Large Time Behavior on the Linear Self-Interacting Diffusion Driven by Sub-Fractional Brownian Motion II: Self-Attracting Case

Rui Guo¹, Han Gao²*, Yang Jin³ and Litan Yan³

¹College of Information Science and Technology, Donghua University, Shanghai, China, ²College of Fashion and Art Design, Donghua University, Shanghai, China, ³Department of Statistics, College of Science, Donghua University, Shanghai, China

In this study, as a continuation to the studies of the self-interaction diffusion driven by subfractional Brownian motion \( SH \), we analyze the asymptotic behavior of the linear self-attracting diffusion:

\[
dX^H_t = dS^H_t - \theta \left( \int_0^t (X^H_t - X^H_s) ds \right) dt + \nu dt, \quad X^H_0 = 0,
\]

where \( \theta > 0 \) and \( \nu \in \mathbb{R} \) are two parameters. When \( \theta < 0 \), the solution of this equation is called self-repelling. Our main aim is to show the solution \( X^H_t \) converges to a normal random variable \( \xi_\infty^H \) with mean zero as \( t \) tends to infinity and obtain the speed at which the process \( X^H_t \) converges to \( \xi_\infty^H \) as \( t \) tends to infinity.

Keywords: subfractional Brownian motion, self-attracting diffusion, law of large numbers, Malliavin calculus, asymptotic distribution

1 INTRODUCTION

In a previous study (I) (see [12]), as an extension to classical result, we considered the linear self-interacting diffusion as follows:

\[
X^H_t = S^H_t - \theta \int_0^t \left( \int_0^s (X^H_s - X^H_u) du \right) ds + \nu t, \quad t \geq 0, \tag{1}
\]

with \( \theta \neq 0 \), where \( \theta \) and \( \nu \) are two real numbers, and \( S^H \) is a sub-fBm with the Hurst parameter \( \frac{1}{2} \leq H < 1 \). The solution of Eq. (1) is called self-repelling if \( \theta < 0 \) and is called self-attracting if \( \theta > 0 \). When \( \theta < 0 \), in a previous study (I), we showed that the solution \( X^H_t \) diverges to infinity as \( t \) tends to infinity and

\[
J^H_0 (t; \theta, \nu) := t e^{\theta t} X^H_t \to \xi_\infty^H - \frac{\nu}{\theta}
\]

and

\[
J^H_n (t; \theta, \nu) := \theta t^2 \left( J^H_{n-1} (t; \theta, \nu) - (2n - 3)!! \left( \xi_\infty^H - \frac{\nu}{\theta} \right) \right) \to (2n - 1)!! \left( \xi_\infty^H - \frac{\nu}{\theta} \right)
\]

in \( L^2 \) and almost surely, for all \( n = 1, 2, \ldots \), where \( (-1)!! = 1 \) and
\[ \rho_{\infty}^H = \int_0^\infty e^{2t/\lambda_{H,\theta}} ds_t^H. \]

In the present study, we consider the case \( \theta > 0 \) and study its large time behaviors.

Let us recall the main results concerning the system (Eq. 1). When \( H = \frac{1}{2} \), as a special case of path-dependent stochastic differential equations, in 1995, Cranston and Le Jan [8] introduced a linear self-attracting diffusion (Eq. 1) with \( \theta > 0 \). They showed that the process \( X_t \) converges in \( L^2 \) and almost surely as \( t \) tends infinity. This path-dependent stochastic differential equation was first developed by Durrett and Rogers [10] introduced in 1992 as a model for the shape of a growing polymer (Brownian polymer). The general form of this kind of model can be expressed as follows:

\[ X_t = X_0 + B_t + \int_0^t \int_0^s f(X_u - X_v) du dv, \quad t \geq 0 \tag{2} \]

where \( B \) is a \( d \)-dimensional standard Brownian motion and \( f \) is Lipschitz continuity. \( X_t \) corresponds to the location of the end of the polymer at time \( t \). Under some conditions, they established asymptotic behavior of the solution of the stochastic differential equation. The model is a continuous analog of the notion of edge (respectively, vertex) self-interacting random walk (see, e.g., Pemantle [22]). By using the local time of the solution process \( X \), we can make it clear how the process \( X \) interacts with its own occupation density. In general, Eq. 2 defines a self-interacting diffusion without any assumption on \( f \). We call it self-repelling (respectively, self-attracting) if, for all \( x \in \mathbb{R}^d, x \cdot f(x) \geq 0 \) (respectively, \( \leq 0 \)). More examples can be found in Benaim et al. [2, 3], Cranston and Mountford [9], Gan and Yan [11], Gauthier [13], Herrmann and Roynet [14], Herrmann and Scheutzow [15], Mountford and Tarr [20], Sun and Yan [26, 27], Yan et al [34], and the references therein.

In this present study, our main aim is to expound and prove the following statements:

(I) For \( \theta > 0 \) and \( \frac{1}{2} < H < 1 \), the random variable

\[ X_\infty^H = \int_0^\infty h_0(s) ds + \int_0^\infty h_0(s) ds \]

exists as an element in \( L^2 \), where the function is defined as follows:

\[ h_0(s) = 1 - \theta s e^{2s^2/\lambda_{H,\theta}} \int_s^\infty e^{-2u^2/\lambda_{H,\theta}} du, \quad s \geq 0 \]

with \( \theta > 0 \).

(II) For \( \theta > 0 \) and \( \frac{1}{2} < H < 1 \), we have

\[ X_t^H \to X_\infty^H \]

in \( L^2 \) and almost surely as \( t \to \infty \).

(III) For \( \theta > 0 \) and \( \frac{1}{2} < H < 1 \), we have

\[ \frac{t^{H/2}}{\lambda_{H,\theta}} (X_t^H - X_\infty^H) \to N(0, 1) \]

in distribution as \( t \to \infty \), where

\[ \lambda_{H,\theta} = \frac{1}{2} \Gamma(2H + 1) \theta^{-2H}. \]

(IV) For \( \theta > 0 \) and \( \frac{1}{2} < H < 1 \), we have

\[ Y_t^H = \int_0^t (X_u^H - X_v^H) du, \quad t \geq 0. \]

Then the convergence

\[ \lim_{t \to \infty} \frac{1}{t^{2-2H}} \int_0^t (Y_u^H)^2 du \to \frac{H}{3 - 2H} \theta^{-2H} \Gamma(2H) \]

holds in \( L^2 \) as \( T \) tends to infinity.

This article is organized as follows. In Section 2, we present some preliminaries for sub-fBm and Malliavin calculus. In Section 3, we obtain some lemmas. In Section 4, we prove the main results given as before. In Section 5, we give some numerical results.

### 2 PRELIMINARIES

In this section, we briefly recall the definition and properties of stochastic integral with respect to sub-fBm. We refer to Alós et al [1], Nualart [21], and Tudor [31] for a complete description of stochastic calculus with respect to Gaussian processes.

