ON THE EXPONENT OF BOGOMOLOV MULTIPLIERS

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Abstract. We prove that if $G$ is a finite group, then the exponent of its Bogomolov multiplier divides the exponent of $G$ in the following four cases: (i) $G$ is metabelian, (ii) $\exp G = 4$, (iii) $G$ is nilpotent of class $\leq 5$, or (iv) $G$ is a 4-Engel group.

1. Introduction

Let $G$ be a group given by a free presentation $G = F/R$. Then the Schur multiplier of $G$ can be defined via Hopf’s formula as $M(G) = ([F, F] \cap R)/[F, R]$. It is known that if $G$ is a finite group, then $M(G) \cong H_2(G, \mathbb{Z}) \cong H^2(G, \mathbb{Q}/\mathbb{Z})$; see, for example, Beyl and Tappe’s book [3] for details and applications of Schur multipliers.

A significant part of the theory of Schur multipliers consists of estimating their size, rank, or exponent. Focussing on the latter one, it was already Schur [23] who showed that if $G$ is a finite group, then $(\exp M(G))^2$ divides $|G|$. On the other hand, it was shown in [14] that, for every finite group $G$, the exponent of $M(G)$ can be bounded in terms of $\exp G$ only. Good bounds of this kind are however still out of reach. In practice it often happens that $\exp M(G)$ divides $\exp G$. One may conjecture that this is always the case, yet a counterexample of exponent 4 with Schur multiplier of exponent 8 was constructed by Bayes, Kautsky and Wamsley [2]. We mention that there are known examples of odd order groups $G$ with $\exp M(G) > \exp G$, though it seems plausible that they exist. On the other hand, $\exp M(G)$ always divides $\exp G$ at least when $G$ is of one of the following types: nilpotent of class $\leq 3$ [12, 15], powerful $p$-group [13], a $p$-group of maximal class [17]. In addition to that, we have the following.

Theorem 1.1 ([14, 15, 16]). Let $G$ be a group of finite exponent.

1. If $G$ is nilpotent of class $\leq 4$, then $\exp M(G)$ divides $2 \cdot \exp G$.
2. If $G$ is a 4-Engel group, then $\exp M(G)$ divides $10 \cdot \exp G$.
3. If $\exp G = 4$, then $\exp M(G)$ divides 8.
4. If $G$ is metabelian, then $\exp M(G)$ divides $(\exp G)^2$.

There are numerous other estimates for $\exp M(G)$, but we do not go further into this here, see, for example, Sambonet’s paper [22] for a short review of such results.

Given a free presentation $G = F/R$, define

\[ B_0(G) = ([F, F] \cap R)/([K(F) \cap R]), \]
\[ G \wr G = [F, F]/([K(F) \cap R]), \]

where $K(F)$ is the set of commutators in $F$. It is shown in [18] that if $G$ is a finite group and $V$ a faithful representation of $G$ over $\mathbb{C}$, then the dual of $B_0(G)$ is naturally isomorphic to the unramified Brauer group $H^2_{nr}(\mathbb{C}(V), \mathbb{Q}/\mathbb{Z})$ introduced...
by Artin and Mumford [1]. This invariant represents an obstruction to Noether’s problem [20] asking as to whether the field \( \mathbb{C}(V)^G \) is purely transcendental over \( \mathbb{C} \). The above mentioned result of [18] is based on a result of Bogomolov [4] who showed that \( H^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z}) \) is naturally isomorphic to the intersection of the kernels of restriction maps \( H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z}) \), where \( A \) runs through all (two-generator) abelian subgroups of \( G \). The latter group is also known as the Bogomolov multiplier of \( G \). Here we use the same name for \( B_0(G) \).

Our main result is the following.

**Theorem 1.2.** Let \( G \) be a group of finite exponent. If \( G \) satisfies one of the following properties, then \( \exp(G \wr G) \) divides \( \exp G \):

1. Nilpotent of class \( \leq 5 \),
2. Metabelian,
3. 4-Engel,
4. \( \exp G = 4 \).

This result thus complements Theorem 1.1. It also implies, that, in the cases listed in Theorem 1.2, it always happens that \( \exp B_0(G) \) divides \( \exp G \). Note that the question of determining \( \exp B_0(G) \) was recently addressed by Garcia-Rodriguez, Jaikin-Zapirain, and Jezernik [6, Theorem 6].

