The Vacuum Structure and Spectrum of
$N = 2$ Supersymmetric $SU(n)$ Gauge Theory

Philip C. Argyres and Alon E. Faraggi
School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540
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Abstract

We present an exact description of the metric on the moduli space of vacua and the spectrum of massive states for four dimensional $N = 2$ supersymmetric $SU(n)$ gauge theories. The moduli space of quantum vacua is identified with the moduli space of a special set of genus $n - 1$ hyperelliptic Riemann surfaces.

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Recently Seiberg and Witten [1] obtained exact expressions for the metric on moduli space and dyon spectrum of \( N = 2 \) supersymmetric \( SU(2) \) gauge theory using a version of Olive-Montonen duality [2]. In this Letter we use this approach to obtain similar information for the \( N = 2 \) supersymmetric \( SU(n) \) gauge theory with no \( N = 2 \) matter.

The \( N = 2 \) Yang-Mills theory involves a single chiral \( N = 2 \) superfield which, in terms of \( N = 1 \) superfields, decomposes into a vector multiplet \( W_\alpha \) and a chiral multiplet \( \Phi \). In components, \( W_\alpha \) includes the gauge field strength \( F_{\mu\nu} \) as well as the Weyl gaugino, while \( \Phi \) includes a Weyl fermion and a complex scalar \( \phi \). All these fields transform in the adjoint representation of \( SU(n) \).

The potential for the complex scalar is \( \text{Tr}[\phi, \phi^\dagger] \), implying (at least classically) an \( n - 1 \)-complex dimensional moduli space of flat directions. Any vev for \( \phi \) can be rotated by a gauge transformation to lie in the Cartan subalgebra of \( SU(n) \). This vev generically breaks \( SU(n) \rightarrow U(1)^{n-1} \). Denote by \( \Phi_i \) and \( W_i \) the components of the chiral superfield \( \Phi \) and the vector superfield \( W \) in the Cartan subalgebra with respect to the same basis (so that \( \Phi_i \) and \( W_i \) are \( N = 1 \) components of the same \( U(1) \ N = 2 \) gauge multiplet). The running of the couplings of the low-energy \( U(1) \)'s induced by the symmetry-breaking scales leads to a low-energy effective action derived from a single holomorphic function \( \mathcal{F}(\Phi_k) \) [3]:

\[
S_{\text{eff}} = \frac{1}{2\pi} \text{Im} \left[ \int d^2\theta d^2\bar{\theta} \Phi_D^i \bar{\Phi}_i + \frac{1}{2} \int d^2\theta \tau^{ij} W_i W_j \right], \tag{1}
\]

where, denoting \( \partial^i = \partial / \partial \Phi_i \),

\[
\Phi_D^i \equiv \partial^i \mathcal{F}, \quad \tau^{ij} \equiv \partial^i \bar{\partial}^j \mathcal{F}. \tag{2}
\]

The real and imaginary parts of the lowest component of \( \tau^{ij} \) are the low energy effective theta angles and coupling constants of the theory respectively. They are functions of the vevs of the \( \phi_i \) fields.

A change in basis of the \( U(1) \) fields corresponding to the transformation \( W_i \rightarrow q_i^j W_j \) by an arbitrary invertible matrix \( q \), could be absorbed in a redefinition \( \tau^{ij} \rightarrow (q^{-1})^k_i(q^{-1})^j_k \) of the effective couplings. This ambiguity can be partially fixed by demanding that the \( W_i \) are normalized so that the charges of fields in the fundamental of \( SU(n) \) form a unit cubic lattice so that the allowed set of electric charges \( n_i^k \) are all the integers. Then the transformations \( q \) are restricted to be integer matrices with determinants \( \pm 1 \). Denoting the magnetic charges of any monopoles or dyons by \( 2\pi n_{m,i} \), the Dirac quantization condition requires the \( n_{m,i} \) to lie in the dual lattice to that of the electric charges, implying that the \( n_{m,i} \) are also integers.

