Bihamiltonian Cohomologies and Integrable Hierarchies I: A Special Case

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Abstract

We present some general results on properties of the bihamiltonian cohomologies associated to bihamiltonian structures of hydrodynamic type, and compute the third cohomology for the bihamiltonian structure of the dispersionless KdV hierarchy. The result of the computation enables us to prove the existence of bihamiltonian deformations of the dispersionless KdV hierarchy starting from any of its infinitesimal deformations.

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1 Introduction

The notion of bihamiltonian cohomologies was introduced in [14] for the study of deformations of bihamiltonian structures of hydrodynamic type and the associated integrable hierarchies, such a class of bihamiltonian integrable hierarchies have important applications in the study of Gromov-Witten theory, singularity theory and some other research fields in mathematical physics, see [33, 24, 8, 10, 11, 14, 21] and references therein. For any semisimple bihamiltonian structure of hydrodynamic type, the knowledge of the first and second bihamiltonian cohomologies are used in [27, 12] to classify their infinitesimal deformations. The purpose of the present and the subsequent papers is to study the problem of existence of deformations of semisimple bihamiltonian structures of hydrodynamic type with a given infinitesimal deformation, the main tool of our study is again provided by the bihamiltonian cohomologies.

In order to give readers a quick introduction of the notion of bihamiltonian cohomologies, let us first recall the notion of Poisson cohomology defined by Lichnerowicz for finite dimensional Poisson manifolds [25, 26], and then introduce the notion of bihamiltonian cohomologies for a Poisson manifold endowed with a second compatible Poisson structure.

Let $M^n$ be an $n$-dimensional smooth manifold endowed with a Poisson bivector $P$. The bivector $P$ is a Poisson bivector means that it satisfies the condition

$$[P, P] = 0,$$

where $[,]$ is the Schouten-Nijenhuis bracket [25] defined on the space of multi-vectors

$$\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \ldots$$

of the manifold $M$. The Poisson bivector $P$ yields a complex $(\Lambda^*, d)$ with the coboundary operator

$$d : \Lambda^k \to \Lambda^{k+1}, \ a \mapsto [P, a].$$

The Poisson cohomology is then defined by

$$H^k(M, P) = \frac{\text{Ker} \ d|_{\Lambda^k}}{\text{Im} \ d|_{\Lambda^{k-1}}}, \ k \geq 0.$$

From this definition it follows that the cohomology $H^0(M, P)$ is given by the space of Casimirs of the Poisson bracket $\{ , \}$ defined on $C^\infty(M)$ as follows:

$$\{f, g\} := P(df, dg), \ \forall f, g \in C^\infty M.$$

The other Poisson cohomologies have the following descriptions:

i) $H^1(M, P)$: quotient of the algebra of infinitesimal symmetries of the Poisson bracket by the subalgebra of Hamiltonian vector fields on $M$. 

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ii) \( H^2(M, P) \): the quotient of the space of infinitesimal deformations of the Poisson bracket by the subspace of deformations obtained by infinitesimal change of coordinates.

iii) \( H^3(M, P) \): characterizes the obstruction of extending an infinitesimal deformation of the Poisson bracket to a full deformation.

Now let us assume that the Poisson manifold \((M, P)\) is endowed with a second Poisson bivector \(P_2\) which is compatible with \(P_1 = P\), i.e. \([P_1, P_2] = 0\). In this case we say that the manifold \(M\) has a bihamiltonian structure \((P_1, P_2)\). It defines a bicomplex \((\Lambda^*, d_1, d_2)\) with the coboundary operators defined as we did above, they satisfy the conditions

\[
d_2^2 = d_1^2 = d_1 d_2 + d_2 d_1 = 0.
\]

From this bicomplex we obtain an induced complex \((\tilde{\Lambda}^* = \text{Ker} d_1, d_2)\) and we call its cohomologies the bihamiltonian cohomologies. We denote them as

\[
BH^k(M, P_1, P_2) := \frac{\text{Ker} d_2|_{\tilde{\Lambda}^k}}{\text{Im} d_2|_{\tilde{\Lambda}^{k-1}}} = \frac{\text{Ker} d_1|_{\Lambda^k} \cap \text{Ker} d_2|_{\Lambda^k}}{\text{Im} d_2|_{\tilde{\Lambda}^k-1}}, k \geq 0. \tag{1.1}
\]

It is easy to see that the zeroth bihamiltonian cohomology is given by the space of common Casimirs of the Poisson brackets defined by \(P_1, P_2\), and we also have

i) \( BH^1(M, P_1, P_2) \): the quotient of the space of common symmetries of \(P_1\) and \(P_2\) by the space of Hamiltonian vector fields of \(P_2\) generated by Casimirs of the Poisson bracket defined by \(P_1\).

ii) \( BH^2(M, P_1, P_2) \): the quotient of the space of infinitesimal deformations of the bihamiltonian structure \((P_1, P_2)\) keeping \(P_1\) to be fixed by the space of infinitesimal coordinate transformations that preserves \(P_1\).

And the vanishing of \( BH^3(M, P_1, P_2) \) implies that any infinitesimal deformation of \((P_1, P_2)\) that preserves \(P_1\) can be extended to a full deformation of the bihamiltonian structure.

The above setting of Poisson structures and their cohomologies is generalized in [14] to an infinite dimensional version. Poisson structures now live on the formal loop space \(\mathcal{L}(M)\) of \(M\), and they are given by Poisson bivectors on \(\mathcal{L}(M)\), see also [22, 23, 20] and references in [14] for similar formulations of infinite dimensional Poisson structures. Similar to the finite dimensional case, a Poisson bivector defines a Poisson bracket on the space of local functionals of \(\mathcal{L}(M)\). This version of infinite dimensional Poisson structures is formulated for the purpose of studying systems of nonlinear evolutionary PDEs of the form

\[
\frac{\partial u^i}{\partial t} = A^i_k(u) u^k_x + \epsilon \left( B^i_{k,l}(u) u^k_{xx} + C^i_{k,l}(u) u^k_x u^l_x \right) + O(\epsilon^2) \tag{1.2}
\]

which possess bihamiltonian structures with hydrodynamic limit. Here \(\epsilon\) is a dispersion parameter and \(O(\epsilon^2)\) represents the higher order in \(\epsilon\) terms \(\sum_{k \geq 2} R^i_k \epsilon^k\) with coefficients
given by homogeneous differential polynomials in $u^{i,s} = \partial^s_x u^i$ ($s \geq 1$) of degree $k+1$, here we set $\text{deg} u^{i,s} = s$. We also assume summations for repeated upper and lower indices above and in what follows unless otherwise specified. The system of evolutionary PDEs (1.2) possesses a bihamiltonian structure with hydrodynamic limit means the existence of a compatible pair of Poisson brackets of the form

$$\{u^i(x), u^j(y)\}_a = g^{i,j}_{a}(u) \delta'(x-y) + \Gamma^{i,j}_{a;k}(u) u^k_x \delta(x-y) + \sum_{k \geq 1} \epsilon^k \sum_{l=0}^{k+1} P^{i,j}_{a;k,l}(u, u_x, \ldots, u^{(k+l)}) \delta^{(k+1-l)}(x-y), \quad a = 1, 2 \quad (1.3)$$

and Hamiltonians

$$H_a = \int h_a(u(x)) dx + \sum_{k \geq 1} \epsilon^k \int f_{a;k}(u, u_x, \ldots, u^{(k)}) dx, \quad a = 1, 2$$

such that

$$\frac{\partial u^i}{\partial t} = \{u^i(x), H_1\}_1 = \{u^i(x), H_2\}_2. \quad (1.4)$$

Here the matrices $(g^{i,j}_{a}(u), a = 1, 2$ are nondegenerate, and $P^{i,j}_{a;k,l}, f_{a;k}$ are differential polynomials of degree $l$ and $k$ respectively. The bihamiltonian recursion relation given by the second equality of (1.4) enables one to find in general infinitely many Hamiltonians which are in involution w.r.t. both of the Poisson brackets [32, 20, 14, 5], these Hamiltonians yield a hierarchy of bihamiltonian integrable systems which includes the originally given one (1.2). Many important nonlinear integrable hierarchies, including the ones associated to affine Lie algebras constructed by using the Drinfeld-Sokolov reduction procedure [9] and the ones that appear in Gromov-Witten theory and singularity theory (see [33, 24, 8, 10, 11, 14, 21] and references therein), are bihamiltonian systems of the above form.

The leading terms of the r.h.s. of (1.3) obtained by setting $\epsilon = 0$ give a compatible pair of Poisson brackets $(\{, \}_{1}^{[0]}, \{, \}_{2}^{[0]})$ of hydrodynamic type. Such a bihamiltonian structure is called semisimple if the characteristic polynomial $\text{det}(g^{i,j}_{2} - \lambda g^{i,j}_{1})$ has pairwise distinct roots.

A natural problem which is important in the theory of integrable systems and its applications in mathematical physics is to classify the bihamiltonian structures of the form (1.3) and the associated integrable hierarchies. The first step toward solving this problem is done in [27, 12] with the help of the infinite dimensional version of the bihamiltonian cohomologies formulated in [14]. It classifies all the infinitesimal deformations of any given semisimple bihamiltonian structure of hydrodynamic type modulo Miura type transformations of the form

$$u^i \mapsto \tilde{u}^i = F^i_0(u) + \sum_{k \geq 1} \epsilon^k F^i_k(u, u_x, \ldots, u^{(k)}). \quad (1.5)$$

Here $F : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible smooth map, and $F^i_k$ are differential polynomials of degree $k$. It is proved in [14, 27, 12] that under the Miura type transformations
any deformation of a semisimple bihamiltonian structure of hydrodynamic type can be transformed to a bihamiltonian structure of the form (1.3) which contains only terms with even powers in \( \epsilon \), and two deformations are equivalent under a Miura type transformation if and only if they are equivalent up to \( \epsilon^2 \)-order approximation. Moreover, it is proved that any equivalent class (under the Miura type transformations) of deformations of a semisimple bihamiltonian structure of hydrodynamic type is characterized by a set of \( n \) functions \( c_i(\lambda_i) \), \( i = 1, \ldots, n \) depending on the canonical coordinates \( \lambda_i \) (see [14] for their definition) of the bihamiltonian structure. These functions are called the central invariants of the deformed bihamiltonian structure. On the other hand, any set of smooth functions \( c_1(\lambda_1), \ldots, c_n(\lambda_n) \) determines an infinitesimal deformation of a given semisimple bihamiltonian structure of hydrodynamic type – a deformation at the approximation up to \( \epsilon^2 \).

In the present and the subsequent papers we are to consider the problem of whether one can extend any infinitesimal deformation of a semisimple bihamiltonian structure of hydrodynamic type to a genuine deformation of it. To this end, we reformulate in this paper the notion of infinite dimensional Poisson structures in terms of the infinite dimensional jet space \( J^\infty(\hat{M}) \) of a super manifold \( \hat{M} \), where \( M \) is an \( n \)-dimensional manifold. This formulation provides us a more convenient way to study properties of the bihamiltonian cohomologies for a semisimple bihamiltonian structure of hydrodynamic type. In particular, the long exact sequence (4.5) which we prove in Corollary 4.6 provides an important tool for us to compute the bihamiltonian cohomologies. By using this result, we compute in this paper the third bihamiltonian cohomology, denoted by

\[
BH^3(\hat{F}) = \bigoplus_{d \geq 0} BH^3_d(\hat{F})
\]

in Section 4 of the following bihamiltonian structure

\[
\begin{align*}
\{ u(x), u(y) \}_1 &= \delta'(x - y), \\
\{ u(x), u(y) \}_2 &= u(x)\delta'(x - y) + \frac{1}{2} u_x(x)\delta(x - y)
\end{align*}
\]

for the dispersionless KdV hierarchy

\[
\frac{\partial u}{\partial t_p} = \frac{1}{p!} u^p u_x, \quad p \geq 0.
\]

The equation (1.8) of this hierarchy can be represented as a bihamiltonian system of the form (1.4) with

\[
H_{1,p} = \frac{1}{(p + 2)!} \int u(x)^{p+2} dx, \quad H_{2,p} = \frac{2}{(2p + 1)(p + 1)!} \int u(x)^{p+1} dx.
\]

We prove the following theorem in the present paper:

**Theorem 1.1** For the bihamiltonian structure (1.6), (1.7) the bihamiltonian cohomologies \( BH^3_d(\hat{F}) \) are trivial for \( d \geq 4 \).
For the bihamiltonian structure (1.6), (1.7) the canonical coordinate is given by \( \lambda = u \). As a corollary of the above theorem we have

**Theorem 1.2** For any given smooth function \( c(u) \) the bihamiltonian structure (1.6), (1.7) has a unique equivalent class of deformations with a representative of the form

\[
\begin{align*}
\{u(x), u(y)\}_1 &= \delta'(x - y), \\
\{u(x), u(y)\}_2 &= u(x)\delta'(x - y) + \frac{1}{2} u(x)\delta(x - y) \\
&\quad + \epsilon^2 \left(3c(u)\delta''(x - y) + \frac{9}{2} c'(u)u_x\delta''(x - y) + \frac{3}{2} c''(u)u_x^2\delta'(x - y) \\
&\quad + \frac{3}{2} c'(u)u_{xx}\delta'(x - y) \right) + \sum_{g \geq 2} \epsilon^{2g} \sum_{k=0}^{2g+1} A_{g,k}(u, u_x, \ldots, u^{(k)})\delta^{(2g+1-k)}(x - y).
\end{align*}
\]

Here \( A_{g,k} \) are differential polynomials of degree \( k \).

