We consider a boundary value problem for a general second-order linear equation in a domain with a fine perforation. The latter is made by small cavities; both the shapes of the cavities and their distribution are arbitrary. The boundaries of the cavities are subject either to a Dirichlet or a nonlinear Robin condition. On the perforation, certain rather weak conditions are imposed to ensure that under the homogenization, we obtain a similar problem in a non-perforated domain with an additional potential in the equation usually called a strange term. Our main results state the convergence of the solution of the perturbed problem to that of the homogenized one in $W^{1,2}$ and $L^2$-norms uniformly in $L^2$-norm of the right hand side in the equation. The estimates for the convergence rates are established, and their order sharpness is discussed.

**KEYWORDS**
non-periodic perforation, operator estimates, order sharp estimates, perforated domain, strange term

**MSC CLASSIFICATION**
35B27, 35B40

## 1 | INTRODUCTION

Nowadays, a new direction in the homogenization theory devoted to so-called operator estimates is quite intensively developed. In contrast to classical results in the homogenization theory on strong and weak convergence of the solutions, here, the studies are aimed on proving the norm resolvent convergence and obtaining estimates for the convergence rates; the latter are often called operator estimates. Recently, such results were obtained in few papers for problems in domains with fine perforation distributed along entire domain. Problems in such perforated domains are classical in the homogenization theory; see, for instance, [1–6] and the references therein, and there are many results describing the convergence in a strong or weak sense in $L^2$ and $W^{1,2}$ for fixed right hand sides in the equations and boundary conditions. There are also many works on problems with finitely many small holes in a domain; here, we cite only one of classical works [7] and few recent ones [8–10], see also the references therein.

In [11–16], the classical homogenization results were improved, and operator estimates were established for several cases of periodic and almost periodic perforation in arbitrary domains. The case of the Neumann condition was addressed in [14–16], the sizes of the cavities were of the same order as the distances between them, and the perforation was purely periodic. In [13], on the boundaries of the cavities, the Dirichlet condition was imposed, and the sizes of these cavities were assumed to satisfy certain relation with respect to the size of the periodicity cell. All cavities were of the same shapes up to an arbitrary rotation, and its location in the periodicity cell was also quite arbitrary. In [11, 12], the perforation was pure periodic, and it was made by small balls with the Dirichlet or Neumann [11] or Robin [12] condition on the boundaries. The main results of the cited papers were the formulation of the homogenized problems and various operator estimates; their order sharpness was not established.
A non-periodic perforation was studied in [17]. Here, the domain was a manifold with a perforation made by arbitrary cavities with the Dirichlet or Neumann condition on their boundaries, and the operator was the Laplacian. The main results were again homogenized problems and operator estimates, and they were established under the validity of certain local upper bounds for $L_2$-norm in terms of $W^1_2$-norms. And these bounds were main tools in proving the convergence and operator estimates. Then several cases of possible homogenized problems were addressed, and as an example, it was shown that the developed scheme worked for the perforation by small balls.

We also mention several recent papers on operator estimates for domain perforated along a given manifold [18–20]. The perforation was non-periodic and formed by arbitrary cavities and distribution. The homogenized problems were classified, and a series of operator estimates was established. In some cases, these estimates turned out to be order sharp.

In this paper, we study a boundary value problem for a linear second-order elliptic equation in a perforated domain. The differential expression is general with complex-valued varying coefficients and is not formally symmetric. The perforation is arbitrary and non-periodic and is assumed to satisfy natural geometric conditions. On the boundaries of the cavities, we impose the Dirichlet or a nonlinear Robin condition; both types of conditions can be simultaneously present on different cavities. Then we impose additional rather weak conditions on the perforation to describe the case, when the homogenization produces a so-called strange term, namely, when an additional potential appears the homogenized equation. Our main results state the convergence of the perturbed solution to the homogenized one in $L_2$- and $W^1_2$-norms uniformly in $L_2$-norm of the right hand side in the equation. We also establish the estimates for the convergence rates, and some terms in these estimates are shown to be order sharp. An important feature of our results is that our assumptions are rather weak and do not a priori require any local estimates like in [17]. Instead of this, we prove that similar estimates are guaranteed by our assumptions. One more advantage of our study is that we can deal with a nonlinear Robin condition.

In conclusion, we mention that a similar problem was studied in a very recent paper [21] but in the situation, when the solution to the perturbed problem vanishes as the perforation becomes finer. Operator estimates in such case were obtained, and the convergence rates were shown to be order sharp. The case of the Neumann condition on the boundaries of the cavities was treated in [22]. It was shown that in this case, the cavities disappear under the homogenization, and appropriate operator estimates were established.

2 | PROBLEM AND MAIN RESULTS

2.1 | Formulation of problem

Let $x = (x_1, \ldots, x_n)$ be Cartesian coordinates in $\mathbb{R}^n$ and $\Omega$ be an arbitrary domain in $\mathbb{R}^n$; if its boundary is non-empty, we suppose that its smoothness is $C^2$. The domain $\Omega$ can be both bounded or unbounded. In this domain, we choose a family of points $M_k^\varepsilon$, $k \in M^\varepsilon$, where $\varepsilon$ is a small positive parameter and $M^\varepsilon$ is some at most countable set of indices. We also choose a family of bounded non-empty domains $\omega_{k,\varepsilon} \subset \mathbb{R}^d$, $k \in M^\varepsilon$, with $C^2$-boundaries. Then we define

$$\omega_k^\varepsilon := \{x : (x - M_k^\varepsilon)\varepsilon^{-1}\eta^{-1}(\varepsilon) \in \omega_{k,\varepsilon}\} \, , \, k \in M^\varepsilon \, , \, \theta^\varepsilon := \bigcup_{k \in M^\varepsilon} \omega_k^\varepsilon, \quad (2.1)$$

where $\eta = \eta(\varepsilon)$ is some function obeying $0 < \eta(\varepsilon) \leq 1$. We shall formulate rigorously the assumptions on the domains $\omega_{k,\varepsilon}$ later; now, we just say that they are assumed to be approximately of the same size (but not the shapes!), and there is a minimal distance between the points $M_k^\varepsilon$, which ensures that the domains $\omega_k^\varepsilon$ are mutually disjoint. The mentioned minimal distance is proportional to $\varepsilon$, while the size of the cavities $\omega_k^\varepsilon$ is of order $\varepsilon \eta(\varepsilon)$.

By means of the domains $\omega_k^\varepsilon$, we introduce a perforation of the domain $\Omega$ as $\Omega^\varepsilon := \Omega\setminus\theta^\varepsilon$. In the perforated domain $\Omega^\varepsilon$, we consider a boundary value problem for an elliptic equation with the coefficients $A_{ij} = A_{ij}(x)$, $A_j = A_j(x)$, $A_0 = A_0(x)$ defined in the non-perforated domain $\Omega$, which are supposed to satisfy the conditions:

$$A_{ij} \in W^1_\infty(\Omega), \quad A_j, A_0 \in L_\infty(\Omega), \quad (2.2)$$

$$A_{ij} = A_{ji}, \quad \sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j \geq c_1 \sum_{j=1}^n |\xi_j|^2, \quad x \in \Omega \, , \, \xi_i \in \mathbb{C}, \quad (2.3)$$

where $c_1 > 0$ is some fixed constant independent of $\xi$ and $x$. The functions $A_{ij}$ are real-valued, while the functions $A_j$ and $A_0$ are complex-valued.
The boundaries of the cavities \( o_k^\varepsilon \) are subject to either the Dirichlet condition or a nonlinear Robin condition. In order to introduce them, we first partition arbitrarily the set \( \theta^\varepsilon \):

\[
\theta^\varepsilon_D := \bigcup_{k \in M^\varepsilon_D} o_k^\varepsilon, \quad \theta^\varepsilon_R := \bigcup_{k \in M^\varepsilon_R} o_k^\varepsilon, \quad M^\varepsilon_D \cup M^\varepsilon_R = M^\varepsilon, \quad M^\varepsilon_D \cap M^\varepsilon_R = \emptyset.
\]

For \( x \in \partial\theta^\varepsilon_D \) and \( u \in \mathbb{C} \) by \( a^\varepsilon = a^\varepsilon(x, u) \), we denote a measurable complex-valued function, which will serve as a nonlinear term in the Robin condition. This function is non-zero, has at most linear growth in \( u \), and should satisfy some additional conditions describing its dependence on \( \varepsilon \); we shall formulate all conditions rigorously later.

The main object of our study is the following boundary value problem:

\[
(L - \lambda)u = f \text{ in } \Omega^\varepsilon, \quad u = 0 \text{ on } \partial\Omega \cup \partial\theta^\varepsilon_R, \quad \frac{\partial u}{\partial \nu} + a^\varepsilon(x, u) = 0 \text{ on } \partial\theta^\varepsilon_R.
\]

Here, \( L \) and \( \frac{\partial}{\partial \nu} \) are a differential expression and a conormal derivative:

\[
L := -\text{div} AV + \sum_{j=1}^n A_j \frac{\partial}{\partial x_j} + A_0, \quad \frac{\partial}{\partial \nu} = \nu \cdot AV,
\]

\[
A(x) := \begin{pmatrix}
A_{11}(x) & \cdots & A_{1n}(x) \\
\vdots & \ddots & \vdots \\
A_{n1}(x) & \cdots & A_{nn}(x)
\end{pmatrix},
\]

\( f \in L_2(\Omega^\varepsilon) \) is an arbitrary function, \( \lambda \in \mathbb{C} \) is a fixed constant, and \( \nu \) is the unit normal to \( \partial\theta^\varepsilon_R \) directed inside \( \theta^\varepsilon_R \).

Our main aim is analyze the behavior of a generalized solution to problem (2.4) as \( \varepsilon \to +0 \). Namely, we address two questions: How does a homogenized problem for (2.4) read and whether the operator estimates hold? If so, what are the corresponding convergence rates? It is very well known that the form of the homogenized problem depends very much on the distribution and shapes of the cavities as well as on their sizes and the distances between them. In this paper, we consider the case when a so-called strange term appears and we ensure such situation by a few assumptions on the cavities and the nonlinearity \( a^\varepsilon \) in the Robin condition. All of them will be formulated later; now, we just say that the sizes of cavities, controlled by the function \( \eta(\varepsilon) \), depends on the small parameter \( \varepsilon \), governing the distances between the holes, as follows:

\[
\varepsilon^{-2}\eta^{n-2}(\varepsilon)\kappa^{-1}(\varepsilon) \to \gamma, \quad \varepsilon \to +0,
\]

where \( \gamma \) is a nonnegative constant and

\[
\kappa(\varepsilon) := |\ln(\eta(\varepsilon))| + 1 \text{ as } n = 2, \quad \kappa(\varepsilon) := 1 \text{ as } n \geq 3.
\]

Convergence (2.5) is the key point guaranteeing the appearance of the strange term in our model. Namely, we show that under our assumptions, the homogenized problem reads as

\[
(L + \gamma Y \beta - \lambda)u_0 = f \text{ in } \Omega, \quad u_0 = 0 \text{ on } \partial\Omega,
\]

where \( \beta \in L_\infty(\Omega) \) is some function determined by the shapes and distribution of the cavities \( o_k^\varepsilon \) and \( Y \) is just a fixed function:

\[
Y(x) := \sqrt{\det A(x) \text{ meas}_{n-1} \partial B_1(0)}, \quad x \in \Omega;
\]

hereinafter by \( B_r(M) \), we denote a ball in \( \mathbb{R}^n \) of a radius \( r \) centered at a point \( M \), while \( \text{meas}_{n-1} \) denotes the \( (n - 1) \)-dimensional measure on surfaces. We observe that in view of ellipticity condition (2.3), the matrix \( A(x) \) is symmetric, positive, and bounded uniformly in \( x \), and this is why the function \( Y \) is well-defined. The assumed smoothness of the functions \( A_{ij} \) implies that \( Y \in W^1_\infty(\Omega) \).
2.2 | Main assumptions

In this subsection, we formulate our main assumptions. We begin with a geometric assumption on the cavities \( \omega_k^\varepsilon \).

In the vicinity of the boundaries \( \partial \omega_k^\varepsilon \), we define a local variable \( \tau \) being the distance measured along the normal vector \( v \) to \( \partial \omega_k^\varepsilon \).

**Assumption A.1.** The points \( M_k^\varepsilon \) and the domains \( \omega_k^\varepsilon \) obey the conditions:

\[
B_{R_1}(y_k^\varepsilon) \subseteq \omega_k^\varepsilon \subset B_{R_2}(0), \quad B_{R_2}(M_k^\varepsilon) \cap B_{R_2}(M_j^\varepsilon) = \emptyset, \quad \text{dist}(M_k^\varepsilon, \partial \Omega) \geq R_3 \varepsilon, \quad k \neq j, \quad k, j \in \mathbb{M}^\varepsilon, \tag{2.8}
\]

where \( y_k^\varepsilon \) are some points and \( R_1 < R_2 < R_3 \) are some fixed constants independent of \( \varepsilon, \eta, k, \) and \( j \). The sets \( B_{R_2}(0) \setminus \omega_k^\varepsilon \) are connected. For each \( k \in \mathbb{M}^\varepsilon \), there exist local variables \( s \) on \( \partial \omega_k^\varepsilon \) such that the variables \((\tau, s)\) are well-defined at least on \( \{ x \in \mathbb{R}^n : \text{dist}(x, \partial \omega_k^\varepsilon) \leq \tau_0 \} \subseteq B_{R_2}(0), \) where \( \tau_0 \) is a fixed constant independent of \( k \in \mathbb{M}^\varepsilon \) and \( \varepsilon \). The Jacobians corresponding to passing from variables \( x \) to \((\tau, s)\) are separated from zero and bounded from above uniformly in \( \varepsilon, k \in \mathbb{M}^\varepsilon \), and \( x \) as \( \text{dist}(x, \partial \omega_k^\varepsilon) \leq \tau_0 \). The derivatives of \( x \) with respect to \((\tau, s)\) and of \((\tau, s)\) with respect to \( x \) up to the second order are bounded uniformly in \( \varepsilon, k \in \mathbb{M}^\varepsilon \), and \( x \) as \( \text{dist}(x, \partial \omega_k^\varepsilon) \leq \tau_0 \).

The first relation in (2.8) means that all domains \( \omega_k^\varepsilon \) are approximately of the same sizes: We can inscribe a fixed ball of the radius \( R_1 \) inside each domain, which in its turn is contained in a fixed ball \( B_{R_2}(0) \). The second condition in (2.8) guarantees that each two neighboring cavities do not intersect, and there is a minimal distance \( 2R_3 \varepsilon \) between each two neighboring points \( M_k^\varepsilon \), while the third condition says that the cavities are not too close to the boundary of \( \Omega \); see Figure 1. It should be said that we suppose only the existence of the minimal distance between the points \( M_k^\varepsilon \) and impose no conditions for the maximal distance. This allows us to consider a large class of possible distributions of points \( M_k^\varepsilon \). For example, the mutual distances between neighboring points \( M_k^\varepsilon \) can be proportional to \( \varepsilon \) and then the perforation becomes finer as \( \varepsilon \) goes to zero: Each cavity and the distances between them become smaller, while the total number of cavities grow. At the same time, the situation of finitely many cavities located near a fixed set of points is also allowed; in such case, the mutual distance between the points \( M_k^\varepsilon \) is finite. It is also possible to consider various intermediate situations, in which at some parts of the domain the cavities are located very densely, while in other parts, their distribution is sparse.

The connectedness of the domains \( B_{R_2}(0) \setminus \omega_k^\varepsilon \) is also a natural condition meaning that the perforation produces no new isolated connected components in the domain \( \Omega \). The rest of Assumption A.1 postulates a regularity of the boundaries \( \partial \omega_k^\varepsilon \) uniformly in \( k \) and \( \varepsilon \). We stress that the domains \( \omega_k^\varepsilon \) can have different shapes, and moreover, they are allowed to

![FIGURE 1](image-url) Domains \( \omega_k^\varepsilon \) (indicated by solid black lines) and points \( M_k^\varepsilon \) (indicated by gray color). Dotted lines show (rescaled) the balls \( B_{R_1}(M_k^\varepsilon + \varepsilon y_k^\varepsilon) \) and \( B_{R_2}(M_k^\varepsilon) \) from the first condition in (2.8) and the balls \( B_{R_3}(M_k^\varepsilon) \) from the second condition.
depend on $\epsilon$. These domains are not necessarily simply connected, and each such domain can consist of several connected components. The domains $\omega_{k,\epsilon}$ also do not necessarily contain zero, and this means that while defining domain $\omega_k$ in (2.1), the domain $\omega_{k,\epsilon}$ can be rescaled with respect to some external point not necessarily contained in $\omega_{k,\epsilon}$.

Our second assumptions concern the function $a^\epsilon$. We first suppose that

$$\Re(a^\epsilon(x, u_1) - a^\epsilon(x, u_2))(u_1 - u_2) \geq -\mu_0(\epsilon)|u_1 - u_2|^2,$$

$$|a^\epsilon(x, u_1) - a^\epsilon(x, u_2)| \leq c_2|u_1 - u_2|, \quad a^\epsilon(x, 0) = 0,$$  \quad (2.9)

where $c_2$ is some constant independent of $\epsilon$, $x \in \partial \omega_R$, and $u_1, u_2 \in \mathbb{C}$ and $\mu_0(\epsilon)$ is some nonnegative function such that

$$\epsilon \eta(\epsilon) \kappa(\epsilon) \mu_0(\epsilon) \to +0, \quad \epsilon \to +0.$$ \quad (2.10)

These conditions describe the class of admissible nonlinearities, and in view of the technique we use, they guarantee the unique solvability of problem (2.4). In particular, these conditions mean that the nonlinearity in $u$ of the function $a^\epsilon$ is rather weak and the growth in $u$ is at most linear. This is postulated by the inequalities in (2.9). The identity $a^\epsilon(x, 0) = 0$ is needed to ensure the uniqueness of the solution to problem (2.4), and this identity is immediately satisfied once $a^\epsilon$ is linear in $u$.

A more important assumption for $a^\epsilon$ is as follows; it is needed to ensure that the homogenized problem is indeed (2.6).

**Assumption A.2.** The set $M_{R}^\epsilon$ is partitioned into two disjoint subsets $M_{R,1}^\epsilon$ and $M_{R,2}^\epsilon$ obeying the following conditions:

$$\Re(a^\epsilon(x, u)\bar{u} \geq \mu_1(\epsilon)|u|^2, \quad x \in \partial \omega_k^\epsilon, \quad u \in \mathbb{C}, \quad k \in M_{R,1}^\epsilon, \quad (2.11)$$

$$a^\epsilon(x, u) = \epsilon^{-1} \eta^{-1}(\epsilon)b_k^\epsilon ((x - M_k^\epsilon) \epsilon^{-1} \eta^{-1}(\epsilon) u + \tilde{a}_k^\epsilon(x, u), \quad x \in \partial \omega_k^\epsilon, \quad u \in \mathbb{C}, \quad k \in M_{R,2}^\epsilon, \quad (2.12)$$

where $\mu_1 = \mu_1(\epsilon)$ is a fixed function independent of $k \in M_{R,1}^\epsilon$, the functions $b_k^\epsilon(\xi, \epsilon), \xi \in \partial \omega_k^\epsilon$, belong to $C^1(\partial \omega_k^\epsilon)$, and $\tilde{a}_k^\epsilon(x, u), (x, u) \in \partial \omega_k^\epsilon \times \mathbb{C}$, are complex-valued and measurable in $x$ and $u$ for each $\epsilon$ and the relations

$$\epsilon \eta(\epsilon) \kappa^{-1}(\epsilon) \mu_1(\epsilon) \to +\infty, \quad \epsilon \to +0, \quad (2.13)$$

$$\|b_k^\epsilon(\cdot, \epsilon)\|_{C^1(\partial \omega_k^\epsilon)} \leq c_4, \quad \Re b_k^\epsilon(\xi, \epsilon) \geq c_5, \quad \xi \in \partial \omega_k^\epsilon, \quad (2.14)$$

hold with some positive constants $c_3, c_4$ independent of $\epsilon$, $\xi$ and $k$ and $\mu_2$ is some function independent of $k$.

This condition says that we deal with two main types of the nonlinear Robin condition. The first is imposed for $k \in M_{R,1}^\epsilon$, and here, the nonlinear term $a^\epsilon$ is sign definite and large in the sense of inequality (2.11) and convergence (2.13). These conditions ensure that the corresponding cavities, for $k \in M_{R,1}^\epsilon$, behave similar to ones with the Dirichlet condition for $k \in M_{D}^\epsilon$. Namely, the traces of the function $u_\epsilon$ on $\partial \omega_k^\epsilon$ for $k \in M_{R,1}^\epsilon$ tend to zero as $\epsilon \to +0$.

The second type of the Robin condition is imposed for $k \in M_{R,2}^\epsilon$, and here, the function $a^\epsilon$ is linear in the leading term as it is described by (2.12) and (2.14). The coefficient $\epsilon^{-1} \eta^{-1}$ at the functions $b_k^\epsilon$ indicates the growth of the linear term in the Robin condition. We stress that one of the sets $M_{R,1}^\epsilon$ and $M_{R,2}^\epsilon$ or even both of them can be empty.

All cavities contribute to the function $\beta$ in the strange term in (2.6). The contribution of each cavity is made via certain constants, which are related with the following boundary value problems:

$$\text{div}_x A_k^\epsilon \nabla_x X_k^\epsilon = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \overline{\omega_k^\epsilon}, \quad k \in M_{R}^\epsilon, \quad A_k^\epsilon := \Lambda(M_k^\epsilon), \quad (2.15)$$

$$X_k^\epsilon = 0 \quad \text{on} \quad \partial \omega_k^\epsilon, \quad k \in M_{D}^\epsilon \cup M_{R,1}^\epsilon, \quad (2.16)$$

$$\nu_x \cdot A_k^\epsilon \nabla_x X_k^\epsilon + b_k^\epsilon(\xi, \epsilon)X_k^\epsilon = 0 \quad \text{on} \quad \partial \omega_k^\epsilon, \quad k \in M_{R,2}^\epsilon, \quad (2.17)$$
$$X_{k,\varepsilon}(\xi) = \begin{cases} 
+K_{k,\varepsilon}|A_{k,\varepsilon}^{-2}\xi|^{-n+2} + O\left(|\xi|^{-n+1}\right), & \xi \to \infty \text{ as } n \geq 3, \\
\ln|A_{k,\varepsilon}^{-2}\xi| + K_{k,\varepsilon} + O\left(|\xi|^{-1}\right), & \xi \to \infty \text{ as } n = 2, 
\end{cases} \quad (2.18)$$

where $K_{k,\varepsilon}$ are some constants and $\nu_\xi$ is the unit normal to $\partial\omega_{k,\varepsilon}$ directed inside $\omega_{k,\varepsilon}$. We shall show in Lemma 4.1 that these problems are uniquely solvable and have classical solutions belonging to $C^2(\mathbb{R}^d \setminus \omega_{k,\varepsilon}) \cap C^\infty(\mathbb{R}^d \setminus \omega_{k,\varepsilon})$. As $n \geq 3$, the constants $K_{k,\varepsilon}$ are certain capacities of the cavities $\omega_{k,\varepsilon}$. Indeed, the change of the variables $\xi = A_{k,\varepsilon}^{-2}\xi$ transforms equation (2.15) to the Laplace equation, and it also transforms appropriately the domain $\omega_{k,\varepsilon}$. In case of boundary condition (2.16), we then see that the constant $(n-2)K_{k,\varepsilon}\operatorname{meas}_{n-1}\partial B_1(0)$ is exactly the harmonic capacity of the aforementioned image of the domain $\omega_{k,\varepsilon}$. In case of boundary condition (2.17), the constants $K_{k,\varepsilon}$ have a similar meaning, but now, the quadratic form in the standard definition of the capacity involves an additional boundary integral over $\partial\omega_{k,\varepsilon}$.

We introduce an auxiliary function:

$$\beta^\varepsilon(x) := \begin{cases} \frac{(2-n)K_{k,\varepsilon}}{R_1^2\operatorname{meas}_n B_1(0)} & \text{on } B_{R_1}(M_k^\varepsilon), \ k \in \mathbb{M}^\varepsilon, \text{ as } n \geq 3, \\
\frac{1}{R_1^2\operatorname{meas}_n B_1(0)} & \text{on } B_{R_1}(M_k^\varepsilon), \ k \in \mathbb{M}^\varepsilon, \text{ as } n = 2, \\
0 & \text{on } \Omega \setminus \bigcup_{k \in \mathbb{M}^\varepsilon} B_{R_1}(M_k^\varepsilon), 
\end{cases} \quad (2.19)$$

where $\operatorname{meas}_n$ stands for the Lebesgue measure in $\mathbb{R}^n$. It will be shown, see Lemma 4.4, that the constants $K_{k,\varepsilon}$ for $n \geq 3$ are bounded uniformly in $k$ and $\varepsilon$. Then it follows from the above definition that the family of functions $\beta^\varepsilon$ belongs to $L_\infty(\Omega)$ and is bounded uniformly in $\varepsilon$ in this space. Our third assumption says that the function $\beta^\varepsilon$ converges to some limit $\beta$ as $\varepsilon \to +0$ in an appropriate space of multipliers; this is how the strange term in (2.6) appears and how the cavities contribute to this term. The mentioned space of multipliers is denoted by $\mathfrak{W}$, and this is the space of the functions $F$ defined on $\Omega$ such that for each $u \in \dot{W}^1_2(\Omega) \cap W^2_2(\Omega)$, the function $Fu$ is a continuous antilinear functional on $\dot{W}^1_2(\Omega)$; here, $\dot{W}^1_2(\Omega)$ is the space of the functions from $W^1_2(\Omega)$ with the zero trace on $\partial \Omega$. The norm in $\mathfrak{W}$ is introduced as

$$\|F\|_{\mathfrak{W}} = \sup_{u \in \dot{W}^1_2(\Omega) \setminus W^2_2(\Omega)} \frac{|\langle Fu, v \rangle|}{\|u\|_{\dot{W}^1_2(\Omega)} \|v\|_{W^2_2(\Omega)}}, \quad (2.20)$$

where $\langle Fu, v \rangle$ stands for the action of the functional $Fu$ on a function $v$. The space $L_\infty(\Omega)$ is a subset of $\mathfrak{W}$ due to the identity $\langle Fu, v \rangle = \langle Fu, v \rangle_{L_\infty(\Omega)}$ for $F \in L_\infty(\Omega)$.

