NON-SIMPLE PURELY INFINITE STEINBERG ALGEBRAS
WITH APPLICATIONS TO KUMJIAN-PASK ALGEBRAS

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Abstract. In this paper, we characterize properly purely infinite Steinberg algebras $A_K(G)$ for strongly effective, ample Hausdorff groupoids $G$. As an application, when $\Lambda$ is a strongly aperiodic $k$-graph, we show that the notions of pure infiniteness and proper pure infiniteness are equivalent for the Kumjian-Pask algebra $KP_K(\Lambda)$, which may be determined by the proper infiniteness of vertex idempotents. In particular, for unital cases, we give simple graph-theoretic criteria for the (proper) pure infiniteness of $KP_K(\Lambda)$.

Furthermore, since the complex Steinberg algebra $A_C(G)$ is a dense subalgebra of the reduced groupoid $C^*$-algebra $C^*_r(G)$, we focus on the problem that “when does the proper pure infiniteness of $A_C(G)$ imply that of $C^*_r(G)$ in the $C^*$-sense?”. In particular, we show that if the Kumjian-Pask algebra $KP_C(\Lambda)$ is purely infinite, then so is $C^*(\Lambda)$ in the sense of Kirchberg-Rørdam.

1. Introduction

Inspired by the Kirchberg-Rørdam’s definition of non-simple purely infinite $C^*$-algebras [18], Aranda Pino, Goodearl, Perera, and Siles Molina introduced the notion of purely infinite and properly purely infinite properties for (not necessarily simple) rings [6] as the nonsimple generalization of [4] (see Section 3 for definitions). In particular, it is shown in [6, Proposition 3.17] that if a $C^*$-algebra is purely infinite in the sense of [6], then it is purely infinite in the Kirchberg-Rørdam’s sense. These concepts are closely related, such that for s-unital rings, the properly purely infinite property implies the purely infinite one [6, Lemma 3.4(i)], and we know that the converse is also true if the ring is either exchange [6, Corollary 2.9] or simple [6, Corollary 5.8] for example.

Let $K$ be a field. Following [12], in this paper we work with strongly effective ample groupoids $G$ and associated Steinberg algebras $A_K(G)$. Our aim here is to investigate (non-simple) purely infinite Steinberg algebras and Kumjian-Pask algebras. Recall that the theory of Steinberg algebras were independently introduced in [27] and [13] as a pure algebraic analogue to Renault’s groupoid $C^*$-algebras [24]. There has been a lot of interest...
in the study of Steinberg algebras, partly because they include, and give a

groupoid approach for, many interesting classes of algebras such as Leavitt

path [1], Kumjian-Pask [5], and inverse semigroup algebras [27] among oth-
ers. Moreover, many structural properties of these algebras can be described

by that of the underlying groupoids; such as, simplicity [8], primeness [28],

primitivity [29], simple pure infiniteness [9], and ideal structure [12], among

others.

The contents of this article are as follows. We review in Section 2 some

preliminaries about groupoids, Steinberg algebras and Kumjian-Pask alge-
bras which will be used throughout. In Section 3, we prove that a Steinberg

algebra \( A_K(G) \) is properly purely infinite if and only if its characteristic

functions on compact open bisections are properly infinite. Then we give a

sufficient condition for this property. In Section 4, we apply this result to the

Kumjian-Pask algebras of strongly aperiodic \( k \)-graphs, and derive interest-

ing consequences. In particular, we show that purely infinite and properly

purely infinite Kumjian-Pask algebras coincide, which may be characterized

by verifying the proper infiniteness of their vertex idempotents. This is the

generalization of [6, Theorem 7.4] which is for the class of Leavitt path alge-
bras, but our proof, even in the Leavitt path algebras setting, is different and

simpler. Moreover, we see that a unital Kumjian-Pask algebras is (properly)
purely infinite if and only if every vertex in the underlying \( k \)-graph receives

at least one nontrivial path.

For any locally compact Hausdorff groupoid \( G \), we know that the complex

Steinberg algebra \( A_C(G) \) is a dense subalgebra of the reduced groupoid \( C^* \)

-algebra \( C^*_r(G) \) (see [13, Proposition 4.2]). In [9], the authors considered [6,

Problem 8.4] for simple groupoid algebras, and proved that for any second-
countable ample groupoid \( G \), if the complex Steinberg algebra \( A_C(G) \) is

purely infinite simple, then so is the \( C^* \)-algebra \( C^*_r(G) \) in the sense of [16].

In Section 5, we consider second-countable and amenable groupoids \( G \) and
generalize this result to not necessarily simple groupoid algebras with a

shorter proof. To do this, we first show that a groupoid \( C^* \)-algebra \( C^*_r(G) \)
is \( C^* \)-purely infinite (in the sense of [18]) if and only if all its characteristic
functions are \( C^* \)-properly infinite, which is interesting in its own right. Since
the groupoid \( G_\Lambda \) associated to each \( k \)-graph \( \Lambda \) is always amenable and second-
countable, we conclude that the pure infiniteness of complex Kumjian-Pask
algebra \( KP_C(\Lambda) \) implies that of the \( C^* \)-algebra \( C^*(\Lambda) \) for strongly aperiodic
\( \Lambda \)'s.

2. Preliminaries

Here, we review some definitions and basic properties of groupoids, Stein-
berg algebras and Kumjian-Pask algebras.

2.1. Groupoids. By groupoid, we mean a small category \( G \) endowed with a

composition \( G^{(2)} \subseteq G \times G \to G \), by \( (\alpha, \beta) \mapsto \alpha \beta \), and an inversion \( \alpha \mapsto \alpha^{-1} \)
on \( G \) satisfying the following conditions:
(1) if $(\alpha, \beta, (\beta, \gamma) \in \mathcal{G}^{(2)}$, then both $(\alpha\beta, \gamma)$ and $(\alpha, \beta\gamma)$ are composable and we have $(\alpha\beta)\gamma = \alpha(\beta\gamma)$;

(2) $(\alpha^{-1}, \alpha) \in \mathcal{G}^{(2)}$ for all $\alpha \in \mathcal{G}$; and

(3) if $(\alpha, \beta) \in \mathcal{G}^{(2)}$, then $\alpha^{-1}(\alpha\beta) = \beta$ and $(\alpha\beta)\beta^{-1} = \alpha$.

Then the source and range maps on $\mathcal{G}$ are $s(\alpha) = \alpha^{-1}\alpha$ and $r(\alpha) = \alpha\alpha^{-1}$ for all $\alpha \in \mathcal{G}$. In particular, we have $(\alpha, \beta) \in \mathcal{G}^{(2)}$ if and only if $s(\alpha) = r(\beta)$.

