Extracting information from the signature of a financial data stream

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1 Introduction

The frequency of market events such as order placement and order cancellation in most trading venues has increased dramatically since the introduction of computer based trading. During certain time intervals, orders can be submitted or modified within milliseconds or even faster. It is highly challenging to understand and track the characteristics of a market at a given moment. Quantitative analysts are eager to find methods to compress high frequency data streams without losing information that is relevant for their objectives.

This paper presents financial examples in which we classify streams of data. The classification is based on features of streams that are specified by a property of sample paths known as the signature of the path.

At an abstract mathematical level, the notion of a signature as an informative transform of a multidimensional time series was established by Ben Hambly and Terry Lyons [7], meanwhile Hao Ni and Terry Lyons [12] have introduced the possibility of its use to understand financial data and pointed to the power this approach has for machine learning and prediction.

We evaluate and refine these theoretical suggestions against some real world data and practical examples. In each of the analysed cases, we identify particular features, that is a low dimensional part of the signature that captures information relevant for the characterisation of various features.

In section 2 we recall the definition and key properties of the signature of a data stream. Moreover, we present a few motivating examples which demonstrate what information the signature preserves with special attention on properties which are not captured by the traditional statistical indicators.

Section 3 gives a description of the input data as well as the linear regression-based classification method.

Finally, in section 4 three numerical experiments are presented and the classification accuracy is analysed. In the first experiment we characterise the typical market behaviour within standard 30-minute time buckets sampled from the WTI crude oil future market (NYMEX). The second and third experiments aim to characterise the market impact of parent orders generated by two different trade execution algorithms on the FTSE 100 Index futures market listed on NYSE Liffe.
2 Signature of data streams

2.1 Origins of signature

The mathematical properties of iterated integrals of piece-wise regular multi-dimensional paths were first studied by K.T. Chen \[2, 3\]. B. Hambly and T. Lyons \[7\] extended these results to continuous paths of bounded variation.

Iterated integrals of continuous multi-dimensional paths naturally arise in the Taylor-expansion of controlled ordinary differential equations \([10, 13, 11]\). Moreover, several numerical approximations of the solution to stochastic differential equations (SDEs) are based on the iterated (stochastic) integrals of Brownian motion (the reader is referred to \[8\] and the references therein).

Informally, the signature as data transform maps multi-dimensional paths to the sequence of their iterated integrals \(1\), where the sequence is equipped with some algebraic operations (ref. \([10, 13, 11]\)). The signatures of continuous paths play a central role in the theory of rough paths \([10, 13, 11, 4]\). Moreover, the algebraic properties of the signature have been extensively used in the construction of numerical methods for approximation of solutions to SDEs \([9, 14, 6]\) as well as for the approximation of solutions to differential equations driven by rough paths \([4, 5]\).

The signature of time series and its application have been established by H. Ni and T. Lyons \([12]\).

2.2 Intuitive introduction

Let \(X\) be a continuous function of finite length defined on \([0, T]\) and taking values in \(\mathbb{R}^d\), where \(X_t = (X^1_t, \ldots, X^d_t)\). Moreover, let \(\mathcal{I}\) denote the set of all multi-indices \((i_1, \ldots, i_k)\), where \(k \geq 0, i_j \in \{1, \ldots, d\}\) for \(j = 1, \ldots, k\). We introduce the notation for \(0 \leq s < t \leq T\):

\[
X^I_{s,t} = \int_{s < u_1 < \cdots < u_k < t} dX^{i_1}_{u_1} \cdots dX^{i_k}_{u_k}, \tag{1}
\]

where \(I = (i_1, \ldots, i_k)\) is a multi-index with \(i_j \in \{1, 2, \ldots, d\}\) for \(j = 1, \ldots, k\).

The signature \(S_{s,t}(X)\) of \(X\) over the time interval \([s, t]\) maps \(X\) to the sequence \((X^I_{s,t})_{I \in \mathcal{I}}\).

