Semi-Fredholmness of Weighted Singular Integral Operators with Shifts and Slowly Oscillating Data

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Abstract. Let \( \alpha, \beta \) be orientation-preserving homeomorphisms of \([0, \infty]\) onto itself, which have only two fixed points at 0 and \( \infty \), and whose restrictions to \( \mathbb{R}_+ = (0, \infty) \) are diffeomorphisms, and let \( U_\alpha, U_\beta \) be the corresponding isometric shift operators on the space \( L^p(\mathbb{R}_+) \) given by \( U_\mu f = (\mu')^{1/p} (f \circ \mu) \) for \( \mu \in \{ \alpha, \beta \} \). We prove sufficient conditions for the right and left Fredholmness on \( L^p(\mathbb{R}_+) \) of singular integral operators of the form \( A_+ P^+_\gamma + A_- P^-_\gamma \), where \( P^\pm_\gamma = (I \pm S_\gamma)/2 \), \( S_\gamma \) is a weighted Cauchy singular integral operator, \( A_+ = \sum_{k \in \mathbb{Z}} a_k U^k_\alpha \) and \( A_- = \sum_{k \in \mathbb{Z}} b_k U^k_\beta \) are operators in the Wiener algebras of functional operators with shifts. We assume that the coefficients \( a_k, b_k \) for \( k \in \mathbb{Z} \) and the derivatives of the shifts \( \alpha', \beta' \) are bounded continuous functions on \( \mathbb{R}_+ \) which may have slowly oscillating discontinuities at 0 and \( \infty \).

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1. Introduction

Let \( B(X) \) denote the Banach algebra of all bounded linear operators acting on a Banach space \( X \). Recall that an operator \( A \in B(X) \) is said to be left invertible (resp. right invertible) if there exists an operator \( B \in B(X) \) such that \( BA = I \) (resp. \( AB = I \)) where \( I \in B(X) \) is the identity operator on \( X \). The operator \( B \) is called a left (resp. right) inverse of \( A \). An operator

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A ∈ ℬ(X) is said to be invertible if it is left invertible and right invertible simultaneously. We say that A is strictly left (resp. right) invertible if it is left (resp. right) invertible, but not invertible. If the operator A is invertible only from one side, then the corresponding inverse is not uniquely defined. We refer to [10] Section 2.5] for further properties of one-sided invertible operators acting on Banach spaces.

Let ℋ(X) be the closed two-sided ideal of all compact operators in ℬ(X), and let \( ℬ^\pi(X) := ℬ(X)/ℋ(X) \) be the Calkin algebra of the cosets \( A^\pi := A + ℋ(X) \) where \( A ∈ ℬ(X) \). Following [4] Chap. XI, Definition 2.3], an operator \( A ∈ ℬ(X) \) is said to be left Fredholm (resp., right Fredholm) if the coset \( A^\pi \) is left invertible (resp., right invertible) in the Calkin algebra \( ℬ^\pi(X) \). An operator \( A ∈ ℬ(X) \) is said to be semi-Fredholm if it is left or right Fredholm. We will write \( A \simeq B \) if \( A − B ∈ ℋ(X) \).

Let \( C_b(\mathbb{R}_+) \) denote the \( C^\ast \)-algebra of all bounded continuous functions on the positive half-line \( \mathbb{R}_+ := (0, +\infty) \). Following Sarason [30] p. 820, a function \( f ∈ C_b(\mathbb{R}_+) \) is called slowly oscillating (at 0 and \( \infty \)) if

\[
\lim_{r \to s} \sup_{t, \tau \in [r, 2r]} |f(t) − f(\tau)| = 0 \quad \text{for} \quad s ∈ \{0, \infty\}.
\]

The set \( SO(\mathbb{R}_+) \) of all slowly oscillating functions is a \( C^\ast \)-algebra. This algebra properly contains \( C(\mathbb{R}_+) \), the \( C^\ast \)-algebra of all continuous functions on the two-point compactification \( \mathbb{R}_+ := [0, +\infty] \) of \( \mathbb{R}_+ \).

Suppose \( \alpha \) is an orientation-preserving homeomorphism of \( \mathbb{R}_+ \) onto itself, which has only two fixed points 0 and \( \infty \), and whose restriction to \( \mathbb{R}_+ \) is a diffeomorphism. We say that \( \alpha \) is a slowly oscillating shift if \( \log \alpha' \) is bounded and \( \alpha' ∈ SO(\mathbb{R}_+) \). The set of all slowly oscillating shifts is denoted by \( SOS(\mathbb{R}_+) \). By [15] Lemma 2.2, an orientation-preserving diffeomorphism \( \alpha \) of \( \mathbb{R}_+ \) onto itself belongs to \( SO(\mathbb{R}_+) \) if and only if \( \alpha(t) = te^{\omega(t)} \) for \( t ∈ \mathbb{R}_+ \) and a real-valued function \( \omega ∈ SO(\mathbb{R}_+) ∩ C^1(\mathbb{R}_+) \) is such that the function \( \psi \) given by \( \psi(t) := t\omega'(t) \) also belongs to \( SO(\mathbb{R}_+) \) and \( \inf_{t ∈ \mathbb{R}_+} (1 + t\omega'(t)) > 0 \). The real-valued slowly oscillating function

\[
\omega(t) := \log[\alpha(t)/t], \quad t ∈ \mathbb{R}_+,
\]

is called the exponent function of \( \alpha ∈ SOS(\mathbb{R}_+) \).

Through the paper, we will suppose that \( 1 < p < \infty \) and will use the following notation:

\[
\mathcal{B} := ℬ(L^p(\mathbb{R}_+)), \quad \mathcal{K} := ℋ(L^p(\mathbb{R}_+)).
\]

It is easily seen that if \( \alpha ∈ SOS(\mathbb{R}_+) \), then the weighted shift operator defined by

\[
U_\alpha f := (\alpha')^{1/p}(f ∘ \alpha)
\]

is an isometric isomorphism of the Lebesgue space \( L^p(\mathbb{R}_+) \) onto itself. It is clear that \( U_{\alpha}^{-1} = U_{\alpha^{-1}} \), where \( \alpha^{-1} \) is the inverse function to \( \alpha \). For \( k ∈ \mathbb{N} \), we denote by \( U_{\alpha}^{-k} \) the operator \( (U_{\alpha}^{-1})^k \). Let \( W_{\alpha,p}^{SO} \) denote the collection of all
operators of the form
\[ A = \sum_{k \in \mathbb{Z}} a_k U_k^\alpha \]  
where \( a_k \in SO(\mathbb{R}_+) \) for all \( k \in \mathbb{Z} \) and
\[ \|A\|_{W^{SO}_{\alpha,p}} := \sum_{k \in \mathbb{Z}} \|a_k\|_{C_b(\mathbb{R}_+)} < +\infty. \]  
(1.2)

The set \( W^{SO}_{\alpha,p} \) is, actually, a Banach algebra with respect to the usual operations and the norm (1.2). By analogy with the Wiener algebra of absolutely convergent Fourier series, we will call \( W^{SO}_{\alpha,p} \) the Wiener algebra.

Let \( \Re \gamma \) and \( \Im \gamma \) denote the real and imaginary part of \( \gamma \in \mathbb{C} \), respectively. If \( \gamma \in \mathbb{C} \) satisfies
\[ 0 < 1/p + \Re \gamma < 1, \]  
(1.3)

then the operator
\[ (S_\gamma f)(t) := \frac{1}{\pi i} \int_{\mathbb{R}_+} \left( \frac{t}{\tau} \right)^\gamma \frac{f(\tau)}{\tau - t} d\tau, \]  
(1.4)

where the integral is understood in the principal value sense, is bounded on the Lebesgue space \( L^p(\mathbb{R}_+) \) (see, e.g., [29, Proposition 4.2.11]). Put
\[ P^\pm_\gamma := \frac{1}{2} \left( I \pm S_\gamma \right). \]

This paper is a continuation of our recent works [8, 9, 19, 20] (see also references therein). Let \( \alpha, \beta \) belong to \( SO(\mathbb{R}_+) \) and \( a_k, b_k \in SO(\mathbb{R}_+) \) for all \( k \in \mathbb{Z} \). In [19, 20] we found criteria for the Fredholmness and a formula permitting to calculate the index of the weighted singular integral operator of the form
\[ M := (a_0 I + a_1 U_\alpha)P^+_\gamma + (b_0 I + b_1 U_\beta)P^-_\gamma. \]

In this paper we assume that
\[ A_+ := \sum_{k \in \mathbb{Z}} a_k U_k^\alpha \in W^{SO}_{\alpha,p}, \quad A_- := \sum_{k \in \mathbb{Z}} b_k U_k^\beta \in W^{SO}_{\beta,p} \]  
(1.5)

and consider the weighted singular integral operator of the form
\[ N := A_+ P^+_\gamma + A_- P^-_\gamma. \]  
(1.6)

Criteria for the Fredholmness of the operator \( N \) in the particular case of \( \alpha = \beta \) and \( \gamma = 0 \) were obtained in [9]. The proof of the sufficiency portion is based on the Allan-Douglas local principle and follows ideas of [15]. In this paper we will show that the localization technique is flexible enough to treat also the case of the left and right Fredholmness for arbitrary shifts \( \alpha, \beta \) and arbitrary \( \gamma \) satisfying (1.3), provided that there are one-sided inverses of \( A_+ \) and \( A_- \) belonging to the Wiener algebras \( W^{SO}_{\alpha,p} \) and \( W^{SO}_{\beta,p} \), respectively. We show that the required result on one-sided inverses can be obtained from [8].

By \( M(\mathfrak{M}) \) we denote the maximal ideal space of a unital commutative Banach algebra \( \mathfrak{M} \). Identifying the points \( t \in \mathbb{R}_+ \) with the evaluation functionals \( t(f) = f(t) \) for \( f \in C(\mathbb{R}_+) \), we get \( M(C(\mathbb{R}_+)) = \mathbb{R}_+ \). Consider the
fibers
\[ M_s(SO(\mathbb{R}^+)) := \{ \xi \in M(SO(\mathbb{R}^+)) : \xi|_{C(\mathbb{R}^+)} = s \} \]
of the maximal ideal space \( M(SO(\mathbb{R}^+)) \) over the points \( s \in \{0, \infty\} \). By \cite[Proposition 2.1]{22}, the set
\[ \Delta := M_0(SO(\mathbb{R}^+)) \cup M_\infty(SO(\mathbb{R}^+)) \]
cointides with \( \text{clos}_{SO(\mathbb{R}^+) \setminus \mathbb{R}^+} \), where \( \text{clos}_{SO(\mathbb{R}^+)} \) is the weak-star closure of \( \mathbb{R}^+ \) in the dual space of \( SO(\mathbb{R}^+) \). Then \( M(SO(\mathbb{R}^+)) = \Delta \cup \mathbb{R}^+ \). In what follows we write \( a(\xi) := \xi(a) \) for every \( a \in SO(\mathbb{R}^+) \) and every \( \xi \in \Delta \).