The low-energy effective action (1) is left invariant by an \( Sp(2n-2, \mathbb{Z}) \) group of duality transformations. The action of the duality group on the fields is realized as follows [4]. Define the \((2n-2)\)-component vectors \( \hat{\Phi} = (\Phi_D^i, \Phi_i) \) and \( \hat{W} = (W_D^i, W_i) \), where \( W_D^i \) are the dual \( U(1) \) field strengths. Then a \((2n-2) \times (2n-2)\) matrix \( M \in Sp(2n-2, \mathbb{Z}) \) acts as \( \Phi \rightarrow M \cdot \Phi, \ W \rightarrow M \cdot W \), and \( \tau \rightarrow (A \cdot \tau + B)(C \cdot \tau + D)^{-1} \) where \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) and \( \tau \) denotes the matrix \( \tau^{ij} \) of effective couplings. With this action one can show [4] that any \( Sp(2n-2, \mathbb{Z}) \) duality transformation can be generated by a change of basis of the \( U(1) \) generators, the symmetry under discrete shifts in the theta angles \( \tau^{ij} \rightarrow \tau^{ij} + 1 \), and the \( \tau^{ij} \rightarrow - (\tau^{ij})^{-1} \) electric-magnetic duality transformation.
Due to the structure of the $N = 2$ supersymmetry algebra $\mathfrak{B}$, a dyon of magnetic and electric charges $n = (n_{m,i}, n'_e)$ has a mass saturating the Bogomol’nyi bound $[\mathfrak{B}]$

$$M = \sqrt{2} \left| \mathbf{a} \cdot \mathbf{n} \right|,$$

where $\mathbf{a} = (a'_D, a_i)$ is the vector of vevs of the scalar component of the chiral superfield and its dual: $a_i = \langle \phi_i \rangle$ and $a'_D = \langle \phi'^*_i \rangle$. This formula is invariant under the $Sp(2n - 2, \mathbb{Z})$ duality transformations since the electric and magnetic charges transform oppositely to the scalar vevs: $n \rightarrow M^{-1} \cdot n$ if $a \rightarrow M \cdot a$.

As discussed in Ref. $[\mathbb{H}]$, the combination of the requirements of analyticity of the superpotential $F$ and positivity of the Kähler metric $\text{Im} \, \tau$, together with the form of the superpotential at weak coupling, imply that there must be singularities in the moduli space around which the theory has non-trivial monodromies lying in $Sp(2n - 2, \mathbb{Z})$. Since there is a region of the $SU(n)$ moduli space where $SU(n)$ is broken at a large scale down to $SU(n - 1)$, if follows that at sufficiently weak coupling a copy of $SU(n - 1)$ moduli space will be embedded in the $SU(n)$ moduli space. We will essentially use these facts to find an exact description of the $SU(n)$ moduli space by induction in $n$. First, though, we assemble some facts about the classical $SU(n)$ moduli space.

**Classical Moduli Space.** The moduli space of the $SU(n)$ theory is most conveniently described in a basis associated with the $U(n)$ Lie algebra, where the tracelessness constraint is not imposed. For this reason we adopt the convention that upper-case indices $I, J, K, \ldots$ run from 1 to $n$ and lower-case indices $i, j, k, \ldots$ run from 1 to $n - 1$. Use a basis $\{H^I, E^+_{IJ} (I > J)\}$ for the generators of the $U(n)$ Lie algebra where the $n \times n$ matrices $[H^I]_{AB} = \delta^I_A \delta_J^B$ span the Cartan subalgebra. Then the $SU(n)$ vector superfield $W = W_I H^I + W^\pm_{IJ} E^\pm_{IJ}$ will satisfy the tracelessness condition

$$\sum_I W_I = 0. \tag{4}$$

If we everywhere substitute for $W_n$ in terms of the $W_i$’s using the tracelessness constraint, we can choose the $W_i$ as a basis of the Cartan subalgebra of $SU(n)$. This basis respects the requirement imposed in the last section that the charges of fields in the fundamental of $SU(n)$ generate a unit cubic lattice.

The vev of the complex scalar $\phi$ can always be rotated by a gauge transformation to lie in the Cartan subalgebra of $SU(n)$: $\langle \phi \rangle = a_I H^I$, where the $a_I$ must also satisfy the tracelessness constraint

$$\sum_I a_I = 0. \tag{5}$$

If we denote the space of independent complex $a_I$’s by $\mathcal{T}_n = \{a_I| \sum_I a_I = 0\} \simeq \mathbb{C}^{n-1}$, then the classical moduli space is $\mathcal{T}_n$ up to gauge equivalences. The only $SU(n)$ elements which act non-trivially on the Cartan subalgebra are the elements of the Weyl group, isomorphic to the permutation group $S_n$, which acts by permuting the $a_I$’s. Thus, the classical moduli space of the $SU(n)$ theory is $\mathcal{M}_n = \mathcal{T}_n / S_n$.

The Higgs mechanism gives the $W^\pm_{IJ}$ bosons masses proportional to $|a_I - a_J|$. The Weyl group $S_n$ does not act freely on $\mathcal{T}_n$: a submanifold of partial symmetry-breaking to
$SU(m)$ is fixed by $S_m \subset S_n$, since $m$ of the $a_I$’s are equal there. Classically $\mathcal{M}_n$ has singularities along these submanifolds since extra $W_{IJ}^\pm$ bosons become massless there. Since the theory is strongly coupled in the vicinity of these submanifolds, one expects that quantum mechanically the classical moduli space given above is modified in these regions.

