LEAFWISE FIXED POINTS FOR $C^0$-SMALL
HAMILTONIAN FLOWS AND LOCAL COISOTROPIC
FLOER HOMOLOGY

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Consider a closed coisotropic submanifold $N$ of a symplectic manifold $(M,\omega)$ and a Hamiltonian diffeomorphism $\varphi$ on $M$. The main result of this article is that $\varphi$ has a leafwise fixed point w.r.t. $N$, provided that it is the time-1-map of a Hamiltonian flow whose restriction to $N$ stays $C^0$-close to the inclusion $N \to M$. This appears to be the first leafwise fixed point result in which neither $\varphi|_N$ is assumed to be $C^1$-close to the inclusion $N \to M$, nor $N$ to be of contact type or regular (i.e., “fibering”).

The method of proof of this result leads to a local coisotropic version of Floer homology.

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1. Introduction and main result

Consider a symplectic manifold $(M,\omega)$ and a coisotropic submanifold $N \subseteq M$. This means that for every $x \in N$ the symplectic complement of $T_x N$,

$$T_x N^\omega := \{ v \in T_x M \mid \omega(v, w) = 0, \forall w \in T_x N \},$$

is contained in $T_x N$. It follows that $TN^\omega = \bigsqcup_{x \in N} T_x N^\omega$ is an involutive distribution on $N$. By Frobenius’ Theorem such a distribution gives rise to a foliation on $N$. The leaves of this foliation are called isotropic leaves.

Let $\varphi : M \to M$ be a Hamiltonian diffeomorphism. A leafwise fixed point for $\varphi$ is a point $x \in N$ for which $\varphi(x)$ lies in the isotropic
leaf through $x$. We denote by $\text{Fix}(\varphi, N)$ the set of such points. A fundamental problem in symplectic geometry is the following:

**Problem.** *Find conditions under which $\text{Fix}(\varphi, N)$ is non-empty.*

In the extreme case $N = M$ the set $\text{Fix}(\varphi, N)$ consists of the usual fixed points of $\varphi$. Such points correspond to periodic orbits of Hamiltonian systems. Starting with a famous conjecture by V. I. Arnold [Ar], the above problem has been extensively studied in this case.

On the opposite extreme, consider the case in which $N$ is Lagrangian, i.e., has half the dimension of $M$. Then

$$\text{Fix}(\varphi, N) = N \cap \varphi^{-1}(N)$$

(provided that $N$ is connected). In this situation, based on seminal work by A. Floer [Fl], the above problem has given rise to Lagrangian Floer homology and the Fukaya category.

As an intermediate case, coisotropic submanifolds of codimension 1 arise in classical mechanics as energy level sets for an autonomous Hamiltonian. If $\varphi$ is the time-one flow of a time-dependent perturbation of the Hamiltonian, then $\text{Fix}(\varphi, N)$ corresponds to the set of points on the level set whose trajectory is changed only by a phase shift, under the perturbation.

For a coisotropic submanifold $N$ of general codimension several solutions to the above problem have been found under restrictive assumptions on $\varphi$ or $N$, see [Mo, Ban] [EH], [Ho2], [Gi] [Dr] [Gü] [Zi1] [Zi2] [AF1] [AF2] [AF3] [AMo] [AMc] [Bae] [Ka] [MMP]. In all results $\varphi|_N$ is assumed to be $C^1$-close to the inclusion $N \to M$ or $N$ to be of contact type or regular (i.e., “fibering”).

The main result of this article is that the conditions on $N$ can be removed altogether, if $\varphi$ is the time-1-map of a Hamiltonian flow whose restriction to $N$ stays $C^0$-close to the inclusion $N \to M$:

**Theorem 1** (leafwise fixed points). *Let $(M, \omega)$ be a symplectic manifold and $N \subseteq M$ be a closed 1 coisotropic submanifold. Then there exists a $C^0$-neighbourhood $\mathcal{U} \subseteq C(N, M)$ of the inclusion $N \to M$ with the following property. If $(\varphi^t)_{t \in [0,1]}$ is a Hamiltonian flow on $M$ satisfying $\varphi^t|_N \in \mathcal{U}$, for every $t \in [0,1]$, then

$$\text{Fix}(\varphi^1, N) \neq \emptyset.$$*

**Remark** ($C^0$-closeness needed). *The condition that $\varphi^t|_N \in \mathcal{U}$ in this result cannot be dropped in general. As an example, let $(M, \omega)$ be the

\footnote{This means that $N$ is compact and has no boundary.}
product of a symplectic manifold and $\mathbb{R}^2$, equipped with the standard symplectic structure. Then for every closed coisotropic submanifold $N \subseteq M$ there exists a Hamiltonian diffeomorphism that displaces $N$ from itself.\footnote{To see this, we choose a Hamiltonian diffeomorphism $\psi$ of $\mathbb{R}^2$ that displaces the image of $N$ under the canonical projection onto $\mathbb{R}^2$. The product $\text{id} \times \psi$ has the required properties.} Such a diffeomorphism has no leafwise fixed points.

