A BCS Gap on the Lattice

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Monte Carlo simulations of the 3+1 dimensional NJL model are performed with baryon chemical potential \( \mu > 0 \). For \( \mu > \Sigma_0 \), the constituent quark mass in vacuum, chiral symmetry is restored and a diquark condensate \( \langle qq_+ \rangle \) forms. We analyse the fermion propagator and find evidence for particle-hole mixing in the vicinity of the Fermi surface and an energy gap \( \Delta > 0 \), both of which provide evidence for superfluidity at high baryon density induced by a BCS mechanism. At \( \mu a = 0.8 \) the ratio between the BCS gap and the vacuum quark mass is \( \Delta / \Sigma_0 = 0.15(2) \).

The Nambu–Jona-Lasinio (NJL) model has served for many years as a model of strong interactions [1]; in particular with the addition of a non-zero baryon chemical potential \( \mu \) it can be used to model the relativistic quark matter anticipated in dense systems such as neutron star cores. Phenomena such as the BCS-condensation of quark pairs at the Fermi surface, suggested as a mechanism for color superconductivity and baryon number superfluidity, can be modelled in this way [2]. From a lattice theorist’s perspective, the attractive feature of the NJL model is that its action in Euclidean metric remains real even once \( \mu \neq 0 \), so that standard Monte Carlo simulation techniques can be employed. Moreover, such simulations expose a chiral symmetry restoration transition at a critical \( \mu_c \approx \Sigma_0 \), where \( \Sigma_0 \) is the constituent quark scale, in qualitative agreement with analytic approaches such as the Hartree approximation [1].

In previous work [3] we have simulated the NJL model in an attempt to identify diquark condensation for \( \mu > \mu_c \). In continuum notation the Lagrangian is

\[
\mathcal{L} = \bar{\psi} \left( \partial \! \! \! / + m + \mu \gamma_0 \right) \psi - \frac{g^2}{\pi} \left[ (\bar{\psi}\psi)^2 - (\bar{\psi} \gamma_5 \otimes \tau \otimes \gamma_5 \psi)^2 \right] + \frac{1}{2} \left[ (\bar{\psi}, \psi^{\text{tr}}) \left( \begin{array}{c} \bar{j} \\ j \end{array} \right) C \gamma_5 \otimes \tau_2 \otimes C \gamma_5 \left( \begin{array}{c} \bar{\psi}^{\text{tr}} \\ \psi \end{array} \right) \right].
\]

When formulated using staggered fermions, the resulting lattice model has \( N_f = 2 \) isospin degrees of freedom, and \( N_c = 8 \) global ‘colors’ (there are no gluons). The third term in (1) is a U(1)\(_B\)-violating source term which enables the identification of a superfluid diquark condensate \( \langle qq_+ \rangle \) for source strengths \( j = \bar{j} = j^* \neq 0 \) on a finite volume.

At \( \mu = 0 \) the model exhibits breaking of SU(2)\(_L\) \( \otimes \) SU(2)\(_R\) into SU(2)\(_I\) by the spontaneous generation of a chiral condensate and a constituent quark mass \( \Sigma_0 \gg m \). Since in 3+1 dimensions, the NJL model is only an effective theory, the lattice parameters must be chosen to match low energy phenomenology. Using the large-\( N_c \) (Hartree) approximation we determine a suitable set, leading to physically reasonable results, to be

\[
\begin{align*}
ma &= 0.006 & \Sigma_0 &= 400 \text{MeV} \\
\frac{a^2 g^{-2}}{2} &= 0.495 & f_\pi &= 93 \text{MeV} \\
\frac{a^{-1}}{2} &= 720 \text{MeV} & m_\pi &= 138 \text{MeV}.
\end{align*}
\]

We use a standard HMC algorithm with a quark kinetic matrix \( M \) which is a functional of auxiliary boson fields \( \sigma \) and \( \pi \). The source strength \( j \) is set to zero in the HMC update, but is allowed to vary over a range of values in partially quenched measurements on valence quarks. If we define an antisymmetric matrix

\[
A = \frac{1}{2} \left( \begin{array}{cc}
\bar{j} \tau_2 & M \\
-M^{\text{tr}} & j \tau_2
\end{array} \right),
\]

\[\tag{3}\]
then e.g. the diquark condensate is given by
\[ \langle qq^+ \rangle = \frac{1}{2V} \frac{\partial \ln Z}{\partial j} = \frac{1}{8V} (\text{tr} \tau_2 A^{-1}). \] (4)

The results show that at \( \mu_a \approx 0.6 \) there is a transition at which the chiral condensate decreases sharply until by \( \mu a = 0.8 \) it has just 5% of its vacuum value, and at the same point the baryon density \( n_B = (2V)^{-1} \partial \ln Z/\partial \mu \) begins to rise from zero. Results for \( \langle qq^+ \rangle \), first extrapolated linearly in \( L_t^{-1} \) to \( T = 0 \) from \( 12^4, 16^4 \) and \( 20^4 \) lattices, and then linearly in \( j \to 0 \) using data with \( j a \in (0.3, 1.0) \), suggest that a superfluid condensate forms for \( \mu a \gtrsim 0.6 \) and then increases monotonically until \( \mu a \approx 1.0 \), where saturation artifacts become apparent 3.