The following properties of iterated integrals and the signature are relevant in this paper:\footnote{For the rigorous treatment of these properties the reader is referred to \([11, 7]\)}

(i) **Invariance to time re-parameterisation:** for any continuous and monotone increasing function \(\sigma : [0, T] \to [S, U]\), \((i_1, \ldots, i_k) \in \mathcal{I}\) and \(0 \leq s < t \leq T\):

\[
\int_{s < u_1 < \cdots < u_k < t} dX^{i_1}_{u_1} \cdots dX^{i_k}_{u_k} = \int_{\sigma(s) < u_1 < \cdots < u_k < \sigma(t)} dX^{i_1}_{\sigma^{-1}(u_1)} \cdots dX^{i_k}_{\sigma^{-1}(u_k)}
\]

(ii) **Linearity:** for all \(I, J \in \mathcal{I}\) there exist a set \(\mathcal{I}_{I,J} \subset \mathcal{I}\), such that

\[
X^I_{s,t} X^J_{s,t} = \sum_{K \in \mathcal{I}_{I,J}} X^K_{s,t},
\]

that is any polynomial of iterated integrals can be represented by a linear combination of iterated integrals (ref. \([11]\)).
(iii) **Signature of linear paths**: if $X_t = a + bt$ for some $a, b \in \mathbb{R}^d$ and for all $t \in [0, T]$, moreover $I = (i_1, \ldots, i_k)$, then

$$X_{s,t}^I = \frac{(t - s)^k}{k!} \prod_{j=1}^{k} b_{i_j}.$$ 

(iv) **Multiplicative property**: there exists a binary operation $\otimes$, such that for any $0 \leq s < t < u \leq T$

$$S_{s,t}(X) \otimes S_{t,u}(X) = S_{s,u}(X).$$

The operation is specified in [10, 11].

(v) **Uniqueness**: The main result of [7] is that $S_{s,t}(X)$ determines the function $u \mapsto X_u - X_s$ for $u \in [s, t]$ up to tree-like equivalence. A sufficient condition for the uniqueness is the existence of $i \in \{1, \ldots, d\}$ such that $X_{i}^t$ is strictly monotone increasing. Moreover, typically, the first few terms of the signature contain most of the information on the path.

In particular, the uniqueness property (v) highlights the relation between paths and their signatures: the truncated signature of a path is regarded as a projection of the path to a lower dimensional space. Furthermore, (iii) enables us to apply linear regression based machine learning techniques when estimating functionals of paths. Points (iii) and (iv) are relevant for the computation of the truncated signatures of data streams. Finally, (i) is used in section 2.5, where we claim that the lead-transform and the lag-transform both preserve the signature of data streams.

### 2.3 Interpretations of the terms in the signature

The lower order terms in the signature have simple interpretations. Recursively, one can derive the interpretation of the higher order terms. Here, we focus on some terms of order at most three.

**Increments.** For $i = 1, \ldots, d$

$$X_{s,t}^{(i)} = X_{t}^{i} - X_{s}^{i},$$

that is the first order terms determine the increments of the components.

Figure 1 shows two trajectories in two dimensions. The left-hand-side plot, represents a volume profile in which the traded volume is concentrated towards the end of the time interval, whereas on the right-hand-side plot the traded volume is concentrated at the beginning of the time interval. Both curves have unit increments in both of their time and cumulative volume components.

**Higher order terms.** Higher order iterated integrals can be interpreted as generalised polynomials of paths. Linear regression on these generalised polynomials can be interpreted as polynomial regression directly on the input paths. In order to obtain better accuracy in machine learning, we explore the relevance of each of these terms.
Figure 1: Back-loaded and front-loaded volume profiles.

