Ward Identities for Affine-Virasoro Correlators

M.B. Halpern

Department of Physics, University of California
Theoretical Physics Group, Lawrence Berkeley Laboratory
Berkeley, California 94720
USA

N.A. Obers

Physikalisches Institut der Universität Bonn
Nußallee 12, D-5300 Bonn 1
Germany

Abstract

Generalizing the Knizhnik-Zamolodchikov equations, we derive a hierarchy of non-linear Ward identities for affine-Virasoro correlators. The hierarchy follows from null states of the Knizhnik-Zamolodchikov type and the assumption of factorization, whose consistency we verify at an abstract level. Solution of the equations requires concrete factorization ansätze, which may vary over affine-Virasoro space. As a first example, we solve the non-linear equations for the coset constructions, using a matrix factorization. The resulting coset correlators satisfy first-order linear partial differential equations whose solutions are the coset blocks defined by Douglas.

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†e-mail: HALPERN%THEORM.HEPNET@LBL.GOV, THEORY::HALPERN
‡e-mail: OBERS@PIB1.PHYSIK.UNI-BONN.DE, 13581::OBERS
1 Introduction

Affine-Virasoro constructions are Virasoro operators constructed with the currents $J_a$, $a = 1, \ldots, \dim g$ of affine $g$. Here is a brief history of these constructions.

Affine Lie algebra, or current algebra on $S^1$, was discovered independently in mathematics [1] and physics [2]. The first concrete representation [2] was untwisted $SU(3)_1$, obtained with world-sheet fermions [2, 3] in the construction of current-algebraic spin and internal symmetry on the string [2]. Examples of affine-Sugawara constructions [4, 5] and coset constructions [2, 5] were also given in the first string era, as well as the vertex operator construction of fermions and untwisted $SU(n)_1$ from compactified spatial dimensions [6, 7]. The generalization of these constructions [8, 9, 10, 11] and their application to the heterotic string [12] mark the beginning of the present era. See Refs. [13, 14] for further historical remarks on these early affine-Virasoro constructions.

The general affine-Virasoro construction [15, 16]

$$ T(L) = L^{ab} J_a J_b $$

is summarized by the Virasoro master equation [15, 16] for the inverse inertia tensor $L^{ab} = L^{ba}$. A generalized master equation including $\partial J$ terms [15] has also been obtained, as well as the superconformal master equation [17], which collects the superconformal solutions of the Virasoro master equation.

Here is an overview of the solution space, called affine-Virasoro space, of the Virasoro master equation.

a) The standard rational conformal field theories are contained in the affine-Sugawara nests [18], which include the affine-Sugawara constructions [2, 5], [19, 18], the coset constructions [2, 5, 11] and the nested coset constructions [13, 18].

b) The master equation has a very large number of solutions, e.g. approximately 1/4 billion on each level of affine $SU(3)$, and exponentially larger.

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1 The Sugawara-Sommerfeld model [4] was in four dimensions on the algebra of fields. The first affine-Sugawara constructions, on affine Lie algebra, were given by Bardakç and Halpern [2, 3] in 1971.
numbers on larger manifolds [18]. Most of these constructions are new, and many new solutions have been found in closed form, including large numbers of unitary solutions with irrational central charge [18, 20, 21, 22, 23, 24, 25, 26, 27]. As examples, the value at level 5 of $SU(3)$ [22]

$$c\left(\left(SU(3)\right)_{D(1)}^\#\right) = 2\left(1 - \frac{1}{\sqrt{61}}\right) \simeq 1.7439$$

(1.2)

is the lowest unitary irrational central charge yet observed, while the simplest exact unitary irrational level-families yet obtained are the $rs$-superconformal set with central charge [25]

$$c(SU(n)_{x}^\#[m(N = 1), rs]) = \frac{6nx}{nx + 8\sin^2(rs\pi/n)}$$

(1.3)

where $r, s \in \mathbb{N}$ and $x$ is the level of affine $SU(n)$. Ref. [27] gives the most recent list of exact unitary solutions with irrational central charge. Large classes of unitary irrational solutions have also been studied by high-level expansion [22], which remains the most powerful tool so far developed for the study of the space.

c) The generic conformal field theory has irrational central charge, and rational central charge is rare in the space of unitary conformal field theories. Indeed, the standard rational conformal field theories live in the much larger space of Lie $h$-invariant conformal field theories [27], which are themselves quite rare. Large numbers of candidates for new rational conformal field theories, beyond the affine-Sugawara nests, have also been found [28, 29, 30].

d) Partial classification of affine-Virasoro space has been achieved with graph theory and generalized graph theories [23, 24, 17, 31, 31, 25, 26, 27]. Moreover, the master equation generates the graph theories on the group manifolds in such a way that unsuspected Lie group structure, called Generalized Graph Theory on Lie $g$ [24, 26], is seen in each of the graph theories. The interested reader should consult Ref. [26], which axiomatizes the subject.

e) Large as they are, the graph theories so far cover only very small regions of affine-Virasoro space. Enough has been learned, however, to see that
all known exact solutions are special cases with relatively high symmetry, whereas the generic solution is completely asymmetric [23]. In this circumstance, an exact general solution of the master equation seems beyond hope.

We also mention a number of other developments in the program, including geometric identification [32] of the master equation as an Einstein-like system on the group manifold, a world-sheet action [33] for the generic affine-Virasoro construction on simple $g$, and the exact C-function [34] and C-theorem on affine-Virasoro space. See also Ref. [35], which gives an introductory review of developments in the Virasoro master equation.

It is clear that the Virasoro master equation is the first step in the study of irrational conformal field theory, but how are we to obtain the correlators of such theories? Most of the computational methods of conformal field theory [36, 37] are based on chiral algebras [38, 39] and their corresponding chiral null states, a situation of relatively high symmetry which cannot be generic in affine-Virasoro space. On the other hand, each affine-Virasoro construction has null states of the Knizhnik-Zamolodchikov (KZ) type [10], if only we can learn to exploit them.

The context for this development was given in the 1989 paper “Direct Approach to Operator Conformal Constructions” [14], by one of the present authors. The central point is that affine-Virasoro constructions come in commuting K-conjugate pairs [4, 5, 11, 15], which naturally form biconformal field theories. In these systems, the natural analogues of Virasoro primary fields are the Virasoro biprimary fields, which are simultaneously Virasoro primary under each of the two commuting Virasoro operators. These fields were called bitensor fields in the original paper, and, although they were originally given only for the coset constructions, their form is the same for all affine-Virasoro constructions.

In this paper, we combine the three elements

1. Virasoro biprimary fields [14] in biconformal field theory
2. KZ-type null states [10] of each affine-Virasoro construction
3. Factorization [10, 11, 14, 23] to a K-conjugate pair of ordinary conformal field theories
to derive a hierarchy of non-linear Ward identities for affine-Virasoro correlators (see eq.(8.2)). The Ward identities properly follow from the first two elements, and become non-linear differential equations on the assumption of factorization, which we argue is consistent at an abstract level.

The abstract form of the Ward identities is only a first step toward the correlators, however, because solution of the equations requires specific factorization ansätze, which may vary over affine-Virasoro space.

As a first example, we have solved the non-linear equations for the simplest non-trivial K-conjugate pairs, \( h \subset g \) and the \( g/h \) coset constructions, using a matrix factorization. The resulting coset correlators solve first-order linear partial differential equations, with flat connections, whose solutions are the coset blocks defined by Douglas [11].

The Conclusion speculates on other possible factorization ansätze, and, following our clue in the coset constructions, we speculate briefly about flat connections for all affine-Virasoro constructions.

2 The Virasoro Master Equation

In this section, we review the Virasoro master equation and some features of the system which will be useful below.

The general construction begins with the currents of untwisted affine \( g \) \[1, 2\]

\[
J_a(z) = \sum_m J_a^{(m)} z^{-m-1}, \quad a = 1, \ldots, \dim g, \quad m, n \in \mathbb{Z} \quad (2.1a)
\]

\[
J_a(z) J_b(w) = \frac{G_{ab}}{(z-w)^2} + if_{ab}^c \left( \frac{1}{z-w} + \frac{1}{2} \partial_w \right) J_c(w) + T_{ab}(w) + O(z-w) \quad (2.1b)
\]

where \( f_{ab}^c \) and \( G_{ab} \) are respectively the structure constants and general Killing metric of \( g \). The current algebra (2.1) is completely general since \( g \) is not necessarily compact or semisimple. In particular, to obtain level \( x_I = 2k_I/\psi_I^2 \) of \( g_I \) in \( g = \oplus_I g_I \) with dual Coxeter number \( \tilde{h}_I = Q_I/\psi_I^2 \), take

\[
G_{ab} = \oplus_I k_I \eta_{ab}^I, \quad f_{ac}^d f_{bd}^c = - \oplus_I Q_I \eta_{ab}^I \quad (2.2)
\]

where \( \eta_{ab}^I \) and \( \psi_I \) are respectively the Killing metric and the highest root of \( g_I \).
Next, consider the class of operators quadratic in the currents

\[ T(z) = L^{ab} J_a(z) J_b(z) = \sum_m L^{(m)} z^{-m-2} \]  

where \( T_{ab} = J_a J_b = T_{ba} \) is the composite two-current operator in (2.1b). The set of coefficients \( L^{ab} = L^{ba} \) is called the inverse inertia tensor, in analogy with the spinning top. The requirement that \( T(z) \) is a Virasoro operator

\[ T(z)T(w) = \frac{c/2}{(z-w)^4} + \left( \frac{2}{(z-w)^2} + \frac{\partial w}{z-w} \right) T(w) + \text{reg.} \]  

restricts the values of the inverse inertia tensor to those which solve the Virasoro master equation [15, 16]

\[ L^{ab} = 2L^{ac}G_{cd}L^{db} - L^{cd}L^{ef}f_{ce}^{\ a} f_{df}^{\ b} - L^{cd}f_{ce}^{\ f} f_{df}^{\ (a} L^{b)e} \]  

\[ c = 2G_{ab}L^{ab} \]  

The Virasoro master equation has been identified [32] as an Einstein-like system on the group manifold, with \( L^{ab} \) the inverse metric on tangent space and \( c = \dim g - 4R \), where \( R \) is the Einstein curvature scalar.

Some general features of the Virasoro master equation include:

1. The affine-Sugawara construction [2, 3, 4, 5] \( L_g \) is

\[ L_g^{ab} = \oplus I \frac{\eta^{ab}_I}{2k_I + Q_I} , \quad c_g = \sum I \frac{x_I \dim g_I}{x_I + \tilde{h}_I} \]  

for arbitrary level of any \( g \), and similarly for \( L_h \) when \( h \subset g \). In what follows, we refer to the affine-Sugawara constructions as the A-S constructions.

2. K-conjugation covariance [2, 3, 4, 5, 6]. When \( L \) is a solution of the master equation on \( g \), then so is the K-conjugate partner \( \tilde{L} \) of \( L \),

\[ \tilde{L}^{ab} = L^{ab}_g - L^{ab} , \quad \tilde{c} = c_g - c \]  

and the corresponding stress tensors form a K-conjugate pair of commuting Virasoro algebras

\[ \tilde{T}(z) = \sum_m \tilde{L}^{(m)} z^{-m-2} \]
\[
\hat{T}(z)\hat{T}(w) = \frac{\tilde{c}/2}{(z-w)^4} + \left(\frac{2}{(z-w)^2} + \frac{1}{z-w}\partial_w\right)\hat{T}(w) + \text{reg.} \quad (2.8b)
\]

\[
T(z)\hat{T}(w) = \text{reg.} \quad . \quad (2.8c)
\]

The affine-Virasoro stress tensors \(T\) and \(\tilde{T}\) are quasi-primary under the A-S stress tensor \(T_g = T + \tilde{T}\).

The simplest K-conjugate pairs are the subgroup constructions \(L_h\) and the corresponding \(g/h\) coset constructions \([2, 5, 11]\)

\[
L_{g/h}^{ab} = L_{g}^{ab} - L_{h}^{ab}, \quad c_{g/h} = c_{g} - c_{h} \quad (2.9)
\]

while repeated K-conjugation on embedded subgroup sequences generates the nested coset constructions \([19, 18]\) and the affine-Virasoro nests \([18]\).

3. Non-chiral versions of affine-Virasoro constructions are formed as usual by left-right doubling \([33, 35]\). Because the constructions come in commuting K-conjugate pairs, the action formulation of the generic theory \([33]\) is a gauge theory, in which a given construction is gauged by the Virasoro generators of its K-conjugate partner. The simplest gauge choice for the physical Hilbert space of the generic theory \(L\) is the set of states which are Virasoro primary under the K-conjugate construction

\[
L_{m>0}^{m>0}|L \text{ physical} \rangle = 0 \quad (2.10)
\]

and vice versa for the \(\tilde{L}\) theory. This is not a complete gauge fixing for the coset constructions, nor for the more general affine \(h\)-invariant conformal field theories \([27]\), all of which are special cases with some residual affine Lie symmetry.

3 L-Bases and Virasoro Biprimary States

In this section we review and extend the construction of the \(L^{ab}\)-broken biprimary states \([14, 18]\), which are Virasoro primary under both \(T\) and \(\tilde{T}\). These states were called simultaneous affine-conformal highest-weight (ACHW) states in the original work.
We begin with the vacuum of affine $g$

$$J_a^{m\geq 0}|0\rangle = L^{m\geq -1}|0\rangle = \bar{L}^{m\leq -1}|0\rangle = 0$$ (3.1)

and introduce the affine primary field $R_g^I(\mathcal{T})$

$$J_a(z)R_g^I(\mathcal{T}, w) = \left(\frac{1}{z-w} + \frac{\partial_w}{2\Delta_g(\mathcal{T})}\right) R_g^I(\mathcal{T}, w)(\mathcal{T}_a)_I^J + (R_g)_a^I(\mathcal{T}, w) + \mathcal{O}(z-w)$$ (3.2a)

$$J_a^{m\geq 0}R_g^I(\mathcal{T}, 0)|0\rangle = \delta_{m,0}R_g^I(\mathcal{T}, 0)|0\rangle(\mathcal{T}_a)_I^J, \quad I, J = 1, \ldots, \dim \mathcal{T}$$ (3.2b)

which corresponds to matrix representation $\mathcal{T}$ of $g$. In the finite part of (3.2a), $\Delta_g(\mathcal{T})$ is the conformal weight of representation $\mathcal{T}$ under the A-S construction $T_g$ and $(R_g)_a^I = \chi^I_a R_g J_a^J$ is a composite field defined by the OPE.

Consider the action of the affine-Virasoro construction $T = L^{ab} J_a J_b$ on the affine primary states. It is easily verified with (3.2b) that

$$L^{(0)} R_g^I(\mathcal{T}, 0)|0\rangle \chi^J_I(\mathcal{T}) = \Delta_\alpha(\mathcal{T}) R_g^I(\mathcal{T}, 0)|0\rangle \chi^\alpha_I(\mathcal{T})$$ (3.3a)

$$L^{ab} (\mathcal{T}_a \mathcal{T}_b)_I^J \chi^\alpha_J(\mathcal{T}) = \Delta_\alpha(\mathcal{T}) \chi^\alpha_I(\mathcal{T}), \quad \alpha, \beta = 1, \ldots, \dim \mathcal{T}$$ (3.3b)

where $\Delta_\alpha(\mathcal{T})$ and $\chi^\alpha_I(\mathcal{T})$ are the eigenvalues and eigenvectors of the conformal weight matrix $\Delta(\mathcal{T}) = L^{ab} \mathcal{T}_a \mathcal{T}_b$. It is conventional [18] to call $\Delta_\alpha(\mathcal{T})$ the $L^{ab}$-broken conformal weights of the affine primary state, and it is convenient to work directly in the eigenbasis of the conformal-weight matrix, which we call an $L$-basis of representation $\mathcal{T}$,

$$R_g^\alpha(\mathcal{T}, z) \equiv R_g^I(\mathcal{T}, z) \chi^\alpha_I(\mathcal{T})$$ (3.4a)

$$(\mathcal{T}_a)_\alpha^\beta \equiv (\chi^J_a(\mathcal{T}))_I^J (\mathcal{T}_a)_I^J \chi^\beta_J(\mathcal{T})$$ (3.4b)

$$\Delta(\mathcal{T})_\alpha^\beta = L^{ab} (\mathcal{T}_a \mathcal{T}_b)_\alpha^\beta = \delta_\alpha^\beta \Delta_\alpha(\mathcal{T})$$ (3.4c)

$$J_a(z) R_g^\alpha(\mathcal{T}, w) = \frac{R_g^\beta(\mathcal{T}, w)}{z-w} (\mathcal{T}_a)_\beta^\alpha + \text{reg.} \quad .$$ (3.4d)

In what follows, we refer to the eigenbasis of any relevant conformal weight matrix as an $L$-basis. Including the higher modes of $T$ and $\bar{T}$, one now verifies that

$$L^{m\geq 0} R_g^\alpha(\mathcal{T}, 0)|0\rangle = \delta_{m,0} \Delta_\alpha(\mathcal{T}) R_g^\alpha(\mathcal{T}, 0)|0\rangle$$ (3.5a)

$$\bar{L}^{m\geq 0} R_g^\alpha(\mathcal{T}, 0)|0\rangle = \delta_{m,0} \Delta_\alpha(\mathcal{T}) R_g^\alpha(\mathcal{T}, 0)|0\rangle$$ (3.5b)
\[ \Delta_\alpha(\mathcal{T}) + \tilde{\Delta}_\alpha(\mathcal{T}) = \Delta_g(\mathcal{T}) \quad . \] (3.5c)

It is clear that, in an \( L \)-basis, the \( L^{ab} \)-broken affine primary states are Virasoro biprimary states with conformal weights \( (\Delta_\alpha(\mathcal{T}), \tilde{\Delta}_\alpha(\mathcal{T})) \) under the K-conjugate partners \( T \) and \( \tilde{T} \).

