A Lattice Coding Scheme for Secret Key Generation from Gaussian Markov Tree Sources

Shashank Vatedka and Navin Kashyap
Department of Electrical Communication Engineering
Indian Institute of Science, Bengaluru, India
email: {shashank, nkashyap}@ece.iisc.ernet.in

Abstract—In this article, we study the problem of secret key generation in the multiterminal source model, where the terminals have access to correlated Gaussian sources. We assume that the sources form a Markov chain on a tree. We give a nested lattice-based key generation scheme whose computational complexity is polynomial in the number, N, of independent and identically distributed samples observed by each source. We also compute the achievable secret key rate and give a class of examples where our scheme is optimal in the fine quantization limit. However, we also give examples that show that our scheme is not always optimal in the limit of fine quantization.

I. INTRODUCTION

We study secret key (SK) generation in the multiterminal source model, where m terminals possess correlated Gaussian sources. Each terminal observes N independent and identically distributed (iid) samples of its source. The terminals have access to a noiseless public channel of infinite capacity, and their objective is to agree upon a secret key by communicating across the public channel. The key must be such that any eavesdropper having access to the public communication must not be able to guess the key. In other words, the key must be independent (or almost independent) of the messages communicated across the channel. A measure of performance is the secret key rate that can be achieved, which is the number of bits of secret key generated per (source) sample. On the other hand, the probability that any terminal is unable to reconstruct the key correctly should be arbitrarily small.

The discrete setting — the case where the correlated sources take values in a finite alphabet — was studied by Csiszár and Narayan [2]. They gave a scheme for computing a secret key in this setting and found the secret key capacity, i.e., the maximum achievable secret key rate. This was later generalized by Nitinawarat and Narayan [9] to the case where the sources are obtained by observing some natural parameters, e.g., temperature in a field. In other words, the underlying source is continuous. However, for the purposes of storage and computation, these sources must be quantized, and only the quantized source can be used for secret key generation. If each terminal uses a scalar quantizer, then we get the discrete source model studied in [2]. However, we could do better and instead use a vector quantizer to obtain a higher secret key rate. Nitinawarat and Narayan [9] found the secret key capacity for correlated Gaussian sources in such a setting.

However, to approach capacity, the quantization rate and the rate of public communication at each terminal must approach infinity. In practice, it is reasonable to have a constraint on the quantization rate at each terminal. The terminals can only use the quantized source for secret key generation. Nitinawarat and Narayan [9] studied a two-terminal version of this problem, where a quantization rate constraint was imposed on only one of the terminals. They gave a nested lattice coding scheme and showed that it was optimal, i.e., no other scheme can give a higher secret key rate. In related work, Watanabe and Oohama [12] characterized the maximum secret key rate achievable under a constraint on the rate of public communication in the two-terminal setting. More recently, Ling et al. [7] gave a lattice coding scheme for the public communication-constrained problem and were able to achieve a secret key rate within 1/2 nats of the maximum in [12].

We consider a multiterminal generalization of the two-terminal version studied by [9] where quantization rate constraints are imposed on each of the terminals. Terminal i has access to N iid copies of a Gaussian source X_i(1), X_i(2), ..., X_i(N). The sources are correlated across the terminals. We assume that the joint distribution of the sources has a Markov tree structure [2] Example 7], which is a generalization of a Markov chain. Let us define this formally. Suppose that T = (V,E) is a tree and {X_i : i ∈ V} is a collection of random variables indexed by the vertices. Consider any two disjoint subsets I and J of V. Let v be any vertex such that removal of v from T disconnects I from J (Equivalently, for every i ∈ I and j ∈ J, the path connecting i and j passes through v). For every such I, J, v, if {X_i : i ∈ I} and {X_j : j ∈ J} are conditionally independent given X_v, then we say that {X_i : i ∈ T} form a Markov chain on T. Alternatively, we say that {X_i : i ∈ V} is a Markov tree source.

The contributions of this paper are the following. We study the problem of secret key generation in a Gaussian Markov tree source model with individual quantization rate constraints imposed at each terminal. We give a nested lattice-based scheme and find the achievable secret key rate. For certain classes of Markov trees, particularly homogeneous Markov trees we show that our scheme achieves the secret key...
capacity as the quantization rates go to infinity. However, we also give examples where our scheme does not achieve the key capacity. A salient feature of our scheme is that the overall computational complexity required for quantization and key generation is polynomial in the number of samples \( N \). It is also interesting to note that unlike the general schemes in \([2, 9]\), we give a scheme where at least one terminal remains silent (does not participate in public communication), and omniscience is not attained.

II. NOTATION AND DEFINITIONS

If \( \mathcal{I} \) is an index set and \( \{ A_i : i \in \mathcal{I} \} \) is a class of sets indexed by \( \mathcal{I} \), then their Cartesian product is denoted by \( \times_{i \in \mathcal{I}} A_i \). Given two sequences indexed by \( n \in \mathbb{N} \), \( f(n) \) and \( g(n) \), we say that \( f(n) = O(g(n)) \) if there exists a constant \( c \) such that \( f(n) \leq cg(n) \) for all sufficiently large \( n \). Furthermore, \( f(n) = o_n(1) \) if \( f(n) \to 0 \) as \( n \to \infty \).

Let \( G = (V, E) \) be a graph. The distance between two vertices \( u \) and \( v \) in \( G \) is the length of the shortest path between \( u \) and \( v \). Given a rooted tree \( T = (V, E) \) with root \( r(T) \), we say that a vertex \( u \) is the parent of \( v \neq r(T) \), denoted \( u = \text{par}(v) \), if \( u \) lies in the shortest path from \( r(T) \) to \( v \) and the distance between \( u \) and \( v \) is 1. Furthermore, for every \( v \in V \), we define \( \mathcal{N}_T(v) \) to be the set of all neighbours of \( v \) in \( T \).

III. SECRET KEY GENERATION FROM CORRELATED GAUSSIAN SOURCES

A. The Problem

We now formally define the problem. We consider a multiterminal Gaussian source model \([9]\), which is described as follows. There are \( m \) terminals, each having access to \( N \)-independent and identically distributed (iid) copies of a correlated Gaussian source, i.e., the \( i \)th terminal observes \( X_i(1), X_i(2), \ldots, X_i(N) \) which are iid. Without loss of generality, we can assume that \( X_i(i) \) has mean zero and variance 1. We can always subtract the mean and divide by the variance to ensure that this is indeed the case. The joint distribution of \( \{X_i(i) : 1 \leq i \leq m\} \) can be described by their covariance matrix \( \Phi \).