As we pointed out in the previous study (I) (see [12]), the sub-fBm \( S_t^H \) is a rather special class of self-similar Gaussian processes such that \( S_t^H = 0 \) and

\[ R^H(t, s) := E[S_t^H S_s^H] = s^{2H} + t^{2H} - \frac{1}{2} \left[ (s + t)^{2H} + |t - s|^{2H} \right] \quad (3) \]

for all \( s, t \geq 0 \). For \( H = 1/2 \), \( S_t^H \) coincides with the standard Brownian motion \( B_t \). \( S_t^H \) is neither a semimartingale nor a Markov process unless \( H = 1/2 \), so many of the powerful techniques from stochastic analysis are not available when dealing with \( S_t^H \). As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to \( S_t^H \). The sub-fBm appeared in Bojdecki et al [4] in a limit of occupation time fluctuations of a system of independent particles moving in \( \mathbb{R}^d \) according a symmetric \( \alpha \)-stable Lévy process. More examples for sub-fBm and related processes can be found in Bojdecki et al. [4–7], Li [16–19], Shen and Yan [23, 24], Sun and Yan [25], C. A. Tudor [32], Tudor [28–31], C. A. Tudor [33], Yan et al [33, 35, 36], and the references therein.

The normality and Hölder continuity of the sub-fBm \( S_t^H \) imply that \( t \to S_t^H \) admits a bounded \( p_{H,\theta} \) variation on any finite interval with \( p_{H,\theta} > \frac{1}{\lambda_{H,\theta}} \). As an immediate result, one can define the Young integral of a process \( u = u(s, t) \geq 0 \) with respect to a sub-fBm \( S_t^H \)

\[ \int_0^t u_t ds_t^H \]

as the limit in probability of a Riemann sum. Clearly, when \( u \) is of bounded \( q_{H,\theta} \) variation on any finite interval with \( q_{H,\theta} > 1 \) and \( \frac{1}{p_{H,\theta}} + \frac{1}{q_{H,\theta}} > 1 \), the integral is well-defined and
\[ u_t S_t^H = \int_0^t u_s ds^H + \int_0^t S_t^H du_s \]

for all \( t \geq 0 \).

Let \( H \) be the completion of the linear space \( E \) generated by the indicator functions \( 1_{[0,t]} \), \( t \in [0, T] \) with respect to the inner product:

\[ \langle 1_{[0,t]}, 1_{[0,s]} \rangle_H = R^H(t,s) \]

for \( s, t \in [0, T] \). For every \( \varphi \in \mathcal{H} \), we can define the Wiener integral with respect to \( S^H \), denoted by

\[ S^H(\varphi) = \int_0^T \varphi(s) ds^H \]
as a linear (isometric) mapping from \( \mathcal{H} \) onto \( S^H \) by using the limit in probability of a Riemann sum, where \( S^H \) is the Gaussian Hilbert space generating by \( S^H \) and

\[ \| \varphi \|_H^2 = E\left( \int_0^T \varphi(s) ds^H \right)^2 \]

for any \( \varphi \in \mathcal{H} \). In particular, when \( \frac{1}{2} < H < 1 \), we can show that

\[ \| \varphi \|_H^2 = \int_0^T \int_0^T \varphi(t) \varphi(s) \psi_H(t,s) dsdt, \quad \forall \varphi \in \mathcal{H}, \]

where

\[ \psi_H(t,s) = \frac{\partial^2}{\partial t \partial s} R^H(t,s) = H(2H-1)(|t-s|^{2H-2} - |t+s|^{2H-2}) \]

as a linear mapping from \( \mathcal{H} \) onto \( S^H \) by using the limit in probability of a Riemann sum, where \( S^H \) is the Gaussian Hilbert space generating by \( S^H \) and

\[ \| \varphi \|_H^2 = E\left( \int_0^T \varphi(s) ds^H \right)^2 \]

for any \( \varphi \in \mathcal{H} \). In particular, when \( \frac{1}{2} < H < 1 \), we can show that

\[ \| \varphi \|_H^2 = \int_0^T \int_0^T \varphi(t) \varphi(s) \psi_H(t,s) dsdt, \quad \forall \varphi \in \mathcal{H}, \]

where

\[ \psi_H(t,s) = \frac{\partial^2}{\partial t \partial s} R^H(t,s) = H(2H-1)(|t-s|^{2H-2} - |t+s|^{2H-2}) \]
for $s, t \in [0, T]$. Thus, when $\frac{1}{2} < H < 1$ if for every $T > 0$, the integral
\[ \int_0^T \varphi(s)dS^H_t \]
exists in $L^2$ and
\[ \int_0^\infty \int_0^\infty \varphi(t)\varphi(s)\psi_H(t, s)dtds < \infty, \]
we can define the integral as follows:
\[ \int_0^\infty \varphi(s)dS^H_t \]
and
\[ E\left( \int_0^\infty \varphi(s)dS^H_t \right)^2 = \int_0^\infty \int_0^\infty \varphi(t)\varphi(s)\psi_H(t, s)dtds. \]

Let now $D$ and $\delta$ be the (Malliavin) derivative and divergence operators associated with the sub-fBm $S^H$. And let $\mathbb{D}^{1,2}$ denote the Hilbert space with respect to the norm as follows:
\[ \|F\|_{1,2} := \sqrt{E[F]^2 + E[DF]^2_H}. \]
Then the duality relationship
\[ E[F\delta(u)] = E(DF, u)_H \]
holds for any $F \in \mathbb{D}^{1,2}$ and $\mathbb{D}^{1,2} \subset \text{Dom}(\delta)$. Moreover, for any $u \in \mathbb{D}^{1,2}$, we have
\[ E[\delta(u)^2] = E[u]_H^2 + E(Du, (Du)^\gamma)_{\gamma H t} \]
\[ = E[u]_H^2 + E\left[ \int_{[0, T]} D\varphi\psi_H(\xi, s)d\xi d\eta \right]. \]
where \((Du)^*\) is the adjoint of \(Du\) in the Hilbert space given as follows: \(\mathcal{H} \otimes \mathcal{H}\). We denote

\[
\delta(u) = \int_0^T u_t dS_t^H
\]

for an adapted process \(u\), and it is called the Skorohod integral. By using Alós et al. [1], we can obtain the relationship between the Skorohod and the Young integral as follows:

\[
\int_0^T u_t dS_t^H = \int_0^T u_t dS_t^H + \int_0^T D_t u_t \psi_H(t, s) ds dt,
\]

provided \(u\) has a bounded \(q\) variation with \(1 \leq q < \frac{1}{H}\) and \(u \in D_{1,2}\) such that

\[
\int_0^T \int_0^T D_t u_t \psi_H(t, s) ds dt < \infty.
\]

