BOSONIZATION OF SUPERALGEBRA $U_q(\widehat{\mathfrak{sl}}(N|1))$
FOR AN ARBITRARY LEVEL

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Abstract

We give a bosonization of the quantum affine superalgebra $U_q(\widehat{\mathfrak{sl}}(N|1))$ for an arbitrary level $k \in \mathbb{C}$. The bosonization of level $k \in \mathbb{C}$ is completely different from those of level $k = 1$. From this bosonization, we induce the Wakimoto realization whose character coincides with those of the Verma module. We give the screening that commute with $U_q(\widehat{\mathfrak{sl}}(N|1))$. Using this screening, we propose the vertex operator that is the intertwiner among the Wakimoto realization and typical realization. We study non-vanishing property of the correlation function defined by a trace of the vertex operators.
1 Introduction

Bosonizations provide a powerful method to construct correlation function of exactly solvable models. We construct a bosonization of the quantum affine superalgebra \( U_q(\hat{sl}(N|1)) \) for an arbitrary level \( k \in \mathbb{C} \) \[1\]-\[2\]. For the special level \( k = 1 \), bosonizations have been constructed for the quantum affine algebra \( U_q(g) \) in many cases \( g = (ADE)^{(r)}, (BC)^{(1)}, G_2^{(1)}, \hat{sl}(M|N), osp(2|2)^{(2)} \) \[3\]-\[10\]. Bosonizations of level \( k \in \mathbb{C} \) are completely different from those of level \( k = 1 \). For an arbitrary level \( k \in \mathbb{C} \) bosonizations have been studied only for \( U_q(\hat{sl}_N) \) \[11\]-\[12\] and \( U_q(\hat{sl}(N|1)) \) \[1\]-\[2\]. Our construction is based on the ghost-boson system. We need more consideration to get the Wakimoto realization whose character coincides with those of the Verma module. Using \( \xi-\eta \) system we construct the Wakimoto realization \[13\]-\[14\] from our level \( k \) bosonization. For an arbitrary level \( k \neq -N + 1 \) we construct the screening current that commutes with \( U_q(\hat{sl}(N|1)) \) modulo total difference. By using Jackson integral and the screening current, we construct the screening that commute with \( U_q(\hat{sl}(N|1)) \) \[13\]-\[15\]. We propose the vertex operator that is the intertwiner among the Wakimoto realization and typical realization. By using the Gelfand-Zetlin basis, we have checked the intertwining property of the vertex operator for rank \( N = 2, 3, 4 \) \[15\]. We balance the background charge of the vertex operator by using the screening and propose the correlation function by a trace of them, which gives quantum and super generalization of Dotsenko-Fateev theory \[10\].

The paper is organized as follows. In section 2 we review bosonizations of \( U_q(\hat{sl}_2) \). In section 3 we construct a bosonization of \( U_q(\hat{sl}(N|1)) \) for an arbitrary level \( k \in \mathbb{C} \). We induce the Wakimoto realization by \( \xi-\eta \) system. In section 4 we construct the screening that commute with \( U_q(\hat{sl}(N|1)) \) for an arbitrary level \( k \neq -N + 1 \). We propose the vertex operator and the correlation function.

2 Bosonization : Level \( k = 1 \) vs. Level \( k \in \mathbb{C} \)

In this section we review the bosonization of the quantum affine algebra \( U_q(\hat{sl}_2) \). The purpose of this section is to make readers understand that the bosonization of level \( k \in \mathbb{C} \) is complete different from those of level \( k = 1 \). In what follows let \( q \) be a generic complex number \( 0 < |q| < 1 \). We use the standard \( q \)-integer notation:

\[
[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.
\]

First we recall the definition of \( U_q(\hat{sl}_2) \). We recall the Drinfeld realization of the quantum affine algebra \( U_q(\hat{sl}_2) \).

**Definition** \[17\]. The generators of the quantum affine algebra \( U_q(\hat{sl}_2) \) are \( x_{i,n}^\pm, h_m, h, c \ (n \in \mathbb{Z}, m \in \mathbb{Z}_{\neq 0}) \). Defining relations are

\[
c : \text{central, } [h, h_m] = 0, \quad [h_m, h_n] = \delta_{m+n,0} \frac{[2m]_q[cm]_q}{m},\]

where \( [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}} \) and \( [cm]_q = \frac{q^m - q^{-m}}{q - q^{-1}} \).
\[ h, x^\pm(z) = \pm 2x^\pm(z), \]
\[ [h, x^\pm(z)] = \pm \frac{[2m]_q [m]_q}{m} q^{\frac{m}{2}} z^m x^\pm(z), \]
\[ (z_1 - q^{1±2}z_2) x^\pm(z_1) x^\pm(z_2) = (q^{1±2}z_1 - z_2) x^\pm(z_2) x^\pm(z_1), \]
\[ [x^+(z_1), x^-(z_2)] = \frac{1}{(q - q^{-1}) z_1 z_2} \]
\[ \times \left( \delta(q^{-c} z_1/z_2) \psi^+(q^{-c} z_2) - \delta(q^{c} z_1/z_2) \psi^-(q^{-c} z_2) \right). \]

where we have used \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \). We have set the generating function
\[ x^\pm(z) = \sum_{n \in \mathbb{Z}} x^\pm z^{-n-1}, \]
\[ \psi^\pm(q^{1±2} z) = q^{±h} e^{ \pm (q-q^{-1}) \sum_{m>0} h_{\pm m} z^m} \]

When the center \( c \) takes the complex number \( c = k \in \mathbb{C} \), we call it the level \( k \) representation. We call the realization by the differential operators the bosonization. Frenkel-Jing [3] constructed the level \( k = 1 \) bosonization of the quantum affine algebra \( U_q(sl_2) \) for simply-laced \( g = (ADE)^{(1)} \). Here we recall the level \( k = 1 \) bosonization of \( U_q(sl_2) \). We introduce the boson \( a_n \ (n \in \mathbb{Z}) \) and the zero-mode operator \( \partial, \alpha \) by
\[ [a_m, a_n] = \frac{[2m]_q [m]_q}{m} \delta_{m+n,0}, \quad [\partial, \alpha] = 2. \]

In what follows, in order to avoid divergences, we restrict ourselves to the Fock space of the bosons.

