POSITIVE DEFINITE KERNELS AND LATTICE PATHS

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ABSTRACT. We discuss the structure of positive definite kernels in terms of operator models. In particular, we introduce two models, one of Hessenberg type and another one that we call near tridiagonal. These models produce parametrizations of the kernels and we describe the combinatorial nature of these parametrizations in terms of lattice paths of Dyck and Lukasiewicz type.

1. Introduction

In this paper we are interested in parametrizations and combinatorial descriptions of positive definite kernels on the set $\mathbb{N}_0$ of non-negative integers. Positive definite kernels are complex valued maps $K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ with the property that for each $n > 0$ and each choice of elements $p_1, \ldots, p_n$ in $\mathbb{N}_0$ and complex numbers $\lambda_1, \ldots, \lambda_n$, we have

$$\sum_{k,j=1}^{n} K(p_k, p_j) \lambda_j \lambda_k \geq 0. \tag{1.1}$$

A fundamental result of Kolmogorov, [2] provides a Hilbert space interpretation of positive definite kernels as Gram kernels, that is, there exists a Hilbert space $\mathcal{H}$ and elements $v(n) \in \mathcal{H}, n \geq 0$, such that

$$K(i, j) = \langle v(j), v(i) \rangle. \tag{1.2}$$

Two of the best known examples of positive definite kernels are those of Toeplitz type, for which $K(i + l, j + l) = K(i, j), i, j, l \in \mathbb{N}_0$, and those of Hankel type, for which $K(i, j + l) = K(i + l, j), i, j, l \in \mathbb{N}_0$. In both these cases the representation (1.2) can be improved by some more specialized descriptions that might be called operator models of the kernels. Thus, if $K$ is Toeplitz (for simplicity we assume $K(0,0) = 1$ and all the inequalities in (1.1) are strict) then there exists an isometric operator $W_0$ written in upper Hessenberg form, such that

$$K(i, j) = e_0 W^j e_0^*, \tag{1.3}$$

where $e_0 = [1 \quad 0 \quad \ldots]$. Likewise, if $K$ is Hankel (with same simplifications as above) then there exists a symmetric operator $J$, written in tridiagonal form, such that

$$K(i, j) = e_0 J^{i+j} e_0^*. \tag{1.4}$$

Our goal is to extend both models (1.3) and (1.4) to arbitrary positive definite kernels on $\mathbb{N}_0$ without Toeplitz or Hankel assumptions. These models will produce parametrizations of the kernels and we will give combinatorial descriptions of these parametrizations in terms of lattice paths.
2. Isometric Hessenberg models and Dyck paths

In this section we show that any positive definite kernel on $\mathbb{N}_0$ has a Hessenberg model and then we show how to relate this model to the set of Dyck paths. In order to simplify the notation we consider only positive definite kernels for which all the inequalities in (1.1) are strict, when we say that the kernel is strictly positive definite, and we also assume $K(l,l) = 1$ for all $l \geq 0$. Both these assumptions can be easily removed. In addition, all our considerations can be easily adapted to kernels $K : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathcal{L(\mathcal{E})}$, where $\mathcal{L(\mathcal{E})}$ denotes the set of linear bounded operators on the Hilbert space $\mathcal{E}$.

We now introduce the elements necessary in the presentation of the results. For a complex number $\gamma$ with $|\gamma| \leq 1$ we define its defect by $d_\gamma = (1 - |\gamma|^2)^{1/2}$ and its Julia matrix by

$$J(\gamma) = \begin{bmatrix} \gamma & d_\gamma \\ d_\gamma & -\frac{1}{\gamma} \end{bmatrix}.$$

We note that the Julia matrix is unitary and this construction can be extended to certain families of complex numbers as follows. Let $\Gamma = \{\gamma_{k,j}\}_{0 \leq k < j}$ be a family of complex numbers such that $|\gamma_{k,j}| < 1$ for all $k < j$. For simplicity we will write $d_{k,j}$ instead of $d_{\gamma_{k,j}}$. We can now describe the Hessenberg model. First we define for $k < j$,

$$V_{k,j}(\Gamma) = (J(\gamma_{k,k+1}) \oplus I_{j-k-1})(I_1 \oplus J(\gamma_{k,k+2}) \oplus I_{j-k-2}) \ldots (I_{j-k-1} \oplus J(\gamma_{k,j})), $$

where $I_l$ denotes the $l \times l$ identity matrix. Then we introduce the operators $W_k(\Gamma)$ on the Hilbert space $l^2(\mathbb{N}_0)$ of square-summable sequences by the formula:

