On a Deformation of $sl(2)$ with Paragrassmannian Variables

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Abstract

We propose a new structure $\mathcal{U}_q^r(sl(2))$. This is realized by multiplying $\delta$ ($q = e^\delta$, $\delta \in \mathbb{C}$) by $\theta$, where $\theta$ is a real nilpotent -paragrassmannian- variable of order $r$ ($\theta^{r+1} = 0$) that we call the order of deformation, the limit $r \to \infty$ giving back the standard $\mathcal{U}_q(sl(2))$. In particular we show that, for $r = 1$, there exists a new $R$-matrix associated with $sl(2)$. We also proof that the restriction of the values of the parameters of deformation give nonlinear algebras as particular cases.
1 Introduction

During the last few years, $q$-deformations [1] ($q = e^\delta$, $\delta \in \mathbb{C}$) of the universal enveloping algebra of Lie algebras have attracted a wide attention. They are indeed remarkable mathematical structures known as Hopf algebras and they have been proved to be connected to Conformal Field Theory, in particular, as they have been figuring in 2d-solvable model $S$-matrices and solutions to their Yang-Baxter factorization equations (See Ref. [2] and references therein).

The pioneering papers [3] devoted to the specific $\mathcal{U}_q(sl(2))$ case have been extended by various authors. Let us just mention here the Roček proposal [4] (based on generalized nonlinear deformations) providing a new algebraic description of the Morse and modified P¨oschl-Teller Hamiltonians [5]. Despite of its physical interest, the Roček deformation has been rarely exploited, compared to the Drinfeld-Jimbo one, because of its mathematical defect: its Hopf characteristics (coproduct, counit, antipode) have not yet been pointed out.

In this letter, we answer the following question: Is it possible to obtain the nonlinear algebras as particular restrictions of the quantum deformation?

Our purpose is then twofold. First, we introduce the nilotent algebra $\mathcal{U}_q^\theta(sl(2))$ by multiplying $\delta$ by $\theta$, where $\theta$ is a real nilotent -paragrassmannian- variable [6] of order $r$ ($\theta^{r+1} = 0$). Second, we discuss the connection of this new structure to some particular nonlinear deformations of $sl(2)$ whose Hopf characteristics are introduced.

In section 2, we briefly review the Drinfeld-Jimbo deformation of $sl(2)$. Then, in section 3, we introduce the quantization with one paragrassmannian variable and its Hopf structure. The quantization with two paragrassmannian variables is given in section 4. In section 5, we give the connection of these structures to particular nonlinear deformations of $sl(2)$. Finally, we conclude in section 6 with some comments.

2 The $\mathcal{U}_q(sl(2))$ algebra.

The standard Drinfeld-Jimbo deformation [1] of the Lie algebra $sl(2)$ generated by $H$, $J_+$, $J_-$ is characterized by the relations

\[
[ J_+, J_- ] = \frac{q^H - q^{-H}}{q - q^{-1}} = \frac{\sinh(\delta H)}{\sinh(\delta)},
\]

\[
[ H, J_{\pm} ] = \pm 2J_{\pm}.
\]

(2.1)

It is completed by the additional operations, coproduct $\triangle : \mathcal{U}_q(sl(2)) \to \mathcal{U}_q(sl(2)) \otimes \mathcal{U}_q(sl(2))$, counit $\varepsilon : \mathcal{U}_q(sl(2)) \to \mathbb{C}$ and the antipode $S : \mathcal{U}_q(sl(2)) \to \mathcal{U}_q(sl(2))$ such that

\[
\triangle(H) = H \otimes 1 + 1 \otimes H,
\]
\[ \Delta(J_\pm) = J_+ \otimes e^{\delta H/2} + e^{-\delta H/2} \otimes J_\pm, \]
\[ \varepsilon(1) = 1, \quad \varepsilon(J_\pm) = \varepsilon(H) = 0, \]
\[ S(1) = 1, \quad S(H) = -H, \quad S(J_\pm) = -e^{\pm \delta} J_\pm \quad (2.2) \]

where \( \Delta \) and \( \varepsilon \) are homomorphisms while \( S \) is an algebra antihomomorphism

\[ \Delta(a \ b) = \Delta(a) \ \Delta(b), \]
\[ \varepsilon(a \ b) = \varepsilon(a) \varepsilon(b), \]
\[ S(a \ b) = S(b) S(a). \quad (2.3) \]

