Deviations for Jumping Times of a Branching Process Indexed by a Poisson Process

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Consider a continuous time process \(\{Y_t = Z_{N_t}, t \geq 0\}\), where \(\{Z_n\}\) is a supercritical Galton–Watson process and \(\{N_t\}\) is a Poisson process which is independent of \(\{Z_n\}\). Let \(\tau_n\) be the \(n\)-th jumping time of \(\{Y_t\}\), we obtain that the typical rate of growth for \(\{\tau_n\}\) is \(n/\lambda\), where \(\lambda\) is the intensity of \(\{N_t\}\). Probabilities of deviations \(\{|n^{-1}\tau_n - \lambda^{-1}| > \delta\}\) are estimated for three types of positive \(\delta\).

1. Statements of the Main Results

The model of Poisson randomly indexed branching process (PRIBP) \(\{Y_t = Z_{N_t}, t \geq 0\}\) was introduced by [1] to study the evolution of stock prices and its statistical investigation was done in [2].

In a recent manuscript [3] the authors there consider the asymptotic properties of \(\log Y_t\). Let \(\{p_k, k \geq 0\}\) be the offspring distribution of the branching process with mean \(m = \sum_k kp_k \in (1, \infty)\); we distinguish between the Shröder case and the Böttcher case depending on whether \(p_0 + p_1 > 0\) or \(p_0 + p_1 = 0\). In Böttcher case, it was proved in [3] that \(\log Y_t\) have similar asymptotic results to the Poisson process \(\{N_t\}\). But differences appeared in Shröder case; see [4]. For subcritical and critical PRIBP one can see [5, 6] for details.

In this paper, we deal with the asymptotic theory for the jumping times of PRIBP defined as follows. For any \(\omega\), define

\[
\tau_{\infty}(\omega) = \inf \left\{ t : t > 0; \lim_{s \uparrow t} Y_s(\omega) = \infty \right\},
\]

where \(\inf \emptyset = \infty\); then

\[
\{\tau_{\infty} < \infty\} = \bigcup_{l=1}^{\infty} \{\tau_{\infty} < l\}.
\]

Note that \(\{Z_n\}\) is independent of \(\{N_t\}\); one has

\[
P(\tau_{\infty} < l) \leq P(Y_l = \infty) = \sum_n P(Z_n = \infty) P(N_l = n) = 0,
\]

and thus \(P(\tau_{\infty} = \infty) = 1\). Define \(\tau_0 = 0\) and \(\{\tau_n, n \geq 1\}\) as the successive times of jump of the PRIBP \(\{Y_t, t \geq 0\}\).

In Böttcher case, the jumping times of \(\{Y_t\}\) coincide with that of \(\{N_t\}\). Let \(\{T_n\}\) be the successive times of jump of \(\{N_t\}\); then both \(\tau_n\) and \(T_n\) have a gamma distribution with parameters \(n\) and \(\lambda\). But when \(p_0 + p_1 > 0\), at the jumping time of \(N_t\), PRIBP can have no jump, since an individual can replicate himself at this time. So the jumping times of \(\{Y_t\}\) are likely to be delayed; see Figure 1 for example. In the path of Figure 1, \(\tau_1 = T_1\) and \(\tau_2 = T_3, \ldots\).

Although \(\tau_n \geq T_n\) for all \(n\), the growth rate of \(\tau_n\) is not too fast as that of \(T_n\). In fact, the typical growth rate of \(T_n\) is \(n/\lambda\) by the law of large numbers and we can show that the typical growth rate of \(\tau_n\) is

\[
\sum_{k=1}^{n} \frac{1}{\lambda (1 - p_k)} \leq \frac{n}{\lambda} + \frac{1}{\lambda (1 - p_1)}
\]

and see the proof of Theorem 1. Thus, for almost all the path of Shröder case PRIBP, \(\tau_n/n\) has a limit \(\lambda^{-1}\) when \(n \to \infty\).
In the rest of this paper, we always assume that our branching process belongs to the Shröder case, $p_0 = 0$ and $Z_0 = 1$.

