THE POSITIVE CONTRACTIVE PART OF A NONCOMMUTATIVE
\(L^p\)-SPACE IS A COMPLETE JORDAN INVARIANT

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Abstract. Let \(1 \leq p \leq +\infty\). We show that the positive part of the closed unit ball of a non-commutative \(L^p\)-space, as a metric space, is a complete Jordan *-invariant for the underlying von Neumann algebra.

1. Introduction

Given a von Neumann algebra \(M\), celebrated results of R. V. Kadison showed that several partial structures of \(M\) can recover the von Neumann algebra up to Jordan *-isomorphisms. In particular, each of the following is a complete Jordan *-invariant of \(M\): the Banach space structure of the self-adjoint part \(M_{sa}\) of \(M\) ([3, Theorem 2]), the ordered vector space structure of \(M_{sa}\) ([3, Corollary 5]) and the topological convex set structure of the normal state space of \(M\) ([6, Theorem 4.5]).

Let \(p \in [1, +\infty]\), and let \(L^p(M)\) be the non-commutative \(L^p\)-space associated to \(M\) with the canonical cone \(L^p(M)_+\). If \(M\) is semi-finite, P.-K. Tam showed in [15] that the ordered Banach space \((L^p(M)_{sa}, L^p(M)_+)\) characterises \(M\) up to Jordan *-isomorphisms. In the case when \(M\) is \(\sigma\)-finite (but not necessarily semi-finite) and \(p = 2\), the corresponding result follows from a result of A. Connes (namely, [3 Théorème 3.3]). On the other hand, extending results of B. Russo ([12]) and F. J. Yeadon ([16]), D. Sherman showed in [13] that the Banach space \(L^p(M)\) is also a complete Jordan *-invariant for a general von Neumann algebra \(M\) when \(p \neq 2\).

Along this line, we show in this article that the underlying metric space structure of the positive contractive part
\[L^p(M)_+^1 := L^p(M)_+ \cap L^p(M)^1\]
(1 \(\leq p \leq +\infty\)) of \(L^p(M)\) is also a complete Jordan *-invariant of \(M\), where \(L^p(M)^1\) is the closed unit ball. More precisely, we will show in Theorem 3.11 that two arbitrary von Neumann algebras \(M\) and \(N\) are Jordan *-isomorphic whenever there exist a bijective isometry \(\Phi\) from \(L^p(M)^1_+\) onto \(L^p(N)^1_+\), i.e.,
\[\|\Phi(x) - \Phi(y)\| = \|x - y\| \quad (x, y \in L^p(M)^1_+).\]
Notice that the closed unit ball \(L^2(M)^1_+\) itself is not a complete Jordan *-invariant (since for any infinite dimensional von Neumann algebra \(M\) with a separable predual, one has \(L^2(M) \cong l^2\)), but its positive part is a Jordan *-invariant.

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2. Preliminaries

Throughout this article, if $E$ is a subset of a normed space $X$ and $\lambda > 0$, we set

$$E^\lambda := \{ x \in E : \| x \| \leq \lambda \}. $$

In the following, we will briefly recall (mainly from [11]) notations concerning non-commutative $L^p$-spaces. Let $\mathcal{M}$ be a (complex) von Neumann algebra on a (complex) Hilbert space $\mathcal{H}$ and $\alpha : \mathbb{R} \to \text{Aut}(\mathcal{M})$ be the modular automorphism group. Then the von Neumann algebra crossed product $\mathcal{M} := \mathcal{M} \rtimes_\alpha \mathbb{R}$ is semi-finite and we fix a normal faithful semi-finite trace $\tau$ on $\mathcal{M}$. The measure topology on $\mathcal{M}$ (as introduced by E. Nelson in [9]) is given by a neighborhood basis at $0$ of the form

$$U(\epsilon, \delta) := \{ x \in \mathcal{M} : \| xp \| \leq \epsilon \text{ and } \tau(1 - p) \leq \delta, \text{ for a projection } p \in \mathcal{M} \}. $$

The completion, $L_0(\mathcal{M}, \tau)$, of $\mathcal{M}$ with respect to this topology is a $^*$-algebra extending the $^*$-algebra structure on $\mathcal{M}$.