In view of the above result and further extensive computatio nal evidence we pose the following conjecture:

**Conjecture 1.3.** Let \( G \) be a finite group. Then \( \exp B_0(G) \) divides \( \exp G \), and thus, in particular, \( \exp M(G) \) divides \( (\exp G)^2 \).

In order to justify this conjecture, we first mention that Saltman [21] showed that for every \( p > 2 \) there exists a \( p \)-group of exponent \( p \) with non-trivial Bogomolov multiplier of exponent \( p \), thus the bound given in the conjecture is sharp. On the other hand, if \( \exp G \) is not prime, then \( B_0(G) \) is usually small (or of small exponent), compared to \( G \). Fernandez-Alcober and Jezernik [5] recently constructed examples of finite \( p \)-groups \( G \) of maximal class with \( \exp B_0(G) \approx \sqrt{\exp G} \), and it appears that this might be close to the worst case. In fact, we have not been able to find a group \( G \) with \( \exp G \leq \exp G_n \) so it may be possible that even the stronger conjecture with \( B_0(G) \) replaced by \( G \wr G \) might hold true.

The proof of Theorem 1.2 goes roughly as follows. First, one may assume without loss of generality that \( G \) is a finite group. By [11], there exists a so-called CP cover \( H \) of \( G \) (see Section 2 for the details) whose main feature is that \( G \wr G \) is isomorphic to \( [H, H] \), and all commutator relations of \( G \) lift to commutator relations in \( H \). The calculations are then performed in \([H, H] \). In the metabelian case, the proof is fairly straightforward. In the remaining cases, it relies on a careful examination of the power structure of the lower central series of \( H \), which uses information on the free groups in a given variety. For class \( \leq 5 \) groups, this can be obtained using elementary commutator calculus. In the exponent 4 and 4-Engel case, we mainly use M. Hall’s description of the free 3-generator group of exponent 4 [8], and Nickel’s computations of free 4-Engel groups of low ranks [19], along with Havas and Vaughan-Lee’s proof of local nilpotency of 4-Engel groups [9].

2. Preliminaries

2.1. **CP covers.** Let \( G \) be a group and \( Z \) a \( G \)-module. Denote by \( e = (\chi, H, \pi) \) the extension

\[
1 \longrightarrow Z \xrightarrow{\chi} H \xrightarrow{\pi} G \longrightarrow 1
\]

of \( Z \) by \( G \). Following [18], we say that \( e \) is a **CP extension** if commuting pairs of elements of \( G \) have commuting lifts in \( H \). A stem central CP extension of \( Z \) by
$G$, where $|Z| = |B_0(G)|$, is called a CP cover of $G$. CP covers are analogs of the usual covers in the theory of Schur multipliers. It is proved in [11] that every finite group has a CP cover. It also follows from [11] that if $\epsilon = (\chi, H, \pi)$ of the above form is a CP cover of $G$, then $Z^1 \cap K(H) = 1$. This in particular implies that any commutator law satisfied by $G$ is also satisfied by $H$.

### 2.2. Collection process.

We recall [7, Theorem 11.2.4] that Hall’s Basis Theorem implies that if $F$ is a free nilpotent group of class $c$ and $a, b \in F$, then the word $(ab)_n$, where $n$ is a non-negative integer, can be written uniquely as a product $c_1^{n_1}c_2^{n_2} \cdots c_t^{n_t}$, where $c_i$ are basic commutators in $\{a, b\}$ of weights $1, 2, \ldots, c$, and $n_i = b_1(n_1) + b_2(n_2) + \cdots + b_r(n_r)$, where $r$ is the weight of $c_i$ and $b_i$ are non-negative integers not depending on $n$. Specifically, we will be interested in the case when $F$ is free nilpotent of class 6. We need to determine the coefficients $b_i$ explicitly, and this can be done using the collection process described in [7, Section 12.3]. We omit the details regarding calculations, and only record the values of $b_i$ for all basic commutators of weight $\leq 6$ in Table 1.

