Sensitivity of Global Dynamics on the microscopic details of a network of dynamically coupled maps

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Here we analyze the behavior of dynamically coupled maps, based on those introduced in a series of papers by Ito & Kaneko (Phys. Rev. Lett., 88, 2002 & Phys. Rev. E, 67, 2003). We show how the microscopic coupling mechanism changes the behavior of the system both by affecting the stability of fixed points and through a more subtle effect in the crossover behavior between different regions of the parameter space. This makes it necessary to choose very carefully the exact manner in which one couples maps if they are to be used as a general model of composite systems.

I. Introduction

It is generally accepted that the specific details of microscopic couplings are irrelevant for the collective behavior of globally coupled maps (GCMs) (see [3] and references therein). There are a multitude of arbitrarily different ways in which maps may be coupled together, with none being clearly and generically better. We show here that the exact mechanism of coupling can lead to distinct behaviors of the system. Thus making it necessary to choose very carefully the exact manner in which one couples maps if they are to be used as a general model for composite systems.

Synchronization in systems of coupled maps has been widely observed [1, 2, 4, 5, 6, 7]. They are therefore of immense interest as models for understanding synchronization phenomena as observed in the real-world such as fireflies [8], heart pacemaker cells [9] etc. However, many of the models to date have their connections prescribed a priori and as such they are fixed in time. This is in contrast to real-world systems where both the units and the connections are dynamic elements. It is therefore desirable to extend these coupled systems so as to allow the interactions to evolve along with the node dynamics.

Models of interacting iterative maps fall into two general classes: The first consists of those systems where the interactions to evolve along with the node dynamics. Thus making it necessary to choose very carefully the exact manner in which one couples maps if they are to be used as a general model for composite systems.

Alternatively, the nonlinear transformation may be applied prior to the coupling, as in

\[ x_{n+1}^i = (1 - \epsilon) f(x_n^i) + \epsilon \sum_j h(x_n^j) \]  

(1)

which shall be referred to as internal coupling (see for example [1, 6]).

II. Model Details

The models studied here are based on those introduced by Ito & Kaneko in [1, 2]. When external coupling is employed, the system is described by

\[ x_{n+1}^i = (1 - \epsilon) f(x_n^i) + \epsilon \sum_{j=1}^{N} w_{ij}^N h(x_n^j) \]  

(3)

\[ w_{n+1}^{ij} = \frac{[1 + \delta \cdot g(x_n^i, x_n^j)]w_n^{ij}}{\sum_{j'=1}^{N} [1 + \delta \cdot g(x_n^i, x_n^{j'})]w_n^{ij'}} \]  

(4)

\[ g(x_n^i, x_n^j) = 1 - 2|x_n^i - x_n^j| \]  

(5)

The functional form of \( g \) was chosen so as to employ Hebbian dynamics [11], although the results shown here are qualitatively unchanged if anti-Hebbian dynamics are employed by using \( g(x_n^i, x_n^j) = |x_n^i - x_n^j| \) instead of Eq. (5) (compare Figures 4 and 6). \( \delta \) is a constant that governs the plasticity of the network and is set to 0.1 throughout the present work.

The equivalent model with internal coupling is defined

\[ w_{n+1}^{ij} = \frac{[1 + \epsilon \cdot g_h(x_n^i, x_n^j)]w_n^{ij}}{\sum_{j'=1}^{N} [1 + \epsilon \cdot g_h(x_n^i, x_n^{j'})]w_n^{ij'}} \]  

where \( g_h \) is the functional form of the coupling mechanism.
by the same equations, only with Eq. (3) replaced by

\[ x_{n+1}^{i} = f \left[ (1 - c)x_{n}^{i} + c \sum_{j=1}^{N} w_{ij} x_{n}^{j} \right] \] (6)

As with many GCM studies, we use the logistic map, \( f(x) = ax(1 - x) \) as the underlying nonlinear dynamics. This is one of the simplest functions that can display such distinct behavior, from stationary-state, through periodic to chaotic. There has also been recent evidence to suggest that functions with the form of the logistic map may indeed be of direct relevance in neuroscience [12]. We will consider two different forms of \( h(x) \),

\[ h(x) = x \] (7)

\[ h(x) = f(x) = ax(1 - x) \] (8)

When we use \( h(x) = f(x) \) in the externally coupled system, the model is similar to many externally coupled systems in the literature. Yet, this corresponds to interactions between the units occurring instantaneously which is obviously not possible in real systems. We therefore introduce a degree of causality to the model by simply using \( h(x) = x \) instead.

Each system has two basic parameters governing the dynamics: the nonlinear parameter \( a \) and the coupling strength \( c \). This \( a-c \) parameter space is known to have a series of regions of qualitatively different behavior (see [11] for a detailed discussion of these). These regions are present in both the internally and externally coupled systems studied, although the boundaries occur at different actual values and with different characteristics as will be shown in Section III C.

