Varying $k$-Lipschitz Constraint for Generative Adversarial Networks

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Abstract

Generative Adversarial Networks (GANs) are powerful generative models, but suffer from training instability. The recent proposed Wasserstein GAN with gradient penalty (WGAN-GP) makes progress toward stable training. Gradient penalty acts as the role of enforcing a Lipschitz constraint. Further investigation on gradient penalty shows that gradient penalty may impose restriction on the capacity of discriminator. As a replacement, we introduce varying $k$-Lipschitz constraint. Proposed varying $k$-Lipschitz constraint witness better image quality and significantly improved training speed when testing on GAN architecture. Besides, we introduce an effective convergence measure, which correlates well with image quality.

1 Introduction

Generative Adversarial Networks (GANs) have led to significant improvements on unsupervised learning. GANs are architecture around two functions: the generator $G(z)$, which maps a sample $z$ to the data distribution, and the discriminator $D(x)$, which is trained to distinguish real samples of a dataset from fake samples produced by generators. The Kellback-Leibler (KL) divergence is widely used as the loss function. However, the KL divergence suffers from vanishing gradients and instable training. Wasserstein GAN and its improved model WGAN-GP [4] introduced Wasserstein distance as its loss function, which witness stable training and avoid vanishing gradients. Weight clip and gradient penalty are proposed to enforce the Lipschitz constraint to determine the Wasserstein distance.

Our investigations on Lipschitz constraint find that gradients penalty is not the perfect way to enforce Lipschitz constraint. Gradient penalty may impose restriction on the capacity of discriminator. As a result, we introduce varying $k$-Lipschitz constraint as a replacement of gradient penalty. In this paper, we make the following contributions:

- On toy datasets, we demonstrate how gradient penalty impose restriction on the behavior of discriminator.
- We propose varying $k$-Lipschitz ($VKL$) constraint as well as a new kind of convergence measure. $VKL$ constraint is demonstrated to be more powerful than gradient penalty. In addition, proposed convergence measure correlates well with image quality.

2 Related work

DCGAN The idea of DCGAN is that it applied KL divergence to assess how well a probabilistic model explains observed data. If the data $x$ is from real distribution, then the probability $D(x)$ will be close to 1. If the data $x$ is generated, then probability $D(x)$ will be close to 0. Formally, the objective of the generator and discriminator can be defined by:

$$L_D = \log(D(x)) + \log(1 - \log(D(g(z))))$$

$$L_G = \log(D(g(z)))$$

Wasserstein GAN (WGAN) is mainly motivated by the vanishing gradient problem caused by
Kullback-Leibler or Jensen-Shannon divergence. As a result, Wasserstein distance along with Lipschitz constraint are introduced to solve this problem:

$$W(P_r, P_g) = \sup_{f \in \text{Lip}} E[f(x)] - E[f(z)]$$  \hspace{1cm} (3)

The implementation of Lipschitz constraint is the key to determine Wasserstein distance. In WGAN, weight clip acts as the role of Lipschitz constraint, which was later proven to be unsatisfactory by WGAN-GP. To impose the Lipschitz constraint, WGAN-GP introduced a gradient penalty term to form the following objective:

$$W(P_r, P_g) = E[f(x)] - E[f(z)] + \alpha E[\| \nabla f(y) \| - 1]^2$$  \hspace{1cm} (4)

3 Proposed Method

The implementation of \( k \)-Lipschitz constraint is the key to determine Wasserstein distance. We introduce varying \( k \)-Lipschitz constraint as the replacement of gradient penalty. Tests on toy distributions show that gradient penalty suffers from limitation on the capacity of discriminator. So that the discriminator trained with penalty gradients ignores higher moments of the data distribution and models very simple approximations to optimal functions. In contrast, our approach obtain much better results. Furthermore, we propose convergence measure, which correlates well with image quality.

