On differential completions and compactifications of a differential space

Diana Dziewa-Dawidczyk* Zbigniew Pasternak-Winiarski†

January 18, 2013

Abstract

Differential completions and compactifications of differential spaces are introduced and investigated. The existence of the maximal differential completion and the maximal differential compactification is proved. A sufficient condition for the existence of a complete uniform differential structure on a given differential space is given.

Key words and phrases: differential space, differential structure.

2000 AMS Subject Classification Code 58A40.

1 Introduction

This article is the third of the series of papers concerning integration of differential forms and densities on differential spaces (the first two are [4] and [5]). We describe differential completions and differential compactifications of differential spaces which are used in our theory of integration.

Section 2 of the paper contains basic definitions and the description of preliminary facts concerning theory of differential spaces. In Section 3 we give basic definitions and describe the standard facts concerning theory of uniform spaces. We introduce the notion of a differential completion of a differential space. We construct differential completions of a differential space using families of generators of its differential structure (Proposition 3.7, Definition 3.12). Section 4 is devoted to the investigation of properties of differential completions. We define some natural order in the set of all differential completions of a given differential space. We prove that for any differential space \((M, C)\) there exists the maximal differential completion with respect to this order (Theorem 4.1). If for the uniform structure defined by some family of generators the

*Faculty of Mathematics and Information Science, Warsaw University of Technology, Pl. Politechniki 1, 00-661 Warsaw, POLAND, e-mail: ddziewa@mini.pw.edu.pl
†Faculty of Mathematics and Information Science, Warsaw University of Technology, Pl. Politechniki 1, 00-661 Warsaw, POLAND, Institute of Mathematics, University of Białystok Akademicka 2, 15-267 Białystok, Poland, e-mail: Z.Pasternak-Winiarski@mini.pw.edu.pl
space $M$ is complete then the appropriate differential completion of $(M, C)$ is maximal and coincides with $(M, C)$ (Theorem 4.2). At the end we prove that if a differential structure $C$ posses a countable family of generators then it coincides with its maximal differential completion (Theorem 4.3). As a corollary we obtain general topological result about the existence on a given topological space a uniform structure defining the initial topology (Corollary 4.1). In Section 5 we introduce and investigate the notion of a differential compactification of a differential space. Similarly as in Section 4 we prove the existence of the maximal differential compactification of a given differential space with respect to the suitable order.

Without any other explanation we use the following symbols: \( \mathbb{N} \)-the set of natural numbers; \( \mathbb{R} \)-the set of reals.

### 2 Differential spaces

Let $M$ be a nonempty set and let $C$ be a family of real valued functions on $M$. Denote by $\tau_C$ the weakest topology on $M$ with respect to which all functions of $C$ are continuous.

A base of the topology $\tau_C$ consists of sets:

$$(\alpha_1, \ldots, \alpha_n)^{-1}(P) = \bigcap_{i=1}^{n} \{m \in M : a_i < \alpha_i(m) < b_i\},$$

where $n \in \mathbb{N}, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}, a_i < b_i, \alpha_1, \ldots, \alpha_n \in C, P = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; a_i < x_i < b_i, i = 1, \ldots, n\}$.

**Definition 2.1** A function $f : M \to \mathbb{R}$ is called a local $C$-function on $M$ if for every $m \in M$ there is a neighborhood $V$ of $m$ and $\alpha \in C$ such that $f|_V = \alpha|_V$. The set of all local $C$-functions on $M$ is denoted by $C_M$.

Note that any function $f \in C_M$ is continuous with respect to the topology $\tau_C$. Then $\tau_{C_M} = \tau_C$ (see [4], [5]).

**Definition 2.2** A function $f : M \to \mathbb{R}$ is called $C$-smooth function on $M$ if there exist $n \in \mathbb{N}, \omega \in C^\infty(\mathbb{R}^n)$ and $\alpha_1, \ldots, \alpha_n \in C$ such that

$$f = \omega \circ (\alpha_1, \ldots, \alpha_n).$$

The set of all $C$-smooth functions on $M$ is denoted by $scC$.

Since $C \subset scC$ and any superposition $\omega \circ (\alpha_1, \ldots, \alpha_n)$ is continuous with respect to $\tau_C$ we obtain $\tau_{scC} = \tau_C$ (see [4], [5]).

**Definition 2.3** A set $C$ of real functions on $M$ is said to be a (Sikorski’s) differential structure if: (i) $C$ is closed with respect to localization i.e. $C = C_M$; (ii) $C$ is closed with respect to superposition with smooth functions i.e. $C = scC$. 

2
In this case a pair \((M, C)\) is said to be a \((\text{Sikorski's})\) differential space (see [3]). Any element of \(C\) is called a smooth function on \(M\) (with respect to \(C\)).

**Proposition 2.1.** The intersection of any family of differential structures defined on a set \(M \neq \emptyset\) is a differential structure on \(M\).

For the proof see [4], [5], Proposition 2.1. □

Let \(\mathcal{F}\) be a set of real functions on \(M\). Then, by Proposition 2.1, the intersection \(\mathcal{C}\) of all differential structures on \(M\) containing \(\mathcal{F}\) is a differential structure on \(M\). It is the smallest differential structure on \(M\) containing \(\mathcal{F}\). One can easy prove that \(\mathcal{C} = \{\text{sc} \mathcal{F}\}_M\) (see [9]). This structure is called the differential structure generated by \(\mathcal{F}\) and is denoted by \(\text{gen}(\mathcal{F})\). Functions of \(\mathcal{F}\) are called generators of the differential structure \(\mathcal{C}\). We have also \(\tau_{\{\text{sc} \mathcal{F}\}_M} = \tau_{\text{sc} \mathcal{F}} = \tau_{\mathcal{F}}\) (see remarks after Definitions 2.1 and 2.2).

Let \((M, C)\) and \((N, D)\) be differential spaces. A map \(F : M \to N\) is said to be smooth if for any \(\beta \in D\) the superposition \(\beta \circ F \in C\). We will denote the fact that \(F\) is smooth writing

\[
F : (M, C) \to (N, D).
\]

If \(F : (M, C) \to (N, D)\) is a bijection and \(F^{-1} : (N, D) \to (M, C)\) then \(F\) is called a diffeomorphism.

If \(A\) is a nonempty subset of \(M\) and \(C\) is a differential structure on \(M\) then \(C_A\) denotes the differential structure on \(A\) generated by the family of restrictions \(\{\alpha_{|A} : \alpha \in C\}\). The differential space \((A, C_A)\) is called a differential subspace of \((M, C)\). One can easy prove the following

**Proposition 2.2.** Let \((M, C)\) and \((N, D)\) be differential spaces and let \(F : M \to N\). Then \(F : (M, C) \to (N, D)\) iff \(F : (M, C) \to (F(M), F(M)_D)\).

If the map \(F : (M, C) \to (F(M), F(M)_D)\) is a diffeomorphism then we say that \(F : M \to N\) is a diffeomorphism onto its range (in \((N, D)\)). In particular the natural embedding \(A \ni m \mapsto i(m) := m \in M\) is a diffeomorphism of \((A, C_A)\) onto its range in \((M, C)\).