**Theorem 2.2** [3] A bosonization of the quantum affine algebra \( U_q(sl_2) \) for the level \( k = 1 \) is given as follows.

\[ c = 1, \quad h = \partial, \quad h_n = a_n, \]
\[ x^\pm(z) = e^{± \sum_{n \neq 0} \frac{a_n}{m} q^{1±2} z^{-n(\alpha+\partial)}} :. \]

We have used the normal ordering symbol :
\[ :a_k a_l: = \begin{cases} a_k a_l & (k < 0), \\ a_l a_k & (k > 0), \end{cases} :\alpha \partial : = : \partial \alpha : = : \alpha \partial :. \]

Next we recall the level \( k \) bosonization of the quantum affine algebra \( U_q(sl_2) \) [11]. We introduce the bosons and the zero-mode operator \( a_n, b_n, c_n, Q_\alpha, Q_b, Q_c \ (n \in \mathbb{Z}) \) as follows.
\[ [a_m, a_n] = \delta_{m+n,0} \frac{[2m]_q [m]_q}{m} (k + 2), \quad [\tilde{a}_0, Q_\alpha] = 2(k + 2), \]
\[ [b_m, b_n] = -\delta_{m+n,0} \frac{[2m]_q [2m]_q}{m} \]
\[ [c_m, c_n] = \delta_{m+n,0} \frac{[2m]_q [2m]_q}{m} \]
\[ \tilde{b}_0, Q_b] = -4, \]
\[ [\tilde{c}_0, Q_c] = 4, \]
\[ \tilde{a}_0 = \frac{2 - q^{-1}}{2 \log q} a_0, \quad \tilde{b}_0 = \frac{2 - q^{-1}}{2 \log q} b_0, \quad \tilde{c}_0 = \frac{2 - q^{-1}}{2 \log q} c_0. \]

It is convenient to introduce the generating function \( a(N|z; \alpha) \).
\[ a(N|z; \alpha) = - \sum_{n \neq 0} \frac{a_n}{[N]_q} q^n z^{-n} + \tilde{a}_0 \frac{\log z + Q_\alpha}{N}. \]
In what follows, in order to avoid divergences, we restrict ourselves to the Fock space of the bosons.

**Theorem 2.3** A bosonization of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ for the level $k \in \mathbb{C}$ is given as follows.

\[
c = k \in \mathbb{C}, \quad h = a_0 + b_0,
\]

\[
h_m = q^{2m-|m|}a_m + q^{(k+2)m-\frac{k+2}{2}|m|}b_m,
\]

\[
x^+(z) = \frac{-1}{(q-q^{-1})z} \left(e^{-b(2|q^{k-2}z;1|)-c(2|q^{k-1}z;0)} ; -e^{-b(2|q^{-k-2}z;1|)-c(2|q^{-k-3}z;0)} \right),
\]

\[
x^-(z) = \frac{1}{(q-q^{-1})z} \left(e^{a(k+2|q^kz,-\frac{1}{2}|) - a(k+2|q^{-2}z;1| + b(2|z;1| + c(2|q^{-1}z;0))} ; -e^{a(k+2|q^{-k-4}z;1| - b(2|q^{-2k+4}z;1| + c(2|q^{-2k-3}z;0))} \right).
\]

The level $k = 1$ bosonization is given by "monomial". The level $k \in \mathbb{C}$ bosonization is given by "sum". They are completely different.

## 3 Bosonization of Quantum Superalgebra $U_q(\hat{\mathfrak{sl}}(N|1))$

In this section we study the bosonization of the quantum superalgebra $U_q(\hat{\mathfrak{sl}}(N|1))$ for an arbitrary level $k \in \mathbb{C}$.

### 3.1 Quantum Superalgebra $U_q(\hat{\mathfrak{sl}}(N|1))$

In this section we recall the definition of the quantum superalgebra $U_q(\hat{\mathfrak{sl}}(N|1))$. We fix a generic complex number $q$ such that $0 < |q| < 1$. The Cartan matrix $(A_{i,j})_{0 \leq i,j \leq N}$ of the affine Lie algebra $\hat{\mathfrak{sl}}(N|1)$ is given by

\[
A_{i,j} = (\nu_i + \nu_{i+1})\delta_{i,j} - \nu_i\delta_{i,j+1} - \nu_{i+1}\delta_{i+1,j}.
\]

Here we set $\nu_1 = \cdots = \nu_N = +, \nu_{N+1} = \nu_0 = -$. We introduce the orthonormal basis $\{\epsilon_i| i = 1, 2, \cdots, N + 1\}$ with the bilinear form, $(\epsilon_i|\epsilon_j) = \nu_i\delta_{i,j}$. Define $\bar{\epsilon}_i = \epsilon_i - \frac{\nu_i}{N+1} \sum_{j=1}^{N+1} \epsilon_j$. Note that $\sum_{j=1}^{N} \bar{\epsilon}_j = 0$. The classical simple roots $\bar{\alpha}_i$ and the classical fundamental weights $\Lambda_i$ are defined by $\bar{\alpha}_i = \nu_i\epsilon_i - \nu_{i+1}\epsilon_{i+1}$, $\Lambda_i = \sum_{j=1}^{i} \bar{\epsilon}_j$ ($1 \leq i \leq N$). Introduce the affine weight $\Lambda_0$ and the null root $\delta$ satisfying $(\Lambda_0|\Lambda_0) = (\delta|\delta) = 0$, $(\Lambda_0|\epsilon_i) = 0$, $(\delta|\epsilon_i) = 0$, $(1 \leq i \leq N)$. The other affine weights and the affine roots are given by $\alpha_0 = \delta - \sum_{j=1}^{N} \bar{\alpha}_j$, $\alpha_i = \bar{\alpha}_i$, $\Lambda_i = \Lambda_i + \Lambda_0$, $(1 \leq i \leq N)$. Let $P = \oplus_{j=1}^{N} \mathbb{Z} \Lambda_j \oplus \mathbb{Z} \delta$ and $P^* = \oplus_{j=1}^{N} \mathbb{Z} \bar{h}_j \oplus \mathbb{Z} \bar{d}$ the affine $\hat{\mathfrak{sl}}(N|1)$ weight lattice and its dual lattice, respectively.

