Dynamical breaking of conformal symmetry in the massless Thirring model

M. Faber* and A. N. Ivanov†‡

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Atominstitut der Österreichischen Universität en, Arbeitsbereich Kernphysik und Nukleare Astrophysik, Technische Universität Wien, Wiedner Hauptstr. 8-10, A-1040 Wien, Österreich

Abstract

We discuss conformal invariance of the massless Thirring model. We show that conformal symmetry of the massless Thirring model is dynamically broken due to the constant of motion caused by the equations of motion. This confirms the existence of the chirally broken phase in the massless Thirring model (Eur. Phys. J. C 20, 723 (2001), which is accompanied by the appearance of massless (pseudo)scalar Goldstone bosons (Eur. Phys. J. C 24, 653 (2002), [hep-th/0210104 and hep-th/0305174]).

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*E-mail: faber@kph.tuwien.ac.at, Tel.: +43–1–58801–14261, Fax: +43–1–58801–14299
†E-mail: ivanov@kph.tuwien.ac.at, Tel.: +43–1–58801–14261, Fax: +43–1–58801–14299
‡Permanent Address: State Polytechnical University, Department of Nuclear Physics, 195251 St. Petersburg, Russian Federation
1 Introduction

The massless Thirring model \([1]\) is a theory of a self–coupled Dirac field \(\psi(x)\)

\[
\mathcal{L}_{\text{Th}}(x) = \bar{\psi}(x)i\gamma^\mu \partial_\mu \psi(x) - \frac{1}{2} g \bar{\psi}(x)\gamma^\mu \psi(x)\bar{\psi}(x)\gamma_\mu \psi(x),
\]

(1.1)

where \(g\) is a dimensionless coupling constant that can be both positive and negative as well. The field \(\psi(x)\) is a spinor field with two components \(\psi_1(x)\) and \(\psi_2(x)\), \(x\) is a 2–vector \(x^\mu = (x^0, x^1)\), where \(x^0\) and \(x^1\) are time and spatial components. The \(\gamma\)–matrices are defined by \(\gamma^0 = \sigma_1\), \(\gamma^1 = -i\sigma_1\) and \(\gamma^5 = \gamma^0\gamma^1 = \sigma_3\), where \(\sigma_i (i = 1, 2, 3)\) are \(2 \times 2\) Pauli matrices. These \(\gamma\)–matrices obey the relations \([2]\)

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad \gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = 0,
\]

(1.2)

where \(\varepsilon^{\mu\nu}\) is the Levy–Civita tensor defined by \(\varepsilon^{01} = -\varepsilon_{01} = 1\). The Lagrangian (1.1) is invariant under the chiral group \(U_V(1) \times U_A(1)\)

\[
\psi(x) \xrightarrow{V} \psi'(x) = e^{i\alpha_V} \psi(x), \\
\psi(x) \xrightarrow{A} \psi'(x) = e^{i\alpha_A} \gamma^5 \psi(x),
\]

(1.3)

where \(\alpha_V\) and \(\alpha_A\) are real parameters defining global rotations. Due to invariance under the chiral group \(U_V(1) \times U_A(1)\) the vector and axial–vector current \(j^\mu(x)\) and \(j_5^\mu(x)\), induced by vector (V) and axial–vector (A) rotations and defined by

\[
\begin{align*}
j^\mu(x) &= \bar{\psi}(x)\gamma^\mu \psi(x), \\
j_5^\mu(x) &= \bar{\psi}(x)\gamma^\mu \gamma^5 \psi(x),
\end{align*}
\]

(1.4)

are conserved \(\partial_\mu j^\mu(x) = \partial_\mu j_5^\mu(x) = 0\). Recall, that in 1+1–dimensional field theories the vector and axial–vector currents are related by \(j_5^\mu(x) = -\varepsilon^{\mu\nu} j_\nu(x)\) due to the properties of Dirac matrices.

