Degeneracy of zero modes of the Dirac operator in three dimensions

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Abstract
One of the key properties of Dirac operators is the possibility of a degeneracy of zero modes. For the Abelian Dirac operator in three dimensions the question whether such multiple zero modes may exist has remained unanswered until now. Here we prove that the feature of zero mode degeneracy indeed occurs for the Abelian Dirac operator in three dimensions, by explicitly constructing a class of Dirac operators together with their multiple zero modes. Further, we discuss some implications of our results, especially a possible relation to the topological feature of Hopf maps.

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1 Introduction

Fermionic zero modes of the Dirac operator \( D_A = \gamma^\mu(\partial_\mu - iA_\mu) \) are of importance in many places in quantum field theory and mathematical physics \([1, 2, 3]\). They are the ingredients for the computation of the index of the Dirac operator and play a key rôle in understanding anomalies. In Abelian gauge theories, which is what we are concerned with here, they affect crucially the behaviour of the Fermion determinant \( \det(D_A) \) in quantum electrodynamics. The nature of the QED functional integral depends strongly on the degeneracy of the bound zero modes.

In three dimensions – which is the case which we want to study here – the first examples of such zero energy Fermion bound states have been obtained only in 1986 \([4]\), and some further results have been found recently \([5]\). In both articles no degeneracy of these zero modes has been observed, because, by their very methods, the authors of \([4]\) and of \([5]\) could only construct one zero mode per gauge field. Hence, the question of a possible degeneracy of zero energy bound states of the Abelian Dirac operator in three dimensions has remained completely unanswered up to now.

It should be emphasized here that the problem of the existence and degeneracy of zero modes of the Abelian Dirac operator in three dimensions, in addition to being interesting in its own right, has some deep physical implications. The authors of \([4]\) were mainly interested in these zero modes because in an accompanying paper \([6]\) it was proven that one-electron atoms with sufficiently high nuclear charge in an external magnetic field are unstable if such zero modes of the Dirac operator exist.

Further, there is an intimate connection between the existence and number of zero modes of the Dirac operator for strong magnetic fields on the one hand, and the non-perturbative behaviour of the three dimensional Fermionic determinant (for massive Fermions) in strong external magnetic fields on the other hand. The behaviour of this determinant, in turn, is related to the paramagnetism of charged Fermions, see \([7, 8]\). So, a thorough understanding of the zero modes of the Dirac operator is of utmost importance for the understanding of some deep physical problems as well.

In addition, it is speculated in \([8]\) that the existence and degeneracy of zero modes for \( QED_3 \) may have a topological origin as it does in \( QED_2 \) \([9]–[13]\) – cf. \([8]\) for details and an account of the situation for \( QED_{2,3,4} \).

It is the purpose of this letter to address the question of a possible degeneracy of zero energy bound states of the Abelian Dirac operator in three dimensions. We shall prove that the phenomenon of a degeneracy of zero modes does indeed occur, by constructing a special class of gauge fields that lead to an arbitrary number of square-integrable zero modes of the corresponding Dirac operators. In addition, we shall discuss how our results are related to some rigorous bounds on the number of zero modes, and we shall indicate a possible relation of our results to some underlying topological properties of the special class of gauge fields that we provide.
2 Construction of the zero modes

We are interested in solutions of the three-dimensional, Abelian Dirac equation (the Pauli equation)

\[-i\sigma_i \partial_i \Psi(x) = A_i(x)\sigma_i \Psi(x).\]  

(1)

Here \(\vec{x} = (x_1, x_2, x_3)^T\), \(i, j, k = 1 \ldots 3\), \(\Psi\) is a two-component, square-integrable spinor on \(\mathbb{R}^3\), \(\sigma_i\) are the Pauli matrices and \(A_i\) is an Abelian gauge field. The authors of [4] observed that a solution to this equation could be obtained from a solution to the simpler equation

\[-i\vec{\sigma}\vec{\nabla}\Psi = h\Psi\]  

(2)

for some scalar function \(h(x)\). In this case the corresponding gauge field that obeys the Dirac equation (1) together with the spinor (2) is given by

\[A_i = h\frac{\Psi^\dagger \sigma_i \Psi}{\Psi^\dagger \Psi}.\]  

(3)

In addition, they gave the following explicit example

\[\Psi = (1 + r^2)^{-\frac{3}{2}}(1 + i\vec{x}\vec{\sigma})\Phi_0\]  

(4)

where \(\Phi_0\) is the constant unit spinor \(\Phi_0 = (1, 0)^T\). The spinor (4) obeys

\[-i\vec{\sigma}\vec{\nabla}\Psi = \frac{3}{1 + r^2}\Psi\]  

(5)

and is, therefore, a zero mode for the gauge field

\[\vec{A} = \frac{3}{1 + r^2} \frac{\Psi^\dagger \vec{\sigma} \Psi}{\Psi^\dagger \Psi} = \frac{3}{(1 + r^2)^2} \left( \begin{array}{c} 2x_1x_3 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{array} \right).\]  

