Bound-State Instability of the Chiral Luttinger Liquid in One-Dimension.

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We have developed a new boot-strap method for solving a class of interacting one-dimensional chiral fermions. The conventional model for interacting right-moving electrons with spin has an $SO(4)$ symmetry, and can be written as four interacting Majorana fermions, each with the same velocity. We have found a method for solving some cases when the velocities of these Majorana fermions are no longer equal. We demonstrate in some detail the remarkable result that corrections to the non skeleton self-energy identically vanish for these models, and this enables us to solve them exactly. For the cases where the model can be solved by bosonization, our method can be explicitly checked. However, we are also able to solve some new cases where the excitation spectrum differs qualitatively from a Luttinger liquid.

Of particular interest, is the so-called $SO(3)$ model, where a triplet of Majorana fermions moving at one velocity, interact with a single Majorana fermion moving at another velocity. We show using our method, that a sharp bound (or anti-bound) state splits off from the original Luttinger liquid continuum, cutting off the X-ray singularity to form a broad incoherent excitation with a lifetime that grows linearly with frequency.

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\section{I. INTRODUCTION}

The anomalous normal state behavior discovered in the cuprate superconductors has stimulated enormous interest in the possibility of new kinds of electronic fluid that might provide an alternative to Fermi liquid behavior. The classic model for non-Fermi liquid behavior is provided by the one-dimensional electron gas, where the generic fixed point behavior is a Luttinger Liquid.\cite{1} Thanks to a wide array of non-perturbative techniques, there is a rather solid understanding of the Non-Fermi Liquid properties in such 1\textit{d} systems. Motivated by an early suggestion of Anderson,\cite{2} many authors have attempted to generalize the Luttinger liquid concept to higher dimensions.\cite{3,4}

The Luttinger Liquid in 1\textit{d} is truly special in that it has no quasi-particle poles but a branch cut singularity; its correlation functions are scale-invariant, with an associated beta function that is zero to all orders in perturbation theory for a wide range in the coupling:

$$\beta(g) = 0.$$ 

That the beta function is zero is not in itself special to the Luttinger Liquid. For example, in the absence of nesting, or a Cooper instability, the beta function associated with Landau’s Fermi Liquid fixed point is also zero for the forward scattering channel.\cite{5}

The profound differences between the Luttinger liquid and Landau Fermi liquid fixed points originate in the special kinematics of one dimension. In 1\textit{d}, the Fermi surface consists of just two points $\pm k_f$ where the electrons interact very strongly, and asymptotically near these Fermi points, energy and momentum conservation impose a single constraint on scattering processes, giving rise to a qualitative enhancement in scattering phase space. This causes the electron to lose its eigen-state status to the collective spin and charge density bosonic modes. Luttinger liquid behavior requires the absence of umklapp interactions, and in this case, left- and right-moving particles are separately conserved. The spin and charge current densities of the right (or left) moving particles are then simply proportional to the corresponding spin and charge densities:

$$J^R_c = v_c \rho^R_c,$$

$$J^R_s = v_s \rho^R_s,$$

so that the continuity equation assumes a special form

$$(\partial_x - iv_{s,c} \partial_x) \rho^R_{s,c} = 0.$$ 

As noted long ago by Dzyaloshinskii and Larkin,\cite{6,7} these conservation laws lead to the vanishing of the $N$-point connected current correlation functions for $N > 2$ ("Loop Cancellation Theorem", see Section IV), which leads to a Gaussian theory for the spin and charge bosons in the Tomonaga Luttinger model, and also for the low energy effective theory of the Hubbard model in 1\textit{d}.

Unfortunately, the special kinematics of one dimension do not survive in higher dimensions, and largely for this reason, attempts to generalize the Luttinger Liquid to $d \geq 2$ with strictly local interactions have been unsuccessful. In one dimension, energy and momentum conservation impose a single constraint on the forward scattering processes, whereas in higher dimensions, they impose independent constraints on the scattering processes. These additional constraints eliminate many of the potentially

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dangerous singularities present in one dimensional scattering processes, stabilizing the Fermi liquid in two or higher dimensions. Lin et. al arrived at the same conclusion, making the passage from one, to two dimensions by coupling $N$ Hubbard chains together and taking the limit $N \to \infty$. Whilst it is possible to circumvent the Fermi liquid in two dimensions by introducing long-range or singular interactions and a route to non-Fermi liquid behavior in 2d that involves strictly local interactions has not yet been found.

An alternative approach has however been advocated by Anderson who noted that higher dimensional non-Fermi liquid behavior might derive from the formation of bound, or anti-bound states above and below the single-particle continuum. Such bound-states play an important role in the formation of the one dimensional Luttinger liquid, where they give rise a finite scattering phase shift at the Fermi energy, driving the formation of X-ray singularities in the spinon-holon continuum.

