Full proof of Kwapien’s theorem on representing bounded mean zero functions on $[0, 1]$

by

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Abstract. In 1984, Kwapien announced that every mean zero function $f \in L_{\infty}[0, 1]$ can be written as a coboundary $f = g \circ T - g$ for some $g \in L_{\infty}[0, 1]$ and some measure preserving transformation $T$ of $[0, 1]$. Whereas Kwapien’s original proof holds for continuous functions, there is a serious gap in the proof for functions with discontinuities. In this article we fill in this gap and establish Kwapien’s result in full generality. Our method also allows us to improve the original result by showing that for any given $\varepsilon > 0$ the function $g$ can be chosen to satisfy $\|g\|_{\infty} \leq (1 + \varepsilon)\|f\|_{\infty}$.

Introduction. In this article we prove the following strengthening of a theorem announced by Kwapien [K].

THEOREM 0.1. Let $f \in L_{\infty}[0, 1]$ be a real-valued mean zero function. Let $\varepsilon > 0$. Then there exists a $g \in L_{\infty}[0, 1]$ with $\|g\|_{\infty} \leq (1 + \varepsilon)\|f\|_{\infty}$ and a mod 0 measure preserving transformation $T$ of $[0, 1]$ such that $f = g \circ T - g$.

This theorem holds for all measure spaces which are mod 0 isomorphic to the interval $[0, 1]$ equipped with Lebesgue measure.

The proof given in [K] is incomplete for discontinuous functions. On the other hand, in the last 20 years, Kwapien’s Theorem 0.1 has been used in the theories of symmetric functionals (see e.g. [FK]) and singular traces (see e.g. [LSZ]) and featured in some measure theory treatises (see [B2], p. 335, Exercise 9.12.68). It is therefore important to obtain its full proof and this is the main objective of the present article. Our proof complements some deep ideas and interesting technical approaches from [K].

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Now, we briefly explain the nature of the gap in \([K]\) (a more detailed explanation is given in Appendix A below). The original proof suggested in \([K]\) is based on the usage of Luzin’s theorem (see Theorem 1.2 below), which guarantees the existence of disjoint sets \(A_n \subseteq [0,1]\) for \(n \geq 1\) such that \(\lambda([0,1] \setminus \bigcup_{n=1}^{\infty} A_n) = 0\) and the following hold:

(1) \(A_n\) is a closed subset, homeomorphic to the Cantor set.
(2) \(f\) restricted to \(A_n\) is a continuous function.
(3) \(\lambda(A_n) > 0\) and \(\int_{A_n} f \, d\lambda = 0\), where \(\lambda\) is Lebesgue measure.

It is further stated in \([K]\) that for each \(n \geq 1\), there exists a homeomorphism from \(A_n\) to the Cantor set \([0,1]\) that maps the measure \(\lambda/\lambda(A_n)\) to the Cantor measure \(\mu = \prod_{i=1}^{\infty} \mu_i\) where \(\mu_i\) is the probability measure on \([0,1]\) given by \(\mu_i(\{0\}) = \mu_i(\{1\}) = 1/2\). However, as the counterexample given in Appendix A shows, this is not in general possible and hence the proof given in \([K]\) has a gap.

We outline the paper’s structure. In Section 1, we set notation and state some known results that will be used throughout this article. In Section 2, we treat some special class of measurable bounded functions \(f\) and adapt the proof in \([K]\) to this class. More precisely, we require that for a bounded mean zero function \(f\), there exists a \(\kappa \in \mathbb{R}\) such that \(\lambda(f^{-1}(\{y\})) = 0\) for all \(y < \kappa\) and \(f|_{\{f \geq \kappa\}} = \kappa\). In that section, we present a modified method of constructing the sets \(A_n\) used in \([K]\) which allows us to treat this subclass.

Next, in Section 3, we prove Kwapień’s Theorem for countably valued, mean zero functions \(f \in L_\infty[0,1]\). Finally, in Section 4, we combine the results from Sections 2 and 3 and obtain a complete proof of Kwapień’s Theorem. The motivation and short outline of the arguments presented in Sections 2–4 are given at the beginning of each section.

We note that [AR1] (see also [AR2]) contains an alternative approach to some version of Kwapień’s Theorem. However, their method does not provide the important bound \(\|g\|_\infty \leq C\|f\|_\infty\) for an absolute constant \(C\), which is needed in some applications (see e.g. [LSZ]).

1. Preliminaries. In this section we set the notation and state the known results that we need. Throughout this article we will equip a Lebesgue measurable set \(K \subset [0,1]\) with the Borel \(\sigma\)-algebra \(\mathcal{B}(K)\) and the (induced) Lebesgue measure \(\lambda\), unless otherwise stated. We then let \(L_\infty(K)\) denote the space of essentially bounded real-valued functions under the equivalence relation of equality almost everywhere. We denote the essential supremum by \(\|\cdot\|_\infty\).

We will use the following notion of a measure preserving transformation.

**Definition 1.1.** Let \((\Omega, \mathcal{A}, \mu)\) and \((\Omega', \mathcal{A}', \mu')\) be measure spaces. We define a mod 0 measure preserving transformation between the measure spaces
as a bijection $T : \Omega \setminus N \to \Omega' \setminus N'$ for some null sets $N \in \mathcal{A}$ and $N' \in \mathcal{A}'$ such that both $T$ and $T^{-1}$ are measurable mappings and $\mu'(T(A)) = \mu(A)$ for all $A \subseteq \Omega \setminus N$ in $\mathcal{A}$.

We shall use the classical Luzin’s Theorem [B2 Theorem 7.1.13].

**Theorem 1.2** (Luzin’s Theorem). Let $D \subseteq [0, 1]$ be a Lebesgue measurable set and let $f : D \to \mathbb{R}$ be measurable. If $\varepsilon > 0$, then there is a compact subset $K \subseteq A$ such that $\mu(A \setminus K) < \varepsilon$ and $f|_K$ is continuous.

The following theorem is obtained by combining Theorems 9.3.4 and 9.5.1 in [B1].

**Theorem 1.3.** Let $A, B \subseteq [0, 1]$ be subsets of equal positive measure. Then there exists a mod 0 measure preserving transformation $T$ between $A$ and $B$.

We shall also need the following two lemmas, taken from [K Lemma] and [LSZ Lemma 5.2.3] respectively. For the convenience of the reader, we give a short proof of the second lemma.

**Lemma 1.4.** Let $(a_{i,j})_{n \times m}$ be a real matrix such that $|a_{i,j}| \leq C$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ and $\sum_{j=1}^{m} a_{i,j} = 0$ for $i = 1, \ldots, n$. Then there exist permutations $\sigma_1, \ldots, \sigma_n$ of $\{1, \ldots, m\}$ such that

$$\left| \sum_{i=1}^{k} a_{i,\sigma_i(j)} \right| \leq 2C \quad \text{for all } k = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$

**Lemma 1.5.** Let $a_1, \ldots, a_n \in \mathbb{R}$ with $\sum_{k=1}^{n} a_k = 0$. Then there exists a permutation $\sigma$ of $\{1, \ldots, n\}$ such that $|\sum_{k=1}^{m} a_{\sigma(k)}| \leq \max_{k=1}^{n} |a_k|$ for every $m \in \{1, \ldots, n\}$.

**Proof of Lemma 1.5.** Set $\sigma(1) = 1$ and assume that the sought-for permutation is already defined for $1, \ldots, m$, so that $|\sum_{k=1}^{m} a_{\sigma(k)}| \leq \max_{k=1}^{n} |a_k|$. Since $\sum_{k=1}^{n} a_k = 0$, among the remaining indices (excluding $\sigma(1), \ldots, \sigma(m)$) there exists an index $j$, such that $\sum_{k=1}^{m} a_{\sigma(k)}$ and $a_j$ have opposite signs. Set $\sigma(m+1) = j$. Then it follows from the obvious implication $a \leq 0 \leq b \Rightarrow a \leq a + b \leq b$ that $-\max_{k=1}^{n} |a_k| \leq \min(\sum_{k=1}^{m} a_{\sigma(k)}, a_{\sigma(m+1)}) \leq \sum_{k=1}^{m+1} a_{\sigma(k)} \leq \max(\sum_{k=1}^{m} a_{\sigma(k)}, a_{\sigma(m+1)}) \leq \max_{k=1}^{n} |a_k|$. \[\square\]

2. **Theorem 0.1** for almost nowhere constant functions. In this section we shall always assume that

(i) $K \subseteq [0, 1]$ is a Lebesgue measurable set with $\lambda(K) > 0$ and we consider the measure space $(K, \lambda)$ equipped with Lebesgue measure;

(ii) $f \in L_\infty(K)$ is a real-valued mean zero function such that there exists a constant $\kappa \in \mathbb{R}$ with $\lambda(f^{-1}(\{y\})) = 0$ for $y < \kappa$ and $f|_{\{f \geq \kappa\}} = \kappa$.

In this section, we will prove the following special case of Theorem 0.1.
Theorem 2.1. Suppose that $K$ and $f$ satisfy assumptions (i) and (ii) above. For every $\varepsilon > 0$, there is a $g \in L_\infty(K)$ with $\|g\|_\infty \leq (1+\varepsilon)\|f\|_\infty$ and a mod 0 measure preserving transformation $T$ of $(K, \lambda)$ such that $f = g \circ T - g$.

We prove Theorem 2.1 by adapting and adjusting the method suggested by Kwapien [K]. We modify the construction of sets $A_n$, $n \geq 1$, used in [K] for continuous functions to make them suitable for functions $f$ as above.

Below, we outline our arguments and the main modifications compared to [K].

To construct the sets $A_n$ we start with a function $f \in L_\infty(D)$ and $\varepsilon > 0$ where $D$ and $f$ satisfy (i) and (ii) above. There are five main steps in our argument.

1. We need to restrict $f$ to a slightly smaller compact subset $K$ on which $f\vert_K$ is continuous and mean zero. For this we need Lemma 2.2 which says that for a small enough $\varepsilon' > 0$, for a compact subset $E \subseteq D$ with $\lambda(D \setminus E) < \varepsilon'$ for which $f\vert_E$ is continuous, we can find a compact set $K \subseteq E$ on which $f$ has mean zero and $\lambda(D \setminus K) \leq c\varepsilon'$ where $c$ is some constant depending only on $f$. Using this lemma together with Luzin’s Theorem yields the required restriction.

2. In order to do step (3) below, we need Lemma 2.5 which asserts that for a compact set $K$ and a continuous $f \in L_\infty(K)$ satisfying (i) and (ii), and for small enough $\varepsilon'' > 0$, denoting $K^L = K \cap [0, (\inf K + \sup K)/2]$ and $K^R = K \cap [(\inf K + \sup K)/2, 1]$, there exists a subset $\tilde{K} := K^L \cup K^R \subseteq K \setminus \{\inf K, \sup K\}$, where $\tilde{K}^L \subseteq K^L$ and $\tilde{K}^R \subseteq K^R$ are compact, such that $\lambda(\tilde{K}) = c$, where $c$ is an arbitrary scalar selected in advance from $(0, \lambda(K))$, and further $\int_{\tilde{K}} f \, d\lambda = 0$ and $\lambda(\tilde{K}^L)/\lambda(\tilde{K})$ is rational. The proof of Lemma 2.5 is technical and requires Lemmas 2.3 and 2.4.

3. Next, in Lemma 2.6 we will find a compact subset $C \subseteq K$ of positive measure such that $\int_C f \, d\lambda = 0$ and $C$ is homeomorphic to a Cantor set $\prod_{j=0}^\infty \{1, \ldots, m_j\}$ for some integer sequence $(m_j)_{j \geq 0}$ with $m_j \geq 2$ for $j = 0, 1, \ldots$, and the homeomorphism maps the measure $\lambda/\lambda(C)$ to the associated probability measure on $\prod_{j=0}^\infty \{1, \ldots, m_j\}$.

In order to construct $C$, we work layer by layer. We assume that we already have compact subsets $C_n \subseteq \cdots \subseteq C_1 \subseteq K$ and integers $m_j \geq 2$ for $j = 0, 1, \ldots, n-1$ satisfying $\int_{C_j} f \, d\lambda = 0$ and such that $C_j$ is partitioned from left to right into compact subsets $(K_a)_{a \in \mathcal{E}_j}$ of equal measure that only overlap at their endpoints. Here $\mathcal{E}_j$ is an index set with $|\mathcal{E}_j| = \prod_{i=0}^{j-1} m_i$. Moreover the partition of $C_j$ refines the partition of $C_{j-1}$, $j \geq 1$.