Specifically, we assume that the sources form a Markov tree, defined in Sec. I. Let \( T = (V, E) \) be a tree having \( |V| = m \) vertices, which defines the conditional independence structure of the sources. For \( u, v \in V \), let us define \( \rho_{uv} := \mathbb{E}[X_u X_v] \).

We can therefore write
\[
X_u = \rho_{uv} X_v + \sqrt{1 - \rho_{uv}^2} Z_{uv}
\]
where \( Z_{uv} \) is a zero-mean, unit-variance Gaussian random variable which is independent of \( X_v \). Similarly,
\[
X_v = \rho_{uv} X_u + \sqrt{1 - \rho_{uv}^2} Z_{vu}
\]
where \( Z_{vu} \) is also a zero-mean, unit-variance Gaussian random variable which is independent of \( X_u \) (and different from \( Z_{uv} \)).

Our objective is to generate a secret key using public communication. For \( v \in V \), let \( X_v^N := (X_v(1), X_v(2), \ldots, X_v(N)) \) denote the \( N \) iid copies of \( X_v \) available at terminal \( v \). Each terminal uses a vector quantizer \( Q_v : \mathbb{R}^N \to \mathcal{X}_v \) of rate \( R_v := \frac{1}{N} \log_2 |\mathcal{X}_v| \). Terminal \( v \) transmits \( F_v^N \in \mathcal{F}_v^N \) — which is a (possibly randomized) function of \( Q_v(X_v^N) \) — across a noiseless public channel that an eavesdropper may have access to. Using the public communication and their respective observations of the quantized random variables, \( Q_v(X_v^N) \), the terminals must generate a secret key \( \mathbf{K}(N) \in \mathcal{K}(N) \) which is concealed from the eavesdropper. Let \( \mathcal{F}_G := \times_{v \in V} \mathcal{F}_v^N \).

Fix any \( \epsilon > 0 \). We say that \( \mathbf{K}(N) \) is an \( \epsilon \)-secret key (\( \epsilon \)-SK) if there exist functions \( f_v : (\mathcal{X}_v, \mathcal{F}_v \to \mathcal{K}(N) \) such that:
\[
\Pr \left[ f_v(Q_v(X_v^N), (F_u^N : u \in V)) \neq \mathbf{K}(N) \right] < \epsilon,
\]
\[
\log_2 |\mathcal{K}(N)| - H(\mathbf{K}(N)) < \epsilon,
\]
and
\[
I \left( \{F_v^N : v \in V\} : \mathbf{K}(N) \right) < \epsilon.
\]

We say that \( R_{\text{key}} \) is an achievable secret key rate if for every \( \epsilon > 0 \), there exist quantizers \( \{Q_v\} \), a scheme for public communication, \( \{F_v^N\} \), and a secret key \( \mathbf{K}(N) \), such that for all sufficiently large \( N \), \( \mathbf{K}(N) \) is an \( \epsilon \)-SK, and \( \frac{1}{N} \log_2 |\mathcal{K}(N)| \geq R_{\text{key}} - \epsilon \).

Consider the following procedure to obtain a class of rooted subtrees of \( T \):

- Identify a vertex \( v \in V \) as the root. The tree \( T \) with \( v \) as the root is a rooted tree. Call this \( T_v^r \).
- Delete all the leaves of \( T_v^r \). Call the resulting rooted subtree \( T_v^* \).

In this work, we only consider noninteractive communication, i.e., the public communication is only a function of the source and not of the prior communication.
Let $T^* := \{ T^*_v : v \in V \}$ denote the set of all rooted subtrees of $T$ obtained in the above manner. Fig. 1 illustrates this for a tree having four vertices. Note that there are $|V|$ trees in $T^*$, one corresponding to each vertex. For any such rooted subtree $T^* = (V^*, E^*)$ in $T^*$, let $x(T^*)$ denote the root of $T^*$. We will see later that it is only the terminals that correspond to $T^*$ that participate in the public communication while the other terminals remain silent. For any $v \in V^*$, let $N_T(v)$ denote the set of all neighbours of $v$ in $T$ (not $T^*$). Recall that each terminal $v$ operates under a quantization rate constraint of $R_q^{(v)}$. For every $T^* = (V^*, E^*)$, we define

$$R_{\text{ent}} = R_q^{(x(T^*))} + \sum_{u \in V^* \setminus x(T^*)} \frac{1}{2} \log_2 \left( (e^{2R_q^{(v)}} - 1)(1 - \rho_{uv}^2) + 1 \right)$$

and

$$R_{\text{com}} = \sum_{v \in V^*} \max_{u \in N_T(v)} \frac{1}{2} \log_2 \left( (e^{2R_q^{(v)}} - 1)(1 - \rho_{uv}^2) + 1 + \frac{\rho_{uv}^2 e^{2R_q^{(v)}}}{e^{2R_q^{(v)}} - 1} \right).$$

We will show that the joint entropy of the quantized sources is at least $R_{\text{ent}}$ and the sum rate of public communication is at most $R_{\text{com}}$ in our scheme. Also, the public communication that achieves $R_{\text{com}}$ requires only the terminals in $T^*$ to participate in the communication; the terminals in $V \setminus V^*$ are silent. Let us also define

$$\alpha := \frac{\max_{u \in V^*} R_q^{(u)}}{\min_{v \in V^*} R_q^{(v)}}.$$  

Our aim is to prove the following result

**Theorem 1.** For a fixed quantization rate constraint $\{R_q^{(v)} : v \in V\}$, a secret key rate of

$$R_{\text{key}} = \max_{T^* \in T^*} \left\{ R_{\text{ent}} - R_{\text{com}} \right\}$$

is achievable using a nested lattice coding scheme whose computational complexity grows as $O(N^{\alpha+1})$.

Note that if all terminals have identical quantization rate constraints, then the complexity is $O(N^2)$. Sec. V describes the scheme and contains the proof of the above theorem.

We now discuss some of the implications of the result. Letting the quantization rates $R_q^{(v)}$ in (4) go to infinity, i.e., as $R_q^{(v)} \to \infty$ for all $v$, we get that

**Corollary 2.** In the fine quantization limit, a secret key rate of

$$R_{\text{key}} = \max_{T^* \in T^*} \left\{ \min_{v \in N_T(x(T^*))} \frac{1}{2} \log_2 \left( \frac{1}{1 - \rho_{uv}^2} \right) + \sum_{u \in V^* \setminus x(T^*)} \min_{v \in N_T(u)} \frac{1}{2} \log_2 \left( \frac{1}{1 - \rho_{uv}^2} \right) \right\}$$

(5)

is achievable.

