SOME NEW INTEGRAL INEQUALITIES FOR TWICE DIFFERENTIABLE CONVEX MAPPINGS

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Abstract. In this paper, we establish several new inequalities for some twice differentiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality. Some applications for special means of real numbers are also provided.

1. Introduction

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [8]):

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}
\]

where \( f : I \subset \mathbb{R} \to \mathbb{R} \) is a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). A function \( f : [a, b] \subset \mathbb{R} \to \mathbb{R} \) is said to be convex if whenever \( x, y \in [a, b] \) and \( t \in [0, 1] \), the following inequality holds

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).
\]

This definition has its origins in Jensen’s results from [3] and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them. We say that \( f \) is concave if \((-f)\) is convex.

A largely applied inequality for convex functions, due to its geometrical significance, is Hadamard’s inequality, (see [1],[2], [4] -[9]) which has generated a wide range of directions for extension and a rich mathematical literature.

In [1], Dragomir and Agarwal established the following results connected with the right part of (1.1) as well as to apply them for some elementary inequalities for real numbers and numerical integration:

**Theorem 1.** Let \( f : I^o \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \), \( a, b \in I^o \) with \( a < b \), and \( f' \in L(a, b) \). If the mapping \( |f'| \) is convex on \([a, b]\), then the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a) \left( \frac{|f'(a)| + |f'(b)|}{8} \right).
\]

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In [7], Pearce and Pečarić proved the following theorem:

**Theorem 2.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \), \( a, b \in I^o \) with \( a < b \). If the mapping \( |f'|^q \) is convex on \([a, b]\) for some \( q \geq 1 \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{b - a}{4} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}
\]

and

\[
\left| \frac{1}{b - a} \int_a^b f(x) dx - f\left( \frac{a + b}{2} \right) \right| \leq \frac{b - a}{8} \left( |f'(a)| + |f'(b)| \right).
\]

Also, in [4], Kırmacı obtained the following inequality for differentiable mappings which are connected with Hermite-Hadamard’s inequality:

**Theorem 3.** Let \( f : I^o \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \), \( a, b \in I^o \) with \( a < b \). If the mapping \( |f'|^q \) is convex on \([a, b]\), then we have

\[
\left| \frac{1}{b - a} \int_a^b f(x) dx - f\left( \frac{a + b}{2} \right) \right| \leq \frac{b - a}{8} \left( |f'(a)| + |f'(b)| \right).
\]

In [9], Sarikaya et. al. established inequalities for twice differentiable convex mappings which are connected with Hadamard’s inequality, and they used the following lemma to prove their results:

**Lemma 1.** Let \( f : I^o \subset \mathbb{R} \to \mathbb{R} \) be twice differentiable function on \( I^o \), \( a, b \in I^o \) with \( a < b \). If \( f'' \in L_1[a, b] \), then

\[
\frac{1}{b - a} \int_a^b f(x) dx - f\left( \frac{a + b}{2} \right) = \frac{(b - a)^2}{2} \int_0^1 m(t) \left[ f''(ta + (1 - t)b) + f''(tb + (1 - t)a) \right] dt,
\]

where

\[
m(t) := \begin{cases} 
  t^2, & t \in [0, \frac{1}{2}] \\
  (1 - t)^2, & t \in [\frac{1}{2}, 1].
\end{cases}
\]

Also, the main inequalities in [9], pointed out as follows:

**Theorem 4.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be twice differentiable function on \( I^o \) with \( f'' \in L_1[a, b] \). If \( |f''| \) is convex on \([a, b]\), then

\[
\left| \frac{1}{b - a} \int_a^b f(x) dx - f\left( \frac{a + b}{2} \right) \right| \leq \frac{(b - a)^2}{24} \left[ |f''(a)| + |f''(b)| \right].
\]