One may identify $L_0(\mathcal{M}, \tau)$ with a collection of closed and densely defined operators on $L^2(\mathbb{R}; \mathcal{H})$ affiliated with $\mathcal{M}$. More precisely, suppose that $T$ is such a closed operator on $L^2(\mathbb{R}; \mathcal{H})$ and that $|T|$ is the absolute value of $T$ with the spectral measure $E_{|T|}$. Then $T$ corresponds (uniquely) to an element in $L_0(\mathcal{M}, \tau)$ if and only if $\tau(1 - E_{|T|}([0, \lambda])) < +\infty$ when $\lambda$ is large enough. In this case, the $^*$-operation on $L_0(\mathcal{M}, \tau)$ coincides with the adjoint. Moreover, the addition and the multiplication on $L_0(\mathcal{M}, \tau)$ are the closures of the corresponding operations for densely defined closed operators. We denote by $L_0(\mathcal{M}, \tau)_+$ the set of all positive self-adjoint (but not necessarily bounded) operators in $L_0(\mathcal{M}, \tau)$.

The dual action $\hat{\alpha} : \mathbb{R} \to \text{Aut}(\mathcal{M})$ of $\alpha$ extends to an action on $L_0(\mathcal{M}, \tau)$ by $^*$-automorphisms. For any $p \in [1, +\infty]$, we set, as in the literature,

$$L^p(\mathcal{M}) := \{ T \in L_0(\mathcal{M}, \tau) : \hat{\alpha}_s(T) = e^{-sp}T, \text{ for all } s \in \mathbb{R} \}. $$

Denote by $L^p(\mathcal{M})_{sa}$ the set of all self-adjoint operators in $L^p(\mathcal{M})$ and put

$$L^p(\mathcal{M})_+ := L^p(\mathcal{M}) \cap L_0(\mathcal{M}, \tau)_+. $$

If $T \in L_0(\mathcal{M}, \tau)$ and $T = u|T|$ is the polar decomposition, then $T \in L^p(\mathcal{M})$ if and only if $u \in \mathcal{M}$ and $|T| \in L^p(\mathcal{M})$.

In the case when $p \in (1, +\infty)$, the map that sends $x \in \mathcal{M}_+$ to $x^p$ extends to a map

$$\Lambda_p : L_0(\mathcal{M}, \tau)_+ \to L_0(\mathcal{M}, \tau)_+. $$

For any $T \in L_0(\mathcal{M}, \tau)_+$, one has $T \in L^p(\mathcal{M})$ if and only if $\Lambda_p(T) \in L^1(\mathcal{M})$. There is a canonical identification of $M_+$ with $L^1(\mathcal{M})$ that sends the positive part $M_+^+$ of $M_+$ onto $L^1(\mathcal{M})_+$, and this induces a Banach space norm $\| \cdot \|_1$ on $L^1(\mathcal{M})$. The function defined by

$$(2.1) \quad \| T \|_p := \| \Lambda_p(|T|) \|_1^{1/p} $$

is a norm on $L^p(\mathcal{M})$ that turns it into a Banach space. On the other hand, one may identify $\mathcal{M}$ with $L^\infty(\mathcal{M})$ (as ordered Banach spaces) through the canonical inclusion $\mathcal{M} \subseteq \tilde{\mathcal{M}} \subseteq L_0(\mathcal{M}, \tau)$.
3. Results and Questions

3.1. The case of $p = +\infty$.

Proposition 3.1. If $\Phi : M^1_+ \rightarrow N^1_+$ is a bijective isometry, then $\Psi : x \mapsto \Phi(x + \frac{1}{2}) - \frac{1}{2}$ extends to a linear isometry from $M_{sa}$ onto $N_{sa}$.

