SOME REMARKS ON
DIMENSION-FREE ESTIMATES FOR THE
DISCRETE HARDY–LITTLEWOOD MAXIMAL FUNCTIONS

BY

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*Dariusz Kosz and Paweł Plewa were supported by funds of Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, #049U/0052/19.
**Mariusz Mirek was partially supported by Department of Mathematics at Rutgers University.
†Mariusz Mirek and Błażej Wróbel were supported by the National Science Centre, Poland, grant Opus 2018/31/B/ST1/00204.
Received October 14, 2020 and in revised form August 30, 2021.
ABSTRACT
Dependencies of the optimal constants in strong and weak type bounds will be studied between maximal functions corresponding to the Hardy–Littlewood averaging operators over convex symmetric bodies acting on $\mathbb{R}^d$ and $\mathbb{Z}^d$. Firstly, we show, in the full range of $p \in [1, \infty]$, that these optimal constants in $L^p(\mathbb{R}^d)$ are always not larger than their discrete analogues in $\ell^p(\mathbb{Z}^d)$; and we also show that the equality holds for the cubes in the case of $p = 1$. This in particular implies that the best constant in the weak type $(1, 1)$ inequality for the discrete Hardy–Littlewood maximal function associated with centered cubes in $\mathbb{Z}^d$ grows to infinity as $d \to \infty$, and if $d = 1$ it is equal to the largest root of the quadratic equation $12C^2 - 22C + 5 = 0$. Secondly, we prove dimension-free estimates for the $\ell^p(\mathbb{Z}^d)$ norms, $p \in (1, \infty]$, of the discrete Hardy–Littlewood maximal operators with the restricted range of scales $t \geq C_qd$ corresponding to $q$-balls, $q \in [2, \infty)$. Finally, we extend the latter result on $\ell^2(\mathbb{Z}^d)$ for the maximal operators restricted to dyadic scales $2^n \geq C_qd^{1/q}$.

1. Introduction

1.1. A brief overview of the paper. Throughout this paper $d \in \mathbb{N}$ always denotes the dimension of the Euclidean space $\mathbb{R}^d$, and $G$ denotes a convex symmetric body in $\mathbb{R}^d$, which is a bounded closed and symmetric convex subset of $\mathbb{R}^d$ with nonempty interior. We shall consider convex bodies in two contexts, continuous and discrete. Therefore, in order to avoid unnecessary technicalities, we always assume that $G \subset \mathbb{R}^d$ is closed (whereas in the literature it is usually assumed to be open). One of the most classical examples of convex symmetric bodies are the $q$-balls $B^q \subset \mathbb{R}^d$, $q \in [1, \infty]$, defined for $q \in [1, \infty)$ by

$$B^q := B^q(d) := \left\{ x \in \mathbb{R}^d : |x|_q := \left( \sum_{i=1}^{d} |x_i|^q \right)^{1/q} \leq 1 \right\},$$

and for $q = \infty$ by

$$B^\infty := B^\infty(d) := \{ x \in \mathbb{R}^d : |x|_\infty := \max_{1 \leq i \leq d} |x_i| \leq 1 \}.$$

If $p = 2$ then $B^2$ is the closed unit Euclidean ball in $\mathbb{R}^d$ centered at the origin, and if $p = \infty$ then $B^\infty$ is the cube in $\mathbb{R}^d$ centered at the origin and of side length 2.
We associate with a convex symmetric body \( G \subset \mathbb{R}^d \) the families of continuous \((M_t^G)_{t>0}\) and discrete \((\mathcal{M}_t^G)_{t>0}\) averaging operators given respectively by
\[
M_t^G F(x) := \frac{1}{|G_t|} \int_{G_t} F(x-y) \, dy, \quad F \in L^1_{\text{loc}}(\mathbb{R}^d)
\]
and
\[
\mathcal{M}_t^G f(x) := \frac{1}{|G_t \cap \mathbb{Z}^d|} \sum_{y \in G_t \cap \mathbb{Z}^d} f(x-y), \quad f \in \ell^\infty(\mathbb{Z}^d),
\]
where \( G_t := \{ y \in \mathbb{R}^d : t^{-1}y \in G \} \) is the dilate of \( G \subset \mathbb{R}^d \). Moreover, we define the corresponding maximal functions by
\[
M_*^G F(x) := \sup_{t>0} |M_t^G F(x)| \quad \text{and} \quad \mathcal{M}_*^G f(x) := \sup_{t>0} |\mathcal{M}_t^G f(x)|.
\]

It is well known that both maximal functions are of weak type \((1, 1)\) and of strong type \((p, p)\) for any \( p \in (1, \infty] \). Moreover, neither of these maximal functions is of strong type \((1, 1)\). Our primary interest is focused on determining whether the constants arising in the weak and strong type inequalities can be chosen independently of the dimension \( d \). Moreover, we shall compare the best constants in such inequalities for \( M_*^G \) and \( \mathcal{M}_*^G \), respectively.

For \( p \in (1, \infty] \) we denote by \( C(G, p) \) the smallest constant \( 0 < C < \infty \) for which the following strong type inequality holds:
\[
\|M_*^G F\|_{L^p(\mathbb{R}^d)} \leq C\|F\|_{L^p(\mathbb{R}^d)}, \quad F \in L^p(\mathbb{R}^d).
\]
Similarly, \( C(G, 1) \) will stand for the smallest constant \( 0 < C < \infty \) satisfying
\[
\sup_{\lambda>0} \lambda |\{ x \in \mathbb{R}^d : M_*^G F(x) > \lambda \}| \leq C\|F\|_{L^1(\mathbb{R}^d)}, \quad F \in L^1(\mathbb{R}^d).
\]
Analogously to \( C(G, p) \), we define \( C(G, p) \) for any \( p \in [1, \infty] \), referring to \( \mathcal{M}_*^G \) in place of \( M_*^G \).

Our main result of this paper can be formulated as follows.

**Theorem 1:** Fix \( d \in \mathbb{N} \) and let \( G \subset \mathbb{R}^d \) be a convex symmetric body. Then for each \( p \in [1, \infty] \) we have
\[
C(G, p) \leq C(G, p).
\]
Moreover, for the \( d \)-dimensional cube \( B^\infty(d) \subset \mathbb{R}^d \) one has
\[
C(B^\infty(d), 1) = C(B^\infty(d), 1).
\]
Some comments are in order:

(i) Clearly, $C(G, \infty) = C(G, \infty) = 1$, since we have been working with averaging operators.

(ii) Theorem 1 gives us a quantitative dependence between $C(G, p)$ and $C(G, p)$. Inequality (1.3) coincides with a well known phenomenon in harmonic analysis, which states that it is harder to establish bounds for discrete operators than the bounds for their continuous counterparts.

(iii) Formula (1.4) was observed by the first author in his master thesis. However, it has not been published before. In particular, it yields that $C(\mathcal{B}_\infty(d), 1) \rightarrow \infty$ in view of the result of Aldaz [1] asserting that the optimal constant $C(\mathcal{B}_\infty(d), 1)$ in the weak type $(1, 1)$ inequality grows to infinity as $d \rightarrow \infty$. Quantitative bounds for the constant $C(\mathcal{B}_\infty(d), 1)$ were given by Aubrun [2], who proved $C(\mathcal{B}_\infty(d), 1) \gtrsim (\log d)^{1-\varepsilon}$ for every $\varepsilon > 0$, and soon after that by Iakovlev and Strömberg [11] who considerably improved Aubrun’s lower bound by showing that $C(\mathcal{B}_\infty(d), 1) \gtrsim d^{1/4}$. The latter result also ensures in the discrete setup that $C(\mathcal{B}_\infty(d), 1) \gtrsim d^{1/4}$.

(iv) If $d = 1$, then (1.4) combined with the result of Melas [12] implies that $C(\mathcal{B}_\infty(1), 1)$ is equal to the larger root of the quadratic equation $12C^2 - 22C + 5 = 0$.

(v) The product structure of the cubes $\mathcal{B}_\infty(d)$ and the fact that one works with continuous/discrete norms for $p = 1$ are essential to prove $C(\mathcal{B}_\infty(d), 1) = C(\mathcal{B}_\infty(d), 1)$.

At this moment it does not seem that our method can be used to attain the equality in (1.3) for $G = \mathcal{B}_\infty(d)$ with $p \in (1, \infty)$. In the general case, as we shall see later in this paper, inequality (1.3) cannot be reversed.

(vi) The proof of inequality (1.3) relies on a suitable generalization of the ideas described in the master thesis of the first author. The details are presented in Section 2.

Systematic studies of dimension-free estimates in the continuous case were initiated by Stein [15], who showed that $C(\mathcal{B}_2^2, p)$ is bounded independently of the dimension for all $p \in (1, \infty]$. Shortly afterwards Bourgain [3] proved that $C(G, 2)$ can be estimated by an absolute constant independent of the dimension and the convex symmetric body $G \subset \mathbb{R}^d$. This result was extended to the range $p \in (3/2, \infty]$ in [4] and independently by Carbery in [9]. It is
conjectured that one can estimate $C(G,p)$ by a dimension-free constant for all $p \in (1, \infty]$. This was verified for the $q$-balls $B^q$, $q \in [1, \infty)$, by Müller [13] and for the cubes $B^\infty$ by Bourgain [5]. The latter result exhibits an interesting phenomenon, which shows that the dimension-free estimates on $L^p(\mathbb{R}^d)$ for $p \in (1, \infty]$ cannot be extended to the weak type $(1, 1)$ endpoint. Namely, the optimal constant $C(B^\infty(d), 1)$ in the weak type $(1, 1)$ inequality, as Aldaz [1] proved, grows to infinity with the dimension. Additionally, if the range of scales $t$ in the definition of the maximal operator $M^G_t$ is restricted to the dyadic values ($t \in \mathbb{D} := \{2^n : n \in \mathbb{Z}\}$), then the constants in the strong type $(p,p)$ inequalities with $p \in (1, \infty]$ are bounded uniformly in $d$ for any $G$; see [9]. For a more detailed account of the subject in the continuous case, its history and extensive literature, we refer to [10] or [7].

Surprisingly, in the discrete setting there is no hope for estimating $C(G,p)$ independently of the dimension and the convex symmetric body. Fixing $1 \leq \lambda_1 < \cdots < \lambda_d < \cdots < \sqrt{2}$ and examining the ellipsoids

$$E(d) := \left\{ x \in \mathbb{R}^d : \sum_{j=1}^d \lambda_j^2 x_j^2 \leq 1 \right\},$$

it was proved in [6, Theorem 2] that for every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that for all $d \in \mathbb{N}$ we have

$$C(E(d), p) \geq \sup_{\|f\|_{\ell^p(\mathbb{Z}^d)} \leq 1} \sup_{0 < t \leq d} \|M^E_t f\|_{\ell^p(\mathbb{Z}^d)} \geq C_p (\log d)^{1/p}. \quad (1.6)$$

This inequality shows that if $p \in (3/2, \infty)$, then for sufficiently large $d$ the inequality in (1.3) with $G = E(d)$ is strict, since from [4] and [9] we know that there exists a finite constant $C_p > 0$ independent of the dimension such that

$$C(E(d), p) \leq C_p. \quad (1.7)$$

On the other hand, for cubes $G = B^\infty(d)$, it was also proved [6, Theorem 3] that for every $p \in (3/2, \infty]$ there is a finite constant $C_p > 0$ such that for every $d \in \mathbb{N}$ one has

$$C(B^\infty(d), p) \leq C_p. \quad (1.8)$$

For $p \in (1, 3/2]$ it still remains open whether $C(B^\infty(d), p)$ can be estimated independently of the dimension. In view of the second part of Theorem 1 interpolation does not help, since $C(B^\infty(d), 1) \xrightarrow{d \to \infty} \infty$. 

Inequalities (1.6) and (1.8) illustrate that the dimension-free phenomenon in the discrete setting contrasts sharply with the situation that we know from the continuous setup. However, as it was shown in [6, Theorem 2], if the dimension-free estimates fail in the discrete setting it may only happen for small scales; see also (1.6). To be more precise, define the discrete restricted maximal function

$$M_{*,>D}^G f(x) := \sup_{t>D} |M_t^G f(x)|; \quad x \in \mathbb{Z}^d, \quad D \geq 0,$$

corresponding to the averages from (1.2). For $D = 0$ we see that $M_{*,>D}^G = M_*^G$. Then by [6, Theorem 1] one has for each convex symmetric body $G \subset \mathbb{R}^d$ that there exists $c(G) > 0$ such that

$$\|M_{*,>c(G)d}^G f\|_{\ell^p(\mathbb{Z}^d)} \leq e^6 C(G, p) \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d).$$

Specifically, $\frac{1}{2} d^{1/2} \leq c(E(d)) \leq d^{1/2}$ for the ellipsoids (1.5), and consequently by (1.9), we also have

$$\|M_{*,>d^{3/2}}^E f\|_{\ell^p(\mathbb{Z}^d)} \leq e^6 C(E(d), p) \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d),$$

which ensures dimension-free estimates for any $p \in (3/2, \infty]$ thanks to (1.7).

In the case of $q$-balls $G = B^q(d), \ q \in [1, \infty]$, in view of [5] and [13], inequality (1.9) comes down to

$$\|M_{*,>d^{1+1/q}}^G f\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,q} \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d),$$

for all $p \in (1, \infty]$, where $C_{p,q}$ denotes a constant that depends on $p$ and $q$ but not on the dimension $d$. We close this discussion by gathering a few conjectures that arose upon completing [6], [7] and [8].

