On the SuperConformal Quantum Mechanics in the Nonlinear Realizations Approach

V.P. Akulov\textsuperscript{a,b} *, Oktay Cebecioglu\textsuperscript{b,c}† and A. Pashnev\textsuperscript{d}‡

\textsuperscript{a}Department of Natural Sciences, Baruch College of the City University of New York, New York, NY 10010, USA

\textsuperscript{b}Graduate School and University Center, the City University of New York, 365 Fifth Ave, New York, NY 10016-4309, USA

\textsuperscript{c}Department of Physics, Kocaeli University, Ataturk Bulvari, Anitpark Yanl, Kocaeli, Turkey

\textsuperscript{d}Bogoliubov Laboratory of Theoretical Physics, JINR Dubna, 141980, Russia

Abstract

In the framework of nonlinear realizations we rederive the action of the $N = 2$ SuperConformal Quantum Mechanics (SCQM). We propose also the WZNW – like construction of interaction term in the lagrangian with the help of Cartan’s Omega forms.
1 Introduction

The Conformal Quantum Mechanics (CQM)[1] as well as its supersymmetrical generalization – SCQM [2]-[3] are the simplest theories for developing the methods of investigation of more complicated higher dimensional field theories. The interest to the SCQM’s with extended supersymmetry [4]-[9] is connected also with the fact that these theories with anticommuting variables are exactly solvable not only on the classical level [1, 2, 3][10][11][5], but on the quantum one as well, where superconformal group plays the role of dynamical symmetry group. One should also note that in spite of its simplicity, the SCQM describes the physical objects like a particle near horizons of black holes [12] etc.¹

The geometrical meaning of CQM and SCQM can be understood in the framework of nonlinear realizations of the symmetry groups, underlying both theories - the group SL(2, R) and its supersymmetrical generalization SU(1, 1|1) respectively[11][5]. In this approach the N = 4 SCQM was also constructed[4] (see also [6]). Some other variants of N = 2 and N = 4 SCQM were analyzed in [8],[9]. So, the nonlinear realizations method leads to the lagrangians, which may be constructed in the framework of the usual superfield approach in which supersymmetry is realized linearly in the standard manner.

In deriving of these results from the nonlinear realizations approach the Cartan’s Omega-forms technics is usually supplied by the so called inverse Higgs effect [13]. It is very powerful approach which gives the possibility of covariant reduction of the number of the variables by expressing some of them in terms of other. However, in some cases the role of inverse Higgs effect can play the equations of motion for some auxiliary variables like momenta in the Hamiltonian formulation of the action integral. In general the question of interrelations of these two approaches is not investigated yet.

One of the goals of the present paper is to show that the consistent application of the nonlinear realizations approach gives the possibility of constructing both the kinetic and interaction terms for CQM and N = 2 SCQM in the superfield approach without using the inverse Higgs effect.

In Section 2 we reproduce results of [11] for CQM using the natural matrix representation for the group SL(2, R). In the framework of nonlinear realizations method we construct invariant actions with the help of Cartan’s Omega-forms. We give also the method of construction of some additional invariant actions from the Cartan’s Omega-forms, though in this case we do not found new invariants - it reproduces only old ones, constructed straightforwardly. These invariants plays the role of the well known WZNW terms and are constructed as integrals over some additional parameter σ of Cartan’s Omega-forms components.

After that we apply the same technics to N = 2 SCQM using the matrix realization for the group SU(1, 1|1). We consider the nonlinear realization of the coset SU(1, 1|1)/U(1), construct Cartan’s Omega-forms and show, how to build the kinetic part of the superfield action from the invariant coefficients of these Omega-forms. The application of the method developed in the first part of the paper gives us the possibility to construct from the Cartan’s Omega-forms the interaction part of the superfield action as well.