If there are no constraints on the quantization rates, then from [9, Theorem 3.1] and [2, Example 7], we know that the maximum achievable secret key rate is

$$C_{\text{key}}^{(\infty)} = \min_{(u,v) \in E} \frac{1}{2} \log_2 \left( \frac{1}{1 - \rho_{uv}^2} \right).$$

(6)

IV. REMARKS ON THE ACHIEVABLE SECRET KEY RATE

A. The Two-User Case

Consider the two-user case with terminals $u$ and $v$. Let us define

$$R(u, v) := \frac{1}{2} \log_2 \left( \frac{e^{2R_q^{(v)}}}{(e^{2R_q^{(v)}} - 1)(1 - \rho_{uv}^2) + 1 + \rho_{uv}^2 e^{2R_q^{(v)}}} \right).$$

As we will see later, the above SK rate is achieved with $u$ participating in the public communication and $v$ remaining silent. The achievable secret key rate, (4), is equal to $\max \{ R(u, v), R(v, u) \}$. A simple calculation reveals that

$$e^{-2R(u,v)} - e^{-2R(v,u)} = \rho_{uv}^2 \left( \frac{1}{e^{2R_q^{(v)}} - 1} - \frac{1}{e^{2R_q^{(v)}} - 1} \right).$$

Hence, if $R_q^{(v)} > R_q^{(u)}$, then $R(u, v) > R(v, u)$. This means that in order to obtain a higher secret key rate using our scheme, the terminal with the lower quantization rate must communicate, while the other must remain silent.

If we let $R_q^{(v)}$ in $R(u, v)$ go to infinity, then we get the rate $R_{NN}$ achieved in [9], which was shown to be optimal when we only restrict the quantization rate of one terminal.

$$R_{NN} = \frac{1}{2} \log_2 \left( \frac{e^{2R_q^{(v)}}}{(e^{2R_q^{(v)}} - 1)(1 - \rho_{uv}^2) + 1} \right).$$

Fig. 2 illustrates the behaviour of the achievable rate for different sum-rate constraints ($R_q^{(u)} + R_q^{(v)} = R$). The rate achieved by the scheme of Nitinawarat and Narayan [9], $R_{NN}$, is also shown.

B. Optimality of $R_{\text{key}}$ in the Fine Quantization Limit

We present a class of examples where $R_{\text{key}}$ is equal to the secret key capacity $C_{\text{key}}^{(\infty)}$ in the fine quantization limit. One
such example is the class of homogeneous Markov trees, where $\rho_{uv} = \rho$ for all edges $(u, v)$. In this case,
\[
\min_{v \in N_T(u)} \frac{1}{2} \log_2 \left( \frac{1 - \rho_{uv}^2 \text{par}(u)}{1 - \rho_{uv}^2} \right) = 0,
\]
and hence, by Corollary 2
\[
R_{\text{key}} = \frac{1}{2} \log_2 \left( \frac{1}{1 - \rho^2} \right) = C_{\text{key}}^{(\infty)}.
\]

This property holds for a wider class of examples. Consider the case where $T$ has a rooted subtree $T^*$ such that for every $u \in V^*$, $\arg \min_{v \in N_T(u)} \rho_{uv} = \text{par}(u)$. Once again, we have
\[
\min_{v \in N_T(u)} \frac{1}{2} \log_2 \left( \frac{1 - \rho_{uv}^2 \text{par}(u)}{1 - \rho_{uv}^2} \right) = 0.
\]

Moreover, the edge $(u, v) \in E$ with the minimizing $\rho_{uv}$ (and therefore, the minimizing mutual information) is incident on $r(T^*)$. Hence, $R_{\text{key}} = C_{\text{key}}^{(\infty)}$.

C. Suboptimality of $R_{\text{key}}$ in the Fine Quantization Limit

We can give several examples for which $R_{\text{key}}$ in the fine quantization limit is strictly less than $C_{\text{key}}^{(\infty)}$. Note that
\[
\min_{v \in N_T(u)} \frac{1}{2} \log_2 \left( \frac{1 - \rho_{uv}^2 \text{par}(u)}{1 - \rho_{uv}^2} \right) \leq 0,
\]
and if these terms are nonzero for every $T^* \in T^*$, then the scheme is suboptimal. As a specific example, consider the Markov chain of Fig. 3 where $\rho_{23} > \max\{\rho_{12}, \rho_{34}\}$. Let us further assume that $\rho_{12} = \rho_{34}$. The secret key capacity is
\[
C_{\text{key}}^{(\infty)} = \frac{1}{2} \log_2 \frac{1}{1 - \rho_{12}}.
\]

Irrespective of which $T^* \in T^*$ we choose (see Fig. 3), we have
\[
\min_{v \in N_T(r(T^*))} \frac{1}{2} \log_2 \left( \frac{1}{1 - \rho_{r(T^*)v}^2} \right) = C_{\text{key}}^{(\infty)}.
\]

Furthermore, the second term in (5) is negative for every $T^*$. This is because every $T^*$ has some $u \neq r(T^*)$ for which $\arg \min_{v \in V^*} \rho_{uv} \neq \text{par}(v)$.

V. The Secret Key Generation Scheme

We now describe the lattice coding scheme that achieves the promised secret key rate. Our scheme is very similar to the scheme given by Nitinawarat and Narayan [9] for the two-terminal case.

We use a block encoding scheme just like the one in [9]. Recall that each terminal $v$ has a quantization rate constraint of $R_{qv}^{(n)}$. The total blocklength $N$ is partitioned into $N_{\text{out}}$ blocks of $n$ samples each, i.e., $N = nN_{\text{out}}$, where $N_{\text{out}} = \min_{2^k R_{qv}^{(n)}} - 1$. The secret key generation scheme comprises two phases: an information reconciliation phase, and a privacy amplification phase. The reconciliation phase is divided into two subphases: a lattice coding-based analog phase, which is followed by a Reed-Solomon coding-based digital phase. The privacy amplification phase employs a linear mapping to generate the secret key from the reconciled information. This uses the results of Nitinawarat and Narayan [9]. The digital phase is also inspired by the concatenated coding scheme used in [11] in the context of channel coding for Gaussian channels.