Our third assumption reads as follows.

**Assumption A.3.** The family of the functions $\beta^\varepsilon$ converges in $\mathfrak{W}$.

This assumption means that there exists a function $\beta \in \mathfrak{W}$ such that $\|\beta^\varepsilon - \beta\|_{\mathfrak{W}} \to 0$ as $\varepsilon \to +0$. The function $\beta^\varepsilon$ is determined by the distribution of the points $M_k^\varepsilon$ and also by the shapes of the cavities as $n \geq 3$. Namely, each cavity associated with the point $M_k^\varepsilon$ is replaced by a single bump of height $(2-n)K_{k,\varepsilon}/R_1^2\operatorname{meas}_n B_1(0)$ or $1/R_1^2\operatorname{meas}_n B_1(0)$; see (2.19). The location of such bumps reflects the distribution of the points $M_k^\varepsilon$. In the case $n \geq 3$, the heights of the bumps also implicitly describes the shapes of the cavities via the constants $K_{k,\varepsilon}$. In general situation, the function $\beta^\varepsilon$ is fast and non-periodically oscillating since the mutual distances $M_k^\varepsilon$ are small. Then Assumption A.3 states that this fast oscillating function $\beta^\varepsilon$ should converge in an appropriate norm, and this statement should be regarded as an appropriate homogenization of the function $\beta^\varepsilon$. Namely, Assumption A.3 in fact implies that the operator $L + \gamma \beta^\varepsilon Y$ with a fast oscillating coefficient potential $\gamma \beta^\varepsilon Y$ converges in a norm resolvent sense to $L + \gamma \beta Y$. In other words, our fine perforation can be replaced first by a fast oscillating potential $\gamma \beta^\varepsilon Y$, and then this potential is to be homogenized in an appropriate sense. The possibility of the latter step is ensured by Assumption A.3. This is why this assumption and the definition of the function $\beta^\varepsilon$ describe the class of non-periodic perforations, which we can treat. We shall discuss the convergence in the space $\mathfrak{W}$ as well as possible examples of the perforations in a separate section (Section 6). In particular, it will be shown that the limit $\beta$ is necessary an element of the space $L_\infty(\Omega)$. 
2.3 Main results

Here, we formulate our main results. They involve a special boundary corrector generated by the cavities, and this corrector is introduced as follows. We denote

\[ E_{k,ε} := \left\{ \xi : |A_{k,ε}^{-1} ξ| < R_4 η^{-1} \right\}, \quad E_k^ε := \left\{ x : |A_{k,ε}^{-1} (x - M_k^ε)| < R_4 ε \right\}, \]

where \( R_4 > 0 \) is some fixed positive constant independent of \( ε, k, \) and \( η \) such that

\[ \alpha_k^ε \subseteq B_{r_4 R_k^ε}(M_k^ε) \subseteq E_k^ε \subset B_{r_4 R_k^ε}(M_k^ε). \]

Since convergence (2.5) yields that \( η(ε) \to 0 \) as \( ε \to +0, \) in view of Assumption A.1 and conditions (2.2) and (2.3), such constant \( R_4 \) obviously exists. We consider one more family of boundary value problem similar to (2.15), (2.16), (2.17), and (2.18):

\[ \text{div}_n A_{k,ε} \nabla Z_{k,ε} = 0 \text{ in } E_{k,ε} \setminus \overline{α_k^ε}, \quad k \in M^ε, \]

\[ Z_{k,ε} = \begin{cases} 1 + K_{k,ε} R_4^{-n+2} η^{n+2} & \text{as } n \geq 3, \\ |\ln η| + \ln R_4 + K_{k,ε} & \text{as } n = 2, \end{cases} \quad \text{on } \partial E_{k,ε}, \]

with boundary conditions (2.16) and (2.17) on \( \partial α_{k,ε}. \) We shall show in Lemma 4.5 that these problems are uniquely solvable and possess classical solutions belonging to \( C^2(\overline{E_{k,ε} \setminus α_{k,ε}}) \cap C^0(\overline{E_{k,ε} \setminus α_{k,ε}}). \) The aforementioned corrector is introduced as follows:

\[ \Xi_{k,ε}(x) := \begin{cases} Z_{k,ε} ((x - M_k^ε) ε^{-1} η^{-1}) & \text{in } E_k^ε \setminus α_k^ε, \quad k \in M^ε, \quad n \geq 3, \\ \frac{1 + K_{k,ε} R_4^{-n+2} η^{n-2}}{|\ln η| + \ln R_4 + K_{k,ε}} & \text{in } E_k^ε \setminus α_k^ε, \quad k \in M^ε, \quad n = 2, \\ \frac{1}{n} & \text{in } Ω \setminus \bigcup_{k \in M^ε} E_k^ε. \end{cases} \]

Our first main result states the convergence of the solution of perturbed problem to that of the limiting one in \( L^2(Ω^ε) \) uniformly in \( L^2(Ω) \)-norm of the right hand side; the employing of the corrector \( Ξ_{k,ε} \) allows us to state a similar convergence in \( W^1_2(Ω^ε) \).

**Theorem 2.1.** Let Assumption A.1 and (2.5) be satisfied. In the case \( M^ε_ R \neq \emptyset, \) suppose also that Assumption A.2 holds true, and if \( γ \neq 0, \) let Assumption A.3 holds as well. Then there exists a fixed \( λ_0 \in \mathbb{R} \) independent of \( ε \) such that as \( Re λ \leq λ_0, \) problems (2.4) and (2.6) are uniquely solvable for each \( f \in L^2(Ω), \) and the solutions satisfy the estimates:

\[ \| u_{ε} - u_0 \Xi_{ε} \|_{W^1_2(Ω^ε)} \leq C \left( ε + |η^{n-2} ε^{-2} λ_{-1} - γ| + γ \| β^ε - β \|_{\mathcal{Y}} + (ε η λ_{1})^{-\frac{1}{2}} \right) \| f \|_{L^2(Ω)}, \]

\[ \| u_{ε} - u_0 \|_{L^2(Ω^ε)} \leq C \left( ε + |η^{n-2} ε^{-2} λ_{-1} - γ| + γ \| β^ε - β \|_{\mathcal{Y}} + (ε η λ_{1})^{-\frac{1}{2}} \right) \| f \|_{L^2(Ω)}, \]

where \( C \) is some constant independent of \( ε \) and \( f. \) If the set \( M^ε_ R \) is empty, then the terms \( (ε η λ_{1})^{-\frac{1}{2}} \) and \( ε η λ_{2} \) can be omitted in the above estimates. The terms \( ε, |η^{n-2} ε^{-2} λ_{-1} - γ|, \| β^ε - β \|_{\mathcal{Y}} \) and \( ε η λ_{2} \) in estimate (2.24) are order sharp. The terms \( |η^{n-2} ε^{-2} λ_{-1} - γ| \) and \( \| β^ε - β \|_{\mathcal{Y}} \) in (2.25) are order sharp.

In a particular case \( γ = 0, \) the assumptions on the perforation can be weakened; this case is treated in the following theorem.

**Theorem 2.2.** Let condition (2.5) hold with \( γ = 0 \) and Assumption A.1 be satisfied. Then there exists a fixed \( λ_0 \in \mathbb{R} \) independent of \( ε \) such that as \( Re λ \leq λ_0, \) problems (2.4) and (2.6) with \( β = 0 \) are uniquely solvable for each \( f \in L^2(Ω), \) and the solutions satisfy the estimates.
\[ \|u_\varepsilon - u_0\|_{W_\varepsilon^2(\Omega')} \leq C \left( \varepsilon + \varepsilon^{-1} \eta^{-1} x^{-\frac{1}{2}} + \varepsilon^{-1} \eta \right) \|f\|_{L_2(\Omega)}, \quad (2.26) \]

and if, in addition, \( A_j \in W^1_\infty(\Omega), \) then

\[ \|u_\varepsilon - u_0\|_{L_2(\Omega)} \leq C \left( \varepsilon^2 + \varepsilon^{-2} \eta^{-2} x^{-1} + \varepsilon \eta \right) \|f\|_{L_2(\Omega)}, \quad (2.27) \]

where \( C \) are some constants independent of \( \varepsilon \) and \( f \). The term \( \eta^{-1} \varepsilon^{-1} x^{-\frac{1}{2}} \) in (2.26) is order sharp.

Let us discuss briefly the problem and main result. There are several main features of our problem. The first is that we consider a general perforation of a rather arbitrary non-periodic structure. Assumption A.1 is very natural and rather weak, and as it has been discussed above, it allows a wide class of various non-periodic perforations.

The second feature is that the boundaries of the cavities can be subject either to the Dirichlet condition or to the nonlinear Robin condition; both types of these conditions can be simultaneously present on the boundaries of the cavities. The structure of the Robin condition is described by Assumption A.2, and this is the only serious restriction. As we have already said, conditions (2.9) and (2.10) are needed only to ensure the unique solvability of problem (2.4), and they hold immediately once we deal with the classical linear Robin condition, that is, as \( a^*(\xi, u) = a_c(\xi)u \) with an appropriate function \( a_c(\xi) \). The third feature of our model is that we consider a general second linear elliptic equation and the differential expression \( \mathcal{L} \) is not supposed to be formally symmetric. The coefficients \( A_j \) and \( A_0 \) are allowed to be complex-valued.

Our main theorem states that under the above discussed conditions, the homogenized problem is (2.6), and the convergence in \( W^1_2(\Omega') \) and \( L_2(\Omega') \) holds uniformly in the right hand side \( f \); the estimates for the convergence rates are our main results. In the case when the Robin condition is present and is linear, these are operator estimates describing the norm resolvent convergence of the perturbed operator to the homogenized one. Similar operator estimates were already established in the case of only Dirichlet condition on the boundaries of the cavities, when the solution vanishes in the limit [21], and also in the case of the Neumann condition on the boundaries of the cavities [22], when the cavities disappear under the homogenization. And in the present paper, we show that the operator estimates are also present in a much more nontrivial situation, when a strange term appears. This name is usually used for additional potential appearing under the homogenization in the equation, and this is exactly the potential \( \gamma \beta Y \) in problem (2.6).

Estimate (2.24) says that the solution \( u_\varepsilon \) to problem (2.4) can be approximated by \( u_0 \Xi_\varepsilon \) in \( W^1_2(\Omega') \), and the estimate for the convergence rate is provided. The corrector \( \Xi_\varepsilon \) can be omitted, and then we have a similar result but only in \( L_2(\Omega') \) with the same convergence rate. The terms involving \( \mu_1 \) and \( \mu_2 \) are generated only due to the presence of the nonlinear Robin condition. If the set \( \mathcal{N}_{\mu'} \) is empty, they can be removed from the estimates.

It is also shown that all terms except for \( (\varepsilon \eta \mu_1)^{-\frac{1}{2}} \) are order sharp in (2.24). In particular, this implies that the term \( \|\beta^* - \beta\|_{W_\varepsilon^1(\Omega')} \) cannot be omitted, and hence, the same concerns Assumption A.3. In estimate (2.25), two terms in the convergence rate are also order sharp. It is unclear to us whether other terms in (2.24) and (2.25) are also order sharp or the estimate could be improved by using some additional techniques. This question remains open.

As \( \gamma = 0 \), it is possible to omit Assumptions A.2 and A.3 and to prove similar results only under Assumption A.1, see estimates (2.26) and (2.27). We stress that in (2.26), the corrector is absent in comparison with (2.24), but the price we pay for this and for omitting additional assumptions is a worse convergence rate. However, estimating then in \( L_2(\Omega') \)-norm, the order of the convergence rate can be improved twice; see (2.27). The term \( \varepsilon^{-1} \eta^{-1} x^{-\frac{1}{2}} \) is shown to be order sharp in (2.26). The sharpness of the other terms in (2.26) and of all terms in (2.27) remains an open question.

We observe that the sharpness of the term \( \varepsilon^{-1} \eta^{-1} x^{-\frac{1}{2}} \) does not contradict the sharpness of a similar term in (2.24) since in (2.26), we have omitted the corrector. We also stress that if \( \gamma \neq 0 \), then by omitting the corrector in (2.24), we destroy the convergence in \( W^1_2(\Omega') \); namely, it turns out that \( \|u_\varepsilon - u_0\|_{W^1_2(\Omega')} \) is just of order \( O(1) \) once \( \gamma \neq 0 \).

3 | AUXILIARY LEMMATA

In this section, we provide a series of auxiliary lemmata, which will be employed then in the proofs of Theorems 2.1 and 2.2.
Lemma 3.1. Under Assumption A.1 for all \( k \in \mathbb{M}^e \) and all \( u \in W^1_2(B_{\epsilon R_k}(M_k^e)\setminus \omega_k^e) \), the estimates

\[
||u||^2_{L^2(\partial \omega_k^e)} \leq C \left( \epsilon \eta x ||\nabla u||^2_{L^2(B_{\epsilon R_k}^e(M_k^e)\setminus \omega_k^e)} + \epsilon^{-1} \eta^{-1} ||u||^2_{L^2(B_{\epsilon R_k}^e(\partial \omega_k^e))} \right),
\]

(3.1)

\[
||u||^2_{L^2(B_{\epsilon R_k}^e(M_k^e)\setminus \omega_k^e)} \leq C \left( \epsilon^2 \eta^2 ||\nabla u||^2_{L^2(B_{\epsilon R_k}^e(M_k^e)\setminus \omega_k^e)} + \epsilon \eta ||u||^2_{L^2(\partial \omega_k^e)} \right)
\]

(3.2)

hold, where \( C \) are constants independent of \( k, \epsilon, \eta \) and \( u \). If, in addition,

\[
\int_{B_{\epsilon R_k}^e(M_k^e)\setminus \omega_k^e} u \, dx = 0,
\]

(3.3)

then the estimate

\[
||u||^2_{L^2(B_{\epsilon R_k}^e(\partial \omega_k^e))} \leq C \epsilon \eta ||\nabla u||^2_{L^2(B_{\epsilon R_k}^e(M_k^e)\setminus \omega_k^e)}
\]

(3.4)

holds, where \( C \) is a constant independent of \( k, \epsilon, \eta \), and \( u \).

Proof. Inequality (3.1) was proved in [21, Lm. 3.6], while inequality (3.4) was established in [21, Lm. 3.5]. Given an arbitrary \( k \in \mathbb{M}^e \) and \( u \in W^1_2(B_{\epsilon R_k}^e(M_k^e)\setminus \omega_k^e) \), we denote

\[
\langle u \rangle_k := \frac{1}{\text{meas}_n B_{\epsilon R_k}^e(M_k^e)\setminus \omega_k^e} \int_{B_{\epsilon R_k}^e(M_k^e)\setminus \omega_k^e} u \, dx, \ u_k := u - \langle u \rangle_k.
\]

(3.5)

The function \( u_k \) satisfies condition (3.3), and by (3.4), we have

\[
||u_k||^2_{L^2(B_{\epsilon R_k}^e(M_k^e)\setminus \omega_k^e)} \leq C \epsilon^2 \eta^2 ||\nabla u||^2_{L^2(B_{\epsilon R_k}^e(M_k^e)\setminus \omega_k^e)};
\]

hereinafter, in the proof, we denote by \( C \) inessential constants independent of \( \epsilon, \eta, k \), and \( v \). By [21, Lm. 3.5], we also have

\[
||u_k||^2_{L^2(\partial \omega_k^e)} \leq C \epsilon \eta ||\nabla u||^2_{L^2(B_{\epsilon R_k}^e(\partial \omega_k^e))}.
\]

(3.6)

Employing the first condition in (2.8), we argue as follows:

\[
\int_{\partial \omega_k^e} |u|^2 \, ds = |\langle u \rangle_k|^2 \text{meas}_{n-1} \partial \omega_k^e + 2 \text{Re} \langle u \rangle_k \int_{\partial \omega_k^e} u_k \, ds + \int_{\partial \omega_k^e} |u_k|^2 \, ds \\
\geq (R_1 \epsilon \eta)^{n-1} |\langle u \rangle_k|^2 \text{meas}_{n-1} \partial B_1(0) - 2|\langle u \rangle_k| \int_{\partial \omega_k^e} |u_k| \, ds \\
\geq \frac{1}{2} (R_1 \epsilon \eta)^{n-1} |\langle u \rangle_k|^2 \text{meas}_{n-1} \partial B_1(0) - C(\epsilon \eta)^{n+1} \left( \int_{\partial \omega_k^e} |u_k| \, ds \right)^2 \\
\geq \frac{1}{2} (R_1 \epsilon \eta)^{n-1} |\langle u \rangle_k|^2 \text{meas}_{n-1} \partial B_1(0) - C ||u_k||^2_{L^2(\partial \omega_k^e)}.
\]

Hence, by (3.6),

\[
(\epsilon \eta)^n |\langle u \rangle_k|^2 \leq C \left( \epsilon \eta ||u||^2_{L^2(\partial \omega_k^e)} \, ds + \epsilon^2 \eta^2 ||\nabla u||^2_{L^2(B_{\epsilon R_k}^e(\partial \omega_k^e))} \right).
\]
Using this estimate and (3.6), we obtain

\[ ||u||_{L^2(B_{\eta \eta}((M_k^c) \backslash \omega_k^c))}^2 \leq 2|\langle u \rangle_k|^2 \text{meas}_k(B_{\eta \eta}((M_k^c) \backslash \omega_k^c)) + 2||u_k||_{L^2(B_{\eta \eta}((M_k^c) \backslash \omega_k^c))}^2 \]
\[ \leq C \left( \epsilon \eta ||u||_{L^2(\partial \omega_k^c)}^2 + \epsilon^2 \eta^2 ||\nabla u||_{L^2(\partial \omega_k^c)}^2 \right). \]

This proves (3.2).

We note that under convergence (2.5), we have

\[ \eta'^{-1} = \epsilon^2 \eta \times \epsilon^{-2} \eta'^{-2} \leq C \epsilon^2 \eta, \]
where C is some fixed constant independent of \( \epsilon \) and \( \eta \). Then estimate (3.1) can be rewritten as

\[ ||u||_{L^2(\partial \omega_k^c)}^2 \leq C \epsilon \eta \chi ||u||_{W^1_2(\partial \omega_k^c)}^2, \]

(3.8)

We also observe that estimates (3.1) and (3.2) imply

\[ ||u||_{L^2(B_{\eta \eta}((M_k^c) \backslash \omega_k^c))}^2 \leq C \left( \epsilon^2 \eta^2 \chi ||\nabla u||_{L^2(\partial \omega_k^c)}^2 + \eta' ||u||_{L^2(B_{\eta \eta}((M_k^c) \backslash \omega_k^c))}^2 \right) \]

(3.9)

with a constant C independent of \( k, \epsilon, \) and \( u \).

**Lemma 3.2.** Under Assumption A.1 for all \( k \in M_k^c \) and all \( u \in W^1_2(B_{\eta \eta}((M_k^c) \backslash \omega_k^c)) \) obeying the identity

\[ \int_{B_{\eta \eta}((M_k^c) \backslash \omega_k^c)} u(\chi) d\chi = 0, \]

(3.10)

the estimate

\[ ||u||_{L^2(B_{\eta \eta}((M_k^c) \backslash \omega_k^c))}^2 \leq C \epsilon^2 ||\nabla u||_{L^2(\partial \omega_k^c)}^2, \]

(3.11)

holds, where C is a constant independent of the parameters \( k, \epsilon, \eta \) and the function \( u \).

**Proof.** Given an arbitrary \( u \in W^1_2(B_{\eta \eta}((M_k^c) \backslash \omega_k^c)) \) obeying (3.10), we introduce a function \( \tilde{u}(\xi) := u(M_k^c + \epsilon \xi) \), which belongs to \( W^1_2(B_{\eta \eta}((M_k^c) \backslash \eta \omega_k, \kappa)) \) and satisfies the condition

\[ \int_{B_{\eta \eta}((M_k^c) \backslash \eta \omega_k, \kappa)} \tilde{u}(\xi) d\xi = 0. \]

By Lemma 3.4 in [21], we then have

\[ ||\tilde{u}||_{L^2(B_{\eta \eta}((M_k^c) \backslash \eta \omega_k, \kappa))}^2 \leq C ||\nabla \tilde{u}||_{L^2(B_{\eta \eta}((M_k^c) \backslash \eta \omega_k, \kappa))}^2, \]

where C is a constant independent of \( \epsilon, \eta, \) and \( u \). Returning back to the function \( u \), we arrive at the statement of the lemma. The proof is complete.

We recall that a generalized solution to problem (2.4) is a function \( u \in \tilde{W}^1_2(\Omega^c, \partial \Omega \cup \partial \Omega_D^c) \) satisfying the integral identity

\[ h(u, v) + (a^c(\cdot, u), v)_{L^2(\partial \Omega_D^c)} - \lambda(u, v)_{L^2(\Omega^c)} = (f, v)_{L^2(\Omega^c)} \]

(3.12)

for each \( v \in \tilde{W}^1_2(\Omega^c, \partial \Omega \cup \partial \Omega_D^c) \), where

\[ h(u, v) := (A \nabla u, \nabla v)_{L^2(\Omega^c)} + \sum_{j=1}^n \left( A_j \frac{\partial u}{\partial x_j}, v \right)_{L^2(\Omega^c)} + (A_0 u, v)_{L^2(\Omega^c)}. \]
A generalized solution to problem (2.6) is defined in a similar way. The next lemma ensures the unique solvability of problems (2.4) and (2.6).

Lemma 3.3. Under Assumption A.1 and also under Assumption A.2 if \( V^e_R \neq \emptyset \), there exists \( \lambda_0 \in \mathbb{R} \) independent of \( \epsilon \) such that for \( \text{Re} \lambda < \lambda_0 \), problems (2.4) and (2.6) are uniquely solvable for each \( f \in L^2(\Omega^e) \), and for their solutions, the estimates

\[
\|u_\epsilon\|_{W^1_2(\Omega^e)} \leq C(\lambda)\|f\|_{L^2(\Omega^e)},
\]

\[
\|u_0\|_{W^2_2(\Omega)} \leq C(\lambda)\|f\|_{L^2(\Omega)}
\]

hold, where \( C(\lambda) \) are some constants independent of \( \epsilon \) and \( f \).

Proof. Assumption A.2 guarantees that estimate (2.5) in [21] is satisfied, and hence, by Lemma 3.7 in [21], there exists \( \lambda_0 \in \mathbb{R} \) independent of \( \epsilon \) such that for \( \text{Re} \lambda < \lambda_0 \), problem (2.4) is solvable in \( \bar{W}^1_2(\Omega^e, \partial \Omega \cup \partial \Theta^e) \) for each \( f \in L^2(\Omega^e) \). This is why we just need to check the uniqueness of the solution.

