Conformal Designs based on Vertex Operator Algebras

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Abstract

We introduce the notion of a conformal design based on a vertex operator algebra. This notation is a natural analog of the notion of block designs or spherical designs when the elements of the design are based on self-orthogonal binary codes or integral lattices, respectively. It is shown that the subspaces of fixed degree of an extremal self-dual vertex operator algebra form conformal 11-, 7-, or 3-designs, generalizing similar results of Assmus-Mattson and Venkov for extremal doubly-even codes and extremal even lattices. Other examples are coming from group actions on vertex operator algebras, the case studied first by Matsuo. The classification of conformal 6- and 8-designs is investigated. Again, our results are analogous to similar results for codes and lattices.

1 Introduction

In the past, it has been a fruitful approach to generalize concepts known for codes and lattices to vertex operator algebras and then to show that analogous results hold in this context. Important examples are the construction of the Moonshine module [Bor86, FLM88] or the modular invariance of the genus-1 correlation functions [Zhu90]. Other examples involving vertex operator super algebras are given in [Höh95, Höh97]. A comprehensive analysis of these analogies can be found in [Höh03b], where also a fourth step in this analogy, codes over the Kleinian four-group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), was introduced.

Two well-studied combinatorial structures are block designs and spherical designs and many examples of such designs are related to codes and lattices. A notion analogous to block designs and spherical designs in the context of vertex operator algebras however has been missing so far. In this paper, we introduce the notion of a conformal design based on a vertex operator algebra and prove several results analogous to known results for block and spherical designs.

The paper is organized as follows. In the rest of the introduction we discuss the concepts of block and spherical \( t \)-designs to motivate our definition of conformal \( t \)-designs. Details of the definition will be discussed in the next section. Section 2 also contains several basic results. In particular, we prove that a vertex operator algebra

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V with a compact automorphism group $G$ leads to conformal designs if the fixed point vertex operator algebra $V^G$ has large minimal weight (Theorem 2.5). The minimal weight of a vertex operator super algebra $V$ is the smallest $m > 0$ with $V_m \neq 0$. We also describe the construction of derived designs (Theorem 2.10). In Section 3, we prove that the homogeneous parts of an extremal vertex operator algebra are conformal 11-, 7-, or 3-designs depending on the residue class mod 24 of the central charge of $V$. Extremal vertex operator algebras have been introduced in [Hoh95] and are vertex operator algebras $V$ with only one irreducible module up to isomorphism such that the first coefficients of the graded trace of $V$ are as small as possible. The Moonshine module is an example of an extremal vertex operator algebra supporting 11-designs. This theorem is the analog of theorems of Assmus-Mattson [AM69] (see also [Bac99]) and Venkov [Ven84] for extremal doubly-even codes and extremal even lattices, respectively. In the final section, we study the classification of conformal 6- and 8-designs $V_m$ supported by vertex operator super algebras $V$ of minimal weight $m \leq 2$.

Our result for conformal 6-designs (Theorem 4.1) is analogous to (but a little weaker than) similar results for codes [LL01] and lattices [Mar01]: For $m = 1/2$ and $m = 1$, the only examples are the single Fermion vertex operator super algebra $V_{\text{Fermi}}$ and the two lattice vertex operator algebras $V_A$ and $V_E$, respectively. For $m = 3/2$, the allowed central charges are $c = 16$ and $c = 23 \frac{1}{2}$. In the later case the shorter Moonshine module $VB^2$ [Hoh95] is a known example. For $m = 2$, the allowed central charges are $c = 8, 16, 23 \frac{3}{2}, 23 \frac{1}{2}, 24, 32, 32 \frac{1}{2}, 33 \frac{1}{2}, 34 \frac{1}{2}, 40, 1496$ and examples are known for $c = 8, 16, 23 \frac{1}{2}, 24$ and 32. For conformal 8-designs, one obtains (Theorem 4.2) $m = 2$ and $c = 24$ with the Moonshine module $V^2$ the only known example.

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A block design $X$ of type $t$-$(n, k, \lambda)$ (or $t$-design, for short) is a set of size-$k$-subsets of a set $M$ of size $n$ such that each size-$t$-subset of $M$ is contained in the same number $\lambda$ of sets from $X$. A subset $K$ of $M$ can be identified with the vector $(x_q)_{q \in M} \in \mathbf{F}_2^M$, where $x_q = 1$ if $q \in K$ and $x_q = 0$ otherwise. Let $\text{Hom}(m)$ be the complex vector space of homogeneous polynomial functions of degree $m$ on $\mathbf{F}_2^M$ in the variables $t_p$, $p \in M$, where $t_p((x_q)_{q \in M}) = x_p$ for $(x_q)_{q \in M} \in \mathbf{F}_2^M$. A basis of $\text{Hom}(m)$ is formed by the monomials $\prod_{p \in N} t_p$ where $N$ is a size-$m$-subset of $M$.

There is a natural action of $\text{Aut}M \cong S_n$, the symmetric group of degree $n$, on $M$, $\mathbf{F}_2^M$, and $\text{Hom}(m)$. The condition for a set $X$ of size-$k$-subsets of $M$ to be a $t$-design is now equivalent to (one may choose for $f$ the monomial $\prod_{p \in N} t_p$ where $N \subset M$ with $|N| = s$) that for all $0 \leq s \leq t$ and $f \in \text{Hom}(s)$ and for all $g \in S_n$ the following equation holds

$$\sum_{B \in X} f(B) = \sum_{B \in X} f(g(B)).$$

(1)

The vector spaces $\text{Hom}(m)$ can be decomposed into a direct sum of irreducible components under the action of $S_n$. Let $\pi$ be the projection of $\text{Hom}(m)$ onto the trivial component. By averaging equation (1) over all $g \in S_n$ one finds that the above
condition on $X$ is equivalent to that for all $0 \leq s \leq t$ and $f \in \text{Hom}(s)$ one has

$$\sum_{B \in X} f(B) = \sum_{B \in X} \pi(f)(B).$$

(2)

The decomposition of $\text{Hom}(m)$ into irreducible components can be done explicitly: The kernel of $\Delta : \text{Hom}(m) \rightarrow \text{Hom}(m-1)$, $f \mapsto \sum_{p \in M} \frac{\partial f}{\partial p}$, forms the irreducible constituent $\text{Harm}(m)$ of “discrete” harmonic polynomials of degree $m$. One has the decomposition $\text{Hom}(m) = \bigoplus_{i=0}^{m} \text{Harm}'(i)$, where $\text{Harm}'(i)$ is the image of $\text{Harm}(i)$ under the multiplication with $(\sum_{p \in M} t_{p})^{m-i}$. This gives the following characterization of $t$-designs: A set $X$ of size-$k$-subsets of $M$ is a $t$-design if and only if for all $0 \leq s \leq t$ and $f \in \text{Harm}(s)$ one has

$$\sum_{B \in X} f(B) = 0.$$  

(3)

A spherical $t$-design (cf. [DGS77, GS79]) is a finite subset $X$ of a sphere $S(r)$ of radius $r$ around 0 in $\mathbb{R}^n$ (usually the unit sphere) such that $X$ can be used to integrate all polynomials $f$ of degree $s \leq t$ on $S(r)$ exactly by averaging their values on $X$, i.e., the following equation holds:

$$\frac{1}{|X|} \sum_{B \in X} f(B) = \frac{1}{\text{Vol}(S(r))} \int_{S(r)} f \omega,$$

(4)

where $\omega$ is the canonical volume form on the Riemannian manifold $S(r)$.

Let $\text{Hom}(m)$ be the complex vector space of homogeneous polynomials of degree $m$ in the variables $x_i$, where the $x_i$, $i = 1, \ldots, n$, are identified with the standard coordinate functions on $\mathbb{R}^n$. One has $\text{dim} \text{Hom}(m) = \binom{n+m-1}{n-1}$. There is a natural action of the orthogonal group $\text{O}(n)$ on $\mathbb{R}^n$ and $\text{Hom}(m)$. The condition for a set $X \subset S(r)$ to be a spherical $t$-design is now equivalent to (the right hand side of equation (4) is obviously $\text{O}(n)$-invariant) that for all $0 \leq s \leq t$ and $f \in \text{Hom}(s)$ and for all $g \in \text{O}(n)$ the following equation holds

$$\sum_{B \in X} f(B) = \sum_{B \in X} f(g(B)).$$

(5)

The vector spaces $\text{Hom}(m)$ can again be decomposed into a direct sum of irreducible components under the action of $\text{O}(n)$. Let $\pi$ be the projection of $\text{Hom}(m)$ onto the trivial component. By averaging equation (5) over all $g \in \text{O}(n)$ using an invariant measure, one finds that the above condition on $X$ is equivalent to that for all $0 \leq s \leq t$ and $f \in \text{Hom}(s)$ one has

$$\sum_{B \in X} f(B) = \sum_{B \in X} \pi(f)(B).$$

(6)

The decomposition of $\text{Hom}(m)$ into irreducible components can also be done explicitly: The kernel of $\Delta : \text{Hom}(m) \rightarrow \text{Hom}(m-2)$, $f \mapsto \sum_{l=1}^{n} \frac{\partial^2 f}{\partial x_l \partial x_l}$, forms the irreducible constituent $\text{Harm}(m)$ of harmonic polynomials of degree $m$. One has the
decomposition \( \text{Hom}(m) = \bigoplus_{i=0}^{[m/2]} \text{Harm}'(m - 2i) \), where \( \text{Harm}'(m - 2i) \) is the image of \( \text{Harm}(m - 2i) \) in \( \text{Hom}(m) \) under the multiplication with \( (\sum_{i=1}^{n} x_i^2)^l \). This gives the following characterization of spherical \( t \)-designs: A finite subset \( X \subset S(r) \) is a spherical \( t \)-design if and only if for all \( 1 \leq s \leq t \) and \( f \in \text{Harm}(s) \) one has

\[
\sum_{B \in X} f(B) = 0.
\]  

(7)

In the definition of a conformal \( t \)-design based on a vertex operator algebra \( V \) we will replace the groups \( S_n \) and \( O(n) \) by the Virasoro algebra of central charge \( c \). The role of \( \text{Hom}(m) \) will be the degree-\( m \)-part \( V_m \) of a vertex operator algebra \( V \). For \( X \), we will take a homogeneous part of a module of \( V \) and the evaluation \( f[X] = \sum_{B \in X} f(X) \) will be replaced by

\[
f[X] = \text{tr}|_{\text{O}(f)},
\]

where \( f \in V_m \) and \( \text{O}(f) \) is the coefficient \( v_{(k)} \) of the vertex operator \( Y(v, z) = \sum_{n \in \mathbb{Z}} v_{(n)} z^{-n-1} \) which maps \( X \) into itself. The definition of a conformal \( t \)-design is then the same as in [2] and [4] or, equivalently, as in [3] and [7].

2 Definition and basic properties

The Virasoro algebra is the complex Lie algebra spanned by \( L_n, n \in \mathbb{Z} \), and the central element \( C \) with Lie bracket

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0} C
\]

(8)

where \( \delta_{k,0} = 1 \) if \( k = 0 \) and \( \delta_{k,0} = 0 \) otherwise.

For a pair \( (c, h) \) of real numbers the Verma module \( M(c, h) \) is a representation of the Virasoro algebra generated by a highest weight vector \( v \in M(c, h) \) with \( Cv = c \), \( L_0v = hv \) and \( L_nv = 0 \) for \( n \geq 1 \). The number \( c \) is called the central charge and \( h \) is called the conformal weight of the module. The set

\[
\{L_{-m_1} \cdots L_{-m_k} v \mid k, m_1, \ldots, m_k \in \mathbb{Z}_{\geq 0}, m_1 \geq \cdots \geq m_k \geq 1\}
\]

forms a basis of \( M(c, h) \). The vector space \( M(c, h) \) is graded by the eigenvalues of \( L_0 \) and the vector \( L_{-m_1} \cdots L_{-m_k} v \) is homogeneous of degree \( m_1 + \cdots + m_k + h \).

For \( h = 0 \), the module \( M(c, 0) \) has a quotient isomorphic to \( M(c, 0)/M(c, 1) \) with a basis which can be identified with

\[
\{L_{-m_1} \cdots L_{-m_k} v \mid k, m_1, \ldots, m_k \in \mathbb{Z}_{\geq 0}, m_1 \geq \cdots \geq m_k \geq 2\}.
\]

A vector \( w \) of \( M(c, h) \) or \( M(c, 0)/M(c, 1) \) is said to be a singular vector if \( L_mw = 0 \) for all \( m \geq 1 \). The Kac-determinant formula shows that a module \( M(c, h) \) for \( h \neq 0 \) contains a singular vector of degree up to \( n \) if \( h_{p,q}(c) = h \) for \( pq \leq n \) where

\[
h_{p,q}(c) = \frac{(m + 1)p - mq)^2 - 1}{4m(m + 1)} \quad \text{with} \quad m = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{25 - c}{1 - c}}.
\]
The modules $M(c,0)/M(c,1)$ contain a singular vector of degree up to $n$ if and only if the central charge $c$ is a zero of the following normalized polynomial $D_n(c)$:

\[
\begin{align*}
D_2(c) &= c \\
D_4(c) &= c(5c + 22) \\
D_6(c) &= c(2c - 1)(5c + 22)(7c + 68) \\
D_8(c) &= c(2c - 1)(3c + 46)(5c + 3)(5c + 22)(7c + 68).
\end{align*}
\]

(9)

A vertex operator algebra $V$ over the field of complex numbers is a complex vector space equipped with a linear map $Y : V \to \text{End}(V)[[z, z^{-1}]]$ and two nonzero vectors $1$ and $\omega$ in $V$ satisfying certain axioms; cf. [Bor86, FLM88]. For $v \in V$ one writes

\[
Y(v, z) = \sum_{n \in \mathbb{Z}} a_n(z) z^{-n-1}.
\]

The vacuum vector vector $1$ satisfies $1_{(-1)} = \text{id}_V$ and $1_{(n)} = 0$ for $n \neq -1$. The coefficients $L_n = \omega_{n+1}$ for the Virasoro vector $\omega$ are satisfying the Virasoro relation [B] with $C = c \cdot \text{id}_V$ for a complex number $c$ called the central charge of $V$.

The operator $L_0$ give rise to a grading $V = \bigoplus_{n \in \mathbb{Z}} V_n$, where $V_n$ denotes the eigenspace of $L_0$ with eigenvalue $n$, called the degree. $V_n$ is supposed to be finite-dimensional and we assume $V_n = 0$ for $n < 0$ and $V_0 = C \cdot 1$. For $v \in V_n$ the operator $v_{(n)}$ is homogeneous of degree $0$. We define $o(v) = v_{(n)}$.

We will make use several times of the associativity relation

\[
(u(m)v)_{(n)} = \sum_{i \geq 0} (-1)^i \binom{m}{i} (u(m-i)v_{(n+i)} - (-1)^m v_{(m+n-i)} u_{(i)})
\]

for elements $u, v \in V$.

For the notion of admissible and (ordinary) module we refer to [DLM98]. A vertex operator algebra is called rational if every admissible module is completely reducible. In this case there are only finitely many irreducible admissible modules up to isomorphism and every irreducible admissible module is an ordinary module. A vertex operator algebra is called simple if it is irreducible as a module over itself.

For an irreducible module $W$ there exists an $h$ such that $W = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} W_{n+h}$, where the degree $n$ subspace $W_n$ is again the eigenspace of $L_0$ of eigenvalue $n$, with $W_h \neq 0$. We call $h$ the conformal weight of the module $W$.

