Low-momentum interactions in three- and four-nucleon scattering

A. Deltuva and A. C. Fonseca
Centro de Física Nuclear da Universidade de Lisboa, P-1649-003 Lisboa, Portugal

S. K. Bogner
National Superconducting Cyclotron Laboratory and Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824

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Low momentum two-nucleon interactions obtained with the renormalization group method and the similarity renormalization group method are used to study the cutoff dependence of low energy 3N and 4N scattering observables. The residual cutoff dependence arises from omitted short-ranged 3N (and higher) forces that are induced by the renormalization group transformations, and may help to estimate the sensitivity of various 3N and 4N scattering observables to short-ranged many-body forces.

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I. INTRODUCTION

Modern few- and many-body calculations of nuclear structure and reactions are based on the picture of point-like nucleons interacting via two- and three-nucleon potentials. For this purpose, a number of high precision ($\chi^2$/datum $\simeq 1$) but phenomenological meson exchange models of the two-nucleon (2N) force such as the Nijmegen [1], Argonne V18 (AV18) [2] and CD-Bonn [3] potentials have been developed over the past decade. However, with phenomenological models it is not clear how to construct consistent three-nucleon (3N) forces and other operators. The lack of a systematic organization or counting scheme results in model-dependent predictions, as there is no way to make controlled comparisons between the different force models. More recently, substantial progress has been made in constructing nuclear interactions from chiral effective field theory (EFT) [4, 5], which is based on the most general local Lagrangian with nucleon and pion fields and all possible interactions consistent with the (broken) chiral symmetry of QCD. In contrast to phenomenological interaction models, the EFT approach is universal and provides a model-independent framework with a systematic organization of consistent 2N, 3N and higher-body forces (and other operators) prescribed by the power counting.

For both phenomenological and EFT potentials, nuclear few- and many-body calculations are complicated by strong short-range repulsion and tensor forces that necessitate highly correlated trial wave functions, non-perturbative resummations, and slowly convergent basis expansions. However, the non-perturbative nature of inter-nucleon interactions is strongly scale dependent and can be radically softened by using the renormalization group (RG) to lower the momentum cutoff that is present in all nuclear interactions. A consequence is that many-body calculations become much more tractable at lower resolutions, resulting in calculations that are amenable to straightforward perturbative methods, simple variational ans" atze, and rapidly convergent basis expansions [6, 7, 8, 9]. The RG approach has the important advantage of being able to vary the cutoff as a tool to optimize and probe the quality of the many-body solution, and to provide estimates of omitted terms in the Hamiltonian.

The above considerations have motivated the construction of low-momentum potentials $V_{\text{low}}$ through the renormalization group (RG) method [10, 11] and, more recently, by the similarity renormalization group (SRG) method [6, 7]. Both methods serve to eliminate the strong coupling between low- and high-momentum modes in the Hamiltonian such that low-energy observables are preserved. In the RG method, one integrates out the problematic high-momentum components of the input interaction above a momentum cutoff $\Lambda$, leading to a new energy independent potential $V_{\text{low}}$ that has the same low-energy on-shell transition matrix (t-matrix) as the input potential. In the original approach $\Lambda$ constitutes a sharp cutoff above which the t-matrix is zero; the method has since been generalized to include a smooth momentum-space regulator to avoid technical difficulties stemming from the sharp cutoff [8]. On the other hand, the SRG method uses a continuous sequence of unitary transformations that weakens off-diagonal matrix elements, driving the Hamiltonian towards a band-diagonal form [6, 7]. In contrast to the RG method, SRG preserves both low- and high-energy observables independent of the value of the flow parameter $\lambda$ that provides a measure of the spread of off-diagonal strength. However, as with the standard RG, the calculation of low-energy observables is decoupled from the high-momentum physics with SRG-evolved potentials (i.e., one can truncate intermediate state summations to low momenta without distorting low-energy observables).