A global $U(1)_R$ symmetry of the $SU(n)$ theory is broken down to $Z_{4n}$ by anomalies. Since the scalar field $\Phi$ has charge 2 under this symmetry, only a $Z_{2n}$ bosons become massless there. Since the theory is strongly coupled in the vicinity of these submanifolds, one expects that quantum mechanically the classical moduli space given above is modified in these regions.

A basis of gauge-invariant coordinates covering $\mathcal{M}_n$ at weak coupling are given by $u_\alpha = \langle \text{Tr}(\phi^\alpha) \rangle = \sum_I a_I^\alpha$, for $\alpha = 2, \ldots, n$. The $Z_{2n}$ symmetry acts on these coordinates by $u_\alpha \to e^{i\pi \alpha/n} u_\alpha$. A more convenient set of gauge-invariant coordinates is given classically by the elementary symmetric polynomials in the $a_I$’s

$$s_\alpha \equiv (-)^\alpha \sum_{I_1 < \cdots < I_\alpha} a_{I_1} \cdots a_{I_\alpha}, \quad \alpha = 1, \ldots, n. \tag{6}$$

These symmetric coordinates can be expressed as polynomials in terms of the $u_\alpha$’s (thus defining them quantum mechanically). These polynomials are generated by Newton’s formula

$$rs_r + \sum_{\alpha=0}^r s_{r-\alpha} u_\alpha = 0, \quad r = 1, 2, 3, \ldots \tag{7}$$

where $s_0 \equiv 1$, $u_0 \equiv 0$, and $s_1 = u_1 = 0$ by the tracelessness constraint.

The $SU(n)$ Curve. The effective couplings $\tau$ transform under $Sp(2n-2, \mathbb{Z})$ and $\text{Im}\tau$ must be positive definite for the theory to be unitary. The period matrix of a genus $n-1$ Riemann surface has precisely these properties, so it is natural to guess that the moduli space of the $SU(n)$ theory be identified with the moduli space of the Riemann surface. Indeed, the solution of the $SU(2)$ case is of just this form [1]. However, for $n > 2$, the dimension of the moduli space of Riemann surfaces of genus $n-1$ is too large, so the $SU(n)$ theory must correspond only to special Riemann surfaces. A relatively simple set of Riemann surfaces are the hyperelliptic ones [7], described by the complex curve

$$y^2 = \prod_{\ell=1}^{2n} (x - e_\ell), \tag{8}$$

which is the double-sheeted cover of the Riemann sphere branched at $2n$ points $e_\ell$. The $SU(n)$ curve should also have a $Z_{2n}$ symmetry, reflecting the $U(1)_R$ symmetry broken by instantons in the $SU(n)$ theory. This symmetry fits naturally with the hyperelliptic surfaces if we assign $\mathcal{R}$-charge 1 to $x$ and $n$ to $y$.

We now assume, following [3], that the coefficients of the polynomial in $x$ defining the $SU(n)$ curve are themselves polynomials in the gauge-invariant coordinates $s_\alpha$ (or $u_\alpha$) and $\Lambda_n^{2n}$, where $\Lambda_n$ is the renormalization scale of the $SU(n)$ theory. The power of $\Lambda_n^{2n}$ ensures that it has the quantum numbers of a one-instanton amplitude.
In the weak coupling limit there are non-trivial monodromies around the regions of moduli space where extra gauge symmetries are restored. These regions lie around the submanifolds where a pair or more of the $a_I$ take the same values. So, as $\Lambda_n \to 0$, the $SU(n)$ curve should be singular along these submanifolds. A curve is singular whenever a pairs or more of its branch points $\epsilon_I$ coincide. A polynomial in $x$ which has the required property is $F(x) = \prod_{I=1}^{n} (x - a_I)$. As we will shortly see, there is also a monodromy of the $SU(n)$ theory at weak coupling which does not correspond to any classical singularity of the moduli space. Thus, in the weak coupling limit the $SU(n)$ curve should be singular for all values of the $a_I$’s. This can be achieved by simply squaring the polynomial $F(x)$, so that all its zeros are doubled. Also, it then has the right degree in $x$ to describe a hyperelliptic curve as in (3). There is then only one way to add in the instanton contributions (terms dependent on $\Lambda_n$) consistent with our assignment of the $R$-charges: $y^2 = F^2(x) - \Lambda_n^{2n}$. The coefficient of $\Lambda_n^{2n}$ is arbitrary as it reflects a choice of renormalization group prescription.