For every two subsets $A, B$ of $\mathcal{G}$, we define the product of $A$ and $B$ as $AB := \{\alpha\beta : \alpha \in A, \beta \in B, s(\alpha) = r(\beta)\}$. We will let $\mathcal{G}^{(0)}$ the unit space of $\mathcal{G}$, that is $\mathcal{G}^{(0)} = \{\alpha^{-1}\alpha : \alpha \in \mathcal{G}\}$. Moreover, the isotropy subgroupoid of $\mathcal{G}$ is defined by $\text{Iso}(\mathcal{G}) := \{\alpha \in \mathcal{G} : s(\alpha) = r(\alpha)\}$.

We say that a subset $U \subseteq \mathcal{G}^{(0)}$ is invariant if, for $\alpha \in \mathcal{G}$, $s(\alpha) \in U$ implies $r(\alpha) \in U$, under which we have $r(s^{-1}(U)) = s^{-1}(U)$. Note that in case $U$ is an invariant subset of $\mathcal{G}^{(0)}$, then $\mathcal{G}_U := s^{-1}(U)$ is a subgroupoid of $\mathcal{G}$ (with the unit space $U$), which coincides with the restriction $\mathcal{G}_U := \{\alpha \in \mathcal{G} : s(\alpha), r(\alpha) \in U\}$.

A groupoid $\mathcal{G}$ endowed with a topology is called a topological groupoid in case the composition and inverse maps are continuous. Let $\mathcal{G}$ be a topological groupoid. A subset $B \subseteq \mathcal{G}$ is a bisection if the restrictions $r|_B$ and $s|_B$ are homeomorphisms. Then we say that $\mathcal{G}$ is ample if $\mathcal{G}$ has a basis of compact open bisections. In this case, $\mathcal{G}^{(0)}$ is an open and totally disconnected subset of $\mathcal{G}$.

Definition 2.1. A topological groupoid $\mathcal{G}$ is called effective if the interior of $\text{Iso}(\mathcal{G})$ is $\mathcal{G}^{(0)}$. Also, we say that $\mathcal{G}$ is strongly effective in case for every nonempty closed invariant subset $V \subseteq \mathcal{G}^{(0)}$, the restricted groupoid $\mathcal{G}_V$ is effective.

Clearly, when $\mathcal{G}$ is strongly effective, then it is effective as well because $\mathcal{G}^{(0)}$ is a closed invariant set. In this paper, we only consider the strongly effective ample Hausdorff groupoids.

2.2. Steinberg Algebras. Let $\mathcal{G}$ be an ample groupoid and $K$ a field. The Steinberg algebra $A_K(\mathcal{G})$ associated to $\mathcal{G}$ consists of all locally constant and compactly supported functions from $\mathcal{G}$ into $K$. Then it has $K$-algebra structure by considering the pointwise addition as usual and the convolution multiplication as $(fg)(\alpha) = \sum_{r(\beta) = r(\alpha)} f(\beta)g(\beta^{-1}\alpha)$ $(f, g \in A_K(\mathcal{G})$ and $\alpha \in \mathcal{G})$.

Since $\mathcal{G}$ is assumed to be ample, we have $A_K(\mathcal{G}) = \text{span}_K \{1_B : B$ is a compact open bisection$\}$,
where \( 1_B \) is the characteristic function on \( B \). In particular, \( A_K(\mathcal{G}) \) has local units, in the sense that for each \( f \in A_K(\mathcal{G}) \) there is an idempotent \( p \in A_K(\mathcal{G}) \) such that \( fp = pf = f \), and so is \( s \)-unital. Examples of Steinberg algebras of ample groupoids include Leavitt path algebras \([13]\) as well as Kumjian-Pask algebras \([13, 15]\) (see Section 4 for details).

By ideal, we always mean a two-sided one. According to \([12, \text{Theorem 3.1}]\), when \( \mathcal{G} \) is a strongly effective ample Hausdorff groupoid, then every ideal of \( A_K(\mathcal{G}) \) is of the form

\[
I_U := \{ f \in A_K(\mathcal{G}) : \text{supp} f \subseteq \mathcal{G}_U \},
\]

for some open invariant subset \( U \) of \( \mathcal{G}(0) \). Moreover, if \( U \) is an open invariant subset of \( \mathcal{G}(0) \) and \( D := \mathcal{G}(0) \setminus U \), then \([12, \text{Lemma 3.6}]\) says that the restrictive map \( q(f) = f\mid_{\mathcal{G}_D} \) is an epimorphism from \( A_K(\mathcal{G}) \) into \( A_K(\mathcal{G}_D) \) such that \( \ker(q) = I_U \); thus we have \( A_K(\mathcal{G})/I_U \cong A_K(\mathcal{G}_D) \). We will use this result in Section 3 because the property of proper infiniteness is closely related to the structure of ideals and quotients.

### 2.3. Higher-rank graphs and their Kumjian-Pask algebras.

Here, we recall some basic definitions of graphs of rank \( k \geq 1 \) and associated Kumjian-Pask algebras from \([19, 5, 14]\), which will be needed in the results of Section 4. Let \( \mathbb{N} = \{0,1,2,\ldots\} \). For fixed integer \( k \geq 1 \), we consider the additive semigroup \( \mathbb{N}^k \) with the identity \( 0 := (0,\ldots,0) \). We write each element \( n \in \mathbb{N}^k \) as \( n = (n_1,\ldots,n_k) \), and the generators of \( \mathbb{N}^k \) by \( e_1,\ldots,e_k \), where \( e_{ij} = \delta_{i,j} \) for \( 1 \leq i,j \leq k \). We may put a partial order \( \leq \) on \( \mathbb{N}^k \) by \( m \leq n \iff m_i \leq n_i \) for all \( 1 \leq i \leq k \), and denote \( m \lor n \) and \( m \land n \) for the coordinate-wise maximum and minimum, respectively.

**Definition 2.2 ([19]).** A \( k \)-graph (or higher-rank graph) is a countable small category \( \Lambda = (\Lambda^0, \Lambda, r, s) \), where \( \Lambda^0 \) is the objects, \( \Lambda \) morphisms, \( r, s : \Lambda \to \Lambda^0 \) the range and source maps, which is equipped with a degree functor \( d : \Lambda \to \mathbb{N}^k \) satisfying the unique factorisation property: for each \( \lambda \in \Lambda \) and \( 0 \leq n \leq d(\lambda) \), there exists a unique factorisation \( \lambda = \lambda(0,n)\lambda(n,d(\lambda)) \) for \( \lambda \) such that \( d(\lambda(0,n)) = n, d(\lambda(n,d(\lambda))) = d(\lambda) - n \).

