Rarita Schwinger Type Operators on Cylinders

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Abstract

Here we define Rarita-Schwinger operators on cylinders and construct their fundamental solutions. Further the fundamental solutions to the cylindrical Rarita-Schwinger type operators are achieved by applying translation groups. In turn, a Borel-Pompeiu Formula, Cauchy Integral Formula and a Cauchy Transform are presented for the cylinders. Moreover we show a construction of a number of conformally inequivalent spinor bundles on these cylinders. Again we construct Rarita-Schwinger operators and their fundamental solutions in this setting. Finally we study the remaining Rarita-Schwinger type operators on cylinders.

This paper is dedicated to the memory of Jaime Keller

1 Introduction

The Rarita-Schwinger operators are generalizations of the Dirac operator which in turn is a natural generalization of the Cauchy-Riemann operator. They have been studied in Euclidean space in [BSSV, BSSV1, DLRV, Va, Va1] and on spheres and real projective spaces in [LRV].

Conformally flat manifolds are manifolds with atlases whose transition functions are Möbius transformations. They can be constructed by factoring out a subdomain $U$ of either the sphere $S^n$ or $\mathbb{R}^n$ by a Kleinian subgroup $\Gamma$ of the Möbius group where $\Gamma$ acts strongly discontinuously on $U$. This gives rise to the conformally flat manifold $U\setminus \Gamma$. Real projective spaces are examples of conformally flat manifolds of type $S^n \setminus \{\pm 1\}$. Other simple examples are $\mathbb{R}^n \setminus \mathbb{Z}^l$, where $\mathbb{Z}^l$ is an integer lattice and $1 \leq l \leq n$. We call these manifolds cylinders. In case $l = n$ we have the $n$-torus. For cylinders and tori the conformal structure is given by translations.

In this paper we define Rarita-Schwinger operators on cylinders and construct their fundamental solutions. Further the fundamental solutions to the cylindrical Rarita-Schwinger type operators are achieved by applying translation groups. In turn, Borel-Pompeiu Formula, Cauchy Integral Formula and a Cauchy Transform are presented for
the cylinders. We also show a construction of a number of conformally inequivalent spinor bundles on these cylinders. Again we construct Rarita-Schwinger operators and their fundamental solutions in this setting. The Rarita Schwinger type operators on tori are left for future research.

In [KR] R. Krausshar and J. Ryan introduce Clifford analysis on cylinders and tori making use of the fact that the universal covering space of all of these manifolds is $\mathbb{R}^n$. So provided the functions and kernels are $l$-periodic for some $l \in \{1, \ldots, n\}$ then R. Krausshar and J. Ryan use the projection map to obtain the equivalent function or kernel on those manifolds. Following them, we take the Rarita-Schwinger kernel in $\mathbb{R}^n$ and use the translation group to construct new kernels that are $l$-fold periodic, which are projected on the cylinders. In order to prove that the new kernels are well defined we adapt the Eisenstein series argument developed in [K]. We also show a construction of a number of conformally inequivalent spinor bundles on these cylinders. In the last section, we discuss the remaining Rarita-Schwinger operators studied in [LR] and determine their kernels on cylinders. We finally establish some basic integral formulas associated with the remaining Rarita-Schwinger operators on cylinders.

2 Preliminaries

A Clifford algebra, $Cl_n$, can be generated from $\mathbb{R}^n$ by considering the relationship

$$x^2 = -\|x\|^2$$

for each $x \in \mathbb{R}^n$. We have $\mathbb{R}^n \subseteq Cl_n$. If $e_1, \ldots, e_n$ is an orthonormal basis for $\mathbb{R}^n$, then $x^2 = -\|x\|^2$ tells us that $e_ie_j + e_je_i = -2\delta_{ij}$. Let $A = \{j_1, \ldots, j_r\} \subseteq \{1, 2, \ldots, n\}$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq n$. An arbitrary element of the basis of the Clifford algebra can be written as $e_A = e_{j_1} \cdots e_{j_r}$. Hence for any element $a \in Cl_n$, we have $a = \sum_A a_Ae_A$, where $a_A \in \mathbb{R}$. For $a \in Cl_n$, we will need the anti-involution called Clifford conjugation:

$$\bar{a} = \sum_A (-1)^{|A|(|A|+1)/2}a_Ae_A,$$

where $|A|$ is the cardinality of $A$. In particular, we have $\overline{e_{j_1} \cdots e_{j_r}} = (-1)^r e_{j_r} \cdots e_{j_1}$ and $\overline{ab} = \overline{b}\overline{a}$ for $a, b \in Cl_n$.

