Extended phase space thermodynamics and $P - V$ criticality of charged black holes in Brans–Dicke theory

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Abstract Motivated by conformal relation between dilaton gravity and Brans–Dicke theory, in this paper, we are taking into account extended phase space thermodynamics to investigate phase transition of charged black holes. We regard spherically symmetric charged black hole solutions in the presence of a scalar field in both Einstein and Jordan frames and calculate related conserved and thermodynamic quantities. Then, we study the analogy of the black hole solution with the Van der Waals liquid–gas system in the extended phase space by considering the cosmological constant proportional to thermodynamical pressure. We obtain critical values of thermodynamic coordinates and plot $P - V$ and $G - T$ diagrams to study the phase transition points and compare the results of dilaton gravity and Brans–Dicke theory.

Keywords Brans–Dicke theory · Extended phase space thermodynamics · Charged black hole

1 Introduction

In the context of astrophysics, scalar tensor gravity theories are well-known models to explain, successfully, various theoretical problems of Einstein general relativity such
as galaxy rotation curves. In addition, there has been a renewed interest in examining
the coupling of scalar field with general relativity ever since black hole solutions
have been introduced in the context of string theory [1,2]. The analytical (charged)
black hole solutions of Einstein-dilaton gravity have particular interest. For example,
it was found that in the presence of Liouville-type dilaton potential, which is the
generalization of the cosmological constant, dilaton field changes the asymptotical
behavior of the solutions [3]. Moreover, it has been shown that the action of Brans–
Dicke (BD) theory is conformally related to the dilaton gravity action [4–8].

Einstein constructed the theory of general relativity that describes the dynamics of
our solar system well enough, but it probably does not describe gravity accurately at
all scales. One of the problems that general relativity faced is that, it does not accom-
modate either Mach’s principle or Dirac’s large-number hypothesis. It is also unable
to describe the accelerated expansion of the universe without fine-tuning. Herein cos-
omologists explored various alternative theories of gravity (see [9] for a good review).
Brans and Dicke [10] were pioneers in studying these alternative theories and they
developed another relativistic theory known as BD theory. This theory can be regarded
as an economic modification of Einstein general relativity which describes gravitation
in terms of metric as well as a scalar field and it accommodates both Mach’s princi-
ple and Dirac’s large-number hypothesis. Due to the importance of black holes and
gravitational collapse in both classical and quantum gravity, authors have investigated
various aspects of them in BD theory [11–14]. It has been proved that in four dimen-
sions, the stationary and vacuum BD solution is just the Kerr solution with a constant
scalar field [15]. In order to investigate the distinction between the BD theory and Ein-
stein theory Cai and Myung proved that the black hole solution in the BD–Maxwell
(BDM) theory in four dimensions is just the Reissner–Nordström (RN) solution with a
trivial scalar field [5]. In higher dimensions, however, it would be the RN solution with
a non-trivial scalar field [5,6]. This is because the stress energy tensor of Maxwell field
is not traceless in higher dimensions and the action of Maxwell field is not invariant
under the conformal transition.

On the other hand, thermodynamic properties of the black holes have been a fas-
cinating subject for many years. It was found out that black holes along all assigned
thermodynamic variables also have rich phase structure in complete analogy with
non-gravitational thermodynamic system similar to van der Waals gas system. With
the conception of expecting the cosmological constant term to arise from the vac-
uum expectation value of a quantum field, we can assume that it can vary. Hence, we
can treat the cosmological constant and its conjugate as thermodynamical pressure
and volume of a black hole system, respectively [16–27]. Regarding this extension
of thermodynamic quantities, one finds that the mass of the black holes is equated
with enthalpy rather than the internal energy [19]. Studying the thermodynamics of
black holes in AdS spacetime has exhibited various phase transitions with the same
critical behavior as van der Waals model, qualitatively (for a complete improvement
of the VdP analogy between a black hole and van der Waals liquid/gas system, see
[22,23]). The paper of Hawking and Page was the initial studies on the subject of phase
transition [28]. They pointed out there is a thermal radiation (black hole first order
phase transition) for Schwarzschild-AdS black hole spacetime. Adding charge and/or
radiation will result a behavior similar to a van der Waals liquid/gas [29,30] and the
analogy will improve by being in the extended phase space where the cosmological constant is interpreted as thermodynamical pressure.

In this paper, we want to investigate the thermodynamic phase transition of charged black holes in BD theory by using the analogy between our system and the van der Waals liquid/gas.

The outline of our paper is as follows. Section 2 is devoted to brief review of BDM field equations (Jordan frame) and their relation with dilaton gravity (Einstein frame) by a conformal transformation. In Sect. 3, we obtain charged black hole solutions of dilaton gravity. Next, we extend the phase space by considering cosmological constant proportional to thermodynamic pressure and calculate critical values. After that we use the conformal transformation to obtain charged black hole solutions in Jordan frame and repeat thermodynamical calculations for BD solutions. In order to investigate phase transition of black hole solutions, we present four tables and plot diagrams of $P - V$ and $G - T$ for both Einstein and Jordan frames. In next section, we give a detailed discussion regarding tables and diagrams. We finish our paper with some closing remarks.