In this paper, we are motivated by this discussion to examine whether such singularities are robust against the removal of some of the special kinematic symmetries of one dimension. By modifying the 1d kinematics, we show that it is possible to actually split-off bound-states from the spinon-holon continuum giving rise to a new type of one-dimensional non-Fermi liquid that does not rely on the special 1d symmetries mentioned above. The key to our idea is as follows. The electron fluid on the Fermi surface is made up of spin-up and down electrons and holes. Borrowing from the Dirac equation, we can rewrite the electrons and holes as charge-conjugation eigen-states:

$$c_\uparrow = \frac{1}{\sqrt{2}}(\Psi^{(1)} - i\Psi^{(2)}), \quad c_\downarrow = -\frac{1}{\sqrt{2}}(\Psi^{(3)} + i\Psi^{(0)}).$$

where the $\Psi^{(a)} (a = (0, 1, 2, 3))$ represent four chiral Majorana fermions such that $\Psi^{(a)}(x) = \Psi^{(a)}(x)$. Instead of changing the interaction, we modify the scattering kinematics by making one of the Majorana fermions to have a different velocity to the others. In the classic Tomonaga Luttinger model, all four Majorana fermions have the same velocity, (exhibiting the full $SO(4)$ symmetry) and this leads to the special 1d kinematics mentioned. But in our model (with the reduced $SO(3)$ symmetry), lifting the velocity degeneracy causes the energy and momentum conservation to be distinct constraints in scattering phase space. We shall show that in this case, the reduced (relative to the Luttinger model) scattering cuts off the X-ray catastrophe associated with the Luttinger liquid behavior. The “horn-like” feature in the spectral weight of the Luttinger liquid is then split into a sharp bound (or anti-bound) state that co-exists with an incoherent spin-charge decoupled continuum. We summarize these results in Fig.1.

While the main motivation of our model has been to find new fixed point behaviour in 1d, our model also has physical relevance to two recent work:

- The transport phenomenology of the cuprates suggests that electrons near the Fermi surface might divide up into two Majorana modes with different scattering rates and dispersion. To date, this kind of behavior has only been realized in impurity models and their infinite dimensional generalization. We shall show that by breaking the velocity degeneracy of the original chiral Luttinger model, we obtain a one-dimensional realization of this behavior: a sharp Majorana mode intimately co-existing with an incoherent continuum of excitations, reminiscent of the higher dimensional phenomenology.

- Frahm et. al have recently proposed that the low energy effective Hamiltonian of an integrable spin-1 Heisenberg chain doped with mobile spin-1/2 holes is given by (3), with one Majorana fermion $\tilde{\Psi}^{(0)}$ describing a slow moving excitation coming from the dopant, interacting with three rapidly-moving Majorana fermions that describe the spin-1 excitations of the spin-chain. (See Discussion.) Such doped spin-chain models may be relevant to certain experimental systems such as $Y_{2-x}Ca_xBaNiO_5$.

Whereas the $SO(4)$ model can be treated by bosonization by changing the velocity of a single Majorana fermion we introduce a non-linear term into the bosonized Hamiltonian that preclude a separation in terms of Gaussian spin and charge bosons. (See Discussion.)

To tackle this $SO(3)$ model, we have developed a new fermionic bootstrap method, that has its basis the diagrammatic approach of Dzyaloshinskii and Larkin (1974). Their method depends crucially on the existence of...
of conserved currents to eliminate large sets of diagrams, leading to a closed set of equations that can be solved analytically for the Green function. On first glance, the reduced number of conserved currents in the \( SO(3) \) model (compared to the \( SO(4) \) model) causes the Dzyaloshinskii and Larkin method to be inapplicable, because one has to deal with non-conserved current vertices that involve the singlet Majorana fermion of different velocity. We have found however, that by dealing directly with fermionic propagators and the four-legs fermionic vertex, bypassing the intermediate currents, there are enough conservation laws after all to eliminate all vertex corrections to the skeleton self-energy (Fig.2), allowing us to write down a compact set of coupled equations involving only the fully renormalized skeleton self-energy and the exact Green function of the theory.

The plan of the paper is as follows. In Section II, we define the class of models of interest here. In Section III, we describe our modification of the classic Dzyaloshinskii and Larkin\cite{10} diagrammatic method for solving one-dimensional fermionic systems, to deal with our case where not all the velocities are the same. In Section IV, we take advantage of the purely chiral nature of our model\cite{10} to write down a scaling form to simplifying considerably the bootstrap equations derived in Section III. In Section V, we derive asymptotic solutions for frequencies near the spectral weight singularities, and demonstrate our results with numerical solutions. In Section VI, we discuss the nature of this new fixed point. Some of the results have appeared in a brief form in Ho and Coleman\cite{1}.

II. MODEL

The class of model we study here is:

\[
H = \int dx \left\{ -i \sum_{a=0}^{3} v_a \bar{\Psi}^{(a)}(x) \partial_x \Psi^{(a)}(x) + g \bar{\Psi}^{(0)}(x) \Psi^{(1)}(x) \Psi^{(2)}(x) \Psi^{(3)}(x) \right\},
\]

(1)

where \( \Psi^{(a)} \) are real (Majorana) fermions such that \( \Psi^{(a)}(x) = \Psi^{(a)\dagger}(x) \). The fermions are chiral (right-movers, say): this is a crucial property that allows the system stays gapless, and allows for exact solutions in a number of cases.