(6)

The magnetic field \(\vec{B} = \vec{\nabla} \times \vec{A}\) for the gauge field (6) reads

\[\vec{B} = \frac{12}{(1 + r^2)^3} \left( \begin{array}{c} 2x_1x_3 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{array} \right).\]  

(7)

Now assume that a function \(\chi\) exists such that

\[(-i\sigma_j \partial_j \chi)(1 + i\vec{x}\vec{\sigma})\Phi_0 = 0\]  

(8)

then \(\chi^n \Psi, n \in \mathbb{Z}\) (where \(\Psi\) is the zero mode (4)), are additional formal zero modes for the same gauge field (6). Condition (8) implies

\[\det(-i\vec{\sigma}\chi) = \sum_{i=1}^3 \chi_i \chi_i = 0,\]  

(9)
therefore, \( \chi \) necessarily must be complex. Indeed, such a function \( \chi \) fulfilling (8) exists,

\[
\chi =: S e^{i\sigma} = \frac{2(x_1 + ix_2)}{2x_3 - i(1 - r^2)}
\]

\[
S^2 = \frac{4(r^2 - x_3^2)}{4x_3^2 + (1 - r^2)^2}, \quad \sigma = \text{atan} \frac{x_2}{x_1} + \text{atan} \frac{1 - r^2}{2x_3}.
\]

as may be checked easily. Here we expressed \( \chi \) by its modulus \( S \) and phase \( \sigma \) for later convenience. For the formal zero modes \( \chi^n \Psi \) we observe the following two points. Firstly, \( n \) has to be integer, because only integer powers of \( \chi \) lead to a single-valued spinor \( \chi^n \Psi \). Secondly, \( \chi^n \Psi \) is singular for all \( n \in \mathbb{Z} \setminus \{0\} \), because \( \chi \) is singular along the circle \( \{ \vec{x} \in \mathbb{R}^3 \setminus x_3 = 0, x_1^2 + x_2^2 = 1 \} \) and zero along the \( x_3 \) axis. Therefore, the formal zero modes \( \chi^n \Psi \), with \( \Psi \) given in (4), are not acceptable. However, we shall find some zero modes, different from (4), where multiplication with \( \chi^n \) will lead to acceptable new zero modes for some \( n \neq 0 \). For this purpose, let us first review some more results of [4]. The authors of [4] observed that, in addition to their simplest solution (4), they could find similar solutions to eq. (2) with higher angular momentum. Using instead of the constant spinor \( \Phi_0 = (1,0)^T \) the spinor

\[
\Phi_{l,m} = \begin{pmatrix} \sqrt{l + m + 1/2} Y_{l,m-1/2} \\ -\sqrt{l - m + 1/2} Y_{l,m+1/2} \end{pmatrix}
\]

(12)

(where \( m \in [-l-1/2,l+1/2] \) and \( Y \) are spherical harmonics), they found the solutions

\[
\Psi_{l,m} = r^l(1 + r^2)^{-l-\frac{1}{2}}(1 + i\vec{x} \vec{\sigma}) \Phi_{l,m}
\]

\[
\vec{A}_{l,m} = (2l + 3)(1 + r^2)^{-3/2} \frac{\Psi_{l,m}^\dagger \vec{\sigma} \Psi_{l,m}}{\Psi_{l,m} \dagger \Psi_{l,m}}.
\]

Specifically, for maximal magnetic quantum number \( m = l + 1/2 \), these solutions read

\[
\Psi_l := \Psi_{l,l+1/2} = \frac{Y_{l,l} r^l}{(1 + r^2)^{l+3/2}}(1 + i\vec{x} \vec{\sigma}) \Phi_0
\]

\[
\vec{A}^{(l)} = \frac{3 + 2l}{(1 + r^2)^2} \begin{pmatrix} 2x_1 x_3 - 2x_2 \\ 2x_2 x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}
\]

\[
\vec{B}^{(l)} = \frac{4(3 + 2l)}{(1 + r^2)^3} \begin{pmatrix} 2x_1 x_3 - 2x_2 \\ 2x_2 x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}
\]

(where we have omitted an irrelevant constant factor in (15)). Hence, \( \Psi_l \) is proportional to the simplest zero mode (4) and is, therefore, still an eigenvector of the matrix \(-i\sigma_j \partial_j \chi\)
with eigenvalue zero. Further, the zero mode $\Psi_l$ may be rewritten as (again, we ignore irrelevant constant factors)

$$\Psi_l = e^{il\varphi} \frac{S^l}{(1 + S^2)^{l/2}} (1 + r^2)^{-3/2} (1 + i\vec{x}\cdot\vec{\sigma})\Phi_0$$

where we introduced polar coordinates $(x_1, x_2, x_3) \to (r, \theta, \varphi)$, $S$ is the modulus (11), and

$$Y_{l,l} = e^{il\varphi} \sin l \theta = e^{il\varphi} \frac{(r^2 - x_3^2)^{l/2}}{r^l} = e^{il\varphi} \frac{(1 + r^2)^l}{r^l} \frac{S^l}{(1 + S^2)^{l/2}}.$$ 