However, not all terms are required for the unique representation of a path in terms of its iterated integrals. In particular, some of the iterated integrals are determined by other iterated integrals as the following examples show.

\[
X_{s,t}^{(i,i)} = \frac{\langle X_i \rangle_t^2}{2} \text{ for } i \in \{1, \ldots, d\}
\]

\[
X_{s,t}^{(i,j)} = X_{s,t}^{(i)} X_{s,t}^{(j)} - X_{s,t}^{(j,i)} \text{ for } i, j \in \{1, \ldots, d\}
\]

These identities are implied by property (ii) in section 2.2. The minimal set of information that uniquely determines all the iterated integrals of a path through non-linear operations is called the log-signature of the path (ref. [10, 11, 4]). The components of the log-signature consist of certain linear combinations of the iterated integrals. We briefly introduce two of these terms – the area and a second order area – below.

**Area.** Now, we give a possible interpretation of certain linear combinations of the higher order iterated integrals. For \(1 \leq i < j \leq d\), the terms

\[
A_{s,t}^{i,j} := \frac{1}{2} \left( \int_{s < u_1 < u_2 < t} dX_{u_1}^1 dX_{u_2}^2 - \int_{s < u_1 < u_2 < t} dA_{u_1}^{1,2} dX_{u_2}^1 \right)
\]

determine the signed area between the curve \(u \mapsto (X_u^i, X_u^j)\) for \(u \in [s,t]\) and the chord that connects the points \((X_s^i, X_s^j)\) and \((X_t^i, X_t^j)\). For example the blue (respectively red) shaded area in figure 2 indicates negative (positive) area. As noted above, the plotted curves have unit increments in both of their components, however one has a relatively large positive area, whereas the other has a relatively large negative area.

**Second order area.** The upper graphs of figure 3 show two paths in two dimensions – the mid-loaded and the front-and-back-loaded volume profiles. Both trajectories have unit increments in both their components, and both have zero signed areas. In order to tell the difference between these two curves through their signature, one has to look into the third order terms. In particular, on the bottom graphs in figure 3 the curve \(u \mapsto (X_u^1, A_{u,0}^{1,2})\) is plotted; the area under this curve is negative for the mid-loaded volume profile, and positive for the front-and-back-loaded volume profile. This second order area is given by the formula

\[
A_{s,t}^{1,(1,2)} := \frac{1}{2} \left( \int_{s < u_1 < u_2 < t} dX_{u_1}^1 dA_{u_2}^{1,2} - \int_{s < u_1 < u_2 < t} dA_{u_1}^{1,2} dX_{u_2}^1 \right) - \frac{1}{4} (X_{s,t}^{(1,1,2)} - X_{s,t}^{(2,1,1)})
\]
where the last step is implied by property [4] (the reader is referred to section 2.2.3 of [11] for a rigorous proof).

Using similar recursive arguments, one can identify higher order areas and represent them in terms of linear combinations of iterated integrals.

**Lead-lag relationship.** As the examples of this section show, the terms in the signature capture the lead-lag relationship. In particular, if an increase (respectively decrease) of the component $X^1$ is typically followed by an increase (decrease) in the component $X^2$, then the area $A_{1,2}$ is positive. If a move in $X^1$ is followed by a move in $X^2$ to the opposite direction, the area is negative.

### 2.4 Signature of data streams

The previous section introduced the signature of continuous trajectories. In financial time series analysis, one can observe data streams, i.e. trajectories defined at and indexed by finitely many time points: $(\hat{X}_{t_i})_{i=0}^N$ in $\mathbb{R}^d$.

We use the approach of Ni and Lyons [12] when we define signature of such data streams. In particular, given $(\hat{X}_{t_i})_{i=0}^N$, we introduce the continuous axis path $X : [0, 2N) \to \mathbb{R}^+ \times \mathbb{R}^d$ as follows.

$$X_u = \begin{cases} (t_i + (t_{i+1} - t_i)(2i - u); \hat{X}_{t_i}) & \text{if } u \in [2i - 2, 2i - 1) \\ (t_{i+1}; \hat{X}_{t_i} + (\hat{X}_{t_{i+1}} - \hat{X}_{t_i})(2i + 1 - u)) & \text{if } u \in [2i - 1, 2) \end{cases}$$

Figure 4 gives the axis path representation of a possible stream.

