GEOMETRIC ESSENCE OF “COMPACT” OPERATORS ON HILBERT C*-MODULES

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Abstract. We introduce a uniform structure on any Hilbert C*-module \( N \) and prove the following theorem: suppose, \( F : \mathcal{M} \to \mathcal{N} \) is a bounded adjointable morphism of Hilbert C*-modules over \( \mathcal{A} \) and \( \mathcal{N} \) is countably generated. Then \( F \) belongs to the Banach space generated by operators \( \theta_{x,y}, \theta_{x,y}(z) := x\langle y, z \rangle, x \in \mathcal{N}, y, z \in \mathcal{M} \) (i.e. \( F \) is \( \mathcal{A} \)-compact, or “compact”) if and only if \( F \) maps the unit ball of \( \mathcal{M} \) to a totally bounded set with respect to this uniform structure (i.e. \( F \) is a compact operator).

Introduction

The equivalence of two definitions of compactness of operators on a Hilbert space (to be approximated by finite-dimensional operators and to map bounded sets to totally bounded sets) is an extremely useful source for the study of compact operators.

That is why it is interesting to obtain some similar equivalence in the case of Hilbert C*-modules. For a long time this problem was considered as not having a reasonable solution, because, roughly speaking, \( \mathcal{A} \)-compact operators are far from C-finite-dimensional ones.

Up to our knowledge, the only attempt to obtain some results in this direction was made very recently by D. Kečkić and Z. Lazović in [6]. Namely, they have introduced some system of pseudo-metrics (related to known topologies [13, 2], see also [9]) on the standard Hilbert C*-module \( \ell_2(\mathcal{A}) \) and the corresponding notion of a totally bounded set, where \( \mathcal{A} \) is a von Neumann algebra. In other words, they have suggested to consider total boundedness with respect to a uniform structure, which is not induced by the norm of the Hilbert C*-module under consideration. Unfortunately, their approach does not give a solution of the above problem, because it works only for \( W^* \)-algebras and for \( \mathcal{N} = \ell_2(\mathcal{A}) \) and they prove the equivalence of \( \mathcal{A} \)-compactness (they name it “compactness”) of an adjointable operator \( F : \ell_2(\mathcal{A}) \to \ell_2(\mathcal{A}) \) and total boundedness of \( F(B) \), where \( B \) is the unit ball of \( \ell_2(\mathcal{A}) \), for \( \mathcal{A} = B(H) \), the algebra of all operators on a Hilbert space \( H \). Unfortunately, in general, even for commutative algebras, they are not equivalent, but “compactness” implies compactness.

Quite recently, Z. Lazović [8] has involved unital C*-algebras in the context, but the other above listed unsatisfactory moments remain.

Our approach (namely, a choice of some other system of pseudo-metrics to define a uniform structure) seems to be giving a solution to the problem overcoming these difficulties, in particular, the corresponding notion of total boundedness is defined and has good properties: for any C*-algebra (not only unital), and for any Hilbert C*-module over \( \mathcal{A} \) (not only the standard one) and it gives the equivalence of \( \mathcal{A} \)-compactness of an adjointable operator \( F \) and total boundedness of \( F(B) \) for all \( \mathcal{M} \) and \( \mathcal{N} \) (with the only restriction: \( \mathcal{N} \) is supposed to
be countably generated; this is a very natural restriction by Lemma 1.10). This equivalence is our main result (Theorem 2.5).

The paper is arranged in the following way. In Section 1 we recall some facts and definitions from the theory of \( C^* \)-algebras and Hilbert \( C^* \)-modules. Also we give a technical definition of relatively \( \mathcal{A} \)-compact operators (Definition 1.11) and prove a couple of properties of \( \mathcal{A} \)-compact operators to be used later.

In Section 2 we define our uniform structure, prove some properties of the related total boundedness, and formulate the main result.

In Section 3 we prove an important particular case (more precisely, a variant of the main result in a particular case) to be used in the proof of the general case.

In Section 4 we prove the main result using a circle of implications. Some necessary facts are proved as separate lemmas.

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1. Preliminaries

We start with a couple of statements about states. The first one is very well known [12, Theorem 3.3.2].

**Lemma 1.1.** For any state \( \varphi \) on \( \mathcal{A} \) and any \( a \in \mathcal{A} \) one has \( |\varphi(a)|^2 \leq \varphi(a^*a) \).

The following “inverse” statement is also known, but we have not found an appropriate reference.

**Lemma 1.2.** For any \( a \in \mathcal{A} \) there is a state \( \varphi \) such that \( \|a\| \leq 2|\varphi(a)| \).

**Proof.** Decompose: \( a = \frac{1}{2}(a + a^*) + i \cdot \frac{1}{2i}(a - a^*) \). Then, by [12, Theorem 3.3.6], for some states \( \varphi_1 \) and \( \varphi_2 \),

\[
\|a\| \leq \frac{1}{2}\|a + a^*\| + \frac{1}{2}\|i(a - a^*)\| = \frac{1}{2}(\varphi_1(a + a^*) + \varphi_2(i(a - a^*))) \leq \frac{1}{2}(|\varphi_1(a) + \varphi_1(a)| + |\varphi_2(a) - \varphi_2(a)|) \leq 2 \sup_{\varphi \text{ is a state}} |\varphi(a)|.
\]

Since the set of states is \(*\)-weakly compact ([12, Theorem 5.1.8]), the continuous function \( \varphi \mapsto |\varphi(a)| \) reaches its maximum. \( \Box \)

Now we will give some basic facts about Hilbert \( C^* \)-modules over \( \mathcal{A} \) and \( \mathcal{A} \)-compact operators. Details and proofs can be found in books [7, 11] and survey paper [10]. Some other directions joining Hilbert \( C^* \)-modules and operator theory can be found in [3, 14, 1, 15].

**Definition 1.3.** A (right) pre-Hilbert \( C^* \)-module over a \( C^* \)-algebra \( \mathcal{A} \) is an \( \mathcal{A} \)-module equipped with an \( \mathcal{A} \)-inner product \( \langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A} \) being a sesquilinear form on the underlying linear space and restricted to satisfy:

1. \( \langle x, x \rangle \geq 0 \) for any \( x \in \mathcal{M} \);
2. \( \langle x, x \rangle = 0 \) if and only if \( x = 0 \);
3. \( \langle y, x \rangle = \langle x, y \rangle^* \) for any \( x, y \in \mathcal{M} \);
4. \( \langle x, y \cdot a \rangle = \langle x, y \rangle a \) for any \( x, y \in \mathcal{M}, \ a \in \mathcal{A} \).
A pre-Hilbert $C^*$-module over $\mathcal{A}$ is a *Hilbert $C^*$-module* if it is complete w.r.t. its norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

A Hilbert $C^*$-module $\mathcal{M}$ is *countably generated* if there exists a countable set of its elements with dense set of $\mathcal{A}$-linear combinations.

We will denote by $\oplus$ the Hilbert sum of Hilbert $C^*$-modules in an evident sense.

We have the following Cauchy-Schwartz inequality [13] (see also [11, Proposition 1.2.4]) for any $x, y \in \mathcal{M}$

$$\langle x, y \rangle \langle y, x \rangle \leq \|y\|^2 \langle x, x \rangle.$$  \hfill (1)

**Definition 1.4.** The *standard Hilbert $C^*$-module* $\ell_2(\mathcal{A})$ (also denoted by $H_\mathcal{A}$) is the set of all infinite sequences $a = (a_1, a_2, \ldots, \infty)$, $a_i \in \mathcal{A}$, such that the series $\sum_i (a_i)^* a_i$ is norm-convergent in $\mathcal{A}$. It is equipped with the inner product $\langle a, b \rangle = \sum_i (a_i)^* b_i$, where $b = (b_1, b_2, \ldots)$.

If $\mathcal{A}$ is unital, then $\ell_2(\mathcal{A})$ is countably generated.

One of the most nice properties of countably generated modules is the following theorem [4] (see [11, Theorem 1.4.2]). We mean that an isomorphism preserves the $C^*$-Hilbert structure.

**Theorem 1.5** (Kasparov stabilization theorem). *For any countably generated Hilbert $C^*$-module $\mathcal{M}$ over $\mathcal{A}$, there exists an isomorphism of Hilbert $C^*$-modules $\mathcal{M} \oplus \ell_2(\mathcal{A}) \cong \ell_2(\mathcal{A})$.

