Some preliminary results on
the set of principal congruences of a finite lattice

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To the memory of our friend of more than 50 years, Bjarni Jónsson

Abstract. In the second edition of the congruence lattice book, Problem 22.1 asks
for a characterization of subsets $Q$ of a finite distributive lattice $D$ such that there is
a finite lattice $L$ whose congruence lattice is isomorphic to $D$ and under this isomor-
phism $Q$ corresponds the the principal congruences of $L$.

In this note, we prove some preliminary results.

1. Introduction

For a finite lattice $K$, let $J(K)$ and $\text{Princ } K$ denote the (ordered) set of join-
irreducible elements of $K$ and the principal congruences of $K$, respectively, and let

$$J^+(K) = \{0, 1\} \cup J(K).$$

Then for a finite lattice $L$,

$$J^+(\text{Con } L) \subseteq \text{Princ } L \subseteq \text{Con } L,$$

since every join-irreducible congruence is generated by a prime interval; furt-
thermore, $0 = \text{con}(x, x)$ for any $x \in L$ and $1 = \text{con}(0, 1)$.

This paper continues G. Grätzer [11] (see also Section 10-6 of [17] and Part
VI of [12]), whose main result is the following statement.

Theorem 1. Let $P$ be a bounded ordered set. Then there is a bounded lattice $K$
such that $P \cong \text{Princ } K$. If the ordered set $P$ is finite, then the lattice $K$
can be chosen to be finite.

The bibliography lists a number of papers related to this result.

Let $D$ be a finite distributive lattice and let $Q \subseteq D$. We call $Q$ principal
congruence representable (representable, for short), if there is a finite lattice $L$
such that the following two conditions are satisfied:

(i) there is an isomorphism $\psi$ of $\text{Con } L$ and $D$;
(ii) $\text{Princ } L$ corresponds to $Q$ under $\psi$.

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gruence, principal congruence representable set.
By (1), if \( Q \subseteq D \) is representable, then
\[
J^+(D) \subseteq Q \subseteq D.
\] (2)

We call \( Q \subseteq D \) a candidate for principal representability (candidate for short) if (2) holds.

**Problem.** Characterize representable sets for finite distributive lattices.

This is Problem 22.1 in the book \([12]\). In this paper, we prove some preliminary results. We hope that the techniques developed here will have further applications.

The first two results deal with the two extremal cases: \( Q = D \) and \( Q = J^+(D) \).

**Theorem 2.** Let \( D \) be a finite distributive lattice. Then \( Q = D \) is representable.

**Theorem 3.** Let \( D \) be a finite distributive lattice with a join-irreducible unit element. Then \( Q = J^+(D) \) is representable.

Let us call a finite distributive lattice \( D \) fully representable, if every candidate in \( D \) is representable.

The third result finds the smallest distributive lattice that is not fully representable.

**Theorem 4.** The smallest finite distributive lattice \( D \) that is not fully representable has 8 elements.

We use the notation as in \([12]\). You can find the complete Part I. A Brief Introduction to Lattices and Glossary of Notation of \([12]\) at tinyurl.com/lattices101

For a bounded ordered set \( Q \), the ordered set \( Q^- \) is \( Q \) with the bounds removed.

**Outline.** Theorems 2 and 3 are proved in Section 2 and 3 respectively. Section 4 states (but does not prove) that all distributive lattices of size \( \leq 7 \) are fully representable; then it proves that the eight element Boolean lattice is not fully representable.

The version of this paper on arXive.org and Research Gate contain the full Section 4 providing the diagrams for all distributive lattices of size 4–7, enumerating the candidates for representation, and representing each with a lattice.

There is an Addendum with some very recent results.
2. Proving Theorem 2

We need the following easy lemma from the folklore.