Let us briefly outline the protocol for secret key generation. Each terminal $v$ uses a chain of nested lattices $(\Lambda_v, \Lambda_v^{(1)}, \Lambda_v^{(2)})$ in $\mathbb{R}^n$, where $\Lambda_v^{(2)} \subset \Lambda_v^{(1)} \subset \Lambda_v$. The Gaussian input $x_v$ at terminal $v$ is processed blockwise, with $n$ samples collected to form a block. Suppose $x_v = (x_v^{(1)}, \ldots, x_v^{(N_{\text{out}})})$ where $x_v^{(i)}$ denotes the $i$th block of length $n$. Each terminal $v$ also generates random dithers $(d_v^{(i)} : 1 \leq i \leq N_{\text{out}})$, which are all uniformly distributed over the fundamental Voronoi region of $\Lambda_v$, and independent of each other. These are assumed to be
known to all terminals. The protocol for secret key generation is as follows.

- **Quantization:** Terminal \( v \in V \) computes
  \[
  y_v^{(i)} = [Q_{\Lambda_v}(x_v^{(i)} + d_v^{(i)}) - d_v^{(i)}] \mod \Lambda_v^{(2)}.
  \]

- **Information reconciliation: Analog phase:** Let \( T^* = (V^*, E^*) \) be the rooted subtree which achieves the maximum in (4). Terminal \( v \in V^* \) broadcasts
  \[
  w_v^{(i)} = [y_v^{(i)}] \mod \Lambda_v^{(1)}.
  \]
  across the public channel. Terminal \( u \) has access to \( y_u^{(i)} \) and for every \( v \in N_T(u) \), it estimates
  \[
  \tilde{y}_v^{(i)} = w_v^{(i)} + Q_{\Lambda_v^{(1)}} \left( \rho_{uv} y_u^{(i)} - w_v^{(i)} \right).
  \]

Having estimated \( y_v^{(i)} \) for all neighbours \( v \), it estimates \( y_v^{(i)} \) for all \( v \) which are at a distance \( 2 \) from \( u \), and so on, till it has estimated \( \{y_v^{(i)} : v \in V^*, 1 \leq i \leq N_{\text{out}} \} \).

- **Information reconciliation: Digital phase:** To ensure that all \( N_{\text{out}} \) blocks can be recovered at all terminals with an arbitrarily low probability of error, we use a Slepian-Wolf scheme using Reed-Solomon codes. Each terminal uses an \( (N_{\text{out}}, K_{\text{out}}) \) Reed-Solomon code over \( \mathbb{F}_{2^m} \), where the parameters \( K_{\text{out}} \) and \( p^{K_{\text{out}}} \) will be specified later. The syndrome corresponding to \( (y_v^{(1)}, \ldots, y_v^{(N_{\text{out}})}) \) in the code is publicly communicated by terminal \( v \). We show that this can be used by the other terminals to estimate all the \( y_v^{(i)} \)'s with a probability of error that decays exponentially in \( N \).

- **Key generation:** We use the result [9, Lemma 4.5] that there exists a linear transformation of the source symbols (viewed as elements of a certain finite field) that can act as the secret key. Since all terminals can estimate \( \{y_v^{(i)} : v \in V^*, 1 \leq i \leq N_{\text{out}} \} \) reliably, they can all compute the secret key with an arbitrarily low probability of error.

Before we go into the details of each step, we describe some specifics of the coding scheme. We want the nested lattices that form the main component of our protocol to satisfy certain “goodness” properties. We begin by describing the features that the lattices must possess.

### A. Nested Lattices

Some basic definitions and relevant results on lattices have been outlined in Appendix A. Given a lattice \( \Lambda \), we let \( \mathcal{V}(\Lambda) \) be the fundamental Voronoi region of \( \Lambda \), and \( \sigma^2(\Lambda) \) denotes the second moment per dimension of \( \Lambda \). Furthermore, we define \( \text{vol}(\Lambda) := \text{vol}(\mathcal{V}(\Lambda)) \).

Each terminal \( v \) uses a chain of \( n \)-dimensional nested lattices \( (\Lambda_v, \Lambda_v^{(1)}, \Lambda_v^{(2)}) \), with \( \Lambda_v^{(2)} \subset \Lambda_v^{(1)} \subset \Lambda_v \). These are all Construction-A lattices [3, 4] obtained from linear codes of blocklength \( n \) over \( \mathbb{F}_p \), with \( p \) chosen large enough to ensure that these lattices satisfy the required goodness properties. Furthermore, \( \Lambda_v^{(1)} \) and \( \Lambda_v^{(2)} \) are obtained from subcodes of linear codes that generate \( \Lambda_v \). Fix any \( \delta > 0 \). The lattices are chosen so that

\[
\frac{1}{n} \log_2 \frac{\text{vol}(\Lambda_v \cap \mathcal{V}(\Lambda_v^{(2)}))}{\text{vol}(\mathcal{V}(\Lambda_v))} = \frac{1}{n} \log_2 \frac{\text{vol}(\Lambda_v^{(2)})}{\text{vol}(\Lambda_v)} = \frac{k_v}{n} \log_2 p = R_v^{(v)},
\]

and

\[
\frac{\text{vol}(\Lambda_v^{(1)})}{2\pi e} = \max_{u \in N_T(v)} (1 - \rho_{uv}^2 + \sigma^2(\Lambda_v) + \rho_{uv}^2 \sigma^2(\Lambda_u)) \times (1 + \delta).
\]

Furthermore, these lattices satisfy the following “goodness” properties [3]:

- \( \Lambda_v \) is good for covering.
- \( \Lambda_v^{(1)} \) and \( \Lambda_v^{(2)} \) are good for AWGN channel coding.

### B. Quantization

Terminal \( v \) observes \( N \) samples \( x_v = (x_v(1), \ldots, x_v(N)) \). As mentioned earlier, the quantizer operates on blocks of \( n \) samples each, and there are \( N_{\text{out}} \) such blocks. We can write \( x_v \) as a sequence of \( N_{\text{out}} \) blocks. Each terminal uses Reed-Solomon codes. Each terminal uses

\[
y_v^{(i)} = [Q_{\Lambda_v}(x_v^{(i)} + d_v^{(i)}) - d_v^{(i)}] \mod \Lambda_v^{(2)}
\]

and

\[
\text{denote the output of the lattice quantizer at terminal } v. \text{ The terminals can only use } y_v := (y_v^{(1)}, y_v^{(2)}, \ldots, y_v^{(N_{\text{out}})}) \text{ for the secret key generation protocol. From (8) and (9), we can see that the quantization rates satisfy}
\]

\[
R_v^{(v)} = \frac{1}{2} \log_2 \left( 1 + \frac{1}{\sigma^2(\Lambda_v)} \right) + \log_2(1 + \delta) + o_n(1).
\]

### C. Information Reconciliation: The Analog Phase

Let \( T^* = (V^*, E^*) \) denote the rooted tree in \( T^* \) which achieves the maximum in (4). The terminals in \( V^* \) are the only ones that communicate across the public channel. Terminal \( v \in V^* \) broadcasts

\[
w_v^{(i)} := [y_v^{(i)}] \mod \Lambda_v^{(1)}
\]

for \( 1 \leq i \leq N_{\text{out}} \) across the public channel. Prior to the analog phase, terminal \( u \in V \) only has access to \( y_u := (y_u^{(1)}, y_u^{(2)}, \ldots, y_u^{(N_{\text{out}})}) \). At the end of the information reconciliation phase, every terminal \( u \) will be able to recover \( \{y_v : v \in V^*\} \) with low probability of error. The analog phase
ensures that every $y^{(i)}_v$ can be individually recovered with low probability of error. The digital phase guarantees that the entire block $y_v$ can be recovered reliably.

