GLOBAL DYNAMICS OF A MICROORGANISM FLOCCULATION MODEL WITH TIME DELAY

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Abstract. In this paper, we consider a microorganism flocculation model with time delay. In this model, there may exist a forward bifurcation/backward bifurcation. By constructing suitable positively invariant sets and using Lyapunov-LaSalle theorem, we study the global stability of the equilibria of the model under certain conditions. Furthermore, we also investigate the permanence of the model, and an explicit expression of the eventual lower bound of microorganism concentration is given.

1. Introduction. Microorganisms are widely used in many fields such as environmental protection industry, pharmaceutical industry, food industry, and energy industry, etc (see, e.g., [2, 26]). Hence, the study about continuous cultivation, collection and extraction, degradation of microorganisms has been drawn much attention (see, e.g., [9, 17, 18, 20, 21, 24, 28, 29]). Generally speaking, the physical method of flocculation to collect and extract microorganisms, which is one of the most efficient methods in the world, is playing an important role in production and application in microbial industry.

Recently, models of continuous cultivation of microorganisms have been proposed by many authors (see, e.g., the models with time delay [1, 3, 4, 15, 14, 24], ones without time delay [6, 11, 12, 16]). They investigated the stability or permanence of these models by applying some theories in delay differential equations [5, 7, 10, 13, 22, 27]. In this paper, we consider the permanence and the global stability for...
the following time-delayed model of microorganism flocculation which is proposed in [24].

\[
\begin{align*}
\dot{x}(t) &= dx^0 - dx(t) - \tilde{h}_1 x(t) y(t), \\
\dot{y}(t) &= h x(t - \tau) y(t - \tau) - dy(t) - \tilde{h}_2 y(t) z(t), \\
\dot{z}(t) &= dz^0 - dz(t) - \tilde{h}_3 y(t) z(t),
\end{align*}
\]  

where \(x(t), y(t), z(t)\) represent the concentrations of nutrient, microorganisms and flocculant at time \(t\), respectively. The positive constant \(d\) represents the same velocities of inflow and outflow of nutrient and flocculant as well as the velocity of outflow of microorganisms. The constants \(x^0 > 0, z^0 > 0\) indicate the input concentrations of nutrient for cultivating microorganisms and flocculant for precipitating microorganisms, respectively. \(\tilde{h}_1, h, \tilde{h}_2, \tilde{h}_3\) are positive constants standing for the consumption rate of nutrient, the growth rate of microorganisms, the flocculating rate of microorganisms, and the consumption rate of flocculant, respectively. The nonnegative constant \(\tau\) is the time delay.

Let \(C := C([-\tau, 0], \mathbb{R}^3)\) denote the Banach space of continuous functions mapping from the interval \([-\tau, 0]\) to \(\mathbb{R}^3\) equipped with the sup-norm, and \(C^+ := C([-\tau, 0], \mathbb{R}^3_+)\) stands for the nonnegative cone of \(C\) with \(\mathbb{R}^3_+ := [0, \infty)\). For simplicity, the dimensionless system for (1) is given by

\[
\begin{align*}
\dot{x}(t) &= 1 - x(t) - h_1 x(t) y(t), \\
\dot{y}(t) &= r x(t - \tau) y(t - \tau) - y(t) - h_2 y(t) z(t), \\
\dot{z}(t) &= 1 - z(t) - h_3 y(t) z(t),
\end{align*}
\]  

where

\[
h_1 = \frac{\tilde{h}_1}{d}, \quad r = \frac{h_2}{d}, \quad h_2 = \frac{\tilde{h}_2}{d}, \quad h_3 = \frac{\tilde{h}_3}{d},
\]

with initial condition

\[
x(\theta) = \phi_1(\theta), \quad y(\theta) = \phi_2(\theta), \quad z(\theta) = \phi_3(\theta), \quad \theta \in [-\tau, 0],
\]

where \(\phi = (\phi_1, \phi_2, \phi_3)^T \in C^+\). It can be seen from [24] that (2) with the initial condition (3) is well-posed with its solutions \((x(t), y(t), z(t))^T\) nonnegative and bounded on \([0, \infty)\), and the local asymptotic stability of equilibria is analyzed in detail under some suitable assumptions (see, Figs. 1 and 2). We define \(u_t \in C^+, t \geq 0\) as \(u_t(\theta) = u(t + \theta), \theta \in [-\tau, 0]\), where \(u : [-\tau, \infty) \to \mathbb{R}^3_+\) is continuous. It is not difficult to find that the closed set

\[
G = \{\phi = (\phi_1, \phi_2, \phi_3)^T \in C^+ : ||\phi_1|| \leq 1, ||\phi_3|| \leq 1\}
\]

is a positively invariant set of (2) which attracts all its solutions. Hence, we confine our discussion to the set \(G\).