To construct the next layer we employ Lemma 2.5 which asserts the existence of compact sets

$$\tilde{K}_a \subseteq K_a \setminus \{\inf K_a, \sup K_a\}$$
of a specified measure such that
\[ \frac{1}{\lambda(\tilde{K}_a)} \int f \, d\lambda = \frac{1}{\lambda(K_a)} \int f \, d\lambda \]
and
\[ \lambda(\tilde{K}_a \cap [0, (\inf K_a + \sup K_a)/2]) \]
is rational. We then set
\[ C_{n+1} = \bigcup_{a \in \mathcal{E}_n} \tilde{K}_a, \]
and verify that \( f \) has mean zero on \( C_{n+1} \). Furthermore, we choose an integer \( m_n \) such that the product
\[ m_n \cdot \frac{\lambda(\tilde{K}_a \cap [0, (\inf K_a + \sup K_a)/2])}{\lambda(\tilde{K}_a)} \]
is an integer for all partition sets \((K_a)_{a \in \mathcal{E}_n}\) of \( C_n \). This is the main reason why we require that the ratio above is rational.

We then partition the sets \( \tilde{K}_a \) from left to right into compact subsets \((\tilde{K}_a^i)_{i=1}^{m_n}\) of equal measure, overlapping only at their endpoints. By the choice of \( m_n \) the diameters of these subsets are at least halved compared to the diameter of \( \tilde{K}_a \). This gives us the partition of the next layer and finishes the inductive construction.

We finally set \( C = \bigcap C_n \) and verify that \( f \) has mean zero on \( C \). Since the diameter of the partition sets goes to zero, and since those sets are compact, \( C \) is homeomorphic to the Cantor set. Furthermore, since the partition subsets of each layer have equal measure, the homeomorphism also maps the measure \( \lambda/\lambda(C) \) to the desired probability measure.

(4) By slightly adapting Kwapien’s proof for continuous functions on \([0, 1] \), in Lemma 2.7 we select \( g \in L_\infty(C) \) with
\[ \|g\|_\infty \leq (1 + \varepsilon)\|f\|_\infty \]
and a mod 0 measure preserving transformation \( T \) of \( C \), such that \( f|_C = g \circ T - g \). To this end, we exploit the properties of \( C \). First, we construct step functions \( h_0 \) and \( h_k \) for \( k \geq 1 \) living on \( C \) such that \( f = h_0 + \sum_{k=1}^{\infty} h_k \) with \( \|h_0\|_\infty \leq \|f\|_\infty \) and \( \sum_{k=1}^{\infty} \|h_k\|_\infty \) arbitarily small. The functions \( h_0 \) and \( h_k \) are so constructed that they are constant on the partition sets of \( C_n \) for some \( n \). Next, inductively for the functions \( h_0 \) and \( h_k \), we construct functions \( g_k \) with \( \|g_0\|_\infty \leq \|h_0\|_\infty \) and \( \|g_k\|_\infty \leq 4\|h_k\|_\infty \) for \( k \geq 1 \), and mod 0 measure preserving transformations \( T_k \) of \( C \) such that
\[ h_0 = g_0 \circ T_0 - g_0 \quad \text{and} \quad h_k = g_k \circ T_k - g_k, \quad k \geq 1. \]
The functions $g_0$ and $g_k$ are step functions, and $T_k$ is a permutation of partition sets of some layer of the Cantor set. Moreover, each $T_k$ in fact extends $T_{k-1}$. This trick is performed via using the permutations afforded by Lemmas 1.4 and 1.5. The application of these lemmas is crucial for obtaining the stated bounds on $\|g_0\|_\infty$ and $\|g_k\|_\infty$. It should also be pointed out that the construction is feasible due to the fact that all partition sets of the same layer of the Cantor set are selected to have equal measures and hence $T_k$ can be chosen as a permutation of those sets. Finally, we set $g = \sum_{k=0}^\infty g_k$, and $T$ is defined as the pointwise limit of $T_k$.

(5) In step (4) above we have established that for some set $C \subseteq D$, there exists $g \in L_\infty(C)$ with $\|g\|_\infty \leq (1 + \varepsilon)\|f\|_\infty$ and a mod 0 measure preserving transformation $T$ of $C$ such that $f|_C = g \circ T - g$. Now, $D \setminus C$ and $f|_{D \setminus C}$ also satisfy assumptions (i) and (ii) of Theorem 2.1 and therefore we can essentially reiterate the argument and extend it to the function $f$ on $D$. We do this in Subsection 2.4 using the Zorn Lemma. There, we define a partition of the domain into sets $A_n$, $n \geq 1$, for which we can find $g_n \in L_\infty(A_n)$ with $\|g_n\|_\infty \leq (1 + \varepsilon)\|f\|_\infty$ and mod 0 measure preserving transformations $T_n$ of $A_n$ such that $f|_{A_n} = g_n \circ T_n - g_n$. This is the final step in the proof of Theorem 2.1.

2.1. Preserving mean zero condition for subsets. As explained in step (4) above, we need the following result which is to be used in combination with Luzin’s Theorem in order to slightly shrink the domain of $f$ to a compact set $E$ such that $f|_E$ is continuous and has mean zero.

**Lemma 2.2.** Let $D \subseteq [0,1]$ and let $f \in L_\infty(D)$ be a real-valued function satisfying $f \neq 0$ and $\int_D f \, d\lambda = 0$. Then $\tau^+ = \lambda(\{ f > \frac{1}{2}\|f^+\|_\infty \}) > 0$ and $\tau^- = \lambda(\{ f < -\frac{1}{2}\|f^-\|_\infty \}) > 0$. Let

$$0 < \varepsilon < \frac{1}{4} \min \left\{ \tau^+ \frac{\|f^+\|_\infty}{\|f\|_\infty}, \tau^- \frac{\|f^-\|_\infty}{\|f\|_\infty} \right\}.$$

Then, for any compact subset $E \subseteq D$ with $\lambda(E) \geq \lambda(D) - \varepsilon$ and $f|_E$ continuous, there is a compact subset $K \subseteq E$ with $\int_K f \, d\lambda = 0$ and

$$\lambda(K) \geq \lambda(D) - \left( 1 + 2\|f\|_\infty \max \left\{ \frac{1}{\|f^+\|_\infty}, \frac{1}{\|f^-\|_\infty} \right\} \right) \varepsilon.$$

**Proof.** Set $\bar{\varepsilon} = \int_E f \, d\lambda$. As $f \neq 0$ and $\int_D f \, d\lambda = 0$ we see that $f^+, f^- \neq 0$, hence $\tau^+, \tau^- > 0$. We have

$$|\bar{\varepsilon}| = \left| \int_E f \, d\lambda \right| = \left| \int_{D \setminus E} f \, d\lambda \right| \leq \lambda(D \setminus E)\|f\|_\infty \leq \varepsilon\|f\|_\infty.$$

We will further suppose that $\bar{\varepsilon} \geq 0$. The set $f|_E^{-1}(\frac{1}{2}\|f^+\|_\infty, \infty)$ is open in $E$ since $f|_E$ is continuous. Now, for $0 \leq r \leq 1$ consider the measurable set
We conclude that there exists $H_r := f|_E^{-1}((\frac{1}{2}\|f^+\|_\infty, \infty)) \cap [0, r) \subseteq E$ and let $F : [0, 1] \to \mathbb{R}$ be given by $F(r) = \int_{E \setminus H_r} f \, d\lambda$. Since $|F(r_1) - F(r_2)| \leq \|f\|_\infty |r_1 - r_2|$, it follows that $F$ is continuous. Further $F(0) = \tilde{\epsilon} \geq 0$ by assumption. Since $\lambda(R_1) = \lambda(f^{-1}((\frac{1}{2}\|f^+\|_\infty, \infty)) \cap E) \geq \tau^+ - \lambda(D \setminus E) \geq \tau^+ - \epsilon \geq \frac{1}{2}\tau^+$ it follows that
\[
F(1) = \int_E f \, d\lambda - \int_{R_1} f \, d\lambda \leq \tilde{\epsilon} - \lambda(R_1)\frac{1}{2}\|f^+\|_\infty \leq \tilde{\epsilon} - \tau^+\frac{1}{2}\|f^+\|_\infty
\]
\[
\leq \left(\epsilon - \frac{1}{4}\tau^+\frac{\|f^+\|_\infty}{\|f\|_\infty}\right)\|f\|_\infty < 0.
\]
We conclude that there exists $r_0 \in [0, 1]$ with $F(r_0) = 0$. Now set $K = E \setminus R_{r_0} \subseteq D$. Since $R_{r_0}$ is open in $E$, it follows that $K$ is compact. Further, $\int_K f \, d\lambda = F(r_0) = 0$. Finally,$$
\tilde{\epsilon} = \int_E f \, d\lambda = \int_{R_{r_0}} f \, d\lambda \geq \lambda(R_{r_0})\frac{1}{2}\|f^+\|_\infty,
$$which immediately yields$$
\lambda(R_{r_0}) \leq \frac{2\tilde{\epsilon}}{\|f^+\|_\infty} \leq \frac{2\|f\|_\infty}{\|f^+\|_\infty} \epsilon.
$$Hence,$$
\lambda(K) = \lambda(E) - \lambda(R_{r_0}) \geq (\lambda(D) - \epsilon) - \frac{2\|f\|_\infty}{\|f^+\|_\infty} \epsilon = \lambda(D) - \left(1 + \frac{2\|f\|_\infty}{\|f^+\|_\infty}\right) \epsilon
\geq \lambda(D) - \left(1 + 2\|f\|_\infty \max\left\{\frac{1}{\|f^+\|_\infty}, \frac{1}{\|f^-\|_\infty}\right\}\right) \epsilon.
$$The case $\tilde{\epsilon} < 0$ follows by replacing $f$ by $-f$. This completes the proof. \H

The next lemma shows that if we have a compact set $K$ and a continuous function $f$ on $K$ satisfying assumptions (i) and (ii) of Theorem 2.1 then for any $0 < c < \lambda(K)$ we can find a compact subset $E$ of $K$ such that $\lambda(K \setminus E) = c$, $\int_E f \, d\lambda = 0$, and $E$ contains no endpoint of $K$. This is later used in the proof of Lemma 2.5.

**Lemma 2.3.** Let $K \subseteq [0, 1]$ be a compact measurable set with $\lambda(K) > 0$ and let $f \in L_\infty(K)$ be a continuous, real-valued, mean zero function satisfying assumptions (i) and (ii) of Theorem 2.1. For every $c \in (0, \lambda(K))$, there is a compact subset $E \subseteq K \cap (\inf K, \sup K)$ with $\lambda(E) = \lambda(K) - c$ and $\int_E f \, d\lambda = 0$.

**Proof.** If $f = 0$ we can simply take $E = K \cap [\inf K + r, \sup K - r]$ for some $r > 0$ such that $\lambda(E) = \lambda(K) - c$. Hence, without loss of generality, we assume that $\lambda(f^{-1}(\{0\})) = 0$. Setting $u^+ = \lambda(\{f^+ > 0\})$
we observe that $v^\pm > 0$ since $f$ has mean zero and does not vanish identically. Now for $0 \leq r \leq v^+$, we set

$$A_r = \{f > 0\} \cap ([0, \inf K + a_r) \cup (\sup K - a_r, 1]),$$

where $a_r \geq 0$ satisfies $\lambda(A_r) = r$. Furthermore, for $0 \leq r \leq v^-$, let

$$B_r = \{f < 0\} \cap ([0, \inf K + b_r) \cup (\sup K - b_r, 1]),$$

where $b_r \geq 0$ satisfies $\lambda(B_r) = r$. We have $A_{r_1} \subset A_{r_2}$ and $B_{r_1} \subset B_{r_2}$ whenever $r_1 < r_2$. Let $F_\pm : [0, v^\pm] \to \mathbb{R}^+$ be given by

$$F_+(r) = \int_{A_r} f \, d\lambda \quad \text{and} \quad F_-(r) = -\int_{B_r} f \, d\lambda.$$

The functions $F_\pm$ are continuous, strictly increasing and satisfy

$$F_+(0) = F_-(0) = 0 \quad \text{and} \quad F_+(v^+) = F_-(v^-)$$

because $f$ has mean zero on $K$.

Let $G : [0, v^+] \times [0, v^-] \to \mathbb{R}$ be given by $G(t, r) = F_-(r) - F_+(t)$ so that

$$G(t, 0) = F_-(0) - F_+(t) \leq 0$$

and

$$G(t, v^-) = F_-(v^-) - F_+(t) = F_+(v^+) - F_+(t) \geq 0$$

for all $t \in [0, v^+]$. The function $G(t, \cdot)$ is continuous, and therefore there is $0 \leq x \leq v^-$ with $G(t, x) = 0$. Further, since $G(t, \cdot)$ is strictly increasing, the value of $x$ is uniquely determined. Now we can define

$$H : [0, v^+] \to [0, v^-]$$

by letting $H(t) \in [0, v^-]$ be unique such that $G(t, H(t)) = 0$.