Now consider any $v \in V^*$ and $u \in N_T(v)$ (not necessarily in $V^*$). Suppose that some terminal $u'$ (not necessarily $u$) has a reliable estimate of $y^{(i)}_v$. From $y^{(i)}_v$ and $w^{(i)}_v$, terminal $u'$ can estimate $y^{(i)}_v$ as follows:

$$\hat{y}^{(i)}_v = w^{(i)}_v + Q_{v}^{(i)}(\rho_{vu}y^{(i)}_u - w^{(i)}_v).$$ \hfill (14)

The following proposition is proved in Appendix B-A.

**Proposition 3.** Fix a $\delta > 0$. For every $1 \leq i \leq N_{\text{out}}$, we have

$$\mathbb{E}_{y^{(i)}} \Pr[\hat{y}^{(i)}_v \neq y^{(i)}_v] \leq e^{-nE_{\text{out}}(\delta)}$$ \hfill (15)

where $E_{\text{out}}$ is a quantity which is positive for all positive $\delta$ and all sufficiently large $n$, as long as

$$\frac{(\text{vol}(\Lambda^{(1)}_v))^{2/n}}{2\pi e} > \max_{a \in N_T(v)} \left(1 - \rho_{va}^2 + \sigma^2(\Lambda_v) + \sigma^2(\Lambda_v) \right) \times (1 + \delta),$$ \hfill (16)

and

$$\frac{(\text{vol}(\Lambda^{(2)}_v))^{2/n}}{2\pi e} > (1 + \sigma^2(\Lambda_v))(1 + \delta),$$ \hfill (17)

and

$$\frac{(\text{vol}(\Lambda^{(2)}_v))^{2/n}}{2\pi e} > (1 + \sigma^2(\Lambda_v))(1 + \delta).$$ \hfill (18)

Since terminal $u$ has $y^{(i)}_v$, it can (with high probability) recover the corresponding quantized sources of its neighbours. Assuming that these have been recovered correctly, it can then estimate the quantized sources of all terminals at distance two from $u$, and so on, till all $y^{(i)}_v$ for $v \in V^*$ have been recovered.

Using the union bound, we can say that the probability that terminal $u$ correctly recovers $\{y^{(i)}_v : v \in V^*\}$ is at least $1 - \sum_{u \in V^*} \max_{a \in N_T(u)} e^{-nE_{\text{out}}(\delta)}$.

For all terminals to be able to agree upon the key, we must ensure that every terminal can recover all blocks $\{y^{(i)}_v : v \in V^*\}$ with low probability of error. Since $N_{\text{out}}$ is exponential in $n$, the analog phase does not immediately guarantee this. For that, we use the digital phase.

### D. Information Reconciliation: The Digital Phase

Observe that $y^{(i)}_v \in \Lambda_v \cap \mathcal{V}(\Lambda^{(2)}_v)$, where both $\Lambda_v$ and $\Lambda^{(2)}_v$ are Construction-A lattices obtained by linear codes over $\mathbb{F}_p$. As a result, $|\Lambda_v \cap \mathcal{V}(\Lambda^{(2)}_v)|$ is always an integer power of $p$ \cite{3}. Let

$$|\Lambda_v \cap \mathcal{V}(\Lambda^{(2)}_v)| = p^h v.$$ \hfill (19)

Let $c_{\text{out}}^{N_{\text{out}}} \in C_v$, and $c_{\text{out}}^{\text{err}}$ is a minimum Hamming weight representative of the coset to which $y^{(i)}_v$ belongs in $\mathbb{F}_p^{N_{\text{out}}}$. Terminal $v$ broadcasts $s^{\text{out}}_v$ across the public channel. This requires a rate of public communication of at most

$$\frac{1}{N} \log_2 |\mathbb{F}_p^{N_{\text{out}}}| = \frac{N_{\text{out}}}{N} \log_2(p^h v) = 2\delta R(v).$$ \hfill (20)

From $s^{\text{out}}_v$ and $\hat{y}^{\text{out}}_v$, terminal $u$ can compute

$$\hat{c}^{\text{out}}_v \neq \hat{y}^{\text{out}}_v - s^{\text{out}}_v = c^{\text{out}}_v + e^{\text{out}}_v.$$ \hfill (21)

For sufficiently large $n$ the probability that $y^{(i)}_v$ is estimated incorrectly is less than $\delta$, and terminal $u$ can recover $c^{\text{out}}_v$ with high probability using the decoder for the Reed-Solomon code.

**Proposition 4 (Theorem 2, \cite{11}).** The probability that the Reed-Solomon decoder incorrectly decodes $c^{\text{out}}_v$ from $\hat{c}^{\text{out}}_v$ decays exponentially in $N$.

Having recovered $c^{\text{out}}_v$ reliably, the terminals can obtain $y^{\text{out}}_v$ using (20). Therefore, at the end of the digital phase, all terminals can recover $\{y^{\text{out}}_v : v \in V^*\}$ with a probability of error that decays exponentially in $N$.

### E. Secret Key Generation

Let $k := \sum_{v \in V^*} k_{vu}$. There exists a (set) bijection $\phi$ from $\mathbb{F}_p^{N_{\text{out}}}$ to $\mathbb{F}_p^k$. Let $y^{(i)}_v : v \in V^*$. We use the following result by Nitinawarat and Narayan \cite{6}, which says that there exists a linear function of the sources that can act as the secret key.

**Lemma 5 (Lemma 4.5, \cite{6}).** Let $Y$ be a random variable in a Galois field $\mathbb{F}_p$ and $D$ be an $\mathbb{R}^n$-valued random variable jointly distributed with $Y$. Consider $N_{\text{out}}$ iid repetitions of $(Y, D)$, namely $(Y^{\text{out}}, D^{\text{out}}) = ((Y_1, D_1), \ldots, (Y_{N_{\text{out}}}, D_{N_{\text{out}}}))$.