In addition to chiral invariance the massless Thirring model is invariant under the conformal group \([3, 4]\), which contains the Poincaré group supplemented by dilatations \(x_\mu \rightarrow x'_\mu = x_\mu / \rho\) and inversions \(x_\mu \rightarrow x'_\mu = c x_\mu / x^2\), where \(\rho\) and \(c\) are parameters of the transformations \([4, 6]\).

If the conformal invariance of the massless Thirring model can be broken neither explicitly nor spontaneously, the spontaneous breaking of chiral symmetry of the massless Thirring model found in \([2, 7]\), leading to the breakdown of conformal symmetry of the massless Thirring model, becomes never possible.

Below we show that conformal symmetry of the massless Thirring model becomes dynamically broken due to the constant of motion following from the equations of motion for the massless Thirring fermion fields \([2]\). This allows the existence of the chirally broken phase in the massless Thirring model obtained in \([2, 7]\). Since dilatations are a part of the conformal group, the dynamical breaking of dilatational invariance of the massless Thirring model should testify the dynamical breaking of a conformal symmetry.
Under dilatations $x_\mu \to x'_\mu = x_\mu/\rho$ the massless Thirring fermion fields behave as follows \[8\] (see also \[6\])

\[ U(\rho) \psi(x) U^\dagger(\rho) = \rho^d \psi(\rho x), \tag{1.5} \]

where $d$ is a dimension of the field $\psi(x)$. A unitary operator $U(\rho)$ is defined by \[6\]

\[ U(\rho) = e^{i \ell n \rho D}, \tag{1.6} \]

where $D$ is the generator of the scale transformations (dilatations). For the infinitesimal dilatation $\rho = 1 + \epsilon$ the operator $D$ satisfies the commutation relation

\[ i[D, \psi(x)] = (d + x^\mu \partial_\mu) \psi(x). \tag{1.7} \]

Under scale transformation (1.5) the Lagrangian (1.1) transforms as follows

\[ \mathcal{L}_{\text{Th}}(x) \to \mathcal{L}_{\text{Th}}(\rho x) = \rho^{2d+1/2} \bar{\psi}(\rho x) i \gamma^\mu \partial_\mu \psi(\rho x) - \rho^{d - 1/2} \frac{1}{2} g \bar{\psi}(\rho x) \gamma^\mu \psi(\rho x) \bar{\psi}(\rho x) \psi(\rho x). \tag{1.8} \]

The action $S_{\text{Th}}[\psi, \bar{\psi}]$ we determine by

\[ S_{\text{Th}}[\psi, \bar{\psi}] = \int d^2x \mathcal{L}_{\text{Th}}(x). \tag{1.9} \]

Due to the scale transformation (1.5) it changes as

\[ S_{\text{Th}}[\psi, \bar{\psi}] = \int d^2x \mathcal{L}_{\text{Th}}(x) \to \rho^{2d-1} \int d^2(\rho x) \left( \bar{\psi}(\rho x) i \gamma^\mu \partial_\mu \psi(\rho x) \right) - \rho^{2d-1} \frac{1}{2} g \bar{\psi}(\rho x) \gamma^\mu \psi(\rho x) \bar{\psi}(\rho x) \psi(\rho x). \tag{1.10} \]

It is seen that the action is invariant under the scale transformations (1.5) if the dimension of the massless Thirring fermion field is equal to $d = 1/2$. This is the so–called canonical dimension of a fermion field in 1+1–dimensional space–time.

For $d = 1/2$ the Lagrangian $\mathcal{L}_{\text{Th}}(\rho x)$ differs from the Lagrangian $\mathcal{L}_{\text{Th}}(x)$ by a constant factor $\rho^2$. This means that the equations of motion of the massless Thirring fermion fields should be invariant under scale transformations.

As has been shown in \[2\], the equations of motion for the massless Thirring fermion fields lead to a constant of motion, which is not invariant under scale transformations $x_\mu \to x'_\mu = x_\mu/\rho$ and (1.5). This should testify a dynamical breaking of the dilatational invariance of the Thirring model.

