NON-RELATIVISTIC LIMIT OF THE COMPRESSIBLE
NAVIER-STOKES-FOURIER-P1 APPROXIMATION MODEL
ARISING IN RADIATION HYDRODYNAMICS

SONG JIANG, FUCAI LI, AND FENG XIE

Abstract. As is well-known that the general radiation hydrodynamics models include two mainly coupled parts: one is macroscopic fluid part, which is governed by the compressible Navier-Stokes-Fourier equations; another is radiation field part, which is described by the transport equation of photons. Under the two physical approximations: “gray” approximation and P1 approximation, one can derive the so-called Navier-Stokes-Fourier-P1 approximation radiation hydrodynamics model from the general one. In this paper we study the non-relativistic limit problem for the Navier-Stokes-Fourier-P1 approximation model due to the fact that the speed of light is much larger than the speed of the macroscopic fluid. Our results give a rigorous derivation of the widely used macroscopic model in radiation hydrodynamics.

1. Introduction and Main Results

The key aim of the radiation hydrodynamics is to include the radiation effects into hydrodynamics. And hydrodynamics with explicit account of the radiation energy and momentum contributions constitutes the main charter of “radiation hydrodynamics”. If the viscosity and heat-conductivity of macroscopic fluids are also considered, the general radiation hydrodynamics equations can be written as the following compressible Navier-Stokes-Fourier system with additional radiation terms [14]:

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \nabla (\rho \mathbf{u} \otimes \mathbf{u} + P I_n + P \mathbf{I}) &= \nabla \cdot \Psi(\mathbf{u}), \\
\partial_t (\rho E + E_v) + \nabla (\rho (E + P) + F_v) &= \nabla \cdot (\Psi(\mathbf{u}) \cdot \mathbf{u}) + \nabla (\kappa \nabla \theta).
\end{align*}
\]

Here the unknowns \( \rho, \mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n(n = 2, 3) \) and \( \theta \) denote the density, the velocity, and the temperature of the fluid, respectively; \( \Psi(\mathbf{u}) \) is the viscous stress tensor given by

\[
\Psi(\mathbf{u}) = 2\mu \nabla \cdot \mathbf{u} + \lambda \nabla \mathbf{u} \otimes \mathbf{u}, \quad \nabla \cdot \mathbf{u} = (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)/2,
\]

where \( I_n \) denotes the \( n \times n \) identity matrix, and \( \nabla \mathbf{u}^\top \) the transpose of the matrix \( \nabla \mathbf{u} \). The pressure \( P = P(\rho, \theta) \) and the internal energy \( e = e(\rho, \theta) \) are smooth functions of \( \rho \) and \( \theta \) and satisfy the Gibbs relation

\[
\theta dS = de + P d\left(\frac{1}{\rho}\right)
\]

2000 Mathematics Subject Classification. 35Q30, 35Q70, 35B25.
Key words and phrases. Radiation Hydrodynamics, Compressible Navier-Stokes-Fourier system, P1 approximation, “Gray” approximation, Non-relativistic limit.
for some smooth function (entropy) \( S = S(\rho, \theta) \), which expresses the first law of the thermodynamics. \( E = e + \frac{\mu^2}{2} \) denotes the total energy. The viscosity coefficients \( \mu \) and \( \lambda \) of the fluid satisfy \( \mu > 0 \) and \( 2\mu + n\lambda > 0 \). The parameter \( \kappa > 0 \) is the heat conductivity. For simplicity we assume that \( \mu, \lambda, \) and \( \kappa \) are constants.

Now we consider the radiation energy \( E_r \), the radiation flux \( F_r \) and the radiation pressure \( P_r \) appearing in (1.1)–(1.3) which can be defined by

\[
E_r = \frac{1}{c} \int_0^\infty d\nu \int_{S^{n-1}} I(\nu, \omega) d\omega,
\]

\[
F_r = \int_0^\infty d\nu \int_{S^{n-1}} \omega I(\nu, \omega) d\omega,
\]

\[
P_r = \frac{1}{c} \int_0^\infty d\nu \int_{S^{n-1}} \omega \otimes \omega I(\nu, \omega) d\omega.
\]

Here the radiation intensity \( I = I(t, x, \nu, \omega) \), depending on the direction vector \( \omega \in S^{n-1} \) and the frequency \( \nu > 0 \), is determined by solving the linear Boltzmann type equation:

\[
\frac{1}{c} \partial_t I + \omega \cdot \nabla I = S(\nu) - \sigma_a(\nu) I + \int_0^\infty d\nu' \int_{S^{n-1}} \left[ \frac{\nu}{\nu'} \sigma_a(\nu' \to \nu) I(\nu', \omega') - \sigma_s(\nu \to \nu') I(\nu, \omega) \right] d\omega'.
\]

The emission term \( S(\nu) \) can be taken as the well-known Planck function, i.e.,

\[
S(\nu) = 2\hbar \nu^3 c^{-2} (e^{\hbar \nu/k\theta} - 1)^{-1}.
\]

In general the absorbing coefficient \( \sigma_a \) and the scattering coefficient \( \sigma_s \) depend on the frequency \( \nu \), the density \( \rho \), and the temperature \( \theta \) of the macroscopic fluid.

In the present paper, we focus on the “gray” approximation case such that the transport coefficients \( \sigma_a \) and \( \sigma_s \) are independent of the frequency \( \nu \). Consequently, the radiation quantities both \( I \) and \( S \) can be integrated on frequency. For \( S \), we have

\[
\int_0^\infty S(\nu) d\nu = \int_0^\infty 2\hbar \nu^3 c^{-2} (e^{\hbar \nu/k\theta} - 1)^{-1} d\nu = \bar{C} \theta^4,
\]

for some positive constant \( \bar{C} \). To deal with \( I \), for simplification, we further assume that both \( \sigma_a \) and \( \sigma_s \) are two positive constants in the following derivation. It should be noted that the general case \( \sigma_a = \sigma_a(\rho, \theta) \) and \( \sigma_s = \sigma_s(\rho, \theta) \) can also be dealt with similarly. In this way, the equation for the integration of \( I \) with respect to frequency \( \nu \), still denoted by \( I \), can be written as

\[
\frac{1}{c} \partial_t I + \omega \cdot \nabla I = \bar{C} \theta^4 - \sigma_a I + \sigma_s S^{n-1} |\langle I \rangle - I | 
\]

with the average \( \langle I \rangle := \frac{1}{|S^{n-1}|} \int_{S^{n-1}} I(t, x, \omega) d\omega \).

In addition, when the distribution of photons is almost isotropic, one can take the P1 hypothesis by choosing the ansatz

\[
I = I_0 + I_1 \cdot \omega,
\]

where \( I_0 \) and \( I_1 \) do not depend on \( \omega \), \( I_1 \cdot \omega \) is regarded as a correction term of the main term \( I_0 \). Inserting the ansatz (1.7) into (1.6) gives

\[
\frac{1}{c} \partial_t \langle I_0 \rangle + \omega \cdot \nabla \langle I_0 \rangle = \frac{1}{c} \partial_t \langle I_1 \rangle + \omega \cdot \nabla \langle I_1 \rangle.
\]
Taking gradient to \((1.18)\), one gets
\[
\frac{1}{c} \partial_t I_0 + \frac{1}{n|S^{n-1}|} \text{div } x I_1 = \bar{C}\theta^4 - \sigma_a I_0,
\]
(1.9)

Hence we have
\[
\frac{1}{c} \partial_t I_1 + \nabla x I_0 = -(\sigma_a + \sigma_s |S^{n-1}|) I_1.
\]
(1.10)

Moreover, by the equation \((1.6)\) and the definitions of \(E_r, F_r\) and \(P_r\), we obtain that
\[
\frac{1}{c^2} \partial_t F_r + \text{div } P_r = -\frac{1}{c} \left( I_0 - \sigma_a \right) \frac{1}{n} |S^{n-1}| I_1,
\]
(1.11)

\[
\partial_t E_r + \text{div } F_r = \bar{C} |S^{n-1}| \theta^4 - |S^{n-1}| \sigma_a I_0.
\]
(1.12)

One can also refer to the Chapter 3 in the book [14] by Pomraning for the above derivation in details.

In general, the speed of light \(c\) can be regarded as a very large number such that the reciprocal of the light speed \(\frac{1}{c}\) is very small. If we take \(\epsilon = \frac{1}{c}\) and ignore the influence of other constants, we obtain the following compressible Navier-Stokes-Fourier-P1 approximation model via the system \((1.1) - (1.3), (1.9) - (1.12)\):

\[
\partial_t \rho + \text{div } (\rho \mathbf{u}) = 0,
\]
(1.13)

\[
\partial_t (\rho \mathbf{u}) + \text{div } (\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I}_n) = \text{div } \Psi(\mathbf{u}) + \epsilon \mathbf{I}_1,
\]
(1.14)

\[
\partial_t (\rho E) + \text{div } (\rho \mathbf{u} (E + P)) = \text{div } (\Psi(\mathbf{u}) \cdot \mathbf{u}) + \text{div } (\kappa \nabla \theta) + I_0 - \theta^4,
\]
(1.15)

\[
\epsilon \partial_t I_0 + \text{div } \mathbf{I}_1 = \theta^4 - I_0,
\]
(1.16)

\[
\epsilon \partial_t \mathbf{I}_1 + \nabla I_0 = -\mathbf{I}_1.
\]
(1.17)

In this paper we consider the non-relativistic limit \(\epsilon \to 0\) for the system \((1.13) - (1.17)\). Formally, letting \(\epsilon = 0\) in \((1.16)\) and \((1.17)\), we obtain that
\[
\text{div } \mathbf{I}_1 = \theta^4 - I_0, \quad -\mathbf{I}_1 = \nabla I_0.
\]

Hence we have
\[
-\Delta I_0 = \theta^4 - I_0.
\]
(1.18)

Taking gradient to \((1.18)\), one gets
\[
-\nabla \text{div } (\nabla I_0) = \nabla \theta^4 - \nabla I_0.
\]
(1.19)

Setting \(\mathbf{q} = -\nabla I_0\), we can rewrite \((1.19)\) as
\[
-\nabla \text{div } \mathbf{q} + \mathbf{q} + \nabla \theta^4 = 0.
\]
(1.20)

Therefore, we can formally obtain the following limit system from \((1.13) - (1.17)\) as \(\epsilon \to 0:\)

\[
\partial_t \rho + \text{div } (\rho \mathbf{u}) = 0,
\]
(1.21)

\[
\partial_t (\rho \mathbf{u}) + \text{div } (\rho \mathbf{u} \otimes \mathbf{u} + P \mathbb{I}_n) = \text{div } \Psi(\mathbf{u}),
\]
(1.22)

\[
\partial_t (\rho E) + \text{div } (\rho \mathbf{u} (E + P)) = \text{div } (\Psi(\mathbf{u}) \cdot \mathbf{u}) + \text{div } (\kappa \nabla \theta) - \text{div } \mathbf{q},
\]
(1.23)

The system \((1.20) - (1.23)\) with \(\mu, \lambda\) and \(\kappa\) being zero are widely used in \([4, 5, 7, 9, 13, 17, 19]\) to describe the dynamics of the fluid in radiation hydrodynamics. For
the case that \( \mu, \lambda \) and \( \kappa \) are not zero, one can refer to [18][20] and references cited therein.