We define signature of the stream $(\hat{X}_{t_i})_{i=0}^N$ as

$$S_{t_{0..N}}(\hat{X}) := S_{0..2N}(X).$$

### 2.5 Quadratic variation

We have shown that the terms of the signature – the iterated integrals – measure various path-dependent quantities. However, the quadratic variation of the process is not directly captured. Since this quantity has a high relevance in financial applications, we have found
it crucial to incorporate quadratic variation into the signature. This is possible through the lead-lag transformation of streams.

In particular, given a stream $\left(\hat{X}_{t_i}\right)_{i=0}^N$ in $\mathbb{R}^d$, we define its lead-transformed stream $\left(\hat{X}^{\text{lead}}_j\right)_{j=0}^{2N}$ by

$$\hat{X}^{\text{lead}}_j = \begin{cases} \hat{X}_{t_i} & \text{if } j = 2i \\ \hat{X}_{t_{i-1}} & \text{if } j = 2i - 1 \end{cases}$$

Moreover, we define its lag-transformed stream $\left(\hat{X}^{\text{lag}}_j\right)_{j=0}^{2N}$ by

$$\hat{X}^{\text{lag}}_j = \begin{cases} \hat{X}_{t_{i+1}} & \text{if } j = 2i \\ \hat{X}_{t_i} & \text{if } j = 2i + 1 \end{cases}$$

Finally, the lead-lag-transformed stream takes values in $\mathbb{R}^{2d}$ and is defined by the paired stream

$$\left(\hat{X}^{\text{lead-lag}}_j\right)_{j=0}^{2N} = (\hat{X}^{\text{lead}}_j, \hat{X}^{\text{lag}}_j)_{j=0}^{2N}.$$ 

Note that the axis path $X^{\text{lead}}$ corresponding to the lead-transform stream is a time-reparameterisation of the axis path $X$, hence

$$S_{t_0,t_N}(\hat{X}) = S_{0,2N}(\hat{X}^{\text{lead}}),$$

Figure 3: Mid-loaded (on the left) and front-and-back-loaded (on the right) volume profiles
and similarly

\[ S_{t_0, t_N}(\hat{X}) = S_{0, 2N}(\hat{X}^{\text{lag}}). \]

Furthermore, the (signed) area between the \(i\)th component of the lead-transform and the \(j\)th component of the lag-transform equals to the quadratic cross-variation of the trajectories \(\hat{X}^i\) and \(\hat{X}^j\).

In practice, we use a partial lead-lag-transform, that is we take the lead-transform of the input stream, and pair it with the lag-transform of only those components of which the quadratic variation is relevant (see section 3.1).

For instance, in figure 5, two representations of the axis path corresponding to the lead-

![Figure 4: Axis path representation of a data stream.](image)

![Figure 5: Lead-lag transformed mid-price](image)

(a) Lead-lag transformed path as function of time  (b) Lead-lag transformed path in the lead-lag space

Figure 5: Lead-lag transformed mid-price
lag transform of a simulated mid-price stream are shown. The signed area between the
curve and the cord on the right-hand side plot equals to the quadratic variation of the mid-
price.

3 Methodology

3.1 Input data

Our numerical tests are focused on two futures markets: WTI crude oil (NYMEX) and
FTSE 100 index (NYSE Liffe). We work with sub-sampled level-one order book data of the
front-month (closest maturity) futures as raw data, in particular:

- $P^a$: best ask price,
- $P^b$: best bid price,
- $V^a$: number of outstanding orders at best ask price,
- $V^b$: number of outstanding orders at best bid price,
- $C$: cumulative traded volume.