We also define the carrier-space metric \( \eta_{\alpha\beta}(\mathcal{T}) \) in the \( L \)-basis of \( \mathcal{T} \), which is used to raise and lower indices. The inverse of the carrier-space metric appears in the A-S two-point correlators

\[ \langle R^\alpha_g(\mathcal{T}^1, z) R^\beta_g(\mathcal{T}^2, w) \rangle = \frac{\eta^{\alpha\beta}(\mathcal{T}^1) \delta(\mathcal{T}^2, \tilde{T}^1)}{(z - w)^{2\Delta_g(\mathcal{T}^1)}} \] (3.6a)

\[ (\mathcal{T}_a)_\alpha^\beta = -\eta_{\alpha\rho}(\mathcal{T}_a)^\sigma_\rho = -\mathcal{T}_a^{*\alpha\beta} \] (3.6b)

where the Kronecker-delta on the right of (3.6a) records that the correlators vanish except when the second representation \( \mathcal{T}^2 \) is the complex conjugate of the first, as defined in (3.6b). The two-point A-S correlators satisfy the usual global Ward identity

\[ \langle R_g(\mathcal{T}^1, z) R_g(\mathcal{T}^2, w)(\mathcal{T}^1_a + \mathcal{T}^2_a) \rangle = 0 \quad , \quad a = 1, \ldots, \dim g \] (3.7)

which implies with (3.4c) that the carrier-space metric satisfies

\[ \eta^{\alpha\beta}(\mathcal{T})(\Delta_\alpha(\mathcal{T}) - \Delta_\beta(\mathcal{T})) = \eta_{\alpha\beta}(\mathcal{T})(\Delta_\alpha(\mathcal{T}) - \Delta_\beta(\mathcal{T})) = 0 \] (3.8a)

\[ \eta^{\alpha\beta}(\mathcal{T})(\tilde{\Delta}_\alpha(\mathcal{T}) - \tilde{\Delta}_\beta(\mathcal{T})) = \eta_{\alpha\beta}(\mathcal{T})(\tilde{\Delta}_\alpha(\mathcal{T}) - \tilde{\Delta}_\beta(\mathcal{T})) = 0 \] (3.8b)

in each \( L \)-basis. These identities are trivial for the A-S constructions, but will be useful below in the general case.

In the appropriate \( L \)-bases, affine secondary states are also Virasoro biprimary. As an example, we give the results for the \( L^{ab} \)-broken one-current states of the affine vacuum module

\[ \langle J_A(z)J_B(w) \rangle = \frac{G_{AB}}{(z - w)^2} \] (3.9a)

\[ L^{m \geq 0} J_A(0)|0\rangle = \delta_{m,0} \Delta_A J_A(0)|0\rangle \] (3.9b)

\[ \tilde{L}^{m \geq 0} J_A(0)|0\rangle = \delta_{m,0} \tilde{\Delta}_A J_A(0)|0\rangle \] (3.9c)

\[ \Delta_A + \tilde{\Delta}_A = 1 \] (3.9d)
\[ M^B_A(L) = 2G^C_A L^{CB} + f^{E}_{AC} L^{CD} f^B_D = \Delta_A \delta^B_A \]  
(3.9e)
\[ G_{AB}(\Delta_A - \Delta_B) = G^{AB} (\tilde{\Delta}_A - \tilde{\Delta}_B) = 0 \]  
(3.9f)
where \( A, B = 1, \ldots, \dim g \) labels the currents in an \( L \)-basis of the conformal weight matrix \( M^B_A \). The identities (3.9f) follow in an \( L \)-basis because \( M^c_a G_{cb} \) is \( a, b \) symmetric in any basis $^{[34]}$.

### 4 Virasoro Biprimary Fields

Virasoro biprimary fields were first constructed in Ref. $^{[14]}$, where they were called bitensor fields. We review and extend this development in the language of OPE’s, incorporating an observation due to Schrans $^{[42]}$.

Let \( \phi^g_\alpha(z) \) be a Virasoro primary field under the A-S construction on \( g \)

\[ T_g(z) \phi^g_\alpha(w) = \left( \frac{\Delta_g}{(z-w)^2} + \frac{\partial_w}{z-w} \right) \phi^g_\alpha(w) + \text{reg.} \]  
(4.1)

where an \( L \)-basis for \( \phi_g \) is assumed, so that \( \phi^g_\alpha(0)|0\rangle \) is biprimary under \( T \) and \( \tilde{T} \). In what follows, we refer to \( \{ \phi^g_\alpha \} \) as the A-S fields, examples of which include the affine primary fields and the currents

\[ T_g(z) R^\alpha_g (T, w) = \left( \frac{\Delta_g(T)}{(z-w)^2} + \frac{\partial_w}{z-w} \right) R^\alpha_g (T, w) + \text{reg.} \]  
(4.2a)
\[ T_g(z) J_A(w) = \left( \frac{1}{(z-w)^2} + \frac{\partial_w}{z-w} \right) J_A(w) + \text{reg.} \]  
(4.2b)

It should be noted that, although (4.2a) is usually assumed $^{[10]}$ for the affine primary fields, the form is strictly correct only for integer level of affine compact \( g \). This subtlety is discussed in Appendix A, which finds an extra zero-norm operator contribution for non-unitary A-S constructions.

Because the affine-Virasoro stress tensors \( T \) and \( \tilde{T} \) are quasi-primary fields under \( T_g \), we may infer quite generally that $^{[12]}$

\[ T(z) \phi^g_\alpha(w) = \frac{\Delta_g \phi^g_\alpha(w)}{(z-w)^2} + \frac{\partial_w \phi^g_\alpha(w) + \delta \phi^g_\alpha(w)}{z-w} + \text{reg.} \]  
(4.3a)
\[ \tilde{T}(z) \phi^g_\alpha(w) = \frac{\tilde{\Delta}_g \phi^g_\alpha(w)}{(z-w)^2} + \frac{\partial_w \phi^g_\alpha(w) + \tilde{\delta} \phi^g_\alpha(w)}{z-w} + \text{reg.} \]  
(4.3b)
\[ \Delta_\alpha + \tilde{\Delta}_\alpha = \Delta_g \]  
\[ \partial \phi_g^\alpha + \delta \phi_g^\alpha + \tilde{\delta} \phi_g^\alpha = 0 \]

where \((\Delta_\alpha, \tilde{\Delta}_\alpha)\) are the \((T, \tilde{T})\) conformal weights of the biprimary states \(\phi_g^\alpha(0)|0\rangle\) and \((\delta \phi_g^\alpha, \tilde{\delta} \phi_g^\alpha)\) are extra terms which are non-vanishing in the general case. As an example of the characteristic OPE (4.3), we have the known OPE for the currents [15]:

\[ T(z)J_A(w) = \Delta_A \left( \frac{1}{(z-w)^2} + \frac{\partial_w}{z-w} \right) J_A(w) + \frac{2iL^{BC}f_{BA}^D T_{CD}(w)}{z-w} + \text{reg.} \]

\[ \delta J_A^\alpha = (\Delta_A - 1) \partial J_A + 2iL^{BC}f_{BA}^D T_{CD} \]

where \(T_{AB} = \ast J_A J_B \ast \) is the composite operator \(T_{ab}\) in an \(L\)-basis of the currents. The corresponding form of \(\tilde{\delta} J_A^\alpha\) is obtained from (4.4b) with \(\Delta \to \tilde{\Delta}\) and \(L \to \tilde{L}\).

As another example, we have computed

\[ T(z)R^\alpha_g(\mathcal{T}, w) = \Delta_\alpha(\mathcal{T}) \left( \frac{1}{(z-w)^2} + \frac{1}{\Delta_g(\mathcal{T})} \frac{\partial_w}{z-w} \right) R^\alpha_g(\mathcal{T}, w) \]

\[ + \frac{2L^{ab}(R^\alpha_g)_a(\mathcal{T}, w)(R^\mathcal{B}_b)_{\beta}^\alpha}{z-w} + \text{reg.} \]

\[ \delta R^\alpha_g = \left( \frac{\Delta_\alpha(\mathcal{T})}{\Delta_g(\mathcal{T})} - 1 \right) \partial R^\alpha_g(\mathcal{T}) + 2L^{ab}(R^\alpha_g)_a(\mathcal{T})(R^\mathcal{B}_b)_{\beta}^\alpha \]

for the affine primary fields \(R^\alpha_g(\mathcal{T})\), where the composite operator \((R^\alpha_g)_a(\mathcal{T}) = \ast J_A R^\alpha_g(\mathcal{T}) \ast\) is defined in eq.(3.2a). Further details of this computation are given in Appendix A.

A number of results follow from the characteristic OPE (4.3) by standard manipulations. First, we have the equivalent forms

\[ [L^{(m)}, \phi_g^\alpha(z)] = z^m [z \partial_z \phi_g^\alpha(z) + (m+1)\Delta_\alpha \phi_g^\alpha(z) + z \delta \phi_g^\alpha(z)] \]

\[ [\tilde{L}^{(m)}, \phi_g^\alpha(z)] = z^m [z \partial_z \phi_g^\alpha(z) + (m+1)\tilde{\Delta}_\alpha \phi_g^\alpha(z) + z \tilde{\delta} \phi_g^\alpha(z)] \]

from which one recovers that \(\phi_g^\alpha\) creates the Virasoro biprimary states

\[ L^{m \geq 0} \phi_g^\alpha(0)|0\rangle = \delta_{m,0} \Delta_\alpha \phi_g^\alpha(0)|0\rangle \]

\[ \tilde{L}^{m \geq 0} \phi_g^\alpha(0)|0\rangle = \delta_{m,0} \tilde{\Delta}_\alpha \phi_g^\alpha(0)|0\rangle \]
as discussed in the previous section. Moreover, the relations
\[
\delta \phi^\alpha_g(z) = -[\bar{L}^{-1}, \phi^\alpha_g(z)], \quad \check{\delta} \phi^\alpha_g(z) = -[L^{-1}, \phi^\alpha_g(z)]
\] (4.8)
also follow from (4.6) and (4.3d), so the extra terms \(\delta \phi^\alpha_g, \check{\delta} \phi^\alpha_g\) in (4.3) are directly linked to the existence of a non-trivial K-conjugate theory. See also Section 7, where the extra terms in (4.3) are understood as a consequence of factorization.

Finally, we recover from (4.6) the generalized stability conditions \[2, 14\]
\[
[z^{-m} \bar{L}^{(m)} - L^{(0)}, \phi^\alpha_g(z)] = m \Delta^\alpha \phi^\alpha_g(z) \quad (4.9a)
\]
\[
[\bar{z}^{-m} \bar{L}^{(m)} - \bar{L}^{(0)}, \phi^\alpha_g(z)] = m \bar{\Delta}^\alpha \phi^\alpha_g(z) \quad (4.9b)
\]
which are independent of the extra terms, and whose form at \(z = 1\) was central in the original construction of the biprimary fields.

We turn now to the Virasoro biprimary fields \(\phi^\alpha(\bar{z}, z)\), which satisfy
\[
T(z)\phi^\alpha(\bar{w}, w) = \left( \frac{\Delta^\alpha}{(z - w)^2} + \frac{\partial_w}{z - w} \right) \phi^\alpha(\bar{w}, w) + \text{reg.} \quad (4.10a)
\]
\[
\bar{T}(\bar{z})\phi^\alpha(\bar{w}, w) = \left( \frac{\bar{\Delta}^\alpha}{(\bar{z} - \bar{w})^2} + \frac{\partial_{\bar{w}}}{\bar{z} - \bar{w}} \right) \phi^\alpha(\bar{w}, w) + \text{reg.} \quad . \quad (4.10b)
\]
These fields were called “bitensor fields” in the original paper \[14\], and they have a number of equivalent forms
\[
\phi^\alpha(\bar{z}, z) = z^{-\Delta^\alpha} \bar{z}^{\Delta^\alpha} \phi^\alpha_g(\bar{z}) \quad (4.11a)
\]
\[
= \left( \frac{\bar{z}}{z} \right)^{\Delta^\alpha} \phi^\alpha_g(z) \left( \frac{z}{\bar{z}} \right)^{\bar{\Delta}^\alpha} \quad (4.11b)
\]
\[
= \left( \frac{z}{\bar{z}} \right)^{\Delta^\alpha} \phi^\alpha_g(\bar{z}) \left( \frac{\bar{z}}{z} \right)^{\bar{\Delta}^\alpha} \quad (4.11c)
\]
\[
= e^{(\bar{z}-z)\bar{L}^{-1}} \phi^\alpha_g(z) e^{(z-\bar{z})L^{-1}} \quad (4.11d)
\]
\[
= e^{(z-\bar{z})L^{-1}} \phi^\alpha_g(\bar{z}) e^{(\bar{z}-z)L^{-1}} \quad (4.11e)
\]
each of which is an \(SL(2)\) boost of the A-S field \(\phi^\alpha_g\).

The first line (4.11a) is the original form of the biprimary fields, but the equality of all the forms in (4.11) may be verified with the A-S boost identities
\[
\phi^\alpha_g(z\bar{z}) = z^{-\Delta^\alpha} \bar{z}^{\Delta^\alpha} \phi^\alpha_g(\bar{z}) \quad (4.12a)
\]
\[ \phi_g^\alpha(z + \bar{z}) = e^{zL_g^{(-1)}} \phi_g^\alpha(\bar{z}) e^{-zL_g^{(-1)}} \]  

(4.12b)

and the boost identity

\[
\partial_z \left( e^{(\bar{z} - z)L^{(-1)}} \left( \frac{z}{\bar{z}} \right)^{L^{(0)}} \phi_g^\alpha(\bar{z}) \left( \frac{\bar{z}}{z} \right)^{L^{(0)} + \Delta_\alpha} e^{(z - \bar{z})L^{(-1)}} \right) = 0
\]

(4.13)

which itself follows from the stability condition (4.9a).

Following the original arguments [14], a check of the form (4.11a) is given in Appendix B. For the present discussion, the simplest check of the OPE (4.10a) uses (4.3a), (4.8) and the form in (4.11d),

\[
T(z)\phi^\alpha(\bar{w}, w) = e^{(\bar{w} - w)\tilde{L}^{(-1)}} T(z)\phi^\alpha(w) e^{(w - \bar{w})\tilde{L}^{(-1)}}
\]

\[
= e^{(\bar{w} - w)\tilde{L}^{(-1)}} \left( \frac{\Delta_\alpha \phi^\alpha_g(w)}{(z - w)^2} + \frac{\partial_w \phi^\alpha_g(w) - [\tilde{L}^{(-1)}, \phi^\alpha_g(w)]}{z - w} \right) + \text{reg.} 
\]

\[
= \left( \frac{\Delta_\alpha}{(z - w)^2} + \frac{\partial_w}{z - w} \right) \phi^\alpha(\bar{w}, w) + \text{reg.} 
\]

(4.14)

Similarly, the simplest check of the OPE (4.10b) follows the same steps from the form in (4.11e).

Other useful properties of the biprimary fields include

\[
\langle \phi^\alpha(\bar{z}, z) \rangle = 0
\]

(4.15a)

\[
\phi^\alpha(z, z) = \phi^\alpha_g(z)
\]

(4.15b)

\[
\partial_z \phi^\alpha(\bar{z}, z)|_{\bar{z}=z} = [L^{(-1)}, \phi^\alpha_g(z)]
\]

(4.15c)

\[
\partial_z \phi^\alpha(\bar{z}, z)|_{\bar{z}=z} = [\tilde{L}^{(-1)}, \phi^\alpha_g(z)]
\]

(4.15d)

\[
(\partial_z + \partial_{\bar{z}}) \phi^\alpha(\bar{z}, z)|_{\bar{z}=z} = \partial_z \phi^\alpha_g(z)
\]

(4.15e)

\[
\partial_{\bar{z}} \partial_z \phi^\alpha(\bar{z}, z)|_{\bar{z}=z} = (\text{ad} \tilde{L}^{(-1)})^q (\text{ad} L^{(-1)})^p \phi^\alpha_g(z)
\]

(4.15f)

\[
\phi^\alpha(0, 0)|0\rangle = \phi^\alpha_g(0)|0\rangle = \text{biprimary state}
\]

(4.15g)

See also Section 7, where the \(SL(2)\)-boost form (4.11) of the biprimary fields is understood by factorization.
5 Biconformal Field Theory

It is clear from the discussion above that K-conjugate pairs of affine-Virasoro constructions naturally form biconformal field theories, whose chiral form includes two commuting Virasoro algebras. This is the viewpoint emphasized in the Direct Approach to Operator Conformal Constructions [14], the Generic Affine-Virasoro Action [33], and the review in Ref. [35].

In biconformal field theories, the biprimary fields \( \phi^\alpha(\bar{z},z) \) have conformal weights \((\Delta^\alpha, \tilde{\Delta}^\alpha), \Delta^\alpha + \tilde{\Delta}^\alpha = \Delta_g\), under the K-conjugate pair of commuting stress tensors \( (T, \tilde{T}), T + \tilde{T} = T_g \), and create the biprimary states \( \phi^\alpha(0,0)|0\rangle \) with the same conformal weights. The biprimary fields are in 1-1 correspondence with the A-S fields \( \phi^\alpha_g(z) = \phi^\alpha(z,z) \), which are Virasoro primary under the A-S construction. Similarly, the correlators of the biprimary fields, called biconformal correlators, reduce to their corresponding A-S correlators when \( \bar{z}_i = z_i \) for all points \( i \) in the correlator. Biconformal secondary fields have the form \( \partial^n_z \partial^n_{\bar{z}} \phi^\alpha(\bar{z},z) \) and biconformal secondary states can be obtained with negative modes of \( T \) or \( \tilde{T} \), as usual.

The reader may enjoy the structural analogue between the chiral biconformal field theories and ordinary non-chiral (or closed string) conformal field theories, which also have two commuting Virasoro algebras. Indeed, it is to highlight this analogy that we have written the argument of the biprimary fields as \( (\bar{z}, z) \), though it is not necessary here to think of \( \bar{z} \) as the complex conjugate of \( z \).

As examples in this paper, we will focus on the biprimary fields of the \( L^{ab} \)-broken affine primary fields

\[
R^a(T, \bar{z}, z) = e^{(\bar{z}-z)\tilde{L}(-1)} R^a_g(T, z) e^{(z-\bar{z})\tilde{L}(-1)}
\]

\[
\Delta_a(T) + \tilde{\Delta}_a(T) = \Delta_g(T)
\]

and the biprimary fields of the \( L^{ab} \)-broken currents

\[
J_A(\bar{z}, z) = e^{(\bar{z}-z)\tilde{L}(-1)} J_A(z) e^{(z-\bar{z})\tilde{L}(-1)}
\]

\[
\Delta_A + \tilde{\Delta}_A = 1
\]
Two- and three-point biconformal correlators and leading-term OPE’s of the biprimary fields are easily determined from the principles above, for example

\[ \langle R^\alpha (T^1, \bar{z}, z) R^\beta (T^2, \bar{w}, w) \rangle = \frac{\eta^{\alpha \beta}(T^1) \delta(T^2, \bar{T}^1)}{(z-w)^{2\Delta_a(T^1)}(\bar{z}-\bar{w})^{2\Delta_a(T^1)}} \]  

(5.3a)

\[ \langle \mathcal{J}_A(\bar{z}, z) \mathcal{J}_B(\bar{w}, w) \rangle = \frac{G_{AB}}{(z-w)^{2\Delta_A(\bar{z}-\bar{w})^{2\Delta_A}}} . \]  

(5.3b)

In both cases, the denominators are fixed first by \( SL(2) \times SL(2) \) covariance and the numerators are then fixed by comparison with the corresponding A-S correlators (3.6a) and (3.9a) at \( \bar{z} = z \) and \( \bar{w} = w \). Recalling eqs.(3.8) and (3.9f) in the form

\[ \eta^{\alpha \beta}(T) = 0 \quad \text{when} \quad \Delta_a(T) \neq \Delta_\beta(T) \]  

(5.4a)

\[ G_{AB} = 0 \quad \text{when} \quad \Delta_A \neq \Delta_B \]  

(5.4b)

we see that the biconformal correlators (5.3) are non-zero only when the \( L^{ab} \)-broken conformal weights of the two biprimary fields are equal.