The purpose of this paper is to give a rigorous derivation of the system (1.20)–(1.23) from the Navier-Stokes-Fourier-P1 approximation model (1.13)–(1.17) as \( \epsilon \) tends to zero. For the sake of simplicity and clarity of presentation, we shall focus on the fluids obeying the perfect gas relations

\[
P = \Re \rho \theta, \quad e = c_V \theta, \tag{1.24}
\]

where the parameters \( \Re > 0 \) and \( c_V > 0 \) are the gas constant and the heat capacity at constant volume. We consider the system (1.13)–(1.17) in the whole space \( \mathbb{R}^n \) or the torus \( \mathbb{T}^n = (\mathbb{R}/(2\pi \mathbb{Z}))^n \), which will be denoted by \( \Omega \).

In what follows, for simplicity of presentation, we take the physical constants \( \Re \) and \( c_V \) to be one. To emphasize the unknowns depending on the small parameter \( \epsilon \), we rewrite the system (1.13)–(1.17), (1.4), (1.5), (1.24) as

\[
\dot{\rho} + \div (\rho \mathbf{u}^\epsilon) = 0, \tag{1.25}
\]

\[
\rho^s (\dot{\mathbf{u}}^\epsilon + \mathbf{u}^\epsilon \cdot \nabla \mathbf{u}^\epsilon) + \nabla (\rho^s \mathbf{u}^\epsilon) = \div \Psi (\mathbf{u}^\epsilon) + \mathbf{I}_1^t, \tag{1.26}
\]

\[
\rho^s (\dot{\mathbf{u}}^\epsilon + \mathbf{u}^\epsilon \cdot \nabla \mathbf{u}^\epsilon) + \rho \mathbf{u}^\epsilon \div \mathbf{u}^\epsilon = \kappa \Delta \theta^\epsilon + \Psi (\mathbf{u}^\epsilon) : \nabla \mathbf{u}^\epsilon + I_0 - (\theta^\epsilon)^4, \tag{1.27}
\]

\[
\epsilon \dot{\mathbf{I}}_1 + \mathbf{I}_1 = (\theta^\epsilon)^4 - I_0, \tag{1.28}
\]

where \( \Psi (\mathbf{u}^\epsilon) \) is defined through (1.4) with \( \mathbf{u} \) replaced by \( \mathbf{u}^\epsilon \). The symbol \( \Psi (\mathbf{u}^\epsilon) : \nabla \mathbf{u}^\epsilon \) denotes the scalar product of two matrices:

\[
\Psi (\mathbf{u}^\epsilon) : \nabla \mathbf{u}^\epsilon = \sum_{i,j=1}^n \frac{\mu}{2} \left( \frac{\partial u_i^\epsilon}{\partial x_j} + \frac{\partial u_j^\epsilon}{\partial x_i} \right)^2 + \lambda |\div \mathbf{u}^\epsilon|^2
\]

\[
= 2\mu |\nabla (\mathbf{u}^\epsilon)|^2 + \lambda |\nabla^2 (\mathbf{u}^\epsilon)|^2. \tag{1.30}
\]

The system (1.20)–(1.23), (1.4), (1.5), (1.24) are supplemented with initial data

\[
(\rho^0, \mathbf{u}^0, \theta^0, I_0, I_1)|_{t=0} = (\rho^0_0(x), \mathbf{u}^0_0(x), \theta^0_0(x), I^0_0(x), I^0_1(x)), \quad x \in \Omega. \tag{1.31}
\]

We also rewrite the limit equations (1.20)–(1.23), (1.4), (1.5), (1.24) (recall that \( \Re = c_V = 1 \)) as

\[
\dot{\rho}^0 + \div (\rho^0 \mathbf{u}^0) = 0, \tag{1.32}
\]

\[
\rho^0 (\dot{\mathbf{u}}^0 + \mathbf{u}^0 \cdot \nabla \mathbf{u}^0) + \nabla (\rho^0 \mathbf{u}^0) = \div \Psi (\mathbf{u}^0), \tag{1.33}
\]

\[
\rho^0 (\dot{\mathbf{u}}^0 + \mathbf{u}^0 \cdot \nabla \mathbf{u}^0) + \rho^0 \mathbf{u}^0 \div \mathbf{u}^0 = \kappa \Delta \theta^0 + \Psi (\mathbf{u}^0) : \nabla \mathbf{u}^0 - \nabla \mathbf{q}^0 + \mathbf{q}^0 + \nabla (\theta^0)^4 = 0. \tag{1.34}
\]

where \( \Psi (\mathbf{u}^0) \) and \( \Psi (\mathbf{u}^0) : \nabla \mathbf{u}^0 \) are defined through (1.4) and (1.30) with \( \mathbf{u} \) and \( \mathbf{u}^\epsilon \) replaced by \( \mathbf{u}^0 \), respectively. The system (1.32)–(1.35) are equipped with initial data

\[
(\rho^0, \mathbf{u}^0, \theta^0)|_{t=0} = (\rho^0_0(x), \mathbf{u}^0_0(x), \theta^0_0(x)), \quad x \in \Omega. \tag{1.36}
\]

We first state a result on the local existence of smooth solutions to the problem (1.32)–(1.36), one can refer to [20] for a similar proof in details.

**Proposition 1.1.** Let \( s > n/2 + 2 \) be an integer and assume that the initial data \( (\rho_0^0, \mathbf{u}^0_0, \theta^0_0) \) satisfy

\[
\rho_0^0, \mathbf{u}^0_0, \theta^0_0 \in H^{s+2}(\Omega),
\]
0 < \tilde{\rho} = \inf_{x \in \Omega} \rho_0^0(x) \leq \rho_0^0(x) \leq \bar{\rho} = \sup_{x \in \Omega} \rho_0^0(x) < +\infty,
0 < \tilde{\theta} = \inf_{x \in \Omega} \theta_0^0(x) \leq \theta_0^0(x) \leq \bar{\theta} = \sup_{x \in \Omega} \theta_0^0(x) < +\infty

for some positive constants \( \tilde{\rho}, \bar{\rho}, \tilde{\theta}, \) and \( \bar{\theta} \). Then there exist positive constants \( T_* \) (the maximal time interval, \( 0 < T_* \leq +\infty \)), and \( \hat{\rho}, \tilde{\rho}, \tilde{\theta}, \) such that the problem (1.32) – (1.36) has a unique classical solution \((\rho^0, u^0, \theta^0, q^0)\) satisfying

\[
\rho^0 \in C^l([0, T_*), H^{s+2-1}(\Omega)), \quad u^0, \theta^0 \in C^l([0, T_*), H^{s+2-2}(\Omega)), \quad l = 0, 1;
\]

\[
q^0 \in C^l([0, T_*), H^{s+1-2}(\Omega)), \quad l = 0, 1;
\]

0 < \hat{\rho} = \inf_{(x, t) \in \Omega \times [0, T_*)} \rho^0(x, t) \leq \rho^0(x, t) \leq \bar{\rho} = \sup_{(x, t) \in \Omega \times [0, T_*)} \rho^0(x, t) < +\infty,
0 < \hat{\theta} = \inf_{(x, t) \in \Omega \times [0, T_*)} \theta^0(x, t) \leq \theta^0(x, t) \leq \bar{\theta} = \sup_{(x, t) \in \Omega \times [0, T_*)} \theta^0(x, t) < +\infty.

Our convergence results can be stated as follows.

**Theorem 1.2.** Let \( s > n/2 + 2 \) be an integer and \((\rho^0, u^0, \theta^0, q^0)\) be the unique classical solution to the problem (1.32) – (1.36) given in Proposition 1.1. Suppose that the initial data \((\rho_0^0, u_0^0, \theta_0^0, I_{00}, I_{10})\) satisfy

\[
\rho_0^0, u_0^0, \theta_0^0, I_{00}, I_{10} \in H^s(\Omega), \quad \inf_{x \in \Omega} \rho_0^0(x) > 0, \quad \inf_{x \in \Omega} \theta_0^0(x) > 0,
\]

and

\[
\left\| (\rho^c - \rho_0^0, u^c - u_0^0, \theta^c - \theta_0^0) \right\| \leq L_0 \epsilon \quad (1.37)
\]

for some constant \( L_0 > 0 \). Then, for any \( T_0 \in (0, T_*) \), there exist a constant \( L > 0 \), and a sufficient small constant \( \epsilon_0 > 0 \), such that for any \( \epsilon \in (0, \epsilon_0) \), the problem (1.25) – (1.31) has a unique smooth solution \((\rho^c, u^c, \theta^c, I^c_{0}, I^c_{1})\) on \([0, T_0] \) enjoying

\[
\left\| (\rho^c - \rho_0^0, u^c - u_0^0, \theta^c - \theta_0^0) \right\| \leq L \epsilon, \quad t \in [0, T_0]. \quad (1.38)
\]

Here \( q_0^0 \) is defined via the initial datum \( \theta_0^0 \) in the following way:

\[
q_0^0 = \left( \frac{-\Delta}{\Delta - I} \right) \nabla (\theta_0^0)^4
\]

and \( \| \cdot \|_s \) denotes the norm of Sobolev space \( H^s(\Omega) \).

**Remark 1.1.** If the domain \( \Omega \) in the Proposition 1.1 is the whole space \( \mathbb{R}^n \), then the conditions \( \rho_0^0, u_0^0, \theta_0^0 \in H^{s+2}(\Omega) \) should be replaced by \( \rho_0^0 - \hat{\rho}, u_0^0, \theta_0^0 - \bar{\theta} \in H^{s+2}(\Omega) \) for some positive constants \( \hat{\rho} \) and \( \bar{\theta} \). At the same time, the conditions \( \rho_0^0, u_0^0, \theta_0^0, I_{00}, I_{10} \in H^s(\Omega) \) in Theorem 1.2 are required to be changed into \( \rho_0^0 - \hat{\rho}, u_0^0, \theta_0^0 - \bar{\theta}, I_{00} - \hat{\theta}, I_{10} \in H^s(\Omega) \) accordingly. The corresponding proof is essentially unchanged and can be modified in a direct way.