In the special case where all velocities are the same, this model has an \( SO(4) \) symmetry, where the four Majorana modes can be associated with the spin up and down, electron and hole excitations of the Fermi surface. To see this, write \( c_\uparrow = \frac{1}{\sqrt{2}} (\Psi^{(1)} - i \Psi^{(2)}) \), \( c_\downarrow = -\frac{1}{\sqrt{2}} (\Psi^{(3)} + i \Psi^{(0)}) \), where \( c_\alpha \) are the usual Dirac fermions, and the \( SO(4) \) model is just the conventional one-branch spin-1/2 Luttinger model:

\[
H_{SO(4)} = \int dx \left\{ \sum_{\alpha, \sigma} c_\sigma^\dagger(x) i v \partial_x c_\alpha(x) + H.c. \right\}.
\]

FIG. 2. Renormalized “skeleton self-energy” (SSE), where double lines represent full propagators.

\[-g \left\{ c_\uparrow^\dagger(x)c_\uparrow(x) - 1/2 \right\} \left\{ c_\downarrow^\dagger(x)c_\downarrow(x) - 1/2 \right\}. \]

(2)

This \( SO(4) \) model can be shown by bosonization to be a Luttinger Liquid\cite{10}.

We shall mostly focus on the \( SO(3) \) model where \( v_1 = v_2 = v_3 = v \neq v_0 \):

\[
H = \int dx \left\{ H_{\text{kin}}(x) + g \bar{\Psi}^{(0)}(x) \Psi^{(1)}(x) \Psi^{(2)}(x) \Psi^{(3)}(x) \right\},
\]

\[
H_{\text{kin}}(x) = -iv \sum_{a=1}^{3} \bar{\Psi}^{(a)}(x) \partial_x \Psi^{(a)}(x) - i v_0 \bar{\Psi}^{(0)}(x) \partial_x \Psi^{(0)}(x)
\]

(3)

Note that this model reduces to the single-impurity model of Coleman et. al\cite{10} when the mode \( \Psi^{(0)} \) is made to localize at the impurity site, and Ho and Coleman have studied the same lattice \( SO(3) \) model in high dimensions\cite{10}. We will show that, by making the velocity of one Majorana fermion different, the scattering phase space decreases drastically, leading to this singlet splitting off from the Luttinger continuum to form a sharp bound-state/anti-bound-state. Thus this is a system that has two qualitatively distinct relaxation rates, a dramatic departure from the Luttinger Liquid scenario.

The \( SO(2) \times SO(2) \) model where \( v_0 = v_1 \neq v_2 = v_3 \) is also solvable by bosonization, and interestingly, our bootstrap method also works here. (See the sections Results and Discussion.)

Finally, we shall also briefly look at the \( SO(2) \) model where \( v_0 \neq v_1 \neq v_2 = v_3 \). While we do not know if our method works here, we expect that due to the separate energy and momentum conservation, there is still a restriction of scattering phase space, and the theme of split-off sharp bound/anti-bound-states continues.

Note that the number of degrees of freedom and the interaction are the same in all the cases; the variety of behavior seen is due solely to changes in the scattering phase space, when the velocities of the fermions are made to be different.

III. METHOD: PHILOSOPHY

Our approach is based on the observation that for the \( SO(4) \) and \( SO(3) \) models (and possibly others too), the renormalized skeleton self-energy (SSE) containing full propagators, but no vertex corrections, (Fig.2) is exact, so that
\[ \Sigma_a(x, \tau) = g^2 G_b(x, \tau) G_c(x, \tau) G_d(x, \tau), \]  
\hspace*{1cm} (4)

where the \( G_a \) are the exact, interacting Greens functions and \( \{a, b, c, d\} \) is a cyclic permutation of \( \{0, 1, 2, 3\} \). These equations close with the usual relations:

\[ \Sigma_a(k, \omega) = (i\omega - v_a k) - G_a(k, \omega)^{-1}. \hspace*{1cm} (a = 0, 1, 2, 3) \]  
\hspace*{1cm} (5)

Equations (4,5) together define a boot-strap method to solve the problem.

To show that there are no vertex corrections to the renormalized skeleton self-energy, we first review and then extend Dzyaloshinskii and Larkin’s method. Provided that we have a minimal SO(3) symmetry, then the three current densities \( j^\alpha(x) = -i\epsilon_{\alpha \beta \gamma} \Psi^{(\beta)}(x) \bar{\Psi}^{(\gamma)}(x) \) \((a, b \in \{1, 2, 3\})\) are conserved classically. Following Dzyaloshinskii and Larkin, since charge and current are the same in a chiral model, the continuity equation guarantees that the N-point connected current-current correlation functions vanish for \( N > 2 \): \( \langle x_i \rangle = (x_i, \tau_i) \)

\[ \langle j^\alpha(x_1)j^\beta(x_2) \ldots j^\gamma(x_N) \rangle_C = 0, \hspace*{1cm} (N > 2) \]  
\hspace*{1cm} (6)

For the non-interacting system, this result leads to the “loop cancellation theorem”: for the amplitude associated with a closed fermion loop with \( N > 2 \) conserved current insertions, the sum over all possible permutations of \( \{x_i\} \) of the current operators must give zero. In Appendix A, for illustration, we give a derivation for the \( N = 4 \) case and also for odd \( N \). Dzyaloshinskii and Larkin used this cancellation to eliminate all diagrams that contain such closed loops, considerably simplifying the vertex function and polarization bubbles.