Every \( k \)-graph \( \Lambda \) has an edge-colored graph as its 1-skeleton, so we usually like to refer the objects of \( \Lambda^0 \) as vertices and the morphisms of \( \Lambda \) as paths. If \( H \subseteq \Lambda^0 \) and \( E \subseteq \Lambda_1 \), we write

\[
HE := \{ \lambda \in E : r(\lambda) \in H \} \quad \text{and} \quad EH := \{ \lambda \in E : s(\lambda) \in H \},
\]

and when \( H = \{v\} \) is a singleton, we simply write \( vE \) and \( Ev \), respectively. For any \( n \in \mathbb{N}^k \), we define \( \Lambda^n := \{ \lambda \in \Lambda : d(\lambda) = n \} \) and

\[
\Lambda^{\leq n} := \{ \lambda \in \Lambda : d(\lambda) \leq n, \quad \text{and} \quad d(\lambda) + e_i \leq n \implies s(\lambda)\Lambda^{e_i} = \emptyset \}. \]

Given \( \mu, \nu \in \Lambda \), the set of minimal common extensions for \( \mu \) and \( \nu \) is denoted by \( \text{MCE}(\mu, \nu) \), which is

\[
\text{MCE}(\mu, \nu) := \left\{ \lambda \in \Lambda^{d(\mu) + d(\nu)} : \mu\alpha = \lambda = \nu\beta \right\},
\]

for some \( \alpha, \beta \in \Lambda \).
A $k$-graph $\Lambda$ is called row-finite if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Also, we say $\Lambda$ is locally convex in case for every $\lambda \in \Lambda^e$ and $1 \leq j \neq i \leq k$, $r(\lambda)\Lambda^{e_j} \neq \emptyset$ implies $s(\lambda)\Lambda^{e_i} \neq \emptyset$ [22, Definition 3.10]. In this article, we consider only locally convex and row-finite $k$-graphs.

**Example 2.3.** Let $m \in (\mathbb{N} \cup \{\infty\})^k$ for $k \geq 1$. Then the set $\Omega_{k,m} = \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m\}$ equipped with the degree map as $d(p,q) = q - p$, and the range and source maps as $r(p,q) = (p,p)$ and $s(p,q) = (q,q)$ is a row-finite $k$-graph.

A boundary path of $\Lambda$ is a degree preserving functor $x : \Omega_{k,m} \to \Lambda$ such that for every $p \in \mathbb{N}^k$ and $1 \leq i \leq k$, $p \leq m$ and $p_i = m_i$ imply that $x(p,p)\Lambda^{e_i} = \emptyset$. Then we define $d(x) = m$ and usually call $x(0,0)$ range of $x$, written by $r(x)$. The set of all boundary paths of $\Lambda$ is denoted by $\Lambda^{\leq \infty}$.

There are some definitions for aperiodicity in the literature, which are equivalent for locally convex row-finite $k$-graphs. Following [22], we say that $\Lambda$ is aperiodic if for each $v \in \Lambda^0$, there exists $x \in v\Lambda^{\leq \infty}$ such that $\alpha \neq \beta \in \Lambda v$ implies $\alpha x \neq \beta x$.

**Definition 2.4 ([5]).** Let $\Lambda$ be a locally convex, row-finite $k$-graph and $K$ a field. The Kumjian-Pask algebra associated to $\Lambda$ is the universal $K$-algebra $\text{KP}_K(\Lambda)$ generated by a family $\{s_\lambda, s_\lambda^* : \lambda \in \Lambda\}$ (which is called a Kumjian-Pask $\Lambda$-family) satisfying the following conditions:

(KP1) $s_v s_w = \delta_{v,w} s_v$ for all $v, w \in \Lambda^0$.

(KP2) $s_\lambda s_\mu = s_{\lambda \mu}$ and $s_\mu^* s_\lambda^* = s_{(\lambda \mu)^*}$ for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$.

(KP3) $s_{\lambda^*} s_\mu = \delta_{\lambda, \mu} s_\lambda$ for all $\lambda, \mu \in \Lambda$.

(KP4) $s_v = \sum_{\lambda \in v \Lambda^{\leq n}} s_\lambda s_{\lambda^*}$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

It is show in [5, Theorem 3.4] and [14, Theorem 3.7(a)] that the $K$-algebra $\text{KP}_K(\Lambda)$ exists and is unique up to isomorphism.

A subset $H \subseteq \Lambda^0$ is called hereditary if for every $\lambda \in \Lambda$, $r(\lambda) \in H$ implies $s(\lambda) \in H$, and is called saturated if for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, $\{s(\lambda) : \lambda \in v\Lambda^{\leq n}\} \subseteq H$ implies $v \in H$. Note that if $H$ is a saturated hereditary subset of $\Lambda^0$, then $\Lambda \setminus \Lambda H$ is a locally convex row-finite $k$-graph [22, Theorem 5.2 (b)]. We say that $\Lambda$ is strongly aperiodic in case all such quotient $k$-graphs are aperiodic. Recall from [5, Corollary 5.7] that, when $\Lambda$ is strongly aperiodic, every (two-sided) ideal of $\text{KP}_K(\Lambda)$ is of the form

$$I_H := \text{span}_K \{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu) \in H\}$$

for some saturated hereditary subset $H$ of $\Lambda^0$ (see also [14, Theorem 9.4]), and we have $\text{KP}_K(\Lambda)/I_H \cong \text{KP}_K(\Lambda \setminus \Lambda H)$ [5, Proposition 5.5].

### 3. Properly purely infinite Steinberg algebras

In this section, we characterize properly purely infinite Steinberg algebras $A_{\mathcal{K}}(\mathcal{G})$ by giving necessary and sufficient conditions for it. Our result is the algebraic analogue of [7, Theorem 4.1] and the non-simple version of [9, Theorem 3.1]. Since the concept of proper purely infiniteness is closely
related to the ideal and quotient structure [6, Corollary 2.9], we consider here strongly effective groupoids to apply the results of [12, Section 3]. However, verifying this concept for general groupoids seems to be more complicated, even for the \(k\)-graph’s ones.

Let us first recall some definitions and terminology from [6]. Suppose that \(R\) is a ring. If \(a \in M_m(R)\) and \(b \in M_n(R)\) are two square matrices over \(R\), we denote
\[
a \oplus b := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{m+n}(R).
\]
We define \(a \lesssim b\) if there exist \(x \in M_{m,n}(R)\) and \(y \in M_{n,m}(R)\) such that \(a = xby\). Note that we may consider \(a, b, x,\) and \(y\) as elements in \(M_{m+n}(R)\) by enlarging them by zeros. For \(a, b \in R\), we write \(a \sim b\) if there exist \(r, s \in R\) such that \(rs = a\) and \(sr = b\), \(a \leq b\) if \(ab = ba = a\), and \(a < b\) if \(a \leq b\) and \(a \neq b\). A nonzero element \(a \in R\) is called \(\text{infinite}\) in case \(a \oplus x \lesssim a\) for some \(x \in R\), and \(\text{properly infinite}\) in case \(a \oplus a \lesssim a\) (or equivalently, \(a \oplus a \lesssim a \oplus 0\)). As explained after [6, Remark 2.7], if \(p \in R\) is an idempotent, \(p\) is infinite if and only if there is a subidempotent \(p < q\) such that \(p \sim q\).

