Abstract. The first part of this paper deals with electrical networks and symplectic reductions. We consider two operations on electrical networks (the "trace map" and the "gluing map") and show that they correspond to symplectic reductions. We also give several general properties about symplectic reductions, in particular we study the singularities of symplectic reductions when considered as rational maps on Lagrangian Grassmannians. This is motivated by [23] where a renormalization map was introduced in order to describe the spectral properties of self-similar lattices. In this text, we show that this renormalization map can be expressed in terms of symplectic reductions and that some of its key properties are direct consequences of general properties of symplectic reductions (and the singularities of the symplectic reduction play an important role in relation with the spectral properties of our operator). We also present new examples where we can compute the renormalization map.

Introduction

In [23], we introduced a renormalization map in order to describe the spectral properties of Laplace operators on finitely ramified self-similar lattices. This map is rational and defined on a Lagrangian Grassmannian. The aim of this text is to present this map from a different point of view. We insist on the aspects of symplectic geometry, and in particular on the role played by symplectic reductions. In this respect, we take inspiration from the works of Colin de Verdière ([5], [6]), where Lagrangian compactifications and symplectic reductions are related to operations on electrical networks. One of the main goals of this text is to show that the crucial properties of the map introduced in [23] are consequences of general properties of symplectic reductions. These properties, which seem to be new, are proved in section 3 and essentially concern the singularities of the symplectic reduction, when considered as a rational map. We also show that the symplectic reduction can be lifted, in a natural way, to a linear map, through the Plücker embedding. This is
a key feature in [23]. This leads us to introduce a class of rational maps, which is a natural generalization of the maps that appear in the context of self-similar structures, cf. section 6.

We also present new examples, in particular an example related to the spectrum of Schreier graphs of some automatic groups (cf. [1], [14]), and explain, with more detail than in [23], how to explicitly compute the renormalization map in the “nested fractal like” cases (which have many symmetries).

Let us now briefly explain the main ideas of this work. In sections 1 and 2, we are concerned about two operations on electrical networks and their translation in terms of symplectic reductions. If $F = \{1, \ldots, K\}$ is a finite set, an electrical network on $F$, is a family of non-negative reals $(\rho_{i,j})_{i,j \in F}$, such that $\rho_{i,j} = \rho_{j,i}$, and a family of non-negative reals $(\rho_i)_{i \in F}$. The $\rho_{i,j}$ are called the conductances, and the $\rho_i$ are the dissipative terms. We say that the network is conservative when $\rho_i = 0$ for all $i$ in $F$.

For any function $f : F \to \mathbb{R}$ (which represents the potential on the poles of $F$), the energy dissipated by the network is

$$E_{\rho}(f, f) = \frac{1}{2} \sum_{i,j \in F} \rho_{i,j}(f(i) - f(j))^2 + \sum_{i \in F} \rho_i(f(i))^2. $$

(We denote by $E_{\rho}(f, g)$ the bilinear form obtained by polarization). For any potential $f : F \to \mathbb{R}$, the current is the element $I_{\rho}^f$ of the dual space $(\mathbb{R}^F)^*$, defined by

$$I_{\rho}^f(h) = E_{\rho}(f, h), \quad \forall h \in \mathbb{R}^F.$$ 

Physically, $I_{\rho}^f$ represents the current flowing through the poles of $F$, when the potential $f$ is imposed on $F$.

In section 2, we consider two natural operations on electrical networks. The first one is the so-called “trace map” (this terminology comes from Dirichlet forms). One way to present this operation is the following. Consider a subset $\partial F$ of $F$ ($\partial F$ is often viewed as a boundary set for $F$, which justifies this notation), then there is a unique electrical network on $\partial F$, denoted $(\rho_{\partial F}^i)_{i \in \partial F}$, such that for any function $h : \partial F \to \mathbb{R}$,

$$E_{\rho_{\partial F}}(h, h) = \inf_{h : F \to \mathbb{R}} E_{\rho}(h, h).$$

When the electrical network $\rho$ is irreducible (cf. section 1), this infimum is attained at a unique point, denoted $H_f$, which is the harmonic extension of $f$ with respect to $E_{\rho}$. Physically, $E_{\rho_{\partial F}}(f, f)$ is the energy dissipated by the network when the potential $f$ is imposed on the poles of $\partial F$. It is clear that the current $I_{H_f}$ induced by the potential $H_f$ is supported by $\partial F$.

The second operation we consider is the following: suppose that $(F, \rho)$ is an electrical network and that $\mathcal{R}$ is an equivalence relation on $F$. There is a natural way to define an electrical network $\rho^\mathcal{R}$ on the quotient set $F/\mathcal{R}$ by

$$\rho^\mathcal{R}_{x,y} = \sum_{i,j \in F} \rho_{\pi(i)=x, \pi(j)=y}, \quad \rho^\mathcal{R}_{x,x} = \sum_{i \in F, \pi(i)=x} \rho_i,$$

where $\pi$ is the canonical surjection $\pi : F \to F/\mathcal{R}$.

For some reason that will appear later, it is important to define these two maps not only on electrical networks, but on the larger set of symmetric matrices. Let us
denote by \(\text{Sym}_F\) the space of \(F \times F\) symmetric matrices (we take the coefficients in \(\mathbb{R}\) in this introduction). For an electrical network \(\rho\), we denote by \(Q_\rho\) the element of \(\text{Sym}_F\) defined by
\[
\mathcal{E}_\rho(f, h) = \langle Q_\rho f, h \rangle, \quad \forall f, h \in \mathbb{R}^F,
\]
(where \(<\cdot, \cdot>\) is the canonical scalar product on \(\mathbb{R}^F\)). It is clear that \(Q_\rho\) determines completely \(\rho\), and that the subset \(\{Q_\rho, \rho \text{ elec. net.}\}\) is a cone of \(\text{Sym}_F\), with non-empty interior. The maps \(\rho \mapsto \rho^{\mathbb{R}}\) and \(\rho \mapsto \rho^{F/\mathbb{R}}\), naturally induce two maps from the cone \(\{Q_\rho\}\) to respectively \(\text{Sym}_{\mathbb{R}}\) and \(\text{Sym}_{F/\mathbb{R}}\). Furthermore, it is easy to check that the coefficients of \(Q_\rho^{\mathbb{R}}\) and \(Q_\rho^{F/\mathbb{R}}\) are rational in the coefficients of \(Q_\rho\) (cf. section 2.1). Hence, we can extend these maps into rational maps on \(\text{Sym}_F\), that we denote by
\[
\text{Sym}_F \rightarrow \text{Sym}_{\mathbb{R}} \quad \text{and} \quad \text{Sym}_F \rightarrow \text{Sym}_{F/\mathbb{R}}.
\]
(Explicit expressions for these maps are given in section 2).

An electrical network can be considered as a Lagrangian subspace. Let us consider \(V_F = \mathbb{R}^F \oplus (\mathbb{R}^F)^*\), where \((\mathbb{R}^F)^*\) is the dual space of \(\mathbb{R}^F\). We consider the bilinear symplectic form on \(V_F \times V_F\), defined by
\[
\omega((x, \xi), (x', \xi')) = \xi'(x) - \xi(x'),
\]
for any \((x, \xi)\) and \((x', \xi')\) in \(V_F \simeq \mathbb{R}^F \times (\mathbb{R}^F)^*\). Let \(W\) be a linear subspace of \(V_F\). We denote by \(W^\circ\) the orthogonal of \(W\) with respect to \(\omega\). By definition, the subspace \(W\) is isotropic if \(W \subset W^\circ\), and coisotropic if \(W^\circ \subset W\). A Lagrangian subspace is a maximal isotropic subspace of \(V_F\) (which, thus is also coisotropic and of dimension \(K = |F|\)). We denote by \(L_F\) the set of Lagrangian subspaces of \(V_F\). The set \(L_F\) has the structure of a smooth projective variety of dimension \(\dim \text{Sym}_F = K(K - 1)/2\) (cf. section 1.2). If \(W\) is a coisotropic subspace, then the symplectic form \(\omega\) induces a symplectic form on \(W/W^\circ\), and if \(L\) is a Lagrangian subspace of \(V_F\), then \((L \cap W)/W^\circ\) is a Lagrangian subspace of \(W/W^\circ\) (cf. section 1.3). The symplectic reduction is defined as the map \(t_W : L_F \rightarrow L_{W/W^\circ}\) given by \(t_W (L) = (L \cap W)/W^\circ\) (where \(L_{W/W^\circ}\) is the variety of Lagrangian subspaces of \(W/W^\circ\)). The map \(t_W\) is defined everywhere on \(L_F\), but is not everywhere smooth. The singularities of this map play an important role in relation with the operations on electrical networks we have described (and section 3 of this paper is devoted to the study of the singularities of the symplectic reduction, when considered as a rational map).

With any electrical network \(\rho\), we associate the subspace
\[
L_\rho = \{f + I_\rho f, \ f \in \mathbb{R}^F\} \subset V_F,
\]
which is a Lagrangian subspace of \(V_F\) and determines \(\rho\) completely. More generally, if \(Q\) is in \(\text{Sym}_F\), then we can define the subspace
\[
L_Q = \text{Span}\{e_i + \left(\sum_{j=1}^K Q_{ij} e^*_j\right)\}_{i=1, \ldots, K},
\]
which is a Lagrangian subspace of \(V_F\) \((e_i)\) is the canonical basis of \(\mathbb{R}^F = \mathbb{R}^K\), and \((e^*_i)\) the dual basis). It is clear with these notations that \(L_\rho = L_{Q_\rho}\). The map
Q ↦→ L_Q defines an embedding of Sym_F into the variety L_F, such that L_F \ Sym_F is the a subvariety of L_F of codimension 1 given by

\[ L_F \setminus \text{Sym}_F = \{ L \in L_F, \ L \cap (0 \oplus (\mathbb{R}^F)^*) \neq \{0\} \} \]

Hence, L_F defines a compactification of Sym_F (which is in general different from the compactification by the projective space of dimension K(K-1)/2).

Let us come back to the operations of restriction and gluing we have defined. The trace map Q ↦→ Q_{\partial F} and the gluing map Q ↦→ Q_{F/\mathbb{R}} naturally induce the maps L_Q ↦→ L_Q_{\partial F} and L_Q ↦→ L_Q_{F/\mathbb{R}} on Sym_F ⊂ L_F. The main point of section 2, is to show that these two maps coincide with symplectic reductions. More precisely, this means that we can find some explicit coisotropic subspaces of V_F, W_{\partial F} and W_{F/\mathbb{R}}, such that W_{\partial F}/(W_{\partial F})^o \simeq V_{\partial F} and W_{F/\mathbb{R}}/(W_{F/\mathbb{R}})^o \simeq V_{F/\mathbb{R}} and such that t_{W_{\partial F}}(L_Q) = L_{Q_{\partial F}}, and t_{W_{F/\mathbb{R}}}(L_Q) = L_{Q_{F/\mathbb{R}}} (for the trace map, this was proved by Colin de Verdière in [5]). The main interest of these formulas is to give an explicit expression of the extension of the trace map and the gluing map to the Lagrangian compactification L_F.

This work is motivated by the spectral analysis of self-similar lattices. Let us describe in this introduction the simple case of the Sierpinski gasket. Let F = F_{<0>}, be the set of vertices of a regular triangle, and F_{<1>} be the (non-disjoint) union of 3 copies of F, as shown on figure 1. Formally, it means that F_{<1>} = \{1, 2, 3\} \times F/\mathbb{R}, where R is a certain equivalence relation which represents the connexions in F_{<1>}. We denote by \partial F_{<1>} \simeq F, the boundary points of F_{<1>} (the circled points on figure 1). Then, F_{<2>} is constructed as 3 copies of F_{<1>}, glued together by the boundary points \partial F_{<1>}, as shown on figure 1. The boundary set of F_{<2>}, \partial F_{<2>}, is the set of circled points on figure 1.

Repeating this operation, we construct a sequence of lattices F_{<n>}, together with their boundary sets \partial F_{<n>} (consisting of the 3 vertices of the larger triangle). Let us now consider an electrical network \rho on F. Then, we can naturally define an electrical network \rho_{<n>} on F_{<n>}: \rho_{<1>} is constructed from \rho by first making three copies of (F, \rho), and then (F_{<1>}, \rho_{<1>}) is obtained by the gluing procedure described at the beginning of the introduction (considering that F_{<1>} is a quotient of \{1, 2, 3\} \times F). The electrical network \rho_{<n+1>} is defined similarly from \rho_{<n>}.

If b is a positive measure on F, then we can construct a self-similar positive measure
b_{<n>} on F_{<n>}, in a natural way (the details are in section 4). Let \( H_{<n>} \) be the self-adjoint operator on \( L^2(b_{<n>}) \) defined by
\[
\mathcal{E}^{<n>} (f,h) = - \int_{F_{<n>}} (H^{<n>} f) h \rho_{<n>}(x), \quad \forall f,h \in \mathbb{R}^{F_{<n>}}.
\]
The operator \( H_{<n>} \) is a self-similar Schrödinger operator on the sequence of self-similar lattices \( F_{<n>} \) (\( H_{<n>} \) is of “Laplace type” when \( \rho \) is conservative). In [23], [24], and in this work we are interested in the spectral properties of this operator. Remark that \( f : F_{<n>} \to \mathbb{R} \) is an eigenvalue of \( H_{<n>} \), with eigenvalue \( \lambda \), if and only if
\[
\mathcal{E}^{<n>} (f,h) = - \lambda \int_{F_{<n>}} f h \rho_{<n>}, \quad \forall h \in \mathbb{R}^{F_{<n>}}.
\]
As shown in [23], the spectral properties of \( H_{<n>} \) are related to the dynamics of a certain renormalization map that we describe now. Let us denote by \( \tilde{F}_{<1>} = \{1,2,3\} \times F \), three copies of \( F \). If \( Q \) is a symmetric \( F \times F \) matrix, then \( Q_{<1>} \) is defined as the block diagonal \( \tilde{F}_{<1>} \times \tilde{F}_{<1>} \) matrix obtained by making three copies of \( Q \) on each subset \( \{i\} \times F \subseteq \tilde{F}_{<1>} \). Then \( Q_{<1>} \) is in \( \text{Sym}_{\tilde{F}_{<1>}} \) obtained from \( Q_{<1>} \) by the gluing map we have described. Then we define \( TQ \) as the element of \( \text{Sym}_{\tilde{F}_{<1>}} \) obtained by the trace map:
\[
TQ = (Q_{<1>})_{\tilde{F}_{<1>}}.
\]
Since there is a natural identification between \( \partial F_{<1>} \) and \( F \), we see that \( T \) is a map from \( \text{Sym}_F \) to \( \text{Sym}_F \) (the coefficients of \( TQ \) are rational in the coefficients of \( Q \)). We see that the map \( T \) is the composition of three maps
\[
T : Q \xrightarrow{\text{copies}} \tilde{Q}_{<1>} \xrightarrow{\text{gluing}} Q_{<1>} \xrightarrow{\text{trace map}} TQ = (Q_{<1>})_{\tilde{F}_{<1>}}.
\]
The last two operations correspond to symplectic reductions on the Lagrangian compactification. Since a composition of two symplectic reductions is a symplectic reduction (cf. section 1.3), we see that the extension of the map \( T \) to the Lagrangian compactification \( \mathbb{L}_F \) has the simple expression
\[
g : \mathbb{L}_F \xrightarrow{\text{copies}} \mathbb{L}_{\tilde{F}_{<1>}} \xrightarrow{\text{symplectic reduction}} \mathbb{L}_F,
\]
(0.1)
\[
L \xrightarrow{\text{symp. red.}} \tilde{L}_{<1>} \xrightarrow{\text{gluing}} g(L) = t_{W_{<1>}} (\tilde{L}_{<1>})
\]
where \( W_{<1>} \) is a certain coisotropic subspace of \( V_{\tilde{F}_{<1>}} \) (which is made explicit in section 4). This renormalization map is crucial in the understanding of the spectral properties of the operator \( H_{<n>} \); in particular it is crucial to understand the behavior of
\[
g^n (L_{Q_{<n>}+\lambda I_b})
\]
where \( I_b \) is the diagonal \( F \times F \) matrix with diagonal terms \( (I_b)_{x,x} = b(x) \). The reason is that \( g^n (L_{Q_{<n>}+\lambda I_b}) \) is equal to the following Lagrangian subspace of \( V_{\partial F_{<n>}} \simeq V_F \): consider the functions \( f : \mathbb{R}^{F_{<n>}} \to \mathbb{R} \) such that
\[
\mathcal{E}^{<n>} (f,h) + \lambda \int_{F_{<n>}} f h \rho_{<n>} = 0, \quad \forall h \in \mathbb{R}^{F_{<n>}} \text{ s.t. } h|_{\partial F_{<n>}} = 0.
\]
For such a function we denote by \( I^n_f^{<n>}\lambda \) the element of \(( \mathbb{R}^{F_{<n>}})^*\) such that
\[
I^n_f^{<n>}\lambda (h) = \mathcal{E}^{<n>} (f,h) + \lambda \int_{F_{<n>}} f h \rho_{<n>}, \quad \forall h \in \mathbb{R}^{F_{<n>}}.
\]
By (0.2), $I_f^{\rho<n>\lambda}$ is supported by $\partial F_{<n>}$ and hence lies in $(\mathbb{R}^{\partial F_{<n>}})^*$. Then,

$$g^n(L_{Q_n} + \lambda I_k) = \left\{ f|_{\partial F_{<n>}} + I_f^{\rho<n>\lambda}, \ f \text{ solution of (0.2)} \right\}.$$  

**Remark 0.1.** Otherwise stated, it means that we consider the solutions of $(H_{<n>} - \lambda)f = 0$ on $F_{<n>} \setminus \partial F_{<n>}$, and that $I_f^{\rho<n>\lambda}$ plays the role of a kind of discrete derivative on $\partial F_{<n>}$. Hence, $g^n(L_{Q_n} + \lambda I_k)$ is the subspace generated by the boundary values of the space of solutions of $(H_{<n>} - \lambda)f = 0$ on $F_{<n>} \setminus \partial F_{<n>}$. This formula is useful to understand the role played by the renormalization map $g$. Indeed, if $f$ is an eigenfunction of $H_{<n>}$ with eigenvalue $\lambda$, we see that it is a solution of (0.2) with $I_f^{\rho<n>\lambda} = 0$. Hence, if $f|_{\partial F_{<n>}} \neq 0$, it means that

$$g^n(L_{Q_n} + \lambda I_k) \cap (\mathbb{R}^F \oplus 0)$$

is a non trivial subspace of $V_F$. Similarly, the intersection

$$g^n(L_{Q_n} + \lambda I_k) \cap (0 \oplus (\mathbb{R}^F)^*)$$

is related to the Dirichlet eigenfunctions of $H_{<n>}$, with eigenvalues $\lambda$. Hence, if $C^+ = \mathbb{R}^F \oplus 0$ and $C^- = 0 \oplus (\mathbb{R}^F)^*$, we see that the Neumann (resp. Dirichlet) spectrum is related to the intersection of the curve $\lambda : L_{Q_n} + \lambda I_k$ with the hypersurface $f^{-n}(C^+)$ (resp. $f^{-n}(C^-)$). Technically, to count these eigenvalues with multiplicities, we consider the current of integration on $C^+$ (resp. $C^-$) and its pull-back by $f^n$ (cf. section 4.6).

The last point we want to insist on in this introduction deals with the relation between the singularities of the renormalization map $g$ and a certain type of eigenfunctions on $F_{<n>}$. These special eigenfunctions are the so-called “Neumann-Dirichlet” eigenfunctions (N-D for short): a function $f : F_{<n>} \to \mathbb{R}$ is a N-D eigenfunction with eigenvalue $\lambda$, if

$$H_{<n>}f = \lambda f, \ \text{and} \ f|_{\partial F_{<n>}} = 0.$$ 

Hence, $f$ is an eigenfunction with both Neumann and Dirichlet boundary conditions (and actually, with any mixed boundary condition). Remark now that the boundary values (i.e. $f|_{\partial F_{<n>}}$ and $I_f^{\rho<n>\lambda}$) of these eigenfunctions vanish, and thus do not contribute to the Lagrangian subspace $g^n(L_{Q_n} + \lambda I_k)$ (cf. formula (0.3)). Actually, these eigenfunctions appear, with multiplicities, as the singularities of the map $g$ and its iterates $g^n$. This was proved in [23], but in this text, we clarify this point by a systematic analysis of the singularities of symplectic reductions.

These are the main ideas underlying this work. Part of them were already presented in [23], but compared to [23], the main goals are

- To explain the relations between operations on electrical networks and symplectic reductions (sections 1 and 2).
- To describe the singularities of symplectic reductions. We also give several general results about symplectic reductions, which are the bases of some of the key properties of our renormalization map. In particular, we show that symplectic reductions can be lifted to the exterior product $\bigwedge^K V$ by a linear map, using the Plücker embedding of $\mathbb{L}_V$ into the projective space $\mathcal{P}(\bigwedge^K V)$. This generalizes to symplectic reductions one of the main arguments of [23]. This is done in section 3. Let us stress that this section is more or less self-contained and does not appeal to such notions as electrical networks or self-similar lattices.
We present the renormalization map introduced in [23] from the point of view of symplectic geometry. More precisely, we give an explicit expression of the renormalization map on the Lagrangian compactification in terms of symplectic reduction. This is new compared to [23]. We also use several general results obtained in section 3, to recover some of the key results of [23]. This is done in section 4.

In section 6, we propose a class of rational maps on Lagrangian Grassmannians with a simple and natural definition (which essentially reproduces the figure (0.1)) and which shares the same basic properties as the renormalization maps of self-similar lattices.

Finally, we present some new examples (cf. [25] for other examples). In particular, we show that one of the rational maps appearing in relation with some automatic groups in the works of Grigorchuk, Bartholdi and Zuk (cf. [1, 14]) can be handled in our framework (section 7). In section 7, we also try to clarify how to proceed to make explicit computations when the structure has a large group of symmetries.

1. Electrical networks, Lagrangian compactification and Plücker embedding

1.1. Electrical networks. Let $F = \{1, \ldots, K\}$ be a finite set. We denote by $\text{Sym}_F(\mathbb{C})$, $\text{Sym}_K(\mathbb{R})$ (or $\text{Sym}_K(\mathbb{C})$, $\text{Sym}_K(\mathbb{R})$) the set of symmetric $K \times K$ matrices with coefficients in $\mathbb{R}$ or $\mathbb{C}$. By abuse of notation, we identify a $K \times K$ matrix with the linear operator induced on $\mathbb{R}^F$ or $\mathbb{C}^F$.

We call dissipative electrical network a family $(\rho_{i,j})$, $i \neq j$, $i, j \in F$, and a family $(\rho_i)$, $i \in F$, such that

i) $\rho_{i,j} = \rho_{j,i}$, $i \neq j$,

ii) $\rho_i$, $\rho_{i,j}$ are non-negative reals.

The terms $(\rho_{i,j})$ are called the conductances, and the terms $(\rho_i)$ are the dissipative terms. We say that the electrical network is irreducible when the graph defined by the strictly positive $\rho_{i,j}$ is connected. We say that $\rho$ is conservative when $\rho_i = 0$ for all $i$.