2 Field equations and conformal transformations

The action of $(n + 1)$-dimensional BDM theory with a scalar field $\Phi$ and a self-interacting potential $V(\Phi)$ can be written as [5,6]

$$I_G = -\frac{1}{16\pi} \int_\mathcal{M} d^{n+1}x \sqrt{-g} \left( \Phi \mathcal{R} - \frac{\omega}{\Phi} (\nabla \Phi)^2 - V(\Phi) - F_{\mu \nu} F^{\mu \nu} \right),$$  

(1)

where $\mathcal{R}$ is the scalar curvature, the factor $\omega$ is the coupling constant, $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field and $A_\mu$ is the electromagnetic potential. Varying the action (1) with respect to the gravitational field $g_{\mu \nu}$, the scalar field $\Phi$ and the gauge field $A_\mu$, one can obtain equations of motion with the following explicit forms [5,6]

$$G_{\mu \nu} = \frac{\omega}{\Phi^2} \left( \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu \nu} (\nabla \Phi)^2 \right) - \frac{V(\Phi)}{2\Phi} g_{\mu \nu} + \frac{1}{\Phi} \left( \nabla_\mu \nabla_\nu \Phi - g_{\mu \nu} \nabla^2 \Phi \right)$$

$$+ \frac{2}{\Phi} \left( F_{\mu \lambda} F^{\lambda \nu} - \frac{1}{4} F_{\rho \sigma} F^{\rho \sigma} g_{\mu \nu} \right),$$  

(2)

$$\nabla^2 \Phi = -\frac{n - 3}{2 [(n - 1) \omega + n]} F^2$$

$$+ \frac{1}{2 [(n - 1) \omega + n]} \left[ (n - 1) \Phi \frac{dV(\Phi)}{d\Phi} - (n + 1) V(\Phi) \right],$$  

(3)

$$\nabla_\mu F^{\mu \nu} = 0,$$  

(4)

where $G_{\mu \nu}$ and $\nabla_\mu$ are, respectively, the Einstein tensor and covariant derivative of manifold $\mathcal{M}$ with metric $g_{\mu \nu}$. Due to the appearance of inverse powers of the scalar field on the right hand side of (2), solving the field equations (2)–(4), directly, is
a non-trivial task. Using a suitable conformal transformation, one can remove this difficulty. Indeed, via the conformal transformation \([5,6]\) the BDM theory can be transformed into the Einstein–Maxwell theory with a minimally coupled scalar dilaton field. Suitable conformal transformation can be shown as

\[
\bar{g}_{\mu\nu} = \Phi^{2/(n-1)} g_{\mu\nu},
\]

\[
\bar{\Phi} = \frac{n-3}{4\alpha} \ln \Phi,
\]

where

\[
\alpha = (n-3)/\sqrt{4(n-1)\omega + 4n}.
\]

It is notable that all functions and quantities in Jordan frame \((g_{\mu\nu}, \Phi, F_{\mu\nu})\) can be transformed into Einstein frame \((\bar{g}_{\mu\nu}, \bar{\Phi}, \bar{F}_{\mu\nu})\). Applying the mentioned conformal transformation on the BD action (1), one can obtain action of dilaton gravity

\[
\bar{I}_G = -\frac{1}{16\pi} \int_M d^{n+1}x \sqrt{-\bar{g}} \left\{ \bar{\mathcal{R}} - \frac{4}{n-1}(\bar{\nabla} \bar{\Phi})^2 - \bar{V}(\bar{\Phi}) \right\} - \exp\left(-\frac{4\alpha \bar{\Phi}}{(n-1)}\right) \bar{F}_{\mu\nu} \bar{F}^{\mu\nu},
\]

where \(\bar{\mathcal{R}}\) and \(\bar{V}\) are, respectively, the Ricci scalar and covariant derivative corresponding to the metric \(\bar{g}_{\mu\nu}\), and \(\bar{V}(\bar{\Phi})\) is

\[
\bar{V}(\bar{\Phi}) = \Phi^{-(n+1)/(n-1)} V(\Phi).
\]

Regarding \((n+1)\)-dimensional Einstein–Maxwell–dilaton (EMd) action (7), \(\alpha\) is an arbitrary constant that governs the strength between the dilaton and Maxwell fields. One can obtain the equations of motion by varying this action (7) with respect to \(\bar{g}_{\mu\nu}\), \(\bar{\Phi}\) and \(\bar{F}_{\mu\nu}\)

\[
\bar{\mathcal{R}}_{\mu\nu} = \frac{4}{n-1} \left( \bar{\nabla}_\mu \bar{\Phi} \bar{\nabla}_\nu \bar{\Phi} + \frac{1}{4} \bar{\nabla}^2 \bar{g}_{\mu\nu} \right) + 2e^{-4\alpha \bar{\Phi}/(n-1)} \left( \bar{F}_{\mu\lambda} \bar{F}^{\lambda}_{\nu} - \frac{1}{2(n-1)} \bar{F}_{\rho\sigma} \bar{F}^{\rho\sigma} \bar{g}_{\mu\nu} \right),
\]

\[
\bar{\nabla}^2 \bar{\Phi} = \frac{n-1}{8} \frac{d \bar{V}}{d \Phi} - \frac{\alpha}{2} e^{-4\alpha \bar{\Phi}/(n-1)} \bar{F}_{\rho\sigma} \bar{F}^{\rho\sigma},
\]

\[
\partial_\mu \left[ \sqrt{-\bar{g}} e^{-4\alpha \bar{\Phi}/(n-1)} \bar{F}^{\mu\nu} \right] = 0.
\]

By assuming the \((\bar{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \bar{\Phi})\) as solutions of Eqs. (9)–(11) with potential \(\bar{V}(\bar{\Phi})\) and comparing Eqs. (2)–(4) with Eqs. (9)–(11) we find the solutions of Eqs. (2)–(4) with potential \(V(\Phi)\) can be written as

\[
[g_{\mu\nu}, F_{\mu\nu}, \Phi] = \left[ \exp\left(-\frac{8\alpha \bar{\Phi}}{(n-1)(n-3)}\right) \bar{g}_{\mu\nu}, \bar{F}_{\mu\nu}, \exp\left(\frac{4\alpha \bar{\Phi}}{n-3}\right) \right].
\]
3 Charged black hole solutions in dilaton gravity