### 3 SOME BASIC ESTIMATES

For simplicity, we throughout let \(C\) stand for a positive constant which depends only on its superscripts, and its value may be different in different appearances, and this assumption is also suitable to \(c\). Recall that the linear self-attracting diffusion with sub-fBm \(S^H\) is defined by the following stochastic differential equation:

\[
X_t^H = S_t^H - \theta \int_0^t (X_s^H - X_u^H) du + \nu t, \quad t \geq 0
\]
TABLE 1 | Data of $X^H_t$ with $\theta = 1$ and $H = 0.7$

| $t$  | $X^H_t$ | $t$  | $X^H_t$ |
|------|--------|------|--------|
| 0.000 | 0.000  | 0.3438 | -0.1216 | 0.6875 | -0.0979 |
| 0.015 | 0.0067 | 0.3594 | -0.1290 | 0.7031 | -0.0968 |
| 0.031 | 0.0113 | 0.3750 | -0.1406 | 0.7188 | -0.107 |
| 0.046 | 0.0160 | 0.3906 | -0.1467 | 0.7344 | -0.109 |
| 0.062 | -0.0153 | 0.4063 | -0.1469 | 0.7500 | -0.108 |
| 0.078 | -0.0238 | 0.4219 | -0.1579 | 0.7656 | -0.118 |
| 0.093 | -0.0229 | 0.4375 | -0.1624 | 0.7813 | -0.116 |
| 0.109 | -0.0270 | 0.4531 | -0.1666 | 0.7969 | -0.112 |
| 0.125 | -0.0335 | 0.4688 | -0.1701 | 0.8125 | -0.123 |
| 0.145 | -0.0523 | 0.4843 | -0.1717 | 0.8281 | -0.140 |
| 0.156 | -0.0370 | 0.5000 | -0.1738 | 0.8438 | -0.146 |
| 0.171 | -0.0498 | 0.5175 | -0.1774 | 0.8594 | -0.155 |
| 0.185 | -0.0544 | 0.5313 | -0.1766 | 0.8750 | -0.160 |
| 0.203 | -0.0593 | 0.5469 | -0.1713 | 0.8906 | -0.170 |
| 0.218 | -0.0765 | 0.5625 | -0.1667 | 0.9063 | -0.174 |
| 0.234 | -0.0850 | 0.5781 | -0.1664 | 0.9219 | -0.178 |
| 0.250 | -0.0981 | 0.5938 | -0.1521 | 0.9375 | -0.179 |
| 0.265 | -0.1062 | 0.6094 | -0.1422 | 0.9531 | -0.180 |
| 0.281 | -0.1127 | 0.6250 | -0.1395 | 0.9688 | -0.178 |
| 0.296 | -0.1132 | 0.6406 | -0.1282 | 0.9844 | -0.193 |
| 0.312 | -0.1140 | 0.6563 | -0.1234 | 1.0000 | -0.198 |
| 0.328 | -0.1214 | 0.6719 | -0.1054 | |

TABLE 2 | Data of $X^H_t$ with $\theta = 10$ and $H = 0.7$

| $t$  | $X^H_t$ | $t$  | $X^H_t$ |
|------|--------|------|--------|
| 0.000 | 0.0000 | 0.3438 | -0.0983 | 0.6875 | -0.1109 |
| 0.015 | 0.0067 | 0.3594 | -0.1104 | 0.7031 | -0.1121 |
| 0.031 | 0.0113 | 0.3750 | -0.1108 | 0.7188 | -0.1126 |
| 0.046 | 0.0160 | 0.3906 | -0.1098 | 0.7344 | -0.1034 |
| 0.062 | -0.0179 | 0.4063 | -0.1119 | 0.7500 | -0.0991 |
| 0.078 | -0.0177 | 0.4219 | -0.1106 | 0.7656 | -0.0901 |
| 0.093 | -0.0242 | 0.4375 | -0.1128 | 0.7813 | -0.0890 |
| 0.109 | -0.0319 | 0.4531 | -0.1170 | 0.7969 | -0.0894 |
| 0.125 | -0.0306 | 0.4688 | -0.1185 | 0.8125 | -0.0909 |
| 0.140 | -0.0416 | 0.4844 | -0.1205 | 0.8281 | -0.0857 |
| 0.156 | -0.0523 | 0.5000 | -0.1311 | 0.8438 | -0.0851 |
| 0.171 | -0.0577 | 0.5156 | -0.1068 | 0.8594 | -0.0951 |
| 0.187 | -0.0637 | 0.5313 | -0.1067 | 0.8750 | -0.0909 |
| 0.203 | -0.0690 | 0.5469 | -0.1137 | 0.8906 | -0.0890 |
| 0.218 | -0.0708 | 0.5625 | -0.1105 | 0.8963 | -0.0949 |
| 0.234 | -0.0670 | 0.5781 | -0.1101 | 0.9219 | -0.0976 |
| 0.250 | -0.0630 | 0.5938 | -0.1078 | 0.9375 | -0.1006 |
| 0.265 | -0.0744 | 0.6094 | -0.1078 | 0.9531 | -0.0998 |
| 0.281 | -0.0831 | 0.6250 | -0.1069 | 0.9688 | -0.0941 |
| 0.296 | -0.0865 | 0.6406 | -0.1059 | 0.9844 | -0.0933 |
| 0.312 | -0.0881 | 0.6563 | -0.1085 | 1.0000 | -0.0928 |
| 0.328 | -0.0962 | 0.6719 | -0.1107 | |

with $\theta > 0$. The kernel $(t, s) \rightarrow h_0(t, s)$ is defined as follows:

$$h_0(t, s) = \begin{cases} 1 - \theta \text{sech}^H t & t \geq s, \\ 0 & \text{otherwise} \end{cases}$$

for $s, t \geq 0$. By the variation of constants method (see, Cranston and Le Jan [8]) or Itô’s formula, we may introduce the following representation:

$$X^H_t = \int_0^t h_0(t, s)ds^H_t + \nu \int_0^t h_0(t, s)ds$$

for $t \geq 0$.

The kernel function $(t, s) \rightarrow h_0(t, s)$ with $\theta > 0$ admits the following properties (these properties are proved partly in Cranston and Le Jan [8]):

- For all $s \geq 0$, the limit
Let $\frac{1}{2} < H < 1$ and $\theta > 0$. Then the random variable

$$X^H = \int_0^\infty h_0(s) dS^H + v \int_0^\infty h_0(s) ds$$

exists as an element in $L^2$.

**Proof.** This is a simple calculus exercise. In fact, we have

$$h_0(s) = \lim_{t \to 0} h_0(t, s) = 1 - \theta e^{\frac{1}{2} \theta t} \int_s^\infty e^{-\frac{1}{2} \theta t} du$$

exists.