With $\rho$, we associate the element $Q_\rho$ in $\text{Sym}_F(\mathbb{R})$, by

$$(Q_\rho)_{i,j} = \begin{cases} -\rho_{i,j}, & i \neq j, \\ \rho_i + \sum_{k \neq i} \rho_{i,k}, & i = j. \end{cases}$$

The energy dissipated by the network, for the potential $f : F \to \mathbb{R}$, is given by the quadratic form:

$$\mathcal{E}_\rho(f, f) = \langle Q_\rho f, f \rangle = \sum_{i \in F} f(i)^2 \rho_i + \frac{1}{2} \sum_{i \neq j} \rho_{i,j}(f(i) - f(j))^2,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^F$ (the bilinear form $\mathcal{E}_\rho(f, h)$ on $\mathbb{R}^F \times \mathbb{R}^F$ is defined by polarization).

The electrical current associated with a potential $f$ is the element $I_f^\rho$ of the dual space $(\mathbb{R}^F)^*$ of $\mathbb{R}^F$, defined by

$$I_f^\rho(h) = \mathcal{E}_\rho(f, h), \quad \forall h \in \mathbb{R}^F.$$
It is clear with these notations, that for any subspace \( \perp \)
We denote by \( i \)
We denote by \( L \)
Of course, if \( (e_i) \) is the canonical basis of \( \mathbb{R}^F \), and \( (e_i^*) \) the dual basis, then we have
\[
I'_F = \sum_{i \in F} \left( \sum_{j \in F} (Q_\rho)_{i,j} f(j) \right) e_i^*.
\]
We denote by \( D_F \subset \text{Sym}_F(\mathbb{R}) \) the positive cone of real symmetric operators of the type \( Q_\rho \), for \( \rho \equiv ((\rho_{i,j}), (\rho_j)) \) a dissipative electrical network. We denote by \( D_F^0 \)
the subcone of \( D_F \) consisting of elements of the type \( Q_\rho \) for conservative electrical networks.

**Probabilistic interpretation.** When \( Q_\rho \in D_F \), the bilinear form \( \mathcal{E}_\rho(\cdot, \cdot) \) is a Dirichlet form on the set \( F \) (cf. [11]). If \( b \) is a positive measure on the set \( F \), the symmetric operator \( H_{\rho,b} \), on \( \mathbb{R}^F \), defined by
\[
< Q_\rho f, h > = - \int H_{\rho,b} f \cdot h db, \quad \forall f, h \in \mathbb{R}^F,
\]
is the infinitesimal generator of a discrete Markov process behaving as follows: the process waits an exponential time of parameter \( \frac{1}{\in(b)}(\rho_i + \sum_{j \neq i} \rho_{i,j}) \) at a point \( i_0 \) and then is killed with probability \( \frac{\rho_{i_0}}{\rho_{i_0} + \sum_{j \neq i_0} \rho_{i_0,j}} \) or jumps to a point \( j_0 \neq i_0 \) with probability \( \frac{\rho_{i_0,j_0}}{\rho_{i_0} + \sum_{j \neq i_0} \rho_{i_0,j}} \). The set \( D_F^0 \) corresponds to conservative Dirichlet forms. In this case there is no killing part.

### 1.2. Lagrangian compactification.
We set \( E = \mathbb{C}^F \), and denote by \( E^* = (\mathbb{C}^F)^* \) the dual space. We denote by \( (\epsilon_i)_{i \in F} \) the canonical basis of \( E \) and by \( (\epsilon_i^*)_{i \in F} \) the dual basis. Let us set \( V_F = E \oplus E^* \) (sometimes we write \( V_K \) or simply \( V \) when no ambiguity is possible), and denote by \( (\cdot, \cdot) \) the canonical symmetric bilinear form, and by \( <\cdot,\cdot> \), the canonical Hermitian scalar product on \( V_F \), given by
\[
(X, Y) = \sum_{i=1}^{2K} X_i \bar{Y}_i, \quad <X, Y> = \sum_{i=1}^{2N} \bar{X}_i Y_i,
\]
where \( X_i, Y_i \) are the coordinates of \( X, Y \) in the basis \(((\epsilon_i), (\epsilon_i^*))\). When we consider the real part, we write \( E_{\mathbb{R}} \) for \( \mathbb{R}^F \) and \( E^*_{\mathbb{R}} = (\mathbb{R}^F)^* \). Let \( \omega \) be the canonical symplectic bilinear form on \( V_F \times V_F \) given by
\[
\omega((x, \xi), (x', \xi')) = \xi'(x) - \xi(x'),
\]
for all \((x, \xi)\) and \((x', \xi')\) in \( V_F \cong E \times E^* \). We denote by \( \perp_\omega \) the orthogonality relation for the bilinear form \( \omega \). For any subspace \( L \subset V \), we denote by \( L^o \) the orthogonal subspace of \( L \) for the bilinear form \( \omega \).

Let \( J \) be the antisymmetric operator on \( V = E \oplus E^* \) defined by block by
\[
J = \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}.
\]
Clearly, we have
\[
(1.1) \quad \omega(X, Y) = (JX, Y), \quad \omega(X, Y) = <J\bar{X}, Y>.
\]
We denote by \( \perp_{<,>} \), the orthogonality relation for \( <,> \). For a subspace \( L \subset V \), we denote by \( L^\perp \) its orthogonal complement, for the Hermitian scalar product \( <,> \). It is clear with these notations, that for any subspace \( L \subset V \), we have
\[
L^o = \overline{JL^\perp}, \quad L^\perp = \overline{JL^o}.
\]
(N.B.: Here and in the following, $\overline{JL}$ and $\overline{JL}$ represent the complex conjugation of the linear spaces $JL$ and $JL$.)

Indeed, $\overline{JL}$ has the right dimension and $\overline{JL} \perp L$, using formula (1.2).

**Definition 1.1.** A vector subspace $L \subset V$ is isotropic (resp. coisotropic) if $L \subset L^\circ$ (resp. $L^\circ \subset L$). We say that $L$ is Lagrangian if $L^\circ = L$. Lagrangian subspaces have dimension equal to $\dim E = |F| = K$.

If $L$ is Lagrangian, then clearly, $L^\perp = JL$.

We denote by $\mathbb{L}_F$, resp. $\mathbb{L}_{F,R}$, the set of Lagrangian subspaces of $V$, resp. of real Lagrangian subspaces of $V$. The set $L_F$ has the structure of a smooth subvariety of $G_{\mathbb{C}}(K,2K)$, the complex Grassmannian of $K$-dimensional subspaces of $\mathbb{C}^{2K}$ (indeed, $\mathbb{L}_F$ is isomorphic to $Sp(K, \mathbb{C})/P_K$ where $Sp(K, \mathbb{C})$ is the symplectic linear group and $P_K$ a maximal parabolic subgroup, cf. [23, appendix E]). The tangent space at a point is isomorphic to $\text{Sym}_K(\mathbb{C})$ and we now give an explicit local parameterization of $\mathbb{L}_F$. Let $L$ be a Lagrangian subspace of $V$. Let $(v_1, \ldots, v_K)$ be an orthonormal basis of $L$ and set $(v_1^*, \ldots, v_K^*) = J(v_1, \ldots, v_K)$, which is an orthonormal basis of $L^\perp = JL$. For $Q$ in $\text{Sym}_K(\mathbb{C})$, we set

$$v_i^Q = v_i + \sum_{j=1}^K Q_{i,j} v_j^*.$$ 

The subspace generated by the family $\{v_i^Q\}_{i=1}^K$, is Lagrangian. The map

$$\text{Sym}_K(\mathbb{C}) \rightarrow \mathbb{L}_F$$

$$(Q \rightarrow \text{Vect}\{v_i^Q\}_{i=1}^K)$$

defines a local set of coordinates. Indeed, it is easy to check that any Lagrangian subspace in a neighborhood of $L$ can be represented in such a form.

Considering this local parameterization at the point $E \oplus 0$, with the basis $(v_1, \ldots, v_K) = (e_1, \ldots, e_K)$, gives a natural embedding of $\text{Sym}_F(\mathbb{C})$ in $\mathbb{L}$. More precisely, with any point $Q$ in $\text{Sym}_F(\mathbb{C})$ we associate the subspace $L_Q$ in $\mathbb{L}_F$ given by

$$L_Q = \text{Span}\{e_i + \sum_j Q_{i,j} e_j^*\}_{i \in F},$$

where we recall that $(e_1, \ldots, e_K, e_1^*, \ldots, e_K^*)$ is the canonical basis of $V = E \oplus E^*$. With this embedding, the set $\mathbb{L}_F \setminus \text{Sym}_F(\mathbb{C})$ is exactly the set

$$\mathbb{L}_F \setminus \text{Sym}_F(\mathbb{C}) = \{L \in \mathbb{L}_F, \ L \cap (0 \oplus E^*) \neq \{0\}\}$$

which is an analytic subvariety of codimension 1 in $\mathbb{L}_F$. Hence, $\mathbb{L}_F$ is a compactification of $\text{Sym}_F(\mathbb{C})$. If $\rho$ is an electrical network, we see that

$$L_{Q \rho} = \{f + I^\rho_f, \ f \in \mathbb{R}^F\},$$

where $I^\rho_f$ is the current defined in section 1.1.

**Remark 1.2.** The set of Dirichlet forms $D_F$ is thus a subset of $\mathbb{L}_{F,R}$ and the closure of $D_F$ in $\mathbb{L}_F$ gives a compactification of $D_F$, which is described in theorem 5 of [2].
1.3. Symplectic reduction. Let $W$ be a coisotropic subspace of $V$, with dimension $K + p$, $p \geq 0$ (the dimension of $W$ is necessarily greater or equal to $K$ since $W^o \subset W$ and $\dim W + \dim W^o = \dim V = 2K$). The quotient space $W/W^o$ has dimension $2p$ and the symplectic form $\omega$ on $V$ naturally induces a symplectic form $\omega_{W/W^o}$ on $W/W^o$. Indeed, if $X, Y$ are in $W/W^o$, we define

$$ (1.4) \quad \omega_{W/W^o}(X, Y) = \omega(\tilde{X}, \tilde{Y}), $$

where $\tilde{X}$ and $\tilde{Y}$ are any representatives in $W$ of the quotient class $X$ and $Y$. (The right-hand side does not depend on the choice of $\tilde{X}$ and $\tilde{Y}$ since $W^o \cap W = \{0\}$). The bilinear form $\omega_{W/W^o}$ is antisymmetric and non-degenerate (indeed, the $\omega_{W/W^o}$-orthogonal of $W/W^o$ is $\{0\}$, by construction), thus it is a symplectic form.

For any subspace $L$ of $W$, we set

$$ t_W(L) = (L \cap W)/W^o, $$

(more precisely, we mean that $t_W(L)$ is the subspace equal to the projection of $L \cap W$ to $W/W^o$). We have

$$ (t_W(L))^o = t_W(L^o), $$

where on the left-hand side, the term $(t_W(L))^o$ stands for the $\omega_{W/W^o}$-orthogonal of $t_W(L)$. Indeed, we have

$$ (L \cap W)^o \cap W = (L^o + W^o) \cap W = L^o \cap W + W^o, $$

and

$$ (t_W(L))^o = (L \cap W/W^o)^o $$

$$ = ((L \cap W)^o \cap W)/W^o $$

$$ = (L^o \cap W + W^o)/W^o $$

$$ = (L^o \cap W)/W^o = t_W(L^o). $$

It implies that if $L$ is isotropic (resp. coisotropic, resp. Lagrangian), then $t_W(L)$ is an isotropic (resp. coisotropic, resp. Lagrangian) subspace of $W/W^o$. In this text, $L$ will always be Lagrangian, and we will consider the symplectic reductions $t_W$ as a map from $\mathbb{L}_{\mathbb{F}}$ to $\mathbb{L}_{W/W^o}$ (where $\mathbb{L}_{W/W^o}$ is the Grassmannian of Lagrangian subspaces of $W/W^o$).

Composition of symplectic reductions.

If $W'$ is a subspace of $W/W^o$, we denote by $W'+W^0$ the preimage of $W'$ by the canonical projection $W \mapsto W/W^o$. Remark that we have

$$ (W' + W^0)^0 = ((W')^o + W^0). $$

(Indeed, we have $(W' + W^0)^0 \subset W$, and the previous formula comes from the definition of $\omega_{W/W^o}$, cf. [13.].) In particular, if $W'$ is $\omega_{W/W^o}$-coisotropic, then $W' + W^0$ is $\omega$-coisotropic and we have the following formula

$$ (1.5) \quad t_{W'} \circ t_W = t_{W'+W^0}. $$

Indeed, if $L$ is a subspace of $V$, then we have

$$ ((L \cap W/W^o) \cap W')/(W')^0 = ((L \cap (W' + W^0))/W^o)/(W')^0 $$

$$ = L \cap (W' + W^0)/(W')^0 $$

$$ = t_{W'+W^0}(L). $$
1.4. Plücker embedding. As a subvariety of the Grassmannian $G_C(K, 2K)$, $L_F$ can be embedded in a projective space by the Plücker embedding. It will often be useful to consider this embedding since it gives a set of homogeneous coordinates to represent the points of $L_F$. In particular, it will be useful in order to represent some rational maps on $L_F$ by homogeneous polynomial maps through the Plücker embedding.

We consider the exterior product

$$\wedge^K (E \oplus E^*) \simeq \bigoplus_{k=0}^K (\wedge^k E) \otimes (\wedge^{K-k} E^*),$$

and denote by $P(\wedge^K (E \oplus E^*))$ the associated projective space and by $\pi : \wedge^K (E \oplus E^*) \to P(\wedge^K (E \oplus E^*))$ the canonical projection. Classically, the manifold $L_F$ can be embedded in the projective space $P(\wedge^K (E \oplus E^*))$ by the Plücker embedding $L \rightarrow P(\wedge^K (E \oplus E^*))$,

$$L = \text{Vect}\{x_1, \ldots, x_K\} \mapsto \pi(x_1 \wedge \cdots \wedge x_K).$$

(1.6)

Grassmann algebra. When there is a canonical splitting of the space $V$, as is the case here with $V = E \oplus E^*$, then it is sometimes easier to represent this embedding a bit differently. Let $(\eta_i)_{i \in F}$ and $(\eta_j)_{j \in F}$ be two sets of variables, and consider the Grassmann algebra generated by these variables, i.e. the $\mathbb{C}$-algebra generated by $(\eta_i)_{i \in F}$ and $(\eta_j)_{j \in F}$ with the anticommuting relations

$$\eta_i \eta_j = -\eta_j \eta_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad \eta_i \eta_j = -\eta_j \eta_i.$$

We denote by $\mathcal{A}$ the subalgebra generated by the monomials containing the same number of variables $\eta$ and $\eta$ (clearly, $\mathcal{A}$ is isomorphic to $\oplus_{k=0}^K \wedge^k E \otimes \wedge^k E$). A canonical basis of $\mathcal{A}$ is

$$(1, \eta_1 \cdots \eta_k, \eta_j \cdots \eta_j, i_1 < \cdots < i_k, j_1 < \cdots < j_k, 1 \leq k \leq K).$$

We endow $\mathcal{A}$ with $< \cdot, \cdot >$, the Hermitian scalar product which makes this basis an orthonormal basis (with the convention that it is linear on the right and sesquilinear on the left).

If $Q$ is a $K \times K$ matrix, then we denote $\eta Q \eta$ the element of $\mathcal{A}$:

$$\eta Q \eta = \sum_{i,j \in F} Q_{i,j} \eta_i \eta_j.$$

We will be particularly interested in terms of the type

$$\exp(\eta Q \eta) = \sum_{k=0}^K \frac{1}{k!} \left( \sum_{i,j} Q_{i,j} \eta_i \eta_j \right)^k$$

$$= \sum_{k=0}^K \sum_{i_1 < \cdots < i_k} \det(Q_{i_1 \cdots i_k}) \eta_1 \eta_j \cdots \eta_k \eta_j,$$

where $(Q)_{i_1 \cdots i_k}$ is the $k \times k$ matrix obtained from $Q$ by keeping only the lines $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$. 


The algebra \( \mathcal{A} \) is clearly isomorphic to \( \bigwedge^K (E \oplus E^*) \) by the isomorphism \( \tau : \mathcal{A} \to \bigwedge^K (E \oplus E^*) \) given on the elements of the basis by

\[ \tau(\eta_{i_1} \eta_{j_1} \cdots \eta_{i_k} \eta_{j_k}) = e_1 \wedge \cdots \wedge e^*_j \wedge \cdots \wedge e^*_{j_k} \wedge \cdots \wedge e_K, \]

for all \( k \leq K, i_1 < \cdots < i_k, j_1 < \cdots < j_k \), and where we write \( \cdots \wedge e^*_j \wedge \cdots \) for the element obtained by replacing the term \( e_i \) by \( e^*_j \) in the monomial \( e_1 \wedge \cdots \wedge e_K \).

It is clear from formula (1.7), that, for \( Q \) in \( \text{Sym}_F(\mathbb{C}) \), we have

\[ \tau(\exp \eta Q \eta) = (e_1 + \sum_{j=1}^K Q_{1,j} e^*_j) \wedge \cdots \wedge (e_K + \sum_{j=1}^K Q_{K,j} e^*_j). \]

Hence, through the isomorphism \( \tau \), the subset \( \text{Sym}_F(\mathbb{C}) \) is embedded in the projective space \( P(\mathcal{A}) \) by

\[ \text{Sym}_F \hookrightarrow P(\mathcal{A}), \quad Q \mapsto \pi(\exp(\eta Q \eta)). \]

2. Trace map, gluing, and symplectic reduction

2.1. Trace map. Let \( \partial F \) be a non-empty subset of \( F \) (\( \partial F \) plays the role of a boundary set for \( F \)). Here, we describe an operator which plays a key role in the analysis of self-similar Schrödinger operators on pcf self-similar sets. Following the terminology of Dirichlet forms we called this operator the “trace operator” in [23], but it bears several different names and appears in several different fields of mathematics: for example, it is called Neumann to Dirichlet operator in the theory of differential operators or “answer of a network” in the context of electrical networks (cf. [5]), but also trace operator in the theory of Dirichlet forms (cf. [11], part 6) and Schubert’s complement in linear algebra (cf. for example [19], [4]).

We first describe this map on the set of real, symmetric, non-negative \( K \times K \) matrices. Let \( Q \) be in \( \text{Sym}_F(\mathbb{R}) \) and non-negative. We denote by \( Q_{\partial F} \) the real symmetric operator on \( \mathbb{R}_{\partial F} \) defined by the following variational problem

\[ <Q_{\partial F} f, f> = \inf_{g \in \mathbb{R}_F, g|_{\partial F} = f} <Q g, g>, \quad \forall f \in \mathbb{R}_{\partial F}, \]

where \( <\cdot, \cdot> \) denotes the usual scalar product respectively on \( \mathbb{R}_{\partial F} \) and \( \mathbb{R}_F \). We take from [23], proposition 2.1, the following simple properties.

\[ Q_{\partial F} = Q|_{\partial F} - B(Q|_{F \setminus \partial F})^{-1} B^t, \]

when \( Q \) has the following block decomposition on \( \partial F \) and \( F \setminus \partial F \):

\[ Q = \begin{pmatrix} Q|_{\partial F} & B \\ B^t & Q|_{F \setminus \partial F} \end{pmatrix}. \]

Therefore the map \( Q \mapsto Q_{\partial F} \) is rational in the coefficients of \( Q \) with poles included in the set \( \{ \det(Q|_{F \setminus \partial F}) = 0 \} \).
ii) If \( \det(Q_{F \setminus \partial F}) \neq 0 \), then for any function \( f \) in \( \mathbb{C}^{\partial F} \), we denote by \( Hf \), the function of \( \mathbb{C}^F \) given by

\[
\begin{cases}
Hf = f & \text{on } \partial F, \\
Hf = -(Q_{F \setminus \partial F})^{-1}Bf & \text{on } F \setminus \partial F.
\end{cases}
\]

We call \( Hf \) the harmonic extension of \( f \) with respect to \( Q \) and we have \( Q_{\partial F}(f) = (Q(Hf))_{|\partial F} \).

**Remark 2.2.** The trace map sends the cone of Dirichlet forms \( \mathcal{D}_F \) (resp. \( \mathcal{D}^0_{\partial F} \)) to the cone of Dirichlet forms \( \mathcal{D}_{\partial F} \) (resp. \( \mathcal{D}^0_{\partial F} \)). It means that if \( \rho \) is a dissipative (resp. conservative) electrical network, then there exists a dissipative (resp. conservative) electrical network \( \rho_{\partial F} \) on \( \partial F \) such that \( (Q_{\rho})_{\partial F} = Q_{\rho_{\partial F}} \) (cf. for example [20], proposition 1.9). This operation has a probabilistic interpretation in terms of the underlying Markov process, cf. [14], theorem 6.2.1.

Let us now describe the extension of this map to \( \mathbb{L}_F \). As shown in [5], this corresponds to a symplectic reduction. We set

\[ W_{\partial F} = \mathbb{C}^F \oplus (\mathbb{C}^{\partial F})^* \subset E \oplus E^*. \]

(In this section we sometimes simply write \( W \) to simplify notations). The \( \omega \)-orthogonal subspace of \( W \) is \( W^\omega = \mathbb{C}^{F \setminus \partial F} \oplus 0 \). Remark first that \( W/W^\omega \) can be identified with \( V_{\partial F} = \mathbb{C}^{\partial F} \oplus (\mathbb{C}^{\partial F})^* \), and that the restriction of \( \omega \) to \( W \) naturally induces the canonical symplectic form \( w_{\partial F} \) on \( W/W^\omega \sim \mathbb{C}^{\partial F} \oplus (\mathbb{C}^{\partial F})^* \) (cf. section 1.3).

If \( L \) is a Lagrangian subspace of \( E \oplus E^* \), then we set

\[
(2.2) \quad t_{F \to \partial F}(L) = t_{W_{\partial F}}(L) = (L \cap W)/W^\omega \subset W/W^\omega.
\]

We know, from section 1.3, that \( t_{F \to \partial F}(L) \) is a Lagrangian subspace of \( V_{\partial F} \). We have the following proposition (cf. [5], section 5.1).

**Proposition 2.3.** The map \( t_{F \to \partial F} \) coincides with the map \( Q \mapsto Q_{\partial F} \) on the set \( \text{Sym}_F(\mathbb{C}) \setminus \{ \det Q_{F \setminus \partial F} = 0 \} \), through the embedding of \( \text{Sym}_F(\mathbb{C}) \) and \( \text{Sym}_{\partial F}(\mathbb{C}) \) respectively in \( \mathbb{L}_F \) and \( \mathbb{L}_{\partial F} \) (as described in section 1.2). Otherwise stated, this means that

\[
t_{F \to \partial F}(L_Q) = L_{Q_{\partial F}},
\]

on the set \( \{ \det(Q_{F \setminus \partial F}) \neq 0 \} \).

**Remark 2.4.** The map \( t_{F \to \partial F} \) is not everywhere smooth. In section 3, we describe its discontinuities.