The paper is organized as follows. In Section 2 we analyse the equations of motion for classical massless Thirring fermion fields and derive the constant of motion. We show that the constant of motion breaks dynamically dilatational and conformal symmetries. In Section 3 we derive the equations of motion for the quantum massless Thirring fermion fields and the quantum version of the constant of motion breaking dilatational and conformal symmetries in the quantum massless Thirring model. In the Conclusion we discuss the obtained results in connection with the existence of the chirally broken phase of the massless Thirring model pointed out in \[2\] \[7\].
2 Constant of motion and dynamical breaking of dilatational invariance. Classical fermion fields

In this section we deal with classical Thirring fermion fields and show that the classical equations of motion lead to the constant of motion found in [2]. For the analysis of the evolution of the classical Thirring fermion fields the Lagrangian (1.1) does not need to be taken in the normal–ordered form [2]. The equations of motion read [2]

\[
\begin{align*}
  i \gamma^\mu \partial_\mu \psi (x) &= g j^\mu(x) \gamma_\mu \psi (x), \\
  - i \partial_\mu \bar{\psi} (x) \gamma^\mu &= g \bar{\psi} (x) \gamma_\mu j^\mu (x).
\end{align*}
\] (2.1)

Due to the peculiar properties of 1+1–dimensional quantum field theories of fermion fields [11] the equations of motion (2.1) are equivalent to [2]

\[
\begin{align*}
  i \partial_\mu \psi (x) &= a j_\mu (x) \psi (x) + b \epsilon_{\mu \nu} j_\nu (x) \gamma^5 \psi (x), \\
  - i \partial_\mu \bar{\psi} (x) &= a \bar{\psi} (x) j_\mu (x) + b \bar{\psi} (x) \gamma^5 j_\nu (x) \epsilon_{\nu \mu},
\end{align*}
\] (2.2)

where the parameters \(a\) and \(b\) are equal to [2]

\[
\begin{align*}
  a &= \frac{1}{2} \left( g + \frac{1}{c} \right), \\
  b &= \frac{1}{2} \left( g - \frac{1}{c} \right).
\end{align*}
\] (2.3)

where \(c\) is the Schwinger term [2].

The equations of motion (2.2) can be transcribed into the equations of motion for the scalar and pseudoscalar fermion densities \(\bar{\psi} (x) \psi (x)\) and \(\bar{\psi} (x) i \gamma^5 \psi (x)\). They read

\[
\begin{align*}
  \partial_\mu [\bar{\psi} (x) \psi (x)] &= - 2 b \epsilon_{\mu \nu} j^\nu (x) [\bar{\psi} (x) i \gamma^5 \psi (x)], \\
  \partial_\mu [\bar{\psi} (x) i \gamma^5 \psi (x)] &= + 2 b \epsilon_{\mu \nu} j^\nu (x) [\bar{\psi} (x) \psi (x)].
\end{align*}
\] (2.4)

Multiplying the first equation by \(\bar{\psi} (x) \psi (x)\) and the second by \(\bar{\psi} (x) i \gamma^5 \psi (x)\) and summing up the obtained expressions we arrive at the relation [2]

\[
\frac{\partial}{\partial x^\mu} \left( [\bar{\psi} (x) \psi (x)]^2 + [\bar{\psi} (x) i \gamma^5 \psi (x)]^2 \right) = 0.
\] (2.5)

Hence, the expression in the parentheses is the constant of motion

\[
[\bar{\psi} (x) \psi (x)]^2 + [\bar{\psi} (x) i \gamma^5 \psi (x)]^2 = C,
\] (2.6)

where \(C\) is a constant. As has been shown in [2], this is equal to \(C = M^2 / g^2\), where \(M\) is the dynamical mass of Thirring fermion fields.

Applying a Fierz transformation and multiplying both sides by \(g / 2\) we transform the constant of motion (2.6) to the form [2]

\[
\frac{1}{2} g \bar{\psi} (x) \gamma^\mu \psi (x) \bar{\psi} (x) \gamma_\mu \psi (x) = - \frac{M^2}{2 g} = \frac{2 \pi}{g} \mathcal{E} [M],
\] (2.7)
where $E[M] = -M^2/4\pi$ is the minimum of the energy density of the ground state of the Thirring fermion fields in the chirally broken phase \[2\].

