Fermionic Sum Representations for Conformal Field Theory Characters

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Abstract

We present sum representations for all characters of the unitary Virasoro minimal models. They can be viewed as fermionic companions of the Rocha-Caridi sum representations, the latter related to the (bosonic) Feigin-Fuchs-Felder construction. We also give fermionic representations for certain characters of the general \( \frac{(G^{(1)})_k \times (G^{(1)})_{k+1}}{(G^{(1)})_{k+1}} \) coset conformal field theories, the non-unitary minimal models \( \mathcal{M}(p, p + 2) \) and \( \mathcal{M}(p, kp + 1) \), the \( N=2 \) superconformal series, and the \( \mathbb{Z}_N \)-parafermion theories, and relate the \( q \to 1 \) behaviour of all these fermionic sum representations to the thermodynamic Bethe Ansatz.

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1. Introduction

Recently it was found [1] that characters (or branching functions) of the coset conformal field theories \( \frac{(G(1))_1 \times (G(1))_1}{(G(1))_2} \), \( G \) a simply-laced Lie algebra, can be represented in the form

\[
\sum_{m} Q \frac{q^{1/2} m B m'}{(q)_{m_1} \cdots (q)_{m_r}},
\]

where \( B \) is twice the inverse Cartan matrix of the algebra \( G \), \( r \) is the rank of the algebra, \( \mathbf{m} = (m_1, m_2, \ldots, m_r) \),

\[
(q)_m = \prod_{k=1}^{m} (1 - q^k),
\]

and \( Q \) indicates certain restrictions on the summation variables \( \mathbf{m} \) which depend on the character under consideration. These expressions extend the results of Lepowsky and Primc [2] for the case \( G = A_r \) to all simply-laced Lie algebras. It was also shown in [1] that these sums can be interpreted as partition functions of massless fermionic quasi-particles with non-trivial lower bounds on the single-particle momenta, which depend on the number, \( m_a \), and type, \( a \), of quasi-particles present in a given state. This interpretation was based on an analysis [3] of Bethe’s equations for the (gapless) antiferromagnetic 3-state Potts chain whose continuum limit is the \( c=1 \) conformal field theory of \( \mathbf{Z}_4 \)-parafermions, which corresponds to the case \( G = A_3 \) in the above notation.

These fermionic sum representations have widespread applicability. The case of the Virasoro minimal models \( \mathcal{M}(2, 2r + 3) \) was considered in [4][5], and the coset models \( \frac{(G(1))_k}{U(1)^r} \) in [6][7]. These results are all of the form (1.1). Here we generalize (1.1) to a form which encompasses a much wider class of conformal field theories. Specifically, we provide fermionic sum representations for:

- All of the characters of the unitary Virasoro minimal models \( \mathcal{M}(n + 2, n + 3) \), which correspond to the cosets \( \frac{(A_1^{(1)})_n \times (A_1^{(1)})_1}{(A_1^{(1)})_{n+1}} \);
- Certain characters of \( \frac{(G(1))_k \times (G(1))_l}{(G(1))_{k+l}} \) for \( G \) a simply-laced Lie algebra;
- Certain characters of non-unitary minimal models \( \mathcal{M}(p, p + 2) \) and \( \mathcal{M}(p, kp + 1) \);
- The identity character in the unitary \( N=2 \) superconformal series and in the \( \mathbf{Z}_N \)-parafermion theories.

As with the previous work [1] which led to (1.1), we are again motivated by an analysis of Bethe’s equations for the 3-state Potts chain, this time for the ferromagnetic case [8].
which is related through an orbifold construction \([10]\) to the \(r=6\) RSOS model at the III/IV boundary. The corresponding partition function is the non-diagonal \((D\text{-series})\) modular invariant combination \([10]\ [11]\) of \(c = \frac{4}{5}\) Virasoro characters.

2. Sum Representations for Virasoro Characters of \(\mathcal{M}(n+2, n+3)\)

The characters \(\chi_{r,s}^{(p,p')} (q)\) of the irreducible highest-weight representations with \(\Delta_{r,s}^{(p,p')} = \frac{(rp'-sp)^2-(p'-p)^2}{4pp'}\) of the minimal model \(\mathcal{M}(p,p')\) of central charge \(c_{p,p'} = 1 - \frac{6(p-p')^2}{4pp'}\) are \([12]\ [13]\)

\[
\chi_{r,s}^{(p,p')} (q) = \frac{q^{-\Delta_{r,s}^{(p,p')}}}{(q)_\infty} \sum_{k \in \mathbb{Z}} \left( q^{\Delta_{r+2kp,s}^{(p,p')}} - q^{\Delta_{r+2kp,-s}^{(p,p')}} \right). \tag{2.1}
\]

The normalization is chosen such that the \(\chi_{r,s}^{(p,p')} (q)\) are power series in \(q\) that start out with 1. Whenever \(p'\) in formulas below is suppressed, it will be understood as being equal to \(p + 1\).

