A BIJECTION BETWEEN THE SETS OF \((a, b, b^2)\)-GENERALIZED MOTZKIN PATHS AVOIDING uvv-PATTERNS AND uvu-PATTERNS

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Abstract. A generalized Motzkin path, called G-Motzkin path for short, of length \(n\) is a lattice path from \((0, 0)\) to \((n, 0)\) in the first quadrant of the XOY-plane that consists of up steps \(u = (1, 1)\), down steps \(d = (1, -1)\), horizontal steps \(h = (1, 0)\) and vertical steps \(v = (0, -1)\). An \((a, b, c)\)-G-Motzkin path is a weighted G-Motzkin path such that the \(u\)-steps, \(h\)-steps, \(v\)-steps and \(d\)-steps are weighted respectively by 1, \(a\), \(b\) and \(c\). Let \(\tau\) be a word on \(\{u, d, v, d\}\), denoted by \(G_\tau^n(a, b, c)\) the set of \(\tau\)-avoiding \((a, b, c)\)-G-Motzkin paths of length \(n\) for a pattern \(\tau\). In this paper, we consider the uvv-avoiding \((a, b, b^2)\)-G-Motzkin paths and provide a direct bijection \(\sigma\) between \(G_{uvw}^n(a, b, b^2)\) and \(G_{uvu}^n(a, b, b^2)\). Finally, the set of fixed points of \(\sigma\) is also described and counted.

Keywords: G-Motzkin path, Catalan number.

2020 Mathematics Subject Classification: Primary 05A15, 05A19; Secondary 05A10.

1. Introduction

A generalized Motzkin path, called G-Motzkin path for short, of length \(n\) is a lattice path from \((0, 0)\) to \((n, 0)\) in the first quadrant of the XOY-plane that consists of up steps \(u = (1, 1)\), down steps \(d = (1, -1)\), horizontal steps \(h = (1, 0)\) and vertical steps \(v = (0, -1)\). Other related lattice paths with various steps including vertical steps permitted have been considered by [4, 5, 6, 13, 11, 12]. See Figure 1 for a G-Motzkin path of length 25.

![Figure 1. A G-Motzkin path of length 25.](image)

An \((a, b, c)\)-G-Motzkin path is a weighted G-Motzkin path \(P\) such that the \(u\)-steps, \(h\)-steps, \(v\)-steps and \(d\)-steps of \(P\) are weighted respectively by 1, \(a\), \(b\) and \(c\). The weight of \(P\), denoted by \(w(P)\), is the product of the weight of each step of \(P\). For example, \(w(uhuduuvvdh) = a^3b^2c^2\). The weight of a subset \(A\) of the set of weighted G-Motzkin paths, denoted by \(w(A)\), is the sum of the total weights of all paths in \(A\). Denoted by \(G_n(a, b, c)\) the weight of the set \(G_n(a, b, c)\) of all \((a, b, c)\)-G-Motzkin paths of length \(n\). Let \(\tau\) be a word on \(\{u, d, v, d\}\), denoted by \(G_\tau^n(a, b, c)\) the weight of the set \(G_\tau^n(a, b, c)\) of all \(\tau\)-avoiding \((a, b, c)\)-G-Motzkin paths of length \(n\), that is the weight of the subset of all \((a, b, c)\)-G-Motzkin paths of length \(n\)

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avoiding the pattern $\tau$. Figure 1 is an example of a G-Motzkin paths of length 25 avoiding the pattern $uvv$, but not avoiding the pattern $uvu$.

Recently Sun et al. [11, 12] have derived the generating functions of $G_n(a, b, c)$ and $G_{uvu}^n(a, b, c)$ as follows

$$G(a, b, c; x) = \sum_{n=0}^{\infty} G_n(a, b, c)x^n = \frac{1 - ax - \sqrt{(1 - ax)^2 - 4x(b + cx)}}{2x(b + cx)} = \frac{1}{1 - ax} C \left( \frac{x(b + cx)}{(1 - ax)^2} \right),$$

$$G_{uvu}(a, b, c; x) = \sum_{n=0}^{\infty} G_{uvu}^n(a, b, c)x^n = \frac{(1 - ax)(1 + bx) - \sqrt{(1 - ax)^2(1 + bx)^2 - 4x(1 + bx)(b + cx)}}{2x(b + cx)} = \frac{1}{1 - ax} C \left( \frac{x(b + cx)}{(1 - ax)^2(1 + bx)} \right),$$

where

$$C(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is the generating function for the well-known Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$, counting the number of Dyck paths of length of $2n$ [9, 10].