**Remark** (Hofer-closeness). The condition that $\varphi^1|_N \in \mathcal{U}$ does not imply that $\varphi^1$ is Hofer-close to the identity. In fact the map $\varphi^1$ may have arbitrarily high Hofer norm. More precisely, assume that $M$ is closed and $\pi_2(M) = 0$. Then for every non-empty open subset $U \subseteq M$ and every constant $C \geq 0$ there exists a Hamiltonian flow $(\varphi^t)_{t \in [0,1]}$ on $M$ with support in $U$, such that the Hofer norm of $\varphi^1$ is at least $C$. This follows from the proof of [Os, Theorem 1.1]. The flow $(\varphi^t)$ stays arbitrarily $C^0$-close to the identity, provided that $U$ is chosen small enough.

**Remark 2** (multiple leafwise fixed points). The proof of Theorem 7 given below shows that in the setting of this result

$$|\text{Fix}(\varphi^1, N)| \geq 1 + \text{cup-length of } N.$$  

It also shows that

$$|\text{Fix}(\varphi^1, N)| \geq \sum_{i=0}^{\dim N} b_i(N),$$

provided that $U$ is chosen as in this proof, $(\varphi^t)$ satisfies the conditions in Theorem 1 and the pair $(\varphi^1, N)$ is nondegenerate in the sense of [Zi, p. 105]. Here $b_i(N)$ denotes the $i$-th Betti number of $N$ with $\mathbb{Z}_2$-coefficients.

**Remark** (extreme cases). In the extreme cases $N = M$ and $N$ Lagrangian A. Weinstein [We1, We2] proved lower bounds on $|\text{Fix}(\varphi, N)|$ for the time-one flow $\varphi$ of a $C^0$-small Hamiltonian vector field. Such a $\varphi$ satisfies a stronger condition than the one in Theorem 1.

The idea of proof of Theorem 1 is to construct a symplectic submanifold $\tilde{M}$ of the presymplectic manifold $M \times N$ that contains the diagonal embedding $\tilde{N}$ of $N$ as a Lagrangian submanifold. The result will then follow from the fact that the zero section of the cotangent bundle $T^*\tilde{N}$ is not displaceable in a Hamiltonian way. The manifold $\tilde{M}$ is constructed by using a smooth family of local slices in $N$ that are transverse to the isotropic distribution $TN^\omega$. Such a family can
be viewed as a substitute for the symplectic quotient of $N$, which is in
general not well-defined.

The proof refines an approach from [Zi1] that only works in the
regular case. (See Remark 6 below.) It suggests a construction of a
local coisotropic version of Floer homology. (See the next section.)

2. Local coisotropic Floer homology

In this section the construction of a local coisotropic version of Floer
homology is outlined, which is expected to reproduce the lower bound
(1) on $|\text{Fix}(\varphi, N)|$ in Remark 2. Details will be carried out elsewhere.
For the extreme cases $N = M$ and $N$ Lagrangian, local versions of Floer
homology were developed in [Fl2, Oh1, Oh2, CFHW, Po, GG]; see
also the book [Oh3, Chapter 17.2].

This section will not be used in the proof of Theorem 1, and it is not
needed in the proofs of the estimates mentioned in Remark 2. (Hence
the impatient reader may immediately proceed to Section 3.)

Potentially a (more) global version of coisotropic Floer homology
may be defined, so that the $C^0$-condition on $\varphi$ in Theorem 1 can be
relaxed. This may also yield a lower bound on $|\text{Fix}(\varphi, N)|$ that is
higher than the sum of the Betti numbers of $N$, for a suitably generic
pair $(\varphi, N)$.

The local coisotropic Floer homology may play a role in mirror sym-
metry, as physicists have realized that the Fukaya category should be
enlarged by coisotropic submanifolds, in order to make homological
mirror symmetry work, see e.g. [KO].

To explain the definition of this homology, consider a symplectic
manifold $(M, \omega)$, a closed coisotropic submanifold $N \subseteq M$, a diffeo-
morphism $\varphi$ of $M$, and an $\omega$-compatible almost complex structure $J$
on $M$. Assume that $\varphi$ is the time-1 map of a Hamiltonian flow on $M$
whose restriction to $N$ stays “$C^0$-close” to the inclusion $N \to M$, and
that $(\varphi, N)$ is nondegenerate in the sense of [Zi1, p. 105].

Heuristically, we define the local Floer homology $HF(N, \varphi, J)$ as fol-
lows. Its chain complex is generated by the points $x \in \text{Fix}(N, \varphi)$, for
which there is a “short” path from $x$ to $\varphi(x)$ within the isotropic leaf
through $x$.

To explain the boundary operator $\partial = \partial_{N, \varphi, J}$, we equip $M \times M$
with the symplectic form $\omega \oplus (-\omega)$. We choose a symplectic submanifold $\tilde{M}$

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3In [Al] a Lagrangian Floer homology was constructed that is “local” in a dif-
ferent sense.
of $M \times N$ that contains the diagonal

$$\tilde{N} := \{(x, x) \mid x \in N \}.$$  

(Such a submanifold exists by Lemma 3 below.) Then $\tilde{N}$ is a Lagrangian submanifold of $\tilde{M}$. We shrink $\tilde{M}$, so that it is a Weinstein neighbourhood of $\tilde{N}$. The map $\varphi$ induces a Hamiltonian diffeomorphism $\tilde{\varphi}$ between two open neighbourhoods of $\tilde{N}$ in $\tilde{M}$. The structure $J$ induces an almost complex structure $\tilde{J}$ on $\tilde{M}$ that is $\tilde{\omega}$-compatible.

