ON A CLASS OF KÄHLER MANIFOLDS
WHOSE GEODESIC FLOWS ARE INTEGRABLE

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ABSTRACT. We study \( n \)-dimensional Kähler manifolds whose geodesic flows possess \( n \) first integrals in involution that are fibrewise hermitian forms and simultaneously normalizable. Under some mild assumption, one can associate with such a manifold an \( n \)-dimensional commutative Lie algebra of infinitesimal automorphisms. This, combined with the given \( n \) first integrals, makes the geodesic flow integrable. If the manifold is compact, then it becomes a toric variety.

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Introduction

It is known that the geodesic flow of the complex projective space \( CP^n \) equipped with the standard Kähler metric is (completely) integrable in the sense of symplectic geometry (or in Liouville’s sense) (cf. Thimm [13], see also [9], [10], [5]). The first integrals are given as follows: Let \( c_0, \ldots, c_n \) be constants such that \( 1 = c_0 > c_1 > \cdots > c_n = 0 \), and let \((z_0, \ldots, z_n)\) be the homogeneous coordinate system. Then, by putting \( \partial_i = \partial/\partial z_i \),

\[
\sum_{0 \leq j \leq n \atop j \neq i} \frac{(z_i \bar{\partial}_j - z_j \bar{\partial}_i)(\bar{z}_i \partial_j - \bar{z}_j \partial_i)}{c_j - c_i} \quad (1 \leq i \leq n - 1),
\]

\[
\sum_{i, j} (z_i \bar{\partial}_j - z_j \bar{\partial}_i)(\bar{z}_i \partial_j - \bar{z}_j \partial_i), \quad \sqrt{-1}(z_i \partial_i - \bar{z}_i \bar{\partial}_i) \quad (1 \leq i \leq n)
\]

are well-defined symmetric tensor fields and vector fields on \( CP^n \). Regarded as functions on the cotangent bundle \( T^*CP^n \), they become a system of first integrals.
in involution of the geodesic flow (note that the first integrals given here are slightly different from those in [13]).

In this paper, we shall define a class of Kähler manifolds whose geodesic flows are integrable by a set of first integrals possessing similar properties to those for \( \mathbb{C}P^n \), and study the properties of such manifolds. We shall call them Kähler-Liouville manifolds (of type (A)). The precise definition is as follows. Let \( M \) be an \( n \)-dimensional complete Kähler manifold, \( I \) its complex structure, and \( E \) its energy function (the hamiltonian of the geodesic flow). Let \( \mathcal{F} \) be an \( n \)-dimensional vector space of sections of \( S^2TM \) (the symmetric tensor product over \( \mathbb{R} \) of two copies of the tangent bundle \( TM \)). Then we say that \( (M, \mathcal{F}) \) is Kähler-Liouville manifold if it satisfies the following conditions:

1. \( E \in \mathcal{F} \);
2. \( \{F, H\} = 0 \) for any \( F, H \in \mathcal{F} \);
3. every \( F \in \mathcal{F} \) is "hermitian";
4. \( \mathcal{F}_p = \{F_p \mid F \in \mathcal{F}\} \) is simultaneously normalizable for every \( p \in M \);
5. \( \mathcal{F}_p \) is \( n \)-dimensional at some \( p \in M \).

Here, \( F_p \in S^2T_pM \) is the value of \( F \) at \( p \in M \). Since sections of \( S^2TM \) are naturally regarded as functions on the cotangent bundle \( T^*M \), the Poisson bracket in (2) makes sense. Also, (3) means that the function \( F \), restricted to each fibre \( T^*_pM \), is a hermitian form. We say that two Kähler-Liouville manifolds \( (M, \mathcal{F}) \) and \( (M', \mathcal{F}') \) are mutually isomorphic if there is an isomorphism \( \phi : M \to M' \) of Kähler manifolds that maps \( \mathcal{F} \) to \( \mathcal{F}' \).

As is immediately seen, only \( n \) first integrals are given in the definition. Nevertheless, it will turn out that other \( n \) first integrals appear automatically if some non-degeneracy condition is assumed. A Kähler-Liouville manifold satisfying this assumption is called of type (A) (for the precise definition, see Section 1). The main purpose of this paper is to investigate local and global properties of Kähler-Liouville manifolds of type (A). Compact, 2-dimensional Kähler-Liouville manifolds were studied by Igarashi [4], in which he adopted another type of non-degeneracy condition and obtained similar results to ours in this case. The results indicate that the condition adopted in [4] is equivalent to our condition of type (A).

We now explain the various results in this paper. In the following, Kähler-Liouville manifolds are assumed to be of type (A), unless otherwise stated.

**1 (cf. Proposition 1.9).** A finite, partially ordered set \( A \) is naturally associated with \( (M, \mathcal{F}) \). Also, a positive integer \( |\alpha| \) is assigned to each \( \alpha \in A \) so that \( \sum_{\alpha \in A} |\alpha| = n \). For any \( \alpha \in A \), the subset \( \{\beta \in A \mid \beta < \alpha\} \) is totally ordered.

**2 (cf. Proposition 1.16, Theorems 3.1, 3.2).** An \( n \)-dimensional commutative Lie algebra \( \mathfrak{k} \) of infinitesimal automorphisms of \( (M, \mathcal{F}) \) is naturally defined, possessing the property that \( \mathfrak{k} \) and \( \mathcal{F} \) are elementwise commutative with respect to the Poisson bracket. With \( \mathfrak{k} \) and \( \mathcal{F} \) the geodesic flow of \( M \) becomes integrable.

Up to now, further results are obtained only for compact Kähler-Liouville manifolds. In this case, we obtain the results below. Put \( \mathfrak{g} = \mathfrak{k} + i\mathcal{F} \), and let \( K \) and \( G \) be the transformation group of \( M \) generated by \( \mathfrak{k} \) and \( \mathfrak{g} \) respectively.

**3 (Theorems 4.2, 4.18).** \( K \) and \( G \) are isomorphic to \( U(1)^n \) and \( (\mathbb{C}^*)^n \) respectively. With the action of \( G \), \( M \) becomes a toric variety.

The structure of \( M \) as toric variety is completely investigated in Section 4, and as a consequence we obtain the notion of “toric variety of KL-A type” (cf. Section 3).
The toric variety associated with a Kähler-Liouville manifold of type (A) is necessarily of KL-A type. Conversely, we have the following result.

4 (cf. Theorem 8.3). Suppose that a toric variety of KL-A type is given. Then there exists a Kähler-Liouville manifold of type (A) such that the associated toric variety is isomorphic to the given one.

Thanks to the general theory for toric varieties (cf. [1], [2], [3], [11]), we can know what kind of complex manifold $M$ is. For the detail, see Section 5. In particular, we have the following bundle structures.

5 (Proposition 5.4). There is a holomorphic principal fibre bundle

$$\prod_{\alpha \in A} (\mathbb{C}^{[\alpha]+1} - \{0\}) \to M$$

whose structure group is isomorphic to $(\mathbb{C}^*)^\#A$.

6 (Proposition 5.5, Theorems 6.3, 6.4, 6.5). Let $A'$ be a subset of $A$ possessing the property that if $\alpha \in A'$ and $\beta \prec \alpha$, then $\beta \in A'$. Put $A'' = A - A'$. Then, associated with $A'$, there naturally exist Kähler-Liouville manifolds $(M', F')$, $(M'', F'')$, and a holomorphic fibre bundle $\pi : M \to M''$ whose typical fibre is $M'$. They possess the following properties: (1) $(M'', F'')$ is of type (A) and the associated partially ordered set is $A''$; (2) if $(M', F')$ is of type (A), then the associated partially ordered set is $A'$; (3) there is a homomorphism $G \to G''$ so that $\pi : M \to M''$ is equivariant, where $G$ and $G''$ denote the algebraic tori acting on $M$ and $M''$ respectively; (4) even if $(M', F')$ is not of type (A), $M'$ possess the structure of toric variety inherited from $M$ so that the maximal compact subgroup of the algebraic torus acting on $M'$ preserves the metric and each element of $F'$.

The property (4) stated above indicates that the geodesic flow of $(M', F')$ is integrable even if it is not of type (A). In this case we shall say that the Kähler-Liouville manifold $(M', F')$ is of type (B) (cf. Section 6). Such a manifold will be necessary for the study of Kähler-Liouville manifold of type (A) only when $\dim M' = 1$. Using the result above successively, we consequently obtain a family of Kähler-Liouville manifolds $(M_\alpha, F_\alpha)$ $(\alpha \in A)$ such that the partially ordered set associated with $(M_\alpha, F_\alpha)$ consists of one element $\{\alpha\}$. In this case, it turns out that $M_\alpha$ is isomorphic to $\mathbb{C}P^{[\alpha]}$ as toric variety. It also turns out that it is of type (A) if and only if $|\alpha| \geq 2$.

The result 4 mentioned above is actually given in much finer form in Theorem 8.3. There, besides the toric variety $M$, we prescribe the Kähler-Liouville manifolds $(M_\alpha, F_\alpha)$ $(\alpha \in A)$ and some constants, and prove the uniqueness of $(M, F)$ as well as the existence. The reason for formulating the “existence theorem” in this form is that Kähler-Liouville manifolds such that the associated partially ordered sets consist of one element are easily understandable by using our work [6] on (real) Liouville manifolds. The result is roughly stated as follows.

7 (cf. Theorem 7.2). The isomorphism classes of Kähler-Liouville manifolds such that $\#A = 1$ are completely classified by means of several constants and a function on a circle.

We now briefly explain the organization of the paper. In Section 1 we first formulate some non-degeneracy condition (depending on points) on a Kähler-Liouville manifold in order to establish the existence of a Kähler-Liouville manifold of type (A) which is isomorphic to the given toric variety. Then we describe the properties of the Kähler-Liouville manifold which is obtained by solving these non-degeneracy conditions. In Section 2, we briefly recall the basic notions and results concerning toric varieties and Kähler-Liouville manifolds. In Section 3, we prove several results on the general theory of toric varieties and Kähler-Liouville manifolds. In Section 4, we give the proof of the “existence theorem” for the Kähler-Liouville manifold of type (A) that was mentioned above. In Section 5, we give the proof of the “existence theorem” for the Kähler-Liouville manifold of type (B) that was mentioned above. In Section 6, we prove several results on the general theory of toric varieties and Kähler-Liouville manifolds. In Section 7, we give the proof of the “existence theorem” for the Kähler-Liouville manifold of type (C) that was mentioned above. In Section 8, we prove several results on the general theory of toric varieties and Kähler-Liouville manifolds. In Section 9, we give the proof of the “existence theorem” for the Kähler-Liouville manifold of type (D) that was mentioned above. In Section 10, we prove several results on the general theory of toric varieties and Kähler-Liouville manifolds. In Section 11, we give the proof of the “existence theorem” for the Kähler-Liouville manifold of type (E) that was mentioned above.
manifold \((M, \mathcal{F})\). Denoting by \(M^1\) the set of points where the condition is satisfied, we say that \((M, \mathcal{F})\) is of type (A) if \(M^1 \neq \emptyset\). We perform local calculus on \(M^1\), and introduce almost all basic quantities related to \((M, \mathcal{F})\). In Sections 2 and 3 we sum up the local data given in Section 1, and describe properties of the basic quantities and the Lie algebra \(\mathfrak{k}\) in global form. The result 2 stated above is proved in Section 3.

Through Sections 4–8 we assume that \(M\) is compact. In Section 4 we prove the result 3 stated above and the results that determine the structure of \(M\) as toric variety. In particular, we show that \(M^1\) is the unique open \(G\)-orbit, and \(M - M^1\) is the union of \(n + \# \mathcal{A}\) closed hypersurfaces that are \(G\)-invariant and totally geodesic.

In Section 5 we describe the various properties of the toric variety \(M\). There we specify the fan of \(M\), and give the definition of toric variety of KL-A type. This section contains 3 subsections; “The fan of \(M\)”, “Fibre bundles associated with \(M\)”, and “Line bundles”. In Section 6 we prove the result 5 stated above. There we also prove its converse (Theorem 6.11), which plays a crucial role in the proof of Theorem 8.3.

Section 7 is devoted to the proof of Theorem 7.2 (see the result 7 above). We establish the one-to-one correspondence between the isomorphism classes of Kähler-Liouville manifolds of type (A) such that \(\# \mathcal{A} = 1\) and the isomorphism classes of Liouville manifolds of rank one and type (B) that satisfy a certain condition. The definition and the classification of Liouville manifolds of rank one are given in [6]. By using them, the theorem is proved. In Section 8 we prove Theorem 8.3 mentioned above. It is no longer hard by virtue of Theorems 6.11 and 7.2.

**Notations and preliminary remarks**

Let \(M\) be an \(n\)-dimensional Kähler manifold, and let \(g\) and \(I\) be its Kähler metric and complex structure respectively. Then the Kähler form \(\omega\) and the energy function \(E\) are given as follows:

\[
\omega(X, Y) = g(IX, Y) \quad (X, Y \in T_pM, p \in M) \quad E(\lambda) = \frac{1}{2} |\lambda|^2 \quad (\lambda \in T^*M),
\]

where \(|\cdot|\) denotes the norm function on \(T^*M\) associated with the metric \(g\). The energy function \(E\) is the hamiltonian of the geodesic flow of \(M\). For a function \(h\) on \(M\) we define vector fields \(\text{grad} \, h\) and \(s\text{grad} \, h\) by the following formulae:

\[
i_{\text{grad} \, h} g = dh, \quad i_{s\text{grad} \, h} \omega = -dh,
\]

where \(i\) denotes the inner derivation. We have \(s\text{grad} \, h = I\text{grad} \, h\).

Let \(p \in M\), and let \(S^2T_pM\) be the symmetric tensor product of two copies of the tangent space \(T_pM\). Let \(F\) be an element of \(S^2T_pM\), and suppose that, regarded as a quadratic form on \(T^*_pM\), \(F\) is a hermitian form. Then there is an orthonormal basis \(V_j, IV_j (1 \leq j \leq n)\) of \(T_pM\) and constants \(a_1, \ldots, a_n\) such that

\[
F = \sum_j a_j (V_j^2 + (IV_j)^2).
\]

We define the endomorphism \(F^e\) of \(T_pM\) by

\[
F^e(V_j) = a_j V_j, \quad F^e(IV_j) = a_j IV_j.
\]
Clearly it is independent of the choice of \( \{V_i\} \). Regarding \( T_p M \) as a complex vector space (by identifying \( I \) with \( \sqrt{-1} \)), \( F^e \) becomes \( \mathbb{C} \)-linear. We define \( \text{tr} F \) and \( \det F \) by the trace and the determinant of \( F^e \) over \( \mathbb{C} \) respectively (\( \text{tr} F = \sum_j a_j \), \( \det F = \prod_j a_j \)).

Let \( (M, \mathcal{F}) \) be a Kähler-Liouville manifold. Then the condition (4) in the definition is equivalent to that \( \{F^e_p \mid F \in \mathcal{F}\} \) is commutative with respect to the composition of endomorphisms for every \( p \in M \).

We shall use the term “smooth” in the same meaning as “of class \( C^\infty \)”.

1. Local calculus on \( M^1 \)

Let \( (M, \mathcal{F}) \) be a Kähler-Liouville manifold of dimension \( n \). Put

\[
M^0 = \{ p \in M \mid \dim \mathcal{F}_p = n \} \quad M^s = M - M^0.
\]

Then \( M^0 \) is an open subset of \( M \), which is not empty because of the condition (5) in the definition of Kähler-Liouville manifold. Let \( F_1, \ldots, F_n \) be a basis of \( \mathcal{F} \), and let \( p \) be a point of \( M^0 \). Then there are an orthonormal frame \( V_i, IV_i \) \((i = 1, \ldots, n)\) and \( n^2 \) functions \( f_{ij} \) around \( p \) such that

\[
F_i = \sum_{j=1}^{n} f_{ij} (V_j^2 + (IV_j)^2).
\]

Putting \( (a_{ij}) = (f_{ij})^{-1} \), we have

\[
\sum_{j=1}^{n} a_{ij} F_j = V_i^2 + (IV_i)^2.
\]

Let \( D_i \) be the subbundle of \( TM \) defined around \( p \) spanned by \( V_i \) and \( IV_i \).

**Proposition 1.1.** There are positive functions \( a_1, \ldots, a_n \) around the point \( p \) such that, putting

\[
b_{ij} = \frac{a_{ij}}{a_i}, \quad W_i = \frac{V_i}{\sqrt{a_i}},
\]

(1.1)

\[
W_j b_{ik} = (IW_j) b_{ik} = 0 \quad (j \neq i, \text{any } k),
\]

(1.2)

\[
\{W_i^2 + (IW_i)^2, W_j^2 + (IW_j)^2\} = 0 \quad (\text{any } i, j).
\]

The function \( a_i \) can be chosen to be one of \( |a_{i1}|, \ldots, |a_{in}| \) that is non-zero around \( p \). Moreover, if \( \{a'_i\} \) possess the same properties as above, then

\[
V_j \frac{a'_i}{a_i} = (IV_j) \frac{a'_i}{a_i} = 0 \quad (j \neq i).
\]

**Proof.** We have

\[
\{V_i^2 + (IV_i)^2, V_j^2 + (IV_j)^2\} = \sum_{k,l=1}^{n} \{\{a_{ik}, F_l\} a_{jl} F_k + \{F_k, a_{jl}\} a_{ik} F_l\}
\]

(1.3)

\[
= \sum_{k,l=1}^{n} \{a_{ik}, V_j^2 + (IV_j)^2\} F_k + \sum_{k,l=1}^{n} \{V_i^2 + (IV_i)^2, a_{jl}\} F_l
\]
Note that each term in the formula above is a homogeneous polynomial of degree 3 in the variables $V_k, IV_k$ ($1 \leq k \leq n$). Since the left-hand side belongs to the ideal $(V_iV_j, V_iIV_j, V_jIV_i, IV_iIV_j)$ of the polynomial algebra, and since $F_k$ are linear combinations of $V_l^2 + (IV_l)^2$ ($1 \leq l \leq n$), it follows that

\begin{equation}
\sum_k \{a_{ik}, V_j\}F_k = c_{ij}(V_i^2 + (IV_i)^2), \quad \sum_k \{a_{ik}, IV_j\}F_k = d_{ij}(V_i^2 + (IV_i)^2)
\end{equation}

for some functions $c_{ij}$ and $d_{ij}$, provided $i \neq j$. Hence we have

\begin{equation}
\{a_{ik}, V_j\} = c_{ij}a_{ik}, \quad \{a_{ik}, IV_j\} = d_{ij}a_{ik}.
\end{equation}

Let $a_i$ be one of the functions $|a_{i1}|, \ldots, |a_{in}|$ that does not vanish around the point $p$. Then by (1.5) we have

\[
\left\{ \frac{a_{ik}}{a_i}, V_j \right\} = \left\{ \frac{a_{ik}}{a_i}, IV_j \right\} = 0 \quad (i \neq j).
\]

This implies that

\[
\sum_k \{a_{ik}, V_j^2 + (IV_j)^2\}F_k = \frac{1}{a_i} \{a_{ik}, V_j^2 + (IV_j)^2\}(V_i^2 + (IV_i)^2).
\]

Hence, by (1.3) we obtain

\[
\left\{ \frac{1}{a_i}(V_i^2 + (IV_i)^2), \frac{1}{a_j}(V_j^2 + (IV_j)^2) \right\} = 0.
\]

The remaining part is clear. □

**Proposition 1.2.** $[V_i, IV_i] \equiv \text{sgrad} \ (\log a_i) \mod D_i$.

**Proof.** We use the Kähler form $\omega$. We have

\[
0 = d\omega(V_i, IV_i, V_j) = -\omega([V_i, IV_i], V_j) + \omega([V_i, V_j], IV_i) - \omega([IV_i, V_j], V_i).
\]

Since (1.2) implies

\begin{equation}
[W_i, W_j] = \alpha_{ij}IW_i - \alpha_{ji}IW_j, \quad [IW_i, IW_j] = \beta_{ij}W_i - \beta_{ji}W_j,
\end{equation}

\[
[W_i, IW_j] = -\beta_{ij}IW_i + \alpha_{ji}W_j, \quad [IW_i, W_j] = -\alpha_{ij}W_i + \beta_{ji}IW_j
\]

for some functions $\alpha_{ij}$ and $\beta_{ij}$ ($i \neq j$), it follows that

\[
\omega([V_i, V_j], IV_i) = -\omega([IV_i, V_j], V_i) = -V_j \log \sqrt{a_i}.
\]

Hence we have

\begin{equation}
\omega([V_i, IV_i], V_j) = -V_j \log a_i,
\end{equation}

provided $j \neq i$. Replacing $V_j$ with $IV_j$, we also have

\begin{equation}
\omega([V_i, IV_i], IV_j) = -IV_j \log a_i
\end{equation}
From (1.7) and (1.8) it thus follows that

\[ [V_i, IV_i] \equiv \sum_{j \neq i} (-IV_j \log a_i)V_j + (V_j \log a_i)IV_j \mod D_i. \]

\[ \square \]

We now consider the following condition for points on \( M^0 \):

(1.9) For any \( i \) there is some \( j \) such that \( da_j|_{D_i} \neq 0 \) at \( p \).

Note that this condition is independent of the choice of \( \{a_i\} \). Put

\[ M^1 = \{ p \in M^0 | (1.9) \text{ holds at } p \}. \]

We shall say that a Kähler-Liouville manifold \((M, F)\) is of type (A) if

\[ M^1 \neq \emptyset. \]

From now on (until the end of Section 4) Kähler-Liouville manifolds are assumed to be of type (A), unless otherwise stated.

Let \( p \in M^1 \).

**Proposition 1.3.** Let \( j, k \neq i \). If \( d \log a_j|_{D_i} \neq 0 \) and \( d \log a_k|_{D_i} \neq 0 \) at \( q \in M^1 \) near \( p \), then they are linearly dependent at \( q \).

**Proof.** Note first that \( \sum_j a_{jk} \) is constant for every \( k \), because \( E \in F \). Since \( a_{jk} = a_jb_{jk} \), this implies that \( \{a_i\} \) and \( \{a_{ij}\} \) are written as rational functions of \( \{b_{ij}\} \). Hence for each \( i \) there is some \( l \) such that \( db_{il} \neq 0 \) around \( p \). On the other hand, the kernel of \( d \log a_j \) on \( D_i \) is spanned by

\[ -(IV_j \log a_j)V_i + (V_i \log a_j)IV_j = [V_j, IV_j]|_{D_i}, \]

where the right-hand side denotes the \( D_i \)-component of \([V_j, IV_j]\). Since \( V_jb_{il} = IV_jb_{il} = 0 \), we also have

\[ 0 = [V_j, IV_j]b_{il} = [V_j, IV_j]|_{D_i}b_{il}. \]

Hence the kernel of \( d \log a_j \) on \( D_i \) coincides with that of \( db_{il} \) on \( D_i \). Since the latter does not depend on \( j \), the proposition follows. \( \square \)

By virtue of the proposition above we can take the orthonormal frame \( V_i, IV_i \) \((i = 1, \ldots, n)\) around \( p \in M^1 \) so that

\[ d \log a_j(IV_i) = 0 \quad \text{for any } j \neq i. \]

Note that \( V_i \) are uniquely determined up to sign (and the numbering). We shall assume that \( V_i \) are taken in this way. Let \( D^+ \) (resp. \( D^- \)) be the subbundle of \( TM \) spanned by \( V_1, \ldots, V_n \) (resp. \( IV_1, \ldots, IV_n \)). \( D^+ \) and \( D^- \) are well-defined over \( M^1 \);

\[ TM^1 = D^+ \oplus D^- \].
Proposition 1.4. (1) \( da_i, da_{ij}, db_{ij} \) are zero on \( D^- \).
(2) For any \( i \) there is some \( j \) such that \( W_i b_{ij} \neq 0 \).
(3) \([W_i, W_j] = [IW_i, IW_j] = [W_i, IW_j] = 0 \) \((i \neq j)\). In particular, \( D^+ \) and \( D^- \) are integrable.

Proof. (1) and (2) are clear from the proof of Proposition 1.3. Suppose that \( W_i b_{ik} \neq 0 \). Then by (1.6),

\[
0 = [IW_i, IW_j]b_{ik} = \beta_{ij} W_i b_{ik}, \quad 0 = [IW_i, W_j]b_{ik} = -\alpha_{ij} W_i b_{ik}.
\]

Hence \( \alpha_{ij} = \beta_{ij} = 0 \), and (3) follows. \( \square \)

Proposition 1.5. \([W_i, IW_i] \in D^- \).

Proof. By virtue of Propositions 1.2 and 1.4, it suffices to prove that the \( D_i \)-component of \([W_i, IW_i] \) belongs to \( D^- \). Choose \( j(\neq i) \) such that \( d \log a_j|_{D_i} \neq 0 \). Then the \( D_i \)-component of \([W_j, IW_j] \) is not zero. Describing

\[
[W_j, IW_j] = \alpha W_j + \beta IW_j + \sum_{k \neq j} \gamma_k IW_k,
\]

we have

\[
0 = [W_i, [W_j, IW_j]]_{D_i} = (W_i\gamma_i)IW_i + \gamma_i[W_i, IW_i]_{D_i}.
\]

Since \( \gamma_i \neq 0 \), the proposition follows. \( \square \)

Proposition 1.6. Maximal integral manifolds of \( D^+ \) are (locally) totally geodesic.

Proof. Since \( IW_k < W_i, W_j >= 0 \) and \([W_i, IW_k] \in D^- \), it follows that

\[
< \nabla W_i W_j, IW_k >= 0
\]

for any \( i, j, k \), where \( \nabla \) denotes the Levi-Civita covariant derivative. Hence the proposition follows. \( \square \)

By virtue of Proposition 1.5 the \( D_i \)-component of \([W_i, IW_i] \) is of the form \( c_i IW_i \), \( c_i \) being the function around \( p \).

Proposition 1.7. (1) For \( i, j \) such that \( i \neq j \) and \( W_i a_j \neq 0 \),

\[
c_i = -W_i \log a_i + W_i \log a_j - \frac{W_i^2 \log a_j}{W_i \log a_j}.
\]

(2) \( (IW_k)c_i = 0 \) \( \text{(any } k) \).
(3) \( W_k c_i = -(W_k \log a_i)(W_i \log a_k) \) \( (k \neq i) \).
(4) \( W_j W_i \log a_i = (W_i \log a_j)(W_j \log a_i) \) \( (i \neq j) \).
(5) \( W_i W_j \log a_k = 0 \) \( (i \neq j \neq k \neq i) \).
(6) \( (W_i \log a_k)(W_j \log a_k) = (W_i \log a_j)(W_j \log a_k) + (W_j \log a_i)(W_i \log a_k) \) \( (i \neq j \neq k \neq i) \).

Proof. By Propositions 1.2 and 1.4 we have

\[
[W_i, IW_i] = c_i IW_i + \sum \frac{a_j}{a_i} (W_j \log a_i) IW_j.
\]
Then, computing \([W_k, [W_i, IW_i]]\) for \(k \neq i\), we obtain

\[
0 = (W_k c_i + (W_k \log a_i)(W_i \log a_k)) IW_i \\
+ (c_k \frac{a_k}{a_i} W_k \log a_i + W_k (\frac{a_k}{a_i} W_k \log a_i)) IW_k \\
+ \sum_{j \neq i, k} \left( W_k \left( \frac{a_j}{a_i} W_j \log a_i \right) + \frac{a_k}{a_i} \left( W_k \log a_i \right) (W_j \log a_k) \right) IW_j.
\]

This formula implies (1), (3), and

\[
(1.10) \quad W_k W_j \log a_i = (W_k \log a_i)(W_j \log a_i) \\
- (W_k \log a_j)(W_j \log a_i) - (W_j \log a_k)(W_k \log a_i)
\]

for mutually distinct \(i, j, k\). Also, (2) follows from (1) and Proposition 1.4. To prove (4), (5), and (6), we recall that \(\sum_i a_i b_{il}\) are constants. Differentiating these by \(W_j\) and \(W_k\) successively (\(j \neq k\)) we have

\[
\sum_i (W_j \log a_i) a_i b_{il} + a_j W_j b_{jl} = 0, \\
\sum_i (W_k W_j \log a_i) a_i b_{il} + (W_j \log a_k) a_k W_k b_{kl} \\
+ \sum_i (W_j \log a_i)(W_k \log a_i) a_i b_{il} + (W_k \log a_j) a_j W_j b_{jl} = 0.
\]

Thus,

\[
(1.11) \quad W_k W_j \log a_i = (W_j \log a_k)(W_k \log a_i) \\
+ (W_k \log a_j)(W_j \log a_k) - (W_k \log a_i)(W_j \log a_i).
\]

From (1.10) and (1.11) the formulae (5) and (6) follows. Also, since \(i\) is arbitrary in the formula (1.11), (4) follows by putting \(i = k\) in (1.11). \(\square\)

We now define the binary relation \(\preceq\) on the set of indices \(\{1, \ldots, n\}\):

\[
i \preceq j \iff i \neq j \text{ and } W_i \log a_j \neq 0 \text{ at } p \in M^1, \text{ or } i = j.
\]

When it is necessary to clarify the point-dependence, we shall say that \(i \preceq j\) at \(p\). Also, we write \(i \sim j\) if \(i \preceq j\) and \(j \preceq i\).

Lemma 1.8. (1) If \(i \preceq j\) and \(j \preceq k\), then \(i \preceq k\).

(2) The relation \(\sim\) is an equivalence relation.

Proof. Assume that \(i \npreceq k\) and \(j \neq i, k\). Then, by Proposition 1.7 (6) we have

\[
(W_i \log a_j)(W_j \log a_k) = 0.
\]

Hence \(i \npreceq j\) or \(j \npreceq k\), and (1) follows. (2) is an immediate consequence of (1). \(\square\)

Let \(\mathcal{A}\) be the set of the equivalence classes. It is clear from Lemma 1.8 (1) that the relation \(\preceq\) induces a binary relation (denoted by the same symbol) on the set \(\mathcal{A}\). For \(\alpha \in \mathcal{A}\), let \(|\alpha|\) denote the number of indices contained in the equivalence class \(\alpha\).
Proposition 1.9. Let $\alpha, \beta, \gamma \in \mathcal{A}$.

1. If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$. Namely, the relation $\leq$ is a partial order on $\mathcal{A}$.

2. If $\alpha \leq \gamma$ and $\beta \leq \gamma$, then $\alpha \leq \beta$ or $\beta \leq \alpha$. Namely, for any fixed $\gamma$, the set of $\delta \in \mathcal{A}$ such that $\delta \leq \gamma$ is a totally ordered subset of $\mathcal{A}$.

3. If $\alpha$ is a maximal element, then $|\alpha| \geq 2$.

Proof. (1) is clear from Lemma 1.8 (1).

(2) Let $i \in \alpha$, $j \in \beta$, and $k \in \gamma$, and assume that $\alpha \not< \beta$ and $\beta \not< \alpha$. Then $W_i \log a_j = W_j \log a_i = 0$. Hence by Proposition 1.7 (6), we have either $W_i \log a_k = 0$ or $W_j \log a_i = 0$.