The validity of this theorem was conjectured by Lorenzoni in [31]. He also gave the approximation of the deformation up to \( \epsilon^4 \). In [1], Arsie and Lorenzoni further extend the approximation up to \( \epsilon^8 \).

The bihamiltonian structures in the above theorem for the particular choices of \( c(u) = \frac{1}{24} \) and \( c(u) = \frac{u^2}{24} \) are well known in the theory of integrable systems. They provide bihamiltonian structures for the KdV hierarchy and the Camassa-Holm hierarchy respectively [27]. For other choices of the central invariant \( c(u) \) the above bihamiltonian structures and the associated integrable hierarchies are new. In the present paper we will consider in some detail the bihamiltonian structure and the associated integrable hierarchy when \( c(u) \) is inversely proportional to \( u \).

The paper is organized as follow. In Section 2 we define the differential polynomial algebra and the Schouten bracket on the infinite jet space of a super manifold. In Section 3 we formulate the infinite dimensional Poisson structure and their cohomologies in terms of the notions established in Section 2. In Section 4 we define the notion of bihamiltonian cohomologies, and present an important method to compute the bihamiltonian cohomologies. In Section 5 we compute the third bihamiltonian cohomology of the bihamiltonian structure (1.6), (1.7), and prove Theorem 1.1 and Theorem 1.2. In Section 6 we consider examples of the bihamiltonian structures corresponding to some particular choices of the central invariant \( c(u) \). The final section is for conclusion.

### 2 The differential polynomial algebra and the Schouten bracket

Let \( M \) be a smooth manifold of dimension \( n \), \( \tilde{M} \) be the super manifold \( \Pi(T^*M) \), i.e. the cotangent bundle of \( M \) with fiber’s parity reversed. It is well known that the
algebra of multi-vectors, i.e. sections of the exterior algebra bundle $\Lambda(TM)$ of the tangent bundle $TM$, can be regarded as the algebra of smooth functions on $\hat{M}$, that is $C^\infty(\hat{M}) = \Gamma(\Lambda(TM))$.

With the help of this identification, the Schouten-Nijenhuis bracket between multi-vectors can be computed via the Poisson bracket of the canonical (super) symplectic structure on $\hat{M}$. Suppose $P \in \Gamma(\Lambda^p(TM))$, $Q \in \Gamma(\Lambda^q(TM))$ are two multi-vectors with degree $p$ and $q$ respectively, their Schouten-Nijenhuis bracket $[P, Q]$, which is a multi-vector of degree $p + q - 1$, has the following expression on a local trivialization $\hat{U} = U \times \mathbb{R}^{0|n}$ of $\hat{M}$

$$[P, Q] = \frac{\partial P}{\partial \theta^i} \frac{\partial Q}{\partial u^i} + (-1)^p \frac{\partial P}{\partial u^i} \frac{\partial Q}{\partial \theta^i},$$  \hspace{1cm} (2.1)$$

where $u^1, \ldots, u^n$ are coordinates on $U$, and $\theta_1, \ldots, \theta_n$ are the corresponding dual coordinates on the fiber $\mathbb{R}^{0|n}$.

In this section, we introduce the infinite dimensional analog of (2.1) on the infinite jet space $J^\infty(\hat{M})$.

Let $J^k(\hat{M}) (k \geq 1)$ be the $k$-th jet space of $\hat{M}$. The manifold $J^k(\hat{M})$ is, by definition, a fiber bundle with fibers being the spaces of $k$-th Taylor polynomials of germs of curves on $\hat{M}$. Let $\hat{U}$ be a local chart of $\hat{M}$ with local coordinates $(u^i, \theta^i) (i = 1, \ldots, n)$, and $\hat{U} \times \mathbb{R}^{nk|nk}$ be a local trivialization of $J^k(\hat{M})$ over $\hat{U}$, we can take the coordinates on the fiber $\mathbb{R}^{nk|nk}$ as the values of higher derivatives of curves

$$u^{i,s} = \frac{d^s}{dx^s} u^i(x) \bigg|_{x=0}, \quad \theta^s = \frac{d^s}{dx^s} \theta_i(x) \bigg|_{x=0}, \quad s = 1, \ldots, k.$$  \hspace{1cm} (2.2)$$

It is easy to see that the transition functions of $J^k(\hat{M})$, induced by the change of coordinates on $\hat{M}$ from $(u^i, \theta_i)$ to $(\tilde{u}^i, \tilde{\theta}_i)$ with

$$\tilde{u}^i = \tilde{u}^i(u^1, \ldots, u^n), \quad \tilde{\theta}_i = \frac{\partial u^i}{\partial \tilde{u}^j} \theta_j,$$

are given by the chain rule of higher derivatives as follows:

$$\tilde{u}^{i,s} = \sum_{t \geq 0} \frac{\partial \tilde{u}^{i,s-1}}{\partial u^j} u^{j,t+1}, \quad \tilde{\theta}^s = \sum_{t=0}^{s} \binom{s}{t} \frac{\partial u^{j,t}}{\partial \tilde{u}^i} \theta^{s-t}_j.$$  \hspace{1cm} (2.2)$$
For example, when $s = 1, 2$ we have
\[
\dot{u}^{i,1} = \frac{\partial \dot{u}^i}{\partial u^j} u^{j,1}, \\
\dot{\theta}^1_i = \frac{\partial \dot{\theta}^i}{\partial \theta^j} \theta^{j,1} + \frac{\partial \dot{\theta}^i}{\partial u^j} u^{j,1} \theta^{1,j}, \\
\dot{u}^{i,2} = \frac{\partial \dot{u}^i}{\partial u^j} u^{j,2} + \frac{\partial^2 \dot{u}^i}{\partial u^j \partial u^m} u^{j,1} u^{m,1}
\]
\[
\dot{\theta}^2_i = \frac{\partial \dot{\theta}^i}{\partial u^j} \theta^{j,1} + 2 \frac{\partial^2 \dot{\theta}^i}{\partial u^j \partial \dot{u}^l} \dot{u}^{l,1} \theta^{j,l}
\]
\[
+ \left( \frac{\partial \dot{\theta}^i}{\partial \dot{u}^j \partial \dot{u}^l} \dot{u}^{l,2} + \frac{\partial^3 \dot{\theta}^i}{\partial \dot{u}^j \partial \dot{u}^l \partial \dot{u}^m} \dot{u}^{l,1} \dot{u}^{m,1} \right) \theta^j.
\]
Note that these transition functions are always polynomials of the jet variables $u^{i,s}, \theta^s_i$ ($s = 1, 2, \ldots$) with coefficients being smooth functions on $M$.

For $k \geq l$, there exists a natural forgetful map
\[
\pi_{k,l} : J^k(\hat{M}) \to J^l(\hat{M}),
\]
which just forgets the derivatives with orders greater than $l$. It is easy to see that the system
\[
\left( \{J^k(\hat{M})\}_{k \geq 1}, \{\pi_{k,l}\}_{k \geq l \geq 1} \right)
\]
forms a projective system, so we can take its projective limit
\[
J^\infty(\hat{M}) = \lim_{\leftarrow k} J^k(\hat{M}),
\]
which is called the infinite jet space of $\hat{M}$.

The algebras of smooth functions on $\{J^k(\hat{M})\}_{k \geq 1}$ and the pullback maps $\{\pi_{k,l}^*\}_{k \geq l \geq 1}$ among them form an inductive system, so we can define its inductive limit
\[
C^\infty(J^\infty(\hat{M})) = \lim_{\rightarrow k} C^\infty(J^k(\hat{M})),
\]
which is called the algebra of smooth functions on $J^\infty(\hat{M})$.

A function $f \in C^\infty(J^\infty(\hat{M}))$ is called a differential polynomial if it depends on the jet variables polynomially in certain local coordinate system. Due to the form of transition functions [22], this definition is independent of the choice of local coordinate systems. All differential polynomials form a subalgebra of $C^\infty(J^\infty(\hat{M}))$, we denote this subalgebra by $\hat{A}$. Locally, we can regard $\hat{A}$ as
\[
C^\infty(\hat{U})[u^{i,s}, \theta^s_i \mid i = 1, \ldots, n, \ s = 1, 2, \ldots],
\]
where $\hat{U}$ is a chart on $\hat{M}$ with local coordinates $(u^i, \theta_i) \ (i = 1, \ldots, n)$. 

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The space $\hat{A}$ is not big enough to define, say, the inverse of a Miura-type transformation, so we need to enlarge it. First we introduce a gradation on $\hat{A}$ by defining
\[
\deg u^{i,s} = \deg \theta^s_i = s \text{ for } s > 0, \quad \text{and } \deg f = 0 \text{ for } f \in C^\infty(\hat{M}),
\]
then for any element $f \in \hat{A}$, one can decompose it into a direct sum of homogeneous components
\[
f = f_0 + f_1 + \ldots, \quad \text{where } \deg f_k = k.
\]
This gradation induces the valuation
\[
\nu(f) = \begin{cases} 
\min \{k \mid f_k \neq 0\}, & f \neq 0 \\
\infty, & f = 0
\end{cases}
\]
We denote the distance induced by this valuation by
\[
d(f, g) = e^{-\nu(f-g)}
\]
and complete $\hat{A}$ by using this distance. By abuse of notations, we still denote the completion of $\hat{A}$ by $\hat{A}$, and still call it the differential polynomial algebra of $\hat{M}$. Locally, the completed $\hat{A}$ looks like
\[
C^\infty(\hat{U})[[u^{i,s}, \theta^s_i \mid i = 1, \ldots, n, \ s = 1, 2, \ldots]].
\]
Note that the topology on $C^\infty(\hat{U})$ is, by definition, the discrete one, so only finite sum of smooth functions are allowed. Elements of $\hat{A}$ are still called differential polynomials, though they may be formal series of homogeneous differential polynomials with strict increasing degrees. We call the above defined gradation the standard gradation, and denote the degree $d$ component of $\hat{A}$ w.r.t this gradation by $\hat{A}_d$.

Sometimes we use a formal parameter $\epsilon$ to count the degree of homogeneous terms, i.e. we represent $f \in \hat{A}$ in the form
\[
f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \ldots, \quad \text{where } f_k \in \hat{A}_k.
\]
In this way, the topology on $\hat{A}$ is just the $\epsilon$-adic topology.

The super variables in $\hat{A}$ induce another gradation
\[
\deg \theta^s_i = 1, \quad \deg u^{i,s} = \deg f = 0.
\]
We call it super gradation, and denote the degree $p$ component of $\hat{A}$ w.r.t this gradation by $\hat{A}_p$. We also use the notation $\hat{A}_d^p = \hat{A}_d \cap \hat{A}_p$. The degree $0$ component $\hat{A}_d^0$ is denoted by $\hat{A}$, and is called the differential polynomial algebra on $\hat{M}$. It’s easy to see that $\hat{A}_d^0 = C^\infty(\hat{M})$.

By using the form of the transition functions given in (2.2), one can show that the vector field
\[
\partial = \sum_{s>0} \left( u^{i,s+1} \frac{\partial}{\partial u^{i,s}} + \theta^{s+1}_i \frac{\partial}{\partial \theta^s_i} \right)
\]
is globally defined on $J^\infty(\hat{M})$, where $u^{i,0} = u^i$ and $\theta^0_i = \theta_i$. It defines a derivation on $\hat{A}$, whose restriction on $\hat{A}$ also yields a derivation on $\hat{A}$, so we have the following definition.
**Definition 2.1** We denote \( \hat{\mathcal{F}} = \hat{\mathcal{A}}/\partial \hat{\mathcal{A}} \) and call its elements local functionals on \( \hat{M} \). We denote its subspace \( \hat{\mathcal{F}}^0 = \mathcal{A}/\partial \mathcal{A} \) by \( \mathcal{F} \) and call its elements local functional on \( M \).

In what follows we will use \( \int \) to denote the projection \( \mathcal{A} \to \mathcal{F} \) and \( \hat{\mathcal{A}} \to \hat{\mathcal{F}} \).

**Definition 2.2** We denote by \( \mathcal{E} = \text{Der}(\mathcal{A})^0 \) the centralizer of \( \partial \) in the Lie algebra of continuous derivations of \( \mathcal{A} \) and call its elements evolutionary vector fields on \( M \).

