Exceptional holonomy and Einstein metrics constructed from Aloff–Wallach spaces

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ABSTRACT

We investigate cohomogeneity-1 metrics whose principal orbit is an Aloff–Wallach space SU(3)/U(1). In particular, we are interested in metrics whose holonomy is contained in Spin(7). Complete metrics of this kind which are not product metrics have exactly one singular orbit. We prove classification results for metrics on tubular neighbourhoods of various singular orbits. Since the equation for the holonomy reduction has only few explicit solutions, we make use of power series techniques. In order to prove the convergence and the smoothness near the singular orbit, we apply methods developed by Eschenburg and Wang. As a by-product of these methods, we find many new examples of Einstein metrics of cohomogeneity 1.

1. Introduction

Metrics with holonomy Spin(7) are an active area of research in differential geometry and in M-theory (see [1]). Most of the explicitly known examples (see [4–6, 9, 13, 14, 21, 24, 25]) are of cohomogeneity 1. The advantage of cohomogeneity-1 metrics is that the equation for the holonomy reduction is equivalent to a system of first-order ordinary differential equations. Among those metrics, the metrics with an Aloff–Wallach space as the principal orbit are an interesting subclass. An Aloff–Wallach space is a coset space \( N^{k,l} := SU(3)/U(1)_{k,l} \) where \( U(1)_{k,l} := \{ \text{diag}(e^{kit}, e^{lit}, e^{-i(k+l)t}) \mid t \in \mathbb{R} \} \). The spaces \( N^{1,0} \) and \( N^{1,1} \) have a different geometry from the other ones and are called exceptional Aloff–Wallach spaces. Any principal orbit of a manifold carrying a parallel cohomogeneity-1 Spin(7)-structure, is equipped with a cocalibrated homogeneous \( G_2 \)-structure. We therefore will describe how a connected component of the space of all cocalibrated SU(3)-invariant \( G_2 \)-structures on \( N^{k,l} \) looks like. This problem can be solved by means of representation theory. After that we are able to deduce a system of ordinary differential equations which is equivalent to the holonomy reduction. For reasons of simplicity, we carry out this program only for metrics which are diagonal with respect to a certain basis.

If a cohomogeneity-1 metric with holonomy Spin(7) or a smaller group is complete and not a product, it has exactly one singular orbit. We therefore fix the initial values of our differential equations at the singular orbit. Unfortunately, these initial value problems have an explicit solution only in some special cases [4, 6, 14, 21]. Furthermore, the equations for the holonomy reduction degenerate near the singular orbit. More precisely, some of the summands behave like \( \frac{0}{0} \). We therefore cannot apply the theorem of Picard–Lindelöf. In the literature [14, 24, 25] there are indeed examples, where the solutions do not only depend on the metric on the singular orbit but also on an initial condition of second or third order which can be chosen freely.

In order to solve these problems, we make a power series ansatz for the metric. We have to check if the power series converges and how many free parameters of higher order there are.
Another problem is that not every solution of the differential equations corresponds to a metric which can be smoothly extended to the singular orbit. There are certain smoothness conditions which have to be satisfied and are in some cases a serious obstacle. If, for example, the principal orbit is \( N^{1,1} \) and the singular orbit is \( SU(3)/U(1)^2 \), we have to replace the principal orbit by a quotient \( N^{1,1}/\mathbb{Z}_2 \) in order to satisfy the smoothness conditions.

The above problems were addressed by Eschenburg and Wang \[16\] in the context of cohomogeneity-1 Einstein metrics. Although metrics with exceptional holonomy are Ricci-flat and thus Einstein metrics, we have to adapt the methods of \[16\] to our situation. After that we are finally able to prove the existence of metrics whose holonomy is a subgroup of \( \text{Spin}(7) \) on a tubular neighbourhood of the singular orbit. At this point we are able to apply the main theorem of Eschenburg and Wang \[16\] and can also show the existence of Einstein metrics of cohomogeneity 1. With the exception of an \( SU(3) \)-invariant Einstein metric on \( \mathbb{HP}^2 \) which can be found in \[28\], all of the Einstein metrics are to the best knowledge of the author new.

The main results of the paper can be summarized as follows. We find a two-parameter family of smooth non-homothetic cohomogeneity-1 metrics with holonomy a subgroup of \( \text{Spin}(7) \). All of them have \( N^{1,1}/\mathbb{Z}_2 \) as principal orbit and \( SU(3)/U(1)^2 \) as singular orbit. Among them, there is a one-parameter family of metrics with holonomy \( SU(4) \). At the border of the moduli space the metric converges to the Calabi metric (see \[10\]) on \( T^4 \mathbb{CP}^2 \), which has holonomy \( \text{Sp}(2) \).

First evidence for the above metrics can be found in \[28\]. There exists a further two-parameter family of non-homothetic metrics with that orbit structure which was found by Bazaikin \[5\]. We also construct these metrics and show that the family can be parametrized by two free parameters of third order. Moreover, we prove for all of our cases with the help of the methods of Eschenburg and Wang \[16\] that there are no other free parameters except the known ones, that the smoothness conditions are satisfied and that the power series solutions converge. We also prove that under certain conditions that, for example, the metric is diagonal, and there are no further cohomogeneity 1 metrics whose holonomy is contained in \( \text{Spin}(7) \). Finally, we prove that the holonomy of the metrics whose principal orbit is not \( N^{1,1} \) is all of \( \text{Spin}(7) \).

Any solution of the equations for the holonomy reduction that satisfies the smoothness conditions and is defined on all of \([0, \infty)\) yields a complete metric whose holonomy is a subgroup of \( \text{Spin}(7) \). The solutions with singular orbit \( S^5 \) and those with singular orbit \( \mathbb{CP}^2 \) and another Aloff–Wallach space as \( N^{1,1} \) as principal orbit are studied in \[14\]. There is numerical evidence that they are defined on all of \([0, \infty)\) for an open subset of all possible initial values. If this is indeed true, our results guarantee the existence of smooth complete metrics with holonomy \( \text{Spin}(7) \). Moreover, we have found an independent proof that the complete metrics of Bazaikin and Malkovich \[4, 6\] actually exist.

The paper is organized as follows. In Section 2, we collect some basic facts on \( G_2 \)- and \( \text{Spin}(7) \)-structures. In Section 3, cohomogeneity-1 manifolds and the methods of Eschenburg and Wang \[16\] are introduced. Section 4 deals with the geometry of the Aloff–Wallach spaces. Our metrics with cohomogeneity 1 are constructed in the remaining three sections. In each section, we deal with metrics that have one particular singular orbit. Section 5 is about metrics with singular orbit \( SU(3)/U(1)^2 \), Section 6 is about \( S^5 \), and Section 7 is about metrics with
$\mathbb{CP}^2$ as singular orbit. At the end of the last three sections we comment how our results relate to those of other researchers and what is known about the asymptotics of our metrics.

2. $G_2$- and Spin(7)-structures

In this section, we define some terms which we shall often use later on. Let $(dx^1, \ldots, dx^7)$ be the standard basis of one-forms on $\mathbb{R}^7$. Furthermore, let

$$\omega := dx^{123} + dx^{145} - dx^{167} + dx^{246} + dx^{257} + dx^{347} - dx^{356},$$

(2.1)

where $dx^{i_1i_2\cdots i_k}$ denotes $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. The Hodge dual of $\omega$ is

$$* \omega = -dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567}.$$  

(2.2)

We supplement $(dx^1, \ldots, dx^7)$ with $dx^0$ to a basis of one-forms on $\mathbb{R}^8$ and define

$$\Omega := * \omega + dx^0 \wedge \omega = dx^{0123} + dx^{0145} - dx^{0167} + dx^{0246} + dx^{0257} + dx^{0347} - dx^{0356} - dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567}.$$  

(2.3)

A $G_2$-structure on a seven-dimensional manifold is a three-form that can be identified via local frames with $\omega$. Analogously, a Spin(7)-structure on an eight-dimensional manifold is a four-form that can be identified at each point with $\Omega$. To any $G_2$-structure or Spin(7)-structure we can associate a canonical metric $g$ and an orientation. We call a $G_2$-structure or Spin(7)-structure $\omega$ or $\Omega$ parallel if $\nabla g \omega = 0$ or $\nabla g \Omega = 0$. A pair $(N, \omega)$ or $(M, \Omega)$ of a seven-dimensional or eight-dimensional manifold and a parallel $G_2$-structure or Spin(7)-structure is called a $G_2$-manifold or Spin(7)-manifold. Those manifolds have the following interesting properties.

**Theorem 2.1** (see [7, 17, 18]). (i) Let $N$ be a seven-dimensional manifold with a $G_2$-structure $\omega$. Then, the following statements on $\omega$ are equivalent:

(a) $\omega$ is parallel;
(b) $d\omega = d*\omega = 0$;
(c) the holonomy of the associated metric $g$ is contained in $G_2$.

Conversely, if $(N, g)$ is a Riemannian manifold with holonomy a subgroup of $G_2$, then there exists a parallel $G_2$-structure on $N$ such that its associated metric is $g$. If any of the above conditions is satisfied, then $g$ is Ricci-flat.

(ii) Let $M$ be an eight-dimensional manifold with a Spin(7)-structure $\Omega$. Then, the following statements on $\Omega$ are equivalent:

(a) $\Omega$ is parallel;
(b) $d\Omega = 0$;
(c) the holonomy of the associated metric $g$ is contained in Spin(7).

Conversely, if $(M, g)$ is a Riemannian manifold with holonomy a subgroup of Spin(7), then there exists a parallel Spin(7)-structure on $N$ such that its associated metric is $g$. If any of the above conditions is satisfied, then $g$ is Ricci-flat.

Finally, we introduce the following types of non-parallel $G_2$-structures, which we shall need later on.

**Definition 2.2.** A $G_2$-structure $\omega$ is called

(i) nearly parallel if $d\omega = \lambda * \omega$ for some $\lambda \in \mathbb{R} \setminus \{0\}$;
(ii) cocalibrated if $d * \omega = 0$. 
3. Cohomogeneity-1 manifolds

The set of all Spin(7)-structures on a manifold $M$ does not define a vector subbundle of $\wedge^4 T^* M$. Therefore, the condition $d\Omega = 0$ should be considered as a non-linear partial differential equation. Explicit solutions of this equation are hard to find. Among them are the first examples of complete metrics with holonomy Spin(7) by Bryant and Salamon [9]. The existence of the non-explicit metrics of Bryant [8] and of Joyce [23] can only be proved by sophisticated analytic arguments. Our problem becomes a lot simpler if we assume that $M$ admits an action of cohomogeneity 1.

**Definition and Lemma 3.1** (see [27] and references therein). (i) Let $M$ be an $n$-dimensional connected manifold with a smooth action by a Lie group $G$. The action of $G$ is called a cohomogeneity-1 action if there exists an orbit with dimension $n - 1$.

(ii) An orbit $O$ of a cohomogeneity-1 action is called a principal orbit if there is an open subset $U$ of $M$ with the following properties: $O \subseteq U$ and $U$ is $G$-equivariantly diffeomorphic to $O \times (-\epsilon, \epsilon)$, where $\epsilon > 0$. It can be proved that this condition is equivalent to $\dim O = n - 1$. All principal orbits are $G$-equivariantly diffeomorphic to each other, and the union of all principal orbits is an open dense subset of $M$.

(iii) A Spin(7)-manifold $(M, \Omega)$ is called cohomogeneity 1 if there exists a cohomogeneity-1 action on $M$ which preserves $\Omega$ (and thus the associated metric).

Since any Ricci-flat homogeneous metric is flat (see [2]), spaces of cohomogeneity 1 are the most symmetric manifolds which may admit metrics with exceptional holonomy. For the following considerations, we fix some notation.

**Convention 3.2.** Let $G$ be a compact Lie group that acts with cohomogeneity 1 on a Spin(7)-manifold $(M, \Omega)$. The associated metric on $M$ we denote by $g$. We identify any orbit of $G$ with the quotient of $G$ by the isotropy group. The principal orbit shall be $G/H$ and $G/K$ shall be a non-principal orbit. The union of all principal orbits will be denoted by $M^0$. After conjugation, $H$ has to be a subgroup of $K$. We denote the Lie algebras of $G$, $H$, and $K$ by $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{k}$, respectively. Let $q$ be an auxiliary $\text{Ad}_K$-invariant metric on $\mathfrak{g}$. We identify the tangent space of $G/H$ with the $q$-orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$. The tangent space of $G/K$ can be identified with the complement $\mathfrak{p}$ of $\mathfrak{k}$. We denote the normal space of the orbit $G/K$ by $\mathfrak{p}^\perp$. On any cohomogeneity-1 manifold, there exists a geodesic that intersects all orbits perpendicularly. We fix such a geodesic $\gamma$ and parametrize it by arclength. The parameter of $\gamma$ we denote by $t$.

The following theorem of Mostert [27] gives us some information on the shape of $M$.

**Theorem 3.3.** (i) If $G$ acts isometrically on a Riemannian cohomogeneity-1 manifold $(M, g)$, then $M/G$ is homeomorphic to the circle $S^1$, $[0, 1]$, $[0, \infty)$, or $\mathbb{R}$. The inner points of $M/G$ correspond to principal orbits and the endpoints of the intervals to non-principal orbits.

(ii) Let $G/H$ be a principal and $G/K$ be a non-principal orbit. The quotient $K/H$ is a sphere.

(iii) Any sufficiently small tubular neighbourhood of the non-principal orbit $G/K$ is a disc bundle over $G/K$. The projection map maps a point $gH$ of the principal orbit to $gK$.

**Remark 3.4.** Spin(7)-manifolds with certain kinds of singularities are important in M-theory (see [1]). Therefore, we also consider the cases where $M$ is not a manifold but an
or orbifold. If $K/H$ is a quotient of a sphere by a discrete group $\Gamma$, then the tubular neighbourhood of $G/K$ is an $\mathbb{R}^{\dim K/H+1}/\Gamma$-bundle over $G/K$ and thus $M$ is an orbifold. In this section, we state our theorems for manifolds only. Nevertheless, it is easily possible to adapt them to the orbifold case.

Since the volume of the metric on $K/H$ shrinks to zero as we approach the singular orbit, we shall refer to $K/H$ as the collapsing sphere or, if $\dim K/H = 1$, as the collapsing circle. We can restrict the topology of $M$ even further. If $M/G = \mathbb{R}$, then $(M, g)$ contains a complete geodesic that minimizes the length between any of its points. It follows from the Cheeger–Gromoll splitting theorem that $M$ is a Riemannian product of $\mathbb{R}$ and a seven-dimensional manifold.

Since then the holonomy would be a subgroup of $G_2$, we will not consider this case. If $M/G$ was $S^1$ or $[0, 1]$, then $M$ would be compact. Since it is Ricci-flat, all Killing vector fields are parallel and commute with each other. $G/H$ thus is a flat torus. It is easy to see that $M$ has to be flat, too.

There are two kinds of non-principal orbits. If $K/H = S^0 = \mathbb{Z}_2$, then the orbit $G/K$ is called an exceptional orbit. Otherwise, it is a singular orbit. If there is exactly one exceptional orbit, then $M$ would be two-fold covered by a space $\tilde{M}$ with $\tilde{M}/G = \mathbb{R}$. Motivated by the above considerations, we assume from now on that there is exactly one singular orbit and all other orbits are principal.

On any principal orbit, there exists a canonical $G_2$-structure $\omega$ which is related to $\Omega$ by $\Omega := *\omega + dt \wedge \omega$. The equation $d\Omega = 0$ can be written in terms of $\omega$.

**Theorem 3.5.** Let $G/H$ be a seven-dimensional homogeneous space and $\omega$ be a $G$-invariant cocalibrated $G_2$-structure on $G/H$. Then there exists an $\epsilon > 0$ and a one-parameter family $(\omega_t)_{t \in (-\epsilon, \epsilon)}$ of $G$-invariant $G_2$-structures on $G/H$ such that the initial value problem

$$\frac{\partial}{\partial t} *_{G/H} \omega_t = d_{G/H} \omega_t, \tag{3.1}$$

$$\omega_0 = \omega \tag{3.2}$$

has a unique solution on $G/H \times (-\epsilon, \epsilon)$. In the above formula, $\frac{\partial}{\partial t}$ denotes the Lie derivative in $t$-direction. The index $G/H$ of $d$ and * emphasizes that we consider the exterior derivative on $G/H$ instead of $G/H \times (-\epsilon, \epsilon)$. If $\epsilon$ is sufficiently small, then $\omega_t$ is, for all $t \in (-\epsilon, \epsilon)$, a $G_2$-structure and we have $d_{G/H} *_{G/H} \omega_t = 0$. The four-form $\Omega := *_{G/H} \omega + dt \wedge \omega$ is a $G$-invariant parallel $\text{Spin}(7)$-structure on $G/H \times (-\epsilon, \epsilon)$.

Conversely, let $\Omega$ be a parallel Spin(7)-structure preserved by a cohomogeneity-1 action of a Lie group $G$. We identify the union of all principal orbits $G$-equivariantly with $G/H \times I$, where the metric on $I$ is $dt^2$. In this situation, the $G_2$-structures on the principal orbits are cocalibrated and satisfy equation (3.1).

**Remark 3.6.** (i) The above theorem was proved by Hitchin [22] for the more general case, where $\omega$ is a (not necessarily homogeneous) cocalibrated $G_2$-structure on a compact manifold.

(ii) If $\omega$ is nearly parallel, then the maximal solution of (3.1) describes a cone over $G/H$.