**Conjecture 1:** Let $d \in \mathbb{N}$ and let $B^q(d) \subset \mathbb{R}^d$ be a $q$-ball.

1. **(Weak form)** Is it true that for every $p \in (1, \infty)$ and $q \in [1, \infty]$ there exist constants $C_{p,q} > 0$ and $t_q > 0$ such that for every $d \in \mathbb{N}$ we have

$$\sup_{\|f\|_{\ell^p(\mathbb{Z}^d)} \leq 1} \|M_{*,>t_q d}^{B^q(d)} f\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,q}?$$

2. **(Strong form)** Is it true that for every $p \in (1, \infty)$ and $q \in [1, \infty]$ there exists a constant $C_{p,q} > 0$ such that for every $d \in \mathbb{N}$ we have

$$C(B^q(d), p) \leq C_{p,q}?
A few comments about these conjectures are in order.

(i) The first conjecture arose on the one hand in view of the inequalities (1.6) and (1.10), and on the other hand in view of the result from [7], where it had been verified for the Euclidean balls. Indeed, if \( q = 2 \) then following [7, Section 5] one can see that (1.10) holds with \( ad \) in place of \( d^{1+1/q} \), where \( a > 0 \) is a large absolute constant independent of \( d \). Since the dimension-free phenomenon may only break down for small scales, the conjectured threshold from which we can expect dimension-free estimates in the discrete setup is at the level of a constant multiple of \( d \).

(ii) The second conjecture says that one should expect dimension-free estimates for \( C(B^q(d), p) \) corresponding to \( q \)-balls as we have for their continuous counterparts \( C(B^q(d), p) \). It was verified [6] in the case of cubes \( C(B^\infty(d), p) \) with \( p \in (3/2, \infty] \) as we have seen in (1.8). Moreover, in [8] the second and fourth authors in collaboration with Bourgain and Stein proved that the discrete dyadic Hardy–Littlewood maximal function

\[
\sup_{n \in \mathbb{N}} |M_{2^n}^{B^2} f|
\]

over the Euclidean balls \( B^2(d) \) has dimension-free estimates on \( \ell^2(\mathbb{Z}^d) \). Although this can be thought of as the first step towards establishing (1.11), the general case seems to be very difficult even for the Euclidean balls or cubes for \( p \in (1, 3/2] \). This will surely require new methods.

In our second main result of this paper we verify the first conjecture for the balls \( B^q \) for all \( q \in (2, \infty) \).

**Theorem 2:** For every \( q \in (2, \infty) \) and each \( a > 0 \) and \( p \in (1, \infty) \) there exists \( C(p, q, a) > 0 \) independent of the dimension \( d \in \mathbb{N} \) such that for all \( f \in \ell^p(\mathbb{Z}^d) \) we have

\[
\| \sup_{N \geq ad} |M_N^{B^q} f|\|_{\ell^p(\mathbb{Z}^d)} \leq C(p, q, a)\|f\|_{\ell^p(\mathbb{Z}^d)}.
\]

The proof of Theorem 2 is presented in Section 3; it relies on the methods developed in [7, Section 5], Hanner’s inequality (3.1) and Newton’s generalized binomial theorem. It follows from [7, Theorem 2] that Theorem 2 remains true for \( q = 2 \), but only for \( a > 0 \), which is large enough. Our proof of Theorem 2 can
be easily adapted to yield the same result. We point out necessary changes in the proof. Since we rely on Hanner’s inequality our proof of Theorem 2 does not carry over to \( q \in [1, 2) \). This is because for such values of \( q \) the inequality (3.1) is reversed.

Our final result concerns the dyadic maximal operator associated with \( q \)-balls.

**Theorem 3:** Fix \( q \in [2, \infty) \). Let \( C_1, C_2 > 0 \) and define
\[
\mathbb{D}_{C_1, C_2} := \{ N \in \mathbb{D} : C_1 d^{1/q} \leq N \leq C_2 d \}.
\]

Then there exists a constant \( C_q > 0 \) independent of the dimension such that for every \( f \in \ell^2(\mathbb{Z}^d) \) we have
\[
\| \sup_{N \in \mathbb{D}_{C_1, C_2}} |M_{B_N^q} f|\|_{\ell^2(\mathbb{Z}^d)} \leq C_q \| f \|_{\ell^2(\mathbb{Z}^d)}.
\]
(1.12)

Theorem 3 is an incremental step towards establishing the second conjecture. By adapting the ideas developed in [8] we are able to obtain dimension-free estimates for the discrete restricted dyadic Hardy–Littlewood maximal functions over \( q \)-balls for all \( q \in [2, \infty) \). Theorem 3 generalizes [6, Theorem 2.2], which was stated for \( q = 2 \). The proof of inequality (1.12) as in [8] exploits the invariance of \( B_N^q \cap \mathbb{Z}^d \) under the permutation group of \( \mathbb{N}_d \). Then we can efficiently use probabilistic arguments on a permutation group corresponding to \( \mathbb{N}_d \) that reduce the matter to the decrease dimension trick as in [8]. The proof of Theorem 3 is a technical elaboration of the methods from [8]. However, for the convenience of the reader, mainly due to intricate technicalities we decided to provide necessary details in Section 4. We remark that the condition \( q \in [2, \infty) \) cannot be dropped in our proof of Theorem 3 as it is required in the estimate at the origin from Proposition 4.1.

1.2. Notation. The following basic notation will be used throughout the paper.

1. We will write \( A \lesssim_\delta B \) (\( A \gtrsim_\delta B \)) to say that there is an absolute constant \( C_\delta > 0 \) (which depends on a parameter \( \delta > 0 \)) such that \( A \leq C_\delta B \) (\( A \geq C_\delta B \)). We will write \( A \asymp_\delta B \) when \( A \lesssim_\delta B \) and \( A \gtrsim_\delta B \) hold simultaneously. We shall abbreviate subscript \( \delta \) if irrelevant.

2. Let \( \mathbb{N} := \{1, 2, \ldots\} \) be the set of positive integers and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), and \( \mathbb{D} := \{2^n : n \in \mathbb{Z}\} \) will denote the set of dyadic numbers. We set \( \mathbb{N}_N := \{1, 2, \ldots, N\} \) for any \( N \in \mathbb{N} \).
(3) For a measurable set $A \subseteq \mathbb{R}^d$ we denote by $|A|$ the Lebesgue measure of $A$ and by $|A \cap \mathbb{Z}^d|$ the number of lattice points in $A$.

(4) The Euclidean space $\mathbb{R}^d$ is endowed with the standard inner product

$$x \cdot \xi := \langle x, \xi \rangle := \sum_{k=1}^{d} x_k \xi_k$$

for every two vectors $x = (x_1, \ldots, x_d)$ and $\xi = (\xi_1, \ldots, \xi_d)$ from $\mathbb{R}^d$. Let $|x|_2 := \sqrt{\langle x, x \rangle}$ denote the Euclidean norm of a vector $x \in \mathbb{R}^d$. The Euclidean open ball centered at the origin with radius one will be denoted by $B^2$. We shall abbreviate $B^2$ to $B$ and $| \cdot |_2$ to $| \cdot |$.

(5) Let $(X, \mu)$ be a measure space $X$ with a $\sigma$-finite measure $\mu$. The space of all measurable functions whose modulus is integrable with $p$-th power is denoted by $L^p(X)$ for $p \in (0, \infty)$, whereas $L^\infty(X)$ denotes the space of all measurable essentially bounded functions. The space of all measurable functions that are weak type $(1, 1)$ will be denoted by $L^{1,\infty}(X)$. In our case we will usually have $X = \mathbb{R}^d$ or $X = \mathbb{T}^d$ equipped with Lebesgue measure, and $X = \mathbb{Z}^d$ endowed with counting measure. If $X$ is endowed with counting measure we will abbreviate $L^p(X)$ to $\ell^p(X)$ and $L^{1,\infty}(X)$ to $\ell^{1,\infty}(X)$. If the context causes no confusions we will also abbreviate $\| \cdot \|_{L^p(\mathbb{R}^d)}$ to $\| \cdot \|_{L^p}$ and $\| \cdot \|_{\ell^p(\mathbb{Z}^d)}$ to $\| \cdot \|_{\ell^p}$.

(6) Let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}^d$ defined for any function $f \in L^1(\mathbb{R}^d)$ as

$$\mathcal{F} f(\xi) := \int_{\mathbb{R}^d} f(x) e^{2\pi i \xi \cdot x} \, dx \quad \text{for any } \xi \in \mathbb{R}^d.$$ 

For $f \in \ell^1(\mathbb{Z}^d)$ we define its discrete Fourier transform by setting

$$\hat{f}(\xi) := \sum_{x \in \mathbb{Z}^d} f(x) e^{2\pi i \xi \cdot x} \quad \text{for any } \xi \in \mathbb{T}^d,$$

where $\mathbb{T}^d$ denotes the $d$-dimensional torus, which will be identified with

$$Q := [-1/2, 1/2]^d.$$ 

To simplify notation we will denote by $\mathcal{F}^{-1}$ the inverse Fourier transform on $\mathbb{R}^d$ or the inverse Fourier transform (Fourier coefficient) on the torus $\mathbb{T}^d$. It will cause no confusions since their meaning will be always clear from the context.
2. Transference of strong and weak type inequalities: Proof of Theorem 1

Here we elaborate the arguments from the master thesis of the first author to prove Theorem 1. The general idea behind the proof of (1.3) is as follows. We fix a nonnegative bump function $F: \mathbb{R}^d \to \mathbb{R}$ for which the constant in the corresponding maximal inequality is almost $C(G,p)$. Since dilations are available in the continuous setting, $F$ can be taken to be very slowly varying. Then we sample the values of $F$ at lattice points to produce $f: \mathbb{Z}^d \to \mathbb{R}$. Because $F$ is regular, the norms of $F$ and $f$ are almost the same. Moreover, we deduce that $M^G F$ cannot be essentially larger than $M^G f$. Indeed, for $F$ being slowly varying its maximal function is slowly varying as well. Also, given $n \in \mathbb{Z}^d$ we see that $M^G_t F(n)$ is certainly not much greater than $f(n)$, unless $t$ is very large. For large values of $t$, in turn, the sets $G_t \cap \mathbb{Z}^d$ are regular, making the quantities $M^G_t F(n)$ and $M^G_t f(n)$ comparable to each other. The constant in the maximal inequality associated with $f$ is then at least not much smaller than $C(G,p)$. Thus, (1.3) holds.

**Proof of Theorem 1.** Let $G \subset \mathbb{R}^d$ be a convex symmetric body. Let $r \in (0,1)$ and $R > 1$ be real numbers such that $B_r \subset G \subset B_R$, where $B_t$ is the Euclidean ball centered at the origin with radius $t > 0$. We may assume that $p \in [1, \infty)$, otherwise there is nothing to do. We now distinguish three cases. In the first two cases we prove (1.3) for arbitrary $G$ and all $p \in [1, \infty)$. In the third case we show that the equality is attained in (1.3) if $G = B^\infty$ and $p = 1$.

**Case 1, when $p \in (1, \infty)$** Fix $\eta \in (0,1)$ and take $F \in C^\infty_c(\mathbb{R}^d)$ such that $F \geq 0$ and

\[(2.1) \quad \|M^G F\|_{L^p(\mathbb{R}^d)} \geq (1 - \eta)C(G,p)\|F\|_{L^p(\mathbb{R}^d)}.\]

For each $K \in \mathbb{N}$ let us define $F_K$ by setting $F_K(x) := F(\frac{x}{K})$. Note that

\[\|F_K\|_{L^p(\mathbb{R}^d)} = K^{d/p}\|F\|_{L^p(\mathbb{R}^d)}.\]

Moreover, since $M^G F_K(x) = M^G F(\frac{x}{K})$, we have

\[\|M^G F_K\|_{L^p(\mathbb{R}^d)} = K^{d/p}\|M^G F\|_{L^p(\mathbb{R}^d)}.\]
Next, we define $f_K: \mathbb{Z}^d \to [0, \infty)$ by setting $f_K(n) := F_K(n)$ for $n \in \mathbb{Z}^d$. Then we immediately have

$$\|F\|_{L^p(\mathbb{R}^d)} = \lim_{K \to \infty} \left( \frac{1}{K^d} \sum_{n \in \mathbb{Z}^d} f_K(n)^p \right)^{1/p}. \quad (2.2)$$

Thus for all sufficiently large $K \in \mathbb{N}$ (say $K \geq K_1$) we see that

$$\|f_K\|_{\ell^p(\mathbb{Z}^d)} \leq (1 + \eta) K^{d/p} \|F\|_{L^p(\mathbb{R}^d)}. \quad (2.3)$$

Choose $N \in \mathbb{N}$ such that

$$\|M_G F \cdot 1_{[-N,N]^d}\|_{L^p(\mathbb{R}^d)} \geq (1 - \eta) \|M_G F\|_{L^p(\mathbb{R}^d)}. \quad (2.4)$$

In a similar way as in (2.2), we conclude that there exists $K_2$ such that for all $K \geq K_2$ we have

$$\left( \frac{1}{K^d} \sum_{n \in \mathbb{Z}^d \cap [-NK,NK]^d} M_G F_K(n)^p \right)^{1/p} \geq (1 - \eta) \|M_G F \cdot 1_{[-N,N]^d}\|_{L^p(\mathbb{R}^d)}. \quad (2.5)$$