Similarly, we have

\[ \langle R^{\alpha_1}(T^1, z_1, z_1) R^{\alpha_2}(T^2, z_2, z_2) R^{\alpha_3}(T^3, z_3, z_3) \rangle = \frac{Y_{g}^{\alpha_1 \alpha_2 \alpha_3}}{z_1^{\gamma_1} z_2^{\gamma_2} z_3^{\gamma_3}} \]  

(5.5)

where \( z_{ij} = z_i - z_j \), \( \bar{z}_{ij} = \bar{z}_i - \bar{z}_j \) and

\[ \gamma_{ij} = \Delta_{\alpha_i}(T^i) + \Delta_{\alpha_j}(T^j) - \Delta_{\alpha_k}(T^k) \]  

(5.6a)

\[ \bar{\gamma}_{ij} = \bar{\Delta}_{\alpha_i}(\bar{T}^i) + \bar{\Delta}_{\alpha_j}(\bar{T}^j) - \bar{\Delta}_{\alpha_k}(\bar{T}^k) \]  

(5.6b)

The coefficient \( Y_g \) is the invariant A-S three-point correlator which satisfies the usual global Ward identity

\[ Y_{g}^{\beta}(\sum_{i=1}^{3} T_{i}^a)_{\alpha} = 0 \quad , \quad a = 1, \ldots, \dim g \]  

(5.7)

where \( \alpha \) and \( \beta \) are a shorthand for the index sets \( \alpha_1 \alpha_2 \alpha_3 \) and \( \beta_1 \beta_2 \beta_3 \).

For the \( L^{ab} \)-broken currents, we obtain in the same way

\[ \langle \mathcal{J}_A(\bar{z}_1, z_1) \mathcal{J}_B(\bar{z}_2, z_2) \mathcal{J}_C(\bar{z}_3, z_3) \rangle \]

\[ = \frac{if_{AB}^{DC} G_{DC}^{\bar{z}_1 \bar{z}_2 \bar{z}_3}}{\bar{z}_{12} \bar{z}_{13} \bar{z}_{23}} \Delta_{A} + \Delta_{B} - \Delta_{C} \quad \Delta_{A} + \Delta_{C} - \Delta_{B} \quad \Delta_{B} + \Delta_{C} - \Delta_{A} \]  

(5.8)
because $\gamma^g_{ij} = \Delta^g = 1$. Then we also have the most singular terms of the $L^a b$-broken bilocal current algebra

$$J_A(z, z) J_B(w, w) = \frac{G_{AB}}{(z - w)^{2\Delta_A (z - w)^{2\Delta_B} \cdots}} \quad (5.9)$$

which follows from (5.8), (5.3b) and (3.9f). Similarly, we have

$$\langle J_A(z_1, z_1) R_{\alpha_2} (T^2, z_2, z_2) R_{\alpha_3} (T^3, z_3, z_3)\rangle = \frac{(T_A^2)^{\alpha_2 \alpha_3} \delta (T^2, \bar{T}^3)}{z_{12} z_{13} z_{23}^{2\Delta_g(T^2)} - 1} \left( \frac{\bar{z}_{12}}{z_{12}} \right)^{\Delta_A + \Delta_{\alpha_2} - \Delta_{\alpha_3}} \left( \frac{\bar{z}_{13}}{z_{13}} \right)^{\Delta_A + \Delta_{\alpha_3} - \Delta_{\alpha_2}} \left( \frac{\bar{z}_{23}}{z_{23}} \right)^{\Delta_{\alpha_2} + \Delta_{\alpha_3} - \Delta_A} \quad (5.10a)$$

$$J_A(z, z) R_\beta (T, \bar{w}, w) = \sum_\beta \frac{R_\beta (T, \bar{w}, w) (T_A)^{\alpha_2}}{(z - w)^{\Delta_A + \Delta_\alpha - \Delta_\beta (\bar{z} - \bar{w})} + \Delta_\alpha - \Delta_\beta} \quad (5.10b)$$

where $\Delta_\alpha = \Delta_\alpha(T^i)$ and $\Delta_\alpha = \bar{\Delta}_\alpha(T^i)$. In these bilocal OPE’s, the corresponding biconformal correlators can be used to determine the contributions of the biconformal secondaries $\partial_\alpha z \partial_\beta \phi^n (z, z)$ on the right.

More generally, we have the $SL(2) \times SL(2)$ decomposition

$$\langle \phi^1_n (z_1, z_1, \ldots, \phi^a_n (z_n, z_n) \rangle \equiv \Phi^{\alpha_1 \alpha_n} (\bar{z}, z) = \frac{Y^{\alpha_1 \ldots \alpha_n} (\bar{u}, u)}{\prod_{i < j} \gamma_{ij} (z_{ij})} \quad (5.11a)$$

$$\sum_{j \neq i} \gamma_{ij} (\alpha) = 2\Delta_\alpha , \quad \sum_{j \neq i} \bar{\gamma}_{ij} (\alpha) = 2\bar{\Delta}_\alpha \quad (5.11b)$$

$$\gamma_{ij} (\alpha) + \bar{\gamma}_{ij} (\alpha) = \gamma^g_{ij} , \quad \sum_{j \neq i} \gamma^g_{ij} = 2(\Delta_\alpha + \bar{\Delta}_\alpha) = 2\Delta^g \quad (5.11c)$$

$$Y^{\alpha_1 \ldots \alpha_n} (u, u) = Y^g_{\alpha_1 \ldots \alpha_n} (u) \quad (5.11d)$$

where $\{\bar{u}\} \{u\}$ are the sets of independent cross-ratios constructed from $\{z_i\}$ and $\{\bar{z}_i\}$ respectively, and $Y^{\alpha}_{\alpha} (u)$ is the invariant correlator of the corresponding A-S fields $\phi_\alpha^\alpha$. In general, the A-S correlator $\Phi^\alpha_g (z, z)$ and the corresponding invariant A-S correlator $Y^\alpha_g$ also satisfy a global Ward identity, for example

$$Y^\beta_g (\sum_{i=1}^n T_i^i) \alpha = 0 \quad , \quad a = 1, \ldots, \dim g \quad (5.12)$$

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when broken affine primary fields $\phi^\alpha(\bar{z}, z) = R^\alpha(T, \bar{z}, z)$ are chosen for the correlator.

We finally note that, up to this point, our development applies as well to the interacting bosonic models [43, 44, 28, 29], which include the affine-Virasoro constructions in principle, and may be more general. Because they exhibit K-conjugation covariance, these models also form biconformal field theories: The analogues of the A-S fields $\phi_\gamma$ are sums of vertex operators which satisfy the characteristic OPE (4.3), and the corresponding biprimary fields $\phi(\bar{z}, z)$ are the $SL(2)$ boosts (4.11) of the vertex operator sums. In what follows, however, a more central role is played by the organization of our constructions on affine Lie algebra.

6 Ward Identities

The form (4.11) of the biprimary fields indicates that the biconformal correlators can be constructed as power series expansions in A-S correlators about the A-S point

$$\langle \phi_1^{\alpha_1}(\bar{z}_1, z_1) \ldots \phi_n^{\alpha_n}(\bar{z}_n, z_n) \rangle |_{\bar{z}=z} = \langle \phi_{g,1}^{\alpha_1}(z_1) \ldots \phi_{g,n}^{\alpha_n}(z_n) \rangle$$

and we have organized this expansion as a sequence of computations of the Knizhnik-Zamolodchikov [10] type, expressed in the language of OPE’s. The equivalent language of KZ-type null states is discussed in Section 8.

For example, we know

$$\partial_i \langle \phi_1^{\alpha_1}(\bar{z}_1, z_1) \ldots \phi_n^{\alpha_n}(\bar{z}_n, z_n) \rangle$$

$$= \oint_{z_i} \frac{dw_i}{2\pi i} \langle T(w_i) \phi_1^{\alpha_1}(\bar{z}_1, z_1) \ldots \phi_n^{\alpha_n}(\bar{z}_n, z_n) \rangle$$

$$= \oint_{z_i} \frac{dw_i}{2\pi i} \oint_{\eta_i} \frac{d\eta}{2\pi i} \frac{1}{\eta - w_i} \langle L^{ab} J_a(\eta) J_b(w_i) \phi_1^{\alpha_1}(\bar{z}_1, z_1) \ldots \phi_n^{\alpha_n}(\bar{z}_n, z_n) \rangle$$

where $\partial_i = \partial_{z_i}$, and so, at the A-S point we have

$$\partial_i \langle \phi_1^{\alpha_1}(\bar{z}_1, z_1) \ldots \phi_n^{\alpha_n}(\bar{z}_n, z_n) \rangle |_{\bar{z}=z}$$

$$= \oint_{z_i} \frac{dw}{2\pi i} \oint_{\eta} \frac{d\eta}{2\pi i} \frac{1}{\eta - w} \langle L^{ab} J_a(\eta) J_b(w) \phi_{g,1}(z_1) \ldots \phi_{g,n}(z_n) \rangle$$

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where the right side of (6.3) is an A-S correlator. Similarly, we have

\[ \partial_i \partial_j \langle \phi_1^{\alpha_1}(\bar{z}_1, z_1) \ldots \phi_n^{\alpha_n}(\bar{z}_n, z_n) \rangle |_{\bar{z}=z} \]

\[ = \oint \frac{dw_i}{2\pi i} \oint \frac{d\eta_i}{2\pi i} \frac{1}{\eta_i - w_i} \oint \frac{dw_j}{2\pi i} \oint \frac{d\eta_j}{2\pi i} \frac{1}{\eta_j - w_j} \]

\[ \cdot \langle L^{ab} J_a(\eta) J_b(w_i) L^{cd} J_a(\eta) J_c(w_j) \phi_1^{\alpha_1}(z_1) \ldots \phi_n^{\alpha_n}(z_n) \rangle \]

(6.4)

and so on uniformly for any number of derivatives. To obtain derivatives with respect to one or more barred variables, say \( \partial_i \to \bar{\partial}_i = \partial_{\bar{z}_i} \), replace \( L \to \tilde{L} \) in the \( i \)th current bilinear on the right.

The right sides of these equations can be evaluated by standard dispersive methods, using only the singular terms of the OPE’s of the currents with the A-S fields \( \phi_g^{\alpha} \). In what follows, we call these relations the Ward identities of the biconformal field theories.

It is clear that the simplest Ward identities will be obtained for the biprimary fields \( \phi^{\alpha}(\bar{z}, z) = R^{\alpha}(\mathcal{T}, \bar{z}, z) \) of the \( L^{ab} \)-broken affine primaries \( \phi_g^{\alpha}(z) = R_g^{\alpha}(\mathcal{T}, z) \): In this case, the simple algebra of \( J \) and \( R_g \) in (3.4d) will guarantee that the right sides of these identities are proportional to the A-S correlators themselves, whereas extra inhomogeneous terms are generally obtained for broken affine secondaries. In what follows, we focus on the broken affine primaries, although we have collected some of the corresponding results for \( L^{ab} \)-broken currents in Appendix C.

We present our results for the \( L^{ab} \)-broken affine primaries in the simplified notation

\[ A^{\alpha}(\bar{z}, z) \equiv \langle R^{\alpha_1}(\mathcal{T}, 1, \bar{z}_1, z_1) \ldots R^{\alpha_n}(\mathcal{T}, n, \bar{z}_n, z_n) \rangle \]

(6.5a)

\[ A_g^{\alpha}(z) = A^{\alpha}(z, z) \equiv \langle R_g^{\alpha_1}(\mathcal{T}, 1, z_1) \ldots R_g^{\alpha_n}(\mathcal{T}, n, z_n) \rangle \]

(6.5b)

where the subscript \( g \) labels the A-S correlator. In this notation, the Ward identities take the form

\[ \bar{\partial}_1 \ldots \bar{\partial}_q \partial_1 \ldots \partial_p A^{\alpha}|_{\bar{z}=z} = A_g^{\beta}(W_{j_1 \ldots j_q i_1 \ldots i_p})^{\alpha}_\beta \]

(6.6)

where \( W_{j_1 \ldots j_q i_1 \ldots i_p} \) is a hierarchy of affine-Virasoro connections, defined for each K-conjugate pair of affine-Virasoro constructions. By construction, the connections are symmetric under exchange of any pair of indices, for example \( W_{ji} = W_{ij} \) and \( W_{ji} = W_{ij} \).
We have computed the first- and second-order connections, which correspond to first- and second-order derivatives. The first-order identities are

\[
\partial_i A^\alpha|_{\bar{z}=z} = A_g^\beta (W_i)_\beta^\alpha \quad \text{,} \quad W_i = 2L_{ab}^{ab} \sum_{j \neq i} \frac{T^i_a T^j_b}{z_{ij}} \tag{6.7a}
\]

\[
\tilde{\partial}_i A^\alpha|_{\bar{z}=z} = A_g^\beta (W_i)_\beta^\alpha \quad \text{,} \quad W_i = 2\tilde{L}_{ab}^{ab} \sum_{j \neq i} \frac{T^i_a T^j_b}{z_{ij}} \tag{6.7b}
\]

where the first-order connections \( W_i, W_i \) have the form of the A-S connection

\[
W_i^g = 2L_{ab}^{ab} \sum_{j \neq i} \frac{T^i_a T^j_b}{z_{ij}} \tag{6.8}
\]

which is also obtained from \( W_i \) when \( L = L_g \).

The second-order identities are

\[
\partial_i \partial_j A^\alpha|_{\bar{z}=z} = A_g^\beta (W_{ij})_\beta^\alpha \quad \text{,} \quad W_{ij} = \partial_i W_j + \frac{1}{2} (W_i, W_j)_+ + E_{ij} \tag{6.9a}
\]

\[
\tilde{\partial}_i \tilde{\partial}_j A^\alpha|_{\bar{z}=z} = A_g^\beta (W_{ij})_\beta^\alpha \quad \text{,} \quad W_{ij} = \tilde{\partial}_i W_j + \frac{1}{2} (W_i, W_j)_+ + E_{ij} \tag{6.9b}
\]

\[
\tilde{\partial}_i \partial_j A^\alpha|_{\bar{z}=z} = A_g^\beta (W_{ij})_\beta^\alpha \quad \text{,} \quad W_{ij} = W_i W_j + E_{ij} \tag{6.9c}
\]

where the extra terms \( E \) in the second-order connections are

\[
E_{ij} = \begin{cases} 
2iL^{da} L^{c(b f_{de})} \left\{ \frac{T^j_a T^i_b}{z_{ij}} + \sum_{k \neq i,j} \frac{T^k_a T^i_b T^j_a}{z_{ik} z_{jk}} \right\} + (i \leftrightarrow j) & , \ i \neq j \\
-2iL^{da} L^{e(b f_{dc})} \sum_{l \neq i} \left\{ \frac{T^i_a T^l_a T^j_a + T^j_a T^l_a T^i_a}{z_{il}} + \sum_{k \neq i,l} \frac{T^k_a T^l_a T^i_a}{z_{il} z_{ik}} \right\} & , \ i = j 
\end{cases} \tag{6.10a}
\]

\[
E_{ij} = E_{ij}|_{L \rightarrow \tilde{L}} \tag{6.10b}
\]

\[
E_{ij} = \begin{cases} 
-2iL^{da} L^{e(b f_{de})} \left\{ \frac{T^j_a T^i_b + T^i_a T^j_b}{z_{ij}} - 2 \sum_{k \neq i,j} \frac{T^k_a T^i_b T^j_a}{z_{ik} z_{jk}} \right\} & , \ i \neq j \\
2iL^{da} L^{c(b f_{dc})} \sum_{l \neq i} \left\{ \frac{T^i_a T^l_a T^j_a + T^j_a T^l_a T^i_a}{z_{il}} + \sum_{k \neq i,l} \frac{T^k_a T^l_a T^i_a}{z_{il} z_{ik}} \right\} & , \ i = j 
\end{cases} \tag{6.10c}
\]

We note that the Virasoro master equation (2.5a) was employed in this computation to obtain the terms \( \partial_i W_j \) and \( \partial_i W_j \) in (6.9), which are linear in \( L \) and \( \tilde{L} \). The results (6.10) may also be collected in the simpler form

\[
E_{ij} = -\frac{1}{2} (E_{ij} + E_{ji}) \quad , \quad E_{ij} = E_{ij}|_{L \rightarrow \tilde{L}} \quad , \quad E_{ii} = -\sum_{j \neq i} E_{ij} \tag{6.11}
\]
where $E_{ij}, i \neq j$ is given in (6.10c).

Using the results (6.7), (6.9) and (6.11), it is not difficult to verify the first two orders of the general sum rules

$$
\sum_{i_k} W_{i_{i_1}...i_{i_p}} = \sum_{i_k} W_{\bar{i}_{i_1}...\bar{i}_{i_p}} = 0
$$

(6.12)

which follow from the $SL(2) \times SL(2)$ decomposition (5.11).

Many other consistency relations must also hold among the affine-Virasoro connections because the Ward identities (6.6) can be used to express the biconformal correlators (6.5) in two ways

$$
A^\alpha (\bar{z}, z) = A^\beta (\bar{z}, z) = (\bar{\phi}(\bar{z})\phi(z))^\alpha \quad (6.13a)
$$

$$
= A^\beta (z) \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{i_1...i_p} (z_{i_1} - \bar{z}_{i_1})... (z_{i_p} - \bar{z}_{i_p})(W_{i_1...i_p}(\bar{z}))^\beta \quad (6.13b)
$$

Then, re-expansion of (6.13a) about $\bar{z} = z$ implies (by comparison with (6.13b)) that the barred connections $W_{j_1...j_q}$ can be expressed in terms of the unbarred connections $W_{i_1...i_p}$ and the A-S connection $W_g^\alpha$. In Section 8, we will discuss these relations as members of a larger set of relations which is necessary for the consistency of factorization.

7 Factorization

In this section, we begin to examine the consistency and implications of factorization [40, 41, 14, 29], which separates the biconformal field theories into their corresponding K-conjugate pairs of ordinary conformal field theories. Intuitively, each K-conjugate pair is a pair of “square-roots” of the affine-Sugawara construction.