**Lemma 5.** Let \( L \) be a finite sectionally complemented lattice. Then every congruence of \( L \) is principal.

**Proof.** For a congruence \( \alpha \), let \( a \) be the largest element in \( L \) satisfying \( a \equiv 0 \) (mod \( \alpha \)). Then \( \text{con}(a, 0) \leq \alpha \). Conversely, the congruence \( u \equiv v \) (mod \( \alpha \)) in \( L \) holds if \( w \equiv 0 \) (mod \( \alpha \)), where \( w \) is a sectional complement of \( u \land v \) in \( u \lor v \). Since \( w \leq a \), it follows that \( \text{con}(a, 0) = \alpha \) and so \( \alpha \) is principal. \( \square \)

This result is implicit in G. Grätzer [8] (see also G. Grätzer and E. T. Schmidt [16]), since in a sectionally complemented lattice \( L \) every congruence is a standard congruence and if \( L \) is finite, then every standard congruence is of the form \( \text{con}(0, s) \), where \( s \) is a standard element. See also [10, Section III.2].

We need now the main result of G. Grätzer and E. T. Schmidt [16] (see also Theorem 8.5 in [12]).

**Theorem 6.** Every finite distributive lattice \( D \) can be represented as the congruence lattice of a finite sectionally complemented lattice \( L \).

Theorem 6 and a reference to Lemma 5 completes the proof of Theorem 2. Rewriting (2) for \( D = \text{Con} L \), we get

\[
J^+(\text{Con} L) \subseteq Q \subseteq \text{Con} L. \tag{3}
\]

3. Proving Theorem 3

We prove the following result.

**Theorem 7.**

(i) Let \( P \) be a finite ordered set with zero and unit. Then there is a finite lattice \( L \) such that \( P \cong \text{Princ} L \).

(ii) This finite lattice \( L \) can be constructed such that

(a) the unit congruence, \( 1 \), is join-irreducible;

(b) for every congruence \( \alpha > 0 \) of \( L \), if \( \alpha \) is principal, then it is join-irreducible.

**Proof.** The first statement is the finite case of Theorem 1. The second statement follows from the way the lattice \( L \) is constructed; to verify it, we have to briefly review this construction.

For a finite bounded ordered set \( P \), we construct the lattice \( L \) as follows. Let 0 and 1 denote the zero and unit of \( P \), respectively.

We first construct the lattice \( F \), the frame, consisting of the elements \( o, i \) and the elements \( a_p, b_p \) for every \( p \in P \), where \( a_p \neq b_p \) for every \( p \in P^- \) and
Figure 1. The frame lattice $F$

$a_0 = b_0$, $a_1 = b_1$. These elements are ordered and the lattice operations are formed as in Figure 1.

The lattice $L$ is as an extension of $F$. We add five elements to the sublattice

$\{o, a_p, b_p, a_q, b_q, i\}$

of $F$ for $p < q \in P^-$, as illustrated in Figure 2 to form the sublattice $S(p, q)$. For $p \in P^d$, let $C_p = \{o < a_p < b_p < i\}$ be a four-element chain. Let $L$ be the lattice $F$ with all the $S(p, q)$ for $p < q \in P^-$ inserted.

There are two crucial observations about $L$.

(i) For any element $x \in L - \{o, i, a_0, a_1\}$, the sublattice $\{o, i, a_0, a_1, x\}$ is isomorphic to $M_3$.

Figure 2. The lattice $S = S(p, q)$
(ii) For incomparable pairs \( x, y \in L \) that belong to an \( S(p, q) \), it holds that \( x, y \) are complementary.

It follows that the map

\[
p \mapsto \begin{cases} 
\text{con}(a_p, b_p) & \text{for } p \in P^-; \\
0 & \text{for } p = 0; \\
1 & \text{for } p = 1 
\end{cases}
\]

is an isomorphism between \( P \) and \( \text{Princ} \). Observe that \( a_p \prec b_p \) in \( L \) and therefore \( \text{con}(a_p, b_p) \) is join-irreducible in \( \text{Con} \). Also \( 0 = \text{con}(o, o) \) and \( 1 = \text{con}(o, i) \), verifying statement (b).