Let $\kappa > 0$ be such that $M_G F_K(n) \geq \kappa$ for each $K \in \mathbb{N}$ and $n \in \mathbb{Z}^d \cap [-NK,NK]^d$. We fix $\varepsilon \in (0, \eta \kappa / 2)$ and take $\delta > 0$ for which $|x - y| < \delta$ implies $|F(x) - F(y)| < \varepsilon$. Since $G_t \subset B_{tR}$, we obtain

$$M_t^G F(x) \leq F(x) + \varepsilon, \quad t \in (0, \delta R^{-1}),$$

or, equivalently,

$$M_t^G F_K(x) \leq F_K(x) + \varepsilon, \quad t \in (0, K \delta R^{-1}).$$

Our goal is to prove that

$$\mathcal{M}_*^G f_K(n) \geq (1 - \eta) M_*^G F_K(n), \quad n \in \mathbb{Z}^d \cap [-NK,NK]^d, \quad (2.6)$$

if $K$ is large enough. To this end we shall show separately that

$$\mathcal{M}_*^G f_K(n) \geq M_t^G F_K(n) - \eta \kappa, \quad t \in (0, K \delta R^{-1}),$$

and

$$\mathcal{M}_*^G f_K(n) \geq (1 - \eta/2) M_t^G F_K(n) - \eta \kappa/2, \quad t \geq K \delta R^{-1}.$$
Hence, we are reduced to proving the second estimate for $M^G_t F_K(n)$ in the case $t \geq K \delta R^{-1}$. Let $\rho \in (0, 1)$ be such that $|G_{1+2\rho}| \leq (1 - \eta/2)^{-1}|G|$ and assume that $K \geq K_3 := \sqrt{dR/(r\delta \rho)}$. Therefore, for each $t \geq K \delta R^{-1}$ we have

$$t + \sqrt{d}/r \leq (1 + \rho)t.$$ 

Let $Q_m := m + B_{1/2}^{\infty}$ be the cube centered at $m \in \mathbb{Z}^d$ and of side length 1. If $Q_m \cap G_t \neq \emptyset$ for some $m \in \mathbb{Z}^d$, then one can easily see that

$$Q_m \subseteq G_{t+\sqrt{d}/r} \subseteq G_{(1+\rho)t},$$

provided $t \geq K_3 \delta R^{-1}$. Consequently, we conclude that

$$M^G_{t+\sqrt{d}/r} f_K(n) = \frac{1}{|G_{t+\sqrt{d}/r} \cap \mathbb{Z}^d|} \sum_{m \in G_{t+\sqrt{d}/r} \cap \mathbb{Z}^d} f_K(n-m) \geq \frac{1}{|G_{t+\sqrt{d}/r} \cap \mathbb{Z}^d|} \sum_{m \in G_{t+\sqrt{d}/r} \cap \mathbb{Z}^d} \left( \int_{Q_m} F_K(n-x) \, dx - \varepsilon \right) \geq \left( \frac{1}{|G_{t+\sqrt{d}/r} \cap \mathbb{Z}^d|} \int_{G_t} F_K(n-x) \, dx \right) - \varepsilon \geq \frac{|G_t|}{|G_{(1+\rho)t} \cap \mathbb{Z}^d|} M^G_t F_K(n) - \eta \kappa/2,$$

where in the first inequality we have used that $K \geq K_3 \geq \sqrt{d}/\delta$. Hence, it remains to show that

$$|G_t| \geq (1 - \eta/2) |G_{(1+\rho)t} \cap \mathbb{Z}^d|.$$ 

We notice that if $m \in G_{(1+\rho)t} \cap \mathbb{Z}^d$, then

$$Q_m \subseteq G_{(1+\rho)t+\sqrt{d}/r} \subseteq G_{(1+2\rho)t}.$$ 

Thus we obtain

$$|G_{(1+\rho)t} \cap \mathbb{Z}^d| \leq |G_{(1+2\rho)t}| \leq (1 - \eta/2)^{-1} |G_t|.$$ 

Finally, combining (2.1), (2.3), (2.4), (2.5), and (2.6), we have

$$\|M^G_* f_K\|_{\ell^p(\mathbb{Z}^d)} \geq \frac{(1 - \eta)^4}{1 + \eta} C(G, p) \|f_K\|_{\ell^p(\mathbb{Z}^d)}$$

for any $K \geq \max\{K_1, K_2, K_3\}$. Hence, since $\eta \in (0, 1)$ was arbitrary, we conclude that

$$C(G, p) \geq C(G, p).$$
Case 2, when $p = 1$ The inequality $C(G, 1) \geq C(G, 1)$ can be deduced in a similar way as was done for $p \in (1, \infty)$. We only describe necessary changes. Namely, as in (2.1) we fix $\eta \in (0, 1)$ and take $F \in C_c^\infty(\mathbb{R}^d)$ such that $F \geq 0$ and

$$
\|M_*^G F\|_{L^1,\infty(\mathbb{R}^d)} \geq (1 - \eta)C(G, 1)\|F\|_{L^1(\mathbb{R}^d)}.
$$

Then we choose $N \in \mathbb{N}$ such that

$$
\|M_*^G F \cdot 1_{[-N,N]^d}\|_{L^1,\infty(\mathbb{R}^d)} \geq (1 - \eta)\|M_*^G F\|_{L^1,\infty(\mathbb{R}^d)}.
$$

It is easy to see that for each $x_0, y_0 \in \mathbb{R}^d$ one has

$$
|M_*^G F(x_0) - M_*^G F(y_0)| \leq \sup_{|x - y| = |x_0 - y_0|} |F(x) - F(y)|.
$$

This allows us to deduce for sufficiently large $K \in \mathbb{N}$ that

$$
\|M_*^G F_K \cdot 1_{Z^d \cap [-NK,NK]^d}\|_{\ell^1,\infty(Z^d)} \geq (1 - \eta)K^d\|M_*^G F \cdot 1_{[-N,N]^d}\|_{L^1,\infty(\mathbb{R}^d)}
$$

with the function $F_K$ as in the previous case. From now on we may proceed in much the same way as in the previous case to establish (2.6). Once (2.6) is proved we combine (2.7), (2.3) (with $p = 1$), (2.8), (2.9) and (2.6) to obtain

$$
\|M_*^G f_K\|_{\ell^1,\infty(Z^d)} \geq \frac{(1 - \eta)^4}{1 + \eta} C(G, 1)\|f_K\|_{\ell^1(Z^d)}
$$

for any $K \geq \max\{K_1, K_2, K_3\}$. Since $\eta \in (0, 1)$ was arbitrary, we conclude that $C(G, 1) \geq C(G, 1)$ as desired. This completes the first part of Theorem 1. We now turn our attention to the case $G = B^\infty$ and show that the last inequality can be reversed.

Case 3, when $p = 1$ and $G = B^\infty = [-1, 1]^d$ Given $\eta \in (0, 1)$ consider $f \in \ell^1(Z^d)$ and $\lambda > 0$ such that $f \geq 0$ and

$$
\lambda |\{n \in Z^d : M_*^{B^\infty} f(n) > \lambda\}| \geq (1 - \eta)C(B^\infty, 1)\|f\|_{\ell^1(Z^d)}.
$$

Let $Q_\delta(x) := x + B^\infty$ denote the cube centered at $x \in \mathbb{R}^d$ and of side length $\delta > 0$. For $\delta \in (0, 1)$, we set

$$
F_\delta(x) := \sum_{n \in Z^d} f(n) |Q_\delta(n)|^{-1} 1_{Q_\delta(n)}(x).
$$

Clearly $\|F_\delta\|_{L^1(\mathbb{R}^d)} = \|f\|_{\ell^1(Z^d)}$, this is the place where it is essential that we are working with $p = 1$. 

We show that for each $n \in \mathbb{Z}^d$ one has
\begin{equation}
M_{B}^{\infty} F_{\delta}(x) \geq M_{N}^{\infty} f(n), \quad x \in Q_{1-\delta}(n).
\end{equation}

To prove (2.10) we note that
\[ M_{B}^{\infty} f(n) = \sup_{t > 0} M_{B}^{\infty} f(t) = \sup_{N \in \mathbb{N}_0} M_{N}^{\infty} f(n), \]
since $|B_{t}^{\infty} \cap \mathbb{Z}^d| = |B_{\lfloor t \rfloor}^{\infty} \cap \mathbb{Z}^d| = (2 \lfloor t \rfloor + 1)^d$, where $M_{0}^{\infty} f := f$.

If $N = 0$, then for each $n \in \mathbb{Z}^d$ and $x \in Q_{1-\delta}(n)$ we obtain
\[ M_{0}^{\infty} f(n) = f(n) = \int_{Q_{1}(x)} F_{\delta}(y) dy \leq M_{B}^{\infty} F_{\delta}(x). \]

If $N \in \mathbb{N}$, then $n + B_{N+1-\delta/2}^{\infty} \subseteq x + B_{N+1-\delta/2}^{\infty}$ for each $n \in \mathbb{Z}^d$ and $x \in Q_{1-\delta}(n)$. Therefore,
\[ M_{N}^{\infty} f(n) = \frac{1}{|n + B_{N}^{\infty} \cap \mathbb{Z}^d|} \sum_{k \in n + B_{N}^{\infty} \cap \mathbb{Z}^d} f(k) \leq \frac{1}{|n + B_{N+1/2}^{\infty} \cap \mathbb{Z}^d|} \int_{x + B_{N+1/2}^{\infty}} F_{\delta}(y) dy \leq M_{B}^{\infty} F_{\delta}(x), \]
since $|n + B_{N}^{\infty} \cap \mathbb{Z}^d| = (2N + 1)^d = |x + B_{N+1/2}^{\infty}|$. Hence (2.10) follows, and consequently we obtain
\[ \lambda \{ x \in \mathbb{R}^d : M_{B}^{\infty} F_{\delta}(x) > \lambda \} \geq \lambda (1 - \delta)^d \{ n \in \mathbb{Z}^d : M_{N}^{\infty} f(n) > \lambda \} \geq (1 - \eta)(1 - \delta)^d C(B^{\infty}, 1) \| F_{\delta} \|_{L^1(\mathbb{R}^d)}. \]

Since $\eta$ and $\delta$ were arbitrary, we conclude that $C(B^{\infty}, 1) \geq C(B^{\infty}, 1)$. Finally, combining this inequality with the previous result we obtain
\[ C(B^{\infty}, 1) = C(B^{\infty}, 1) \]
and the proof of Theorem 1 is completed.

3. Discrete maximal operator for $B^q$ and large scales: Proof of Theorem 2

The purpose of this section is to prove Theorem 2. We will follow the ideas from [7, Section 5]. From now on we assume that $q \in (2, \infty)$ is fixed. By $Q := B_{1/2}^{\infty}$ we mean the unit cube centered at the origin.
LEMMA 3.1: Let \( \tilde{C}_q := \max \{ \sum_{k=1}^{\infty} |(\frac{q}{2k})| 2^{-2k}, (\frac{3}{2})^q \} \). If \( N \geq d^{\frac{d}{2} + \frac{1}{4}} \), then

\[
|B_N^q \cap \mathbb{Z}^d| \leq 2|B_N^q| \leq 2e^{\frac{\tilde{C}_q}{N}}|B_N^q|,
\]

where \( N_1 := N(1 + d^{-1} \tilde{C}_q)^{\frac{1}{4}} \).

Proof. Since \( q > 2 \), by Hanner’s inequality for \( x \in B_N^q \cap \mathbb{Z}^d \) and \( y \in Q \) we obtain

\[
|x + y|^q + |x - y|^q \leq (|x|_q + |y|_q)^q + |x|_q - |y|_q|^q.
\]

Moreover, for every \( x \in B_N^q \) we have

\[
|\{ y \in Q : |x + y|_q > |x - y|_q \}| = |\{ y \in Q : |x + y|_q < |x - y|_q \}|
\]

which implies \( |\{ y \in Q : |x + y|_q \leq |x - y|_q \}| \geq 1/2 \). Hence,

\[
|B_N^q \cap \mathbb{Z}^d| = \sum_{x \in B_N^q \cap \mathbb{Z}^d} 1 \leq 2 \sum_{x \in B_N^q \cap \mathbb{Z}^d} \int_Q \mathbf{1}_{\{ y \in \mathbb{R}^d : |x + y|_q \leq |x - y|_q \}}(z) \, dz
\]

\[
\leq 2 \sum_{x \in B_N^q \cap \mathbb{Z}^d} \int_Q \mathbf{1}_{\{ y \in \mathbb{R}^d : 2|x + y|_q \leq (|x|_q + |y|_q)^q + |x|_q - |y|_q^q \}}(z) \, dz.
\]

