Maximal $L_p$-regularity of non-local boundary value problems

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Received: 6 November 2013 / Accepted: 8 July 2014 / Published online: 30 July 2014
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Abstract We investigate the $\mathcal{R}$-boundedness of operator families belonging to the Boutet de Monvel calculus. In particular, we show that weakly and strongly parameter-dependent Green operators of nonpositive order are $\mathcal{R}$-bounded. Such operators appear as resolvents of non-local (pseudodifferential) boundary value problems. As a consequence, we obtain maximal $L_p$-regularity for such boundary value problems. An example is given by the reduced Stokes equation in waveguides.

Keywords $\mathcal{R}$-boundedness · Maximal regularity · Boundary value problems · Pseudodifferential operators · Boutet de Monvel’s calculus · Waveguide

Mathematics Subject Classification 35K35 · 35S15 · 47G30 · 46B20

1 Introduction

During the last decade, the theory of maximal $L_p$-regularity turned out to be an important tool in the theory of nonlinear partial differential equations and boundary value problems. Roughly speaking, maximal regularity in the sense of well-posedness of
the linearized problem is the basis for a fixed-point approach to show (local in time) unique solvability for the nonlinear problem. Here, the setting of $L_p$-Sobolev spaces with $p \neq 2$ is helpful in treating the nonlinear terms, due to better Sobolev embedding results. Meanwhile, a large number of equations from mathematical physics has been successfully treated by this method, in particular in fluid dynamics and for free boundary problems. Let us mention only Amann [2] for the general concept of maximal regularity and Escher, Prüss, Simonett [8] for one of the first applications in fluid mechanics.

A densely defined closed operator $A : \mathcal{D}(A) \subset X \to X$ in a Banach space $X$ is said to have maximal $L_p$-regularity, $1 < p < \infty$, in the interval $I = (0, T)$ with $0 < T \leq \infty$ if the Cauchy problem

$$u'(t) + Au(t) = f(t) \quad (t \in I), \quad u(0) = 0,$$

has, for any right-hand side $f \in L_p(I, X)$, a unique solution $u$ satisfying

$$\|u'\|_{L_p(I, X)} + \|Au\|_{L_p(I, X)} \leq C\|f\|_{L_p(I, X)}$$

with a constant $C$ independent of $f$. Here, $W^1_p(I, X)$ refers to the standard $X$-valued first-order Sobolev space. If $I$ is finite or $A$ is invertible an equivalent formulation is that the map

$$\frac{d}{dt} + A : \mathfrak{D}(A) \cap L_p(I, X) \to L_p(I, X)$$

is an isomorphism, where $W^1_p(I, X)$ denotes the space of all elements in $W^1_p(I, X)$ with vanishing time trace at $t' = 0$. Note that non-zero initial values can be treated by an application of related trace theorems. A standard approach to prove maximal regularity is based on operator-valued Mikhlin type results due to Weis [22] and the concept of $\mathcal{R}$-boundedness (see Denk et al. [5], Kunstmann and Weis [17]). For a short introduction to $\mathcal{R}$-boundedness, see Sect. 2 of this paper.

In many applications, the operator $A$ is given as the $L_p$-realization of a differential boundary value problem. Under appropriate ellipticity and smoothness assumptions, maximal regularity is known to hold in this case (see, for example, Denk et al. [5]). However, several applications demand for generalizations to non-local (pseudodifferential) operators and boundary value problems. For instance, the Dirichlet-to-Neumann map in a bounded domain leads to a pseudodifferential operator on the boundary, i.e. on a closed manifold. An example for a non-local boundary value problem is obtained by the pseudodifferential approach to the Stokes equation as developed by Grubb and Solonnikov [15] (see also Grubb [13] and Grubb and Kokholm [14]), which was also one of our motivations.

In the present note we analyze the $\mathcal{R}$-boundedness of operator families belonging to the so-called Boutet de Monvel calculus with parameter. This is a pseudodifferential calculus containing, in particular, the resolvents to a large class of non-local boundary value problems which allows to describe in great detail the micro-local fine structure of
such resolvents. A typical application of the calculus is the following theorem (which, in fact, is a simplified version of Theorem 3.2.7 of Grubb [12]):

**Theorem 1** Let \( A(\mu), \mu \in \Sigma \) (an angular subsector of the complex plane), be a parameter-dependent second order differential operator on a compact manifold \( M \) with smooth boundary, and let \( G(\mu) \) be a weakly parameter-dependent Green operator of order and type less than or equal to 2 and regularity at least \( 1/2 \). Let \( \gamma_0 \) and \( \gamma_1 \) denote Dirichlet and Neumann boundary conditions, respectively. If the parameter-dependent boundary value problem

\[
\begin{align*}
(A(\mu) + G(\mu)) \gamma_j : H^s_p(M) &\longrightarrow H^{s-2}_{p}(M) \\
 &\bigoplus_{B_{pp}^{s-j-1/p}(\partial M)} , \\
\end{align*}
\]

with \( p \in (1, \infty) \) is parameter-elliptic then it is an isomorphism for \(|\mu| \) sufficiently large, and

\[
\left(A(\mu) + G(\mu)\right)^{-1} = (P(\mu) K(\mu)),
\]

with \( P(\mu) \in B^{-2,0 \nu}(M; \Sigma) \) and a parameter-dependent Poisson operator \( K(\mu) \) of order \(-j\).

The involved operator classes as well as the meaning of parameter-ellipticity will be explained in the sequel; the mentioned Green operators are certain non-local operators that are smoothing in the interior of \( M \), but on the whole manifold with boundary have a finite order. As a consequence of (1),

\[
A(\mu) + G(\mu) : \left\{ u \in H^2_p(M) \mid \gamma_j u = 0 \right\} \subset L_p(M) \longrightarrow L_p(M)
\]

is invertible for large \( \mu \) with inverse \( P(\mu) \in B^{-2,0 \nu}(M; \Sigma) \). Making use of this specific pseudodifferential structure we shall derive, in particular, that \( \{(1 + |\mu|)^2 P(\mu) \mid \mu \in \Sigma \} \subset \mathcal{L}(L_p(M)) \) is \( \mathcal{R} \)-bounded, cf. Theorem 11. For the proof we also adopt a tensor-product argument first used in Denk and Krainer [6] in the analysis of the \( \mathcal{R} \)-boundedness of parameter-dependent families of “scattering” or “SG-pseudodifferential” operators (which, roughly speaking, allows to reduce considerations to constant coefficient operators) and use general results of Kalton et al. [16] on the behaviour of \( \mathcal{R} \)-boundedness under interpolation and duality.

There are different versions of Boutet de Monvel’s calculus, one with a strong, the other with a weak parameter-dependence. The first calculus is essentially designed to handle fully differential problems, and is described, for example, in Schrohe and Schulze [20]. The second is a broader calculus developed by Grubb allowing the investigation of certain non-local problems, see Grubb [12] and Grubb and Kokholm [14] for instance. Actually, we shall blend these two versions and consider operator
families depending on two parameters, where one enters in the strong way and the other only weakly; for details see Sect. 3. Though this combination cannot be found explicitly in the literature, we shall use it freely and avoid giving any proofs, since these are quite standard (though laborious if done with all necessary details). Our main result is Theorem 11 stating that such operator families are $R$-bounded as operator families in the $L_p$-space of the bounded manifold. An application is provided in Sect. 5 where we consider a resolvent problem for the Stokes operator in a wave guide (i.e. cylindrical domain) with compact, smoothly bounded cross-section.

Boutet de Monvel’s calculus can also be exploited to demonstrate existence of bounded imaginary powers and even of a bounded $H_\infty$-calculus, cf. Duong [7], Abels [1] and Coriasco et al. [4] for example; as it turns out, the strategy of proof we use in the present work is closely related to that of Abels [1]. On one hand, bounded $H_\infty$-calculus is stronger than $R$-boundedness, on the other hand the concept of $R$-boundedness applies to operator-families more general than the resolvent of a fixed operator.

2 A short review of $R$-boundedness

We will briefly recall the definition of $R$-boundedness and some results that will be important for our purpose. For more detailed expositions we refer the reader to Denk et al. [5] and Kunstmann and Weis [17]. Throughout this section, let $X$, $Y$, $Z$ denote Banach spaces.

A set $T \subset \mathcal{L}(X, Y)$ is called $R$-bounded if there exists a $q \in [1, \infty)$ such that

$$R_q(T) := \sup \left\{ \left( \sum_{z_1, \ldots, z_N = \pm 1} \left( \sum_{j=1}^N z_j A_j x_j \right)^q \right)^{1/q} \left( \sum_{z_1, \ldots, z_N = \pm 1} \left( \sum_{j=1}^N z_j x_j \right)^q \right)^{-1/q} \right\}$$

is finite, where the supremum is taken over all $N \in \mathbb{N}$, $A_j \in T$ and $x_j \in X$ (for which the denominator is different from zero, of course). The number $R_q(T)$ is called the $R$-bound of $T$. It is a consequence of Kahane’s inequality that finiteness of $R_q(T)$ for a particular $q$ implies finiteness for any other choice of $q \geq 1$. Therefore $q$ is often suppressed from the notation. Clearly an $R$-bounded set is norm bounded and its norm-bound is majorized by its $R$-bound. In case both $X$ and $Y$ are Hilbert spaces, $R$-boundedness is equivalent to norm-boundedness.

If $S, T \subset \mathcal{L}(X, Y)$ and $R \subset \mathcal{L}(Y, Z)$ are $R$-bounded then $S + T$ and $RS$ are $R$-bounded, too, with

$$R(S + T) \leq R(S) + R(T), \quad R(RS) \leq R(R)R(S).$$

Under mild assumptions on the involved Banach spaces $R$-boundedness behaves well under duality and interpolation. The following two results can be found in Kalton–Kunstmann–Weis [16], Proposition 3.5 and Proposition 3.7, respectively.
Theorem 2 Let $T$ be an $\mathcal{R}$-bounded subset of $\mathcal{L}(X, Y)$ and assume that $X$ is $B$-convex$^1$. Then

$$T' := \{A' | A \in T\} \quad \text{(set of dual operators)}$$

is an $\mathcal{R}$-bounded subset of $\mathcal{L}(Y', X')$ with $\mathcal{R}(T') \leq C \mathcal{R}(T)$ with a constant $C \geq 0$ not depending on $T$.