Supposing that there are two solutions \( u_\epsilon \) and \( \tilde{u}_\epsilon \) for some \( f \), the difference \( \hat{u}_\epsilon := u_\epsilon - \tilde{u}_\epsilon \) then solves the boundary value problem

\[
(\mathcal{L} - \lambda) \hat{u}_\epsilon = 0 \quad \text{in} \quad \Omega^e, \quad \hat{u}_\epsilon = 0 \quad \text{on} \quad \partial \Omega \cup \partial \Theta^e, \quad \frac{\partial \hat{u}_\epsilon}{\partial \nu} + a^e(x, u_\epsilon) - a^e(x, \tilde{u}_\epsilon) = 0 \quad \text{on} \quad \partial \Theta^e.
\]

Writing the corresponding integral identity with \( \hat{u}_\epsilon \) as the test function, we immediately get:

\[
\mathfrak{h}(\hat{u}_\epsilon, \hat{u}_\epsilon) - \lambda \|\hat{u}_\epsilon\|^2_{L^2(\Omega^e)} + (a^e(\cdot, u_\epsilon) - a^e(\cdot, \tilde{u}_\epsilon), \hat{u}_\epsilon)_{L^2(\omega^e)} = 0.
\]

Thanks to the lower bound in the two-sided inequality in (2.9) and also to (3.1), the second term in the above identity satisfies the estimate:

\[
\text{Re}(a^e(\cdot, u_\epsilon) - a^e(\cdot, \tilde{u}_\epsilon), \hat{u}_\epsilon)_{L^2(\omega^e)} \geq -\mu_0(\epsilon)\|u_\epsilon - u_\tilde{\epsilon}\|^2_{L^2(\omega^e)}
\]

\[
\geq -C \mu_0 \sum_{k \in M^e_\omega} (\epsilon \eta\|\nabla u\|^2_{L^2(B_{R_{\epsilon}}(M^e_\omega) \setminus \Omega^e)})
\]

\[
- C \mu_0 \sum_{k \in M^e_\omega} \left( +\epsilon^{-1} \eta^{-1-1} \|u\|^2_{L^2(\Omega^e)} \right)
\]

\[
\geq C \left( \mu_0 \eta \|\nabla u\|^2_{L^2(\Omega^e)} + \mu_0 \epsilon^{-1} \eta^{-1-1} \|u\|^2_{L^2(\Omega^e)} \right).
\]

By (2.10) and (2.5), we have

\[
\mu_0 \epsilon^{-1} \eta^{-1-1} = \mu_0 \eta \kappa \cdot \kappa^{-2} \cdot \epsilon^{-2} \eta^{-2} \kappa^{-1} \rightarrow +0, \quad \epsilon \rightarrow +0.
\]

Conditions (2.2) and (2.3) and the Cauchy-Schwarz inequality imply that

\[
\text{Re}(u, u) \geq \frac{3c_1}{4} \|\nabla u\|^2_{L^2(\Omega^e)} - C\|u\|^2_{L^2(\Omega^e)}
\]

for all \( u \in W^1_2(\Omega^e) \), where \( C \) is some absolute constant independent of \( \epsilon \) and \( u \in W^1_2(\Omega^e) \). This estimate and (2.10), (3.15), (3.16), and (3.17) then yield

\[
0 = \text{Re}(\hat{u}_\epsilon, \hat{u}_\epsilon) - \text{Re}(\lambda\|\hat{u}_\epsilon\|^2_{L^2(\Omega^e)} + \text{Re}(a^e(\cdot, u_\epsilon) - a^e(\cdot, \tilde{u}_\epsilon), \hat{u}_\epsilon)_{L^2(\omega^e)}
\]

\[
\geq \frac{c_1}{2} \|\nabla \hat{u}_\epsilon\|^2_{L^2(\Omega^e)} - (\text{Re}\lambda + C)\|\hat{u}_\epsilon\|^2_{L^2(\Omega^e)}
\]

for \( \epsilon \) small enough, where \( C \) is some fixed constant independent of \( \epsilon \) and \( \hat{u}_\epsilon \). Hence, as \( \text{Re}\lambda < -C - 1 \), we necessarily have \( \hat{u}_\epsilon = 0 \), and this proves the uniqueness of solution to problem (2.4). Writing the integral identity corresponding to problem (2.4) with \( u_\epsilon \) as the test function and proceeding as above, we prove easily estimate (3.13).
Problem (2.6) can be treated as a resolvent equation for the operator generated by the differential expression $L + Y \beta$ in $L_2(\Omega)$ subject to the Dirichlet condition. Conditions (2.2) and (2.3) imply easily that such operator is $m$-sectorial, and this is why the unique solvability is just a standard fact from the theory of $m$-sectorial operators. This operator is bounded as that from $W^2_2(\Omega) \cap W^1_1(\Omega)$ into $L_2(\Omega)$, and hence, by the Banach theorem, its resolvent is a bounded operator from $L_2(\Omega)$ into $W^2_2(\Omega) \cap W^1_1(\Omega)$. This implies estimate (3.14). The proof is complete.

4 PROPERTIES OF CORRECTOR

In this section, we prove the solvability of problems (2.15), (2.16), (2.17), (2.18), (2.21), and (2.22) and study certain properties of their solutions. Throughout this section, we suppose that Assumption A.1 is satisfied.

The main point in this study is an appropriate Kelvin transform reducing problems (2.15), (2.16), (2.17), and (2.18) to ones in bounded domains. This transform is defined as

$$
\tilde{\xi} := \frac{A^{-1}_{k,\varepsilon}(\xi - y_{k,\varepsilon})}{|A^{-1}_{k,\varepsilon}(\xi - y_{k,\varepsilon})|^2},
$$

(4.1)

with the points $y_{k,\varepsilon}$ introduced in Assumption A.1. By $\tilde{\omega}_{k,\varepsilon}$, we denote the images of the domains $\omega_{k,\varepsilon}$ arising while passing to the variable $\tilde{\xi}$. It is clear that the domains $\tilde{\omega}_{k,\varepsilon}$ are unbounded, namely, $\tilde{\omega}_{k,\varepsilon} \supset \mathbb{R}^d \setminus B_{R_1}(0)$, the boundaries of these domains are smooth, and the domains $\mathbb{R}^d \setminus \tilde{\omega}_{k,\varepsilon}$ are bounded. It is also easy to see that the origin does not belong to $\tilde{\omega}_{k,\varepsilon}$, and moreover, $B_{2R_1}(0) \cap \tilde{\omega}_{k,\varepsilon} = \emptyset$.

We seek a solution to problem (2.15), (2.16), (2.17), and (2.18) as

$$
X_{k,\varepsilon}(\xi) = \frac{1}{|A^{-1}_{k,\varepsilon}(\xi - y_{k,\varepsilon})|^{n-2}} \tilde{X}_{k,\varepsilon} \left( \frac{A^{-1}_{k,\varepsilon}(\xi - y_{k,\varepsilon})}{|A^{-1}_{k,\varepsilon}(\xi - y_{k,\varepsilon})|^2} \right) + \begin{cases} 
1 & \text{as } n \geq 3, \\
\ln |A^{-1}_{k,\varepsilon}(M^e_k)(\xi - y_{k,\varepsilon})| & \text{as } n = 2.
\end{cases}
$$

(4.2)

Then for the functions $\tilde{X}_{k,\varepsilon}$, we obtain the boundary values problems:

$$
\begin{align*}
\Delta_2 \tilde{X}_{k,\varepsilon} &= 0 \text{ in } \mathbb{R}^d \setminus \tilde{\omega}_{k,\varepsilon}, \quad k \in \mathbb{M}^e, \\
\tilde{X}_{k,\varepsilon} &= \tilde{\phi}_{k,\varepsilon}(\xi) \text{ on } \partial \tilde{\omega}_{k,\varepsilon}, \quad k \in \mathbb{M}^e_D \cup \mathbb{M}^e_{R,1}, \\
\left( \frac{\partial}{\partial \tilde{\nu}} + \tilde{b}_{k,\varepsilon}(\tilde{\xi}) \right) \tilde{X}_{k,\varepsilon} &= \tilde{\phi}_{k,\varepsilon}(\xi) \text{ on } \partial \tilde{\omega}_{k,\varepsilon}, \quad k \in \mathbb{M}^e_{R,2},
\end{align*}
$$

(4.3)

where $\tilde{b}_{k,\varepsilon}$ and $\tilde{\phi}_{k,\varepsilon}$ are some complex-valued functions. These functions are the elements of the following spaces:

$$
\tilde{\phi}_{k,\varepsilon} \in C^2(\partial \tilde{\omega}_{k,\varepsilon}), \quad k \in \mathbb{M}^e_D \cup \mathbb{M}^e_{R,1}, \\
\tilde{\phi}_{k,\varepsilon}, \tilde{b}_{k,\varepsilon} \in C^1(\partial \tilde{\omega}_{k,\varepsilon}), \quad k \in \mathbb{M}^e_{R,2},
$$

and they are bounded uniformly in $k$ and $\varepsilon$ in the norms of these spaces. The functions $\tilde{b}_{k,\varepsilon}$ also satisfy the estimate

$$
\text{Re} \tilde{b}_{k,\varepsilon} \geq \tilde{c}_3,
$$

(4.4)

where $\tilde{c}_3$ is some fixed positive constant independent of $k$ and $\varepsilon$, while $\tilde{\nu}$ is the normal to $\partial \tilde{\omega}_{k,\varepsilon}$ directed inside $\tilde{\omega}_{k,\varepsilon}$.

The first lemma in this section establishes the solvability of problems (2.15), (2.16), (2.17), and (2.18).

**Lemma 4.1.** Under Assumption A.1 and conditions (2.2), (2.3), and (2.14), problems (2.15), (2.16), (2.17), and (2.18) are uniquely solvable in $C^\omega(\mathbb{R}^d \setminus \tilde{\omega}_{k,\varepsilon}) \cap W^2_2(\mathbb{R}^d \setminus \tilde{\omega}_{k,\varepsilon})$.

**Proof.** We treat problems (4.3) in the generalized sense seeking their solutions in $W^2_2(\mathbb{R}^d \setminus \tilde{\omega}_{k,\varepsilon})$. We consider homogeneous problems (4.3) with $\tilde{\phi}_{k,\varepsilon} = 0$, write the corresponding integral identities, and use inequality (4.4) for $k \in \mathbb{M}^e_{R,2}$. Then we see easily that these homogeneous problems can have only trivial solutions. Hence, problems (4.3) are uniquely solvable in $W^2_2(\mathbb{R}^d \setminus \tilde{\omega}_{k,\varepsilon})$. By standard smoothness improving estimates, we immediately conclude that these
functions are the elements of $W^2_2(\mathbb{R}^d \setminus \partial_{h_k} \omega_{k})$ and are infinitely differentiable in $\mathbb{R}^d \setminus \partial_{h_k} \omega_{k}$. Since the functions $X_{k,x}$ are infinitely differentiable in the vicinity of the origin and are represented there by their Taylor series; in particular,

$$X_{k,x}(\tilde{\xi}) = K_{k,x} + O(|\tilde{\xi}|), \quad \tilde{\xi} \to 0,$$

where $K_{k,x}$ are some constants. Recovering then functions $X_{k,x}$ by formulae (4.2), we complete the proof. □

The second lemma states a uniform boundedness of the functions $\tilde{X}_{k,x}$.

**Lemma 4.2.** For all $k \in M^r$, the functions $\tilde{X}_{k,x}$ belong to $L_\infty(\mathbb{R}^d \setminus \partial_{h_k} \omega_{k})$ and satisfy the uniform estimates:

$$\|\tilde{X}_{k,x}\|_{L_\infty(\mathbb{R}^d \setminus \partial_{h_k} \omega_{k})} \leq C,$$

where $C$ is some constant independent of $k$ and $\varepsilon$.

**Proof.** As $k \in M^r_{\partial_{h_k}} \cup M^r_{\partial_{R_k,1}}$, by the weak maximum principle [23, Ch. 8, Sect. 8.1, Thm. 8.1] applied to the real and imaginary parts of the function $X_{k,x}$ and by the uniform boundedness of the functions $\tilde{\phi}_{k,x}$ in $C(\partial \omega_{k,x})$, we immediately get the statement of the lemma for such $k$.

The case $k \in M^r_{\partial_{R_k,2}}$ requires a more detailed study. We first state that for all $u \in W^2_2(\mathbb{R}^d \setminus \partial_{h_k} \omega_{k})$ and all $k \in M^r$, the estimate holds:

$$\|u\|_{L^2(\mathbb{R}^d \setminus \partial_{h_k} \omega_{k})} \leq C \|u\|_{W^2_2(\mathbb{R}^d \setminus \partial_{h_k} \omega_{k})},$$

where $C$ is some constant independent of $u$, $\varepsilon$, and $k \in M^r$. This is a standard estimate for the $L^2$-norm of the trace of a function from $W^2_2$-norm, and we additionally state that it is uniform on $\varepsilon$ and $k$. The latter statement is implied by a similar estimate for $u \in B_{R_k}(0) \setminus \omega_{k,x}$ established in the proof of Lemma 3.5 in [21]. The mentioned estimate from [21] was established by passing to local variables $(\tau, s)$ in the vicinity of the boundaries $\partial \omega_{k,x}$, the existence of which is ensured by Assumption A.1, and then the proof follows standard lines. In this proof, also Lemma 3.2 from [21] is to be used, which provides an estimate

$$\|u\|_{L^2(\partial \omega_{k,x})} \leq C \|\nabla u\|_{L^2(\partial \omega_{k,x})},$$

with a constant $C$ independent of $\varepsilon$, $k$, and $u \in W^2_2(B_{R_k}(0) \setminus \partial_{h_k} \omega_{k,x})$ with zero trace on $\partial B_{R_k}(0)$.

We write the integral identity corresponding to problem (4.3) with $\tilde{X}_{k,x}$ as the test function and take then the real part of this identity. In view of (4.4) and (4.6) and the uniform boundedness of $\tilde{\phi}_{k,x}$, this gives the estimate

$$\|\nabla \tilde{X}_{k,x}\|_{L^2(\mathbb{R}^d \setminus \partial_{h_k} \omega_{k})}^2 + \|\tilde{X}_{k,x}\|_{L^2(\partial \omega_{k,x})}^2 \leq C;$$

(4.7)

hereinafter, till the end of the proof by $C$, we denote inessential constants independent of $k$ and $\varepsilon$.

Now, we use the technique from [24, Ch. III, Sect. 13]. We choose an arbitrary $\phi > 0$ and write the integral identity for problem (4.3) with the test function $\tilde{X}_{k,x}$, $Q_{\min,\phi} := \min \{|\tilde{X}_{k,x}|^2, \phi\}$. Taking then the real part of the obtained identity and using (4.4) and the uniform boundedness of $\tilde{\phi}_{k,x}$, after some simple arithmetical calculations, we get the following:

$$\int_{\mathbb{R}^d \setminus \partial \omega_{k,x}} \left( |\nabla \tilde{X}_{k,x}|^2 Q_{\min,\phi} + \frac{1}{2} |\nabla \tilde{X}_{k,x}|^2 \right) \, dx + \tilde{c}_3 \int_{\partial \omega_{k,x}} |\tilde{X}_{k,x}|^2 Q_{\min,\phi} \, ds \leq C \int_{\partial \omega_{k,x}} |\tilde{X}_{k,x}| Q_{\min,\phi} \, ds \leq \frac{\tilde{c}_3}{2} \int_{\partial \omega_{k,x}} |\tilde{X}_{k,x}|^2 Q_{\min,\phi} \, ds + C \int_{\partial \omega_{k,x}} Q_{\min,\phi} \, ds,$$

where the constants $C$ are independent of $\phi$, $\tilde{X}_{k,x}$, $k$, and $\varepsilon$. Passing to the limit as $\phi \to +\infty$ and using (4.7), we get

$$\int_{\mathbb{R}^d \setminus \partial \omega_{k,x}} ( |\nabla \tilde{X}_{k,x}|^2 + |\nabla \tilde{X}_{k,x}|^2 ) \, dx \leq \int_{\partial \omega_{k,x}} |\tilde{X}_{k,x}|^4 \, ds \leq C,$$
where $C$ is a constant independent of $\varepsilon$ and $k$.

We once again choose an arbitrary $\rho > 0$ and write the integral identity for problem (4.3) with the test function $X_k, Q_{max, o}, Q_{max, o} := \max\{|X_k|^2 - \rho, 0\}$, taking then the real part of the obtained identity. After simple estimates in the integrals over $\partial \tilde{\omega}_{k, \varepsilon}$ in this identity, in view of the uniform boundedness of $\phi_{k, \varepsilon}$ and (4.4), we get the following:

$$\int_{\{\xi : |X_k|^2 \geq \rho\}} \left( |\nabla \tilde{X}_k|^2 Q_{max, o} + \frac{1}{2} |\nabla \tilde{X}_k|^2 \right) dx + \frac{\varepsilon_3}{2} \int_{\partial \tilde{\omega}_{k, \varepsilon}} |\tilde{X}_k|^2 Q_{max, o} ds \leq C \int_{\partial \tilde{\omega}_{k, \varepsilon}} |\tilde{X}_k|^2 Q_{max, o} ds + C \int_{\partial \tilde{\omega}_{k, \varepsilon}} Q_{max, o} ds$$

and hence,

$$\int_{\partial \tilde{\omega}_{k, \varepsilon}} \left( C - \frac{\varepsilon_3}{2} |\tilde{X}_k|^2 \right) Q_{max, o} ds \geq \frac{1}{2} \int_{\{\xi : |X_k|^2 \geq \rho\}} |\nabla \tilde{X}_k|^2 dx.$$

Since the first integral in the left hand side of the above inequality is non-negative and $|X_k|^2 \geq \rho$ on $\text{supp}Q_{max, o}$, we conclude that

$$\left( C - \frac{\varepsilon_3}{2} \rho \right) \int_{\partial \tilde{\omega}_{k, \varepsilon} \cap \{\xi : |X_k|^2 \geq \rho\}} Q_{max, o} ds \geq \frac{1}{2} \int_{\{\xi : |X_k|^2 \geq \rho\}} |\nabla \tilde{X}_k|^2 dx.$$

As $\rho > \frac{2C \varepsilon_3}{\varepsilon_5}$, the above inequality is possible only if $\text{meas}_n \{\xi : |X_k|^2 \geq \rho\} = 0$. Hence, the function $\tilde{X}_{k, \varepsilon}$ belongs to $L_{\infty}(\mathbb{R}^d \setminus \tilde{\omega}_{k, \varepsilon})$ and satisfies (4.5). The proof is complete. \hfill \Box

The third lemma states the uniform boundedness of the derivatives of the functions $\tilde{X}_{k, \varepsilon}$.

**Lemma 4.3.** The functions $\tilde{X}_{k, \varepsilon}$ belong to $W^2_{2n+1}(\mathbb{R}^d \setminus \tilde{\omega}_{k, \varepsilon})$ and satisfy the estimates:

$$\|\tilde{X}_{k, \varepsilon}\|_{W^2_{2n+1}(\mathbb{R}^d \setminus \tilde{\omega}_{k, \varepsilon})} \leq C,$$

where $C$ is a constant independent of $\varepsilon$ and $k$.

**Proof.** For $k \in \mathcal{M}^e_{\mathbb{R}} \cup \mathcal{M}^e_{\mathbb{R}, 1}$ by [24, Ch. III, Sect. 15, Thm. 15.1], we conclude that $\tilde{X}_{k, \varepsilon} \in W^2_{2}(\mathbb{R}^d \setminus \tilde{\omega}_{k, \varepsilon})$. Then we observe that in view of definition (4.1) of the Kelvin transform and Assumption A.1, the boundaries of the domains $\tilde{\omega}_{k, \varepsilon}$ have the same regularity as described in this assumption. This allows us to reproduce the proof of the a priori estimate from [25, Ch. 15, Thm. 15] controlling at the same time the dependence of the constants on the boundaries; this is done while making the standard unity partition. Then, in view of Lemma 4.2 and the uniform boundedness of $\phi_{k, \varepsilon}$, we have the following:

$$\|\tilde{X}_{k, \varepsilon}\|_{W^2_{2n+1}(\mathbb{R}^d \setminus \tilde{\omega}_{k, \varepsilon})} \leq C \left( \|\phi_{k, \varepsilon}\|_{C^\infty(\tilde{\omega}_{k, \varepsilon})} + \|\tilde{X}_{k, \varepsilon}\|_{L_{2n+1}(\mathbb{R}^d \setminus \tilde{\omega}_{k, \varepsilon})} \right) \leq C.$$

For $k \in \mathcal{M}^e_{R, \mathbb{R}}$, the above a priori estimate also holds true; one just should use the norm $\|\phi_{k, \varepsilon}\|_{C^\infty(\tilde{\omega}_{k, \varepsilon})}$. This is why, to complete the proof, we need to show that $\tilde{X}_{k, \varepsilon}$ is an element of $W^2_{2n+1}(\mathbb{R}^d \setminus \tilde{\omega}_{k, \varepsilon})$. This can be done by using an approximation technique from the proof of Theorem 8.3.4 in [23, Ch. 8, Sect. 8.11]. Namely, the domains $\tilde{\omega}_{k, \varepsilon}$ are to be approximated by a sequence of domains $\tilde{\omega}_{k, \varepsilon, m}$ with $C^3$-boundaries; we can simply assume that the boundaries are described by the equations $\tilde{r} = \alpha_m(\tilde{s})$, where $\tilde{r}$ is the distance along the normal vector $\tilde{v}$ to $\tilde{\omega}_{k, \varepsilon}$, the symbol $\tilde{s}$ denotes local variables on $\tilde{\omega}_{k, \varepsilon}$, and $\alpha_m$ is a sequence of some functions converging to zero in $C^2(\tilde{\omega}_{k, \varepsilon})$ as $m \to \infty$. The functions $\tilde{b}_{k, \varepsilon}$ then are also extended to the surfaces $\tilde{r} = \alpha_m(\tilde{s})$ just by assuming that they are independent of $\tilde{r}$, and on each such surface, the function $\tilde{b}_{k, \varepsilon}$ is approximated by a $C^2$-function $\tilde{b}_{k, \varepsilon, m}$, which, in the sense of the above translation along $\tilde{r}$, converges to $\tilde{b}_{k, \varepsilon}$ in $C^1$-norm as $m \to \infty$. The functions $\tilde{\phi}_{k, \varepsilon}$ are also approximated in the same way by a sequence of $C^2$-functions $\tilde{\phi}_{k, \varepsilon, m}$ converging to $\tilde{\phi}_{k, \varepsilon}$ in $C^1(\tilde{\omega}_{k, \varepsilon})$. Then we consider problems similar to (4.3)
functions are elements of the space $W^2_{2n+1}(\mathbb{R}^d \setminus \omega_{k,\varepsilon})$, and they also satisfy uniform bounds

$$\|X_{k,\varepsilon,m}\|_{W^2_{2n+1}(\mathbb{R}^d \setminus \omega_{k,\varepsilon})} \leq C$$

with some constant $C$ independent of $m$. Then, as in the proof of Theorem 8.34 in [23, Ch. 8, Sect. 8.11], we easily show that the sequence $X_{k,\varepsilon,m}$ contains a subsequence weakly converging in $W^2_{2n+1}(\mathbb{R}^d \setminus \omega_{k,\varepsilon})$ and its weak limit coincides with $X_{k,\varepsilon}$. The proof is complete.

The fourth lemma describes the behavior of the functions $X_{k,\varepsilon}$ at infinity.

**Lemma 4.4.** There exists a fixed positive constant $C_0 > 0$ independent of $\varepsilon$ and $k \in \mathbb{M}^r$ such that for $|\xi| \geq C_0$, the functions $X_{k,\varepsilon}$ satisfy the representations:

$$X_{k,\varepsilon}(\xi) = 1 + \frac{K_{k,\varepsilon}}{|A_{k,\varepsilon}^{-1} \xi|^n} + \sum_{j=1}^{n} \frac{K^{(j)}_{k,\varepsilon}}{|A_{k,\varepsilon}^{-1} \xi|^j} X_{k,\varepsilon}(\xi), \quad \xi \to \infty \text{ as } n \geq 3,$$

and

$$X_{k,\varepsilon}(\xi) = \ln |A_{k,\varepsilon}^{-1} \xi| + K_{k,\varepsilon} + \sum_{j=1}^{2} \frac{K^{(j)}_{k,\varepsilon}}{|A_{k,\varepsilon}^{-1} \xi|^j} X_{k,\varepsilon}(\xi), \quad \xi \to \infty \text{ as } n = 2,$$

where $K^{(j)}_{k,\varepsilon}$ are some constants, $(A_{k,\varepsilon}^{-1} \xi)$ is the $j$th component of the vector $A_{k,\varepsilon}^{-1} \xi$, while $X_{k,\varepsilon}$ are some infinitely differentiable functions such that

$$\left| \frac{\partial^\theta X_{k,\varepsilon}(\xi)}{\partial \xi^\theta} \right| \leq C_1 |\xi|^{n-|\theta|}, \quad |\xi| \geq C_0,$$

where $C_1$ is some constant independent of $k, \varepsilon, \xi$ and $\theta \in \mathbb{Z}^n_+$ is a multi-index, $|\theta| \leq 3$. The estimates hold:

$$|K_{k,\varepsilon}| \leq C, \quad |K^{(j)}_{k,\varepsilon}| \leq C, \quad j = 1, \ldots, n,$$

$$|X_{k,\varepsilon}(\xi)| \leq C, \quad n \geq 3, \quad |X_{k,\varepsilon}(\xi) - \ln |A_{k,\varepsilon}^{-1} \xi|| \leq C, \quad n = 2,$$

and

$$|\nabla_\xi X_{k,\varepsilon}(\xi)| \leq C,$$

where $C$ are some constants independent of $k, \varepsilon, \xi$.

**Proof.** By Lemma 4.3 and the definition of the Kelvin transform in (4.1), the functions $X_{k,\varepsilon}$ are the elements of the space $W^2_{2n+1}(B_{R_1}(0) \setminus \omega_{k,\varepsilon})$ and are bounded in this space uniformly in $k$ and $\varepsilon$. Using the regularity of the boundaries $\partial \omega_{k,\varepsilon}$ postulated in Assumption A.1, we continue the function $X_{k,\varepsilon}$ into $\omega_{k,\varepsilon}$ as follows:

$$X_{k,\varepsilon}(\tau, s) = X_{k,\varepsilon}(-\tau, s) \chi_1(\tau),$$

where $\chi_1 = \chi_1(\tau)$ is an infinitely differentiable function vanishing as $|\tau| > \frac{2n}{3}$ and equalling to one as $|\tau| < \frac{2n}{3}$. In the same way, we continue each derivative $\frac{\partial X_{k,\varepsilon}}{\partial \tau_j}$, $j = 1, \ldots, n$. It is clear that after such continuation, the obtained functions are elements of $W^1_{2n+1}(B_{R_1}(0))$ bounded in this space uniformly in $\varepsilon$ and $k$. Applying then Sobolev theorem [24, Ch. II, Sect. 2, Thm. 2.2], we see that $X_{k,\varepsilon}, \frac{\partial X_{k,\varepsilon}}{\partial \tau_j} \in C^1(\overline{B_{R_1}(0) \setminus \omega_{k,\varepsilon}})$, and the estimate holds:

$$\|X_{k,\varepsilon}\|_{C^1(\overline{B_{R_1}(0) \setminus \omega_{k,\varepsilon}})} \leq C,$$
where \( C \) is a constant independent of \( k \) and \( \epsilon \).