For each $a = a_0 + \cdots + a_{1\ldots n}e_1 \cdots e_n \in Cl_n$ the scalar part of $aa$ gives the square of the norm of $a$, namely $a_0^2 + \cdots + a_{1\ldots n}^2$.

We recall that if $y \in S^{n-1} \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, then $yxy$ gives a reflection of $x$ in the $y$ direction, because $yxy = yx^\|y + x^\perp y = -x^\| + x^\perp$ where $x^\|$ is the projection of $x$ onto $y$ and $x^\perp$ is perpendicular to $y$.

The Dirac Operator in $\mathbb{R}^n$ is defined to be

$$D := \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}.$$
Let $\mathcal{M}_k$ denote the space of $\text{Cl}_n$ valued polynomials, homogeneous of degree $k$ and such that if $p_k \in \mathcal{M}_k$ then $Dp_k = 0$. Such a polynomial is called a left monogenic polynomial homogeneous of degree $k$. Note if $h_k \in \mathcal{H}_k$, the space of $\text{Cl}_n$ valued harmonic polynomials homogeneous of degree $k$, then $Dh_k \in \mathcal{M}_{k-1}$. But $D\text{up}_{k-1}(u) = (-n - 2k + 2)p_{k-1}(u)$, so

$$\mathcal{H}_k = \mathcal{M}_k \bigoplus u\mathcal{M}_{k-1}, h_k = p_k + up_{k-1}.$$ 

This is the so-called Almansi-Fischer decomposition of $\mathcal{H}_k$. See [BDS].

Suppose $U$ is a domain in $\mathbb{R}^n$. Consider a function of two variables

$$f : U \times \mathbb{R}^n \longrightarrow \text{Cl}_n$$

such that for each $x \in U, f(x, u)$ is a left monogenic polynomial homogeneous of degree $k$ in $u$. Consider the action of the Dirac operator:

$$D_x f(x, u).$$

As $\text{Cl}_n$ is not commutative then $D_x f(x, u)$ is no longer monogenic in $u$ but it is still harmonic and homogeneous of degree $k$ in $u$. So by the Almansi-Fischer decomposition, $D_x f(x, u) = f_{1, k}(x, u) + uf_{2, k-1}(x, u)$ where $f_{1, k}(x, u)$ is a left monogenic polynomial homogeneous of degree $k$ in $u$ and $f_{2, k-1}(x, u)$ is a left monogenic polynomial homogeneous of degree $k - 1$ in $u$. Let $P_k$ be the left projection map

$$P_k : \mathcal{H}_k \rightarrow \mathcal{M}_k,$$

then $R_k f(x, u)$ is defined to be $P_k D_x f(x, u)$. The left Rarita-Schwinger equation is defined to be (see [BSSV])

$$R_k f(x, u) = 0.$$

For an integer $l$, $1 \leq l \leq n$ we define the $l$-cylinder $C_l$ to be the $n$-dimensional manifold $\mathbb{R}^n / \mathbb{Z}^l$, where $\mathbb{Z}^l$ denote the $l$-dimensional lattice defined by $\mathbb{Z}^l := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_l$. We denote its members $m_1e_1 + \cdots + m_le_l$ for each $m_1, \ldots, m_l \in \mathbb{Z}$ by a bold letter $\mathbf{m}$. When $l = n$, $C_l$ is the $n$-torus, $T_n$. For each $l$ the space $\mathbb{R}^n$ is the universal covering of the cylinder $C_l$. Hence there is a projection map $\pi_l : \mathbb{R}^n \rightarrow C_l$.