In this section, we briefly discuss the \((n + 1)\)-dimensional charged dilaton gravity solutions. To do so, we assume that the metric has the following form

\[
d\bar{s}^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r)d\Omega^2_k,
\]

where

\[
d\Omega^2_k = \begin{cases} 
  d\theta_1^2 + \sum_{i=2}^{n-1} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = 1 \\
  d\theta_1^2 + \sinh^2 \theta_1 d\theta_2^2 + \sinh^2 \theta_1 \sum_{i=3}^{n-1} \prod_{j=2}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = -1 \\
  \sum_{i=1}^{n-1} d\phi_i^2 & k = 0
\end{cases}
\]

represents the line element of an \((n - 1)\)-dimensional hypersurface with constant curvature \((n - 1)(n - 2)k\) and volume \(\sigma_{n-1}\). It is notable that the constant \(k\) indicates the boundary of \(t = \text{constant}\) and \(r = \text{constant}\), in which it can be a positive (elliptic boundary), zero (flat boundary) or negative (hyperbolic boundary). Taking into account metric (13) and integrating the Maxwell equation (11), we can obtain the nonzero electric field \(\bar{F}_{tr}\) with the following form

\[
\bar{F}_{tr} = \frac{q}{(rR)^{n-1}} \exp \left( \frac{4\alpha \bar{\Phi}}{n - 1} \right).
\]

In order to obtain metric function with consistent dilaton field, we should consider a suitable potential \(\bar{V}(\bar{\Phi})\). One of the appropriate potential, which we consider in this paper, is Liouville-type potential with the following explicit form

\[
\bar{V}(\bar{\Phi}) = 2\Lambda \exp \left( \frac{4\alpha \bar{\Phi}}{n - 1} \right) + \frac{k(n - 1)(n - 2)\alpha^2}{c^2(\alpha^2 - 1)} e^{\frac{4\bar{\Phi}}{(n - 1)\sigma}},
\]

where in the absence of dilaton field \((\alpha & \bar{\Phi} = 0)\), it reduces to \(2\Lambda\). Taking into account the metric (13) with potential (16) and Maxwell field (15), the consistent solutions of (9) and (10) are

\[
f(r) = -\frac{k(n - 2)(\alpha^2 + 1)^2 e^{-2\nu} r^{2\nu}}{(\alpha^2 + n - 2)(\alpha^2 - 1)} + \frac{2\Lambda(\alpha^2 + 1)^2 e^{2\nu}}{(n - 1)(\alpha^2 - n)} r^{2(1 - \gamma)} - \frac{m}{r^{(n - 2)}} r^{(n - 1)\gamma} + \frac{2q^2(\alpha^2 + 1)^2 e^{-2(n - 2)\nu}}{(n - 1)(\alpha^2 + n - 2)r^{2(n - 2)(1 - \gamma)}},
\]

\[
R(r) = \exp \left( \frac{2\alpha \bar{\Phi}}{n - 1} \right) = \left( \frac{c}{r} \right)^\gamma,
\]
where $m$ is an integration constant which is related to the total mass, $c$ is another arbitrary constant related to the scalar field and $\gamma = \alpha^2 / (1 + \alpha^2)$. Calculating curvature scalars shows that there is a curvature singularity located at $r = 0$ which can be covered with an event horizon and therefore one can interpret it as black hole. In Ref. [3,5], it was shown that although the asymptotical behavior of the solutions is neither flat nor AdS, the solutions have well-defined causal structure and well-defined asymptotical behavior. In other words, there is only one curvature singularity located at the origin ($r = 0$) and all curvature invariants are finite for $r \neq 0$. Hereafter, we take into account positive curvature constant boundary ($k = 1$) for investigating phase transition.

### 3.1 Extended phase space and $P - V$ criticality of dilatonic black holes

In order to regard phase space extension of thermodynamics, we should obtain conserved and thermodynamic quantities. Using the surface gravity interpretation ($\kappa$), one finds the Hawking temperature

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \sqrt{-\frac{1}{2} \left( \nabla_\mu \chi^\nu \right) \left( \nabla^\mu \chi^\nu \right)},$$

(20)

where $\chi = \partial / \partial t$ is the null Killing vector of the horizon $r_+$. Regarding $\chi^\nu = (1, 0, 0, \ldots, 0)$, one can obtain $\chi_\nu = (-f(r_+), 0, 0, \ldots, 0)$ and therefore $(\nabla_\mu \chi^\nu) (\nabla^\mu \chi^\nu) = -\frac{1}{2} [f'(r_+)]^2$. Thus, we can calculate the following explicit form of the Hawking temperature

$$T = \frac{f'(r_+)}{4\pi},$$

(21)

which leads to

$$T = \frac{(\alpha^2 + 1)}{2\pi (n - 1)} \left[ -k (n - 1) (n - 2) \left( \frac{c}{r_+} \right)^{2\gamma} - A r_+ \left( \frac{c}{r_+} \right)^{2\gamma} - \frac{q^2}{r_+^{2n-3}} \left( \frac{c}{r_+} \right)^{-2\gamma(n-2)} \right].$$

(22)

In addition, the finite mass and the entropy of the black hole can be obtained with the following forms [31]

$$M = \frac{\sigma_{n-1} c^{(n-1)\gamma}}{16\pi} \left( \frac{n - 1}{1 + \alpha^2} \right) m,$$

(23)

$$S = \frac{\sigma_{n-1} c^{(n-1)\gamma}}{4} r_+^{(n-1)(1-\gamma)}.$$
This relation may be generalized in the presence of dilaton field with the following form

\[ P = -\frac{\Lambda}{8\pi} \left( \frac{c}{r_+} \right)^{2\gamma}, \]  

