Dynamical Origin of Power Spectra

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Abstract

We discuss a possible origin of Tsallis’ statistics from the correlation among constituents which reduces the phase space of the system. We also show that in the system of coupled linear harmonic oscillators can exhibit a Tsallis type behavior.
I. INTRODUCTION

If we want to determine precisely the dynamical behavior of a macroscopic system out of equilibrium, we need in general to deal with highly coupled equations of motion related to the full microscopic degrees of freedom of the system. However, when we are interested in knowing only the dynamics of macroscopic observables, we can sometimes expect a large amount of reduction of degrees of freedom relevant to the dynamics of these macroscopic observables. They should be described by a closed set of simpler coarse-grained equations. Such a situation happens when the space-time scales for our macroscopic observables are clearly separated from those of the microscopic ones. The hydrodynamics is such an example.

Hydrodynamics is derived by assuming the local thermal equilibrium: at any point of a fluid, there exists a finite domain (fluid cell) in which the fluid can be regarded as homogeneous and in thermal equilibrium, so that the usual thermodynamic relations hold for this piece of fluid. The physical properties of the fluid are determined by these thermodynamic relations. Furthermore, although finite, the size of the domain is considered to be negligible compared to the typical scale of the system, so that the thermodynamic variables can be considered as continuously varying functions of position and time.

On the other hand, strictly speaking, the thermodynamic description of a matter is only possible when the time scale in question is much larger than the relaxation time of the matter needed to attain thermodynamic equilibrium. In other words, for hydrodynamic description, the fluid cell should have the minimum scale to realize local thermal equilibrium. For example, for a gas, such a domain should at least be several times larger than the mean-free path of the gas particles, otherwise fluctuations are too large and the thermodynamic description of the gas property will fail. It is thus reasonable to consider that hydrodynamic approach is not applicable when the size of the system is relatively small so that the typical space-time inhomogeneity scales are comparable or smaller than those required for the local thermalization of the system. In fact, it is believed that the deviations from the usual thermodynamics is an important factor in physics associated with molecular scales such as biological systems [1].

A parameter commonly used to evaluate quantitatively the applicability of hydrodynamic approach is the Knudsen number $K$, defined as the ratio, $K = \frac{\lambda_{meso}}{\lambda_{hydro}}$, where $\lambda_{meso}$ is the mesoscopic scale of the microscopic structure (in the case of a gas, it is the mean-free
path of the gas particles) and $\lambda_{\text{hydro}}$ is the typical hydrodynamical inhomogeneity scale. If $K \ll 1$ everywhere, a hydrodynamic description will be applicable as the dynamics of the system. This also means that the time scale of the macroscopic (hydrodynamic) scales are much larger than the thermodynamic relaxation time. In general, the hydrodynamic time scale is of the order of $\lambda_{\text{hydro}}/v_{\text{sound}}$, where $v_{\text{sound}}$ is the velocity of the sound of the fluid.

On the other hand, there are several counter examples where hydrodynamic models seem to work for $K$ values apparently not so small. One of these is the physics of relativistic heavy-ion collisions [2]. To analyse the behaviors of huge amount of produced particles in the relativistic heavy-ion collisions, we simply assume that the many-body dynamics can be approximately given by relativistic hydrodynamics (in particular, perfect fluid). Since the typical dimension of the whole system is initially from few $fm$ ($10^{-15}m$) (or even less if we consider the Lorentz contraction due to the relativistic effects) and expands violently. For a hydrodynamic description to be valid from the beginning of this expansion, we expect that the fluid cell size should at least be less than one order of magnitude of $1fm$. That is, we should use the cell size of the order of $0.1fm$, which is already comparable, if not smaller, to a typical value of mean-free path of the constituent particles, partons. The value, $0.1fm$ is even closer to the scale of quantum correlation length. In spite of these facts, the collective motion of the matter produced in the relativistic heavy ion collision is well described by the hydrodynamic model. The hydrodynamic model is a very important tool to understand the experimental data.

Then how should we understand such a situation? Here, we have to recall that the local thermodynamic equilibrium is a sufficient condition for the thermodynamic relations to be valid, but not a necessary condition. For example, let us consider an ideal Boltzmann gas. If the system is isotropic in momentum space at any position, the energy density and pressure are given as

$$\varepsilon (\vec{r}) = \int d^3 \vec{p} \ E (\vec{p}) \ f (\vec{p}, \vec{r}) , \quad (1)$$

and

$$P (\vec{r}) = \frac{1}{3} \int d^3 \vec{p} \ (\vec{p} \cdot \vec{v}) \ f (\vec{p}, \vec{r}) , \quad (2)$$

where $f (\vec{p}, \vec{r})$ is the one-particle phase space distribution function and $\vec{v} = \vec{p}/E$. If these particles are massless (or ultra-relativistic), then $|\vec{p}| = E (\vec{p})$, so that we always have

$$\varepsilon = 3P , \quad (3)$$
independent of the form of the distribution function $f$.

