INDEX MAP, $\sigma$-CONNECTIONS, AND CONNES-CHERN CHARACTER IN THE SETTING OF TWISTED SPECTRAL TRIPLES

RAPHAËL PONGE AND HANG WANG

Abstract. Twisted spectral triples are a twisting of the notion of spectral triple aiming at dealing with some type III geometric situations. In the first part of the paper, we give a geometric construction of the index map of a twisted spectral triple in terms of $\sigma$-connections on finitely generated projective modules. This makes it more transparent the analogy with the indices of Dirac operators with coefficients in vector bundles. In the second part, we give a direct construction of the Connes-Chern character of a twisted spectral, both in the invertible and non-invertible cases. Combining these two parts we obtain an analogue the Atiyah-Singer index formula for twisted spectral triples.

1. Introduction

Motivated by type III geometric situations, e.g., the action of an arbitrary group of diffeomorphisms on a manifold, Connes-Moscovici [CM4] introduced the notion of a twisted spectral triple. This is a modification of the usual definition of a spectral triple $(A, \mathcal{H}, D)$, where the boundedness of commutators $[D, a]$, $a \in A$, is replaced by that of twisted commutators $[D, a]_\sigma = Da - \sigma(a)D$, where $\sigma$ is a given automorphism of the algebra $A$. Examples include the following:

- Conformal deformations of ordinary spectral triples [CM4].
- Twistings of ordinary spectral triples by scaling automorphisms [Mo2].
- Conformal Dirac spectral triples $(C^\infty(M) \rtimes G, L^2_g(M, \mathcal{S}), D_g)_\sigma$, where $D_g$ is the Dirac operator acting on spinors and $G$ is a group of conformal diffeomorphisms [CM4].
- Spectral triples over noncommutative tori associated to conformal weights [CT, CM5].
- Spectral triples over noncommutative tori associated to various quantum statistical systems, including Connes-Bost systems, graphs, and supersymmetric Riemann gas [GMT].

We refer to Section 2 for a review of the first and third examples. Connes-Moscovici [CM4] showed that, as for ordinary spectral triples, the datum of a twisted spectral $(A, \mathcal{H}, D)_\sigma$ gives rise to a well defined index map $\text{ind}_D : K_0(A) \to \frac{1}{2}\mathbb{Z}$. Moreover, in the $p$-summable case, this index map is computed by the pairing of the $K$-theory $K_0(A)$ with a Connes-Chern character in ordinary cyclic cohomology.

One goal of this paper is to present a geometric interpretation of the index map of a twisted spectral triple. First, instead of compressing idempotents by $D$ and its inverse as in [CM4] (see also [FK1]), we define the index map in terms of Fredholm indices of the following operators,

$\sigma(e)D : e\mathcal{H}^o \to \sigma(e)\mathcal{H}^o$, $e \in M_q(A)$, $e^2 = e$.

This definition is totally analogous to the definition of the index map of an ordinary spectral triple mentioned in [Mo1].

In the case of an ordinary spectral, the index map is usually defined in terms of selfadjoint idempotents, since any idempotent is equivalent to a selfadjoint idempotent. For a twisted spectral triple $(A, \mathcal{H}, D)_\sigma$ the relevant notion of selfadjointness is meant with respect to the $\sigma$-involution $a \mapsto \sigma(a)^\ast$. We shall say that such an idempotent is $\sigma$-selfadjoint. In general, it is not clear that an idempotent is a $\sigma$-selfadjoint idempotent. For this reason, it is important to define the index map for arbitrary idempotents. As a result, for a twisted spectral triple the index map a priori takes values in $\frac{1}{2}\mathbb{Z}$. Nevertheless, when the automorphism has a suitable square root, it

---

R.P. was partially supported by Research Resettlement Fund and Foreign Faculty Research Fund of Seoul National University and Basic Research Grant 2013R1A1A2008802 of National Research Foundation of Korea (South Korea).
can be shown that any idempotent is equivalent to a \( \sigma \)-selfadjoint idempotent and the index map is integer-valued (Lemma 1.6). The precise condition is called ribbon (see Definition 4.5) and is satisfied by all the main examples of twisted spectral triples.

As it turns out, the aforementioned construction of the index map is only a special case of a more geometric construction in terms of couplings of the operator \( D \) with \( \sigma \)-connections on finitely generated projective modules. We refer to Section 5 for the precise definition of a \( \sigma \)-connection. This is the twisted analogue of the usual notion of a connection. The two notions actually agrees when \( \sigma = \text{id} \). Given a \( \sigma \)-connection on a finitely generated projective module \( \mathcal{E} \), the definition of the coupled operator \( D_{\nabla^\mathcal{E}} \) is similar to the coupling of a Dirac operator with a connection on an auxiliary vector bundle (see Section 5 for the precise definition). In the special case \( \mathcal{E} = e\mathcal{A}^q \) we recover the operator \( \sigma(e)De \) by using the so-called Grassmannian \( \sigma \)-connection, which is the twisted analogue of the Grassmannian connection. We then show that the operator \( D_{\nabla^\mathcal{E}} \) is Fredholm and we have

\[
\text{ind}_{D,\sigma}[\mathcal{E}] = \text{ind} D_{\nabla^\mathcal{E}}.
\]

This provides us with a geometric interpretation of the index map of a twisted spectral triple. In the case of an ordinary spectral triple we recover the geometric interpretation of the index map mentioned in [Mo1]. The above formula exhibits a close analogy with the definition of the standard Fredholm index map of a Dirac operator (the construction of which is recalled in Section 3). In particular, we recover the latter in the special case of an ordinary Dirac spectral triple (see the discussion on this point at the end of Section 5).

Another goal of this paper is to give a direct construction of the Connes-Chern character of a \( p \)-summable twisted spectral triple \( (\mathcal{A}, \mathcal{H}, D)_\sigma \). In [CM1] the Connes-Chern character is defined as the difference of Connes-Chern characters of a pair of bounded Fredholm modules canonically associated to the twisted spectral triple. This is the same passage as in [Co1] from the unbounded Fredholm module picture to the bounded Fredholm module picture. One advantage of our definition of the index map is the following index formula (Proposition 7.2):

\[
\text{ind} \sigma(e)De = \frac{1}{2} \text{Str} \left( (D^{-1}[D,e]_\sigma)^{2k+1} \right), \quad e = e^2 \in M_q(\mathcal{A}),
\]

where \( k \) is any integer \( \geq \frac{1}{2}(p-1) \) and \( D \) is assumed to be invertible. It is immediate that the right-hand side is the pairing of \( e \) is with the cochain given by

\[
\tau_{2k}^D(a^0, \ldots, a^{2k}) = c_k \text{Str} \left( D^{-1}[D,a^0]_\sigma \cdots D^{-1}[D,a^{2k}]_\sigma \right), \quad a^j \in \mathcal{A},
\]

where \( c_k \) is a normalization constant. This is the same cochain used in [CM1] to define the Connes-Chern character of a twisted spectral triple. We give a direct proof that \( \tau_{2k}^D \) is a normalized cyclic cocycle whose class in periodic cyclic cohomology is independent of \( k \) (Proposition 7.6). The Connes-Chern character \( \text{Ch}(D)_\sigma \) is then defined as the class in periodic cyclic cohomology of any cocycles \( \tau_{2k}^D \).

We also use some care to define the Connes-Chern character when \( D \) is non-invertible by passing to the unital invertible double, which we define as a twisted spectral triple over the augmented unital algebra \( \tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C} \). In the invertible case, we thus obtain two definitions of the Connes-Chern character, but these two definitions agree (see Proposition 7.14). This uses the homotopy invariance of the Connes-Chern character, a detailed proof of which is given in Appendix C.

With the use of the Connes-Chern character and the geometric interpretation 151 of the index map we obtain the following index formula: for any finitely generated projective module \( \mathcal{E} \) and \( \sigma \)-connection \( \nabla^\mathcal{E} \) on \( \mathcal{E} \), it holds that

\[
\text{ind} D_{\nabla^\mathcal{E}} = \langle \text{Ch}(D)_\sigma,[\mathcal{E}] \rangle.
\]

This is the analogue for twisted spectral triples of the Atiyah-Singer index formula.

We have attempted to give very detailed accounts on the constructions of the index map and Connes-Chern character of twisted spectral triples. It is hoped that the details of these constructions should also be helpful to readers who are primarily interested in understanding these constructions in the setting of ordinary spectral triples.
The paper is organized as follows. In Section 2, we review some important definitions and examples regarding twisted spectral triples. In Section 3, we present the construction of the Fredholm index map of a Dirac operator. In Section 4, we present the construction of the index map of a twisted spectral triple and single out a simple condition ensuring us it is integer-valued. In Section 5, we give a geometric description of the index map of a twisted spectral triple in terms of positive and negative spinors. In Section 6, we review the main definitions and properties of cyclic cohomology and periodic cyclic cohomology and their pairings with K-theory. In Section 7, we give a direct construction of the Connes-Chern character of a twisted spectral triple for both the invertible and non-invertible cases. In Appendix A and Appendix B, we present proofs of technical lemmas from Section 5. In Appendix C, we give a detailed proof of the homotopy invariance of the Connes-Chern character of a twisted spectral triple.

Acknowledgements

The authors would like to thank the following institutions for their hospitality during the preparation of this manuscript: Seoul National University (HW), Mathematical Sciences Center of Tsinghua University, Kyoto University (Research Institute of Mathematical Sciences and Department of Mathematics), and the University of Adelaide (RP), Australian National University, Chern Institute of Mathematics of Nankai University, and Fudan University (RP+HW).

2. Twisted Spectral Triples.

In this section, we review various definitions and examples regarding twisted spectral triples.

Definition 2.1. A spectral triple \((A, \mathcal{H}, D)\) consists of the following data:

1. A \(\mathbb{Z}_2\)-graded Hilbert space \(\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-\).
2. An involutive unital algebra \(A\) represented by bounded operators on \(\mathcal{H}\) preserving its \(\mathbb{Z}_2\)-grading.
3. A selfadjoint unbounded operator \(D\) on \(\mathcal{H}\) such that for all \(a \in A\),
   a. \(D\) maps \(\text{dom}(D) \cap \mathcal{H}^\pm\) to \(\mathcal{H}^\mp\).
   b. The resolvent \((D + i)^{-1}\) is a compact operator.
   c. \(a\ \text{dom}(D) \subset \text{dom}(D)\) and \([D, a]\) is bounded for all \(a \in A\).

Example 2.2. The paradigm of a spectral triple is given by a Dirac spectral triple,
\[(C^\infty(M), L^2_H(M, S), \mathcal{D}_g),\]
where \((M^n, g)\) is a compact spin Riemannian manifold (\(n\) even) and \(\mathcal{D}_g\) is its Dirac operator acting on the spinor bundle \(S\). In this case the \(\mathbb{Z}_2\)-grading of \(L^2(M, S)\) arises from the \(\mathbb{Z}_2\)-grading \(S = S^+ \oplus S^-\) of the spinor bundle in terms of positive and negative spinors.

The definition of a twisted spectral triple is similar to that of an ordinary spectral triple, except for some “twist” given by the conditions (3) and (4)(b) below.

Definition 2.3. A twisted spectral triple \((A, \mathcal{H}, D)_{\sigma}\) consists of the following:

1. A \(\mathbb{Z}_2\)-graded Hilbert space \(\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-\).
2. An involutive unital algebra \(A\) represented by even bounded operators on \(\mathcal{H}\).
3. An automorphism \(\sigma : A \to A\) such that \(\sigma(a)^\ast = \sigma^{-1}(a^\ast)\) for all \(a \in A\).
4. An odd selfadjoint unbounded operator \(D\) on \(\mathcal{H}\) such that
   a. The resolvent \((D + i)^{-1}\) is compact.
   b. \(a\ \text{dom}(D) \subset \text{dom}(D)\) and \([D, a]_{\sigma} := Da - \sigma(a)D\) is bounded for all \(a \in A\).

Remark 2.4. The condition that \(\sigma(a)^\ast = \sigma^{-1}(a^\ast)\) for all \(a \in A\) exactly means that the map
\(a \to \sigma(a)^\ast\) is an involutive antilinear automorphism of \(A\).
Remark 2.5. Throughout the paper we shall further assume that the algebra $A$ is closed under holomorphic functional calculus. This implies that an element $a \in A$ is invertible if and only if it is invertible in $\mathcal{L}(H)$. This also implies that all the algebras $M_q(A)$, $q \in \mathbb{N}$, are closed under holomorphic functional calculus.

Remark 2.6. The boundedness of twisted commutators naturally appear in the setting of quantum groups, but in the attempts of constructing twisted spectral over quantum groups the compactness of the resolvent of $D$ seems to fail (see [DA, KS, KW]). We also refer to [KW] for relationships between twisted spectral triples and Woronowicz’s covariant differential calculi.

2.2. Conformal deformations of ordinary spectral triples. An important class of examples of twisted spectral triples arises from conformal deformations (i.e., inner twistings) of ordinary spectral triples.

Let us start with a Dirac spectral triple $(C^\infty(M), L^2_g(M, \mathcal{S}), \mathcal{D}_g)$ associated to a compact Riemannian spin oriented manifold $(M^n, g)$ of even dimension. Consider a conformal change of metric,

$$\hat{g} = k^{-2}g, \quad k \in C^\infty(M), \; k > 0.$$ 

We then can form a new Dirac spectral triple $(C^\infty(M), L^2_{\hat{g}}(M, \mathcal{S}), \mathcal{D}_{\hat{g}})$. Bearing this in mind, note that the inner product of $L^2_g(M, \mathcal{S})$ is given by

$$\langle \xi, \eta \rangle_g := \int_M (\xi(x), \eta(x)) \sqrt{g(x)} dx, \quad \xi, \eta \in L^2_g(M, \mathcal{S}),$$

where $(\cdot, \cdot)$ is the Hermitian metric of $\mathcal{S}$ (and $n = \dim M$). Consider the linear isomorphism $U : L^2_g(M, \mathcal{S}) \to L^2_{\hat{g}}(M, \mathcal{S})$ given by

$$U \xi = k^{1/2} \xi \quad \forall \xi \in L^2_g(M, \mathcal{S}).$$

We observe that $U$ is a unitary operator since, for all $\xi \in L^2_g(M, \mathcal{S})$, we have

$$\langle U \xi, U \eta \rangle_g = \int_M (k(x)^{1/2} \xi(x), k(x)^{1/2} \eta(x)) \sqrt{k(x)^{-1}g(x)} dx = \langle \xi, \eta \rangle_g.$$ 

Moreover, the conformal invariance of the Dirac operator (see, e.g., [HH]) means that

$$\mathcal{D}_{\hat{g}} = k^{1/4} \mathcal{D}_g k^{-1/4}.$$ 

Thus,

$$U^* \mathcal{D}_{\hat{g}} U = k^{-1/2} \left( k^{1/4} \mathcal{D}_g k^{-1/4} \right) k^{1/2} = \sqrt{k} \mathcal{D}_g \sqrt{k}.$$ 

Therefore, we obtain the following result.

Proposition 2.7. The spectral triples $(C^\infty(M), L^2_g(M, \mathcal{S}), \mathcal{D}_g)$ and $(C^\infty(M), L^2_{\hat{g}}(M, \mathcal{S}), \sqrt{k} \mathcal{D}_g \sqrt{k})$ are unitarily equivalent.

Remark 2.8. Whereas the definition of $(C^\infty(M), L^2_g(M, \mathcal{S}), \mathcal{D}_g)$ requires $k$ to be smooth, in the definition of $(C^\infty(M), L^2_{\hat{g}}(M, \mathcal{S}), \sqrt{k} \mathcal{D}_g \sqrt{k})$ it is enough to assume that $k$ is a positive Lipschitz function.

More generally, let $(A, \mathcal{H}, D)$ be an ordinary spectral and $k$ a positive element of $A$. If we replace $D$ by its conformal deformation $kDk$ then, when $A$ is noncommutative, the triple $(A, \mathcal{H}, kDk)$ need not be an ordinary spectral triple. However, as the following result shows, it always gives rise to a twisted spectral triple.

Proposition 2.9 (CM1). Consider the automorphism $\sigma : A \to A$ defined by

$$\sigma(a) = k^2 ak^{-2} \quad \forall a \in A.$$ 

Then $(A, \mathcal{H}, kDk)_\sigma$ is a twisted spectral triple.
We also denote by \( \phi \), \( \phi \in \mathcal{A} \). This follows from the equalities,
\[
[kDk, a]_\sigma = (kDk)a - (k^2ak^{-2})(kDk) = k(D(kak^{-1}) - (kak^{-1})D)k = k[D, kak^{-1}]k.
\]

Remark 2.11. We refer to [PW3] for a generalization of the above construction in terms “pseudo-inner twistings” of ordinary spectral triples. We note that this construction also encapsulates the construction of twisted spectral triples over noncommutative tori associated to conformal weights of \( C_\mathbb{T} \).

2.3. Conformal Dirac spectral triple. The conformal Dirac spectral triple of \( C_{\mathbb{M}} \) is a nice illustration of the geometric relevance of twisted spectral triples. Let \( \Gamma \) be the diffeomorphism group of a compact manifold \( M \). In order to study the action of \( \Gamma \) on \( M \) noncommutative geometry suggests to seek for a spectral triple over the crossed-product algebra \( C^\infty(M) \rtimes \Gamma \), i.e., the algebra with generators \( f \in C^\infty(M) \) and \( u_\varphi, \varphi \in \Gamma \), with relations,
\[
u_\varphi u_\psi = u_\varphi u_\psi, \quad u_\varphi f = (f \circ \varphi)u_\varphi.
\]
The first set of relations implies that any unitary representation of \( C^\infty(M) \rtimes \Gamma \) induces a unitary representation of \( \Gamma \). The second set of relations shows the compatibility with the pushforward of functions of by diffeomorphisms.

The manifold structure is the only diffeomorphism-invariant differentiable structure on \( M \), so in particular \( M \) does not carry a diffeomorphism-invariant metric. This prevents us from constructing a unitary representation of \( \Gamma \) in a \( L^2 \)-space of tensors or differential forms or a first-order (pseudo)differential operator \( D \) with a \( \Gamma \)-invariant principal symbol (so as to ensure the boundedness of commutators \( [D, u_\varphi] \)). As observed by Connes [Co2] we can bypass this issue by passing to the total space of the metric bundle \( P \to M \) (seen as a ray subbundle of the bundle \( T^*M \otimes T^*M \) of symmetric 2-tensors). As it turns out, the metric bundle \( P \) carries a wealth of diffeomorphism-invariant structures, including a diffeomorphism-invariant Riemannian structure. The construction of a spectral triple over \( C^\infty(P) \rtimes \Gamma \) was carried out by Connes-Moscovici [CM1] who also computed its Connes-Chern character [CM2, CM3]. The passage from the base manifold \( M \) to the metric bundle \( P \) is the geometric counterpart of the well known passage from type III factors to type II factors by taking crossed-products with the action of \( \mathbb{R} \).

Even if there is a Thom isomorphism \( K_*C^\infty(P) \rtimes \Gamma \simeq K_*C^\infty(M) \rtimes \Gamma \), it would be desirable to work directly with the base manifold. As mentioned above there are obstructions to do so when dealing with the full group of diffeomorphism. However, as observed by Connes-Moscovici, if we restrict our attention to a group of diffeomorphisms preserving a conformal structure, then we are able to construct a spectral triple provided we relax the definition of ordinary spectral triple to that of a twisted spectral triple. This construction can be explained as follows.

Let \( M^n \) be a compact (closed) spin oriented manifold of even dimension \( n \) equipped with a conformal structure \( \mathcal{C} \), i.e., a conformal class of Riemannian metrics. We denote by \( G \) (the identity component of) the group of (smooth) orientation-preserving diffeomorphisms of \( M \) preserving the conformal and spin structures. Let \( g \) be a metric in the conformal class \( \mathcal{C} \) with associated Dirac operator \( D_g : C^\infty(M, \mathcal{S}) \to C^\infty(M, \mathcal{S}) \) acting on the sections of the spinor bundle \( \mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^- \).

We also denote by \( L^2_g(M, \mathcal{S}) \) the corresponding Hilbert space of \( L^2 \)-spinors.

If \( \phi : M \to M \) is a diffeomorphism preserving the conformal class \( \mathcal{C} \), then there is a unique function \( k_\phi \in C^\infty(M) \), \( k_\phi > 0 \), such that
\[
\phi_*g = k_\phi^2 g.
\]
(2.4)
In addition, \( \phi \) uniquely lifts to a unitary vector bundle isomorphism \( \phi^\mathcal{S} : \mathcal{S} \to \phi_*\mathcal{S} \), i.e., a unitary section of \( \operatorname{Hom}(\mathcal{S}, \phi_*\mathcal{S}) \) (see [BG]). We then let \( V_\phi : L^2_g(M, \mathcal{S}) \to L^2_g(M, \mathcal{S}) \) be the bounded operator given by
\[
V_\phi u(x) = \phi^\mathcal{S} (u \circ \phi^{-1})(x) \quad \forall u \in L^2_g(M, \mathcal{S}) \land x \in M.
\]
(2.5)
The map \( \phi \to V_\phi \) is a representation of \( G \) in \( L^2_g(M, \mathcal{S}) \), but this is not a unitary representation. In order to get a unitary representation we need to take into account the Jacobian \( |\phi'(x)| = k_\phi(x)^n \)
of $\phi \in G$. This is achieved by using the unitary operator $U_\phi : L^2_s(M, \mathcal{S}) \to L^2_s(M, \mathcal{S})$ given by
\begin{equation}
U_\phi = k^\pm_\phi V_\phi, \quad \phi \in G.
\end{equation}
Then $\phi \to U_\phi$ is a unitary representation of $G$ in $L^2_s(M, \mathcal{S})$. This enables us to represent the elements of the crossed-product algebra $C^\infty(M) \rtimes G$ as linear combinations of operators $fU_\phi$ on $L^2_s(M, \mathcal{S})$, where $\phi \in G$ and $f \in C^\infty(M)$ acts by scalar multiplication. These operators are subject to the relations,
\begin{equation}
U_{\phi^{-1}} = U_\phi^{-1} = U_\phi^* \quad \text{and} \quad U_\phi f = (f \circ \phi^{-1})U_\phi.
\end{equation}
We then let $\sigma_\phi$ be the automorphism of $C^\infty(M) \rtimes G$ given by
\begin{equation}
\sigma_\phi(fU_\phi) := k_\phi fU_\phi \quad \forall f \in C^\infty(M) \forall \phi \in G.
\end{equation}

Proposition 2.12 (CM [Moo]). The triple $(C^\infty(M) \rtimes G, L^2_s(M, \mathcal{S}), \mathcal{D}_g)_{\sigma_\phi}$ is a twisted spectral triple.