The sampling frequency is specified in the description of the particular tests.

In each test, the input variables are streams of level-one order book data; the fea-
tures used by the classification algorithm are the terms of the truncated signatures of
these streams. The lengths of the input streams depend on the particular test. How-
ever, in each case, we transform and normalise the data as follows. For each stream
$\hat{Y} := ((P^a, P^b, V^a, V^b, C))^N_{i=0}$, we define the components:

- the normalised time: $u_{t_i} := \frac{t_i - t_0}{t_N - t_0}$,
- the normalised logarithm of mid-price:

$$p_{t_i} := C_p \log \frac{P^a_{t_i} + P^b_{t_i}}{2},$$

where for each stream, $C_p$ is chosen to satisfy $\text{StDev}(\Delta p) = 1$,

- the standardised spread: $s_{t_i} := C_s(P^a_{t_i} - P^b_{t_i})$ where for each stream, $C_s$ is chosen to
  satisfy $\text{StDev}(s) = 1$,

- the normalised imbalance $d_{t_i} := \frac{V^a_{t_i} - V^b_{t_i}}{V^a_{t_i} + V^b_{t_i}},$

- the normalised cumulative volume: $c_{t_i} := \frac{C_{t_i}}{C_N}.$

Using these components and the time stamps, each $\hat{Y}$ stream is transformed to $\hat{X} :=$
$(u_{t_i}, p_{t_i}, s_{t_i}, d_{t_i}, c_{t_i})^N_{i=0}.$

In order to capture the quadratic variation of the log-price, the lead-transform $\hat{X}^{\text{lead}}$ is
computed and is paired with the lag-transform of $p$; hence, the input streams are of the
form:

$$(\hat{Z}_{s_i})^2_{i=0} := \left( (u_{s_i}^{\text{lead}}, p_{s_i}^{\text{lead}}, s_{s_i}^{\text{lead}}, d_{s_i}^{\text{lead}}, c_{s_i}^{\text{lead}}, p_{s_i}^{\text{lag}}) \right)^2_{i=0}.$$  (2)
We compute the signature of each stream up to depth 4, that is for all iterated integrals that are indexed with multi-indices of length at most 4. The signatures are computed with the **sigtools** Python package which is based on the the **libalgebra** library of the CoRoPa project[^2].

### 3.2 Linear regression-based classification

In each numerical test (ref. sections 4.1, 4.2 and 4.3) a set of input streams is given, which we partition into two categories along some properties of the streams. The aim of the tests is to predict the category of a stream using its signature, that is when the independent variables – or features – are the terms in the truncated signatures of the input streams. A certain percentage of the streams in each category is used as a learning set and the remaining streams are used for out-of-sample testing. The value of the dependent variable – or response – is defined to be 0 along the streams of one category and 1 along the streams of the other category. The classification method is linear regression combined with LASSO (least absolute shrinkage and selection operator) shrinkage (ref. [1]) in order to identify the relevant terms in the signature. That is, we minimise the following objective function:

\[
\min_\beta \left[ \sum_{k=1}^{K} \left( \sum_{I \in I_m} \beta_I Z_{0,2N_k}^I (k) - y(k) \right)^2 + \alpha \sum_{I \in I_m} |\beta_I| \right],
\]

where \(I_m\) denotes the set of multi-indices of length at most \(m\), \(\beta = (\beta_I)_{I \in I_m}\), \(K\) denotes the cardinality of the learning set, \(Z(k)\) denotes the axis path corresponding to the \(k\)th input stream \((\hat{Z}_i)_{i=0}^{2N_k}(k)\), \(y(k)\) denotes the category (0 or 1) of the \(k\)th input stream, and \(\alpha\) is the shrinkage parameter.

Since the LASSO shrinkage is sensitive to the rescaling of variables, we centralise and normalise each independent variable across streams based on first and second moments that are estimated using the learning set of streams.