**Theorem 5.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be twice differentiable function on \( I^o \) such that \( f'' \in L_1[a, b] \) where \( a, b \in I \), \( a < b \). If \( |f''|^q \) is convex on \([a, b]\), \( q > 1 \), then

\[
\left| \frac{1}{b - a} \int_a^b f(x) dx - f\left( \frac{a + b}{2} \right) \right| \leq \frac{(b - a)^2}{8 (2p + 1)^{1/p}} \left[ |f''(a)|^q + |f''(b)|^q \right]^{1/q}
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).
Theorem 6. Let \( f : I \subset [0, \infty) \to \mathbb{R} \) be twice differentiable mapping on \( I^0 \) such that \( f'' \in L_1[a, b] \) where \( a, b \in I \) with \( a < b \). If \( |f''| \) is \( s \)-convex on \([a, b]\) for some fixed \( s \in (0, 1] \) and \( q \geq 1 \), then the following inequality holds:

\[
(1.8) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{(b - a)^2}{2.67} \left[ \frac{|f''(a)|^q + |f''(b)|^q}{(s + 2)(s + 3)} \right]^\frac{1}{q}
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Remark 1. If we take \( s = 1 \) in (1.8), then we have

\[
(1.9) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{(b - a)^2}{12} \left[ \frac{|f''(a)|^q + |f''(b)|^q}{2} \right]^\frac{1}{q}.
\]

In [6], Kirmanci proved the generalization identity connected with Hermite-Hadamard integral inequality for differentiable convex functions and established the following results:

Theorem 7. Let \( f : I^0 \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \) \( a, A, c, B, b \in I^0 \) with \( a \leq A \leq c \leq B \leq b \) and \( p > 1 \). If the mapping \( |f^p|^{p/(p-1)} \) is convex on \([a, b]\), then we have

i. \( f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a} \int_a^b f(x) dx = (a-b) \int_0^1 S(t)f(ta+(1-t)b)dt \)

ii. \( \frac{1}{b-a} \left| \int_a^b f(x) dx \right| \)

\[
\leq \left[ \frac{Ap^+ + (c-A)^p+1}{p+1} \right] ^{1/p} \left[ \frac{(2c^2|f'(a)|^q + (2c-c^2)|f'(b)|^q)}{2} \right] + \left[ \frac{(B-c)^p+1 + (B-c)^p+1}{p+1} \right] ^{1/p} \left[ \frac{(1-c^2)|f'(a)|^q + (1-c^2)|f'(b)|^q}{2} \right]
\]

where \( S(t) = \begin{cases} t-A, & t \in [0,c] \\ t-B, & t \in (c,1] \end{cases} \).

Corollary 1. Under the assumptions of Theorem 7 with \( A = B = c = \frac{1}{2} \), we have

\[
(1.11) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b - a)}{4(p+1)^{1/p}} \left\{ \left( \frac{|f''(a)|^q + 3|f''(b)|^q}{4} \right)^{1/q} + \left( \frac{3|f''(a)|^q + |f''(b)|^q}{4} \right)^{1/q} \right\}.
\]
and if we take $A = 0$, $B = 1$, $c = \frac{1}{2}$ in Theorem 7 then it follows that

\begin{equation}
\left| \frac{1}{b-a} \int_a^b f(x)dx - f \left( \frac{a+b}{2} \right) \right| \\
\leq \frac{(b-a)}{4(p+1)^{1/p}} \left\{ \left( \frac{|f''(a)|^q + 3|f''(b)|^q}{4} \right)^{1/q} + \left( \frac{3|f''(a)|^q + |f''(b)|^q}{4} \right)^{1/q} \right\}.
\end{equation}

In this article, using functions whose twice derivatives absolute values are convex, we obtained new inequalities related to the left and right hand side of Hermite-Hadamard inequality. Finally, we gave some applications for special means of real numbers.

2. Main Results

Throughout, we suppose $I$ is an interval on $\mathbb{R}$ and $a, b, c, d, y \in I$ with $a \leq c \leq y \leq d \leq b$, $(y \neq a, b)$. We start with the following lemma:

**Lemma 2.** Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on $I^o$ with $f'' \in L_1[a, b]$, then

\begin{equation}
\frac{(d-y)^2 - (c-y)^2}{2} (a-b) f'(ya + (1-y)b) + \frac{c^2 f'(b) - (d-1)^2 f'(a)}{2} (a-b) \\
+ (d-c) f(ya + (1-y)b) + (cf(a) - (d-1) f(b)) + \frac{1}{b-a} \int_a^b f(x)dx
\end{equation}

\begin{equation}
= (b-a)^2 \int_0^1 k(t) f''(ta + (1-t)b)dt
\end{equation}

where

\begin{equation}
k(t) := \begin{cases}
\frac{(c-t)^2}{2}, & t \in [0, y) \\
\frac{(d-t)^2}{2}, & t \in [y, 1].
\end{cases}
\end{equation}