**Remark 1.2.** As a consequence of our result, we obtain the local existence of solutions to the primitive system (1.25) – (1.29), and the life-span of which is independent of \( \epsilon \). Furthermore, the inequality (1.38) implies that the sequences \((\rho^0, u^0, \theta^0)\) converge strongly to \((\rho^c, u^c, \theta^c)\) in \( L^\infty(0, T; H^s(\Omega)) \) and \((I^c_{0}, I^c_{1})\) converge strongly to \( ((-\Delta)^{-1}\text{div } q_0^0, q_0^0) \) in \( L^\infty(0, T; H^s(\Omega)) \) but with different convergence rates.
Remark 1.3. In the local existence for the problem (1.32)--(1.35), the regularity assumption on initial data \((\rho_0^0, \mathbf{u}_0^0, \theta_0^0, q_0^0) \in H^s(\Omega)\) for \(s > n/2 + 2\) is in fact sufficient. Here we have added more regularity assumption in Proposition 1.1 in order to obtain more regular solutions which are needed in the proof of Theorem 1.2.

Remark 1.4. The viscosity and heat conductivity terms in the system (1.25)--(1.29) play a crucial role in our uniformly bounded estimates (in order to control some undesirable higher-order terms). For the case of \(\lambda = \mu = \kappa = 0\), we should use the quasilinear symmetric structure of the hyperbolic part in (2.1)--(2.5). That is, the positively definite matrix \(A^e\) in (2.5) is essentially used in the energy estimation. By using arguments similar to those in showing boundedness of high norms in Chapter 2 in \([10]\), the main Theorem 1.2 is also valid for the Euler-P1 system. Precisely speaking, the smooth solutions to (1.25)--(1.29) with \(\mu = \lambda = \kappa = 0\) converge to the smooth solutions to (1.32)--(1.35) with \(\mu = \lambda = \kappa = 0\). The limit equations (1.32)--(1.35) with \(\mu = \lambda = \kappa = 0\) are indeed considered in \([4, 5, 7--9, 17--19]\). In this case the local existence of solutions to limit system is referred to \([4, 5]\). Here we mention the related nonrelativistic limit process for a simplified model in radiation hydrodynamics in \([16]\).

We give some comments on the proof of Theorem 1.2. The main difficulty in dealing with our non-relativistic limit is the oscillatory behavior of \(I_{\epsilon}^0\) and \(I_{\epsilon}^1\). The time derivatives of \(I_{\epsilon}^0\) and \(I_{\epsilon}^1\) in (1.28)--(1.29) are multiplied by a small parameter, hence the uniform energy estimates are obtained from the relaxation terms rather than from the time-derivative terms. Besides the singularity in (1.28)--(1.29), there exists an extra singularity caused by the coupling of \(I_{\epsilon}^0\) and \(I_{\epsilon}^1\) in the momentum and temperature equations. In this paper, we shall overcome all these difficulties by adopting and modifying the elaborate nonlinear energy method developed in [2,3]. First, we derive the error system (2.1)--(2.5) by utilizing the original system (1.25)--(1.29) and the limit system (1.32)--(1.35). In this step, we need to find the suitable quantities from the limit system, which are related to \(I_{\epsilon}^0\) and \(I_{\epsilon}^1\). Next, we study the estimates of \(H^s\)-norm to the error system. To do so, we shall make full use of the special structure of the error system, the Sobolev imbedding and the Moser-type inequalities, and the regularity of the limit equations. In particular, a very refined analysis is carried out to deal with the higher order nonlinear terms in the system (2.1)--(2.5). It is noted that the damping terms in equations (2.3)--(2.5) also play a crucial role in controlling the nonlinear coupled terms. Finally, we combine these obtained estimates and apply the Gronwall inequality to get the desired results. In addition, we should remark that for fixed \(\epsilon\), the global in time existence of solutions to the barotropic case of the equations (1.25)--(1.29) is achieved in the critical Besov spaces by Danchin and Ducomet recently in \([15]\). As is pointed out in \([11,14]\) that the energy exchange between the hydrodynamics and the radiation field sometime plays a leading role. This is the key reason why we include the energy equation into the system (1.25)--(1.29). In this way it will greatly increase difficulties of mathematical analysis. To our best knowledge, for fixed \(\epsilon\) the global existence of strong solutions to the equations (1.25)--(1.29) is still open, because the analysis of the spectrum for the linearized system is very complicated and is left for our future study.

Before ending this introduction, we give some notations and recall some basic facts which will be frequently used throughout this paper.
(1) We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $L^2(\Omega)$ with $\langle f, f \rangle = \| f \|^2$, by $H^k$ the standard Sobolev space $W^{k,2}$ with norm $\| \cdot \|_k$. The notation $\| (A_1, A_2, \ldots, A_l) \|_k$ means the summation of $\| A_i \|_k$ from $i = 1$ to $i = l$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, we denote $\partial^\alpha_x = \partial_{\alpha_1} \cdots \partial_{\alpha_n}$ and $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$. For an integer $m$, the symbol $D_x^\alpha$ denotes the summation of all terms $\partial^\alpha_x$ with the multi-index $\alpha$ satisfying $|\alpha| = m$. We use $C_i$, $\delta_i$, $K_i$, and $K$ to denote the constants which are independent of $\epsilon$ and may change from line to line. We also omit the spatial domain $\Omega$ in integrals for convenience.

(2) We shall frequently use the following Moser-type calculus inequalities (see [6]):

(i) For $f, g \in H^s(\Omega) \cap L^\infty(\Omega)$ and $|\alpha| \leq s$, $s > n/2$, it holds that

$$
\| \partial^\alpha_x (f g) \| \leq C_s(\| f \|_{L^\infty(\Omega)} \| D^{\alpha, s}_x g \| + \| g \|_{L^\infty(\Omega)} \| D^{\alpha, s}_x f \|).
$$

(1.39)

(ii) For $f \in H^s(\Omega)$, $D_x^1 f \in L^\infty(\Omega)$, $g \in H^{s-1}(\Omega) \cap L^\infty(\Omega)$ and $|\alpha| \leq s$, $s > n/2 + 1$, it holds that

$$
\| \partial^\alpha_x (f g) - f \partial^\alpha_x g \| \leq C_s(\| D^1_x f \|_{L^\infty(\Omega)} \| D^{\alpha, s-1}_x g \| + \| g \|_{L^\infty(\Omega)} \| D^{\alpha, s}_x f \|).
$$

(1.40)

(3) Let $s > n/2$, $f \in C^\alpha(\Omega)$, and $u \in H^s(\Omega)$, then for each multi-index $\alpha$, $1 \leq |\alpha| \leq s$, we have (6.12):

$$
\| \partial^\alpha_x (f u) \| \leq C(1 + \| u \|_{L^\infty(\Omega)}) \| u \|_{|\alpha|}.
$$

Moreover, if $f(0) = 0$, then (1)

$$
\| \partial^\alpha_x (f u) \| \leq C(\| u \|_s) \| u \|_s.
$$

(1.42)

This paper is organized as follows. In Section 2, we utilize the primitive system [1.26]–[1.29] and the target system [1.32]–[1.35] to derive the error system and state the local existence of the solution. In Section 3 we give the a priori energy estimates of the error system and present the proof of Theorem 1.2.

2. Derivation of the error system

In this section we first derive the error system from the original system [1.26]–[1.29] and the limiting equations [1.32]–[1.35], then we state the local existence of solution to this error system.

Setting $N^\epsilon = \rho^\epsilon - \rho^0$, $U^\epsilon = u^* - u^0$, $\Theta^\epsilon = \theta^* - \theta^0$, $J_0^\epsilon = J_0^0 - (\Delta)^{-1}\text{div } q^0$, and $J_1^\epsilon = I_1^0 - q^0$, and utilizing the system [1.26]–[1.29] and the system [1.32]–[1.35], we obtain that

$$
\partial_t N^\epsilon + (N^\epsilon + \rho^0) \text{div } U^\epsilon + \langle U^\epsilon + u^0 \rangle \cdot \nabla N^\epsilon = -N^\epsilon \text{div } u^0 - \nabla \rho^0 \cdot U^\epsilon,
$$

(2.1)

$$
\partial_t U^\epsilon + [(U^\epsilon + u^0) \cdot \nabla] U^\epsilon + \nabla \Theta^\epsilon + \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} \nabla N^\epsilon - \frac{1}{N^\epsilon + \rho^0} \text{div } \Psi(U^\epsilon)

= -(U^\epsilon \cdot \nabla) u^0 - \left[ \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} - \frac{\theta^0}{\rho^0} \right] \nabla \rho^0 + \left[ \frac{1}{N^\epsilon + \rho^0} - \frac{1}{\rho^0} \right] \text{div } \Psi(u^0)

+ \frac{\epsilon}{N^\epsilon + \rho^0} (J_1^\epsilon + q^0),
$$

(2.2)

$$
\partial_t \Theta^\epsilon + [(U^\epsilon + u^0) \cdot \nabla] \Theta^\epsilon + (\Theta^\epsilon + \theta^0) \text{div } U^\epsilon - \frac{\kappa}{N^\epsilon + \rho^0} \Delta \Theta^\epsilon

= -(U^\epsilon \cdot \nabla) \theta^0 - \Theta^\epsilon \text{div } u^0 + \left[ \frac{\kappa}{N^\epsilon + \rho^0} - \frac{\kappa}{\rho^0} \right] \Delta \theta^0
$$
\[ + \frac{2\mu}{N^e + \rho^0}|\mathbb{D}(U^e)|^2 + \frac{\lambda}{N^e + \rho^0}|\text{tr}\mathbb{D}(U^e)|^2 \]
\[ + \frac{4\mu}{N^e + \rho^0}\mathbb{D}(U^e) : \mathbb{D}(u^0) + \frac{2\lambda}{N^e + \rho^0}[\text{tr}\mathbb{D}(U^e)\text{tr}\mathbb{D}(u^0)] \]
\[ + \frac{2\mu}{N^e + \rho^0}\left[\frac{2\mu}{\rho^0}\right]|\mathbb{D}(u^0)|^2 + \left[\frac{\lambda}{N^e + \rho^0} - \frac{\lambda}{\rho^0}\right](\text{tr}\mathbb{D}(u^0))^2 \]
\[ + \frac{1}{N^e + \rho^0}\left\{ J_0^e - (\Theta^e)^4 - 4(\Theta^e)^3\theta^0 - 6(\Theta^e)^2(\theta^0)^2 - 4\Theta^e(\theta^0)^3 \right\} \]
\[ - \left[\frac{1}{N^e + \rho^0} - \frac{1}{\rho^0}\right] \text{div} q^0, \]  
(2.3)

\[ \epsilon \tilde{c}_t J_0^e + \text{div} J_1^e = (\Theta^e)^4 + 4(\Theta^e)^3\theta^0 + 6(\Theta^e)^2(\theta^0)^2 + 4\Theta^e(\theta^0)^3 \]
\[ - J_0^e - \epsilon \tilde{c}_t (-\Delta)^{-1} \text{div} q^0, \]  
(2.4)

\[ \epsilon \tilde{c}_t J_1^e + \nabla J_0^e = -J_1^e - \epsilon \tilde{c}_t q^0, \]  
(2.5)

with initial data

\[ (N_e, U_e, \Theta^e, J_0^e, J_1^e)|_{t=0} = (N_0^e, U_0^e, \Theta_0^e, J_{00}^e, J_{10}^e) \]
\[ = (\rho_0^e - \rho_0, u_0^e - u_0^0, \theta_0^0 - \theta_0^0, I_0^0 - (-\Delta)^{-1}\text{div} q_0^0, I_{10}^0 - q_0^0). \]  
(2.6)

We remark that in (2.6) we have used the fact that

\[ q^0 = \nabla \text{div} q^0 - \nabla(\theta^0)^4 = \left(\frac{-\Delta}{I - \Delta} - I\right)\nabla(\theta^0)^4 = \frac{-\nabla}{I - \Delta}(\theta^0)^4 \]

is a gradient.