### III. Results

As was found for systems with fixed couplings [11] [13], we find distinct differences in the dynamics of the three systems: The system with external coupling and \( h(x) = x \) is found to enter a stationary-state after an initial transient time, see Fig. 1. This model could be thought of as representing systems where interactions occur at finite speed and there is therefore a time lag involved in the updating. The value of the node-states in the stationary state corresponds to the fixed point of the logistic map \( \bar{x} = \frac{a-1}{a} \).

In contrast, both the system with internal coupling and that with external coupling but \( h(x) = f(x) \) are never found to evolve into this state. The behavior in the coherent region appears to vary at random, see Fig. 2.

This distinction is not immediately obvious since the fixed point of the logistic map, \( \bar{x} = \frac{a-1}{a} \) is a fixed point of both the externally and internally coupled map systems. It must therefore be the stability of this fixed point that is crucially dependant on the coupling mechanism employed.

In addition, the boundaries between the different phases of parameter space occur at slightly different values of the parameters and with different characteristics. This can be seen by comparing Figs. 7, 8 and 9. Once again there is a stark difference between the behavior of the externally coupled system with \( h(x) = x \) and the others. The trend with increasing system size for the external system and \( h(x) = x \) is for the boundary between ordered and coherent behavior to occur at lower coupling strength for larger systems. This is the opposite of what happens with increasing system size for both the internally coupled system and the externally coupled system when \( h(x) = f(x) \) where the boundary between ordered and coherent behavior shifts to higher coupling strength for larger systems.

#### A. Fixed Point Stability: Analytical Results

The stability of a fixed point of a dynamical system can be analyzed through a simple perturbation analysis. Even to first order we can see where the distinction be-
FIG. 2: Same as Fig. 1, but for the model with internal coupling and $h(x) = f(x)$ for $a = 3.97, c = 0.6, N = 100$.

For the externally coupled system with $h(x) = x$, to first order we have

$$\delta_{n+1} \bigg|_{h(x)=x} \simeq (2-a)(1-c)\delta_n + (2-a)c\left(\sum_j w_{ijn}\delta_j^i\right)$$

For the externally coupled system with $h(x) = f(x)$, to first order we have

$$\delta_{n+1} \bigg|_{h(x)=f(x)} \simeq (2-a)(1-c)\delta_n + (2-a)c\left(\sum_j w_{ijn}\delta_j^i\right)$$

For the internally coupled system with $h(x) = x$, to first order we have

$$\delta_{n+1} \bigg|_{h(x)=x} \simeq (2-a)(1-c)\delta_n + (2-a)c\left(\sum_j w_{ijn}\delta_j^i\right)$$

Consider Eq. (10): The pre-factor $(2-a)$ to the second term plays a vital role in the region we are interested in, $a > 2$. Here, $(2-a) < 0$ so the first and second terms will be of the same sign and therefore add constructively causing $\delta_n^i$ to oscillate about the fixed point. Namely, that if $\delta_n^i < 0$ then $\delta_{n+1}^i > 0$. Whereas if $\delta_n^i > 0$ then $\delta_{n+1}^i < 0$.

In contrast, for the system with $h(x) = x$, Eq. (11), there is no negative pre-factor to the second term. Thus, the two terms will be of opposite sign and so add destructively. This cancelation of one another is what leads to the reduction in $\delta_n^i$ and therefore the stability of the fixed point.

We have analyzed these terms through numerical simulations and found the results (not shown here) to confirm this conjecture of the terms canceling one another for the external system with $h(x) = x$. For this system, the two terms were found to be of comparable amplitude and opposite sign and thus the fixed point of the system is stable to perturbations.

B. Fixed Point Stability: Numerical Results

FIG. 3: A bifurcation diagram for the system with external coupling and $h(x)=x$. The plot shows the behavior of all nodes for 100 time steps at various $c$-values for fixed $a = 3.97$. For each time step, each $x_n^i$ is plotted. For higher coupling values when the system is synchronized, these will all coincide and there will therefore appear to be less points plotted.

Through numerical simulations of the three systems we have been able to characterize the behavior displayed by each system. Through plotting bifurcation diagrams, we can see how this behavior is dependant upon the coupling strength parameter, $c$.

Fig. 3 shows us that the externally coupled system with $h(x) = x$ is in fact stable on the fixed point for a range of coupling strengths ($c$-values). This is in contrast to the other systems whose bifurcation diagrams show that the systems do not evolve onto a fixed point at any coupling strength, see Figs. 4 and 5.

When the system is reduced to the standard GCM with all-to-all couplings that are constant in time, these distinguishing behaviors of the systems are effectively un-
FIG. 4: A bifurcation diagram for the system with external coupling and \( h(x) = f(x) \). The plot shows the behavior of all nodes for 100 time steps at various \( c \)-values for fixed \( a = 3.97 \). The points become markedly less dense after \( c \sim 0.5 \) because the system is synchronized in this region. Thus, each time-step results in only one point on the graph as opposed to 100 (one for each node) as is the case when the system is unsynchronized.

FIG. 5: A bifurcation diagram for the system with internal coupling. The plot shows the behavior of all nodes for 100 time steps at various \( c \)-values for fixed \( a = 3.97 \).