Remark 2.13. The bulk of the proof is showing the boundedness of the twisted commutators $[\mathcal{D}_g, U_\phi]_{\sigma_\phi}, \phi \in G$. We remark that
\begin{equation}
U_\phi \mathcal{D}_g U_\phi^* = k_\phi^\pm (V_\phi \mathcal{D}_g^0 V_{\phi^{-1}})k_\phi^{-\frac{1}{2}} = k_\phi^\pm \mathcal{D}_g k_\phi^{-\frac{1}{2}} = k_\phi^\pm \mathcal{D}_g k_\phi^{-\frac{1}{2}}.
\end{equation}
Thus, using the conformal invariance law (2.1), we get
\begin{equation}
U_\phi \mathcal{D}_g U_\phi^* = k_\phi^\pm \left( k_\phi^{-\frac{1}{2}} \mathcal{D}_g k_\phi^{-\frac{1}{2}} \right) k_\phi^{-\frac{1}{2}} = k_\phi^{-\frac{1}{2}} \mathcal{D}_g k_\phi^{-\frac{1}{2}}.
\end{equation}
Using this we see that the twisted commutator $[\mathcal{D}_g, U_\phi]_{\sigma_\phi} = \mathcal{D}_g U_\phi - k_\phi U_\phi \mathcal{D}_g$ is equal to
\begin{equation}
\left( \mathcal{D}_g k_\phi^\pm - k_\phi (U_\phi \mathcal{D}_g U_\phi^0) k_\phi \right) k_\phi^{-\frac{1}{2}} U_\phi = \left( \mathcal{D}_g k_\phi^\pm - k_\phi \mathcal{D}_g \right) k_\phi^{-\frac{1}{2}} U_\phi = [\mathcal{D}_g, k_\phi^\pm] k_\phi^{-\frac{1}{2}} U_\phi.
\end{equation}
This shows that $[\mathcal{D}_g, U_\phi]_{\sigma_\phi}$ is bounded.

3. The Fredholm Index Map of a Dirac Operator

In this section, we recall how the datum of a Dirac operator gives rise to an additive index map in $K$-theory. In the next two sections we shall generalize this construction to arbitrary twisted spectral triples.

Let $(M^n, g)$ be a compact spin oriented Riemannian manifold of even dimension $n$ and let $\mathcal{D}_g : C^\infty(M, \mathcal{S}) \to C^\infty(M, \mathcal{S})$ be the associated Dirac operator acting on sections of the spinor bundle. As $n$ is even, the spinor bundle splits as $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, where $\mathcal{S}^+$ (resp., $\mathcal{S}^-$) is the bundle of positive (resp., negative) spinors. The Dirac operator is odd with respect to this $\mathbb{Z}_2$-grading, and so it takes the form,
\begin{equation}
\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}, \quad \mathcal{D}^\pm : C^\infty(M, \mathcal{S}^\pm) \to C^\infty(M, \mathcal{S}^\pm).
\end{equation}
Let $E$ be a Hermitian bundle over $M$ and $\nabla^E : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ be a Hermitian connection on $E$. The operator $\mathcal{D}_{\nabla^E} : C^\infty(M, \mathcal{S} \otimes E) \to C^\infty(M, \mathcal{S} \otimes E)$ is defined by
\begin{equation}
\mathcal{D}_{\nabla^E} = \mathcal{D} \otimes 1_E + c(\nabla^E),
\end{equation}
where $c(\nabla^E)$ is given by the composition,
\begin{equation}
C^\infty(M, \mathcal{S} \otimes E) \xrightarrow{1_E \otimes \nabla^E} C^\infty(M, \mathcal{S} \otimes T^*M \otimes E) \xrightarrow{c \otimes 1_E} C^\infty(M, \mathcal{S} \otimes E),
\end{equation}
where $c : \mathcal{S} \otimes T^*M \to \mathcal{S}$ is the Clifford action of $T^*M$ on $\mathcal{S}$. With respect to the splitting $\mathcal{S} \otimes E = (\mathcal{S}^+ \otimes E) \oplus (\mathcal{S}^- \otimes E)$, the operator $\mathcal{D}_{\nabla^E}$ takes the form,
\begin{equation}
\mathcal{D}_{\nabla^E} = \begin{pmatrix} 0 & D^-_{\nabla^E} \\ D^+_{\nabla^E} & 0 \end{pmatrix}, \quad \mathcal{D}^\pm_{\nabla^E} : C^\infty(M, \mathcal{S}^\pm \otimes E) \to C^\infty(M, \mathcal{S}^\pm \otimes E).
\end{equation}
As $\nabla^E$ is a Hermitian connection, the operator $\nabla^E \phi$ is formally selfadjoint, i.e., $(\nabla^E)^* = \nabla^E$. Moreover, $\nabla^E$ is an elliptic differential operator, and hence is Fredholm. We then define its Fredholm index by

$$\text{ind} \nabla^E := \dim \ker \nabla^E - \dim \ker \nabla^E.$$

This index is computed by the local index formula of Atiyah-Singer [AS1, AS2],

$$\text{ind} \nabla^E = (2i\pi)^{-}\int_M \hat{A}(R^M) \wedge \text{Ch}(F^E),$$

where $\hat{A}(R^M) = \det \left( \frac{R^M/2}{\sinh(R^M/2)} \right)$ is the (total) $\hat{A}$-form of the Riemann curvature $R^M$ and $\text{Ch}(F^E) = \text{Tr} \left( e^{-F^E} \right)$ is the (total) Chern form of the curvature $F^E$ of the connection $\nabla^E$.

We observe that even without using the Atiyah-Singer index formula, it is not difficult to see that its value is independent of the choice of the Hermitian structure of $E$ and the Hermitian connection $\nabla^E$, since the principal symbol of $\nabla^E$ does not depend on these data. Second, let $\phi : E \to E'$ be a vector bundle isomorphism. We pushforward the Hermitian metric of $E$ to a Hermitian metric on $E'$, so that pushforwarding the connection $\nabla^E$ we get a Hermitian connection on $E'$. Then $\nabla^E' = (1 \otimes \phi) \nabla^E (1 \otimes \phi^*)$, so that $\ker \nabla^E \simeq \ker \nabla^E'$, and hence $\text{ind} \nabla^E = \text{ind} \nabla^E'$. In addition, let $F$ be another Hermitian vector bundle equipped with a Hermitian connection $\nabla^F$. We equip $E \oplus F$ with the connection $\nabla^{E \oplus F} = \nabla^E \oplus \nabla^F$. Then, with respect to the splitting $\mathcal{S} \otimes (E \oplus F) = (\mathcal{S} \otimes E) \oplus (\mathcal{S} \otimes F)$, it holds that $\nabla^E \oplus \nabla^F = \nabla^E \oplus \nabla^F$, so that $\ker \nabla^E \oplus \nabla^F = \ker \nabla^E \oplus \ker \nabla^F$. Thus,

$$\text{ind} \nabla^E \oplus \nabla^F = \text{ind} \nabla^E + \text{ind} \nabla^F.$$

It follows from all this that $\text{ind} \nabla^E$ depends only on the $K$-theory class of $E$, and there actually is a well-defined additive map,

$$\text{ind}_\nabla : K^0(M) \to \mathbb{Z},$$

such that, for any Hermitian vector bundle $E$ equipped with a Hermitian connection $E$, it holds that

$$\text{ind}_\nabla [E] = \text{ind} \nabla^E.$$

Let $H_{[0]}(M, \mathbb{C}) = \bigoplus_{i \geq 0} H_{2i}(M, \mathbb{C})$ be the even de Rham homology of $M$ and $H^{[0]}(M, \mathbb{C}) = \bigoplus_{i \geq 0} H^{2i}(M, \mathbb{C})$ its even de Rham cohomology. Composing the natural duality pairing between $H_{[0]}(M, \mathbb{C})$ and $H^{[0]}(M, \mathbb{C})$ with the Chern character map $\text{Ch} : K^0(M) \to H^{[0]}(M, \mathbb{C})$, we obtain a bilinear pairing,

$$\langle \cdot, \cdot \rangle : H_{[0]}(M, \mathbb{C}) \times K^0(M) \to \mathbb{C},$$

so that, for any closed even de Rham current $C$ and any vector bundle $E$ over $M$, we have

$$\langle [C], [E] \rangle = \langle C, \text{Ch}(F^E) \rangle,$$

where $F^E$ is the curvature of any connection on $E$. Then the Atiyah-Singer index formula can be rewritten as

$$\text{ind} D^E = (2i\pi)^{-} \left\langle \hat{A}(R^M)^\wedge, \text{Ch}(F^E) \right\rangle = (2i\pi)^{-} \left\langle \left[ \hat{A}(R^M)^\wedge \right], [E] \right\rangle,$$

where $[\hat{A}(R^M)^\wedge]$ is the homology class of the Poincaré dual of the $\hat{A}$-form $\hat{A}(R^M)$.

Finally, we stress out that the definition of $\nabla^E \phi$ does not require the connection $\nabla^E$ to be Hermitian, and so the construction of $\nabla^E$ holds for any connection $\nabla^E$ on $E$. In this general case, the operator $\nabla^E$ need not be selfadjoint, but it still is Fredholm and of the form $\gamma^E$. 

A priori we could consider the two Fredholm indices $\text{ind} \nabla^E$ and $\text{ind} \nabla^E$ separately. When $\nabla^E$ is Hermitian, we have $\text{ind} \nabla^E = \text{ind}(\nabla^E)^* = -\text{ind} \nabla^E$. The value of these indices are
independent of the the choice of Hermitian connection, so we see that \( \text{ind}(\mathcal{D}^+_{\nabla E}) = -\text{ind}(\mathcal{D}^-_{\nabla E}) \) even when \( \nabla E \) is not Hermitian. In any case, we equivalently could define the index of \( \mathcal{D}_{\nabla E} \) by

\[
(3.6) \quad \text{ind}(\mathcal{D}_{\nabla E}) = \frac{1}{2} \left( \text{ind}(\mathcal{D}^+_{\nabla E}) - \text{ind}(\mathcal{D}^-_{\nabla E}) \right).
\]

4. The Index Map of a Twisted Spectral Triple

Let \((A, \mathcal{H}, D)\) be a twisted spectral triple. As observed by Connes-Moscovici \cite{CM}, the datum of \((A, \mathcal{H}, D)\) gives rise to a well-defined index map \( \text{ind}_{D, \sigma} : K_0(A) \to \frac{1}{2} \mathbb{Z} \). The definition of the index map in \cite{CM} is based on the observation that the phase \( F = D|D|^{-\frac{1}{2}} \) defines an ordinary Fredholm module over \( A \) (namely, the pair \((\mathcal{H}, F)\)). The index map is then defined in terms of compressions of \( F \) by idempotents. As we shall now explain, we also can define the index map by using twisted versions of the compression of the operator \( D \) by idempotents. This construction is actually a special case of the coupling of \( D \) by \( \sigma \)-connections which will be described in the next section. We still need to deal with this special case in order to carry out the more general construction of the next section.

Let \( e \) be an idempotent in \( \mathcal{M}_q(A), q \in \mathbb{N} \). We regard \( e \mathcal{H}^q \) as a closed subspace of the Hilbert space \( \mathcal{H}^q \), so that \( e \mathcal{H}^q \) is a Hilbert space with the induced inner product. As the action of \( A \) on \( \mathcal{H}^q \) is by \( \sigma \)-operators \( e \mathcal{H} \cap (\mathcal{H}^q) = e(\mathcal{H}^q)^q \), and so we have the orthogonal splitting \( e \mathcal{H}^q = e(\mathcal{H}^+)^q \oplus e(\mathcal{H}^-)^q \). In addition, the action of \( A \) preserves the domain of \( D \), so we see that \( e(\text{dom}(D))^q = e(\text{dom}(D))^q \cap e \mathcal{H}^q \). We then let \( D_{e, \sigma} \) be the unbounded operator from \( e \mathcal{H}^q \) to \( \sigma(e) \mathcal{H}^q \) given by

\[
(4.1) \quad D_{e, \sigma} := e(\mathcal{D} \oplus 1_q), \quad \text{dom}(D_{e, \sigma}) = e(\text{dom}(D))^q.
\]

We note that, as \( D \) is an odd operator, with respect to the orthogonal splitting \( \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \) it takes the form,

\[
(4.2) \quad D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}, \quad D^\pm : \text{dom}(D) \cap \mathcal{H}^\pm \to \mathcal{H}^\mp.
\]

Incidentally, with respect to the orthogonal splittings \( e \mathcal{H}^q = e(\mathcal{H}^+)^q \oplus e(\mathcal{H}^-)^q \) and \( \sigma(e) \mathcal{H}^q = \sigma(e)(\mathcal{H}^+)^q \oplus \sigma(e)(\mathcal{H}^-)^q \) the operator \( D_{e, \sigma} \) takes the form,

\[
D_{e, \sigma} = \begin{pmatrix} 0 & D_{e, \sigma}^- \\ D_{e, \sigma}^+ & 0 \end{pmatrix}, \quad D_{e, \sigma}^\pm = \sigma(e)(D^\pm \oplus 1_q).
\]

In order to determine the adjoint of \( D_{e, \sigma} \), we make the following observation.

**Lemma 4.1.** Let \( S_e : e \mathcal{H}^q \to e^* \mathcal{H}^q \) be the restriction to \( e \mathcal{H}^q \) of \( e^* \) (which we represent as an operator on \( \mathcal{H}^q \)). Then \( S_e \) is a linear isomorphism from \( e \mathcal{H}^q \) onto \( e^* \mathcal{H}^q \) such that

\[
(4.3) \quad \langle S_e \xi_1, \xi_2 \rangle = \langle \xi_1, \xi_2 \rangle \quad \forall \xi_j \in e \mathcal{H}^q. 
\]

**Proof.** Let \( \xi_j \in e \mathcal{H}^q, j = 1, 2 \). Then

\[
\langle S_e \xi_1, \xi_2 \rangle = \langle e^* \xi_1, \xi_2 \rangle = \langle \xi_1, e \xi_2 \rangle = \langle \xi_1, \xi_2 \rangle.
\]

In particular, when \( \xi_2 = \xi_1 \), we get \( \langle S_e \xi_1, \xi_1 \rangle = ||\xi_1||^2 \), which shows that \( S_e \) is one-to-one.

Let \( \eta \in e^* \mathcal{H}^q \). Then \( \langle \eta, \cdot \rangle_{e \mathcal{H}^q} \) is a continuous linear form on \( e \mathcal{H}^q \), so there exists \( \tilde{\eta} \in \mathcal{H}^q \) such that \( \langle \eta, \xi \rangle = \langle \tilde{\eta}, \xi \rangle \) for all \( \xi \in e \mathcal{H}^q \). Therefore, for all \( \xi \in \mathcal{H}^q \),

\[
\langle \eta, \xi \rangle = \langle e^* \eta, \xi \rangle = \langle \eta, e \xi \rangle = \langle \tilde{\eta}, e \xi \rangle = \langle e^* \tilde{\eta}, \xi \rangle.
\]

Thus \( \eta = e^* \tilde{\eta} = S_e \tilde{\eta} \). This shows that \( S_e \) is onto. As \( S_e \) is one-to-one we then deduce that \( S_e \) is linear isomorphism. The proof is complete. \( \square \)

The above lemma holds for the idempotent \( \sigma(e) \) as well. In what follows, we denote by \( S_{\sigma(e)} \) the linear isomorphism from \( \sigma(e) \mathcal{H}^q \) to \( \sigma(e)^* \mathcal{H}^q \) induced by \( \sigma(e)^* \).

**Lemma 4.2.** Let \( D_{e, \sigma}^* \) be the adjoint of \( D_{e, \sigma} \). Then

\[
(4.4) \quad D_{e, \sigma}^* = S_e^{-1} D_{\sigma(e)^*, \sigma(e)}.
\]
Proof. Let $D_{s,e}^1$ be the operator given by the following graph,

$$G(D_{s,e}^1) = \{(ξ, η) ∈ σ(e)^∗H^q × e^∗H^q; (ξ, D_{s,e}^1 ζ) = (η, ζ) \quad ∀ζ ∈ \text{dom}(D_{s,e})\}.$$ 

We note that the graph of $D_{s,e}^*$ is

$$G(D_{s,e}^*) = \{(ξ, η) ∈ σ(e)H^q × eH^q; (ξ, D_{s,e}^* ζ) = (η, ζ) \quad ∀ζ ∈ \text{dom}(D_{s,e})\}.$$ 

It then follows from (4.3) that a pair $(ξ, η) ∈ σ(e)H^q × eH^q$ is contained in $G(D_{s,e}^*)$ if and only if $(S_{s,e}(ξ), S_{s,e}η)$ lies in $G(D_{s,e}^1)$. That is, $S_{s,e} D_{s,e}^* = D_{s,e}^1 S_{s,e}$. Therefore, showing that $D_{s,e}^* = S_{s,e}^{-1} D_{s,e}^1 S_{s,e}$ is equivalent to showing that the operators $D_{s,e}^1$ and $D_{s,e}^*$ agree.

Let $(ξ, η) ∈ G(D_{s,e}^1)$. For all $ζ ∈ \text{dom}(D_{s,e})$, we have

$$⟨D_{s,e}^1 ζ, η⟩ = ⟨σ(e)^∗(D ⊗ 1_q)ζ, η⟩ = (ξ, σ(e)(D ⊗ 1_q)ζ) = ⟨ξ, D_{s,e} ζ⟩.$$ 

Thus $(ξ, D_{s,e}^* η)$ belongs to $G(D_{s,e}^1)$. Thus $G(D_{s,e}^*)$ is contained in $G(D_{s,e}^1)$, i.e., $D_{s,e}^*$ is an extension of $D_{s,e}^1$.

Let $(ξ, η) ∈ G(D_{s,e}^1)$ and set $R := σ(e)(D ⊗ 1_q)(1 − e)$. We note that

$$R = σ(e)(D ⊗ 1_q)(1 − e) = σ(e)\{(1 − σ(e))(D ⊗ 1_q) − [D ⊗ 1_q], e\} = −σ(e)[D ⊗ 1_q, e]σ.$$ 

Thus, $R$ is a bounded operator. Incidentally, its adjoint $R^*$ is a bounded operator as well. Set $\tilde{η} = η + (1 − e^∗)R^∗S_{s,e}^{-1} ξ$ and let $ζ ∈ (\text{dom}(D)^q)$. As $eζ ∈ \text{dom}(D_{s,e})$ and the subspaces $e^∗H^q$ and $(1 − e)H^q$ are orthogonal to each other, we have

$$⟨\tilde{η}, ζ⟩ = ⟨η, eζ⟩ + ⟨η, (1 − e)ζ⟩ + \left\{(1 − e^∗)R^∗S_{s,e}^{-1} ξ, ζ\right\} = ⟨ξ, D_{s,e} η⟩ + \left\{S_{s,e}^{-1} ξ, R((1 − e)ζ)\right\}.$$ 

Moreover, as $ξ ∈ σ(e)^∗H^q$ and $R((1 − e)ζ) ∈ σ(e)H^q$, using (4.3) we see that $\left\{S_{s,e}^{-1} ξ, R((1 − e)ζ)\right\}$ agrees with $⟨ξ, R((1 − e)ζ)⟩$. Therefore, $⟨\tilde{η}, ζ⟩$ is equal to

$$⟨ξ, D_{s,e} η⟩ + R((1 − e)ζ) = ⟨ξ, σ(e)(D ⊗ 1_q)ζ⟩ = ⟨σ(e)^∗ ξ, (D ⊗ 1_q)ζ⟩ = ⟨ξ, (D ⊗ 1_q)ζ⟩.$$ 

This shows that $(ξ, η)$ lies in the graph of the operator $(D ⊗ 1_q)^*$, which agrees with $D ⊗ 1_q$ since $D$ is selfadjoint. Thus $(ξ, η)$ lies in the graph of $D ⊗ 1_q$. Therefore, we see that $ξ$ is contained in both $(\text{dom}(D))^q$ and $σ(e)^∗H^q$, so it lies in $(\text{dom}(D))^q ∩ σ(e)^∗H^q = σ(e)^∗(\text{dom}(D))^q = \text{dom}(D_{s,e}^*)$. This shows that $D_{s,e}^*$ is contained in $\text{dom}(D_{s,e})$. As $D_{s,e}^1$ is an extension of $D_{s,e}^*$, we then deduce that the two operators agree. As explained above this proves that $D_{s,e}^* = S_{s,e}^{-1} D_{s,e}^1 S_{s,e}$. The proof is complete. \[\square\]

**Lemma 4.3.** The operator $D_{s,e}$ is closed and Fredholm, and we have

$$\text{ind } D_{s,e}^± = \dim \text{ker } D_{s,e}^± - \dim \text{ker } D_{s,e}^∗.$$ 

Proof. Substituting $σ(e)^∗ = σ^{-1}(e^∗)$ for $e$ in (4.3) shows that $D_{s,e}^1 = S_{s,e}^{-1} D_{s,e} S_{s,e}^∗$, i.e., $D_{s,e}^* = S_{s,e} D_{s,e}^* S_{s,e}^∗$. As $D_{s,e}^*$ is a closed operator and the operators $S_{s,e}^∗$ and $S_{s,e}^{-1}$ are bounded, we see that $D_{s,e}^*$ is a closed operator.