For each test case, the level of shrinkage parameter \(\alpha\) is set to match the ratio of correct classification in the learning set and the out of sample test as closely as possible.

### 3.3 Statistical indicators

In order to measure the accuracy of the classification, for each test case we compute the following measures based on the regressed values.

(i) The Kolmogorov-Smirnov distance of the distributions of regressed values for each category computed using the learning set of streams.

(ii) The Kolmogorov-Smirnov distance of the distributions of regressed values for each category computed using the out-of-sample streams.

(iii) Setting the decision boundary at the level where the Kolmogorov-Smirnov distance is attained[^3] in \([0, 1]\), the number of true positives, false positives, true negatives, false negatives and the ratio of correct classification are computed.

[^2]: [http://coropa.sourceforge.net/](http://coropa.sourceforge.net/)

[^3]: This choice of threshold maximises the ratio of correct predictions.
Table 1: The 14:00-14:30 EST time bucket compared some other intervals.
“LS”: learning set, “OS”: out-of-sample set.

| bucket         | Kolmogorov-Smirnov distance | Area under ROC curve | Ratio of correct classification | Number of relevant variables |
|----------------|-----------------------------|----------------------|---------------------------------|----------------------------|
| 9:30-10:00     | 0.84                        | 0.96                 | 0.92                            | 3                          |
| 10:30-11:00    | 0.80                        | 0.95                 | 0.90                            | 4                          |
| 11:30-12:00    | 0.88                        | 0.98                 | 0.94                            | 3                          |
| 13:30-14:00    | 0.77                        | 0.95                 | 0.89                            | 5                          |

(iv) We plot the receiver operating characteristic (ROC) curve and compute the area under the ROC curve for the regressed values calculated using the learning set of streams.

(v) We repeat (iv) using the score values computed using the out-of-sample streams.

4 Numerical tests

In this section, we describe various test cases and the numerical results.

4.1 Characterising standard time buckets of a futures market

As an initial exercise, we explore to what extent the signatures of 30-minute data streams determine the time bucket they are sampled from. In particular, we run binary classification on various pairs of time intervals, and explore which time buckets show similar behaviour and which time buckets have unique characteristics.

Since there are recurring patterns in the market activity, the data standardisation described in section 3.1 is done in order to hide the obvious features. Primarily, the order of magnitude of the trading volume or the level of volatility are all discarded through the standardisation process. The information that remains is what is captured by the first and higher order areas (see section 2.3).

Input streams The underlying instrument is the front-month WTI crude oil future (NYMEX). The input streams are sampled by the minute from standard 30-minute time intervals on each trading date between 1st January 2011 and 1st February 2013. From each time-bucket 75% of the streams are randomly selected for learning and the remaining 25% are used for out of sample testing.

14:00-14:30 Despite hiding the obvious features, the 14:00-14:30 EST time bucket shows very distinctive characteristics. Since the open outcry trading closes at 14:30 EST, this result – after all – is not surprising.

In figure 6, we present the classification results (the estimated densities of regressed values and the ROC curve) when we compare the 9:30-10:00 time bucket to 14:00-14:30 time bucket. Similarly, in figure 7, the results of the binary classification of the 10:30-11:00 and 14:00-14:30 intervals are shown.

Table 1 contains some more results. Apart from the Kolmogorov-Smirnov distance, area under ROC curve and ratio of correct classification, the table shows the number of relevant
variables identified by the LASSO shrinkage algorithm. In particular, the most relevant terms are fourth order iterated integrals corresponding to these multi-indices:

\[(1, 5, 1, 5), \ (1, 5, 5, 5), \ (5, 1, 1, 1), \text{ and } (6, 2, 5, 5),\]

where 1, 2, 5 and 6 are the indices of \(t^{\text{lead}}\), \(p^{\text{lead}}\), \(c^{\text{lead}}\) and \(p^{\text{lag}}\) respectively. The first three terms are involved in the higher order characteristics of the cumulative volume profile. The last term includes the lead-lag-transformed log-mid price, hence it is related to the quadratic variation of the mid-price. In figures 8, 9 and 10, the signatures of the input streams are projected on to two-dimensional subspaces that are spanned by pairs of the most relevant features. The projections demonstrate how the particular fourth order terms capture the difference between the two time buckets.