Let \(R = (r - 1)/h_2\). (2) always has a boundary equilibrium \(E_0 = (1, 0, 1)^T\). In addition, for the positive equilibria, we have

(I) If \(h_2h_3 \leq h_1h_2 + h_1\) and \(R > 1\) (i.e. (2) undergoes a forward bifurcation, see Fig. 1), then there exists a unique positive equilibrium \(E^* = (x^*, y^*, z^*)^T\).

(II) Assume that \(h_2h_3 > h_1h_2 + h_1\). Then the following three propositions hold:

(i) If \(R \geq 1\), then there exists a unique positive equilibrium \(E^* = (x^*, y^*, z^*)^T\).

(ii) If \(w > R < 1\) (i.e. (2) undergoes a backward bifurcation, see Fig. 2), then there exist two positive equilibria \(E^* = (x^*, y^*, z^*)^T\) and \(E^{**} = (x^{**}, y^{**}, z^{**})^T\), where

\[
w = \frac{h_1h_2 - h_1 + 2\sqrt{h_1h_2(h_3 - h_1)}}{h_2h_3}.
\]
(iii) If $R = w$, then there exists a unique positive equilibrium $E^* = (x^*, y^*, z^*)^T$.
(III) Otherwise, in any other cases, (2) has no positive equilibria.

![Fig. 1. Forward bifurcation.](image1)

![Fig. 2. Backward bifurcation.](image2)

The purpose of this paper is to study the global stability of the equilibria of (2) under some conditions by constructing suitable positively invariant sets and using Lyapunov-LaSalle theorem. In addition, we also investigate the permanence of (2), and give a specific estimation of the eventual lower bound of microorganism concentration by using a thorough analysis (also see [8]) which differs from traditional methods.

2. Permanence of the system. We will consider the permanence of (2) in this section. We have

**Theorem 2.1.** If $R > 1$, then (2) is permanent in $X = \{\phi \in G : \phi_2(0) > 0\}$, and the solution $(x(t), y(t), z(t))^T$ of (2) with any $\phi \in X$ fulfills

$$\liminf_{t \to \infty} y(t) \geq \frac{q}{h_1(1 + h_2)} e^{-(1 + h_2)(T_0 + \tau)} \equiv \nu,$$

where $q \in (0, r - 1 - h_2)$ and

$$T_0 = -\frac{1 + h_2}{1 + h_2 + q} \ln \frac{r - 1 - h_2 - q}{r}.$$

**Proof.** We can first obtain that $X$ is positively invariant for (2). Let $u_t = (x_t, y_t, z_t)^T$ be the solution of (2) with any $\phi \in X$, and let

$$s(t) = \frac{r}{h_1} x_t(-\tau) + y_t(0) = \frac{r}{h_1} x(t - \tau) + y(t), \ t \geq \tau.$$

Then we have $\dot{s}(t) \leq r/h_1 - s(t)$ for $t \geq \tau$. Hence, $\limsup_{t \to \infty} y(t) \leq r/h_1$, it follows immediately from (2) that

$$\liminf_{t \to \infty} x(t) \geq \frac{1}{r + 1}, \ \liminf_{t \to \infty} z(t) \geq \frac{h_1}{h_1 + rh_3}.$$  

Thus, we only need to show that (4) holds. Define the functional $V$ as follows,

$$V(\phi) = \phi_2(0) + r \int_{-\tau}^0 \phi_1(\theta) \phi_2(\theta) d\theta, \ \phi \in X.$$  

The derivative of $V$ along this solution $u_t$ is taken as

$$\dot{V}(u_t) = (rx(t) - 1 - h_2z(t))y(t) \geq (rx(t) - 1 - h_2) y(t).$$

By (5), there exists $T = T(\phi) > 0$ such that $x(t) \geq 1/(r + 1)$ for all $t \geq T$. Let $y_1 = q/h_1(1 + h_2) > 0$, where $q \in (0, r - 1 - h_2)$ and let $A = (1 + h_2)/(1 + h_2 + q)$. 