If \(\{(M_i, C_i)\}_{i \in I}\) is an arbitrary family of differential spaces then we consider the Cartesian product \(\prod_{i \in I} M_i\) as a differential space with the differential structure \(\bigotimes_{i \in I} C_i\) generated by the family of functions \(\mathcal{F} := \{\alpha_i \circ \text{pr}_i : i \in I, \alpha_i \in C_i\}\), where \(\prod_{i \in I} M_i \ni (m_i) \mapsto \text{pr}_j((m_i)) := m_j \in M_j\) for any \(j \in I\). The topology \(\tau_{\bigotimes_{i \in I} C_i}\) coincides with the standard product topology on \(\prod_{i \in I} M_i\). We will denote the differential structure \(\bigotimes_{i \in I} C^\infty(\mathbb{R})\) on \(\mathbb{R}^I\) by \(C^\infty(\mathbb{R}^I)\). In the case when \(I\) is an \(n\)-element finite set the differential structure \(C^\infty(\mathbb{R}^I)\) coincides with the ordinary differential structure \(C^\infty(\mathbb{R}^n)\) of all real-valued functions on \(\mathbb{R}^n\) which posses partial derivatives of any
order (see [8]). In any case a function \( \alpha : \mathbb{R}^I \rightarrow \mathbb{R} \) is an element of \( C^\infty(\mathbb{R}^I) \) iff for any \( a = (a_i) \in \mathbb{R}^I \) there are \( n \in \mathbb{N} \), elements \( i_1, i_2, \ldots, i_n \in I \), a set \( U \) open in \( \mathbb{R}^n \) and a function \( \omega \in C^\infty(\mathbb{R}^n) \) such that \( a \in U[i_1, i_2, \ldots, i_n] := \{ (x_i) \in \mathbb{R}^I : (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \in U \} \) and for any \( x = (x_i) \in U[i_1, i_2, \ldots, i_n] \) we have

\[
\alpha(x) = \omega(x_{i_1}, x_{i_2}, \ldots, x_{i_n}).
\]

Let \( F \) be a family of generators of a differential structure \( C \) on a set \( M \). The generator embedding of the differential space \((M, C)\) into the Cartesian space defined by \( F \) is a mapping \( \phi_F : (M, C) \rightarrow (\mathbb{R}^F, C^\infty(\mathbb{R}^F)) \) given by the formula

\[
\phi_F(m) = (\alpha(m))_{\alpha \in F}
\]

(for example if \( F = \{ \alpha_1, \alpha_2, \alpha_3 \} \) then \( \phi_F(m) = (\alpha_1(m), \alpha_2(m), \alpha_3(m)) \in \mathbb{R}^3 \cong \mathbb{R}^F \)). If \( F \) separates points of \( M \) the generator embedding is a diffeomorphism onto its image. On that image we consider a differential structure of a subspace of \((\mathbb{R}^F, C^\infty(\mathbb{R}^F))\) (see [5], Proposition 2.3).

Let \( M \) be a group (a ring, a field, a vector space over the field \( K \)). A differential structure \( C \) on \( M \) is said to be a group (ring, field, vector space) differential structure if the suitable group (ring, field, vector space) operations are smooth with respect to \( C, C \otimes C \) and \( C_K \), where \( C_K \) is a field differential structure on \( K \). In this case the pair \((M, C)\) is called a differential group (ring field, vector space). If \( K = \mathbb{R} \) or \( K = \mathbb{C} \) we take \( C_K = C^\infty(\mathbb{K}) \) as a standard field differential structure.

**Proposition 2.3.** Let \( V \) be a vector space over \( \mathbb{R} \) and let \( F \) be a family of constant functions and linear functionals defined on \( V \). Then the differential structure \( C \) generated by \( F \) on \( V \) is a vector space differential structure.

For the proof see [4], Proposition 2.3. \( \square \)

**Definition 2.4.** By a tangent vector to a differential space \((M, C)\) at a point \( m \in M \) we call an \( \mathbb{R} \)-linear mapping \( v : C \rightarrow \mathbb{R} \) satisfying the Leibniz condition:

\[
v(\alpha \cdot \beta) = \alpha(m)v(\beta) + \beta(m)v(\alpha)
\]

for any \( \alpha, \beta \in C \). We denote by \( T_mM \) the set of all vectors tangent to \((M, C)\) at the point \( m \in M \) and call it the tangent space to \((M, C)\) at the point \( m \). The union \( TM := \bigcup_{m \in M} T_mM \) is called the tangent space to \((M, C)\).

The set \( TM \) can be endowed with a differential structure in the following standard way. We define the projection \( \pi : TM \rightarrow M \) such that for any \( m \in M \) and any \( v \in T_mM \)

\[
\pi(v) = m.
\]

For any \( \alpha \in C \) we define the differential (or the exterior derivative) of \( \alpha \) as a map \( d\alpha : TM \rightarrow \mathbb{R} \) given by the following formula

\[
d\alpha(v) := v(\alpha), \quad v \in TM.
\]
Then we define $TC$ as the differential structure on $TM$ generated by the family of functions $TC_0 := \{ \alpha \circ \pi : \alpha \in \mathcal{C} \} \cup \{ d\alpha : \alpha \in \mathcal{C} \}$. From now on we will consider $TM$ as a differential space with the differential structure $TC$.

For any $m \in M$ we will denote by $d\alpha_m$ the restriction $d\alpha_{|T_mM}$. It is clear that $d\alpha_m$ is a linear functional on $T_mM$.

We have also that $\pi : (TM, TC) \to (M, C)$. Then $\pi$ is continuous and for any $U \in \tau_C$ the set $TU := \bigcup_{m \in U} T_mM = \pi^{-1}(U)$ is open in $TM$ ($TU \in \tau_{TC}$). It can be proved that $TU$ is (isomorphic to) a tangent space to the differential space $(U, C_U)$.

**Theorem 2.1.** If $(M, C)$ is a differential space then for any $m \in M$ the pair $(T_mM, TC_{T_mM})$ is a differential vector space and $T_mM$ is a Hausdorff space (with respect to the topology induced by $TC_{T_mM}$).

For the proof see [4], Theorem 3.1. □

**Proposition 2.4.** Let $(M, C)$ and $(N, D)$ be differential spaces and let $F : (M, C) \to (N, D)$. Then for any $v \in TM$ the linear functional $TF(v) : D \to \mathbb{R}$ given by the formula
\[
[TF(v)](\beta) := v(\beta \circ F), \quad \beta \in D,
\]
is an element of $T_{F(\pi_M(v))}$, where $\pi_M : TM \to M$ is the natural projection.

**Proof.** We should show that $TF(v)$ fulfils Leibniz condition at the point $F(m)$, where $m = \pi_M(v)$.

For any $\beta, \gamma \in D$ we have
\[
[TF(v)](\beta \gamma) = v((\beta \gamma) \circ F) = v((\beta \circ F) (\gamma \circ F)) = v(\beta \circ F) (\gamma \circ F) = [TF(v)](\beta) \cdot (\gamma \circ F)(m) + \beta(\gamma(m)) \cdot TF(v)(\gamma).
\]
□

**Definition 2.5** Let $(M, C)$ and $(N, D)$ be differential spaces and let $F : (M, C) \to (N, D)$. The map $TF : TM \to TN$ given in Proposition 2.4 by the formula (1) is called the map tangent to $F$.

**Proposition 2.5.** If $F : (M, C) \to (N, D)$ then $TF : (TM, TC) \to (TN, TD)$ and $\pi_N \circ TF = F \circ \pi_M$, where $\pi_M : TM \to M$ and $\pi_N : TN \to N$ are natural projections.

**Proof.** The second part of the thesis follows immediately from Proposition 2.4 because for any $v \in TM$ the vector $TF(v) \in T_{F(\pi_M(v))}$. Then $\pi_N(TF(v)) = F(\pi_M(v))$. For the proof of the first part of the thesis it enough to show that for any $\kappa \in TD_0$ the superposition $\kappa \circ TF \in TC$. Let us consider the case $\kappa = \beta \circ \pi_N$. We have
\[
\kappa \circ TF = \beta \circ (\pi_N \circ TF) = (\beta \circ F) \circ \pi_M.
\]
Since $(\beta \circ F) \in C$ we obtain $\kappa \circ TF \in TC$. 

5
Now suppose that \( \kappa = d\beta \), where \( \beta \in D \). For any \( v \in TM \) we have
\[ \kappa \circ TF(v) = d\beta(TF(v)) = [TF(v)](\beta) = v(\beta \circ F) = d(\beta \circ F)(v). \]
Hence \( \kappa \circ TF = d(\beta \circ F) \in TC \).