The graded trace of an element $v \in V_n$ on a module $W$ of conformal weight $h$ is defined by

\[
\chi_W(v, q) = q^{-c/24} \sum_{k \in \mathbb{Z}_{\geq 0}} \text{tr}|_{W_{k+h}} o(v) k^h.
\]

For $v = 1$, we call $\chi_W(q) = \chi_W(1, q)$ the character of $W$. If $V$ is assumed to be rational and satisfying the $C_2$-cofiniteness condition $V/\text{Span}\{x_{(-2)}y \mid x, y \in V\} < \infty$ it is a result of Zhu that $\chi_W(v, q)$ is a holomorphic function on the complex upper half plane in the variable $\tau$ for $q = e^{2\pi i \tau}$. We assume in this paper that the $C_2$-cofiniteness condition is satisfied.
Furthermore, for \( v \in V_n \) a highest weight vector for the Virasoro algebra, the family \( \{ \chi_W(v, q) \}_W \), where \( W \) runs through the isomorphism classes of irreducible \( V \)-modules \( W \), transforms as a vector-valued modular form of weight \( n \) for the modular group \( \text{PSL}_2(\mathbb{Z}) \) acting on the upper half plane in the usual way.

Given a vertex operator algebra \( V \), Zhu defined a new vertex operator algebra on the same underlying vector space with vertex operator \( Y[v, z] = Y(v, e^{-1}e^{z\omega_1}) \) for homogeneous elements \( v \in V \). The vacuum element is the same as the original one and the new Virasoro element is \( \bar{\omega} = \omega - \frac{c}{24} \). We let \( \bar{Y}[\bar{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2} \).

The new Virasoro algebra generator \( L[0] \) introduces a new grading \( V = \bigoplus_{n=0}^{\infty} V_n \) on \( V \) and similar for modules \( W \). One has \( \bigoplus_{n \leq N} W_n = \bigoplus_{n \leq N} W[n] \) for all \( N \in \mathbb{R} \).

In particular, the Virasoro highest weight vectors for both Virasoro algebras are the same.

It was shown by H. Li that a vertex operator algebra \( V \) as above has a unique normalized invariant bilinear form \( (\cdot, \cdot) \) given by

\[(u, v)_{1} = \text{Res}_{z=0} (z^{-1}Y(e^{zL_1}(-z^{-2})L(0))u, z^{-1})v\]

for elements \( u, v \in V \) with normalization \((1, 1) = 1\) provided that \( L_1V_1 = 0 \).

We assume that the vertex operator algebras \( V \) in this paper are isomorphic to a direct sum of highest weight modules for the Virasoro algebra, i.e., one has

\[V = \bigoplus_{i \in I} M_i,\]

where each \( M_i \) is a quotient of a Verma modules \( M(c, h) \) with \( h \in \mathbb{Z}_{\geq 0} \). One has therefore a natural decomposition

\[V = \bigoplus_{h=0}^{\infty} \overline{M}(h) \tag{10}\]

where \( \overline{M}(h) \) is a direct sum of finitely many quotients of the Verma module \( M(c, h) \). The module \( \overline{M}(0) \) is the vertex operator subalgebra of \( V \) generated by \( \omega \) which we denote also by \( V_\omega \) and is therefore a quotient of \( M(c, 0)/M(c, 1) \). The smallest \( h > 0 \) for which \( \overline{M}(h) \neq 0 \) is called the minimal weight of \( V \) and denoted by \( \mu(V) \). (If no such \( h > 0 \) exists, we let \( \mu(V) = \infty \).) We note that our assumption implies that \( L_1V_1 = 0 \).

In particular, the decomposition (10) gives us the natural projection map

\[\pi : V \rightarrow V_\omega\]

with kernel \( \bigoplus_{h > 0} \overline{M}(h) \).

The decomposition \( V = \bigoplus_{h=0}^{\infty} \overline{M}(h) \) is the same for the modified Virasoro algebra with generators \( L[n] \).

In Section 4, we study vertex operator super algebras. For the full definition we refer to [Höh95, Kac97]. We note that a vertex operator super algebra is a super vector space \( V = V(0) \oplus V(1) \) where the even part \( V(0) = \bigoplus_{n=0}^{\infty} V_n \) is a vertex operator algebra and the odd part \( V(1) = \bigoplus_{n=0}^{\infty} V_{n+1/2} \) is a \( V(0) \)-module with a conformal
weight \( h \in \mathbb{Z}_{\geq 0} + \frac{1}{2} \). The minimal weight \( \mu(V) \) of \( V \) is defined to be the minimum of \( \mu(V_{(0)}) \) and the conformal weight \( h \) of \( V_{(1)} \). A vertex operator super algebra \( V \) is called rational in this paper if the even vertex operator subalgebra \( V_{(0)} \) is rational.

As explained in the introduction, the following definition is motivated by analogous definitions of block designs and spherical designs.

**Definition:** Let \( V \) be vertex operator algebra of central charge \( c \) and let \( X \) be a degree \( h \) subspace of a module of \( V \). For a positive integer \( t \) one calls \( X \) a conformal design of type \( t \)-\((c, h)\) or conformal \( t \)-design, for short, if for all \( v \in V_n \) where \( 0 \leq n \leq t \) one has

\[
\text{tr}|_X o(v) = \text{tr}|_X o(\pi(v)).
\]

The following two observations are clear:

**Remark 2.1** If \( X \) is a conformal \( t \)-design based on \( V \), it is also a conformal \( t \)-design based on an arbitrary vertex operator subalgebra of \( V \). □

**Remark 2.2** A conformal \( t \)-design is also a conformal \( s \)-design for all integers \( 1 \leq s < t \). □

**Theorem 2.3** Let \( X \) be the homogeneous subspace of a module of a vertex operator algebra \( V \). The following conditions are equivalent:

(i) \( X \) is a conformal \( t \)-design.

(ii) For all homogeneous \( v \in \ker \pi = \bigoplus_{h > 0} \mathcal{M}(h) \) of degree \( n \leq t \), one has \( \text{tr}|_X o(v) = 0 \).

**Proof.** For \( v \in V \) one has \( v - \pi(v) \in \bigoplus_{h > 0} \mathcal{M}(h) \). □

If \( V \) has minimal weight \( \mu \), then the homogeneous subspaces of any module of \( V \) are conformal \( t \)-designs for all \( t = 1, 2, \ldots, \mu - 1 \). We call such a conformal \( t \)-design trivial. In particular, if \( V \) is isomorphic to \( V_\omega \), i.e., has minimal weight infinity, all conformal designs based on \( V \) are trivial.

**Theorem 2.4** Let \( V \) be a vertex operator algebra and let \( N \) be a \( V \)-module graded by \( \mathbb{Z} + h \). The following conditions are equivalent:

(i) The homogeneous subspaces \( N_n \) of \( N \) are conformal \( t \)-designs based on \( V \) for \( n \leq h \).

(ii) For all Virasoro highest weight vectors \( v \in V_s \) with \( 0 < s \leq t \) and all \( n \leq h \) one has

\[
\text{tr}|_{N_n} o(v) = 0.
\]

**Proof.** It is clear that (i) implies (ii). Assume now that (ii) holds. Let \( v \in V \) be a Virasoro highest weight vector and let \( w = L_{[-i_1]} \cdots L_{[-i_k]}v \) with positive integers \( i_1, \ldots, i_k \) be in the Virasoro highest weight module generated by \( v \). In [Zhu90], Lemma 4.4.4, it is proven that

\[
\chi_N(w, q) = \sum_{i=0}^k g_i(q) \left( \frac{d}{dq} \right)^i \chi_N(v, q)
\]
Example 2.6

The lattice vertex operator algebra $V_{A_1}$ associated to the root lattice $A_1$ has the complex Lie group $\text{PSU}_2(\mathbb{C})$ as automorphism group and $V_{A_1}$ is a highest weight representation of level 1 for the affine Kac-Moody algebra of type $A_1$. The graded character of $V_{A_1}$ as a $\text{PSU}_2(\mathbb{C})$-module is

$$
\chi_V = \left(1 + \sum_{n=1}^{\infty} (\lambda^n + \lambda^{-n}) q^{n^2}\right) \div \left(q^{1/24} \prod_{n=1}^{\infty} (1-q^n)\right).
$$

The characters of the irreducible $\text{PSU}_2(\mathbb{C})$-modules are $\sum_{k=-i}^{i} \lambda^k$ with non-negative integers $i$ and it follows immediately that the graded multiplicity of the trivial $\text{PSU}_2(\mathbb{C})$-representation in $V_{A_1}$ is $q^{-1/24} (\prod_{n=2}^{\infty} (1-q^n))^{-1}$. This equals the graded character of the Virasoro vertex operator algebra of central charge 1. By restricting $\text{PSU}_2(\mathbb{C})$ to the compact group $\text{PSU}_2(\mathbb{R})$, Theorem 2.5 shows that $V_{A_1}$ is of class $\mathcal{S}^t$ for all $t$ and the homogeneous subspaces $(V_{A_1})_n$ are conformal $t$-designs for all $t$. By using the two-fold cover $\text{SU}_2(\mathbb{R})$ of $\text{PSU}_2(\mathbb{R})$ it follows that the homogeneous subspaces of the irreducible $V_{A_1}$-module $V_{A_1, 1/4}$ of conformal weight 1/4 are also conformal $t$-designs for all $t$. 

Let $G$ be a group of automorphisms acting on a vertex operator algebra $V$. We say that a module $N$ of $V$ is $G$-invariant, if there exists a central extension $\tilde{G}$ of $G$ acting on $N$ such that $g^{-1} Y_N(\tilde{g}v, z) g = Y_N(v, z)$ for all $g \in \tilde{G}$ and $v \in V$. Here, $\tilde{g}$ denotes the image of $g$ in $G$.

The following result is a generalization of [Mat01], Lemma 2.8.

**Theorem 2.5** Let $V$ be a vertex operator algebra and $G$ be a compact Lie group of automorphisms of $V$. Let $X$ be a homogeneous subspace of a $G$-invariant module of $V$. If the minimal conformal weight of $V^G$ is larger or equal to $t+1$, then $X$ is a conformal $t$-design.

The analogous result that $t$-homogeneous permutation groups $G$ lead to block $t$-designs is trivial. An analogous result relating the invariants of a real representation of a finite group $G$ with spherical designs is due to Sobolev [Sob02].

**Proof.** For all $g \in \tilde{G}$ and $v \in V$ we have $\text{tr}_{|X} o(\tilde{g}v) = \text{tr}_{|X} o(v)$ as $o(\tilde{g}v) = g o(v)g^{-1}$. Let $\tilde{v}$ be the average of the $\tilde{g}v$, i.e., $\tilde{v} = \frac{1}{|\tilde{G}|} \sum_{\tilde{g} \in \tilde{G}} \tilde{g}v$ for $G$ finite and $\tilde{v} = \int_{G} \bar{g}v \mu / \int_{G} 1 \mu$ with $\mu$ the Haar measure on $G$ in general. Then $\text{tr}_{|X} o(\tilde{v}) = \text{tr}_{|X} o(v)$. Since $\tilde{v} \in V_G$ and the minimal weight of $V^G$ is larger or equal to $t+1$ and the action of $G$ on $V_G$ is trivial, one gets $\tilde{v} = \pi(v)$ and hence $\text{tr}_{|X} o(\pi(v)) = \text{tr}_{|X} o(v)$ for $v \in V_s$, $0 \leq s \leq t$. 

A $t$-design $X$ based on $V$ as in the previous theorem is trivial as a conformal $t$-design based on $V^G$.

In [Mat01], a vertex operator algebra $V$ is said to be of class $\mathcal{S}^n$ if the minimal weight of $V^G$ is larger or equal to $n$, where $G = \text{Aut} V$ is the automorphism group of $V$. (Matsuo assumes in addition $V_1 = 0$, but the definition clearly works without this assumption.)
Example 2.7 The lattice vertex operator algebra \( V_{E_8} \) associated to the root lattice \( E_8 \) has the complex Lie group \( E_8(\mathbb{C}) \) as automorphism group and \( V_{E_8} \) is the unique highest weight representation of level 1 for the affine Kac-Moody algebra of type \( E_8 \). It can be deduced from the Weyl-Kac character formula that the minimal weight of the fixed point vertex operator algebra \( V_{E_8}(\mathbb{C}) \) is 8. Hence, \( V_{E_8} \) is of class \( S^7 \) and the homogeneous subspaces \( (V_{E_8})_n \) are conformal 7-designs since \( E_8(\mathbb{C}) \) can be restricted to the compact group \( E_8(\mathbb{R}) \) without changing the fixpoint vertex operator algebra.

Example 2.8 It was noted in [Hohl93], Sect. 5.1, that Borcherds’ proof of the Moonshine conjectures implies that the minimal weight of \((V^2)^M\), where \( V^2 \) is the Moonshine module with the Monster \( M \) as automorphism group, is equal to 12. Hence, \( V^2 \) is of class \( S^{11} \) and the homogeneous subspaces \( V_n^2 \) are conformal 11-designs.

Example 2.9 Using the proof of the generalized Moonshine conjecture in [Hohl02], it was shown loc. cit. that the even part \( V_{B(0)}^2 \) of the odd Moonshine module \( V^3 \) with the Baby Monster \( B \) as automorphism group [Hohl02] is of class \( S^7 \). Hence, the homogeneous subspaces of \( V_{B(0)}^2 \) and of its two further \( B \)-invariant irreducible modules \( V_{B(1)}^2 \) and \( V_{B(2)}^2 \) are conformal 7-designs.

We will give a more direct proof for Examples 2.8 and 2.7 in the next section and for Example 2.9 after the next theorem.

Let \( V \) be a vertex operator algebra of central charge \( c \) and \( N \) be a \( V \)-module graded by \( \mathbb{Z} + h \). Let \( U = V_{\omega'} \) be a Virasoro vertex operator subalgebra of \( V \) of central charge \( c' \) with Virasoro element \( \omega' \), let \( W \) be the commutant of \( U \) in \( V \) and assume that \( \omega^* = \omega - \omega' \) is a Virasoro element of \( W \) of central charge \( c^* = c - c' \). Assume also that as \( U \otimes W \)-module one has a finite decomposition \( N = \bigoplus_{h' + h^* = h} M(h') \otimes K(h^*) \), where \( M(h') \) is a module of \( V_{\omega'} \) generated by highest weight vectors of degree \( h' \) and the \( K(h^*) \) are \( W \)-modules graded by \( \mathbb{Z} + h^* \). For the homogeneous degree-\( h \)-part \( X \) of \( N \) one gets the finite decomposition

\[
X = \bigoplus_{h' + h^* = h} \bigoplus_{k \in \mathbb{Z}_{\geq 0}} X^*_{h', h' + k} \otimes X^*_{h^*, h^* - k}
\]

with homogeneous parts \( X^*_{h', h' + k} = M(h')_{h' + k} \) and \( X^*_{h^*, h^* - k} = K(h^*)_{h^* - k} \).

**Definition:** The homogeneous subspaces \( X^*_{h^*, h^* - k} \) of the \( W \)-modules \( K(h^*) \) are called the derived parts of \( X \) with respect to \( \omega' \). We also define \( S = \{ h' \mid M(h') \neq 0 \} \) and \( s^* = |S| \).

**Theorem 2.10** Let \( V \) be a vertex operator algebra of central charge \( c \) and let \( N \) be a \( V \)-module graded by \( \mathbb{Z} + h \). Assume that \( \omega', \omega - \omega' \in V_2 \) generate two commuting Virasoro vertex operator subalgebras of central charge \( c \) and \( c^* = c - c' \), respectively. Assume further that the homogeneous parts \( N_n \) for \( n \leq h \) are conformal designs of type \( t(c, n) \) based on \( V \). Then the derived parts \( X^*_{h^*, h^* - k} \) of \( X = N_h \) with respect to \( \omega' \) are conformal designs of type \( t^*(c^*, h^* - k) \) based on the commutant \( W \) of \( V_{\omega'} \) in \( V \) with \( t^* = t + 2 - 2s^* \).
Example 2.11

By taking for decomposition (cf. [Hö95], Sect. 4.1)

\[ V_{\omega'} \otimes W \hookrightarrow V \]

where the upper horizontal arrow is the natural injection and the other arrows are the projections onto the relevant Virasoro vertex operator subalgebras it follows that if \( w \in \ker \pi^* \), then \( u \otimes w \in \ker \pi \) for all \( u \in V_{\omega'} \).