(25)

where one can obtain Eq. (25) from the concept of energy–momentum tensor. It is worthwhile to mention that the known relation \( P = -\frac{\Lambda}{8\pi} \) is recovered for \( \alpha = 0 \), as we expect. Considering the relation between cosmological constant and thermodynamical pressure, it was shown that the interpretation of mass will become enthalpy [19] and therefore, one can calculate the generalized volume with the following form

\[ V = \left( \frac{\partial H}{\partial P} \right)_{S, Q} = \left( \frac{\partial M}{\partial P} \right)_{S, Q} = \frac{\sigma_{n-1} (\alpha^2 + 1) r_+^n}{(n - \alpha^2)} \left( \frac{c}{r_+} \right)^{\gamma(n-1)}, \]  

(26)

where in the absence of dilaton field one obtains \( V = \frac{\sigma_{n-1} r_+^n}{n} \) [22,23,25–27].

Before studying of phase transition, we would like to examine the first law of thermodynamics with related Smarr formula. To do so, we should obtain electric charge, \( Q \), and potential, \( U \). Following Refs. [5,6], one can find

\[ Q = \frac{q}{4\pi}, \]  

(27)

\[ U = \frac{qc^{(3-n)\gamma}}{[(3-n)\gamma + n - 2] r^{[(3-n)\gamma + n - 2]}}. \]  

(28)

Regarding mass, entropy, electric charge and pressure, one can find \( T \), \( U \) and \( V \) with the following relations

\[ T = \left( \frac{\partial M}{\partial S} \right)_{Q, P}, \quad U = \left( \frac{\partial M}{\partial Q} \right)_{S, P}, \quad V = \left( \frac{\partial M}{\partial P} \right)_{S, Q}. \]  

(29)

It is straightforward to show that obtained quantities in Eq. (29) coincide with Eqs. (22), (28) and (25), and therefore, the solutions obey the first law of black hole thermodynamics in an extended phase space

\[ dM = T dS + U dQ + V dP. \]  

(30)

In addition, using scaling (dimensional) argument [19], the corresponding (generalized) Smarr relation is

\[ M = \frac{n - 1}{\alpha^2 + n - 2} TS + QU + \frac{2(\alpha^2 - 1)}{\alpha^2 + n - 2} PV, \]  

(31)

where in the absence of dilaton field, it reduces to Smarr formula of RN black holes [23].
Here, we are interested in studying the phase transition of this black hole. The equation of state of the black hole can be obtained by using Eqs. (20), (25) and (26) as \((k = 1)\)

\[
P = \frac{(n - 1)(n - 2)\varepsilon^{2(\gamma - 1)}}{16\pi (\alpha^2 - 1) c^2\gamma V^{\frac{2(1 - \gamma)}{n - \gamma(n - 1)}}} - \frac{(n - 1)T}{4(1 + \alpha^2)\varepsilon V^{\frac{1}{n - \gamma(n - 1)}}} + \frac{\varepsilon^{2\gamma(n - 2) - 2(n - 1)}q^2}{8\pi c^2\gamma V^{\frac{2(n - 1) - 2\gamma(n - 2)}{n - \gamma(n - 1)}}},
\]

in which

\[
\varepsilon = \left(\frac{\alpha^2 + 1}{n - \alpha^2}\right)^\frac{1}{\gamma(n - 1) - \pi}.
\]

We can investigate the existence of phase transition and critical behavior of this black hole by plotting and analyzing the graphs of \(P - V\) and \(G - T\) diagrams. One may use the inflection point properties

\[
\left(\frac{\partial P}{\partial V}\right)_T = 0,
\]

\[
\left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0,
\]

to obtain the critical values for the temperature, pressure and volume. Due to the difficulties of solving these equations analytically, we use the numerical method to obtain critical values.

According to extended first law of black hole thermodynamics and the interpretation of \(M\) as the black hole enthalpy \([19,34]\) the Gibbs free energy of black hole can be written as

\[
G = H - TS = M - TS,
\]

where after some simplifications one obtains the following explicit form of Gibbs free energy per unit volume \(\sigma_{n-1}\)

\[
G = \frac{P}{(n - 1)(n - 2)} \frac{\alpha^4 - 1}{n - \alpha^2} \frac{r_+^n}{(n - 1)} \frac{c}{r_+} \gamma(n - 1) \frac{K}{16\pi (\alpha^2 + n - 2)} \frac{r_+^{n - 2}}{n - \gamma(n - 1)} \frac{c}{r_+} \gamma(n - 3) + \frac{\alpha^2}{8\pi (n - 1)(\alpha^2 + n - 2) r_+^{n - 2}} \frac{c}{r_+} \gamma(n - 3).
\]

The behavior of Gibbs free energy with respect to temperature may be investigated by plotting the graph of \(G - T\). We will see the characteristic swallow-tail behavior which guarantees the existence of the phase transitions.
4 Charged BD black holes solutions

In this section, we focus on the main goal of the paper which is studying of charged black hole phase transition in BD theory. Applying the conformal transformation (5), the potential $V(\Phi)$ of BD theory with $n \geq 4$ becomes

$$V(\Phi) = 2A\Phi^2 + \frac{k(n-1)(n-2)\alpha^2}{c^2(\alpha^2 - 1)}\Phi^{[(n+1)(1+\alpha^2) - 4]/[(n-1)\alpha^2]}. \quad (37)$$

With the dilatonic black hole solutions at hand, we are in a position to obtain the solutions of Eqs. (2)–(4) by using the conformal transformation. Considering the following line element