**Definition 1.6.** An *operator* is a bounded $\mathcal{A}$-homomorphism. An operator having an adjoint (in an evident sense) is *adjointable* (see [11, Section 2.1]). We will denote the Banach space of all operators $F : \mathcal{M} \to \mathcal{N}$ by $L(\mathcal{M}, \mathcal{N})$ and the Banach space of adjointable operators by $L^*(\mathcal{M}, \mathcal{N})$. The space $L(\mathcal{M}, \mathcal{M})$ is a Banach algebra and $L^*(\mathcal{M}, \mathcal{M})$ is a $C^*$-algebra.

**Definition 1.7.** An *elementary $\mathcal{A}$-compact* operator $\theta_{x,y} : \mathcal{M} \to \mathcal{N}$, where $x \in \mathcal{N}$ and $y \in \mathcal{M}$, is defined as $\theta_{x,y}(z) := x(y,z)$. Then the Banach space $K(\mathcal{M}, \mathcal{N})$ of $\mathcal{A}$-compact operators is the closure of the subspace generated by all elementary $\mathcal{A}$-compact operators in $L(\mathcal{M}, \mathcal{N})$.

Since $(\theta_{x,y})^* = \theta_{y,x}$, $\mathcal{A}$-compact operators are adjointable.

Since $T\theta_{x,y} = \theta_{Tx,y}$ if $T \in L(\mathcal{N}, \mathcal{N}')$, and $\theta_{x,y}S = \theta_{x,S'y}$ if $S \in L(\mathcal{M}', \mathcal{M})$, we have the following statement (see also [11, Section 2.2]).

**Proposition 1.8.** The set of $\mathcal{A}$-compact operators on a Hilbert $C^*$-module is a (closed two-sided self-adjoint) ideal in the $C^*$-algebra of adjointable endomorphisms.

If $F$ is an $\mathcal{A}$-compact operator $F : \mathcal{M} \to \mathcal{N}$ and $F_1 : \mathcal{M} \to \mathcal{M}$ and $F_2 : \mathcal{N} \to \mathcal{N}$ are adjointable, then $F \circ F_1$ and $F_2 \circ F$ are $\mathcal{A}$-compact.

If $F$ is an adjointable operator $F : \mathcal{M} \to \mathcal{N}$ and $K_1 : \mathcal{M} \to \mathcal{M}$ and $K_2 : \mathcal{N} \to \mathcal{N}$ are $\mathcal{A}$-compact, then $F \circ K_1$ and $K_2 \circ F$ are $\mathcal{A}$-compact.

**Lemma 1.9.** Let $F : \mathcal{M} \to \mathcal{N}_1 \oplus \mathcal{N}_2$ be an adjointable operator. Then $F$ is $\mathcal{A}$-compact if and only if $p_1 F$ and $p_2 F$ are $\mathcal{A}$-compact, where $p_1 : \mathcal{N}_1 \oplus \mathcal{N}_2 \to \mathcal{N}_1$ and $p_2 : \mathcal{N}_1 \oplus \mathcal{N}_2 \to \mathcal{N}_2$ are orthogonal projections.

**Proof.** ‘Only if’ follows immediately from Proposition 1.8.

If $p_1 F$ is approximated by a combination of $\theta_{x_1,y_1}$, $x_1 \in \mathcal{N}_1$, $y_1 \in \mathcal{M}$, and $p_2 F$ is approximated by a combination of $\theta_{x_2,y_2}$, $x_2 \in \mathcal{N}_2$, $y_2 \in \mathcal{M}$, then $F = p_1^* p_1 F + p_2^* p_2 F$ is approximated by the sum of appropriate combinations of $\theta_{p_1^*(x_1),y_1}$ and $\theta_{p_2^*(x_2),y_2}$. \(\square\)
Lemma 1.10. Let $F : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{A}$-compact operator. Then its image $F(\mathcal{M})$ is contained in a countably generated module.

Proof. Indeed, suppose $\varepsilon_n \to 0$ as $n \to \infty$. Then for each $n$ there exists a combination of $\theta_{y,z}$, which is $\varepsilon_n$-close to $F$:

$$
\|\theta_{y(n,1),z(n,1)}(x) + \cdots + \theta_{y(n,k(n)),z(n,k(n))}(x) - F(x)\| < \varepsilon_n \|x\|.
$$

The Hilbert $C^*$-module generated by all $y(i,j)$ contains $F(\mathcal{M})$, since, for any $x \in \mathcal{M}$ and any $\varepsilon > 0$, taking $\varepsilon_n < \varepsilon / \|x\|$ we have

$$
\|\theta_{y(n,1),z(n,1)}(x) + \cdots + \theta_{y(n,k(n)),z(n,k(n))}(x) - F(x)\| < \varepsilon_n \|x\| < \varepsilon.
$$

\qed

For technical reasons (not for the formulation of the results) we will need a modification of the above definition.

Definition 1.11. We say that an operator $F : \mathcal{M} \to \mathcal{N}$ is $\mathcal{A}$-compact relatively a submodule $\mathcal{N}^0 \subset \mathcal{N}$ if $F(\mathcal{M}) \subset \mathcal{N}^0$ and $F$ is $\mathcal{A}$-compact as an operator from $\mathcal{M}$ to $\mathcal{N}^0$. Roughly speaking this means that $y$ in $\theta_{y,z}$ can be taken from $\mathcal{N}^0$. Denote the set of these operators by $K(\mathcal{M}, \mathcal{N}; \mathcal{N}^0)$.

Proposition 1.8 implies the following lemma.

Lemma 1.12. Suppose $F \in K(\mathcal{M}, \mathcal{N}; \mathcal{N}^0)$, $G \in L(\mathcal{N}_1, \mathcal{N}_1)$ and $G(\mathcal{N}^0) \subset \mathcal{N}_1^0$. Then $GF \in K(\mathcal{M}, \mathcal{N}_1; \mathcal{N}_1^0)$.

Suppose $F \in K(\mathcal{M}, \mathcal{N}; \mathcal{N}^0)$ and $G \in L^*(\mathcal{M}_1, \mathcal{M})$. Then $FG \in K(\mathcal{M}_1, \mathcal{N}; \mathcal{N}^0)$.

2. Definitions and formulation of the main theorem

Now we pass to the definition of the desired uniform structure and totally bounded sets.

Definition 2.1. Any uniform structure on a non-empty space $X$ can be defined by a system of pseudo-metrics (see [5, p. 188]), i.e. functions $d_{\alpha} : X \times X \to [0, +\infty)$ restricted to satisfy:

1) the symmetry property $d_{\alpha}(x, y) = d_{\alpha}(y, x)$;
2) the triangle inequality $d_{\alpha}(x, z) \leq d_{\alpha}(x, y) + d_{\alpha}(y, z)$;
3) the separation property: if $x \neq y$, then $d_{\alpha}(x, y) > 0$ for some $\alpha$.

We will define totally bounded sets directly in our case in Definition 2.4 below.

Definition 2.2. Let $\mathcal{N}$ be a Hilbert $C^*$-module over $\mathcal{A}$. A countable system $X = \{x_i\}$ of its elements is called admissible for a submodule $\mathcal{N}_0 \subset \mathcal{N}$ (or $\mathcal{N}^0$-admissible) if for each $x \in \mathcal{N}^0$ partial sums of the series $\sum \langle x, x_i \rangle \langle x_i, x \rangle$ are bounded by $\langle x, x \rangle$ and the series is convergent. In particular, $\|x_i\| \leq 1$ for any $i$.

Example 2.3. For the standard module $\ell_2(\mathcal{A})$ over a unital algebra $\mathcal{A}$ one can take for $X$ the natural base $\{e_i\}$. In the case of $\ell_2(\mathcal{A})$ over a general algebra $\mathcal{A}$, one can take $x_i$ having only the $i$-th component nontrivial and of norm $\leq 1$. The other important example is $X$ with only finitely many non-zero elements in any module and an appropriate normalization.