Theorem 3 Let $(X_0, X_1)$ and $(Y_0, Y_1)$ be two interpolation couples with both $X_0$ and $X_1$ being $B$-convex. Let $T \subset \mathcal{L}(X_0 + X_1, Y_0 + Y_1)$ such that $T \subset \mathcal{L}(X_j, Y_j)$ is $\mathcal{R}$-bounded with $\mathcal{R}$-bound $\kappa_j$ for $j = 0, 1$. Then

$$T \subset \mathcal{L}((X_0, X_1)_{\theta, p}, (Y_0, Y_1)_{\theta, p}), \quad 0 < \theta < 1, \ 1 < p < \infty,$$

is $\mathcal{R}$-bounded with $\mathcal{R}$-bound $\kappa \leq \kappa_0^{1-\theta} \kappa_1^\theta$, where $(\cdot, \cdot)_{\theta, p}$ refers to the real interpolation method.

The following statement (Proposition 3.3 in Denk et al. [5]) is very useful in analyzing the $\mathcal{R}$-boundedness of families of integral operators.

Theorem 4 Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p < \infty$, and assume that

$$(K_0 f)(\omega) = \int_\Omega k_0(\omega, \omega') f(\omega') \, d\omega'$$

defines an integral operator $K_0 \in \mathcal{L}(L_p(\Omega))$. Let $\{k_\lambda : \Omega \times \Omega \to \mathcal{L}(X, Y) | \lambda \in \Lambda\}$ be a family of measurable integral kernels and $T = \{K_\lambda | \lambda \in \Lambda\}$ be the set of associated integral operators. If

$$\mathcal{R}_p \left(\{k_\lambda(\omega, \omega') | \lambda \in \Lambda\}\right) \leq k_0(\omega, \omega') \quad \text{for all } \omega, \omega' \in \Omega$$

then $T \subset \mathcal{L}(L_p(\Omega, X), L_p(\Omega, Y))$ is $\mathcal{R}$-bounded with

$$\mathcal{R}_p \left(\{K_\lambda | \lambda \in \Lambda\}\right) \leq \|K_0\|_{\mathcal{L}(L_p(\Omega))}.$$
Theorem 5 Let both $X$ and $Y$ have properties $(\mathcal{H}T)$ and $(\alpha)$.\footnote{For the definition of these properties we refer the reader to [17] or [5]. For us it is sufficient to know that scalar-valued $L_p$-spaces, $1 < p < \infty$, have these properties.} Let $T \subset \mathcal{L}(X, Y)$ be $\mathcal{B}$-bounded. Then

$$\left\{ \text{op}(a) | a \in \mathcal{C}^{\ell} (\mathbb{R}_\eta^\ell \setminus \{0\}, \mathcal{L}(X, Y)) \text{ with } \eta^\alpha D_\eta^\alpha a(\eta) \in T \text{ for all } \eta \neq 0 \text{ and } \alpha \in \{0, 1\}^\ell \right\}$$

is an $\mathcal{B}$-bounded subset of $\mathcal{L}(L_p(\mathbb{R}^\ell, X), L_p(\mathbb{R}^\ell, Y))$ with $\mathcal{B}$-bound less than or equal to $C \mathcal{B}(T)$ for some constant $C$ not depending on $T$.

In other words, this Theorem of Girardi and Weis is the operator-valued generalization of the classical theorem of Lizorkin on the continuity of Fourier multipliers in $L_p$-spaces. As an immediate consequence one obtains:

Corollary 1 Denote by $S^d_{\mathcal{B}}(\mathbb{R}^\ell; X, Y)$, $d \in \mathbb{R}$, the space of all smooth functions $a : \mathbb{R}^\ell_\eta \rightarrow \mathcal{L}(X, Y)$ such that $T_\alpha(a) := \{ (\eta)^{-d+|\alpha|} D_\eta^\alpha a(\eta) | \eta \in \mathbb{R}^\ell \}$ is an $\mathcal{B}$-bounded subset of $\mathcal{L}(X, Y)$ for any choice of the multi-index $\alpha$. As shown in Denk and Krainer [6], this is a Fréchet space, by taking as semi-norms the $\mathcal{B}$-bounds of $T_\alpha(a)$. If both $X$ and $Y$ have properties $(\mathcal{H}T)$ and $(\alpha)$ then op induces a continuous mapping

$$S^0_{\mathcal{B}}(\mathbb{R}^\ell; X, Y) \longrightarrow \mathcal{L} \left( L_p(\mathbb{R}^\ell, X), L_p(\mathbb{R}^\ell, Y) \right).$$

For the interested reader, we refer to Portal and Strkalj [18] for a more general result on the $L_p$-continuity of pseudodifferential operators with symbols in operator-valued $S^0_{\varrho,\delta}$-classes of Hörmander type.

3 Boutet de Monvel’s calculus with parameters

In this section, we will present some elements of a parameter-dependent version of Boutet de Monvel’s calculus [3] which we use to describe solution operators of parameter-elliptic boundary value problems subject to homogeneous boundary conditions. The elements of this calculus are operators of the form

$$P(\tau, \mu) = A^\tau(\tau, \mu) + G(\tau, \mu) : \mathcal{S}(\mathbb{R}_+^n) \longrightarrow \mathcal{S}(\mathbb{R}_+^n)$$

(2)

(Extending by continuity to Sobolev spaces), where $k, \ell \in \mathbb{N}$ are some natural numbers, $\mathbb{R}_+^n$ denotes the half-space

$$\mathbb{R}_+^n = \left\{ x = (x', x_n) \in \mathbb{R}^n | x_n > 0 \right\}$$

and $\mathcal{S}(\mathbb{R}_+^n)$ consists of all functions obtained by restricting rapidly decreasing functions from $\mathbb{R}^n$ to the half-space $\mathbb{R}_+^n$ (this space is a Fréchet space by identification
with the quotient space $\mathcal{S}(\mathbb{R}^n)/N$, where $N := \{ u \in \mathcal{S}(\mathbb{R}^n) \mid u = 0 \text{ if } x_n < 0 \}$ is a closed subspace of $\mathcal{S}(\mathbb{R}^n)$.

In (2), $A_+^\tau(\tau, \mu)$ is a parameter-dependent pseudodifferential operator and $G(\tau, \mu)$ a so-called parameter-dependent Green operator (one also speaks of singular Green operators; however for convenience we omit the term ‘singular’). We shall consider two classes of Green operators which are weakly and strongly parameter-dependent, respectively.

In the following, we let $\Sigma$ denote a closed sector in the two-dimensional plane with vertex at the origin. We call a function smooth on $\Sigma$ provided all partial derivatives exist in the interior and extend continuously to $\Sigma$.

We shall frequently make use of pseudodifferential symbols taking values in Fréchet spaces. To this end, let us give the following definition:

**Definition 1** Let $E$ be a Fréchet space with a system $\{ p_j \mid j \in \mathbb{N} \}$ of semi-norms determining its topology. We let $S_d^d(\mathbb{R}^m; E)$, $d \in \mathbb{R}$, denote the space of all smooth functions $a : \mathbb{R}^m \to E$ satisfying uniform estimates $q_{j, \alpha}(a) := \sup_{y \in \mathbb{R}^m} p_j \left( y^{\lvert \alpha \rvert - d} D_y^\alpha a(y) \right) < \infty$ for every $j$ and every multi-index $\alpha$. These semi-norms make $S_d(\mathbb{R}^m; E)$ a Fréchet space. In case $E = \mathbb{C}$ we suppress $E$ from the notation.

The subspace $S_{cl}^d(\mathbb{R}^m; E)$ consists, by definition, of those symbols that have an expansion into homogeneous components in the following sense: there exist $a_{(d-\ell)} \in \mathcal{C}^\infty(\mathbb{R}^m \setminus \{0\}, E)$ satisfying

$$a_{(d-\ell)}(ty) = t^{d-\ell} a_{(d-\ell)}(y), \quad t > 0, \quad y \neq 0,$$

such that

$$R_N(a)(y) := a(y) - \sum_{\ell=0}^{N-1} \chi(y)a_{(d-\ell)}(y) \in S_d^{d-N}(\mathbb{R}^m; E)$$

for any $N \in \mathbb{N}$, where $\chi$ denotes an arbitrary zero-excision function.

The space of smooth positively homogeneous functions $\mathbb{R}^m \setminus \{0\} \to E$ of a fixed degree is canonically isomorphic to $\mathcal{C}^\infty(\mathbb{S}^{m-1}, E)$, the smooth $E$-valued functions on the unit-sphere in $\mathbb{R}^m$. We then equip $S_{cl}^d(\mathbb{R}^m; E)$ with the projective topology with respect to the maps

$$a \mapsto a_{(d-\ell)} : S_{cl}^d(\mathbb{R}^m; E) \to \mathcal{C}^\infty(\mathbb{S}^{m-1}, E),$$

$$a \mapsto R_N(a) : S_{cl}^d(\mathbb{R}^m; E) \to S_{cl}^{d-N}(\mathbb{R}^m; E),$$

where $N$ and $\ell$ run through the non-negative integers. It will be of some importance for us that

$$S_{cl}^d(\mathbb{R}^m; E) = S_{cl}^d(\mathbb{R}^m) \hat{\otimes}_\pi E,$$  \( \hat{\otimes} \) Springer
where $F \hat{\otimes}_\pi E$ denotes the completed projective tensor-product of the two Fréchet spaces $E$ and $F$, see for example Trèves [21]. In other words, $S^d_{cl}(\mathbb{R}^m; E)$ can be identified with the closure of the algebraic tensor product

$$S^d_{cl}(\mathbb{R}^m) \otimes E = \left\{ \sum_{i=1}^{N} a_ie_i \middle| N \in \mathbb{N}, \ a_i \in S^d_{cl}(\mathbb{R}^m), \ e_i \in E \right\}$$

with respect to the system of semi-norms

$$\tilde{q}_{j,\alpha}(a) = \inf \left\{ \sum_{i=1}^{N} q_{\alpha}(a_i) p_j(e_i) \middle| a = \sum_{i=1}^{N} a_ie_i \right\},$$

where $q_{\alpha}$ is as in (3) with $E = \mathbb{C}$.

### 3.1 Parameter-dependent pseudodifferential operators

Let us denote by

$$S^d_{\text{const}}(\mathbb{R}^n \times \mathbb{R} \times \Sigma), \quad d \in \mathbb{R},$$

the space of all smooth functions $a : \mathbb{R}^n_x \times \mathbb{R}_\xi \times \Sigma_{\mu} \longrightarrow \mathbb{C}$ satisfying

$$\sup_{(\xi, \tau, \mu) \in \mathbb{R}^n_x \times \mathbb{R}_\xi \times \Sigma} \left| D_{\xi}^{\alpha} D_{\tau}^{\beta} D_{\mu}^{\gamma} a(\xi, \tau, \mu) \right| \langle \xi, \tau, \mu \rangle^{|\alpha|+|\beta|+|\gamma|+k-d} < \infty$$

for every order of derivatives. This is a Fréchet space and we can define

$$S^d(\mathbb{R}^n_x \times \mathbb{R}^n_x \times \mathbb{R} \times \Sigma) := S^d_{\text{const}}(\mathbb{R}^n_x \times \mathbb{R}^n_x \times \mathbb{R}_\tau \times \Sigma_{\mu}). \quad (5)$$

**Remark 1** Let us emphasise that a symbol $a(x, \xi, \tau, \mu)$ from (5) does not only satisfy the standard uniform symbol estimates

$$\sup_{(x, \xi, \tau, \mu) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \Sigma} \left| D_{x}^{\alpha} D_{\xi}^{\beta} D_{\tau}^{\gamma} D_{\mu}^{\delta} a(x, \xi, \tau, \mu) \right| \langle \xi, \tau, \mu \rangle^{|\alpha|+|\beta|+|\gamma|+|\delta|+k-d} < \infty$$

but also has an expansion into homogeneous components of decreasing degree with respect to the $x$-variable. In particular, if $a$ satisfies the above estimates and has compact $x$-support, it belongs to the space (5). Symbols of the latter type typically arise when working on compact manifolds, by using local coordinate systems and subordinate partitions of unity.

