COUNTING FINITE LANGUAGES BY TOTAL WORD LENGTH

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Abstract. We investigate the number of sets of words that can be formed from a finite alphabet, counted by the total length of the words in the set. An explicit expression for the counting sequence is derived from the generating function, and asymptotics for large alphabet respectively large total word length are discussed. Moreover, we derive a Gaussian limit law for the number of words in a random finite language.

1. Introduction and Basic Properties

Let \( f_n = f_n(m) \) denote the number of languages (i.e., sets of words) with total word length \( n \) over an alphabet with \( m \geq 2 \) symbols \([1, I.37]\). For instance, \( f_2(2) = 5 \) and \( f_3(2) = 16 \), as seen from the listings \{a, b\}, \{aa\}, \{ab\}, \{ba\}, \{bb\} respectively \{a, aa\}, \{a, ab\}, \{a, ba\}, \{a, bb\}, \{aab\}, \{aba\}, \{a, ba\}, \{a, bb\}, \{bab\}, \{bba\}, \{bbb\}.

Another value is \( f_2(3) = 12 \), illustrated by \{aa\}, \{ab\}, \{ac\}, \{ba\}, \{bb\}, \{bc\}, \{ca\}, \{cb\}, \{cc\}, \{a, b\}, \{a, c\}, \{b, c\}.

The sequence \( f_n(2) \) is number A102866 of Sloane’s On-Line Encyclopedia of Integer Sequences\(^1\). In the present note, we will derive an explicit expression for \( f_n(m) \) (Theorem 1 below), establish asymptotics (Sections 2–4), and derive a limit law for the number of words in a random finite language (Section 5).

The ordinary generating function (ogf) \([1, I.37]\)

\[
F(z) := \sum_{n=0}^{\infty} f_n z^n = \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{mz^k}{1-mz^k} \right)
\]

(1)

can be obtained by a standard procedure (the “power set construction” \([1, I.2]\); finite languages are sets of sequences built from alphabet elements). Its first terms are

\[
F(z) = 1 + mz + \frac{1}{2} m (3m - 1) z^2 + m (\frac{1}{6} m^2 - \frac{1}{2} m + \frac{1}{3}) + O(z^4).
\]

(2)

Note that

\[
F(z) = \exp \left( \frac{mz}{1-mz} \right) \phi(z),
\]

where

\[
\phi(z; m) = \phi(z) := \exp \left( \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} \frac{mz^k}{1-mz^k} \right)
\]

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\(^1\)http://www.research.att.com/~njas/sequences/
is analytic for \( |z| < 1/\sqrt{m} \). (Indeed, for \( 0 < \varepsilon < 1/\sqrt{m} \), \( |z| \leq 1/\sqrt{m} - \varepsilon \), and \( k \geq 2 \), we have
\[
m|z|^k \leq m|z|^2 \leq m(m^{-1/2} - \varepsilon)^2 = 1 - \varepsilon \sqrt{m}(2 - \varepsilon \sqrt{m}) = 1 - \varepsilon',
\]
whence
\[
\left| m z^k \frac{1}{1 - m z^k} \right| \leq m|z|^k.
\]
The dominating singularity of \( F(z) \) is thus located at \( z = 1/m \), leading to the rough approximation \( f_n(m) \approx m^n \). Clearly (consider languages consisting only of one word), we have \( f_n(m) > m^n \) for \( m, n \geq 2 \). We will see in Theorem 3 below that the ratio \( f_n(m)/m^n \) is \( e^{\sqrt{m}+O(\log n)} \).

Our first result is an explicit expression for \( f_n(m) \), which can be obtained from (1). To state it, we write \( i + n \), if the vector \( i = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n \) represents a partition of \( n \), in the sense that \( i_1 + 2i_2 + \cdots + ni_n = n \).