It is now easy to extend this curve to strong coupling in $SU(n)$. The coefficients of the polynomial $F(x)$ are precisely the elementary symmetric functions $s_n$ of the $a_I$’s (3), which are defined away from weak coupling by Eq. (7). We make the assumption that the $s_n$ remain good global coordinates on the $SU(n)$ moduli space even at strong coupling. Then the proposed $SU(n)$ curve is

$$y^2 = \left( \sum_{\alpha=0}^{n} s_{\alpha} x^{n-\alpha} \right)^2 - \Lambda_n^{2n}. \tag{9}$$

The remainder of this Letter describes various consistency checks on this proposed curve. For brevity’s sake, we confine ourselves to checking properties that depend only on the conjugacy class of the monodromies in $Sp(2n - 2, \mathbb{Z})$. A more detailed exposition involving explicit choices of bases will be given elsewhere (2).

Weak Coupling Monodromies. The first check we perform is to show that (3) has all the right monodromies at weak coupling. We constructed it only by demanding that it have singularities at the right places, so computing the monodromies around those singularities is an independent check.

Note that in the limit where $SU(n)$ is strongly broken down to $SU(n-1)$, e.g. $a_i \sim a$ and $a_n \sim (1 - n)a$ where $|a| >> \Lambda_n$, then shifting $x$ to $x + a$ in (3) will send two of the branch points to $\sim -na$ in the $x$ plane while leaving the rest clustered around the origin. From the usual renormalization group matching $\Lambda_{n}^{2n} \sim a^2 \Lambda_{n-1}^{2(n-1)}$, so taking the limit $a \to \infty$ while leaving $\Lambda_{n-1}$ fixed sends the two branch points at $-na$ to infinity, and rescaling $y$ by $(x + na)^{-1}$, we recover the curve (3) again, but now for $SU(n-1)$ instead of $SU(n)$. Thus the $SU(n)$ curve at weak coupling automatically contains all $SU(n-1)$ monodromies. This fact allows us to proceed by induction in $n$.

First consider the $SU(2)$ curve $y^2 = (x^2 - \frac{1}{2} u)^2 - \Lambda^4$ (where we have used $-2s_2 = u_2 \equiv u$). This can easily be shown to be equivalent to the $SU(2)$ curve found in [8], $\tilde{y}^2 = \tilde{x}(\tilde{x}^2 + 2u\tilde{x} + \Lambda^4)$, by a fractional linear transformation on the $\tilde{x}$ variable. The point is simply that the automorphisms of the Riemann sphere allow us to fix three of the branch points arbitrarily by an $SL(2, \mathbb{C})$ transformation. The $SU(2)$ curve of Ref. [8] has branch points fixed at 0 and infinity, whereas the curve (4) does not.

Next consider the $SU(3)$ curve. We know that along an $SU(2)$ direction at weak coupling it degenerates to the $SU(2)$ curve, and so gives the correct monodromies. However, as
mentioned above, the $SU(3)$ curve has another singularity at weak coupling corresponding to the limit where all the $a_I$'s scale together by some large factor (or, equivalently, where the $a_I$'s are held fixed at some generic values and $\Lambda_n \to 0$). If the special $SU(3)$ monodromy around this singularity agrees with the answer calculated from perturbation theory, then all the weak coupling monodromies of $SU(3)$ will have been checked, and the induction can proceed to $SU(4)$, etc. So, in general, we will need to compute just one special monodromy for each $SU(n)$ curve.

We are free to pick a convenient curve along which to measure this monodromy. Since the special monodromies are not associated with any coincidences of the $a_I$'s, let us look in a direction in moduli space along which the $a_I$'s are maximally separated: $a_I = \omega^I a$ where $\omega \equiv e^{2\pi i/n}$. This is the direction along which classically all the $s_n$'s except $s_n$ vanish identically. The monodromy in question is obtained upon traversing a large circle at weak coupling in the $s_n$ complex plane. In this plane the $SU(n)$ curve \( \text{(9)} \) factorizes for $|s_n| >> \Lambda_n^n$ as

$$y^2 = \prod_{j=1}^{n} \left( x - \omega^j s_n^{1/n} [1 + s_n^{-1} \Lambda_n^n] \right) \left( x - \omega^j s_n^{1/n} [1 - s_n^{-1} \Lambda_n^n] \right). \tag{10}$$

The branch points are arranged in $n$ pairs with a pair at each $n$th root of unity times $s_n^{1/n}$. As $s_n \to e^{2\pi i s_n}$, these pairs are rotated into one another in a counter-clockwise sense, and each pair also revolves once about its common center in a clockwise sense.