Denote

\[ W^e = \begin{pmatrix} N^e \\ U^e \\ \Theta^e \\ J_0^e \\ J_1^e \end{pmatrix}, \quad W_0^e = \begin{pmatrix} N_0^e \\ U_0^e \\ \Theta_0^e \\ J_{00}^e \\ J_{10}^e \end{pmatrix}, \quad D^e = \begin{pmatrix} \mathbb{I}_{n+2} & 0 \\ \epsilon & 0 \\ 0 & \epsilon \mathbb{I}_n \end{pmatrix}, \]

\[ A^e_i = \begin{pmatrix} (U^e + u^0)_i & (N^e + \rho^0)e_i^T \\ (\Theta^e + \theta^0)e_i^T & (U^e + u^0)_i \\ 0 & (U^e + u^0)_i \end{pmatrix}, \]

\[ A^e_{ij} = \begin{pmatrix} 0 & e_i^T e_j \\ 0 & e_i^T e_j \\ 0 & e_i^T e_j \end{pmatrix}, \]

\[ S^e(W^e) = \begin{pmatrix} -N^e \text{div} u^0 - \nabla \theta^0 \cdot U^e \\ R_1^e \\ R_2^e \\ R_3^e \\ -J_1^e - \epsilon \tilde{c}_t q^0 \end{pmatrix}, \]
where $R'_1, R'_2,$ and $R'_3$ denote the right-hand side of (2.2), (2.3), and (2.4), respectively; $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$ and $y_i$ denotes the $i$-th component of $y \in \mathbb{R}^n$.

Using these notations we can rewrite the problem (2.1)–(2.6) in the form:

\[
\left\{ \begin{array}{l}
D^t \hat{e}_i W^\varepsilon + \sum_{i=1}^{n} A^\varepsilon_i \hat{e}_x, W^\varepsilon + \sum_{i,j=1}^{n} A^\varepsilon_{ij} \hat{e}_{x,x_j} W^\varepsilon = S^\varepsilon(W^\varepsilon), \\
W^\varepsilon|_{t=0} = W^0. 
\end{array} \right. 
\]  

(2.7)

It is not difficult to see that the system for $W^\varepsilon$ in (2.7) can be reduced to a quasilinear symmetric hyperbolic-parabolic one. In fact, if we introduce

\[
A^\varepsilon = \left( \begin{array}{ccc}
\Theta^\varepsilon + \rho^0 & 0 & 0 \\
0 & \mathbb{I}_n & 0 \\
0 & 0 & 1
\end{array} \right),
\]

(2.8)

which is positively definite when $\|N^\varepsilon\|_{L^2_t L^2_x} \leq \hat{\rho}/2$ and $\|\Theta^\varepsilon\|_{L^2_t L^2_x} \leq \hat{\theta}/2$, then $A^\varepsilon = A^\varepsilon D^t_0$ is positive symmetric and $A^\varepsilon = A^\varepsilon D^t_0$ are symmetric on $[0, T]$ for all $1 \leq i \leq n$. Moreover, the assumptions that $\mu > 0, 2\mu + n\lambda > 0$, and $\kappa > 0$ imply that

$A^\varepsilon = \sum_{i,j=1}^{n} A^\varepsilon A^\varepsilon_{ij} \hat{e}_{x,x_j} W^\varepsilon$

is an elliptic operator. Thus, for fixed $\varepsilon > 0$, we can apply the result of Vol’pert and Hudjaev [21] to obtain the following local existence for the problem (2.7).

**Proposition 2.1.** Let $s > n/2 + 2$ be an integer and $(\rho^0, u^0, \theta^0, q^0)$ satisfy the conditions in Proposition 1.1. Assume that the initial data $(N^0_0, U^0_0, \Theta^0_0, J^0_0, J^1_0)$ satisfy

$N^0_0, U^0_0, \Theta^0_0, J^0_0, J^1_0 \in H^s(\Omega)$ and $\|(N^0_0, U^0_0, \Theta^0_0, J^0_0, J^1_0)\|_s \leq \delta$

for some small constant $\delta > 0$. Then there exist positive constants $T^\varepsilon (0 < T^\varepsilon \leq +\infty)$ and $K$, such that the Cauchy problem (2.7) has a unique classical solution $(N^\varepsilon, U^\varepsilon, \Theta^\varepsilon, J^0_\varepsilon, J^1_\varepsilon)$ satisfying

$N^\varepsilon, J^0_\varepsilon, J^1_\varepsilon \in C^l([0, T^\varepsilon), H^{s-l}), U^\varepsilon, \Theta^\varepsilon \in C^l([0, T^\varepsilon), H^{s-2l}), l = 0, 1$;

$\|(N^\varepsilon(t), U^\varepsilon(t), \Theta^\varepsilon(t), J^0_\varepsilon(t), J^1_\varepsilon(t))\|_s \leq K \delta$, $t \in [0, T^\varepsilon)$.

Note that if $\|N^\varepsilon\|_{L^2_t L^2_x} \leq \hat{\rho}/2$ and $\|\Theta^\varepsilon\|_{L^2_t L^2_x} \leq \hat{\theta}/2$, then for smooth solutions, the system (1.25)–(1.29) with initial data (1.31) are equivalent to (2.4)–(2.6) or (2.7) on $[0, T^\varepsilon], T < \min\{T^\varepsilon, T^*_\varepsilon\}$. Usually, the life-span $T^\varepsilon$ depends on $\varepsilon$ and may shrink to zero as as $\varepsilon \to 0$. Therefore, in order to avoid this situation and to obtain the convergence of system (1.25)–(1.29) to the system (1.32)–(1.35), we only need to establish the uniform decay estimates with respect to the parameter $\varepsilon$ of the solution to the error system (2.7). This will be achieved by the elaborate energy method presented in next section.
3. Uniform energy estimates and proof of Theorem 1.2

In this section we derive the uniform decay estimates with respect to the parameter $\epsilon$ of the solution to the problem (2.1)–(2.5) and justify rigorously the convergence of the system (1.25)–(1.29) to the system (1.32)–(1.35). Here we adopt and modify some techniques developed in [2, 3] and put main efforts on the estimates of higher order nonlinear terms.

We first establish the convergence rate of the error system by establishing the a priori estimates uniformly in $\epsilon$. For conciseness of presentation, we define

$$
\|E^\epsilon(t)\|_s^2 := \|(N^\epsilon, U^\epsilon, \Theta^\epsilon)(t)\|_s^2,
$$

$$
\|E^\epsilon(t)\|_s^2 := \|E^\epsilon(t)\|_s^2 + \epsilon \|(J_0^\epsilon, J_1^\epsilon(t))\|_2^2,
$$

$$
\|E^\epsilon\|_{s,T} := \sup_{0 < t < T} \|E^\epsilon(t)\|_s.
$$

The crucial estimate of this paper is the following decay result on the error system (2.1)–(2.5).

**Proposition 3.1.** Let $s > n/2 + 2$ be an integer and assume that the initial data $(N_0^\epsilon, U_0^\epsilon, \Theta_0^\epsilon, J_0^\epsilon, J_1^\epsilon)$ satisfy

$$
\||(N_0^\epsilon, U_0^\epsilon, \Theta_0^\epsilon)\|_s^2 + \epsilon \|(J_0^\epsilon, J_1^\epsilon)\|_2^2 = \|E^\epsilon(t = 0)\|_s^2 \leq M_0 \epsilon^2
$$

for sufficiently small $\epsilon$ and some constant $M_0 > 0$ independent of $\epsilon$. Then, for any $T_0 \in (0, T_s)$, there exist two constants $M_1 > 0$ and $\epsilon_1 > 0$ depending only on $T_0$, such that for all $\epsilon \in (0, \epsilon_1)$, it holds that $T^\ast \geq T_0$ and the solution $(N^\epsilon, U^\epsilon, \Theta^\epsilon, J_0^\epsilon, J_1^\epsilon)$ of the problem (2.1)–(2.5), well-defined in $[0, T_0]$, enjoys that

$$
\|E^\epsilon\|_{s,T_0} \leq M_1 \epsilon.
$$

Once this proposition is established, the proof of Theorem 1.2 is a direct procedure. In fact, we have

**Proof of Theorem 1.2.** Suppose that Proposition 3.1 holds. According to the definition of the error functions $(N^\epsilon, U^\epsilon, \Theta^\epsilon, J_0^\epsilon, J_1^\epsilon)$ and the regularity of $(\rho^0, u^0, \theta^0, q^0)$, the error system (2.1)–(2.5) and the primitive system (1.25)–(1.29) are equivalent on $[0, T]$ for some $T > 0$. Therefore the assumption (3.1) in Theorem 1.2 implies the assumption (3.1) in Proposition 3.1 and hence (3.2) gives (3.3).

Therefore, our main goal next is to prove Proposition 3.1 which can be approached by the following a priori estimates. For some given $T < 1$ and any $\tilde{T} \leq T$ independent of $\epsilon$, we denote $T = T_s = \min\{\tilde{T}, T^\ast\}.

**Lemma 3.2.** Let the assumptions in Proposition 3.1 hold. Then, for all $0 < t < T$ and sufficiently small $\epsilon$, there exist two positive constants $\delta_1$ and $\delta_2$, such that

$$
\|E^\epsilon(t)\|_s^2 + \int_0^t \left\{ \delta_1 \|\nabla U^\epsilon\|_s^2 + \delta_2 \|\nabla \Theta^\epsilon\|_s^2 + \frac{1}{4} \|J_0^\epsilon\|_s^2 + \frac{1}{4} \|J_1^\epsilon\|_s^2 \right\}(\tau) d\tau
$$

$$
\leq \|E^\epsilon(t = 0)\|_s^2 + C \int_0^t \left\{ (\|E^\epsilon\|_s^{2(\epsilon+1)} + 1) \|E^\epsilon\|_s^2 \right\}(\tau) d\tau + C \epsilon^2.
$$

**Proof.** Let $0 \leq |\alpha| \leq s$. In the following arguments the commutators will disappear in the case of $|\alpha| = 0$. 