11:30-12:00 As another example, we also discuss how the 11:30-12:00 EST time bucket is distinguished from the other time-buckets.

The results are summarised in table 2. The 11:30-12:00 time bucket shows more similarities with the reference time intervals, however it is still distinguishable, although more variables are required. Similarly as in the previous case, all the relevant variables correspond to certain fourth order iterated integrals; apart from the higher order characteristics of the cumulative volume profile and the quadratic variation, iterated integrals that contain the spread are also identified.

12:00-12:30 versus 12:30-13:00 Naturally, the market shows similar behaviour in certain pairs of time intervals. For instance, the signatures of the data streams observed in the 12:00-12:30 EST and 12:30-13:00 EST intervals indicate no significant difference. Figure 11 demonstrates the low-accuracy classification results which is based on 27 independent variables.

4.2 Characterising the traces of trade execution algorithms

The second numerical experiment aims to characterise the traces of trade execution algorithms. For this purpose, we use data streams sampled from the FTSE 100 index futures market (NYSE Liffe), which we transform as described in section 3.1. The start and finish time of the sampled streams are determined by the start and completion time of parent orders generated by two different trade execution algorithms. We refer to these algorithms as \textit{algorithm A} and \textit{algorithm B}. In the observed time period, the execution team randomly
selected a trade execution algorithm for each trading signal. There is no significant difference in distributions of size and start time of parent orders that have been executed with algorithm A and algorithm B respectively.

Note that the time variable is normalised in each data stream, hence the length of the execution period of the parent orders is not visible by the signatures. Besides the start and completion time of the parent orders, we have also been given the size of the parent order (in terms of number of lots) and the direction of the trade (sell or buy). We did not use any other specifics of the parent orders; in particular no information on the order submission, order cancellations and realised trades were used.

Investors in general aim to minimise their losses when trading. In particular, they aim to trade at favourable prices and minimise their market impact. This objective is particularly crucial for market participants who trade in large volumes. In the light of these facts, we approached this numerical experiment with moderately high expectations. Moreover, many trading algorithms operate simultaneously on the market; some of these algorithms might possess similar characteristics to the algorithms A and B in the focus of our experiment. This can results in noisy observations.

**Input streams** The data set contains 606 and 594 data streams that are subsampled from algorithm A and algorithm B parent orders respectively over the time period 1/4/2012-1/1/2013; 85% of each category is randomly selected as a learning set and the remaining 15% is used for out-of-sample testing. The experiment is focused on parent orders of a trade size between 5 and 100 contracts.

**Classification results** The classification results are summarised in figure 12. The signature and linear regression-based classification classifies 76% of the out-of-sample streams correctly. In particular, when a stream is predicted to contain the traces of algorithm A, the prediction is correct in 89% of the test cases. When an stream is predicted to contain the traces of algorithm B, the prediction is correct in 67% of the test cases.

The LASSO shrinkage helped to identify 12 relevant variables. The multi-indices that specify the relevant iterated integrals together with the estimated regression coefficients are shown in table 3.

| Multi-index | coefficient | Multi-index | coefficient |
|-------------|-------------|-------------|-------------|
| (5,6,2,5)   | 0.1956      | (1,1,6,2)   | 0.1522      |
| (1,2,6,5)   | -0.0960     | (6,4,5)     | 0.0778      |
| (6,4,1,5)   | -0.0747     | (6,4)       | 0.0519      |
| (1,5,6,2)   | -0.0515     | (6,4,6,2)   | -0.0480     |
| (1,2,6,1)   | -0.0452     | (6,4,5,5)   | 0.0392      |
| (1,6,2,5)   | -0.0109     | (4,6)       | -0.0001     |