$$W_k(\Gamma) = s - \lim_{j \to \infty} (V_{k,j}(\Gamma) \oplus 0), \quad k \geq 0,$$

where $s - \lim$ denotes the strong operator limit. It is easily seen that each $W_k(\Gamma)$ is an isometry with upper Hessenberg matrix with respect to the standard basis of $l^2(\mathbb{N}_0)$, that is, if $(W_k(\Gamma))_{i,j}$ denotes the $(i,j)$ entry of $W_k(\Gamma)$, then $(W_k(\Gamma))_{i,j} = 0$ for $i \geq j + 1$. It is also useful to consider the unitary matrices $U_{k,j}(\Gamma)$ defined recursively by $U_{k,k}(\Gamma) = I_1$ and for $k < j$,

$$U_{k,j}(\Gamma) = V_{k,j}(\Gamma) (U_{k+1,j}(\Gamma) \oplus I_1).$$

We can now describe the existence of an isometric Hessenberg model for any strictly positive definite kernel on $\mathbb{N}_0$.

**Theorem 2.1.** If $K$ is a strictly positive definite kernel on $\mathbb{N}_0$ with $K(l,l) = 1$ for all $l \geq 0$, then there exists a family $\{W_k\}_{k \geq 0}$ of isometric Hessenberg operators such that for $j > i$,

$$K(i,j) = e_0 W_i(\Gamma) W_{i+1}(\Gamma) \ldots W_{j-1}(\Gamma) e_0^*, \quad (2.1)$$

Proof. This is just a reformulation of Theorem 2.3 in [2] (see also [3], Chapter 1). Thus, by Theorem 1.3 in [2], there exists a uniquely determined family $\Gamma = \{\gamma_{k,j}\}_{0 \leq k < j}$ of complex numbers such that $K(i,j) = (U_{i,j}(\Gamma))_{0,0}$ for $i < j$. Then it is easily seen from the definitions that

$$e_0 W_i(\Gamma) W_{i+1}(\Gamma) \ldots W_{j-1}(\Gamma) e_0^* = (U_{i,j}(\Gamma))_{0,0}. \quad \square$$
When $K$ is a Toeplitz kernel, then (2.1) reduces to (1.3) and the parameters $\gamma_{k,j}$ satisfy $\gamma_{i+t,j+l} = \gamma_{i,j}$ for $i < j$, $l \geq 1$. The numbers $\gamma_{n} = \gamma_{0,n}$, $n \geq 1$, are called the Szegö parameters of $K$ (other names, like Schur parameters, reflection coefficients, or Verblunski parameters are currently used in the literature), and they play a central role in the theory of orthogonal polynomials on the unit circle and its many applications, [8] and, for a recent account, [6] (which also contains a detailed discussion of the Hessenberg model in the Toeplitz case).

Next we explain the connection between Hessenberg models and Dyck paths. A Dyck path of length $2k$ is a path in the positive quadrant of the lattice $\mathbb{Z}^2$ which starts at $(0,0)$, ends at $(2k,0)$, and consists of rise steps $\nearrow$ and fall steps $\searrow$ (see Figure 1). For more information on Dyck paths and their combinatorics, see [7].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{dyck_path.png}
\caption{A Dyck path of length 8 and height 3}
\end{figure}

Let $D_k$ be the set of Dyck paths of length $2k$ and let $A_k$ be the set of points $(l,q)$, $q > 0$, with the property that there exists $p \in D_k$ with $(l,q) \in p$. It is seen that

\[ A_k = \{(j+i,j-i) \mid 0 \leq i < j \leq k\}. \]

Also, we notice that if $p \in D_k$ and $x = (l,q) \in p$, then there are only four types of behaviour of $p$ about $x$: (I) a rise step followed by a fall step; (II) a fall step followed by a rise step; (III) two consecutive rise steps; (IV) two consecutive fall steps (see Figure 2).

\begin{figure}[h]
\centering
\begin{tabular}{ccccc}
\(x\) & \(x\) & \(x\) & \(x\) \\
(I) & (II) & (III) & (IV) \\
\end{tabular}
\caption{Behaviour of a Dyck path about a vertex $x \in A_k$}
\end{figure}

