Contact topology and \( CR \) geometry in three dimensions

Jih-Hsin Cheng
Institute of Mathematics, Academia Sinica
Taipei, R.O.C.(Taiwan)
E-mail: cheng@math.sinica.edu.tw

Abstract

We study low-dimensional problems in topology and geometry via a study of contact and Cauchy-Riemann (\( CR \)) structures. A contact structure is called spherical if it admits a compatible spherical \( CR \) structure. We will talk about spherical contact structures and our analytic tool, an evolution equation of \( CR \) structures. We argue that solving such an equation for the standard contact 3-sphere is related to the Smale conjecture in 3-topology. Furthermore, we propose a contact analogue of Ray-Singer’s analytic torsion. This ”contact torsion” is expected to be able to distinguish among ”spherical space forms” \( \{\Gamma \backslash S^3\} \) as contact manifolds. We also propose the study of a certain kind of monopole equation associated with a contact structure. In view of the recently developed theory of contact homology algebras, we will discuss its overall impact on our study.

1 Spherical contact structures

Let \((M^3, \xi)\) denote a contact 3-manifold with the contact structure \( \xi \). (assume \( M^3 \) oriented and \( \xi \) cooriented if necessary) We call an almost complex structure \( J \) on \( \xi \) a \( CR \) (stands for Cauchy-Riemann) structure (compatible with \( \xi \)). That is to say, an endomorphism \( J : \xi \to \xi \) such that \( J^2 = -Id_\xi \). There are no local invariants for \((M^3, \xi)\) according to a well known theorem of Darboux. Also for closed \( M^3 \), two nearby contact structures are isotopy-equivalent by a theorem of Gray. ([Gr], [Ham]) Therefore a contact structure on a closed \( M^3 \) has no continuous moduli. On the other
hand, we do have local invariants for a CR 3-manifold \((M^3, \xi, J)\). Namely, we can talk about "curvature". Our strategy of studying 3-topology is via a study of contact topology and CR geometry.

There are distinguished CR structures \(J\), called spherical, if \((M^3, \xi, J)\) is locally CR equivalent to the standard 3-sphere \((S^3, \hat{\xi}, \hat{J})\), or equivalently if there are contact coordinate maps into open sets of \((S^3, \hat{\xi})\) so that the transition contact maps can be extended to holomorphic transformations of open sets in \(C^2\). In 1930’s, Elie Cartan ([Ca], [CL1]) obtained a geometric quantity, denoted as \(Q_J\), by solving the local equivalence problem for the CR structure so that the vanishing of \(Q_J\) characterizes \(J\) to be spherical. We will call \(Q_J\) the Cartan (curvature) tensor. A contact structure \(\xi\) is called spherical if there is a spherical CR structure compatible with it.

Our main concern is the existence problem of spherical contact structures. For instance, we ask if any homology 3-sphere admits a spherical contact structure, or does there exist a nonspherical contact homology 3-sphere? Notice that a spherical (contact) homotopy 3-sphere is contact-diffeomorphic to \((S^3, \hat{\xi})\).

It has been believed that for closed \(M^3\), a spherical (contact) structure is tight. (for open \(M^3\), Eliashberg gave counterexamples) We probably can prove this conjecture by showing that the contact homology (recently developed by Eliashberg, Givental and Hofer, [EGH]) of a spherical structure does not vanish. So, most likely, \{spherical structures\} is a restricted class of tight contact structures, which we can apply more analytic tools to study.

### 2 The Cartan flow

The tool we’d like to use is the so-called Cartan flow, an evolution equation for CR structures \(J_{(t)}\) on \((M^3, \xi)\):

\[
\partial_t J_{(t)} = Q_{J_{(t)}}.
\]

(2.1)

Namely, we deform a CR structure in the direction of its Cartan tensor. And we hope that the limit CR structure has the vanishing Cartan tensor, therefore is spherical.
First note that (2.1) is a system of 4-th order nonlinear subparabolic equations (up to an action of contact diffeomorphisms). ([CL1]) Second, we’ll mention some topological and geometrical implications of solving such an equation. Before doing that, let’s see what have been known.

In the late 1980’s, it was observed that (2.1) is a downward (negative) gradient flow. In fact, D. Burns and C. Epstein (also J. Lee and myself) ([BE], [CL1]) found an energy functional $\mu_\xi$ defined on a certain space of $CR$ structures (assuming trivial holomorphic tangent bundle for instance) so that

$$\delta \mu_\xi(J) = -Q_J$$

(meaning $D\mu_\xi(J)(E) = -< Q_J, E >$ for any tangent vector $E$ at $J$, in which $< , >$ is the inner product induced by the Levi form).