**Proof:** Let \( p = |\partial F| \). For \( x \) in \( \partial F \) we set \( g_x = H(\epsilon_x) \). The vector \( g_x \) can be written \( \sum_{y \in F} c^y_x \epsilon_y \) and we set

\[
g^*_x = \sum_{y \in F} c^y_x \sum_{z \in F} Q_{y,z}^* \epsilon^*_z.
\]

By definition, \( g_x + g^*_x \) is in \( L_Q \cap W \) for all \( x \) in \( \partial F \) and

\[
p_{W/W^*} (\text{Span} \{ g_x + g^*_x \}_{x \in \partial F})
\]

has dimension \( p \). This immediately implies that

\[
t_{F \to \partial F}(L_Q) = \text{Span} \{ \epsilon_x + (g^*_x)_{|\partial F} \}_{x \in \partial F}
\]

and hence that \( t_{F \to \partial F}(L_Q) = L_{Q_{\partial F}} \).
Remark 2.5. When the Lagrangian subspace is of the type $L_Q$, it is easy to
describe $t_{F \to \partial F}(L_Q)$. Indeed, consider the space of solutions $f : F \to \mathbb{C}$
of (2.3)
$$(Qf)_{|F \setminus \partial F} = 0.$$ 
For any solution $f$ of the previous equation, the current $I_f^Q$, defined as the element
of $(\mathbb{C}^F)^*$ such that $I_f^Q(h) = \langle Qf, h \rangle$ for all $h \in \mathbb{C}^F$, is supported by $\partial F$ and
hence is an element of $(\mathbb{C}^{\partial F})^*$. Then
$$(2.4) \quad t_{F \to \partial F}(L_Q) = \{ f_{|\partial F} + I_f^Q, \text{ for all } f \text{ solution of } (2.3) \}.$$ 
This expression is interesting when, for an electrical network $\rho$ and a positive measure
$b$ on $F$, we consider $Q = Q_{\rho} + \lambda I_b$, where $I_b$ is the diagonal matrix with
diagonal terms $(I_b)_{x,x} = b(\{x\})$. In this case the solutions of (2.3) are the solutions
of a discrete Schrödinger equation
$$((H_{\rho,b} - \lambda)f)_{|F \setminus \partial F} = 0.$$ 
(N.B.: $H_{\rho,b}$ is defined in section 1.1.) In particular, if $f$ is an eigenfunction of
$H_{\rho,b}$ with eigenvalue $\lambda$, which does not vanish on the boundary (i.e. such that
$f_{|\partial F} \neq 0$), then $I_f^Q = 0$ and $t_{F \to \partial F}(L_Q)$ intersects the Lagrangian subspace $\mathbb{C}^{\partial F} \oplus 0$
non-trivially. Similarly, the intersection
$$(2.5) \quad t_{F \to \partial F}(L_Q) \cap (0 \oplus (\mathbb{C}^{\partial F})^*)$$
is related to the eigenfunctions with Dirichlet boundary conditions. This will play
an important role in relation with the renormalization map we introduce in section
4. Remark also that $t_{F \to \partial F}(L_Q)$ is in $\text{Sym}_{\partial F}$ if and only if the intersection (2.5) is
$\{0\}$. This is true if $\ker(Q_{F \setminus \partial F}) = \{0\}$, and this is coherent with the fact that the set
of singularities of the trace map $Q \to Q_{\partial F}$ is included in the set $\{\det(Q_{F \setminus \partial F}) = 0\}$.

2.2. Gluing. Suppose now that $\mathcal{R}$ is an equivalence relation on $F$. We denote
by $\pi : F \to F/\mathcal{R}$ the canonical surjection and by $s : \mathbb{C}^{F/\mathcal{R}} \to \mathbb{C}^F$ the linear map
given by:
$$s(f) = f \circ \pi, \quad \forall f \in \mathbb{C}^{F/\mathcal{R}}.$$ 
We denote by $s^t : \mathbb{C}^F \to \mathbb{C}^{F/\mathcal{R}}$ the transposed map and by $s^* : (\mathbb{C}^F)^* \to (\mathbb{C}^{F/\mathcal{R}})^*$
the dual map given by
$$s^t(e_x) = e_{\pi(x)}, \quad s^*(e_x^*) = e_{\pi(x)}^*, \quad \forall x \in F;$$
where we recall that $(e_x)_{x \in F}$ and $(e_x^*)_{x \in F}$ (resp. $(e_x)_{x \in F/\mathcal{R}}$ and $(e_x^*)_{x \in F/\mathcal{R}}$) are the
canonical basis of $\mathbb{C}^F$ and $(\mathbb{C}^F)^*$ (resp. $\mathbb{C}^{F/\mathcal{R}}$ and $(\mathbb{C}^{F/\mathcal{R}})^*$). If $Q$ is a symmetric
operator on $\mathbb{C}^F$, it is natural to define the linear operator $Q_{F/\mathcal{R}}$ on $\mathbb{C}^{F/\mathcal{R}}$ by
$$Q_{F/\mathcal{R}} = s^t \circ Q \circ s.$$ 
It is clear that if $Q$ is in $\mathcal{D}_F$ and associated with an electrical network $\rho$, then $Q_{F/\mathcal{R}}$
is in $\mathcal{D}_{F/\mathcal{R}}$ and associated with an electrical network $\rho^{F/\mathcal{R}}$ given by
$$\rho_{x,y}^{F/\mathcal{R}} = \sum_{x',y',t} \rho_{x',y',t} \quad \rho_x^{F/\mathcal{R}} = \sum_{x',t} \rho_{x',t}.$$
As previously, the extension of this map to $\mathbb{L}_F$ is a symplectic reduction. Indeed,
let us consider the subspace $W_{F/\mathcal{R}} \subset V_F$ (we sometimes simply write $W$ in
this section):
$$W_{F/\mathcal{R}} = \text{Im}(s) \oplus (\mathbb{C}^F)^* = \mathbb{C}^{F/\mathcal{R}} \oplus (\mathbb{C}^F)^*,$$
where we considered $\mathbb{C}^{F/R}$ as the subset \{ $f \in \mathbb{C}^{F}$, $f(y) = f(y')$ if $\pi(y) = \pi(y')$ \} of $\mathbb{C}^{F}$. We have,

$$W^o = 0 \oplus (\ker s^*) = \{(0, \xi), \xi \in (\mathbb{C}^{F})^* \text{ s. t. } \sum_{y, \pi(y) = x} \xi(e_y) = 0, \forall x \in F\}.$$ 

Clearly, $W/W^o$ can be identified with $\mathbb{C}^{F/R} \oplus (\mathbb{C}^{F/R})^*$ and the restriction $\omega|_{W}$ induces the canonical symplectic form $\omega_{F/R}$ on $W/W^o$. We define the map $t_{F \to F/R} : L_F \to L_{F/R}$ by

$$t_{F \to F/R}(L) = t_{W_{F/R}}(L) = L \cap W/W^o.$$ 

**Proposition 2.6.** The map $t_{F \to F/R}$ coincides with the map $Q \mapsto Q_{F/R}$ on the set $\text{Sym}_{F}(\mathbb{C})$, i.e.

$$t_{F \to F/R}(L_Q) = L_{Q_{F/R}}.$$ 

Proof: It is simple and left to the reader.\(\diamondsuit\)

### 3. Properties of the symplectic reduction

#### 3.1. Singularities of the symplectic reduction

The trace map and the gluing map correspond to symplectic reductions. The symplectic reduction is not everywhere continuous, and in [23], the singularities play an important role in the understanding of the spectral properties of the operator. In this section, we consider the symplectic reduction from the algebraic point of view, as a rational map, and determine explicitly its indeterminacy points and its blow-up. We also generalize proposition 2.3 of [23], which describes the zeros of the corresponding map defined through the Plücker embedding.

Let us first recall the definition of a rational map between algebraic varieties (cf. for example [13], pp. 490-493). Let $X$ and $Y$ be two algebraic varieties. We denote by $\pi_1$ (resp. $\pi_2$) : $X \times Y \to X$ (resp. $Y$) the two canonical projections. A rational map $g$ from $X$ to $Y$ is defined by its graph

$$\Gamma_g \subset X \times Y,$$

when $\Gamma_g$ is an irreducible algebraic subvariety of $X \times Y$ such that for all $x$ in the complement of a non-trivial analytic subset of $X$, $\pi_1^{-1}(\{x\}) \cap \Gamma_g$ is a singleton. The subset $I \subset X$ where $\pi_1^{-1}(\{x\}) \cap \Gamma_g$ is not a singleton is called the set of indeterminacy points of $g$. It is an analytic subset of codimension (strictly) bigger than 1. (This comes from the fact that the graph $\Gamma_g$ is assumed to be irreducible.) The image of a point $x$ in $X$ is defined by

$$g(x) = \pi_2(\pi_1^{-1}(\{x\})).$$

Hence, $g(x)$ is a single point for $x$ in $X \setminus I$.

Let $f : X \to Y$ and $g : Y \to Z$ be two rational maps with indeterminacy sets $I_f$ and $I_g$, and such that $f(X \setminus I_f)$ is not included in $I_g$. We define the composition $g \circ f$ as the rational map defined by its graph

$$\Gamma_{g \circ f} = \text{closure} \{(x, g(f(x))), x \in X \setminus I_f, f(x) \in Y \setminus I_g\}.$$ 

**Remark 3.1.** The graph $\Gamma_{g \circ f}$ is not necessarily equal to the graph

$$\Gamma_g \circ \Gamma_f = \{(x, z) \in X \times Z, \exists y \in Y \text{ s.t. } (x, y) \in \Gamma_f, (y, z) \in \Gamma_g\}.$$
This equality is true only when $\Gamma_p \circ \Gamma_L$ is irreducible. This plays an important role in relation with the degrees of the iterates $g^n$, when $g$ is a rational map from $X$ to itself (cf. for example [8] in the 2-dimensional case, or [9]).

Let us recall that $V = \mathbb{C}^K \oplus (\mathbb{C}^K)^*$, that $\omega$ is the canonical symplectic form on $V$, and that $\mathbb{L}_V$ is the Grassmannian of Lagrangian subspaces of $V$. Let $W$ be a coisotropic subspace of $V$, with dimension $K + p$, and $W^\circ$ its $\omega$-orthogonal. We first claim that

**Proposition 3.2.** The symplectic reduction $t_W : \mathbb{L}_V \to \mathbb{L}_{W/W^\circ}$ is analytic on $\mathbb{L}_V \setminus \{L, L \cap W^\circ \neq \{0\}\}$ and can be extended into a rational map $\tilde{t}_W$ given by the graph

$$\Gamma_{\tilde{t}_w} = \{(L, L') \in \mathbb{L}_V \times \mathbb{L}_{W/W^\circ}, \dim(t_W(L) \cap L') \geq p - \dim(L \cap W^\circ)\}.$$  

In particular, the set of indeterminacy points is $\{L \in \mathbb{L}_V, L \cap W^\circ \neq \{0\}\}$, and for $L$ in the set of indeterminacy points, $t_W(L)$ is a particular point of the set $\tilde{t}_W(L)$.

**Remark 3.3.** This means that $\Gamma_{\tilde{t}_w}$ is the closure of the graph of $t_W$.

**Remark 3.4.** Even if this question seems natural, we could not find such a result anywhere in the literature.

**Proof:** We first recall that $\overline{W} = (W^\circ)^\perp$. Thus, there is a natural $\subset$, $>$-orthogonal decomposition

$$W = W \cap \overline{W} \oplus W^\circ.$$

Hence, we can canonically identify $W/W^\circ$ with $W \cap \overline{W}$, and we do so in the following.

We remark now that, with this canonical identification, $\Gamma_{\tilde{t}_w}$ can be written in the equivalent form

(3.1) \hspace{1cm} \Gamma_{\tilde{t}_w} = \{(L, L') \in \mathbb{L}_V \times \mathbb{L}_{W/W^\circ}, \dim(L \cap (L' \oplus W^\circ)) \geq p\}.

Indeed, denote by $p_{W/W^\circ} : W \to W/W^\circ$ the canonical projection. We have

$$\dim((L' \oplus W^\circ) \cap (L \cap W)) = \dim(p_{W/W^\circ}^{-1}(L') \cap (L \cap W))
= \dim(L \cap \ker(p_{W/W^\circ})) + \dim(L' \cap p_{W/W^\circ}(L \cap W))
= \dim(L \cap W^\circ) + \dim(L' \cap t_W(L)).$$

This immediately implies formula (3.1).

The fact that $t_W$ is analytic in $\{L \in \mathbb{L}_V, L \cap W^\circ = \{0\}\}$ is easy. Indeed, when $L \cap W^\circ = \{0\}$, the vector subspaces $L$ and $W$ are in generic position, thus the application $L \to L \cap W$ is analytic from $G(K, V)$ to $G(p, W)$, respectively the Grassmannian of $K$ dimensional subspaces of $V$ and the Grassmannian of $p$-dimensional subspaces of $W$. Then, the application $L \cap W \mapsto p_{W/W^\circ}(L \cap W)$ is analytic on the set where $(L \cap W) \cap \ker p_{W/W^\circ} = \{0\}$.

Thus, the only thing we have to prove is that $\Gamma_{\tilde{t}_w}$, defined in proposition 3.2, is equal to the closure

$$\{(L, t_W(L)), L \in \mathbb{L}_V, \text{ s.t. } L \cap W^\circ = \{0\}\}.$$  

Using the representation (3.1), we already know that this closure is included in $\Gamma_{\tilde{t}_w}$. Indeed, the dimension of the intersection of two subspaces is semi-continuous from below.
We prove now that for any \((L, L')\) in \(\Gamma_{\mathcal{W}}\), such that \(\dim(L \cap W^o) = n_0 > 0\), we can find \(L_0\) in a small neighborhood of \(L\) such that \(L_0 \cap W^o = \{0\}\) and \(t_\mathcal{W}(L) = L'\). Let us first prove this for \(L' = t_\mathcal{W}(L)\). We remark first that we have the following \(<,>\)-orthogonal decomposition

\[
L = L \cap W^o \oplus (L \cap W) \cap (L \cap W^o)^\perp \oplus L \cap (L \cap W)^\perp.
\]

Take some orthonormal basis \(f_1, \ldots, f_{n_0}\) of \(L \cap W^o\), \(f_{n_0+1}, \ldots, f_{n_0+n_0}\) of \((L \cap W) \cap (L \cap W^o)^\perp\), and \(f_{n_0+n_0+1}, \ldots, f_K\) of \(L \cap (L \cap W)^\perp\). Define now \((f_1^*, \ldots, f_K^*) = J(f_1, \ldots, f_K)\), and \(f_i^*\) by \(f_i^* = f_i + \epsilon f_i^+\) for \(i \leq n_0\), and \(f_i^* = f_i\) for \(i \geq n_0 + 1\). Then it is clear (cf. section 1.2) that \(L_\epsilon\), the vector space generated by the family \((f_i^*)\), is Lagrangian and satisfy both \(L \cap W^o = \{0\}\) and \(t_\mathcal{W}(L_\epsilon) = t_\mathcal{W}(L)\).

Let us consider now any \((L, L')\) in \(\Gamma_{\mathcal{W}}\) such that \(\dim(L \cap W^o) = n_0 > 0\). We just have to prove that in any small neighborhood of \(L\), we can find \(L_0\) such that \(t_\mathcal{W}(L_0) = L'\) (indeed, by a small modification, we can have the extra property \(L_0 \cap W^o = \{0\}\)). We set

\[
\tilde{L} = L' \cap t_\mathcal{W}(L).
\]

The subspace \(\tilde{L}\) has dimension \(n_1 \geq p - n_0\), by hypothesis. Define \(f_1, \ldots, f_K\) as before. We can always suppose that \(f_{n_0+1}, \ldots, f_{n_0+n_1}\) are such that

\[
p_{\mathcal{W}/W^o}(\text{Span}\{f_{n_0+1}, \ldots, f_{n_0+n_1}\}) = \tilde{L}.
\]

We set

\[
L'_1 = L' \cap \tilde{L}^\perp.
\]

Take now any surjective linear map \(T : L \cap W^o \to L'_1\) (there exists such a map since \(\dim(L \cap W^o) \geq \dim L'_1\)). For \(\epsilon > 0\) we set \(f_i^* = f_i + \epsilon f_i^+\) for \(i \leq n_0\), and \(f_i^* = f_i\) for \(n_0 + 1 \leq i \leq n_0 + n_1\). Then we define \(K_\epsilon = \text{Span}\{f_i^*, 1 \leq i \leq n_0 + n_1\}\). It is clear that \(p_{\mathcal{W}/W^o}(K_\epsilon) = L'\). This implies that \(K_\epsilon\) is isotropic, since \(L'\) is Lagrangian in \(W/W^o \sim W \cap J\mathcal{W}\). Hence, we can always construct a symplectic transformation \(S_\epsilon \in Sp(V)\), close to the identity for \(\epsilon\) small, such that \(S_\epsilon(K_0) = K_\epsilon\). Let us define \(L_\epsilon = S_\epsilon(L)\), which is a Lagrangian subspace of \(V\), close to \(L\) for small \(\epsilon\). By construction \(K_\epsilon \subset L_\epsilon \cap W\), hence \(t_\mathcal{W}(L_\epsilon) = L'\).

### 3.2. Linear lift by the Plücker embedding

Remind that \(L_{\mathcal{V}}\) is embedded in the projective space \(\mathcal{P}(\Lambda^K V)\) by the Plücker embedding (cf. section 1.4). Similarly, \(L_{\mathcal{W}/W^o}\) is embedded in \(\mathcal{P}(\Lambda^K W/W^o)\). In this section we construct an explicit linear map \(R_{\mathcal{W}} : \Lambda^K V \to \Lambda^K W/W^o\) which lifts the symplectic reduction \(t_\mathcal{W}\).

We recall that \((W^o)^\perp = J\mathcal{W}, W^\perp = J\mathcal{W}^o\) and that we have the orthogonal decomposition \(W = W \cap J\mathcal{W} \oplus W^o\), which gives a canonical isomorphism between \(W/W^o\) and \(W \cap J\mathcal{W}\). We choose an orthonormal basis \((g_1, \ldots, g_K - p)\) of \(W^o\) and set \((g_1^*, \ldots, g_K - p) = J(g_1, \ldots, g_K - p)\), which gives an orthonormal basis of \(W^\perp = J\mathcal{W}^o\).

For \(l \leq K\) and \(Y\) in \(\bigwedge^l V\), we denote by \(i_Y : \bigwedge^K V \to \bigwedge^{K-l} V\), the interior product defined as the linear map on \(\bigwedge^K V\) such that

\[
< Z, i_Y(X) > = < Y \wedge Z, X >, \quad \forall X \in \bigwedge^K V, \quad \forall Z \in \bigwedge^{K-l} V.
\]
where $<,>$ is the Hermitian product induced by the canonical Hermitian product on $V$. The interior product $i_{g_1^* \wedge \cdots \wedge g_{K-p}^*}$ sends $\bigwedge^K V$ to $\bigwedge^p W$ and we set

$$R_W : \bigwedge^K V \to \bigwedge^p W/W^\circ,$$

$$X \mapsto (\bigwedge^p p_{W/W^\circ}) \circ i_{g_1^* \wedge \cdots \wedge g_{K-p}^*}(X),$$

where $p_{W/W^\circ} : W \to W/W^\circ$ is the orthogonal projection on $W \cap \overline{JW} \simeq W/W^\circ$.

**Remark 3.5.** The expression of $R_W$ is not very simple, but in the special cases of the trace map and the gluing map, the expression is quite simple and natural (cf. the end of the section).

**Remark 3.6.** Up to a sign, the value of $R_W$ does not depend on the particular choice of the orthonormal basis $(g_1, \ldots, g_{K-p})$.

Let us give a definition: if $f$ is a holomorphic function from a domain $D \subset \mathbb{C}^n$ to $\mathbb{C}^m$, then we denote by $\ord(f, x_0)$ the order of vanishing of $f$ at the point $x^0 \in D$, i.e. the maximal integer $p$ such that one can find an open set $U$ containing $x_0$ and holomorphic functions $h_{i_1}, \ldots, h_{i_p}$, $1 \leq i_1 \leq \cdots \leq i_p \leq n$ on $U$ such that

$$f = \sum_{i_1 \leq \cdots \leq i_p} (x_{i_1} - x^0_{i_1}) \cdots (x_{i_p} - x^0_{i_p}) h_{i_1, \ldots, i_p}(x), \quad \text{on } U.$$

Let us finally recall that we denote by $\pi$, both the canonical projection $\pi : \bigwedge^K V \to \mathcal{P}(\bigwedge^K V)$ and $\pi : \bigwedge^p W/W^\circ \to \mathcal{P}(\bigwedge^p W/W^\circ)$.

**Proposition 3.7.** i) If $L \in \mathbb{L}_V$ is such that $L \cap W^\circ = \{0\}$, and $X_L \in \bigwedge^K V \setminus \{0\}$ such that $\pi(X_L) = L$, then $R_W(X_L) \neq 0$ and

$$\pi(R_W(X_L)) = t_W(L).$$

ii) If $L \in \mathbb{L}_V$ is such that $\dim(L \cap W^\circ) = n_0$, and $s$ is a local holomorphic section of $\pi$ on an open subset $U \subset \mathbb{L}_V$ containing $L$, then

$$\ord(R_W \circ s, L) = n_0.$$

**Remark 3.8.** Otherwise stated, (i) means that the following diagram commutes on the subset where all the maps are well-defined.

$$\begin{array}{ccc}
\pi^{-1}(\mathbb{L}_V) & \xrightarrow{R_W} & \pi^{-1}(\mathbb{L}_V) \\
\downarrow \pi & & \downarrow \pi \\
\mathbb{L}_V & \xrightarrow{t_W} & \mathbb{L}_V
\end{array}$$

**Remark 3.9.** This is a generalization of proposition 2.2, formula (30), and proposition 2.3 of [23], to general symplectic reductions. Remark also that in proposition 2.3 of [23], this result was proved only for real $Q_0$. Actually, this restriction is not necessary, as shown in the previous proposition. The proof we give here is also simpler than the proof of [23].