As we saw above, what we need to arrive at Eq. (3) is only the condition of *local isotropization* of the particles in the momentum space, and not the *thermal equilibrium*. It may well be possible that the process of local isotropization in momentum space can be attained much faster than the complete thermalization. In fact, for a quark-gluon plasma which is now believed to be produced by relativistic heavy-ion collisions, such a dynamical mechanism has been studied in analogy with the plasma instabilities. This is an example where the thermodynamic relations become valid very quickly even the system is not in a real thermal equilibrium. It is also shown that the nonequilibrium dynamics of quantum scalar field exhibits such a prethermalization behavior. These considerations indicate that there are situations where hydrodynamical description can still be valid even for a system where the usual criteria in terms of Knudsen number is apparently violated. From now on, we refer such a state of matter to “prethermalized” state, for which functional relations among macroscopic observables (such as equation of state) emerge, although the true thermalization is not yet attained.

Now, two basic questions arise here. One is in which situations such a prethermalized state occur, and the second, whether the thermodynamics properties (equation of states) of the matter stay the same as those of the real thermal equilibrium, or not. Either prethermalization or real thermalization, the emergence of functional relations among macroscopic observables indicates the reduction of the other microscopic degrees of freedom irrelevant to the description of the physical properties of the matter. This must be due to a kind of randomization processes of these irrelevant degrees of freedom (irrelevant variables) where the information of the initial condition is lost by random interactions among them. The difference between the real thermalization and pre-thermalization is that, in the case of pre-thermalization, there still exist some degrees of freedom which still depends on the initial condition.

If such a prethermalization scenario happens, we can expect some universal behavior such as the Boltzmann-Gibbs statistics which leads to the thermodynamic relations. As a matter of fact, the Boltzmann-Gibbs statistics is not a unique possibility to be consistent with thermodynamics. In 1988, Tsallis proposed a new statistical mechanics where entropy is non-additive. So far, extensive studies have been developed and the applications of Tsallis’ entropy extend to biological systems, high-energy physics, and cosmology. Because
of the non-additivity of the entropy, Tsallis statistics is considered to be important for describing correlated systems. From our prethermalization scenario, the Tsallis statistics may be attained before the real thermalization is achieved and the later relaxation process after the Tsallis statistics may be described by the time evolution of $q$, which is the parameter of Tsallis statistics, and the Tsallis statistics is reduced to the Boltzmann-Gibbs statistics for $q = 1$ [7, 8].

To make the idea clear on the effect of correlation on the relaxation time of a prethermalization mechanism, let us consider a problem of finding a minimum of a function of two variables, as shown in Fig. 1. When the function has a narrow long valley, starting from some arbitrary initial point, the usual steepest descent method always first goes to the closest point of the valley instead of going directly to the real minimum, as indicated by the dashed curve. The narrow valley means that there exists a strong correlation between the two variables. In other words, if there exist any strong correlations among variables, the minimizing path always tries to satisfy the correlation first (fast relaxation stage), then follows downstream to the true minimum along the valley generated by the correlation (slow relaxation stage).

The above consideration suggests that a prethermal state may correspond to a kind of narrow valley of the free energy generated by dynamical correlations. If such a local minimum stays for a very long time, we may consider the system as in a kind of equilibrium, but the microscopic occupation probability is not that of the Boltzmann-Gibbs statistics, due to the constraint created by the correlation. Then what characterizes such a local equilibrium? We show that there exists a class of dynamical correlation for which the minimisation of the energy under the constraints lead exactly to the general form of Tsallis’ statistics. In
this image, the dynamics of the slow relaxation stage may be described by the evolution of $q$ to arrive at the real thermal equilibrium described by the Boltzmann-Gibbs statistics corresponds to $q = 1$.

Note that in the above analogy, the correlation is rather static one. However the dynamical constraints are not necessarily determined only from the Hamiltonian itself, but may depend on the state of the system in a dynamical way. In other words, for a given system, depending on the initial condition, the correlation and hence the constraints among microscopic degrees of freedom may change. In this sense, it is possible that the value of $q$ is not a quantity determined from the beginning even when the system is specified.