Denote by $\Phi$ a countable collection $\{\varphi_1, \varphi_2, \ldots\}$ of states on $\mathcal{A}$. For each pair $(X, \Phi)$ with an $\mathcal{N}^0$-admissible $X$, consider the following pseudo-metrics

$$
(2) \quad d_{X,\Phi}(x, y)^2 := \sup_k \sum_{i=k}^{\infty} |\varphi_k(\langle x - y, x_i \rangle)|^2, \quad x, y \in \mathcal{N}^0.
$$
First, remark that this is a finite non-negative number. Indeed, by Lemma 1.1
\[
\sum_{i=k}^{s} |\varphi_k (\langle x - y, x_i \rangle)|^2 = \sum_{i=k}^{s} |\varphi_k (\langle x_i, x - y \rangle)|^2 \leq \varphi_k \left( \sum_{i=k}^{s} \langle x - y, x_i \rangle \langle x_i, x - y \rangle \right)
\]
\[
\leq \left\| \sum_{i=k}^{s} \langle x - y, x_i \rangle \langle x_i, x - y \rangle \right\| \leq \|x - y\|^2.
\]
Since in (2) we have a series of non-negative numbers, this estimation implies its convergence and the estimation
\[
d_{X,\Phi}(x, y) \leq \|x - y\|.
\]
For \( x \neq y \) there exists \( (X, \Phi) \) such that \( d_{X,\Phi}(x, y) > \frac{1}{2} \|x - y\| \). Indeed, take \( X \) with \( x_1 = \frac{x - y}{\|x - y\|} \) and other \( x_i = 0 \), and \( \varphi_1 \) such that \( \varphi_1 (\langle x - y, x - y \rangle) > \frac{1}{2} \|x - y\|^2 \). Then for any \( z \), inequality (1) implies
\[
\sum_{i} \langle z, x_i \rangle \langle x_i, z \rangle = \langle z, x_1 \rangle \langle x_1, z \rangle \leq \|x_1\|^2 \langle z, z \rangle \leq \langle z, z \rangle
\]
and
\[
d_{X,\Phi}(x, y) \geq |\varphi_1 (\langle x - y, x_1 \rangle)| = \frac{|\varphi_1 (\langle x - y, x - y \rangle)|}{\|x - y\|} > \frac{1}{2} \|x - y\|.
\]
Let us verify the triangle inequality:
\[
d_{X,\Phi}(u, v) \leq d_{X,\Phi}(u, w) + d_{X,\Phi}(w, v),
\]
which can be rewritten as
\[
d_{X,\Phi}(u - v, 0) \leq d_{X,\Phi}(u - w, 0) + d_{X,\Phi}(w - v, 0),
\]
and thus it is sufficient to prove
\[
d_{X,\Phi}(z + 0, 0) \leq d_{X,\Phi}(z, 0) + d_{X,\Phi}(0, 0).
\]
Take an arbitrary \( \varepsilon > 0 \) and choose \( k \) and \( m \) such that
\[
d_{X,\Phi}(z + x, 0) < \sqrt{\sum_{i=k}^{m} |\varphi_i (\langle z + x, x_i \rangle)|^2 + \varepsilon}.
\]
We have
\[
\sqrt{\sum_{i=k}^{m} |\varphi_i (\langle z + x, x_i \rangle)|^2} \leq \sqrt{\sum_{i=k}^{m} (|\varphi_i (\langle z, x_i \rangle)| + |\varphi_i (\langle x, x_i \rangle)|)^2}
\]
Consider in \( \mathbb{C}^{m-k+1} \) with the standard inner product \( \langle ., . \rangle_\mathbb{C} \) and the norm \( \| . \|_\mathbb{C} \) the following vectors with non-negative coordinates:
\[
\vec{a} = (a_1, \ldots) := (|\varphi_k (\langle z, x_k \rangle)|, \ldots, |\varphi_m (\langle z, x_m \rangle)|),
\]
\[
\vec{b} = (b_1, \ldots) := (|\varphi_k (\langle x, x_k \rangle)|, \ldots, |\varphi_m (\langle x, x_m \rangle)|).
\]
Then by the triangle inequality for \( \| . \|_\mathbb{C} \),
\[
\sqrt{\sum_{i=k}^{m} (|\varphi_i (\langle z, x_i \rangle)| + |\varphi_i (\langle x, x_i \rangle)|)^2} = \sqrt{\sum_{j} (a_j + b_j)^2} = \|\vec{a} + \vec{b}\|_\mathbb{C} \leq \|\vec{a}\|_\mathbb{C} + \|\vec{b}\|_\mathbb{C}
\]
where

Suppose, the set

Lemma 2.6. If \( \mathcal{N} \) is a countably generated submodule. Then \( \mathcal{N} \) is countably generated. Then

Theorem 2.5 (Main Theorem).

Proof. If \( p_j \mathcal{N} \) is contained in a countably generated submodule \( \mathcal{N}_j \subset \mathcal{N}_j, j = 1, 2 \), then \( Y \) is contained in the countably generated module \( \mathcal{N}_1 \oplus \mathcal{N}_2 \). Conversely, if \( Y \) is contained in a countably generated submodule \( \mathcal{N}_1 \), then \( p_j \mathcal{N} \) is countably generated and \( p_j \mathcal{N} \subset p_j \mathcal{N} \), \( j = 1, 2 \).

Denote by \( J_j = p_j^* \) the corresponding inclusions \( J_j : \mathcal{N}_j \hookrightarrow \mathcal{N}_1 \oplus \mathcal{N}_2, j = 1, 2 \).

Suppose \( Y \) is \( (\mathcal{N}, \mathcal{N}_0) \)-totally bounded and \( X = \{x_i\} \) is an admissible system for a countably generated submodule \( \mathcal{N}_1 \subset \mathcal{N}_1 \). Then \( J_1 X = \{J_1(x_i)\} \) is admissible for \( \mathcal{N}_0 \) because

\[
\langle x, J_1(x_i) \rangle \langle J_1(x_i), x \rangle = \langle p_1 x, x_i \rangle \langle x_i, p_1 x \rangle.
\]

Let \( y_1, \ldots, y_s \) be an \( \varepsilon \)-net in \( Y \) for \( d_{J_1, \Phi} \). Then \( p_1 y_1, \ldots, p_1 y_s \) is an \( \varepsilon \)-net in \( p_1 Y \) for \( d_{X, \Phi} \).

Indeed, consider an arbitrary \( z \in p_1 Y \). Then \( z = p_1 y \) for some \( y \in Y \). Find \( y_k \) such that

\[
\sum_{i=k}^{m} |\varphi_i(z, x_i)|^2 + \sum_{i=k}^{m} |\varphi_i(x, x_i)|^2 \leq d_{X, \Phi}(z, 0) + d_{X, \Phi}(x, 0).
\]

Since \( \varepsilon \) in (6) is arbitrary, together with (7) the last estimation gives (5) and hence (4).

Also, we have the following version of the triangle inequality:

\[
d_{X, \Phi}(x + y, u + v) \leq d_{X, \Phi}(x, u) + d_{X, \Phi}(y, v),
\]

since by (5),

\[
d_{X, \Phi}((x - u) + (y - v), 0) \leq d_{X, \Phi}(x - u, 0) + d_{X, \Phi}(y - v, 0).
\]

So, we have verified that \( d_{X, \Phi} \) satisfy the conditions of Definition 2.1 and thus define a uniform structure on the unit ball of \( \mathcal{N}_0 \).

Definition 2.4. A set \( Y \subset \mathcal{N}_0 \subset \mathcal{N} \) is totally bounded with respect to this uniform structure, if for any \( (X, \Phi) \), where \( X \subset \mathcal{N} \) is \( \mathcal{N}_0 \)-admissible, and any \( \varepsilon > 0 \) there exists a finite collection \( y_1, \ldots, y_n \) of elements of \( Y \) such that the sets

\[
\{y \in Y \mid d_{X, \Phi}(y_i, y) < \varepsilon\}
\]

form a cover of \( Y \). This finite collection is an \( \varepsilon \)-net in \( Y \) for \( d_{X, \Phi} \).

If so, we will say briefly that \( Y \) is \( (\mathcal{N}, \mathcal{N}_0) \)-totally bounded.

Now we are able to formulate our main result.

Theorem 2.5 (Main Theorem). Suppose, \( F : \mathcal{M} \to \mathcal{N} \) is an adjointable operator and \( \mathcal{N} \) is countably generated. Then \( F \) is \( \mathcal{A} \)-compact if and only if \( F(B) \) is \( (\mathcal{N}, \mathcal{N}) \)-totally bounded, where \( B \) is the unit ball of \( \mathcal{M} \).

We will complete this section with the following property.