We use the loop cancellation theorem in a new way, to show that the vertex corrections to the skeleton self-energy (SSE) (Fig.2) identically vanish. Unlike Dzyaloshinskii and Larkin, we discard the intermediate currents and the associated current vertices, and deal only with fermionic propagators and the four-leg interaction vertex. The Loop Cancellation Theorem is the same. This method has the advantage that it is more compact (only the self-energy and the Green functions are involved), and treats all propagators in a symmetric manner. To illustrate the idea, consider the self-energy of the singlet Majorana mode in the SO(3) model. Figs. list all such diagrams at order \( g^4 \). The Feynman diagrams contributing to the skeleton self-energy are constructed by combining loops with two insertions. This is clearly true for the second order diagram, and we illustrate this using the first non-trivial order: the fourth order diagram in Fig. (A), but it holds to all orders in perturbation theory. Non-skeleton contributions to the self energy involve diagrams with loops containing more than two current insertions. In these diagrams, the sum over all permutations of the current insertions into the loops is automatically zero, as illustrated to order \( g^4 \) in Fig. (B). A convenient way to represent these diagrams is to split each diagram into a backbone which is the same in all three diagrams, and the 4-insertions loop. Inserting the 4 vertices of the 4-loop in various ways into the 4 vertices of the backbone gives the three diagrams in Fig. (B). Note that this method of generating the diagrams give rise to the correct degeneracy for each of the diagram types (i,ii,iii).

To generalize these results to higher order graphs, it is more convenient to look at the set of diagrams for the free-energy. Cutting a \( \Psi^{(0)} \) line gives back the singlet self-energy \( \Sigma_0 \). We first note that only even orders in \( g \) occur in the free-energy expansion, because the bare Majorana propagators are diagonal in the Majorana flavor index. Next, there is always a closed loop with \( n \) propagators (not necessary of the same type) in any of the free-energy diagrams of order \( g^n \). Otherwise, improper and/or disconnected self-energy diagrams would be generated. Then, at order \( g^6 \) for example, we have the following classes of diagrams listed in Fig. that might generate non-SSE diagrams.

The Loop Cancellation Theorem applies to each case where there is a closed loop with more than 2 propagators of the same kind. Thus case (III) is the only one left. Yet, case (III) generates either SSE diagrams, improper self-energy diagrams (where cutting one of the lines lead to 2 disconnected parts), or else, diagrams that have already been counted in the other cases. The last observation follows from the fact one can always find a closed 6-loop or 4-loop buried in the diagram. Hence, all potential non-SSE generating diagrams disappear! One can clearly generalize the same reasoning to higher order diagrams. We only need to check that this method deals with the combinatoric factors correctly, i.e. all the degeneracies of the diagrams are such that there are no non-SSE diagrams left over. Here, we appeal to the fact that in the SO(4) model, there must also be the correct loop-cancellations, because our method gives the same exact answer as Dzyaloshinskii and Larkin’s method. Even though we have drawn the diagrams treating the triplet lines as identical, these triplet lines actually must carry a Majorana flavor index, and to generate all possible diagrams whether distinct under SO(3) or not, we must draw all possible diagrams with proper indexing of each of the lines. Listing all diagrams this way is independent of which symmetry we are dealing with, and consequently, combinatoric factors will automatically be taken care of in doing loop cancellation with these Majorana indices on the propagator lines. In particular, the symmetry or combinatoric factors for each diagram must be just right to allow loop cancellation to work in the SO(4) case, and hence for the SO(3) case too.

Thus we can show that the vertex corrections to the self-energy \( \Sigma_0 \) of the singlet Majorana fermion cancel to all orders, leaving the fully renormalized SSE as the only remaining contribution. Intriguingly, this argument fails for the \( SO(2) \times SO(2) \) model, because each vertex has two “fast” legs and two “slow” legs, unlike in the SO(3) case where there are only one of the singlet leg. Thus, for example, the non-SSE diagrams in Fig.2
(i) + (ii) + (iii) = 0  (Loop Cancellation Theorem)

FIG. 3. (A) Illustrating how the only non-vanishing singlet self-energy at order $g^4$ is constructed by combining a propagator back-bone with loops containing two vertex insertions. Dotted lines indicate bare propagator for the singlet Majorana fermion $\Psi^{(0)}$. Full lines indicate bare propagator for the triplet Majorana fermions. (B) Illustrating how the non-skeleton self-energy at order $g^4$ is constructed by combining a propagator back-bone with loops containing four vertex insertions.

FIG. 4. List of all classes of free-energy diagrams that generate non-SSE diagrams at order $g^6$. 
IV. METHOD-DETAILS

We now apply this result, using the limiting case of the \( SO(4) \) model to check the validity of our results. Our equations are dramatically simplified by seeking solutions to (8) which satisfy a scaling form

\[
G_a(x, \tau) = \frac{1}{2\pi i x} G_a(\tau/ix),
\]

This form is motivated by the observation that chirality prevents space from acquiring an anomalous dimension. Under a Fourier Transform, this scaling form is self dual,

\[
\frac{1}{2\pi i x} G_a(\tau/ix) \frac{\partial}{\partial \tau} \frac{1}{i\omega} G_a(k/\omega),
\]

where the same function \( G_a \) appears on both sides. Inserting Eqn. (10) into (8) and Fourier transforming,

\[
\Sigma_a(x, \tau) = -\frac{1}{2\pi i x} \frac{d^2}{du^2} [1 - v_a u - 1/G_a(u)]_{u=-\tau/ix}.
\]

Since the bare Green function scaling form is \( 1/G_a^0(u) = 1 - v_a u \), it does not contribute to the self-energy. Combining Eqsns. (4) and (11),

\[
\frac{d^2}{du^2} [G_a(u)]^{-1} = -(g/2\pi)^2 G_b(u) G_c(u) G_d(u)
\]

where \( \{a, b, c, d\} \) are cyclic permutations of \( \{0, 1, 2, 3\} \). The boundary conditions are:

\[
G_a(0) = 1,
\]

\[
G_a'(0) = v_a,
\]

derived from the physical requirement that at high frequencies, the fermions are free particles, moving with the bare velocity \( v_a \). Equations (12) and (13) are the scaling form version of our bootstrap method equations (8). Notice that the differential equation (12) like Eqn. (10) is independent of the sign of the coupling \( g \). Also, (12) has no information on which model of the class \( \{4\} \) it refers to, the symmetry of the model (ie. the velocities) only come in through the boundary conditions (13).