Of course, prethermalization involves a broad range of phenomena, and has been studied for a long time in several works [9]. In particular, the relaxation to a quasi-stationary state under long-range interactions is strongly suggested in the Hamiltonian mean field model [10]. Here we develop a general phenomenological argument of how the states described by the Tsallis nonextensive statistics can be regarded as a prethermalized states.

II. DYNAMICAL CLUSTERS AND TSALLIS DISTRIBUTION FUNCTION

In this section, we show that the Tsallis distribution function can be obtained when there exists a class of correlations among particles in each single particle energy levels.

Let us consider a system composed of $N$ particles where the single-particle energy spectrum and the corresponding occupation numbers are given by \{\varepsilon_{\alpha}\} and \{N_{\alpha}\}, respectively.

Suppose there exists a strong $q$-body correlation among $N_{\alpha}$ particles in any single-particle state $\alpha$. In general, $q$ is a function of $\alpha$. However, we assume, for simplicity, the number of correlated particles are independent of state $\alpha$. Due to this correlation among particles, the rules to determine the occupation numbers \{N_{\alpha}\} affects the statistics as described below.

For a given value of $q$, the number of possible ways of forming correlated $q$-body clusters in each $\alpha$ state is given by

\[
\binom{N_{\alpha}}{q} = \frac{N_{\alpha}!}{q!(N_{\alpha} - q)!} \approx \frac{N_{\alpha}^q}{\Gamma(q + 1)},
\]

for $N_{\alpha} \gg 1$. Here, we used Stirling’s formula for the asymptotic values of $\Gamma$-function. Note that $q$ is the average number of particles to form a cluster and does not have to be integer. We call these correlated subsystems $q$-clusters. If the $q$-clusters are formed with equal a
priori probability, the number of such clusters $M_\alpha$ should be proportional to the number of ways of forming them. Thus
\[ M_\alpha = \text{Const} \times N_\alpha^q, \]  
and the total number of the $q$-clusters in the whole system is then
\[ M = \sum_\alpha M_\alpha. \]  

We may estimate the energy associated with the cluster formation. Since a $q$-cluster in the energy level $\alpha$ carries the single particle energy, $q \times \varepsilon_\alpha$, we have
\[ E_q = q \sum_\alpha M_\alpha \varepsilon_\alpha. \]  
Note that, for $q \to 1$, the expression above formally leads to the total energy of the system, except for a constant factor. This fact will be useful for the recovering the usual Boltzmann-Gibbs distribution. See the later discussion.

It is natural to think that, once the number of the correlated particles are fixed by $q$, the most probable configuration will be realized by minimizing the correlation energy, that is, the energy associated with these clusters,
\[ \delta E_q = 0, \]  
with
\[ M = \text{Const}. \]  
where the variation should be taken with the occupation numbers, $\{N_\alpha\}$. This is equivalent to the variational problem of the following quantity,
\[ \delta \left[ \sum_\alpha \varepsilon_\alpha N_\alpha^q + \lambda \sum_\alpha N_\alpha^q - \mu \sum_\alpha N_\alpha \right] = 0, \]  
That is, we consider the following variational problem,
\[ \delta \left[ \sum_\alpha \varepsilon_\alpha N_\alpha^q + \lambda \sum_\alpha N_\alpha^q - \mu \sum_\alpha N_\alpha \right] = 0, \]
for all $N_\alpha$’s. We get

$$
(\varepsilon_\alpha + \lambda) N_\alpha^{q-1} - \mu = 0,
$$

(14)

If we introduce the occupation probability

$$
p_\alpha = \frac{N_\alpha}{N},
$$

(15)

then the above solution can be expressed as

$$
p_\alpha = \left[ \frac{\mu'}{q} \left( \frac{1}{1 + \varepsilon_\alpha/\lambda} \right) \right]^{\frac{1}{q-1}},
$$

(16)

where $\mu' = \mu/N^{q-1}$. Calling $\lambda = (q - 1)T$, this distribution can equivalently parametrised in terms of $q$-exponential function as,

$$
\frac{p(E)}{p(0)} = \left( \frac{1}{1 + \frac{E}{q-1}T} \right)^{\frac{1}{q-1}} = \text{Exp}_q\left( -\frac{E}{T} \right).
$$

(17)

This form of parametrisation is convenient since it reduces to the the Boltzmann-Gibbs distribution function in the limit of $q \to 1$.