With a symbol $a$ from (5) we associate a family of pseudodifferential operators

$$A(\mu, \tau) = \text{op}(a)(\mu, \tau) : \mathcal{S}(\mathbb{R}^n_x) \rightarrow \mathcal{S}(\mathbb{R}^n_x)$$
in the standard way, i.e.,

$$[A(\mu, \tau)u](x) = \int e^{ix_\xi} a(x, \xi, \tau, \mu) \hat{u}(\xi) d\xi.$$ 

This map can be extended to a map $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ in the space of tempered distributions. Now let

$$e_+: \mathcal{S}(\mathbb{R}^n_+) \to \mathcal{S}'(\mathbb{R}^n_+), \quad r+: \mathcal{S}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n_+),$$

be the operators of extension by 0 and restriction to the half-space, respectively. For $A(\mu, \tau)$ as above we set

$$A_+(\mu, \tau) = \text{op}(a)_+(\tau, \mu) := r_+ \circ A(\mu, \tau) \circ e_+.$$

This gives rise to a map $\mathcal{S}(\mathbb{R}^n_+) \to \mathcal{C}_\infty(\mathbb{R}^n_+)$, for example. If $d = 0$ it induces maps

$$A_+(\mu, \tau) : L_p(\mathbb{R}^n_+) \to L_p(\mathbb{R}^n_+), \quad 1 < p < \infty.$$ (6)

It is this mapping (6) we will be most interested in, and we shall analyze it below for the symbol class we have just introduced. However, for motivations of the calculus (for example, to ensure that $A_+(\tau, \mu)$ preserves the space $\mathcal{S}(\mathbb{R}^n_+)$ and that the operators behave nicely under standard operations like composition) one actually needs to require an additional property of the symbols: the so-called two-sided transmission condition with respect to the boundary of $\mathbb{R}^n_+$. For a symbol $a$ of order $d$ as above, the condition requires that, for any choice of $k \in \mathbb{N}_0$,

$$\left. \mathcal{F}^{-1} \frac{-D^k}{\xi_n} D^k_{\xi'_n} p(x', 0, \xi', (\xi', \tau, \mu)\xi_n, \tau, \mu) \right|_{\pm z > 0} \in S^d_{\text{cl}} \left( \mathbb{R}_{x'_n}^{n-1} ; S^d \left( \mathbb{R}_{\xi'}^{n-1} \times \mathbb{R}_{\tau} \times \Sigma \mu; \mathcal{S}(\mathbb{R}_{\pm z}) \right) \right),$$

i.e., the restriction of the distribution $\mathcal{F}^{-1} \frac{-D^k}{\xi_n} D^k_{\xi'_n} p(x', 0, \xi', (\xi', \tau, \mu)\xi_n, \tau, \mu) \in \mathcal{S}'(\mathbb{R}_z)$ to the half-space $\mathbb{R}_+$ or $\mathbb{R}_-$ can be extended to a rapidly decreasing function on $\mathbb{R}$, and the other variables enter as parameters in the indicated specific way (cf. Definition 2.3 in Schrohe [19], replacing there $\xi'$ by $(\xi', \tau, \mu)$ and passing to the inverse Fourier transform; note that the inverse Fourier transform of a polynomial has support in the origin $z = 0$ and thus is eliminated by restriction to the half-line $\pm z > 0$). Here, $(\xi', \tau, \mu) := (1 + |\xi'|^2 + \tau^2 + |\mu|^2)^{1/2}$. Symbols with the transmission condition form a closed subspace of $S^d(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \Sigma)$ that we shall denote by

$$S^d_{\text{tr}}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \Sigma).$$

**Remark 2** The operator $A_+(\mu, \tau) = \text{op}(a)_+(\tau, \mu)$ does not depend on the values of the symbol $a$ for $x_n < 0$. Hence, if we define $S^d_{\text{tr}}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \Sigma)$ as the closed
subspace of symbols from $S^d_{tr}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \Sigma)$ whose $x$-support is contained in half-space \( \{ x \in \mathbb{R}^n | x_n \leq 0 \} \), then the class of operators is isomorphic to the quotient

\[
S^d_{tr}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \Sigma) / S^d_{-}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \Sigma),
\]

yielding a natural Fréchet topology.

3.2 Parameter-dependent Green operators

We shall use the splitting $\mathbb{R}^n_+ = \mathbb{R}^{n-1}_+ \times \mathbb{R}^+_n$ and write $x = (x', x_n)$ and, correspondingly, $\xi = (\xi', \xi_n)$ for the covariable $\xi$ to $x$. Roughly speaking, Green operators in tangential direction (i.e., on $\mathbb{R}^{n-1}$) act like pseudodifferential operators while in normal direction (i.e., on $\mathbb{R}_+$) they act like integral operators with smooth kernel. However, there is a certain twisting between the two directions which reflects in a specific structure of the operators. To describe this structure we shall need the function $\varrho$ defined by

\[
\varrho(\xi', \tau, \mu) := \langle \mu \rangle \langle \xi', \tau, \mu \rangle - 1.
\]

Note that $0 < \varrho \leq 1$. Now let

\[
R^{d,v}_{const}(\mathbb{R}^{n-1} \times \mathbb{R} \times \Sigma), \quad d \in \mathbb{R}, \quad v \geq 0,
\]

denote the space of all smooth scalar-valued functions $k(\xi', \tau, \mu; x_n, y_n)$ satisfying uniform estimates

\[
\left\| x_n^\ell D_{x_n}^\ell y_n^m D_{y_n}^m k(\xi', \tau, \mu; x_n, y_n) \right\|_{L^2(\mathbb{R}_+ \times \mathbb{R}_+ \times \Sigma)} \leq C_{\ell, \mu} \left( \varrho(\xi', \tau, \mu)^{v-|\ell|+|\ell'|+|m|+|m'|+|\alpha'|+k+1} \right) \times \langle \xi', \tau, \mu \rangle^{d-\ell+\ell'-m+m'-|\alpha'|-k-|\gamma'|},
\]

for every order of derivatives and any $\ell, m \in \mathbb{N}_0$; here $[s]_+ = \max(s, 0)$ for any real number $s$. We call such a $k$ a weakly parameter-dependent symbol kernel of order $d$ and regularity $v$ (with constant coefficients), see also [12]. The best constants define a system of semi-norms, yielding a Fréchet topology. We set

\[
R^{d,v}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \Sigma) = S^0_{cl}(\mathbb{R}^{n-1}; R^{d,v}_{const}(\mathbb{R}^{n-1} \times \mathbb{R} \times \Sigma)).
\]

The class of strongly parameter-dependent symbol kernels

\[
R^{d}_{const}(\mathbb{R}^{n-1} \times \mathbb{R} \times \Sigma) := \bigcap_{v \in \mathbb{R}} R^{d,v}_{const}(\mathbb{R}^{n-1} \times \mathbb{R} \times \Sigma)
\]

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consists of those symbol kernels satisfying the uniform estimates

\[
\left\| x_\ell D_{x_n}^\ell y_\mu D_{y_n}^\mu D_{\xi'} D_{\tau}^\tau D_{\mu}^\mu k(\xi', \tau, \mu; x_n, y_n) \right\|_{L^2(\mathbb{R}_+; \mathbb{R}_+)} \leq C_{\alpha', \ell, \ell', m, m'}(\langle k \rangle^d - \ell + \ell' - m + m' - |\alpha| - |\gamma| - k)
\]

for any order of derivatives and any \( \ell, m \in \mathbb{N}_0 \). Again this is a Fréchet space, and we have

\[
R^d(\mathbb{R}_n^{-1} \times \mathbb{R}_n^{-1} \times \mathbb{R} \times \Sigma) := \bigcap_{v \in \mathbb{R}} R^{d,v}(\mathbb{R}_n^{-1} \times \mathbb{R}_n^{-1} \times \mathbb{R} \times \Sigma)
\]

\[= S^0_{\text{cl}}(\mathbb{R}_n^{-1}; R^d_{\text{const}}(\mathbb{R}_n^{-1} \times \mathbb{R} \times \Sigma)). \tag{9} \]

Let us point out once more the dependence on \( x' \) as a classical symbol of order 0 and not only as a function bounded with all its derivatives, cf. Remark 1. In particular, the class of Green symbols defined above is a subclass of that defined in [14].

**Definition 2** A weakly parameter-dependent Green operator of order \( d \in \mathbb{R} \), type \( r = 0 \), and regularity \( v \) is of the form

\[
[G(\tau, \mu)u](x) = \int e^{ix' \xi'} \int_0^\infty k(x', \xi', \tau, \mu; x_n, y_n) \mathcal{F}_{y' \to \xi'} u(\xi', y_n) dy_n d\xi'. \tag{10}\]

where \( k \in R^{d,v}(\mathbb{R}_n^{-1} \times \mathbb{R}_n^{-1} \times \mathbb{R} \times \Sigma) \) is a weakly parameter-dependent symbol kernel of order \( d \) and regularity \( v \) as introduced above; we occasionally shall write \( G(\tau, \mu) = \text{op}(k)(\tau, \mu) \). Parameter-dependent Green operators of order \( d \in \mathbb{R} \), type \( r \in \mathbb{N} \), and regularity \( v \) have the form

\[
G(\tau, \mu) = G_0(\tau, \mu) + \sum_{j=1}^r G_j(\tau, \mu) D_{x_n}^j \tag{11}\]

where each \( G_j \) has order \( d - j \), type 0, and regularity \( v \). We shall denote this class of operators by \( G^{d,r,v}(\mathbb{R}_n^d; \mathbb{R} \times \Sigma) \). Analogously, we obtain the classes \( G^{d,r}(\mathbb{R}_n^d; \mathbb{R} \times \Sigma) \) of strongly parameter-dependent Green operators, using strongly parameter-dependent symbols kernel. The subclasses \( G_{\text{const}}^{d,r,v} \) and \( G_{\text{const}}^{d,r} \) refer to symbol kernels that do not depend on the \( x' \)-variable.