A Dyck path of length $2n$ is a G-Motzkin path of length $2n$ with no $h$-steps or $v$-steps. A Motzkin path of length $n$ is a G-Motzkin path of length $n$ with no $v$-steps. An $(a, b)$-Dyck path is a weighted Dyck path with $u$-steps weighted by 1, $d$-steps in $ud$-peaks weighted by $a$ and other $d$-steps weighted by $b$. An $(a, b)$-Motzkin path of length $n$ is an $(a, 0, b)$-G-Motzkin path of length $n$. A Schröder path of length $2n$ is a path from $(0, 0)$ to $(2n, 0)$ in the first quadrant of the XOY-plane that consists of up steps $u = (1, 1)$, down steps $d = (1, -1)$ and horizontal steps $H = (2, 0)$. An $(a, b)$-Schröder path is a weighted Schröder path such that the $u$-steps, $H$-steps and $d$-steps are weighted respectively by 1, $a$ and $b$.

Let $C_n(a, b), M_n(a, b)$ and $S_n(a, b)$ be respectively the sets of $(a, b)$-Dyck paths of length $2n$, $(a, b)$-Motzkin paths of length $n$ and $(a, b)$-Schröder paths of length $2n$. Let $C_n(a, b), M_n(a, b)$ and $S_n(a, b)$ be their weights with $C_0(a, b) = M_0(a, b) = S_0(a, b) = 1$ respectively. It is not difficult to deduce that [2]

$$C_n(a, b) = \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k-1} \binom{n}{k} a^k b^{n-k},$$

$$M_n(a, b) = \sum_{k=0}^{n} \binom{n}{2k} C_k a^{n-2k} b^k,$$

$$S_n(a, b) = \sum_{k=0}^{n} \binom{n+k}{2k} C_k a^{n-k} b^k,$$
and their generating functions

\[ C(a, b; x) = \sum_{n=0}^{\infty} C_n(a, b)x^n = 1 - (a - b)x - \frac{\sqrt{(1 - (a - b)x)^2 - 4bx}}{2bx}, \]

\[ M(a, b; x) = \sum_{n=0}^{\infty} M_n(a, b)x^n = 1 - ax - \frac{\sqrt{(1 - ax)^2 - 4bx^2}}{2bx^2}, \]

\[ S(a, b; x) = \sum_{n=0}^{\infty} S_n(a, b)x^n = 1 - ax - \frac{\sqrt{(1 - ax)^2 - 4bx}}{2bx}. \]

There are closely relations between \( C_n(a, b) \), \( M_n(a, b) \) and \( S_n(a, b) \). Exactly, Chen and Pan \cite{2} derived the following equivalent relations

\[ S_n(a, b) = C_n(a + b, b) = (a + b)M_{n-1}(a + 2b, (a + b)b) \]

for \( n \geq 1 \) and provided some combinatorial proofs. Sun et al. \cite{12} obtained that

\[ G_n^{uvu}(a, b, b^2) = S_n(a, b) \]

for \( n \geq 0 \) and presented bijections between the sets \( G_n^{uvu}(a, b, b^2) \) and \( S_n(a, b) \) as well as the set \( C_n(a + b, b) \).

In the literature, there are many papers dealing with \((a, b)\)-Motzkin paths. For examples, Chen and Wang \cite{11} explored the connection between noncrossing linked partitions and \((3, 2)\)-Motzkin paths, established a one-to-one correspondence between the set of noncrossing linked partitions of \( \{1, \ldots, n+1\} \) and the set of large \((3, 2)\)-Motzkin paths of length \( n \), which leads to a simple explanation of the well-known relation between the large and the little Schröder numbers. Yan \cite{14} found a bijective proof between the set of restricted \((3, 2)\)-Motzkin paths of length \( n \) and the set of the Schröder paths of length \( 2n \).

In the present paper we concentrate on the \( uvv \)-avoiding G-Motzkin paths, that is, the G-Motzkin paths with no \( uvv \) patterns. Precisely, the next section considers the enumerations of the set of \( uvv \)-avoiding \((a, b, c)\)-G-Motzkin paths and the set of \( uvv \)-avoiding \((a, b, c)\)-G-Motzkin paths with no \( h \)-steps on the \( x \)-axis, and find that \( G_n^{uvv}(a, b, b^2) = G_n^{uvu}(a, b, b^2) \).

The third section provides a direct bijection \( \sigma \) between the set \( G_n^{uvv}(a, b, b^2) \) of \( uvv \)-avoiding \((a, b, b^2)\)-G-Motzkin paths and the set \( G_n^{uvu}(a, b, b^2) \) of \( uvv \)-avoiding \((a, b, b^2)\)-G-Motzkin paths. Finally, the set of fixed points of \( \sigma \) is also described and counted.