We may then conclude from [5, Theorem 2] that $x \mapsto \Psi(1)\Psi(x)$ is a Jordan $^*$-isomorphism. In order to establish this proposition, we need the following stronger version of the Mazur-Ulam theorem, which was first proved in [8, Theorem 2] (see also [1, Theorem 14.1]).

Lemma 3.2. Let $U$ be a non-empty open connected subset of a normed space $X$ and $W$ be an open subset of a normed space $Y$. Then every isometry from $U$ onto $W$ can be extended uniquely to an affine isometry from $X$ onto $Y$.

Proof of Proposition 3.1. Let us first note that for any $x \in M_{sa}$, one has $x \in M^1_+$ if and only if $\|x - \frac{1}{2}\| \leq \frac{1}{2}$ (by considering the $C^*$-subalgebra generated by $x$ and $1$). Thus, $x \mapsto x - \frac{1}{2}$ is a bijective isometry from $M^1_+$ onto $M^2_{sa}$ and the map $\Psi$ in the statement is a bijective isometry from $M^2_{sa}$ onto $N^2_{sa}$.

If $x \in M^2_{sa}$, then $\|x\| = \frac{1}{2}$ if and only if there exists $x' \in M^2_{sa}$ with $\|x - x'\| = 1$. This implies
\[
\Psi(\{x \in M_{sa} : \|x\| = 1/2\}) = \{y \in N_{sa} : \|y\| = 1/2\}.
\]
Consequently, $\Psi(0) = 0$ and $\Psi$ will send the interior, $B_M(0, \frac{1}{2})$, of $M^2_{sa}$ onto the interior of $N^2_{sa}$. By Lemma 3.2, $\Psi|_{B_M(0, \frac{1}{2})}$ extends to a linear isometry $\tilde{\Psi}$ from $M_{sa}$ onto $N_{sa}$ and the continuity of $\Psi$ tells us that $\tilde{\Psi}|_{M^2_{sa}} = \Psi$.

Example 3.3. Let $M = \mathbb{C} \oplus \mathbb{R} \oplus \mathbb{C}$. The set $M^1_+$ equals the square in $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ with vertices $(0, 0), (0, 1), (1, 1)$ and $(1, 0)$. If $\Phi_0 : \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R}$ is the clockwise rotation by 90 degree about the center $(\frac{1}{2}, \frac{1}{2})$, then the restriction of $\Phi_0$ on $M^1_+$ is a bijective isometry onto $M^1_+$ that sends $(0, 0)$ to $(0, 1)$. Hence, $\Phi$ itself cannot be extended to a linear map. However, if $\Phi$ is as defined in Proposition 3.1 then $\Psi(1, 1) = \Phi(\frac{1}{2}, \frac{1}{2}) - (\frac{1}{2}, \frac{1}{2}) = (1, -1)$ and the map
\[
(x, y) \mapsto \Psi(1, 1)\Psi(x, y) = (1, -1)(\Phi_0(x + 1/2, y + 1/2) - (1/2, 1/2)) = (y, x)
\]
is a $^*$-automorphism of $M$.

3.2. The case of $p = 1$.

Proposition 3.4. If there exists a bijective isometry $\Phi$ from $M_{s,+}^1$ onto $N_{s,+}^1$, then $M$ and $N$ are Jordan $^*$-isomorphic.

Note, first of all, that one cannot use Lemma 3.2 for this case, since the interior of $M_{s,+}^1$ could be an empty set, e.g. when $M = L^\infty([0, 1])$.

For any $\mu \in M_{s,+}$, we denote by $\text{supp} \mu$ the support projection of $\mu$ in $M$. Recall that for any $\mu, \nu \in M_{s,+}$, we have
\[
\|\mu - \nu\| = \|\mu\| + \|\nu\| \quad \text{if and only if} \quad \text{supp} \mu \cdot \text{supp} \nu = 0.
\]
In order to obtain Proposition 3.3 we need the following lemma.
Lemma 3.5. If $N$ contains three non-zero orthogonal projections $q_1$, $q_2$ and $q_3$, then the bijective isometry $\Phi$ in Proposition 3.4 will send $0$ to $0$.