With the operators \( A_\pm \) defined by (1.5), we associate the functions \( a_\pm \) defined on \( \mathbb{R}^+ \times \mathbb{R} \) by
\[
a_+(t, x) := \sum_{k \in \mathbb{Z}} a_k(t) e^{i k \omega(t)x}, \quad a_-(t, x) := \sum_{k \in \mathbb{Z}} b_k(t) e^{i k \eta(t)x}, \quad (1.7)
\]
where \( \omega, \eta \in SO(\mathbb{R}^+) \) are the exponent functions of \( \alpha, \beta \), respectively. Since the series in (1.5) converge absolutely, we have \( a_\pm(\cdot, x) \in SO(\mathbb{R}^+) \) for all \( x \in \mathbb{R} \). With the operator \( N \) we associate the function \( n \) defined on \( \mathbb{R}^+ \times \mathbb{R} \) by
\[ n(t, x) = a_+(t, x)p_\gamma^+(x) + a_-(t, x)p_\gamma^-(x), \]
where
\[ p_\gamma^+(x) := (1 + s_\gamma(x))/2, \quad s_\gamma(x) := \coth[\pi(x + i/p + i\gamma)], \quad x \in \mathbb{R}. \quad (1.8)\]
Since \( a_\pm(\cdot, x), n(\cdot, x) \in SO(\mathbb{R}^+) \) for every \( x \in \mathbb{R} \), taking the Gelfand transform of \( n(\cdot, x) \), we obtain for \( (\xi, x) \in (\Delta \cup \mathbb{R}^+) \times \mathbb{R} \),
\[ n(\xi, x) := a_+(\xi, x)p_\gamma^+(x) + a_-(\xi, x)p_\gamma^-(x), \quad (1.9)\]
which gives extensions of the functions \( n(\cdot, x) \) to \( M(SO(\mathbb{R}^+)) \).

**Theorem 1.1 (Main result).** Let \( 1 < p < \infty \) and let \( \gamma \in \mathbb{C} \) satisfy (1.3). Suppose \( a_k, b_k \in SO(\mathbb{R}^+) \) for all \( k \in \mathbb{Z} \) and \( \alpha, \beta \in SO(\mathbb{R}^+) \). If
(i) the functional operators
\[ A_+ := \sum_{k \in \mathbb{Z}} a_k U^k_\alpha \in W^{SO}_{\alpha, p}, \quad A_- := \sum_{k \in \mathbb{Z}} b_k U^k_\beta \in W^{SO}_{\beta, p} \]
are left (resp., right) invertible on the space \( L^p(\mathbb{R}^+) \);
(ii) for every \( \xi \in \Delta \), the function \( n \) defined by (1.7)–(1.9) satisfies the inequality
\[ \inf_{x \in \mathbb{R}} |n(\xi, x)| > 0; \quad (1.10)\]
then the operator \( N = A_+ P_\gamma^+ + A_- P_\gamma^- \) is left (resp., right) Fredholm on the space \( L^p(\mathbb{R}^+) \).

We conjecture that conditions (i) and (ii) of Theorem 1.1 are also necessary for the one-sided Fredholmness of the operator \( N \).

The paper is organized as follows. Section 2 contains some auxiliary results. In Section 3 on the basis of recent results from \cite{8}, we show that if an operator \( A \in W^{SO}_{\alpha, p} \) is left (resp., right) invertible, then at least one of its left (resp., right) inverses belongs to the same algebra \( W^{SO}_{\alpha, p} \).
Section 4 is devoted to the algebra $\mathcal{A}$ generated by the identity operator $I$ and the operator $S_0$. We recall that $\mathcal{A}$ is the smallest Banach subalgebra of $\mathcal{B}$ that contains all operators similar to Mellin convolution operators with continuous symbols. In particular, the algebra $\mathcal{A}$ contains the operator $S_\gamma$ and the operator $R_\gamma$ with fixed singularities defined by

$$ \langle R_\gamma f \rangle (t) := \frac{1}{\pi i} \int_{\mathbb{R}^+} \left( \frac{t}{\tau} \right)^\gamma \frac{f(\tau)}{\tau + t} d\tau, \tag{1.11} $$

where the integral is understood in the principal value sense. We also recall the description of the maximal ideal space of the algebra $\mathcal{A}^\pi := (\mathcal{A} + \mathcal{K})/\mathcal{K}$.

In Section 5 we recall a version of the Allan-Douglas local principle suitable for the study of one-sided invertibility in subalgebras of the Calkin algebra $B^\pi = B/\mathcal{K}$ (see [3, Theorem 1.35(a)]). Following [15, Section 6], we consider the algebra $\mathcal{Z}$ generated by $\mathcal{A} \cup \mathcal{K}$ and the operators of the form $cI_0$, where $c \in SO(\mathbb{R}_+)$. We recall that the maximal ideal space of $\mathcal{Z}^\pi := \mathcal{Z}/\mathcal{K}$ is homeomorphic to the set $(-\infty, +\infty) \cup (\Delta \times \mathbb{R})$. Further, we consider the algebra $\Lambda$ of operators of local type that consists of all operators $A \in B$ such that $AC - CA \in \mathcal{K}$ for all $C \in \mathcal{Z}$. Since $\mathcal{Z}^\pi$ is a commutative central subalgebra of $\Lambda^\pi := \Lambda/\mathcal{K}$, we can apply the Allan-Douglas local principle to $\Lambda^\pi \subset B^\pi$ and its central subalgebra $\mathcal{Z}^\pi$. In particular, an operator $T \in \Lambda$ is left (resp., right) Fredholm if certain cosets $T^\pi + J_{\xi,x}^\pi$, $T^\pi + J_{\pi,\infty}^\pi$, and $T^\pi + J_{-\infty}^\pi$ are left (resp., right) invertible in the corresponding local algebras $\Lambda_{\xi,x}^\pi$, $\Lambda_{+\infty}^\pi$, and $\Lambda_{-\infty}^\pi$. Here $\langle \xi, x \rangle$ runs through $\Delta \times \mathbb{R}$. This result is applicable to the operator $N$ because it belongs to the algebra $F_{\alpha, \beta}$ generated by the operators $S_0, U_{\alpha}^{\pm 1}, U_{\beta}^{\pm 1}$ and the multiplication operators $cI$ with $c \in SO(\mathbb{R}_+)$. In turn, this algebra is contained in the algebra $\Lambda$ of operators of local type.

In Section 6 we recall the definition of the algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ of slowly oscillating functions on $\mathbb{R}_+$ with values in the algebra $V(\mathbb{R})$ of all absolutely continuous functions of finite total variation. This algebra is important for our purposes because Mellin pseudodifferential operators with symbols in $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ commute modulo compact operators. Moreover, if $\alpha \in SO(\mathbb{R}_+)$, then $U_\alpha R_\gamma$ is similar to a Mellin pseudodifferential operator with symbol in $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ up to a compact operator. These results are important ingredients of the proof of two-sided invertibility of the cosets $N^\pi + J_{\xi,x}^\pi$ in the quotient algebras $\Lambda_{\xi,x}^\pi$ for $(\xi, x) \in \Delta \times \mathbb{R}$ under condition (ii) of Theorem 1.1.

In Section 7 we prove Theorem 1.1. Since, according to Section 6 there are left/right inverses of $A_+$ (resp., $A_-$) belonging to $W_{\alpha, p}^{SO} \subset \Lambda$ (resp., $W_{\alpha, p}^{SO} \subset \Lambda$), the left/right invertibility of $A_\pm$ implies the left/right invertibility of the coset $A_+^\pi + J_{\pi,\infty} = N^\pi + J_{\pi,\infty}^\pi$ in the local algebra $\Lambda_{\pm, \infty}^\pi$. Finally, with the aid of the results of Section 6 we show that condition (ii) of Theorem 1.1 is sufficient for the two-sided invertibility of the cosets $N^\pi + J_{\xi,x}^\pi$ in the local algebras $\Lambda_{\xi,x}^\pi$ for all $(\xi, x) \in \Delta \times \mathbb{R}$. To complete the proof of Theorem 1.1 it remains to apply the Allan-Douglas local principle (see Section 5).
Finally, in Section 8 we formulate criteria for the two-sided and one-sided invertibility of a binomial functional operator with shift in the form $A = aI - bU_\alpha$, which were obtained in [18]. These results together with Theorem 1.1 imply more effective sufficient conditions for the left, right, and two-sided Fredholmness of the operator $(aI - bU_\alpha)P_+ + (cI - dU_\beta)P_-$ with $a, b, c, d \in SO(\mathbb{R}_+)$ and $\alpha, \beta \in SOS(\mathbb{R}_+)$. 

2. Auxiliary results

2.1. One-sided invertibility of operators on Hilbert spaces

Lemma 2.1. Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$.

(a) The operator $A$ is left invertible on the space $\mathcal{H}$ if and only if the operator $A^*A$ is invertible on the space $\mathcal{H}$. In this case, one of the left inverses of $A$ is given by $A^L = (A^*A)^{-1}A^*$.

(b) The operator $A$ is right invertible on the space $\mathcal{H}$ if and only if the operator $AA^*$ is invertible on the space $\mathcal{H}$. In this case, one of the right inverses of $A$ is given by $A^R = A^*(AA^*)^{-1}$.

This statement is known, although we are not able to provide a precise reference. The proof of the sufficiency portion of part (a) is a trivial computation. Now assume that $\langle \cdot, \cdot \rangle$ is the inner product of $\mathcal{H}$ and $A^L \in \mathcal{B}(\mathcal{H})$ is a left inverse of $A$. Then for every $f \in \mathcal{H}$,

$$\|f\|^2 \leq \|A^L\|^2 \|Af\|^2 = \|A^L\|^2 |\langle Af, Af \rangle| = \|A^L\|^2 |\langle A^*Af, f \rangle|.$$ 

In view of the previous inequality, the invertibility of the operator $A^*A$ follows from the Lax-Milgram theorem (see, e.g., [33, Chap. III, Section 7]). This completes the proof of part (a). The proof of part (b) is reduced to the previous one by passing to adjoint operators.

Another proof of the above lemma can be obtained from general results for $C^*$-algebras contained in [24, § 23, Corollaries 2–3].

Lemma 2.1 can also be deduced from more general results on the Moore-Penrose invertibility of operators on a Hilbert space (see [13, Example 2.16] or [2, Theorem 4.24]). Notice that the operator $A^L$ (resp., $A^R$) is the Moore-Penrose inverse of the operator $A$.

2.2. Fundamental property of slowly oscillating functions

Lemma 2.2 ([22, Proposition 2.2]). Let $\{a_k\}_{k=1}^\infty$ be a countable subset of $SO(\mathbb{R}_+)$ and $s \in \{0, \infty\}$. For each $\xi \in M_s(SO(\mathbb{R}_+))$ there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \to s$ as $n \to \infty$ and

$$a_k(\xi) = \lim_{n \to \infty} a_k(t_n) \quad \text{for all} \quad k \in \mathbb{N}. \quad (2.1)$$

Conversely, if $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ is a sequence such that $t_n \to s$ as $n \to \infty$ and the limits $\lim_{n \to \infty} a_k(t_n)$ exist for all $k \in \mathbb{N}$, then there exists a functional $\xi \in M_s(SO(\mathbb{R}_+))$ such that (2.1) holds.
2.3. Properties of iterations of slowly oscillating shifts

In this subsection we collect some properties of iterations of slowly oscillating shifts. For \( t \in \mathbb{R}_+ \) and \( k \in \mathbb{N} \), let

\[
\alpha_0(t) := t, \quad \alpha_k(t) := \alpha[\alpha_{k-1}(t)], \quad \alpha_{-k}(t) := \alpha_{-1}[\alpha_{-k+1}(t)].
\]

**Lemma 2.3** ([16] Corollary 2.5). If \( \alpha \in \text{SOS}(\mathbb{R}_+) \), then \( \alpha_k \in \text{SOS}(\mathbb{R}_+) \) for every \( k \in \mathbb{Z} \).