¹The extended SCQM is closely related with the Calogero model with spin, which has many physical applications.
2 The Conformal Quantum mechanics

2.1 Conformal group and its matrix representation

The conformal group in one dimensional space $SL(2, R)$ is a three-parameter subgroup of the infinite-dimensional reparametrization (diffeomorphisms) group on the line. When the line is parametrized by some parameter $s$ the generators of this group are $L_m = i s^{m+1} \frac{d}{ds}$ and form the Virasoro algebra without central charge

$$[L_n, L_m] = -i(n - m)L_{n+m}. \quad (2.1)$$

If one restricts to the regular at the origin $s = 0$ transformations, it is convenient to parametrize the group element as

$$G = e^{ix_0L_0} \cdot e^{ix_1L_1} \cdot e^{ix_2L_2} \cdot e^{ix_3L_3} \ldots e^{ix_0L_0}. \quad (2.2)$$

The transformation laws of the coordinates in (2.2) under the infinitesimal left action

$$G' = (1 + i\epsilon)G, \quad \epsilon = \epsilon^0L_{-1} + \epsilon^1L_0 + \epsilon^2L_1 + \ldots + \epsilon^mL_m + \ldots, \quad (2.3)$$

are

$$\begin{align*}
\delta \tau &= \varepsilon(\tau) \equiv \epsilon^0 + \epsilon^1\tau + \epsilon^2\tau^2 + \ldots, \quad (2.4) \\
\delta x_0 &= \dot{\varepsilon}(\tau), \quad (2.5) \\
\delta x_1 &= -\ddot{\varepsilon}(\tau)x_1 + \frac{1}{2}\dddot{\varepsilon}(\tau), \quad (2.6) \\
\delta x_2 &= -2\dddot{\varepsilon}(\tau)x_2 + \frac{1}{6}\ddddot{\varepsilon}(\tau), \quad (2.7) \\
\ldots 
\end{align*}$$

In general the coordinate $x_n$ in (2.2) transforms through the infinitesimal transformation function $\varepsilon(\tau)$ and coordinates $x_k, \quad k \leq n$. In addition the transformation law for parameter $x_n$ contains the term with $n + 1$-st derivative of the parameter $\varepsilon(\tau)$.

The simplest transformation law have the dimension-one coordinate $\tau$ which transforms as the coordinate of the one-dimensional space under the reparametrization. The coordinates $x_0$ and $x_1$ transform correspondingly as the dilaton and one-dimensional Cristoffel symbol. At this stage it is natural to consider all parameters as the fields in one-dimensional space parametrized by the coordinate $\tau$. However, in general all fields $x_m(\tau)$ can depend on some coordinate $\sigma$, which plays the role of additional parameter. Even more, such dependence on the additional parameter $\sigma$ can take place only for fields $x_m$ with $m \geq M$ with fixed $M$. This will not lead to any contradictions with the transformation laws (2.3)-(2.7). As we will see, the introduction of such additional coordinates gives the possibility to construct some nontrivial invariants of the transformations (2.3).

As we already mentioned, the conformal group in one dimension is a subgroup of (2.2), namely the ones generated by $L_{-1}, L_0$ and $L_1$

$$G_C = e^{i\tau L_{-1}} \cdot e^{ix_1L_1} \cdot e^{ix_0L_0}. \quad (2.8)$$

\footnote{The parameterizations like (2.2) were firstly introduced in \cite{14, 15} for 2-dimensional (super)conformal groups.}
As was shown in [11] the one-dimensional conformal mechanics introduced in [1] can be described on the language of invariant differential Cartan’s forms connected with the parametrization (2.8) of the conformal group. Moreover, by the linear change of basis of the conformal algebra one can describe on the same footing the “new” conformal mechanics of [12].

In this Section we reproduce the results of [11] using the natural matrix realization for the generators of $SL(2,\mathbb{R})$ group: translation $H = L_{-1}$, dilatation $D = L_0$ and conformal transformation $K = L_1$

$$H = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad D = -\frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(2.9)

Such representation of the generators of the conformal group can be easily generalized to the superconformal case [16], including the extended ones.