The boundary operator $\partial$ is now defined to be the boundary operator of the “local Lagrangian Floer homology” of $(\tilde{M}, \tilde{\omega}, \tilde{N}, \tilde{\varphi}^{-1}(\tilde{N}), \tilde{J})$. This map counts finite energy $\tilde{J}$-holomorphic strips in $\tilde{M}$ that stay “close” to $\tilde{N}$, have Viterbo-Maslov index 1, map the lower and upper boundaries of the strip to $\tilde{N}$ and $\tilde{\varphi}^{-1}(\tilde{N})$, and connect two intersection points of $\tilde{N}$ and $\tilde{\varphi}^{-1}(\tilde{N})$. Such points correspond to points $x \in \text{Fix}(N, \varphi)$, for which there exists a short path from $x$ to $\varphi(x)$ within a leaf.

To understand why heuristically, the boundary operator is well-defined and squares to zero, observe that $\tilde{N}$ intersects $\tilde{\varphi}^{-1}(\tilde{N})$ transversely, since $(\varphi, N)$ is nondegenerate. Therefore, for generic $\tilde{J}^4$, the moduli space of $\tilde{J}$-strips is canonically a 0-dimensional manifold. (Here we divided by the translation action.) It is compact, since index-1-strips generically do not break, and disks or spheres cannot bubble off, because $\tilde{N}$ is an exact Lagrangian in $\tilde{M}$. Here we used our assumption that $\tilde{M}$ is a Weinstein neighbourhood of $\tilde{N}$. For similar reasons we have $\partial^2 = 0$.

Given two choices of symplectic submanifolds $\tilde{M}, \tilde{M}' \subseteq M \times N$ containing $\tilde{N}$, one obtains a symplectomorphism between open neighbourhoods of $\tilde{N}$ in $\tilde{M}$ and $\tilde{M}'$, by sliding $\tilde{M}$ to $\tilde{M}'$ along the isotropic leaves of $N$. This symplectomorphism intertwines the corresponding $\tilde{\varphi}$’s and $\tilde{J}$’s. It follows that the boundary operator does not depend on the choice of $\tilde{M}$, and therefore, heuristically, is well-defined.

To make the outlined Floer homology rigorous, the words “close” and “short” used above, need to be made precise. To obtain an object that does not depend on the choice of “closeness”, the local Floer homology

\footnote{Here one needs to work with a family of almost complex structures depending on the time $t$.}
of \((N, J)\) should really be defined to be the germ of the map
\[
\varphi \mapsto \operatorname{HF}(N, \varphi, J)
\]
around \(\text{id} : M \to M\).

By showing that \(\operatorname{HF}(N, \varphi, J)\) is isomorphic to the singular homology of \(\tilde{N}\), it should be possible to reproduce the lower bound (1) on \(|\text{Fix}(\varphi, N)|\), which was mentioned in Remark 2.

3. Proof of Theorem 1

The proof of Theorem 1 is based on the following lemma. Let \((M, \omega)\) be a symplectic manifold and \(N \subseteq M\) a closed coisotropic submanifold. We equip \(M \times M\) with the symplectic form \(\omega \oplus (-\omega)\) and denote
\[
2n := \dim M, \quad k := \text{codim} N, \quad m := n - k,
\]
and by
\[
\iota_N : N \to M
\]
the canonical inclusion.

**Lemma 3** (existence of a good symplectic submanifold of \(M \times N\)).
There exists a symplectic submanifold (without boundary) \(\tilde{M}\) of \(M \times M\) of dimension \(2n + 2m\) that is contained in \(M \times N\) and contains the diagonal
\[
\tilde{N} := \{(x, x) \mid x \in N\}.
\]

**Remark.** If \(\tilde{M}\) is as in this lemma then it contains \(\tilde{N}\) as a Lagrangian submanifold.

The idea of proof of this lemma is to choose an open neighbourhood \(U \subseteq M\) of \(N\), a retraction \(r : U \to N\), and for each \(y_0 \in N\) a local slice \(S_{y_0} \subseteq N\) through \(y_0\) that is transverse to the isotropic foliation, and to define
\[
\tilde{M} := \{(x, y) \in M \times N \mid y \in S_{r(x)}\}.
\]

**Proof of Lemma 3.** We will choose the submanifold \(\tilde{M}\) to be a subset of the image of the map \(F\) whose existence is stated in the following claim. We choose an open neighbourhood \(U \subseteq M\) of \(N\) and a smooth retraction
\[
r : U \to N.
\]
(This means that \(r\) equals the identity on \(N\).) We denote by
\[
\text{pr}_1 : M \times N \to M, \quad \text{pr}_2 : M \times N \to N
\]
the canonical projections.

**Claim 1** (existence of a good map to $M \times N$). There exist a smooth manifold $\tilde{M}$ of dimension $2n + 2m$, a compact subset $\tilde{N} \subseteq \tilde{M}$, and a smooth map

$$F : \tilde{M} \to M \times N$$

that maps $\tilde{N}$ bijectively to $\tilde{N}$, such that the following holds. Let $\hat{x} \in \tilde{N}$.