(iii) Since $(M, \Omega)$ is of cohomogeneity 1, equation (3.1) is equivalent to a system of ordinary differential equations. In order to ensure that $M$ has the desired topology, we fix the initial conditions at the singular orbit $G/K$.

Before we investigate equation (3.1), we have to choose the principal orbit. The following lemma answers the question if $G/H$ admits a $G$-invariant $G_2$-structure.
Lemma 3.7 (see [30]). Let $G/H$ be a homogeneous space such that $G$ acts effectively on $G/H$. Furthermore, let $p \in G/H$ be arbitrary and let $\rho : H \to \text{GL}(T_p G/H)$ be the isotropy representation of $H$. The space $G/H$ admits a $G$-invariant $G_2$-structure if and only if there exists a vector space isomorphism $\varphi : T_p G/H \to \mathbb{R}^7$ such that $\varphi \rho(H) \varphi^{-1} \subseteq G_2$.

Our next step is to describe the space of all $G$-invariant $G_2$-structures on $G/H$ explicitly. This can be done in two steps. Any $G$-invariant metric on $G/H$ can be identified via $q$ with an $H$-equivariant endomorphism of $\mathfrak{m}$. These endomorphisms can be classified with the help of Schur’s lemma. Since any manifold that admits a $G_2$-structure is orientable and an $\text{SO}(7)$-structure is the same as a metric and an orientation, we have classified all $G$-invariant $\text{SO}(7)$-structures on $G/H$. The classification of all $G$-invariant $G_2$-structures, whose extension to an $\text{SO}(7)$-structure is fixed, can be done with the help of the following lemma.

Lemma 3.8 (see [31]). Let $G/H$ be a seven-dimensional homogeneous space. We assume that $G$ acts effectively and that $G/H$ admits a $G$-invariant $G_2$-structure. Let $\mathcal{G}$ be an arbitrary $G$-invariant $\text{SO}(7)$-structure on $G/H$. The space of all $G$-invariant $G_2$-structures on $G/H$, whose extension to an $\text{SO}(7)$-structure is $\mathcal{G}$, is diffeomorphic to

$$\text{Norm}_{\text{SO}(7)} \varphi \rho(H) \varphi^{-1} / \text{Norm}_{G_2} \varphi \rho(H) \varphi^{-1}. \tag{3.3}$$

In the above formula, $\varphi$ and $\rho$ denote the same maps as in Lemma 3.7 and the normalizer $\text{Norm}_L L'$ of a subgroup $L' \subseteq L$ is defined as $\{ g \in L \mid gL'g^{-1} = L' \}$.

After having determined the space of all $G$-invariant $G_2$-structures $\omega$ on $G/H$, we calculate $d \ast \omega$ and have a description of the space of all cocalibrated invariant $G_2$-structures. In some cases, we shall not be able to describe that space explicitly. In order to find examples of parallel cohomogeneity-1 $\text{Spin}(7)$-structures, it suffices to construct a space of cocalibrated $G_2$-structures which is invariant under equation (3.1).

Not every solution of (3.1) corresponds to a metric with holonomy $\text{Spin}(7)$. The reason for this is that $\Omega$ does not automatically have a smooth extension to the singular orbit. This is the case only if certain smoothness conditions are satisfied, which we describe in detail.

We split the tangent space of $M$ at a point $p \in G/K$ into the $K$-modules $\mathfrak{p}$ and $\mathfrak{p}^\perp$. The orbits of the $K$-action on $\mathfrak{p}^\perp$ except $\{0\}$ are spheres of type $K/H$. Let $\mathcal{B}$ be a vector bundle over the union of all principal orbits which admits a $K$-action on the fibres. For reasons of simplicity we assume that there exist non-negative numbers $s_1$ and $s_2$ such that the fibres of $\mathcal{B}$ are contained in $\bigotimes^{s_1} \mathcal{T} M \otimes \bigotimes^{s_2} T^* M$. Moreover, let $\rho$ be a $G$-invariant section of $\mathcal{B}$. Since $G/K$ is homogeneous, $\rho$ is determined by its values at $p$.

Let $\gamma$ be a geodesic that intersects all orbits perpendicularly. We assume that $\gamma(0) \in G/K$. Since the action of $G$ on $\gamma$ generates all of $M$, it suffices to consider $\rho$ along $\gamma$ only. The metric $g$ is Ricci-flat. It is well known (see [15]) that any Einstein metric is analytic. We therefore assume that $\rho$ is a power series with respect to $t$. The $m$th derivative of $\rho$ in the vertical direction can be considered as a map, which assigns to a tuple $(v_1, \ldots, v_m) \in \mathfrak{p}^\perp$ an element of the fibre $\mathcal{B}_p$. This map can be extended to a map $S^m(\mathfrak{p}^\perp) \to \mathcal{B}_p$, where $S^m(\mathfrak{p}^\perp)$ denotes the $m$th symmetric power of $\mathfrak{p}^\perp$. Since $\rho$ is analytic, the sequence of those maps determines $\rho$. If $\rho$ has a smooth extension to the singular orbit, the above maps are $K$-equivariant. Conversely, we have the following theorem.

Theorem 3.9 (see [16]). Let $(M,g)$ be a Riemannian manifold with an isometric action of cohomogeneity 1 by a Lie group $G$. We assume that there is a singular orbit $G/K$. Let $\mathcal{B} \subseteq \bigotimes^1 \mathcal{T} M \otimes \bigotimes^2 T^* M$ be a vector bundle over $M$ whose fibres at the singular orbit...
are $K$-equivariantly isomorphic to a $K$-module $B$. Let $r : (0, \varepsilon) \to B$, where $\varepsilon > 0$, be a real analytic map with Taylor expansion $\sum_{m=1}^{\infty} r_m t^m$. We can identify $r$ with a tensor field $\rho$ along a geodesic $\gamma$ which intersects all orbits perpendicularly. By the action of $G$, we can extend $\rho$ to the union of all principal orbits. The tensor field $\rho$ is well defined and has a smooth extension to the singular orbit if and only if

$$r_m \in \iota_m(W_m) \quad \forall m \in \mathbb{N}_0.$$  \tag{3.4}

In the above formula, $W_m$ denotes the space of all $K$-equivariant maps

$$W_m := \{ P : S^m(p^\perp) \to B | P \text{ is linear and } K\text{-equivariant} \}$$  \tag{3.5}

and $\iota_m$ is the evaluation map

$$\iota_m : W_m \to B,$$

$$\iota_m(P) := P(\gamma'(0)).$$  \tag{3.6}

In the following, we restrict ourselves to metrics with no ‘mixed coefficients’, that is,

$$g \in S^2(p) \oplus S^2(p^\perp).$$  \tag{3.7}

Instead of $W_m$, it suffices to study the spaces

$$W^h_m := \{ P : S^m(p^\perp) \to S^2(p) | P \text{ is linear and } K\text{-equivariant} \},$$

$$W^v_m := \{ P : S^m(p^\perp) \to S^2(p^\perp) | P \text{ is linear and } K\text{-equivariant} \}$$  \tag{3.8}

in order to prove the smoothness. The reason for assumption (3.7) is that later on we need a result that is proved only if (3.7) is satisfied.

We often write our metric as $g_t + dt^2$ where $g_t \in S^2(m)$ is the restriction of $g$ to a principal orbit. Since $t$ can be considered as a distance function on $p^\perp$ and $g_t|_{p^\perp \times p^\perp}$ describes the metric on the collapsing sphere, $p^\perp$ is equipped with ‘polar’ rather than ‘Euclidean’ coordinates. We therefore have to modify Theorem 3.9 in order to fit our needs.

Remark 3.10. (i) By the choice of our coordinates we have fixed $\| \frac{\partial}{\partial t} \| = 1$ and $g(\frac{\partial}{\partial t}, v) = 0$ for all $v \in m$. The degrees of freedom for the higher derivatives of the vertical part of $g$ will therefore seem to be fewer than Theorem 3.9 predicts.

(ii) Let $v$ be a tangent vector of the collapsing sphere $K/H$. The metric on $K/H$ has to approach the round metric of a sphere of radius $t$. This condition fixes the value of $\frac{\partial}{\partial t}|_{t=0} g_t(v, v)^{1/2}$. If $K/H$ is a sphere, we can compute this value with the help of the fact that the length of any great circle on $K/H$ has to be $2\pi t + O(t^2)$ for $t \to 0$. If $K/H$ is a quotient of a sphere by a discrete group, we can use the estimate $1/t + O(1)$ for the sectional curvature. The above statements are in fact equivalent to the smoothness condition of zeroth order for the vertical part.

(iii) Since the length of $v$ shrinks to zero, any statement on the $m$th derivative of $g$ in the vertical direction translates into a statement on $\frac{\partial^m}{\partial t^m}|_{t=0} (1/t)g_t(v, v)^{1/2}$. Because of l'Hôpital's rule this is essentially a statement on the $(m+1)$th derivative of $g_t$.

We assume that $(M^0, \Omega)$ is of holonomy Spin(7) and that the metric $g$ has a smooth extension to the singular orbit. In this situation, the holonomy of $(M, g)$ equals Spin(7), too. Therefore, there exists a unique smooth Spin(7)-structure $\tilde{\Omega}$ on $M$. Without loss of generality, we can assume that $\Omega$ and $\tilde{\Omega}$ coincide on $M^0$. This observation proves that $\tilde{\Omega}$ is a smooth extension of $\Omega$ to the singular orbit and we do not have to prove the smoothness conditions for $\Omega$. 


If the holonomy $\text{Hol}$ is a smaller group, for example, $\text{Sp}(2)$ or $\text{SU}(4)$, we can prove by similar arguments that there exists a smooth $\text{Hol}$-structure on $M$. Since $G$ acts by isometries, it leaves the holonomy bundle invariant and thus the $\text{Hol}$-structure is $G$-invariant.

The following lemma allows us to decide if a solution of (3.1) describes a complete metric.

**Lemma 3.11.** Let $(M, g)$ be a Riemannian manifold with an isometric cohomogeneity-1 action by a compact Lie group. We assume that $M$ has exactly one singular orbit and that there exists a geodesic of infinite length on $M$ that intersects all orbits perpendicularly. In this situation, $(M, g)$ is geodesically complete.

A proof of this statement can be found in [12] or in [29]. The above lemma tells us that any solution of (3.1) that satisfies the smoothness conditions and is defined on all of $[0, \infty)$ yields a geodesically complete metric.

Since $\dim G/K < \dim G/H$, equation (3.1) sometimes degenerates at the singular orbit. More precisely, it is equivalent to a system that contains equations of type $c'(t) = \ldots + a(t)/b(t) + \ldots$ with $\lim_{t \to 0} a(t) = \lim_{t \to 0} b(t) = 0$. In that situation, we cannot apply the theorem of Picard–Lindelöf, since the right-hand side $f(a, b, c, \ldots)$ is not defined on an open set. There are indeed cases where the solution of our initial value problem depends on initial conditions of higher order which can be chosen freely. In order to classify the solutions of (3.1), we make the power series ansatz

$$\omega = \sum_{m=0}^{\infty} \omega_m t^m \quad \text{with} \quad \omega_m \in \bigwedge^3 \text{Ad}_H\text{-invariant.}$$

In the cases that we shall consider, we fix for any choice of the metric $g_t$ on $G/H$ a single cocalibrated $G_2$-structure $\omega_t$ whose associated metric is $g_t$. The cocalibrated $G_2$-structures with that property are in many cases a discrete set. The set of $G_2$-structures which is obtained from a sufficiently large set of $g_t$ is preserved by (3.1) and our restriction to those $G_2$-structures thus is justified. The three-form $\omega_t$ will always depend analytically on $g_t$, and the cohomogeneity-1 metric $g$ is Ricci-flat. Since any Einstein metric is analytic (see [15]), we are allowed to make the above power series ansatz. Equation (3.1) yields the following system of recursive equations for the $g_m$:

$$L_m(g_m) = P_m(g_0, \ldots, g_{m-1}),$$

where $L_m$ is a linear operator acting on the space of all $\text{Ad}_H$-invariant symmetric bilinear forms on $m$ and $P_m$ is a polynomial. For a fixed choice of the principal and the singular orbit, $L_m$ can be calculated for all $m$. We will see that in each of our cases there exists an $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$, $L_m$ is always invertible. It follows from Theorem 3.13, which we will state below, that there is a deeper reason behind this. By solving (3.10) for all $m < m_0$, we can classify all formal power series which solve equation (3.1). By certain arguments which we shall make explicit when we need them, we can check the smoothness conditions. All that is left to be done is to check if the power series converges. This follows by a theorem of Eschenburg and Wang [16] on cohomogeneity-1 Einstein metrics. Before we state that theorem, we make the following assumption.

**Assumption 3.12.** The tangent space $\mathfrak{p}$ and the normal space $\mathfrak{p}^\perp$ of the singular orbit shall have no $H$-submodule of positive dimension in common.

**Theorem 3.13** (see [16]). Let $M$ be a manifold equipped with a cohomogeneity-1 action by a compact Lie group $G$. We assume that the principal orbits of this action are $G$-equivariantly
diffeomorphic to $G/H$ and that there is a singular orbit $G/K$. Moreover, we assume that Assumption 3.12 is satisfied.

Let $g_0$ be an arbitrary $G$-invariant metric on the singular orbit. Furthermore, let $g'_0 : p^\perp \to S^2(p)$ be a linear, $K$-equivariant map. Finally, let $\lambda \in \mathbb{R}$ be arbitrary. In this situation, there exists a $G$-invariant Einstein metric $g$ on a sufficiently small tubular neighbourhood of the singular orbit which has the following properties.

(i) The metric $g$ has $\lambda$ as Einstein constant.
(ii) The restriction of $g$ to the singular orbit is $g_0$.
(iii) The first derivation of $g$ at the singular orbit in the normal directions is $g'_0$.

The set of all Einstein metrics with the above properties depends on additional initial conditions of higher order, which we can prescribe arbitrarily. The freedom for the $m$th derivative of the metric in the horizontal or vertical direction can be described by

$$
W^h_m/W^h_{m-2} \quad \text{in the horizontal case if } m \geq 2,
$$

$$
W^v_m/W^v_0 \quad \text{in the vertical case if } m = 2
$$

and there are no further free parameters in the vertical direction.

Remark 3.14. (i) Up to constant multiples, there is exactly one $K$-invariant scalar product $h$ on $p^\perp$. The reason for this is that the orbits of the $K$-action on $p^\perp$ are spheres. In particular, we have $\dim W^0_0 = 1$.

(ii) $S^m(p^\perp)$ is embedded into $S^{m+2}(p^\perp)$ by the map $\iota$ with $\iota(P) := h \vee P$, where $\vee$ is the symmetrized tensor product. The map $\iota$ induces canonical embeddings of $W^h_m$ and $W^v_m$ into $W^h_m$ and $W^v_m$, respectively, which we have implicitly used in the formulation of the above theorem.

(iii) Since we assume that Assumption 3.12 is satisfied, the metric is automatically contained in $S^2(p) \oplus S^2(p^\perp)$. Analogously, we have $W_m = W^h_m \oplus W^v_m$.

(iv) For sufficiently large $m$ the chains $W_0 \subseteq W_2 \subseteq W_4 \subseteq \ldots$ and $W_1 \subseteq W_3 \subseteq W_5 \subseteq \ldots$ stabilize. In particular, there are only finitely many initial conditions which we can prescribe.

(v) The cohomogeneity-1 Einstein condition is a system of second-order differential equations. That system yields a recursive equation that is similar to (3.10). The convergence of the power series solutions can be shown with the help of a Picard iteration. More precisely, any power series that satisfies the Einstein condition converges if $p$ and $p^\perp$ have no $H$-submodule in common. In particular, any formal power series solution of (3.10) converges if we assume Assumption 3.12. In some of the cases that we shall consider, this assumption is not satisfied. Nevertheless, the convergence can be proved in those cases, too. In the paper, we restrict ourselves to metrics that are diagonal with respect to a fixed basis of $m$. The metric is therefore an element of $S^2(p) \oplus S^2(p^\perp)$. Moreover, the Ricci tensor of the diagonal metrics on the principal orbits which we consider is diagonal, too. The equations for the Einstein condition thus do not change a diagonal metric into a non-diagonal one and the space $S^2(p) \oplus S^2(p^\perp)$ is invariant under the Picard iteration. This allows us to repeat the arguments of [16] and to prove the convergence.

(vi) In order to deduce the smoothness conditions, we have to describe the spaces $W^h_m$ and $W^v_0$. We can therefore easily apply Theorem 3.13 and obtain new examples of cohomogeneity-1 Einstein metrics.

(vii) The above theorem predicts that certain second derivatives in the vertical direction can be chosen freely. For similar reasons as in Remark 3.10, these derivatives are third derivatives with respect to $t$. As we have already remarked in Remark 3.10, we have $g(\frac{\partial}{\partial t}, v) = 0$ for all $v \in m$ by the choice of our coordinates. We therefore have to ignore the free parameters in the vertical direction which describe the change of $g(\frac{\partial}{\partial t}, v)$.
(viii) In general, the power series converges for small values of $t$ only. The metrics that we construct with the help of the above theorem are thus incomplete. They can be extended to complete metrics if and only if the equation $\frac{\partial}{\partial t} \ast \omega = d\omega$ or $\text{Ric} = \lambda g$ has a solution for all $t \in [0, \infty)$ or there are two singular orbits.

After we have proved the convergence, we construct metrics whose holonomy group is a subgroup of Spin(7) on a tubular neighbourhood of the singular orbit. In order to decide if the holonomy is all of Spin(7), we need the following lemma.