We are interested in the decay rates about the probabilities of

$$\left\{ \omega : \left| \frac{\tau_n(\omega)}{n} - \frac{1}{\lambda} \right| > \delta \right\}$$

for some positive $\delta$. Typically, there are three classes of $\delta$ to be chosen.

The first one is that $\delta = a \sqrt{n}$ for some fixed $a > 0$. In this case, the event in (5) is said to be a normal deviation event. The decay rate of its probability can be characterized by the following central limit theorem.

**Theorem 1.** $\left\{ \tau_n \right\}$ satisfies the law of large number and the central limit theorem; that is, $\tau_n/n \xrightarrow{a.e.} \lambda^{-1}$ and $\sqrt{n}(\tau_n/n - \lambda^{-1}) \xrightarrow{d} N(0,1)$ when $n \to \infty$, where $N(0,1)$ is standard normal distribution.

Next, if $\delta = a$ for some fixed $a > 0$, the event in (5) is said to be a large deviation event whose probability has an exponential convergence rate by the following large deviation principle.

**Theorem 2 (LDP).** For any measurable subset $B$ of $\mathbb{R}$,

$$-\inf_{x \in B^o} \Lambda^* (x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P \left( \frac{\tau_n}{n} \in B \right) \leq \limsup_{n \to \infty} \frac{1}{n} \log P \left( \frac{\tau_n}{n} \in B \right) \leq -\inf_{x \in B} \Lambda^* (x),$$

where $B^o$ denotes the interior of $B$, $\overline{B}$ its closure, and

$$\Lambda^* (x) = \begin{cases} \lambda (1 - p_1) x + \log p_1, & x \geq (\lambda p_1)^{-1}; \\ \lambda x - \log (\lambda x) - 1, & (\lambda p_1)^{-1} > x > 0; \\ +\infty, & x \leq 0. \end{cases}$$

By Cramér’s theorem (see Theorem 2.2.3 of [7]), $T_n/n$ satisfies the large deviation principle with rate function

$$\Psi (x) = \begin{cases} \lambda x - \log (\lambda x) - 1, & x > 0; \\ +\infty, & x \leq 0. \end{cases}$$

By Theorem 2, the rate function of $\tau_n/n$ coincides with that of $T_n/n$ for $x \leq (\lambda p_1)^{-1}$, but differences appeared for large $x$; see Figure 2 for example.

If $\delta = \delta_n \to \infty$ and $\delta_n = o(\sqrt{n})$ as $n \to \infty$, we call the event in (5) a moderate deviation event. Let $\{a_n, n \geq 0\}$ be a family of positive numbers satisfying

$$\frac{a_n}{n} \to 0$$

and $a_n/\sqrt{n} \to \infty$ as $n \to \infty$.

As in the case of large deviation principle, based on the Gärtner-Ellis theorem (see [7], page 44), we have the following moderate deviation principle.

**Theorem 3 (MDP).** For any measurable subset $B$ of $\mathbb{R}$,

$$-\inf_{x \in B^o} \frac{\lambda^2 x^2}{2} \leq \liminf_{n \to \infty} \frac{n}{a_n^2} \log P \left( \frac{\tau_n - \lambda^{-1} n}{a_n^2} \in B \right) \leq \limsup_{t \to \infty} \frac{n}{a_n^2} \log P \left( \frac{\tau_n - \lambda^{-1} n}{a_n^2} \in B \right) \leq -\inf_{x \in B} \frac{\lambda^2 x^2}{2}.$$
2. The Law of Large Number and the Central Limit Theorem

**Proposition 4.** PRIBP is a homogenous continuous time Markov chain in which Q matrix satisfies \( q_i = \lambda (1 - p_i^t) \) for all \( i \geq 1 \).

