OPERATOR REPRESENTATIONS OF FUNCTION ALGEBRAS AND FUNCTIONAL CALCULUS

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Abstract. This paper deals with some operator representations $\Phi$ of a weak*-Dirichlet algebra $A$, which can be extended to the Hardy spaces $H^p(m)$, associated to $A$ and to a representing measure $m$ of $A$, for $1 \leq p \leq \infty$. A characterization for the existence of an extension $\Phi_p$ of $\Phi$ to $L^p(m)$ is given in the terms of a semispectral measure $F_\Phi$ of $\Phi$. For the case when the closure in $L^p(m)$ of the kernel in $A$ of $m$ is a simply invariant subspace, it is proved that the map $\Phi_p|H^p(m)$ can be reduced to a functional calculus, which is induced by an operator of class $C_\rho$ in the Nagy-Foiaş sense. A description of the Radon-Nikodym derivative of $F_\Phi$ is obtained, and the log-integrability of this derivative is proved. An application to the scalar case, shows that the homomorphisms of $A$ which are bounded in $L^p(m)$ norm, form the range of an embedding of the open unit disc into a Gleason part of $A$.

Keywords: weak*-Dirichlet algebra, Hardy space, operator representation, semispectral measure.

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1. INTRODUCTION AND PRELIMINARIES

Let $X$ be a compact Hausdorff space and $C(X)$ the Banach algebra of all complex continuous functions on $X$. Denote by $A$ a function algebra on $X$, that is a closed subalgebra of $C(X)$ which contains the constant functions and separates the points of $X$. $\mathcal{M}(A)$ stands for the set of all non zero complex homomorphisms (or Gelfand spectrum) of $A$. The equivalence classes of $\mathcal{M}(A)$ induced by the relation: $\gamma \sim \varphi$ iff $\|\gamma - \varphi\| < 2$ for $\gamma, \varphi \in \mathcal{M}(A)$, are the Gleason parts of $A$ (see [2,21]).

For $\gamma \in \mathcal{M}(A)$, $A_\gamma$ means the kernel of $\gamma$, and $M_\gamma$ designates the set of all representing measures $m$ for $\gamma$, that is $m$ is a probability Borel measure on $X$ satisfying
such that \( \rho > 0 \) is a scalar that \( \Phi \) is a function algebra on \( X \) which is weak*-Dirichlet in \( L^\infty(m) \), that is \( A + \overline{A} \) is weak* dense in \( L^\infty(m) \), for some fixed \( m \in M \), and \( \gamma \in M(A) \). This concept introduced in [20] is weaker than one of \( \Phi \) being a Dirichlet algebra, which means that \( A + \overline{A} \) is dense in \( C(X) \). For example, the standard algebra \( A(T) \) of all continuous functions \( f \) on the unit circle \( T \) which have analytic extensions to the open unit disc \( D \), is a Dirichlet algebra on \( T \). On the other hand, the subalgebra \( A_1(T) \) of \( A(T) \) of those functions \( f \) satisfying \( f(1) = \overline{f}(0) \) is a weak*-Dirichlet algebra in \( L^\infty(m_0) \), \( m_0 \) being the normalized Lebesgue measure on \( T \), and \( A_1(T) \) is not a Dirichlet algebra.

Let \( \mathcal{H} \) be a complex Hilbert space and \( \mathcal{B}(\mathcal{H}) \) be the Banach algebra of all bounded linear operators on \( \mathcal{H} \).

Any bounded linear and multiplicative map \( \Phi \) of \( A \) in \( \mathcal{B}(\mathcal{H}) \) with \( \Phi(1) = I \) (the identity operator on \( \mathcal{H} \)) is called a representation of \( A \) on \( \mathcal{H} \). When \( \|\Phi\| \leq 1 \) one says that \( \Phi \) is contractive. Here, we only consider a representation \( \Phi \) for which there exist a scalar \( \rho > 0 \) and a system \( \{\mu_x\}_{x \in \mathcal{H}} \) of positive measures on \( X \) with \( \|\mu_x\| = \|x\|^2 \) such that

\[
\langle \Phi(f)x, x \rangle = \int [\rho f + (1 - \rho)\gamma(f)]d\mu_x
\]

for any \( f \in A \) and \( x \in \mathcal{H} \). Such a \( \mu_x \) is called a weak \( \rho \)-spectral measure for \( \Phi \) attached to \( x \) by \( \gamma \). It is known ([8,9]) that the existence of a system of measures \( \{\mu_x\}_{x \in \mathcal{H}} \) as above, is equivalent to the fact that \( \Phi \) satisfies a weaker von Neumann inequality of the form

\[
w(\Phi(f)) \leq \|\rho f + (1 - \rho)\gamma(f)\| \quad (f \in A), \tag{1.1}
\]

where \( w(T) \) means the numeric radius of \( T \in \mathcal{B}(\mathcal{H}) \).

In [10] it was proved that if the representation \( \Phi \) of \( A \) on \( \mathcal{H} \) admits a system \( \{\mu_x\}_{x \in \mathcal{H}} \) of weak \( \rho \)-spectral measures attached by \( \gamma \) such that \( \mu_x \) is \( m - a.c. \) for any \( x \in \mathcal{H} \), then \( \Phi \) has a \( \gamma \)-spectral \( \rho \)-dilation, that is there exists a contractive representation \( \tilde{\Phi} \) of \( C(X) \) on a Hilbert space \( \mathcal{K} \supset \mathcal{H} \) satisfying the relation

\[
\Phi(f) = \rho P_{\mathcal{H}}\tilde{\Phi}(f)|_{\mathcal{H}} \quad (f \in A_\gamma), \tag{1.2}
\]

where \( P_{\mathcal{H}} \) is the orthogonal projection on \( \mathcal{H} \). Moreover, in this case there exists a unique semispectral measure \( F_\Phi : \text{Bor}(X) \to \mathcal{B}(\mathcal{H}) \) such that \( \langle F_\Phi(\cdot)x, x \rangle = \mu_x \), or equivalently

\[
\langle \Phi(f)x, y \rangle = \int [\rho f + (1 - \rho)\gamma(f)]d\langle F_\Phi x, y \rangle \quad (f \in A), \tag{1.3}
\]

for any \( x, y \in \mathcal{H} \). As usual, \( \text{Bor}(X) \) denotes the set of all Borel subsets of \( X \). Using the polarization formula, it follows that all measures \( \langle F_\Phi(\cdot)x, y \rangle \) for \( x, y \in \mathcal{H} \) are \( m - a.c. \).
The relation (1.2) means that the representation \( \tilde{\Phi} \) is a \( \gamma \)-spectral \( \rho \)-dilation of \( \Phi \), and \( F_\Phi \) is obtained as the compression to \( \mathcal{H} \) of the spectral measure of \( \tilde{\Phi} \) (see [21]).

The representations with spectral \( \rho \)-dilations was first studied by D. Gaşpar ([4–6]), and recently by T. Nakazi ([15,16]). Any such representation of the algebra \( \mathcal{A}(\mathbb{T}) \) on \( \mathcal{H} \) reduces to the usual functional calculus with the operators of class \( C_\rho \) in \( \mathcal{B}(\mathcal{H}) \) in the sense of Sz. Nagy-Foiaş [22] (i.e. \( \rho \)-contractions; [1,11]). In the general setting of a weak*-Dirichlet algebra \( \mathcal{A} \), it is natural to find conditions for a representation \( \Phi \) of \( \mathcal{A} \) on \( \mathcal{H} \) under which \( \Phi \) can be reduced to a certain functional calculus with a \( \rho \)-contraction. Recall that in [6] was given an example of a contractive representation of a Dirichlet algebra which cannot be reduced to a functional calculus with contractions.

In the sense of [5,6], the problem of reduction to a functional calculus refers to absolutely continuous representations with respect to representing measures. Thus, we only investigate here the representations \( \Phi \) which have a system of \( m \)-a.c. weak \( \rho \)-spectral measures attached by \( \gamma \). In the sequel \( H^p(m) \) stands for the (weak*, for \( p = \infty \)) closure of \( \mathcal{A} \) into \( L^p(m) \), that is the Hardy space associated to \( \mathcal{A} \) in \( L^p(m) \).