For $t \in [0, v^+]$ we now have $F_-(H(t)) - F_+(t) = 0$, or equivalently

$$\int_{A_t} f \, d\lambda + \int_{B_{H(t)}} f \, d\lambda = 0.$$ 

Hence, setting

$$E_t = K \setminus (A_t \cup B_{H(t)}),$$

we obtain a compact set (since $A_t$ and $B_{H(t)}$ are open in $K$) such that

$$\int_{E_t} f \, d\lambda = 0, \quad \forall t \in [0, v^+].$$

Observing that $A_t \cup B_{H(t)} \subset K$, we have

$$\lambda(E_t) = \lambda(K) - \lambda(A_t \cup B_{H(t)}) = \lambda(K) - (\lambda(A_t) + \lambda(B_{H(t)})) = \lambda(K) - (t + H(t)).$$

The function $G$ is continuous, strictly decreasing in $t$ and strictly increasing in $r$, therefore $H$ is continuous and strictly increasing. Observing that $H(0) = 0$ and

$$v^+ + H(v^+) = v^+ + v^- = \lambda(K)$$
(since $f$ is non-zero almost everywhere on $K$), we infer that there exists $t_0 \in (0, \upsilon^+)$ such that $t_0 + H(t_0) = c$. Setting $E = E_{t_0}$ we have

$$\lambda(E) = \lambda(K) - c \quad \text{and} \quad \int_E f \, d\lambda = 0.$$ 

It remains to verify that $E \subseteq K \cap (\inf K, \sup K)$.

Let $a = \min(a_{t_0}, b_{t_0})$. Then $E \cap \{f \neq 0\}$ intersects neither $(\inf K, \inf K + a)$ nor $(\sup K - a, \sup K)$. Due to the assumption $\lambda(\{f = 0\}) = 0$, we may replace $E$ with $E' = E \cap [\inf K + a, \sup K - a]$. Then $E \setminus E' \subseteq \{f = 0\}$ is a null set, and $E'$ is a compact set avoiding the points $\inf K$ and $\sup K$. This completes the proof. ■

2.2. Rational splitting of the set $K$. In this subsection, we shall assume that the compact set $K \subset [0, 1]$ is represented as a union of two disjoint compact sets $K_1$ and $K_2$. We show that in this case the subset $E$ in the preceding lemma can be chosen such that the numbers $\lambda(E \cap K_1)/\lambda(E)$ and $\lambda(E \cap K_2)/\lambda(E)$ are both rational.

Lemma 2.4. Let $K_1$ and $K_2$ be compact subsets of $[0, 1]$ with $\lambda(K_1 \cap K_2) = 0$ and assume that $K = K_1 \cup K_2$ has positive measure. Let $f \in L_\infty(K)$ be a continuous, real-valued function satisfying assumptions (i) and (ii) of Theorem 2.1. Then for $\varepsilon > 0$ we can find a compact subset $E \subseteq K$ of positive measure such that $\lambda(E) \geq \lambda(K) - \varepsilon$, $\int_E f \, d\lambda = 0$ and

$$\frac{\lambda(E \cap K_1)}{\lambda(E)} = \frac{p}{q} \quad \text{for some integers } p \geq 0 \text{ and } q \geq 1.$$

Proof. We can assume that $\lambda(K_1), \lambda(K_2) > 0$ since otherwise we can take $E = K$. Further, if $f = 0$ we simply set $E = (K_1 \cap [0, r]) \cup K_2$ where $0 \leq r \leq 1$ is chosen to satisfy $\lambda(E) \geq \lambda(K) - \varepsilon$ and

$$\frac{\lambda(K_1 \cap [0, r])}{\lambda(K_1 \cap [0, r]) + \lambda(K_2)} = \frac{\lambda(E \cap K_1)}{\lambda(E)} \in \mathbb{Q}.$$ 

Without loss of generality, we can assume that $f \neq 0$. Writing

$$f_1 = f|_{K_1} \quad \text{and} \quad f_2 = f|_{K_2},$$

we observe that $f_1, f_2 \neq 0$. Hence, letting

$$u_1^+ = \lambda(\{f_1^+ > 0\}) \quad \text{and} \quad u_2^+ = \lambda(\{f_2^+ > 0\}),$$

we see that $u_1^+ > 0$ or $u_1^- > 0$, and likewise $u_2^+ > 0$ or $u_2^- > 0$, because $f_1, f_2 \neq 0$. Now if $u_1^+ > 0$ and $u_2^+ > 0$ then also $u_1^- > 0$ or $u_2^- > 0$, again because $f$ has mean zero. Hence we can assume that $u_1^+, u_2^- > 0$, the case $u_1^-, u_2^+ > 0$ being similar by changing the roles of $K_1$ and $K_2$.

If $\lambda(\{f_1 \geq \kappa\}) < u_1^+$, then we choose $\omega = \lambda(\{0 < f_1 < \kappa\}) > 0$. In this case, we consider the function

$$L : [0, \kappa] \to \mathbb{R} \quad \text{given by} \quad L(a) = \lambda(f_1^{-1}(0, a)).$$
It is non-decreasing with $L(0) = 0$ and $L(\kappa) = \omega$. Since $\lambda(f_1^{-1}(\{y\})) = 0$ for $y < \kappa$ we see that $L$ is continuous. Hence, for $r \in [0, \omega]$ we can find $0 \leq a_r \leq \kappa$ such that the set $A_r := f_1^{-1}(0, a_r)$ has $\lambda(A_r) = L(a_r) = r$. Moreover, we can choose the largest possible $a_r$ with this property, that is, 

$$a_r = \sup\{a : L(a) = r\}.$$ 

Clearly, $L(a_r) = r$ as $L$ is a continuous non-decreasing function.

If $\lambda(\{f_1 \geq \kappa\}) = v_1^+$, then we set 

$$\omega = v_1^+ \quad \text{and} \quad A_r := \{f_1 > 0\} \cap [0, a_r) \quad \text{for } 0 \leq r < \omega,$$

where we choose $0 \leq a_r \leq 1$ such that $\lambda(A_r) = r$. In both cases, $A_r$ is open in $K_1$.

Further, we define 

$$B_r = f_2^{-1}((-\infty, b_r)) \subseteq K_2 \quad \text{for } 0 \leq r \leq v_2^-,$$

where $b_r \leq 0$ is chosen to satisfy $\lambda(B_r) = r$ and $b_r = \sup\{b : L(b) = r\}$; such a choice of $b_r$ is possible thanks to the assumption that $\lambda(f^{-1}(\{y\})) = 0$ for all $y \leq 0$. We now have 

$$A_{r_1} \subset A_{r_2} \quad \text{and} \quad B_{r_1} \subset B_{r_2} \quad \text{whenever } r_1 < r_2,$$

which implies that 

$$(2.1) \quad a_{r_1} < a_{r_2} \quad \text{and} \quad b_{r_1} < b_{r_2} \quad \text{whenever } r_1 < r_2.$$ 

Let $F_1, F_2 : [0, \min\{\omega, v_2^-\}] \rightarrow \mathbb{R}^+$ be given by 

$$F_1(r) = \int_{A_r} f \, d\lambda \quad \text{and} \quad F_2(r) = -\int_{B_r} f \, d\lambda.$$

These are continuous, strictly increasing functions with $F_1(0) = F_2(0) = 0$. Let us compute their derivatives. When $A_r = \{f_1 > 0\} \cap [0, a_r)$, for $h > 0$ we have 

$$\frac{F_1(r + h) - F_1(r)}{h} = \frac{\int_{A_{r+h}\backslash A_r} f \, d\lambda}{h} = \frac{\kappa h}{h} = \kappa.$$ 

Hence, $F'_1(r) = \kappa$. On the other hand, when $A_r = f_1^{-1}(0, a_r)$,

$$\left| \frac{F_1(r + h) - F_1(r)}{h} - a_r \right| = \left| \frac{\int_{A_{r+h}\backslash A_r} (f - a_r) \, d\lambda}{h} \right| \leq \frac{\int_{A_{r+h}\backslash A_r} |f - a_r| \, d\lambda}{h} \leq |a_{r+h} - a_r|.$$ 

Since $a_r$ is maximal such that $\lambda(A_r) = r$, it follows that for any $\varepsilon > 0$ we have 

$$\lambda(f_1^{-1}(0, a_r + \varepsilon)) > \lambda(A_r) = r.$$ 

Hence, if $r < r + h < \lambda(f_1^{-1}(0, a_r + \varepsilon))$ we have $a_r < a_{r+h} < a_r + \varepsilon$, so that $|a_{r+h} - a_r| \rightarrow 0$ as $h \downarrow 0$. Hence, the right derivative $F'_1(r^+) = a_r$ is strictly
increasing. Now, we consider the right derivative of $F_2$. For $h > 0$ we have

$$\left| \frac{F_2(r+h) - F_2(r)}{h} + b_r \right| = \left| \frac{\int_{B_{r+h}\setminus B_r} (b_r - f) \, d\lambda}{h} \right| \leq \frac{\int_{B_{r+h}\setminus B_r} |b_r - f| \, d\lambda}{h} \leq |b_{r+h} - b_r|.$$

Since $b_r$ is maximal such that $\lambda(B_r) = r$, it follows that

$$\lambda(f_2^{-1}(-\infty, b_r + \varepsilon)) > \lambda(B_r) = r, \quad \forall \varepsilon > 0.$$ 

Hence if $r < r + h < \lambda(f_2^{-1}(-\infty, b_r + \varepsilon))$ we have $b_r < b_{r+h} < b_r + \varepsilon$, so that $|b_{r+h} - b_r| \to 0$ as $h \to 0$. This means that for the right derivative we have $F'_2(r^+) = -b_r$, which is strictly decreasing due to (2.1).

Now fix $0 < r_0 < \min\{\omega, v_2^-\}$ and choose $0 < t_0 < \min\{\omega, v_2^-\}$ with $0 < F_1(t_0) < F_2(r_0)$, which can be done since $F_1$ is continuous and $F_1, F_2$ are strictly increasing (indeed, since $\lambda(A_r)$ is strictly increasing and since $f|_{A_r} > 0$ we infer that $F_1(r) = \int_{A_r} f \, d\lambda$ is also strictly increasing; the argument for $F_2$ is the same).

Let

$$G : [0, t_0] \times [0, r_0] \to \mathbb{R} \quad \text{be given by} \quad G(t, r) = F_2(r) - F_1(t)$$

so that

$$G(t, 0) = F_2(0) - F_1(t) \leq 0 \quad \text{and} \quad G(t, r_0) = F_2(r_0) - F_1(t) > 0 \quad \text{for} \ t \in [0, t_0].$$

Observe that $G(t, \cdot)$ is continuous, and therefore there exists $0 \leq x < r_0$ such that $G(t, x) = 0$. Further, since $G(t, \cdot)$ is strictly decreasing, this $x$ is unique. Now define $H : [0, t_0] \to [0, r_0]$ by letting $H(t)$ be the unique value in $[0, r_0)$ satisfying $G(t, H(t)) = 0$.

For $t \in [0, t_0]$ we now have

$$(2.2) \quad F_2(H(t)) - F_1(t) = 0,$$

or equivalently

$$\int_{A_t} f \, d\lambda + \int_{B_{H(t)}} f \, d\lambda = 0.$$

Let us set

$$E_t = K \setminus (A_t \cup B_{H(t)}).$$

It is a compact set since $A_t, B_{H(t)}$ are open in $K_1, K_2$ respectively, and combining the assumption and the preceding display we have

$$\int_{E_t} f \, d\lambda = \int_K f \, d\lambda - 0 = 0.$$

Now $\lambda(E_t) = \lambda(K) - (H(t) + t)$ (indeed, by construction $\lambda(A_t) = t, \lambda(B_{H(t)}) = H(t)$, and the sets $A_t$ and $B_{H(t)}$ are disjoint and sit inside $K$).
Since $G$ is continuous, and strictly increasing in the first variable $r$ and strictly decreasing in the second variable $t$, it follows that $H$ is continuous and strictly increasing. Hence, recalling that $H(0) = 0$, we can select $t_1 > 0$ so small that

$$\lambda(E_{t_1}) \geq \lambda(K) - \varepsilon,$$

that is, $H(t_1) + t_1 \leq \varepsilon$.