From the above definition it follows that any evolutionary vector field takes the form
\[
D_X = \sum_{s \geq 0} \partial^s(X^i) \frac{\partial}{\partial u^i,s}, \text{ where } X^i \in \mathcal{A}.
\] (2.3)

If we define
\[
X = \int X^i \theta_i,
\] (2.4)
then one can verify that \( X \) is globally defined on \( \hat{M} \), so it is in fact an element of \( \hat{\mathcal{F}}^1 \). On the other hand, each element of \( \hat{\mathcal{F}}^1 \) has a representative of the form (2.4), so it corresponds to an element \( D_X \in \mathcal{E} \). Thus we will identify \( \mathcal{E} \) with \( \hat{\mathcal{F}}^1 \) in this way from now on.

We can also associate to any element (2.3) of \( \mathcal{E} \) the following system of evolutionary PDEs
\[
u^i = X^i.
\] (2.5)

If \( u = (u^1, \ldots, u^n) \) is a solution of this system, and \( f \) is a differential polynomial in \( \mathcal{A} \), then we have
\[
f_t = D_X(f).
\]

So a local functional \( F \in \mathcal{F} \) is a conserved quantity of (2.5) if and only if \( D_X(F) = 0 \). Here we used the fact that \( \mathcal{F} \) is a module of the Lie algebra \( \mathcal{E} \).

Now let us define an infinite dimensional analog of the bracket (2.1) on the space \( \hat{\mathcal{F}} \). First we introduce the variational derivatives of \( f \in \hat{\mathcal{A}} \) w.r.t \( \theta_i \) and \( u^i \) as follows:
\[
\delta^i f = \sum_{s \geq 0} (-\partial)^s \frac{\partial f}{\partial \theta_i^s}, \quad \delta_i f = \sum_{s \geq 0} (-\partial)^s \frac{\partial f}{\partial u^i,s}, \quad i = 1, \ldots, n.
\]

It is well known that
\[
\delta^i \partial = 0, \quad \delta_i \partial = 0,
\]
so they induce maps from \( \hat{\mathcal{F}} \) to \( \hat{\mathcal{A}} \). We denote the induced maps by the same notations \( \delta^i \) and \( \delta_i \). Then we define a bracket
\[
[ \quad , \quad ] : \hat{\mathcal{F}}^p \times \hat{\mathcal{F}}^q \to \hat{\mathcal{F}}^{p+q-1}, \quad (P, Q) \mapsto [P, Q],
\]
where
\[
[P, Q] = \int (\delta^i P \delta_i Q + (-1)^p \delta^i P \delta_i Q).
\] (2.6)
Theorem 2.3 ([30]) For any $P \in \hat{F}^p$, $Q \in \hat{F}^q$, $R \in \hat{F}^r$, we have

i) $[P, Q] = (-1)^{pq}[Q, P]$;

ii) $(-1)^{pq}[[P, Q], R] + (-1)^{qr}[[Q, R], P] + (-1)^{qr}[[R, P], Q] = 0$.

The bracket $[,]$ is called the Schouten-Nijenhuis bracket on $J^\infty(\hat{M})$. From now on, we call it the Schouten bracket for short.

In order to compute the bihamiltonian cohomologies, we also need another useful map. First we denote by $\hat{E}^p_d$ the space of super derivations of degree $(p, d)$ over $\hat{A}$. Here a super derivation of degree $(p, d)$ over $\hat{A}$ is a continuous linear map $\Delta : \hat{A} \to \hat{A}$ satisfying

$$\Delta(\hat{A}^p_d) \subset \hat{A}^p_d + \hat{A}^{p+1}_d,$$

and

$$\Delta(f \cdot g) = \Delta(f) \cdot g + (-1)^{pk} f \cdot \Delta(g),$$

where $f \in \hat{A}^k$.

We also denote

$$\hat{E} = \prod_{d \in \mathbb{Z}} \hat{E}^{p,d},$$

Then we define a map

$$D : \hat{F}^p \to \hat{E}^{p-1}, \quad P \mapsto D_P,$$

where

$$D_P = \sum_{s \geq 0} \left( \partial^s(\delta^i P) \frac{\partial}{\partial u^{i,s}} + (-1)^p \partial^s(\delta^i P) \frac{\partial}{\partial \theta^{i,s}} \right).$$

(2.7)

Note that for $P = X \in \mathcal{F}^1$ the operator $D_P$, when it is restricted to $\mathcal{A}$, coincides with the one that is defined in (2.3).

It is well known that the space $\hat{E}$ has a Lie superalgebra structure w.r.t. the super degree, i.e. for $D_1 \in \hat{E}^{p_1}_{d_1}$, $D_2 \in \hat{E}^{p_2}_{d_2}$, the bracket

$$[D_1, D_2] = D_1 D_2 - (-1)^{p_1 p_2} D_2 D_1 \in \hat{E}^{p_1+p_2}_{d_1+d_2}$$

is a super Lie bracket over $\hat{E}$.

Theorem 2.4 ([30]) For any $P \in \hat{F}^p$, $Q \in \hat{F}^q$, we have

i) $[P, Q] = \int D_P(Q)$;

ii) $D_{[P, Q]} = (-1)^{p^{-1}}[D_P, D_Q]$.

Remark 2.5 If we regard $P \in \hat{F}^p$ as a Hamiltonian, then $D_P$ is the corresponding Hamiltonian vector field, and the identity ii) of the above theorem is just the analog of the homomorphism from the Poisson algebra of a symplectic manifold to the Lie algebra of its Hamiltonian vector fields.
3 Hamiltonian structures and their cohomologies

Now we are ready to formulate the infinite dimensional version of Poisson structures which we will also call them Hamiltonian structures.

Definition 3.1 A bivector $P \in \hat{F}^2$ is called a Poisson bivector or a Hamiltonian structure if $[P, P] = 0$. We say that an evolutionary vector field $X \in \mathcal{E}$ has a Hamiltonian structure if there exists a Poisson bivector $P \in \hat{F}^2$ and a local functional $F \in \mathcal{F}$ such that $X = -[P, F]$.

For a given Poisson bivector $P$, we define the associated Poisson bracket as follows:

$$\{ , \}_P : \mathcal{F} \times \mathcal{F} \to \mathcal{F}, \quad (F, G) \mapsto \{F, G\}_P,$$

where $\{F, G\}_P = -[F, [P, G]]$. In particular, if $P$ is given by

$$P = \frac{1}{2} \int P_{ij} \theta_i \theta_j,$$

where $P_{ij}$ are normalized to satisfy the antisymmetric condition

$$P_{ij} \partial^s = (-1)^{s+1} \partial^s P_{ji}, \quad (3.1)$$

then

$$\{F, G\}_P = \int \delta F \left( P_{ij} \partial^s \right) \left( \delta G \right).$$

Example 3.2 Suppose $P \in \hat{F}_0^2$, then $P$ must take the following form

$$P = \frac{1}{2} \int P^{ij}(u) \theta_i \theta_j \quad \text{with} \quad P^{ij} = -P^{ji},$$

so it actually corresponds to the following bivector on $M$:

$$\bar{P} = \frac{1}{2} P^{ij}(u) \frac{\partial}{\partial u^i} \wedge \frac{\partial}{\partial u^j}.$$

It's easy to see that the bracket $[P, P]$ (see (2.6)) coincides with $[\bar{P}, \bar{P}]$ (see (2.1)), so $P$ is a Poisson bivector if and only if $P$ is a Poisson bivector on $M$.

Example 3.3 ([13]) Suppose $P \in \hat{F}_1^2$, then $P$ can be represented as

$$P = \frac{1}{2} \int \left( g^{ij}(u) \theta_i \theta_j + \Gamma^{ij}_k(u) u^k \theta_i \theta_j \right)$$

which satisfies the normalization condition (3.1). We also assume that the matrix $(g^{ij})$ is nondegenerate, then $P$ is a Hamiltonian structure if and only if
i) $g = (g_{ij}) = (g^{ij})^{-1}$ is a flat metric on $M$;

ii) $\Gamma^j_{ik} = -g_{il}\Gamma^l_{jk}$ is the Levi-Civita connection of $g$.

Hamiltonian structures of this form are called of hydrodynamic type.

By using Theorem 2.3 and the definition of Hamiltonian structures, one can easily prove the following lemma.

**Lemma 3.4** Let $P \in \hat{F}^2$ be a Hamiltonian structure, $d_P$ be its adjoint action, i.e.

$$d_P(Q) = [P, Q], \quad \forall Q \in \hat{F}.$$  

Then we have

$$d_P^2 = 0, \quad D_P^2 = 0.$$  

Thus given a Hamiltonian structure $P \in \mathcal{F}^2$ we have two complexes $(\hat{A}, D_P)$ and $(\hat{F}, d_P)$. Let $\mathbb{R}$ be the subalgebra of $\hat{A}$ consists of constant functions on $J^\infty(M)$, then one can show that $\mathbb{R}$ is the kernel of $\partial : \hat{A} \to \hat{A}$. Note that $\mathbb{R}$ is contained in $D_P$’s kernel, so we can define another complex $(\hat{A}/\mathbb{R}, D_P)$.

**Lemma 3.5** The following sequence of complexes is exact

$$0 \to (\hat{A}/\mathbb{R}, D_P) \xrightarrow{\partial} (\hat{A}, D_P) \xrightarrow{\int} (\hat{F}, d_P) \to 0,$$

hence we have a long exact sequence of cohomologies,

$$\cdots \to H^p(\hat{A}/\mathbb{R}) \to H^p(\hat{A}) \to H^p(\hat{F}) \to H^{p+1}(\hat{A}/\mathbb{R}) \to \cdots. \quad (3.2)$$

Now we assume that $P$ is a Hamiltonian structure of hydrodynamic type, so $P \in \hat{F}^2_1$ and $D_P \in \hat{E}^1_1$. Then the homologies of the above three complexes become direct sums of their homogeneous components, and the long exact sequence (3.2) can be written as long exact sequences of homogeneous components:

$$\cdots \to H^p_{d-1}(\hat{A}/\mathbb{R}) \to H^p_d(\hat{A}) \to H^p_d(\hat{F}) \to H^{p+1}_d(\hat{A}/\mathbb{R}) \to \cdots. \quad (3.3)$$

Note that $\mathbb{R} \subset \hat{A}_0^1$, so we have

$$H^p_d(\hat{A}/\mathbb{R}, D_P) = \begin{cases} 
H^p_d(\hat{A}, D_P), & (p, d) \neq (0, 0); \\
H^0_d(\hat{A}, D_P)/\mathbb{R}, & (p, d) = (0, 0).
\end{cases}$$

**Lemma 3.6** [30] Let $P \in \hat{F}^2_1$ be a Hamiltonian structure of hydrodynamic type, then we have

$$H^p_d(\hat{A}, D_P) = \begin{cases} 
0, & d > 0; \\
\Lambda^p(\mathbb{R}^n), & d = 0.
\end{cases} \quad (3.4)$$
From the long exact sequence (3.3) and the above lemma we have the following theorem [30]:

**Theorem 3.7** Let $P \in \mathcal{F}_1^2$ be a Hamiltonian structure of hydrodynamic type, then we have

$$H^p_{>0}(\mathcal{F}, d_P) = 0.$$  \hspace{1cm} (3.5)

This theorem is first proved in [20], and it is also proved in [5, 14] for the $p = 1, 2$ cases. Recently, De Sole and Kac [6, 7] prove some theorems on cohomologies of Poisson vertex algebra, which also imply the above theorem. In particular, the fact that $H^2_{>1}(\mathcal{F}, d_P) = 0$ implies that any deformation of $P$ can be eliminated by a Miura type transformation.

## 4 Bihamiltonian structures and their cohomologies

In this section we give the definition of bihamiltonian cohomologies and show its main properties.

**Definition 4.1** Let $P_1, P_2 \in \mathcal{F}^2$ be two Poisson bivectors, if $[P_1, P_2] = 0$, then we call the pair $(P_1, P_2)$ a bihamiltonian structure.

Let $(P_1, P_2)$ be a bihamiltonian structure, and denote

$$D_1 = D_{P_1}, \quad D_2 = D_{P_2}, \quad d_1 = d_{P_1}, \quad d_2 = d_{P_2}.$$  

Then we have

$$D_1D_2 + D_2D_1 = 0, \quad d_1d_2 + d_2d_1 = 0,$$

so we obtain two double complexes $(\mathcal{A}, D_1, D_2)$ and $(\mathcal{F}, d_1, d_2)$.

**Definition 4.2** Let $(C, \partial_1, \partial_2)$ be the double complex $(\mathcal{A}, D_1, D_2)$ or $(\mathcal{F}, d_1, d_2)$, its bihamiltonian cohomology is defined as

$$BH^p_d(C, \partial_1, \partial_2) = \frac{C^p_d \cap \text{Ker} \partial_1 \cap \text{Ker} \partial_2}{C^p_d \cap \text{Im}(\partial_1 \partial_2)}.$$  \hspace{1cm} (4.1)

We often denote it by $BH^p_d(C)$ for short.