Lemma 2.6. Suppose, the set \( Y \subset \mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2 \) is \( (\mathcal{N}, \mathcal{N}_0) \)-totally bounded, where \( \mathcal{N}_0 \) is a countably generated submodule. Then \( p_1 Y \) and \( p_2 Y \) are \( (\mathcal{N}_1, \mathcal{N}_0) \)- and \( (\mathcal{N}_2, \mathcal{N}_0) \)-totally bounded, respectively, where \( \mathcal{N}_0 = p_1(\mathcal{N}_0) \) and \( \mathcal{N}_2 = p_2(\mathcal{N}_0) \) are countably generated submodules, and \( p_1 : \mathcal{N}_1 \oplus \mathcal{N}_2 \to \mathcal{N}_1 \), \( p_2 : \mathcal{N}_1 \oplus \mathcal{N}_2 \to \mathcal{N}_2 \) are the orthogonal projections.

Conversely, if \( p_1 Y \) and \( p_2 Y \) are \( (\mathcal{N}_1, \mathcal{N}_0) \)- and \( (\mathcal{N}_2, \mathcal{N}_0) \)-totally bounded for some \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \), respectively, then \( Y \) is \( (\mathcal{N}, \mathcal{N}_0 \oplus \mathcal{N}_2) \)-totally bounded. Evidently \( \mathcal{N}_0 \oplus \mathcal{N}_2 \) is countably generated if \( \mathcal{N}_0 \) and \( \mathcal{N}_2 \) are countably generated.

Proof. If \( p_j Y \) is contained in a countably generated submodule \( \mathcal{N}_j \subset \mathcal{N}_j, j = 1, 2 \), then \( Y \) is contained in the countably generated module \( \mathcal{N}_1 \oplus \mathcal{N}_2 \). Conversely, if \( Y \) is contained in a countably generated submodule \( \mathcal{N}_1 \), then \( p_j \mathcal{N}_0 \) is countably generated and \( p_j \mathcal{N} \subset p_j \mathcal{N}_0 \), \( j = 1, 2 \).

Denote by \( J_j = p_j^* \) the corresponding inclusions \( J_j : \mathcal{N}_j \hookrightarrow \mathcal{N}_1 \oplus \mathcal{N}_2, j = 1, 2 \).

Suppose \( Y \) is \( (\mathcal{N}, \mathcal{N}_0) \)-totally bounded and \( X = \{x_i\} \) is an admissible system for a countably generated submodule \( \mathcal{N}_1 \subset \mathcal{N}_1 \). Then \( J_1 X = \{J_1(x_i)\} \) is admissible for \( \mathcal{N}_0 \) because

\[
\langle x, J_1(x_i) \rangle \langle J_1(x_i), x \rangle = \langle p_1 x, x_i \rangle \langle x_i, p_1 x \rangle.
\]

Let \( y_1, \ldots, y_s \) be an \( \varepsilon \)-net in \( Y \) for \( d_{J_1, \Phi} \). Then \( p_1 y_1, \ldots, p_1 y_s \) is an \( \varepsilon \)-net in \( p_1 Y \) for \( d_{X, \Phi} \). Indeed, consider an arbitrary \( z \in p_1 Y \). Then \( z = p_1 y \) for some \( y \in Y \). Find \( y_k \) such that
Indeed, we obtain the convergence and can estimate the sum using (as above) the equality

\[ d^2_{J_1, \Phi}(y, y_k) < \varepsilon. \]

Then

\[ d^2_{J_1, \Phi}(z, p_1y_k) = \sup_k \sum_{i=k}^{\infty} |\varphi_k (\langle z - p_1y_k, x_i \rangle)|^2 = \sum_{i=k}^{\infty} |\varphi_k (\langle p_1(y - y_k), x_i \rangle)|^2 \]

\[ = \sum_{i=k}^{\infty} |\varphi_k (\langle y - y_k, J_1(x_i) \rangle)|^2 = d^2_{J_1, \Phi}(y, y_k) < \varepsilon^2. \]

Similarly for \( j = 2 \).

Conversely, suppose that \( p_jY \) are \((\mathcal{N}_j, \mathcal{N}_j^0)\)-totally bounded, \( j = 1, 2 \). Let \( X = \{x_i\} \) be an admissible system in \( \mathcal{N} \) for \( \mathcal{N}_1^0 \oplus \mathcal{N}_2^0 \) and \( \varepsilon > 0 \) is arbitrary. Then \( X_j := \{p_j(x_i)\} \) is an admissible system in \( \mathcal{N}_j \) for \( \mathcal{N}_j^0 \) and, for \( u, v \in p_jY \),

\[ d_{X, \Phi}(J_ju, J_jv) = d_{X_j, \Phi}(u, v). \]

Indeed, we obtain the convergence and can estimate the sum using (as above) the equality

\[ \sum_{i=1}^{s} \langle u - v, p_jx_i \rangle \langle p_jx_i, u - v \rangle = \sum_{i=1}^{s} \langle J_j(u - v), x_i \rangle \langle x_i, J_j(u - v) \rangle \]

and, quite similarly, (9) follows from the equality

\[ \langle J_jp_ju - J_jv, x_i \rangle = \langle p_ju - v, p_jx_i \rangle, \quad j = 1, 2. \]

Suppose, \( z_1, \ldots, z_m \) is an \( \varepsilon/4 \)-net in \( p_1Y \) for \( d_{X_1, \Phi} \) and \( w_1, \ldots, w_r \) is an \( \varepsilon/4 \)-net in \( p_2Y \) for \( d_{X_2, \Phi} \). Consider \( \{z_k + w_s\}, k = 1, \ldots, m, s = 1, \ldots, r \). Then \( \{J_1z_k + J_2w_s\} \) is an \( \varepsilon/2 \)-net in \( p_1Y \oplus p_2Y \) for \( d_{X, \Phi} \). Indeed, for any \( J_1p_1y_1 + J_2p_2y_2, y_1, y_2 \in Y \), one can find \( z_k \) and \( w_s \) such that

\[ d_{X_1, \Phi}(p_1y_1, z_k) < \varepsilon/4, \quad d_{X_2, \Phi}(p_2y_2, w_s) < \varepsilon/4. \]

Then by (8) and (9)

\[ d_{X, \Phi}(J_1p_1y_1 + J_2p_2y_2, J_1z_k + J_2w_s) \leq d_{X, \Phi}(J_1p_1y_1, J_1z_k) + d_{X, \Phi}(J_2p_2y_2, J_2w_s) \]

\[ = d_{X_1, \Phi}(p_1y_1, z_k) + d_{X_2, \Phi}(p_2y_2, w_s) < \varepsilon/2. \]

Now find a subset \( \{u_l\} \subset \{z_k + w_s\} \) formed by all elements of \( \{z_k + w_s\} \) such that there exists an element \( u^* \in Y \subset p_1Y \oplus p_2Y \) with \( d_{X, \Phi}(u^*, z_k + w_s) < \varepsilon/2 \). Denote these \( u^* \) by \( u^*_l \), \( l = 1, \ldots, L \). So,

1) for any \( y \in Y \), there exists \( l = 1, \ldots, L \) such that \( d_{X, \Phi}(y, u_l) < \varepsilon/2 \);
2) for each \( l = 1, \ldots, L \), we have \( d_{X, \Phi}(u^*_l, u_l) < \varepsilon/2 \).

By the triangle inequality, \( \{u_l^*\} \) is a finite \( \varepsilon \)-net in \( Y \) for \( d_{X, \Phi} \) and we are done.

\[ \square \]

3. The case \( \mathcal{N} \subset \mathcal{A} \)

We will prove the following statement in this section.

**Theorem 3.1.** Suppose that \( G : \mathcal{M} \to \mathcal{A} \) is an adjointable operator such that \( G(\mathcal{M}) \) is contained in a countably generated submodule \( \mathcal{N}^0 \subset \mathcal{A} \). Then

1) if \( G \) is \( \mathcal{A} \)-compact relatively \( \mathcal{N}^0 \), i.e. \( G \in \mathcal{K}(\mathcal{M}, \mathcal{A}; \mathcal{N}^0) \), then \( G(B) \) is \((\mathcal{A}, \mathcal{N}^0)\)-totally bounded;

2) if \( G(B) \) is \((\mathcal{A}, \mathcal{N}^0)\)-totally bounded, then \( G \) is \( \mathcal{A} \)-compact, i.e. \( G \in \mathcal{K}(\mathcal{M}, \mathcal{A}; \mathcal{A}) = \mathcal{K}(\mathcal{M}, \mathcal{A}) \).