Moreover, if \( m : U_q(sl(2)) \otimes U_q(sl(2)) \to U_q(sl(2)) \) stands for the multiplication mapping of \( U_q(sl(2)) \) i.e. \( m(a \otimes b) = a.b \), we have

\[ (id \otimes \Delta) \Delta = (\Delta \otimes id) \Delta, \]
\[ m(id \otimes S) \Delta = m(S \otimes id) \Delta = i \circ \varepsilon, \]
\[ (\varepsilon \otimes id) \Delta = (id \otimes \varepsilon) \Delta = id. \quad (2.4) \]

These are just all the axioms of a Hopf algebra, and so \( U_q(sl(2)) \) endowed with \( \varepsilon, \Delta \) and \( S \) just forms a Hopf algebra.

Let us define the formal series

\[ J_\pm = \sum_{k=0}^{\infty} \delta^k J_\pm^{(k)} \quad (2.5) \]

and

\[ \frac{\sinh(H\delta)}{\sinh(\delta)} = \sum_{k=0}^{\infty} \psi_k(H) \delta^{2k}, \quad (2.6) \]

the second formula being just the result of a Taylor expansion. The generators \( J_\pm^{(k)} \) and \( H \) satisfy the following commutation relations

\[ [H, \ J_\pm^{(k)}] = \pm 2 \ J_\pm^{(k)}, \]
\[ \sum_{m=0}^{2k} [J_+^{(m)}, \ J_-^{(2k-m)}] = \psi_k(H), \quad \sum_{m=0}^{2k+1} [J_+^{(m)}, \ J_-^{(2k+1-m)}] = 0, \]
\[ \sum_{m=0}^{k} [J_+^{(m)}, \ J_-^{(k-m)}] = 0. \quad (2.7) \]

Its Hopf structure is given by

\[ \Delta(H) = 1 \otimes H + H \otimes 1, \]
\[ \triangle(J_{\pm}^{(k)}) = \sum_{m=0}^{k} \frac{1}{2^m m!} ((-1)^m H^m \otimes J_{\pm}^{(k-m)} + J_{\pm}^{(k-m)} \otimes H^m), \]

\[ \varepsilon(H) = \varepsilon(J_{\pm}^{(k)}) = 0, \quad \varepsilon(1) = 1, \]

\[ S(J_{\pm}^{(k)}) = - \sum_{m=0}^{k} \frac{(\pm)^m}{m!} J_{\pm}^{(k-m)}, \quad S(H) = -H, \quad S(1) = 1, \quad (2.8) \]
as it can be verified.

### 3 The U_q(sl(2)) algebra

Let us introduce the real nilpotent -paragrassmannian- variable \( \theta \) of order \( r \), i.e.

\[ \theta^{r+1} = 0, \quad (3.1) \]

being realized, in a simple way, by

\[ \theta = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (3.2) \]

Besides this choice, we want to notice that there are other representations such as that given by

\[ \theta = \sum_{\alpha=1}^{r} \theta^{(\alpha)}, \quad (3.3) \]

where

\[ (\theta^{(\alpha)})^2 = 0, \quad [\theta^{(\alpha)}, \theta^{(\beta)}] = 0, \quad \alpha \neq \beta. \quad (3.4) \]

Then with Eq. (3.2), we propose to generalize the operators (2.5) through

\[ J_{\pm}^{\theta} = \sum_{m=0}^{r} \delta^m \theta^m J_{\pm}^{(m)} \quad (3.5) \]

\[ = \begin{pmatrix} J_{\pm}^{(0)} & 0 & \cdots & 0 \\ \delta J_{\pm}^{(1)} & \delta J_{\pm}^{(0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{r-1} J_{\pm}^{(r-1)} & \delta^{r-1} J_{\pm}^{(r-1)} & \cdots & \delta J_{\pm}^{(1)} \end{pmatrix}. \quad (3.6) \]
Using the commutations relations (2.7), we thus have
\[ [H, J^\theta_{\pm}] = \pm 2 J^\theta_{\pm} \]  
(3.7)
and
\[ [J^\theta_+, J^\theta_-] = \sum_{k=0}^r \delta^k \theta^k \left( \sum_{m=0}^k [J^\theta_+^{(m)} + J^\theta_-^{(k-m)}] \right) \]
\[ = \psi_0(H) + \theta^2 \delta^2 \psi_1(H) + \cdots + \theta^{r/2} \delta^{r/2} \psi_{r/2}(H) \]
\[ = \sum_{k=0}^{[r/2]} \psi_k(H) \theta^{2k} \delta^{2k}, \]
(3.8)
where \([\lambda]\) stands for the integer part of \(\lambda\). Defining the exponential map by
\[ e(x; \theta) = \sum_{k=0}^r \frac{x^k \theta^k}{k!}, \]
(3.9)
we can finally write
\[ [J^\theta_+, J^\theta_-] = \frac{e(H\delta; \theta) - e(-H\delta; \theta)}{e(\delta; \theta) - e(-\delta; \theta)}, \]
\[ [H, J^\theta_{\pm}] = \pm 2 J^\theta_{\pm}. \]
(3.10)