Observables are scale-independent quantities. It is well-known that RG (SRG) transformations generate
short-range many-body forces (in principle, up to $A$-body) that “run” with the cutoff to maintain exact $\Lambda (\lambda)$ independence of $A$-body observables. If the RG transformation is truncated at the 2N level, then the resulting cutoff-dependence in 3N observables may provide an estimate of omitted short-range 3N forces in the Hamiltonian. Along these lines, low-momentum 2N potentials have been recently used in three- and nucleon(4N) bound state calculations [12] as a means to assess the size of omitted higher-body forces by varying the cutoff. There, it was found that the induced 3N forces due to the truncation to low momentum are of the same order as the so-called “bare” 3N forces attributed to integrating out excitations of nucleons. That is, the cutoff-dependence of the 3N binding energies was rather weak, varying by only 1 MeV over a large cutoff range, which is comparable to the 0.7-1 MeV binding provided by the missing “bare” 3N forces in conventional models and EFT calculations. In this sense, the RG evolution to low momentum does not induce strong short-ranged three-body force contributions to these 3N bound state observables. Similar results were obtained in 4N bound state calculations, where the various 2N $V_{\text{low}} k$ calculations did not differ any more from the phenomenological Tjon-line than did calculations using 2N plus adjusted 3N forces.

In the current study, we extend the cutoff-dependence study of Ref. [12] to 3N and 4N scattering observables. In particular, we apply RG- and SRG-evolved 2N interactions to study how the neutron-deuteron ($n-d$) elastic vector analyzing power $A_y$ and the space star cross section in $n-d$ breakup change with the cutoff. These being the two major long-standing failures of realistic interactions in their description of 3N data at low energy, one would like to use cutoff-dependence as a tool to assess the sensitivity of these observables to omitted short-range 3N force effects. Likewise, the same applies to observables in 4N scattering that show large deviations to data, namely the total neutron-triton ($n-t$) cross section $\sigma_t$ around the resonance region at neutron lab energy $E_n = 3.5$ MeV and the $p-^3\text{He}$ $A_y$ that also misses the data by as much as 25 - 40%.

In Section III we study 3N observables and in Section IV 4N observables. Finally in Section IV we present the conclusions.

II. THREE-NUCLEON OBSERVABLES

The results shown in this section are obtained from the solution of the symmetrized Alt, Grassberger and Swanson (AGS) equations [13] for the 3N system using the numerical techniques of Ref. [14]. In order to relate the present work to the findings of Ref. [12] we repeat in Fig. 1 the cutoff dependence of the triton binding energy $\epsilon_t$ for CD-Bonn, AV18 and EFT potential at next-to-next-to-next-to-leading order (N3LO) [8] based $V_{\text{low}} k$ potentials using RG (left side) and SRG (right side) methodologies. In contrast to the calculations of Ref. [12] with a sharp cutoff $\Lambda$, for simpler numerics we use a smooth regulator of the form exp $\left(-k^2/\Lambda^2\right)$. The results are consistent with the ones of Refs. [6, 12]. At first glance, the SRG parameter $\lambda$ that provides a measure of the spread of off-diagonal strength is not obviously related to the cutoff $\Lambda$ in the RG. However, in Ref. [7] it was found that the “decoupling scale” for SRG-evolved interactions was of order $\lambda$. That is, low-energy phase shifts and binding energies are not distorted at high-momentum modes greater than the decoupling scale are set to zero (or any arbitrary value) by hand. Therefore, it is not surprising that the behavior of $\epsilon_t$ in terms of $\Lambda$ or $\lambda$ is qualitatively quite similar. We emphasize that the existence of cutoffs where $\epsilon_t$ agrees with the experimental value does not imply vanishing 3N forces, as they will contribute to other observables.

The neutron analyzing power $A_y$ in $n-d$ elastic scattering at neutron lab energy $E_n = 3$ MeV has a maximum at the center of mass (c.m.) scattering angle $\theta_{\text{c.m.}} = 104$ deg, where the predictions based on realistic interaction models underestimate the experimental value by about 20%. In Fig. 2 we plot the maximum value of $A_y$ as a function of RG cutoff $\Lambda$ and SRG parameter $\lambda$. The cutoff dependence is quite weak, indicating that this observable is not a sensitive probe of short-range force effects. The net variation of $A_y$ over the range of cutoffs is smaller than the discrepancy from experiment of the initial interactions, which implies that short-range 3N forces are not likely to solve the $A_y$ problem.

The cutoff dependence is even weaker for the $n-d$ breakup differential cross section in the space star configuration. We demonstrate that in Fig. 3 for the differential cross section close to the center of the space star configuration at $E_n = 13$ MeV; the values measured in two different experiments are shown as a reference. These flat curves are again an indication that space star cross section is not sensitive to short-range physics as already found in conventional calculations with different 2N interactions or by adding a 3N force [16, 17].
III. FOUR-NUCLEON OBSERVABLES

The results shown in this section are based on the solution of the AGS equations [21] in a symmetrized form following the technical developments expressed in Ref. [21, 22, 23] for all elastic and transfer 4N reactions below three-body breakup threshold.