**Theorem 1.** For \( m \geq 2 \) and \( n \geq 1 \), we have
\[
f_n(m) = \sum_{i\vdash n} \frac{A_1(m)^{i_1} \cdots A_n(m)^{i_n}}{i_1! \cdots i_n!},
\]
where
\[
A_j(m) := \sum_{d|i} (-1)^{d-1} m^{j/d} / d, \quad j \geq 1.
\]

**Proof.** We expand the Lambert series [4] in the exponent of \( F(z) \), using the geometric series formula:
\[
F(z) = \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{j=1}^{\infty} m^j z^k \right) = \exp \left( \sum_{n=1}^{\infty} A_n(m) z^n \right) = 1 + \frac{1}{1!} \left( \sum_{n=1}^{m} A_n(m) z^n \right) + \frac{1}{2!} \left( \sum_{n=1}^{m} A_n(m) z^n \right)^2 + \ldots
\]
The \( k \)-th term here can be expanded as
\[
\left( \sum_{n=1}^{m} A_n(m) z^n \right)^k = \sum_{i_1 + \cdots + i_m = k} \binom{k}{i_1, \ldots, i_m} (A_1 z)^{i_1} (A_2 z^2)^{i_2} \ldots (A_m z^m)^{i_m}
\]
\[
= \sum_{i_1 + 2i_2 + \cdots + mi_m + k \leq m} \binom{k}{i_1, \ldots, i_m} A_1^{i_1} \cdots A_m^{i_m} z^{i_1 + 2i_2 + \cdots + mi_m} + O(z^{m+1})
\]
\[
= \sum_{n=1}^{m} \left( \sum_{i_1 + \cdots + i_n = k} \frac{k}{i_1, \ldots, i_n} A_1^{i_1} \cdots A_n^{i_n} \right) z^n + O(z^{m+1}).
\]
Now (5) follows from (4) and (3).

2. ASYMPTOTICS FOR LARGE ALPHABET SIZE

Next we derive the asymptotics of \( f_n(m) \) as \( m \), the cardinality of the alphabet, tends to infinity. Define \( \kappa_n \) and \( \mu_n = \mu_n(m) \) by
\[
\sum_{n=0}^{\infty} \kappa_n z^n = \exp \left( \frac{z}{1 - z} \right) \quad \text{and} \quad \sum_{n=0}^{\infty} \mu_n z^n = \phi(z).
\]
Note that \( n! \kappa_n \) is Sloane’s A000262 (several combinatorial interpretations are given on that web page), and that \( \kappa_n \) has the representation

\[
\kappa_n = \sum_{i_1 + \cdots + i_n = n} \frac{1}{i_1! \cdots i_n!}, \quad n \geq 1.
\]

Then we can write

\[
f_n/m^n = [z^n] \exp \left( \frac{z}{1-z} \right) \phi(z/m) = \kappa_n + \kappa_{n-1}\mu_1/m + \cdots + \kappa_0\mu_n/m^n.
\]

If the dependence of \( \mu_n \) on \( m \) is not too strong, the first term on the right-hand side should dominate when \( m \to \infty \). This is indeed the case:

**Theorem 2.** If \( n \geq 1 \) is fixed and \( m \to \infty \), we have

\[
f_n(m) \sim \kappa_n m^n.
\]

**Proof.** Since, as \( m \to \infty \),

\[
A_j(m) = m^j + O(m^{j/2}), \quad j \geq 1,
\]

we have

\[
A_j(m)^{i_j} = m^{i_j} (1 + O(m^{-j/2})),
\]

whence, for \( i \vdash n \),

\[
A_1(m)^{i_1} \cdots A_n(m)^{i_n} = m^n (1 + O(m^{-1/2})).
\]

The result thus follows from (3) and (6). \( \square \)

Note that \( \kappa_1 = 1 \), \( \kappa_2 = \frac{3}{2} \), and \( \kappa_3 = \frac{13}{6} \), in line with (2).