Proof: i) Let us consider $L \in \mathbb{L}_V$, such that $L \cap W^\circ = \{0\}$. The subspace $L$ can be decomposed orthogonally in

$$L = L \cap W \oplus (L \cap W)^\perp \cap L.$$
We choose an orthonormal basis \((f_1, \ldots, f_p)\) of \(L \cap W\) and \((f_{p+1}, \ldots, f_K)\) of \((L \cap W)^\perp \cap L\). We consider the orthogonal projection \(p_{W/W^\omega}\) on \(W/W^\omega\). Clearly, \(p_{W/W^\omega}\) is an isomorphism from \((L \cap W)^\perp \cap L\) onto \(W/W^\omega\); since \(\ker(p_{W/W^\omega}) = W\) and since they have the same dimension. Thus, we have
\[
\int_{g_1^\perp \cdots \perp g_{K-p}^\perp} (f_1 \wedge \cdots \wedge f_p \wedge \cdots \wedge f_K) = C f_1 \wedge \cdots \wedge f_p,
\]
where \(C = \int_{g_1^\perp \cdots \perp g_{K-p}^\perp} (f_{p+1} \wedge \cdots \wedge f_K)\) is a non-null complex scalar. It follows that
\[
\pi \left( (\wedge^P_{W/W^\omega}) \circ \int_{g_1^\perp \cdots \perp g_{K-p}^\perp} (f_1 \wedge \cdots \wedge f_K) \right) = \pi (p_{W/W^\omega} (f_1) \wedge \cdots \wedge p_{W/W^\omega} (f_p)) = t_W (L),
\]
which is exactly what we want.

ii) If now \(\dim (L \cap W^\omega) = n_0 > 0\), we have the orthogonal decomposition
\[
L = (L \cap W^\omega) \oplus (L \cap W^\omega)^\perp \cap (L \cap W) \oplus (L \cap W)^\perp \cap L.
\]
We choose orthonormal bases \((f_1^1, \ldots, f_{n_0})\) of \(L \cap W^\omega\), \((f_{n_0+1}, \ldots, f_{n_0+p})\) of \((L \cap W^\omega)^\perp \cap (L \cap W)\), and \((f_{n_0+p+1}, \ldots, f_K)\) of \((L \cap W)^\perp \cap L\). As usual, we set \((f_1^1, \ldots, f_K^1) = J (f_1, \ldots, f_K)\). Recall that \((g_1, \ldots, g_{K-p})\) is the orthonormal basis we chose for \(W^\omega\). We can as well suppose (up to a change of sign in \(R_W\)) that \((f_1^1, \ldots, f_{n_0}) = (g_1, \ldots, g_{n_0})\). For \(i \geq n_0 + p+1\), we can make the orthogonal decomposition \(f_i^1 = f_i^1 + f_i^2\) with \(f_i^1 \in W\), \(f_i^2 \in J^\perp W\). We have \(f_i^n = J (L \cap W^\omega)^\perp \cap J^\perp W\), since
\[
0 = \omega (f_i, f_j) = \omega (f_i^1, f_j^1) = \omega (f_i^1, f_j^2) > 0,
\]
for \(i \leq n_0\) and \(j \geq n_0 + p + 1\). Moreover,
\[
(g_1^1, \ldots, g_{n_0}, f_{n_0+p+1}^1, \ldots, f_K^1)
\]
form a basis of \(J^\perp W\). For \(Q\) in \(\text{Sym}_K (\mathbb{C})\), we set \(f_i^Q = f_i + \sum_{j=1}^n Q_{i,j} f_j^1\). In the neighborhood of \(L\), \(\mathbb{C} L\) can be parametrized by \(\text{Sym}_K (\mathbb{C})\), by
\[
Q \mapsto \text{Span} \{ f_i^Q \}_{i=1}^K.
\]
Then, we have
\[
R_W (f_1^Q \wedge \cdots \wedge f_n^Q) = C \det \left( (Q_{i,j})_{i,j=1}^{n_0} \right) \left( p_{W/W^\omega} (f_{n_0+1}^Q \wedge \cdots \wedge p_{W/W^\omega} (f_{n_0+p}) \right. + \text{terms of higher degree in } Q_{i,j},
\]
where \(C = \int_{g_1^\perp \cdots \perp g_{K-p}^\perp} (f_1 \wedge \cdots \wedge f_{n_0} \wedge f_{n_0+p+1}^1 \wedge \cdots \wedge f_K^1)\) is a non-null complex scalar. This immediately implies ii) of the proposition since \(\bigwedge^P p_{W/W^\omega} (f_{n_0+1}^Q \wedge \cdots \wedge f_{n_0+p})\) is non null.\(\diamondsuit\)

The corresponding map on the Grassmann algebra, in the case of section 2.

We come back to the situation of the trace map and the gluing map, described in section 2. We use the Grassmann algebra \(A_F\) to give the explicit expression of the map \(R_W\), since the expressions are simpler (and have an interpretation in terms of antisymmetric integrals).

Let us first come back to the case of the trace map. We denote by \(A_{\partial F}\) the Grassmann algebra associated with the set \(\partial F\), as in section 1.4. The algebra \(A_{\partial F}\) corresponds also to the subalgebra of \(A_F\) generated by the monomials containing only the variables \(\eta_x\) and \(\eta_x\), for \(x \in \partial F\).
If $Y$ is in $\mathcal{A}$ we denote by $i_Y$ the interior product by $Y$, i.e. the linear operator $i_Y : \mathcal{A} \rightarrow \mathcal{A}$ defined by
\begin{equation}
< Z, i_Y(X) > = < YZ, X >, \quad \forall X, Z \in \mathcal{A}.
\end{equation}
In particular, remark that
\begin{equation}
i_{\Pi_{x \in \mathcal{F}} \eta_x}^\tau(\exp \mathcal{F} Q\eta) = \det Q.
\end{equation}
We define the linear operator
\begin{equation}
R_{F \rightarrow \partial F} : \mathcal{A} \rightarrow \mathcal{A}_{\partial F}
\end{equation}
\begin{equation}
X \mapsto i_{\Pi_{x \in \mathcal{F}} \partial F \eta_x}^\tau(X).
\end{equation}

**Remark 3.10.** The operator $R_{F \rightarrow \partial F}$ is often presented as an antisymmetric integral. More precisely, $R_{F \rightarrow \partial F}(X)$ coincides with the antisymmetric integral of $X$ with respect to $\Pi_{x \in \mathcal{F}} \partial F \eta_x d\eta_x$, i.e. $R_{F \rightarrow \partial F}(X) = \int X \Pi_{x \in \mathcal{F}} \partial F \eta_x d\eta_x$, as defined in $[3]$ (cf. also $[29]$).

**Lemma 3.11.** The operator $R_{F \rightarrow \partial F}$ corresponds, up to a sign, to the operator $R_{\mathcal{W}_F}$ for the coisotropic subspace $\mathcal{W}_F$ defined in section 2.1. More precisely, it means that $R_{\mathcal{W}_F} \circ \tau = \pm \tau \circ R_{F \rightarrow \partial F}$, where $\tau$ is the isomorphism defined in section 1.4.

Proof: We can easily check this on the elements of the canonical basis. Suppose that $|\partial F| = p \leq K$, and that $\partial F = \{K - p + 1, \ldots, K\}$. We have $(\mathcal{W}_F)^o = \mathbb{C}^{F \setminus \partial F} \oplus 0$. We take $(g_1, \ldots, g_{K-p}) = (e_1, \ldots, e_{K-p})$, which is a basis of $(\mathcal{W}_F)^o$. We have $(g^*_1, \ldots, g^*_{K-p}) = (e^*_1, \ldots, e^*_{K-p})$. Consider now an element of the basis of the type (with the notations of section 1.4)
\begin{equation}
e_1 \wedge \cdots \wedge \hat{e}_{i_1}^* \wedge \cdots \wedge \hat{e}_{i_k}^* \wedge \cdots \wedge e_K.
\end{equation}
for $k \leq K$, $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$. It corresponds to $\eta_{i_1} \eta_{j_1} \cdots \eta_{i_k} \eta_{j_k}$ by the isomorphism $\tau$ defined in section 1.4. It is clear that $R_{\mathcal{W}_F}$ is non null on this element if and only if $\{i_1, \ldots, i_k\} \supset \{1, \ldots, K - p\}$, and $\{j_1, \ldots, j_k\} \supset \{1, \ldots, K - p\}$, i.e. $R_{\mathcal{W}_F}$ is non null on the elements of the type
\begin{equation}e^*_1 \wedge \cdots \wedge e^*_{K-p} \wedge e_{K-p+1} \wedge \cdots \wedge e^*_{j_{K-p+1}} \wedge \cdots \wedge e^*_{j_k} \wedge \cdots \wedge e_K.
\end{equation}
The map $R_{\mathcal{W}_F}$ applied to the previous element gives
\begin{equation}e_{K-p+1} \wedge \cdots \wedge e_{j_{K-p+1}}^* \wedge \cdots \wedge e_{j_k}^* \wedge \cdots \wedge e_K,
\end{equation}
which corresponds to the element $\eta_{i_{K-p+1}} \eta_{j_{K-p+1}} \cdots \eta_{i_k} \eta_{j_k}$ in $\mathcal{A}_{\partial F}$. The latter is also equal to
\begin{equation}R_{F \rightarrow \partial F}(\eta_{i_1} \cdots \eta_{K-p} \mid \eta_{K-p+1} \eta_{i_{K-p+1}} \cdots \eta_{i_k} \eta_{j_k}).
\end{equation}
This is exactly the equality we need.

It means that the map $R_{F \rightarrow \partial F}$ lifts the trace map $Q \mapsto Q_{\partial F}$ and the associated symplectic reduction $i_{\mathcal{W}_F}$. This was already proved in $[23]$, where we proved the following formula
\begin{equation}R_{F \rightarrow \partial F}(\exp \mathcal{F} Q\eta) = \det(Q_{F \setminus \partial F}) \exp \mathcal{F} Q_{\partial F} \eta.
\end{equation}
Let us now remark that, in the case of the trace map, for a point of the type $L_Q = \pi(\exp \overline{\eta} \eta)$, the set $L_Q \cap W^o_{\partial F} = \ker^{ND}(Q) \oplus 0$, where

$$\ker^{ND}(Q) = \{ f \in \mathbb{C}^F, \ Qf = 0 \text{ and } f_{|\partial F} = 0 \}.$$ 

Hence, the order of vanishing of $R_{F\rightarrow F}/\partial F$ at $L_Q$ is equal to $\dim \ker^{ND}(Q)$. (N.B.: we write $\ker^{ND}$ for “Neumann-Dirichlet kernel”, in reference to the Neumann-Dirichlet spectrum which plays an important role in section 4.)

Let us now consider the case of the “gluing map”. We denote by $A_{F/\mathcal{R}}$ the Grassmann algebra associated with the set $F/\mathcal{R}$, as in section 1.4. The canonical surjection $\pi : F \rightarrow F/\mathcal{R}$ naturally induces a morphism of algebra $R_{F\rightarrow F/\mathcal{R}} : A_F \rightarrow A_{F/\mathcal{R}}$ defined on generating variables by

$$R_{F\rightarrow F/\mathcal{R}}(\overline{\eta}_x) = \overline{\eta}_{\pi(x)}, \ R_{F\rightarrow F/\mathcal{R}}(\eta_x) = \eta_{\pi(x)}.$$ 

**Lemma 3.12.** The linear map $R_{F\rightarrow F/\mathcal{R}}$ corresponds to the map $R_{W_F/\mathcal{R}}$, up to a sign, for the coisotropic subspace $W_{F/\mathcal{R}}$ introduced in section 2.2. Otherwise stated it means that $R_{W_F/\mathcal{R}} \circ \tau = \pm \tau \circ R_{F\rightarrow F/\mathcal{R}}$.

Proof: it is simple, similar to the previous one, and left to the reader. ♦

Hence, $R_{F\rightarrow F/\mathcal{R}}$ lifts the “gluing map” $Q \mapsto Q_{F/\mathcal{R}}$ to $A_F$. It is actually very easy to check directly this last point since we have the following trivial formula:

$$R_{F\rightarrow F/\mathcal{R}}(\exp \overline{\eta} \eta) = \exp \overline{\eta} Q_{F/\mathcal{R}}.$$ 

For all $Q$ in $\text{Sym}_F$, $L_Q \cap W_{F/\mathcal{R}} = \{ 0 \}$; hence, the symplectic reduction $t_{F/\mathcal{R}}$ is smooth on $\text{Sym}_F \subset L_F$. This is coherent with the fact that $R_{F\rightarrow F/\mathcal{R}}(\exp \overline{\eta} \eta)$ does not vanish for $Q$ in $\text{Sym}_F$ (cf. formula [4.5]).

### 3.3. Intersection of the set of indeterminacy points by a holomorphic curve.

Let $U \subset \mathbb{C}$ be an open subset containing 0. Let $L : U \rightarrow L_F$ be analytic and such that $L(0) = L_0$ is in the set of indeterminacy points of $\hat{t}_W$, i.e. such that $\dim(L_0 \cap W^o) = n_0 > 0$. We suppose that $L(U)$ is not contained in the set of indeterminacy points of $\hat{t}_W$, and we may as well suppose that $L(\lambda)$ intersects the set of indeterminacy points at 0 only, by taking $U$ small enough. Since $L_F$ is compact, $(t_W \circ L)|_{U \setminus \{0\}}$ can be analytically continued to $U$. We choose as in the proof of proposition 3.7 ii), an orthonormal basis $\{ f_1, \ldots, f_K \}$ of $L_0$, such that $\{ f_1, \ldots, f_{n_0} \}$ is a basis of $L_0 \cap W^o$. We can identify the tangent plane of $L_F$ at $L_0$ with $\text{Sym}_K(\mathbb{C})$, thanks to the local parametrization described in section 1.2, associated with the basis $(f_1, \ldots, f_K)$. Let $Z \subset \text{Sym}_K(\mathbb{C})$ be the homogeneous analytic set given by

$$Z = \{ Q \in \text{Sym}_K(\mathbb{C}), \ \det((Q_{i,j})_{i,j=1}^n) = 0 \}.$$ 

**Lemma 3.13.** If $L'(0) \in \text{Sym}_K(\mathbb{C}) \setminus Z$ then

$$\text{ord}(\lambda \mapsto R_W \circ s(L(\lambda)), 0) = n_0,$$

for any local section $s$ of $\pi$, in a neighborhood of $L_0$. Moreover, $(t_W \circ L)|_{U \setminus \{0\}}$ is analytically continued at 0 by $t_W(L_0)$.

**Remark 3.14.** This means in particular, that if the map $\lambda \mapsto L(\lambda)$ intersects the indeterminacy point $L_0$ in a generic direction, then the analytic continuation is given by the point $t_W(L_0) \in \hat{t}_W(L_0)$. This points out the specific role of the symplectic reduction $t_W(L_0)$ in the blow-up $\hat{t}_W(L)$. 
Proof: We take the notations of the proof of proposition 3.7 ii). If $L'(0) = Q_0 \in \text{Sym}_K(\mathbb{C}) \setminus Z$, then in a neighborhood of 0,

$$L(\lambda) = \text{Span}\{f_i^{Q(\lambda)}\}_{i=1}^K,$$

for a holomorphic function $Q(\lambda)$, with $Q(\lambda) = \lambda Q_0 + O(\lambda^2)$. From formula (3.2) we have

$$R_W(f_1^{Q(\lambda)} \wedge \cdots \wedge f_K^{Q(\lambda)}) = C\lambda^{n_0} \det((Q_0)_{i,j=1}^{n_0}) (p_{W/W^*}(f_{n_0+1}) \wedge \cdots \wedge p_{W/W^*}(f_{n_0+p+1})) + O(\lambda^{n_0+1}).$$

But

$$\pi(p_{W/W^*}(f_{n_0+1}) \wedge \cdots \wedge p_{W/W^*}(f_{n_0+p})) = t_W(L_0).$$

Thus, $\lim_{\lambda \to 0} \tilde{t}_W(L(\lambda)) = t_W(L_0).$  

3.4. Siegel upper half-plane. We now prove a specific property of the symplectic reduction when the coisotropic space $W$ is the complexification of a real subspace. Let us first introduce some definitions. The subset $S^+,K$ of $\text{Sym}_K(\mathbb{C})$ defined by

$$S^+,K = \{Q, \text{ Im}(Q) \text{ is positive definite}\},$$

is called the Siegel upper-half plane (cf. [28]), and is a homogeneous space (isomorphic to $\text{sp}(K,\mathbb{R}) \setminus U(K)$).

Let us remark now that for any $X$ in $V$, $\omega(X,X)$ is a pure imaginary number, since $\omega$ is antisymmetric. Let us define the subset $S^+,V \subset \mathbb{L}_V$ by

$$S^+,V = \{L \in \mathbb{L}_V, \quad -i\omega(X,X) > 0, \quad \forall X \in L \setminus \{0\}\}.$$ 

We have then the following simple result.

**Proposition 3.15.** Let $L_1$ be the complexification of a real Lagrangian subspace. Let $v_1, \ldots, v_K$ be a real orthonormal basis of $L_1$ and $(v_1^*, \ldots, v_K^*) = J(v_1, \ldots, v_K)$ be the associated basis of $L_1^* = JL_1$. For any $Q$ in $\text{Sym}_K(\mathbb{C})$ we set

$$v_i^Q = v_i + \sum_{j=1}^K Q_{i,j} v_j^*,$$

and denote by $L_Q(v_1^Q, \ldots, v_K^Q) \in \mathbb{L}_V$ the Lagrangian subspace generated by the family $(v_i^Q)_{i=1}^K$. Then we have

$$S^+,V = \{L_Q(v_1^Q, \ldots, v_K^Q) \in \mathbb{L}_V, \quad Q \in S^+,K\}.$$ 

**Remark 3.16.** In particular, for the canonical decomposition $V = \mathbb{C}^F \oplus (\mathbb{C}^F)^*$ and the canonical basis $(e_1, \ldots, e_K, e_1^*, \ldots, e_K^*)$ this gives a canonical identification of $S^+,V$ with $S^+,K$ given by

$$S^+,K \to S^+,V$$

$$Q \mapsto L_Q,$$

where $Q \mapsto L_Q$ is the embedding described in section 1.2 (with the notations of the previous proposition, we have $L_Q = L_Q^{(e_1, \ldots, e_K)}$). Thus, when no ambiguity is possible, we simply write $S^+$ for $S^+,K \simeq S^+,V$. 

Proof: If $X$ is in $L^v_{W/W}^{(v_1,...,v_K)}$ then

$$X = \sum_{i=1}^{K} c_i(v_i + \sum_{j=1}^{K} Q_{i,j}v_j^*)$$

for some vector $(c_1, \ldots, c_K) \in \mathbb{C}^K$. We easily get that

$$\omega(X,X) = \langle JX, X \rangle$$

(3.8)

$$= 2i \sum_{k,k'=1}^{K} \pi_k \text{Im}(Q)_{k,k'} c_{k'}.$$ 

Hence, if $Q$ is in $S_{+,V}$ then $L^v_{W/W}^{(v_1,...,v_K)}$ is in $S_{+,V}$.

Conversely, we first remark that

$$L_V \setminus \{ L^v_{W/W}^{(v_1,...,v_K)}, Q \in \text{Sym}_K(\mathbb{C}) \} = \{ L \in \mathbb{L}_V, \ L \cap L^1 \neq \{0\} \}.$$ 

Since $L_1$ is the complexification of a real Lagrangian subspace, if $X$ is in $L^1_1 = JL_1$ then so is $\bar{X}$. Thus $\omega(\bar{X},X) = 0$ for all $X$ in $L^1_1$, since $L^1_1$ is Lagrangian. This implies that for any Lagrangian subspace $L$ in $S_{+,V}$, $L \cap L^1_1 = \{0\}$, and thus that any $L$ in $S_{+,V}$ can be written $L^v_{W/W}^{(v_1,...,v_K)}$ for a certain symmetric operator $Q$ in $\text{Sym}_K(\mathbb{C})$. By formula (3.8), we get the result. \diamond

Let $W$ be the complexification of a real coisotropic subspace of dimension $K+p$.

**Proposition 3.17.** The symplectic reduction $t_W$ is analytic on $S_{+,V}$ and $t_W(S_{+,V}) \subset S_{+,W/W^o}$.

Proof: Let us consider $L$ in $S_{+,V}$. As previously, we denote by $p_{W/W^o} : W \rightarrow W/W^o$ the canonical projection. With the identification $W/W^o \cong W \cap JW$, we can write any point $X$ in $L \cap W$ as $X = p_{W/W^o}(X) + X'$ with $X' \in W^o$. Using the fact that $W$ is the complexification of a real subspace we have $\overline{p(X)} = \overline{p(X)}$ and $\overline{W} = W^o$. Thus, for any vector $X$ in $L \cap W$

$$\omega(X,X) = \omega(p(X),p(X)) + \omega(p(X),X') + \omega(X',p(X)) + \omega(X',X)$$

$$= \omega(p(X),\overline{p(X)}),$$

since $W^o \perp W$. This implies that $-i\omega(p(X),\overline{p(X)}) > 0$ for all $X$ in $L \cap W \setminus \{0\}$. We deduce, firstly, that $p(X) \neq 0$ for all $X \neq 0$ in $L \cap W$, thus that $L \cap W^o = \{0\}$, and that $t_W$ is continuous on $S_{+,V}$. Secondly, we deduce that $t_W(L)$ is in $S_{+,W/W^o}$. \diamond

### 3.5. A special class of holomorphic curves.

When $W$ is real, there is a natural class of applications $\lambda \mapsto L(\lambda)$ which satisfies the hypotheses of lemma \ref{lem:3.13} at any point.

**Lemma 3.18.** Let us suppose that $L : \mathbb{C} \rightarrow \mathbb{L}_F$ is holomorphic on $\mathbb{C}$ and such that

$$L(\mathbb{R}) \subset \mathbb{L}_{F,R}, \text{ and }, L(\{\lambda, \text{Im}\lambda > 0\}) \subset S_{+,V}.$$ 

then $\lambda \mapsto L(\lambda)$ satisfies the hypotheses of lemma \ref{lem:3.13} at any point of $\mathbb{C}$, i.e. $\lambda \mapsto t_W(L(\lambda))$ is holomorphic and

$$\text{ord}(\lambda \mapsto R_W \circ s(L(\lambda)), \lambda_0) = \dim(L(\lambda_0) \cap W^o),$$

if $s$ is a local holomorphic section of $\pi$ on a neighborhood $U \subset \mathbb{L}_F$ of $L(\lambda_0)$.
We denote by $\pi$ and $\partial F$ self-similar structures. These structures appear in relation with finitely ramified groups, cf. section 5.2 and examples, section 7. A generalization of these structures also seems to appear in relation with some automatic self-similar sets (also called p.c.f self-similar sets), cf. section 5.1. A generalization of a subset $F,N$, we call finite self-similar structure, the triplet $(F,N)$. From this finite structure, we can construct a sequence of sets $F_{<n>}$, with an identification of a subset $\partial F_{<n>}$ with $F$, as follows. Suppose that the sequence $(F_{<n>}, \partial F_{<n>})$ is constructed up to level $n$. We consider the set $\{1, \ldots, N\} \times F_{<n>};$ the subset $\{1, \ldots, N\} \times \partial F_{<n>}$ can be identified with $\tilde{F}_{<1>}$. Then we have, cf. formula (3.8),

$$\lambda_{\ast} = \sum_{i=1}^{K} c_i (v_i + \sum_{j=1}^{K} (Q(\lambda))_{i,j} v_j^*).$$

On the other hand, we have locally the following approximation

$$<C,Q(\lambda)C> = \frac{(\lambda - \lambda_0)^n}{n!} <C,Q(n)(\lambda_0)C> + O((\lambda - \lambda_0)^{n+1}),$$

where $Q(n)(\lambda_0)$ is the $n$-th derivative of $Q(\lambda)$ and $n$ the smallest integer such that $<C,Q(n)(\lambda_0)C> \neq 0$ (this $n$ exists, since otherwise $<X,Q(\lambda)X> = 0$ for all $\lambda$).

For all $\lambda$ such that $\Im \lambda > 0$, we must have $\Im <C,Q(\lambda)C> > 0$ since $X_\lambda$ is in $L(\lambda)$, and $L(\lambda)$ in $S_{\ast,V}$. This is possible only if $n = 1$ and $<C,Q'(\lambda_0)C> > 0$. Hence, $Q'(\lambda_0)$ is positive definite. This immediately implies that $\lambda \mapsto L(\lambda)$ satisfies the hypothesis of lemma 3.13 for all $\lambda_0$ in $\mathbb{C} \setminus \mathbb{R}$.