As discussed in Ref. [21], one of the simplest observables in 4N scattering is the total n-^3H cross section σ_t that exhibits a resonance around E_n ≈ 3.5 MeV. This peak of the total cross section results from a complicated interference between ^3P_1 n-^3H partial waves whose relative strength is sensitive to the realistic 2N force one uses. While at threshold we find the usual scaling between σ_t and ε_t (σ_t decreases as |ε_t| increases), at E_n ≈ 3.5 MeV we observe a breakdown of scaling when we use N3LO [21] which is a low-momentum potential when compared with the meson-exchange potentials. There N3LO yields the largest cross section while not having the lowest |ε_t|.

Furthermore, in Ref. [24] it was found that adding the Urbana IX 3N force to AV18 slightly reduces σ_t at the peak while more significantly lowering the cross section at threshold towards the data as expected through scaling.

Therefore, in order to investigate the effect of low-momentum potentials on σ_t we plot in Fig. 4 the total cross section at the peak versus Λ (λ) (left side) and σ_t versus Λ (λ) (right side). The horizontal line at σ_t = 2.45 b is the experimental value from Ref. [23].

In Fig. 5 we split up the total cross section into n-^3H scattering at E_n = 3.5 MeV as function of RG cutoff Λ (left side) and SRG parameter λ (right side). The experimental data are from Ref. [18] (square) and [19] (circle).

FIG. 2: (Color online) Neutron analyzing power A_y for n-d scattering at E_n = 3 MeV and θ_{cm} = 104 deg as function of RG cutoff Λ (left side) and SRG parameter λ (right side). The horizontal line at 10^2 A_y = 5.86 is the experimental value from Ref. [13].

FIG. 3: (Color online) Differential cross section for n-d breakup at E_n = 13 MeV in the space star configuration (50.5°, 50.5°, 120°) at arclength S = 6.25 MeV as function of RG cutoff Λ (left side) and SRG parameter λ (right side). The experimental data are from Ref. [18] (square) and [19] (circle).

FIG. 4: (Color online) Total cross section for n-^3H scattering at E_n = 3.5 MeV as function of RG cutoff Λ (left side) and SRG parameter λ (right side). The horizontal line at σ_t = 2.45 b is the experimental value from Ref. [23].

FIG. 5: (Color online) S- and P-wave contributions to the total cross section for n-^3H scattering at E_n = 3.5 MeV. On the left side they are shown as functions of RG cutoff Λ or SRG parameter λ, while on the right side their correlation with the ^3H binding energy is shown. The SRG and RG interactions are derived from the AV18 potential.
for the one that yields deepest binding. While one finds that one may describe the total neutron cross section over a wide energy range by using $\Lambda \approx 1.25 \text{ fm}^{-1}$ in RG method or $\lambda \approx 1.8 \text{ fm}^{-1}$ in the SRG approach that also yield reasonable values for $\epsilon_y$, we emphasize once again that these particular values of $\Lambda (\lambda)$ do not imply vanishing 3N and 4N forces, as they will contribute to other few- and many-nucleon observables, for example, to the ground state energies of light nuclei that do not match experiment with those “special” choices of $\Lambda (\lambda)$.

There is a clear correlation between maximum values of neutron analyzing power $A_y$ in $p^3\text{He}$ and $n^3\text{H}$ scattering \cite{21,22}; we therefore study only the latter case. Though $A_y$ in $n$$-d$ and $n^3\text{H}$ scattering are also correlated to some extent, their dependence on cutoff is different as shown in Fig. 7. It is considerably stronger for $n^3\text{H}$. The largest increase of $A_y$ value at the maximum, by a factor $1.13$ (N3LO) to $1.21$ (AV18), is observed around cutoff values that yield experimental or deepest binding. However, according to Ref. \cite{22}, the experimental $A_y$ value at the maximum for $p^3\text{He}$ scattering in the same energy region is larger than theoretical predictions by a factor $1.45$ (CD Bonn) to $1.55$ (AV18). In Fig. 8 we use the AV18 potential to show $A_y$ versus $\theta_{\text{c.m.}}$ for the values of $\Lambda (\lambda)$ that fit the experimental triton binding energy and for the one that yields deepest binding.