**Lemma 2.4** ([15] Lemma 2.3). If \( c \in \text{SO}(\mathbb{R}_+) \) and \( \alpha \in \text{SOS}(\mathbb{R}_+) \), then \( c \circ \alpha \) belongs to \( \text{SO}(\mathbb{R}_+) \) and

\[
\lim_{t \to s}(c(t) - c[\alpha(t)]) = 0 \quad \text{for} \quad s \in \{0, \infty\}.
\]

**Lemma 2.5** ([17] Lemma 2.6). Let \( \alpha \in \text{SOS}(\mathbb{R}_+) \) and \( \alpha_{-1} \) be the inverse function to \( \alpha \). If \( \omega \) and \( \omega_{-1} \) are the exponent functions of \( \alpha \) and \( \alpha_{-1} \), respectively, then \( \omega(\xi) = -\omega_{-1}(\xi) \) for all \( \xi \in \Delta \).

**Lemma 2.6.** Let \( \alpha \in \text{SOS}(\mathbb{R}_+) \) and let \( \omega \) be its exponent function. If \( k \in \mathbb{Z} \) and \( \omega_k \) is the exponent function of \( \alpha_k \), then \( \omega_k(\xi) = k\omega(\xi) \) for every \( \xi \in \Delta \).

**Proof.** For \( k = 0, 1 \), the statement is trivial. If \( k > 1 \), then

\[
\omega_k(t) = \log \frac{\alpha_k(t)}{t} = \log \left( \prod_{j=0}^{k-1} \frac{\alpha[\alpha_j(t)]}{\alpha_j(t)} \right) = \sum_{j=0}^{k-1} \omega[\alpha_j(t)], \quad t \in \mathbb{R}_+.
\]

Since \( \omega \in \text{SO}(\mathbb{R}_+) \), we deduce from Lemmas 2.3–2.4 that for every integer \( j \in \{0, \ldots, k-1\} \), the function \( \omega \circ \alpha_j \) belongs to \( \text{SO}(\mathbb{R}_+) \) and

\[
\lim_{t \to s}(\omega(t) - \omega[\alpha_j(t)]) = 0, \quad s \in \{0, \infty\}.
\]

Fix \( s \in \{0, \infty\} \) and \( \xi \in M_s(\text{SO}(\mathbb{R}_+)) \). By Lemma 2.2, there is a sequence \( \{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+ \) such that \( t_n \to s \) as \( n \to \infty \) and

\[
\omega(\xi) = \lim_{n \to \infty} \omega(t_n), \quad (\omega \circ \alpha_j)(\xi) = \lim_{n \to \infty} \omega[\alpha_j(t_n)].
\]

Equalities 2.3–2.4 imply that for \( j \in \{0, \ldots, k-1\} \),

\[
(\omega \circ \alpha_j)(\xi) - \omega(\xi) = \lim_{n \to \infty} (\omega[\alpha_j(t_n)] - \omega(t_n)) = 0.
\]

We derive from 2.2 and the above equalities that

\[
\omega_k(\xi) = \sum_{j=0}^{k-1} (\omega \circ \alpha_j)(\xi) = k\omega(\xi),
\]

which completes the proof for \( k > 1 \).

If \( k < 0 \), then we have \( \omega_{-k}(\xi) = -k\omega(\xi) \) by the statement just proved. On the other hand, we deduce from Lemma 2.3 that \( \omega_k(\xi) = -\omega_{-k}(\xi) \). Thus, \( \omega_k(\xi) = (-k)\omega(\xi) = k\omega(\xi) \) for all \( \xi \in \Delta \). \( \square \)
3. Weak one-sided inverse closedness of the algebra $W_{\alpha,p}^{SO}$

3.1. Inverse closedness of the algebra $W_{\alpha,p}^{SO}$ in the algebra $\mathcal{B}$

Let $\mathfrak{A} \subset \mathcal{B}$ be two Banach algebras with the same unit element. Recall that the algebra $\mathfrak{A}$ is said to be inverse closed in the algebra $\mathcal{B}$ if for every element $a \in \mathfrak{A}$ invertible in the algebra $\mathcal{B}$ its inverse $a^{-1}$ belongs to the algebra $\mathfrak{A}$.

We say that the algebra $\mathfrak{A}$ is weakly left (resp., right) inverse closed in the algebra $\mathcal{B}$ if for every element $a \in \mathfrak{A}$, which is left (resp., right) invertible in the algebra $\mathcal{B}$, there exists at least one its left (resp., right) inverse $a^{(-1)}$ that belongs to the algebra $\mathfrak{A}$.

Theorem 3.1 ([8, Theorem 7.4]). For every $p \in (1, \infty)$, the algebra $W_{p,\alpha}^{SO}$ is inverse closed in the algebra $\mathcal{B}$.

3.2. One-sided inverses belonging to the algebra $W_{\alpha,p}^{SO}$

A function $f \in L^{\infty}(\mathbb{R}^+)$ is said to be essentially slowly oscillating (at 0 and $\infty$) if for each (equivalently, for some) $\lambda \in (0,1)$,

$$\lim_{r \rightarrow s} \text{ess sup}_{t, \tau} |f(t) - f(\tau)| = 0, \quad s \in \{0, \infty\}.$$  

Fix $\alpha \in SOS(\mathbb{R}^+)$ and $\tau \in \mathbb{R}^+$. Consider the semi-segment $\gamma \subset \mathbb{R}^+$ with the endpoints $\tau$ and $\alpha(\tau)$ such that $\tau \in \gamma$ and $\alpha(\tau) \notin \gamma$. Following [8, Section 3.2], let $\mathfrak{G}$ denote the $C^*$-subalgebra of $L^{\infty}(\mathbb{R}^+)$ consisting of all functions on $\mathbb{R}^+$ that are continuous on every semi-segment $\alpha_n(\gamma)$ with $n \in \mathbb{Z}$, have one-sided limits at the points $\alpha_n(\tau)$ for $n \in \mathbb{Z}$, and are essentially slowly oscillating at 0 and $\infty$. Let $W_{\alpha,p}^{SO}$ be the unital Banach algebra of operators of the form (1.1) with $a_k \in \mathfrak{G}$ for all $k \in \mathbb{Z}$ and the norm

$$\|A\|_{W_{\alpha,p}^{SO}} = \sum_{k \in \mathbb{Z}} \|a_k\|_{L^{\infty}(\mathbb{R}^+)} < \infty.$$  

From [8, Theorems 6.3–6.4] we get the following.

Theorem 3.2. Let $1 < p < \infty$, $\alpha \in SOS(\mathbb{R}^+)$, $a_k \in SO(\mathbb{R}^+)$ for all $k \in \mathbb{Z}$, and

$$A = \sum_{k \in \mathbb{Z}} a_k U_{\alpha}^k \in W_{\alpha,p}^{SO}.$$  

If $A$ is left (resp. right) invertible on $L^p(\mathbb{R}^+)$, then there exists a left inverse $A^L$ (resp. right inverse $A^R$) of $A$ such that $A^L \in W_{\alpha,p}^{SO}$ (resp. $A^R \in W_{\alpha,p}^{SO}$).

3.3. Weak one-sided inverse closedness of the algebra $W_{\alpha,p}^{SO}$ in $\mathcal{B}$

We will show that the algebra $W_{\alpha,p}^{SO}$ is weakly left and right inverse closed in the algebra $\mathcal{B}$. For every operator $A \in W_{\alpha,p}^{SO}$ of the form (1.1), define its formally adjoint $A^\circ$ by

$$A^\circ := \sum_{k \in \mathbb{Z}} (\overline{a_k} \circ \alpha_{-k}) U_{\alpha}^{-k} \in W_{\alpha,p}^{SO}.$$  

Theorem 3.3. Let $1 < p < \infty$, $\alpha \in SOS(\mathbb{R}_+)$, $a_k \in SO(\mathbb{R}_+)$ for all $k \in \mathbb{Z}$, and

$$A = \sum_{k \in \mathbb{Z}} a_k U^k_\alpha \in W_{\alpha,p}^{SO}.$$ 

(a) If $A$ is left invertible on $L^p(\mathbb{R}_+)$, then the operator $A^\circ A$ is invertible on the space $L^p(\mathbb{R}_+)$, the operator $A_L := (A^\circ A)^{-1} A^\circ$ is a left inverse of $A$, and $A_L \in W_{\alpha,p}^{SO}$.

(b) If $A$ is right invertible on $L^p(\mathbb{R}_+)$, then the operator $AA^\circ$ is invertible on the space $L^p(\mathbb{R}_+)$, the operator $A_R := A^\circ (AA^\circ)^{-1}$ is a right inverse of $A$, and $A_R \in W_{\alpha,p}^{SO}$.

Proof. Along with the operator $U_\alpha$ acting on $L^p(\mathbb{R}_+)$, consider the operator $U_{\alpha,2}$ acting on $L^2(\mathbb{R}_+)$ and defined by the same rule $U_{\alpha,2} f = (a')^{1/2} (f \circ \alpha)$. Then we can define the canonical isometric isomorphisms of Banach algebras

$$\Psi_\mathcal{B}: W_{\alpha,p}^{\mathcal{B}} \to W_{\alpha,2}^{\mathcal{B}}, \quad \Psi_{SO}: W_{\alpha,p}^{SO} \to W_{\alpha,2}^{SO}$$

by the formulas

$$\Psi_\mathcal{B} \left( \sum_{k \in \mathbb{Z}} c_k U^k_\alpha \right) = \sum_{k \in \mathbb{Z}} c_k U^k_{\alpha,2}, \quad \Psi_{SO} \left( \sum_{k \in \mathbb{Z}} c_k U^k_\alpha \right) = \sum_{k \in \mathbb{Z}} c_k U^k_{\alpha,2},$$

respectively.

If $A \in W_{\alpha,p}^{SO}$ is left invertible on the space $L^p(\mathbb{R}_+)$, then by Theorem 3.2, there exists an operator $\tilde{A}_L \in W_{\alpha,p}^{\mathcal{B}}$ such that $\tilde{A}_L A = I$ on $L^p(\mathbb{R}_+)$. Hence $\Psi_\mathcal{B}(\tilde{A}_L) \Psi_\mathcal{B}(A) = I$ on $L^2(\mathbb{R}_+)$. Therefore, the operator

$$A_2 := \Psi_{SO}(A) = \Psi_\mathcal{B}(A) \in W_{\alpha,2}^{SO}$$

is left invertible on $L^2(\mathbb{R}_+)$. Hence, in view of Lemma 2.1, the operator $A_2^* A_2$ is invertible on $L^2(\mathbb{R}_+)$. Observe that $A_2^* \in W_{\alpha,2}^{SO}$ and $A^\circ = \Psi_{SO}^{-1}(A_2^*) \in W_{\alpha,p}^{SO}$. Since $A_2^* A_2 \in W_{\alpha,2}^{SO}$, we deduce from the inverse closedness of the algebra $W_{\alpha,2}^{SO}$ in the algebra $\mathcal{B}$ (see Theorem 3.1) that $(A_2^* A_2)^{-1} \in W_{\alpha,2}^{SO}$. Then $\Psi_{SO}((A_2^* A_2)^{-1}) \in W_{\alpha,p}^{SO}$. Now it is easy to check that

$$(A^\circ A)^{-1} = \Psi_{SO}^{-1}((A_2^* A_2)^{-1}).$$

Hence $A_L = (A^\circ A)^{-1} A$ is a left inverse to $A$. Part (a) is proved.