**Proof.** For any nonnegative real numbers \( t_0 < t_1 < \cdots < t_{n-1} < s < t + s \) and nonnegative integers \( i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i \leq j \), define

\[
A = \{ Y(s) = i, Y(t_{n-1}) = i_{n-1}, \ldots, Y(t_0) = i_0 \}. \tag{11}
\]

For any nonnegative integers \( k_0 \leq k_1 \leq \cdots \leq k_{n-1} \leq k_s \), define

\[
B(k_0, \ldots, k_{n-1}, k_s) = \{ N(t_0) = k_0, \ldots, N(t_{n-1}) = k_{n-1}, N(s) = k_s, N(t + s) = k_{t+s} \}. \tag{12}
\]

Since Poisson process \( \{ N_i \} \) is independent of the Galton-Watson process \( \{ Z_n \} \),

\[
P(Y(t+s) = j | A) = \sum_{k_s, k_{n-1}, k_s} P(B(k_0, \ldots, k_{n-1}, k_s))
\cdot P(Z_{k_{s}} = j | Z_{k_s} = i, Z_{k_{n-1}} = i_{n-1}, \ldots, Z_{k_s} = i_0). \tag{13}
\]

Note that the Galton-Watson process is a Markov chain with \( n \) step transition probabilities \( P_n(i, j) \), and summing \( k_0, \ldots, k_{n-1} \), one has

\[
P(Y(t+s) = j | A) = \sum_{k_s, k_{n-1}, k_s} P(N(s) = k_s, N(t+s) = k_{t+s})
\cdot P(Z_{k_{s}} = j | Z_{k_s} = i)
\leq \sum_{k_s, k_{n-1}, k_s} P(N(s) = k_s, N(t+s) = k_{t+s}) P_{k_{t+s} - k_s}(i, j)
\leq E(P_{N(t+s) - N(s)}(i, j)) = E(P_{N(t)}(i, j)). \tag{14}
\]

Similarly,

\[
P(Y(t+s) = j | Y(s) = i) = E(P_{N(t)}(i, j))
\leq P(Y(t+s) = j | A), \tag{15}
\]

which means that PRIBP is a homogenous continuous time Markov chain.

Next, note that \( N_i \) has a Poisson distribution with parameter \( \lambda t > 0 \); we have

\[
E(P_{N(t)}(i, i)) = e^{-\lambda t} + \lambda t e^{-\lambda t} + \sum_{n \geq 2} \frac{\lambda^n}{n!} P(N(t) = n), \tag{16}
\]

which implies \( q_i = \lambda (1 - p_i^t) \).

**Proof of Theorem 1.** Define \( X_n(\omega) = Y(\tau_n(\omega)) (\omega) \); then \( \{ X_n \} \) is a homogeneous discrete-time Markov chain. Define \( \rho_n = \tau_n - \tau_{n-1} \) for \( n \geq 1 \); then the conditional distribution of \( \rho_n \) relative to \( X_1, X_2, \ldots, X_{n-1} \) equals exponential distribution with parameter \( \alpha_{X_{n-1}} \), where \( \alpha_i = \lambda (1 - p_i^t) \), see page 259 of [8] for example. So

\[
\tau_n = \tau_n - \sum_{k_1 = 1}^n \frac{1}{\lambda (1 - p_1^{X_{k-1}})} = \sum_{k_1 = 1}^n \left( \rho_k - \frac{1}{\lambda (1 - p_1^{X_{k-1}})} \right)
\]

is a square-integrable martingale adapted to the \( \sigma \)-fields \( \sigma(X_1, \ldots, X_n) \). Consequently, there exists a random variable \( Z \) such that \( \tau_n \to Z \) as \( n \to \infty \).