**Remark 4.3** Due to Lemma 3.6 and Theorem 3.7, the above definition of bihamiltonian cohomologies is in agreement with the one given in (1.1) for finite dimensional bihamiltonian structures when $d \geq 2$. 

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Lemma 4.4 Let \((C, \partial_1, \partial_2)\) be the double complex \((\bar{A}, D_1, D_2)\) or \((\bar{F}, d_1, d_2)\). Define
\[
C_\lambda = C \otimes \mathbb{R}[\lambda], \quad \partial_\lambda = \partial_2 - \lambda \partial_1,
\]
then \((C_\lambda, \partial_\lambda)\) is a complex. Moreover, if \(P_1\) is a Hamiltonian structure of hydrodynamic type, then for any \(p \in \mathbb{N}\) and \(d \geq 2\), we have
\[
BH^p_d(C, \partial_1, \partial_2) \cong H^p_d(C_\lambda, \partial_\lambda).
\]

Proof We embed \(C\) into \(C_\lambda\) in the natural way. It’s easy to see that if \(z \in C^p_d \cap \text{Ker} \partial_1 \cap \text{Ker} \partial_2\), then \(z \in (C_\lambda)^p_d \cap \text{Ker} \partial_\lambda\), so there is an embedding
\[
j : C^p_d \cap \text{Ker} \partial_1 \cap \text{Ker} \partial_2 \to (C_\lambda)^p_d \cap \text{Ker} \partial_\lambda.
\]
We are to show that this embedding induces the isomorphism \((4.2)\).

Let \(\pi\) be the projection
\[
\pi : (C_\lambda)^p_d \cap \text{Ker} \partial_\lambda \to H^p_d(C_\lambda, \partial_\lambda),
\]
we first show that the composition \(\pi j\) is surjective. Let \(x \in (C_\lambda)^p_d \cap \text{Ker} \partial_\lambda\), we need to prove that there exists \(y \in (C_\lambda)^{p-1}_{d-1}\) such that \(x - \partial_\lambda y \in \text{Im} j\). Suppose \(x\) takes the following form
\[
x = x_0 + x_1 \lambda + \cdots + x_m \lambda^m,
\]
where \(x_k \in C^p_d\), the condition \(\partial_\lambda x = 0\) implies that
\[
0 = \partial_2 x_0, \quad \partial_1 x_0 = \partial_2 x_1, \quad \cdots, \quad \partial_1 x_{m-1} = \partial_2 x_m, \quad \partial_1 x_m = 0.
\]
Since \(H^p_d(C, \partial_1) \cong 0\), we know that there exists \(y_{m-1} \in C^{p-1}_{d-1}\) such that
\[
x_m = -\partial_1 y_{m-1},
\]
then we have \(\partial_1 x_{m-1} = \partial_1 \partial_2 y_{m-1}\), so there exists \(y_{m-2} \in C^{p-1}_{d-1}\) such that
\[
x_{m-1} = \partial_2 y_{m-1} - \partial_1 y_{m-2}.
\]
By induction, one can show the existence of \(y_{k-1} \in C^{p-1}_{d-1}\) such that
\[
x_k = \partial_2 y_k - \partial_1 y_{k-1}, \quad k = 1, \ldots, m - 1.
\]
Now let
\[
y = y_0 + y_1 \lambda + \cdots + y_{m-1} \lambda^{m-1} \in (C_\lambda)^{p-1}_{d-1},
\]
then from \((4.3)\), \((4.4)\) it follows that
\[
x - \partial_\lambda y = x_0 - \partial_2 y_0 \in \text{Im} j.
\]
The surjectivity is proved.
Next we need to show that $\text{Ker}(\pi_j) = C^p_d \cap \text{Im}(\partial_1 \partial_2)$. If $z = \partial_1 \partial_2 w$, then
\[ z = \partial_\lambda(-\partial_1 w), \]
so $C^p_d \cap \text{Im}(\partial_1 \partial_2) \subset \text{Ker}(\pi_j)$. Conversely, suppose $z \in C^p_d \cap \text{Ker} \partial_1 \cap \text{Ker} \partial_2$ is exact in $H^p_d(\Lambda, \partial_\lambda)$, i.e. there exists $y \in (C^p_d)_{p-1}^{-1}$ such that $z = \partial_\lambda y$. By using the fact that $H^{p-1}_{d-1}(C, \partial_1) \cong 0$ and a similar argument as we used above in the proof of the surjectivity of $\pi_j$, it can be shown that we can choose this $y$ such that it belongs to $C$. Then we have
\[ z = \partial_2 y, \quad 0 = \partial_1 y. \]
Since $H^{p-1}_{d-1}(C, \partial_1) \cong 0$, there exists $w \in C^{p-2}_{d-2}$ such that $y = -\partial_1 w$. From this it follows that $z = \partial_1 \partial_2 w$, so we have $\text{Ker}(\pi_j) \subset \text{Im}(\partial_1 \partial_2)$.

Finally, $H^p_d(\Lambda, \partial_\lambda) = \text{Im}(\pi_j) \cong \frac{C^p_d \cap \text{Ker} \partial_1 \cap \text{Ker} \partial_2}{\text{Ker}(\pi_j)} = BH^p_d(C, \partial_1, \partial_2)$.

The lemma is proved. \[ \square \]

**Remark 4.5** A similar description of $BH^p_d(C, \partial_1, \partial_2)$ was given in [2], in which the role of the complex $(\Lambda, \partial_\lambda)$ is replaced by a bicomplex $C^{**}$.

**Corollary 4.6** If $P_1$ is a Hamiltonian structure of hydrodynamic type, then for any $p \in \mathbb{N}$ and $d \geq 2$ there exists a long exact sequence:
\[ \cdots \rightarrow BH^p_{d-1}(\hat{A}) \rightarrow BH^p_d(\hat{A}) \rightarrow BH^p_d(\hat{F}) \rightarrow BH^{p+1}_d(\hat{A}) \rightarrow \cdots. \] (4.5)

**Proof** Denote
\[ \hat{A}_\lambda = \hat{A} \otimes \mathbb{R}[\lambda], \quad D_\lambda = D_2 - \lambda D_1, \]
\[ \hat{F}_\lambda = \hat{F} \otimes \mathbb{R}[\lambda], \quad d_\lambda = d_2 - \lambda d_1, \]
then we have a short exact sequence
\[ 0 \rightarrow (\hat{A}_\lambda / \mathbb{R}[\lambda], D_\lambda) \xrightarrow{\partial} (\hat{A}_\lambda, D_\lambda) \xrightarrow{f} (\hat{F}_\lambda, d_\lambda) \rightarrow 0. \]
This short exact sequence implies a long exact sequence
\[ \cdots \rightarrow H^p_{d-1}(\hat{A}_\lambda / \mathbb{R}[\lambda]) \rightarrow H^p_d(\hat{A}_\lambda) \rightarrow H^p_d(\hat{F}_\lambda) \rightarrow H^{p+1}_d(\hat{A}_\lambda / \mathbb{R}[\lambda]) \rightarrow \cdots, \]
which is isomorphic to (4.5), according to Lemma 1.4. \[ \square \]

For a semisimple bihamiltonian structure $(P_1, P_2)$ of hydrodynamic type, it is proved in [27, 12] that the second bihamiltonian cohomologies has the following property:
\[ BH^2_d(\hat{F}) \cong 0, \text{ if } d \geq 2 \text{ and } d \neq 3, \]
This knowledge of the second bihamiltonian cohomologies enables us to classify the infinitesimal deformations of \((P_1, P_2)\). In order to study the existence of a genuine deformation of \((P_1, P_2)\) with a given infinitesimal one, we need to compute the third cohomologies \(BH_3^{\geq 6}(\mathcal{F})\) and to show its triviality.

The computation carried out in [27, 12] to prove the above result on the second bihamiltonian cohomologies is quite involved. This is mainly due to the fact that the definition of the differentials \(d_1, d_2\) on \(\hat{F}\) are complicated, so it is difficult to perform similar computations to study the third bihamiltonian cohomologies. Fortunately, the long exact sequence (4.5) given in Corollary 4.6 helps us to overcome this difficulty. From this long exact sequence it follows that the triviality of \(BH_3^{d_2 \geq 6}(\hat{F})\) can be proved by showing that \(BH_3^{\geq 6}(\hat{A}) \cong 0\) and \(BH_4^{d_2 \geq 6}(\hat{A}) \cong 0\). Note that the definition of the differentials \(D_1, D_2\) on \(\hat{A}\) are much more simple than that of the differentials \(d_1, d_2\) on \(\hat{F}\), so it is easier to compute the bihamiltonian cohomologies \(BH_p^{d}(\hat{A})\) than to compute \(BH_p^{d}(\hat{F})\).

In the following section, we illustrate the above mentioned method of computing bihamiltonian cohomologies by proving Theorem 1.1 and Theorem 1.2 for the bihamiltonian structure (1.6), (1.7) of the dispersionless KdV hierarchy.

## 5 The third bihamiltonian cohomology for (1.6), (1.7)

The following subsections are devoted to prove Theorem 1.1 and Theorem 1.2.

### 5.1 Some preparations

For the bihamiltonian structure of the dispersionless KdV hierarchy we have \(M = \mathbb{R}\), so we can omit the index \(i\) in the notations such as \(u^{i,s}, \theta^i_s\). The bihamiltonian structure (1.6), (1.7) can be represented by the bivectors

\[
P_1 = \frac{1}{2} \int \theta \theta^1, \quad P_2 = \frac{1}{2} \int u \theta \theta^1,
\]

so we have

\[
D_1 = \sum_{s \geq 0} \theta^{s+1} \frac{\partial}{\partial u^s},
\]

\[
D_2 = \sum_{s \geq 0} \left( \partial^s \left( u \theta^1 + \frac{1}{2} u_1 \theta \right) \frac{\partial}{\partial u^s} + \partial^s \left( \frac{1}{2} \theta \theta^1 \right) \frac{\partial}{\partial \theta^s} \right).
\]

The following identities are useful and easy to prove, so we list them here and omit the proof.
Lemma 5.1

\[
\begin{align*}
\left[ \frac{\partial}{\partial u^k}, D_1 \right] &= 0, \\
\left[ \frac{\partial}{\partial \theta^k}, D_1 \right] &= \frac{\partial}{\partial u^{k-1}}, \\
\left[ \frac{\partial}{\partial u^k}, D_2 \right] &= \sum_{l \geq k-1} \left( \binom{l}{k} + \frac{1}{2} \binom{l}{k-1} \right) \theta^{l+1-k} \frac{\partial}{\partial u^l}, \\
\left[ \frac{\partial}{\partial \theta^k}, D_2 \right] &= \sum_{l \geq k-1} \left( \binom{l}{k-1} + \frac{1}{2} \binom{l}{k} \right) u^{l+1-k} \frac{\partial}{\partial u^l} \\
&\quad + \frac{1}{2} \sum_{l \geq k-1} \left( \binom{l}{k} - \binom{l}{k-1} \right) \theta^{l+1-k} \frac{\partial}{\partial \theta^l}.
\end{align*}
\]

Suppose \( Q \in \hat{A}_d^p \ (d \geq 2) \) satisfies

\[
D_1 Q = 0, \quad D_2 Q = 0,
\]

to prove the triviality of \( BH_d^p(\hat{A}) \), we need to find \( X \in \hat{A}_d^{p-2} \) such that

\[
Q = D_1 D_2 X.
\]

According to Lemma 3.6 there exists \( F \in \hat{A}_d^{p-1} \), such that

\[
Q = D_1 F, \quad D_1 D_2 F = 0.
\]

So in order to prove Theorem 1.1 we need to do the following:

For any given \((p, d) = (3, \geq 6)\) or \((4, \geq 6)\) and \( F \in \hat{A}_d^{p-1} \) satisfying \( D_1 D_2 F = 0 \), to find \( X \in \hat{A}_d^{p-2} \) such that \( D_1 F = D_1 D_2 X \).