We let

$$R(t) = \frac{\lambda(E_t \cap K_1)}{\lambda(E_t)} = \frac{\lambda(K_1) - t}{\lambda(K) - (t + H(t))} \quad \text{for } t \in [0, t_1].$$

We have

$$R(0) = \frac{\lambda(K_1)}{\lambda(K)}.$$

Now, suppose that $R$ is constant on $[0, t_1]$. In this case, substituting to the left hand side of (2.3) the value of $R(0)$ and solving for $H(t)$, we obtain

$$H(t) = \frac{\lambda(K) - \lambda(K_1)}{\lambda(K_1)} t, \quad H'(t) = \frac{\lambda(K) - \lambda(K_1)}{\lambda(K_1)}, \quad t \in [0, t_1].$$

Further, recalling (2.2), we have $F_2(H(t)) = F_1(t)$ so that differentiating this equality from the right yields

$$F'_2(H(t))H'(t) = F'_1(t), \quad t \in [0, t_1].$$

Since $F'_2 \circ H$ is strictly decreasing and $F'_1$ is strictly increasing (see (2.1)), and since $H'$ is a positive constant, this yields a contradiction with the equality

$$F'_2(H(t))H'(t) = F'_1(t).$$

This contradiction shows that $R$ is not constant on $[0, t_1]$.

Hence, since $R$ is continuous we can find a $t_2 \in [0, t_1]$ such that $R(t_2) = p/q$ for some integer $p \geq 0$ and some positive integer $q$. Setting $E = E_{t_2}$, we obtain

$$\lambda(E) \geq \lambda(K) - \varepsilon, \quad \int_E f \, d\lambda = 0, \quad \frac{\lambda(E \cap K_1)}{\lambda(E)} = R(t_2) = \frac{p}{q}.$$
exists a compact subset

\[ E \subset K \cap (\inf K, \sup K) \quad \text{with} \quad \frac{\lambda(E \cap K^L)}{\lambda(E)} = \frac{p}{q} \]

for some integer \( p \geq 0 \) and some positive integer \( q \) and such that

\[ \lambda(E) = \lambda(K) - c \quad \text{and} \quad \int_E f \, d\lambda = 0. \]

**Proof.** If \( \lambda(K^L) = 0 \), we apply Lemma 2.3 to \( K^R \) and obtain a compact subset \( E \subset K \cap (\inf K, \sup K) \) such that

\[ \lambda(E) = \lambda(K) - c \quad \text{and} \quad \int_E f \, d\lambda = 0. \]

Moreover \( \lambda(E \cap K^L)/\lambda(K) = 0 \) so that \( E \) satisfies the assertion of the lemma. A similar argument holds when \( \lambda(K^R) = 0 \) via interchanging the roles of \( K^L \) and \( K^R \).

We can thus assume that \( \lambda(K^L), \lambda(K^R) > 0 \). Now we apply Lemma 2.4 when

\[ K = K^L \cup K^R \]

with \( f \) and \( c/2 \) to obtain a compact subset

\[ \tilde{K} \subseteq K \quad \text{with} \quad \int_{\tilde{K}} f \, d\lambda = 0 \quad \text{and} \quad \frac{\lambda(\tilde{K} \cap K^L)}{\lambda(\tilde{K})} = \frac{p}{q} \]

and

\[ \lambda(\tilde{K}) \geq \lambda(K) - c/2. \]

We set

\[ \tilde{K}^L = \tilde{K} \cap K^L \quad \text{and} \quad \tilde{K}^R = \tilde{K} \cap K^R. \]

We also set

\[ \Delta = \lambda(K) - \lambda(\tilde{K}) \]
so that $0 < \Delta < c$. Furthermore we have

$$\lambda(\tilde{K}^L) = \lambda(K^L) - \lambda(K^L \cap (K \setminus \tilde{K})) \geq \lambda(K^L) - \Delta$$

and likewise

$$\lambda(\tilde{K}^R) \geq \lambda(K^R) - \Delta.$$ 

By our choice,

$$0 < c - \Delta < \min\{\lambda(K^L) - \Delta, \lambda(K^R) - \Delta\} \leq \min\{\lambda(\tilde{K}^L), \lambda(\tilde{K}^R)\}.$$ 

Now set

$$(2.4) \quad h^L = f|_{\tilde{K}^L} - \frac{1}{\lambda(\tilde{K}^L)} \int_{\tilde{K}^L} f \, d\lambda \quad \text{and} \quad h^R = f|_{\tilde{K}^R} - \frac{1}{\lambda(\tilde{K}^R)} \int_{\tilde{K}^R} f \, d\lambda.$$ 

Since $f$ satisfies assumption (ii) of Theorem 2.1, so do $h^L$ and $h^R$. This shows that we can apply Lemma 2.3 to the set $\tilde{K}^L$, the function $h^L$ and the scalar $\frac{p}{q}(c - \Delta)$ and also to the set $\tilde{K}^R$, the function $h^R$ and the scalar $(1 - \frac{p}{q})(c - \Delta)$. This yields compact subsets

$$E^L \subseteq \tilde{K}^L \cap (\inf \tilde{K}^L, \sup \tilde{K}^L) \quad \text{and} \quad E^R \subseteq \tilde{K}^R \cap (\inf \tilde{K}^R, \sup \tilde{K}^R)$$

with

$$(2.5) \quad \lambda(E^L) = \lambda(\tilde{K}^L) - (c - \Delta)\frac{p}{q} \quad \text{and} \quad \lambda(E^R) = \lambda(\tilde{K}^R) - (c - \Delta)\left(1 - \frac{p}{q}\right)$$

and furthermore

$$\int_{E^L} h^L \, d\lambda = \int_{E^R} h^R \, d\lambda = 0.$$ 

Substituting the definitions of $h^L$ and $h^R$ from (2.4) into the equalities above, we arrive at

$$\int_{E^L} f \, d\lambda = \frac{\lambda(E^L)}{\lambda(\tilde{K}^L)} \int_{\tilde{K}^L} f \, d\lambda \quad \text{and} \quad \int_{E^R} f \, d\lambda = \frac{\lambda(E^R)}{\lambda(\tilde{K}^R)} \int_{\tilde{K}^R} f \, d\lambda.$$ 

Now, we define a compact set $E$ by setting

$$E = E^L \cup E^R \subseteq K \cap (\inf K, \sup K).$$
We have
\[
\int_E f\,d\lambda = \frac{\lambda(E_L)}{\lambda(K_L)} \int_{\tilde K_L} f\,d\lambda + \frac{\lambda(E_R)}{\lambda(K_R)} \int_{\tilde K_R} f\,d\lambda
\]
\[
= \left(1 - (c - \Delta)\frac{p/q}{\lambda(K_L)}\right) \int_{\tilde K_L} f\,d\lambda + \left(1 - (c - \Delta)\frac{1 - p/q}{\lambda(K_R)}\right) \int_{\tilde K_R} f\,d\lambda
\]
\[
= \left(1 - \frac{c - \Delta}{\lambda(K)}\right) \int_{\tilde K_L} f\,d\lambda + \left(1 - \frac{c - \Delta}{\lambda(K)}\right) \int_{\tilde K_R} f\,d\lambda
\]
\[
= \left(1 - \frac{c - \Delta}{\lambda(K)}\right) \int_{\tilde K_L \cup \tilde K_R} f\,d\lambda = 0.
\]
Furthermore
\[
\lambda(E) = \lambda(\tilde K_L) - (c - \Delta)\frac{p}{q} + \lambda(\tilde K_R) - (c - \Delta)\left(1 - \frac{p}{q}\right) = \lambda(\tilde K) - (c - \Delta)
\]
\[
= \lambda(K) - c.
\]
Finally, we claim that
\[
\frac{\lambda(E_L)}{\lambda(K_L)} = \frac{\lambda(E_R)}{\lambda(K_R)}.
\]
Indeed, by (2.5), we have
\[
\frac{\lambda(E_L)}{\lambda(K_L)} = \frac{\lambda(\tilde K_L) - (c - \Delta)p/q}{\lambda(K_L)} = \frac{\lambda(\tilde K) - (c - \Delta)}{\lambda(K)}
\]
due to the equality \(\lambda(\tilde K_L) = \lambda(\tilde K)p/q\). Similarly,
\[
\frac{\lambda(E_R)}{\lambda(K_R)} = \frac{\lambda(K) - (c - \Delta)}{\lambda(K)}.
\]
Hence, recalling that \(E \cap K_L = E_L\), we have
\[
\frac{\lambda(E_L)}{\lambda(E)} = \frac{\lambda(E_L)}{\lambda(E_L) + \lambda(E_R)} = \frac{\lambda(\tilde K_L)}{\lambda(\tilde K_L) + \lambda(\tilde K_R)} = \frac{\lambda(\tilde K)}{\lambda(\tilde K_L)} = \frac{p}{q}.
\]

2.3. Constructing towers of the sets \(K_a\). In this subsection, for a given set \(K\) satisfying the assumptions of Lemma 2.5 and for a given \(\varepsilon \in (0, \lambda(K))\), we shall build the measurable set \(C\) from step (3) of the proof of Theorem 2.1, which is later used in step (4) to construct a function \(g \in L_\infty(C)\) and a mod 0 measure preserving transformation \(T\) of \(C\) such that \(f|_C = g \circ T - g\).

Below, we shall use the following notation. Fix a sequence \((m_n)_{n=0}^\infty\) of natural numbers such that \(m_n \geq 2\). For every \(n \geq 1\) denote \(\mathcal{E}_n = \prod_{j=1}^n \{1, \ldots, m_j-1\}\). With every element \(a \in \mathcal{E}_n\), we shall link a measurable
set \( K_a \) (a subset of a fixed measurable set \( K \)) and consider the collection \( \{K_a\}_{a \in \mathcal{E}_n} \) for \( n \in \mathbb{N} \). Further, we denote

\[
C_n = \bigcup_{a \in \mathcal{E}_n} K_a \quad \text{and} \quad C = \bigcap_{n=1}^{\infty} C_n
\]

and define

\[
f_n = \sum_{a \in \mathcal{E}_n} \frac{1}{\lambda(K_a \cap C)} \int_{K_a \cap C} f \, d\lambda \cdot \chi_{K_a \cap C} \in L_\infty(C).
\]

**Lemma 2.6.** Suppose that the set \( K \subseteq [0,1] \) and the function \( f \) satisfy the assumptions of Lemma 2.5. Then for every \( \varepsilon \in (0, \lambda(K)) \) there exists a sequence \((m_n)_{n=0}^{\infty}\) of natural numbers as above such that the following properties hold for every \( n \geq 1 \):

1. For each \( a \in \mathcal{E}_n \) the set \( K_a \) is a compact subset of \([0,1]\). For \( a_1, a_2 \in \mathcal{E}_n \) distinct we have either \( \sup K_{a_1} \leq \inf K_{a_2} \) or \( \sup K_{a_2} \leq \inf K_{a_1} \).
2. If \( a \in \mathcal{E}_n \) and \( b \in \mathcal{E}_{n+1} \) with \( a_j = b_j \) for \( 1 \leq j \leq n \), then \( K_b \subseteq K_a \).
3. For \( a,b \in \mathcal{E}_n \) distinct the sets \( K_a \cap C \) and \( K_b \cap C \) are disjoint.
4. For \( a,b \in \mathcal{E}_n \) the sets \( K_a \) and \( K_b \) have the same positive measure \( M_n := \lambda(K_a) = \lambda(K_b) > 0 \). Moreover, \( \lambda(K_a \cap C) = \lambda(C)/|\mathcal{E}_n| \).
5. \( \int_{C_n} f \, d\lambda = 0 \) and \( \int_C f \, d\lambda = 0 \).
6. \( \lambda(C_n) \geq \lambda(K) - (1 - 2^{-n})\varepsilon > \lambda(K) - \varepsilon \) and \( \lambda(C) \geq \lambda(K) - \varepsilon \).
7. For every chain \( K_{c_1} \supseteq K_{c_2} \supseteq \cdots \) with \( c_j \in \mathcal{E}_j \) we have \( \text{diam}(K_{c_j}) \to 0 \) as \( j \to \infty \).
8. The set \( \{K_a \cap C : a \in \mathcal{E}_n \text{ for some } n \geq 1\} \) generates the Borel \( \sigma \)-algebra \( \mathcal{B}(C) \).
9. \( \|f_j - f\|_\infty \to 0 \) as \( j \to \infty \).

**Proof.** We will construct \((m_n)_{n=0}^{\infty}\) and \( \{K_a\}_{a \in \mathcal{E}_n} \) inductively. For convenience we first set \( \mathcal{E}_0 = \{\varepsilon\} \) where \( \varepsilon \) denotes the empty tuple, and we define \( K_\varepsilon = K \). We see that properties (1)–(6) hold for \( n = 0 \). Now fix \( n \geq 0 \) and suppose that the sets \( K_a \) with \( a \in \mathcal{E}_n \) have been defined so that (1)–(6) hold.