In Section 2 we characterize in terms of \( F_\Phi \) the representations \( \Phi \) which have bounded linear extensions \( \Phi_p \) to the space \( L^p(m) \) for \( 1 \leq p \leq \infty \). In Section 3 we prove the main result which says that, under some hypothesis on an invariant subspace of \( H^p(m) \) when \( 1 \leq p \leq 2 \), the map \( \Phi_p|H^p(m) \) is given by a functional calculus with a \( \rho \)-contraction with the spectrum in \( D \), the functional calculus being induced by a Hoffman type [7] naturally associated to the corresponding invariant subspace. In this case, the Radon-Nikodym derivative of \( F_\Phi \) is an essentially bounded function on \( X \) and its logarithm belongs to \( L^1(m) \). The scalar case is considered in Section 4 where we refer to the homomorphisms in \( \mathcal{M}(\mathcal{A}) \) which are bounded in the \( L^p(m) \)-norm. Our main result is a version of Wermer’s embedding theorem ([1,7,21]) for weak*-Dirichlet algebras, which prove that the set of above quoted homomorphisms corresponds to an analytic disc in the Gleason part which contains \( \gamma \).

2. EXTENSION OF A REPRESENTATION TO THE SPACE \( L^p(m) \)

We characterize below some representations \( \Phi \) of \( \mathcal{A} \) on \( \mathcal{H} \) which can be linearly and boundedly extended to the space \( L^p(m) \) for \( 1 \leq p \leq \infty \). Our characterization is given in the terms of the Radon-Nikodym derivative with respect to \( m \) of the corresponding \( \mathcal{B}(\mathcal{H}) \)-valued semispectral measure \( F_\Phi \). In the sequel we put \( \varphi_{x,y}dm = d\langle F_\Phi(\cdot)x,y \rangle \) for \( x,y \in \mathcal{H} \).

**Theorem 2.1.** Let \( \Phi \) be a representation of \( \mathcal{A} \) on \( \mathcal{H} \) which admits a system of \( m \)-a.c. weak \( \rho \)-spectral measures attached by \( \gamma \). Then \( \Phi \) has a bounded linear extension \( \Phi_p \) from \( L^p(m) \) into \( \mathcal{B}(\mathcal{H}) \) for \( 1 \leq p \leq \infty \), if and only if \( \varphi_{x,y} \in L^q(m) \) and there exists a constant \( c > 0 \) such that

\[
\|\varphi_{x,y}\|_q \leq c\|x\|\|y\| \quad (x,y \in \mathcal{H}),
\]
where $\frac{1}{2} + \frac{1}{2} = 1$. In this case, $\Phi_p$ is uniquely determined and it satisfies for $h \in L^p(m)$ and $x, y \in \mathcal{H}$ the relation

$$\langle \Phi_p(h)x, y \rangle = \int [\rho h + (1 - \rho)] \int hdm |\varphi_{x,y}dm. \tag{2.2}$$

Furthermore, for $h \in L^2(m)$ and $x \in \mathcal{H}$ we have the inequality

$$\| \Phi_2(h)x \|^2 \leq \int [\rho h + (1 - \rho)] \int hdm |\varphi_{x,y}dm. \tag{2.3}$$

Hence, if $\{h_n\} \subset L^\infty(m)$ is a bounded net such that $\{h_n\}$ converges a.e. (m) to $h \in L^\infty(m)$, then $\{\Phi_p(h_n)\}$ strongly converges to $\Phi_p(h)$ in $\mathcal{B}(\mathcal{H})$, for $p \geq 2$.

**Proof.** Suppose firstly that $\varphi_{x,y} \in L^q(m)$ and that the inequality (2.1) is satisfied. Since for $f \in A$, $g \in A_\gamma$, and $x, y \in \mathcal{H}$ we have

$$\langle (\Phi(f) + \Phi(g)^*)x, y \rangle = \int [\rho(f + g) + (1 - \rho)\gamma(f + g)] |\varphi_{x,y}dm,$$

we infer that

$$|\langle (\Phi(f) + \Phi(g)^*)x, y \rangle| \leq \rho |(f + g)| \varphi_{x,y}dm + |(1 - \rho) |(f + g)| \varphi_{x,y}dm| \leq \rho + |1 - \rho| ||f + g||p ||\varphi_{x,y}||q.$$

Since $A + \mathcal{A}_\gamma$ is weak* dense in $L^\infty(m)$, the closure of $A + \mathcal{A}_\gamma$ in $L^p(m)$ is just $L^p(m)$, for $1 \leq p < \infty$, (see [20]). Thus, the previous relations prove that for any $x, y \in \mathcal{H}$ there exists a bounded linear functional $\Phi_{x,y}$ on $L^p(m)$ satisfying for $f \in A, g \in A_\gamma, h \in L^p(m),$

$$\Phi_{x,y}(f + g) = \langle (\Phi(f) + \Phi(g)^*)x, y \rangle,$$

and

$$\Phi_{x,y}(h) = \int [\rho h + (1 - \rho)] \int hdm |\varphi_{x,y}dm.$$

Also we have $\Phi_{x,y} = \Phi_{y,x}$ and using (2.1) we obtain

$$||\Phi_{x,y}|| \leq c(\rho + |1 - \rho|) ||x|| ||y||.$$

It follows that for every $h \in L^p(m)$, the map $(x, y) \mapsto \Phi_{x,y}(h)$ is a bounded bilinear functional on $\mathcal{H} \times \mathcal{H}$, hence there exists an operator $\Phi_p(h) \in \mathcal{B}(\mathcal{H})$ such that

$$\langle \Phi_p(h)x, y \rangle = \Phi_{x,y}(h), \quad x, y \in \mathcal{H}$$

and

$$||\Phi_p(h)|| \leq c(\rho + |1 - \rho|) ||h||.$$

Then $\Phi_p : h \mapsto \Phi_p(h)$ is a bounded linear map from $L^p(m)$ into $\mathcal{B}(\mathcal{H})$, which extends $\Phi$ and also satisfies the relation (2.2). Using also (2.2) with $h = f + g$ for $f \in A, g \in A_\gamma$ one can see that $\Phi_p$ is the unique bounded linear extension of $\Phi$ to $L^p(m)$.
Now, let \( \tilde{\Phi} \) be the \( \gamma \)-spectral \( \rho \)-dilation of \( \Phi \) (from (1.2)), corresponding to the Naimark dilation (as a spectral measure) of the semispectral measure \( F_\Phi \) (see [6,21]). Then for \( f \in A, \ g \in A, \) and \( x \in \mathcal{H} \) we have \( \Phi(g) x = \rho P_\gamma \tilde{\Phi}(\gamma)x \) and

\[
\|\Phi_2(f + \gamma)x\|^2 = \|(\Phi(f) + \Phi(g)^\ast)x\|^2 = \|P_\gamma \Phi(\rho f + \gamma) + (1 - \rho)\gamma(f + \gamma)x\|^2 \leq \\
\leq \langle \tilde{\Phi}(\rho(f + \gamma) + (1 - \rho)\gamma(f + \gamma))^2 \rangle^x \leq \\
= \int |\rho(f + \gamma) + (1 - \rho)\gamma(f + \gamma)|^2 \varphi_{x,x} \ dm.
\]

Since \( A + A \) is dense in \( L^2(m) \), by the continuity of \( \Phi_2 \) one obtains from this inequality just the inequality (2.3).