So, in the purely bosonic case the element of the conformal group in one dimension can be parametrized as a product of three matrix multipliers

$$K_C = G_C = \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -ix_1 & 1 \end{pmatrix}, \quad \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} = \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} x & 0 \\ ip & 1/x \end{pmatrix}$$

(2.10)

The parameters in (2.8) and (2.10) are connected by the relations $t = -\tau$, $x = e^{x_0/2}$ and $p = -x_1 x$. The conformal group transformation of these new variables are

$$t' = \frac{at+b}{ct+d}, \quad x' = \frac{x}{ct+d}, \quad p' = (ct+d)p - cx,$$

(2.11)

where parameters of the transformation are constrained by the unimodularity condition $ad - bc = 1$. Using the representation (2.2) one can calculate also the transformations of functions $x(\tau)$ and $p(\tau)$ under the most general (finite) reparametrization:

$$t \rightarrow t' = f(t),$$

(2.12)

$$x(t) \rightarrow x'(t') = (\dot{f}(t))^{1/2}x(t)$$

(2.13)

$$p(t) \rightarrow p'(t') = \frac{1}{(\dot{f}(t))^{1/2}}p(t) + \frac{\ddot{f}(t)}{2(\dot{f}(t))^{3/2}}x(t).$$

(2.14)

The invariant differential Cartan’s form, calculated with the help of (2.10) is

$$\Omega_C = K_C^{-1}dK_C = \begin{pmatrix} (dx - pdt)/x & idt/x^2 \\ i(xdp - pdx + p^2 dt) & -(dx - pdt)/x \end{pmatrix} = \begin{pmatrix} \omega_D & i\omega_H \\ i\omega_K & -\omega_D \end{pmatrix}$$

(2.15)

All matrix elements in (2.15) are invariant under the transformations (2.11). One can recognize among them the einbein differential form $\omega_H$. 

3
2.2 The action integral for Conformal Mechanics

All these differential forms can be used for construction of an invariant action. The simplest one is the linear combination

$$ S = -\frac{1}{2} \int \omega_K + \alpha \int \omega_D - \Lambda \int \omega_H = \int dt \left( -1/2(x\dot{p} - p\dot{x} + p^2) + \alpha(\dot{x}/x - p/x) - \frac{\Lambda}{x^2} \right). $$

The first term in this expression is appropriately normalized to get the correct kinetic term. The parameter $\Lambda$ plays the role of cosmological constant.

One can find $p$ by solving its equation of motion, insert it back in the lagrangian and get the action of De Alfaro, Fubini and Furlan

$$ S = \frac{1}{2} \int dt \left( \dot{x}^2 - \frac{\gamma}{x^2} \right) $$

with the coupling constant $\gamma = \Lambda + \alpha^2/2$. So, the parameter $\alpha$ simply renormalizes the cosmological constant $\Lambda$.

At this point we should note that the integrand in the action $S$ is invariant only up to the total derivative, though it was deduced from the expression (2.16) in which the integrand is strictly invariant because it was constructed out of invariant Cartan’s Omega forms $\omega_K$, $\omega_D$ and $\omega_H$. The reason of this lies in the utilization of some equations of motion and partial integration when (2.16) is transformed into (2.17). The same situation will take place also in the more complicated case of $N = 2$ SCQM.

The additional invariant actions can be constructed as

$$ S_F = \int \omega_H F(\frac{\omega_K}{\omega_H}, \frac{\omega_D}{\omega_H}) $$

with arbitrary function $F$ of two invariant variables which are the coefficients in the expressions of invariant differential one-forms $\omega_K$ and $\omega_D$ in terms of only one (in dimension one) independent invariant one-form $\omega_H$. Some of these actions will have form (2.17), but in general the actions (2.18) will include the higher degrees of the velocity $\dot{x}$.