Next, we prove \( \lambda t_n^{1/2} / \sqrt{N} \to N(0, 1) \). Let \( \eta_{n,i} = \lambda n^{-1/2} [ \rho_i - (\lambda (1 - p_1^{X_{n-1}}))^{-1} ] \). \( I_{f_i} \) be the field generated by \( X_1, \ldots, X_{n-1} \); by Hölder's inequality, one has

\[
E(\eta_{n,i}^2 | I_{f_i})^{1/2} [E(\eta_{n,i}^3 | I_{f_i})]^{1/3}, \tag{19}
\]

where \( I_A \) is the indicator function of \( A \).

Note that the conditional distribution of \( \rho_n \) relative to \( \rho_1, \ldots, \rho_{n-1} \) equals exponential distribution with parameter \( \lambda (1 - p_1^{X_{n-1}}) \); one has

\[
E(\eta_{n,i}^3 | I_{f_i}) = (\lambda n)^{-3/2} E(\left(\rho_i - \left(\lambda (1 - p_1^{X_{n-1}}))^{-1}\right)^3 | I_{f_i}) \leq 4n^{-3/2} (\lambda^{1/2} (1 - p_1^{X_{n-1}}))^{-3} a.s., \tag{20}
\]

which implies \( \tau_n / n \to \lambda^{-1} \) as \( n \to \infty \).
According to Markov's inequality,
\[
E \left( I_{\{\eta_{n,i} > \epsilon \}} \mid \mathcal{F}_{i-1} \right) \leq e^{-2} E \left( \eta_{n,i}^2 \mid \mathcal{F}_{i-1} \right) \leq \frac{1}{n^2} \left( \lambda - \theta \right)^{1/2} a.s.,
\]
(21)

By formulas (19)-(21) we have
\[
\sum_{i=1}^{n} E \left( \eta_{n,i}^2 I_{\{\eta_{n,i} > \epsilon \}} \mid \mathcal{F}_{i-1} \right) \leq C n^{-1/3} a.e.
\]
(22)

for some positive constant C. Similarly,
\[
\sum_{i=1}^{n} E \left( \eta_{n,i}^2 \mid \mathcal{F}_{i-1} \right) = \sum_{i=1}^{n} \left( 1 - p_i \right)^{-2} \rightarrow 1.
\]
(23)

According to (22), (23), and Corollary 3.1 of [9], one has \( \lambda r_n^2 / \sqrt{n} \rightarrow N(0,1) \). Note that
\[
\left| \frac{\lambda r_n^2}{\sqrt{n}} - \lambda \sqrt{n} \left( \frac{r_n}{n} - \lambda^{-1} \right) \right| \leq \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( 1 - p_i \right)^{-2} \lambda^{-1} \rightarrow 0,
\]
and the central limit theorem follows from Theorem 6 in page 39 of [10].

\[\square\]

### 3. Large Deviation Principle

Let us begin with some lemmas to show the conditions of Gärtner-Ellis Theorem (see Appendix) are satisfied. Define \( \Lambda_n(\theta) = \log E[\epsilon_{\theta r_n}^{n}] \).

**Lemma 5.** For any \( \theta \in \mathbb{R} \), \( \Lambda(\theta) = \lim_{n \to \infty} (1/n) \Lambda_n(\theta) \) exists and satisfies

\[
\Lambda(\theta) = \begin{cases} 
\log \lambda - \log(\lambda - \theta), & \theta < \lambda(1 - p_1) \\
\infty, & \theta \geq \lambda(1 - p_1) \end{cases}
\]
(25)

Particularly, \( 0 \in D_\Lambda = [\theta : \Lambda(\theta) < \infty] \).