Here \( B \) is the unit ball of \( \mathcal{M} \) as above.
We need the following statement.

**Lemma 3.2.** Let \( F : \mathcal{M} \to \mathcal{A} \) be a bounded adjointable, but not an \( \mathcal{A} \)-compact operator. Suppose, \( K > 0 \) is a constant. Then there exists \( \delta > 0 \) such that for any \( z \in \mathcal{A} \) there exists an element \( x \in \mathcal{M} \) with \( \|x\| \leq 1 \) such that \( \|z \alpha - F(x)\| > \delta \) for any \( \alpha \in \mathcal{A} \) with \( \|\alpha\| \leq K \). Taking \( \alpha = 0 \) gives \( \|F(x)\| > \delta \).

**Proof.** Suppose the opposite: for any \( \varepsilon > 0 \) there exists an element \( z \in \mathcal{A} \) such that for any \( x \) of norm 1 there exists \( \alpha_x \), \( \|\alpha_x\| \leq K \), \( \|z \alpha_x - F(x)\| < \varepsilon \).

Choose an element of an approximate unit \( \omega \) of \( \mathcal{A} \), such that \( \|z - \omega \| < \varepsilon \), \( 0 \leq \omega \leq 1 \). Then for any \( x \in \mathcal{M} \) of norm 1,

\[
\|\theta_{\omega, \omega}(F(x)) - F(x)\| \leq \|\theta_{\omega, \omega}(z \alpha_x) - F(x)\| + \|\theta_{\omega, \omega}(z \alpha_x - F(x))\| + \varepsilon \\
\leq \|\omega z \alpha_x - F(x)\| + \varepsilon = \|z \alpha_x - F(x)\| + \|(z - \omega \alpha_x)\| + \varepsilon \\
\leq \|\omega \alpha_x - F(x)\| + K \varepsilon + \varepsilon < (2 + K) \varepsilon.
\]

This means that \( F \) can be approximated by \( \mathcal{A} \)-compact operators \( \theta_{\omega, \omega} \circ F \) (see Proposition 1.8). Hence, \( F \) is \( \mathcal{A} \)-compact. A contradiction. \( \square \)

**Proof of Theorem 3.1.** Let \( a_1, a_2, \ldots \) be a countable system of generators for \( \mathcal{N}^0 \). We have \( G(\mathcal{M}) \subseteq \mathcal{N}^0 \), hence \( G(\mathcal{M}) \subseteq \mathcal{N}^0 \). Consider a separable \( C^* \)-subalgebra \( \mathcal{A}_0 \subseteq \mathcal{A} \) generated by these elements and its increasing countable approximate unit \( \omega_i \), such that \( \omega_i \leq 1 \), \( \omega_i \leq \omega_j \), if \( j < j \), and

\[
\omega_j \omega_i = \omega_i, \quad j \geq i, \tag{10}
\]

\[
(\omega_j - \omega_i)^2 \leq \omega_j - \omega_i, \quad j \geq i. \tag{11}
\]

Suppose, that \( G(\mathcal{B}) \) is \( (\mathcal{A}, \mathcal{N}^0) \)-totally bounded, but \( G \) is not \( \mathcal{A} \)-compact. Then, for \( K = \|G\| \), Lemma 3.2 implies that for some \( \delta > 0 \) and each \( \omega_i \), there exists an element \( z_i = G(x_i) \), with \( \|x_i\| \leq 1 \), such that \( \|z_i - \omega_i \beta\| > \delta \) for any \( \beta \in \mathcal{A} \) with \( \|\beta\| \leq K \). Also, \( \|z_i\| > \delta \) (cf. the last line of the formulation of Lemma 3.2). In particular, for \( \beta = z_i \),

\[
\|(1 - \omega_i) z_i\| = \|z_i - \omega_i z_i\| > \delta \tag{12}
\]

(in the unitalization). Now choose some sub-sequence \( i(j) \) of \( i \) in such a way that

\[
\|\omega_i(j+1) z_i(j) - z_i(j)\| < \delta/2. \tag{13}
\]

This is possible to do, because one can approximate \( z_i(j) \) with a finite linear combination \( a_1 \alpha_1 + \cdots + a_N \alpha_N \), while \( \{\omega_i\} \) is an approximate unit for each of \( a_i \).

From (12) and (13) we obtain the estimation:

\[
\|\omega_i(j+1) - \omega_i(j)\| \geq \|\omega_i(j) z_i(j) - z_i(j)\| - \|z_i(j) - \omega_i(j+1) z_i(j)\| > \delta - \delta/2 = \delta/2. \tag{14}
\]

Suppose that \( G(\mathcal{B}) \) is totally bounded and consider a semi-norm \( d_{X, \Phi} \) defined by \( X = \{x_j\} := \{\omega_i(j+1) - \omega_i(j)\} \) and \( \Phi = \{\varphi_1, \varphi_2, \ldots \} \), where

\[
|\varphi_j((z_i(j), x_j))| = |\varphi_j((z_i(j))^* x_j)| = |\varphi_j(x_j z_i(j))| \geq \frac{\delta}{4}. \tag{15}
\]

This \( X \) is admissible for \( \mathcal{N}^0 \) because we can estimate the partial sums by (11):

\[
\sum_{j=1}^{s} (\omega_i(j+1) - \omega_i(j))^2 \leq \sum_{j=1}^{s} \omega_i(j+1) - \omega_i(j) = \omega_i(s+1)
\]
and hence, for any \( x \in \mathcal{A} \),
\[
\sum_{j=1}^{s} \langle x, x_j \rangle \langle x_j, x \rangle = x^* \left( \sum_{j=1}^{s} (\omega_{i(j+1)} - \omega_{i(j)})^2 \right) x \leq x^* \omega_{i(s+1)} x \leq x^* x = \langle x, x \rangle.
\]
Similarly, the convergence for \( x \in \mathcal{N}^0 \) follows from the estimation
\[
\sum_{j=k}^{s} \langle x, x_j \rangle \langle x_j, x \rangle = x^* \left( \sum_{j=k}^{s} (\omega_{i(j+1)} - \omega_{i(j)})^2 \right) x \leq x^* (\omega_{i(s+1)} - \omega_{i(k)}) x
\]
\[
\leq \|x\| \cdot \| (\omega_{i(s+1)} - \omega_{i(k)}) x \| \leq \|x\| \cdot \| x - \omega_{i(k)} x \| \to 0 \quad (k \to \infty).
\]
Also, these \( \varphi_j \) satisfying (15) do exist by (14) and Lemma 1.2.

Then there exist \( y_1, \ldots, y_D \in G(B) \) such that for any \( y \in G(B) \) there exists \( k \in \{1, \ldots, D\} \) such that \( d_{X, \Phi}(y, y_k) < \delta/8 \). One can find, as above (cf. the argument after (13)) a number \( j_0 \) such that
\[
\|(1 - \omega_{i(j)}) y_k \| < \delta/8, \quad j \geq j_0, \quad k = 1, \ldots, D,
\]
and hence, for \( j \geq j_0, k = 1, \ldots, D, \)
\[
(16) \quad \delta/8 > \|\omega_{i(j+1)}(1 - \omega_{i(j)}) y_k \| = \| (\omega_{i(j+1)} - \omega_{i(j)}) y_k \| = \| \langle x_j, y_k \rangle \| = \| \langle y_k, x_j \rangle \|.
\]
Then, for all \( k = 1, \ldots, D \) and \( y := z_{i(j)} \),
\[
d_{X, \Phi}(y, y_k) \geq |\varphi_{j+1}(\langle y - y_k, x_j \rangle)| > |\varphi_j(\langle z_{i(j)}, x_j \rangle)| - \delta/8 \geq \delta/4 - \delta/8 = \delta/8.
\]
by (16) and (15). A contradiction with the choice of \( y_1, \ldots, y_D \) and the supposition that \( G(B) \) is \( (\mathcal{A}, \mathcal{N}^0) \)-totally bounded. This proves Theorem 3.1 in one direction.