Fix \( a \in \mathcal{E}_n \) and set

\[
K_a^L := K_a \cap \left[ \inf K_a, \frac{\inf K_a + \sup K_a}{2} \right],
\]

\[
K_a^R := K_a \cap \left[ \frac{\inf K_a + \sup K_a}{2}, \sup K_a \right].
\]

Observe that \( \text{diam}(K_a^L) \leq \frac{1}{2} \text{diam}(K_a) \) and \( \text{diam}(K_a^R) \leq \frac{1}{2} \text{diam}(K_a) \). Choose
\[ \varepsilon_n > 0 \text{ with} \]
(2.6) \[ \varepsilon_n < \min \left\{ \frac{\varepsilon}{2n|\mathcal{E}_n|}, M_n \right\}, \]
(2.7) \[ \varepsilon_n < \min \{\lambda(K_c^L), \lambda(K_c^R)\} \text{ for } c \in \mathcal{E}_n \text{ for which } \lambda(K_c^L), \lambda(K_c^R) > 0. \]
Further set
\[
h_a = f - \frac{1}{\lambda(K_a)} \int_{K_a} f \, d\lambda \quad \text{and} \quad \kappa_a = \kappa - \frac{1}{\lambda(K_a)} \int_{K_a} f \, d\lambda.
\]
Then \( h_a \) is a continuous function on \( K \) with \( h_a|_{\{h_a \geq \kappa_a\}} = \kappa_a \) constant. Hence, by the choice of \( \varepsilon_n \), we can now apply Lemma 2.5 to the set \( K_a = K_a^L \cup K_a^R \), the function \( h_a \) and the scalar \( \varepsilon_n \) to obtain a compact subset \( \tilde{K}_a \subseteq K_a \) with \( \lambda(\tilde{K}_a) = \lambda(K_a) - \varepsilon_n, \)
\[ \int_{\tilde{K}_a} h_a = 0 \text{ and such that if we set } \tilde{K}_a^L = \tilde{K}_a \cap K_a^L \text{ and } \tilde{K}_a^R = \tilde{K}_a \cap K_a^R \text{ we have } \lambda(\tilde{K}_a^L)/\lambda(\tilde{K}_a) = p_a/q_a \text{ for some integer } p_a \geq 0 \text{ and positive integer } q_a. \]
Now, set
\[
m_n = 2 \prod_{a \in \mathcal{E}_n} q_a, \quad k_a = \frac{m_n p_a}{q_a}.
\]
We select points
\[
x_a^0 < x_a^1 < \cdots < x_a^{k_a} = \frac{\inf K_a + \sup K_a}{2} < \cdots < x_a^{m_n}
\]
in \( K_a \) so that for \( 1 \leq i \leq m_n \) the sets \( K_a^i := \tilde{K}_a \cap [x_a^{i-1}, x_a^i] \) all have equal measure,
\[
\lambda(K_a^i) = \frac{\lambda(\tilde{K}_a)}{m_n} = \frac{\lambda(K_a) - \varepsilon_n}{m_n} = \frac{M_n - \varepsilon_n}{m_n},
\]
and moreover
\[
K_a^i \subseteq \tilde{K}_a^L, \quad \forall i \leq k_a,
\]
\[
K_a^i \subseteq \tilde{K}_a^R, \quad \forall k_a < i \leq m_n.
\]
If \( b = (a_1, \ldots, a_n, i) \) with \( 1 \leq i \leq m_n \) then we define \( K_b = K_a^i. \)
Observe that the definitions of \( K_a^i \) and of \( K_b \) guarantee that assertions (1) and (2) of Lemma 2.6 hold.
This completes the construction.
Before we show that all the stated properties hold, we first give an intuitive idea of what we have done. We first shrank \( K_a \) to some compact subset \( \tilde{K}_a \) so as to have
\[
\frac{1}{\lambda(K_a)} \int_{K_a} f \, d\lambda = \frac{1}{\lambda(\tilde{K}_a)} \int_{\tilde{K}_a} f \, d\lambda
\]
and keep the ratio
\[
\frac{\lambda(K_a^L \cap \tilde{K}_a)}{\lambda(\tilde{K}_a)}
\]
rational. Thereafter we could choose \(m_n\), and divide the set \(K_a\) from left to right into sets \(K_i^a\). Due to the choice of \(m_n\), the subsets \(K_i^a\) are each contained either in \(K_a^L\) or in \(K_a^R\). Then, for \(b = (b_1, \ldots, b_n, i)\) we defined \(K_b\) to be some \(K_i^a\). This guarantees that \(\text{diam}(K_b) \leq \frac{1}{2}\text{diam}(K_a)\) and thus the diameter of the tower goes to zero, which is assertion (7).

It is also important to emphasize that Lemma 2.5 guarantees that the sets \(\tilde{K}_a^L\) and \(\tilde{K}_a^R\) do not contain \(\inf K_a\) or \(\sup K_a\). Therefore, \(\inf K_a, \sup K_a \notin \bigcup_{b \in E_{n+1}} K_b\), and thus the family \(\{K_a \cap C_{n+1} : a \in E_n\}\) consists of pairwise disjoint sets. This observation guarantees that assertion (3) of Lemma 2.6 holds.

\[
K_a = K_a^L \cup K_a^R
\]
\[
\tilde{K}_a = \tilde{K}_a^L \cup \tilde{K}_a^R
\]
\[
K_{(a_1, \ldots, a_n)} \quad \ldots \quad \ldots \quad K_{(a_1, \ldots, a_n)_{m_n}}
\]
\[
K_{(a_1, \ldots, a_n, 1)} \quad \ldots \quad \ldots \quad K_{(a_1, \ldots, a_n, m_n)}
\]

Fig. 2. Visualisation of the subdivision of \(K_a\) for some \(a = (a_1, \ldots, a_n) \in E_n\). The lines mean that the lower set is included in the upper set. A subset \(K_b \subseteq K_a\) with \(b \in E_{n+1}\) is set equal to \(K_i^a\) for some \(1 \leq i \leq m_n\). Further, \(K_b\) is fully contained either in \(K_a^L\) or in \(K_a^R\), hence its diameter is less than half the diameter of \(K_a\). This ensures that the diameter of elements of \(n\)-level sets in the tower tends to zero as \(n \to \infty\).

Now we shall verify the remaining assertions of Lemma 2.6 (recall that we have verified (1)–(3) and (7) above).

(4) Fix \(n \geq 0\) and observe that all sets \(\{K_g\}_{g \in E_n}\) have equal positive measure \(M_n\). Choose \(a, b \in E_n\) and \(c, d \in E_{n+1}\) such that \(K_c \subseteq K_a\) and \(K_d \subseteq K_b\). Then
\[
\lambda(K_c) = \frac{M_n - \varepsilon_n}{m_n} = \lambda(K_d).
\]
Further, since \(\varepsilon_n < M_n\) this measure is positive. Hence, by induction all sets
\{K_a\}_{a \in \mathcal{E}_{n+1}} have equal positive measure. Finally, for any \(a \in \mathcal{E}_n\),

\[
\lambda(K_a \cap C) = \lim_{N \to \infty} \lambda(K_a \cap C_N) = \lim_{N \to \infty} \frac{\lvert \mathcal{E}_N \rvert}{\lvert \mathcal{E}_n \rvert} M_N = \lim_{N \to \infty} \frac{\lambda(C_N)}{\lambda(C)} = \frac{\lambda(C)}{\lambda(C)}.
\]

(5) For \(n = 0\), we have \(\int_{C_0} f \, d\lambda = \int_K f \, d\lambda = 0\). Now choosing an integer \(n \geq 0\) and assuming that \(\int_{C_n} f \, d\lambda = 0\), we have (the second equality below follows from (2.8)):

\[
\int_{C_{n+1}} f \, d\lambda = \sum_{a \in \mathcal{E}_n} \int_{\widetilde{K}_a} f \, d\lambda = \sum_{a \in \mathcal{E}_n} \frac{\lambda(\widetilde{K}_a)}{\lambda(K_a)} \int_{K_a} f \, d\lambda
\]

\[
= \sum_{a \in \mathcal{E}_n} \frac{\lambda(K_a) - \varepsilon_n}{\lambda(K_a)} \int_{K_a} f \, d\lambda
\]

\[
= \frac{M_n - \varepsilon_n}{M_n} \sum_{a \in \mathcal{E}_n} \int_{K_a} f \, d\lambda = \frac{M_n - \varepsilon_n}{M_n} \int_{C_n} f \, d\lambda = 0.
\]

Hence inductively \(\int_{C_n} f \, d\lambda = 0\) for all \(n \in \mathbb{N}\). Moreover we have \(\lvert \int_{C_n} f \, d\lambda \rvert \leq \lvert \int_{C_n} f \, d\lambda \rvert + \lambda(C_n \setminus C) \lVert f \rVert_\infty = \lambda(C_n \setminus C) \lVert f \rVert_\infty \to 0\) as \(n \to \infty\).

(6) We have \(\lambda(C_0) = \lambda(K)\). Now, choose \(n \geq 0\) and assume \(\lambda(C_n) \geq \lambda(K) - (1 - 2^{-n})\varepsilon\). Then for \(a \in \mathcal{E}_n\) we have \(\lambda(K_a) \geq \frac{\lambda(K) - (1 - 2^{-n})\varepsilon}{\lvert \mathcal{E}_n \rvert}\). Now for \(K_b \subseteq K_a\) with \(b \in \mathcal{E}_{n+1}\) we have (due to the assumption on \(\varepsilon_n\))

\[
M_{n+1} = \lambda(K_b) = \frac{\lambda(\widetilde{K}_b)}{m_n} = \frac{\lambda(K_a) - \varepsilon_n}{m_n} \geq \frac{\lambda(K_a) - \varepsilon}{2^{n+1} \lvert \mathcal{E}_n \rvert m_n}
\]

\[
= \frac{\lambda(K) - (1 - 2^{-n})\varepsilon}{m_n \lvert \mathcal{E}_n \rvert} - \frac{2^{-n-1}\varepsilon}{m_n \lvert \mathcal{E}_n \rvert} = \frac{\lambda(K) - (1 - 2^{-(n+1)})\varepsilon}{\lvert \mathcal{E}_{n+1} \rvert}.
\]

Hence

\[
\lambda(C_{n+1}) = \lvert \mathcal{E}_{n+1} \rvert M_{n+1} \geq \lambda(K) - (1 - 2^{-(n+1)})\varepsilon.
\]

This proves the first claim by induction. Furthermore,

\[
\lambda(C) = \inf_{n \in \mathbb{N}} \lambda(C_n) \geq \lambda(K) - \varepsilon.
\]

(8) We show that \(\mathcal{A} = \{K_a \cap C : a \in \mathcal{E}_a, n \geq 0\}\) generates \(\mathcal{B}(C)\). First of all, since for \(n \geq 0\) the sets \(\{K_a \cap C\}_{a \in \mathcal{E}_n}\) are compact, they are in \(\mathcal{B}(C)\). Now choose \(u \in [0, 1]\); we show that \([0, u) \cap C\) is generated by \(\mathcal{A}\). Let \(x \in [0, u) \cap C\). We have

\[
\text{diam}(K_a) \to 0, \quad a \in \mathcal{E}_n, \quad n \to \infty.
\]

Hence, there exist \(N \in \mathbb{N}\) and \(a_x \in \mathcal{E}_N\) such that \(K_{a_x}\) contains \(x\) and \(\sup K_{a_x} < u\). Let \(\mathcal{A}_0 := \{K_{a_x} \cap C : x \in C \cap [0, u]\} \subseteq \mathcal{A}\), which is countable, since every \(\mathcal{E}_n\) is finite. Then \(\bigcup_{A \in \mathcal{A}_0} A = C \cap [0, u)\) is generated by \(\mathcal{A}\), and since \(\{[0, u) \cap C : u \in [0, 1]\}\) generates \(\mathcal{B}(C)\), so does \(\mathcal{A}\).
Since \( f \) is continuous on \( K \) and \( K \) is compact, \( f \) is uniformly continuous on \( K \). Hence, for \( \varepsilon' > 0 \) we can find a \( \delta > 0 \) such that \( |f(x) - f(y)| < \varepsilon' \) whenever \( |x - y| < \delta \). Appealing to \( (2.9) \), we can find an \( N \in \mathbb{N} \) such for \( n \geq N \) we have \( \text{diam}(K_a) < \delta \) for all \( a \in \mathcal{E}_n \). Hence \( |f(x) - f(y)| < \varepsilon' \) for \( x, y \in K_a \) and \( a \in \mathcal{E}_n \). Now, for \( x \in K_a \cap C \),

\[
|f_n(x) - f(x)| = \left| \frac{1}{\lambda(K_a \cap C)} \int_{K_a \cap C} (f(t) - f(x)) dt \right| \\
\leq \frac{1}{\lambda(K_a \cap C)} \int_{K_a \cap C} |f(t) - f(x)| dt \leq \varepsilon'.
\]

Since this holds for all \( x \in C \) and \( \varepsilon' > 0 \), it follows that \( \|f_n - f|_C\|_\infty \to 0 \) as \( n \to \infty \).