- For all $t \geq s \geq 0$, we have $h_0(s) \leq h_0(t, s)$, and

$$0 \leq h_0(s) \leq C_0 \min\left\{1, \frac{1}{s^2}\right\}, \quad e^{-\frac{1}{2} \theta (s^2 - t^2)} \leq h_0(t, s) \leq 1;$$

- For all $t \geq s$, $r \geq 0$ and $\theta \neq 0$, we have

$$h_0(t, 0) = h_0(t), t = 1, \quad \int_s^t h_0(t, u) du = e^{\frac{1}{2} \theta t} \int_s^t e^{-\frac{1}{2} \theta u} du$$

and

$$|h_0(t, s) - h_0(s)| = |h_0(t, r) - h_0(r)| \leq \frac{1}{t^2} \text{sec} \left(\frac{1}{2} \theta (s^2 - t^2)\right) e^{-\theta t};$$

- For all $t > 0$, we have

$$\left|\int_0^t \left[ h_0(t, s) - h_0(s) \right] ds \right| \leq \frac{1}{\theta t}$$

**Lemma 3.1.** Let $\frac{1}{2} < H < 1$ and $\theta > 0$. Then the random variable

$$X^H = \int_0^\infty h_0(s) dS^H + v \int_0^\infty h_0(s) ds$$

exists as an element in $L^2$.

**Proof.** This is a simple calculus exercise. In fact, we have

$$\int_0^\infty h_0(s) ds dS^H = \int_0^\infty h_0(s) ds dS^H + v \int_0^\infty h_0(s) ds$$

for all $\theta > 0$ and $\frac{1}{2} < H < 1$. Clearly, Eq. 10 implies that

$$\int_0^1 h_0(s) h_0(r) \left( (s - r)^{2H - 2} - (r + s)^{2H - 2} \right) dr ds,$$
\[
\int_{1}^{\infty} \int_{0}^{\infty} h_0(s)h_0(r)(s - r)^{2H-2}drds \\
\leq (C_0)^2 \int_{1}^{\infty} \int_{1}^{\infty} (rs)^{-\theta}((s-r)^{2H-2}-(r+s)^{2H-2})drds \\
\leq (C_0)^2 \int_{1}^{\infty} \int_{1}^{\infty} (rs)^{-\theta}((s-r)^{2H-2}-(r+s)^{2H-2})drds \\
= (C_0)^2 \int_{1}^{\infty} \int_{1}^{\infty} r^{2H-2}x^{-\theta}(x-1)^{2H-2}-(1+x)^{2H-2})dxdr < \infty
\]
for all \( \theta > 0 \) and \( \frac{1}{2} < H < 1 \). These show that the random variable \( X_H^t \) exists as an element in \( L^2 \).

**Lemma 3.2.** Let \( \theta > 0 \). We then have
\[
\lim_{t \to \infty} t e^{t\theta} \left( \int_{0}^{t} h_0(t,s)ds - \int_{0}^{\infty} h_0(s)ds \right) = -\frac{1}{\theta}
\]

**Proof.** This is a simple calculus exercise. In fact, we have
\[
\int_{0}^{t} h_0(t,s)ds - \int_{0}^{\infty} h_0(s)ds = \int_{0}^{t} [h_0(t,s) - h_0(s)]ds - \int_{0}^{\infty} h_0(s)ds \\
= t \theta e^{t\theta} \left( \int_{1}^{\infty} e^{-\theta x^2}dx - \int_{1}^{\infty} e^{-\theta x^2}dx \right)ds - \int_{0}^{\infty} h_0(s)ds \\
= (e^{t\theta} - 1) \int_{0}^{\infty} e^{-\theta x^2}dx - \int_{0}^{\infty} h_0(s)ds.
\]
for all \( t \geq 0 \) and \( \theta > 0 \). Noting that
\[
\lim_{t \to \infty} t \left( e^{t\theta} - 1 \right) \int_{0}^{\infty} e^{-\theta x^2}dx = \lim_{t \to \infty} \frac{1}{t} \int_{0}^{\infty} e^{t\theta}dx = \frac{1}{\theta}
\]
and
\[
\lim_{t \to \infty} t \int_{0}^{\infty} h_0(s)ds = \lim_{t \to \infty} \frac{1}{t} t \int_{0}^{\infty} h_0(s)ds \\
= \lim_{t \to \infty} t h_0(t) = \lim_{t \to \infty} t \left( 1 - \theta e^{t\theta} \right) \int_{0}^{\infty} e^{-\theta x^2}dx = \frac{1}{\theta}
\]
we see that
\[
\lim_{t \to \infty} t e^{t\theta} \left( \int_{0}^{t} h_0(t,s)ds - \int_{0}^{\infty} h_0(s)ds \right) \\
= \lim_{t \to \infty} \frac{1}{t} e^{t\theta} \left( e^{t\theta} - 1 \right) \int_{0}^{\infty} e^{-\theta x^2}dx - \int_{0}^{\infty} h_0(s)ds = -\frac{1}{\theta}.
\]
by L’Hôpital’s rule.

**Lemma 3.3.** Let \( \theta > 0 \). We then have
\[
\left| \frac{d}{dt} h_0(t) \right| \leq C_0 \min \left\{ 1, \frac{1}{t^2} \right\}
\]
for all \( t \geq 0 \).

**Lemma 3.4.** Let \( \theta > 0 \) and \( \frac{1}{2} < H < 1 \). We then have
\[
\lim_{t \to \infty} \frac{1}{t^{2H-2}} e^{-\frac{t}{2H}} t \int_{0}^{t} s e^{\frac{t}{2H}} \psi_H(s,r)dsdr = \frac{1}{4} \theta^{2H-1}(2H+1).
\]

**Proof.** By L’Hôpital’s rule and the change of variable \( \frac{1}{2} \theta (t^2 - r^2) = x \), it follows that
\[
\lim_{t \to \infty} \frac{1}{t^{2H-2}} e^{-\frac{t}{2H}} t \int_{0}^{t} s e^{\frac{t}{2H}} \psi_H(s,r)dsdr \\
= \lim_{t \to \infty} \frac{1}{2\theta t^{2H-2}} \int_{0}^{t} e^{\frac{t}{2H}} \psi_H(t,r)dr \\
= \lim_{t \to \infty} \frac{H(2H-1)}{2\theta t^{2H-2}} \int_{0}^{t} e^{\frac{t}{2H}} \left( (t-r)^{2H-2} - (t+r)^{2H-2} \right)dr \\
= \lim_{t \to \infty} \frac{H(2H-1)}{2\theta t^{2H-2}} \int_{0}^{t} e^{-\frac{t}{2H}} \left( t - \sqrt{t^2 - \frac{2x}{\theta}} \right)^{2H-2}dx \\
= \frac{1}{2} \theta^{2H} H(2H-1) \Gamma(2H-1) = \frac{1}{4} \theta^{2H-1}(2H+1),
\]
where we have used the equation
\[
\lim_{t \to \infty} \frac{1}{t^{2H-2}} e^{-\frac{t}{2H}} t \int_{0}^{t} e^{\frac{t}{2H}} (t+r)^{2H-2}dr = 0.
\]
This completes the proof.