Let $B = B^{(\text{out})} \in B^{(\text{out})}$ be a finite-valued rv with a given joint distribution with $(Y^{\text{out}}, D^{\text{out}})$. \hfill (22)
Then, for every $\delta > 0$ and every
$$R < H(Y|D) - \frac{1}{N_{out}} \log |\mathcal{B}(N_{out})| - 2\delta,$$
there exists a $\left\lfloor \frac{N_{out}R}{\log q} \right\rfloor \times N_{out}$ matrix $L$ with $\mathbb{F}_q$-valued entries such that
$$N_{out}R - H(LY^{N_{out}}) + I(LY^{N_{out}}; D^{N_{out}}, B)$$
vanishes exponentially in $N_{out}$.

In other words, $LY^{N_{out}}$ is an $\epsilon$-SK for suitable $\epsilon$. Let $q = p^k$ and $B = (w^y, s^y_{\text{out}} : y \in V^*)$. Then, the above lemma guarantees the existence of an $\mathbb{F}_q$-valued matrix $L$, so that $L(y^{(1)} , \ldots, y^{(N_{out})})^T$ is a secret key with a rate of
$$R_{\text{key}} = \frac{1}{N} H (y_{\mathbf{f}} : y \in V^* | d_{\mathbf{f}} : y \in V^*) - \sum_{y \in V^*} R_{\text{com}}(y),$$
where $R_{\text{com}}(y)$ denotes the total rate of communication of terminal $y$. We give a lower bound on $R_{\text{key}}$ by bounding $H (y_{\mathbf{f}} : y \in V^* | d_{\mathbf{f}} : y \in V^*)$ in the next section.

F. Joint Entropy of the Quantized Sources

The proof of the following lemma is given in Appendix B-B.

**Lemma 6.** Fix a $\delta > 0$, and let $D_i := \{d_{\mathbf{f}}^{(i)} : y \in V^*\}$. For all sufficiently large $n$, we have
$$\frac{1}{n} H (y_{\mathbf{f}}^{(i)} : y \in V^* | d_{\mathbf{f}}) \geq \frac{1}{2} \log_2 \left( 1 + \frac{1}{\sigma^2 (\Lambda_{\mathbf{f}}^T)} \right) \geq \frac{1}{2} \log_2 \left( 1 + \frac{1 - \rho^2_{\mathbf{u}, \text{par}} (n)}{\sigma^2 (\Lambda_{\mathbf{f}})} \right) - \delta$$
(23)

For every $y$, $\{y_{\mathbf{f}}^{(i)} : 1 \leq i \leq N_{out}\}$ are independent and identically distributed. If $D := \{D_i : 1 \leq i \leq N_{out}\}$, then $H (y_{\mathbf{f}} : y \in V^* | D) = N_{out} H (y_{\mathbf{f}}^{(i)} : y \in V^* | D_i)$. Substituting for $\sigma^2 (\Lambda_{\mathbf{f}})$ from (12) in (23), we get
$$\frac{1}{N} H (y_{\mathbf{f}} : y \in V^* | D) \geq R_{\text{ent}} - g(\delta) - o_n(1),$$
where $R_{\text{ent}}$ is defined in (1), and $g(\delta)$ is a quantity that goes to 0 as $\delta \to 0$.

G. Achievable Secret Key Rate and Proof of Theorem 7

Lemma 6 guarantees the existence of a strong secret key which is a linear transformation of $(y^{(1)}, \ldots, y^{(N_{out})})$. From Propositions 3 and 4 all terminals are able to recover $(y^{(1)}, \ldots, y^{(N_{out})})$ with a probability of error that decays exponentially in $N = nN_{out}$.

During the analog phase, each terminal $y$ in $V^*$ publicly communicates
$$R_{\text{analog}}(y) = \frac{1}{n} \log_2 \frac{\nu (\Lambda_{\mathbf{f}}^{(1)})}{\nu (\Lambda_{\mathbf{f}})} \leq \max_{u \in N_{V^*} (y)} \frac{1}{2} \log_2 \left( \frac{1 - \rho^2_{\mathbf{u}, \text{par}} (n)}{\sigma^2 (\Lambda_{\mathbf{f}})} \right) + o_n(1) + \delta.$$
by the TCS Research scholarship programme, and that of the second author by a Swarnajayanti fellowship awarded by the Department of Science and Technology (DST), India.

APPENDIX A

LATICE CONCEPTS

In this appendix, we briefly review basic lattice concepts that are relevant to this work. We direct the interested reader to [1, 3, 4, 13] for more details. Let $\Lambda$ denote a full-rank $n \times n$ matrix with real entries. Then the set of all integer-linear combinations of the columns of $\Lambda$ is called a lattice in $\mathbb{R}^n$, and $\Lambda$ is called a generator matrix of the lattice. Given a lattice $\Lambda$ in $\mathbb{R}^n$, we define $Q_{\Lambda} : \mathbb{R}^n \rightarrow \Lambda$ to be the lattice quantizer that maps every point in $\mathbb{R}^n$ to the closest (in terms of Euclidean distance) point in $\Lambda$, with ties being resolved according to a fixed rule. The fundamental Voronoi region, $V(\Lambda)$, is the set of all points in $\mathbb{R}^n$ for which $0$ is the closest lattice point, i.e., $V(\Lambda) := \{ x \in \mathbb{R}^n : Q_{\Lambda}(x) = 0 \}$. For any $x \in \mathbb{R}^n$, we define $[x] \mod \Lambda := x - Q_{\Lambda}(x)$. We also define $\text{vol}(\Lambda) := \text{vol}(V(\Lambda))$. The coverings radius of $\Lambda$, denoted $r_{\text{cov}}(\Lambda)$ is the radius of the smallest closed ball in $\mathbb{R}^n$ centered at $0$ that contains $V(\Lambda)$. Similarly, the effective radius, $r_{\text{eff}}(\Lambda)$, is the radius of a ball in $\mathbb{R}^n$ having volume $\text{vol}(\Lambda)$.