| Commutator $c_i$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $b_6$ |
|------------------|-------|-------|-------|-------|-------|-------|
| $a$              | 1     |       |       |       |       |       |
| $b$              | 1     |       |       |       |       |       |
| $[b, a]$         |       | 1     |       |       |       |       |
| $[b, a, a]$      |       |       | 1     |       |       |       |
| $[b, a, b]$      |       |       | 2     | 2     |       |       |
| $[b, a, a, a]$   |       |       |       | 1     |       |       |
| $[b, a, a, b]$   |       |       |       | 3     | 3     |       |
| $[b, a, b, b]$   |       |       |       | 2     | 3     |       |
| $[b, a, a, [b, a]]$ |       |       |       | 1     | 7     | 6     |
| $[b, a, b, [b, a]]$ |       |       |       | 6     | 18    | 12    |
| $[b, a, a, a, a]$ |       |       |       | 1     |       |       |
| $[b, a, a, a, b]$ |       |       |       |       | 3     | 4     |
| $[b, a, a, b, b]$ |       |       |       |       | 1     | 6     |
| $[b, a, b, b, b]$ |       |       |       |       |       | 4     |
| $[b, a, a, [b, a, a]]$ | 4   | 21    | 36    | 20    |
| $[b, a, a, a, [b, a]]$ | 3   | 13    | 10    |
| $[b, a, a, b, [b, a]]$ | 2   | 24    | 52    | 30    |
| $[b, a, b, b, [b, a]]$ | 3   | 27    | 54    | 30    |
| $[b, a, a, a, a, a]$ |       |       |       |       | 1     |
| $[b, a, a, a, a, b]$ |       |       |       |       | 4     |
| $[b, a, a, a, b, b]$ |       |       |       |       | 3     |
| $[b, a, a, b, b, b]$ |       |       |       |       |       |

Table 1. Coefficients in exponents of $c_i$.

### 3. Proof of Theorem 1.2

In what follows, $G$ will be a group of finite exponent satisfying one of the properties listed in Theorem 1.2. In each of those cases, $G$ is then locally finite. As $B_0$ commutes with direct limits [18], one may assume without loss of generality that $G$ is a finite group; furthermore, Bogomolov’s results [4] imply that we can restrict ourselves to the case when $G$ is a finite $p$-group. Let

$$1 \rightarrow Z \rightarrow H \xrightarrow{\pi} G \rightarrow 1$$
be a CP cover of $G$, where $Z$ is a central subgroup of $H$ with the property that $Z \cong B_0(G)$ and $Z \cap K(H) = 1$. From here on the proof goes on by considering each case separately.

3.1. Metabelian groups. The case of metabelian groups is easy:

**Theorem 3.2.** Let $G$ be a metabelian group of finite exponent. Then the exponent of $G \times G$ divides $\exp G$.

**Proof.** Put $\exp G = e$. Note that $H$ is also metabelian, hence it suffices to prove that $[x, y]^e = 1$ for all $x, y \in H$. We expand $1 = [x, y]^e = [x, y]^e \prod_{k=2}^e [x, k, y]^{(2)}$. Observe that $\prod_{k=2}^e [x, k, y]^{(2)} \in Z$. Furthermore,

$$\prod_{k=2}^e [x, k, y]^{(2)} = \left[ \prod_{k=2}^e [x, k-1, y]^{(2)}, y \right] \in K(H),$$

therefore $[x, y]^e = 1$, as required. □

3.3. Exponent 4. At first we list some properties of groups of exponent 4 that will be used in the proof of this case.

**Lemma 3.4.** Let $G$ be a group of exponent 4 and $a, b, c \in G$.

(a) The group $\langle a, b \rangle$ is nilpotent of class $\leq 5$, $\langle a, b, c \rangle$ is nilpotent of class $\leq 7$, and $\langle a, b, c \rangle$ is nilpotent of class $\leq 4$,

(b) $[a, b, b, a] = 1$,

(c) $[a, b, a, a^2, [a, b]] = 1$,

(d) $[c, [a, b], [a, b], [a, b]] = 1$.