**Lemma 3.5.** Let \( \theta > 0 \) and \( \frac{1}{2} < H < 1 \). We then have
\[
c(t-s)^{2H} \leq E \left[ (X_H^t - X_H^s)^2 \right] \leq C(t-s)^{2H}
\]
for all \( 0 \leq s < t \leq T \), where \( C \) and \( c \) are two positive constants depending only on \( H, \theta, \upsilon \) and \( T \).

**Proof.** The lemma is similar to Lemma 3.5 in the previous study (I).
On the other hand, by Lemma (3.5), 3.3 and the equation \( \sqrt{t_t} \to 0 \) almost surely as \( t \) tends to infinity, we find that
\[
\int_t^\infty S_d^t dh_\theta(s) \leq C_\theta \int_t^\infty |S_d^t| \frac{ds}{s^2} \to 0,
\]
as \( t \) tends to infinity. It follows from the integration by parts that
\[
\int_t^\infty h_\theta(s) dS^t_d = -h_\theta(t) S_d^t - \int_t^\infty S_d^t dh_\theta(s) \to 0
\]
almost surely as \( t \) tends to infinity.

4 SOME LARGE TIME BEHAVIORS

In this section, we consider the long time behaviors for \( X^t \) with \( \frac{1}{2} < H < 1 \) and \( \theta > 0 \) and our objects are to prove the statements given in Section 1.

Theorem 4.1. Let \( \theta > 0 \) and \( \frac{1}{2} \leq H < 1 \). Then the convergence
\[
X^{t_s} \to X^t_{\infty}
\]
holds in \( L^2 \) and almost surely as \( t \) tends to infinity.

Proof. When \( H = \frac{1}{2} \), the convergence is obtained in Cranston-Le Jan [8]. Consider the decomposition
\[
X^t - X^t_{\infty} = \int_0^t [h_\theta(t, s) - h_\theta(s)] dS^t_d + \int_t^\infty h_\theta(s) dS^t_d + \sum_{n=1}^\infty \sigma^2 n \int_{s}^{t} (h_\theta(t, s) - h_\theta(s)) dS^t_d \equiv Y^t_t + \sum_{n=1}^\infty \sigma^2 s \int_{s}^{t} h_\theta(s) dS^t_d + \sum_{n=1}^\infty \sigma^2 n \int_{s}^{t} h_\theta(s) dS^t_d
\]
for all \( t \geq 0 \).

We first check that Eq. 19 holds in \( L^2 \). By Lemma 3.6 and Lemma 3.2, we only need to prove \( Y^t_t \) converges to zero in \( L^2 \). It follows from the equation
\[
\int_t^\infty e^{-\frac{1}{2} \theta t} d\tau = \frac{1}{\theta} e^{-\frac{1}{2} \theta t}
\]
for all \( \theta > 0 \) as \( t \) tends to infinity and Lemma 3.4 that
\[
E[Y^t_t] = \int_0^t \left| |h_\theta(t, s) - h_\theta(s)| h_\theta(t, r) - h_\theta(r) | \psi_{H}(s, r) d\sigma dr + \sum_{n=1}^\infty \sigma^2 n \int_{s}^{t} (h_\theta(t, s) - h_\theta(s)) dS^t_d + \sum_{n=1}^\infty \sigma^2 n \int_{s}^{t} h_\theta(s) dS^t_d
\]
for all \( \theta > 0 \) as \( t \) tends to infinity and Lemma 3.4 that
\[
E[Y^t_t] = \int_0^t \left| |h_\theta(t, s) - h_\theta(s)| h_\theta(t, r) - h_\theta(r) | \psi_{H}(s, r) d\sigma dr + \sum_{n=1}^\infty \sigma^2 n \int_{s}^{t} (h_\theta(t, s) - h_\theta(s)) dS^t_d + \sum_{n=1}^\infty \sigma^2 n \int_{s}^{t} h_\theta(s) dS^t_d
\]
for all \( \theta > 0 \) and \( \frac{1}{2} < H < 1 \) as \( t \) tends to infinity, which implies that Eq. 19 holds in \( L^2 \).

We now check that Eq. 19 holds almost surely as \( t \) tends to infinity. By Lemma 3.6, we only need check that \( Y^t_t \) converges to zero almost surely as \( t \) tends to infinity. We have
\[
Y^t_t = \int_0^t [h_\theta(t, s) - h_\theta(s)] dS^t_d
\]
\[
= \left( \int_t^\infty e^{-\frac{1}{2} \theta t} d\tau \right) \int_0^t \theta e^{\frac{1}{2} \theta t} dS^t_d = \frac{1}{t} e^{-\frac{1}{2} \theta t} \int_0^t \theta e^{\frac{1}{2} \theta t} dS^t_d
\]
for all \( \theta > 0 \) and \( \frac{1}{2} < H < 1 \) as \( t \) tends to infinity. To obtain the convergence, we define the random sequence
\[
Z_{n,k} = Y^t_t, \quad k = 0, 1, 2, \ldots, n
\]
for every integer \( n \geq 1 \). Then \( \{Z_{n,k}, k = 0, 1, 2, \ldots, n\} \) is Gaussian for every integer \( n \geq 1 \). It follows from Lemma 3.4 that
\[
\sigma^2 (n) := E\left[ (Z_{n,k})^2 \right] \sim \frac{1}{(n + k)^2} e^{-\theta (n + k)^2} E\left[ \int_0^{n+1} \sigma^2 s \int_{s}^{t} h_\theta(s) dS^t_d \right]^2
\]
\[
\leq \frac{1}{(n + k)^2} e^{-\theta (n + k)^2} \int_0^{n+1} \int_0^{n+1} \sigma^2 \int_{s}^{t} h_\theta(s) dS^t_d \int_{s}^{t} h_\theta(r) dS^t_d
\]
for every integer \( n \geq 1 \) and \( 0 \leq k \leq n \), which implies that
\[
P\left( \left| Z_{n,k} \right| > \epsilon \right) = \int_0^\infty \frac{1}{\sqrt{2\pi \sigma(n)}} \frac{e^{-\frac{1}{2} \sigma(n)^2 x^2}}{\sqrt{2\pi \sigma(n)}} dx \leq \frac{1}{\epsilon} \int_0^\infty x e^{-\frac{1}{2} \sigma(n)^2 x^2} dx
\]
\[
= \frac{\sigma(n)}{\epsilon} \int_{\epsilon \sigma(n)}^{\infty} \frac{e^{-\frac{1}{2} y^2}}{\sqrt{2\pi \sigma(n)^2}} dy \leq \frac{\sigma(n)}{\epsilon} \int_{\epsilon \sigma(n)}^{\infty} \frac{e^{-\frac{1}{2} y^2}}{\sqrt{2\pi \sigma(n)^2}} dy
\]
\[
\leq \frac{\sigma(n)}{\epsilon} \int_{\epsilon \sigma(n)}^{\infty} \frac{e^{-\frac{1}{2} y^2}}{\sqrt{2\pi \sigma(n)^2}} dy
\]
for any \( \epsilon > 0 \), every integer \( n \geq 1 \) and \( 0 \leq k \leq n \). On the other hand, for every \( s \in (0, 1) \), we denote
\[
R_s = Y^t_t - Y^t_{t-1}
\]
Then \( \{R_s, 0 \leq s \leq 1\} \) is also Gaussian for every integer \( n \geq 1 \) and \( 0 \leq k \leq n \). It follows that
\[
E\left[ (R_s^t - R_s^{t'})^2 \right] \leq \frac{C}{n^{2H}} E\left[ (S^t - S^{t'})^2 \right]
\]
for all \( s \), \( s' \in [0, 1] \). Thus, for any \( \epsilon > 0 \), by Slepian’s theorem and Markov’s inequality, one can get
\[
P\left( \sup_{s \in [0, 1]} \left| R_s^t \right| > \epsilon \right) \leq P\left( C \frac{1}{n^{2H}} E\left[ (S^t - S^{t'})^2 \right] \right)
\]
for every integer \( n \geq 1 \) and \( 0 \leq k \leq n \). Combining this with the Borel–Cantelli lemma and the relationship
\[
\left\{ \sup_{s \in [0, 1]} \left| Y^t_t \right| > \epsilon \right\} \subset \left\{ \left| Z_{n,k} \right| > \epsilon/2 \right\} \cup \left\{ \sup_{s \in [0, 1]} \left| R_s^t \right| > \epsilon/2 \right\}
\]
we show that $Y_t^H \to 0$ almost surely as $t$ tends to infinity. This completes the proof.