**Proof.** It suffices to note that

\begin{equation}
I = \int_0^1 k(t) f''(ta + (1-t)b)dt
\end{equation}

\begin{equation}
= \int_0^y \frac{(c-t)^2}{2} f''(ta + (1-t)b)dt + \int_y^1 \frac{(d-t)^2}{2} f''(ta + (1-t)b)dt
\end{equation}

\begin{equation}
= I_1 + I_2
\end{equation}
Similarly, we observe that by integration by parts, we have the following identity

\[
I_1 = \int_0^y \frac{(c - t)^2}{2} f''(ta + (1 - t)b) dt
\]

\[
= \frac{(c - t)^2}{2(a - b)} f'(ta + (1 - t)b) \bigg|_0^y + \frac{1}{a - b} \int_0^y (c - t) f'(ta + (1 - t)b) dt
\]

\[
= \frac{(c - y)^2}{2(a - b)} f'(ya + (1 - y)b) - \frac{c^2}{2(a - b)} f'(b) + \frac{1}{a - b} \int_0^y (c - t) f'(ta + (1 - t)b) dt
\]

\[
= \frac{(c - y)^2}{2(a - b)} f'(ya + (1 - y)b) - \frac{c^2}{2(a - b)} f'(b)
\]

\[
+ \frac{1}{a - b} \left[ \frac{c - t}{a - b} f(ta + (1 - t)b) \bigg|_0^y + \frac{1}{a - b} \int_0^y f(ta + (1 - t)b) dt \right]
\]

\[
= \frac{(c - y)^2}{2(a - b)} f'(ya + (1 - y)b) - \frac{c^2}{2(a - b)} f'(b)
\]

\[
+ \frac{c - y}{(b - a)^2} f(ya + (1 - y)b) - \frac{c}{(b - a)^2} f(b) + \frac{1}{(b - a)^2} \int_0^y f(ta + (1 - t)b) dt.
\]

Similarly, we observe that

\[
I_2 = \frac{(d - 1)^2}{2(a - b)} f'(a) - \frac{(d - y)^2}{2(a - b)} f'(ya + (1 - y)b)
\]

\[
+ \frac{d - 1}{(b - a)^2} f(a) - \frac{d - y}{(b - a)^2} f(ya + (1 - y)b) + \frac{1}{(b - a)^2} \int_0^1 f(ta + (1 - t)b) dt.
\]

Thus, we can write

\[
I = I_1 + I_2
\]

\[
= \frac{(c - y)^2 - (d - 1)^2}{2(a - b)} f'(ya + (1 - y)b) + \frac{(d - 1)^2}{2(a - b)} f'(a) - \frac{c^2}{2(a - b)} f'(b)
\]

\[
+ \frac{c - d}{(b - a)^2} f(ya + (1 - y)b) + \frac{(d - 1) f(a) - cf(b)}{(b - a)^2} + \frac{1}{(b - a)^2} \int_0^1 f(ta + (1 - t)b) dt.
\]

Using the change of the variable \( x = ta + (1 - t)b \) for \( t \in [0, 1] \) and by multiplying the both sides by \( (b - a)^2 \) which gives the required identity (2.1). \( \square \)

Now, by using the above lemma, we prove our main theorems:

**Theorem 8.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be twice differentiable function on \( I^o \) such that \( f'' \in L_1[a, b] \) where \( a, b \in I, a < b \). If \( |f''| \) is convex on \( [a, b] \) then the following
inequality holds:

\[
\left| \frac{(c - y)^2 - (d - y)^2}{2} (a - b) f'(ya + (1 - y)b) + \frac{(d - 1)^2 f'(a) - c^2 f'(b)}{2} (a - b) \right|
\]

\[
+ (c - d) f(ya + (1 - y)b) + [(d - 1) f(a) - cf(b)] + \frac{1}{b-a} \int_a^b f(x)dx \right|
\]

\[
\leq \frac{(b - a)^2}{24} (A |f''(a)| + B |f''(b)|)
\]

where

\[
A = 6d^2 - 8d + 3 + (6c^2 - 6d^2) y^2 + (8d - 8c) y^3
\]

and

\[
B = 6d^2 - 4d + 1 + (12c^2 - 12d^2) y + (12d + 6d^2 - 12c - 6c^2) y^2 + (8c - 8d) y^3.
\]