The short time solution can be proved by adding a gauge-fixing term to the right-hand side of (2.1). The linearization of the resulting equation is subparabolic with the leading space term of the form $-(\text{const})L^*\alpha L\alpha u$. Here $L_\alpha$ is the generalized Folland-Stein operator and subelliptic if $\alpha$ is not an odd integer. In our case, $\alpha = 4 + i\sqrt{3}$. ([CL1])

Now we come back to the first potential application in 3-topology by solving (2.1) just for $(M^3, \xi) = (S^3, \hat{\xi})$. This will confirm the so-called Smale conjecture: $\text{Diff}(S^3) \approx O(4)$ (”$\approx$” means ”homotopy equivalent”) as first pointed out by Eliashberg in the early 1990’s. In fact, Hatcher ([Hat]) gave a combinatorial proof in 1983. But people are still seeking for more geometric proofs. We can argue that the solution for $(S^3, \hat{\xi})$ implies the Smale conjecture as follows. Since $\hat{J}$ is the unique spherical $CR$ structure on $(S^3, \hat{\xi})$, any other $J$ will converge to $\hat{J}$ through the Cartan flow. This means a certain marked $CR$ moduli space $\mathcal{X}/\mathcal{C}$ is contractible. But $\mathcal{X}$, the space of certain marked $CR$ structures, is contractible too. So $\mathcal{C}$, the group of certain marked contact diffeomorphisms, is contractible too. It follows that $\text{Diff}(S^3) \approx O(4)$ by the relation between $\text{Diff}(S^3)$ and $\mathcal{C}$.

We remark that $\hat{J}$ is a strict local minimum for $\mu_\xi$. ([CL2]) The solution to (2.1) for $(S^3, \hat{\xi})$ will imply that $\hat{J}$ is actually a global minimum. Because there seems to be no suitable maximum principle available for 4-th order subelliptic operators, a proof of the solution to (2.1) would probably have to be based on a priori integral estimates.
in place of the usual pointwise estimates for 2nd order parabolic flows. To learn more analytic techniques, we have been working on a comparatively easier flow. Let us define an energy $e_J$ for a contact form $\theta$ as follows:

$$e_J(\theta) = \int_{M^3} (W_{J,\theta})^2 \theta \wedge d\theta.$$ 

Here $W_{J,\theta}$ denotes the Tanaka-Webster curvature associated with $(J, \theta)$. ([Ta], [We]) We consider the downward gradient flow of $e_J$. If we write $\theta(t) = e^{2\lambda(t)} \hat{\theta}$ with respect to a fixed background contact form $\hat{\theta}$, then the equation can be expressed in $\lambda(t)$ as

$$(2.2) \quad \partial_t \lambda(t) = \Delta_b W_{J,\theta(t)}.$$ 

Here $\Delta_b$ denotes the (positive) sublaplacian. (notice the sign difference for $\Delta_b$ in [Lee]) The equation (2.2) is a 4-th order subparabolic, but scalar, flow. (while (2.1) is a "vector" flow with two independent real unknowns) It is easy to see that the volume $\int_{M^3} \theta(t) \wedge d\theta(t) = \int_{M^3} e^{4\lambda(t)} \hat{\theta} \wedge d\hat{\theta}$ is preserved under the flow (2.2). Under certain conditions, we can establish the following integral estimate: ([CCg])

$$(2.3) \quad \partial_t \int_{M^3} e^{5\lambda(t)} \hat{\theta} \wedge d\hat{\theta} \leq C.$$ 

Here the constant $C$ may or may not depend on the maximum time according to the applied situations. The idea of estimating an integral such as the one in (2.3) comes from the study of a certain metric flow related to general relativity. The involved integral quantity is known as the Bondi mass. We wonder if there are Bondi-mass type estimates for the Cartan flow (2.1).

A $CR$ manifold is embeddable if it can be "realized" as the boundary of a compact complex manifold. (with the $CR$ structure being the one induced from the complex structure) The embeddability is a special property for 3-dimensional $CR$ manifolds since any closed $CR$ manifold of dimension $\geq 5$ is embeddable. ([BdM]) Now it is natural to ask the following question:

**Is the embeddability preserved under the Cartan flow (2.1)?**

By a direct construction of an integrable almost complex structure, we can show that if $J_{(0)}$ is embeddable with the torsion $L_T J_{(0)} = 0$ and $W_{J_{(0)}, \theta} > 0$ (or $< 0$), then
\( J_{(t)} \) stays embeddable (for a short time). ([Ch2]) Here \( T \) denotes the Reeb vector field associated with \( \theta \). In fact, the torsion stays zero under the flow. Also the existence of a \( CR \) vector field \( T \) is sufficient to imply the embeddability of the \( CR \) structure as pointed out by László Lempert. ([Lem]) So the condition on the Tanaka-Webster curvature is redundant. We conjecture that the embeddability is preserved under the Cartan flow without any conditions.