**Proof.** Note that \( r_n = p_1 + \cdots + p_n \), where the conditional distribution of \( \rho_n \) relative to \( p_1, \ldots, p_{n-1} \) is the same as that relative to \( X_0, \ldots, X_{n-1} \) and equals exponential distribution with parameter \( \lambda(1 - p_1) \) (see page 259 of [8]); one has
\[
E \left[ \epsilon_{\theta r_n}^{n} \right] = E \left[ \epsilon_{\theta \sum_{i=1}^{n} \rho_i} \right] = E \left[ E \left[ \epsilon_{\theta \sum_{i=1}^{n} \rho_i} \mid \rho_1, \ldots, \rho_{n-1} \right] \right] = E \left[ \epsilon_{\theta \sum_{i=1}^{n} \rho_i} E \left( \epsilon_{\theta \rho_1} \mid \rho_1, \ldots, \rho_{n-1} \right) \right] = E \left[ \epsilon_{\theta \sum_{i=1}^{n} \rho_i} E \left( \epsilon_{\theta \rho_1} \mid X_0, \ldots, X_{n-1} \right) \right].
\]
(26)

If \( \theta < \lambda(1 - p_1) \), we have
\[
E \left( \epsilon_{\theta \rho_n} \mid X_0, \ldots, X_{n-1} \right) = \frac{\lambda(1 - p_1)}{\lambda(1 - p_1) - \theta}.
\]
(27)

If \( \theta \leq 0 \), note that \( X_{n-1} \geq n \); by (27) one has
\[
\frac{\lambda(1 - p_1)}{\lambda(1 - p_1) - \theta} \leq E \left( \epsilon_{\theta \rho_n} \mid X_0, \ldots, X_{n-1} \right) \leq \frac{\lambda}{\lambda - \theta}.
\]
(28)

By (26), (28) and induction, we have
\[
\prod_{i=1}^{n} \frac{\lambda(1 - p_i)}{\lambda(1 - p_i) - \theta} \leq E \left[ \epsilon_{\theta r_n}^{n} \right] \leq \frac{\lambda^n}{(\lambda - \theta)^n},
\]
(29)

which means \( \Lambda(\theta) = \log \lambda - \log(\lambda - \theta) \).

If \( 0 < \theta < \lambda(1 - p_1) \), note that \( X_{n-1} \geq n \); by (27) one has
\[
\frac{\lambda}{\lambda - \theta} \leq E \left( \epsilon_{\theta \rho_n} \mid X_0, \ldots, X_{n-1} \right) \leq \frac{\lambda(1 - p_1)}{\lambda(1 - p_1) - \theta}.
\]
(30)

We can get \( \Lambda(\theta) = \log \lambda - \log(\lambda - \theta) \) similarly.

If \( \theta \geq \lambda(1 - p_1) \), note that \( X_0 = Z_0 = 1 \) and the conditional distribution of \( \rho_1 \) relative to \( X_0 \) equals exponential distribution with parameter \( \lambda(1 - p_1) \); one has
\[
E \left[ \epsilon_{\theta r_n}^{n} \right] \geq E \left[ \epsilon_{\theta \rho_1} \right] = E \left[ E \left( \epsilon_{\theta \rho_1} \mid X_0 \right) \right] = +\infty
\]
(31)

for all \( n \geq 1 \). Thus \( \Lambda(\theta) = +\infty \).

\[\square\]

**Lemma 6.** Let \( \Lambda^* \) be the Fenchel-Legendre transform of \( \Lambda \); then
\[
\Lambda^* (x) = \sup_{\theta \in \mathbb{R}} \{ \theta x - \Lambda(\theta) \}
\]
(32)

\[
= \begin{cases} 
\lambda(1 - p_1) x + \log p_1, & x \geq (\lambda p_1)^{-1} - 1 \\
\lambda x - \log(\lambda x) - 1, & (\lambda p_1)^{-1} > x > 0 \\
+\infty, & x \leq 0.
\end{cases}
\]

In addition, the set of exposed points (see Appendix) of \( \Lambda^* \) is \( \mathcal{E} > (0, +\infty) \).