Let now \( w \in \ker \pi^* \cap W_s \) where \( s \leq t^* \). Denote with \( L_n' \) and \( L_n^* \) the generators of the Virasoro algebra associated to \( \omega' \) and \( \omega^* \), respectively. For \( 0 \leq i \leq s^* - 1 \) we define \( v^i \in V_{s+2i} \) by \( v^i = (L_{-2}^i)1 \otimes w \). For the graded traces one gets

\[
\text{tr}|_N o(v^i)q^{L_0} = \sum_{h' \in S} (\text{tr}|_{M(h')} o((L_{-2}^i)1) q^{L_0}) \cdot (\text{tr}|_{K(h-h')} o(w) q^{L_0}).
\]

The matrix valued power series

\[
(\text{tr}|_{M(h')} o((L_{-2}^i)1) q^{L_0-h'^*}), \quad h' \in S, \quad i = 0, 1, \ldots, s^* - 1,
\]

has as constant term the matrix \((n_{h', P_i(h')})\), where \( n_{h'} \) is the dimension of the lowest degree subspace \( M(h')_{h'} \) and \( P_i(h') \) is a monic polynomial of degree \( i \) in \( h' \). This matrix is invertible since its determinant is of Vandermonde type and therefore the matrix valued power series is invertible, too. Since the \( N_n \) are conformal \( t \)-designs for \( n \leq h \), the coefficients of \( \text{tr}|_N o(v^i)q^{L_0} = \sum_{Z+h} a_n^i q^n \) vanish for \( n \leq h \) and for all \( i \) as \( s + 2i \leq t^* + 2(s^* - 1) = t \). Hence the coefficients of \( \text{tr}|_{K(h-h')} o(w) q^{L_0+h'} = \sum_{k \in Z+h-h'} b_{h',k} q^{h'+k} \) also vanish for \( k \leq h-h' = h^* \) and all \( h' \in S \). This implies that the \( X_{h^*,h'-h}^* \) are conformal \( t \)-designs based on \( W \).

We give two examples, both based on the Moonshine module \( V^2 \). As already mentioned before, all its homogeneous subspaces \( V^2_n \) are conformal \( 11 \)-designs.

Example 2.12

The Leech lattice \( \Lambda \) has a sublattice isomorphic to \( \Lambda_8 \oplus \Lambda_{16} \), where \( \Lambda_8 \) is the \( E_8 \)-root lattice rescaled by the factor \( \sqrt{2} \) and \( \Lambda_{16} \) is the Barnes-Wall lattice of rank 16. This sublattice defines a vertex operator subalgebra \( V_{\Lambda_8}^+ \otimes V_{\Lambda_{16}}^+ \) of \( V^2 \). The vertex operator algebra \( V_{\Lambda_8}^+ \) was studied by Griess in [Gri98] and has \( O(10,2)^+ \) as automorphism group. It can be easily seen from [DGH98] that \( V_{\Lambda_8}^+ \) and \( V_{\Lambda_{16}}^+ \) are
both framed vertex operator algebras. In fact, \( V_{16}^+ \) is isomorphic to the framed vertex operator algebra \( V_C \) where the binary code \( C \) is equal to the Hamming code \( H_{16} \) of length 16 and the code \( D \) is the zero code. The irreducible modules of a framed vertex operator algebra with \( D = 0 \) are described in [Miy98]. One finds that \( V_{16} \) has \( 2^{10} \) modules \( L_\mu, \mu \in \Xi \), having the conformal weights 0, \( \frac{1}{2} \) or 1. Therefore one has a decomposition

\[
V^\sharp = \bigoplus_{\mu \in \Xi} L_\mu \otimes K_\mu
\]

(11)
of the Moonshine module into \( V_{16}^+ \otimes V_{16}^+ \)-modules. The graded traces \( \chi_{L_\mu}(q) \) of the modules \( L_\mu \) are easily computed. It turns out that they depend only on the conformal weight of \( L_\mu \), i.e., they give isomorphic \( V_{\omega'} \)-modules where \( \omega' \) is the Virasoro element of \( V_{16}^+ \). From decomposition (11), we obtain now the following decomposition into \( V_{\omega'} \otimes V_{16}^+ \)-modules:

\[
V^\sharp = \bigoplus_{m=0}^\infty L'(m) \otimes K(0) \oplus \bigoplus_{m=0}^\infty L''(m + \frac{1}{2}) \otimes K(\frac{3}{2}) \oplus \bigoplus_{m=0}^\infty L'''(m + 1) \otimes K(1),
\]

(12)

where the \( L'(h) \), \( L''(h) \), and \( L'''(h) \) are direct sums of highest weight representations of highest weight \( h \) for the Virasoro algebra of central charge 8 associated to \( \omega' \) and \( K(0) = V_{16}^+, K(\frac{3}{2}) \) and \( K(1) \) are \( V_{16}^+ \)-modules of conformal weight 0, \( \frac{3}{2} \) and 1, respectively. Although Theorem 2.10 is not directly applicable, the argument given in its proof shows that the homogeneous subspaces of \( K(0) = V_{16}^+, K(\frac{3}{2}) \), and \( K(1) \) are conformal 7-designs based on \( V_{16}^+ \). Moreover, one can switch the role of \( V_{16}^+ \) and \( V_{16}^+ \) in the preceding discussion and one obtains that the homogeneous subspaces of \( V_{16}^+ \) and the direct sum of the \( V_{16}^+ \)-modules \( L_\mu \) of conformal weight \( \frac{1}{2} \) and 1, respectively, are also conformal 7-designs.

The application range of Theorem 2.10 is somewhat restricted since the minimal weight of the underlying vertex operator algebra cannot be larger than 2 and this leaves only a limited set of examples of \( t \)-designs with \( t \geq 6 \) as the classification Theorem 4.11 of Section 4 shows.

One may ask if the vertex operator algebras \( V_L \) associated to even integral lattices \( L \) besides the root lattices \( A_1 \) and \( E_8 \) lead to interesting conformal \( t \)-designs for larger values of \( t \). The next theorem shows that a necessary condition is that one starts with a spherical \( t \)-design. We recall that the irreducible modules of a lattice vertex operator algebra associated to a lattice \( L \) are parametrized by the elements of the discriminant group \( L^*/L \).

**Theorem 2.13** Let \( V_L \) be the lattice vertex operator algebra associated to an even integral lattice \( L \subset \mathbb{R}^k \) of rank \( k \) and let \( N = \sum \lambda_i \lambda_i \in L^*/L, \lambda_i \in L^*/L, \) be a module of \( V_L \). If the degree-\( n \)-subspace \( X = N_n \) of \( N \) is a conformal \( t \)-design based on \( V_L \) then the set of vectors of norm \( 2n \) in \( \bigcup \lambda \lambda_i \lambda_i \) must form a spherical \( t \)-design.

The proof uses the following result ([DMN01], Th. 3):

**Proposition 2.14 (Dong-Mason-Nagatomo)** Let \( P \) be a homogeneous spherical harmonic polynomial on \( \mathbb{R}^k \) and let \( V_L \) be the lattice vertex operator algebra associated
to an even integral lattice $L$ of rank $k$. Then there exists a Virasoro highest weight vector $v_P$ with the property that

$$\chi_{V_L}(v_P, q) = \left( \sum_{x \in L} P(x) q^{(x,x)/2} \right) / \left( q^{1/24} \prod_{i=1}^{\infty} (1 - q^i) \right)^k.$$  

The vector $v_P$ is given by $v_P = P(h_1^1, \ldots, h_k^1) 1$, where $\{h_1, \ldots, h_k\}$ is an orthonormal basis of $R \otimes Z L \subset (V_L)_1$. It can immediately be seen from its proof that the proposition remains valid if one replaces $L$ by a coset $L + \lambda$ and $V_L$ by the $V_L$-module $V_{L+\lambda}$.

Proof of Theorem 2.13. As explained in the introduction, the set $\tilde{L}_{2n} = \{x \in \bigcup_i L + \lambda_i \mid (x, x) = 2n\}$ on a sphere in $R^k$ around 0 is a spherical $k$-design if and only if $\sum_{x \in \tilde{L}_{2n}} P(x) = 0$ for all harmonic polynomials $P$ homogeneous of degree $s$ with $1 \leq s \leq t$. The result follows now directly from the mentioned generalization of Proposition 2.14.

It will follow from Theorem 4.1 part (b) of Section 4 that for a conformal $t$-design based on a lattice vertex operator algebra $V_L$ the largest $t$ one can hope for is $t = 5$ if $L \neq A_1, E_8$.

One of the main results of [Mat01] are certain formulae for traces of the form

$$\text{tr}|_{V_0} o(v_1) o(v_2) \cdots o(v_k)$$

with $v_1, \ldots, v_k \in V_2$ and $V$ a vertex operator algebra of class $S^{2k}$, $k \leq 5$, with $V_1 = 0$ and $\text{Aut} V$ finite (Theorem 2.1 for $n = 2$ and Theorem 5.1 for $k \leq 2$ and general $n$). For the Moonshine module $V^4$ and $n = 2$ they were first obtained by S. Norton [Nor96]. The proof given in [Mat01] remains valid, if one replaces the assumption that $V$ is a vertex operator algebra of class $S^{2k}$ and $\text{Aut} V$ is finite with the assumption that $V_l$ for $l \leq n$ is a conformal $2k$-design based on $V$.

One can study similar trace identities for conformal $t$-designs supported by a module of $V$, without the assumption $V_1 = 0$, and with $v_1, \ldots, v_k$ homogeneous elements of $V$ not necessarily in $V_2$.

We end this section with an open problem: It is known that there exist non-trivial block $t$-designs [Tei87] and spherical $t$-designs [SZS93] for arbitrary large $t$ and arbitrary large length respectively dimension. The same result for block designs supported by self-orthogonal codes and for spherical designs supported by integral lattices seems to be open (and less likely). The example of the lattice vertex operator algebra $V_{A_1}$ (Example 2.6) shows that there exist non-trivial $t$-designs for arbitrary large values of $t$. However, this case may be considered exceptional because the central charge $c = 1$ of $V_{A_1}$ is small. We ask therefore if there exist non-trivial conformal $t$-designs for arbitrary large values of $t$ and arbitrary large central charge.

3 Conformal designs associated to extremal vertex operator algebras

A rational vertex operator algebra $V$ is called self-dual (other authors use also the names holomorphic or meromorphic) if the only irreducible module of $V$ up to iso-
morphism is \( V \) itself. The central charge \( c \) of a self-dual vertex operator algebra is of the form \( c = 8l \) where \( l \) is a positive integer. Its character \( \chi_V \) is a weighted homogeneous polynomial of weight \( c \) in the polynomial ring over the rationals generated by the character of the self-dual lattice vertex operator algebra \( V_{E_8} \) associated to the \( E_8 \) lattice (given the weight 8) and the character of the self-dual Moonshine module \( V^\natural \) (given the weight 24); see [Hoh95, Thm. 2.1.2]. Since \( k \) with \( k = \frac{c}{24} \) one can use alternatively \( \sqrt[3]{j} \) (weight 8) and the constant function 1 (weight 24) as generators; see [Hoh95, Sect. 2.1].

In [Hoh95], extremal vertex operator algebras were defined. A self-dual vertex operator algebra \( V \) of central charge \( c \) is called extremal if its minimal weight \( \mu(V) \) satisfies \( \mu(V) > \left[ \frac{c}{24} \right] \), i.e., a Virasoro primary highest weight vector of \( V \) different from a multiple of the vacuum has at least the conformal weight \( \left[ \frac{c}{24} \right] + 1 \). It follows from the above description of the character of a self-dual vertex operator algebra, that the character of an extremal vertex operator algebra \( V \) has the form

\[
\chi_V = \chi_{V^\natural} \cdot (1 + A_{k+1} q^{k+1} + A_{k+2} q^{k+2} + \cdots),
\]

with \( k = \left[ \frac{c}{24} \right] \) and constants \( A_k \) independent of \( V \). It can be shown that \( A_{k+1} > 0 \) (see [Hoh95, Thm. 5.2.2]), i.e., the minimal weight of an extremal vertex operator algebra is in fact equal to \( \left[ \frac{c}{24} \right] + 1 \).

**Theorem 3.1** Let \( V \) be an extremal vertex operator algebra of central charge \( c \). Then the degree \( n \) subspace \( V_n \) of \( V \) is a conformal \( t \)-design with \( t = 11 \) for \( c \equiv 0 \) (mod 24), \( t = 7 \) for \( c \equiv 8 \) (mod 24), and \( t = 3 \) for \( c \equiv 16 \) (mod 24).

**Proof.** Let \( v \in V_s \) be a Virasoro highest weight vector of conformal weight \( s \), where \( 0 < s \leq t \). It follows from Zhu, [Zhu90] Thm. 5.3.3, that

\[
\chi_V(v, q) = q^{-c/24} \sum_{n=0}^{\infty} \text{tr}|_{V_n} o(v) q^n
\]

is a meromorphic modular form of weight \( s \) for \( \text{PSL}_2(\mathbb{Z}) = \langle S, T \rangle \) with character \( \rho \) given by \( \rho(S) = 1 \) and \( \rho(T) = e^{-2\pi i c/24} \). Here, \( S = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Since \( V \) is assumed to be an extremal vertex operator algebra, one has

\[
\text{tr}|_{V_n} o(v) = \text{tr}|_{(V^\natural)_n} o(v) = 0
\]

for \( n = 0, \ldots, k \), where \( k = \left[ \frac{c}{24} \right] \). For the last equal sign in (13), one uses the skew-symmetry identity \( Y(v, z)u = e^{-L_z} Y(u, -z)v \). This shows that for \( u \in (V^\natural)_n \), the product \( o(v)u \in V_n \) is contained in the Virasoro highest-weight module generated by \( v \in V_s \). Since \( V \) is extremal, one has \( s > k \geq n \) and therefore \( o(v)u = 0 \).

Let \( \Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \) be the unique normalized cusp form of weight 12 for \( \text{PSL}_2(\mathbb{Z}) \). One has \( \sqrt[3]{\Delta} = \eta \), the Dedekind eta-function. Since \( \eta(-1/\tau) = \eta(\tau) \) and
\[ \eta(\tau + 1) = e^{2\pi i/24} \eta(\tau) \] it follows that \( \chi_V(v, q) \cdot \eta^c \) is a holomorphic modular form for \( \text{PSL}_2(\mathbb{Z}) \) of weight \( c/2 + s \) and trivial character for which the first \( k+1 \) coefficients of its \( q \)-expansion vanish. Such a modular form is of the form \( \Delta^{k+1} f \) for some holomorphic modular form \( f \) of weight

\[ c/2 + s - 12(k+1) = \begin{cases} 
  s - 12, & \text{for } c = 24k, \\
  s - 8, & \text{for } c = 24k + 8, \\
  s - 4, & \text{for } c = 24k + 16.
\end{cases} \]

Using the fact that there is no non-zero holomorphic modular form of negative weight, one concludes that \( f = 0 \) if \( s \leq t \) and \( t \) as in the theorem. This gives \( \chi_V(v, q) = 0 \) and so \( \text{tr}_{V_n} o(v) = 0 \) for any \( n \). The result follows now from Theorem 2.3.

Remark: Under the same assumption as in the theorem, our proof shows that also \( \text{tr}_{V_n} o(v) = 0 \) for any Virasoro highest weight vector \( v \) of conformal weight \( s \) where \( s = 13, 14, \) or \( 15 \) for \( c \equiv 0 \pmod{24} \), \( s = 9, 10 \) or \( 11 \) for \( c \equiv 8 \pmod{24} \) and \( s = 5, 6, 7 \) for \( c \equiv 16 \pmod{24} \) if we use the fact that there is no non-zero holomorphic modular form of weight 1, 2, or 3.