Proof. Suppose on the contrary that $\Phi(0) \neq 0$. Let us first show that $\text{supp } \Phi(0) = 1$. Indeed, if it is not the case, one can find $\mu \in M_{1,+}^1$ such that $\|\Phi(\mu)\| = 1$ and $\text{supp } \Phi(\mu) \leq 1 - \text{supp } \Phi(0)$, which, together with (3.1), gives the contradiction that $1 \geq \|\mu - 0\| = \|\Phi(\mu) - \Phi(0)\| = \|\Phi(\mu)\| + \|\Phi(0)\| > 1$.

As a result, $\Phi(0)(q_k) > 0$ for $k = 1, 2, 3$. We may also assume, without loss of generality, that $\Phi(0)(q_1) \leq \|\Phi(0)\|/3$ because $3 \sum_{k=1}^3 \Phi(0)(q_k) \leq \|\Phi(0)\|$.

Now, pick any $\nu \in M_{1,+}^1$ with $\|\Phi(\nu)\| = 1$ and $\text{supp } \Phi(\nu) \leq q_1$. Since $1 - 2q_1$ is a unitary and $\|\Phi(\nu) - \Phi(0)\| = \|\nu\| \leq 1$, one arrives at the following contradiction:

$$1 \geq |(\Phi(\nu) - \Phi(0))(q_1 - (1 - q_1))| = |1 - \Phi(0)(q_1) + \Phi(0)(1 - q_1)| = 1 + \|\Phi(0)\| - 2\Phi(0)(q_1) > 1.$$

Consequently, if $N$ contains three non-zero orthogonal projections, then $\Phi$ induces an isometric bijection from the normal state space of $M$ to that of $N$, and hence, we may conclude that $M$ and $N$ are Jordan $*$-isomorphic by using [7, Theorem 3.4]. For the benefit of the readers, we will instead go through briefly the argument of [7, Theorem 3.4] by recalling the following two lemmas. These two lemmas are also needed in the case of $p \in (0, +\infty)$ below.

Let us recall that a bijection $\Gamma$ from the lattice of projections in $M$ to that of $N$ is called an orthoisomorphism if for any projections $p$ and $q$ in $M$, one has

$$pq = 0 \quad \text{if and only if} \quad \Gamma(p)\Gamma(q) = 0.$$

Lemma 3.6. ([7, Lemma 3.1(a)]) Suppose that $\Psi$ is a bijection from the normal state space of $M$ to that of $N$, which is biorthogonality preserving in the sense that for any normal states $\mu$ and $\nu$ of $M$, one has

$$\text{supp } \mu \cdot \text{supp } \nu = 0 \quad \text{if and only if} \quad \text{supp } \Psi(\mu) \cdot \text{supp } \Psi(\nu) = 0.$$

Then there is an orthoisomorphism $\Psi$ from the lattice of projections in $M$ to that of $N$ satisfying $\Psi(\text{supp } \mu) = \text{supp } \Psi(\mu)$ for any normal state $\mu$ on $M$.

A second lemma that we need is the following possibly well-known variant of a theorem of H. A. Dye in [4] (see e.g. [7, Lemma 2.2(a)]). Note that an assumption of not having type $I_2$ summand is needed for the original version of Dye’s theorem. However, the variant here has a weaker conclusion and does not need the assumption concerning the absence of type $I_2$ summand.

Lemma 3.7. If there exists an orthoisomorphism from the lattice of projections in $M$ to that of $N$, then $M$ and $N$ are Jordan $*$-isomorphic.

Proof of Proposition 3.4. Let us first consider the case when $N$ contains three non-zero orthogonal projections. Then by Lemma 3.5, the map $\Phi$ restricts to an isometric bijection $\Psi$ from
the normal state space of $M$ to that of $N$. Moreover, \[3.1\] implies that $\Psi$ is biorthogonality preserving. Now, the conclusion follows from Lemmas \[3.6\] and \[3.7\].