We again use the Kelvin transform introduced in the proof of Lemma 4.1, and even for \( k \in M^R_{r,2} \) we treat \( X_{k,x} \) as solutions to the Dirichlet problem with appropriate \( \Phi_{k,x} \). Estimate (4.13) allows us to say that these functions \( \Phi_{k,x} \) are bounded uniformly in \( k, \epsilon, \) and \( \xi \). The well-known estimates for the derivatives of the harmonic function, see, for instance, [26, Ch. IV, Sect. 3.2, Lm. 3], then yield

\[
\frac{\partial^2 X_{k,x}}{\partial \tilde{\xi} \theta}(\tilde{\xi}) \leq C, \quad \tilde{\xi} \in B_{(3R_j)^{-1}}(0), \quad \theta \in \mathbb{Z}_n, \quad |\theta| \leq 3, \]

where \( C \) are some constants independent of \( k, \epsilon, \) and \( \xi \). Using Lemma 4.2, writing the Taylor series for \( X_{k,x} \) at zero and returning back to the function \( X_{k,x} \) by formula (4.2), we get (4.8), (4.9), (4.10), (4.11), and (4.12). The proof is complete. \( \square \)

The fifth lemma states the unique solvability of problem (2.21), (2.22), (2.16), and (2.17) and provides estimates for the difference \( X_{k,x} - Z_{k,x} \).

**Lemma 4.5.** Problems (2.21), (2.22), (2.16), and (2.17) are uniquely solvable in \( W^2_2(E_{k,x} \setminus \omega_{k,x}) \cap C^1(\overline{E_{k,x}} \setminus \omega_{k,x}) \) and satisfy the estimates

\[
|X_{k,x} - Z_{k,x}| \leq C \eta^{-1}, \quad |\nabla X_{k,x} - \nabla Z_{k,x}| \leq C \eta \quad \text{in} \quad \overline{E_{k,x}} \setminus \omega_{k,x},
\]

(4.14)

where \( C \) is a constant independent of \( \epsilon, \eta, k, \) and \( \xi \).

**Proof.** The unique solvability in \( W^2_2(E_{k,x} \setminus \omega_{k,x}) \) is easily checked in the same way how a similar fact was established in the proof of Lemma 4.1; here, we even do not need to make the Kelvin transform since the domains \( E_{k,x} \setminus \omega_{k,x} \) are bounded. By the standard smoothness improving theorems, we also see that \( Z_{k,x} \in C^0(\overline{E_{k,x}} \setminus \omega_{k,x}) \).

We let

\[
\psi^+_{k,x}(\xi) := X_{k,x}(\xi) - \sum_{j=1}^n K^{(j)}_{k,x} R^{n_j}_4 \left( \frac{1}{A_{k,x}^{1/2} \xi} \right) - \left( 1 + K_{k,x} \right) A_{k,x}^{1/2} |\xi|^{-n+2} \quad \text{as} \quad n \geq 3,
\]

\[
\ln |A_{k,x}^{1/2} \xi| + K_{k,x} \quad \text{as} \quad n = 2,
\]

and by (4.8) and (4.9), we see that

\[
\| \psi^+_{k,x} \|_{C^0(\partial B_{k,x})} \leq C \eta^n \quad \text{(4.15)}
\]

hereinafter, in the proof by \( C \), we denote inessential constants independent \( \epsilon, k, \) and \( \eta \). By \( T^0_{k,x}(\xi) \), we denote the solution to the problem:

\[
\Delta \xi T^0_{k,x} = 0 \quad \text{in} \quad B_{R_{k,x}^{-1}}(0), \quad T^0_{k,x} = \psi^+_{k,x} \left( A_{k,x}^{1/2} \xi \right) \quad \text{on} \quad \partial B_{R_{k,x}}(0).
\]

(4.16)

This problem is uniquely solvable. We reproduce the proof of the Schauder estimate [24, Ch. III, Sect. 1, 2] for problem (4.16) covering \( B_{R_{k,x}^{-1}}(0) \) by balls of a fixed radius, and in view of (4.15), we conclude that \( T^0_{k,x} \in C^2(\overline{B_{R_{k,x}^{-1}}(0)}) \) and

\[
\| T^0_{k,x} \|_{C^2(\overline{B_{R_{k,x}^{-1}}(0)})} \leq C \eta^n.
\]

(4.17)

The functions

\[
T_{k,x}(\xi) := X_{k,x}(\xi) - Z_{k,x}(\xi) - \sum_{j=1}^n K^{(j)}_{k,x} R^{n_j}_4 \left( \frac{1}{A_{k,x}^{1/2} \xi} \right) - T^0_{k,x} \left( A_{k,x}^{1/2} \xi \right)
\]

(4.18)

solve the boundary value problems

\[
\begin{align*}
\text{div}_\xi A_{k,x} \nabla_\xi T_{k,x} &= 0 \quad \text{in} \quad E_{k,x} \setminus \omega_{k,x}, & T_{k,x} &= 0 \quad \text{on} \quad \partial E_{k,x} \\
T_{k,x} &= \psi^-_{k,x}, & k &\in M^R_{r,2} \cup M^R_{r,1}, & \nabla_\xi \cdot A_{k,x} \nabla_\xi T_{k,x} + b_k T_{k,x} &= \psi^-_{k,x}, & k &\in M^R_{r,2} \setminus \omega_{k,x} \quad \text{on} \quad \partial \omega_{k,x},
\end{align*}
\]

(4.19)
where
\[
\psi_{k,x}(\xi) := -\sum_{j=1}^{n} \frac{K_{j,x}^{(j)} n_j}{R^4_j} \left( A_{k,x}^{-1} \xi_j \right)_j - T_{k,x}^0, \quad k \in \mathcal{M}_{D}^r \cup \mathcal{M}_{R,1}^r,
\]
\[
\psi_{k,x}(\xi) := -\left( v_{2} \cdot A_{k,x} \nabla_{2} T_{k,x} + b_{k} T_{k,x} \right) \sum_{j=1}^{n} \frac{K_{j,x}^{(j)} n_j}{R^4_j} \left( A_{k,x}^{-1} \xi_j \right)_j + T_{k,x}^0, \quad k \in \mathcal{M}_{R,2}^r.
\]

It follows from (4.17) that
\[
\|\psi_{k,x}\|_{C^1(\partial\omega_{k,x})} \leq C\eta^n, \quad k \in \mathcal{M}_{D}^r \cup \mathcal{M}_{R,1}^r, \quad \|\psi_{k,x}\|_{C^1(\partial\omega_{k,x})} \leq C\eta^n, \quad k \in \mathcal{M}_{R,2}^r.
\]

Writing then integral identities associated with problem (4.19), we easily obtain:
\[
\|\nabla_{2} T_{k,x}\|_{L^2(B_{R_r}(0) \setminus \omega_{k,x})} + \|T_{k,x}\|_{L^2(\partial\omega_{k,x})} \leq C\eta^n,
\]
where the second term in the left hand side obviously vanishes for \( k \in \mathcal{M}_{D}^r \cup \mathcal{M}_{R,1}^r \). By the standard smoothness improving theorems, we then get
\[
\|T_{k,x}\|_{C^1(\overline{B_{R_r}(0) \setminus \partial B_{R_r}(0)})} \leq C\eta^n.
\] (4.20)

Let \( \chi_2 = \chi_2(\xi) \) be an infinitely differentiable cut-off function equalling to one as \( |\xi| < (R_2 + R_3)/2 \) and vanishing as \( |\xi| > R_3 \). Then the functions \( T_{k,x} \chi_2 \) solve the boundary value problems
\[
\text{div}_{2} A_{k,x} \nabla_{2} T_{k,x} \chi_2 = 2\nabla_{2} \chi_2 \cdot A_{k,x} \nabla_{2} T_{k,x} + T_{k,x} \text{div}_{2} A_{k,x} \nabla_{2} \chi_2 \quad \text{in} \ B_{R_r}(0) \setminus \omega_{k,x},
\]
\[
T_{k,x} \chi_2 = 0 \quad \text{on} \ \partial B_{R_r}(0)
\]
with boundary conditions (2.16) and (2.17). These problems can be studied following the lines of the proofs of Lemmata 4.2 and 4.3 and employing also estimate (4.20) and the inequality [21, Lm. 3.1]:
\[
\|u\|_{L^2(B_{R_r}(0) \setminus \omega_{k,x})} \leq C\|\nabla_{2} u\|_{L^2(B_{R_r}(0) \setminus \omega_{k,x})} \quad \text{for all} \ u \in W^1_{2}(B_{R_r}(0) \setminus \omega_{k,x}, \partial B_{R_r}(0))
\]
with a constant \( C \) independent of \( k, \varepsilon, \) and \( u \). As a result, we obtain the following:
\[
\|T_{k,x}\|_{C^1(\overline{B_{R_r}(0) \setminus \omega_{k,x})}) \leq C\eta^n.
\] (4.21)

We also conclude that \( T_{k,x} \in C^1(\overline{B_{R_r}(0) \setminus \omega_{k,x})} \). Since the function \( T_{k,x}(A_{k,x}^{-1}(M_{k,x}^r)\xi) \) is obviously harmonic, by the classical maximum principle for the harmonic functions and the first estimate in (4.15), we immediately obtain
\[
\|T_{k,x}\|_{C^1(\overline{B_{R_r}(0) \setminus \omega_{k,x})}) \leq C\eta^n.
\] (4.22)
For \( k \in \mathbb{M}_R \), we also have an appropriate maximum principle. Namely, the real and imaginary parts of \( \mathcal{T}_{k,e} (A^i (M'_k) \xi) \) are harmonic functions, and by the mean value theorem,

\[
\mathcal{T}_{k,e} (A^i (M'_k) \xi) = \frac{1}{\text{meas}_d B_d (\xi)} \int_{B_d (\xi)} \mathcal{T}_{k,e} (A^i (M'_k) y) dy
\]

for each \( \xi \in E_{k,e} \setminus \omega_{k,e} \) and each ball \( B_d (\xi) \) such that \( \left\{ A^i (M'_k) y : y \in B_d (\xi) \right\} \subset E_{k,e} \setminus \omega_{k,e} \). This identity implies

\[
|\mathcal{T}_{k,e} (A^i (M'_k) \xi)| \leq \max_{y \in B_d (\xi)} |\mathcal{T}_{k,e} (A^i (M'_k) y)|.
\]

If \( \xi \in E_{k,e} \setminus \omega_{k,e} \) is a point of the global maximum of \( |\mathcal{T}_{k,e}| \), then the above inequality implies that \( |\mathcal{T}_{k,e}| \) is constant in \( B_d (\xi) \). Hence, the function \( |\mathcal{T}_{k,e}| \) attains its global maximum on the boundary \( \partial \omega_{k,e} \) or on \( \partial E_{k,e} \). It follows from the boundary condition for \( \mathcal{T}_{k,e} \) that

\[
\frac{\partial |\mathcal{T}_{k,e}|^2}{\partial \nu} + \text{Re } b_{k,e} |\mathcal{T}_{k,e}|^2 = 0.
\]

If a point of the global maximum of \( \mathcal{T}_{k,e} \) is located on \( \partial \omega_{k,e} \), then \( \frac{\partial |\mathcal{T}_{k,e}|^2}{\partial \nu} \geq 0 \), and the above identity due to the positivity of \( \text{Re } b_k \), see (2.14), implies that \( \mathcal{T}_{k,e} = 0 \). Hence, the function \( |\mathcal{T}_{k,e}| \) attains its maximum on \( \partial E_{k,e} \), and this gives estimate (4.22) for \( k \in \mathbb{M}_R \).

We consider the function \( \mathcal{T}_{k,e} \) as a solution of the equation from (4.19) but on \( E_{k,e} \setminus B_{R_k} (0) \) subject to the boundary condition on \( \partial E_{k,e} \) from (4.19). Taking into consideration then (4.20) and reproducing again the proof of the Schauder estimate with covering by balls of a fixed radius, we obtain

\[
\|\mathcal{T}_{k,e}\|_{C^r (E_{k,e} \setminus B_{R_k} (0))} \leq C \eta^2.
\]

This estimate and (4.21) yield

\[
\|\mathcal{T}_{k,e}\|_{C^r (E_{k,e} \setminus \omega_{k,e})} \leq C \eta^n.
\]

Returning back then to the function \( X_{k,e} \) by formula (4.18) and using estimates (4.17) and (4.10), we arrive at (4.14). The proof is complete.

The above lemmata implies several properties of the function \( \Xi_e \). We first observe that estimates (4.8), (4.9), and (4.14) imply

\[
|\Xi_e (x) - 1| \leq C \text{ in } B_{R_k} (M'_k) \setminus \omega_k.
\]  

(4.23)

Employing (4.8) and (4.9), by straightforward calculations, we find

\[
A_{k,e} \nabla X_{k,e} \cdot \nu = (2 - n) K_{k,e} \frac{\eta^{n-2}}{R_k^n} |A^{-1} (M'_k) \xi|^{-1} + O (\eta^n) \quad \text{as } n \geq 3,
\]

\[
A_{k,e} \nabla X_{k,e} \cdot \nu = |A^{-1} (M'_k) \xi|^{-1} + O (\eta^2) \quad \text{as } n = 2,
\]

on \( \partial E_{k,e} \), where \( \nu \) is the outward normal to \( \partial E_{k,e} \) and the \( O \)-terms are uniform in \( \xi, \epsilon, k, \) and \( \eta \). Using then (2.5), (4.14), and the definition of \( \Xi_e \), we obtain

\[
A_{k,e} \nabla \Xi_e \cdot \nu = (2 - n) K_{k,e} \frac{\eta^{n-2}}{R_k^n} |A^{-1} (M'_k) (x - M'_k) |^{-1} + O (\eta^{n-1} \epsilon^{-1}) \quad \text{as } n \geq 3,
\]  

(4.24)

\[
A_{k,e} \nabla \Xi_e \cdot \nu = |\ln \eta|^{-1} |A^{-1} (M'_k) (x - M'_k) |^{-1} + O (\epsilon^{-1} \ln^{-2} \eta) \quad \text{as } n = 2.
\]  

on \( \partial E'_k \), where the \( O \)-terms are uniform in \( x, \epsilon, k, \) and \( \eta \).
Our sixth lemma provides auxiliary estimates for the corrector $\Xi$, and these estimates will play one of key roles in proving our main results.

**Lemma 4.6.** The estimates hold

$$\| (\Xi - 1) u \|_{L^2(B_{2R}(M^e) \setminus o^e_k)} \leq C \| u \|_{W^2_2(B_{2R}(M^e) \setminus o^e_k)}$$

(4.25)

for all $u \in W^1_2(B_{2R}(M^e) \setminus o^e_k)$ and

$$\| u \nabla \Xi \|_{L^2(B_{2R}(M^e) \setminus o^e_k)} \leq C \left( \eta^{n-1} - 1 + \epsilon \eta^{n-1} \right) \| u \|_{W^2_2(B_{2R}(M^e) \setminus o^e_k)},$$

(4.26)

$$\| (\Xi - 1) u \|_{L^2(B_{2R}(M^e) \setminus o^e_k)} \leq C (\epsilon^{2} + \eta) \| u \|_{W^2_2(B_{2R}(M^e) \setminus o^e_k)}$$

(4.27)

for all $u \in W^2_2(B_{2R}(M^e) \setminus o^e_k)$, where $C$ are some constants independent of the parameters $\epsilon$, $k$ and the function $u$.

**Proof.** We fix $k \in M^e$, and for a given $u \in W^2_2(B_{2R}(M^e) \setminus o^e_k)$, we denote

$$\langle u \rangle := \frac{1}{\text{meas}_n B_{2R}(M^e) \setminus o^e_k} \int_{B_{2R}(M^e) \setminus o^e_k} u(x) \, dx, \quad u^1 := u - \langle u \rangle. \quad (4.28)$$

Then, in view of the fact that $\Xi$ is identically one in $B_{2R}(M^e) \setminus E^e_k$,

$$\| (\Xi - 1) u \|_{L^2(B_{2R}(M^e) \setminus o^e_k)}^2 \leq 2 |\langle u \rangle|^2 \| (\Xi - 1) u \|_{L^2(E^e_k \setminus o^e_k)}^2 + 2 \| (\Xi - 1) u^1 \|_{L^2(B_{2R}(M^e) \setminus o^e_k)}^2. \quad (4.29)$$

Since the function $u^1$ obeys condition (3.10), by Lemma 3.2, it satisfies estimate (3.11), and by (4.23), we immediately get

$$\| (\Xi - 1) u^1 \|_{L^2(B_{2R}(M^e) \setminus o^e_k)}^2 \leq C \epsilon^{2} \| u \|_{W^2_2(B_{2R}(M^e) \setminus o^e_k)}^2; \quad (4.30)$$

hereinafter, in the proof by $C$, we denote various inessential constants independent of $\epsilon$, $k$, and $u$. Passing then to the variables $\xi = (x - M^e_k) e^{-1} \eta^{-1}$, by (4.8), (4.9), (4.10), (4.14), and (4.23) for $n \geq 3$, we find

$$\| \Xi - 1 \|_{L^2(E^e_k \setminus o^e_k)}^2 \leq C \epsilon^{n} \eta^{n} \| Z_{k,e} - 1 - K_{k,e} R^{-n+2} \eta^{-2} \|_{L^2(E^e_k \setminus o^e_k)}^2$$

$$\leq C \epsilon^{n} \eta^{2n-2} + C \epsilon^{n} \eta^{n} \int_C \left( (r^{-n+2} - R^{-n+2} \eta^{-2})^2 + r^{2n+2} \right) r^{n+1} \, dr$$

$$\leq C \epsilon^{n} \eta^{n} \left\{ \begin{array}{ll}
1 + \eta^{n-4}, & n \neq 4, \\
1 + |\ln \eta|, & n = 4.
\end{array} \right.$$  

As $n = 2$, we estimate along the same lines:

$$\| \Xi - 1 \|_{L^2(E^e_k \setminus o^e_k)}^2 \leq C \epsilon^{2} \eta^{2} \ln^{-2} \eta \| Z_{k,e} - \ln R \eta^{-1} - K_{k,e} \|_{L^2(E^e_k \setminus o^e_k)}^2$$

$$\leq C \epsilon^{2} \eta^{2} \ln^{-2} \eta + C \epsilon^{2} \eta^{2} \ln^{-2} \eta \| X_{k,e} - \ln R \eta^{-1} - K_{k,e} \|_{L^2(E^e_k \setminus o^e_k)}^2$$

$$\leq C \epsilon^{2} \eta^{2} \ln^{-2} \eta \leq C \epsilon^{2} \eta^{2} \ln^{-2} \eta \int_C \left( \ln^{2} \frac{r}{R \eta^{-1}} + r^{2} \right) r \, dr \leq C \epsilon^{2} \ln^{-2} \eta.$$
Hence, in view of convergence (2.5),

$$\|\Xi - 1\|^2_{L^2(E_k')} \leq C\varepsilon^{n+2}. \tag{4.31}$$

We also see easily that

$$|\langle u \rangle|^2 \leq C\varepsilon^{-n}\|u\|^2_{L^2(E_k')}, \tag{4.32}$$

Employing this estimate and (4.29), (4.30), (4.31), and (2.5), we obtain (4.25). In the same way, we prove easily the estimate:

$$\|\Xi - 1\|^2_{L^2(E_k')} = C\varepsilon\|u\|_{W^2_2(E_k')}. \tag{4.33}$$

We proceed to proving (4.26) and (4.27). We begin with simpler relations:

$$\|u\Xi\|^2_{L^2(E_k')} = \|u\Xi\|^2_{L^2(E_k')} \leq C(A_{k,\varepsilon} u \nabla \Xi, u \nabla \Xi)_{L^2(E_k')} \tag{4.34}$$

Then we integrate by parts employing the definition of the function \(\Xi\):

$$\left(A_{k,\varepsilon} u \nabla \Xi, u \nabla \Xi \right)_{L^2(E_k')} = \text{Re} \int_{E_k'} |u|^2 A_{k,\varepsilon} \nabla \Xi \cdot \nu \, ds + \text{Re} \int_{\partial E_k'} \Xi |u|^2 A_{k,\varepsilon} \nabla \Xi \cdot \nu \, d\omega_k$$

$$- \text{Re} \int_{E_k'} \nabla \text{div} A_{k,\varepsilon} |u|^2 \nabla \Xi \, dx$$

$$= \text{Re} \int_{E_k'} |u|^2 A_{k,\varepsilon} \nabla \Xi \cdot \nu \, ds + \text{Re} \int_{\partial E_k'} \Xi |u|^2 A_{k,\varepsilon} \nabla \Xi \cdot \nu \, d\omega_k$$

$$- \frac{1}{2} \int_{E_k'} A_{k,\varepsilon} \nabla(|\Xi|^2 - 1) \cdot \nabla |u|^2 \, dx \tag{4.35}$$

$$= \text{Re} \int_{E_k'} |u|^2 A_{k,\varepsilon} \nabla \Xi \cdot \nu \, ds + \text{Re} \int_{\partial E_k'} \Xi |u|^2 A_{k,\varepsilon} \nabla \Xi \cdot \nu \, d\omega_k$$

$$- \frac{1}{2} \int_{\partial E_k'} (|\Xi|^2 - 1)A_{k,\varepsilon} \nabla |u|^2 \cdot \nu \, ds$$

$$+ \frac{1}{2} \int_{\partial E_k'} (|\Xi|^2 - 1)\text{div} A_{k,\varepsilon} |u|^2 \, dx,$$

where \(\nu\) denotes the unit normal to \(\partial E_k'\) directed outside \(E_k'\) and also the unit normal to \(\partial \omega_k'\) directed inside \(\omega_k'\). In view of the boundary conditions for \(X_{k,\varepsilon}\) in (2.16) and (2.17) and the inequality for \(b_k\) in (2.14), we also have

$$\text{Re} \int_{\partial \omega_k'} \Xi |u|^2 A_{k,\varepsilon} \nabla \Xi \cdot \nu' \, ds = 0, \quad k \in \mathcal{M}^I_{R,1},$$

$$\text{Re} \int_{\partial \omega_k'} \Xi |u|^2 A_{k,\varepsilon} \nabla \Xi \cdot \nu' \, ds = -\varepsilon^{-1} \eta^{-1} \text{Re} b_k(\varepsilon) \|u\|^2_{L^2(\partial \omega_k')} \leq 0, \quad k \in \mathcal{M}^I_{R,2},$$
and this allows us to omit the second term in the final right hand side in (4.35). Then by relations (3.8) with \( \partial\omega^k \) replaced by \( \partial E^k \), (4.23), (4.33), (4.34), (2.5), and (4.24) and the definition of the function \( \Xi \), we get

\[
\|u\nabla\Xi\|_{L^2(B_{R_k}(M^e_k)\setminus\omega^k)}^2 \leq C \left( \eta^{n-2} \varepsilon^{-1} \eta^{-1}\|u\|_{L^2(\partial E^k)}^2 + \|\nabla u\|_{L^2(\partial\omega^k)} \|u\|_{L^2(\partial\omega^k)} \right) \\
+ C \|u\|_{W^2_2(\Xi \setminus 1)u\|_{L^2(\partial E^k)}^2} + C \left\| \Xi - 1 \right\|^2_{L^2(\partial E^k)} + C \varepsilon^2 \|u\|_{W^2_2(\Xi \setminus 1)u\|_{L^2(\partial E^k)}^2}.
\]

This inequality gives an opportunity to improve (4.30) for \( u \in W^2_2(B_{R_k}(M^e_k)\setminus\omega^k) \). Namely, we integrate by parts and estimate then using (4.36) with \( u = u^\perp \):

\[
\|\Xi - 1\|_{L^2(\partial\omega^k)}^2 = \frac{1}{n} \int_{L^2(\partial\omega^k)} \Xi - 1 \|u\|^2 \text{div} x \text{dx}
\]

It also follows from (3.1) with \( \eta = 1, \partial\omega^k = \partial E^k, u = u^\perp \) and (3.11) that

\[
\|u^\perp\|_{L^2(\partial E^k)}^2 \leq C \varepsilon \|\nabla u\|_{L^2(B_{R_k}(M^e_k)\setminus\omega^k)}^2.
\]

Therefore, due to (3.8), (4.25), and (3.7),

\[
\|\Xi - 1\|_{L^2(\partial\omega^k)}^2 \leq C \varepsilon^2 \eta^2 \varepsilon + \varepsilon^4 + \varepsilon^3 \eta \varepsilon \|u\|_{W^2_2(\Xi \setminus 1)u\|_{L^2(\partial E^k)}^2} \varepsilon^2 \eta^2 \varepsilon \|u\|_{W^2_2(\Xi \setminus 1)u\|_{L^2(\partial E^k)}^2}
\]

This estimate and (4.29), (4.31), and (4.32) yield (4.27). Substituting then (4.27) into the right hand side of (4.36) and using (3.1) with \( \eta = 1, \partial\omega^k = \partial E^k \), we arrive at (4.26). The proof is complete.