**Lemma 3.15** (see [31]).

(i) Let $M$ be an eight-dimensional manifold that carries a parallel SU(4)-structure $\mathfrak{G}$. We denote the space of all parallel Spin(7)-structures on $M$ which are an extension of $\mathfrak{G}$ and have the same extension to an SO(8)-structure as $\mathfrak{G}$ by $\mathcal{S}$. Any connected component of $\mathcal{S}$ is diffeomorphic to a circle.

(ii) Let $M$ be an eight-dimensional manifold that carries a one-parameter family $\mathcal{S}$ of parallel Spin(7)-structures. Moreover, let the extension of the Spin(7)-structures to an SO(8)-structure always be the same and let $\mathcal{S}$ be diffeomorphic to a circle. Then, there also exists a parallel SU(4)-structure on $M$.

With the help of the facts that we have collected in this section we are now able to construct examples of Spin(7)-manifolds.

4. The geometry of the Aloff–Wallach spaces

Before we construct cohomogeneity-1 metrics with an Aloff–Wallach space as the principal orbit, we have to study those spaces in detail. The Aloff–Wallach spaces are certain homogeneous spaces which were introduced by Aloff and Wallach [3] in order to study metrics with positive sectional curvature. Let $i_{k,l}: U(1) \rightarrow SU(3)$,

$$i_{k,l}(e^{i\varphi}) := \begin{pmatrix} e^{ik\varphi} & 0 & 0 \\ 0 & e^{il\varphi} & 0 \\ 0 & 0 & e^{-i(k+l)\varphi} \end{pmatrix} \quad \text{with } k, l \in \mathbb{Z}.$$  \hfill (4.1)

We denote the image of $U(1)$ with respect to $i_{k,l}$ by $U(1)_{k,l}$. Any one-dimensional subgroup of SU(3) is conjugate to a $U(1)_{k,l}$. The Aloff–Wallach space $N^{k,l}$ is defined as the quotient $SU(3)/U(1)_{k,l}$. Without loss of generality, we assume that $k$ and $l$ are coprime. Let $\sigma$ be a permutation of the triple $(k, l, -k - l)$. It is easy to see that the spaces $N^{k,l}$ and $N^{\sigma(k),\sigma(l)}$ are SU(3)-equivariantly diffeomorphic. Let $2u(1)$ be the Cartan subalgebra of all diagonal matrices in $su(3)$ and let $u(1)_{k,l}$ be the Lie algebra of $U(1)_{k,l}$. The Weyl group of $su(3)$ is isomorphic to the permutation group $S_3$ and acts on $2u(1)$. The action of a $\sigma \in S_3$ on $2u(1)$ changes $u(1)_{k,l}$ into $u(1)_{\sigma(k),\sigma(l)}$. Therefore, the $S_3$-action on $(k, l, -k - l)$ can be identified with the action of the Weyl group. Since $N^{k,l}$ and $N^{-k,-l}$ are the same manifold, we introduce the following convention.

**Convention 4.1.** When we consider an Aloff–Wallach space $N^{k,l}$, we assume that $k \geq l \geq 0$. Later on, we shall turn to another convention which will be introduced at that point.

The Aloff–Wallach spaces $N^{1,0}$ or $N^{1,1}$ are called exceptional and the other ones are called generic. Since there are many differences between the exceptional and the generic Aloff–Wallach
spaces, we often have to treat $N^{1,0}$ or $N^{1,1}$, and the generic $N^{k,l}$ as separate cases. There are infinitely many homotopy types of Aloff–Wallach spaces. This follows from the fact that

$$H^4(N^{k,l}, Z) = \mathbb{Z}_{k^2+l^2}.$$  \hspace{1cm} (4.2)

Some of the $N^{k,l}$ are homeomorphic but not diffeomorphic to each other. Examples of this fact can be found in [26]. On the Aloff–Wallach spaces $N^{k,l}$ there exist two nearly parallel $G_2$-structures, which depend on $k$ and $l$ (see [14]). The holonomy of the cones over those $G_2$-structures therefore is contained in Spin(7). We will see that the Aloff–Wallach spaces are not covered by a sphere. Therefore, the cones have a singularity at the tip, which is not an orbifold singularity.

Our next step is to describe all SU(3)-invariant metrics on the Aloff–Wallach spaces. In order to do this, we fix the following basis $(e_i)_{1 \leq i \leq 8}$ of $\mathfrak{su}(3)$:

$$e_1 := E^2_1 - E^1_1, \quad e_2 := iE^2_1 + iE^1_1, \quad e_3 := E^3_1 - E^1_3,$$
$$e_4 := iE^3_1 + iE^1_3, \quad e_5 := E^3_2 - E^2_3, \quad e_6 := iE^3_2 + iE^2_3,$$
$$e_7 := (2l + k)iE^1_1 + (-2k - l)iE^2_2 + (k - l)iE^3_3,$$
$$e_8 := kiE^1_1 + liE^2_2 - (k + l)iE^3_3,$$  \hspace{1cm} (4.3)

where $E^i_j$ denotes the $3 \times 3$ matrix with a 1 in the $i$th row and $j$th column and zeroes elsewhere. The Lie algebra $\mathfrak{u}(1)_{k,l}$ is generated by $e_8$. The equation $q(X, Y) := -\text{tr}(XY)$ defines a biinvariant metric on $\mathfrak{su}(3)$ and $(e_1, \ldots, e_7)$ is a basis of the $q$-orthogonal complement $\mathfrak{m}$ of $\mathfrak{u}(1)_{k,l}$. The isotropy action of $\mathfrak{u}(1)_{k,l}$ splits $\mathfrak{m}$ into the following irreducible submodules:

$$V_1 := \text{span}(e_1, e_2), \quad V_2 := \text{span}(e_3, e_4),$$
$$V_3 := \text{span}(e_5, e_6), \quad V_4 := \text{span}(e_7).$$  \hspace{1cm} (4.4)

The weights of the first three submodules are

$$k - l, \quad 2k + l, \quad k + 2l.$$  \hspace{1cm} (4.5)

If $N^{k,l}$ is generic, then $V_1$, $V_2$, and $V_3$ are pairwise inequivalent. If $(k, l) = (1, 0)$, then $V_1$ and $V_3$ are equivalent and $V_2$ is not equivalent to the two other modules. In the case where $k = l = 1$, $V_1$ is trivial, and $V_2$ and $V_3$ are equivalent to each other. Any SU(3)-invariant metric $g$ on $N^{k,l}$ can be identified via $q$ with a $\mathfrak{u}(1)_{k,l}$-equivariant endomorphism of $\mathfrak{m}$. We can therefore classify the invariant metrics with the help of Schur’s lemma. If $N^{k,l}$ is generic, the matrix representation $g_{ij} := g(e_i, e_j)$ of $g$ is

$$
\begin{pmatrix}
  a^2 & 0 & b^2 & 0 \\
  0 & a^2 & 0 & c^2 \\
  b^2 & 0 & c^2 & 0 \\
  0 & b^2 & 0 & f^2
\end{pmatrix}
$$

with $a, b, c, f \in \mathbb{R} \setminus \{0\}$.  \hspace{1cm} (4.6)

The coefficients of the Spin(7)-structure, which we shall construct, contain odd powers of $a$, $b$, $c$, and $f$. Therefore, we allow these numbers to be negative, too, although this does not change the metric. Next, we assume that $k = 1$ and $l = 0$. The matrix representation of $g$ with respect
to the basis \((e_1, e_2, e_5, e_6, e_3, e_4, e_7)\) is

\[
\begin{pmatrix}
0 & a^2 & 0 & \beta_{1.5} & \beta_{1.6} \\
0 & 0 & a^2 & -\beta_{1.6} & \beta_{1.5} \\
\beta_{1.5} & -\beta_{1.6} & c^2 & 0 & 0 \\
\beta_{1.6} & \beta_{1.5} & 0 & c^2 & 0 \\
b^2 & 0 & b^2 & f^2 & 0 \\
0 & b^2 & 0 & 0 & b^2 \\
f^2 & 0 & b^2 & 0 & 0
\end{pmatrix}
\] (4.7)

with \(a, b, c, f, \beta_{1.5}, \beta_{1.6} \in \mathbb{R}, \ a^2 c^2 \geq \beta_{1.5}^2 + \beta_{1.6}^2, \ b \neq 0, \) and \(f \neq 0.\) If \(k = l = 1,\) the matrix representation of \(g\) with respect to \((e_1, e_2, e_7, e_3, e_4, e_5, e_6)\) is

\[
\begin{pmatrix}
a_1 \beta_{1.2} & \beta_{1.7} & b^2 \\
\beta_{1.2} & a_2 \beta_{2.7} & 0 \\
\beta_{1.7} & \beta_{2.7} & f^2 \\
b^2 & 0 & \beta_{3.5} \\
0 & \beta_{3.6} & \beta_{3.5} \\
\beta_{3.5} & -\beta_{3.6} & c^2 \\
\beta_{3.5} & \beta_{3.5} & 0
\end{pmatrix}
\] . (4.8)

As in the other cases, the above matrix has to be positive definite. Throughout the paper we assume that the metric on the principal orbit is diagonal with respect to \((e_1, \ldots, e_7).\) This assumption simplifies our calculations and we, nevertheless, obtain interesting results. Moreover, some of the non-diagonal metrics can be changed by the action of the normalizer \(\text{Norm}_{SU(3)}U(1)_{k,l}\) into diagonal ones.

In [16, 20, 32], it is explained how the Einstein condition \(\text{Ric} = \lambda g\) for a cohomogeneity-1 manifold can be rewritten as a system of ordinary differential equations for the coefficient functions of \(g.\) The Ricci tensor of a generic \(N^{k,l}\) with an arbitrary \(SU(3)\)-invariant metric was calculated by Wang [33]. If we put the results of the above papers together, then we see that the Einstein condition for a generic \(N^{k,l}\) is equivalent to

\[
-\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'}{a} \left(2 \frac{a'}{a} + 2 \frac{b'}{b} + 2 \frac{c'}{c} + \frac{f'}{f}\right) + \frac{6}{a^2} - \frac{1}{2 \left(k^2 + lk + l^2\right)^2} \frac{f^2}{a^4}
\]

\[
+ \frac{a^4 - b^4 - c^4}{a^2 b^2 c^2} = \lambda,
\]

\[
-\frac{b''}{b} + \frac{b'^2}{b^2} - \frac{b'}{b} \left(2 \frac{a'}{a} + 2 \frac{b'}{b} + 2 \frac{c'}{c} + \frac{f'}{f}\right) + \frac{6}{b^2} - \frac{1}{2 \left(k^2 + lk + l^2\right)^2} \frac{f^2}{b^4}
\]

\[
+ \frac{b^4 - a^4 - c^4}{a^2 b^2 c^2} = \lambda,
\]

\[
-\frac{c''}{c} + \frac{c'^2}{c^2} - \frac{c'}{c} \left(2 \frac{a'}{a} + 2 \frac{b'}{b} + 2 \frac{c'}{c} + \frac{f'}{f}\right) + \frac{6}{c^2} - \frac{1}{2 \left(k^2 + lk + l^2\right)^2} \frac{f^2}{c^4}
\]

\[
+ \frac{c^4 - a^4 - b^4}{a^2 b^2 c^2} = \lambda,
\]

\[
-\frac{f''}{f} + \frac{f'^2}{f^2} - \frac{f'}{f} \left(2 \frac{a'}{a} + 2 \frac{b'}{b} + 2 \frac{c'}{c} + \frac{f'}{f}\right) + \frac{1}{2 \left(k^2 + lk + l^2\right)^2} \frac{f^2}{a^4}
\]

\[
+ \frac{1}{2 \left(k^2 + lk + l^2\right)^2} \frac{f^2}{b^4} + \frac{1}{2 \left(k^2 + lk + l^2\right)^2} \frac{f^2}{c^4} = \lambda,
\]

\[
-2 \frac{a''}{a} - 2 \frac{b''}{b} - 2 \frac{c''}{c} - \frac{f''}{f} = \lambda.
\]
If the principal orbit is $N^{1,0}$ and carries a diagonal metric, the Einstein condition is equivalent to (4.9) with $k = 1$ and $l = 0$. In particular, a diagonal metric cannot be changed by the equations of (4.9) into a non-diagonal one for a different value of $t$. If $k = l = 1$, then we do not necessarily have $g(e_1, e_1) = g(e_2, e_2)$ and $\text{Ric} = \lambda g$ becomes

$$-rac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'}{a_1} \left( \frac{a_1}{a_2} + \frac{a'^2}{b} + 2c' + f'' \right) + \frac{6}{a_1^2} - \frac{2f^2}{9a_1^2a_2^2}$$

$$+ 18 \frac{a_1 - a_2}{a_1 a_2 f^2} + \frac{a_1 - b^2 - c^2}{a_1 b^2 c} = \lambda,$$

$$- \frac{a''}{a_2} + \frac{a'^2}{a_2^2} - \frac{a'}{a_2} \left( \frac{a_1}{a_2} + \frac{a'^2}{b} + 2c' + f'' \right) + \frac{6}{a_2^2} - \frac{2f^2}{9a_1 a_2^2}$$

$$+ 18 \frac{a_2 - a_1}{a_1 a_2 f^2} + \frac{a_2 - b^2 - c^2}{a_2 b^2 c} = \lambda,$$

$$- \frac{b''}{b} + \frac{b'^2}{b^2} - \frac{b'}{b} \left( \frac{a_1}{a_1} + \frac{a'^2}{b} + 2c' + f'' \right) + \frac{6}{b_2} - \frac{1}{18b_4}$$

$$+ \frac{b^4 - a_1^2 - c^4}{2a_1^2 b^2 c^2} + \frac{b^4 - a_2^2 - c^4}{2a_2^2 b^2 c^2} = \lambda,$$

$$- \frac{c''}{c} + \frac{c'^2}{c^2} - \frac{c'}{c} \left( \frac{a_1}{a_1} + \frac{a'^2}{b} + 2c' + f'' \right) + \frac{6}{b} - \frac{1}{18c^2}$$

$$+ \frac{c^4 - a_1^2 - b^4}{2a_1^2 b^2 c^2} + \frac{c^4 - a_2^2 - b^4}{2a_2^2 b^2 c^2} = \lambda,$$

$$- \frac{f''}{f} + \frac{f'^2}{f^2} - f'' \left( \frac{a_1}{a_1} + \frac{a'^2}{b} + 2c' + f'' \right) + \frac{36}{f^2} - \frac{18}{a_1^2 f^2} - \frac{18}{a_2^2 f^2}$$

$$+ \frac{2f^2}{9a_1^2 a_2^2} - \frac{1}{18b_2} + \frac{1}{18c^3} = \lambda,$$

$$- \frac{a''}{a_1} - \frac{a''}{a_2} - \frac{b''}{b} - \frac{2c''}{c} - \frac{f''}{f} = \lambda.$$

As in the previous case, the differential equations in (4.10) preserve the space of all diagonal metrics on $N^{k,l}$. According to Remark 3.14(v), any formal power series that solves one of the above systems and satisfies the smoothness conditions converges.

Our next step is to deduce a system of ordinary differential equations which is equivalent to $d\Omega = 0$. Let $M$ be a cohomogeneity-1 manifold whose principal orbit is a generic Aloff–Wallach space. The following basis of the tangent space induces an SU(3)-invariant Spin(7)-structure on $M$ whose associated metric restricted to a principal orbit is (4.6):

$$f_0 := \frac{\partial}{\partial t}, \quad f_1 := \frac{1}{a} e_7, \quad f_2 := \frac{1}{a} e_1, \quad f_3 := \frac{1}{a} e_2,$$

$$f_4 := \frac{1}{b} e_4, \quad f_5 := \frac{1}{b} e_3, \quad f_6 := \frac{1}{c} e_6, \quad f_7 := \frac{1}{c} e_5.$$ (4.11)

The matrix representation of the isotropy action of $u(1)_{k,l}$ with respect to the basis (4.11) can be identified with a one-dimensional subgroup of Spin(7). The $SU(3)$-invariant four-form $\Omega$, which is determined by (4.11), is thus well defined. Since $(f_i)_{1 \leq i \leq 7}$ is orthonormal with respect to (4.6), $(f_i)_{1 \leq i \leq 7}$ defines a $G_2$-structure on $N^{k,l}$ whose associated metric is an arbitrary $SU(3)$-invariant one. If we wrote down that $G_2$-structure explicitly, we would see that it contains odd powers of $a, b, c,$ and $f$, as we have remarked above. We calculate $d\Omega$ and see that $d\Omega = 0$ is
the metric on the principal orbit as an arbitrary diagonal one. If $k \neq 0$, too, and (4.12) has also been deduced by Kanno and Yasui [24]. This convention will be maintained throughout the paper. We remark that the system (4.11) has different coefficients, since we may have $g$ equivalent to $g(4.11)$. In that situation, we can choose the metric on the principal orbit as an arbitrary diagonal one. Let $\Delta$ denote $k^2 - l$, then $f = \frac{1}{k} e_7$, $f_2 := \frac{1}{a_1} e_1$, $f_3 := \frac{1}{a_2} e_2$, $f_4 := \frac{1}{b} e_4$, $f_5 := \frac{1}{b} e_5$, $f_6 := \frac{1}{c} e_6$, $f_7 := \frac{1}{c} e_7$.