**Definition 3.1** $[18]$ The quantum affine superalgebra $U_q(\hat{\mathfrak{sl}}(N|1))$ are generated by the generators $h_i, e_i, f_i$ ($0 \leq i \leq N$). The $\mathbb{Z}_2$-grading of the generators are $|e_0| = |f_0| = |e_N| = |f_N| = 1$ and zero otherwise. The defining relations are given as follows.
The Cartan-Kac relations: For $N \geq 2$, $0 \leq i, j \leq N$, the generators subject to the following relations.

$$[h_i, h_j] = 0, \ [h_i, e_j] = A_{i,j} e_j, \ [h_i, f_j] = -A_{i,j} f_j, \ [e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}.$$  

The Serre relations: For $N \geq 2$, the generators subject to the following relations for $1 \leq i \leq N - 1$, $0 \leq j \leq N$ such that $|A_{i,j}| = 1$.

$$[e_i, [e_i, e_j]_{q^{-1}}]_q = 0, \ [f_i, [f_i, f_j]_{q^{-1}}]_q = 0.$$  

For $N \geq 2$, the generators subject to the following relations for $0 \leq i, j \leq N$ such that $|A_{i,j}| = 0$.

$$[e_i, e_j] = 0, \ [f_i, f_j] = 0.$$  

For $N \geq 3$, the Serre relations of fourth degree hold.

$$[e_N, [e_0, [e_N, e_{N-1}]_{q^{-1}}]_q]_q = 0, \ [e_0, [e_1, [e_0, e_N]_{q^{-1}}]_q] = 0,$$

$$[f_N, [f_0, [f_N, f_{N-1}]_{q^{-1}}]_q]_q = 0, \ [f_0, [f_1, [f_0, f_N]_{q^{-1}}]_q] = 0.$$  

For $N = 2$, the extra Serre relations of fifth degree hold.

$$[e_2, [e_0, [e_2, [e_0, e_1]_q]]_{q^{-1}}] = [e_0, [e_2, [e_0, e_1]_q]]_{q^{-1}},$$

$$[f_2, [f_0, [f_2, [f_0, f_1]_q]]_{q^{-1}}] = [f_0, [f_2, [f_0, f_1]_q]]_{q^{-1}}.$$  

Here and throughout this paper, we use the notations

$$[X, Y]_{\xi} = XY - (-1)^{|X||Y|} \xi YX.$$  

We write $[X, Y]_1$ as $[X, Y]$ for simplicity.

The quantum affine superalgebra $U_q(\hat{\mathfrak{sl}}(N|1))$ has the $\mathbb{Z}_2$-graded Hopf-algebra structure. We take the following coproduct

$$\Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i, \ \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i, \ \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,$$

and the antipode

$$S(e_i) = -q^{-h_i} e_i, \ S(f_i) = -f_i q^{h_i}, \ S(h_i) = -h_i.$$  

The coproduct $\Delta$ satisfies an algebra automorphism $\Delta(XY) = \Delta(X)\Delta(Y)$ and the antipode $S$ satisfies a $\mathbb{Z}_2$-graded algebra anti-automorphism $S(XY) = (-1)^{|X||Y|} S(Y) S(X)$. The multiplication rule for the tensor product is $\mathbb{Z}_2$-graded and is defined for homogeneous elements $X, Y, X', Y' \in U_q(\hat{\mathfrak{sl}}(N|1))$ and $v \in V, w \in W$ by $X \otimes Y \cdot X' \otimes Y' = (-1)^{|Y||X'|} XX' \otimes YY'$ and $X \otimes Y \cdot v \otimes w = (-1)^{|Y||v|} X v \otimes Y w$, which extends to inhomogeneous elements through linearity.

**Definition 3.2** The quantum superalgebra $U_q(\hat{\mathfrak{sl}}(N|1))$ is the subalgebra of $U_q(\hat{\mathfrak{sl}}(N|1))$, that is generated by $e_1, e_2, \cdots, e_N, f_1, f_2, \cdots, f_N$, and $h_1, h_2, \cdots, h_N$.

We recall the Drinfeld realization of $U_q(\hat{\mathfrak{sl}}(N|1))$, that is convenient to construct bosonizations.
Definition 3.3 [13] The generators of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ are $x_{i,n}^\pm$, $h_{i,m}$, $c$ ($1 \leq i \leq N, n \in \mathbb{Z}, m \in \mathbb{Z}_\neq 0$). Defining relations are

\begin{align*}
c : & \text{ central, } [h_i, h_{j,m}] = 0, \\
[h_{i,m}, h_{j,n}] & = \frac{[A_{i,j}m]q_{c,m}q^{-e[1]}m\delta_{m+n,0}}, \\
[h_i, x_j^+(z)] & = \pm A_{i,j}x_j^+(z), \\
[h_{i,m}, x_j^+(z)] & = \frac{[A_{i,j}m]q_{c,m}z^m x_j^+(z)}, \\
[h_{i,m}, x_j^-(z)] & = -\frac{[A_{i,j}m]q_{c,m}z^m x_j^-(z)}, \\
(x_1 - q^{A_{i,j}Z_+}z_2)x_i^+(z_1) & = (q^{A_{i,j}}z_1 - z_2)x_j^+(z_2)x_i^+(z_1) \text{ for } |A_{i,j}| \neq 0, \\
x_i^+(z_1), x_j^+(z_2) & = 0 \text{ for } |A_{i,j}| = 0, \\
x_i^+(z_1), x_j^-(z_2) & = \frac{\delta_{i,j}}{(q - q^{-1})z_1z_2} (\delta(q^{-e[1]}z_1z_2)\psi_i^+(q^{\frac{1}{2}}z_2) - \delta(q^{e[1]}z_1z_2)\psi_i^-(q^{-\frac{1}{2}}z_2)), \\
(x_i^+(z_1)x_i^+(z_2) - (q + q^{-1})x_i^+(z_1)x_i^+(z_2)) & = x_j^+(z_1)x_j^+(z_2) + x_j^-(z_1)x_j^+(z_2)) + (z_1 \leftrightarrow z_2) = 0 \text{ for } |A_{i,j}| = 1, i \neq N,
\end{align*}

where we have used $\delta(z) = \sum_{m \in \mathbb{Z}} z^m$. Here we have used the generating function

\begin{align*}
x_j^\pm(z) & = \sum_{m \in \mathbb{Z}} x_j^\pm z^{-m-1}, \\
\psi_i^\pm(q^{\frac{1}{2}}z) & = q^{h_i e_i \pm (q-q^{-1})} \sum_{m > 0} h_{i,\pm m} z^m.
\end{align*}

The relation between two definitions of $U_q(\widehat{sl}(N|1))$ are given by

\begin{align*}
h_0 & = c - (h_1 + \cdots + h_N), \quad e_i = x_{i,0}^+, \quad f_i = x_{i,0}^- \text{ for } 1 \leq i \leq N, \\
e_0 & = (-1)[x_{N,0}, x_{N-1,0}, \cdots, x_{2,0}, x_{1,0}]_{q^{-1}} \cdots q^{-h_1 - h_2 - \cdots - h_N}, \\
f_0 & = q^{h_1 h_2 + \cdots + h_N} \cdots [x_{1,1}^+, x_{1,0}^+, x_{2,0}^+, x_{3,0}^+, \cdots x_{N,0}^+].
\end{align*}

For instance we have the coproduct as follows.

\begin{align*}
\Delta(h_{i,m}) & = h_{i,m} \otimes q^m a_{i,n}^d + q^{-m} \otimes h_{i,m} (m > 0), \\
\Delta(h_{i,-m}) & = h_{i,-m} \otimes q^{-m} a_{i,n}^d + q^m \otimes h_{i,-m} (m > 0).
\end{align*}