By Theorem 7, \( Q = J^+(\text{Con} \ L) = J^+(D) = \text{Princ} \ L \) is representable.

There are many related constructions. We mention two.

(i) Let the finite lattice \( K \) represent \( Q \subseteq D = \text{Con} \ L \). Form the lattice \( L \) that is an \( M_3 \) with \( K \) replacing one of the atoms. Then \( L \) represents \( Q + C_1 \) in \( D + C_1 \).

(ii) Similarly, with \( Q, D, K \) as in (i), form \( L' \) that is \( C_2^2 \) with \( K \) replacing one of the atoms. Then \( L' \) represents \( Q \subseteq D \) with \( C_2^2 \) glued to the top.

4. Small distributive lattices

In this section, we will settle Problem 1 for distributive lattices of size \( \leq 8 \). We start with a few easy statements.

Let \( D \) be a finite distributive lattice and let \( Q \subseteq D \). We call a candidate proper, if \( Q \subset D \).

The following statement is evident.

**Lemma 8.** Let \( L \) be a finite lattice. If \( J^+(\text{Con} \ L) = \text{Con} \ L \), then there is no proper candidate \( Q \) for \( \text{Con} \ L \). Therefore, \( D \) is fully representable.

**Corollary 9.** Let \( D \) be a finite chain. Then there is only one candidate \( Q \) for \( D \), and it is representable.

**Lemma 10.** Let \( D \) be a finite distributive lattice. We assume that the unit element of \( D \) is join-irreducible and that \( D \) has a unique join-reducible element \( d \in D^- \). Then \( D \) has a unique proper candidate and it is representable.

**Proof.** \( Q = D - \{d\} \) is the only proper candidate. Since \( Q = J^+(D) \), it is representable by Theorem 3.

So let \( D \) be a finite distributive lattice of size \( \leq 8 \). If \( D \) satisfies \( J^+(D) = D \), then the Problem is settled by Lemma 8 so we can assume that

\[
J^+(D) \subset D.
\]

(5)
4.1. Distributive lattices of size $\leq 7$. In this section, we state the first half of Theorem 4.

**Theorem 11.** Let $D$ be a distributive lattice of size $\leq 7$. Then $D$ is fully representable.

We illustrate the proof with the distributive lattice $D = D_{7,5}$, see Figure 3. Let $u \prec v$ be the two join-reducible elements of $D$. Then there are three proper candidates, $Q_{u,v} = D - \{u, v\}$, $Q_u = D - \{u\}$ and $Q_v = D - \{v\}$. Since the unit element of $D$ is join-irreducible, $Q_{u,v}$ is done by Theorem 3.

We represent the candidates $Q_u$ and $Q_v$ in Figures 4 and 5. For the full proof Theorem 11, see this paper in Research Gate: http://tinyurl.com/principalcongruencesI

![Figure 3](image-url)

**Figure 3.** The distributive lattice $D = D_{7,5}$ and $P = J(D) - \{1\}$

![Figure 4](image-url)

**Figure 4.** Representing $Q_u$ for $D = D_{7,5}$; the congruence $q \lor r$ is principal, $p \lor r$ is not.
4.2. A distributive lattice of size 8. Let $L$ be a finite lattice with congruence lattice $\text{Con} L = B_3$, where $B_3$ is the eight-element Boolean lattice with atoms $\alpha_1, \alpha_2, \alpha_3$.