Let $D^{-1}$ be the partial inverse of $D$ and $P_0$ be the orthogonal projection onto $\text{ker } D$. Set $Q_{e,s} := e(D^{-1} ⊗ 1_q)$, which we regard as a bounded operator from $σ(e)H^q$ to $eH^q$. Note that $Q_{e,s}$ is a compact operator. Moreover, on $σ(e)H^q$ we have

$$D_{s,e} Q_{e,s} = σ(e)(D ⊗ 1_q)e(D^{-1} ⊗ 1_q) = σ(e) + σ(e)[D ⊗ 1_q, e]σ(D^{-1} ⊗ 1_q) − σ(e)(P_0 ⊗ 1_q) = 1 + e[D ⊗ 1_q, e](D^{-1} ⊗ 1_q) − σ(e)(P_0 ⊗ 1_q).$$

Likewise, on $e(\text{dom}(D))^q$ we have

$$Q_{e,s} D_{s,e} = e(D^{-1} ⊗ 1_q)σ(e)(D ⊗ 1_q) = 1 − eD^{-1}[D ⊗ 1_q, e]σ − e(P_0 ⊗ 1_q)e.$$ 

As $D^{-1}, P_0$ are compact operators and $[D ⊗ 1_q, e]$ is bounded, we see that $Q_{e,s}$ inverts $D_{s,e}$ modulo compact operators. It then follows that $D_{s,e}$ is a Fredholm operator.
We note that $S_e$ and $S_{\sigma(e)}$ are even operators, so the equality \(4.4\) means that $(D^\pm_{e,\sigma})^* = S_e^{-1}D^\mp_{\sigma(e)^*,\sigma}S_{\sigma(e)}^{-1}$. Therefore, the operator $S_e$ induces an isomorphism $\ker D^\mp_{\sigma(e)^*,\sigma} \simeq \ker (D^\pm_{e,\sigma})^*$. Thus,

\[
\text{ind } D^\pm_{e,\sigma} = \dim \ker D^\pm_{e,\sigma} - \dim \ker (D^\pm_{e,\sigma})^* = \dim \ker D^\pm_{e,\sigma} - \dim \ker (D^\mp_{\sigma(e)^*,\sigma})^*.
\]

The proof is complete. \(\square\)

We define the index of $D_{e,\sigma}$ by

\[(4.6)\quad \text{ind } D_{e,\sigma} := \frac{1}{2} \left( \text{ind } D^+_{e,\sigma} - \text{ind } D^-_{e,\sigma} \right).
\]

Thanks to \(4.5\) we have

\[(4.7)\quad \text{ind } D_{e,\sigma} = \frac{1}{2} \left( \dim \ker D^+_{e,\sigma} + \dim \ker D^+_{\sigma(e)^*,\sigma} - \dim \ker D^-_{e,\sigma} - \dim \ker D^-_{\sigma(e)^*,\sigma} \right).
\]

In particular, when $\sigma(e)^* = e$ we get

\[
\text{ind } D_{e,\sigma} = \dim \ker D^+_{e,\sigma} - \dim \ker D^-_{e,\sigma}.
\]

Let $g \in \text{Gl}_q(A)$ and set $\hat{e} = g^{-1}eg$. On $(\text{dom}(D))^q$ the operator $\sigma(\hat{e})(D \otimes 1_q)\hat{e}$ agrees with

\[
\sigma(g)^{-1}\sigma(e)\sigma(g)(D \otimes 1_q)g^{-1}eg = \sigma(g)^{-1}\sigma(e)\sigma(g) \sigma(g^{-1})(D \otimes 1_q)eg + \sigma(g)^{-1}\sigma(e)\sigma(g)[D \otimes 1_q, g^{-1}]_\sigma eg
\]

\[
= \sigma(g)^{-1}D_{e,\sigma}g + \sigma(g)^{-1}\sigma(e)\sigma(g)[D \otimes 1_q, g^{-1}]_\sigma eg.
\]

As $[D \otimes 1_q, g^{-1}]_\sigma$ is a bounded operator, we see that $D^\pm_{e,\sigma}$ and $\sigma(g)^{-1}(D^\pm_{\sigma(e)^*,\sigma})g$ agree up to a bounded operator. It then follows that $D^\pm_{e,\sigma}$ and $D^\pm_{\sigma(e)^*,\sigma}$ have the same Fredholm index. Thus,

\[(4.8)\quad \text{ind } D_{e,\sigma} = \text{ind } D_{\sigma(e)^*,\sigma} \quad \forall g \in \text{Gl}_q(A).
\]

Moreover, if $e' \in M_q(A)$ is another idempotent, then, with respect to the splittings $(e \oplus e')(H^\pm)^q = e(H^\pm)^q \oplus e'(H^\pm)^q$ and $\sigma(e \oplus e')(H^\pm)^q = \sigma(e)(H^\pm)^q \oplus \sigma(e')(H^\pm)^q$, we have $D^\pm_{e \oplus e',\sigma} = D^\pm_{e,\sigma} \oplus D^\pm_{e',\sigma}$.

We then see that $\text{ind } D^\pm_{e \oplus e',\sigma} = \text{ind } D^\pm_{e,\sigma} + \text{ind } D^\pm_{e',\sigma}$. Thus,

\[
\text{ind } D^\pm_{e \oplus e',\sigma} = \text{ind } D_{e,\sigma} + \text{ind } D_{e',\sigma}.
\]

Therefore, we arrive at the following statement.

**Proposition 4.4 (CMH).** There is a unique additive map $\text{ind}_{D,\sigma} : K_0(A) \to \frac{1}{2} \mathbb{Z}$ such that

\[(4.9)\quad \text{ind}_{D,\sigma}[e] = \text{ind } D_{e,\sigma} \quad \forall e \in M_q(A), \ e^2 = e.
\]

As pointed out in Remark \(2.4\) the fact that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in A$ means that the map $a \to \sigma(a)^*$ is an involutive antilinear automorphism of $A$, which we shall call the $\sigma$-involution. An element $a \in A$ is selfadjoint with respect to this involution if and only if $\sigma(a)^* = a$. As \(4.7\) shows that when $\sigma(e)^* = e$ the index of $D_{e,\sigma}$ is an integer. While an idempotent in $M_q(A)$ is always conjugate to a selfadjoint idempotent, in general it need not be conjugate to an idempotent which is selfadjoint with respect to the $\sigma$-involution. Nevertheless, this property holds under a further assumption on the automorphism $\sigma$.

**Definition 4.5.** The automorphism $\sigma$ is called ribbon when it has a square root in the sense there is an automorphism $\tau : A \to A$ such that

\[(4.10)\quad \sigma(a) = \tau(\tau(a)) \quad \text{and} \quad \tau(a)^* = \tau^{-1}(a^*) \quad \text{for all } a \in A.
\]

**Lemma 4.6.** Assume that the automorphism $\sigma$ is ribbon. Then

(i) Any idempotent $e \in M_q(e), \ q \in \mathbb{N}$, is conjugate to an idempotent which is selfadjoint with respect to the $\sigma$-involution.

(ii) The index map $\text{ind}_{D,\sigma}$ is integer-valued.
Proof. The 2nd part follows by combining the first part with (4.8) and (4.5). Therefore, we only need to prove the first part. In addition, without any loss of generality, we may assume that \( q = 1 \) in the first part. Thus let \( e \) be an idempotent element of \( \mathcal{A} \).

Let us briefly recall how we construct a selfadjoint idempotent in \( \mathcal{A} \) which is conjugate to \( e \) (see, e.g., [Bl, Prop. 4.6.2] for more details). Set \( a = e - e^\ast \) and \( b = 1 + aa^\ast \). Observing that \( b \) is an invertible element of \( \mathcal{A} \) which commutes with \( e \) and \( e^\ast \), define \( p = ee^\ast b^{-1} \). It can be checked that \( p^2 = p^* = p \), i.e., \( p \) is a selfadjoint idempotent of \( \mathcal{A} \). Moreover, if we set \( g = 1 - p + e \), then \( g \) has inverse \( g^{-1} = 1 + p - e \) and \( g^{-1}pg = e \).

We remark that the above construction holds verbatim if we replace the involution \( a \to a^\ast \) by any other involutive antilinear automorphism of \( \mathcal{A} \), provided it can be shown that the corresponding operator \( b \) is invertible (in \( \mathcal{A} \)). Thus, if we substitute \( \sigma(e)^\ast \) for \( e^\ast \) and we assume that \( b := 1 + \sigma(a)^\ast a \) is invertible, where \( a = e - \sigma(e)^\ast \), then \( p := e\sigma(e)^\ast b^{-1} \) is such that \( p^2 = \sigma(p)^\ast = p \) and \( g^{-1}pg = e \) where \( g := 1 - p + e \) has inverse \( g^{-1} = 1 + p - e \). Therefore, the main question at stake is to show that \( b \) is invertible.

Let \( \tau \) be a square root of \( \sigma \) in the sense of (4.10). Then \( \tau(\sigma(a)^\ast) = \tau(\sigma^{-1}(a^\ast)) = \tau^{-1}(a^\ast) = \tau(a)^\ast \). Thus,

\[
\tau(b) = 1 + \tau(\sigma(a)^\ast a) = 1 + \tau(\sigma(a)^\ast)\tau(a) = 1 + \tau(a)^\ast \tau(a).
\]

As \( \tau(a)^\ast \tau(a) \) is a positive element of \( \mathcal{A} \) we see that \( \tau(b) \) is invertible, and hence \( b \) is invertible as well. The proof is complete. \( \square \)

As we shall now see, the ribbon condition (4.10) is satisfied by the automorphisms occurring in the main examples of twisted spectral triples.

**Example 4.7.** Assume that \( \sigma(a) = kak^{-1} \) where \( k \) is a positive invertible element of \( \mathcal{A} \). Then \( \sigma \) has the square root \( \tau(a) = k^\frac{\sigma}{2}a k^{-\frac{\sigma}{2}} \). We note that \( k^\frac{\sigma}{2} \) is an element of \( \mathcal{A} \) since \( \mathcal{A} \) is closed under holomorphic functional calculus.

More generally, we have the following.

**Example 4.8.** Suppose that \( \sigma \) agrees with the value at \( t = -i \) of the analytic extension of a strongly continuous one-parameter group of isometric \( * \)-isomorphisms \( \sigma_t \in \mathbb{R} \). This condition is called (1PG) in [CM4]. In this case \( \sigma \) is ribbon with square root \( \tau := \sigma_t|_{t=-i/2} \).

**Remark 4.9.** By a result of Bost [Bo] the analytic extension of a strongly continuous one-parameter group of isometric isomorphisms on an involutive Banach algebra always exists on a dense subalgebra which is closed under holomorphic functional calculus.

**Remark 4.10.** Connes-Moscovici [CM4] proved that under the (1PG) condition, it holds that \( \text{ind } D_{e,\sigma}^+ = - \text{ind } D_{e,\sigma}^- \), and so in this the index map \( \text{ind}_{D_{e,\sigma}} \) is integer valued.

**Example 4.11.** The ribbon condition is also satisfied by the automorphism \( \sigma_g \) appearing in the construction of the conformal Dirac spectral triple. From the definition (2.7) of \( \sigma_g \) we see that a square root satisfies the ribbon condition (4.10). Indeed, a square root satisfying (4.10) is given by the automorphism \( \tau_g \) defined by

\[
\tau_g(fU_\phi) := \sqrt{k_\phi}fU_\phi \quad \forall f \in C^\infty(M) \forall \phi \in G.
\]

In fact, \( \sigma_g \) satisfies the (1PG) condition with respect to the one-parameter group of isometric \( * \)-isomorphisms \( \sigma_t \), \( t \in \mathbb{R} \), given by \( \sigma_t(fU_\phi) = k_t^\sigma fU_\phi \).

5. Index Map and \( \sigma \)-Connections

In this section, we present a more geometric description of the index map of a twisted spectral triple in terms of couplings by \( \sigma \)-connections on finitely generated projective modules (i.e., noncommutative vector bundles). As we shall explain in the next section, this description makes it much more transparent the analogy with the construction of the index map in the commutative case in terms of Dirac operators coupled with connections (see, e.g., [BGV]). We refer to [Mo1] for a similar description of the index map in the case of ordinary spectral triples.
Throughout this section we let \((A, \mathcal{H}, D)\) be a twisted spectral triple. Let \(E\) be a finitely generated projective right module over \(A\), i.e., \(E\) is the direct summand of a finite rank free module \(E_0 \cong A^q\). Let \(\phi : E_0 \to A^q\) be a right module isomorphism. The image of \(E\) by \(\phi\) is a right module of the form \(eA^q\) for some idempotent \(e \in M_q(A)\). Set \(E^\sigma := \phi^{-1}(\sigma(e)A^q)\); this is a direct summand of \(E_0\). The isomorphism \(\phi\) induces isomorphisms of right modules,

\[
\phi : E \to eA^q \quad \text{and} \quad \phi^\sigma : E^\sigma \to \sigma(e)A^q.
\]

In addition, the automorphism \(\sigma\) lifts to \(A^q\) by

\[
\sigma(\xi) = (\sigma(\xi)) \quad \forall \xi = (\xi_j) \in A^q.
\]

We observe that \(\sigma\) maps \(eA^q\) to \(\sigma(e)A^q\). Set \(\sigma^\phi = \phi^{-1} \circ \sigma \circ \phi\). Then \(\sigma^\phi\) induces a linear isomorphism \(\sigma^\phi : E \to E^\sigma\) such that

\[
\sigma \circ \phi = \phi^\sigma \circ \sigma^\phi \quad \text{and} \quad \sigma^\phi(\xi a) = \sigma^\phi(x)\sigma(a) \quad \forall \xi, a \in A \text{ and } \xi \in E.
\]

Thus \(\sigma^\phi\) is a right module isomorphism from \(E\) onto \(E^\sigma\), where \(E^\sigma\) is \(E^\sigma\) equipped with the action \((\xi, a) \to \xi \sigma(a)\).

This leads us to the following definition.

**Definition 5.1.** Let \(E\) be a finitely generated projective right module over \(A\). We say that a finitely generated projective right module \(E^\sigma\) is a \(\sigma\)-translate for \(E\) when there exist

(i) A linear isomorphism \(\sigma^\phi : E \to E^\sigma\).

(ii) An idempotent \(e \in M_q(A)\), \(q \in \mathbb{N}\).

(iii) Right module isomorphisms \(\phi : E \to eA^q\) and \(\phi^\sigma : E^\sigma \to \sigma(e)A^q\) such that

\[
(5.1) \quad \phi^\sigma \circ \sigma^\phi = \sigma \circ \phi.
\]

**Remark 5.2.** The condition (5.1) implies that

\[
\sigma^\phi(\xi a) = \sigma^\phi(\xi)\sigma(a) \quad \forall \xi \in E \text{ and } a \in A.
\]

**Remark 5.3.** When \(\sigma = \text{id}\) we shall always take \(E^\sigma = E\) as \(\sigma\)-translate of \(E\). When \(E = eA^q\) with \(e = e^2 \in M_q(A)\) we will always take \(E^\sigma = \sigma(e)A^q\) as \(\sigma\)-translate of \(eA^q\). In this case \(\sigma^\phi\) agrees on \(eA^q\) with the lift of \(\sigma\) to \(A^q\).

**Lemma 5.4.** Suppose that the automorphism \(\sigma\) is ribbon in the sense of \((4.10)\). Then, in Definition 5.1 we may choose the idempotent \(e \in M_q(A)\) so that \(\sigma(e) = e^\sigma\).

**Proof.** Let \(E\) be a finitely generated projective module with \(\sigma\)-translate \(E^\sigma\). Thus there are an idempotent \(e \in M_q(A)\) and right-module isomorphisms \(\phi : E \to eA^q\) and \(\phi^\sigma : E^\sigma \to \sigma(e)A^q\) such that \(\phi^\sigma \circ \sigma^\phi = \sigma \circ \phi\). By Lemma 1.10 there is \(g \in \text{GL}_q(A)\) such that the idempotent \(f := geg^{-1}\) obeys \(\sigma(f) = f^\sigma\). We then have a right module isomorphism \(\psi : E \to fA^q\) given by

\[
\psi(\xi) = g\phi(\xi) \quad \forall \xi \in E.
\]

As \(\sigma(e) = \sigma(g)\sigma(f)\sigma(g)^{-1}\), we also have a right-module isomorphism \(\psi^\sigma : E^\sigma \to \sigma(f)A^q\) given by

\[
\psi^\sigma(\xi) = \sigma(g)\phi^\sigma(\xi) \quad \forall \xi \in E^\sigma.
\]

Furthermore, for all \(\xi \in E\), we have

\[
(5.2) \quad \sigma \circ \psi(\xi) = \sigma(\phi(\xi)) = \sigma(g)\sigma(\phi(\xi)) = \sigma(g)\phi^\sigma(\sigma(\phi(\xi))) = \psi^\sigma \circ \sigma^\phi(\xi).
\]

Therefore the triple \(\{f, \psi, \psi^\sigma\}\) satisfies the conditions (i)–(iii) of Definition 5.1. This proves the lemma. \(\square\)

Following [CM4] we consider the space of twisted 1-forms,

\[
\Omega^1_{D,\sigma}(A) = \{ \Sigma a^i[D, b^j]_\sigma : a^i, b^j \in A\}.
\]

This is naturally an \((A, A)\)-bimodule, since

\[
a^i(a^j[D, b^k]_\sigma) - a^i a^j[D, b^k]_\sigma = a^i a^j[D, b^k]_\sigma - a^i a^j[D, b^k]_\sigma = a^i a^j[D, b^k]_\sigma \quad \forall a^i, b^j \in A.
\]

We also have a “twisted” differential \(d_\sigma : A \to \Omega^1_{D,\sigma}(A)\) defined by

\[
d_\sigma a := [D, a]_\sigma \quad \forall a \in A.
\]
This is a $\sigma$-derivation, in the sense that
\[(5.3) \quad d_\sigma(ab) = (d_\sigma a)b + \sigma(a)d_\sigma b \quad \forall a, b \in A.\]

Let $E$ be a finitely generated projective right module over $A$ and $E^\sigma$ a $\sigma$-translate of $E$.

**Definition 5.5.** A $\sigma$-connection on $E$ is a $\mathbb{C}$-linear map $\nabla : E \to E^\sigma \otimes_A \Omega^1_{D,\sigma}(A)$ such that
\[(5.4) \quad \nabla(\xi) = (\nabla\xi)a + \sigma^\varepsilon(\xi) \otimes d_\sigma a \quad \forall \xi \in E \forall a \in A.\]

**Example 5.6.** Suppose that $E = eA^q$ with $e = e^2 \in M_q(A)$. Then a natural $\sigma$-connection on $E$ is the Grassmannian $\sigma$-connection $\nabla_0^E$ defined by
\[(5.5) \quad \nabla_0^E \xi = \sigma(e)(d_\sigma \xi) \quad \text{for all } \xi = (\xi_j) \in E.\]

**Lemma 5.7.** The set of $\sigma$-connections on $E$ is an affine space modeled on $\text{Hom}_A(E, E^\sigma \otimes \Omega^1_{D,\sigma}(A))$.

**Proof.** It follows from (5.4) that two $\sigma$-connections on $E$ differ by an element of $\text{Hom}_A(E, E^\sigma \otimes \Omega^1_{D,\sigma}(A))$. Therefore, the only issue at stake is to show that the set of $\sigma$-connections is nonempty. This is a true fact when $E = eA^q$ with $e = e^2 \in M_q(A)$ since in this case there is always the Grassmannian $\sigma$-connection.

In general, there are an idempotent $e \in M_q(A)$ and right module isomorphisms $\phi : E \to eA^q$ and $\phi^\varepsilon : E^\sigma \to \sigma(e)A^q$ satisfying (5.1). We then can pullback to $E$ any connection $\nabla$ on $A^q$ to the linear map $\nabla^E : E \to E^\sigma \otimes_A \Omega^1_{D,\sigma}(A)$ defined by
\[\nabla^E = \left( (\phi^\varepsilon)^{-1} \otimes 1_{\Omega^1_{D,\sigma}(A)} \right) \circ \nabla \circ \phi.\]

For $\xi \in E$ and $a \in A$ we have
\[\nabla^E(\xi a) = \left( (\phi^\varepsilon)^{-1} \otimes 1_{\Omega^1_{D,\sigma}(A)} \right) \circ \nabla(\phi(\xi)a) = \left( (\phi^\varepsilon)^{-1} \circ \sigma \circ \phi \right) \nabla^E(\xi) a + \left( (\phi^\varepsilon)^{-1} \circ \sigma \circ \phi \right) \sigma^\varepsilon(\xi) \otimes d_\sigma a = \nabla^E(\xi) a + \sigma^\varepsilon(\xi) \otimes d_\sigma a.\]

Thus $\nabla^E$ is a $\sigma$-connection on $E$. The proof is complete. \qed

In what follows we denote by $E'$ the dual $A$-module $\text{Hom}_A(E, A)$.

**Definition 5.8.** A Hermitian metric on $E$ is a map $(\cdot, \cdot) : E \times E \to A$ such that
1. $(\cdot, \cdot)$ is $A$-sesquilinear, i.e., it is $A$-antilinear with respect to the first variable and $A$-linear with respect to the second variable.
2. $(\cdot, \cdot)$ is positive, i.e., $(\xi, \xi) \geq 0$ for all $\xi \in E$.
3. $(\cdot, \cdot)$ is nondegenerate, i.e., $\xi \to (\xi, \cdot)$ is an $A$-antilinear isomorphism from $E$ onto $E'$.

**Remark 5.9.** Using (2) and a polarization argument it can be shown $(\xi_1, \xi_2) = (\xi_2, \xi_1)^*$ for all $\xi_j \in A$.

**Example 5.10.** The canonical Hermitian structure on the free module $A^q$ is given by
\[(5.6) \quad (\xi, \eta)_0 = \xi^\varepsilon \eta_1 + \cdots + \xi^\varepsilon_0 \eta_q \quad \text{for all } \xi = (\xi_j) \text{ and } \eta = (\eta_j) \in A^q.\]

**Lemma 5.11.** Suppose that $E = eA^q$ with $e = e^2 \in M_q(A)$. Then the canonical Hermitian metric of $A^q$ induces a Hermitian metric on $E$.

**Proof.** See Appendix [A]. \qed

**Remark 5.12.** Let $\phi : E \to F$ be an isomorphism of finitely generated projective modules and assume $F$ carries a Hermitian metric $(\cdot, \cdot)_F$. Then using $\phi$ we can pullback the Hermitian metric of $F$ to the Hermitian metric on $E$ given by
\[\xi_1, \xi_2)_E := (\phi(\xi_1), \phi(\xi_2))_F \quad \forall \xi_j \in E.\]

In particular, if we take $F$ to be of the form $eA^q$ with $e = e^2 \in M_q(A)$, then we can pullback the canonical Hermitian metric $(\cdot, \cdot)_0$ to a Hermitian metric on $E$. \qed
From now on we assume that $E$ carries a Hermitian metric. We denote by $\mathcal{H}(E)$ the pre-Hilbert space consisting of $E \otimes_A H$ equipped with the Hermitian inner product,

$$(5.7) \quad (\xi_1 \otimes \zeta_1, \xi_2 \otimes \zeta_2) := (\zeta_1, (\xi_1, \xi_2) \zeta_2), \quad \xi_j, \zeta_j \in H,$$

where $(\cdot, \cdot)$ is the Hermitian metric of $E$.

**Lemma 5.13.** $\mathcal{H}(E)$ is a Hilbert space whose topology is independent of the choice of the Hermitian inner product of $E$.

**Proof.** See Appendix B. \hfill $\square$

**Remark 5.14.** In [Mo] the Hilbert space $\mathcal{H}(E)$ as the completion of $E \otimes_A H$ with respect to the Hermitian inner product $(5.7)$. As Lemma 5.13 shows this per-Hilbert space is already complete.