Thus, it follows that for \( x \geq 1886 \) SONGBAI GUO AND WANBIAO MA we also have

\[
\dot{x}(t) = 1 - (1 + h_1 y(t)) x(t) \geq 1 - (1 + h_1 y_1) x(t) = 1 - \frac{1}{A} x(t),
\]

which implies that

\[
x(t) \geq A + (x(t_0) - A) e^{-(t-t_0)/A} \geq A + \left[ \frac{1}{2(r+1)} - A \right] e^{-(t-t_0)/A} > A - A e^{-(t-t_0)/A} = \frac{1}{2(r+1)} - A.
\]

Thus, it follows that for \( t \geq t_0 + T_0 \),

\[
x(t) \geq A + \left[ \frac{1}{2(r+1)} - A \right] e^{-T_0/A} \equiv \rho > A - A e^{-T_0/A} = \frac{1}{2(r+1)} - A
\]  

where \( 1/2(r+1) - A < 0 \) is used. Then from (8), we obtain \( \rho \geq 1 \) and we also have \( \dot{V}(u_2) \geq (\rho p - 1 - h_2) y(t) \) for \( t \geq t_0 + T_0 \) by (7).

Let \( y_m = \min_{t \in [t_0 + T_0 + \tau]} y(t) \) and \( y_m \geq y_m \) for all \( t \geq t_0 + T_0 \). In fact, otherwise, there is a \( T_1 \geq 0 \) such that \( y(t) \geq y_m \) for \( t_0 + T_0 \leq t \leq t_0 + T_0 + \tau + T_1 \), \( y(t_0 + T_0 + \tau + T_1) = y_m \) and \( y(t_0 + T_0 + \tau + T_1) \leq 0 \), then we get from the second equation of (2) and (8) that, for \( \bar{t} = t_0 + T_0 + \tau + T_1 \),

\[
\dot{y}(\bar{t}) \geq (r x (\bar{t} - \tau) - 1 - h_2) y_m \geq (r \rho - 1 - h_2) y_m > 0.
\]

Clearly, this is a contradiction. Hence, \( y(t) \geq y_m \) for all \( t \geq t_0 + T_0 \). In consequence, we have that for all \( t \geq t_0 + T_0 \),

\[
\dot{V}(u_1) \geq (\rho p - 1 - h_2) y(t) \geq (\rho p - 1 - h_2) y_m,
\]

which implies that \( V(u_1) \to \infty \) as \( t \to \infty \). This contradicts with the boundedness of \( V(u_1) \). The claim is proved.

As a result, there remain two cases to be considered. The first case is that \( y(t) \geq y_1 \) for all large \( t \), which contributes straightforward to get \( \liminf_{t \to \infty} y(t) \geq \nu \). The other case is that \( y(t) \) oscillates about \( y_1 \) for all large \( t \). Let \( t_1, t_2 \geq 0 \) be sufficiently large such that \( y(t_1) = y(t_2) = y_1, y(t) < y_1 \) for \( t_1 < t < t_2 \). If \( t_2 - t_1 \leq T_0 + \tau \), from the second equation of (2), it follows that \( \dot{y}(t) \geq -(1 + h_2) y(t) \). Hence, for \( t_1 \leq t \leq t_2 \), we have

\[
y(t) \geq y_1 e^{-(1 + h_2)(t-t_1)} \geq y_1 e^{-(1 + h_2)(T_0 + \tau)} \geq \nu.
\]

If \( t_2 - t_1 > T_0 + \tau \), it can be easily obtained that \( y(t) \geq \nu \) for \( t \in [t_1, t_1 + T_0 + \tau] \). Then, proceeding exactly as the proof above, it shows that \( y(t) \geq \nu \) for \( t \in [t_1 + T_0 + \tau, t_2] \). In fact, if not, there exists a \( T_2 \geq 0 \) such that \( y(t) \leq \nu \) for \( t_1 \leq t \leq t_1 + T_0 + \tau + T_2 \), \( y(t_1 + T_0 + \tau + T_2) = \nu \) and \( y(t_1 + T_0 + \tau + T_2) \leq 0 \). On the other hand, it follows from the second equation of (2) and (8) that, for \( \bar{t} = t_1 + T_0 + \tau + T_2 \),

\[
\dot{y}(ar{t}) \geq (r x (\bar{t} - \tau) - 1 - h_2) \nu \geq (r \rho - 1 - h_2) \nu > 0.
\]