Choose cuts and a standard basis for the independent cycles on the $SU(n)$ surface as shown (for $SU(3)$) in Fig. 1. Thus, $\gamma_1$ and $\gamma_2$ are independent non-intersecting cycles, similarly for $\gamma^i_D$, and their intersection form is $(\gamma^i_D, \gamma_j) = \delta^i_j$. Note that $\gamma_3$ is not independent of the $\gamma_i$'s: a simple contour deformation shows that $\sum_I \gamma_I = 0$. The generalization to the $SU(n)$ curve should be clear. As $s_n \to e^{2\pi i s_n}$ the $\gamma_I$ are simply dragged around the circle so that $\gamma_i \to \gamma_{i+1} \equiv P^i_{i+1} \gamma_j$, where $P^i_{i+1} = \delta^i_{i+1} - \delta^i_{i+1}$ is an $(n-1) \times (n-1)$ matrix representation of the $\pi = (1 \ldots n)$ permutation.

The monodromies of the $\gamma^i_D$ cycles can be determined as follows. From the monodromies
of the $\gamma_i$'s and the defining properties of symplectic matrices, it follows that the monodromy $\gamma \to M \cdot \gamma$ in $Sp(2n-2, \mathbb{Z})$ of $\gamma = (\gamma_D, \gamma_i)$ can be written in the block form

$$M = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{P}^{-1} & 0 \\ 0 & \mathbf{P} \end{pmatrix}$$

(11)

where $\mathbf{P}$ is the permutation matrix found above, and $N$ is some symmetric matrix which we wish to determine. Now, if $N \mathbf{P} = \mathbf{P}^{-1} N$, so that the two matrices in Eq. (11) commute, then $M^n = \begin{pmatrix} 1 & nN \\ 0 & 1 \end{pmatrix}$ since $\mathbf{P}^n = 1$. But $M^n$ is easy to compute: as $s_n \to e^{2\pi i n} s_n$, the $\gamma_i$ cycles are simply dragged back to themselves and similarly for the $\gamma_D$ cycles except that their ends get wound $n$ times (in a clockwise sense) around each cut that they pass through. As illustrated in Fig. 2, each such winding can be deformed to give two of the associated $\gamma_i$'s. Keeping track of the signs, one finds $\gamma^i_D \to \gamma^i_D - 2n \gamma_i + 2n \gamma_n \equiv \delta^i_j \gamma^j_D + n N^{ij} \gamma_j$, where

$$N^{ij} = -2(\delta^{ij} + 1).$$

(12)

Since (12) satisfies $N \mathbf{P} = \mathbf{P}^{-1} N$, it follows that it is, in fact, the matrix $N$ of Eq. (11).

![FIG. 2. Contour unwinding after a long day.](image)

The Special Monodromies in Perturbation Theory. Since pure $N = 2$ $SU(n)$ gauge theory is asymptotically free, there is a weak coupling region where perturbation theory is reliable when $SU(n)$ is completely broken at a high enough scale so that all the $|a_I - a_J| \gg \Lambda_n$. We calculate in perturbation theory the leading behavior of the couplings of the low energy effective action for the massless $U(1)^{n-1} \subset SU(n)$ gauge bosons. Denote the effective coupling of the $W_I$ with $W_J$ fields by $\tilde{\tau}^{IJ}$, so the effective $N = 1$ gauge action is $S_{\text{eff}} \sim \int \tilde{\tau}^{IJ} W_I W_J$. The one-loop result for the running of the couplings is $\tilde{\tau}^{IJ} = (i/2\pi)[\delta^{IJ} \sum_K \ln(a_{IK} a_{KI}) - \ln(a_{IJ} a_{JI})]$ where $a_{IJ} \equiv a_I - a_J$. The tracelessness constraint (4) implies $\tilde{\tau}^{ij} = \tilde{\tau}^{ij} - \tilde{\tau}^{in} - \tilde{\tau}^{nj} + \tilde{\tau}^{nn}$, or

$$\tilde{\tau}^{ij} = \frac{i}{2\pi} \left\{ \delta^{ij} \sum_k \ln(a_{ik} a_{ki}) + \delta^{ij} \ln(a_{in} a_{ni}) - \ln(a_{ij} a_{ji}) + \sum_k (\delta^{ik} + \delta^{jk} + 1) \ln(a_{kn} a_{nk}) \right\}.$$

(13)

From Eq. (2) it follows that

$$a^i_D = \tau^{ij} a_j.$$

(14)

A possible constant term in the $a^i_D$ can be shown [1] to be zero by matching to the full $SU(n)$ theory.