### 3. Asymptotics for Large Total Word Length

**Theorem 3.** For large total word length \( n \), the sequence \( f_n = f_n(m) \) has the asymptotics

\[
f_n \sim \phi(1/m) \frac{m^n e^{2\sqrt{\pi}/n^{3/4}}}{2\sqrt{\pi}} \quad \text{as} \quad n \to \infty.
\]

More precisely, there is a full asymptotic expansion of the form

\[
f_n \sim \phi(1/m) \frac{m^n e^{2\sqrt{\pi}/n^{3/4}}}{2\sqrt{\pi}} \left( 1 + \sum_{j \geq 1} c_j n^{-j/4} \right), \quad n \to \infty.
\]

**Proof.** The proof of Theorem 3 is similar to the saddle point analysis [1] Example VIII.7] of \( \exp(z/(1-z)) \), the ogf of \( \kappa_n \), slightly perturbed by the presence of the factor \( \phi(z) \). The ogf \( F(z) \) is actually Hayman-admissible [1] [2], but carrying out the saddle point method explicitly gives access to a full asymptotic expansion, and will be useful for the refined expansions required in Sections 4 and 5. Let us shift the dominating singularity from \( z = 1/m \) to \( z = 1 \). Then the integrand in Cauchy’s formula

\[
f_n = f_n(m) = \frac{m^n}{2\pi i} \oint \frac{F(z/m)}{z^{n+1}} \, dz
\]

has an approximate saddle point at \( z = \hat{z} := 1 - 1/\sqrt{n} \). We write \( z = \hat{z} e^{i\theta} \), where \( \theta = \arg(z) \) is constrained by

\[
|\theta| < n^{-\alpha}, \quad \frac{3}{4} < \alpha < \frac{3}{2},
\]
so that $z$ lies in a small arc around the saddle point. In this range we have the uniform expansion
\[
z^{-n-1} = \exp \left( -(n+1) \log \left( 1 - \frac{1}{\sqrt{n}} \right) - i\theta + O(n^{-\alpha}) \right)
\]
(12)
\[
= \exp \left( \sqrt{n} + \frac{1}{2} - i\theta + O(n^{-1/2}) \right), \quad n \to \infty.
\]
Furthermore
\[
1 - z = n^{-1/2} (1 - i\theta \sqrt{n} + O(n^{-\alpha})),
\]
hence
\[
(13)
\]
\[
\frac{1}{1 - z} = \sqrt{n} + i\theta n - n^{3/2} \theta^2 + O(n^{1/2 - \alpha}).
\]
From (12) and (13) we get
\[
z^{-n-1} \exp \left( \frac{z}{1 - z} \right) = \exp \left( -\frac{1}{2} + 2\sqrt{n} - n^{3/2} \theta^2 \right) \times (1 + O(n^{1/2 - \alpha})).
\]
Since $\phi(z/m)$ is analytic at $z = 1$, the local expansion of the integrand in (10) at the saddle point $\hat{z}$ is
\[
(14)
\]
\[
F(z/m)_{z^{n+1}} = \phi(1/m) \exp \left( -\frac{1}{2} + 2\sqrt{n} - n^{3/2} \theta^2 \right) \times (1 + O(n^{1/2 - \alpha})),
\]
valid as $n \to \infty$, uniformly w.r.t. $\theta$ in the range (11). Note that
\[
\int_{-\pi}^{\pi} e^{-n^{3/2} \theta^2} d\theta \sim \sqrt{\pi} n^{-3/4},
\]
so that integrating (14) from $-n^{-\alpha}$ to $n^{-\alpha}$ yields the right-hand side of (8). To prove (8), it remains to show that the integral from $n^{-\alpha}$ to $\pi$ grows slower (the other half of the tail is handled by symmetry). There is a $C > 0$ such that
\[
(15)
\]
\[
\left| \frac{F(z/m)}{z^{n+1}} \right| \leq C |z|^{-n} \exp \Re \left( \frac{1}{1 - z} \right), \quad |z| < 1.
\]
If $z = \hat{z} e^{i\theta}$ lies on the integration contour, then the factor $|z|^{-n}$ in (15) is $O(e^{\sqrt{n}})$. The remaining factor $\exp \Re (1/(1 - z))$ decreases if $|\theta| = |\arg(z)|$ increases, hence
\[
\int_{-\pi}^{\pi} \exp \Re \left( \frac{1}{1 - \hat{z} e^{i\theta}} \right) d\theta \leq \pi \exp \Re \left( \frac{1}{1 - \hat{z} e^{i\theta}} \right)_{\theta=n^{-\alpha}}
\]
\[
= \exp \left( \sqrt{n} - n^{3/2 - 2\alpha} + O(1) \right).
\]
(The last line is obtained by recapitulating the derivation of (14), with $\theta = n^{-\alpha}$.) Hence
\[
(16)
\]
\[
\left| \oint_{|z|^{n^{-\alpha}} < \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot ...)
Taking more terms in (13) and in the expansion of the analytic function $\phi$, we readily see that the full local expansion of the integrand in (10) is of the form