(b) The proof of part (b) is reduced to the previous one by passing to adjoint operators. \qed

4. Algebra $\mathcal{A}$ of singular integral operators

4.1. Fourier and Mellin convolution operators

Let $\mathcal{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform,

$$(\mathcal{F} f)(x) := \int_\mathbb{R} f(y) e^{-ixy} dy, \quad x \in \mathbb{R},$$

and let $\mathcal{F}^{-1}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the inverse of $\mathcal{F}$. A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $L^p(\mathbb{R})$ if the mapping $f \mapsto \mathcal{F}^{-1} a \mathcal{F} f$ maps
$L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ into itself and extends to a bounded operator on $L^p(\mathbb{R})$. The latter operator is then denoted by $W^0(a)$. We let $\mathcal{M}_p(\mathbb{R})$ stand for the set of all Fourier multipliers on $L^p(\mathbb{R})$. One can show that $\mathcal{M}_p(\mathbb{R})$ is a Banach algebra under the norm

$$\|a\|_{\mathcal{M}_p(\mathbb{R})} := \|W^0(a)\|_{\mathcal{B}(L^p(\mathbb{R}))}.$$ 

Let $d\mu(t) = dt/t$ be the (normalized) invariant measure on $\mathbb{R}_+$. Consider the Fourier transform on $L^2(\mathbb{R}_+, d\mu)$, which is usually referred to as the Mellin transform and is defined by

$$M : L^2(\mathbb{R}_+, d\mu) \to L^2(\mathbb{R}), \quad (Mf)(x) := \int_{\mathbb{R}_+} f(t)t^{-ix} \frac{dt}{t}.$$ 

This operator is invertible, with inverse given by

$$M^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+, d\mu), \quad (M^{-1}g)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} g(x)t^{ix} \, dx.$$ 

Let $E$ be the isometric isomorphism

$$E : L^p(\mathbb{R}_+, d\mu) \to L^p(\mathbb{R}), \quad (Ef)(x) := f(e^x), \quad x \in \mathbb{R}. \quad (4.1)$$

Then the map

$$A \mapsto E^{-1}AE \quad (4.2)$$

transforms the Fourier convolution operator $W^0(a) = \mathcal{F}^{-1}a\mathcal{F}$ to the Mellin convolution operator

$$\text{Co}(a) := M^{-1}aM$$

with the same symbol $a$. Hence the class of Fourier multipliers on $L^p(\mathbb{R})$ coincides with the class of Mellin multipliers on $L^p(\mathbb{R}_+, d\mu)$.

4.2. Continuous and piecewise continuous multipliers

We denote by $PC$ the $C^*$-algebra of all bounded piecewise continuous functions on $\mathbb{R} = \mathbb{R} \cup \{\infty\}$. By definition, $a \in PC$ if and only if $a \in L^\infty(\mathbb{R})$ and the one-sided limits

$$a(x_0 - 0) := \lim_{x \to x_0 - 0} a(x), \quad a(x_0 + 0) := \lim_{x \to x_0 + 0} a(x)$$

exist for each $x_0 \in \mathbb{R}$. If a function $a$ is given everywhere on $\mathbb{R}$, then its total variation is defined by

$$V(a) := \sup \sum_{k=1}^n |a(x_k) - a(x_{k-1})|,$$

where the supremum is taken over all $n \in \mathbb{N}$ and

$$-\infty < x_0 < x_1 < \cdots < x_n < +\infty.$$ 

If $a$ has a finite total variation, then it has finite one-sided limits $a(x - 0)$ and $a(x + 0)$ for all $x \in \mathbb{R}$, that is, $a \in PC$ (see, e.g., [25 Chap. VIII, Sections 3 and 9]). The following theorem gives an important subset of $\mathcal{M}_p(\mathbb{R})$. Its proof can be found, e.g., in [11 Theorem 17.1] or [6 Theorem 2.11].
Theorem 4.1 (Stechkin’s inequality). If \( a \in PC \) has finite total variation \( V(a) \), then \( a \in M_p(\mathbb{R}) \) and
\[
\|a\|_{M_p(\mathbb{R})} \leq \|S_a\|_{B(L^p(\mathbb{R}))} (\|a\|_{L^\infty(\mathbb{R})} + V(a)),
\]
where \( S_a \) is the Cauchy singular integral operator on \( \mathbb{R} \).

According to [6] or [1, p. 325], let \( PC \) be the closure in \( M_p(\mathbb{R}) \) of the set of all functions \( a \in PC \) with finite total variation on \( \mathbb{R} \). Following [1, p. 331], put
\[
C_p(\mathbb{R}) := PC \cap C(\mathbb{R}),
\]
where \( \mathbb{R} := [-\infty, +\infty] \). It is easy to see that \( PC_p \) and \( C_p(\mathbb{R}) \) are Banach algebras.

4.3. Maximal ideal space of the algebras \( A \) and \( A^\pi \)

Let \( \mathfrak{A} \) be a Banach algebra and \( \mathfrak{C} \) be a subset of \( \mathfrak{A} \). Following [3, Section 3.45], we denote by \( \text{alg}_{\mathfrak{A}} \mathfrak{C} \) the smallest closed subalgebra of \( \mathfrak{A} \) containing \( \mathfrak{C} \) and by \( \text{id}_{\mathfrak{A}} \mathfrak{C} \) the smallest closed two-sided ideal of \( \mathfrak{A} \) containing \( \mathfrak{C} \).

Put \( A := \text{alg}_{\mathfrak{B}} \{I, S_0\} \). Obviously, the algebra \( A \) is commutative. Consider the isometric isomorphism
\[
\Phi : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+, d\mu), \quad (\Phi f)(t) := t^{1/p} f(t), \quad t \in \mathbb{R}_+.
\]

The following result is well known. It is essentially due to Duduchava [6, 7] and Simonenko, Chin Ngok Minh [31]. It can be found with a proof in [5, Section 1.10.2], [12, Section 2.1.2], [29, Sections 4.2.2-4.2.3].

Theorem 4.2. (a) The algebra \( A \) is the smallest closed subalgebra of \( \mathfrak{B} \) that contains the operators \( \Phi^{-1}\text{Co}(a)\Phi \) with \( a \in C_p(\mathbb{R}) \).
(b) The maximal ideal space of the commutative Banach algebra \( A \) is homeomorphic to \( \mathbb{R} \). In particular, the operator \( \Phi^{-1}\text{Co}(a)\Phi \) with \( a \in C_p(\mathbb{R}) \) is invertible if and only if \( a(x) \neq 0 \) for all \( x \in \mathbb{R} \). Thus \( A \) is an inverse closed subalgebra of \( \mathfrak{B} \).
(c) The operator \( \Phi^{-1}\text{Co}(a)\Phi \) with \( a \in C_p(\mathbb{R}) \) belongs to \( \text{id}_{\mathfrak{A}} \{R_0\} \) if and only if \( a(-\infty) = a(+\infty) = 0 \).
(d) If \( \gamma \in \mathbb{C} \) satisfies (1.3), then the function \( s_\gamma \) given by (1.8) and the function \( r_\gamma \) defined by
\[
r_\gamma(x) := 1/\sinh[\pi(x + i/p + i\gamma)], \quad x \in \mathbb{R},
\]
belong to \( C_p(\mathbb{R}) \) and
\[
S_\gamma = \Phi^{-1}\text{Co}(s_\gamma)\Phi, \quad R_\gamma = \Phi^{-1}\text{Co}(r_\gamma)\Phi.
\]

Let us describe the quotient algebra
\[
A^\pi = (A + \mathcal{K})/\mathcal{K}.
\]

By [29, Proposition 4.2.14], a Mellin convolution operator is Fredholm on the space \( L^p(\mathbb{R}_+, d\mu) \) if and only if it is invertible on this space. Hence, Theorem 4.2 implies the following.
Corollary 4.3. (a) The algebra $A^\pi$ is commutative and its maximal ideal space is homeomorphic to $R$.
(b) The Gelfand transform of the coset $(\Phi^{-1}Co(a)\Phi)^\pi \in A^\pi$ for $a \in C_p(R)$ is given by

$$[(\Phi^{-1}Co(a)\Phi)^\pi](x) = a(x) \quad \text{for} \quad x \in R.$$  

4.4. Some operator relations

Lemma 4.4 ([14, Lemma 2.4], [20, Lemma 4.2]). Let $1 < p < \infty$ and $\gamma, \delta \in C$ be such that $0 < 1/p + \Re \gamma < 1$ and $0 < 1/p + \Re \delta < 1$. Then

$$P_\delta^+ - P_\gamma^+ = P_\gamma^- - P_\delta^- = \frac{1}{2} \sinh[\pi i(\gamma - \delta)] R_\gamma R_\delta, \quad P^-_\gamma P^+ = -\frac{e^{i\pi(\delta - \gamma)}}{4} R_\gamma R_\delta.$$  

4.5. Compactness of commutators of singular integral operators and functional operators

Fix $\alpha, \beta \in SOS(R_+)$ and consider the Banach algebra of functional operators with shifts and slowly oscillating data defined by

$$FO_{\alpha, \beta} := \text{alg}_B \{U_\alpha, U^{-1}_\alpha, U_\beta, U^{-1}_\beta, cI : c \in SO(R_+)\}.$$  

Lemma 4.5 ([17, Lemma 2.8]). Let $\alpha, \beta \in SOS(R_+)$. If $A \in FO_{\alpha, \beta}$ and $B \in A$, then $AB \simeq BA$.

5. Allan-Douglas localization

5.1. The Allan-Douglas local principle

Let $\mathfrak{A}$ be a Banach algebra with identity. A subalgebra $\mathfrak{Z}$ of $\mathfrak{A}$ is said to be a central subalgebra of $\mathfrak{A}$ if $za = az$ for all $z \in \mathfrak{Z}$ and all $a \in \mathfrak{A}$.

The proof of the following result is contained, e.g., in [3, Theorem 1.35(a)].

Theorem 5.1 (Allan-Douglas). Let $\mathfrak{A}$ be a Banach algebra with identity $e$ and let $\mathfrak{Z}$ be a closed central subalgebra of $\mathfrak{A}$ containing $e$. Let $M(\mathfrak{Z})$ be the maximal ideal space of $\mathfrak{Z}$, and for $\omega \in M(\mathfrak{Z})$, let $\mathfrak{Z}_\omega$ refer to the smallest closed two-sided ideal of $\mathfrak{A}$ containing the ideal $\omega$. Then an element $a$ is left (resp., right, two-sided) invertible in $\mathfrak{A}$ if and only if $a + \mathfrak{Z}_\omega$ is left (resp., right, two-sided) invertible in the quotient algebra $\mathfrak{A}/\mathfrak{Z}_\omega$ for all $\omega \in M(\mathfrak{Z})$.

The algebra $\mathfrak{A}/\mathfrak{Z}_\omega$ is referred to as the local algebra of $\mathfrak{A}$ at $\omega \in M(\mathfrak{Z})$.