Now, we shall pass to step (4) of the proof of Theorem 2.1 and do the construction of a function \( g \) and a mod 0 measure preserving transformation \( T \) of the set \( C \) constructed in Lemma 2.6.

**Lemma 2.7.** Suppose that the set \( K \subseteq [0,1] \) and the function \( f \) satisfy the assumptions of Lemma 2.5 (and Lemma 2.6). Take \( \varepsilon \in (0, \lambda(K)) \) and take the set \( C \) from Lemma 2.6. Then we can find a function \( g \in L_\infty(C) \) with \( \|g\|_\infty \leq (1 + \varepsilon)\|f\|_\infty \) and a mod 0 measure preserving transformation \( T \) of \( C \) such that \( f|_C = g \circ T - g \).

**Proof.** We shall use the notation introduced at the beginning of this subsection and in the formulation of Lemma 2.6. We let

\[
v_n : \mathcal{E}_n \to \{1, \ldots, |\mathcal{E}_n|\}
\]

be the function that arranges the elements in \( \mathcal{E}_n \) in lexicographical order. Further, for \( i \in \{1, \ldots, |\mathcal{E}_n|\} \) let

\[
I_i^n = K_{v_n^{-1}(i)} \cap C,
\]

which is compact (see Lemma 2.6(1)). Since \( \|f_n - f|_C\|_\infty \to 0 \) as \( n \to \infty \) (see Lemma 2.6(9)), there exists a sequence \( (n_k)_{k \geq 0} \) of natural numbers such that for \( n \geq n_k \) we have

\[
\|f_n - f|_C\|_\infty \leq 2^{-k-3}\varepsilon \|f\|_\infty.
\]

Setting

\[
h_k = f_{n_k} - f_{n_{k-1}}, \quad \text{so that} \quad \|h_k\|_\infty \leq 2^{-k-2}\varepsilon \|f\|_\infty, \quad k \geq 1,
\]

we have

\[
f = f_{n_0} + \sum_{k=1}^{\infty} h_k.
\]

Now, let \( a_i \) be the value of \( f_{n_0} \) taken on \( I_i^n \) for \( 1 \leq i \leq |\mathcal{E}_n| \). As \( \int_C f \, d\lambda = 0 \) we have \( \sum_{i=1}^{|\mathcal{E}_n|} a_i = 0 \) so that we can use Lemma 1.5 to ob-
tain a permutation \( \sigma \) of \( \{1, \ldots, |E_{n_0}|\} \) such that
\[
\sum_{i=1}^{m} a_{\sigma(i)} \leq \max\{a_i : 1 \leq i \leq |E_{n_0}|\} \leq \|f_{n_0}\|_\infty
\]
for \( 0 \leq m \leq |E_{n_0}| \). Denote by \( T_0 \) the mod 0 measure preserving cyclic transformation of \( C \) sending \( I_{\sigma(i)}^{n_0} \) to \( I_{\sigma(i+1)}^{n_0} \) for \( 1 \leq i \leq |E_{n_0}| - 1 \) and sending \( I_{\sigma(n)}^{n_0} \) to \( I_{\sigma(1)}^{n_0} \). Such a transformation exists by Theorem 1.3, since all sets \( I_i^{n_0} \) for \( i = 1, \ldots, |E_n| \) have equal measure. We now denote by \( g_0 : C \to \mathbb{R} \) the function taking the value
\[
\sum_{i=1}^{l-1} a_{\sigma(i)} \text{ on } I_{\sigma(l)} \text{ for } l = 2, \ldots, |E_{n_0}| \text{ and taking the value } 0 \text{ on } I_{\sigma(1)} \text{. Then }
\|g_0\|_\infty \leq \|f_{n_0}\|_\infty \leq \|f\|_\infty \text{ and for } l = 2, \ldots, |E_n| \text{ and } t \in I_{\sigma(i)} \text{ we have }
\]
\[
g_0(T_0(t)) - g_0(t) = \sum_{i=1}^{l} a_{\sigma(i)} - \sum_{i=1}^{l-1} a_{\sigma(i)} = a_{\sigma(l)} = f_{n_0}(t). \]
When \( l = 1 \) and \( t \in I_{\sigma(1)} \), we have \( g_0(T_0(t)) - g_0(t) = \sum_{i=1}^{l} a_{\sigma(i)} - 0 = a_{\sigma(1)} = f_{n_0}(t) \).

Using the same argument as in \([K]\), for each \( k \geq 1 \) we denote \( J_k = \{I_i^{n_k} : 1 \leq i \leq |E_{n_k}|\} \), and define a sequence \( \{T_k\}_{k=1}^\infty \) of mod 0 measure preserving transformations \( T_k \) of \( C \) and functions \( \{g_k\}_{k=1}^\infty \) with \( g_k \in L_\infty(C) \) such that:

(i) \( T_k \) is a cyclic rearrangement of the sets of \( J_k \).

(ii) \( T_{k+1} \) extends \( T_k \) in the sense that if \( I \in J_k \), \( I' \in J_{k+1} \) and \( I' \subseteq I \) then \( T_{k+1}(I') \subseteq T_k(I) \).

(iii) \( \|g_k\|_\infty \leq 4\|h_k\|_\infty \).

(iv) \( g_k \) is constant on each \( I \in J_k \).

(v) \( h_k = g_k \circ T_k - g_k \) on \( C \).

Suppose that the transformations \( T_1, \ldots, T_k \) and functions \( g_1, \ldots, g_k \) with the stated properties have already been defined. For convenience we set \( n = |J_k| \) and \( m = |J_{k+1}|/|J_k| \). Let \( I_1, \ldots, I_n \) be the sets from \( J_k \), enumerated so that \( T_k(I_i) = I_{i+1} \) when \( i < n \) and \( T_k(I_n) = I_1 \), which can be done since \( T_k \) is a cyclic rearrangement of the sets of \( J_k \). Furthermore, for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) denote by \( I_{i,j} \) all sets from \( J_{k+1} \) which are contained in \( I_i \). Denote by \( a_{i,j} \) the value of \( h_{k+1} \) on \( I_{i,j} \). Since
\[
\int_{I_i} h_{k+1} \, d\lambda = \sum_{j=1}^{m} \int_{I_{i,j}} (f_{n_{k+1}} - f_{n_k}) \, d\lambda = 0, \quad \forall I_i \in J_k,
\]
it follows that \( \sum_{j=1}^{m} a_{i,j} = 0 \) for all \( i = 1, \ldots, n \). In addition, \( |a_{i,j}| \leq \|h_{k+1}\|_\infty \) for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \). Therefore, by Lemma 1.4 there exist permutations \( \sigma_1, \ldots, \sigma_n \) of \( \{1, \ldots, m\} \) such that
\[
\left| \sum_{i=1}^{k} a_{i,\sigma_i(j)} \right| \leq 2\|h_{k+1}\|_\infty
\]
for \( k = 1, \ldots, n \) and \( j = 1, \ldots, m \). Define \( T_{k+1} \) by setting
\[
T_{k+1}(I_{i,\sigma_i(j)}) = I_{i+1,\sigma_{i+1}(j)}, \quad i = 1, \ldots, n-1, \quad j = 1, \ldots, m.
\]
We set
\[ b_j = \sum_{i=1}^{n} a_{i,\sigma_i(j)}, \quad j = 1, \ldots, m. \]

Since \( \sum_{j=1}^{m} b_j = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} = 0 \) and \( |b_j| \leq 2\|h_{k+1}\|_{\infty} \), Lemma 1.5 yields the existence of a permutation \( \sigma_0 \) of \( \{1, \ldots, m\} \) such that
\[
\left| \sum_{j=1}^{l} b_{\sigma_0(j)} \right| \leq 2\|h_{k+1}\|_{\infty}, \quad \forall l = 1, \ldots, m.
\]

Set
\[ T_{k+1}(I_{n,\sigma_n(\sigma_0(j))}) = I_{1,\sigma_1(\sigma_0(j)+1)}, \quad \forall j = 1, \ldots, m - 1, \]
and set
\[ T_{k+1}(I_{n,\sigma_n(\sigma_0(m))}) = I_{1,\sigma_1(\sigma_0(1))}. \]

Observe that \( T_{k+1} \) is a mod 0 measure preserving transformation due to Theorem 1.3 (taking into account that the sets \( I_{i,j} \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) are of equal positive measure).

Let us explain in a simpler language what we have just done. The matrix \( (a_{i,j})_{n \times m} \) is transformed into the matrix \( (a_{i,\sigma_i(\sigma_0(j))})_{n \times m} \) in such a way that in every column the modulus of the sum of the first \( k \) elements does not exceed \( 2\|h_{k+1}\|_{\infty} \) and the sum of the first \( l \) columns does not exceed \( 2\|h_{k+1}\|_{\infty} \). Next, we have built the transformation \( T_{k+1} \) which “scans” the matrix columnwise: the first column from top to bottom, then the second column from top to bottom, etc.

Hence,
\[
\left| \sum_{r=0}^{l-1} h_{k+1}(T_{k+1}^{r}(t)) \right| = \left| \sum_{j=1}^{p-1} b_{\sigma_0(j)} + \sum_{i=1}^{q} a_{i,\sigma_i(\sigma_0(p))} \right| \leq 4\|h_{k+1}\|_{\infty},
\]
where \( l+1 = (p-1)n+q \), for every \( t \in I_{1,\sigma_1(\sigma_0(1))} \) and every \( l = 0, \ldots, nm-1 \).

Now, let us define the function \( g_{k+1} \) by setting its value on \( T_{k+1}^{l}(I_{1,\sigma_1(\sigma_0(1))}) \) equal to \( \sum_{r=0}^{l-1} h_{k+1}(T_{k+1}^{r}(t)) \), where \( t \in I_{1,\sigma_1(\sigma_0(1))} \) for \( l = 1, \ldots, nm-1 \) and setting \( g_{k+1}(I_{1,\sigma_1(\sigma_0(1))}) = 0 \). Then \( \|g_{k+1}\|_{\infty} \leq 4\|h_{k+1}\|_{\infty} \).

Let \( t \in I_{1,\sigma_1(\sigma_0(1))} \). If \( 0 < l < nm - 1 \), then
\[
g_{k+1}(T_{k+1}(T_{k+1}^{l}(t))) - g_{k+1}(T_{k+1}^{l}(t)) = \sum_{r=0}^{l} h_{k+1}(T_{k+1}^{r}(t)) - \sum_{r=0}^{l-1} h_{k+1}(T_{k+1}^{r}(t)) = h_{k+1}(T_{k+1}^{l}(t)),
\]
and further
\[
g_{k+1}(T_{k+1}(t)) - g_{k+1}(t) = h_{k+1}(t) - 0 = h_{k+1}(t),
\]
finally yielding
\[
g_{k+1}(T_{k+1}(T_{k+1}^{mn-1}(t)))) - g_{k+1}(T_{k+1}^{mn-1}(t)) = 0 - \sum_{r=0}^{nm-2} h_{k+1}(T_{k+1}^r(t)) = h_{k+1}(T_{k+1}^{mn-1}(t)).
\]

Thus, for every \( t \in C \) we have
\[
g_{k+1}(T_{k+1}(t)) - g_{k+1}(t) = h_{k+1}(t).
\]

This completes the construction of the functions \( \{g_k\}_{k=1}^\infty \) and the transformations \( \{T_k\}_{k=1}^\infty \) with the required properties.

It follows from the construction that \( T_{k+1} \) satisfies condition (ii). Hence the sequences \( T_k \) and \( g_k \) satisfy (i)–(v). Observe that the inverse mappings \( T_k^{-1} \) also satisfy (ii).

It follows from (iii) that the series \( \sum_{k=0}^{\infty} g_k \) converges in \( L_\infty(C) \) to some function \( g \) and \( \|g\|_\infty \leq \|g_0\|_\infty + \sum_{k=1}^{\infty} \|g_k\|_\infty \leq \|f\|_\infty + \epsilon \|f\|_\infty = (1+\epsilon)\|f\|_\infty \).

Next, it follows from (ii) that for almost all \( t \in C \) the sequence \( T_k(t) \) is Cauchy and hence it converges. We then set \( T(t) = \lim_{k \to \infty} T_k(t) \in C \). Now, if \( x, y \in C \) with \( x \neq y \) then there is an \( N \) such that \( x \in I \) and \( y \in \tilde{I} \) for some \( I, \tilde{I} \in J_N \) with \( I \neq \tilde{I} \). Hence, \( T(x) \in T(I) \subseteq T_k(I) \) and \( T(y) \in T(\tilde{I}) \subseteq T_k(\tilde{I}) \) so that \( T(x) \neq T(y) \). Hence, \( T \) is injective.

Denote
\[ A_k = \bigcup_{I \in J_k} T_k^{-1}(I), \quad k \geq 1. \]

For each \( k \geq 1 \) the set \( A_k \) has full measure in \( C \). Hence, \( \bigcap_{k=1}^{\infty} A_k \) also has full measure in \( C \).