2. \( uvv \)-AVOIDING \((a, b, c)\)-G-MOTZKIN PATHS

In this section, we first consider the \( uvv \)-avoiding \((a, b, c)\)-G-Motzkin paths which involve some classical structures as special cases, and count the set of \( uvv \)-avoiding \((a, b, c)\)-G-Motzkin paths with no \( h \)-steps on the \( x \)-axis.

Let \( G_n^{uvv}(a, b, c; x) = \sum_{n=0}^{\infty} G_n^{uvv}(a, b, c)x^n \) be the generating function for the \( uvv \)-avoiding \((a, b, c)\)-G-Motzkin paths. According to the method of the first return decomposition \cite{8}, any \( uvv \)-avoiding \((a, b, c)\)-G-Motzkin path \( P \) can be decomposed as one of the following four forms:

\[ P = \varepsilon, \ P = h_0Q_1, \ P = uQ_2d_1Q_1 \text{ or } P = uQ_3v_1Q_1, \]

where \( x_t \) denotes the \( x \)-steps with weight \( t \), \( Q_1 \) and \( Q_2 \) are (possibly empty) \( uvv \)-avoiding \((a, b, c)\)-G-Motzkin paths, and \( Q_3 \) is any \( uvv \)-avoiding \((a, b, c)\)-G-Motzkin paths with no \( uv \)-step at the end of \( Q_3 \). Then we get the relation

\[ G^{uvv}(a, b, c; x) = 1 + axG^{uvv}(a, b, c; x) + cx^2G^{uvv}(a, b, c; x)^2 \]

\[ + bx(G^{uvv}(a, b, c; x) - bxG^{uvv}(a, b, c; x))G^{uvv}(a, b, c; x) \]

\[ = 1 + axG^{uvv}(a, b, c; x) + (b + (c - b^2)x)G^{uvv}(a, b, c; x)^2. \]
Solve this, we have
\[
G_{uvv}(a, b, c; x) = \frac{1 - ax - \sqrt{(1 - ax)^2 - 4x(b + (c - b^2)x)}}{2x(b + (c - b^2)x)}
\]
\[
= \frac{1}{1 - ax}C\left(\frac{x(b + (c - b^2)x)}{(1 - ax)^2}\right).
\]
(2.2)

When \(a = b = c = 1\), \(G_{uvv}(1, 1, 1; x) = \frac{1-x-\sqrt{(1-x)^2-4x}}{2x}\) is just the generating function of the large Schröder numbers [8].

By (1.3), taking the coefficient of \(x^n\) in \(G_{uvv}(a, b, c; x)\), we derive that

**Proposition 2.1.** For any integer \(n \geq 0\), there holds
\[
G_{uvv}^n(a, b, c) = \sum_{k=0}^{n} \sum_{j=0}^{n-k-j} (1)^\ell \binom{k}{\ell} \binom{n + k - j - \ell}{\ell} C_k a^{n-k-j} b^{k+\ell-j} (c - b^2)\]
\[
= \sum_{k=0}^{n} \sum_{j=0}^{n-k-j} (1)^{n-k-j-\ell} \binom{k}{\ell} 2^{k+\ell} \binom{n-j-\ell}{n-k-j-\ell} C_k a^{\ell} b^n 2j-\ell (c - b^2)\]
Set \(T = xG(a, b, c; x)\), (2.1) produces
\[
T = x \frac{1 + aT + (c - b^2)T^2}{1 - bT},
\]
using the Lagrange inversion formula [7], taking the coefficient of \(x^{n+1}\) in \(T\) in three different ways, we derive that

**Proposition 2.2.** For any integer \(n \geq 0\), there holds
\[
G_{uvv}^n(a, b, c) = \frac{n+1}{n+1} \sum_{k=0}^{n-2} \sum_{j=0}^{n-k-j} \binom{n + 1}{k} \binom{n + 1 - k}{j} 2^{n-2k-j} (c - b^2)^k a^j b^{n-2k-j} (c - b^2)^j
\]
\[
= \frac{n+1}{n+1} \sum_{k=0}^{n-2} \sum_{j=0}^{n-k-j} \binom{n + 1}{k} \binom{n + 1 - k}{j} 2^{n-2k-j} (c - b^2)^j a^j b^{n-2k-j} (c - b^2)^j
\]
\[
= \frac{n+1}{n+1} \sum_{k=0}^{n-2} \sum_{j=0}^{n-k-j} \binom{n + 1}{k} \binom{n + 1 - k}{j} 2^{n-2k-j} (c - b^2)^j a^j b^{n-2k-j} (c - b^2)^j.
\]