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2H^2(r)d\Omega_k^2, \quad (38)$$

with Eqs. (2)–(4), we find that the functions $A(r)$ and $B(r)$ are

$$A(r) = \frac{-k(n-2)(\alpha^2 + 1)^2}{(\alpha^2 + n - 2)(\alpha^2 - 1)}\left(\frac{c}{r}\right)^{-2\gamma\left(\frac{n-4}{n-3}\right)} + \frac{2A(\alpha^2 + 1)^2c^2\gamma\left(\frac{n-5}{n-3}\right)}{(n-1)(\alpha^2 - n)}r^{2\left[1 - \frac{\gamma(n-5)}{n-3}\right]}, \quad (39)$$

$$B(r) = \frac{-k(n-2)(\alpha^2 + 1)^2}{(\alpha^2 + n - 2)(\alpha^2 - 1)}\left(\frac{c}{r}\right)^{-2\gamma\left(\frac{n-5}{n-3}\right)} + \frac{2A(\alpha^2 + 1)^2c^2\gamma\left(\frac{n-5}{n-3}\right)}{(n-1)(\alpha^2 - n)}r^{2\left[1 - \frac{\gamma(n-1)}{n-3}\right]}, \quad (40)$$

$$H(r) = \left(\frac{c}{r}\right)^{\gamma\left(\frac{n-5}{n-3}\right)}, \quad (41)$$

$$\Phi(r) = \left(\frac{c}{r}\right)^{\frac{2\gamma(n-1)}{n-3}}. \quad (42)$$

It is also notable that obtained solutions are just the charged solutions of Einstein gravity (Reissner–Nordström AdS black hole) as $\omega \to \infty \ (\alpha \to 0).$ It was previously shown that the mentioned solutions can be interpreted as black holes [5,6]. For investigating the causal structures and dynamics of BD field, we refer the reader to Refs. [32,33].

Using the conformal transformation, the electromagnetic field becomes

$$F_{tr} = \frac{qc^{(3-n)\gamma}}{r(n-3)(1-\gamma)+2}, \quad (43)$$

where it becomes zero as $r \to \infty.$ In addition, in the absence of dilaton field, $F_{tr}$ reduces to that of Reissner–Nordström black hole.
4.1 Extended phase space and $P - V$ criticality of BD black holes

At first, we obtain the Hawking temperature of a BD black holes on the outer horizon $r_+$. Regarding Eq. (20) with contravariant Killing vector $\chi^\nu = (1, 0, 0, \ldots, 0)$, one finds $\chi^\nu = (-A(r_+), 0, 0, \ldots, 0)$ and hence $(\nabla_\mu \chi^\nu)(\nabla^\mu \chi^\nu) = -\frac{B(r)}{A(r)} A'(r_+)^2$. Now, we can write

$$T = \frac{1}{4\pi} \sqrt{\frac{B(r)}{A(r)}} A'(r_+) = \frac{1}{4\pi} \left(\frac{c}{r_+}\right)^{4\gamma/(n-3)} A'(r_+), \quad (44)$$

where after some algebraic manipulations, we obtain

$$T = \frac{(\alpha^2 + 1)}{2\pi (n-1)} \left[ \frac{-k(n-1)(n-2)}{2(\alpha^2 - 1)r_+} \left(\frac{c}{r_+}\right)^{-2\gamma} - \Lambda r_+ \left(\frac{c}{r_+}\right)^{2\gamma} - \frac{q^2}{r_+^{2\gamma - 3}} \left(\frac{c}{r_+}\right)^{-2\gamma(n-2)} \right], \quad (45)$$

The finite mass and the entropy of the black hole can be obtained by using the Euclidian action [5,6]

$$M = \frac{\omega_{n-1} c^{(n-1)\gamma}}{16\pi} \left(\frac{n-1}{1+\alpha^2}\right) m, \quad (46)$$

$$S = \frac{\omega_{n-1} c^{(n-1)\gamma}}{4} r_+^{(n-1)(1-\gamma)}. \quad (47)$$

Using the mentioned approach, one can regard the energy–momentum tensor of BD theory to obtain generalized pressure and its corresponding volume. It is easy to show that

$$P = -\frac{\Lambda}{8\pi} \left(\frac{c}{r_+}\right)^{2\gamma(n-1)/n-3}, \quad (48)$$

and

$$V = \left(\frac{\partial H}{\partial P}\right)_{S,Q} = \left(\frac{\partial M}{\partial P}\right)_{S,Q} = \frac{\omega_{n-1} (1+\alpha^2) r_+^n}{n-\alpha^2} \left(\frac{c}{r_+}\right) \frac{\gamma(n^2-4n-1)}{n-3} r_+^{\gamma(n^2-4n-1)/n-3}. \quad (49)$$

where in the absence of dilaton field, as we mentioned before in Einstein frame, one easily obtains $V = \frac{\omega_{n-1} r_+^n}{n}[22,23,25–27]$.

Regarding Refs. [5,6], one finds that the electric charge, $Q$, and potential, $U$ are the same as those in dilaton gravity (Eqs. 27, 28). Taking into account the mentioned mass, entropy, electric charge and pressure, we can find $T$, $U$ and $V$ with the following equations.
\[ T = \left( \frac{\partial M}{\partial S} \right)_{Q,P}, \quad U = \left( \frac{\partial M}{\partial Q} \right)_{S,P}, \quad V = \left( \frac{\partial M}{\partial P} \right)_{S,Q}. \]  

(50)

Since obtained quantities in Eq. (50) coincide with Eqs. (45), (28) and (48), we find that the BD solutions obey the first law of thermodynamics in an extended phase space

\[ dM = T dS + U dQ + V dP. \]  

(51)

Moreover, applying scaling argument [19], the generalized Smarr relation for the charged BD black holes is the same as that in dilaton gravity (Eq. 31).