4. Application to the renormalization map of finite self-similar structures

4.1. Finite self-similar structures. Here, we introduce the notion of finite self-similar structures. These structures appear in relation with finitely ramified self-similar sets (also called p.c.f self-similar sets), cf. section 5.1. A generalized version of these structures also seems to appear in relation with some automatic groups, cf. section 5.2 and examples, section 7.

Let $F = \{1, \ldots, K\}$ be a finite set and $N$ an integer, $N \geq 2$. We set

$$\tilde{F}_{<1>} = \{1, \ldots, N\} \times F,$$

and $F_{<1>,i} = \{i\} \times F \subset \tilde{F}_{<1>}$. We suppose given an equivalence relation $\mathcal{R}$ on $\tilde{F}_{<1>}$ and we set $F_{<1>} = \tilde{F}_{<1>} / \mathcal{R}$. We denote by $\pi : \tilde{F}_{<1>} \rightarrow F_{<1>}$ the canonical projection. Finally, we suppose that a subset $\partial F_{<1>}$ is specified in $F_{<1>}$, together with a bijective map between $F$ and $\partial F_{<1>}$, which gives a canonical identification between $F$ and $\partial F_{<1>}$ (cf. the example of the Sierpinski gasket in the introduction and in section 7).

We call finite self-similar structure, the triplet $(F,N,\mathcal{R})$ together with the identification of a subset $\partial F_{<1>} \subset F_{<1>}$ with $F$.

From this finite structure, we can construct a sequence of sets $F_{<n>}$, with an identification of a subset $\partial F_{<n>}$ with $F$, as follows. Suppose that the sequence $(F_{<n>}, \partial F_{<n>})$ is constructed up to level $n$. We consider the set $\{1, \ldots, N\} \times F_{<n>}$; the subset $\{1, \ldots, N\} \times \partial F_{<n>}$ can be identified with $\tilde{F}_{<1>}$. 

Proof: Clearly, $L(\lambda)$ may intersect the set of indeterminacy points of $t_U$ at real points only. Hence, $\lambda \mapsto L(\lambda)$ satisfies the hypotheses of lemma 3.13 on $\mathbb{C} \setminus \mathbb{R}$. Let $\lambda_0$ be real, and $(v_1, \ldots, v_K)$ be a real orthonormal basis of $L(\lambda_0)$, and $(v_1', \ldots, v_K') = J(v_1, \ldots, v_K)$. In a neighborhood of $\lambda_0$, we have $L(\lambda) = L_{Q(\lambda)}(v_1', \ldots, v_K')$, with the notations of proposition 3.15 and with a holomorphic $Q(\lambda)$. Since $L(\mathbb{R}) \subset L_{F,R}$, $Q'(\lambda_0)$ is real. It is also positive definite. Indeed, let $C = (c_i)$ be in $\mathbb{R}^K \setminus \{0\}$ and consider $X_\lambda$ in $V$ given by

$$X_\lambda = \sum_{i=1}^{K} c_i (v_i + \sum_{j=1}^{K} (Q(\lambda))_{i,j} v_j^*).$$

Then we have, cf. formula (3.8),

$$-i\omega(X_\lambda, X_\lambda) = 2\text{Im} <C,Q(\lambda)C>.$$
then inside \( \{1, \ldots, N\} \times F_{<n>}, \) we glue together the points of \( \{1, \ldots, N\} \times \partial F_{<n>} \) according to the relation \( R. \) This gives a set \( F_{<n+1>} \) which contains a copy of \( F_{<1>}: \) thus, we define \( \partial F_{<n+1>} \) as the boundary set \( \partial F_{<1>}, \) when \( F_{<1>} \) is considered as the subset \( \{1, \ldots, N\} \times \partial F_{<n>}/R. \) Remark that \( F_{<n>} \) can also be considered as the quotient of

\[
\tilde{F}_{<n>} = \{1, \ldots, N\}^n \times F
\]

by an equivalence relation that we denote \( R_{<n>} \) (but which we do not describe explicitly here). We denote by \( \tilde{F}_{<n>,i_1,\ldots,i_n} \) (resp. \( F_{<n>,i_1,\ldots,i_n} \)) the subset of \( \tilde{F}_{<n>} \) (resp. of \( F_{<n>} \)) of the type \( \{(i_1,\ldots,i_n)\} \times F \) (resp. \( \{(i_1,\ldots,i_n)\} \times F/R_{<n>}). \)

### 4.2. Self-similar Schrödinger operators.

Let \( \rho = ((\rho_{i,j})_{i\neq j}, (\rho_i)_{i \in F}), \) be an electrical network on \( F, \) as defined in section 1.1. We define an electrical network \( \tilde{\rho}_{<n>} \) on \( \tilde{F}_{<n>} = \{1, \ldots, N\} \times F \) by making a copy of \( \rho \) on each \( \tilde{F}_{<n>,i_1,\ldots,i_n}. \) Otherwise stated, it means that

\[(4.1) \quad \tilde{\rho}|_{\tilde{F}_{<n>,i_1,\ldots,i_n}} = \rho, \quad \forall i_1, \ldots, i_n,
\]

and that the conductances between two different subsets \( F_{<n>,i_1,\ldots,i_n} \) and \( F_{<n>,i_1',\ldots,i_n'} \) are null. Then we define \( \rho_{<n>} \) as the electrical network on \( F_{<n>} \) obtained from \( \tilde{\rho}_{<n>} \) by the gluing map described in section 2.2 (considering that \( F_{<n>} = \tilde{F}_{<n>}/R_{<n>}). \)

Similarly, if \( b \) is a positive measure on \( F, \) it induces a positive measure \( \tilde{b}_{<n>} \) on \( \tilde{F}_{<n>} \) equal to \( b \) on each subset \( F_{<n>,i_1,\ldots,i_n}, \) and a positive measure \( b_{<n>} \) on \( F_{<n>}, \) image of \( \tilde{b}_{<n>} \) by the canonical projection \( \tilde{F}_{<n>} \to F_{<n>}. \)

Let \( H_{<n>} = H_{<n>,\rho,b} \) be the Schrödinger operator defined from \( \rho_{<n>}, b_{<n>} \) by

\[
\langle Q_{\rho_{<n>}}, f, h \rangle = -\int H_{<n>}((f)(x)h(x))db_{<n>}(x), \quad \forall f, h \in \mathbb{R}^{F_{<n>}}.
\]

N.B.: \( <, > \) is the usual scalar product on \( \mathbb{R}^{F_{<n>}}. \)

We denote by \( \nu_{<n>}^+ \) the counting measure of the eigenvalues of \( H_{<n>}. \) We denote by \( \nu_{<n>}^- \) the counting measure of the Dirichlet eigenvalues of \( H_{<n>}, \) i.e. of the eigenvalues of the restriction of \( H_{<n>} \) to

\[
\mathcal{D}_{<n>} = \{f \in \mathbb{R}^{F_{<n>}}, f|_{\partial F_{<n>}} = 0\}.
\]

Remark that \( f \in \mathbb{R}^{F_{<n>}} \) is an eigenfunction of \( H_{<n>} \) with eigenvalue \( \lambda \) if and only if

\[(4.2) \quad (Q_{\rho_{<n>}} + \lambda I_{b_{<n>}}) f = 0,
\]

where \( I_{b_{<n>}} \) is the diagonal operator with diagonal terms \( (I_{b_{<n>}})_{x,x} = b_{<n>}(x) \) for all \( x \) in \( F_{<n>}. \) Similarly, \( f \) is a Dirichlet eigenfunction with eigenvalue \( \lambda \) if and only if

\[
(4.3) \quad (Q_{\rho_{<n>}} + \lambda I_{b_{<n>}}) f|_{F_{<n>} \setminus \partial F_{<n>}} = 0,
\]

\[(4.4) \quad f|_{\partial F_{<n>}} = 0.
\]

We denote by \( \mu \) the following limit

\[
\mu = \lim_{n \to \infty} \frac{1}{N^n} \nu_{<n>}^\pm,
\]

when it exists and does not depend on the boundary condition \( \pm. \) The measure \( \mu \) is called the density of states (the existence of this limit was first proved in [12], [17]).
Let us now define the so-called Neumann-Dirichlet eigenvalues. A function $f : F_{<n>} \rightarrow \mathbb{R}$ is a Neumann-Dirichlet (N-D for short) eigenfunction of $H_{<n>}$ with eigenvalue $\lambda$ if $H_{<n>} f = \lambda f$ and $f|_{\partial F_{<n>}} = 0$. This means that $f$ is both a Neumann and Dirichlet eigenfunction. Obviously, $f$ is a N-D eigenfunction of $H_{<n>}$ with eigenvalue $\lambda$ if and only if $f$ satisfies (4.2) and (4.4). We denote by $\nu_{<n>}$ the counting measure (with multiplicity) of the N-D eigenvalues. It is clear that if $f$ is a N-D eigenfunction on $F_{<n>}$, then we can make $N$ independent copies of $f$ on each subcell of $F_{<n+1>}$ (cf. [23]). Thus,

$$\nu_{<n+1>}^N \geq N \nu_{<n>}^N,$$

and the limit

$$\mu^N = \lim_{n \rightarrow \infty} \frac{1}{N^n} \nu_{<n>}^N,$$

exists and is called the density of Neumann-Dirichlet states. These two measures play an important role in the understanding of the spectral properties of some infinite self-similar lattices, cf. [24], and section 5.

### 4.3. The renormalization map

In [23], we have introduced a renormalization map defined on the Lagrangian Grassmannian $L_F$. We recall its definition and give its expression in terms of a symplectic reduction. In particular, we give an explicit expression of the zeros of the associated map on the Grassmann algebra, and this can be useful for applications.

Firstly, this renormalization map can be defined on $\operatorname{Sym}_F(\mathbb{C})$ as follows. Let $Q$ be in $\operatorname{Sym}_F(\mathbb{C})$. We denote by $Q_{<1>}$ the block-diagonal symmetric operator on $\mathbb{C}^{\hat{F}_{<1>}}$ defined by

$$\hat{Q}_{<1>} = Q \oplus \cdots \oplus Q,$$

on the decomposition $\mathbb{C}^{\hat{F}_{<1>}} = \mathbb{C}^{\hat{F}_{<1>}_{<1>}} \oplus \cdots \oplus \mathbb{C}^{\hat{F}_{<1>}_{<N>}}$. We denote by $Q_{<1>}$ the symmetric operator defined on $\mathbb{C}^{F_{<1>}}$ by the gluing operation described in section 1.2. Then we take the trace operator $(Q_{<1>})|_{\partial F_{<1>}}$, which is a symmetric operator on $\mathbb{C}^{\partial F_{<1>}} \sim \mathbb{C}^{F_{<1>}}$.

We denote by $T : \operatorname{Sym}_F(\mathbb{C}) \rightarrow \operatorname{Sym}_F(\mathbb{C})$ the map given by

$$T(Q) = (Q_{<1>})|_{\partial F_{<1>}}.$$

The coefficients of $T(Q)$ are rational functions of the coefficients of $Q$ and the poles are included in the set $\det(Q_{|F_{<1>} \cap \partial F_{<1>}}) = 0$. It is clear that the iterate $T^n$ has the following expression: let $Q_{<n>}$ be the block-diagonal symmetric operator on $\mathbb{C}^{\hat{F}_{<n>}}$ defined by

$$\hat{Q}_{<n>} = Q \oplus \cdots \oplus Q,$$

on the decomposition $\mathbb{C}^{\hat{F}_{<n>}} = \oplus_{i_1, \ldots, i_n} \mathbb{C}^{\hat{F}_{<n>_{<i_1, \ldots, i_n>}}}$ (cf. the notations of section 4.1), and $Q_{<n>}$ the element of $\operatorname{Sym}_{F_{<n>}}$ obtained by gluing from $\hat{Q}_{<n>}$ (considering that $F_{<n>}$ is a quotient of $\hat{F}_{<n>}$). Then we have $T^n(Q) = (Q_{<n>})|_{\partial F_{<n>}}$.

The map $T$ is the composition of three operations: the map $Q \mapsto \hat{Q}_{<1>}$, the gluing map $\hat{Q}_{<1>} \mapsto Q_{<1>}$, and the trace map $Q_{<1>} \mapsto (Q_{<1>})|_{\partial F_{<1>}}$. The last two operations correspond to symplectic reductions on the Lagrangian compactification. Since the composition of two symplectic reductions is a symplectic reduction we see that the extension of this map to the Lagrangian compactification must be represented by the composition of the “copies” map with a symplectic reduction.
Let us describe precisely this map on the Lagrangian compactification. We set $V_{\tilde{F}_{<1>}} = \mathbb{C}_{\tilde{F}_{<1>}} \oplus (\mathbb{C}_{\tilde{F}_{<1>}})^*$ and we have the decomposition

$$V_{\tilde{F}_{<1>}} = V_{F_{<1>},1} \oplus \cdots \oplus V_{F_{<1>},N},$$

with obvious notations. For any $L$ subspace of $V_{\tilde{F}_{<1>}}$ equal to $L_{<1>} = L \oplus \cdots \oplus L$. It is clear that the map $L \mapsto \tilde{L}_{<1>}$ extends the map $Q \mapsto \tilde{Q}_{<1>}$ to the Lagrangian compactifications $L_{F} \to \mathbb{L}_{\tilde{F}_{<1>}}$. Considering section 2.1 and 2.2, we see that the map $g : L_{F} \to L_{\partial F_{<1>}} \simeq L_{F}$ defined by

$$g(L) = \tilde{t}_{F_{<1>}} \circ \tilde{t}_{F_{<1>}} \circ (L \mapsto \tilde{L}_{<1>}),$$

extends the map $T$ to the Lagrangian compactification.

**Remark 4.1.** Formally, the map $g$ is defined as the rational map obtained as a composition of rational maps, as defined at the beginning of section 3.1. Remark that this composition is well-defined since the Siegel upper half-space extends the map $T$ defined by $g(L) = \tilde{t}_{F_{<1>}} \circ \tilde{t}_{F_{<1>}} \circ (L \mapsto \tilde{L}_{<1>})$.

From section 1.3, we know that the composition $t_{F_{<1>}} \circ \partial F_{<1>} \circ t_{F_{<1>}}$ can be expressed directly as a symplectic reduction. Let us denote by $s : (\mathbb{C}_{\tilde{F}_{<1>}}) \to \mathbb{C}_{\tilde{F}_{<1>}}$ the canonical injection given by $s(f) = f \circ \pi$, and by $s^* : ((\mathbb{C}_{\tilde{F}_{<1>}})^*) \to ((\mathbb{C}_{\tilde{F}_{<1>}})^*)$ the dual linear map (i.e. the map given by $s^*(e_x^*) = e_{\pi(x)}^*$). We consider the subspace $W_{<1>} \subset V_{\tilde{F}_{<1>}}$ defined by

$$W = \text{Im}(s) \oplus (s^*)^{-1}((\mathbb{C}_{\partial F_{<1>}})^*) \simeq \mathbb{C}_{\tilde{F}_{<1>}} \oplus ((\mathbb{C}_{\partial F_{<1>}})^* \oplus \ker(s^*)) .$$

It is clear that the $\omega$-orthogonal subspace $W_{<1>}$ is equal to

$$W_{<1>} = s((\mathbb{C}_{\tilde{F}_{<1>}} \setminus \partial F_{<1>}) \oplus \ker(s^*) \simeq \mathbb{C}_{\tilde{F}_{<1>}} \setminus \partial F_{<1>}) \oplus \ker(s^*) .$$

Hence, $W_{<1>}$ is coisotropic and $W_{<1>}/W_{<1>}$ is isomorphic to the symplectic structure $V_{\partial F_{<1>}} \simeq V_{F}$. From formula (1.5), we know that $t_{F_{<1>}} \circ \partial F_{<1>} \circ t_{F_{<1>}} = t_{W_{<1>}}$, and thus that $\tilde{t}_{F_{<1>}} \circ \partial F_{<1>} \circ \tilde{t}_{F_{<1>}} = \tilde{t}_{W_{<1>}}$. Hence, $g$ has the following expression

$$g : L_{F} \to L_{F} \quad L \mapsto \tilde{t}_{W} \circ (L \mapsto \tilde{L}_{<1>}).$$

The iterates $g^n$ can be described in a similar fashion. we have the decomposition

$$V_{\tilde{F}_{<n>}} = \oplus_{i_1, \ldots, i_n} V_{\tilde{F}_{<n>, i_1, \ldots, i_n}}.$$ 

For $L$ in $L_{F}$ we define $\tilde{L}_{<n>}$ as the element of $V_{\tilde{F}_{<n>}}$ given by

$$\tilde{L}_{<n>} = L \oplus \cdots \oplus L .$$

Then it is easy to see that

$$g^n(L) = \tilde{t}_{F_{<n>}} \circ \partial F_{<n>} \circ \tilde{t}_{F_{<n>}} \circ \partial F_{<n>} \circ (L \mapsto \tilde{L}_{<n>}) .$$
As previously, we denote by \( s_{<n>} : \mathbb{C}^{F_{<n>}} \to \mathbb{C}^{F_{<n>}} \) the natural linear injection, and by \( s_{<n>}^* \) the dual linear operator. We set

\[
W_{<n>} = \text{Im}(s_{<n>}) \oplus (s_{<n>}^*)^{-1}\left(\left(\mathbb{C}^{\partial F_{<n>}}\right)^*\right)
\]

\[
\simeq \mathbb{C}^{F_{<n>}} \oplus \left(\left(\mathbb{C}^{\partial F_{<n>}}\right)^* \oplus \ker(s_{<n>}^*)\right).
\]

The subspace \( W_{<n>} \) is coisotropic and we have

\[
W_{<n>}^o = s(\mathbb{C}^{F_{<n>}}\setminus \partial F_{<n>}) \oplus \ker(s_{<n>}^*).
\]

As previously, we have

\[
g^n(L) = \tilde{t}_{W_{<n>}} \circ (L \mapsto \tilde{L}_{<n>}).
\]

The corresponding map on the Grassmann algebra

We recall that \( A_F \) is the Grassmann algebra associated with the set \( F \), described in section 1. The smooth manifold \( L_F \) is a projective variety and is embedded in \( \mathcal{P}(A_F) \), cf. section 1.4. The map \( g \) can be lifted to a homogeneous polynomial map on \( \pi^{-1}(L_F) \subset A_F \), that we describe now. If \( X \) is in \( A_F \) we denote by \( \tilde{X}_{<1>} \) the element of \( A_{\tilde{F}_{<1>}} \) defined by

\[
\tilde{X}_{<1>} = X_{<1>,1} \cdots X_{<1>,N},
\]

where \( X_{<1>,i} \) is the element of \( A_{\tilde{F}_{<1>,i}} \) corresponding to \( X \) in \( A_F \) (indeed, \( A_{\tilde{F}_{<1>,i}} \simeq A_F \)). Then we set

\[
R : A_F \to A_{\partial F_{<1>} \sim A_F}
\]

\[
X \mapsto \left(R_{F_{<1>} \to \partial F_{<1>}} \circ R_{\tilde{F}_{<1>} \to F_{<1>}}\right)(\tilde{X}_{<1>}),
\]

where \( R_{F_{<1>} \to \partial F_{<1>}} \) and \( R_{\tilde{F}_{<1>} \to F_{<1>}} \) are the maps defined in section 2.1 and 2.2 respectively. It is clear that the map \( X \mapsto \tilde{X}_{<1>} \) is a homogeneous polynomial map of degree \( N \) in the coefficients of \( X \) and that it lifts the map \( L \mapsto \tilde{L}_{<1>} \) to \( \pi^{-1}(L_F) \), i.e. that \( \pi(\tilde{X}_{<1>,i}) = L_{<1>} \). It is clear by section 2 and 3, that \( R_{F_{<1>} \to \partial F_{<1>}} \circ R_{\tilde{F}_{<1>} \to F_{<1>}} \) corresponds on \( A_F \) to the map \( R_{W_{<1>}} \) defined in section 3, and that it lifts the symplectic reduction \( \tilde{t}_{W_{<1>}} \) to \( \pi^{-1}(L_{\tilde{F}_{<1>}}) \).

Hence, we have shown the commutation of the following diagram

\[
\begin{array}{ccc}
\pi^{-1}(L_F) & \xrightarrow{R} & \pi^{-1}(L_F) \\
\downarrow{\pi} & & \downarrow{\pi} \\
L_F & \xrightarrow{g} & L_F
\end{array}
\]

on the set where these maps are well-defined. Let us finally remark that the map \( R \) is a homogeneous polynomial map of degree \( N \), since \( R_{F_{<n>} \to \partial F_{<n>}} \) and \( R_{\tilde{F}_{<n>} \to F_{<n>}} \) are linear. Similarly, we have \( R^n(X) = R_{F_{<n>} \to \partial F_{<n>}} \circ R_{\tilde{F}_{<n>} \to F_{<n>}}(\tilde{X}_{<n>}) \), where \( \tilde{X}_{<n>} \) is the product \( \tilde{X}_{<n>} = \prod_{i_1,\ldots,i_n} X_{<n>,i_1,\ldots,i_n} \) (with obvious notations). This formula is simple (and proved in detail in [23, proposition 3.1, ii]).

4.4. Group of symmetries. Most of the classical examples have a natural group of symmetries, that we denote \( G \). In these cases it is natural to restrict our analysis to \( G \)-invariant objects, i.e. \( G \)-invariant electrical networks, \( G \)-invariant measures and to consider the restriction of the map \( g \) to \( G \)-invariant Lagrangian
structure depends on the type and the multiplicities of the real representation of $L$ comes from the map on $F$, where the action of $L$ might be very different on the points of $G$. Suppose given a finite group $G$ together with an action of $G$ on $\{1, \ldots, N\} \times F$, i.e. that

$$(g \cdot i, g \cdot x)R(g \cdot j, g \cdot y) \iff (i, x)R(i, y).$$

This induces an action of $G$ on the quotient set $F_{<1>$, and we suppose that the subset $\partial F_{<1>$ is left invariant by the action of $G$ and that the identification between $F$ and $\partial F_{<1>}$ commutes with the action of $G$ on $F$ and $\partial F_{<1>}$.

This implies that the action of $G$ can be defined on all $F_{<n>$, by the action of $G$ on $\{1, \ldots, N\} \times F$ (cf. the example of the Sierpinski gasket).

We denote by $\text{Sym}^G_F(C)$, the subspace of $G$-invariant elements of $\text{Sym}_F(C)$, i.e. the space of symmetric matrices $Q$ which satisfies

$$Q_{g \cdot i, g \cdot j} = Q_{i, j}, \quad \forall g \in G.$$  

The space $\text{Sym}^G_F(C)$ is embedded in $L_F$, since $\text{Sym}_F(C)$ is embedded in $L_F$. We denote by $L^G_F$, the closure in $L_F$ of $\text{Sym}^G_F(C)$. As shown in [23], appendix E, $L^G_F$ is included in the subset of $G$-invariant Lagrangian subspaces of $F$, and is a smooth projective variety, which can be locally parametrized by $\text{Sym}^G_F(C)$. Obviously, the map $T$ sends $\text{Sym}^G_F(C)$ to itself, since the whole structure is $G$-invariant, and thus, $g : L_F \to L_F$ leaves the subvariety $L^G_F$ invariant. If we are only interested in $G$-invariant Schrödinger operators, then it is much better to consider only the map on the invariant subvariety $L^G_F$. In particular, some values like the asymptotic degree of the map might be very different on $L_F$ and on $L^G_F$, and the significant value comes from the map on $L^G_F$. In the following, we will always consider the restricted map $g : L^G_F \to L^G_F$.