The equation $d\Omega = 0$ is equivalent to the slightly more complicated system

\[
\begin{align*}
\frac{a'}{a} &= \frac{b^2 + c^2 - a^2}{a_1 b c} + 3 \frac{a_1^2 - a_2^2}{a_1 a_2 f} - \frac{1}{3} \frac{f}{a_1 a_2}, \\
\frac{b'}{b} &= \frac{b^2 + c^2 - a^2}{a_2 b c} + 3 \frac{a_2^2 - a_1^2}{a_1 a_2 f} - \frac{1}{3} \frac{f}{a_1 a_2}, \\
\frac{c'}{c} &= \frac{b^2 + c^2 - a^2}{a_1 b c} + 3 \frac{b^2 + c^2 - a^2}{a_1 b c} - \frac{1}{3} \frac{f}{a_1 a_2}, \\
\frac{f'}{f} &= -\frac{3}{a_1 a_2 f} + \frac{1}{3} \frac{f}{a_1 a_2 f} - \frac{1}{3} \frac{f}{a_1 a_2 f} - \frac{1}{3} \frac{f}{a_1 a_2 f}.
\end{align*}
\] (4.14)

In the above equations, we again have replaced $t$ by $-t$. The system (4.14) was also deduced by Kanno and Yasui [25]. We want to know if there are any SU(3)-invariant $G_2$-structures on the Aloff–Wallach spaces except those that are determined by a basis of type (4.11) or (4.13). Let $g$ be an arbitrary homogeneous metric on $N^{k,l}$. In Lemma 3.8, we have proved that the space of all SU(3)-invariant $G_2$-structures on $N^{k,l}$, whose associated metric is $g$ and whose orientation is fixed, can be described by

\[
\text{Norm}_{SO(7)} I_{k,l} / \text{Norm}_{G_2} I_{k,l}.
\] (4.15)

In the above formula, $I_{k,l}$ denotes the one-dimensional subgroup $\varphi (U(1))_{k,l}$ of GL$(7, \mathbb{R})$. As in Lemma 3.7, $\rho$ is the isotropy representation of $U(1)_{k,l}$ and $\varphi$ is the map that identifies $\mathfrak{m}$ with $\mathbb{R}^7$. We first investigate the problem on the Lie algebra level. Let $G \in \{G_2, SO(7)\}$, $\mathfrak{g}$ be the Lie algebra of $G$, and $i_{k,l}$ be the Lie algebra of $I_{k,l}$. The tangent space of $\text{Norm}_{G_2} I_{k,l}$ is

\[
\text{Norm}_{\mathfrak{g}} i_{k,l} := \{ x \in \mathfrak{g} | [z, x] \in i_{k,l} \text{ for all } z \in i_{k,l} \}.
\] (4.16)
The Lie algebra $i_{k,l}$ is generated by a single $z \in \mathfrak{gl}(7, \mathbb{R})$. Let $\kappa$ be the Killing form of $\mathfrak{g}$. An element $x$ is contained in $\text{Norm}_x i_{k,l}$ if and only if

$$[z, x] = \lambda z \quad \text{for some } \lambda \in \mathbb{R}. \quad (4.17)$$

From this relation, it follows that

$$0 = \kappa(x, [z, z]) = \kappa([x, z], z) = -\lambda \kappa(z, z). \quad (4.18)$$

The above equation is satisfied only if $\lambda = 0$ and thus we have

$$\text{Norm}_x i_{k,l} = \{ x \in \mathfrak{g} | [z, x] = 0 \} =: C_x i_{k,l}. \quad (4.19)$$

We are going to determine the centralizer $C_x i_{k,l}$. First, we work with the complexification of $C_x i_{k,l}$, since this will simplify some of our arguments. Any $x \in \mathfrak{g} \otimes \mathbb{C}$ has a Cartan decomposition

$$x = h + \sum_{\alpha \in \Phi} \mu_\alpha x_\alpha \quad \text{with } \mu_\alpha \in \mathbb{C}. \quad (4.20)$$

In the above formula, $h$ is an element of a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g} \otimes \mathbb{C}$. Furthermore, $\Phi$ is the root system of $\mathfrak{g} \otimes \mathbb{C}$ and $x_\alpha$ is a suitable generator of the root space $L_\alpha$ of $\alpha$. We assume without loss of generality that $z \in \mathfrak{h}$. With this notation, the centralizer can be described as follows:

$$C_{\mathfrak{g} \otimes \mathbb{C}} (i_{k,l} \otimes \mathbb{C}) = \left\{ x \in \mathfrak{g} \otimes \mathbb{C} \left| [z, x] = \sum_{\alpha \in \Phi} \alpha(z) \mu_\alpha x_\alpha = 0 \right\}. \quad (4.21)$$

Let $\Phi' := \{ \alpha \in \Phi | \alpha(z) = 0 \}$. The above formula can be simplified to

$$C_{\mathfrak{g} \otimes \mathbb{C}} (i_{k,l} \otimes \mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi'} L_\alpha. \quad (4.22)$$

We specialize to the case $\mathfrak{g} = \mathfrak{so}(7, \mathbb{C})$. Let $(\theta_1, \theta_2, \theta_3)$ be a basis of the dual $\mathfrak{h}^*$ of $\mathfrak{h}$ such that

$$\Phi = \{ \pm \theta_i | 1 \leq i \leq 3 \} \cup \{ \pm \theta_i \pm \theta_j | 1 \leq i < j \leq 3 \}. \quad (4.23)$$

For reasons of simplicity we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ by the Killing form. The action of $z$ on $\mathfrak{m}$ is described by the weights $(4.5)$. For a suitable choice of $z$ and $(\theta_1, \theta_2, \theta_3)$, we thus have

$$z = (k - l)\theta_1 + (2k + l)\theta_2 + (k + 2l)\theta_3. \quad (4.24)$$

Let $\alpha = \sum_{i=1}^3 \alpha_i \theta_i \in \Phi$. The equation $\alpha(z) = 0$ is equivalent to

$$(k - l)\alpha_1 + (2k + l)\alpha_2 + (k + 2l)\alpha_3 = 0. \quad (4.25)$$

The number $\theta_i(z)$ vanishes if and only if the $i$th term of the three coefficients $k - l, 2k + l$, and $k + 2l$ equals zero. Analogously, the root $\pm \theta_i \pm \theta_j$ is contained in $\Phi'$ if and only if the $i$th and the $j$th coefficient coincide up to the sign. We are now able to describe the normalizer in each of the three cases.

(1) Let $N_{k,l}^i$ be a generic Aloff–Wallach space. Since the set $\Phi'$ is empty, $\text{Norm}_{\mathfrak{so}(7, \mathbb{C})} (i_{k,l} \otimes \mathbb{C}) = \mathfrak{h}$. The normalizer $\text{Norm}_{\mathfrak{so}(7)} i_{k,l}$ therefore has to be isomorphic to $3a(1)$. More precisely, it has the following matrix representation with respect to the basis $(e_i)_{1 \leq i \leq 7} := (e_i / \|e_i\|)_{1 \leq i \leq 7}$ of $\mathfrak{m}$:

$$\begin{pmatrix}
0 & a & 0 \\
-a & 0 & b \\
0 & -b & 0 \\
0 & b & 0 \\
0 & c & 0 \\
c & 0 & 0 \\
0 & -c & 0
\end{pmatrix} \quad \text{for } a, b, c \in \mathbb{R}. \quad (4.26)$$
(2) If $k = 1$ and $l = 0$, then the set $\Phi'$ consists of the two roots $\pm(\theta_1 - \theta_3)$ and thus is a root system of type $A_1$. Since $\text{Norm}_{a_0(7)}i_{1,0}$ is the compact real form of $\text{Norm}_{a_0(7,C)}(i_{1,0} \otimes \mathbb{C})$, it is isomorphic to $\text{su}(2) \oplus 2u(1)$. The semisimple part of the normalizer acts irreducibly on $V_1 \oplus V_3$ such that $(e'_1, e'_2, e'_3, e'_6)$ is identified with the standard basis of $\mathbb{C}^2$.

(3) If $k = l = 1$, then $\Phi'$ consists of the two pairs $\pm(\theta_1 \pm \theta_2 - \theta_3)$. Since $A_1 \times A_1$ is the Dynkin diagram of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, the real Lie algebra $\text{Norm}_{a_0(7)}i_{1,1}$ has to be isomorphic to $2\mathfrak{su}(2) \oplus u(1)$. One of the two simple summands acts by its three-dimensional representation on $\text{span}(e'_1, e'_2, e'_3)$. The other one acts irreducibly on $V_2 \oplus V_3$ such that we can identify $(e'_3, e'_4, e'_5, e'_6)$ with the standard basis of $\mathbb{C}^2$.

By intersecting $\text{Norm}_{a_0(7)}i_{k,l}$ with $\mathfrak{g}_2$ we are able to determine $\text{Norm}_{a_2,1}i_{k,l}$. Again, we consider the three cases separately.

(1) Let $N^{k,l}$ be a generic Aloff–Wallach space. $\text{Norm}_{a_2,1}i_{k,l}$ is isomorphic to the two-dimensional subalgebra of (4.26) which is a Cartan subalgebra of $\mathfrak{g}_2$.

(2) Let $k = 1$ and $l = 0$. By an explicit calculation, we see that the Lie algebra action of the semisimple part of $\text{Norm}_{a_0(7)}i_{1,0}$ on $\omega \in \mathbb{C}^2$ is trivial. The semisimple part therefore is a subalgebra of $\mathfrak{g}_2$. Since $\mathfrak{g}_2$ is of rank 2, $\text{Norm}_{a_2,1}i_{1,0}$ is isomorphic to $\mathfrak{su}(2) \oplus u(1)$.

(3) Finally, let $k = l = 1$. Since $\text{Norm}_{a_2,1}i_{1,1}$ contains the Cartan subalgebra of $\mathfrak{g}_2$, it is of rank 2. There are four subalgebras of $2\mathfrak{su}(2) \oplus u(1)$ which have rank 2. One of them is $2\mathfrak{su}(2)$. The other three are isomorphic to $\mathfrak{su}(2) \oplus u(1)$ where the semisimple part is either an ideal of $2\mathfrak{su}(2)$ or diagonally embedded. By similar techniques as in the previous case, we see that the semisimple part is diagonally embedded. More precisely, it acts irreducibly on $\text{span}(e'_1, e'_2, e'_3)$ and $\text{span}(e'_3, e'_4, e'_5, e'_6)$.

We are now able to describe both normalizers of the Lie group $I_{k,l}$. For the following considerations, we denote the identity component of a Lie group $G$ by $G_e$. $(\text{Norm}_{\text{SO}(7)}I_{k,l})_e$ acts transitively and almost freely on any connected component of $\text{Norm}_{\text{SO}(7)}i_{1,0}$, or $\text{Norm}_{\text{SO}(7)}i_{1,1}$ has to be isomorphic to $\mathfrak{su}(2) \oplus u(1)$.

(1) If $N^{k,l}$ is generic, then we have

$$(\text{Norm}_{\text{SO}(7)}I_{k,1})_e/\text{Norm}_{\text{G}^2}I_{k,1})_e \cong U(1)^3/U(1)^2 \cong S^1. \quad (4.27)$$

Let $T$ be a one-dimensional connected Lie subgroup of the maximal torus of SO(7) whose Lie algebra is (4.26). We choose $T$ in such a way that $G_2 \cap T$ is discrete. The action of $T$ on a fixed homogeneous $G_2$-structure $\omega$ generates a connected component of the space of all SU(3)-invariant $G_2$-structures which have the same associated metric and orientation as $\omega$.

(2) If $k = 1$ and $l = 0$, then we have

$$(\text{Norm}_{\text{SO}(7)}I_{k,1})_e/\text{Norm}_{\text{G}^2}I_{k,1})_e \cong (U(2) \times U(1))/U(2) \cong S^1. \quad (4.28)$$

The space of all SU(3)-invariant $G_2$-structures with a fixed associated metric and orientation can therefore be described as in the previous case.

(3) The semisimple part of $\text{Norm}_{a_0(7)}i_{1,1}$ is diagonally embedded into the semisimple part $2\mathfrak{su}(2)$ of $\text{Norm}_{a_0(7)}i_{1,1}$. We denote the ideal of $2\mathfrak{su}(2)$ that acts non-trivially on $\text{span}(e'_1, e'_2, e'_3)$ by $\mathfrak{su}(2)'$. The abelian part of $\text{Norm}_{a_0(7)}i_{1,1}$ is not a subgroup of $\mathfrak{su}(2)'$. We can conclude that the Lie group which corresponds to $\mathfrak{su}(2)'$ is isomorphic to SO(3) and acts transitively and freely on $(\text{Norm}_{\text{SO}(7)}I_{k,1})_e/\text{Norm}_{\text{G}^2}I_{k,1})_e$. The set of all SU(3)-invariant $G_2$-structures on $N^{1,1}$ whose associated metric and orientation is fixed can therefore be generated by an SO(3)-action on a single $G_2$-structure.

Until now, we have only determined the type of the connected components of $\text{Norm}_{a_0(7)}I_{k,1}/\text{Norm}_{\text{G}^2}I_{k,1}$, but not their number. Therefore cannot rule out that there are further homogeneous $G_2$-structures that cannot be obtained by the above $U(1)$- or SO(3)-actions. It may be possible that the additional $G_2$-structures could be extended to new examples of parallel cohomogeneity-1 Spin(7)-structures. Nevertheless, we will not discuss this issue.
further. There are three reasons for our decision. First, the study of the Spin(7)-structures whose restriction to a principal orbit is one of the $G_2$-structures which we have constructed so far is already a rewarding project even if further examples existed. Second, it may be possible that we switch to another connected component of $\text{Norm}_{SO(7)}I_{k,l}/\text{Norm}_{G_2}I_{k,l}$ if we change the sign of some of the functions $a$, $a_1$, $a_2$, $b$, $c$, and $f$. Therefore, we may already describe more than one or even all connected components by our ansatz for $(f_i)_{1 \leq i \leq 7}$. A third reason is that the results which we have found are sufficient to make statements on the holonomy of our metrics.

We start with the generic Aloff–Wallach spaces $N^{k,l}$. Let $g$ be a fixed metric on $N^{k,l}$ and $\omega$ be a $G_2$-structure whose associated metric is $g$. By an explicit calculation we see that $\omega$ is cocalibrated only if it is induced by the basis $(f_i)_{1 \leq i \leq 7}$ of $m$ which is defined by (4.11). Let $\tilde{g}$ be a cohomogeneity-1 metric whose holonomy is contained in Spin(7). We assume that there is a principal orbit such that the restriction of $\tilde{g}$ to that orbit is $g$. Since the holonomy bundle is $SU(3)$-invariant, any parallel Spin(7)-structure $\Omega$ whose associated metric is $\tilde{g}$ is $SU(3)$-invariant, too. Any parallel Spin(7)-structure induces a cocalibrated $G_2$-structure on the principal orbit. Since the set of all $SU(3)$-invariant cocalibrated $G_2$-structures is discrete, the set of all invariant parallel Spin(7)-structures is discrete, too. According to Lemma 3.15, the holonomy of $\tilde{g}$ cannot be SU(4) or one of its subgroups.

If the holonomy was $G_2$, then there would exist a parallel vector field $X$ on the manifold. Moreover, the space of all parallel vector fields would be one-dimensional. Since that space is $SU(3)$-invariant, $X$ is invariant, too, and of type $c_1(t) \cdot (\partial/\partial t) + (c_2(t)/f(t)) \cdot e_7$. The dual $c_1(t) \cdot dt + c_2(t) \cdot f(t) \cdot e_7$ of $X$ has to be a closed one-form. By calculating the exterior derivative, we see that $c_2$ has to be constant, too. If $d/dt$ was a parallel vector field, we would have $a' = b' = c' = f' = 0$.

If the principal orbit is $N^{1,0}$, it follows by the same arguments that the holonomy is all of Spin(7). In the case where $k = l = 1$, the situation is more complicated, since the space of all $G_2$-structures with a fixed associated metric is three-dimensional. Let $\omega$ be the $G_2$-structure on $N^{1,1}$ which is induced by the subfamily $(f_i)_{1 \leq i \leq 7}$ of basis (4.13). If the holonomy is contained in SU(4), then there exists a map

$$\tilde{\omega} : [0, \epsilon) \longrightarrow \bigwedge^3 m^*$$

such that $\tilde{\omega}(0) = \omega$, $d * \tilde{\omega}(s) = 0$ for all $s \in [0, \epsilon)$, and the metric that is associated to $\tilde{\omega}(s)$ is the same as of $\omega$. We have proved that $\tilde{\omega}(s)$ can be obtained by the action of an $A(s) \in SO(3)$ on $(e_1', e_2', e_7')$. We consider the case where $A(s)$ is of type

$$\begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

(4.30)

With the help of a short MAPLE program, we can show that if $s \notin \pi \mathbb{Z}$, then $d * \tilde{\omega}(s) = 0$ is equivalent to $a_1(t) + a_2(t) = 0$. Under this assumption, it follows from (4.14) that $a_1(t)^2 = b(t)^2 + c(t)^2$. Under these assumptions (4.14) is explicitly solvable and we obtain the metrics of Bazaikin and Malkovich [6]. In [6] it is proved that the holonomy is all of SU(4), except in a limiting case where we obtain Calabi’s [10] hyperkähler metric on $T^* \mathbb{CP}^2$.

If we replace the $A$ from (4.30) by another one-parameter subgroup of SO(3), we obtain more complicated conditions on the Spin(7)-structures, which may or may not be satisfiable. The question if there are further metrics whose holonomy is a proper subgroup of Spin(7) is a subject of future research.
At the end of this section, we classify all possible singular orbits of a cohomogeneity-1 manifold whose principal orbit is an Aloff–Wallach space. This classification can also be found in Gambioli [19]. Our results are summarized by the following lemma.