Now suppose that \( G \) is an \( \mathcal{A} \)-compact operator relatively \( \mathcal{N}^0 \). Denote
\[
c := \max\{1, \|G\|\}.
\]
Consider \( d_{X, \Phi} \) for some \( X = \{x_i\} \) (admissible for \( \mathcal{N}^0 \)) and \( \Phi = \{\varphi_i\} \). Consider arbitrary small \( \varepsilon > 0 \). We can suppose that \( \varepsilon < 1 \). Then, one can approximate \( G \) with a finite combination of \( \theta_{u, \varepsilon} \): for any \( x \in \mathcal{M} \),
\[
\|\theta_{u(1), v(1)}(x) + \cdots + \theta_{u(n), v(n)}(x) - G(x)\| < \frac{\varepsilon}{55c^2} \|x\|,
\]
\[
\|u(1)v(1)(x) + \cdots + u(n)v(n)(x) - G(x)\| < \frac{\varepsilon}{55c^2} \|x\|, \quad u(i) \in \mathcal{N}^0.
\]
We can replace \( u(i) \) by some arbitrary close \( \mathcal{A} \)-linear combinations of generators \( a_j \) of \( \mathcal{N}^0 \) and obtain (for simplicity of notation we take successively all \( a_1, \ldots, a_D \) for some \( D \))
\[
\|a_1w_1(x) + \cdots + a_Dw_D(x) - G(x)\| < \frac{\varepsilon}{54c^2} \|x\|.
\]
Now, for a sufficiently small \( \tau > 0 \), namely, \( \tau < \frac{\varepsilon^2}{54c^4D^2} (\sup_{j} \|w_j\|)^2 \), consider \( a := a_1(a_1)^* + \cdots + a_D(a_D)^* \) and \( b := a(\tau + a)^{-1} \), so \( \|b\| \leq 1 \). We have \( a \in \mathcal{N}^0 \), and hence, \( b \in \mathcal{N}^0 \). For any \( j = 1, \ldots, D \) we have
\[
(b - 1)a_ja_j^*(b - 1) \leq (a(\tau + a)^{-1} - 1)a(a(\tau + a)^{-1} - 1) \leq \frac{\tau}{2},
\]
because we have the following estimation for positive numbers \( t \),
\[
\left( \frac{t}{\tau + t} - 1 \right)^2 t = \frac{\tau^2 t}{(\tau + t)^2} \leq \frac{\tau^2 t}{2\tau t} = \frac{\tau}{2}.
\]
Hence,
\begin{equation}
(b - 1) a_j \leq \sqrt{\tau/2} \leq \frac{\varepsilon}{54c^2 \cdot D \cdot \sup_j \|w_j\|}.
\end{equation}

Thus,
\begin{equation}
\|G - bG\| \leq \|\theta_{a_1, w_1} + \cdots + \theta_{a_D, w_D} - G\| + \|\theta_{a_1, w_1} - b\theta_{a_1, w_1}\| + \cdots + \|b\| \cdot \|\theta_{a_1, w_1} + \cdots + \theta_{a_D, w_D} - G\|
\leq \frac{\varepsilon}{54c^2} + D \cdot \sup_j \| (1 - b) a_j \| \cdot \|w_j\| + \frac{\varepsilon}{54c^2} \leq \frac{\varepsilon}{18c^2}.
\end{equation}

By the triangle inequality, (3), and (18), for \( r, t \in B \), we have
\begin{equation}
d_{X, \Phi}(G(r), G(t)) \leq d_{X, \Phi}(bG(r), bG(t)) + d_{X, \Phi}(G(r), bG(r)) + d_{X, \Phi}(G(t), bG(t)) 
\leq d_{X, \Phi}(bG(r), bG(t)) + \frac{\varepsilon}{9c^2}
\end{equation}
and
\begin{equation}
d_{X, \Phi}(G(r), G(t))^2 \leq d_{X, \Phi}(bG(r), bG(t))^2 + \frac{\varepsilon^2}{81c^4} + 2 \frac{\varepsilon}{9c^2} \cdot \|G\|^2
\leq d_{X, \Phi}(bG(r), bG(t))^2 + \frac{\varepsilon}{81} + \frac{2\varepsilon}{9}
\leq d_{X, \Phi}(bG(r), bG(t))^2 + \frac{\varepsilon}{3}.
\end{equation}

Since \( b \in N^0 \), the series \( \sum_i \langle b, x_i \rangle \langle x_i, b \rangle \) is convergent by the definition, and we can find a sufficiently large \( K \) such that
\begin{equation}
\left\| \sum_{i=K+1}^{\infty} \langle b, x_i \rangle \langle x_i, b \rangle \right\| < \frac{\varepsilon}{12 \|G\|^2}.
\end{equation}

Thus for any \( x \in M \),
\begin{equation}
\left\| \sum_{i=K+1}^{\infty} \langle bG(x), x_i \rangle \langle x_i, bG(x) \rangle \right\| = \left\| (G(x))^* \left( \sum_{i=K+1}^{\infty} \langle b, x_i \rangle \langle x_i, b \rangle \right) G(x) \right\|
\leq \|G(x)\|^2 \left\| \sum_{i=K+1}^{\infty} \langle b, x_i \rangle \langle x_i, b \rangle \right\| < \frac{\varepsilon}{12} \|x\|^2.
\end{equation}

Taking into the account Lemma 1.1 and (20) we have for \( k > K \),
\begin{equation}
\sum_{i=k}^{\infty} |\varphi_k (\langle bG(r - t), x_i \rangle) |^2 \leq \varphi_k \left( \sum_{i=k}^{\infty} \langle bG(r - t), x_i \rangle \langle x_i, bG(r - t) \rangle \right) \leq \frac{\varepsilon}{12} \cdot 2^2 = \frac{\varepsilon}{3}.
\end{equation}
Thus, by (21) and (19),
\begin{equation}
d_{X, \Phi}(G(r), G(t))^2 \leq \sup_{k \leq K} \sum_{i=k}^{K} |\varphi_k (\langle bG(r - t), x_i \rangle) |^2 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.
\end{equation}
The first summand can be considered as the calculation of the distance in the finite-dimensional complex space \( C^d \) of dimension \( d = K + (K - 1) + \ldots + 1 \) with the norm
\begin{equation}
\|(u_{1,1}, \ldots, u_{1,K}, u_{2,1}, \ldots, u_{2,K-1}, \ldots, u_{K,K})\|^2
\end{equation}
\begin{equation*}
= \sup \left\{ |u_{1,1}|^2 + \cdots + |u_{1,K}|^2, |u_{2,1}|^2 + \cdots + |u_{2,K-1}|^2, \ldots, |u_{K,K}|^2 \right\}
\end{equation*}

between the images of \( G(r) \) and \( G(t) \) under the following bounded \( \mathbb{C} \)-linear map
\[
R : G(B) \to \mathbb{C}^d, \quad y \mapsto \{ \varphi_k(\langle x_i, by \rangle) \}, \quad k = 1, \ldots, K, \quad i = k, \ldots, K.
\]

(We have transposed entries to have not an anti-linear, but a linear map.) Thus one can find an \( \varepsilon/3 \)-net in \( G(B) \) for the semi-norm \( \gamma \), being the composition of \( R \) and (23). Namely, find an \( \varepsilon/3 \)-net \( R(G(t_1)), \ldots, R(G(t_s)) \), \( t_i \in B \), in \( \mathbb{C}^d \) for the bounded set \( R(G(B)) \) and the norm (23). Then \( G(t_1), \ldots, G(t_s) \) will be an \( \varepsilon/3 \)-net for \( \gamma \). By (22) \( G(t_1), \ldots, G(t_s) \) will be an \( \varepsilon \)-net for \( d_{X, \Phi} \).

\[\square\]

4. THE GENERAL CASE

**Definition 4.1.** Denote by \( q_n \) the orthogonal projection in \( \ell_2(A) \) onto the \( n \)-th summand of the standard decomposition of \( \ell_2(A) \).

Denote by \( Q_n := q_1 + \cdots + q_n \) the orthogonal projection onto the module \( L_n \cong A^n \) formed by the first \( n \) standard summands.

The proof of the general case of Main Theorem 2.5 will be done in the following way and uses, in particular, a reduction to Theorem 3.1.

**Proof of Theorem 2.5.** Denote by \( S \) the Kasparov stabilization \( S : \mathcal{N} \to \mathcal{N} \oplus \ell_2(A) \cong \ell_2(A) \) and prove the theorem moving along the following cycle of statements:

\begin{enumerate}
\item \( F : \mathcal{M} \to \mathcal{N} \) is an \( A \)-compact operator and \( \mathcal{N} \) is a countably generated \( A \)-module.