3.1 Varying \( k \)-Lipschitz (VKL) constraint

WGAN utilized Wasserstein distance as its loss function. The mapping function \( f \) should obey the \( k \)-Lipschitz constraint, which can be expressed as:

$$\| f(x_r) - f(x_g) \| \leq k \| x_r - x_g \|$$  \hspace{1cm} (5)

**Proposition 1.** Let \( P_r \) and \( P_g \) be the distributions of real images and generated images in \( X \), a compact metric space, and \( P_r, P_g \) have support contained on compact subsets \( \mathcal{M} \) and \( \mathcal{P} \), respectively. Let mapping function \( f : X \rightarrow \mathbb{R}^d \) satisfy \( k \)-Lipschitz constraint, then there is supremum:

$$\| f_{y_1 \in \mathcal{M} \cup \mathcal{P}}(y_1) - f_{y_2 \in \mathcal{M} \cup \mathcal{P}}(y_2) \| \leq m$$  \hspace{1cm} (6)

**Proof** Because \( f \) satisfy \( k \)-Lipschitz constraint, then

$$\| f_{y_1 \in \mathcal{M} \cup \mathcal{P}}(y_1) - f_{y_2 \in \mathcal{M} \cup \mathcal{P}}(y_2) \| \leq k \| y_1 - y_2 \|$$  \hspace{1cm} (7)

\( \| y_1 - y_2 \| \) is the actual distance between \( y_1 \) and \( y_2 \) in \( X \). There is a strong empirical and theoretical evidence to believe that \( P_r \) and \( P_g \) are indeed extremely concentrated on a low dimensional manifold. And the actual distance between \( y_1 \) and \( y_2 \) should be finite. That is:

$$\| y_1 - y_2 \| \leq Y$$  \hspace{1cm} (8)

Then we can obtain:

$$\| f_{y_1 \in \mathcal{M} \cup \mathcal{P}}(y_1) - f_{y_2 \in \mathcal{M} \cup \mathcal{P}}(y_2) \| \leq k \| y_1 - y_2 \| \leq kY = m$$  \hspace{1cm} (9)

**Proposition 2.** Given a specific distribution \( P_r \) and \( P_g \) in the compact metric space \( X \), and let \( f \) be the mapping function :\( X \rightarrow \mathbb{R}^d \), and for any \( y \) in \( \mathcal{M} \cup \mathcal{P} \), \( f \) satisfy:

$$\| f_{y_1 \in \mathcal{M} \cup \mathcal{P}}(y_1) - f_{y_2 \in \mathcal{M} \cup \mathcal{P}}(y_2) \| \leq m$$  \hspace{1cm} (10)

Then the mapping function satisfy \( k \)-Lipschitz.

**Proof** For any \( y \) in \( \mathcal{M} \cup \mathcal{P} \), we have
\[ \| f_{y_1 \in M \cup P}(y_1) - f_{y_2 \in M \cup P}(y_2) \| \leq m = \frac{m}{\| y_1 - y_2 \|} \| y_1 - y_2 \| \]  

(11)

There must exist a \( k \) to satisfy:

\[ k = \frac{m}{\max(\| y_1 - y_2 \|)} \]  

(12)

and then:

\[ \| f_{y_1 \in M \cup P}(y_1) - f_{y_2 \in M \cup P}(y_2) \| \leq k \| y_1 - y_2 \| \]  

(13)

**Proposition 3.** Give a specific distribution \( P_r \) and \( P_g \), which represents the distributions of real image and generated images in the compact metric space \( X \), and let \( \Omega \) be the set of all \( f : X \rightarrow \mathbb{R}^d \) which satisfy:

\[ \Omega : \{ f \mid \| f_{y_1 \in M \cup P}(y_1) - f_{y_2 \in M \cup P}(y_2) \| \leq m \} \]  

(14)

There must exist \( k \), and then we have:

\[ kW(P_r, P_g) = \sup_{f \in \Omega} E_{x \sim P_r}[f(x)] - E_{z \sim P_g}[f(z)] \]  

(15)

**Proof** The set \( \Omega \) is actually all the \( k \)-Lipschitz functions \( f : X \rightarrow \mathbb{R}^d \) (Proposition 2). According to Kantorovich-Rubinstein duality, the supremum over all the function in \( \Omega \) is the \( k \cdot W(P_r, P_g) \).