Now, let \( \omega \in \bigcap_{k=1}^{\infty} A_k \). Then, for \( k \geq 1 \) we can find \( I^k_\omega \in J_k \) with \( \omega \in I^k_\omega \). Therefore, for all such \( \omega \), the set \( \bigcap_{k=1}^{\infty} T_k^{-1}(I^k_\omega) \) is non-empty. Pick \( x \in \bigcap_{k=1}^{\infty} T_k^{-1}(I^k_\omega) \subseteq C \), so that \( T_k(x) \in I^k_\omega \) for all \( k \geq 1 \). Now, as \( \text{diam}(I^k_\omega) \to 0 \) (see Lemma 2.6(7)) we must have \( T(x) = \omega \). This means that \( T \) is a bijection between two subsets of \( C \) of full measure: one is \( \bigcap_{k=1}^{\infty} A_k \) and the other is its image under \( T \).

Let us verify that \( T \) is measure preserving. Fix \( k \geq 1 \). Then for \( I \in J_k \) we have \( T(I) = \tilde{I} \) for some \( \tilde{I} \in J_k \). Hence,
\[
\lambda(T(I)) = \lambda(\tilde{I}) = \frac{\lambda(C)}{|J_k|} = \lambda(I), \quad \forall I \in J_k.
\]

As \( \bigcup_{n=1}^{\infty} J_k \) generates the Borel \( \sigma \)-algebra on \( C \) (see Lemma 2.6(8)), this equality holds for all sets in \( \mathcal{B}(C) \). Thus \( T \) is a mod 0 measure preserving transformation of \( C \). Now, for \( k \geq 1 \) we have
\[
g_k(T(x)) - g_k(x) = g_k(T_k(x)) - g_k(x) = h_k(x).
\]
Hence, \[ g(T(x)) - g(x) = \sum_{k=0}^{\infty} g_k(T(x)) - g_k(x) = f_n + \sum_{k=1}^{\infty} h_k = f. \]

This completes the proof of Lemma 2.7. \[ \blacksquare \]

2.4. Completing the proof of Theorem 2.1 We can now complete the proof of Theorem 2.1. Let \( K \subseteq [0,1] \) and \( f \) satisfy the assumptions of Theorem 2.1. Further choose \( \varepsilon > 0 \). We will define a mod 0 measure preserving transformation \( T \) of \( K \) and a function \( g \in L_\infty(K) \) with \( \|g\|_\infty \leq (1 + \varepsilon)\|f\|_\infty \) such that \( f = g \circ T - g \). We will do this by considering a family \( \{A_i\}_{j \in J} \) of measurable subsets of \( K \), where \( J \) is some index set, such that:

1. \( A_i, A_j \) are disjoint for distinct \( i, j \in J \).
2. \( \int_{A_j} f \, d\lambda = 0 \) and \( \lambda(A_j) > 0 \) for all \( j \in J \).
3. \( f|_{A_j} \) is continuous for all \( j \in J \).
4. For each \( j \in J \) we can find a function \( g_j \in L_\infty(A_j) \) with \( \|g_j\|_\infty \leq (1 + \varepsilon)\|f\|_\infty \) and a mod 0 measure preserving transformation \( T_j \) of \( A_j \) such that \( f|_{A_j} = g_j \circ T_j - g_j \).

We partially order the set of such families \( \{A_j\}_{j \in J} \) by inclusion. Suppose that we have a chain \( \{\{A_i\}_{i \in I}, \{i \in I \} \) for some index set \( I \). Then if we let \( J = \bigcup_{i \in I} J_i \), we obtain an upper bound \( \{A_j\}_{j \in J} \) for the chain. Hence, by Zorn’s Lemma we can choose a family \( \{A_j\}_{j \in J} \) that is maximal. Now, let \( D = K \setminus \bigcup_{j \in J} A_j \) and suppose \( \lambda(D) > 0 \). We have \( \int_D f \, d\lambda = \int_K f \, d\lambda - \sum_{j \in J} \int_{A_j} f \, d\lambda = 0 \). We set

\[ \tau^\pm = \lambda(\{f|_D \geq \frac{1}{2}\|f|_D\}\}), \quad z = 1 + 2\|f|_D\|_\infty \max\left\{\frac{1}{\|f|_D^+\|_\infty}, \frac{1}{\|f|_D^-\|_\infty}\right\}. \]

We choose \( \varepsilon_1 > 0 \) with

\[ \varepsilon_1 < \min\left\{\frac{\lambda(D)}{z}, \tau^+, \frac{\|f^+D\|_\infty}{\|f\|_\infty}, \tau^- \frac{\|f^-D\|_\infty}{\|f\|_\infty}\right\}. \]

Now, we apply Luzin’s Theorem 1.2 to get a compact set \( E \subseteq D \) with \( \lambda(E) \geq \lambda(D) - \varepsilon_1 \) such that \( f \) is continuous on \( E \). By the bound on \( \varepsilon_1 \) we can apply Lemma 2.2 on \( E \subseteq D \) with \( f|_D \) and \( \varepsilon_1 \) to select a compact subset \( K \subseteq E \) with \( \lambda(K) \geq \lambda(D) - \varepsilon_1 > 0 \) such that \( \int_K f \, d\lambda = 0 \). Applying Lemma 2.6 together with Lemma 2.7 to \( K \), \( f \) and \( \min\{\varepsilon_1, \frac{1}{2}\lambda(K)\} \) we obtain a compact subset \( C \subseteq K \) with \( \lambda(C) \geq \lambda(K) - \frac{1}{2}\lambda(K) > 0 \) such that \( \int_C f \, d\lambda = 0 \), a function \( g \in L_\infty(C) \) with \( \|g\|_\infty \leq (1 + \varepsilon)\|f\|_\infty \) and a mod 0 measure preserving transformation \( T \) of \( C \) such that \( f|_C = g \circ T - g \). We now see that \( \{C\} \cup \{A_j\}_{j \in J} \) has properties (1)–(4) above so that \( \{A_j\}_{j \in J} \) is not maximal, which is a contradiction. We conclude that \( \lambda(D) = 0 \).
Having established that \( K \setminus \bigcup_{j \in \mathcal{J}} A_j \) has measure zero, we can define a final transformation \( T \) of \( K \) as \( T|_{A_j} = T_j \) and \( T(x) = x \) for \( x \in K \setminus \bigcup_{j \in \mathcal{J}} A_j \), and likewise define a function \( g \) as \( g|_{A_j} = g_j \). Then \( T \) is a mod 0 measure preserving transformation of \( K \) and \( g \in L_\infty(K) \) with \( \|g\|_\infty = \sup \{\|g_j\| : j \in \mathcal{J}\} \leq (1 + \varepsilon)\|f\|_\infty \) and \( f = g \circ T - g \), which completes the proof of Theorem 2.1.

3. Kwapieni’s Theorem for elementary functions. In this section we prove Theorem 0.1 for mean zero functions taking only countably many values. More precisely, we establish the following result.

**Theorem 3.1.** Let \( K \subseteq [0, 1] \) be measurable. Let \( f \in L_\infty(K) \) be a mean zero real-valued function taking at most countably many values. Then there exists a mod 0 measure preserving transformation \( T \) of \( K \) and a function \( g \in L_\infty(K) \) with \( \|g\|_\infty \leq \|f\|_\infty \) such that \( f = g \circ T - g \).

**Proof.** We will consider several cases, increasing the level of generality.

1. Let \( a \in [0, 1] \) and \( f_a = (1 - a)\chi_{[0,a]} - a\chi_{[a,1)} \). Clearly, \( \int_0^1 f_a \, d\lambda = 0 \) and \( \|f_a\|_\infty = \max(a, 1 - a) \geq 1/2 \). Set \( g(t) = t - 1/2 \), \( T(t) = \{t - a\} \) (by \( \{t\} \) we denote the fractional part of \( t \)). Then \( T \) is measure preserving, \( g(T(t)) - g(t) = f_a(t) \) for all \( t \in [0,1] \) and \( \|g\|_\infty = 1/2 \leq \|f_a\|_\infty \).

2. Let now \( a \in [0,1] \), \( f = \alpha \chi_{[0,a]} + \beta \chi_{[a,1]} \) and \( \int_0^1 f \, d\lambda = 0 \). Then \( \alpha a + \beta (1 - a) = 0 \) and \( f = \frac{\alpha}{1 - a} f_a \). Therefore, this case can be reduced to the preceding one.

3. Let \( a, b \in [0,1] \), \( a < b \), \( f = \alpha \chi_{[0,a]} + \beta \chi_{[a,b]} \) and \( \int_0^1 f \, d\lambda = 0 \). This case can be reduced to the preceding one as follows. We define \( \tilde{f}(t) = f(bt) \) so that \( \tilde{f} = \alpha \chi_{[0,a/b]} + \beta \chi_{[a/b,1]} \) and \( \int_0^1 \tilde{f} \, d\lambda = 0 \). Hence we find \( \tilde{g} \) with \( \|\tilde{g}\|_\infty \leq \|\tilde{f}\|_\infty \) and a measure preserving transformation \( \tilde{T} \) of \( [0,1] \) such that \( \tilde{f} = \tilde{g} \circ \tilde{T} - \tilde{g} \).

Now define \( g(t) = \tilde{g}(t/b) \) and \( T(t) = b\tilde{T}(t/b) \) for \( t \leq b \) and \( g(t) = 0 \) and \( T(t) = t \) for \( t > b \). Then for \( t \leq b \) we get \( f(t) = \tilde{f}(t/b) = \tilde{g}(\tilde{T}(t/b)) - \tilde{g}(t/b) = g(T(t)) - g(t) \) and for \( t > b \) we obtain \( f(t) = 0 = g(T(t)) - g(t) \). Moreover, \( \|g\|_\infty = \|\tilde{g}\|_\infty \leq \|\tilde{f}\|_\infty = \|f\|_\infty \).

4. Let \( A, B \) be disjoint measurable sets. Let \( f = \alpha \chi_A + \beta \chi_B \in L_\infty[0,1] \) have mean zero. We set \( C = [0,1] \setminus (A \cup B) \). By Theorem 1.3 there exists a mod 0 measure preserving transformation \( S \) of \( [0,1] \) such that \( S(A) = [0, \lambda(A)) \), \( S(B) = [\lambda(A), \lambda(A) + \lambda(B)] \), \( S(C) = (\lambda(A) + \lambda(B), 1] \).

Letting \( \tilde{f} = f \circ S^{-1} \), we obtain

\[
\tilde{f} = \alpha \chi_{[0,\lambda(A))] + \beta \chi_{[\lambda(A),\lambda(A)+\lambda(B)]}.
\]

Hence, appealing to the preceding case, we find \( \tilde{g} \in L_\infty[0,1] \) with \( \|\tilde{g}\|_\infty \leq \|\tilde{f}\|_\infty \) and a measure preserving transformation \( \tilde{T} \) of \( [0,1] \) such that
\( \tilde{f} = \tilde{g} \circ T - \tilde{g} \). Furthermore, \( \tilde{T} \) is the identity on \([\lambda(A) + \lambda(B), 1]\). Now define \( T = S^{-1} \circ \tilde{T} \circ S \) which is a mod 0 measure preserving transformation of \([0, 1]\) and define \( g = \tilde{g} \circ S \in L_\infty[0, 1] \). We have \( f = \tilde{f} \circ S = g \circ T - g \).

Moreover \( \|g\|_\infty = \|\tilde{g}\|_\infty \leq \|\tilde{f}\|_\infty = \|f\|_\infty \). Further noting that \( T \) is the identity on \([0, 1] \setminus (A \cup B) \) we can also consider \( T \) as a mod 0 measure preserving transformation of \( A \cup B \). Hence \( f|_{A \cup B} = g|_{A \cup B} \circ T|_{A \cup B} - g|_{A \cup B} \).

5. Let \( K \subseteq [0, 1] \) be measurable. Let \( f \in L_\infty(K) \) have mean zero and take at most countably many values. Without loss of generality, we write \( f = \sum_{i=1}^\infty \alpha_i \chi_{A_i} \) for some scalars \( \alpha_i \in \mathbb{R} \setminus \{0\} \) and some pairwise disjoint measurable sets \( A_i \subseteq K \).

Let \( \{B_j^+, B_j^-\}_{j \in J} \) be a collection of pairs with \( J \) being a countable index set, such that all sets \( B_j^+, B_j^- \) are disjoint, of positive measure and for each \( j \in J \) there exist \( i_1, i_2 \in \mathbb{N} \) with \( B_j^+ \subseteq A_{i_1} \) and \( B_j^- \subseteq A_{i_2} \) such that \( \alpha_{i_1} > 0 > \alpha_{i_2} \) and \( \alpha_{i_1} \lambda(B_j^+) + \alpha_{i_2} \lambda(B_j^-) = 0 \).