However, it should be stressed that the thermodynamic properties of the prethermal state proposed here are different from that of the Boltzmann-Gibbs statistics. In usual derivation of the Boltzmann distribution, we have to maximise the total number of microstates accessible for a given total energy and number of particles. The variation principle Eq.(13) in its form does not lead to the Boltzmann-Gibbs variational equation for maximising the entropy of the system, since for $q \to 1$, the second term and the third term degenerates if $\lambda$ and $\mu$ are finite constant, independent of $q$.

On the other hand, we may re-arrange these Lagrange multipliers $\lambda$ and $\mu$ in terms of new constants, $\beta$ and $\tilde{\mu}$ as

$$
\delta \left[ \sum_\alpha \varepsilon_\alpha N_\alpha^q + \frac{1}{\beta (q-1)} \left( N - \sum_\alpha N_\alpha^q \right) - \tilde{\mu} \sum_\alpha N_\alpha \right] = 0,
$$

(18)

where

$$
\frac{1}{\beta (q-1)} = \lambda
$$

(19)

$$
\tilde{\mu} = \mu + \frac{1}{\beta (q-1)}
$$

(20)
and we have used
\[ \sum_{\alpha} N_{\alpha} = N. \] (21)

The second term
\[ \frac{1}{\beta (q-1)} \left( N - \sum_{\alpha} N_{\alpha}^q \right) = \frac{1}{\beta} \sum_{\alpha} \frac{1 - N_{\alpha}^{q-1}}{q-1} N_{\alpha} \]
\[ = \frac{1}{\beta} \sum_{\alpha} \ln_q (N_{\alpha}) N_{\alpha} \] (22)

where
\[ \ln_q (N_{\alpha}) \equiv \frac{1 - N_{\alpha}^{q-1}}{1 - q} \] (23)

and in the limit of \( q \to 1 \), we have
\[ \ln_q (N_{\alpha}) \to \ln (N_{\alpha}). \]

Thus, for \( q \to 1 \), the variational principle, Eq.(18) leads to
\[ \delta \left[ \sum_{\alpha} \epsilon_{\alpha} N_{\alpha} - \frac{1}{\beta} \sum_{\alpha} N_{\alpha} \ln N_{\alpha} - \bar{\mu} \sum_{\alpha} N_{\alpha} \right] = 0, \] (24)

which is exactly the same equation we get in the Boltzmann-Gibbs statistics when we maximise the entropy
\[ S_{BG} = k \ln W, \] (25)

with
\[ \ln W = \ln \left( \frac{N!}{\prod_{\alpha} N_{\alpha}!} \right) \] (26)
\[ \approx - \sum_{\alpha} N_{\alpha} \ln N_{\alpha} + \text{const.} \] (27)
after using Stirling’s formula.

All the discussion above can be done in terms of occupation probability, \( p_{\alpha} \) from the beginning. The variational problem, Eq.(10) together with the constraints, Eqs.(11) and (12) is equivalent to
\[ \delta \left( \sum_{\alpha} \epsilon_{\alpha} p_{\alpha}^{\epsilon_{\alpha}} \right) = 0 \] (28)

with
\[ \sum_{\alpha} p_{\alpha} = 1 \] (29)
and

$$\sum_\alpha p_\alpha^n = B,$$  \hspace{1cm} (30)

where the variation is now taken with respect to \(\{p_\alpha\}\). The corresponding equation to Eq.(18) is

$$\delta \left( \sum_\alpha \varepsilon_\alpha p_\alpha^n - \frac{1}{\beta} \sum_\alpha p_\alpha \ln_q (p_\alpha) + \mu \sum_\alpha p_\alpha \right) = 0$$  \hspace{1cm} (31)

where the second term is nothing but Tsallis’ entropy

$$S_{Tsallis} = - \sum_\alpha p_\alpha \ln_q (p_\alpha).$$  \hspace{1cm} (32)

Thus, we have shown formally that our system is a physical realization of Tsallis’ formulation of the generalised statistics.