On the other hand, the zero torsion condition reduces the complexity of our flow a lot. It seems to be a good starting point. We are in a situation analogous to Hamilton’s Ricci flow. Namely given a closed contact 3-manifold \((M^3, \xi)\). Suppose there is a pseudohermitian structure \((J, \theta)\) with vanishing torsion and positive Tanaka-Webster curvature. Then can we conclude that \( \xi \) is spherical? A possible proof is to apply the Cartan flow to show that the limit \( CR \) structure (together with the fixed contact form \( \theta \)) has the positive constant Tanaka-Webster curvature. (recall that the torsion stays zero for all time) Therefore it has the vanishing Cartan tensor. So it is spherical.

3 Spherical space forms

Since the linearization \( \delta Q_J \) of the Cartan tensor \( Q_J \) is subelliptic modulo the action of the contact diffeomorphism group \( C_\xi \) ([CL1]), the kernel of \( \delta Q_J \) is finite-dimensional modulo the action of \( C_\xi \). So the ”virtual” dimension of the moduli space of spherical \( CR \) structures is finite-dimensional. In this section, we will just consider a class of examples for the 0-dimensional case. Let \( \Gamma \) denote a fixed point free finite subgroup of the \( CR \) automorphism group of the standard \( S^3 \) (which is isomorphic to \( PU(2, 1) \)). Then the quotient space \( \Gamma \backslash S^3 \) inherits a (spherical) contact structure from \( (S^3, \hat{\xi}) \). It’s natural to work on the following problem.

Problem: Classify \( \{\Gamma \backslash S^3\} \) as contact manifolds.

It has been believed that \( \Gamma_1 \backslash S^3 \) is contact-diffeomorphic to \( \Gamma_2 \backslash S^3 \) if and only if they are \( CR \)-diffeomorphic to each other in analogy with the conformal case. Thus to deal with the above problem, we borrow ideas from quantum physics to find a potential invariant in terms of \( CR \) geometry.
If we view \( \mu_\xi \) as a Lagrangian (action, more accurately) in 2+1 dimensions, spherical CR-structures are just classical fields. Therefore, “quantum fluctuations” should give us refined invariants. In practice, we compute the partition function heuristically:

\[
Z_k = \int_{\mathcal{M}/\mathcal{C}_\xi} \mathcal{D}[J] e^{ik\mu_\xi([J])} = k^{-\frac{\dim}{2}} (Z_{sc} + O(k^{-1})) \quad (k \text{ large}),
\]

in which \( Z_{sc} \) is called the semi-classical approximation. Note that only classical fields make contributions to \( Z_{sc} \). By imitating the finite dimensional case, we can compute the modulus of \( Z_{sc} \):

\[
|Z_{sc}| = \lim_{k \to \infty} |Z_k|^k = \Sigma_{J: \text{spherical}} |\frac{\det \Box_J}{\det' \delta Q_J}|^{\frac{1}{2}},
\]

in which \( \Box_J \) is a fourth-order subelliptic self-adjoint operator related to the \( C_\xi \)-action, and \( \delta Q_J \), the second variation of \( \mu_\xi \), is also a fourth-order subelliptic self-adjoint operator modulo the \( C_\xi \)-action. We can regularize two determinants via zeta functions. (\( \det' \) means taking a regularized determinant under a certain gauge-fixing condition.) (see [Ch1] for more details)

**Conjecture:** If \( J \) is spherical,

\[
\text{Tor}(J) \equiv \left| \frac{\det \Box_J}{\det' \delta Q_J} \right|^{\frac{1}{2}}
\]

is independent of any choice of contact form, i.e., a CR invariant.

We expect to use \( \text{Tor}(J) \) to distinguish among spherical space forms \( \{ \Gamma \setminus S^3 \} \). And we note that \( \text{Tor}(J) \) is a contact-analogue of Ray-Singer’s analytic torsion while no contact-analogue is known for the Reidemeister torsion. Also we speculate that if the contact homology of \( \Gamma \setminus S^3 \) ([EGH]) can distinguish \( \Gamma \setminus S^3 \)'s, it may be possible to identify \( \text{Tor}(J) \) with a certain quantity composed of elements in the contact homology of \( \Gamma \setminus S^3 \).