Consequently, for each pair $i,j$ with $0 \leq i < j \leq k$ we define the function $a_{i,j} : D_k \to \mathbb{C}$,

\[ a_{i,j}(p) = \begin{cases} 
1 & \text{if } x = (j+i,j-i) \notin p; \\
\gamma_{i,j} & \text{if } x = (j+i,j-i) \in p \text{ and (I) holds;}
\neg\gamma_{i,j} & \text{if } x = (j+i,j-i) \in p \text{ and (II) holds;}
d_{i,j} & \text{if } x = (j+i,j-i) \in p \text{ and either (III) or (IV) holds.}
\end{cases} \]

Let $p$ be a Dyck path in $D_k$ with the property that $(2l,0) \in p$. The restriction of $p$ from $(2l,0)$ to $(2k,0)$ is called a Dyck subpath starting at $(2l,0)$ in $D_k$. The set of all possible Dyck subpaths starting at $(2l,0)$ in $D_k$ is denoted by $D_{k-l}^l$ and there exists a bijection between $D_{k-l}^l$ and $D_{k-l}$. This implies that the number of elements in $D_{k-l}^l$ is given by the Catalan number $C_{k-l} = 1 + \frac{1}{k-l+1} \binom{2(k-l)}{k-l}$; also, $D_{k-l}^0 = D_k$. If

\[ C_{k-l} = 1 + \frac{1}{k-l+1} \binom{2(k-l)}{k-l}. \]
q ∈ ℓ

 then there could be many Dyck paths whose restrictions at (2l, 0) coincide
with q, however if p_1 and p_2 are two such Dyck paths then a_i,j(p_1) = a_i,j(p_2) for
j + i > 2l. We will write a_i,j(q) in order to denote this common value.

We now describe the structure of the strictly positive definite kernels on N_0.

**Theorem 2.2.** The kernel K on N_0 with K(l, l) = 1 for all l is strictly positive
definite if and only if there is a family \{γ_{k,j}\}_{0≤k<j} of complex numbers, |γ_{k,j}| < 1
for all k < j, such that

\[
K(l, m) = \sum_{q \in ℓ^m \cap l \leq i < j \leq m} \prod_{i,j} a_{i,j}(q).
\]

**Proof.** Half of this result was proved in [2], but we give some details here for com-
pleteness. Assume that K is strictly positive definite. By Theorem 1.3 and Theo-
rem 2.3 in [2] there exists a uniquely determined family Γ = \{γ_{k,j}\}_{0≤k<j}
of complex numbers such that K(l, m) = (U_{l,m}(Γ))_{0,0}, the (0, 0) entry of the matrix U_{l,m}(Γ).
It is convenient to visualize this relation by means of a so-called transmission line,
as showed in Figure 3 for K(0, 2) and K(0, 3).

![Figure 3. Transmission line for K(0, 3)](image)

Thus, if 1 is the input at A then at B we read off the expression of K(0, 3) in terms
of the parameters γ_{0,1}, γ_{0,2}, γ_{0,3}, γ_{1,2}, γ_{1,3}, γ_{2,3} and their defects,

\[
K(0, 3) = γ_{0,1}γ_{1,2}γ_{2,3} + γ_{0,1}d_{1,2}γ_{1,3}d_{2,3} + d_{0,1}γ_{0,2}d_{1,2}γ_{2,3}
+ d_{0,1}γ_{0,2}d_{1,2}γ_{2,3}.
\]

Likewise, if the input at C is 1, then the output at B is now the expression of
K(0, 2),

\[
K(0, 2) = γ_{0,1}γ_{1,2} + d_{0,1}γ_{0,2}d_{1,2}
\]

(for more details see [2]). Each path in the transmission line contributes an additive
term in K(l, m). Going from a path in the transmission line to a Dyck path is easy,
each box associated with a Julia matrix corresponds to a point in A_k, see Figure 4.
It is also clear that each additive term in K(l, m) is given by \prod_{l \leq i < j \leq m} a_{i,j}(q) for
some q ∈ ℓ^m. This gives (2.2).

Conversely, given a family Γ = \{γ_{k,j}\}_{0≤k<j} of complex numbers with |γ_{k,j}| < 1
for all k < j, we define

\[
K(l, m) = \sum_{q \in ℓ^m \cap l \leq i < j \leq m} \prod_{4} a_{i,j}(q).
\]
By the first part of the proof, this gives \( K(l, m) = (U_{l,m}(\Gamma))_{0,0} \), and it remains to show that \( K \) is a strictly positive definite kernel on \( \mathbb{N}_0 \). By Theorem 2.1,

\[
K(i, j) = e_0 W_i(\Gamma) W_{i+1}(\Gamma) \cdots W_{j-1}(\Gamma) e_0^*
\]

for \( i < j \). This relation implies that \( K \) is a positive definite kernel. Also, by Proposition 1.7 in [2],