This is a contradiction. Thus, \( y(t) \geq \nu \) for \( t \in [t_1, t_2] \). Since this kind of interval \( [t_1, t_2] \) is chosen in an arbitrary way, \( y(t) \geq \nu \) holds for any sufficiently large \( t \). Therefore, \( \liminf_{t \to \infty} y(t) \geq \nu \). \( \Box \)
3. Global stability of equilibria. In this section, we shall consider the global stability of equilibria under some conditions. According to Figs. 1 and 2, for the global stability of the boundary equilibrium, we only need to consider two cases: (1) $h_2h_3 \leq h_1h_2 + h_1$, $R \leq 1$ and (2) $h_2h_3 > h_1 + h_1h_2$, $R < w$; for the global stability of the positive equilibrium, we only need to consider $R > 1$. First, for the boundary equilibrium, we have

**Theorem 3.1.** If $r < \left[ h_3 - h_1 + \sqrt{(h_1 + h_3)^2 + 4h_1h_2h_3} \right] / 2h_3 \equiv \bar{r}$, then $E_0$ is globally asymptotically stable for any $\tau \geq 0$ in $G$.

**Proof.** Obviously, $1 < \bar{r} < 1 + h_2$, and then $r < \bar{r}$ implies $R < 1$. If $h_2h_3 > h_1 + h_1h_2$, then $r < \bar{r}$ also implies $R < w < 1$. Consider the functional $V$ as defined in (6) on $G$. It can be seen that $V$ is continuous on $G$. Since $r < \bar{r}$ if and only if $r < 1 + h_1h_2/(h_1 + rh_3)$, there exists $r > 1$ such that $r - 1 - h_1h_2/\varepsilon (h_1 + rh_3) < 0$.

Let $u_t = (x_t, y_t, z_t)^T$ be the solution of (2) with any $\phi \in G$. By (5), there is a $T = T(\phi) > 0$ such that $z(t) \geq h_1/\varepsilon (h_1 + rh_3)$ for all $t \geq T$. The derivative of $V$ along this solution of (2) for $t \geq T$ is given as

$$
\dot{V}(u_t) = (rx(t) - 1 - h_2z(t))y(t) \leq \left[r - 1 - \frac{h_1h_2}{\varepsilon (h_1 + rh_3)}\right] y(t) \leq 0.
$$

Denote $E = \{ \phi = (\phi_1, \phi_2, \phi_3)^T \in G : \dot{V}(\phi) = 0 \}$, and let $M$ be the largest invariant set with respect to (2) in $E$. It can be found that $O_T(\phi) := \{ u_t : t \geq T \} \subset G$ is positively invariant for (2), and $V$ is a Lyapunov functional on $O_T(\phi)$. From [24, Theorem 2.1], it follows that $O_T(\phi)$ is bounded. Thus, by the Lyapunov-LaSalle theorem (see, e.g., [10, Theorem 5.3.1] or [13, Theorem 2.5.3]), the solution $u_t$ tends to $M$ as $t$ tends to $\infty$.

Now, let us show that $M = \{ E_0 \}$. Let $u_t = (x_t, y_t, z_t)^T$ be the solution of (2) with any $\psi \in M$. Note that $M$ is invariant for system (2), consequently, by contradiction, if there exists some $t_1 \in \mathbb{R}$ such that $y(t_1) > 0$, then from the second equation of (2), it follows that for $t \geq t_1$,

$$
y(t) \geq y(t_1)e^{-\int_{t_1}^{t} (1 + h_2z(s))ds} > 0.
$$

This contradicts $y(t) = 0$ for $t \geq T(\phi)$. Thus, we have $y(t) = 0$ for $t \in \mathbb{R}$. The first and the third equations of (2) together with the invariance of $M$ yield that $x(t) = z(t) = 1$ for $t \in \mathbb{R}$. Hence, $M = \{ E_0 \}$ and then $E_0$ is globally attractive. With the local stability of $E_0$ established in [24, Theorem 3.1], we thus prove the global stability of $E_0$.

**Theorem 3.2.** If $R > 1$, $4h_1 \geq rh_3$, then $E^*$ is globally asymptotically stable for any $\tau \geq 0$ in $X = \{ \phi \in G : \phi_2(0) > 0 \}$.