**Remark 4.2.** This section is not necessary to understand the rest of the text (with the exception of section 7), and we advise a reader not familiar with the subject to skip it and forget any reference about the group $G$, upon a first reading.

**Remark 4.3.** Remark that $\mathbb{L}^G_F$ is not contained in the set of indeterminacy points of $g$ since $L^G_F \cap S_{+, V} \neq \emptyset$. Hence, the restriction of $g$ to $L^G_F$ is well-defined.

In [23], appendix E, we have described explicitly the structure of $L^G_F$. This structure depends on the type and the multiplicities of the real representation of $\mathbb{R}^F$. Indeed, the space $\mathbb{R}^F$ is the sum of $r$ distinct irreducible real representations $W_0, \ldots, W_r$, with multiplicities $n_0, \ldots, n_r$. In [23], we have proved that $L^G_F$ is isomorphic to the product

$$L_0 \times \cdots \times L_r,$$

where the $L_i$ are Grassmannians of one of the following three types: Lagrangian Grassmannian (as defined in section 1), Grassmannian of $k$-dimensional subspaces in $\mathbb{C}^{2k}$, or orthogonal Grassmannian (cf. [23]). The type of $L_i$ depends on the type of the representation $W_i$, and the dimension depends on the multiplicity $n_i$. The main point is that, independently of the type and the multiplicity of $W_i$, we have

$$\dim(H^1(L_i)) = 0, \quad \dim(H^2(L_i)) = 1,$$

where $H^k(L_i)$ is the $k$-th cohomology group of $L_i$. 


4.5. Properties of the map \( g \) and \( R \). The following properties are easy consequences of section 3.

**Proposition 4.4.**

i) The set of indeterminacy points of \( g \) does not intersect \( S_{+,V} \) and
\[
g(S_{+,V}) = S_{+,V}.
\]

ii) For any \( L_0 \) in \( \mathbb{L}_F^{G,\mathbb{R}} \)
\[
\text{ord}(R^n \circ s, L_0) = \dim((\widetilde{L}_0)_{<n>} \cap W^0_{<n>}),
\]
where \( s \) is a holomorphic section of the projection \( \pi \) on a neighborhood \( U \subset \mathbb{L}_F^{G} \) of \( L_0 \).

iii) Let \( L : \mathbb{C} \to \mathbb{L}_F^{G,\mathbb{R}} \) be holomorphic and such that
\[
L(\mathbb{R}) \subset \mathbb{L}_F^{G,\mathbb{R}}, \quad \text{and} \quad L(\{\lambda, \text{Im} \lambda > 0\}) \subset S_+.
\]

(4.5) At any \( \lambda_0 \) in \( \mathbb{C} \), we have
\[
\text{ord}(\lambda \mapsto R^n \circ L(\lambda), \lambda_0) = \dim((\widetilde{L}(\lambda_0))_{<n>} \cap W^0_{<n>}).
\]
Moreover, the map \( \lambda \mapsto t_{W_{<n>}}((\widetilde{L}(\lambda))_{<n>}) \) gives a holomorphic extension of \( \lambda \mapsto g^n(L(\lambda)) \) at any point where \( L(\lambda) \) intersects the set of indeterminacy points of \( g^n \).

**Remark 4.5.** By the theorem of resolution of singularities, since \( \mathbb{L}_F \) is compact, we know that \( g^n(L) = \widetilde{t}_{W_{<n>}}((\widetilde{L}(\lambda))_{<n>}) \) has a holomorphic extension to \( \mathbb{C} \). The point iii) says that at any point of indeterminacy of \( g^n \) this holomorphic extension is given by the symplectic reduction \( t_{W_{<n>}}((\widetilde{L}(\lambda))_{<n>}) \) (recall that the symplectic reduction is defined everywhere).

**Remark 4.6.** A priori, the order of vanishing on the left hand side of (4.5) and (4.6) is bigger than the right-hand side, from proposition 3.7. For real points and for the restriction to some specific curves, we see that there is actually equality. The type of holomorphic curves that appears in ii) is exactly the type of curve that we will encounter later on.

Proof: It is clear that if \( L \) is in \( S_{+,V} \), then \( \widetilde{L}_{<1>} \) is in \( S_{+,V<1>} \). This implies that i) and iii) are direct consequences of proposition 3.17 and lemma 3.18. ii) Let \( (v_1, \ldots, v_K) \) be a real orthonormal basis of \( L_0 \). The map \( \lambda \mapsto L(\lambda) = L^{(v_1, \ldots, v_K)} \text{Id} \) satisfies hypothesis of iii). Thus, the left hand side of (4.6) is smaller than the right-hand side. It is also bigger or equal by proposition 3.17.

**Remark 4.7.** It is trivial from the definition, that \( T \) is 1-homogeneous, i.e. that \( T(\alpha Q) = \alpha T(Q) \), for all \( \alpha \in \mathbb{C} \). This 1-homogeneity of \( T \) induces the following invariance property of \( g \)
\[
(4.7) \quad \tau_\alpha \circ g = g \circ \tau_\alpha,
\]
for any non-null complex number \( \alpha \) (N.B.: \( \tau_\alpha \) is defined in 2.3). This commutation property is no longer true for some natural generalizations of these models, cf. section 5.2.2.
4.6. Counting measures, density of states and Green current. The iterates of the map $g$ are used in [23] to describe the counting measures $\nu_{<n>}^\pm$ of the self-similar Schrödinger operator $H_{<n>}$ on $F_{<n>}$, and the counting measures $\nu_{<n>}^{ND}$ of the Neumann-Dirichlet eigenvalues. We don’t want to go to much into the details here; we just want to explain the ideas behind the main results of [23], and we try to insist on the new perspectives given by the interpretation in terms of symplectic reduction.

We will be interested in the holomorphic curve

$$\mathbb{C} \rightarrow L_F$$

$$\lambda \mapsto L_{g^2 + \lambda b}.$$

Let us first remark that this map satisfies the hypotheses of proposition (4.4), iii), since $\text{Im}(Q_\rho + \lambda I_b) = (\text{Im}\lambda) I_b$. We also introduce the map

$$\phi : \mathbb{C} \rightarrow A_F$$

$$\lambda \mapsto \exp(\eta(Q_\rho + \lambda I_b))\eta,$$

which lifts $L_{Q_\rho + \lambda I_b}$ to $A_F$, i.e.

$$\pi(\phi(\lambda)) = L_{Q_\rho + \lambda I_b}.$$

Let us now explain why the spectrum of $H_{<n>}$ is related to the iterates $g^n(L_{Q_\rho + \lambda I_b})$ and $R^n(\phi(\lambda))$. We first describe the Lagrangian subspace $g^n(L_{Q_\rho + \lambda I_b})$. We already know that $t_{F_{<n>}} \rightarrow F_{<n>}((L_{Q_\rho + \lambda I_b})_{<n>}) = L_{Q_{rho} + \lambda I_b}$. For $\lambda$ in $\mathbb{C}$, we consider the following equation on $F_{<n>}$

$$((Q_{rho} + \lambda I_b) f)|_{F_{<n>} \cap \partial F_{<n>}} = 0.$$

For $f$ solution of the previous equation, we define $I_f^{\rho_{<n>}, \lambda}$ as the element of $(\mathbb{C}F_{<n>})^*$ given by

$$I_f^{\rho_{<n>}, \lambda}(h) = (Q_{rho} + \lambda I_b) f, h >, \forall h \in \mathbb{R}F_{<n>}.$$  

Clearly, $I_f^{\rho_{<n>}, \lambda}$ is supported by $\partial F_{<n>}$, and thus, in $(\mathbb{C}^0F_{<n>})^*$. From formula [22], it is clear that

$$g^n(L_{Q_\rho + \lambda I_b}) = \{f|_{\partial F_{<n>}} + I_f^{\rho_{<n>}, \lambda}, \ f \text{ solution of (4.8)} \}.$$

Hence, we see that $g^n(L_{Q_\rho + \lambda I_b})$ contains information about the boundary values of the space of solutions of (1.3). In particular, we see that the number of Neumann-only eigenfunctions (i.e. Neumann eigenfunctions which are not N-D eigenfunctions), with eigenvalue $\lambda_0$ is given by

$$\nu_{<n>}^+(\{\lambda_0\}) - \nu_{<n>}^{ND}(\{\lambda_0\}) = \dim \left(g^n(L_{Q_\rho + \lambda_0 I_b}) \cap (\mathbb{C}F + 0)\right).$$

Similarly, we have

$$\nu_{<n>}^-(\{\lambda_0\}) - \nu_{<n>}^{ND}(\{\lambda_0\}) = \dim \left(g^n(L_{Q_\rho + \lambda_0 I_b}) \cap (0 + (\mathbb{C}F)^*)\right),$$

for the Dirichlet spectrum. On the other hand, the number of N-D eigenfunctions with eigenvalue $\lambda$ is obtained as the order of vanishing of $R^n \circ \phi$ at $\lambda$. Indeed, from proposition (4.4) we know that the order of vanishing of $R^n(\phi(\lambda))$ is equal to the dimension of

$$(L_{Q_\rho + \lambda I_b})_{<n>} \cap W_{<n>}.$$
But, considering that
\[(LQ_{p<\nu} + \lambda I_{b<\nu}) \cap W_{\partial F<\nu}^o = \{0\}\]
since \(t_{F<\nu} \to F_{<\nu}\) is smooth on \(\text{Sym}_{F<\nu}\), we know that
\[(LQ_{p<\nu} + \lambda I_{b<\nu}) \cap W_{\partial F<\nu}^o \simeq LQ_{p<\nu} + \lambda I_{b<\nu} \cap W_{\partial F<\nu}^o\]
(indeed \(t_{F<\nu} \to F_{<\nu}\) \(\Rightarrow \) \(LQ_{p<\nu} + \lambda I_{b<\nu}\)). We have
\[LQ_{p<\nu} + \lambda I_{b<\nu} \cap W_{\partial F<\nu}^o = \ker^{ND}(Q_{p<\nu} + \lambda I_{b<\nu}) \oplus 0,\]
where
\[\ker^{ND}(Q_{p<\nu} + \lambda I_{b<\nu}) = \{f \in \mathbb{R}^{F<\nu}, (Q_{p<\nu} + \lambda I_{b<\nu}) f = 0 \text{ and } f|_{\partial F<\nu} = 0\}\]
is the subspace generated by the N-D eigenfunctions of \(H_{<\nu}\) with eigenvalue \(\lambda\). Hence, we have
\[\text{ord}(\lambda \mapsto R^n(\phi(\lambda)), \lambda_0) = \nu^{ND}_{<\nu}(\{\lambda_0\}).\]
Hence, we see that the maps \(g^n\) and \(R^n\) contain a lot of information about the spectrum of \(H_{<\nu}\). Remark that the relation between \(R^n\) and the spectrum of \(H_{<\nu}\) can also be seen from the following simple formulae:
\[
\det(Q_{p<\nu} + \lambda I_{b<\nu}) = <R^n \circ \phi(\lambda), X^+>,
\]
\[
\det((Q_{p<\nu} + \lambda I_{b<\nu})|_{F<\nu} \setminus \partial F_{<\nu}) = <R^n \circ \phi(\lambda), X^->,
\]
where
\[X^+ = \Pi_{x \in F} \eta_x, \quad X^- = 1.\]
(These formulas are direct consequences of the definition of \(R^n\)). Hence, we see that \(\nu^+(\{\lambda_0\})\) (resp. \(\nu^-(\{\lambda_0\})\)) corresponds to the order of vanishing of \(\lambda \mapsto <R^n \circ \phi(\lambda), X^+>\) (resp. \(\lambda \mapsto <R^n \circ \phi(\lambda), X^->\)) at \(\lambda_0\). In particular we have the following expression of \(\nu^{\pm}_{<\nu}\):
\[\nu^{\pm}_{<\nu} = \frac{1}{2\pi} \Delta \ln |<R^n \circ \phi(\lambda), X^\pm>|,\]
where \(\Delta\) is the Laplacian, in the sense of distributions (remark that the function \(\ln |<R^n \circ \phi(\lambda), X^\pm>|\) is subharmonic). To understand the asymptotics of \(\nu^{\pm}_{<\nu}\), we have to understand the asymptotics of the subharmonic functions
\[\frac{1}{N^n} \ln |<R^n \circ \phi(\lambda), X^\pm>|.\]
This is related to the asymptotics of the Green function and the Green current of \(R\). We do not want to enter too much into these details here, and we just sum-up the main results of [23]. In particular, we refer to [27], [8], or to the appendix of [23], for the definitions related to pluricomplex analysis, rational dynamics.

We set
\[G_n : \pi^{-1}(\mathbb{L}_F^\nu) \to \mathbb{R} \cup \{-\infty\},\]
\[X \mapsto \ln \|R^n(X)||,\]
\((G_n)\) can be defined on all \(\mathcal{A}_F\), but we are only interested in its restriction to \(\pi^{-1}(\mathbb{L}_F^\nu))\). The functions \(G_n\) are plurisubharmonic (which roughly mean that the restriction to any holomorphic curve is subharmonic, cf. appendix of [23]). The first main result of [23] is the following convergence result
\[\lim_{n \to \infty} \frac{1}{N^n} \ln |<R^n \circ \phi(\lambda), X^\pm>| = G \circ \phi(\lambda),\]
where the convergence is in the sense of $L^1_{\text{loc}}(\mathbb{C})$. This implies the following formula for $\mu$.

$$\mu = \lim_{n \to \infty} \frac{1}{N^n} \nu^+_n = \frac{1}{2\pi} \Delta G \circ \phi.$$ 

The interest of this formula lies in the fact that $G$ contains a lot of information about the dynamics of the map $g$ (cf. for example [27]).

**Remark 4.8.** This formula can be seen as a generalization of the classical Thouless formula combined with an explicit expression for the Lyapounov exponent in terms of the Green function of the map $R$: indeed, in the example of the self-similar Sturm-Liouville operator on $\mathbb{R}$, the function $G \circ \phi$ coincides with the value of the Lyapounov exponent $\zeta(\lambda)$ of the propagator of the underlying ODE. Thus, this formula has two components: first $\zeta(\lambda) = G \circ \phi(\lambda)$ and $\mu = \frac{1}{2\pi} \Delta \zeta$. This last equality is exactly the Thouless formula.

In terms of currents, this result has the following meaning. We denote by $S_n$ the closed positive (1,1)-current with potential $G_n$, i.e. the current given locally by $(S_n)_U = dd^c G_n \circ s$ for any local holomorphic section $s$ of $\pi$ on $U \subset L^F$ (cf. for example [23], section A.3). The current $S_0$ is the restriction to $L^F$ of the Fubini-Study form on $\mathcal{P}(\mathcal{A}_F)$, and hence is a Kähler form on $L^F$. We define similarly $S$, the current with potential $G$.

We define the hypervarieties $C^\pm$ by

$$C^+ = \{ L \in L^F, \ L \cap (\mathbb{C}^F \oplus 0) \neq \{0\}, \}$$

$$C^- = \{ L \in L^F, \ L \cap (0 \oplus (\mathbb{C}^F)^*) \neq \{0\}, \}.$$

We denote by $S^+_n$ the closed positive (1,1) current on $L^F$ with potential $X \mapsto \ln | < R^n(X), X^\pm > |$ on $\pi^{-1}(L^F)$. It is clear that $S^+_n$ are supported by $C^\pm$. (Indeed, if $s$ is any local holomorphic section of $\pi$ on $U \subset L^F$, then $C^\pm \cap U = \{ L \in U, < s(L), X^\pm >= 0 \}.$)

**Remark 4.9.** If the action of $G$ is trivial, i.e. if $L^F = L_0$, then it is not difficult to check that $S^+_n$ is exactly the current of integration on the hypersurface $C^\pm$. This may be wrong if $G$ is not trivial, cf. examples.

The formula for $\nu^+_n$, (4.9), can be rephrased as follows

$$\nu^+_n = (\pi \circ \phi)^*(S^+_n).$$

N.B: $(\pi \circ \phi)^*(S^+_n)$ is the pull-back of the current $S^+_n$ by $\pi \circ \phi$, defined using the local potential, cf. for example [23], A.6.

Similarly, the formula for $\mu$ can be translated in terms of current by

$$\mu = (\pi \circ \phi)^* S.$$

**Remark 4.10.** This formula says that the density of states is equal to the section of the current $S$ by the curve $\pi \circ \phi(\lambda)$.

**More general boundary conditions.** We can generalize the previous results to more general boundary conditions. Let $B$ be the complexification of a real Lagrangian subspace of $V$. The Lagrangian subspace $B$ plays the role of boundary condition: $B = \mathbb{C}^F \oplus 0$ for Neumann boundary conditions, and $B = 0 \oplus (\mathbb{C}^F)^*$ for Dirichlet boundary conditions. We denote by $C^B$ the hypervariety $C^B = \{ L \in L^F, \ L \cap B \neq \}$
Let $B^\perp = JB$, be the orthogonal (Lagrangian) subspace of $B$, and $X_B$ a point in $A_F \setminus \{0\}$ such that $\pi(X_B) = B^\perp$. We denote by $S_n^B$ the closed positive $(1,1)$ current on $\mathcal{L}_F^G$ with potential

$$\ln | < R^n(X), X_B > |,$$

on $\pi^{-1}(\mathcal{L}_F^G)$. Clearly, $S_n^B$ is supported by $C^B$. In [23], appendix D, remark A.7, we have proved the convergence of $(\pi \circ \phi)^* (S_n^{B_\infty})$ to the density of states $\mu = (\pi \circ \phi)^* (S)$, for any sequence of boundary conditions $B_n$. (Actually, this result was proved only for boundary conditions of the type $B_n = L_Q$, for a sequence $Q_n$ in $\text{Sym}_F(\mathbb{R})$, but it is not difficult to adapt the proof to a general sequence $B_n$.)

**Remark 4.11.** When $B = L_Q$, for a real symmetric operator $Q$, the measure $(\pi \circ \phi)^* (S_n^{B_\infty})$ is equal to the counting measure $\nu^Q_{\lambda_{n>}}$, of the spectrum of $(Q_{\lambda_{n>}}, b_{\lambda_{n>}})$ with boundary condition $Q$, i.e. $\nu^Q_{\lambda_{n>}}$ is the counting measure of the eigenvalues of the operator associated with the quadratic form $<Q_{\lambda_{n>}} + Q>, \cdot >$ and the measure $b_{\lambda_{n>}}$, as in section 4.2.

For any $L$ in $\mathcal{L}_F^G$ we denote by $\rho_n(L)$ the order of vanishing of $R^n \circ s$ at $L$, for any local holomorphic section $s$ of the projection $\pi$ on an open subset $U \subset \mathcal{L}_F^G$ containing $L$. We set

$$\rho_\infty(L) = \lim_{n \to \infty} \frac{1}{N^n} \rho_n(L).$$

**Remark 4.12.** Remark that if $G'$ is a subgroup of $G$, then $\mathcal{L}_F^{G'} \subset \mathcal{L}_F^G$ and for any real Lagrangian subspace $L \in \mathcal{L}_F^{G'}$ the value of $\rho_n(L)$ is the same when we consider the map on $\mathcal{L}_F^G$ or $\mathcal{L}_F^{G'}$, since it is equal to $\dim(\mathcal{L}_{G'} \cap \mathcal{L}_{G'}^{\perp})$ by proposition 4.14 ii).

It is clear by proposition 4.13 ii), that

$$\nu^N_{\lambda_{n>}} (\{\lambda\}) = \rho_n(L),$$

and that

$$\mu^N = \sum_{\lambda \in \mathbb{C}} \rho_\infty(\pi \circ \phi(\lambda)) \delta_\lambda,$$

where $\delta_\lambda$ is the Dirac mass at the point $\lambda$. (N.B.: remark that the sum on the right is finite except for a denumerable set of reals $\lambda$.)

**Remark 4.13.** Since $\pi \circ \phi$ satisfies the hypothesis of proposition 4.13 iii), $\rho_n(\pi \circ \phi(\lambda))$ is also the order of vanishing of $R^n \circ \phi(\lambda)$ at the point $\lambda$. Hence, $\nu^N_{\lambda_{n>}} (\{\lambda\})$ is equal to the Lelong number of $(\pi \circ \phi)^* (S_n)$ at $\lambda$ (and also of $S_n$ at $\pi \circ \phi(\lambda)$). Yet, it is not clear wether this equality can be pushed to the limit, i.e. whether $\rho_\infty(\pi \circ \phi(\lambda))$ is the Lelong number of $(\pi \circ \phi)^* (S)$ at $\lambda$. This would imply that $\mu^N - \mu^N$ has no atom. (Relations between the Lelong number of the Green current and multiplicity of the indeterminacy points of the map have been established in [9], but they are not sufficient to get this result.) Actually, this last equality is proved in [17], using completely different arguments, based on the renewal theorem.
4.7. Asymptotic degree, and the dichotomy theorem. The relations between the maps $g^n$ and $R^n$ is quite subtle and the problem of the asymptotic degree and the dichotomy theorem are essentially related to this. The main idea is that $g^n$ does not always contain all the useful information about $R^n$: when $R^n$ vanishes on a full projective hypervariety of $\pi^{-1}(L^G_F)$, a term can be locally factorized in $R^n$ and this hypervariety of annulation does not appear in $g^n$. This is responsible for the phenomenon of decrease of degrees (which is easier to understand first in the case of rational maps on the projective space, cf. \cite{27}, or appendix B of \cite{23}). The dichotomy theorem gives the consequences for the spectrum of the operators $H_{<n>}$ of the fact that such a factorization appears or not.

We set 

$$\hat{I}_{<n>} = \{ L \in \mathbb{L}_F^G, \tilde{L}_{<n>} \cap W_{<n>}^G \neq \{0\} \}.$$ 

The analytic set $\pi^{-1}(\hat{I}_{<n>})$ is also the set of zeros of $R^n$. The set $I_n$ of indeterminacy points of $g^n$ corresponds to the analytic subset of zeros of $R^n$ of codimension bigger than 1 (cf. \cite{23}, part 4, for more details). Let $\{D_{n,1}, \ldots, D_{n,k_n}\}$ be the set of irreducible components of codimension 1 of $\hat{I}_{<n>}$ (which maybe empty) and $c_{n,j}$ be the generic order of vanishing of $R^n$ on $D_{n,j}$. We denote by $D_n$ the divisor 

$$D_n = \sum_{j=1}^{k_n} c_{n,j}D_{n,j},$$

and by $[D_n]$ the current of integration on the divisor $D_n$, i.e. $[D_n] = \sum_j c_{n,j}[D_{n,j}]$. We proved in \cite{23}, part 4, (but it is almost immediate) that 

$$S_n = (g^n)^*(S_0) + [D_n].$$

In \cite{23}, we proved, for the class of self-similar structures described there

**Theorem 4.14.** i) If $D_{n_0} \neq 0$ for a $n_0 > 0$, then

$$S = \lim_{n \to \infty} \frac{1}{N^n}[D_n].$$

The current $S$ is a countable sum of currents of integration on hypervarieties, and $\mu^{ND} = \mu$ for all choices of $G$-invariant $\rho$ and $b$ on $F$.

ii) If $D_n = 0$ for all $n$, then

$$S = \lim_{n \to \infty} \frac{1}{N^n}(g^n)^*(S)$$

is the Green current of $g$. In particular, $S$ is null on the Fatou set of $g$. Moreover, for a generic choice of $G$-invariant $\rho$ and $b$, we have $\mu^{ND} = 0$.