All the previously introduced spaces of Green operators inherit a Fréchet topology from the underlying spaces of symbol kernels (factoring out the ambiguity of representing Green operators as different linear combinations or, in other words, forming the non-direct sum of Fréchet spaces).

The definition of Green operators in Definition 2 is in the spirit of Schrohe and Schulze [20] and, at the first glance, differs from the original approach in Boutet de Monvel [3], cf. also Grubb and Kokholm [14]. However, both approaches are equivalent as was shown in Lemma 2.2.14 of Schrohe and Schulze [20] in case of strong parameter-dependence; weak parameter-dependence can be treated similarly.
Below we shall make use of an alternative characterisation of strongly parameter-dependent Green operators, see Theorem 3.7 in Schrohe [19], for instance (our variables \(x'\) and \((\xi', \tau, \mu)\) correspond to \(y\) and \(\eta\), respectively, in [19]; the assumption in [19] that both \(y\) and \(\eta\) belong to some \(\mathbb{R}^q\) is only devoted to the context and can be relaxed without any difficulty to \(y \in \mathbb{R}^p\) and \(\eta \in \mathbb{R}^q\) with different dimensions \(p\) and \(q\)).

**Proposition 1** Any strongly parameter-dependent Green operator of order \(d\) and type 0 has a symbol kernel of the form

\[
k(x', \xi', \tau, \mu; x_n, y_n) = \tilde{k}(x', \xi', \tau, \mu; \langle \xi', \tau, \mu \rangle x_n, \langle \xi', \tau, \mu \rangle y_n).
\]

Here,

\[
\tilde{k}(x', \xi', \tau, \mu; s_n, t_n) \in S^0_{\text{cl}} \left( \mathbb{R}^{n-1}_x; S^{d+1}(\mathbb{R}^{n-1}_x \times \mathbb{R}_\tau \times \Sigma; \mathcal{J}(\mathbb{R}_+, s_n \times \mathbb{R}_+, t_n)) \right),
\]

where \(\mathcal{J}(\mathbb{R}_+ \times \mathbb{R}_+) = \mathcal{J}(\mathbb{R}^2)|_{\mathbb{R}_+ \times \mathbb{R}_+}\) and \(S^d(\mathbb{R}^{n-1}_x \times \mathbb{R} \times \Sigma; E)\) for a Fréchet space \(E\) is defined as in Definition 1, replacing \(\mathbb{R}^m\) by \(\mathbb{R}^{n-1}_x \times \mathbb{R} \times \Sigma\).

### 3.3 Some elements of the calculus

Having described parameter-dependent pseudodifferential and Green operators let us introduce the spaces

\[
B^{d,r}_{(\text{const})} (\mathbb{R}^n_+, \mathbb{R} \times \Sigma), \quad B^{d,r,v}_{(\text{const})} (\mathbb{R}^n_+, \mathbb{R} \times \Sigma)
\]

consisting of operators \(A_+ (\tau, \mu) + G(\tau, \mu)\) with a parameter-dependent pseudodifferential operator of order \(d \in \mathbb{Z}\) as in Sect. 3.1 and a—strongly or weakly—parameter-dependent Green operator of order \(d\), type \(r \in \mathbb{N}_0\), and regularity \(v \geq 0\) as described in Sect. 3.2. Using the topologies of both pseudodifferential operators and Green operators introduced above we obtain natural topologies as non-direct sums of Fréchet spaces. Considering the parameter-dependent operators as families of operators \(\mathcal{J}(\mathbb{R}^n_+) \to \mathcal{J}(\mathbb{R}^n_+)\), the following results hold (Theorem 5.1 and Theorem 5.3, respectively, in Grubb and Kokholm [14]).

**Theorem 6** The pointwise composition of operators induces continuous mappings

\[
B^{d_0,r_0,v_0}(\mathbb{R}^n_+; \mathbb{R} \times \Sigma) \times B^{d_1,r_1,v_1}(\mathbb{R}^n_+; \mathbb{R} \times \Sigma) \longrightarrow B^{d,r,v}(\mathbb{R}^n_+; \mathbb{R} \times \Sigma)
\]

where

\[
d = d_0 + d_1, \quad r = \max\{r_1, r_0 + d_1\}, \quad v = \min\{v_0, v_1\}.
\]

Moreover, the subclass of Green operators forms an ideal, i.e., is preserved under composition from the left or the right by operators of the full class. Similar statements.
hold for the classes of strongly parameter-dependent operators (by passing to the intersection over all regularities $\nu \geq 0$).

**Theorem 7** If $d \leq 0$, taking the (formal) adjoint with respect to the $L_2(\mathbb{R}_+^n)$-inner products induces continuous mappings

$$B^{d,0;\nu}(\mathbb{R}_+^n; \mathbb{R} \times \Sigma) \longrightarrow B^{d,0;\nu}(\mathbb{R}_+^n; \mathbb{R} \times \Sigma).$$

The subclasses of Green operators are preserved under taking adjoints. Similar statements hold for the classes of strongly parameter-dependent operators.

It has been shown in Grubb and Kokholm [14] that the operators extend by continuity from the spaces of Schwarz functions to $L_p$-Sobolev spaces. In fact, if we set, with $s \in \mathbb{R}$ and $1 < p < \infty$,

$$H^s_p(\mathbb{R}_+^n) = \left\{ u|_{\mathbb{R}_+^n} : u \in H^s_p(\mathbb{R}^n) \right\} \cong H^s_p(\mathbb{R}^n)/N^s_p,$$

where

$$N^s_p := \left\{ u \in H^s_p(\mathbb{R}^n) | \text{supp} u \subset \mathbb{R}^{n-1} \times (-\infty, 0) \right\},$$

then any element of $B^{d,r,\nu}(\mathbb{R}_+^n; \mathbb{R} \times \Sigma)$ induces pointwise (i.e., for any fixed value of $(\tau, \mu)$) mappings

$$H^s_p(\mathbb{R}_+^n) \longrightarrow H^{s-d}_p(\mathbb{R}_+^n), \quad s > r - 1 + \frac{1}{p}. \quad (12)$$

If we let $B^{d,r}(\mathbb{R}_+^n)$ be the Fréchet space of operators not depending on the parameters $\tau, \mu$ (which is obtained as above by eliminating everywhere the parameters), we have

$$B^{d,r,\nu}(\mathbb{R}_+^n; \mathbb{R} \times \Sigma) \hookrightarrow C^\infty(\mathbb{R} \times \Sigma) \otimes_{\pi} B^{d,r}(\mathbb{R}_+^n);$$

in particular, whenever $s > r - 1 + \frac{1}{p}$,

$$B^{d,r,\nu}(\mathbb{R}_+^n; \mathbb{R} \times \Sigma) \hookrightarrow C^\infty \left( \mathbb{R} \times \Sigma, \mathcal{L}(H^s_p(\mathbb{R}_+^n), H^{s-d}_p(\mathbb{R}_+^n)) \right). \quad (13)$$

**4 $\mathcal{R}$-boundedness of families from Boutet de Monvel’s calculus**

Due to (12), operators of non-positive order and type zero induce families of continuous operators in $L_p$-spaces. We are now going to analyze the $\mathcal{R}$-boundedness of these families. First we consider strongly parameter-dependent Green operators. They can be treated using their particular symbol kernel structure exhibited in Proposition 1.
Theorem 8 Let \( d \leq 0 \) and \( 1 < p < \infty \). Then

\[
G^{d,0}(\mathbb{R}^n_+; \mathbb{R} \times \Sigma) \hookrightarrow S^d_{\mathcal{A}}(\mathbb{R} \times \Sigma; L_p(\mathbb{R}^n_+), L_p(\mathbb{R}^n_+))
\]

(where the latter space is defined as in Corollary 1, replacing \( \mathbb{R}^\ell \) by \( \mathbb{R} \times \Sigma \)).

Proof For convenience we shall use the short-hand notations \( G^{d,0} \), \( G^{d,0}_{\text{const}} \), and \( S^d_{\mathcal{A}} \).

Step 1. We first consider operators with symbol kernel independent of \( x' \). Let \( G \in G^{d,0}_{\text{const}} \) have symbol kernel \( k \). Define

\[
g(\xi', \tau, \mu) : L_p(\mathbb{R}^n_+) \to L_p(\mathbb{R}^n_+)
\]

by

\[
g(\xi', \tau, \mu)u(x_n) = \int_0^\infty k(\xi', \tau, \mu; x_n, y_n)u(y_n)\,dy_n. \tag{14}
\]

Then \( G(\tau, \mu) \) can be understood as the Fourier multiplier with symbol \( g(\cdot, \tau, \mu) \). In view of Theorem 5 it suffices to show that

\[
\{(\tau, \mu)^{-d+k+|\gamma'|}D^{\alpha}_\xi D^k D^\gamma\mu g(\xi', \tau, \mu) | (\xi', \tau, \mu) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \Sigma\}
\]

is an \( \mathcal{A} \)-bounded subset of \( \mathcal{L}(L_p(\mathbb{R}^n_+)) \). Since

\[
(\tau, \mu)^{-d+k+|\gamma'|}D^{\alpha}_\xi D^k D^\gamma\mu g(\xi', \tau, \mu) \leq (\tau, \mu)^{-d+|\alpha|+k+|\gamma'|}.
\]

this follows with Kahane’s contraction principle\(^3\) if we show the \( \mathcal{A} \)-boundedness of

\[
\{(\xi', \tau, \mu)^{-d+|\alpha|+k+|\gamma'|}D^{\alpha}_\xi D^k D^\gamma\mu g(\xi', \tau, \mu) | (\xi', \tau, \mu) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \Sigma\}. \tag{15}
\]

Since \((\xi', \tau, \mu)^{-d+|\alpha|+k+|\gamma'|}D^{\alpha}_\xi D^k D^\gamma\mu g\) is a finite linear combination of symbols like \( g \) we may assume without loss of generality that \(|\alpha| = k = |\gamma'| = 0\). Then we can estimate

\[
|k(\xi', \tau, \mu; x_n, y_n)| = |\tilde{k}(\xi', \tau, \mu; \langle \xi', \tau, \mu \rangle x_n, \langle \xi', \tau, \mu \rangle y_n)| \\
\leq C \langle \xi', \tau, \mu \rangle^{d+1} (\langle \xi', \tau, \mu \rangle (x_n + y_n))^{-1} \\
\leq C \langle \xi', \tau, \mu \rangle^d \frac{1}{x_n + y_n}, \tag{16}
\]

since \( \tilde{k} \) behaves like a symbol of order \( d + 1 \) in \( (\xi', \tau, \mu) \) and is rapidly decreasing in \((s_n, t_n)\). Now the \( \mathcal{A} \)-boundedness of (15) follows from Theorem 4.