3.2 Bosonization

In this section we construct bosonizations of quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbb{C}$. We introduce the bosons and the zero-mode operators $a_{i,m}^d, Q_{a,m}^j (m \in \mathbb{Z}, 1 \leq j \leq N), b_{i,m}^d, Q_{b,m}^j (m \in \mathbb{Z}, 1 \leq i < j \leq N)$ which satisfy

\begin{align*}
[a_{i,m}^d, a_{n,m}^d] & = \frac{[(k+N-1)m]_q [A_{i,j}m]_2 q^{-m+n,0}}, \quad [a_{0,m}^d, Q_{a,m}^j] = (k+N-1)A_{i,j}, \\
[b_{i,m}^d, b_{n,m}^d] & = -\nu_{i,j} \frac{[m]^2_2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [b_{0,m}^d, Q_{b,m}^j] = -\nu_{i,j} \delta_{i,i'} \delta_{j,j'},
\end{align*}
Theorem 3.4 [2] A bosonization of the quantum superalgebra

We impose the restriction $p \leq N+1 \delta_{j',N+1} \pi \sqrt{-1} (i, j) \neq (i', j')$.

Other commutation relations are zero. In what follows we use the standard symbol of the normal orderings $::$. It is convenient to introduce the generating function $b^{i,j}(z), c^{i,j}(z), b^{i,j}_\pm(z), a^{j}_\pm(z)$ and $\left( \frac{\gamma_1 \cdots \gamma_z}{\beta_1 \cdots \beta_r} \alpha^i \right)(z|\alpha)$ given by

$$
\begin{align*}
b^{i,j}(z) &= -\sum_{m \neq 0} \frac{b^{i,j}_m}{[m]_q} z^m + Q_b^{i,j} + b_0^{i,j} \log z, \\
c^{i,j}(z) &= -\sum_{m \neq 0} \frac{c^{i,j}_m}{[m]_q} z^m + Q_c^{i,j} + c_0^{i,j} \log q, \\
b^{i,j}_\pm(z) &= \pm (q - q^{-1}) \sum_{m \geq 0} b^{i,j}_m z^m \pm b_0^{i,j} \log q, \\
a^{j}_\pm(z) &= \pm (q - q^{-1}) \sum_{m \geq 0} a^{i}_m z^m \pm a_0^j \log q,
\end{align*}
$$

$$
\left( \frac{\gamma_1 \cdots \gamma_z}{\beta_1 \cdots \beta_r} \alpha^i \right)(z|\alpha) = -\sum_{m \neq 0} \frac{[\gamma_1 m]_q \cdots [\gamma_z m]_q}{[\beta_1 m]_q \cdots [\beta_r m]_q [m]_q} a^{i}_m q^{-\alpha|m|} z^{-m} + \frac{\gamma_1 \cdots \gamma_z}{\beta_1 \cdots \beta_r}(Q_a + a_0^j \log z).
$$

In order to avoid divergence we work on the Fock space defined below. We introduce the vacuum state $|0\rangle \neq 0$ of the boson Fock space by

$$
a^{i}_m|0\rangle = b^{i,j}_m|0\rangle = c^{i,j}_m|0\rangle = 0 \quad (m \geq 0).
$$

For $p^{i}_a \in \mathbb{C}$ ($1 \leq i \leq N$), $p^{i,j}_b \in \mathbb{C}$ ($1 \leq i < j \leq N+1$), $p^{i,j}_c \in \mathbb{C}$ ($1 \leq i < j \leq N$), we set

$$
|p_a, p_b, p_c\rangle = e^{\sum_{i,j=1}^{N} \min(\gamma_{ij}, (N+1 - \min(\gamma_{ij}, j)) \pi \sqrt{-1}) p^{i,j}_a Q^{i,j}_a} \times e^{-\sum_{1 \leq i < j \leq N+1} p^{i,j}_b Q^{i,j}_b + \sum_{1 \leq i < j \leq N} p^{i,j}_c Q^{i,j}_c} |0\rangle.
$$

It satisfies

- $a^{i}_0|p_a, p_b, p_c\rangle = p^{i}_a|p_a, p_b, p_c\rangle$,
- $b^{i,j}_0|p_a, p_b, p_c\rangle = p^{i,j}_b|p_a, p_b, p_c\rangle$, $c^{i,j}_0|p_a, p_b, p_c\rangle = p^{i,j}_c|p_a, p_b, p_c\rangle$.

The boson Fock space $F(p_a, p_b, p_c)$ is generated by the bosons $a^{i}_m, b^{i,j}_m, c^{i,j}_m$ on the vector $|p_a, p_b, p_c\rangle$. We set the space $F(p_a)$ by

$$
F(p_a) = \bigoplus_{p^{i,j}_b \in \mathbb{Z}, (1 \leq i < j \leq N)} F(p_a, p_b, p_c).
$$

We impose the restriction $p^{i,j}_b = -p^{i,j}_c \in \mathbb{Z}$ ($1 \leq i < j \leq N$). We construct a bosonization on the space $F(p_a)$.

**Theorem 3.4** [2] A bosonization of the quantum superalgebra $U_q(\mathfrak{sl}(N|1))$ for an arbitrary level $k \in \mathbb{C}$.
is given as follows.

\[ c = k \in \mathbb{C}, \]

\[ h_i = a_0^i + \sum_{l=1}^i (b_0^{l,i} - b_0^{l,i+1}) + \sum_{l=i+1}^N (b_0^{l,i} - b_0^{l+1,i}) + b_0^{N+1,i} - b_0^{0,i+1,N+1}, \]

\[ h_N = a_0^N - \sum_{l=1}^{N-1} (b_0^{l,N} + b_0^{l,N+1}), \]

\[ h_{i,m} = \sum_{l=1}^i (q^{-\frac{1}{2}(l+1)}|m|b^{l,i}_m - q^{-\frac{1}{2}(l+1)}|m|b^{l,i}_m) \]

\[ + \sum_{l=i+1}^N (q^{-\frac{1}{2}(l+1)}|m|b^{l,i}_m - q^{-\frac{1}{2}(l+1)}|m|b^{l+1,i}_m) \]

\[ + q^{-\frac{1}{2}(N+1)}|m|b^{i,N+1}_m - q^{-\frac{1}{2}(N+1)}|m|b^{i+1,N+1}_m, \]

\[ h_{N,m} = \sum_{l=1}^{N-1} (q^{-\frac{1}{2}(l+1)}|m|b^{l,N}_m + q^{-\frac{1}{2}(l+1)}|m|b^{l,N+1}_m), \]