By [2],
\[ J^+(B_3) = \{ \alpha_1, \alpha_2, \alpha_3, 0, 1 \} \subseteq \text{Princ} L. \]

**Lemma 12.** There is no finite lattice $L$ with $\text{Con} L = B_3$ and $J^+(B_3) = \text{Princ} L$.

**Proof.** Let us assume that $L$ is a finite lattice with $\text{Con} L = B_3$ and $J^+(B_3) = \text{Princ} L$. Then in $L$, $0 \equiv 1 \pmod{1}$ and so $0 \equiv 1 \pmod{\alpha_1 \lor \alpha_2 \lor \alpha_3}$. It follows that there is a finite chain $0 = x_0 < x_1 < \cdots < x_n = 1$ such that for every $0 \leq i < n$, there is $1 \leq j_i \leq 3$ satisfying $x_i \equiv x_{i+1} \pmod{\alpha_{j_i}}$ and $j_0 \neq j_1 \neq \cdots \neq j_{n-1}$. Without loss of generality, let $j_0 = 1$ and $j_1 = 2$.

Observe that
\[ 0 < \text{con}(x_0, x_1) \leq \alpha_1 \]
and so $\text{con}(x_0, x_1) = \alpha_1$ because $\alpha_1$ is an atom. Since $n > 1$, it follows that
\[ \alpha_1 \lor \alpha_2 = \text{con}(x_0, x_2) \in \text{Princ} L, \]
contrary to the assumption that $J^+(B_3) = \text{Princ} L$. \qed

Using the notation in the proof of Lemma [12] we can prove more.

Since $0 \equiv 1 \pmod{\alpha_1 \lor \alpha_2}$ fails, there must be a $2 \leq k < n$ so that $x_k \equiv x_{k+1} \pmod{\alpha_3}$. As before, $\text{con}(x_k, x_{k+1}) = \alpha_3$. Then $j_{k-1} = 1$ or
\[ j_{k-1} = 2, \text{ say, } j_{k-1} = 1. \] It follows that \( \text{con}(x_{k-1}, x_k) = \alpha_1 \) and 
\[ \alpha_1 \lor \alpha_3 = \text{con}(x_{k-1}, x_{k+1}) \in \text{Princ}\ L. \]

So we conclude

**Lemma 13.** Let \( L \) be a finite lattice with \( \text{Con}\ L = B_3 \). Then 
\[ |B_3 - \text{Princ}\ L| \leq 1. \]

The converse is also true.

**Theorem 14.** Let \( L \) be a finite lattice with \( \text{Con}\ L = B_3 \). Let \( J^+(B_3) \subseteq Q \subseteq B_3 \). Then \( Q \) is representable iff \( |B_3 - Q| \leq 1 \).

**Proof.** If \( Q \) is representable, then \( |B_3 - Q| \leq 1 \) by Lemma 13.

Conversely, let \( |B_3 - Q| \leq 1 \). Then either \( B_3 = Q \) or \( B_3 = Q \cup \{x\} \), where \( x \) is a dual atom of \( B_3 \). In the first case, we represent \( Q \) by \( L = B_3 \). In the second case, we represent \( Q \) by the chain \( C_4 \). \( \square \)

This example is easy to modify to yield the following result.

**Corollary 15.** Let \( Q \) be representable. Then \( Q^- = Q - \{1\} \) is not necessarily a down set.

Note the odd role planarity is playing in this paper. All distributive lattices proven to be fully representable are planar; the lattices we construct in this section are planar. The smallest distributive lattice that is not fully representable is not planar.

**Addendum, June 20, 2017**

After this paper was accepted for publication, we proved two theorems concerning the “odd role planarity is playing” mentioned in the last paragraph of the previous section. These results were independently obtained by Gábor Czédli, and are presented here with his permission.

**Theorem 16.** Let \( D \) be a finite distributive lattice. If \( D \) is fully representable, then it is planar.

**Proof.** Assume that \( D \) is fully representable but not planar. Then \( J(D) \) contains a three-element antichain \( \alpha_0, \alpha_1, \alpha_2 \). Let \( \alpha = \alpha_0 \lor \alpha_1 \lor \alpha_2 \). Using that our antichain is in \( J(D) \), we obtain that \( \alpha \neq \alpha_i \lor \alpha_j \) for any \( i, j \in \{0, 1, 2\} \). Since \( D \) is fully representable, there is a finite lattice \( L \) representing \( Q = J(D) \cup \{\alpha, 0, 1\} \). In particular, \( \alpha \) is principal in \( L \), that is, \( \alpha = \text{con}(a, b) \).