**Proof.** It is not difficult to show that $X$ is positively invariant for (2). Let $(x_t, y_t, z_t)^T$ be the solution of (2) with any $\phi \in X$ and define $p(t) := rx(t)(1 + y(t)) + h_2(1 - z(t))/h_3$. Then it follows that $(x_t, y_t)^T$ is a solution of the following nonautonomous system:

\[
\begin{aligned}
\dot{x}(t) &= 1 - x(t) - h_1x(t)y(t), \\
\dot{y}(t) &= rx(t)(t) y(t) - \frac{r}{h_1}x(t) y(t) - h_3y^2(t) + (h_3p(t) - 1 - h_2) y(t).
\end{aligned}
\]
By (2), it is clear to find that for $t \geq \tau$,
\[ \dot{p}(t) = \frac{r}{h_1} - p(t). \]
Hence, we obtain $\lim_{t \to \infty} p(t) = r/h_1$. This implies that (9) has the following limiting system:
\[
\begin{align*}
&\dot{x}_1(t) = 1 - x_1(t) - h_1 x_1(t) y_1(t), \\
&\dot{y}_1(t) = r x_1(t - \tau) y_1(t - \tau) - \frac{r h_3}{h_1} x_1(t - \tau) y_1(t) - h_3 y_1^2(t) + \left( \frac{r h_3}{h_1} - 1 - h_2 \right) y_1(t).
\end{align*}
\]
(10)
Accordingly, the nonautonomous solution semiflow of (9) is asymptotic to the autonomous solution semiflow of (10) on $X_1 = \{ \varphi = (\varphi_1, \varphi_2)^T \in C([-\tau, 0], \mathbb{R}^2_+) : ||\varphi|| \leq 1, \varphi_2(0) > 0 \}$ (see, e.g., [19, 25]).

Next, we define
\[ \omega = \{ \psi \in X_1 : \lim_{n \to \infty} (x_{t_n}, y_{t_n})^T = \psi \text{ for some } t_n \to \infty \}, \]
where $X_1$ is the closure of $X_1$. In virtue of Theorem 2.1, clearly, $\omega \subset X_1$ is nonempty and compact. By [10, Theorem 2.2.3], the solution $u_t = (x_{t+}, y_{t+})^T$ of (10) with any $\varphi \in X_1$ is unique on its maximal interval $[0, \varepsilon_\varphi)$ of existence. It follows from [23, Theorem 5.2.1] that the solution $u_t$ is nonnegative on $[0, \varepsilon_\varphi)$. Let $(x_1, y_1) = (x_1(t), y_1(t))$. Since $\dot{x}_1 \leq 1 - x_1$, we have that $x_1$ is bounded on $[0, \varepsilon_\varphi)$. It $t \in [0, \tau] \cap [0, \varepsilon_\varphi)$, then there exists a constant $M_\varphi$ such that
\[ \dot{y}_1 \leq r x_1(t - \tau) y_1(t - \tau) + \frac{r h_3}{h_1} y_1 \leq M_\varphi + \frac{r h_3}{h_1} y_1. \]
Hence, $y_1$ is bounded on $[0, \tau] \cap [0, \varepsilon_\varphi)$. Whence $\varepsilon_\varphi > \tau$. We define $q := r x_1(t - \tau)/h_1 + y_1$. Then it follows that for $t \in [\tau, \varepsilon_\varphi)$,
\[ \dot{\theta} = \frac{r}{h_1} - q - h_3 y_1 q - h_2 y_2 + \frac{r h_3}{h_1} y_1. \]
If $q > r/h_1$ for $t \in [\tau, \varepsilon_\varphi)$, then $\dot{\theta} \leq r/h_1 - q$, and $q$ is bounded on $[0, \varepsilon_\varphi)$; if there is some $t_0 \in [\tau, \varepsilon_\varphi)$ such that $q(t_0) \leq r/h_1$, then it follows from [23, Remark 5.2.1] that $q(t) \leq r/h_1$ for $t \in [t_0, \varepsilon_\varphi)$, and $q$ is also bounded on $[0, \varepsilon_\varphi)$. Thus, $y_1$ is bounded on $[0, \varepsilon_\varphi)$. By the continuation theorem of solutions for delay differential equations (see, e.g., [10, 13]) that $\varepsilon_\varphi = \infty$. Moreover, we can obtain that $(x_{11}(0), y_{11}(0))^T = (x_1(t), y_1(t))^T$ is strongly positive on $(0, \infty)$. Thus, $X_1$ is positively invariant for (10), and we can show that solutions of (10) are ultimately bounded and uniformly in $X_1$.