5.2. Algebras of singular integral operators with shifts and algebras of operators of local type

Following [15, Section 6.3], we consider the following sets:

$$\mathcal{Z} := \text{alg}_B \{I, S_0, cR_0, K : c \in SO(R_+), K \in K\},$$

$$\Lambda := \{A \in B : AC - CA \in K \quad \text{for all} \quad C \in \mathcal{Z}\}.$$  

By [15, Lemma 6.7(a)], the set $\Lambda$ is a closed unital subalgebra of the algebra $B$, which is usually called the algebra of operators of local type.
For $\alpha, \beta \in \text{SOS}(\mathbb{R}_+)$, put
\[ \mathcal{F}_{\alpha,\beta} := \text{alg}_B(\{S_0\} \cup \mathcal{F}_{\alpha,\beta}). \]

By a minor modification of the proof of [15, Theorem 6.8] with the aid of Lemma 4.3, we get the following.

**Theorem 5.2.** We have $\mathcal{K} \subset \mathcal{Z} \subset \mathcal{F}_{\alpha,\beta} \subset \Lambda$.

**5.3. Maximal ideal space of the algebra $\mathcal{Z}^\pi$**

It follows from Theorem 5.2 that the quotient algebras $\mathcal{Z}^\pi := \mathcal{Z}/\mathcal{K}$ and $\Lambda^\pi := \Lambda/\mathcal{K}$ are well defined. Clearly, $\mathcal{Z}^\pi$ lies in the center of $\Lambda^\pi$.

**Theorem 5.3 ([15, Theorem 6.11]).** For the commutative Banach algebra $\mathcal{Z}^\pi$ the following statements hold:

(a) the maximal ideal space $M(\mathcal{Z}^\pi)$ of $\mathcal{Z}^\pi$ is homeomorphic to the set 
\[ \{-\infty, +\infty\} \cup (\Delta \times \mathbb{R}); \]
(b) any coset $Z^\pi \in \mathcal{Z}^\pi$ is of the form
\[ Z^\pi = (c_+P_0^+)\pi + (c_-P_0^-)\pi + \lim_{n \to \infty} \sum_{k=1}^{m_n} (c_{n,k}H_{n,k})\pi \]  
where $c_{\pm} \in \mathbb{C}$, $c_{n,k} \in \text{SO}(\mathbb{R}_+)$, $H_{n,k} \in \text{id}_{\mathcal{A}}\{R_0\}$, and $m_n \in \mathbb{N}$;
(c) the Gelfand transform of the coset $Z^\pi \in \mathcal{Z}^\pi$ defined by (5.1) is given for a point $(\xi, x) \in \Delta \times \mathbb{R}$ by
\[ (Z^\pi)(\xi,x) = c_+p_0^+(x) + c_-p_0^-(x) + \lim_{n \to \infty} \sum_{k=1}^{m_n} c_{n,k}(\xi)(H_{n,k}^\pi)^\pi(x), \]
where $(H_{n,k}^\pi)^\pi(x)$ is the Gelfand transform of a coset $H_{n,k}^\pi \in \mathcal{A}^\pi$, which is calculated in Corollary 4.1(b).

**5.4. Semi-Fredholmness of operators of local type**

Let $\mathcal{J}_{\pi}^\pi$, $\mathcal{J}_{\pi}^\pi$, and $\mathcal{J}_{\pi}^\pi$ for $(\xi, x) \in \Delta \times \mathbb{R}$ be the closed two-sided ideals of the Banach algebra $\Lambda^\pi$ generated, respectively, by the maximal ideals
\[ \mathcal{I}_{+\pi} := \text{id}_{\mathcal{Z}^\pi}\{(P_0^-)\pi, (gR_0)^\pi : g \in \text{SO}(\mathbb{R}_+)\}, \]
\[ \mathcal{I}_{-\pi} := \text{id}_{\mathcal{Z}^\pi}\{(P_0^+)\pi, (gR_0)^\pi : g \in \text{SO}(\mathbb{R}_+)\}, \]
\[ \mathcal{I}_{\pi,\pi} := \{Z^\pi \in \mathcal{Z}^\pi : (Z^\pi)(\xi,x) = 0\} \]
of the algebra $\mathcal{Z}^\pi$, and let
\[ \Lambda_{+\pi} := \Lambda^\pi/\mathcal{J}_{+\pi}^\pi, \quad \Lambda_{-\pi} := \Lambda^\pi/\mathcal{J}_{-\pi}^\pi, \quad \Lambda_{\pi,\pi} := \Lambda^\pi/\mathcal{J}_{\pi,\pi}^\pi \]
be the corresponding quotient algebras (see also [15, Section 6.6]).

Obviously, an operator $T \in \Lambda$ is left Fredholm (resp., right Fredholm) on the space $L^p(\mathbb{R}_+)$ if the coset $T^\pi := \Lambda + \mathcal{K}$ is left invertible (resp., right invertible) in the quotient Banach algebra $\Lambda^\pi \subset \mathcal{B}^\pi$. Applying now Theorem 5.4 with $\mathcal{A}\Lambda = \Lambda^\pi$ and $\mathcal{Z} = \mathcal{Z}^\pi$, we immediately obtain the following.

**Theorem 5.4.** An operator $T \in \Lambda$ is left (resp., right) Fredholm on the space $L^p(\mathbb{R}_+)$ if the following three conditions are fulfilled:
(i) the coset $T^\pi_+ + J^\pi_+\infty$ is left (resp., right) invertible in the quotient algebra $\Lambda^\pi_+\infty$;
(ii) the coset $T^\pi_+ + J^\pi_-\infty$ is left (resp., right) invertible in the quotient algebra $\Lambda^\pi_-\infty$;
(iii) for every $(\xi, x) \in \Delta \times \mathbb{R}$, the coset $T^\pi_+ + J^\pi_\xi, x$ is left (resp., right) invertible in the quotient algebra $\Lambda^\pi_\xi, x$.

It follows from Theorems 4.2(d) and 5.2 that $N \in F_{\alpha, \beta} \subset \Lambda$. Thus, Theorem 5.4 is applicable to $N$. Hence, our next aim is to study one-sided invertibility of the cosets $N^\pi_+ + J^\pi_+\infty$, $N^\pi_+ + J^\pi_-\infty$ and $N^\pi_+ + J^\pi_\xi, x$ in the corresponding local algebras $\Lambda^\pi_+\infty$, $\Lambda^\pi_-\infty$ and $\Lambda^\pi_\xi, x$ for all $(\xi, x) \in \Delta \times \mathbb{R}$.

6. Mellin pseudodifferential operators and their symbols

6.1. Mellin PDO’s: overview

Mellin pseudodifferential operators are generalizations of Mellin convolution operators. Let $a$ be a sufficiently smooth function defined on $\mathbb{R}_+ \times \mathbb{R}$. The Mellin pseudodifferential operator (shortly, Mellin PDO) with symbol $a$ is initially defined for smooth functions $f$ of compact support by the iterated integral

$$[\text{Op}(a)f](t) = [\mathcal{M}^{-1}a(t, \cdot)\mathcal{M}f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} dx \int_{\mathbb{R}_+} a(t, x) \left( \frac{t}{\tau} \right)^{ix} f(\tau) \frac{d\tau}{\tau} \quad \text{for} \quad t \in \mathbb{R}_+. \quad (6.1)$$

Obviously, if $a(t, x) = a(x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, then the Mellin pseudodifferential operator $\text{Op}(a)$ becomes the Mellin convolution operator

$$\text{Op}(a) = \text{Co}(a).$$

In 1991 Rabinovich [26] (see also [27]) proposed to use Mellin pseudodifferential operators with $C^\infty$ slowly oscillating symbols to study singular integral operators with slowly oscillating coefficients on $L^p$ spaces. Namely, he considered symbols $a \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ such that

$$\sup_{(t, x) \in \mathbb{R}_+ \times \mathbb{R}} \left| (t\partial_t)^j \partial_x^k a(t, x) \right| (1 + x^2)^{k/2} < \infty \quad \text{for all} \quad j, k \in \mathbb{Z}_+ \quad (6.2)$$

and

$$\lim_{t \to s, x \in \mathbb{R}} \sup_{(t, x) \in \mathbb{R}_+ \times \mathbb{R}} \left| (t\partial_t)^j \partial_x^k a(t, x) \right| (1 + x^2)^{k/2} = 0 \quad \text{for all} \quad j \in \mathbb{N}, \quad k \in \mathbb{Z}_+, \quad (6.3)$$

where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $s \in \{0, \infty\}$. Here and in what follows $\partial_t$ and $\partial_x$ denote the operators of partial differentiation with respect to $t$ and to $x$. Notice that (6.2) defines nothing but the Mellin version of the Hörmander class $S_{0,0}^1(\mathbb{R})$ (see, e.g., [23] Chap. 2, Section 1) for the definition of the Hörmander classes $S_{0,\delta}^m(\mathbb{R}^n)$. If $a$ satisfies (6.2), then the Mellin PDO $\text{Op}(a)$ is bounded on the spaces $L^p(\mathbb{R}_+, dm)$ for $1 < p < \infty$ (see, e.g., [32] Chap. VI, Proposition 4) for the corresponding Fourier PDO’s. Condition (6.3) is the Mellin
version of Grushin’s definition of symbols slowly varying in the first variable (see, e.g., [11, 23, Chap. 3, Definition 5.11]).

The idea of application of Mellin PDO’s with considered class of symbols was exploited in a series of papers by Rabinovich and coauthors (see [28, Sections 4.6–4.7] for the complete history up to 2004). On the other hand, the smoothness conditions imposed on slowly oscillating symbols are very strong. In particular, they are not applicable directly to the problem we are dealing with in the present paper.

In 2005 the second author [21] developed a Fredholm theory for the Fourier pseudodifferential operators with slowly oscillating symbols of limited smoothness in the spirit of Sarason’s definition [30, p. 820] of slow oscillation adopted in the present paper (much less restrictive than in [27] and in the works mentioned in [28]). Necessary for our purposes results from [21] were translated to the Mellin setting, for instance, in [16] with the aid of the transformation defined by (4.1)–(4.2). For the convenience of readers, we reproduce the results required in what follows exactly in the same form as they were stated in [16], where more details on their proofs can be found.

6.2. Boundedness of Mellin PDO’s

If $a$ is an absolutely continuous function of finite total variation on $\mathbb{R}$, then its derivative belongs to $L^1(\mathbb{R})$ and

$$V(a) = \int_{\mathbb{R}} |a'(x)| \, dx$$

(see, e.g., [25, Chap. VIII, Sections 3 and 9; Chap. XI, Section 4]). The set $V(\mathbb{R})$ of all absolutely continuous functions of finite total variation on $\mathbb{R}$ forms a Banach algebra when equipped with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a) = \|a\|_{L^\infty(\mathbb{R})} + \int_{\mathbb{R}} |a'(x)| \, dx. \quad (6.4)$$

Let $C_b(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach algebra of all bounded continuous $V(\mathbb{R})$-valued functions on $\mathbb{R}_+$ with the norm

$$\|a(\cdot, \cdot)\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} = \sup_{t \in \mathbb{R}_+} \|a(t, \cdot)\|_V.$$ 

As usual, let $C_0^\infty(\mathbb{R}_+)$ be the set of all infinitely differentiable functions of compact support on $\mathbb{R}_+$.