Next, let \( \{h_\alpha\} \subset L^\infty(m) \), be a bounded net which converges a.e. \( (m) \) to \( h \in L^\infty(m) \). Then using (2.3) we obtain

\[
\|(\Phi_2(h_\alpha) - \Phi_2(h))x\|^2 \leq \\
\leq \int |\rho(h_\alpha - h) + (1 - \rho)\int(h_\alpha - h) dm|^2 \varphi_{x,x} \ dm \leq \\
\leq 2|\rho|^2 \int |h_\alpha - h|^2 \varphi_{x,x} \ dm + |1 - \rho|^2 \int |h_\alpha - h| dm \int \varphi_{x,x} \ dm \leq \\
\leq 2|\rho|^2 \int |h_\alpha - h|^2 \varphi_{x,x} \ dm + |1 - \rho|^2 \int |h_\alpha - h|^2 \ dm \int \varphi_{x,x} \ dm = \\
= 2 \int |h_\alpha - h|^2 (\rho^2 \varphi_{x,x} + |1 - \rho|^2 \|x\|^2) dm \rightarrow_\alpha 0.
\]

The convergence to 0 is assured by Lebesgue’s theorem, because \( \mu = \varphi_\rho^2(m) \) is a \( m - a.c. \) positive measure on \( X \), where \( \varphi_\rho^2 = \rho^2 \varphi_{x,x} + |1 - \rho|^2 \|x\|^2 \). We infer that \( \Phi_2(h_\alpha)x \rightarrow \Phi_2(h)x \) in \( \mathcal{H} \) for any \( x \in \mathcal{H} \), and since \( \Phi_p = \Phi_2 \|L^p(m) \) we have that \( \{\Phi_p(h_\alpha)\} \) strongly converges to \( \Phi_p(h) \) in \( \mathcal{B(H)} \), for \( p \geq 2 \) (including and the case \( p = \infty \) because \( \Phi_\infty = \Phi_2 \|L^\infty(m) \) for \( p < \infty \).

For the converse statement, we suppose now that \( \Phi \) admits a bounded linear extension \( \Psi \) to \( L^p(m) \) with \( 1 \leq p < \infty \). For \( x, y \in \mathcal{H} \) the functional \( \langle \Psi(x), y \rangle \) is bounded linear on \( L^p(m) \), so there exists \( \psi_{x,y} \in L^p(m) \) such that

\[
\langle \Psi(h), x \rangle = \int h\psi_{x,y} dm \quad (h \in L^p(m)).
\]

Since \( \Psi|A = \Phi \) we have for \( f \in A \) and \( g \in A, \)

\[
\int (f + \gamma) \psi_{x,y} dm = \langle \Psi(f + \gamma)x, y \rangle = \langle (\Phi(f) + \Phi(g)^\ast)x, y \rangle = \\
= \int |\rho(f + \gamma) + (1 - \rho)\gamma(f + \gamma)| \varphi_{x,y} dm = \\
= \int (f + \gamma)(\rho \varphi_{x,y} + (1 - \rho)\langle x, y \rangle) dm.
\]
Using the weak* density of $A + A^\gamma$ in $L^\infty(m)$ we obtain

$$\int h\psi_{x,y}dm = \int h(\rho\varphi_{x,y} + (1 - \rho)(x,y))dm$$

for any $h \in L^\infty(m)$, hence $\psi_{x,y} = \rho\varphi_{x,y} + (1 - \rho)(x,y)$. This implies $\varphi_{x,y} \in L^q(m)$ and also

$$\|\varphi_{x,y}\|_q = \frac{1}{\rho}\|\psi_{x,y} + (\rho - 1)(x,y)\|_q \leq \left(\frac{1}{\rho}\|\Psi\| + |1 - \frac{1}{\rho}|\right)\|x\||y||,$$

for any $x, y \in H$. Thus, $\varphi_{x,y}$ satisfies (2.1) and this proves the converse statement when $p < \infty$. If $p = \infty$ that is we assume that $\Phi$ has a bounded linear extension $\Psi$ to $L^\infty(m)$, then clearly we have

$$\langle \Psi(h)x, y \rangle = \int (\rho h + (1 - \rho)\int hdm)\varphi_{x,y}dm$$

for all $h \in L^\infty(m)$ and $x, y \in H$. Since $\varphi_{x,y} \in L^1(m)$ we get

$$\|\varphi_{x,y}\|_1 = \sup_{g \in L^\infty(m), \|g\|\leq 1} \left|\int g\varphi_{x,y}dm\right| = \sup_{g \in L^\infty(m), \|g\|\leq 1} \left|\langle \Psi\left(\frac{1}{\rho} + \left(1 - \frac{1}{\rho}\right)\int hdm\right)x, y \rangle\right| \leq \|\Psi\|\left(\frac{1}{\rho} + |1 - \frac{1}{\rho}|\right)\|x\||y||,$$

and so $\varphi_{x,y}$ also satisfies (2.1) when $p = \infty$. This ends the proof. \hfill \Box

**Remark 2.2.** The equivalent conditions of Theorem 2.1 imply

$$\|\Phi\|_p := \sup_{f \in A, \|f\|_p \leq \frac{1}{\rho}} \|\Phi(f)\| < \infty.$$  \hspace{1cm} (2.4)

It is easy to see that the condition (2.4) is equivalent to the existence of a bounded linear extension $\hat{\Phi}_p$ of $\Phi$ to $H^p(m)$. In this case, $\hat{\Phi}_p$ is uniquely determined and it satisfies the relation (2.2) for $g \in H^p(m)$. In addition, the following property holds.

**Proposition 2.3.** Let $\Phi$ be a representation of $A$ on $H$ as in Theorem 2.1 such that $\|\Phi\|_p < \infty$. Then

$$\hat{\Phi}_p(fg) = \hat{\Phi}_p(f)\hat{\Phi}_p(g) \quad (f \in H^\infty(m), \ g \in H^p(m))$$  \hspace{1cm} (2.5)

and, in particular, $\hat{\Phi} := \hat{\Phi}_p|H^\infty(m)$ is a representation of $H^\infty(m)$ on $H$. Moreover, if $\{f_n\} \subset H^\infty(m)$ is a bounded net which converges a.e. $(m)$ to $f \in H^\infty(m)$, then $\{\hat{\Phi}(f_n)\}$ strongly converges to $\hat{\Phi}(f)$ in $B(H)$. 

Proof. Let \( g \in H^p(m) \) and \( f, g_n \in A \) such that \( g_n \to g \) in \( L^p(m) \). Then \( fg_n \to fg \) in \( L^p(m) \), so

\[
\hat{\Phi}_p(fg) = \lim_n \Phi(fg_n) = \Phi(f)\hat{\Phi}_p(g).
\]

Now, if \( f \in H^\infty \) and \( \{f_n\} \subset A \) is a net which converges to \( f \) in the weak* topology of \( L^\infty(m) \) then for \( g, g_n \) as above and \( x, y \in \mathcal{H} \) one has

\[
\langle \hat{\Phi}_p(fg)x, y \rangle = \int (\rho fg + (1 - \rho) \int fgdm)\varphi_{x,y}dm = \\
= \lim_n \int (\rho f_n + (1 - \rho) \int f_n g_n dm)\varphi_{x,y}dm = \\
= \lim_n \int \Phi(f_n g_n)\varphi_{x,y}dm = \\
= \lim_n \int (\rho f_n + (1 - \rho) \int f_n g_n dm)\varphi_{x,y}dm = \\
= \lim_n \int (\rho f + (1 - \rho) \int fdm)\varphi_{x,y}dm = \\
= \lim \langle \hat{\Phi}_p(f)\Phi(g_n)x, y \rangle = \langle \hat{\Phi}_p(f)\hat{\Phi}_p(g)x, y \rangle.
\]

So, property (2.5) is proved. This also gives that \( \hat{\Phi}_p \) is multiplicative on \( H^\infty(m) \), therefore \( \hat{\Phi} := \hat{\Phi}_p|H^\infty(m) \) is a representation of \( H^\infty(m) \) on \( \mathcal{H} \).