Here, there are important remarks. Although we have shown that the energy minimum principle of correlated clusters lead to the same results as Tsallis’ generalised statistics, the physical significance of our example is very different from that of Tsallis’ approach. In our approach, we imposed the condition to minimise the energy of systems of \(q\)-clusters, keeping the total number of clusters. This energy is not the total energy of the system, but that of a subsystem of the system which are strongly correlated. When we neglect the interaction energies, the total energy of the system is the sum of single particle energies so that

$$E_{total} = \sum_\alpha \varepsilon_\alpha N_\alpha.$$  \hspace{1cm} (33)

In this case, if we require conservation of the total energy of the system, we have to add the constraint,

$$\delta E_{total} = 0$$  \hspace{1cm} (34)

in addition to Eqs.(11), (12). By introducing a new Lagrange multipliers, we obtain instead of Eq.(8),

$$p(E) = \left( \frac{1 - E/\varepsilon_{max}}{1 + E/T_{q-1}} \right)^{1/(q-1)}$$  \hspace{1cm} (35)

where now \(p(0), T\) and \(\varepsilon_{max}\) should be determined from the constraints, Eqs.(11), (12) and (33). As expected, the energy spectrum has the maximum cut-off energy, \(\varepsilon_{max}\) which is a natural consequence of the conservation of energy. For \(E \ll \varepsilon_{max}\), the above expression reduces to Eq.(17) as expected.
In general, the number of the $q$-clusters and the coefficient of the probability of the cluster formation will depend on $\alpha$. In this case, we may write the number of clusters as

$$M_\alpha = K_\alpha \frac{N_\alpha^{q\alpha}}{\Gamma(q\alpha + 1)}. \quad (36)$$

The variational problem is then

$$\delta \left[ \sum_\alpha q_\alpha K_\alpha \varepsilon_\alpha p_\alpha^{q\alpha} + \lambda \sum_\alpha K_\alpha p_\alpha^{q\alpha} - \mu \sum_\alpha p_\alpha \right] = 0.$$ 

Then the distribution function is

$$p(E) = \left( \frac{K_0/K_E}{1 + \frac{q_\alpha}{q_\alpha-1} \frac{E}{E_T}} \right)^{\frac{1}{q_\alpha-1}} \quad (37)$$

### III. TOY MODEL OF CORRELATED PARTICLES

To check our prethermalization scenario, let us consider a very simple toy model described below.

1. Initially $N$ particles are distributed as $\{N_0, N_1, ..., N_{\alpha}, ...\}$ over equally spaced energy levels. Let us denote the energy of the $i$-th particle as $E_i$.

2. Choose randomly a pair of particles, say, $i$ and $j$.

3. Energies of $i$ and $j$ are updated according to one of the following alternatives:

   (a) The new energies are set to the lower one of $E_i$ and $E_j$, that is,

   $$E_i, E_j \rightarrow E_i' = E_j' = \min(E_i, E_j), \quad (38)$$

   then choose another $k$-th ($k \neq i, j$) particle randomly and attribute

   $$E_k' = E_k + \max(E_i, E_j) - \min(E_i, E_j) \quad (39)$$

   to conserve the total energy.

   (b) Change the energies as

   $$E_i \rightarrow E_i' = E_i \pm \Delta E,$$

   $$E_j \rightarrow E_j' = E_j \mp \Delta E, \quad (40)$$

   where $\Delta E$ is the level spacing. Here, if one of $E_i', E_j'$ becomes negative, then this step is skipped. That is, we have to keep always $E_i \geq 0$. 

11
4. The alternatives (a) and (b) are chosen randomly, but the ratio of the average frequency of (a) to (b) is kept as a constant, \( r \).

Here, the collision of type (a) is the process of the formation of a kind of clusters, while the collision of type (b) is the energy exchange process and may destroy the formed clusters. It is well-known that for \( r = 0 \) (no type (a) collision) in the above model, the final single-particle spectrum will become the Boltzmann distribution. By the competition of the two collision processes, we expect the number of correlated particles will reach some stationary value, and, by construction, the energy of a correlated pair always tends to diminish. In this way, we may expect that the above system will lead to the situation described by Eq. (16). In Fig. 2, we show the results of simulations, for several values of \( r \) from 0.00001 to 0.1. It is interesting to note that \( r > 0 \) leads to a non-Boltzmann distribution which is well approximated by the Tsallis distribution, \( p_a = C/[1 + \beta (q-1) \langle E_a \rangle^{\frac{1}{1-q}}] \), where \( \beta = 1/(3-2q) \langle E \rangle \) and \( C = (2-q)/(3-2q) \langle E \rangle \) are determined from the normalization condition and conservation of energy. One parameter fits with respect to \( q \) were performed and the results are indicated by the continuous curves in this figure. For \( r \to 0 \), the spectrum converges to the Boltzmann distribution \( p_a \to e^{-\frac{E_a}{\langle E \rangle}} \) as expected. Note that for a one dimensional case like the present model, the Tsallis distribution is valid only for \( q < 3/2 \), otherwise the energy expectation value diverges. For larger values of \( r \), the fitted value of \( q \) tends to this limiting value, but the distribution begins to deviate substantially from the Tsallis distribution in the low energy region.