Since the l.h.s. of (2.7) is a potential of the self–coupled Thirring fermions, one can conclude that relation (2.7) testifies the evolution of the Thirring fermions with a constant potential energy in the proximity of the minimum of the energy density of the ground state of the massless Thirring model in the chirally broken phase.

Under scale transformation the l.h.s. of (2.6) changes as follows

$$
\rho^4 \left( [\bar{\psi}(\rho x) \psi(\rho x)]^2 + [\bar{\psi}(\rho x) i\gamma^5 \psi(\rho x)]^2 \right) = C.
$$

(2.8)

Since $[\bar{\psi}(x) \psi(x)]^2 + [\bar{\psi}(x) i\gamma^5 \psi(x)]^2$ is a constant of motion, it is obvious that

$$
[\bar{\psi}(\rho x) \psi(\rho x)]^2 + [\bar{\psi}(\rho x) i\gamma^5 \psi(\rho x)]^2 = [\bar{\psi}(x) \psi(x)]^2 + [\bar{\psi}(x) i\gamma^5 \psi(x)]^2 = C.
$$

(2.9)

Substituting (2.9) in (2.8) we obtain the relation

$$
\rho^{4d} C = C
$$

(2.10)

which is obviously broken for $\rho \neq 1$ for $C \neq 0$, even if the dimension $d$ of the Thirring fermion fields is equal to the canonical dimension $d = 1/2$.

The constraint (2.10) is valid only for $C = 0$. However, the vanishing constant of motion entails the trivial solution of the equations of motion of the massless Thirring model, $\psi(x) = 0$. Hence, for any non–trivial solution dilatational symmetry as well as conformal symmetry becomes dynamically broken due to the constant of motion.

This means that the massless Thirring fermion fields evolve conserving the constant of motion and breaking scale invariance. Dynamical breaking of the scale invariance is equivalent to the dynamical breaking of conformal symmetry \[5\]. This agrees with the assertion by Fradkin and Palchik \[4\] that conformal symmetry in the massless Thirring model can be spontaneously broken.

### 3 Constant of motion and dynamical breaking of dilatational invariance. Quantum fermion fields

In this section we analyse the existence of the constant of motion (2.6) for quantum massless Thirring fermion fields. Since the constant of motion is the consequence of the equations of motion, we have to derive the quantum equations of motion. For the derivation of quantum equations of motion the Lagrangian (1.1) should be taken in the normal–ordered form \[2\]

$$
\mathcal{L}_{Th}(x) =: \bar{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) : - \frac{1}{2} g : \bar{\psi}(x) \gamma^\mu \psi(x) \bar{\psi}(x) \gamma^\mu \psi(x) :,
$$

(3.1)

where $: \ldots :$ indicates the normal ordering.

For further transformations we decompose conventionally Thirring fermion fields $\psi(x)$ and $\bar{\psi}(x)$ into positive and negative frequency parts \[12\]

$$
\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x),
$$

$$
\bar{\psi}(x) = \bar{\psi}^{(+)}(x) + \bar{\psi}^{(-)}(x),
$$

(3.2)
Due to non–linearity the operators $\psi^{(+)}(x)(\bar{\psi}^{(-)}(x))$ and $\psi^{(-)}(x)(\bar{\psi}^{(+)}(x))$ do not annihilate a fermion (anti–fermion) or create of an anti–fermion (fermion). However, the normal–ordering assumes that all operators $\psi^{(-)}(x)(\bar{\psi}^{(+)}(x))$ should stand to left from the operators $\psi^{(+)}(x)(\bar{\psi}^{(-)}(x))$.