Our proof is in the spirit of spectral sequence, especially the one of filtered complex. We first introduce a filtration on \( \hat{A} \)

\[
0 \subset \hat{A}^{(0)} \subset \hat{A}^{(1)} \subset \hat{A}^{(2)} \subset \cdots \subset \hat{A},
\]

where \( \hat{A}^{(k)} \) is the differential polynomial algebra on \( J^k(\hat{M}) \). Suppose

\[
Q \in \text{Ker } D_1 \cap \text{Ker } D_2 \cap \hat{A}_d^{p(N+1)},
\]

if we can find \( X \in \hat{A}_d^{p-2(N-1)} \) such that

\[
\hat{Q} = Q - D_1 D_2 X \in \hat{A}_d^{p(N)},
\]

then \( \hat{Q} \) gives a new representative of the cohomology class of \( Q \), which depends on less variables. By induction on \( N \), we will show that one can choose a representative \( Q \) with smallest \( N \) (in the \( BH^3 \) case, the smallest \( N \) is 1, while in the \( BH^4 \) case it is 2). For this \( Q \), the equations \( D_1 Q = 0 \) and \( D_2 Q = 0 \) are easy to solve, and in this way one
is able to compute the bihamiltonian cohomologies. So the most important step in our computation of the bihamiltonian cohomologies is the following:

For any given \((p, d) = (3, \geq 6) \text{ or } (4, \geq 6)\) and \(F \in \hat{A}_{d-1}^{p-1}(N)\) satisfying \(D_1 D_2 F = 0\), to find \(X \in \hat{A}_{d-2}^{p-2}(N-1)\) such that \(D_1 F - D_1 D_2 X \in \hat{A}_d^{p}(N)\).

In this subsection, we prove some general results first. They will be used in the next two subsections.

**Lemma 5.2** If \(F \in \hat{A}^{(N)}\) with \(N \geq 1\) satisfies
\[
\frac{\partial F}{\partial u^N} = 0,
\]
then there exists \(X \in \hat{A}^{(N-1)}\) such that \(F - D_1 X \in \hat{A}^{(N-1)}\).

*Proof* First we write \(F\) as
\[
F = \theta^N F_N + F_0,
\]
where \(F_N, F_0 \in \hat{A}^{(N-1)}\). Then one can check that
\[
X = \partial_{u^{N-1}}^{-1} F_N
\]
fulfills the condition \(F - D_1 X \in \hat{A}^{(N-1)}\). Here the notation \(\partial_{u^{N-1}}^{-1} A(t)\) means
\[
\partial_{\tau}^{-1} A(t) = \int_0^t A(\tau) \, d\tau.
\]
Note that \(F_N\) is always a formal power series of \(\epsilon\) whose coefficients are smooth function or polynomials of \(u^{N-1}\), so the operator \(\partial_{u^{N-1}}^{-1}\) is well defined. \(\square\)

**Lemma 5.3** If \(F \in \hat{A}^{(N)}\) with \(N \geq 1\) satisfies
\[
D_1 D_2 F \in \hat{A}^{(N+1)},
\]
then there exists \(X \in \hat{A}^{(N-1)}\) such that \(\tilde{F} = F - D_1 X\) has the form \(\tilde{F} = \theta A + B\), where \(A \in \hat{A}^{(N)}, B \in \hat{A}^{(N-1)}\).

*Proof* By using the condition \(F \in \hat{A}^{(N)}\), one can obtain that
\[
\frac{\partial}{\partial u^{N+2}} (D_1 D_2 F) = 0, \quad \frac{\partial}{\partial \theta^{N+2}} (D_1 D_2 F) = \frac{1}{2} \theta \frac{\partial F}{\partial u^N}.
\]
We write \(F\) as \(F = \theta A + B_1\), where \(B_1\) is independent of \(\theta\), then the above results and the condition \(D_1 D_2 F \in \hat{A}^{(N+1)}\) imply that
\[
\frac{\partial B_1}{\partial u^N} = 0.
\]
According to Lemma 5.2, there exists \(X \in \hat{A}^{(N-1)}\) such that \(B = B_1 - D_1 X \in \hat{A}^{(N-1)}\). The lemma is proved. \(\square\)
Lemma 5.4 Let $R$ be an $\mathbb{R}$-algebra, $g \in R[x_1, \ldots, x_\ell]$, then for any $a_0, \ldots, a_\ell \in \mathbb{R}_{>0}$, the equation
\[
a_0 f + a_1 x_1 \frac{\partial f}{\partial x_1} + \cdots + a_\ell x_\ell \frac{\partial f}{\partial x_\ell} = g
\]
has a unique solution $f \in R[x_1, \ldots, x_\ell]$.

Proof We prove the existence first. Every element of $R[x_1, \ldots, x_\ell]$ can be written as finite sum of monomials, so we only need to prove the case that $g$ is a monomial. Suppose 
\[g = r x_1^{m_1} \cdots x_\ell^{m_\ell},\]
where $r \in R$, $m_1, \ldots, m_\ell \in \mathbb{N}$, then the equation (5.2) has a solution 
\[f = \frac{r}{a_0 + m_1 a_1 + \cdots + m_\ell a_\ell} x_1^{m_1} \cdots x_\ell^{m_\ell}.\]
Here we have used the condition that $a_k > 0$ to ensure that the denominator does not vanish.

To prove the uniqueness, we only need to show that the homogeneous equation of (5.2) has only zero solution. Suppose $f \in R[x_1, \ldots, x_\ell]$ is a nonzero solution to (5.2) with $g = 0$, then we can write $f$ as 
\[f = \sum_{m_1, \ldots, m_\ell \geq 0} r_{m_1, \ldots, m_\ell} x_1^{m_1} \cdots x_\ell^{m_\ell}.\]
Let $K$ be the smallest non-negative integer such that $m_1 + \cdots + m_\ell > K$ implies $r_{m_1, \ldots, m_\ell} = 0$, then we know that there exist $m_1, \ldots, m_\ell$ such that 
\[m_1 + \cdots + m_\ell = K, \quad r_{m_1, \ldots, m_\ell} \neq 0.\]
We denote $\underline{m} = (m_1, \ldots, m_\ell)$, and 
\[\partial^{\underline{m}} = \partial_{x_1}^{m_1} \cdots \partial_{x_\ell}^{m_\ell}, \quad x^{\underline{m}} = x_1^{m_1} \cdots x_\ell^{m_\ell},\]
then compute the action of $\partial^{\underline{m}}$ on the equation (5.2). According to the choice of $\underline{m}$, we obtain that 
\[(a_0 + m_1 a_1 + \cdots + m_\ell a_\ell) r_{m_1, \ldots, m_\ell} \partial^{\underline{m}}(x^{\underline{m}}) = 0.\]
This is impossible, since every term on the left hand side does not vanish, so the lemma is proved. $\square$

Lemma 5.5 Suppose $F = \theta A + B \in \hat{A}^{(N)}$ with $N \geq 1$, where 
\[\frac{\partial A}{\partial u^N} \in \hat{A}^{(N-1)}, \quad \frac{\partial A}{\partial \theta^N} = 0, \quad B \in \hat{A}^{(N-1)},\]
then there exists $X, Y \in \hat{A}^{(N-1)}$ such that $F - D_2 X + D_1 Y \in \hat{A}^{(N-1)}$. 20
Proof. The proof is similar to the one for Lemma 5.2, so we only present

\[ X = \partial_{u_{N-1}} \left( 2 \frac{\partial A}{\partial u^N} \right), \quad Y = -\partial_{u_{N-1}} \left( \frac{1}{2} \theta \frac{\partial X}{\partial u^N} - u \frac{\partial X}{\partial u^N} \right), \]

and omit the details. □

Now we give the proof of triviality of \( BH^2(\hat{A}) \) to illustrate the usage of these lemmas.

**Theorem 5.6** For the second bihamiltonian cohomologies we have

\[
BH^2_d(\hat{A}) = \begin{cases} 
0, & d = 0, \\
\{a(u)\theta\theta'\}, & d = 1, \\
0, & d \geq 2.
\end{cases}
\]

Proof. In the \( d = 0 \) or \( 1 \) case, since there is no \( d - 2 \) degree component in \( \hat{A} \), so \( BH^2_d(\hat{A}) \) is just \( \text{Ker } D_1 \cap \text{Ker } D_2 \cap \hat{A}_d^2 \), which is easy to compute. So we only give the proof for the \( d \geq 2 \) cases.

Suppose \( F \in \hat{A}^1_{d,2} \) satisfies \( D_1 D_2 F = 0 \). Due to Lemma 5.3 we can assume that \( F \) has the following form:

\[ F = a + b \]

where \( a \in \hat{A}^{0,(N)} = A^{(N)}, b \in \hat{A}^{1,(N-1)} \). A simple calculation shows that

\[
\frac{\partial^2}{\partial \theta^N \partial \theta^{N+1}} (D_1 D_2 F) = \left( N + \frac{1}{2} \right) u^1 \theta \frac{\partial^2 a}{\partial u^N \partial u^N},
\]

so from the vanishing of \( D_1 D_2 F \) it follows that \( a \) must take the form

\[ a = a_0(u, \ldots, u^{N-1}) u^N + a_1(u, \ldots, u^{N-1}), \]

thus \( F \) satisfies the conditions of Lemma 5.3. By using this lemma, we can find \( X, Y \in \hat{A}^{1,(N-1)} \) such that \( \tilde{F} = F - D_2 X + D_1 Y \in \hat{A}^{1,(N-1)} \).

By induction on \( N \), we can reduce \( F \) to an element of the space \( \hat{A}^{1,0} \). But \( \hat{A}^{1,0} = \hat{A}_0^1 \), so \( Q = D_1 F \in \hat{A}_d^2 \) cannot be an element of \( \hat{A}_{d \geq 2}^2 \) unless it vanishes. The theorem is proved. □

### 5.2 The bihamiltonian cohomology \( BH^3(\hat{A}) \)

We are to prove the following theorem in this subsection.

**Theorem 5.7** The bihamiltonian cohomology \( BH^3_d(\hat{A}) \) is given by

\[
BH^3_d(\hat{A}) = \begin{cases} 
0, & d = 0, 1, 2, \\
\{a(u)\theta\theta'\theta^2\}, & d = 3, \\
0, & d \geq 4.
\end{cases}
\]
Proof  For any given $Q \in \text{Ker } D_1 \cap \text{Ker } D_2 \cap \hat{A}^{3,(N+1)}$, $N \geq 1$, we need to prove the existence of $X \in \hat{A}^1$ such that $Q - D_1D_2X$ has the expression $a(u)\theta^1\theta^2$. To this end, let us first find $F \in \hat{A}^{2,(N)}$ such that $Q = D_1F$, then $F$ satisfies $D_1D_2F = 0$. We are to show that, by modifying $F$ in the way $F \mapsto F - D_2X + D_1Y$ with certain $X, Y \in \hat{A}^1$, we can reduce $F$ to an element of the space $\hat{A}^{2,(1)}$.

Denote $Z = D_1D_2F$, then by using the fact that $Z = 0$ and the result of Lemma 5.3 we can reduce $F$ to the form

$$F = \sum_{s=1}^{N} A_s \theta^s + B,$$

where $A_s \in A^{(N)}$, $B \in \hat{A}^{2,(N-1)}$.

Let us assume $N \geq 2$ and denote the coefficients of $\theta^{N-s}\theta^N\theta^{N+1}$ in $Z$ by

$$Z_s = \frac{\partial^4 Z}{\partial \theta^{N+1}\partial \theta^N\partial \theta^{N-s}\partial \theta^s}, \quad s = 1, 2, \ldots, N-1,$$

then by using Lemma 5.1 one can easily obtain that

$$Z_1 \equiv \frac{N^2}{2} u^2 \frac{\partial^2 A_N}{\partial u^{N-1}\partial u^N} \pmod{u^1}.$$  

Since $Z = 0$, we arrive at

$$u^1 \bigg| \frac{\partial^2 A_N}{\partial u^{N-1}\partial u^N}.$$  

By using this fact, one can further obtain that

$$0 = Z_2 \equiv \frac{(N-1)^2}{2} u^2 \frac{\partial^2 A_N}{\partial u^{N-2}\partial u^N} \pmod{u^1},$$

so we have

$$u^1 \bigg| \frac{\partial^2 A_N}{\partial u^{N-1}\partial u^N}.$$  

By induction on $s$, one can prove that

$$u^1 \bigg| \frac{\partial^2 A_N}{\partial u^{N-s+1}\partial u^N}, \quad s = 1, \ldots, N-2.$$  

When $s = N-1$, we have

$$0 = Z_{N-1} \equiv 2u^2 \frac{\partial^2 A_N}{\partial u^2\partial u^N} + (N-1)\frac{\partial A_N}{\partial u^N} \pmod{u^1}.$$  

Denote $f = \frac{\partial A_N}{\partial u^N}$, then

$$2u^2 \frac{\partial f}{\partial u^2} + (N-1)f \equiv 0 \pmod{u^1},$$

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By using Lemma 5.3 and the above equation we know that
\[ u^1 \frac{\partial A_N}{\partial u^N}, \]
so the differential polynomial \( A_N \) can be put into the form
\[ A_N = u^1 a + b, \quad \text{with } a \in \mathcal{A}^{(N)}, \ b \in \mathcal{A}^{(N-1)}. \]

Then Lemma 5.2 enables us to modify the term \( b \theta \theta^N \) of \( F \) by adding certain \( D_1 X \in \hat{\mathcal{A}}^{2,(N)} \) such that the resulting bivector belongs to \( \hat{\mathcal{A}}^{2,(N-1)} \), and we can combine it into the term \( B \) in the expression (5.3) of \( F \), so we can assume that
\[ A_N = u^1 a. \]