To begin, we return to the biprimary fields, for which we assume the abstract factorization

$$
\phi^\alpha (\bar{z}, z) = (\tilde{\phi}(\bar{z})\phi(z))^\alpha \quad (7.1a)
$$

$$
\phi^\alpha_{\tilde{z}} (z) = \phi^\alpha (z, z) = (\tilde{\phi}(z)\phi(z))^\alpha \quad (7.1b)
$$

where $\phi(z)$ and $\tilde{\phi}(z)$ are the proper fields of the $L$ theory and the $\tilde{L}$ theory respectively. More precisely, $\phi$ and $\tilde{\phi}$ are $(\Delta_\alpha, 0)$ and $(0, \tilde{\Delta}_\alpha)$ Virasoro primary
fields respectively under the K-conjugate stress tensors \((T, \tilde{T})\), where the zeroes mean that \(T(z)\tilde{\phi}(w) = \text{reg} \) and \(\tilde{T}(z)\phi(w) = \text{reg} \). By abstract factorization, we mean that we do not yet specify any particular group index assignment for \(\phi(z)\) or \(\tilde{\phi}(\bar{z})\) separately, and much of our effort below concerns the consistency of factorization in this abstract form. Ultimately, we must also study concrete factorizations - or factorization ansätze - which may vary over affine-Virasoro space. As an example, we will see below that the matrix factorization
\[
\phi^\alpha(\bar{z}, z) = \phi^{\beta}_{g/h}(\bar{z})(\phi_h(z))_\beta^\alpha
\] (7.2)
is selected by the \(g/h\) coset constructions, but, more generally, it may be necessary to consider other ansätze, such as the symmetric factorization
\[
\phi^\alpha(\bar{z}, z) = \sum_\nu \tilde{\phi}^{\alpha}_\nu(\bar{z})\phi^\alpha_\nu(z)
\] (7.3)
where \(\nu\) is a conformal-block index whose range is to be determined.

We note first that the abstract factorization (7.1b) of the general A-S field \(\phi_g\) provides both a consistency check and a deeper understanding of the characteristic OPE (4.3):
\[
T(z)\phi^\alpha_g(w) = (\tilde{\phi}(w)T(z)\phi(w))^\alpha \\
= (\tilde{\phi}(w) \left( \frac{\Delta_\alpha}{(z-w)^2} + \frac{\partial_w}{z-w} \right) \phi(w))^\alpha + \text{reg.} \hspace{1cm} (7.4a)
\]
\[
= \frac{\Delta_\alpha}{(z-w)^2} \phi^\alpha_g(w) + \frac{\partial_w \phi^\alpha_g(w) + \delta \phi^\alpha_g(w)}{z-w} + \text{reg.} \\
\delta \phi^\alpha_g(w) = -(\partial_w \tilde{\phi}(w)\phi(w))^\alpha = -[\tilde{L}^{(-1)}, \phi^\alpha_g(w)] \hspace{1cm} (7.4b)
\]
In the same way, the K-conjugate relations
\[
\tilde{\delta} \phi^\alpha_g(w) = -(\bar{\phi}(w)\partial_w \phi(w))^\alpha = -[L^{(-1)}, \phi^\alpha_g(w)] \hspace{1cm} (7.5)
\]
are obtained for the characteristic OPE with \(\tilde{T}(z)\), so the extra terms \(\delta \phi^\alpha_g, \tilde{\delta} \phi^\alpha_g\) are required by factorization when the K-conjugate theory is non-trivial. The argument of this paragraph was obtained with D. Gepner.

Similarly, the \(SL(2)\)-boost forms (4.11) of the biprimary fields are consistent with factorization [14], and factorization of the A-S field implies factorization of
the biprimary field: Beginning with the factorized A-S field in (7.1b), we have
\[\phi^\alpha(\bar{z}, z) = e^{(\bar{z}-z)L(-1)}\phi^\beta_g(z)e^{(z-\bar{z})\tilde{L}(-1)}}\phi^\alpha\]
\[= (e^{(\bar{z}-z)L(-1)}\tilde{\phi}(z)e^{(z-\bar{z})\tilde{L}(-1)}\phi(z))^\alpha\]
where a standard $SL(2)$ boost in the $\tilde{L}$ theory, analogous to (4.12b), was employed in the last step.

8 Factorized Ward Identities

We consider the corresponding abstract factorization of the biconformal correlators
\[A^\alpha(\bar{z}, z) = (\bar{A}(\bar{z})A(z))^\alpha\]
where $A, \bar{A}$ are the affine-Virasoro correlators, that is, the proper correlators of the $L$ theory and the $\tilde{L}$ theory respectively. Then the Ward identities (6.6) become non-linear differential equations for the affine-Virasoro correlators
\[(\partial_{\bar{j}_1} \cdots \partial_{\bar{j}_q}\bar{A}\partial_{i_1} \cdots \partial_{i_p} A)^\alpha = A^\beta_g(W_{j_1 \cdots j_q i_1 \cdots i_p})^\alpha\]
\[A^\alpha_g(z) = (\bar{A}(z)A(z))^\alpha\]
\[A^\beta_g(\sum_{i=1}^{n} T^i_a)_{\beta}^\alpha = 0\quad a = 1, \ldots, \text{dim } g\]
So long as factorization is held at this abstract level, the form of these factorized Ward identities is universal across all the conformal field theories of affine-Virasoro space. In this and the following section, we study the consistency of these general systems at the abstract level.

To familiarize the reader with these equations, we begin by discussing the first-order system
\[(\bar{A} \partial_{\bar{i}} A)^\alpha = A^\beta_g(W_{i})^\alpha\]
\[(\partial_{\bar{i}} \bar{A} A)^\alpha = A^\beta_g(W_{\bar{i}})^\alpha\]
whose connections are given in (6.7). As a first exercise, note that the first-order equations and the global Ward identity (8.2c) on $A_g$ guarantee $SL(2)$ covariance.
of the factorized correlators

\[(\bar{A} \sum_i \partial_i A)^\alpha = 0\]
\[(\bar{A} \sum_i (z_i \partial_i + \Delta_{\alpha_i}(T^i)) A)^\alpha = 0\]
\[(\bar{A} \sum_i (z_i^2 \partial_i + 2z_i \Delta_{\alpha_i}(T^i)) A)^\alpha = 0\]

and similarly on \(\bar{A}\) with \(\Delta_{\alpha_i}(T^i) \rightarrow \tilde{\Delta}_{\alpha_i}(T^i)\). Verification of these identities follows essentially standard lines, e.g.

\[(\bar{A} \sum_i z_i \partial_i A)^\alpha = A^\beta g \sum_i \sum_{j \neq i} L^{ab}_{ij} (T^i_a T^j_b)^\beta\]
\[= -A^\beta g \sum_i L^{ab}(T^i_a T^i_b)^\beta = -A^\beta g \sum_i \Delta_{\alpha_i}(T^i)\]

where we have recalled in the last step that the conformal-weight matrices \(L^{ab}_{ij} T^i_a T^j_b\) are diagonal in their respective \(L\)-bases.

A more important feature of the first-order equations is that they imply the KZ-equations \[10\]

\[\partial_i A^\alpha = \partial_i (\bar{A} A)^\alpha = A^\beta (W_i^g)^\beta\]
\[W_i + W_i = W_i^g\]  

where \(W_i^g\) is the A-S connection in (6.8). We shall see that the relation (8.6b) is the first of a hierarchy of factorization relations necessary for the consistency of factorization.

Moving on, we consider the second-order system

\[(\tilde{A} \partial_i \partial_j A)^\alpha = A^\beta (W_{ij})^\beta, \quad (\partial_i \partial_j \tilde{A} A)^\alpha = A^\beta (W_{ij})^\beta\]  
\[(\partial_i \tilde{A} \partial_j A)^\alpha = A^\beta (W_{ij})^\beta\]

whose connections are given in (6.9). These equations give a non-trivial check of factorization because differentiation of the first-order equations requires a second set of factorization relations

\[W_{ji} + W_{ji} = (\partial_i + W_i^g) W_j\]
\[W_{ji} + W_{ji} = (\partial_i + W_i^g) W_j\]
among the first and second-order connections. With some algebra and the identity
\[ [W_i, W_j] = E_{ij} - E_{ji} \] (8.9)
we have verified that the factorization relations (8.8) are satisfied by the explicit forms (6.7), (6.9) of the connections.

More generally, let
\[ W\{n\} = \{W_{i_1...i_q...i_n} | 0 \leq q \leq n\} , \quad W_0 \equiv 1 \] (8.10)
be the set of \( n + 1 \) connection types of order \( n \). Among the \( n \)th-order connections, we find by differentiation exactly \( n \) factorization relations which may be summarized in the form
\[ W\{(n-1)\}_{i_n} + W\{(n-1)\}_{\bar{i}_n} = (\partial_{i_n} + W_{i_n}^g)W\{(n-1)\} \] (8.11)
The relations (8.6b) and (8.8) are included in this set when \( n = 1 \) and 2 respectively.

In fact, we can prove from the biprimary fields that all the factorization relations among all the connections are satisfied: For example, the differentiation relation (4.15e) implies that the correlators satisfy
\[ \{(\partial_i + \bar{\partial}_i)A^\alpha(\bar{z}, z)\}|_{\bar{z}=z} = \partial_i A^\alpha_g(z) \] (8.12)
and, taken with (6.7), this identity establishes the first factorization relation (8.6b). Similarly, we find the generalization of (8.12)
\[ \{(\partial_{i_n} + \bar{\partial}_{i_n})(\bar{\partial}_{i_1}...\bar{\partial}_{i_q}\partial_{i_{q+1}}...\partial_{i_{n-1}})A^\alpha(\bar{z}, z)\}|_{\bar{z}=z} = \partial_{i_n}\{(\bar{\partial}_{i_1}...\bar{\partial}_{i_q}\partial_{i_{q+1}}...\partial_{i_{n-1}})A^\alpha(\bar{z}, z)\}|_{\bar{z}=z} \] (8.13)
by repeated differentiation of the biprimary fields. Taken with the defining relations (6.6) of the general connection, these identities establish the general factorization relations (8.11). We conclude that the factorized Ward identities (8.2) are consistent at the abstract level.

One consequence of the factorization relations is that, given the A-S connection, there is only one independent connection type at each order, so that all the connections can be expressed in terms of a canonical set, say
\[ W^g_i , \ W_i , \ W_{ij} , \ W_{ijk} , \ W_{ijkl} , ... \] (8.14)
As an illustration, we have \( W_i = W_i^g - W_i \) at first order and
\[
\begin{align*}
W_{ij} &= (\partial_i + W_i^g)W_j - W_{ij} \quad (8.15a) \\
W_{ij} &= (\partial_i + W_i^g)(W_j^g - W_j) - (\partial_j + W_j^g)W_i + W_{ij} \quad (8.15b)
\end{align*}
\]
at second order. In this form, the \( n \)th-order connections involve up to \( n - 1 \) powers of the A-S covariant derivative \((\partial_i + W_i^g)\). The expressions obtained in this way for the barred connections \( \bar{W}_{ij} \) are the consistency relations which follow by Taylor expansion of (6.13a) into (6.13b).

Another consequence of the factorization relations is that, given the KZ equations, all the Ward identities are solved by solving only one equation at each order. As an example, the complete set of Ward identities
\[
\begin{align*}
\partial_i A^\alpha_g &= A^\beta_g(W_i^g)_\beta^\alpha \quad , \quad (\bar{A} \partial_i \ldots \partial_i A)^\alpha = A^\beta_g(W_1 \ldots i_p)_\beta^\alpha \quad , \quad W_0 = 1 \quad (8.16a) \\
A^\beta_g(\sum_{i=1}^n T^i_a)_\beta^\alpha &= 0 \quad , \quad a = 1, \ldots, \dim g \quad (8.16b)
\end{align*}
\]
involves only the canonical set of connections in (8.14). Combining the complete set (8.16) with eqs.(6.13a) and (8.2b), we find
\[
A^\alpha(z, \bar{z}) = (\bar{A}(\bar{z})A(z))^\beta \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \sum_{i_1 \ldots i_p} \left( z_{i_1} - \bar{z}_{i_1} \right) \ldots \left( z_{i_p} - \bar{z}_{i_p} \right) (W_{i_1 \ldots i_p}(\bar{z}))_\beta^\alpha
\]
\[
= (\bar{A}(\bar{z})A(z))^\alpha \quad (8.17)
\]
so that, as we saw for the biprimary fields, factorization of the biconformal correlators follows from factorization of the A-S correlators.

We finally emphasize that the factorized Ward identities (8.2) are natural generalizations of the KZ equations (8.6). We have seen that the identities imply the KZ equations, and, in an equivalent language, the identities follow from KZ-type null states. As an example, the identities
\[
|\chi^\alpha\rangle = \hat{L}^{(-1)} R^\alpha_g(\mathcal{T}, 0)|0\rangle - 2\hat{L}^{ab} J^{(-1)}_a R^\beta_g(\mathcal{T}, 0)|0\rangle (T_b)_\beta^\alpha = 0 \quad (8.18a) \\
\langle 0 | R^a_{g1}(\mathcal{T}^1, z_1) \ldots R^a_n(\mathcal{T}^n, z_n)|\chi^\alpha\rangle = 0 \quad (8.18b) \\
[J^{(-1)}_a, R^\alpha_g(\mathcal{T}, z)] = z^{-1} R^\beta_g(\mathcal{T}, z)(T_a)_\beta^\alpha \quad (8.18c)
\]
[\tilde{L}^{(-1)}, R^\alpha_g(T, z)] = \bar{\partial}_z R^\alpha(T, \bar{z}, z)|_{\bar{z}=z} = (\partial_z \bar{R}(T, z) R(T, z))^\alpha \tag{8.18d}

imply the first-order factorized Ward identity (8.3b), while the unfactorized Ward identity (6.7b) is obtained by stopping one step short in (8.18d). Similarly, the K-conjugate Ward identities (8.3a) and (6.7a) follow by $\tilde{L} \rightarrow L$.

To obtain the A-S correlators as solutions of the Ward identities, consider the K-conjugate pair

$$\tilde{L} = L_g \ , \ L = 0 \ , \ W_i = W^g_i \ , \ W_i = 0 \tag{8.19}$$

in (8.18) and (8.3), and the concrete factorization

$$A^\alpha(\bar{z}, z) = A^\beta(\bar{z})A^\alpha_\beta = A^\alpha(\bar{z}) \tag{8.20}$$

in (8.3), where $A^\alpha_\beta = \delta^\alpha_\beta$ are the correlators of the trivial theory. Then (8.3a) is trivially satisfied and (8.3b) is the KZ equation on $g$. In this simple case, we obtain all the higher-order connections

$$W_{j_1 \ldots j_q} = (\partial_{j_1} + W^g_{j_1}) \ldots (\partial_{j_{q-1}} + W^g_{j_{q-1}})W^g_{j_q} \ , \ W_{\{q-1\}i_q} = 0 \tag{8.21}$$

by iteration of the KZ equations and comparison with the Ward identities in (8.2). The induced connections (8.21) agree with the by-hand second-order connections in (6.9) (because $L^c(b f_{de} c) = 0$), and satisfy the factorization relations (8.11) on inspection.

9 **Invariant Equations for Four-Point Correlators**

As further checks on the factorized Ward identities (8.2), we have substituted the two- and three-point correlators (5.3a) and (5.5) into the first- and second-order equations (8.3) and (8.7). Using the global Ward identity (8.2c), we find that the equations are satisfied identically before or after factorization.

To study the four-point correlators at this level, we introduce the $SL(2) \times SL(2)$ decomposition (5.11), which involves the invariant correlators

$$Y^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(\bar{u}, u) = Y^{\alpha}(\bar{u}, u) = (\bar{Y}(\bar{u})Y(u))^\alpha \tag{9.1a}$$

$$u = \frac{z_{12}z_{34}}{z_{14}z_{32}} \ , \ \bar{u} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{14}\bar{z}_{32}} \tag{9.1b}$$
and we choose the KZ gauge \( \gamma_{12} = \gamma_{13} = \gamma_{12} = \gamma_{13} = 0 \) for simplicity. With this decomposition, the factorized Ward identities (8.2) reduce to the one-dimensional invariant differential equations

\[
(\partial^\alpha \tilde{Y}(u) \partial^\beta Y(u))^\alpha = Y_\beta^\alpha (u)(W_{qp}(u))^\alpha
\]

(9.2a)

\[
Y_\alpha^\beta (u) = (\tilde{Y}(u)Y(u))^\alpha
\]

(9.2b)

\[
Y_\beta^\beta (\sum_{i=1}^4 T^i_{\alpha})^\alpha = 0 \quad , \quad a = 1, \ldots, \dim g
\]

(9.2c)

where \( W_{qp} \) are the invariant connections of order \( q + p \). The explicit form of the invariant connections through second order

\[
W_{00} = 1
\]

(9.3a)

\[
W_{01} = \frac{2}{u} L^{ab} T^1_a T^2_b + \frac{2}{u - 1} L^{ab} T^1_a T^3_b , \quad W_{10} = \frac{2}{u} L^{ab} T^1_a T^2_b + \frac{2}{u - 1} L^{ab} T^1_a T^3_b
\]

(9.3b)

\[
W_{02} = \partial W_{01} + W_{01}^2 + E_{02} , \quad W_{20} = \partial W_{10} + W_{10}^2 + E_{20}
\]

(9.3c)

\[
W_{11} = W_{10} W_{01} - E_{02} = W_{01} W_{10} - E_{20}
\]

(9.3d)

\[
E_{02} = -2i L^{da} L^{b} [f_{de} (\frac{1}{u^2} T^1_a T^2_b T^3_c + T^2_a T^1_b T^3_c)]
\]

(9.3e)

\[
+ \frac{1}{(u - 1)^2} [T^1_a T^3_b T^3_c + T^3_a T^1_b T^3_c] + \frac{2}{u(u - 1)} T^1_a T^2_b T^3_c
\]

(9.3f)

was obtained by using the global Ward identity (9.2c) in the quadratic form

\[
Y_\beta^\beta (\sum_{j \neq i} (L^{ab} T^i_a T^j_b)_\beta^\alpha + \delta_\beta^\alpha \Delta_\alpha (T^i)) = 0 \quad , \quad i = 1, 2, 3, 4
\]

(9.4)

to eliminate the fourth representation \( T^4 \).

The first-order invariant equations imply the invariant KZ equations

\[
\partial Y_\alpha^\beta = \partial (\tilde{Y} Y)^\alpha = Y_\beta^\alpha (W^g)^\alpha
\]

(9.5a)

\[
W^g = W_{01} + W_{10} = \frac{2}{u} L^{ab} T^1_a T^2_b + \frac{2}{u - 1} L^{ab} T^1_a T^3_b
\]

(9.5b)
as in the previous section, where (9.5b) is the first factorization relation on the invariant connections. We have also checked that the second-order factorization relations

\[ W_{02} + W_{11} = (\partial + W^g)W_{01} \]  
\[ W_{20} + W_{01} = (\partial + W^g)W_{10} \]

are satisfied identically by the invariant connections (9.3). The general factorization relations among the invariant connections

\[ W_{q+1,p} + W_{q,p+1} = (\partial + W^g)W_{qp} \]

include the special cases (9.5b) and (9.6), and are presumably guaranteed by the general factorization relation (8.11) among the connections.