Table 3: Estimated regression coefficients

In table 3 the indices 1, 2, 3, 4, 5 and 6 correspond to the variables $u^{\text{lead}}$, $p^{\text{lead}}$, $s^{\text{lead}}$, $d^{\text{lead}}$, $c^{\text{lead}}$ and $p^{\text{lag}}$ respectively. In particular, certain cross-terms of the lead-lag transformed price, the cumulative volume, time and the normalised imbalance are relevant. Moreover, the standardised spread is not used in the classification.
4.3 Localising and testing robustness of classification

In order to test the robustness of the signature and linear regression-based classification, we apply it in a sequential prediction context. The same input data is used as in section 4.2, however here we localise the classification to 60 day time windows. In each window, the first 85% of the streams is used for classification and the last 15% is used for out-of-sample testing. The total sample size in each time window varies between 192 and 347 streams. The linear regression in each time window is based on the 12 features – expressed in terms of iterated integrals of the input streams – that are identified in section 4.2.

Classification results The classification results are summarised in figure 13. The signature and linear regression-based classification performs reasonably well in a sequential setting. The ratio of correct classification of the out-of-sample streams is typically above 70%, in some time windows it is close to 80%, although on one occasion – in the 1/7/2012-30/8/2012 period – it is as low as 60%.

The area under the ROC curve measure is even more robust, it varies between 0.72 and 0.92.

In summary, in the observed time windows, the linear regression based on the chosen features characterises the market impact of the underlying trade execution algorithms with a reasonably high level of accuracy.

5 Conclusions

We use the terms in the signature of a data stream to define features for supervised learning algorithms. In particular, we extract information from noisy market data and classify data streams along various properties with a degree of high accuracy. The numerical examples demonstrate that the signature-based classification has great potential in machine learning.
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(a) **Learning set**: Estimated densities of the regressed values, K-S distance: 0.84, correct classification: 92%

(b) **Out of sample**: Estimated densities of the regressed values, K-S distance: 0.83, correct classification: 91%

(c) **ROC curve**. Area under ROC – learning set: 0.96, out of sample: 0.97

Figure 6: 14:00-14:30 EST versus 9:30-10:00 EST
(a) **Learning set**: Estimated densities of the regressed values, K-S distance: 0.8, correct classification: 90%

(b) **Out of sample**: Estimated densities of the regressed values, K-S distance: 0.84, correct classification: 89%

(c) **ROC curve**. Area under ROC – learning set: 0.95, out of sample: 0.96

Figure 7: 14:00-14:30 EST versus 10:30-11:00 EST
Figure 8: Projection of the signatures (i) - 14:00-14:30 versus 10:30-11:00

Figure 9: Projection of the signatures (ii) - 14:00-14:30 versus 10:30-11:00
Figure 10: Projection of the signatures (iii) - 14:00-14:30 versus 10:30-11:00
(a) **Learning set**: Estimated densities of the regressed values, K-S distance: 0.29, correct classification: 55%

(b) **Out of sample**: Estimated densities of the regressed values, K-S distance: 0.07, correct classification: 48%

(c) **ROC curve**. Area under ROC – learning set: 0.67, out of sample: 0.49

Figure 11: 12:00-12:30 EST versus 12:30-13:00 EST
(a) **Learning set**: Estimated densities of the regressed values, K-S distance: 0.55, correct classification: 77%

(b) **Out of sample**: Estimated densities of the regressed values, K-S distance: 0.46, correct classification: 76%

(c) **ROC curve.** Area under ROC – learning set: 0.84, out of sample: 0.80

Figure 12: Algo A versus Algo B (1/4/2012-1/10/2012)
Figure 13: Algo A vs Algo B – localised to 60 day time windows