An open subset $U$ of the space $\mathbb{R}^n$ is called $l$–fold periodic if for each $x \in U$ the point $x + \mathbf{m} \in U$. So $\pi_l(U) = U'$ is an open subset of the $l$-cylinder $C_l$.

Suppose that $U \subset \mathbb{R}^n$ is a $l$–fold periodic open set. Let $f(x, u)$ be a function defined on $U \times \mathbb{R}^n$ with values in $\text{Cl}_n$, and such that $f$ is a monogenic polynomial homogeneous of degree $k$ in $u$. Then we say that $f(x, u)$ is a $l$-fold periodic function if for each $x \in U$ we have that $f(x, u) = f(x + \mathbf{m}, u)$.

Now if $f : U \times \mathbb{R}^n \rightarrow \text{Cl}_n$ is a $l$–fold periodic function then the projection $\pi_l$ induces a well defined function $f' : U' \times \mathbb{R}^n \rightarrow \text{Cl}_n$, where $f'(x', u) = f(\pi_l^{-1}(x'), u)$ for each $x' = \pi_l(x) \in U'$. Moreover, any function $f' : U' \times \mathbb{R}^n \rightarrow \text{Cl}_n$ lifts to a $l$–fold periodic function $f : U \times \mathbb{R}^n \rightarrow \text{Cl}_n$, where $U = \pi_l^{-1}(U')$. 

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The projection map \( \pi_l \) induces a projection of the Rarita-Schwinger operator \( R_k \) to an operator \( R_k^{C_l} \) acting on domains on \( C_l \times \mathbb{R}^n \) which is defined by \( P_k D' \), where \( D' \) is the projection of the Dirac operator \( D \). That is

\[
R_k^{C_l} f'(x', u) = P_k D'_x f'(x', u).
\]

We call the operator \( R_k^{C_l} \) a \( l \)-cylindrical Rarita-Schwinger type operator and the solutions of the equation

\[
R_k^{C_l} f'(x', u) = 0
\]

\( l \)-cylindrical Rarita-Schwinger functions.

As

\[
I - P_k : \mathcal{H}_k \to u\mathcal{M}_{k-1},
\]

where \( I \) is the identity map, then we can define the remaining Rarita-Schwinger operators

\[
Q_k := (I - P_k)D_x : u\mathcal{M}_{k-1} \to u\mathcal{M}_{k-1} \quad ug(x, u) :\to (I - P_k)D_x ug(x, u).
\]

See [BSSV, LR].

The remaining Rarita-Schwinger equation is defined to be \( (I - P_k)D_x ug(x, u) = 0 \) or \( Q_k ug(x, u) = 0 \), for each \( x \) and \((x, u) \in U \times \mathbb{R}^n\), where \( U \) is a domain in \( \mathbb{R}^n \) and \( g(x, u) \in \mathcal{M}_{k-1} \).

Suppose that \( g : U \times \mathbb{R}^n \to Cl_n \) is a \( l \)-fold periodic function then the projection \( \pi_l \) induces a well defined function \( g' : U' \times \mathbb{R}^n \to Cl_n \), where \( g'(x', u) = g(\pi_l^{-1}(x'), u) \) for each \( x' = \pi_l(x) \in U' \). The projection map \( \pi_l \) also induces a projection of the remaining Rarita-Schwinger operator \( Q_k \) to an operator \( Q_k^{C_l} \) acting on domains on \( C_l \times \mathbb{R}^n \) which is defined by \( (I - P_k)D' \). That is

\[
Q_k^{C_l} g'(x', u) = P_k D'_x g'(x', u).
\]

We call the operator \( Q_k^{C_l} \) a \( l \)-cylindrical remaining Rarita-Schwinger type operator and the solutions of the equation

\[
Q_k^{C_l} g'(x', u) = 0
\]

\( l \)-cylindrical remaining Rarita-Schwinger functions.