\(^3\) This principle states that the inequality

\[
\sum_{z_1, \ldots, z_N = \pm 1} \left| \sum_{j=1}^N z_j \alpha_j x_j \right|^q \leq 2^q \sum_{z_1, \ldots, z_N = \pm 1} \left| \sum_{j=1}^N z_j \beta_j x_j \right|^q
\]

holds true whenever \( \alpha_j, \beta_j \in \mathbb{C} \) with \(|\alpha_j| \leq |\beta_j|\) and \( x_1, \ldots, x_N \in X \) with arbitrary \( N \).
Since $G_{\text{const}}^{d,0}$ is continuously embedded in $\mathcal{C}(\mathbb{R} \times \Sigma; \mathcal{L}(L_p(\mathbb{R}_+^n)))$, cf. (13), the closed graph theorem implies the continuity of the embedding into $S_{\mathcal{B}}^d$.

**Step 2.** Due to Step 1, $G_{\text{const}}^{d,0} \hookrightarrow S_{\mathcal{B}}^d$. In other words, for any semi-norm $p(\cdot)$ of $S_{\mathcal{B}}^d$ there exists a semi-norm $q(\cdot)$ of $G_{\text{const}}^{d,0}$ such that $p(G) \leq q(G)$ for any $G \in G_{\text{const}}^{d,0}$.

For a function $f \in S_{\text{cl}}^0 := S_{\text{cl}}^0(\mathbb{R}_+^n \times \mathbb{R}_+^n)$ let $M_f$ denote the operator of multiplication, $M_f \in \mathcal{L}(L_p(\mathbb{R}_+^n))$. By (9) and (4) we have the identification $G_{\text{const}}^{d,0} = S_{\text{cl}}^0 \otimes \pi G_{\text{const}}^{d,0}$. Now let

$$G = \sum_{j=1}^N M_f j G_j, \quad f_j \in S_{\text{cl}}^0, \quad G_j \in G_{\text{const}}^{d,0}.$$ 

Then $G$ belongs to $S_{\mathcal{B}}^d$ and, with $p(\cdot)$ and $q(\cdot)$ as above,

$$p(G) \leq \sum_{j=1}^N p(M_f j G_j) = \sum_{j=1}^N \|f_j\|_\infty p(G_j) \leq \sum_{j=1}^N \|f_j\|_\infty q(G_j),$$

where $\|\cdot\|_\infty$ is the supremum-norm. By passing to the infimum over all possibilities to represent $G$ as such a linear combination we get

$$p(G) \leq \inf \left\{ \sum_{j=1}^N \|f_j\|_\infty q(G_j) \bigg| G = \sum_{j=1}^N M_f j G_j \right\} =: \widehat{q}(G).$$

However, $\widehat{q}(\cdot)$ induces a continuous semi-norm on the projective tensor product $S_{\text{cl}}^0 \otimes \pi G_{\text{const}}^{d,0}$, cf. the discussion after (4). Since $p(\cdot)$ was arbitrary, we conclude that $S_{\text{cl}}^0 \otimes \pi G_{\text{const}}^{d,0} \hookrightarrow S_{\mathcal{B}}^d$. \hfill $\Box$

It seems that the direct proof of Theorem 8 does not generalize to the case of weakly parameter-dependent Green operators; for example, estimate (16) in the weak case is only valid in case of regularity $\nu \geq 1$. In fact, for a function $f \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}_+)$, the identity

$$|f(x_n, y_n)|^2 = \int_{x_n}^{\infty} \int_{y_n}^{\infty} \partial_u \partial_v \left( f(u, v) \overline{f(u, v)} \right) dudv$$

gives the estimate

$$|f(x_n, y_n)|^2 \leq 2\|f\|_{L_2} \|D_{x_n} D_{y_n} f\|_{L_2} + 2\|D_{x_n} f\|_{L_2} \|D_{y_n} f\|_{L_2},$$

where the norm is that of $L_2(\mathbb{R}_+ \times \mathbb{R}_+)$. Combining this with the estimates of (7), for a symbol kernel $k \in R_{\text{const}}^{d,\nu}(\mathbb{R}_-^{n-1} \times \mathbb{R} \times \Sigma)$, $1/2 \leq \nu < 1$, we obtain only

$$|k(\xi', \tau, \mu; x_n, y_n)| \leq C \langle \mu \rangle^{\frac{\nu}{2}} \langle \xi', \tau, \mu \rangle^d + \frac{1}{x_n + y_n},$$

which is weaker than the estimate (16).
Thus we proceed differently, combining results of Grubb and Kokholm \cite{14} on mapping properties of Green operators in weighted $L_2$-spaces and the stability of $\mathcal{R}$-boundedness under interpolation. To this end, we shall make use of the spaces

$$L_2^\delta(\mathbb{R}+) = L_2(\mathbb{R}+, t^{2\delta} dt), \quad \delta \in \mathbb{R}$$

and the following embeddings (Theorem 1.9 of Grubb and Kokholm \cite{14}).

**Theorem 9** Let $p \geq 2$ be given. Then, for any choice of $0 < \delta' < \frac{1}{2} - \frac{1}{p} < \delta < 1$,

$$\left( H_2^{\delta'}(\mathbb{R}+), H_2^{\delta}(\mathbb{R}+) \right)_{\theta,p} \hookrightarrow L_p(\mathbb{R}+) \hookrightarrow \left( L_2^{-\delta'}(\mathbb{R}+), L_2^{-\delta}(\mathbb{R}+) \right)_{\theta,p}$$

where $\theta$ is chosen such that $\theta \delta + (1 - \theta) \delta' = \frac{1}{2} - \frac{1}{p}$ and $(\cdot, \cdot)_{\theta,p}$ refers to real interpolation.

Moreover, let us introduce the Fréchet space $S_{\mathcal{R},w}^d(\mathbb{R} \times \Sigma; X, Y)$ of smooth functions $a : \mathbb{R} \times \Sigma \to \mathcal{L}(X, Y)$ for which the sets

$$T_{k,\gamma}(a) := \left\{ (\tau, \mu)^{-d} \langle \tau^k \langle \mu \rangle^{|\gamma|} D_{\xi'}^\alpha D_{\mu}^\gamma a(\tau, \mu) | (\tau, \mu) \in \mathbb{R} \times \Sigma \right\}$$

are $\mathcal{R}$-bounded for any choice of $k$ and $\gamma$. The semi-norms are defined as the $\mathcal{R}$-bounds of the sets $T_{k,\gamma}$.

**Theorem 10** If $d \leq 0$, $\nu \geq 1/2$ and $1 < p < \infty$ then

$$G_{d,0;\nu}(\mathbb{R}_+^n; \mathbb{R} \times \Sigma) \hookrightarrow S_{\mathcal{R},w}^d(\mathbb{R} \times \Sigma; L_p(\mathbb{R}_+^n), L_p(\mathbb{R}_+^n)).$$

**Proof** By Theorems 2 and 7 we may assume that $p \geq 2$. Using a tensor product argument as in the second step of the proof of Theorem 8 reduces the proof to showing that $G_{d,0;\nu}^\text{const} \hookrightarrow S_{\mathcal{R},w}^d$. Thus let $G \in G_{d,0;\nu}^\text{const}$. We have to show that

$$\left\{ (\tau, \mu)^{-d} \langle \tau^k \langle \mu \rangle^{|\gamma|} D_{\xi'}^\alpha D_{\mu}^\gamma G(\tau, \mu) | (\tau, \mu) \in \mathbb{R} \times \Sigma \right\}$$

is an $\mathcal{R}$-bounded subset of $\mathcal{L}(L_p(\mathbb{R}_+^n))$. To this end represent $G$ as a Fourier multiplier with symbol $g(\xi', \tau, \mu)$ as done in the proof of Theorem 8. Due to Theorem 5 it suffices to show that

$$\left\{ (\tau, \mu)^{-d} \langle \tau^k \langle \mu \rangle^{|\gamma|} \langle \xi' \rangle^{|\alpha|} D_{\xi'}^\alpha D_{\mu}^\gamma g(\xi', \tau, \mu) | (\xi', \tau, \mu) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \Sigma \right\}.$$

is an $\mathcal{R}$-bounded subset of $\mathcal{L}(L_p(\mathbb{R}_+^n))$. Observing that

$$\langle \tau, \mu \rangle^{-d} \langle \tau^k \langle \mu \rangle^{|\gamma|} \langle \xi' \rangle^{|\alpha|} \leq \langle \xi', \tau, \mu \rangle^{-d+|\gamma|} \langle \xi', \tau \rangle^{k+|\alpha|},$$
that $\langle \xi', \tau, \mu \rangle^{|x|} D^{\xi}_{\mu} g$ has the same structure as $g$, and using Kahane’s contraction principle, we may assume $\gamma = 0$ and show that

$$M_{\alpha,k} := \left\{ \langle \xi', \tau, \mu \rangle^{\alpha} D^{\xi}_{\mu} D^{k}_{\tau} \psi(\xi', \tau, \mu) \mid (\xi', \tau, \mu) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \Sigma \right\}$$

is an $\mathcal{R}$-bounded subset of $\mathcal{L}(L_p(\mathbb{R}^+))$. We know from Theorem 4.1.(5) of Grubb and Kokholm [14] (see actually (4.15) in its proof) that for any $0 < \varepsilon < 1/2$

$$M_{\alpha,k} \subset \mathcal{L}(L_{2-\varepsilon}^p(\mathbb{R}^+), H^2_2(\mathbb{R}))$$

is a bounded set. Since the involved spaces are Hilbert spaces, boundedness coincides with $\mathcal{R}$-boundedness. Then using Theorem 9 (with $\varepsilon = \delta$ and $\varepsilon = \delta'$ where $0 < \delta' < \frac{1}{2}$, respectively) and Theorem 3 we obtain the $\mathcal{R}$-boundedness of $M_{\alpha,k}$ in $\mathcal{L}(L_p(\mathbb{R}^+))$.

Since from Grubb and Kokholm [14] we know that the norm-bound of $M_{\alpha,k}$ can be estimated in terms of semi-norms of $G$, an application of the closed graph theorem yields the continuity of the embedding.