We denote

1. $\tilde{V} := T_{F(\hat{x})}(M \times N)$, $\tilde{\Omega} := (\omega \oplus (-t_\bar{x}^*\omega))_{F(\hat{x})}$, $\tilde{W} := dF(\hat{x})T_{\hat{x}}\tilde{M}$.

Then

2. $\tilde{W} + \tilde{V} \tilde{\Omega} = \tilde{V}$.

(Here we denote by $\tilde{V} \tilde{\Omega}$ the $\tilde{\Omega}$-complement of $\tilde{V}$.)

**Proof of Claim 1.** We will use the next claim, which roughly states that there exists a smooth family of local slices on $N$ that are transverse to the isotropic foliation. We denote

$$T_yN_\omega := T_yN/T_yN_\omega, \forall y \in N, \quad T_N := TN/TN_\omega$$

**Claim 2** (local slices on $N$). There exists a smooth map

$$f : TN_\omega \to N,$$

such that, defining

$$f_y := f(y, \cdot),$$

we have

1. $f_y(0) = y$, 
2. $df_y(0)(T_yN_\omega) + T_yN_\omega = T_yN,$

for every $y \in N$. Here in (6) we canonically identified $T_0(T_yN_\omega)$ with $T_yN_\omega$.

**Proof of Claim 2.** We choose a Riemannian metric $g$ on $N$ and denote by $\perp$ the orthogonal complement w.r.t. $g$. The map

$$TN \supseteq (TN_\omega)^\perp \ni (y, v) \mapsto (y, v + T_yN_\omega) \in TN_\omega,$$

is an isomorphism of vector bundles over $N$. We denote by $\Phi$ the inverse of this map and by exp the exponential map w.r.t. $g$, and define

$$f := \exp \circ \Phi : TN_\omega \to N.$$. 
Let \( y \in N \). Condition (5) is satisfied. Furthermore, since \( f_y = \exp_y \circ \Phi_y \), we have
\[
d f_y(0) T_y N_\omega = d \exp_y(0) (T_y N^\omega)^\perp = (T_y N^\omega)^\perp.
\]
Equality (6) follows. This proves Claim 2. \( \square \)

We choose \( f \) as in Claim 2. We define \( \hat{M} := \{ (x, \overline{v}) \mid x \in U, \overline{v} \in T_{r(x)} N_\omega \} \).
(Recall here from (2) the choice of the retraction \( r \).) This is a submanifold of \( M \times T N_\omega \) of dimension \( 2n + 2m \). We define \( \hat{N} := \{ (y, 0_y) \mid y \in N \} \subseteq \hat{M} \), \( F : \hat{M} \to M \times N \), \( F(x, \overline{v}) := (x, f(r(x), \overline{v})) \).
This map is smooth. Because of (5) it maps \( \hat{N} \) bijectively to \( \tilde{N} \).

To show (4), we define \( R : U \to M \times N \), \( R(x) := (x, r(x)) \), \( \hat{R} : U \to \hat{M} \), \( \hat{R}(x) := (x, 0_{r(x)}) \).

Let \( \hat{x} = (y, 0_y) \in \hat{N} \). We have \( R = F \circ \hat{R} \), hence
\[
d R(y) = d F(\hat{x}) d \hat{R}(y),
\]
and therefore,
\[
d R(y) T_y M \subseteq \tilde{W} = d F(\hat{x}) T_{\hat{x}} \hat{M}.
\]
On the other hand, defining \( \iota : T_y N_\omega \to \hat{M} \), \( \iota(\overline{v}) := (y, \overline{v}) \),
we have \( F \circ \iota(\overline{v}) = (y, f_y(\overline{v})) \), hence
\[
d F(\hat{x}) d \iota(0) = (0, df_y(0)) : T_0(T_y N_\omega) = T_y N_\omega \to T F(\hat{x})(M \times N),
\]
and therefore,
\[
\{0\} \times df_y(0) T_y N_\omega \subseteq \tilde{W}.
\]
Combining this with (7) and the fact \( \hat{V} = \{0\} \times T_y N^\omega \), it follows that
\[
d R(y) T_y M + \{0\} \times (df_y(0) T_y N_\omega + T_y N^\omega) \subseteq \tilde{W} + \hat{V}.
\]
Using (6) and the fact that \( d R(y) T_y M \) is the graph of a linear map (namely of \( dr(y) \)), the left hand side of (8) equals \( \tilde{V} \). Equality (4) follows. This completes the proof of Claim 3. \( \square \)
We choose \( \hat{M}, \hat{N}, F \) as in this claim. We show that the image \( \hat{M} \) under \( F \) of a suitable subset of \( \hat{M} \) is a symplectic submanifold of \( M \times N \) that contains \( \hat{N} = F(\hat{N}) \), as desired. For this we show that \( F \) is a symplectic immersion along \( \hat{N} \).

Let \( \hat{x} \in \hat{N} \). We define \( \hat{V}, \hat{\Omega}, \hat{W} \) as in (3).

Claim 3. The map \( dF(\hat{x}) \) is injective and \( \hat{W} = dF(\hat{x})(T_{\hat{x}}\hat{M}) \) is a symplectic linear subspace of \( \hat{V} \) w.r.t. \( \hat{\Omega} \).