$\frac{F(z/m)}{z^{n+1}} = \phi(1/m) \exp\left(-\frac{1}{2} + 2\sqrt{n} - n^{3/2}\theta^2\right) \times \left(1 + \sum_{k,j} c_{k,j} n^{-k/2}\theta^j\right),$

where each term in the sum is $o(1)$. The resulting Gaussian integrals are of the kind

$$\int_{-n^{-\alpha}}^{n^{-\alpha}} \theta^M e^{-n^{3/2}\theta^2} d\theta = \frac{n^{-3/4}}{\sqrt{\pi}} \int_{-n^{-3/4-\alpha}}^{n^{-3/4-\alpha}} \left(\frac{y}{n^{3/4}}\right)^M e^{-y^2} dy \sim \text{const} \times n^{-3(M+1)/4}, \quad M \text{ even}. \quad (\text{Those with odd } M \text{ vanish.})$$

This finishes the proof of (11). \qed

4. Joint Asymptotics

Note that the limits $m \to \infty$ and $n \to \infty$ commute in the following sense: Since we have $\kappa_n \sim 1/(2\sqrt{e\pi}) e^{2\sqrt{\pi}/n^{3/4}}$ [11 Prop. 8.4], the right-hand side of (7) has, as $n \to \infty$, the same asymptotics as the right-hand side of (5) for $m \to \infty$. We will now show that letting $m$ and $n$ tend to infinity simultaneously yields the same result, regardless of their respective speeds.

**Theorem 4.** If both the word length and the alphabet size tend to infinity, we have

$$f_n(m) \sim \frac{1}{2\sqrt{\pi}} \times \frac{m^n e^{2\sqrt{\pi}}}{n^{3/4}}, \quad m, n \to \infty.$$

**Proof.** The result can be obtained by an adaption of the proof of Theorem 3. Again we use Cauchy’s formula, with the same saddle point contour as before:

$$f_n(m) = \frac{m^n}{2\pi} \int_{-\pi}^{\pi} F(\hat{z}e^{i\theta}/m) e^{-i(n+1)\theta} d\theta \quad \sim \frac{m^n}{2\pi} \int_{-\pi}^{\pi} \exp\left(\frac{\hat{z}e^{i\theta}}{1-\hat{z}e^{i\theta}}\right) \phi\left(\frac{\hat{z}e^{i\theta}}{m}; m\right) e^{-i(n+1)\theta} d\theta. \quad (17)$$

We will show that

$$\phi\left(\frac{\hat{z}e^{i\theta}}{m}; m\right) \to 1, \quad m, n \to \infty, \quad \text{uniformly w.r.t. } \theta \in [-\pi, \pi]. \quad (18)$$

Assuming this we are done. Indeed, assertion (18) shows at the same time the validity of the local expansion (14), with $\phi(1/m)$ replaced by 1, and the persistence of the tail estimate (16).

To prove (18), notice that

$$\phi\left(\frac{\hat{z}e^{i\theta}}{m}; m\right) = \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} m^{1-k} \hat{z}^k e^{k\theta}\right). \quad (19)$$

We have $|m^{1-k} \hat{z}^k e^{k\theta}| < \frac{1}{m}$ for $m \geq 2$, hence

$$\left|\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} m^{1-k} \hat{z}^k e^{k\theta}\right| \leq \sum_{k=2}^{\infty} m^{1-k} \hat{z}^k = \sum_{k=2}^{\infty} m^{1-k} \left(1 - \frac{1}{\sqrt{n}}\right)^k = \frac{(1 - 1/\sqrt{n})^2}{m(1 - 1/m + 1/(m\sqrt{n}))}.$$