Finally, let us consider a family of pseudodifferential operators

$$A_+(\tau, \mu) = \text{op}_+(a)(\tau, \mu) : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$$

with a symbol $a \in S^d(\mathbb{R}^n \times \mathbb{R} \times \Sigma)$ with $d \leq 0$, cf. (5). Since we consider the operator between $L_p$-spaces only (and not between Sobolev spaces of higher regularity) it is now not necessary to require the transmission property for $a$. We will show that

$$A_+ \in S^d(\mathbb{R} \times \Sigma; L_p(\mathbb{R}^n), L_p(\mathbb{R}^n)).$$

Since $\text{op}_+(a) = r_+ \text{op}(a) e_+$ with the continuous operators $e_+ : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ and $r_+ : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ of extension and restriction, respectively, it suffices to show that

$$\text{op}(a) \in S^d(\mathbb{R} \times \Sigma; L_p(\mathbb{R}^n), L_p(\mathbb{R}^n)).$$

Again by a tensor product argument analogous to that of Step 2 in the proof of Theorem 8, we can assume that $a$ has constant coefficients, i.e., $a \in S^d_{\text{const}}$. However, then the statement follows immediately from Theorem 5, choosing there $X = Y = \mathbb{C}$. Thus we can conclude:

**Theorem 11** Let $d \leq 0$, $\nu \geq 1/2$, and $1 < p < \infty$. Then

$$B^{d,0}(\mathbb{R}^n; \mathbb{R} \times \Sigma) \hookrightarrow S^d(\mathbb{R} \times \Sigma; L_p(\mathbb{R}^n), L_p(\mathbb{R}^n)), \quad B^{d,0;\nu}(\mathbb{R}^n; \mathbb{R} \times \Sigma) \hookrightarrow S^d_{\text{w}}(\mathbb{R} \times \Sigma; L_p(\mathbb{R}^n), L_p(\mathbb{R}^n)).$$
Recalling the definition of the spaces $\mathcal{S}^{d}_{\mathcal{R}}$ and $\mathcal{S}^{d}_{\mathcal{R},w}$ this means that if $P(\tau, \mu) \in B^{d,0}(\mathbb{R}^{n}_{+}; \mathbb{R} \times \Sigma)$ and $Q(\tau, \mu) \in B^{d,0;v}(\mathbb{R}^{n}_{+}; \mathbb{R} \times \Sigma)$ then

$$\left\{ \langle \tau, \mu \rangle^{-d+k+|\gamma|} D^{\gamma}_{\mu} D^{\gamma}_{\tau} P(\tau, \mu)(\tau, \mu) \in \mathbb{R} \times \Sigma \right\},$$

$$\left\{ \langle \tau, \mu \rangle^{-d} \langle \tau \rangle^{k} \langle \mu \rangle^{\gamma} D^{\gamma}_{\tau} D^{\gamma}_{\mu} Q(\tau, \mu)(\tau, \mu) \in \mathbb{R} \times \Sigma \right\}$$

are $\mathcal{R}$-bounded subsets of $\mathcal{L}(L_{p}(\mathbb{R}^{n}_{+})).$

**Corollary 2** Let $P(\tau, \mu) \in B^{d,0}(\mathbb{R}^{n}_{+}; \mathbb{R} \times \Sigma)$ or $P(\tau, \mu) \in B^{d,0;v}(\mathbb{R}^{n}_{+}; \mathbb{R} \times \Sigma)$ with $d \leq 0, \nu \geq 1/2,$ and $p \in (1, \infty).$ Define $(op_{\tau}P)(\mu) := \mathcal{F}_{\tau}^{-1} P(\tau, \mu) \mathcal{F}_{\tau}.$ Then we have

$$(op_{\tau}P) \in S^{d}_{\mathcal{R}} \left( \Sigma; L_{p}(\mathbb{R}^{n+1}_{+}, L_{p}(\mathbb{R}^{n+1}_{+})) \right).$$

**Proof** Due to Theorem 5, we have to show, for $k = 0, 1$ and all $\gamma \in \mathbb{N}_{0},$ the $\mathcal{R}$-boundedness of the set

$$\left\{ \langle \mu \rangle^{-d+|\gamma|} D^{\gamma}_{\mu} D^{\gamma}_{\tau} P(\tau, \mu) \mid \tau \in \mathbb{R} \setminus \{0\}, \mu \in \Sigma \right\}.$$ 

In both cases, this follows from Kahane’s inequality and Theorem 11, since $\langle \mu \rangle^{-d+|\gamma|} \tau^{k} \leq \langle \tau, \mu \rangle^{-d+k+|\gamma|}$ and $\langle \tau, \mu \rangle^{-d} \langle \tau \rangle^{k} \langle \mu \rangle^{\gamma} \leq \langle \tau, \mu \rangle^{-d+k+|\gamma|}.$ $\Box$

In applications, the complex parameter $\mu$ is related to the spectral parameter $\lambda$ appearing in the resolvent of the $L_{p}$-realization of a non-local boundary value problem. We included a second parameter $\tau \in \mathbb{R}$ in order to be able to treat additional parameters arising from the problem itself, e.g., in the form of a covariable in the unbounded direction of a waveguide. In this case, Corollary 2 leads to maximal $L_{p}$-regularity by an application of the Theorem of Weis [22].

### 5 Maximal $L_{p}$-regularity for non-local boundary value problems in a wave-guide

We will study non-local boundary value problems in a wave-guide, i.e., on a cylinder $\mathbb{R} \times M$ whose cross-section is a smooth compact manifold $M$ with boundary $\partial M.$ For this, we need to provide some material on Boutet de Monvel’s calculus on manifolds and the corresponding concept of parameter-ellipticity. We follow Grubb [12] and Grubb and Kokholm [14]. As an application, we study the reduced Stokes problem in a waveguide in Sect. 5.2.

#### 5.1 Manifolds with boundary and parameter-ellipticity

In this section we indicate how the calculus can be modified to cover domains with smooth boundary and how it is used to describe solution operators of certain non-local
boundary value problems. In the sequel we let $M$ denote a smooth compact manifold with boundary. In view of the formulation of parameter-ellipticity given below, we need to describe a refined subclass of the class of Green operators introduced in Sect. 3.2 as well as to introduce another type of operators, the so-called Poisson operators.

5.1.1 Polyhomogeneous Green operators

Let $G(\tau, \mu)$ be a weakly parameter-dependent Green operator of order $d$, type 0, and regularity $\nu$ as described in Definition 2. We call $G(\tau, \mu)$ polyhomogeneous or classical if there exists a sequence of Green operators $G_{d-j}(\tau, \mu)$, $j \in \mathbb{N}_0$, such that, for any $N \in \mathbb{N}_0$,

$$G(\tau, \mu) - \sum_{j=0}^{N-1} G_{d-j}(\tau, \mu) \in G_{d-N,0,\nu-N}(\mathbb{R}_n^+; \mathbb{R} \times \Sigma),$$

and if $k_{d-j}$ are the symbol kernels associated with $G_{d-j}$ as in (10) and (7) (with $d$ replaced by $d-j$) it holds

$$k_{d-j}(x', t\xi', t\tau, t\mu; x_n/t, y_n/t) = t^{d-j} k_{d-j}(x', \xi', \tau, \mu; x_n, y_n)$$

whenever $t \geq 1$ and $|(|\xi'|, \tau)| \geq 1$. Extension by homogeneity allows us to associate with $k_{d-j}$ a symbol kernel $k_{d-j}^h$ defined for $(\xi', \tau) \neq 0$ and satisfying (17) whenever $t > 0$ and $(\xi', \tau) \neq 0$. With this symbol kernel we associate an operator-valued function $g_{d-j}(x', \xi', \tau, \mu)$, $(\xi', \tau) \neq 0$, as in (14). The component of highest degree, $g_{d-j}^h$, is called the principal boundary symbol of $G$.

If $G$ is strongly parameter-dependent, the previous definitions are slightly modified, asking the equality in (17) to hold whenever $t \geq 1$ and $|(|\xi'|, \tau, \mu)| \geq 1$. Then all $g_{d-j}^h(x', \xi', \tau, \mu)$ are defined for $(\xi', \tau, \mu) \neq 0$. We denote the resulting classes by $G_{d,0,\nu}^r(\mathbb{R}_n^+; \mathbb{R} \times \Sigma)$ and $G_{d,0}^r(\mathbb{R}_n^+; \mathbb{R} \times \Sigma)$, respectively.

Forming finite sums as in (11) yields operators of type $r \in \mathbb{N}$. In this case the principal boundary symbol is given by

$$g_{d}^h(x', \xi', \tau, \mu) = g_{0,d}^h(x', \xi', \tau, \mu) + \sum_{j=1}^{r} g_{j,d-j}^h(x', \xi', \tau, \mu) D_{x_{n}}^j.$$

**Definition 3** A weakly parameter-dependent negligible Green operator $C$ of type $r = 0$ and regularity $\nu' \in \mathbb{R}$ on $M$ is an integral-operator with kernel

$$k(\tau, \mu; x, x') \in \mathcal{C}_c(\mathbb{R} \times \Sigma \times M \times M)$$

(smoothness up to the boundary) that satisfies estimates

$$p\left(D_{\xi}^r D_{\mu}^\alpha k(\tau, \mu; \cdot, \cdot)\right) \leq C_{pakN}(\mu)^{\frac{1}{2}-\nu' - |\alpha|} |\tau|^{-N}$$
for any continuous semi-norm $p$ of $\mathcal{C}^\infty(M \times M)$, all orders of derivatives and all $N \in \mathbb{N}$. In case of strong parameter-dependence we ask that $k$ is rapidly decreasing in $(\tau, \mu)$,

$$k(\tau, \mu; x, x') \in \mathcal{S}'(\mathbb{R} \times \Sigma, \mathcal{C}^\infty(M \times M)).$$

Negligible operators of general type $r \in \mathbb{N}$ are of the form

$$C(\tau, \mu) = \sum_{j=0}^{r} C_j(\tau, \mu) D^j,$$

where the $C_j$ are negligible of type 0 and regularity $\nu'$ and $D$ denotes a first order differential operator on $M$ which in a collar neighborhood of the boundary coincides with the derivative in normal direction.

Using a covering of $M$ with local coordinate systems and a subordinate partition of unity, we can now define the classes of (global) parameter-dependent Green operators $G_{d,r,\nu}^c(M; \mathbb{R} \times \Sigma)$ and $C_{d,r}^c(M; \mathbb{R} \times \Sigma)$, using the corresponding classes on the half-space and the negligible operators of the previous definition, where in case of finite regularity $\nu$ the negligible remainders are required to have regularity $\nu' = \nu - d$.

With any such operator we can associate a principal boundary symbol, using the local principal boundary symbols, which is defined on $(T^* \partial M \setminus \{0\}) \times \mathbb{R} \times \Sigma$ in case of weak parameter-dependence and on $(T^* \partial M \times \mathbb{R} \times \Sigma) \setminus \{0\}$ in case of strong parameter-dependence. Here, $T^* \partial M$ denotes the cotangent bundle of $\partial M$.