We shall estimate \( (|x|_q + |y|_q)^q + |x|_q - |y|_q^q \) for \( x \in B_N^q \cap \mathbb{Z}^d \) and \( y \in Q \). Let us firstly assume that \( |x|_q \geq 2|y|_q \). By Newton’s generalized binomial theorem we have

\[
(|x|_q + |y|_q)^q + |x|_q - |y|_q^q = \sum_{k=0}^{\infty} \left( \begin{array}{c} q \\ k \end{array} \right) |y|_q^k |x|_q^{q-k} + \sum_{k=0}^{\infty} \left( \begin{array}{c} q \\ k \end{array} \right) (-1)^k |y|_q^k |x|_q^{q-k}
\]

\[
= 2 \sum_{k=0}^{\infty} \left( \begin{array}{c} q \\ 2k \end{array} \right) |y|_q^{2k} |x|_q^{q-2k}
\]

\[
\leq 2 \left( |x|_q^q + |x|_q^{q-2} |y|_q^2 \sum_{k=1}^{\infty} \left( \begin{array}{c} q \\ 2k \end{array} \right) 2^{-2k+2} \right)
\]

\[
\leq 2 \left( N^q + N^{q-2} d^{\frac{2}{q}} \sum_{k=1}^{\infty} \left( \begin{array}{c} q \\ 2k \end{array} \right) 2^{-2k} \right)
\]

\[
\leq 2N^q \left( 1 + d^{-1} \sum_{k=1}^{\infty} \left( \begin{array}{c} q \\ 2k \end{array} \right) 2^{-2k} \right).
\]
On the other hand, if $|x|_q \leq 2|y|_q$, then

\[
(|x|_q + |y|_q)^q + ||x| - |y|_q|^q \leq 2(3|y|_q)^q \leq 2\left(\frac{3}{2}\right)^q d
\]
\[
\leq 2N^q d^{-\frac{q}{2}}\left(\frac{3}{2}\right)^q \leq 2N^q d^{-1}\left(\frac{3}{2}\right)^q.
\]

Combining the above gives

\[
(|x|_q + |y|_q)^q + ||x| - |y|_q|^q \leq 2N^q(1 + d^{-1}\tilde{C}_q) = 2N^q_1.
\]

Finally, by (3.2) and (3.3) we obtain

\[
|B^q_N \cap \mathbb{Z}^d| \leq 2 \sum_{x \in B^q_N \cap \mathbb{Z}^d} \int_Q \mathbb{1}_{\{y \in \mathbb{R}^d : |x+y|_q \leq N(1+d^{-1}\tilde{C}_q)^{1/q}\}}(z) \, dz
\]
\[
= 2 \sum_{x \in B^q_N \cap \mathbb{Z}^d} \int_Q \mathbb{1}_{B^q_N(1+d^{-1}\tilde{C}_q)^{1/q}}(x+z) \, dz
\]
\[
\leq 2|B^q_{N_1}|
\]
\[
= 2(1 + d^{-1}\tilde{C}_q)^{\frac{d}{q}}|B^q_N|
\]
\[
\leq 2e^{\tilde{C}_2^q/2}|B^q_N|,
\]

which finishes the proof.  

We remark that Lemma 3.1 holds for $q = 2$ without any changes with $\tilde{C}_2 = 9/4$, so that $e^{\tilde{C}_2^q/2} = e^{9/8}$.

**Lemma 3.2:** Let $a > 0$. Take $N \geq ad$, $0 \leq j \leq N - 1$, and $x \in \mathbb{R}^d$ such that

\[
N\left(1 + \frac{j}{N}\right)^{\frac{q}{N}} \leq |x|_q \leq N\left(1 + \frac{j+1}{N}\right)^{\frac{q}{N}}.
\]

If $d \in \mathbb{N}$ is sufficiently large (depending only on $a$ and $q$), then

\[
|Q \cap (B^q_N - x)| = |\{y \in Q : x + y \in B^q_N\}| \leq e^{-\frac{x}{128q^2}j^2}.
\]

**Proof.** Note that the claim is trivial for $j = 0$. Therefore, let $1 \leq j \leq N - 1$ and take $x \in \mathbb{R}^d$ such that (3.4) holds. Assume that $y \in Q$ and $x + y \in B^q_N$. Then

\[
|x|_q^q - |x + y|_q^q \geq N^q\left(1 + \frac{j}{N}\right) - N^q = jN^{q-1}.
\]
We shall also estimate the expression on the left hand side of the above inequality from above. Let \( I_i := |x_i|^q - |x_i + y_i|^q \), then
\[
|x_i|^q - |x + y|^q = \sum_{i=1}^{d} I_i.
\]

For \(|x_i| > 2|y_i|\), by Newton’s generalized binomial theorem we have
\[
I_i = |x_i|^q - |x_i| + \text{sgn}(x_i)y_i|^q = |x_i|^q - \sum_{k=0}^{\infty} \binom{q}{k} |x_i|^{q-k} \text{sgn}(x_i)^k y_i^k
\]
\[
\leq |x_i|^q - |x_i|^q + q|x_i|^{q-1} \text{sgn}(x_i) y_i + \sum_{k=2}^{\infty} \binom{q}{k} |x_i|^{q-k} |y_i|^k
\]
\[
\leq -q|x_i|^{q-1} \text{sgn}(x_i) y_i + \sum_{k=2}^{\infty} \binom{q}{k} |x_i|^{q-k} |y_i|^k
\]
\[
\leq -q|x_i|^{q-1} \text{sgn}(x_i) y_i + |x_i|^{q-2}|y_i|^2 \sum_{k=2}^{\infty} \binom{q}{k} |2^{2-k}.
\]

On the other hand, if \(|x_i| \leq 2|y_i| \leq 1\), and \( A_q := 1 + (\frac{3}{2})^q + q\), then
\[
I_i \leq (2|y_i|)^q + (3|y_i|)^q \leq 1 + \left(\frac{3}{2}\right)^q \leq -q|x_i|^{q-1} \text{sgn}(x_i) y_i + A_q.
\]

Let \( \tilde{x} = (|x_1|^{q-1} \text{sgn}(x_1), \ldots, |x_d|^{q-1} \text{sgn}(x_d))\). Combining the above we obtain by Hölder’s inequality
\[
|x|^q - |x + y|^q \leq -q\langle \tilde{x}, y \rangle + A_q d + \sum_{k=2}^{\infty} \binom{q}{k} 2^{2-k} d \sum_{i=1}^{d} |x_i|^{q-2}|y_i|^2
\]
\[
\leq -q\langle \tilde{x}, y \rangle + A_q d + \sum_{k=2}^{\infty} \binom{q}{k} 2^{2-k}|x|^q |y|^2_q
\]
\[
\leq -q\langle \tilde{x}, y \rangle + A_q d + N^{q-2} d^{\frac{2}{q}} \sum_{k=2}^{\infty} \binom{q}{k} 2^{-k+1}
\]
\[
\leq -q\langle \tilde{x}, y \rangle + \left( A_q + \sum_{k=2}^{\infty} \binom{q}{k} 2^{-k+1} \right) N^{q-2} d^{\frac{2}{q}};
\]
to ensure the validity of the last inequality above we take \( d \) so large that \( ad \geq d^{1/q} \).
By (3.5) and the previous display, since \( q > 2 \), we obtain
\[
(3.6) \quad -q \langle \tilde{x}, y \rangle \geq jN^{q-1} - \left( A_q + \sum_{k=2}^{\infty} \left( \binom{q}{k} \right) 2^{-k+1} \right) N^{q-2} \left( \frac{2}{d} \right)^{\frac{3}{2}} \geq \frac{1}{2} jN^{q-1},
\]
provided that \( d \) is so large that
\[
(3.7) \quad \left( A_q + \sum_{k=2}^{\infty} \left( \binom{q}{k} \right) 2^{-k+1} \right) a^{-1} \left( \frac{2}{d} \right)^{\frac{3}{2}} \leq \frac{1}{2}.
\]
Note that
\[
|\tilde{x}|_2 = |x|_2^{q-1} \leq |x|_q^{q-1} \leq 2N^{q-1}.
\]
Hence, for \( y \in Q \) and \( x + y \in B_q^N \), we obtain
\[
\langle -\tilde{x}, y \rangle \geq \frac{j}{4q}.
\]
We know from [7, Inequality (5.6)] that for every unit vector \( z \in \mathbb{R}^d \) and for every \( s > 0 \) we have
\[
|\{ y \in Q : \langle z, y \rangle \geq s \}| \leq e^{-\frac{1}{\pi} \frac{s^2}{2}}.
\]
Applying this inequality for \( z = -\frac{\tilde{x}}{|\tilde{x}|_2} \) and \( s = j/(4q) \) we arrive at
\[
|\{ y \in Q : x + y \in B_N^q \}| \leq \left| \left\{ y \in Q : \langle -\frac{\tilde{x}}{|\tilde{x}|_2}, y \rangle \geq \frac{j}{4q} \right\} \right| \leq e^{-\frac{\pi j^2}{128q^2}}.
\]
This concludes the proof of the lemma. \( \blacksquare \)

A comment is in order here. For \( q = 2 \), a version of Lemma 3.2 holds for all \( d \in \mathbb{N} \) under the restriction
\[
a \geq 2 \left( 1 + \left( \frac{3}{2} \right)^2 + 2 \sum_{k=2}^{\infty} \left( \binom{2}{k} \right) 2^{-k+1} \right) = 2(1 + (3/2)^2 + 2 + 1/2) = \frac{23}{2}.
\]
The above ensures that (3.7) (and hence also (3.6)) is satisfied for all \( d \in \mathbb{N} \).

**Lemma 3.3:** Let \( a > 0 \). If \( d \in \mathbb{N} \) is sufficiently large (depending on \( a \) and \( q \)), then there exists a constant \( C'_q > 0 \) depending only on \( q \) such that for all \( N \geq ad \) one has
\[
|B_N^q| \leq C'_q |B_N^q \cap \mathbb{Z}^d|.
\]
Proof. For each $a > 0$ we take $J := J_{q,a} \in \mathbb{N}$ satisfying

$$\sum_{j \geq J} e^{-\frac{\tau_j^2}{128q^2}} e^{2(j+1)/(aq)} \leq \frac{1}{4} e^{-\tilde{C}_q/q},$$

where $\tilde{C}_q > 0$ is the constant specified in Lemma 3.1. It suffices to show that for every $M \geq \frac{ad}{2}$ we have

$$|B^q_M| \leq 2|B^q_M(1+J/M)^{1/q} \cap \mathbb{Z}^d|.$$  

Indeed, take $d$ so large that $ad \geq 2J$. Then for $N \geq ad$ we can find $M \in \left[\frac{ad}{2}, N\right]$ such that $N = M(1 + J/M)^{1/q}$ and thus (3.9) gives

$$|B^q_N| = (1 + J/M)^{d/q}|B^q_M| \leq 2e^{2J/(aq)}|B^q_N \cap \mathbb{Z}^d|.$$

Our aim now is to prove (3.9). Define

$$U_j = \left\{ x \in \mathbb{R}^d : M\left(1 + \frac{j}{M}\right)^{1/q} < |x| \leq M\left(1 + \frac{(j+1)}{M}\right)^{1/q} \right\}$$

for $j \geq 0$ and observe that

$$|B^q_M| = \sum_{x \in \mathbb{Z}^d} \int_Q 1_{B^q_M}(x+y)dy$$

$$= \sum_{x \in B^q_M \cap \mathbb{Z}^d} \int_Q 1_{B^q_M}(x+y)dy + \sum_{0 \leq j < J} \sum_{x \in U_j \cap \mathbb{Z}^d} \int_Q 1_{B^q_M}(x+y)dy$$

$$+ \sum_{j \geq J} \sum_{x \in U_j \cap \mathbb{Z}^d} \int_Q 1_{B^q_M}(x+y)dy$$

$$\leq |B^q_M(1+J/M)^{1/q} \cap \mathbb{Z}^d| + \sum_{j=J}^{M-1} \sum_{x \in U_j \cap \mathbb{Z}^d} |Q \cap (B^q_M - x)|.$$

Indeed, if $j \geq M$, then

$$|Q \cap (B^q_M - x)| = 0$$

holds for each $x \in U_j$; here we take $d$ so large that $ad(2^{1/q} - 1) \geq d^{1/q}$. Clearly $M \geq \frac{ad}{2} \geq d^{2+\frac{2}{q}}$, if $d$ is large enough. Now applying Lemma 3.2 (with $M$ in place of $N$ and $a/2$ in place of $a$) together with Lemma 3.1 we see
that for sufficiently large \( d \) one has
\[
\sum_{j=J}^{M-1} \sum_{x \in U_j \cap \mathbb{Z}^d} |Q \cap (B^q_M - x)| \leq \sum_{j=J}^{M-1} \sum_{x \in U_j \cap \mathbb{Z}^d} e^{-\frac{j^2}{128q^2} e^{2(j+1)/(aG)}} e^{-\frac{j}{128q^2} e^{2(j+1)/(aG)}}.
\]
Hence, by (3.8) we have
\[
|B^q_M| \leq |B^q_M (1+J/M)^{1/q} \cap \mathbb{Z}^d| + \frac{1}{2} |B^q_M|,
\]
which finishes the proof.

We make a similar remark as below Lemma 3.2. For \( q = 2 \), the claim of Lemma 3.3 holds for all \( d \in \mathbb{N} \) provided that the \( a > 0 \) is large enough. In this case it suffices to take
\[
a \geq \max(23, 2J_{2,23}),
\]
where \( J := J_{2,23} \) is a nonnegative integer satisfying
\[
\sum_{j \geq J} e^{-\frac{j^2}{128q^2} e^{2(j+1)/23}} \leq \frac{1}{4} e^{-\frac{q}{8}}.
\]
Then the implied constant \( C'_2 \) equals \( 2e^{J/23} \). Finally, in the proof of Theorem 2 below it suffices to take \( N \geq C'_2 d \). We leave the details to the interested reader.