**Theorem 4.2.** Let $\theta > 0$ and $\frac{1}{2} \leq H < 1$. Then the convergence

$$t^H (X_t^H - X_0^H) \rightarrow \mathcal{N}(0, \lambda_{H, \theta})$$  \hspace{1cm} (21)

holds in distribution, where $\mathcal{N}$ is a central normal random variable with its variance

$$\lambda_{H, \theta} = \frac{1}{2} \Gamma (2H + 1) \theta^{-2H}.$$  

**Proof.** When $H = \frac{1}{2}$, this result also is unknown. We only consider the case $\frac{1}{2} < H < 1$ and similarly one can prove the convergence for $H = \frac{1}{2}$ by Eq. 20, Slutsky’s theorem, and Lemma 3.2, we only need to show that

$$t^H \int_0^\infty h_0(s)dS_t^H \to 0 \quad (t \to \infty)$$  \hspace{1cm} (22)

in probability and

$$t^H Y_t^H \rightarrow \mathcal{N}(0, \lambda_{H, \theta}) \quad (t \to \infty).$$  \hspace{1cm} (23)

in distribution.

First, Eq. 22 follows from Eq. 10 and

$$t^H E \left[ \int_0^\infty h_0(s)dS_t^H \right]^2 = t^{2H} \int_0^\infty \int_0^\infty h_0(s)h_0(r)\psi_H(s, r)ds dr$$

$$\leq \frac{4t^{2H}}{\theta^2} \int_0^\infty \int_0^\infty \frac{1}{(sr)^2} \psi_H(s, r)ds dr$$

$$= \frac{4t^{2H-4}}{\theta^2} \int_0^\infty \int_0^\infty \frac{1}{(xy)^2} \psi_H(x, y)dxdy \to 0$$

for all $\theta > 0$ and $\frac{1}{2} < H < 1$ as $t$ tends to infinity.

We now obtain convergence (23). By the equation

$$\int_0^\infty e^{+i\omega u} du \sim \frac{1}{\theta^2} e^{+i\omega u^2},$$

as $t$ tends to infinity and Lemma 3.4, we get

$$t^{2H} E [Y_t^H]^2 = t^{2H} \int_0^\infty \int_0^\infty \psi_H(s, r)ds dr$$

$$= t^{2H} \left( \int_0^\infty e^{+i\omega u} du \right)^2 \int_0^\infty \int_0^\infty sre^{+i\omega s^2} \psi_H(s, r)ds dr$$

$$\sim \frac{2}{t^{2H}} e^{+i\omega t} \int_0^t \int_0^\infty \frac{1}{r} e^{+i\omega s^2} \psi_H(s, r)ds dr \to \frac{1}{2} \Gamma (2H + 1) \theta^{-2H}$$

for all $\theta > 0$ and $\frac{1}{2} < H < 1$ as $t$ tends to infinity. Thus, convergence (23) follows from the normality of $t^H Y_t^H$ for all $\frac{1}{2} < H < 1$ and the theorem follows.

At the end of this section, we obtain a law of large numbers. Consider the process $Y_t^H$ defined by

$$Y_t^H = \int_0^t (X_s^H - X_0^H)ds, \quad t \geq 0.$$

Then the self-attracting diffusion $X^H$ satisfies

$$X_t^H = S_t^H - \theta \int_0^t Y_s^H ds + v_t, \quad t \geq 0$$  \hspace{1cm} (24)

and

$$Y_t^H = tX_t^H - \int_0^t X_s^H ds = \int_0^t s dX_s^H$$

by integration by parts. It follows that

$$dY_t^H = -\theta Y_t^H dt + \frac{1}{2} \theta Y_t^H dt + v_t dt$$  \hspace{1cm} (25)

for all $\frac{1}{2} \leq H < 1$ and $t \geq 0$. By the variation of constant method, we can give the explicit representation of $Y_t^H$ as follows:

$$Y_t^H = e^{-\frac{1}{2} \int_0^t \theta s^2 ds} Y_{0}^H + \frac{\theta}{2} (1 - e^{-\frac{1}{2} \theta t^2}), \quad t \geq 0.$$  \hspace{1cm} (26)

**Lemma 4.1.** Let $\frac{1}{2} \leq H < 1$ and $\theta > 0$. Then we have

$$\frac{1}{T} \int_0^T Y_t^H dt \to \frac{\theta}{\theta^2}$$  \hspace{1cm} (27)

almost surely and in $L^2$ as $T$ tends to infinity.

**Proof.** This lemma follows from Eq. 24 and the estimates

$$E \left( \left( \frac{1}{T} \int_0^T Y_t^H dt - \frac{\theta}{\theta^2} \right)^2 \right) = \frac{1}{T^2} E \left( \left( \frac{Y_T^H}{T} - \frac{\theta}{\theta^2} \right)^2 \right)$$

$$\leq \frac{2}{\theta^2} \left( \frac{E (Y_T^H)^2}{T^2} + \frac{E (Y_T^H)^2}{T^2} \right) \to 0,$$

as $T$ tends to infinity.