Let us consider the differential space \((\mathbb{R}^I, C^\infty(\mathbb{R}^I))\). The differential structure \(C^\infty(\mathbb{R}^I)\) is generated by the family of projections \( F := \{ pr_i \}_{i \in I} \), where
\[ pr_j((x_i)) := x_j, \quad (x_i) \in \mathbb{R}^I, \quad j \in I. \]
For any \( x = (x_i), v = (v_i) \in \mathbb{R}^I \) the functional \( \vec{v} : C^\infty(\mathbb{R}^I) \to \mathbb{R} \) given by the formula
\[ \vec{v}(\alpha) := \sum_{i \in I} v_i \frac{\partial \alpha}{\partial x_i}(x) \]
is well defined (in some neighbourhood of \( x \) the function \( \alpha \) depends on finite number of variables \( x_i \)) and is a vector tangent to \( \mathbb{R}^I \) at \( x \). On the other hand, if \( u \in T_x\mathbb{R}^I \) and for any \( i \in I \) we denote \( v_i := u(pr_i) \) then for any \( \alpha \in C^\infty(\mathbb{R}^I) \) we have \( \vec{v}(\alpha) = u(\alpha) \). Then we identify the set \( T_x\mathbb{R}^I \) with \( \{ x \} \times \mathbb{R}^I \). Consequently we identify the set \( T\mathbb{R}^I \) with \( \mathbb{R}^I \times \mathbb{R}^I \). In this case the differential structure \( T C^\infty(\mathbb{R}^I) \) is generated by the family of functions \( TF := \{ pr_i \circ \pi \}_{i \in I} \cup \{ dpr_i \}_{i \in I} \), where
\[ \pi(x,v) = x, \quad (x,v) \in \mathbb{R}^I \times \mathbb{R}^I. \]
Hence for any \( j \in I \)
\[ pr_j \circ \pi((x_i),(v_i)) = x_j \quad \text{and} \quad dpr_j((x_i),(v_i)) = v_j. \]
It means that \( TC^\infty(\mathbb{R}^I) = C^\infty(\mathbb{R}^I \times \mathbb{R}^I) \) and consequently for any \( x \in \mathbb{R}^I \) the differential structure \( TC^\infty(\mathbb{R}^I)_{T_x\mathbb{R}^I} \) is generated by the family of projections \( \{ pr_i' \}_{i \in I} \), where
\[ pr_i'(x,(v_i)) = v_j. \]
Then we can identify \( TC^\infty(\mathbb{R}^I)_{T_x\mathbb{R}^I} \) with \( C^\infty(\mathbb{R}^I) \).

Let \( \phi_F : (M,C) \to (\mathbb{R}^F, C^\infty(\mathbb{R}^F)) \) be the generator embedding of the differential Hausdorff space \((M,C)\) defined by some family of generators \( F \). Then we can identify differential spaces \((M,C)\) and \((\phi_F(M), C^\infty(\mathbb{R}^F)_{\phi_F(M)})(\phi_F \text{ is a diffeomorphism}). We also identify tangent spaces \( T_mM \) and \( T_{\phi_F(m)}\phi_F(M) \) using the tangent map \( T\phi_F \) (for any \( \alpha \in C^\infty(\mathbb{R}^F)_{\phi_F(M)} \)).

**Theorem 2.2.** Let \( I \) be an arbitrary nonempty set and let \( X \) be a nonempty subset of the Cartesian space \( \mathbb{R}^I \). Then for any \( x = (x_i) \in X \) the space \( T_xX \) tangent to the differential space \( (X, C^\infty(\mathbb{R}^I)_X) \) at the point \( x \) is a closed subspace of the space \( T_x\mathbb{R}^I \) tangent to the differential space \( (\mathbb{R}^I, C^\infty(\mathbb{R}^I)) \) at \( x \).

For the proof see 4, Theorem 3.2.

A map \( X : M \to TM \) such that for any \( m \in M \) the value \( X(m) \in T_mM \) is called a vector field on \( M \). A vector field \( X \) on \( M \) is smooth if \( X : (M,C) \to (TM, TC) \).
3 Uniform structures and completions of a differential space defined by families of generators

For the general theory of uniform structures and completions see [6], Chapter 8 or [1]. It is also described in [7] and [5]. Here we start with the definition of the uniform structure given on a differential space by a family $\mathcal{F}$ of generators of its differential structure.

Let $\mathcal{F}$ be a family of real-valued functions on a set $M$ and let $(M, C)$ be a differential space such that $C = (\text{sc} \mathcal{F})_M$ and $(M, \tau_C)$ is a Hausdorff space (the last is true iff the family $C$ separates points in $X$ iff the family $\mathcal{F}$ separates points in $X$). On the set $M$ the family $\mathcal{F}$ defines the uniform structure $U$ such that the base $B$ of $U$ is given as follows:

$$B = \{ V(f_1, \ldots, f_k, \varepsilon) \subset M \times M; k \in \mathbb{N}; f_1, \ldots, f_k \in \mathcal{F}, \varepsilon > 0 \},$$ (2)

where

$$V(f_1, \ldots, f_k, \varepsilon) = \{ (x, y) \in M \times M : \forall 1 \leq i \leq k \ | f_i(x) - f_i(y) | < \varepsilon \}$$ (see [5], Proposition 3.1).

**Definition 3.1** The uniform structure $U$ on a set $M$ is said to be a differential uniform structure on the differential space $(M, C)$ if there exist a family $\mathcal{F}$ of generators of $C$ such that $U = U^F$, where $U^F$ is defined by the base (2). The uniform space $(M, U^F)$ is said to be the uniform space given by the family of generators $\mathcal{F}$.

If we have two different families $\mathcal{F}_1$ and $\mathcal{F}_2$ of generators of a differential space $(M, C)$, then the uniform structures $U^{F_1}$ and $U^{F_2}$ can be different too.

**Example 3.1** Let $M = \mathbb{R}$, $C = C^\infty(\mathbb{R})$, $\mathcal{F}_1 = \{ id_\mathbb{R} \}$ and $\mathcal{F}_2 = \{ id_\mathbb{R}, f \}$, where

$$id_\mathbb{R}(x) = x, \text{ and } f(x) = x^2, \quad x \in \mathbb{R}.$$ 

Then does not exists $\varepsilon > 0$ such that $V(id_\mathbb{R}, \varepsilon) \subset V(f, 1)$. Hence $V(f, 1) \not\in U^{F_1}$ and $U^{F_1} \neq U^{F_2}$. □

If $\mathcal{F}$ is a family of generators of a differential structure $C$ on a set $M$ then we define a uniform structure $U_{T \mathcal{F}}$ on the space $TM$ tangent to the differential space $(M, C)$ using the family of real-valued functions

$$T \mathcal{F} = \{ f \circ \pi : f \in \mathcal{F} \} \cup \{ df : f \in \mathcal{F} \},$$

where $\pi : TM \to M$ is the natural projection and $df : TM \to \mathbb{R}$, $df(v) = v(f)$. As we know from the previous section, the family $T \mathcal{F}$ generates the natural differential structure $T^C$ on $TM$. The base $D$ of $U_{T \mathcal{F}}$ is given by:

$$D = \{ V(\pi \circ f_1, \ldots, \pi \circ f_k, df_{k+1}, \ldots, df_m, \varepsilon) \subset TM \times TM ; k, m \in \mathbb{N},$$

7
Let \( (X, \mathcal{U}) \) and \( (Y, \mathcal{V}) \) be uniform spaces.

**Definition 3.2** A mapping \( f : X \to Y \) is said to be uniform with respect to uniform structures \( \mathcal{U} \) and \( \mathcal{V} \) if

\[
\forall V \in \mathcal{V} \exists U \in \mathcal{U} \forall x, x' \in X \quad |x - x'| < U \Rightarrow |f(x) - f(x')| < V.
\]

In other words for every \( V \in \mathcal{V} \) there is \( U \in \mathcal{U} \) such that \( U \subset (f \times f)^{-1}(V) \). We denote it by

\[
f : (X, \mathcal{U}) \to (Y, \mathcal{V}).
\]

It is easy to prove that:

(i) any uniform mapping \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is continuous with respect to topologies \( \tau_{\mathcal{U}} \) and \( \tau_{\mathcal{V}} \);

(ii) a superposition of uniform mappings is a uniform mapping.