Thus the exponent in (19) is uniformly $o(1)$, which establishes (18). \qed
5. The Distribution of the Number of Words

A natural parameter to consider is the number \( W_n \) of words in a random finite language of total word length \( n \). (The alphabet size \( m \geq 2 \) is fixed throughout this section.) The appropriate bivariate ogf, with \( z \) marking total word length and \( u \) marking number of words, is given by

\[
F(z, u) := \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1} m^k u^k}{1 - m^k} \right).
\]

The expected number of words is then

\[
\mathbb{E}[W_n] = f_n^{-1} \left[ z^n \right] \frac{\partial}{\partial u} F(z, u) \bigg|_{u=1}.
\]

Notice that

\[
\frac{\partial}{\partial u} F(z, u) \bigg|_{u=1} = F(z) \sum_{k=1}^{\infty} \frac{(-1)^{k-1} m^k}{1 - m^k},
\]

so that the asymptotic analysis of \( [z^n] \frac{\partial}{\partial u} F(z, u) \bigg|_{u=1} \) is an easy extension of the one of \( f_n \) in Section 3: Close to the saddle point, the new factor resulting from the right-hand side of (21) is

\[
\frac{1}{1 - z} = \sqrt{n} (1 + o(1)).
\]

Hence \( [z^n] \frac{\partial}{\partial u} F(z, u) \bigg|_{u=1} \sim \sqrt{n} f_n \), so that, by (20), the expectation of \( W_n \) satisfies

\[
\mathbb{E}[W_n] \sim \sqrt{n}, \quad n \to \infty.
\]

Similarly, one can obtain the asymptotics \( \sigma(W_n) \sim n^{1/4}/\sqrt{2} \) for the standard deviation.

**Theorem 5.** The number of words \( W_n \) in a random finite language admits a Gaussian limit law:

\[
\frac{W_n - a_n}{b_n} \to \mathcal{N}(0, 1), \quad n \to \infty,
\]

in distribution, where the scaling constants satisfy \( a_n \sim \sqrt{n} \) and \( b_n \sim n^{1/4}/\sqrt{2} \).

**Proof.** As is well known, combinatorial limit laws can often be obtained by an asymptotic analysis of the probability generating function

\[
\mathbb{E}[u^{W_n}] = f_n^{-1} [z^n] F(z, u).
\]

Again, we adapt the proof of Theorem 3. If \( u \) ranges in a fixed small neighbourhood of \( u = 1 \), the expansion (24) generalizes to the uniform local expansion

\[
\frac{F(z/m, u)}{z^{n+1}} = \phi(1/m, u; m) \exp \left\{ -\frac{1}{2} u + 2\sqrt{u n} - u^{-1/2} m^{3/2} \theta^2 \right\} \times \left( 1 + O(n^{1/2-\alpha}) \right),
\]

where

\[
\phi(z, u; m) := \exp \left( \sum_{k=2}^{\infty} \frac{(-1)^{k-1} m^k u^k}{k \left( 1 - m^k \right)} \right).
\]

Integrating from \( \theta = -n^{-\alpha} \) to \( n^{-\alpha} \), and taking into account (3), we infer that (22) has the uniform asymptotics

\[
\mathbb{E}[u^{W_n}] \sim \exp(h_n(u)), \quad n \to \infty,
\]

with

\[
h_n(u) := 2(\sqrt{u} - 1) \sqrt{n} + \frac{1}{2} \log u + \log \phi(1/m, u; m) \phi(1/m; m). \]
Note that, for $n \to \infty$,
\begin{align*}
h'_n(1) &= \sqrt{n} + O(1), \\
h''_n(1) &= -\frac{1}{2}\sqrt{n} + O(1), \\
h'''_n(1) &= \frac{3}{4}\sqrt{n} + O(1),
\end{align*}
so that the function $h_n(u)$ satisfies the conditions of [1, Theorem 9.13], itself taken from [3]. We conclude that
\[
\frac{W_n - h'_n(1)}{(h'_n(1) + h''_n(1))^{1/2}}
\]
converges in distribution to a standard normal random variable. □

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