In terms of the positive and negative frequency parts of the fermion fields $\psi(x)$ and $\bar{\psi}(x)$ the Lagrangian acquires the form

$$
\mathcal{L}_{\text{Th}}(x) = \bar{\psi}^{(+)}_{a}(x)(i\gamma^{\mu})_{ab}\partial_{\mu}\psi^{(+)}_{b}(x) - (i\gamma^{\mu})_{ab}\partial_{\mu}\bar{\psi}^{(-)}_{a}(x)\psi^{(-)}_{b}(x) + \bar{\psi}^{(+)}_{a}(x)(i\gamma^{\mu})_{ab}\partial_{\mu}\psi^{(-)}_{b}(x) + \bar{\psi}^{(-)}_{a}(x)(i\gamma^{\mu})_{ab}\partial_{\mu}\psi^{(+)}_{b}(x) - \frac{1}{2} g (\gamma^{\mu})_{ab}(\gamma_{\mu})_{cd}\{-\bar{\psi}^{(+)}_{a}(x)\bar{\psi}^{(+)}_{c}(x)\psi^{(+)}_{b}(x)\psi^{(+)}_{d}(x) + \bar{\psi}^{(+)}_{a}(x)\psi^{(+)}_{d}(x)\bar{\psi}^{(+)}_{b}(x) - \bar{\psi}^{(-)}_{a}(x)\psi^{(-)}_{d}(x)\bar{\psi}^{(-)}_{b}(x) + \bar{\psi}^{(-)}_{a}(x)\psi^{(-)}_{d}(x)\bar{\psi}^{(-)}_{b}(x)\}.
$$

The quantum equations of motion can be obtained by differentiating the Lagrangian with respect to $\bar{\psi}^{(+)}_{a}(x)$. This gives

$$
i(\gamma^{\mu})_{ab}\partial_{\mu}\psi^{(+)}_{b}(x) + i(\gamma^{\mu})_{ab}\partial_{\mu}\psi^{(-)}_{b}(x) = \frac{1}{2} g (\gamma^{\mu})_{ab}(\gamma_{\mu})_{cd}\{-\bar{\psi}^{(+)}_{c}(x)\psi^{(+)}_{d}(x)\psi^{(+)}_{a}(x) + \bar{\psi}^{(-)}_{c}(x)\psi^{(-)}_{d}(x)\psi^{(-)}_{a}(x) - \bar{\psi}^{(+)}_{a}(x)\psi^{(+)}_{d}(x)\bar{\psi}^{(+)}_{c}(x) + \bar{\psi}^{(-)}_{a}(x)\psi^{(-)}_{d}(x)\bar{\psi}^{(-)}_{c}(x) - \bar{\psi}^{(+)}_{a}(x)\psi^{(+)}_{d}(x)\bar{\psi}^{(+)}_{c}(x) + \bar{\psi}^{(-)}_{a}(x)\psi^{(-)}_{d}(x)\bar{\psi}^{(-)}_{c}(x)\}.
$$

These equations of motion can be rewritten in the usual form

$$
i\gamma^{\mu}\partial_{\mu}\psi(x) = \frac{1}{2} g :\bar{\psi}(x)\gamma^{\mu}\psi(x)\gamma_{\mu}\psi(x) + \gamma_{\mu}\psi(x)\bar{\psi}(x)\gamma^{\mu}\psi(x) :=
$$

$$
\quad = g :\{\bar{\psi}(x)\gamma^{\mu}\psi(x), \gamma_{\mu}\psi(x)\} :.
$$

For the Dirac conjugate fermion field the equations of motion read

$$
- i\partial_{\mu}\bar{\psi}(x)\gamma^{\mu} = \frac{1}{2} g :\bar{\psi}(x)\gamma_{\mu}\bar{\psi}(x)\gamma^{\mu}\psi(x) + \bar{\psi}(x)\gamma^{\mu}\psi(x)\bar{\psi}(x)\gamma_{\mu} :=
$$

$$
\quad = g :\{\bar{\psi}(x)\gamma_{\mu}, \bar{\psi}(x)\gamma^{\mu}\psi(x)\} :.
$$