(3) Suppose $\alpha$ is maximal and $i \in \alpha$. Then by the condition (1.9) we see that there is some $j(\neq i)$ such that $W_i \log a_j \neq 0$. Since $\alpha$ is maximal, it follows that $j \in \alpha$. □

Let $W_i^* (1 \leq i \leq n)$ be 1-forms such that $W_i^*(IW_j) = 0$ and $W_i^*(W_j) = \delta_{ij}$. Then Propositions 1.4 and 1.5 imply that $dW_i^* = 0$. Hence there is a system of functions $(x_1, \ldots, x_n)$ such that $dx_i = W_i^*$. Clearly, $a_i$ are functions of $(x_1, \ldots, x_n)$, and $W_j \log a_i$ is nothing but the derivative of $\log a_i$ with respect to the variable $x_j$. For simplicity, we shall write $\partial_j$ instead of $\partial/\partial x_j$.

Lemma 1.10. For any $i, j$ ($i \neq j$), there are functions $h_{ij}(x_i)$ such that $h_{ij} - h_{ji} \neq 0$ at $p$, and

$$\partial_i \log a_j = -\partial_j \log |h_{ij} - h_{ji}|. $$

Moreover, if $\partial_i \log a_j \neq 0$ and $\partial_i \log a_k \neq 0$ at $p$ ($j, k \neq i$), then there are constants $c(\neq 0)$ and $d$ such that

$$h_{ik} = ch_{ij} + d. $$

Proof. By Proposition 1.7 (5) the function $\partial_i \log a_j$ depends only on the variables $x_i$ and $x_j$. Since

$$\partial_j \partial_i \log a_j = \partial_i \partial_j \log a_i = (\partial_i \log a_j)(\partial_j \log a_i), $$

it follows that there is a positive function $H = H(x_i, x_j)$ such that

$$\partial_i \log a_j = \partial_i \log H, \quad \partial_j \log a_i = \partial_j \log H, \quad \partial_i \partial_j \log H = (\partial_i \log H)(\partial_j \log H). $$

The last formula implies that

$$\partial_i \partial_j \left( \frac{1}{H} \right) = 0. $$

Hence there are functions $h_{ij}(x_i)$ and $h_{ji}(x_j)$ such that $H^{-1} = |h_{ij} - h_{ji}|$.

Now, let us assume that $\partial_i \log a_j \neq 0$ and $\partial_i \log a_k \neq 0$ at $p$. Then the derivatives $h'_{ij}$ and $h'_{ik}$ do not vanish at $p$. By virtue of Proposition 1.7 (1), the function

$$\partial_i \log a_j - \frac{\partial_i \log a_j}{\partial_j \log a_j} = -\frac{h''_{ij}}{h''_{ij}} $$

does not depend on $j$. Hence it follows that

$$\frac{h''_{ij}}{h''_{ij}} = \frac{h''_{ik}}{h''_{ik}}, $$

which proves the latter half of the lemma. □
Proposition 1.11. For any \( p \in M^1 \) there is a neighborhood \( U \) of \( p \) such that the relation \( \preceq \) is stable on \( U \). Namely,

\[
i \preceq j \text{ at } p \iff i \preceq j \text{ at } q
\]

for any \( q \in U \).

Proof. Let \( U(\subset M^1) \) be a connected neighborhood of \( p \) such that every \( \partial_i \log a_j \) \((i \neq j)\) that does not vanish at \( p \) also does not vanish everywhere on \( U \), and that all the functions \( h_{ij} \) are defined there.

Now, let us assume that \( i \not\preceq j \) at \( p \) and \( i \preceq j \) at some \( q \in U \). Let \( k(\neq i) \) be a number such that \( i \preceq k \) at \( p \). Then \( i \preceq k \) at \( q \), and by Lemma 1.10,

\[
(1.12) \quad h'_{ij} = ch'_{ik}
\]
on a neighborhood of \( q \) for some constant \( c \neq 0 \). Let \( U' \) be the set of all \( q' \in U \) such that the formula (1.12) is valid at \( q' \). Clearly, \( U' \) is closed in \( U \). Moreover, if \( q' \in U' \), then \( h'_{ij} \neq 0 \) at \( q' \). This implies \( i \preceq j \) at \( q' \) by Lemma 1.10, and thus (1.12) holds on a neighborhood of \( q' \). Hence \( U' \) is open. Since \( q \in U' \) and \( U \) is connected, it follows that \( U' = U \). But since \( p \not\in U' \), it is a contradiction. \( \square \)

We shall write \( \alpha \prec \beta \) \((\alpha, \beta \in A)\) if \( \alpha \preceq \beta \) and \( \alpha \neq \beta \).

Proposition 1.12. Let \( p \in M^1 \), and let \( U \) be a small neighborhood of \( p \). Then there are functions \( h_i \ (1 \leq i \leq n) \) on \( U \) and constants \( e_{\alpha\beta} \) \((\alpha, \beta \in A, \alpha \prec \beta)\) satisfying the following four conditions:

1. \( \partial h_i|_{D^-} = \partial h_i|_{D^+} = 0 \quad (i \neq j), \quad W_i h_i \neq 0 \) everywhere on \( U \);
2. \( e_{\alpha\beta} = e_{\alpha\gamma} \quad (\alpha \prec \beta \preceq \gamma) \);
3. \( h_i \neq h_j \quad (i \sim j, \ i \neq j), \quad h_i + e_{\alpha\beta} \neq 0 \quad (i \in \alpha, \alpha \prec \beta) \) everywhere on \( U \);
4. the functions \( a_i \) can be taken in the form

\[
a_i = \left| \prod_{j \in \alpha, \ j \neq i} (h_j - h_i) \right|^{-1} \left| \prod_{\gamma \prec \alpha} \prod_{k \in \gamma} (h_k + e_{\gamma\alpha}) \right|^{-1} (i \in \alpha).\]

If \( \{\tilde{h}_i\} \) and \( \{\tilde{e}_{\alpha\beta}\} \) also satisfy the conditions above, then there are constants \( c_{\alpha} \neq 0 \) and \( d_{\alpha} \) such that

\[
\tilde{h}_i = c_{\alpha} h_i - d_{\alpha}, \quad \tilde{e}_{\alpha\beta} = c_{\alpha} e_{\alpha\beta} + d_{\alpha} \quad (i \in \alpha).
\]

Proof. We determine the functions \( h_i \) and the constants \( e_{\alpha\beta} \) inductively with respect to the partial order \( \prec \). Let \( h_{ij}(x_i) \ (i \neq j) \) and \( U \) be as in Lemma 1.10 and Proposition 1.11 respectively.

Let \( \alpha \in A \) be a minimal element. Then the functions \( a_i \ (i \in \alpha) \) depend only on the variables \( x_j \ (j \in \alpha) \). Hence, if \(|\alpha| = 1\), then we can put \( a_i = 1 \) (cf. Proposition 1.1). In this case the function \( h_i \) does not appear yet. Also, if \(|\alpha| = 2\), then putting \( h_i = h_{ij} \) and \( h_j = h_{ji} \ (i, j \in \alpha) \), we can define \( a_i \) and \( a_j \) as

\[
a_i = a_j = \frac{1}{|\{i, j\}|}.
\]
Now, suppose that $|\alpha| \geq 3$. By Lemma 1.10 we know that the ratios $h'_{ij}/h'_{ik}$ are non-zero constants for any mutually distinct $i, j, k \in \alpha$. Hence there is a function $h_i(x_i)$ and non-zero constants $c_{ij}$ such that

$$h'_{ij} = c_{ij}h'_i$$

for any $i, j \in \alpha$ ($i \neq j$). Then by Proposition 1.7 (6) we have

$$c_{ij}c_{jk}c_{ki} = c_{ji}c_{kj}c_{ik} \quad (i, j, k \in \alpha)$$

Namely, $\{c_{ij}/c_{ji}\}$ satisfies the cocycle condition. An easy calculation shows that it is actually a coboundary, i.e., there are non-zero constants $c_i$ ($i \in \alpha$) such that $c_{ij}/c_{ji} = c_i/c_j$. Then, putting

$$\tilde{h}_{ij} = \frac{c_i}{c_{ij}}h_{ij}, \quad \tilde{h}_i = c_i h_i,$$

we obtain

$$\partial_i \tilde{h}_{ij} = \partial_i \tilde{h}_i, \quad \tilde{h}_{ij} - \tilde{h}_{ji} = \frac{c_i}{c_{ij}}(h_{ij} - h_{ji}).$$

Putting $\tilde{h}_{ij} - \tilde{h}_i = d_{ij}$ (a constant), and using Proposition 1.7 (6) again, we have

$$d_{ij} + d_{jk} + d_{ki} = d_{ji} + d_{kj} + d_{ik}.$$ 

Put

$$d_i = |\alpha|^{-1} \sum_{j \in \alpha \atop j \neq i} (d_{ij} - d_{ji}) \quad (i \in \alpha).$$

Then we have

$$d_{ij} - d_{ji} = d_i - d_j, \quad \tilde{h}_{ij} - \tilde{h}_{ji} = (\tilde{h}_i + d_i) - (\tilde{h}_j + d_j)$$

for any $i, j \in \alpha$ ($i \neq j$). Hence, by redefining $h_i$ as

$$h_i = \tilde{h}_i + d_i \quad (i \in \alpha),$$

the difference

$$\log a_i - \sum_{j \in \alpha \atop j \neq i} \log |h_j - h_i| \quad (i \in \alpha)$$

becomes a function of the single variable $x_i$. Thus we can take $a_i$ ($i \in \alpha$) as

$$a_i = \prod_{j \in \alpha \atop j \neq i} |h_j - h_i|.$$ 

Now, let $\alpha$ be a non-minimal element of $\mathcal{A}$, and let $\beta \in \mathcal{A}$ ($\beta \prec \alpha$) be a unique element such that there is no $\gamma \in \mathcal{A}$ satisfying $\beta \prec \gamma \prec \alpha$. We assume that the functions $h_i$ are defined for all $i \in \gamma$ and $\gamma \prec \alpha$ ($\gamma \prec \beta$ if $|\beta| = 1$) and the constants $c_{\gamma\delta}$ are defined for all $\gamma$ and $\delta$ ($\gamma \prec \delta \prec \alpha$) so that the conditions (1), (2), (3), and
(4) in the proposition are satisfied. Under this assumption we shall define suitable functions \( h_i \ (i \in \alpha) \) when \(|\alpha| \geq 2\), \( h_j \ (j \in \beta) \) when \(|\beta| = 1\), and a constant \( e_{\beta\alpha} \).

Let \( i \in \alpha, j \in \beta, \) and \( k \in \gamma(\leq \beta) \). Then the assumption implies that
\[
\partial_k \log a_j = \begin{cases} 
-\partial_k \log |h_k - h_j| & (\gamma = \beta, \ k \neq j) \\
-\partial_k \log |h_k + e_{\gamma\beta}| & (\gamma < \beta).
\end{cases}
\]

Since
\[
(\partial_j \log a_i)(\partial_k \log a_i) = (\partial_j \log a_k)(\partial_k \log a_i) + (\partial_k \log a_j)(\partial_j \log a_i),
\]
we have \( \partial_k \log a_i = \partial_k \log a_j \) in case \( \gamma < \beta \). Hence in this case, putting \( e_{\gamma\alpha} = e_{\gamma\beta} \), we have
\[
(1.13) \quad \partial_k \log |a_i(h_k + e_{\gamma\alpha})| = 0.
\]

Also, in case \( \gamma = \beta \) and \( j \neq k \) (hence \(|\beta| \geq 2\)), we have
\[
-h_k + h_k' (h_{ki} - h_{ik}) = -h_j + h_j' (h_{ji} - h_{ij}).
\]

Since the left- and the right-hand side of the formula above are functions of \( x_k \) and \( x_j \) respectively, it follows that they are constants. Let \( e_{\beta\alpha} \) be this common constant. Then, in case \(|\beta| \geq 2\), we have
\[
(1.14) \quad h_j + e_{\beta\alpha} = c_{ji}(h_{ji} - h_{ij})
\]
for any \( j \in \beta \), where \( c_{ji} \) are non-zero constants. Moreover, we put
\[
(1.15) \quad h_j = |h_{ji} - h_{ij}|, \quad e_{\beta\alpha} = 0
\]
in case \(|\beta| = 1\). Then, by (1.13), (1.14), and (1.15), we see that
\[
\partial_k \log \left( a_i \prod_{\gamma < \alpha} \prod_{l \in \gamma} |h_l + e_{\gamma\alpha}| \right) = 0
\]
for every \( k \in \gamma \) and \( \gamma < \alpha \).

Now, applying the same argument as for minimal element, we obtain functions \( h_l \ (l \in \alpha) \) so that the function
\[
\log \left( a_i \prod_{\gamma < \alpha} \prod_{l \in \gamma} |h_l + e_{\gamma\alpha}| \right) + \log \prod_{l \in \alpha, l \neq i} |h_l - h_i|
\]
depends only on \( x_i \) for every \( i \in \alpha \) (if \(|\alpha| = 1\), the second term does not appear, and \( h_i \ (i \in \alpha) \) remains undetermined). Hence \( a_i \ (i \in \alpha) \) can be taken so that this function is equal to zero. This completes the induction. The remaining part of the proposition is easy. □

We now assume that each equivalence class \( \alpha \in \mathcal{A} \) consists of successive numbers:
\[
\alpha = \{s(\alpha), s(\alpha) + 1, \ldots, t(\alpha)\}.
\]

Let \( m(i) \ (i \in \alpha) \) be the number of functions \( h_j \ (j \in \beta) \) such that \( h_i > h_j \) at \( p \). Also, let \( m(\alpha, \beta) \) be the number of negative functions in
\[
\{h_i + e_{\alpha\beta} \mid i \in \alpha\}.
\]

Let \( n(\alpha) \) be the set of \( \beta \in \mathcal{A} \) such that \( \alpha < \beta \) and there is no \( \gamma \) between \( \alpha \) and \( \beta \). Also, put
\[
u_{\alpha} = \prod_{\gamma < \alpha} \prod_{l \in \gamma} (h_l + e_{\gamma\alpha}).\]
Proposition 1.13. For a suitably chosen basis $F_1, \dots, F_n$ of $\mathcal{F}$, the functions $b_{ij} = a_{ij}/a_i$ ($i \in \alpha$) are given by:

$$b_{ij} = \begin{cases} (-1)^{m(i)}(-h_i)^{j-s(\alpha)} & (j \in \alpha) \\ (-1)^{m(i)-1+m(\alpha, \beta)}(h_i + e_{\alpha\beta})^{-1} & (j = t(\beta), \beta \in n(\alpha)) \\ 0 & (\text{otherwise}). \end{cases}$$

Moreover,

$$\sum_{\beta \geq \alpha} \sum_{j \in \beta} a_j b_{jk} = \begin{cases} |u_\alpha|^{-1} & (k = t(\alpha)) \\ 0 & (k \neq t(\alpha)). \end{cases}$$

Proof. Put

$$\tilde{b}_{ij} = \begin{cases} (-1)^{m(i)}(-h_i)^{j-s(\alpha)} & (j \in \alpha) \\ (-1)^{m(i)-1+m(\alpha, \beta)}(h_i + e_{\alpha\beta})^{-1} & (j = t(\beta), \beta \in n(\alpha)) \\ 0 & (\text{otherwise}). \end{cases}$$

Then a direct computation shows that

$$\sum_{\beta \geq \alpha} \sum_{j \in \beta} a_j \tilde{b}_{jk} = \begin{cases} |u_\alpha|^{-1} & (k = t(\alpha)) \\ 0 & (k \neq t(\alpha)). \end{cases}$$

Let $B$ be the $n \times n$ matrix $(\tilde{b}_{ij})$, and let $B_i$ be the $(n-1) \times (n-1)$ matrix obtained by deleting $i$-th row and $t(\alpha)$-th column from $B$, where $i \in \alpha$.

Lemma 1.14. (1) $\det B \neq 0$.

(2) $\det B_i \neq 0$.

Proof. It is easily seen that $\det B$ is equal to $\prod_{\alpha \in A} \det B^\alpha$, where $B^\alpha = (\tilde{b}_{ij})_{i,j \in \alpha}$. Since $\det B^\alpha \neq 0$ by Vandermonde’s formula, (1) follows. (2) is similar. □

Define functions $c_{kj}$ ($1 \leq k, j \leq n$) by the formula

$$b_{ij} = \sum_k \tilde{b}_{ik} c_{kj}. \quad (1.16)$$

To prove Proposition 1.13 it suffices to show that $c_{kj}$ are constants. Let $\alpha \in A$ and $l \in \alpha$. We claim that $\partial_l c_{t(\alpha), j} = 0$ for any $j$. In fact, since $\sum_i a_i b_{ij}$ are constants, we obtain

$$0 = \partial_l \sum_i a_i b_{ij} = \partial_l \sum_{\beta \geq \alpha} \sum_{i \in \beta} a_i b_{ij}. \quad (1.16)$$

By (1.16) we also have

$$\sum_{\beta \geq \alpha} \sum_{i \in \beta} a_i b_{ij} = |u_\alpha|^{-1} c_{t(\alpha), j}. \quad (1.16)$$

Thus, it follows that $\partial_l c_{t(\alpha), j} = 0$.

Moreover, the formula (1.16) implies that

$$0 = \sum_{k \neq t(\alpha)} \tilde{b}_{ik} \partial_l c_{kj} = 0,$$

provided $i \neq l$ and $l \in \alpha$. It then follows from Lemma 1.14 that

$$\partial_l c_{kj} = 0 \quad (l \in \alpha, \ k \neq t(\alpha)).$$

This completes the proof. □

Correspondingly, $(f_{\alpha}) = (a_{\alpha})^{-1}$ is given as follows.
Proposition 1.15. Suppose \( i \in \alpha \). Then:

\[
\tilde{f}_{ij} = \begin{cases} 
|u_\alpha|S_{t(\alpha)-i}(h_l ; l \in \alpha - \{j\}) & (j \in \alpha) \\
|u_\alpha|\sum_{m=0}^{t(\alpha)-i} e_{\alpha \beta}^m S_{t(\alpha)-i-m}(h_l ; l \in \alpha) & (j \in \gamma \geq \beta, \beta \in \mathfrak{n}(\alpha)) \\
0 & (\text{otherwise})
\end{cases}
\]

where \( S_m(h_l ; l \in \alpha) \) stands for the elementary symmetric function of degree \( m \) \((0 \leq m \leq |\alpha|)\) with respect to \(|\alpha|\) functions \( \{h_l | l \in \alpha\} \).

The proof is straightforward. Put

\[ v_i = u_\alpha S_{t(\alpha)-i+1}(h_l ; l \in \alpha) \quad (i \in \alpha, \alpha \in \mathcal{A}), \]

and let \( \mathcal{V} \) be the vector space of functions on \( U \) spanned by constant functions and \( v_i \) \((1 \leq i \leq n)\). Put

\[ \mathfrak{k} = \{\text{sgrad}(v) | v \in \mathcal{V}\}. \]

Proposition 1.16.

(1) \([Y, W_j] = [Y, IW_j] = 0 \quad (Y \in \mathfrak{k}, \text{any } j)\).

(2) \(\{Y, F\} = 0 \quad (Y \in \mathfrak{k}, F \in \mathcal{F})\).

(3) \(\mathfrak{k}\) is the commutative Lie algebra of infinitesimal automorphisms of the Kähler manifold \( M \) on \( U \).

Proof. Put \( Y = \text{sgrad } v_m \). Then

\[ Y = \sum_j a_j(W_j v_m)IW_j. \]

Hence, by Propositions 1.4 and 1.5 we have \([Y, IW_j] = 0\) for any \( j \). Since

\[ [W_i, IW_i] = \left(-\partial_i \log a_i - \frac{h''_i}{h'_i}\right) IW_i + \sum_{j \neq i} \frac{a_j}{a_i} (\partial_j \log a_i) IW_j, \]

it follows that

\[ [W_i, Y] = a_i \left( \partial^2 v_m - (\partial_i v_m) \frac{h''_i}{h'_i} \right) IW_i \]

\[ + \sum_{j \neq i} a_j(\partial_i \partial_j v_m + (\partial_i \log a_j)(\partial_j v_m) + (\partial_j \log a_i)(\partial_i v_m))IW_j. \]

Then, it is easily seen that each term in the right-hand side of the formula above vanishes. Thus (1) follows.

From (1) it follows that

\[ 0 = \{Y, W_i^2 + (IW_i)^2\} = \sum_j b_{ij}\{Y, F_j\}. \]

This indicates (2). In particular, we have \(\{Y, E\} = 0\), which implies that \( Y \) is an infinitesimal isometry. The property (1) also implies that \((\mathcal{L}_Y I)W_i = (\mathcal{L}_Y I)IW_i = 0\) for any \( i \), where \( \mathcal{L}_Y \) denotes the Lie derivative with respect to \( Y \). Hence it follows that \( \mathcal{L}_Y I = 0 \). Moreover, putting \( Y' = \text{sgrad } v_l \) \((l \neq m)\), we have

\[ i_{[Y, Y']}\omega = -\mathcal{L}_Y(dv_l) = -d(Y v_l) = 0. \]

Hence \([Y, Y'] = 0\), and (3) follows. □
2. Summing up the local data

In the previous section we have given, for each \( p \in M^1 \), the neighborhood \( U \), the constants \( e_{\alpha\beta} \), the functions \( h_i, a_i, b_{ij}, v_i \), and the basis \( F_1, \ldots, F_n \) of \( F \), as well as their numbering. From now on, to clarify the point-dependence, we shall write \( U(p) \), \( h_i(p) \), \( F_i(p) \), etc. instead. We assume that each neighborhood \( U(p) \) is taken to be a small distance ball centered at \( p \) so that it is convex.

Take a point \( p_0 \in M^1 \) and fix it. Let \( M^{1,0} \) be the connected component of \( M^1 \) containing \( p_0 \). Let \( p \in M^{1,0} \), and let \( \gamma(t) \) \((0 \leq t \leq 1)\) be a curve in \( M^{1,0} \) such that \( \gamma(0) = p_0 \), \( \gamma(1) = p \). Along the curve \( \gamma \) there is a unique numbering of \( \{D_i\} \) so that \( t \mapsto (D_i)_{\gamma(t)} \) is continuous.

Since the relations \( i \preceq j \) are locally stable, we have the following

**Lemma 2.1.** The relation \( \preceq \) on \( \{1, \ldots, n\} \) is constant along the curve \( \gamma \). In particular, the partially ordered set \( \mathcal{A} \) is constantly defined along \( \gamma \).

We put

\[
F^{(q)}_{\alpha}(\lambda) = \sum_{i \in \alpha} (-\lambda)^{i-s(\alpha)} F^{(q)}_i.
\]

As is easily seen,

\[
F^{(q)}_{\alpha}(\lambda) = |u^{(q)}_{\alpha}| \sum_{j \in \alpha} \prod_{k \in \alpha, k \neq j} (h^{(q)}_k - \lambda) \cdot (V_j^2 + (IV_j)^2) + |u^{(q)}_{\alpha}| \sum_{\beta \in \mathcal{A}(\alpha)} \frac{\prod_{l \in \alpha} (h^{(q)}_l + e^{(q)}_{\alpha\beta}) - \prod_{l \in \alpha} (h^{(q)}_l - \lambda)}{e^{(q)}_{\alpha\beta} + \lambda} \sum_{\gamma \geq \beta, j \in \gamma} (V_j^2 + (IV_j)^2).
\]

**Proposition 2.2.** There are constants \( c_{\alpha} \neq 0 \) and \( d_{\alpha} \) such that

\[
F^{(p)}_{\alpha}(c_{\alpha} \lambda - d_{\alpha}) = \left( c^{\alpha|\alpha|-1}_{\alpha} \prod_{\beta \leq \alpha} c^{\alpha|\beta|}_{\beta} \right) F^{(p_0)}_{\alpha}(\lambda).
\]

If \( \{h^{(p)}_i\} \) and \( e^{(p)}_{\alpha\beta} \) are suitably chosen, then those constants become

\[
c_{\alpha} = 1, \quad d_{\alpha} = 0 \quad (\text{any } \alpha).
\]

**Proof.** Let \( I_{\gamma(t)} \) be the connected component of the intersection of \( U(\gamma(t)) \) and the image of \( \gamma \) containing the point \( \gamma(t) \). Suppose that \( I_{\gamma(t_1)} \cap I_{\gamma(t_2)} \neq \emptyset \). Then, by Proposition 1.11 there are constants \( \tilde{c}_{\alpha}(\neq 0) \) and \( \tilde{d}_{\alpha} \) such that

\[
h^{(\gamma(t_1))}_i = \tilde{c}_{\alpha} h^{(\gamma(t_2))}_i - \tilde{d}_{\alpha}, \quad e^{(\gamma(t_1))}_{\alpha\beta} = \tilde{c}_{\alpha} e^{(\gamma(t_2))}_{\alpha\beta} + \tilde{d}_{\alpha}
\]

on \( U(\gamma(t_1)) \cap U(\gamma(t_2)) \). This implies that

\[
F^{(\gamma(t_1))}_{\alpha}(\tilde{c}_{\alpha} \lambda - \tilde{d}_{\alpha}) = \left( c^{\alpha|\alpha|-1}_{\alpha} \prod_{\beta \leq \alpha} c^{\alpha|\beta|}_{\beta} \right) F^{(\gamma(t_2))}_{\alpha}(\lambda).
\]
Taking a finite number of points on \( \gamma \) and iterating this argument successively, we obtain the former half of the proposition.

Now, putting
\[
\tilde{h}_i^{(p)} = c_\alpha^{-1}(h_i^{(p)} + d_\alpha), \quad \tilde{e}_{\alpha\beta}^{(p)} = c_\alpha^{-1}(e_{\alpha\beta}^{(p)} - d_\alpha),
\]
and denoting by \( \tilde{F}_\alpha^{(p)}(\lambda) \) the corresponding polynomial, we have
\[
\tilde{F}_\alpha^{(p)}(\lambda) = \left( c_{\alpha}^{-|\alpha|-1} \prod_{\beta \prec \alpha} c_{\beta}^{|\beta|} \right)^{-1} F_\alpha^{(p)}(c_\alpha \lambda - d_\alpha) = F_\alpha^{(p_0)}(\lambda).
\]
\( \Box \)

Now, let us suppose that \( p = p_0 \), i.e., \( \gamma \) is a loop. Let \( \nu \) be the permutation of the indices \( 1, \ldots, n \) defined by
\[
(D_i)_{\gamma(1)} = (D_{\nu(i)})_{\gamma(0)}.
\]

**Proposition 2.3.** \( \nu \) is the identity.

**Proof.** Since \( \nu \) preserves the relation \( \leq \), it induces an automorphism of the partially ordered set. We also denote it by \( \nu \). Then, by Proposition 2.2 there are constants \( c_\alpha \neq 0 \) and \( d_\alpha \) such that
\[
F_{\nu(\alpha)}^{(p_0)}(c_\alpha \lambda - d_\alpha) = \left( c_{\alpha}^{-|\alpha|-1} \prod_{\beta \prec \alpha} c_{\beta}^{|\beta|} \right) F_\alpha^{(p_0)}(\lambda).
\]

This formula clearly indicates that \( \nu(\alpha) = \alpha \) and \( F_i^{(p)} \) \( (i \in \alpha) \) is written as a linear combination of \( F_j^{(p)} \) \( (i \leq j \leq t(\alpha)) \). Thus \( \nu(i) = i \) for every \( i \in \alpha \). \( \Box \)

This proposition implies that the subbundles \( D_i \) are globally defined on \( M^{1,0} \).

**Proposition 2.4.** Suppose that \( \{h_i^{(q)}\} \) and \( e_{\alpha\beta}^{(q)} \) \( (q \in M^{1,0}) \) are taken so that \( F_\alpha^{(q)}(\lambda) = F_\alpha^{(p_0)}(\lambda) \) for all \( \alpha \in A \). Then for any \( p, q \in M^{1,0} \) such that \( U^{(p)} \cap U^{(q)} \neq \emptyset \), \( e_{\alpha\beta}^{(p)} = e_{\alpha\beta}^{(q)} \) and \( h_i^{(p)} = h_i^{(q)} \) for any \( i \) on the intersection. Hence, there are functions \( \{h_i\} \) on \( M^{1,0} \) such that \( h_i = h_i^{(p)} \) on \( U^{(p)} \).

**Proof.** Since \( F_\alpha^{(p)}(\lambda) = F_\alpha^{(q)}(\lambda) \), the proposition follows from the proof of Proposition 2.2. \( \Box \)

### 3. Structure of \( M - M^1 \)

In the previous section we have obtained the constants \( e_{\alpha\beta} \), the functions \( h_i, a_i, b_{ij}, f_{ij} \) on \( M^{1,0} \) and the basis \( \{F_i\} \) of \( \mathcal{F} \). Also, the functions \( v \in \mathcal{V} \) and the vector fields \( Y \in \mathfrak{f} \) are now defined on \( M^{1,0} \), and the properties described in Proposition 1.16 hold on \( M^{1,0} \). Since for each \( \alpha \) the functions \( h_i \) \( (i \in \alpha) \) take mutually distinct values at every point in \( M^{1,0} \), we may assume that the numbering of \( \{D_i\} \) is chosen so that
\[
h_1 > h_2 > \cdots > h_{|\alpha|} \quad (\alpha \in \mathcal{A}).
\]
on $M^{1,0}$. We put

$$v_\alpha(\lambda) = u_\alpha \prod_{i \in \alpha} (h_i - \lambda)$$

$$= \sum_{m=0}^{|\alpha|-1} (-\lambda)^m v_{s(\alpha)+m} + (-\lambda)^{|\alpha|} u_\alpha.$$ 

The main purpose of this section is to prove the following

**Theorem 3.1.** $M - M^1$ is equal with a locally finite union of closed, totally geodesic, complex hypersurfaces $L$. In particular $M^1$ is connected and dense in $M$. Moreover, the functions $h_i$ are continuously extended to the whole $M$ and the subbundles $D_i$ are smoothly extended to $M^0$, and they possess the following properties:

1. $h_i + e_{\alpha\beta}$ ($i \in \alpha$, $\alpha < \beta$) are everywhere nonzero on $M$;
2. $h_{s(\alpha)} > h_{s(\alpha)+1} > \cdots > h_{t(\alpha)}$ on $M^0$ ($\alpha \in \mathcal{A}$);
3. $h_i$ are $C^\infty$ functions on $M^0$;
4. $S_m(h_i; i \in \alpha)$ $(1 \leq m \leq |\alpha|)$ are $C^\infty$ functions on $M$;
5. $D_\alpha = \sum_{i \in \alpha} D_i$ ($\alpha \in \mathcal{A}$) are extended to the whole $M$ as $C^\infty$ subbundles of $TM$;
6. The vector fields $Y \in \mathfrak{k}$ are globally defined and of $C^\infty$ on $M$;
7. For each hypersurface $L$ there are $\alpha \in \mathcal{A}$ and $c \in \mathbb{R}$ such that a connected component of the set of zeros of $\text{sgrad} v_\alpha(c) \in \mathfrak{k}$ coincides with $L$, and $v_\alpha(c)$ vanishes on $L$.