Exactly, by (1.1), (1.2) and (2.2), it can be deduced that
\[
G_{uvv}^n(a, b, b^2 + c; x) = G(a, b, c; x), \quad G_{uvv}^n(a, b, b^2; x) = G_{uvu}^n(a, b, b^2; x).
\]
That is \(G_{uvv}^n(a, b, b^2 + c) = G_n(a, b, c)\) and \(G_{uvv}^n(a, b, b^2) = G_{uvu}^n(a, b, b^2)\). The first identity has a direct combinatorial interpretation if one notices that each \(d_{b^2+c}\)-step of \(P \in G_{uvv}^n(a, b, b^2 + c)\) can be regarded equivalently as the corresponding \(d_c\)-step and \(uv_{b^2}v_{b}\)-step of \(P' \in G_n(a, b, c)\). The combinatorial interpretation of the second identity will be given in the next section.

When \((a, b, c)\) is specialized, \(G_{uvv}^n(a, b, c; x)\) and \(G_{uvv}^n(a, b, c)\) reduce to some well-known generating functions and classical combinatorial sequences involving the Catalan numbers \(C_n\), Motzkin numbers \(M_n\), the large Schröder numbers \(S_n\), (\(a+b\), \(b\))-Catalan number \(C_n(a+b, b)\), (\(a, b\))-Motzkin number \(M_n(a, b)\) and (\(a, b\))-Schröder number \(S_n(a, b)\). See Table 2.1 for example.
\[
\begin{array}{|c|c|c|c|}
\hline
(a, b, c) & G_{uvv}(a, b, c; x) & G_n^{uvv}(a, b, c) & \text{Senquences} \\
\hline
(0, 1, 1) & C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} & C_n & \square \text{ A000108} \\
(1, 0, 1) & M(x) = \frac{1 - x - \sqrt{1 - 2x^2 - 3x^2}}{2x} & M_n & \square \text{ A001006} \\
(1, 1, 1) & S(x) = \frac{1 - x - \sqrt{1 - 6x + 2x^2}}{2x} & S_n & \square \text{ A006318} \\
(1, 0, 2) & \frac{1 - x - \sqrt{1 - 2x - 7x^2}}{4x} & a_n & \square \text{ A025235} \\
(-3, 4, 16) & \frac{1 + 3x - \sqrt{1 - 10x + 9x^2}}{8x} & a_n & \square \text{ A059231} \\
(a, 0, b) & \frac{1 - ax - \sqrt{(1 - ax)^2 - 4bx^2}}{2bx^2} & M_n(a, b) \\
(a, b, b^2) & \frac{1 - ax - \sqrt{(1 - ax)^2 - 4bx^2}}{2bx} & C_n(a + b, b) \text{ or } S_n(a, b) \\
\hline
\end{array}
\]

Table 2.1. The specializations of \(G_{uvv}(a, b, c; x)\) and \(G_n^{uvv}(a, b, c)\).

Denoted by \(G_n^{uvv}(a, b, c)\) the weight of the set \(\tilde{G}_n^{uvv}(a, b, c)\) of all \(uvv\)-avoiding \((a, b, c)\)-G-Motzkin paths of length \(n\) such that the paths have no \(h\)-steps on the \(x\)-axis. Set \(\tilde{G}_n^{uvv}(a, b, c) = \bigcup_{n \geq 0} G_n^{uvv}(a, b, c)\).

Let \(\tilde{G}_{uvv}(a, b; c; x) = \sum_{n=0}^{\infty} \tilde{G}_n^{uvv}(a, b, c)x^n\) be the generating function for the \(uvv\)-avoiding \((a, b, c)\)-G-Motzkin paths in \(\tilde{G}_n^{uvv}(a, b, c)\). According to the method of the first return decomposition, any paths \(P \in \tilde{G}_{uvv}(a, b, c)\) can be decomposed as one of the following three forms:

\[P = \varepsilon, \quad P = uQ_2d_cQ_1 \text{ or } P = uQ_3v_dQ_1,\]

where \(Q_1 \in \tilde{G}_{uvv}(a, b, c), Q_2 \in G_{uvv}(a, b, c)\) and \(Q_3 \in G_{uvv}(a, b, c)\) has no \(uv\)-step at the end of \(Q_3\). Then we get the relation