The second moment per dimension of a lattice, $\sigma^2(\Lambda)$ is defined as

$$\sigma^2(\Lambda) = \frac{1}{n \text{vol}(\Lambda)} \int_{x \in V(\Lambda)} ||x||^2 dx,$$

and is equal to the second moment per dimension of a random vector uniformly distributed over $V(\Lambda)$. The normalized second moment per dimension of $\Lambda$ is defined as

$$G(\Lambda) := \frac{\sigma^2(\Lambda)}{\text{vol}(\Lambda)^{2/n}}.$$

If $\Lambda$ and $\Lambda_0$ are two lattices in $\mathbb{R}^n$ that satisfy $\Lambda_0 \subset \Lambda$, then we say that $\Lambda_0$ is a sublattice of $\Lambda$, or $\Lambda_0$ is nested in $\Lambda$. Furthermore,

$$|\Lambda \cap V(\Lambda_0)| = \frac{\text{vol}(\Lambda_0)}{\text{vol}(\Lambda)}.$$

We say that a lattice $\Lambda$ (or more precisely, a sequence of lattices $\{\Lambda\}$ indexed by the dimension $n$) is good for mean squared error (MSE) quantization if

$$\lim_{n \rightarrow \infty} G(\Lambda) = \frac{1}{2\pi e}.$$

A useful property is that if $\Lambda$ is good for MSE quantization, then $\text{vol}(\Lambda)^{2/n} \rightarrow 2\pi e \sigma^2(\Lambda)$ as $n \rightarrow \infty$. We say that $\Lambda$ is good for covering (or covering-good or Rogers-good) if $r_{\text{cov}}(\Lambda)/r_{\text{eff}}(\Lambda) \rightarrow 1$ as $n \rightarrow \infty$. It is a fact that if $\Lambda$ is good for covering, then it is also good for MSE quantization.

Let $Z$ be a zero-mean $n$-dimensional white Gaussian vector having second moment per dimension equal to $\sigma^2$. Let

$$\mu := \frac{\text{vol}(V(\Lambda))^{2/n}}{\sigma^2}.$$

Then we say that $\{\Lambda\}$ is good for AWGN channel coding (or AWGN-good or Poltyrev-good) if the probability that $Z$ lies outside the fundamental Voronoi region of $\Lambda$ is upper bounded by

$$\Pr[Z \notin V(\Lambda)] \leq e^{-n(\mathcal{E}_U(\mu) - o_n(1))}$$

for all $\sigma^2$ that satisfy $\mu \geq 2\pi e$. Here, $\mathcal{E}_U(\cdot)$, called the Poltyrev exponent is defined as follows:

$$\mathcal{E}_U(\mu) = \begin{cases} \frac{\mu}{16\pi e} & \text{if } 8\pi e \leq \mu \\ \frac{\mu}{4\pi e} - \frac{1}{2} \ln \frac{\mu}{2\pi e} & \text{if } 2\pi e \leq \mu \leq 4\pi e. \end{cases}$$

Suppose that we use a subcollection of points from an AWGN-good lattice $\Lambda$ as the codebook for transmission over an AWGN channel. Then, as long as

$$\frac{\text{vol}(V(\Lambda))^{2/n}}{\sigma^2} \geq 2\pi e,$$

the probability that a lattice decoder decodes to a lattice point other than the one that was transmitted, decays exponentially in the dimension $d$, with the exponent given by (26).

Lattices that satisfy the above “goodness” properties were shown to exist in [4]. Moreover, such lattice can be constructed from linear codes over prime fields. Let $p$ be a prime number, and $C$ be an $(n, k)$ linear code over $\mathbb{F}_p$. In other words, $C$ has blocklength $n$ and dimension $k$. Let $\phi$ be the natural embedding of $\mathbb{F}_p$ in $\mathbb{Z}$, and for any $x \in \mathbb{F}_p^n$, let $\phi(x)$ be the $n$-length vector obtained by operating $\phi$ on each component of $x$. The set $\Lambda := \phi(C) + p\mathbb{Z}^n := \{ \phi(x) + py : x \in C, y \in \mathbb{Z}^n \}$ is a lattice, and is called the Construction-A lattice obtained from the linear code $C$. With a slight abuse of notation, we will call any scaled version of $\Lambda$, i.e., $\alpha \Lambda$ for any $\alpha > 0$, a Construction-A lattice obtained from $C$. A useful fact is that $\Lambda$ always contains $p\mathbb{Z}^n$ as a sublattice, and the nesting ratio $\Lambda/p\mathbb{Z}^n = p^k$. It was shown in [4] that if $k$ and $p$ are appropriately chosen functions of $n$, then a randomly chosen Construction-A lattice over $\mathbb{F}_p$ is good for covering and AWGN channel coding with probability tending to $1$ as $n \rightarrow \infty$.

We use the nested lattice construction in [3, 6] to obtain good nested lattices. Let $\Lambda_0$ be a Construction-A lattice which is good for covering and AWGN, and let $\Lambda$ be a generator matrix for $\Lambda_0$. Then, if $\Lambda'$ is another Construction-A lattice, then $\Lambda = \Lambda_0 + \Lambda'$ is a lattice that contains $\Lambda_0$ as a sublattice. It was shown in [6] that if $\Lambda$ and $\Lambda_0$ are chosen at random, then they are both simultaneously good for AWGN channel coding and covering with probability tending to $1$ as $n \rightarrow \infty$ (provided that $k$ and $p$ are suitably chosen).
APPENDIX B

TECHNICAL PROOFS

A. Proof of Proposition 4

Recall that
\begin{align}
y_u^{(i)} &= [Q_{\Lambda_u} (x_u^{(i)} + d_u^{(i)}) - d_u^{(i)}] \mod \Lambda_u^{(2)} \\
&= \left[ x_u^{(i)} - [x_u^{(i)} + d_u^{(i)}] \mod \Lambda_u \right] \mod \Lambda_u^{(2)} \\
&= [x_u^{(i)} + \tilde{d}_u^{(i)}] \mod \Lambda_u^{(2)},
\end{align}
where \(\tilde{d}_u^{(i)}\) is uniformly distributed over \(\mathcal{V}(\Lambda_u)\) and is independent of \(x_u^{(i)}\) [3 Lemma 1]. Since \(\Lambda_u\) is good for MSE quantization, \(\Lambda_u^{(2)}\) is good for AWGN and [18] is satisfied, we can use [4] Theorem 4 to assert that\footnote{Note that there is a slight difference here since \(\tilde{d}_u^{(i)}\) is not Gaussian. However, the arguments in [3] Theorem 5 can be used to show that \(x_u^{(i)} + \tilde{d}_u^{(i)}\) can be approximated by a Gaussian since \(\Lambda_u\) is good for MSE quantization.}\(\tilde{d}_u^{(i)}\) the probability
\begin{equation}
\Pr[y_u^{(i)} \neq x_u^{(i)} + \tilde{d}_u^{(i)}] \leq e^{-n(E_1(\delta) - o_n(1))}
\end{equation}
where \(E_1(\delta) > 0\) for all \(\delta > 0\). Similarly, we can write
\begin{equation}
y_v^{(i)} = [Q_{\Lambda_v} (x_v^{(i)} + \tilde{d}_v^{(i)}) - \tilde{d}_v^{(i)}] \mod \Lambda_v^{(2)},
\end{equation}
where \(\tilde{d}_v^{(i)}\) is independent of \(x_v^{(i)}\), and
\begin{equation}
\Pr[y_v^{(i)} \neq x_v^{(i)} + \tilde{d}_v^{(i)}] \leq e^{-n(E_2(\delta) - o_n(1))}
\end{equation}
where \(E_2(\delta) > 0\) for all \(\delta > 0\).