The quantum analogies of the equations read

$$
i\partial_{\mu}\psi(x) = a :\{\bar{\psi}(x)\gamma_{\mu}\psi(x), \psi(x)\} : + b \varepsilon_{\mu\nu} :\{\bar{\psi}(x)\gamma^{\nu}\psi(x), \gamma^{5}\psi(x)\} :,$$

$$
- i\partial_{\mu}\bar{\psi}(x) = a :\{\bar{\psi}(x), \bar{\psi}(x)\gamma_{\mu}\psi(x)\} : - b \varepsilon_{\mu\nu} :\{\bar{\psi}(x)\gamma^{5}, \bar{\psi}(x)\gamma^{\nu}\psi(x)\} :.
$$
For the quantum versions of (2.4) we get
\[
\partial_\mu [\bar{\psi}(x)\psi(x)] = -i a \bar{\psi}(x) : \{ \bar{\psi}(x)\gamma_\mu \psi(x), \psi(x) \} : +i a : \{ \bar{\psi}(x), \bar{\psi}(x)\gamma_\mu \psi(x) \} : \psi(x) \\
- i b \varepsilon_{\mu\nu} \bar{\psi}(x) : \{ \bar{\psi}(x)\gamma^\nu \psi(x), \gamma^5 \psi(x) \} : - i b \varepsilon_{\mu\nu} : \{ \bar{\psi}(x)\gamma^\nu \psi(x), \gamma^5 \psi(x) \} : \psi(x),
\]
\[
\partial_\mu [\bar{\psi}(x)i\gamma^5\psi(x)] = a \bar{\psi}(x)\gamma^5 : \{ \bar{\psi}(x)\gamma_\mu \psi(x), \psi(x) \} : -a : \{ \bar{\psi}(x), \bar{\psi}(x)\gamma_\mu \psi(x) \} : \gamma^5 \psi(x) \\
+ b \varepsilon_{\mu\nu} \bar{\psi}(x)\gamma^5 : \{ \bar{\psi}(x)\gamma^\nu \psi(x), \gamma^5 \psi(x) \} : + b \varepsilon_{\mu\nu} : \{ \bar{\psi}(x)\gamma^\nu \psi(x), \gamma^5 \psi(x) \} : \gamma^5 \psi(x). \tag{3.8}
\]
For the transformation of the r.h.s. of (3.8) we would use Wick’s theorem [12, 13]. This yields
\[
\partial_\mu [\bar{\psi}(x)\psi(x)] = +(a + b) \lim_{y \to x} \bar{\psi}(x)i\gamma^5 \psi(x) : \text{tr} \{ \gamma_\mu \gamma^5 S_F(x - y) \},
\]
\[
\partial_\mu [\bar{\psi}(x)i\gamma^5 \psi(x)] = -(a + b) \lim_{y \to x} \bar{\psi}(x)\psi(x) : \text{tr} \{ \gamma_\mu \gamma^5 S_F(x - y) \}, \tag{3.9}
\]
where \( S_F(x - y) \) is the exact causal two–point Green function of the massless Thirring fermion fields which we define in the form of the Källen–Lehmann representation [13]
\[
S_F(x - y) = i \langle 0 | T(\bar{\psi}(x)\psi(y)) | 0 \rangle = \\
- \frac{1}{2\pi} \frac{\gamma_\mu}{\varepsilon(\varepsilon(x - y))} \frac{\partial}{\partial x_\mu} \int_0^\infty dm^2 \rho(m^2) K_0(m\sqrt{-\varepsilon(x - y)^2 + i\varepsilon} =
\]
\[
= \frac{1}{2\pi} \frac{\gamma_\mu(x - y)_\mu}{\varepsilon(x - y)^2 + i\varepsilon} \int_0^\infty dm^2 \rho(m^2) m K_1(m\sqrt{-\varepsilon(x - y)^2 + i\varepsilon}), \tag{3.10}
\]
where \( K_0(z) \) and \( K_0(z) \) are McDonald’s functions and \( \rho(m^2) \) is the Källen–Lehmann spectral function.