We can give criteria of the uniformity:

**Theorem 3.1** Let \( f : X \to Y \) and let \( \mathcal{U} \) and \( \mathcal{V} \) be uniform structures on \( X \) and \( Y \) respectively. Then the following conditions are equivalent:

(a) \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}). \)

(b) If \( \mathcal{B} \) and \( \mathcal{D} \) are bases of \( \mathcal{U} \) and \( \mathcal{V} \) respectively then for each \( V \in \mathcal{D} \) there exists \( U \in \mathcal{B} \) such that \( U \subset (f \times f)^{-1}(V) \).

(c) For every pseudometric \( \varrho \) in \( Y \) uniform with respect to \( \mathcal{V} \), a pseudometric \( \sigma \) in \( X \) given by the formula

\[
\sigma(x, y) = \varrho(f(x), f(y)), \quad x, y \in X,
\]

is uniform with respect to the uniform structure \( \mathcal{U} \).

For the proof see [6].

A mapping \( f \), that is uniform with respect to uniform structures \( \mathcal{U} \) and \( \mathcal{V} \) could not be uniform with respect to another uniform structures \( \mathcal{U}' \) and \( \mathcal{V}' \) defined on \( X \) and \( Y \) respectively even if topologies \( \tau_{\mathcal{U}} = \tau_{\mathcal{U}'} \) and \( \tau_{\mathcal{V}} = \tau_{\mathcal{V}'} \).

**Example 3.2** Let \( M = \mathbb{R}, \mathcal{C} = C^\infty(\mathbb{R}), \mathcal{F}_1 = \{id_{\mathbb{R}}\}, \mathcal{F}_2 = \{id_{\mathbb{R}}, f\} \), where \( f(x) = x^2, x \in \mathbb{R} \). Let \( \mathcal{V} \) be a standard uniform structure on \( \mathbb{R} \). Then the map \( f \) is uniform with respect to \( \mathcal{U}_{\mathcal{F}_1} \) and \( \mathcal{V} \), but it is not uniform with respect to \( \mathcal{U}_{\mathcal{F}_2} \) and \( \mathcal{V} \). In fact, if \( V = \{(x, y) \in \mathbb{R} : |x - y| < \varepsilon\} \in \mathcal{V} \), then does not exists \( U \in \mathcal{U}_{\mathcal{F}_1} \) such that \( U \subset (f \times f)^{-1}(V) \).
DEFINITION 3.3 A bijective mapping \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is a uniform homeomorphism if \( f^{-1} \) is a uniform mapping. Then we say that \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) are uniformly homeomorphic.

By (i) it is obvious that if \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is a uniform homeomorphism then \( f \) is a homeomorphism of the topological spaces \((X, \tau_{\mathcal{U}})\) and \((Y, \tau_{\mathcal{V}})\).

Let \((X, \mathcal{U})\) be a uniform space and let \( A \subset X \). Then the family \( \mathcal{U}_A := \{(A \times A) \cap U : U \in \mathcal{U}\} \) is a uniform structure on \( A \). The uniform space \((A, \mathcal{U}_A)\) is called the uniform subspace of the uniform space \((X, \mathcal{U})\). Note that if \( \mathcal{F} \) is a family of generators of a differential structure \( C \) on a set \( M, A \subset M \) and \( \mathcal{F}_|A = \{f|_A : f \in \mathcal{F}\} \), then the uniform space \((A, \mathcal{U}_\mathcal{F}_|A)\) is a uniform subspace of the uniform space \((M, \mathcal{U}_\mathcal{F})\).

DEFINITION 3.4 Let \((X, \mathcal{D})\) be a uniform space and \( V \in \mathcal{D} \). A set \( U \subset X \) is said to be small of rank \( V \) if \( \exists x \in U \forall y \in U \ [(x, y) \in V] \) (see [5], Definition 2.2).

If we define the ball \( K(x, V) \) as a set:
\[
K(x, V) = \{ y \in X : (x, y) \in V \}
\]
then a set \( U \subset X \) is small of rank \( V \) iff \( \exists x \in U \forall y \in U \ [(x, y) \in V] \).

If \( F \subset X \) and \( V \in \mathcal{D} \) we define the \( V \)-neighbourhood of \( F \) as a set
\[
K(F, V) := \bigcup_{x \in F} K(x, V) = \{ y \in X : \exists x \in F \ [(x, y) \in V] \}.
\]

DEFINITION 3.5 A nonempty family \( \mathcal{F} \) of subsets of a set \( X \) is said to be a filter on \( X \) if:

(F1) \( F \in \mathcal{F} \land F \subset U \subset X \) \( \Rightarrow \) \( U \in \mathcal{F} \);

(F2) \( F_1, F_2 \in \mathcal{F} \) \( \Rightarrow \) \( F_1 \cap F_2 \in \mathcal{F} \);

(F3) \( \emptyset \notin \mathcal{F} \).

DEFINITION 3.6 A filtering base on \( X \) is a nonempty family \( \mathcal{B} \) of subsets of \( X \) such that

(FB1) \( \forall A_1, A_2 \in \mathcal{B} \exists A_3 \in \mathcal{B} \ [A_3 \subset A_1 \cap A_2] \);

(FB2) \( \emptyset \notin \mathcal{B} \).

If \( \mathcal{B} \) is a filtering base on \( X \) then
\[
\mathcal{F} = \{ F \subset X : \exists A \in \mathcal{B} \ [A \subset F] \}.
\]
is a filter on \( X \). It is called the filter defined by \( B \) and \( B \) is called the base of \( F \).

**Proposition 3.1** If \( \{ F_i \}_{i \in I} \) is the family of filters on the set \( X \) then the intersection \( \bigcap_{i \in I} F_i \) is a filter on \( X \).

*Proof.* It is obvious that \( \bigcap_{i \in I} F_i \) fulfils (F3). Suppose now that \( F \in \bigcap_{i \in I} F_i \). Then \( F \in F_i \) for any \( i \in I \) and if \( F \subset U \subset X \) we obtain \( U \in F_i \). Hence \( U \in \bigcap_{i \in I} F_i \). It means that \( \bigcap_{i \in I} F_i \) fulfils (F1).

Let us consider now two arbitrary elements \( F, G \in \bigcap_{i \in I} F_i \). For any \( i \in I \) we have \( F, G \in F_i \) and therefore (by (F2)) \( F \cap G \in F_i \). Hence \( F \cap G \in \bigcap_{i \in I} F_i \). It means that \( \bigcap_{i \in I} F_i \) fulfils (F2). \( \square \)

**Definition 3.7** Let \( X \) be a topological space. We say that a filter \( F \) on \( X \) is convergent to \( x \in X \) (\( F \to x \)) if for any neighbourhood \( U \) of \( x \) there exists \( F \in F \) such that \( F \subset U \) (i.e. \( U \in F \)).

**Proposition 3.2** If \( X \) is a topological space and for any \( i \in I \) the filter \( F_i \to x \) then \( \bigcap_{i \in I} F_i \to x \).