**Lemma 4.2.** Let $U(1)$ be embedded into $SU(3)$ as $U(1)_{1,l}$. Furthermore, let $K$ be a connected closed group with $U(1)_{1,l} \subseteq K \subseteq SU(3)$. We denote the Lie algebra of $K$ by $\mathfrak{k}$. In this situation, we have infinitely many submodules of $V$ and the statements of the lemma are satisfied. If $\mathfrak{k}$ is closed under the Lie bracket, there may exist a two-dimensional space $W$ as an arbitrary $\mathfrak{k}$-equivariantly isomorphic, there may exist a two-dimensional space $W$ such that $\mathfrak{k}$ is closed under the Lie bracket. It turns out that the possible $\mathfrak{k}$ are precisely

$$\mathfrak{k} = \text{span}(e_8, 1/2 \sqrt{2} e_1 + 1/2 \sqrt{2} e_3, -1/2 \sqrt{2} e_2 - 1/2 \sqrt{2} e_6)$$

(4.31)

and its conjugates with respect to $U(1)_{1,l}$. Since $\mathfrak{k}$ is conjugate to the standard embedding of $so(3)$ into $su(3)$, we have $K \cong SO(3)$, $K/U(1)_{1,-1} \cong S^2$ and the singular orbit is the symmetric space $SU(3)/SO(3)$.

If $k = l = 1$, then $\mathfrak{k}$ again cannot be a Lie algebra.

**Type 3:** $\mathfrak{k} = u(1)_{1,l} \oplus W$, $\dim W = 3$. Let $N^{k,l}$ be a generic Aloff–Wallach space. In this situation, we have $W = V_i \oplus V_j$ for an $i \in \{1, 2, 3\}$. As a Lie algebra $\mathfrak{k}$ is isomorphic to $u(2)$. We assume without loss of generality that $i = 1$. The Lie group $K$ is given by $S(U(2) \times U(1))$. In the above table, $\Gamma$ denotes an arbitrary discrete subgroup of $O(8)$ and the group $\mathbb{Z}/(k+l)$, by which we divide $S^3$, is explicitly described by (4.32).
We analyse the topology of $K/U(1)_{k,l}$. Let $\pi : SU(2) \rightarrow K/U(1)_{k,l}$ be the map with $\pi(h) := hU(1)_{k,l}$, where $SU(2)$ is embedded into $SU(3)$ such that its Lie algebra is $u(1)_{1,-1} \oplus V_1$. It is easy to see that $\pi$ is a covering map. Its kernel is $SU(2) \cap U(1)_{k,l}$. Since $k$ and $l$ are coprime, this intersection is, except for $(k,l) = (1,1)$,
\[
\left\{ \begin{array}{c}
e^{2\pi i(m/(k+l))} & 0 \\
0 & e^{-2\pi i(m/(k+l))} \end{array} \right\} \quad m \in \mathbb{Z}.
\] (4.32)
The quotient $K/U(1)_{k,l}$ thus is the lens space $L(k+l,1)$. For reasons of brevity, we will denote $K/U(1)_{k,l}$ by $S^3/Z_{k+l}$, since $Z_{k+l}$ will always be the above discrete group. The quotient $SU(3)/K$ is diffeomorphic to $\mathbb{CP}^2$.

Next, we consider the exceptional case $k = l = 1$. The Lie algebra $\mathfrak{f}$ has to be isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. As above, we decompose $\mathfrak{f}$ into the direct sum $\mathfrak{h} \oplus \mathfrak{w}'$ of $u(1)_{1,1}$-modules, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{f}$ which contains $u(1)_{1,1}$ and $\mathfrak{w}'$ is its $\mathfrak{g}$-orthogonal complement. Since $\text{span}(e_1, e_2, e_7)$ is the centralizer of $u(1)_{1,1}$, it follows that $\mathfrak{h}$ can be any of the following spaces:
\[
\mathfrak{h} = u(1)_{1,1} \oplus \text{span}(x) \quad \text{with } x \in \text{span}(e_1, e_2, e_7) \setminus \{0\}.
\] (4.33)
The module $\mathfrak{w}'$ is either a submodule of $\text{span}(e_1, e_2, e_7)$ or of $V_2 \oplus V_3$. In the first case, it has to be the complement of $\text{span}(x)$ in $\text{span}(e_1, e_2, e_7)$ and $\mathfrak{f}$ therefore is $2u(1) \oplus V_1$, which yields $K/U(1)_{1,1} = S^3/Z_2$ and $SU(3)/K = \mathbb{CP}^2$.

We turn to the case where $\mathfrak{w}' \subseteq V_2 \oplus V_3$. There exists an
\[
A \in \left\{ \left( \begin{array}{c}
A' \\
1
\end{array} \right) \in \text{SU}(2) \right\} \quad \text{with } A \mathfrak{h} A^{-1} = 2u(1).
\] (4.34)
We conjugate our decomposition of $\mathfrak{f}$ by $A$ and obtain
\[
A \mathfrak{f} A^{-1} = 2u(1) \oplus \mathfrak{w}'A^{-1}.
\] (4.35)
Since $\mathfrak{w}'A^{-1}$ is a $2u(1)$-module, we can prove by the same arguments as in the case $(k,l) = (1,0)$ that $\mathfrak{w}'A^{-1}$ is either $V_2$ or $V_3$. In both cases we obtain $K/U(1)_{1,1} = S^3$ and $SU(3)/K = \mathbb{CP}^2$.

**Type 4:** $\mathfrak{f} = u(1)_{1,1} \oplus \mathfrak{w}', \dim \mathfrak{w} \in \{4,6\}$. Since, for any root $\alpha$ of $\mathfrak{f}$, $-\alpha$ is a root, too, we have $\dim \mathfrak{f} \equiv \text{rank } \mathfrak{f} \pmod{2}$. Therefore, the rank of $\mathfrak{f}$ has to be odd. Since rank $\mathfrak{su}(3) = 2$, the only possibility for rank $\mathfrak{f}$ is in fact 1. Since the only Lie algebras of rank 1 that belong to a compact Lie group are $u(1)$ and $\mathfrak{su}(2)$, we can exclude these cases.

**Type 5:** $\mathfrak{f} = u(1)_{1,1} \oplus \mathfrak{w}', \dim \mathfrak{w} = 5$. The only six-dimensional Lie algebra of rank less than or equal to 2 that belongs to a compact Lie group is $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. There is no Lie subalgebra of $\mathfrak{su}(3)$ of this type and we therefore can exclude this case.

**Type 6:** $\mathfrak{f} = u(1)_{1,1} \oplus \mathfrak{w}', \dim \mathfrak{w} = 7$. In this case, $\mathfrak{f}$ is all of $\mathfrak{su}(3)$. Since $H^4(N_{k,l}, \mathbb{Z}) = \mathbb{Z}_{k^2+kl+2}$, it follows that $N_{k,l}$ is not a quotient of $S^7$ by a discrete subgroup $\Gamma \subseteq O(8)$. □

**Convention 4.3.** We want to treat the three Lie algebras $2u(1) \oplus V_i$ with $i \in \{1,2,3\}$ in a uniform manner. Therefore, we will drop the convention $k \geq l \geq 0$ whenever we consider the case $\mathfrak{f} \equiv u(2)$. This will allow us to fix $\mathfrak{f}$ as $2u(1) \oplus V_1$. Since we can replace $(k,l)$ by $(-k,-l)$, we can still assume that $k \geq l$. We have implicitly used this convention in the statement of
the lemma when we described the topology of $K/U(1)_{k,l}$ as $S^3/\mathbb{Z}_{|k+l|}$ instead of $S^3/\mathbb{Z}_{|k|}$ or $S^3/\mathbb{Z}_{|l|}$, which would be the case if $i \in \{2, 3\}$.

The results of this section can be summarized as follows.

**Theorem 4.4.** Let $(M, \Omega)$ be a Spin(7)-orbifold with a cohomogeneity-1 action of SU(3) which preserves $\Omega$ and whose principal orbit is an Aloff–Wallach space. We denote the metric associated to $\Omega$ by $g$. In this situation, the following statements are true.

(i(a) If $N^{k,l}$ is generic, then the matrix representation of $g$ with respect to the basis $(e_1, \ldots, e_7)$, which is defined by (4.3), is of type (4.6).

(ii) If $k = 1$ and $l = 0$, then the matrix representation of $g$ with respect to the basis $(e_1, e_2, e_5, e_6, e_3, e_4, e_7)$ is of type (4.7).

(iii) If $k = 1$ and $l = 1$, then the matrix representation of $g$ with respect to the basis $(e_1, e_2, e_7, e_3, e_4, e_5, e_6)$ is of type (4.8).

We assume that $N^{k,l}$ is generic or $N^{1,0}$ and that $g$ is diagonal with respect to the basis (4.3). Let $S'$ be a connected component of the space $S$ of all SU(3)-invariant Spin(7)-structures on $M$ with the same associated metric as $\Omega$. We assume that $S'$ contains a four-form $\tilde{\Omega}$ which is obtained from a basis of type (4.11). In this situation, $\tilde{\Omega}$ can be parallel only if it coincides with $\Omega$. In other words, (4.11) is up to other connected components of $S$ the only possible basis for a diagonal metric with holonomy contained in Spin(7).

(iii(a)) If $N^{k,l}$ is generic or $N^{1,0}$ and $\Omega$ is determined by a basis of type (4.11), then $\Omega$ is parallel if and only if the system (4.12) is satisfied. In that case, the holonomy of $g$ is all of Spin(7).

(iii(b)) If $k = l = 1$ and $\Omega$ is determined by a basis of type (4.13), then $\Omega$ is parallel if and only if the system (4.14) is satisfied. In this case, the holonomy of $g$ is a subgroup of Spin(7). If we also have $a_1(t) + a_2(t) = 0$ for all $t$, the holonomy of $g$ is SU(4) except in the case where we obtain the Calabi metric which has holonomy Sp(2).

If $M$ has a singular orbit, which has to be the case if $(M, \Omega)$ is parallel and complete, then it can be found in the table of Lemma 4.2. For any choice of the principal and the singular orbit, the orbifold $M$ is a manifold, except in the case where the singular orbit is $\mathbb{C}P^2$ and $|k + l| \neq 1$. In that case, $M$ is a $D^4/\mathbb{Z}_{|k+l|}$-bundle over $\mathbb{C}P^2$.

In the following three sections, we search for cohomogeneity-1 metrics with special holonomy on the spaces of the above theorem. We have to treat each possible combination of a principal and a singular orbit as a separate case, since the equations for the holonomy reduction or their initial conditions will change.

5. SU(3)/U(1)$^2$ as singular orbit

5.1. The generic Aloff–Wallach spaces as principal orbit

Throughout this and the following sections we will assume that the singular orbit is at $t = 0$. In the situation of this subsection, the Lie algebra of the isotropy group at the singular orbit is given by $u(1)_{k,l} \oplus V_4$. We therefore have $f(0) = 0$ and $a(0), b(0), c(0) \neq 0$ as initial conditions. We take a look at the equation

$$f' = -\frac{k - l}{2\Delta} \frac{f^2}{a^2} - \frac{l}{2\Delta} \frac{f^2}{b^2} - \frac{k}{2\Delta} \frac{f^2}{c^2}$$

(5.1)

from the system (4.12). We see that $f'(0) = 0$ and by a complete induction we can prove that all higher derivatives of $f$ at $t = 0$ vanish, too. Since the metric has to be analytic,
it follows that \( f \) vanishes identically. If we insert \( f \equiv 0 \) into \((4.12)\), we obtain the equations for a cohomogeneity-1 metric with principal orbit \( \text{SU}(3)/\text{U}(1)^2 \) and holonomy \( G_2 \). Since those equations were investigated by Clelton and Swann [11], we will not discuss this issue further.

Our next aim is to classify all cohomogeneity-1 Einstein metrics with our fixed orbit structure. In order to apply Theorem 3.13, we have to decompose certain \( 2\mathfrak{u}(1) \)-modules. The fact that we need these decompositions in the later sections anyway is a further motivation for our task. As in Section 3, we denote the orbit of the \( \mathfrak{u}(1) \)-action on a point \( \text{U} \) with a loop in the space \( \text{SU}(3)/\text{U}(1) \), which is an integer. That number is the same as the number of \( \mathfrak{u}(1) \)-equivariant maps from \( \text{U}(1)^{2\mathfrak{u}(1)} \) into \( \text{U}(1) \), where \( \text{U}(1)^2 \) as usual denotes the subgroup of all diagonal matrices in \( \text{SU}(3) \), can be considered as a subset of \( \mathfrak{p}^\perp \). Moreover, the orbit of the \( \text{U}(1)^\perp \)-action on a point \( p \in \mathfrak{p}^\perp \setminus \{0\} \) can be identified with a loop in the space \( \text{U}(1)^{2\mathfrak{u}(1)} \), where \( \text{U}(1)^2 \) is even, \( \text{U}(1)^1 \) is odd, and \( \text{U}(1)^0 \) is trivial with respect to \( \mathfrak{p}^\perp \). With the help of relations \((5.5)\), \((5.6)\), \((5.7)\), we can conclude that

\[
\mathfrak{p}^\perp = \mathfrak{V}_{2(2l+k^2)\mathfrak{u}(1)\mathfrak{u}(1)},.0.
\]

We are now able to decompose \( S^m(\mathfrak{p}^\perp) \) into irreducible \( 2\mathfrak{u}(1) \)-submodules and obtain

\[
S^m(\mathfrak{p}^\perp) = \begin{cases} 
\mathfrak{V}_{2m(k^2+l^2)\mathfrak{u}(1)\mathfrak{u}(1)},.0 \oplus \mathfrak{V}_{2(m-2)(l^2+k^2)\mathfrak{u}(1)\mathfrak{u}(1)},.0 \\
\quad \oplus \ldots \oplus \mathfrak{V}_{4(k^2+l^2)\mathfrak{u}(1)\mathfrak{u}(1)},.0 \oplus \mathbb{R} & \text{if } m \text{ is even,} \\
\mathfrak{V}_{2m(k^2+l^2)\mathfrak{u}(1)\mathfrak{u}(1)},.0 \oplus \mathfrak{V}_{2(m-2)(k^2+l^2)\mathfrak{u}(1)\mathfrak{u}(1)},.0 \\
\quad \oplus \ldots \oplus \mathfrak{V}_{2(k^2+l^2)\mathfrak{u}(1)\mathfrak{u}(1)},.0 & \text{if } m \text{ is odd.}
\end{cases}
\]

With the help of the decompositions of \( S^2(\mathfrak{p}) \) and \( S^m(\mathfrak{p}^\perp) \) as well as Schur’s lemma, it follows that

\[
\dim W^h_m = \begin{cases} 
3 & \text{if } m \text{ is even} \\
0 & \text{if } m \text{ is odd}
\end{cases} \quad \text{and} \quad \dim W^r_2 = 3.
\]

Since \( \mathfrak{p}^\perp \) is trivial with respect to \( \text{U}(1)^{k,l} \) and \( N^{k,l} \) is not exceptional, \( \mathfrak{p} \) and \( \mathfrak{p}^\perp \) have no non-trivial \( \text{U}(1)^{k,l} \)-submodule in common and Assumption 3.12 is satisfied. We can therefore apply Theorem 3.13 and Remark 3.14 to our situation and thus have proved the following theorem.
There are two initial conditions of third order which we can choose freely. Since the summand is a singular orbit at \( t = 0 \), \( 0 \), we have proved the following theorem.

(i) There exists no SU(3)-invariant metric on \( M \) which has holonomy Spin(7).
(ii) For any choice of \( a_0, b_0, c_0 \in \mathbb{R} \setminus \{0\} \) and \( f_3, \lambda \in \mathbb{R} \), there exists a unique SU(3)-invariant Einstein metric on a tubular neighbourhood of SU(3)/U(1)^2 such that:

(a) \( a(0)^2 = a_0^2, b(0)^2 = b_0^2, c(0)^2 = c_0^2 \);
(b) \( f''(0) = f_3 \);
(c) the Einstein constant is \( \lambda \).

5.2. \( N^{1,0} \) as principal orbit

In this situation, the only trivial \( U(1)_{1,0} \)-submodule of \( m \) again is \( V_4 \) and we have \( f(0) = 0 \) as the initial condition. Since we restrict ourselves to diagonal metrics, we can work as before with the system (4.12). Therefore, we can prove by the same arguments as in the previous case that \( f \) vanishes and the metric is degenerate.