Example 3.2 Extremal vertex operator algebras are known to exist for \( c = 8, 16, 24, 32 \) and 40; see [Hö95], Sect. 5.2.

For the \( c = 8 \), the only example is \( V_{E_8} \) and we know already from Example 2.7 that its homogeneous subspaces are conformal 7-designs. For \( c = 16 \), the two self-dual vertex operator algebras \( V_{E_8}^2 \) and \( V_{D_{16}}^+ \) are both extremal and their homogeneous subspaces are therefore conformal 3-designs.

The known examples for \( c = 24, 32, 40 \) are \( \mathbb{Z}_2 \)-orbifolds of lattice VOAs, where the lattice is an even unimodular lattice of rank \( c \) without vectors of squared length 2, i.e., an extremal lattice. For \( c = 24 \), this gives only the Moonshine module \( V_{24} \). Using Theorem 2.5 we have already seen that its subspaces of fixed degree are conformal 11-designs. For \( c = 32 \), there exist at least \( 10^7 \) [Kin03] extremal even lattices. Our theorem shows in particular that the degree subspace \( V_2 \) of the \( \mathbb{Z}_2 \)-orbifold of the associated lattice vertex operator algebra is a conformal 7-design. In the case of the Barnes-Wall lattice of rank 32, it was observed by R. Griess and the author that the automorphism group of the \( \mathbb{Z}_2 \)-orbifold vertex operator algebra is likely \( 2^{27}:E_6(2) \) and one may use in this case also Theorem 2.5 to derive the conformal 7-design property. There are at least 12579 extremal doubly-even codes [Kin01] of length 40. Using orbifold constructions (cf. [DGH98], Sect. 4) one sees that there are at least so many extremal even lattices of rank 40 and at least so many extremal vertex operator algebras of central charge 40. The homogeneous subspaces of those vertex operator algebras, in particular \( V_2 \), are conformal 3-designs. Characters of extremal vertex operator algebras for small \( c \) are given in [Hö95], Table 5.1. For \( c = 32 \), one has \( \text{dim} V_2 = 139504 \); for \( c = 40 \), one has \( \text{dim} V_2 = 20620 \).

Since it is unknown if any extremal vertex operator algebra of central charge 48, 72, \ldots exists, we do not get currently any other conformal 11-designs from our theorem besides the ones from the Moonshine module.

The minimal weight of an extremal vertex operator algebra grows linearly with \( c \) and therefore the conformal designs as in Theorem 3.1 become trivial for \( c \geq 264, 176, \) or 88, if \( c \equiv 0, 8, \) or 16 (mod 24), respectively.
One can ask for a similar theorem for vertex operator super algebras. In [Höh95, Sect. 5.3], it has been shown that the minimal weight of a self-dual vertex operator super algebra (under certain natural conditions called “very nice”) satisfies \( \mu(V) \leq \frac{1}{2} \left\lfloor \frac{c}{2} \right\rfloor + \frac{1}{2} \). Self-dual vertex operator super algebras meeting that bound are called extremal. Since that time, the analogous bounds for the minimal weight respectively length of even self-dual codes respectively unimodular lattices have been improved to the same bounds as one knows for doubly-even self-dual codes and even unimodular lattices (with the exception of codes of length \( n \equiv 22 \pmod{24} \) and the shorter Leech Lattice of rank 23). So one may expect that for self-dual vertex operator super algebras the analogous bound \( \mu(V) \leq \left\lfloor \frac{c}{2} \right\rfloor + 1 + r \) holds with \( r = \begin{cases} \frac{1}{2}, & \text{for } c \equiv 23\frac{1}{2} \pmod{24}, \\ 0, & \text{else.} \end{cases} \) For the case of the exceptional lengths \( n \equiv 22 \pmod{24} \), it was proven in [LL01], that codes meeting the improved bound lead to block 3-designs. The analogous result for vertex operator super algebras would be that for a self-dual vertex operator super algebra \( V \) of central charge \( c = 24k + 23\frac{1}{2} \) and minimal weight \( \geq k + 3/2 \) the homogeneous subspaces of \( V_{(0)} \)-modules are conformal 7-designs. For the proof, one has to analyze the singular part of the vector valued modular functions associated to \( V \). However, in the case of lattices the analogous theorem only applies to the shorter Leech lattice leading to spherical 7-designs. The only known — and likely the only — example for vertex operator super algebras would be the shorter Moonshine module \( VB^{\natural} \) for which we have already shown in the previous section that the homogeneous subspaces of \( VB^{\natural}_{(0)} \)-modules are conformal 7-designs.

4 Classification results

In this section, we investigate vertex operator algebras and super algebras of minimal weight \( m \leq 2 \) whose degree-\( m \)-part form a conformal 6- or 8-design.

4.1 Statement of results

For conformal 6-designs, we have the following classification result:

**Theorem 4.1** Let \( V \) be a simple vertex operator super algebra of central charge \( c \) and minimal weight \( m \) and assume that \( V \) has a real form such that the invariant bilinear form is positive-definite. Denote with \( V_{(0)} \) the even vertex operator subalgebra of \( V \). If the degree-\( m \)-subspace \( V_m \) is a conformal 6-design, one has:

(a) If \( m = \frac{1}{2} \), then \( V \) is isomorphic to the self-dual “single fermion” vertex operator super algebra \( V_{\text{Fermi}} \cong L_{1/2}(0) \oplus L_{1/2}(\frac{1}{2}) \) of central charge 1/2.

(b) If \( m = 1 \), then \( V \) is isomorphic to the lattice vertex operator algebra \( V_{A_1} \) of central charge 1 associated to the root lattice \( A_1 \) or the lattice vertex operator algebra \( V_{E_8} \) of central charge 8 associated to the root lattice \( E_8 \).

(c) If \( m = \frac{3}{2} \) and the additional assumption \( \dim V_2 > 1 \) holds, the central charge of \( V \) is either \( c = 16 \) or \( c = 23\frac{1}{2} \).

(d) For \( m = 2 \) and the additional assumptions that \( V \) is rational and that for any \( V_{(0)} \)-module of conformal weight \( h \) there exists a \( V_{(0)} \)-module of the same conformal weight whose lowest degree subspace is a conformal 6-design, it follows...
Table 1: Possible central charges and conformal weights of additional \(V(0)\)-modules in case of Theorem 4.1 (d)

| cent. charge \(c\) | \(8\) | \(16\) | \(24\) | \(32\) | \(40\) | \(1496\) |
|-----------------|-----|------|------|------|------|--------|
| dim \(V_2\)     | \(156\) | \(2296\) | \(63428\) | \(96256\) | \(196884\) | \(90118\) |
| conf. weight \(h\) | \(\frac{1}{2}, 1\) | \(1, \frac{3}{2}\) | \(103, 67\) | \(3, 31\) | \(-\) | \(-\) |

that \(V\) is a vertex operator algebra and there are at most 11 possible cases for the central charge. The allowed values of \(c\), \(\text{dim} \ V_2\) and conformal weights \(h\) of a possible additional irreducible \(V(0)\)-modules are given in Table 1. In particular, if \(c \in \{24, 32, 40, 1496\}\), \(V\) has to be self-dual.

For all the cases in which examples are known, one gets in fact conformal 7-designs. This theorem is analogous to similar results of Martinet [Mar01] on integral lattices whose vectors of minimal norm \(m \leq 4\) form a spherical 7-design and of Lalauze-Labayle [LL01] on binary self-orthogonal codes for which the set of words of minimal weight \(m \leq 8\) form a block 3-design.

The additional assumption in Theorem 4.1 (d) that the lowest degree subspaces of certain \(V\)-modules are also conformal 6-designs seems quite strong. It was introduced since it is necessary to apply Proposition 4.3. The analogue of Proposition 4.3 for codes and lattices holds without an analogous assumption, so that this may be also the case for vertex operator algebras. If one assumes only that for at least one irreducible \(V\)-module with \(h \neq 0\) the lowest degree subspace is a conformal 6-design, then one gets the same values for \(c\) and \(\text{dim} \ V_2\) as in Table 1 but no further conditions on the allowed values for \(h\). If one drops the assumption on \(V(0)\)-modules completely, all values of \(c\) occurring in Lemma 4.15 are allowed.

By requiring that \(V\) has a real form such that the restriction \((\ , \ )\) of the natural invariant bilinear form is positive-definite, it follows that the central charge \(c\) of \(V\) and the central charge \(e\) of any vertex operator subalgebra of \(V\) with Virasoro element \(\omega' \in V_2\) with \(L_1 \omega' = 0\) is positive since \(0 < \langle \omega', \omega' \rangle = \omega'(3)\omega' + (L_1 \omega')(2)\omega' = \omega'(3)\omega' = e/2\cdot 1\).

In addition, we obtain for conformal 8-designs:

**Theorem 4.2**

(i) Let \(V\) be a simple vertex operator super algebra of central charge \(c \neq \frac{1}{2}, 1\) and minimal weight \(m \leq 2\) with \(\text{dim} \ V_2 > 1\) having a real form such that the natural invariant bilinear form is positive-definite. If the weight-\(m\)-part \(V_m\) is a conformal 8-design, then \(V\) is a vertex operator super algebra of central charge 24 and minimal weight 2 with \(\text{dim} \ V_2 = 196884\).

(ii) If in addition, we assume that for any \(V(0)\)-module of conformal weight \(h\) there exists a \(V(0)\)-module of the same conformal weight whose lowest degree subspace is a conformal 8-design, then \(V\) is a self-dual vertex operator algebra with the same conformal character as the Moonshine module \(V^\natural\).

Analogous results for spherical 11-designs and block 5-designs characterizing the Leech lattice and the Golay code can again be found in [Mar01] and [LL01], respectively.

Part (i) of Theorem 4.2 for \(m = 2\) was proven by Matsuo [Mat01]. He also considered the case (d) of Theorem 4.1 with weaker assumptions resulting in more
In the next subsection, we will prove two Propositions which give relations between
$\omega'$ and $\omega''$ for conformal 6-designs. In the subsequent four subsections, we will
prove the four cases of Theorem 4.1. In subsection 4.7, we will prove Theorem 4.2. In
the final subsection 4.8, we will discuss the examples respectively candidates of vertex
operator super algebras occurring in Theorem 4.1 and 4.2 and compare our results
with the situation for codes and lattices.

### 4.2 Conditions for 6-designs

In this subsection, we assume that $V$ is a vertex operator algebra of central charge $c$
and $\omega'$ and $\omega''$ are two elements in $V_2$ generating two commuting Virasoro vertex
operator algebras $U_{\omega'}$ and $U_{\omega''}$ of central charge $c$ and $c-e$, respectively. These
two Virasoro vertex operator subalgebras are isomorphic to a quotient of the Verma
module quotient $M(x,0)/M(x,1)$ where $x = e$ or $c-e$. We decompose now the
subalgebra $U_{\omega'} \otimes U_{\omega''}$ of $V$ as a module for the Virasoro vertex operator subalgebra
$V_\omega$ of $V$.

**Lemma 4.3** Let $\bigoplus_h \overline{M}(h)$ be the decomposition of $U_{\omega'} \otimes U_{\omega''}$ into isotypical
components as a module for the Virasoro vertex operator algebra $V_\omega$ of central charge $c$.
Assume that $c, c-e \neq 0$, $-\frac{22h}{7}, -\frac{68}{7}, \frac{1}{2}$. Then the multiplicity $\mu_h$ of $M(h)$ in $\overline{M}(h)$
is 0 for $h = 1, 3, 5, 7$ and is $1, 1, 1, 2$ for $h = 0, 2, 4, 6$, respectively. If also $e, c-e \neq -\frac{46}{7}, -\frac{3}{2}$, then, in addition, the multiplicity of $M(8)$ in $\overline{M}(8)$ is 3.

If $e = \frac{3}{2}$ and $U_{\omega'}$ is assumed to be simple and the other assumptions hold, then the
multiplicity of $M(6)$ in $\overline{M}(6)$ is 1, the multiplicity of $M(8)$ in $\overline{M}(8)$ is 2 and the other
multiplicities are the same.

**Proof.** Let $U(x)$ be a Virasoro vertex operator algebra of central charge $x$. For $x \neq 0$,
$-\frac{22}{7}, -\frac{68}{7}, \frac{1}{2}$ (and $-\frac{46}{7}, -\frac{3}{2}$) one has $\dim U(x)_n = \dim(M(x,0)/M(x,1))_n$ for $n \leq 7$
(respectively, $n \leq 8$) because the formula (10) shows that under these conditions there
are no singular vectors of degree $n$ in $M(x,0)/M(x,1)$. If we know that the Virasoro
module $U_{\omega'} \otimes U_{\omega''}$ does not contain any additional singular vectors of degree $n \leq 6$
(respectively, $n \leq 8$) besides the one in $M(c,0)$, the result follows from

\[
q^x \chi_{U_{\omega'} \otimes U_{\omega''}} = \left( \prod_{n=2}^{\infty} (1-q^n)^{-1} \right)^2 + O(q^9)
\]

\[
= \prod_{n=2}^{\infty} (1-q^n)^{-1} + (9 + 2q^6 + 3q^8)^\infty (1-q^n)^{-1} + O(q^9) = \sum_n \mu_h q^h \chi_{M(h)}.
\]

In the case $e = \frac{3}{2}$ with simple $U_{\omega'}$ one has to form a quotient module of $M(1/2,0)/M(1/2,1)$ by dividing out additional singular vectors and one gets $U_{\omega'} \cong L(1/2,0)$
with character

\[
\chi_{U_{\omega'}} = q^{-1/48} (1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 5q^8 + 5q^9 + \cdots).
\]

Under the same assumption as before it follows that the multiplicity of $M(h)$ in $\overline{M}(h)$
does not change for $h < 6$ or $h = 7$, but the multiplicity of $M(6)$ in $\overline{M}(6)$ is only 1
and the multiplicity of $M(8)$ in $\overline{M}(8)$ is 2.
The space of Virasoro highest weight vectors of degree $h$ in $U_{\omega'} \otimes U_{\omega''}$ can also be computed explicitly for $n \leq 8$ by using the characterization $L_1 v = L_2 v = 0$ for highest weight vectors $v \in \mathcal{M}(h)$. Its dimension equals indeed the given values for $\mu_h$.

We choose highest weight vectors $v^{(2)} \in \mathcal{M}(2)$, $v^{(4)} \in \mathcal{M}(4)$ and two linear independent highest weight vectors $v^{(6)}_a$, $v^{(6)}_b \in \mathcal{M}(6)$.

We list here only $v^{(2)}$ and $v^{(4)}$ as the expressions for $v^{(6)}_a$ and $v^{(6)}_b$ are quite long:

\begin{align*}
v^{(2)} &= b_{-2} 1 - \frac{(c-e)}{e} a_{-2} 1, \\
v^{(4)} &= \frac{-3}{5} b_{-4} 1 + b_{2}^2 1 - \frac{3}{5} \frac{(22 + 5 c - 5 e)}{e} (c-e) a_{-4} 1 - \frac{2}{5} \frac{(22 + 5 c - 5 e)}{e} a_{-2} b_{-2} 1 + \frac{(22 + 5 c - 5 e)}{e} (c-e) a_{-2}^2 1.
\end{align*}

Here, $a_n$ and $b_n$ denote the usual generators of the Virasoro algebras of $U_{\omega'}$ and $U_{\omega''}$, respectively. The expressions for $v^{(6)}_a$ and $v^{(6)}_b$ are not well-defined for $e = \frac{1}{2}$ or $c-e = \frac{1}{2}$.