We note there is a natural $\mathbb{Z}_2$-grading on $\mathcal{H}(E)$ given by

$$(5.8) \quad \mathcal{H}(E)^{\pm} := \mathcal{H}^\pm(E) \oplus \mathcal{H}^\mp(E), \quad \mathcal{H}^\pm(E) := E \otimes_A H^\pm.$$

In what follows we let $E^\sigma$ be a $\sigma$-translate of $E$ equipped with a Hermitian metric. We form the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}(E^\sigma)$ as above. In addition, we let $\nabla^E$ be a $\sigma$-connection on $E$. Regarding $\Omega^1_{D,\sigma}(A)$ as a subalgebra of $\mathcal{L}(H)$ we have a natural left-action $c : \Omega^1_{D,\sigma}(A) \otimes_A H \rightarrow H$ given by

$$c(\omega \otimes \zeta) = \omega(\zeta) \quad \text{for all } \omega \in \Omega^1_{D,\sigma}(A) \text{ and } \zeta \in H.$$

We denote by $c(\nabla^E)$ the composition $(1_{E^\sigma} \otimes c) \circ (\nabla^E \otimes 1_H) : E \otimes H \rightarrow E^\sigma \otimes H$. Thus, for $\xi \in E$ and $\zeta \in H$, and upon writing $\nabla^E \xi = \sum \xi_\alpha \otimes \omega_\alpha$ with $\xi_\alpha \in E^\sigma$ and $\omega_\alpha \in \Omega^1_{D,\sigma}(A)$, we have

$$(5.9) \quad c(\nabla^E)(\xi \otimes \zeta) = \sum \xi_\alpha \otimes \omega_\alpha(\zeta).$$

In what follows we regard the domain of $D$ as a left $A$-module, which is possible since the action of $A$ on $H$ preserves dom$(D)$.

**Definition 5.15.** The coupled operator $D_{\nabla^E} : E \otimes_A \text{dom}(D) \rightarrow \mathcal{H}(E^\sigma)$ is defined by

$$(5.10) \quad D_{\nabla^E}(\xi \otimes \zeta) := \sigma^E(\xi) \otimes D\zeta + c(\nabla^E)(\xi \otimes \zeta) \quad \text{for all } \xi \in E \text{ and } \zeta \in \text{dom}(D).$$

**Remark 5.16.** The operators $\sigma^E$, $D$ and $\nabla^E$ are not module maps we need to check the compatibility of $(5.10)$ with the action of $A$. This is a consequence of $(5.2)$. Indeed, if $\xi \in E$ and $\zeta \in \text{dom}(D)$, then, for all $a \in A$,

$$c(\nabla^E)(\xi a \otimes \zeta) = (1 \otimes c)(\nabla^E(\xi a) \otimes \zeta) = (1 \otimes c)((\nabla^E \xi) a \otimes \zeta + \sigma^E(\xi) \otimes d_\sigma(a)(\zeta) + \sigma^E(\xi) \otimes d_\sigma(a)(\zeta))$$

Thus,

$$D_{\nabla^E}(\xi a \otimes \zeta) - D_{\nabla^E}(\xi \otimes a\zeta) = \sigma^E(\xi a) \otimes D\zeta + \sigma^E(\xi) \otimes d_\sigma(a)(\zeta) - \sigma^E(\xi) \otimes D(a\zeta)$$

which shows that $D_{\nabla^E}(\xi \otimes a\zeta) = D_{\nabla^E}(\xi \otimes \zeta)$ in $E^\sigma \otimes_A H$.

**Remark 5.17.** With respect to the $\mathbb{Z}_2$-gradings $(5.8)$ for $\mathcal{H}(E)$ and $\mathcal{H}(E^\sigma)$ the operator $D_{\nabla^E}$ takes the form,

$$D_{\nabla^E} = \begin{pmatrix} 0 & D_{\nabla^E}^- \\ D_{\nabla^E}^+ & 0 \end{pmatrix}, \quad D_{\nabla^E}^\pm : E \otimes_A \text{dom}(D) \rightarrow \mathcal{H}^\pm(E^\sigma).$$

That is, $D_{\nabla^E}$ is an odd operator.

Suppose that $E = eA^q$ with $e = e^2 \in M_q(A)$. Then there is a canonical isomorphism $U_e$ from $\mathcal{H}(E)$ to $e\mathcal{H}^q$ given by

$$U_e(\xi \otimes \zeta):= (\zeta_1 \cdots \zeta_q)_{1\leq j \leq q} \quad \text{for all } \xi = (\xi_1) \in E \text{ and } \zeta \in H,$$

where $E = eA^q$ is regarded as a submodule of $A^q$. The inverse of $U_e$ is given by

$$U_e^{-1}((\zeta_1)) = \sum e_j \otimes \zeta_j \quad \text{for all } (\zeta_j) \in e\mathcal{H}^q,$$

where $(\cdot, \cdot)$ is the Hermitian metric of $E$. 

[14]
where \( e \mathcal{H}^q \) is regarded as a subspace of \( \mathcal{H}^q \) and \( e_1, \ldots, e_q \) are the column vectors of \( e \). We also note that \( U_e \) is a graded isomorphism.

**Lemma 5.18.** Suppose that \( \mathcal{E} = e \mathcal{A}^q \) as above and let \( \nabla^e \mathcal{E} \) be the Grassmannian \( \sigma \)-connection of \( \mathcal{E} \). Then

\[
U_{\sigma(e)} D_{\nabla^e \mathcal{E}} U_e^{-1} = D_{e,\sigma},
\]

where \( D_{e,\sigma} \) is defined in (5.2).

**Proof.** The image of \( \mathcal{E} \otimes \mathcal{A} \) dom(D) under \( U_e \) is \( e(\text{dom}(D))^q = \text{dom}(D_{e,\sigma}) \). Let \( \zeta \in \text{dom}(D) \) and let \( \xi = (\xi_j) \) be in \( \mathcal{E} \subset \mathcal{A}^q \). Then

\[

\begin{align*}
& c(\nabla^e \mathcal{E}) (\xi \otimes \zeta) = \sum \sigma(e_j) \otimes (d_\sigma \xi_j) \zeta = \sum \sigma(e_j) \otimes D(\xi_j \zeta) - \sum \sigma(e_j) \otimes \sigma(\xi_j) D\zeta. 
\end{align*}

\]

The fact that \( \xi \in \mathcal{E} \) means that \( \xi = \xi_j \otimes \xi_j \). Thus,

\[
\sum \sigma(e_j) \otimes \sigma(\xi_j) D\zeta = \sum \sigma(e_j) \zeta \otimes D\zeta = \sigma^2(\xi) \otimes \zeta.
\]

Therefore,

\[

\begin{align}
D_{\nabla^e \mathcal{E}} (\xi \otimes \zeta) &= \sigma^2(\xi) \otimes D\eta + c(\nabla^e \mathcal{E}) (\xi \otimes \zeta) = \sum \sigma(e_j) \otimes D(\xi_j \zeta). 
\end{align}

\]

For \( j = 1, \ldots, q \) set \( \zeta_j = \xi_j \xi_j \), so that \( U_e (\xi \otimes \zeta) = (\xi_j \zeta_j)_{1 \leq j \leq q} = (\xi_j)_{1 \leq j \leq q} \). From (5.11) we get

\[
U_{\sigma(e)} D_{\nabla^e \mathcal{E}} (\xi \otimes \zeta) = \sum_j \sigma(e_j) D(\zeta_j).
\]

In view of the definition of the operator \( D_{e,\sigma} \) in (5.11) we see that

\[
U_{\sigma(e)} D_{\nabla^e \mathcal{E}} (\xi \otimes \zeta) = (\sigma(e) (D\xi_j))_{1 \leq j \leq q} = D_{e,\sigma} ((\xi_j)_{1 \leq j \leq q}) = D_{e,\sigma} U_e (\xi \otimes \zeta).
\]

This proves the lemma. \( \square \)

**Remark 5.19.** As \( U_e \) and \( U_{\sigma(e)} \) are graded isomorphisms, we further have

\[
U_{\sigma(e)} D_{\nabla^e \mathcal{E}} (U_e^{-1})^{-1} = D_{e,\sigma}.
\]

**Lemma 5.20.** Let \( \mathcal{E} \) be a Hermitian projective module with Hermitian \( \sigma \)-translate \( \mathcal{E}^\sigma \). Then, for any \( \sigma \)-connection \( \nabla^\mathcal{E} \) on \( \mathcal{E} \), the operator \( D_{\nabla^\mathcal{E}} \) is Fredholm

**Proof.** Let us first assume that \( \mathcal{E} = e \mathcal{A}^q \) and \( \mathcal{E}^\sigma = \sigma(e) \mathcal{A}^q \) with \( e = e^2 \in M_q(\mathcal{A}) \). As shown in Section [3], the operator \( D_{e,\sigma} \) is Fredholm, so it follows from Lemma 5.18 that \( D_{\nabla^\mathcal{E}} \) is a Fredholm operator as well. By Lemma 5.7 the difference \( \nabla^\mathcal{E} - \nabla^e \mathcal{E} \) lies in \( \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E}^\sigma) \otimes \Omega^1_{D,\sigma}(\mathcal{A}) \). This implies that \( D_{\nabla^\mathcal{E}} - D_{\nabla^e \mathcal{E}} \) is a bounded odd operator. As \( D_{\nabla^e \mathcal{E}} \) is Fredholm, it then follows that so is \( D_{\nabla^\mathcal{E}} \).

In general, we can find an idempotent \( e \in M_q(\mathcal{A}) \) and right module isomorphisms \( \phi : \mathcal{E} \to e \mathcal{A}^q \) and \( \phi^\sigma : \mathcal{E}^\sigma \to \sigma(e) \mathcal{A}^q \) satisfying (5.1). We equip \( e \mathcal{A}^q \) and \( \sigma(e) \mathcal{A}^q \) with the Hermitian metrics,

\[
(\cdot, \cdot)_e := (\phi^{-1}(\cdot), \phi^{-1}(\cdot))_\mathcal{E} \quad \text{and} \quad (\cdot, \cdot)_{\phi^\sigma} := ((\phi^\sigma)^{-1}(\cdot), (\phi^\sigma)^{-1}(\cdot))_{\mathcal{E}^\sigma},
\]

where \( (\cdot, \cdot)_\mathcal{E} \) (resp., \( (\cdot, \cdot)_{\mathcal{E}^\sigma} \)) is the Hermitian metric of \( \mathcal{E} \) (resp., \( \mathcal{E}^\sigma \)). We then have unitary operators \( U_\phi : \mathcal{H}(\mathcal{E}) \to \mathcal{H}(e \mathcal{A}^q) \) and \( U_{\phi^\sigma} : \mathcal{H}(\mathcal{E}^\sigma) \to \sigma(e) \mathcal{A}^q \) given by

\[
U_\phi := \phi \otimes 1_{\mathcal{H}} \quad \text{and} \quad U_{\phi^\sigma} := \phi^\sigma \otimes 1_{\mathcal{H}}.
\]

In addition, we denote by \( \nabla^{\phi,\mathcal{E}} \) the \( \sigma \)-connection on \( e \mathcal{A}^q \) defined by

\[
\nabla^{\phi,\mathcal{E}} := (\phi^\sigma \otimes 1_{\Omega^1_{D,\sigma}(\mathcal{A})}) \circ \nabla^{\mathcal{E}} \circ \phi^{-1}.
\]

Let \( \xi \in \mathcal{E} \) and \( \zeta \in \text{dom}(D) \). Set \( \eta = \phi(\xi) \) and \( \nabla^\mathcal{E} \xi = \sum \xi_\alpha \otimes \omega_\alpha \) with \( \xi_\alpha \in \mathcal{E}^\sigma \) and \( \omega_\alpha \in \Omega^1_{D,\sigma}(\mathcal{A}) \). Then \( \nabla^{\phi,\mathcal{E}} \eta = (\phi^\sigma \otimes 1_{\Omega^1_{D,\sigma}(\mathcal{A})}) (\nabla^{\mathcal{E}} \xi) = \sum (\phi^\sigma(\xi_\alpha) \otimes \omega_\alpha) \). Thus,

\[
\begin{align}
& c(\nabla^{\phi,\mathcal{E}}) (\eta \otimes \zeta) = \sum \phi^\sigma(\xi_\alpha) \otimes \omega_\alpha(\zeta) = U_{\phi^\sigma} \circ c(\nabla^{\mathcal{E}}) (\eta \otimes \zeta). 
\end{align}
\]
We also note that $\sigma(\eta) = \sigma \circ \phi(\xi) = \phi^\sigma \circ \sigma^\xi(\xi)$, and hence
\begin{equation}
\sigma(\eta) \otimes D^\xi = (\phi^\sigma \otimes 1_U) (\sigma^\xi(\xi) \otimes D^\xi) = U_{\phi^\sigma} (\sigma^\xi(\xi) \otimes D^\xi).
\end{equation}

Combining (5.13) and (5.14) we get
\begin{equation}
D_{\nabla^\xi \varepsilon} U_{\phi}(\xi \otimes \xi) = D_{\nabla^\xi \varepsilon} (\eta \otimes \xi) = \sigma(\eta) \otimes D^\xi + c \left( \nabla^\phi, \varepsilon \right) (\eta \otimes \xi) = U_{\phi^\sigma} D_{\nabla^\xi \varepsilon} (\xi \otimes \xi).
\end{equation}

This shows that
\begin{equation}
D_{\nabla^\xi \varepsilon} = U_{\phi^\sigma}^{-1} D_{\nabla^\xi \varepsilon} U_{\phi}.
\end{equation}

By the first part of the proof we know that $D_{\nabla^\xi \varepsilon}$ is Fredholm. As $U_{\phi}$ and $U_{\phi^\sigma}$ are isomorphisms we then deduce that $D_{\nabla^\xi \varepsilon}$ is a Fredholm operator. The proof is complete.

In analogy with (5.6) and (4.9) we make the following definition.

**Definition 5.21.** Let $E$ be a Hermitian projective module with Hermitian $\sigma$-translate $E^\sigma$ and let $\nabla^E$ be a $\sigma$-connection on $E$. The index of $D_{\nabla^E \varepsilon}$ is defined by

$$
\text{ind } D_{\nabla^E \varepsilon} = \frac{1}{2} (\text{ind } D^+_E - \text{ind } D^-_E).
$$

We are now in a position to prove the main result of this section.

**Theorem 5.22.** Let $E$ be a Hermitian finitely generated projective module with Hermitian $\sigma$-translate $E^\sigma$. Then, for any $\sigma$-connection $\nabla^E$ on $E$, it holds that
\begin{equation}
\text{ind}_{D,\sigma}[E^\sigma] = \text{ind } D_{\nabla^E \varepsilon}.
\end{equation}

**Proof.** Let us first assume that $E = e A^q$ with $e = e^2 \in M_q(A)$ and let $\nabla^E_0$ be the $\sigma$-Grassmannian connection. As shown in the proof of Lemma 5.20 the Fredholm operators $D_{\nabla^E \varepsilon}$ and $D_{\nabla^E_0}$ differ by a bounded odd operator, and so $\text{ind } D^E_{\nabla^E} = \text{ind } D^E_{\nabla^E_0}$. Moreover, it follows from (5.12) that $\text{ind } D^E_{\nabla^E_0} = \text{ind } D^E_{c,\sigma}$. Thus,
\begin{equation}
\text{ind } D^E_{\nabla^E_0} = \text{ind } D^E_{\nabla^E} = \text{ind } D^E_{c,\sigma} = \text{ind } D_{c,\sigma}[e].
\end{equation}

In general, we can find an idempotent $e \in M_q(A)$ and right module isomorphisms $\phi : E \rightarrow e A^q$ and $\phi^\sigma : E^\sigma \rightarrow \sigma(e) A^q$ satisfying (5.1). We equip $e A^q$ and $\sigma(e) A^q$ with the inner products and let $\nabla^\phi, \varepsilon$ be the $\sigma$-connection given by (5.13). Then (5.17) shows that
\begin{equation}
D_{\nabla^E \varepsilon} = U_{\phi^\sigma}^{-1} D_{\nabla^E_0} U_{\phi}.
\end{equation}

where $U_{\phi} : \mathcal{H}(E) \rightarrow \mathcal{H}(e A^q)$ and $U_{\phi^\sigma} : \mathcal{H}(E^\sigma) \rightarrow \mathcal{H}(\sigma(e) A^q)$ are the unitary operators given by (5.13). As $U_{\phi}$ and $U_{\phi^\sigma}$ are even isomorphisms we see that $\text{ind } D^E_{\nabla^E} = \text{ind } D^E_{c,\sigma}$. Combining this with (5.19) we then get
\begin{align*}
\text{ind } D_{\nabla^E \varepsilon} &= \text{ind } D_{\nabla^E_0} [e] = \text{ind } D_{c,\sigma}[E^\sigma].
\end{align*}

The proof is complete.

We conclude this section by looking at the index formula (5.18) in the example of a Dirac spectral triple $(C^\infty(M), \nu^2(M, \mathcal{S}), \mathcal{D})$, where $(M, g)$ is a compact Riemannian spin oriented manifold of even dimension. Let $E$ be a vector bundle over $M$ and $\nabla^E$ a connection on $E$. Then $E : C^\infty(M, E)$ is a finitely generated projective module over the algebra $A := C^\infty(M)$. We observe that, $A$ is a commutative algebra we can identify left and right modules over $A$. It would be more convenient to work with left modules instead of right modules as we have been doing so far. This provides us with a natural identification of $A$-modules $E_1 \otimes_A E_2 \simeq E_2 \otimes_A E_1$ for the tensor products of two modules $E_1$ and $E_2$: the isomorphism is given by the flip map $\xi_1 \otimes \xi_2 \rightarrow \xi_2 \otimes \xi_1$.

Let $c : A_{\nu^2} T^* M \rightarrow \text{End} \mathcal{S}$ be the Clifford representation. Then, for all $a$ and $b$ in $A$,
\begin{equation}
a[\mathcal{D}, b] = ac(db) = \omega(ab).
\end{equation}

Therefore, we see that
\begin{equation}
\Omega_{\mathcal{D}}(A) = \text{Span } \{ c(\omega) : \omega \in C^\infty(M, \nu^2 M) \}.
\end{equation}
Note that $\nabla^E$ is a linear map from $E = C^\infty(M, E)$ to $C^\infty(M, T^*M \otimes E) = C^\infty(M, T^*_C M) \otimes A E$.
Consider the linear map $\nabla^E$ from $E$ to $\Omega^1_{\mathcal{D}_g T} (A) \otimes A E \approx E \otimes A \Omega^1_{\mathcal{D}_g} (A)$ defined by
\[
\nabla^E := (c \otimes 1_E) \circ \nabla^E.
\]
Let $\xi \in E$ and $a \in A$. Using (5.20) we get
\[
\nabla^E (a \xi) = (c \otimes 1_E) (da \otimes \xi + a \nabla^E \xi) = c(da) + a \nabla^E \xi = [D_g, a] \xi + a \nabla^E \xi.
\]
Therefore, we see that $\nabla^E$ is a connection on the finitely generated projective module $E$.
As $\nabla^E$ is a connection on $E$, we can form the operator $\mathcal{D}_{\nabla^E} := (\nabla^E)^* \nabla^E$. Set $\mathcal{H} = L^2(M, E)$. In what follows we identify $\mathcal{H}(E) = E \otimes A \mathcal{H}$ with $\mathcal{H} \otimes A E \approx L^2(M, S \otimes E)$, so that we can regard $\mathcal{D}_{\nabla^E}$ as an unbounded operator of $L^2(M, S \otimes E)$. Let $\zeta \in C^\infty(M, S)$ and $\xi \in E$. Let us write $\nabla^E \xi = \sum \omega_\alpha \otimes \xi_\alpha$, where $\omega_\alpha \in C^\infty(M, T^*_C M)$ and $\xi_\alpha \in E$. For each $\alpha$ let us also write $\omega_\alpha = \sum a_{\alpha \beta} \delta b_{\alpha \beta}$, with $a_{\alpha \beta}$ and $b_{\alpha \beta}$ in $C^\infty(M)$. Then, using (5.30) and (5.20), we see that $\mathcal{D}_{\nabla^E} (\zeta \otimes \xi)$ is equal to
\[
\mathcal{D}_{\nabla^E} (\zeta \otimes \xi) = \sum \alpha, \beta \left( a_{\alpha \beta} [\mathcal{D}_g, b_{\alpha \beta}] \right) \zeta \otimes \xi_\alpha = \mathcal{D}_g \zeta \otimes \xi + \sum \alpha \left( c(\omega_\alpha) \zeta \right) \otimes \xi_\alpha = \mathcal{D}_{\nabla^E} (\zeta \otimes \xi).
\]
Thus, under the identification $\mathcal{H}(E) \approx L^2(M, S \otimes E)$, the operator $\mathcal{D}_{\nabla^E}$ and $\mathcal{D}_{\nabla^E}$ agree. Combining this with (5.11) we then deduce that, for $\sigma = \text{id}$,
\[
\text{ind}_{\mathcal{D}_g, \sigma} [E] = \text{ind} \mathcal{D}_{\nabla^E} = \text{ind} \mathcal{D}_{\nabla^E} = \text{ind}_{\mathcal{D}_g} [E],
\]
where the second index map is the Fredholm index map (5.2). Thus under the Serre-Swan isomorphism $K_0(M) \approx K_0(C^\infty(M))$ this Fredholm index map agrees with the index map (5.10).

6. CYCLIC COHOMOLOGY

In this section, we review the main definitions and properties regarding cyclic cohomology and its pairing with $K$-theory. Cyclic cohomology was discovered by Connes [Co1] and Tsygan [Ts] independently. For more details on cyclic cohomology we refer to [Co1, Lo]. Throughout this section we let $A$ be a unital algebra over $\mathbb{C}$.

The Hochschild cochain complex of $A$ is defined as follows. The space of $k$-cochains $C^k(A)$, $k \in \mathbb{N}_0$, consists of $(k + 1)$-linear maps $\varphi : A^{k+1} \rightarrow \mathbb{C}$. For $j = 0, \ldots, k$, let $b_j : C^k(A) \rightarrow C^{k+1}(A)$ be the linear map defined by
\[
b_j \varphi(a^0, \ldots, a^{k+1}) = \varphi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{k+1}), \quad a^j \in A.
\]
We also define $b_{k+1} : C^k(A) \rightarrow C^{k+1}(A)$ by
\[
b_{k+1} \varphi(a^0, \ldots, a^{k+1}) = \varphi(a^{k+1} a^0, \ldots, a^k), \quad a^j \in A.
\]
Then the Hochschild coboundary $b : C^k(A) \rightarrow C^{k+1}(A)$ is given by
\[
b = b_0 - b_1 + \cdots + (-1)^{k+1} b_{k+1}.
\]
It can be checked that $b^2 = 0$. A cochain $\varphi \in C^k(A)$ is called cyclic when $T \varphi = \varphi$, where the operator $T : C^k(A) \rightarrow C^k(A)$ is given by
\[
T \varphi(a^0, \ldots, a^k) = (-1)^{k} \varphi(a^1, \ldots, a^k, a^0), \quad a^j \in A.
\]
We denote by $C^k_\text{c}(A)$ the space of cyclic $k$-cochains. The cyclic cochains form a subcomplex of the Hochschild complex since it can be shown that $b(C^k_\text{c}(A)) \subset C^{k+1}_\text{c}(A)$. We denote by $HC^k(A)$, $k = 0, 1, \ldots$, the associated cohomology groups.