Now, let us define a functional $L$ on $X_2 := \{ \varphi \in X_1 : \varphi \gg 0 \}$ as follows,
\[ L(\varphi) = L_1(\varphi) + h_1 x^* y^* L_2(\varphi) + L_3(\varphi), \]
(11)
where
\[ L_1(\varphi) = \varphi_1(0) - x^* - x^* \ln \frac{\varphi_1(0)}{x^*} + \frac{h_1}{r} \left( \varphi_2(0) - y^* - y^* \ln \frac{\varphi_2(0)}{y^*} \right), \]
and
\[ L_2(\varphi) = \int_{-\tau}^{0} \left( \frac{\varphi_1(\theta) \varphi_2(\theta)}{x^* y^*} - 1 - \ln \frac{\varphi_1(\theta) \varphi_2(\theta)}{x^* y^*} \right) d\theta, \]
\[ L_3(\varphi) = \int_{-\tau}^{0} (\varphi_1(\theta) - x^*)^2 d\theta. \]
It can be seen that $L$ is continuous on the subset $X_2$ of $X_1$. Taking the derivative of $L_1$ along this solution of (10) for $t > \tau$, it follows that

$$\dot{L}_1(u_t) = \left(1 - \frac{x^*}{x_1}\right) (1 - x_1 - h_1 x_1 y_1) + \left(1 - \frac{y^*}{y_1}\right) l,$$

where

$$l = h_1 x_1 (t - \tau) y_1 (t - \tau) - h_3 x_1 (t - \tau) y_1 - \frac{h_1 h_3}{r} y_1^2 + \left(h_3 - \frac{h_1}{r} - \frac{h_1 h_2}{r}\right) y_1.$$ 

It is easy to obtain that for $t > \tau$,

$$\dot{L}_2(u_t) = \frac{x_1 y_1}{x^* y^*} - \frac{x_1 (t - \tau) y_1 (t - \tau)}{x^* y^*} + \ln \frac{x_1 (t - \tau) y_1 (t - \tau)}{x_1 y_1},$$

$$\dot{L}_3(u_t) = (x_1 - x^*)^2 - (x_1 (t - \tau) - x^*)^2.$$ 

By using $1 = x^* + h_1 x^* y^*$ and $h_3 - h_1/r - h_1 h_2/r = h_3 x^* - h_1 x^* + h_1 h_3 y^*/r$, we have that for $t > \tau$,

$$\dot{L}(u_t) = \dot{L}_1(u_t) + h_1 x^* y^* \dot{L}_2(u_t) + \dot{L}_3(u_t)$$

$$= -\left(\frac{x_1 - x^*}{x_1}\right)^2 - \frac{h_1 h_3}{r} (y_1 - y^*)^2$$

$$+ h_1 x^* y^* \left(2 + \ln \frac{x_1 (t - \tau) y_1 (t - \tau)}{x^* y^*} + \ln \frac{x^*}{x_1} - \frac{x_1 (t - \tau) y_1 (t - \tau)}{x^* y^*} - \frac{x^*}{x_1}\right)$$

$$+ h_3 (x^* - x_1 (t - \tau)) (y_1 - y^*)$$

$$+ (x_1 - x^*)^2 - (x_1 (t - \tau) - x^*)^2$$

$$\leq -\left(\frac{h_1 h_3}{r} - h_3\right) \left(\left[(x_1 (t - \tau) - x^*) (y_1 - y^*)\right] + h_1 x^* y^* \left(2 + \ln \frac{x_1 (t - \tau) y_1 (t - \tau)}{x^* y^*} + \ln \frac{x^*}{x_1} - \frac{x_1 (t - \tau) y_1 (t - \tau)}{x^* y^*} - \frac{x^*}{x_1}\right)\right)$$

$$\leq -\left(2 \sqrt{\frac{h_1 h_3}{r} - h_3}\right) \left[(x_1 (t - \tau) - x^*) (y_1 - y^*)\right]$$