\[
\tilde{G}_{uvv}(a, b, c; x) = 1 + cx^2\tilde{G}_{uvv}(a, b, c; x)\tilde{G}_{uvv}(a, b, c; x) \\
+ bx(\tilde{G}_{uvv}(a, b, c; x) - bx\tilde{G}_{uvv}(a, b, c; x))\tilde{G}_{uvv}(a, b, c; x),
\]

which, by (2.1) and (2.3), leads to

\[
x\tilde{G}_{uvv}(a, b, c; x) = \frac{x}{1 - (b + (c - b^2)x)x\tilde{G}_{uvv}(a, b, c; x)} = \frac{xG_{uvv}(a, b, c; x)}{1 + axG_{uvv}(a, b, c; x)} = \frac{T}{1 + aT}.
\]

By the Lagrange inversion formula, taking the coefficient of \(x^{n+1}\) in \(x\tilde{G}_{uvv}(a, b, c; x)\) in three different ways, we derive that
Proposition 2.3. For any integer $n \geq 0$, there holds

$$G_{n}^{uvu}(a, b, c) = \sum_{i=0}^{n+1} (-1)^{i} \frac{i+1}{n+1} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=0}^{n-2k} \binom{n+1}{k} \binom{n+1-k}{j} \left( \frac{2n-i-2k-j}{n-i-2k-j} \right) a^{i+j} b^{n-i-2k-j} (c-b)^{k}$$

$$= \sum_{i=0}^{n+1} (-1)^{i} \frac{i+1}{n+1} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=0}^{n-k} \binom{n+1}{k} \binom{n+1-k}{j} \left( \frac{2n-i-k-j}{n-i-k-j} \right) a^{i+k} b^{n-i-k-j} (c-b)^{j}$$

In this section, we give a direct bijection between the set $G_{n}^{uvu}(a, b, b^2)$ of $uvu$-avoiding $(a, b, b^2)$-G-Motzkin paths and the set $G_{n}^{uvu}(a, b, b^2)$ of $uvu$-avoiding $(a, b, b^2)$-G-Motzkin paths.

Theorem 3.1. There exists a bijection $\sigma$ between $G_{n}^{uvu}(a, b, b^2)$ and $G_{n}^{uvu}(a, b, b^2)$ for any integer $n \geq 0$.

Proof. Given any $Q \in G_{n}^{uvu}(a, b, b^2)$ for $n \geq 0$, when $n = 0, 1$ and 2, we define

$$\sigma(\varepsilon) = \varepsilon, \quad \sigma(h) = h, \quad \sigma(uv) = uv.$$  

For $n \geq 2$, $Q$ is $uvu$-avoiding, there are six cases to be considered to define $\sigma(Q)$ recursively.

Case 1. When $Q = h_{n}Q$ with $Q' \in G_{n-1}^{uvu}(a, b, b^2)$, we define $\sigma(Q) = h_{n} \sigma(Q')$.

Case 2. When $Q = uv_{b}h_{n}Q'$ with $Q' \in G_{n-2}^{uvu}(a, b, b^2)$, we define $\sigma(Q) = uv_{b}h_{n} \sigma(Q')$.

Case 3. When $Q = uv_{b}Q''Q'$ such that $Q'' \in G_{k}^{uvu}(a, b, b^2)$ is primitive and $Q' \in G_{n-k-1}^{uvu}(a, b, b^2)$ for certain $1 \leq k \leq n-1$, we define $\sigma(Q) = u \sigma(Q'' \varepsilon) v_{b} \sigma(Q')$. In this case, one can notice that there exist $uvu$’s in $Q$, but not in $\sigma(Q)$.

Case 4. When $Q = u^{i}d_{b_{j}}u_{j}v_{j}Q'$ with $Q' \in G_{n-i}^{uvu}(a, b, b^2)$ for $0 \leq i \leq n-2$, we define

$$\sigma(Q) = \left\{ \begin{array}{ll} u^{i}uv_{b}d_{j}^{i+1} \sigma(Q'), & \text{if } i = 2j - 1 \geq 1, \\ u^{i+1}d_{j}^{i+1} \sigma(Q'), & \text{if } i = 2j \geq 0. \end{array} \right.$$  

Case 5. When $Q = u^{i}uQ''d_{b_{j}}v_{j}Q'$ such that $Q'' \in G_{k}^{uvu}(a, b, b^2)$ is nonempty and $Q' \in G_{n-k-1}^{uvu}(a, b, b^2)$ for certain $1 \leq k \leq n-i$ and $0 \leq i \leq n-2$, we define

$$\sigma(Q) = \left\{ \begin{array}{ll} u^{i} \sigma(Q'' \varepsilon) d_{j}^{i} \sigma(Q'), & \text{if } i = 2j - 1 \geq 1, \\ u^{i+1} \sigma(Q'' \varepsilon) v_{b} d_{j}^{i+1} \sigma(Q'), & \text{if } i = 2j \geq 0. \end{array} \right.$$
Conversely, the inverse procedure can be handled as follows. Given any $n$ for $n$ and the following assertions hold:

- In the case 3, $\sigma(Q'')$ is primitive and must not be $uuv_bv_b$ since $Q''$ is primitive;
- In the case 5, $\sigma(Q''/uv_b)$ has the form $P_1uuv_bv_b$ or $P_2uv_b$ since $Q''$ is nonempty, where both $P_1 \in G_k^{uvu}(a, b, b^2)$ and $P_2 \in G_r^{uvu}(a, b, b^2)$ must not end with $uv_b$ for certain $r \geq 1$;
- In the case 6, $\sigma(Q'')$ is not primitive and does not end with $uv_b$ or $uuv_bv_b$ since $Q''$ is not primitive and does not end with $uv_b$.

Conversely, the inverse procedure can be handled as follows. Given any $P \in G_n^{uvu}(a, b, b^2)$ for $n \geq 0$, when $n = 0, 1$, we define

$$\sigma^{-1}(\varepsilon) = \varepsilon, \quad \sigma^{-1}(h_a) = h_a, \quad \sigma^{-1}(uv_b) = uv_b.$$  

For $n \geq 2$, there are five cases to be considered to define $\sigma^{-1}(P)$ recursively.

**Case I.** When $P = h_aP'$ with $P' \in G_{n-1}^{uvu}(a, b, b^2)$, we define $\sigma^{-1}(P) = h_a\sigma^{-1}(P')$.

**Case II.** When $P = uv_bh_aP'$ with $P' \in G_{n-2}^{uvu}(a, b, b^2)$, we define $\sigma^{-1}(P) = uv_bh_a\sigma^{-1}(P')$.

**Case III.** When $P = uP''v_bP'$ such that $P'' \in G_k^{uvu}(a, b, b^2)$ and $P' \in G_{n-1-k}^{uvu}(a, b, b^2)$ for certain $1 \leq k \leq n - 1$, we define

$$\sigma^{-1}(P) = \begin{cases} 
  
  uuv_b\sigma^{-1}(P'')\sigma^{-1}(P'), & \text{if } P'' \text{ is primitive and } P'' \neq uuv_bv_b, \\
  u\sigma^{-1}(P'')v_b\sigma^{-1}(P'), & \text{if } P''(\neq \varepsilon) \text{ is not primitive and does not end with } uuv_bv_b \text{ or } uv_b, \\
  u\sigma^{-1}(P_1uv_b)d_{\geq 2}\sigma^{-1}(P'), & \text{if } P'' = P_1uuv_bv_b, \\
  u\sigma^{-1}(P_2)d_{\geq 2}\sigma^{-1}(P'), & \text{if } P'' = P_2uv_b, 
\end{cases}$$

where both $P_1 \in G_{k-1}^{uvu}(a, b, b^2)$ and $P_2 \in G_{r-1}^{uvu}(a, b, b^2)$ must not end with $uv_b$ for certain $r \geq 1$, since $P$ is $uvu$-avoiding.

**Case IV.** When $P = uP''d_{\geq 2}P'$ such that $P'' \in G_k^{uvu}(a, b, b^2)$ and $P' \in G_{n-2j-k}^{uvu}(a, b, b^2)$ for certain $0 \leq k \leq n - 2j$ and the maximum $j \geq 1$, we define

$$\sigma^{-1}(P) = \begin{cases} 
  
  u^{2j-2}u_d^{2j-2}\sigma^{-1}(P'), & \text{if } P'' = \varepsilon, \\
  u^{2j-1}u\sigma^{-1}(P_1uv_b)d_{\geq 2}v_b^{2j-1}\sigma^{-1}(P'), & \text{if } P'' = P_1uuv_bv_b, \\
  u^{2j-1}u\sigma^{-1}(P_2)d_{\geq 2}v_b^{2j-1}\sigma^{-1}(P'), & \text{if } P'' = P_2uv_b, \\
  u^{2j}\sigma^{-1}(P'')v_b^{2j}\sigma^{-1}(P'), & \text{if } P''(\neq \varepsilon) \text{ is not primitive and does not end with } uuv_bv_b \text{ or } uv_b, 
\end{cases}$$