Let us consider the case that \( \Gamma = I^* \), the binary icosahedral group. It is known that \( I^* \setminus S^3 \) is just the Poincaré homology sphere \( P \). Therefore its contact structure is
spherical. We know that two spherical CR manifolds can be glued together to form the spherical connected sum by an orientation-preserving gluing map. (the gluing map is given by a CR inversion defined on the Heisenberg group $H$ minus the origin in view of a spherical CR manifold being CR equivalent to $H$ locally. In coordinates $(t,z)$ where $t \in \mathbb{R}, z \in \mathbb{C}$, the CR inversion $I$ defined by $I(t,z) = (-t/|w|^2, z/w)$ in which $w = t + i|z|^2$ satisfies $I^*\theta_0 = |w|^{-2}\theta_0$ where $\theta_0 = dt + izd\bar{z} - i\bar{z}dz$ is the standard contact form on $H$. It is easy to verify that $I$ is orientation preserving and interchanges the surfaces defined by $|w| = 2$ and $|w| = 1/2$, respectively). It follows that the connected sum $P \# P$ of $P$ and itself is spherical too. On the other hand, we have the following conjecture

**Conjecture:** There does not exist any spherical contact structure on $P \# \bar{P}$.

Here $\bar{P}$ denotes $P$ with the reverse orientation. In [EH], Etnyre and Honda proved that there does not exist any tight contact structure on $P \# \bar{P}$, either positive or negative. So if a spherical contact structure on a closed 3-manifold is tight (a previous conjecture that we mentioned in the end of section 1), then the above-mentioned conjecture holds in view of Etnyre and Honda’s result. If so, we then have a homology 3-sphere that does not admit any spherical contact structure.

### 4 Monopoles and contact structures

Given a contact 3-manifold $(M^3, \xi)$ and a background pseudohermitian structure $(J, \theta)$, we can discuss a canonical spin$^c$-structure $c_\xi$ on $\xi^*$. ([CCu]) With respect to $c_\xi$, we will consider the equations for our “monopole” $\Phi$ coupled to the “gauge field” $A$. Here, $A$, the spin$^c$-connection, is required to be compatible with the pseudohermitian connection on $M^3$. The Dirac operator $D_\xi$ relative to $A$ is identified with a certain boundary $\bar{\partial}$-operator $\sqrt{2}(\bar{\partial}_0^a + (\bar{\partial}_0^a)^*)$. In terms of the components $(\alpha, \beta)$ of $\Phi$, our equations read as

\begin{equation}
\begin{cases}
(\bar{\partial}_0^a + (\bar{\partial}_0^a)^*)(\alpha + \beta) = 0 \\
\text{(or } \alpha_{1,1}^a = 0, \beta_{1,1}^a = 0) \\
d_a(e_1, e_2) - W_{J,\theta} = |\alpha|^2 - |\beta_1|^2,
\end{cases}
\end{equation}

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where \( A = A_{can} + iaI \) and \( W_{J,\theta} \) denotes the Tanaka-Webster curvature. Our first step in understanding (4.1) is as follows:

Suppose the torsion \( L_T J = 0 \) (\( T \) is the Reeb vector field). Also, suppose \( \xi \) is symplectically semifillable, and that the Euler class \( e(\xi) \) is not a torsion class. Then (4.1) has nontrivial solutions (i.e., \( \alpha \) and \( \beta \) are not identically zero simultaneously). ([CCu])

On the other hand, the Weitzenbock-type formula gives a nonexistence result for \( W_{J,\theta} > 0 \). Together with the above existence result, we can conclude the following:

Suppose the torsion vanishes and the Tanaka-Webster curvature \( W_{J,\theta} > 0 \). Then, either \( \xi \) is not symplectically semifillable, or \( e(\xi) \) is a torsion class. ([CCu])

We remark that Rumin ([Ru]) proved that \( M^3 \) must be a rational homology sphere under the conditions given above using a different method. Originally we were hoping to define contact invariants from the solution space of (4.1). But since \( D_\xi \) (also \( da(e_1, e_2) \)) is not elliptic (not even subelliptic), the solution space might be infinite dimensional. To distinguish such spaces, it seems that we need to know more structures about the solution space. On the other hand, the contact homology algebras recently developed in [EGH] seem to provide such a structure from the algebraic point of view.