Instead of using Wasserstein distance as the objective of the discriminator, we could utilize \( k \)-Wasserstein distance alternatively:

\[ k \cdot W(P_r, P_g) = \max_{f \in \Omega} E_{x \sim P_r}[f(x)] - E_{z \sim P_g}[f(g(z))] \]  

(16)

where \( f \) is in the set of \( \Omega \). \( k \) corresponds to given \( P_r \) and \( P_g \). In another word, for every given \( P_r \) and \( P_g \), there is a corresponding \( k \). The objective for the generator is:

\[ \min_{\theta} E_{x \sim P_r}[f(x)] - E_{z \sim P_g}[f(g(z))] \]  

where \( \theta \) is the parameters of the generator. Theorem 3 in [4] tell us that,

\[ \nabla_{\theta} kW(P_r, P_g) = - E_{z \sim P_g}[\nabla_{\theta} f(g(z))] \]  

(18)

\( P_g \) is varying in the training process, resulting in a varying \( k \) in each iteration of the generator (Proposition 2). However, during the iteration of generator (equation 18), \( k \) is a constant. Updating \( \theta \) through equation 18 is always toward the decrease of \( W(P_r, P_g) \). Eventually, \( P_g \) is closer and closer to \( P_r \). The procedure is described in Algorithm 1.

A more intuitional way to explain the training process is that in an iteration we use \( k_1 \)-Lipschitz constraint, and obtain \( k_1 \cdot W(P_r, P_g) \). We update the generator using equation 18. \( k_1 \) is just a constant, which won’t affect the iteration of the generator. In the next iteration, we use \( k_2 \)-Lipschitz constraint instead and repeat the updating process. The choice of \( k \)-Lipschitz is hidden in the constraint that \( f \in \Omega \), which is unknown to us and have no effect on iteration.

**3.2 Implementation of VKL Constraint**

There are various ways to implement the Varying \( k \)-Lipschitz (VKL) constraint. Mainly in this paper, we use a boundary to enforce the VKL constraint. The boundary can be expressed as:
where $m$ is a constant. Usually, we take $m$ as 1. It’s obvious that for any $f$ in the set $\Psi$, $f$ will satisfy the constraint in equation (14). The VKL constraint in the loss function can be expressed as:

$$P_{skl} = \max(\| E_{x \sim P}\{ f(x) \} \| -m, 0) + \max(\| E_{z \sim p(z)}[ f( g(z) ) ] \| -m, 0)$$

Then, the new objective is:

$$L_d = \min_{f} \{ E_{x \sim P}[ f(x) ] - E_{z \sim p(z)}[ f( g(z) ) ] + P_{skl} \}$$

$$L_g = \min_{g} E_{z \sim p(z)}[ f( g(z) ) ]$$

**Capacity comparison** We applied the toy model in [4] to demonstrate the capacity of VKL constraint. The toy models hold the generator distribution $P_g$ at the real distribution plus unit-variance Gaussians noise. Figure1 shows the value surface of the discriminator. For an excellent discriminator, the contour line around the orange points should be dense, so that the discriminator can easily distinguish the true distribution away from the fake distribution. In the process of iteration, the contour lines in the model of VKL constraint are getting more and more dense, showing excellent performance as a discriminator. This is not found in the model, which applied gradient penalty. What’s more, the VKL constraint model learns much higher moments of the data distribution. During the iteration, model with gradient penalty tried to learn more details inside the distributions, but it is not stable and failed eventually. It does not converge to optimal results, however VKL constraint model does. VKL constraint model could distinguish the real data from the fake data perfectly. Take 25 Gaussians Distribution as an example. The contour lines are dense and distributed almost around the 25 points, which shows good capacity of the VKL constraint. If the capacity of the discriminator is limited, the generator may converge toward the wrong minima when using Wasserstein gradient.