If $e = \frac{1}{2}$ and $c-e \neq \frac{1}{2}$, the Virasoro module $M(1/2,0)/M(1/2,1)$ contains the singular vector $s_6 = (-108 a_{-6} - 264 a_{-4} a_{-2} + 93 a_{-2}^3 + 64 a_{-3}^2) 1$ of degree 6 and $(U_{\omega'})_6 \cong (L_{1/2}(0))_6 = (M(1/2,0)/M(1/2,1))_6/\mathbb{C} s_6$. A representative $v^{(6)}$ for a non-zero highest weight vector in $\mathcal{M}(6)$ is given by

\begin{align*}
v^{(6)} &= \big((-2/3)(2373 + 1657 c + 20 e^2) b_{-6} - (1/3)(12387 + 7301 c + 112 c^2) b_{-4} b_{-2} + (1/3)(6051 + 854 + 70 e^2) b_{-3}^2 + (501 + 1099 c) b_{-2}^2, \\
&\quad -(5/48)(-1 + 2 c)(-5031 + 3195 c + 1696 c^2 + 140 c^3) a_{-6} \\
&\quad -(11/3)(-5031 + 3195 c + 1696 c^2 + 140 c^3) a_{-4} b_{-2} - (2/3)(-5031 + 3195 c + 1696 c^2 + 140 c^3) a_{-3} b_{-3} \\
&\quad + (1/3)(-102555 + 89361 c + 12970 c^2 + 224 c^3) a_{-2} b_{-4} - 407(-129 + 115 c + 14 c^2) a_{-2} b_{-2} \\
&\quad -(7/24)(-1 + 2 c)(-5031 + 3195 c + 1696 c^2 + 140 c^3) a_{-4} a_{-2} \\
&\quad + (35/192)(-1 + 2 c)(-5031 + 3195 c + 1696 c^2 + 140 c^3) a_{-2}^2 \\
&\quad + 7(-5031 + 3195 c + 1696 c^2 + 140 c^3) a_{-2} b_{-2} 1.\big)
\end{align*}

Assume that $W$ is a module of $V$ of conformal weight $h$. Using the associativity relation for a vertex operator algebra and its modules one can evaluate the trace of $o(v)$ for an element $v = a_{p_1} \cdots a_{p_n} b_{q_1} \cdots b_{q_l} 1 \in U_{\omega'} \otimes U_{\omega''}$ on the lowest degree part $W_h$. One obtains $\text{tr}|_{W_h} o(v) = \sum_{i=0}^{\infty} \alpha_i m_i^* \ast 1$ for some $n$, where $m_i^* = \text{tr}|_{W_h} a_{p_i}^\ast b_{q_i}$, and the $\alpha_i$ are explicit constants depending on $c$, $e$ and $h$. We also define $d^* = m_0^* = \dim W_h$. The traces of $v^{(2)}$, $v^{(4)}$, $v^{(6)}_a$ and $v^{(6)}_b$ and $v^{(6)}$ can now be computed. Again, we list only the traces for $v^{(2)}$ and $v^{(4)}$ explicitly:

\begin{align*}
\text{tr}|_{W_h} o(v^{(2)}) &= h d^* - \frac{c}{e} m_1^*,
\end{align*}

\[18\]
\[ \text{tr}|_{W_h} o(v^{(4)}) = \frac{h}{2} + h^2 d + \frac{(22 + 5c)(c - 2(e + 22h + 5eh))}{5e(22 + 5e)} m^*_1 + \frac{968 + 330c + 25c^2}{5e(22 + 5e)} m^*_2. \] (18)

**Proposition 4.4** Let \( V \) be a vertex operator algebra of central charge \( c \neq -24, -15, -\frac{49}{3}, \frac{15}{19}, 1 \) which contains elements \( \omega', \omega - \omega' \in V_2 \) generating two commuting Virasoro vertex operator algebras of central charge \( e \) and \( c - e \), respectively, with \( e, c - e \neq -\frac{49}{3} \), \(-\frac{15}{19}\), \(0\). If \( e = \frac{1}{2} \) or \( e = c - \frac{1}{2} \), assume that the Virasoro vertex operator algebra generated by \( \omega' \) or \( \omega - \omega' \), respectively, is simple and the other assumptions hold. If there exists a module \( W \) of \( V \) of conformal weight \( h \neq 0 \) such that the lowest degree part \( W_h \) is a conformal 6-design, then

\[ 4 + 7c + c^2 - 124h - 31ch + 248h^2 = 0. \]

**Proof.** Assume first that \( e \neq \frac{1}{2}, c - \frac{1}{2} \). In this case the conditions on the central charges guarantee that the Virasoro vertex operator algebras generated by \( \omega' \) and \( \omega - \omega' \) are isomorphic to \( M(x, 0)/M(x, 1) \) up to degree 6, where \( x = e \) or \( c - e \), respectively. By assumption, the degree \( h \) subspace \( W_h \) is a conformal 6-design. Therefore, the highest weight vectors \( v^{(2)}, v^{(4)} \), \( v^{(6)}_a \) and \( v^{(6)}_b \) give the trace identities

\[ \text{tr}|_{W_h} o(v^{(2)}) = \text{tr}|_{W_h} o(v^{(4)}) = \text{tr}|_{W_h} o(v^{(6)}) = 0. \] (19)

By evaluating the traces, the equations (19) lead to a homogeneous system of linear equations for \( d^* = m^*_0, m^*_1, m^*_2 \) and \( m^*_3 \). Since \( d^* > 0 \), the system has to be singular and its determinant

\[ \Delta = \frac{15(c + 24)(c + 15)(c + \frac{44}{7})(c - \frac{44}{7})(4 + 7c + c^2 - 124h - 31ch + 248h^2)(c - e)h}{8(e + \frac{44}{7})(e + \frac{44}{7})^2 c^2(e - \frac{1}{2})} \]

has to vanish. The proposition follows in this first case.

If \( e = \frac{1}{2} \) or \( e = c - \frac{1}{2} \), we can, without loss of generality, assume that \( e = \frac{1}{2} \) and \( c - e \neq \frac{1}{2} \) because otherwise one can switch the role of \( \omega' \) and \( \omega - \omega' \) since \( c \neq \frac{19}{15} \).

Let \( d^*_i \) be the dimension of the \( a_0 = \omega^{(1)}_0 \) eigenspace for the eigenvalue \( l \). One has \( m^*_3 = d^* = \dim W_h = d^*_0 + d^*_1/2 + d^*_1/16 \) and \( m^*_1 = d^*_1/2 \cdot (1/2)^1 + d^*_1/16 \cdot (1/16)^1 \), for \( i = 1, 2, 3 \). The homogeneous system

\[ \text{tr}|_{W_h} o(v^{(2)}) = \text{tr}|_{W_h} o(v^{(4)}) = \text{tr}|_{W_h} o(v^{(6)}) = 0 \] (20)

of linear equations for \( d^*_0, d^*_1/2, d^*_1/16 \) has to be singular since \( d^* > 0 \). Hence the determinant

\[ \Delta = \frac{15(c + 24)(c + 15)(c + \frac{44}{7})(c - \frac{44}{7})(4 + 7c + c^2 - 124h - 31ch + 248h^2) h}{512} \]

of the system must vanish. The proposition follows also in this case. \( \square \)

Using the associativity relation of a vertex operator algebra one can also evaluate the trace of \( o(v) \) for \( v = a_{p_1} \cdots a_{p_k} b_{q_1} \cdots b_{q_l} 1 \) on \( V_2 \). One obtains now \( \text{tr}|_{V_2} o(v) = \)
β + \sum_{i=0}^{n} \alpha_i m_i \) for some \( n \), with \( m_i = \text{tr}|V_2 a_0^i \), and the \( \alpha_i \) and \( \beta \) are explicit constants depending on \( c \) and \( e \). We also define \( d = m_0 = \dim V_2 \). The traces of \( v^{(2)}, v^{(4)}, v_a^{(6)}, v_b^{(6)} \) and \( v^{(6)} \) can now be computed. Again, we list only the traces for \( v^{(2)} \) and \( v^{(4)} \) explicitly:

\[
\text{tr}|V_2 o(v^{(2)}) = 2d - \frac{c}{e} m_1, \tag{21}
\]
\[
\text{tr}|V_2 o(v^{(4)}) = \frac{22}{5} d + \frac{22 + 5c}{5e} (c - 22 (4 + e)) m_1 + \frac{968 + 330c + 25c^2}{5e (22 + 5e)} m_2. \tag{22}
\]

**Proposition 4.5 (Matsuo)** Let \( V \) be a vertex operator algebra of central charge \( c \neq -24, -15, -\frac{44}{5}, \frac{55 + \sqrt{33}}{2}, 1 \) with \( V_1 = 0 \) which contains elements \( \omega', \omega - \omega' \in V_2 \) generating two commuting Virasoro vertex operator algebras of central charge \( e \) and \( c - e \), respectively, with \( e, c - e \neq -\frac{56}{7}, -\frac{22}{7}, 0 \). If \( e = \frac{1}{2} \) or \( e = c - \frac{1}{2} \), assume that the Virasoro vertex operator algebra generated by \( \omega' \) or \( \omega - \omega' \), respectively, is simple and the other assumptions hold. If \( V_2 \) forms a conformal 6-design then

\[
d = \frac{c (2388 + 955c + 70c^2)}{2 (748 - 55c + c^2)}. \]

**Proof.** Assume first \( e \neq \frac{1}{2}, c - \frac{1}{2} \). As in the previous proposition, the conditions on the central charges guarantee that \( v^{(2)}, v^{(4)}, v_a^{(6)} \) and \( v_b^{(6)} \) are defined. By assumption, \( V_2 \) is a conformal 6-design. Therefore one has for the highest weight vectors \( v^{(2)}, v^{(4)}, v_a^{(6)} \) and \( v_b^{(6)} \) the trace identities

\[
\text{tr}|V_2 o(v^{(2)}) = \text{tr}|V_2 o(v^{(4)}) = \text{tr}|V_2 o(v_a^{(6)}) = \text{tr}|V_2 o(v_b^{(6)}) = 0. \tag{23}
\]

The equations (23) form a system of linear equations for \( d, m_1, m_2 \) and \( m_3 \). This system is non-singular since its determinant

\[
\Delta = \frac{15(c + 24)(c + 15)(c + \frac{44}{5})(c - \frac{44}{5})(c - \frac{55 + \sqrt{33}}{2})(c - \frac{55 - \sqrt{33}}{2})(c - e)}{4(e + \frac{68}{7})(e + \frac{22}{7})^2 c^2 (c - \frac{1}{2})}
\]

is not 0. In this case there is a unique solution for \( d, m_1, m_2 \) and \( m_3 \). The solution for \( d \) is given in the proposition and does not depend on \( e \).

If \( e = \frac{1}{2} \) or \( e = c - \frac{1}{2} \), we can, without loss of generality, assume that \( e = \frac{1}{2} \) and \( c - e \neq \frac{1}{2} \) because otherwise one can switch the role of \( \omega' \) and \( \omega - \omega' \) since \( c \neq 1 \).

Similar as in the proof of Proposition 4.2, let \( d_l \) be the dimension of the \( a_0 = \omega' \) eigenspace for the eigenvalue \( l \). One has \( m_0 = d = \dim V_2 = 1 + d_0 + d_{1/2} + d_{1/16} \) and \( m_i = 2^i + d_{1/2} \cdot (1/2)^i + d_{1/16} \cdot (1/16)^i \), for \( i = 1, 2, 3 \). The system

\[
\text{tr}|V_2 o(v^{(2)}) = \text{tr}|V_2 o(v^{(4)}) = \text{tr}|V_2 o(v^{(6)}) = 0
\]

of linear equations for \( d_0, d_{1/2}, d_{1/16} \) is non-singular since its determinant

\[
\Delta = \frac{15(c + 24)(c + 15)(c + \frac{44}{5})(c - \frac{44}{5})(c - \frac{55 + \sqrt{33}}{2})(c - \frac{55 - \sqrt{33}}{2})}{256}
\]

is not 0. In this case there is a unique solution for \( d_0, d_{1/2}, d_{1/16} \).
is not 0. The solution for \( d \) is again the one given in the proposition.

The last proposition was first obtained by Matsuo \[\text{Mat01}\] using a different argument.

To apply Proposition \[4.3\] and \[4.5\] to a vertex operator algebra as in Theorem \[4.1\] and \[4.2\] one has to find suitable elements \( \omega' \in V_2 \) such that \( \omega' \) and \( \omega - \omega' \) generate commuting Virasoro vertex operator algebras. By Theorem 5.1 of \[\text{FZ92}\], this is the case if \( \omega' \) generates a Virasoro vertex operator algebra and \( L_1 \omega' = 0 \). The last condition will be automatically satisfied if either \( V_1 = 0 \) or \( \omega' \) is the Virasoro element of an affine Kac-Moody vertex operator algebra \( U < V \) or a Clifford vertex operator super algebra \( U < V \). In the case of a Kac-Moody vertex operator subalgebra, \( \omega' \) is given by the Sugawara expression \( \omega' = \sum_i \alpha_i a_i^{(-1)} a_i \), where the \( a_i \) form an orthonormal basis of a non-degenerated invariant bilinear form on \( U_1 \). For a Virasoro highest weight vector \( v \) of conformal weight \( h \) one has \( [L_m, v_n] = ((h-1)m-n)v_{m+n} \) with \( v_k = v_{(k-h+1)} \) (cf. \[\text{Kac97}\], Cor. 4.10). For \( u \in U_1 \) one gets therefore
\[
L_1 u_{(-1)} u = [L_1, u_{-1}] u + u_{-1} L_1 u = u_0 u = 0
\]
and the claim follows. Similarly, for \( b \in U_{1/2} \) and \( \omega' = \frac{1}{2} b_{-3/2} b_{-1/2} 1 = \frac{1}{2} b_{-3/2} b \) one gets
\[
L_1 b_{-3/2} b = [L_1, b_{-3/2}] b + b_{-3/2} L_1 b = b_{-1/2} b = 0
\]
and the claim follows also in the case of Clifford vertex operator super algebras.

### 4.3 Minimal weight \( \frac{1}{2} \)

Let \( V \) be a vertex operator super algebra as in part (a) of Theorem \[4.1\] and let \( V_{(0)} \) be the even vertex operator subalgebra. The vertex operator super subalgebra of \( V \) generated by the degree-1/2-part \( V_{1/2} \) is known to be isomorphic to the vertex operator super algebra \( V_{\text{Fermi}} \otimes d^* \), where \( d^* = \dim V_{1/2} \) and \( V_{\text{Fermi}} \cong L_{1/2}(0) \oplus L_{1/2}(\frac{1}{2}) \) is the so-called single fermion vertex operator super algebra of central charge 1/2. By assumption one has \( \mu(V) = \frac{1}{2} \) and hence \( d^* = \dim V_{1/2} \geq 1 \).

Recall that the central charge \( c \) of \( V \) is positive and assume first \( c \neq \frac{1}{2}, \frac{34}{35}, 1 \). The Virasoro element \( \omega' \) of a subalgebra \( V_{\text{Fermi}} \subset V \otimes d^* \) generates a Virasoro algebra of central charge \( c = \frac{1}{2} \). Since \( V_{1/2} \) is assumed to be a conformal 6-design, one can apply now Proposition \[4.3\] to the \( V_{(0)} \)-module \( V_{(1)} \) of conformal weight \( h = \frac{1}{2} \). For \( h = \frac{1}{2} \), the proposition gives for the central charge either \( c = \frac{1}{2} \) or \( c = 8 \).

If \( c = 8 \), one gets as in the proof of Proposition \[4.3\] by using the equations \[19\] that
\[
d_0 = \frac{255 d^*}{496}, \quad d_{1/2} = \frac{d^*}{496}, \quad d_{1/16} = \frac{15 d^*}{31}.
\]
This implies that \( 496 d^* \) and so \( d^* \geq 496 \), contradicting \( d^*/2 \leq c \) which follows from \( V_{\text{Fermi}} \subset V \) and the fact that the Virasoro element \( \omega^* \) of \( V_{\text{Fermi}} \) satisfies \( L_1 \omega^* = 0 \).