Proof of Claim 3. Condition (4) implies that

\[
\dim \hat{W} \geq \dim \hat{V} - \dim \hat{\Omega} = 2n + 2m.
\]

Since \( \dim T_{\hat{x}}\hat{M} = 2n + 2m \), it follows that \( dF(\hat{x}) \) is injective. This proves the first assertion.

Since \( \hat{W} \) is the image of \( T_{\hat{x}}\hat{M} \) under the linear map \( dF(\hat{x}) \), we have

\[
\dim \hat{W} \leq \dim T_{\hat{x}}\hat{M} = 2n + 2m = \dim \hat{V} - \dim \hat{V}^\Omega.
\]

Therefore, equality (4) implies that

\[
\hat{W} \cap \hat{V}^\Omega = \{0\}.
\]

The same equality also implies that

\[
\hat{W}^\Omega = (\hat{W} + \hat{V}^\Omega)^\Omega = \hat{V}^\Omega.
\]

Combining this with (11), it follows that \( \hat{W} \) is a symplectic linear subspace of \( \hat{V} \). This proves Claim 3. \( \square \)

Using Claim 3 and injectivity of \( F|_{\hat{U}} \), by Lemma 7 below there exists an open neighbourhood \( \hat{U} \subseteq \hat{M} \) of \( \hat{N} \), such that \( F|_{\hat{U}} \) is an embedding. We define

\[
\hat{M} := \{ \hat{x} \in F(\hat{U}) \mid T_{\hat{x}}(F(\hat{U})) \text{ symplectic} \}.
\]

This is an open subset of \( F(\hat{U}) \), and therefore a symplectic submanifold of \( M \times N \) of dimension \( 2n + 2m \). Claim 3 and the fact \( F(\hat{N}) = \hat{N} \) imply that \( \hat{M} \) contains \( \hat{N} \). Hence \( \hat{M} \) has the required properties. This proves Lemma 3. \( \square \)

Let \( \tilde{M} \subseteq M \times N \) be a symplectic submanifold as in Lemma 3. The next lemma relates the Hamiltonian vector field of a function on \( M \) to its lift on \( \tilde{M} \). This will be used in the proof of Theorem 1 to relate the corresponding Hamiltonian flows.
We denote by $\iota_{\tilde{M}}$ the inclusion of $\tilde{M}$ into $M \times M$, and define

$$\tilde{\omega} := \iota_{\tilde{M}}^* \left( \omega \oplus (-\omega) \right).$$

This is a symplectic form on $\tilde{M}$. We denote by $X_H$ the Hamiltonian vector field generated by a smooth function $H$ on a symplectic manifold. Let $H \in \mathcal{C}^\infty(M, \mathbb{R})$. We denote

$$\tilde{H} := H \circ \text{pr}_1 : \tilde{M} \to \mathbb{R}.$$

**Lemma 4** (Hamiltonian vector field on symplectic submanifold $\tilde{M}$). We have

$$d\text{pr}_1 X_{\tilde{H}} = X_H \circ \text{pr}_1,$$

$$d\text{pr}_2 X_{\tilde{H}} \in T N^\omega. \quad (13)$$

**Proof of Lemma 4.** To see (12), observe that

$$\tilde{\omega}(X_{\tilde{H}}, \cdot) = d\tilde{H} = dHd\text{pr}_1 = \omega(X_H, d\text{pr}_1 \cdot).$$

Hence, for every $\tilde{x} = (x, y) \in \tilde{M}$ and $v \in T_x M$, we have

$$\omega(X_H(x), v) = \tilde{\omega}(X_{\tilde{H}}(\tilde{x}), (v, 0)) = \omega(d\text{pr}_1 X_{\tilde{H}}(\tilde{x}), v),$$

and therefore,

$$X_H(x) = d\text{pr}_1 X_{\tilde{H}}(\tilde{x}).$$

This proves (12).

We prove (13). Let $\tilde{x} = (x, y) \in \tilde{M}$. We define

$$H_y := \{ w \in T_y N \mid (0, w) \in T_{\tilde{x}} \tilde{M} \},$$

$$\tilde{V} := T_{\tilde{x}}(M \times N), \quad \tilde{\Omega} := \left( \omega \oplus (-\iota_{\tilde{N}}^* \omega) \right)_{\tilde{x}}.$$

Since $T_{\tilde{x}} \tilde{M}$ is a symplectic subspace of $\tilde{V}$ of maximal dimension $2n+2m$, we have

$$\tilde{V} = T_{\tilde{x}} \tilde{M} + \tilde{V}_{\tilde{\Omega}}.$$

Combining this with the fact $\tilde{V}_{\tilde{\Omega}} = \{0\} \times T_y N^\omega$, it follows that

$$(14) \quad T_y N \subseteq H_y + T_y N^\omega.$$ 

For every $w \in H_y$ we have

$$0 = dHd\text{pr}_1(\tilde{x})(0, w)$$

$$= d\tilde{H}(\tilde{x})(0, w)$$

$$= \tilde{\omega}(X_{\tilde{H}}(\tilde{x}), (0, w))$$

$$= -\omega(d\text{pr}_2 X_{\tilde{H}}(\tilde{x}), w).$$
Because of (14) it follows that \( \omega(d\text{pr}_2X_\tilde{H}(\tilde{x})), w) = 0 \), for every \( w \in T_yN \). This shows (13) and completes the proof of Lemma 4. □

The next lemma produces leafwise fixed points for a Hamiltonian diffeomorphism on \( M \) out of Lagrangian intersection points of \( \tilde{N} \) and its translate under the lifted Hamiltonian diffeomorphism.