The another sort of invariants in the action can be constructed by introducing the dependence of the group element (2.8) or (2.10) on some parameter $\sigma$. Indeed, one can consider the special dependence of the group element (2.10) on some new parameter $\sigma$ such that $t$ does not depend on it, whereas functions $x_0(t, \sigma)$ and $x_1(t, \sigma)$ are subject to the following boundary conditions

$$ x_0(t, 0) = x_1(t, 0) = 0, \quad x_0(t, 1) = x_0(t), \quad x_1(t, 1) = x_1(t). $$

So, the boundary group elements in (2.10) are

$$ K_C(\sigma = 0) = \begin{bmatrix} 1 & it \\ 0 & 1 \end{bmatrix}, \quad K_C(\sigma = 1) = K_C, $$

and

$$ K_C(\sigma = 0) = \begin{bmatrix} 1 & it \\ 0 & 1 \end{bmatrix}, \quad G_0, $$

(2.20)
where \( G_0 \) is the identity element of the group. The parameters \( \epsilon^n \) (in our case \( n = 0, 1, 2 \)) of transformation (2.3) are assumed to be independent of \( \sigma \). It means that condition (2.21) will be the same for transformed quantities. On the other hand the group element \( G_0 \) in the boundary condition (2.20) will vary. But this variation is very simple, as one can see from the transformation laws of \( x_0 \) and \( x_1 \). Moreover, both of them are total derivatives.

All this can serve as an argumentation of the following construction, leading in general to some integral invariants on the group. In our case there are two independent invariant differential one-forms - \( \omega_H \) and \( d\sigma \). So, the coefficients in expanding of all other Cartan’s forms in terms of these two forms are invariant as well. For example, such one is

\[
I = \frac{\omega_H}{d\sigma} = \frac{1}{x} \frac{dx}{d\sigma}
\]  

(2.22)

In the presence of new coordinate \( \sigma \) the invariant integration measure is

\[
\int dv = \int \omega_H d\sigma = \int \frac{dtd\sigma}{x^2}.
\]  

(2.23)

The result of integration over \( \sigma \)

\[
S_1 = \int dv I = \int dtd\sigma \frac{1}{x^3} \frac{dx}{d\sigma} = -\frac{1}{2} \int dt \frac{1}{x^2} |_{\sigma=1} + \frac{1}{2} \int dt \frac{1}{x^2} |_{\sigma=0}
\]  

(2.24)

is the difference of two terms at points \( \sigma = 1 \) and \( \sigma = 0 \). The last one is invariant by virtue of the transformation laws (2.5)-(2.6) of \( x_0 \) and \( x_1 \) near the identity element which corresponds to \( x_0 = 0 \) and \( x_1 = 0 \). Indeed, they transform as total derivatives. Because by construction \( S_1 \) is invariant, the term at the point \( \sigma = 1 \) should be also invariant, though the integrand in this term transforms as a total derivative. In the case under consideration it is not wondering, because this invariant simply reproduces the already known third term in (2.10). Nevertheless, as we will see later, such procedure of constructing can lead to new invariants in more complicated cases.

3 The \( N = 2 \) SuperConformal Quantum mechanics

3.1 The matrix representation of the \( N = 2 \) SuperConformal Group

The \( N = 2 \) SuperConformal group in one dimensional space is an eight-parameter subgroup of the infinitendimensional \( N = 2 \) Super Virasoro group with the following algebra of its generators

\[
\begin{align*}
[L_n, L_m] &= -i(n - m)L_{n+m}, \\
[L_n, G_r] &= -i\left(\frac{n}{2} - r\right)G_{n+r}, \quad [L_n, \bar{G}_r] = -i\left(\frac{n}{2} - r\right)\bar{G}_{n+r}, \\
\{G_r, \bar{G}_q\} &= -2L_{r+q} - 2(r - q)U_{r+q}, \\
[U_n, G_r] &= -i\frac{1}{2}G_{n+r}, \quad [U_n, \bar{G}_r] = i\frac{1}{2}\bar{G}_{n+r}.
\end{align*}
\]  