For the \( SO(4) \) model, where \( G_a(u) = G(u) \) (\( a = 0, \ldots , 3 \)), Eqsns. (12) reduce to a single differential equation:

\[
\frac{d^2}{du^2} [G(u)]^{-1} = -(g/2\pi)^2 [G(u)]^3,
\]

for which the solution satisfying the boundary conditions \( G(0) = 1 \), \( G'(0) = v \) is:

\[
G(x, \tau) = \frac{1}{2\pi i x} [1 - v_+ \tau/ix]^{-1/2} [1 - v_- \tau/ix]^{-1/2}
\]

where \( v_+ = v \pm (g/2\pi) \) and \( v \) is the bare velocity. Identical results are obtained by bosonization where \( v_+ \) and \( v_- \) are in fact the velocity of the spin-boson and the charge-boson. Thus this confirms that the skeleton self-energy is exact for the Luttinger model.

V. RESULTS

In the \( SO(4) \) model, the electron spectral weight displays two classic X-ray singularities associated with the
decay of the electron into a spinon and holon continuum. (Fig.6) We now show that if $\Delta v = v - v_0$ is finite, one of these X-ray edge singularities is completely eliminated. If $v_0 < v$, we find that low velocity “horn”, originally with velocity $v_-$, develops a sharp bound-state pole in the singlet channel, and a broad incoherent excitation in the triplet channel with a lifetime growing linearly in energy. If $v_0 > v$, the high velocity “horn” splits off a singlet anti-bound-state and the triplet channel develops a high-velocity incoherent excitation. (Fig.7). The sharp bound-state in the singlet channel develops once a velocity difference is introduced, because energy and momentum conservation now provide distinct constraints to scattering (unlike in the $SO(4)$ model), leading to much less phase space for $\Psi^{(0)}$ to decay into.

To see this, we must analyze Eqn. (13) for the $SO^a$ model.

$$\frac{d^2}{du^2}G_3^{-1} = -(g/2\pi)(G_3)^2G_0,$$

$$\frac{d^2}{du^2}G_0^{-1} = -(g/2\pi)(G_3)^3.$$ (16)

A very convenient way to discuss these equations is to map them onto a central force problem. If we write $r = (G_3^{-1}, G_0^{-1})$, $F = -(gG_3/2\pi)^2(G_0, G_3)$, then $\dot{r} = F$, where $\dot{r} \equiv \frac{du}{dt}$, i.e., $u$ is like “time”. By inspection, $r \times F = 0$, so the force is radial, thus the “angular momentum” $r \times \dot{r} = \Delta v$ is a constant. If we use polar co-ordinates, $(G_3^{-1}, G_0^{-1}) = r(\cos \theta, \sin \theta)$ the equations for the Green-function resemble the motion of a fictitious particle under the influence of an anisotropic central force:

$$\ddot{r} - \frac{\Delta v^2}{r^3} = -(g/2\pi)^2 \frac{1}{r^3 \cos^3 \theta \sin \theta}.$$ (17)

The velocity difference $\Delta v = v - v_0$ provides a repulsive centrifugal force. The boundary conditions (13) mean that the “particle” starts out at $r(0) = \sqrt{2}, \theta(0) = \pi/4$, and with a slope change $\dot{\theta}(0) = \Delta v/2$.

Without loss of generality, let $\Delta v \leq 0$. For $\Delta v > 0$ simply replace $v_+ \rightarrow v_-$, $g \rightarrow -g$. When $\Delta v = 0$, the “particle” falls directly into the origin, and both $G_3$ and $G_0$ diverge with X-ray singularities when the particle first hit the origin at “time” $u = 1/v_+$. Then the particle goes purely imaginary in both coordinates, which gives rise to the Luttinger continuum in the spectral weight, until the time $u = 1/v_-$ when the particle goes back to the origin, leading to the other v-x-ray singularities for both $G_3$ and $G_0$. From then on, the particle stays in the real plane. (Fig.8)

However, once $\Delta v < 0$ is finite, $\dot{\theta}(0) = \Delta v$ causes the orbit to miss the origin at $u \sim 1/v_+$. Instead, $\theta \rightarrow 0$ at some finite “time” $u = 1/v_0$ (Fig.9), at which $r = C$ and $\dot{\theta} = \Delta v/C^2$. For $u \sim 1/v_0$, it follows that $(r, \theta) = (C, \theta(u-1/v_0^2))$, from which we can read off the following asymptotics:
integrating Eqn.17 with the approximation
\[
\tilde{\zeta} \int \frac{\Delta v}{u^2} du = \frac{\pi}{4} \int_0^{1/v_0} \frac{\Delta v}{u^2} du, \quad C \approx \tilde{r}(1/v_0^*). \tag{20}
\]
After doing the integral, this estimate gives (for \( |\Delta v| << |g|/2\pi\)):
\[
v_0^* = v_1 + \frac{g}{\pi} \exp - \left| \frac{g}{2\Delta v} \right|, \tag{21}
\]
\[
Z = \left| \frac{\sqrt{2g}}{\pi \Delta v} \right| \exp - \left| \frac{g}{4\Delta v} \right| \tag{22}
\]
indicating that the formation of the sharp anti-bound-state is non-perturbative in the velocity difference.