Given the factorization relations (9.7) and the invariant connection \( W^g \), the steps of Section 8 show that there is only one independent invariant connection at each order, and all the invariant connections can be expressed in terms of, say, the canonical set \( W^g \) and \( W_{0p} \). Similarly, given the invariant KZ equations, only one invariant Ward identity must be solved at each order, so that, for example, the collection

\[ \partial Y^{\alpha}_{\beta} = Y^{\beta}(W^g)_{\beta}^{\alpha} , \quad (\bar{Y} \partial Y)^{\alpha} = Y^{\beta}(W_{0p})_{\beta}^{\alpha} , \quad W_{00} = 1 \]
\[ Y^{\beta}_{\alpha} \left( \sum_{i=1}^{4} T_i \right)_{\beta}^{\alpha} = 0 , \quad a = 1, \ldots, \dim g \]

is a complete set of invariant Ward identities.

Finally, we follow the reasoning at the end of Section 8 to obtain the invariant A-S correlators and their associated invariant connections

\[ Y^{\alpha}(\bar{u}, u) = Y^{\beta}_g(\bar{u})Y^{\alpha}_g = Y^{\alpha}_g(\bar{u}) \]
\[ W_{0p} = \delta_{p,0} , \quad W_{qp} = \delta_{p,0}(\partial + W^g)^{q-1}W^g , \quad q \geq 1 \]

as solutions of the invariant Ward identities (9.2), where \( Y^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} \) are the invariant correlators of the trivial theory.
10 Matrix Factorization of $h$ and $g/h$

As a first non-trivial example, we solve the new Ward identities for the correlators of the coset constructions [2, 5, 11]. We will see that the coset correlators satisfy first-order linear differential equations whose solutions are the coset blocks defined by Douglas [11].

To begin, we collect some special properties of $h \subset g$ and $g/h$, which we choose as $T = T_h$ and $\bar{T} = T_{g/h}$. First, the generators of $h$ commute with $T_h$ and $T_{g/h}$, so the conformal weights of the broken affine primary fields satisfy

$$[T_a, \Delta_h(T)] = [T_a, \Delta_{g/h}(T)] = 0 \quad a \in h \quad (10.1a)$$

$$(T_a)_\alpha^\beta (\Delta_h^a(T) - \Delta_h^b(T)) = (T_a)_\alpha^\beta (\Delta_{g/h}^a(T) - \Delta_{g/h}^b(T)) = 0 \quad (10.1b)$$

in an $L$-basis of $T$ (see Section 3). Second, the biconformal fields (5.1a) of the broken affine primaries satisfy

$$J_a(T, \bar{w}, w) = \frac{R^\beta(T, \bar{w}, w)}{z - w} (T_a)_\alpha^\beta + \text{reg.} \quad a \in h \quad (10.2)$$

because the currents of $h$ commute with $\bar{T} = T_{g/h}$. Third, we will need the explicit form of the $h$ and $g/h$ connections through second order

$$W_i^h \equiv W_i = 2L_h^{ab} \sum_{j \neq i} \frac{T_a^j T_a^j}{z_{ij}} \quad W_i^{g/h} \equiv W_i = 2L_{g/h}^{ab} \sum_{j \neq i} \frac{T_a^j T_a^j}{z_{ij}} \quad (10.3a)$$

$$W_{ij} = (\partial_i + W_i^h) W_j^h \quad W_{ij}^{g/h} = W_j^{g/h} W_i^h \quad (10.3b)$$

$$W_{ij}^{g/h} = \partial_i W_j^{g/h} + W_i W_j^{g/h} - W_j^{g/h} W_i^h \quad (10.3c)$$

which follow from (6.9), (6.10) and (8.8) because $L_h^{da} L_h^{eb} f_{de(c)} = E_{ij} = 0$. Finally, we will study the matrix factorization

$$R^\alpha(T, \bar{z}, z) = (R_{g/h}(T, \bar{z}) R_h(T, z))^\alpha = \bar{R}_{g/h}^\beta(T, \bar{z}) (R_h(T, z))^\beta \quad (10.4a)$$

$$A^\alpha(\bar{z}, z) = (\bar{A}_{g/h}(\bar{z}) A_h(z))^\alpha = \bar{A}_{g/h}^\beta(\bar{z}) (A_h(z))^\beta \quad (10.4b)$$

announced in Section 7, where $\bar{A}_{g/h}$ are the coset correlators.

The equations for $h$ and $g/h$ are then

$$\partial_{j_1} \ldots \partial_{j_q} \bar{A}_{g/h}^{\beta} \partial_{i_1} \ldots \partial_{i_p} (A_h)_\beta^\alpha = A_{g}^{\beta}(W_{j_1} \ldots j_{q} j_{1} \ldots i_{p})_\beta^\alpha \quad (10.5a)$$

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\[ A^\alpha_g(z) = \tilde{A}^\beta_{g/h}(z)(A_h(z))_\beta^\alpha \]  
(10.5b)

\[ A^\beta_g\left(\sum_{i=1}^n T^i_a\right)_\beta^\alpha = 0 \quad a = 1, \ldots, \dim g \]  
(10.5c)

\[ \tilde{A}^\beta_{g/h}(z)(A_h(z))_\beta^\gamma\left(\sum_{i=1}^n T^i_a\right)_\gamma^\alpha = 0 \quad a = 1, \ldots, \dim h \]  
(10.5d)

where (10.5a-c) are the factorized Ward identities (8.2) and the extra \( h \)-global Ward identity in (10.5d) is implied by (10.2).

To solve this system, we focus on the first-order equations in the form

\[ \tilde{A}^\beta_{g/h}(\partial_i A_h - A_h W^h_i)_\beta^\alpha = 0 \]  
(10.6a)

\[ \partial_i \tilde{A}^\beta_{g/h}(A_h)_\beta^\alpha = \tilde{A}^\beta_{g/h}(A_h)_\beta^\gamma(W^g/h_\gamma)_\gamma^\alpha \quad . \]  
(10.6b)

The first of these equations is solved by taking \( A_h \) to be the (invertible) evolution operator of the \( h \)-connection

\[ \partial_i(A_h)_\beta^\alpha = (A_h)_\beta^\gamma(W^h_i)_\gamma^\alpha \quad , \quad \partial_i(A^{-1}_h)_\beta^\alpha = -(W^h_i)_\beta^\gamma(A^{-1}_h)_\gamma^\alpha \]  
(10.7)

that is, the KZ equations of \( h \) embedded in \( g \). Then (10.5b) gives the coset correlators

\[ \tilde{A}^\alpha_{g/h} = A^\beta_g(A^{-1}_h)_\beta^\alpha \]  
(10.8)

which solve (10.6b) in the form

\[ \partial_i \tilde{A}^\alpha_{g/h} = \tilde{A}^\beta_{g/h}(W_i[g/h])_\beta^\alpha \]  
(10.9a)

\[ W_i[g/h] = A_h W^g/h A^{-1}_h \quad . \]  
(10.9b)

In what follows, we refer to the first-order linear partial differential equations (10.9) as the coset equations, and we will call \( W_i[g/h] \) the dressed coset connections. Moreover, we will often suppress the group indices on our equations, as in (10.9b).

Before proceeding, we emphasize that the dressed coset connections are flat connections

\[ F_{ij}(W[g/h]) = A_h\{F_{ij}(W^g) - F_{ij}(W^h)\}A^{-1}_h = 0 \]  
(10.10a)

\[ F_{ij}(W) \equiv \partial_i W_j - \partial_j W_i + [W_i, W_j] \]  
(10.10b)
as they must be, by construction from (10.8). On the other hand, the dressed connections are complicated by the $h$-dressing, and, at least generically, do not fall in the class of connections associated to the classical Yang-Baxter equation.

To see this explicitly, we consider high-level expansion of the inverse inertia tensors on simple $g$, restricting the correlators to “low-spin” representations with $\Delta(T^i) = O(k^{-1})$ at high-level. Then, it is not difficult to obtain the first few terms of $W_i^{g/h}$, $A_h$ and the dressed coset connections

$$W_i^{g/h} = \sum_{j \neq i} \frac{r_{ij}(z)}{z_{ij}} = W_i^{g/h} + O(k^{-2}) \quad (10.11a)$$

$$r_{ij}(z) = \frac{P_{ab}^{g/h}}{k} T^i_a T^j_b + \frac{1}{2k^2} (P^h_{ab} Q_h - \eta^{ab} Q_g) T^i_a T^j_b$$

$$- \frac{i}{k^2} P_{g/h}^{ab} P_{h e f}^{cd} f^{e f} (\sum_{s \neq i} \ln \left( \frac{z_{is}}{z_{ij}} \right) T^i_a T^j_e T^s_d + \sum_{s \neq j} \ln \left( \frac{z_{js}}{z_{ij}} \right) T^i_a T^j_e T^s_d) + O(k^{-3}) \quad (10.11b)$$

where we have chosen the simplest initial condition $A_h(z_0) = 1$ for the evolution operator and defined the projection operators $P^h_{ab}$ and $P_{g/h}^{ab}$ onto $h$ and $g/h$. Since $r_{ij} \neq r_{ij}(z_{ij})$, the classical Yang-Baxter equation is excluded.

Moving on, we consider our solution vis-à-vis the higher-order Ward identities (10.5). The $h$-evolution equation (10.7) and the coset equations (10.9) induce the all-order connections

$$W_{j_1 \ldots j_q i_1 \ldots i_p} = W_{j_1 \ldots j_q}^{g/h} W^{h}_{i_1 \ldots i_p} \equiv W_{\{q\}}^{g/h} W_{\{p\}}^{h} \quad (10.12a)$$

$$W_{(p) i_{p+1}}^{h} = (\partial_{i_{p+1}} + W_{i_{p+1}}^{h}) W_{(p)}^{h} \quad (10.12b)$$

$$W_{\{q\} j_{q+1}}^{g/h} = \partial_{j_{q+1}} W_{\{q\}}^{g/h} + W_{j_{q+1} \{q\}}^{g/h} - W_{j_{q+1} \{q\}}^{g/h} W_{j_{q+1}}^{h} \quad (10.12c)$$

by comparison with the Ward identities (10.5). At second order, the induced connections (10.12) agree precisely with our by-hand connections in (10.3b,c), so our solution solves the second-order Ward identities. At higher order, it is not difficult to check that the induced connections (10.12) solve the general factorization relations (8.11).

Next, we verify that our solution solves the $h$-global Ward identity (10.5d). To see this, note first that the coset correlators satisfy an $h$-global identity

$$\tilde{A}_{g/h} \sum_i T^i_a = A_g A_h^{-1} \sum_i T^i_a = A_g \sum_i T^i_a A_h^{-1} = 0 \quad , \quad a = 1, \ldots, \dim h \quad (10.13)$$
because $A_h$ and $A_h^{-1}$ commute with the generators of $h$, while the A-S construction $A_g$ satisfies the $g$-global Ward identity (10.5c). By the same reasoning, we have

$$\bar{A}_{g/h}(\bar{z})A_h(z) \sum_i T^i_a = \bar{A}_{g/h}(\bar{z}) \sum_i T^i_a A_h(z) = 0, \quad a = 1, \ldots, \dim h \quad (10.14)$$

as required by (10.5d).

We turn now to verify the $SL(2)$ covariance of our solution, starting with $g/h$. The coset equations and the $g$-global identity (10.5c) guarantee the $SL(2)$ covariance of the coset correlators,

$$\sum_i \partial_i \bar{A}^\alpha_{g/h} = \sum_i (z_i \partial_i + \Delta^g_{\alpha_i}) \bar{A}^\alpha_{g/h} = \sum_i (z_i^2 \partial_i + 2z_i \Delta^g_{\alpha_i}) \bar{A}^\alpha_{g/h} = 0 \quad \text{for } \alpha = 1, \ldots, \dim h \quad (10.15)$$

but the last two identities require a special trick, e.g.

$$\sum_i z_i \partial_i \bar{A}^\alpha_{g/h} = -\sum_{\beta, \gamma} A^\beta_g \sum_i (L^a_{g/h} T^i_a T^i_b)_{\beta} A^{-1}_{h} \Delta^g_{\alpha_i} \bar{A}^\alpha_{g/h} = -\sum_{\beta} A^\beta_g \sum_i \Delta^g_{\beta_i} (A^{-1}_{h})_\beta = -\bar{A}^\alpha_{g/h} \sum_i \Delta^g_{\alpha_i} \quad (10.16)$$

In the last step, we used one of the $\Delta$-exchange identities

$$(A_h)_\alpha^\beta (\Delta^g_{\alpha_i} (T^i) - \Delta^g_{\beta_i} (T^i)) = (A_h)_\alpha^\beta (\Delta^h_{\alpha_i} (T^i) - \Delta^h_{\beta_i} (T^i)) = 0 \quad (10.17a)$$

$$(A_h^{-1})_\alpha^\beta (\Delta^g_{\alpha_i} (T^i) - \Delta^g_{\beta_i} (T^i)) = (A_h^{-1})_\alpha^\beta (\Delta^h_{\alpha_i} (T^i) - \Delta^h_{\beta_i} (T^i)) = 0 \quad (10.17b)$$

which follow from (10.1) because $A_h$ and $A_h^{-1}$ are matrix-valued functions of $\{T_a, \ a \in h\}$.

It is instructive to compare the statement of $SL(2)$ covariance in (10.15) with the general statement (8.4) in this case, e.g.

$$\left(\sum_i (z_i \partial_i + \tilde{\Delta}_{\alpha_i}) A\right)_{\alpha} = \sum_{\beta} \sum_i (z_i \partial_i + \Delta^g_{\alpha_i}) \bar{A}^\beta_{g/h} (A_h)_{\beta} = 0 \quad (10.18)$$

This relation is verified from the coset equations by stopping one step short in (10.16) and multiplying by $A_h$; equivalently, the two forms in (10.15) and (10.18) differ by a $\Delta$-exchange identity. It is clear that the matrix factorization (10.4) of $h$ and $g/h$ introduced an $\alpha_i \leftrightarrow \beta_i$ mismatch in the general identity (10.18), which is corrected by the $\Delta$-exchange. Since the $\Delta$-exchange identities...
are special to $h$ and $g/h$, it seems unlikely that matrix factorization will suffice in the general case.

To see the $SL(2)$ covariance of the $h$ theory, we use the $h$-invariance (10.13) of the coset correlators to rewrite the biconformal correlators as

$$A^\beta_{g/h}(\bar{z})(A_h(z))_\beta^\alpha = \bar{A}_{g/h}^M(\bar{z})(A_h(z))_M^\alpha \quad (10.19a)$$

$$A^\alpha_{g/h} = v_M^\alpha(h)A_{g/h}^M \quad (A_h)_M^\alpha = v_M^\beta(h)(A_h)^\beta_\alpha \quad (10.19b)$$

$$v_M^\beta(h)(\sum_i T_{a}^i)_\beta^\alpha = 0 \quad , \quad a = 1, \ldots, \dim h \quad , \quad M = 1, \ldots, \dim v(T, h) \quad (10.19c)$$

where $v_M^{\alpha_1 \ldots \alpha_n}(h)$ in (10.19c) are the $h$-invariant tensors of $T^1 \otimes \ldots \otimes T^n$. The projections $(A_h)_M^\alpha$ of the evolution operator are the physical $h$-correlators because they satisfy both the KZ equations of $h$ and an $h$-global Ward identity,

$$\partial_i (A_h)_M^\alpha = (A_h)_M^\beta(W^h_i)_\beta^\alpha \quad , \quad (A_h)_M^\beta(\sum_i T_{a}^i)_\beta^\alpha = 0 \quad , \quad a = 1, \ldots, \dim h \quad (10.20)$$

from which the usual $SL(2)$ identities are inferred.

11 Coset Blocks

To study the invariant four-point coset correlators, we introduce the $SL(2)$ decompositions

$$(A_h(z))_M^\alpha = (Y_h(u))_M^\alpha \prod_{i<j} z_{ij}^{\alpha} \quad , \quad \bar{A}_{g/h}^\alpha(z) = \frac{Y_{g/h}(u)}{\prod_{i<j} z_{ij}^{\alpha}} \quad (11.1a)$$

$$(Y_h)_M^\beta(\sum_{i=1}^4 T_{a}^i)_\beta^\alpha = \bar{Y}_{g/h}^\beta(\sum_{i=1}^4 T_{a}^i)_\beta^\alpha = 0 \quad , \quad a = 1, \ldots, \dim h \quad (11.1b)$$

where the $h$-global Ward identities on the invariant correlators follow from (10.20) and (10.13) by $\Delta$-exchange identities. This corresponds to the invariant factorization

$$Y^\alpha(\bar{u}, u) = (\bar{Y}_{g/h}(\bar{u})Y_h(u))^\alpha = \bar{Y}_{g/h}^\beta(\bar{u})(Y_h(u))_\beta^\alpha = C^M(\bar{u})(Y_h(u))_M^\alpha \quad (11.2a)$$

$$(Y_h)_M^\alpha = v_M^\beta(h)(Y_h)_\beta^\alpha \quad , \quad \bar{Y}_{g/h}^\alpha = v_M^\alpha(h)C^M \quad (11.2b)$$
and hence the factorized invariant Ward identities

\[ \partial^\alpha \bar{Y}_{g/h}^{\beta} \partial^\beta (Y_h)_\gamma^\alpha = Y_g^\beta (W_{\bar{q}p})_\beta^\alpha \]  
(11.3a)

\[ Y_g^\alpha = \bar{Y}_{g/h}(Y_h)_\beta^\alpha \]  
(11.3b)

\[ Y_g^\beta \left( \sum_{i=1}^4 T_i^a \right)_\gamma^\alpha = 0, \quad a = 1, \ldots, \text{dim } g \]  
(11.3c)

The invariant \( h \) and \( g/h \) connections through order two

\[ W^h \equiv W_{01} = \frac{2}{u} L_{h}^{ab} T_a^1 T_b^2 + \frac{2}{u - 1} L_{g/h}^{ab} T_a^1 T_b^2 \]  
(11.4a)

\[ W^{g/h} \equiv W_{10} = \frac{2}{u} L_{g/h}^{ab} T_a^1 T_b^2 + \frac{2}{u - 1} L_{g/h}^{ab} T_a^1 T_b^2 \]  
(11.4b)

\[ W_{02} = (\partial + W^h)W^h, \quad W_{11} = W^{g/h}W^h \]  
(11.4c)

\[ W_{20} = \partial W^{g/h} + W^g W^{g/h} - W^{g/h}W^h \]  
(11.4d)

also follow from Section 9 because \( L_{h}^{da} L_{h}^{e(b} f_{de)} = E_{02} = 0 \).