In the case when $M$ contains three non-zero orthogonal projections, one obtains the same conclusion by considering the bijective isometry $\Phi^{-1}$.

Suppose that neither $M$ nor $N$ contains three non-zero orthogonal projections. Then $M$ and $N$ can only be $\mathbb{C}$, $\mathbb{C} \oplus \mathbb{C}_\infty$ or $M_2(\mathbb{C})$. Observe that the Hausdorff dimensions of the quasi-state space of $\mathbb{C}$, $\mathbb{C} \oplus \mathbb{C}_\infty$ and $M_2(\mathbb{C})$ are 1, 2 and 4 respectively. Since a bijective isometry preserves Hausdorff dimensions, we conclude that $M$ and $N$ are $^*$-isomorphic. \hfill $\square$

3.3. A preparation for the case of $p \in (1, +\infty)$.

**Proposition 3.8.** Let $p \in (1, +\infty)$. Suppose that $M$ and $N$ are two von Neumann algebras such that either $M \neq \mathbb{C}$ or $N \neq \mathbb{C}$. Then any bijective isometry $\Phi : L^p(M)_{sa} \to L^p(N)_{sa}$ extends to a linear isometry from $L^p(M)_{sa}$ onto $L^p(N)_{sa}$.

Notice that $L^p(M)_{sa}$ and $L^p(N)_{sa}$ are strictly convex Banach spaces for $p \in (1, +\infty)$ (see e.g., Section 5 of [10]). We recall the following well-known fact concerning strictly convex Banach spaces.

**Lemma 3.9.** Let $X_1$ and $X_2$ be Banach spaces such that $X_2$ is strictly convex. Then every isometry from a convex subset $K$ of $X_1$ into $X_2$ is automatically an affine map.

In fact, we only need to verify that $f((x+y)/2) = (f(x) + f(y))/2$, for any $x \neq y$ in $K$. By “shifting” $K$ and $f$, one may assume that $y = 0$ and $f(0) = 0$. Under this assumption, we have $\|f(z)\| = \|z\|$ $(z \in K)$ and

$$\|f(x) - f(x/2)\| = \|x - x/2\| = \|f(x)/2\| = \|f(x)/2\| - \|x/2\| = \|f(x)\| - \|f(x/2)\|.$$  

The strict convexity of $X_2$ gives $f(x) - f(x/2) \in \mathbb{R} : f(x/2)$. This, together with the last two equalities in (3.2), will produce $f(x) = 2f(x/2)$.

The following lemma is an analogue of [7] Proposition 3.7]. Note that we consider in this lemma bijective isometries between the contractive parts instead of those between the norm-one parts of $K_1$ and $K_2$ as in [7]. Moreover, we have a more general setting here.

**Lemma 3.10.** Let $X_1$ and $X_2$ be strictly convex real Banach spaces of dimensions at least two (could be infinite). Let $\mathcal{F} : \mathbb{R}_+^4 \to \mathbb{R}_+$ be a function satisfying

$$\mathcal{F}(t, 0, t, 0) = 0 \quad (t \in \mathbb{R}_+).$$

For $k = 1, 2$, suppose that $K_k \subseteq X_k$ is a closed and proper cone in $X_k$ which is $\mathcal{F}$-generating, in the sense that for any $x \in X_k$, there exist unique elements $x_+, x_- \in K_k$ with

$$x = x_+ - x_- \quad \text{and} \quad \mathcal{F}(\|x\|, \|x_+\|, \|x_-\|, \|x_+ + x_-\|) = 0.$$  

Then there are canonical bijective correspondences amongst the following (given by restrictions):

- the set $\mathcal{J}$ of real linear isometries from $X_1$ onto $X_2$ that send $K_1$ onto $K_2$,
- the set $\mathcal{J}_B$ of bijective isometries from $K_1^1$ onto $K_2^1$,
- the set $\mathcal{J}_K$ of bijective isometries from $K_1$ onto $K_2$. 
Proof. If \( \Phi \in \mathcal{I} \), then obviously \( \Phi|_{K_1^i} \in \mathcal{I}_B \). The assignment \( \Phi \mapsto \Phi|_{K_1^i} \) is an injection from \( \mathcal{I} \) to \( \mathcal{I}_B \) because \( K_1^i \) linearly spans \( X_1 \).