Applying the operator $\partial_x^\alpha$ to (2.1), multiplying the resulting equation by $\partial_x^\alpha N^\varepsilon$, and integrating over $\Omega$, we obtain that

$$\frac{1}{2} \frac{d}{dt} \langle \partial_x^\alpha N^\varepsilon, \partial_x^\alpha N^\varepsilon \rangle = -\langle \partial_x^\alpha \left(\left(\nabla \cdot (\nabla N^\varepsilon)\right), \partial_x^\alpha N^\varepsilon\right) \rangle \quad \text{(3.4)}$$

where the commutator

$$[\partial_x^\alpha N^\varepsilon, \partial_x^\alpha N^\varepsilon] = \langle \left(\left(\nabla \cdot (\nabla N^\varepsilon)\right), \partial_x^\alpha N^\varepsilon\right), \partial_x^\alpha N^\varepsilon \rangle$$

can be estimated as follows. Using the Moser-type and Cauchy-Schwarz’s inequalities, we obtain that

$$\langle \partial_x^\alpha \left(\left(\nabla \cdot (\nabla N^\varepsilon)\right), \partial_x^\alpha N^\varepsilon\right) \rangle = \langle \left(\left(\nabla \cdot (\nabla N^\varepsilon)\right), \partial_x^\alpha N^\varepsilon\right), \partial_x^\alpha N^\varepsilon \rangle$$

where the commutator

$$[\partial_x^\alpha N^\varepsilon, \partial_x^\alpha N^\varepsilon] = \langle \left(\left(\nabla \cdot (\nabla N^\varepsilon)\right), \partial_x^\alpha N^\varepsilon\right), \partial_x^\alpha N^\varepsilon \rangle$$

can be estimated as follows. Using the Moser-type and Cauchy-Schwarz’s inequalities, and the regularity of $u^0$ and Sobolev’s imbedding inequalities, we obtain that

$$\left\| H^{(1)} \right\| \leq C \| D_1^1 (U^\varepsilon + u^0) \|_{L^\infty} \| D_2^1 N^\varepsilon \|_s \| D_1^1 (U^\varepsilon + u^0) \|_s \| D_2^1 N^\varepsilon \|_s$$

Similarly, the second term on the right-hand side of (3.3) can bounded as follows.

$$\left\| H^{(2)} \right\| \leq C \| D_1^1 (U^\varepsilon + u^0) \|_{L^\infty} \| D_2^1 N^\varepsilon \|_s \| D_1^1 (U^\varepsilon + u^0) \|_s \| D_2^1 N^\varepsilon \|_s \| D_2^1 N^\varepsilon \|_s$$
Applying the operator $\partial_x^\alpha$ to $[22]$, multiplying the resulting equations by $\partial_x^\alpha U^\epsilon$, and integrating over $\Omega$, we obtain that
\[
\frac{1}{2} \frac{d}{dt} \left( \partial_x^\alpha U^\epsilon, \partial_x^\alpha U^\epsilon \right) + \left( \partial_x^\alpha \left( [(U^\epsilon + u^0) \cdot \nabla] U^\epsilon \right), \partial_x^\alpha U^\epsilon \right) \\
+ \left( \partial_x^\alpha \nabla \Theta^\epsilon, \partial_x^\alpha U^\epsilon \right) + \left( \partial_x^\alpha \left( \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} \nabla N^\epsilon \right), \partial_x^\alpha U^\epsilon \right) \\
- \left( \partial_x^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \text{div} \Psi(U^\epsilon) \right), \partial_x^\alpha U^\epsilon \right) \\
= - \left( \partial_x^\alpha \left( (U^\epsilon \cdot \nabla) u^0 \right), \partial_x^\alpha U^\epsilon \right) \\
- \left( \partial_x^\alpha \left( \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} \nabla \rho^0 \right), \partial_x^\alpha U^\epsilon \right) \\
+ \left( \partial_x^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \text{div} \Psi(u^0) \right), \partial_x^\alpha U^\epsilon \right) \\
+ \left( \partial_x^\alpha \left( \frac{\epsilon}{N^\epsilon + \rho^0} (J^\epsilon + q^0) \right), \partial_x^\alpha U^\epsilon \right) \\
:= \sum_{i=1}^{4} R^{(i)}. \tag{3.11}
\]

The last four terms on the left-hand side of (3.11) can be estimated as follows. Similar to (3.10), we infer that
\[
\left( \partial_x^\alpha \left( [(U^\epsilon + u^0) \cdot \nabla] U^\epsilon \right), \partial_x^\alpha U^\epsilon \right) \\
= \left( [(U^\epsilon + u^0) \cdot \nabla] \partial_x^\alpha U^\epsilon, \partial_x^\alpha U^\epsilon \right) + \left( \mathcal{H}^{(3)}, \partial_x^\alpha U^\epsilon \right) \\
= - \frac{1}{2} \left( \text{div} (U^\epsilon + u^0) \partial_x^\alpha U^\epsilon, \partial_x^\alpha U^\epsilon \right) + \left( \mathcal{H}^{(3)}, \partial_x^\alpha U^\epsilon \right) \\
\leq C(\|\mathcal{E}^\epsilon(t)\|_s + 1) \|\partial_x^\alpha U^\epsilon\|^2 + \|\mathcal{H}^{(3)}\|^2, \tag{3.12}
\]
where the commutator
\[
\mathcal{H}^{(3)} = \partial_x^\alpha \left( [(U^\epsilon + u^0) \cdot \nabla] U^\epsilon \right) - [(U^\epsilon + u^0) \cdot \nabla] \partial_x^\alpha U^\epsilon
\]
can be bounded by
\[
\|\mathcal{H}^{(3)}\| \leq C(\|D_x^2 (U^\epsilon + u^0)\|_{L^\infty} \|D_x^2 U^\epsilon\| + \|D_x^4 U^\epsilon\|_{L^\infty} \|D_x^4 (U^\epsilon + u^0)\|) \\
\leq C\|\mathcal{E}^\epsilon(t)\|_s^2 + C\|\mathcal{E}^\epsilon(t)\|_s. \tag{3.13}
\]

By Hölder’s inequality, we have
\[
\left( \partial_x^\alpha \nabla \Theta^\epsilon, \partial_x^\alpha U^\epsilon \right) \leq \eta_2 \|\partial_x^\alpha \nabla \Theta^\epsilon\|^2 + C\eta_2 \|\partial_x^\alpha U^\epsilon\|^2 \tag{3.14}
\]
for any $\eta_2 > 0$. For the fourth term on the left-hand side of (3.11), similar to (3.11), we integrate it by parts to deduce that
\[
\left( \partial_x^\alpha \left( \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} \nabla N^\epsilon \right), \partial_x^\alpha U^\epsilon \right) \\
= \left( \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} \partial_x^\alpha \nabla N^\epsilon, \partial_x^\alpha U^\epsilon \right) + \left( \mathcal{H}^{(4)}, \partial_x^\alpha U^\epsilon \right) \\
= - \left( \partial_x^\alpha N^\epsilon, \text{div} \left( \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} \partial_x^\alpha U^\epsilon \right) \right) + \left( \mathcal{H}^{(4)}, \partial_x^\alpha U^\epsilon \right)
Recalling that for the fifth term on the left-hand side of (3.11), we have
\[ \mathcal{H}^{(4)} = \mathcal{H}(x) \left( \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} \nabla N^\epsilon \right) - \mathcal{H}(x) \left( \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} \partial^\alpha \nabla N^\epsilon \right) \]
can be bounded as follows, using (1.40), (1.41) and Cauchy-Schwarz’s inequality:
\[ \| \mathcal{H}^{(4)} \| \leq C \left( \left\| D_x^1 \left( \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} \right) \right\|_{L^\infty} \| D_x^1 N^\epsilon \|_{L^\infty} \| D_x^1 \left( \frac{\Theta^\epsilon + \theta^0}{N^\epsilon + \rho^0} \right) \|_{L^\infty} \right) \]
\[ \leq C(\| \mathcal{E}^\epsilon(t) \|_{s}^2 + \| \mathcal{E}^\epsilon(t) \|_{s}^{2(s+1)} + \| \mathcal{E}^\epsilon(t) \|_{s}). \quad (3.16) \]

For the fifth term on the left-hand side of (3.11), we have
\[ - \left\langle \partial^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \text{div} \Psi(U^\epsilon) \right), \partial^\alpha U^\epsilon \right\rangle = - \left\langle \frac{1}{N^\epsilon + \rho^0} \partial^\alpha \text{div} \Psi(U^\epsilon), \partial^\alpha U^\epsilon \right\rangle - \left\langle \mathcal{H}^{(5)}, \partial^\alpha U^\epsilon \right\rangle, \quad (3.17) \]
where the commutator
\[ \mathcal{H}^{(5)} = \partial^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \text{div} \Psi(U^\epsilon) \right) - \frac{1}{N^\epsilon + \rho^0} \partial^\alpha \text{div} \Psi(U^\epsilon). \]

By the Moser-type and Cauchy-Schwarz inequalities, the regularity of \( \rho^0 \) and the positivity of \( N^\epsilon + \rho_0 \), the definition of \( \Psi(U^\epsilon) \) and Sobolev’s imbedding theorem, we find that
\[ \left| \left\langle \mathcal{H}^{(5)}, \partial^\alpha U^\epsilon \right\rangle \right| \leq \| \mathcal{H}^{(5)} \| \cdot \| \partial^\alpha U^\epsilon \| \]
\[ \leq C \left( \left\| D_x^1 \left( \frac{1}{N^\epsilon + \rho^0} \right) \right\|_{L^\infty} \| \text{div} \Psi(U^\epsilon) \|_{s-1} + \| \text{div} \Psi(U^\epsilon) \|_{L^\infty} \left\| \frac{1}{N^\epsilon + \rho^0} \right\|_{s} \right) \| \partial^\alpha U^\epsilon \| \]
\[ \leq \eta_{4} \| \nabla U^\epsilon \|_{s}^2 + C_{\eta_{4}} (\| \mathcal{E}^\epsilon(t) \|_{s}^2 + 1) \| \partial^\alpha U^\epsilon \|_{s}^2 + C \| \mathcal{E}^\epsilon(t) \|_{s}^2 (1 + \| \mathcal{E}^\epsilon(t) \|_{s}^2) \quad (3.18) \]
for any \( \eta_{4} > 0 \), where we have used the assumption \( s > \frac{n}{2} + 2 \) and the imbedding \( H^l(\Omega) \hookrightarrow L^\infty(\mathbb{R}^n) \) for \( l > \frac{n}{2} \). By virtue of the definition of \( \Psi(U^\epsilon) \) and integration by parts, the first term on the right-hand side of (3.17) can be rewritten as
\[ - \left\langle \frac{1}{N^\epsilon + \rho^0} \partial^\alpha \text{div} \Psi(U^\epsilon), \partial^\alpha U^\epsilon \right\rangle = 2\mu \left\langle \partial^\alpha \text{div} \Psi(U^\epsilon), \partial^\alpha \text{div} \Psi(U^\epsilon) \right\rangle + \lambda \left\langle \frac{1}{N^\epsilon + \rho^0} \partial^\alpha U^\epsilon, \partial^\alpha \text{div} U^\epsilon \right\rangle \]
\[ + 2\mu \left\langle \nabla \left( \frac{1}{N^\epsilon + \rho^0} \partial^\alpha \Psi(U^\epsilon) \right) \otimes \partial^\alpha U^\epsilon, \partial^\alpha \text{div} U^\epsilon \right\rangle \]
\[ + \lambda \left\langle \nabla \left( \frac{1}{N^\epsilon + \rho^0} \partial^\alpha U^\epsilon \right) \cdot \partial^\alpha U^\epsilon, \partial^\alpha \text{div} U^\epsilon \right\rangle \]
\[ := \sum_{i=1}^{4} \mathcal{I}^{(i)}. \quad (3.19) \]