The algebra \(\{ J^\theta_{\pm}, H \}\) described by the commutation relations (3.10) is just the deformation of \(sl(2)\) with one paragrassmannian variable and is denoted by \(U^q_r(sl(2))\). This algebra is isomorphic to \(U^q(sl(2))/\langle \delta^{r+1} U^q(sl(2)) \rangle\), i.e.
\[ U^q_r(sl(2)) \cong U^q(sl(2))/\langle \delta^{r+1} U^q(sl(2)) \rangle. \]

In order to define a Hopf structure for \(U^q_r(sl(2))\), we need the following definition

**Definition 1** Let
\[ a = a_0 + a_1 \theta + \cdots + a_r \theta^r, \quad b = b_0 + b_1 \theta + \cdots + b_r \theta^r, \]
(3.11)
the tensor product between \(a\) and \(b\) is defined by
\[ a \tilde{\otimes} b = \sum_{m=1}^r \sum_{n=1}^r a^{(m)} \otimes b^{(n)} \theta^{m+n}, \]
(3.12)
and
\[ (a \tilde{\otimes} b)(c \tilde{\otimes} d) = (ac \tilde{\otimes} bd). \]
(3.13)
This operation is called the paragrassmannian tensor product.
When the paragrassmannian order $r \to \infty$, this operation is equivalent to the standard one. This paragrassmannian tensor product is compatible with

$$\mathcal{U}_q^r(sl(2)) \otimes \mathcal{U}_q^r(sl(2)) \equiv \mathcal{U}_q^r(so(4))$$ (3.14)

and with the inclusion

$$\mathcal{U}_q^r(sl(2)) \subset \mathcal{U}_q^r(sl(3)) \subset \cdots \mathcal{U}_q^r(sl(N-1)) \subset \mathcal{U}_q^r(sl(N)).$$ (3.15)

We are now able to claim that

**Proposition 1** The Hopf structure associated to the $\mathcal{U}_q^r(sl(2))$ is given by

$$\Delta(H) = H \otimes 1 + 1 \otimes H,$$

$$\Delta(J^\theta_{\pm}) = J^\theta_{\pm} \otimes e\left(\frac{H\delta}{2}; \theta\right) + e\left(-\frac{H\delta}{2}; \theta\right) \otimes J^\theta_{\pm},$$

$$\varepsilon(J^\theta_{\pm}) = \varepsilon(H) = 0, \quad \varepsilon(1) = 1,$$

$$S(H) = -H, \quad S(J^\theta_{\pm}) = -e\left(\pm\delta; \theta\right) J^\theta_{\pm}, \quad S(1) = 1,$$

$$\Delta(e\left(\frac{H\delta}{2}; \theta\right)) = e\left(\frac{H\delta}{2}; \theta\right) \otimes e\left(\frac{H\delta}{2}; \theta\right).$$ (3.16)

The following axioms are then satisfied

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta,$$

$$m(id \otimes S)\Delta = m(S \otimes id)\Delta = i \circ \varepsilon,$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id,$$ (3.17)

with the coproduct $\Delta : \mathcal{U}_q^r(sl(2)) \to \mathcal{U}_q^r(sl(2)) \otimes \mathcal{U}_q^r(sl(2))$, counit $\varepsilon : \mathcal{U}_q^r(sl(2)) \to \mathbb{C}[\theta]$, the antipode $S : \mathcal{U}_q^r(sl(2)) \to \mathcal{U}_q^r(sl(2))$ and $m : \mathcal{U}_q^r(sl(2)) \otimes \mathcal{U}_q^r(sl(2)) \to \mathcal{U}_q^r(sl(2))$, where the operations $\Delta$, $S$ and $\varepsilon$ only act on $H$ and $J^\theta_{\pm}$.