The second statement of the proposition can be inferred as in the previous proof.

\[\square\]

**Remark 2.4.** If the representation \( \Phi \) in Theorem 2.1 is contractive, that is \( \rho = 1 \) and \( \|\Phi\| = 1 \) (because \( \Phi(1) = I \)), then its extension \( \hat{\Phi}_p \) is also contractive, in the case when it exists. Indeed, if \( \hat{\Phi} \) is as in the proof of Theorem 2.1, we have for \( f \in A, \ g \in A_\gamma \) and \( x, y \in \mathcal{H} \),

\[
\left| \int (f + \overline{g})\varphi_{x,y}dm \right| = \|\Phi(f) + \Phi(g^*)\varphi_{x,y}\| \leq \|\Phi(f + \overline{g})\varphi_{x,y}\| \leq \|f + \overline{g}\|\|x\|\|y\|,
\]

because \( \hat{\Phi} \) is a contractive representation of \( C(X) \). From this inequality we infer by the density of \( A + \overline{A} \) in \( L^p(m) \) that

\[
\left| \int h\varphi_{x,y}dm \right| \leq \|h\|\|x\|\|y\| \quad (h \in L^p(m)),
\]

hence \( \|\varphi_{x,y}\| \leq \|x\|\|y\| \). Thus, we can take \( c = 1 \) in (2.1) and from the proof of Theorem 2.1 we deduce (the case \( \rho = 1 \)) that \( \|\Phi_p\| \leq 1 \), and finally \( \|\Phi_p\| = 1 \) because \( \Phi_p(1) = I \).
In the sequel we denote by $H^p_0(m)$ the closure (weak*, if $p = \infty$) of $A_\gamma$ in $L^p(m)$, that is

$$H^p_0(m) = \left\{ f \in H^p(m) : \int f \, dm = 0 \right\}.$$  

We say ([17, 20, 21]) that $H^p_0(m)$ is simply invariant if the closure of $A_\gamma H^p_0(m)$ in $L^p(m)$ is strictly contained in $H^p_0(m)$. By Theorem 4.1.6 [20] (see also [17, 21]) if $H^p_0(m)$ is simply invariant then there exists a function $Z \in H^\infty_0(m)$ with $|Z| = 1$ a.e. $(m)$ such that $H^p_0(m) = ZH^p(m)$.

As in Theorem 3 [14] one can prove that, if $m_0$ is the normalized Lebesgue measure on $\mathbb{T}$, there exists an isometric $*$-isomorphism $\tau$ of $L^p(m_0)$ onto a closed subspace of $L^p(m)$, taking $H^p(m_0)$ onto a closed subspace of $H^p(m)$, for $1 \leq p \leq \infty$. In fact, $\tau$ is defined by

$$(\tau h)(s) = h(Z(s))$$

for $h \in L^p(m_0)$ and a.e. $(m)$ $s \in X$.

The following main result shows that under the simple invariance of $H^p_0(m)$ with $1 \leq p \leq 2$, the representations from Theorem 2.1 and their extensions to $H^p(m)$ can be reduced to functional calculus. For this we need to define the operator $S : H^p(m) \rightarrow L^p(m)$ by

$$Sg = Z(g - \int g \, dm) \quad (g \in H^p(m)).$$

(3.1)

Also, for $T \in \mathcal{B}(\mathcal{H})$ we denote by $r(T)$ the spectral radius of $T$.

**Theorem 3.1.** Suppose that $H^p_0(m)$ is a simply invariant subspace for $1 \leq p < \infty$, and let $\Phi$ be a representation of $A$ on $\mathcal{H}$ satisfying Theorem 2.1. Then $r(\Phi(Z)) < 1$, and if $1 \leq p \leq 2$ one has

$$\hat{\Phi}_p(g) = \sum_{n=0}^{\infty} \hat{g}(n)\hat{\Phi}(Z)^n \quad (g \in H^p(m)),$$

(3.2)

where $\hat{g}(n) = \int \mathcal{Z}^n g \, dm$ for $n \in \mathbb{N}$, the series being absolutely convergent in $\mathcal{B}(\mathcal{H})$. Moreover, the relation (3.2) is also true when $2 < p < \infty$, for $g \in H^p(m)$ such that $\{S^ng\}$ is a bounded sequence in $H^p(m)$, $S$ being the operator from (3.1).

**Proof.** The assumption on $\Phi$ means that $\varphi_{x,y}$ satisfies (2.1) for any $x, y \in \mathcal{H}$. As a bounded linear functional on $L^p(m)$, $\varphi_{x,y}$ induces, by the isomorphism $\tau$, a bounded linear functional on $L^p(m_0)$, that is there exists $\varphi_{x,y}^0 \in L^q(m_0)$ satisfying

$$\int h \varphi_{x,y}^0 dm_0 = \int (\tau h) \varphi_{x,y} dm \quad (h \in L^p(m_0)).$$

(3.3)

Since $\tau$ is an isometry we find

$$\|\varphi_{x,y}\|_q = \sup_{\|h\|_p = 1} \left| \int h \varphi_{x,y} \, dm_0 \right| = \sup_{\|h\|_p = 1} \left| \int (\tau h) \varphi_{x,y} \, dm \right| \leq \|\varphi_{x,y}\|_q \leq c\|x\|\|y\|,$$

with $c$ as in (2.1).
Now from (3.3) and (2.2) we infer, for any analytic polynomial $P$, that
\[
\int [\rho P + (1 - \rho)P(0)]\varphi_{x,g}^0 dm_0 = \int [\rho(P \circ Z) + (1 - \rho)P(0)]\varphi_{x,g} dm = 
= \langle \Phi_p(P \circ Z)x, y \rangle = \langle P(\Phi_p(Z))x, y \rangle.
\]
So, using the previous inequality we get
\[
|\langle P(\Phi_p(Z))x, y \rangle| \leq \|\rho P + (1 - \rho)P(0)\|_p\|\varphi_{x,g}^0\|_q \leq c(\rho + |1 - \rho|)\|P\|_p|x||y|,
\]
and putting $c_p = c(\rho + |1 - \rho|)$ one obtains
\[
\|P(\Phi_p(Z))\| \leq c_p\|P\|_p.
\]
This means that the operator $\Phi_p(Z)$ is polynomially bounded. On the other hand, taking $P(\lambda) = \lambda^n$ for $n \in \mathbb{N}$ in the above equality, we obtain
\[
\langle \Phi_p(Z)^n x, y \rangle = \rho \int \lambda^n\varphi_{x,g}^0 dm_0
\]
and so it follows that for $x, y \in H$ there exists $\psi_{x,g} \in L^q(m_0)$ such that
\[
\langle \Phi_p(Z)^n x, y \rangle = \int \lambda^n\psi_{x,g} dm_0 \quad (n \in \mathbb{N}).
\]
This yields that the operator $\Phi_p(Z)^*$ is absolutely continuous, and since $\psi_{x,g} \in L^q(m_0)$ with $q > 1$ (by the choose of $p$), from Lebow’s theorem [13] we infer that $r(\Phi_p(Z))^* < 1$.

The assumption that $H_0^q(m) = ZH^p(m)$ assures that the range of operator $S$ from (3.1) is contained in $H^p(m)$, so $S \in B(H^p(m))$. In addition, for $g \in H^p(m)$ we have
\[
\int Sg dm = \int Zg dm = \hat{g}(1),
\]
therefore $S^2 g = Z(Sg - \hat{g}(1))$, or $Sg = \hat{g}(1) + Z(S^2 g)$. This also gives
\[
g = \int g dm + Z(Sg) = \hat{g}(0) + \hat{g}(1)Z + Z^2(Sg).
\]
Assume now that $g = \sum_{j=0}^{n-1} \hat{g}(j)Z^j + Z^nS^n g$ for $n > 1$. Then
\[
S^n g = Z^n g - \sum_{j=0}^{n-1} \hat{g}(j)Z^{n-j},
\]
whence we get $\int S^n g dm = \hat{g}(n)$. So, we have $S^{n+1} g = Z(S^n g - \hat{g}(n))$, or $S^n g = \hat{g}(n) + Z(S^{n+1} g)$, and by our assumption on $g$ we obtain
\[
g = \sum_{j=0}^{n} \hat{g}(j)Z^j + Z^{n+1}(S^{n+1} g) \quad (g \in H^p(m)).
\]

(3.4)
Considering the extension \( \hat{\Phi}_p = \Phi_p|H^p(m) \) of \( \Phi \) to \( H^p(m) \) (as in Proposition 2.3) we get by (3.4) that
\[
\|\hat{\Phi}_p(g) - \sum_{j=0}^{n} \hat{g}(j)\hat{\Phi}(Z^j)\| = \|\hat{\Phi}(Z^{n+1})\hat{\Phi}_p(S^{n+1}g)\| \leq \|\hat{\Phi}_p\|\|S^{n+1}g\|\|\hat{\Phi}(Z)^{n+1}\|, \tag{3.5}
\]
for any \( g \in H^p(m) \).