For each \( r > 0 \) and \( x \in \Omega \), we denote

\[
E_r(x) := \left\{ \xi \in \mathbb{R}^d : |\nabla^{-1}(x)\xi| < r \right\}.
\]

In the next lemma, we prove an auxiliary identity for the function \( Y \), which, we recall, was defined in (2.7).
Lemma 4.7. For all $x \in \Omega$, the identity holds:

$$\int_{\partial E(x)} \frac{ds}{|A^{-\frac{1}{2}}(x)\xi|} = Y(x).$$

Proof. We first consider the case $n \geq 3$. Given $x \in \Omega$, we choose a sufficiently large $r > 0$, and denoting by $\nu$ the unit outward normal to $\partial E(x)$, we integrate once by parts as follows:

$$0 = (2 - n)^{-1} \int_{\partial E(x)} |A^{-\frac{1}{2}}(x)\xi|^{-n+2} \text{div}_x A(x) \nabla_x |A^{-\frac{1}{2}}(x)\xi|^{-n+2} \, d\xi$$

$$= (2 - n)^{-1} \int_{\partial E(x)} |A^{-\frac{1}{2}}(x)\xi|^{-n+2} A(x) \nabla_x |A^{-\frac{1}{2}}(x)\xi|^{-n+2} \cdot \nu \, d\xi$$

$$- (2 - n)^{-1} \int_{\partial E(x)} |A^{-\frac{1}{2}}(x)\xi|^{-n+2} A(x) \nabla_x |A^{-\frac{1}{2}}(x)\xi|^{-n+2} \cdot \nu \, d\xi$$

$$- (2 - n)^{-1} \int_{\partial E(x)} A(x) \nabla_x |A^{-\frac{1}{2}}(x)\xi|^{-n+2} \cdot \nabla_x |A^{-\frac{1}{2}}(x)\xi|^{-n+2} \, d\xi$$

$$= \int_{\partial E(x)} \frac{ds}{r^{2n-4}|A^{-1}(x)\xi|^2} - \int_{\partial E(x)} \frac{ds}{|A^{-1}(x)\xi|^2} - (2 - n) \int_{E(x)} \frac{d\xi}{|A^{-\frac{1}{2}}(x)\xi|^{2n-2}}.$$

Making the change of the variables $y = A^{-\frac{1}{2}}(x)\xi$ in the integral over $E(x) \setminus E_1(x)$ and passing then to the limit as $r \to +\infty$, we obtain

$$\int_{\partial E_1(x)} \frac{ds}{|A^{-1}(x)\xi|^2} = (n - 2) \lim_{r \to +\infty} \int_{E(x) \setminus E_1(x)} \frac{d\xi}{|A^{-1}(x)\xi|^{2n-2}} = (n - 2) \sqrt{\det A(x)} \int_{\mathbb{R}^n \setminus B_1(0)} \frac{dy}{|y|^{2n-2}} = Y(x)$$

and this proves the needed formula for $n \geq 3$. For $n = 2$, we integrate in a similar way:

$$0 = \int_{E(x) \setminus E_1(x)} \ln |A^{-\frac{1}{2}}(x)\xi| \text{div}_x A(x) \nabla_x \ln |A^{-\frac{1}{2}}(x)\xi| \, d\xi = \ln r \int_{E(x)} \frac{ds}{|A^{-1}(x)\xi|^2} - \int_{E(x) \setminus E_1(x)} \frac{d\xi}{|A^{-\frac{1}{2}}(x)\xi|^2}$$

$$= \ln r \int_{E(x)} \frac{ds}{|A^{-1}(x)\xi|^2} - \sqrt{\det A(x)} \int_{B_1(0) \setminus B_0(0)} \frac{dy}{|y|^2} = \ln r \int_{E(x)} \frac{ds}{|A^{-1}(x)\xi|^2} - 2\pi \sqrt{\det A(x)} \ln r$$

and we arrive at the statement of the lemma for $n = 2$. The proof is complete. \(\square\)

Estimates (4.10) allow us to prove one more auxiliary lemma, which will be used then in the proof of our main theorems.

Lemma 4.8. The family $\beta^\varepsilon$ is uniformly bounded in $L^\infty(\Omega)$. Under Assumption A.3, the limit $\beta$ of the family $\beta^\varepsilon$ in the space $\mathcal{M}$ is an element of $L^\infty(\Omega)$.

Proof. The family $\beta^\varepsilon$ is uniformly bounded in $L^\infty(\Omega)$ due to its definition (2.19) and estimates (4.10). Since the space $L^\infty(\Omega)$ is dual to $L_1(\Omega)$, there exists a sequence $\varepsilon'$ such that $\beta^\varepsilon'$ converges weakly in $L^\infty(\Omega)$ to some limit $\bar{\beta} \in L^\infty(\Omega)$. Hence,

$$(\beta^\varepsilon'u, v)_{L^1(\Omega)} = \int_\Omega \beta^\varepsilon'u \nabla \Delta v = \int_\Omega \bar{\beta}u \nabla v, \quad \varepsilon' \to 0,$$

for all $u, v \in C^\infty(\Omega)$ vanishing on $\partial \Omega$. At the same time, it follows from definition (2.20) of the norm in $\mathcal{M}$ that $(\beta^\varepsilon'u, v)_{L^1(\Omega)} \to \langle \beta u, v \rangle$, $\varepsilon' \to 0$. Hence, $\langle \beta u, v \rangle = (\bar{\beta}u, v)_{L^1(\Omega)}$, and owing to the density of the functions from $C^\infty(\Omega)$ vanishing on $\partial \Omega$ in $\tilde{W}^1_2(\Omega)$ and $\tilde{W}^1_2(\Omega) \cap W^2_2(\Omega)$, we get the identity $\beta = \bar{\beta}$, and this proves the lemma. \(\square\)
5 | OPERATOR ESTIMATES

In this section, we prove Theorems 2.1 and 2.2. The proofs consist of three main steps. At the first step, we prove estimates (2.24) and (2.26). At the second step, we establish estimates (2.25) and (2.27). And at the third step, we show the order sharpness of the certain terms in the estimates.

5.1 | \(W^1\)-estimates: general case

In this subsection, we prove estimate (2.24). We choose an arbitrary ordersharpness of the certainterms in the estimates. mates (2.24) and (2.26). At thesecond step, we establish estimates (2.25) and (2.27). And at the third step, we show the part of the resulting relation:

where
\[
\text{Re } h(u_t) := h^{(1)} + h^{(3)} + h^{(4)},
\]
\[
h^{(1)} := (f, (1 - \overline{\Xi}) v_t)_L^2(\Omega^\prime),
\]
\[
h^{(2)} := - \sum_{k \in M^\prime} \left( (A - A_k) u_0 \nabla \overline{\Xi}, v_t \right)_{L^2(\Omega^\prime \setminus \Omega^k)}
\]
\[
h^{(3)} := \sum_{k \in M^\prime} \left( A_j u_0 \frac{\partial \overline{\Xi}}{\partial x_j}, v_t \right)_{L^2(\Omega^\prime \setminus \Omega^k)}^{(\theta_{R,k})}
\]
\[
h^{(4)} := \sum_{k \in M^\prime} \left( A_k \nabla u_0 \cdot v^*, v_t \overline{\Xi} \right)_{L^2(\Omega^\prime \setminus \Omega^k)} - \left( A_k \nabla u_0, v_t \overline{\Xi} \right)_{L^2(\Omega^\prime \setminus \Omega^k)}^{(\theta_{R,k})}.
\]

It follows from the definition of the form \(h\) that
\[
\text{Re } h(u_t) = \text{Re } h(\cdot, u_t)_L^2(\Omega^\prime) + \text{Re } (\alpha^t(\cdot, u_t), v_t)_L^2(\Omega^\prime, \overline{\Xi}) = \text{Re } h_t,
\]

Having this identity and (5.1) in mind, we calculate the difference of identities (5.2) and (5.3) and take then the real part of the resulting relation:
\[
\text{Re } h(u_t) - \text{Re } \lambda \|v_t\|_{L^2(\Omega^\prime)}^2 + \text{Re } (\alpha^t(\cdot, u_t) - \alpha^t(\cdot, u_0 \Xi), v_t)_{L^2(\Omega^\prime, \overline{\Xi})} = \text{Re } h_t,
\]

where we recall that \(v^*\) is the unit normal to \(\partial \Omega^k\) directed inside \(\Omega^k\).

Letting \(\theta^*_{R,i} := \bigcup_{k \in M^*} \alpha^k, i = 1, 2\), we observe that the function \(\Xi\) vanishes on \(\partial \theta^*_{R,i}\). By Assumption A2 and (2.9), we find:
\[
\text{Re } (\alpha^t(\cdot, u_t) - \alpha^t(\cdot, u_0 \Xi), v_t)_{L^2(\Omega^\prime, \overline{\Xi})} = \text{Re } (\alpha^t(\cdot, u_t), v_t)_{L^2(\theta^*_{R,i})} + \text{Re } (\alpha^t(\cdot, u_t) - \alpha^t(\cdot, u_0 \Xi), v_t)_{L^2(\theta^*_{R,i})}
\]
\[
\geq \mu_1 \|v_t\|_{L^2(\theta^*_{R,i})}^2 - \mu_0 \|v_t\|_{L^2(\theta^*_{R,i})}^2.
\]
Using then inequalities (3.18) and (3.8) and Assumption A.2, we find a lower bound for the left hand side of identity (5.5):

$$
\text{Re}(v_x, v_y) - \text{Re} \lambda ||v||^2_{L^2(\Omega)} + \text{Re}(a^t(\cdot, u), v_x)_{L_2(\Omega)} \geq C ||v||_{W^2_2(\Omega)}^2 + \mu_1 ||v||_{L_2(\Omega)}^2.
$$

(5.6)

Hereinafter in this section by $C$, we denote inessential constants independent of $\epsilon, x, k, u_x, u_y, v_x, f$ but, generally speaking, depending on $\lambda$. Our next key step is to estimate the right hand side in (5.5) and to get in this way a bound for $||v||_{W^2_2(\Omega)}$.

Taking into consideration (4.25) and Lemma 4.8, we can estimate the function $h^{(1)}_{e,k}$:

$$
|h^{(1)}_{e,k}| \leq \sum_{k \in \mathbb{N}} \left| (f, (1 - \Xi_x) v_x)_{L_2(\mathbb{E}_k') \cap \Omega} \right| + \gamma \sum_{k \in \mathbb{N}} \left| (\beta Y u_0, (1 - \Xi_x) v_x)_{L_2(\mathbb{E}_k') \cap \Omega} \right|

\leq \sum_{k \in \mathbb{N}} \|f\|_{L_2(\mathbb{E}_k') \cap \Omega} \| (\Xi_x - 1)v_x \|_{L_2(\mathbb{E}_k') \cap \Omega} + C \sum_{k \in \mathbb{N}} \| u_0 \|_{L_2(\mathbb{E}_k') \cap \Omega} \| (\Xi_x - 1)v_x \|_{L_2(\mathbb{E}_k') \cap \Omega}

\leq C \epsilon \sum_{k \in \mathbb{N}} \left( \|f\|_{L_2(\mathbb{E}_k') \cap \Omega} + \| u_0 \|_{L_2(\mathbb{E}_k') \cap \Omega} \right) \|v_x\|_{W^2_2(\mathbb{E}_k') \cap \Omega} \|v_x\|_{W^2_2(\Omega)}.
$$

(5.7)

The assumed smoothness of the functions $A_{ij}$ implies the inequality:

$$
|A(x) - A_{k,e}| \leq C|\chi| \leq C \epsilon \text{ a.e. in } \mathbb{E}_k',
$$

and this is why by (4.26) and (2.5), we get the following:

$$
|h^{(2)}_{e,k}| \leq C \left( \eta^{\frac{1}{2}} + \eta^{\frac{1}{2}} \epsilon^2 \right) ||u||_{W^2_2(\Omega)} \| \nabla v_x \|_{L_2(\Omega)}.
$$

(5.8)

In order to estimate $h^{(3)}_{e,k}$, we first integrate by parts using the properties of the function $\Xi_x$:

$$
\begin{align*}
\delta^{(3)}_{e,k} &= - \sum_{j=1}^{n} \left( A_j u_0 \frac{\partial (\Xi_x - 1)}{\partial x_j}, v_x \right)_{L_2(\mathbb{E}_k') \cap \Omega} + \left( (A + A_{k,e}) \nabla u_0, v_x \nabla (\Xi_x - 1) \right)_{L_2(\mathbb{E}_k') \cap \Omega}

- \left( A_{k,e} \nabla u_0 \cdot v, v_x \Xi_x \right)_{L_2(\mathbb{E}_k') \cap \Omega} = h^{(5)}_{e,k} + h^{(6)}_{e,k},
\end{align*}
$$

(5.9)

where $v_j$ are the components of the unit normal $v$. Inequality (4.25) applied with $v = u_0$ and $v = v_x$ allows us to estimate $h^{(6)}_{e,k}$:

$$
|h^{(6)}_{e,k}| \leq C \epsilon ||u_0||_{W^2_2(\mathbb{E}_k') \cap \Omega} \|v_x\|_{W^2_2(\mathbb{E}_k') \cap \Omega} \|v_x\|_{W^2_2(\Omega)}.
$$

(5.10)

Inequalities (3.8) and (4.23) give rise to a similar estimate for $h^{(5)}_{e,k}$:

$$
|h^{(5)}_{e,k}| \leq C \epsilon \eta |\chi||u_0||_{W^2_2(\mathbb{E}_k') \cap \Omega} \|v_x\|_{W^2_2(\mathbb{E}_k') \cap \Omega} \|v_x\|_{W^2_2(\Omega)}.
$$

(5.11)
We proceed to estimating the function $h_{r,k}^{(4)}$, which is one of the most non-trivial steps in the proof. As above, we first integrate by parts in $h_{r,k}^{(4)}$ taking into consideration the definition of $\Xi$ and the equation for $Z_{k,e}$:

$$h_{r,k}^{(4)} = -(A_{k,e}u_0 \nabla \Xi_e \cdot \nu, v_e)_{L^2(\partial E^e_k)} - (A_{k,e}u_0 \nabla \Xi_e \cdot \nu, v_e)_{L^2(\partial E^e_k)} - (\alpha' \cdot \nabla \Xi_e, v_e)_{L^2(\partial E^e_k)}, \quad (5.12)$$

where $\nu$ stands for the unit outward normal to $\partial E^e_k$. We fix $k \in M^e_k$ and represent the functions $u_0$ and $v_e$ as

$$u_0 = \langle u_0 \rangle + u^1_0, \quad v_e = \langle v_e \rangle + v^1_e, \quad (5.13)$$

where the operations $\langle \cdot \rangle$ and $\cdot ^\perp$ have been defined in (4.28). Then by inequality (3.1) with $\partial \omega^e_k = \partial E^e_k$ and $\eta = 1$ and by Lemma 3.2, the functions $u_0, u^1_0, v^1_e$ satisfy the estimates

$$\|u_0\|_{L^2(\partial E^e_k)} \leq C\varepsilon^{-\frac{1}{2}} \|u_0\|_{W^2_2(B_{R^4_k}(M^e_k))}, \quad \|u^1_0\|_{L^2(\partial E^e_k)} \leq C\varepsilon^{-\frac{1}{2}} \|\nabla u_0\|_{L^2(B_{R^4_k}(M^e_k))}, \quad \|v^1_e\|_{L^2(\partial E^e_k)} \leq C\varepsilon^{-\frac{1}{2}} \|\nabla v_e\|_{L^2(B_{R^4_k}(M^e_k))}. \quad (5.14)$$

By identities (5.13), we rewrite the first term in formula (5.12) as

$$(A_{k,e}u_0 \nabla \Xi_e \cdot \nu, v_e)_{L^2(\partial E^e_k)} = \langle u_0 \rangle \langle \nabla v_e \rangle \int_{\partial E^e_k} A_{k,e} \nabla \Xi_e \cdot \nu \, ds + (A_{k,e}u_0 \nabla \Xi_e \cdot \nu, v^1_e)_{L^2(\partial E^e_k)}$$

$$+ \langle v_e \rangle \int_{\partial E^e_k} A_{k,e}u^1_0 \nabla \Xi_e \cdot \nu \, ds. \quad (5.15)$$

The second term and the third term in the right hand side can be estimated by means of (4.23), (4.24), (5.14), and (3.7):

$$\left| (A_{k,e}u_0 \nabla \Xi_e \cdot \nu, v^1_e)_{L^2(\partial E^e_k)} \right| \leq C\varepsilon \|u_0\|_{W^2_2(B_{R^4_k}(M^e_k))} \|\nabla v^1_e\|_{L^2(B_{R^4_k}(M^e_k))}, \quad (5.16)$$

$$\left| \langle v_e \rangle \int_{\partial E^e_k} A_{k,e}u^1_0 \nabla \Xi_e \cdot \nu \, ds \right| \leq C\varepsilon^{-\frac{1}{2}} \|\nabla v_e\|_{L^2(B_{R^4_k}(M^e_k))} \|\nabla v^1_e\|_{L^2(B_{R^4_k}(M^e_k))}.$$
Hence,
\[ \int_{\partial E^k_i} A_{k,x} \nabla e \cdot v \, ds = \epsilon^n \gamma K_{k,x} (2 - n) Y(M^k_i) + O \left( \epsilon^n |\eta|^{-2} - \gamma + \epsilon^{n+2} \eta^2 \right) \quad \text{as } n \geq 3, \]
\[ \int_{\partial E^k_i} A_{k,x} \nabla e \cdot v \, ds = \epsilon^2 \gamma Y(M^k_i) + O \left( \epsilon^2 |\epsilon^{-2} \ln \eta - 3| + \epsilon^4 \eta \right) \quad \text{as } n = 2. \]

(5.17)

It is clear that
\[
(Yu_0, v_\varepsilon)_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} = \langle u_0 \rangle_{(\nabla e)} \int_{B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k} Y \, dx + \langle \nabla e \rangle \int_{B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k} Y u_0 \, dx
\]
\[
+ (Yu_0, v_\varepsilon^\perp)_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)},
\]
\[
\left| \int_{B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k} Y \, dx - Y(M^k_i) \meas B_{\varepsilon R_3}(M^k_i) \right| \leq C \epsilon^{n+1}.
\]

Then by Lemma 3.2, the estimates
\[
\left| (Yu_0, v_\varepsilon^\perp)_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} - (u_0)_{(\nabla e)} \right| \leq C \epsilon \| u_0 \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} \| \nabla e \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)},
\]
\[
\left| \langle \nabla e \rangle \int_{B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k} Y u_0 \, dx \right| \leq C \epsilon \| u_0 \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} \| u_0 \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)}
\]
\[
\leq C \epsilon \| \nabla u_0 \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} \| v_\varepsilon \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)},
\]

hold true, and therefore,
\[
\left| (Yu_0, v_\varepsilon)_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} - (u_0)_{(\nabla e)} \right| \leq C \epsilon \| u_0 \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} \| v_\varepsilon \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)}.
\]

The above estimate and (5.15), (5.16), and (5.17) imply
\[
(A_{k,x} v, \nabla e \cdot v, v_\varepsilon)_{L^2(\partial E^k_i)} = h^{(7)}_{x,k} + h^{(8)}_{x,k},
\]
(5.18)

where
\[
h^{(7)}_{x,k} := - \frac{K_{k,x}}{R_3^2 \meas B_1(0)} (Yu_0, v_\varepsilon)_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} \quad \text{as } n \geq 3,
\]
\[
h^{(7)}_{x,k} := - \frac{\gamma}{R_3^2 \meas B_1(0)} (Yu_0, v_\varepsilon)_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} \quad \text{as } n = 2,
\]

while the functions \(h^{(8)}_{x,k}\) obey the estimates:
\[
|h^{(8)}_{x,k}| \leq C \epsilon \| u_0 \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} \| v_\varepsilon \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)}
\]
\[
+ C \left| \eta^{n-2} \epsilon^{-2} - \gamma \right| \| u_0 \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)} \| v_\varepsilon \|_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)}.
\]

(5.19)

We observe that
\[
h^{(7)}_{x,k} = - \gamma (Y \beta u_0, v_\varepsilon)_{L^2(B_{\varepsilon R_3}(M^k_i) \setminus \omega^e_k)}.
\]

(5.20)
We proceed to estimating two other terms in the right hand side of (5.12). We first of all note that as \( k \in M_{D}^{e} \), this term vanishes since \( \nu_{c} = 0 \) on \( \partial \omega_{k}^{c} \) for such \( k \). This is why we need to estimate it only for \( k \in M_{R}^{e} \). We first consider the case \( k \in M_{R,1}^{e} \). The function \( \Xi_{c} \) vanishes on \( \partial \omega_{k}^{c} \), and hence, in view of the identity in (2.9), the function \( a^{c} (\cdot, \Xi_{c}, \nu_{c}) \) vanishes and the same is true for the third term in the right hand side of (5.12). Since \( Z_{k}^{c} = 0 \) on \( \partial \omega_{k}^{c} \) and \( Z_{k}^{e} \in C^{1}(\overline{B_{R}(M_{R}^{e}) \setminus \omega_{k}^{c}}) \), we have

\[
\frac{\partial Z_{k}^{c}}{\partial x_{j}} = \frac{\partial \tau}{\partial x_{j}} \frac{\partial Z_{k}^{c}}{\partial \tau} \quad \text{on} \quad \partial \omega_{k}^{c},
\]

here, \( \tau \) is the distance measured along the unit normal to \( \partial \omega_{k}^{c} \); see Assumption A.1. According to this assumption, the derivatives of \( \tau \) in \( x_{j} \) are bounded uniformly in \( k, \epsilon \) and the spatial variables. By (4.12) and (4.14), we then obtain

\[
\left| \frac{\partial Z_{k}^{c}}{\partial x_{j}} \right| \leq C(\epsilon \eta)^{-1} \quad \text{on} \quad \partial \omega_{k}^{c}, \quad k \in M_{R}^{e}.
\]

Hence, by (3.8) and (2.9),

\[
\left| (A_{k,c} u_{0} \nabla \Xi_{c} \cdot \nu, v_{c})_{L_{2}(\omega_{k}^{c})} \right| \leq C(\epsilon \eta \kappa)^{-\frac{1}{2}} \| u_{0} \|_{w_{2}^{1}(\overline{B_{R}(M_{R}^{e}) \setminus \omega_{k}^{c}})} \| v_{c} \|_{L_{2}(\omega_{k}^{c})},
\]

(5.21)

as \( k \in M_{R,1}^{e} \). For \( k \in M_{R,2}^{e} \), by boundary condition (2.18) and the definition of \( \Xi_{c} \), we see that

\[
A_{k,c} \nabla \Xi_{c} \cdot \nu = \epsilon^{-1} n^{-1} b_{k} \Xi_{c} \quad \text{on} \quad \partial \omega_{k}^{c},
\]

and therefore, in view of (2.12),

\[
-(A_{k,c} u_{0} \nabla \Xi_{c} \cdot \nu, v_{c})_{L_{2}(\omega_{k}^{c})} - (a^{c} (\cdot, u_{0} \Xi_{c}), v_{c})_{L_{2}(\omega_{k}^{c})} = - \left( \tilde{a}_{k}^{c} (\cdot, u_{0} \Xi_{c}), v_{c} \right)_{L_{2}(\omega_{k}^{c})},
\]

Using then the estimate for \( \tilde{a}_{k}^{c} \) in (2.14) as well as (3.8) and (4.23), we get

\[
\left| (A_{k,c} u_{0} \nabla \Xi_{c} \cdot \nu, v_{c})_{L_{2}(\omega_{k}^{c})} + (a^{c} (\cdot, u_{0} \Xi_{c}), v_{c})_{L_{2}(\omega_{k}^{c})} \right| \leq C(\epsilon \eta \kappa)^{-\frac{1}{2}} \| u_{0} \|_{w_{2}^{1}(\overline{B_{R}(M_{R}^{e}) \setminus \omega_{k}^{c}})} \| v_{c} \|_{L_{2}(\omega_{k}^{c})}.
\]

(5.22)

Summing up the above estimates over \( k \in M_{R,2}^{e} \), relations (5.21) over \( k \in M_{R,1}^{e} \), identities (5.18) and (5.20), and inequalities (5.19) over \( k \in M_{R}^{e} \), by (3.2), we finally obtain the following:

\[
h_{k}^{(4)} = \gamma ((\beta - \beta^{e}) u_{0}, Y v_{c})_{L_{2}(\Omega^{e})} + h_{k}^{(9)},
\]

(5.23)

\[
|h_{k}^{(9)}| \leq C(\epsilon + \epsilon \eta \kappa)^{-\frac{1}{2}} \| u_{0} \|_{w_{2}^{1}(\Omega^{e})} \| v_{c} \|_{w_{2}^{2}(\Omega^{e})} + C(\epsilon \eta \kappa)^{-\frac{1}{2}} \| u_{0} \|_{w_{2}^{2}(\Omega^{e})} \| v_{c} \|_{L_{2}(\omega_{k}^{c})}
\]