To present the fermionic representations for the characters (2.1) we recall the definition of the \(q\)-binomial coefficient, that for integers \(n\) and \(m\)

\[
\left[ \begin{array}{c} n \\ m \end{array} \right]_q = \begin{cases} 
\frac{(q)_n}{(q)_m(q)_{n-m}} & \text{if } 0 \leq m \leq n \\
0 & \text{otherwise,} \end{cases} \tag{2.2}
\]

where \((q)_m\) is defined by \([1.2]\) (with \((q)_0=1\)). We also set \([\infty]_m \equiv 1/(q)_m\) for \(m \geq 0\). The \(q\)-binomial coefficients are polynomials in \(q\), known as gaussian polynomials \([14]\).

Our main result is that the characters (2.1), for \(p = p' = n + 2\) with \(n\) a positive integer, can all be written as special cases of the following general sum:

\[
S_n^{[Q]_A} (u|q) = \sum_{m \in (2\mathbb{Z}_{\geq 0})^n + Q} q^{\frac{1}{2} \frac{mC_n}{m} + \frac{1}{2} \frac{A \cdot m}{(q)_{m_1}} \prod_{a=2}^n \left[ \frac{\frac{1}{2} (mI_n + u)_a}{m_a} \right]_q}, \tag{2.3}
\]

where \(A, u \in \mathbb{Z}^n, A \cdot m = \sum_{a=1}^n A_a m_a\), \(Q \in (\mathbb{Z}_2)^n\) with \((QI + u)_a \in 2\mathbb{Z}\) for \(a = 2, \ldots, n\), and \(I_n\) and \(C_n = 2 - I_n\) are the incidence and Cartan matrix, respectively, of the Lie algebra \(A_n\). Explicitly,

\[
(I_n)_{ab} = \delta_{a,b+1} + \delta_{a,b-1} \quad \text{for } a, b = 1, \ldots, n. \tag{2.4}
\]

Note that \(S_n^{[Q]_A} (u|q)\) does not depend on \(u_1\); it will prove instructive to view the factor \(1/(q)_{m_1}\) in (2.3) as \([\frac{(m_2 + u_1)/2}{m_1}]_q\) with \(u_1 = \infty\).
Due to the definition (2.2), the sum in (2.3) is actually restricted to non-negative values of $m_a$, and furthermore for any given $m_1$ there is only a finite number of nonvanishing terms. In fact, it is easy to see that the support in the space of $m$ of the summand in (2.3) is bounded by the region defined by $m_1 \geq 0$ and $0 \leq m_a \leq \frac{n-a+1}{n-a+2}$ for $a = 2, \ldots, n$, where $D_{ab} = \prod_{j=a}^{b} \frac{n-j+1}{n-j+2}$ for $b \geq a$ and $0$ otherwise.

To specify the full set of characters (2.1) denote the $n$-dimensional unit vector in the $a$-direction by $e_a$ (i.e. $(e_a)_b = \delta_{ab}$), set $e_a = 0$ for $a \not\in \{1, \ldots, n\}$, and let $\rho = e_1 + \ldots + e_n$. For fixed $n \geq 1$ define

$$Q_{r,s} = (s-1)\rho + (e_{r-1} + e_{r-3} + \ldots) + (e_{n+3-s} + e_{n+5-s} + \ldots) ,$$

(2.5)

where addition of the components of the vectors on the rhs is modulo 2. Our result is that the Virasoro characters (2.1) with $p-2 = p' - 3 = n$ can be expressed as

$$\chi^{(p)}_{r,s}(q) = q^{-\frac{1}{2}(s-r)(s-r-1)} S_n \left[ \frac{Q_{r,s}}{e_{n+2-s}} \right] (e_r + e_{n+2-s} | q) .$$

(2.6)

(Note that terms in $u$ which are proportional to $e_1$ can be ignored.) These representations have been conjectured on the basis of the results of [8] and have been verified in many cases to order $q^{100}$ or more using Mathematica.

Due to the symmetry $(r, s) \leftrightarrow (n+2-r, n+3-s)$ of the conformal grid, another representation must also exist, namely

$$\chi^{(p)}_{s-1,r}(q) = q^{-\frac{1}{2}(s-r)(s-r-1)} S_n \left[ \frac{Q_{s-1,r}}{e_{s-1}} \right] (e_{s-1} + e_{n+2-r} | q) ,$$

(2.7)

where we used $Q_{n+2-r,n+3-s} = Q_{s-1,r}$ (in $(\mathbb{Z}_2)^n$), which follows from the definition (2.3).