If \( p = 2 \), the operator \( S \) is a contraction on \( H^2(m) \) that is
\[
\|Sg\| = \|g - \int gdm\| \leq \|g\|,
\]
because \( g - \int gdm \) is the orthogonal projection of \( g \) on \( H^2_0(m) \) for \( g \in H^2(m) \). In this case, in (3.5) we have \( \|S^{n+1}g\| \leq \|g\| \) for any \( n \in \mathbb{N} \), and since \( \hat{\Phi}(Z)^{n} \to 0 \) \( (n \to \infty) \) by a remark before, it follows that the representation (3.2) holds true for \( g \in H^2(m) \).

Suppose now \( 1 \leq p < 2 \). As \( H^2(m) \) is dense in \( H^p(m) \), for \( g \in H^p(m) \) and every \( \varepsilon > 0 \) there exists \( g_\varepsilon \in H^2(m) \) with \( \|g - g_\varepsilon\|_p < \varepsilon \). Since \( \|\hat{g}(n)\| \leq \|g\|_p \) for \( n \in \mathbb{N} \), the series from (3.2) is absolutely convergent in \( \mathcal{B}(\mathcal{H}) \) and applying the previous remark to \( g_\varepsilon \) we obtain
\[
\|\sum_{n=0}^{\infty} \hat{\Phi}_p(g)\| = \|\sum_{n=0}^{\infty} (\hat{g}(n) - \hat{g}_\varepsilon(n))\hat{\Phi}(Z)^n\| + \|\Phi_\varepsilon(g_\varepsilon - g)\| \leq \|g - g_\varepsilon\|_p \|\hat{\Phi}_p\| + \sum_{n=0}^{\infty} \|\hat{\Phi}(Z)^n\| < \varepsilon M
\]
for some constant \( M > 0 \). Thus, the representations (3.2) occurs for any \( g \in H^p(m) \), if \( p \leq 2 \). When \( p > 2 \), from the inequality (3.5) we infer that the equality (3.2) is also true for \( g \in H^p(m) \) for which \( \{S^n g\} \) is a bounded sequence in \( H^p(m) \). The proof is finished. \( \square \)

**Remark 3.2.** By (3.4) we have that the sequence \( \{S^n g\}_n \) is bounded if and only if the sequence \( \{\sum_{j=0}^{n} \hat{g}(j)Z^j\}_n \) is bounded in \( H^p(m) \), and in particular, this happens if \( S \) is a power bounded operator in \( \mathcal{B}(H^p(m)) \). But, even if the second sequence before converges, its limit is not necessary the function \( g \). In fact, one has (by (3.4)) \( g = \sum_{j=0}^{\infty} \hat{g}(j)Z^j \) in \( H^p(m) \) if and only if \( S^n g \to 0 \) \( (n \to \infty) \); but this condition is false, in general, as we can see in the following

**Example 3.3.** Let \( A \) be the algebra of all continuous functions \( f \) on \( \mathbb{T}^2 \) having the Fourier coefficients
\[
c_{ij} = \int_{\mathbb{T}^2} \lambda^i \eta^j f(\lambda, \eta) d\mu_2 \quad (i, j \in \mathbb{Z})
\]
such that $c_{ij} = 0$ if either $j < 0$, or $j = 0$ and $i < 0$. Then $A$ is a Dirichlet algebra on $\mathbb{T}^2$, while the normalized Lebesgue measure $m_2$ on $\mathbb{T}^2$ is the representing measure for the homomorphism of evaluation in $(0, 0)$ of $A$. Here the function $Z \in H^\infty(m_2)$ is given by $Z(\lambda, w) = \lambda, \lambda, w \in \mathbb{T}$. On the other hand, for the function $g_0 \in H^\infty(m_2)$ defined by $g_0(\lambda, w) = w$, we have $(S^n g_0)(\lambda, w) = \lambda^nw$ and $\|S^n g_0\|_p = 1$, for any $n \in \mathbb{N}, \lambda, w \in \mathbb{T}$. Hence $\{S^n g_0\}$ is a bounded sequence which is not convergent to $0$, in $H^p(m_2)$ for $1 \leq p \leq \infty$. Clearly, $\hat{g}_0(n) = 0$ for any $n \geq 0$, therefore $\sum_{j=0}^n \hat{g}_0(j) \lambda^j = 0$ for $n \geq 0$, what justifies the last assertion of Remark 3.2.

This example also provides that, in general under the hypothesis of Theorem 3.1, the space $H^p(m)$ is not spanned by $\{Z^n\}_{n \in \mathbb{N}}$, even if the operator $S$ is power bounded. For instance, $S$ is always a contraction on $H^2(m)$, but $\{Z^n\}_{n \in \mathbb{N}}$ becomes an orthonormal basis in $H^2(m)$ if and only if $H^\infty(m)$ is a maximal weak* closed algebra in $L^\infty(m)$, when $m$ is the unique representing measure for $\gamma$, while $\{\gamma\}$ is not a Gleason part of $A$ (see [1,6]).

If $H^p(m)$ is spanned by $\{Z^n\}_{n \in \mathbb{N}}$ then for any $g \in H^p(m)$ the representation (3.2) holds (by Remark 3.2), which means that the map $\Phi_p$ is reduced to a functional calculus. Theorem 3.1 shows that this fact occurs for $\Phi$ satisfying (2.1) for $2 \leq q \leq \infty$, but we cannot prove (3.2) in the case $2 < p \leq \infty$ (when $1 \leq q < 2$), the boundedness condition (2.1) for $\varphi_{x,y}$, being weakened in this case.

We see now that, from the point of view of the semispectral measure $F_\Phi$, the cases when $p$ belongs to the range $1 \leq p \leq 2$ are not essentially different, in Theorem 3.1.

**Theorem 3.4.** Suppose $1 \leq p \leq 2$ and that $H^p_\Phi(m)$ is a simply invariant subspace in $H^p(m)$. Let $\Phi$ be a representation of $A$ on $\mathcal{H}$ satisfying Theorem 2.1. Then the semispectral measure $F_\Phi$ has the form $F_\Phi = \theta(\cdot)m$ where the function $\theta : X \to \mathcal{B}(\mathcal{H})$ is given by

$$\theta(s) = \sum_{n=-\infty}^{\infty} Z^n(s) \hat{\Phi}(Z)^{(n)},$$

while the series converges absolutely and uniformly a.e. (m) for $s \in X$. Moreover, $\theta$ is a bounded function a.e. (m) on $X$.

**Proof.** Since $r(\hat{\Phi}(Z)) < 1$ (by Theorem 3.1) one can define the function

$$\theta_+(s) = \sum_{n=0}^{\infty} Z^n(s) \hat{\Phi}(Z)^n,$$

the series being absolutely and uniformly convergent a.e. (m) for $s \in X$. In addition, one has

$$\|\theta_+(s)\| \leq \sum_{n=0}^{\infty} \|\hat{\Phi}(Z)^n\| \quad \text{(a.e. (m) s \in X)}.$$
Then for \( g \in H^p(m) \) and \( x \in \mathcal{H} \) the function \( g(\theta_+ \cdot x, x) \) belongs to \( L^p(m) \), and we have (by (3.2) and (2.2)),

\[
\int g(\theta_+ \cdot x, x) dm = \sum_{n=0}^{\infty} \hat{g}(n) (\hat{\Phi}(Z)^n x, x) = (\sum_{n=0}^{\infty} \hat{g}(n) \hat{\Phi}(Z)^n x, x) = (\Phi_{\rho}(g)x, x) = \int (pg + (1 - \rho)) \int g dm) \varphi_{x,x} dm.
\]

Equivalently, taking \( \frac{1}{\rho}g + (1 - \frac{1}{\rho}) \int g dm \) instead of \( g \) in this relation, we obtain

\[
\int g \varphi_{x,x} dm = \int \left[ \frac{1}{\rho}g(s) + \left( 1 - \frac{1}{\rho} \right) \int g dm \right] (\theta_+(s)x, x) dm = \int g(s) \left[ \frac{1}{\rho}(\theta_+(s)x, x) + \left( 1 - \frac{1}{\rho} \right) \int (\theta_+(s)x, x) dm \right] dm = \int g(s) \left[ \frac{1}{\rho}(\theta_+(s)x, x) + \left( 1 - \frac{1}{\rho} \right) ||x||^2 \right] dm = \left( 1 - \frac{1}{\rho} \right) ||x||^2 + \frac{1}{\rho} \int g \varphi_{x,x} dm,
\]

where the function \( \theta \) is defined as in (3.6), that is

\[
\theta(s) = \theta_+(s) + \theta_+(s)^* - I \quad (\text{a.e. } (m) \ s \in X).
\]

Clearly, we used before that \( \int gZ^n dm = 0 \) for \( n > 0 \).