At this point, the invariant solution

\[ \partial(Y_h)_\gamma^\beta (W^h)_\gamma^\alpha, \quad \partial(Y_h^{-1})_\gamma^\beta = -(W^h)_\gamma^\beta (Y_h^{-1})_\gamma^\alpha \]  
(11.5a)

\[ \bar{Y}_{g/h}^\alpha = Y_g^\beta (Y_h^{-1})_\beta^\alpha \]  
(11.5b)

\[ \partial \bar{Y}_{g/h}^\alpha = \bar{Y}_{g/h}^\beta W_{g/h}^\beta \]  
(11.5c)

can be inferred from (11.1) and the results of Section 10, or, equivalently, from (11.3), following the arguments of Section 10. In what follows, we will refer to (11.5b) as the invariant coset correlators and to (11.5c) as the invariant coset equations. We also remark that the induced invariant connections

\[ W_{q}^{q} = W_{q}^{g/h}W_{p}^{h} \]  
(11.6a)

\[ W_{p}^{h} = (\partial + W^h)W_p^h, \quad W_{q}^{g/h} = \partial W_{q}^{g/h} + W^g W_{q}^{g/h} - W_{q}^{g/h}W^h \]  
(11.6b)

agree with the by-hand second-order invariant connections in (11.4c,d), and satisfy the general factorization relations (9.7).
Two steps are needed to make the transition from the group basis to a more familiar basis. First, we introduce a \((g, h)\) invariant tensor basis

\[
Y_{g}^{\alpha} = v_{m}^{\alpha}(g) G^{m}, \quad Y_{g/h}^{\alpha} = v_{M}^{\alpha}(h) C^{M}
\]

(11.7a)

\[
(Y_{h})_{M}^{\alpha} = v_{M}^{\beta}(h)(Y_{h})_{\beta}^{\alpha} = \mathcal{H}_{M}^{N} v_{N}^{\alpha}(h)
\]

(11.7b)

\[
(Y^{-1}_{h})_{M}^{\alpha} \equiv v_{M}^{\beta}(h)(Y^{-1}_{h})_{\beta}^{\alpha} = (\mathcal{H}^{-1})_{M}^{N} v_{N}^{\alpha}(h)
\]

(11.7c)

\[
(W^{g})_{m}^{\alpha} \equiv v_{M}^{\beta}(g)(W^{g})_{\beta}^{\alpha} = (W^{g})_{m}^{n} v_{n}^{\alpha}(g)
\]

(11.7d)

\[
(W^{h})_{M}^{\alpha} \equiv v_{M}^{\beta}(h)(W^{h})_{\beta}^{\alpha} = (W^{h})_{M}^{N} v_{N}^{\alpha}(h)
\]

(11.7e)

where \(C^{M}\) are the invariant coset correlators in the tensor basis and \(\{v_{m}^{\alpha}(g)\} \subset \{v_{M}^{\alpha}(h)\}\) are the \(g\)-invariant tensors of \(T_{1} \otimes \ldots \otimes T_{n}\),

\[
v_{m}^{\alpha}(g) \left(\sum_{i} T_{i}^{i}_{a}\right)_{\beta}^{\alpha} = 0 , \quad a = 1, \ldots, \dim g , \quad m = 1, \ldots, \dim v(T, g)
\]

(11.8)

which may be chosen to satisfy \(v_{m}^{\alpha}(g) = v_{m}^{\alpha}(h)\). It follows that

\[
\mathcal{C}^{M} \mathcal{H}_{M}^{N} = G^{m} \delta_{n}^{N}
\]

(11.9a)

\[
\mathcal{C}^{M} = G^{m}(\mathcal{H}^{-1})_{m}^{M}
\]

(11.9b)

\[
\partial G^{m} = G^{n}(W^{g})_{n}^{m}
\]

(11.9c)

\[
\partial \mathcal{H}_{M}^{N} = \mathcal{H}_{M}^{L} (W^{h})_{L}^{N} , \quad \partial (\mathcal{H}^{-1})_{M}^{N} = -(W^{h})_{M}^{L} (\mathcal{H}^{-1})_{L}^{N}
\]

(11.9d)

and we will choose the initial condition

\[
(\mathcal{H}(u_{0}))_{M}^{N} = (\mathcal{H}^{-1}(u_{0}))_{M}^{N} = \delta_{M}^{N}
\]

(11.10)

for the evolution operators \(\mathcal{H}\) and \(\mathcal{H}^{-1}\) in (11.9d).

Second, we introduce a block basis,

\[
\mathcal{G}^{m}(u) = d^{r}(\mathcal{F}_{g}(u))^{m}_{r}
\]

(11.11a)

\[
(\mathcal{H}(u))_{M}^{N} = (\mathcal{F}_{h}^{-1}(u_{0}))_{M}^{R} (\mathcal{F}_{h}(u))_{R}^{N}
\]

(11.11b)

\[
(\mathcal{H}^{-1}(u))_{M}^{N} = (\mathcal{F}_{h}^{-1}(u))_{M}^{R} (\mathcal{F}_{h}(u_{0}))_{R}^{N}
\]

(11.11c)
where \((\mathcal{F}_g)_r^m\) and \((\mathcal{F}_h)_R^M\) are the usual conformal blocks of \(g\) and \(h\), chosen so that the left indices \(r\) and \(R\) label the blocks by \(g\) and \(h\) representations in the \(s\) channel \((u \to 0)\). Then

\[
\mathcal{C}^M(u) = d^r(\mathcal{F}_g(u))_r^n(\mathcal{F}_h^{-1}(u))_n^R(\mathcal{F}_h(u_0))_R^M
\]

and we finally obtain

\[
\bar{Y}^\alpha_{gh}(u) = d^r \mathcal{C}(u)_r^R w^\alpha_R(u_0, h)
\]
\[
w^\alpha_R(u_0, h) = (\mathcal{F}_h(u_0))_R^M v^\alpha_M(h)
\]
\[
\mathcal{C}(u)_r^R = (\mathcal{F}_g(u))_r^n(\mathcal{F}_h^{-1}(u))_n^R
\]
\[
\partial \mathcal{C}_r^R = \mathcal{C}_r^S W[g/h]_S^R
\]

\[
W[g/h]_R^S = (\mathcal{F}_h)_R^M [s_M^m(W^g)_m^L \delta^L_i - (W^h)_M^L](\mathcal{F}_h^{-1})_L^S
\]

where \(\mathcal{C}_r^R\) in (11.13c) are the coset blocks, and (11.13d) are the invariant coset equations in the block basis. With (10.19c) and (11.8), we count \(\text{dim } \nu(\mathcal{T}, h) \cdot \dim \nu(\mathcal{T}, g)\) coset blocks for general integrable representations \(\{\mathcal{T}^i\}\) of \(g\) and general \(g/h\).

The coset blocks in (11.13c) are essentially those defined by Douglas [41], who argued that they can be used to define consistent non-chiral conformal field theories.

12 Simplification on \((g \times g)/g\)

Among the coset constructions and representations of \(g\), we distinguish a particularly simple class: For the cosets, we choose the type II symmetric spaces

\[
\frac{g}{h} = \frac{\mathfrak{g}_{x_1} \times \mathfrak{g}_{x_2}}{\mathfrak{g}_{x_1+x_2}}
\]

and we consider the integrable representations of \(\mathfrak{g}_{x_1}\)

\[
(\mathcal{T}_a^i)^{\hat{\alpha}_i}_{\hat{\beta}_i} = ((\mathcal{T}_a^i)^{\hat{\beta}_i}_{\hat{\alpha}_i}, 0) , \quad a = 1, \ldots, \text{dim } g
\]

In this case, we have the simplifications

\[
Y^\alpha_g(u) = Y^\hat{\alpha}_{\mathfrak{g}_{x_1}}(u) , \quad \mathcal{G}^M = d^R(\mathcal{F}_{\mathfrak{g}_{x_1}})_R^M , \quad Y^\alpha_h(u) = Y^\hat{\alpha}_{\mathfrak{g}_{x_1+x_2}}(u)
\]
\[ v^\alpha_m(g) = v^\alpha_M(h) = v^\hat{\alpha}_M(g) , \quad m = M = 1, \ldots, \dim v(T, g) \quad (12.3b) \]
\[ Y^\alpha_{g/h}(u) = Y^\hat{\alpha}_{g/h}(u) = d^R C(u)_R^S w^\hat{\alpha}_S(u_0, g) \quad (12.3c) \]
\[ w^\hat{\alpha}_R(u_0, g) = (F_{\mathfrak{g}_{x_1+x_2}}(u_0)) R^M v^\hat{\alpha}_M(g) \quad (12.3d) \]
\[ C(u)_R^S = (F_{\mathfrak{g}_{x_1}}(u)) R^L (F_{\mathfrak{g}_{x_1+x_2}}(u))_L^S \quad (12.3e) \]

where \( \dim v(T, g) \) is the number of \( g \)-invariant tensors and \( (\dim v(T, g))^2 \) is the number of coset blocks in the square matrix \( C \). In (12.3), the blocks of \( g_x \) satisfy

\[ \partial F_{\mathfrak{g}_x} = F_{\mathfrak{g}_x} W^{\mathfrak{g}_x} , \quad \partial F^{-1}_{\mathfrak{g}_x} = -W^{\mathfrak{g}_x} F^{-1}_{\mathfrak{g}_x} \quad (12.4a) \]
\[ W^{\mathfrak{g}_x} = \lambda_{\mathfrak{g}_x} \left( \frac{P}{u} + \frac{Q}{u-1} \right) , \quad \lambda_{\mathfrak{g}_x} = \frac{1}{x + \tilde{h}_g} \quad (12.4b) \]
\[ v^\hat{\beta}_M(g) 2\psi^\beta_g - (\eta^{ab}_g T^1_a T^2_b)_{\hat{\beta}} = P^M N v^\hat{\alpha}_M(g) \quad (12.4c) \]
\[ v^\hat{\beta}_M(g) 2\psi^\beta_g - (\eta^{ab}_g T^1_a T^3_b)_{\hat{\beta}} = Q^M N v^\hat{\alpha}_M(g) \quad (12.4d) \]

where \( \tilde{h}_g \) and \( \psi_g \) are the dual Coxeter number and highest root of \( g \) respectively, and the square matrices \( P \) and \( Q \) are defined in (12.4c,d). Finally, the iterated coset equations

\[ \partial^n C = C W_q[g/h] , \quad W_q[g/h] = F_{\mathfrak{g}_{x_1+x_2}} W^{g/h}_q F^{-1}_{\mathfrak{g}_{x_1+x_2}} \quad (12.5a) \]
\[ W_0^{g/h} = 1 , \quad W_1^{g/h} = (\lambda_{\mathfrak{g}_{x_1}} - \lambda_{\mathfrak{g}_{x_1+x_2}}) \left( \frac{P}{u} + \frac{Q}{u-1} \right) \quad (12.5b) \]
\[ \partial W^{g/h}_q + W^{\mathfrak{g}_x} W^{g/h}_q - W^{g/h}_q W^{\mathfrak{g}_x} = (12.5c) \]

follow from (10.12c) and (12.4).

### 13 An Example on \( (\text{SU}(n) \times \text{SU}(n))/\text{SU}(n) \)

As an example on \( (g \times g)/g \), we specify \( (SU(n)_{x_1} \times SU(n)_{x_2})/SU(n)_{x_1+x_2} \) and we choose the \( g_{x_1} = SU(n)_{x_1} \) representations as

\[ T^1 = T^4 = T_n , \quad T^2 = T^3 = T_n \quad (13.1) \]
where $\mathcal{T}(n)$ is the fundamental representation of $SU(n)$ and $\bar{\mathcal{T}}(n)$ is its complex conjugate, defined in (3.6b). These choices correspond to the correlators and conformal weights

$$A^\hat{\alpha}_g = \langle R_{\Delta g/z}(\mathcal{T}, z_1) R_{\Delta g/z}(\mathcal{T}, z_2) R_{\Delta g/z}(\mathcal{T}, z_3) R_{\Delta g/z}(\mathcal{T}, z_4) \rangle = \frac{Y^\hat{\alpha}_g(u)}{(z_1 z_2 z_3 z_4)^{2\Delta g/z}}$$  \hspace{1cm} (13.2a)

$$\Delta_{g/z}(\mathcal{T}(n)) = \Delta_{g/z} = \frac{n^2 - 1}{2n(x + n)}$$  \hspace{1cm} (13.2b)

$$A^\hat{\alpha}_{g/h} = \langle R_{\Delta g/h}(\mathcal{T}, z_1) R_{\Delta g/h}(\mathcal{T}, z_2) R_{\Delta g/h}(\mathcal{T}, z_3) R_{\Delta g/h}(\mathcal{T}, z_4) \rangle = \frac{\bar{Y}^\hat{\alpha}_{g/h}(u)}{(z_1 z_2 z_3 z_4)^{2\Delta g/h}}$$  \hspace{1cm} (13.2c)

$$\Delta_{g/h}(\mathcal{T}(n)) = \Delta_{g/h} = \Delta_{g/h}^{x_1} - \Delta_{g/h}^{x_2} = \frac{x_2(n^2 - 1)}{2n(x_1 + n)(x_1 + x_2 + n)}$$  \hspace{1cm} (13.2d)

of $SU(n)_x$ and $g/h$ respectively. The $n$ conformal weights of each coset field are degenerate in this case, so that any basis is an $L$-basis, and we choose the Cartesian basis of Gell-Mann for simplicity.

In this basis, Knizhnik and Zamolodchikov [10] have provided us with the required data on $SU(n)_x$,

$$v^\hat{\alpha}_g = \delta\hat{\alpha}_1 \hat{\alpha}_2 \delta\hat{\alpha}_3 \hat{\alpha}_4 , \quad v^\hat{\alpha}_g = \delta\hat{\alpha}_1 \hat{\alpha}_3 \delta\hat{\alpha}_2 \hat{\alpha}_4 , \quad \dim v(g) = 2$$  \hspace{1cm} (13.3a)

$$P = -\frac{1}{n} \begin{pmatrix} n^2 - 1 & 0 \\ n & -1 \end{pmatrix} , \quad Q = -\frac{1}{n} \begin{pmatrix} -1 & n \\ 0 & n^2 - 1 \end{pmatrix} , \quad \lambda_{g/z} = \frac{1}{x + n}$$  \hspace{1cm} (13.3b)

$$\mathcal{F}_{g/z} = \begin{pmatrix} (\mathcal{F}_{g/z})_V^1 \\ (\mathcal{F}_{g/z})_A^1 \end{pmatrix} , \quad \Delta_{g/z}^A = \frac{n}{x + n}$$  \hspace{1cm} (13.3c)

$$(\mathcal{F}_{g/z}(u))_V^1 = u^{-2\Delta g/z}(1 - u)^{\Delta_{g/z}^A - 2\Delta_{g/z}^A} F(\lambda_{g/z}, -\lambda_{g/z}, 1 - n\lambda_{g/z}; u)$$

$$(\mathcal{F}_{g/z}(u))_A^1 = u^{-\Delta_{g/z}^A - 2\Delta_{g/z}^A}(1 - u)^{\Delta_{g/z}^A - 2\Delta_{g/z}^A} F((n - 1)\lambda_{g/z}, (n + 1)\lambda_{g/z}, 1 + n\lambda_{g/z}; u)$$

$$(\mathcal{F}_{g/z}(u))_V^2 = \frac{1}{x} u^{-2\Delta g/z + 1}(1 - u)^{\Delta_{g/z}^A - 2\Delta_{g/z}^A} F(1 + \lambda_{g/z}, 1 - \lambda_{g/z}, 2 - n\lambda_{g/z}; u)$$

$$(\mathcal{F}_{g/z}(u))_A^2 = -nu^{-\Delta_{g/z}^A - 2\Delta_{g/z}^A}(1 - u)^{\Delta_{g/z}^A - 2\Delta_{g/z}^A} F((n - 1)\lambda_{g/z}, (n + 1)\lambda_{g/z}, n\lambda_{g/z}; u)$$  \hspace{1cm} (13.3d)

where $V, A$ label the vacuum and adjoint blocks in the $u \to 0$ channel, $\Delta_{g/z}^A$ is the conformal weight of the adjoint representation $\mathcal{T}(A)$ and $F(a, b, c; u)$ is the
hypergeometric function. We will also need the inverse A-S blocks
\[ F^{-1}_{g_e} = \frac{1}{n} (u(1-u))^{4\Delta_{g_e} - \Delta^A_{g_e}} \left[ \begin{array}{c c}
\left( \frac{F_{g_e}}{A} \right)^2 & -(\frac{F_{g_e}}{A})^1
\end{array} \right]
\]
and the crossing symmetry of the blocks
\[ F_{g_e}(u) = X_{g_e} F_{g_e}(1-u) \sigma_1, \quad F^{-1}_{g_e}(u) = \sigma_1 F^{-1}_{g_e}(1-u) X_{g_e} \] (13.5a)
\[ (X_{g_e})^V = n \frac{\Gamma(n \lambda_{g_e}) \Gamma(-n \lambda_{g_e})}{\Gamma(\lambda_{g_e}) \Gamma(-\lambda_{g_e})}, \quad (X_{g_e})^A = -n \frac{\Gamma(n \lambda_{g_e})^2}{\Gamma((n-1) \lambda_{g_e}) \Gamma((n+1) \lambda_{g_e})} \] (13.5b)
\[ \text{Tr} X_{g_e} = 0, \quad \det X_{g_e} = -1, \quad X^{-1}_{g_e} = X_{g_e} \] (13.5c)
where \( \sigma_1 \) is the first Pauli matrix and \( \Gamma \) is the gamma function.