### 5.1.2 Poisson operators

Parameter-dependent Poisson operators on the half-space are of the form

$$[K(\tau, \mu)u](x) = \int e^{ix',\xi'} k(x', \xi', \tau, \mu; x_n) \hat{u}(\xi') d\xi',$$

where $u(x')$ is defined on the boundary of $\mathbb{R}_+^n$ and the symbol kernel has a specific structure. Poisson operators have an order $d$ and a regularity $\nu$, but there is no type involved. The mentioned structure of a Poisson operator of order $d$ and finite or infinite regularity $\nu$ is obtained by repeating all the constructions of Sect. 3.2 concerning Green operators of type $r = 0$ and regularity $\nu$ by simply eliminating the $y_n$-variable and replacing $d$ by $d - 1/2$. Such a Poisson operator induces (pointwise, for each $(\tau, \mu)$) continuous maps

$$B_{pp}^{s+d-1/p} (\mathbb{R}_+^{n-1}) \rightarrow H^s_p(\mathbb{R}_+^n), \quad s \in \mathbb{R}.$$

To obtain polyhomogeneous Poisson operators one needs to repeat the construction of the previous Sect. 5.1.1, again cancelling the $y_n$-variable.

Again these constructions can be generalized to the case of a manifold $M$, using local coordinate systems and a partition of unity as well as an analogue of Definition 3,
replacing $C^\infty(M \times M)$ by $C^\infty(M \times \partial M)$. The resulting classes we shall denote by $P^{d,\nu}_{cl}(M; \mathbb{R} \times \Sigma)$ and $P^{d}_{cl}(M; \mathbb{R} \times \Sigma)$, respectively.

### 5.1.3 Parameter-elliptic boundary value problems

Let

$$A(\tau, \mu) = \sum_{j+k+\ell+|\alpha|\leq 2} a_{jk\ell\alpha}(x', x_n) \tau^k \mu^\ell D^{\alpha}_{x'} D^j_{x_n}, \quad (\tau, \mu) \in \mathbb{R} \times \Sigma,$$

be a parameter-dependent differential operator on the half-space $\mathbb{R}^n_+$ with coefficients that are smooth up to the boundary. We associate with $A(\tau, \mu)$ two principal symbols, the usual homogeneous principal symbol

$$\sum_{j+k+\ell+|\alpha|\leq 2} a_{jk\ell\alpha}(x', x_n) \tau^k \mu^\ell \xi x' \xi D^j_{x_n}, \quad (\xi, \tau, \mu) \neq 0,$$

and the principal boundary symbol

$$a^h_2(x', \xi', \tau, \mu) = \sum_{j+k+\ell+|\alpha|\leq 2} a_{jk\ell\alpha}(x', 0) \tau^k \mu^\ell \xi x' \xi D^j_{x_n}, \quad (\xi', \tau, \mu) \neq 0.$$

We call $A$ (interior) parameter-elliptic if its usual homogeneous principal symbol is pointwise invertible. Similar constructions make sense on the manifold $M$, leading to principal symbols on $(T^*M \times \mathbb{R} \times \Sigma) \setminus \{0\}$ and on $(T^*\partial M \times \mathbb{R} \times \Sigma) \setminus \{0\}$, respectively.

The following theorem is a very special version of results due to Grubb. We have chosen to only state this special version, since it suffices for our application to the reduced Stokes problem in the next section and since in this way we can keep the exposition shorter. In fact, one may admit more general classes of pseudodifferential operators $A(\tau, \mu)$ of arbitrary order acting between vector bundles as well as more general boundary conditions. For details we refer to Grubb [12] and Grubb and Kokholm [14].

**Theorem 12** Let $A(\tau, \mu)$ be a second order parameter-dependent differential operator on $M$, $G(\tau, \mu) \in G^{2,r,\nu}_{cl}(M; \mathbb{R} \times \Sigma)$ a weakly parameter-dependent polyhomogeneous Green operator of type $r \leq 2$ and regularity $\nu \geq 1/2$. Let $\gamma_0$ and $\gamma_1$ denote Dirichlet and Neumann boundary conditions on $M$, respectively. The boundary value problem

$$\left( A(\tau, \mu) + G(\tau, \mu) \right)_{\gamma_j} : H^s_p(M) \longrightarrow H^{s-2}_p(M) \oplus B^s_{pp}(-1/p)(\partial M), \quad s > 1 + 1/p,$$

satisfies

$$\sum_{j+k+\ell+|\alpha|\leq 2} a_{jk\ell\alpha}(x', x_n) \tau^k \mu^\ell \xi x' \xi D^j_{x_n}, \quad (\xi, \tau, \mu) \neq 0.$$
is called parameter-elliptic if $A(\tau, \mu)$ is interior parameter-elliptic and, whenever $\xi' \neq 0$, the initial value problem

$$
(a^h_2(x', \xi', \tau, \mu) + b^h_2(x', \xi', \tau, \mu))u = 0 \quad \text{on } \mathbb{R}_+,
$$

$$(1 - j)u(0) + j u'(0) = 0$$

has only the trivial solution $u = 0$ in $\mathcal{D}(\mathbb{R}_+)$. In this case (18) is an isomorphism for $|\tau, \mu|$ sufficiently large, and

$$
\left( A(\tau, \mu) + G(\tau, \mu) \right)^{-1} = (P_j(\tau, \mu) K_j(\tau, \mu)),
$$

with an operator $P_j(\tau, \mu) \in B^{-2,0,v}(M; \mathbb{R} \times \Sigma)$ as described in Sect. 3.3 and a Poisson operator $K_j(\tau, \mu) \in P^{-j,v}_{\text{cl}}(M; \mathbb{R} \times \Sigma)$.

**Corollary 3** In the situation of Theorem 12, assume that $A(\tau, \mu) = \mu^2 + \tilde{A}(\tau)$ and that $G(\tau, \mu) = G(\tau)$ is independent of $\mu$. Let $1 < p < \infty$ and $T > 0$ be finite. Define the operator $A$ in $L_p(Z)$ with $Z := \mathbb{R} \times M$ by

$$
\mathcal{D}(A) := \left\{ u \in W^2_p(Z) \mid \gamma_j u = 0 \text{ on } \partial Z \right\},
$$

$$
Au := \text{op}_\tau \tilde{A}(\tau) u + \text{op}_\tau G(\tau) u \quad (u \in \mathcal{D}(A)).
$$

If the boundary value problem (18) is parameter-elliptic in the sector $\Sigma := \{ \mu \in \mathbb{C} \setminus \{0\} : |\arg \mu| \leq \frac{\pi}{4} \} \cup \{0\}$, then $A$ has maximal $L_q$-regularity for every $q \in (1, \infty)$, i.e., the mapping

$$
\partial_t + A : W^1_q((0, T); L_p(Z)) \cap L_q((0, T); \mathcal{D}(A)) \to L_q((0, T); L_p(Z))
$$

is an isomorphism of Banach spaces.

**Proof** Define the operator $A_M(\tau)$ for $\tau \in \mathbb{R}$ by $\mathcal{D}(A_M(\tau)) := \{ v \in W^2_p(M) : \gamma_j v = 0 \}$ and $A_M(\tau)v := \tilde{A}(\tau) + G(\tau)$. By Theorem 12, the resolvent $(\mu^2 + A_M(\tau))^{-1}$ exists for sufficiently large $\mu \in \Sigma$ and is given by $P_j(\tau, \mu) \in B^{-2,0,v}(M; \mathbb{R} \times \Sigma)$. Choosing $\lambda_0 > 0$ sufficiently large, we obtain

$$
\mu^2 \left( \mu^2 + \lambda_0 + A_M(\tau) \right)^{-1} \in B^{0,0,v}(M; \mathbb{R} \times \Sigma).
$$

Setting $\lambda = \mu^2$, Corollary 2 yields

$$
\lambda (\lambda + \lambda_0 + A)^{-1} = \text{op}_\tau \left[ \mu^2 (\mu^2 + \lambda_0 + A_M(\tau))^{-1} \right] \in S^0_q(\Sigma; L_p(Z), L_p(Z)).
$$

---

The operator class on $M$ instead of the half-space is again obtained by using a covering by local coordinate systems and a subordinate partition of unity and taking into account the global smoothing remainders defined in Definition 3.
By the Theorem of Weis [22], $A + \lambda_0$ has maximal $L_q$-regularity for all $1 < q < \infty$. As the time interval $(0, T)$ is assumed to be finite, this gives maximal $L_q$-regularity for $A$. □

As indicated at the end of Sect. 4, the analog results hold for $\tau$-independent operators, i.e., for parameter-elliptic boundary value problems of the form

$$(\lambda + A + G)u = f \text{ in } M,$$

$$\gamma_j u = 0 \text{ on } \partial M,$$

where $A$ and $G$ are (parameter-independent) pseudodifferential and Green operators, respectively.

5.2 The reduced Stokes problem

Let $\Sigma := \{ \mu \in \mathbb{C} \setminus \{0\} : |\arg \mu| \leq \theta \} \cup \{0\}$ with $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. For $\mu \in \Sigma$, we consider in the waveguide $Z := \mathbb{R} \times M$ the resolvent problem

$$\mu^2 u - \Delta u + \nabla p = f \text{ in } Z,$$
$$\text{div } u = 0 \text{ in } Z,$$
$$\gamma_0 u = 0 \text{ on } \partial Z,$$

(19)

where $\gamma_0$ denotes the operator of restriction to the boundary. We write the Laplacian $\Delta$ on $Z$ and the inner normal $\nu$ of $Z$ as $\Delta = \partial_r^2 + \Delta_{M}$ and $\nu = (0, \nu_M)$, respectively, where $r$ denotes the variable of $\mathbb{R}$ and the subscript $M$ indicates the corresponding objects on $M$. We define the boundary operators $\gamma_0$ and $\gamma_1$ on $Z$ by $\gamma_0 u = \nu \cdot \gamma_0 u$ and $\gamma_1 p = \nu (\nabla p)$, respectively. Moreover, let us write $u = (u_1, u)$ with $u_1 : Z \to \mathbb{R}$ and $u : Z \to \mathbb{R}^n$ and analogously $f = (f_1, f)$.