**Proof.** All the above properties can be deduced immediately from a polycyclic presentation of $B(3, 4)$, see also [8]. □

**Theorem 3.5.** Let $G$ be a group of exponent 4. Then the exponent of $G \times G$ divides 4.

**Proof.** As noted above, we may assume without loss of generality that $G$ is a finite group. Choose $x, y, z \in H$.

First note that $[[x, y]^2, x] \in Z \cap K(H) = 1$ by Lemma 3.4, therefore

(3.5.1) $1 = [x, y, x]^2[x, y, x, [x, y]].$

Take $w \in \{x, y\}$. As $\langle x, y \rangle$ is nilpotent of class $\leq 5$, we get $1 = [[x, y]^2, x, w] = [x, y, x, w]^2$. From here it follows that

(3.5.2) $\gamma_4((x, y))^2 = 1.$

We also have that $[x, y, z]^4 = 1$ by [14, Proof of Theorem 2.6]. Now we expand $1 = [x^4, y]$ using [15, Lemma 9]:

$$1 = [x^4, y]$$
$$= [x, y]^4[x, y, x]^6[x, y, x, x]^4[x, y, x, x, x][x, y, x, [x, y]]^{14}$$
$$= [x, y]^4[x, y, x]^2[x, y, x, x].$$

Lemma 3.4 implies that $[x, y, x, x]^2[x, y] = 1$. Expanding this using the class restriction, we obtain $1 = [x, y, x, x]^2[x, y, x, x][x, y, x, [x, y]]$, and this implies $[x, y, x, x, x] = [x, y, x, [x, y]]$. From (3.5.1) and the above expansion we get that $[x, y]^4 = 1$. 


By Lemma 3.4, the group $\langle [x, y], z \rangle$ is metabelian and nilpotent of class $\leq 4$, and we also have that $[z, [x, y], [x, y], [x, y]] = 1$. We now expand $([x, y]z)^4$ using Subsection 2.2 and (3.5.2):

$$([x, y]z)^4 = z^4 [z, [x, y]]^2 [z, [x, y], [x, y], z] [z, [x, y], z, z].$$

Denote $w = [z, [x, y]]^2 [z, [x, y], [x, y], z] [z, [x, y], z, z]$ and consider the following words:

$w_1 = [x, z, z, x^2 z^2 [z, y, x] [z, x, y, y]],$

$w_2 = [y^2 z^2, [z, y, z]],$

$w_3 = [y^2 z^2, [y, x, x] [z, x, x, x]],$

$w_4 = [y^2 z^2, [z, x, z, z] [z, y, y, x, x]],$

$w_5 = [[z, y] [z, x] z^2, z^4 [z, x] [z, y] [z, x, y] [z, y, x] [z, x, x, x] [z, y, y, x, x]],$

$w_6 = [[z, y] [z, x] z^2, [z, y, z] [z, y, z, y] [z, y, y, x, x]],$

$w_7 = [[z, y] [z, x] z^2, [z, x, z, z]].$

The subgroup $(x, y, z)$ is an image of $K = \langle a, b, c | \text{class 7 laws} [x^4, x^2] = [x_1, x_2]^4 = [[x_1, x_2]^2, x_1] = [x_1, 3 [x_2, x_3]] = 1 \rangle$. Expansions of the above defined words in $K$ into products of basic commutators reveals that $w = w_1 w_2 \cdots w_7$. On the other hand, inspection of the presentation of $B(3, 4)$ shows that $w_i \in K(H) \cap Z = 1$ for all $i = 1, 2, \ldots, 7$, therefore $w = 1$. This immediately implies $([x, y]z)^4 = z^4$ for all $x, y, z \in H$. From here it is not difficult to conclude that $\exp \gamma_2(H)$ divides 4, and this finishes the proof. \hfill $\square$

3.6. 4-Engel groups. The aim of this section is to prove

**Theorem 3.7.** Let $G$ be a 4-Engel group of finite exponent. Then the exponent of $G \times G$ divides $\exp G$.

As 4-Engel groups are locally nilpotent [9], the situation can be easily reduced to the case when $G$ is a finite $p$-group. If $p \neq 2, 5$, then it follows from [16] that even $\exp(G \times G)$ divides $\exp G$. This is no longer true when $p = 2$ or $p = 5$. In the case when $p = 5$, there is a short proof of Theorem 3.7. Let $H$ be a $CP$ cover of $G$ and denote $\exp G = 5^e$. Note that $H$ is a 4-Engel 5-group, hence it is regular [10]. It follows from [16] that if $x, y \in H$, then $[x, y]^{5^e} = 1$. Regularity now implies that $\gamma_2(H)$ has exponent dividing $5^e$.

We are thus left with 4-Engel 2-groups. The argument here is more involved. We start with some preliminaries.