\[
\det [K(l, m)]_{l,m=0}^n = \prod_{0 < i < j \leq n} d_{i,j}^2 > 0,
\]

so that \( K \) is a strictly positive definite kernel on \( \mathbb{N}_0 \). □

**Figure 4.** From a path in a transmission line to a Dyck path

**Remarks**

(a) It is quite simple to remove the two restrictions on \( K \) considered in Theorem 2.2. First, formula (2.2) still provides a one-to-one correspondence between the set of positive definite kernels on \( \mathbb{N}_0 \) with \( K(l, l) = 1 \) for all \( l \) and the set \( S \) of families \( \{\gamma_{k,j}\}_{0 \leq k < j} \) of complex numbers with the properties: \( |\gamma_{k,j}| \leq 1 \) for all \( k < j \); if \( |\gamma_{k,j}| = 1 \) for some pair \( (k, j) \), then \( \gamma_{l,j} = 0 \) for \( l < k \) and \( \gamma_{k,m} = 0 \) for \( m > j \).

(b) If we remove the assumption that \( K(l, l) = 1 \) for all \( l \), then the diagonal elements \( K(l, l) \) of the kernel \( K \) could be considered as parameters and there is a one-to-one correspondence between the positive definite kernels on \( \mathbb{N}_0 \) and the set \( S_+ \) of pairs \( (\{f_l\}_{l \geq 0}, \{\gamma_{k,j}\}_{0 \leq k < j}) \), where \( f_l \geq 0 \) for all \( l \geq 0 \) and \( \{\gamma_{k,j}\}_{0 \leq k < j} \) is an element of \( S \) with the additional property that if \( f_l = 0 \) for some \( l \geq 0 \), then \( \gamma_{k,l} = 0 \) and \( \gamma_{l,m} = 0 \) for \( k < l \) and \( m > l \).

Formula (2.2) has to be replaced in this case with:

\[
K(l, m) = f_1^{1/2} f_m^{1/2} \sum_{q \in D_m} \prod_{l \leq i < j \leq m} a_{i,j}(q).
\]

(c) In case \( K \) is a Toeplitz kernel we noticed already that \( \gamma_{i+l,j+l} = \gamma_{i,j} \) for \( i < j \), \( l \geq 1 \) and we denoted \( \gamma_n = \gamma_{0,n}, n \geq 1 \). We conclude that \( a_{i+l,j+l} = a_{i,j} \) for \( i < j \), \( l \geq 1 \) and formula (2.2) reduces to

\[
K(0, n) = \sum_{p \in D_n} \prod_{0 \leq i < j \leq n} a_{i,j}(p).
\]

We can compare this result with a classical formula of Verblunsky, according to which there exists a polynomial \( V^{(n)} = V^{(n)}(\gamma_1, \ldots, \gamma_{n-1}; d_1, \ldots, d_{n-1}) \) with integer coefficients so that

\[
K(0, n) = \gamma_n \prod_{k=1}^{n-1} d_k^2 + V^{(n)}
\]
We see that the term $\gamma_n \prod_{k=1}^{n-1} d_k^2$ corresponds to the path $p_0$ made of $n$ consecutive rise steps followed by $n$ consecutive fall steps. Consequently, we deduce from (2.4) that

$$V(n) = \sum_{p \in D_n - \{p_0\}} \prod_{0 \leq i < j \leq n} a_{i,j}(p),$$

an explicit formula that explains some of the features of $V(n)$. □

3. Near tridiagonal models

In this section we show that positive definite kernels do not have tridiagonal models. Instead we introduce a near tridiagonal model and then we show how this model is related to the set of Łukasiewicz paths. Again, in order to simplify the notation we consider only strictly positive definite kernels $K$ and we assume $K(0,0) = 1$. We denote by $D \subset l^2(N_0)$ the vector space generated by the standard basis of $l^2(N_0)$ and we call tridiagonal model of $K$ a family $\{J_n\}_{n \geq 0}$ of tridiagonal operators (not necessarily bounded),

$$J_n = \begin{bmatrix}
   b_0(n) & c_1(n) & 0 \\
   a_1(n) & b_1(n) & c_2(n) \\
   0 & a_2(n) & b_2(n) \\
   \vdots & \ddots & \ddots \\
   0 & \cdots & \cdots
\end{bmatrix},$$

such that $J_0 = I$ and

$$K(i, j) = e_0 J_n^* \cdots J_{i}^* J_j \cdots J_1 e_0^*, \quad i, j \geq 0.$$  \hspace{1cm}(3.1)

Also, in analogy with the Hankel case, we ask $a_k(n) > 0$, $k, n \geq 1$. Since each $J_n$ is tridiagonal, $J_n D \subset D$ so (3.1) makes sense. However we have the following result.