Recalling that \( \mu > 0 \) and \( 2\mu + n\lambda > 0 \), and the positivity of \( N^\epsilon + \rho_0 \), the first two terms \( \mathcal{I}^{(1)} \) and \( \mathcal{I}^{(2)} \) can be bounded as follows.
\[ \mathcal{I}^{(1)} + \mathcal{I}^{(2)} = \int \frac{1}{N^\epsilon + \rho^0} \left\{ 2\mu |\partial^\alpha \text{div} \Psi(U^\epsilon)|^2 + \lambda |\partial^\alpha \text{div} \Psi(U^\epsilon)|^2 \right\} dx \]
where and for any \( \eta \) for some constant \( C \), by Cauchy-Schwarz’s inequality and (1.42), we see that
\[
\rho \text{ regularity of the first two terms, we obtain that }
\]
If we make use of the Moser-type inequality, (1.41) and the regularity of \( \rho \), we utilize the positivity of \( \rho \) to deduce that
\[
|\mathcal{R}^{(1)}| + |\mathcal{R}^{(2)}| \leq C(|\mathcal{E}^s(t)|^2 + \|\mathcal{E}^0(t)\|^2) + C\|\hat{\varphi}_{\eta}^0 \mathcal{U}^0\|^2.
\]
for some constant \( C \), where (1.42). In view of the regularity of \( \rho^0, u^0, q^0 \), the positivity of \( N^\epsilon + \rho^0 \) and Cauchy-Schwarz’s inequality, the first two terms \( \mathcal{R}^{(1)} \) and \( \mathcal{R}^{(2)} \) can be controlled by
\[
|\mathcal{R}^{(1)}| + |\mathcal{R}^{(2)}| \leq C\|\mathcal{E}^s(t)\|^2 + \|\mathcal{E}^0(t)\|^2 + C\|\hat{\varphi}_{\eta}^0 \mathcal{U}^0\|^2.
\]
For the term \( \mathcal{R}^{(3)} \), by the regularity of \( \rho^0 \) and \( u^0 \), the positivity of \( N^\epsilon + \rho^0 \), Cauchy-Schwarz’s inequality and (1.42), we see that
\[
|\mathcal{R}^{(3)}| \leq C\|\mathcal{E}^s(t)\|^2 + C\|\hat{\varphi}_{\eta}^0 \mathcal{U}^0\|^2.
\]
For the last term \( \mathcal{R}^{(4)} \), we utilize the positivity of \( N^\epsilon + \rho^0 \) to deduce that
\[
|\mathcal{R}^{(4)}| = \left| \left\langle \frac{\epsilon}{N^\epsilon + \rho^0} \hat{\varphi}_{\eta}^0 \mathcal{J}_1 \cdot \hat{\varphi}_{\eta}^0 \mathcal{U}^0 \right\rangle \right| + \left| \left\langle \mathcal{H}^{(6)}, \hat{\varphi}_{\eta}^0 \mathcal{U}^0 \right\rangle \right| + |\mathcal{R}^{(4)}|
\]
\[
\leq \frac{1}{16} |\mathcal{J}_1 \cdot \hat{\varphi}_{\eta}^0 \mathcal{U}^0|^2 + C|\hat{\varphi}_{\eta}^0 \mathcal{U}^0|^2 + \left| \left\langle \mathcal{H}^{(6)}, \hat{\varphi}_{\eta}^0 \mathcal{U}^0 \right\rangle \right| + |\mathcal{R}^{(4)}|,
\]
where
\[
\mathcal{H}^{(6)} = \epsilon \hat{\varphi}_{\eta}^0 \left\langle \mathcal{J}_1 \left( \frac{\mathbf{J}_1}{N^\epsilon + \rho^0} \right) - \frac{\epsilon}{N^\epsilon + \rho^0} \hat{\varphi}_{\eta}^0 \mathcal{J}_1 \right\rangle
\]
and
\[
\mathcal{R}^{(4)} = \epsilon \left\langle \hat{\varphi}_{\eta}^0 \left( \frac{\mathbf{q}^0}{N^\epsilon + \rho^0} \right), \hat{\varphi}_{\eta}^0 \mathcal{U}^0 \right\rangle.
\]
If we make use of the Moser-type inequality, (1.41) and the regularity of \( \rho^0 \) and \( q^0 \), we obtain that
\[
|\left\langle \mathcal{H}^{(6)}, \hat{\varphi}_{\eta}^0 \mathcal{U}^0 \right\rangle| \leq \|\mathcal{H}^{(6)}\| \cdot \|\hat{\varphi}_{\eta}^0 \mathcal{U}^0\|^2
\]
\[
\leq C \left[ \left| D_{\hat{\varphi}_{\eta}^0} \left( \frac{1}{N^\epsilon + \rho^0} \right) \right| \|J_1\|_{H^s-1} + \|J_1\|_{H^\infty} \left| \frac{1}{N^\epsilon + \rho^0} \right| \|\hat{\varphi}_{\eta}^0 \mathcal{U}^0\|^2 \right].
\]
inequality, we find that

\[
\eta_6 \left| \mathbf{J}_x \right|^2 \leq C \eta_6 \left( \left| \mathbf{E}'(t) \right|_2^{2(s+1)} + 1 \right) \left| \mathbf{\tilde{C}}_2 \mathbf{U}' \right|^2
\]

(3.26)

for any \( \eta_6 > 0 \). Recalling the regularity of \( \mathbf{u}^0 \) and \( \mathbf{q}^0 \), (1.39) and (1.41) and Hölder’s inequality, we find that

\[
\left| R^{(4)} \right| \leq C \left( \left| \mathbf{E}'(t) \right|_2^{2s} + 1 \right) \left| \mathbf{\tilde{C}}_2 \mathbf{U}' \right|^2 + C \epsilon^2.
\]

(3.27)

Substituting (3.23)–(3.27) into (3.22), we conclude that

\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt} \left( \mathbf{\tilde{C}}_2^0 \mathbf{U}'^{\alpha}, \mathbf{\tilde{C}}_2^0 \mathbf{U}'^{\beta} \right) + \int \frac{\mu}{N^r + \rho^0} \left| \nabla \mathbf{\tilde{C}}_2^r \mathbf{U}' \right|^2 dx - \left( \eta_3 + \eta_4 + \eta_5 \right) \left| \nabla \mathbf{U}' \right|^2 dx \\
&\leq \tilde{C}_\eta \left( \left| \mathbf{E}'(t) \right|_2^{2(s+1)} + 1 \right) \left| \mathbf{E}'(t) \right|_2^2 \\
&+ \eta_6 \left| \mathbf{\tilde{C}}_2^r \nabla \mathbf{\Theta}' \right|^2 + \left( \eta_6 + \frac{1}{16} \right) \left| \mathbf{J}_x \right|^2 + C \epsilon^2
\end{align*}
\]

(3.28)

for some constant \( \tilde{C}_\eta > 0 \) depending on \( \eta_i \) (\( i = 2, \ldots, 6 \)).

Applying the operator \( \mathbf{\tilde{C}}_2^0 \) to (2.3), multiplying the resulting equation by \( \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \), and integrating over \( \Omega \), we arrive at

\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt} \left( \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}'^{\alpha}, \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}'^{\beta} \right) + \left( \mathbf{\tilde{C}}_2^0 \{ \left[ (\mathbf{U}' + \mathbf{u}^0) \cdot \nabla \right] \mathbf{\Theta}' \}, \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&+ \left( \mathbf{\tilde{C}}_2^0 \{ (\mathbf{\Theta}' + \theta^0) \text{ div } \mathbf{u}' \}, \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) - \left( \mathbf{\tilde{C}}_2^0 \left( \frac{\kappa}{N^r + \rho^0} \Delta \mathbf{\Theta}' \right), \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&= - \left( \mathbf{\tilde{C}}_2^0 \{ (\mathbf{U}' \cdot \nabla) \theta^0 - \mathbf{\Theta}' \text{ div } \mathbf{u}' \}, \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&+ \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{\kappa}{N^r + \rho^0} \Delta \theta^0 \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&+ \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{2\mu}{N^r + \rho^0} - \frac{2\mu}{\rho^0} \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&+ \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{\lambda}{N^r + \rho^0} - \frac{\lambda}{\rho^0} \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&+ \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{1}{N^r + \rho^0} - \frac{1}{\rho^0} \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&+ \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{2\mu}{N^r + \rho^0} \text{ div } \mathbf{u}^0 \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&+ \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{4\mu}{N^r + \rho^0} \Delta \mathbf{u}' \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&+ \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{2\lambda}{N^r + \rho^0} \text{ div } \mathbf{u}' \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&+ \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{J_0^2}{N^r + \rho^0} \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) - \left( \mathbf{\tilde{C}}_2^0 \left( \frac{(\mathbf{\Theta}')^4}{N^r + \rho^0} \right), \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&- \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{4(\mathbf{\Theta}')^3}{N^r + \rho^0} \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) - \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{6(\mathbf{\Theta}')^2(\theta^0)^2}{N^r + \rho^0} \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right) \\
&- \left( \mathbf{\tilde{C}}_2^0 \left[ \frac{4(\mathbf{\Theta}')(\theta^0)^3}{N^r + \rho^0} \right], \mathbf{\tilde{C}}_2^0 \mathbf{\Theta}' \right)
\end{align*}
\]
\[
: = \sum_{i=1}^{13} S^{(i)}. \tag{3.29}
\]