Let us now turn to some specific examples.

**Example. 1.** The $r = 0$ case is characterized by

$$\theta = 0, \quad J^\theta_{\pm} = J^{(0)}_{\pm}$$

and

$$[H, J^\theta_{\pm}] = \pm 2J^\theta_{\pm},$$

$$[J^\theta_{\pm}, J^\theta_{\pm}] = H.$$ (3.18)

Thus, the $\mathcal{U}_q^0(sl(2))$ algebra is nothing but $sl(2)$, endowed as usual with

$$\Delta(H) = 1 \otimes H + H \otimes 1,$$

$$\Delta(J^\theta_{\pm}) = J^\theta_{\pm} \otimes 1 + 1 \otimes J^\theta_{\pm},$$ etc.
**Example. 2.** The \( r = 1 \) case is characterized by

\[
\theta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad J^\theta_\pm = \begin{pmatrix} J^{(0)}_\pm & 0 \\ \delta J^{(1)}_\pm & J^{(0)}_\pm \end{pmatrix}
\]

and the \( sl(2) \) algebra (3.18) but now supplemented by a non cocommutative coproduct

\[
\Delta(H) = H \otimes 1 + 1 \otimes H,
\]

\[
\Delta(J^\theta_\pm) = J^\theta_\pm \otimes (1 + \frac{\theta \delta}{2} H) + (1 - \frac{\theta \delta}{2} H) \otimes J^\theta_\pm.
\] (3.19)

**Example. 3.** When \( r = 2 \), i.e.

\[
\theta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad J^\theta_\pm = \begin{pmatrix} J^{(0)}_\pm & 0 & 0 \\ \delta J^{(1)}_\pm & J^{(0)}_\pm & 0 \\ \delta^2 J^{(2)}_\pm & \delta J^{(1)}_\pm & J^{(0)}_\pm \end{pmatrix},
\]

we obtain

\[
[H, J^\theta_\pm] = \pm 2J^\theta_\pm,
\]

\[
[J^\theta_+, J^\theta_-] = H + \theta^2 \frac{\delta^3}{3!} (H^3 - H).
\] (3.20)

The coproduct is given by

\[
\Delta(H) = H \otimes 1 + 1 \otimes H,
\] (3.21)

\[
\Delta(J^\theta_\pm) = J^\theta_\pm \otimes (1 + \frac{\theta \delta}{2} H + \frac{(\theta \delta)^2}{8} H^2) + (1 - \frac{\theta \delta}{2} H + \frac{(\theta \delta)^2}{8} H^2) \otimes J^\theta_\pm.
\]

Such a structure is discussed in [7] in connection with the Higgs algebra, characterized by

\[
[H, J_\pm] = \pm 2J_\pm,
\]

\[
[J_+, J_-] = H + c H^3,
\] (3.22)

c being an arbitrary constant. This algebra is of special interest as it is the one of dynamical symmetries for the harmonic oscillator and the Kepler problem in a two-dimensional curved space [8].

**Example. 4.** The \( r \to \infty \) case (\( \theta \) is equivalent to a real variable) is characterized by

\[
J^\theta_\pm = \sum_{m=0}^{\infty} J^{(m)}_\pm \delta^m \theta^m
\]

\[
= \sum_{m=0}^{\infty} J^{(m)}_\pm \zeta^m
\]

\[
J^\theta_\pm := \tilde{J}_\pm
\] (3.23)
and
\[
[H, \tilde{J}_\pm] = \pm 2\tilde{J}_\pm,
\]
\[
[\tilde{J}_+, \tilde{J}_-] = \frac{e^{\zeta H} - e^{-\zeta H}}{e^\zeta - e^{-\zeta}},
\]
(3.24)
where \(\zeta = \theta \delta\). We thus recover the Drinfeld-Jimbo structure \(\mathcal{U}_{\zeta}(sl(2))\) as a particular case of \(\mathcal{U}^\infty_q(sl(2))\).