The \((\dim v(T, g))^2 = 4\) coset blocks of the \(g/h\) correlator in (13.2c)
\[ C(u)^V = \frac{1}{n} u^{-2\Delta_{g/h}} (1-u)^{-2\Delta_{g/h} + \Delta^A_{{g/h}}} \]
\[ [nF(\lambda_g, -\lambda_g, 1-n\lambda_g; u)F((n-1)\lambda_h, (n+1)\lambda_h, n\lambda_h; u)
+ \sum_{x_1} u \frac{F(1+\lambda_g, 1-\lambda_g, 2-n\lambda_g; u)F((n-1)\lambda_h, (n+1)\lambda_h, 1+n\lambda_h; u)}{x_1 + x_2}] \]
\[ C(u)^A = \frac{1}{n} u^{-2\Delta_{g/h} + \Delta^A_{{g/h}}} (1-u)^{-2\Delta_{g/h} + \Delta^A_{{g/h}}} \]
\[ [\sum_{x_1} \frac{1}{x_1} F(\lambda_g, -\lambda_g, 1-n\lambda_g; u)F(1+\lambda_h, 1-\lambda_h, 2-n\lambda_h; u)
- \sum_{x_1} \frac{1}{x_1} F(1+\lambda_g, 1-\lambda_g, 2-n\lambda_g; u)F(\lambda_h, -\lambda_h, 1-n\lambda_h; u)] \]
\[ C(u)^A = u^{-2\Delta_{g/h} + \Delta^A_{{g/h}}} (1-u)^{-2\Delta_{g/h} + \Delta^A_{{g/h}}} \]
\[ [F((n-1)\lambda_g, (n+1)\lambda_g, 1+n\lambda_g; u)F((n-1)\lambda_h, (n+1)\lambda_h, n\lambda_h; u)
- F((n-1)\lambda_g, (n+1)\lambda_g, n\lambda_g; u)F((n-1)\lambda_h, (n+1)\lambda_h, 1+n\lambda_h; u)] \]
\[ C(u)^A = \frac{1}{n} u^{-2\Delta_{g/h} + \Delta^A_{{g/h}}} (1-u)^{-2\Delta_{g/h} + \Delta^A_{{g/h}}} \]
\[ [\sum_{x_1 + x_2} \frac{u}{x_1} F((n-1)\lambda_g, (n+1)\lambda_g, 1+n\lambda_g; u)F(1+\lambda_h, 1-\lambda_h, 2-n\lambda_h; u)
+ nF((n-1)\lambda_g, (n+1)\lambda_g, n\lambda_g; u)F(\lambda_h, -\lambda_h, 1-n\lambda_h; u)] \] (13.6a)
\[ \lambda_g \equiv \lambda_{g_{x_1}} = \frac{1}{x_1 + n}, \quad \lambda_h \equiv \lambda_{g_{x_1 + x_2}} = \frac{1}{x_1 + x_2 + n} \quad (13.6b) \]

\[ \Delta_{g/h}^{V,V} \equiv 0 \]

\[ \Delta_{g/h}^{V,A} \equiv 1 - \Delta_{g_{x_1 + x_2}}^{A} = \frac{x_1 + x_2}{x_1 + x_2 + n} \quad (13.6c) \]

\[ \Delta_{g/h}^{A,V} \equiv \Delta_{g_{x_1}}^{A} + 1 = \frac{x_1 + 2n}{x_1 + n} \]

\[ \Delta_{g/h}^{A,A} \equiv \Delta_{g_{x_1}}^{A} - \Delta_{g_{x_1 + x_2}}^{A} = \frac{nx_2}{(x_1 + n)(x_1 + x_2 + n)} \]

are now computed by substitution of (13.3) and (13.4) into the result (12.3e). Note that

\[ C(u)_{R}^{S} \sim u^{-2\Delta_{g/h}^{V,V}+\Delta_{g/h}^{R,S}} \quad (13.7) \]

so the results in (13.6c) are the conformal weights\(^1\) of the s-channel coset blocks.

The crossing symmetry of the coset blocks

\[ C(u) = X_{g_{x_1}}C(1-u)X_{g_{x_1 + x_2}} \quad (13.8) \]

follows from (13.5), so the coset blocks in the t channel \((u \to 1)\) are the same as the s-channel. In the u channel \((u \to \infty)\), we find four coset blocks with conformal weights\(^1\)

\[ \Delta_{g/h}^{a,a} \equiv \Delta_{g_{x_1}}^{a} - \Delta_{g_{x_1 + x_2}}^{a} , \quad \Delta_{g/h}^{a,s} \equiv \Delta_{g_{x_1}}^{a} - \Delta_{g_{x_1 + x_2}}^{s} + 1 \quad (13.9a) \]

\[ \Delta_{g/h}^{s,s} \equiv \Delta_{g_{x_1}}^{s} - \Delta_{g_{x_1 + x_2}}^{s} , \quad \Delta_{g/h}^{s,a} \equiv \Delta_{g_{x_1}}^{s} - \Delta_{g_{x_1 + x_2}}^{a} + 1 \quad (13.9b) \]

\[ \Delta_{g_{x}}^{a} = \frac{(n - 2)(n + 1)}{n(x + n)} , \quad \Delta_{g_{x}}^{s} = \frac{(n + 2)(n - 1)}{n(x + n)} \quad (13.9c) \]

where \(\Delta_{g_{x}}^{a}\) and \(\Delta_{g_{x}}^{s}\) are the conformal weights of the antisymmetric and symmetric representations \(T_{(a)}\) and \(T_{(s)}\) in \(n \otimes n = (a) + (s)\). On \(SU(2)_{x}\) and

\(^1\)The conformal weights \(\Delta_{g/h}^{V,V}\) and \(\Delta_{g/h}^{A,A}\) correspond to the affine primary fields \((0,0,0)\) and (adjoint, 0, adjoint), where \(\left( T_{g_{x_1}}, T_{g_{x_2}}, T_{g_{x_1 + x_2}} \right)\) denotes the branching of \(T_{g_{x_1}} \otimes T_{g_{x_2}}\) into representations of \(g_{x_1 + x_2}\). The fields of the other two conformal weights in (13.6c) are apparently affine secondary in general.

\(^3\)The fields with conformal weights \(\Delta_{g/h}^{a,a}\) and \(\Delta_{g/h}^{s,s}\) are the affine primary fields \((a,0,a)\) and \((s,0,s)\) respectively, while the fields with \(\Delta_{g/h}^{a,s}\) and \(\Delta_{g/h}^{s,a}\) are apparently affine secondary in general.
(SU(2)_{x_1} \times SU(2)_{x_2})/SU(2)_{x_1+x_2} the u-channel blocks are the same as the s-channel blocks,
\[ R^\Delta_{g/z} (\mathcal{T}_2) \sim R^\Delta_{g/z} (\mathcal{T}_2), \quad R^\Delta_{g/h} (\mathcal{T}_2) \sim R^\Delta_{g/h} (\mathcal{T}_2) \]  
(13.10a)
\[ \Phi_{a,a} \sim \Phi_{V,V}, \quad \Phi_{a,s} \sim \Phi_{V,A}, \quad \Phi_{s,a} \sim \Phi_{A,V}, \quad \Phi_{s,s} \sim \Phi_{A,A} \]  
(13.10b)
\[ \Delta^a_{g/z} = \Delta^A_{g/z}, \quad \Delta^a_{g/z} = 0 \]  
(13.10c)
\[ \Delta^a_{g/h} = \Delta^V_{g/h}, \quad \Delta^a_{g/h} = \Delta^A_{g/h}, \quad \Delta^s_{g/h} = \Delta^V_{g/h}, \quad \Delta^s_{g/h} = \Delta^A_{g/h} \]  
(13.10d)
because \( \mathcal{T}_2 \sim \mathcal{\tilde{T}}_2 \), \( \mathcal{T}_s = \mathcal{T}_A \) and \( \mathcal{T}_a = 0 \) for SU(2).

Following the usual analyticity and crossing arguments [36, 10], we may also construct the crossing-symmetric non-chiral coset correlators
\[ \langle R^\Delta_{g/h} (z_1, z_1^*) R^\Delta_{g/h} (z_2, z_2^*) R^\Delta_{g/h} (z_3, z_3^*) R^\Delta_{g/h} (z_4, z_4^*) \rangle = \frac{N Y(u, u^*)}{|z_{14} z_{23}|} \]  
(13.11a)
\[ Y(u, u^*) = C(u)_V C(u^*)_V + f(\lambda_h) C(u)_V A C(u^*)_V A \]
\[ + f(\lambda_g)^{-1} C(u)_A C(u^*)_A V + f(\lambda_h) C(u)_A A C(u^*)_A A ] \]  
(13.11b)
\[ f(\alpha) \equiv n^2 \left( \frac{\Gamma(n\alpha)}{\Gamma(1-n\alpha)} \right)^2 \frac{\Gamma(1-(n-1)\alpha) \Gamma(1-(n+1)\alpha)}{\Gamma((n-1)\alpha) \Gamma((n+1)\alpha)} \]  
(13.11c)
where \( N \) is a normalization and \( z^*, u^* \) are the complex conjugates of \( z, u \).

From the non-chiral correlators we may infer the coset fusion rules for \( x_1 \neq 1 \)
\[ [\Phi] \times [\Phi] = [\Phi_{V,V}] + [\Phi_{V,A}] + [\Phi_{A,V}] + [\Phi_{A,A}] \]  
(13.12a)
\[ [\Phi] \times \Phi = [\Phi_{a,a}] + [\Phi_{a,s}] + [\Phi_{s,a}] + [\Phi_{s,s}] \]  
(13.12b)
where \([\Phi]\) and \([\tilde{\Phi}]\) are the conformal blocks of \( R_{g/h}(T) \) and \( R_{g/h}(\tilde{T}) \) respectively.

At \( x_1 = 1 \), however, only the coset blocks \( C^R V \), \( R = V, A \) contribute because
\[ f(\lambda_{g_{x_1=1}})^{-1} = f \left( \frac{1}{n+1} \right)^{-1} = 0 \]  
(13.13)
and we obtain the truncated coset fusion rules
\[ [\Phi] \times [\Phi] = [\Phi_{V,V}] + [\Phi_{V,A}] \]  
(13.14a)
\[ [\Phi] \times [\Phi] = [\Phi_{a,a}] + [\Phi_{a,s}] \]  
(13.14b)
for \((SU(n)_1 \times SU(n)_{x_2})/SU(n)_{x_2+1}\). This consistent truncation of the coset blocks is equivalent to a consistent affine cutoff on the blocks of \(g_{x_1} = SU(n)_{x_1}\),

\[
d^A = 0, \quad \mathcal{G}^M(u) = d^V(\mathcal{F}_{g_{x_1}=1}(u))_{V^M}, \quad \mathcal{Y}_{g/h}^\alpha(u) = d^V C(u)_V \mathcal{S}^{\alpha}(u_0, g)
\]

in which only the vacuum blocks \((\mathcal{F}_{g_{x_1}})_V^M, M = 1, 2\) of the A-S construction are included at level one \([10]\).

Since the coset blocks \((13.6)\) are unfamiliar and the non-chiral correlators \((13.11)\) are largely new, we have checked our results against the literature in a number of cases.

Consider first the two-block solutions at \(x_1 = 1\), whose coset blocks

\[
\mathcal{C}(u)_V^A = -\frac{x_2}{n(x_2 + 1)} u^{-2\Delta_{g/h} + \Delta_{V,A}^V} F(1 - u, 1 - \lambda_h, 2 - n\lambda_h; u)
\]

\[
\Delta_{g/h} = \frac{x_2(n^2 - 1)}{2n(n + 1)(x_2 + n + 1)}, \quad \Delta_{V,h}^V = 0, \quad \Delta_{V,A}^V = \frac{x_2 + 1}{x_2 + n + 1}
\]

(13.16a)

(13.16b)

satisfy the truncated fusion rules \((13.14)\). The simple form of these blocks, with only one hypergeometric function, is obtained from \((13.6)\) because \(F(a, b, a; u) = (1 - u)^{-b}\).

As a first check, the two-block solutions reduce to the correct results \([36]\)

\[
R^\alpha_{g/h}(T(2)) \sim \Phi_{1,2}
\]

\[
\Delta_{g/h} = \Delta_{g/h}(T(2)) = \Delta_{1,2} = \frac{m}{4(m + 3)}
\]

\[
\Delta_{V,h}^V = \Delta_{1,1} = 0, \quad \Delta_{V,A}^V = \Delta_{1,3} = \frac{m + 1}{m + 3}
\]

\([\Phi_{1,2}] \times [\Phi_{1,2}] = [\Phi_{1,1}] + [\Phi_{1,3}]
\]

(13.17c)

(13.17d)

for the Virasoro minimal models \((SU(2)_1 \times SU(2)_m)/SU(2)_{m+1}\), where the minimal model conformal weights \(\Delta_{p,q}\) and the \(SU(2)\) equivalences \((13.10)\) were used to make the identifications. Moreover, the explicit form of the coset blocks in \((13.16)\) agrees with the blocks obtained by Dotsenko and Fateev \([37]\) in this case.

Finally, we have checked on \((SU(3)_1 \times SU(3)_m)/SU(3)_{m+1}\) that the blocks in \((13.16)\) are the known \([38, 13]\) two-block solutions in the \(W_3\) minimal models.
More generally, the iterated coset equations \( \partial \mathcal{C}_V^R = \mathcal{C}_V^S W_{q/h} s^R \) in (12.5) give the second-order differential equation

\[
\left\{ \frac{n + 1}{n [1 + \frac{2(n^2 - 1)}{n(n-1)} \Delta_{g/h}]} \partial^2 + \left[ \frac{1}{u} + \frac{1}{u - 1} \right] \partial - \Delta_{g/h} \left[ \frac{2}{n} \frac{[1 + \frac{2}{n-1} \Delta_{g/h}]}{n [1 + \frac{2(n^2 - 2)}{n(n-1)} \Delta_{g/h}]} \left( \frac{1}{u^2} + \frac{1}{(u - 1)^2} \right) - \frac{2}{u(u - 1)} \right] \right\} C(u)_{V}^R = 0
\]

(13.18)

for the entire set of two-block solutions, where \( \Delta_{g/h} \) is given in (13.16b). On \((SU(2)_1 \times SU(2)_m)/SU(2)_{m+1}\), this is the equation derived by BPZ from the chiral null state

\[
[(L_{g/h}^{-1})^2 - \frac{2}{3} (1 + 2 \Delta_{g/h} (T_{2})) L_{g/h}^{(-2)}] R_{g/h}^\alpha (T_{2}, 0) |0\rangle = 0 , \quad R_{g/h}^\alpha (T_{2}, 0) |0\rangle = |\Phi_{1,2}\rangle
\]

(13.19)

where \( \Delta_{g/h}(T_{2}) = \Delta_{1,2} \) is given in (13.17b).

As a check on the four-block solutions, we recover Douglas’ conclusion for the Virasoro minimal models

\[
R_{g/h}^\alpha (T_{2}) \sim \Phi_{2,2}
\]

(13.20a)

\[
\Delta_{g/h} = \Delta_{2,2} = \frac{3}{4(m + 2)(m + 3)}
\]

(13.20b)

\[
\Delta_{g/h}^{V,V} = \Delta_{1,1} = 0 , \quad \Delta_{g/h}^{V,A} = \Delta_{1,3} = \frac{m + 1}{m + 3}
\]

(13.20c)

\[
\Delta_{g/h}^{A,V} = \Delta_{3,1} = \frac{m + 4}{m + 2} , \quad \Delta_{g/h}^{A,A} = \Delta_{3,3} = \frac{2}{(m + 2)(m + 3)}
\]

(13.20d)

\[
[\Phi_{2,2}] \times [\Phi_{2,2}] = [\Phi_{1,1}] + [\Phi_{1,3}] + [\Phi_{3,1}] + [\Phi_{3,3}]
\]

(13.20e)

when \( n = 2, x_1 = m \neq 1 \) and \( x_2 = 1 \). Moreover, the explicit form of the coset blocks in (13.6) agrees with the blocks obtained by Zamolodchikov [46] in this case. Finally, we have checked that the four-block solutions are in agreement with the known [47, 39] conformal weights and fusion rules in the N=1 superconformal series \((n = 2 \text{ and } x_1 = 2, x_2 = m \text{ or } x_1 = m \neq 1, x_2 = 2)\) and \( W_3 \) minimal models \((n = 3 \text{ and } x_1 = m \neq 1, x_2 = 1)\).
For completeness, we finally give the fourth-order differential equation of the general four-block solutions, obtained\(^*\) from the iterated coset equations in (12.5):

\[
\left\{ \partial^4 + a_1 \left[ \frac{1}{u} + \frac{1}{u-1} \right] \partial^3 + \left[ a_2 \left( \frac{1}{u^2} + \frac{1}{(u-1)^2} \right) + \frac{a_3}{u(u-1)} \right] \partial^2 \\
+ a_4 \left( \frac{1}{u^3} + \frac{1}{(u-1)^3} \right) + a_5 \left( \frac{1}{u^2(u-1)} + \frac{1}{u(u-1)^2} \right) \partial \\
+ a_6 \left( \frac{1}{u^4} + \frac{1}{(u-1)^4} \right) + a_7 \left( \frac{1}{u^3(u-1)} + \frac{1}{u(u-1)^3} \right) \\
+ \frac{a_8}{u^2(u-1)^2} \right\} C(u)_R^s = 0 \quad (13.21a)
\]

\[
a_1 = 4 \left[ 1 + \frac{n^2 - 2}{n^2 - 1} \Delta_{g/h} \right]
\]

\[
a_2 = 2 \left[ 1 + \frac{2(2n^2 - 3)}{n^2 - 1} \Delta_{g/h} + \frac{2(n^4 - 6n^2 + 6)}{(n^2 - 1)^2} \Delta_{g/h} - \frac{n^2 \Delta_{g/h}}{c_{g/h} + 2n^2 \Delta_{g/h}} \right]
\]

\[
a_3 = 2 \left[ 5 + \frac{10(n^2 - 2)}{n^2 - 1} \Delta_{g/h} + \frac{4(2n^4 - 7n^2 + 6)}{(n^2 - 1)^2} \Delta_{g/h} + \frac{2(n^2 - 2) \Delta_{g/h}}{c_{g/h} + 2n^2 \Delta_{g/h}} \right]
\]

\[
a_4 = \frac{2n^2 \Delta_{g/h}}{n^2 - 1} \left[ 1 + \frac{2(n^2 - 2)}{n^2 - 1} \Delta_{g/h} - \frac{8(n^2 - 2)}{n^2(n^2 - 1)} \Delta_{g/h}^2 \\
- \frac{n^2 - 1 + 2(n^2 - 2) \Delta_{g/h}}{c_{g/h} + 2n^2 \Delta_{g/h}} \right]
\]

\[
a_5 = 2 \left[ 1 + \frac{5(n^2 - 2)}{n^2 - 1} \Delta_{g/h} + \frac{2(5n^4 - 20n^2 + 16)}{(n^2 - 1)^2} \Delta_{g/h}^2 \\
+ \frac{8(n^2 - 2)(n^2 - 3)}{(n^2 - 1)^2} \Delta_{g/h}^3 + \frac{(n^2 - 2)}{n^2 - 1} \Delta_{g/h} - \frac{2(n^2 - 4) \Delta_{g/h}}{c_{g/h} + 2n^2 \Delta_{g/h}} \right] \Delta_{g/h} \]
\]

\[
a_6 = -\frac{4 \Delta_{g/h}^2}{n^2 - 1} \left[ 1 + \frac{4}{n^2 - 1} \Delta_{g/h} - \frac{4}{n^2 - 1} \Delta_{g/h}^2 - \frac{2n^2 \Delta_{g/h}}{c_{g/h} + 2n^2 \Delta_{g/h}} \right] \quad (13.21b)
\]

\[
a_7 = \frac{4 \Delta_{g/h}^2}{n^2 - 1} \left[ n^2 + \frac{2n^2(n^2 - 2)}{n^2 - 1} \Delta_{g/h} - \frac{8(n^2 - 2)}{n^2 - 1} \Delta_{g/h}^2 \\
- (n^2 - 2) \frac{(n^2 - 1) + 2(n^2 + 2) \Delta_{g/h}}{c_{g/h} + 2n^2 \Delta_{g/h}} \right]
\]

\*The coefficients in (13.21) were obtained by G. Rivlis using Mathematica.
\[ a_8 = -\frac{4\Delta^2_{g/h}}{n^2 - 1} \left[ \frac{(n^2 + 2) + 8\Delta_{g/h} - \frac{4(n^4 - 4n^2 + 6)}{n^2 - 1}\Delta^2_{g/h}}{\frac{2(n^2 - 1)(n^2 - 2) + 4(n^4 - 3n^2 + 4)\Delta_{g/h}}{c_{g/h} + 2n^2\Delta_{g/h}} \right] . \]

Bearing in mind that we already have the solutions to this equation, its form illustrates the technical superiority of the coset block approach over chiral null state computations.