**Remark 4.15.** This equality $\mu^{ND} = \mu$ has strong implications on the nature of the spectrum of the operator defined on some infinite lattices constructed from $F_{<n>}$ (cf. \cite{24}, and section 5).

**Remark 4.16.** The class of self-similar structures considered in \cite{23} is a bit more restrictive. But, it is clear from the proof of proposition 4.2 that this theorem still holds in our setting when the group of symmetries is trivial, i.e. when $L^G_F = L^F_F$. When the group $G$ is not trivial, the result is based on lemma 4.2 of \cite{23}, which depends on the particular self-similar structure we defined there.
Asymptotic degree. When $G$ is trivial, then the $(1,1)$ Dolbeault cohomology group of $L_G^F = L^F$ is of dimension 1 and the degree of $g^n$ is defined as the integer $d_n$ such that

$$(g^n)^*\{S_0\} = d_n\{S_0\},$$

where $\{S_0\}$ is the cohomology class of the current $S_0$. Since $\{S_n\} = N^n\{S_0\}$, we have $d_n = N^n$ if $D_n = 0$, and $d_n < N^n$ if $D_n \neq 0$. The asymptotic degree is defined by

$$d_\infty = \lim_{n \to \infty} (d_n)^{1/n}.$$  

N.B. The sequence $d_n$ is submultiplicative, cf. [23]. In [23], there is a mistake and $d_\infty$ is wrongly defined by $\lim \frac{1}{n} \ln d_n$ (which is obviously equal to $\ln d_\infty$).

The case i) of the previous theorem holds when $d_\infty < N$, and the case ii) holds when $d_\infty = N$.

When $G$ is not trivial, then the $(1,1)$ Dolbeault cohomology group of $L_G^F$ is of dimension $r + 1$ ($r + 1$ is the number of irreducible representations of $G$ contained in $\mathbb{R}^F$). The degree of the map $g^n$ can be represented by a matrix with positive integral coefficients. But the asymptotic degree can be defined anyway, and the same result holds, cf. [23], section 4.3.

Remark 4.17. This asymptotic degree is also the asymptotic degree of some rational maps, birationally equivalent to $g$, defined on $\mathbb{P}^k$, cf. section 4.5 of [23]. These maps on $\mathbb{P}^n$ are useful for practical reasons, since they are sometimes easier to compute.

4.8. Spectral analysis of continuous self-similar Laplace operators.

In general, the finite self-similar structures defined in section 4.1, come from a finitely ramified self-similar set $X$, cf. section 5.1. Under some conditions, cf. [16], [20], there exists a (unique) natural self-similar Dirichlet form $a$ and self-similar measure $m$ on $X$. The previous results can be generalized to the spectrum of the infinitesimal generator associated with $(a, m)$. In this case, the function $\phi$ defined in section 4.6 must be replaced by the meromorphic function $\phi : \mathbb{C} \to \mathcal{A}_F$ given by

$$\phi(\lambda) = \exp \int \eta A(\lambda) \eta,$$

where $A(\lambda)$ is defined as the trace of the Dirichlet form $a(f,g) + \lambda \int_X fg dm$ on $X$, cf. [23], section 3.3.

The crucial but simple fact we want to emphasize here is that $\pi \circ \phi : \mathbb{C} \to L_G^F$ is a holomorphic function, which again satisfies the hypothesis of proposition 4.4 iii).

5. Post-critically finite self-similar sets and generalizations

5.1. Post critically finite self-similar sets. We briefly recall the definition of p.c.f self-similar sets of Kigami (cf. [16]) to explain where the finite self-similar structures defined in section 4 appear.

A self-similar set is a compact metric set $X$, together with a family $(\psi_1, \ldots, \psi_N)$ of $N$ $\kappa$-Lipschitz functions from $X$ to $X$, for a $\kappa < 1$, and such that

$$X = \bigcup_{i=1}^N \psi_i(X).$$

It is clear, and well-known, that for any sequence $(i_k)_{k=1}^\infty \in \{1, \ldots, N\}^\mathbb{N}$, the limit

$$\lim_{n \to \infty} \Psi_{i_1} \circ \cdots \circ \Psi_{i_n}(x),$$

exists and is independent of the choice of the sequence $(i_k)$.
converges and that the limit does not depend on the particular choice of the point \( x \).

We denote by \( \pi((i_k)) \) this limit. Thus we have the following commutative diagram

\[
\begin{array}{ccc}
\{1, \ldots, N\}^N & \xrightarrow{\tau_i} & \{1, \ldots, N\}^N \\
\downarrow \pi & & \downarrow \pi \\
X & \xrightarrow{\Psi_i} & X
\end{array}
\]

where \( \tau_i \) is the map of \( \{1, \ldots, N\}^N \) given by

\[
\tau_i((i_1, \ldots, i_n, \ldots)) = (i, i_1, \ldots, i_n, \ldots).
\]

We call critical set the set \( C \subset \{1, \ldots, N\}^N \) given by

\[
C = \bigcup_{i \neq j} \pi^{-1}(\Psi_i(X) \cap \Psi_j(X)),
\]

and post-critical set the set

\[
P = \bigcup_{n>0} \sigma^n(C),
\]

where \( \sigma \) is the shift map \( \sigma((i_1, i_2, \ldots)) = (i_2, \ldots) \). The set \( X \) is called post-critically finite if \( P \) is finite. In this case we set

\[
F = \pi(P),
\]

and it is clear that \( F \) satisfies

\[
\Psi_i(X) \cap \Psi_j(X) = \Psi_i(F) \cap \Psi_j(F), \quad \forall i \neq j,
\]

\[
F \subset \bigcup_{i=1}^N \Psi_i(F).
\]

This naturally induces a finite self-similar structure if we define \( R \) as the equivalence relation on \( \{1, \ldots, N\} \times F \) given by

\[
(i, x)R(j, y) \text{ if and only if } \Psi_i(x) = \Psi_j(y).
\]

The set \( \bigcup_{i=1}^N \Psi_i(F) \) can be clearly identified with \( F_{\geq 1} = \{1, \ldots, N\} \times F/R \), and contains \( F \) as a subset. This gives a natural identification of a subset of \( \partial F_{\geq 1} \subset F_{\geq 1} \) with \( F \).

We set

\[
F^{(n)} = \bigcup_{j_1, \ldots, j_n=1}^N \Psi_{j_1} \circ \cdots \circ \Psi_{j_n}(F),
\]

which can clearly be identified with \( F_{\geq n} \), as defined in section 4.1. The sequence \( F^{(n)} \) is an approximating sequence of \( X \).

**Blow-up of the structure.**

To simplify the notations, we suppose here that \( X \subset \mathbb{R}^d \) and that \( \Psi_1, \ldots, \Psi_N \) are defined on all of \( \mathbb{R}^d \), injective and \( \kappa \)-Lipschitz (but its is not necessary, cf. [23]).

We fix an element \( \omega \) of \( \{1, \ldots, N\}^N \), called the blow-up. For \( n \in \mathbb{N} \) we set

\[
X_{\leq n}(\omega) = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(X), \quad \partial X_{\leq n}(\omega) = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(F),
\]

and

\[
F_{\leq n}(\omega) = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(F^{(n)}), \quad \partial F_{\leq n}(\omega) = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(F).
\]

Clearly, the sequences \( X_{\leq n}(\omega) \) and \( F_{\leq n}(\omega) \) are increasing and we set

\[
X_{\leq \infty}(\omega) = \bigcup_{n \in \mathbb{N}} X_{\leq n}(\omega), \quad F_{\leq \infty}(\omega) = \bigcup_{n \in \mathbb{N}} F_{\leq n}(\omega),
\]

and \( \partial X_{\leq \infty}(\omega) = \partial F_{\leq \infty}(\omega) = \bigcap_{n \in \mathbb{N}} \partial F_{\leq n}(\omega) \). Let us remark that the structures at finite level \( F_{\leq n}(\omega) \) (and \( X_{\leq n}(\omega) \)) are "isomorphic" for different \( \omega \), but the
unbounded lattice $F_{<\infty>}(\omega)$ (and the unbounded set $X_{<\infty>}(\omega)$) are not in general (cf. [24]).

The self-similar Schrödinger operators defined from $(Q_{\rho_{<\infty>}}, b_{<\infty>})$ can be naturally extended to the unbounded lattice $F_{<\infty>}(\omega)$ (cf. [24]). The measures $\mu$ and $\mu^{\cup D}$ play an important role in relation with the spectral properties of these operators. In particular, we have proved in [24] that $\text{supp} \mu$ is the spectrum of the operator for almost all blow-up $\omega$ (actually, this is true for any $\omega$ such that $\partial F_{<\infty>}(\omega) = \partial X_{<\infty>}(\omega) = \emptyset$), and that the equality $\mu = \mu^{\cup D}$ implies that for almost all $\omega$, the spectrum on $F_{<\infty>}(\omega)$ is pure point with compactly supported eigenfunctions. In [25], we have given some examples where the role of the blow-up sequence $\omega$ is crucial with respect to the spectral properties of the operator.

The map $g$ we defined in section 4 plays also an important role in the construction of a self-similar Dirichlet form on $X$. Indeed, the subset $D^g$, cf. section 1.1, is left invariant by $g$ and the existence of a non-degenerate self-similar Dirichlet form on $X$ (cf. [20] for the appropriate definitions) is related to the existence of a fixed point of $g$ in the set of irreducible elements of $D^g$.

5.2. Generalization to weighted self-similar operators and weak connections.

5.2.1. Weighted self-similar operators. Let $(\alpha_1, \ldots, \alpha_N)$ and $(b_1, \ldots, b_n)$ be two $N$-tuples of positive reals. We could generalize the previous setting by considering weighted electrical networks on $F_{<\infty>}$, obtained by defining $\tilde{\rho}_{<\infty>}$ by

$$ (\tilde{\rho}_{<\infty>})_{|F_{<\infty>}, i_1, \ldots, i_n} = (\alpha_{i_1} \cdots \alpha_{i_n})^{-1} \rho, $$

instead of formula [24]. Similarly, $\tilde{b}_{<\infty>$} is defined on $\tilde{F}_{<\infty>}$ as the sum of the measures $(b_{i_1} \cdots b_{i_n})b$ on $F_{<\infty>, i_1, \ldots, i_n}$, and then $b_{<\infty>}$ is the image of $\tilde{b}_{<\infty>}$ by the projection $\tilde{F}_{<\infty>} \rightarrow F_{<\infty>}$. Then, we make the following hypothesis.

(H) The values $\gamma_i = (\alpha_{i} b_{i})^{-1}$ does not depend on $i$. We denote by $\gamma_i$ the common value of the $\gamma_i$.

Under this hypothesis, the Schrödinger operator $H_{<\infty>, \rho, b}$ associated with $(\rho_{<\infty>}, b_{<\infty>})$ is “locally invariant by translation” (cf. [22]).

5.2.2. Weak connections. In the definition of section 4.1, the set $F_{<\infty>}$ is defined as a quotient $\tilde{F}_{<\infty>} = \{1, \ldots, N\}^n \times F$. From the electrical point of view, this means that we add an infinite conductance between the points of $\tilde{F}_{<\infty>}$ which are connected in $F_{<\infty>}$. We could instead, connect different points in $\tilde{F}_{<\infty>}$ by positive, but finite, conductance, or mix the two types of connections. This generalization seems to be useful for applications, in particular, it seems that Schreier graphs of certain automatic groups belong to this setting (cf. [11, 14]).

Formally, this means that we fix a certain electrical network (eventually dissipative) $\tilde{\rho}_1$ on $\tilde{F}_{<1>}$, and an equivalence relation $\mathcal{R}$ on $\tilde{F}_{<1>}$. We define $\tilde{\rho}_{<1>}$ as previously, and then $\rho_{<1>}$ is defined as the electrical network on $F_{<1>}$ obtained by gluing from $\tilde{\rho}_{<1>} + \tilde{\rho}_1$.

Remark 5.1. When we introduce weak connection, we loose the homogeneity property. Indeed, it is no longer true that if we change $\rho$ into $\lambda \rho$, then $\rho_{<1>}$
is changed in $\lambda_{\rho_{<1>}}$. Similarly, for the renormalization map $g$, the invariance property is no longer valid.

### 5.2.3. Addition and scaling

To extend the construction of $g$ and $R$ to the case of weighted self-similar operators and to the case of weak connections, we have to consider two operations on electrical networks that we describe now.

Let $\alpha$ be a non-zero complex number. The linear operator $Q \mapsto \alpha Q$ on $\text{Sym}_F(\mathbb{C})$, can be continued to the compactification $L_F$ by the linear operator on $V$ given by blocks by

$$\tau_\alpha = \begin{pmatrix} \text{Id} & 0 \\ 0 & \alpha \text{Id} \end{pmatrix}.$$  

(Indeed, clearly $\omega(\tau_\alpha X, \tau_\alpha Y) = \alpha \omega(X, Y)$, and thus, for $\alpha \neq 0$, $\tau_\alpha$ acts on Lagrangian subspaces of $V$.) On the Grassmann algebra, $\tau_\alpha$ can be lifted by the linear map, that we also denote $\tau_\alpha$, defined on monomials by

$$\tau_\alpha(\pi_{i_1}(\eta_{j_1}) \cdots \pi_{i_k}(\eta_{j_k})) = \alpha^k \pi_{i_1}(\eta_{j_1}) \cdots \pi_{i_k}(\eta_{j_k}),$$

i.e. we have $\tau_\alpha(\pi(X)) = \pi(\tau_\alpha(X))$, for any $X$ in $\pi^{-1}(L_F)$.

In the case of weighted self-similar operators, we have to define $\hat{Q}_{<1>}$ as the block diagonal operator on $\hat{F}_{<1>}$ equal to $\alpha Q$ on the block $\{i\} \times F$. Hence, on the Lagrangian compactification it means that $\hat{L}_{<1>}$ is defined by

$$\hat{L}_{<1>} = \tau_{\alpha_1}(L) \oplus \cdots \oplus \tau_{\alpha_N}(L),$$

with obvious notations. The map $R$ is defined as in the usual case, except that $\hat{X}_{<1>}$ is defined by

$$\hat{X}_{<1>} = (\tau_{\alpha_1}(X))_{<1>,1} \cdot \cdots \cdot (\tau_{\alpha_N}(X))_{<1>,N},$$

where $(\tau_{\alpha_i}(X))_{<1>,i}$ is the copy of $\tau_{\alpha_i}(X)$ on $\mathcal{A}_{\hat{F}_{<1>,i}}$.

Let us now consider an element $Q_0$ of $\text{Sym}_F(\mathbb{C})$, and the linear operator $Q \mapsto Q + Q_0$ on $\text{Sym}_F(\mathbb{C})$. It is clear that this operator can be extended to $L_F$ by the symplectic transformation

$$\tau_{Q_0} = \begin{pmatrix} \text{Id} & 0 \\ Q_0 & \text{Id} \end{pmatrix}. $$

(Indeed, we have $\tau_{Q_0}(L_Q) = L_{Q+Q_0}$.) On $\pi^{-1}(L_F) \subset \mathcal{A}_F$, this operation is lifted by the multiplication

$$X \mapsto \exp(\eta_{Q_0}\eta)X.$$  

In the case of weak connexions, the definition of $g$ and $R$ must be modified as follows:

$$g(L) = \hat{t}_{F_{<1>}} \circ \partial F_{<1>} \circ \hat{t}_{\hat{F}_{<1>}} \circ F_{<1>} \circ \tau_{Q_0}(\hat{L}_{<1>}),$$

and

$$R(X) = R_{F_{<1>}} \circ \partial F_{<1>} \circ R_{\hat{F}_{<1>}} \circ F_{<1>} \circ (\exp(\eta_{Q_0}\eta)\hat{X}_{<1>}).$$

### 6. A class of rational maps on $L_K$

In view of the construction of the map $g$, it is natural to introduce the following class of rational maps, which contains all the previous examples and share the same basic properties. Unfortunately, we know nearly nothing about the dynamics of these maps.
We consider \( V = \mathbb{C}^K \oplus (\mathbb{C}^K)^* \), equipped with its canonical symplectic form \( \omega \), and \( N \in \mathbb{N} \), \( N > 1 \). We denote be \( \tilde{V}_{<n>} \) the direct sum of \( N^n \) copies of \( V \):
\[
\tilde{V}_{<n>} = \oplus_{i_1, \ldots, i_n=1}^N \tilde{V}_{<n>_{i_1, \ldots, i_n}}.
\]
We also denote by \( \omega \) the symplectic structure on \( \tilde{V}_{<n>} \) induced by \( \omega \) on \( V \).

Let us fix a real coisotropic subspace \( W \) of \( \tilde{V}_{<1>} \), with dimension \((N+1)K\), and denote as usual by \( W^\circ \) its \( \omega \)-orthogonal subspace. The space \( W/W^\circ \) has dimension \( 2K \), and \( \omega \) induces a symplectic structure on \( W/W^\circ \). We suppose given an isomorphism (of symplectic structure) between \( W/W^\circ \) and \( V \). Then we define the map \( g \) as the composition
\[
g = \tilde{t}_W \circ (L \mapsto \tilde{L}_{<1>}),
\]
where, as previously, \( \tilde{L}_{<1>} \) is the Lagrangian subspace of \( \tilde{V}_{<1>} \) equal to \( L \oplus \cdots \oplus L \) for the decomposition \((6.1)\), and \( \tilde{t}_W \) is the rational map defined by the closure of the graph of the symplectic reduction \( t_W \), as defined in section 3. The map \( g \) is rational from \( \mathbb{P}^V \) to \( \mathbb{P}_{W/W^\circ} \simeq \mathbb{P} \).

It is easy to check that the main properties of the map \( g \) described in proposition \[44\] remain valid for this class of maps. In particular, the subset \( S_{V, +} \) is invariant by \( g \), and hence is contained in the Fatou set of \( g \), since it is hyperbolic and hyperbolic embedded. The degree of \( g \) is smaller than \( N \), and equal to \( N \) if and only if the set \( \{L, L_{<1>} \cap W^\circ \neq \{0\} \} \) has codimension bigger than \( 1 \). As previously, the iterates \( g^n \) can also be defined as the composition of the map \( L \mapsto \tilde{L}_{<n>} \) and \( t_{W_{<n>}} \) for a certain subset \( W_{<n>} \subset \tilde{V}_{<n>} \).

It might be interesting to classify this class of maps, up to isomorphism, in the simplest, but yet non-trivial, case where \( K = 2 \), and \( N = 2 \). In this case the Lagrangian Grassmannian \( L_F \) is 3-dimensional, and isomorphic to the following subvariety of \( \mathbb{P}^5 \):
\[
\{ [Z, a, d, q, D] \in \mathbb{P}^5, \quad ad - q^2 = DZ \},
\]
cf. \[24\], section 5.2.

7. Practical implementation and new examples

It is not always easy to compute the maps \( g \) and \( R \) since the dimension of \( A_F \) and \( L_{\mathbb{C}} \) might be big, and since \( L_{\mathbb{C}} \) is not a projective space. The aim of this section is to describe more precisely than in \[24\] how to proceed, in practice in a very symmetric case. We suppose here that \( C^r \) has the following decomposition: \( C^r = W_0 \oplus \cdots \oplus W_r \), where \( W_0, \ldots, W_r \) are \( r + 1 \) distinct \( \mathbb{C} \)-irreducible representations of \( G \), realizable in \( \mathbb{R} \). (Hence, \( \mathbb{R}^F \) has the same decomposition \( \mathbb{R}^F = W_0 \oplus \cdots \oplus W_r \).) This is the case for example for nested fractals, \[21\]. By Schur’s lemma, \( \text{Sym}^2_F(\mathbb{C}) \simeq \mathbb{C}^{r+1} \) and any element \( Q \) can be written
\[
Q = u_0 p_{W_0} + \cdots + u_r p_{W_r},
\]
where \( (u_0, \ldots, u_r) \in \mathbb{C}^{r+1} \) and \( p_{W_i} \) is the Hermitian projection on \( W_i \). We denote by \( Q_0^{u_0, \ldots, u_r} \) the element of \( \text{Sym}^2_F(\mathbb{C}) \) of the previous form. The map \( T \) can be represented in coordinates \((u_0, \ldots, u_r)\); it is rational, and thus can be written under the form
\[
T ((u_0, \ldots, u_r)) = \left( \frac{\mathcal{T}_0}{Q_0}, \ldots, \frac{\mathcal{T}_r}{Q_r} \right),
\]
where \( P_i, Q_i \) are polynomials in the variables \((u_0, \ldots, u_r)\) (in the usual case of strong connections, i.e. except in the case of section 5.2.2, the fraction \( \frac{P_i}{Q_i} \) is homogeneous of degree 1 in the variables \((u_0, \ldots, u_r)\)).

**Remark 7.1.** The map \( T \) is in general not too difficult to compute, even by hand, since it is just a minimization, and since symmetry arguments can reduce the number of parameters.

It is easy to check that, in this case, the compactification is \( L_F^G \simeq P^1 \times \cdots \times P^1 \).

The compactification \( g \) of \( T \) on \( P^1 \times \cdots \times P^1 \) can be lifted by a polynomial map
\[
\tilde{R} : C^2 \times \cdots \times C^2 \to C^2 \times \cdots \times C^2,
\]
of the form
\[
\tilde{R}((u_0, v_0), \ldots, (u_r, v_r)) = ((P_0, Q_0), \ldots, (P_r, Q_r)),
\]
where \( P_i, Q_i \) are polynomials in the variables \((u_j, v_j)\) and homogeneous (with the same degree) in each of the couple \((u_j, v_j)\), and such that the following diagram is commutative
\[
\begin{array}{ccc}
C^2 \times \cdots \times C^2 & \xrightarrow{\tilde{R}} & C^2 \times \cdots \times C^2 \\
\downarrow \pi \times \cdots \pi & & \downarrow \pi \times \cdots \pi \\
P^1 \times \cdots \times P^1 & \xrightarrow{\tilde{g}} & P^1 \times \cdots \times P^1
\end{array}
\]
We denote by \( d_{1, i, j} \) the degree of homogeneity of the polynomials \((P_i, Q_i)\) in the variables \((u_j, v_j)\).

**Remark 7.2.** The polynomials \((P_0, Q_0)\) are obtained from the polynomials \((P_0, Q_0)\) simply by homogeneization (cf. examples, section 7.2).