Suppose that \( \Psi \in \mathcal{I}_B \). We put
\[
S_i := \{ u \in K_i : \| u \| = 1 \} \quad (i = 1, 2).
\]
The set of extreme points of \( K_1^i \) is \( S_i \cup \{ 0 \} \) (since \( X_i \) is strictly convex). By Lemma 3.9, the map \( \Psi \) is affine and hence \( \Psi(0) \in S_2 \cup \{ 0 \} \). If \( \Psi(0) \in S_2 \), then there is a sequence \( \{ u_i \}_{i \in \mathbb{N}} \) in \( S_2 \setminus \{ \Psi(0) \} \) with \( \| v_i - \Psi(0) \| \to 0 \) (as \( \dim X_2 > 1 \)), and hence \( \{ \Psi^{-1}(u_i) \}_{i \in \mathbb{N}} \) is a sequence in \( S_1 \) norm-converging to 0, which is absurd. Thus, we know that \( \Psi(0) = 0 \). Define \( \hat{\Psi} : K_1 \to K_2 \) by
\[
(3.3) \quad \hat{\Psi}(0) := 0 \quad \text{and} \quad \hat{\Psi}(u) := \| u \| \Psi(\frac{u}{\| u \|}) \quad (u \in K_1 \setminus \{ 0 \}).
\]
As \( \Psi \) is an affine map sending 0 to 0, we see that \( \hat{\Psi} \) extends \( \Psi \) and that \( \hat{\Psi}(tu) = t \hat{\Psi}(u) \) \( (u \in K_1, t \in \mathbb{R}_+) \). For any \( u, v \in K_1 \), if \( \lambda := \| u \| + \| v \| + 1 \), then
\[
\left\| \hat{\Psi}(u) - \hat{\Psi}(v) \right\| = \left\| \lambda \Psi \left( \frac{u}{\lambda} \right) - \lambda \Psi \left( \frac{v}{\lambda} \right) \right\| = \lambda \left\| \frac{u}{\lambda} - \frac{v}{\lambda} \right\| = \| u - v \|.
\]
Consequently, \( \hat{\Psi} \in \mathcal{I}_K \). The assignment \( \Psi \mapsto \hat{\Psi} \) is clearly injective.