For a time-dependent smooth function \( H \) on a symplectic manifold we denote by \( \phi_t^H \) its Hamiltonian time-\( t \)-flow. Let

\[ H \in C^\infty([0, 1] \times M, \mathbb{R}). \]

We denote \( H_t := H(t, \cdot) \) and define

\[ \tilde{H}_t := H_t \circ \text{pr}_1 : \tilde{M} \to \mathbb{R}, \quad \tilde{H} := (\tilde{H}_t)_{t \in [0, 1]} \cdot \]

Lemma 5 (Lagrangian intersection points and leafwise fixed points).

For every

\[ \tilde{x} \in \tilde{N} \cap (\varphi_H^1)^{-1}(\tilde{N}) \]

we have

\[ x := \text{pr}_1(\tilde{x}) \in \text{Fix}(\varphi_H^1, N). \]

**Remark.** Recall that the domain of the flow \( \varphi_H^1 \) is an open subset \( \tilde{U} \) of \( \tilde{M} \). Consequently, in (12) \( (\varphi_H^1)^{-1}(\tilde{N}) \) is a subset of \( \tilde{U} \).

**Proof of Lemma 5.** Since \( \tilde{x} \) lies in the domain of \( \varphi_H^1 \), equality (12) in Lemma 4 implies that for every \( t \in [0, 1] \), \( x \) lies in the domain of \( \varphi_H^1 \) and

\[ \varphi_H^1(x) = \text{pr}_1 \circ \varphi_H^1(\tilde{x}). \]

We denote

\[ y : [0, 1] \to N, \quad y(t) := \text{pr}_2 \circ \varphi_H^1(\tilde{x}). \]

Since \( \tilde{x} \in \tilde{N} \), we have

\[ y(0) = \text{pr}_2(\tilde{x}) = \text{pr}_1(\tilde{x}) = x \in N. \]

By condition (13) in Lemma 4 we have

\[ y(t) \in T_{y(t)}N^\omega, \quad \forall t \in [0, 1]. \]

It follows that \( y \) stays inside the isotropic leaf of \( x \). Equality (16) and our assumption that \( \tilde{x} \in (\varphi_H^1)^{-1}(\tilde{N}) \) imply that

\[ \varphi_H^1(x) = \text{pr}_1 \circ \varphi_H^1(\tilde{x}) = \text{pr}_2 \circ \varphi_H^1(\tilde{x}) = y(1). \]

Since \( y(1) \) lies in the isotropic leaf of \( x \), it follows that \( x \) is a leafwise fixed point for \( \varphi \). This proves Lemma 5. □
We are now ready for the proof of the main result.

Proof of Theorem. We choose a submanifold $\widetilde{M}$ as in Lemma. Shrinking $\widetilde{M}$ if necessary, by Weinstein’s neighbourhood theorem we may assume w.l.o.g. that there exists a symplectomorphism between $\widetilde{M}$ and an open neighbourhood of the zero section in $T^*\widetilde{N}$, that is the identity on $\widetilde{N}$.

In order to construct the desired $C^0$-neighbourhood of the inclusion $N \to M$, we need control over the “isotropic part” of the lifted Hamiltonian flow on $\widetilde{M}$. For this the idea is to work with local coordinates on $M \times N$, in which this part disappears. (It will in fact suffice to choose coordinates on $N$, but not on $M$.) The following claim will imply that our coordinates are well-defined.

Let $y \in N$. We choose an open neighbourhood $V = V_y \subseteq N$ of $y$ and a foliation chart $\varphi = \varphi_y : V \to \mathbb{R}^{2m} \times \mathbb{R}^k$, such that $\varphi(y) = 0$. We denote by

$$\pi : \mathbb{R}^{2m} \times \mathbb{R}^k \to \mathbb{R}^{2m}$$

the canonical projection onto the first factor. We define

$$\widetilde{\varphi} := \varphi_y : \widetilde{M} \cap (M \times V) \to M \times \mathbb{R}^{2m}, \quad \widetilde{x} := (y, y).$$

Claim 1. The derivative

$$\widetilde{\Phi} := d\widetilde{\varphi}(\widetilde{x}) : T_{\widetilde{x}}\widetilde{M} \to T_yM \times \mathbb{R}^{2m}$$

is invertible.

Proof of Claim. We denote

$$\Phi := d\pi d\varphi(y) : T_yN \to \mathbb{R}^{2m}.$$

We equip $\mathbb{R}^{2m}$ with the unique linear symplectic form that pulls back to $(\iota^*_N \omega)_y$ under $\Phi$. (Here we use that $\varphi$ is a foliation chart.) We have

$$\widetilde{\Phi} = \text{id} \times \Phi : V := T_{\widetilde{x}}\widetilde{M} \subseteq T_yM \times T_yN \to W := T_yM \times \mathbb{R}^{2m}.$$

$V$ and $W$ are symplectic vector spaces of dimension $2n + 2m$, and the map $\widetilde{\Phi}$ is symplectic. It follows that $\widetilde{\Phi}$ is an isomorphism. This proves Claim. 