**Remark 7.3.** These degrees \( d_{1, i, j} \) corresponds to the matrix of degrees of \( g \). If we denote by \( \nu_i \) the pull-back of the Fubini-Study form on \( P^1 \) by the projection on the \( i \)th-factor of \( P^1 \times \cdots \times P^1 \), then this means that the degrees \( (d_{1, i, j}) \) corresponds to the equation in cohomology
\[
g^*(\{\nu_i\}) = \sum_{j=0}^r d_{1, i, j} \{\nu_j\},
\]
where \( \{\nu_j\} \) is the cohomology class of \( \nu_j \) (the family \( \{\nu_0, \ldots, \nu_r\} \) is a basis of the \((1, 1)\) cohomology of \( L_F^G \), cf. [23]).

At this point we have just described the map \( g \), but this is not enough to describe the currents \( S^a_0 \) and \( S_a \), if there is a non trivial divisor \( D_1 \). Let us come back to \( \mathcal{A}_F \) now. We have the canonical projections \( \pi \times \cdots \times \pi : C^2 \times \cdots \times C^2 \to P^1 \times \cdots \times P^1 \) and \( \pi : \pi^{-1}(L_F^G) \to L_F^G \). We describe a map that lifts the isomorphism \( L_F^G \simeq P^1 \times \cdots \times P^1 \). We set \( p_i = \text{dim}(W_i) \), and we denote by \((f_{i}^1, \ldots, f_{i}^{p_i})\) a real orthonormal basis of \( W_i \). Each \( f_{i}^k \) can be written \( f_{i}^k = \sum_{x \in F} c_{x} e_x \) and we denote by \( \xi_{i}^k = \sum_{x \in F} c_x \eta_x \) and \( \bar{\xi}_{i}^k = \sum_{x \in F} c_x \bar{\eta}_x \) the corresponding vectors in the Grassmann algebra generated by \((\eta_x, \bar{\eta}_x)\). We denote by \( \hat{s} : C^2 \times \cdots \times C^2 \to \mathcal{A}_F \) the map given by
\[
\hat{s}((u_0, v_0), \ldots, (u_r, v_r)) = \prod_{k=0}^{r} \prod_{i=0}^{p_i-k} (v_i + u_l \xi_{i}^k) \bar{\xi}_{i}^k).
\]
It is clear with these notations that
\[ \hat{s}((u_0,1),\ldots,(u_r,1)) = \exp\pi Q^{u_0\cdots u_r} \eta, \]
and that \( \hat{s} \) takes its values in \( \pi^{-1}(L^2_F) \). The map \( \hat{s} \) is clearly polynomial and homogeneous in each couple of variables \((u_j,v_j)\) with degree \( p_j \), i.e.
\[ \hat{s}(\lambda_0(u_0,v_0),\ldots,\lambda_r(u_r,v_r)) = \left( \prod_{j=0}^r \lambda_j^{p_j} \right) \hat{s}((u_0,v_0),\ldots,(u_r,v_r)). \]

Since \( R \) is polynomial homogeneous of degree \( N \), then \( R^n \circ \hat{s} \) is polynomial homogeneous of degree \( N^np_j \) in \((u_j,v_j)\).

Remark 7.4. This means, in particular, that the cohomology class of \( S_0 \), the current with potential \( \ln ||X|| \) on \( \pi^{-1}(L^2_F) \), is \( \{S_0\} = \sum_{j=0}^r p_j \{\nu_j\} \), and that \( \{S_n\} = N^n \{S_0\} = \sum_{j=0}^r N^n d_j \{\nu_j\} \).

Since \( \tilde{R} \) and \( R \) induce the same map on \( \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \) then \( R \circ \hat{s} \) is of the form
\[ R \circ \hat{s} = H \hat{s} \circ \tilde{R}, \]
where \( H \) is a polynomial homogeneous in each couple \((u_j,v_j)\), and we denote by \( h_{1,j} \) the homogeneity degree of \( H \) in \((u_j,v_j)\). The divisor \( D_1 \) is then the divisor associated with the zeros of \( H \). Let us remark that, by homogeneity, these degrees must satisfy
\[ (7.1) \quad Np_i = (\sum_{j=0}^r d_{1,i,j}p_j) + h_{1,i}. \]

In practice, the strategy is first to compute \( T \), which is in general quite simple. This gives us the rational map \( \tilde{R} \) by homogenization. Then, to have all the information about \( R \), we must compute the polynomial \( H \), which gives the hypersurfaces of zeroes of \( R \). Either we can guess what are the hypersurfaces of \( L^2_F \) where \( \tilde{L}_{<1>} \cap W^* \neq \{0\} \), with multiplicity, and then check that we found all the factors of \( H \) thanks to formula \( (7.1) \) (in simple examples it is quite easy, using symmetry arguments). Or we can use formula \( (3.6) \), and compute \( \det(Q|_{F_{<1>}\setminus \partial F_{<1>}}) \) in coordinates \((u_0,\ldots,u_r)\).

Remark 7.5. Formula \( (7.1) \) corresponds to the following equation in cohomology: \( \{S_1\} = g^\ast\{(S_0)\} + \{[D_1]\} \).

Once \( \tilde{R} \) and \( H \) are computed, we have just to iterate \( \tilde{R} \). At this step, one just has to compute the iterates of \( \tilde{R} \). The iterates \( \tilde{R}^n \) can be written
\[ \tilde{R}^n = \left( \tilde{H}_{n,0} \times (P_{n,0},Q_{n,0}),\ldots,\tilde{H}_{n,r} \times (P_{n,r},Q_{n,r}) \right), \]
where \( H_{n,j} \) are polynomials, homogeneous in the variables \((u_j,v_j)\), and where each of the \((P_{n,i},Q_{n,i})\) are polynomials with no common factors, and homogeneous in each couple \((u_j,v_j)\) with the same degree of homogeneity that we denote \( d_{n,i,j} \). This matrix of degrees \( (d_{n,i,j}) \) corresponds to the degrees of \( g^n \). We set
\[ \tilde{R}_n = ((P_{n,0},Q_{n,0}),\ldots,(P_{n,r},Q_{n,r})). \]
Then clearly $R_n \circ \hat{s}$ can be written

$$R_n \circ \hat{s} = (\prod_{k=0}^{n-1} (H \circ \tilde{R}^k)N^{n-k}) \circ \tilde{R}^n$$

$$= (\prod_{k=0}^{n-1} (H \circ \tilde{R}^k)N^{n-k}) (\prod_{i=0}^{r} \tilde{R}_{n,i}^{p_i}) \circ \tilde{R}_n.$$ 

We denote by $H_n$ the polynomial in factor in the last expression. The divisor $D_n$ is the divisor of $H_n$.

Let us now describe explicitly the spectrum of the operators $H_{<n>}$. We denote by $u_0^\rho, \ldots, u_r^\rho$ the coordinates of the initial operator $Q_\rho$. The measure $b$ is necessarily a uniform measure on $F$ (indeed, $G$ acts transitively on the set $F$, since the trivial representation $W_0$ has multiplicity 1) hence up to a constant, $I_b = \text{Id}$. We have

$$\phi(\lambda) = \hat{s}((u_j^\rho + \lambda, 1)_{j=0,\ldots,r}),$$

and the Neumann spectrum of $H_{<n>}$ is equal to the zeros of the polynomial

$$\lambda \mapsto H_n(\prod_{i=0}^{r} ||(P_{n,i}, Q_{n,j})||^{p_i}) ((u_j^\rho + \lambda, 1)_{j=0,\ldots,r}),$$

counted with multiplicities. The Dirichlet spectrum corresponds to the zeros of the same polynomials where $P_{n,j}$ is replaced by $Q_{n,j}$. The Neumann-Dirichlet spectrum is obtained by considering the order of vanishing of

$$\lambda \mapsto \left| H_n \prod_{i=0}^{r} ||(P_{n,i}, Q_{n,j})||^{p_i} \right| ((u_j^\rho + \lambda, 1)_{j=0,\ldots,r}).$$

### 7.1. The Sierpinski gasket

In this case the connections are described in the following figure.

The group $G$ is the group $G \simeq S_3$ of permutations of $F$. The subspace $W_0 = \mathbb{C} \cdot 1$ of constant functions and its orthogonal complement $W_1$ are the 2 $\mathbb{C}$-irreducible representations of $G$ contained in $\mathbb{C}^F$, and they are realizable in $\mathbb{R}$. In coordinates $(u_0, u_1)$ we have

$$T(u_0, u_1) = 3 \left( \frac{u_0 u_1}{2 u_0 + u_1}, \frac{u_1 (u_0 + u_1)}{5 u_1 + u_0} \right),$$

and thus

$$\tilde{R}((u_0, v_0), (u_1, v_1)) = \left((3u_0 u_1, 2u_0 v_1 + u_1 v_0), (3u_1 (u_0 + u_1 v), 5u_1 v_1 v_1 + u_0 v_1^2)\right),$$
which means that the map $g$ is represented in homogeneous coordinates by

$$g ([u_0, v_0], [u_1, v_1]) = ([3u_0u_1, 2u_0v_1 + u_1v_0], [3u_1v_1 + u_1v_0], 5u_1v_0v_1 + u_0v_1^2]),$$

$([x, y]$ represents the point of $\mathbb{P}^1$, corresponding to $(x, y)$ in $\mathbb{C}^2$).

The matrix of degrees is

$$d_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The polynomial $H$ is, up to a constant,

$$H = v_1.$$

Indeed, the following vector of $V_{\tilde{F}_{<1>}} = \mathbb{C}^{\tilde{F}_{<1>}} \oplus (\mathbb{C}^*)^{\tilde{F}_{<1>}}$

is clearly in $(\tilde{L})_{<1>} \cap W^\circ$ if $L$ is a Lagrangian subspace in $\mathbb{L}_{\tilde{F}^F}$ corresponding to a point of the type $([u_0, v_0], [1, 0])$ in $\mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, for any $[u_0, v_0] \in \mathbb{P}^1$. Thus, $\dim(\tilde{L}_{<1>} \cap W^\circ)$ is generically at least 1, on the hypersurface $\{v_1 = 0\} = \mathbb{P}^1 \times [1, 0]$. Equation (7.1) tells us that we found all the factors of $H$.

Remark 7.6. In [23], we described explicitly the current $S$ using a 1-dimensional rational map.

7.2. An example coming from group theory. In [11], [14], Grigorchuk, Bartholdi and Zuk, considered several examples of fractal groups acting on rooted trees, and computed their spectrum using a renormalization equation involving a 2-dimensional rational map. It seems that in some particular cases, these computation can be performed in our context, using the generalization of section 5.2.2. We present here one example: the group $\Gamma$ of [11]. We don’t explain where this example comes from, but just how it can be described in our context.

The initial cell is 3 points, $F = \{1, 2, 3\}$. We fix some real constants $r$ and $v$. Then, we consider the following self-similar structure, with weak connections, as in section 5.2.2.
This means that in \( \tilde{F}_{<1>} = \tilde{F}_{<1>,1} \sqcup \tilde{F}_{<1>,2} \sqcup \tilde{F}_{<1>,3} \) we connect the points according to the previous figure, where the bolded liaisons labelled \( r \) represent points connected by a conductance \( r \). At each connecting point we put a dissipative term \( v \), as represented on the figure. This means that if we label the connecting points as on the following figure

\[
\begin{array}{c}
\circ \\
\downarrow \\
x \quad y
\end{array}
\]

\[
\begin{array}{c}
\circ \\
\downarrow \\
x' \quad y'
\end{array}
\]

\[
\begin{array}{c}
\circ \\
\downarrow \\
x'' \quad y''
\end{array}
\]

then \( Q_{<1>} = Q_{<1>,1} + Q_{<1>,2} + Q_{<1>,3} + Q_{\hat{\rho}_1} \), where \( Q_{\hat{\rho}_1} \) is the matrix associated with the electrical network \( \hat{\rho}_1 \) given by

\[
(\hat{\rho})_{x,x'} = (\hat{\rho}_1)_{x',x''} = \cdots = (\hat{\rho})_{y,y'} = \cdots = r, \quad (\hat{\rho})_{x} = (\hat{\rho}_1)_{x'} = \cdots = (\hat{\rho})_{y} = \cdots = v.
\]

**Remark 7.7.** The example of \( \mathbb{H} \) corresponds to the case \( r = -t, v = 2t \), for a real \( t \), so that the diagonal terms of \( Q_{\rho} \) are cancelled (and actually \( t = -1 \) in \( \mathbb{H} \)).

The circled points represent the points of \( \partial F_{<1>} \). It is clear that the structure is invariant by the group \( G \), the group of isometries leaving the triangle \( F \) invariant, \( G \sim D_3 \sim S_3 \). As for the Sierpinski gasket, we have \( \mathbb{R}^F = W_0 \oplus W_1 \), where \( W_0 \) is the space of constant functions and \( W_1 \) its orthogonal supplement. Any \( Q \) in \( \text{Sym}^G_F(\mathbb{C}) \) can be written

\[
Q = u_0 p_{|W_0} + u_1 p_{|W_1},
\]
where \((u_0, u_1) \in \mathbb{C}^2\) and \(p_{W_0}\) and \(p_{W_1}\) are the orthogonal projections on \(W_0\) and \(W_1\). A simple computation gives

\[
T((u_0, u_1)) = \left( \frac{3u_0u_1 + z_0u_0 + 2z_0u_1}{2u_0 + u_1 + 3z_0}, \frac{3u_0u_1 + z_1u_0 + 2z_1u_1}{2u_0 + u_1 + 3z_1} \right).
\]

with

\[z_0 = v, \quad z_1 = 3v + v.\]

**Remark 7.8.** In the case of \(\mathbb{P}_1\), we have \(z_0 = 2t, z_1 = -t\), which gives

\[
T((u_0, u_1)) = \left( \frac{3u_0u_1 + 2tu_0 + 4u_1}{2u_0 + u_1 + 6t}, \frac{3u_0u_1 - tu_0 - 2tu_1}{2u_0 + u_1 + 3t} \right).
\]

We could compute the map \(T\) in different coordinates. For example, one can represent any \(Q\) in Sym\(_2\)(\(\mathbb{C}\)) in the following form

\[Q = \lambda P - \mu Id,
\]

where \(P\) is the matrix null on the diagonal, and equal to 1 on any off-diagonal term. (NB: this means \(P_{i,i} = 0\) and \(P_{i,j} = 1, i \neq j\).) In these coordinates, \(T\) has the form

\[
T((\lambda, \mu)) = \left( \frac{2\lambda^2 t}{(\lambda - 2t - \mu)(\mu - t - \lambda)}, \frac{2\lambda^2}{(\lambda - \mu + t)(\lambda - 2t - \mu)} \right).
\]

We remark that this map, when \(t = -1\), is exactly the map which appears in the renormalization equation of lemma 4.14 of \(\mathbb{P}_1\). But this set of variables is not the best suited to the problem, as we shall see later.

From equation (7.2), we see that the polynomial map \(\tilde{R} : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2\) induced by \(T\) is given by

\[
\tilde{R}((u_0, v_0), (u_1, v_1)) = ((3u_0u_1 + z_0u_0v_1 + 2z_0u_1v_0, 2u_0v_1 + u_1v_0 + 3z_0v_0v_1),
\]

\[(3u_0u_1 + z_1u_0v_1 + 2z_1u_1v_0, 2u_0v_1 + z_1u_1v_0 + 3z_1v_0v_1)).
\]

Thus, the matrix of degrees is

\[
d_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

This means in particular that the asymptotic degree \(d_\infty\) is smaller than 2, and that we are in case ii) of theorem 4.14.

**Remark 7.9.** If we compactify in \(\mathbb{P}^2\) then we get a rational map of degree 3. This means that we cannot see that \(d_\infty < 3\) at this level on this compactification. This also means that there will be a decrease in the degree of the iterates, in this compactification, cf. section 4.5 of \(\mathbb{P}_3\). Remark also that if we consider the compactification in \(\mathbb{P}^1 \times \mathbb{P}^1\) for the coordinates \((\lambda, \mu)\) (which is a complete nonsense), then the degree of the map, i.e. the largest eigenvalue of the matrix of degrees, is bigger than 3.

**The divisor \(D_1\).**

It is not difficult to check that, up to a multiplicative constant, the polynomial \(H\) is equal to

\[H = (u_1 + z_0)(u_1 + z_1)^2.
\]

Indeed, for \(u_1 = -z_0\) and any \(u_0\), it is easy to check that the following function on \(F_{<1>}\) is in \(\ker^{ND}((Q_{u_0, u_1})_{<1>})\)
Furthermore, for $u_1 = -z_1$ and any $u_0$, the following function

and the function obtained by a rotation of $2\pi/3$ are in $\ker^{ND}((Q^{u_0,u_1})_{<1>})$. Hence, on the hypersurface $u_1 + z_0 v_1 = 0$, $\dim(\tilde{L}_{<1>} \cap W^o)$ is at least 1, and on $u_1 + z_1 v_1 = 0$ it is at least 2, i.e. we have $h_{1,1} \geq 3$, and equation \ref{eq:kernel} tells us that there is equality, thus that we found all the factors of $H$. This implies that the current $[D_1]$ is equal to

$$[D_1] = 2\{(u_1 + tv_1 = 0)\} + \{|u_1 - 2tv_1 = 0|\}.$$  

7.3. A semi-symmetric version of the previous example. We consider now the following example, with weak connection:
As previously, the links labelled $r$ or $r'$, represent conductances $r$ and $r'$, and the point labelled $v$ and $v'$, have dissipative terms $v$ and $v'$. We see that now the symmetry group is no longer $G \simeq S_3$, the group of isometries of the triangle (since the connecting network is not invariant by reflections) but the symmetry group $G' \simeq \mathbb{Z}/3\mathbb{Z}$ of rotations of the triangle. But $\text{Sym}_F^G(\mathbb{C}) = \text{Sym}_F^{G'}(\mathbb{C}) \simeq \mathbb{C}^2$, since obviously, any $G'$-invariant $Q$ is of the form $Q = u_0 p_{|W_0} + u_1 p_{|W_1}$.

A computation gives

$$T((u_0, u_1)) = \begin{pmatrix} 
\frac{3u_0u_1^2 + s_0u_1(2u_0 + u_1) + p_0(u_0 + 2u_1)}{2u_0u_1 + u_1^2 + 3p_0}, \\
\frac{3u_0u_1^2 + s_1u_1(2u_0 + u_1) + p_1(u_0 + 2u_1)}{2u_0u_1 + s_1(u_0 + 2u_1) + 3p_1} 
\end{pmatrix},$$

where

$$s_0 = z_0 + z'_0, \quad p_0 = z_0 z'_0, \quad s_1 = z_1 + z'_1, \quad p_1 = z_1 z'_1,$$

$$z_0 = v, \quad z'_0 = v', \quad z_1 = r + v, \quad z'_1 = r' + v'.$$

Indeed, the previous formula is obtained as follows. Consider the first component of $T$. We need to compute $T(Q_{u_0, u_1}')((1))$, where 1 is the constant function 1 on $F$. The harmonic continuation of 1 is, by symmetry, necessarily of the form

$$1 = \frac{a}{\bar{a}} = \frac{b}{\bar{b}} = \frac{\bar{a}}{a} = \frac{\bar{b}}{b},$$

where $d = 2u_0u_1 + u_1^2 + s_0(u_0 + 2u_1) + 3p_0$.

A simple computation gives

$$T((u_0, u_1)) = \begin{pmatrix} 
a = \frac{(u_1 - u_0)(u_1 + z_0)}{d}, \\
b = \frac{(u_1 - u_0)(u_1 + z_0)}{d} 
\end{pmatrix},$$

where $d = 2u_0u_1 + u_1^2 + s_0(u_0 + 2u_1) + 3p_0$.

To compute the second coordinates of $T$, we consider the function

$$j = e^{2i\pi/3}.$$
A simple computation gives the same formula for $a$ and $b$ as formula (7.5), if we replace $z_0, z'_0, s_0, p_0$ by $z_1, z'_1, s_1, p_1$.

Remark 7.10. When $r = r'$, and $v = v'$, then we are in the situation of the previous example, where the group of symmetries is $G \cong S_3$. In this case, there are simplifications in both terms of the formula for $T$. Precisely, the components of $T$ can be written

\[
\begin{align*}
&\frac{(u_1 + z_0)(3u_0u_1 + z_0u_0 + 2z_0u_1)}{(u_1 + z_0)(2u_0 + u_1 + 3z_0)}, \quad \frac{(u_1 + z_1)(3u_0u_1 + z_1u_0 + 2z_1u_1)}{(u_1 + z_1)(2u_0 + u_1 + 3z_1)}
\end{align*}
\]

(where we used that $s_0^2 = 4p_0$ and $s_1^2 = 4p_1$). We see that we recover the formulas of 7.2, and that Neumann-Dirichlet eigenfunctions come from the simplifications in these formulas (indeed, we can remark that the factors in these equations are exactly the factors which enter the polynomial $H$: the multiplicities corresponds to the factor $p_j$, which corresponds to the dimension of the representation $W_j$, cf. section 7). This is exactly what is predicted by the general theory.

We see that the associated homogeneous polynomial on $\mathbb{C}^2 \times \mathbb{C}^2$ is

\[
R((u_0, v_0), (u_1, v_1)) = \begin{pmatrix}
(3u_0u_1^2 + s_0u_1(2u_0v_1 + u_1v_0) + p_0(u_0v_1^2 + 2u_1v_0v_1),
2u_0u_1v_1 + u_1^2v_0 + s_0(u_0v_1^2 + 2u_1v_0v_1) + 3p_0v_0v_1^2),
(3u_0u_1^2 + s_1u_1(2u_0v_1 + u_1v_0) + p_1(u_0v_1^2 + 2u_1v_0v_1),
2u_0u_1v_1 + u_1^2v_0 + s_1(u_0v_1^2 + 2u_1v_0v_1) + 3p_1v_0v_1^2)
\end{pmatrix}.
\]

One can easily check that there is no common factor in $R$, hence that the matrix of degrees is

\[
\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.
\]

Obviously \( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) is an eigenvector with eigenvalue 3. This means that the polynomial $H$ has degree 0. It means that the current $[D_1]$ is null, hence that there is no hypersurface of Neumann-Dirichlet eigenvalue at level 1. We did not check that $[D_n]$ is null for all $n$ (i.e. that there is no factorization in the iterates $R^n$), but it seems very probable that it is the case (and it is certainly possible to verify it, using a formal computation to identify the contracting curves).
Let us finally mention that the map \( f \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \), associated with the homogeneous polynomial \( R \) has four indeterminacy points, which are in \( \mathbb{C}^2 \) and given by
\[
(u_0, u_1) = (z_0, z_0), \quad (u_0, u_1) = (z'_0, z'_0), \quad (u_0, u_1) = (z_1, z_1), \quad (u_0, u_1) = (z'_1, z'_1).
\]
We can also remark easily that the diagonal \( \{ (z, z), z \in \mathbb{P}^1 \} \subset \mathbb{P}^1 \times \mathbb{P}^1 \) is invariant point by point (i.e. \( f \) is the identity on the diagonal).

The general theory also predicts that the upper half-plane \( \{ \text{Im} u_0 > 0, \text{Im} u_1 > 0 \} \subset \mathbb{C}^2 \) is invariant by \( f \) (which does not seem to be easy to check directly on the formulas).

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