Now, consider the set of all such collections partially ordered by inclusion. Suppose we have some chain \( \{\{B_j^+, B_j^-\}_{j \in J_i}\}_{i \in I} \), where \( I \) is some index set. Then if we set \( J = \bigcup_{i \in I} J_i \), we find an upper bound \( \{B_j^+, B_j^-\}_{j \in J} \) for the chain. An appeal to Zorn’s Lemma yields a maximal element in the set of collections, \( \{B_j^+, B_j^-\}_{j \in J} \) for some countable set \( J \). Let us set

\[ Z = \bigcup_{j \in J} (B_j^+ \cup B_j^-) \]

and suppose that \( \lambda(\text{supp} f \setminus Z) \neq 0 \).

Taking into account that \( \int_{B_j^+ \cup B_j^-} f \, d\lambda = 0 \) for all \( j \in J \), we infer that \( \int_{\text{supp} f \setminus Z} f \, d\lambda = 0 \), and hence we can select \( i_1, i_2 \in \mathbb{N} \) and sets \( B^+ \subseteq A_{i_1} \cap (\text{supp} f \setminus Z) \) and \( B^- \subseteq A_{i_2} \cap (\text{supp} f \setminus Z) \) of positive measure such that \( B^+, B^- \) are disjoint from all sets \( B_j^+, B_j^- \) for \( j \in J \), and \( \alpha_{i_1} > 0 > \alpha_{i_2} \). This means that we can find \( B^+ \subseteq B_j^+ \) and \( B^- \subseteq B_j^- \) of positive measure such that \( \alpha_{i_1} \lambda(B^+) + \alpha_{i_2} \lambda(B^-) = \int_{B^+ \cup B^-} f \, d\lambda = 0 \). But this means that the collection \( \{B_j^+, B_j^-\}_{j \in J} \) is not maximal, which is a contradiction. Hence

\[ \lambda(\text{supp} f \setminus Z) = 0. \]

Thus, by referring to the preceding case, for each \( j \in J \) we can find a \( g_j \in L_\infty(B_j^+ \cup B_j^-) \) with \( \|g_j\|_\infty \leq \|f\|_\infty \) and a mod 0 measure preserving transformation \( T_j \) of \( B_j^+ \cup B_j^- \) such that \( f|_{B_j^+ \cup B_j^-} = g_j \circ T_j - g_j \) on \( B_j^+ \cup B_j^- \).

Hence, defining \( T \) and \( g \) as \( T_j \) and \( g_j \) respectively on \( B_j^+ \cup B_j^- \) for \( j \in J \) and setting \( T(x) = x \) and \( g(x) = 0 \) for \( x \) in the null set \( K \setminus Z \) yields a mod 0 measure preserving transformation \( T \) of \( K \) and a function \( g \in L_\infty(K) \) satisfying \( f = g \circ T - g \). Moreover \( \|g\|_\infty \leq \sup_{j \in J} \|g_j\|_\infty \leq \|f\|_\infty \). The proof of Theorem 3.1 is complete.
4. Completing the proof of Theorem 0.1. We can now complete the proof of Kwapięń’s Theorem 0.1. For convenience, we restate it below.

**Theorem 4.1.** Let \( f \in L_\infty[0, 1] \) be a real-valued mean zero function. For any \( \varepsilon > 0 \) there exists a mod 0 measure preserving transformation \( T \) of \([0, 1]\) and a function \( g \in L_\infty[0, 1] \) with \( \|g\|_\infty \leq (1+\varepsilon)\|f\|_\infty \) such that \( f = g \circ T - g \).

**Proof.** We will partition \([0, 1]\) into subsets on which \( f \) has mean zero. Set 
\[
\tilde{D}' = \{ y \in \mathbb{R} : \lambda(f^{-1}(\{y\})) > 0 \} \quad \text{and} \quad D = f^{-1}(D').
\]
The function \( f \) takes only countably many values on \( D \), since every value on \( D \) is taken on a set of positive measure. We set 
\[
D^\pm = \{ f^\pm \geq 0 \} \cap D.
\]
We assume \( \int_D f \, d\lambda \geq 0 \); the case \( \int_D f \, d\lambda < 0 \) follows by considering \(-f\). Let 
\[
C = (D^+ \cap [0, R]) \cup D^- 
\]
for some \( 0 \leq R \leq 1 \) such that \( \int_C f \, d\lambda = 0 \). Further, we consider the sets 
\[
C' = [0, 1] \setminus C, \quad C_1 = C' \cap D, \quad C_2 = C' \setminus D.
\]
As \( C_1 \subseteq D \setminus D^- \) we have \( f|_{C_1} > 0 \) and hence \( \int_{C_1} f \, d\lambda \geq 0 \). Further, as \( C_2 \subseteq [0, 1] \setminus D \) we have \( \lambda(f|_{C_2}^{-1}(\{y\})) = 0 \) for all \( y \in \mathbb{R} \). As \( f \) has mean zero on \([0, 1]\) and on \( C \), we have 
\[
\int_{C_1} f \, d\lambda + \int_{C_2} f \, d\lambda = \int_{C'} f \, d\lambda = -\int_C f \, d\lambda = 0.
\]
Hence, \( \int_{C_2} f \, d\lambda \leq 0 \). We further denote 
\[
C_2^\pm = \{ f^\pm \geq 0 \} \cap C_2
\]
and define 
\[
B_0 = C_2^+ \cup (C_2^- \cap [0, R])
\]
for some \( 0 \leq R \leq 1 \) such that 
\[
\int_{B_0} f \, d\lambda = 0.
\]
Now, finally, we let 
\[
\tilde{C}_2 = C_2 \setminus B_0.
\]
As \( \tilde{C}_2 \subseteq C_2^- \setminus C_2^+ \) we have \( f|_{\tilde{C}_2} < 0 \). We further have 
\[
\int_{\tilde{C}_2} f \, d\lambda = \int_{C_2} f \, d\lambda = -\int_{C_1} f \, d\lambda.
\]
We let \((y_i)_{i \geq 1}\) be an enumeration of \( f(C_1) \) (finite or infinite) and we set 
\[
A_i = f^{-1}(\{y_i\}) \cap C_1.
\]
Further, we let \( r_0 = 0 \). Now, as
\[
\int_{C_1} f \, d\lambda + \int_{\tilde{C}_2} f \, d\lambda = 0
\]
we can recursively choose \( r_i \) for \( i \geq 1 \) such that \( r_i \geq r_{i-1} \) and
\[
\int_{[r_{i-1}, r_i] \cap \tilde{C}_2} f \, d\lambda + \int_{A_i} f \, d\lambda = 0.
\]
We then set
\[
B_i = [r_{i-1}, r_i] \cap \tilde{C}_2.
\]
We have thus partitioned \([0, 1]\) into the sets \( C, B_0, \) and \( A_i \cup B_i \) for \( i \geq 1 \) (or possibly for finitely many \( i \)). On each of these sets, \( f \) has mean zero.

Now, as \( f \) takes only countably many values on \( C \), we can use Theorem 3.1 to get a mod 0 measure preserving transformation \( T_C \) of \( C \) and a function \( g_C \in L_\infty(C) \) with \( \|g_C\|_\infty \leq \|f\|_\infty \) such that \( f|_C = g_C \circ T_C - g_C \).

On \( E_0 := B_0 \) we have \( \lambda(f|_{E_0}^{-1}(\{y\})) = 0 \) for all \( y \in \mathbb{R} \), hence we can use Theorem 2.1 to obtain a mod 0 measure preserving transformation \( T_{E_0} \) of \( E_0 \) and a function \( g_{E_0} \in L_\infty(E_0) \) with \( \|g_{E_0}\|_\infty \leq (1 + \varepsilon)\|f\|_\infty \) such that \( f|_{E_0} = g_{E_0} \circ T_{E_0} - g_{E_0} \). Last, we use Theorem 2.1 on \( E_i : = A_i \cup B_i \) for \( i \geq 1 \) with \( \kappa = y_i \) to obtain a mod 0 measure preserving transformation \( T_{E_i} \) of \( E_i \) and a function \( g_{E_i} \in L_\infty(E_i) \) with \( \|g_{E_i}\|_\infty \leq (1 + \varepsilon)\|f\|_\infty \) such that \( f|_{E_i} = g_{E_i} \circ T_{E_i} - g_{E_i} \).

Finally, we define a mod 0 measure preserving transformation \( T \) of \([0, 1]\) by setting \( T|_C = T_C \) and \( T|_{E_i} = T_{E_i} \) for \( i \geq 0 \), and on the remaining null set we define \( T \) as the identity. Likewise, we define \( g \in L_\infty[0, 1] \) by setting \( g|_C = g_C \), and \( g|_{E_i} = g_{E_i} \) for \( i \geq 0 \). We then have \( f = g \circ T - g \) as well as the bound \( \|g\|_\infty \leq \sup\{\|g_C\|_\infty\} \cup \{\|g_{E_i}\|_\infty : i \geq 1\} \leq (1 + \varepsilon)\|f\|_\infty \). This completes the proof. \( \blacksquare \)

**Appendix. Gap in the original proof from [K].** Now, we explain the nature of the gap in [K] and present a counterexample. The original proof is based on the usage of Luzin’s Theorem 1.2, which guarantees the existence of disjoint sets \( A_n \subseteq [0, 1] \) for \( n \geq 1 \) such that \( \lambda([0, 1] \setminus \bigcup_{n=1}^{\infty} A_n) = 0 \) and such that:

1. \( A_n \) is a closed subset, homeomorphic to the Cantor set.
2. \( f \) restricted to \( A_n \) is a continuous function.
3. \( \lambda(A_n) > 0 \) and \( \int_{A_n} f \, d\lambda = 0 \), where \( \lambda \) is Lebesgue measure.

It is further stated in [K] that for each \( n \geq 1 \), there exists a homeomorphism from \( A_n \) onto the Cantor set \( \{0, 1\}^\mathbb{N} \) that maps the measure \( \lambda/\lambda(A_n) \) to the Cantor measure \( \mu \) (the product measure \( \prod_{i=1}^{\infty} \mu_i \) where \( \mu_i \) is the probability measure on \( \{0, 1\} \) given by \( \mu_i(\{0\}) = \mu_i(\{1\}) = 1/2 \).
Here is our counterexample. Fix an irrational scalar $\alpha \in (0,1)$ and set $f = (1 - \alpha)\chi_{[0,\alpha]} - \alpha\chi_{(\alpha,1]}$, so $\int_{[0,1]} f \, d\lambda = 0$. Let $A \subseteq [0,1]$ satisfy (1)–(3). Since $f|_{A}$ must be continuous and $A$ must be compact, it follows that either $(\alpha, \alpha + \varepsilon)$ or $(\alpha, \alpha - \varepsilon)$ is disjoint from $A$ for some $\varepsilon > 0$. Suppose, for definiteness, that $(\alpha, \alpha + \varepsilon) \cap A = \emptyset$ for some $\varepsilon > 0$ (the argument for the other case is the same). We set $C_1 = A \cap [0,\alpha]$ and $C_2 = A \cap [\alpha + \varepsilon,1]$ so that $(1 - \alpha)\lambda(C_1) - \alpha\lambda(C_2) = \int_A f \, d\lambda = 0$. This means that $\lambda(C_2) = \frac{1 - \alpha}{\alpha} \lambda(C_1)$ and hence $\frac{\lambda(C_1)}{\lambda(A)} = \alpha$ is irrational. Let $\varphi : A \to \{0,1\}^\mathbb{N}$ be a homeomorphism. Since $C_1$ is open and closed in $A$, the set $\varphi(C_1)$ is open and compact. Hence $\varphi(C_1) = \bigcup_{U \in A} U$ for some subset $A$ of the basis $\mathcal{B} = \{\{x \in \{0,1\}^\mathbb{N} : (x_i)_{i=1}^N = (a_i)_{i=1}^N \} : (a_i)_{i=1}^N \in \{0,1\}^N \text{ for some } N \in \mathbb{N} \}$ of the topology of $\{0,1\}^\mathbb{N}$. By compactness, $\varphi(C_1) = \bigcup_{i=1}^M U_i$ for some $M \in \mathbb{N}$ and some open sets $U_i \in A$ for $i = 1, \ldots, M$. After a further subdivision, $\varphi(C_1) = \bigcup_{i=1}^{M'} B_i$ for some $M' \in \mathbb{N}$ and some pairwise disjoint open sets $B_i \in \mathcal{B}$ for $i = 1, \ldots, M'$. This implies that $\mu(\varphi(C_1)) = j/2^l$ for some $j,l \in \mathbb{N}$ so that $\varphi$ does not map $\lambda/\lambda(A)$ to $\mu$. This contradiction explains that Kwapien’s construction fails in this particular case.

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