A Well-balanced Positivity Preserving Central-upwind Scheme for Shallow Water Flows over Uneven Topography

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Abstract. In this study, a well-balanced positivity preserving numerical model based on shallow water equation is developed to simulate flood propagation over uneven topography. The central-upwind scheme is adopted to calculate the flux at the interface of computational mesh. The bed slope source term is discretized using a central difference method to ensure the well-balanced property of the model. The model is able to preserve the positivity of water depth if the Courant number is kept less than 0.5. The proposed model is verified against a benchmark test and then applied to simulate a dam break experiment carried out in CADAM project. The numerical results are in good agreement with observations, which show that the model is capable of predicting the flood propagation over complex topography with a good accuracy.

Keywords: finite volume method; shallow water equation; central-upwind scheme; well-balance; uneven topography.

1. Introduction
With the development of computer technology and numerical schemes, shallow water equations are widely used to simulate water flows over irregular topography. A good numerical method for shallow water equations should have the following two characteristics: 1) preserves the steady state at the discrete level, which is called well-balanced property. 2) Guarantee the positivity of water depth at wet/dry fronts. Recently the Godunov-type central-upwind scheme, which was introduced as a universal Riemann-problem-solver-free method for hyperbolic conservation laws, satisfied the aforementioned two requirements and attracted tremendous attentions due to its simplicity, robustness and efficiency [1-4].

In this paper, a numerical model based on shallow water equation is developed to simulate flood propagation over uneven topography. The central-upwind scheme is adopted to calculate the flux at the interface of the cells. The model is not only well-balanced, but also preserves the positivity of water depth when the Courant number is less than 0.5.

2. One-dimensional shallow water equations
The vector form of 1D shallow water equations is written as:
\[
\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = S
\]

\[
q = \begin{bmatrix} \eta \\ hu \end{bmatrix}, \quad f = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2} gh^2 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & \partial h \partial z b - S_f \end{bmatrix}
\]  

(1)

Where \( q, f \) and \( S \) are respectively the vectors representing conserved flow variables, fluxes and source terms. \( t \) is time; \( x \) is horizontal coordinate; \( u \) is depth-averaged velocity; \( g \) is the gravitational acceleration; \( \eta \) is the water surface elevation; \( h \) is water depth; \( z_b \) is bed elevation over the datum; \( S_f \) is the friction term, which can be expressed as:

\[
S_f = \frac{g u |u| n^2}{h^{1/3}}
\]  

(2)

where \( n \) is the Manning coefficient.

3. Numerical scheme

3.1. Discretization using finite volume method.

As shown in Fig.1, the model uses the stagger grids, in which the conserved variables including water surface elevation \( \eta \), water depth \( h \) and depth-averaged velocity \( u \) are defined at the cell center, and the bed elevation \( z_b \) is defined at the cell corner. Assuming the bed elevation \( z_b \) is a piecewise linear function, the bed elevation at cell center is given by \( (z_b)_i = (z_b)_{i+1/2} + (z_b)_{i-1/2})/2 \).

The governing Eq. (1) can be discretized using finite volume method and discretization takes the form:

\[
q_{i}^{k+1} = q_{i}^{k} - \frac{\Delta t}{\Delta x} (f_{i+1/2} - f_{i-1/2}) + \Delta t S_i
\]  

(3)

where the superscript \( k \) is the time level, subscripts \( i \) is the index of the cell, \( \Delta t \) is the time step and \( \Delta x \) is the cell size. \( f_{i+1/2} \) and \( f_{i-1/2} \) are the flux at the right and left interface of the cell, respectively. \( S_i \) represents the source terms evaluated at the cell center.