It is important to note that these distributions are the stationary and stable ones. We confirm that starting from any different initial conditions, the final distribution converges uniquely for a given \( r \) parameter. Furthermore, the cluster formation through type (a) collisions accelerate the speed of the convergence. For example, for \( r = 0.01 \) the distribution converges 10 times faster than \( r = 0 \) case.

In the above toy model, the time reversal is violated for the collision type (a). However, this is not the crucial factor to obtain the non-Boltzmann distribution. We have checked this in a more elaborate model which has time-reversal invariance.
FIG. 2: Energy spectrum after a large number of collisions per particle, starting from a distribution peaked at $\langle E \rangle = 22.5$. The probability of cluster-forming collisions ranges from $r = 0.1$ (top) to $r = 0.00001$ (bottom). Tsallis distributions fitted to the calculated spectra are also shown, together with the corresponding $q$-values.

IV. INTERACTING LINEAR HARMONIC OSCILLATORS

As an another simple example in which a single particle spectrum deviates from a simple exponential form, let us consider $N$ identical harmonic oscillators $\vec{q}(t)$, described by the following Hamiltonian,

$$ H = \frac{1}{2} \left( \frac{d\vec{q}}{dt} \right)^2 + \frac{1}{2} \vec{q}^2 + \frac{1}{2} \vec{q}^T C \vec{q}. $$

(41)
Here

\[ \bar{q} = \begin{pmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_N(t) \end{pmatrix}, \]

and \( C \) is an \( N \times N \) symmetric matrix, representing the linear interactions among these \( N \) harmonic oscillators. The constant \( G \) represents the magnitude of coupling. It is well-known that such a system is formally solvable. Let \( U \) the orthogonal transformation which diagonalize the matrix \( C \),

\[ UCU^T = \Lambda, \]

where \( \Lambda \) is the diagonal matrix whose diagonal elements are eigenvalues of \( C \).

\[ \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N). \]

Then by introducing the new vector,

\[ \bar{\xi}(t) = U\bar{q}(t) , \]

we have

\[ H = \frac{1}{2} \left( \frac{d\bar{\xi}}{dt} \right)^2 + \frac{1}{2} \bar{\xi}^2 + \frac{1}{2} g\bar{\xi}^T \Lambda \bar{\xi} \]

so that the general solution for \( \bar{\xi} \) is given by

\[ \xi_i(t) = \frac{1}{\omega_i} A_i \sin(\omega_i t) + B_i \cos(\omega_i t) , \quad i = 1, \ldots, N \]

where

\[ \omega_i = \sqrt{1 + g \lambda_i} \]

and \( \bar{A} = (A_i) \) and \( \bar{B} = (B_i) \) can be determined from the initial condition given for \( \bar{q} \) and \( \frac{d\bar{q}}{dt} \) as

\[ \bar{B} = U\bar{q}(0) \]

and

\[ \bar{A} = U \frac{d\bar{q}(0)}{dt}. \]

Eq.(47), together with Eqs.(49) and (50) shows that the whole system never realize the thermally equilibrated state, which is a very known fact. On the other hand, a single particle
trajectory in a reduced phase space \((q, p)\) is very complicated and even ergodic if the coupling matrix \(C\) is sufficiently complex. It is then interesting to see how the long-time average of single particle phase space distribution of any of \(q_i(t)\)'s behaves. Let us define

\[
f(q, \dot{q}) \Delta q \Delta \dot{q} = \lim_{T \to \infty} \frac{1}{T} \frac{1}{N} \sum_i \int_0^T dt \Theta^{(2)} [(q_i(t), \dot{q}_i(t)); \Delta \Omega],
\]

where \(\Theta^{(2)}\) is 2-dimensional Heaviside's step function defined by

\[
\Theta^{(2)} [(q_i(t), \dot{q}_i(t)); \Delta \Omega] = \begin{cases} 1 & \text{if } (q_i(t), \dot{q}_i(t)) \in \Delta \Omega, \\ 0 & \text{otherwise}, \end{cases}
\]

and \(\Delta \Omega\) is an infinitesimal domain around the phase-space point \((q, \dot{q})\) whose volume is given by \(\Delta q \Delta \dot{q}\). In the above, we assume that any of harmonic oscillators are identical, so we consider the inclusive single particle distribution of these \(N\) oscillators.