To illustrate these results further, we have carried out numerical solutions of the differential equations (16) for intermediate values of the coupling constant \( \zeta \), using a standard adaptive integration routine. Results are summarized in Figs.10.

While we have not established the validity of our method to models of lower symmetry (but see Discussion), we believe that the method captures the essence of the kinematic constraints imposed by energy and momentum conservation, at least for weak coupling. Thus, we have also performed numerical calculations for the \( SO(2) \times SO(2) \) and \( SO(2) \) models.

For the \( SO(2) \times SO(2) \) model, the pair \( \Psi^{(0)}, \Psi^{(1)} \) with the same bare velocity can combine together to form a boson, and similarly for \( \Psi^{(2)}, \Psi^{(3)} \). This leads back to a Luttinger liquid form, but with asymmetric power law singularities at the renormalized velocities \( v_+ \) and \( v_- \) (Fig.11). (Also see Eqn.29 later in Discussion, for the exact analytical solution for this model.)

As we progress to the \( SO(2) \) case, when \( v_0 < v_1, v_2 < v_3 \), we see a sharp pole for the fermion which has the extremal velocity different to all the others, while the Luttinger continuum turns into wide peaks linear in energy for the fermion(s) with intermediate velocities, see Fig.13. This illustrates once more our contention that making one Majorana degree of freedom to have a different (extremal) velocity causes drastic collapse of the scattering phase space for this fermion.

VI. DISCUSSION AND CONCLUSION

A. The 1d Majorana \( SO(3) \) Model

In summary, we have demonstrated that by breaking the velocity degeneracy of a system of interacting chiral fermions we restrict the scattering phase space in a way which causes a sharp bound or anti-bound state to
split off from the spin-charge continuum, leading to a system with two qualitatively distinct spectral peaks and scattering rates. This is a significant departure from the Luttinger liquid scenario and demonstrates a new class of one-dimensional fixed point behavior.

This new fixed point exhibits properties in common with both Luttinger and Fermi liquids, and is perhaps closest in character to the Marginal Fermi liquid phenomenology introduced in the context of cuprate metals. Like the Fermi liquid, there is a sharp quasiparticle bound-state, but this co-exists with a Luttinger-liquid-like continuum which is bounded by two extremal velocities.

As mentioned, the SO(3) model cannot be solved by conventional bosonization, forcing us to introduce this new bootstrap method. Two immediate questions arise:

- the nature of the SO(3) fixed point, and
- the range of validity of the bootstrap method.

In the SO(4) model, the fermionic spectral weight has X-ray singularities at the velocities \( v_+ , v_- \) (see Methods-Details). By bosonization, the model can be mapped onto a theory of free bosons (the spin-boson and charge-boson) moving at \( v_+, v_- \), where for \( g > 0 \), \( v_{\text{spin}} = v_- , \ v_{\text{charge}} = v_+ \), and for \( g < 0 \), the role of \( v_+, v_- \) are swapped round. This is a direct consequence of separate charge and spin conservation in the model. We can demonstrate this in the Majorana fermionic representation. The classically conserved densities are:

\[
J_{01}(x) = -i\Psi^{(0)}(x)\Psi^{(1)}(x), \\
J_{23}(x) = -i\Psi^{(2)}(x)\Psi^{(3)}(x).
\]  

(23)

(By the SO(4) symmetry, we can define other combinations also.) Using the commutation relations listed in Appendix B, we get the equations of motion:

\[
(-\partial_x - vq)J_{01}(q) = \frac{g}{2\pi}qJ_{23}(q), \\
(-\partial_x + vq)J_{23}(q) = \frac{g}{2\pi}qJ_{01}(q).
\]  

(24)

The right hand side of the equations is not zero (as would be expected for conserved currents) because of the anomalous commutator (Appendix B):

\[
[J_{01}(p), J_{01}(q)] = [J_{23}(p), J_{23}(q)] = p\delta(p + q),
\]  

(25)

which is the \( SU(2) \) level 2 Kac-Moody algebra anomaly. Fortunately, by diagonalizing the system (24), the linear combinations \( J_-(q) = J_{01}(q) - J_{23}(q) \) and \( J_+(q) = J_{01}(q) + J_{23}(q) \) do satisfy the continuity equation:

\[
(-\partial_x - v_+ q)J_+(q) = 0, \\
(-\partial_x - v_- q)J_-(q) = 0
\]  

(26)

where \( v_\pm \) is as before in Eqn. (13), indicating that these new densities \( J_\pm \) are proportional to the spin-boson and charge-boson. This then leads to sharp poles in the charge and spin susceptibilities.