(3.1-3.4)

The indices \( n, m \) are integer and \( q, r \) - halfinteger ones. The \( N = 2 \) SuperConformal algebra contains in addition to the generators of Conformal algebra (translation \( H = L_{-1}, \) dilatation
$D = L_0$ and conformal transformation $K = L_1$ the $U(1)$ generator $U \equiv U_0$ and generators of Poincaré ($Q = G_{-1/2}, \bar{Q} = \bar{G}_{-1/2}$) and Conformal ($S = G_{1/2}, \bar{S} = \bar{G}_{1/2}$) supersymmetries. All of these generators can be realized in terms of $3 \times 3$ graded matrices with vanishing supertrace $(\text{Str} \, M = M_{11} + M_{33} - M_{22})$:

$$H = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad D = -\frac{i}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad U = -\frac{i}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{Q} = \sqrt{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{S} = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}$$

Our parametrization for coset space of $N = 2$ SuperConformal group over the $U(1)$ subgroup generated by $U$ is:

$$K_{\text{SC}} = \frac{G_{\text{SC}}}{U(1)} = \begin{pmatrix} 1 & \theta & i(\theta+\bar{\theta})/2 \\ 0 & 1 & \bar{\theta} \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.7)$$

The transformation laws of the coset space parameters under the infinitesimal left shift

$$K'_{\text{SC}} = (1 + i\epsilon)K_{\text{SC}} = \begin{pmatrix} 1 + b/2 & \epsilon & ia \\ \lambda & 1 & \bar{\epsilon} \\ ic \lambda & \bar{\lambda} & 1 - b/2 \end{pmatrix} \cdot K_{\text{SC}}$$

are:

$$\delta t = a + bt + ct^2 - \frac{i}{2} \epsilon \bar{\theta} - \frac{i}{2} \epsilon \theta + \frac{1}{2}(\lambda \theta - \bar{\lambda} \bar{\theta})t,$$

$$\delta \theta = \left( \frac{b}{2} + ct - \frac{1}{2} \bar{\lambda} \bar{\theta} \right) \theta + \epsilon - i \bar{\lambda} t,$$

$$\delta \psi = \left( -\frac{b}{2} - ct + \frac{i}{2} c \theta \bar{\theta} + \bar{\lambda} \bar{\theta} \right) \psi + \lambda - ic \bar{\theta},$$

$$\delta x = \left( \frac{b}{2} + ct - \frac{1}{2} \theta \lambda + \frac{1}{2} \bar{\theta} \bar{\lambda} \right) x,$$

$$\delta (x_1) = \left( -2bt + \bar{\lambda} \bar{\theta} + \theta \lambda \right)x_1 + b - \frac{i}{2}(\bar{\lambda} + ib \theta) \psi - \frac{i}{2}(\lambda - ib \bar{\theta}) \bar{\psi}.$$ 

One can see that in the point $(x = 1, x_1 = \psi = 0)$ the variables $(x, x_1, \psi)$ transform as a total derivatives.

The differential Cartan’s form, calculated with the help of (3.7) is

$$\Omega_C = K_C^{-1} dK_C = \begin{vmatrix} \omega_D + i\omega_U & \omega_Q & i\omega_H \\ \omega_S & 2i\omega_U & \omega_Q \\ i\omega_K & \omega_S & -\omega_D + i\omega_U \end{vmatrix}$$