If a combination of \( \theta_{x,y} \) approximates \( F \), where \( x \in \mathcal{N}, \ y \in \mathcal{M} \), then the same combination of \( \theta_{S(x),y} \) approximates \( F' = S \circ F \). Thus, we obtain:

\item The image of its Kasparov stabilization \( F' : \mathcal{M} \to \mathcal{N} \subset \mathcal{N} \oplus \ell_2(A) \) is contained in a countably generated module \( S(\mathcal{N}) \) and \( F' \) is \( A \)-compact relatively \( S(\mathcal{N}) \).

By Lemma 4.3 below this implies:

\item For arbitrary \( \varepsilon > 0 \) there exists \( D \) such that \( \|F' - Q_DF'\| < \varepsilon \), the image of \( Q_DF' \) is contained in a countably generated module \( Q_DS(\mathcal{N}) \) and \( Q_DF' \) is \( A \)-compact relatively \( Q_DS(\mathcal{N}) \).

By Lemma 1.12 we obtain:

\item For arbitrary \( \varepsilon > 0 \) there exists \( D \) such that \( \|F' - Q_DF'\| < \varepsilon \), the image of \( q_iF' \) is contained in a countably generated module \( q_iS(\mathcal{N}) \) and \( q_iF' \) is \( A \)-compact relatively \( q_iS(\mathcal{N}) \), \( i = 1, \ldots, D \).

By Theorem 3.1 we arrive to:

\item For arbitrary \( \varepsilon > 0 \) there exists \( D \) such that \( \|F' - Q_DF'\| < \varepsilon \), the image of \( q_iF' \) is contained in a countably generated module \( q_iS(\mathcal{N}) \oplus \cdots \oplus q_DS(\mathcal{N}) \) and \( Q_DF'(B) \subset \mathcal{A} \oplus \cdots \oplus \mathcal{A} \) is \( (\mathcal{A} \oplus \cdots \oplus \mathcal{A}, q_1S(\mathcal{N}) \oplus \cdots \oplus q_Ds(\mathcal{N})) \)-totally bounded.

Apply inductively Lemma 2.6, keeping in mind that \( q_iQ_D = q_i \) and obtain:

\item For arbitrary \( \varepsilon > 0 \) there exists \( D \) such that \( \|F' - Q_DF'\| < \varepsilon \), the image of \( Q_DF' \) is contained in a countably generated module \( q_1S(\mathcal{N}) \oplus \cdots \oplus q_Ds(\mathcal{N}) \) and \( Q_DF'(B) \subset \mathcal{A} \oplus \cdots \oplus \mathcal{A} \) is \( (\mathcal{A} \oplus \cdots \oplus \mathcal{A}, q_1S(\mathcal{N}) \oplus \cdots \oplus q_Ds(\mathcal{N})) \)-totally bounded.

By Lemma 4.4 below we have:
The image of $F^\prime : \mathcal{M} \to \ell_2(\mathcal{A})$ is contained in a countably generated module $\mathcal{N}^1 := q_1S(\mathcal{N}) \oplus \cdots \oplus q_nS(\mathcal{N}) \oplus \cdots$ (where the $C^\ast$-Hilbert sum supposes taking the closure) and $F^\prime(B)$ is $(\ell_2(\mathcal{A}), \mathcal{N}^1)$-totally bounded.

We have $S^\ast(\mathcal{N}^1) = \mathcal{N}$. Indeed, for an arbitrary small $\varepsilon$ and any $x \in S(\mathcal{N})$ we can find $Q_n$ such that $\|Q_n x - x\| < \varepsilon$, where we have $Q_n(x) = q_i x + \cdots + q_n x \in \mathcal{N}^1$. Since $\mathcal{N}^1$ is closed, this implies that $S$ maps injectively $\mathcal{N} \to \mathcal{N}^1$. Then $S(\mathcal{N})$ has an orthogonal complement $\mathcal{N}^2$ in $\mathcal{N}^1$, namely, $\mathcal{N}^2 = \mathcal{N}^1 \cap (S(\mathcal{N}))^\perp$, where $(S(\mathcal{N}))^\perp$ is the orthogonal complement in $\ell_2(\mathcal{A})$.

Thus, we have $\mathcal{N}^1 = S(\mathcal{N}) \oplus \mathcal{N}^2$. Since $S^\ast S(y) = y$ and $S^\ast(z) = 0$ if $z \in \mathcal{N}^2$, $S^\ast : \mathcal{N}^1 \to \mathcal{N}$ is a surjection. Now we can apply Lemma 2.6 to the direct sum $\mathcal{N}^1 = S(\mathcal{N}) \oplus \mathcal{N}^2$ to obtain:

$F(B)$ is $(\mathcal{N}, \mathcal{N})$-totally bounded.

We can apply Lemma 2.6 “in the opposite direction”, because evidently $(1 - P)F^\prime(B) = 0$ is totally bounded, where $P = SS^\ast$ is the orthogonal projection onto $S(\mathcal{N})$. Then we have:

$F^\prime(B)$ is $(\ell_2(\mathcal{A}), S(\mathcal{N}))$-totally bounded.

By Lemma 4.4 below we obtain:

For arbitrary $\varepsilon > 0$ there exists $D$ such that $\|F^\prime - Q_D F^\prime\| < \varepsilon$, the image of $Q_D F^\prime$ is contained in a countably generated module $Q_D S(\mathcal{N})$ and $Q_D F^\prime(B) \subseteq \mathcal{A} \oplus \cdots \oplus \mathcal{A}$ is $(\mathcal{A} \oplus \cdots \oplus \mathcal{A}, Q_D S(\mathcal{N}))$-totally bounded.

Apply inductively Lemma 2.6 (keeping in mind that $q_i Q_D = q_i$) to obtain:

For arbitrary $\varepsilon > 0$ there exists $D$ such that $\|F^\prime - Q_D F^\prime\| < \varepsilon$, the image of $q_i F^\prime$ is contained in a countably generated module $q_i S(\mathcal{N})$ and $q_i F^\prime(B) \subseteq \mathcal{A}$ is $(\mathcal{A}, p_i S(\mathcal{N}))$-totally bounded, $i = 1, \ldots, D$.

By Theorem 3.1 we obtain:

For arbitrary $\varepsilon > 0$ there exists $D$ such that $\|F^\prime - Q_D F^\prime\| < \varepsilon$, the image of $q_i F^\prime$ is contained in a countably generated module $q_i S(\mathcal{N})$ and $q_i F^\prime : \mathcal{M} \to \mathcal{A}$ is $\mathcal{A}$-compact, $i = 1, \ldots, D$.

Applying inductively Lemma 1.9 we arrive to:

For arbitrary $\varepsilon > 0$ there exists $D$ such that $\|F^\prime - Q_D F^\prime\| < \varepsilon$ and $Q_D F^\prime : \mathcal{M} \to L_D$ is $\mathcal{A}$-compact.

The operator $F^\prime$ is approximated by $\mathcal{A}$-compact operators $Q_D F^\prime$. Thus, $F^\prime$ is $\mathcal{A}$-compact (find more detail in Lemma 4.2):

$F^\prime$ is $\mathcal{A}$-compact.

Since $F = S^\ast \circ F^\prime$, Proposition 1.8 implies:

$F : \mathcal{M} \to \mathcal{N}$ is $\mathcal{A}$ compact. \hfill \Box

**Lemma 4.2.** Suppose $F^\prime : \mathcal{M} \to \ell_2(\mathcal{A})$ is an adjointable operator and for any $\varepsilon > 0$ there exists an integer $D$ such that

1. $\|F^\prime - Q_D F^\prime\| < \varepsilon$,
2. $Q_D F^\prime$ is an $\mathcal{A}$-compact operator (as an operator $\ell_2(\mathcal{A}) \to L_D$).

Then $F^\prime$ is $\mathcal{A}$-compact.
Proof. Since $L_D$ has an orthogonal complement in $\ell_2(\mathcal{A})$, the operator $Q_D F'$ is $\mathcal{A}$-compact as an operator $\ell_2(\mathcal{A}) \to \ell_2(\mathcal{A})$. Hence, $F'$ is approximated by $\mathcal{A}$-compact operators and $F'$ is $\mathcal{A}$-compact itself. \hfill \square

**Lemma 4.3.** Suppose the image of an adjointable operator $F' : \mathcal{M} \to \ell_2(\mathcal{A})$ is contained in a countably generated module $\mathcal{N}^0$ and $F'$ is an $\mathcal{A}$-compact operator relatively $\mathcal{N}^0$. Then for any $\varepsilon > 0$ there exists an integer $D$ such that

1) $\|F' - Q_D F'\| < \varepsilon$, 
2) $Q_k F'$ is an $\mathcal{A}$-compact operator relatively $Q_k \mathcal{N}^0$ for any $k$.