\[ x_+^i(z) = \frac{1}{(q - q^{-1})z} \left\{ \sum_{j=1}^i e^{(b+c)^{j+i}(q^{-1}z)} + \sum_{j=1}^{i-1} (b^{j,i+1}_+ (q^{-1}z) - b^{j,i}_+(q^i z)) \times \right\} \]

\[ \times \left\{ e^{b^{j,i+1}_+(q^{-1}z)} - (b+c)^{j+i+1}(q^i z) - e^{b^{j,i}_+(q^{-1}z)} - (b+c)^{j+i}(q^i z) \right\} \right. \}

\[ x_+^N(z) = \sum_{j=1}^N e^{(b+c)^{j+N}(q^{-1}z)} + \sum_{j=1}^{N-1} (b^{j+N}_+(q^{-1}z) + b^{j+N}_+(q^i z)) \times \]

\[ x_+^-(z) = q^{k+N-1} : e^{a^+_N(q^{k+N-1}z)} - \sum_{l=1}^N (q^{k+N-1}z) - \sum_{l=1}^{N-1} (q^{k+N-1}z) - \sum_{l=1}^N (q^{k+N-1}z) - \sum_{l=1}^{N-1} (q^{k+N-1}z) \times \]

\[ + \frac{1}{(q - q^{-1})z} \left\{ \sum_{j=1}^{i-1} e^{a^+_N(q^{k+N-1}z)} + (b+c)^{j+i+1}(q^{-k-j}z) + b^{j,N+1}_+(q^{-k-j}z) - b^{j,N+1}_+(q^{-k-j}z) \times \right\} \]

\[ + e^{a^+_N(q^{k+N-1}z)} + (b+c)^{j+i+1}(q^{-k-j}z) \times \]

\[ \sum_{l=1}^N (b^{1}_{-}(q^{k-N}z) - b^{1}_{-}(q^{-k+j+1}z)) + b^{1}_{-}(q^{k-N}z) - b^{1}_{-}(q^{k-N}z) \times \]

\[ - e^{a^+_N(q^{k+N-1}z)} + (b+c)^{j+i+1}(q^{k+j}z) \times \]

\[ \sum_{l=1}^N (b^{1}_{+}(q^{k+N}z) - b^{1}_{+}(q^{k+N}z)) + b^{1}_{+}(q^{k+N}z) - b^{1}_{+}(q^{k+N}z) \times \]

\[ - e^{a^+_N(q^{k+N-1}z)} + (b+c)^{j+i+1}(q^{k+j}z) \times \]

\[ \sum_{j=i+1}^N e^{a^+_N(q^{k+N-1}z)} + (b+c)^{j+i+1}(q^{k+j}z) \times \]

\[ e^{b^{i+1}_+(q^{k+j}z)} - (b+c)^{i+1}_+(q^{k+j}z) \times \]

\[ e^{b^{i+1}_+(q^{k+j}z)} - (b+c)^{i+1}_+(q^{k+j}z) \times \]

\[ e^{b^{i+1}_+(q^{k+j}z)} - (b+c)^{i+1}_+(q^{k+j}z) \times \]

\[ \sum_{j=i+1}^N \left\{ \sum_{l=1}^N e^{a^+_N(q^{k+N-1}z)} - b^{i+N}_+(q^{-k+j}z) - b^{i+1}_+(q^{k+j}z) \times \right\} \]
\begin{align*}
&\times q^{j-1} \left( e^{-b_{i,j}^{N}(q^{-k-j} z)} - (b+c)^{j-N} (q^{-k-j+1} z) - e^{-b_{j,i}^{N}(q^{-k-j} z)} \right) \\
&+ q^{N-1} \left( e^{a_{x}^{N}(q^{\frac{k-N}{2}} z)} - b^{N, N+1}(q^{N-N} z) - e^{a_{x}^{N}(q^{\frac{k-N}{2}} z)} \right) \}.
\end{align*}

### 3.3 Replacement from $U_q(sl(N|1))$ to $U_q(\widehat{sl(N|1)})$

In this section we study the relation between $U_q(sl(N|1))$ and $U_q(\widehat{sl(N|1)})$. Let us recall the Heisenberg realization of quantum superalgebra $U_q(sl(N|1))$ \[\text{(III)}\]. We introduce the coordinates $x_{i,j}$, $(1 \leq i < j \leq N+1)$ by

$$x_{i,j} = \begin{cases} z_{i,j} & (1 \leq i < j \leq N), \\ \theta_{i,j} & (1 \leq i \leq N, j = N+1). \end{cases} \quad (3.1)$$

Here $z_{i,j}$ are complex variables and $\theta_{i,N+1}$ are the Grassmann odd variables that satisfy $\theta_{i,N+1}\theta_{i,N+1} = 0$ and $\theta_{i,N+1}\theta_{j,N+1} = -\theta_{j,N+1}\theta_{i,N+1}$, $(i \neq j)$. We introduce the differential operators $\vartheta_{i,j} = x_{i,j} \frac{\partial}{\partial x_{i,j}}$, $(1 \leq i < j \leq N+1)$.

**Theorem 3.5** \[\text{(III)}\] We fix parameters $\lambda_i \in \mathbb{C}$ $(1 \leq i \leq N)$. The Heisenberg realization of $U_q(sl(N|1))$ is given as follows.

\begin{align*}
\hat{h}_i &= \sum_{j=1}^{i-1} (\nu_i \vartheta_{j,i} - \nu_{i+1} \vartheta_{j,i+1}) + \lambda_i - (\nu_i + \nu_{i+1}) \vartheta_{i,i+1} + \sum_{j=i+1}^{N} (\nu_{i+1} \vartheta_{i+1,j+1} - \nu_i \vartheta_{i,j+1}), \\
\hat{e}_i &= \sum_{j=1}^{i} \frac{x_{j,i}}{x_{j,i+1}} [\vartheta_{j,i+1}] q^{\sum_{l=1}^{j-1} (\nu_l \vartheta_{l,i+1} - \nu_{l+1} \vartheta_{l,i+1})}, \\
\hat{f}_i &= \sum_{j=1}^{i-1} \frac{x_{j,i+1}}{x_{j,i}} [\vartheta_{j,i}] q^{\sum_{l=1}^{j-1} (\nu_{i+1} \vartheta_{i+1,l} - \nu_i \vartheta_{i,l}) - \lambda_i + (\nu_i + \nu_{i+1}) \vartheta_{i,i+1} + \sum_{l=i+2}^{N+1} (\nu_{i+1} \vartheta_{i+1,l} - \nu_i \vartheta_{i,l})} \\
&+ x_{i,i+1} \left[ \lambda_i - \nu_i \vartheta_{i,i+1} - \sum_{l=i+2}^{N+1} (\nu_l \vartheta_{l,i} - \nu_{l+1} \vartheta_{l+1,i}) \right] q^{\lambda_i + \sum_{l=i+2}^{N+1} (\nu_{i+1} \vartheta_{i+1,l} - \nu_i \vartheta_{i,l})}. \\
\end{align*}

Here we read $x_{i,i} = 1$ and, for Grassmann odd variables $x_{i,j}$, the expression $\frac{1}{x_{i,j}}$ stands for the derivative $\frac{1}{x_{i,j}} = \frac{\partial}{\partial x_{i,j}}$.