From expression (18) for $\Psi_l$ it follows easily that the spinors

$$\Psi_{n,l} = \chi^{-n} \Psi_l = e^{i(l\varphi - n\sigma)} \frac{S^{l-n}}{(1 + S^2)^{l/2}} \Psi, \quad n = 0, \ldots, l$$

(20)

(where $\Psi$ is the spinor (4) and $\sigma$ is the phase (11)), are non-singular, square-integrable zero modes for the same gauge field $\vec{A}^{(l)}$ and, therefore, the Dirac operator with gauge field $\vec{A}^{(l)}$ given by (16) has $l + 1$ square-integrable zero modes.

### 3 Discussion

We explicitly constructed the gauge fields $\vec{A}^{(l)}$, (16), and showed that the corresponding Dirac operator has $l + 1$ non-singular, square-integrable zero energy bound states (20). Hence we proved by explicit construction that the phenomenon of zero mode degeneracy does occur for the Abelian Dirac operator in three dimensions, which has been unknown until now.

Before closing, we want to further comment on some points. Firstly, the magnetic fields (17) for higher $l$ are just multiples of the simplest magnetic field (7). Therefore, the number of zero modes $N_l$ for strong magnetic fields (i.e. large $l$) behaves like

$$N_l = l + 1 \sim c \int d^3x |\vec{B}^{(l)}|$$

(21)

(it holds that $\lim_{|x|\to\infty} |\vec{B}^{(l)}| \sim r^{-4}$, therefore the integral in (21) exists), i.e., $N_l$ grows linearly with the strength of the magnetic field (here $c$ is some constant). This is well within the rigorous upper bound on the possible growth of the number of zero modes

$$N \sim c' \int d^3x |\vec{B}|^{3/2}$$

(22)

that was first stated in [4] and later derived in [5] (here $c'$ is a constant).

Secondly, there is in fact a relation between our magnetic fields (17) and the topological feature of Hopf maps. Hopf maps are maps $S^3 \to S^2$, and they fall into distinct homotopy classes that are labelled by the integers (the Hopf index). Hopf maps may be represented by complex functions $\chi : \mathbb{R}^3 \to \mathbb{C}$ provided that $\chi(|\vec{x}| = \infty) = \chi_0 = \text{const}.$
Here the coordinates in $\mathbb{R}^3$ and $\mathbb{C}$ are interpreted as stereographic coordinates of the $S^3$ and $S^2$, respectively. Further, a magnetic field $\vec{B}$ (the Hopf curvature) is related to each Hopf map, where $\vec{B}$ is tangent to the closed curves $\chi = \text{const}$, see [14]–[17] for details.

The observation which we want to make here is that the function $\chi$ in (10) is just the simplest standard Hopf map with Hopf index 1, and the magnetic field (7) is related to the corresponding Hopf curvature of $\chi$. More precisely, after the addition of a fixed prescribed background magnetic field, the magnetic field (7) is precisely equal to the Hopf curvature of the simplest Hopf map $\chi$ (see [17]). Further, as the higher $\vec{B}^{(l)}$ are related to the simplest $\vec{B}$ by integer coefficients, (17), they are related to the Hopf curvatures of higher Hopf maps in the same way (i.e., they are equal to higher Hopf curvatures after the addition of the same fixed background magnetic field). The corresponding Hopf index $H_l$ may be expressed by the number of zero modes $N_l$ as

$$H_l = ((N_l + 1)/2)^2$$  \hspace{1cm} (23)

(when the conventions of [17] for the Hopf curvature are used). Eq. (23) will be proved and discussed in more detail elsewhere, [18]. As $l$ is related to the angular momentum, (13), this indicates an interesting connection between angular momentum, the number of zero modes and the Hopf index.

The above observation leads, of course, to the question whether the topological feature of Hopf maps is related to the existence and degeneracy of zero modes in the general case. This question will be investigated further in future publications. Here we just want to mention that there exist more zero modes that are proportional the simplest zero mode (4) up to a scalar function, see e.g. [17]. For these zero modes it remains true that additional formal zero modes of the same Dirac operator may be constructed by multiplication with the complex function $\chi$, as we did above. In addition, the number of square-integrable zero modes remains related to the Hopf index of the corresponding magnetic field (when interpreted as a Hopf curvature) as in (23), [18].

Anyhow, we think that our results will be relevant for some future developments in mathematical physics, as well as for the understanding of non-perturbative aspects of quantum electrodynamics, especially in three dimensions.

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