Now, we are going to investigate the phase transition of BD black holes. Using Eqs. (45), (48) and (49), the equation of state of the black hole can be written as \( (k = 1) \)

\[ P = \frac{(n - 1) (n - 2) \xi^{2} \gamma(n - 5 - n + 3)}{16 \pi (\alpha^{2} - 1) c^{3} V^{\gamma(n - 4n - 1) - n(n - 3)}} + \frac{2^{2y(n - 5) - n + 3} \xi^{2} \gamma(n - 5) V^{\gamma(n - 4n - 1) - n(n - 3)}}{4 (1 + \alpha^{2}) \xi^{4y(n - 3) - n + 3} V^{\gamma(n - 4n - 1) - n(n - 3)}} + \frac{2^{2y(n - 1)(n - 4)} q^{2}}{8 \pi c^{2} V^{\gamma(n - 4n - 1) - n(n - 3)}} \]  

(52)

where \( \xi \) is

\[ \xi = \left( (1 + \alpha^{2}) c^{\frac{2y(n - 1)(n - 4)}{n - 3}} V^{\gamma(n - 4n - 1) - n(n - 3)} \right)^{\gamma(n - 3) - n(n - 3)} \]

As we mentioned before, in order to investigate the existence of phase transition, we can use the inflection point properties. Taking into account Eqs. (33) and (34), one can find the critical values for the temperature, pressure and volume. We employ numerical method for calculating critical values and plot various related figures. In addition, regarding extended first law of black hole thermodynamics and the interpretation of \( M \) (total mass of black hole) as \( H \) (the black hole enthalpy) [19,34], the Gibbs free energy of black hole can be written as

\[ G = H - T S = M - T S \]

where after some algebraic manipulations, one obtains

\[ G = P (\alpha^{4} - 1) (n - 1) r_{+}^{n} \left( \frac{c}{r_{+}} \right)^{\gamma(n - 4n - 1) - n(n - 3)} + \frac{1 + \alpha^{2} (n - 2) r_{+}^{n - 2}}{16 \pi (\alpha^{2} + n - 2)} \left( \frac{c}{r_{+}} \right)^{\gamma(n - 3)} + \frac{(1 + \alpha^{2})(\alpha^{2} + 2n - 3) q^{2}}{8 \pi (n - 1) (\alpha^{2} + n - 2) r_{+}^{n - 2}} \left( \frac{c}{r_{+}} \right)^{-\gamma(n - 3)}. \]  

(53)
Hereafter, we plot some $P - V$ and $G - T$ figures related to Jordan frame (BDM solutions) and Einstein frame (EMd solutions) and give more discussions.

5 Discussion on the results of diagrams

Thermodynamical behavior of the system in both Einstein and Jordan frames is shown in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10. Figures with odd numbers are related to BD–Maxwell theory while even number figures are corresponding to EMd gravity. In order to investigate the critical points we also present four tables (see Tables 1, 2, 3, 4).

![Figure 1](image1.png)

Fig. 1 BDM $P - V$ (left), $G - T$ (right) diagrams for $n = 4$, $q = c = 1$ and $\omega = 100$. $P - V$ diagram, from up to bottom $T = 2T_C$, $T = 1.5T_C$, $T = T_C$, $T = 0.5T_C$ and $T = 0.25T_C$, respectively. $G - T$ diagram, from up to bottom $P = 1.5P_C$, $P = P_C$ and $P = 0.5P_C$, respectively.

![Figure 2](image2.png)

Fig. 2 EMd $P - V$ (left), $G - T$ (right) diagrams for $n = 4$, $q = c = 1$ and $\omega = 100$. $P - V$ diagram, from up to bottom $T = 2T_C$, $T = 1.5T_C$, $T = T_C$, $T = 0.5T_C$ and $T = 0.25T_C$, respectively. $G - T$ diagram, from up to bottom $P = 1.5P_C$, $P = P_C$ and $P = 0.5P_C$, respectively.
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Fig. 3  BDM $P - V$ (left), $G - T$ (right) diagrams for $n = 6$, $q = c = 1$ and $\omega = 100$. $P - V$ diagram, from up to bottom $T = 2T_C$, $T = 1.5T_C$, $T = T_C$, $T = 0.5T_C$ and $T = 0.25T_C$, respectively. $G - T$ diagram, from up to bottom $P = 1.5P_C$, $P = P_C$ and $P = 0.5P_C$, respectively.

Fig. 4  EMd $P - V$ (left), $G - T$ (right) diagrams for $n = 6$, $q = c = 1$ and $\omega = 100$. $P - V$ diagram, from up to bottom $T = 2T_C$, $T = 1.5T_C$, $T = T_C$, $T = 0.5T_C$ and $T = 0.25T_C$, respectively. $G - T$ diagram, from up to bottom $P = 1.5P_C$, $P = P_C$ and $P = 0.5P_C$, respectively.

Considering figures and tables together we understand that system in both Einstein and Jordan frames has similar behavior.

We know that the phase transition occurs at critical points, which represents critical pressure, critical volume and critical temperature. Accordingly, we can see through the figures that the obtained values which are shown in tables are critical values.

Studying the $G - T$ diagrams of Figs. 5, 6, 7 and 8 and tables, it is evident that by increasing the BD coupling constant ($\omega$), the critical values of temperature and Gibbs free energy decrease. Therefore we can conclude that with larger values of $\omega$, the energy that system needs in order to have phase transition would decrease.
Fig. 5 BDM $P - V$ diagram for $T = T_C$ (left) and $G - T$ diagram, for $P = 0.5P_C$ (right) with $n = 4$, $q = c = 1$, and $\omega = 60$ (solid line), $\omega = 100$ (dotted line) and $\omega = 150$ (dashed line)

Fig. 6 EMd $P - V$ diagram for $T = T_C$ (left) and $G - T$ diagram, for $P = 0.5P_C$ (right) with $n = 4$, $q = c = 1$, and $\omega = 60$ (solid line), $\omega = 100$ (dotted line) and $\omega = 150$ (dashed line)

Studying the $P - V$ diagrams of Figs. 5, 6, 7 and 8, and related tables indicates that, increasing the BD coupling constant would make an increment in the critical value of pressure and on contrary a reduction in critical value of volume. On the other hand it is worth mentioning that thorough the relation between pressure and cosmological constant in both Einstein and Jordan frames, Eqs. (25) and (48), it is evident that due to the increment that increasing $\omega$ would cause in pressure, there will be a reduction in value of the cosmological constant.