Let us choose
\[ X = -(N + \frac{1}{2})^{-1} \partial u^{-1}_N (a) \theta, \quad Y = uX, \]
then it is easy to see that after the modification \( F \mapsto F - D_2X + D_1Y \) we can reduce \( F \) to the form (5.3) with \( A_N = 0 \) and \( B \in \hat{\mathcal{A}}^{2,(N-1)} \). Now the functions \( Z_s \) defined in (5.4) have the expression
\[ 0 = Z_s = -(N - 1)^2 \frac{2}{2} u^1 \frac{\partial^2 A_{N-s}}{\partial u^N \partial u_N}, \quad s = 1, \ldots, N - 1, \]
so the vector \( A = \sum_{s=1}^{N-1} A_s(u, u^1, \ldots, u^N) \theta^s \) satisfies the condition
\[ \frac{\partial A}{\partial u^N} \in \hat{\mathcal{A}}^{(N-1)}, \quad \frac{\partial A}{\partial \theta^N} = 0, \]
and it follows from Lemma 5.5 that we can find \( X, Y \in \hat{\mathcal{A}}^{1,(N-1)} \) such that
\[ F - D_2X + D_1Y \in \hat{\mathcal{A}}^{2,(N-1)}. \]

By induction on \( N \), we can prove the existence of \( X, Y \in \hat{\mathcal{A}}^1 \) such that \( F - D_2X - D_1Y \in \hat{\mathcal{A}}^{2,1} \), so we can reduce \( F \) to the form
\[ F = A_1(u, u^1) \theta \theta^1. \]

Then the trivector \( D_1F \) has the expression \( D_1F = \frac{\partial A_1}{\partial u^1} \theta \theta^1 \theta^2 \). Denote \( \frac{\partial A_1}{\partial u^1} = u^1b(u, u^1) + a(u) \), and define \( X = -\frac{2}{3} \partial^{-2}_u (b(u, u^1)) \theta \), then we have
\[ Q \sim D_1F - D_1D_2X = a(u) \theta \theta^1 \theta^2. \]
Thus we proved the theorem. \( \square \)
5.3 The bihamiltonian cohomology $BH^4(\hat{A})$

In this subsection, we continue to compute the bihamiltonian cohomology $BH^4(\hat{A})$.

**Theorem 5.8** The bihamiltonian cohomology $BH^4(\hat{A})$ is trivial.

**Proof** Given $Q \in \text{Ker} D_1 \cap \text{Ker} D_2 \cap \hat{A}^{4,(N+1)}$, $N \geq 2$, we need to prove the existence of $X \in \hat{A}^2$ such that $Q = D_1 D_2 X$. As we did in the computation of the bihamiltonian cohomology $BH^3(\hat{A})$, we can find $F \in \hat{A}^{3,(N)}$, $N \geq 2$, such that $Q = D_1 F$, and $F$ satisfies $D_1 D_2 F = 0$. Let us prove the theorem in three steps. In the first step, we prove that when $N \geq 4$ we can reduce $F \in \hat{A}^{3,(N)}$, by modifying it in the way $F \mapsto F - D_2 X + D_1 Y$ with certain $X, Y \in \hat{A}^2$, to an element of the space $\hat{A}^{3,(3)}$. In the second step, we prove that an element $F$ of $\hat{A}^{3,(3)}$ satisfying $D_1 D_2 F = 0$ can be reduced to an element of the space $F \in \hat{A}^{3,(2)}$. In the last step, we prove that for any element $F$ of $\hat{A}^{3,(2)}$ satisfying $D_1 D_2 F = 0$, one can always represent $D_1 F$ as $D_1 D_2 X$ for a certain $X \in \hat{A}^2$.

Step 1: For the case when $N \geq 4$.

Denote $Z = D_1 D_2 F$, then by using Lemma 5.3 and $Z = 0$ we know that $F$ can be assumed to have the form

$$F = \sum_{1 \leq p < q \leq N} A_{p,q} \theta^p \theta^q + B,$$

where $A_{p,q} \in \mathcal{A}^{(N)}$, $B \in \hat{A}^{3,(N-1)}$. We define

$$Z_s = \frac{\partial^4 Z}{\partial \theta^{N+1} \partial \theta^{N} \partial \theta^{N-s} \partial \theta }, \quad s = 1, 2, \ldots, N - 1,$$

then by using Lemma 5.1 and induction on $p$ we have

$$\frac{\partial Z_1}{\partial \theta^p} = \frac{(p+1)(p+2)(2p+3)}{12} u^3 \frac{\partial^2 A_{N-1,N}}{\partial u^{p+2} \partial u^N} \quad (\text{mod } u^1, u^2), \quad 2 \leq p \leq N - 2,$$

$$\frac{\partial Z_1}{\partial \theta^1} = \frac{5 u^3 \partial^2 A_{N-1,N}}{2 \partial u^3 \partial u^N} + \frac{3N - 4}{2} \frac{\partial A_{N-1,N}}{\partial u^N} \quad (\text{mod } u^1, u^2).$$

So we have $\frac{\partial A_{N-1,N}}{\partial u^N} \equiv 0 \quad (\text{mod } u^1, u^2)$. A similar argument as we gave in the proof of Theorem 5.7 shows that $A_{N-1,N}$ can be assumed to have the form

$$A_{N-1,N} = u^1 B_1 + u^2 B_2, \quad B_1, B_2 \in \mathcal{A}^{(N)}.$$

Now we assume $X, Y \in \hat{A}^2$ have the form

$$X = x_0 \theta^N + x_1 \theta^{N-1}, \quad Y = uX, \quad x_0, x_1 \in \mathcal{A}^{(N)},$$

and we consider the following modification of $F$

$$\tilde{F} = F - D_2 X + D_1 Y.$$
Then $\tilde{F} \in \hat{A}^{3,(N)}$ and the coefficient of $\theta^{N-1} \theta^N$ in $\tilde{F}$ has the expression

$$\frac{\partial^3 \tilde{F}}{\partial \theta \partial \theta^{N-1} \partial \theta^N} = u^1 \left( \frac{2N + 1}{2} \frac{\partial x_1}{\partial u^N} - \frac{2N - 1}{2} \frac{\partial x_0}{\partial u^{N-1}} - B_1 \right) - u^2 \left( \frac{N^2}{2} \frac{\partial x_0}{\partial u^N} + B_2 \right).$$

From the above formula it follows the existence of differential polynomials $x_0, x_1$ such that $\frac{\partial^3 \tilde{F}}{\partial \theta \partial \theta^{N-1} \partial \theta^N} = 0$ and we have reduced $F$ to the form (5.5) with $A_{N-1,N} = 0$.

We are to prove by induction that $F$ can actually be reduced to the form (5.5) with $A_{p,N} = 0$, $p = 1, 2, \ldots, N - 1$. Assume that $F$ has been reduced to the form

$$F = \sum_{1 \leq p < q \leq N - 1} A_{p,q} \theta^p \theta^q + \sum_{1 \leq p \leq m} A_{p,N} \theta^p \theta^N + B,$$

with $A_{p,q} \in A^{(N)}$ and $B \in \hat{A}^{3,(N-1)}$ for a certain $1 \leq m \leq N - 2$. Then the vectors $Z_s$ defined in (5.6) have the following properties (which can be proved by induction on $s$)

$$0 = \frac{\partial Z_s}{\partial \theta^p} \equiv - \frac{(N - s + 1)^2}{2} u^2 \frac{\partial^2 A_{p,N}}{\partial u^{N-s+1} \partial u^N} \pmod{u^1},$$

$s = 1, \ldots, N - m - 1$, $p = 1, \ldots, m$.

These relations imply that

$$u_1 \left| \frac{\partial^2 A_{p,N}}{\partial u^{N-s+1} \partial u^N} \right|, \quad s = 1, \ldots, N - m - 1$, $p = 1, \ldots, m$, (5.8)

from which it follows that (here an induction on $p$ is needed)

$$\frac{\partial Z_{N-m}}{\partial \theta^p} \equiv (p + 1)(p + 2)(2p + 3) \frac{12}{u^4} \frac{\partial^2 A_{m,N}}{\partial u^{p+2} \partial u^N} \pmod{u^1, u^2}, \quad 2 \leq p \leq m - 1,$$

$$\frac{\partial Z_{N-m}}{\partial \theta^1} \equiv \frac{5}{2} \frac{u^3}{u^3 \partial u^{p+2} \partial u^N} + \frac{2N + m - 3}{2} \frac{\partial A_{m,N}}{\partial u^N} \pmod{u^1, u^2}.$$

The last equation together with Lemma 5.4 imply that

$$\frac{\partial A_{m,N}}{\partial u^N} \equiv 0 \pmod{u^1, u^2}.$$

So we can represent the differential polynomial $A_{m,N}$ in the form

$$A_{m,N} = u^1 B_1(u, u^1, \ldots, u^N) + u^2 B_2(u, u^2, \ldots, u^N),$$

after eliminating a remainder term that does not depend on $u^N$ by using Lemma 5.2. Note that the differential polynomial $B_2$ is chosen to be independent of $u^1$. By using (5.8) with $p = m$ we know that

$$\frac{\partial^2 B_2}{\partial u^{m+2} \partial u^N} = \cdots = \frac{\partial^2 B_2}{\partial u^N \partial u^N} = 0.$$  
(5.10)
Now let us try to find $X, Y \in \hat{A}^2$ of the form
\[
X = x_0 \theta^N + x_1 \theta^m + x_2 \theta^{m+1}, \quad x_0, x_1, x_2 \in \mathcal{A}^{(N)},
\]
\[
Y = uX - \sum_{k=m}^{N-1} y_k \theta^k, \quad y_k \in \mathcal{A}^{(N-1)},
\]
such that the following modification of $F$
\[
\tilde{F} = F - D_2 X + D_1 Y \quad (5.11)
\]
leads a reduction of $F$ to the form (5.7) with $A_{m,N} = 0$.

By a direct calculation we obtain
\[
\frac{\partial^3 \tilde{F}}{\partial \theta^N \partial \theta^m \partial \theta} = \sum_{s=1}^{N-p+1} \left( \frac{(p + s - 1)}{s} \right) u^s \frac{\partial x_0}{\partial \theta^{s+1}} - \frac{\partial y_p}{\partial \theta^{N-1}}
\]
\[
- \delta_{p,m+1} \left( N + \frac{1}{2} \right) u^1 \frac{\partial x_2}{\partial \theta^{N}}, \quad m + 1 \leq p \leq N - 1,
\]
\[
\frac{\partial^3 \tilde{F}}{\partial \theta^N \partial \theta^m \partial \theta} = \sum_{s=3}^{N-m+1} \left( \frac{(m + s - 1)}{s} \right) u^s \frac{\partial x_0}{\partial \theta^{m+s-1}}
\]
\[
+ \frac{1}{2} \delta_{m,1} x_0 + u^1 \left( \frac{2m + 1}{2} \frac{\partial x_0}{\partial \theta^m} - \frac{2N + 1}{2} \frac{\partial x_1}{\partial \theta^N} + B_1 \right)
\]
\[
+ u^2 \left( \frac{(m + 1)^2}{2} \frac{\partial x_0}{\partial \theta^{m+1}} + B_2 \right) - \frac{\partial y_m}{\partial \theta^{N-1}}. \quad (5.13)
\]

We first proceed to fix the differential polynomials $x_s = x_s(u, u^1, \ldots, u^N), \ s = 0, 1$ and $y_m$. In order to do so, we consider the cases $m \geq 2$ and $m = 1$ separately.

i) The $m \geq 2$ case. In this case we can find a solution $x_0$ of the equation
\[
\frac{(m + 1)^2}{2} \frac{\partial x_0}{\partial \theta^{m+1}} = -B_2
\]
so that it also satisfies the following additional requirements
\[
\frac{\partial^2 x_0}{\partial \theta^{m+2} \partial \theta^N} = \cdots = \frac{\partial^2 x_0}{\partial \theta^{N} \partial \theta^N} = 0. \quad (5.14)
\]

This can be achieved due to the properties of $B_2$ given in (5.10). With such a choice of the function $x_0$, the first term that appears in the r.h.s. of (5.13) does not depend on $u^N$, so this term can be canceled by the last term $-\frac{\partial y_m}{\partial \theta^{N-1}}$ with an appropriately chosen $y_m \in \mathcal{A}^{(N-1)}$. Now we can fix $x_1$ by the equation
\[
\frac{2N + 1}{2} \frac{\partial x_1}{\partial \theta^N} = \frac{2m + 1}{2} \frac{\partial x_0}{\partial \theta^m} + B_1. \quad (5.15)
\]

Thus the above choice of $x_0, x_1$ and $y_m$ enables us to reduce the r.h.s. of (5.13) to zero.
ii) The $m = 1$ case. In this case the r.h.s. of (5.13) has the expression

$$\frac{1}{2}x_0 + \sum_{s=1}^{N} \left(1 + \frac{s}{2}\right) u^s \frac{\partial x_0}{\partial u^s} + u^2 B_2 + u^1 \left(B_1 - \frac{2N + 1}{2} \frac{\partial x_1}{\partial u^N}\right) - \frac{\partial y_1}{\partial u^{N-1}}.$$ 

By using Lemma 5.4 and the condition (5.10) we can find a unique solution $x_0 = x_0(u, u^2, \ldots, u^n)$ satisfying the equation

$$\frac{1}{2}x_0 + \sum_{s=1}^{N} \left(1 + \frac{s}{2}\right) u^s \frac{\partial x_0}{\partial u^s} + u^2 B_2 = 0$$

and the condition (5.14). We then fix the function $x_1$ by the equation

$$B_1 - \frac{2N + 1}{2} \frac{\partial x_1}{\partial u^N} = 0.$$ 

After setting $y_1 = 0$ we thus reduce the r.h.s. of (5.13) to zero.