In what follows we shall say that a cochain $\varphi \in C^k(A)$ is normalized when
\[
\varphi(a^0, \ldots, a^k) = 0 \quad \text{whenever } a^j = 1 \text{ for some } j \geq 1.
\]
We denote by $C^k_\text{nc}(A)$ the space of normalized $k$-cochains. It can be verified that $b$ maps $C^k_\text{nc}(A)$ to $C^{k+1}_\text{nc}(A)$. Moreover, if $\varphi \in C^k_\text{nc}(A)$ is cyclic and normalized, then the cyclicity and normalization conditions imply that, for all $a^j \in A$,
\[
\varphi(1, a^0, \ldots, a^{2k}) = \varphi(a^2 k, 1, a^1, \ldots, a^{2k-1}) = 0.
\]
For $k \geq 1$, the boundary operator $B : C^k_0(A) \to C^{k-1}_0(A)$ is defined by
\begin{equation}
B := AB_0, \quad A := 1 + T + \cdots + T^k,
\end{equation}
where $B_0 : C^k(A) \to C^{k-1}(A)$ is given by
\begin{equation}
B_0 \varphi(a^0, \ldots, a^{k-1}) = \varphi(1, a^0, \ldots, a^{k-1}), \quad a^i \in A.
\end{equation}
It follows from (6.5) that $B_0$ and $B$ are annihilated by normalized cyclic cochains. It also can be verified that $B^2 = 0$ and $Bb + Bb = 0$. For $i = 0, 1$, we define
\begin{equation}
C^i_0(A) = \bigoplus_{k=0}^{\infty} C^{2k+i}_0(A).
\end{equation}
Regarding $b$ and $B$ as operators acting between the vector spaces $C^0_0(A)$ and $C^1_0(A)$, we see that $(b + B)^2 = 0$. The cochain complex $\{C^i_0(A), b + B\}$ is called the periodic cyclic cochain complex of $A$ and its cohomology groups are denoted $H^i_0(A), i = 0, 1$. Thus a periodic cyclic coycle $(\xi_0, \xi_1, \ldots, \xi_n)$ is called the periodic cyclic cocycle $(\xi_0, \xi_1, \ldots, \xi_n)$, where $(\xi_0, \xi_1, \ldots, \xi_n)$ is defined as follows. Let $\varphi_C$ be a periodic cyclic cocycle of degree $2k+i$, where $(2k+i)$-cochain $\varphi_C$ is defined by
\begin{equation}
\varphi_C(f^0, \ldots, f^{2k+i}) = (C^k_{2k+i}, f^0 df^1 \wedge \cdots \wedge df^{2k}), \quad f^j \in C^\infty(M).
\end{equation}
It is immediate to check that $b \varphi_C$ satisfies the normalization condition. It can be verified that $b \varphi_C = 0$ and $B \varphi_C = \varphi_C$, where $d'$ is the de Rham boundary for current. Therefore, the map $C \to \varphi_C$ descends to a linear map,
\begin{equation}
\beta : H^i_0(M, \mathbb{C}) \to H^i_0(C^\infty(M)), \quad H^i_0(M, \mathbb{C}) := \bigoplus_{k \geq 0} H^k_{2k+i}(M, \mathbb{C}),
\end{equation}
where $H^k_{2k+i}(M, \mathbb{C})$ is the de Rham homology of $M$ in degree $2k+i$. As proved by Connes we actually obtain an isomorphism if we define the periodic cyclic cohomology of $C^\infty(M)$ in terms of continuous cochains, where $C^\infty(M)$ is endowed with its standard Fréchet-space topology.

The pairing between $H^i_0(A)$ and $K_0(A)$ is defined as follows. Let $\varphi = (\varphi_2k)$ be an even cochain and $e \in M_0(A)$ an idempotent. We define
\begin{equation}
\langle \varphi, e \rangle := \text{tr} \# \varphi_0(e) + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{k!} \text{tr} \# \varphi_{2k} \left( e - \frac{1}{2}, e, \ldots, e \right),
\end{equation}
where $\text{tr} \# \varphi_{2k}$ is the $2k$-cochain on $M_0(A) = M_0(A) \otimes A$ given by
\[
\text{tr} \# \varphi_{2k}(\mu^0, \ldots, \mu^{2k}) = \text{tr} \left[ \mu^0 \cdots \mu^{2k} \right] \varphi_{2k}(a^0, \ldots, a^{2k}), \quad \mu^j \in M_0(A), \quad a^j \in A.
\]
It is worth checking that if $\varphi$ is an even periodic coycle, then the value of $\langle \varphi, e \rangle$ depends only on the class of $\varphi$ in $H^0(A)$ and the class of $e$ in $K_0(A)$. We thus obtain a bilinear pairing,
\begin{equation}
\langle \cdot, \cdot \rangle : H^0_0(A) \times K_0(A) \to \mathbb{C}.
\end{equation}

Example 6.2. Suppose now that $A = C^\infty(M)$, where $M$ is a closed manifold, and let $e \in M_0(C^\infty(M))$, where $e^2 = e$. Consider the vector bundle $E = \text{ran} e$, which we regard as a subbundle of the trivial vector bundle $E_0 = M \times \mathbb{C}^k$. We note that by Serre-Swan theorem any vector bundle over $M$ is isomorphic to a vector bundle of this form. We equip $E$ with the Grassmannian connection $\nabla^E$ defined by $e$, so that
\[
\nabla^E \xi = e(X \xi) \quad \text{for all } X \in \mathfrak{X}(M, TM) \text{ and } \xi = (\xi_j) \in \mathcal{E} = C^\infty(M, E).
\]
The curvature of $\nabla^E$ is $F^E = e(de)^2 = e(de)^2e$, and so its Chern form is given by

$$\text{Ch}(F^E) = \sum (-1)^k \frac{1}{k!} \text{tr} [e(de)^{2k}] \in \Omega^{(0)}(M).$$

Let $C = (C_{2k})$ be a closed even de Rham current and denote by $\varphi_C$ the associated cocycle defined by \([6.15]\). Then

$$\langle C, \text{Ch}(F^E) \rangle = \sum (-1)^k \frac{1}{k!} \langle C_{2k}, \text{tr} [e(de)^{2k}] \rangle = \sum (-1)^k \frac{(2k)!}{k!} \varphi_{C_{2k}}(e, \ldots, e).$$

Noting that $B_0 \varphi_{C_{2k}} = 0$, we then deduce that

$$\langle \{C\}, [E]\rangle = \langle C, \text{Ch}(F^E) \rangle = \langle \{\varphi_C\}, [e]\rangle.$$  

Thus the pairing \([6.11]\) between even periodic cyclic cohomology and $K$-theory reduces to the classical pairing \([6.10]\) between de Rham homology and $K$-theory.

The above description of the periodic cyclic cohomology is given in terms of normalized cochains, which simplifies the formulas for the operator $B$ and the description of the pairing with $K$-theory. As done originally by Connes \([\text{Co}1]\), it is also possible to define periodic cohomology in terms of unnormalized cochains by replacing the operator $B$ by the operator $\overline{B} : C^m(A) \to C^{m-1}(A)$ defined by

$$\overline{B} = AB_0(1 - T).$$

Note this is the definition of operator $B$ in \([\text{Co}1, \text{Co}4]\). It can be checked that $\overline{B}^2 = 0$ and $\overline{B}B_0 + B\overline{B} = 0$. Therefore, setting $C^i(A) := \bigoplus C^{2k+i}(A), i = 0, 1$, we obtain an unnormalized complex $(C^\bullet(A), b + \overline{B})$ whose cohomology groups are denoted by $\overline{\text{HP}}(A), i = 0, 1$. As $\overline{B}$ agrees with $B$ on normalized cochains, there is a natural embedding of the normalized periodic complex $(C^\bullet(A), b + B)$ into $(C^\bullet(A), b + \overline{B})$. As it turns out (see \([\text{Lo}, \text{Corollary 2.1.10}]\)), at the level of cohomology this gives rise to a canonical identification,

$$\text{HP}^i(A) = \text{HP}^i(A).$$

One advantage of using unnormalized cochains is the clarification of the relationship with cyclic cohomology $HC^\bullet(A)$. As $\overline{B}$ is annihilated by cyclic cochains, any cyclic $m$-cocycle $\varphi$ is naturally identified with the unnormalized cocycle $(0, \ldots, 0, \varphi, 0, \ldots) \in C^i(A)$, where $i$ is the parity of $m$. This gives rise to an embedding,

$$HC^m(A) \hookrightarrow \text{HP}^m(A).$$

More can be said by using Connes’s periodicity operator $S : C^m(A) \to C^{m+2}(A)$. Up to normalization, this operator is given by the cup product with the unique cyclic $2$-cocycle on $\mathbb{C}$ taking the value $1$ at $(1, 1, 1)$. Equivalently,

$$S = \frac{1}{(m + 1)(m + 2)} \sum_{j=1}^{m+1} (-1)^j S_j,$$

where, for $j = 1, \ldots, m$, the operator $S_j : C_\chi^m(A) \to C_\chi^{m+2}(A)$ is given by

$$S_j \varphi(a^0, \ldots, a^{m+2}) = \sum_{0 \leq l \leq j - 2} (-1)^l \varphi(a^0, \ldots, a^l a^{l+1}, \ldots, a^j a^{j+1}, \ldots, a^{m+2})$$

$$+ (-1)^{j+1} \varphi(a^0, \ldots, a^{j-1} a^j a^{j+1}, \ldots, a^{m+2}).$$

Our normalization for the operator $S$ is chosen so that if $\psi \in C^{m+1}(A)$ is such that $b\psi$ is cyclic, then by \([\text{Co}4, \text{Lemma III.1.23}]\) the cochain $\overline{B}\psi$ is a cyclic cocycle and we have

$$S\overline{B}\psi = -b\psi \quad \text{in} \quad HC^{m+2}(A).$$

It can be checked that $S(bC_{\chi}^{m-1}) \subset bC_{\chi}^{m+1}(A)$, so that $S$ descends to a linear map,

$$S : HC^m(A) \to HC^{m+2}(A).$$

Note also that Moreover, Connes \([\text{Co}1, \text{Co}4]\) proved that

$$\lim_{i \to} (HC^{2i+1}(A), S) = \text{HP}^i(A), \quad i = 0, 1,$$
where the left-hand side is the inductive limit of the direct system \((\text{HC}^{2k+i}(A), S)\). A consequence of this, which is especially relevant to the construction of the Chern-Character in the next section, is the following. Let \(\varphi\) be a cyclic \(m\)-cocycle and \(\psi\) a cyclic \((m+2)\)-cocycle, where \(m\) has parity \(i\). Then

\[
\psi = \varphi \text{ in } \text{HP}(A) \iff \psi = S\varphi \text{ in } \text{HC}^{m+2}(A).
\]

If \(\varphi\) is a normalized cyclic \(2k\)-cocycle and we regard it as an even cocycle, then thanks to \((6.10)\) the pairing \((6.10)\) reduces to

\[
\langle \varphi, e \rangle = (-1)^k \frac{(2k)!}{k!} \text{tr} \#\varphi_{2k}(e, \ldots, e).
\]

In fact, the above right-hand side makes sense even if \(\varphi\) is not normalized and depends only of the class of \(\varphi\) in \(\text{HC}^{2k}(A)\) and the class of \(e\) in \(K_0(A)\). We thus get a bilinear pairing,

\[
\langle \cdot, \cdot \rangle : \text{HC}^{2k}(A) \times K_0(A) \to \mathbb{C}.
\]

It can be checked that the above pairing is invariant under the action of \(S\). Therefore, using \((6.17)\) we obtain a bilinear pairing,

\[
\langle \cdot, \cdot \rangle : \text{HC}^{2k}(A) \times K_0(A) \to \mathbb{C}.
\]

Under the identification \((6.13)\) this pairing agrees with the pairing \((6.11)\).

7. Connes-Chern Character and Index Formula

In this section, we give a direct construction of the Connes-Chern character of a twisted spectral triple. Combining with the results of Section 5, we shall obtain a reformulation of the Atiyah-Singer index formula \((6.5)\) for twisted spectral triples.

Throughout this section we let \((A, \mathcal{H}, D)\) be a twisted spectral triple. For \(p \geq 1\) we denote by \(L^p(\mathcal{H})\) the Schatten ideal of operators \(T \in \mathcal{L}(\mathcal{H})\) such that \(\text{Tr}|T|^p < \infty\). We recall that \(L^p(\mathcal{H})\) is a Banach ideal with respect to the \(p\)-norm,

\[
\|T\|_p = (\text{Tr}|T|^p)^{\frac{1}{p}}, \quad T \in L^p(\mathcal{H}).
\]

In what follows we assume that the twisted spectral triple \((A, \mathcal{H}, D)\) is \(p\)-summable, i.e.,

\[
D^{-1} \in \mathcal{L}^p(\mathcal{H}).
\]

7.1. Invertible Case. We start by assuming that \(D\) is invertible. We shall explain later how to remove this assumption. We recall the following result.

Lemma 7.1 [H5 Lemma 7.1; see also [501, p. 304]]. Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be Hilbert spaces and \(T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)\) a Fredholm operator. Let \(S \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)\) be such that \(1 - ST \in \mathcal{L}^p(\mathcal{H}_1)\) and \(1 - TS \in \mathcal{L}^p(\mathcal{H}_2)\). Then

\[
\text{ind} T = \text{Tr}((1 - ST)^q) - \text{Tr}((1 - TS)^q) \quad \forall q \geq p.
\]

The main impetus for our construction of the Connes-Chern character is the following index formula.

Proposition 7.2. Let \(e \in M_q(A), q \geq 1\), be an idempotent. Then, for any integer \(k \geq \frac{1}{2}p\),

\[
\text{ind} D_{e, \sigma} = \frac{1}{2} \text{Str} \left( (D^{-1}[D, e]_e)^{2k+1} \right).
\]

Proof. We defined \(D_{e, \sigma}\) as an unbounded operator from \(e\mathcal{H}^q\) to \(\sigma(e)\mathcal{H}^q\). Alternatively, let \(\mathcal{H}_1\) be the Hilbert space given by the vector space \(\text{dom}(D)\) equipped with the Hermitian inner product,

\[
\langle \xi, \eta \rangle_1 := \langle \xi, \eta \rangle + \langle D\xi, D\eta \rangle, \quad \xi, \eta \in \text{dom}(D).
\]

We denote by \(\| \cdot \|_1\) the norm of \(\mathcal{H}_1\). This norm is complete since \(D\) is a closed operator. We then can regard \(D\) as an invertible bounded operator from \(\mathcal{H}_1\) to \(\mathcal{H}_1\). Let \(e \in A\) and \(\xi \in \text{dom}(D)\). Upon writing \(D\xi = aD\xi + [D, a]\xi\) we see that

\[
\|a\xi\|_1^2 = \|a\xi\|^2 + \|D\xi\|^2 \leq \|a\xi\|^2 + 2\|a\xi\|^2 + \|D\xi\|^2 \leq 2\|a\xi\|^2 + \|D\xi\|^2 \leq 2\|a\xi\|^2 + \|D\xi\|_1^2.
\]
Thus $a$ induces a bounded operator on $H_1$. It then follows that $e$ induces a bounded operator on $H_1^q$, so that $eH_1^q$ is a closed subspace of $H_1^q$. We regard $D_{e,\sigma}$ as a bounded operator from $eH_1^q$ to $\sigma(e)H_1^q$. Set $Q = eD^{-1}\sigma(e) \in \mathcal{L}(\sigma(e)H_1^q, eH_1^q)$. Then the product $D_{e,\sigma}Q$ is equal to

\begin{equation}
(7.2) \quad \sigma(e)DeD^{-1}\sigma(e) = \sigma(e)D[D_{e,\sigma}D^{-1}\sigma(e) = 1 + \sigma(e)[D_{e,\sigma}D^{-1}\sigma(e),
\end{equation}

where we have used the fact that $e = 1$ on $eH_1^q$. Likewise, the operator $QD_{e,\sigma}$ is equal to

\begin{equation}
(7.3) \quad eD^{-1}\sigma(e)De = eD^{-1}(De - [D_{e,\sigma}D^{-1}\sigma(e),
\end{equation}

We observe that $D_{e,\sigma}D^{-1}[D_{e,\sigma}D^{-1}\sigma(e),
\end{equation}

Similarly, the operator $\sigma(e)[D_{e,\sigma}D^{-1}[D_{e,\sigma}D^{-1}\sigma(e),
\end{equation}

As $D_{e,\sigma} \in \mathcal{L}(eH_1^q)$ and $[D_{e,\sigma}]$ is bounded, using (7.3) we see that $eD_{e,\sigma}D^{-1}[D_{e,\sigma}D^{-1}\sigma(e),
\end{equation}

The $Z_2$-grading $H^\ddagger = H^\ddagger_1 \oplus H^\ddagger_2$ induces a $Z_2$-grading $H_1 = H_1^\ddagger_1 \oplus H_1^\ddagger_2$ with $H^\ddagger_1 = H_1 \cap H^\ddagger$. This gives rise to splittings $eH_1^q = e(H_1^\ddagger_1)^q \oplus e(H_1^\ddagger_2)^q$ and $\sigma(e)H_1^q = \sigma(e)(H^\ddagger_1)^q \oplus \sigma(e)(H^\ddagger_2)^q$. With respect to these splittings, the operator $Q = eD\sigma(e)$ takes the form

\[ Q = \begin{pmatrix} 0 & Q^- \\ Q^+ & 0 \end{pmatrix} \]

where $Q^\pm := e(D^\mp)^{-1}\sigma(e)$. Therefore, (7.2) and (8.3) can be rewritten as

\[ D_{e,\sigma}Q^\ddagger = 1 + \sigma(e)[D_{e,\sigma}(D^\ddagger)^{-1}\sigma(e), \quad Q^\ddagger D_{e,\sigma} = 1 - e(D^\mp)^{-1}\sigma(e), \]

where the first equality holds in $\mathcal{L}(\sigma(e)H_1^q)$ and the second holds in $\mathcal{L}(eH_1^q)$. As shown above the operators $\sigma(e)[D_{e,\sigma}(D^\ddagger)^{-1}\sigma(e)$ and $e[D_{e,\sigma}(D^\ddagger)^{-1}\sigma(e)$ are in the Schatten classes $\mathcal{L}_{\infty}(\sigma(e)H^\ddagger_1)$ and the second holds in $\mathcal{L}_{\infty}(eH^\ddagger_1)$, respectively. Therefore, we may apply Lemma (4.1) to obtain that, for all $k \geq \frac{1}{2}p$,

\[ \text{ind} D_{e,\sigma} = \text{Tr} (e[D_{e,\sigma}(D^\ddagger)^{-1}\sigma(e)\sigma(e)k. \]

Thus,

\[ 2\text{ind} D_{e,\sigma} = \text{Str} (e[D_{e,\sigma}D^{-1}\sigma(e)k - \text{Str} (-\sigma(e)(D^\ddagger)^{-1}[D_{e,\sigma}(D^\ddagger)^{-1}\sigma(e)k. \]

Combining this with (7.3) and (7.3) we then get

\[ 2\text{ind} D_{e,\sigma} = \text{Str} ((D^{-1}[D_{e,\sigma}]^2k + \text{Str} (\sigma(e)[D_{e,\sigma}D^{-1}]^2k. \]

We observe that $\text{Str}(\sigma(e)[D_{e,\sigma}D^{-1}]^2k$ is equal to

\[ \text{Str} (\sigma(e)(D^{-1}[D_{e,\sigma}]^2k = - \text{Str} (D^{-1}\sigma(e)(D^{-1}[D_{e,\sigma}]^2k. \]

Therefore, we obtain

\[ \text{ind} D_{e,\sigma} = \frac{1}{2} \text{Str} ((e - D^{-1}\sigma(e)D)(D^{-1}[D_{e,\sigma}]^2k = \frac{1}{2} \text{Str} ((D^{-1}[D_{e,\sigma}]^2k+1). \]

The lemma is thus proved. \hfill \Box

**Definition 7.3.** For $k \geq \frac{1}{2}(p - 1)$ let $\tau_k^n$ be the 2k-cochain on $A$ defined by

\begin{equation}
(7.6) \quad \tau_k^n(a_0^0, \ldots, a_2^{2k}) = c_k \text{Str} (D^{-1}[D_{e,\sigma}]^0 \cdot \ldots \cdot D^{-1}[D_{e,\sigma}]^n a_2^{2k} \sigma) \quad \forall a_j \in A,
\end{equation}

where we have set $c_k = \frac{1}{2}(-1)^{k} \frac{A}{(2k)!}$. \hfill \Box
We note that \( \tau^D_k \) is a normalized cyclic cochain. Moreover, using (6.10), for \( k \geq \frac{1}{2}p \), we can rewrite the index formula in the form,

\[
\text{ind } D_{e, \sigma} = \langle \tau^D_k, e \rangle \quad \forall e \in M_q(A), \quad e^2 = e.
\]

For \( m \geq p \), we let \( \varphi_m \) and \( \psi_m \) be the normalized \( m \)-cochains on \( A \) defined by

\[
\varphi_m(a_0, \ldots, a^m) = \text{Str}(a^0 D^{-1}[D, a^1]_\sigma \cdots D^{-1}[D, a^m]_\sigma), \tag{7.8}
\]

\[
\psi_m(a_0, \ldots, a^m) = \text{Str}(a^0 \sigma(a^0) D^{-1} \cdots [D, a^m]_\sigma D^{-1}), \quad a^0 \in A. \tag{7.9}
\]

We observe that \( \psi_m(a^0, \ldots, a^m) \) is equal to

\[
\text{Str}(D^{-1}D^{-1}[D, a^1]_\sigma \cdots D^{-1}[D, a^m]_\sigma) = -\varphi_m(D^{-1} \sigma(a^0) D, a^1, \ldots, a^m). \tag{7.10}
\]

Using the equality \( D^{-1}[D, a^0]_\sigma = a^0 - D^{-1} \sigma(a^0) D \), we then see that, for \( k \geq \frac{1}{2}p \),

\[
c_k^{-1} \tau^D_k(a^0, \ldots, a^{2k}) = \text{Str}
\left[
\left.
\sum_{j=0}^{k-1} a^0 D^{-1}[D, a^1]_\sigma \cdots D^{-1}[D, a^{2j}]_\sigma \right)
\right.
\]

\[
= \varphi_{2k}(a^0, \ldots, a^{2k}) + \psi_{2k}(a^0, \ldots, a^{2k}) \tag{7.11}
\]

**Lemma 7.4.** Let \( k \geq \frac{1}{2}(p-1) \). Then

\[
B \varphi_{2k+1} = -B \psi_{2k+1} = (2k+1)c_k^{-1} \tau^D_{2k}. \tag{7.12}
\]

**Proof.** It follows from (7.5) and (7.9) that \( B \varphi_{2k+1} = -B \psi_{2k+1} = c_k^{-1} \tau^D_{2k} \). As \( \tau^D_{2k} \) is a cyclic cochain and by definition \( B = AB_0 \), we then deduce that \( B \varphi_{2k+1} = -B \psi_{2k+1} = c_k^{-1} \tau^D_{2k} = (2k+1)c_k^{-1} \tau^D_{2k} \), proving the lemma.