The same embedding is also present at the level of the Hopf structure with
\[
\triangle(H) = H \otimes 1 + 1 \otimes H,
\]
\[
\triangle(\tilde{J}_\pm) = \tilde{J}_\pm \otimes e^{\zeta H/2} + e^{-\zeta H/2} \otimes \tilde{J}_\pm,
\]
\[
\varepsilon(\tilde{J}_\pm) = \varepsilon(H) = 0,
\]
\[
S(H) = -H, \quad S(\tilde{J}_\pm) = -e^{\pm \zeta} \tilde{J}_\pm.
\]
(3.25)

4 The \(\mathcal{U}_{q_1,q_2}^{r_1,r_2}(sl(2))\) algebra

Let us now introduce, for example, two real paragraphmannian variables \(\theta_1\) and \(\theta_2\) respectively of order \(r_1\) and \(r_2\), i.e.
\[
\theta_1^{r_1+1} = 0, \quad \theta_2^{r_2+1} = 0,
\]
\[
\theta_1 \theta_2 + \theta_2 \theta_1 = 0.
\]
(4.1)

Using the Campbell-Baker-Hausdorff expansion
\[
(\exp A)(\exp B) = \exp C,
\]
(4.2)

where
\[
C = A + B + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{(m+1)!} (ad A)^m(B) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{(m+1)!} (ad B)^m(A),
\]
\[
(ad A)^m(B) = [A, [A, \ldots, [A, B]\ldots]],
\]
\[
(ad B)^m(A) = [B, [B, \ldots, [B, A]\ldots]],
\]
(4.3)

we propose to define
\[
J_{\pm}^{(\theta_1, \theta_2)} = \sum_{m=0}^{\infty} \theta^m J_{\pm}^{(m)}
\]
(4.4)

where
\[
\theta = \theta_1 \delta_1 + \theta_2 \delta_2 + \frac{1}{2} \sum_{m=1}^{r_1} \frac{\delta_1 \delta_2^m}{(m+1)!} \theta_1^m \theta_2 + \frac{1}{2} \sum_{m=1}^{r_2} \frac{\delta_1^m \delta_2^m}{(m+1)!} \theta_1^m \theta_2
\]
\[
\exp(\theta_1 \delta_1) \exp(\theta_2 \delta_2) = \exp \theta.
\]
(4.5)
Using (4.4), we deduce that

\[
\left[ J_{\pm}^{(\theta_1, \theta_2)}, J_{\mp}^{(\theta_1, \theta_2)} \right] = \frac{e(H\delta_1; \theta_1)e(H\delta_2; \theta_2) - e(-H\delta_2; \theta_2)e(-H\delta_1; \theta_1)}{e(\delta_1; \theta_1)e(\delta_2; \theta_2) - e(-\delta_2; \theta_2)e(-\delta_1; \theta_1)},
\]

\[
[H, J_{\pm}^{(\theta_1, \theta_2)}] = \pm 2J_{\pm}^{(\theta_1, \theta_2)},
\]

(4.6)

The algebra \( \{ J_{\pm}^{(\theta_1, \theta_2)}, H \} \) described by the commutation relations (4.6) is just the quantization of \( sl(2) \) with two paragrassmannian variables and is denoted by \( \mathcal{U}_{\theta_1, \theta_2}^{(\delta_1, \delta_2)}(sl(2)) \).

The \( \mathcal{U}_{\theta_1, \theta_2}^{(\delta_1, \delta_2)}(sl(2)) \) algebra is equipped with the following Hopf structure

\[
\Delta(H) = H \otimes 1 + 1 \otimes H,
\]

\[
\Delta(J_{\pm}^{(\theta_1, \theta_2)}) = J_{\pm}^{(\theta_1, \theta_2)} \otimes e \left( \frac{H\delta_1}{2}; \theta_1 \right) e \left( \frac{H\delta_2}{2}; \theta_2 \right) + e \left( -\frac{H\delta_2}{2}; \theta_2 \right) e \left( -\frac{H\delta_1}{2}; \theta_1 \right) \otimes J_{\pm}^{(\theta_1, \theta_2)},
\]

\[
\varepsilon(J_{\pm}^{\theta}) = \varepsilon(H) = 0, \quad \varepsilon(1) = 1,
\]

\[
S(H) = -H, \quad S(1) = 1,
\]

\[
S(J_{\pm}^{(\theta_1, \theta_2)}) = -e \left( \frac{H\delta_1}{2}; \theta_1 \right) e \left( \frac{H\delta_2}{2}; \theta_2 \right) J_{\pm}^{(\theta_1, \theta_2)} e \left( -\frac{H\delta_2}{2}; \theta_2 \right) e \left( -\frac{H\delta_1}{2}; \theta_1 \right),
\]

\[
\Delta(e \left( \frac{H\delta_1}{2}; \theta_1 \right) e \left( \frac{H\delta_2}{2}; \theta_2 \right)) = e \left( \frac{H\delta_1}{2}; \theta_1 \right) e \left( \frac{H\delta_2}{2}; \theta_2 \right) \otimes e \left( \frac{H\delta_1}{2}; \theta_1 \right) e \left( \frac{H\delta_2}{2}; \theta_2 \right).
\]