Using the claim, by the Inverse Function Theorem there exist open neighbourhoods $U := U_y \subseteq M$ of $y$ and $W := W_y \subseteq \mathbb{R}^{2m}$ of $0$, such that

$$\widetilde{\varphi} : \widetilde{U} := \widetilde{U}_y := \widetilde{\varphi}^{-1}(U \times W) \to U \times W$$
is a diffeomorphism. (Note that
\[ \tilde{U} = (U \times (\pi \circ \varphi)^{-1}(W_1)) \cap \tilde{M}. \]
We choose a compact neighbourhood \( K_y \) of \( y \) that is contained in \( U \).
Since \( N \) is compact, there exists a finite subset \( S \subseteq N \) such that
\[ N \subseteq \bigcup_{y \in S} \overset{\circ}{K}_y. \]
(Here \( \overset{\circ}{K}_y \) denotes the interior of \( K_y \).) We define
\[ U := \{ \psi \in C(N, M) \mid \psi(N \cap K_y) \subseteq U_y, \forall y \in S \}. \]
This is a \( C^0 \)-neighbourhood of the inclusion \( \iota_N : N \to M \).

We show that \( U \) has the desired property. Let \( (\varphi^t)_{t \in [0,1]} \) be a Hamiltonian flow on \( M \) satisfying \( \varphi^t|_N \in U \), for every \( t \in [0,1] \). The next claim will be used to show that \( \tilde{N} \) intersects its translate under the lifted flow on \( \tilde{M} \).

We choose a function \( H \in C^\infty([0,1] \times M, \mathbb{R}) \) that generates \( (\varphi^t) \), and define
\[ \tilde{H}_t := H_t \circ \text{pr}_1 : \tilde{M} \to \mathbb{R}. \]

**Claim 2.** The domain of the Hamiltonian time-t-flow \( \varphi^t_{\tilde{H}} \) of \( \tilde{H} \) contains \( \tilde{N} \), for every \( t \in [0,1] \).

*Proof of Claim 2.* Let \( \tilde{x}_0 = (y_0, y_0) \in \tilde{N} \). We choose \( y' \in S \) such that \( y_0 \in K_{y'} \). Let \( \tilde{x} = (x, y) \in \tilde{U} := \tilde{U}_{y'} \). Since \( \varphi := \varphi_{y'} \) is a foliation chart, the derivative \( d(\pi \circ \varphi)(y) = d\pi(\varphi(y))d\varphi(y) \) vanishes on \( T_y N^\omega \). Hence condition (13) in Lemma 4 implies that
\[ d(\pi \circ \varphi \circ \text{pr}_2) X_{\tilde{H}_t}(\tilde{x}) = 0. \]
Using condition (12) in Lemma 4, it follows that
\[ d\tilde{\varphi}(\tilde{x}) X_{\tilde{H}_t}(\tilde{x}) = (X_{H_t}(x), 0). \]
(Here \( \tilde{\varphi} = \tilde{\varphi}_{y'} \) is defined as in (17).) Let \( t \in [0,1] \). Since \( y_0 \in K_{y'} \) and by assumption \( \varphi^t|_N \in U \), we have \( \varphi^t(y_0) \in U_{y'} \), and therefore, \( (\varphi^t(y_0), 0) \in \tilde{\varphi}(\tilde{U}_{y'}) \). Using equality (18), it follows that for every \( t \in [0,1] \), \( \tilde{x}_0 = \tilde{\varphi}_{\tilde{H}}^{-1}(y_0, 0) \) lies in the domain of \( \tilde{\varphi}_{\tilde{H}} \), and
\[ \tilde{\varphi} \circ \tilde{\varphi}_{\tilde{H}}^{-1}(\tilde{x}_0) = (\varphi^t_{\tilde{H}}(y_0), 0). \]
This proves Claim 2. \( \square \)
By Lemma 8 below there exists a function $\hat{H} \in C^\infty([0, 1] \times \widetilde{M}, \mathbb{R})$ with compact support, such that
\[
\varphi_t^{\hat{H}} = \varphi_t^H \text{ on } \widetilde{N}, \quad \forall t \in [0, 1].
\]
By Claim 2 this condition makes sense. By our assumption there exists an open neighbourhood $U' \subseteq T^*\widetilde{N}$ of the zero section and a symplectomorphism $\psi : U' \to \tilde{M}$ that equals the identity on $\widetilde{N}$. We define the function $H'_t : T^*\widetilde{N} \to \mathbb{R}$ to be equal to $\hat{H}_t \circ \psi$ on $U'$ and 0 outside $U'$.