For the SO(3) model, using the same definitions (23), we find:

\[
(-\partial_x - v_0 q)J_{01}(q) = \frac{g}{2\pi}qJ_{23}(q) + (v_3 - v_0)K_{01}(q), \\
(-\partial_x - v_3 q)J_{23}(q) = \frac{g}{2\pi}qJ_{01}(q).
\]  

(27)

where \( K_{01}(q) = -i\sum_k k\Psi^{(0)}(q-k)\Psi^{(1)}(k) \). This extra term came from the commutator of \( J_{01} \) and the kinetic energy, and causes the set (27) not to close, and bosonization in terms of free spin and charge-bosons (or any linear combinations) is impossible. In short, because of the anomaly, the classically conserved SO(3) density \( J_{23} \) is admixed with the classically non-conserved \( J_{01} \), leading to the loss of a sharp pole for the susceptibility corresponding to \( J_{23} \). Thus it is unlikely that the model can be mapped to a model of free bosons, which makes it very different to the conformally invariant fixed points of the SO(4) model and the \( SO(2) \times SO(2) \) model (see below). Also the presence of a sharp pole in the fermionic spectral weight indicates that there is at least one fermionic degree of freedom in the diagonalized system.

Now, Frahm et al.\(^[3] \) has conjectured that this SO(3) model is the low-energy effective theory of an integrable model of a spin-1 chain doped with spin-1/2 mobile holes. Using Thermodynamic Bethe Ansatz, they have shown that the spin and charge sectors of the doped holes become decoupled at low temperatures, and have calculated the low temperature free energy of the spin contribution to be:

\[
F_{\text{spin}} = -\frac{\pi T^2}{6v_0}\left(\frac{1}{2} - \frac{3A}{4\pi} \ln A \right) \\
-\frac{\pi T^2}{6v} \left(\frac{3}{2} + \frac{3A}{4\pi} \ln A \right) + \ldots
\]  

(28)
where $A > 0$ is a constant that depends on the doping only. With $v_1 = v_2 = 0$ (undoped case), the first term has been interpreted as coming from a single Majorana fermion of velocity $v_0$, while the second term come from a triplet of massless Majorana fermions with velocity $v$ that represent the $SU(2)$-level 2 WZNW model, which has been shown by Affleck to be the low energy effective theory of the gapless integrable spin-1 chain. Naively, one expects that a system of fermions with two velocities cannot be conformally invariant, unless the two species do not interact with each other and thus form two decoupled sectors that are individually conformally invariant. If the free energy (28) is indeed the $SO(3)$ model low temperature free energy, the form of the free energy suggests that the $SO(3)$ model is again conformally invariant asymptotically, and hence the diagonalized (presumably fermionic) basis consists of two decoupled sectors. Note that the diagonalized basis is unlikely to coincide with the bare Majorana fermions of Eqn. (3) because the coupling is marginal, at least up to $O(g^3)$.

As for the range of validity of the bootstrap method we introduced to solve the model, we note that if we change two Majorana velocities at the same time, so that $v_0 = v_1$ and $v_2 = v_3$, we would have reduced the symmetry still further, to an $SO(2) \times SO(2)$ symmetry. We can solve the differential equations (12) with the results

$$G_3(x, \tau) = \frac{1}{2\pi ix} \left[ 1 - \frac{v_+ \tau}{ix} \right]^{-\frac{1}{2} + \gamma} \left[ 1 - \frac{v_- \tau}{ix} \right]^{-\frac{1}{2} - \gamma},$$

$$v_\pm = \frac{1}{2} \left[ v_0 + v_3 \pm \sqrt{(v_3 - v_0)^2 + (g/\pi)^2} \right],$$

and $\gamma = \frac{1}{2} (v_3 - v_0)(v_3 - v_0)^2 + (g/\pi)^2)^{-\frac{1}{2}}$. Interestingly, this model can be bosonized to a model of free bosons, and the bosonization result agrees exactly with [24]. This is surprising because as far as we can see, the closed-loop cancellation is not sufficient in the case of the $SO(2) \times SO(2)$ model to cancel all vertex corrections. This suggests that a more general cancellation principle is at work, and that the range of validity of our solution may even extend to models with a still smaller, $SO(2)$ symmetry. To date, we have not been able to prove this result.

We also wish to point out that our differential version (13) of the bootstrap equations (14) are of such a simple form only because we have a purely chiral system. If we allow Left- and Right-currents to interact, the scaling form (14) no longer applies [24], and we have not found a different scaling form that allows similar simplifications. However, we expect the bootstrap method to work still, as long as there are separate conservation of Left- and Right- currents. This is true at least for the $SO(4)$ model, because Dzyaloshinskii and Larkin have shown that their method works also for such systems, and our method is a generalization of theirs.

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### VII. APPENDIX A

| $v_0 = v_1 = v_2 = v_3$ | 1 d | 2 d |
|-------------------------|-----|-----|
| Luttinger Liquid | Fermi Liquid |
| spinon-holon continuum | sharp quasi-particles |
| "Majorana Liquid" | ? |
| (anti-) bound-states | ? |

**FIG. 12.**

**B. Broader Issues: higher dimensions?**

Our work raises the question whether this kind of non-Fermi Liquid behavior might survive in dimensions higher than one. In higher dimensions energy conservation and momentum conservation are distinct constraints on scattering phase space, and the Luttinger liquid reverts to a Fermi liquid, at least for short-range interactions. By contrast, the $SO(3)$ model cannot be solved by bosonization, and its unusual properties have reduced reliance on the special kinematics in 1d. Thus, this kind of behavior might be more robust in higher dimensions. In fact, near infinite dimensions, two lifetime behavior persists in the $SO(3)$ model, but here, the thermodynamics near zero temperature is that of a Fermi Liquid. The case of small, but finite dimensions is however, still open.