![FIG. 6: (Color online) Total cross section for $n^3\text{H}$ scattering as function of neutron lab energy for different values of RG cutoff $\Lambda$ (top) and SRG parameter $\lambda$ (bottom). All results are derived from the AV18 potential. The predictions of the original AV18 potential (dashed curves) are also shown. The experimental data are from Ref. \cite{25}.](image)

![FIG. 7: (Color online) Maximum of the neutron analyzing power $A_y$ for $n^3\text{H}$ scattering at $E_n = 3.5$ MeV as function of RG cutoff $\Lambda$ (left side) and SRG parameter $\lambda$ (right side). All results are derived from the AV18 potential. The predictions of the original AV18 potential (dashed curves) are also shown.](image)

![FIG. 8: (Color online) Neutron analyzing power $A_y$ for $n^3\text{H}$ scattering at $E_n = 3.5$ MeV as function of c.m. scattering angle for different values of RG cutoff $\Lambda$ (left side) and SRG parameter $\lambda$ (right side). All results are derived from the AV18 potential. The predictions of the original AV18 potential (dashed curves) are also shown.](image)

**IV. CONCLUSIONS**

In order to probe the sensitivity of 3N and 4N scattering observables to short-range physics, we used AV18, CD Bonn and N3LO based $V_{\text{low } k}$ potentials that are generated through the RG (SRG) method to study their evolution with the cutoff $\Lambda (\lambda)$. Truncating the RG (SRG) equations to the two-body level amounts to neglecting short-ranged 3N (and higher) forces that are generated to preserve exact cutoff independence. Therefore, one expects to find residual cutoff dependence in few-body observables when only 2N low momentum interactions are used. That cutoff dependence may provide a measure of the sensitivity of a given observable to omitted short-ranged 3N (and higher) forces since the RG evolution does not distort the long-ranged forces arising from pion exchange, provided $\Lambda$ (or $\lambda$) is well above the pion mass, and it is only short-ranged operators that “run” to maintain cutoff independence.

Comparing the results shown in Figs. 2-3 with those in Figs. 4-7 one cannot help noticing that the cutoff de-
dependence of 3N observables is much weaker than the one observed for 4N observables. Clearly for the 3N observables, the cutoff dependence is rather weak, which seems to imply that short-ranged 3N forces are not likely to fix the two long-standing discrepancies with data mentioned above. This is indeed what has been found when the leading missing “bare” 3N force, which contains both long- and short-ranged operators, is added. Nucleon-deuteron \( A_p \) in elastic scattering and the space star differential cross section for breakup barely change by adding a two-\( \pi \)-exchange 3N force \( [16, 26, 27] \), or an effective 3N force due to the explicit \( \Delta \)-isobar excitation \( [17, 28] \), or the more recent leading 3N force from chiral EFT \( [29] \). However, there is hope that the subleading long-range 3N forces from chiral EFT might be important for the resolution of these problems due to their novel space, spin, and isospin structures.

On the contrary 4N scattering observables seem to be more sensitive to omitted short-ranged many-body forces as demonstrated by the more pronounced dependence on the cutoff. In \( n-^3\text{H} \) scattering at low energy the total cross section \( \sigma_t \) is dominated by \( S \) and \( P \) waves in the relative \( n-^3\text{H} \) motion. The \( S \) waves \( (^1S_0, ^3S_1) \) are Pauli repulsive and therefore simply scale with \( \epsilon_t \) over the whole energy region shown in Fig. 6. Therefore sensitivity to 2N forces comes through the \( P \) waves \( (^3P_0, ^3P_1 - ^1P_1, ^3P_2) \) which, in the resonance region, have a very complex behavior with the cutoff parameter, leading to breaking of scaling with \( \epsilon_t \). This is consistent with the previous findings \( [21] \) obtained with various 2N potentials. It also indicates that the \( \sigma_t \) discrepancy may be sensitive to missing short or intermediate range 3N forces, in contrast to the \( p-^3\text{He} \) puzzle \( [22] \).

From these studies one may conclude that 4N scattering observables are more sensitive to short range physics than the 3N observables where, at low energy, they seem to be constrained, to a large extent, by on-shell 2N scattering and three-particle unitarity, as was expressed long ago by Brayshaw \( [30] \). Recent developments \( [26, 27] \) indicate that one needs to fit triton binding energy or neutron-deuteron doublet scattering length to constrain some other 3N observables that, unlike \( A_p \), are sensitive to scaling. Nevertheless, this is already fine tuning on top of results that are already very close to the experimental data. This is not the case for low-energy 4N observables.

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