We shall call $\{h_i\}$ and $\{e_{\alpha\beta}\}$ the fundamental functions and the conjunction constants of the Kähler-Liouville manifold $(M, \mathcal{F})$ (of type (A)) respectively. Note that if $\{h'_i\}$ and $\{e'_{\alpha\beta}\}$ are other choice, then there are constants $c_\alpha \neq 0$ and $d_\alpha$ such that $e'_{\alpha\beta} = c_\alpha e_{\alpha\beta} + d_\alpha$ and for $i \in \alpha$,

$$h'_i = \begin{cases} c_\alpha h_i - d_\alpha & (c_\alpha > 0) \\ c_\alpha h_{t(\alpha)+s(\alpha)-i} - d_\alpha & (c_\alpha < 0) \end{cases}$$

The following theorem is an immediate consequence of the theorem above and Proposition 1.16.

**Theorem 3.2.** The geodesic flow of a Kähler-Liouville manifold of type (A) is integrable with respect to the first integrals in $\mathcal{F}$ and $\mathfrak{k}$.

We begin the proof of Theorem 3.1 with a characterization of the functions $v_i$ in terms of $F_j$.

**Lemma 3.3.** Let $i \in \alpha$. If $i \neq s(\alpha)$, the function $v_i$ is equal with a linear combination of $\text{tr} F_{i-1}$, $\ldots$, $\text{tr} F_{t(\alpha)}$ and $\text{tr} F_j$ ($j \in \beta$, $\beta < \alpha$). If $i = s(\alpha)$ and $\alpha$ is not a maximal element, then $v_i$ is equal with a linear combination of $\text{tr} F_j$ ($j \in \beta$, $\beta \preceq \alpha$) and $\text{tr} F_{t(\gamma)}$, where $\gamma$ is any element of $\mathfrak{n}(\alpha)$. Finally, if $i = s(\alpha)$ and $\alpha$ is a maximal element, then $v_i u_\alpha^{s(\alpha)-1}$ is a polynomial of $\det F_{t(\alpha)-1}$ and $\text{tr} F_j$ ($j \in \beta$, $\beta \preceq \alpha$).

The proof is straightforward. The lemma above implies that there is a $C^\infty$ function on $M$ expressed by the traces of $F_j$ that coincides with $v_i$ on $M^{1,0}$, unless $i = s(\alpha)$ and $\alpha$ is maximal. Though the expression is not unique in general, we take...
one and fix it. We denote the extended function by the same symbol $v_i$. Note that every $u_\beta$ ($\beta \in \mathcal{A}$) is a linear combination of those functions. Similarly, if $i = s(\alpha)$ and $\alpha$ is maximal, then there is a $C^\infty$ function on $M$ that coincides with $v_i u_\alpha |^{\alpha-1}$ on $M^{1,0}$.

Let $M^{0,0}$ be the connected component of $M^0$ that contains $M^{1,0}$. For technical reason we shall first show that $M^{0,0} \cap M^1$ is connected and dense in $M^{0,0}$, and that the functions $h_i$ are smoothly extended to $M^{0,0}$. Let $c(t)$ be a geodesic such that

$$p = c(0) \in M^{1,0}, \quad c([0, t_0)) \subset M^{1,0}, \quad q = c(t_0) \in M^{0,0} - M^{1,0}.$$  

Since $q \in M^0$, every simultaneous eigenspace of $\{ F_q^e \mid F \in \mathcal{F} \}$ is (complex) one-dimensional. Hence there is a neighborhood $U$ of $q$, and there are subbundles $D_i$ of $TM$ on $U$ such that each $D_i$ coincides with that on $M^{1,0}$ on the connected component of $U \cap M^{1,0}$ containing a curve segment of the form $c((t_0 - \epsilon, t_0))$, $\epsilon > 0$.

Since the polynomial $v_\alpha(\lambda)$ has $|\alpha|$ real roots at each point in $M^{1,0}$, so does at $q$ if $u_\alpha(q) \neq 0$. In this case, denoting those roots by $h_i(q)$ ($i \in \alpha$), $h_s(\alpha)(q) \geq \cdots \geq h_t(\alpha)(q)$, we have the continuous extension of $h_i$ up to $q$.

**Lemma 3.4.** (1) $u_\alpha(q) \neq 0$ for any $\alpha$.

(2) $h_s(\alpha)(q) > \cdots > h_t(\alpha)(q)$.

**Proof.** If $u_\alpha(q) = 0$ for some $\alpha$, then we have $F_{t(\alpha)} = 0$ at $q$, contradicting $q \in M^0$. Hence $u_\alpha(q) \neq 0$ for every $\alpha$, and the functions $h_i$ are well-defined at $q$. Since the eigenvalues of the endomorphism $F_i$ at each point in $M^{1,0}$ are given by $f_{ij}$ described in Proposition 1.15, so are at $q$ by continuity. Therefore, if $h_i(q) = h_{i+1}(q)$ for some $i, i+1 \in \alpha$, then one can easily see that $D_i$ and $D_{i+1}$ cannot be separated by means of the eigenvalues, which contradicts the facts that every simultaneous eigenspace is one-dimensional. □

By virtue of the lemma above, we see that the polynomial $v_\alpha(\lambda)$ has $|\alpha|$ distinct real roots on $U$, if $U$ is taken small enough. Denoting those roots again by $h_i$ ($i \in \alpha$), $h_s(\alpha) > \cdots > h_t(\alpha)$, we obtain the smooth extension of the functions $h_i$ to $M^{1,0} \cup U$.

**Lemma 3.5.** There is some $h_i$ such that $dh_i = 0$ at $q$. In this case we also have:

(1) The hessian $\text{Hess} h_i$ of $h_i$ at $q$ is given by

$$\text{Hess} h_i(X, Y) = ag([X]_{D_i}, [Y]_{D_i}), \quad X, Y \in T_q M,$$

where $a \in \mathbb{R}$, $a \neq 0$, and $[X]_{D_i}$ denotes the $D_i$-component of $X$;

(2) $\dot{c}(t_0)$ is not orthogonal to $D_i$.

**Proof.** Put $U' = M^{1,0} \cap U$, and let $\{ \tilde{a}_i \}$ be functions around $q$ given in Proposition 1.1. Then we have

$$d \log \tilde{a}_i \equiv -d \log \left| u_\alpha \prod_{j \in \alpha \setminus \{i\}} |h_j - h_i| \right| \mod D_i$$

on $U'$. Clearly, it is also valid on the closure of $U'$ in $U$ by continuity. Hence, if every derivatives $h_i'$ does not vanish at $q$, then it follows that $q \in M^1$, a contradiction. Thus there is some $i$ such that $h_i' = 0$ at $q$. 

Put $b = h_i(q)$ and suppose $i \in \alpha$. Then $dv_\alpha(b) = 0$ and

$$
Hess v_\alpha(b) = \tilde{a} \text{Hess} h_i
$$

at $q$, where $\tilde{a}$ is a non-zero constant. Put $Y = \text{sgrad} v_\alpha(b)$. Then we have

$$
Hess v_\alpha(b)(X, Z) = g(\nabla_X Y, IZ).
$$

Moreover, since $Y$ is an infinitesimal automorphism of the Kähler manifold $M$ on $U'$, we have $\nabla_{IX}Y = I\nabla_X Y$. Hence $\text{Hess} v_\alpha(b)$ is a hermitian form on $U'$, and so is at $q$ by continuity. Since $(\text{Hess} h_i)(X, Z) = 0$ at $q$ if $X$ or $Z$ is orthogonal to $D_i$, it follows that

$$(\text{Hess} h_i)(X, Z) = (\text{Hess} h_i)([X]_{D_i}, [Z]_{D_i}) = a g([X]_{D_i}, [Z]_{D_i})$$

at $q$ for some constant $a$.

Now, we show $(\text{Hess} v_\alpha(b))(\dot{c}(t_0), \cdot) \neq 0$, which will prove that $a \neq 0$, and (2). Since $Y$ is a Jacobi field along $c(t)$ ($0 \leq t < t_0$), it satisfies the equation of Jacobi field up to $q = c(t_0)$ by continuity. Hence, $Y_q$ being $0$, we have $\nabla_{\dot{c}(t_0)} Y \neq 0$ at $q$. From this it follows that

$$(\text{Hess} v_\alpha(b))(\dot{c}(t_0), \cdot) \neq 0.$$

\[\square\]

**Lemma 3.6.** There is a constant $\epsilon > 0$ such that $c((t_0, t_0 + \epsilon)) \subset M^{0, 0}$.

**Proof.** Let $S$ be the subspace of $T_qM$ spanned by $\dot{c}(t_0)$ and $I\dot{c}(t_0)$, and let $B$ be the image of the $\epsilon$-ball $\{V \in S \mid |V| < \epsilon\}$ in $S$ by the exponential mapping $\text{Exp}_q : T_qM \to M$. Then, it follows from the previous lemma that $h'_i \neq 0$ at $q'$ for every $i$ and $q' \in B - \{q\}$, provided $\epsilon$ small enough. It also follows from the proof of the previous lemma that if $q' \in U$ lies on the boundary of $U'$, then some $h'_i$ vanishes at $q'$. Hence we have $B - \{q\} \subset M^{1, 0}$. \[\square\]

**Proposition 3.7.** $M^{0, 0} \cap M = M^{1, 0}$, and it is dense in $M^{0, 0}$. Also, the functions $h_i$ and the subbundles $D_i$ extend smoothly to $M^{0, 0}$ and satisfies

$$
h_{s(\alpha)} > \cdots > h_{t(\alpha)} \ (\alpha \in A), \quad dh_i|_{D_j} = 0 \quad \text{if} \ i \neq j.
$$

Moreover, $D_i$ are the simultaneous eigenspaces of the endomorphisms $\{F^* \mid F \in F\}$ at each point in $M^{0, 0}$.

**Proof.** Let $N$ be the closure of $M^{1, 0}$ in $M^{0, 0}$, and assume that $N \neq M^{0, 0}$. Let $q \in N$ be a boundary point of $N$, and let $U$ be an open distance ball centered at $q$, which is small enough so that it is convex and contained in $M^{0, 0}$. Let $p_0$ and $q_0$ be two points in $U$ such that $p_0 \in M^{1, 0}$ and $q_0 \notin N$. Let $c(t)$ ($0 \leq t \leq T$) be the minimal geodesic from $p_0$ to $q_0$, and let $t = t_1$ be the earliest time when $c(t)$ meets the boundary of $N$.

Applying the argument above to the geodesic $c(t)$ and the point $p_1 = c(t_1)$, we see that there is $\epsilon > 0$ such that $c((t_1, t_1 + \epsilon)) \subset M^{1, 0}$. Iterating this procedure successively, we obtain a sequence of times $0 = t_0 < t_1 < t_2 < \cdots < T$ such that

$$
c(t_{b-1}) \subset M^{1, 0}, \quad c(t_b, t_{b+1}) \subset M^{1, 0} \quad (b \geq 1)
$$
We claim that the number of such $t_k$ is finite. In fact, suppose that it is not the case, and put $t_\infty = \lim_{k \to \infty} t_k \leq T$. Then, as seen above, any vector fields of the form
\[ Y = \text{sgrad} \, v_\alpha(b) \quad (\alpha \in \mathcal{A}, \, b \in \mathbb{R}) \]
are Jacobi fields along $c(t)$ ($0 \leq t \leq t_\infty$), and among them there is $Y_k$ that vanish at $c(t_k)$ for every $k \geq 1$. Let $\mathcal{F}$ be the vector space of Jacobi fields spanned by such $Y$. Since $\mathcal{F}$ coincides with $\mathcal{F}$ on $c([0,T]) \cap M^{1,0}$, it follows that it is $n$-dimensional, and satisfies
\[ g(Y, \nabla_c Y') = g(\nabla_c Y, Y') \quad Y, Y' \in \mathcal{F}. \]

From this property one can easily conclude that the set of points $t$ such that some non-zero $Y \in \mathcal{F}$ vanishes at $c(t)$ is discrete. On the other hand, choosing a subsequence if necessary, we obtain $Y_\infty \in \mathcal{F} - \{0\}$ as a limit of (constant multiples of) $\{Y_k\}$. Since $Y_\infty$ vanishes at $c(t_\infty)$, it is a contradiction. Hence there are only finitely many $t_k; \, t_1, \ldots, t_l$.

Now, again by Lemma 3.6 we see that $c((t_l, T)) \subset M^{1,0}$. However, this contradicts the fact that $q_0 = c(T) \notin N$. Thus we conclude that $N = M^{0,0}$, and $M^{0,0} \cap M^1 = M^{1,0}$. The remaining part is obvious. \(\square\)

Next, we shall prove that $M^0$ is connected and dense in $M$. Let $q$ be a boundary point of $M^{0,0}$, and assume that $u_\alpha(q) \neq 0$ for every $\alpha$. Then in the same way as above the functions $h_i$ are continuously extended to $M^{0,0} \cup \{q\}$. Also, $T_q M$ is decomposed to the simultaneous eigenspaces of the endomorphisms $\{F^e | F \in \mathcal{F}\}$. Clearly those subspaces are uniquely extended as $\mathcal{C}^\infty$ subbundles around $q$ so that they are sums of simultaneous eigenspaces at each point.

**Lemma 3.8.** Let $q \in M^s \cap \overline{M^{0,0}}$, and assume that $u_\alpha(q) \neq 0$ for any $\alpha$. Let $D$ be one of the simultaneous eigenspaces of $\{F^e | F \in \mathcal{F}\}$ at $q$, and let $\dim D = m$. Then there is $\alpha \in \mathcal{A}$ and its subset $\alpha'$ consisting of successive numbers $i, \ldots, i + m - 1$ such that the extended subbundle (also denoted by $D$) is equal with $\sum_{j \in \alpha'} D_j$ on $M^{0,0}$ near $q$. Moreover, there is a constant $c$ such that:

1. $h_i(q) = \cdots = h_{i+m-1}(q) = c, \quad h_j(q) \neq c \quad (j \in \alpha - \alpha')$;
2. The functions $S_k(h_j; j \in \alpha')$ on $M^{0,0}$ can be smoothly extended around $q$ $(1 \leq k \leq m)$;
3. $dv_\alpha(c) = 0$ at $q$ if $m \geq 2$;
4. The hessian of $v_\alpha(c)$ vanishes at $q$ if $m \geq 3$.

Also, there exists at least one simultaneous eigenspace $D$ with $\dim D \geq 2$.

**Proof.** Let $U$ be a neighborhood of $q$ where the subbundle $D$ is defined. Since $D$ is a sum of simultaneous eigenspaces at each point, it is a sum of $D_j$ on $U \cap M^{0,0}$. Let us consider the endomorphisms $F^e_{t(\alpha)}$. It is $|u_\alpha|$ times the identity on $\sum_{\beta \geq \alpha} D_\beta$, and 0 on the orthogonal complement. Therefore the subbundles $D_\alpha$ are continuously extended to $q$ so that $\sum_{\beta \geq \alpha} D_\beta$ is still the eigenspace of $F^e_{t(\alpha)}$ corresponding to the eigenvalue $|u_\alpha(q)|$. Since $D_\alpha$ is a sum of simultaneous eigenspaces at $q$, it is consequently extended on $U$ as the subbundle of $TM$. Clearly $D \subset D_\alpha$ for some $\alpha$. Also, the endomorphism $F^e_j$ at $q$ is a constant multiple of the identity on $D_\alpha$ if $j \notin \alpha$, and the eigenvalues of $F^e_j$ ($j \in \alpha$) on $D_\alpha$ are

\[ |u_\alpha(q)| S_{\beta \geq \alpha} (b \cdot (q); l \in \alpha - \{b\}) \quad (b \in \alpha). \]
at \( q \). This implies that there is a subset \( \alpha' \) of \( \alpha \) consisting of successive numbers \( i, \ldots, i + m - 1 \) such that

\[
D = \sum_{j \in \alpha'} D_j \quad \text{on } U \cap M^{0,0}
\]

\[
h_i(q) = \cdots = h_{i+m-1}(q) = c
\]

\[
h_j(q) \neq c \quad \text{for } j \in \alpha - \alpha'.
\]

To prove (2) we note that \( \sum_{j \in \alpha} h_j \) is extended as the \( C^\infty \) function around \( q \), because \( v_{t(\alpha)} \) and \( u_\alpha \) are of \( C^\infty \), and \( u_\alpha(q) \neq 0 \). Hence the symmetric functions of the eigenvalues of the endomorphisms

\[
u_\alpha^{-1} \left( \left( \sum_{j \in \alpha} h_j \right) F_{t(\alpha)}^e - F_{t(\alpha)-1}^e \right)
\]

on \( D \), which are

\[
S_l \left( \sum_{j \in \alpha' \setminus \{k\}} h_j; k \in \alpha' \right),
\]

on \( M^{0,0} \), are \( C^\infty \) functions around \( q \). This proves (2).

To prove the assertions (3) and (4), assume \( m \geq 2 \), and let \( G(\lambda) \) be the endomorphism of \( D_\alpha \) defined to be the restriction of

\[
F_{t(\alpha)-1}^e - \left( \sum_{j \in \alpha} h_j(q) - \lambda \right) F_{t(\alpha)}^e
\]

to \( D_\alpha \). Then \( \det G(\lambda) \) is a \( C^\infty \) function on \( U \). Since \( G(c) = 0 \) on \( D \) at \( q \), the order of the zero \( q \) of the function \( \det G(c) \) is not less than \( m \). On the other hand, we have

\[
\det G(\lambda) = |u_\alpha|^{|\alpha|} \prod_{j \in \alpha} \left( \sum_{k \in \alpha} h_k - \sum_{k \in \alpha} h_k(q) - (h_j - \lambda) \right)
\]

\[
= \epsilon^{|\alpha|} \sum_{l=0}^{|\alpha|} (-1)^l u_\alpha^l S_l(h_j - \lambda; j \in \alpha) \left( v_{t(\alpha)} - u_\alpha \sum_{k \in \alpha} h_k(q) \right)^{|\alpha|-l}
\]

on \( U \cap \overline{M^{0,0}} \), where \( \epsilon \) is the sign of \( u_\alpha \). Note that \( u_\alpha S_l(h_j - \lambda; j \in \alpha) \) is expressed as a linear combination of \( v_j \ (j \in \alpha) \) and \( u_\alpha \). Hence the last formula described above also expresses a \( C^\infty \) function on \( U \) that coincides with \( \det G(\lambda) \) on \( U \cap \overline{M^{0,0}} \). In particular those two functions coincides at \( q \) up to infinite order. Since \( d(\det G(c)) = 0 \) at \( q \), and since

\[
S_{|\alpha|-1}(h_j - c; j \in \alpha) = v_{t(\alpha)} - u_\alpha \sum_{k \in \alpha} h_k(q) = 0
\]

at \( q \), it follows that

\[
d(\alpha; S_{|\alpha|-1}(h_j - c; j \in \alpha)) = 0
\]
at \( q \). Moreover, if \( m \geq 3 \), considering the function \((d/d\lambda) \det G(\lambda)|_{\lambda=c}\), we also have

\[
d(u_\alpha S_{\alpha|-1}(h_j - c; j \in \alpha)) = 0
\]

at \( q \). Then observing the function \( \det G(c) \) again, we see that the hessian of the function

\[
u_\alpha S_{\alpha}(h_j - c; j \in \alpha) = \nu_\alpha(c)
\]

vanishes at \( q \).

Finally, assume that every simultaneous eigenspace is of dimension one. Then as is easily seen, the \( |\alpha| \) functions \( h_i \ (i \in \alpha) \) take mutually different values at \( q \) for any \( \alpha \). Hence the functions \( a_{ij} = a_i b_{ij} \) can be continuously extended to \( q \), which implies the linear independence of \( F_1, \ldots, F_n \) at \( q \). □

**Proposition 3.9.** \( M^0 \) is connected and dense in \( M \).

**Proof.** Assume that \( M^0 \) is not connected, and let \( p, p' \in M^0 \) such that \( p \in M^{1,0} \) and \( p' \) lies in another component. Let \( c(t) \ (0 \leq t \leq t_0) \) be a geodesic from \( p \) to \( p' \). A slight modification of the geodesic \( c \) and the point \( p' \) enables us to assume that \( D_i \)-component of \( \dot{c}(0) \in T_p M \) is not zero for any \( i \). Let \( t_1 \) be the time such that \( q = c(t_1) \in M^s \) and \( c(t) \in M^{0,0} \) for any \( t \in [0, t_1] \).

We first claim that every (extended) function \( u_\alpha \ (\alpha \in A) \) does not vanish at \( q \). In fact, assume that \( u_\alpha(q) = 0 \). Since

\[
\text{tr} F_{t(\alpha)} = u_\alpha \sum_{\beta > \alpha} |\beta|
\]

and since \( F_{t(\alpha)} \) is positive semi-definite on \( M^{0,0} \), it follows that \( F_{t(\alpha)} = 0 \) at \( q \). Let \( \zeta_t \) be the geodesic flow and \( \pi : T^*M \to M \) the bundle projection. Then \( c(t) = \pi(\zeta_t \lambda_0) \), where \( \lambda_0 \in T_p^*M \) is given by

\[
\lambda_0(X) = g(\dot{\zeta}(0), X) \quad X \in T_p M,
\]

and we have

\[
F_{t(\alpha)}(\lambda_0) = F_{t(\alpha)}(\zeta_t \lambda_0) = 0.
\]

This implies that \( \dot{c}(0) \) is orthogonal to \( D_j \) for any \( j \in \beta, \beta \geq \alpha \), a contradiction. Hence \( u_\alpha(q) \neq 0 \) for every \( \alpha \).

Therefore, as stated above, the functions \( h_i \) is extended up to \( q \), and Lemma 3.8 is applicable. Let \( D \) be a simultaneous eigenspace of the endomorphisms \( \{F_e \mid F \in \mathcal{F}\} \) at \( q \), and let \( \alpha \) and \( \alpha' \) be as in Lemma 3.8. Suppose \( m = \dim D \geq 2 \), and put \( h_j(q) = a \ (j \in \alpha') \). Then \( Y = \text{sgrad} \nu_\alpha(a) \) is the Jacobi field along the geodesic \( c_{[0,t_1]} \), and vanishes at \( q = c(t_1) \). Hence \( \nabla_{\dot{c}(t_1)} Y \neq 0 \), which implies that the hessian of \( \nu_\alpha \) at \( q \) does not vanish. Thus we have \( m = 2 \) by Lemma 3.8.

Put \( \mathcal{F}' = \{ F \in \mathcal{F} \mid F_q = 0 \} \), and let \( \dim \mathcal{F}' = k \). For each 2-dimensional simultaneous eigenspace \( D \), put

\[
H_D = F_\alpha(a) - \sum_{\beta \in \alpha} \epsilon_\beta (e_{\alpha\beta} + a)^{-1} F_{t(\beta)},
\]

where \( \epsilon_\beta \) is the sign of \( \prod_{l \in \alpha}(h_l + e_{\alpha\beta}) \). As is easily seen, the elements \( H_D \) form a basis of \( \mathcal{F}' \). For \( F \in \mathcal{F} \), let \( X_F \) be the vector field on \( T^*M \) defined by

\[
\frac{d}{dt} X_F = X_F
\]
Lemma 3.10. \( (1) \) \((X_F)_{c_{t_1}, \lambda_0} \neq 0 \) for any \( F \in \mathcal{F} - \{0\} \).

\( (2) \) For each 2-dimensional simultaneous eigenspace \( D \), there is a unit vector \( V_D \in D \) such that \( \Lambda \) is given by

\[ \Lambda = \{ \lambda \in S_q^* M \mid \lambda(V_D) = \lambda(IV_D) = 0 \text{ for some } D \} \]

In particular the complement of \( \Lambda \) in \( S_q^* M \) is connected and dense in \( S_q^* M \).

**Proof.** Describing

\[ F_p = \sum_j b_j(V_j^2 + (IV_j)^2), \]

we have

\[ \pi_*((X_F)_{\lambda_0}) = 2 \sum_j b_j(g(V_j, \dot{c}(0))V_j + g(IV_j, \dot{c}(0))IV_j). \]

Since the right-hand side does not vanish because of the assumption on \( \dot{c}(0) \), it follows that

\[ (X_F)_{c_{t_1}, \lambda_0} = c_{t_1}^*((X_F)_{\lambda_0}) \neq 0. \]

As is easily seen, the endomorphism \( H_D^c \) on the orthogonal complement \( D^\perp \) of \( D \) vanishes up to order 2 at \( q \). Hence, taking an orthonormal frame \( \tilde{V}_i, \tilde{I}\tilde{V}_i \) \((i = 1, 2)\) of \( D \) around \( q \), we see that \( X_{H_D}|_{T_q^* M} \) is of the form

\[ X_{H_D}|_{T_q^* M} = f_1\tilde{V}_1^* + f_2\tilde{V}_2^* + f_3(\tilde{I}\tilde{V}_1)^* + f_4(\tilde{I}\tilde{V}_2)^*, \]

where \( \tilde{V}_1^*, \ldots, (\tilde{I}\tilde{V}_2)^* \) are covectors (constant vector fields on \( T_q^* M \)) that vanish on \( D^\perp \), and dual to \( \tilde{V}_1, \ldots, \tilde{I}\tilde{V}_2 \); also, \( f_i \) are linear combinations of the hermitian forms

\[ \tilde{V}_i^2 + (\tilde{I}\tilde{V}_i)^2 \quad (i = 1, 2), \quad \tilde{V}_1\tilde{V}_2 + (\tilde{I}\tilde{V}_1)(\tilde{I}\tilde{V}_2), \quad \tilde{V}_1\tilde{I}\tilde{V}_2 - \tilde{V}_2\tilde{I}\tilde{V}_1. \]

Then, since

\[ 0 = X_{H_D}F_{\ell(\alpha)} = |u(\alpha)(q)|X_{H_D}\sum_{i=1}^2(\tilde{V}_i^2 + (\tilde{I}\tilde{V}_i)^2), \]

we can choose \( \tilde{V}_1 \) and \( \tilde{V}_2 \) so that

\[ aX_{H_D} = (\tilde{V}_1^2 + (\tilde{I}\tilde{V}_1)^2)\tilde{V}_2^* - (\tilde{V}_1\tilde{V}_2 + (\tilde{I}\tilde{V}_1)(\tilde{I}\tilde{V}_2))\tilde{V}_1^* \]

\[ + (\tilde{V}_1\tilde{I}\tilde{V}_2 - \tilde{V}_2\tilde{I}\tilde{V}_1)(\tilde{I}\tilde{V}_1)^*, \]

where \( a \) is a non-zero constant, and the covectors \( \tilde{V}_i^*, (\tilde{I}\tilde{V}_i)^* \) are identified with the constant vector fields on \( T_q^* M \). Hence, the zero set of \( X_{H_D} \) on \( T_q^* M \) is the vector subspace defined by \( \tilde{V}_1 = \tilde{I}\tilde{V}_1 = 0 \). By putting \( V_D = \tilde{V}_1 \), \( (2) \) follows. \( \Box \)

Let \( \Lambda_1 \) be the set of points \( \lambda \in S_q^* M \) such that

\[ \dim((X_F)_\lambda \mid F \in \mathcal{F}) < n. \]
Lemma 3.11.  (1) $S_q^* M - \Lambda_1$ is connected and dense in $S_q^* M$.

(2) For $\lambda \in S_q^* M - \Lambda_1$ the set of $t \in R$ such that $\pi(\zeta_t \lambda) \notin M^0$ is discrete. In particular there is a constant $\epsilon > 0$ such that $\pi(\zeta_t \lambda) \in M^{0,0}$ for $t$ satisfying $|t| < \epsilon, t \neq 0$.

Proof. We define $V_D$ for each 1-dimensional simultaneous eigenspace $D$ as a unit vector in $D$. Let $\Lambda_2$ be the set of $\lambda \in S_q^* M$ such that

$$\lambda(V_D) = \lambda(IV_D) = 0$$

for some simultaneous eigenspace $D$ (of dimension 1 or 2). Then the complement of $\Lambda_2$ in $S_q^* M$ is still connected and dense in $S_q^* M$. We prove that

$$S_q^* M - \Lambda_2 \subset S_q^* M - \Lambda_1,$$

which will indicate (1).

Let $\lambda \in S_q^* M - \Lambda_2$. Since $\lambda \notin \Lambda$, the previous lemma implies that $(X_F)_{\lambda}$ $(F \in F')$ form a $k$-dimensional subspace of $T_{\lambda}(T_q^* M)$. On the other hand, we know that $\{F_q \mid F \in F\}$ is $(n-k)$-dimensional, and spanned by

$$\tilde{V}_1^2 + \tilde{V}_2^2 + (I\tilde{V}_1)^2 + (I\tilde{V}_2)^2 \quad (D, 2\text{-dimensional}),$$

$$(V_D)^2 + (IV_D)^2 \quad (D, 1\text{-dimensional}),$$

where $\tilde{V}_1, \ldots, I\tilde{V}_2$ is an orthonormal basis of $D$. Hence, it follows that

$$\{\pi_*((X_F)_{\lambda}) \mid F \in F\}$$

is $(n-k)$-dimensional subspace of $T_q M$, provided $\lambda \notin \Lambda_2$. From these two facts we see that

$$\dim\{(X_F)_{\lambda} \mid F \in F\} = n \quad (\lambda \in S_q^* M - \Lambda_2).$$

Hence $\lambda \in S_q^* M - \Lambda_1$.

Now, let $\lambda \in S_q^* M - \Lambda_1$, and put $Z_F(t) = \pi_*((X_F)_{\zeta_t \lambda})$. Then, $Z_F(t)$ are Jacobi fields along the geodesic $c_1(t) = \pi(\zeta_t \lambda)$, and satisfy

$$g(Z_F, \nabla_{\zeta_t} Z_F) = g(\nabla_{\zeta_t} Z_F, Z_F) \quad (F, \tilde{F} \in F).$$

This implies that the set of $t$ at which $Z_F(t) = 0$ for some $F \in F - \{0\}$ is discrete. Since $F_{c_1(t)} = 0$ implies $Z_F(t) = 0$ for each $t$, it follows that $\pi(\zeta_t \lambda) \in M^0$ except discrete $t$'s. □

We now continue the proof of Proposition 3.9. The lemmas above imply that there are only finite number of $t$ on the interval $(0, t_0)$ such that $c(t) \notin M^0$. Let $t_1, \ldots, t_l$ ($t_1 < \cdots < t_l < t_0$) be those points. Then the previous lemma also implies that $c((t_1, t_2)) \subset M^{0,0}$. Since Lemmas 3.10 and 3.11 are still valid for the point $c(t_2)$, it also follows that $c((t_2, t_3)) \subset M^{0,0}$. Iterating this procedure successively, we consequently see that $p' = c(t_0) \in M^{0,0}$, which shows the connectedness of $M^0$.