**Proof of Theorem 2.** Fix \( p \in (1, \infty) \). It is well known that for any \( q \in (2, \infty) \) and any \( d \in \mathbb{N} \) one has
\[
\| \sup_{N > 0} |\mathcal{M}^B_N f| \|_{\ell^p(\mathbb{Z}^d)} \leq C_d(p, q) \| f \|_{\ell^p(\mathbb{Z}^d)},
\]
with a constant \( C_d(p, q) \) which depends on the dimension \( d \). Thus we may assume that the dimension \( d \) is large enough, in fact larger than any fixed number \( d_0 \) (which may depend on \( q \in (2, \infty) \) and \( a > 0 \)).
Let \( f : \mathbb{Z}^d \rightarrow \mathbb{C} \) be a nonnegative function. Define \( F : \mathbb{R}^d \rightarrow \mathbb{C} \) by setting
\[
F(x) := \sum_{y \in \mathbb{Z}^d} f(y) \mathbbm{1}_{y + Q}(x).
\]
Clearly,
\[
\|F\|_{L^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{Z}^d)}.
\]

Fix \( a > 0 \) and let \( \tilde{C}_q > 0 \) be the constant specified in Lemma 3.1. Take \( N \geq ad \) and define
\[
N_1 := N(1 + d^{-1}\tilde{C}_q)^{1/q}.
\]
Observe that \( ad \geq d^{\frac{1}{2} + \frac{1}{q}} \) when \( d \) is large enough. Hence, by (3.3) and (3.1), for \( z \in Q \) and \( y \in B^q_N \) we have the estimate
\[
|y + z|_q \leq N_1
\]
on the set \( \{ z \in Q : |y + z|_q \leq |y - z|_q \} \), and the Lebesgue measure of this set is at least \( 1/2 \). Then by Lemma 3.3 for sufficiently large dimension \( d \) and all \( x \in \mathbb{Z}^d \) we obtain
\[
\mathcal{M}^{B^q_N}_{N_1} f(x) = \frac{1}{|B^q_N \cap \mathbb{Z}^d|} \sum_{y \in B^q_N \cap \mathbb{Z}^d} f(x + y) \mathbbm{1}_{B^q_N}(y)
= \frac{1}{|B^q_N \cap \mathbb{Z}^d|} \sum_{y \in B^q_N \cap \mathbb{Z}^d} f(x + y) \int_Q \mathbbm{1}_{B^q_N}(y) \, dz
\leq q \frac{1}{|B^q_N|} \sum_{y \in \mathbb{Z}^d} f(x + y) \int_Q \mathbbm{1}_{B^q_{N_1}}(y + z) \, dz
= \frac{1}{|B^q_N|} \sum_{y \in \mathbb{Z}^d} f(y) \int_{x + B^q_{N_1}} \mathbbm{1}_{y + Q}(z) \, dz
= \frac{1}{|B^q_{N_1}|} \int_{x + B^q_{N_1}} F(z) \, dz
= \left( \frac{N_1}{N} \right)^d \frac{1}{|B^q_{N_1}|} \int_{B^q_{N_1}} F(x + z) \, dz
\leq q \frac{1}{|B^q_{N_1}|} \int_{B^q_{N_1}} F(x + z) \, dz
= M^{B^q}_{N_1} f(x).
\]
Let us now take
\[ N_2 := N_1 (1 + d^{-1} \tilde{C}_q)^{1/q}. \]

Similarly as above, for \( y \in Q \) and \( z \in B_{N_1}^q \) we have
\[ |y + z|_q \leq N \]
on the set \( \{y \in Q : |y + z|_q \leq |y - z|_q \} \), and the Lebesgue measure of this set is at least 1/2. Therefore, Fubini’s theorem leads to
\[
M_{B_{N_1}^q} F(x) = \frac{1}{|B_{N_1}^q|} \int_{B_{N_1}^q} F(x + z) \, dz \\
\leq \frac{2}{|B_{N_1}^q|} \int_{\mathbb{R}^d} F(x + z) \mathbb{1}_{B_{N_1}^q}(z) \int_Q \mathbb{1}_{B_{N_2}^q}(z + y) \, dy \, dz \\
= 2 \left( \frac{N_2}{N_1} \right)^d \int_Q \frac{1}{|B_{N_1}^q|} \int_{\mathbb{R}^d} F(x + z - y) \mathbb{1}_{B_{N_1}^q}(z - y) \mathbb{1}_{B_{N_2}^q}(z) \, dz \, dy \\
\leq \int_{x + Q} \frac{1}{|B_{N_1}^q|} \int_{\mathbb{R}^d} F(z - y) \mathbb{1}_{B_{N_2}^q}(z) \, dz \, dy \\
= \int_{x + Q} M_{B_{N_2}^q} F(y) \, dy. \\
\tag{3.11}
\]

Denote \( C_{d,q} := a (1 + d^{-1} \tilde{C}_q)^{2/q_d} \). Combining (3.10) with (3.11) and applying Hölder’s inequality, we obtain
\[
\| \sup_{N \geq d} M_{B_N^q} f \|_{\ell^p(\mathbb{Z}^d)} \lesssim_q \sum_{x \in \mathbb{Z}^d} \left( \int_{x + Q} \sup_{N \geq C_{d,q}} |M_{B_N^q} F(y)| \right)^p \, dy \\
\leq \sum_{x \in \mathbb{Z}^d} \int_{x + Q} \sup_{N \geq C_{d,q}} |M_{B_N^q} F(y)|^p \, dy \\
= \| \sup_{N \geq C_{d,q}} |M_{B_N^q} F| \|_{L^p(\mathbb{R}^d)}^p. 
\]

By the dimension-free \( L^p(\mathbb{R}^d) \) boundedness of the maximal operator \( M_{B_N^q} \) (proved in [13]), we obtain
\[
\| \sup_{N \geq C_{d,q}} |M_{B_N^q} F| \|_{L^p(\mathbb{R}^d)} \lesssim_q \| F \|_{L^p(\mathbb{R}^d)} = \| f \|_{\ell^p(\mathbb{Z}^d)}. 
\]
This proves Theorem 2. \( \blacksquare \)
4. Decrease dimension trick: Proof of Theorem 3

We now prove Theorem 3 by adapting the methods introduced in [8, Section 2] to the case of \( q \)-balls. In fact this section is a technical elaboration of [8, Section 2]. However, we have decided to provide necessary details due to intricate technicalities. Throughout this section we abbreviate \( \| \cdot \|_{\ell^p(\mathbb{Z}^d)} \) to \( \| \cdot \|_{\ell^p} \) and \( \| \cdot \|_{L^p(\mathbb{R}^d)} \) to \( \| \cdot \|_{L^p} \). We fix \( q \in [2, \infty) \) and recall that \( M_{B^q_N} \) is the operator whose multiplier is given by

\[
m_{B^q_N}(\xi) := \frac{1}{|B_N^q \cap \mathbb{Z}^d|} \sum_{x \in B_N^q \cap \mathbb{Z}^d} e^{2\pi i \xi \cdot x}, \quad \xi \in \mathbb{T}^d \equiv [-1/2, 1/2)^d.
\]

For each \( \xi \in \mathbb{T}^d \) we will write

\[
\| \xi \|^2 := \| \xi_1 \|^2 + \cdots + \| \xi_d \|^2,
\]

where \( \| \xi_j \| := \text{dist}(\xi_j, \mathbb{Z}) \) for any \( j \in \mathbb{N}_d \). Since we identify \( \mathbb{T}^d \) with \( [-1/2, 1/2)^d \), it is easy to see that the norm \( \| \cdot \| \) coincides with the Euclidean norm \( | \cdot | \) restricted to \( [-1/2, 1/2)^d \). It is also very well known that

\[
\| \eta \| \asymp | \sin(\pi \| \eta \|) |
\]

for every \( \eta \in \mathbb{T} \), since \( | \sin(\pi \| \eta \|) | = \sin(\pi \| \eta \|) \) and for \( 0 \leq | \eta | \leq 1/2 \) we have

\[
(4.1) \quad 2|\eta| \leq | \sin(\pi \| \eta \|) | \leq \pi |\eta|.
\]

The proof of Theorem 3 is based on Proposition 4.1 and Proposition 4.2, which give respectively estimates of the multiplier \( m_{B^q_N}(\xi) \) at the origin and at infinity. These estimates will be described in terms of the proportionality constant

\[
\kappa(d, N) := \kappa_q(d, N) := Nd^{-1/q}.
\]

**Proposition 4.1:** For every \( d, N \in \mathbb{N} \) and for every \( \xi \in \mathbb{T}^d \) we have

\[
(4.2) \quad |m_{B^q_N}(\xi) - 1| \leq 2\pi^2 \kappa(d, N)^2 \| \xi \|^2.
\]

**Proposition 4.2:** There is a constant \( C_q > 0 \) such that for any \( d, N \in \mathbb{N} \) if \( 10 \leq \kappa(d, N) \leq 50qd^{1-1/q} \), then for all \( \xi \in \mathbb{T}^d \) we have

\[
(4.3) \quad |m_{B^q_N}(\xi)| \leq C_q((\kappa(d, N)\| \xi \|)^{-1} + \kappa(d, N)^{-\frac{1}{q}}).
\]

Before we prove Proposition 4.1 and Proposition 4.2 we show how (1.12) follows from (4.2) and (4.3).
Proof of Theorem 3. Since $\mathcal{D}_{C_1, C_2}$ is a subset of the dyadic set $\mathcal{D}$ we can assume, without loss of generality, that $C_1 = C_2 = 10$. For every $t > 0$ let $P_t$ be the semigroup with the multiplier

$$p_t(\xi) := e^{-t \sum_{i=1}^{d} \sin^2(\pi \xi_i)}, \quad \xi \in \mathbb{T}^d.$$ 

It follows from [14] (see also [6] for more details) that for every $p \in (1, \infty)$ there is $C_p > 0$ independent of $d \in \mathbb{N}$ such that for every $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\|\sup_{t > 0} |P_t f|\|_{\ell^p} \leq C_p \|f\|_{\ell^p}. $$ 

It suffices to compare the averages $\mathcal{M}_{N}^{B^q_N}$ with $P_{N^2/d^2/q}$. Namely, the proof of (1.12) will be completed if we obtain the dimension-free estimate on $\ell^2(\mathbb{Z}^d)$ for the following square function

$$S_f(x) := \left( \sum_{N \in \mathcal{D}_{C_1, C_2}} |\mathcal{M}_N f(x) - P_{N^2/d^2/q} f(x)|^2 \right)^{1/2}, \quad x \in \mathbb{Z}^d. $$ 

By Plancherel’s formula, (4.2), and (4.3), we can estimate $\|S(f)\|_{\ell^2}^2$ by

$$C'_q \int_{\mathbb{T}^d} \left( \sum_{m \in \mathbb{Z}} \min \left\{ \frac{2^{2m}}{d^2/q} \|\xi\|_2^2, \frac{2^{2m}}{d^2/q} \|\xi\|_2^{-2} \right\} \right) \|\hat{f}(\xi)\|^2 \, d\xi,$$

where $C'_q$ is a constant that depends only on $q$. To complete the proof we note that the integral above is clearly bounded by $C \|f\|_{\ell^2}^2$ for a suitable constant $C > 0$ independent of $d$. 

The rest of this section is devoted to proving Proposition 4.1 and Proposition 4.2. We emphasize that in the proof of Proposition 4.1 the assumption $q \geq 2$ is crucial.

Proof of Proposition 4.1. Since the balls $B^q_N$ are symmetric under permutations and sign changes we may repeat the proof of [8, Proposition 2.1] reaching

$$|m_{N}^{B^q_N}(\xi) - 1| \leq \frac{2}{|B^q_N \cap \mathbb{Z}^d|} \sum_{j=1}^{d} \sin^2(\pi \xi_j) \sum_{x \in B^q_N \cap \mathbb{Z}^d} \frac{|x|^2}{d}. $$
Observe that $|x|^2 \leq |x_q|^2 \cdot d^{1-2/q}$. Indeed, for $q = 2$ this is simply an equality, and for $q > 2$ it suffices to apply Hölder’s inequality for the pair $(\frac{q}{2}, \frac{q}{q-2})$.

Consequently,
\[
m_{N}^{q}(\xi) - 1 \leq \frac{2}{|B_{N}^{q} \cap \mathbb{Z}^d|} \sum_{j=1}^{d} \sin^2(\pi \xi_j) \sum_{x \in B_{N}^{q} \cap \mathbb{Z}^d} \frac{|x|^2}{d^{2/q}} \leq 2\pi^2 \kappa(d, N)^2 \|\xi\|^2,
\]

which gives the claim. ■

The proof of Proposition 4.2 can be deduced from a series of auxiliary lemmas, which we formulate and prove below. In what follows, for an integer $1 \leq r \leq d$ and a radius $R > 0$ we let
\[
B_{R}^{q}(r) = \{x \in \mathbb{R}^r : |x|_q \leq R\} \quad \text{and} \quad S_{R}^{q}(r) = \{x \in \mathbb{R}^r : |x|_q = R\}
\]
be the $r$-dimensional ball and sphere of radius $R > 0$, respectively.