We apply the methods of [16] in order to describe the cohomogeneity-1 Einstein metrics with the orbit structure of this subsection. Since none of the \( U(1)_{1,0} \)-modules \( V_1, V_2, \) and \( V_3 \) is trivial, Assumption 3.12 is satisfied and we are in the situation of Theorem 3.13. If \( k = 1 \) and \( l = 0 \), the decompositions of \( S^2(p) \) and \( S^m(p^+) \) which we have found specialize to

\[
S^2(p) = V_{6,2} \oplus V_{0,4} \oplus V_{-6,2} \\
\quad \quad \quad \quad \oplus V_{3,3} \oplus V_{3,-1} \oplus V_{0,2} \oplus V_{6,0} \\
\quad \quad \quad \quad \oplus V_{-3,3} \oplus V_{3,1} \oplus 3R
\]

and

\[
S^m(p^+) = \begin{cases} 
V_{2m,0} \oplus V_{2(m-2),0} \oplus \cdots \oplus V_{4,0} \oplus R & \text{if } m \text{ is even}, \\
V_{2m,0} \oplus V_{2(m-2),0} \oplus \cdots \oplus V_{2,0} & \text{if } m \text{ is odd},
\end{cases}
\]

where \( S^2(p) \) contains a summand which is isomorphic to \( V_{6,0} \). Therefore, we have \( \dim W^h_m = 2 \) if \( m \) is odd and \( \geq 3 \). We calculate \( \dim W^h_m \) for all \( m \in \mathbb{N}_0 \) and obtain

\[
\dim W^h_m = \begin{cases} 
3 & \text{if } m \text{ is even}, \\
0 & \text{if } m = 1, \\
2 & \text{if } m \geq 3 \text{ is odd}.
\end{cases}
\]

There are two initial conditions of third order which we can choose freely. Since the summand \( V_{6,0} \) is contained in \( V_1 \oplus V_3 \), the two parameters of third order describe how the metric on the singular orbit, which is diagonal, changes into a non-diagonal one. Analogously to the previous subsection we have proved the following theorem.

5.2. \( N^{1,0} \) as principal orbit

In the situation of Theorem 4.4, let \( N^{1,0} \) be the principal orbit and let SU(3)/U(1)^2 be a singular orbit at \( t = 0 \).

(i) There exists no SU(3)-invariant metric on \( M \) that is diagonal with respect to the basis (4.3) and has holonomy Spin(7).
(ii) For any choice of \( a_0, b_0, c_0 \in \mathbb{R} \setminus \{0\} \) and \( \beta, \bar{\beta}, f_3, \lambda \in \mathbb{R} \), there exists a unique SU(3)-invariant Einstein metric on a tubular neighbourhood of SU(3)/U(1)^2 such that:

(a) \( a(0)^2 = a_0^2, b(0)^2 = b_0^2, c(0)^2 = c_0^2 \);
(b) \( \beta'_{\alpha\beta}(0) = \beta, \beta'_{\gamma\delta}(0) = \bar{\beta} \);
(c) \( f''(0) = f_3 \);
(d) the Einstein constant is \( \lambda \).
5.3. \( N^{1,1} \) as principal orbit

We make the same assumptions as in Theorem 4.4 and thus can work with the system (4.14). The isotropy algebra \( \mathfrak{k} \) of the SU(3)-action on the singular orbit is spanned by \( e_8 \) and an arbitrary \( x \in V_1 \oplus V_4 \setminus \{0\} \). We assume that \( x = e_2 \) or equivalently that \( f(0) = 0 \). Since the length of the collapsing circle shall be \( 2\pi + O(t^2) \) for small \( t \), we need \( f'(0) \neq 0 \). The differential equation for \( f' \) contains the additional term \(-3(a_1 - a_2)^2/(a_1 a_2)\), which is not there if the principal orbit is generic. We therefore have \( f'(0) \neq 0 \) if and only if \( a_1(0) = a_0 = -a_2(0) \) for an \( a_0 \in \mathbb{R} \setminus \{0\} \). The values \( b_0 \) of \( b(0) \) and \( c_0 \) of \( c(0) \) can be chosen arbitrarily. We thus have formulated an initial value problem for the first-order system (4.14).

We take a short look at the case where \( x = \alpha e_1 + \beta e_2 + \gamma e_7 \) for some coefficients \( \alpha, \beta, \gamma \in \mathbb{R} \). Since we assume that the metric is diagonal, it is nevertheless an element of \( \mathcal{S}^3 \mathcal{P}(SU(3)U(1))_{1,1} \) which maps \( e_1 \) or \( e_2 \) to \( e_7 \). Metrics, which are described by solutions of (4.14) with \( a_1(0) = 0 \) or \( a_2(0) = 0 \), can be therefore be mapped by \( \tau \) to metrics with \( f(0) = 0 \). Diagonal metrics with \( a_1(0) = 0 \) or \( a_2(0) = 0 \) can be mapped to non-diagonal metrics, which we will not consider in the paper. Nevertheless, the above observation is a motivation to restrict ourselves to the case \( f(0) = 0 \).

We make a power series ansatz for (4.14) with the initial values \( a_0, b_0, \) and \( c_0 \) and obtain

\[
\begin{align*}
   a_1(t) &= a_0 - \frac{1}{2} a_0^2 - b_0^2 - c_0^2 + \frac{1}{8} \frac{13a_0^4 - 2a_0^2b_0^2 - 2a_0^2c_0^2 - b_0^4 + 14b_0^2c_0^2 - c_0^4}{a_0 b_0^2c_0^2} t^2 + \ldots, \\
   a_2(t) &= -a_0 + \frac{1}{2} a_0^2 - b_0^2 - c_0^2 t - \frac{1}{8} \frac{13a_0^4 - 2a_0^2b_0^2 - 2a_0^2c_0^2 - b_0^4 + 14b_0^2c_0^2 - c_0^4}{a_0 b_0^2c_0^2} t^2 + \ldots, \\
   b(t) &= b_0 + 0 \cdot t - \frac{1}{4} \frac{a_0^2 - 6a_0^2b_0^2 - b_0^4 + c_0^4}{a_0^2b_0^2c_0^2} t^2 + \ldots, \\
   c(t) &= c_0 + 0 \cdot t - \frac{1}{4} \frac{a_0^2 - 6a_0^2b_0^2 + b_0^4 - c_0^4}{a_0^2b_0^2c_0^2} t^2 + \ldots, \\
   f(t) &= 0 + 12t + 0 \cdot t^2 + \ldots.
\end{align*}
\]

The next issue which we discuss is how the smoothness conditions from Theorem 3.9 translate into conditions on the coefficients of the above power series. The modules \( p = \mathbb{V}_{6,0} \oplus \mathbb{V}_{3,3} \oplus \mathbb{V}_{-3,3} \) and \( p^\perp = \mathbb{V}_{6,0} \) have one submodule in common, and Assumption 3.12 is thus not satisfied. Since we assume that the metric is diagonal, it is nevertheless an element of \( S^2(p) \oplus S^2(p^\perp) \). We can therefore still check the smoothness conditions by describing \( W^h_m \) and \( W^v_m \). The module \( S^2(p) \) decomposes as

\[
S^2(p) = \mathbb{V}_{12,0} \oplus \mathbb{V}_{6,6} \oplus \mathbb{V}_{-6,6} \\
\oplus \mathbb{V}_{9,3} \oplus \mathbb{V}_{3,9} \oplus \mathbb{V}_{3,3} \oplus \mathbb{V}_{9,-3} \\
\oplus \mathbb{V}_{6,0} \oplus \mathbb{V}_{6,0} \oplus 3\mathbb{R}.
\]

For \( S^m(p^\perp) \), we obtain

\[
S^m(p^\perp) = \left\{ \begin{array}{ll}
\mathbb{V}_{6m,0} \oplus \mathbb{V}_{6(m-2),0} & \text{if } m \text{ is even,} \\
\mathbb{V}_{6m,0} \oplus \mathbb{V}_{6(m-2),0} \oplus \ldots \oplus \mathbb{V}_{6,0} & \text{if } m \text{ is odd.}
\end{array} \right.
\]

The dimensions of \( W^h_m \) and \( W^v_m \) can be calculated with the help of Schur’s lemma:

\[
\dim W^h_m = \left\{ \begin{array}{ll}
3 & \text{if } m = 0, \\
5 & \text{if } m \geq 2 \text{ and even,} \\
2 & \text{if } m \text{ odd}
\end{array} \right. \quad \text{and} \quad \dim W^v_m = \left\{ \begin{array}{ll}
1 & \text{if } m = 0, \\
3 & \text{if } m \geq 2 \text{ and even,} \\
0 & \text{if } m \text{ odd.}
\end{array} \right.
\]

The three dimensions of \( W^h_0 \) correspond to the coefficients of \( e^1 \otimes e^1 + e^2 \otimes e^2, e^3 \otimes e^3 + e^4 \otimes e^4, \) and \( e^5 \otimes e^5 + e^6 \otimes e^6 \). The two additional dimensions of \( W^h_m \), where \( m \) is even and \( \geq 2 \),
are caused by the submodule $\mathcal{V}_{12,0}$ of $S^2(V_1)$. They therefore describe the coefficients of $e^1 \otimes e^1 - e^2 \otimes e^2$ and $e^1 \otimes e^2 + e^2 \otimes e^1$. Since we consider only diagonal metrics, we can ignore the freedom that we have for the second coefficient. The two dimensions of $W^h_m$, where $m$ is odd, describe $K$-equivariant maps $S^m(p^\perp) \to S^2(p)$, which are non-zero only on $V_2 \otimes V_3$ and can be ignored for the same reasons as above.

The two additional dimensions of $W^v_m$, where $m$ is even and greater than 2, describe the freedom of $g(\frac{\partial}{\partial f}, \frac{\partial}{\partial m})$ and $g(\frac{\partial}{\partial f}, e_7)$ and can be ignored, too (see Remark 3.10). As we have remarked at the same place, the value of $|f'(0)|$ has to be chosen in such a way that the length of the collapsing circle is $2\pi t + O(t^2)$ for small $t$. Since $\exp(e_7t)$ intersects $U(1)_{1,1}$ at $t = \pi/3$ for the first time, the length of the collapsing circle is

$$\int_0^{\pi/3} \sqrt{g(e_7, e_7)} ds = \frac{\pi}{3} |f(t)|. \quad (5.15)$$

We therefore obtain $|f'(0)| = 6$ as a smoothness condition. By translating our statements on $W^h_m$ and $W^v_m$ into conditions on power series expansion of the metric, we finally see that an analytic diagonal metric of type (4.8) is smooth if and only if

1. $a^2_1(0) = a^2_2(0)$;
2. $a^2_1, a^2_2, b^2,$ and $c^2$ are even functions;
3. $f(0) = 0, |f'(0)| = 6$;
4. $f$ is odd.

The power series (5.11) does not satisfy these conditions. Nevertheless, there is a method to obtain smooth cohomogeneity-1 metrics whose holonomy is contained in Spin(7) from (5.11). Let

$$h := \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{SU}(3). \quad (5.16)$$

The above matrix acts by $h.gU(1)_{1,1} := hgh^{-1}U(1)_{1,1}$ on $N^{1,1}$ and stabilizes the $G_2$-structures that are determined by a basis of type (4.13). Since $h^2 \in U(1)_{1,1}$, it follows that $N^{1,1}$ is a double cover of the quotient of SU(3) by the group that is generated by $U(1)_{1,1}$ and $h$. We denote this quotient simply by $N^{1,1}/\mathbb{Z}_2$. On SU(3)/U(1)$^2$, $h$ acts trivially. We divide the cohomogeneity-1 manifold with principal orbit $N^{1,1}$ and singular orbit SU(3)/U(1)$^2$ by the group that is generated by the simultaneous action of $h$ on all orbits. Any (smooth or non-smooth) Spin(7)-structure on the old manifold, which is induced by (4.13) is mapped by the quotient map to a new one.

It is easy to see that any circle that was wrapped $r$ times around the origin of the old normal space is wrapped $2r$ times around the origin of the new normal space. We reconsider the arguments which we have made and see that in this new situation we have to require $|f'(0)| = 12$ instead of $|f'(0)| = 6$ in order to make the metric smooth at the singular orbit. Since $U(1)^2$ now acts on $p^\perp$ as $\mathcal{V}_{12,0}$ rather than $\mathcal{V}_{6,0}$, we have

$$\dim W^h_m = \begin{cases} 3 & \text{if } m \text{ even}, \\ 2 & \text{if } m \text{ odd}. \end{cases} \quad (5.17)$$

Since the values of $\dim W^h_m$ have changed, the smoothness conditions on the functions $a_1, a_2, b,$ and $c$ have changed, too. The meaning of the three dimensions in the even case is the same as before and the two dimensions in the odd case now correspond to the components of the metric in $S^2(V_1)$. The dimensions of $W^v_m$ stay the same and we have found the following new smoothness conditions:

1. $a^2_1(t) = a^2_2(-t)$;
2. $b^2$ and $c^2$ are even;
3. $f(0) = 0, |f'(0)| = 12$;
4. $f$ is odd.
In particular, these conditions are satisfied if \( a_1(t) = -a_2(-t), b(t) = b(-t), \) and \( c(t) = c(-t). \) The first coefficients of (5.11) obviously satisfy this new set of conditions.

By an explicit calculation, we can prove that any power series solution of (4.14) with the same initial conditions as in this subsection is uniquely determined by \( a_0, b_0, \) and \( c_0. \) The system (4.14) is preserved if we replace \( (a_1(t), a_2(t), b(t), c(t), f(t)) \) by \((-a_2(-t), -a_1(-t), b(-t), c(-t), -f(-t))\), respectively. Therefore, both sets of functions are the unique solutions of the same initial value problem and the power series thus satisfies the smoothness conditions. The matrix \( h \) acts trivially on \( p^+ \) and non-trivially on \( V_1, V_2, \) and \( V_3. \) For this reason, Assumption 3.12 is satisfied and we can show with the help of Theorem 3.13 that the series converges near the singular orbit. Moreover, we can use the dimensions of \( W^m_n \) and \( W^m_n \) to calculate the number of cohomogeneity-1 Einstein metrics near the singular orbit.

We take another look at the power series (5.11). The condition \( a_1(t) + a_2(t) = 0 \) can only be satisfied if \( a_0^2 = b_0^2 + c_0^2. \) In that situation, we obtain the metrics of Bazaikin and Malkovich [6], which have holonomy SU(4). We finally have proved the following theorem.

**Theorem 5.3.** Let \( M \) be a cohomogeneity-1 manifold whose principal orbit is \( N^{1,1}/\mathbb{Z}_2, \) where \( \mathbb{Z}_2 \) is generated by (5.16). We assume that \( M \) has exactly one singular orbit of type \( SU(3)/U(1)^2, \) which is at \( t = 0. \)

(i) For any \( a_0, b_0, c_0 \in \mathbb{R} \setminus \{0\} \) there exists a unique \( SU(3) \)-invariant parallel \( Spin(7) \)-structure \( \Omega \) on a tubular neighbourhood of \( SU(3)/U(1)^2, \) which is determined by a basis of type (4.13) and satisfies \( f(0) = 0, a_1(0) = -a_2(0) = a_0, b(0) = b_0, \) and \( c(0) = c_0. \)

(ii) The metric associated to \( \Omega \) is a diagonal metric of type (4.8). If \( a_0^2 = b_0^2 + c_0^2, \) then its holonomy is \( SU(4). \)

(iii) For any choice of \( a_0, b_0, c_0 \in \mathbb{R} \setminus \{0\} \) and \( a_1, \beta, f_3, \lambda \in \mathbb{R} \), there exists a unique \( SU(3)-\)invariant Einstein metric on a tubular neighbourhood of \( SU(3)/U(1)^2 \) such that:

(a) \( f(0) = 0, a_1(0)^2 = a_2(0)^2 = a_0^2, b(0)^2 = b_0^2, c(0)^2 = c_0^2; \)

(b) \( (a_1 - a_2)'(0) = a_1, \beta_1^2(0) = \beta; \)

(c) \( f''(0) = f_3; \)

(d) the Einstein constant is \( \lambda. \)

**Remark 5.4.** We have proved that any solution of (4.14) that satisfies the initial conditions from the above theorem automatically satisfies the smoothness conditions. According to Lemma 3.11, such a solution describes a complete metric if it is defined on all of \([0, \infty). \)

For special cases the global behaviour of the functions that solve (4.14) is known. In [4], Bazaikin investigates a geometric evolution equation on a 3-Sasakian manifold, which yields a parallel \( Spin(7) \)-structure. If the 3-Sasakian manifold is \( N^{1,1}/\mathbb{Z}_2, \) then his system is equivalent to (4.14) with the additional constraint \( b(t) = c(t) \) for all \( t. \)

If \( b(0) > a_2(0) > 0, \) then Bazaikin proves that the metric is asymptotically locally conical (ALC) and has holonomy \( Spin(7). \) If we have \( b(0) = a_2(0), \) then the metric is asymptotically conical (AC) and has holonomy \( SU(4). \) In particular, both kinds of metrics are complete. Any other solution with \( b(0) = c(0) \) is either incomplete or isometric to the known ones.

The metrics from our theorem with holonomy a subgroup \( SU(4) \) are described explicitly in a paper by Bazaikin and Malkovich [6]. All of them are AC and the special example from [4] with holonomy \( SU(4) \) is among them. The authors of [6] prove that the holonomy of their metrics is all of \( SU(4) \) except in the limiting case where \( b_0 \to 0 \) and the metric degenerates into the hyperkähler metric of Calabi [10] on \( T^*\mathbb{C}P^2. \) In [4, 6], the authors deduce the same smoothness conditions as in this paper. Similar observations as Bazaikin on the metrics with holonomy \( SU(4) \) have been made by Kanno and Yasui [25].

We remark that the authors of [6] announce a proof that the metrics from Theorem 5.3 are either incomplete or isometric to the metrics from [4, 6]. Although many facts are known about
this subject, our methods for the calculation of the smoothness conditions and for the proof of the holonomy reduction in the case \( a_1(t) + a_2(t) = 0 \) are new contributions of the author.