**Lemma 3.1.** Let \(R\) be a ring and let \(p\) and \(q\) be two idempotents in \(R\). If \(p \sim q\) and \(p\) is properly infinite, then so is \(q\).

**Proof.** Suppose that \(p = xy\) and \(q = yx\) for some \(x, y \in R\). Let \(p \oplus p = ApB\) where \(A \in M_{2,1}(R)\) and \(B \in M_{1,2}(R)\). Then we have
\[
q \oplus q = q^2 \oplus q^2 = (y \oplus y)(xy \oplus xy)(x \oplus x) = (y \oplus y)(p \oplus p)(x \oplus x) = (y \oplus y)(ApB)(x \oplus x) = ((y \oplus y)Ax)q(yB(x \oplus x)),
\]
and therefore \(q \oplus q \lesssim q\) as desired. \(\square\)

**Definition 3.2 ([6, Definition 3.1]).** Let \(R\) be a ring. We say that \(R\) is purely infinite if
1. each quotient of \(R\) is not a division ring, and
2. for every \(a \in R\), \(b \in RaR\) implies \(b \lesssim a\).

Also, we say that \(R\) is \(\text{properly purely infinite}\) in case all nonzero elements of \(R\) are properly infinite.

According to [6, Lemma 3.4], every properly purely infinite \(s\)-unital ring is purely infinite as well, and we know at least that the converse is also true for exchange cases [6, Corollary 5.8]. Moreover, it follows from [6, Theorem 7.4] that these notions are equivalent for the class of Leavitt path algebras, and in the next section, we will prove a similar result for the class of Kumjian-Pask algebras.

We use the following lemma to prove Theorem 3.4 below.
Lemma 3.3. Let $\mathcal{G}$ be an ample Hausdorff groupoid and $K$ a field. Then the following are equivalent.

1. $\mathcal{G}$ is effective.
2. Every nonzero ideal of $A_K(\mathcal{G})$ contains a nonzero idempotent $1_V$ for some compact open subset $V \subseteq \mathcal{G}^{(0)}$.
3. Every nonzero one-sided ideal of $A_K(\mathcal{G})$ contains a nonzero idempotent $p$ such that $p \sim 1_V$ for some compact open subset $V \subseteq \mathcal{G}^{(0)}$.

Proof. (1) $\implies$ (3). Let $I$ be a nonzero right ideal of $A_K(\mathcal{G})$. Fix a nonzero element $a \in I$. Then, by [11], Lemma 2.2], we may write

$$a = \sum_{B \in F} r_B 1_B$$

where $F$ is a finite collection of mutually disjoint compact open bisections. We may follow the proof of [9, Theorem 3.1] to find an idempotent $p \in I$ and a compact open $V \subseteq \mathcal{G}^{(0)}$ such that $p \sim 1_V$, as desired. For left ideals, we can argue similarly.

(3) $\implies$ (2). Let $I$ be a nonzero ideal of $A_K(\mathcal{G})$. According to (3), $I$ contains a nonzero element $p$ such that $p \sim 1_V$ for some compact open $V \subseteq \mathcal{G}^{(0)}$. If $p = xy$ and $1_V = yx$ for some $x, y \in A_K(\mathcal{G})$, then we have

$$1_V = 1_V^2 = y(xy)x = ypx \in I,$$

giving (2).

(2) $\implies$ (1). This follows from [12, Lemma 3.4].

Theorem 3.4. Let $\mathcal{G}$ be a strongly effective, ample and Hausdorff groupoid and let $K$ be a field. Suppose that $\mathcal{B}$ is a basis of compact open sets for $\mathcal{G}^{(0)}$. Then the following are equivalent:

1. $A_K(\mathcal{G})$ is properly purely infinite.
2. For every $V \in \mathcal{B}$, $1_V$ is properly infinite.
3. For every invariant open subset $U \subseteq \mathcal{G}^{(0)}$ and any nonempty relatively compact open subset $V \subseteq D := \mathcal{G}^{(0)} \setminus U$, $1_V$ is an infinite idempotent in $A_K(\mathcal{G}_D)$.
4. Every nonzero one-sided ideal in any quotient of $A_K(\mathcal{G})$ contains an infinite idempotent.

In particular, if one of the above statements holds (and so all), then $A_K(\mathcal{G})$ is purely infinite as well.

Proof. (1) $\implies$ (2) is immediate.

(2) $\implies$ (3). Let $U$ be an invariant open subset of $\mathcal{G}^{(0)}$ and write $D := \mathcal{G}^{(0)} \setminus U$. Given any relatively compact open $V \subseteq D$, there is an open subset $V \subseteq \mathcal{G}^{(0)}$ such that $V = V \cap D$. We may also suppose that $V$ is compact. Indeed, since $\mathcal{B}$ is a basis for $\mathcal{G}^{(0)}$, we can write $V = \bigcup_{C \in \mathcal{C}} C$ for some subcollection $\mathcal{C} \subseteq \mathcal{B}$. Then $\mathcal{V} = \bigcup_{C \in \mathcal{C}} C \cap D$. As $\mathcal{V}$ is compact, there are finitely many $C_1, \ldots, C_n$ in $\mathcal{C}$ such that $\mathcal{V} \subseteq \bigcup_{i=1}^n C_n \cap D$. Thus, we can
replace $V$ with $\bigcup_{i=1}^{n} C_n$ and assume that $V$ is a compact and open subset of $G^{(0)}$.

Now part (2) says that $1_V$ is a properly infinite idempotent in $A_K(G)$. Thus, since $1_{\tilde{\varphi}}$ is the image of $1_V$ in the quotient $A_K(G_D) \cong A_K(G)/I_U$, [6, Corollary 2.9] implies that $1_{\tilde{\varphi}}$ is an infinite idempotent in $A_K(G_D)$.

(3) $\iff$ (4). Let $I$ be an ideal of $A_K(G)$. By [12, Theorem 3.1], there is an open invariant $U \subseteq G^{(0)}$ such that $I = I_U$, and we have $A_K(G)/I \cong A_K(G_D)$ for $D := G^{(0)} \setminus U$ [12, Lemma 3.6].

Now fix an arbitrary nonzero one-sided ideal $J$ of $A_K(G_D)$. Since $G_D$ is effective, Lemma 3.3 implies that $J$ contains a nonzero idempotent $p$ such that $p \sim 1_V$ for some relatively compact open subset $V$ of $D$. Since $1_V$ is infinite in $A_K(G_D)$ [6, Corollary 2.9], then so is $p$ as desired.