We study how to recover the bosonization of the affine superalgebra $U_q(\widehat{sl(N|1)})$ from the Heisenberg realization of $U_q(sl(N|1))$. We make the following replacement with suitable argument.

\begin{align*}
\vartheta_{i,j} &\rightarrow -b_{i,j}^{N}(z)/\log q \quad (1 \leq i < j \leq N+1), \\
[\vartheta_{i,j}]_q &\rightarrow \begin{cases} q^{-b_{i,j}^{N}(z)} - e^{b_{i,j}^{N}(z)} (q - q^{-1})z & (j \neq N+1) \\ 1 & (j = N+1). \end{cases}
\end{align*}
We have a direct sum decomposition.

\[
\lambda_i \rightarrow a^i(z)/\log q \quad (1 \leq i \leq N),
\]

\[
|\lambda_i|_q \rightarrow \frac{e^{\pm a^i(z)} - e^{-\pm a^i(z)}}{(q - q^{-1})z} \quad (1 \leq i \leq N).
\]

From the above replacement, the element \(h_i\) of the Heisenberg realization is replaced as following.

\[
q^{h_i} \rightarrow \begin{cases} 
q^{a^i(z) + \sum_{i=1}^N (b^{i+1}_z(z) - b^i_z(z)) + \sum_{i=1}^N (b^{i+1}_z(z) - b^i_z(z))} & (1 \leq i \leq N - 1), \\
q^{a^i_z(z) - \sum_{i=1}^{N-1} (b^{i}_z(z) + b^{i+1}_z(z))} & (i = N).
\end{cases}
\]

We impose \(q\)-shift to variable \(z\) of the operators \(a^i_\pm(z), b^i_\pm(z)\). For instance, we have to replace \(a^i_\pm(z) \rightarrow a^i(z) - \frac{2\mp q^{i+\frac{1}{2}}}{q - q^{-1}} z\). Bridging the gap by the \(q\)-shift, we have the bosonizations \(\psi^\pm_i(\mp q^{1/2}z) \in U_q(\tilde{sl}(N|1))\) from \(q^{h_i} \in U_q(sl(N|1))\).

\[
\psi^\pm_i(\mp q^{1/2}z) = e^{\pm a^i_\pm(\mp q^{1/2}z)} + \sum_{i=1}^N (b^{i+1}_z(z) - b^i_z(z)) \frac{q^{1/2}z}{z^2}.
\]

In this replacement, one element \(q^{h_i}\) goes to two elements \(\psi^\pm_i(\mp q^{1/2}z)\). Hence this replacement is not a map. Replacements from \(e_i, f_i\) to \(x^\pm_i(z)\) are given by similar way, however they are more complicated. See details in [\ref{ref}].

### 3.4 Wakimoto Realization

In this section we give the Wakimoto realization \(F(p_a)\) whose character coincides with those of the Verma module [\ref{ref}]. We introduce the operators \(\xi^{i,j}_m\) and \(\eta^{i,j}_m\) (\(1 \leq i < j \leq N, m \in \mathbb{Z}\)) by

\[
\eta^{i,j}_m(z) = \sum_{m \in \mathbb{Z}} \eta^{i,j}_m z^{-m-1} = e^{-\epsilon^{i,j}(z)}:\xi^{i,j}_m(z) = \sum_{m \in \mathbb{Z}} \xi^{i,j}_m z^{-m} = e^{\epsilon^{i,j}(z)}:.
\]

The Fourier components \(n^{i,j}_m = \oint \frac{dz}{2\pi i z} e^{z\epsilon^{i,j}_m(z)}\), \(c^{i,j}_m = \oint \frac{dz}{2\pi i z} z^{m-1} \xi^{i,j}_m(z)\) (\(m \in \mathbb{Z}\)) are well defined on the space \(F(p_a)\). We focus our attention on the operators \(\eta^{i,j}_0, \xi^{i,j}_0\) satisfying \((\eta^{i,j}_0)^2 = 0, (\xi^{i,j}_0)^2 = 0\). They satisfy

\[
\text{Im}(\eta^{i,j}_0) = \text{Ker}(\eta^{i,j}_0), \quad \text{Im}(\xi^{i,j}_0) = \text{Ker}(\xi^{i,j}_0), \quad \eta^{i,j}_0 \xi^{i,j}_0 + \xi^{i,j}_0 \eta^{i,j}_0 = 1.
\]

We have a direct sum decomposition,

\[
F(p_a) = \eta^{i,j}_0 \xi^{i,j}_0 F(p_a) \oplus \xi^{i,j}_0 \eta^{i,j}_0 F(p_a),
\]

\[
\text{Ker}(\eta^{i,j}_0) = \eta^{i,j}_0 F(p_a), \quad \text{Coker}(\eta^{i,j}_0) = \xi^{i,j}_0 F(p_a) = F(p_a)/(\eta^{i,j}_0 \xi^{i,j}_0 F(p_a)).
\]

We set the operator \(\eta_0, \xi_0\) by

\[
\eta_0 = \prod_{1 \leq i < j \leq N} \eta^{i,j}_0, \quad \xi_0 = \prod_{1 \leq i < j \leq N} \xi^{i,j}_0.
\]
Definition 3.6 [13] We introduce the subspace \( \mathcal{F}(p_a) \) by

\[
\mathcal{F}(p_a) = \eta_0 \xi_0 F(p_a).
\]

We call \( \mathcal{F}(p_a) \) the Wakimoto realization.

4 Screening and Vertex Operator

In this section we give the screening that commutes with the quantum superalgebra \( U_q(\hat{\mathfrak{sl}}(N|1)) \). We propose the vertex operators and the correlation functions.

4.1 Screening

In this section we give the screening \( Q_i \) (\( 1 \leq i \leq N \)) that commutes with the quantum superalgebra \( U_q(\hat{\mathfrak{sl}}(N|1)) \) for an arbitrary level \( k \neq -N + 1 \) [15]. The Jackson integral with parameter \( p \in \mathbb{C} \) (\( |p| < 1 \)) and \( s \in \mathbb{C}^* \) is defined by

\[
\int_0^{s \infty} f(z)dz = s(1 - p) \sum_{m \in \mathbb{Z}} f(sp^m)p^m.
\]

In order to avoid divergence we work in the Fock space.