6. \( S^5 \) as singular orbit

The sphere \( S^5 \) is a possible singular orbit only if \( k \cdot l \cdot (-k - l) = 0 \). We assume in this section that \( (k, l) = (1, -1) \), since this implies the initial conditions \( a(0) = 0 \) and \( b(0), c(0), f(0) \neq 0 \). The only difference to the case \( (k, l) = (1, 0) \) is that the off-diagonal coefficients of the metric are contained in \( V_2 \otimes V_3 \oplus V_3 \otimes V_2 \) instead of \( V_1 \otimes V_3 \oplus V_3 \otimes V_1 \) and hence will be denoted by \( \beta_{3,5} \) and \( \beta_{3,6} \) instead of \( \beta_{1,5} \) and \( \beta_{1,6} \), respectively. We want the right-hand side of the second and third equations of (4.12) to converge to a real number for \( t \to 0 \). This is only possible if \( b(0)^2 = c(0)^2 \). Since we may replace \( c \) by \( -c \), \( f \) by \( -f \), and \( t \) by \( -t \) without changing the system (4.12), we can assume that \( b(0) = c(0) \). We denote \( b(0) \) by \( b_0 \) and \( f(0) \) by \( f_0 \). By a power series ansatz, we obtain the following solution of the system (4.12):

\[
\begin{align*}
a(t) &= 0 + 2t + 0 \cdot t^2 - \frac{1}{b_0^3} \cdot \frac{36b_0^2 - f_0^2}{27} \cdot t^4 + \ldots, \\
b(t) &= b_0 - \frac{1}{6} \left( \frac{f_0}{b_0} - 1 \right) + \frac{72b_0^2 - 5f_0^2}{72} \cdot t^2 + \frac{1}{6480} \left( \frac{f_0(504b_0^2 - 167f_0^2)}{b_0^3} \right) \cdot t^3 + \ldots, \\
c(t) &= b_0 + \frac{1}{6} \left( \frac{f_0}{b_0} - 1 \right) + \frac{72b_0^2 - 5f_0^2}{72} \cdot t^2 - \frac{1}{6480} \left( \frac{f_0(504b_0^2 - 167f_0^2)}{b_0^3} \right) \cdot t^3 + \ldots, \\
f(t) &= f_0 + 0 \cdot t + \frac{1}{6} \left( \frac{f_0}{b_0} - 1 \right) \cdot t^2 + 0 \cdot t^3 + \ldots.
\end{align*}
\]

Our next step is to decompose \( S^m(p^\perp) \) and \( S^2(p) \) into \( \text{SU}(2) \)-submodules. This is necessary in order to deduce the smoothness conditions. We denote the complex irreducible representation of \( \text{SU}(2) \) with weight \( r \) by \( \mathbb{V}_r^C \). We denote the real irreducible representation with the same weight by \( \mathbb{V}_r^R \). We recall that \( \text{dim} \mathbb{V}_r^R = r + 1 \) if \( r \) is even and \( \text{dim} \mathbb{V}_r^R = 2(r + 1) \) if \( r \) is odd.

The orbits of the \( \text{SU}(2) \)-action on the three-dimensional space \( p^\perp \) are spheres. Therefore, it is isomorphic to \( \mathbb{V}_2^R \). In order to decompose \( S^m(p^\perp) \), we first consider the space \( S_m^R(\mathbb{C}^2) \) of all homogeneous polynomials of \( m \)-th order with real coefficients on \( \mathbb{C}^2 \). The subscript \( R \) means that we consider \( \mathbb{C}^2 \) as a four-dimensional real vector space and we have \( \text{dim} S_m^R(\mathbb{C}^2) = 10 \), \( \text{dim} S_2^R(\mathbb{C}^2) = 20 \), etc. The complexification \( S_m^C(\mathbb{C}^2) \otimes \mathbb{C} \) consists of all polynomials depending on \( z_1, z_2, z_\bar{1}, z_\bar{2} \). Let \( \mathbb{V}_{s,m-s} \) be the subspace of all trace-free polynomials depending on \( s \) complex and \( m - s \) conjugate complex variables. The vector spaces \( S_{s,m-s}^C(\mathbb{C}^2) \otimes \mathbb{C} \) and \( \mathbb{V}_{s,m-s} \) are obviously \( \text{SU}(2) \)-modules and we have the following decomposition

\[
S_{s,m-s}^C(\mathbb{C}^2) \otimes \mathbb{C} = \bigoplus_{p=0}^{\lfloor m/2 \rfloor} \bigoplus_{s=0}^{m-2p-s} \mathbb{V}_{s,m-2p-s}^C.
\]

The submodules \( \mathbb{V}_{s,m-2p-s} \) are irreducible and by calculating their dimension we see that

\[
S_{s,m-2p}^C(\mathbb{C}^2) \otimes \mathbb{C} = \bigoplus_{p=0}^{\lfloor m/2 \rfloor} (m - 2p + 1) \mathbb{V}_{s,m-2p}^C,
\]

where \( (m - 2p + 1) \mathbb{V}_{s,m-2p}^C \) denotes the direct sum of \( m - 2p + 1 \) copies of \( \mathbb{V}_{s,m-2p}^C \). The module \( S_{s}^C(\mathbb{C}^2) \) consists of exactly those elements of \( S_{s}^C(\mathbb{C}^2) \otimes \mathbb{C} \) which are invariant with respect to the conjugation map \( \tau \). The conjugation \( \tau \) maps \( \mathbb{V}_{s,m-2p-s} \) into \( \mathbb{V}_{m-2p-s} \) and vice versa. If \( m \) is even and \( s \neq (m - 2p)/2 \), then the subspace of \( \mathbb{V}_{s,m-2p-s} \oplus \mathbb{V}_{m-2p-s,\tau} \) is invariant under \( \tau \) and decomposes into two real submodules of the same dimension. Both submodules are irreducible and equivalent to \( \mathbb{V}_{s,m-2p}^R \). The module \( \mathbb{V}_{(m-2p)/2,(m-2p)/2} \) is a real one and is
isomorphic to $\mathcal{V}^R_{m-2p}$. If $m$ is odd, then the subspace of $\mathcal{V}^R_{m-2p-s} \oplus \mathcal{V}^R_{m-2p-s,t}$ is invariant under $\tau$, irreducible, and equivalent to $\mathcal{V}^R_{m-2p}$. All in all, we have

$$S^m_R(\mathbb{C}^2) = \begin{cases} \bigoplus_{p=0}^{m/2} (2p+1)\mathcal{V}^R_{2p} & \text{if } m \text{ is even}, \\ \bigoplus_{p=0}^{(m-1)/2} (p+1)\mathcal{V}^R_{2p+1} & \text{if } m \text{ is odd}. \end{cases} \quad (6.4)$$

$\mathcal{V}^R_{2}$ can be identified with $\mathcal{V}_{1,1}$. The module $S^m(\mathcal{V}^R_{2})$ therefore is a submodule of $\mathcal{V}_{m,m}$. With the help of the above considerations, we obtain

$$S^m(\mathcal{V}^R_{2}) = \bigoplus_{p=0}^{[m/2]} \mathcal{V}_{m-2p,m-2p} = \bigoplus_{p=0}^{[m/2]} \mathcal{V}^R_{2m-4p}. \quad (6.5)$$

We are now able to calculate the dimension of $W^h_m$ and $W^v_m$. By a short calculation we see that

$$p = \mathcal{V}^C_1 \oplus \mathcal{V}^R_0. \quad (6.6)$$

With the help of (6.4), we conclude that

$$S^2(p) = S^2(\mathcal{V}^C_1) \oplus (\mathcal{V}^C_1 \oplus \mathcal{V}^R_0) \oplus S^2(\mathcal{V}^R_0) = 3\mathcal{V}_2^R \oplus \mathcal{V}^C_1 \oplus 2\mathcal{V}^R_0 \quad (6.7)$$

and finally obtain

$$\dim W^h_m = \begin{cases} 2 & \text{if } m \text{ is even}, \\ 3 & \text{if } m \text{ is odd}. \end{cases} \quad (6.8)$$

We interpret these numbers and start with the case $m = 0$. Any SU(2)-invariant metric on the singular orbit is diagonal and satisfies $b(0)^2 = c(0)^2$. The fact that $\dim W^h_0 = 2$ therefore simply means that we can choose the initial values $b_0$ and $f_0$ freely. Let $b_m$, $c_m$, and $f_m$ denote the $m$th coefficients of the power series for $b$, $c$, and $f$. Analogously to the case $m = 0$, $\dim W^h_m = 2$ for even $m$ means that $b_m = c_m$.

There is a suitable $h \in \text{SU}(2)$ that acts on $\text{span}(e_3, e_4, e_5, e_6) \subseteq p$ in the same way as $j \in \text{Sp}(1)$ by right-multiplication on $\mathbb{H}$. Since $p^\perp$ is three-dimensional, $j$ acts as a rotation around an angle of $\pi$. Moreover, it is a rotation around an axis perpendicular to $\partial/\partial t$ and thus turns $\partial/\partial t$ into $-\partial/\partial t$. Since the metric $g$ in invariant under $h$, it follows that

$$g_t(e_3, e_3) = g_{-t}(e_5, e_5),$$
$$g_t(e_3, e_5) = -g_{-t}(e_5, e_3),$$
$$g_t(e_3, e_6) = -g_{-t}(e_5, e_4) = -g_{-t}(e_3, e_6),$$
$$g_t(e_7, e_7) = g_{-t}(e_7, e_7). \quad (6.9)$$

This translates into $b_m = -c_m$ and $f_m = 0$ if $m$ is odd. The space of all $U(1)_{1,1}$-invariant elements of $S^2(p)$ satisfying these conditions has dimension 3 which equals $\dim W^h_m$. Therefore, there are no further conditions on the horizontal part and the horizontal part of an analytic metric is smooth if and only if

$$b(t) = c(-t) \quad \text{and} \quad f, \beta_{3,5}, \beta_{3,6} \text{ are even}. \quad (6.10)$$

We turn to the vertical component of the metric, which is determined by $a$. It follows from (6.5) that

$$S^2(p^+) = \mathcal{V}^R_4 \oplus \mathcal{V}^R_0. \quad (6.11)$$

Since $\dim \text{Hom}_{\text{SU}(2)}(\mathcal{V}^R_4, \mathcal{V}^R_4) = 1$, we have

$$\dim W^v_m = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \text{ is odd}, \\ 2 & \text{if } m \geq 2 \text{ and even}. \end{cases} \quad (6.12)$$
For similar reasons as in Subsection 5.3, this means that \( a \) has to be an odd function. The only missing smoothness condition is that the length \( \ell(t) \) of any great circle of the collapsing sphere \( SU(2)/U(1)_{1,-1} \) has to be \( 2\pi t + O(t^2) \) for small \( t \). The Lie group that is generated by \( e_1 \) intersects \( U(1)_{1,-1} \) twice. Therefore, \( \ell(t) = \sqrt{g_\pi(e_1, e_1)} = |a(t)|\pi \). By the same argument as in Subsection 5.3, it follows that \( |a'(0)| \) has to be 2, which is indeed the case.

As in the previous case, we make an explicit calculation and prove that, for any choice of \( b_0, f_0 \in \mathbb{R} \setminus \{0\} \), there exists a unique power series solution of (4.12). The change of variables \( (a(t), b(t), c(t), f(t)) \rightarrow (-a(-t), c(-t), b(-t), f(-t)) \) is a symmetry of (4.12) if \( (k, l) = (1, -1) \).

As in the previous subsection, it follows that the power series satisfies the smoothness conditions.

Unfortunately, Assumption 3.12 is not satisfied, since \( p \) and \( p^\perp \) both contain a trivial submodule, namely \( \text{span}(e_7) \) or \( \text{span}(\frac{\partial}{\partial t}) \), respectively. Nevertheless, we always have, due to the choice of our coordinates, \( g(e_7, \frac{\partial}{\partial t}) = 0 \). Moreover, we have \( \text{Ric}(e_7, \frac{\partial}{\partial t}) = 0 \) (see Grove, Ziller [20]). Therefore, \( g, \text{Ric} \), and the steps of the Picard iteration in Eschenburg, Wang [16] do not leave the space \( S^2(p) \oplus S^2(p^\perp) \). The arguments of the proof in [16] therefore remain unchanged, and we can conclude that the power series converges. After that, we are finally able to determine the number of the Einstein metrics.

**Theorem 6.1.** Let \( M \) be a cohomogeneity-1 manifold whose principal orbit is \( N^{1,1} \). We assume that \( M \) has exactly one singular orbit of type \( S^5 \), which is at \( t = 0 \).

- (i) In this situation, any cohomogeneity-1 metric satisfies \( a(0) = 0 \).
- (ii) For any \( b_0, f_0 \in \mathbb{R} \setminus \{0\} \) there exists a unique \( SU(3) \)-invariant parallel \( \text{Spin}(7) \)-structure \( \Omega \) on a tubular neighbourhood of \( S^5 \) which is determined by a basis of type (4.11) and satisfies \( b(0) = c(0) = b_0 \) and \( f(0) = f_0 \). The holonomy of the associated metric is all of \( \text{Spin}(7) \).
- (iii) For any choice of \( b_0, f_0 \in \mathbb{R} \setminus \{0\} \) and \( b_1, \beta, \tilde{\beta}, a_3, \lambda \in \mathbb{R} \), there exists a unique \( SU(3) \)-invariant Einstein metric on a tubular neighbourhood of \( S^5 \) such that:
  - (a) \( b(0)^2 = c(0)^2 = b_0^2, f(0)^2 = f_0^2 \);
  - (b) \( b - c)'/(0) = b_1, \beta_{3,5}(0) = \beta, \beta_{3,6}'(0) = \tilde{\beta} \);
  - (c) \( a'''(0) = a_3 \);
  - (d) the Einstein constant is \( \lambda \).

**Remark 6.2.**

- (i) The metrics with singular orbit \( S^5 \) and holonomy \( \text{Spin}(7) \) were also considered by Cvetiˇc et al. [14]. The authors solve equation (4.12) numerically. There is a critical value of \( \|f_0/b_0\| \) such that the metric is AC. If \( |f_0|/|b_0| \) is smaller, the metric is ALC, and for bigger values it is incomplete. However, there is up to now no explicit proof of this statement. The discussion of the smoothness conditions, the convergence of the power series and the existence of the Einstein metrics are new results of this paper.
- (ii) If the principal orbit is \( N^{1,1} \), then the singular orbit can also be a space of type \( SU(3)/SO(3) \). However, it is impossible that the metric on \( SU(3)/SO(3) \) is positive and the volume of \( SO(3)/U(1)_{1,-1} \) shrinks to zero if the metric is diagonal with respect to \( (e_i)_{i=1}^7 \). Although it is possible to construct spaces with singular orbit \( SU(3)/SO(3) \) by considering non-diagonal metrics, we shall not investigate this case further.

**7. \( \mathbb{CP}^2 \) as singular orbit**

We assume that the principal orbit is a generic Alff–Wallach space \( N^{k,l} \). In this situation, the isotropy algebra \( \mathfrak{k} \) of the \( SU(3) \)-action on \( \mathbb{CP}^2 \) is either \( u(1)_{k,l} \oplus V_1 \oplus V_4, u(1)_{k,l} \oplus V_2 \oplus V_4, \) or \( u(1)_{k,l} \oplus V_3 \oplus V_4 \). As we have remarked in Convention 4.3, it suffices to work with the case \( \mathfrak{k} = u(1)_{k,l} \oplus V_1 \oplus V_4 \) if we consider all pairs \( (k, l) \) of coprime integers with \( k \geq l \).
If \((k, l) = (1, -1)\), then \(u(1)_{k,l} \oplus V_1 \oplus V_4\) is not a possible choice of \(\mathfrak{t}\) since \(K/U(1)_{1,-1}\) is diffeomorphic to \(S^2 \times S^1\). Nevertheless, the two related cases \((k, l) \in \{(1,0), (0,-1)\}\) are still possible.

If \(k = l = 1\) and the metric is diagonal, then \(u(1)_{1,1} \oplus V_1 \oplus V_4\), \(u(1)_{1,1} \oplus V_2 \oplus V_4\), and \(u(1)_{1,1} \oplus V_3 \oplus V_4\) are still the only possibilities for \(\mathfrak{t}\). If the complement of \(u(1)_{1,1}\) was transversely embedded into \(V_1 \oplus V_2 \oplus V_3\), then the null space of the degenerate metric \(g_0 \in S^2(m)\) could not be \(\mathfrak{t}\). Moreover, \(u(1)_{1,1} \oplus V_2 \oplus V_4\) and \(u(1)_{1,1} \oplus V_3 \oplus V_4\) can be obtained by the action of an element of \(\text{Norm}_{SU(3)}U(1)_{1,1}\) from each other. We therefore have to consider only one of these cases. All in all, we have to consider the following three initial values problems.

1. The system (4.12) with \(a(0) = f(0) = 0\) and \((k, l) \notin \{(1, -1), (1, 1), (1, -2), (2, -1)\}\).
2. The system (4.14) with \(a_1(0) = a_2(0) = f(0) = 0\).
3. The system (4.14) with \(b(0) = f(0) = 0\).