**V. Diagrammatic method**

In this appendix, we prove by diagrammatic method, the Loop Cancellation Theorem for a loop with four current insertions. It is easiest to prove this in $x, \tau$ space. (For a proof in momentum-frequency space, see Kopietz et al.) Let the four insertions be at $x_i = (x_i, \tau_i)$, $i = 1, \ldots, 4$. Each leg of the loop is a free propagator:

$$G_{ij} \equiv G(x_i, \tau_i; x_j, \tau_j) = \frac{1}{v(\tau_i - \tau_j) + i(x_i - x_j)}.$$  (30)
Denote by $[1234]$ the loop where going clockwise starting from $x_1$, we encounter successively $x_1, x_2, x_3, x_4$, i.e.,

$$[1234] = G_{13}G_{32}G_{21}G_{14}. \quad (31)$$

Without loss of generality, we can fix $x_1$ and sum over permutations of the other three vertices. The Loop Cancellation Theorem then says:

$$[1234] + [1243] + [1342] + [1324] + [1423] + [1432] = 0 \quad (32)$$

But for even number of propagators in a loop, going clockwise is the same as going anti-clockwise, hence, e.g. $[1243] = [1342]$. So, we only need to prove:

$$[1234] + [1243] + [1324] = 0. \quad (33)$$

To do this, we need the important identity:

$$G_{ij}G_{jk} = G_{ik}(G_{ij} + G_{ik}), \quad (34)$$

which can be proven simply by substituting in Eqn (30). Use this to rewrite the loops:

$$[1234] = G_{14}G_{13}G_{32}G_{21} = G_{13}(G_{14} + G_{34})G_{32}G_{21}$$
$$[1243] = G_{21}G_{12}G_{43}G_{21} = G_{21}(G_{34} + G_{43})G_{32}G_{21}$$
$$[1342] = G_{14}G_{13}G_{32}G_{21} = G_{13}(G_{14} + G_{34})G_{32}G_{21}$$

and it is clear that they do all cancel, since $G_{ij} = -G_{ji}$. From this example, we can see that it is important for the cancellation of loops with even number of current insertions, that all the propagators be of the same type, to use the identity (34). In our context, this means all the propagators are for fermions of the same velocity.

For an odd number of insertions, the identity (34) is not needed, because time-reversal invariance guarantees the cancellation: a loop $[1ijkl...xyz]$ will be cancelled by the counter-clockwise partner $[1xyz...lkji]$, thanks to $G_{ij} = -G_{ji}$ and a total of odd number of propagators. (This is the analogue of Furry’s theorem in QED, see eg. Peskin and Schroeder).

We note in passing that this identity (34) can also be used to prove the Ward Identity by diagrammatic methods, order by order (For further information on the Ward Identity and how it is used for diagrammatic methods for finding the exact Green Function in some one-dimensional systems, see Metzner et al).

**VIII. APPENDIX B**

Here we list some commutation relations used to derive the equations of motion of the various (classically) conserved densities. Start from the canonical anti-commutation relation for the Majorana fermions:

$$\{\psi^{(a)}(x), \psi^{(b)}(y)\} = \delta_{ab}\delta(x-y). \quad (36)$$

With the definitions $\delta_{ab}$, and with the $SO(3)$ Hamiltonian $H = H_0 + H_{int}$:

$$H_0 = \int dx - i \sum_{a=1}^{3} v\psi^{(a)}(x)\partial_x \psi^{(a)}(x) - iv\psi^{(n)}(x)\partial_x \psi^{(n)}(0),$$

$$H_{int} = -g \int dx J_0(x)J_{23}(x) \quad (37)$$

we can get straightforwardly:

$$[J_0(p), H_0] = vp J_0(p) + (v - v_0)K_0(p),$$

$$[J_{23}(p), H_0] = vp J_{23}(p). \quad (38)$$

We can recover $SO(4)$ results by setting $v = v_0$. Ordinarily, we would expect $[J_0(p), H_{int}] = 0$, but this is spoilt by the $SU(2)$ level 2 anomalous commutator:

$$[J_0(p), J_0(q)] = [J_{23}(p), J_{23}(q)] = p\delta(p + q). \quad (39)$$

One can derive this by eg. a diagrammatic method, see Ch. 13 of the book by Tselvilius. This then leads to the only non-trivial commutation relations:

$$[J_0(p), H_{int}] = \frac{g}{2\pi}p J_{23}(p),$$

$$[J_{23}(p), H_{int}] = \frac{g}{2\pi}p J_0(p). \quad (40)$$

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28 For a proof of the Ward Identity in QED using diagrammatic methods analogous to ours here, see eg. M.E. Peskin and D.V. Schroeder, “Introduction to Quantum Field Theory”, Addison Wesley (1995).
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31 This is in fact the case for the two bosonizable models mentioned here: the $SO(4)$ model can be bosonized to a free spin-boson with velocity $v_{\text{spin}}$ and a free charge-boson with $v_{\text{charge}}$. For the $SO(2) \times SO(2)$ model, where $v_0 = v_1$ and $v_2 = v_3$, this model bosonizes to two free bosons again, with velocities $v_+$ and $v_-$ as mentioned in DISCUSSION.
32 Ho and Coleman, unpublished.