Suppose that \( \varphi \in \mathcal{I}_K \). Again, Lemma 3.9 implies that \( \varphi \) is affine, and will send extreme points of \( K_1 \) to extreme points of \( K_2 \). However, as \( K_1 \) is a proper cone, the only extreme point in \( K_1 \) is zero \( (i = 1, 2) \) and we have \( \varphi(0) = 0 \). This means that \( \varphi \) is additive and positively homogeneous on \( K_1 \). Hence,
\[
(3.4) \quad \| \varphi(u) + \varphi(v) \| = \| \varphi(u + v) \| = \| u + v \| \quad (u, v \in K_1).
\]
Let us define \( \bar{\varphi} : X_1 \to X_2 \) by
\[
\bar{\varphi}(x) := \varphi(x_+) - \varphi(x_-) \quad (x \in X_1).
\]
Since \( \mathcal{I}(t, t, 0, t) = 0 \) for all \( t \in \mathbb{R}_+ \), one has \( x_+ = x \) and \( x_- = 0 \) whenever \( x \in K_1 \). Hence, \( \bar{\varphi} \) extends \( \varphi \). Moreover, Relation (3.4) as well as \( \| \bar{\varphi}(x) \| = \| x \| \) \( (x \in X_1) \) implies
\[
\mathcal{I}(\| \bar{\varphi}(x) \|, \| \varphi(x_+) \|, \| \varphi(x_-) \|, \| \varphi(x_+) + \varphi(x_-) \|) = \mathcal{I}(\| x \|, \| x_+ \|, \| x_- \|, \| x_+ + x_- \|) = 0,
\]
and the uniqueness of \( \bar{\varphi}(x) \) ensures that \( \bar{\varphi}(x) = \varphi(x_+) \) \( (x \in X_1) \). Furthermore, for any \( x, y \in X_1 \), one has
\[
\| \bar{\varphi}(x) - \bar{\varphi}(y) \| = \| \varphi(x_+) - \varphi(x_-) - \varphi(y_+) + \varphi(y_-) \| = \| \varphi(x_+ + y_-) - \varphi(x_- + y_+) \| = \| x - y \|.
\]
Applying the same arguments to
\[
\psi := \varphi^{-1},
\]
we will obtain a map \( \hat{\psi} \) from \( X_2 \) into \( X_1 \) satisfying \( \hat{\psi}(z) = \psi(z) \) \( (z \in X_2) \). For any \( z \) in \( X_2 \), if we set \( x := \hat{\psi}(z) \in X_1 \), then
\[
\hat{\varphi}(x) = \varphi(x_+) - \varphi(x_-) = \varphi(\psi(z_+)) - \varphi(\psi(z_-)) = z.
\]
This ensures the surjectivity of \( \hat{\varphi} \). Hence, \( \hat{\varphi} \) is a bijective isometry sending 0 to 0, and the Mazur-Ulam theorem tells us that \( \hat{\varphi} \in \mathcal{I} \). It is easy to see that the canonical extension of \( \hat{\varphi}|_{K_1^i} \) to \( K_1 \) as in (3.3) coincides with \( \varphi \). This completes the proof. \[ \square \]

For any \( T \in L^p(M)_{sa} \), we denote by \( \supp T \) the support projection of \( T \), i.e. \( \supp T \) is the smallest projection \( p \) in \( M \) satisfying \( T \cdot p = T \) (or equivalently, \( p \cdot T = T \)). Let us recall the following statements concerning \( S, T \in L^p(M)_{+} \) from Fact 1.2 and Fact 1.3 of [11]:
S1). $\text{supp} \Lambda_p(T) = \text{supp} T$;
S2). $S \cdot T = 0$ if and only if $S \cdot \text{supp} T = 0$;
S3). if $S \cdot \text{supp} T = 0$, then $\|S + T\|_p^p = \|S - T\|_p^p = \|S\|_p^p + \|T\|_p^p$;
S4). if $p \neq 2$ and $\|S + T\|_p^p = \|S - T\|_p^p = \|S\|_p^p + \|T\|_p^p$, then $\text{supp} S \cdot \text{supp} T = 0$.

**Proof of Proposition 3.8.** We note, first of all, that if $M = \mathbb{C}$, then there are only two extreme points of $L^p(M)_{1+}^+$ and hence the strict convexity of $L^p(N)_{sa}$ implies that there are only two extreme points of $L^p(N)_{1+}$ (see the argument of Lemma 3.10), which gives $N = \mathbb{C}$. Therefore, the hypothesis actually implies that the dimensions of both $M_{sa}$ and $N_{sa}$ are at least two.

Let us first consider the case when $p \neq 2$, and define a map $\mathcal{F}_p : \mathbb{R}^{(4)}_+ \to \mathbb{R}_+$ by

$$
\mathcal{F}_p(a, b, c, d) := |a^p - b^p - c^p| + |d^p - b^p - c^p|.
$$

For any $T \in L^p(M)_{sa}$, we know that $|T| \in L^p(M)_+$. We denote by $T_+$ and $T_-$ the positive and the negative part of the self-adjoint operator $T$ respectively. As a closed operator, $T_{\pm}$ is the closure of $\frac{|T| + T}{2}$. Hence, $T_{\pm} = \frac{|T| + T}{2}$ as elements in $L_0(\mathcal{M}, \tau)$. This means that $T_{\pm} \in L^p(M)_+$ and satisfies $T_+T_- = T_-T_+ = 0$. Now, we know from (S2) and (S3) that