Theorem 4.1 [13] The screening \( Q_i \) commutes with the quantum superalgebra.

\[
[Q_i, U_q(\hat{\mathfrak{sl}}(N|1))] = 0 \quad (1 \leq i \leq N).
\]

We have introduced the screening operators \( Q_i \) (\( 1 \leq i \leq N \)) as follows.

\[
Q_i = \int_0^{s \infty} e^{-(\frac{a^i}{1 - p^1})}(|z|) \tilde{S}_i(z) : d_p z, \quad (p = q^{2(k+N-1)}).
\]

Here we have set the bosonic operators \( \tilde{S}_i(z) \) (\( 1 \leq i \leq N \)) by

\[
\tilde{S}_i(z) = \frac{1}{(q - q^{-1})z} \sum_{j=i+1}^{N} : e^{-b_i^{i,j}(q^{N-1-j}z)}(1 - (b+c)^{i,j}(q^{N-j}z)) \cdot e^{-b_i^{i,j}(q^{N-1-j}z)} - e^{-b_i^{i,j}(q^{N-1-j}z)} : \cdot e^{(b+c)^{i,j}(q^{N-1-j}z)+\sum_{j'=j+1}^{N}(b_{i,j'}^{i,j}(q^{N-1-j}z)-b_{i,j}^{i,j}(q^{N-1-j}z)+b_{i,j}^{i,j+1,N+1}(z)-b_{i,j}^{i,j}(q^{-1}z))} \cdot q^1 : e^{b_i^{i,N+1}(z)+b_{i+1,j}^{i+1,N+1}(z)-b_{i+1,j}^{i+1,N}(qz)} : \quad (1 \leq i \leq N - 1),
\]

\[
\tilde{S}_N(z) = -q^{-1} : e^{b_i^{i,N+1}(z)} :.
\]

4.2 Vertex Operator

In this section we introduce the vertex operators \( \Phi(z), \Phi^*_a(z) \) [15]. Let \( \mathcal{F} \) and \( \mathcal{F}' \) be \( U_q(\hat{\mathfrak{sl}}(N|1)) \) representation for an arbitrary level \( k \neq -N + 1 \). Let \( V_\alpha \) and \( V_\alpha^\star \) be \( 2^N \)-dimensional typical representation
with a parameters \( \alpha \) \[21\]. Let \( \{ v_j \}_{j=1}^{2^N} \) be the basis of \( V_\alpha \). Let \( \{ v^*_j \}_{j=1}^{2^N} \) be the dual basis of \( V_\alpha^* \), satisfying \( (v_i | v^*_j) = \delta_{i,j} \). Let \( V_{\alpha,z} \) and \( V_{\alpha,z}^* \) be the evaluation module and its dual of the typical representation. For instance, the 8-dimensional representation \( V_{\alpha,z} \) of \( U_q(\hat{sl}(3|1)) \) is given by

\[
\begin{align*}
h_1 &= E_{3,3} - E_{4,4} + E_{5,5} - E_{6,6}, \\
h_2 &= E_{2,2} - E_{3,3} + E_{6,6} - E_{7,7}, \\
h_3 &= \alpha(E_{1,1} + E_{2,2}) + (\alpha + 1)(E_{3,3} + E_{4,4} + E_{5,5} + E_{6,6}) + (\alpha + 2)(E_{7,7} + E_{8,8}), \\
e_1 &= E_{3,4} + E_{5,6}, \\
e_2 &= E_{2,3} + E_{6,7}, \\
e_3 &= \sqrt{\alpha} q E_{1,2} - \sqrt{\alpha + 1} q (E_{4,5} + E_{4,6}) + \sqrt{\alpha + 2} q E_{7,8}, \\
f_1 &= E_{4,3} + E_{6,5}, \\
f_2 &= E_{3,2} + E_{7,6}, \\
f_3 &= \sqrt{\alpha} q E_{2,1} - \sqrt{\alpha + 1} q (E_{5,3} + E_{6,4}) + \sqrt{\alpha + 2} q E_{8,7}, \\
h_0 &= -\alpha(E_{1,1} + E_{4,4}) - (\alpha + 1)(E_{2,2} + E_{3,3} + E_{6,6} + E_{7,7}) - (\alpha + 2)(E_{8,5} + E_{8,8}), \\
e_0 &= -z(\sqrt{\alpha} q E_{4,1} - \sqrt{\alpha + 1} q (E_{6,2} + E_{7,3}) + \sqrt{\alpha + 2} q E_{8,5}), \\
f_0 &= z^{-1}(\sqrt{\alpha} q E_{1,4} - \sqrt{\alpha + 1} q (E_{6,2} + E_{3,7}) + \sqrt{\alpha + 2} q E_{8,5}).
\end{align*}
\]

Consider the following intertwiners of \( U_q(\hat{sl}(N|1)) \)-representation \[20\].

\[
\Phi(z) : \mathcal{F} \rightarrow \mathcal{F}' \otimes V_{\alpha,z}, \quad \Phi^*(z) : \mathcal{F} \rightarrow \mathcal{F}' \otimes V_{\alpha,z}^*.
\]

They are intertwiners in the sense that for any \( x \in U_q(\hat{sl}(N|1)) \),

\[
\Phi(z) \cdot x = \Delta(x) \cdot \Phi(z), \quad \Phi^*(z) \cdot x = \Delta(x) \cdot \Phi^*(z).
\]

We expand the intertwining operators.

\[
\Phi(z) = \sum_{j=1}^{2^N} \Phi_j(z) \otimes v_j, \quad \Phi^*(z) = \sum_{j=1}^{2^N} \Phi^*_j(z) \otimes v_j^*.
\]

We set the \( \mathbb{Z}_2 \)-grading of the intertwiner be \( |\Phi(z)| = |\Phi^*(z)| = 0 \). For \( l_a = (l^1_a, l^2_a, \cdots, l^N_a) \in \mathbb{C}^N \) and \( \beta \in \mathbb{C} \), we set the bosonic operator \( \delta^b(z|\beta) \) by

\[
\delta^b(z|\beta) := e^{\sum_{i,j=1}^{N} \left( \frac{t^1_{ij} - \min(1,1) N^* - \max(1,1) a}{a} \right) (z|\beta)(z|\beta)}.
\]

In order to balance the background charge of the vertex operators, we introduce the product of the screenings \( Q^t \) for \( t = (t_1, t_2, \cdots, t_N) \in \mathbb{N}^N \).

\[
Q^t = Q_{t_1}^1 Q_{t_2}^2 \cdots Q_{t_N}^N.
\]

The screening operator \( Q^t \) give rise to the map,

\[
Q^t : \mathcal{F}(p_a) \rightarrow \mathcal{F}(p_a + \hat{t}).
\]
Here \( \bar{t} = (t_1, t_2, \ldots, t_N) \) where \( t_i = \sum_{j=1}^{N} A_{i,j} t_j \).