The equality of the rhs’s of (2.6) and (2.7) amounts to an interesting set of identities between sums (2.3) with different “characteristics”. (For $(r, s) = (1, 1)$ and $(r, s) = (n+1, 1)$, labeling the corners of the conformal grid, these identities are trivial). The simplest example occurs already at $n=1$ for the character of the spin field of the Ising model, where (2.6) and (2.7) give

$$\chi^{(3)}_{1,2}(q) = \chi^{(3)}_{2,2}(q) = \sum_{m=0}^{\infty} q^{m(m-1)/2} \frac{(q)_m}{(q)_m} = \sum_{m=1}^{\infty} q^{m(m-1)/2} \frac{(q)_m}{(q)_m} .$$

(2.8)

The equality of the two sums here is noted in the list of Slater [15], eqs. (84)–(85). (It can be simply proved using the methods of the next section, in particular eq. (3.12) with $M=\infty$.)
3. Fermionic Quasi-Particle Interpretation

The sum forms (1.1) have a direct interpretation as the partition functions of fermionic quasi-particles [1]. We now present such an interpretation for the form (2.3). Our treatment follows the reverse of the procedure used in [8] to derive the characters for the special case $n = 3$.

A quasi-particle form for an energy spectrum is a representation of the energy levels (above the ground state) in the form

$$E(\{P\}) - E_{GS} = \sum_{a=1}^{n} \sum_{j_{a}=1}^{m_{a}} e_{a}(P_{a,j_{a}})$$  \hspace{1cm} (3.1)

with the total momentum given by

$$P = \sum_{a=1}^{n} \sum_{j_{a}=1}^{m_{a}} P_{a,j_{a}} ,$$  \hspace{1cm} (3.2)

where the $P_{a,j_{a}}$ are chosen to satisfy a set of rules. When one of the rules is

$$P_{a,j_{a}} \neq P_{a,k_{a}} \text{ for } j_{a} \neq k_{a} ,$$  \hspace{1cm} (3.3)

the spectrum is said to be fermionic.

The characters (2.6) may be viewed as partition functions constructed from the energies (3.1) as

$$Z(q) = \sum e^{-(E(\{P\}) - E_{GS})/kT} .$$  \hspace{1cm} (3.4)

We will show that

$$Z(q) = S_{n} \left[ \frac{Q}{A} \right](\mathbf{u}|q)$$  \hspace{1cm} (3.5)

if

$$e_{a}(P) = vP ,$$  \hspace{1cm} (3.6)

where $v$ is the “speed of sound (or light),

$$q = e^{-\frac{2\pi v}{LkT}} ,$$  \hspace{1cm} (3.7)

and the $P_{a,j_{a}} (j_{a} = 1, 2, \ldots, m_{a} \text{ with } m_{a} \equiv Q_{a} \text{ (mod 2)})$ obey the exclusion principle (3.3) but are otherwise freely chosen from the sets

$$P_{1,j_{1}} \in \left\{ P_{1}^{\min}(\mathbf{m}), P_{1}^{\min}(\mathbf{m}) + \frac{2\pi}{L}, P_{1}^{\min}(\mathbf{m}) + \frac{4\pi}{L}, \ldots \right\} ,$$  \hspace{1cm} (3.8)
and for $2 \leq a \leq n$

$$P_{a,j} \in \left\{ P_{a}^{\text{min}}(m), P_{a}^{\text{min}}(m) + \frac{2\pi}{L}, \ldots, P_{a}^{\text{max}}(m) \right\}.$$  \hfill (3.9)

The vectors $\mathbf{P}^{\text{min,max}} = \{P_{a}^{\text{min,max}}\}$ here are

$$\mathbf{P}^{\text{min}}(m) = - \frac{2\pi}{L} \left( \frac{1}{2} m \mathbf{I}_n + \mathbf{A} - \mathbf{r} \right)$$  \hfill (3.10)

where $\mathbf{r}$ denotes the $n$-dimensional vector $(1, 1, \ldots, 1)$, and for $a \geq 2$

$$P_{a}^{\text{max}}(m) = - P_{a}^{\text{min}}(m) + \frac{2\pi}{L} \left( \frac{u}{2} - \mathbf{A}_a \right).$$  \hfill (3.11)