5 General discussion

About the Cartan flow (2.1), one would like to know under what conditions the solution to (2.1) exists for all time and converges as \( t \to +\infty \) to a spherical \( CR \) structure. This will be impossible in general. Even if our manifold is the sphere, if we start with an overtwisted contact structure, the solution to (2.1) can not converge since the limit \( CR \) structure would perforce be diffeomorphic to the standard one (which is tight). Hence the solution must blow up at a finite time. We then ask what the shape of the blow-up set looks like.

In [Go], W. Goldman obtained some topological obstruction to the existence of spherical (contact) structures. In particular, \( T^3 \) does not admit any spherical struc-
tures. We hope to be able to obtain some contact topological obstruction in terms of contact homology algebras. To do this, we have to analyze how the contact homology changes under covering and developing maps associated with a spherical structure. K. Mohnke [Mo] has studied the contact homology of certain coverings. His work should be useful for our study. Haven’t obtained some contact obstruction, we can then answer the nonexistence problem of spherical structures in a more refined way. For instance, we might be able to determine which ones among those known tight contact structures are nonspherical for Brieskorn homology spheres $\Sigma(2, 3, 6n - 1), n \geq 2$. Also we can then easily confirm the following previously mentioned conjecture by showing that the contact homology of a spherical structure does not vanish.

**Conjecture:** A spherical structure on a closed 3-manifold is tight.

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**References**

[BdM] Boutet de Monvel, L., Integration des equations de Cauchy-Riemann induites formelles, Seminar Goulaouic-Lions-Schwartz (1974-75) IX.1-IX.13.

[BE] Burns, D. and Epstein, C., A global invariant for three dimensional $CR$-manifolds, Invent. Math., 92 (1988) 333-348.

[Ca] Cartan, E., Sur la géométrie pseudo-conforme des hypersurfaces de l’espace de deux variables complexe I, Ann. Mat., 11 (1932a) 17-90; Sur la géométrie pseudo-conforme des hypersurfaces de l’espace de deux variables complexe II, Ann. Sc. Norm. Sup. Pisa, 1 (1932b) 333-354.
[CCg] Chang, S.-C. and Cheng, J.-H., The Calabi flow on CR 3-manifolds, in preparation.

[CCu] Cheng, J.-H. and Chiu, H.-L., Monopoles and contact 3-manifolds, to appear in Quademi di Matematica, topical volume: CR and pseudohermitian geometry. (math.DG/9905066)

[Ch1] Cheng, J. H., The geometry of Cauchy-Riemann manifolds in dimension three, Proceedings of the Second Asian Mathematics Conference 1995 (Nakhon Ratchasima), World Sci. Publishing, River Edge, NJ (1998) 201-208.

[Ch2] ———, The Cartan flow and embeddability of CR structures, in preparation.

[CL1] Cheng, J. H. and Lee, J. M., The Burns-Epstein invariant and deformation of CR structures, Duke Math. J., 60 (1990) 221-254.

[CL2] Cheng, J. H. and Lee, J. M., A local slice theorem for 3-dimensional CR structures, Amer. J. Math., 117 (1995) 1249-1298.

[EGH] Eliashberg, Y., Givental, A. and Hofer, H., Introduction to Symplectic Field Theory, math.SG/0010059.

[EH] Etnyre, J. and Honda, K., On the non-existence of tight contact structures, math.GT/9910115.

[Go] Goldman, W., Conformally flat manifolds with nilpotent holonomy and the uniformization problem for 3-manifolds, Trans. Amer. Math. Soc., 278 (1983) 573-583.

[Gr] Gray, J. W., Some global properties of contact structures, Ann. Math., 69 (1959) 421-450.

[Ham] Hamilton, R. S., The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc., 7 (1982) 65-222.

[Hat] Hatcher, A. E., A proof of the Smale conjecture, Diff(S^3) ≃ O(4), Ann. Math.,117 (1983) 553-607.
[Lee] Lee, J. M., The Fefferman metric and pseudohermitian invariants, Trans. Amer. Math. Soc., 296 (1986) 411-429.

[Mo] Mohnke, K., Contact homology of coverings, preprint.

[Ru] Rumin, M., Formes différentielles sur les variétés de contact, J. Diff. Geom., 39 (1994) 281-330.

[Ta] Tanaka, N., A Differential Geometric Study on Strongly Pseudo-Convex Manifolds, Kinokuniya Co. Ltd., Tokyo (1975)

[We] Webster, S. M., Pseudohermitian structures on a real hypersurface, J. Diff. Geom., 13 (1978) 25-41.