In order to compute the monodromies in the $a^i_D$ along a closed path in $\mathcal{M}_n$ at weak coupling, we must first lift the path to a path in $\mathcal{T}_n$. Since $\mathcal{M}_n = \mathcal{T}_n/S_n$ is formed by identifying points in $\mathcal{T}_n$ which differ by a permutation of their coordinates $a_I$, in general there will be a non-trivial monodromy along any path in $\mathcal{T}_n$ which connects a point with
its image under the action of a non-trivial permutation \( \pi \in S_n \). With one exception, the different possible choices of permutation \( \pi \) reflect the pattern of symmetry-breaking of \( SU(n) \) at high energies. For example, the monodromy associated to \( \pi = (23 \ldots n) \) winds around a region of moduli space where \( SU(n) \rightarrow SU(n-1) \) at high energies. The exception is the monodromy associated to the conjugacy class of cyclic permutations of all \( n \) elements, \( \pi = (1 \ldots n) \), which does not correspond to any special symmetry breaking pattern. This is the monodromy special to \( SU(n) \).

As in the computation from the curve, we choose the path realizing the special monodromy to be \( a_i(t) = \omega^{j+i}a \) for \( 0 \leq t \leq 1 \), where \( |a| \) is some large scale and \( \omega = e^{2\pi i/n} \). This path precisely traverses a large circle in the \( s_n \) complex plane. The monodromy of the \( a_i \)'s along this path is clearly \( a_i \rightarrow P_j^ia_j \), where \( P \) is the same permutation found above from the curve. The logarithms in Eq. (13) contribute a shift to the \( \tau^{ij} \) monodromy, \( \tau^{ij} \rightarrow \tau^{ij} + N^{ij} \), where \( N \) is easily computed to be equal to the \( N^{ij} \) of Eq. (12). The \( a_i \)'s then transform as \( a_i \rightarrow \tau^{ij}P_j^ka_k + N^{ij}P_j^ka_k \) from (14). Now, either from the defining properties of symplectic matrices, or from the fact that the effective action is completely symmetric among all the low energy \( U(1) \)'s in the \( s_n \) plane (since \( SU(n) \rightarrow U(1)^{n-1} \) at a single scale), it follows that \( \tau P = P^{-1}\tau \), and so the monodromy of the scalar vevs \( a = (a_D, a_i) \) indeed agrees with the monodromy (11) computed from the \( SU(n) \) curve. This completes our check that the monodromies of the curve (11) agree with all the monodromies of the \( SU(n) \) theory at weak coupling.

**Metric on Moduli Space and Dyon Spectrum.** The identification of the metric and spectrum—that is to say, \( a_i \) and \( a_D \), as functions of the moduli \( s_a \)—closely parallels the discussion of Ref. [1]. Choosing a basis of cycles \( (\gamma_D^i, \gamma_i) \) of the \( SU(n) \) curve with the canonical intersection form \( (\gamma_D^i, \gamma_j) = \delta^i_j \), we identify \( a_i \) and \( a_D^i \) as sections of a flat \( Sp(2n-2, \mathbb{Z}) \) bundle over moduli space given by

\[
a_i = \oint_{\gamma_i} \lambda, \quad a_D^i = \oint_{\gamma_D^i} \lambda, \quad (15)
\]

where \( \lambda \) is some meromorphic one form on the curve with no residues. There is a \( 2n-2 \) dimensional space of such forms spanned by the \( n-1 \) holomorphic one forms \( (x^{i-1}/y)dx \), and the \( n-1 \) meromorphic one forms \( x^{n-1} \). The one-form \( \lambda \) defining our solution can be written as a linear combination of these basis one-forms (with coefficients that can depend on the \( s_a \) and \( \Lambda_n \) up to a possible total derivative.

Since the period matrix of the Riemann surface defined by the \( SU(n) \) curve has a positive definite imaginary part, transforms in the same way as \( \tau^{ij} \) under \( Sp(2n-2, \mathbb{Z}) \), and has the same monodromies as \( \tau^{ij} \) does, it follows that they should be identified. Now, the period matrix, or \( \tau^{ij} \), is defined by \( \sum_j \tau^{ij}(f_{ij} \lambda_k) = f_{ij}^* \lambda_k \). Since also \( \tau^{ij}(\partial a_j/\partial s_a) = (\partial a_D^i/\partial s_a) \), by (2), it is natural to guess that

\[
\frac{\partial a_i}{\partial s_a} = \oint_{\gamma_i} \lambda, \quad \frac{\partial a_D^i}{\partial s_a} = \oint_{\gamma_D^i} \lambda, \quad (16)
\]

where the \( \lambda_a \) are some as yet undetermined basis of holomorphic one forms. Eqs. (15) and (16) imply a set of differential equations for \( \lambda \). In the \( SU(2) \) case they can be easily solved
to find \( \lambda \propto 2x^2(dx)/y \), since \( d\lambda/ds_2 = -(dx)/y + d(x/y) \). The generalization to \( SU(n) \) is\(^\dagger\)

\[
\lambda \propto \left( \sum_{\alpha=0}^{n} (n - \alpha)s_{\alpha}x^{n-\alpha} \right) \frac{dx}{y}, \tag{17}
\]

since \( \partial \lambda/\partial s_{\alpha} = -x^{n-\alpha}(dx)/y + d(x^{n+1-\alpha}/y) \). The overall constant normalization of \( \lambda \) can be determined only by making a choice of basis cycles and matching to perturbation theory.