We first bound the terms on the left-hand side of (3.29). Similar to (3.30), we have
\[
\langle \tilde{c}_x^\alpha ([(U^\nu + u^0) \cdot \nabla] \Theta^\nu), \tilde{c}_x^\alpha \Theta^\nu \rangle \\
= \langle [(U^\nu + u^0) \cdot \nabla] \tilde{c}_x^\alpha \Theta^\nu, \tilde{c}_x^\alpha \Theta^\nu \rangle + \langle \mathcal{H}^{(7)}, \tilde{c}_x^\alpha \Theta^\nu \rangle \\
= -\frac{1}{2} \langle \text{div} (U^\nu + u^0) \tilde{c}_x^\alpha \Theta^\nu, \tilde{c}_x^\alpha \Theta^\nu \rangle + \langle \mathcal{H}^{(7)}, \tilde{c}_x^\alpha \Theta^\nu \rangle \\
\leq C(\|\mathcal{E}^\nu (t)\|_s + 1) \|\tilde{c}_x^\alpha \Theta^\nu\|^2 + \|\mathcal{H}^{(7)}\|^2, \tag{3.30}
\]
where the commutator
\[
\mathcal{H}^{(7)} = \tilde{c}_x^\alpha ([(U^\nu + u^0) \cdot \nabla] \Theta^\nu) - [(U^\nu + u^0) \cdot \nabla] \tilde{c}_x^\alpha \Theta^\nu
\]
can be bounded by
\[
\|\mathcal{H}^{(7)}\| \leq C(\|D^1_\nu (U^\nu + u^0)\|_{L^\infty} \|D^2_x \Theta^\nu\| + \|D^1_\nu \Theta^\nu\|_{L^\infty} \|D^2_x (U^\nu + u^0)\|) \\
\leq C\|\mathcal{E}^\nu (t)\|^2_s + C\|\mathcal{E}^\nu (t)\|_s. \tag{3.31}
\]
The second term on the left-hand side of (3.29) can bounded, similarly to (3.31), as follows.
\[
\begin{align*}
& \leq \left| \langle \tilde{c}_x^\alpha ((\Theta^\nu + \rho^0) \text{div} U^\nu), \tilde{c}_x^\alpha \Theta^\nu \rangle \right| \\
& \leq \left| \langle (\Theta^\nu + \rho^0) \tilde{c}_x^\alpha \text{div} U^\nu, \tilde{c}_x^\alpha \Theta^\nu \rangle \right| + \left| \langle \mathcal{H}^{(8)}, \tilde{c}_x^\alpha \Theta^\nu \rangle \right| \\
& \leq \eta \|\nabla \tilde{c}_x^\alpha U^\nu\|^2 + C \eta \|\tilde{c}_x^\alpha \Theta^\nu\|^2 + \|\mathcal{H}^{(8)}\|^2 \quad \text{for any } \eta > 0, \tag{3.32}
\end{align*}
\]
where the commutator
\[
\mathcal{H}^{(8)} = \tilde{c}_x^\alpha ((\Theta^\nu + \rho^0) \text{div} U^\nu) - (\Theta^\nu + \rho^0) \tilde{c}_x^\alpha \text{div} U^\nu
\]
can be controlled by
\[
\|\mathcal{H}^{(8)}\| \leq C(\|D^1_\nu (\Theta^\nu + \rho^0)\|_{L^\infty} \|D^2_x U^\nu\| + \|D^1_\nu U^\nu\|_{L^\infty} \|D^2_x (\Theta^\nu + \rho^0)\|) \\
\leq C\|\mathcal{E}^\nu (t)\|^2_s + C\|\mathcal{E}^\nu (t)\|_s. \tag{3.33}
\]
For the fourth term on the left-hand side of (3.29), we integrate it by parts to deduce that
\[
-\kappa \left< \tilde{c}_x^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \Delta U^\nu \right), \tilde{c}_x^\alpha \Theta^\nu \right> \\
= -\kappa \left< \frac{1}{N^\epsilon + \rho^0} \Delta \tilde{c}_x^\alpha \Theta^\nu, \tilde{c}_x^\alpha \Theta^\nu \right> - \kappa \langle \mathcal{H}^{(9)}, \tilde{c}_x^\alpha \Theta^\nu \rangle \\
= \kappa \left< \frac{1}{N^\epsilon + \rho^0} \nabla \tilde{c}_x^\alpha \Theta^\nu, \nabla \tilde{c}_x^\alpha \Theta^\nu \right> \\
+ \kappa \left< \nabla \left( \frac{1}{N^\epsilon + \rho^0} \right) \Delta \tilde{c}_x^\alpha \Theta^\nu, \tilde{c}_x^\alpha \Theta^\nu \right> - \kappa \langle \mathcal{H}^{(9)}, \tilde{c}_x^\alpha \Theta^\nu \rangle, \tag{3.34}
\]
where
\[
\mathcal{H}^{(9)} = \tilde{c}_x^\alpha \left( \frac{1}{N^\epsilon + \rho^0} \Delta \Theta^\nu \right) - \frac{1}{N^\epsilon + \rho^0} \Delta \tilde{c}_x^\alpha \Theta^\nu.
\]
By the Moser-type and H"older inequalities, the regularity of $\rho^0$, the positivity of $N^\epsilon + \rho_0$ and (1.11), we find that
\[
\langle H(9), \frac{1}{N^\epsilon + \rho^0} \rangle \leq \|H(9)\| \cdot \|\frac{1}{N^\epsilon + \rho^0}\|
\]
\[
\leq C \left( \left\| D^1 \left( \frac{1}{N^\epsilon + \rho^0} \right) \right\| L^2 \left\| \Delta \Theta^\epsilon \right\|_{L^2} + \left\| \Delta \Theta^\epsilon \right\|_{L^2} \left\| \frac{1}{N^\epsilon + \rho^0} \right\|_{L^2} \langle \Theta^\epsilon, \Theta^\epsilon \rangle \right)
\leq \eta_8 \|\nabla \Theta^\epsilon\|^2 + C_{\eta_9} (\|E^\epsilon(t)\|^2 + 1)\|E^\epsilon(t)\|^2
\]
(3.35)
and
\[
\|\nabla \left( \frac{1}{N^\epsilon + \rho^0} \right) \nabla \Theta^\epsilon, \Theta^\epsilon \rangle \|
\leq \eta_9 \|\nabla \Theta^\epsilon\|^2 + C_{\eta_9} \left\| \nabla \left( \frac{1}{N^\epsilon + \rho^0} \right) \right\| L^2 \|\Theta^\epsilon\|^2
\leq \eta_9 \|\nabla \Theta^\epsilon\|^2 + C_{\eta_9} (\|E^\epsilon(t)\|^2 + 1)\|\Theta^\epsilon\|^2
\]
(3.36)
for any $\eta_9 > 0$ and $\eta_9 > 0$, where we have used the assumption $s > n/2 + 2$ in the derivation of (3.35) and the imbedding $H^1(\Omega) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $l > n/2$.

Now, we estimate every term on the right-hand side of (3.29). By virtue of the regularity of $\Theta^0$ and $u^0$, and Cauchy-Schwarz’s inequality, the first term $S(1)$ can be estimated as follows.
\[
|S(1)| \leq C (\|E^\epsilon(t)\|^2 + 1) (\|\varphi^\alpha \Theta^\epsilon\|^2 + \|\tilde{\varphi}^\alpha U^\epsilon\|^2).
\]
(3.37)
For the terms $S(i)$ ($i = 2, 3, 4, 5$), we utilize the regularity of $\rho^0$, $u^0$ and $q^0$, the positivity of $N^\epsilon + \rho^0$, Cauchy-Schwarz’s inequality and (1.12) to deduce that
\[
|S(2)| + |S(3)| + |S(4)| + |S(5)| \leq C (\|E^\epsilon(t)\|^2 + \|E^\epsilon(t)\|^2) + C \|\varphi^\alpha \Theta^\epsilon\|^2,
\]
while for the sixth term $S(6)$, we integrate by parts, use Cauchy-Schwarz’s inequality and the positivity of $\Theta^\epsilon + \rho^0$ to obtain that
\[
|S(6)| = \left| \int \varphi^\alpha \left( \frac{2\mu}{N^\epsilon + \rho^0} [\partial \xi^\epsilon]^2 + \frac{\lambda}{N^\epsilon + \rho^0} [\partial \epsilon^\xi^\epsilon]^2 \right) , \varphi^\alpha \Theta^\epsilon \right|
\leq \eta_{10} \|\nabla \varphi^\alpha \Theta^\epsilon\|^2 + C_{\eta_{10}} (\|E^\epsilon(t)\|^2 + \|E^\epsilon(t)\|^2 + 1) + \|E^\epsilon(t)\|^2 + 1)
\]
(3.39)
for any $\eta_{10} > 0$, where $\alpha = (1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$. Similarly, we have
\[
|S(7)| + |S(8)| \leq \eta_{11} \|\nabla \varphi^\alpha \Theta^\epsilon\|^2 + C_{\eta_{11}} (\|E^\epsilon(t)\|^2 + \|E^\epsilon(t)\|^2 + 1)
\]
(3.40)
for any $\eta_{11} > 0$.
For the ninth term $S(9)$, similar to $R(4)$, we have
\[
|S(9)| \leq \sigma_0 \|\varphi^\alpha J^0_0\|^2 + C (\|E^\epsilon(t)\|^2 + 1) (\|\varphi^\alpha N^\epsilon\|^2 + \|\varphi^\alpha \Theta^\epsilon\|^2).
\]
(3.41)
for any $\sigma_0 > 0$. For the tenth term $S(10)$, we have
\[
S(10) = \left\langle \frac{1}{N^\epsilon + \rho^0} \varphi^\alpha \Theta^\epsilon^4 , \varphi^\alpha \Theta^\epsilon \right\rangle + \langle H(10), \varphi^\alpha \Theta^\epsilon \rangle,
\]
where
\[
H(10) = \varphi^\alpha \left\langle \left( \frac{\Theta^\epsilon}{N^\epsilon + \rho^0} \right)^4 - \frac{1}{N^\epsilon + \rho^0} \right\rangle.
\]
By the positivity of $\Theta^r + \rho^0$, and the properties of $\Theta^r$ obtained in Proposition 2.1, we have
\[ \left| \frac{1}{N^r + \rho^0} \tilde{\Theta}^a_{x} (\Theta^r)^4, \tilde{\Theta}^a_{x} \Theta^r \right| \leq C (|\Theta^r(t)|^8_s + |\tilde{\Theta}^a_{x} \Theta^r|^2). \]