*Proof.* For any neighbourhood \( U \) of \( x \) and any \( i \in I \) we have \( U \in F_i \). Hence \( U \in \bigcap_{i \in I} F_i \). It means that \( \bigcap_{i \in I} F_i \to x \). \( \square \)

**Definition 3.8** Let \((X, U)\) be a uniform space. A filter \( F \) on \( X \) is a Cauchy filter if

\[
\forall V \in U \\exists F \in F \quad [F \times F \subset V].
\]

We say that two Cauchy filters \( F_1 \) and \( F_2 \) are in the relation \( R \) if

\[
\forall V \in U \\exists F_1 \in F_1, F_2 \in F_2 \quad [F_1 \times F_2 \subset V].
\]

**Proposition 3.3** Two filters \( F_1 \) and \( F_2 \) on the uniform space \((X, U)\) are in the relation \( R \) iff \( F_1, F_2 \) and \( F_1 \cap F_2 \) are Cauchy filters on \( X \).

*Proof.* (\( \Rightarrow \)) Suppose \( F_1 \) and \( F_2 \) to be in the relation \( R \) and fix \( V \in U \). Let \( W \in U \) be such that \( 4W \subset V \). There exist \( F_1 \in F_1 \) and \( F_2 \in F_2 \) such that \( F_1 \times F_2 \subset W \). Then \( F_2 \subset K(F_1, W) \) which implies that \( K(F_1, W) \in F_2 \). Since \( F_1 \subset K(F_1, W) \) we have \( K(F_1, W) \in F_1 \). Hence \( K(F_1, W) \in F_1 \cap F_2 \). On the other hand, for any \( y_1, y_2 \in K(F_1, W) \) there are \( x_1, x_2 \in F_1 \) such that \( (y_1, x_1), (y_2, x_2) \in W \). For an arbitrary chosen \( z \in F_2 \) we have \( (z, x_1), (z, x_2) \in W \). Hence \( (y_1, y_2) \in 4W \subset V \). I means that \( K(F_1, W) \times K(F_1, W) \subset V \). Since \( K(F_1, W) \) is an element of \( F_1, F_2 \) and \( F_1 \cap F_2 \) we obtain all this filters to be Cauchy filters.

(\( \Leftarrow \)) Suppose \( F_1 \cap F_2 \) to be Cauchy filter on \( X \). Fix \( V \in U \), choose \( F \in F_1 \cap F_2 \) such that \( F \times F \subset V \) and put \( F_1 := F_2 := F \). Then \( F_1 \in F_1 \), \( F_2 \in F_2 \) and
Proposition 3.4 The relation $R$ described in Definition 3.8 is an equivalence relation on the set $\mathcal{CF}(X)$ of all Cauchy filters on the uniform space $(X, \mathcal{U})$.

Proof. It is obvious that for any Cauchy filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on $X$ we have $\mathcal{F}_1 R \mathcal{F}_2$, and if $\mathcal{F}_1 R \mathcal{F}_2$ then $\mathcal{F}_2 R \mathcal{F}_1$. Suppose now $\mathcal{F}_3$ to be such a Cauchy filter on $X$ that $\mathcal{F}_1 R \mathcal{F}_3$ and $\mathcal{F}_2 R \mathcal{F}_3$. Fix $V \in \mathcal{U}$ and choose $W \in \mathcal{U}$ such that $2W \subset V$. There exist $F_1 \in \mathcal{F}_1$, $F_2', F_3'' \in \mathcal{F}_2$ and $F_3 \in \mathcal{F}_3$ such that $F_1 \times F_2' \subset W$ and $F_2'' \times F_3 \subset W$. Let $F_2 := F_2' \cap F_2''$. Then $F_2 \in \mathcal{F}_2$, $F_1 \times F_2 \subset W$ and $F_2 \times F_3 \subset W$. Since $2W \subset V$ we obtain $F_1 \times F_3 \subset V$. Hence $\mathcal{F}_1 R \mathcal{F}_3$. □

For any Cauchy filter $\mathcal{F}$ on $X$ we denote by $[\mathcal{F}]$ the equivalence class of $\mathcal{F}$ with respect to the equivalence relation $R$ given in Definition 3.8.

Proposition 3.5 If $\{\mathcal{F}_i\}_{i \in I}$ is a family of Cauchy filters on an uniform space $X$ contained in an equivalence class $[\mathcal{F}]$ then $\bigcap_{i \in I} \mathcal{F}_i \in [\mathcal{F}]$ i.e. $\bigcap_{i \in I} \mathcal{F}_i$ is a Cauchy filter and it is equivalent to $\mathcal{F}$.

Proof. Let $V$ be an arbitrary element of the uniform structure $\mathcal{U}$ on $X$. Let $W \in \mathcal{U}$ be such that $4W \subset V$. Choose $F \in \mathcal{F}$ such that $F \times F \subset W$. Similarly as in the proof of Proposition 3.3 we obtain that $K(F, W) \times K(F, W) \subset 3W \subset V$. For any $i \in I$ there are $F_i \in \mathcal{F}_i$ and $G_i \in \mathcal{F}$ such that $F_i \times G_i \subset W$. Hence $F_i \times (G_i \cap F) \subset W$ and therefore $F_i \subset K(F, W)$. Consequently $K(F, W) \in \mathcal{F}_i$ for any $i \in I$. Then $K(F, W) \in \bigcap_{i \in I} \mathcal{F}_i$ and moreover $K(F, W) \times K(F, W) \subset V$. It means that $\bigcap_{i \in I} \mathcal{F}_i$ is a Cauchy filter on $X$. Since $K(F, W) \in \mathcal{F}$ ($F \subset K(F, W))$ we obtain that $\bigcap_{i \in I} \mathcal{F}_i$ is equivalent to $\mathcal{F}$. □

Corollary 3.1 If $\mathcal{F}$ is a Cauchy filter on $X$ then $\bigcap_{\mathcal{G} \in [\mathcal{F}]} \mathcal{G}$ is a Cauchy filter on $X$ equivalent to $\mathcal{F}$. Since for any $\mathcal{F}_1 \in [\mathcal{F}]$ we have $\bigcap_{\mathcal{G} \in [\mathcal{F}]} \mathcal{G} \subset \mathcal{F}_1$ we obtain $\bigcap_{\mathcal{G} \in [\mathcal{F}]} \mathcal{G}$ is the minimal element of $[\mathcal{F}]$ with respect to the ordering relation $\subset$ on the family of all filters on a set $X$. □

Definition 3.9 For any Cauchy filter $\mathcal{F}$ on $X$ the Cauchy filter $\bigcap_{\mathcal{G} \in [\mathcal{F}]} \mathcal{G}$ is called the minimal Cauchy filter on $X$ defined by (smaller then) $\mathcal{F}$.

Definition 3.10 A uniform space $(X, \mathcal{U})$ is said to be complete if each Cauchy filter on $X$ is convergent in $\tau_\mathcal{U}$.

Theorem 3.2 If $(X, \mathcal{U})$ is a complete uniform space and $M$ is a closed subset of the topological space $(X, \tau_\mathcal{U})$ then a uniform space $(M, \mathcal{U}_M)$ is complete. Conversely, if $(M, \mathcal{U}_M)$ is a complete uniform subspace of some (not necessarily complete) uniform space $(X, \mathcal{U})$, then $M$ is closed in $X$ with respect to $\tau_\mathcal{U}$. 

11
For the proof see [1], [6] or [7].

It is well known that the uniform space of reals \((\mathbb{R}, \mathcal{U})\) with the standard uniform structure \(\mathcal{U} = \{id_\mathbb{R}\}\) defined by the one element family of functions \(\{id_\mathbb{R}\}\) (or by the standard metric) is complete. We have also more general

**PROPOSITION 3.6** For any set \(I\) the uniform space \((\mathbb{R}^I, \mathcal{U}_G)\), where \(G = \{pr_i\}_{i \in I}\) is the set of all natural projections \(pr_i : \mathbb{R}^I \to \mathbb{R}\),

\[
pr_i(f) = f(i), \quad f \in \mathbb{R}^I,
\]

for any \(i \in I\), is complete.