Recall that \(w_v^{(i)} = [y_v^{(i)}] \mod \Lambda_v^{(1)}\). We can write
\begin{align}
\tilde{y}_v &= w_v + Q_{\Lambda_v^{(1)}} (\rho_{uv} y_u^{(i)} - w_v) \\
&= w_v + Q_{\Lambda_v^{(1)}} (\rho_{uv} y_u^{(i)} - y_v^{(i)} + y_v^{(i)}) \\
&= w_v + Q_{\Lambda_v^{(1)}} (y_v^{(i)} + \rho_{uv} y_u^{(i)} - y_v^{(i)}) \\
&= y_v^{(i)} + Q_{\Lambda_v^{(1)}} (\rho_{uv} y_u^{(i)} - y_v^{(i)})
\end{align}
From (28) and (29), we know that \(y_v^{(i)} = x_v^{(i)} + \tilde{d}_v^{(i)}\) and \(y_v^{(i)} = x_v^{(i)} + \tilde{d}_v^{(i)}\) with high probability. Now,
\begin{equation}
\rho_{uv} y_u^{(i)} - x_v^{(i)} = -\sqrt{1 - \rho_{uv}^2} z_{sv}^{(i)},
\end{equation}
and again using the AWGN goodness property of \(\Lambda_v^{(1)}\) and (16), we have
\begin{equation}
\Pr \left[ Q_{\Lambda_v^{(1)}} (-\sqrt{1 - \rho_{uv}^2} z_{sv}^{(i)} + \tilde{d}_u^{(i)} - \tilde{d}_v^{(i)} \neq 0 \right] \leq e^{-n(E_3(\delta) - o_n(1))}
\end{equation}
where \(E_3(\delta) > 0\) for \(\delta > 0\). Using (28), (29) and (31), we get that
\begin{equation}
\Pr[\tilde{y}_v^{(i)} \neq y_v^{(i)}] \leq \sum_{i=1}^{3} e^{-n(E_1(\delta) - o_n(1))}
\end{equation}
which completes the proof of the proposition.\[\square\]

B. Proof of Lemma 6

We prove the result by expanding the joint entropy using the chain rule, and then use a lower bound on the entropy of a quantized Gaussian. To do this, we will expand the joint entropy in a particular order. Let \(S\) be any (totally) ordered set containing the vertices of \(T^*\) and satisfying the following properties:
\begin{itemize}
  \item \(\max_v S = r(T^*)\), i.e., \(r(T^*) \geq v\) for all \(v \in S\).
  \item \(v > u\) if the distance between \(v\) and \(r(T^*)\) is less than that between \(u\) and \(r(T^*)\).
\end{itemize}
Essentially, \(v > u\) if \(v\) is closer to \(r(T^*)\) than \(u\), and we do not care how the vertices at the same level (vertices at the same distance from \(r(T^*)\)) are ordered. Let \(D = (d_v^{(i)} : v \in V^*)\).

Then,
\begin{align}
H(y_v^{(i)} : v \in V^*) &= H(y_v^{(i)} : v \in V^*) + \sum_{v \in V^* \setminus r(T^*)} H(y_v^{(i)} : D, y_v^{(i)} : u > v) \\
&\geq H(y_v^{(i)} : r(T^*)) + \sum_{v \in V^* \setminus r(T^*)} H(y_v^{(i)} : D, x_v^{(i)} : u > v)
\end{align}
where (32) follows from the data processing inequality. We would like to remark that (33) is the only place where we use Markov tree assumption. The rest of the proof closely follows [9 Lemma 4.3], and we give an outline. The idea is to find the average mean squared error distortion in representing \(x_v^{(i)}\) by \(y_v^{(i)}\) (with or without the side information \(x_v^{(i)}_{\text{par}(v)}\)), and then argue that the rate of such a quantizer must be greater than or equal to the rate-distortion function.

Claim B.1.
\begin{equation}
\frac{1}{n} H(y_v^{(i)} : r(T^*)) \geq \frac{1}{2} \log_2 \left( 1 + \frac{1}{\sigma^2(L_{r(T^*)})} \right) - o_n(1).
\end{equation}

Making minor modifications to the proof of [9 Lemma 4.3], we can show that conditioned on \(D\), the average MSE distortion (averaged over \(D\)) between \(x_v^{(i)}_{r(T^*)}\) and \(y_v^{(i)}_{r(T^*)}\)
\begin{equation}
x_v^{(i)}_{r(T^*)} = \frac{1}{1 + \sigma^2(L_{r(T^*)})} y_v^{(i)}_{r(T^*)}
\end{equation}
is at most \(\frac{\sigma^2(L_{r(T^*)})}{\sigma^2(L_{r(T^*)})} + o_n(1)\). Since any rate-distortion code for quantizing \(y_v^{(i)}_{r(T^*)}\) must have a rate at least as much as the rate-distortion function, we can show that (again following the proof of [9 Lemma 4.3]) Claim B.1 is true.
Claim B.2.

\[
\frac{1}{n} H(\mathbf{y}_v(1)|D, \mathbf{x}_{\text{par}(v)}^{(i)}) \geq \frac{1}{2} \log_2 \left( 1 + \frac{1 - \rho_{\text{par}(v)}^2}{\sigma^2(\Lambda_v)} \right) - o_n(1)
\]

(35)

The proof of the above claim also follows the same technique. We can show that conditioned on \( D \) and \( \mathbf{x}_{\text{par}(v)}^{(i)} \), the average MSE distortion between \( \sqrt{1 - \rho_{uv}^2} \mathbf{z}_{\text{vu}}^{(i)} \) and

\[
\mathbf{z}_{\text{vu}}^{(i)} = (1 - \rho_{uv}^2)(\mathbf{y}_v^{(i)} - \rho_{uv} \mathbf{x}_{\text{par}(v)}^{(i)}) \mod \Lambda_v
\]

is \((1 - \rho_{uv}^2)^2(\Lambda_v) + o_n(1)\). Arguing as before, the claim follows.

Finally, using (34) and (35) in (33) completes the proof of Lemma 6.

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