where both $P_1 \in G_{k-1}^{uvu}(a, b, b^2)$ and $P_2 \in G_{r-1}^{uvu}(a, b, b^2)$ must not end with $uv_b$ for certain $r \geq 1$, since $P$ is $uvu$-avoiding.
Case V. When $P = u^i u P'' v_d j P'$ such that $P'' \in G^u v w (a, b, b^2)$ and $P' \in G^u v w (a, b, b^2)$ for certain $0 \leq k \leq n - 2j - 1$ and the maximum $j \geq 1$, we define

$$
\sigma^{-1}(P) = \begin{cases} 
    u^{2j-1} d v_w v_b^{2j-1} \sigma^{-1}(P') & \text{if } P'' = \varepsilon, \\
    u^{2j} \sigma^{-1}(P_1 u v_w v_b) d v_w v_b^{2j} \sigma^{-1}(P') & \text{if } P'' = P_1 u v_w v_b, \\
    u^{2j} u^{j-1} (P_2) d v_w v_b^{2j} \sigma^{-1}(P') & \text{if } P'' = P_2 u v_w, \\
    u^{2j+1} \sigma^{-1}(P') v_b^{2j+1} \sigma^{-1}(P') & \text{if } P''(\neq \varepsilon) \text{ is not primitive and does not end with } u v w v_b \text{ or } u v b,
\end{cases}
$$

where both $P_1 \in G^u v w (a, b, b^2)$ and $P_2 \in G^u v w (a, b, b^2)$ must not end with $u v b$ for certain $r \geq 1$, since $P$ is $u v b$-avoiding.

It is not difficult to verify that $\sigma^{-1} \sigma = \sigma \sigma^{-1} = 1$, both $\sigma$ and $\sigma^{-1}$ are two weight-keeping mappings and $\sigma^{-1}(P)$ is $u v v$-avoiding by induction on the length of $P$. Hence, $\sigma$ is a desired bijection between $G^u v w (a, b, b^2)$ and $G^u v w (a, b, b^2)$. This completes the proof of Theorem 3.1.

In order to give a more intuitive view on the bijection $\sigma$, a pictorial description of $\sigma$ is presented for $Q = u^3 d_b d_v v_b^2 u^2 d_v v_b u^5 v_b d_l z v_b^3 h_a u^2 h_a d_v v_b u^3 v_b h_a u v_b u^2 v_b d_l z v_b^2 \in G^u v w (a, b, b^2)$, we have

$$
\sigma(Q) = u^2 d_b^2 u^2 v_b d_l z u^4 v_b^2 d_l z h_a u h_a u v_b u d_l z u^3 v_b h_a u^2 h_a u d_l z v_b^2 \in G^u v w (a, b, b^2).
$$

See Figure 2 for detailed illustrations.

![Figure 2](image_url)

**Figure 2.** An example of the bijection $\sigma$ described in the proof of Theorem 3.1.
4. Counting the set of fixed points of the bijection $\sigma$

In this section, we will count the set of fixed points of the bijection $\sigma$ presented in Section 3. Let $F_n = \{ Q \in G_n^{uvv}(a, b, b^2) | \sigma(Q) = Q \}$ and $F = \bigcup_{n \geq 0} F_n$, set $F_n = |F_n|$. It is easy to verify the few initial values for $F_n$, see Table 4.1.

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| $F_n$ | 1  | 2  | 5  | 13 | 39 | 125 | 421 | 1478 | 5329 | 19658 | 73783 |

Table 4.1. The first values of $F_n$.

According to the definition of $\sigma$, any $Q \in F_n$ must belong to $G_n^{uvv, uvu}(a, b, b^2)$, the set of $(a, b, b^2)$-G-Motzkin paths avoiding both the $uvv$ and $uvu$ patterns, since $Q$ is $uvv$-avoiding and $\sigma(Q)$ is $uvu$-avoiding. But there exists $P \in G_n^{uvv, uvu}(a, b, b^2)$ such that $\sigma(P) \neq P$. For example, $\sigma(u, uv, uv, vu, ud)$ is not primitive and does not end with $uv$. Equivalently, one can derive that

- In Case 1, $Q = h_aQ' \in F_n$ if and only if $Q' \in F_{n-1}$;
- In Case 2, $Q = uvb_aQ' \in F_n$ if and only if $Q' \in F_{n-2}$;
- In Case 4 when $i = 0$, $Q = ud_{b^2}Q' \in F_n$ if and only if $Q' \in F_{n-2}$;
- In Case 6 when $i = 1$, $Q = uQ''v_bQ' \in F_n$ if and only if $Q'' \in F_k$ and $Q' \in F_{n-k-1}$ for certain $1 \leq k \leq n - 1$ such that $Q''(\neq \varepsilon)$ is not primitive and does not end with $uv$.