**Proof.** If \(F\) is a Cauchy filter on \(\mathbb{R}^I\) then for any \(i \in I\) the set \(pr_i(F) = \{pr_i(F) : F \in F\}\) is a filtering base of some Cauchy filter on \(\mathbb{R}\). Then the Cauchy filter corresponding to \(pr_i(F)\) converges to some \(y_i \in \mathbb{R}\). Putting \(f(i) := y_i, \ i \in I\) we obtain function \(f \in \mathbb{R}^I\) such that \(F \to f\). \(\square\)

Any uniform space can be treated as a uniform subspace of some complete uniform space. We have the following

**THEOREM 3.3** For each uniform space \((X, \mathcal{U})\):
(i) there exists a complete uniform space \((\tilde{X}, \tilde{\mathcal{U}})\) and a set \(A \subset \tilde{X}\) dense in \(\tilde{X}\) (with respect to the topology \(\tau_{\tilde{\mathcal{U}}}\)) such that \((X, \mathcal{U})\) is uniformly homeomorphic to \((A, \tilde{\mathcal{U}}_A)\);
(ii) if the complete uniform spaces \((\tilde{X}_1, \tilde{\mathcal{U}}_1)\) and \((\tilde{X}_2, \tilde{\mathcal{U}}_2)\) satisfies condition of the point (i) then they are uniformly homeomorphic.

For the details of the proof see [1] or [7]. Here we only want to describe the construction of \((\tilde{X}, \tilde{\mathcal{U}})\).

Let \(\tilde{X}\) be the set of all minimal Cauchy filters in \(X\). For every \(V \in \mathcal{U}\) we denote by \(\tilde{V}\) the set of all pairs \((\mathcal{F}_1, \mathcal{F}_2)\) of minimal Cauchy’s filters, which have a common element being a small set of rank \(V\). We define a family \(\tilde{\mathcal{U}}\) of subsets of \(\tilde{X} \times \tilde{X}\) as the smallest uniform structure on \(X\) containing all sets from the family \(\{\tilde{V} : V \in \mathcal{U}\}\).

**EXAMPLE 3.3** Let us consider two uniform structures \(\mathcal{U}_{(f)}\) and \(\mathcal{U}_{(g)}\) on the differential space \((\mathbb{R}, C^\infty)\), where

\[
f(x) = x, \quad g(x) = \arctan x, \quad x \in \mathbb{R}.
\]

Then \((\mathbb{R}, \mathcal{U}_{(f)})\) is the complete space i.e. \(\tilde{\mathcal{U}}_{(f)} = \mathcal{U}_{(f)}\) while \((\mathbb{R}, \mathcal{U}_{(g)})\) is not complete and we have \(X \simeq [-\frac{\pi}{2}; \frac{\pi}{2}]\). Consequently \(\mathcal{U}_{(f)} \neq \mathcal{U}_{(g)}\).

Let \(N\) be a set, \(M \subseteq N, M \neq \emptyset\), \(C\) be a differential structure on \(M\).

**DEFINITION 3.11.** The differential structure \(\mathcal{D}\) on \(N\) is an extension of the differential structure \(C\) from the set \(M\) to the set \(N\) if \(C = D_M\) (if we get the structure \(C\) by localization of the structure \(\mathcal{D}\) to \(M\)).
For the sets \( N, M \) and the differential structure \( \mathcal{C} \) on \( M \) we can construct many different extensions of the structure \( M \) to \( N \).

**Example 3.4.** If for each function \( f \in \mathcal{C} \) we assign \( f_0 \in \mathbb{R}^N \) such that \( f_0|_M = f \) and \( f_0|_{N \setminus M} \equiv 0 \), then the differential structure generated on \( N \) by the family of functions \( \{ f_0 \}_{f \in \mathcal{C}} \) is an extension of \( \mathcal{C} \) from \( M \) to \( N \). Similarly, if for each function \( f \in \mathcal{C} \) we assign the family \( \mathcal{F}_f := \{ g \in \mathbb{R}^N : g|_M = f \} \), then the differential structure on \( N \) generated the family of functions \( \mathcal{F} := \bigcup_{f \in \mathcal{C}} \mathcal{F}_f \) is an extension of \( \mathcal{C} \) from \( M \) to \( N \). If the set \( N \setminus M \) contain at least two elements, then the differential structures generated by the families \( \{ f_0 \}_{f \in \mathcal{C}} \) and \( \mathcal{F} \) are different.

**Definition 3.12.** If \( \tau \) is a topology on the set \( N \), then the extension \( \mathcal{D} \) of the differential structure \( \mathcal{C} \) from \( M \) to \( N \) is **continuous with respect to** \( \tau \) if each function \( f \in \mathcal{D} \) is continuous with respect to \( \tau \) (\( \tau_{\mathcal{D}} \subset \tau \)).

If on the set \( N \) there exists a continuous with respect to some topology \( \tau \) extension of the differential structure \( \mathcal{C} \) from the set \( M \subset N \), then the structure \( \mathcal{C} \) is said to be **extendable from the set \( M \) to the topological space \( (N, \tau) \).**

**Example 3.5.** The differential structure \( C^\infty(\mathbb{R}) \) is extendable from the set of rationals to the set of reals. The continuous extensions are e.g. \( C^\infty(\mathbb{R}) \) and the structure \( \mathcal{D} \) generated on \( \mathbb{R} \) by the family of the functions \( C^\infty(\mathbb{R}) \cup \{ f \} \), where \( f : \mathbb{R} \to \mathbb{R}, f(x) := |x - \sqrt{2}|, x \in \mathbb{R} \).

**Proposition 3.7** Let \( M \neq \emptyset, (M, \mathcal{C}) \) be a differential space and \( \mathcal{G} \) be a family of generators of \( \mathcal{C} \), i.e. \( \mathcal{C} = \text{gen}(\mathcal{G}) \). Let \( (\hat{M}, \mathcal{U}_{\hat{G}}) \) be the completion of the uniform space \( (M, \mathcal{U}_G) \). Then any function \( g \in \mathcal{G} \) poses the continuous extension \( \hat{g} : \hat{M} \to \mathbb{R} \). If \( \hat{G} \) is the family of all continuous extensions of elements of \( \mathcal{G} \) to \( \hat{M} \) then the differential structure \( \mathcal{D} = \text{gen}(\hat{G}) \) is an continuous extension of the differential structure \( \mathcal{C} \) from the set \( M \) to the set \( \hat{M} \). Moreover \( \tau_\mathcal{D} = \tau_{\mathcal{U}_{\hat{G}}} \).

**Proof.** Let \( \phi_\mathcal{G} \) be the generator embedding of the differential space \( (M, \mathcal{C}) \) into the Cartesian space \( (\mathbb{R}^\mathcal{G}, C^\infty(\mathbb{R}^\mathcal{G})) \) defined by \( \mathcal{G} \). Then the closure \( \phi_\hat{G}(\hat{M}) \) is a complete subspace of the complete uniform space \( \mathbb{R}^\hat{G} \) (see Theorem 3.2 and Proposition 3.6). We know that \( \mathbb{R}^\mathcal{G} : (M, \mathcal{C}) \to (\phi_\mathcal{G}(M), C^\infty(\mathbb{R}^\mathcal{G})_{\phi_\mathcal{G}(M)}) \) is a diffeomorphism and \( pr_g \circ \phi_\mathcal{G} = g \) for any \( g \in \mathcal{G} \). Moreover \( \phi_\mathcal{G}(M) \) is dense in \( \phi_\mathcal{G}(\hat{M}) \). Then identifying any \( g \in \mathcal{G} \) with \( \pi_{g|_{\phi_\mathcal{G}(M)}} \) with \( C^\infty(\mathbb{R}^\mathcal{G})_{\phi_\mathcal{G}(M)} \) and putting \( \hat{M} := \phi_\mathcal{G}(\hat{M}) \) we obtain that \( \hat{g} \) should be identify with \( pr_g|_{\phi_\mathcal{G}(\hat{M})} \) and \( \mathcal{D} = C^\infty(\mathbb{R}^\mathcal{G})_{\phi_\mathcal{G}(\hat{M})} \). We have also \( \tau_\mathcal{D} = \tau_{\mathcal{U}_{\hat{G}}} \), where \( \tau_{\mathcal{U}_{\hat{G}}} \) is the topology of \( \hat{M} \) treated as a topological subspace of \( \mathbb{R}^\hat{G} \). \( \square \)