**Proof.** Approximate $F'$:

$$||F' - (\theta_{x_1,y_1} + \cdots + \theta_{x_r,y_r})|| < \varepsilon/3, \quad x_i, y_i \in \mathcal{N}^0.$$ 

Find a sufficiently large $D$ such that

$$\|x_j - Q_D x_j\| < \frac{\varepsilon}{3r\|y_j\|}, \quad j = 1, \ldots, r.$$ 

Then

$$\|\theta_{x_j,y_j} - Q_D \theta_{x_j,y_j}\| < \frac{\varepsilon}{3r}, \quad j = 1, \ldots, r,$$

and

$$\|F' - Q_D F'\| \leq \|F' - (\theta_{x_1,y_1} + \cdots + \theta_{x_r,y_r})\| + \|Q_D (F' - (\theta_{x_1,y_1} + \cdots + \theta_{x_r,y_r}))\|$$

$$+ \sum_{j=1}^{r} \|\theta_{x_j,y_j} - Q_D \theta_{x_j,y_j}\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + r \frac{\varepsilon}{3r} = \varepsilon.$$

Once again use the first inequality of the proof and obtain for an arbitrary $k$ and $\varepsilon/3$:

$$\varepsilon/3 > \|Q_k (F' - (\theta_{x_1,y_1} + \cdots + \theta_{x_r,y_r}))\| = \|Q_k F' - (\theta_{Q_k(x_1),y_1} + \cdots + \theta_{Q_k(x_r),y_r})\|.$$ 

Since $Q_k(x_j) \in Q_k(\mathcal{N}^0)$, we are done. \hfill \square

**Lemma 4.4.** Suppose $F' : \mathcal{M} \to \ell_2(\mathcal{A})$ is an adjointable operator. If $F'(B)$ is totally bounded relatively a countably generated submodule $\mathcal{N}^0$ if and only if

1) for any $\varepsilon > 0$ there exists an integer $D$ such that $\|F' - Q_D F'\| < \varepsilon$, 
2) $Q_k F'(\mathcal{M})$ is contained in a countably generated submodule $\mathcal{N}_k^0$, satisfying

$$Q_k \mathcal{N}_{k+1}^0 = \mathcal{N}_k^0$$

for any $k$. 
3) $Q_k F'(B)$ is totally bounded relatively $\mathcal{N}_k^0$ for any $k$.

Conversely, if 1), 2), and 3) have place, then $F'$ is totally bounded relatively the countably generated submodule $\mathcal{N}^0 := \bigcup_k \mathcal{N}_k^0$ in $\ell_2(\mathcal{A})$.

**Proof.** Suppose that $F'(B)$ is totally bounded relatively a countably generated submodule $\mathcal{N}^0$. Take $\mathcal{N}_k^0$ to be $Q_k \mathcal{N}^0$. Then evidently, 2) takes place, while 3) follow from Lemma 2.6 for mutually complement projections $Q_k$ and $1 - Q_k$. Suppose that 1) is not true. Then there exists $\delta > 0$ and a sequence of elements $z_i \in B$ and a corresponding increasing sequence of numbers $j(i) \to \infty$ such that $\|q_j(i) F'(z_i)\| > \delta$. Let $\mu_i$ be an element of an approximate unit of $\mathcal{A}$ (generally uncountable) such that

$$\|\mu_i q_j(i) F'(z_i)\| > \frac{3}{4} \delta.$$
Suppose \( x_i \in \ell_2(\mathcal{A}) \) has \( \mu_i \) at the \( j(i) \)-th place and zeros on the remaining ones. In other words,
\[
q_{j(i)} x_i = \mu_i, \quad (1 - q_{j(i)}) x_i = 0.
\]
Rewrite (25) as
\[
\left\| \frac{\mu_i q_{j(i)} F'(z_i)}{\| F' \|} \right\| > \frac{3\delta}{4 \| F' \|}
\]
and use Lemma 1.2 to find a state \( \varphi_i \) such that
\[
\left| \varphi_i \left( \frac{\mu_i q_{j(i)} F'(z_i)}{\| F' \|} \right) \right| > \frac{1}{2} \frac{3\delta}{4 \| F' \|},
\]
or
\[
(26) \quad \left| \varphi_i (\mu_i q_{j(i)} F'(z_i)) \right| > \frac{3\delta}{8 \| F' \|}.
\]
For this data there exists \( d_{X,\Phi} \) with \( X = \{ x_i \} \) and \( \Phi = \{ \varphi_i \} \), because \( X \) is evidently admissible even for the entire \( \ell_2(\mathcal{A}) \) (cf. Example 2.3). Thus one can find a finite collection \( \{ y_1, \ldots, y_n \} \) of elements of \( B \) such that for any \( y \in B \) there exists \( k \in \{ 1, \ldots, n \} \) with
\[
(27) \quad d_{X,\Phi}(F'(y), F'(y_k)) < \frac{\delta}{8 \| F' \|}.
\]
There exists a sufficiently large \( D \) such that
\[
(28) \quad \| (1 - Q_D) F'(y_k) \| < \frac{\delta}{8 \| F' \|}, \quad k = 1, \ldots, n.
\]
Then for \( j(i) > D \) and any \( k \in \{ 1, \ldots, n \} \), by (26) and (28) we have
\[
d_{X,\Phi}(F'(z_i), F'(y_k)) \geq |\varphi_i(\langle F'(z_i) - F'(y_k), x_i \rangle)| = |\varphi_i(\langle x_i, F'(z_i) - F'(y_k) \rangle)|
\]
\[
= |\varphi_i(\mu_i q_{j(i)} F'(z_i)) - \varphi_i(\mu_i q_{j(i)} F'(y_k))| \geq \frac{3\delta}{8 \| F' \|} - \frac{\delta}{8 \| F' \|} = \frac{\delta}{4 \| F' \|}.
\]
A contradiction with (27).

Conversely, items 1) and 2) imply that \( F'(\mathcal{M}) \) is contained in the above defined module \( \mathcal{N}^0 \), which is a countably generated submodule with a set of generators being a union of countable generating sets for all \( Q_k F'(\mathcal{M}) \) (cf. the proof of Lemma 1.10).

Let \( d_{X,\Phi} \) be a seminorm for some admissible \( X \in \ell_2(\mathcal{A}) \) for this \( \mathcal{N}^0 \). Consider an arbitrary \( \varepsilon > 0 \). Choose a sufficiently large \( n \) such that
\[
(29) \quad \| Q_n F' - F' \| < \frac{\varepsilon}{3}.
\]
As in the proof of Lemma 2.6, we see that \( X' := \{ Q_n x_i \} \) is an admissible set for \( Q_n \mathcal{N}^0 = \mathcal{N}^0_n \) (by (24)) and
\[
(30) \quad d_{X',\Phi}(Q_n u, Q_n z) = d_{X,\Phi}(Q_n u, Q_n z).
\]
Thus by item (2), we can find an \( \varepsilon/3 \)-net \( \{ Q_n F'(y_1), \ldots, Q_n F'(y_s) \} \) in \( Q_n F'(\mathcal{M}) \) for \( d_{X',\Phi} \). We claim that \( \{ F'(y_1), \ldots, F'(y_s) \} \) is an \( \varepsilon \)-net in \( F'(\mathcal{M}) \) for \( d_{X,\Phi} \). Indeed, for any \( y \in B \) find a number \( k \in \{ 1, \ldots, s \} \) such that
\[
d_{X',\Phi}(Q_n F'(y_k), Q_n F'(y)) < \varepsilon/3.
\]
Then by (30) and (3),
\[
d_{X,\Phi}(F'(y_k), F'(y)) \leq d_{X,\Phi}(F'(y_k), Q_n F'(y_k)) + d_{X,\Phi}(Q_n F'(y_k), Q_n F'(y))
\]
\[ +d_{X,\Phi}(Q_n F'(y), F'(y)) \]
\[ \leq \| F'(y_k) - Q_n F'(y_k) \| + d_{X,\Phi}(Q_n F'(y_k), Q_n F'(y)) + \| Q_n F'(y) - F'(y) \| \]
\[ < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \]

This completes the proof. \qed

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