Theorem 3.1. There are strictly positive definite kernels with no tridiagonal model.

Proof. We consider a strictly positive definite kernel $K$ with

$$K(0, 1) = K(0, 2) = K(1, 2) = 0, \quad K(0, 3) \neq 0$$

(it is easy to construct such a kernel using, for instance, Theorem 2.4). Let $\{J_n\}_{n \geq 1}$ be a tridiagonal model of $K$, then we deduce that $b_0(1) = c_1(2) = b_1(2) = 0$, which implies

$$K(0, 3) = e_0 J_3 J_2 J_1 e_0^*$$

$$= b_0(3) (b_0(2) b_0(1) + c_1(2) a_1(1)) + c_1(3) (a_1(2) b_0(1) + b_1(2) a_1(1)) = 0,$$

a contradiction showing that $K$ has no tridiagonal model. □

We are now trying to find a model as close as possible of being tridiagonal, which should reduce to (1.4) in case the kernel $K$ is Hankel. Thus, we consider operators (not necessarily bounded) with matrix still of Hessenberg form

$$J_n = \begin{bmatrix}
   b_0(1) & c_{0,1}(n) & c_{0,2}(n) \\
   a_1(1) & b_1(2) & c_{1,2}(n) \\
   0 & a_2(2) & b_2(3) \\
   \vdots & \ddots & \ddots \\
   0 & \cdots & \cdots
\end{bmatrix}, \quad n \geq 1,$$  \hspace{1cm}(3.2)
with respect to the standard basis of \( L^2(\mathbb{N}_0) \), and with the additional conditions:

\[
\begin{align*}
    a_k(k) > 0, & \quad k \geq 1, \\
    c_{i,j}(1) = 0, & \quad j \geq 2, \quad 0 \leq i < j - 1, \\
    c_{i,j}(n) = 0, & \quad j \geq n, \quad 0 \leq i < j - 1, \\
    c_{k-1,k}(n) = a_k(k), & \quad k \geq n, \\
    c_{i,j}(n) = c_{i,j}(n-1), & \quad j < n - 1, \quad 0 \leq i < j - 1.
\end{align*}
\]

(3.3)

We see that \( J_n D \subset D \) for each \( n \geq 1 \). We call such a family \( \{J_n\}_{n \geq 1} \) a near tridiagonal model of the kernel \( K \) provided that

\[
K(i,j) = e_0 J_1^* \ldots J_i^* J_j \ldots J_1 e_0^*, \quad i, j \geq 0.
\]

(3.4)

**Theorem 3.2.** Any strictly positive definite kernel has a near tridiagonal model.

**Proof.** Let \( K \) be a kernel and \( K_n = [K(i,j)]_{0 \leq i,j \leq n} \). Then \( K \) is strictly positive definite if and only if \( K_n > 0, \ n \geq 1 \) (as already mentioned we can assume without loss of generality that \( K(0,0) = 1 \)). Let \( D_n = [d_{i,j}(n)]_{0 \leq i,j \leq n} \) be the upper triangular Cholesky factor of \( K_n \), therefore \( K_n = D_n^* D_n \) and \( d_{i,i}(n) > 0 \). The uniqueness of the Cholesky factor implies that

\[
D_{n+1} = \begin{bmatrix} D_n & l_{n+1} \\ 0 & d_{n+1,n+1}(n+1) \end{bmatrix},
\]

and \( d_{n,n}(k+1) = d_{n,n}(k) \) for \( k \geq n \), so we can drop the label \( n \) in \( d_{i,j}(n) \). We now construct the near tridiagonal model of \( K \). Thus we prove by induction on \( n \) that there exist numbers \( a_k(k), b_k-1(k), c_{i,k-1}(k), 0 \leq i \leq k - 1, k \leq n \), such that (3.3) holds and

\[
D_n = \begin{bmatrix} 1 & J_{2,1} & J_{3,2} J_{2,1} & \ldots & J_{n+1,n} \ldots J_{2,1} \\ 0_n & 0_{n-1} & 0_{n-2} & \ldots & 0_0 \end{bmatrix},
\]

(3.5)

where

\[
J_{2,1} = \begin{bmatrix} b_{0}(1) \\ a_{1}(1) \end{bmatrix}, \quad J_{k+1,k} = \begin{bmatrix} b_{0}(1) & c_{0,1}(2) & \ldots & c_{0,k-1}(k) \\ a_{1}(1) & b_{1}(2) & \ldots & \ldots \\ 0 & a_{2}(2) & \ldots & \ldots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \ldots & a_{k}(k) \end{bmatrix},
\]

and \( 0_k \) denotes the column with \( k \) zero entries.