We can also find the effects of dimensionality on phase transition and critical values of the system from Figs. 9 and 10 and related tables. The $G - T$ diagrams of Figs. 9 and 10 show that increasing the dimensions of the spacetime would cause an increment and reduction in critical values of temperature and Gibbs free energy, respectively. On


\textbf{Fig. 7} BDM \( P - V \) diagram for \( T = T_C \) (left) and \( G - T \) diagram for \( P = 0.5 P_C \) (right) with \( n = 6 \), \( q = c = 1 \), and \( \omega = 60 \) (solid line), \( \omega = 100 \) (dotted line) and \( \omega = 150 \) (dashed line).

\textbf{Fig. 8} EMd \( P - V \) diagram for \( T = T_C \) (left) and \( G - T \) diagram for \( P = 0.5 P_C \) (right) with \( n = 6 \), \( q = c = 1 \), and \( \omega = 60 \) (solid line), \( \omega = 100 \) (dotted line) and \( \omega = 150 \) (dashed line).

The other hand, in higher dimensions the total finite mass (or enthalpy) of the system would increase, which causes an increment in energy of the system too. Through the \( P - V \) graphs of Figs. 9 and 10, we can see that in higher dimensions the critical volume (pressure) decreases (increases).

To sum up, we can see that as the BD coupling constant increases (dilatonic coupling constant decreases), we have an increment in critical value of pressure and a reduction in the critical value of volume and temperature and also system needs less energy (mass) in order to have phase transition. On the other hand, in higher dimensions the critical value of volume decreases and in return the critical value of pressure and temperature increase, which result into system needing to absorb more energy (mass) in order to have phase transition.
Fig. 9  BDM $P - V$ diagram for $T = T_C$ (left) and $G - T$ diagram for $P = 0.5P_C$ (right) with $\omega = 100$, $q = c = 1$, and $n = 4$ (solid line), $n = 5$ (dotted line) and $n = 6$ (dashed line).

Fig. 10  EMd $P - V$ diagram for $T = T_C$ (left) and $G - T$ diagram for $P = 0.5P_C$ (right) with $\omega = 100$, $q = c = 1$, and $n = 4$ (solid line), $n = 5$ (dotted line) and $n = 6$ (dashed line).

Table 1  Critical quantities of BD–Maxwell for $q = c = 1$ and $n = 4$

| $\omega$   | $\alpha$ | $V_c$  | $T_c$  | $P_c$  | $P_cV_c/T_c$ |
|------------|----------|--------|--------|--------|--------------|
| 60.000000 | 0.03686  | 1.25281| 0.17032| 0.03528 | 0.25951      |
| 100.000000| 0.02868  | 1.25170| 0.17030| 0.03540 | 0.26019      |
| 150.000000| 0.02347  | 1.25114| 0.17030| 0.03546 | 0.26051      |
| 300.000000| 0.01663  | 1.25027| 0.17029| 0.03552 | 0.26079      |
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Table 2  Critical quantities of BD–Maxwell for $q = c = 1$ and $n = 6$

| $\omega$  | $\alpha$  | $V_c$  | $T_c$  | $P_c$  | $P_cV_c/T_c$ |
|---------|---------|-------|-------|-------|-------------|
| 60.00000 | 0.08575 | 0.51932 | 0.47149 | 0.21457 | 0.23634     |
| 100.00000| 0.06668 | 0.51759 | 0.47045 | 0.21612 | 0.23778     |
| 150.00000| 0.05455 | 0.51671 | 0.46993 | 0.21692 | 0.23851     |
| 300.00000| 0.03865 | 0.51583 | 0.46941 | 0.21772 | 0.23925     |

Table 3  Critical quantities of Einstein Maxwell Dilaton for $q = c = 1$ and $n = 4$

| $\omega$  | $\alpha$  | $V_c$  | $T_c$  | $P_c$  | $P_cV_c/T_c$ |
|---------|---------|-------|-------|-------|-------------|
| 60.00000 | 0.03686 | 1.25234 | 0.17070 | 0.03555 | 0.26081     |
| 100.00000| 0.02868 | 1.25142 | 0.17054 | 0.03556 | 0.26094     |
| 150.00000| 0.02347 | 1.25095 | 0.17045 | 0.03557 | 0.26105     |
| 300.00000| 0.01663 | 1.25047 | 0.17037 | 0.03558 | 0.26115     |

Table 4  Critical quantities of Einstein Maxwell Dilaton for $q = c = 1$ and $n = 6$

| $\omega$  | $\alpha$  | $V_c$  | $T_c$  | $P_c$  | $P_cV_c/T_c$ |
|---------|---------|-------|-------|-------|-------------|
| 60.00000 | 0.08575 | 0.51989 | 0.47353 | 0.21705 | 0.23830     |
| 100.00000| 0.06668 | 0.51793 | 0.47169 | 0.21764 | 0.23898     |
| 150.00000| 0.05455 | 0.51694 | 0.47076 | 0.21794 | 0.23932     |
| 300.00000| 0.03865 | 0.51594 | 0.46983 | 0.21824 | 0.23966     |

6 Conclusions

In this paper, we considered both dilaton gravity and BD theory in the presence of electromagnetic field and studied their thermodynamical behaviors and phase structures. We extended the phase space by considering cosmological constant proportional to thermodynamical pressure and its conjugate variable as volume, and regarded the interpretation of total mass of black hole as the enthalpy of the system.