## 3 Fundamental solutions of \( R_k^{C_l} \)

Let \( U \) a domain in \( \mathbb{R}^n \). We recall the fundamental solution of the Rarita-Schwinger operator \( R_k \) in \( \mathbb{R}^n \), see [DLRV]:

\[
E_k(x, u, v) = \frac{1}{\omega_n c_k \|x\|^n} Z_k \left( \frac{xu}{\|x\|^2}, v \right) ,
\]

where \( c_k = \frac{n - 2}{n + 2k - 2} \), \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \),

\[
Z_k(u, v) := \sum_{\sigma} P_{\sigma}(u) V_{\sigma}(v) v ,
\]

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where $P_\sigma(u) = \frac{1}{k!} \sum_{\sigma} (u_{i_1} - u_1 e_1^{-1} e_{i_1}) \ldots (u_{i_k} - u_1 e_1^{-1} e_{i_k})$, $V_\alpha(v) = \frac{\partial^k G(v)}{\partial v_{i_2} \ldots \partial v_{i_k}}$, $k_2 + \ldots + k_n = k$, $i_k \in \{2, \ldots, n\}$ and the summation is taken over all permutations of the monomials without repetition. See [BDS].

Now we study the convergence of the series summing only over those lattice paravectors $z$ and the ball $\bar{B}$ that satisfy
\[ \sum_{k=1}^n \alpha \in \mathbb{N}^{k+1}, |\alpha| \geq 1, \alpha = (0, \alpha_1, \ldots, \alpha_k), \text{ the following estimate holds for all } z \in \mathbb{N}^{k+1} \backslash \{0\}: \]
\[ \| \frac{\partial^{|\alpha|}}{\partial z^{|\alpha|}} q_0^{(s)}(z) \| \leq \frac{(k + 1 - s)(k + 2 - s) \ldots (k + |\alpha| - s)}{\|z\|^{k+|\alpha|+1-s}}, \]
where $q_0(z) := \frac{z}{\|z\|^{k+1}} \frac{\partial^{|\alpha|}}{\partial z^{|\alpha|}} q_0(z) := \frac{\partial^{\alpha_1 + \ldots + \alpha_k}}{\partial x_{i_1}^{\alpha_1} \ldots \partial x_{i_k}^{\alpha_k}} q_0(z)$ and $q_0^{(s)}(z)$ is the kernel of the $s$th-power of the Dirac operator.

**Proposition 1.** Let $N_0$ be the set of non-negative integers and $A_{k+1}$ be the space of paravectors $z = x_0 + x$ with $Sc(z) = x_0$ and $Vec(z) = x \in \mathbb{R}^k$. Let $s \in \{1, \ldots, k\}$. For all multi-indices $\alpha \in \mathbb{N}^{k+1}$, $|\alpha| \geq 1$, $\alpha = (0, \alpha_1, \ldots, \alpha_k)$, the following estimate holds for all $z \in A_{k+1} \backslash \{0\}$:
\[ \| \frac{\partial^{|\alpha|}}{\partial z^{|\alpha|}} q_0^{(s)}(z) \| \leq \frac{(k + 1 - s)(k + 2 - s) \ldots (k + |\alpha| - s)}{\|z\|^{k+|\alpha|+1-s}}, \]
where $q_0(z) := \frac{z}{\|z\|^{k+1}} \frac{\partial^{|\alpha|}}{\partial z^{|\alpha|}} q_0(z) := \frac{\partial^{\alpha_1 + \ldots + \alpha_k}}{\partial x_{i_1}^{\alpha_1} \ldots \partial x_{i_k}^{\alpha_k}} q_0(z)$ and $q_0^{(s)}(z)$ is the kernel of the $s$th-power of the Dirac operator.

**Remark 1.** In the vector formalism in $\mathbb{R}^n$, $q_0(x) := -\frac{x}{\|x\|^n}$. From now on we are working only in the case $s = 1$.

Now we have a particular case of Proposition 2.2 appearing in [K].

**Proposition 2.** Let $p \in \mathbb{N}$ with $1 \leq p \leq n-2$. Let $\mathbb{Z}^p$ be the $p$-dimensional lattice. Then the series
\[ \sum_{m \in \mathbb{Z}^p} q_0(x + m) \]
converges normally in $\mathbb{R}^n \backslash \mathbb{Z}^p$.