We make a power series ansatz for all of the above initial value problems and start with the first one. As in Section 6, we can assume without loss of generality that \(b(0) = c(0) =: b_0\). The Taylor expansion of any solution of our initial value problem begins with

\[
a(t) = 0 + t + 0 \cdot t^2 - \frac{1}{24b_0^2} \frac{12\Delta + q(k + l)}{\Delta} t^3 + 0 \cdot t^4 + \ldots,
\]

\[
b(t) = b_0 + 0 \cdot t + \frac{4k + 5l}{6b_0} t^2 + 0 \cdot t^3 + \frac{1}{288b_0^3} \frac{(-104k^2 - 224kl - 140l^2)\Delta + q(-k^3 - k^2l + kl^2 + l^3)}{\Delta^2} t^4 + \ldots,
\]

\[
c(t) = b_0 + 0 \cdot t + \frac{5k + 4l}{6b_0} t^2 + 0 \cdot t^3 + \frac{1}{288b_0^3} \frac{(-140k^2 - 224kl - 140l^2)\Delta + q(k^3 + k^2l - kl^2 - l^3)}{\Delta^2} t^4 + \ldots,
\]

\[
f(t) = 0 + \frac{2\Delta}{k + l} t + 0 \cdot t^2 + \frac{q}{6b_0} t^3 + 0 \cdot t^4 + \ldots.
\]

The parameter \(q\) of third order can be chosen freely. Any solution of (4.14) with \(a_1(0) = a_2(0) = f(0)\) also has to satisfy \(b(0)^2 = c(0)^2\). We again can assume that \(b(0) = c(0) =: b_0\), since \((c(t), f(t)) \mapsto (-c(-t), -f(-t))\) is a symmetry of the system (4.14), and obtain the following Taylor expansion:

\[
a_1(t) = 0 + t + 0 \cdot t^2 + \frac{q_1}{6b_0^2} t^3 + 0 \cdot t^4 + \frac{2q_1^2 - 3q_1 q_2 - 3q_2^2 - 3q_1 - 18q_2}{60b_0^4} t^5 + \ldots,
\]

\[
a_2(t) = 0 + t + 0 \cdot t^2 + \frac{q_2}{6b_0^2} t^3 + 0 \cdot t^4 + \frac{-3q_1^2 - 3q_1 q_2 + 2q_2^2 - 18q_1 - 3q_2}{60b_0^4} t^5 + \ldots,
\]

\[
b(t) = b_0 + 0 \cdot t + \frac{3}{4b_0} t^2 + 0 \cdot t^3 - \frac{39}{96b_0^3} t^4 + 0 \cdot t^5 + \ldots,
\]

\[
c(t) = b_0 + 0 \cdot t + \frac{3}{4b_0} t^2 + 0 \cdot t^3 - \frac{39}{96b_0^3} t^4 + 0 \cdot t^5 + \ldots,
\]

\[
f(t) = 0 + 3t + 0 \cdot t^2 - \frac{6 + q_1 + q_2}{2b_0^2} t^3 + 0 \cdot t^4 + \frac{2q_1^2 + 7q_1 q_2 + 2q_2^2 + 27q_1 + 27q_2 + 90}{20b_0^4} t^5 + \ldots.
\]

Analogously to the previous case, there are two parameters \(q_1\) and \(q_2\) which can be chosen freely. The functions \(b\) and \(c\) coincide up to fifth order. Later on, we will prove that actually \(b(t) = c(t)\) for all values of \(t\). Next we consider equation (4.14) under the assumption that \(b(0) = f(0) = 0\). We necessarily have \(c(0)^2 = a_1(0)^2 = a_2(0)^2\). Let \(a_0 := a_1(0)\). Since there are
four possibilities for the signs of \(a_2(0)\) and \(c(0)\), there are four kinds of initial value problems. Because of the symmetry of (4.14), we can assume that the sign of \(a_1(0)\) and \(c(0)\) is the same. Therefore, the only two subcases that we have to consider are \(a_1(0) = a_2(0)\) and \(a_1(0) = -a_2(0)\). The initial condition \(a_1(0) = a_2(0)\) yields the following power series:

\[
\begin{align*}
a_1(t) &= a_0 + 0 \cdot t + \frac{1}{a_0} t^2 + 0 \cdot t^3 - \frac{q + 21}{24a_0^3} t^4 + 0 \cdot t^5 + \ldots, \\
a_2(t) &= a_0 + 0 \cdot t + \frac{1}{a_0} t^2 + 0 \cdot t^3 - \frac{q + 21}{24a_0^3} t^4 + 0 \cdot t^5 + \ldots, \\
b(t) &= 0 + t + 0 \cdot t^2 + \frac{q}{6a_0^2} t^3 + 0 \cdot t^4 - \frac{8q^2 + 42q + 9}{120a_0^4} t^5 + \ldots, \\
c(t) &= a_0 + 0 \cdot t + \frac{1}{2a_0} t^2 + 0 \cdot t^3 - \frac{q - 6}{24a_0^3} t^4 + 0 \cdot t^5 + \ldots, \\
f(t) &= 0 - 6t + 0 \cdot t^2 + \frac{2(q + 3)}{a_0^2} t^3 + 0 \cdot t^4 - \frac{11q^2 + 54q + 123}{10a_0^4} t^5 + \ldots,
\end{align*}
\]

where \(q\) is a free parameter. Later on it will be proved that \(a_1(t) = a_2(t)\) for all \(t\). Next, we study (4.14) under the assumption that \(b(0) = f(0) = 0\) and \(a_1(0) = -a_2(0)\). We obtain a system of quadratic equations for the first derivatives \((a_1'(0), \ldots, f'(0))\). The only two meaningful solutions of that system are \((0, 0, 1, 0, 6)\) and \((0, 0, -1, 0, 6)\). The change of variables \((a_1(t), a_2(t), b(t), c(t), f(t)) \mapsto (-a_2(-t), -a_1(-t), b(-t), c(-t), -f(-t))\) is another symmetry of (4.14). Since it maps a solution with \(b'(0) = 1\) into a solution with \(b'(0) = -1\), we only need to consider the case \(b'(0) = 1\) and obtain

\[
\begin{align*}
a_1(t) &= a_0 + 0 \cdot t + \frac{1}{a_0} t^2 + 0 \cdot t^3 + \frac{3q - 23}{24a_0^3} t^4 + 0 \cdot t^5 + \ldots, \\
a_2(t) &= -a_0 + 0 \cdot t + \frac{q - 2}{a_0} t^2 + 0 \cdot t^3 - \frac{12q^2 - 25q - 7}{24a_0^3} t^4 + 0 \cdot t^5 + \ldots, \\
b(t) &= 0 + t + 0 \cdot t^2 - \frac{1}{6a_0^2} t^3 + 0 \cdot t^4 - \frac{39q^2 - 114q + 25}{240a_0^5} t^5 + \ldots, \\
c(t) &= a_0 + 0 \cdot t + \frac{q}{2a_0} t^2 + 0 \cdot t^3 + \frac{3q^2 - 13q + 3}{24a_0^3} t^4 + 0 \cdot t^5 + \ldots, \\
f(t) &= 0 + 6t + 0 \cdot t^2 - \frac{4}{a_0^2} t^3 + 0 \cdot t^4 + \frac{3q^2 - 18q + 175}{20a_0^4} t^5 + \ldots.
\end{align*}
\]

As usual, \(q\) can be chosen freely. Our aim is to prove that the three initial value problems have a unique smooth solution for any choice of the parameters \(a_0, b_0, q, q_1,\) and \(q_2\). We therefore have to check the smoothness conditions for the above power series. In [31], we have proved that an analytic diagonal metric \(g = g_t + dt^2\) of cohomogeneity 1 has a smooth extension to a singular orbit at \(t = 0\) if

1. \(g_t\) converges for \(t \to 0\) to a degenerate bilinear form that is invariant with respect to the cohomogeneity-1 action;
2. the sectional curvature of the collapsing sphere behaves as \(1/t + O(1)\) for \(t \to 0\);
3. the coefficient functions of the horizontal part are even;
4. the coefficient functions of the vertical part are odd.

We remark that this result can also be applied to orbifold metrics. The metric on the singular orbit \(\mathbb{C}P^2\) is in all three cases the Fubini study metric and thus \(SU(3)\)-invariant.

In order to check the second smoothness condition, which we have also mentioned in Remark 3.10, we have to search for the metric \(h\) with constant sectional curvature 1 on \(K/U(1)_{k,l}\). Let \(h\) with \(h(X, Y) = -\tfrac{1}{2} \text{tr}(XY)\) for all \(X, Y \in \mathfrak{su}(2)\) be the metric with
sectional curvature 1 on SU(2). We embed SU(2) into SU(3) such that its Lie algebra becomes \( u(1)_{-1} \oplus V_1 \). The map \( \pi : SU(2) \to K/U(1)_{k,l} \) with \( \pi(k) := kU(1)_{k,l} \) is a covering map. The metric \( h \) therefore has to satisfy
\[
\|d\pi(X)\|_h = \|X\|_h. 
\] (7.5)

With the help of this formula, we see that

1. in the case where \( a(0) = f(0) = 0 \), \( \|e_1\|_q = \|e_2\|_q = 1 \), and \( \|e_\gamma\|_q = |2\Delta/(k + l)| \); we therefore obtain \( |a'(0)| = 1 \) and \( |f'(0)| = |2\Delta/(k + l)| \);
2. in the case where \( a_1(0) = a_2(0) = f(0) = 0 \), \( \|e_1\|_q = \|e_2\|_q = 1 \), and \( \|e_\gamma\|_q = 3 \); we therefore obtain \( |a'_1(0)| = |a'_2(0)| = 1 \) and \( |f'(0)| = 3 \);
3. in the case where \( b(0) = f(0) = 0 \), \( \|e_3\|_q = \|e_4\|_q = 1 \), and \( \|e_\gamma\|_q = 6 \); we therefore obtain \( |b'(0)| = 1 \) and \( |f'(0)| = 6 \).

Here, \( q \) denotes the biinvariant metric on \( su(3) \) with \( q(X,Y) = -\frac{1}{4}tr(XY) \) which we have introduced earlier.

The power series (7.1), (7.2), (7.3), and (7.4) obviously satisfy the first two smoothness conditions. As in the previous two sections, we can prove that the initial value problems have a unique power series solution for any choice of \( a_0, b_0, q, q_1 \), and \( q_2 \). We are now able to prove the remaining smoothness conditions by means of symmetry arguments.

1. The system (4.12) is invariant under \( (a(t), b(t), c(t), f(t)) \mapsto (-a(-t), b(-t), c(-t), -f(-t)) \) is a symmetry of (4.12). Any solution of that system with \( a(0) = f(0) = 0 \), \( b(0) = c(0) = b_0 \), and \( f''(0) = q/b_0^2 \) is mapped by the symmetry to another solution with the same initial values. The functions \( a \) and \( f \) are therefore odd and \( b \) and \( c \) are even. The power series (7.1) thus satisfies all smoothness conditions.

2. The smoothness of (7.2) can be proved with the help of the symmetry \( (a_1(t), a_2(t), b(t), c(t), f(t)) \mapsto (-a_1(-t), -a_2(-t), b(t), c(t), -f(-t)) \) of (4.14). The relation \( b(t) = c(t) \) follows with the help of the symmetry \( (b,c) \mapsto (c,b) \).

3. The smoothness of (7.3) can be proved with the help of the symmetry \( (a_1(t), a_2(t), b(t), c(t), f(t)) \mapsto (a_1(-t), a_2(-t), b(-t), c(-t), -f(-t)) \) of (4.14), and the relation \( a_1(t) = a_2(t) \) follows with the help of the symmetry \( (a_1, a_2) \mapsto (a_2, a_1) \).

4. The smoothness of (7.4) can be proved with the help of the symmetry \( (a_1(t), a_2(t), b(t), c(t), f(t)) \mapsto (a_2(-t), -a_1(-t), b(-t), c(t), -f(-t)) \) of (4.14).

We finally have to prove that the power series converges. In the setting of this section, Assumption 3.12 is not always satisfied. If \( k = l = 1 \) and \( b(0) = f(0) = 0 \), then \( p \) and \( p^\perp \) both contain a trivial \( U(1)_{1,1} \)-submodule. The spaces \( p \) and \( p^\perp \) also contain a common submodule if \( k = 1, l = 0 \), and \( a(0) = f(0) = 0 \). We can nevertheless prove the convergence with the help of the arguments that we have made in Remark 3.14(v).

Since Assumption 3.12 is in some cases not satisfied and we have not described the spaces \( W_h^m \) and \( W_2^y \) explicitly, we will not study the existence of Einstein metrics on a tubular neighbourhood of \( \mathbb{CP}^2 \). We conclude this section by summarizing our results on metrics with special holonomy.

**Theorem 7.1.** Let \( M \) be a cohomogeneity-1 orbifold whose principal orbit is \( N^{k,l} \). We assume that \( M \) has exactly one singular orbit of type \( \mathbb{CP}^2 \), which is at \( t = 0 \).

1. Let \( N^{k,l} \) be not \( SU(3) \)-equivariantly diffeomorphic to \( N^{1,1} \) and let \( k + l \neq 0 \). For any \( b_0 \in \mathbb{R} \setminus \{0\} \) and \( q \in \mathbb{R} \), there exists a unique \( SU(3) \)-invariant parallel \( Spin(7) \)-structure \( \Omega \) on a tubular neighbourhood of \( \mathbb{CP}^2 \) which is determined by a basis of type (4.11) and satisfies \( a(0) = f(0) = 0 \), \( b(0) = c(0) = b_0 \), and \( f''(0) = q/b_0^2 \). The holonomy of the associated metric is all of \( Spin(7) \).

2. Let \( k = l = 1 \). For any \( b_0 \in \mathbb{R} \setminus \{0\} \) and \( q_1, q_2 \in \mathbb{R} \), there exists a unique \( SU(3) \)-invariant parallel \( Spin(7) \)-structure \( \Omega \) on a tubular neighbourhood of \( \mathbb{CP}^2 \) which is determined by a
basis of type \((4.13)\) and satisfies \(a_1(0) = a_2(0) = f(0) = 0, b(0) = c(0) = b_0, a_1''(0) = q_1/b_0^2,\) and \(a_2''(0) = q_2/b_0^2.\) Moreover, we have \(b(t) = c(t)\) for all values of \(t.\)

(ii) Let \(k = l = 1.\) For any \(a_0 \in \mathbb{R} \setminus \{0\}\) and \(q \in \mathbb{R},\) there exists a unique SU(3)-invariant parallel Spin(7)-structure \(\Omega\) on a tubular neighbourhood of \(\mathbb{C}P^2,\) which is determined by a basis of type \((4.13)\) and satisfies \(b(0) = f(0) = 0, a_1(0) = a_2(0) = c(0) = a_0,\) and \(b''(0) = \frac{q}{a_0^2}.\) Moreover, we have \(a_1(t) = a_2(t)\) for all values of \(t.\)

(iv) Let \(k = l = 1.\) For any \(a_0 \in \mathbb{R} \setminus \{0\}\) and \(q \in \mathbb{R}\) there exists a unique SU(3)-invariant parallel Spin(7)-structure \(\Omega\) on a tubular neighbourhood of \(\mathbb{C}P^2\) which is determined by a basis of type \((4.13)\) and satisfies \(b(0) = f(0) = 0, a_1(0) = -a_2(0) = c(0) = a_0,\) \(b'(0) = 1,\) and \(c''(0) = q/a_0.\)

Remark 7.2. (i) The first class of metrics from the above theorem is studied by Cvetic et al. [14] for the first time. In particular, the authors observe the free parameter \(q\) of third order. Numerical arguments suggest that there exists a critical value \(q_0 > 0\) which depends on \(k\) and \(l\) such that for \(q > q_0\) the metric is ALC, for \(q = q_0\) it is AC, and for \(q < q_0\) it is incomplete. The authors of [14] also find an explicit solution of \((4.12),\) which is ALC if \(2k > l \geq 0.\) Those explicit solutions are studied independently of [14] by Kanno and Yasui [24] and for the special case \((k, l) = (1, 0)\) the solution is constructed by Gukov and Sparks [21].

(ii) The second class of metrics was discovered independently of the author by Bazaikin [5]. Bazaikin has proved that for all sufficiently small \(\epsilon > 0\) and all \((\lambda_1, \lambda_2, \lambda_3)\) with \(\lambda_1, \lambda_2, \lambda_3 > 0\) and \(\lambda_1^2 + \lambda_2^2 + \frac{1}{3} \lambda_3^2 = \epsilon\) there exists a \(t_0 > 0\) with

\[a_1(t_0) = \lambda_1, \quad a_2(t_0) = \lambda_2, \quad f(t_0) = \lambda_3.\] (7.6)

Moreover, it is shown in [5] that the metric is AC if the three numbers \(\lambda_1, \lambda_2,\) and \(\frac{1}{3} \lambda_3\) coincide and that it is ALC if the two smallest ones coincide. All other metrics of this kind are incomplete.

In this paper, we have described this class of metrics in terms of the two initial conditions \(q_1\) and \(q_2\) of third order. Moreover, we have proved that no further metrics with \(b(t) \neq c(t)\) exist.

We remark that the metric with \(q_1 = q_2 = -2\) satisfies \(f(t) = -3c(t)\) for all \(t.\) This metric is AC and has an explicit description (see [4, 13, 24]).

(iii) In the second paper of Kanno and Yasui [25], a power series ansatz for the third and the fourth class of metrics was made. However, our proofs of the smoothness and the convergence of the power series are new. Up to now, nothing is known about the completeness and global structure of these metrics.

(iv) In the cases where \(N^{k,l}\) is generic or \(k = l = 1\) and \(a_1(0) = a_2(0) = f(0) = 0\) Assumption 3.12 is satisfied. By calculating \(W_2^2/W_0^0,\) we see that the free parameters \(q, q_1,\) and \(q_2\) are indeed a subset of the free parameters from Theorem 3.13. Although we do not know if Assumption 3.12 is necessary for Theorem 3.13 to be true, we can make a similar observation in the other two cases from the above theorem. All in all, we hope to have shed some light on the origin of these parameters.

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