**Lemma 7.5.** Let \( k \geq \frac{1}{2}(p+1) \). Then

\[
b_0 \varphi_{2k-1} = \varphi_k \quad \text{and} \quad b_0 \psi_{2k-1} = -\psi_k. \tag{7.13}
\]

**Proof.** For \( j = 1, \ldots, 2k \) let \( \theta'_j \) and \( \theta''_j \) be the \( 2k \)-cochains on \( A \) defined by

\[
\theta'_j(a^0, \ldots, a^{2k}) = \text{Str}
\left[
\left.
\sum_{l=0}^{j-1} a^0 D^{-1}[D, a^1]_\sigma \cdots D^{-1}[D, a^{2l}]_\sigma \right)
\right.
\]

\[
= \varphi_{2k}(a^0, \ldots, a^{2k}). \tag{7.14}
\]

We note that

\[
\theta''_j(a^0, \ldots, a^{2k}) = \text{Str}
\left[
\left.
\sum_{l=0}^{j-1} a^0 D^{-1}[D, a^1]_\sigma \cdots D^{-1}[D, a^{2l}]_\sigma \right)
\right.
\]

\[
= \varphi_{2k}(a^0, \ldots, a^{2k}). \tag{7.15}
\]

Using the equality \( D^{-1}[D, a^j a^{j+1}]_\sigma = D^{-1}[D, a^j]_\sigma a^{j+1} + D^{-1}(a^j) D \cdot D^{-1}[D, a^{j+1}]_\sigma \) we also find that

\[
b_j \varphi_{2k-1}(a^0, \ldots, a^{2k}) = \text{Str}
\left[
\left.
\sum_{l=0}^{j-1} a^0 D^{-1}[D, a^1]_\sigma \cdots D^{-1}[D, a^{2l}]_\sigma \right)
\right.
\]

\[
= \theta''_{j+1}(a^0, \ldots, a^{2k}) + \theta''_j(a^0, \ldots, a^{2k}). \tag{7.16}
\]

Thus \( \sum_{j=1}^{2k-1} (-1)^j b_j \varphi_{2k-1} \) is equal to

\[
\sum_{j=1}^{2k-1} (-1)^j \left( \theta''_{j+1} - \theta''_j \right) = -\theta''_1 + \sum_{j=2}^{2k-1} (-1)^{j-1} \left( \theta''_{j-1} - \theta''_j \right) - \theta''_{2k}
\]

\[
= -\theta''_1 + \sum_{j=2}^{2k-1} (-1)^{j-1} \varphi_{2k} - \theta''_{2k}
\]

\[
= -\theta''_1 - \theta''_{2k}. \tag{7.17}
\]

We also note that

\[
b_0 \varphi_{2k-1}(a^0, \ldots, a^{2k}) = \text{Str}
\left[
\left.
\sum_{l=0}^{j-1} a^0 a^1 D^{-1}[D, a^2]_\sigma \cdots D^{-1}[D, a^{2l}]_\sigma \right)
\right.
\]

\[
= \theta'_1(a^0, \ldots, a^{2k}). \tag{7.18}
\]
Moreover, the cochain $b_{2k} \varphi_{2k-1}(a^0, \ldots, a^{2k})$ is equal to
\[
\text{Str}(a^{2k}a^1D^{-1}[D,a^1]_\sigma \cdots D^{-1}[D,a^{2k-1}]_\sigma) = \text{Str}(a^0D^{-1}[D,a^1]_\sigma \cdots D^{-1}[D,a^{2k-1}]_\sigma a^{2k}) = \theta'_{2k}(a^0, \ldots, a^{2k}).
\]
Therefore, we find that
\[
b_{2k} \varphi_{2k-1} = \sum_{j=0}^{2k} (-1)^j b_j \psi_{2k-1} = b_0 \varphi_{2k-1} - \theta''_1 - \theta''_{2k} = \theta'_1 - \theta''_1 = \varphi_{2k}.
\]
As $\psi_{2k-1}(a^0, \ldots, a^{2k-1}) = -\varphi_{2k-1}(D^{-1}\sigma(a^0)D, a^1, \ldots, a^{2k-1})$, using (7.12) we get
\[
\sum_{j=1}^{2k-1} (-1)^j b_j \psi_{2k-1} = \theta''_1(D^{-1}\sigma(a^0)D, a^1, \ldots, a^{2k}) + \theta''_{2k}(D^{-1}\sigma(a^0)D, a^1, \ldots, a^{2k}).
\]
We observe that $b_0 \psi_{2k-1}(a^0, \ldots, a^{2k})$ is equal to
\[
-\text{Str}(D^{-1}a^0a^1D^{-1}[D,a^1]_\sigma \cdots D^{-1}[D,a^{2k-1}]_\sigma) = -\theta''_1(D^{-1}\sigma(a^0)D, a^1, \ldots, a^{2k-1}).
\]
Moreover, we have
\[
b_{2k} \psi_{2k-1}(a^0, \ldots, a^{2k}) = -\text{Str}(D^{-1}\sigma(a^{2k})\sigma(a^0)D \cdot D^{-1}[D,a^1]_\sigma \cdots D^{-1}[D,a^{2k-1}]_\sigma)
\]
\[
= -\theta''_1(D^{-1}\sigma(a^0)D, a^1, \ldots, a^{2k}).
\]
Therefore, we see that $b \psi_{2k-1}(a^0, \ldots, a^{2k}) = \sum_{j=0}^{2k} (-1)^j b_j \psi_{2k-1}(a^0, \ldots, a^{2k})$ is equal to
\[
\theta''_{2k}(D^{-1}\sigma(a^0)D, a^1, \ldots, a^{2k}) = \varphi_{2k}(D^{-1}\sigma(a^0)D, a^1, \ldots, a^{2k}) = -\psi_{2k}(a^0, a^1, \ldots, a^{2k}).
\]
The proof is complete. \qed

**Proposition 7.6 (CM4).** Let $k \geq \frac{1}{4}(p-1)$. Then
(1) The cochain $\tau_{2k}^D$ in (7.6), is a normalized cyclic cocycle.
(2) The class of $\tau_{2k}^D$ in $\text{HP}^0(\mathcal{A})$ is independent of the value of $k$.

**Proof.** We already know that $\tau_{2k}^D$ is a cyclic normalized cochain. We also note that
\[
c_{k+1} = \frac{1}{2}(-1)^{k+1} \frac{(k+1)!}{(2k+2)!} = -\frac{c_k}{2(2k+1)}.
\]
Combining this with Lemma 7.3 we get
\[
\tau_{2k}^D = \frac{c_k}{2(2k+1)} B(\varphi_{2k+1} - \psi_{2k+1}) = -c_{k+1} B(\varphi_{2k+1} - \psi_{2k+1}).
\]
Using (7.13) and the fact that $bB = -Bb$ we then see that
\[
b_{2k} \tau_{2k}^D = -c_{k+1} B(\varphi_{2k+1} - \psi_{2k+1}) = c_{k+1} B(\varphi_{2k+1} - \psi_{2k+1}).
\]
Moreover, using (7.11) and Lemma 7.5 we get
\[
\tau_{2k+2}^D = c_{k+1}(\varphi_{2k+2} + \psi_{2k+2}) = c_{k+1} b(\varphi_{2k+1} - \psi_{2k+1}).
\]
As $B$ is annihilated by cyclic cochains we then deduce that
\[
b_{2k} \tau_{2k+2} = B(c_{k+1} b(\varphi_{2k+1} - \psi_{2k+1})) = B \tau_{2k+2} = 0.
\]
That is, $\tau_{2k}^D$ is a cocycle. We also see that
\[
\tau_{2k+2}^D - \tau_{2k}^D = c_{k+1} b(\varphi_{2k+1} - \psi_{2k+1}) + c_{k+1} B(\varphi_{2k+1} - \psi_{2k+1}).
\]
This shows that $\tau_{2k+2}^D = \tau_{2k}^D$ in $\text{HP}^0(\mathcal{A})$. It then follows that the class of $\tau_{2k}^D$ in $\text{HP}^0(\mathcal{A})$ is independent of the value of $k$. The proof is complete. \qed
Remark 7.7. The proof of Lemma 7.4 uses the fact that the unit of \( A \) is represented by the identity of \( \mathcal{H} \). Otherwise the equalities \( B_0 \varphi_{2k+1} = - B_0 \psi_{2k+1} = c_k^{-1} \tau_{2k}^D \) need not hold. Therefore, the unitality of \( A \) is a crucial ingredient of the proof of Proposition 7.6.

**Definition 7.8 (CMA).** Let \((A, \mathcal{H}, D)\) be a p-summable twisted spectral triple with \( D \) invertible. Then its Connes-Chern character, denoted \( \text{Ch}(D)_{\sigma} \), is the class in \( H^{p0}(A) \) of any of the cocycles \( \tau_{2k}^D \), \( k \geq \frac{1}{2}(p-1) \).

We are now in a position to reformulate the Atiyah-Singer index formula (5.3) for twisted spectral triples in the invertible case.

**Theorem 7.9.** Let \((A, \mathcal{H}, D)_{\sigma}\) be a p-summable twisted spectral triple with \( D \) invertible. Then, for any Hermitian finitely generated projective module \( \mathcal{E} \) over \( A \) and any \( \sigma \)-connection on \( \mathcal{E} \),

\[
\text{ind} D_{\mathcal{E}, \sigma} \equiv (\text{Ch}(D)_{\sigma}, [\mathcal{E}])
\]

**Proof.** Thanks to Theorem 7.2 we know that \( \text{ind} D_{\mathcal{E}, \sigma} = \text{ind}_D [\mathcal{E}] \). Let \( e \) be an idempotent in some matrix algebra \( M_n(A) \) such that \( \mathcal{E} \simeq e A \). Then (7.7) shows that, for \( k \geq \frac{1}{2}p \),

\[
\text{ind}_{D_{\sigma}} [e] = \text{ind}_{D_{\sigma}} (\tau_{2k}^D, e) = (\text{Ch}(D)_{\sigma}, [\mathcal{E}])
\]

As \( \mathcal{E} \) and \( e \) defines the same class in \( K_0(A) \) we then deduce that

\[
\text{ind} D_{\mathcal{E}, \sigma} \equiv (\text{Ch}(D)_{\sigma}, [\mathcal{E}])
\]

The proof is complete. \( \square \)

7.2. **General case.** The assumption on the invertibility of \( D \) can be removed by passing to the unital invertible double as follows. Consider the Hilbert space \( \hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H} \), which we equip with the \( \mathbb{Z}_2 \) grading given by

\[
\tilde{\gamma} = \left( \begin{array}{cc} \gamma & 0 \\ 0 & -\gamma \end{array} \right)
\]

where \( \gamma \) is the grading operator of \( \mathcal{H} \). On \( \hat{\mathcal{H}} \) consider the selfadjoint operators,

\[
\tilde{D}_0 = \left( \begin{array}{cc} D & 0 \\ 0 & -D \end{array} \right), \quad J = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \tilde{D} = \tilde{D}_0 + J = \left( \begin{array}{cc} D & 1 \\ 1 & -D \end{array} \right),
\]

where the domain of \( \tilde{D}_0 \) and \( \tilde{D} \) is \( \text{dom}(D) \oplus \text{dom}(D) \). As \( \tilde{D}_0 J + T \tilde{D}_0 = 0 \) and \( J^2 = 1 \) we get

\[
\tilde{D}^2 = \tilde{D}_0^2 + 1 = \left( \begin{array}{cc} D^2 + 1 & 0 \\ 0 & D^2 + 1 \end{array} \right).
\]

It then follows that \( \tilde{D} \) is invertible. Moreover,

\[
\text{Tr} |\tilde{D}|^{-p} = \text{Tr}(D^2)^{-\frac{p}{2}} = 2 \text{Tr}(D^2 + 1)^{-\frac{p}{2}} \leq 2 \text{Tr} |D|^{-p} < \infty.
\]

That is, \( \tilde{D}^{-1} \in \mathcal{L}^p(\hat{\mathcal{H}}) \).

Let \( \pi : A \to \mathcal{L}(\hat{\mathcal{H}}) \) be the linear map given by

\[
\pi(a) = \left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) \quad \forall a \in A.
\]

We note that \( \pi \) is multiplicative, but as \( \pi(1) \neq 0 \) this is not a representation of the unital algebra \( A \). As mentioned in Remark 7.7 the representation of the unit \( 1 \) by the identity of \( \hat{\mathcal{H}} \) is essential to the construction of the Connes-Chern character in the invertible case. To remedy to the lack of unitality of \( \pi \) we pass to the \( * \)-algebra \( \tilde{A} = A \oplus \mathbb{C} \) with product and involution given by

\[
(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda \mu), \quad (a, \lambda)^* = (a^*, \overline{\lambda}), \quad a, b \in A, \quad \lambda, \mu \in \mathbb{C}.
\]

The unit of \( \tilde{A} \) is \( 1_{\tilde{A}} = (0,1) \). Thus, identifying any element \( a \in A \) with \( (a, 0) \), any element \( \tilde{a} = (a, \lambda) \in \tilde{A} \) can be uniquely written as \( (a, \lambda) = a + \lambda 1_{\tilde{A}} \). We extend \( \pi \) into the (unital) representation \( \tilde{\pi} : \tilde{A} \to \mathcal{L}(\hat{\mathcal{H}}) \) given by

\[
\tilde{\pi}(a + \lambda 1_{\tilde{A}}) = \pi(a) + \lambda \quad \forall (a, \lambda) \in A \times \mathbb{C}.
\]
We also extend the automorphism $\sigma$ into the automorphism $\tilde{\sigma} : \tilde{A} \to \tilde{A}$ given by
$$\tilde{\sigma}(a + \lambda 1_{\tilde{A}}) = \sigma(a) + \lambda 1_{\tilde{A}} \quad \forall (a, \lambda) \in A \times \mathbb{C}.$$ 

For $a \in A$ and $\lambda \in \mathbb{C}$, the twisted commutator $[\tilde{D}, \tilde{\pi}(a + \lambda 1_{\tilde{A}})]$ is equal to
$$\begin{pmatrix} D & 1 \\ 1 & -D \end{pmatrix} \begin{pmatrix} a + \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \sigma(a) + \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} D & 1 \\ 1 & -D \end{pmatrix} = \begin{pmatrix} [D, a]_\sigma & -\sigma(a) \\ 0 & 0 \end{pmatrix} \in \mathcal{L} (\tilde{H}).$$

Combining all this we obtain the following statement.

**Proposition 7.10.** $(\tilde{A}, \tilde{\mathcal{H}}, \tilde{D})_\alpha$ is a $p$-summable twisted spectral triple.

As $\tilde{D}$ is invertible, we can form the normalized cyclic cocycles $\tau_{2k}^\tilde{D}, \quad k \geq \frac{1}{2} (p - 1)$, as in Definition [6.3].

We note that if $\tilde{\varnothing}$ is a cyclic $m$-cochain on $\tilde{A}$, then its restriction $\tilde{\varnothing}$ to $A^{m+1}$ is a cyclic cochain on $A$. Moreover, it follows from the formulas (6.13) and (6.14) for the operators $b$ and $S$ that

$$\tilde{\varnothing} = b\varnothing \quad \text{and} \quad S\varnothing = S\varnothing.$$ 

If in addition $\varnothing$ is normalized, then the normalization condition and (6.15) imply that

$$\varnothing (a^0 + \lambda^0 1_A, \ldots, a^m + \lambda^m 1_A) = \varnothing (a^0, \ldots, a^m) \quad \forall a^j \in A \quad \forall \lambda^j \in \mathbb{C}.$$ 

Thus $\tilde{\varnothing}$ is uniquely determined by its restriction $\tilde{\varnothing}$ to $A^{m+1}$. Conversely, any cyclic $m$-cochain $\varnothing$ on $A$ uniquely extends to a normalized cyclic $m$-cochain $\tilde{\varnothing}$ on $A$ satisfying (7.18).

**Definition 7.11.** Let $k \geq \frac{1}{2} (p - 1)$. Then $\tau_{2k}^\tilde{D}$ is the cyclic $2k$-cochain on $A$ given by the restriction of $\tau_{2k}^D$ to $A^{2k+1}$.

**Proposition 7.12.** Let $k \geq \frac{1}{2} (p - 1)$. Then

1. The $2k$-cochain $\tau_{2k}^\tilde{D}$ is a cyclic cochain whose class in $\text{HT}^k (A)$ is independent of $k$.
2. For any idempotent $e \in M_q (A)$, we have

$$\text{ind } D_{e, \sigma} = \begin{pmatrix} \tau_{2k}^D, e \end{pmatrix}.$$ 

**Proof.** It follows from (7.17) that $b\tau_{2k}^\tilde{D} = b\tau_{2k}^D = 0$, so $\tau_{2k}^\tilde{D}$ is a cocycle. By Proposition 7.6 the cocycles $\tau_{2k}^D$ and $\tau_{2k+2}^D$ are cohomologous in $\text{HT}^k (A)$. Therefore, by (6.15) there is a cyclic $(2k+1)$-cochain $\theta$ on $A$ such that $S\tau_{2k}^D = \tau_{2k+2}^D + b\theta$. Let $\tilde{\theta}$ be cyclic $(2k + 1)$-cochain on $A$ given by the restriction of $\theta$ to $A^{2k+2}$. Using (7.17) we get

$$S\tau_{2k}^\tilde{D} = S\tau_{2k}^D = \tau_{2k+2}^D + b\tilde{\theta} = \tau_{2k+2}^D - b\tilde{\theta}.$$ 

Therefore, the cycles $S\tau_{2k}^D$ and $\tau_{2k+2}^D$ are cohomologous in $\text{HC}^{2k+2} (A)$. Combining this with (6.18) we then deduce that $\tau_{2k}^D$ and $\tau_{2k+2}^D$ are cohomologous in $\text{HT}^k (A)$. It then follows that the class of $\tau_{2k}^D$ in $\text{HT}^k (A)$ is independent of $k$.

Let $e \in M_q (A), \quad e^2 = e$. We regard $e$ as an idempotent in $M_q (\tilde{A})$. Then

$$\tilde{\pi}(e) \tilde{\mathcal{H}}^q = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} (\mathcal{H}^q \oplus \mathcal{H}^q) = e(\mathcal{H}^q) \oplus \{ 0 \} \simeq e\mathcal{H}^q.$$ 

Likewise, $\tilde{\pi}(\sigma(e)) \tilde{\mathcal{H}}^q = \sigma(e)(\mathcal{H}^q) \oplus \{ 0 \} \simeq \sigma(e)\mathcal{H}^q$. Moreover,

$$\sigma(e) \tilde{D} e = \begin{pmatrix} \sigma(e) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & 1 \\ 1 & -D \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma(e) D e & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Thus, under the identifications $\tilde{\pi}(e) \tilde{\mathcal{H}}^q \simeq e\mathcal{H}^q$ and $\tilde{\pi}(\sigma(e)) \tilde{\mathcal{H}}^q \simeq \sigma(e)\mathcal{H}^q$ above, the operators $\tilde{D}_{e, \sigma}$ and $D_{e, \sigma}$ agree. Therefore, using (7.20) we get

$$\text{ind } D_{e, \sigma} = \text{ind } \tilde{D}_{e, \sigma} = \begin{pmatrix} \tau_{2k}^D, e \end{pmatrix} = \begin{pmatrix} \tau_{2k}^\tilde{D}, e \end{pmatrix}.$$ 

The proof is complete. \qed
Definition 7.13. Let \((\mathcal{A}, \mathcal{H}, D)_\sigma\) be a \(p\)-summable twisted spectral triple. Then its Connes-Chern character, denoted \(\text{Ch}(D)_\sigma\), is the class in \(\prod^0(\mathcal{A}) \simeq \text{HP}^0(\mathcal{A})\) of any of the cocycles \(\tau^{\tilde{D}}_{2k}\), \(k \geq \frac{1}{2}(p-1)\).

Assume now that \(D\) is invertible. We then have two possible definitions of the Connes-Chern character: one in terms of the cocycles \(\tau^D_{2k}\) and the other in terms of the cocycles \(\tau^{\tilde{D}}_{2k}\). We shall now show that these definitions are equivalent.

Consider the homotopy of operators,

\[ \tilde{D}_t = \tilde{D}_0 + tJ, \quad 0 \leq t \leq 1. \]

In the same way as in (7.15) we have

\[ \tilde{D}_t^2 = \tilde{D}_0^2 + t^2 = \begin{pmatrix} D^2 + t^2 & 0 \\ 0 & D^2 + t^2 \end{pmatrix}, \]

which shows that \(\tilde{D}_t\) is invertible for all \(t \in [0, 1]\). Moreover, as in (7.16) we have

\[ \text{Tr} |\tilde{D}_0|^{-p} = 2 \text{Tr}(D^2 + t^2)^{-\frac{p}{2}} \leq 2 \text{Tr}|D|^{-p}. \]

Thus \((\tilde{D}_t)^{-1})_{0 \leq t \leq 1}\) is a bounded family in \(L^p(\tilde{H})\). Therefore, the family \((\tilde{D}_t)_{0 \leq t \leq 1}\) satisfies the assumption of Proposition [C.1] in Appendix C on the homotopy invariance of the Connes-Chern character. Therefore, \((\tilde{A}, \mathcal{H}, \tilde{D}_t)_\sigma\) is a \(p\)-summable twisted spectral triple for all \(t \in [0, 1]\) and, for \(k \geq \frac{1}{2}(p+1)\) the cocycles \(\tau^D_{2k}\) and \(\tau^{\tilde{D}}_{2k}\) are cohomologous in \(\text{HC}^{2k}(\tilde{A})\). Let \(\tau^D_{2k}\) be the restriction to \(\tilde{A}^{2k+1}\) of \(\tau^D_{2k}\). Then using (7.17) we see that \(\tau^D_{2k}\) and \(\tau^{\tilde{D}}_{2k}\) are cohomologous in \(\text{HC}^{2k}(\mathcal{A})\).