(5.24)

\[
+ C \left| \eta^{-\frac{1}{2}} \epsilon^{-\frac{1}{2} - \kappa^{-1}} \right| \| u_{0} \|_{L_{2}(\Omega^{e})} \| v_{c} \|_{L_{2}(\Omega^{e})}.
\]

If \( \gamma \neq 0 \), then the first term in the left hand side of (5.23) is, generally speaking, non-zero, and we need Assumption A.3 to estimate it. In order to apply this assumption, we first continue the function \( v_{c} \) inside \( \omega_{k}^{c} \) to make it defined on the entire domain \( \Omega \). We first let

\[
v_{c} := 0 \quad \text{in} \quad \omega_{k}^{c} \quad \text{for} \quad k \in M_{D}^{e}.
\]

(5.25)

For \( k \in M_{R}^{e} \), we introduce the quantities \( \langle v_{c} \rangle_{k} \) and the functions \( v_{c,k} \) by formulae (3.5). By Lemma 3.2 with \( \epsilon \) replaced by \( \epsilon \eta \), we have

\[
\| v_{c,k} \|_{L_{2}(B_{R}(M_{R}^{e}) \setminus \omega_{k}^{c})}^{2} \leq C \epsilon^{2} \eta^{2} \| \nabla v_{c} \|_{L_{2}(B_{R}(M_{R}^{e}) \setminus \omega_{k}^{c})}^{2}.
\]

(5.26)
Then for \( k \in M_R^e \), we define the continuation of the function \( v_i \) inside \( \omega_k^o \) in terms of the local variables \((r, s)\) as follows:

\[
\begin{align*}
v_i(r, s) &:= (v_i)_k + v_{i,k}(-r, s) \chi_1(r\varepsilon^{-1} \eta^{-1}) & \text{for } x \in \omega_k^o, \text{ dist}(x, \partial \omega_k^o) \leq \eta \tau_0, \\
v_i(r, s) &:= (v_i)_k + v_{i,k}(r, s) \text{ for } x \in \omega_k^o, \text{ dist}(x, \partial \omega_k^o) > \eta \tau_0,
\end{align*}
\]

(5.27)

where \( \chi_1 \) is the cut-off function introduced in the proof of Lemma 4.4. It is obvious that this continuation gives a function in \( W^1_2(B_{\tau_0}(M_k^e)) \), which in view of estimates (3.4) and (5.26) satisfy the inequalities:

\[
\begin{align*}
\|v_i\|_{L^2(\omega_k^o)}^2 &\leq C\left( \|(v_i)_k\|^2 \text{meas } \omega_k^o + \|v_{i,k}\|_{L^2(B_{\tau_0}(M_k^e) \setminus \omega_k^o)}^2 \right) \\
&\leq C\|v_i\|_{W^1_2(B_{\tau_0}(M_k^e) \setminus \omega_k^o)}^2 \leq C\varepsilon^2 \eta^2 \varepsilon \|v_i\|_{L^2_2(B_{\tau_0}(M_k^e) \setminus \omega_k^o)},
\end{align*}
\]

(5.28)

\[
\begin{align*}
\|\nabla v_i\|_{L^2(\omega_k^o)}^2 &\leq C\left( \|\nabla v_i\|_{L^2_2(B_{\tau_0}(M_k^e) \setminus \omega_k^o)}^2 + \varepsilon^{-2} \eta^{-2} \|v_{i,k}\|_{L^2_2(B_{\tau_0}(M_k^e) \setminus \omega_k^o)}^2 \right) \\
&\leq C\|v_i\|_{L^2_2(B_{\tau_0}(M_k^e) \setminus \omega_k^o)}^2.
\end{align*}
\]

(5.29)

Due to these inequalities and identity (5.25), the continued function \( v_i \), regarded as defined on the entire domain \( \Omega \), is an element of \( \dot{W}^1_2(\Omega) \) and

\[
\|v_i\|_{L^2_2(\Omega)} \leq C\varepsilon^2 \eta^2 \varepsilon \|v_i\|_{L^2_2(\omega_k^o)}^2, \quad \|\nabla v_i\|_{L^2_2(\Omega)} \leq C\|\nabla v_i\|_{L^2_2(\omega_k^o)}^2.
\]

(5.30)

These inequalities allow us to rewrite the scalar product in the right hand side of (5.23) as

\[
((\beta - \beta^o)u_0, Y v_i)_{L^2_2(\Omega)} = ((\beta - \beta^o)u_0, Y v_i)_{L^2_2(\Omega)} + h^{(10)},
\]

(5.31)

where \( h^{(10)} \) is a function satisfying the estimate

\[
|h^{(10)}| \leq C\|u_0\|_{L^2_2(\Omega)}\|v_i\|_{L^2_2(\Omega)} \leq C\varepsilon^2 \eta^2 \varepsilon \|u_0\|_{W^2_2(\Omega)}\|v_i\|_{W^2_2(\Omega)}.
\]

(5.32)

In the scalar product in the right hand of (5.31), the function \( Y v_i \) is an element of \( \dot{W}^1_2(\Omega) \), and then the function \((\beta - \beta^o)u_0\) can be regarded as a functional on this space. Then by formula (2.20), Assumption A.3, and inequalities (5.30), we can estimate this scalar product as follows:

\[
\begin{align*}
|((\beta - \beta^o)u_0, Y v_i)_{L^2_2(\Omega)}| &\leq C\|\beta - \beta^o\|_{H^1_2(\Omega)}\|u_0\|_{W^2_2(\Omega)}\|v_i\|_{W^2_2(\Omega)} \leq C\|\beta - \beta^o\|_{H^1_2(\Omega)}\|u_0\|_{W^2_2(\Omega)}\|v_i\|_{W^2_2(\Omega)} \\
&\leq C\|\beta - \beta^o\|_{H^1_2(\Omega)}\|u_0\|_{W^2_2(\Omega)}\|v_i\|_{W^2_2(\Omega)}.
\end{align*}
\]

(5.33)

The above estimate and (5.7), (5.8), (5.11), (5.23), (5.24), (5.31), (5.32), and (3.14) yield a final estimate for the right hand side in (5.5):

\[
\begin{align*}
|\Re h_{\varepsilon}| &\leq |h_{\varepsilon}| \leq C\|f\|_{L^2_2(\Omega)} \left( \varepsilon + \varepsilon \eta \|\mu\| \right)\|v_i\|_{W^2_2(\Omega)} + \left| \eta^{\varepsilon/2} \varepsilon^{-2} \varepsilon^{-1} - \gamma \right| \|v_i\|_{L^2_2(\Omega)} \\
&\quad + \gamma \varepsilon \|\beta^e - \beta\|_{H^1_2(\Omega)}\|v_i\|_{W^2_2(\Omega)} + (\varepsilon \eta)^{-\frac{1}{2}} \|v_i\|_{L^2_2(\partial \omega_{k^1}^e)} \\
&\leq \delta \left( \|v_i\|_{W^2_2(\Omega)}^2 + \mu_1\|v_i\|_{L^2_2(\partial \omega_{k^1}^e)}^2 \right) \\
&\quad + C(\delta) \left( \varepsilon^2 + (\varepsilon \eta \|\mu\|)^2 + \left| \eta^{\varepsilon/2} \varepsilon^{-2} \varepsilon^{-1} - \gamma \right|^2 + (\varepsilon \eta \|\mu_1\|)^2 \right) \|f\|_{L^2_2(\Omega)}^2,
\end{align*}
\]

(5.34)

where \( \delta > 0 \) is arbitrary but fixed, while \( C(\delta) \) is a constant independent of \( \varepsilon, f, u_0, \) and \( v_i \). Substituting the above estimate with a sufficiently small \( \delta \) into the left hand side of (5.5) and employing then (5.6), we obtain the following:

\[
\begin{align*}
\|v_i\|_{W^2_2(\Omega)}^2 + \mu_1\|v_i\|_{L^2_2(\partial \omega_{k^1}^e)}^2 &\leq C \left( \varepsilon^2 + (\varepsilon \eta \|\mu\|)^2 + \left| \eta^{\varepsilon/2} \varepsilon^{-2} \varepsilon^{-1} - \gamma \right|^2 + (\varepsilon \eta \|\mu_1\|)^2 + \varepsilon \|\beta^e - \beta\|_{H^1_2(\Omega)}^2 \right) \|f\|_{L^2_2(\Omega)}^2.
\end{align*}
\]

(5.35)

This estimate implies (2.24).
If the set \( M_R \) is empty, then the second term in (5.12) is zero for \( k \in M_R \) just because the function \( v_x \) vanishes on \( \partial \Omega_k \). Then estimates (5.21) and (5.22) are no longer needed. Estimate (5.24) also simplifies the following:

\[
|h_x| \leq C \|u_0\|_{W^2_2(\Omega)} \|v_x\|_{W^2_2(\Omega)} + C \left| \eta^{n-2} \epsilon^{-2} \lambda^{-1} - \gamma \right| \|u_0\|_{L^2(\Omega)} \|v_x\|_{L^2(\Omega)}.
\]

We also do not need continuation (5.27) and, hence, Assumption A.2. Estimate (5.30) are also omitted in the considered case. Estimates (5.33) and (5.34) then become

\[
|((\beta - \beta') u_0, Y v_x)_{L^2(\Omega)}| \leq C \|\beta' - \beta\| W_2^2(\Omega) \|u_0\|_{W^2_2(\Omega)} \|v_x\|_{W^2_2(\Omega)},
\]

\[
|\text{Re } h_x| \leq \delta \|v_x\|_{W^2_2(\Omega)}^2 + C(\delta) \left( \epsilon^2 + \left| \eta^{n-2} \epsilon^{-2} \lambda^{-1} - \gamma \right|^2 + \|\beta' - \beta\|_{W_2^2(\Omega)}^2 \right) \|f\|_{L^2(\Omega)}^2.
\]

All other above arguing remain the same, and we arrive at estimate (2.24) without the terms \((\epsilon \eta \mu \lambda)^{-\frac{1}{2}}\) and \(\epsilon \eta \lambda \mu_2\).

### 5.2 \( W^1_2 \)-estimates: \( \gamma = 0 \)

Here, we prove estimate (2.26). Assume that \( \gamma = 0 \), and only Assumption A.1 holds. In this case, the function \( h_x \) can be written as \( h_x = g_x[v_x] \), where

\[
g_x[v] := (f, (1 - \overline{\tau}_x)v)_{L^2(\Omega)} - \sum_{k \in M} \left( (A u_0 \nabla \tau_x, \nabla v)_{L^2(\Omega)} - (A \nabla u_0, v \nabla \tau_x)_{L^2(\Omega)} \right)
\]

\[
\quad - \sum_{k \in M} (A \nabla u_0 \cdot v', v_x \overline{\tau}_x)_{L^2(\partial \Omega_k)} - \sum_{k \in M} \sum_{j=1}^n \left( A_{ij} u_0 \frac{\partial \tau_x}{\partial x_j}, v \right)_{L^2(\Omega_k)}
\]

(5.36)

We also integrate by parts:

\[
\sum_{k \in M} \left( A \nabla u_0, v \nabla \tau_x \right)_{L^2(\Omega_k)} = - \sum_{k \in M} \int (\tau_x - 1) \text{div } A \nabla u_0 \, dx + \sum_{k \in M} \left( A \nabla u_0 \cdot v', (\overline{\tau}_x - 1)v \right)_{L^2(\partial \Omega_k)}
\]

and this allows us to rewrite (5.36):

\[
g_x[v] := (f, (1 - \overline{\tau}_x)v)_{L^2(\Omega)} - \sum_{k \in M} \sum_{j=1}^n \left( A_{ij} u_0 \frac{\partial \tau_x}{\partial x_j}, v \right)_{L^2(\Omega_k)} + \sum_{k \in M} \left( A \nabla u_0 \cdot v', (\overline{\tau}_x - 1)v \right)_{L^2(\partial \Omega_k)}
\]

\[
\quad - \sum_{k \in M} (A u_0 \nabla \tau_x, \nabla v)_{L^2(\Omega_k)} - \sum_{k \in M} \int (\tau_x - 1) \text{div } A \nabla u_0 \, dx.
\]

Then the function \( h_x \) can be directly estimated by means of inequalities (3.14), (4.23), and (3.1) and Lemma 4.6:

\[
|h_x| = |g_x[v_x]| \leq C \left( \epsilon \|f\|_{L^2(\Omega)} + \left( \eta^{\frac{n-1}{2}} \epsilon^{-1} \lambda^{-\frac{1}{2}} + \epsilon^{\frac{1}{2}} \eta^{\frac{1}{2}} + \epsilon \right) \|u_0\|_{W^2_2(\Omega)} \right) \|v_x\|_{W^2_2(\Omega)}
\]

\[
\leq C \left( \eta^{\frac{n-1}{2}} \epsilon^{-1} \lambda^{-\frac{1}{2}} + \epsilon^{\frac{1}{2}} \eta^{\frac{1}{2}} + \epsilon \right) \|f\|_{L^2(\Omega)} \|v_x\|_{W^2_2(\Omega)}.
\]

Substituting this estimate into the right hand side of (5.5) and using (5.6), we obtain

\[
\|v_x\|_{W^2_2(\Omega)} \leq C \left( \eta^{\frac{n-1}{2}} \epsilon^{-1} \lambda^{-\frac{1}{2}} + \epsilon^{\frac{1}{2}} \eta^{\frac{1}{2}} + \epsilon \right) \|f\|_{L^2(\Omega)}.
\]

(5.37)
By Lemma 4.6 and inequality (3.14), we also have the following:

\[ \| (\Xi - 1)u_0 \|_{L^2(\Omega')} \leq C(\epsilon^2 + \epsilon \eta) \| u_0 \|_{W^2_1(\Omega')} \leq C \epsilon \| f \|_{L^2(\Omega)}, \]
\[ \| \nabla (\Xi - 1)u_0 \|_{L^2(\Omega')} \leq \| (\Xi - 1)\nabla u_0 \|_{L^2(\Omega')} + \| u_0 \nabla \Xi \|_{L^2(\Omega')} \]
\[ \leq C \left( \eta^{\frac{3}{2}} - 1 \eta - \frac{1}{2} + \frac{1}{2} \eta^2 + \epsilon \right) \| f \|_{L^2(\Omega)}. \]  

(5.38)

These estimates and (5.37) and an obvious identity

\[ u_\epsilon - u_0 = v_\epsilon + (1 - \Xi_\epsilon)u_0 \]  

(5.39)

prove (2.26).

5.3 | \( L_2 \)-estimates

Here, we prove inequalities (2.25) and (2.27). The former is implied immediately by identity (5.39) and estimates (4.25), (3.14), and (2.24).

In the proof of (2.27), we use an approach based on duality arguments; see [27–31], with a slight modification proposed recently in [32]. Namely, we first introduce a differential expression:

\[ \mathcal{L}^* := - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} A_{ij}^e \frac{\partial}{\partial x_j} - \sum_{j=1}^n \frac{\partial}{\partial x_j} A_j^e + A_0 \]

and consider an auxiliary boundary value problem:

\[ (\mathcal{L}^* - \lambda)w = f_\epsilon \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega, \]  

(5.40)

where \( f_\epsilon = v_\epsilon \) in \( \Omega' \) and \( f_\epsilon = 0 \) in \( \theta^e \); here, we use the notations from Section 5.1. Since \( A_j^e \in W^1_\infty(\Omega) \), this problem is of the same nature as (2.6). This is why it is solvable for \( \Re \lambda \leq \lambda_0 \), and its solution belongs to \( W^2_2(\Omega) \) and satisfies the estimate

\[ \| w \|_{W^2_2(\Omega)} \leq C \| v_\epsilon \|_{L^2(\Omega')} \]  

(5.41)

Hereinafter, by \( C \), we denote inessential constants independent of \( \epsilon, k, f, v_\epsilon, \) and \( w \).

In what follows, the function \( v_\epsilon \) is supposed to be continued inside \( \theta^e \) in accordance with (5.25) and (5.27) and thus is regarded as an element of \( W^2_2(\Omega) \). We then write an integral identity associated with problem (5.40) choosing \( v_\epsilon \) as a test function:

\[ (v_\epsilon, f_\epsilon)_{L^2(\Omega')} = \| v_\epsilon \|^2_{L^2(\Omega')} = \mathcal{J}(v_\epsilon, w) - \lambda (v_\epsilon, w)_{L^2(\Omega')} + (A \nabla w, \nabla v_\epsilon)_{L^2(\Omega')} \]
\[ + \sum_{j=1}^n \left( A_j \frac{\partial w}{\partial x_j}, v_\epsilon \right)_{L^2(\Omega')} + (A_0 w, v_\epsilon)_{L^2(\Omega')} - \lambda (w, v_\epsilon)_{L^2(\Omega')} \]  

(5.42)

By estimates (5.28), (5.29), (5.37), and (5.41), we then immediately obtain the following:

\[ \left| \sum_{j=1}^n \left( A_j \frac{\partial w}{\partial x_j}, v_\epsilon \right)_{L^2(\Omega')} + (A_0 w, v_\epsilon)_{L^2(\Omega')} - \lambda (w, v_\epsilon)_{L^2(\Omega')} \right| \leq C \epsilon \eta^{\frac{3}{2}} \| w \|_{W^2_2(\Omega)} \| v_\epsilon \|_{W^2_2(\Omega')} \]
\[ \leq C \epsilon \eta^{\frac{3}{2}} \left( \eta^{\frac{3}{2}} - 1 \eta - \frac{1}{2} + \frac{1}{2} \eta^2 + \epsilon \right) \| f \|_{L^2(\Omega)} \| v_\epsilon \|_{L^2(\Omega')} \]  

(5.43)

By \( \Pi_{\epsilon} \), we denote a particular case of function \( \Xi_\epsilon \) in the case when on the boundaries of all cavities, the Dirichlet condition is imposed. In other words, only the functions \( X_{\kappa,\epsilon} \) satisfying Dirichlet condition (2.16) are used in (2.23) for all \( k \in \mathbb{M}^e \) while defining \( \Pi_{\epsilon} \). The function \( \Pi_{\epsilon} \) vanishes on \( \partial \theta^e \) and is real.
We write identities (5.2), (5.3), and (5.4) replacing there \( v_r \) by \( \Pi_r w \), and then we take the difference of the obtained analogues of (5.2) and (5.3). This gives

\[
L(v_r, \Pi_r w) - \lambda(v_r, \Pi_r w)w_{L_2(\Omega')} = \left( f, (1 - \Xi)\Pi_r w \right)_{L_2(\Omega')} + (A\nabla u_0, \Pi_r w\nabla \Xi)_{L_2(\Omega')} + \left( A_j u_0 \frac{\partial \Xi}{\partial x_j}, \Pi_r w \right)_{L_2(\Omega')}.
\]

(5.44)

Let us estimate the right hand side of this identity.

Since the function \( \Pi_r \) is a particular case of \( \Xi_r \), it possesses the same properties, namely, relations (4.23) and (4.24) and Lemma 4.6 hold true for \( \Pi_r \). Then by (4.25) and (5.41), we obtain

\[
\left| \left( f, (1 - \Xi)\Pi_r w \right)_{L_2(\Omega')} \right| \leq C \| f \|_{L_2(\Omega)} \| (1 - \Xi)w \|_{L_2(\Omega')} \leq C \epsilon \| f \|_{L_2(\Omega)} \| v_r \|_{L_2(\Omega')}.
\]

(5.45)

Using the definition of the functions \( \Xi_r \) and \( \Pi_r \), we integrate by parts as follows:

\[
\sum_{j=1}^n \left( A_j u_0 \frac{\partial \Xi}{\partial x_j}, \Pi_r w \right)_{L_2(\Omega')} = \sum_{j=1}^n \left( A_j u_0 \frac{\partial (\Xi - 1)}{\partial x_j}, \Pi_r w \right)_{L_2(\Omega')} = \sum_{j=1}^n \int_{\Omega'} (\Xi - 1) \frac{\partial}{\partial x_j} A_j u_0 \Pi_r \dot{w} \, dx.
\]

Hence, by estimates (4.23), (4.25), and (4.26) for the functions \( \Xi_r \) and \( \Pi_r \) and by estimates (3.14) and (5.41), we obtain

\[
\left| \sum_{j=1}^n \left( A_j u_0 \frac{\partial \Xi}{\partial x_j}, \Pi_r w \right)_{L_2(\Omega')} \right| \leq C \| (\Xi - 1)u_0 \|_{L_2(\Omega')} \left( \| \nabla w \|_{L_2(\Omega')} + \| \nabla \Pi_r \|_{L_2(\Omega')} \right)
\]

\[
+ C \| (\Xi - 1)w \|_{L_2(\Omega')} \| u_0 \|_{L_2(\Omega)} \leq C(\epsilon^2 + \epsilon \eta) \| u_0 \|_{L_2(\Omega)} \| w \|_{L_2(\Omega)}.
\]

(5.46)

We rewrite two remaining terms in the right hand side of (5.44) as

\[
(A\nabla u_0, \Pi_r w\nabla \Xi)_{L_2(\Omega')} = (A\nabla u_0, \nabla \Xi)_{L_2(\Omega')} - (A\nabla u_0, \nabla \Xi)_{L_2(\Omega')}
\]

(5.47)

By Lemma 4.6 and estimates (3.14) and (5.41), we see that

\[
\left| (A\nabla u_0, (\Pi_r - 1)w\nabla \Xi)_{L_2(\Omega')} - (A\nabla u_0, \nabla \Xi)_{L_2(\Omega')} \right| \leq C \left( \| (\Pi_r - 1)\nabla u_0 \|_{L_2(\Omega')} \| w \nabla \Xi \|_{L_2(\Omega')} + \| u_0 \nabla \Xi \|_{L_2(\Omega')} \| \nabla \Pi_r \|_{L_2(\Omega')} \right)
\]

\[
\leq C(\eta \alpha^2 + \epsilon^2 + \epsilon \eta) \| u_0 \|_{L_2(\Omega)} \| w \|_{L_2(\Omega)}.
\]

In the other two terms in the right hand side of (5.47), we integrate by parts using the definition of the functions \( \Xi_r \) and \( \Pi_r \):

\[
(A\nabla u_0, w\nabla \Xi)_{L_2(\Omega')} - (A\nabla u_0, \nabla \Xi)_{L_2(\Omega')} = \int_{\Omega} (\Xi - 1) (\div A\nabla \dot{w} - \div A\dot{w}u_0) \, dx
\]

\[
= \int_{\Omega} (\Xi - 1) (u_0 \div A\nabla \dot{w} - \dot{w} \div A\nabla u_0) \, dx.
\]
By estimates (3.14), (4.23), (4.25), (5.41), and (3.9), we then obtain the following:

\[
\left| (A \nabla u_0, w \nabla \bar{z})_{L_2(\Omega')} - (A u_0 \nabla \bar{z}, \nabla w)_{L_2(\Omega')} \right| \\
\leq C \left\| (\bar{z} - 1) u_0 \right\|_{L_2(\Omega)} \| w \|_{W^2_2(\Omega)} + C \left\| (\bar{z} - 1) w \right\|_{L_2(\Omega)} \| u_0 \|_{W^2_2(\Omega)} \\
\leq C \left( \left\| \bar{z} - 1 \right\|_{L_2(\Omega')} + \| u_0 \|_{L_2(\Omega')} \right) \| w \|_{W^2_2(\Omega)} \\
+ C \left( \left\| \bar{z} - 1 \right\|_{L_2(\Omega')} + \left\| \bar{z} - 1 \right\|_{L_2(\Omega')} \right) \| u_0 \|_{W^2_2(\Omega)} \\
\leq C(\epsilon^2 + \epsilon \eta) \| u_0 \|_{W^2_2(\Omega)} \| w \|_{W^2_2(\Omega)} \\
\leq C(\epsilon^2 + \epsilon \eta) \| f \|_{L_2(\Omega)} \| v \|_{L_2(\Omega')}.
\]

These estimates and (5.44), (5.45), (5.46), (5.47), and (5.48) yield

\[
\left| b(v, \Pi w) - \lambda(v, \Pi w) \right|_{L_2(\Omega')} \leq C \left( \eta^{n-2} \epsilon^2 \chi^{-1} + \epsilon^2 + \epsilon \eta \right) \| f \|_{L_2(\Omega)} \| v \|_{L_2(\Omega')}.
\]  
\[
(5.49)
\]

It also follows from Lemma 4.6 and estimate (4.23) for the function \( \Pi \) and from (5.37) that

\[
\left| b(v, (1 - \Pi \epsilon \omega) w) - \lambda(v, (1 - \Pi \epsilon \omega) w) \right|_{L_2(\Omega')} \leq C \left\| v \right\|_{W^2_2(\Omega')} \left( \left\| (1 - \Pi \epsilon \omega) \nabla w \right\|_{L_2(\Omega')} + \| w \Pi \right\|_{L_2(\Omega')} \\
\leq C \left( \eta^{n-2} \epsilon^2 \chi^{-1} + \epsilon^2 + \epsilon \eta \right) \| f \|_{L_2(\Omega)} \| w \|_{W^2_2(\Omega)} \\
\leq C \left( \eta^{n-2} \epsilon^2 \chi^{-1} + \epsilon^2 + \epsilon \eta \right) \| f \|_{L_2(\Omega)} \| v \|_{L_2(\Omega')}.
\]

These inequalities and (5.49) and (5.43) allow us to estimate the left hand side in (5.42):

\[
\left\| v \right\|_{L_2(\Omega')}^2 \leq C \left( \eta^{n-2} \epsilon^2 \chi^{-1} + \epsilon^2 + \epsilon \eta \right) \| f \|_{L_2(\Omega)} \| v \|_{L_2(\Omega')}
\]

and hence,

\[
\left\| v \right\|_{L_2(\Omega')} \leq C \left( \eta^{n-2} \epsilon^2 \chi^{-1} + \epsilon^2 + \epsilon \eta \right) \| f \|_{L_2(\Omega)}.
\]

Employing now estimate (4.25) with \( u = u_0 \) and identity (5.39), we arrive at (2.27).