## 14 Conclusion

Following a review of the Virasoro master equation \[15, 16\] and Virasoro biprimary fields \[14\] in biconformal field theory, we have derived a hierarchy of non-linear Ward identities for affine-Virasoro correlators. The hierarchy follows from KZ-type null states \[10\] and the assumption of factorization \[10, 11, 14, 29\], whose consistency was verified at an abstract level.

The abstract form of the Ward identities is only a first step toward the correlators, however, because solution of the equations requires specific factorization ansätze, which may vary over affine-Virasoro space.

In this paper, we solved the non-linear equations only for the simple case of \( h \subset g \) and the \( g/h \) coset constructions \[2, 5, 11\], using a matrix factorization: The resulting coset correlators satisfy first-order linear partial differential equations, called the coset equations, whose solutions are the coset blocks defined by Douglas \[41\]. The coset equations exhibit a class of flat connections, called the dressed coset connections, which are not in the class of connections associated to the classical Yang-Baxter equation.

Beyond the coset constructions, we have noted in Sections 7 and 10 that other factorization ansätze may be required, such as the symmetric factorization

\[ R^\alpha(T, \bar{z}, z) = (\bar{R}(T, \bar{z})R(T, z))^\alpha = \sum_\nu \bar{R}^\alpha_\nu(T, \bar{z})R^\alpha(T, z) \]  \hspace{1cm} (14.1a)

\[ A^\alpha(\bar{z}, z) = (\bar{A}(\bar{z})A(z))^\alpha = \sum_\nu \bar{A}^\alpha_\nu(\bar{z})A^\alpha(z) \]  \hspace{1cm} (14.1b)

\[ Y^\alpha(\bar{u}, u) = (\bar{Y}(\bar{u})Y(u))^\alpha = \sum_\nu \bar{Y}^\alpha_\nu(\bar{u})Y^\alpha(u) \]  \hspace{1cm} (14.1c)

where \( \nu \) is a conformal-block index to be determined by the equations.
A more conservative direction keeps the matrix factorization and follows the flat-connection clue provided by the coset constructions. By high-level expansion for low-spin representations on simple $g$, flat connections $W_i[L]$ can be associated to every high-level smooth affine-Virasoro construction $L$. The leading term of the flat connections

$$W_i[L] = W_i + \mathcal{O}(k^{-2}) = 2L^{ab}\sum_{j \neq i} \frac{T_a^i T_b^j}{z_{ij}} + \mathcal{O}(k^{-2})$$  \hspace{1cm} (14.2a)

$$L^{ab} = \frac{P^{ab}}{2k} + \mathcal{O}(k^{-2})$$  \hspace{1cm} (14.2b)

is abelian-flat, where $P^{ab}$ is the high-level projection operator of the construction. In the all-order expansion around (14.2), one finds a sequence of abelian Bianchi identities which are satisfied so long as the leading term is abelian-flat. These flat connections are correct for the A-S and coset constructions, and, for all $L$, the corresponding high-level correlators

$$A^\alpha[L] = v^\beta(g) \left( \delta^\alpha_\beta + \frac{P^{ab}}{k} \sum_{i<j} \ln \left( \frac{z_{ij}}{z_{0ij}} \right) (T^i_a T^j_b)_{\beta}^\alpha \right) + \mathcal{O}(k^{-2})$$  \hspace{1cm} (14.3)

satisfy $SL(2)$-covariance and the Ward identities at the indicated order. We are presently investigating these properties at higher order.

We finally remark on an open question posed by Douglas for the coset constructions. We saw in Section 13 that the set of coset blocks is sometimes larger than the set of chiral blocks, as defined by chiral null-state differential equations. The question posed by Douglas concerns the precise relation between the two sets of blocks. Based on our examples, it is a reasonable conjecture that the two sets are the same when the A-S blocks, and hence the coset blocks, are restricted to the integrable blocks,

$$C_{\hat{r}}^{\hat{R}} = (\mathcal{F}_g)_{\hat{r}}{^n} (\mathcal{F}_h^{-1})_n^{\hat{R}}$$  \hspace{1cm} (14.4)

where $\hat{r}$ and $\hat{R}$ are the integrable representations of $g$ and $h$. This truncation to the integrable blocks is always consistent (because affine Lie algebra is a chiral construction), and ordinary differential equations for the integrable blocks are obtainable in principle from the coset equations. Without a better characterization of the set of all chiral blocks, however, the conjecture seems difficult to prove or disprove.
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Appendix A: Stress tensors and affine primary fields

The results of this appendix were obtained with E. Kiritsis.

We begin with the defining relations

\[ J_a(z)R^I_g(\mathcal{T}, w) = \left( \frac{1}{z - w} + \frac{1}{2\Delta_g(\mathcal{T})} \partial_w \right) R^J_g(\mathcal{T}, w)(\mathcal{T}_a)_J^I + (R_g)_a^I(\mathcal{T}, w) + \mathcal{O}(z - w) \]  \hfill (A.1a)

\[ \langle R^I_g(\mathcal{T}, z)R^J_g(\tilde{\mathcal{T}}, w) \rangle = \frac{\eta^{IJ}(\mathcal{T})}{(z - w)^{2\Delta_g(\mathcal{T})}} \]  \hfill (A.1b)

for the affine primary field whose matrix representation is \( \mathcal{T} \). \( \tilde{\mathcal{T}} \) is the complex conjugate representation defined in (3.6b), and indices are raised and lowered with the metric on carrier space \( \eta_{IJ}(\mathcal{T}) \), so that \( \tilde{\mathcal{T}} = -\mathcal{T}^T = -\mathcal{T}^* \).

With (A.1), we may compute the correlators of any number of currents with two primary fields \( R_g(\mathcal{T})R_g(\bar{\mathcal{T}}) \). From these, we obtain the correlators

\[ \langle T_{ab}(z_1)R^I_g(\mathcal{T}, z_2)R^J_g(\tilde{\mathcal{T}}, z_3) \rangle = \frac{\frac{1}{2}(\mathcal{T}_a, \mathcal{T}_b)^{IJ}}{z_1^{2\Delta_a(\mathcal{T})}z_2^{2\Delta_a(\tilde{\mathcal{T}})}z_3^{2\Delta_a(\mathcal{T})} - 2} \]  \hfill (A.2a)

\[ \langle (R_g)_a^I(\mathcal{T}, z)R^J_g(\tilde{\mathcal{T}}, w) \rangle = 0 \]  \hfill (A.2b)

\[ \langle (R_g)_a^I(\mathcal{T}, z)(R_g)_b^J(\tilde{\mathcal{T}}, w) \rangle = \frac{\left( G_{ab}\eta(\mathcal{T}) + \frac{2\Delta_a(\mathcal{T}) - 1}{2\Delta_a(\tilde{\mathcal{T}})}(\mathcal{T}_b + \mathcal{T}_a) \right)^{IJ}}{(z - w)^{2\Delta_g(\mathcal{T}) + 2}} \]  \hfill (A.2c)

\[ \langle T_{ab}(z_1)R^I_g(\mathcal{T}, z_2)(R_g)_c^J(\tilde{\mathcal{T}}, z_3) \rangle = \frac{(G_{ac}\mathcal{T}_b + G_{bc}\mathcal{T}_a - (\mathcal{T}_a\mathcal{T}_b + \mathcal{T}_b\mathcal{T}_a) + \frac{2\Delta_a(\mathcal{T}) - 1}{2\Delta_a(\tilde{\mathcal{T}})}(\mathcal{T}_c + \mathcal{T}_a, \mathcal{T}_b))^{IJ}}{z_1^{2\Delta_a(\mathcal{T})}z_2^{2\Delta_a(\tilde{\mathcal{T}})}z_3^{2\Delta_a(\mathcal{T})} - 1} \]  \hfill (A.2d)

where \( T_{ab} = \ast J_a J_b \ast \) and \( (R_g)_a^I(\mathcal{T}) = \ast R_g^I(\mathcal{T}) \ast \). These correlators determine the OPE

\[ T_{ab}(z)R^I_g(\mathcal{T}, w) = \left( \frac{1}{(z - w)^2} + \frac{1}{\Delta_g(\mathcal{T})} \frac{\partial_w}{z - w} \right) R^J_g(\mathcal{T}, w) \frac{1}{2}(\mathcal{T}_a, \mathcal{T}_b)^{IJ} + \frac{1}{z - w}(R_g)_a^I(\mathcal{T}, w)(\mathcal{T}_b)_J^I + \text{reg.} \]  \hfill (A.3)

which we take in an \( L \)-basis of representation \( \mathcal{T} \) (see Section 3). Then, multiplication by \( L^{ab}_a \) gives the OPE of the affine-Virasoro contraction \( T = L^{ab}_a J_a J_b \ast \) with the affine primary field

\[ T(z)R^\alpha_g(\mathcal{T}, w) = \Delta_\alpha(\mathcal{T}) \left( \frac{1}{(z - w)^2} + \frac{1}{\Delta_g(\mathcal{T})} \frac{\partial_w}{z - w} \right) R^\alpha_g(\mathcal{T}, w) \]
\[
+ \frac{2L^{ab}(R_g)^\beta_a(T, w)(T_b)^\alpha_\beta}{z-w} + \text{reg.} \tag{A.4}
\]

where \(\Delta_\alpha(T)\) are the \(L^{ab}\)-broken conformal weights of representation \(T\). The OPE with \(\bar{T} = \bar{L}^{ab} J_a J_b\) is obtained from (A.4) by the replacement \(T \rightarrow \bar{T}, \Delta \rightarrow \bar{\Delta}\) and \(L \rightarrow \bar{L}\).

The OPE (A.4) shows the characteristic affine-Virasoro form (4.3) with the extra term \(\delta R_g^\alpha\) given in (4.5b).

The non-leading term in (A.4), proportional to the composite field \((R_g)^\alpha_a\), was first seen (for A-S and coset constructions) in Ref. [14]. We emphasize here the apparent persistence of the term in the special case of the A-S construction

\[
T_g(z) R_g^\alpha(T, w) = \left(\frac{\Delta_g(T)}{(z-w)^2} + \frac{\partial_w}{z-w}\right) R_g^\alpha(T, w) + \frac{2L^{ab}(R_g)^\beta_a(T, w)(T_b)^\alpha_\beta}{z-w} + \text{reg.} \tag{A.5}
\]

where, unless the operator is zero, it contradicts the conventional wisdom that affine primary fields are Virasoro primary under the A-S construction.

In fact, it is not difficult to check from (3.3b) and (A2.c) that

\[
\langle L^{ab}(R_g)^\beta_a(T, z)(T_b)^\alpha_\beta L^{cd}(R_g)^\sigma_c(\bar{T}, w)(\bar{T}_d)^\rho_d\rangle = 0 \tag{A.6}
\]

so the offending operator is zero for unitary representations of affine compact \(g\). Although it creates only null states, the composite field term apparently persists for non-unitary constructions, and deserves further study.

**Appendix B: Biprimary fields and the stability condition**

The original form of the Virasoro biprimary fields [14]

\[
\phi^\alpha(\bar{z}, z) = z^{L(0)} \bar{z}^{\bar{L}(0)} \phi^\alpha_g(1) z^{-L(0)} \bar{z}^{-\bar{L}(0)} z^{-\Delta^\alpha} \bar{z}^{-\bar{\Delta}^\alpha} \tag{B.1}
\]

was given only for \(h\) and \(g/h\), since the Virasoro master equation had not yet been found. Here, we follow the steps of the original argument to verify that \(\phi^\alpha(\bar{z}, z)\) is biprimary for any K-conjugate pair.

We begin with the identities

\[
z\partial_z \phi^\alpha(\bar{z}, z) = [L(0), \phi^\alpha(\bar{z}, z)] - \Delta^\alpha \phi^\alpha(\bar{z}, z) \tag{B.2a}
\]
\[ z^{-L(0)}L^{(m)}z^{L(0)} = L^{(m)}z^m \]  
(B.2b)

\[ [L^{(m)} - L^{(0)}, \phi^\alpha_g(1)] = m\Delta_a\phi^\alpha_g(1) \]  
(B.2c)

where (B.2c) is the \( L \)-stability condition [2, 14], which follows from (4.9a). Then, \( \phi^\alpha(\bar{z}, z) \) is a Virasoro primary field under \( T(z) \) because

\[
[L^{(m)}, \phi^\alpha(\bar{z}, z)] = z^{L(0)}\bar{z}^{L(0)}[L^{(m)}, \phi^\alpha_g(1)]z^{-L(0)}\bar{z}^{-L(0)}z^m\Delta_a\bar{z}^{-\tilde{\Delta}_a} 
\]  
(B.3a)

\[
= z^{L(0)}\bar{z}^{L(0)}\{[L^{(0)}, \phi^\alpha_g(1)] + m\Delta_a\phi^\alpha_g(1)\}z^m-L(0)-\Delta_a\bar{z}^{-L(0)-\tilde{\Delta}_a} 
\]  
(B.3b)

\[
= z^m(\bar{z}\partial_{\bar{z}} + (m + 1)\Delta_a)\phi^\alpha(\bar{z}, z) 
\]  
(B.3c)

where the \( L \)-stability condition was used in the second step. Similarly, \( \phi^\alpha(\bar{z}, z) \) is Virasoro primary under \( \tilde{T}(z) \) because

\[
[L^{(m)}, \phi^\alpha(\bar{z}, z)] = \bar{z}^m(\bar{z}\partial_{\bar{z}} + (m + 1)\tilde{\Delta}_a)\phi^\alpha(\bar{z}, z) 
\]  
(B.4)

is obtained from (B.1) with the \( \bar{L} \)-analogues of (B.2).

### Appendix C: Ward identities for \( L^{ab} \)-broken currents

We give the first-order Ward identities for the \( L^{ab} \)-broken current correlators. Define

\[
\langle \tilde{J}(\bar{z})\mathcal{J}(z) \rangle_{A_1..A_n} \equiv \langle \mathcal{J}_{A_1}(\bar{z}_1, z_1) \ldots \mathcal{J}_{A_n}(\bar{z}_n, z_n) \rangle 
\]  
(C.1a)

\[
\langle \tilde{J}(z)\mathcal{J}(z) \rangle_{A_1..A_n} \equiv \langle J_{A_1}(z_1) \ldots J_{A_n}(z_n) \rangle 
\]  
(C.1b)

where \( \mathcal{J}_{A}(\bar{z}, z) \) are the biprimary fields (5.2a) of the \( L^{ab} \)-broken currents \( J_A \). Then the factorized Ward identities

\[
\langle \tilde{J}(z)\partial_i\mathcal{J}(z) \rangle_{A_1..A_n} = -2L^{CD}_{\, \, \, ij} \sum_{j \neq i} \left\{ \frac{f^{B_i}_{\, \, \, B_j}f^{B_j}_{\, \, \, B_i}}{z_{ij}^2} \langle \tilde{J}(z)\mathcal{J}(z) \rangle_{A_1..B_i..B_j..A_n} + \frac{G_{CA_i}f^{B_j}_{\, \, \, B_i}}{z_{ij}^3} \langle \tilde{J}(z)\mathcal{J}(z) \rangle_{A_1..\hat{A}_i..B_j..A_n} - \frac{G_{CA_j}f^{B_i}_{\, \, \, B_i}}{z_{ij}^3} \langle \tilde{J}(z)\mathcal{J}(z) \rangle_{A_1..B_i..\hat{A}_j..A_n} \right\} 
\]  
(C.2)

are obtained with (C.1) by choosing \( \phi^\alpha_g = J_A \) in (6.3). Here, hatted indices indicate currents which are not present in the A-S correlators, and the corresponding right side of \( \langle \partial_i\tilde{J} \mathcal{J} \rangle \) is obtained from (C.2) by \( L \rightarrow \bar{L} \).
Since \((T^\text{adjoint}_A)^C_B = -if_{AB}^C\), the first term on the right side of (C.2) is analogous to the right side of the first-order Ward identity (8.3a) for the broken affine primaries. The extra inhomogeneous terms, with hatted indices, arise from the central term of the affine Lie algebra (2.1b), and such terms are expected generically in the Ward identities of broken affine secondaries.

These equations are solved identically, before or after factorization, by the two- and three-point biconformal current correlators in (5.3b) and (5.8). For the invariant four-point correlators (see eq.(5.11)), we choose the cross ratios \(u\) and \(\bar{u}\) in (9.1b), and the KZ gauge

\[
\gamma_{12} = \gamma_{13} = 0 \quad , \quad \gamma_{14} = 2\Delta_A \quad , \quad \gamma_{23} = \Delta_A + \Delta_B - \Delta_C - \Delta_D
\]

\[
\gamma_{24} = -\Delta_A + \Delta_B + \Delta_C - \Delta_D \quad , \quad \gamma_{34} = -\Delta_A - \Delta_B + \Delta_C + \Delta_D \quad (C.3)
\]

where \(\Delta_A\) is the \(L^{ab}\)-broken conformal weight of the current \(J_A\) and \(\bar{\Delta}_A + \Delta_A = 1\). Then, we obtain the one-dimensional non-linear equations for the invariant current correlators

\[
(Y \partial Y)_{A_1A_2A_3A_4} = -\Delta_{A_1} \left( \frac{2}{u^3} G_{A_1A_2} G_{A_3A_4} + \frac{2}{(u-1)^3} G_{A_1A_3} G_{A_2A_4} \right)
\]

\[
+ \frac{1}{u^2} f_{A_1A_2}^C f_{A_1A_3C} + \frac{1}{(u-1)^2} f_{A_1A_3}^C f_{A_2A_4C}
\]

\[
+ \frac{1}{u(u-1)} \sum_C \left[ \frac{1}{u} (\Delta_{A_2} - \Delta_C) f_{A_1A_2 C} f_{A_1A_3 C} + \frac{1}{u-1} (\Delta_{A_3} - \Delta_C) f_{A_1A_3 C} f_{A_2A_4 C} \right.
\]

\[
-(\Delta_{A_4} - \Delta_C) f_{A_1A_4 C} f_{A_2A_3 C} \right] \quad (C.4)
\]

which we have expressed entirely in terms of the \(L^{ab}\)-broken conformal weights of the currents. Similarly, the right side of \((\partial Y Y)_A\) is obtained from (C.4) by \(\Delta_A \rightarrow \bar{\Delta}_A\).
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