**Lemma 3.8.** Let $G$ be a 4-Engel group of exponent $2^e$ and $a, b, c \in G$.

(a) $\gamma_2((a, b)) = \gamma_8((a, b, c))^2 = \gamma_9((a, b, c)) = 1$,

(b) $[a, b, a]^{2^{e-1}} = [a, b, b]^{2^{e-1}} = 1$,

(c) $\gamma_4((a, b, c))^{2^{e-1}} = 1$.

**Proof.** It follows from [19] that if $(a, b, c)$ is a 4-Engel group, then $\gamma_7((a, b)) = (\gamma_8((a, b, c))/\gamma_9((a, b, c)))^{30} = \gamma_9((a, b, c))^{3} = 1$. This proves (a). The fact that the exponent of $\gamma_4((a, b, c))$ divides $2^{e-1}$ is proved in [16, Lemma 4.6], whereas the proof of that Lemma also yields (b). \hfill $\square$

We will also use the following:

**Lemma 3.9** (cf Lemma 4.4 of [16]). Let $G$ be a 4-Engel group, $a, b \in G$ and $n$ a non-negative integer. Then

$$[a^n, b] = [a, b]^n [a, b, a]^{(1)} [a, b, a]^{(2)} [a, b, a]^{(1)} [a, b, a]^{(2)} [a, b, a]^{(1)} + 2^{(1)}.$$
Referring to a polycyclic presentation of the free 4-Engel group with two or three generators obtained in [19], we have:

**Lemma 3.10.** Let $G$ be a 4-Engel group and $a, b \in G$.

(a) $[a, b, a, [b, a, a]] = [b, a, a, a] = 1$,
(b) $[b, a, b, a, b] = [b, a, a, [a, [a, b]]/[b, a, b, a, b]]_2 = 1$,
(c) If $G$ has no elements of order 3, then $[b, a, b, a, b, a] \in \gamma_3((a, b, c)) \gamma_2((a, b, c))$.

**Proposition 3.11.** Let $G$ be 4-Engel of exponent $2^r$. Then

(a) $[[a, b]^{2^{r-1}}, a] = 1$,
(b) $[[c, [a, b], [a, b], [a, b]]]^{2^{r-2}} = 1$.

**Proof.** We may assume that $e > 2$. Let us expand $(ab)^{2^e} = 1$ using Subsection 2.2 and Lemma 3.8. We obtain

\[
[a, a, b, a, b, a] = ([b, a, a, a][b, a, a, b][b, a, b, b][b, a, a, [b, a]]/[b, a, a, a][b, a, a, a])[b, a, a, a, a, b]
\]

× $[b, a, b, [b, a, a][b, a, a, b, b]](2^e + 2^e)$.

We commute this with $a$ and apply class restriction and Lemma 3.10 (a):

\[
[[a, b]^{2^e}, a] = ([b, a, a, b][b, a, b, b][b, a, a, b][b, a, b, a])(2^e).
\]

Using Lemma 3.8 and Lemma 3.9, we obtain after a short calculation that

\[
[[a, b]^{2^r}, a] = [b, a, b, a]^{2^{r-1}}.
\]

The equations (3.11.2) and (3.11.3), together with Lemma 3.10 (b), give $[b, a, b, a, b, a]^{2^{r-2}} = 1$. This immediately yields $\gamma_3((a, b, c))^{2^{r-2}} = 1$. Now replace $b$ by $ab$ in (3.11.1) and use (3.11.1). Expansion under given class restriction gives

\[
1 = ([b, a, b, a][b, a, a, a])^{2^{r-2}}.
\]

If we further replace $b$ by $ab$ in (3.11.4) and apply (3.11.3), we obtain $[b, a, b, a, b, a]^{2^{r-2}} = 1$. Replacing $a$ by $ba$ in this identity, we conclude that also $[b, a, b, a, b, a]^{2^{r-2}} = 1$. Equation (3.11.2) now gives $[[a, b]^{2^e}, a] = 1$. This proves (a), whereas (b) follows directly from Lemma 3.10 (c) and Lemma 3.8 (c), as $e > 2$.