**Strong Coupling Monodromies.** The singularities of the curve (9) occur along submanifolds of the moduli space where a pair or more of the branch points coincide. As we have argued above, these submanifolds all lie at strong coupling. However, physically, singularities in the moduli space are expected to occur where a dyon in the spectrum becomes massless. The renormalization group flow of the low-energy \( U(1) \)'s to weak coupling at small scales is cut off at the mass of the lightest charged particle in the spectrum. But at those points in moduli space where a dyon becomes massless, the \( U(1) \)'s that couple to them flow to zero coupling, and are well-described by perturbation theory. Thus, there will be a dual description of the physics near the singular submanifolds which is weakly coupled, and so can be used to check these limits of the curve (9) as well.

Consider the case where \( m \) dyons become massless at a point \( P \) in \( \mathcal{M}_n \). The low energy theory is by definition local, so all \( m \) massless dyons must be mutually local. This implies their charge vectors \( \mathbf{n}^a \) are symplectically orthogonal: \( \mathbf{n}^a \cdot \mathbf{I} \cdot \mathbf{n}^b = 0 \) for all \( a, b = 1, \ldots, m \), where \( \mathbf{I} \) is the symplectic form \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). This can only be satisfied for \( m \leq n - 1 \) linearly independent vectors since there exists a symplectic transformation to dual fields where each dyon is described as an electron charged with respect to only one dual low energy \( U(1) \). In this dual description the physics near the point \( P \) is weakly coupled, since \( m \) independent electrons are becoming massless there.

The above symplectic transformation also specifies the dual scalar vevs \( \tilde{a}_a \) which are good coordinates on moduli space near \( P \) since, by (3), as \( P \) is approached, \( \tilde{a}_a \to 0 \). This means that locally in moduli space, a single dyon, say the one with dual electric charge \( \tilde{n}_e^1 \), becomes massless along a hypersurface of complex co-dimension 1, given by the solution to the (complex) equation \( \tilde{a}_1 = 0 \). Two dyons become massless at the intersection of two such surfaces, which is locally described as a submanifold of \( \mathcal{M}_n \) of complex codimension 2, and so forth. The maximum number \( n - 1 \) of dyons becoming massless at once will generically occur at an isolated point in moduli space. Note that if \( m < n - 1 \), then \( n - m - 1 \) of the \( U(1) \)'s may still be strongly coupled, and cannot be reliably calculated using perturbation theory.

Along these hypersurfaces the effective action is singular, and so can lead to nontrivial monodromies for paths looping around them. The one-loop effective couplings near \( P \) are \( \tilde{\tau}^{ij} = (-i/2\pi)\delta^{ij}(\tilde{n}_e^i)^2 \ln(\tilde{n}_e^i \tilde{a}_i) \), where \( \tilde{n}_e^i \) denotes the charge of the \( i \)th electron. It is straightforward to compute the monodromy \( \mathbf{M}_i \) of a path \( \gamma_i \) winding around the \( \tilde{a}_i = 0 \) hypersurface to be

\[
\mathbf{M}_i = \begin{pmatrix} 1 & (\tilde{n}_e^i)^2 \mathbf{e}_{ii} \\ 0 & 1 \end{pmatrix}, \tag{18}
\]

\(^\dagger\)Special thanks to R. Plesser who derived this formula.
where $e_{ii}$ is an $(n - 1) \times (n - 1)$ matrix of zeros except for a 1 in the $i$th position along the diagonal. A strong coupling test of the curve $\Pi$ is that its monodromies around intersecting singular submanifolds all be conjugate to the above $M_i$ monodromies corresponding to mutually local dyons.

This test for the $SU(2)$ curve is trivially satisfied since the only singular submanifolds are the two isolated points found in Ref. [1]. They each are conjugate to the monodromy $([8]$ with $\tilde{n}_e = 1$, corresponding to the conjugacy class associated with the classically stable spectrum of $SU(2)$ dyons.