For \( n = 1 \) we define

\[
b_{0}(1) = d_{0,1} \quad \text{and} \quad a_{1}(1) = d_{1,1} > 0,
\]

so that \( D_1 = \begin{bmatrix} 1 & J_{2,1} \\ 0 & 0 \end{bmatrix} \). Assume the statement is true up to \( n \). We determine the numbers \( a_{n+1}(n+1) > 0, \ b_n(n+1), \) and \( c_{k,n}(n+1), \ k = 0, \ldots, n, n - 1 \), such that

\[
\begin{bmatrix} l_{n+1} \\ d_{n+1,n+1} \end{bmatrix} = J_{n+2,n+1} J_{n+1,n+1} \ldots J_{2,1}.
\]

By the induction hypothesis

\[
J_{n+1,n+1} \ldots J_{2,1} = \begin{bmatrix} l_n \\ d_{n,n} \end{bmatrix},
\]

(3.4)
so that we must have

\[
\begin{bmatrix}
    l_{n+1} \\
    d_{n+1,n+1}
\end{bmatrix}
= \begin{bmatrix}
    x_0 + c_{0,n}(n + 1)d_{n,n} \\
    x_1 + c_{1,n}(n + 1)d_{n,n} \\
    \vdots \\
    x_n + b_n(n + 1)d_{n,n} \\
    a_{n+1}(n + 1)d_{n,n}
\end{bmatrix},
\]

where \(x_0, \ldots, x_n\) are numbers uniquely determined by \(a_k(k) > 0, b_{k-1}(k)\), and \(c_{i,l}(k), i < l < k \leq n\). Since \(d_{n,n} > 0\), \(3.6\) uniquely determine the numbers \(c_{0,n}(n + 1), \ldots, c_{n-1,n}(n + 1), b_n(n + 1)\) and \(a_{n+1}(n + 1) = \frac{d_{n+1,n+1}}{d_{n,n}} > 0\) such that \(3.5\) holds for \(n + 1\).

Now we can use all the numbers \(a_k(k), b_{k-1}(k), c_{i,l}(k)\) in order to define \(J_n\) by \(3.2\) and then \(3.3\) shows that \(\{J_n\}_{n \geq 1}\) is a near tridiagonal model of \(K\).

We notice that the label \(n\) of the numbers \(a_n(n), b_{n-1}(n), c_{i,l}(n)\) is superfluous due to the conditions \(3.3\). We used it in order to have a uniform definition of \(J_n\) in \(3.2\) but we will drop it from now on. The proof of Theorem 3.2 gives a one-to-one correspondence between the set of strictly positive definite kernels on \(\mathbb{N}_0\) with \(K(0, 0) = 1\) and the set \(\mathcal{J}\) of families of numbers \(\{a_k, b_{k-1}, c_{i,l} \mid k \geq 1, \ 0 \leq i < l\}\). We will call these numbers the *Jacobi parameters of \(K\).* In addition, we can easily characterize the strictly positive definite Hankel kernels by the additional conditions on the Jacobi parameters:

\[
\begin{align*}
    c_{i,l} &= 0, \quad l > i + 1, \\
    c_{k,k+1} &= a_{k+1}, \quad k \geq 0.
\end{align*}
\]

In this case the near tridiagonal model reduces to \(1.3\).

The next task is to establish an explicit formula for the Cholesky factors \(D_n\) in terms of the Jacobi parameters. First, we obtain a recursive relation for \(D_n\).

**Lemma 3.3.** For \(n \geq 1\),

\[
D_n = F_n(1 \oplus D_{n-1}),
\]

where

\[
F_n = \begin{bmatrix}
    1 & b_0 & c_{0,1} & c_{0,2} & \ldots & c_{0,n-1} \\
    0 & a_1 & b_1 & c_{1,2} & \cdots & c_{1,n-1} \\
    0 & 0 & a_2 & b_2 & \cdots & \\
    0 & 0 & 0 & a_3 & \cdots & \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \\
    0 & \cdots & 0 & 0 & \cdots & a_n
\end{bmatrix}.
\]