**Proof.** By Lemma 5, if \( x \leq 0 \),
\[
\Lambda^* (x) = \sup_{\theta \in \lambda(1 - p_1)} \{ \theta x - \log \lambda + \log(\lambda - \theta) \}
\]
(33)

\[
= \lim_{\theta \to -\infty} (\theta x - \log \lambda + \log(\lambda - \theta)) = +\infty.
\]

Next, if \( (\lambda p_1)^{-1} > x > 0 \), then \( \lambda x - x^{-1} < \lambda(1 - p_1) \) and
\[
\Lambda^* (x) = \sup_{\theta \in \lambda(1 - p_1)} \{ \theta x - \log \lambda + \log(\lambda - \theta) \}
\]
(34)

\[
= \lambda x - \log(\lambda x) - 1.
\]

\[\square\]
Finally, if \( x \geq (\Lambda p_1)^{-1} \), then \( \lambda - x^{-1} \geq \lambda(1 - p_1) \). Note that
\[
\frac{d}{d \theta} \left( \theta x - \log \lambda + \log(\lambda - \theta) \right) = x - \frac{1}{\lambda - \theta} > 0
\]  
(35)
for all \( \theta < \lambda(1 - p_1) \); we have
\[
\Lambda^*(x) = \sup_{\theta : \lambda(1-p_1)} \{ \theta x - \log \lambda + \log(\lambda - \theta) \} = \lambda (1 - p_1) x + \log p_1.
\]  
(36)
Equation (32) follows from (33), (34), and (36).

In addition, for any \( \theta < \lambda, \Lambda'(\theta) = (\lambda - \theta)^{-1} \), so the range of \( \Lambda'(\theta) \) for \( \theta < \lambda \) is \((0, +\infty)\), the set of exposed points of \( \Lambda^* \); see Lemma 2.3.9 of [7].

**Proof of Theorem 2.** Note that for any \( x \leq 0, \Lambda^*(x) = +\infty \) and the set of exposed points \( \mathcal{E} = (0, +\infty) \), then for any open set \( G \),
\[
\inf_{x \in \mathcal{E} \cap G} \Lambda^*(x) = \inf_{x \in \mathcal{E} \cap G} \Lambda^*(x) = \inf_{x \in G} \Lambda^*(x).
\]  
(37)
Consequently, Theorem 2 follows from Lemma 5, Lemma 6, and the Gärtner-Ellis theorem (see Appendix).

**4. Moderate Deviation Principle**

In this section, we deal with the proof of Theorem 3. Define
\[
\Lambda_n(\theta) = \log E \left[ \exp \left\{ \theta - \frac{\lambda^{-1}}{a_n} n \right\} \right].
\]  
(38)

**Lemma 7.** For each \( \theta \in R \), one has,
\[
\Delta(\theta) = \lim_{n \to \infty} \frac{n}{a_n^2} \Lambda_n(\theta) = \theta^2 \frac{\lambda^2 x^2}{2}.
\]  
(39)
Particularly, \( 0 \in D_\Delta = \{ \theta : \Delta(\theta) < \infty \} \). In addition, let \( \Delta^* \) be the Fenchel-Legendre transform of \( \Delta \); then
\[
\Delta^*(x) = \frac{\lambda^2 x^2}{2},
\]  
(40)
and the set of exposed points of \( \Delta^* \) is \( \mathcal{F} = R \).