The denseness of $M^0$ in $M$ is now clear, because any geodesic $c(t)$ emanating from the point $p$ meets $M^s$ only at discrete values of $t$, if every $D_t$-component of $\dot{c}(0)$ is nonzero. This completes the proof of Proposition 3.9. □
We have proved that \( M^0 \) and \( M^1 \) are connected and dense in \( M \), and \( u_\alpha \) does not vanish everywhere on \( M^0 \) for any \( \alpha \). Now we shall show that \( u_\alpha \neq 0 \) everywhere on \( M \). Assume that \( u_\alpha(q) = 0 \) for some \( \alpha \in A \) and \( q \in M^\circ \). Then \( du_\alpha = 0 \) at \( q \), because \( u_\alpha \) is either non-negative or non-negative on \( M \). Since the functions \( u_\beta \) are globally defined and smooth on \( M \), the vector fields sgrad \( u_\beta \) are globally defined infinitesimal automorphisms of the Kähler manifold \( M \). Therefore the connected component \( L \) of the set of zeros of sgrad \( u_\alpha \) containing \( q \) is a totally geodesic complex submanifold of \( M \) (see [8], for instance). Also, the tangent space of \( L \) at \( q \) coincides with the kernel of the hessian of \( u_\alpha \) at \( q \).

Let \( U \) be a small neighborhood of \( q \), and let \( p \in U \cap M^1 \) that does not belong to any such submanifolds \( L \) corresponding to the vector fields sgrad \( u_\beta \) vanishing at \( q \). Let \( c(t) \) be the minimal geodesic joining \( q \) and \( p \); \( c(0) = q \), \( c(t_0) = p \). Fix \( \alpha \) such that \( u_\alpha(q) = 0 \), and put

\[
v = v_{t(\alpha)} - \left( \sum_{i \in \alpha} h_i(p) \right) u_\alpha.
\]

Note that the function \( v \) is well-defined and smooth on the whole \( M \).

**Lemma 3.12.** Hess \( v(\dot{c}(0), \dot{c}(0)) = 0 \).

**Proof.** Taking \( U \) small enough, we may assume that \( u_\gamma(c(t)) \neq 0 \) for any \( \gamma \in A \). Hence, as we have seen before, there is an open and dense subset \( J \) of the interval \( (0, t_0] \) such that \( c(t) \in M^1 \) for \( t \in J \). Since \( F_{t(\alpha)} \) is semi-positive everywhere, and since \( tr F_{t(\alpha)} \) is a non-zero multiple of \( u_\alpha \), we have \( F_{t(\alpha)} = 0 \) at \( q \). Hence \( F_{t(\alpha)}(\zeta, \lambda) = 0 \) for any \( t \in \mathbb{R} \), \( c(t) = \pi(\zeta, \lambda) \), and this implies that

\[
\dot{c}(t) \in \sum_{\gamma \notin A} D_{\gamma} \quad (t \in J).
\]

Hence \( \sum_{i \in \alpha} h_i(c(t)) \) is constant on each connected component of \( J \). So, by continuity we have

\[
v(c(t)) = u_\alpha(c(t)) \left( \sum_{i \in \alpha} h_i(c(t)) - \sum_{i \in \alpha} h_i(p) \right) = 0
\]

for all \( t \in [0, t_0] \), and it therefore follows that Hess \( v(\dot{c}(0), \dot{c}(0)) = 0 \). \( \square \)

It is clear that a slight modification of the initial vector \( \dot{c}(0) \in T_qM \) does not affect the conclusion of Lemma 3.12. Thus we have Hess \( v = 0 \) at \( q \), which contradicts the fact that sgrad \( v \) is the non-trivial Killing vector field on \( M \). Hence it has been shown that \( u_\alpha \) does not vanish everywhere on \( M \) for every \( \alpha \).

We now prove the remaining part of Theorem 3.1. We have already shown that the subbundles \( D_i \) are well-defined and of \( C^\infty \) on \( M^0 \). Also, (1), (2), and (3) have been verified. Since the functions \( u_\alpha \) are everywhere non-zero, (4) and (6) also follows. Let \( q \) be a point in \( M - M^1 \). If \( c(t) \) is a geodesic passing through \( q \) and a point in \( M^1 \), then the set of \( t \in \mathbb{R} \) such that \( \{ Y_{c(t)} | Y \in T \} \) is not \( n \)-dimensional is discrete, as we have already seen. Hence Lemma 3.5 and the argument in the proof of Proposition 3.9 is applicable. Thus if \( q \in M^0 - M^1 \), there are \( i(\in \alpha) \) and \( a, b \in \mathbb{R}, b \neq 0 \) such that \( d(v_\alpha(a)) = 0 \) at \( q \) and

\[
\text{Hess } v_\alpha(a)(X, Y) = h_0(\alpha) \quad [X, Y] \quad X, Y \in T_q M
\]
Therefore the connected component \( L \) of the zeros of the infinitesimal automorphism \( \text{sgrad} v_\alpha(a) \) passing through \( q \) is the complex submanifold of codimension one.

Now let \( q \in M^s \). Then it follows that every simultaneous eigenspace of \( \{ F^c_q \mid F \in \mathcal{F} \} \) is of dimension one or two, and they are contained in some \( D_\alpha \). Hence (5) follows. Let \( D \) be a simultaneous eigenspace of dimension two, and let \( \alpha, \alpha' = \{ i, i + 1 \} \) and \( c \in R \) be as in Lemma 3.8. From the proof of Lemma 3.8, it easily follows that \( d(v_\alpha(c)) = 0 \) and \( dv \neq 0 \) at \( q \), where

\[
v = v_\alpha s_{|\alpha| - 1}(h_j - c; j \in \alpha)
\]

This implies that the exterior derivative of the function \( h_i + h_{i+1} \), which is of \( C^\infty \) around \( q \), does not vanish at \( q \). Since \( d((h_i - c)(h_{i+1} - c)) = 0 \) at \( q \), it follows that \( \text{grad} v \) is contained in the kernel of \( \text{Hess} v_\alpha \) at \( q \). Since \( dv_\alpha(c)(\text{grad} v) = 0 \) everywhere, it therefore follows that the connected component \( L \) of the set of zeros of \( \text{sgrad} v_\alpha(c) \) passing through \( q \) is also \((n - 1)\)-dimensional.

Thus \( M - M^1 \) is equal with the union of such hypersurfaces \( L \). The local finiteness is clear. This complete the proof of Theorem 3.1.

Also, we have just proved the following

**Proposition 3.13.** Let \( q \in M^s \). Then the simultaneous eigenspaces \( D \) of the linear endomorphisms \( \{ F^c_q \mid F \in \mathcal{F} \} \) are of dimension one or two. They are smoothly extended to a neighborhood \( U \) of \( q \) as the subbundles of \( TM \) in the following way: If \( \dim D = 1 \), then \( D = D_i \) on \( U \cap M^0 \) for some \( i \); if \( \dim D = 2 \), then there is \( \alpha \in \mathcal{A} \) and \( i, i + 1 \in \alpha \) such that \( D = D_i + D_{i+1} \) on \( U \cap M^0 \). In the first case, the function \( h_i \) is smooth around \( q \), and if \( i \in \alpha \),

\[
h_j(q) \neq h_i(q) \quad (j \in \alpha, j \neq i).
\]

In the second case,

\[
h_i(q) = h_{i+1}(q), \quad h_j(q) \neq h_i(q) \quad (j \in \alpha, j \neq i, i + 1),
\]

and the functions \( h_i \) and \( h_{i+1} \) are not differentiable at \( q \). Moreover, \( h_i + h_{i+1} \) and \( h_i h_{i+1} \) are smooth functions around \( q \) such that their exterior derivatives are non-zero and \( d((h_i - h_i(q))(h_{i+1} - h_i(q))) = 0 \) at \( q \).

**4. Torus action and the invariant hypersurfaces**

In the rest of the paper we shall assume that the Kähler-Liouville manifold \( M \) is compact. Let \( \mathfrak{k} \) be as before, and put

\[
\mathfrak{g} = \mathfrak{k} + I\mathfrak{k},
\]

which is a commutative Lie algebra of infinitesimal holomorphic transformations of \( M \). Let \( K \) and \( G \) be the Lie transformation group of \( M \) generated by \( \mathfrak{k} \) and \( \mathfrak{g} \) respectively. Note that \( \mathfrak{g} \) is naturally regarded as a complex Lie algebra. Accordingly, \( G \) is regarded as a complex Lie group so that the action \( G \times M \rightarrow M \) is holomorphic. In this section we shall investigate the properties of the action of those groups and the hypersurfaces contained in \( M - M^1 \).

We first prove the following
Proposition 4.1.  
(1) \( G \) preserves \( M^1 \) and each hypersurface \( L \subset M - M^1 \).
(2) The action of \( G \) on \( M^1 \) is simply transitive.

Proof. To prove (1), recall that \( L \) is a connected component of the set of zeros of \( \text{sgrad} \ v \in \mathfrak{k} \) for some \( v \in V \). Therefore, every \( Y \in \mathfrak{k} \) is tangent to \( L \). Since \( L \) is a complex submanifold, any \( Z \in \mathfrak{g} \) is also tangent to \( L \). Thus \( G \) preserves \( L \). Since \( M^1 \) is the complement of the union of such hypersurfaces, it is also preserved by \( G \). To prove (2), note that \( \{ Y_p \mid Y \in \mathfrak{g} \} \) is real \( 2n \)-dimensional for every point \( p \in M^1 \). Hence the \( G \)-action on \( M^1 \) is transitive. Suppose \( gp = p \) for some \( g \in G \) and \( p \in M^1 \). Then \( gg = q \) for every \( q \in M^1 \), because \( G \) is abelian. Hence \( g \) should be the identity transformation of \( M \) by continuity. \( \square \)

Theorem 4.2. The Lie group \( K \) is isomorphic to \( U(1)^n = U(1) \times \cdots \times U(1) \) \( (n \text{ times}) \), where \( U(1) \) is the group of unit complex numbers. Also, \( G \) is isomorphic to \( (\mathbb{C}^\times)^n \) as complex Lie group, where \( \mathbb{C}^\times \) is the multiplicative group of non-zero complex numbers.

To prove this theorem we need several lemmas. Let \( L \) be a complex hypersurface contained in \( M - M^1 \). As observed before, there is \( v_\alpha(c) \in V \) such that \( L \) coincides with a connected component of the set of zeros of \( \text{sgrad} \ v_\alpha(c) \), and \( v_\alpha(c) \) vanishes on \( L \). In this case we shall call \( v_\alpha(c) \) the function that determines \( L \).

Lemma 4.3. Let \( L \) and \( v_\alpha(c) \) be as above. Then the vector field \( \text{sgrad} \ v_\alpha(c) \) generates a circle action on \( M \).

Proof. Put \( Y = \text{sgrad} \ v_\alpha(c) \), and let \( \text{ad} Y \) be the linear endomorphism of \( T_p M \) \( (p \in L) \) given by the formula
\[
(\text{ad} Y)(X) = [Y, X] = -\nabla_X Y, \quad X \in T_p M.
\]
Clearly, \( \text{ad} Y \) preserves the normal space \( N_p L \). Since \( N_p L \) is complex 1-dimensional and \( Y \) is the infinitesimal isometry, there is \( a \in \mathbb{R} \) such that
\[
(\text{ad} Y)(X) = aIX \quad \text{for any} \quad X \in N_p L.
\]
We claim that \( a \) is independent of \( p \in L \). In fact, let \( Z \in T_p L \), and let \( X \) be a unit normal vector field along \( L \). Then
\[
-\nabla_Z (\nabla_X Y) = (Za)IX + aI\nabla_Z X.
\]
Since \( Y \) is a Killing vector field, we have
\[
\nabla_Z (\nabla_X Y) = \nabla_{\nabla_Z X} Y + \frac{1}{2} (\text{R}(Z, X)Y - \text{R}(Y, Z)X + \text{R}(X, Y)Z).
\]
Hence \( Za = 0 \), which shows that \( a \) is constant on \( L \).

The lemma is now clear, because the linear isotropy action of the 1-parameter group \( \phi_t \) generated by \( Y \) has the least period \( 2\pi/|a| \) at every point \( p \in L \), and so is \( \phi_t \) itself via the exponential mapping \( \text{Exp}: NL \to M \). \( \square \)
**Lemma 4.4.** Fix $\alpha \in A$, and take a constant $b$ so that $h_i + b > 0$ on $M$ for any $i \in \alpha$. Put

$$ v = u_{\alpha} \prod_{i \in \alpha} (h_i + b) \in \mathcal{V}, $$

and let $p \in M$ be a point where $|v|$ attains the maximum. Then the functions $h_j$ are smooth around $p$, and $dh_j = 0$ at $p$ for any $j$ such that $j \in \beta$, $\beta \preceq \alpha$.

**Proof.** If $p \in M^0$, then the assertion is clear, because $dh_i(D_j) = 0$ for any $i, j$ such that $i \neq j$. Similarly, if there is no 2-dimensional simultaneous eigenspace of $\{F_p^e | F \in F\}$ contained in $\sum_{\beta \preceq \alpha} D_{\beta}$, the assertion is also clear. Now, assume that $p \in M^s$ and there are $\beta \preceq \alpha$ and $i', i + 1 \in \beta$ such that $h_i(p) = h_{i+1}(p)$. Let $D$ be the subbundle of $TM$ defined on a neighborhood $U$ of $p$ that coincides with $D_i + D_{i+1}$ on $M^0 \cap U$. Then, putting

$$ X = \text{grad} \ (h_i + h_{i+1}), $$

we have $X \in D$ and $X \neq 0$ by virtue of Proposition 3.13. Since

$$ d((h_i - h_i(p))(h_{i+1} - h_i(p))) = 0 $$

at $p$, it therefore follows that

$$ d((h_i + b)(h_{i+1} + b))(X) \neq 0 \quad \text{at } p. $$

However, this implies that $dv(X) \neq 0$ at $p$, a contradiction. $\square$

**Proof of Theorem 4.2.** Fix $\alpha \in A$, and let $p \in M$ be as in Lemma 4.4. Clearly the function $v_{\alpha}(h_i(p)) \ (i \in \alpha)$ determines a hypersurface $L_i$ contained in $M - M^1$ that pass through $p$. Hence the vector fields

$$ Y_i = \text{sgrad} \ v_{\alpha}(h_i(p)) \quad (i \in \alpha) $$

generate circle actions on $M$ by Lemma 4.3. Executing the same procedure for all $\alpha$ we thus obtain $n$ vector fields $Y_i \in \mathfrak{g}$ each of which generates a circle action on $M$. As is easily seen, those vector fields form a basis of $\mathfrak{g}$. Hence $K$ is compact, and is isomorphic to $U(1)^n$.

Now, fix a point $q \in M^1$, and let $\phi_i : \mathfrak{g} \to \mathbb{R}$ be the mapping defined by

$$ \phi_i(Z) = v_{\alpha}(h_i(p))((\exp Z)q). $$

Put $\Phi = (\phi_i) : \mathfrak{g} \to \mathbb{R}^n$. Then we have $\Phi(Z + Y) = \Phi(Z)$ for any $Y \in \mathfrak{g}$ and $Z \in \mathfrak{g}$, and

$$ \frac{\partial}{\partial t_j} \phi_i(\sum_k t_k IY_k) = -g(Y_i, Y_j). $$

From this formula it easily follows that the inner product of two vectors

$$ \Phi(\sum_k t_k IY_k) - \Phi(\sum_k s_k IY_k) \quad \text{and} \quad t - s = (t_k - s_k) $$

in $\mathbb{R}^n$ is negative, provided $t \neq s$. Hence $\Phi|_{\mathfrak{g}}$ is injective.

These facts indicate that the kernel of the homomorphism $\exp : \mathfrak{g} \to G$ is equal to that of $\exp : \mathfrak{g} \to K$. This proves the latter half of the theorem. $\square$

Our next goal is to determine all the hypersurfaces contained in $M - M^1$. For this purpose we need deeper information on the fundamental functions $h_i$. Let $L$ be a hypersurface in $M - M^1$ determined by the function $v_{\alpha}(c) \in \mathcal{V}$. The following lemma is easy.
Lemma 4.5. Let \( p \in L \). Then there are neighborhoods \( W \) and \( U \) of \( p \) in \( L \) and \( M \) respectively, a neighborhood \( V \) of 0 in \( C = \{(z)\} \), and a holomorphic diffeomorphism \( \phi : W \times V \to U \) such that

1. \( \phi(q, 0) = q \) for any \( q \in W \),
2. \( \phi_* z \frac{\partial}{\partial z} = (2a)^{-1}(-iY + \sqrt{-1}Y) \),

where \( Y = \text{sgrad} \alpha(c) \) and \( a \) is the eigenvalue of \( I \circ \text{ad} Y \) on \( NL \).

Note that the hessian of \( \alpha(c) \) on the normal bundle \( NL \) is equal to \( a \) times the metric \( g \), \( a \) being the constant in Lemma 4.5. Hence there is a neighborhood \( U \) of \( L \) such that \( \alpha(c) > 0 \) (resp. \( < 0 \)) on \( U - L \) if \( a > 0 \) (resp. \( a < 0 \)).

Lemma 4.6. \( \alpha(c) > 0 \) (resp. \( < 0 \)) on \( M - L \) if \( a > 0 \) (resp. \( a < 0 \)).

Proof. Suppose \( a > 0 \). Let \( \psi_t \) be the one-parameter group of transformations generated by \( -\text{grad} \alpha(c) \). Then the previous lemma implies that \( \psi_t(q) \) converges to a point in \( L \) as \( t \to \infty \) for any \( q \in U \), provided \( U (\supset L) \) small enough. Now, fix \( q_0 \in U \), and let \( q \in M^1 \) be an arbitrary point. Then there is \( g \in G \) such that \( q = g q_0 \). Since \( \psi_t(q) = g \psi_t(q_0) \) and \( gL = L \), it follows that \( \psi_t(q) \in U \) for sufficiently large \( t \). Hence we have

\[
\alpha(c)(q) > \alpha(c)(\psi_t(q)) > 0.
\]

Let \( b > 0 \) be the minimal value of the function \( \alpha(c) \) on the boundary of \( U \), and let \( q' \) be a point in \( M - (M^1 \cup U) \). Then one can take a point \( q \in M^1 - U \) arbitrary near \( q' \). Since \( \alpha(c)(q) > b \), it follows that \( \alpha(c)(q') \geq b \), proving the lemma. The case \( a < 0 \) is similar.

Lemma 4.7. Let \( \alpha \in A \), and let \( i, i + 1 \in \alpha \). Then

\[
\min h_i \geq \max h_{i+1},
\]

where the minimum and the maximum are taken on \( M \).

Proof. Suppose that \( h_i \) takes its minimal value \( c \) at \( p \in M \). In this case we have \( h_{i-1}(p) > c \) if \( i - 1 \in \alpha \). In fact, if \( i - 1 \in \alpha \) and \( h_{i-1}(p) = c \), then \( c \) is also the minimal value of \( h_{i-1} \). Hence \( d(h_{i-1} + h_i) = 0 \) at \( p \), contradicting Proposition 3.13. Then, there are two cases: (1) \( h_{i+1}(p) < c \), or (2) \( h_i(p) = h_{i+1}(p) = c \). If the case (1) occurs, then the function \( h_i \) is smooth around \( p \). Hence the function \( \alpha(c) \) determines a hypersurface \( L \subset M - M^1 \). Since \( h_{i+1} \neq c \) on \( M - L \) by the previous lemma, we have \( \max h_{i+1} \leq c \).

Now, suppose that the case (2) occurs. Then by virtue of Proposition 3.13 the function \( \alpha(c) \) again determines a hypersurface \( L \subset M - M^1 \). Proposition 3.13 also implies that there is a point \( q \in L \) near \( p \) such that \( h_{i+1}(q) < c \) and \( h_i(q) = c \). Hence we again conclude that \( \max h_{i+1} \leq c \).

Proposition 4.8. A function in \( V \) of the form \( \alpha(c) \) determines a hypersurface \( L \) in \( M - M^1 \) if and only if \( c \) is the maximal or minimal value of \( h_i \) for some \( i \in \alpha \).

Proof. The “if” part has been indicated in the proof of the previous proposition. Now, suppose that a function \( \alpha(c) \) determines a hypersurface \( L \in M - M^1 \), and let \( p \in L \). Then there is \( i \in \alpha \) such that \( h_i(p) = c \). Since \( h_i \neq c \) on \( M - L \), \( c \) should be the maximal value or the minimal value of \( h_i \).
Corollary 4.9. Suppose that the function \( h_i \) is smooth around a point \( p \in M \), and \( dh_i = 0 \) at \( p \). Then \( h_i \) takes its maximum or minimum at \( p \).

Proof. The assumption implies that the function \( v_\alpha(h_i(p)) \) determines a hypersurface in \( M - M^1 \) (\( i \in \alpha \)). Thus the corollary follows from Proposition 4.8. \( \square \)

We now prove the following theorem.

Theorem 4.10. For any \( \alpha \in A \) and \( i, i + 1 \in \alpha \),

\[
\min h_i = \max h_{i+1}.
\]

The following corollary is an immediate consequence of Theorem 4.10 and Proposition 4.8.

Corollary 4.11. Put

\[
c_{\alpha,0} = \max h_{s(\alpha)}, \quad c_{\alpha,|\alpha|} = \min h_{t(\alpha)},
\]

\[
c_{\alpha,\nu} = \min h_{s(\alpha)+\nu-1} = \max h_{s(\alpha)+\nu} \quad (1 \leq \nu \leq |\alpha| - 1),
\]

and let \( L_{\alpha,\nu} \) be the hypersurface in \( M - M^1 \) determined by \( v_\alpha(c_{\alpha,\nu}) \). Then the hypersurfaces \( L_{\alpha,\nu} \) are mutually distinct, and the set

\[
\{ L_{\alpha,\nu} \mid \alpha \in A, \ 0 \leq \nu \leq |\alpha| \}
\]

coincides with the set of all closed hypersurfaces contained in \( M - M^1 \).

Let us recall that the fundamental functions \( h_i \) (\( i \in \alpha \)) and the conjunction constants \( e_{\alpha\beta} \) (\( \alpha \prec \beta \)) may be replaced with

\[
k_\alpha h_i - l_\alpha \quad (k_\alpha > 0) \quad \text{or} \quad k_\alpha h_{t(\alpha)+s(\alpha)-i} - l_\alpha \quad (k_\alpha < 0)
\]

and \( k_\alpha e_{\alpha\beta} + l_\alpha \) respectively, where \( k_\alpha \neq 0 \) and \( l_\alpha \) are constants. Hence it is always possible to choose \( h_i \) and \( e_{\alpha\beta} \) so that

\[
1 = c_{\alpha,0} > c_{\alpha,1} > \cdots > c_{\alpha,|\alpha|} = 0.
\]

In this case we also have

\[
e_{\alpha\beta} > 0 \quad \text{or} \quad e_{\alpha\beta} < -1.
\]

Under this condition the only possible alternative choice of \( h_i \) (\( i \in \alpha \)) and \( e_{\alpha\beta} \) are given by

\[
h_i' = 1 - h_{t(\alpha)+s(\alpha)-i} \quad (i \in \alpha), \quad e_{\alpha\beta}' = -1 - e_{\alpha\beta}.
\]

In the rest of the paper we shall always assume that the fundamental functions \( \{h_i\} \) and the conjunction constants \( \{e_{\alpha\beta}\} \) are chosen so that the condition (4.1) is satisfied. Also, we shall call \( \{c_{\alpha,\nu}\} \) the fundamental constants.

Proof of Theorem 4.10. Assume that \( i, i + 1 \in \alpha \) and

\[
\min h_i = c_2 > c_3 = \max h_{i+1},
\]

and put \( c_1 = \max h_i, \ c_4 = \min h_{i+1} \). Let \( L_\mu \) be the hypersurface in \( M - M^1 \) determined by \( v_\alpha(c_\mu) \), and put

\[
Y_\mu = \text{sgrad} \ v_\alpha(c_\mu) \quad (\mu = 1, \ldots, 4).
\]

Also, put

\[
b_j = \begin{cases} 
\max h_j & (j < i) \\
\min h_j & (j > i + 1)
\end{cases}
\]

for \( i \in \alpha, i \neq i, i + 1 \).
Lemma 4.12. There are four points \( p_{13}, p_{14}, p_{23}, p_{24} \) such that

1. \( p_{\mu\nu} \in L_\mu \cap L_\nu \quad (\mu = 1, 2, \nu = 3, 4) \),
2. \( h_j \) is smooth and \( dh_j = 0 \) at the four points for every \( \beta \leq \alpha \) and every \( j \in \beta \),
3. \( h_j = b_j \) at the four points for every \( j \in \alpha, j \neq i, i + 1 \).

Proof. First we show that \( L_\mu \cap L_\nu \neq \emptyset \ (\mu = 1, 2, \nu = 3, 4) \). Let \( b \) be the maximal value of the function \( h_i \) on the hypersurface \( L_3 \), and suppose \( h_i(q) = b, \ q \in L_3 \). Then the similar argument as the proof of Lemma 4.4 implies that the function \( h_i \) is smooth and \( dh_i = 0 \) at \( q \). Hence \( b = c_1 \) or \( c_2 \) by Corollary 4.9. Since the case \( b = c_2 \) contradicts the choice of \( q \), we have \( b = c_1 \). Thus \( L_1 \cap L_3 \neq \emptyset \). The other cases are similar.

Now, choose a constant \( d \) such that \( c_2 > d > c_3 \), and let \( p_{\mu\nu} \in L_\mu \cap L_\nu \) be a point where the function \( |v_\alpha(d)| \), restricted to \( L_\mu \cap L_\nu \), takes the maximal value \( (\mu = 1, 2, \nu = 3, 4) \). Then the similar argument as above clearly indicates that the conditions (2) and (3) are satisfied. \( \square \)

Let us consider the following identity (the decomposition to linear fractions):

\[
\frac{v_\alpha(\lambda)}{\prod_{j \in \alpha'} (b_j - \lambda) \prod_{\mu = 1}^3 (c_\mu - \lambda)} = \sum_{j \in \alpha'} \frac{1}{b_j - \lambda} \frac{v_\alpha(b_j)}{\prod_{k \in \alpha', k \neq j} (b_k - b_j) \prod_{\mu = 1}^3 (c_\mu - b_j)} + \sum_{\mu = 1}^3 \frac{1}{c_\mu - \lambda} \frac{v_\alpha(c_\mu)}{\prod_{j \in \alpha'} (b_j - c_\mu) \prod_{1 \leq \nu \leq 3} (c_\nu - c_\mu)},
\]

(4.3)

where \( \alpha' = \alpha - \{i, i + 1\} \). Multiplying both sides by \( -\lambda \) and taking the limit \( \lambda \to \infty \), we obtain the identity:

\[
u_\alpha = \sum_{j \in \alpha'} \frac{v_\alpha(b_j)}{\prod_{k \in \alpha', k \neq j} (b_k - b_j) \prod_{\mu = 1}^3 (c_\mu - b_j)} + \sum_{\mu = 1}^3 \frac{v_\alpha(c_\mu)}{\prod_{j \in \alpha'} (b_j - c_\mu) \prod_{1 \leq \nu \leq 3} (c_\nu - c_\mu)}.
\]

This formula gives the linear dependence of the skew-gradient vector fields \( Y_1, Y_2, Y_3, \) \( \text{sgrad} \ v_\alpha(b_j) \ (j \in \alpha') \), and \( \text{sgrad} \ u_\alpha \). Let \( d_\mu \) be the positive number so that the least period of the one-parameter group generated by \( Y_\mu \) is \( 2\pi/d_\mu \ (\mu = 1, \ldots, 4) \).

We now consider the linear isotropy action of the one-parameter groups generated by those vector fields at the point \( p_{13} \) (Note that those vector fields vanish at \( p_{13} \)). There we have the decomposition

\[
T_{p_{13}}M = N_{p_{13}}L_1 \oplus N_{p_{13}}L_3 \oplus T_{p_{13}}(L_1 \cap L_3).
\]

Clearly, the linear isotropy action of the one-parameter groups generated by \( Y_3, \) \( \text{sgrad} \ u_\alpha \), and \( \text{sgrad} \ v_\alpha(b_j) \ (j \in \alpha') \) are trivial on \( N_{p_{13}}L_1 \). Hence the linear endomorphism

\[
\exp \left( \frac{tY_2}{\prod_{1 \leq \nu \leq 3} (c_\nu - c_2)} \right) \exp \left( \frac{tY_1}{\prod_{1 \leq \mu \leq 1} (c_\mu - c_1)} \right)
\]

(4.4)
of $N_{p_1} L_1$ is trivial for all $t \in \mathbb{R}$. Substituting
\[
t = 2\pi d_2^{-1}(c_1 - c_2)(c_3 - c_2) \prod_{j \in \alpha'} (b_j - c_2)
\]
in (4.4), we conclude that
\[
(4.5) \quad 2\pi d_2^{-1}(c_1 - c_2)(c_3 - c_2) \prod_{j \in \alpha'} (b_j - c_2) = m \cdot 2\pi d_1^{-1}(c_2 - c_1)(c_3 - c_1) \prod_{j \in \alpha'} (b_j - c_1).
\]

The similar argument at the point $p_{23}$ gives the formula
\[
2\pi d_1^{-1}(c_2 - c_1)(c_3 - c_1) \prod_{j \in \alpha'} (b_j - c_1) = m' \cdot 2\pi d_2^{-1}(c_2 - c_1)(c_3 - c_2) \prod_{j \in \alpha'} (b_j - c_2)
\]
for some $n' \in \mathbb{Z}$. Therefore $m = \pm 1$. Replacing $c_3$ by $c_4$ in the formula (4.3), and considering the linear isotropy actions at $p_{14}$ and $p_{24}$, we also obtain the formula
\[
(4.6) \quad 2\pi d_2^{-1}(c_1 - c_2)(c_4 - c_2) \prod_{j \in \alpha'} (b_j - c_2) = \pm 2\pi d_1^{-1}(c_2 - c_1)(c_4 - c_1) \prod_{j \in \alpha'} (b_j - c_1).
\]

From (4.5) and (4.6) we have
\[
(c_3 - c_2)(c_4 - c_1) = \pm (c_4 - c_2)(c_3 - c_1),
\]
which contradicts the inequality; $c_1 > c_2 > c_3 > c_4$. This completes the proof of Theorem 4.10. \[\square\]

In the rest of the section we shall observe the detail of the action of $G$ on $M$. In particular we shall show that $M$ is a toric variety. Let $c_{\alpha,\nu}$ and $L_{\alpha,\nu}$ be as in Corollary 4.11, and put
\[
\mathcal{J} = \{(\alpha, \nu) \mid \alpha \in \mathcal{A}, \ 0 \leq \nu \leq |\alpha|\}.
\]

Let $d_{\alpha,\nu}$ be the non-zero constant such that
\[
\text{Hess } v_{\alpha}(c_{\alpha,\nu})(X,X) = d_{\alpha,\nu} g(X,X) \quad (X \in NL_{\alpha,\nu}),
\]
and put
\[
Y_{\alpha,\nu} = d_{\alpha,\nu}^{-1} \text{sggrad } v_{\alpha}(c_{\alpha,\nu}) \in \mathfrak{k}.
\]

Also, let $\mathcal{I}$ be the set of sections of the mapping $\mathcal{J} \to \mathcal{A}$ ($(\alpha, \nu) \mapsto \alpha$), that is, $\iota \in \mathcal{I}$ is an assignment of an index $\iota(\alpha)$ ($0 \leq \iota(\alpha) \leq |\alpha|$) to each $\alpha \in \mathcal{A}$. Put
\[
\mathcal{J}(\iota) = \{(\alpha, \nu) \in \mathcal{J} \mid \nu \neq \iota(\alpha)\} \quad (\iota \in \mathcal{I}).
\]
Lemma 4.13. (1) For any $\alpha$, $\bigcap_{\nu=0}^{\alpha} L_{\alpha,\nu} = \emptyset$.