Proof. From Lemma 2 and by the definition \( k(t) \), we get,

\[
\left| \frac{(c - y)^2 - (d - y)^2}{2} (a - b) f'(ya + (1 - y)b) + \frac{(d - 1)^2 f'(a) - c^2 f'(b)}{2} (a - b) \right|
\]

\[
+ (c - d) f(ya + (1 - y)b) + [(d - 1) f(a) - cf(b)] + \frac{1}{b-a} \int_a^b f(x)dx \right|
\]

\[
\leq \frac{(b - a)^2}{2} \left\{ \int_0^y (c - t)^2 |f''(a + (1 - t)b)| dt + \int_1^y (d - t)^2 |f''(a + (1 - t)b)| dt \right\}.
\]

By the convexity of \( |f''| \), we get

\[
\left| \frac{(c - y)^2 - (d - y)^2}{2} (a - b) f'(ya + (1 - y)b) + \frac{(d - 1)^2 f'(a) - c^2 f'(b)}{2} (a - b) \right|
\]

\[
+ (c - d) f(ya + (1 - y)b) + [(d - 1) f(a) - cf(b)] + \frac{1}{b-a} \int_a^b f(x)dx \right|
\]

\[
\leq \frac{(b - a)^2}{2} \left\{ \int_0^y (c - t)^2 t |f''(a)| dt + \int_0^y (c - t)^2 (1 - t) |f''(b)| dt \\
+ \int_1^y (d - t)^2 t |f''(a)| dt + \int_1^y (d - t)^2 (1 - t) |f''(b)| dt \right\}
\]
= (b - a)^2 \left\{ \frac{6c^2 y^2 - 8cy^3 + 3y^4}{12} |f''(a)| \\
+ \frac{12c^2 y + (-12c - 6c^2) y^2 + (4 + 8c) y^3 - 3y^4}{12} |f''(b)| \\
+ \frac{6d^2 - 8d + 3 - 6d^2 y^2 + 8dy^3 - 3y^4}{12} |f''(a)| + \\
+ \frac{6d^2 - 4d + 1 - 12d^2 y + (12d + 6d^2) y^2 + (4 - 8d) y^3 + 3y^4}{12} |f''(b)| \right\}

= \frac{(b - a)^2}{2} \left\{ \frac{6d^2 - 8d + 3 + (6c^2 - 6d^2) y^2 + (8d - 8c) y^3}{12} |f''(a)| \\
+ \frac{6d^2 - 4d + 1 + (12c^2 - 12d^2) y + (12d + 6d^2 - 12c - 6c^2) y^2 + (8c - 8d) y^3}{12} |f''(b)| \right\}.

which completes the proof.

\[ \square \]

**Corollary 2.** Under the assumptions Theorem 8 with \( y = \frac{1}{2}, \ c = 0, \ d = 1, \) we have

\[
\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{48} (|f''(a)| + |f''(b)|)
\]

and if we take \( y = c = d = \frac{1}{2} \) and \( f'(a) = f'(b) \) in Theorem 8 we have

\[
\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+f(b)}{2}\right) \right| \leq \frac{(b-a)^2}{48} (|f''(a)| + |f''(b)|).
\]

**Remark 2.** We note that the obtained midpoint inequalities (2.2) and (2.3) are better than the inequalities (1.5) and (1.2), respectively.