**Lemma 4.3:** For all $d, N \in \mathbb{N}$ we have
\[
(2 \lfloor \kappa(d, N) \rfloor + 1)^d \leq |B_{N}^{q} \cap \mathbb{Z}^d| \leq |B_{N+d^{1/q}}^{q} d^2/q| = \frac{2^d \Gamma(1 + \frac{1}{q})^d}{\Gamma(1 + \frac{2}{q})} (N + d^{1/q})^d.
\]

**Proof.** The lower bound follows from the inclusion
\[
[-\kappa(d, N), \kappa(d, N)]^d \cap \mathbb{Z}^d \subseteq B_{N}^{q} \cap \mathbb{Z}^d,
\]
while the upper bound is a simple consequence of the triangle inequality. ■

**Lemma 4.4:** Given $\varepsilon_1, \varepsilon_2 \in (0, 1]$ we define for every $d, N \in \mathbb{N}$ the set
\[
E = \{x \in B_{N}^{q} \cap \mathbb{Z}^d : |\{i \in \mathbb{N}_d : |x_i| \geq \varepsilon_2 \kappa(d, N)\}| \leq \varepsilon_1 d\}.
\]
If $\varepsilon_1, \varepsilon_2 \in (0, 1/(10q)]$ and $\kappa(d, N) \geq 10$, then we have
\[
|E| \leq 2e^{-\frac{d}{10q}} |B_{N}^{q} \cap \mathbb{Z}^d|.
\]

**Proof.** As in [8, Lemma 2.4] we can estimate
\[
|E| \leq (2\varepsilon_2 \kappa(d, N) + 1)^d + \sum_{1 \leq m \leq \varepsilon_1 d} \binom{d}{m} (2\varepsilon_2 \kappa(d, N) + 1)^{d-m} |B_{N}^{q}(m) \cap \mathbb{Z}^m|.
\]
Since $\kappa(d, N) \geq 10$ and $\varepsilon_2 \leq 1/10$, using the lower bound from Lemma 4.3 gives
\[
(2\varepsilon_2 \kappa(d, N) + 1)^d \leq e^{-\frac{d}{10q}} |B_{N}^{q} \cap \mathbb{Z}^d|.
\]
as in the case $q = 2$. On the other hand, the upper bound from Lemma 4.3 can be applied to estimate the sum appearing in (4.4) by

$$\sum_{1 \leq m \leq \varepsilon_1d} \frac{d^m}{m!} (2\varepsilon_2\kappa(d, N) + 1)^{d-m} \frac{2^m}{\Gamma(1 + \frac{m}{q})} d^{m/q}(\kappa(d, N) + 1)^m. \tag{4.6}$$

Now observe that

$$\Gamma(1 + \frac{m}{q}) \leq \frac{1}{\Gamma(1 + \frac{m}{q})} \leq \frac{4q^{m/q}}{(m/(2q))^{-1+m/q}}. \tag{4.7}$$

Indeed, if $m/q \geq 2$, then

$$\frac{1}{\Gamma(1 + \frac{m}{q})} \leq \frac{1}{\Gamma(m/q)} \leq \frac{e^{m/q}}{(m/(2q))^{-1+m/q}},$$

where in the second inequality we have used that $\frac{1}{n!} \leq e^n/n^n$ holds for any $n \in \mathbb{N}$. If $m/q \leq 2$, then

$$\frac{1}{\Gamma(1 + \frac{m}{q})} \leq 2 \leq \frac{4qm}{2q} (2e)^{m/q} (m/q)^{-m/q} = \frac{4qe^{m/q}}{(m/(2q))^{-1+m/q}}.$$

In the first inequality we used the fact that the gamma function is estimated from below by $1/2$ on the interval $[1, 3]$.

Applying (4.7) we get

$$\frac{d^{m+m/q}2^m}{m! \Gamma(1 + \frac{m}{q})} \leq \frac{d^{m+m/q}2^m q^{m/q}}{m^m (m/(2q))^{-1+m/q}} \leq \left(\frac{2de}{m}\right)^{m(1+1/q)} 2mq^{m/q} \leq \left(\frac{2de}{m}\right)^{m(1+1/q)}.$$

Combining this with (4.4), (4.5), and (4.6), and repeating the argument used in [8, (2.16)], we arrive at

$$|E| \leq e^{-\frac{16d}{19}} |B_N^q \cap \mathbb{Z}^d|$$

$$+ \sum_{m=1}^{\lfloor \varepsilon_1d \rfloor} \left(\frac{2de}{m}\right)^{m(1+1/q)} (2\varepsilon_2\kappa(d, N) + 1)^{d-m}(\kappa(d, N) + 1)^m$$

$$\leq \left(e^{-\frac{16d}{19}} + \sum_{m=1}^{\lfloor \varepsilon_1d \rfloor} \left(\frac{2de}{m}\right)^{m(1+1/q)} (2\varepsilon_2\kappa(d, N) + 1)^{d-m} \left(2\lfloor\kappa(d, N)\rfloor + 1\right)^{-m/q}\right) |B_N^q \cap \mathbb{Z}^d|$$

$$\leq \left(e^{-\frac{16d}{19}} + e^{-\frac{72d}{95}} \sum_{m=1}^{\lfloor \varepsilon_1d \rfloor} e^{\varphi(m)}\right) |B_N^q \cap \mathbb{Z}^d|,$$

where $\varphi(x) := (1 + 1/q)x \log(\frac{2eqd}{x})$, $x \geq 0$. 
For $x \in [0, d/(10q)]$ we have
\[
\varphi'(x) = (1 + 1/q) \log \left( \frac{2eqd}{x} \right) - (1 + 1/q) \geq \log \left( \frac{2qd}{x} \right) \geq \log 3.
\]
Hence, $\varphi$ is increasing on $[0, \lfloor \varepsilon_1 d \rfloor]$, and arguing as in the proof of [8, Lemma 2.4] we get
\[
\sum_{m=1}^{\lfloor \varepsilon_1 d \rfloor} e^{\varphi(m)} \leq e^{\varphi\left( \frac{d}{10q} \right)} \sum_{m=1}^{\lfloor \varepsilon_1 d \rfloor} e^{-\left( \lfloor \varepsilon_1 d \rfloor - m \right) \log 3} \\
\leq \frac{3}{2} e^{\varphi\left( \frac{d}{10q} \right)} \leq \frac{3}{2} (20eq^{2d}/(5q)) \\
\leq \frac{3}{2} e^{\frac{2d}{5e}} e^{\frac{4d}{5q}} < \frac{3}{2} e^{\frac{2d}{5}},
\]
since $q^{1/q} \leq e^{1/e}$ and $20 < e^3$. Then (4.8) gives
\[
\frac{|E|}{|B_N^q \cap \mathbb{Z}^d|} \leq e^{-\frac{16d}{19}} + e^{-\frac{72d}{95}} \sum_{m=1}^{\lfloor \varepsilon_1 d \rfloor} e^{\varphi(m)} \\
\leq e^{-\frac{16d}{19}} + \frac{3}{2} e^{-\frac{4d}{19}} \leq e^{-\frac{d}{19}} \left( e^{-\frac{4d}{19}} + \frac{3}{2} \right) \\
\leq 2e^{-\frac{d}{19}},
\]
which finishes the proof. \[\square\]

Recall that $\text{Sym}(d)$ denotes the permutation group on $\mathbb{N}_d$. We will also write $\sigma \cdot x = (x_{\sigma(1)}, \ldots, x_{\sigma(d)})$ for every $x \in \mathbb{R}^d$ and $\sigma \in \text{Sym}(d)$. Let $P$ be the uniform distribution on the symmetry group $\text{Sym}(d)$, i.e., $P(A) = |A|/d!$ for any $A \subseteq \text{Sym}(d)$, since $|\text{Sym}(d)| = d!$. The expectation $E$ will be always taken with respect to the uniform distribution $\mathbb{P}$ on the symmetry group $\text{Sym}(d)$. We will need two lemmas from [8].

**Lemma 4.5:** Assume that $I, J \subseteq \mathbb{N}_d$ and $|J| = r$ for some $0 \leq r \leq d$. Then
\[
\mathbb{P}[\{\sigma \in \text{Sym}(d) : |\sigma(I) \cap J| \leq r|I|/(5d)\}] \leq e^{-\frac{r|I|}{10d}}.
\]
In particular, if $\delta_1, \delta_2 \in (0, 1]$ satisfy $5\delta_2 \leq \delta_1$ and $\delta_1 d \leq |I| \leq d$, then we have
\[
\mathbb{P}[\{\sigma \in \text{Sym}(d) : |\sigma(I) \cap J| \leq \delta_2 r\}] \leq e^{-\frac{\delta_1 r}{10}}.
\]
LEMMA 4.6: Assume that we have a finite decreasing sequence
\[ 0 \leq u_d \leq \cdots \leq u_2 \leq u_1 \leq (1 - \delta_0)/2 \]
for some \( \delta_0 \in (0, 1) \). Suppose that \( I \subseteq \mathbb{N}_d \) satisfies \( \delta_1 d \leq |I| \leq d \) for some \( \delta_1 \in (0, 1] \). Then for every \( J = (d_0, d] \cap \mathbb{Z} \) with \( 0 \leq d_0 \leq d \) we have
\[
E \left[ \exp \left( - \sum_{j \in \sigma(I) \cap J} u_j \right) \right] \leq 3 \exp \left( - \frac{\delta_0 \delta_1}{20} \sum_{j \in J} u_j \right).
\]

LEMMA 4.7: For \( d, N \in \mathbb{N} \), \( \varepsilon \in (0, 1/(50q)] \) and an integer \( 1 \leq r \leq d \) we define
\[
E = \left\{ x \in B_N^q \cap \mathbb{Z}^d : \sum_{i=1}^r x_i^q < \varepsilon^{q+1} \kappa(d, N) q r \right\}.
\]
If \( \kappa(d, N) \geq 10 \), then we have
\[
(4.9) \quad |E| \leq 4 e^{-\frac{\delta_1}{20}} |B_N^q \cap \mathbb{Z}^d|.
\]
As a consequence, \( B_N^q \cap \mathbb{Z}^d \) can be written as a disjoint sum
\[
(4.10) \quad B_N^q \cap \mathbb{Z}^d = \bigcup_{\varepsilon^{q+1} \kappa(d, N) q r \leq 1} (B_{l/q}^{q,r} \cap \mathbb{Z}^r) \times (S_{d,N-l/q}^{q,d-r} \cap \mathbb{Z}^{d-r}) \bigcup E',
\]
where \( E' \subset \mathbb{Z}^d \) satisfies \( |E'| \leq 4 e^{-\frac{\delta_1}{100}} |B_N^q \cap \mathbb{Z}^d| \).

Proof. Let \( \delta_1 \in (0, 1/(10q)] \) be such that \( \delta_1 \geq 5 \varepsilon \), and define
\[
I_x = \{ i \in \mathbb{N}_d : |x_i| \geq \varepsilon \kappa(d, N) \}.
\]
We have \( E \subseteq E_1 \cup E_2 \), where
\[
E_1 = \left\{ x \in B_N^q \cap \mathbb{Z}^d : \sum_{i \in I_x \cap \mathbb{N}_r} |x_i|^q < \varepsilon^{q+1} \kappa(d, N) q r \text{ and } |I_x| \geq \delta_1 d \right\},
\]
\[
E_2 = \left\{ x \in B_N^q \cap \mathbb{Z}^d : |I_x| < \delta_1 d \right\}.
\]
By Lemma 4.4 (with \( \varepsilon_1 = \delta_1 \) and \( \varepsilon_2 = \varepsilon \)) we have \( |E_2| \leq 2 e^{-\frac{\delta_1}{100}} |B_N^q \cap \mathbb{Z}^d| \), provided that \( \kappa(d, N) \geq 10 \). Observe that
\[
|E_1| = \sum_{x \in B_N^q \cap \mathbb{Z}^d} \frac{1}{d!} \sum_{\sigma \in \operatorname{Sym}(d)} \mathbb{1}_{E_1}(\sigma^{-1} \cdot x)
\]
\[
= \sum_{x \in B_N^q \cap \mathbb{Z}^d} \mathbb{P} \left[ \left\{ \sigma \in \operatorname{Sym}(d) : \sum_{i \in \sigma(I_x) \cap \mathbb{N}_r} |x_{\sigma^{-1}(i)}|^q < \varepsilon^{q+1} \kappa(d, N) q r \text{ and } |\sigma(I_x)| \geq \delta_1 d \right\} \right],
\]
since $I_{\sigma^{-1}x} = \sigma(I_x)$. Now by Lemma 4.5 (with $J = \mathbb{N}_r$, $\delta_2 = \frac{\delta_1}{50}$ and $\delta_1$ as above) we obtain, for every $x \in B^q_N \cap \mathbb{Z}^d$ such that $|I_x| \geq \delta_1 d$, the estimate
\[
P\left\{ \sigma \in \text{Sym}(d) : \sum_{i \in \sigma(I_x) \cap \mathbb{N}_r} |x_{\sigma^{-1}(i)}|^q < \varepsilon^{q+1} \kappa(d,N)^q r \text{ and } |\sigma(I_x)| \geq \delta_1 d \right\} \leq \varepsilon^{q+1} \kappa(d,N)^q r,
\]
since
\[
\left\{ \sigma \in \text{Sym}(d) : \sum_{i \in \sigma(I_x) \cap \mathbb{N}_r} |x_{\sigma^{-1}(i)}|^q < \varepsilon^{q+1} \kappa(d,N)^q r \text{ and } |\sigma(I_x)| \geq \delta_1 d \right\} \cap \left\{ \sigma \in \text{Sym}(d) : |\sigma(I_x) \cap \mathbb{N}_r| > \delta_2 r \right\} = \emptyset.
\]
Thus $|E_1| \leq 2e^{-\frac{|\theta|}{10} |B^q_N \cap \mathbb{Z}^d|}$, which proves (4.9). To prove (4.10) we write
\[
B^q_N \cap \mathbb{Z}^d = \bigcup_{l=0}^{N^q} (B^q_{l/q} \cap \mathbb{Z}^r) \times (S^d_{l/q} \cap \mathbb{Z}^d).
\]
Then we see that
\[
\left( \bigcup_{l=0}^{N^q} (B^q_{l/q} \cap \mathbb{Z}^r) \times (S^d_{l/q} \cap \mathbb{Z}^d) \right) \cap E^c
\]
\[
= \left( \bigcup_{l \geq \varepsilon^{q+1} \kappa(d,N)^q r} (B^q_{l/q} \cap \mathbb{Z}^r) \times (S^d_{l/q} \cap \mathbb{Z}^d) \right) \cap E^c,
\]
and hence we obtain (4.10) with some $E' \subseteq E$. The proof is completed.