(2) For any $i \in I$, $\bigcap_{(\alpha,\nu) \in J(i)} L_{\alpha,\nu} \neq \emptyset$.

Proof. (1) Let $p \in \bigcap_{\nu=0}^{\alpha} L_{\alpha,\nu}$. First, since $p \in L_{\alpha,0}$, it follows that $h_{s(\alpha)}(p) = c_{\alpha,0}$. Next, the condition that $p \in L_{\alpha,1}$ implies that $h_{s(\alpha)+1}(p) = c_{\alpha,1}$, and so on. Consequently, we have $h_{s(\alpha)+\nu}(p) = c_{\alpha,\nu}$ ($0 \leq \nu \leq |\alpha| - 1$) from the condition that $p \in \bigcap_{\nu=0}^{\alpha-1} L_{\alpha,\nu}$. However, since $p \in L_{\alpha,|\alpha|}$, we also have $h_{t(\alpha)}(p) = c_{\alpha,|\alpha|}$, a contradiction.

(2) Put

$$v = \prod_{\alpha \in A} \prod_{i \in \alpha} (h_i - c_{\alpha,i(\alpha)}),$$

and let $p \in M$ be a point where the function $|v|$ takes the maximum. Then, as in the proof of Lemma 4.4, we see that $p \in M^0$ and $dh_i = 0$ at $p$ for every $i$. Hence

$$\{h_i(p) \mid i \in \alpha\} = \{c_{\alpha,\nu} \mid 0 \leq \nu \leq |\alpha|, \nu \neq i(\alpha)\}.$$

This indicates that $p \in L_{\alpha,\nu}$ for every $(\alpha,\nu) \in J(i)$. $\Box$

Let $U$ be the neighborhood of $L_{\alpha,\nu}$ given in the proof of Lemma 4.6, and put

$$U_{\alpha,\nu} = \bigcup_{t \in \mathbb{R}} \psi_t(U),$$

where $\psi_t$ is the one-parameter group generated by $-IY_{\alpha,\nu}$. It is clear from Lemma 4.6 that $U_{\alpha,\nu}$ is $G$-invariant, and $\psi_t(q)$ converges to a point in $L_{\alpha,\nu}$ as $t \to -\infty$ for any $q \in U_{\alpha,\nu}$. Define the mapping $\rho_{\alpha,\nu} : U_{\alpha,\nu} \to L_{\alpha,\nu}$ by

$$\rho_{\alpha,\nu}(q) = \lim_{t \to -\infty} \psi_t(q).$$

Proposition 4.14. (1) $U_{\alpha,\nu} = M - \cap_{0 \leq \mu \leq |\alpha|} L_{\alpha,\nu}$.

(2) $\rho_{\alpha,\nu} : U_{\alpha,\nu} \to L_{\alpha,\nu}$ is the holomorphic fibre bundle with typical fibre $C$. Also, $\rho_{\alpha,\nu} : U_{\alpha,\nu} - L_{\alpha,\nu} \to L_{\alpha,\nu}$ is the principal $C^\times$-bundle.

Proof. Put

$$S = \cap_{0 \leq \mu \leq |\alpha|} L_{\alpha,\nu}.$$

Since $S$ is $G$-invariant, and $S \cap L_{\alpha,\nu} = \emptyset$, it follows that $U_{\alpha,\nu} \subset M - S$. Now, we shall show the reversed inclusion. Put

$$v = \prod_{i \in \alpha} (h_i - c_{\alpha,\nu}) = u_{\alpha}^{-1} v_{\alpha}(c_{\alpha,\nu}).$$

Then, the function $|v|$ is positive on $M - L_{\alpha,\nu}$ and takes its maximal value

$$\prod_{0 \leq \mu \leq |\alpha|} |c_{\alpha,\mu} - c_{\alpha,\nu}|$$

on $S$. Also, we have

$$\frac{d}{dt} |v(\psi_t(q))| = |\text{grad} v|^2 \cdot \frac{d_{\alpha,i} v_{\alpha}(c_{\alpha,\nu})}{|v|} > 0$$

for every $q \in U_{\alpha,\nu} - L_{\alpha,\nu}$ and $t \in \mathbb{R}$.

Let $q \in M - S$. Let $p_j = \psi_{t_j}(q)$ ($0 \geq t_j \to -\infty$) be a converging sequence, and let $p \in M$ be its limit point. Since $|v(p)| < |v(q)|$, it follows that $p \in M - S$. Also, we have $(d|v|)_p = 0$. As is easily seen, the set of critical points of the function $|v|$ is equal to $L_{\alpha,\nu} \cup S$. Hence it follows that $p \in L_{\alpha,\nu}$. This implies that $\psi_{-t}(q) \in U$ for sufficiently large $t$. Thus $q \in U_{\alpha,\nu}$, completing the proof of (1). (2) is the immediate consequence of Lemma 4.5. $\Box$
Proposition 4.15. Let $J_0$ be a subset of $J$ such that $\cap_{(\alpha,\nu) \in J_0} L_{\alpha,\nu} \neq \emptyset$. Let $\rho_0$ be the composition of all $\rho_{\alpha,\nu}$, $(\alpha,\nu) \in J_0$, and put

$$S_0 = \bigcup_{\alpha \in A} \cap_{0 \leq \nu \leq |\alpha|} L_{\alpha,\nu}.$$ \vspace{5pt}

Then the mapping $\rho_0 : M - S_0 \to \cap_{(\alpha,\nu) \in J_0} L_{\alpha,\nu}$ is well-defined, and is a fibre bundle with typical fibre $C^k$, where $k = |J_0|$. In particular, $\cap_{(\alpha,\nu) \in J_0} L_{\alpha,\nu}$ is connected.

Proof. We shall prove the proposition by induction on $k$. Let $J_0$, $\rho_0$, and $S_0$ be as in the statement. Suppose that $(\beta, \mu) \notin J_0$, and put

$$J_1 = J_0 \cup \{(\beta, \mu)\}.$$ \vspace{5pt}

We assume that $\cap_{(\alpha,\nu) \in J_1} L_{\alpha,\nu} \neq \emptyset$. $\rho_1$ and $S_1$ are similarly defined. We then have the following commutative diagram:

$$\begin{array}{ccc}
M - S_1 & \xrightarrow{\rho_0} & \cap_{(\alpha,\nu) \in J_0} L_{\alpha,\nu} - \cap_{\nu \neq \mu} L_{\beta,\nu} \\
\rho_{\beta,\mu} \downarrow & & \downarrow \rho_{\beta,\mu} \\
L_{\beta,\mu} - S_0 & \xrightarrow{\rho_0} & \cap_{(\alpha,\nu) \in J_1} L_{\alpha,\nu}
\end{array}$$

Hence $\rho_1 = \rho_0 \circ \rho_{\beta,\mu}$ is well-defined. From the induction assumption and the previous proposition, the rows and the columns in the diagram are fibre bundles with fibre $C^k$ and $C$ respectively. Let $q \in M - S_1$, and let $p = \rho_1(q)$. Then, there is a neighborhood $U$ of $p$ in $\cap_{(\alpha,\nu) \in J_1} L_{\alpha,\nu}$ such that $\rho_{\beta,\mu}^{-1}(U)$ and $\rho_0^{-1}(U)$ are isomorphic to $U \times C$ and $U \times C^k$ respectively. Moreover, for each $r \in \rho_0^{-1}(p)$ the mapping

$$\rho_0 : \rho_{\beta,\mu}^{-1}(r) \to \rho_{\beta,\mu}^{-1}(p)$$

is an isomorphism, because it commutes with the $C^\times$-action generated by $Y_{\beta,\mu}$ and $IY_{\beta,\mu}$. Therefore the mapping

$$\rho_1^{-1}(U) \to U \times C \times C^k$$

defined by $\rho_0$ and $\rho_{\beta,\mu}$ is isomorphic, and it gives the local triviality of $\rho_1$. $\square$

The following corollary is an immediate consequence of Proposition 4.15.

Corollary 4.16. Let $i \in I$, and fix a point $p_0 \in M^1$. Then there is a $G$-equivariant holomorphic isomorphism from $M - \cup_{\alpha \in A} L_{\alpha,i(\alpha)}$ to $C^n = \{(z^{(i)}_{\alpha,\nu}; (\alpha,\nu) \in J(i))\}$ such that $p_0$ corresponds to the point given by $z^{(i)}_{\alpha,\nu} = 1$ for every $(\alpha,\nu) \in J(i)$. Here the $G$-action on $C^n$ is given by

$$\exp \left( \sum_{(\beta,\mu) \in J(i)} (-t_{\beta,\mu} IY_{\beta,\mu} + s_{\beta,\mu} Y_{\beta,\mu}) \right) (z^{(i)}_{\alpha,\nu}) = \left( e^{t_{\alpha,\nu} + \sqrt{-1}s_{\alpha,\nu}} z^{(i)}_{\alpha,\nu} \right).$$

Let $\Gamma$ be the lattice in $\mathfrak{k}$ such that $2\pi \Gamma$ is equal to the kernel of the homomorphism $exp: \mathfrak{k} \to K$ of the abelian groups. Clearly $Y_{\alpha,\nu} \in \Gamma$ for every $(\alpha,\nu) \in J$. 

Proposition 4.17. For any \( \iota \in I \), the elements \( Y_{\alpha,\nu} \) \((\alpha, \nu) \in J(\iota)\) form a \(Z\)-basis of \( \Gamma \).

Proof. Fix \( \iota \in I \). Then, by virtue of Lemma 4.13 (2) there is a point \( p \in L_{\alpha,\nu} \) for every \((\alpha, \nu) \in J(\iota)\). Since the associated endomorphisms \( \text{ad} Y_{\alpha,\nu} \) of \( T_p M \) are linearly independent, it follows that \( Y_{\alpha,\nu} \) \((\alpha, \nu) \in J(\iota)\) form a \(R\)-basis of \( \kappa \). Let \( Y \) be any element of \( \Gamma \), and let

\[
Y = \sum_{(\alpha, \nu) \in J(\iota)} a_{\alpha, \nu} Y_{\alpha, \nu}, \quad a_{\alpha, \nu} \in R.
\]

We recall that the linear isotropy action of the one-parameter group \( \exp(tY_{\alpha,\nu}) \) on \( N_p L_{\alpha',\nu'} \) is trivial if \((\alpha, \nu) \neq (\alpha', \nu')\), and has the least period \(2\pi\) if \((\alpha, \nu) = (\alpha', \nu')\). Since \( \exp(2\pi Y) \) is the identity, it thus follows that \( a_{\alpha, \nu} \in Z \) for all \((\alpha, \nu) \in J(\iota)\).

\(\square\)

Theorem 4.18. \( M \) is a toric variety with respect to the action of \( G \).

Proof. By virtue of Corollary 4.16, \( M \) is covered by the open sets \( U_{\iota} = M - \bigcup_{\alpha \in A} L_{\alpha, \iota(\alpha)} \) \((\iota \in I)\) each of which is holomorphically isomorphic to \( C^n \). As is easily seen, the coordinate change on \( U_{\iota} \cap U_{\iota'} \) is given by Laurent monomials whose exponents are equal to the coefficients of the base change of \( \Gamma \): \( Y_{\alpha,\nu} \) \((\alpha, \nu) \in J(\iota)\) to \( Y_{\alpha,\nu} \) \((\alpha, \nu) \in J(\iota')\). Hence \( M \) is an algebraic variety. Corollary 4.16 also indicates that the action of the “algebraic torus” \( G \) on \( M \) is algebraic.

\(\square\)

The next several propositions will give the information on the structure of the toric variety \( M \).

Proposition 4.19. For each \( \alpha \in A \) the value

\[
d_{\alpha,\nu} \prod_{0 \leq \mu \leq |\alpha|, \mu \neq \nu} (c_{\alpha,\mu} - c_{\alpha,\nu})
\]

does not depend on \( \nu \) \((0 \leq \nu \leq |\alpha|)\).

Proof. In the same way as the proof of Lemma 4.12 we have

\[
u_{\alpha} = \sum_{\nu=0}^{|\alpha|} \frac{v_\alpha(c_{\alpha,\nu})}{\prod_{0 \leq \mu \leq |\alpha|, \mu \neq \nu} (c_{\alpha,\mu} - c_{\alpha,\nu})}.
\]

Taking the skew gradient vector fields of both sides, we then have

\[
sgrad u_\alpha = \sum_{\nu=0}^{|\alpha|} \frac{d_{\alpha,\nu}}{\prod_{0 \leq \mu \leq |\alpha|, \mu \neq \nu} (c_{\alpha,\mu} - c_{\alpha,\nu})} Y_{\alpha,\nu}.
\]

(4.7)

We claim here that \( sgrad u_\alpha \) is written as a linear combination of \( Y_{\gamma,\nu} \) \((\gamma \prec \alpha, 1 \leq \nu \leq |\gamma|)\). In fact, it is clear from the definition of \( u_\alpha \) that \( sgrad u_\alpha \) is written as a linear combination of \( Y_{\beta,\nu} \) \((0 \leq \nu \leq |\beta|)\), where \( \beta \) is the maximal element of the totally ordered set \( \{ \gamma \in A \mid \gamma \prec \alpha \} \). Hence, by the formula (4.7) (replaced \( \alpha \) with \( \beta \)) \( sgrad u_\alpha \) is written as a linear combination of \( sgrad u_\beta \) and \( Y_{\beta,\nu} \) \((1 \leq \nu \leq |\beta|)\). Thus the claim follows by induction on \( \beta \).
Hence, it has been shown that each \( Y_{\alpha,\nu} \) is written as a linear combination of \( Y_{\alpha,\mu} \) (\( \mu \neq \nu \)), \( Y_{\gamma,\mu} \) (\( \gamma \prec \alpha, 1 \leq \mu \leq |\gamma| \)), which are part of a basis of \( \Gamma \). Since \( Y_{\alpha,\nu} \) is a primitive element, i.e., there is no integer \( m > 1 \) such that \( m^{-1}Y_{\alpha,\nu} \in \Gamma \), the coefficients are integers. This being true for every \( \nu \), we have

\[
\frac{d_{\alpha,\nu}}{\prod_{0 \leq \mu \leq |\alpha|} (c_{\alpha,\mu} - c_{\alpha,\nu})} = \pm \frac{d_{\alpha,\nu'}}{\prod_{0 \leq \mu \leq |\alpha|} (c_{\alpha,\mu} - c_{\alpha,\nu'})}
\]

for any \( \nu \) and \( \nu' \).

Note the sign of \( d_{\alpha,\nu} \) is equal to the sign of the function \( v_\alpha(c_{\alpha,\nu}) \) on \( M - L_{\alpha,\nu} \). This implies that the sign of

\[
\frac{d_{\alpha,\nu}}{\prod_{0 \leq \mu \leq |\alpha|} (c_{\alpha,\mu} - c_{\alpha,\nu})}
\]

is equal to the sign of \( u_\alpha \). In particular it does not depend on \( \nu \). This completes the proof of the proposition.

We put

\[
d_\alpha = \frac{d_{\alpha,\nu}}{\prod_{0 \leq \mu \leq |\alpha|} (c_{\alpha,\mu} - c_{\alpha,\nu})} \quad (0 \leq \nu \leq |\alpha|).
\]

Then, \( d_\alpha^{-1} \text{sgrad} u_\alpha = \sum_{\nu=0}^{|\alpha|} Y_{\alpha,\nu} \in \Gamma \). Put

\[
Z_\alpha = \sum_{\nu=0}^{|\alpha|} Y_{\alpha,\nu} \in \Gamma
\]

Note that \( Z_\alpha = 0 \) if \( \alpha \) is minimal. We shall call \( d_\alpha \) (\( \alpha \in A \)) the scaling constants.

For convenience we shall use two symbols \( p(\alpha) \) and \( n(\alpha) \): For each non-minimal \( \alpha \in A \), \( p(\alpha) \) denotes the maximal element of the totally ordered subset \( \{ \gamma \in A \mid \gamma \prec \alpha \} \); for each non-maximal \( \alpha \in A \), \( n(\alpha) \) denotes the subset of \( A \) defined by

\[
n(\alpha) = \{ \gamma \in A \mid p(\gamma) = \alpha \}
\]

(\( n(\alpha) \) is identical with the one defined before). For non-minimal elements \( \alpha \in A \) we define constants \( m_{\alpha,\nu} \) (\( 0 \leq \nu \leq |p(\alpha)| \)); putting \( \beta = p(\alpha) \),

(4.8)

\[
m_{\alpha,\nu} = \frac{d_\beta}{d_\alpha} \prod_{0 \leq \mu \leq |\beta|} (c_{\beta,\mu} + e_{\beta,\alpha}).
\]

**Proposition 4.20.**

\[
Z_\alpha = m_{\alpha,0}Z_{p(\alpha)} + \sum_{\nu=1}^{|p(\alpha)|} (m_{\alpha,\nu} - m_{\alpha,0})Y_{p(\alpha),\nu}
\]

\[
= \sum_{\beta \text{non-minimal}} \left( \prod_{\beta \prec \alpha} m_{\gamma,0} \right)^{|p(\beta)|} \sum_{\nu=1}^{|p(\beta)|} (m_{\beta,\nu} - m_{\beta,0})Y_{p(\beta),\nu}.
\]
Proof. Putting \( \lambda = -e_{\beta \alpha} \) in the identity

\[
\frac{v_{\beta}(\lambda)}{\prod_{\nu=0}^{\beta} (c_{\beta,\nu} - \lambda)} = \sum_{\nu=0}^{\beta} \frac{1}{c_{\beta,\nu} - \lambda} \frac{v_{\beta}(c_{\beta,\nu})}{\prod_{\mu \neq \nu} (c_{\beta,\mu} - c_{\beta,\nu})},
\]

and taking the skew gradient of both sides, we have

\[
Z_\alpha = \left( \frac{d_{\beta}}{d_{\alpha}} \prod_{1 \leq \mu \leq |\beta|} (c_{\beta,\mu} + e_{\beta \alpha}) \right) Z_\beta
\]

\[
+ \sum_{\nu=1}^{\beta} \left( \frac{d_{\beta}}{d_{\alpha}} \prod_{0 \leq \mu \leq |\beta| \atop \mu \neq \nu} (c_{\beta,\mu} + e_{\beta \alpha}) - \frac{d_{\beta}}{d_{\alpha}} \prod_{1 \leq \mu \leq |\beta|} (c_{\beta,\mu} + e_{\beta \alpha}) \right) Y_{\beta,\nu}.
\]

This proves the first equality. The second one is immediate. \( \square \)

**Proposition 4.21.** The constants \( m_{\alpha,\nu} (\alpha, \text{not minimal}, 0 \leq \nu \leq p(\alpha)) \) possess the following properties.

1. \( m_{\alpha,\nu} - m_{\alpha,0} \in \mathbb{Z} \).
2. \( m_{\alpha,\nu} \in \mathbb{Q} \) if \( p(\alpha) \) is not minimal.
3. \[
\left( \prod_{\beta < \gamma \leq \alpha} m_{\gamma,0} \right) (m_{\beta,\nu} - m_{\beta,0}) \in \mathbb{Z}
\]
   for any non-minimal \( \beta \) and \( \alpha (> \beta) \).
4. Either
   \[
   0 < m_{\alpha,0} < \ldots < m_{\alpha,|p(\alpha)|}
   \]
   or
   \[
   m_{\alpha,0} > \ldots > m_{\alpha,|p(\alpha)|} > 0.
   \]
5. If \( \alpha, \alpha' \in n(\beta), \alpha \neq \alpha' \), then for any \( \nu, 1 \leq \nu \leq |\beta| \),
   \[
   \frac{m_{\alpha,0} m_{\alpha,|\beta|} - m_{\alpha,\nu}}{m_{\alpha,\nu} m_{\alpha,|\beta|} - m_{\alpha,0}} = \frac{m_{\alpha',0} m_{\alpha',|\beta|} - m_{\alpha',\nu}}{m_{\alpha',\nu} m_{\alpha',|\beta|} - m_{\alpha',0}}.
   \]

**Proof.** (1), (2), and (3) are immediately obtained from the second equality in Proposition 4.20. To prove \( m_{\alpha,\nu} > 0 \), note that the sign of \( d_{\alpha} \) is equal to the sign of \( u_{\alpha} \). This implies that the sign of \( d_{p(\alpha)} d_{\alpha}^{-1} \) is equal to the sign of everywhere non-zero function \( \prod_{i \in p(\alpha)} (h_i + e_{p(\alpha),\alpha}) \). Thus we have \( m_{\alpha,\nu} > 0 \). The remaining inequalities in (4) follows from the inequality

\[
1 = c_{p(\alpha),0} > \ldots > c_{p(\alpha),|p(\alpha)|} = 0
\]

and the fact that either \( e_{p(\alpha),\alpha} > 0 \) or \( e_{p(\alpha),\alpha} < -1 \). To prove (5), note that \( d_{\beta} d_{\alpha}^{-1} \), \( c_{\beta,\nu} (1 \leq \nu \leq |\beta| - 1) \), and \( e_{\beta \alpha} \) are uniquely determined from \( m_{\alpha,\nu} (0 \leq \nu \leq |\beta|) \), where \( \alpha \in n(\beta) \). In particular we have

\[
e_{\beta \alpha} = \frac{m_{\alpha,0}}{m_{\alpha,1}}, \quad e_{\beta \alpha} = \frac{m_{\alpha,0} m_{\alpha,|\beta|} - m_{\alpha,\nu}}{m_{\alpha,\nu} m_{\alpha,|\beta|} - m_{\alpha,0}}.
\]
Hence (5) follows. □

Remark. If \( h_i \ (i \in \beta) \) and \( e_{\beta \alpha} \ (\beta \prec \alpha) \) are replaced with \( 1 - h_{s(\beta) + t(\beta) - i} \) and \( -1 - e_{\beta \alpha} \) respectively for a non-maximal \( \beta \), then, (1) the order of \( Y_{\beta,0}, \ldots, Y_{|\beta|} \) and \( m_{\alpha,0}, \ldots, m_{|\alpha|,|\beta|} \) are reversed (\( \alpha \in n(\beta) \)), and (2) \( d_\gamma \) is replaced with \( (-1)^{|\beta|} d_\gamma \) (\( \beta \prec \gamma \)).

If \( \mathcal{A} \) is totally ordered, it is therefore possible to choose \( \{h_i\} \) and \( \{e_{\beta \alpha}\} \) so that every \( e_{\beta \alpha} \) is positive and \( m_{\alpha,0} < \cdots < m_{|\alpha|,|p(\alpha)|} \) for every non-minimal \( \alpha \). But, in general it is impossible.

5. Properties as a toric variety

In the previous section we have proved that \( M \) is a toric variety with respect to the action of \( G \). In this section we shall specify the fan of the toric variety \( M \), and describe some properties that are useful for the “existence problem”. Throughout this section we shall refer to Fulton [3] and Oda [11] on the general theory for toric varieties.

The fan of \( M \)

As a toric variety, \( M \) is constructed from the lattice \( \Gamma \subset \mathfrak{k} \) and a set \( \Delta \) of polyhedral cones in the Lie algebra \( \mathfrak{k} \). The pair \((\Gamma, \Delta)\) (or the set \( \Delta \) if \( \Gamma \) is known) is called the fan of \( M \). In our case, the invariant hypersurfaces \( L_{\alpha, \nu} \ (((\alpha, \nu) \in J)) \) and the information on their intersections will determine \( \Delta \). We first describe \((\Gamma, \Delta)\) in terms of the partially ordered set \( \mathcal{A} \) and the numbers \( |\alpha|, m_{\alpha, \nu} \), and then prove that it is the fan of \( M \).

Let \( \tilde{\mathfrak{k}} \) be the real vector space of dimension \( n + \# \mathcal{A} \) equipped with the basis \( \tilde{Y}_{\alpha, \nu} \ (((\alpha, \nu) \in J)) \), and let \( \tilde{\Gamma} \) be the lattice in \( \tilde{\mathfrak{k}} \) generated by \( \tilde{Y}_{\alpha, \nu} \ (((\alpha, \nu) \in J)) \). Define \( \tilde{Z}_\alpha \in \tilde{\Gamma} \ (\alpha \in \mathcal{A}) \) by

\[
\tilde{Z}_\alpha = \sum_{\beta \text{ non-minimal}} \left( \prod_{\beta \prec \gamma \prec \alpha} m_{\gamma,0} \right) \sum_{\mu=1}^{|p(\beta)|} (m_{\beta, \mu} - m_{\beta,0}) \tilde{Y}_{p(\beta), \mu}
\]

if \( \alpha \) is non-minimal, and by \( \tilde{Z}_\alpha = 0 \) if \( \alpha \) is minimal. Let \( \Gamma_0 \) be the subgroup of \( \tilde{\Gamma} \) generated by

\[
R_\alpha = \sum_{\nu=0}^{|\alpha|} \tilde{Y}_{\alpha, \nu} - \tilde{Z}_\alpha \quad (\alpha \in \mathcal{A}),
\]

and let \( \mathfrak{t}_0 \) be the subspace of \( \tilde{\mathfrak{k}} \) spanned by \( R_\alpha \ (\alpha \in \mathcal{A}) \). Then we have the exact sequences

\[
0 \to \mathfrak{t}_0 \to \tilde{\mathfrak{k}} \overset{\rho}{\to} \mathfrak{k} \to 0,
\]

\[
0 \to \Gamma_0 \to \tilde{\Gamma} \overset{\rho}{\to} \Gamma \to 0,
\]

where \( \rho \) is the homomorphism defined by \( \rho(\tilde{Y}_{\alpha, \nu}) = Y_{\alpha, \nu} \ (((\alpha, \nu) \in J)) \). Namely, \( \Gamma \) and \( \mathfrak{k} \) are isomorphic to \( \tilde{\Gamma}/\Gamma_0 \) and \( \tilde{\mathfrak{k}}/\mathfrak{t}_0 \) respectively.
For each $i \in I$, let $\sigma_i$ be the $n$-dimensional cone in $k$ generated by $Y_{\alpha,\nu}$ ($(\alpha, \nu) \in J(i)$), i.e.,

$$\sigma_i = \left\{ \sum_{(\alpha, \nu) \in J(i)} a_{\alpha, \nu} Y_{\alpha, \nu} \mid a_{\alpha, \nu} \geq 0 \right\}.$$ 

Let $\Delta$ be the set of the cones $\sigma_i$ ($i \in I$) and all the faces of them. Here, a face of the cone $\sigma_i$ means a cone $\sigma$ generated by $Y_{\alpha,\nu}$ ($(\alpha, \nu) \in J_0$) for some subset $J_0 \subset J(i)$. Hence the assignment $\sigma \rightarrow J_0$ gives the one-to-one correspondence between the cones in $\Delta$ and the subsets of $J$ contained in some $J(i)$. The 0-cone $\{0\}$ is supposed to be the face of every cone, which corresponds to the empty subset of $J$. It is easily seen that $(\Gamma, \Delta)$ satisfies the conditions that a fan should satisfy, and the resulting toric variety, denoted by $X(\Delta)$, is compact and non-singular (see [3] Sections 2.4 and 2.5).

Notice that the construction above only needs a partially ordered set $A$, integers $|\alpha|$ ($\alpha \in A$), $m_{\alpha,\nu} - m_{\alpha,0}$ ($\alpha$; non-minimal, $0 \leq \nu \leq |p(\alpha)|$), and rational numbers $m_{\alpha,0}$ ($\alpha$, $p(\alpha)$; non-minimal) satisfying Proposition 1.8 (2) and Proposition 4.21. Namely, only the differences $m_{\alpha,\nu} - m_{\alpha,0}$ are used for such $\alpha$ that $p(\alpha)$ is minimal, and the condition (3) of Proposition 1.8 on the integers $|\alpha|$ for maximal $\alpha$ are not necessary for the construction. Generally, fans and toric varieties obtained in such a way from those data will be called of KL type. If Proposition 1.8 (3) is satisfied, then they will be called of KL-A type. If not, then they will be called of KL-B type.

**Remark.** Since the fan of a toric variety is unique (cf. [12] Theorem 4.1), and since $\Delta$ determines the sets of elements $\{Y_{\alpha,\nu}\}$ and $\{Z_\alpha\}$ of $\Gamma$, it follows that the partially ordered set $A$ and the integers $|\alpha|$ ($\alpha \in A$) are uniquely associated with a toric variety of KL type. Also, for each $\alpha$ there are only two possibilities for the ordering of $Y_{\alpha,0}, \ldots, Y_{\alpha,|\alpha|}$ so that the corresponding numbers $m_{\alpha,\nu}$ satisfy Proposition 4.21 (4); the alternative is the reversed order.

**Proposition 5.1.** $(\Gamma, \Delta)$ is the fan of $M$.

Proposition 5.1 will be proved by giving an explicit identification of the toric variety $X(\Delta)$ (of KL-A type) with $M$. So, let us first review the construction of $X(\Delta)$.

For each $\sigma \in \Delta$ we define the semigroup $S_\sigma$ by

$$S_\sigma = \{ \eta \in \Gamma^* \mid <\eta, Y> \geq 0 \text{ for any } Y \in \sigma \cap \Gamma \},$$

where $\Gamma^*$ denotes the dual lattice of $\Gamma$. Regarding $C$ as the multiplicative semigroup, we put

$$U_\sigma = \{ u : S_\sigma \rightarrow C, \text{ a semigroup homomorphism} \}.$$ 

Here, homomorphisms are assumed to map unit to unit. If $\tau \subset \sigma$, then it is easily seen that

$$S_\sigma \subset S_\tau, \quad U_\sigma \subset U_\tau.$$ 

For the 0-cone, $S_{\{0\}} = \Gamma^*$ and

$$U_{\{0\}} = \{ u : \Gamma^* \rightarrow C^\times, \text{ a group homomorphism} \} = \Gamma \otimes C^\times.$$
which is an algebraic torus isomorphic to $(C^\times)^n$. We shall denote it by $\mathcal{T}_G$. The group $\mathcal{T}_G$ naturally acts on each $U_\sigma$ by

$$(u_0 u)(\eta) = u_0(\eta) u(\eta), \quad u_0 \in \mathcal{T}_G, \ u \in U_\sigma, \ \eta \in S_\sigma.$$  

It follows from the definition that $U_\sigma$ is an affine variety with coordinate ring $C[S_\sigma]$ (the semigroup ring). If the cone $\sigma$ is $k$-dimensional, then $U_\sigma$ is isomorphic to $C^k \times (C^\times)^{n-k}$. Then $X(\Delta)$ is obtained by gluing all $U_\sigma$ with the relations

$$U_\sigma \supset U_{\sigma \cap \tau} \subset U_\tau.$$

The action of $\mathcal{T}_G$ on $X(\Delta)$ is also well-defined.