(3.14)
where

\[ \omega_D = dx/x + \frac{1}{2} d\theta \psi + \frac{1}{2} \bar{\psi} d\bar{\theta} - x_1 dT \]  
(3.15)

\[ \omega_Q = (d\theta + i\bar{\psi}dT)/x \]  
(3.16)

\[ \omega_{\bar{Q}} = (d\bar{\theta} - i\psi dT)/x \]  
(3.17)

\[ \omega_H = dT/x^2 \]  
(3.18)

\[ \omega_S = x (d\psi + x_1 dT \psi + (ix_1 + 1/2 \bar{\psi} \psi) d\bar{\theta}) \]  
(3.19)

\[ \omega_{\bar{S}} = x (d\bar{\psi} + x_1 dT \bar{\psi} + (-ix_1 + 1/2 \bar{\psi} \psi) d\theta) \]  
(3.20)

\[ \omega_K = x^2 \left( dx_1 - \frac{i}{2} d\bar{\psi} \psi + \frac{i}{2} \bar{\psi} d\psi + x_1^2 dT - x_1 (d\theta \psi + \bar{\psi} d\bar{\theta}) \right) \]  
(3.21)

\[ \omega_U = \frac{1}{2} \bar{\psi} \psi dT - \frac{i}{2} d\theta \psi + \frac{i}{2} \bar{\psi} d\bar{\theta} \]  
(3.22)

and

\[ dT = dt - \frac{i}{2} d\theta \bar{\theta} + \frac{i}{2} \theta d\bar{\theta} \]  
(3.23)

Due to the fact that we consider the coset space \( G_{SC}/U(1) \) instead of the whole group \( G_{SC}/U(1) \), not all of Cartan’s forms are invariant. Namely, \( \omega_Q, \omega_{\bar{Q}}, \omega_S \) and \( \omega_{\bar{S}} \) transform homogeneously as linear representations of \( U(1) \). As one can easily see the forms \( \omega_Q, \omega_{\bar{S}} \) carry the same charge under the \( U(1) \) transformations, whereas \( \omega_{\bar{Q}} \) and \( \omega_S \) carry the opposite equal charge. In turn, \( \omega_U \) transform as a total differential.

### 3.2 The action integral for \( N = 2 \) SuperConformal Mechanics

All Cartan’s forms can be expanded in terms of three independent ones \( \omega_H, \omega_Q, \omega_{\bar{Q}} \) using the formula

\[ df(t, \theta, \bar{\theta}) = x \omega_Q Df + x \omega_{\bar{Q}} \bar{D}f + x^2 \omega_H (\dot{f} - i \bar{\psi} Df + i \psi \bar{D}f), \]  
(3.24)

where \( D \) and \( \bar{D} \) are flat covariant derivatives

\[ D = \frac{\partial}{\partial \theta} + \frac{i}{2} \theta \frac{\partial}{\partial t}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} + \frac{i}{2} \bar{\theta} \frac{\partial}{\partial t}. \]  
(3.25)

The coefficients in such expansions of \( \omega_H, \omega_D, \omega_S \) and \( \omega_K \) are invariant under the transformations (3.9)-(3.13) or transform as some linear representation of \( U(1) \). Two of these expansions which are useful in the construction of invariants are

\[ \omega_D = \omega_H (xdx/dt - x_1 x^2 - i \bar{\psi} x Dx + i \psi x \bar{D}x) + \omega_Q (Dx + x/2 \psi) + \omega_{\bar{Q}} (\bar{D}x - x/2 \bar{\psi}), \]  
(3.26)

\[ \omega_S = x^2 \omega_H (x \dot{\psi} - ix \bar{\psi} D\psi + ix \psi \bar{D}\psi) + x^2 \omega_Q D\psi + \omega_{\bar{Q}} (x^2 D\psi + ix_1 x^2 + 1/2 x^2 \bar{\psi}). \]  
(3.27)

Since \( \omega_D \) and \( \omega_H \) are invariant, the coefficient

\[ I_{D/H} = \frac{\omega_D}{\omega_H} = x dx/dt - x_1 x^2 - i \bar{\psi} x Dx + i \psi x \bar{D}x \]  
(3.28)

is invariant as well. At the same time the coefficients

\[ I_{D/Q} = \frac{\omega_D}{\omega_Q} = Dx + x/2 \psi \]  
(3.29)
and

\[- I_{D/\bar{Q}} = - \frac{\omega_D}{\omega_{\bar{Q}}} = - \bar{D}x + x/2 \bar{\psi} \]  

(3.30)

are mutually conjugated and transform with the opposite phase under the transformations (3.9)-(3.13). So, their product \( L_1 = I_{D/\bar{Q}} I_{D/\bar{Q}} \) as well as \( L_2 = I_{D/H} \) may be used for construction of invariant lagrangians.