In Fig. 3, we show the results of numerical calculation for the system with \(N = 100\) and the symmetric matrix \(C\) is created by a homonegeous random number \(C_{ij} \in [-1, 1]\). Also a random initial condition,

\[
q_i(0) \in [0, 1], \\
\dot{q}_i(0) \in [-1, 1],
\]

has been specified. The result seems an Gaussian distribution in \((q, p)\) plane. To see more precisely, we show the average of single particle spectra

\[
P(E) = \frac{1}{N} \sum_i P_i(E),
\]

where

\[
P_i(E) dE = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \Theta \left( \left( \frac{dE}{2} \right)^2 - (E_i(t) - E)^2 \right),
\]

and \(E_i(t)\) is the single particle energy without interaction term,

\[
E_i = \frac{1}{2} p_i^2 + \frac{1}{2} q_i^2 + \frac{1}{2} gC_{ii}q_i^2,
\]

which corresponds to the single particle oscillator energy with renormalised frequency [13].

In Fig. 4, we show the spectrum corresponding to the example of Fig. 3 which in fact, the inclusive single particle behavior is that of the thermal equilibrium. In Fig. 4, the
FIG. 3: Phase Space distribution of single particle states. The abscissa represents the $q$ coordinate, and the ordinate represents the $\dot{q}$ coordinate.

triangles are those calculated from the expression (53), whereas the dashed line indicates the Boltzmann distribution,

$$P_B (E) \propto e^{-E/\langle E \rangle},$$

where $\langle E \rangle = E_{Tot}/N$. We see that the inclusive energy spectrum of harmonic oscillators is essentially described by the Boltzmann distribution. However, we should note that if we see more carefully individual spectra for distinct oscillators, they fluctuate around the
FIG. 4: Energy spectra corresponds to the example of Fig. 3. The numerical results (the triangles) are fitted by the Boltzmann distribution function (The dashed line). Thin lines are spectra for several individual oscillators.

Boltzmann distribution (thin lines).

Such a behavior can be seen in various initial conditions and $C$ matrix. The above example corresponds to the situation where all the oscillators couple each other, a typical case of long range interaction. We may consider more local interaction, that is, the matrix
is semi-diagonal. Let us consider, for example,

\[ C_{ij} = [R] e^{-\alpha|i-j|} \]  

(57)

where \([R]\) is a uniform random number between \([-1,1]\) and \(\alpha\) is a parameter related to the locality of the interaction. In Fig. 5 we show the inclusive spectrum for the case of \(\alpha = 0.2\). Surprisingly, the inclusive spectrum (triangles) deviates appreciably from the expected Boltzmann distribution. Instead, the form of the spectrum can be fitted very well in terms of \(\text{Exp}_q(-E/T)\) function, with \(q = 1.05\) and \(T = 0.9 \langle E \rangle\).

The reason why the inclusive spectrum deviates from the Boltzmann to Tsallis (\(\text{Exp}_q\)) distribution is not obvious. However, we may note several key aspects indicating this result. First, the semi-diagonal nature of the matrix \(C\) means that there are more correlations among oscillators \(\vec{q}\) than they are uniformly coupled among them. Second, the individual spectra for different oscillators fluctuate much more compared to the case of uniformly interacting case (Fig. 4). Third, the effective temperature \(T\) is smaller than the Boltzmann case. The last, but not least, important point is that the inclusive spectrum we are looking in this coupled harmonic oscillators does not account for the interaction energy. That is, the sum of single particle energies defined in Eq.(55) does not give the total energy of the system. The cross terms as \(q_i C_{ij} q_j\) for \(i \neq j\) are not counted. These terms are omitted in the definition of single particle energies, expecting that these are “irrelevant” information and cancels out for the behavior of single particle energy in average. However, these effects become relevant in the presence of correlations.

Of course, there are many cases where the inclusive spectrum is neither Boltzmann nor Tsallis. For example, if we generate \(C_{ij}\) as homogeneous random numbers between \([0,1]\) (that is, all \(C_{ij} > 0\)) or between \([-1,0]\) (all \(C_{ij} < 0\)), the inclusive spectrum are completely different.

It is then interesting to study the variation of the inclusive spectrum with respect to the form of the coupling matrix \(C\). However, it is not a trivial task to choose the coupling matrix \(C\) in an arbitrary manner, since we have to have all the eigenvalues real and positive to guarantee the stability of the system. From this reason, we may generate the interaction matrix starting from its eigenvalues. First we choose the set of \(N\) positive eigenvalues. Then we construct a random unitary matrix \(U\) to define

\[ C = U^T \Lambda U \]  

(58)
FIG. 5: The energy spectra in the case of the semi-diagonal random coupling. The triangles correspond to the calculated inclusive spectrum and dashed-dot line is the fit by $\text{Exp}_q$ function. Thin lines are spectra for several individual oscillators.