It is obvious that the equations (3.9) can be written as
\[
\partial_\mu ( [\bar{\psi}(x)\psi(x)]^2 + [\bar{\psi}(x)i\gamma^5 \psi(x)]^2 : ) = (a + b) \lim_{y \to x} [ : \bar{\psi}(x)\psi(x) : , : \bar{\psi}(x)i\gamma^5 \psi(x) : ]
\times \text{tr} \{ \gamma_\mu \gamma^5 S_F(x - y) \}. \tag{3.11}
\]
Since the scalar and pseudoscalar fermion densities commute, we arrive at the equation
\[
\partial_\mu ( [\bar{\psi}(x)\psi(x)]^2 + [\bar{\psi}(x)i\gamma^5 \psi(x)]^2 : ) = 0. \tag{3.12}
\]
Thus the quantum version of the constant of motion reads
\[
: [\bar{\psi}(x)\psi(x)]^2 + [\bar{\psi}(x)i\gamma^5 \psi(x)]^2 : = C, \tag{3.13}
\]
where \( C = M^2/g^2 \). It differs from the constant of motion of the classical equations of motion (2.6) only by the normal ordering of the field operators. By a Fierz transformation we obtain
\[
: \bar{\psi}(x)\gamma_\mu \psi(x)\bar{\psi}(x)\gamma^\mu \psi(x) : = -C. \tag{3.14}
\]
In the component form the constant of motion reads
\[
: \psi_1^\dagger(x)\psi_1(x)\psi_2^\dagger(x)\psi_2(x) : = \frac{1}{4} C. \tag{3.15}
\]
This constant of motion for \( C \neq 0 \) breaks dynamically both dilatational and conformal symmetry of the massless Thirring model.

It is obvious that for \( C = 0 \), that is demanded by dilatational and conformal invariance, there can be only a trivial solution of the quantum equations of motion, i.e. \( \psi_1(x) = \psi_2(x) = 0 \). Hence, the conformal invariant massless Thirring model does not exist.
4 Conclusion

We have discussed the constant of motion for the evolution of the massless Thirring fermion fields in connection with dynamical breaking of dilatational and conformal symmetries of the massless Thirring model. We have derived the constant of motion for both classical and quantum massless Thirring fermion field. The existence of the constant of motion for the evolution of the massless Thirring fermion fields has been recently confirmed within the free massless boson field representation of the massless Thirring fermion fields [14].

We have shown that the constant of motion of the massless Thirring fermion fields breaks dynamically both dilatational and conformal symmetries of the massless Thirring model. This agrees with Fradkin and Palchik [4]. The vanishing constant of motion, compatible with dilatational and conformal invariance of the massless Thirring model, leads to a trivial solution of the classical and quantum equations of motion, \( \psi(x) = 0 \). This means that the massless Thirring model, invariant under dilatational and conformal symmetry, does not exist.

It is well–known from low–energy hadronic physics that spontaneous breaking of dilatational symmetry entails spontaneous breaking of chiral symmetry [6]. Thus, the dynamical breaking of dilatational and conformal symmetries of the massless Thirring model, due to the constant of motion, testifies the validity of our results concerning the existence of the chirally broken phase of the massless Thirring model pointed out in [2, 7].

We would like to emphasize that the dynamical breaking of conformal symmetry as well as the spontaneous breaking of chiral symmetry in the massless Thirring model does not contradict to Coleman’s theorem. Indeed, as has been discussed in [7, 9, 10], Coleman’s theorem is applicable only to quantum field theories in 1+1–dimensional space–time with Wightman’s observables defined on the test functions \( h(x) \) from \( \mathcal{S}(\mathbb{R}^2) \), \( h(x) \in \mathcal{S}(\mathbb{R}^2) \), whereas the massless Thirring model and its bosonized version are the quantum field theories with Wightman’s observables defined on the test functions \( h(x) \) from \( \mathcal{S}_0(\mathbb{R}^2) = \{ h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0) = 0 \} \), where \( \tilde{h}(0) \) is the Fourier transform of \( h(x) \) at zero momentum. Due to the vanishing of \( \tilde{h}(0) \), the collective zero–mode, responsible for infrared divergences of the free massless (pseudo)scalar field theory bosonizing the massless Thirring model [7, 14], cannot be measured by Wightman’s observables [7, 9, 10].

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