**Theorem 4.2** [13] For \( k = \alpha \neq 0, -1, -2, \ldots, -N + 1 \), bosonizations of the special components of the vertex operators \( \Phi^{(t)}(z) \) and \( \Phi^*(t)^{(t)}(z) \) are given by

\[
\Phi^{(t)}_{2t}(z) = Q^{(t)}(q) \left( q^{k+N-1} \left| -\frac{k+N-1}{2} \right. \right), \\
\Phi^*(t)^{(t)}(z) = Q^{(t)}(q) \left( q^{k+N-1} \left| -\frac{k+N-1}{2} \right. \right),
\]

where we have used \( \hat{t} = -(0, \ldots, 0, \alpha + N - 1) \), \( \hat{t}^* = (0, \ldots, 0, \alpha) \) and \( t = (t_1, t_2, \ldots, t_N) \in \mathbb{N}^N \). The other components \( \Phi^{(t)}_j(z) \) and \( \Phi^*(t)^{(t)}_j(z) \) \( (1 \leq j \leq 2^N) \) are determined by the intertwining property and are represented by multiple contour integrals of Drinfeld currents and the special components \( \Phi^{(t)}_{2t}(z) \) and \( \Phi^*(t)^{(t)}(z) \). We have checked this theorem for \( N = 2, 3, 4 \).

Here we give additional explanation on the above theorem. The explicit formulae of the intertwining properties \( \Phi^{(t)}(z) \cdot x = \Delta(x) \cdot \Phi^{(t)}(z) \) for \( U_q(\mathfrak{sl}(3|1)) \) are summarized as follows. We have set the \( \mathbb{Z}_2 \)-grading of \( V_\alpha \) as follows: \( |v_1| = |v_5| = |v_6| = |v_7| = 0 \), and \( |v_2| = |v_3| = |v_4| = |v_8| = 1 \).

\[
\Phi^{(t)}_3(z) = [\Phi^{(t)}_3(z), f_1]_q, \quad \Phi^{(t)}_5(z) = [\Phi^{(t)}_5(z), f_1]_q, \\
\Phi^{(t)}_2(z) = [\Phi^{(t)}_2(z), f_2]_q, \quad \Phi^{(t)}_6(z) = [\Phi^{(t)}_6(z), f_2]_q, \\
\Phi^{(t)}_1(z) = -\frac{1}{\sqrt{[\alpha + 1]q}}[\Phi^{(t)}_2(z), f_3]_{q^\alpha}, \quad \Phi^{(t)}_3(z) = -\frac{1}{\sqrt{[\alpha + 1]q}}[\Phi^{(t)}_1(z), f_3]_{q^\alpha - 1}, \\
\Phi^{(t)}_4(z) = -\frac{1}{\sqrt{[\alpha + 2]q}}[\Phi^{(t)}_3(z), f_3]_{q^\alpha - 2}, \quad \Phi^{(t)}_2(z) = -\frac{1}{\sqrt{[\alpha + 2]q}}[\Phi^{(t)}_3(z), f_3]_{q^\alpha - 2}.
\]

The elements \( f_j \) are written by contour integral of the Drinfeld current \( f_j = \oint_{\frac{1}{2\pi \sqrt{-1}}} v_j(w) \). Hence the components \( \Phi^{(t)}_j \) \( (1 \leq j \leq 8) \) are represented by multiple contour integrals of Drinfeld currents \( x_j(w) \) \( (1 \leq j \leq 3) \) and the special component \( \Phi^{(t)}_8(z) \).

### 4.3 Correlation Function

In this section we study the correlation function as an application of the vertex operators. We study non-vanishing property of the correlation function which is defined to be the trace of the vertex operators over the Wakimoto module of \( U_q(\mathfrak{sl}(N|1)) \). We propose the \( q \)-Virasoro operator \( L_0 \) for \( k = \alpha \neq -N + 1 \) as follows.

\[
L_0 = \frac{1}{2} \sum_{i,j=1}^{N} \sum_{m \in \mathbb{Z}} \frac{a_i^m}{m_q} m^2 [\text{Min}(i,j)m]_q (N - 1 - \text{Max}(i,j))m_q |a_i^m|_q + \sum_{i,j=1}^{N} \frac{\text{Min}(i,j)(N - 1 - \text{Max}(i,j))}{(k + N - 1)(N - 1)} a_0^i \]

\[
-\frac{1}{2} \sum_{1 \leq i < j \leq N} \sum_{m \in \mathbb{Z}} b_{i,j}^m m^2 |m_q|^2 b_{i,j}^m + \frac{1}{2} \sum_{1 \leq i < j \leq N} \sum_{m \in \mathbb{Z}} c_{i,j}^m m^2 |m_q|^2 c_{i,j}^m \]

\[
+ \frac{1}{2} \sum_{1 \leq i < N} \sum_{m \in \mathbb{Z}} b_{i,N+1}^m m^2 |m_q|^2 b_{i,N+1}^m + \frac{1}{2} \sum_{1 \leq i < N} b_{i,N+1}^1 b_{i,N+1}^1.
\]
The $L_0$ eigenvalue of $|l_0,0,0\rangle$ is $\frac{1}{2(k+N)}(\bar{\lambda} + 2\bar{\rho})$, where $\bar{\rho} = \sum_{i=1}^{N} \bar{\Lambda}_i$ and $\bar{\lambda} = \sum_{i=1}^{N} l_i^0 \bar{\Lambda}_i$.

**Theorem 4.3** \[15\] For $k = \alpha \neq 0, -1, -2, \ldots, -N + 1$, the correlation function of the vertex operators,

$$\text{Tr}_{\mathcal{F}(l_a)} \left( q^{L_0} \Phi_{i_1}^{x(y_{(1)})}(w_1) \cdots \Phi_{i_m}^{x(y_{(m)})}(w_{m}) \Phi_{j_1}^{x(y_{(s)})}(z_1) \cdots \Phi_{j_n}^{x(y_{(s)})}(z_n) \right) \neq 0,$$

if and only if $x(s) = (x(s),1, x(s),2, \cdots, x(s),N) \in \mathbb{N}^N$ (1 ≤ $s$ ≤ $n$) and $y(s) = (y(s),1, y(s),2, \cdots, y(s),N) \in \mathbb{N}^N$ (1 ≤ $s$ ≤ $m$) satisfy the following condition.

$$\sum_{s=1}^{n} x(s),i + \sum_{s=1}^{m} y(s),i = \frac{(n-m)i}{N-1} \alpha + n \cdot i \quad (1 \leq i \leq N).$$

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