**Theorem 6.1** ([16, Theorem 3.1]). If $a \in C_b(\mathbb{R}_+, V(\mathbb{R}))$, then the Mellin pseudodifferential operator $\text{Op}(a)$, defined for functions $f \in C_0^\infty(\mathbb{R}_+)$ by the iterated integral (6.1), extends to a bounded linear operator on the space $L^p(\mathbb{R}_+, d\mu)$ and there is a number $C_p \in (0, \infty)$ depending only on $p$ such that

$$\|\text{Op}(a)\|_{B(L^p(\mathbb{R}_+, d\mu))} \leq C_p \|a\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))}.$$
6.3. Products of Mellin PDO's
Consider the Banach subalgebra \( SO(\mathbb{R}_+, V(\mathbb{R})) \) of the algebra \( C_b(\mathbb{R}_+, V(\mathbb{R})) \) consisting of all \( V(\mathbb{R}) \)-valued functions \( a \) on \( \mathbb{R}_+ \) that slowly oscillate at 0 and \( \infty \), that is,

\[
\lim_{r \to 0} \max_{s, \tau \in [r, 2r]} \|a(t, \cdot) - a(\tau, \cdot)\|_{L^\infty(\mathbb{R})} = 0, \quad s \in \{0, \infty\}.
\]

Let \( E(\mathbb{R}_+, V(\mathbb{R})) \) be the Banach algebra of all \( V(\mathbb{R}) \)-valued functions \( a \) in the algebra \( SO(\mathbb{R}_+, V(\mathbb{R})) \) such that

\[
\lim_{|h| \to 0} \sup_{t \in \mathbb{R}_+} \|a(t, \cdot) - a^h(t, \cdot)\|_V = 0
\]

where \( a^h(t, x) := a(t, x + h) \) for all \( (t, x) \in \mathbb{R}_+ \times \mathbb{R} \).

**Theorem 6.2** ([16 Theorem 3.3]). If \( a, b \in E(\mathbb{R}_+, V(\mathbb{R})) \), then

\[
\text{Op}(a)\text{Op}(b) \simeq \text{Op}(ab).
\]

**Lemma 6.3** ([16 Lemma 3.4]). If \( a, b, c \in E(\mathbb{R}_+, V(\mathbb{R})) \) are such that \( a \) depends only on the first variable and \( c \) depends only on the second variable, then

\[
\text{Op}(a)\text{Op}(b)\text{Op}(c) = \text{Op}(abc).
\]

6.4. Applications of Mellin pseudodifferential operators
We immediately deduce the following assertion from [14 Lemma 4.1].

**Lemma 6.4.** Suppose \( f \in SO(\mathbb{R}_+) \) and \( \gamma \in \mathbb{C} \) satisfies (1.3). Then the functions

\[
\hat{f}(t, x) := f(t), \quad \hat{s}_\gamma(t, x) := s_\gamma(x), \quad \hat{r}_\gamma(t, x) := r_\gamma(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},
\]

belong to the Banach algebra \( E(\mathbb{R}_+, V(\mathbb{R})) \).

**Lemma 6.5.** If \( b \in E(\mathbb{R}_+, V(\mathbb{R})) \), then the operator \( \Phi^{-1}\text{Op}(b)\Phi \) belongs to the algebra \( \Lambda \).

**Proof.** Let \( c \in SO(\mathbb{R}_+) \). It follows from Lemma 6.3 that the functions

\[
c_0(t, x) := c(t)r_0(x), \quad s_0(t, x) := s_0(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},
\]

belong to the algebra \( E(\mathbb{R}_+, V(\mathbb{R})) \). Since \( E(\mathbb{R}_+, V(\mathbb{R})) \subset C_b(\mathbb{R}_+, V(\mathbb{R})) \), Theorem 6.1 implies that \( B := \Phi^{-1}\text{Op}(b)\Phi \in \Lambda \). We infer from Theorem 6.2 that

\[
\text{Op}(s_0)\text{Op}(b) - \text{Op}(b)\text{Op}(s_0) \in K(L^p(\mathbb{R}_+, d\mu)),
\]

(6.5)

\[
\text{Op}(c_0)\text{Op}(b) - \text{Op}(b)\text{Op}(c_0) \in K(L^p(\mathbb{R}_+, d\mu)).
\]

(6.6)

On the other hand, by Theorem 4.2(d) and Lemma 6.3

\[
S_0 = \Phi^{-1}\text{Co}(s_0)\Phi = \Phi^{-1}\text{Op}(s_0)\Phi, \quad (6.7)
\]

\[
cR_0 = c\Phi^{-1}\text{Co}(r_0)\Phi = \Phi^{-1}\text{Op}(c_0)\Phi. \quad (6.8)
\]

Combining (6.5)–(6.8), we conclude that \( S_0B = BS_0, (cR_0)B = B(cR_0) \in K \). Hence \( B \in \Lambda \).

Applying [14 Lemma 4.4] and making minor modifications in the proof of [16 Lemma 4.5], we get the following.
Lemma 6.6. Let $\gamma \in \mathbb{C}$ satisfy (1.3). Suppose $\alpha \in \text{SOS}(\mathbb{R}_+)$, $\omega$ is its exponent function, and $U_\alpha$ is the associated isometric shift operator on $L^p(\mathbb{R}_+)$. Then the operator $U_\alpha R_\gamma$ can be realized as the Mellin pseudodifferential operator up to a compact operator:

$$U_\alpha R_\gamma \simeq \Phi^{-1}\text{Op}(c)\Phi,$$

where the function $c$, given by

$$c(t, x) := e^{i\omega(t)x}r_\gamma(x) \quad \text{for} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

belongs to the Banach algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$.

7. Sufficient conditions for the semi-Fredholmness

7.1. One-sided invertibility in the quotient algebras $\Lambda^\pi_{+\infty}$ and $\Lambda^\pi_{-\infty}$

The next lemma shows that the operator $N$ can be written as a paired operator with respect to the pair $(P_0^+, P_0^-)$.

Lemma 7.1. Let $1 < p < \infty$ and let $\gamma \in \mathbb{C}$ satisfy (1.3). Suppose $a_k, b_k$ belong to $\text{SO}(\mathbb{R}_+)$ for all $k \in \mathbb{Z}$, $\alpha, \beta$ belong to $\text{SOS}(\mathbb{R}_+)$, the operators $A_+ \in W^{SO}_{\alpha,p}$ and $A_- \in W^{SO}_{\beta,p}$ are given by (1.5), and the operator $N$ is given by (1.6). Then the operator $N$ can be represented in each of the forms

$$N = A_+P_0^+ + C_-P_0^- = C_+P_0^+ + A_-P_0^-,$$

(7.1)

where

$$C_+ := A_+ + 2\sinh(\pi i\gamma)e^{\pi i\gamma}(A_+ - A_-)P_\gamma^-,$$

$$C_- := A_- + 2\sinh(\pi i\gamma)e^{-\pi i\gamma}(A_+ - A_-)P_\gamma^+.$$

Moreover, all operators $DP_0^+ - P_0^+D$ and $DP_0^- - P_0^-D$, where $D$ belongs to the set $\{A_+, A_-, C_+, C_-\}$, are compact on the space $L^p(\mathbb{R}_+)$. 

Proof. Both representations follow from Lemma 4.4. The compactness of the commutators $DP_0^\pm - P_0^\pm D$ is a consequence of parts (a) and (d) of Theorem 4.2 and Lemma 4.5. \Box

The following statement generalizes [15, Theorem 8.1] from the case of two-sided invertible binomial functional operators $A_+$ and $A_-$ to the case of one-sided invertible operators $A_+ \in W^{SO}_{\alpha,p}$ and $A_- \in W^{SO}_{\beta,p}$. This generalization is possible thanks to Theorem 3.3 although the proof follows the same lines as in [15].

Theorem 7.2. Let $1 < p < \infty$ and let $\gamma \in \mathbb{C}$ satisfy (1.3). Suppose $a_k, b_k$ belong to $\text{SO}(\mathbb{R}_+)$ for all $k \in \mathbb{Z}$, $\alpha, \beta$ belong to $\text{SOS}(\mathbb{R}_+)$, the operators $A_+ \in W^{SO}_{\alpha,p}$ and $A_- \in W^{SO}_{\beta,p}$ are given by (1.5), and the operator $N$ is given by (1.6).

(a) If the operator $A_+$ is left (resp., right) invertible on the space $L^p(\mathbb{R}_+)$, then the coset $N^\pi + J^\pi_{+\infty}$ is left (resp., right) invertible in the quotient algebra $\Lambda^\pi_{+\infty}$.
Lemma 7.4. Let \( c \) where \( \omega \) is its exponent function, and \( \xi \) exponent function. If \( b \) then there exists a function \( \omega, \gamma \).

Proof. Recall that \( \mathcal{F} \mathcal{O}_{\alpha, \beta} \subset \mathcal{F}_{\alpha, \beta} \subset \Lambda \) in view of Theorem 5.2. By Lemma 7.1, the operator \( N \) is represented in each of the forms (7.1), where

\[
A_{+} \in W_{\alpha, p}^{SO} \subset \mathcal{F}_{\alpha, \beta} \subset \mathcal{F}_{\alpha, \beta} \subset \Lambda, \quad A_{-} \in W_{\beta, q}^{SO} \subset \mathcal{F}_{\alpha, \beta} \subset \mathcal{F}_{\alpha, \beta} \subset \Lambda,
\]

and \( C_{+}, C_{-} \in \mathcal{F}_{\alpha, \beta} \subset \Lambda \).

(a) Take \( A_{+} \in W_{\alpha, p}^{SO} \). If \( A_{+} \) is left (resp., right) invertible in \( \mathcal{B} \), then it follows from Theorem 3.3 that there exists a left (resp., right) inverse \( A_{+}^{(-1)} \) of \( A_{+} \) such that \( A_{+}^{(-1)} \in W_{\alpha, p}^{SO} \subset \mathcal{F}_{\alpha, \beta} \subset \Lambda \). Hence the coset \( A_{+}^{\pi} = A_{+} + \mathcal{K} \) is left (resp., right) invertible in the quotient algebra \( \Lambda^{\pi} \), which implies the left (resp., right) invertibility of the coset \( A_{+}^{\pi} + \mathcal{J}_{+}^{\pi} \) in the quotient algebra \( \Lambda_{+}^{\pi} \). Hence we infer from (7.1) that

\[
N^{\pi} + \mathcal{J}_{+}^{\pi} = (A_{+}P_{+}^{\pi} + C_{-}P_{0}^{\pi})^{\pi} + \mathcal{J}_{+}^{\pi} = A_{+}^{\pi} + [(C_{-} - A_{+})P_{0}^{\pi}]^{\pi} + \mathcal{J}_{+}^{\pi} = A_{+}^{\pi} + \mathcal{J}_{+}^{\pi}
\]

because \( (P_{0}^{\pi})^{\pi} \in \mathcal{T}_{+}^{\pi} \subset \mathcal{J}_{+}^{\pi} \). Thus, the left (resp. right) invertibility of the operator \( A_{+} \) in \( \mathcal{B} \) implies the left (resp., right) invertibility of the coset \( N^{\pi} + \mathcal{J}_{+}^{\pi} \) in the quotient algebra \( \Lambda_{+}^{\pi} \). Part (a) is proved.

(b) The proof is analogous. \( \Box \)

7.2. Invertibility in the quotient algebras \( \Lambda_{\xi, x}^{\pi} \) with \( (\xi, x) \in \Delta \times \mathbb{R} \)

By a literal repetition with minor modifications of the proof of [15, Lemma 7.4], we get the following.