If \( c = 1 \), one finds again a Virasoro element \( \omega' \) of central charge \( \frac{1}{2} \) for a subalgebra \( V_{\text{Fermi}} \subset V \) and \( \omega - \omega' \) is the Virasoro element for another central charge \( \frac{1}{2} \) Virasoro algebra. Thus \( V \) contains a subalgebra isomorphic to \( L_{1/2}(0) \otimes 2 \). The only irreducible modules of \( L_{1/2}(0) \otimes 2 \) of conformal weight 1/2 are \( L_{1/2}(0) \otimes L_{1/2}(\frac{1}{2}) \) and \( L_{1/2}(\frac{1}{2}) \otimes L_{1/2}(0) \).
Let \( L_{1/2}(0) \). Thus \( d_{1/16} = 0 \). The two vectors \( v^{(2)} \) and \( v^{(4)} \) are also well-defined if 
\( e = c - e = \frac{1}{2} \). Together with the equations 
\[
\text{tr}|W_a o(v^{(2)}) = \text{tr}|W_a o(v^{(4)}) = 0 \]
one gets \( d_0 = d_{1/2} = 0 \) and so \( d^* = 0 \), contradicting \( d^* > 0 \).

The central charge \( c = \frac{24}{k} \) belongs to the unitary minimal series \( c = 1 - 6/(n(n + 1)) \), \( n = 3, 4, \ldots \), for \( n = 14 \) and hence \( L_{34/35}(0) \subset V \). The only irreducible \( L_{34/35}(0) \) modules of half-integral highest weight are \( L_{34/35}(0) \), \( L_{34/35}(2) \), \( L_{34/35}(23) \) and \( L_{34/35}(39) \) of conformal weight 0, 2, 23 and 39, respectively. The simple current module \( L_{34/35}(39) \) is the only module which can be used to extend \( L_{34/35}(0) \) to a simple vertex operator super algebra \( V \). Hence the minimal weight of \( V \) would be at least 39, a contradiction.

For \( c = \frac{1}{2} \) one has \( L_{1/2}(0) \subset V \). The two further irreducible modules of \( L_{1/2}(0) \) have the conformal weights \( \frac{1}{2} \) and \( \frac{1}{3} \). It follows \( V \cong L_{1/2}(0) \oplus L_{1/2}(\frac{1}{2}) \cong V_{\text{Fermi}} \) since \( L_{1/2}(\frac{1}{2}) \) is a simple current for \( L_{1/2}(0) \) and thus any non-trivial simple vertex operator super algebra extension of \( L_{1/2}(0) \) contains the module \( L_{1/2}(\frac{1}{2}) \) with multiplicity 1 and is unique up to isomorphism.

This finishes the proof of Theorem 4.1 part (a).

4.4 Minimal weight 1

We begin with the following identity:

**Lemma 4.6** Let \( V \) be a vertex operator algebra and \( a^1, a^2, \ldots, a^l \) be elements of \( V_1 \). Then
\[
\text{tr} | V | o(a^1_{(-1)}a^2_{(-1)} \cdots a^l_{(-1)}) = \text{tr} | V | a^1_{(0)}a^2_{(0)} \cdots a^l_{(0)} \\
+ \text{tr} | V | a^1_{(-1)}a^2_{(1)}a^3_{(-1)} \cdots a^3_{(0)}a^2_{(0)} + \text{tr} | V | a^1_{(-1)}a^2_{(1)}a^3_{(-1)}a^4_{(1)}a^4_{(-1)}a^3_{(0)}a^2_{(0)} + \cdots \\
+ \text{tr} | V | a^1_{(-1)}a^2_{(1)}a^3_{(2)}a^4_{(-1)}a^4_{(0)}a^3_{(0)}a^2_{(0)}a^1_{(0)}.
\]

**Proof.** The result follows from application of the associativity relation by induction. \( \square \)

For \( l = 4 \) the identity can be found in [Hur02] in the proof of Lemma 5.2.

Let \( V \) now be a vertex operator super algebra as in Theorem 4.1 part (b). We will use that under our assumptions \( V_1 \) is a reductive Lie algebra under the product \( x(0)y \) for \( x, y \in V_1 \) and that the vertex operator subalgebra \( \langle V_1 \rangle \) generated by \( V_1 \) is an integrable highest weight representation of the associated affine Kac-Moody Lie algebra (cf. [Hoh95] for vertex operator algebras with positive-definite bilinear form on a real form; see [DM04, DM06] for a result which applies to rational vertex operator algebras).

**Lemma 4.7** The central charge of the vertex operator subalgebra \( \langle V_1 \rangle \) spanned by \( V_1 \) equals the central charge of \( V \).

**Proof.** Let \( \omega' \) be the Virasoro element of \( \langle V_1 \rangle \) and denote the central charge of \( \langle V_1 \rangle \) by \( e \). One has \( e > 0 \) as \( V_1 \neq 0 \). Assume that \( e < c \). Then the vector \( v_2 \) given in [14]
is a Virasoro highest weight vector of $V$ and equation (19) together with the 6-design property of $V_1$ gives $\text{tr}|_{V_1} o(v_2) = \dim V_1 - \frac{c}{\pi} \dim V_1 = 0$ and so $e = c$, a contradiction.

As reductive Lie algebra, $V_1$ can be decomposed into a Lie algebra direct sum

$$ V_1 = t \oplus g_1 \oplus \cdots \oplus g_m $$

with abelian Lie algebra $t$ and simple Lie algebras $g_1, \ldots, g_m$. Let $h$ be a Cartan subalgebra of $V_1$ and let $\{h^1, \ldots, h^k\}$ be an orthonormal basis of $h$ with respect to the invariant form $(\cdot, \cdot)$ on $V_1$ induced by the canonical invariant bilinear form on $V$, i.e., $(x, y) 1 = x_{(-1)} y$ for $x, y \in V_1$. The form $(\cdot, \cdot)$ is an orthogonal sum of a non-degenerate form on $t$ and some nonzero multiples of the Killing forms on each of the simple factors of $V_1$.

**Lemma 4.8** Let $P(x_1, \ldots, x_k)$ be a complex harmonic polynomial in variables $x_1, \ldots, x_k$. Then

$$ \text{tr}|_{V_1} o(P(h^1_{(-1)}, \ldots, h^k_{(-1)})) = \sum_{\alpha \in \Phi} P(\alpha(h^1), \ldots, \alpha(h^k)),$$

where $\Phi \subset h^*$ is the root system of $V_1$ corresponding to $h$.

**Proof.** Let $h = \bigoplus_{\alpha \in \Phi} L_\alpha$ be the root space decomposition of $V_1$ with respect to $h$. Each $L_\alpha$ is one-dimensional and $[h, a] = \alpha(h) a$ for $a \in L_\alpha$ and $h \in h$. Hence

$$ \text{tr}|_{V_1} P(h^1_{(0)}, \ldots, h^k_{(0)}) = \sum_{\alpha \in \Phi} P(\alpha(h^1), \ldots, \alpha(h^k)).$$

We claim that the trace of a monomial $T = h^{i_1}_{(-1)} h^{i_2}_{(1)} h^{i_3}_{(0)} \cdots h^{i_k}_{(0)}$ on $V_1$ where $i_1, \ldots, i_k \in \{1, \ldots, k\}$ is 0 for $l \geq 3$: Indeed, for an element $v \in L_\alpha$ one has $T v = \lambda(h^{i_2}, v) h^{i_1} \in h$ with a constant $\lambda$; for $v \in h$ one has $T v = 0$ if $l \geq 3$.

For a harmonic polynomial $P = \sum_{i,j=1}^k a_{ij} x_i x_j$ of degree 2 we have

$$ \text{tr}|_{V_1} \sum_{i,j=1}^k a_{ij} \cdot (h^1_{(-1)} h^j_{(1)}) = \text{tr}|_h \sum_{i,j=1}^k a_{ij} \cdot (h^1_{(-1)} h^j_{(1)}) $$

$$ = \sum_{i,j,m=1}^k a_{ij} \langle h^m, h^j \rangle \langle h^i, h^m \rangle = \sum_{i=1}^k a_{ii} = 0,$$

where the last equality holds because $P$ is assumed to be harmonic. Similarly,

$$ \text{tr}|_{V_1} \sum_{i,j=1}^k a_{ij} \cdot (h^i_{(-1)} h^j_{(1)}) = 0.$$

The result follows now from Lemma 4.6 since the cases $l = 0$ and $l = 1$ are clear.

For $P = x_1^4 - 6x_1^2 x_2^2 + x_2^4$, the previous Lemma can be found in [Hur02], Sect. 6 and 7.
**Lemma 4.9** If $P(x_1, \ldots, x_k)$ is a complex harmonic polynomial, then the vector $v_p = P(h_{(-1)}^1, \ldots, h_{(-1)}^k)1$ is a highest weight vector for the Virasoro algebra of $V$.

**Proof.** Let $V(h)$ be the Heisenberg vertex operator algebra of central charge $k$ generated by $h$ and let $\omega'$ be its Virasoro element. For the Virasoro algebra associated to $\omega'$ it was proven in [DMN01], Lemma 5.1.12, that $v_p$ is a highest weight vector. Since $\omega'' = \omega - \omega'$ lies in the commutant of $V(h)$, one has $\omega''(x)v_p = 0$ and $\omega''(y)v_p = 0$ and therefore also $L_1v_p = \omega(2)v_p = 0$ and $L_2v_p = \omega(3)v_p = 0$, i.e., $v_p$ is also a highest weight vector for the Virasoro algebra of $V$.

**Lemma 4.10** (Hurley [Hur02] and [Hur06], Lemma 6.2) For the root system $\Phi$ of a nonsimple and nonabelian reductive Lie algebra $g$ of rank $n$ there exists a complex harmonic polynomial $Q$ of degree 4 in $n$ variables such that

$$\sum_{\alpha \in \Phi} Q(\alpha(h^1), \ldots, \alpha(h^n)) \neq 0$$

for an orthogonal basis $\{h^1, \ldots, h^n\}$ of a Cartan algebra $h$.

**Proof.** The rank $n$ of $g$ is at least 2. Let $Q$ be the above mentioned harmonic polynomial $x_1^4 - 6x_2^2x_3^2 + x_4^4$. The root system $\Phi$ is the union $\Phi_1 \cup \ldots \cup \Phi_m$ of the root systems of the either simple or abelian components $k_i (i = 1, \ldots, m)$ of $g$. (For the maybe existing abelian component, the root system is empty.) We choose an orthogonal basis $\{h^1, \ldots, h^n\}$ of the real subspace of $h$ such that $h^1$ and $h^2$ lie in the Cartan subalgebras of the components $k_1$ and $k_2$, respectively. Then $\alpha(h^1) = 0$ for any $\alpha \notin \Phi_1$ and $\alpha(h^2) = 0$ for any $\alpha \notin \Phi_2$. We obtain therefore

$$\sum_{\alpha \in \Phi} Q(\alpha(h^1), \ldots, \alpha(h^n)) = \sum_{\alpha \in \Phi_1} \alpha(h^1)^4 + \sum_{\alpha \in \Phi_2} \alpha(h^2)^4. \tag{25}$$

Both sums on the right hand side of equation (25) are real and nonnegative since we have assumed that the $h^i$ are in the real subspace of $h$. Furthermore, at least one of the two root systems $\Phi_1$ and $\Phi_2$ is nonempty and spans the dual Cartan algebra of the corresponding component. Thus we can find a root $\alpha \in \Phi_1 \cup \Phi_2$ such that either $\alpha(h^1) \neq 0$ or $\alpha(h^2) \neq 0$ and so $\sum_{\alpha \in \Phi} Q(\alpha(h^1), \ldots, \alpha(h^n)) \neq 0$.

**Lemma 4.11** For the root system $\Phi$ of a simple Lie algebra of rank $n$ not of type $A_1$ or $E_8$ there exists a complex harmonic polynomial $Q$ of degree less than or equal to 6 in $n$ variables such that

$$\sum_{\alpha \in \Phi} Q(\alpha(h^1), \ldots, \alpha(h^n)) \neq 0.$$ 

**Proof.** One easily checks that for $y \in \mathbb{R}^n$ the following two polynomials of degree 4 and 6 are harmonic:

$$R_4 = (x, y)^4 - \frac{6}{4+n}(x, y)^2 ||x||^2 ||y||^2 - \frac{3}{8+6n+n^2} ||x||^4 ||y||^4,$$

$$R_6 = (x, y)^6 - \frac{15}{8+n}(x, y)^4 ||x||^2 ||y||^2 + \frac{45}{48+14n+n^2} (x, y)^2 ||x||^4 ||y||^4$$

$$- \frac{15}{192+104n+18n^2+n^3} ||x||^6 ||y||^6.$$
Here, \( x = (x_1, \ldots, x_n) \), the standard scalar product on \( \mathbb{R}^n \) is denoted by \((\ldots,\ldots)\), and we let \(|x|^2 = (x,x)\). (These polynomials can be obtained from Gegenbauer polynomials.)

In the simply laced cases, i.e., \( \Phi \) of type \(\Phi_n, D_n, E_6, E_7\) or \(E_8\), we scale \( \Phi \) in such a way that for a root \( \alpha \) one has \((\alpha,\alpha) = 2\) and we let \( y \) be a root. Then

\[
(y,\alpha) \in \{0, \pm 1, \pm 2\} \text{ for } \alpha \in \Phi.
\]

For \( i \in \{0, \pm 1, \pm 2\} \) let \( n_i = |\{\alpha \in \Phi \mid (y,\alpha) = i\}|. \)

One has the obvious relations \( n_{-1} = n_i, n_2 = 1, \) and \( n_0 + 2n_1 + 2n_2 = |\Phi| \). Furthermore (see [Bou88, Chap. VI, §1.11, Prop. 32]), one has \(|\Phi| = nh\) and \( n_1 = 2h - 4\), where \( h \) is the Coxeter number of \( \Phi \). Writing the polynomials \( R_1 \) and \( R_5 \) in the form

\[
R_l = \sum_{k=0}^{l/2} c_l(x,y)^{2k}(|x|/\sqrt{2})^{l-2k}\text{ one obtains}
\]

\[
\sum_{\alpha \in \Phi} R_l(\alpha(h^1), \ldots, \alpha(h^n)) = 2 \sum_{i=0}^{l/2-1} \sum_{k=0}^{l/2} n_k h^{2k} = \begin{cases} 4(h(n-10)+6(n+2)) & \text{for } l = 4, \\ 4(3(n^2-16)+h(n^2-48n+272)) & \text{for } l = 6. \end{cases}
\]

For \( \Phi \) of type \( A_n \), one has \( h = n + 1 \) and therefore \( R_4 = \frac{4(n^2-3n+2)}{n+2} \neq 0 \) for \( n \neq 1, 2 \) and \( R_6 = 12 \neq 0 \) for \( n = 2 \). For the type \( D_n \) one has \( h = 2n - 2 \) and thus \( R_4 = \frac{8(n-4)^2}{n+2} \neq 0 \) for \( n \neq 4 \) and \( R_6 = 24 \neq 0 \) for \( n = 4 \). For the types \( E_6 \) and \( E_7 \), one has \( h = 12 \) and \( h = 18 \), which gives \( R_4 = 0 \), but \( R_6 = 24 \neq 0 \) and \( R_6 = \frac{142}{11} \neq 0 \), respectively.

Letting \( \{e_1, \ldots, e_n\} \) be an orthonormal base, the remaining types of root systems \( \Phi \) can be realized as follows: \( B_n \) by \( \{e_i, \pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \) for \( n \geq 2 \), \( C_n \) by \( \{\pm e_i, \pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \) for \( n \geq 3 \), \( F_4 \) by \( \{e_i, \frac{1}{\sqrt{2}}(\pm e_i \pm e_j \pm e_k \pm e_\ell) \mid 1 \leq i < j < k < \ell \leq 4\} \) and \( G_2 \) by \( \{e_+ = e_1 - e_2, \pm (e_1 - e_2), \pm (e_2 - e_3), \pm (e_1 - e_3), \pm (e_2 - e_1 - e_3), \pm (2e_1 - e_2 - e_3), \pm (2e_2 - e_1 - e_3)\} \).