(4) $\implies$ (1) follows from [6, Proposition 3.13], completing the proof. □

**Definition 3.5** ([2, Definition 2.1]). Let $G$ be an ample Hausdorff groupoid. We say that $G$ is *locally contracting* if for every compact open $V \subseteq G^{(0)}$, there exists a compact open bisection $B \subseteq G$ such that

$$s(B) \subseteq r(B) \subseteq V.$$

Note that if $s(B) \subsetneq r(B)$, then we have

$$1_{s(B)} = 1_B - 1_B = 1_{BB^{-1}} = 1_{BB^{-1}}.$$  

Using $1_{s(B)}1_{r(B)} = 1_{s(B)} = 1_{r(B)}1_{s(B)}$, this follows $1_{s(B)} \leq 1_{r(B)} \sim 1_{s(B)}$, giving that $1_{r(B)}$ is an infinite idempotent in $A_K(G)$.

**Corollary 3.6.** Let $G$ be a strongly effective, ample Hausdorff groupoid and let $K$ be a field. If for every open invariant $U \subseteq G^{(0)}$, the groupoid $G/G^{(0)} \setminus U$ is locally contractive, then $A_K(G)$ is properly purely infinite.

**Proof.** Let $U$ be an invariant open subset of $G^{(0)}$, and let $V$ be a relatively compact open subset of $D := G^{(0)} \setminus U$. Since $G_D$ is locally contractive, there is some compact open bisection $B \subseteq G^{(0)}$ such that $s(B) \subseteq r(B) \subseteq V$. Hence $1_{r(B)}$ is infinite in $A_K(G_D)$ as explained above, and by $1_{r(B)} \leq 1_V$, then so is $1_V$ as well. Now (3) $\implies$ (1) in Theorem 3.4 concludes the result. □

### 4. Applications to Kumjian-Pask algebras

According to [13, Proposition 4.3] and [15, Proposition 5.4], every Kumjian-Pask algebra $KP_K(\Lambda)$ may be considered as the Steinberg algebra $A_K(G_{\Lambda})$ where $G_{\Lambda}$ is the boundary path groupoid of $\Lambda$. So, in view of Theorem 3.4, we may investigate the proper pure infiniteness of Kumjian-Pask algebras by groupoid approach. Moreover, [6, Theorem 7.4] says that the notions of purely infinite and properly purely infinite are equivalent for Leavitt path algebras. Here, we will also generalize this result to the class of Kumjian-Pask algebras by a quite different proof.

Let us first follow [19, Section 2] and briefly review the construction of the boundary path groupoid $G_{\Lambda}$ associated to $\Lambda$. (Although the groupoid
\( G_\Lambda \) of [19, Section 2] is constructed for a row-finite \( k \)-graph with no sources, however we may easily generalize it for any row-finite \( k \)-graph.) So, suppose that \( \Lambda \) is a fixed locally convex row-finite \( k \)-graph. We set
\[
G_\Lambda := \{ (\lambda x, d(\lambda) - d(\mu), \mu x) : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu), x \in s(\lambda) \Lambda^{\leq \infty} \}
\]
and define the range and source maps by \( r(x, m, y) := x \) and \( s(x, m, y) := y \). If we consider the composition as \( (x, m, y) \rangle (y, n, z) := (x, m + n, z) \) and the inversion as \( (x, m, y)^{-1} := (y, -m, x) \), then \( G_\Lambda \) is a groupoid. We usually identify the unit space \( G_\Lambda^{(0)} \) with the boundary path space \( \Lambda^{\leq \infty} \) by \( (x, 0, x) \leftrightarrow x \). Moreover, we may equip \( G_\Lambda \) with a locally compact Hausdorff topology induced from that of \( \Lambda^{\leq \infty} \). Indeed, for each \( \lambda, \mu \in \Lambda \) with \( s(\lambda) = s(\mu) \) we set
\[
Z(\lambda \ast_s \mu) := \{ (\lambda x, d(\lambda) - d(\mu), \mu x) : x \in s(\lambda) \Lambda^{\leq \infty} \}.
\]
When \( \lambda = \mu \), we write simply \( Z(\lambda) \) for \( Z(\lambda \ast_s \lambda) \). Then [19, Proposition 2.8] shows that the collection \( \{ Z(\lambda \ast_s \mu) : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu) \} \) forms a basis of compact open bisections for a Hausdorff topology on \( G_\Lambda \), making it an ample groupoid. Moreover, an application of the graded uniqueness theorem implies that \( KP_K(\Lambda) \cong A_K(G_\Lambda) \), which maps \( s_\lambda \) to \( 1_{Z(\lambda \ast_s \lambda)} \) (cf. [13, Proposition 4.3] and [15, Proposition 5.4]).

Before Theorem 4.2, we state a lemma.

**Lemma 4.1.** Let \( R \) be a purely infinite ring and let \( p \) be an idempotent in \( R \). If there are mutually orthogonal idempotents \( q_1, q_2 \in R \) such that \( p \in Rq_iR \), then \( p \) is properly infinite.

**Proof.** Since \( R \) is purely infinite, \( p \in Rq_iR \) implies \( p \preceq q_i \) for \( i \in \{1, 2\} \). Suppose \( p = a_i q_i b_i \). Putting \( x_i := q_i b_i a_i q_i \), then \( x_1, x_2 \) are orthogonal idempotents in \( R \) such that \( x_1 \sim p \sim x_2 \). Note that if \( x_1 = rs \) and \( x_2 = sr \) for \( r, s \in R \), then \( x_2 = x_2^2 = s(rs)r = rx_1s \). Hence \( x_1 + x_2 \) lies in \( Rx_1R \), and the pure infiniteness forces \( x_1 + x_2 \preceq x_1 \). Moreover, for \( A = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \) and \( B = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \), we have \( AB = x_1 \oplus x_2 \) and \( BA = x_1 + x_2 \). Therefore, we get
\[
p \oplus p \sim x_1 \oplus x_2 \sim x_1 + x_2 \preceq x_1 \sim p,
\]
establishing \( p \oplus p \preceq p \). \( \square \)

**Theorem 4.2.** Let \( \Lambda \) be a locally convex row-finite \( k \)-graph and \( K \) a field. Suppose that \( \Lambda \) is strongly aperiodic. Then the following are equivalent.

1. \( KP_K(\Lambda) \) is purely infinite.
2. \( KP_K(\Lambda) \) is properly purely infinite.
3. For each vertex \( v \in \Lambda^0 \), \( s_v \) is a properly infinite idempotent.