**Remark 3.** We note that the obtained midpoint inequality (2.2) is the same midpoint in inequality (1.6).

Another similar result may be extended in the following theorem
Theorem 9. Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be twice differentiable function on \( I \) such that \( f'' \in L_1[a, b] \) where \( a, b \in I, a < b \). If \( |f''|^q \) is convex on \([a, b], q > 1\), then

\[
\left| \frac{(c - y)^2 - (d - y)^2}{2} (a - b) f'(ya + (1 - y)b) + \frac{(d - 1)^2 f'(a) - c^2 f'(b)}{2} (a - b) \right.
\]
\[
+ (c - d) f(ya + (1 - y)b) + |(d - 1) f(a) - cf(b)| + \frac{1}{b - a} \int_a^b f(x)dx \right|
\]
\[
\leq \frac{(b - a)^2}{2} \left( \frac{1}{2p + 1} \right)^{1/p} \left\{ (c^{2p+1} + (y - c)^{2p+1})^{1/p} \left( \frac{y^2 |f''(a)|^q + (2y - y^2) |f''(b)|^q}{2} \right) \right\}^{1/q}
\]
\[
+ \left( (d - y)^{2p+1} + (1 - d)^{2p+1} \right)^{1/p} \left( \frac{(1 - y^2) |f''(a)|^q + (1 - y)^2 |f''(b)|^q}{2} \right)^{1/q} \}
\]

Proof. From Lemma 2 by the definition \( k(t) \) and using by Hölder’s inequality, it follows that

\[
\left| \frac{(c - y)^2 - (d - y)^2}{2} (a - b) f'(ya + (1 - y)b) + \frac{(d - 1)^2 f'(a) - c^2 f'(b)}{2} (a - b) \right.
\]
\[
+ (c - d) f(ya + (1 - y)b) + |(d - 1) f(a) - cf(b)| + \frac{1}{b - a} \int_a^b f(x)dx \right|
\]
\[
\leq \frac{(b - a)^2}{2} \left\{ \int_0^y (c - t)^2 |f''(ta + (1 - t)b)|dt \right\} dt + \int_y^1 (d - t)^2 |f''(ta + (1 - t)b)|dt \right\} dt
\]
\[
\leq \frac{(b - a)^2}{2} \left\{ \left( \int_0^y |c - t|^{2p} dt \right)^{1/p} \left( \int_0^y |f''(ta + (1 - t)b)|^q dt \right)^{1/q} \right\}
\]
\[
+ \left( \int_y^1 |d - t|^{2p} dt \right)^{1/p} \left( \int_y^1 |f''(ta + (1 - t)b)|^q dt \right)^{1/q} \}
\]

Since \( |f''|^q \) is convex on \([a, b]\), we known that for \( t \in [0, 1]\),

\[
|f''(ta + (1 - t)b)|^q \leq (1 - t) |f''(a)|^q + (1 - t) |f''(b)|^q
\]
\[
\left| \frac{(c - y)^2 - (d - y)^2}{2} \right| (a - b) f'(ya + (1 - y)b) + \frac{(d - 1)^2 f'(a) - c^2 f'(b)}{2} (a - b)
\]

\[
+ (c - d) f(ya + (1 - y)b) + [(d - 1) f(a) - cf(b)] + \frac{1}{b - a} \int_a^b f(x) dx \right|
\]

\[
\leq \frac{(b - a)^2}{2} \left\{ \left( \int_0^y |c - t|^{2p} dt \right)^{1/p} \left( \int_0^y |f''(ta + (1 - t)b)|^q dt \right)^{1/q} \right.
\]

\[
+ \left( \int_y^1 \int_0^y |d - t|^{2p} dt \right)^{1/p} \left( \int_0^y |f''(ta + (1 - t)b)|^q dt \right)^{1/q} \right\}
\]

\[
\leq \frac{(b - a)^2}{2} \left\{ \left( \int_0^y |c - t|^{2p} dt \right)^{1/p} \left( \int_0^y (t |f''(a)|^q + (1 - t) |f''(b)|^q) dt \right)^{1/q} \right.
\]

\[
+ \left( \int_y^1 \left( \int_y^1 \int_0^y |d - t|^{2p} dt \right)^{1/p} \left( \int_y^1 (t |f''(a)|^q + (1 - t) |f''(b)|^q) dt \right)^{1/q} \right\}
\]