**Proof of Proposition 5.1.** Fix a point $p_0 \in M^1$. Let $\iota \in \mathcal{I}$, and let $(z_{\alpha,\iota}; (\alpha, \nu) \in J(\iota))$ be the coordinate system on $M - \cup_{\alpha \in \mathcal{A}}L_{\alpha,\iota(\alpha)}$ given in Corollary 4.16. Also, let $\eta_{\alpha,\iota}' \ ((\alpha, \nu) \in J(\iota))$ be the basis of $\Gamma^*$ dual to $Y_{\alpha,\iota} \ ((\alpha, \nu) \in J(\iota))$ (note that they are generators for $S_{\alpha,\iota})$. Then, there is a natural holomorphic isomorphism

$$(5.2) \quad U_\sigma, \rightarrow M - \cup_{\alpha \in \mathcal{A}}L_{\alpha,\iota(\alpha)} \quad (u \mapsto p)$$

given by

$$(5.3) \quad u(\eta_{\alpha,\iota}'(p)) = z_{\alpha,\iota}(p) \quad ((\alpha, \nu) \in J(\iota)).$$

If $\sigma \in \Delta$ is a face of $\sigma_\iota$, then the mapping (5.2) gives the holomorphic isomorphism

$$(5.4) \quad U_\sigma \rightarrow M - \cup_{(\alpha, \nu) \in J - J_0}L_{\alpha,\nu},$$

where $J_0$ is the subset of $J$ corresponding to $\sigma$;

$$J_0 = \{(\alpha, \nu) \in J \mid Y_{\alpha,\nu} \in \sigma \} \subset J(\iota).$$

It is easily seen that the isomorphism (5.4) is independent of the choice of $\sigma_\iota$ containing $\sigma$. Hence we obtain the holomorphic isomorphism $X(\Delta) \rightarrow M$.

Defining the isomorphism $\mathcal{T}_G \rightarrow G$ by

$$Y_{\alpha,\nu} \otimes e^{t + \sqrt{-1}s} \mapsto \exp(-tY_{\alpha,\nu} + sY_{\alpha,\nu}),$$

we can easily see that the isomorphism $X(\Delta) \rightarrow M$ commutes with the actions of the groups $\mathcal{T}_G$ and $G$. This completes the proof.□

From now on, we shall fix a point $p_0 \in M^1$ (the base point) and identify each $U_\sigma$ with the subset $M - \cup_{(\alpha, \nu) \in J - J_0}L_{\alpha,\nu}$ of $M$ by the isomorphism given in the proof of Proposition 5.1, where $J_0 \subset J$ corresponds to $\sigma$. Also, the group $\mathcal{T}_G$ will be identified with $G$. For $\sigma \in \Delta$ corresponding to $J_0$ we put

$$O_\sigma = U_\sigma \cap \bigcap_{(\alpha, \nu) \in J_0}L_{\alpha,\nu}.$$  

Then, $O_\sigma$ is a $G$-orbit isomorphic to $(C^\times)^{n-k} \ (k = \dim \sigma)$, and

$$M = \cup_{\sigma \in \Delta}O_\sigma \quad U_\sigma = \cup_{\tau \supset \sigma}O_\tau$$

Note that $O_{\{0\}} = M^1$. Let $V(\sigma)$ be the closure of $O_\sigma$ in $M$. Then we have

$$V(\sigma) = \cup_{(\alpha, \nu) \in J_0}L_{\alpha,\nu} = \cup_{\tau \supset \sigma}O_\tau$$

(see [3] Section 3.1).

Since $\Delta$ contains $n$-dimensional cones, we have:

**Corollary 5.2.** $M$ is simply connected.

For the proof, see [3] p.56, Proposition. Actually, one can see more about the topology of $M$: There is a cell-decomposition of $M$ consisting of $\prod_{\alpha}(|\alpha| + 1)$ $\dim \sigma$-dimensional cells; the number of $2k$-dimensional cells is equal to the number of $\iota \in \mathcal{I}$ such that $\sum_{\alpha} t(\alpha) = k$. This is made in a similar way as [3] pp.101-103. Since the result is not used in this paper, we omit the detail.
Fibre bundles associated with $M$

In general, let $(\Gamma_i, \Delta_i) \ (i = 1, 2)$ be two fans, and let $\phi : \Gamma_1 \to \Gamma_2$ be a homomorphism such that the induced linear homomorphism

$$\phi : \Gamma_1 \otimes \mathbb{Z} R \to \Gamma_2 \otimes \mathbb{Z} R$$

maps each $\sigma \in \Delta_1$ into some $\sigma' \in \Delta_2$. Denoting the resulting toric varieties by $X(\Delta_i)$, we have

**Proposition 5.3.** There is a natural holomorphic mapping $\phi^\# : X(\Delta_1) \to X(\Delta_2)$ that is equivariant with respect to the naturally induced homomorphism

$$\phi^\# : T_{\Gamma_1} \to T_{\Gamma_2}$$

of algebraic tori.

For the proof, see [11] p.19, Theorem 1.13 (see also [3] p.41, Exercise). As applications of this general result, we shall obtain two kinds of fibre bundles associated with $M$. We now explain the first one. Let $\tilde{\Gamma}$ and $\tilde{\mathfrak{f}}$ be as in the previous subsection. For each $\sigma \in \Delta$ corresponding to $J_0 \subset \mathcal{J}$, define the cone $\tilde{\sigma}$ in $\tilde{\mathfrak{f}}$ by

$$\tilde{\sigma} = \{ \sum_{(\alpha,\nu) \in J_0} a_{\alpha,\nu} \tilde{Y}_{\alpha,\nu} \mid a_{\alpha,\nu} \geq 0 \},$$

and put $\tilde{\Delta} = \{ \tilde{\sigma} \mid \sigma \in \Delta \}$. As is easily verified, $(\tilde{\Gamma}, \tilde{\Delta})$ is a fan. Then, by Proposition 5.3 the homomorphism $\rho$ induces the equivariant holomorphic mapping $\rho^\# : X(\Delta) \to M$.

**Proposition 5.4.** (1) The toric variety $X(\tilde{\Delta})$ and the algebraic torus $T_{\tilde{\Gamma}}$ are naturally isomorphic to

$$\prod_{\alpha \in A} (C^{|\alpha|+1} - \{0\}) = \{(z_\alpha; \alpha \in A) \mid z_\alpha = (z_{\alpha,0}, \ldots, z_{\alpha,|\alpha|}) \in C^{|\alpha|+1} - \{0\}\}$$

and

$$(C^\times)^{n+\#A} = \{(\lambda_{\alpha,\nu}; (\alpha, \nu) \in \mathcal{J}) \mid \lambda_{\alpha,\nu} \in C^\times \}$$

respectively.

(2) $\rho^\# : X(\Delta) \to M$ is a principal fibre bundle with structure group $T_{\Gamma_0}$.

**Proof.** (1) Let $\tilde{Y}_{\alpha,\nu}^* ((\alpha, \nu) \in \mathcal{J})$ be the basis of $\tilde{\Gamma}^*$ dual to $\tilde{Y}_{\alpha,\nu} ((\alpha, \nu) \in \mathcal{J})$. Then, all the semigroups $S_{\tilde{\sigma}}$ contain the semigroup generated by $\tilde{Y}_{\alpha,\nu}^* ((\alpha, \nu) \in \mathcal{J})$. This implies that all the affine varieties $U_{\tilde{\sigma}}$ are realized in $C^{n+\#A}$;

$$C^{n+\#A} = \{(z_{\alpha,\nu}; (\alpha, \nu) \in \mathcal{J})\},$$

$$U_{\tilde{\sigma}} = \{(z_{\alpha,\nu}) \mid z_{\alpha,\nu} \neq 0 \text{ for } (\alpha, \nu) \notin \mathcal{J}_0\},$$

$\mathcal{J}_0$ corresponding to $\sigma$. From this the assertion easily follows.
(2) We first review how the mapping $\rho^*_\pi$ is constructed: The surjective homomorphism $\rho : \tilde{\Gamma} \to \Gamma$ induces the inclusion $\rho^* : \Gamma^* \to \tilde{\Gamma}^*$. This gives the inclusion $S_\sigma \to \tilde{S}_{\sigma}$ for any $\sigma \in \Delta$. Thus the mapping $\rho^*_\pi : U_{\tilde{\sigma}} \to U_\sigma$ is defined by

$$u \mapsto u|_{S_\sigma} \quad (u \in U_{\tilde{\sigma}}).$$

Now, fix $\iota \in \mathcal{I}$ and define a splitting $j_\iota : \Gamma \to \tilde{\Gamma}$ of the exact sequence (5.1) by $j_\iota(Y_{\alpha,\nu}) = \tilde{Y}_{\alpha,\nu} (((\alpha, \nu) \in \mathcal{J}(\iota))$. Let $\Gamma_\iota$ be its image. Then we have the direct sum decompositions

$$\tilde{\Gamma} = \Gamma_0 + \Gamma_\iota, \quad \tilde{\Gamma}^* = \Gamma^\perp_\iota + \Gamma^*,$$

and accordingly,

$$S_{\tilde{\sigma}_\iota} = \Gamma^\perp_\iota + S_{\sigma_\iota}.$$

Since $\Gamma^\perp_\iota$ is identified with $\Gamma_0^*$ by the natural homomorphism $\tilde{\Gamma}^* \to \Gamma_0^*$, we thus obtain the holomorphic isomorphism

$$(5.5) \quad U_{\tilde{\sigma}_\iota} \to \mathcal{T}_{\Gamma_0} \times U_{\sigma_\iota} \quad (u \mapsto (u|_{\Gamma^\perp_\iota}, u|_{S_{\sigma_\iota}})).$$

Clearly, the mapping above also gives the isomorphism $\mathcal{T}_{\tilde{\Gamma}} \to \mathcal{T}_{\Gamma_0} \times G$ of algebraic tori, with which the isomorphism (5.5) is equivariant. This proves (2). $\square$

Now, let us explain the other kind of fibre bundles that are naturally associated with $M$. For convenience we shall introduce a topology on the set $\mathcal{A}$: A subset $\mathcal{B}$ of $\mathcal{A}$ is open if it possesses the property;

$$(5.6) \quad \text{if } \beta \in \mathcal{B} \text{ and } \gamma \preceq \beta, \text{ then } \gamma \in \mathcal{B}.$$

Let $\mathcal{A}'$ be an open subset of $\mathcal{A}$, and put $\mathcal{A}'' = \mathcal{A} - \mathcal{A}'$. Let $\Gamma'$ be a subgroup of $\Gamma$ generated by $Y_{\alpha,\nu} ((\alpha, \nu) \in \mathcal{J}, \alpha \in \mathcal{A}')$, and let $\mathfrak{t}'$ be the subspace of $\mathfrak{t}$ spanned by those vectors. Put

$$\Gamma'' = \Gamma/\Gamma', \quad \mathfrak{t}'' = \mathfrak{t}/\mathfrak{t'},$$

and let $\pi : \mathfrak{t} \to \mathfrak{t}''$ be the natural projection. Also, put

$$\Delta' = \{ \sigma \in \Delta \mid \sigma \subset \mathfrak{t}' \}, \quad \Delta'' = \{ \pi(\sigma) \mid \sigma \in \Delta \}.$$

Then the pairs $(\Gamma', \Delta')$ and $(\Gamma'', \Delta'')$ become fans. It is clear that $X(\Delta'')$ is a toric variety of KL-A type. For $X(\Delta')$, it can be only said that it is of KL type. Since the homomorphism $\pi$ satisfies the assumption of Proposition 5.3, we have the equivariant mapping

$$\pi^*: M \to X(\Delta'').$$

**Proposition 5.5.** $\pi^*: M \to X(\Delta'')$ is a fibre bundle with typical fibre $X(\Delta')$. More precisely, for each $\iota \in \mathcal{I}$ there is an isomorphism $G \to \mathcal{T}_{\Gamma'_\iota} \times \mathcal{T}_{\Gamma''_\iota}$ of algebraic tori and an equivariant holomorphic isomorphism

$$\pi_{\pi^*}^{-1}(U_{\pi(\sigma_\iota)}) \to X(\Delta') \times U_{\pi(\sigma_\iota)}.$$

*Proof.* Fix $\iota \in \mathcal{I}$, and let $\Gamma_1$ be the subgroup of $\Gamma$ generated by $Y_{\alpha,\nu} ((\alpha, \nu) \in \mathcal{J}(\iota), \alpha \in \mathcal{A}'')$. We then have the direct sum decompositions

$$\Gamma = \Gamma' + \Gamma_1, \quad \Gamma^* = \Gamma^\perp + (\Gamma''^*)^*.$$
and
\[ S_{\sigma_i} = S_{\sigma'_i} + S_{\pi(\sigma_i)}, \]
where \( \sigma'_i = \sigma_i \cap k' \in \Delta' \), and \( \Gamma_1^* \) is identified with \( (\Gamma')^* \) by the projection \( \Gamma^* \to (\Gamma')^* \). This induces the holomorphic isomorphism
\[ U_{\sigma_i} \to U_{\sigma'_i} \times U_{\pi(\sigma_i)}, \]
and the isomorphism
\[ \mathcal{T}_{\Gamma} \to \mathcal{T}_{\Gamma'} \times \mathcal{T}_{\Gamma''} \]
of algebraic tori so that the mapping (5.7) is equivariant. Then, varying \( \iota \in \mathcal{I} \) so that \( \iota(\alpha) \) remains unchanged for any \( \alpha \in \mathcal{A}'' \), and taking the union of both sides of (5.7) with respect to all such \( \iota \), we have the equivariant isomorphism
\[ \pi^*_\sharp : \pi^{-1}(U_{\pi(\sigma_i)}) \to X(\Delta') \times U_{\pi(\sigma_i)}. \]
\[ \square \]

Let \( \mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i \) (disjoint union) be the decomposition of \( \mathcal{A} \) into the connected components. Let \( \Gamma_i \) be the subgroup of \( \Gamma \) generated by \( Y_{\alpha, \nu} \) (\( \alpha \in \mathcal{A}_i \)), and let \( k_i \) be the subspace of \( k' \) spanned by \( \Gamma_i \). Clearly, \( \Gamma \) is the direct sum of those subgroups, and correspondingly,
\[ G = \mathcal{T}_{\Gamma_1} \times \cdots \times \mathcal{T}_{\Gamma_k}. \]

Putting
\[ \Delta_i = \{ \sigma \cap k_i \mid \sigma \in \Delta \}, \]
we obtain fans \( (\Gamma_i, \Delta_i) \).

**Corollary 5.6.** There is a natural holomorphic isomorphism
\[ M \to X(\Delta_1) \times \cdots \times X(\Delta_k) \]
that is equivariant with respect to the identification
\[ G = \mathcal{T}_{\Gamma_1} \times \cdots \times \mathcal{T}_{\Gamma_k}. \]
Moreover, each \( X(\Delta_i) \) naturally becomes a Kähler-Liouville manifold so that \( M \) becomes the product manifold as Kähler-Liouville manifold.

**Proof.** The former half is an immediate consequence of Proposition 5.5. The latter half is then obvious. \( \square \)

We now go back to the situation of Proposition 5.5 and observe the fibre bundle \( \pi^*_\sharp : M \to X(\Delta') \) from another point of view. Let \( \widetilde{\Gamma}' \) and \( \widetilde{\Gamma}'' \) be the subgroups of \( \widetilde{\Gamma} \) generated by \( \widetilde{Y}_{\alpha, \nu} \) (\( (\alpha, \nu) \in \mathcal{J}, \alpha \in \mathcal{A}' \)) and \( \widetilde{Y}_{\alpha, \nu} \) (\( (\alpha, \nu) \in \mathcal{J}, \alpha \in \mathcal{A}'' \)) respectively. We then have
\[ \widetilde{\Gamma} = \widetilde{\Gamma}' + \widetilde{\Gamma}'' \] (direct sum).

Let \( \bar{\pi} : \widetilde{\Gamma} \to \widetilde{\Gamma}'' \) be the projection. The homomorphism \( \rho : \widetilde{\Gamma} \to \Gamma \) induces the homomorphisms \( \bar{\rho}' : \widetilde{\Gamma}' \to \Gamma' \) and \( \bar{\rho}'' : \widetilde{\Gamma}'' \to \Gamma'' \) (the latter is given by \( \bar{\pi} \circ \rho \)). Let
\( \Gamma_0' \) and \( \Gamma_0'' \) be the kernel of those homomorphisms. Thus we have the following commutative diagram whose rows and columns are exact:

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Gamma_0' & \tilde{\Gamma}' & \rho & \Gamma' & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Gamma_0 & \tilde{\Gamma} & \rho & \Gamma & \longrightarrow & 0 \\
\downarrow & \pi & \downarrow & \pi & \downarrow & \pi \\
0 & \Gamma_0'' & \tilde{\Gamma}'' & \rho'' & \Gamma'' & \longrightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

Note that the splitting in the mid column does not induce the splitting in the left one, i.e., \( \Gamma_0 \cap \tilde{\Gamma}''' \neq \Gamma_0' \) in general. Let \( \tilde{\Gamma}''' \) be the subspace of \( \tilde{\Gamma}' \) spanned by \( \tilde{\Gamma}''' \), and put \( \tilde{\Delta}''' = \{ \sigma \in \tilde{\Delta} \mid \sigma \subset \tilde{\Gamma}''' \} \).

Then, \( (\tilde{\Gamma}'', \tilde{\Delta}'') \) becomes a fan, and the homomorphism \( \rho'' \) induces the principal \( T_{\Gamma_0''} \)-bundle \( \rho'' : X(\tilde{\Delta}'') \to X(\Delta'') \).

We now define the homomorphism \( \psi : \Gamma_0'' \to \Gamma' \) as follows: Let \( \tilde{\Gamma} \to \tilde{\Gamma}' \) be the projection with respect to the decomposition (5.8). Restricting it to \( \Gamma_0 \), we have the homomorphism \( \Gamma_0 \to \tilde{\Gamma}' \). The restriction of this mapping to \( \Gamma_0' \) being the identity, we thus obtain the homomorphism

\[
\psi_1 : \Gamma_0'' = \Gamma_0 / \Gamma_0' \to \tilde{\Gamma}' / \Gamma_0' = \Gamma'.
\]

We put \( \psi = -\psi_1 \). The following formula is easily obtained:

\[
\psi(\tilde{\pi}(R_\alpha)) = \left( \prod_{\alpha_0 < \beta \leq \alpha} m_{\beta,0} \right) Z_{\alpha_0} \quad (\alpha \in A''),
\]

where \( \alpha_0 \) is the minimal element of \( A'' \) satisfying \( \alpha_0 \preceq \alpha \). The induced homomorphism \( T_{\Gamma_0''} \to T_{\Gamma'} \) of algebraic tori is also denoted by \( \psi \). Through this homomorphism \( T_{\Gamma_0''} \) acts on \( X(\Delta') \).

**Proposition 5.7.** The fibre bundle \( \pi''_z : M \to X(\Delta'') \) is isomorphic to the fibre product

\[
X(\tilde{\Delta}'') \times_{T_{\Gamma_0''}} X(\Delta') \to X(\Delta'').
\]

**Proof.** Let \( \tilde{\ell}' \) be the subspace of \( \tilde{\ell} \) spanned by \( \tilde{\Gamma}' \), and put \( \tilde{\Delta}' = \{ \sigma \in \tilde{\Delta} \mid \sigma \subset \tilde{\ell}' \} \).

Then \( (\tilde{\Gamma}', \tilde{\Delta}') \) is also a fan, and we have

\[
X(\tilde{\Delta}) = X(\tilde{\Delta}'') \times X(\tilde{\Delta}').
\]
Hence \( M = X(\bar{\Delta})/\mathcal{T}_{\Gamma_0} \) is equal to
\[
((X(\bar{\Delta}'')) \times X(\bar{\Delta}'))/\mathcal{T}_{\Gamma_0}'' = (X(\bar{\Delta}'') \times X(\Delta'))/\mathcal{T}_{\Gamma_0}''.
\]

Since the action of \( \mathcal{T}_{\Gamma_0}'' \) on \( X(\bar{\Delta}'') \times X(\Delta') \) is given by
\[
(p, q)g = (pg, \psi_1(g)q) = (pg, \psi(g^{-1})q),
\]
the proposition follows. □

**Line bundles**

Let \( \text{Pic}(M) \) denote the group of the isomorphism classes of holomorphic line bundles over \( M \). To each \( \xi \in \tilde{\Gamma}^* \) we associate a divisor of \( M \);
\[
\xi \mapsto \sum_{(\alpha, \nu) \in J} <\xi, \tilde{Y}_{\alpha, \nu}> L_{\alpha, \nu},
\]

where \( <, > \) denotes the natural pairing of \( \tilde{\Gamma}^* \) and \( \tilde{\Gamma} \). Let \( Q_\xi \) be the line bundle over \( M \) associated with this divisor. Then we have the homomorphism \( \tilde{\Gamma}^* \to \text{Pic}(M) \) \( (\xi \mapsto Q_\xi) \). Also, the homomorphism \( -\xi : \tilde{\Gamma} \to \mathbb{Z} \) induces the homomorphism \( \chi_{-\xi} : \mathcal{T}_{\Gamma} \to \mathbb{C}^\times \). Restricting it to \( \mathcal{T}_{\Gamma_0} \), we have another line bundle over \( M \) associated with \( \pi_\sharp : X(\bar{\Delta}) \to M \).

**Proposition 5.8.** (1) The following sequence is exact:
\[
0 \to \Gamma^* \overset{\rho^*}{\to} \tilde{\Gamma}^* \to \text{Pic}(M) \to 0.
\]

(2) \( Q_\xi \) is isomorphic to the fibre product \( X(\bar{\Delta}) \times_{X(\Delta') \chi_{-\xi}} \mathbb{C} \), where \( X(\Delta') \chi_{-\xi} \) is regarded as the homomorphism \( \mathcal{T}_{\Gamma_0} \to \mathbb{C}^\times \) by restriction.

(3) The assignment of the first Chern class \( c_1(Q) \) to each \( Q \in \text{Pic}(M) \) gives the isomorphism \( \text{Pic}(M) \to H^2(M, \mathbb{Z}) \).

**Proof.** For (1) and (3), see [3] pp.63-64 and [11] Corollary 2.5. (2) is easy. □

Put
\[
\zeta_\alpha = c_1(Q_{\tilde{Y}_{\alpha, 0}^*}) \in H^2(M, \mathbb{Z}) \quad (\alpha \in \mathcal{A}).
\]

The proposition above implies that the elements \( \zeta_\alpha \ (\alpha \in \mathcal{A}) \) form a basis of \( H^2(M, \mathbb{Z}) \). Its dual basis is given as follows: Let \( \tau(\alpha) \in \Delta \) be the \((n - 1)\)-dimensional cone generated by
\[
\{Y_{\beta, \nu} \mid 1 \leq \nu \leq |\beta| \text{ if } \beta \neq \alpha; \ 2 \leq \nu \leq |\beta| \text{ if } \beta = \alpha\}.
\]

Then \( V(\tau(\alpha)) \) is 1-dimensional (isomorphic to \( \mathbb{C}P^1 \)). Let \([V(\tau(\alpha))]\) denote its fundamental class in \( H_2(M, \mathbb{Z}) \).
Proposition 5.9. \(<\zeta_\alpha, [V(\tau(\beta))] > = \delta_{\alpha\beta},\) where \(<,>\) denotes the natural pairing of \(H^2(M, \mathbb{Z})\) and \(H_2(M, \mathbb{Z})\).

Proof. The assertion easily follows from the fact:
\[
L_{\alpha,0} \cap V(\tau(\beta)) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\
V(\sigma_\iota) = \{\text{a point}\} & \text{if } \alpha = \beta,
\end{cases}
\]
where \(\iota \in I\) is given by \(\iota(\gamma) = 0\) for every \(\gamma \in A\). □

The next theorem specifies the cohomology class \([\omega] \in H^2(M, \mathbb{R})\) of the Kähler form \(\omega\). Let \(A_i (1 \leq i \leq k)\) be the connected components of \(A\), and let \(\alpha_i\) denote the unique minimal element of \(A_i\).

Theorem 5.10.

(1) \[
\int_{V(\tau(\alpha))} \omega = \frac{2\pi}{d_{\alpha}} \prod_{\beta < \alpha} |\beta| \prod_{\nu=1}^{\beta \prec \alpha} (c_{\beta,\nu} + e_{\beta\alpha}).
\]

(2) \[
[\omega] = \sum_{i=1}^{k} \frac{2\pi}{d_{\alpha_i}} \left( \sum_{\alpha_i \succ \alpha} \left( \prod_{\alpha_i \succ \beta \preceq \alpha} m_{\beta,0} \right) (\zeta_{\alpha} + \zeta_{\alpha_i}) \right).
\]

Proof. (1) As is easily seen, the action of the circle group \(\{\exp(tY_{\alpha,0})\} (t \in \mathbb{R}/2\pi \mathbb{Z})\) on \(V(\tau(\alpha))\) has the two fixed points \(q_0\) and \(q_1\):
\[
\{q_0\} = V(\tau(\alpha)) \cap V(\mathbb{R}_{\geq 0} Y_{\alpha,0}), \quad \{q_1\} = V(\tau(\alpha)) \cap V(\mathbb{R}_{\geq 0} Y_{\alpha,1}),
\]
where \(\mathbb{R}_{\geq 0}\) denotes the set of non-negative real numbers. Let \(\gamma(s) (0 \leq s \leq l)\) be a geodesic of unit speed on \(V(\tau(\alpha))\) from \(q_0\) to \(q_1\). Then, parametrizing \(V(\tau(\alpha))\) by \((s, t)\), we have
\[
\int_{V(\tau(\alpha))} \omega = \int_0^l \int_0^{2\pi} \omega(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) dt ds,
\]
and
\[
\omega(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) = d_{\alpha,0}^{-1} \frac{d}{ds} v_\alpha(c_{\alpha,0})(\gamma(s)).
\]
Hence
\[
\int_{V(\tau(\alpha))} \omega = \frac{2\pi}{d_{\alpha,0}} (v_\alpha(c_{\alpha,0})(q_1) - v_\alpha(c_{\alpha,0})(q_0)).
\]
Since \(v_\alpha(c_{\alpha,0})(q_0) = 0\) and
\[
v_\alpha(c_{\alpha,0})(q_1) = \left( \prod_{\beta \prec \alpha} |\beta| \prod_{\nu=1}^{\beta \preceq \alpha} (c_{\beta,\nu} + e_{\beta\alpha}) \right) \prod_{\mu=1}^{\alpha} (c_{\alpha,\mu} - c_{\alpha,0}),
\]
the assertion follows. (2) is immediately obtained from (1). □
6. Bundle structure associated with a subset of $\mathcal{A}$

In the previous section we have proved that an open subset $\mathcal{A}'$ induces the fibre bundle $\pi_{\sharp} : M \to X(\Delta'')$ whose typical fibre is $X(\Delta')$. In this section we shall show that the toric varieties $X(\Delta')$ and $X(\Delta'')$ naturally possess structures of Kähler-Liouville manifold inherited from $M$. Since the numbering $i = 1, \ldots, n$ is inconvenient for the purpose of this section, we shall use $(\alpha, \nu)$ ($\alpha \in \mathcal{A}, 1 \leq \nu \leq |\alpha|$) instead. The correspondence is given by

$$(\alpha, \nu) \leftrightarrow i = s(\alpha) + \nu - 1.$$  

Fix an open subset $\mathcal{A}'$, and let $(\Gamma', \Delta')$ and $(\Gamma'', \Delta'')$ be as in the previous section. $TM$ is naturally decomposed to the sum of (mutually orthogonal) two subbundles: $TM = D' + D''$, where

$$D' = \sum_{\alpha \in \mathcal{A}'} D_{\alpha}, \quad D'' = \sum_{\alpha \in \mathcal{A}''} D_{\alpha}.$$ 

Clearly, $D'$ is integrable, and the maximal integral submanifolds are the fibres of $\pi_{\sharp} : M \to X(\Delta'')$. Let $\mathcal{A}'_1, \ldots, \mathcal{A}_r'$ be the connected components of $\mathcal{A}'$, and let $\alpha_s$ be the (unique) minimal element of $\mathcal{A}_s'$ ($1 \leq s \leq r$). Put

$$D_s'' = \sum_{\alpha \in \mathcal{A}_s''} D_{\alpha} \quad (1 \leq s \leq r).$$ 

Recalling the orthonormal frame $V_{\alpha, \nu}, IV_{\alpha, \nu}$ ($\alpha \in \mathcal{A}, 1 \leq \nu \leq |\alpha|$) over $M^1$, we put

$$V_{\alpha, \nu}' = \sqrt{|u_{\alpha_s}|} V_{\alpha, \nu} \quad (\alpha \in \mathcal{A}_s'').$$ 

The following lemma is easily obtained by using the properties of the vector fields $W_{\alpha, \nu}$ (cf. Proposition 1.4).

**Lemma 6.1.** For $\alpha \in \mathcal{A}''$ and $\beta \in \mathcal{A}'$,

$$[V_{\alpha, \nu}', V_{\beta, \mu}] = [IV_{\alpha, \nu}', V_{\beta, \mu}] = [IV_{\alpha, \nu}', IV_{\beta, \mu}] = [IV_{\alpha, \nu}', IV_{\beta, \mu}] = 0.$$ 

Let us recall the polynomial $F_{\alpha}(\lambda)$ in the indeterminate $\lambda$ whose coefficients are elements of $\mathcal{F}$ (cf. Section 2):

$$(6.1) \quad F_{\alpha}(\lambda) = |u_{\alpha}| \sum_{1 \leq \mu \leq |\alpha|, \mu \neq \nu} (h_{\alpha, \mu} - \lambda) \cdot (V_{\alpha, \nu}^2 + (IV_{\alpha, \nu})^2)$$

$$+ |u_{\alpha}| \sum_{\beta \in \pi(\alpha)} \prod_{\nu=1}^{[\alpha]} (h_{\alpha, \nu} + e_{\alpha\beta}) - \prod_{\nu=1}^{[\alpha]} (h_{\alpha, \nu} - \lambda) \sum_{\gamma \geq \beta \mu=1}^{[\gamma]} (V_{\gamma, \mu}^2 + (IV_{\gamma, \mu})^2).$$

This polynomial is uniquely determined if the fundamental functions and the conjunction constants are specified. We shall call it the *generating polynomial*. The next proposition is an immediate consequence of the lemma above.
Proposition 6.2. (1) The vector fields $V_{\alpha,\nu}^{\prime\prime}$ are $T_{\Gamma'}$-invariant.
(2) The horizontal subbundle $D^{\prime\prime}$ is $T_{\Gamma'}$-invariant.
(3) For any $\alpha \in A''$, the coefficients of $F_{\alpha}(\lambda)$ are $T_{\Gamma'}$-invariant, and are sections of $S^2D^{\prime\prime}$.