For \( \Phi \) of type \( B_n \), we take \( y = e_1 + e_2 \). By using the fact that the long roots of \( \Phi \) form a root system of type \( D_n \) one obtains finally \( R_4 = \frac{8n^2-60n+112}{n+2} \neq 0 \) for \( n \neq 4 \) and \( R_6 = 23 \neq 0 \) for \( n = 4 \).

For the type \( C_n \), we take again \( y = e_1 + e_2 \). Now the short roots of \( \Phi \) form a root system of type \( D_n \) and one obtains finally \( R_4 = \frac{8(n^2-16)}{n+2} \neq 0 \) for \( n \neq 4 \) and \( R_6 = -60 \neq 0 \) for \( n = 4 \).

For the type \( F_4 \), we take also \( y = e_1 + e_2 \). Taking the additional 24 short roots into account compared to \( D_4 \) one obtains \( R_4 = 0 \), but \( R_6 = 21 \neq 0 \).

Finally, for the type \( G_2 \), we take \( y = e_1 - e_2 \) and obtain \( R_4 = 0 \), but \( R_6 = -312 \neq 0 \).

\( \square \)

The last four Lemmas together and Theorem 2.1 show that \( V_1 \) is either abelian or simple of type \( A_1 \) or \( E_8 \).

**Proposition 4.12** The Lie algebra \( V_1 \) is not abelian.

**Proof.** Assume that \( V_1 \) is an abelian Lie algebra. Let \( V(s) \) be the Heisenberg vertex operator algebra of central charge 1 generated by a one-dimensional subalgebra \( s = C \cdot h \) of \( V_1 \), where \( \langle h, h \rangle = 2 \). One checks directly that for

\[
v = (8h_{(-3)}h_{(-1)} - 6h_{(-2)}^2 - 2h_{(-1)}^4)1 \in V(s)_{-1} \subset V_4
\]

one has \( L_0'v = L_2'v = 0 \), where \( L_n' = \frac{1}{2} \sum_{k \in \mathbb{Z}} h_{(n-k)}h_{(k)} \), \( n \in \mathbb{Z} \), are the generators for the Virasoro algebra of \( V(s) \). It follows as at the end of the proof of Lemma 4.9 that \( v \) is also a Virasoro highest weight vector for the Virasoro algebra of \( V \).

25
Using the associativity relation, one finds for the trace of $o(v)$ on $V_1$:
\[
\text{tr}|_{V_1} o(v) = 8 \text{tr}|_{V_1} o(h_{-3})h_{-1} - 6 \text{tr}|_{V_1} o(h_{-2}) - 2 \text{tr}|_{V_1} o(h_{-1})
= 8 \text{tr}|_{V_1}(3h_{-1}h_{1} + h_{0}^2) - 6 \text{tr}|_{V_1}h_{0}^2 - 2 \text{tr}|_{V_1}(h_{0}^2 + 4h_{-1}h_{1}h_{0}^2)
= 24(h, h) \neq 0.
\]

It follows from Theorem 2.4 that $V_1$ cannot be a 6-design. \hfill \Box

We have proven the following result:

**Proposition 4.13** The Lie algebra $V_1$ is either isomorphic to a Lie algebra of type $A_1$ or $E_8$.

\hfill \Box

**Proposition 4.14** The central charge of $V$ is either 1, 8 or 16.

**Proof.** Let $\omega'$ be the Virasoro element of a central charge $c = 1$ Heisenberg vertex operator subalgebra generated by a one-dimensional subspace of the Cartan subalgebra $\mathfrak{h}$ of $V_1$.

Since $V_1$ is assumed to be a conformal 6-design, one can apply Proposition 4.4 to the $U_{\omega'} \otimes U_{\omega'}^*$-module of conformal weight $h = 1$ generated by $V_1$. If $c \neq 1, \frac{31}{30}, \frac{34}{35}$, the proposition gives $c = 8$ or $c = 16$ for the central charge.

The cases $c = \frac{1}{2}$ or $\frac{34}{35}$ can easily be excluded:

- For $c = \frac{1}{2}$ one has $L_{1/2}(0) \subset V$. The only possible extension of $L_{1/2}(0)$ is by the simple current $L_{1/2}(1/2)$ of conformal weight $1/2$. In both cases one has $V_1 = 0$, a contradiction.

- The central charge $c = \frac{34}{35}$ is excluded as in subsection 4.3. \hfill \Box

The proof of Theorem 4.1 part (b) can now be finished. As mentioned before, the vertex operator subalgebra of $V$ generated by $V_1$ is an integrable highest weight representation of the affine Kac-Moody algebra associated to $V_1$. Such a representation is determined by a level $k$, where $k$ is a positive integer and the central charge of $\langle V_1 \rangle$ and hence, by Lemma 4.7 of $V$ is given by $c = k \dim V_1$, where $\check{g}$ is the dual Coxeter number of $V_1$ [ZG02]. For $V_1$ of type $A_1$ one has $c = \frac{3k}{k + 1}$ and for $V_1$ of type $E_8$ one has $c = \frac{248}{18}$. By using Proposition 4.14 one finds that $k = 1$ and $c = 1$ for $V_1$ of type $A_1$ and $k = \frac{1}{2}$ and $c = 8$ for $V_1$ of type $E_8$ are the only possibilities.

For $k = 1$, the vertex operator algebra associated to the level 1 representation of an affine Kac-Moody algebra of type $A_1$ or $E_8$ is isomorphic to the lattice vertex operator algebra $V_{A_1}$ or $V_{E_8}$ associated to the root lattice $A_1$ or $E_8$, respectively.

It follows that $V$ equals $V_{A_1}$ or $V_{E_8}$ since $V_{A_1}$ has besides $V_{A_1}$ only one irreducible module of conformal weight $1/4$ and $V_{E_8}$ is self-dual, i.e., both vertex operator algebras are extended.

This finishes the proof of Theorem 4.1 part (b).

### 4.5 Minimal weight $\frac{5}{2}$

Let $V$ be a vertex operator super algebra as in part (c) of Theorem 4.1 and let $V_{(0)}$ be the even vertex operator subalgebra. By assumption, $\dim V_2 \geq 2$, i.e., the minimal weight of the even vertex operator subalgebra $V_{(0)}$ of $V$ is 2.
Since we assumed that $V$ has a real form such that the invariant bilinear form on $V_2$ is positive definite, it follows from [MN93], and [Miy96], that the Virasoro element $\omega \in V_2$ can be decomposed into the sum of two nonzero elements $\omega', \omega - \omega' \in V_2$ such that, after dividing by a factor 2, one has two commuting idempotents of the algebra $V_2$. The two elements generate commuting Virasoro vertex operator subalgebras $U_{\omega'}$ and $U_{\omega - \omega'}$ of $V$ of central charge $e = 2(\omega', \omega')$ and $c - e$, respectively, with $0 < e < c$ (cf. [Hohn93], Thm. 1.2.2).

For $c \neq \frac{1}{2}, \frac{34}{35}, 1$, one can apply Proposition 4.4 to the module $V(1)$ of $V(0)$ since $V_{3/2}$ is assumed to be a conformal 6-design. For $h = \frac{3}{7}$ one obtains $c = 16$ or $c = 23\frac{1}{2}$ for the central charge.

The cases not covered by Proposition 4.4 are again easily excluded:

For $c = \frac{1}{2}$ one has $L_{1/2}(0) \subset V$. There are no irreducible modules of $L_{1/2}(0)$ of conformal weight $\frac{3}{2}$. Hence $V \cong L_{1/2}(0)$ and the minimal weight of $V$ cannot be $\frac{3}{2}$.

The central charge $c = \frac{34}{35}$ is excluded as in subsection 4.3.

If $c = 1$, the central charges of the Virasoro algebras generated by $\omega'$ and $\omega - \omega'$ both must be $\frac{1}{2}$, since $\frac{1}{2}$ is the smallest possible central charge in the unitary minimal series. Thus $V$ contains a subalgebra isomorphic to $L_{1/2}(0) \otimes 2$. There are no irreducible modules of $L_{1/2}(0) \otimes 2$ of conformal weight $\frac{3}{2}$. Hence $V \cong L_{1/2}(0) \otimes 2$ and the minimal weight of $V$ would be $2 > \frac{3}{2}$, a contradiction.

This finishes the proof of Theorem 4.1 part (c).

### 4.6 Minimal weight 2

Let $V$ be a vertex operator super algebra as in part (d) of Theorem 4.1. All the assumptions for $V$ also hold for $V(0)$, in particular, the minimal weights of $V$ and $V(0)$ are the same. If $V$ is not a vertex operator algebra, we can replace therefore $V$ by $V(0)$.

As the minimal weight of $V$ is 2, the dimension $d = \dim V_2$ of the degree-2-part of $V$ is at least 2. Since we assumed that $V$ has a real form such that the invariant bilinear form on $V_2$ is positive-definite, we can find as in subsection 4.3 an element $\omega'$ in $V_2$ generating a Virasoro algebra of central charge $e$ such that $0 < e < c$.

Assume first that $c \neq \frac{1}{2}, \frac{34}{35}, (55 \pm \sqrt{55})/2, 1$. In this case we can apply Proposition 4.4 and we have $d = \frac{c(2388 + 955c + 70c^2)}{2(748 - 55c + c^2)}$.

Since $V$ is assumed to be rational, the central charge $c$ has to be a rational number (see [DLM00], Thm. 11.3).

**Lemma 4.15** The only positive rational numbers $c$ for which

$$d = \frac{c(2388 + 955c + 70c^2)}{2(748 - 55c + c^2)}$$

is a positive integer are $c = \frac{1}{7}, \frac{8}{7}, \frac{52}{7}, \frac{16}{7}, \frac{132}{7}, \frac{20}{7}, \frac{102}{7}, \frac{748}{7}, \frac{43}{7}, \frac{22}{7}, \frac{808}{7}, \frac{47}{7}, \frac{24}{7}, \frac{170}{7}, \frac{25}{7}, \frac{172}{7}, \frac{152}{7}, \frac{61}{7}, \frac{154}{7}, \frac{220}{7}, \frac{63}{7}, \frac{32}{7}, \frac{164}{7}, \frac{236}{7}, \frac{34}{7}, \frac{36}{7}, \frac{40}{7}, \frac{203}{7}, \frac{44}{7}, \frac{109}{7}, \frac{42}{7}, \frac{68}{7}, \frac{484}{7}, \frac{132}{7}, 1496$.

**Proof.** Let $c = \frac{p}{q}$ with coprime integers $p$ and $q$. The equation for $d$ can be rewritten as

$$2q(748q^2 - 55pq + p^2)d = p(2388q^2 + 955pq + 70p^2).$$

27
It follows successively \( q | p (2388 q^2 + 955 pq + 70 p^2), q | 2388 q^2 + 955 pq + 70 p^2, q | 70 p^2, q | 70, i.e., c = k/70 \) with a positive integer \( k \). This gives
\[
d = \frac{4805}{2} + \frac{k}{2} + \frac{3850 k + k^2}{2(3665200 − 3850 k + k^2)}.
\]
As \( |(−17611286000 + 15001210 k)/2 (2 3665200 − 3850 k + k^2)| < \frac{1}{2} \) for \( k > 70 (107179 + \sqrt{11483743153}) \approx 15 003 885.97 \ldots \), the result follows by computing \( d \) for all \( k < 15 003 886 \).

If the vertex operator algebra \( V \) is a self-dual vertex operator algebra, then \( c \) has to be an integer divisible by \( 8 \) (see [Hoh95], Cor. 2.1.3 and Section 3 above). From Lemma 4.15 it follows that in this case \( c \) is rational. Hence, \( \text{dim} V \) is an integer divisible by \( 8 \) (see [Hoh95], Cor. 2.1.3 and Section 3 above). Since \( c \) is assumed to be rational, the conformal weight \( h \) of an irreducible module has to be a rational number ([DLM00], Thm. 11.3). A direct verification gives:

**Lemma 4.16** The only values of \( c \) listed in Lemma 4.15 for which
\[
h = \frac{124 + 31 c ± \sqrt{368 + 24 c − c^2}}{496}
\]
is rational are \( c = \frac{1}{2}, 8, 16, \frac{808}{33}, \frac{47}{2}, \frac{164}{9}, \frac{236}{23}, \frac{242}{7} \).

By applying Propositions 4.5 and 4.4 and Lemmata 4.15 and 4.16 it follows therefore that, if \( V \) is not self-dual, the only possible values for the central charge are \( c = 8, 16, \frac{808}{33}, \frac{47}{2}, \frac{164}{9}, \frac{236}{23}, \frac{242}{7} \). One also obtains the values for \( \text{dim} V \) and \( h \) as listed in Table 1.

The remaining cases for \( c \) can again easily be excluded:

For \( c = \frac{1}{2} \), one has \( L_{1/2}(0) \subset V \). There are no irreducible modules of \( L_{1/2}(0) \) of conformal weight 2. Hence \( V \cong L_{1/2}(0) \) and the minimal weight of \( V \) is larger than 2, a contradiction.

For \( c = \frac{1}{2} \), the same argument as in subsection 4.3 holds.

If \( c = 1 \), \( V \) must contain a subalgebra isomorphic to \( L_{1/2}(0)^{\otimes 2} \) by the argument given in subsection 4.3. There are no irreducible modules of \( L_{1/2}(0)^{\otimes 2} \) of conformal weight 2. Hence \( V \cong L_{1/2}(0)^{\otimes 2} \). Let \( \omega' \) be the Virasoro element of one subalgebra \( L_{1/2}(0) \). Let \( d_4 \) be the multiplicity of the eigenvalue \( h \) of \( L_0 = \omega'_{(1)} \) acting on \( V \). We have \( d_0 = 1 \) and \( d_{1/2} = d_{1/16} = 0 \). The vectors \( v^{(2)} \) and \( v^{(4)} \) given in (14) and (15) are still well-defined Virasoro highest weight vectors. For the traces of \( o(v^{(4)}) \) on \( V \) one gets
\[
\text{tr}_{V_2} o(v^{(4)}) = \frac{27}{5} + \frac{22}{5} d_0 - \frac{59}{5} d_{1/2} + \frac{571}{320} d_{1/16}
\]
and so \( \text{tr}_{V_2} o(v^{(4)}) = \frac{49}{5} \), a contradiction to the conformal 6-design property of \( V \).

The cases \( c = (55 ± \sqrt{35})/2 \) are excluded because \( c \) is not rational.