### 3.3 Convergence Measure

Gradient penalty and weight clip are the implementation of $l$-Lipschitz constraint. The loss metric
correlates well with the visual quality of generated samples, demonstrating that the loss function acts as an effective convergence measure. This does not apply to VKL constraint. Considering the excellent capacity of VKL constraint, the discriminator could distinguish the real data from fake data. Take equation (20) as an example. $E_{x\sim p_r}(f(x))$ will converge to $-m$ and $E_{x\sim p_g}(f(x))$ will converge to $m$. As a result, the output loss for the discriminator will be around $-2m$ from beginning to end. The loss metric is not able to act as convergence measure. We introduce another way to measure the convergence:

\[
cm = \frac{1}{\| \nabla_x f(x) \|_2} \tag{23}
\]

where $x = \alpha x_r + (1-\alpha)x_g$, $\alpha \in [0,1]$. Given the distribution $P_g$ and $P_r$, VKL constraint is equivalent to a $k$-Lipschitz constraint, where $k$ is a hidden parameter. According to equation (12), $k$ is inversely proportional to $\max||x_{pr}-x_{pg}||$. While $\max||x_{pr}-x_{pg}||$ indicates the difference of the distribution. We could estimate $k$ to investigate the $\max||x_{pr}-x_{pg}||$, which can be regarded as convergence measure. For $k$-Lipschitz constraint, the equivalent form is that

\[
\| \nabla_x f(x) \|_2 \leq k \tag{24}
\]

As described above, $E_{x\sim p_r}(f(x))$ will converge to $-m$ and $E_{x\sim p_g}(f(x))$ will converge to $m$. In another word,

\[
\| f_{x_{\sim p}}(y_1) - f_{x_{\sim p}}(y_2) \| \approx 2m = 2k \| y_1 - y_2 \| \tag{25}
\]

We could obtain:
As a result, we could use \(\|\nabla_x f(x)\|\) as the estimation of \(k\). Finally, \(cm\) can be used to measure the convergence. The coefficient \(1/2\) is omitted because it is of no effect on convergence measure.

### 3.4 Relations to other GANs

**BEGAN** [7] The loss function used in BEGAN is another form of \(VKL\) constraint. BEGAN utilized auto-encoder as its discriminator. The output of discriminator is:

\[
L(x) = |x - D(x)|^\eta
\]

where \(D\) is the auto-encoder: \(X \rightarrow X\), and \(\eta \in \{1,2\}\). As you can see, \(x\) and \(D(x)\) are in the space \(X\) at the same time. \(|x-D(x)|^\eta\) is the corresponding norm. As a result, there exists a supremum for \(|x-D(x)|^\eta\).

\[
|x-D(x)|^\eta \leq m
\]

Instead of regarding the discriminator as an auto-encoder, we could consider it as a mapping function \(L: X \rightarrow R\). All the mapping functions \(L\) are in the set:

\[
\Theta: \{ L | \|L(x)\| \leq m \}
\]

It is similar to the boundary we utilize to enforce \(VKL\) constraint. BEGAN applied auto-encoder as its discriminator, which satisfy \(VKL\) constraint implicitly.