In a system with a Fermi surface, low-energy excitations have \( k = |\vec{k}| \) close to the Fermi momentum \( k_F \). If a BCS condensate forms, we expect an energy gap \( 2\Delta \) to open up between the highest occupied state in the Fermi sea and the lowest excited state. These excitations are probed by standard means via temporal decay of the Euclidean propagator \( \mathcal{G} \equiv A^{-1} \). We write
\[ \mathcal{G}(x,y) = \begin{pmatrix} A(x,y) & N(x,y) \\ N(x,y) & A(x,y) \end{pmatrix}, \] (5)
where we identify both “normal” \( N(x,y) \sim \langle q(x)\bar{q}(y) \rangle \) and “anomalous” \( A(x,y) \sim \langle q(x)\bar{q}(y) \rangle \) propagators. In an isospin-symmetric ground state the only independent components to survive the quantum average are \( N \equiv \text{Re}N_{11} \) and \( A \equiv \text{Im}A_{12} \), where subscripts label isospin components 4. We proceed by studying the time-slice propagator \( \mathcal{G}(\vec{k},t) = \sum_{\vec{r}} \mathcal{G}(0,0;\vec{x},t)e^{-i\vec{k} \cdot \vec{x}} \) on a \( 96 \times 12^2 \times L_t \) lattice (\( L_t = 12, 16, 20 \) and \( 24 \)) with \( \vec{k} = (\pi n/48, 0, 0) \), \( n = 0, 1, \ldots, 24 \). The resulting propagators are fitted with
\[ N(k,t) = A e^{-Et} + B e^{-E(L_t-t)} \]
\[ A(k,t) = C e^{-Et} - e^{-E(L_t-t)} \] (6)
and amplitudes \( A, B, C \), and excitation energy \( E(k, j) \) extracted.

Fig. 1 shows the pole-fit amplitudes as functions of \( k \). For small \( k \) the normal propagator is predominantly forward-moving, since \( A \gg B \). For \( k \gtrsim \mu \), however, the signal becomes predominantly backward-moving. This is a sign that the excitations for small \( k \) are hole-like, and for large \( k \) particle-like 4. Of particular interest, however, is that in this case even in the limit \( j \to 0 \) there is a region of \( k \approx \mu \) where the anomalous amplitude \( C \neq 0 \). This shows that there are propagating states which are particle-hole superpositions with indefinite baryon number, an indirect signal for superfluidity via a BCS mechanism.

In Fig. 2 we show the dispersion relation \( E(k) \) extracted from the anomalous propagator at \( \mu a = 0.8 \). It is important to note the order of extrapolation: first \( T \to 0 \), then \( j \to 0 \). For clarity we have plotted energies for \( k < k_F \) as negative, where the Fermi momentum \( k_F \) is chosen to be the point where \( A = B \) in Fig. 1. We have also plotted the free lattice fermion dispersion relation, which has \( E(\sin^{-1}(\sinh(\mu a)) = 0 \). The interacting data, however, show no sign of a zero energy state; instead there is a a discontinuity at \( k \approx k_F \) signalling an energy gap \( 2\Delta \). This is the first direct observation of a BCS gap in a lattice simulation.

Finally in Fig. 3 we plot both order parameter \( \langle qq^+ \rangle \) and gap \( \Delta \) as functions of \( \mu \). Note that a re-
liable extrapolation to $T \to 0$ requiring data from $L_t = 24$ lattices is only available for $\mu a = 0.8$; simple linear extrapolation through $L_t = 16$ and 20 is completely consistent with this, however, and hence is used for the other $\mu$ values. One physical interpretation of the order parameter is that it counts the number of $qq$ pairs participating in the condensate. On the simple-minded assumption that these pairs occupy a shell of thickness $O(\Delta)$ around the Fermi surface, we surmise the relation

$$\langle qq_+ \rangle \propto \Delta \mu^2.$$  

(7)

Fig. 3 indeed suggests that $\Delta$ is roughly constant for $\mu > \mu_c$, while $\langle qq_+ \rangle$ rises markedly. The ratio $\Delta/\Sigma_0 \approx 0.15$, which using (2) translates into a physical prediction $\Delta \approx 60$MeV, consistent with the model predictions of [2].

In summary, by presenting numerical estimates for both $\langle qq_+ \rangle$ and $\Delta$ we have amassed a reasonable body of evidence for superfluidity via a BCS mechanism in a relativistic quantum field theory, for the first time using a systematic calculational technique. In future work we hope to address the issue of finite volume effects, unusually large in this system [3], which we believe is due to the difficulty of representing a curved shell of states around the Fermi surface on a discrete momentum lattice. It will also be interesting to study the stability of the superfluid phase for $T > 0$, and for $\mu_u \neq \mu_d$, which has the effect of separating the Fermi surfaces of each flavor and hence suppressing isosinglet pairing.

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