**Proof.** For any \( \theta \in R \), we have
\[
\Lambda_n(\frac{\lambda^2}{n} \theta) = \log E \left[ \exp \left\{ \frac{\lambda^2}{n} \theta \cdot \frac{\tau_n - \lambda^{-1}}{a_n} n \right\} \right]
= \log E \left[ \exp \left\{ \frac{\lambda^2}{n} \theta \left( \frac{\tau_n - \frac{1}{\lambda(1 - p_1^{X_{i-1}})}}{a_n} \right) \right\} \right]
\cdot \exp \left\{ \frac{\lambda^2}{n} \theta \left( \frac{\tau_n - \frac{1}{\lambda(1 - p_1^{X_{i-1}})}}{a_n} \right) \right\}
\]  
(41)
For any \( \theta \geq 0 \), note that \( X_{i-1} \geq i-1 \); one has
\[
0 \leq \frac{a_\theta}{n} \cdot \sum_{i=1}^n \frac{p_i^{X_{i-1}}}{\lambda(1 - p_i^{X_{i-1}})} \leq \frac{a_\theta}{n} \cdot \sum_{i=1}^n \frac{p_i^{X_{i-1}}}{\lambda(1 - p_i^{X_{i-1}})}
\]  
(42)
Similarly, for \( \theta < 0 \), the above inequality should be reversed. Thus, by (41), if \( \Delta(\theta) \) exists, one has
\[
\Delta(\theta) = \lim_{n \to \infty} \frac{n}{a_n^2} \log E \left[ \exp \left\{ \frac{\lambda^2}{n} \theta \left( \tau_n - \frac{1}{\lambda(1 - p_1^{X_{i-1}})} \right) \right\} \right].
\]  
(43)
Define \( \beta_i = \rho_i - (1 - p_1^{X_{i-1}})^{-1} \); then
\[
\tau_n - \frac{1}{\lambda(1 - p_1^{X_{i-1}})} = \sum_{i=1}^n \beta_i.
\]  
(44)
Note that the conditional distribution of \( \rho_n \) relative to \( \rho_1, \ldots, \rho_{n-1} \) is the same as that relative to \( X_1, \ldots, X_{n-1} \) and equals exponential distribution with parameter \( \lambda(1 - p_1^{X_{i-1}}) \); one has
\[
E \left[ \exp \left\{ \frac{\lambda^2}{n} \beta_i \right\} \right] = E \left[ \exp \left\{ \frac{\lambda^2}{n} \beta_i \right\} \right]
= E \left[ \exp \left\{ \frac{\lambda^2}{n} \beta_i \right\} \right] \cdot \frac{\lambda(1 - p_1^{X_{i-1}})}{\lambda(1 - p_1^{X_{i-1}}) - a_\theta/n}
\]  
(45)
where \( \mathcal{F}_{i-1} \) is the \( \sigma \)-field generated by \( X_1, \ldots, X_{i-1} \). After a simple calculation,
\[
E \left[ \exp \left\{ \frac{\lambda^2}{n} \beta_i \right\} \mid \mathcal{F}_{i-1} \right]
= \exp \left\{ \frac{\lambda^2}{n(1 - p_1^{X_{i-1}})} \right\} \cdot \frac{\lambda(1 - p_1^{X_{i-1}})}{\lambda(1 - p_1^{X_{i-1}}) - a_\theta/n}
\]  
(46)
For \( \theta > 0 \),
\[
\exp \left\{ \frac{-\theta a_n}{n(1 - p_1^{X_{i-1}})} \right\} \cdot \frac{\lambda}{\lambda - a_\theta/n}
\leq E \left[ \exp \left\{ \frac{\lambda^2}{n} \beta_i \right\} \mid \mathcal{F}_{i-1} \right]
\]  
(47)
According to (45)-(47) and induction, we obtain

\[
I_n(\theta) = \sum_{i=1}^{n} \left[ \exp \left( \frac{-\theta d_{1,n}}{n \lambda (1 - p_1)} \right) - \frac{\lambda}{\lambda - a_i \theta / n} \right] \leq E \left[ \exp \left( \frac{a_i \theta \sigma_i^2}{n \lambda} \right) \right] \leq \sum_{i=1}^{n} \left[ \exp \left( \frac{-\theta d_{1,n}}{n \lambda} \right) - \frac{\lambda}{\lambda - a_i \theta / n} \right] = H_n(\theta).
\]