(4.7)

5 Connection with some Nonlinear Algebras

Let us take in \( \mathcal{U}_{q}^{\theta}(sl(2)) \) the following restriction

\[
q = 1 \quad \text{i.e.} \quad q = e^{2\pi i n},
\]

(5.1)

where \( n \) characterizes the Riemann branch. We have

\[
e(2\pi i n; \theta) = \cos(2\pi n; \theta) + i \sin(2\pi n; \theta),
\]

(5.2)

with

\[
\cos(x; \theta) = \sum_{k=0}^{[r/2]} (-1)^k x^{2k} \theta^{2k} \frac{(2k)!}{(r-1)/2-\frac{1}{2}(1+(-1)^r)},
\]

\[
\sin(x; \theta) = \sum_{k=0}^{(r-1)/2-\frac{1}{2}(1+(-1)^r)} (-1)^k x^{2k+1} \theta^{2k+1} \frac{(2k+1)!}{(2k+1)!}.
\]

(5.3)

Thus, the commutation relations are written as

\[
[H, J_{\pm}^{\theta}] = \pm 2J_{\pm}^{\theta},
\]

\[
[J_{\pm}^{\theta}, J_{\mp}^{\theta}] = \frac{\sin(2\pi n H; \theta)}{\sin(2\pi n; \theta)}.
\]

(5.4)
When \( n \to \infty \), we deduce
\[
\lim_{n \to \infty} \frac{\sin(2\pi n H; \theta)}{\sin(2\pi n; \theta)} = H^{r-\frac{1}{2}(1+(-1)^r)}
\] (5.5)

and
\[
[H, J^0_\pm] = \pm 2J^0_\pm,
[J^0_+, J^0_-] = H^{r-\frac{1}{2}(1+(-1)^r)},
\] (5.6)

the deformation is being a nonlinear one.

Now, if we take in \( U_{r_1,r_2}^{q_1,q_2}(sl(2)) \) \( r_2 \to \infty \) and \( \delta_1 = 2\pi in \) \( n \to \infty \), we deduce the following nonlinear algebra
\[
[J^0_+, J^-] = \frac{H^{r_1}q^H - (-1)^{r_1}q^{-H}H^{r_1}}{q - (-1)^{r_1}q^{-1}},
[H, J^0_\pm] = \pm J^0_\pm.
\] (5.7)

6 Conclusion

We have proposed new deformed structures \( U_r^q(sl(2)) \) and \( U_{r_1,r_2}^{q_1,q_2}(sl(2)) \) obtained by paragrassmannian deformation. When the order of the paragrassmannian variable goes to infinity, we recover the Drinfeld-Jimbo scheme of deformation.

It has also to be noticed that our proposal points out two different Hopf structures for the same deformed algebra. In particular, \( sl(2) \) can be associated with a cocommutative coproduct \( (r = 0) \) or a non-cocommutative one \( (r = 1) \). Then it is possible to get a new \( \mathcal{R} \)-matrix given by
\[
\mathcal{R}_\theta = 1 \otimes 1 + \delta \theta (J_- \otimes J_+ - J_+ \otimes J_-)
= U_\theta U_\theta^+,
\] (6.1)

where
\[
U_\theta = 1 \otimes 1 + \frac{1}{2} \delta \theta (J_- \otimes J_+ - J_+ \otimes J_-)
\] (6.2)

by requiring
\[
U_\theta \triangle_{r=0} (a) = \triangle_{r=1}(a)U_\theta,
\] (6.3)

for any \( a \) belonging to \( sl(2) \). It has also to be noticed that this matrix \( \mathcal{R}_\theta \) satisfies the Yang-Baxter equation. Thus it is the first solution, to our knowledge, depending on a paragrassmannian variable.
We would like to mention that the \( r = 2 \)-case is a particularly interesting one as already mentioned. It is the first case where the deformation is present at the level of the algebra and these deformations are nonlinear ones in the sense of Roček. We have thus defined ad-hoc coproducts, counits and antipodes for such deformations being of physical interest.

Finally, the restriction of the values of the parameters of the deformation gives some nonlinear algebras as particular cases.

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