Now let us continue to fix the differential polynomials $x_2$ and $y_p$, $p = m+1, \ldots, N-1$ by the requirement that the r.h.s. of (5.12) vanish. To this end we first fix $x_2$ by the equation

$$\left(N + \frac{1}{2}\right) \frac{\partial x_2}{\partial u^N} = \left(m + \frac{3}{2}\right) \frac{\partial x_0}{\partial u^{m+1}}. \quad (5.16)$$

We then choose the functions $y_p \in \mathcal{A}^{(N-1)}$ in order to cancel the rest terms in the r.h.s. of (5.12). This can be done since $x_0$ satisfies the condition (5.14).

Thus we have shown that by appropriately choosing the functions $x_0, x_1, x_2$ and $y_p$'s we can reduce $\tilde{F}$ to the form (5.17) with vanishing coefficient $A_{m,N}$. So we finished the induction procedure in order to prove that the originally given $F$ can be reduce to the form (5.5) with $A_{p,N} = 0$, $p = 1, 2, \ldots, N - 1$.

Up to this moment, we have proved the existence of $X, Y \in \hat{\mathcal{A}}^2$ such that after the modification $F \mapsto F - D_2X + D_1Y$ the trivector $F$ can be reduced to the following form

$$F = \sum_{1 \leq p < q \leq N-1} A_{p,q} \theta^p \theta^q + B, \quad (5.17)$$

with $A_{p,q} \in \mathcal{A}^{(N)}$, $B \in \hat{\mathcal{A}}^{3,N-1}$. Now let us consider the vectors $Z_s$ defined in (5.6) again. With the above form of $F$ we have the following identities:

$$0 = \frac{\partial Z_s}{\partial \theta^p} = \left(N + \frac{1}{2}\right) u^1 \frac{\partial^2 A_{p,N-s}}{\partial u^N \partial u^N}, \quad 1 \leq p \leq N - s - 1, \quad 1 \leq s \leq N - 2.$$ 

From these identities we know that $A_{p,q}$'s depend linearly on $u^N$. So by using Lemma 5.5 we can find $X, Y \in \hat{\mathcal{A}}^2$ such that $F - D_2X + D_1Y \in \hat{\mathcal{A}}^{3,(N-1)}$.

By induction on $N$, we thus proved that $F$ can indeed be reduced to an element of the space $\hat{\mathcal{A}}^{3,(3)}$ by modifying it in the way $F \mapsto F - D_2X + D_1Y$ with certain $X, Y \in \hat{\mathcal{A}}^2$. 


Step 2: For the case when $N = 3$.

In this case, one can also use the above procedure to eliminate the term $\theta \theta^{N-1} \theta^N$ of the trivector $F$, so we assume that

$$F = A_{1,2} \theta \theta^1 \theta^2 + A_{1,3} \theta \theta^1 \theta^3 + B$$

with $A_{1,2}, A_{1,3} \in \mathcal{A}^{(3)}$, $B \in \hat{\mathcal{A}}^{3,(2)}$. The vanishing of $D_1 D_2 F$ is equivalent to the condition

$$7u^1 \frac{\partial^2 A_{12}}{\partial u^3 \partial u^3} - 9u^2 \frac{\partial^2 A_{13}}{\partial u^3 \partial u^3} - 5u^1 \frac{\partial^2 A_{13}}{\partial u^2 \partial u^3} = 0,$$

which implies that

$$u \left| \frac{\partial^2 A_{13}}{\partial u^3 \partial u^3} \right.$$

so $A_{1,3}$ has the following form

$$A_{1,3} = u^1 (u^3)^2 b_1 + u^3 b_2 + b_3, \quad b_1 \in \mathcal{A}^{(3)}, \quad b_2, b_3 \in \mathcal{A}^{(2)}.$$ 

(5.19)

We proceed to reduce $F$ by using the bivector $X, Y \in \hat{\mathcal{A}}^2$ of the form

$$X = x_1 \theta \theta^1 + x_2 \theta \theta^2 + x_3 \theta \theta^3, \quad Y = uX - y_1 \theta \theta^1 - y_2 \theta \theta^2,$$

where $x_1, x_2, x_3 \in \mathcal{A}^{(3)}, y_1, y_2 \in \mathcal{A}^{(2)}$. The coefficients of $\theta \theta^1 \theta^3$ and $\theta \theta^2 \theta^3$ in $F - D_2 X + D_1 Y$ are given by

$$R_1 = A_{1,3} - \frac{7}{2} u^1 \frac{\partial x_1}{\partial u^3} - \frac{1}{2} x_3 + \frac{5}{2} u^3 \frac{\partial x_3}{\partial u^3} + 2u^2 \frac{\partial x_3}{\partial u^2} + \frac{3}{2} u^1 \frac{\partial x_3}{\partial u^1} - \frac{\partial y_1}{\partial u^1},$$

$$R_2 = \frac{9}{2} u^2 \frac{\partial x_3}{\partial u^3} + \frac{5}{2} u^1 \frac{\partial x_3}{\partial u^2} - \frac{7}{2} u^1 \frac{\partial x_2}{\partial u^3} - \frac{\partial y_2}{\partial u^2},$$

(5.20)

(5.21)

respectively. Let $c \in \mathcal{A}^{(2)}$ be the unique solution of the equation

$$b_2 + 2c + 2u^2 \frac{\partial c}{\partial u^2} = 0,$$

then one can verify that

$$x_3 = c u^3, \quad x_2 = \frac{5}{7} u^{-1} (\frac{\partial x_3}{\partial u^2})^3, \quad x_1 = \frac{2}{7} u^{-1} (b_1 (u^3)^2 + \frac{3}{2} u^3),$$

$$y_2 = \frac{9}{2} u^{-1} (c u^2), \quad y_1 = \frac{1}{2} u^{-1} b_3$$

satisfy the equation $R_1 = R_2 = 0$, so the term in $F$ with $A_{1,3}$ is eliminated.

Next, we use the condition $D_1 D_3 F = 0$ to derive the linear dependence of $A_{1,2}$ on $u^N = u^3$, then by using Lemma 5.5 we can reduce $F$ to an element of the space $\hat{\mathcal{A}}^{3,(2)}$.

Step 3: For the case when $N = 2$. 

28
In this case $F$ must take the following form

$$F = A \theta^1 \theta^2, \quad A \in \mathcal{A}^{(2)}.$$ 

It is easy to see that this $F$ satisfies $D_1 D_2 F = 0$ automatically. We are to find

$$X = x_1 \theta^1 + x_2 \theta^2, \quad x_1, x_2 \in \mathcal{A}^{(2)}$$

such that

$$0 = D_1 F - D_1 D_2 X$$

$$= \left( - \frac{\partial A}{\partial u^2} - \frac{\partial x_2}{\partial u^2} + \frac{5}{2} u^1 \frac{\partial^2 x_1}{\partial u^2 \partial u^2} - 2 u^2 \frac{\partial^2 x_2}{\partial u^2 \partial u^2} - \frac{3}{2} u^1 \frac{\partial^2 x_2}{\partial u^1 \partial u^2} \right) \theta^1 \theta^2.$$

We rewrite $A$ as

$$A = b_0(u, u^2) + u^1 b_1(u, u^1, u^2),$$

and take

$$x_1 = \frac{2}{5} \partial u^1 b_1, \quad x_2 = h(u, u^2),$$

then we obtain the following equation for the function $h$:

$$- \frac{\partial b_0}{\partial u^2} = \frac{\partial h}{\partial u^2} + 2 u^2 \frac{\partial^2 h}{\partial u^2 \partial u^2},$$

which has a solution due to Lemma 5.4.

We thus proved the theorem. 

\[\square\]

### 5.4 Proof of Theorem 1.1 and Theorem 1.2

With the results of Theorem 5.7 and Theorem 5.8, the proof of Theorem 1.1 follows directly from the long exact sequence (4.5). For the proof of Theorem 1.2, we note that the bivectors correspond to the bihamiltonian structure (1.6), (1.7) are given by

$$P_1^{[0]} = \frac{1}{2} \int \theta \theta^1, \quad P_2^{[0]} = \frac{1}{2} \int u \theta^1,$$

$$P_2^{[1]} = \frac{3}{2} c(u) \theta^0 + \frac{9}{4} c'(u) u^1 \theta^0^2 + \frac{3}{4} \left( c''(u)(u^1)^2 + c'(u) u^2 \right) \theta^1 \theta^1.$$

In order to prove the existence of bihamiltonian structures of the form (1.6), (1.7), we need to find $P_2^{[g]} \in \mathcal{F}_2^{2g+1}$, $g \geq 2$ such that

$$[P_1^{[0]}, P_2^{[0]} + \epsilon^2 P_2^{[1]} + \sum_{g \geq 2} \epsilon^{2g} P_2^{[g]}] = 0, \quad (5.22)$$

$$[P_2^{[0]} + \epsilon^2 P_2^{[1]} + \sum_{g \geq 2} \epsilon^{2g} P_2^{[g]}] = 0, \quad (5.23)$$

29
Let us first consider the existence of \( P_2^{[2]} \in \mathcal{F}_5^2 \), it should be given by a solution of the equations
\[
[P_1^{[0]}, P_2^{[2]}] = 0, \quad [P_2^{[0]}, P_2^{[2]}] = Q,
\]
where \( Q = -[P_2^{[1]}, P_2^{[1]}] \). Since \([P_1^{[0]}, P_2^{[1]}] = [P_2^{[0]}, P_2^{[1]}] = 0\), by using the graded Jacobi identity given in Theorem 2.3 we know that \( Q \in \mathcal{F}_0^1 \cap \ker d_1 \cap \ker d_2 \). So from the triviality of the bihamiltonian cohomology \( BH_d^2(\mathcal{F}, d_1, d_2) \) for \( d \geq 4 \) it follows the existence of \( X \in \mathcal{F}_4^1 \) such that \( Q = d_1 d_2 X \). Thus we obtained the needed bivector \( P_2^{[2]} \) which is defined by \( P_2^{[2]} = -d_1 X \).

By using induction, we can also prove the existence of \( P_2^{[g]} \) for \( g \geq 3 \) in a similar way as we did for the \( g = 2 \) case above, see [27, 1] for details. Thus we proved the theorem.

6 Some examples

In this section, we give some concrete examples of bihamiltonian structures of the form \((1.1), (1.11)\). We will now denote the \( x \)-derivatives of the function \( u = u(x) \) by \( u_x, u_{xx}, u^{(3)} \) and so on, and, unlike in the previous sections, we denote by \( u^m \) the \( m \)-th power of \( u \).

The existence of bihamiltonian structures of the form \((1.1), (1.11)\) was first conjectured by Lorenzoni in [31], he wrote down its approximation up to \( \epsilon^4 \), which are given by the following expressions:

\[
A_{2,0} = 9c(u)c'(u), \quad A_{2,1} = \frac{45}{2} \left[ c'(u)^2 + c(u)c''(u) \right] u_x,
A_{2,2} = 27 \left[ c'(u)^2 + c(u)c''(u) \right] u_{xx} + 18 \left[ 3c'(u)c''(u) + c(u)c^{(3)}(u) \right] u_x^2,
A_{2,3} = 18 \left[ c'(u)^2 + c(u)c''(u) \right] u^{(3)} + 27 \left[ 3c'(u)c''(u) + c(u)c^{(3)}(u) \right] u_x u_{xx}
+ \frac{9}{2} \left[ 3c''(u)^2 + 4c'(u)c^{(3)}(u) + c(u)c^{(4)}(u) \right] u_x^3,
A_{2,4} = \frac{9}{2} \left[ c'(u)^2 + c(u)c''(u) \right] u^{(4)} + 9 \left[ 3c'(u)c''(u) + c(u)c^{(3)}(u) \right] u_x u^{(3)}
+ \frac{9}{2} \left[ 3c''(u)^2 + 4c'(u)c^{(3)}(u) + c(u)c^{(4)}(u) \right] u_x^2 u_{xx}
+ \frac{9}{2} \left[ 3c''(u)^2 + 4c'(u)c^{(3)}(u) + c(u)c^{(4)}(u) \right] u_x^2 u_{xx}.
\]

In a subsequent paper [1], Arsie and Lorenzoni extend the approximation up to \( \epsilon^8 \).