The coefficients

\[ I_{S/Q} = \frac{\omega_S}{\omega_Q} = x^2 \bar{D} \psi + ix_1 x^2 + 1/2 x^2 \bar{\psi} \psi, \]  

(3.31)

\[ I_{\bar{S}/Q} = \frac{\omega_{\bar{S}}}{\omega_Q} = x^2 D \bar{\psi} - ix_1 x^2 + 1/2 x^2 \bar{\psi} \psi \]  

(3.32)

are mutually conjugated and inert under the transformations (3.9)-(3.13). So, their sum \( L_K = I_{S/\bar{Q}} + I_{\bar{S}/Q} \) gives one more possible term in the Lagrangian\(^3\). Indeed, it describes the kinetic term of the SCQM in the superfield formulation.

Using the invariant measure \( dv = dt d\theta d\bar{\theta} \) one can construct the invariant action in the form

\[ S_K = \int dt d\theta d\bar{\theta} \{ I_{S/\bar{Q}} + I_{\bar{S}/Q} \} = \int dt d\theta d\bar{\theta} \{ x^2 \bar{D} \psi + x^2 D \bar{\psi} + x^2 \bar{\psi} \psi \}. \]  

(3.33)

With the help of the equations of motion for \( \psi \) and \( \bar{\psi} \)

\[- \bar{D}x + x/2 \bar{\psi} = 0, \]  

(3.34)

\[ Dx + x/2 \psi = 0, \]  

(3.35)

the action \( S_K \) can be rewritten in the form

\[ S_K = 4 \int dt d\theta d\bar{\theta} \bar{D} x D x, \]  

(3.36)

in which the integrand is invariant only up to the total derivatives (see the remark after the eq. (2.17)).

Note, that the left hand sides of the equations (3.34)-(3.35) coincide with the coefficients (3.29)-(3.30). So, they vanish some part of the Cartan’s Omega form \( \omega_D \), playing the role of inverse Higgs effect \cite{13}. It means that there is no need to use the eq’s - (3.34)-(3.35) as independent ones arising from the inverse Higgs effect as in \cite{6}.

One can construct some other invariant terms for the action, for example

\[ S_0 = \alpha \int dt d\theta d\bar{\theta} \{ (Dx + 1/2 x \psi)(\bar{D}x - 1/2 x \bar{\psi}) \}, \]  

(3.37)

but this term leads simply to redefinition of overall coefficient in (3.36).

The additional invariants in the action can be constructed by the procedure described in the previous Section. We again introduce the dependence of the group element (3.7) on some parameter \( \sigma \) such that \( t, \theta, \bar{\theta} \) do not depend on it, whereas functions \( x_0(t, \sigma), x_1(t, \sigma) \) and \( \psi(t, \sigma) \) are subject to the boundary conditions

\[ \ln x(t, 0) = x_1(t, 0) = \psi(t, 0) = 0, \]  

(3.38)

\[ \ln x(t, 1) = \ln x(t), \ x_1(t, 1) = x_1(t), \ \psi(t, 1) = \psi(t). \]  

\[^3\text{The analogous construction for } N = 2 \text{ Virasoro group gives the conformally invariant superfield description of } N = 2 \text{ spinning particle}\cite{18}.\]
So, the boundary group elements in (3.7) are

\[
K_{SC}(\sigma = 0) = \begin{pmatrix}
1 & \theta & it + \theta \bar{\theta}/2 \\
0 & 1 & \bar{\theta} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

and

\[
K_{SC}(\sigma = 1) = K_{SC}. \tag{3.40}
\]