**Proof.** We have \(D_0 = 1\) and then

\[
D_1 = \begin{bmatrix}
    1 & b_0 \\
    0 & a_1
\end{bmatrix} = F_1 \begin{bmatrix}
    1 & 0 \\
    0 & D_0
\end{bmatrix}.
\]

Assume the statement is true up to \(n\). Then

\[
D_{n+1} = \begin{bmatrix}
    D_n & J_{n+2,n+1} \ldots J_{2,1} \\
    0 & 1
\end{bmatrix}.
\]
By the induction hypothesis,
\[
\begin{bmatrix}
D_n \\
0
\end{bmatrix} = \begin{bmatrix} F_n (1 \oplus D_{n-1}) \end{bmatrix} = F_{n+1} \begin{bmatrix} 1 & 0 \\
0 & D_{n-1}
\end{bmatrix}
\]
and we notice that
\[
J_{n+2,n+1} \ldots J_{2,1} = F_{n+1} \begin{bmatrix} 0 \\
J_{n+1,n} \ldots J_{2,1}
\end{bmatrix},
\]
so that
\[
D_{n+1} = F_{n+1} \begin{bmatrix} 1 & 0 & 0 \\
0 & D_{n-1} & 0 \\
0 & 0 & J_{n+1,n} \ldots J_{2,1}
\end{bmatrix} = F_{n+1} (1 \oplus D_n).
\]
\[
\begin{proof}
\end{proof}
\]

The matrices $F_n$ have a very simple recursive multiplicative structure. Actually it is convenient to make the dependence of $F_n$ on the Jacobi parameters more explicit and introduce

\[
F_{m,k} = \begin{bmatrix} 1 & b_k & c_{k,k+1} & \ldots & c_{k,m-1} \\
0 & a_{k+1} & b_{k+1} & \ldots & c_{k+1,m-1} \\
0 & a_{k+2} & & \ddots & c_{k+2,m-1} \\
& & \ddots & \ddots & \ddots \\
& & & 0 & a_m
\end{bmatrix}
\]

for $m \geq 1$ and $0 \leq k < m-1$. In particular, $F_{n,0} = F_n$. We show that the building blocks of $F_{m,k}$ (consequently, of $D_n$) are the $2 \times 2$ matrices

\[
B_k = \begin{bmatrix} 1 & b_k \\
0 & a_{k+1}
\end{bmatrix}, \ k \geq 0,
\]

and the $(l - k + 2) \times (l - k + 2)$ matrices

\[
C_{k,l} = \begin{bmatrix} 1 & 0 & \ldots & 0 & c_{k,l} \\
0 & 1 & 0 & 0 & \ddots & 0 \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & 0 & 0 & 1
\end{bmatrix}, \ 0 \leq k < l.
\]

**Lemma 3.4.** For $m \geq 1$ and $0 \leq k < m-1$,

\[
F_{m,k} = (1 \oplus F_{m,k+1}) G_{m,k},
\]

where

\[
G_{m,k} = C_{k,m-1} (C_{k,m-2} \oplus 1) \ldots (C_{k,k+1} \oplus 1_{m-k-2}) (B_k \oplus 1_{m-k-1}).
\]

**Proof.** The proof is a straightforward calculation and can be omitted. \[
\]

Once again it is convenient to visualize all those matrix multiplications by means of a transmission line picture similar to the one in Figure 3. Thus, Figure 5 illustrates how to calculate $D_3$ by using Lemma 3.3 and Lemma 3.4. In particular, if 1 is the input at $A$ then at $B$ we read the expression of $K(0,3)$ in terms of the Jacobi parameters,

\[
K(0,3) = b_0^3 + b_0 a_1 c_{0,1} + a_1 c_{0,1} b_0 + a_1 b_1 c_{0,1} + a_1 a_2 c_{0,2}.
\]
Figure 5 suggests a connection with weighted Lukasiewicz paths. A Lukasiewicz path of length $n$ is a path in the positive quadrant of the lattice $\mathbb{Z}^2$ which starts at $(0,0)$, ends at $(n,0)$, and consists of rise unit steps, horizontal unit steps, and fall steps of arbitrary depth. Let $\mathcal{L}_{n,0}$ denote the set of Lukasiewicz paths of length $n$.