**Lemma 4.8:** For $d,N \in \mathbb{N}$ and $\varepsilon \in (0,1/(50q)]$, if $\kappa(d,N) \geq 10$, then for every $1 \leq r \leq d$ and $\xi \in \mathbb{T}^d$ we have
\[
|m^B_{N^q}(\xi)| \leq \sup_{l \geq \varepsilon^{q+1} \kappa(d,N)^q r} |m^B_{l/q}(\xi_1, \ldots, \xi_r)| + 4e^{-\varepsilon r},
\]
where
\[
m^B_{R^q}(\eta) := \frac{1}{|B^q_{R^q} \cap \mathbb{Z}^d|} \sum_{x \in B^q_{R^q} \cap \mathbb{Z}^d} e^{2\pi i \eta \cdot x}, \quad \eta \in \mathbb{T}^r,
\]
is the lower dimensional multiplier with $r \in \mathbb{N}$ and $R > 0$. 

Proof. We identify $\mathbb{R}^d \equiv \mathbb{R}^r \times \mathbb{R}^{d-r}$ and $\mathbb{T}^d \equiv \mathbb{T}^r \times \mathbb{T}^{d-r}$ and we will write $\mathbb{R}^d \ni x = (x^1, x^2) \in \mathbb{R}^r \times \mathbb{R}^{d-r}$ and $\mathbb{T}^d \ni \xi = (\xi^1, \xi^2) \in \mathbb{T}^r \times \mathbb{T}^{d-r}$ respectively. Invoking (4.10) one obtains

$$|m_{N}^{B_{q}^{q}}(\xi)| \leq \frac{1}{|B_{N}^{q} \cap \mathbb{Z}^d|} \sum_{l \geq \varepsilon^{q+1} \kappa(d,N)q} \sum_{x^2 \in S_{(N q^{-1})/q}^{d-r} \cap \mathbb{Z}^d} |B_{l/q}^{q}(r) \cap \mathbb{Z}^r| \frac{1}{|B_{l/q}^{q}(r) \cap \mathbb{Z}^r|} \times \sum_{x^1 \in B_{l/q}^{q}(r) \cap \mathbb{Z}^r} e^{2\pi i \xi^1 \cdot x^1} + 4e^{-\frac{\varepsilon r}{10}} \leq \sup_{l \geq \varepsilon^{q+1} \kappa(d,N)q} |m_{l/q}^{B_{q}^{q}}(\xi_1, \ldots, \xi_r)| + 4e^{-\frac{\varepsilon r}{10}}.$$ 

In the last inequality the disjointness in the decomposition from (4.10) has been used.

The next two lemmas give information on the size difference between the balls $B_{R}^{q}(r)$ and their shifts $z + B_{R}^{q}(r)$ for $z \in \mathbb{R}^r$.

**Lemma 4.9:** Let $R \geq 1$ and let $r \in \mathbb{N}$ be such that $r \leq R^\delta$ for some $\delta \in (0, q/(q+1))$. Then for every $z \in \mathbb{R}^r$ we have

$$||(z + B_{R}^{q}(r)) \cap \mathbb{Z}^r| - |B_{R}^{q}(r)|| \leq |B_{R}^{q}(r)| |r^{(q+1)/q} R^{-1} e^{r^{(q+1)/q}/R} \leq e |B_{R}^{q}(r)| R^{-1 + (q+1)\delta/q}. \tag{4.11}$$

**Proof.** For the proof we refer to [8, Lemma 2.9].

**Lemma 4.10:** Let $R \geq 1$ and let $r \in \mathbb{N}$ be such that $r \leq R^\delta$ for some $\delta \in (0, q/(q+1))$. Then for every $z \in \mathbb{R}^r$ we have

$$|(B_{R}^{q}(r) \cap \mathbb{Z}^r) \triangle ((z + B_{R}^{q}(r)) \cap \mathbb{Z}^r)| \leq 4e(r|z| R^{-1} e^{r^{1/q}} \geq R^{-1 + (q+1)\delta/q} |B_{R}^{q}(r)|) \leq 4e(|z| R^{-1 + \delta} e^{r^{1/q}} \geq R^{-1 + (q+1)\delta/q} |B_{R}^{q}(r)|).$$

**Proof.** For the proof we refer to [8, Lemma 2.10].

We now recall the dimension-free estimates for the multipliers

$$m_{B_{R}^{q}}(\xi) := |B_{R}^{q}|^{-1} \mathcal{F}(\mathbb{1}_{B_{R}^{q}})(\xi) \text{ for } \xi \in \mathbb{R}^d.$$
Lemma 4.11 ([8, Lemma 2.11]): There exist constants $c_q, C > 0$ independent of $d$ and such that for every $R > 0$ and $\xi \in \mathbb{R}^d$ we have

$$|m^{B_q}(\xi)| \leq C(c_q Rd^{-1/q}\|\xi\|)^{-1} \quad \text{and} \quad |m^{B_q}(\xi) - 1| \leq C(c_q Rd^{-1/q}\|\xi\|).$$

Lemma 4.10 and Lemma 4.11 are essential in proving the following estimate.

Lemma 4.12: There exists a constant $C_q > 0$ such that for every $\delta \in (0, 1/2)$ and for all $r \in \mathbb{N}$ and $R > 0$ satisfying $1 \leq r \leq R^\delta$ we have

$$|m^{B_q(r)}(\eta)| \leq C_q(\kappa(r, R)^{-\frac{1}{q} + \frac{1}{3} + \frac{1}{d}} + r\kappa(r, R)^{-\frac{1}{q} + \frac{1}{d}} + (\kappa(r, R)\|\eta\|)^{-1})$$

for every $\eta \in \mathbb{T}^r$.

Proof. The inequality is obvious when $R \leq 16$, so it suffices to consider $R > 16$.

Firstly, we assume that $\max\{\|\eta_1\|, \ldots, \|\eta_r\|\} > \kappa(r, R)^{-\frac{1}{q} + \frac{1}{d}}$. Let

$$M = \lfloor \kappa(r, R)^{2-\delta} \rfloor$$

and assume without loss of generality that $\|\eta_1\| > \kappa(r, R)^{-\frac{1}{q} + \frac{1}{d}}$. Then

$$|m^{B_q(r)}(\eta)| \leq \frac{1}{|B^{q(r)}_R \cap \mathbb{Z}^r|} \sum_{x \in B^{q(r)}_R \cap \mathbb{Z}^r} \frac{1}{M} \sum_{s=1}^M e^{2\pi i (x + se_1) \cdot \eta}$$

$$+ \frac{1}{M} \sum_{s=1}^M \frac{1}{|B^{q(r)}_R \cap \mathbb{Z}^r|} \sum_{x \in B^{q(r)}_R \cap \mathbb{Z}^r} e^{2\pi i x \cdot \eta} - e^{2\pi i (x + se_1) \cdot \eta}. \quad (4.12)$$

Since $\kappa(r, R) \geq 1$ we now see that

$$\frac{1}{M} \sum_{s=1}^M e^{2\pi i (x + se_1) \cdot \eta} \leq M^{-1}\|\eta_1\|^{-1} \leq 2\kappa(r, R)^{-\frac{1}{q} + \frac{1}{d}} \quad (4.13)$$

We have assumed that $r \leq R^\delta$, thus by Lemma 4.10, with $z = se_1$ and $s \leq M \leq \kappa(r, R)^{2-\delta}$, we obtain

$$\frac{1}{|B^{q(r)}_R \cap \mathbb{Z}^r|} \sum_{x \in B^{q(r)}_R \cap \mathbb{Z}^r} e^{2\pi i x \cdot \eta} - e^{2\pi i (x + se_1) \cdot \eta}$$

$$\leq \frac{1}{|B^{q(r)}_R \cap \mathbb{Z}^r|} \left( |B^{q(r)}_R \cap \mathbb{Z}^r| (se_1 + B^{q(r)}_R \cap \mathbb{Z}^r) \right)$$

$$\leq 8e(srR^{-1}e^{sR^{-1}} + e^{srR^{-1}}R^{-1+(q+1)\delta/q})$$

$$\leq 16e^2\kappa(r, R)^{-\frac{1}{q} + \frac{2\delta}{\delta}},$$

$$
\right.$$
since
\[ srR^{-1} \leq \kappa(r, R)^{2-\delta \over 3} R^{-1+\delta} \leq \kappa(r, R)^{-{1 \over 3} + {2\delta \over 3}} \leq 1 \]
and
\[ R^{-1+(q+1)\delta/q} \leq R^{-1+3\delta/2} \leq R^{-{1 \over 3} + {2\delta \over 3}}, \]
and for \( R > 16 \) we also have
\[
|B^q_{R} \cap \mathbb{Z}^r| \geq |B^q_{R-r^{1/2}}| \geq |B^q_{R-r^{1/2}}| \cdot \left(1 - \frac{r^{1/2}}{R}\right)^r \geq |B^q_{R}| \cdot \left(1 - r^{3/2}R^{-1}\right) \geq |B^q_{R}|/2.
\]
Combining (4.12) with (4.13) and (4.14) we obtain
\[
|m^{B^q_{R}}_{R}(\eta)| \leq (16 \varepsilon^2 + 2) \kappa(r, R)^{-{1 \over 3} + {2\delta \over 3}}.
\]
Secondly, we assume that \( \max\{\|\eta_1\|, \ldots, \|\eta_r\|\} \leq \kappa(r, R)^{-{1 \over 3} + {2\delta \over 3}} \). Observe that by (4.11) we have
\[
\left| \frac{1}{|B^q_{R}|} - \frac{1}{|B^q_{R-r^{1/2}}|} \right| \leq \frac{eR^{-1+(q+1)\delta/q}}{|B^q_{R}|} \leq \frac{2e\kappa(r, R)^{-{1 \over 3} + {2\delta \over 3}}}{|B^q_{R}|}.
\]
Then \( |m^{B^q_{R}}_{R}(\eta)| \) is bounded by
\[
|m^{B^q_{R}}_{R}(\eta)| - \frac{1}{|B^q_{R}|} \mathcal{F}(\mathbb{1}_{B^q_{R}})(\eta) \left| \left| \frac{1}{|B^q_{R}|} \mathcal{F}(\mathbb{1}_{B^q_{R}})(\eta) \right| \leq 2e\kappa(r, R)^{-{1 \over 3} + {2\delta \over 3}}
\]
\[
+ \frac{1}{|B^q_{R}|} \left| \sum_{x \in B^q_{R} \cap \mathbb{Z}^r} e^{2\pi i x \cdot \eta} - \int_{B^q_{R}} e^{2\pi iy \cdot \eta} dy \right| \left| \mathcal{F}(\mathbb{1}_{B^q_{R}})(\eta) \right|.
\]
(4.15)
Let $Q^{(r)} = [-1/2, 1/2]^r$ and note that by Lemma 4.10 with $z = t \in Q^{(r)}$ we obtain

$$\frac{1}{|B^{q,(r)}_R \cap \mathbb{Z}^r|} \left| \sum_{x \in B^{q,(r)}_R \cap \mathbb{Z}^r} e^{2\pi i x \cdot \eta} - \int_{B^{q,(r)}_R} e^{2\pi i y \cdot \eta} dy \right|$$

$$= \frac{1}{|B^{q,(r)}_R \cap \mathbb{Z}^r|} \left| \sum_{x \in \mathbb{Z}^r} \int_{Q^{(r)}} e^{2\pi i x \cdot \eta} \mathbb{1}_{B^{q,(r)}_R}(x) - e^{2\pi i (x+t) \cdot \eta} \mathbb{1}_{B^{q,(r)}_R}(x+t) dt \right|$$

(4.16) $$\leq \frac{1}{|B^{q,(r)}_R \cap \mathbb{Z}^r|} \int_{Q^{(r)}} \left| (B^{q,(r)}_R \cap \mathbb{Z}^r) \Delta ((t + B^{q,(r)}_R) \cap \mathbb{Z}^r) \right| dt$$

$$+ \frac{1}{|B^{q,(r)}_R \cap \mathbb{Z}^r|} \sum_{x \in \mathbb{Z}^r} \mathbb{1}_{B^{q,(r)}_R}(x) \int_{Q^{(r)}} \left| e^{2\pi i x \cdot \eta} - e^{2\pi i (x+t) \cdot \eta} \right| dt$$

$$\leq 16e^2 \kappa(r, R) - \frac{1}{3} + \frac{26}{3} + 2\pi \left( \|\eta_1\| + \cdots + \|\eta_r\| \right)$$

$$\leq 16e^2 \kappa(r, R) - \frac{1}{3} + \frac{26}{3} + 2\pi r \kappa(r, R)^{-\frac{1+\delta}{r}}.$$

Finally, by Lemma 4.11 we obtain

$$\frac{1}{|B^{q,(r)}_R|} |\mathcal{F}(\mathbb{1}_{B^{q,(r)}_R})(\eta)| \leq C(c_q \kappa(r, R) \|\eta\|)^{-1}.$$  

Combining this with (4.15) and (4.16) we conclude that

$$|m^{B^q,(r)}_R(\eta)| \leq (16e^2 + 2e) \kappa(r, R)^{-\frac{1}{3} + \frac{26}{3} + 2\pi r \kappa(r, R)^{-\frac{1+\delta}{3}}} + C^{-1}(\kappa(r, R) \|\eta\|)^{-1},$$

which completes the proof. 