To derive this representation, denote by $Q_{m}(N; M)$ the number of partitions of the non-negative integer $N$ into $m$ distinct non-negative integers which are smaller or equal to the positive integer $M$, possibly infinite. (We set $Q_{0}(N; M) = 0$ for $N > 0$, and $Q_{0}(0; M) = 1$.) The generating functions for $Q_{m}(N; M)$ are essentially the $q$-binomial coefficients $[16]$

$$\sum_{N=0}^{\infty} Q_{m}(N; M) q^{N} = q^{\frac{1}{2} m(m-1)} \left[ \frac{M+1}{m} \right]_{q}. \hfill (3.12)$$

Then using (3.12) we can rewrite (2.3) as

$$S_{n} \left[ \frac{Q}{A} \right] (u|q) = \sum_{N \in (\mathbb{Z}_{\geq 0})^{n}} \sum_{m \in (2 \mathbb{Z}_{\geq 0})^{n} + Q} q^{\mathbf{r} \cdot (N + m)} \frac{1}{2\pi} \mathbf{P}^{\text{min}}(m) \times \prod_{a=1}^{n} Q_{m_{a}}(N_{a}; \frac{L}{2\pi}(P_{a}^{\text{max}}(m) - P_{a}^{\text{min}}(m)))$$ \hfill (3.13)

It is readily seen that the partition function $Z(q)$ of (3.4), constructed according to (3.6)–(3.9), is indeed equal to the rhs of (3.13).

From (3.8) we see that there is one quasi-particle excitation with an infrared cutoff at low momentum which depends on the number of quasi-particles in the state. This is exactly the situation which occurred in the computation of the branching functions $[1][3]$ of $\frac{(G^{(1)})_{1} \times (G^{(1)})_{2}}{(G^{(1)})_{2}}$, where there are $r=\text{rank}(G')$ of these quasi-particles. The additional feature of the present case is that the momenta $P_{a,j}$ for $2 \leq a \leq n$ are restricted by (3.3) to take on only a finite number of values for given $m$ (even in the $L \to \infty$ limit). This phenomenon, while not exactly what was originally envisioned in the term quasi-particle, is in fact a common occurrence in quantum spin chains. It results from cancellations seen in the earliest computations of energy levels in the XYZ spin chain $[17]$ and is seen in all
the RSOS models at the boundary of the III/IV regimes \[18\]. It also has a counterpart in thermodynamic Bethe Ansatz analyses of factorizable scattering theories (see sect. 5 for additional details).

Another feature not encountered in \[1,3\] is that the momenta $P_{a,j}$ can now take on negative values, even for $a=1$, and that the ‘single-particle energies’ are $e_a(P) = vP$, not $e_a(P) = v|P|$. A more complete physical characterisation and interpretation of these excitations is expected to be found from a detailed study of the energy levels of the 3-state chiral Potts model in which the spectrum can be continuously varied from a $\mathbb{Z}_4$ spectrum with 3 quasi-particles to the above spectrum with one quasi-particle and two excitations with finite momentum range. Finally, we note that these finite range excitations, whose $O(1/L)$ contribution to the energy was studied here, also affect the degeneracy of the order one energy levels. A more detailed discussion is found in \[3\].

4. Generalizations

There are many generalizations of the results of sect. 2. We will concentrate here mainly on the character of the (extended) primary field creating the ground state in various series of rational conformal field theories. It appears that these characters can always be represented as fermionic sums

$$S_B(u|q) = \sum_m S_B^m(u|q) = \sum_{m} q^{\frac{1}{4}mBm^t} \prod_{a=1}^{n} \left[ \frac{(m(1-B)+\frac{u}{2})_a}{m_a} \right]_q,$$

where $B$ is a real $n$ by $n$ symmetric matrix and $u$ a vector whose entries are either 0 or $\infty$. When $B = \frac{1}{2}C_n$ and $u = (\infty, 0, \ldots, 0)$ the sum (4.1) is of the form (2.3), whereas taking all $u_a = \infty$ it degenerates into the form of (1.1).

We believe that all other characters can be obtained by introducing nontrivial “characteristics” $Q, A, u$ as in (2.3). As an example, by elementary manipulations of the results of \[2\] one can show that up to an overall power of $q$ all the characters of the cosets $(A^{(2)}_{1}\times A^{(1)}_{1})_{1}/(A^{(1)}_{2})$ can be written in the form (1.1) with $m$ in the quadratic form $mC_{r}^{-1}m^t$ replaced by $m + e_a$, where $e_a$ is an $r$-dimensional unit vector (or 0). Extending this observation to the coset $(E^{(3)}_{8})_{1}\times (E^{(1)}_{8})_{1}/(E^{(1)}_{8})_{2}$ which is identical to the minimal model $\mathcal{M}(3,4)$, we obtain $\chi_{1,1}^{(3)} + \chi_{1,2}^{(3)}$ if $m_1$ in $mC_{E_{8}}^{-1}m^t$ (in the basis used in \[1\]) is replaced by $m_1 - \frac{1}{2}$, and $\chi_{1,1}^{(3)} + \chi_{1,2}^{(3)} + \chi_{1,3}^{(3)}$ if $m_2$ is replaced by $m_2 - \frac{1}{2}$. Thus, together with the result \[1\] for $\chi_{1,1}^{(3)}$, we have $E_{8}$-type fermionic representations for all three characters of the Ising conformal field theory. A more detailed analysis is left for future work.
4.1. The cosets \( \left( A^{(1)}_1 \right)_k \times \left( A^{(1)}_1 \right)_l \) \( \left( A^{(1)}_1 \right)_{k+l} \).