By virtue of Proposition 6.2 (3), the coefficients of $(\pi^*_s)_{\ast}F_{\alpha}(\lambda)$ ($\alpha \in A''$) are well-defined sections of $S^2TX(\Delta'')$. Let $F^{\prime\prime}$ be the vector space spanned by those sections. Also, the riemannian metric $g^{\prime\prime}$ on $X(\Delta'')$ is defined by the conditions: The subbundles $D^{\prime\prime}_s$ ($1 \leq s \leq r$) are mutually orthogonal with respect to $\pi^*_s g^{\prime\prime}$, and

$$\pi^*_s g^{\prime\prime} = |u_{\alpha_s}|^{-1} g \text{ on } D^{\prime\prime}_s \quad (1 \leq s \leq r).$$

It is easily seen that $g^{\prime\prime}$ is a Kähler metric. We denote by $M^{\prime\prime}$ the Kähler manifold $(X(\Delta''), g^{\prime\prime})$.

Theorem 6.3. $(M^{\prime\prime}, F^{\prime\prime})$ is a Kähler-Liouville manifold of type (A). It possesses the following properties:

1. The associated partially ordered set is naturally identified with $A''$;
2. the underlying toric variety is identical with $X(\Delta'')$;
3. the fundamental functions $\{h_{\alpha,\nu}^{\prime\prime}\}$ ($\alpha \in A'', 1 \leq \nu \leq |\alpha|$) are given by $\pi^*_s (h^{\prime\prime}_{\alpha,\nu}) = h_{\alpha,\nu}$;
4. the conjunction constants $e^{\prime\prime}_{\alpha\beta}$ ($\alpha \in A'', \alpha \leq \beta$) are given by $e^{\prime\prime}_{\alpha\beta} = e_{\alpha\beta}$;
5. the scaling constants $d^{\prime\prime}_{\alpha}$ ($\alpha \in A''$) are given by $d^{\prime\prime}_{\alpha} = e(\alpha_s)d_{\alpha}$ ($\alpha \in A''_s$), where $e(\alpha_s)$ is the sign of $d_{\alpha_s}$.

Proof. The commutativity of $F^{\prime\prime}$ with respect to the Poisson bracket follows from that of $F$. Since maximal elements of $A''$ are also maximal in $A$, it follows that $|\alpha| \geq 2$ for any maximal element $\alpha$ of $A''$. This implies that $(M^{\prime\prime}, F^{\prime\prime})$ is of type (A). The properties (1), . . . , (5) are easily verified. \(\square\)

Next, let us consider the fibre. Define the Kähler metric $g'(q)$ on the fibre $\pi_1^{-1}(q)$ $q \in M^{\prime\prime}$ by restricting $g$. With this metric we regard $\pi_1^{-1}(q)$ as a Kähler manifold. Also, we define $F'(q)$ as follows: Each $F \in F$ is a section of $S^2D' + S^2D^{\prime\prime}$; so, taking the $S^2D'$-component $F'$ of $F$, we put

$$F'(q) = \{F'|_{\pi_1^{-1}(q)} \mid F \in F\}.$$

Theorem 6.4. (1) $(\pi_1^{-1}(q), F'(q))$ is a Kähler-Liouville manifold for any $q \in M^{\prime\prime}$.
(2) Let $\tilde{X}$ be the horizontal lift (i.e., the lift as a section of $D^{\prime\prime}$) of a vector field $X$ on $M^{\prime\prime}$. Then the one-parameter group $\{\phi_t\}$ of transformations of $M$ generated by $\tilde{X}$ gives the automorphisms $\pi_1^{-1}(q) \to \pi_1^{-1}(\phi_t(q))$ of Kähler manifolds, and preserves $F'$ for each $F \in F$.

Proof. (1) Let $F'^{\prime}_{\alpha}(\lambda)$ ($\alpha \in A'$) be the $S^2D'$-component of $F_{\alpha}(\lambda)$. We have

$$(6.2) \quad F_{\alpha}(\lambda) = F'^{\prime}_{\alpha}(\lambda) + \epsilon(\alpha) \sum_{\beta \in \rho(\alpha)} \frac{u_{\beta} - v_{\alpha}(\lambda)}{e_{\alpha\beta} + \lambda} \sum_{1 < s < r} 2|u_{\alpha_s}|^{-1} \tilde{E}_s^{\prime\prime},$$
where $E''_s$ is the $S^2D''_s$-components of the horizontal lift of the energy function $E''$ of $M''$, and $\epsilon(\alpha)$ denotes the sign of $u_\alpha$. Then, taking the $S^3D'$-components of

$$0 = \{F_\alpha(\lambda), F_\alpha(\mu)\},$$

we obtain

$$0 = \{F'_\alpha(\lambda), F'_\alpha(\mu)\}.$$

Hence $F'$ is commutative.

(2) is an immediate consequence of Lemma 6.1. $\square$

The typical fibre $X(\Delta')$ is naturally identified with the fibre $\pi_\sharp^{-1}(\pi_\sharp(p_0))$ passing through the base point $p_0 \in M^1$. Denoting the Kähler manifold $\pi_\sharp^{-1}(\pi_\sharp(p_0))$ by $M'$, and $F'(\pi_\sharp(p_0))$ by $F'$, we obtain a Kähler-Liouville manifold $(M', F')$. Note that it is of type (A) if and only if every maximal element $\alpha$ of $A'$ satisfies $|\alpha| \geq 2$. The following theorem is immediate.

**Theorem 6.5.** If $(M', F')$ is of type (A), then it possesses the following properties:

1. The associated partially ordered set is naturally identified with $A'$;
2. the underlying toric variety is isomorphic to $X(\Delta')$;
3. the fundamental functions $\{h'_{\alpha,\nu}\}$ ($\alpha \in A'$, $1 \leq \nu \leq |\alpha|$) are given by the restriction of $h_{\alpha,\nu}$ to $M'$;
4. the conjunction constants $e'_{\alpha\beta}$ ($\beta \in A'$, $\alpha \leq \beta$) are given by $e'_{\alpha\beta} = e_{\alpha\beta}$;
5. the scaling constants $d'_{\alpha}$ ($\alpha \in A'$) are given by $d'_{\alpha} = d_{\alpha}$.

In case $(M', F')$ is not of type (A), then the structure of toric variety on $M'$ may be external, i.e., not determined by $(M', F')$ itself. Nevertheless, we have the following

**Proposition 6.6.** (1) The maximal compact subgroup $K'$ of the algebraic torus $T_{T'}$ acts on the Kähler manifold $M'$ as automorphisms and preserves each element of $F'$.

(2) The geodesic flow of $M'$ is integrable by means of $F'$ and the Lie algebra of $K'$.

The proof is clear. We shall say that a compact Kähler-Liouville manifold is of type (B) if it can be realized as the fibre of a fibre bundle obtained from a compact Kähler-Liouville manifold of type (A) and an open subset of the associated partially ordered set, and if it is not of type (A). By the definition, it possesses a structure of toric variety of KL-B type (not necessarily unique). It is another type of Kähler-Liouville manifold whose geodesic flow is integrable. In this paper we shall not mention further about such a manifold except the 1-dimensional case (see Section 8).

In the rest of this section, we shall show that the Kähler-Liouville manifold $(M, F)$ can be reconstructed from the structure of toric variety on $M$ and the Kähler-Liouville manifolds $(M'', F'')$ and $(M', F')$, provided $(M', F')$ is of type (A). By virtue of Corollary 5.6, $M''$ is described as the product $M''_1 \times \cdots \times M''_r$ of Kähler-Liouville manifolds, corresponding to the decomposition of $A''$ into the connected components. Let $\omega''_s$ denote the Kähler form of $M''_s$. Also, let $\alpha_s$ denote the unique minimal element of $A''_s$. 

Put
\[ Q = \cup_{q \in M''}(\text{the unique open orbit of } \mathcal{T}_{\Gamma'} \text{ in the fibre } \pi_s^{-1}(q)). \]

Then \( Q \) is open and dense in \( M \), and \( \pi_s^*: Q \to M'' \) is a principal \( \mathcal{T}_{\Gamma'} \)-bundle. Proposition 5.7 implies that this bundle is isomorphic to
\[ X(\tilde{\Delta}'') \times \psi \mathcal{T}_{\Gamma'} \to M''. \]

Since the horizontal subbundle \( D'' \) is \( \mathcal{T}_{\Gamma'} \)-invariant, it defines the connection on this principal bundle. Let \( \theta \) be the connection form, and \( \Theta \) the \( \mathfrak{g}' \)-valued 2-form on \( M'' \) so that \( \pi_s^* \Theta \) is the curvature form (\( \mathfrak{g}' \) is the Lie algebra of \( \mathcal{T}_{\Gamma'} \)).

**Proposition 6.7.**
\[ \Theta = \sum_{s=1}^{r} d'' \omega''_s \otimes Z_{\alpha_s}. \]

**Proof.** By Propositions 1.2 and 1.10 we have
\[ [V_i, IV_i] \equiv -\text{sgrad} (\log |u_\alpha|) \mod (D_i) \]
for \( \alpha \in \mathcal{A}'' \) and \( i \in \alpha \). This implies
\[ [V''_i, IV''_i] \equiv -\text{sgrad} |u_{\alpha_s}| \mod (D'') \]
for any \( \alpha \in \mathcal{A}'' \) and \( i \in \alpha \). Since \( |d_{\alpha_s}| = d''_{\alpha_s} \), we have
\[ d\theta(V''_i, IV''_i) = d''_{\alpha_s} Z_{\alpha_s} \quad (\alpha \in \mathcal{A}''_s, i \in \alpha). \]

Also, it is easily seen that
\[ d\theta(V''_i, V''_j) = d\theta(V''_i, IV''_j) = d\theta(IV''_i, IV''_j) = 0 \]
for any \( \alpha, \beta \in \mathcal{A}'' \) and \( i \in \alpha, j \in \beta \) \((i \neq j)\). Hence the proposition follows. \( \square \)

From now on, we forget the structure of Kähler-Liouville manifold, and only assume that \( M = X(\Delta) \) is a toric variety of KL-A type. Let \( \mathcal{A} \) be the associated partially ordered set, and let \( \mathcal{A}' \) be an open subset of it. Put \( \mathcal{A}'' = \mathcal{A} - \mathcal{A}' \). Then we have the fibre bundle \( \pi_s: M \to X(\Delta'') \) with typical fibre \( X(\Delta') \) as before, and the principal \( \mathcal{T}_{\Gamma'} \)-bundle \( \pi^*_s: Q \to X(\Delta'') \) as above. Let \((M'', \mathcal{F}'')\) be a Kähler-Liouville manifold of type (A) whose underlying toric variety is isomorphic to \( X(\Delta'') \). We shall identify \( M'' \) with \( X(\Delta'') \). Let \( \mathcal{A}_s, \alpha_s, \) and \( \omega''_s \) \((1 \leq s \leq r)\) be as above. For each non-minimal \( \alpha \in \mathcal{A} \), let \( l_\alpha \) be the largest positive integer satisfying \( l_\alpha Z_\alpha \in \Gamma \).

**Lemma 6.8.** (1) For any \( \alpha \in \mathcal{A}''_s \),
\[ \left( \prod_{\alpha_s < \beta \leq \alpha} m_{\beta, 0} \right) l_{\alpha_s} \in \mathbb{Z}. \]

(2) \[ \left[ \frac{d''_{\alpha_s} l_{\alpha_s} \omega''_s}{2\pi} \right] \in H^2(M''_s, \mathbb{Z}). \]
Proof. (1) By Proposition 4.20 we have

$$Z_\alpha \equiv \left( \prod_{\alpha_s < \beta \leq \alpha} m_{\beta,0} \right) Z_{\alpha_s} \ mod \ (\sum_{\beta \in A'} \sum_{\nu=1}^{|\beta|} Z Y_{\beta,\nu}).$$

Thus the assertion follows. (2) follows from (1) and Theorem 5.10 (2). □

**Proposition 6.9.** There is a unique connection form $\theta$ on the principal bundle $\pi_s : Q \to M''$ such that

1. the associated curvature form is given by $\pi^*_s \Theta$, where

$$\Theta = \sum_{s=1}^{r} d''_{\alpha_s} \omega''_s \otimes Z_{\alpha_s};$$

2. the horizontal distribution defined by the kernel of $\theta$ is invariant with respect to the complex structure $I$.

Accordingly, the $T\Gamma_e$-invariant horizontal subbundle $D''$ of $TM$ with respect to $\pi_s : M \to M''$ is uniquely determined by $\Theta$ so that the connection $D''|Q$ on $Q$ satisfies the conditions above. The connection form $\theta$ and the subbundle $D''$ are invariant under the action of the maximal compact subgroup $K$ of $T\Gamma$.

Proof. First, we shall prove the uniqueness. Let $\theta$ be a connection form satisfying the condition (1) and (2). $\theta$ is $\mathfrak{g}'$-valued, and here $\mathfrak{g}'$ is regarded as a real Lie algebra with the complex structure $I$. Now, we regard it as a complex Lie algebra by replacing $I$ with $\sqrt{-1}$. We shall write $\tilde{\theta}$ (resp. $\tilde{\Theta}$) instead of $\theta$ (resp. $\Theta$) when $\mathfrak{g}'$ are regarded as the complex Lie algebra. By extending it $\mathbb{C}$-linearly to $TQ \otimes \mathbb{C}$; $\tilde{\theta}$ becomes a $(1,0)$-form. Since $\tilde{\Theta}$ is a $(1,1)$-from, we have

$$\partial \tilde{\theta} = 0, \quad \bar{\partial} \tilde{\theta} = \pi^*_s \tilde{\Theta}.$$  

This implies that if $\theta_1$ is another connection form satisfying the conditions (1) and (2), then $\tilde{\theta} - \tilde{\theta}_1$ is a holomorphic 1-form, and is projectable. Hence there is a holomorphic 1-form $\mu$ on $M''$ such that $\tilde{\theta} - \tilde{\theta}_1 = \pi^*_s \mu$. However, since $M''$ is a compact, simply connected Kähler manifold, we have $\mu = 0$. Thus it follows that $\theta = \theta_1$.

Next, we shall prove the existence. Let $P_s$ be a hermitian line bundle over $M''_s$ with the canonical hermitian connection form $\tilde{\theta}_s$ whose first Chern form is equal to

$$-\frac{d''_{\alpha_s} \omega''_s}{2\pi}$$

(for the existence of such a hermitian line bundle, see [7] p.41, Proposition). Put $U_s = \{v \in P_s \mid |v| = 1\}$ and $U = \prod_{s=1}^{r} U_s$. Then $U$ is a principal $U(1)^r$-bundle over $M'' = \prod_{s=1}^{r} M''_s$. Let $\phi : U(1)^r \to T\Gamma_e$ be the homomorphism given by

$$(\lambda_1, \ldots, \lambda_r) \mapsto \prod_{s=1}^{r} (l^{-1}_{\alpha_s} Z_{\alpha_s} \otimes \lambda_s).$$

Then we obtain the associated $T\Gamma_e$-bundle $U \times_{T\Gamma_e} T\Gamma_e \to M''$.  




Lemma 6.10. The principal bundle $U \times _{\phi} T_{\Gamma'} \rightarrow M''$ is naturally identified with the bundle $\pi'_z : Q \rightarrow M''$.

Proof. Let $\chi_s : \Gamma'_0 \rightarrow Z$ ($1 \leq s \leq r$) be the homomorphism given by

$$\chi_s(\pi(R_\alpha)) = \left( \prod_{\alpha_s < \beta \leq \alpha} m_{\beta,0} \right) l_{\alpha_s}.$$

The associated homomorphism $T_{\Gamma'_0} \rightarrow C^\times$ is also denoted by $\chi_s$. Then, by Proposition 5.8 and Theorem 5.10 we see that the line bundle $P_s$ is isomorphic to the fibre product

$$X(\Delta'') \times _{\chi_s} C \rightarrow M''.$$

Moreover, denoting by $K''_0$ the maximal compact subgroup of $T_{\Gamma''_0}$, we have

$$\psi|_{K''_0} = \phi \circ (\chi_1, \ldots, \chi_r)|_{K''_0}.$$

Therefore the lemma follows. $\square$

We now continue the proof of Proposition 6.9. The direct sum of the connection forms $\tilde{\theta}_s$, restricted to $U_s$, is a connection form on $U$. Composing this with the Lie algebra homomorphism associated with $\phi$, we obtain a connection form $\theta$ on the principal bundle $\pi_z : Q \rightarrow M''$. Then we clearly have $d\theta = \pi_z^* \Theta$.

Finally, we prove the $K$-invariance. Let $k \in K$. Then the pull-back $k^* \theta$ is a connection form with the same curvature, because $\Theta$ is preserved by the transformation of $M''$ induced from $k$. Hence by the uniqueness we have $k^* \theta = \theta$. This completes the proof. $\square$

Now, we moreover assume that there is a Kähler-Liouville manifold $(M', F')$ of type (A) whose underlying toric variety is $X(\Delta')$. Then we have the following

Theorem 6.11. There is a unique Kähler-Liouville manifold $(M, F)$ of type (A) satisfying the following conditions:

1. The underlying structure of toric variety is identical with the given one;
2. the given Kähler-Liouville manifolds $(M', F')$ and $(M'', F'')$ are isomorphic with the ones induced from $(M, F)$.

Proof. We first define functions $h_{\alpha, \nu}$ ($\alpha \in A, 1 \leq \nu \leq |\alpha|)$ on $M$. Let $K'$ be the maximal compact subgroup of $T_{\Gamma'}$. Since the fundamental functions $\{h'_{\alpha, \nu}\}$ of $(M', F')$ are $K'$-invariant, they are supposed to be defined on $M = U \times _{U(1)^r} M'$. So, we put

$$h_{\alpha, \nu} = \begin{cases} h'_{\alpha, \nu} & (\alpha \in A') \\ \pi''_z h''_{\alpha, \nu} & (\alpha \in A'') \end{cases}$$

Accordingly, we also put $c_{\alpha, \nu} = c'_{\alpha, \nu}$ if $\alpha \in A'$ and $= c''_{\alpha, \nu}$ if $\alpha \in A''$, where $c'_{\alpha, \nu}$ and $c''_{\alpha, \nu}$ are the fundamental constants.

We choose the ordering of $Y_{\alpha,0}, \ldots, Y_{\alpha,|\alpha|}$ ($\alpha \in A$) so that the ordering of $Y'_{\alpha, \nu} = (\pi'_z)_* Y_{\alpha, \nu}$ ($\alpha \in A''$) and $Y'_{\alpha, \nu} = Y_{\alpha, \nu}$ ($\alpha \in A'$) are equal to the ones induced from the fundamental functions $\{h''_{\alpha, \nu}\}$ and $\{h'_{\alpha, \nu}\}$ respectively (cf. the remark before Proposition 5.1). Hence the numbers $m_{\alpha, \nu}$ ($\alpha, p(\alpha)$, non-minimal) and $m_{\alpha, \nu} - m_{\alpha, 0}$ ($p(\alpha)$, minimal) are uniquely determined. We define the number $m_{\alpha, \nu}$ for $\alpha$ such
that \( p(\alpha) \) is minimal by the formula (4.9) and the value of the constants \( c_{\alpha,\nu} \). Then again by (4.9) the value of \( e_{\beta \alpha} \) is determined for every non-maximal \( \beta \) and \( \alpha \in \mathfrak{n}(\beta) \). It is also defined for all \( \alpha \) and \( \beta \prec \alpha \) so that Proposition 1.10 (2) is satisfied.

Now, let us define the function \( u_\alpha \) on \( M \) by

\[
u_\alpha = \prod_{\beta \prec \alpha} \prod_{\nu = 1}^{\beta} (h_{\beta,\nu} + e_{\beta \alpha}).
\]

Let \( D'' \) be the (horizontal) subbundle of \( TM \) given by Proposition 6.9. Since the Kähler form \( \omega' \) on \( M' \) is supposed to be defined on \( M \) so that the kernel coincides with the horizontal subbundle \( D'' \), we can define a 2-form \( \omega \) on \( M \) by

\[
\omega = \omega' + \sum_{s=1}^r |u_{\alpha_s}| \pi^*_s \omega''_s.
\]

Also, we define \( F_\alpha(\lambda) \) (\( \alpha \in \mathcal{A} \)) by the horizontal lift of the generating polynomial \( F''_\alpha(\lambda) \) of \( (M'', \mathcal{F}'') \) if \( \alpha \in \mathcal{A}'' \), and by the formula (6.2) if \( \alpha \in \mathcal{A}' \), where \( \tilde{E}''_s \) is the horizontal lift of the energy function \( E''_s \) of \( M''_s \), and \( F'_\alpha(\lambda) \) is the generating polynomial of \( (M', \mathcal{F}') \). Let \( \mathcal{F} \) be the vector space spanned by all the coefficients of \( F_\alpha(\lambda) \) (\( \alpha \in \mathcal{A} \)).

Define the orthonormal frame \( \mathcal{V}_{\alpha,\nu}, \mathcal{I}\mathcal{V}_{\alpha,\nu} \) on the open \( \mathcal{T}_\Gamma \)-orbit \( M^1 \) by using the corresponding frames on \( M' \) and \( M'' \) in the obvious manner. Then we have the formula (6.1) and the relations

\[
[W_{\alpha,\nu}, W_{\beta,\mu}] = [W_{\alpha,\nu}, IW_{\beta,\mu}] = [IW_{\alpha,\nu}, IW_{\beta,\mu}] = 0 \quad ((\alpha, \nu) \neq (\beta, \mu))
\]

\[
[V_{\alpha,\nu}, IV_{\alpha,\nu}] \equiv \text{sgrad} u_{\alpha,\nu} \mod D_{\alpha,\nu},
\]

where \( W_{\alpha,\nu} = a_{\alpha,\nu}^{-1/2} V_{\alpha,\nu} \) and

\[
a_{\alpha,\nu}^{-1} = |u_\alpha| \prod_{\mu \neq \nu} |h_{\alpha,\mu} - h_{\alpha,\nu}|.
\]

Hence the arguments in Section 1 imply that \( \omega \) is a Kähler form, and with this Kähler metric \( (M, \mathcal{F}) \) becomes a Kähler-Liouville manifold of type (A). The uniqueness and the property (2) obviously follow from the construction above. To prove (1) we need the following lemma.

**Lemma 6.12.**

\[
Y_{\alpha,\nu} = \tilde{Y}_{\alpha,\nu}^" + \frac{d''_{\alpha,\nu}}{d''_{\alpha,\nu}} \left( \frac{v_\alpha(c_{\alpha,\nu})}{u_{\alpha_s}} \right) Z_{\alpha_s} \quad (\alpha \in \mathcal{A}''),
\]

where \( \tilde{Y}_{\alpha,\nu}^" \) is the horizontal lift of \( Y_{\alpha,\nu}^" \).

**Proof.** Let \( \theta \) be the connection form given by Proposition 6.9. Then we have

\[
i_{Y_{\alpha,\nu}} d\theta = d''_{\alpha,\nu} \pi^2(i_{Y_{\alpha,\nu}^{\nu''}} \omega'') \otimes Z_{\alpha_s} = -\frac{d''_{\alpha,\nu}}{u_{\alpha_s}} \left( \frac{v_\alpha(c_{\alpha,\nu})}{u_{\alpha_s}} \right) \otimes Z_{\alpha_s}.
\]
Since the left-hand side is equal to \(-d(\theta(Y,\nu))\) by Proposition 6.9, we have

\[ Y_{\alpha,\nu} = \tilde{Y}_{\alpha,\nu}'' + \frac{d''_{\alpha,\nu}}{d''_{\alpha,\nu}} \frac{v_{\alpha}(c_{\alpha,\nu})}{u_{\alpha}} Z_{\alpha} + \text{(constant term)}. \]

Then, by comparing both sides at points on \(L_{\alpha,\nu} = V(R_{\geq 0} Y_{\alpha,\nu})\), the lemma is proved. □

The lemma above implies that \(Y_{\alpha,\nu} = d_{\alpha,\nu}^{-1} \operatorname{sgn} v_{\alpha}(c_{\alpha,\nu})\) for any \(\alpha \in \mathcal{A''}\), where \(d_{\alpha,\nu} = \epsilon(\alpha) d''_{\alpha,\nu}\) and \(\epsilon(\alpha)\) is the sign of \(u_{\alpha}\). Since it is also true for \(\alpha \in \mathcal{A'}\) \((d_{\alpha,\nu} = d'_{\alpha,\nu}\) in this case), the condition (1) is therefore satisfied. This completes the proof of Theorem 6.11. □

7. The case where \(#\mathcal{A} = 1\)

In this section we shall classify compact Kähler-Liouville manifolds (of type (A)) such that \(#\mathcal{A} = 1\). Note that such manifolds are isomorphic to the complex projective space \(CP^n\) (with the standard \((C^\times)^n = (C^\times)^{n+1}/C^\times\) action) as toric variety.

Let \((M, \mathcal{F})\) be a compact Kähler-Liouville manifold of type (A) such that the associated partially ordered set \(\mathcal{A}\) consists of one element. In this case we write \(\nu\) instead of \((\alpha, \nu)\), and \(d*\) instead of \(d_{\alpha}\). Put

\[ S = \cap_{\nu=1}^n L_\nu, \quad \{q_1\} = L_0 \cap S, \quad \{q_2\} = L_n \cap S. \]

\(S\) is holomorphically isomorphic to \(CP^1\). We regard \(S\) as a Kähler manifold with the induced metric. Clearly, \(Y_n\) is tangent to \(S\), and its zeros are \(q_1\) and \(q_2\). Let \(\gamma(t)\) \((0 \leq t \leq l/2)\) be a minimal geodesic of unit speed such that \(\gamma(0) = q_1\) and \(\gamma(l/2) = q_2\). Since \(S\) is a surface of revolution, \(\gamma(t)\) is extended to a closed geodesic of the least period \(l\). Recalling the function \(v(\lambda) = \prod_{\nu=1}^n (h_{\nu} - \lambda)\), put

\[ h(t) = \frac{v(c_n)(\gamma(t))}{\prod_{\nu=0}^{n-1}(c_\nu - c_n)} \quad t \in R/lZ. \]

**Proposition 7.1.** \(h \in C^\infty(R/lZ)\) possesses the following properties:

1. \(h(-t) = h(t)\) for any \(t\);
2. \(h(0) = 1, h(l/2) = 0\);
3. \(h'(t) < 0\) if \(0 < t < l/2\);
4. \(h'(T_\nu) = -\sqrt{2d*} c_\nu (1 - c_\nu)\) \((1 \leq \nu \leq n - 1)\), where \(T_\nu\) \((0 < T_\nu < l/2)\) is defined by \(h(T_\nu) = c_\nu\);
5. \(-h''(0) = h''(l/2) = d*\).

**Proof (except (4)).** (1), (2), and (5) are clear. Since \(Y_n \neq 0\) at \(\gamma(t)\) \((0 < t < l/2)\), (3) is also obvious. □

Let \(\mathcal{C} = \mathcal{C}_n\) be the set of elements \((\{c_0, \ldots, c_n\}, d*, l, h)\) such that \(d*\) and \(l\) are positive constants, \(\{c_\nu\}\) are constants satisfying

\[ 1 = c_0 > c_1 > \ldots > c_n = 0, \]

and \(h \in C^\infty(R/lZ)\) satisfies the conditions (1), \ldots, (5) in Lemma 7.1. We say that two elements \((\{c_\nu\}, d*, l, h)\) and \((\{\tilde{c}_\nu\}, \tilde{d*}, \tilde{l}, \tilde{h})\) are equivalent if \(\tilde{d*} = d*, \tilde{l} = l\) and either \(\tilde{c}_\nu = c_\nu, \tilde{h} = h\), or

\[ \tilde{c}_\nu = 1 - c_{n-\nu}, \quad \tilde{h}(t) = 1 - h\left(\frac{t}{2} - t\right). \]
Theorem 7.2. The assignment of \((\{c_\nu\}, d_*, l, h) \in C\) to \((M, F)\) described above gives the one-to-one correspondence between the set of the isomorphism classes of compact Kähler-Liouville manifolds of type (A) satisfying \(#A = 1\) and the equivalence classes of elements of \(C\).

To prove Proposition 7.1 (4) and Theorem 7.2 we shall use the results for (real) Liouville manifolds obtained by the author [6]. First, we prove the following

**Proposition 7.3.** Let \(p_0 \in M^1\) be the base point so that \(M\) is identified with the toric variety \(X(\Delta)\). Put
\[
N = \text{Exp}_{p_0}(D^+). 
\]
Then, \(N\) is a well-defined real submanifold of \(M\), which is totally geodesic and diffeomorphic to \(R\mathbb{P}^n\). Moreover, take the \(S^2TN\)-component \(F'\) of each element \(F \in \mathcal{F}\) and put
\[
\mathcal{F}' = \{F' \mid F \in \mathcal{F}\}. 
\]
Then \((N, \mathcal{F}')\) is a proper Liouville manifold of rank one and type (B), and its core is isomorphic to
\[
(R/\mathbb{Z}, \{[h - c_1], \ldots, [h - c_{n-1}]\}). 
\]

**Proof.** Using the real number field \(R\) instead of \(C\) in the construction of the toric variety \(X(\Delta)\) (cf. Section 5), one obtains a submanifold \(X(\Delta)(R)\) diffeomorphic to the \(n\)-dimensional real projective space \(R\mathbb{P}^n\). Since its tangent space is spanned by \(I_k = D^+\) at each point on \(R\mathbb{P}^n \cap M^1\), Proposition 1.6 implies that \(X(\Delta)(R)\) is totally geodesic and \(N = X(\Delta)(R)\).

It is easy to verify that \((N, \mathcal{F}')\) is a Liouville manifold. Put
\[
G_\nu = \sum_{\xi=1}^{n} \left( \prod_{\mu \neq \xi} (h_\mu - c_\nu) \right) \left( V_\xi^2 + (IV_\xi)^2 \right) \in \mathcal{F} \quad (1 \leq \nu \leq n - 1). 
\]
Then we have
\[
\{p \in N \mid (G'_\nu)_p = 0\} = M^s \cap L_\nu \cap N, 
\]
\[
\{p \in N \mid \text{rank } (G'_\nu)_p \leq 1\} = L_\nu \cap N. 
\]
Also, we have \((dG'_\nu)_p \neq 0\) at some \(\lambda \in T^*_pN\) for every \(p \in M^s \cap L_\nu \cap N\), because \(d(h_\nu + h_{\nu+1})\) does not vanish at \(p\). Hence the Liouville manifold \((N, \mathcal{F}')\) is proper and of rank one. Since \(N\) is diffeomorphic to \(R\mathbb{P}^n\), it is of type (B) (cf. [6] Theorems 3.3.1 and 3.4.1).