**Lemma 4.13:** For every $\delta \in (0, 1/2)$ and $\varepsilon \in (0, 1/(50q)]$ there is a constant $C_{q, \delta, \varepsilon} > 0$ such that for every $d, N \in \mathbb{N}$, if $r$ is an integer such that $1 \leq r \leq d$ and $\max\{1, \varepsilon \frac{(q+1)\delta}{q} \kappa(d, N)^{\delta}/2\} \leq r \leq \max\{1, \varepsilon \frac{(q+1)\delta}{q} \kappa(d, N)^{\delta}\}$, then for every $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{T}^d$ we have

$$|m^{B^q}_N(\xi)| \leq C_{q, \delta, \varepsilon}(\kappa(d, N)^{-\frac{1}{3} + \frac{26}{3} + \kappa(d, N) \|\eta\|})^{-1},$$

where $\eta = (\xi_1, \ldots, \xi_r)$.

**Proof.** If $\kappa(d, N) \leq \varepsilon^{-\frac{q+1}{q}}$, then there is nothing to do, since the implied constant in question is allowed to depend on $q$, $\delta$, and $\varepsilon$. We will assume that $\kappa(d, N) \geq \varepsilon^{-\frac{q+1}{q}}$, which ensures that $\kappa(d, N) \geq 10$. In view of Lemma 4.8 we have

$$|m^{B^q}_N(\xi)| \leq \sup_{R \geq \varepsilon^{(q+1)/q} \kappa(d, N)^{r^1/q}} |m^{B^q,(r)}_R(\eta)| + 4e^{-\frac{\delta}{10}},$$
where \( \eta = (\xi_1, \ldots, \xi_r) \). By Lemma 4.12, since
\[
 r \leq \varepsilon \frac{(q+1)\delta}{q} \kappa(d,N)^{\delta} \leq \kappa(r,R)^{\delta} \leq R^{\delta},
\]
we obtain
\[
|m_{B^q,R}^q(\eta)| \lesssim q \kappa(r,R)^{-\frac{1}{3} + \frac{2\delta}{3}} + r\kappa(r,R)^{-\frac{1+\delta}{3}} + (\kappa(r,R)\|\eta\|)^{-1}.
\]
Combining the two estimates above with our assumptions we obtain the desired claim. \( \blacksquare \)

We have prepared all necessary tools to prove inequality (4.3). We shall be working under the assumptions of Lemma 4.13 with \( \delta = 2/7 \).

**Proof of Proposition 4.2.** Assume that \( \varepsilon = 1/(50q) \). If \( \kappa(d,N) \leq 2^\frac{7}{2} \cdot (50q)^{\frac{q+1}{q}} \) then clearly (4.3) holds. Therefore, we can assume that \( d, N \in \mathbb{N} \) satisfy
\[
2^\frac{7}{2} \cdot (50q)^{\frac{q+1}{q}} \leq \kappa(d,N) \leq 50qd^{1-1/q}.
\]
We choose an integer \( 1 \leq r \leq d \) satisfying
\[
(50q)^{\frac{2(q+1)}{q} \kappa(d,N)^{\frac{2}{7}}} \leq r \leq (50q)^{-\frac{2(q+1)}{q} \kappa(d,N)^{\frac{2}{7}}};
\]
this is possible since
\[
(50q)^{-\frac{2(q+1)}{q} \kappa(d,N)^{\frac{2}{7}}} \geq 2 \quad \text{and} \quad (50q)^{-\frac{2(q+1)}{q} \kappa(d,N)^{\frac{2}{7}}} \leq d^{(1-1/q)/7} \leq d.
\]
By symmetry we may also assume that \( \|\xi_1\| \geq \cdots \geq \|\xi_d\| \) and we shall distinguish two cases. Suppose first that
\[
\|\xi_1\|^2 + \cdots + \|\xi_r\|^2 \geq \frac{1}{4}\|\xi\|^2.
\]
Then in view of Lemma 4.13 (with \( \delta = 2/7 \) and \( r \asymp q \kappa(d,N)^{\frac{2}{7}} \)) we obtain
\[
|m_{B^q}^q(\xi)| \leq C_q(\kappa(d,N)^{-\frac{2}{7}} + (\kappa(d,N)\|\xi\|)^{-1}),
\]
and we are done. So we can assume that
\[
(4.17) \quad \|\xi_1\|^2 + \cdots + \|\xi_r\|^2 \leq \frac{1}{4}\|\xi\|^2.
\]
Let \( \varepsilon_1 = 1/10 \) and assume first that
\[
(4.18) \quad \|\xi_j\| \leq \frac{\varepsilon_1^{1/q}}{10\kappa(d,N)} \quad \text{for all } r \leq j \leq d.
\]
We use the symmetries of \( B^q_N \cap \mathbb{Z}^d \) to write

\[
m^q_N(\xi) = \frac{1}{|B_N^q \cap \mathbb{Z}^d|} \sum_{x \in B_N^q \cap \mathbb{Z}^d} \prod_{j=1}^d \cos(2\pi x_j \xi_j).
\]

Applying the Cauchy–Schwarz inequality, \( \cos^2(2\pi x_j \xi_j) = 1 - \sin^2(2\pi x_j \xi_j) \) and \( 1 - x \leq e^{-x} \), we obtain

\[
|m^q_N(\xi)|^2 \leq \frac{1}{|B_N^q \cap \mathbb{Z}^d|} \sum_{x \in B_N^q \cap \mathbb{Z}^d} \exp \left( - \sum_{j=r+1}^d \sin^2(2\pi x_j \xi_j) \right).
\]

(4.19) For \( x \in B_N^q \cap \mathbb{Z}^d \) we define

\[
I_x = \{ i \in \mathbb{N}_d : \varepsilon \kappa(d, N) \leq |x_i| \leq 2\varepsilon_1^{-1/q} \kappa(d, N) \},
\]

\[
I'_x = \{ i \in \mathbb{N}_d : 2\varepsilon_1^{-1/q} \kappa(d, N) < |x_i| \},
\]

\[
I''_x = \{ i \in \mathbb{N}_d : \varepsilon \kappa(d, N) \leq |x_i| \} = I_x \cup I'_x,
\]

and

\[
E = \{ x \in B_N^q \cap \mathbb{Z}^d : |I_x| \geq \varepsilon_1 d/2 \}.
\]

Observe that

\[
E^c = \{ x \in B_N^q \cap \mathbb{Z}^d : |I_x| < \varepsilon_1 d/2 \} = \{ x \in B_N^q \cap \mathbb{Z}^d : |I''_x| < \varepsilon_1 d/2 + |I'_x| \}
\]

\[
\subseteq \{ x \in B_N^q \cap \mathbb{Z}^d : |I''_x| < \varepsilon_1 d/2 + |I'_x| \text{ and } |I'_x| \leq \varepsilon_1 d/2 \}
\]

\[
\cup \{ x \in B_N^q \cap \mathbb{Z}^d : |I'_x| > \varepsilon_1 d/2 \}.
\]

Then it is not difficult to see that

\[
|E^c| \leq |\{ x \in B_N^q \cap \mathbb{Z}^d : |I''_x| < \varepsilon_1 d \}| \leq 2 e^{-\frac{d}{4}} |B_N^q \cap \mathbb{Z}^d|.
\]

Therefore, by (4.19) we have

\[
|m_N^q(\xi)|^2 \leq \frac{1}{|B_N^q \cap \mathbb{Z}^d|} \sum_{x \in B_N^q \cap \mathbb{Z}^d} \exp \left( - \sum_{j=1}^d \sin^2(2\pi x_j \xi_j) \right) \mathbf{1}_E(x) + 2e^{-\frac{d}{4}},
\]

where \( J_r = \{ r + 1, \ldots, d \} \). Using (4.1) and the definition of \( I_x \) we have

\[
\sin^2(2\pi x_j \xi_j) \geq 16 |x_j|^2 \|\xi_j\|^2 \geq 16 \varepsilon^2 \kappa(d, N)^2 \|\xi_j\|^2, \]

\[
\mathbf{1}_E(x) \leq \frac{1}{|B_N^q \cap \mathbb{Z}^d|} \sum_{x \in B_N^q \cap \mathbb{Z}^d} \exp \left( - \sum_{j=1}^d \sin^2(2\pi x_j \xi_j) \right) \mathbf{1}_E(x).
\]
since $2|x_j||\xi_j| ≤ 1/2$ by (4.18), and consequently we obtain
\[
\frac{1}{|B_N^q \cap \mathbb{Z}^d|} \sum_{x \in B_N^q \cap \mathbb{Z}^d} \exp \left( - \sum_{j \in I_x \cap J_r} \sin^2(2\pi x_j \xi_j) \right) \mathbb{1}_E(x) \leq \frac{1}{|B_N^q \cap \mathbb{Z}^d|} \sum_{x \in B_N^q \cap \mathbb{Z}^d} \exp \left( - 16\varepsilon^2 \kappa(d, N)^2 \sum_{j \in I_x \cap J_r} ||\xi_j||^2 \right) \leq Ce^{-c \kappa(d, N)^2 ||\xi||^2}
\]
for some constants $C, c > 0$. In order to get the last inequality in (4.20) observe that
\[
\frac{1}{|B_N^q \cap \mathbb{Z}^d|} \sum_{x \in B_N^q \cap \mathbb{Z}^d} \exp \left( - 16\varepsilon^2 \kappa(d, N)^2 \sum_{j \in I_x \cap J_r} ||\xi_j||^2 \right) = \frac{1}{|B_N^q \cap \mathbb{Z}^d|} \sum_{x \in B_N^q \cap \mathbb{Z}^d} \frac{1}{d!} \sum_{\sigma \in \text{Sym}(d)} \exp \left( - 16\varepsilon^2 \kappa(d, N)^2 \sum_{j \in \sigma(I_x) \cap J_r} ||\xi_j||^2 \right) = \frac{1}{|B_N^q \cap \mathbb{Z}^d|} \sum_{x \in B_N^q \cap \mathbb{Z}^d} \mathbb{E} \left[ \exp \left( - 16\varepsilon^2 \kappa(d, N)^2 \sum_{j \in \sigma(I_x) \cap J_r} ||\xi_j||^2 \right) \right],
\]
since $\sigma \cdot (B_N^q \cap \mathbb{Z}^d \cap E) = B_N^q \cap \mathbb{Z}^d \cap E$ for every $\sigma \in \text{Sym}(d)$. We now apply Lemma 4.6 with $\delta_1 = \varepsilon_1/2$, $d_0 = r$, $I = I_x$, $\delta_0 = 3/5$, and
\[
u_j = \begin{cases} 
16\varepsilon^2 \kappa(d, N)^2 ||\xi_r||^2 & \text{for } 1 \leq j \leq r, \\
16\varepsilon^2 \kappa(d, N)^2 ||\xi_j||^2 & \text{for } r + 1 \leq j \leq d,
\end{cases}
\]
noting that $16\varepsilon^2 \kappa(d, N)^2 ||\xi_r||^2 \leq 1/5$ by (4.18). We conclude that
\[
\mathbb{E} \left[ \exp \left( - 16\varepsilon^2 \kappa(d, N)^2 \sum_{j \in \sigma(I_x) \cap J_r} ||\xi_j||^2 \right) \right] \leq 3 \exp \left( - c' \kappa(d, N)^2 \sum_{j=r+1}^d ||\xi_j||^2 \right)
\]
holds for some $c' > 0$ and for all $x \in B_N^q \cap \mathbb{Z}^d \cap E$. This proves (4.20) since by (4.17) we obtain
\[
\exp \left( - c' \kappa(d, N)^2 \sum_{j=r+1}^d ||\xi_j||^2 \right) \leq \exp \left( - \frac{c' \kappa(d, N)^2}{4} \sum_{j=1}^d ||\xi_j||^2 \right).
\]
Assume now that (4.18) does not hold. Then
\[
||\xi_j|| \geq \frac{\varepsilon_1^{1/2}}{10\kappa(d, N)} \text{ for all } 1 \leq j \leq r.
\]
Hence
\[ \| \xi_1 \|^2 + \cdots + \| \xi_r \|^2 \geq \varepsilon \frac{100 \kappa(d, N)^2}{\kappa(d, N)^2}, \]

Therefore, we invoke Lemma 4.13 with \( \eta = (\xi_1, \ldots, \xi_r) \) again and obtain
\[ |m^{B_N}(\xi)| \lesssim_q \kappa(d, N)^{-\frac{1}{4}} + (\kappa(d, N)\|\eta\|)^{-1} \lesssim_q \kappa(d, N)^{-\frac{1}{4}}, \]
since \( r \approx_q \kappa(d, N)^{\frac{3}{2}} \). This completes the proof of Proposition 4.2.

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