**Proof.** (1) \( \implies \) (3). Fix \( v \in \Lambda^0 \). In view of [6, Corollary 2.9], it suffices to show that the image of \( s_v \) in any quotient of \( KP_K(\Lambda) \) is either zero or infinite. Moreover, since \( \Lambda \) is strongly aperiodic, every ideal of \( KP_K(\Lambda) \) is of the form of \( IH \) for some saturated hereditary subset \( H \) of \( \Lambda^0 \) [14, Theorem 9.4]. So,
let $I_H$ be an ideal of $\text{KP}_K(\Lambda)$ such that $v \notin H$. Writing $\Gamma := \Lambda \setminus HA$, then $\Gamma$ is an aperiodic $k$-graph and we have $\text{KP}_K(\Lambda)/I_H \cong \text{KP}_K(\Gamma)$, which is purely infinite as well. In the following, for each $\gamma \in \Gamma$, we write $t_\gamma := s_\gamma + I_H$ in $\text{KP}_K(\Gamma)$. We divide our discussion into two cases:

Case I: There are two distinct vertices $v \neq u, w \in \Gamma^0$ such that $v\Gamma w, u\Gamma w \neq \emptyset$. Fix some $\mu_1 \in v\Gamma w$ and $\mu_2 \in u\Gamma w$. Then $q_1 := t_{\mu_1}t_{\mu_1}^*$ and $q_2 := t_{\mu_2}t_{\mu_2}^*$ are two orthogonal idempotents in $\text{KP}_K(\Gamma)$ (because $s_v$ and $s_u$ are) such that

$$q_1 \sim t_{\mu_1}t_{\mu_1} = t_w = t_{\mu_2}^*t_{\mu_2} \sim q_2.$$ 

Now, Lemma 4.1 implies that $t_w$ is properly infinite. Since $t_v \geq q_1 \sim t_{\mu_1}t_{\mu_1} = t_w$, apply [6, Corollary 3.9(iii)] to conclude that $t_v$ is infinite in $\text{KP}_K(\Gamma)$.

Case II: In the other case, there exists $w \in \Gamma^0$ with $v\Gamma w \neq \emptyset$ such that $w\Gamma u = \emptyset$ for all $u \in \Gamma^0 \setminus \{w\}$. If $w\Gamma w = \{w\}$, i.e. $w$ receives no nontrivial paths, then the ideal $I_w = \langle t_w \rangle$ in $\text{KP}_K(\Gamma)$ is a matricial algebra (cf. [20, Lemma 6.2]), which is impossible. Otherwise, since $\Gamma$ is aperiodic, [20, Lemma 6.1] implies that $w$ is the base of an (initial) cycle with an entrance. Thus $t_w$ is infinite (see Lemma 4.4(1) below). Now if $\mu \in v\Gamma w$, then $t_{\mu} \geq t_{\mu}t_{\mu}^* \sim t_{\mu}^*t_{\mu} = t_w$, and hence $t_{\mu}$ would be infinite in each case. Since $I_H$ was arbitrary, this follows that $s_v$ is a properly infinite idempotent in $\text{KP}_K(\Lambda)$ as desired.

(3) $\implies$ (2). To prove this part, we consider the groupoid $G_\Lambda$ associated to $\Lambda$, and then apply Theorem 3.4. According to [19, Proposition 2.8], the collection $\mathcal{B} = \{Z(\lambda) : \lambda \in \Lambda\}$ is a basis of compact and open sets for the topology on $G^{(0)}$. Also, for each $\lambda \in \Lambda$, we have

$$1_{Z(\lambda)} = s_\lambda s_\lambda^* \sim s_\lambda s_\lambda = s_{s(\lambda)},$$

and hence $1_{Z(\lambda)}$ is properly infinite by Lemma 3.1. Since $G_\Lambda$ is strongly effective [15, Proposition 6.3], Theorem 3.4 follows that $\text{KP}_K(\Lambda) = A_K(G_\Lambda)$ is properly purely infinite.

(2) $\implies$ (1) is immediate by [6, Lemma 3.4(i)].

Now, analogous to [20, Theorem 5.4], we may use the notion of generalized cycle to give a criterion under which the vertex idempotents $s_v$ are properly infinite. It is the $k$-graphic version of Corollary 3.6.

**Definition 4.3** ([17]). A generalized cycle in $\Lambda$ is a pair $(\mu, \nu)$ of distinct paths $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$ and $r(\mu) = r(\nu)$ such that $\text{MCE}(\mu, \nu) \neq \emptyset$ for all $\tau \in s(\mu)\Lambda$; or equivalently $Z(\mu) \subseteq Z(\nu)$. Note that if $\mu$ is a cycle in $\Lambda \setminus \Lambda^0$ (in the sense $s(\mu) = r(\mu)$), then $(\mu^i, \nu^j)$ is a generalized cycle for every $j \leq i \in \mathbb{N}$. We say that $\tau \in s(\nu)\Lambda$ is an entrance for $(\mu, \nu)$ if \text{MCE}(\mu, \nu\tau) = \emptyset \ (i.e., \nu\tau \Lambda \subseteq \emptyset \subseteq Z(\nu) \setminus Z(\mu))$.

Following [20], if there is a path from $s(\mu) = s(\nu)$ to $v \in \Lambda^0$ we say that $(\mu, \nu)$ connects to $v$ or $v$ is reached from $(\mu, \nu)$.

**Lemma 4.4.** Let $(\mu, \nu)$ be a generalized cycle in $\Lambda$ with an entrance. Then
(1) $s_{s(\mu)} = s_{s(\nu)}$ is an infinite idempotent in $\text{KP}_K(\Lambda)$.

(2) If $v \in \Lambda^0$ is reached from $(\mu, \nu)$, then $s_v$ is infinite in $\text{KP}_K(\Lambda)$.

Proof. [20, Lemma 4.3] says that $s_\mu s_{\mu^*} < s_\nu s_{\nu^*}$. Thus by

$$s_\nu s_{\nu^*} \sim s_\nu s_\nu = s_{s(\mu)} = s_\mu s_\mu \sim s_\mu s_{\mu^*},$$

$s_\nu s_{\nu^*}$ is an infinite idempotent, and so is $s_\nu s_\nu = s_{s(\nu)} = s_{s(\mu)}$ as well.

Statement (2) follows from (1). Indeed, if $\gamma \in v\Lambda s(\nu)$ then $s_v \geq s_\gamma s_\gamma^* \sim s_\gamma^* s_\gamma = s_{s(\mu)}$, and hence $s_\gamma s_\gamma^*$ and $s_v$ are infinite by part (1). \qed

**Corollary 4.5.** Let $\Lambda$ be a strongly aperiodic, locally convex and row-finite $k$-graph and let $K$ be a field. Suppose that for each saturated hereditary $H \subseteq \Lambda^0$, every vertex $v \in \Lambda^0 \setminus H = (\Lambda \setminus H\Lambda)^0$ is reached from a generalized cycle with an entrance in the $k$-graph $\Lambda \setminus H\Lambda$. Then $\text{KP}_K(\Lambda)$ is properly purely infinite.

Proof. Fix $v \in \Lambda^0$. For any ideal $I_H \subseteq \text{KP}_K(\Lambda)$ with $v \notin I_H$, $v$ is reached from a generalized cycle with an entrance in $\Lambda \setminus H\Lambda$, so $s_v + I_H$ is an infinite idempotent in $\text{KP}_K(\Lambda)/I_H \cong \text{KP}(\Lambda \setminus H\Lambda)$ by Lemma 4.4(2). Therefore, $s_v$ is properly infinite in $\text{KP}_K(\Lambda)$, and Theorem 4.2 concludes the result. \qed

In the case that there are only finitely many vertices connecting to each vertex $v \in \Lambda^0$, we can establish simple graph-theoretic criteria for the pure infiniteness of Kumjian-Pask algebras $\text{KP}_K(\Lambda)$. Note that this case covers all unital Kumjian-Pask algebras.