The generation of the random unitary matrix can be done using the Schmidt orthonormalization method from a set of arbitrary $N$ linearly independent vectors, $\{\vec{e}_i\}$.

In Fig. 6 we show the sequence of inclusive spectra varying the coupling constant $G$ for the eigenvalues of $C$ randomly chosen between $[0, 1]$. The initial condition are taken...
FIG. 6: the sequence of inclusive spectra varying the coupling constant $G$ for the eigenvalues of $C$ randomly chosen between $[0, 1]$.

randomly generated as in the examples before. We can see that all the inclusive spectra are consistent to the Boltzmann distribution up to the energy plotted. (For larger energy, the spectra deviate from the Boltzmann due to the finiteness of $N$ (= 100).

On the other hand, in Fig. 7 we show the sequence of inclusive spectra when the eigenvalues are very inhomogeneous. In this example, we took eigenvalues as a quickly
increasing function of its index,

\[ \lambda_k = \left( \frac{k}{N} \right)^3. \]  \hspace{1cm} (59)

A similar behavior of the inclusive spectrum appears when we take

\[ \lambda_k = e^{\zeta k/N}, \]  \hspace{1cm} (60)

where \( \zeta \) is a constant. We found that all of these spectra are very well fitted by the two parameter Tsallis distribution, \( Exp_q(-E/T) \). The negative curvature spectra also appear, but they can also be fitted by \( q < 1 \).

It is interesting to observe that other simple 2-parameter function, such as exponential of polynomial in \( E \),

\[ \exp(-aE + bE^2) \]  \hspace{1cm} (61)

does not work at all.

Another important fact is that, as we mentioned in the introduction, the correlation among the internal degrees of freedom may depend on the initial condition so that the value of \( q \) can also depend on the initial condition. In fact, the form of inclusive spectrum when we change the initial condition changes but still can be expressed by \( Exp_q(-E/T) \), with different \( q \) and \( T \). Therefore, the parameter \( q \) is rather a kind of state parameter like other thermodynamic quantities and not the parameter specified by the Hamiltonian of the system.

V. SUMMARY AND CONCLUDING REMARKS

In this paper, we discussed the concept of the prethermalized state as the presence of strong correlation in the phase space of the system. We argued that those states described by Tsallis statistics belong to this category. We show that a very specific class of correlation leads naturally to the Tsallis distribution for the occupation probability in energy levels.

We also show an interesting simple example where Tsallis distribution emerges. We studied the inclusive energy spectrum of \( N \) coupled harmonic oscillators. Depending on the coupling, the inclusive spectrum exhibit clearly the Tsallis distribution. The discussion here have been restricted in the framework of classical mechanics. However, the mathematical structure of general coupling matrix is equivalent to a quantum mechanical problem to find
the eigen frequencies if the vector \( \vec{q} \) is considered as a state vector in an appropriate base. The single particle spectra are thus somewhat related to the reduced density matrix defined for each individual oscillators. In this sense, the question treated here can also be related to the problem of decoherence and locality.

The emergence of Tsallis distribution is not deterministic, but probabilistic in the sense that it depends on the choice of the random unitary transformation. There are specific cases
that the inclusive spectrum does not behave neither Boltzmann nor Tsallis, at all. However, when we choose the unitary transformation randomly, then the probability of emerging Boltzmann, or Tsallis type distribution seems to be dominant. For example, if we go from a system where the coupling strength is small and the interaction matrix is uniformly random to the system where there exist stronger interactions or correlations, the inclusive spectrum starts to deviate from the Boltzmann to Tsallis, statistically. This observation deserves a further systematic investigation. It will be interesting if we can formulate the problem from the point of variational problem as discussed in this paper.

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[11] In this paper, we use the natural unit, $c = \hbar = 1$.

[12] More generally, Eq. (3) is valid if the four-energy momentum tensor is traceless (conformal theory) and isotropic in the Landau local rest frame. For a hydrodynamic description, we only need the thermodynamic relations (equation of state) as Eq. (3).

[13] Another possible way to define the single particle energy is to take the part of Hamiltonian depending on $i$-th coordinate, $E_i = \frac{1}{2}p_i^2 + \frac{1}{2}q_i^2 + \frac{1}{2}g \sum_{j=1}^{N} C_{ij}q_iq_j$ such that $\sum_{i=1}^{N} E_i = E_{TOT}$. However, this energy is not positive definite depending on the interaction.