Due to the divergence condition, in (19) we may replace the Laplacian $\Delta$ by $A = \Delta - \nabla \text{div}$ without changing the problem (a ‘trick’ going back to Grubb, Solonnikov [15], eliminating the second order derivatives in the direction normal to the boundary). Doing so, we obtain from (19) that

$$\Delta p = 0 \text{ in } Z,$$
$$\gamma_1 p = \gamma_0 u + \nu f \text{ on } \partial Z,$$

(20)

for any $f$ with $\text{div } f = 0$. If $K$ denotes the operator satisfying $\Delta K = 0$ and $\gamma_1 K = \text{id}$, the first equation in (19) becomes

$$\mu^2 u - \Delta u + \nabla K (\gamma_0 u + \nu f) = f.$$

(21)

$K$ is the Fourier multiplier with symbol $K(\tau)$ satisfying $(\Delta_{M} - \tau^2) K(\tau) = 0$ and $\gamma_{1,M} K(\tau) = \text{id}$ (denoting the co-variable to $r$ by $\tau$). $K(\tau)$ does not belong to Boutet de Monvel’s calculus, but we can say the following:
Lemma 1  There exists a (strongly) parameter-dependent Poisson operator $K_0(\tau) \in P_{cl}^{-1}(M; \mathbb{R})$ and an $\varepsilon = \varepsilon(p) > 0$ such that

$$\nabla(K - K_0)\gamma\nu A : H_p^{2-\varepsilon}(Z) \cap \ker \gamma_0 \rightarrow L_p(Z)$$

continuously, where $K$ and $K_0$ denote the Fourier multipliers with symbol $K(\tau)$ and $K_0(\tau)$, respectively.

In fact, using a partition of unity and local coordinates, the proof of this lemma can be reduced to the model case of $M = \mathbb{R}_+^n$ being the half space. In this case, the symbol kernels of $K(\tau)$ and $K_0(\tau)$ are

$$k(\xi', \tau; x_n) = -\frac{1}{| (\xi', \tau) |} e^{-| (\xi', \tau) | x_n}, \quad k_0(\xi', \tau; x_n) = -\frac{1}{| (\xi', \tau) |} e^{-| (\xi', \tau) | x_n},$$

respectively, where $[\cdot]$ denotes a smooth function that coincides with the usual modulus $|\cdot|$ outside some ball. We shall not go into further details here, but refer the reader to Abels [1] for an analogous construction, in particular to Lemma 4.2 of that paper.

Now, instead of (21), we first consider the problem obtained by replacing $K$ by $K_0$, i.e.

$$\mu^2 u - \Delta u + \nabla K_0 \gamma\nu A u = f' \quad \text{in } Z, \quad \gamma_0 u = 0 \quad \text{on } \partial Z, \quad (22)$$

where $f' = f - \nabla K_0 \gamma\nu f$. The original problem (19) shall be treated below with help of a suitable perturbation argument.

By direct calculation one sees that

$$\gamma\nu A u = -\gamma_{1,M} \partial_r u_1 + \gamma_{\nu M} A_M u, \quad A_M = \Delta_M - \nabla_M \text{div}_M.$$

By writing $\nabla = (\partial_r, \nabla_M)$ and passing to the Fourier transform in $r$, we derive from (22) that

$$\begin{align*}
(\mu^2 + \tau^2 - \Delta_M)(U_1, U) + (i \tau, \nabla_M) K_0(\tau) \left( -\gamma_{1,M} i \tau U_1 + \gamma_{\nu M} A_M U \right) &= (F'_1, F'_1) \\
\text{with boundary conditions } \gamma_{0,M} U_1 &= 0 \quad \text{and} \quad \gamma_{0,M} U = 0. \quad \text{Here, capital letters indicate the Fourier transform of the respective function in the first variable. In particular, the equation for the first component can be written as}
\end{align*}$$

$$C(\tau, \mu) U_1 = B(\tau) U + F'_1, \quad \gamma_{0,M} U_1 = 0,$$

where

$$B(\tau) = -i \tau K_0(\tau) \gamma_{\nu M} A_M, \quad C(\tau, \mu) = \mu^2 + \tau^2 - \Delta_M + \tau^2 K_0(\tau) \gamma_{1,M}.$$
Now, $B(\tau)$ is an operator of Boutet’s calculus of order and type 2 with strong parameter-dependence on $\tau$, while $C(\tau, \mu)$ has order and type 2 as well and is weakly parameter-dependent with regularity $\nu = 1/2$; for the latter see (2.3.55) in Proposition 2.3.14 of [12]. By parameter-ellipticity and Theorem 12 we can find

$$\left( C(\tau, \mu) \right)_{\gamma_0, M}^{-1} =: \left( D(\tau, \mu) \tilde{K}(\tau, \mu) \right)$$

with $D$ being of order $-2$, $\tilde{K}$ of order 0, and both having type 0 and regularity $\nu = 1/2$. Therefore

$$U_1 = E(\tau, \mu)U + D(\tau, \mu)F'_1, \quad E(\tau, \mu) := D(\tau, \mu)B(\tau); \quad (24)$$

note that $E(\tau, \mu)$ is weakly parameter-dependent of zero order, type 2, and regularity $\nu = 1/2$. Inserting this in the equation for the second component in (22), we find the equation

$$(\mu^2 + \tau^2 - \Delta_M)U + G(\tau, \mu)U = F' - D(\tau, \mu)F'_1 \quad (25)$$

for $U$, where

$$G(\tau, \mu) = \nabla_M K_0(\tau)_{\gamma_v, M}A_M - i \tau \nabla_M K_0(\tau)_{\gamma_1, M}E(\tau, \mu)$$

is a weakly parameter-dependent singular Green operator of order and type 2, with regularity $\nu = 1/2$. Using the parameter-ellipticity of $(\mu^2 + \tau^2 - \Delta_M + G(\tau, \mu), \gamma_0, M)$ we can resolve (25) for $U$ and substitute $U$ in (24), resulting in

$$U(\tau, \cdot) = S(\tau, \mu)F'(\tau, \cdot), \quad |(\tau, \mu)| \geq R,$$

for some sufficiently large $R \geq 0$ and with $S(\tau, \mu)$ being an $(n+1) \times (n+1)$-matrix with components belonging to $B^{-2,0,1/2}(M; \mathbb{R} \times \Sigma)$. Passing to the inverse Fourier transform with respect to $\tau$, we see that $u = S(\mu)f'$ is the unique solution of (22) for $\mu \in \Sigma$ sufficiently large, where the solution operator $R(\mu)$ is defined by

$$R(\mu) := \mathcal{F}_{\tau \rightarrow \cdot}^{-1}S(\tau, \mu)\mathcal{F}_{\tau \rightarrow \tau}.$$  

Due to Theorem 11 we have, for sufficiently large $\mu_0 > 0$,

$$\mathcal{R} \left( \left\{ |\mu|^{2+|\alpha|} \partial_\mu^\alpha R(\mu) : \mu \in \Sigma, |\mu| \geq \mu_0 \right\} \right) < \infty. \quad (26)$$

Therefore, for sufficiently large $\mu \in \Sigma$, the problem (22) is uniquely solvable, and (26) gives a resolvent estimate even in the $\mathcal{R}$-bounded version. This is the main step for proving maximal regularity for the Stokes operator:

**Theorem 13** Let $M \subset \mathbb{R}^n$ be a bounded smooth domain, and $Z := \mathbb{R} \times M$. Let $1 < p < \infty$ and $T > 0$ be finite. Let $P_p$ be the Helmholtz projection in $L_p(Z)$ (see Farwig [9]). Define the Stokes operator $A$ by
Here, $L_{p,\sigma}(Z)$ stands for the standard space of solenoidal $L_p$-vector fields. Then $A$ has maximal $L_q$-regularity for every $1 < q < \infty$ in the time interval $(0, T)$.

**Proof** Due to the existence of the Helmholtz decomposition of $L_p(Z)$ (see Farwig [9]), we see that for $f \in L_{p,\sigma}(Z)$ the solvability of $(\lambda + A)u = f$ in $L_{p,\sigma}(Z)$ is equivalent to the solvability of (19) with $\mu^2 = \lambda$. Instead of (19), we can consider the reduced Stokes problem (21) with Dirichlet boundary conditions. Therefore, we have to show maximal regularity (in finite time intervals) for the reduced Stokes operator $A^{(r)}: D(A^{(r)}) \subset L_p(Z) \to L_p(Z)$ which is defined by $D(A^{(r)}) := W_p^2(Z) \cap W_{p,0}^1(Z)$ and

$$A^{(r)} u := -\Delta u + \nabla K_{\gamma_0} Au, \quad u \in D(A^{(r)}).$$

Substituting $K$ by $K_0$ yields the modified reduced Stokes operator $A^{(r)}_0$,

$$A^{(r)}_0 u := -\Delta u + \nabla K_0 \gamma_0 Au, \quad u \in D(A^{(r)}_0) := D(A^{(r)}).$$

We have seen that, for sufficiently large $\lambda = \mu^2$ with $\mu \in \Sigma$, the operator $\lambda + A^{(r)}_0$ is invertible, and that its inverse is given by $(\lambda + A^{(r)}_0)^{-1} = R(\mu)$. Due to the $\mathcal{R}$-boundedness result in (26) and the condition $\theta > \frac{\pi}{4}$, the operator family

$$\left\{ \lambda(\lambda + \lambda_0 + A^{(r)}_0)^{-1} : \Re \lambda \geq 0 \right\} \subset \mathcal{L}(L_p(Z))$$

is $\mathcal{R}$-bounded for sufficiently large $\lambda_0 > 0$.

The reduced Stokes operator $A^{(r)}$ can be seen as a small perturbation of $A^{(r)}_0$. In fact, due to Lemma 1, for the difference $B := A^{(r)} - A^{(r)}_0$ we have $D(B) \supset D(A^{(r)}_0)$, and for every $\delta > 0$ there exists a $C_\delta > 0$ such that

$$\|Bu\|_{L_p(Z)} \leq C \|u\|_{H^{\frac{3}{2}-\epsilon}_p(Z)} \leq \delta \|A^{(r)}_0 u\|_{L_p(Z)} + C_\delta \|u\|_{L_p(Z)}, \quad u \in D(A^{(r)}_0).$$

Here we used the interpolation inequality. Now, the perturbation result in Denk et al. [5, Proposition 4.2], yields the $\mathcal{R}$-boundedness of

$$\left\{ \lambda(\lambda + \lambda_1 + A^{(r)})^{-1} : \Re \lambda \geq 0 \right\} \subset \mathcal{L}(L_p(Z))$$

for some sufficiently large $\lambda_1 > 0$. By the Theorem of Weis, $A^{(r)} + \lambda_1$ has maximal $L_q$-regularity for all $1 < q < \infty$. As the time interval is finite, this gives maximal $L_q$-regularity for the reduced Stokes operator which finishes the proof. \hfill $\square$
Let us finally remark that more general results on maximal regularity for the Stokes operator in cylindrical domains have been obtained, e.g., by Farwig and Ri [10] under weaker smoothness assumptions on the domain. For the existence of a bounded $H^\infty$-calculus (which also implies the statement of Theorem 13), we also refer to Abels [1]. The intention of the present section was not to recover or even improve these results, but to outline that maximal regularity can also be obtained by employing the $\mathcal{R}$-boundedness of operator-families belonging to Boutet de Monvel’s calculus.

Acknowledgments We thank the anonymous referees for their useful comments and one of them for pointing out a gap in the argumentation of Sect. 5.2.

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