Bearing this in mind, we note that, for \(a \in \mathcal{A}\), we have

\[ [\tilde{D}_0, \tilde{\pi}(a)]_\sigma = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \sigma(a) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} = \begin{pmatrix} [D, a]_\sigma & 0 \\ 0 & 0 \end{pmatrix}. \]

Thus,

\[ \tilde{D}_0^{-1}[\tilde{D}_0, \tilde{\pi}(a)]_\sigma = \begin{pmatrix} D^{-1}[D, a]_\sigma & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \tilde{\gamma}\tilde{D}_0^{-1}[\tilde{D}_0, \tilde{\pi}(a)]_\sigma = \begin{pmatrix} \gamma D^{-1}[D, a]_\sigma & 0 \\ 0 & 0 \end{pmatrix}. \]

It then follows that, for \(a^0, \ldots, a^{2k}\) in \(\mathcal{A}\), we have

\[ \tau^D_{2k}(a^0, \ldots, a^{2k}) = \text{Tr} (\gamma D^{-1}[D, a^0]_\sigma \cdots D^{-1}[D, a^{2k}]_\sigma) = \tau^{D}_{2k}(a^0, \ldots, a^{2k}). \]

That is, the restriction of \(\tau^D_{2k}\) to \(\tilde{A}^{2k+1}\) is precisely the cocycle \(\tau^{\tilde{D}}_{2k}\). Therefore, we arrive at the following statement.

Proposition 7.14. If \(D\) is invertible, then, for any \(k \geq \frac{1}{2}(p+1)\), the cyclic cocycles \(\tau^D_{2k}\) and \(\tau^{\tilde{D}}_{2k}\) are cohomologous in \(\text{HC}^{2k}(\mathcal{A})\).

It follows from this that the cocycles \(\tau^D_{2k}\) and \(\tau^{\tilde{D}}_{2k}\) define the same class in \(\prod^0(\mathcal{A})\). Therefore, under the identification \(\prod^0(\mathcal{A}) \simeq \text{HP}^0(\mathcal{A})\), we see that, when \(D\) is invertible, Definition 7.3 and Definition (7.13) provide us with equivalent definitions of the Connes-Chern character of \((\mathcal{A}, \mathcal{H}, D)_\sigma\).

Finally, using Lemma 7.12 and arguing as in the proof of Theorem 7.9 enables us to remove the invertibility assumption in Theorem 7.9. We thus obtain the following index formula.

Theorem 7.15. Let \((\mathcal{A}, \mathcal{H}, D)_\sigma\) be a \(p\)-summable twisted spectral triple. Then, for any Hermitian finitely generated projective over \(\mathcal{E}\) and \(\sigma\)-connection on \(\mathcal{E}\), it holds that

\[ \text{ind} \ D_{\mathcal{VF} \mathcal{E}} = (\text{Ch}(D)_\sigma, [\mathcal{E}] ). \]

Remark 7.16. The cocycles \(\tau^D_{2k}\) and \(\tau^{\tilde{D}}_{2k}\) may be difficult to compute in practice, even in the case of a Dirac spectral triple (see Theorem 6.5 of [Co1, Part I] and [BF]). In the ordinary case, a representation of the Connes-Chern character in entire cyclic cohomology is given by the JLO cocycle [HLO, C53], the existence of which only requires \(\theta\)-summability. We refer to the paper of Quillen [Q] for interpretations of the Connes-Chern character and JLO cocycle in terms of Chern characters of superconnections on cochains.
Remark 7.18. Let $(M^n, g)$ be a compact Riemannian manifold. The Connes-Chern character of the Dirac spectral triple $(\mathcal{C}^\infty(M), L^2(M, \mathcal{S}), \mathcal{D})$ is represented by the CM cocycle. This CM cocycle can be computed by heat kernel techniques [CM1, Po]. We obtain the even cocycle $\varphi = (\varphi_{2k})$ given by

$$\varphi_{2k}(f^0, \ldots, f^{2k}) = \frac{(2i\pi)^{-k}}{(2k)!} \int_M \hat{\mathcal{A}}(R^M) \wedge f^0 df^1 \wedge \ldots \wedge f^{2k}.$$ 

In other words $\varphi = (2i\pi)^{-\frac{k}{2}} \varphi_{\hat{\mathcal{A}}(R^M)^{\wedge}}$ in the sense of (6.8), where $\hat{\mathcal{A}}(R^M)^{\wedge}$ is the Poincaré dual current of the $\mathcal{A}$-form $\hat{\mathcal{A}}(R^M)$. Let us explain how this enables us to recover the Atiyah-Singer index formula. Let $e \in M_q(\mathcal{C}^\infty(M))$, $e^2 = e$, and form the vector bundle $E = \text{ran}e$, which we equip with its Grassmannian connection $\nabla^E$. Then by (5.21) we have

$$\text{ind}\mathcal{D}_{\nabla^E} = \text{ind}\varphi_{\hat{\mathcal{A}}(R^M)^{\wedge}}[e].$$

As $(2i\pi)^{-\frac{k}{2}} \varphi_{\hat{\mathcal{A}}(R^M)^{\wedge}}$ represents the Connes-Chern character, by Theorem 7.13 we have

$$\text{ind}\varphi_{\hat{\mathcal{A}}(R^M)^{\wedge}}[e] = (2i\pi)^{-\frac{k}{2}} \left< \varphi_{\hat{\mathcal{A}}(R^M)^{\wedge}}, e \right>.$$

Moreover, using (6.12) we have

$$\left< \varphi_{\hat{\mathcal{A}}(R^M)^{\wedge}}, e \right> = \left< \hat{\mathcal{A}}(R^M)^{\wedge}, \text{Ch}(F^E) \right> = \int_M \hat{\mathcal{A}}(R^M) \wedge \text{Ch}(F^E),$$

where $F^E$ is the curvature of $\nabla^E$. Therefore, we obtain

$$\text{ind}\mathcal{D}_{\nabla^E} = (2i\pi)^{-\frac{k}{2}} \int_M \hat{\mathcal{A}}(R^M) \wedge \text{Ch}(F^E),$$

which is the Atiyah-Singer index formula.

Remark 7.19. It remains an open question to construct a version of the CM cocycle for twisted spectral triples. Moscovici [Mo2] devised an Ansatz for such a cocycle and verified it in the case of twistings of ordinary spectral triples by scaling automorphisms. To date this seems to be the only known example of twisted spectral triples satisfying Moscovici’s Ansatz. It would already be interesting to have a version of Connes’s Hochschild character formula [Co4]. We refer to [FK1] for a Hochschild character formula in the special case of twistings of ordinary spectral triples by scaling automorphisms.

Remark 7.20. We refer to [PW1, PW2] for the computation of the Connes-Chern character of the conformal Dirac spectral triple of [CM1] (the construction of which is recalled in Section 2).

Appendix A. Proof of Lemma 5.11

It is immediate that the restriction of $(\cdot, \cdot)_0$ is $\mathcal{A}$-sesquilinear and positive. The only issue at stake is nondegeneracy.

Lemma A.1. Set $\mathcal{E}^* = e^* \mathcal{A}$. Then the restriction of $(\cdot, \cdot)$ to $\mathcal{E}^* \times \mathcal{E}$ is nondegenerate.
Proof of Lemma A.1. We need to show that $\Phi : \mathcal{E}^* \ni \xi \to (\xi, \cdot)_0 |_{\xi} \in \mathcal{E}'$ is an $A$-antilinear isomorphism. Let $\xi = (\xi_j) \in \mathcal{E}^*$. Then

$$(\xi, e\xi)_0 = (e^*\xi, \xi)_0 = (\xi, \xi)_0 = \sum \xi_j^2 \xi_j.$$ 

It then follows that if $(\xi, \cdot)_0$ vanishes on $\mathcal{E}$, then all the positive operators $\xi_j^2 \xi_j$ vanish on $\mathcal{H}$ and hence $\xi = 0$. This shows that $\Phi$ is injective.

Let $\varphi \in \mathcal{E}'$ and let $\hat{\varphi} \in (A^q)'$ be defined by $\hat{\varphi}(\xi) = \varphi(e\xi)$ for all $\xi \in A^q$. The nondegeneracy of $(\cdot, \cdot)_0$ implies there is $\tilde{\eta} \in A^q$ such that $\hat{\varphi}(\xi) = (\tilde{\eta}, \xi)_0$ for all $\xi \in A^q$. Set $\eta = e^*\tilde{\eta} \in \mathcal{E}^*$. Then, for all $\xi \in \mathcal{E}$,

$$\varphi(\xi) = \hat{\varphi}(e\xi) = (\tilde{\eta}, e\xi)_0 = (e^*\tilde{\eta}, \xi)_0 = (\eta, \xi)_0.$$ 

Thus $\varphi = \Phi(\eta)$. This shows that $\Phi$ is surjective. Therefore, $\Phi$ is an $A$-antilinear isomorphism. Likewise, $\Psi : \mathcal{E} \ni \eta \to (\cdot, \eta)_0 |_{\xi} \in (\mathcal{E}^*)_0$ is an $A$-linear isomorphism. This completes the proof of the lemma.

Lemma A.2. Denote by $t : \mathcal{E} \to \mathcal{E}^*$ the $A$-linear map defined by

$$t\xi = e^*\xi \quad \forall \xi \in \mathcal{E}.$$ 

Then $t$ is an $A$-linear isomorphism from $\mathcal{E}$ onto $\mathcal{E}^*$.

Proof of Lemma A.2. If $F$ is a right submodule of $A^q$ we shall denote by $F^\perp$ its orthogonal with respect to the canonical Hermitian metric of $A^q$. For $a \in M_q(A)$ we shall identify $a$ with the associated $A$-linear map $A^q \to A^q$. We observe that with this convention $a^*$ is identified with the adjoint of $a$ with respect to $(\cdot, \cdot)_0$, i.e.,

$$(a^*\xi, \eta)_0 = (\xi, a\eta)_0 \quad \forall \xi, \eta \in A^q.$$ 

We deduce from this that, for any idempotent $f \in M_q(A)$,

$$(A.1) \quad (fA^q)^\perp = (\text{ran } f)^\perp = \ker f^* = \text{ran}(1 - f^*) = (1 - f^*)A^q.$$ 

We note this implies that $((fA^q)^\perp)^\perp = fA^q$.

Using (A.1) we get

$$\ker t = \ker e^* \cap \text{ran } e = (\text{ran } e)^\perp \cap \text{ran } e = \{0\},$$

which shows that $t$ is one-to-one. Moreover, as $A$ is closed under holomorphic functional calculus there is $g \in \text{GL}_q(A)$ such that $f := e^*eg$ is a selfadjoint idempotent which is similar to $e^*$ (cf. [Bl]). Thus,

$$\text{ran } e^*e = \text{ran } f = \left((fA^q)^\perp\right)^\perp = \left((\text{ran } ee^*)^\perp\right)^\perp = (\ker e^*e)^\perp.$$ 

Obviously $\ker e \subset \ker e^*e$. As $(e\xi, e\xi)_0 = (e^*e\xi, \xi)$ for all $\xi \in A^q$, we see that $\ker e^*e$ is contained in $\ker e$, and so the two submodules agree. Thus,

$$\text{ran } e^*e = (\ker e^*e)^\perp = (\ker e)^\perp = \text{ran } e^*.$$ 

This shows that $t(\mathcal{E}) = e^*(\text{ran } e^*) = \text{ran } e^*e = \text{ran } e^* = \mathcal{E}^*$, that is, $t$ is onto. Therefore, the $A$-linear map $t$ is an isomorphism.

Let us go back to the proof of Lemma A.1. For all $\xi_1$ and $\xi_2$ in $\mathcal{E}$, we have

$$(\xi_1, \xi_2)_0 = (\xi_1, e\xi_2)_0 = (e^*\xi_1, \xi_2)_0 = (t\xi_1, t\xi_2)_0.$$ 

As $(\cdot, \cdot)_0$ is nondegenerate on $\mathcal{E}^* \times \mathcal{E}$ by Lemma A.1 and $t$ is an $A$-linear isomorphism by Lemma A.2 we then deduce that $(\cdot, \cdot)_0$ is nondegenerate on $\mathcal{E} \times \mathcal{E}$. This completes the proof of Lemma A.1.
Appendix B. Proof of Lemma 5.13

Let us first assume that $\mathcal{E} = A^q$ for some $q \in \mathbb{N}_0$. Let us denote by $\mathcal{H}(A^q)_0$ the pre-Hilbert space associated to the canonical Hermitian metric $(\cdot, \cdot)_0$ on $A^q$. There is a canonical isomorphism $U_0 : A^q \otimes_A \mathcal{H} \to \mathcal{H}^q$ given by

$$U_0(\xi \otimes \zeta) = (\xi_1, \ldots, \xi_q)$$

for all $\xi = (\xi_j) \in A^q$ and $\zeta \in \mathcal{H}$.

The inverse of $U_0$ is given by

$$U^*(\zeta_1, \ldots, \zeta_q) = \varepsilon_1 \otimes \zeta_1 + \cdots + \varepsilon_q \otimes \zeta_q,$$

where $\varepsilon_1, \ldots, \varepsilon_q$ is the canonical basis of $A^q$. We also observe that, for $\xi \in A^q$ and $\zeta \in \mathcal{H}$,

$$\|U_0(\xi \otimes \zeta)\|^2 = \sum_i \langle \xi_i, \zeta_i \rangle = \sum_i \langle \zeta_i, \xi_i \rangle = \langle \zeta, \xi \rangle = \|\xi \otimes \zeta\|^2,$$

where $\| \cdot \|_0$ is the norm of $\mathcal{H}(A^q)_0$. This shows that $U_0$ is an isometric isomorphism from $\mathcal{H}(A^q)_0$ onto $\mathcal{H}^q$. As $\mathcal{H}^q$ is complete, we deduce that so is $\mathcal{H}(A^q)_0$, i.e., $\mathcal{H}(A^q)_0$ is a Hilbert space.

Let $(\cdot, \cdot)$ be a Hermitian metric on $A^q$. We denote by $\mathcal{H}(E)$ the associated pre-Hilbert space and by $(\cdot, \cdot)$ its inner product. The nondegeneracy of $(\cdot, \cdot)$ and $(\cdot, \cdot)_0$ implies there is a selfadjoint element $g \in \mathrm{GL}_q(A)$ such that

$$\langle \xi, \eta \rangle = (g \xi, \eta)_0 \quad \forall \xi, \eta \in A^q.$$

We also denote by $g$ the representation of $g$ as a selfadjoint bounded operator of $\mathcal{H}^q$. Let $\xi = (\xi_j)$ and $\zeta = (\zeta_j)$ be in $A^q$ and let $\zeta$ and $\zeta'$ be in $\mathcal{H}$. Then

$$\langle \xi \otimes \zeta, \xi' \otimes \zeta' \rangle = \langle \zeta, \xi, \zeta' \rangle = \langle \zeta, (g \xi, \xi')_0 \rangle = \sum_{i,j} \langle \zeta, g \xi_j, \xi_i \rangle \langle \zeta, \xi_j \rangle = \sum_{i,j} \langle g \xi_j, \xi_i \rangle \langle \zeta, \xi_j \rangle$$

$$= (g U_0(\xi \otimes \zeta), U_0(\xi' \otimes \zeta')).$$

By bilinearity it then follows that $\langle \eta, \eta' \rangle = \langle g U_0 \eta, U_0 \eta' \rangle$ for all $\eta$ and $\eta'$ in $\mathcal{H}(A^q)$. Thus, for all $\eta \in \mathcal{H}(A^q)$ and $\zeta \in \mathcal{H}^q$,

\begin{equation}
\|\eta\|^2 = \langle g U_0 \eta, U_0 \eta \rangle \quad \text{and} \quad \langle g \zeta, \zeta \rangle = \|U_0^{-1} \zeta\|^2.
\end{equation}

The 2nd equality in (B.1) shows that $g$ is a positive operator of $\mathcal{H}^q$. As $g$ is invertible, we see that its spectrum is contained in an interval $[c^{-1}, c]$ for some $c > 1$, and so, for all $\zeta \in \mathcal{H}^q$,

$$c^{-1} \|\zeta\|^2 \leq \langle g \zeta, \zeta \rangle \leq c \|\zeta\|^2.$$

Combining (B.1) and the fact that $U_0$ is an isometry from $\mathcal{H}(A^q)_0$ onto $\mathcal{H}^q$ we deduce that, for all $\eta \in A^q \otimes_A \mathcal{H}$, we have

$$\|\eta\|^2 = \langle g U_0 \eta, U_0 \eta \rangle \in \left[ c^{-1} \|U_0 \eta\|^2, c \|U_0 \eta\|^2 \right] = \left[ c^{-1} \|\eta\|^2_0, c \|\eta\|^2_0 \right].$$

This shows that the norms $\| \cdot \|$ and $\| \cdot \|_0$ are equivalent on $A^q \otimes_A \mathcal{H}$. Therefore $\mathcal{H}(A^q)$ has same topology as $\mathcal{H}(A^q)_0$. In particular, $\mathcal{H}(A^q)$ is complete, and hence is a Hilbert space. This proves Lemma 5.13 in the special case $E = A^q$.

Let us now assume that $\mathcal{E} = e A^q$ with $e = e^2 \in \mathcal{M}_q(A)$. By Lemma 5.13, the canonical Hermitian metric of $A^q$ induces a Hermitian metric on $\mathcal{E}$. We denote by $(\cdot, \cdot)$ and $\mathcal{H}(\mathcal{E})$ the associated inner product and pre-Hilbert space. We also denote by $e$ the representation of $e$ as a bounded operator on $\mathcal{H}^q$. We note that as $e$ is idempotent $e \mathcal{H}^q$ is a closed subspace of $\mathcal{H}^q$.

Let $\xi = (\xi_j) \in A^q$ and $\zeta \in \mathcal{H}$. For $i = 1, \ldots, q$, we have

$$U_0(e \xi \otimes \zeta)_i = (e \xi)_i \zeta = \sum_j e_{ij} \xi_j \zeta = \sum_j e_{ij} U_0(\xi \otimes \zeta)_j = (e U_0(\xi \otimes \zeta))_i.$$

That is,

$$U_0(e \xi \otimes \zeta) = e U_0(\xi \otimes \zeta).$$

As $U_0$ is an isometric isomorphism from $\mathcal{H}(A^q)_0$ onto $\mathcal{H}^q$ we see that $U_0$ induces an isometric isomorphism from $\mathcal{H}(\mathcal{E})_0$ onto $e \mathcal{H}^q$. As $e \mathcal{H}^q$ is complete (since this is a closed subspace of $\mathcal{H}^q$) we deduce that $\mathcal{H}(\mathcal{E})_0$ is a Hilbert space.
Let \((\cdot, \cdot)\) be a Hermitian metric on \(E\). Thanks to the nondegeneracy of \((\cdot, \cdot)_0\) and \((\cdot, \cdot)\) there is a unique \(A\)-linear isomorphism \(a : E \to E\) such that
\[
(\xi, \eta) = (a\xi, \eta)_0 \quad \text{for all } \xi \text{ and } \eta \in E.
\]
We then extend \((\cdot, \cdot)\) into the \(A\)-sesquilinear form on \(A^q\) defined by
\[
(B.3) \quad (\xi, \eta) := (ae\xi, \eta)_0 + ((1 - e)\xi, (1 - e)\eta)_0 \quad \text{for all } \xi \text{ and } \eta \in A^q.
\]
We note that \((\cdot, \cdot)\) is positive on \(A^q\), and
\[
(\xi, \eta) = (g\xi, \eta)_0 \quad \text{for all } \xi \text{ and } \eta \in A^q.
\]
where we have set \(g = e^*ae + (1 - e)^*(1 - e)\).

By Lemma A.14 we know that \(a^*\) induces an \(A\)-linear isomorphism from \(eA^q\) onto \(e^*A^q\) and \((1 - e)A^q\) onto \((1 - e)^*A^q\). As \(a\) is an isomorphism from \(E = eA^q\) onto itself we deduce that \(g\) is a right-module isomorphism from \(A^q\) onto itself. Combining this with (B.3) we then see that \((\cdot, \cdot)\) is nondegenerate on \(A^q\times A^q\). Thus \((\cdot, \cdot)\) is a Hermitian metric on \(A^q\). Therefore, by the first part of the proof, the associated norm on \(A^q \otimes_A \mathcal{H}\) is equivalent to the norm of \(\mathcal{H}(A^q)\). As these norms restrict to the norms of \(\mathcal{H}(E)\) and \(\mathcal{H}(E)_0\) on \(E = eA^q\), we then deduce that the norms of \(\mathcal{H}(E)\) and \(\mathcal{H}(E)_0\) are equivalent. This proves Lemma A.13 in the special case \(E = eA^q\), \(e = e^2 \in M_0(A)\).

Let us now prove Lemma A.13 when \(E\) is an arbitrary finitely generated projective module, i.e., it is the direct summand of a free module \(E_0\). Let \(\phi : E_0 \to A^q\) be an \(A\)-linear isomorphism. Then \(\phi(E) = eA^q\) for some idempotent \(e \in M_0(A)\). If \((\cdot, \cdot)\) is a Hermitian metric on \(E\), then we define a Hermitian metric \((\cdot, \cdot)_\phi\) on \(eA^q\) by
\[
(\xi, \eta)_\phi = (\phi^{-1}(\xi), \phi^{-1}(\eta)) \quad \text{for all } \xi \text{ and } \eta \in eA^q.
\]
We denote by \((\cdot, \cdot)_\phi\) and \(\mathcal{H}(eA^q)_\phi\) the associated inner product and Hilbert space.

Set \(U_\phi := \phi \otimes 1_\mathcal{H}\). This a vector bundle isomorphism from \(E \otimes_\mathcal{H} \mathcal{H}\) onto \((eA^q) \otimes_\mathcal{H} \mathcal{H}\). Let \(\xi\) and \(\xi'\) be in \(E\) and let \(\zeta\) and \(\zeta'\) be in \(\mathcal{H}\). Then
\[
(U_\phi(\xi \otimes \zeta), U_\phi(\xi' \otimes \zeta'))_\phi = \langle \zeta, (\phi(\xi), \phi(\xi'))_\phi \zeta' \rangle = \langle \zeta, (\xi, \xi') \zeta' \rangle = \langle \xi \otimes \zeta, \xi' \otimes \zeta' \rangle.
\]
Thus \(U_\phi\) is an isometric isomorphism from \(\mathcal{H}(E)\) and \(\mathcal{H}(eA^q)_\phi\). As \(\mathcal{H}(eA^q)_\phi\) is a Hilbert space, we then deduce that \(\mathcal{H}(E)\) is a Hilbert space as well.