The characters of these coset models are given in [13]. We conjecture that the identity character can be represented in the form (4.1) with \( B = \frac{1}{2} C_n \), \( n = k + l - 1 \), and \( u_l = \infty \), all other \( u_a \) being 0. For \( l = 2 \) the series of theories labeled by \( k \) is the unitary \( N=1 \) superconformal series whose characters are given in [20]. We find that the character corresponding to the identity superfield in these models is obtained by summing over \( m_1 \in \mathbb{Z}, m_a \in 2\mathbb{Z} \) for \( a = 2, \ldots, k + 1 \).

4.2. The cosets \( \left( G^{(1)}_1 \right)_k \times \left( G^{(1)}_1 \right)_l \) \( G^{(1)}_k \) with simply-laced \( G \).

The corresponding characters can be found in [21] [22] (in [21] the coset models with \( l = 1 \) are denoted by \([G_r(h+k)]\) where \( r \) is the rank and \( h \) the Coxeter number of \( G \)). In this case we take \( B = C_G^{-1} \otimes C_n \), \( n = k + l - 1 \), where \( C_G \) is the Cartan matrix of \( G \) and \( C_n \equiv C_{A_n} \) as before. Using a double-index notation

\[
B_{ab}^{\alpha\beta} = (C_G^{-1})_{\alpha\beta} (C_n)_{ab} \quad \alpha, \beta = 1, \ldots, r = \text{rank}(G), \quad a, b = 1, \ldots, n, \quad (4.2)
\]

we set \( u_1^a = \infty \) for all \( a \) and 0 otherwise. (Cf. [6] [7] where \( B = C_G \otimes C_n^{-1} \) with all \( u_a^a = 0 \) is used to produce fermionic sum representations of the form (1.1) for the identity characters in the cosets \( G^{(1)}_k \).

As a non-trivial example consider the coset \( \left( E^{(1)}_8 \right)_2 \times \left( E^{(1)}_8 \right)_1 \left( E^{(1)}_8 \right)_3 \) of central charge \( c = \frac{21}{22} \), which is identified [23] as the Virasoro minimal model \( \mathcal{M}(11,12) \) of \( (E_6, A_{10}) \) type [11]. We verified to order \( q^{100} \) that the corresponding sum (4.1) with all \( m_a^a \in \mathbb{Z} \) agrees with \( \chi^{(11)}_{1,1} (q) + q^8 \chi^{(11)}_{1,7} (q) \), which is the extended identity character of this model.

4.3. Non-unitary Virasoro Minimal Models \( \mathcal{M}(p,p+2) \) (\( p \) odd).

The character \( \chi^{(p,p+2)}_{(p-1)/2,(p+1)/2} (q) \) (see (2.1)) of the field with lowest conformal dimension \( \Delta^{(p,p+2)}_{(p-1)/2,(p+1)/2} = -\frac{3}{4p(p+2)} \) in this model is given by (1.1) with \( B = \frac{1}{2} C'_{(p-1)/2} \), where \( C'_n \) is the Cartan matrix of the tadpole graph of \( n \) nodes (i.e. it differs from \( C_n \) only in one element, which is \( (C'_n)_{nn} = 1 \)), \( u_1 = \infty \) and \( u_a = 0 \) for \( a = 2, \ldots, \frac{p-1}{2} \), and \( m_a \in 2\mathbb{Z} \).

Note that for the first model in this series, \( \mathcal{M}(3,5) \), the sum (1.1) reduces to the form of (4.1),

\[
\chi^{(3,5)}_{1,2} (q) = \sum_{m=0}^{m \text{ even}} \frac{q^{m^2/2}}{(q)_m}. \quad (4.3)
\]
If the summation on the rhs of \( (4.3) \) is performed over odd instead of even \( m \), then one obtains \( q^{1/4} \chi^{(3,5)}_{1,3}(q) \). Furthermore

\[
\chi^{(3,5)}_{1,1}(q) = \sum_{m=0 \atop m \text{ even}} q^{(m^2+2m)/4} (q)_m ,
\]

and \( q^{3/4} \chi^{(3,5)}_{1,4}(q) \) is obtained if one sums over odd \( m \) in \( (4.4) \). The above four sums, representing all characters of \( \mathcal{M}(3,5) \), occur in [13] and have been encountered by Baxter [24] (see eq. (14.5.50)) in regime IV of the hard hexagon model.