Fig. 1 Sketch of computational grid and the location of conserved variables and bed elevation

3.2. Flux calculation using central-upwind scheme.

The central-upwind scheme proposed by Kurganov and Petrova[1] is used to calculate the interface flux. This scheme does not require characteristic decomposition of the hyperbolic system or expensive Riemann solvers. The flux at the interface can be calculated as:

\[
\begin{align*}
    f_{i+1/2} &= \frac{a_i^+ f_{i+1/2} + a_i^- f_{i-1/2}}{a_i^+ + a_i^-} + \frac{a_i^+ a_i^-}{a_{i+1/2} - a_{i-1/2}} \left( q_{i+1/2} - q_{i+1/2} \right) \\
    a_i^+ &= a_i^{+1/2} a_{i+1/2} + a_i^{+1/2} a_{i+1/2} \\
    a_i^- &= a_i^{-1/2} a_{i-1/2} + a_i^{-1/2} a_{i+1/2}
\end{align*}
\]  

(4)

where \( a_i^{+1/2} \) and \( a_i^{-1/2} \) are the one-sided local wave speeds, which are expressed as:
\[ a_{i+1/2}^+ = \max\{u_{i+1/2,R} + \sqrt{gh_{i+1/2,R}} + \sqrt{gh_{i+1/2,L}}, 0\} \]
\[ a_{i+1/2}^- = \min\{u_{i+1/2,R} + \sqrt{gh_{i+1/2,R}} + \sqrt{gh_{i+1/2,L}}, 0\} \]

3.3. Discretization of the source term.

The source term can be split into the bed slope terms and friction terms. The bed slope terms are discretized by using the central difference method, which is given by:
\[ gh \frac{\partial z_b}{\partial x} = g \frac{(\eta_{i+1/2} - (z_b)_{i+1/2}) + (\eta_{i-1/2} - (z_b)_{i-1/2})}{2\Delta x} \]

The semi-implicit scheme is used for the friction terms, which is given by:
\[ S_f = \frac{\rho u|u|^2}{h^{1/3}} = \frac{g|u|^2}{h^{4/3}} (hu)^{k+1} \]

Substituting the Eq. (7) and Eq. (8) into Eq. (3), the values of flow variable at the end of time step can be obtained.

3.4. Stability criteria.

The current numerical scheme is explicit and its stability is governed by the Courant-Friedrichs-Lewy (CFL) condition. In order to preserve the positivity of water depth and the stability of the model, the Courant number has to be kept less than 0.5, which has been proved in [8]. Therefore, the CFL restriction for the current model is:
\[ N_{cfl} = \max_i \frac{\Delta t}{\Delta x} (|u| + \sqrt{gh}) \leq 0.5 \]

The Courant number is set to 0.25 in all the examples in this paper.

4. Testing and Verification of Model

In this section, three cases are simulated to test the performance of proposed model. Comparing the calculated results with the corresponding theoretical solutions or laboratory measurement data, the accuracy and stability of the model are analyzed. For all the tests, \( g=9.81\,\text{m/s}^2 \) and \( \rho=1000\,\text{kg/m}^3 \).

4.1. 1D steady flow over a bump.

This benchmark test proposed by Goutal and Maurel[5] is considered to investigate the accuracy of the model. The flow occurs in a 25 m x 1 m frictionless channel with the bed defined as:
\[ z_b(x) = \begin{cases} 
0.2 - 0.05(x - 10)^2 & \text{if } 8 < x < 12 \\
0.0 & \text{other} 
\end{cases} \]

According to the initial and boundary conditions, the steady flow over bump can be supercritical, transcritical with or without shock, or subcritical. Transcritical flow with shock and subcritical flow cases are chosen to test the model’s accuracy. The computational domain is approximated by 200 cells. The unit discharge of 0.18 \( \text{m}^3/\text{s} \) and 4.42 \( \text{m}^3/\text{s} \) are imposed at the upstream boundaries for transcritical flow and subcritical flow cases, respectively. The downstream water levels are set to be 0.33 m and 2.0 m, respectively. Fig. 2 shows the calculated water level and discharge compared with the theoretical solutions. It can be found that the simulated results agree with the theoretical solutions quite well. For the discharge, the maximum difference between the numerical prediction and analytical solution at a grid point near the middle of the domain is about 5%. Therefore, the model is able to simulate the flow over uneven topography accurately.
4.2. Moving shorelines on parabolic bottom topography.
This benchmark test was first proposed by Sampson[6] and widely used to validate the wetting and drying procedure in the numerical model. As shown in Fig. 3, the oscillatory flow is assumed to take place in a parabolic container with the bed elevation defined by:

$$z_b(x) = 10(x/3000)^2$$  \hspace{1cm} (11)

where $x$ is horizontal coordinate, $x \in [-5000\text{m},5000\text{m}]$.