Calculation of critical values through two different types of phase diagrams ($P - V$ and $G - T$) resulted into phase transition taking place in the critical values. Studying $P - V$ and $G - T$ diagrams in both Einstein and Jordan frames exhibits similar behavior near critical points to their corresponding diagrams in Van der Waals liquid/gas.

The results indicated that conformal transformation did not have significant effect on the total behavior of phase diagrams. In other words, we found that the corresponding phase diagrams in both frames are very similar. We also presented four tables with critical values which confirmed that although critical values of volume, temperature and pressure are not exactly the same for both frames, general behavior of the phase structure in Einstein frame is very close to that of Jordan frame. Considering the variations of $\frac{P_cV_c}{T_c}$, we found that this ratio is an increasing function of $\omega$ and a decreasing
function of dimensions. Although it was indicated that this ratio is not, considerably, changed under conformal transformation, we found that its value in Einstein frame is slightly greater than that in Jordan frame.

Regarding the obtained values of tables with their related figures, we found that for large values of coupling constant the system needs less energy absorption to have phase transition, due to the fact that as coupling constant increases the critical temperature decreases. On the other hand, studying the effects of dimensionality showed that for higher dimensional black holes, phase transition takes place in higher temperature and lower Gibbs free energy.

Finally considering BD theory with various models of nonlinear electrodynamics, it would be interesting to analyze the effects of nonlinearity on extended phase space thermodynamics and \( P - V \) criticality of black hole solutions. Moreover, following the work of Cvetic et al. [35], one may examine the (reverse) isoperimetric inequality condition for the black hole solutions in both Einstein and Jordan frames. We left these issues for the forthcoming work.

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References

1. Gibbons, G.W., Maeda, K.: Nucl. Phys. B 298, 741 (1988)
2. Garfinkle, D., Horowitz, G.T., Strominger, A.: Phys. Rev. D 43, 3140 (1991)
3. Chan, K.C.K., Horne, J.H., Mann, R.B.: Nucl. Phys. B 447, 441 (1995)
4. Chiba, T., Soda, J.: Prog. Theor. Phys. 96, 567 (1996)
5. Cai, R.G., Myung, Y.S.: Phys. Rev. D 56, 3466 (1997)
6. Dehghani, M.H., Pakravan, J., Hendi, S.H.: Phys. Rev. D 74, 104014 (2006)
7. Hendi, S.H.: J. Math. Phys. 49, 082501 (2008)
8. Hendi, S.H., Katebi, R.: Eur. Phys. J. C 72, 2235 (2012)
9. Clifton, T., Ferreira, P.G., Padilla, A., Skordis, C.: Phys. Rep. 513, 1 (2012)
10. Brans, C., Dicke, R.: Phys. Rev. 124, 925 (1961)
11. Scheel, M.A., Shapiro, S.L., Teukolsky, S.A.: Phys. Rev. D 51, 4208 (1995)
12. Scheel, M.A., Shapiro, S.L., Teukolsky, S.A.: Phys. Rev. D 51, 4236 (1995)
13. Kang, G.: Phys. Rev. D 54, 7483 (1996)
14. de Oliveira, H.P., Cheb-Terrab, E.S.: Class. Quantum Gravit. 13, 425 (1996)
15. Hawking, S.W.: Commun. Math. Phys. 25, 167 (1972)
16. Teitelboim, Phys. Lett. B 158, 293 (1985)
17. Creighton, J., Mann, R.B.: Phys. Rev. D 52, 4569 (1995)
18. Padmanabhan, T.: Class. Quantum Gravit. 19, 5387 (2002)
19. Kastor, D., Ray, S., Traschen, J.: Class. Quantum Gravit. 26, 195011 (2009)
20. Delcate, T., Mann, R.B.: J. High Energy Phys. 02, 070 (2015)
21. Rajagopal, A., Kubiznak, D., Mann, R.B.: Phys. Lett. B 737, 277 (2014)
22. Dolan, B.P.: Class. Quantum Gravit. 28, 125020 (2011)
23. Kubiznak, D., Mann, R.B.: J. High Energy Phys. 07, 033 (2012)
24. Dolan, B.P.: Class. Quantum Gravit. 28, 235017 (2011)
25. Hendi, S.H., Vahidinia, M.H.: Phys. Rev. D 88, 084045 (2013)
26. Hendi, S.H., Panahiyan, S., Eslam Panah, B.: Int. J. Mod. Phys. D (accepted). arXiv:1410.0352
27. Hendi, S.H., Panahiyan, S., Momennia, M.: Int. J. Mod. Phys. D (accepted). arXiv:1503.03340
28. Hawking, S.W., Page, D.N.: Commun. Math. Phys. 87, 577 (1983)
29. Chamblin, A., Emparan, R., Johnson, C., Myers, R.: Phys. Rev. D 60, 64018 (1999)
30. Niu, C., Tian, Y., Wu, X.N.: Phys. Rev. D 85, 24017 (2012)
31. Sheykhi, A.: Phys. Rev. D 76, 124025 (2007)
32. Hansen, J., Yeom, D.H.: J. High Energy Phys, 10, 040 (2014)
33. Hwang, D.I., Yeom, D.H.: Class. Quantum Gravit. 27, 205002 (2010)
34. Dolan, B.P.: Mod. Phys. Lett. A 30, 1540002 (2015)
35. Cvetic, M., Gibbons, G.W., Kubiznak, D., Pope, C.N.: Phys. Rev. D 84, 024037 (2011)