### 5.4 Sharpness of estimates

In this subsection, we study that the sharpness of the terms in the right hand sides of inequalities (2.24), (2.25), (2.26), and (2.27) are order sharp. First, let us show that the term \( \| \beta' - \beta \|_{W^1} \) in (2.24) and (2.25) is order sharp.

We choose \( L := -\Delta + 1 \), and we impose only the Dirichlet condition on the boundaries of the cavities, that is, \( W^1_{L_2} = \emptyset \).

In this case, the function \( \beta' \) is non-negative. For \( n = 2 \), this fact is implied by definition (2.19), while for \( n \geq 3 \), it follows from a simple integration by parts:

\[
0 = \lim_{R \to +\infty} \int_{\{\xi : A_{k, \xi} < R, \xi \notin w_{k, \xi}\}} X_{k, \xi} \cdot \text{div}_{\xi} A_{k, \xi} \nabla_{\xi} X_{k, \xi} \, d\xi \\
= \lim_{R \to +\infty} \int_{\{\xi : A_{k, \xi} = R\}} X_{k, \xi} A_{k, \xi} \nabla_{\xi} X_{k, \xi} \cdot v \, ds - \int_{\mathbb{R}^n \setminus w_{k, \xi}} A_{k, \xi} \nabla_{\xi} X_{k, \xi} \cdot \nabla_{\xi} X_{k, \xi} \, d\xi \\
= (2 - n)K_{k, \xi} \cdot \text{mes}_{n-1} \partial B_1(0) \det A_{k, \xi} - \int_{\mathbb{R}^n \setminus w_{k, \xi}} A_{k, \xi} \nabla_{\xi} X_{k, \xi} \cdot \nabla_{\xi} X_{k, \xi} \, d\xi.
\]
Definition (2.20) of the norm in \( \mathcal{M} \) yields that

\[
\int_{\Omega} \beta^\varepsilon \phi \, dx \to \int_{\Omega} \beta \phi \, dx, \quad \varepsilon \to +0, \quad \text{for all } \phi \in C_0^\infty(\Omega).
\]

Choosing then non-negative functions \( \phi \), we see that \( \beta \geq 0 \) almost everywhere in \( \Omega \).

We also assume that \( \eta \) is so that \( \eta^m - 2\varepsilon^{-2}x^{-1} = \gamma > 0 \) for all \( \varepsilon \) and the domain \( \Omega \) is bounded. Then the choice \( \lambda = \lambda_0 = 0 \) ensures the solvability of both perturbed and limiting problems (2.4) and (2.6) as well as of the following auxiliary boundary problem:

\[
(\mathcal{L} + \gamma \beta^\varepsilon)\bar{u}_0 = f \text{ in } \Omega, \quad \bar{u}_0 = 0 \text{ on } \partial \Omega. \quad (5.50)
\]

Since the function \( \beta^\varepsilon \) is piece-wise constant, non-negative, and is uniformly bounded due its definition and (4.10), the above problem is solvable in \( W^2_2(\Omega) \), and its solution also satisfies estimate (3.14). Then we can replace the function \( u_0 \) and problem (2.6) by \( \bar{u}_0 \) and problem (5.50) and reproduce all calculations in Section 5.1 up to (5.23) and (5.24) taking into consideration that \( M_R^\varepsilon = \emptyset \). In the right hand side of identity (5.23), then the first term vanishes, and this removes the term \( \|\beta^\varepsilon - \beta\|_{\mathcal{M}} \) from (5.34) and (5.35). Using then (5.38) and (5.39), we get modifications of estimates (2.24) and (2.25):

\[
\|u_\varepsilon - \Xi_\varepsilon \bar{u}_0\|_{W^2_2(\Omega)} \leq C\varepsilon \|f\|_{L_2(\Omega)}, \quad \|u_\varepsilon - \bar{u}_0\|_{L_2(\Omega)} \leq C\varepsilon \|f\|_{L_2(\Omega)}.
\]

Therefore, it is sufficient to prove that

\[
\|\bar{u}_0 - u_0\|_{L_2(\Omega)} \geq C\|\beta^\varepsilon - \beta\|_{\mathcal{M}} \|f\|_{L_2(\Omega)}, \quad (5.51)
\]

for some \( f \) to show the sharpness of the term \( \|\beta^\varepsilon - \beta\|_{\mathcal{M}} \) in (2.24) and (2.25). If \( \|\beta^\varepsilon - \beta\|_{\mathcal{M}} = 0 \), then the above inequalities are obvious, and this is why in what follows we assume that \( \|\beta^\varepsilon - \beta\|_{\mathcal{M}} \neq 0 \).

Assume that \( \|\beta^\varepsilon - \beta\|_{\mathcal{M}} \neq 0 \). We rewrite definition (2.20) of the norm \( \|\beta^\varepsilon - \beta\|_{\mathcal{M}} \) as

\[
\|\beta^\varepsilon - \beta\|_{\mathcal{M}} = \sup_{u \in W^2_2(\Omega) \cap \bar{W}^1_2(\Omega)} \frac{1}{\|u\|_{W^2_2(\Omega)}} \sup_{v \in W^2_2(\Omega)} \left| \langle (\beta^\varepsilon - \beta) u, v \rangle_{L_2(\Omega)} \right| \quad (5.52)
\]

and conclude that there exists a non-zero function \( u^\varepsilon \in W^2_2(\Omega) \cap \bar{W}^1_2(\Omega) \) such that

\[
\left| \langle (\beta^\varepsilon - \beta) u^\varepsilon, v \rangle_{L_2(\Omega)} \right| \geq \frac{1}{2} \|\beta^\varepsilon - \beta\|_{\mathcal{M}} \|u^\varepsilon\|_{W^2_2(\Omega)} \|v\|_{W^2_2(\Omega)} \text{ for all } v \in \bar{W}^1_2(\Omega). \quad (5.53)
\]

We choose \( f := (\mathcal{L} + \gamma \beta^\varepsilon)u^\varepsilon \), and we see that \( \bar{u}_0 = u^\varepsilon \) solves problem (5.50) and

\[
\|f\|_{L_2(\Omega)} \leq C\|u^\varepsilon\|_{W^2_2(\Omega)} \quad (5.54)
\]

with some fixed constant \( C \) independent of \( \varepsilon \) and \( u^\varepsilon \). Using the corresponding solution \( u_0 \) of problem (2.6) with \( \lambda = 0 \), we define \( \phi^\varepsilon := u^\varepsilon - u_0 \). The latter function solves the boundary value problem:

\[
(\mathcal{L} + \gamma \beta)\phi^\varepsilon + \gamma (\beta^\varepsilon - \beta)u^\varepsilon = 0 \text{ in } \Omega, \quad \phi^\varepsilon = 0 \text{ on } \partial \Omega. \quad (5.55)
\]

Let \( \Lambda \) and \( \Phi \) be a positive eigenvalue and an associated normalized in \( L_2(\Omega) \) eigenfunction of a self-adjoint operator in \( L_2(\Omega) \) with the differential expression \( \mathcal{L} + \gamma \beta \) and the Dirichlet condition on \( \partial \Omega \). We write integral identity corresponding to (5.55) with \( \Phi \) as a test function to obtain:

\[
\langle \phi^\varepsilon, \Phi \rangle_{L_2(\Omega)} = -\frac{1}{\lambda} \langle (\beta^\varepsilon - \beta)u^\varepsilon, \Phi \rangle_{L_2(\Omega)}. \]
Hence, by (5.53) and (5.54),

\[
\| \phi' \|_{L^2(\Omega)} \geq \| (\phi', \Phi)_{L^2(\Omega)} \|_{L^2(\Omega)} = \frac{1}{4} \| (\beta' - \beta) u' , \Phi \|_{L^2(\Omega)} \geq \frac{1}{2\Lambda} \| \beta' - \beta \|_{\mathcal{M}} \| u' \|_{W^2_2(\Omega)} \| \Phi \|_{L^2(\Omega)} \geq \frac{1}{2\Lambda} \| \beta' - \beta \|_{\mathcal{M}} \| f \|_{L^2(\Omega)}
\]

and this proves (5.51).

In order to prove the sharpness of the term $|\eta^{n-2} - \gamma|$ in estimates (2.24) and (2.25), we proceed in a similar way. Namely, assuming that $\eta^{n-2} - \gamma$ does not coincide with its limit $\gamma$, now we define $\tilde{u}_0$ as a solution to the problem

\[
(-\Delta + \mu_3 \beta_0) \tilde{u}_0 = f \quad \text{in } \Omega, \quad \tilde{u}_0 = 0 \quad \text{on } \partial \Omega
\]

with $\mu_3 := \eta^{n-2} - \gamma$. Then we can again reproduce the calculations from Section 5.1 skipping just identities (5.17) and replacing $\gamma$ by $\eta^{n-2} - \gamma$ in all relations after (5.17). This gives the estimates:

\[
\| u_\epsilon - \Xi \tilde{u}_0 \|_{W^2_2(\Omega)} \leq C \left( \epsilon + \| \beta' - \beta \|_{\mathcal{M}} + (\epsilon \eta \mu_1)^{-\frac{1}{2}} + \epsilon \eta \mu_2 \right) \| f \|_{L^2(\Omega)},
\]

\[
\| u_\epsilon - \tilde{u}_0 \|_{L^2(\Omega)} \leq C \left( \epsilon + \| \beta' - \beta \|_{\mathcal{M}} + (\epsilon \eta \mu_1)^{-\frac{1}{2}} + \epsilon \eta \mu_2 \right) \| f \|_{L^2(\Omega)}.
\]

At the same time, it is easy to see that the solution of problem (5.56) is analytic in $\mu_3$, and for $\mu_3 = \gamma$, this solution coincides with the solution $u_0$ to homogenized problem (2.6). Hence, in the general situation, the next-to-leading term in the Taylor expansion of $\tilde{u}_0$ in $\mu_3 - \gamma$ is non-zero and the estimates

\[
\| u_0 - \tilde{u}_0 \|_{L^2(\Omega)} \leq C |\mu_3 - \gamma| \| f \|_{L^2(\Omega)}, \quad \| u_0 - \tilde{u}_0 \|_{W^2_2(\Omega)} \leq C |\mu_3 - \gamma| \| f \|_{L^2(\Omega)},
\]

are order sharp. In particular, we can calculate the norms in their left hand sides over the set $\{ x \in \Omega : \Xi_\epsilon(x) = 1 \}$, and this proves that the term $|\eta^{n-2} - \gamma|$ in estimates (2.24) and (2.25) are order sharp.

We proceed to checking the sharpness of the other terms in (2.24) and (2.26). We are going to do this by aduding an appropriate example. We let $\square := [-2, 2]^n$, $\omega := B_1(0)$ and $\Omega = \mathbb{R}^n$. The points $M^\epsilon_k$ are defined as $M^\epsilon_k := \epsilon k, k \in M^\epsilon := 4\mathbb{Z}^n$. In this case, we deal with a periodic perforation in $\Omega$. Each cavity is a ball of the radius $\epsilon \eta$ centered at a point $\epsilon k, k \in \mathbb{Z}^n$, and hence,

\[
\theta^\epsilon = \bigcup_{k \in \mathbb{Z}^n} B_\epsilon(k), \quad \Omega^\epsilon = \mathbb{R}^n \setminus \theta^\epsilon.
\]

The differential expression is chosen to be the negative Laplacian, $\mathcal{L} := -\Delta + 1$. It is clear that we can take $\lambda = \lambda_0 = 0$. We choose $R_1 = 1$, $R_2 := \frac{7}{6}, R_3 := \frac{3}{2}, R_4 := \frac{4}{3}$.

We consider the solution to the problem

\[
(\mathcal{L} + \eta^{n-2} - \gamma) \tilde{u}_0 = f \quad \text{in } \mathbb{R}^d
\]

and hence, as in the first part of section, by reproducing the arguing from Section 5.1, the solution $u_\epsilon$ to the corresponding equation in (2.4) satisfies the modified versions of estimates (2.24) and (2.25):

\[
\| u_\epsilon - \tilde{u}_0 \Xi_\epsilon \|_{W^2_2(\mathbb{R}^n \setminus \theta^\epsilon)} \leq C \left( \epsilon + (\epsilon \eta \mu_1)^{-\frac{1}{2}} + \epsilon \eta \mu_2 \right) \| f \|_{L^2(\Omega)},
\]

\[
\| u_\epsilon - \tilde{u}_0 \|_{L^2(\mathbb{R}^n \setminus \theta^\epsilon)} \leq C \left( \epsilon + (\epsilon \eta \mu_1)^{-\frac{1}{2}} + \epsilon \eta \mu_2 \right) \| f \|_{L^2(\Omega)}.
\]

This is why, to confirm the sharpness of the other terms in (2.24) and (2.25), we need to estimate from below the norms in the left hand sides of the above inequalities.
We first consider the case of only Dirichlet conditions on the boundaries of the cavities, that is, $\mathcal{M}_D^\epsilon = \emptyset$ and $\mathcal{M}_D^\epsilon = \mathcal{M}^\epsilon$. Such choice of the boundary conditions on $\partial \Omega^\epsilon$ removes the terms $(\epsilon \eta \mu_1 \kappa)^{-\frac{1}{2}}$ and $\epsilon \eta \kappa \mu_2$ from inequalities (5.57). The functions $X_{k,\epsilon}$ and $Z_{k,\epsilon}$ for the considered model can be found explicitly:

$$Z_{k,\epsilon}(\xi) = X_{k,\epsilon}(\xi) = \begin{cases} 1 - |\xi|^{-n+2}, & n \geq 3, \\ \ln |\xi|, & n = 2. \end{cases} \quad (5.58)$$

The corresponding function $\beta^\epsilon$ given by (2.19) then is $\epsilon \Box$-periodic and reads as $\beta^\epsilon(x) = \beta_0(x \epsilon^{-1})$, where

$$\beta_0(\xi) := \begin{cases} \frac{(n-2)}{R_\frac{1}{2}^2 \text{meas}_B(0)} & \text{on } B_R(0), \ k \in 4\mathbb{Z}^n, \ n \geq 3, \\
\frac{1}{R_\frac{1}{2}^2 \text{meas}_B(0)} & \text{on } B_R(0), \ k \in 4\mathbb{Z}^n, \ n = 2, \\
0 & \text{on } \mathbb{R}^n \setminus \bigcup_{k \in 4\mathbb{Z}^n} B_R(0). \end{cases} \quad (5.59)$$

By $Y_0 = Y_0(\xi)$, we denote the $\Box$-periodic solution to the following boundary value problem:

$$-\Delta_\xi Y_0 = \beta_0 \text{ on } \mathbb{R}^n \setminus 4\mathbb{Z}^n, \quad Y_0(\xi) = -\frac{1}{|\xi - k|^{n-2}} + O(|\xi - k|), \quad \xi \rightarrow k, \ n \geq 3, \quad Y_0(\xi) = \ln |\xi - k| + O(|\xi - k|), \quad \xi \rightarrow k, \ n = 2. \quad (5.60)$$

It is easy to see that this problem satisfies the standard solvability condition; the uniqueness of the solution is ensured by the order of the error terms in the prescribed asymptotics.

Given an arbitrary infinitely differentiable function $\tilde{u}_0 = \tilde{u}_0(x)$, we denote

$$f := (\mathcal{L} + \eta^{n-2} \kappa^{-2} \kappa^\epsilon \beta^\epsilon)\tilde{u}_0 \in C_0^\infty(\mathbb{R}^d), \quad U_\epsilon(x) := \left(1 + \eta^{n-2}(\epsilon) \kappa^{-1}(\epsilon) Y_0 \left(\frac{x}{\epsilon}\right)\right)\tilde{u}_0(x).$$

It is clear that

$$\|f\|_{L_2(\mathbb{R}^d)} \leq C\|\tilde{u}_0\|_{W_2^2(\mathbb{R}^d)}, \quad (5.61)$$

where a constant $C$ is independent of $\epsilon$, $\eta$, and $\tilde{u}_0$. It also follows from (5.60) that the function $U_\epsilon$ solves the boundary value problem:

$$\mathcal{L}U_\epsilon = f + f_\epsilon \in \mathbb{R}^n \setminus \theta^\epsilon, \quad U_\epsilon = \eta^{n-2} \kappa^{-1} \varphi_\epsilon^\theta \text{ on } \theta^\epsilon, \quad f_\epsilon(x) := \eta^{n-2}(\epsilon) \kappa^{-1}(\epsilon) Y_0 \left(\frac{x}{\epsilon}\right) \mathcal{L}\tilde{u}_0 - 2\eta^{n-2}(\epsilon) \kappa^{-1}(\epsilon) \nabla Y_0 \left(\frac{x}{\epsilon}\right) \cdot \nabla\tilde{u}_0(x), \quad (5.62)$$

$$\varphi_\epsilon^\theta(x) := \left(Y_0 \left(\frac{x}{\epsilon}\right) + \eta^{-n+2}(\epsilon) \kappa(\epsilon)\right)\tilde{u}_0(x).$$

By straightforward calculations, we confirm that

$$f_\epsilon(x) = \eta^{n-2}(\epsilon) \kappa^{-1}(\epsilon) f_{\epsilon,0}(x) - 2\eta^{n-2}(\epsilon) \kappa^{-1}(\epsilon) \sum_{j=1}^n \frac{\partial f_{\epsilon,j}(x)}{\partial x_j},$$

$$f_{\epsilon,0}(x) := Y_0 \left(\frac{x}{\epsilon}\right) (-\mathcal{L} + 2)\tilde{u}_0(x), \quad f_{\epsilon,j}(x) := Y_0 \left(\frac{x}{\epsilon}\right) \frac{\partial\tilde{u}_0}{\partial x_j}(x).$$

By $\chi_3 = \chi_3(\zeta, \eta)$, we denote an infinitely differentiable $\Box$-periodic cut-off function equalling to one as $|\zeta - k| \leq 2\eta$ and vanishing as $|\zeta - k| \geq 3\eta$ for $k \in 4\mathbb{Z}^n$ and obeying the uniform estimate $|\nabla_\zeta \chi_3(\zeta, \eta)| \leq C\eta^{-1}$ with a constant $C$ independent of $\zeta$ and $\eta$. The function
where \( \eta^m \) satisfies the boundary condition in (5.62). Then we consider the solution \( u_e \) to problem (2.4) with the introduced function \( f \), and in a standard way, we get the estimate:

\[
\| U_e - u_e \|_{W^2_2(\mathbb{R}^n, \rho')} \leq C \left( \eta^{n-2} x^{-1} \| \tilde{\varphi}_D^{\varepsilon} \|_{W^2_2(\mathbb{R}^n, \rho')} + \eta^{n-2} x^{-1} \| f_{e,0} \|_{L_{c,0}(\mathbb{R}^n, \rho')} + \eta^{n-2} x^{-1} \varepsilon \sum_{j=1}^n \| f_{e,j} \|_{L_2(\mathbb{R}^n, \rho')} \right), 
\]

where \( C \) is a fixed constant independent of \( \varepsilon, \eta, f_{e,0}, \tilde{\varphi}_D^{\varepsilon} \).

In view of convergence (2.5) and the asymptotics for \( Y_0 \) in (5.60) and the smoothness of this function, by routine straightforward calculations, we find that

\[
\eta^{n-2} x^{-1} \| \tilde{\varphi}_D^{\varepsilon} \|_{W^2_2(\mathbb{R}^n, \rho')} \leq C \eta^{-n} x^{-1} \leq C \varepsilon^2 \eta, 
\]

where \( C \) are some constants independent of \( \varepsilon \) and \( \eta \) but depending on the choice of the function \( \tilde{u}_0 \). In the same way, we find that

\[
\sum_{j=1}^n \| f_{e,j} \|_{L_2(\mathbb{R}^n, \rho')} \leq C \left\{ \begin{array}{ll}
1 + \eta^{\frac{1}{2} - \frac{1}{2}}, & n \neq 4, \\
1 + | \ln \eta |^{\frac{1}{2}}, & n = 4,
\end{array} \right.
\]

where \( C \) is a constant independent of \( \varepsilon \) and \( \eta \) but depending on \( \tilde{u}_0 \). We substitute these estimates and (5.64) into (5.63) and use convergence (2.5) to obtain:

\[
\| U_e - u_e \|_{W^2_2(\mathbb{R}^n, \rho')} \leq C \varepsilon + C \eta \left\{ \begin{array}{ll}
1, & n \neq 4, \\
| \ln \eta |^{\frac{1}{2}} + 1, & n = 4,
\end{array} \right.
\]

where \( C \) is some constant independent of \( \varepsilon \) but depending on \( \tilde{u}_0 \).

The function \( \Xi_e \) defined by (2.23) with the functions \( Z_k \) from (5.58) reads as

\[
\Xi_e(x) := \begin{cases} 
\frac{1 - \left| \frac{x}{\varepsilon} - k \right|^{-n+2} \eta^{-n}}{1 - \left( \frac{3n}{4} \right)^{-n+2}} & \text{in } B_{\frac{3}{2}^\varepsilon}(\varepsilon k) \setminus B_{\varepsilon}(\varepsilon k), \quad k \in 4\mathbb{Z}^n, \\
1 & \text{in } \mathbb{R}^n \setminus \bigcup_{k \in \mathbb{Z}^n} B_{\frac{3}{2}^\varepsilon}(\varepsilon k),
\end{cases}
\]

as \( n \geq 3 \), and

\[
\Xi_e(x) := \begin{cases} 
\frac{\ln \left| \frac{x}{\varepsilon} - k \right| - \ln \eta}{\ln \frac{3n}{4}} & \text{in } B_{\frac{3}{2}^\varepsilon}(\varepsilon k) \setminus B_{\varepsilon}(\varepsilon k), \quad k \in 4\mathbb{Z}^n, \\
1 & \text{in } \mathbb{R}^n \setminus \bigcup_{k \in \mathbb{Z}^n} B_{\frac{3}{2}^\varepsilon}(\varepsilon k),
\end{cases}
\]

as \( n = 2 \). As above, by straightforward calculations, we confirm that

\[
\| \nabla (U_e - \Xi_e \tilde{u}_0) \|_{L_2(\mathbb{R}^n, \rho')} \geq C \eta^{n-2} \varepsilon^{-1} x^{-1}, 
\]

(5.66)

\[
\| \nabla (U_e - \tilde{u}_0) \|_{L_2(\mathbb{R}^n, \rho')} \geq C \eta^{n-1} \varepsilon^{-1} x^{-\frac{1}{2}}, 
\]

(5.67)
provided $|u_0|$ is uniformly separated from zero on some fixed ball. Assuming that $\eta^2 \varepsilon^2 \chi^{-1} \equiv \gamma$, by (5.61), (5.65), and (5.66), we get

$$\|\nabla (U_\varepsilon - \Xi \tilde{u}_0)\|_{L^2(\mathbb{R}^n \setminus \Theta)} \geq C \eta^2 \varepsilon^2 \chi^{-1} \geq C \varepsilon \|f\|_{L^2(\mathbb{R}^d)}$$

with a constant $C$ independent of $\varepsilon$, and this proves the sharpness of the term $\varepsilon$ in the right hand side of (2.24).

As $\gamma = 0$, the solution $u_0$ to the equation

$$Lu_0 = f \text{ in } \mathbb{R}^d$$

obviously satisfies the estimate

$$\|\tilde{u}_0 - u_0\|_{W^2_2(\mathbb{R}^d)} \leq C \eta^2 \varepsilon^2 \chi^{-1}$$

and in view of (5.65), (5.67), and (2.5), we also see that the term $\eta^2 \varepsilon^2 \chi^{-1}$ in the right hand side of (2.26) is order sharp.

Let us show that the term $\varepsilon \eta \varepsilon \mu_2$ in (2.24) is order sharp. Here, we again consider the above example, but on the boundaries of the cavities, we impose the Robin condition

$$\frac{\partial u_\varepsilon}{\partial \nu} + (\varepsilon^{-1} \eta^{-1} + \mu_2) u_\varepsilon = 0 \text{ on } \partial \Theta', \quad a'(x,u) = (\varepsilon^{-1} \eta^{-1} + \mu_2) u, \quad b_k = 1.$$  

Such choice of the boundary conditions removes the term $(\varepsilon \eta \varepsilon \mu_2)^{-1}$ from (5.57). The functions $X_{k,\varepsilon}$ and $Z_{k,\varepsilon}$ can be again found explicitly

$$Z_{k,\varepsilon}(\xi) = X_{k,\varepsilon}(\xi) = \begin{cases} 1 - \frac{|\xi|^{-n+2}}{n-1}, & n \geq 3, \\ \ln |\xi| + 1, & n = 2. \end{cases}$$

The corresponding function $\beta'$ then again reads as $\beta'(x) = \beta_0(\varepsilon^{-1} x)$, where the function $\beta_0$ is defined by the formula

$$\beta_0(\xi) := \frac{(n-2)}{(n-1)^2} \meas_{k-1}(B_1(0))$$

and by the second and third formulae in (5.59). We suppose that $\eta^2 \varepsilon^2 \chi^{-1} \equiv 1$ for all $\varepsilon$.