Using the arguments of the previous Section one can show that the expression

\[
\tilde{S}_1 = \int dt d\theta d\bar{\theta} d\sigma \left\{ \frac{\omega_D}{d\sigma} \right\} = \int dt d\theta d\bar{\theta} d\sigma \frac{1}{x} \left\{ \frac{dx}{d\sigma} \right\}
\]

is invariant and leads to additional invariant term in the action

\[
S_1 = \int dt d\theta d\bar{\theta} \ln x \tag{3.42}
\]

So, the total action \( S = S_0 + S_1 \) reproduces the action of \( N = 2 \) SuperConformal Quantum mechanics \([2]-[3]\) including both the kinetic and potential terms.

4 Conclusions

In this paper, we applied the methods of nonlinear realizations approach for construction of the actions of Conformal and \( N = 2 \) SuperConformal Quantum Mechanics. We have shown that both the kinetic and interaction terms of these models can be constructed by using the invariant Cartan’s Omega-forms. The interaction part of the action looks like the well known WZNW term. We have shown also that the Inverse Higgs Effect in both cases is a consequence of the equations of motion for some variables. It would be interesting to analyze the possibility of such duality between the Inverse Higgs Effect and equation of motions for auxiliary variables in more complicated theories, like \( N = 4 \) SuperConformal Quantum Mechanics. Besides, it is interesting to investigate in this model the possibility getting with the help of the equations of motion of some irreducibility conditions for the basic superfield, which were obtained originally by the inverse Higgs effect \([5]\).

Acknowledgements

A.P. thanks the members of Graduate School and University Center, the City University of New York, where the essential part of this work was done and especially Prof. S. Catto for hospitality and partial financial support. The work of A.P. was partially supported by INTAS grant, project No 00-00254.

References

[1] V. De Alfaro, S. Fubini, G. Furlan, Nuovo Cim. A 34 (1974) 569.
[2] V. Akulov, A. Pashnev, Teor. Mat. Fiz. 56 (1983) 344.
[3] S. Fubini, E. Rabinovici, Nucl. Phys. B 245 (1984) 17.

[4] A. Pashnev, Sov.J.Theor.Math.Phys., 69 (1986) 1172.

[5] E. Ivanov, S. Krivonos, V. Leviant, J. Phys. A: Math. Gen. 22 (1989) 4201

[6] J.A. de Azcarraga, J.M. Izquierdo, J.C. Perez Bueno, P.K. Townsend, Phys. Rev. D 59 (1999) 084015; hep-th/9810230

[7] V. Akulov and M. Kudinov, Phys.Lett. B460 (1999) 365; hep-th/9905070

[8] E. Ivanov and S. Krivonos and J. Niederle, Conformal and Superconformal Mechanics Revisited
  e-Print Archive: hep-th/0210196

[9] E. Ivanov and S. Krivonos and O. Lechtenfeld, New variant of N=4 superconformal mechanics
  hep-th/0212303

[10] V.P.Akulov and D. Volkov, Sov.J.Theor.Math.Phys. 42 (1980) 10

[11] E. Ivanov and S. Krivonos and V. Leviant, J.Phys. A: Math.Gen., 22 (1989) 345

[12] P. Claus, M. Derix, R. Kallosh, J. Kumar, P.K. Townsend, A. Van Proeyen, Phys. Rev. Lett. 81 (1998) 4553; hep-th/9804177

[13] E. Ivanov, V. Ogievetsky, Teor. Mat. Fiz. 25 (1975) 164

[14] E. Ivanov and S. Krivonos, Lett.Math. Phys., 7 (1983) 523; ibid 8 (1984) 39

[15] E. Ivanov and S. Krivonos, J.Phys. A: Math.Gen., 17 (1984) L671

[16] D.V. Volkov and V.P. Akulov, Phys.Lett., B46 (1973) 109

[17] V.P.Akulov and D. Volkov, Sov.JETP Letters 17 (1973) 367

[18] A. Pashnev, Nucl.Phys.Proc.Suppl. 102 (2001) 240