For any given smooth function \( g(u) \) define the function \( f \) by
\[
f(u) = \partial_u^{-2} \left( u g''(u) + \frac{1}{2} g'(u) \right).
\]

Then we have the following bihamiltonian equation
\[
\frac{\partial u}{\partial t} = \{ u(x), H_f \}_{1} = \{ u(x), H_g \}_{2}. \tag{6.1}
\]

30
Here the Hamiltonians are given by

\[ H_f = \int \left[ f(u) - \epsilon^2 c(u) f^{(3)}(u) u_x^2 + \epsilon^4 \left( A_1 u_{xx}^2 + A_2 u_x^4 \right) \right] dx + O(\epsilon^6) \]  

(6.2)

with

\[ A_1 = 3c(u)c'(u) f^{(3)}(u) + \frac{6}{5} c(u)^2 f^{(4)}(u), \]

\[ A_2 = - \left[ \frac{1}{2} c'(u)^2 + c(u) c''(u) \right] f^{(4)}(u) - c(u) c'(u) f^{(5)}(u) - \frac{1}{6} c(u)^2 f^{(6)}(u). \]

The Hamiltonian \( H_g \) is obtained by replace \( f \) with \( g \) in \( H_f \). In particular, if we set

\[ f(u) = u^p + \frac{2}{p + 2} \]

\[ g(u) = 2u^{p + 1} (2p + 1)(p + 1)! \]

then (6.1) gives the bihamiltonian deformation of the dispersionless KdV hierarchy (1.8).

When \( c(u) = \frac{1}{24} \) the bihamiltonian structure given in Theorem 1.2 has the following truncated form

\[ \{ u(x), u(y) \}_1 = \delta'(x - y), \]

(6.3)

\[ \{ u(x), u(y) \}_2 = u(x)\delta'(x - y) + \frac{1}{2} u_x(x)\delta(x - y) + \frac{\epsilon^2}{8} \delta'''(x - y) \]  

(6.4)

which gives the bihamiltonian structure \[ 35, 32 \] for the prototypical nonlinear integrable evolutionary PDE – the KdV equation

\[ \frac{\partial u}{\partial t} = uu_x + \frac{\epsilon^2}{12} u_{xxx} \]

and the associated KdV hierarchy.

When \( c(u) = \frac{u}{24} \), the bihamiltonian structure (1.9), (1.10) is not truncated, however it is equivalent, under a Miura type transformation \( u \mapsto \tilde{u} = u - \frac{\epsilon^2}{16} u_{xx} - \frac{\epsilon^4}{576} u^{(4)} + O(\epsilon^6) \), to the following bihamiltonian structure:

\[ \{ \tilde{u}(x), \tilde{u}(y) \}_1 = \delta'(x - y) - \frac{\epsilon^2}{8} \delta'''(x - y), \]

(6.5)

\[ \{ \tilde{u}(x), \tilde{u}(y) \}_2 = \tilde{u}(x)\delta'(x - y) + \frac{1}{2} \tilde{u}_x(x)\delta(x - y). \]

(6.6)

it gives the bihamiltonian structure of the equation

\[ (m - \frac{\epsilon^2}{8} m_{xx})_t = mm_x - \frac{\epsilon^2}{12} m_{xx} m_{xx} - \frac{\epsilon^2}{24} mm_{xxx} \]  

(6.7)

with

\[ \tilde{u} = m - \frac{\epsilon^2}{8} m_{xx}. \]  

(6.8)
In fact, if we define the Hamiltonians
\[ H_1 = \int \left( \frac{1}{6} m^3 + \frac{\epsilon^2}{48} m^2_x \right) dx, \quad H_2 = \int \left( \frac{1}{3} m^2 + \frac{\epsilon^2}{24} m^2_x \right) dx, \]
then the equation (6.7) can be represented as
\[ \ddot{u}_t = \{ \ddot{u}(x), H_1 \}_1 = \{ \ddot{u}(x), H_2 \}_2. \]

This equation is equivalent to the well-known Camassa-Holm equation which, like the KdV equation, describes the dynamics of shallow water waves \[3,4,17,18,19].

Although the bihamiltonian structures (1.9), (1.10) and the associated integrable hierarchies are represented as infinite power series in the dispersion parameters \( \epsilon \), we hope that apart from the above two well-known examples of the KdV and Camassa-Holm integrable hierarchies these new bihamiltonian hierarchies will find applications.

In the rest part of this section, let us study in some details the bihamiltonian structure (1.9), (1.10) with central invariant \( c(u) \) being inversely proportional to \( u \). We are to give evidence that it is actually related to the integrable hierarchy obtained from the KdV hierarchy by certain reciprocal transformation. To simplify the notations, we will call the family of bihamiltonian equations given by (6.1), (6.2) with \( c(u) = \frac{1}{m} \) the KdV hierarchy. At the approximation up to \( \epsilon^2 \) these equations take the form
\[ \frac{\partial u}{\partial t} = f''(u) u_x + \frac{\epsilon^2}{24} \left( 2f^{(3)}(u) u^{(3)} + 4f^{(4)}(u) u_x u_{xx} + f^{(5)}(u) u_x^3 \right) + O(\epsilon^4). \] (6.9)

Let us take \( f \) to be a specific smooth function \( \rho(u) \) (whose explicit definition will be given later), and denote by \( \tau \) the time variable of the resulting bihamiltonian evolutionary equation (6.1). Denote \( h(u) = \rho''(u) \), then this equation has the expression
\[ \frac{\partial u}{\partial \tau} = h(u) u_x + \frac{\epsilon^2}{24} \left( 2h'(u) u^{(3)} + 4h''(u) u_x u_{xx} + h^{(3)}(u) u_x^3 \right) + O(\epsilon^4). \] (6.10)

Now we perform the reciprocal transformation to the equation (6.9) by replacing the spatial variable \( x \) with the time variable \( \tau \) of (6.10). In order to do so we first obtain from (6.10) the following relation:
\[
\begin{align*}
\quad u_x &= \frac{u_\tau}{h(u)} + \frac{\epsilon^2}{24} \left( -2u_{\tau \tau \tau}h'(u) - \frac{4h''(u) u_\tau u_{\tau \tau}}{h(u)} + \frac{18h'(u)^2 u_\tau u_{\tau \tau}}{h(u)^5} - \frac{h^{(3)}(u) u_x^3}{h(u)^4} - \frac{24h'(u)^3 u_x^3}{h(u)^6} + \frac{14h'(u) h''(u) u_x^3}{h(u)^5} \right) + O(\epsilon^4),
\end{align*}
\] (6.11)

this enables us to represent the equation (6.9) in the form
\[
\begin{align*}
\frac{\partial u}{\partial t} &= p(u) u_\tau + \frac{\epsilon^2}{12 h(u)^2} \left[ p'(u) u_{\tau \tau \tau} + \left( \frac{2p''(u) - 5h'(u) p'(u)}{h(u)} \right) u_\tau u_{\tau \tau} 
\quad + \left( \frac{4h'(u)^2 p'(u)}{h(u)^2} - \frac{5h'(u) p''(u)}{2 h(u)} - \frac{3h''(u) p'(u)}{2 h(u)} + \frac{1}{2} p'''(u) \right) u_x^3 \right] + O(\epsilon^4),
\end{align*}
\] (6.12)
Then after the change of coordinate with and the Hamiltonians the equation \( (6.12) \) has the following bihamiltonian structure. We conjecture that for bihamiltonian equations with \( \epsilon \)-deformations a reciprocal transformation of the above form still preserves the bihamiltonian property. For our particular example of \( (6.12) \), we can employ an algorithm proposed in [28, 29] to obtain its bihamiltonian structure at the approximation up to a certain order of \( \epsilon \). Here we did the computation at the approximation up to \( \epsilon^8 \). Now let us choose the function \( h(u) \) carefully, so that the leading terms of the bihamiltonian structure of \( (6.12) \) coincide, after certain change of coordinates, with \( (1.6), (1.7) \). To this end, we take

\[
h(u) = u^{-\frac{3}{2}}.
\]

Then after the change of coordinate

\[
w = \frac{1}{u}
\]

the equation \( (6.12) \) has the following bihamiltonian structure

\[
\frac{\partial w}{\partial t} = \{w(\tau), H_1\}_1 = \{w(\tau), H_2\}_2, \tag{6.13}
\]

with

\[
\{w(\tau), w(\sigma)\}_1 = \delta'(\tau - \sigma)
\]

\[
+ \epsilon^2 \left[ \frac{5}{8} w(\tau)^2 \delta(\tau - \sigma) - \frac{15}{8} w(\tau)^3 \delta'' + \left( \frac{3}{16} w(\tau)^2 + \frac{w''(\tau)}{2} \right) \delta' + \left( \frac{27}{8} w(\tau)^3 - \frac{27}{8} w(\tau) w''(\tau) + \frac{9}{16} w''(\tau) \right) \delta \right] + O(\epsilon^4), \tag{6.14}
\]

\[
\{w(\tau), w(\sigma)\}_2 = w(\tau)\delta'(\tau - \sigma) + \frac{1}{2} w(\tau)\delta(\tau - \sigma)
\]

\[
+ \epsilon^2 \left[ \frac{1}{2} \delta(\tau - \sigma) - \frac{3}{4} w(\tau)^2 \delta'' + \left( -\frac{47}{32} w(\tau)^3 + \frac{7}{8} w''(\tau) \right) \delta' + \left( \frac{189}{64} w(\tau)^3 - \frac{99}{32} w(\tau) w''(\tau) + \frac{9}{16} w''(\tau) \right) \delta \right] + O(\epsilon^4), \tag{6.15}
\]

and the Hamiltonians

\[
H_1 = \int \left[ q(w(\tau)) + \frac{\epsilon^2}{2} w'(\tau)^2 \left( \frac{9}{16} w(\tau)^3 + \frac{5}{8} w(\tau)^2 + \frac{q''(w(\tau))}{12 w(\tau)} \right) \right] d\tau + O(\epsilon^4),
\]

\[
H_2 = \int \left[ r(w(\tau)) + \frac{\epsilon^2}{2} w'(\tau)^2 \left( \frac{9}{16} w(\tau)^3 + \frac{5}{8} w(\tau)^2 + \frac{r''(w(\tau))}{12 w(\tau)} \right) \right] d\tau + O(\epsilon^4).
\]
Here $\delta^{(k)} = \delta^{(k)}(\tau - \sigma)$ and the function $q(w), r(w)$ are defined by the relations

$$p(w) = q''\left(\frac{1}{w}\right), \quad q'(w) = \frac{r'(w)}{2} + w r'(w).$$

The above bihamiltonian structure (6.14), (6.15) is equivalent to the one given in (1.9), (1.10) with central invariant $c(u) = -\frac{1}{24u}$ under the Miura type transformation

$$\tilde{w} = w + \epsilon^2 \frac{32}{w^3} \left(29 \left(w'\right)^2 - 10 w w''\right) + \mathcal{O}(\epsilon^4).$$

The above calculation is actually done at the approximation up to $\epsilon^8$, which gives us evidence to conjecture that the evolutionary equation (6.11) that is obtained from the KdV hierarchy by the above reciprocal transformation is equivalent, under a Miura type transformation, to the bihamiltonian hierarchy (6.12) with central invariant $c(u) = -\frac{1}{24u}$.

A full description of the above integrable hierarchy and its bihamiltonian structure is still lack. The approach given in [23] may be helpful to this problem.

### 7 Conclusion

In this paper we proved the existence of deformations of the bihamiltonian structure (1.6), (1.7), we hope that apart from the two well-known bihamiltonian structures that are associated to the KdV hierarchy and the Camassa-Holm hierarchy, this class of deformed bihamiltonian structures also contains some new bihamiltonian structures which have important applications in the theory of integrable systems and in mathematical physics. We will continue to study the properties and applications of these bihamiltonian structures in subsequent publications. It is also interesting to compute the bihamiltonian cohomologies $BH_{p,d}^\ast(\mathcal{F})$ for $p \geq 4$, $d \geq 6$ which, as we expected, should be trivial.

In the subsequent paper [15], we are to consider the existence problem for a general semisimple bihamiltonian structure of hydrodynamic type by using our formulation of the infinite dimensional bihamiltonian structures and their cohomologies given in the present paper.

**Acknowledgments.**

The authors are grateful to Boris Dubrovin for his encouragements and advises. This work is partially supported by the NSFC No. 11071135 and No. 11171176 and also by the Marie Curie IRSES project RIMMP.
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