$$+ h_1 x^* y^* \left(1 + \ln \frac{x_1 (t - \tau) y_1 (t - \tau)}{x^* y^*} - \frac{x_1 (t - \tau) y_1 (t - \tau)}{x^* y^*}\right)$$

$$+ h_1 x^* y^* \left(1 + \ln \frac{x^*}{x_1} - \frac{x^*}{x_1}\right)$$

$$\leq 0.$$ 

From the above inequality and (11), it follows that $L(u_t)$ is bounded on $[2\tau, \infty)$, and then there is some $\epsilon = \epsilon(\varphi) > 0$ such that $\lim_{t \to \infty} \inf_{x_1 t} x_{11}(\theta) > \epsilon$, $\lim_{t \to \infty} \inf_{y_1 t} y_{11}(\theta) > \epsilon$ for any $\theta \in [-\tau, 0]$. Define

$$M = \bigcup_{\varphi \in X_1} \omega(\varphi),$$

where $\omega(\varphi)$ is the omega limit set of $\varphi$. Thus, $M \subset X_2 \subset X_1$. By using a similar argument as in the proof of Theorem 3.1, we can obtain that $\dot{L} = 0$ on $M$. Therefore, the derivative of $L$ along the solution $u_t = (x_{1t}, y_{1t})$ of (10) with any $\psi \in M$ satisfies

$$\dot{L}(u_t) \leq h_1 x^* y^* \left(1 + \ln \frac{x^*}{x_1} - \frac{x^*}{x_1}\right) \leq 0.$$
By the invariance of $M$ and the first equation of (10), then we have $M = \{E_1\}$, where $E_1 = (x^*, y^*)^T$. Whence, $E_1$ is globally attractive in $X_1$. We can show that $E_1$ is locally asymptotically stable. Therefore, $E_1$ is a global attractor for the solution semiflow of (10) in $X_1$.

From the continuous-time version of [27, Lemma 1.2.2] (also see [19, Theorem 1.8]), it follows that $\omega \subset X_1$ is an internally chain transitive set for the solution semiflow of (10). It holds that $W^s(E_1) = X_1$, where $W^s(E_1)$ is the stable set of $E_1$ for the solution semiflow of (10). Consequently, $\omega \cap W^s(E_1) \neq \emptyset$. Whence, [27, Theorem 1.2.1 and Remark 1.3.2] with $A = E_1$ imply that $\omega = \{E_1\}$, and then $\lim_{t \to \infty} (x(t), y(t))^T = E_1$. Thus,

$$\lim_{t \to \infty} z(t) = \lim_{t \to \infty} \left[ 1 - \frac{h_3}{h_2} \left( p(t) - \frac{r}{h_1} x(t) - y(t) \right) \right] = \lim_{t \to \infty} \left[ 1 - \frac{h_3}{h_2} \left( \frac{r}{h_1} - \frac{r}{h_1} x^* - y^* \right) \right] = z^*.$$

Therefore, $E^*$ is globally attractive in $X$, which is together with the local stability of $E^*$ established from [24, Theorems 3.2 and 3.3], ensures the global stability of $E^*$.

4. Conclusions. Tai, Ma and Guo et al. [24] proposed (1) (i.e., (2)) on the basis of some practical problems about microorganism continuous culture and flocculation, and they gave a complete analysis on the local stability of the equilibria of (2). In this paper, we consider the global dynamics of (2) based on the construction of suitable positively invariant sets and the application of Lyapunov-LaSalle theorem. Some detailed analyses on the global stability of equilibria of (2) under certain conditions are carried out. It is shown that, when $r < \bar{r}$ (it implies two cases: (i) $R < 1$; (ii) $h_2h_3 > h_1 + h_1h_2$, $R < w$), the boundary equilibrium $E_0$ is globally asymptotically stable for any $\tau \geq 0$. Furthermore, it is also shown that, if $R > 1$ and $4h_1 \geq rh_3$ hold, the positive equilibrium $E^*$ is globally asymptotically stable for any $\tau \geq 0$. The two results for global stability show that, for (2), the time delay has no effect on both global asymptotic properties of the boundary equilibrium $E_0$ and the positive equilibrium $E^*$. For permanence of (2), it is shown that, when $R > 1$, (2) is permanent, which also ensures the global stability of $E^*$. In addition, the explicit lower bound $\nu$ of $\lim_{t \to \infty} y(t)$ is given in Theorem 2.1 by using some analysis techniques.

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