Lemma 7.3. Let \( \gamma \in \mathbb{C} \) satisfy (1.3). Suppose \( \alpha \in \text{SOS}(\mathbb{R}+) \) and \( \omega \) is its exponent function. If \( \xi \in \Delta \) and

\[
a(t, x) := e^{i\omega(t)x}(r_{\gamma}(x))^{2} \quad \text{for} \quad (t, x) \in (\Delta \cup \mathbb{R}+) \times \mathbb{R},
\]

then there exists a function \( b_{\xi} \in \mathcal{E}(\mathbb{R}+, V(\mathbb{R})) \) such that

\[
a(t, x) - a(\xi, x) = (\omega(t) - \omega(\xi))b_{\xi}(t, x)r_{\gamma}(x) \quad \text{for} \quad (t, x) \in \mathbb{R}_{+} \times \mathbb{R}.
\]

Lemma 7.4. Let \( \gamma \in \mathbb{C} \) satisfy (1.3). Suppose \( \alpha \) is a slowly oscillating shift, \( \omega \) is its exponent function, and \( U_{\alpha} \) is the associated isometric shift operator on \( L^{p}(\mathbb{R}+) \). If \( (\xi, x) \in \Delta \times \mathbb{R} \), then

\[
(U_{\alpha}R_{\gamma})^{\pi} - e^{i\omega(\xi)x}(r_{\gamma}(x))^{2}I^{\pi} \in \mathcal{J}_{\xi, x}^{\pi}.
\]

Proof. The proof is developed by analogy with [15, Lemma 8.3]. In view of Lemma 6.6

\[
U_{\alpha}R_{\gamma} \simeq \Phi^{-1}\text{Op}(c_{\omega, \gamma})\Phi,
\]

where \( c_{\omega, \gamma} \in \mathcal{E}(\mathbb{R}+, V(\mathbb{R})) \) is given by

\[
c_{\omega, \gamma}(t, y) := e^{i\omega(t)y}r_{\gamma}(y) \quad \text{for} \quad (t, y) \in \mathbb{R}_{+} \times \mathbb{R}.
\]

On the other hand, in view of Theorem 4.2(d) and Lemma 6.4

\[
R_{\gamma} = \Phi^{-1}\text{Co}(r_{\gamma})\Phi = \Phi^{-1}\text{Op}(r_{\gamma})\Phi,
\]
where \( r_\gamma \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \) is given by
\[
  r_\gamma(t, y) = r_\gamma(y) \quad \text{for} \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}.
\]
It follows from (7.2)–(7.3) and Lemma 6.3 that
\[
  U_\alpha R_\gamma^2 \simeq \Phi^{-1} \text{Op}(a) \Phi,
\]
where
\[
  a(t, y) = c_{\omega, \gamma}(t, y) r_\gamma(t, y) = e^{i\omega(t)y} (r_\gamma(y))^2 \quad \text{for} \quad (t, y) \in \mathbb{R}_+ \times \mathbb{R}.
\]
Since \( a(\cdot, y) \in SO(\mathbb{R}_+) \) for every \( y \in \mathbb{R} \), taking the Gelfand transform of \( a(\cdot, y) \), we obtain
\[
  a(t, y) = e^{i\omega(t)y} (r_\gamma(y))^2 \quad \text{for} \quad (t, y) \in (\Delta \cup \mathbb{R}_+) \times \mathbb{R}.
\]
Fix \((\xi, x) \in \Delta \times \mathbb{R}\). Let us represent the function \( a \) in the form
\[
  a(t, y) = a(t, y) - a(\xi, y) + c_{\omega, \gamma}(\xi, y) r_\gamma(y)
  = [a(t, y) - a(\xi, y)] + [c_{\omega, \gamma}(\xi, y) - c_{\omega, \gamma}(\xi, x)] r_\gamma(t, y)
  + c_{\omega, \gamma}(\xi, x)[r_\gamma(t, y) - r_\gamma(t, x)] + a(\xi, x),
\]
where \((t, y) \in \mathbb{R}_+ \times \mathbb{R}\). Further, we deduce from Lemma 7.3 that there exists a function \( b_\xi \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \) such that
\[
  a - a(\xi, \cdot) = (\omega - \omega(\xi)) b_\xi r_\gamma.
\]
Hence, we infer from the above equality and Lemmas 6.4 and 6.3 that
\[
  \Phi^{-1} \text{Op}(a - a(\xi, \cdot)) \Phi = \Phi^{-1} \text{Op}((\omega - \omega(\xi)) b_\xi r_\gamma) \Phi
  = (\Phi^{-1} \text{Op}(\omega - \omega(\xi)) \Phi) \cdot (\Phi^{-1} \text{Op}(b_\xi) \Phi) \cdot (\Phi^{-1} \text{Op}(r_\gamma) \Phi).
\]
The latter equality, Theorem 6.2 and equality (7.3) imply that
\[
  \Phi^{-1} \text{Op}(a - a(\xi, \cdot)) \Phi \simeq (\omega - \omega(\xi)) R_\gamma (\Phi^{-1} \text{Op}(b_\xi) \Phi).
\]
Applying Theorem 5.3(c) and Corollary 4.3(b), we obtain
\[
  [(\omega - \omega(\xi)) R_\gamma]^\pi \gamma(\xi, x) = (\omega(\xi) - \omega(\xi)) r_\gamma(x) = 0.
\]
Therefore \([(\omega - \omega(\xi)) R_\gamma]^\pi \in \mathcal{I}_\pi^\gamma \xi, x\). On the other hand, since \( b_\xi \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \), we conclude from Lemma 6.5 that \( \Phi^{-1} \text{Op}(b_\xi) \Phi \in \Lambda \). Then, taking into account (7.6) and the definition of the ideal \( \mathcal{J}_\pi^\gamma \xi, x \), we infer that
\[
  [\Phi^{-1} \text{Op}(a - a(\xi, \cdot)) \Phi]^\pi = [(\omega - \omega(\xi)) R_\gamma (\Phi^{-1} \text{Op}(b_\xi) \Phi)]^\pi \in \mathcal{J}_\pi^\gamma \xi, x.
\]
Taking into account the definition of the norm (6.4) in the algebra \( V(\mathbb{R}) \), it is easy to see that the function \( c_{\omega, \gamma}(\xi, \cdot) \) belongs to \( V(\mathbb{R}) \), where
\[
  c_{\omega, \gamma}(\xi, y) = e^{i\omega(\xi)y} r_\gamma(y) \quad \text{for} \quad y \in \mathbb{R}_+.
\]
Then, by Stechkin’s inequality (Theorem 4.1), \( c_{\omega, \gamma}(\xi, \cdot) \in C_p(\mathbb{R}) \). Hence, it follows from Theorem 4.2(a) that \( \Phi^{-1} \text{Co}(c_{\omega, \gamma}(\xi, \cdot)) \Phi \in \mathcal{A} \). By Theorem 5.3(c)
and Corollary 4.3(b),
\[ ([\Phi^{-1}\text{Op}(c_{\omega,\gamma}(\xi, \cdot) - c_{\omega,\gamma}(\xi, x))\Phi]^\pi)^\sim(\xi, x) = \]
\[ = ([\Phi^{-1}\text{Co}(c_{\omega,\gamma}(\xi, \cdot))\Phi]^\pi)^\sim(\xi, x) - ([c_{\omega,\gamma}(\xi, x)I]^\pi)^\sim(\xi, x) = e^{i\omega(\xi)x}r(\gamma, x) - e^{i\omega(\xi)x}r(\gamma, x) = 0. \]
Therefore
\[ [\Phi^{-1}\text{Op}(c_{\omega,\gamma}(\xi, \cdot) - c_{\omega,\gamma}(\xi, x))\Phi]^\pi \in I_{\xi,x}. \]
By this observation, Lemma 6.3 and equality (7.3), we obtain
\[ [\Phi^{-1}\text{Op}((c_{\omega,\gamma}(\xi, \cdot) - c_{\omega,\gamma}(\xi, x))\Phi)]^\pi = \]
\[ = [\Phi^{-1}\text{Op}(c_{\omega,\gamma}(\xi, \cdot))\Phi]^\pi[\Phi^{-1}\text{Op}(r(\gamma)\Phi)]^\pi \]
\[ = [\Phi^{-1}\text{Op}(c_{\omega,\gamma}(\xi, \cdot) - c_{\omega,\gamma}(\xi, x))\Phi]^\pi R(\gamma)^\pi \in J_{\xi,x}^\pi. \] (7.8)
Finally, in view of (7.3), Theorem 5.3(c) and Corollary 4.3(b), we deduce that
\[ ([\Phi^{-1}\text{Op}(r(\gamma) - r(\gamma)(x))\Phi]^\pi)^\sim(\xi, x) = ([R(\gamma) - r(\gamma)(x)I]^\pi)^\sim(\xi, x) - r(\gamma)(x) - r(\gamma)(x) = 0. \]
Hence
\[ c_{\omega,\gamma}(\xi, x)[\Phi^{-1}\text{Op}(r(\gamma) - r(\gamma)(x))\Phi]^\pi \in I_{\xi,x}^\pi \subset J_{\xi,x}^\pi. \] (7.9)
Combining (7.4)–(7.5) with (7.1)–(7.9), we arrive at the relation
\[ (U_\alpha R(\gamma)^2) - e^{i\omega(\xi)x}(r(\gamma)(x))^2 I^\pi = [\Phi^{-1}\text{Op}(a)\Phi - a(\xi, x)I]^\pi \in J_{\xi,x}^\pi, \]
which completes the proof. \(\Box\)

Now we are in a position to prove that condition (ii) of Theorem 1.1 is sufficient for the invertibility of the coset \(N^\pi + J_{\xi,x}^\pi\) in the quotient algebra \(\Lambda_{\xi,x}^\pi\).

**Theorem 7.5.** Let \(1 < p < \infty\) and let \(\gamma \in \mathbb{C}\) satisfy (1.3). Suppose \(a_k, b_k\) belong to \(SO(\mathbb{R}_+)\) for all \(k \in \mathbb{Z}\), \(\alpha, \beta\) belong to \(SOS(\mathbb{R}_+)\), the operators \(A_+ \in W_{\alpha, p}^{SO}\) and \(A_- \in W_{\beta, p}^{SO}\) are given by (1.5), and the operator \(N\) is given by (1.6). If \(n(\xi, x) \neq 0\) for some \((\xi, x) \in \Delta \times \mathbb{R}\), where the function \(n\) is defined by (1.7)–(1.9), then the coset \(N^\pi + J_{\xi,x}^\pi\) is two-sided invertible in the quotient algebra \(\Lambda_{\xi,x}^\pi\).