**LSGAN** [8] There are distance term and \(VKL\) constraint term in the loss function (equation 20). Is there any way that we could combine distance term and \(VKL\) constraint term together? Actually, there is. This is exactly what LSGAN does. The loss function used in LSGAN is:

\[
L_D = \min_f \{ \frac{1}{2} E_{x \sim p}\left[(D(x) - 1)^2\right] + \frac{1}{2} E_{z \sim p(z)}\left[(D(g(z)) + 1)^2\right] \}
\]

Given enough iteration for the discriminator, \(D(x)\) will converge to 1, and \(D(g(z))\) will converge to

![Figure 2: (left) proposed convergence measure during training on dataset CELEBA. It correlates well with image quality. With the training process going on, image quality improves, while the value of convergence measure decreases. When the training almost converge, the image quality decrease if we keep training. We use a trick to improve this. we update the generator per 10 iteration instead. (right) the loss metric of discriminator in DCGAN with gradient penalty.](image-url)
1. $L_D$ acts as the role of evaluating the distance between $D(x)$ and $D(g(z))$. On the other hand, $L_D$ can also constraint $D(x)$ in a limited values range. LSGAN is a rather special case, which has an implicit connection with $VKL$ constraint.

4. Experiments

**No batch normalization in the Discriminator**

For better evaluation of the ability of proposed $VKL$ constraint, we make comparison with gradient penalty. We test their performance on DCGAN architectures by running image generation on CELEBA. The batch normalization and the sigmoid function are removed. The rest keeps the same as the DCGAN. We train the model with the same optimizer (Adam) and learning rate. The only difference is that we train one using gradient penalty, while training the other with $VKL$ constraint. Without batch normalization in the discriminator, DCGAN with $VKL$ constraint witness stable training. As shown in Figure 3, produced images using $VKL$ constraint have comparable sample quality as original DCGAN, and better quality over DCGAN with gradient penalty. What’s more, DCGAN with $VKL$ constraint benefits from faster convergence speed. After about 15 epochs, it converges (Figure 2). During the training, convergence measure ($cm$) acts as an efficient way to show the image quality. With less oscillation during the training process, proposed convergence measure shows better performance as the indicator of image quality over loss metric in DCGAN with gradient penalty.

There is a rather interesting phenomenon in models using $VKL$ constraint. When the distribution of generated image is close enough to the real distribution, the value of convergence measure ($cm$) increases instead of going down. One of the reasons is that when generated distribution and real distribution are close enough, the discriminator may not be able to converge to a relatively optimal solution during limited iterations. Then, the discriminator will give the wrong minima, leading to the wrong update of generator, which produce images with decreasing image quality. We use a trick to improve this. In the latter iterations, we update the generator every 10 iterations instead, giving the discriminator sufficient time to converge. This avoids the dramatically increasing of convergence measure and the decreasing of image quality. Nevertheless, image quality stops increasing. Currently, we think it converges. This problem may need further investigation; its solution may help improve image quality. Overall, DCGAN with $VKL$ constraint performs well. Moreover, proposed convergence measure acts as effective indicator of image quality.

![Figure 3: (left) samples from DCGAN with $VKL$ constraint. (right) samples from DCGAN with gradient penalty.](image-url)
No batch normalization in the generator

In the generator of DCGAN, Tanh function in the last layer acts as the constraint to enforce the output in the range between -1 and 1. We remove the batch normalization and Tanh function in the generator, and apply the $VKL$ constraint to the loss metric of generator. Then, the new loss metric for the generator is:

$$L_G = \min_g \{ E_{z \sim p(z)} [ f(g(z))] + \max(\|g(z)-1\|_0) \}$$  (30)

Without batch normalization and Tanh function in the generator, DCGAN with $VKL$ constraint can also produce images with comparable quality.

Figure 4: (left) samples from DCGAN with $VKL$ constraint applied in the generator. (right) convergence measure.

5. Conclusion

We introduced varying $k$-Lipschitz constraint to enforce Lipschitz constraint as a replacement of gradient penalty in WGAN. Tests on toy distribution show that $VKL$ constraint model learns much higher moments of the data distribution. Further investigation on image generation demonstrate that DCGAN model with $VKL$ constraint benefit from better image quality over gradient penalty and faster convergence speed over original DCGAN. As you can see, $VKL$ constraint is more powerful than gradient penalty.

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