**Theorem 4.3.** Let $\frac{1}{2} \leq H < 1$ and $\theta > 0$. Then we have

$$\frac{1}{T^{2H}} \int_0^T (Y_t^H)^2 dt \to \frac{H}{3 - 2H} \theta^{-2H} \Gamma (2H)$$  \hspace{1cm} (28)

in $L^2$ as $T$ tends to infinity.

**Proof.** Given $\frac{1}{2} < H < 1$ and $\theta > 0$,

$$\Delta_t = \frac{\theta}{\theta^2} (1 - e^{-\frac{1}{2} \theta t^2}), \quad \eta_t^H = e^{-\frac{1}{2} \theta t^2} \int_0^t s e^{\frac{1}{2} \theta s^2} ds$$

for all $t \geq 0$. Then

$$Y_t^H = \eta_{t\Delta_t} + \Delta_t$$

for all $t \geq 0$. We now prove the lemma in three steps.

**Step 1.** We claim that

$$\lim_{t \to \infty} \frac{1}{T^{3-2H}} \int_0^T E \left( (Y_t^H)^2 \right) dt \to \frac{H}{3 - 2H} \theta^{-2H} \Gamma (2H),$$  \hspace{1cm} (29)

as $t$ tends to infinity. Clearly, we have

$$\lim_{t \to \infty} \frac{1}{T^{3-2H}} \int_0^T \Delta_t^2 dt = 0.$$

Thus, 29 is equivalent to
\[
\frac{1}{T^{3-2H}} \int_0^T E \left[ (\eta^H_t)^2 \right] dt \to \frac{H}{3-2H} \theta^{2H}(2H). \tag{30}
\]

By L'Hôpital's rule and Lemma 3.4, it follows that
\[
\lim_{T \to \infty} \frac{1}{T^{3-2H}} \int_0^T E \left[ (\eta^H_t)^2 \right] dt = \lim_{T \to \infty} \frac{1}{T^{3-2H}} \int_0^T \psi_H(u, v) u dv du
\]
\[
= \lim_{T \to \infty} e^{-2t^{2H}} \int_0^T \psi_H(u, v) u dv du
\]
\[
= \frac{1}{2} (3 - 2H) \theta^{2H}(2H + 1) = \frac{H}{3-2H} \theta^{2H}(2H)
\]
for all \( \frac{1}{2} < H < 1 \).

**Step II.** We claim that
\[
\frac{1}{T^{3-2H}} E \left( \int_0^T \Delta \eta^H_t dt \right)^2 = \frac{1}{T^{3-2H}} \int_0^T \Delta \Delta \eta^H_t ds dt \to 0,
\]
as \( T \) tends to infinity. We have that
\[
E(\eta^H_t \eta^H_s) = e^{-2t^{2H}} \int_0^T \eta^H_s dS_t \int_0^T \eta^H_s dS_t
\]
\[
eq e^{-2t^{2H}} \int_0^T \psi_H(u, v) u dv du
\]
\[
= \frac{H}{2} (2H - 1) e^{-2t^{2H}} \int_0^T \psi_H(u, v) u dv du
\]
\[
+ \frac{H}{2} (2H - 1) e^{-2t^{2H}} \int_0^T \psi_H(u, v) u dv du
\]
\[
= \frac{H}{2} (2H - 1) [\Lambda_t(H; t, s) + \Lambda_t(H; t, s)]
\]
for all \( t > s > 0 \). An elementary calculation may show that
\[
\Lambda_t(H; t, s) \leq e^{-2t^{2H}} \int_0^t u (u - s)^{2H-2} e^{t^{2H}} \left( \int_0^t u e^{-2t^{2H}} dS \right) du
\]
\[
\leq \frac{1}{\theta} e^{-2t^{2H}} (e^{t^{2H}} - 1) \int_0^t u (u - s)^{2H-2} e^{t^{2H}} du
\]
\[
\leq \frac{1}{\theta} e^{-2t^{2H}} (1 - e^{-2t^{2H}}) \int_0^t u (u - s)^{2H-2} e^{t^{2H}} du
\]
\[
\leq \frac{1}{2\theta} e^{-2t^{2H}} \int_0^{t^{2H}} \left( \frac{t^{2H}}{t^{2H}} + x - s \right)^{2H-1} e^{t^{2H}} dx
\]
\[
\leq \frac{1}{2\theta} e^{-2t^{2H}} \int_0^{t^{2H}} x^{2H-2} \left( t^{2H} + x + s \right)^{2H-1} e^{t^{2H}} dx
\]
\[
\leq \frac{1}{2\theta} (t + s)^{2-2H} e^{-2t^{2H}} \int_0^{t^{2H}} x^{2H-2} dx
\]
for all \( t > s > 0 \). It follows from the equation
\[
\int_0^T t^{2H} dS_t \sim \chi^H(1 + \alpha) e^\alpha + \chi^H(1 + \beta) e^\beta + \chi^H(1 + \gamma) e^\gamma
\]
with \( x \geq 0 \) and \( \beta > -1 \) that
\[
\Lambda_t(H; t, s) \leq C (t - s)^{2H-2} (1 \wedge (t^2 - s^2))
\]
\[
\leq C (t - s)^{2H-2} (1 \wedge (t^2 - s^2)) \tag{33}
\]
for all \( t > s > 0 \) and \( 0 \leq \alpha \leq 1 \). For the term \( \Lambda_t(H; t, s) \), by the proof of Lemma 3.4, we find that
\[
\lim_{T \to \infty} \frac{1}{T^{3-2H}} \int_0^T \int_0^T \psi_H(u, v) u dv du
\]
\[
= \frac{1}{\theta} \theta^{2H}(2H + 1)
\]
for all \( \frac{1}{2} < H < 1 \). Combining this with the equation
\[
\int_0^T \int_0^T \psi_H(u, v) u dv du = C \in (0, \infty)
\]
and the equation \( e^{-2t^{2H}} \leq \frac{1}{\theta} \leq \frac{1}{\theta} \) with \( x > 0 \) and \( 0 < q < 1 \), we get
\[
\Lambda_t(H; t, s) = 2 \psi_H(u, v) u dv du = C \in (0, \infty)
\]
\[
\leq C e^{-2t^{2H}} \left( \frac{t^2 - s^2}{2H(2H - 1)} \right)^{2H-2}
\]
\[
\leq \frac{C}{(t^2 - s^2)^{2H-2}}
\]
for all \( t > s > 0 \), \( \frac{1}{2} < H < 1 \) and \( 0 \leq y \leq 2 - 2H \). Thus, we have shown that the estimate
\[
E(\eta^H_t \eta^H_s) \leq C \psi_H(t - s)^{2H-2} (1 \wedge (t^2 - s^2))
\]
\[
\leq (t^2 - s^2)^{2H-2}
\]
holds for all \( t > s > 0 \). In particular, we have
\[
E(\eta^H_t \eta^H_s) \leq C \psi_H(t - s)^{2H-2}
\]
for all \( t, s \geq 0 \). As a corollary, we get
\[
\frac{1}{T^{3-2H}} E \left( \left( \int_0^T \psi_H^2 dS_t \right)^2 \right) \to \frac{H}{3-2H} \theta^{2H}(2H)
\]
\[
\to \frac{C \psi_H}{T^{3-2H}} \to 0,
\]
as \( T \) tends to infinity.