FIG. 6: The same as Figure 4 but employing Anti-Hebbian dynamics in the evolution of the connection strengths. The plot shows the behavior of all nodes for 70 time steps.

changed. Namely, that the externally coupled system with \( h(x) = x \) has a stable fixed point whereas the other systems do not. This result should be expected since the stability as explained through linear perturbation analysis is not a result of the network topology.

C. Phase Boundaries

As was shown by Ito & Kaneko in [2] the \( a-c \) parameter space consists of distinct regions which are qualitatively different. The regions are essentially those first described in [4]; namely that for sufficiently high coupling strength there is a region of \( a-c \) parameter space where all nodes synchronize, displaying coherent behavior. For lower coupling strength, this global synchronization is lost and the system splits into distinct clusters, this region of \( a-c \) parameter space is known as the ordered region. For yet lower values of the coupling strength there is a disordered region, where there is no synchronous behavior between any pair of nodes.

We have found that as one crosses the boundary from ordered to coherent regions, the different systems show qualitatively different transitions. This can be seen by looking at the behavior of \( \sigma^2 \) as we cross the boundary by changing \( c \) at fixed \( a \)-value. Where, \( \sigma^2 \) is as defined in [13]. Namely,

\[
\sigma^2 = \frac{1}{N} \langle \sum_{i} [x_n^i - \langle x_n \rangle]^2 \rangle_t
\]  

(12)
where \( < x_n > \) denotes the average over all node-states \( x^i \) at timestep \( n \) and \( \langle \cdots \rangle_t \) denotes the average over all timesteps.

FIG. 7: Here we see the transition to synchronization and how it changes with the system size for the internally coupled system. For ease of comparison, we re-scale the data and plot \( \sigma^2 \) versus the coupling strength \( c \). This data is the average over 1000 random initial conditions. Here we show the transition for different system sizes, \( N = 10 (~\Box) \), \( N = 100 (~\bigcirc) \), \( N = 500 (~\times) \), \( N = 1000 (~+) \) and \( N = 2000 (~\triangle) \).

Figs. 7, 8 & 9 show that there is a fundamental difference in the crossover from ordered (distinct synchronized clusters) to coherent (one giant synchronized cluster) regions between the three systems. The externally coupled system with \( h(x) = f(x) \) and the internally coupled system have qualitatively similar behavior in \( \sigma^2 \); with increasing system size the transition moves to higher \( c \)-values and the sharpness increases. There is however one local defining feature for the internally coupled system, at \( c = 0.5 \): there is a localized decrease in \( \sigma^2 \). In contrast to either of these two systems, for the externally coupled system with \( h(x) = x \), the transition moves to lower \( c \)-values for larger system sizes. If these results are extrapolated to the limit of infinite system size, this suggests that only the externally coupled system with \( h(x) = x \) would be able to synchronize globally.

One feature that is common to all three systems however is the fact that the nature of the transition from one phase to another (ordered to coherent) is fundamentally different to other types of phase transition. A hallmark of phase transitions in equilibrium statistical mechanics is a discontinuity in the order parameter, in the limit of an infinite system size. Numerically, this can be seen by a sharpening of the transition with increasing system

FIG. 8: Same as Figure 7 but for the externally coupled system with \( h(x) = f(x) \).

FIG. 9: Same as Figure 7 but for the externally coupled system with \( h(x) = x \).
size. However, here the sharpening as seen in Figs. 7 & 8 possesses a subtle distinction since it is a probabilistic measure of the size of the basin of attraction for the coherent state.

For a given point in the $a$-$c$ parameter space and set of initial conditions, the system will evolve into the coherent state say. For an arbitrarily small change in the initial conditions we see that the system is attracted to the ordered state of several clusters. This apparent *riddling* of the basin of attraction of the coherent region is ordinarily not observed in standard phase transitions and gives rise to a broad and structured boundary between the coherent and ordered phases in the $a$-$c$ parameter space. Indeed, such a riddling has been found in other coupled map systems [10]. It has also been long since conjectured that riddled basins of attraction could be extremely common in natural systems [16]. In such a scenario, the parameter space phase diagrams can be misleading since they give no indication of this broad parameter region that is neither wholly in one phase nor the other.

IV. Conclusions

We have shown that the microscopic details of dynamic couplings have a dramatic effect on the macroscopic dynamics displayed by systems of globally coupled maps. Systems with asynchronous coupling are, in some cases, able to stabilize a fixed point of the underlying map. This stabilizing effect of coupling shown here is the same as was previously shown for coupled map lattices with fixed coupling or synchronously updated external coupling. If these models are to be viewed as prototype models of real world synchronization phenomena, we may use this result to gain insight into the mechanisms governing real systems.

Further, we have shown that this stabilizing of fixed points is not the only macroscopic difference between the systems. The crossover from one region of parameter space to another has been shown to be highly variable between the different systems studied. Although all systems show non-trivial mechanisms as one crosses from one phase to another. The boundary maintains a finite width with a great deal of structure to it in all three systems studied.

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