**Theorem 3.12.** Let $G$ be a 4-Engel group of exponent $2^r$. Then the exponent of $G \triangleleft G$ divides $2^r$.

**Proof.** Note that $H$ is a 4-Engel group. Take $x, y, z \in H$ and let $a = x^e, b = y^e, c = z^e$. Proposition 3.11 implies

\[
1 = ([x, y]^{2^{r-1}}, x]
\]

$=[x, y, x]^{2^{r-1}}[x, y, x]^{2^{r-1}}(2^e).$

Equation (3.11.3) implies $[x, y, x, [x, y]]^{2^{r-1}} = 1$, therefore $[x, y, x, [x, y]]^{2^{r-1}} = [x, y, x, [x, y]]^{2^{r-1}} = (2^e)$.

This gives $[x, y, x]^{2^{r-1}} = 1$. From Lemma 3.9 we get

\[
1 = [x^{2^e}, y] = [x, y]^{2^e}[x, y, x]^{2^e}[x, y, x]^{2^e} = [x, y, x]^{2^e} = 1.
\]

Using the above equations, we see that this identity implies $[x, y]^{2^e} = 1$. Now note that the subgroup $\langle x, y, z \rangle$ is nilpotent of class $\leq 5$, since $H$ is 4-Engel. We expand $(x, y, z)^{2^e}$ using collection process (see Subsection 2.2):

\[
([x, y])^{2^e} = z^e = (x, y, z)^{2^e}. 
\]
Note that \([z, [x, y], [x, y], [x, y]](\zeta_3) = [[z, [x, y]](\zeta_3), [x, y], [x, y]] \in K(H)\), and Proposition 3.11 implies that \([z, [x, y], [x, y], [x, y]](\zeta_4) \in Z\). This immediately shows that \([z, [x, y], [x, y], [x, y]](\zeta_4) = 1\). Thus \(([z, [x, y], [x, y], z][z, [x, y], z, z])'(\zeta_4) \in Z\).

Furthermore, the class restriction yields that \(((z, [x, y], [x, y], z'][z, [x, y], z, z])'(\zeta_4) = ([z, [x, y], [x, y], [z, [x, y], z]]'(\zeta_4), z) \in K(H)\), therefore we conclude that \(((z, [x, y], [x, y], z][z, [x, y], z, z])'(\zeta_4) = 1\). Equation (3.12.1) thus gives \((x, y, z)'^2 = z^2\), and induction on the commutator length shows that \(\exp H'\) divides \(2^e\). \(\square\)

### 3.13. Groups of nilpotency class \(\leq 5\).

We will prove the following:

**Theorem 3.14.** Let \(G\) be a group of finite exponent and class \(\leq 5\). Then the exponent of \(G \times G\) divides \(\exp G\).

Again, we may assume that \(G\) is a finite \(p\)-group of class \(\leq 5\) and exponent \(p^e\). The CP cover \(H\) of \(G\) is then also nilpotent of class \(\leq 5\). Let \(x, y \in H\). Assume first that \(p > 2\). Then it follows from the proof of [15, Theorem 13] that \([x, y]^{p^e} = 1\).

As \([H, H]\) is nilpotent of class \(\leq 2\), it is regular, hence \(\exp [H, H]\) divides \(p^e\). Thus we are left with the case when \(G\) is a 2-group. Without loss we can assume that \(e > 2\).

Take \(g, h, k \in G\). Then the expansion of \((gh, k)^{2^e} = 1\) yields

\[1 = [h, k, g](\zeta_2) [h, k, g, [h, k]](\zeta_4 + 2(\zeta_3))[h, k, g, g, g](\zeta_4).\]

If we replace \(h\) by a commutator \([h_1, h_2]\) in the above equation, we get, after renaming the variables, that

\[[h_1, h_2, h_3, h_4]^{2^e - 1} = 1,\]

therefore \(\gamma_4(G)^{2^e - 1} = 1\). By the class restriction this implies \(\gamma_4(H)^{2^e - 1} = 1\). Take now \(x, y, z \in H\). Then

\[1 = [[x, y], z] = [x, y, z]^{2^e}[x, y, z, [x, y]](\zeta_4) = [x, y, z]^{2^e},\]

hence \(\gamma_3(H)^{2^e} = 1\). As \([H, H]\) is nilpotent of class \(\leq 2\), it suffices to prove that \([x, y]^{2^e} = 1\) for all \(x, y \in H\), and then Theorem 3.14 follows.