4.4. Minimal Models \( \mathcal{M}(p, kp + 1) \).

For \( k=1 \) these are the unitary models considered in sect. 2. Here we consider the cases \( k = 2, 3, \ldots \) with \( p > 2 \) (when \( p=2 \) the models are those treated in [4] [5]). The character \( \chi^{(p,kp+1)}_{1,k}(q) \) of the field of lowest conformal dimension in the model is identified as \( (1.1) \) with \( n = k+p-3, \ m_1, \ldots, m_{k-1} \in \mathbb{Z} \) and \( m_k, \ldots, m_{k+p-3} \in 2\mathbb{Z} \), \( u_a=\infty \) for \( a = 1, \ldots, k \) and 0 otherwise. The nonzero elements of \( B \) are given by \( B_{ab} = 2(C'_{k-1})_{ab} \) and \( B_{ka}=B_{ak}=a \) for \( a, b = 1, 2, \ldots, k-1 \), and \( B_{ab} = \frac{1}{2} [(C_{p-2})_{ab} + (k-1)\delta_{ak}\delta_{bk}] \) for \( a, b = k, k+1, \ldots, k+p-3 \).

As in sect. 4.3, the case \( p=3 \) is special in that the fermionic sums are of the form \( (1.1) \) for any \( k \). We found that a slight modification of the matrix \( B \) appropriate for \( \mathcal{M}(3,3k+1) \), namely just setting \( B_{kk} = \frac{k}{2} \) while leaving all other elements unchanged, gives the character \( \chi^{(3,3k+2)}_{1,k}(q) \) in \( \mathcal{M}(3,3k+2) \).

4.5. Unitary \( N=2 \) superconformal series.

Expressions for the characters of these models, of central charge \( c = \frac{3k}{k+2} \) where \( k \) is a positive integer, can be found in [25]. The identity character, given by \( \chi^{0(0)}_0(q) + \chi^{0(2)}_0(q) \) in the notation of [25], can be obtained from \( (1.1) \) if one takes \( B = \frac{1}{2} C_{D_{k+2}}, \ u_k=\infty \) and all other \( u_a \) set to zero (in the basis where the two nodes on the fork of the \( D_{k+2} \) Dynkin diagram are labeled by \( k+1 \) and \( k+2 \), and the junction node is labeled by \( k \) ), and \( m_{k+1}, m_{k+2} \in \mathbb{Z} \) while \( m_a \in 2\mathbb{Z} \) for all other \( a \).
4.6. $Z_N$ parafermions.

The characters of these models are the branching functions $b_m^l$ of the cosets \( (A_{N-1}^{(1)})^1 \times (A_{N-1}^{(1)})^1 / (A_{N-1}^{(2)})^2 \), for which one type of fermionic sum representation is \[1\] of the form (4.1). The results of sect. 2 for the case $n=3$ provide an alternative fermionic representation for the characters of the $Z_3$-parafermion theory which coincides with the minimal model $\mathcal{M}(5,6)$ with the $D$-series partition function. (The $b_m^l$ in the case $N=3$ are linear combinations of the $\chi^{(5)}_{r,s}$.) This latter representation generalizes to arbitrary $N$; in particular, the identity character $b_0^0$ of the $Z_N$-parafermions is obtained from (4.1) by setting $B = \frac{1}{2} C_{D_N}$, $u_N=\infty$ (in the basis used in sect. 4.5) and all other $u_a$ set to zero, with $m_{N-1}, m_N$ running over all integers such that $m_{N-1} + m_N$ is even while $m_a \in 2\mathbb{Z}$ for $a = 1, \ldots, N-2$.

5. Behaviour as $q \to 1$ and Relation to Thermodynamic Bethe Ansatz

Modular covariance relates the behaviour of a character as $q \to 1$ to that of $q \to 0$, where the leading term is fixed by the effective central charge [10] [27] [28]

$$\bar{c} = c - 24\Delta_{\text{min}}$$

of the corresponding conformal field theory. Here $c$ is the central charge and $\Delta_{\text{min}}$ the lowest conformal dimension in the theory. The $q \to 1$ behaviour of sum forms like (1.1) and (2.3) must be consistent with this. It was noticed in [1] [3] [5] that the asymptotic analysis of the sums of the form (1.1) considered there leads to the same equations for $\bar{c}$ which had previously appeared in thermodynamic Bethe Ansatz computations of specific heats. We here extend these observations to the sum forms (2.3) and (4.1). In fact, the results below inspired many of the generalizations in sect. 4.