**Definition 3.12.** The differential space \( (\hat{M}, \mathcal{D}) \) constructed in Proposition 3.7 will be called the **differential completion** of the differential space \( (M, \mathcal{C}) \) defined by the family of generators \( \mathcal{G} \). The set \( \hat{M} \) will be denoted by \( \text{compl}_G M \) and the differential structure \( \mathcal{D} \) will be denoted by \( \text{compl}_G \mathcal{C} \).
4 The maximal differential completion

Let us consider two families $\mathcal{G}$ and $\mathcal{H}$ of generators of a differential structure $C$ on a set $M \neq \emptyset$. If $\mathcal{G} \subset \mathcal{H}$ then for uniform structures $U_\mathcal{G}$ and $U_\mathcal{H}$ we have: $U_\mathcal{G} \subset U_\mathcal{H}$. Consequently any Cauchy filter with respect to $U_\mathcal{H}$ is a Cauchy filter with respect to $U_\mathcal{G}$. In particular any minimal Cauchy filter with respect to $U_\mathcal{H}$ is a Cauchy (but not necessarily minimal Cauchy) filter with respect to $U_\mathcal{G}$. This defines the natural map $\iota_{\mathcal{GH}}: \text{compl}_\mathcal{H}M \to \text{compl}_\mathcal{G}M$ as follows: for any $F \in \text{compl}_\mathcal{H}M$ the value $\iota_{\mathcal{GH}}(F) \in \text{compl}_\mathcal{G}M$ is the minimal Cauchy filter equivalent to $F$ with respect to the uniform structure $U_\mathcal{G}$.

**Proposition 4.1** For any two families $\mathcal{G}$ and $\mathcal{H}$ of generators of a differential structure $C$ on a set $M \neq \emptyset$ such that $\mathcal{G} \subset \mathcal{H}$ the map $\iota_{\mathcal{GH}}: \text{compl}_\mathcal{H}M \to \text{compl}_\mathcal{G}M$ defined above is smooth with respect to differential structures $\text{compl}_\mathcal{H}C$ and $\text{compl}_\mathcal{G}C$.

**Proof.** For smoothness of $\iota_{\mathcal{GH}}$ it is enough to prove that for any $g \in \mathcal{G}$ the function $\tilde{g}_G \circ \iota_{\mathcal{GH}} \in \text{compl}_\mathcal{H}C$, where $\tilde{g}_G$ denotes the continuous extension of $g$ onto $\text{compl}_\mathcal{G}M$. Since any $g \in \mathcal{G}$ is an element of $\mathcal{H}$ we have for each Cauchy filter $F \in \text{compl}_\mathcal{H}M$

$$\tilde{g}_G \circ \iota_{\mathcal{GH}}(F) = \lim g(\iota_{\mathcal{GH}}(F)) = \lim g(F) = \tilde{g}_H(F),$$

where $\tilde{g}_H$ denotes the continuous extension of $g$ onto $\text{compl}_\mathcal{H}M$. Hence $\tilde{g}_G \circ \iota_{\mathcal{GH}} = \tilde{g}_H \in \text{compl}_\mathcal{H}C$.

**Remark 4.1** In general the image $\iota_{\mathcal{GH}}(\text{compl}_\mathcal{H}M) \neq \text{compl}_\mathcal{G}M$ and it is not complete in $\text{compl}_\mathcal{G}M$. For example let $M = (0; \frac{\pi}{2}), C = C^\infty(M)$ and $\mathcal{G} = \{x \cos(\tan x), x \sin(\tan x)\}, \mathcal{H} = \{x \cos(\tan x), x \sin(\tan x), \tan x\}$.

Then we obtain the following

**Theorem 4.1** For any differential space $(M, C)$ the differential completion $(\text{compl}_C M, \text{compl}_C C)$ has the following properties:
(i) for any differential completion $(N, D)$ of $(M, C)$ (where $M \subset N$) there exist the map

$$\iota_D : (\text{compl}_C M, \text{compl}_C C) \to (N, D)$$

such that $\iota_D|_M = \text{id}_M$;
(ii) for any function $g \in C$ there exists uniquely defined extension $\hat{g} \in \text{compl}_C C$.

In the set of all differential completions of the space $(M, C)$ we can define an ordering relation $\preceq$ such that:

$$\text{compl}_G C \preceq \text{compl}_H M \iff \mathcal{G} \subset \mathcal{H},$$

where $\mathcal{G}$ and $\mathcal{H}$ are families of generators of the structure $C$.

The above theorem says that $\text{compl}_C M$ is the maximal with the respect to the order $\preceq$ completion of $M$ which can be constructed using a set of generators of the
Definition 4.1. We will call the differential space \((\text{compl}_C M, \text{compl}_C C)\) the maximal differential completion of the differential space \((M, C)\).

Let us consider the situation when for some family of generators \(G\) the uniform space \((M, U_G)\) is complete.

Theorem 4.2 Let \((M, C)\) be a differential space and \(G\) be a family of generators of \(C\). If the uniform space \((M, U_G)\) is complete then for any family \(H\) of generators of \(C\) such that \(G \subseteq H\) we have

\[
\text{compl}_H M = M \quad (3)
\]
and

\[
\text{compl}_H C = C. \quad (4)
\]

In particular \(\text{compl}_C M = M\) and \(\text{compl}_C C = C\).

Proof. Any element of \(\text{compl}_H M\) is represented by some filter \(\mathcal{F}\) in \(M\) which is a Cauchy filter with respect to \(U_H\). Then \(\mathcal{F}\) is a Cauchy filter with respect to \(U_G\) and therefore \(\mathcal{F}\) can be identify with its limit in \(M\). Then \(\text{compl}_H M \subset M\) On the other hand for any element \(p \in M\) the filter \(\mathcal{F}_p\) of all neighbourhoods of \(p\) is a Cauchy filter with respect to \(U_H\). Hence we can write \(M \subset \text{compl}_H M\).

The equality \((4)\) is an immediate consequence of the definition of \(\text{compl}_H C\) and the equality \((3)\). \(\square\)

Let us consider the case when \(G = C\).

Theorem 4.3 Let \((M, C)\) be a differential space If there exists a finite or countable family of generators of \(C\) then the uniform space \((M, U_C)\) is complete.