Take \(x, y \in H\). Then

\[(3.14.1)\]

\[1 = [x^2, y] = [x, y, x][x, y, x, x, x](\zeta_4).\]

If we interchange \(x\) and \(y\) in (3.14.1), we get

\[(3.14.2)\]

\[1 = [x, y][x, y, y](\zeta_4) = [x, y, y, y, x](\zeta_4).\]

Now we replace \(x\) by \(yx\) in (3.14.1) and apply (3.14.1) and (3.14.2). After a short calculation we obtain

\[(3.14.3)\]

\[[x, y]^{2^e} = ([x, y, x, x, y][x, y, x, x, y][x, y, x, y, y][x, y, y, x, x][x, y, y, y, x][x, y, y, y, x])(\zeta_4).\]

As \(H\) is nilpotent of class \(\leq 5\), we have that \([x, y, x, y, x] = [x, y, x, x, x]\) and \([x, y, x, y, y] = [x, y, y, x, x]\). Thus (3.14.3) can be rewritten as

\[(3.14.4)\]

\[[x, y]^{2^e} = ([x, y, x, y, x][x, y, y, x])(\zeta_4).\]

Denote \(f = (\zeta_4)\), and

\[u = [y, x][y, x, y],\]

\[v = [y, x]^{-1}[y, x, y]^{-1}[y, x, x, x][y, x, y, x],\]

\[w = [y^f u, y^f v].\]
We expand $w$:

$$w = [y^f u, v] [y^f u, y^{-f}]^v = [y^f, v]^u [u, y^{-f}]^v = [y^f, v] [y^f, u] [u, y^{-f}] [u, y^{-f}] [u, u, y^{-f}] [u, v] = [y^f, v] [u, y^{-f}] [u, v] ([y, v, u] [u, y, v]^{-1})^f .$$

Note that $[u, v] = [(y, x), [y, x, x]^{-1}] [(y, x, x)]^{-1} [(y, x, x)]^{-1} = [(y, x, x)]^{-1} [(y, x, x)]^{-1} = 1,$

and the Hall-Witt identity gives $1 = [y, v, u] [u, y, v] [u, y, v] = [y, v, u] [u, y, v].$ Thus $w = [y^f, v] [u, y^{-f}]$. Now,

$$[u, y^{-f}] = ([y^{-1}]^f, [y^{-1}]^f, [y^{-1}]^f, [y^{-1}]^f, [y^{-1}]^f, [y^{-1}]^f, [y^{-1}]^f, [y^{-1}]^f).$$

Quick calculation shows that $[y^{-1}, u] = [y, x, y], [y^{-1}, u, y^{-1}] = [y, x, y, y][y, x, y]^{-1},$

and $[y^{-1}, u, y^{-1}, y^{-1}] = [y, x, y, y].$ Therefore

$$[u, y^{-f}] = [y, x, y]^{-f} [y, x, y, y]^{-f} [y, x, y, y]^{-f} [y, x, y, y]^{-f} [y, x, y, y]^{-f} [y, x, y, y]^{-f} [y, x, y, y]^{-f} [y, x, y, y]^{-f} .$$

On the other hand, we easily get that

$$[y, v] = [y, x, y] [y, x, y, x, x] [y, x, y, x, y, x][y, x, y, x, y][y, x, y, x, y][y, x, y, x, y][y, x, y, x, y][y, x, y, x, y][y, x, y, x, y].$$

and thus

$$[y^f, v] = [y, v]^f [y, v, y] [y, v, y] [y, v, y] [y, v, y] [y, v, y] [y, v, y] [y, v, y] [y, v, y] [y, v, y] [y, v, y] [y, v, y] [y, v, y] [y, v, y] .$$

We thus get

$$w = [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} .$$

Note that $[y, x, y, y]^{f + 2} [y, x, y, y]^{f + 2} = 1$, since $f^2$ is divisible by $2^{2e - 4} \geq 2^{e - 1}$. As $H$ has class $\leq 5$, we also have $[y, x, y, x, y, y, y] = [y, x, y, x, y, x, y, x].$ This, together with (3.14), implies $w = ([x, y, x, x, y][x, y, y, x, y, x])^f = [x, y]^{2f}$. We conclude that $[x, y]^{2f} \in K(H) \cap Z = 1,$ and this proves Theorem 3.14.

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