The initial velocity $u = 0 \text{ m/s}$ and water surface elevation is given by:

$$\eta(x) = 8.7404 - 0.00236x$$ \hspace{1cm} (12)

The bed friction parameter $\tau = (g|u|n^{5/3})/h^{4/3} = 0.001 \text{s}^{-1}$. The water in the parabolic container will oscillate and reach stationary state due to the effect of bed friction.

The simulation lasting 5000 s is carried out on a uniform grid with a size of 100 m. Fig. 3 shows the excellent agreement between the numerical results and analytical solutions of water surface profile, indicating that the model can simulate the moving wet/dry fronts accurately.

Fig. 3 Water surface profile in parabolic container at different times
4.3. Dam break experiment.

The dam break experiment was conducted by the European research community in the Concerted Action on Dam-Break Modeling (CADAM) project. The experiment configuration is shown in Fig. 4. The channel was 38 m long and a dam was located 15.5 m away from the upstream end. The water level in the upstream reservoir was 0.75 m and the downstream floodplain is dry initially. A triangle obstacle with 0.4 m height and 6 m length is installed at the downstream 13 m away from the dam. Time series of water depth are recorded at six gauges, which were located in the downstream 4 m (G4), 8 m (G8), 10 m (G10), 11 m (G11), 13 m (G13) and 20 m (G20) away from the dam. The Manning coefficient \( n \) of the whole area is set to be 0.0125. The total simulation time is 90 s based on a grid with a uniform cell size of 0.05 m and adaptive time step is adopted.

At \( t=0 \) s, the dam is removed instantaneously, the flood begins to flow downstream and reaches the G13 in about 3 s. At \( t=5 \) s, the flood will cross the triangular obstacle and continue to move forward. However, due to the blocking effect of the triangle obstacle, the reflected wave moving upstream will be generated, which will cause the first peak value of the G4 and G8. At \( t=24 \) s, reflection flow reaches the upstream boundary. Because the upstream is the closed boundary, the reflected wave moving downstream, causing the water depth of the G4 and G8 to form a second peak. Meanwhile, since the downstream is a free outlet boundary, the water crossing the triangle obstacle will gradually flow out of the experimental area and the water depth of the G13 and G20 is gradually reduced to zero until the reflected flow generated by upstream end cross the triangle obstacle again. Fig. 5 shows the computed water depth at different gauges compared with the observation data. It can be found that the wave arrival time and water depth are predicted accurately at G4, G8, G10, G11 and G13. However, the water depth in gauge G20 is overpredicted, which was also reported in Liang and Marche [7]. A possible explanation is that the flow between the tip of obstacle and outlet is very complex and the shallow water equations based on the hydrostatic situation are not suitable for this situation. In general, the proposed model is capable of simulating dam break flow over uneven bed with a satisfied accuracy.
5. Conclusion
A well-balanced positivity preserving numerical model based on shallow water equation is developed to simulate flood propagation over uneven topography. The central-upwind scheme is adopted to calculate the flux at the interface of computational mesh. The bed slope source term is discretized using a central difference method to ensure the well-balanced property of the model. The model is able to preserve the positivity of water depth if the Courant number is kept less than 0.5. The developed model is validated against two benchmark tests and a dam break flow experiment. The numerical results agree with the analytical solutions and measurement data quite well, which indicate the model is capable of simulating the flow over uneven topography accurately.

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