3) and (G4) with this interpretation (see the Remark at the end of Appendix G1), we can close Eqs. (G30)–(G32) by using the Lynden-Bell pressure from Eq. (65). This leads to Eqs. (72)–(74) and, finally, to Eq. (100). Note, however, that the friction term, which represents a form of nonlinear Landau damping, does not explicitly appear in the formalism.

4. Fermions

In the case of fermions, the pressure tensor \( P_{ij} \) takes into account the Pauli exclusion principle but, without further assumption, it cannot be explicitly evaluated. Now, if we view \( \sum_{\alpha} \) as a sum over the modes with \( p_{\alpha} \) being the probability of mode \( \alpha \) (see the Remark at the end of Appendix G1), we can close Eqs. (G30)–(G32) by using the Lynden-Bell pressure from Eq. (65) which, in the case of fermions, coincides with the Fermi-Dirac pressure. This leads to the hydrodynamic equations
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \] (G34)
\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla P_{\text{LB/ FD}} - \frac{1}{m} \nabla Q - \nabla \Phi, \] (G35)
\[ \Delta \Phi = 4\pi G \rho, \] (G36)

and, finally, to Eq. (158) with the same comment as above concerning the absence of friction term. We note that the Lynden-Bell (or Fermi-Dirac) pressure has a statistical origin (it depends on the specification of \( p_{\alpha} \)) while the Slater pressure has a purely quantum nature.
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