Finally, we observe that pushforwarding norms by \(U_\phi\) gives rise to a one-to-one correspondence between norms on \(E \otimes_\mathcal{H} \mathcal{H}\) and \((eA^q) \otimes_\mathcal{H} \mathcal{H}\) arising from Hermitian metrics on \(E\) and \(eA^q\). As all those norms on \(eA^q\) are equivalent to each other, we then deduce that the same result holds on \(E\). That is, the topology of \(\mathcal{H}(E)\) is independent of the choice of the Hermitian metric. The proof of Lemma A.13 is complete.

**Appendix C. Homotopy Invariance of the Connes-Chern Character**

In this appendix, we give a proof of the homotopy invariance of the Connes-Chern character in the following form.

**Proposition C.1.** Let \((A, \mathcal{H}, D)\) be a \(p\)-summable twisted spectral triple. Consider an operator homotopy of the form,
\[
D_t = D + V_t, \quad 0 \leq t \leq 1,
\]
where \((V_t)_{0 \leq t \leq 1}\) is a \(C^1\) selfadjoint family in \(\mathcal{L}(\mathcal{H})\) such that \(D_t\) is invertible for all \(t \in [0, 1]\) and \((D_t^{-1})_{0 \leq t \leq 1}\) is a bounded family in \(\mathcal{L}^p(\mathcal{H})\). Then

1. \((A, \mathcal{H}, D_t)\) is a \(p\)-summable twisted spectral triple for all \(t \in [0, 1]\).
2. For any \(k \geq \frac{1}{2}(p + 1)\), the cocycles \(\tau_{2k}D_0\) and \(\tau_{2k}D_t\) are cohomologous in \(HC^{2k}(A)\).

By assumption the resolvent \(D_t^{-1}\) lies in \(\mathcal{L}^p(\mathcal{H})\). Moreover, for all \(a \in A\),
\[
(C.1) \quad [D_t, a]_\sigma = [D, a]_\sigma + (V_t a - \sigma(a)V_t) \in \mathcal{L}(\mathcal{H}).
\]
Therefore \((A, H, D_t)\) is a p-summable twisted spectral triple, and so, for any integer \(k \geq \frac{1}{2}(p - 1)\), we can form the cyclic \(2k\)-cocycle,

\[
\tau_{2k}^{D_t}(a^0, \ldots, a^{2k}) = c_k \text{Str} \left( D_t^{-1} [D_t, a^0] \cdots D_t^{-1} [D_t, a^{2k}] \right), \quad a^j \in A.
\]

The rest of the proof is devoted to comparing the cocycles \(\tau_{2k}^{D_t}\) and \(\tau_{2k}^{D_0}\).

In what follows, we set \(\dot{V}_t = \frac{d}{dt} V_t\) and, for \(a \in A\), we define

\[
\delta_t(a) = D_t^{-1}[\dot{V}_t, D_t^{-1}, \sigma(a)]D_t.
\]

We note that

\[
\delta_t(a) = [D_t^{-1} \dot{V}_t, D_t^{-1}, \sigma(a)]D_t = [D_t^{-1} \dot{V}_t, a] - [D_t^{-1} \dot{V}_t, [D_t, a]]\sigma.
\]

As \((C.1)\) shows that \([[D_t, a]]_{0 \leq t \leq 1}\) is a continuous family in \(L(\mathcal{H})\) and \((D_t^{-1} \dot{V}_t)_{0 \leq t \leq 1}\) is a continuous family in \(L^p(\mathcal{H})\), therefore \((\delta_t(a))_{0 \leq t \leq 1}\) is a continuous family in \(L^p(\mathcal{H})\). For \(j = 1, \ldots, 2k + 1\), we let \(\eta_j^t\) be the \((2k + 1)\)-cochain on \(A\) defined by

\[
\eta_j^t(a^0, \ldots, a^{2k+1}) = \text{Str} \left( \alpha_j(a^0) D_t^{-1} [D_t, a^1] \cdots \delta_t(a^j) \cdots D_t^{-1} [D_t, a^{2k+1}] \right), \quad a^j \in A,
\]

where \(\alpha_j(a) = a\) if \(j\) is even and \(\alpha_j(a) = D_t^{-1} \sigma(a) D_t\) if \(j\) is odd.

In what follows we shall say that a family \((\varphi^t)_{0 \leq t \leq 1} \subset C^m(\mathcal{A})\) is \(C^m\), \(\alpha \geq 0\), when, for all \(a^0, \ldots, a^m\), the function \(t \to \varphi^t(a^0, \ldots, a^m)\) is \(C^m\) on \([0, 1]\). Given a \(C^1\)-family \((\varphi^t)_{0 \leq t \leq 1}\) of \(m\)-cochains, we define \(m\)-cochains \(\frac{d}{dt} \varphi^t\), \(t \in [0, 1]\), by

\[
\left( \frac{d}{dt} \varphi^t \right)(a^0, \ldots, a^m) := \frac{d}{dt} \left( \varphi^t \right)(a^0, \ldots, a^m), \quad a^j \in A.
\]

Given a \(C^0\)-family \((\psi^t)_{0 \leq t \leq 1}\) of \(m\)-cochains we define the integral \(\int \psi^t\) as the \(m\)-cochain given by

\[
\left( \int_0^1 \psi^t dt \right)(a^0, \ldots, a^m) := \int_0^1 \psi^t(a^0, \ldots, a^m) dt, \quad a^j \in A.
\]

If \(F\) is any of the operators \(b, A, B_0\) or \(B\), then

\[
F \left( \frac{d}{dt} \varphi^t \right) = \frac{d}{dt} \left( F \varphi^t \right) \quad \text{and} \quad F \left( \int_0^1 \psi^t dt \right) = \int_0^1 F \psi^t dt.
\]

Moreover, we have

\[
\int_0^1 \left( \frac{d}{dt} \varphi^t \right) dt = \varphi^1 - \varphi^0.
\]

**Lemma C.2.** The family \((\tau_{2k}^{D_t})_{0 \leq t \leq 1}\) is a \(C^1\)-family of \(2k\)-cochains and we have

\[
\frac{d}{dt} \tau_{2k}^{D_t} = \frac{c_k}{2k + 1} \sum_{j=1}^{2k+1} B_t \eta_j^t.
\]

**Proof.** It follows from \((C.1)\) that \([[D_t, a]]_{0 \leq t \leq 1}\) is a \(C^1\)-family in \(L(H)\) and we have

\[
\frac{d}{dt} [D_t, a]_{\sigma} = \dot{V}_t a - \dot{V}_t \sigma(a).
\]

By assumption the family \((D_t^{-1})_{0 \leq t \leq 1}\) is bounded in \(L^p(\mathcal{H})\). Moreover,

\[
D_t^{-1} - D_0^{-1} = -D_t^{-1} (D_t + s - D_t) D_t^{-1} = -D_t^{-1} (V_t s - V_t) D_t^{-1}.
\]

We then deduce that \((D_t^{-1})_{0 \leq t \leq 1}\) is a continuous family in \(L^p(\mathcal{H})\). Combining this with the above equality then shows that \((D_t^{-1})_{0 \leq t \leq 1}\) is a differentiable family in \(L^p(\mathcal{H})\) with

\[
\frac{d}{dt} D_t^{-1} = -D_t^{-1} \dot{V}_t D_t^{-1}.
\]

As the above right-hand side is a continuous family in \(L^p(\mathcal{H})\) we eventually see that \((D_t^{-1})_{0 \leq t \leq 1}\) is a \(C^1\) family in \(L^p(\mathcal{H})\). The product in \(L(H)\) induces a continuous bilinear map from \(L^p(\mathcal{H}) \times L^p(\mathcal{H})\)
to $\mathcal{L}^p(\mathcal{H})$. Therefore, we deduce that $(D_t^{-1}[D_t, a])_{0 \leq t \leq 1}$ is a $C^1$-family in $\mathcal{L}^p(\mathcal{H})$, and using (C.4) and (C.6) we obtain

$$
\frac{d}{dt} D_t^{-1}[D_t, a]_\sigma = -D_t^{-1} \dot{V} D_t^{-1}[D_t, a] + D_t^{-1} \left( \dot{V} a - \sigma(a) \dot{V} \right)
$$

(C.6)

Let $a^0, \ldots, a^{2k}$ be in $\mathcal{A}$. As $2k + 1 \geq p$ the product of $\mathcal{L}(\mathcal{H})$ induces a continuous $(2k + 1)$-linear map from $\mathcal{L}^p(\mathcal{H})^{2k+1}$ to $\mathcal{L}^1(\mathcal{H})$. Therefore, the map $t \mapsto D_t^{-1}[D_t, a^0]_\sigma \cdots D_t^{-1}[D_t, a^{2k}]_\sigma$ is a $C^1$-map from $[0, 1]$ to $\mathcal{L}^1(\mathcal{H})$. Composing it with the supertrace on $\mathcal{L}^1(\mathcal{H})$ we then deduce that the function $t \mapsto \tau_{2k} D_t^{-1}(a^0, \ldots, a^{2k})$ is $C^1$ on $[0, 1]$. Moreover, using (C.2) we get

$$
\frac{d}{dt} \tau_{2k} D_t(a^0, \ldots, a^{2k}) = c_k \sum_{j=0}^{2k} \text{Str} \left( D_t^{-1}[D_t, a^0]_\sigma \cdots \delta_i(a^j) \cdots D_t^{-1}[D_t, a^{2k}]_\sigma \right).
$$

Noting that $\alpha_j(1) = 1$ we see that

$$
\text{Str} \left( D_t^{-1}[D_t, a^0]_\sigma \cdots \delta_i(a^j) \cdots D_t^{-1}[D_t, a^{2k}]_\sigma \right) = B_0 \eta_{j+1}(a^0, \ldots, a^{2k}).
$$

Therefore, we see that $(\tau_{2k} D_t)_{0 \leq t \leq 1}$ is a $C^1$-family of cochains and $\frac{d}{dt} \tau_{2k} = c_k \sum_{j=0}^{2k+1} B_0 \eta_{j}$. As $\tau_{2k}$ is a cyclic cocycle, using (C.2) we get

$$
\frac{d}{dt} \tau_{2k} = \frac{1}{2k+1} \frac{d}{dt} A \tau_{2k} = \frac{1}{2k+1} A \left( \frac{d}{dt} \tau_{2k} \right) = \frac{c_k}{2k+1} \sum_{j=1}^{2k+1} B_0 \eta_{j} = \frac{c_k}{2k+1} \sum_{j=1}^{2k+1} B_0 \eta_{j}.
$$

The proof is complete.

**Lemma C.3.** For $t \in [0, 1]$ and $j = 1, \ldots, 2k+1$ the cochain $\eta_j$ is a Hochschild cocycle, i.e., $b \eta_j = 0$.

**Proof.** Let $\beta$ and $\gamma$ be the $(2k + 2)$-cochains on $\mathcal{A}$ given by

$$
\beta(a^0, \ldots, a^{2k+2}) = \text{Str} \left( \alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots \delta_i(a^{j+1}) \cdots D^{-1}[D, a^{2k+2}]_\sigma \right),
$$

$$
\gamma(a^0, \ldots, a^{2k+2}) = \text{Str} \left( \alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots \delta_i(a^j) \cdots D^{-1}[D, a^{2k+2}]_\sigma \right), \quad a^j \in \mathcal{A}.
$$

For $l = 1, \ldots, j$ we let $\beta_l$ and $\beta_l'$ be the $(2k + 2)$-cochains defined by

$$
\beta_l(a^0, \ldots, a^{2k+2}) = \text{Str} \left( \alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots a^l \cdots \delta_i(a^{j+1}) \cdots D^{-1}[D, a^{2k+2}]_\sigma \right),
$$

$$
\beta_l'(a^0, \ldots, a^{2k+2}) = \text{Str} \left( \alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots \delta_i(a^j) \cdots D^{-1}[D, a^{2k+2}]_\sigma \right).
$$

We note that $\beta_l'(a^0, \ldots, a^{2k+2}) - \beta_l''(a^0, \ldots, a^{2k+2})$ is equal to

(C.7) $$
\text{Str} \left( \alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots (a^j - D^{-1}[D, a^j]_\sigma D) \cdots \delta_i(a^{j+1}) \cdots D^{-1}[D, a^{2k+2}]_\sigma \right) = \beta(a^0, \ldots, a^{2k+2}).
$$

Moreover, from the equality $D^{-1}[D, a^{l+1}]_\sigma = D^{-1}[D, a^1]_\sigma a^{l+1} + D^{-1}[D, a^l] D \cdot D^{-1}[D, a^{l+1}]_\sigma$ we deduce that

(C.8) $$
b \eta_j = \beta_{l+1} + \beta_l''.
$$

For $l = j + 1, \ldots, 2k + 1$ we let $\gamma_j'$ and $\gamma_j''$ be the $(2k + 2)$-cochains on $\mathcal{A}$ defined by

(C.9) $$
\gamma_j'(a^0, \ldots, a^{2k+2}) = \text{Str} \left( \alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots \delta_i(a^j) \cdots D^{-1}[D, a^{2k+2}]_\sigma \right),
$$

(C.10) $$
\gamma_j''(a^0, \ldots, a^{2k+2}) = \text{Str} \left( \alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots \delta_i(a^j) \cdots D^{-1}[D, a^{2k+2}]_\sigma \right).
$$

As in (C.7) and (C.8) we have

(C.11) $$
\gamma_j' - \gamma_j'' = \gamma \quad \text{and} \quad b \eta_j = \gamma_{j+1} + \gamma_i.
$$
In addition, using the equality $\delta_t(a^ja^{j+1}) = \delta_t(a^j)D^{-1}\sigma(a^j)D + D^{-1}a^jD\delta_t(a^{j+1})$ we find that

$$b_1\eta_j^0(a^0, \ldots, a^{2k+2}) = \text{Str} \left( \alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots \delta_t(a^{j+1}) \cdots D^{-1}[D, a^{2k+2}]_\sigma \right)$$

(C.12)

$$= \gamma_{j+1}'(a^0, \ldots, a^{2k+2}) + \beta_j'(a^0, \ldots, a^{2k+2}).$$

Using (C.7)–(C.12) we obtain

$$\sum_{l=1}^{2k+1} b_l\eta_j^l = \sum_{l=1}^{2k+1} \left( b_l\eta_j^l - \beta_l'' \right) - \frac{1}{2} \left( 1 + (-1)^j \right) \beta + \frac{1}{2} \left( 1 - (-1)^j \right) \gamma + b_{2k+2}\eta_j^0 - \gamma_{2k+1}'.'$$

We note that $b_0\eta_j^0(a^0, \ldots, a^{2k+2}) - \beta_1''(a^0, \ldots, a^{2k+2})$ is equal to

(C.13) \[ \text{Str} \left( \alpha_j(a^0)(\alpha_j(a^1) - D^{-1}\sigma(a^1)D) \cdots \delta_t(a^{j+1}) \cdots D^{-1}[D, a^{2k+2}]_\sigma \right). \]

We also observe that

$$b_{2k+2}\eta_j^l(a^0, \ldots, a^{2k+2}) = \text{Str} \left( \alpha_j(a^{2k+2})\alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots \delta_t(a^{j+1}) \cdots D^{-1}[D, a^{2k+2}]_\sigma \right)$$

$$= \text{Str} \left( \alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots \delta_t(a^{j+1}) \cdots D^{-1}[D, a^{2k+2}]_\sigma \alpha_j(a^{2k+2}) \right).$$

Thus $b_{2k+2}\eta_j^l(a^0, \ldots, a^{2k+2}) - \beta_{2k+1}(a^0, \ldots, a^{2k+2})$ is equal to

(C.15) \[ \text{Str} \left( \alpha_j(a^0)D^{-1}[D, a^1]_\sigma \cdots \delta_t(a^{j+1}) \cdots D^{-1}[D, a^{2k+2}]_\sigma \left( \alpha_j(a^{2k+2}) - a^{2k+2} \right) \right). \]

Suppose that $j$ is even, so that $\alpha_j(a) = a$. Then (C.15) shows that $b_{2k+2}\eta_j^0 - \beta_{2k+1} = 0$. Moreover, $\alpha_j(a) - D^{-1}\sigma(a)D = D^{-1}[D, a]_\sigma$, and so using (C.14) we see that $b_0\eta_j^0 - \beta_1'' = \beta$. Therefore, in this case (C.13) gives

$$b_0\eta_j^0 = \beta - \frac{1}{2} \left( 1 + 1 \right) \beta + \frac{1}{2} \left( 1 - 1 \right) \gamma + 0 = 0.$$

When $j$ is odd, $\alpha_j(a) = D^{-1}\sigma(a)D$, and we similarly find that $b_0\eta_j^0 - \beta_1'' = 0$ and $b_{2k+2}\eta_j^0 - \beta_{2k+1} = -\gamma$. Thus, in this case (C.15) gives

$$b_0\eta_j^0 = 0 - \frac{1}{2} \left( 1 - 1 \right) \beta + \frac{1}{2} \left( 1 + 1 \right) \gamma = 0.$$

In any case, $\eta_j^0$ is a Hochschild cocycle. The proof is complete. \qed

In the same way as in the proof of Lemma C.2 it can be shown that each family $(\eta_j^0)_{0 \leq t \leq 1}$ is a continuous family of cochains. Note also that these cochains are normalized. Let $\eta$ be the $(2k + 1)$-cochain defined by

$$\eta = \sum_{j=1}^{2k+1} \int_0^1 \eta_j^0 dt.$$

It follows from (C.2)–(C.3) and Lemma C.2 that

(C.16) \[ B\eta = \int_0^1 \left( \sum_{j=1}^{2k+1} B\eta_j^0 \right) dt = (2k + 1)c_k^{-1} \int_0^1 \left( \frac{d}{dt} \tau_{2k+1} \right) dt = (2k + 1)c_k^{-1} \left( \tau_{2k} \right) - \tau_{2k}. \]
Moreover, using (C.2) and Lemma C.3 we get
\[ b\eta = \sum_{j=1}^{2k+1} \int_0^1 b\eta_j^k \, dt = 0. \]

In particular, as \( \eta \) is a normalized cochain and \( b\eta \) is cyclic, we may apply (6.16) to get
\[ SB\eta = S\Sigma b\eta = -b\eta = 0 \quad \text{in } \text{HC}^{2k+2}(A). \]

As \( \tau_{2k}^D \) and \( \tau_{2k+2}^D \) are cohomologous in \( \text{HP}^{0}(A) \), using (6.18) we see that \( \tau_{2k+2}^D = S\tau_{2k}^D \) in \( \text{HC}^{2k+2}(A) \). Thus,
\[ \tau_{2k+2}^D - \tau_{2k}^D = S(\tau_{2k}^D - \tau_{2k}^D) = SB\eta = 0 \quad \text{in } \text{HC}^{2k+2}(A). \]

That is, the cocycles \( \tau_{2k+2}^D \) and \( \tau_{2k}^D \) are cohomologous in \( \text{HC}^{2k+2}(A) \). This completes the proof of Proposition C.4.

Remark C.4. By using the bounded Fredholm module pairs associated to a twisted spectral triple in [CM4], we also can deduce Proposition C.4 from the homotopy invariance of the Connes-Chern character of a bounded Fredholm module in [Co1, Part I, §5].

References

[At] Atiyah, M.: Global aspects of the theory of elliptic differential operators. Proc. Internat. Congr. Math. (Moscow, 1966), pp. 57–64, Izdat. “ Mir”, Moscow, 1968.

[AS1] Atiyah, M., Singer, I.: The index of elliptic operators. I. Ann. of Math. (2) 87 (1968), 484–530.

[AS2] Atiyah, M., Singer, I.: The index of elliptic operators. III. Ann. of Math. (2) 87 (1968), 546–604.

[BF] Blackadar, B.: K-theory for operator algebras, Mathematical Sciences Research Institute Publications Vol 5, 2nd edition, Cambridge University Press, 1998.

[Bl] Blackadar, B.: K-theory for operator algebras, Mathematical Sciences Research Institute Publications Vol 5, 2nd edition, Cambridge University Press, 1998.

[Bo] Bost, J.B.: Principe d’Oka, K-théorie et systèmes dynamiques non commutatifs. Inv. Math. 101 (1990), 261–333.

[BG] Bourguignon, J.-P.; Gauduchon, P. Spineurs, opérateurs de Dirac et variations de métriques. Comm. Math. Phys. 144 no. 3, (1992), 581–599.

[Co1] Connes, A.: Noncommutative differential geometry. Inst. Hautes Études Sci. Publ. Math. 62 (1985), 257–360.

[Co2] Connes, A.: Cyclic cohomology and the transverse fundamental class of a foliation. Geometric methods in operator algebras (Kyoto, 1983), pp. 52–144, Pitman Res. Notes in Math. 123 Longman, Harlow (1986).

[Co3] Connes, A.: Entire cyclic cohomology of Banach algebras and characters of \( \theta \)-summable Fredholm modules. K-Theory 1 (1988), no. 6, 519–548.

[Co4] Connes, A.: Noncommutative geometry. Academic Press, San Diego, 1994.

[CM1] Connes, A., Moscovici, H.: The local index formula in noncommutative geometry. Geom. Funct. Anal. 5 (1995), 174–243.

[CM2] Connes, A.; Moscovici, H.: Hopf algebras, cyclic cohomology and the transverse index theorem. Comm. Math. Phys. 198 (1998), no. 1, 199–246.

[CM3] Connes, A.; Moscovici, H.: Differentiable cyclic cohomology Hopf algebraic structures in transverse geometry. Monographie 38 de L’Enseignement Mathématique, pp. 217–256, 2001.

[CM4] Connes, A., Moscovici, H.: Type III and spectral triples. Traces in Geometry, Number Theory and Quantum Fields, Aspects of Mathematics E38, Vieweg Verlag 2008, 57–71.

[CM5] Connes, A., Moscovici, H.: Modular curvature for noncommutative two-tori. E-print, arXiv, Oct. 2011, 43 pages.

[CT] Connes, A.; Tretkoff, P.: The Gauss-Bonnet theorem for the noncommutative two torus. Noncommutative geometry, arithmetic, and related topics, pp. 141–158, Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[DA] D’Andrea, F.: Quantum Groups and Twisted Spectral Triples. E-print, arXiv, February 2007.

[FK1] Fathizadeh, F.; Khalkhali, M.: Twisted Spectral Triples and Connes’ Character Formula. Perspectives on Noncommutative Geometry, Fields Institute Communications Series Vol. 61, 2011, pp. 79–102.

[FK2] Fathizadeh, F.; Khalkhali, M.: The Gauss-Bonnet Theorem for Noncommutative Two Tori With a General Conformal Structure. J. Noncommut. Geom. 6 (2012), 457–480.

[GMT] Greenfield, M.; Marcotte, M.; Teh, K.: Type III \( \sigma \)-spectral triples and quantum statistical mechanical systems. E-print, arXiv, May 2013.