Our analysis follows that of [3] [29]. It is easy to see that the leading behaviour of a sum like $S_n[A](u|q)$ in (2.3) as $q \to 1$ is independent of $Q$ and $A$. An asymptotic dependence on $u$ exists only if some of its components are infinite. Without loss of generality we can therefore consider sums $S_B(u|q)$ of the form (1.1).

Let $q = e^{2\pi i \tau}$ and $\tilde{q} = e^{-2\pi i / \tau}$, with $\text{Im} \tau > 0$. Then if the coefficients in the series for $S_B(u|q) = \sum s_M q^M$ behave for large $M$ like $s_M \sim e^{2\pi \sqrt{\gamma M/6}}$, $\gamma > 0$, the series $S_B(u|q)$ diverges like

$$S_B(u|q) \sim \tilde{q}^{-\gamma/24} \quad \text{as} \quad q \to 1^-.$$

(5.2)
Here $\gamma$ must equal the effective central charge (5.1) of the corresponding conformal field theory.

The large $M$ behaviour of $s_M$ is found by considering

$$s_{M-1} = \oint_0 dq \frac{1}{2\pi i} q^{-M} S_B(u|q) = \sum_{m \geq 0} \oint_0 dq \frac{1}{2\pi i} q^{-M} S_B^m(u|q), \quad (5.3)$$

where the contour of integration is a small circle around 0. The behaviour of the integral is now analyzed using a saddle point approximation. We first approximate

$$\ln\left(q^{-M} S_B^m(u|q)\right) \simeq \left(\frac{1}{2} m B m^t - M\right) \ln q$$

$$+ \sum_{a=1}^n \left(\int_0^{(m(1-B)+\frac{m}{2})_a} - \int_0^{(-mB+\frac{m}{2})_a} - \int_0^{m_a}\right) dt \ln(1-q^t), \quad (5.4)$$

for large $m$, and set the derivatives of this expression with respect to the $m_a$ to zero in order to find the saddle point. Introducing $x_a = \frac{(1-w_a)v_a}{1-v_a w_a}$ and $y_a = \frac{1-w_a}{1-v_a w_a}$ where $v_a = q^{m_a}$ and $w_a = q^{-(mB+\frac{m}{2})_a}$, these extremum conditions reduce to

$$1 - x_a = \prod_{b=1}^n x_b^{B_{ab}}, \quad 1 - y_a = \sigma_a \prod_{b=1}^n y_b^{B_{ab}} \quad (5.5)$$

where we define $\sigma_a = 0$ if $u_a = \infty$ and 1 otherwise, ensuring $y_a = 1$ for $u_a = \infty$.

At the extremum point with respect to the $m_a$ we have

$$\ln\left(q^{-M} S_B^m(u|q)\right) \bigg|_{\text{ext}} \simeq -M \ln q$$

$$+ \frac{1}{\ln q} \left\{ \frac{1}{2} \ln v^B \ln v^t - \sum_{a=1}^n \left[ \mathcal{L}(1-v_a) + \mathcal{L}(1-w_a) - \mathcal{L}(1-z_a) \right] \right\}$$

$$- \frac{1}{2} \left[ \ln v \cdot \ln(1-v) + \ln w \cdot \ln(1-w) - \ln z \cdot \ln(1-z) \right] \quad (5.6)$$

with $(\ln v)_a = \ln v_a$ and $z_a = v_a w_a$, where

$$\mathcal{L}(z) = -\frac{1}{2} \int_0^z dt \left[ \ln t + \frac{\ln(1-t)}{t} \right] = - \int_0^z dt \frac{\ln(1-t)}{t} + \frac{1}{2} \ln z \ln(1-z) \quad (5.7)$$

is the Rogers dilogarithm function \[30\]. Now using (5.3) we see that the first term inside the braces in (5.6) cancels against the last. Then using the five-term relation for the dilogarithm \[30\]

$$\mathcal{L}(1-v) + \mathcal{L}(1-w) - \mathcal{L}(1-vw) = \mathcal{L}(1-x) - \mathcal{L}(1-y), \quad (5.8)$$
where \( x = \frac{(1-w)\nu}{1-vw} \) and \( y = \frac{1-w}{1-vw} \), we obtain

\[
\ln \left( q^{-M} S_B^m(u|q) \right) \bigg|_{\text{ext}} \simeq -M \ln q - \frac{\pi^2 \tilde{c}}{6 \ln q}
\]  

(5.9)

with

\[
\tilde{c} = \frac{6}{\pi^2} \sum_{a=1}^{n} \{ \mathcal{L}(1-x_a) - \mathcal{L}(1-y_a) \} .
\]  

(5.10)

Finally the value of \( q \) at the saddle point is determined by extremizing (5.9) with respect to \( q \), which leads to \( s_M \sim e^{2\pi \sqrt{\tilde{c}M/6}} \) and consequently to (5.2) with \( \gamma = \tilde{c} \) of (5.10).