$$
\mathcal{F}_p(||T||, ||T_+||, ||T_-||, ||T_+ + T_-||) = 0.
$$

Conversely, suppose that $T \in L^p(M)_{sa}$ and $R, S \in L^p(M)_+$ such that $T = R - S$ and

$$
\mathcal{F}_p(||T||, ||R||, ||S||, ||R + S||) = 0.
$$

Then by (S2) and (S4), we have $RS = 0$. Therefore, $(R + S)^2 = (R - S)^2 = T^2$, which implies that $R + S = |T|$ (because $R + S$ is a positive self-adjoint operator; see e.g. [2] Theorem 12). Consequently,

$$
R = T_+ \quad \text{and} \quad S = T_-.
$$

This means that $L^p(M)_+$ is a $\mathcal{F}_p$-generating cone of $L^p(M)_{sa}$ and we may apply Lemma 3.10 to extend $\Phi$ to a real linear isometry from $L^p(M)_{sa}$ onto $L^p(N)_{sa}$.

For $p = 2$, we know from the proof of Lemma 3.10 that $\Phi(0) = 0$ and hence $\Phi$ restricts to a bijective isometry from the set of norm-one elements in $L^2(M)_+$ onto that of $L^2(N)_+$. Now, the conclusion follows from [7] Proposition 3.7. □

### 3.4. The proof for the case $p \in (1, +\infty)$ and the presentation of the main result.

**Theorem 3.11.** Let $M$ and $N$ be two von Neumann algebras and let $p \in [1, +\infty]$. If there is a bijective isometry $\Phi : L^p(M)_{1+}^+ \to L^p(N)_{1+}^+$, then $M$ and $N$ are Jordan $^*$-isomorphic.

**Proof.** The cases of $p = +\infty$ and $p = 1$ are proved in Proposition 3.3 (together with [3] Theorem 2) and Proposition 3.4, respectively (through the canonical identifications of $L^1(M)$ and $L^\infty(M)$ with $M_*$ and $M$). Moreover, the case of $p = 2$ is already established in [7] Corollary 3.11 (due to Proposition 3.8 and [7] Proposition 3.7).

Now, we consider $p \in (1, +\infty) \setminus \{2\}$. Without loss of generality, we assume that either $M \neq \mathbb{C}$ or $N \neq \mathbb{C}$. By Proposition 3.8, $\Phi$ is an affine map with $\Phi(0) = 0$. Furthermore, it follows from Relation (2.1) that $\Lambda_p$ induces a bijection from $L^p(M)_{1+}^+$ onto $L^1(M)_{1+}^+$ that sends the norm one part of $L^p(M)_+$ onto the norm one part, $\mathcal{G}(M)$, of $L^1(M)_+$. Hence, $\Phi$ induces a bijection...
\( \Phi : \mathcal{S}(M) \to \mathcal{S}(N) \) with \( \Phi(A) = \Lambda_p(\Phi(\Lambda_p^{-1}(A))) \). For any \( A, B \in \mathcal{S}(M) \), it follows from (S1), (S3) and (S4) that

\[
\text{supp } A \cdot \text{supp } B = 0 \quad \text{if and only if} \quad \left\| \frac{\Lambda_p^{-1}(A)}{2} + \frac{\Lambda_p^{-1}(B)}{2} \right\|_p^p = \left\| \frac{\Lambda_p^{-1}(A)}{2} - \frac{\Lambda_p^{-1}(B)}{2} \right\|_p^p = 2^{1-p}.
\]

As \( \Phi \) is both isometric and affine (as well as \( \Phi(0) = 0 \)), we know that \( \hat{\Phi} \) is a biorthogonality preserving bijection between the normal state spaces of \( M \) and \( N \) (through the identification \( L^1(M) = M_* \)). The conclusion now follows from Lemmas 3.6 and 3.7. \( \square \)

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