By the Cauchy-Schwarz, Moser-type inequalities and Sobolev embedding theorem, we infer that
\[
\left| \langle H^{(10)}, \tilde{\Theta}^a_{x} \Theta^r \rangle \right| \leq \| H^{(10)} \| \cdot \| \tilde{\Theta}^a_{x} \Theta^r \|
\leq C \left[ \left\| D_{x} \left( \frac{1}{N^r + \rho^0} \right) \right\| \| (\Theta^r)^4 \|_{L^\infty} \left( \frac{1}{N^r + \rho^0} \right) \right] \left\| \tilde{\Theta}^a_{x} \Theta^r \right\|
\leq C (|\Theta^r(t)|^8_s + |\tilde{\Theta}^a_{x} \Theta^r|^2 + |\tilde{\Theta}^a_{x} \Theta^r|^2). \]

Hence, we obtain
\[ |S^{(10)}| \leq C (|\Theta^r(t)|^8_s + |\tilde{\Theta}^a_{x} \Theta^r|^2 + |\tilde{\Theta}^a_{x} \Theta^r|^2). \]

The terms $S^{(11)}$, $S^{(12)}$ and $S^{(13)}$ can be bounded in a way similar to that for $S^{(10)}$, and hence we get
\[ |S^{(11)}| + |S^{(12)}| + |S^{(13)}| \leq C (|\Theta^r(t)|^8_s + |\tilde{\Theta}^a_{x} \Theta^r|^2 + |\tilde{\Theta}^a_{x} \Theta^r|^2). \]

Substituting (3.30)–(3.33) into (3.29), we conclude that
\[ \frac{1}{2} \frac{d}{dt} \left( \tilde{\Theta}^a_{x} \Theta^r, \tilde{\Theta}^a_{x} \Theta^r \right) + \kappa \left( \frac{1}{N^r + \rho^0} \nabla \tilde{\Theta}^a_{x} \Theta^r, \nabla \tilde{\Theta}^a_{x} \Theta^r \right)
\leq C_{0} \left( |\tilde{\Theta}^a_{x} \Theta^r|^2 + |\tilde{\Theta}^a_{x} \Theta^r|^2 \right) + |\tilde{\Theta}^a_{x} \Theta^r|^2 + \eta \| \nabla \tilde{\Theta}^a_{x} \Theta^r \|^2 \]

for some constant $C_{0} > 0$ depending on $\eta_i (i = 8, 9, 10, 11)$.

Applying the operator $\tilde{\Theta}^a_{x}$ to (2.4) and (2.5), multiplying the resulting equations by $\tilde{\Theta}^a_{x} J_0$ and $\tilde{\Theta}^a_{x} J_1$ respectively, and integrating them over $\mathbb{T}^n$, one obtains that
\[ \frac{1}{2} \frac{d}{dt} \langle \tilde{\Theta}^a_{x} \Theta^r, \tilde{\Theta}^a_{x} \Theta^r \rangle + \langle \tilde{\Theta}^a_{x} \Theta^r \rangle \tilde{\Theta}^a_{x} \Theta^r + \langle \tilde{\Theta}^a_{x} \Theta^r \rangle \tilde{\Theta}^a_{x} \Theta^r \]
\[ \leq C_{1} \left( |\tilde{\Theta}^a_{x} \Theta^r|^2 + |\tilde{\Theta}^a_{x} \Theta^r|^2 \right) + |\tilde{\Theta}^a_{x} \Theta^r|^2 + \eta \| \nabla \tilde{\Theta}^a_{x} \Theta^r \|^2 \]

for any $\sigma_1 > 0$. Similarly, we have
\[ |T^{(2)}| + |T^{(3)}| + |T^{(4)}| \leq \sigma_2 \| \tilde{\Theta}^a_{x} \Theta^r \|^2 + C_{\sigma_2} (|\Theta^r(t)|^6_s + |\Theta^r(t)|^2_s). \]

for any $\sigma_2 > 0$. 

For the term $T^{(5)}$, one gets from the regularity of $\text{div} q$ and Cauchy-Schwarz’s inequality that
\[
|T^{(5)}| \leq C \epsilon^2 + C \|\partial_x^\alpha J_0^\epsilon\|^2. \tag{3.48}
\]
Pulling (3.46)–(3.48) into (3.45), we find that
\[
\frac{1}{2} \frac{d}{dt}(1) + (1 - \sigma_1 - \sigma_2)\|\partial_x^\alpha J_0^\epsilon\|^2 + \|\partial_x^\alpha J_1^\epsilon\|^2 \\
\leq C(\|\mathcal{E}^\epsilon(t)\|_s^8 + \|\mathcal{E}^\epsilon(t)\|_s^2) + C\epsilon^2. \tag{3.49}
\]
Combining (3.10), (3.28), and (3.44) with (3.49), summing up $\alpha$ with $0 \leq |\alpha| \leq s$, using the fact that $N^\epsilon + \rho^0 \geq N + \tilde{\rho} > 0$, choosing $\eta_i$ ($i = 1, \ldots, 11$), and then $\sigma_0, \sigma_1, \sigma_2$ sufficiently small, and noticing that $s > n/2 + 2$ is an integer, we obtain (3.3). This completes the proof of Lemma 3.2.

With the estimate (3.3) in hand, we can now prove Proposition 3.1.

**Proof of Proposition 3.1.** As in [2,3], we introduce an $\epsilon$-weighted energy functional
\[
\Gamma^\epsilon(t) = \|\mathcal{E}^\epsilon(t)\|_s^2.
\]
Then, it follows from (3.3) that there exists a constant $\epsilon_1 > 0$ depending only on $T$, such that for any $\epsilon \in (0, \epsilon_1]$ and any $t \in (0, T]$,
\[
\Gamma^\epsilon(t) \leq C T^\epsilon(t = 0) + C \int_0^t \left\{\left(\Gamma^\epsilon\right)^{2(s+1)} + 1\right\}\Gamma^\epsilon(\tau)d\tau + C\epsilon^2. \tag{3.50}
\]
Thus, applying the Gronwall lemma to (3.50), and keeping in mind that $\Gamma^\epsilon(t = 0) \leq C\epsilon^2$ and Proposition 3.1 we find that there exist a $0 < T_1 < 1$ and any $\epsilon > 0$, such that $T^\epsilon \geq T_1$ for all $\epsilon \in (0, \epsilon_1]$ and $\Gamma^\epsilon(t) \leq C\epsilon^2$ for all $t \in (0, T_1]$. Therefore, the desired a priori estimate (3.2) holds. Moreover, by the standard continuation induction argument, we can extend $T^\epsilon \geq T_0$ to any $T_0 < T_*$.

**Acknowledgements:** We would like to express our thanks to the anonymous referees for their valuable comments which lead to substantial improvements of the original manuscript. The first author is supported by the National Basic Research Program under the Grant 2011CB309705, NSFC (Grant Nos. 11229101, 11371065), and Beijing Center for Mathematics and Information Interdisciplinary Sciences. The second author is supported in part by NSFC (Grant No. 11271184) and PAPD. The third author is supported by NSFC (Grant No.11171213), Shanghai Rising Star Program No.12QA1401600 and Shanghai Committee of Science and Technology (Grant No. 15XD1502300).

**References**

[1] L. Hörmander, Lectures on Nonlinear Hyperbolic Differential Equations, Springer-Verlag, 1997.

[2] S. Jiang, F.-C. Li, Rigorous derivation of the compressible magnetohydrodynamic equations from the electromagnetic fluid system, Nonlinearity 25 (2012), 1735-1752.

[3] S. Jiang, F.-C. Li, Convergence of the complete electromagnetic fluid system to the full compressible magnetohydrodynamic equations, Asymptot. Anal., accepted.

[4] S. Kawashima, Y. Nikkuni, S. Nishibata, The initial value problem for hyperbolic-elliptic coupled systems and applications to radiation hydrodynamics. Analysis of systems of conservation laws (Aachen, 1997), 87-127, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., 99, Chapman & Hall/CRC, Boca Raton, FL, 1999.
[5] S. Kawashima, Y. Nikkuni, S. Nishibata, Large-time behavior of solutions to hyperbolic-elliptic coupled systems. Arch. Ration. Mech. Anal. 170 (2003), no. 4, 297-329.

[6] S. Klainerman, A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, Comm. Pure Appl. Math., 34 (1981), 481-524.

[7] C. Lattanzio, C. Mascia, D. Serre, Shock waves for radiative hyperbolic-elliptic systems. Indiana Univ. Math. J. 56 (2007), no. 5, 2601-2640.

[8] C.-J. Lin, J.-F. Coulombel, T. Goudon, Shock profiles for non-equilibrium radiating gases. Phys. D 218 (2006), no. 1, 83-94.

[9] C.-J. Lin, Asymptotic stability of rarefaction waves in radiative hydrodynamics. Commun. Math. Sci. 9 (2011), no. 1, 207-223.

[10] A. Majda, Compressible fluid flow and systems of conservation laws in several space variables. Applied Mathematical Sciences, 53. Springer-Verlag, New York, (1984).

[11] D. Mihalas, B. Mihalas, Foundation of Radiation Hydrodynamics, Oxford University Press (1984).

[12] V.B. Moseenkov, Composition of functions in Sobolev spaces, Ukrainian Math. J., 34(1982), 316-319.

[13] T. Nguyen, R.-G. Plaza, K. Zumbrun, Stability of radiative shock profiles for hyperbolic-elliptic coupled systems. Phys. D 239(2010), 428-453.

[14] G. Pomraning, The Equations of Radiation Hydrodynamics, Pergamon Press (1973).

[15] R. Danchin, B. Ducomet, On a simplified model for radiating flows. J. Evol. Equ. 14 (2014), no. 1, 155-195.

[16] C. Rohde, W.-A. Yong, The nonrelativistic limit in radiation hydrodynamics. I. Weak entropy solutions for a model problem. J. Differential Equations 234 (2007), no. 1, 91-109.

[17] C. Rohde, F. Xie, Decay rates to viscous contact waves for a 1D compressible radiation hydrodynamics model. Math. Models Methods Appl. Sci. 23 (2013), no. 3, 441-469.

[18] J. Wang, F. Xie, Asymptotic stability of viscous contact wave for the one-dimensional compressible viscous gas with radiation. Nonlinear Anal. 74 (2011), no. 12, 4138-4151.

[19] J. Wang, F. Xie, Asymptotic stability of viscous contact wave for the 1D radiation hydrodynamics system. J. Differential Equations 251 (2011), no. 4-5, 1030-1055.

[20] W.-J. Wang, F. Xie, The initial value problem for a multi-dimensional radiation hydrodynamics model with viscosity. Math. Methods Appl. Sci. 34 (2011), no. 7, 776-791.

[21] A.I. Vol’pert, S.I. Hudjaev, On the Cauchy problem for composite systems of nonlinear differential equations, Math. USSR-Sb. 16 (1972), 517-544.