This finishes the proof of Theorem 4.1 part (d).
4.7 Conformal 8-designs

For a vertex operator algebra with a module whose lowest degree part is a conformal 8-design one has in addition to Proposition 4.4:

Proposition 4.17 Let $V$ be a vertex operator algebra of central charge $c \neq -31, -\frac{44}{29}, -\frac{184}{105}, \frac{6}{55}, (-47 \pm 5\sqrt{37})/4, 1$ and assume there exists a module $W$ of $V$ of conformal weight $h \neq 0$ such that the lowest degree part $W_h$ is a conformal 8-design. If there exist elements $\omega, \omega - \omega' \in V_2$ generating two commuting Virasoro vertex operator algebras of central charge $e$ and $c-e$, respectively, with $e, c-e \neq -\frac{44}{29}, -\frac{6}{55}, -\frac{22}{3}, 0, \frac{1}{2}$ then

$$(c-24h+12)(10c^3 + (141-615h)c^2 + 2(5740h^2 - 3321h + 171)c$$
$$-24(2870h^3 - 2870h^2 + 451h + 15)) = 0. \quad (26)$$

If $e = \frac{1}{2}$ or $e = c - \frac{1}{2}$ and one assumes that the Virasoro vertex operator algebra generated by $\omega'$ or $\omega - \omega'$, respectively, is simple and the other assumptions hold, then

$$152700c^6 + (3535420 - 2546820h)c^5 + 2(5519040h^2 - 33944388h + 10007663)c^4$$
$$+ (-66228480h^3 + 505357184h^2 - 303807330h + 24963561)c^3$$
$$+ (-2634772224h^3 + 409756928h^2 + 611923251h - 162937170)c^2$$
$$+ 3(4450030592h^3 - 4570094080h^2 + 1282098891h - 49633193)c$$
$$+ 18(120063488h^3 - 120063488h^2 + 34154597h - 1236817) = 0. \quad (27)$$

Proof. The proof is similar to the proof of Proposition 4.4

Assume first that $e, c-e \neq \frac{1}{2}$. The conditions on the central charges guarantee that the Virasoro vertex operator algebras generated by $\omega'$ and $\omega - \omega'$ are isomorphic to $M(x,0)/M(x,1)$ up to degree 8, where $x = e$ or $c-e$, respectively. Lemma 4.3 shows that one can find three linear independent highest weight vectors $v^{(8)}_a, v^{(8)}_b$ and $v^{(8)}_c \in U_{\omega'} \otimes U_{\omega - \omega'}$. By assumption, the degree $h$ subspace $W_h$ is a conformal 8-design. Thus one has the trace identities

$$\text{tr}|W_h \ o(v^{(2)}) = \text{tr}|W_h \ o(v^{(1)}) = \text{tr}|W_h \ o(v^{(8)}_a) = \text{tr}|W_h \ o(v^{(8)}_b) = \text{tr}|W_h \ o(v^{(8)}_c) = 0 \quad (28)$$

which form a homogeneous system of linear equations for $d^* = m^*_0, m^*_1, m^*_2, m^*_3$, and $m^*_4$, where $d^* = \dim W_h$ and $m^*_i = \text{tr}|W_h \ o(a^*_i)$ for $i = 1, 2, 3, 4$. Since $d^* > 0$, the system has to be singular and the determinant has to vanish. This condition gives the proposition in this first case.

The case $e = \frac{1}{2}$ or $e = c - \frac{1}{2}$ is handled similar as in the proof of Proposition 4.4 by choosing an appropriate highest weight vector $v^{(8)} \in U_{\omega'} \otimes U_{\omega - \omega'}$ besides $v^{(2)}$ and $v^{(4)}$.

Let now $V$ be a vertex operator super algebra of minimal weight $m = 1$ or $m = \frac{3}{2}$ satisfying the conditions of Theorem 4.2 (i). The proof of Theorem 4.1 shows that Proposition 4.17 is applicable in these cases with $h = 1$ or $h = \frac{3}{2}$, respectively. First, we assume $e, c-e \neq \frac{1}{2}$. For $h = 1$, equation (26) gives $c = 12$ or $c = (177 + \sqrt{22099})/10;$
for \( h = \frac{3}{2} \), one gets \( c = 24, 10.04101..., 19.13162..., 48.97735... \). These central charges are impossible by Theorem 4.1 part (b) and (c). Now we assume \( e = \frac{1}{3} \) or \( e = c - \frac{1}{2} \). For \( h = 1 \), equation (27) has the positive real solutions \( c = 12 \) and \( c = 2.268296... \); for \( h = \frac{2}{3} \), one gets the positive real solutions \( c = 3.342825... \) and \( c = 18.81561... \). Again, these central charges are excluded by Theorem 4.1 part (b) and (c).

Thus we have proven Theorem 4.2 (i) for \( m = 1 \) or \( m = \frac{3}{2} \). The case \( m = \frac{1}{2} \) follows from Theorem 4.3 (a).

In addition to Proposition 4.18, one has:

**Proposition 4.18**\( \) Let \( V \) be a vertex operator algebra of central charge \( c \neq -31, -44, -\frac{44}{67}, -\frac{36}{7}, -\frac{47}{10}, -\frac{17.58127...}{10}, -\frac{25.84832...}{10}, -\frac{65.47039...}{10}, 1 \) with \( V_1 = 0 \) such that \( V_2 \) forms a conformal 8-design. If there exist elements \( \omega', \omega - \omega' \in V_2 \) generating two commuting Virasoro vertex operator algebras of central charge \( e \) and \( c - e \), respectively, with \( e, c - e \neq -\frac{46}{3}, -\frac{68}{9}, -\frac{22}{3}, -\frac{3}{5}, 0, \frac{1}{2} \), then

\[
   d = \frac{15c (155c^3 + 4133c^2 + 32074c + 88392)}{20c^3 - 2178c^2 + 65956c - 595056}.
\]

If \( e = \frac{1}{2} \) or \( e = c - \frac{1}{2} \) and one assumes that the Virasoro vertex algebra generated by \( \omega' \) or \( \omega - \omega' \), respectively, is simple and also \( c \neq -26.45283... \), \( 0.23461... \), \( 25.60637... \), \( 15.3532... \), \( 13.56755... \), \( -1 \), and the other assumptions hold then

\[
   d = \frac{15c (5734920c^3 + 59136716c^2 + 283246086c^3 + 285884141c^2 + 7908127017c - 2179288566)}{20c^3 - 7840184c^2 + 72948868c + 552859692c - 75371628638c + 17334713996}.
\]

**Proof.** The proof is similar to the proof of Proposition 4.5. Assume first that \( e, c - e \neq \frac{1}{2} \). The conditions on the central charges guarantee again that there exist three linear independent highest weight vectors \( v_a^{(8)} \), \( v_b^{(8)} \) and \( v_c^{(8)} \in U_{\omega'} \otimes U_{\omega - \omega'} \). By assumption, \( V_2 \) is a conformal 8-design. Thus one has the trace identities

\[
   \text{tr}|_{V_2} o(v_2^{(2)}) = \text{tr}|_{V_2} o(v_4^{(4)}) = \text{tr}|_{V_2} o(v_a^{(8)}) = \text{tr}|_{V_2} o(v_b^{(8)}) = \text{tr}|_{V_2} o(v_c^{(8)}) = 0
\]

which form an inhomogeneous system of linear equations for \( d = m_0, m_1, m_2, m_3, m_4 \), where \( d = \dim V_2 \) and \( m_i = \text{tr}|_{V_2} a_i^{(0)} \) for \( i = 1, 2, 3, 4 \). The assumptions on the central charge guarantee that the system is non-singular since its determinant is non-zero. The solution for \( d \) is the one given in the proposition and does not depend on \( e \).

If \( e = \frac{1}{2} \) or \( e = c - \frac{1}{2} \), we choose again an appropriate highest weight vector \( v^{(8)} \in U_{\omega'} \otimes U_{\omega - \omega'} \) besides \( v^{(2)} \) and \( v^{(4)} \) and the result follows as in the proof of Proposition 4.5.

Let now \( V \) be a vertex operator super algebra of minimal weight \( m = 2 \) satisfying the conditions of Theorem 4.2 (i). As explained in subsection 4.1.6 Proposition 4.5 and Proposition 4.18 are applicable provided the conditions on the central charge of \( V \) are satisfied.

Assume first \( e, c - e \neq \frac{1}{2} \). The two expressions given in these two propositions for \( d \) together form an equation for \( c \) with the solutions \( c = -\frac{316}{13}, -\frac{45}{7}, -\frac{23}{5}, 0, 24, \frac{142}{9} \). The case \( c = \frac{142}{9} \) is impossible because \( d = -164081 < 0 \). Also, \( c \) has to be positive.
The cases $c = \frac{1}{2}, \frac{34}{35}$ and 1 are excluded as in subsection 4.6. For $c = \frac{6}{5}, \frac{47+5\sqrt{57}}{4}$, 17.58127..., 25.84832..., 65.47039..., Proposition 4.5 gives a non-integer value for $d$.

This leaves the case $c = 36$, which can be excluded by using the non-singular linear system

$$\text{tr}|_{V_2}(v^{(2)}) = \text{tr}|_{V_2}(v^{(4)}) = \text{tr}|_{V_2}(v^{(6)}) = \text{tr}|_{V_2}(v^{(8)}) = 0$$

leading to $d = -67770 < 0$.

If $e = \frac{1}{2}$ or $e = c - \frac{1}{2}$, one has again two equations for $d$ and one finds for $c$ the real solutions $c = -\frac{22}{5}, 0, \frac{1}{2}, 24, \frac{142}{5}, -8.45952...$. For the cases $c = 0.23416..., 25.60637...$ with $c > 0$ which are not yet excluded, Proposition 4.5 gives non-integer values for $d$.

This finishes the proof Theorem 4.2 (i) for $m = 2$ since for $c = 24$ Proposition 4.5 gives $d = 196884$.

For part (ii) of Theorem 4.2 note that under the extra assumptions there Theorem 4.1 (d) shows that $V$ has to be a self-dual vertex operator algebra of central charge 24. As discussed in section 3, the character of $V$ has therefore to be equal to the character of the Moonshine module $V^\natural$.

4.8 Known examples of conformal 6-designs

We will discuss which vertex operator super algebras satisfying the conditions of Theorem 4.1 are known to support conformal 6-designs. We compare our results for vertex operator super algebras with the analogous results for binary linear codes and integral lattices due to Lalauze-Labayle [LL01] and Martinet [Mar01].

In the case of minimal weight $m = \frac{1}{2}$, the only example is the self-dual vertex operator super algebra $V_{\text{Fermi}}$. The homogeneous subspaces of $V_{\text{Fermi}}$ are trivial conformal $t$-designs for all $t \geq 1$ because $V_{(0)}$ is equal to the Virasoro highest weight module $L_{1/2}(0)$.

The only self-orthogonal binary code of minimal weight 2, whose set of minimal-weight words supports a 3-design is the trivial example of the self-dual code $C_2 \cong \{(0,0), (1,1)\}$.

The only integral lattice of minimal norm 1, whose set of minimal vectors forms a spherical 7-design is the one-dimensional lattice $\mathbb{Z}$ of integers. In fact, the two vectors of any fixed positive integer length form a spherical $t$-design for all $t$.

For $m = 1$, the only examples are the vertex operator algebras $V_{A_1}$ and $V_{E_8}$. As shown in Example 2.6, all the homogeneous subspaces of $V_{A_1}$ are conformal $t$-designs for arbitrary $t$. As shown in Example 2.7 and also in Example 3.2, all the homogeneous subspaces of $V_{E_8}$ are conformal 7-designs. However, Theorem 4.2 (i) shows that $(V_{E_8})_1$ is not a conformal 8-design.

The only self-orthogonal binary code of minimal weight 4, whose set of minimal-weight words supports a 3-design is the doubly-even self-dual Hamming code $H_8$ of length 8. (One may also consider the trivial code $C_2 = \{(0,0)\}$.)

The only integral lattices of minimal norm 2, whose set of minimal vectors are spherical 7-designs are the root lattices $A_1$ of rank 1 and $E_8$ of rank 8.
In the case of minimal weight \( m = \frac{3}{2} \), the shorter Moonshine module \( VB^2 \cong VB_{(0)}^2 \oplus VB^2_{(1)} \) is a vertex operator super algebra of central charge \( 23\frac{1}{2} \) whose homogeneous subspaces are conformal 7-designs by Example 2.12 and 2.11. It follows from [Hoh03a] that \( VB^2_{(0)} \) contains a Baby Monster invariant non-zero Virasoro highest weight vector and Theorem 4.1 (i) shows that \( VB^2_{3/2} \) is not a conformal 8-design.

As seen in Example 2.12, the homogeneous subspaces of the central charge 16 vertex operator algebra \( V_{A_{16}}^+ \) as well as of the module \( K(\frac{3}{2}) \) of conformal weight \( \frac{3}{2} \) are conformal 7-designs. The module \( K(\frac{3}{2}) \) is the direct sum of (all) irreducible \( V_{A_{16}}^+ \) modules \( K_\mu \) of conformal weight \( \frac{3}{2} \) which are simple currents of order 2. (This can be proven by using that \( V_{A_{16}}^+ \) has the structure of a framed vertex operator algebra [DGH98, LY06].) It is not clear (and seems unlikely) that the individual modules \( K_\mu \) give rise to conformal 6-designs. If this is not the case, they cannot be used to extend \( V_{A_{16}}^+ \) to a vertex operator super algebra of the required type.

One may therefore conjecture that the shorter Moonshine module \( VB^2 \) is the only possible example for a vertex operator super algebra as in Theorem 4.1 (c).

The only self-orthogonal binary code of minimal weight 6, whose set of minimal-weight words supports a 3-design is the self-dual shorter Golay code of length 22. The only integral lattice of minimal norm 3 whose set of minimal vectors forms a spherical 7-design is the 23-dimensional unimodular shorter Leech lattice \( O_{23} \).

For \( m = 2 \), the known examples are the vertex operator algebras \( V_{A_8}^+ \) and \( V_{A_{16}}^+ \) of central charge 8 and 16 of Example 2.12, the even part \( VB^2_{(0)} \) of the shorter Moonshine module of central charge \( 23\frac{1}{2} \) from Example 2.11, the short Moonshine module \( V^2 \) of central charge 24 from Example 2.8 and 3.2 and the known extremal self-dual vertex operator algebras of central charge 32 from Example 3.2. In these examples, all homogeneous subspaces are conformal 7-designs; in the case of the Moonshine module, they are even conformal 11-designs. In all examples besides \( V^2 \), Theorem 4.2 gives that the subspace \( V_2 \) is not a conformal 8-design; the Griess algebra \( V_2^2 \) is not a conformal 12-design (see [DM00], Thm. 3 and the following discussion).

For the other values of \( c \), no examples are known. The homogeneous subspaces of extremal vertex operator algebras \( V \) of central charge 40 are by Theorem 4.1 conformal 3-designs and one has \( \dim V_2 = 24620 \). All known examples of such vertex operator algebras are \( \mathbb{Z}_2 \)-orbifolds of lattice vertex operator algebras, where the lattice is an extremal rank 40 lattice with minimal squared length 4. Such vertex operator algebras contain a Virasoro element \( \omega \) generating a Virasoro algebra of central charge 1/2. If \( V_2 \) is a conformal 7-design, one obtains however \( d_0 = \frac{441768}{9} \) which is impossible (cf. [Mat01], Table 3.3). Similarly, one can show that with the assumption \( L_{1/2}(0) \subset V \) the only central charges \( c \) given in Lemma 4.15 for which \( d_0, d_{1/2} \) and \( d_{1/16} \) are nonnegative integers are \( c = \frac{1}{2}, 8, 16, 20, 23\frac{1}{2}, 24, 24\frac{1}{2}, 24\frac{1}{4}, 30\frac{2}{5}, 30\frac{1}{2}, 31\frac{1}{2}, 32, 32\frac{1}{2}, 33\frac{5}{7}, \) and 36.

The only self-orthogonal binary codes of minimal weight 8, whose set of minimal-weight words support a 3-design are the doubly-even code \( C_8 \cong \{(0, 0, 0, 0, 0, 0, 0, 0), (1, 1, 1, 1, 1, 1, 1, 1)\} \) of length 8; the doubly-even Reed-Muller code \( R(1, 4) \) of length 16; the doubly-even subcode of the shorter Golay code of length 22; the doubly-even self-dual Golay code of length 24 and the five extremal doubly-even self-dual codes of length 32.
The only integral lattices of minimal norm 4, whose set of minimal vectors are spherical 7-designs are the even laminated lattices \( \Lambda_8 \) and \( \Lambda_{16} \) of rank 8 and 16; the even sublattice of the shorter Leech Lattice \( O_{23} \) of rank 23; the even unimodular Leech lattice \( \Lambda_{24} \) of rank 24 and the even unimodular extremal lattices of rank 32.

One may therefore conjecture that the examples mentioned above are already all examples of vertex operator super algebras for Theorem 4.1 part (d).

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