**Proposition 4.6** (See [20, Theorem 6.3]). Let $\Lambda$ be a locally convex row-finite $k$-graph and $K$ a field. Suppose that $\Lambda$ is strongly aperiodic, and that $\Lambda^0_{\geq v} := \{w : v\Lambda w \neq \emptyset\}$ is finite for every $v \in \Lambda^0$. Then the following are equivalent.

1. $v\Lambda \neq \{v\}$ for every vertex $v \in \Lambda^0$.

2. Every vertex of $\Lambda$ is reached from a cycle.

3. $\text{KP}_K(\Lambda)$ is properly purely infinite.

4. $\text{KP}_K(\Lambda)$ is purely infinite.

Proof. (1) $\implies$ (2). Fix $v \in \Lambda^0$ and assume $|\Lambda^0_{\geq v}| = t_0$. Using statement (1), for each $t \geq t_0^k$ in $\mathbb{N}$, there exists $\lambda \in v\Lambda$ such that $|\lambda| := d(\lambda)_1 + \cdots + d(\lambda)_k = t$, where $d(\lambda) = (d(\lambda)_1, \ldots, d(\lambda)_k)$. This forces $\lambda(m) = \lambda(n)$ for some $m < n \in \mathbb{N}^k$, and thus the cycle $\lambda(m, n)$ connects to $v$.

(2) $\implies$ (3). Fix some $v \in \Lambda^0$ and an ideal $I_H$ of $\text{KP}_K(\Lambda)$ such that $s_v \notin I_H$. Then $v \notin H$. For convenience, we write $\Gamma := \Lambda \setminus H\Lambda$, which is a strongly aperiodic $k$-graph too. We first claim $v\Gamma \neq \{v\}$. Indeed, if $v\Gamma = \{v\}$ then for each $n \in \mathbb{N}^k \setminus \{0\}$ we have $\{r(\lambda) : \lambda \in v\Lambda \leq n\} \subseteq H$. Thus $v \in H$ by the saturated property, a contradiction. Now implication (1) $\implies$ (2) above yields that $v$ is reached from a cycle in $\Gamma$, and using [20, Lemma 6.1], is also from an (initial) cycle with an entrance. Lemma 4.4 says that $s_v + I_H$ is an infinite idempotent in $\text{KP}_K(\Gamma) = \text{KP}_K(\Lambda)/I_H$. Therefore,
since \( I_H \) was arbitrary, \( s_v \) is properly infinite in \( KP_K(\Lambda) \), and consequently \( KP_K(\Lambda) \) is properly purely infinite by Theorem 4.2.

(3) \iff (4) are obtained from Theorem 4.2.

(3) and (4) \implies (1). If \( v\Lambda = \{v\} \) for some \( v \in \Lambda^0 \), then the ideal \( I_v = \langle s_v \rangle \) in \( KP_K(\Lambda) \) is isomorphic to the matrix algebra \( M_{|v\Lambda|}(K) \) (cf. [20, Lemma 6.2]). But this is impossible because the (proper) pure infiniteness passes to ideals.

\[ \square \]

5. **Complex Steinberg algebras and groupoid \( C^\ast \)-algebras**

In the last section, we investigate the relationship between the purely infinite property of Steinberg algebras and groupoid \( C^\ast \)-algebras. More precisely, since the complex Steinberg algebra \( A_C(\mathcal{G}) \) may be regarded as a dense *-subalgebra of \( C^\ast_p(\mathcal{G}) \) (see [13, Proposition 4.2]), we focus on the conjecture [6, Problem 8.4] for groupoid algebras: If \( A_C(\mathcal{G}) \) is properly purely infinite, is this property inherited by \( C^\ast_p(\mathcal{G}) \)? It is shown in [9, Theorem 4.1] that this conjecture is true for simple groupoid algebras (when \( \mathcal{G} \) is minimal and effective). However, in the nonsimple cases, the properly purely infinite property (in both algebraic and \( C^\ast \)-algebraic senses) is closely related to ideal structure and quotients. Here, we work with amenable groupoids in the sense of [3] to apply [3, Proposition 6.1.8] and the ideal description of [8, Corollary 5.9]. Moreover, since the groupoid \( \mathcal{G}_\Lambda \) associated to each \( k \)-graph \( \Lambda \) is always amenable, we deduce also an interesting result for \( k \)-graph algebras.

We recall the definition of reduced \( C^\ast \)-algebra \( C^\ast_r(\mathcal{G}) \) from [24]. Let \( \mathcal{G} \) be an ample groupoid. Write \( C_c(\mathcal{G}) \) for the complex vector space consisting of compactly supported continuous functions on \( \mathcal{G} \), which is an *-algebra with the convolution multiplication and the involution \( f^*(\alpha) := f(\alpha^{-1}) \). Then \( C^\ast_r(\mathcal{G}) \) is the completion of \( C_c(\mathcal{G}) \) for the left regular representations on \( \ell^2(\mathcal{G}) \). Indeed, for each \( u \in \mathcal{G} \) and \( \mathcal{G}_u := s^{-1}(\{u\}) \), if \( \pi_u : C_c(\mathcal{G}) \to B(\ell^2(\mathcal{G}_u)) \) is the left regular *-representation, defined by

\[
\pi_u(f)\delta_\alpha := \sum_{s(\beta) = r(\alpha)} f(\beta)\delta_\beta \quad (f \in C_c(\mathcal{G}), \ \alpha \in \mathcal{G}_u),
\]

then the reduced \( C^\ast \)-algebra \( C^\ast_r(\mathcal{G}) \) is the completion of \( C_c(\mathcal{G}) \) under the reduced \( C^\ast \)-norm

\[
\|f\|_r := \sup_{u \in \mathcal{G}^{(0)}} \|\pi_u(f)\|.
\]

Moreover, there is a full \( C^\ast \)-algebra \( C^\ast(\mathcal{G}) \) associated to \( \mathcal{G} \), which is the completion of \( C_c(\mathcal{G}) \) taken over all \( \|\cdot\|_{C_c(\mathcal{G})} \)-decreasing representations of \( \mathcal{G} \). So, \( C^\ast_r(\mathcal{G}) \) is a quotient of \( C^\ast(\mathcal{G}) \), and [3, Proposition 6.1.8] follows that they are equal if the underlying groupoid \( \mathcal{G} \) is amenable.

