The Generic Bipartite Graphs of Diameter 3: Their Ages and Their Almost Sure Theories

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Big Question

- When is arbitrarily large finite different than countably infinite?
- If you have an infinite class of finite structures, and this class has a unique countable structure associated with it, when will the large finite structures look like the countably infinite structure?
An *amalgamation class* is a class of finite structures which satisfies three properties:

- Hereditary Property, Joint Embedding Property, Amalgamation Property

An amalgamation class has a unique countable homogeneous structure which embeds all the finite structures—its *Fraïssé limit*

This class is the set of all finite structures which embed into the Fraïssé limit—its *age*
0-1 laws

We’ll stick with graphs, more or less.

- The class of all finite graphs satisfies a 0-1 law: for any first order sentence $\phi$ in the language of graphs,

$$\lim_{n \to \infty} \frac{|G \in G_n : G \models \phi|}{|G_n|} \in \{0, 1\},$$

where $G_n$ is the set of finite graphs on $n$ vertices.

- If we are counting up to isomorphism, then this is an unlabeled 0-1 law.

- If we are counting generally, then this is a labeled 0-1 law.

- The set of sentences which are asymptotically satisfied form the almost sure theory.

- The theory of the Fraïssé limit is the generic theory.
Examples to date

- The class of all graphs satisfies a 0-1 law; its theory matches the theory of the Rado graph
- The class of all triangle-free graphs satisfies a 0-1 law; its theory diverges from the theory of the generic triangle-free graph
Why?
Searching for more examples

- Let’s stick with graphs
- What other graphs are Fraïssé limits?
More Examples

- The class of $K_n$ free graphs; almost all $K_n$ graphs are $(n - 1)$-partite
- ...That’s it. We need to think outside of the box.
- There are other graphs which are Fraïssé limits, but not determined by forbidden graphs
Homogeneity

- A graph is *homogeneous* if every isomorphism between two finite induced subgraphs can be extended to an automorphism of the whole graph.
Homogeneous graphs

Not at all homogeneous  Not homogeneous  Homogeneous
A connected graph is *metrically homogeneous* if, when endowed with the path metric, every finite partial isometry can be extended to a full isometry.

Homogeneous graphs $\subset$ metrically homogeneous graphs.
Metrically homogeneous graphs

- Cherlin has a tentative catalog of the metrically homogeneous graphs
- Many of them are Fraïssé limits
- The configurations they forbid are metric spaces
Graphs as metric spaces

(board work)
Metrically homogeneous graphs: Generic type

All of the known metrically homogeneous graphs of generic type are of the form $\Gamma^\delta_{K_1, K_2, C, C', S}$.

- $(\delta, K_1, K_2, C, C')$ are parameters which restrict the forbidden triangles
- $S$ is a collection of forbidden $(1, \delta)$-spaces.
The parameters

- \( \delta \) is the diameter;
- \( K_1, K_2 \) exclude certain triangles of odd (and not very large) perimeter;
- \( C_0, C_1 \) exclude all triangles of sufficiently large even (respectively, odd) perimeter;
- Alternatively — \( C = \min(C_0, C_1), \quad C' = \max(C_0, C_1) \).
Our graphs

- Cherlin, Amato and MacPherson have found all the metrically homogeneous graphs of diameter 3.
- There are exactly 2 metrically homogeneous graphs of diameter 3 which are bipartite and of generic type.
- I examined these two graphs.
First bipartite graph

- $A^3_{\infty,0,7,8,}, \Gamma^3_{\infty,0,7,8,}$
- Only keeps (112), (123), (222)
- $\Gamma^3_{\infty,0,7,8,}$ can be viewed as a metric space or a graph:

Viewed as a metric space
- Distance 1 is written as a solid black line
- Distance 2 is written as a solid gray line
- Distance 3 is written as a dotted black line

Viewed as a graph
Second bipartite graph

- $\mathcal{A}^3_{\infty,0,7,8}$, $\Gamma^3_{\infty,0,7,8}$
- Only keeps (112), (123), (222), (2, 3, 3)
- $\Gamma^3_{\infty,0,7,8}$ can be viewed as a metric space or a graph:

Viewed as a metric space

Viewed as a graph
Theorem (C.)

Both $A^3_{\infty,0,7,8,\emptyset}$ and $A^3_{\infty,0,7,10,\emptyset}$ have unlabeled and labeled 0-1 laws. The almost sure theory of $A^3_{\infty,0,7,8,\emptyset}$ diverges from its generic theory. The almost sure theory of $A^3_{\infty,0,7,10,\emptyset}$ matches its generic theory.
Why?

Recall that $A^3_{\infty,0,7,8,\emptyset}$ allows $(1,1,2)$ but not $(3,3,2)$

**Figure:** The unique countable model $\Gamma_{as}$ of the almost sure theory of $A^3_{\infty,0,7,8,\emptyset}$
But still, why?

It comes down to that triangle

- In $\mathcal{A}_{3,0,7,8,\emptyset}^3$, each vertex can have at most one 1-edge
- For homogeneity, every vertex has exactly one 1-edge
- The almost sure limit also has vertices without 1-edges
- 1 and 3 were functionally different
- No such difference in $\mathcal{A}_{3,0,7,10,\emptyset}$
More musings

- If we put the class of triangle-free graphs in this context, then it has diameter 2, forbids $(1, 1, 1)$ and allows $(2, 1, 1), (2, 2, 1), \text{ and } (2, 2, 2)$
- $2$ is functionally different than $1$
First I described the spaces in $A^{3\infty,0,7,8,\emptyset}$ and $A^{3\infty,0,7,10,\emptyset}$, then I counted them, both up to isomorphism and generally.

I found axioms which describe them.

I showed that these axioms form a complete theory:

- They have no finite models.
- They are $\mathfrak{c}_0$-categorical.

I showed that these axioms are asymptotically satisfied via a direct count for isomorphism classes for $A^{3\infty,0,7,8,\emptyset}$ and $A^{3\infty,0,7,10,\emptyset}$, also generally for $A^{3\infty,0,7,10,\emptyset}$.

I used a rigidity/asymmetry argument for $A^{3\infty,0,7,10,\emptyset}$.
Thank you for having me!