We also consider $\mathcal{L}_{n,k}$ the set of paths of length $n$ in the positive quadrant, starting at $(0,0)$ and consisting of the same type of steps as above, but ending at $(n,k)$. We introduce a weight on the elements of $\mathcal{L}_{n,k}$ as follows. Let $p \in \mathcal{L}_{n,k}$ consists of steps $p_1, \ldots, p_n$. Then

$$w(p) = \prod_{k=1}^{n} w(p_k)$$

and

$$w(p_k) = \begin{cases} a_l & \text{if } p_k \text{ is a rise step } (j,l) \to (j+1,l+1) \text{ for some } j \geq 0; \\ b_l & \text{if } p_k \text{ is a horizontal step } (j,l) \to (j+1,l) \text{ for some } j \geq 0; \\ c_{k,l} & \text{if } p_k \text{ is a fall step } (j,l) \to (j+1,k) \text{ for some } j \geq l. \end{cases}$$
Theorem 3.5. The Cholesky factor $D_n = [d_{i,j}]_{0 \leq i,j \leq n}$ is given by the formula

$$d_{i,j} = \sum_{p \in \mathcal{L}_{j,i}} w(p), \quad i \leq j, \quad (i,j) \neq (0,0).$$

Proof. We can prove the statement by induction on $n$. For $n \leq 3$, (3.8) is seen from Figure 5. The general induction step is provided by Lemma 3.3. Thus,

$$d_{0,n} = b_0 d_{0,n-1} + c_{0,1} d_{1,n-1} + \ldots + c_{0,n-1} d_{n-1,n-1},$$

$$d_{1,n} = a_1 d_{0,n-1} + b_1 d_{1,n-1} + \ldots + c_{1,n-1} d_{n-1,n-1},$$

and these relations are precisely those obtained by passing from weighted paths of length $n-1$ to weighted paths of length $n$, as shown in Figure 6. \qed

Remarks

(a) For a Hankel kernel $K$, Theorem 3.5 reduces to well-known results in the combinatorial theory for orthogonal polynomials (on the real line), [4], [9]. Indeed, by (3.8) there are no fall steps of depth other than one. In this case, the summation in (3.8) is only over Motzkin paths, which is the classical formula in [4], [9]. It might be interesting to note that the corresponding formula for orthogonal polynomials on the unit circle, (2.4), involves summation over labelled configurations in $\mathbb{Z}^2$ rather than over weighted paths.

(b) There are other significant differences between the two parameterizations discussed in this paper. For instance, the parameters $\{\gamma_{k,j}\}$ have the following inheritance property: the parameters of the kernel $K^{(1)} = [K(l,m)]_{1 \leq l,m}$ are precisely $\{\gamma_{k,j}\}_{1 \leq k < j}$. The Jacobi parameters do not have such a property. Another difference involves computations of determinants. Thus, we have already noticed the formula (we assume $K(0,0) = 1$):

$$\det [K(l,m)]_{l,m=0} = \prod_{k=1}^{n} f_k \prod_{0 \leq i < j \leq n} d_{i,j}^2,$$

and from Lemma 5.3 we deduce

$$\det [K(l,m)]_{l,m=0} = \prod_{k=1}^{n} a_k^{2(n-k)},$$

which does not involve all the Jacobi parameters (up to $n$). So the first determinant formula is much tighter in its parameters.

(c) Theorem 3.5 gives, in particular, that

$$K(0,n) = \sum_{p \in \mathcal{L}_{n,0}} w(p), \quad n \geq 1.$$

Since the Jacobi parameters do not have the inheritance property mentioned above, we cannot have formulae of this type for any $K(l,n)$. Instead we have the following construction.

For $n$ fixed we consider admissible steps in the positive quadrant of the following types: between the vertical lines $x = 0$ and $x = n$ only of Lukasiewicz type are allowed and between the vertical lines $x = n$ and $x = 2n$ only reflections with respect to the line $x = n$ of Lukasiewicz steps are allowed (and they are weighted
with the complex conjugate of the weight of the reflected Lukasiewicz step, see Figure 7).

Figure 7. Admissible steps for \( n = 2 \)

Denote by \( \mathcal{K}_{n,k} \) the set of paths in the positive quadrant of \( \mathbb{Z}^2 \) made of admissible steps, starting at \((0, 0)\) and ending at \((n + k, 0)\), \(0 \leq k \leq n\). In particular, \( \mathcal{K}_{n,0} = \mathcal{L}_{n,0} \). The weights are defined correspondingly. With these elements, we deduce from Theorem 3.5 that

\[
K(l, n) = \sum_{p \in \mathcal{K}_{n,l}} w(p), \quad 0 \leq l \leq n.
\]

(d) We notice that the proof of Theorem 2.2 gives a formula for the Cholesky factor \( D_n \) in terms of Dyck type paths analogous to formula 3.8. \( \square \)

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