Let \( A \) be a \( C^\ast \)-algebra. For positive matrices \( a \in M_m(A)^+ \) and \( b \in M_n(A)^+ \) over \( A \), we write \( a \preceq_C b \) if there exists a sequence \( \{x_n\} \) in \( M_{m,n}(A) \) such that \( x_n b x_n \to a \). As before, for \( a, b \in A \) we denote \( a \oplus b := \text{diag}(a, b) \) in
$M_2(A)$. A nonzero positive element $a \in A$ is called $C^*$-\textit{infinite} if $a \oplus b \precsim_{C^*} a$ for some positive $0 \neq b \in A$, and is called $C^*$-\textit{properly infinite} if $a \oplus a \precsim_{C^*} a$. Following [18], we say that $A$ is $C^*$-\textit{purely infinite} in case every nonzero positive element of $A$ is properly infinite.

\textit{Definition 5.1} (See [24, Chaper 2]). A groupoid $G$ is called \textit{topologically principal} in case the set $\{u \in G^{(0)} : \{\alpha \in G : r(\alpha) = s(\alpha)\} = \{u\}\}$ is dense in $G^{(0)}$. We say that $G$ is \textit{essentially principal} if for every nonempty closed invariant subset $D$ of $G^{(0)}$, the groupoid $G_D$ is topologically principal.

Note that, when $G$ is second countable, [23, Proposition 3.6] follows that $G$ is strongly effective if and only if it is essentially principal.

The following is the $C^*$-analogue of Theorem 3.4 (see also [7, Theorem 4.1]).

\textit{Proposition 5.2.} Let $G$ be a second countable ample groupoid and let $B$ be a basis for $G^{(0)}$ containing compact open sets. Suppose that $G$ is amenable (in the sense of [3]) and essentially principal. Then $C^*(G) = C^*_e(G)$ is $C^*$-purely infinite if and only if $1_V$ is $C^*$-properly infinite for every $V \in B$.

\textit{Remark 5.3.} If $G$ is second countable, amenable and essentially principal (strongly effective), then [8, Corollary 5.9] implies that $C_0(G^{(0)})$ separates closed ideals of $C^*(G)$ in the sense that for every closed ideals $I \subseteq J$ of $C^*(G)$, there exists $f \in C_0(G^{(0)})$ such that $f \in I \setminus J$. Thus, in this case, [21, Proposition 2.14] implies that $C^*$-purely infinite, $C^*$-weakly purely infinite, and $C^*$-strongly purely infinite $C^*$-algebras coincide.

\textit{Proof of Proposition 5.2.} Since the “only if” implication is immediate, we prove the “if” part only. Suppose that all $1_V$, for $V \in B$, are $C^*$-properly infinite in $C^*(G)$. Using [18, Proposition 4.7], it suffices to show that every nonzero hereditary $C^*$-subalgebra in any quotient of $C^*(G)$ contains an infinite projection. For this, let $I$ be a closed ideal of $C^*(G)$ and let $A$ be a nonzero hereditary $C^*$-subalgebra of $C^*(G)/I$. By [8, Corollary 5.9], there is an open invariant $U \subseteq G^{(0)}$ such that $I = I_U$, and whence $C^*(G)/I \cong C^*(G_D)$ where $D := G^{(0)} \setminus U$. So we may assume $A \subseteq C^*(G_D)$. Let $0 \neq a \in A$ be a positive element. Then [10, Lemma 3.2] says that there exists a nonzero positive element $h \in C_0(D)$ such that $h \precsim_{C^*} a$ in $C^*(G_D)$. Now since $\{V \cap D : V \in B\}$ is a basis for the induced topology on $G_D^{(0)}$, there is $V \in B$ such that $V \cap D \neq \emptyset$ and $h(x) > 0$ for all $x \in \overline{V} := V \cap D$. Consider $g \in C_0(D)$ defined by

$$
g(x) = \begin{cases} \frac{1}{\sqrt{h(x)}} & x \in \overline{V} \\
0 & x \in D \setminus \overline{V}. \end{cases}$$

Then we have $ghg = 1_{\overline{V}}$, and thus $1_{\overline{V}} \precsim h \precsim a$ in $C^*(G_D)$. By the argument after [18, Proposition 2.6], we may find $y \in C^*(G_D)$ such that $1_{\overline{V}} = y^*ay^*$. If we set $z := a^{1/2}y$, then $z^*z = y^*ay = 1_{\overline{V}}$ and $zz^* = a^{1/2}yy^*a^{1/2} \in A$, giving...
$zz^* \sim_{C^*} 1_V$. Note that $1_V$ is (properly) infinite because it is the image of $1_V$ in the quotient $C^*(G_D) \cong C^*(G)/I$. Therefore, $zz^*$ is an infinite projection in $A$, as desired.

We now prove the main result of this section, which is the nonsimple generalization of [9, Theorem 4.1] with an easier way.

**Proposition 5.4.** Let $G$ be a second countable ample Hausdorff groupoid. Suppose that $G$ is amenable and strongly effective. If $A_C(G)$ is properly purely infinite (in the sense of Definition 3.2), then $C^*(G) = C^*_r(G)$ is $C^*$-purely infinite.

**Proof.** Suppose that $A_C(G)$ is properly purely infinite. According to Proposition 5.2, it suffices to show that $1_V$ is $C^*$-properly infinite for every nonempty compact open $V \subseteq G^{(0)}$. So fix some such $\emptyset \neq V \subseteq G^{(0)}$. Theorem 3.4 says that $1_V$ is a properly infinite idempotent in $A_C(G)$. Thus there exists $a, b \in M_2(A_C(G))$ such that $a(1_V \oplus 0)b = 1_V \oplus 1_V$. We may consider $M_2(A_C(G))$ as a subalgebra of $M_2(C^*(G))$ in the natural way, and then apply [25, Proposition 2.4 (iii) ⇒ (ii)] to find a sequence $\{r_n\} \subseteq M_2(C^*(G))$ such that $r_n(1_V \oplus 0)r_n^* \rightarrow 1_V \oplus 1_V$. Therefore, we have $1_V \oplus 1_V \prec_{C^*} 1_V \oplus 0$, meaning that $1_V$ is $C^*$-properly infinite. \qed

For any locally convex row-finite $k$-graph $\Lambda$, we may associate a universal $C^*$-algebra $C^*(\Lambda)$ [19, 22], which coincides with the groupoid $C^*$-algebra $C^*_r(G_\Lambda)$ (see [19, Corollary 3.5(i)] for example). Hence, Proposition 5.4 deduces the following result for $k$-graph algebras.

**Corollary 5.5.** Let $\Lambda$ be a locally convex row-finite $k$-graph, which is strongly aperiodic. If $KP_C(\Lambda)$ is purely infinite, then $C^*(\Lambda)$ is $C^*$-purely infinite.

**Proof.** By Theorem 4.2, $A_C(G_\Lambda) = KP_C(\Lambda)$ is properly purely infinite. Since the groupoid $G_\Lambda$ is always second-countable and amenable, Proposition 5.4 follows the result. \qed

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