It follows from [Gr, p. 331, 2.3.B′′ 4 Theorem] or [Ho1, Theorem 2] that the zero section $\widetilde{N}$ intersects $(\varphi_1^{H'})^{-1}(\widetilde{N}) \subseteq T^*\widetilde{N}$. It follows that
\[
\emptyset \neq \psi(\widetilde{N} \cap (\varphi_1^{H'})^{-1}(\widetilde{N})) = \widetilde{N} \cap (\varphi_1^{H'})^{-1}(\widetilde{N}) = \widetilde{N} \cap (\varphi_1^{\hat{H}})^{-1}(\widetilde{N}).
\]

Hence Lemma 8 implies $\text{Fix}(\varphi_1^{H}, N) = \emptyset$. This proves Theorem 1. □

**Remark 6** (method of proof of Theorem 1). The method of proof of Theorem 1 refines the technique used in the proof of [Zi1, Theorem 1.1] in the following sense. Assume that $N$ is regular (i.e., “fibering”) in the sense that there exists a manifold structure on the set $N_\omega$ of isotropic leaves of $N$, such that the canonical projection $\pi_N : N \to N_\omega$ is a smooth submersion. We denote by $\omega_N$ the unique symplectic form on $N_\omega$ that pulls back to $\iota_N^*\omega$ under $\pi_N$. We equip the product $\tilde{M} := M \times N_\omega$ with the symplectic form $\tilde{\omega} := \omega \oplus (-\omega_N)$.

In [Zi1] the symplectic manifold $(\tilde{M}, \tilde{\omega})$ was used to prove a lower bound on $|\text{Fix}(\varphi, N)|$ for a regular coisotropic submanifold $N$. On the other hand, the proof of Theorem 1 is based on the construction of a certain symplectic submanifold $\tilde{M} \subseteq M \times N$ (see Lemma 8), which can be viewed as a local version of $(\tilde{M}, \tilde{\omega})$. More precisely, if $N$ is regular then $\tilde{M}$ can be symplectically embedded into $\tilde{M}$ via the map
\[
(x, y) \mapsto (x, N_y),
\]
where $N_y$ denotes the isotropic leaf of $N$ through $y$.

**Appendix A. Auxiliary results**

In the proof of Lemma 8 we used the following.

**Lemma 7** (local embedding). Let $M$ and $M'$ be manifolds (without boundary), $K \subseteq M$ a compact subset, and $f : M \to M'$ a smooth map whose restriction to $K$ is injective, such that $df(x)$ is injective for every $x \in K$. Then there exists an open neighbourhood $U \subseteq M$ of $K$, such that $f|_U$ is a smooth embedding.
Proof of Lemma 7. We show that there exists an open neighbourhood $U_0$ of $K$ on which $f$ is injective. Since $f$ is continuous and $f|_K$ is injective, the set

$$S := \{(x, y) \in M \times M \mid x = y \text{ or } f(x) \neq f(y)\}$$

is a (possibly non-open) neighbourhood of

$$\{ (x, y) \in K \times K \mid x \neq y \}$$

in $M \times M$. By the Immersion Theorem every point in $K$ admits an open neighbourhood in $M$ on which $f$ is injective. It follows that $S$ is a neighbourhood of

$$\{(x, x) \mid x \in K\},$$

and therefore of $K \times K$.

Claim 1. There exists an open neighbourhood $U_0 \subseteq M$ of $K$ such that $U_0 \times U_0 \subseteq S$.

Proof of Claim 1. We choose a distance function $d$ on $M$ that induces the topology. We define the distance function $\tilde{d}$ on $\tilde{M} := M \times M$ by

$$\tilde{d}((x_1, y_1), (x_2, y_2)) := d(x_1, y_1) + d(x_2, y_2).$$

Since $\tilde{K} := K \times K$ is compact, there exists a constant $\varepsilon > 0$, such that the closed $\varepsilon$-neighbourhood of $\tilde{K}$,

$$\tilde{K}_\varepsilon := \{ \tilde{x} \in \tilde{M} \mid \exists \tilde{y} \in \tilde{K} : \tilde{d}(\tilde{x}, \tilde{y}) \leq \varepsilon \}$$

is compact. The same then holds for $\tilde{K}_\varepsilon \setminus \tilde{S}$. Hence $\tilde{d}$ attains its minimum on $\tilde{K} \times (\tilde{K}_\varepsilon \setminus \tilde{S})$. It follows that the $\tilde{d}$-distance between $\tilde{K}$ and $\tilde{M} \setminus \tilde{S}$ is positive. We denote this distance by $\varepsilon_0$ and define

$$U_0 := \left\{ x \in M \mid \exists y \in K : d(x, y) < \frac{\varepsilon_0}{2} \right\}.$$ 

This set has the desired properties. This proves Claim 1. \Box

We choose $U_0$ as in this claim. The restriction $f|_{U_0}$ is injective. The set

$$U_1 := \{ x \in M \mid df(x) \text{ injective} \}$$

is open and contains $K$. We choose an open subset $U \subseteq M$ whose closure is compact and contained in $U_0 \cap U_1$. The restriction of $f$ to $U$ is proper onto its image. It follows that this restriction is a smooth embedding. This proves Lemma 7. \Box

In the proof of Theorem 1 we used the following.
Lemma 8 (Hamiltonian flow). Let \((M, \omega)\) be a symplectic manifold (without boundary), \(H \in C^\infty([0,1] \times M, \mathbb{R})\), and \(K \subseteq M\) a compact subset that is contained in the domain of the Hamiltonian time-t-flow of \(H\), for every \(t \in [0,1]\). Then there exists a function
\[
H' \in C^\infty([0,1] \times M, \mathbb{R})
\]
with compact support, such that
\[
\varphi^t_H' = \varphi^t_H \text{ on } K, \quad \forall t \in [0,1].
\]

Proof of Lemma 8. This follows from a cut-off argument as in the proof of [SZ] Lemma 35. \(\square\)

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