By definition the core of the Liouville manifold \((N, \mathcal{F}')\) consists of the 1 dimensional riemannian submanifold
\[
C = \cap_{\nu=1}^{n-1} (M^s \cap L_\nu \cap N) 
\]
(called the core submanifold) and the equivalence classes \([\tilde{h}_\nu]\) of the functions \(\tilde{h}_\nu\) on it defined by
\[
(G'_\nu)_p = \tilde{h}_\nu(p)V^2, \quad p \in C 
\]
where \(V\) is the unit normal vector to \(L_\nu \cap N\). The equivalence classes of functions are defined to be the orbits of the affine transformation group on the target space \(R\).
It is easily seen that $C$ is equal to the image of a closed geodesic passing through $q_1$ and $q_2$. Hence we can take $\gamma$ so that its image coincides with $C$. Thus $C$ is isometric to $R/1Z$, and $\hat{h}_\nu(\gamma(t)) = h(t) - c_\nu$. This completes the proof. □

Note that another choice of the base point $p_0$ gives another submanifold, but they are mutually transferred with the action of $K$. Hence the isomorphism class of the Liouville manifold $(N, F')$ is uniquely determined. As was shown in [6], the isomorphism classes of proper Liouville manifolds of rank one are completely classified by means of the isomorphism classes of the cores. In the present case, two cores $(R/lZ, \{[h - c_1], \ldots, [h - c_{n-1}]\})$ and $(R/1Z, \{[h - \tilde{c}_1], \ldots, [h - \tilde{c}_{n-1}]\})$ are isomorphic if and only if $l = \tilde{l}$ and either $h(t) = \hat{h}(t), c_\nu = \tilde{c}_\nu$, or $h(t) = 1 - \hat{h}(-t + l/2), c_\nu = 1 - \tilde{c}_{n-\nu}$. Hence those isomorphism classes corresponds to the equivalence classes of elements of $C$.

By the proof of Theorem 3.3.1 and Theorem 3.4.1 in [6] we obtain a branched covering of $N$ whose covering space is a torus. We now explain it: Put

\begin{equation}
\alpha_\nu = 4 \int_{T_{\nu-1}}^{T_\nu} \frac{dt}{\sqrt{(-1)^{\nu-1} \prod_{\mu=1}^{n-1} (h(t) - c_\mu)}} \quad (1 \leq \nu \leq n),
\end{equation}

where $T_\nu \in [0, l/2]$ is defined by $h(T_\nu) = c_\nu$ ($0 \leq \nu \leq n$). Put

$$R = \prod_{\nu=1}^{n} (R/\alpha_\nu Z),$$

and let $x_\nu \pmod{\alpha_\nu Z}$ be the natural coordinate of $R/\alpha_\nu Z$. Let $H(\simeq (Z/2Z)^n)$ be the transformation group of $R$ generated by $\tau_{2\nu} \circ \tau_{2\nu+1}$ ($1 \leq \nu \leq n - 1$) and $\tau_1 \circ \prod_{\nu=1}^{n-1} \tau_{2\nu}$, where

\begin{align}
\tau_{2\nu-1}(x_1, \ldots, x_n) &= (x_1, \ldots, x_{\nu-1}, \frac{\alpha_\nu}{2} - x_\nu, x_{\nu+1}, \ldots, x_n) \\
\tau_{2\nu}(x_1, \ldots, x_n) &= (x_1, \ldots, x_{\nu-1}, -x_\nu, x_{\nu+1}, \ldots, x_n).
\end{align}

Then we have

**Proposition 7.4 ([6]).** There is a surjective mapping $\Phi : R \rightarrow N$ possessing the following properties:

1. For any $p \in N$, $\Phi^{-1}(p)$ is an $H$-orbit;
2. $\Phi_*(\partial/\partial x_\nu) = \pm W_\nu$;
3. $h_\nu \circ \Phi$ are $C^\infty$ functions;
4. $M^s \cap N = \{p \in N \mid \#\Phi^{-1}(p) < 2^n\}$;
5. $L_\nu \cap N = \{\Phi(x) \mid x_\nu = 0, \alpha_\nu/2 \text{ or } x_{\nu+1} = \pm \alpha_{\nu+1}/4\}$ ($1 \leq \nu \leq n - 1$), $L_0 \cap N = \{\Phi(x) \mid x_1 = \pm \frac{\alpha_1}{2}\}$, $L_n \cap N = \{\Phi(x) \mid x_n = 0, \frac{\alpha_n}{2}\}$;
6. $\Phi \circ \tau_{2\nu-1} = \exp(\pi Y_{\nu-1}) \circ \Phi$, $\Phi \circ \tau_{2\nu} = \exp(\pi Y_\nu) \circ \Phi$.

**Proof of Proposition 7.1 (4).** Since the function $h_\nu \circ \Phi$ depends only on the variable $x_\nu$, we write it $\tilde{h}_\nu(x_\nu)$. Observing the formula

$$\text{Hess } v(c_\nu) = d_* \prod_{\nu \neq \nu} (c_\mu - c_\nu)$$
at a point $p$ such that $h_\nu(p) = c_\nu$ and $h_{\nu+1}(p) \neq c_\nu$, we have
\[
\tilde{h}_\nu'(0) = 0 \quad \text{and} \quad \tilde{h}_\nu''(0) = (-1)^{n-\nu}d_\nu \prod_{\mu \neq \nu} (c_\mu - c_\nu).
\]

Note that the vector fields $V_\nu$ and $W_\nu$ are locally well-defined (up to sign) and smooth as vector fields on $M^0 \cap N$, though they are not determined around $p \notin M^1$ as vector fields on $M$. Since $h(t) = h_\nu(\gamma(t))$ on $[T_{\nu-1}, T_\nu]$, we have
\[
h'(T_\nu)^2 = \lim_{t \to T_\nu} (V_\nu h_\nu)^2(\gamma(t))
= \lim_{t \to T_\nu} \frac{(W_\nu h_\nu)^2(\gamma(t))}{(-1)^{n-\nu} \prod_{\mu \neq \nu} (h_\mu - h_\nu)}
= \lim_{x_\nu \to 0} \frac{(\tilde{h}_\nu'(x_\nu))^2}{(-1)^{n-\nu} \prod_{\mu=1}^{n-1} (c_\mu - \tilde{h}_\nu(x_\nu))}
= 2d_\nu (1 - c_\nu)c_\nu.
\]

\[\square\]

**Proof of Theorem 7.2.** Let $\{(c_\nu), d_\nu, l, h\} \in \mathcal{C}$ be an arbitrary element, and let $(N, \mathcal{F}')$ be a proper Liouville manifold of rank one whose core is isomorphic to
\[
(R/l\mathbb{Z}, \{[h - c_1], \ldots, [h - c_{n-1}]\}).
\]

To prove Theorem 7.2 it suffices to show that there is a unique Kähler-Liouville manifold $(M, \mathcal{F})$ up to isomorphism such that the associated Liouville manifold is isomorphic to $(N, \mathcal{F}')$. To do so, we first review how to construct $(N, \mathcal{F}')$.

Let $\alpha_\nu$, $R$, $\tau_{2\nu-1}$, $\tau_{2\nu}$, and $H$ be as above. It is not hard to see that $R/H$ is homeomorphic to $RP^n$ with the quotient topology. Put $N = R/H$, and let $\Phi : R \to N$ be the quotient mapping. To regard $N$ as differentiable manifold diffeomorphic to $RP^n$, it is necessary to specify coordinate systems around branch points (i.e., points $p \in N$ such that $\#\Phi^{-1}(p) < 2^n$). Let $N^s$ denote the branch locus. Put
\[
I_\nu = \{ \Phi(x) \mid \tau_{2\nu}(x) = x \text{ and } \tau_{2\nu+1}(x) = x \} \quad (1 \leq \nu \leq n - 1)
\]
\[
J_\nu = \{ \Phi(x) \mid \tau_{2\nu}(x) = x \text{ or } \tau_{2\nu+1}(x) = x \} \quad (0 \leq \nu \leq n)
\]

Then $N^s = \bigcup_{\nu=1}^{n-1} I_\nu$. Let $p = \Phi(a) \in N^s$. Then there is a unique subset $K$ of \{1, \ldots, n - 1\} such that $p \in I_\nu$ if and only if $\nu \in K$. Writing $K = \{\nu_1, \ldots, \nu_k\}$, $\nu_1 < \cdots < \nu_k$, we have $\nu_{i+1} - \nu_i \geq 2$. Define functions $y_1, \ldots, y_n$ by
\[
y_{\nu_i} = (x_{\nu_i} - a_{\nu_i})^2 + (x_{\nu_i+1} - a_{\nu_i+1})^2 \quad (1 \leq i \leq k)
y_{\nu_i+1} = 2(x_{\nu_i} - a_{\nu_i})(x_{\nu_i+1} - a_{\nu_i+1}) \quad (1 \leq i \leq k)
y_\nu = x_\nu \quad (\nu, \nu - 1 \notin K)
\]

The system of functions $(y_\nu)$ is then projectable, and becomes a coordinate system around $a$ mentioned above (cf. [6] Proposition 3.3.2).
Define $C^\infty$ mappings

$$R/\alpha_{\nu} \mathbb{Z} \to \begin{cases} [-T_1, T_1] \quad (\nu = 1) \\ [T_{\nu-1}, T_{\nu}] \quad (2 \leq \nu \leq n-1), \quad (x_\nu \mapsto t = t_\nu(x_\nu)) \\ [T_{n-1}, l - T_{n-1}] \quad (\nu = n) \end{cases}$$

by the differential equations

$$t'_\nu(x_\nu)^2 = (-1)^{\nu-1} \prod_{\mu=1}^{n-1} (h(t_\nu) - c_\mu)$$

and the initial conditions

$$t_\nu(0) = T_\nu \quad (1 \leq \nu \leq n), \quad \begin{cases} t'(0) = 0, \ t''(0) < 0 & (1 \leq \nu \leq n-1) \\ t'_n(0) < 0 & (\nu = n). \end{cases}$$

Put $\tilde{h}_\nu(x_\nu) = h(t_\nu(x_\nu))$ and

$$g' = \sum_{\nu=1}^{n} (-1)^{n-\nu} \left( \prod_{\mu \neq \nu} (\tilde{h}_\mu - \tilde{h}_\nu) \right) (dx_\nu)^2$$

$$F'_\nu = \sum_{\mu=1}^{n} \frac{\prod_{\xi \neq \mu} (\tilde{h}_\xi - c_\nu)}{(-1)^{n-\mu} \prod_{\xi \neq \mu} (\tilde{h}_\xi - \tilde{h}_\mu)} \left( \frac{\partial}{\partial x_\mu} \right)^2 \quad (1 \leq \nu \leq n-1).$$

Then $g'$ and $F'_\nu$ are projectable, and define the riemannian metric on $N$ and the sections of $S^2TN$ respectively. Denoting by $E'$ the energy function associated with $g'$ and by $F'$ the vector space spanned by $F'_\nu$ ($1 \leq \nu \leq n-1$) and $E'$, we obtain the proper Liouville manifold $(N, F')$ of rank one and type (B), whose core is isomorphic to the given one.

The functions $\tilde{h}_\nu$ are also projectable, and define the continuous functions $h_\nu$ on $N$. The function $h_\nu$ is smooth outside $I_\nu \cup I_{\nu-1}$. Also, it is easily seen that the symmetric polynomials of $h_1, \ldots, h_n$ are smooth on the whole $N$. Put $v(\lambda) = \prod_{\nu} (h_\nu - \lambda)$, and

$$X_\nu = \frac{1}{d_\nu \prod_{\mu \neq \nu} (c_\mu - c_\nu)} \operatorname{grad} v(c_\nu) \quad (0 \leq \nu \leq n).$$

The following lemma is immediate.

**Lemma 7.5.** (1) $[X_\mu, X_\nu] = 0$ for any $\mu, \nu$.

(2) $v(c_\nu)(p) = 0$, $(X_\nu)_p = 0$ for $p \in J_\nu$.

(3) $\operatorname{Hess} v(c_\nu)$ on the normal bundle $NJ_\nu$ is equal to $d_\nu \prod_{\mu \neq \nu} (c_\mu - c_\nu) g'$.

Let $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ be the natural projection, and let $(w_0, \ldots, w_n)$ be the natural coordinate system of $\mathbb{R}^{n+1}$. 

Proposition 7.6. There is a diffeomorphism \( \phi : N \to \mathbb{R}P^n \) such that
\[
\phi_*(X_\nu) = \pi_* \left( w_\nu \frac{\partial}{\partial w_\nu} \right) \quad (0 \leq \nu \leq n).
\]

Proof. We first construct \( \phi \) on \( N - J_0 \). Noting that \( X_1, \ldots, X_n \) are linearly independent at every point on \( N^1 = N - \cup_{\nu=0}^{n} J_\nu \), we define (closed) 1-forms \( \omega_1, \ldots, \omega_n \) on \( N^1 \) by \( \omega_\nu(X_\mu) = \delta_{\nu \mu} \). It is easily seen that \( \omega_\nu \) is smoothly extended to \( N - (J_0 \cup J_\nu) \). Fix a point \( p_0 \in N^1 \)

Let \( \sigma_\nu \) (\( 0 \leq \nu \leq n \)) be the involution on \( N \) defined by \( \Phi \circ \tau_{2\nu} = \sigma_\nu \circ \Phi \) or \( \Phi \circ \tau_{2\nu+1} = \sigma_\nu \circ \Phi \). Then, \( \sigma_\nu \) is the reflection with respect to \( J_\nu \), and preserves each element of \( F' \). Also, we see that \( N - (J_0 \cup J_\nu) \) has two connected components; one contains \( p_0 \) and the other contains \( \sigma_\nu(p_0) \).

Let \( x_\nu \) (\( 1 \leq \nu \leq n \)) be the function on \( N - (J_0 \cup J_\nu) \) defined by
\[
x_\nu(p) = \begin{cases}
\exp(\int_0^1 \omega_\nu(\dot{c}(t))dt) & (p \simeq p_0) \\
-\exp(\int_0^1 \omega_\nu(\dot{c}(t))dt) & (p \simeq \sigma_\nu(p_0)),
\end{cases}
\]
where \( p \simeq p_0 \) means \( p \) and \( p_0 \) are on the same component, and \( c(t) \) (\( 0 \leq t \leq 1 \)) is a curve in \( N - (J_0 \cup J_\nu) \) from \( p_0 \) or \( \sigma_\nu(p_0) \) to \( p \). Clearly we have
\[
x_\nu(\sigma_\nu(p)) = -x_\nu(p),
\]
\[
x_\nu(p_0) = 1, \quad x_\nu(\sigma_\nu(p_0)) = -1,
\]
\[
\omega_\nu = \frac{dx_\nu}{x_\nu}.
\]

We now claim that \( x_\nu \) is smoothly extended to \( N - J_0 \) by putting \( x_\nu = 0 \) on \( J_\nu - J_0 \). In fact it is an easy consequence of the following lemma.

Lemma 7.7. For each \( p \in J_\nu - J_0 \), there is a neighborhood \( U \) of \( p \) and a \( C^\infty \) function \( u_\nu \) on \( U \) such that \( u_\nu^2 = |\nu(c_\nu)| \).

The lemma above follows from Lemma 7.5. Thus we have obtained the diffeomorphism
\[
(7.3) \quad N - J_0 \to \mathbb{R}P^n \quad (p \mapsto (x_1(p), \ldots, x_n(p)));
\]
which maps \( X_\nu \) to \( x_\nu \partial / \partial x_\nu \) and \( p_0 \) to \( (1, \ldots, 1) \) (the surjectivity follows from the completeness of \( X_\nu \) on \( N^1 \)). Now, making the coordinate functions on \( N - J_\nu \) in the same way, and gluing them together, we consequently obtain the desired diffeomorphism \( \phi : N \to \mathbb{R}P^n \).  \( \square \)

We now continue the proof of Theorem 7.2. By virtue of Proposition 7.6 we may identify \( N \) with \( \mathbb{R}P^n \). Hence \( X_\nu = \pi_* (\partial / \partial w_\nu) \), and \( J_\nu \) is given by \( w_\nu = 0 \). Also, we regard \( \mathbb{R}P^n \) as a submanifold of \( CP^n \) in the natural manner. The projection \( C^{n+1} - \{0\} \to CP^n \) is also denoted by \( \pi \). Let \( K = U(1)^n \) be the torus acting on \( CP^n \) by
\[
((\lambda_1, \ldots, \lambda_n), \pi(w_0, \ldots, w_n)) \to \pi(w_0, \lambda_1 w_1, \ldots, \lambda_n w_n) \quad (|\lambda_\nu| = 1).
\]

The following lemma is immediate.
Lemma 7.8. Let $H$ be a symmetric 2-form on $\mathbb{RP}^n$ invariant with respect to $\sigma_\nu$ ($0 \leq \nu \leq n$). Then there is a unique hermitian form $\tilde{H}$ on $\mathbb{CP}^n$ satisfying the following conditions:

1. $\tilde{H}|_{\mathbb{RP}^n} = H$;
2. $\tilde{H}(X, Y) = 0$ for any $X, Y \in T_p\mathbb{RP}^n$, $p \in \mathbb{RP}^n$;
3. $\tilde{H}$ is $K$-invariant.

By the lemma above the riemannian metric $g'$ extends to a hermitian metric $g$ on $\mathbb{CP}^n$. Let $F' \in \mathcal{F}'$ be an arbitrary element. By using the bundle isomorphism $T\mathbb{RP}^n \to T^*\mathbb{RP}^n$ induced from $g'$, $F'$ is translated to a symmetric 2-form $F'_2$. Extending it to a hermitian form $F_2$ on $\mathbb{CP}^n$, and again translating with respect to $g$, we obtain a section $F$ of $S^2T\mathbb{CP}^n$. Let $\mathcal{F}$ be the vector space (over $\mathbb{R}$) spanned by such $F$. Then direct calculations show that $g$ is a Kähler metric, and with this metric $(\mathbb{CP}^n, \mathcal{F})$ becomes a Kähler-Liouville manifold of type (A) that satisfies $\#\mathcal{A} = 1$. This completes the proof of Theorem 7.2. □

8. Existence theorem

Let $(M, \mathcal{F})$ be a compact Kähler-Liouville manifold of type (A). For each $\alpha \in \mathcal{A}$, we define a Kähler-Liouville manifold $(M_\alpha, \mathcal{F}_\alpha)$ as follows: Define the closed subset $\mathcal{A}_\alpha$ of $\mathcal{A}$ by

$$\mathcal{A}_\alpha = \{ \beta \in \mathcal{A} \mid \alpha \preceq \beta \}.$$

Let $(M'', \mathcal{F}'')$ be the Kähler-Liouville manifold that is the base space of the fibre bundle determined by the open subset $\mathcal{A} - \mathcal{A}_\alpha$. Next, regard $(M'', \mathcal{F}'')$ as the total space, and let $(M_\alpha, \mathcal{F}_\alpha)$ be the Kähler-Liouville manifold that is the typical fibre of the fibre bundle determined by the open subset $\{ \alpha \}$ of $\mathcal{A}_\alpha$.

If $|\alpha| \geq 2$, then $(M_\alpha, \mathcal{F}_\alpha)$ is of type (A), and it possesses the structure of toric variety that is given by the structure of Kähler-Liouville manifold. In case $|\alpha| = 1$, we also regard $M_\alpha$ as a toric variety, whose structure is inherited from that of $M''$. So, in any case $M_\alpha$ is isomorphic to $\mathbb{CP}^{[\alpha]}$ as toric variety.

Let $N$ be a 1-dimensional compact Kähler manifold which is also a toric variety such that the associated $U(1)$-action preserves the metric. We shall simply call it a compact toric Kähler manifold (of dimension 1). To such a manifold we assign positive constants $d_*, l$, and a function $h$ on $\mathbb{R}/l\mathbb{Z}$ as follows: Let $Y$ be an infinitesimal generator of the $U(1)$-action so that the least period of $\exp(sY)$ is $2\pi$. The set of zeros of $Y$ consists of two points, say $q_0$ and $q_1$. We may assume that the endomorphism $\nabla Y$ of $T_{q_1}N$ is equal to the complex structure $I$ (then it is equal to $-I$ at $q_0$). Let $l/2$ be the distance between these two points. Then a minimal geodesic $\gamma(t)$ from $q_0$ to $q_1$ extends to a closed geodesic of least period $l$. Since the 1-form $i_Y\omega$ is closed ($\omega$ is the Kähler form), there is a unique function $\tilde{h}$ on $M$ such that

$$i_Y\omega = -d\tilde{h}, \quad \tilde{h}(q_1) = 0.$$

Put $d_* = \tilde{h}(q_0)^{-1}$ and $h(t) = d_*\tilde{h}(\gamma(t))$.

The following lemma is immediate.

Lemma 8.1. $(d_*, l, h)$ defined above possesses the following properties:

1. $h(-t) = h(t)$ for any $t$;
2. $h(0) = 1$, $h(l/2) = 0$;
3. $h(t)$ is strictly monotone increasing in $[0, l/2)$.

Put $d = h(l/2)^{-1}$ and $\tilde{d} = d_*d$. Let $(\mathcal{M}, \mathcal{F})$ be the Kähler-Liouville manifold that is the total space of the fibre bundle determined by the open subset $\mathcal{A} - \mathcal{A}_\alpha$. Next, regard $(\mathcal{M}_\alpha, \mathcal{F}_\alpha)$ as the total space of the fibre bundle determined by the open subset $\{ \alpha \}$ of $\mathcal{A}_\alpha$.

If $|\alpha| \geq 2$, then $(\mathcal{M}_\alpha, \mathcal{F}_\alpha)$ is of type (A), and it possesses the structure of toric variety that is given by the structure of Kähler-Liouville manifold. In case $|\alpha| = 1$, we also regard $\mathcal{M}_\alpha$ as a toric variety, whose structure is inherited from that of $\mathcal{M}''$. So, in any case $\mathcal{M}_\alpha$ is isomorphic to $\mathbb{CP}^{[\alpha]}$ as toric variety.

Let $N$ be a 1-dimensional compact Kähler manifold which is also a toric variety such that the associated $U(1)$-action preserves the metric. We shall simply call it a compact toric Kähler manifold (of dimension 1). To such a manifold we assign positive constants $d_*, l$, and a function $h$ on $\mathbb{R}/l\mathbb{Z}$ as follows: Let $Y$ be an infinitesimal generator of the $U(1)$-action so that the least period of $\exp(sY)$ is $2\pi$. The set of zeros of $Y$ consists of two points, say $q_0$ and $q_1$. We may assume that the endomorphism $\nabla Y$ of $T_{q_1}N$ is equal to the complex structure $I$ (then it is equal to $-I$ at $q_0$). Let $l/2$ be the distance between these two points. Then a minimal geodesic $\gamma(t)$ from $q_0$ to $q_1$ extends to a closed geodesic of least period $l$. Since the 1-form $i_Y\omega$ is closed ($\omega$ is the Kähler form), there is a unique function $\tilde{h}$ on $M$ such that

$$i_Y\omega = -d\tilde{h}, \quad \tilde{h}(q_1) = 0.$$
Let $\alpha$ be the set of $(d_*, l, h)$ such that $d_*$ and $l$ are positive constants and $h$ is a $C^\infty$ function on $R/lZ$ satisfying the conditions $(1), \ldots, (4)$ in Lemma 8.1. We say that two elements $(d_*, l, h)$ and $(\tilde{d}_*, \tilde{l}, \tilde{h})$ are equivalent if $d_* = \tilde{d}_*$, $l = \tilde{l}$, and either $h(t) = \tilde{h}(t)$ or $h(t) = 1 - \tilde{h}(l/2 - t)$. For consistency with the definition of $C_n$, we shall also write $\{(1,0), d_*, l, h\}$ instead of $(d_*, l, h)$.

**Lemma 8.2.** The assignment above gives the one-to-one correspondence between the set of the isomorphism classes of 1-dimensional compact toric Kähler manifolds and the set of the equivalence classes of elements of $C_1$.

The proof is easy. Now, we state the main theorem in this section, which will imply the existence of compact Kähler-Liouville manifold of type (A) whose structure of toric variety is prescribed. Let $M$ be a toric variety of KL-A type, and let $A$ be the associated partially ordered set. Let $m_{\alpha, \nu}$ ($\alpha \in A$, not minimal, $0 \leq \nu \leq |p(\alpha)|$) be numbers satisfying Proposition 4.21 with which $M$ is defined. Let $c_{\alpha, \nu}$ $(0 \leq \nu \leq |\alpha|$, $\alpha \in A$), $e_{\beta \alpha}$ ($\beta < \alpha$), and $d_{\alpha}$ ($\alpha \in A$) be constants that satisfy the conditions $(4.1)$, $(4.2)$, $(4.8)$, and $(4.9)$. In this case we say that the constants $\{e_{\beta \alpha}, c_{\alpha, \nu}, d_{\alpha}\}$ are compatible with the toric variety $M$ (of KL-A type).

**Remark.** 1. $M$ determines only the differences $m_{\alpha, \nu} - m_{\alpha, 0}$ for $\alpha$ such that $p(\alpha)$ is minimal. Hence for such $\alpha$ one can choose $m_{\alpha, 0}$ arbitrary so that they satisfy Proposition 4.21 (5).

2. $\{m_{\alpha, \nu}\}$ just determine every $e_{\beta \alpha}$, every ratio $d_{p(\alpha)}/d_{\alpha}$, and $\{c_{\alpha, \nu}\}$ for every non-maximal $\alpha$. Hence one can choose $d_{\alpha} > 0$ arbitrary for minimal $\alpha$, and also $c_{\alpha, \nu}$ arbitrary for maximal $\alpha$ so that they satisfy $(4.1)$.

**Theorem 8.3.** Let $M$ be a compact toric variety of KL-A type, and let $A$ be the associated partially ordered set. Let $\{c_{\alpha, \nu}, e_{\beta \alpha}, d_{\alpha}\}$ be constants compatible with $M$. For each $\alpha \in A$, choose $l_{\alpha} > 0$ and $h_{\alpha} \in C^\infty(R/l_\alpha Z)$ so that $\{\{c_{\alpha, \nu}\}, |d_{\alpha}|, l_{\alpha}, h_{\alpha}\} \in C_{|\alpha|}$. Then there is a unique structure of Kähler-Liouville manifold $(M, F_{\alpha})$ of type (A) over the toric variety $M$ possessing the following properties:

1. The associated structure of toric variety is identical with the given one;
2. the fundamental constants, the conjunction constants, and the scaling constants are equal to $\{c_{\alpha, \nu}\}$, $\{e_{\beta \alpha}\}$, and $\{d_{\alpha}\}$ respectively;
3. for each $\alpha \in A$, the induced Kähler-Liouville manifold (the toric Kähler manifold if $|\alpha| = 1$) $(M_{\alpha}, F_{\alpha})$ corresponds to the equivalence class represented by the given element

$$\{\{c_{\alpha, \nu}\}, |d_{\alpha}|, l_{\alpha}, h_{\alpha}\} \in C_{|\alpha|}.$$ 

**Proof.** We prove this theorem by induction on $|A|$. The case $|A| = 1$ follows from Theorem 7.2. Let $k \geq 2$, and assume that the theorem is true for the case where the number of elements of the associated partially ordered set is less than $k$.

Now, let $M$ and $A$ be as above, and suppose $|A| = k$. We may assume that $A$ is connected. Let $\alpha_0 \in A$ be the minimal element, and put $A' = \{\alpha_0\}, A'' = A - A'$. As before, let $A'' = \bigsqcup_{\beta < \alpha} A''_{\beta}$ be the decomposition into connected components, and

\begin{enumerate}
\item $h'(t) < 0$ if $0 < t < l/2$;
\item $-h''(0) = h''(l/2) = d_*$. \hfill (3)
\end{enumerate}
\( \alpha_s \) the minimal elements of \( \mathcal{A}_s' \). Let \( M' \) and \( M'' \) be the associated toric varieties. As is easily seen, the constants \( c_{\alpha, \nu} (\alpha \in \mathcal{A}'') \), \( e_{\beta \alpha} (\beta \in \mathcal{A}'', \beta < \alpha) \), \( \epsilon(\alpha_s) d_\alpha (\alpha \in \mathcal{A}'', 1 \leq s \leq r) \) are compatible with the toric variety \( M'' \) of KL-A type, where \( \epsilon(\alpha_s) \) is the sign of \( d_\alpha \). So, by induction assumption we obtain a unique structure of Kähler-Liouville manifold \( (M'', \mathcal{F}'') \) over the toric variety \( M'' \) possessing the properties stated in the theorem.

Also, by Theorem 7.2 and Lemma 8.2 there is a unique structure of Kähler-Liouville manifold (or toric Kähler manifold if \( |\alpha_0| = 1 \)) \( (M', \mathcal{F}') \) over the toric variety \( M' \) corresponding to the element

\[ \left( \{c_{\alpha, \nu}\}, d_{\alpha_0}, l_{\alpha_0}, h_{\alpha_0} \right) \in \mathcal{C}_{|\alpha_0|}. \]

Then, by Theorem 6.11 we obtain a structure of Kähler-Liouville manifold \( (M, \mathcal{F}) \) over the toric variety \( M \) such that \( (M', \mathcal{F}') \) and \( (M'', \mathcal{F}'') \) are isomorphic to the ones induced from \( (M, \mathcal{F}) \). It is clear that \( (M, \mathcal{F}) \) possesses the properties (1) and (2). (3) follows from the fact that the Kähler-Liouville manifold (or the toric Kähler manifold) \( (M_{\alpha}, \mathcal{F}_{\alpha}) (\alpha \in \mathcal{A}'') \) induced from \( (M, \mathcal{F}) \) is isomorphic to the one induced from \( (M'', \mathcal{F}'') \).

This fact also proves the uniqueness of \( (M, \mathcal{F}) \). In fact, let \( (\widetilde{M}, \widetilde{\mathcal{F}}) \) be another Kähler-Liouville manifold possessing the properties stated in the theorem, and let \( (\widetilde{M}', \widetilde{\mathcal{F}}') \) and \( (\widetilde{M}'', \widetilde{\mathcal{F}}'') \) be the induced ones. Then the fact mentioned above and the induction assumption indicate that \( (\widetilde{M}'', \widetilde{\mathcal{F}}'') \) is isomorphic to \( (M'', \mathcal{F}'') \), and Theorem 7.2 and Lemma 8.2 indicate that \( (\widetilde{M}', \widetilde{\mathcal{F}}') \) is isomorphic to \( (M', \mathcal{F}') \). Hence by Theorem 6.11, \( (\widetilde{M}, \widetilde{\mathcal{F}}) \) is isomorphic to \( (M, \mathcal{F}) \). This completes the proof. \( \square \)

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