For the $SU(3)$ curve we first need to identify the singular submanifolds. They are given by the vanishing of the discriminant $\Delta$ of the polynomial $(x^3 + s_2x + s_3)^2 - \Lambda_3^2$ defining the $SU(3)$ curve. It is convenient to rescale our coordinates on moduli space to $\sigma_3 = \Lambda_3^{-3}s_3$ and $\sigma_2 = 2^{2/3}3^{-1}\Lambda_3^{-2}s_2$. Then the discriminant becomes $\Delta(\sigma_2, \sigma_3) = (\sigma_2^3 + \sigma_3^3)^2 + 2(\sigma_3^3 - \sigma_2^3) + 1$. Possible intersection points of the singular submanifold $\Delta = 0$ are at its singular points where $\partial\Delta/\partial\sigma_i = 0$. There are five such points: the $Z_3$-symmetric triplet of points $\sigma_3^2 = -1$ and $\sigma_3 = 0$, and the $Z_2$ doublet $\sigma_2 = 0$ and $\sigma_3^2 = 1$. The triplet corresponds to a true intersection point since there $|\partial^2\Delta/\partial\sigma_i\partial\sigma_j| \neq 0$. The $Z_2$ points, however, are not intersection points: in terms of local coordinates $\delta\sigma_i$ vanishing at one of the $Z_2$ points, the singular manifold has the equation $(\delta\sigma_2)^3 = (\delta\sigma_3)^2$. This describes a branch point of a single submanifold, instead of the intersection point of two submanifolds. Thus, at this point only one dyon is massless.

We compute the monodromies around the intersecting singular submanifolds at a $Z_3$ point by first expanding the $SU(3)$ curve in local coordinates around one such point: $s_2 \rightarrow -2^{-2/3}3\Lambda_3^2 + s_2$ and $s_3 \rightarrow s_3$, with the new $|s_i| << \Lambda_3^3$. Then the curve approximately factorizes as

$$y^2 \sim (x - 1 + \sqrt{s_2 + s_3})(x - 1 - \sqrt{s_2 + s_3})(x + 1 + \sqrt{s_2 - s_3})(x + 1 - \sqrt{s_2 - s_3})(x^2 - 4)$$

where we have rescaled $\Lambda_3 \rightarrow 2^{1/3}$. Choose a basis of $\gamma_i$ cycles to encircle the pairs of branch points near $-1$ and $+1$, and the $\gamma'_j$'s in the canonical way. Paths which encircle the intersecting singular manifolds are simply a circle in the $s_2 + s_3$ complex plane keeping $s_2 - s_3$ fixed, and vice versa. The resulting monodromies are then easily found to be precisely of the form $([8]$ with $\tilde{n}_e^1 = \tilde{n}_e^2 = 1$. This confirms that there are indeed two different mutually local dyons becoming massless along the two intersecting submanifolds at the $Z_3$ points. Furthermore, their charges are consistent with the semi-classically stable dyon charges in the $SU(2)$ limit. This suggests that, as in the $SU(2)$ case, the spectrum of stable dyon charges remains the semi-classical one all the way down to these strong-coupling singularities.

As a final check of the $SU(3)$ curve, we note that the $Z_3$ intersection points imply the known $N = 1$ $SU(3)$ vacuum structure. Indeed, following the arguments of Ref. [1], add to the microscopic $N = 2$ theory a coupling $\mu$ to the composite $N = 1$ superfield corresponding to $s_2$. This is a mass term for the $N = 1$ chiral superfield $\Phi$. Going to the dual (weakly coupled) description of the physics near a point in the moduli space of the $SU(n)$ theory where $n - 1$ dyons are massless, and using the non-perturbative nonrenormalization theorem of [1], the non-perturbative form of the effective superpotential is found to be $W = \sum_i \tilde{a}_i(s_{s_i})m_i\tilde{m}_i + \mu s_2$, where $m_i$ and $\tilde{m}_i$ are the lowest components of the dyon chiral superfields. Minimizing the superpotential subject to the D-term constraints $|m_i| = |\tilde{m}_i|$ for all $i$ shows that for non-zero $\mu$ the $N = 2$ flat directions are lifted and only the point $\tilde{a}_i = 0$.
where all $n-1$ dyons are massless remains an $N=1$ vacuum. The three $\mathbb{Z}_3$ singularity intersection points of the $SU(3)$ curve found above are just such points, and happily they correspond to the three $N=1$ $SU(3)$ vacua related by a spontaneously broken $\mathbb{Z}_3$.

Computing the discriminant and finding all the strong coupling singularities for the $SU(n)$ curve becomes increasingly difficult for higher $n$.

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argyres@guinness.ias.edu.
faraggi@sns.ias.edu.

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