\[
= \frac{(b - a)^2}{2} \left( \frac{1}{2p + 1} \right)^{1/p} \left\{ \left( c^{2p+1} + (y - c)^{2p+1} \right)^{1/p} \left( \frac{y^2 |f''(a)|^q + (2y - y^2) |f''(b)|^q}{2} \right)^{1/q} \right.
\]

\[
+ \left( (d - y)^{2p+1} + (1 - d)^{2p+1} \right)^{1/p} \left( \frac{(1 - y^2) |f''(a)|^q + (1 - y)^2 |f''(b)|^q}{2} \right)^{1/q} \right\}.
\]

where we have used the facts that

\[
\int_0^y |c - t|^{2p} dt = \int_0^c (c - t)^{2p} dt + \int_y^c (t - c)^{2p} dt = \frac{1}{2p + 1} \left( c^{2p+1} + (y - c)^{2p+1} \right)
\]

\[
\int_y^1 |d - t|^{2p} dt = \int_y^d (d - t)^{2p} dt + \int_1^d (t - d)^{2p} dt = \frac{1}{2p + 1} \left( (d - y)^{2p+1} + (1 - d)^{2p+1} \right),
\]

and

\[
\int_0^y (t |f''(a)|^q + (1 - t) |f''(b)|^q) dt = \frac{y^2 |f''(a)|^q + (2y - y^2) |f''(b)|^q}{2}
\]

\[
\int_y^1 (t |f''(a)|^q + (1 - t) |f''(b)|^q) dt = \frac{(1 - y^2) |f''(a)|^q + (1 - y)^2 |f''(b)|^q}{2},
\]

which completes the proof. □
Corollary 3. Under the assumptions Theorem 4 with $y = \frac{1}{2}$, $c = 0$, $d = 1$, we have
\begin{equation}
\left| \frac{1}{b-a} \int_a^b f(x)dx - f \left( \frac{a+b}{2} \right) \right| \\
\leq \frac{(b-a)^2}{16 (2p+1)^{1/p}} \left\{ \left( \frac{|f''(a)|^q + 3 |f''(b)|^q}{4} \right)^{1/q} + \left( \frac{3 |f''(a)|^q + |f''(b)|^q}{4} \right)^{1/q} \right\}
\end{equation}
and if we take $y = c = d = \frac{1}{2}$ and $f'(a) = f'(b)$ in Theorem 4 we have
\begin{equation}
\left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{f(a) + f(b)}{2} \right|
\leq \frac{(b-a)^2}{16 (2p+1)^{1/p}} \left\{ \left( \frac{|f''(a)|^q + 3 |f''(b)|^q}{4} \right)^{1/q} + \left( \frac{3 |f''(a)|^q + |f''(b)|^q}{4} \right)^{1/q} \right\}.
\end{equation}

Remark 4. We note that the obtained midpoint inequalities (2.4) and (2.5) are better than the inequalities (1.12) and (1.11), respectively.

Theorem 10. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable function on $I^o$ such that $f'' \in L_1[a,b]$ where $a, b \in I$, $a < b$. If $|f''|^q$ is convex on $[a,b]$, $q \geq 1$, then
\begin{equation}
\left| \frac{(c-y)^2 - (d-y)^2}{2} (a-b) f'(ya + (1-y)b) + \frac{(d-1)^2 f'(a) - c^2 f'(b)}{2} (a-b) \right|
+ (c-d) f(ya + (1-y)b) + [(d-1) f(a) - cf(b)] + \frac{1}{b-a} \int_a^b f(x)dx \right|
\leq \frac{(b-a)^2}{6} \left\{ \left( c^3 - (c-y)^3 \right)^{1/p} \left( M |f''(a)|^q + N |f''(b)|^q \right)^{1/q} + \left( (d-y)^3 - (d-1)^3 \right)^{1/p} \left( P |f''(a)|^q + Q |f''(b)|^q \right)^{1/q} \right\}
\end{equation}
where

\begin{align*}
M &= c^4 - (c-y)^3 (c+3y), \quad N = 4c^3 - c^4 + (c-y)^3 (c+3y - 4) \\
P &= (d-y)^3(d+3y) - (d-1)^3 (d+3) \quad \text{and} \quad Q = (d-y)^3 (4-d-3y) + (d-1)^4
\end{align*}
Proof. From Lemma [2] by the definition \( k(t) \) and using by power mean inequality, it follows that