Let \( X \) be a maximal chain \( a = x_0 \prec \cdots \prec x_n = b \) in \([a, b]\).

For each \( i \) with \( 0 \leq i < n \), the interval \([x_i, x_{i+1}]\) is a prime interval in \( L \); thus \( \text{con}(x_i, x_{i+1}) \in J(D) \). Since \( \text{con}(x_i, x_{i+1}) \leq \alpha \), there is then a \( j \in \{0, 1, 2\} \) such that \( \text{con}(x_i, x_{i+1}) \leq \alpha_j \).

Utilizing that \( \bigvee \{\text{con}(x_i, x_{i+1}) \mid 0 \leq i < n\} = \alpha \), for each \( j \in \{0, 1, 2\} \), there exists an \( i_j \) with \( 0 \leq i_j < n \) and \( \alpha_j \leq \text{con}(x_{i_j}, x_{i_j+1}) \). Since \( \{\alpha_0, \alpha_1, \alpha_2\} \) is an
antichain, we conclude that \( \text{con}(x_i, x_{i+1}) = \alpha_j \). We take the largest interval \([u, v] \supseteq [x_i, x_{i+1}]\) of \(X\) with \(\text{con}(u, v) = \alpha_0\) and let \([u', v']\) be an interval of \(X\) that contains \([u, v]\) and one more element of \(X\). Without loss of generality, we can assume that \(u = x_k, v = x_l, k < l, u' = u, v' = x_{l+1}\). By the maximality of \([u, v]\), it follows that \(\text{con}(v, v') \not\leq \alpha_0\). But then \(\text{con}(v, v') \leq \alpha_1\), say \(\text{con}(v, v') \leq \alpha_1\). Thus the congruences \(\text{con}(u, v) = \alpha_0\) and \(\text{con}(v, v')\) are incomparable. Consequently, \(\text{con}(u, v')\) is join-reducible. This is a contradiction, since \(\text{con}(u, v')\) is principal but \(\text{con}(u, v) < \text{con}(u, v') \leq \alpha_0 \lor \alpha_1 < \alpha\), and so \(\text{con}(u, v')\) is not in \(Q\). \(\square\)

**Theorem 17.** The planar distributive lattice \(D = \mathbb{C}_3^2\) is not fully representable.

**Proof.** Let \(D = \mathbb{C}_3^2\). Let \(a < b\) and \(c < d\) be the join irreducible elements of \(D\), and let \(Q = \{0, a, b, c, d, e, 1\}\), where \(e = a \lor c\). We prove that \(D\) is not fully representable because \(Q \subseteq D\) is not representable.

Let us assume that a finite lattice \(L\) represents \(Q \subseteq D\). Let \([x, y]\) be a maximal interval of \(L\) with respect to the property \(\text{con}(x, y) = e\). Since \(e < 1\), it follows that \([x, y] \neq L\). Without loss of generality, we can assume that \(y\) is not the unit element of \(L\). So there is an element \(z \in L\) with \(y < z\). We cannot have \(\text{con}(y, z) \leq e\), because then \(\text{con}(x, z) = e\), in contradiction with the maximality of \([x, y]\). So \(\text{con}(y, z) = b\) or \(d\), say \(b\). Then \(\text{con}(x, z) = b \lor e\), so \(b \lor e\) is principal, contradicting that \(b \lor e \notin Q\). \(\square\)

In the new manuscript [6], Gábor Czédli provides a characterization of fully principal congruence representable distributive lattices as planar distributive lattices in which only one dual atom can be join-reducible.

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