**Proof.** We follow the main lines of the proof of [15, Theorem 8.4].

Fix \((\xi, x) \in \Delta \times \mathbb{R}\) and consider the operators
\[ H_\pm := \frac{p_\pm^\gamma(x)}{|r(\gamma)(x)|^2} R(\gamma)^2. \] (7.10)
Then it follows from Theorem 5.3(c) and Corollary 4.3(b) that
\[ (H_\pm)^\sim(\xi, x) = p_\pm^\gamma(x). \] (7.11)
Therefore, taking into account Corollary 4.3(b) once again, we get
\[ (P_\gamma^\pm - H_\pm)^\pi \in I_{\xi,x}^\pi \]
and
\[ [A_+(P_\gamma^+ - H_+) + A_-(P_\gamma^- - H_-)]^\pi \in J_{\xi,x}^\pi, \]
whence
\[ N^\pi + \mathcal{J}^\pi_{\xi,x} = (A_+H_+ + A_-H_-)^\pi + \mathcal{J}^\pi_{\xi,x}. \]  
(7.12)

We know from Theorem 4.2(d),(a) that \( H_\pm \in \mathcal{A} \). Hence, we infer from Lemma 4.5 that for all \( k \in \mathbb{Z} \),
\[ (U^k_{\alpha}H_+)^\pi = (H_+U^k_{\alpha})^\pi, \quad (U^k_{\beta}H_-)^\pi = (H_-U^k_{\beta})^\pi. \]  
(7.13)

Taking into account (7.11), it is easy to see that for all \( k \in \mathbb{Z} \),
\[ [(a_kH_+)^\pi - (a_k(\xi)H_+)^\pi] \hat{\gamma}(\xi, x) = 0, \quad [(b_kH_-)^\pi - (b_k(\xi)H_-)^\pi] \hat{\gamma}(\xi, x) = 0. \]
Hence
\[ (a_kH_+)^\pi - (a_k(\xi)H_+)^\pi, \quad (b_kH_-)^\pi - (b_k(\xi)H_-)^\pi \in \mathcal{I}^\pi_{\xi,x} \subset \mathcal{J}^\pi_{\xi,x}, \]  
(7.14)

Applying (7.13) and (7.14), we obtain for all \( k \in \mathbb{Z} \),
\[ (a_kU^k_{\alpha}H_+)^\pi - (a_k(\xi)U^k_{\alpha}H_+)^\pi = [(a_kH_+ - a_k(\xi)H_+)U^k_{\alpha}]^\pi \in \mathcal{J}^\pi_{\xi,x}, \]  
(7.15)
\[ (b_kU^k_{\beta}H_-)^\pi - (b_k(\xi)U^k_{\beta}H_-)^\pi = [(b_kH_- - b_k(\xi)H_-)U^k_{\beta}]^\pi \in \mathcal{J}^\pi_{\xi,x}. \]  
(7.16)

Then it follows from (7.12) and (7.15)–(7.16) that
\[ N^\pi + \mathcal{J}^\pi_{\xi,x} = \sum_{k \in \mathbb{Z}} (a_k(\xi)U^k_{\alpha}H_+ + b_k(\xi)U^k_{\beta}H_-)^\pi \in \mathcal{J}^\pi_{\xi,x}. \]  
(7.17)

In view of Theorem 5.3(c) and Corollary 4.3(b), it is easy to see that
\[ ([(r_\gamma(x))^{-2}R_{\gamma}^2 - \hat{I}]^\pi) \hat{\gamma}(\xi, x) = 0. \]
Hence
\[ H_\pm^\pi - p_{r_\gamma}^\pm(x)I^\pi \in \mathcal{I}^\pi_{\xi,x} \subset \mathcal{J}^\pi_{\xi,x}. \]  
(7.18)

By Lemmas 2.3, 2.6 and 7.4 we deduce for all \( k \in \mathbb{Z} \) that
\[ (U^k_{\alpha}R_{\gamma}^2)^\pi - e^{ik\omega(\xi)x}(r_\gamma(x))^2I^\pi, \quad (U^k_{\beta}R_{\gamma}^2)^\pi - e^{ik\eta(\xi)x}(r_\gamma(x))^2I^\pi \in \mathcal{J}^\pi_{\xi,x}. \]

The above inclusions together with (7.10) imply for every \( k \in \mathbb{Z} \) that
\[ (U^k_{\alpha}H_+)^\pi - e^{ik\omega(\xi)x}p^+_\gamma(x)I^\pi \in \mathcal{J}^\pi_{\xi,x}, \]  
(7.19)
\[ (U^k_{\beta}H_-)^\pi - e^{ik\eta(\xi)x}p^-_\gamma(x)I^\pi \in \mathcal{J}^\pi_{\xi,x}. \]  
(7.20)

Combining (7.18)–(7.20), we arrive at the equality
\[ N^\pi + \mathcal{J}^\pi_{\xi,x} = n(\xi, x)I^\pi + \mathcal{J}^\pi_{\xi,x}, \]
where \( n(\xi, x) \) is given by (1.8)–(1.9). If \( n(\xi, x) \neq 0 \), then one can check straightforwardly that \((1/n(\xi, x))I^\pi + \mathcal{J}^\pi_{\xi,x}\) is the inverse of the coset \( N^\pi + \mathcal{J}^\pi_{\xi,x} \) in the quotient algebra \( \Lambda^\pi_{\xi,x} \). \( \square \)
7.3. Proof of Theorem 1.1

The proof is analogous to that of [15, Theorem 1.2]. We know from Theorem 4.2(d) and Theorem 5.2 that \( N \in \Lambda \). If condition (i) of Theorem 1.1 is fulfilled, that is, if the operators \( A_+ \) and \( A_- \) are left (resp., right) invertible, then by Theorem 7.2 the coset \( N^\pi + J^\pi_{\infty} \) is left (resp., right) invertible in the quotient algebra \( \Lambda^\pi_{\infty} \) and the coset \( N^\pi + J^\pi_{\infty} \) is left (resp., right) invertible in the quotient algebra \( \Lambda^\pi_{-\infty} \). On the other hand, if condition (ii) of Theorem 1.1 holds, then in view of Theorem 7.3 the coset \( N^\pi + J^\pi_{\xi,x} \) is two-sided invertible in the quotient algebra \( \Lambda^\pi_{\xi,x} \) for every pair \( (\xi,x) \in \Delta \times \mathbb{R} \). Then, by Theorem 5.4 the operator \( N \in \Lambda \) is left (resp., right) Fredholm. \( \square \)

8. Semi-Fredholmness of weighted singular integral operators
with coefficients being binomial functional operators

8.1. Criteria for the two-sided and strict one-sided invertibility of \( aI - bU_\alpha \)

Suppose \( a, b \in SO(\mathbb{R}_+) \) and \( \alpha \in SOS(\mathbb{R}_+) \). For \( s \in \{0, \infty\} \), put

\[
L_s(s; a, b) := \liminf_{t \to s} |a(t)| - |b(t)|, \quad L_s^*(s; a, b) := \limsup_{t \to s} |a(t)| - |b(t)|.
\]

Fix a point \( \tau \in \mathbb{R}_+ \) and put

\[
\tau_{-, \alpha} := \lim_{n \to -\infty} \alpha_n(\tau), \quad \tau_{+, \alpha} := \lim_{n \to +\infty} \alpha_n(\tau).
\]

Then either \( \tau_{-, \alpha} = 0 \) and \( \tau_{+, \alpha} = \infty \), or \( \tau_{-, \alpha} = \infty \) and \( \tau_{+, \alpha} = 0 \). The points \( \tau_{+, \alpha} \) and \( \tau_{-, \alpha} \) are called attracting and repelling points of \( \alpha \), respectively.

We say that the triple \( \{\alpha, a, b\} \) satisfies conditions (I1), (I2), (LI), (RI) if

(I1) \( L_s(\tau_{-, \alpha}; a, b) > 0 \) and \( L_s^*(\tau_{+, \alpha}; a, b) > 0 \) and \( \inf_{t \in \mathbb{R}_+} |a(t)| > 0 \);

(I2) \( L_s^*(\tau_{-, \alpha}; a, b) < 0 \) and \( L_s^*(\tau_{+, \alpha}; a, b) < 0 \) and \( \inf_{t \in \mathbb{R}_+} |b(t)| > 0 \);

(LI) \( L_s^*(\tau_{-, \alpha}; a, b) < 0 < L_s(\tau_{+, \alpha}; a, b) \) and for every \( t \in \mathbb{R}_+ \) there is an integer \( k_t \) such that \( b[\alpha_k(t)] \neq 0 \) for \( k < k_t \) and \( a[\alpha_k(t)] \neq 0 \) for \( k > k_t \).

(RI) \( L_s^*(\tau_{+, \alpha}; a, b) < 0 < L_s(\tau_{-, \alpha}; a, b) \) and for every \( t \in \mathbb{R}_+ \) there is an integer \( k_t \) such that \( b[\alpha_k(t)] \neq 0 \) for \( k \geq k_t \) and \( a[\alpha_k(t)] \neq 0 \) for \( k < k_t \).

Theorem 8.1 ([15, Theorems 1.1–1.2]). Let \( a, b \in SO(\mathbb{R}_+), \alpha \in SOS(\mathbb{R}_+) \), and let the binomial functional operator \( A \) be given by

\[
A := aI - bU_\alpha.
\]

(a) The operator \( A \) is invertible on the Lebesgue space \( L^p(\mathbb{R}_+) \) if and only if the triple \( \{\alpha, a, b\} \) satisfies either condition (I1), or condition (I2).

(b) The operator \( A \) is strictly left invertible on the space \( L^p(\mathbb{R}_+) \) if and only if the triple \( \{\alpha, a, b\} \) satisfies condition (LI).

(c) The operator \( A \) is strictly right invertible on the space \( L^p(\mathbb{R}_+) \) if and only if the triple \( \{\alpha, a, b\} \) satisfies condition (RI).
8.2. Sufficient conditions for the semi-Fredholmness

Combining Theorem 1.1 and Theorem 8.1, we arrive at the following.

**Corollary 8.2.** Let $1 < p < \infty$ and let $\gamma \in \mathbb{C}$ satisfy (1.3). Suppose $a,b,c,d$ belong to $\text{SO}(\mathbb{R}_+)$, $\alpha,\beta$ belong to $\text{SOS}(\mathbb{R}_+)$, and $\omega,\eta \in \text{SO}(\mathbb{R}_+)$ are the exponent functions of the shifts $\alpha,\beta$, respectively. Consider the operator

$$
M := (aI - bU_\alpha)P^+_{\gamma} + (cI - dU_\beta)P^-_{\gamma}
$$

and the corresponding function $m$ defined for $(\xi,x) \in (\mathbb{R}_+ \cup \Delta) \times \mathbb{R}$ by

$$
m(\xi,x) := (a(\xi) - b(\xi)e^{i\omega(\xi)x})p^+_{\gamma}(x) + (c(\xi) - d(\xi)e^{i\eta(\xi)x})p^-_{\gamma}(x),
$$

where the functions $p^\pm_{\gamma}$ are defined by (1.8).

(a) If each of the triples $\{\alpha,a,b\}$ and $\{\beta,c,d\}$ satisfies either condition (I1) or condition (I2) (but not necessarily the same condition), and

$$
\inf_{x \in \mathbb{R}} |m(\xi,x)| > 0 \quad \text{for every} \quad \xi \in \Delta,
$$

then the operator $M$ is Fredholm on the space $L^p(\mathbb{R}_+)$.

(b) If each of the triples $\{\alpha,a,b\}$ and $\{\beta,c,d\}$ satisfies only one of conditions (I1), (I2), and (II) (but not necessarily the same condition), and condition (8.1) is fulfilled, then the operator $M$ is left Fredholm on the space $L^p(\mathbb{R}_+)$.

(c) If each of the triples $\{\alpha,a,b\}$ and $\{\beta,c,d\}$ satisfies only one of conditions (I1), (I2), and (II) (but not necessarily the same condition), and condition (8.1) is fulfilled, then the operator $M$ is right Fredholm on the space $L^p(\mathbb{R}_+)$.

Another (more involved) proof of Corollary 8.2(a), relying on criteria for the Fredholmness of Mellin pseudodifferential operators (see [21] and [16, Theorem 3.6]), is given in [19, Theorem 1.3]. The converse statement to Corollary 8.2(a) is proved in [20, Theorem 1.2]. The statements of parts (b) and (c) in Corollary 8.2 are new.

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