\[
\left| \frac{(c - y)^2 - (d - y)^2}{2} (a - b) f'(ya + (1 - y)b) + \frac{(d - 1)^2 f'(a) - c^2 f'(b)}{2} (a - b) 
+ (c - d) f(ya + (1 - y)b) + [(d - 1) f(a) - cf(b)] + \frac{1}{b - a} \int_a^b f(x)dx \right|
\]

\[
\leq \frac{(b - a)^2}{2} \left\{ \int_0^y (c - t)^2 |f''(ta + (1 - t)b)| dt + \int_y^1 (d - t)^2 |f''(ta + (1 - t)b)| dt \right\}
\]

\[
\leq \frac{(b - a)^2}{2} \left\{ \left( \int_0^y (c - t)^2 dt \right)^{1/p} \left( \int_0^y (c - t)^2 |f''(ta + (1 - t)b)|^q dt \right)^{1/q}
+ \left( \int_y^1 (d - t)^2 dt \right)^{1/p} \left( \int_y^1 (d - t)^2 |f''(ta + (1 - t)b)|^q dt \right)^{1/q} \right\}.
\]

Since \(|f''|^q\) is convex on \((a, b)\), we known that for \( t \in [0, 1] \),

\[|f''(ta + (1 - t)b)|^q \leq t |f''(a)|^q + (1 - t) |f''(b)|^q,\]

thus, it follows that

\[(2.7) \left| \frac{(c - y)^2 - (d - y)^2}{2} (a - b) f'(ya + (1 - y)b) + \frac{(d - 1)^2 f'(a) - c^2 f'(b)}{2} (a - b) 
+ (c - d) f(ya + (1 - y)b) + [(d - 1) f(a) - cf(b)] + \frac{1}{b - a} \int_a^b f(x)dx \right|
\]

\[
\leq \frac{(b - a)^2}{2} \left\{ \left( \frac{c^3 - (c - y)^3}{3} \right)^{1/p} \left( \int_0^y (c - t)^2 (t |f''(a)|^q + (1 - t) |f''(b)|^q) dt \right)^{1/q}
+ \left( \frac{(d - y)^3 - (d - 1)^3}{3} \right)^{1/p} \left( \int_y^1 (d - t)^2 (t |f''(a)|^q + (1 - t) |f''(b)|^q) dt \right)^{1/q} \right\}.
\]
By simple computation,

$$
\int_0^y (c - t)^2 t \, dt = \frac{c^4 - (c - y)^3(c + 3y)}{12}
$$

$$
\int_0^y (c - t)^2 (1 - t) \, dt = \frac{4c^3 - c^4 + (c - y)^3(c + 3y - 4)}{12}
$$

(2.8)

$$
\int_y^1 (d - t)^2 t \, dt = \frac{(d - y)^3(d + 3y) - (d - 1)^3(d + 3)}{12}
$$

$$
\int_y^1 (d - t)^2 (1 - t) \, dt = \frac{(d - y)^3(4 - d - 3y) + (d - 1)^4}{12}
$$

Substituting (2.8) into (2.7) gives (2.6). □

**Corollary 4.** Under the assumptions Theorem 10 with $y = \frac{1}{2}$, $c = 0$, $d = 1$, we have

(2.9)

$$\left| \frac{1}{b - a} \int_a^b f(x) \, dx - f \left( \frac{a + b}{2} \right) \right| \leq \frac{(b - a)^2}{48} \left\{ \left( \frac{3}{8} |f''(a)|^q + \frac{5}{8} |f''(b)|^q \right)^{1/q} + \left( \frac{5 |f''(a)|^q + 3 |f''(b)|^q}{8} \right)^{1/q} \right\}$$

and if we take $y = c = d = \frac{1}{2}$ and $f'(a) = f'(b)$ in Theorem 10 we have

(2.10)

$$\left| \frac{1}{b - a} \int_a^b f(x) \, dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b - a)^2}{48} \left\{ \left( \frac{|f'(a)|^q + 7 |f'(b)|^q}{8} \right)^{1/q} + \left( \frac{7 |f'(a)|^q + |f'(b)|^q}{8} \right)^{1/q} \right\}.$$  