Proof. Suppose that \((M, U_C)\) is not complete. Then there exists \(x \in \text{compl}_C M \setminus M\). Let \(\chi : C \to \mathbb{R}\) be a functional given by the formula

\[
\chi(g) := \tilde{g}(x), \quad g \in C, \quad (5)
\]

where \(\tilde{g}\) is the continuous extension of \(g\) from \(M\) to \(\text{compl}_C M\). This functional is an element of the spectrum of the algebra \(C\) but it is not an evaluation functional on \(M\) (the algebra \(\text{compl}_C C\) separates points of the space \(\text{compl}_C M\)). Then \(C\) does not posses the spectral property. It is a contradiction with Theorem 1 and Corollary 6 from the work [3] (see also Theorem 2.3 (Twierdzenie 2.3) and Corollary 2.6 (Wniosek 2.6) from [2]). \(\square\)

Corollary 4.1 Let \(X\) be a topological Hausdorff space. If the topology of \(X\) is given by a countable family \(G\) of real-valued functions as the weakest topology on \(X\) with respect to which all elements of \(G\) are continuous then there exists an uniform structure \(\mathcal{U}\) on \(X\) such that the uniform space \((X, \mathcal{U})\) is complete and the topology \(\tau_{\mathcal{U}}\) coincides with the initial topology on \(X\). \(\square\)
5 Compactification of a differential space

Let $(M, C)$ be a differential space such that $C = \text{gen}(G)$. Let $f \in C$, $m \in M$. Then there exist: neighbourhood $U$ of $m$, number $n \in \mathbb{N}$, functions $\alpha_1, \ldots, \alpha_n \in G$ and $\omega \in C^\infty(\mathbb{R}^n)$ such that $f|_U = \omega \circ (\alpha_1, \ldots, \alpha_n)|_U$. We denote $y_\omega := (\alpha_1(m), \ldots, \alpha_n(m))$. Let us take cubes:

$P := (\alpha_1(m) - 1, \alpha_1(m) + 1) \times (\alpha_2(m) - 1, \alpha_2(m) + 1) \times \ldots \times (\alpha_n(m) - 1, \alpha_n(m) + 1)$.

$L := (\alpha_1(m) - 1, \alpha_1(m) + 1) \times (\alpha_2(m) - 2, \alpha_2(m) + 2) \times \ldots \times (\alpha_n(m) - 2, \alpha_n(m) + 2)$.

Let $\eta \in C^\infty(\mathbb{R}^n)$, such that: $\eta|_P \equiv 1$, $\eta|_{\mathbb{R}^n \setminus P} \equiv 0$ and $|\eta| \leq 1$. We mark $\alpha := (\alpha_1, \ldots, \alpha_n)$, $\beta_i := \alpha \cdot \eta(\alpha_1, \ldots, \alpha_n)$, $\beta := (\beta_1, \ldots, \beta_n)$. Let $V := \alpha^{-1}(P)$, $V' := \alpha^{-1}(L)$. Then $V \subset V' \subset M$ and $m \in U \cap V$. For any $x \in U \cap V$ we have: $f(x) = \omega(\alpha_1(x), \ldots, \alpha_n(x)) = \omega(\beta_1(x), \ldots, \beta_n(x))$. We observe that $\forall i \in \{1, \ldots, n\}$ $|\beta_i(x)| \leq \max\{1|\alpha_1(m)| + 2, |\alpha_1(m) - 2|\} =: \mu_i$. Then $f(x) = \omega(\mu_1 \beta_1(x), \mu_2 \beta_2(x), \ldots, \mu_n \beta_n(x)) = \omega(\mu_1 \gamma_1(x), \ldots, \mu_n \gamma_n(x)) = \omega_1(\gamma_1(x), \ldots, \gamma_n(x))$, where: $\forall i \leq n$, $\forall x \in M$ $|\gamma_i(x)| = \frac{\beta_i(x)}{\mu_i}$ and $|\gamma_i(x)| < 1$. Hence we get the following theorem:

**Theorem 5.1** Any differential space $(M, C)$ there exist the family of bounded generators, in particular $C = \text{gen}(G_1)$, where $G_1 = \{\gamma_i\}_{i \in I}$ such that $|\gamma_i| \leq 1 \forall i \in I$.

For any Hausdorff differential space we consider the generators embedding of that space using the family of generators described in Theorem 5.1 (it takes values in the cube $[-1, 1]^I$). So we have the embedding of the differential space into the compact space and we close the image. Hence we get the compact set $\overline{M}_G$.

If we mark $J := [-1, 1]$, then $\overline{M}_G \subset J$. On $J$ there exists the natural differential structure $C^\infty(J) = C^\infty(\mathbb{R})|_J$ generated by the family of projections $\{p_r|_J\}_{r \in I}$, where $p_r : \mathbb{R} \rightarrow \mathbb{R}$ is the projection onto $i$-th coordinate. By localization of that structure to the set $\overline{M}_G$ we get the differential space $(\overline{M}_G, C^\infty(J))$ which is a differential subspace of the space $(J, C^\infty(J))$. We call that differential space the (differential) compactification of the differential space $(M, C)$ by the family of generators $G$ and we denote it by $(\text{compt}_G M, \text{compt}_G C)$. We see that $\text{compt}_G M = \text{compt}_G C$ and $\text{compt}_G C = \text{compt}_G C$. So for the compactification of the differential space we have analogous theorems like for the completion.

**Remark 5.1** Let $G$ and $H$ be the families of bounded generators of the differential structure $C$ on the set $M \not= \emptyset$. If $G \subset H$, that there exists smooth function $\iota_G H : (\text{compt}_H M, \text{compt}_H C) \rightarrow (\text{compt}_G M, \text{compt}_G C)$ such that $\iota_G H|_M = \text{id}_M$.

Let consider the differential space $(M, C)$ and the family $C_0$ of all smooth functions on that space which takes values in $[-1, 1]$. Using the procedure of the compactification, described earlier, we get the differential space that we mark by $(\text{compt}_C M, \text{compt}_C C)$ it means $\text{compt}_C M := \text{compt}_C C_0 M$, $\text{compt}_C C := \text{compt}_C C_0 C$.

**Definition 5.1** The differential space $(\text{compt}_C M, \text{compt}_C C)$ is called the maximal differential compactification of the space $(M, C)$.
Let us assume that the topological space \((M, \tau_C)\) is compact. Than all the functions from \(C\) are limited and by the normalization of each function \(\alpha \in C \setminus \{0\}\) according the formula:

\[
N\alpha(p) = \frac{1}{\sup_{q \in M} |\alpha(q)|} \alpha(p), \ p \in M
\]

we get the family \(NC = \{N\alpha : \alpha \in C\}\) of the generators of the structure \(C\). Then the generators embedding given by \(NC\) converts diffeomorphically \((M, C)\) onto the compact subspace of \((J^{\infty}NC, C^\infty(J^{\infty}NC))\). Similarly, if \(G\) is any family of generators of the structure \(C\), then \(NG = \{N\alpha : \alpha \in G\}\) is the family of generators of \(C\) too, and proper generators embedding is the diffeomorphism of \((M, C)\) onto a compact subspace of \((J^{\infty}NG, C^\infty(J^{\infty}NG))\). We have

\[
\text{compt}_CM = M, \quad \text{compt}_GC = C,
\]

where the equality is the identification of the diffeomorphic spaces and structures.

References

[1] N. Bourbaki, General Topology, Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo 1989.

[2] M. J. Cukrowski, Geometrical properties of function algebras in the category of differential spaces (doctor thesis - in polish), Warsaw University of Technology, Warsaw 2007.

[3] M. J. Cukrowski, Z. Pasternak-Winiarski, W. Sasin On real-valued homomorphisms in countably generated differential structures, arXiv:1103.3597 2011.

[4] D. Dwiewa-Dawidczyk, Z. Pasternak - Winiarski, Differential structures on natural bundles connected with a differential space, in “Singularities and Symplectic Geometry VII” Singularity Theory Seminar (2009), S. Janeczko (ed), Faculty of Mathematics and Information Science, Warsaw University of Technology

[5] D. Dwiewa-Dawidczyk, Z. Pasternak - Winiarski, Uniform structures on differential spaces, arXiv:1103.2799 2011.

[6] R. Engelking, General topology, PWN, Warsaw 1975 (in polish).

[7] Z. Pasternak - Winiarski, Group differential structures and their basic properties (doctor thesis - in polish), Warsaw University of Technology, Warsaw 1981.

[8] R. Sikorski, Introduction to the differential geometry, PWN, Warsaw 1972 (in polish).

[9] W. Waliszewski, Regular and coregular mappings of differential space, Annales Polonici Mathematici XXX, 1975.