Let $A_n$ be the subset of $Q \in F_n$ such that $Q$ is not primitive and does not end with $uv_b$, $B_n$ be the subset of $Q \in F_n$ such that $Q$ ends with $uv_b$, and $C_n$ be the subset of $Q \in F_n$ such that $Q$ is primitive and does not end with $uv_b$. Set $a_n = |A_n|$, $b_n = |B_n|$, $c_n = |C_n|$. Firstly, $F_n$ is the disjoint union of $A_n$, $B_n$, and $C_n$, i.e., $F_n = a_n + b_n + c_n$ for $n \geq 0$; Secondly, $C_0 = C_1 = \emptyset$, $C_2 = \{uh_a, v_b, ud_{b^2} \}$ and $C_n = uA_{n-1}v_b$ for $n \geq 3$, i.e., $c_n = a_{n-1}$ for $n \geq 3$ with $c_0 = c_1 = 0$ and $c_2 = c_3 = 2$; Thirdly, $B_n$ is the disjoint union of $A_{n-1}uv_b$ and $C_{n-1}uv_b$ for $n \geq 1$, i.e., $b_n = a_{n-1} + c_{n-1}$ for $n \geq 1$ with $b_0 = 0$ and $b_1 = b_2 = 1$. These together generate the following Lemma.

**Lemma 4.1.** For any integer $n \geq 4$, there holds

$$F_n = a_n + 2a_{n-1} + a_{n-2}$$

with $a_0 = a_1 = 1, a_2 = 2, a_3 = 7$ and $a_4 = 23$.

On the other hand, the family $F$ can be partitioned into the form:

$$F = \varepsilon + h_aF + uv_bh_aF + ud_{b^2}F + uA'v_bF,$$

where $A' = A - \varepsilon$ and $A = \bigcup_{n \geq 0} A_n$. This leads to the following recurrence for $F_n$.

**Lemma 4.2.** For any integer $n \geq 1$, there holds

$$F_{n+1} = F_n + 2F_{n-1} + \sum_{k=1}^{n} a_k F_{n-k}$$

with $F_0 = 1, F_1 = 2$. 
Let $F(x) = \sum_{n \geq 0} F_n x^n$ and $A(x) = \sum_{n \geq 0} a_n x^n$. By (4.1), we have

$$F(x) = 1 + 2x + 5x^2 + 13x^3 + \sum_{n \geq 4} (a_n + 2a_{n-1} + a_{n-2}) x^n$$

$$= 1 + 2x + 5x^2 + 13x^3 + (A(x) - 1 - x - 2x^2 - 7x^3) + 2x(A(x) - 1 - x - 2x^2) + x^2(A(x) - 1 - x)$$

(4.3)

$$= (1 + x)^2 A(x) - x + x^3.$$

By (4.2), we can obtain

$$F(x) = 1 + 2x + (F(x) - 1) + 2x^2 F(x) + x(A(x) - 1) F(x)$$

(4.4)

$$= 1 + x + 2x^2 F(x) + x A(x) F(x).$$

Eliminating $A(x)$ in (4.3) and (4.4) produces

$$xF(x)^2 - (1 + x)(1 + x - 3x^2 - x^3) F(x) + (1 + x)^3 = 0.$$

Solve this, we have

$$F(x) = \frac{(1 + x)(1 + x - 3x^2 - x^3) - (1 + x) \sqrt{(1 + x - 3x^2 - x^3)^2 - 4x(1 + x)}}{2x}$$

(4.5)

$$= \frac{(1 + x)^2}{1 + x - 3x^2 - x^3} C \left( \frac{x(1 + x)}{(1 + x - 3x^2 - x^3)^2} \right).$$

By (4.5), taking the coefficient of $x^n$ in $F(x)$, we get the explicit formula for the number $F_n$ of the fixed points of the bijection $\sigma$, namely,

**Theorem 4.3.** For any integer $n \geq 0$, there holds

$$F_n = \sum_{k=0}^{n} \sum_{j=0}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \sum_{i=0}^{j} (-1)^{n-k-i} \binom{2k+j}{j} \binom{j}{i} \binom{n-j-i-2}{n-k-2j-i} 3^{j-i} C_k,$$

where $C_k$ is the $k$-th Catalan number.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Acknowledgements**

The authors are grateful to the referees for the helpful suggestions and comments. The Project is sponsored by “Liaoning BaiQianWan Talents Program”.

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