**Remark 5.** If we take $q = 1$ in (2.9) and (2.10), then we have the inequalities (2.2) and (2.3), respectively.

### 3. Applications to Special Means

We shall consider the following special means:

(a) The arithmetic mean: $A = A(a, b) := \frac{a + b}{2}$, $a, b \geq 0$,

(b) The harmonic mean:

$$H = H(a, b) := \frac{2ab}{a + b}, \ a, b > 0,$$

(c) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \ a, b > 0.$$
Proposition 2. Let 

\[ L_p = L_p(a, b) := \begin{cases} \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \bigg/ \frac{b}{a} & \text{if } a \neq b, \\ a & \text{if } a = b \end{cases} \quad \text{, } p \in \mathbb{R} \setminus \{-1, 0\}; a, b > 0. \]

It is well known that \( L_p \) is monotonic nondecreasing over \( p \in \mathbb{R} \) with \( L_{-1} := L \) and \( L_0 := 1 \). In particular, we have the following inequalities

\[ H \leq L \leq A. \]

Now, using the results of Section 2, some new inequalities is derived for the above means.

**Proposition 1.** Let \( a, b \in \mathbb{R}, 0 < a < b \) and \( n \in \mathbb{N}, n > 2 \). Then, we have

\[ |L_n^n(a, b) - A^n(a, b)| \leq n(n-1)\frac{(b-a)^2}{48} (a^{n-2} + b^{n-2}) \]

and

\[ |L_n^n(a, b) - A^n(a^n, b^n)| \leq n(n-1)\frac{(b-a)^2}{48} (a^{n-2} + b^{n-2}). \]

**Proof.** The assertion follows from Corollary 2 applied to convex mapping \( f(x) = x^n, x \in [a, b] \) and \( n \in \mathbb{N} \). \( \square \)

**Proposition 2.** Let \( a, b \in \mathbb{R}, 0 < a < b \) and \( n \in \mathbb{N}, n > 2 \). Then, we have

\[ |L_n^n(a, b) - A^n(a, b)| \leq n(n-1)\frac{(b-a)^2}{16(2p+1)} \left\{ \left( \frac{a^{(n-2)q} + 3b^{(n-2)q}}{4} \right) \right\}^{\frac{1}{p}} + \left( \frac{3a^{(n-2)q} + b^{(n-2)q}}{4} \right)^{\frac{1}{p}} \]

and

\[ |L_n^n(a, b) - A^n(a^n, b^n)| \leq n(n-1)\frac{(b-a)^2}{16(2p+1)} \left\{ \left( \frac{a^{(n-2)q} + 3b^{(n-2)q}}{4} \right) \right\}^{\frac{1}{p}} + \left( \frac{3a^{(n-2)q} + b^{(n-2)q}}{4} \right)^{\frac{1}{p}}. \]

**Proof.** The assertion follows from Corollary 3 applied to convex mapping \( f(x) = x^n, x \in [a, b] \) and \( n \in \mathbb{N} \). \( \square \)

**Proposition 3.** Let \( a, b \in \mathbb{R}, 0 < a < b \). Then, for all \( p > 1 \), we have

\[ |L^{-1}(a, b) - A^{-1}(a, b)| \leq n(n-1)\frac{(b-a)^2}{48} \left\{ \left( \frac{3a^{(n-2)q} + 5b^{(n-2)q}}{8} \right) \right\}^{\frac{1}{p}} + \left( \frac{5a^{(n-2)q} + 3b^{(n-2)q}}{8} \right)^{\frac{1}{p}} \]

and

\[ |L^{-1}(a, b) - H^{-1}(a, b)| \leq n(n-1)\frac{(b-a)^2}{48} \left\{ \left( \frac{a^{(n-2)q} + 7b^{(n-2)q}}{8} \right) \right\}^{\frac{1}{p}} + \left( \frac{7a^{(n-2)q} + b^{(n-2)q}}{8} \right)^{\frac{1}{p}}. \]
Proof. The assertion follows from Corollary 4 applied to the convex mapping $f(x) = 1/x$, $x \in [a, b]$. □

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