Since \( \varphi_{x,x} \) and \( (\theta(\cdot)x, x) \) are real functions, we get that

\[
\int (f + \overline{g}) \varphi_{x,x} dm = \int (f + \overline{g})(\theta(\cdot)x, x) dm
\]

for \( f \in A, \ g \in A_\gamma \), and this gives \( \varphi_{x,x} = (\theta(\cdot)x, x) \) because \( A \) is weak* Dirichlet in \( L^\infty(m) \). Hence \( \theta \) is the Radon-Nikodym derivative of \( F_\Phi \) with respect to \( m \), and \( \theta \) is bounded a.e. \( (m) \) on \( X \), in fact

\[
||\theta(s)|| \leq 1 + \frac{2}{\rho} \sum_{n=0}^{\infty} ||\hat{\Phi}(Z)^n|| \quad (\text{a.e. } (m) \ s \in X).
\]

This ends the proof. \( \square \)

From this theorem it follows that, for \( \Phi \) as in Theorem 2.1, the \( L^q(m) \)-boundedness of \( \varphi_{x,y} \) in the sense of (2.1) for any \( x, y \in \mathcal{H} \) and some \( q \) in the range \( 2 \leq q \leq \infty \), is equivalent to the fact that the Radon-Nikodym derivative of \( F_\Phi \) is a bounded function a.e. \( (m) \) on \( X \), if \( H^p_\Phi(m) \) is simply invariant. In this last case, \( \Phi \) can be extended
to whole $L^1(m)$ as in Theorem 2.1 and one has $\Phi_p = \Phi_1|L^p(m)$ for $1 < p \leq \infty$. Moreover, if $1 \leq p \leq r \leq \infty$ then $\hat{\Phi}_r = \hat{\Phi}_p|H^r(m)$. Hence we infer from Theorem 3.1 the following

**Corollary 3.5.** Suppose that for some $p \in [1, 2]$ the subspace $H^p(m)$ is simply invariant, and let $\Phi$ be a representations of $A$ on $H$ satisfying Theorem 2.1. Then the relation (3.2) holds for $\hat{\Phi}_r$ and any $g \in H^r(m)$ with $p \leq r \leq \infty$.

Notice that the above results extend some facts from [12] where only the case $p = 2$ was considered. Remark also that the assertion $r(\hat{\Phi}(Z)) < 1$ in the corresponding version in [12] of Theorem 3.1 before was obtained in a different way, adapting an argument of M. Schreiber [19].

In turn the Theorem 3.4 shows that the semispectral measure $F_\Phi$ can be described by the operator $\hat{\Phi}(Z)$. Conversely, $\hat{\Phi}(Z)$ can be retrieved from $F_\Phi$ as follows.

**Proposition 3.6.** Suppose that $H^p(m)$ is a simple invariant subspace for some $p \in [1, 2]$, and let $\Phi$ be a representation of $A$ on $H$ satisfying Theorem 2.1. Then $\hat{\Phi}(Z)$ is a $\rho$-contraction on $H$ and we have

$$\hat{\Phi}(Z)^{(n)} = \int Z^n(s)\theta(s)dm \quad (n \in \mathbb{Z}),$$

(3.7)

where $\theta$ is function defined in (3.6).

Moreover, if there exists $s_0 \in X$ and $\lambda \in \mathbb{C}$ such that $\theta(s_0) = \lambda I$ then $\hat{\Phi}(Z)$ is a normal strict contraction.

**Proof.** The relation (3.7) follows immediately because we may integrate the series of $\theta$ term by term (by uniform convergence in norm), having in view that $\int Zdm = 0$. From (3.7) we infer for any analytic polynomial $P$ and $x \in H$ that

$$(P(\hat{\Phi}(Z))x, x) = \int [\rho(P \circ Z)(s) + (1 - \rho)P(0)]\theta(s)x, x)dm = \int [\rho(P \circ Z)(s) + (1 - \rho)P(0)]\varphi_{x,x}(s)dm,$$

the last equality being ensured by Theorem 3.4. So, we obtain

$$|\langle P(\hat{\Phi}(Z))x, x \rangle| \leq \sup_{|\lambda|=1} |\rho P(\lambda) + (1 - \rho)P(0)| \int \varphi_{x,x}dm = \|\rho P + (1 - \rho)P(0)\|\|x\|^2,$$

whence

$$\sup_{\|x\|=1} |\langle P(\hat{\Phi}(Z))x, x \rangle| \leq \|\rho P + (1 - \rho)P(0)\|.$$

This last inequality just means that $\hat{\Phi}(Z)$ is a $\rho$-contraction on $H$ (see [1, 4, 6, 22]).

Suppose now that there exists $s_0 \in X$ and $\lambda \in \mathbb{C}$ such that $\theta(s_0) = \lambda I$. We write $\theta(s_0) = I + T + T^*$ where

$$T = \frac{1}{\rho} \sum_{n=1}^{\infty} Z^n \hat{\Phi}(Z)^n.$$
Then our assumption yields $TT^* = (\lambda - 1)T - T^2 = T^*T$, hence $T$ is a normal operator. Since one has

$$\rho T = [I - Z(s_0)\hat{\Phi}(Z)]^{-1} - I,$$

we get

$$\hat{\Phi}(Z) = Z(s_0)[I - (I + \rho T)^{-1}],$$

therefore $\hat{\Phi}(Z)$ is a normal operator. This also gives $\|\hat{\Phi}(Z)\| = r(\hat{\Phi}(Z)) < 1$, that is $\hat{\Phi}(Z)$ is a strict contraction. This ends the proof.

The converse statement fails for the second assertion of Proposition 3.6, even in the case $\rho = 1$, and this fact was proved in [19, p.189], concerning the contractive representations of the disc algebra.

Theorem 3.4 can be also completed as follows.

**Theorem 3.7.** Suppose that $H^p_0(m)$ is a simply invariant subspace for some $p \in [1, 2]$ and that $H^\infty(m)$ coincides to the weak* closure of the system $\{Z^n\}_{n \in \mathbb{N}}$. Let $\Phi$ be a representation of $A$ on $\mathcal{H}$ satisfying Theorem 2.1. Then the semispectral measure $F_\theta$ is mutually absolutely continuous with respect to $m$, and for every $x \in \mathcal{H}$, $x \neq 0$, the function $\log(\theta(\cdot)x,x)$ belongs to $L^1(m)$, where $\theta$ is defined in (3.6).

**Proof.** Since $F_\theta$ is absolutely continuous with respect to $m$, it remains to prove the converse assertion.

Notice firstly that for $g \in H^\infty(m)$ one has $g = \sum_{n=0}^{\infty} \hat{g}(n)Z^n$, and that $\{Z^n\}_{n \in \mathbb{N}}$ forms an orthogonal basis in $H^2(m)$. Since $L^2(m) = H^2(m) \oplus 
\hat{H}^2(m)$ (the bar meaning the complex conjugate), the isomorphism $\tau$ applies $L^2(m_0)$ onto $L^2(m)$, and $L^\infty(m_0)$ onto $L^\infty(m)$ too.