By $u_\varepsilon$, we define the solution to problem (2.4) for an arbitrary $\mu_2$, while $u^{(0)}_\varepsilon$ is the solution to the same problem for $\mu_2 = 0$. Both these solutions converge to the same solution $u_0$ of the homogenized problem (2.6). The function $u_\varepsilon$ satisfies estimate (2.24), while the function $u^{(0)}_\varepsilon$ satisfies the same estimate with $\mu_2 = 0$. This is why, in order to prove that the term $\varepsilon \eta \varepsilon \mu_2$ is order sharp in estimate (2.24), it is sufficient to find an example of the function $f$ such that

$$\|u_\varepsilon - u^{(0)}_\varepsilon\|_{L^2(\mathbb{R}^d \setminus \Theta')} \geq C \eta \varepsilon \mu_2 \|f\|_{L^2(\mathbb{R}^d)}. \quad (5.68)$$

We choose an arbitrary infinitely differentiable compactly supported in $\mathbb{R}^n$ function $u_0 = u_0(x)$, and we choose the aforementioned function $f$ as $f := (L + \beta) u_0$. Then, we define

$$U_\varepsilon(x) := u^{(0)}_\varepsilon(x) + \varepsilon \eta^{-1}(\varepsilon \mu_2 \varepsilon) Y_2 \left(\frac{x}{\varepsilon}\right) u_0(x),$$

$$Y_2 := \frac{1}{n-1} Y_0, \quad n \geq 3, \quad Y_2 := Y_0 - \ln \eta, \quad n = 2. \quad (5.69)$$

This function solves the following boundary value problem:

$$LU_\varepsilon = f + f_\varepsilon \text{ in } \mathbb{R}^n \setminus \Theta', \quad \frac{\partial U_\varepsilon}{\partial \nu} + (\varepsilon^{-1} \eta^{-1} + \mu_2) U_\varepsilon = \varphi_\varepsilon \text{ on } \partial \Theta',$$
where

\[ f_ε(x) = f_{ε,0}(x) - 2 \sum_{j=1}^{n} \frac{∂ f_{ε,j}}{∂x_j}(x), \]

\[ f_{ε,0}(x) := \varepsilon \eta^{-1} \mu_2 Y_2 \left( \frac{x}{ε} \right) (-L + 2)u_0(x), \]

\[ f_{ε,j}(x) := \varepsilon \eta^{-1} \mu_2 Y_2 \left( \frac{x}{ε} \right) \frac{∂u_0}{∂x_j}(x), \]

\[ φ_R^ε := \varepsilon \eta^{-1} Y_2 \left( \frac{x}{ε} \right) \left( \frac{∂}{∂v} + \mu_2(ε) \right) u_0(x) + \varepsilon \eta^{-1} u_0(x) \left( \frac{∂}{∂v} + \varepsilon^{-1} η^{-1} \right) Y_3 \left( \frac{x}{ε} \right) + \mu_2(ε) \left( u_ε^{(0)}(x) - u_0(x) \right), \]

\[ Y_3(ζ) := \frac{1}{n-1} \left( Y_0(ζ) + η^{-n+2} \right), \quad n ≥ 3, \]

\[ Y_3(ζ) := Y_2(ζ), \quad n = 2. \]

Writing then a problem for \( U_ε - u_0 \) and an associated integral identity with \( U_ε - u_0 \) as the test function and using (3.8), we obtain an analogue of inequality (5.63):

\[ \| U_ε - u_0 \|_{W^2_2(\mathbb{R}^n \setminus Ω)} \leq C \left( \varepsilon \frac{1}{2} η \frac{1}{2} x \frac{1}{2} \| φ_R^ε \|_{L_2(Ω)} + \sum_{j=0}^{n} \| f_{ε,j} \|_{L_2(\mathbb{R}^n \setminus Ω)} \right), \quad (5.70) \]

where \( C \) is some constant independent of \( ε \), \( φ_R^ε \) and \( f_{ε,j} \), \( j = 0, \ldots, n \). Using (3.8), (3.14), the asymptotics for \( Y_0 \) in (5.60), and the definition of the function \( Y_3 \), we find

\[ \| φ_R^ε \|_{L_2(Ω)} \leq C \left( \varepsilon η x \mu_2 + η^{-1} \right) \| u_0 \|_{L_2(Ω)} + \varepsilon \eta \| ∇u_0 \|_{L_2(Ω)} + \frac{1}{2} \| u_ε^{(0)} - u_0 \|_{L_2(Ω)} \]

\[ \leq C(ηx)^{\frac{1}{2}} \left( \varepsilon η x \mu_2 + η \right) \| F \|_{L_2(Ω)} + \frac{1}{2} \| u_ε^{(0)} - u_0 \|_{W^2_2(\mathbb{R}^n \setminus Ω)}, \quad (5.71) \]

where \( C \) is some constant independent of \( ε \), \( u_0 \), \( u_ε^{(0)} \). The functions \( f_{ε,j} \) can be estimated as follows:

\[ \| f_{ε,j} \|_{L_2(\mathbb{R}^n \setminus Ω)} \leq C epsilon \mu_2(η^{-2} + ρ) \| u_0 \|_{C^2(\text{supp} u_0)}, \quad j = 0, \ldots, n, \]

where \( C \) is some constant independent of \( ε \) and \( u_0 \). This estimate and (5.71) and (5.70) yield

\[ \| U_ε - u_ε \|_{W^2_2(\mathbb{R}^n \setminus Ω)} \leq C(\varepsilon η x \mu_2 η^{-2} + ρ) \| u_0 \|_{C^2(\text{supp} u_0)}, \quad j = 0, \ldots, n, \]

(5.72)

where \( C \) is some constant independent of \( ε \), \( u_0 \), \( u_ε^{(0)} \). At the same time, it follows from definition (5.69) of \( U_ε \) that provided \( |u_0| \) is uniformly separated from zero on some fixed ball; the estimate

\[ \| U_ε - u_ε^{(0)} \|_{W^2_2(\mathbb{R}^n \setminus Ω)} \geq C x \eta^{-1} \mu_2 = C epsilon \mu_2 \]

holds with some fixed constant \( C \) independent of \( ε \), where we have also assumed that \( η^{\frac{1}{2}} ε^{-1} x^{-\frac{1}{2}} = 1 \). This estimate, (5.72), and (2.24) for \( u_ε^{(0)} - u_0 \|_{W^2_2(\mathbb{R}^n \setminus Ω)} \) prove (5.68), and hence, the term \( ε η x \mu_2 \) is order sharp in (2.24).

6 | CONVERGENCE IN || · ||_{W^N}-NORM

In this section, we discuss the convergence postulated in Assumption A.3. As a main tool of checking Assumption A.3, we propose the following way. We introduce one more space of multipliers \( \mathfrak{M} \), which consists of the functions \( F \) defined on \( Ω \) such that for each \( u \in \tilde{W}^1_2(Ω) \), the function \( Fu \) is a continuous antilinear functional on \( \tilde{W}^1_2(Ω) \). The norm in \( \mathfrak{M} \) is introduced as

\[ \| F \|_{\mathfrak{M}} = \sup_{u,v \in \tilde{W}^1_2(Ω)} \frac{|⟨Fu,v⟩|}{∥u∥_{\tilde{W}^1_2(Ω)}∥v∥_{\tilde{W}^1_2(Ω)}}. \]
It is clear that $\mathfrak{M} \subset \mathfrak{N}$ and

$$
\|F\|_{\mathfrak{M}} \leq \|F\|_{\mathfrak{N}}. \tag{6.1}
$$

Having this inequality in mind, instead of convergence in the space $\mathfrak{M}$ as it is postulated in Assumption A.3, we propose to check the convergence in the space $\mathfrak{M}$.

It was shown in [33] that the convergence in the space $\mathfrak{M}$ ensures that the operator in $L_2(\Omega)$ with the differential expression $L + \gamma \beta^\varepsilon Y$ subject to the Dirichlet condition converges in the norm resolvent sense to a similar operator but with the differential expression $L + \gamma Y$. The norm resolvent convergence is understood in the sense of the norms of the operators acting from $L_2(\Omega)$ into $W_2^1(\Omega)$.

The convergence in the sense of the norm $\| \cdot \|_{\mathfrak{M}}$ was studied in details in [33], and a simple criterion was established. Namely, we choose an arbitrary lattice $\Gamma$ in $\mathbb{R}^n$ with a periodicity cell $\square$. Given a positive function $\rho_1 = \rho_1(\varepsilon)$, we denote

$$
\Gamma_{\rho_1} := \{ z \in \Gamma : \rho_1 z + \rho_1 \square \subset \Omega \}. \tag{6.2}
$$

The mentioned criterion reads as follows: The function $\beta^\varepsilon$ converges to some function $\beta$ in $\| \cdot \|_{\mathfrak{M}}$-norm if and only if there exist functions $\rho_1 = \rho_1(\varepsilon)$, $\rho_2 = \rho_2(\varepsilon)$ such that

$$
\sup_{z \in \Gamma_{\rho_1}, \rho} \frac{1}{\rho_1(\varepsilon)} \int_{\rho_1(\varepsilon) + \rho_2(\varepsilon) \square} (\beta^\varepsilon(x) - \beta(x)) \, dx \leq \rho_2(\varepsilon), \quad \rho_1(\varepsilon) \to 0, \quad \rho_2(\varepsilon) \to 0, \quad \varepsilon \to +0. \tag{6.3}
$$

Therefore, Assumption A.3 can be guaranteed by condition (6.3), which is very explicit. If condition (6.3) is satisfied, by Theorem 2.4 from [33], we obtain the estimate

$$
\| \beta^\varepsilon - \beta \|_{\mathfrak{M}} \leq C(\rho_2 + \rho_1); \tag{6.4}
$$

hereinafter, in this section by $C$, we denote various constants independent of $\varepsilon$ and spatial variables.

In paper [33], a way for explicit calculation of function $\beta$ for a given $\beta^\varepsilon$ was provided. Namely, let $\omega \subset \mathbb{R}^n$ be a fixed domain and assume that the function

$$
\beta(x) := \lim_{\varepsilon \to +0} \frac{1}{\rho_3(\varepsilon) \text{meas}_n \omega} \int_{x + \rho_3(\varepsilon) \omega} \beta^\varepsilon(y) \, dy
$$

is well-defined in $\Omega$. If the limit in the above formula is uniform in $x$, namely,

$$
\sup_x \left| \beta(x) - \frac{1}{\rho_3(\varepsilon) \text{meas}_n \omega} \int_{x + \rho_3(\varepsilon) \omega} \beta^\varepsilon(y) \, dy \right| \leq \rho_4(\varepsilon), \quad \rho_4(\varepsilon) \to +0, \quad \varepsilon \to +0,
$$

where $\rho_3(\varepsilon)$, $\rho_4(\varepsilon)$ are some functions independent of $x$ and the supremum is taken over $x \in \Omega$ such $x + \rho_3(\varepsilon) \omega \subset \Omega$, then condition (6.3) is satisfied and

$$
\| \beta^\varepsilon - \beta \|_{\mathfrak{M}} \leq C \left( \rho_4 + \rho_3^{\frac{1}{\varepsilon}} \right). \tag{6.5}
$$

Paper [33] provides many particular examples of possible functions obeying condition (6.3) and all of them can be adapted also for our particular function $\beta^\varepsilon$. We do not reproduce here all these examples, but instead, we discuss a few close examples.

The first example is a sparsely distributed perforation. Here, we assume that there exists a function $\rho_5 = \rho_5(\varepsilon)$ such that

$$
\varepsilon \rho_3^{-1}(\varepsilon) \to +0, \quad \varepsilon \to +0, \quad B_{\rho_5}(M^\varepsilon_k) \cap B_{\rho_5}(M^\varepsilon_j) = \emptyset, \quad k \neq j. \tag{6.5}
$$
Then according to the example discussed in Section 3.2 in [33], condition (6.3) is satisfied with $\beta = 0$ and

$$\|\beta^e\|_{g^h} \leq C\left(\epsilon^n \rho_5^{-n} + \rho_5^2\right).$$

This estimate can be even improved for our particular case as the following lemma shows.

**Lemma 6.1.** Suppose that condition (6.5) holds. Then

$$\|\beta^e\|_{g^h} \leq C(\epsilon^n \rho_5^{-n} + \epsilon^2), \quad n \geq 3, \quad \|\beta^e\|_{g^h} \leq C(\epsilon^n \rho_5^{-n} + \epsilon^2 |\ln |\epsilon| |), \quad n = 2,$$

**Proof.** Given an arbitrary function $u \in W_2^1(B_{R_1}(M_k^r))$, by estimate (3.9) with $\omega_k = \emptyset$ and $(\epsilon, \eta)$ replaced by $(\rho_5, \epsilon^{-1})$, we obtain:

$$\|u\|^2_{L_2(B_{R_1}(M_k^r))} \leq C(\epsilon^n \rho_5^{-n} + \epsilon^2)^{\|u\|^2_{W_2^1(B_{R_1}(M_k^r))}}, \quad n \geq 3,$$

$$\|u\|^2_{L_2(B_{R_1}(M_k^r))} \leq C(\epsilon^n \rho_5^{-n} + \epsilon^2 |\ln |\epsilon| |)^{\|u\|^2_{W_2^1(B_{R_1}(M_k^r))}}, \quad n = 2,$$

where $C$ is a constant independent of $\epsilon$, $u$, and $k$. Hence, by the uniform boundedness of $\beta^e$ stated in Lemma 4.8,

$$\left|\langle\beta^e u, v\rangle_{L_2(B_{R_1}(M_k^r))}\right| \leq C(\epsilon^n \rho_5^{-n} + \epsilon^2)^{\|u\|^2_{W_2^1(B_{R_1}(M_k^r))}},$$

where $C$ is a constant independent of $\epsilon$, $u$, and $k$. Summing up this estimate over $k \in M^r$, we complete the proof. \( \square \)

The proven lemma says that if the distances between the cavities are much larger than $\epsilon$, then Assumption A.3 holds with $\beta = 0$. In particular, this is the case when we deal with cavities separated by finite distances.

The second situation describes a perforation, which can be regarded as a general perturbation of a periodically distributed perforations. We choose a fixed lattice $\Gamma$ in $\mathbb{R}^n$ with a periodicity cell $\Box$. Then we define the set $\Gamma$ by formula (6.2) with $\rho_1(\epsilon) = \epsilon$ and in each rescaled cell $\epsilon k + \epsilon \Box$, $k \in \Gamma$, we choose a point $M_k^r$ such that $B_{R_1}(M_k^r) \subset \epsilon k + \epsilon \Box$, $k \in \Gamma$. Then we arbitrary choose the corresponding cavities $\omega_{k,x}$, and in the case $n \geq 3$, we additionally assume that the constants $K_{k,x}$ satisfy the identity

$$2 - n)K_{k,x} = \Psi_x(M_k^r),\quad(6.6)$$

where $\Psi_x \in L_\infty(\Omega) \cap C(\bar{\Omega})$ is some family of functions such that

$$\rho_\theta(\epsilon) := \max_{x \in \Gamma} |\Psi_x(x) - \Psi_0(x)| \rightarrow +0, \quad \epsilon \rightarrow +0, \quad (6.7)$$

and $\Psi_0 \in L_\infty(\Omega) \cap C(\bar{\Omega})$ is some uniformly continuous in $\bar{\Omega}$ function, namely,

$$\rho_\Psi(\epsilon) := \max_{k \in M^r} \max_{x,y \in \epsilon \Box + k} |\Psi(x) - \Psi_0(y)| \rightarrow +0, \quad \epsilon \rightarrow +0. \quad (6.8)$$

We stress that condition (6.6) is imposed only on the constants $K_{k,x}$ and not on the shapes of the corresponding cavities. This means that the cavities corresponding to different $k$ are not necessarily of the same shapes even if the constants $K_{k,x}$ coincide. In the case $n = 2$, we let $\Psi_x(x) := \Psi_0(x) \equiv 1$. These conditions ensure (6.3) with $\rho_1(\epsilon) = \epsilon$ and

$$\rho_1(\epsilon) = \epsilon, \quad \beta = \frac{\Psi_0}{\Box},$$

Indeed,

$$\int_{\epsilon \Box + k} (\beta^e(x) - \beta(x)) \, dx = \epsilon^n \psi_1(M_k^r) - \frac{1}{\Box} \int_{\epsilon \Box + k} \psi_0(x) \, dx = \int_{\epsilon \Box + k} \frac{\psi_1(M_k^r) - \psi_0(M_k^r)}{\Box} \, dx + \int_{\epsilon \Box + k} \frac{\psi_0(M_k^r) - \psi_0(x)}{\Box} \, dx.$$
The right hand side of the above identity can be estimated by means of the functions $\rho_6$ and $\rho_7$ introduced in (6.7) and (6.8), and this yields

$$\frac{1}{\epsilon^n} \left| \int_{|x|\leq \epsilon R} (\beta^e - \beta) \, dx \right| \leq \rho_6 + \rho_7.$$  

This is exactly condition (6.3) for our case, and by (6.4), we obtain

$$\| \beta^e - \beta \|_{\mathcal{M}} \leq C(\epsilon + \rho_6 + \rho_7), \quad n \geq 3, \quad \| \beta^e - \beta \|_{\mathcal{M}} \leq C\epsilon, \quad n = 2.$$  

Our next step is to show how to generate new perforations obeying Assumption A.3 if we are given one already obeying this assumption. The first way is provided by the following lemma.

**Lemma 6.2.** Let a perforation described by the points $M^e_k$ and cavities $\omega_{k,\epsilon}$ obey Assumption A.3. Let $\tilde{M}^e_k, k \in \mathcal{M}^e$, be another set of points satisfying Assumption A.1 with the same constants $R_i, i = 1, \ldots, 4$, as for $M^e_k$, such that

$$|M^e_k - \tilde{M}^e_k| \leq C\epsilon$$

with some constant $C$ independent of $k$ and $\epsilon$. Then the function $\tilde{\beta}_e$ corresponding to the perforation described by the points $\tilde{M}^e_k$ and the same cavities $\omega_{k,\epsilon}$ also obeys Assumption A.3 with the same function $\beta$ and the estimate holds:

$$\| \tilde{\beta}_e - \beta \|_{\mathcal{M}} \leq \| \beta^e - \beta \|_{\mathcal{M}} + C\epsilon^\frac{1}{2},$$

where $C$ is a constant independent of $\epsilon$.

**Proof.** It is clear that

$$\| \tilde{\beta}_e - \beta \|_{\mathcal{M}} \leq \| \beta^e - \beta \|_{\mathcal{M}} + \| \beta^e - \tilde{\beta}_e \|_{\mathcal{M}}$$

and this is why it is sufficient to estimate just the right term in the right hand side of this inequality. We are going to do this by means of condition (6.3). Namely, we let $\rho_1(\epsilon) := \epsilon^\frac{1}{2}$. Then the integral in (6.3) can be rewritten as a sum of the integrals over the balls $B_{\epsilon R_i}(M^e_k)$ and $B_{\epsilon R_i}(\tilde{M}^e_k)$. If for some $k$ both these balls are contained in the cell $\epsilon^\frac{1}{2}z + \epsilon^\frac{1}{2}\partial$, then their contributions to the total integral cancel out just due to the definitions of the points $M^e_k$ and of the function $\tilde{\beta}_e$. Hence, only the balls $B_{\epsilon R_i}(M^e_k)$ and $B_{\epsilon R_i}(\tilde{M}^e_k)$ intersecting with the boundary $\epsilon^\frac{1}{2}z + \epsilon^\frac{1}{2}\partial$ contribute to the considered integral. Then the total number of such balls is proportional to the measure of this boundary, which is of order $\sim \epsilon^\frac{4}{3}$, and the total measure of such balls is obviously estimated by $C\epsilon^\frac{2}{3}$ with some fixed constant $C$. Since the functions $\beta^e$ and $\tilde{\beta}_e$ are uniformly bounded, see Lemma 4.8, we then get the estimate

$$\frac{1}{\epsilon^\frac{1}{2}} \int_{\epsilon^\frac{1}{2}z + \epsilon^\frac{1}{2}\partial} (\beta^e(x) - \beta(x)) \, dx \leq C\epsilon^\frac{1}{2}$$

and we arrive at (6.3) with $\rho_2(\epsilon) = \epsilon^\frac{1}{2}$. Employing then estimate (6.4) for $\| \beta^e - \tilde{\beta}_e \|_{\mathcal{M}}$ and (6.1), we complete the proof. \[\square\]

The proven lemma shows that given a perforation obeying Assumption A.3, we can shift the points $M^e_k$ by the distance of order $O(\epsilon)$ provided the new points satisfy Assumption A.1. This gives an easy way to generate many new non-periodic perforations from a given one keeping Assumption A.3 satisfied.

The second way of generating new perforations obeying Assumption A.3 is as follows. Suppose that we are given two perforations described by $M^e_k, \omega_{k,\epsilon}, k \in \mathcal{M}^e$, and $\tilde{M}^e_k, \tilde{\omega}_{k,\epsilon}, k \in \tilde{\mathcal{M}}^e$. Let these perforations satisfy Assumption A.3, respectively, with the functions $\beta^e, \beta$ and $\tilde{\beta}_e, \tilde{\beta}$. Consider then the union of these perforations formed by the unions of the points and cavities $M^e_k \cup \tilde{M}^e_j, \omega_{k,\epsilon}, \omega_{j,\epsilon}, k \in \mathcal{M}^e, j \in \tilde{\mathcal{M}}^e$, and let this union of the perforations satisfy Assumption A.1.
Then function (2.19) corresponding to this union of the perforations is \( \beta^\varepsilon + \hat{\beta}_\varepsilon \), and it satisfies Assumption A.3 with the limiting function \( \beta + \hat{\beta} \) thanks to the following simple estimate:

\[
\| \beta^\varepsilon + \hat{\beta}_\varepsilon - \beta - \hat{\beta} \|_B \leq \| \beta^\varepsilon - \beta \|_B + \| \hat{\beta}_\varepsilon - \hat{\beta} \|_B.
\]

It is also possible to remove some cavities from a given perforation keeping at the same time Assumption A.3 and this is our third way of producing new perforations. Namely, given a perforation described by the points and cavities \( M^\varepsilon_k, \omega_k^\varepsilon, k \in M^\varepsilon \) and obeying Assumption A.3, suppose that there is a subset \( \hat{M}^\varepsilon_k \subset M^\varepsilon_k \) such that the corresponding perforation satisfies Assumption A.3 with \( \beta = 0 \); the associated function (2.19) is denoted by \( \hat{\beta}_\varepsilon \). Then we consider a difference of perforations corresponding to \( M^\varepsilon_k \setminus \hat{M}^\varepsilon_k \), and we see that its function (2.19) is \( \beta^\varepsilon - \hat{\beta}_\varepsilon \). Hence,

\[
\| \beta^\varepsilon - \hat{\beta}_\varepsilon - \beta \|_B \leq \| \beta^\varepsilon - \beta \|_B + \| \hat{\beta}_\varepsilon \|_B, \quad \| \hat{\beta}_\varepsilon \|_B \rightarrow +0, \ \varepsilon \rightarrow +0,
\]

and the introduced difference of perforations also satisfies Assumption A.3 with the same function \( \beta \).

The fourth way of producing new perforations is to vary the shapes of the cavities. In the dimension \( n = 2 \), the function \( \beta^\varepsilon \) is independent on the shapes of the cavities, and we therefore have a very rich freedom in choosing the shapes of the cavities. As \( n \geq 3 \), the shapes of the cavities are reflected in the constants \( K_k^\varepsilon, \varepsilon \). Then, given a perforation obeying Assumption A.3 with functions \( \beta^\varepsilon \) and \( \beta \), one can deform slightly the shapes of the cavities so that new constants \( \hat{K}^\varepsilon_k \) differ from \( K^\varepsilon_k \) by a small quantity, namely, \( |K^\varepsilon_k - \hat{K}^\varepsilon_k| \leq \rho_8(\varepsilon) \), where \( \rho_8(\varepsilon) \rightarrow +0 \) as \( \varepsilon \rightarrow +0 \). Then it is clear that the new function \( \hat{\beta}_\varepsilon \) corresponding to the constants \( \hat{K}^\varepsilon_k \) satisfies the estimate

\[
\| \hat{\beta}_\varepsilon - \beta \|_B \leq \| \beta^\varepsilon - \beta \|_B + \rho_8(\varepsilon),
\]

which means that Assumption A.3 holds also for modified perforation.

**AUTHOR CONTRIBUTIONS**

Denis I. Borisov: Conceptualization; investigation; funding acquisition; writing—original draft; methodology; validation; visualization; writing—review and editing.

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**CONFLICT OF INTEREST STATEMENT**

The author declares that he has no conflicts of interest.

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