It remains to check that (5.10) indeed gives the expected effective central charge. Consider first the general coset conjecture of sect. 4.2, where \( B \) is given by (4.2). With the \( x_a \) written as \( x_a^\alpha \), where \( a = 1, \ldots, n = k+l-1 \) and \( \alpha = 1, \ldots, r = \text{rank}(G) \), eq. (5.5) becomes

\[
1 - x_a^\alpha = \prod_{\beta=1}^{r} \prod_{b=1}^{n} (x_b^\beta (C_G^{-1})_{\alpha \beta}(C_G)_{ab}) .
\]  

(5.11)

The \( y_a^\alpha \) satisfy the same equations as the \( x_a^\alpha \) except for a factor \( \sigma_a^\alpha \) on the rhs which enforces \( y_l^\alpha = 1 \).

The system of equations (5.10)–(5.11) has been encountered previously in the study of the thermodynamics of RSOS spin chains [31] and in the thermodynamic Bethe Ansatz of integrable perturbations of the \( (G^{(1)})_k \times (G^{(1)})_l \) coset conformal field theories [32] (the particular case \( k=l=1 \) has been treated in [33,34], and the case of \( G=A_1 \) and arbitrary \( k, l \) in [35-37]). In the latter framework these equations arise in an analysis of the corresponding factorizable scattering theories where there are \( r \) particles which are associated with the labels \((\alpha, a=l)\) in (5.11). All other labels, corresponding to the excitations of finite momentum range in the context of sect. 3, can be thought of as associated with fictitious particles which are sometimes referred to as “pseudoparticles” [35] or “magnons” [32].

Finally, in order to reproduce (5.1) the sums of dilogarithms must be evaluated. This subject has been extensively investigated [29-31][34][38-44]. Here we use the sum rule [31,32]

\[
c(G, n) \equiv \frac{6}{\pi^2} \sum_{\alpha=1}^{r} \sum_{a=1}^{n} \mathcal{L}(1-x_a^\alpha) = (n+1)r - c(G^{(1)}_{n+1})
\]  

(5.12)

where

\[
c(G^{(1)}_k) = \frac{k \dim(G)}{k+h}
\]  

(5.13)
is the central charge of the level $k$ WZW model based on $G$, dim($G$) is the dimension of $G$ and $h$ its (dual) Coxeter number. As in [31][32] we therefore find

$$
\tilde{c} = c(G, n) - c(G, k - 1) - c(G, l - 1) = c(G_k^{(1)}) + c(G_l^{(1)}) - c(G_{k+l}^{(1)}) ,
$$

(5.14)
i.e. the central charge of the $\frac{(G_k^{(1)}) \times (G_l^{(1)})}{(G_{k+l}^{(1)})}$ coset conformal field theory. (For these unitary coset models $\Delta_{\text{min}}=0$ and so $\tilde{c} = c$.) Explicitly for $G = A_r$, for instance, (5.14) reads

$$
\tilde{c} = r(r+2)\left(\frac{k}{k+r+1} + \frac{l}{l+r+1} - \frac{k+l}{k+l+r+1}\right).
$$

(5.15)

For the (non-unitary) Virasoro minimal models $\mathcal{M}(p, p')$ the effective central charge is $\tilde{c} = 1 - \frac{6}{pp'}$. For the particular series of sects. 4.3 and 4.4, the dilogarithm sum rules which reproduce the correct $\tilde{c}$ for $\mathcal{M}(p, p + 2)$ and $\mathcal{M}(p, kp + 1)$ have been encountered in the thermodynamic Bethe Ansatz calculations of refs. [15] and [16], respectively (for $k$ odd in the latter case, but the sum rule holds for $k$ even as well). The case $\mathcal{M}(3, 5)$ has also been treated in [17], and we checked that our choice of the matrix $B$ appropriate for $\mathcal{M}(3, 3k + 2)$ (cf. sect. 5.4) leads to the expected effective central charge also for $k > 1$. Finally, the dilogarithm sum rules which are relevant for the theories discussed in sects. 4.5 and 4.6 can be found in [12] and [10], respectively.

There are numerous other examples of systems of equations of the form (5.5),(5.10) which arise in thermodynamic Bethe Ansatz computations, to which the general method of this section can be applied to obtain (conjectures for) fermionic sum representations of characters.

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