Let $\sigma \in \text{Bor}(X)$ and $0 \neq x \in \mathcal{H}$ such that $\langle F_\theta(\sigma)x,x \rangle = 0$. By (3.3) we have ($\chi_\sigma$ being the characteristic function of $\sigma$)

$$\int (\tau^{-1}\chi_\sigma)(\tau^{-1}\theta(\cdot)x,x)dm_0 = \int \chi_\sigma(\theta(\cdot)x,x)dm = \langle F_\theta(\sigma)x,x \rangle = 0.$$

Since one has

$$(\tau^{-1}\theta(\cdot)x,x)(\lambda) = \sum_{n=-\infty}^{\infty} \chi^n\hat{\Phi}(Z)_{\rho}^{(n)}(|\lambda| = 1),$$

this function is just the Radon-Nikodym derivative of the semispectral measure $\hat{\Phi}$ of $\hat{\Phi}(Z)$ with respect to $m_0$ ($\hat{\Phi}(Z)$ being a uniformly stable $p$-contraction, by Theorem 3.1 and Proposition 3.6). So, we have $\int (\tau^{-1}\chi_{\sigma})d(\hat{\Phi}x,x) = 0$, and since the measures $m_0$ and $\langle \hat{\Phi}x,x \rangle$ are equivalent (see [18]), while $\tau^{-1}\chi_{\sigma}$ is a positive function ($(\tau^{-1}\chi_{\sigma})^2 = \tau^{-1}\chi_{\sigma}^2 = \tau^{-1}\chi_{\sigma} \geq 0$), it follows $\int (\tau^{-1}\chi_{\sigma})dm_0 = 0$. Then we obtain

$$m(\sigma) = \int \chi_\sigma dm = \int (\tau^{-1}\chi_{\sigma})dm_0 = 0,$$

hence the measures $m$ and $\langle F_\theta x,x \rangle$ are equivalent.
Now, by (3.3) we also have for $g \in H_0^\infty(m)$,
\[
\int |1 - g(s)|^p \langle \theta(s)x, x \rangle dm = \int |1 - (\tau^{-1}g)(s)|^p d(\hat{F}x, x).
\]

But $\tau^{-1}H_0^\infty(m) = H_0^\infty(m_0)$, and so taking the infimum for $g \in H_0^\infty(m)$ in the previous equality we obtain by Szegö’s Theorem 4.2.2 [20] that
\[
\exp \int \log \langle \theta(s)x, x \rangle dm = \exp \int \log \tau^{-1}\langle \theta(-)x, x \rangle dm_0.
\]

Since the $\rho$-contraction $\hat{\Phi}(Z)$ is completely non unitary, the right side of this equality cannot be 0 (by Theorem 3.8 [18]), hence $\log \langle \theta(-)x, x \rangle \in L^1(m)$. The proof is finished.

Note that the hypothesis on $H^\infty(m)$ in Theorem 3.7 is not verified for the algebra $A$ in Example 3.3., as was proved in [6]. In the case that $H^\infty(m)$ is the weak* closure of $\{Z^n\}_{n \in \mathbb{N}}$, then for any $g \in H^\infty(m)$ we have $g = \sum_{n=0}^{\infty} \hat{g}(n)Z^n$ in $H^2(m)$. In this case, for every $\Phi$ as above, $\hat{\Phi}(g) = \hat{\Phi}_2(g)$ is given by (3.2), and it is easy to see that this means that the representations $\hat{\Phi}$ of $H^\infty(m)$ on $\mathcal{H}$ is reduced to a functional calculus in the sense of Gašpar [4,6]. Finally, let us note that the case $\rho = 1$ of Theorem 3.7 is contained in Theorem 2.3.2 [6].

4. APPLICATION TO THE SCALAR CASE

In this section we consider the case when $\Phi$ is a homomorphism of $A$, this is the one-dimensional case $\mathcal{H} = \mathbb{C}$. In this context, we generalize to a weak* Dirichlet algebra some classical results concerning the function algebra with the uniqueness property for representing measures ([2,7,21]).

**Theorem 4.1.** Suppose that $H_0^p(m)$ is a simply invariant subspace for some $p \in [1,2]$. Then for any homomorphism $\varphi \in \mathcal{M}(A)$ with $\|\varphi\|_p < \infty$ we have $|\hat{\varphi}(Z)| < 1$ and
\[
\varphi_p(g) = \sum_{n=0}^{\infty} \hat{g}(n)\hat{\varphi}(Z)^n \quad (g \in H^p(m)),
\]

where $\varphi_p$ respectively ($\hat{\varphi}$) is the bounded linear extension of $\varphi$ to $H^p(m)$ (respectively, to $H^\infty(m)$), the series being absolutely convergent.

Moreover, the measure
\[
\mu = \frac{1 - |\varphi(Z)|^2}{|Z - \varphi(Z)|^2}m
\]

is a representing measure for $\varphi$. 
Proof. Let \( \varphi \in A\mathcal{M}(A) \) with \( \|\varphi\|_p = \sup\{|\varphi(f)| : f \in A, \|f\|_p \leq 1\} < \infty \). Assume, by contrary, that \( |\hat{\varphi}(Z)| = 1 \). Since \( Z \) is uniquely determined by a scalar \( \lambda \) with \( |\lambda| = 1 \), one can suppose that \( \hat{\varphi}(Z) = 1 \). Then for \( n \geq 1 \) there exists a function \( f_n \in A(T) \) of the form \( f_n(\lambda) = \sum_{j=0}^{\infty} c_j \lambda^j \) with \( f_n(1) = n \) and \( \|f_n\|_p \leq 1 \), because 1 is a Choquet point for the standard algebra \( A(T) \) ([2, 21]). So, \( \tau f_n \in H^p(m) \) and we have

\[
\varphi_p(\tau f_n) = \varphi_p(\sum_{j=0}^{\infty} c_j Z^j) = \sum_{j=0}^{\infty} c_j \hat{\varphi}(Z)^j = \sum_{j=0}^{\infty} c_j = f_n(1) = n
\]

and \( \|\varphi_p\|_p = \|f_n\|_p \leq 1 \), contradicting the fact that \( \varphi \) is bounded on \( H^p(m) \). Hence \( |\hat{\varphi}(Z)| < 1 \).

Now, we can apply Theorem 3.1 for \( \varphi \) to obtain (4.1). Next, since \( |\hat{\varphi}(Z)| < 1 \), the function

\[
\theta_0 = \sum_{n=-\infty}^{\infty} Z^n \hat{\varphi}(Z)^n
\]

is well defined and bounded a.e. \( (m) \) on \( X \). In fact, because

\[
\begin{align*}
\theta_0 &= \sum_{n=0}^{\infty} Z^n \hat{\varphi}(Z)^n + \sum_{n=1}^{\infty} Z^n \hat{\varphi}(Z)^n \\
&= \frac{1}{1 - Z \hat{\varphi}(Z)} + \frac{Z \hat{\varphi}(Z)}{1 - Z \hat{\varphi}(Z)} = \frac{1 - |\hat{\varphi}(Z)|^2}{|Z - \hat{\varphi}(Z)|^2},
\end{align*}
\]

\( \theta_0 \) is positive and \( \int \theta_0 dm = 1 \), hence \( \mu = \theta_0 m \) is a probability measure on \( X \). Clearly, we have by (4.1) for \( f \in A \),

\[
\int f dm = \sum_{n=-\infty}^{\infty} \hat{\varphi}(Z)^n \int Z^n f dm = \sum_{n=0}^{\infty} \hat{f}(n) \hat{\varphi}(Z)^n = \varphi(f),
\]

that is \( \mu \) is a representing measure for \( \varphi \). This ends the proof.

Remark that only boundedness of \( \varphi \) on \( H^p(m) \) assures that \( \varphi \) is \( m - a.c. \) that is \( \varphi \) has a \( m - a.c. \) representing measure, if \( H^p(m) \) is simply invariant. In the general setting of Theorem 3.1, we cannot prove \( r(\hat{\Phi}(Z)) < 1 \) without assuming that \( \Phi \) is \( m - a.c. \).

Concerning the existence of homomorphism of \( A \) which are bounded on \( H^p(m) \), we give the following result which generalize Theorem 6.4 [21] (or Theorem V 7.1, and Theorem VI 7.2 of [1]) in the context of weak* Dirichlet algebras.