Similarly, for \( \theta < 0 \), the above inequality should be reversed. According to (18) and \( \log(1 + x) = x - x^2/2 + o(x^2) \) as \( x \to 0 \), one has

\[
\log I_n(\theta) = \frac{-\theta d_{1,n}}{n \lambda} \sum_{i=1}^{n} \frac{1}{1 - p_1} - n \log \left( 1 - \frac{a_i \theta}{n \lambda} \right)
= \frac{-\theta d_{1,n}}{\lambda} (1 + b_n) + \frac{\theta a_n}{\lambda} + \frac{\theta^2 a_n^2}{2 n \lambda^2} + o \left( \frac{a_n^2}{n} \right)
= \frac{-\theta d_{1,n} b_n}{\lambda} + \frac{\theta^2 a_n^2}{2 n \lambda^2} + o \left( \frac{a_n^2}{n} \right),
\]

where \( b_n \) belongs to \([0,1/(n(1 - p_1^2)])\]. Hence,

\[
\lim_{n \to \infty} \frac{n}{a_n^2} \log I_n(\theta) = \frac{\theta^2}{2 \lambda^2},
\]

and, similarly,

\[
\lim_{n \to \infty} \frac{n}{a_n^2} \log H_n(\theta) = \frac{\theta^2}{2 \lambda^2}.
\]

Equation (39) is followed by (43), (48), (50), and (51). Consequently,

\[
\Delta^*(x) = \sup_{\theta \in R} \left\{ \theta x - \Delta(\theta) \right\} = \sup_{\theta \in R} \left\{ \theta x - \frac{\theta^2}{2 \lambda^2} \right\}
= \frac{\lambda^2 x^2}{2}.
\]

In addition, for any \( \theta \in R \), \( \Delta'(\theta) = \theta / \lambda^2 \); so the range of \( \Delta'(\theta) \) is \( R \), which means \( \mathcal{F} = R \); see Lemma 2.3.9 of [7]. \( \square \)

**Proof of Theorem 2.** Note that the set of exposed points of \( \Delta^* \) is \( R \); Theorem 3 follows from Lemma 7 and the Gärtner-Ellis theorem. \( \square \)

**Appendix**

**The Gärtner-Ellis Theorem**

Consider a stochastic process \( \{S_n\}_{n \geq 0} \), where \( S_n \) possesses the law \( \nu_n \) and logarithmic moment generating function \( \Lambda_n(\theta) := \log E(e^{\theta S_n}) \).

**Assumption A.** For each \( \theta \in R \) and \( 0 < b_n \to \infty \), the logarithmic moment generating function, defined as the limit

\[
\Lambda(\theta) = \lim_{n \to \infty} \frac{1}{b_n} \Lambda_n(b_n \theta)
\]

exists as an extended real number. Further, the origin belongs to the interior of \( \{ \theta : \Lambda(\theta) < \infty \} \).

**Definition.** Let \( \Lambda^* \) be the Fenchel-Legendre transform of \( \Lambda \). \( y \in R \) is an exposed point of \( \Lambda^* \) if for some \( \theta \in R \) and all \( x \neq y \) it is verified that \( \theta y - \Lambda^*(y) > \theta x - \Lambda^*(x) \). \( \theta \) in the previous equation is called an exposing hyperplane. Let \( \mathcal{E} \) be the set of exposed points of \( \Lambda^* \) whose exposing hyperplane belongs to the interior of \( \{ \theta : \Lambda(\theta) < \infty \} \). The following lemma is the Gärtner-Ellis theorem in large deviation theory; see [7] page 44.

**Lemma A.1.** Let Assumption A holds.

(a) For any closed set \( F \),

\[
\limsup_{n \to \infty} \frac{1}{b_n} \log \nu_n(F) \leq -\inf_{x \in F} \Lambda^*(x).
\]

(b) For any open set \( G \),

\[
\liminf_{n \to \infty} \frac{1}{b_n} \log \nu_n(G) \geq -\inf_{x \in G \setminus \mathcal{E}} \Lambda^*(x).
\]

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there is no conflict of interest regarding the publication of this paper.

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