**Step III.** We claim that
\[
\frac{1}{T^{3-2H}} E \left[ \left( \int_0^T \psi_H^2 dS_t \right)^2 \right] \to \frac{H}{3-2H} \theta^{2H}(2H)
\]
as \( t \) tends to infinity. By steps I and II, we find that Eq. 37 is equivalent to
\[
\frac{1}{T^{3-2H}} E \left[ \left( \int_0^T \psi_H^2 dS_t \right)^2 \right] \to \frac{H}{3-2H} \theta^{2H}(2H)
\]
as \( t \) tends to infinity. Noting that the equation
\[
E(\eta^H_t \eta^H_s) = E(\eta^H_t \eta^H_s) + 2 E(\eta^H_t \eta^H_s)
\]
(39)
for all $t, s > 0$, we further find that convergence (38) also is equivalent to

$$
\Lambda(H; T) := \frac{1}{T^{6-4H}} E\left( \int_0^T \left( \eta_t^H \right)^2 - E(\eta_t^H)^2 \right)^2 dt
$$

$$
= \frac{2}{T^{6-4H}} \int_0^T \int_0^t \left( \eta_s^H \eta_t^H \right)^2 dsdt \to 0,
$$

(40)
as $T$ tends to infinity. We now check that convergence (40) in two cases.

**Case 1.** Let $\frac{1}{4} < H < 1$. Clearly, by Eq. 36, we have to

$$
\Lambda(H; T) \leq C_{\delta H} \int_0^T \int_0^1 (t-s)^{4H-4} dsdt
$$

$$
\leq C_{\delta H} T^{4H-2} \to 0 \quad (T \to \infty).
$$

(41)

**Case 2.** Let $\frac{1}{2} < H \leq \frac{1}{3}$. By Eq. 36, we have that

$$
\int_1^T \int_0^{\sqrt{t-1}} \left( E(\eta_t^H \eta_s^H) \right)^2 dsdt \leq C_{\delta H} \int_1^T \int_0^{\sqrt{t-1}} (t-s)^{4H-4} dsdt
$$

$$
\leq C_{\delta H} T^{4H-2}
$$

with $\frac{1}{2} < H < \frac{1}{3}$ and

$$
\int_1^T \int_0^{\sqrt{t-1}} \left( E(\eta_t^H \eta_s^H) \right)^2 dsdt \leq \int_1^T \int_0^{\sqrt{t-1}} \frac{1}{t-s} dsdt \leq CT \log T
$$

with $H = \frac{1}{3}$ for all $T > 1$. Similarly, by Eq. 35, we also have

$$
\int_1^T \int_0^{\sqrt{t-1}} \left( E(\eta_t^H \eta_s^H) \right)^2 dsdt
$$

$$
\leq C_{\delta H} \int_1^T \int_0^{\sqrt{t-1}} (t-s)^{4H-4+2a} dsdt
$$

$$
\leq C_{\delta H} \int_1^T \int_0^{\sqrt{t-1}} t^{2a} (t-s)^{4H-4+2a} dsdt
$$

$$
= C_{\delta H} \int_1^T \int_0^{\sqrt{t-1}} t^{2a} (t-\sqrt{t^2-1})^{4H-3+2a} dt
$$

$$
= C_{\delta H} \int_1^T \int_0^{\sqrt{t-1}} t^{2a} (t+\sqrt{t^2-1})^{4H-3+2a} dt \leq CT^{4-4H}
$$

for all $T > 1$ and $\frac{1}{2} - 2H < \alpha = \gamma < 2 - 2H$ since $0 < t^2 - s^2 < 1$ for $(s, t) \in \{ (s, t) | 1 \leq t \leq T, t^2 - 1 < s < t \}$. Thus, we have shown that

$$
\Lambda(H; T) = \frac{1}{T^{6-4H}} \int_1^T \int_0^{\sqrt{t-1}} \left( E(\eta_t^H \eta_s^H) \right)^2 dsdt
$$

$$
+ \frac{1}{T^{6-4H}} \int_1^T \int_0^{\sqrt{t-1}} \left( E(\eta_t^H \eta_s^H) \right)^2 dsdt + \frac{1}{T^{6-4H}} \int_0^1 \int_0^{\sqrt{t-1}} \left( E(\eta_t^H \eta_s^H) \right)^2 dsdt
$$

$$
\leq C_{\delta H} \frac{T^{4H-2} + T^{4-4H} + 1}{T^2} \to 0
$$

(42)

with $\frac{1}{4} < H < \frac{1}{2}$ and

$$
\Lambda^2(H; T) \leq C_{\delta H} \frac{T^{4H-2} + T^{4-4H} + 1}{T^2} \to 0, \quad \text{as } T \to \infty.
$$

(43)
as $T$ tends to infinity. This shows that convergence (40) holds for all $\frac{1}{4} < H < 1$. Similarly, we can also show the theorem holds for $H = \frac{1}{2}$ and the theorem follows.

**Remark 1.** By using the Borel–Cantelli lemma and Theorem 4.3, we can check that convergence (28) holds almost surely.

## 5 SIMULATION

We have applied our results to the following linear self-attracting diffusion driven by a sub-fBm $\xi_t^H$ with $\frac{1}{2} < H < 1$ as follows:

$$
d\chi_t^H = dS_t^H - \theta \left( \int_0^t (\chi_s^H - \chi_t^H) ds \right) dt + \nu dt, \quad \chi_0^H = 0,
$$

where $\theta > 0$ and $\nu \in \mathbb{R}$ are two parameters. We will simulate the process with $\nu = 0$ in the following cases:

- $H = 0.7: \theta = 1, \theta = 10$ and $\theta = 100$, respectively (see, Figures 1–3, Tables 1–3);
- $H = 0.5: \theta = 1, \theta = 10$ and $\theta = 100$, respectively (see, Figures 4–6, Tables 4–6).

**Remark 2.** From the following numerical results, we can find that it is important to study the estimates of parameters $\theta$ and $\nu$.

## DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article-supplementary material; further inquiries can be directed to the corresponding authors.

## AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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