**Theorem 4.2.** Suppose that \( H^p_0(m) \) is a simple invariant subspace for some \( p \in [1, 2] \). Then the set \( \Delta_p(m) \) of all homomorphisms of \( A \) which are bounded on \( H^p(m) \) is not reduced to \( \{\gamma\} \), and \( \Delta_p(m) \) is contained in the Gleason part of \( A \) which contains \( \gamma \). Moreover, there exists a one to one continuous map \( \Gamma \) from \( D \) into \( A\mathcal{M}(A) \) such that:

(i) \( \Gamma(D) = \Delta_p(m) \), \( \Gamma(0) = \gamma \),

(ii) For any \( f \in A \), the function \( \hat{f} \circ \Gamma \) is analytic on \( D \), where \( \hat{f} \) is the Gelfand transform of \( f \).
Proof. Let $\Delta_p(m) := \{ \varphi \in \mathcal{M}(A) : \|\varphi\|_{H^p(m)} < \infty \}$. For $\varphi \in \Delta_p(m)$ we have by Theorem 4.1 that $|\hat{\varphi}(Z)| < 1$ where $Z \in H^\infty_0(m)$, $|Z| = 1$ a.e. $(m)$ such that $H^p_0(m) = ZH^p(m)$. We define the map $\Gamma_0 : \Delta_p(m) \to \mathbb{D}$ by $\Gamma_0(\varphi) = \hat{\varphi}(Z)$, $\varphi \in \Delta_p(m)$.

Firstly, $\Gamma_0$ is one to one because if $\Gamma_0(\varphi_0) = \Gamma_0(\varphi_1)$ for $\varphi_0, \varphi_1 \in \Delta_p(m)$ then by (4.1) we have $\varphi_0(f) = \varphi_1(f)$ for $f \in A$, so $\varphi_0 = \varphi_1$. $\Gamma_0$ is also onto $\mathbb{D}$. Indeed, for $z \in \mathbb{D}$ we define the linear functional $\varphi_z$ on $A$ by

$$\varphi_z(f) = \sum_{n=0}^\infty \hat{f}(n)z^n \quad (f \in A).$$

Obviously, one has

$$|\varphi_z(f)| \leq \frac{\|f\|_p}{1 - |z|},$$

because $|\hat{f}(n)| \leq \|f\|_p$ for $f \in A$. It is also easy to see (as in the proof of Theorem 6.4 [21]) that $\varphi_z$ is multiplicative on $A$, therefore $\varphi_z \in \mathcal{M}(A)$. From the above estimation we have

$$\|\varphi_z\| \leq \frac{1}{1 - |z|},$$

hence $\varphi_z \in \Delta(m)$, and clearly, $\Gamma_0(\varphi_z) = \hat{\varphi}_z(Z) = z$ that is $\Gamma_0$ is surjective. In addition, by Theorem 4.1 a representing measure for $\varphi_z$ is $m_z$ given by

$$m_z = \frac{1 - |\hat{\varphi}_z(Z)|^2}{|Z - \hat{\varphi}_z(Z)|^2} \cdot m = \frac{1 - |z|^2}{|Z - z|^2} \cdot m.$$

So, the measures $m$ and $m_z$ are mutually absolutely continuous and their corresponding Radon-Nikodym derivatives are bounded a.e. $(m)$ on $X$. This means that $\varphi_z$ belongs to the Gleason part $\Delta(\gamma)$ of $A$ which contains $\gamma$ (see [2, 21]). As $\Gamma_0$ is a bijection from $\Delta_p(m)$ onto $\mathbb{D}$, we infer that

$$\{ \gamma \} \subset \Delta_p(m) = \{ \varphi_z : z \in \mathbb{D} \} \subset \Delta(\gamma).$$

Now, $\Gamma = \Gamma_0^{-1}$ is one to one from $\mathbb{D}$ onto $\Delta(m)$ and for $f \in A$ and $z \in \mathbb{D}$ we obtain by (4.1),

$$(\hat{f} \circ \Gamma)(z) = \hat{\varphi}_z = \varphi_z(f) = \sum_{n=0}^\infty \hat{f}(n)z^n,$$

hence $\hat{f} \circ \Gamma$ is an analytic function on $\mathbb{D}$. Finally, $\Gamma$ is a continuous map on $\mathbb{D}$, relative to the Gelfand topology in $\mathcal{M}(A)$, and $\Gamma(0)(f) = \hat{f}(0) = \int f dm = \gamma(f)$ for $f \in A$, so $\Gamma(0) = \gamma$. This ends the proof. \qed

Remark 4.3. If for a function algebra $A$ on $X$, $m$ is the unique representing measure for $\gamma \in \mathcal{M}(A)$, then $A$ is weak$^*$ Dirichlet in $L^\infty(m)$, and any $\varphi \in \Delta(\gamma)$ has a unique representing measure which is bounded absolutely continuous with respect to $m$ ([2], Cor. IV 1.2). This gives $\|\varphi\|_{H^p(m)} < \infty$ for $\varphi \in \Delta(\gamma)$, hence $\Delta(\gamma) = \Delta_p(m) \neq \{ \gamma \}$ in this case, if $H^p_0(m)$ is simple invariant for some $p \in [1, 2]$. Furthermore, only assumption $\Delta(\gamma) \neq \{ \gamma \}$ assures that $H^p_0(m)$ is simply invariant, in the case of unique representing measure (see Theorem 6.4 [21], or Theorem V 7.2 [1]).
REFERENCES

[1] G. Cassier, N. Suciu, *Mapping theorems and Harnack ordering for ρ-contractions*, Indiana Univ. Math. Journal (2005), 483–524.

[2] T. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.

[3] D. Gaşpar, *On operator representations of function algebras*, Acta Sci. Math. (Szeged) **31** (1970), 339–346.

[4] D. Gaşpar, *Spectral ρ-dilations for representations of function algebras*, Anal. Univ. Timișoara, Ser. Șt. Mat. **3** (1970), 153–157.

[5] D. Gaşpar, *On absolutely continuous representations of function algebras*, Rev. Roum. Math. Pures Appl. **17** (1972) 1, 21–31.

[6] D. Gaşpar, *On the harmonic analysis of representations of function algebras*, Stud. Cerc. Mat. **24** (1972) 1, 7–95 [in Romanian; English summary].

[7] K. Hoffman, *Analytic functions and logmodular Banach algebras*, Acta Math. **108** (1962), 271–317.

[8] A. Juratoni, N. Suciu, *Spectral ρ-dilations for some representations of uniform algebras*, Proc. of the 9th Nat. Conf. Roumanian Math. Soc., Ed. Univ. de Vest Timișoara, (2005), 194–210.

[9] A. Juratoni, *On the weak ρ-spectral representations of function algebras*, Proc. of the 11 Symp. of Math. and Its Appl., Ed. Politehnica, (2006), 153–158.

[10] A. Juratoni, *Some absolutely continuous representations of function algebras*, Surveys in Math. and its Appl. **1** (2006), 51–60.

[11] A. Juratoni, *On uniformly stable ρ-contractions*, Proc. of PAMM Conference, Balatonalmady (Hungary), BAM-CXIII/2008, Nr. 2382-2398, 017–027.

[12] A. Juratoni, *On operator representations of weak*-Dirichlet algebras*, Proc. of the 22nd Conference on Operator Theory, Theta București 2010, 89–98.

[13] A. Lebow, *On von Neumann’s theory of spectral sets*, J. Math. Anal. Appl. **7** (1963), 64–90.

[14] G. Lumer, *H∞ and the embedding of the classical Hp spaces in arbitrary ones*, Function Algebra, Scott, Foresman and Co., 1966, 285–286.

[15] T. Nakazi, *ρ-dilations and hypo-Dirichlet algebras*, Acta Sci. Math. (Szeged) **56** (1992), 175–181.

[16] T. Nakazi, *Some special bounded homomorphisms of uniform algebras*, Contemporary Math. **232** (1999), 243–252.

[17] T. Nakazi, *Invariant subspaces of weak*-Dirichlet algebras*, Pacific J. Math. **69** (1977) 1, 151–167.

[18] A. Racz, *Unitary skew-dilations* (Romanian; English summary), Stud. Cerc. Mat. **26** (1974) 4, 545–621.

[19] M. Schreiber, *Absolutely continuous operators*, Duke Math. **29** (1962), 175–190.
[20] T.P. Srinivasan, Ju-Kwei Wang, *Weak \textit{*}-Dirichlet algebras*, Function Algebras, Scott, Foresman and Co, 1966, 216–249.

[21] I. Suciu, *Function Algebras*, Noordhoff Intern. Publ. Leyden, The Netherlands, 1975.

[22] B.SZ.-Nagy, C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, New York, 1970.

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