**Proof.** We consider an arbitrary compact subset $K \subset \mathbb{R}^n$ and a real number $R > 0$ such that the ball $\bar{B}(0, R)$ covers $K$ completely. Let $x \in \bar{B}(0, R)$, $x = (x_1, \ldots, x_n)$. WLOG we will study the convergence of the series summing only over those lattice $m$ that satisfy $\|m\| > nR \geq \|x\|$. 

\[ \sum_{m \in \mathbb{Z}^p} q_0(x + m) \]

\[ \sum_{m \in \mathbb{Z}^p} q_0(x + m) \]
The function $q_0(x + m)$ is left monogenic in $0 \leq \|x\| < nR$. Hence it is real analytic in $B(0, nR)$ and therefore can be represented in the interior of this ball by its Taylor series, i.e.,

$$q_0(x + m) = \sum_{\nu=0}^{\infty} \left( \sum_{l_1 + \ldots + l_n = \nu} \frac{1}{l_!} x_1^{l_1} \ldots x_n^{l_n} q_l(m) \right),$$

where $l = (l_1, \ldots, l_n)$ and $q_l(m) = \frac{\partial^{\nu}}{\partial m^l} q_0(m)$.

Using Proposition [I] and observing that $\|x\|^\nu \leq R^\nu$, we obtain:

$$\|q_0(x + m)\| \leq \sum_{\nu=0}^{\infty} \left( \sum_{l_1 + \ldots + l_n = \nu} \frac{1}{l_!} \|x\|^\nu |q_l(m)| \right) \leq \sum_{\nu=0}^{\infty} \left( \sum_{l_1 + \ldots + l_n = \nu} \frac{1}{l_!} R^\nu \nu! \prod_{\gamma=1}^{n-2} (\nu + \gamma) \|m\|^{1-\nu} \right).$$

Using the multinomial formula at $\sum_{l_1 + \ldots + l_n = \nu} \nu! l_! \ldots l_n! = \nu!$, we can write the former estimate as

$$\|q_0(x + m)\| \leq \sum_{\nu=0}^{\infty} \prod_{\gamma=1}^{n-2} (\nu + \gamma) \left( \frac{nR}{\|m\|} \right)^\nu \frac{1}{\|m\|^{n-1}}.$$

Since the series $\sum_{k=-s}^{\infty} r^{k+s}$ converges absolutely for $|r| < 1$ to $\frac{1}{1-r}$, we can consider its $s$th-derivative which also converges absolutely for $|r| < 1$ to $\frac{d^s}{dr^s}(\frac{1}{1-r})$. So we obtain

$$\sum_{k=0}^{\infty} \frac{(k + 1)(k + 2) \cdots (k + s)r^k}{nR \|m\|^{n-1}} = \frac{s!}{(1-r)^{s+1}}.$$

Taking this into account and observing that $\|m\| < 1$, we get

$$\sum_{\nu=0}^{\infty} \prod_{\gamma=1}^{n-2} (\nu + \gamma) \left( \frac{nR}{\|m\|} \right)^\nu = \frac{(n-2)!}{(1 - \frac{nR}{\|m\|})^{n-1}}.$$

Therefore we have

$$\|q_0(x + m)\| \leq \frac{(n-2)!}{(1 - \frac{nR}{\|m\|})^{n-1}} \cdot \frac{1}{\|m\|^{n-1}}.$$

Due to Eisenstein’s Lemma (see [E]) a series of the form

$$\sum_{(m_1, \ldots, m_p) \in \mathbb{Z}^p \setminus \{0\}} \|m_1 \omega_1 + \ldots + m_p \omega_p\|^{-(p+\alpha)}$$

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is convergent if and only if $\alpha \geq 1$, where $\omega_i, i = 1, \ldots, p$, are $\mathbb{R}$-linear independent paravectors in $A_n$.

In our case, $1 \leq p \leq n - 2$ which implies $A = \frac{(n - 2)!}{(1 - \|m\|)^{n-1}}$, and therefore observing that also $e_j, j = 1, \ldots, p$, are $\mathbb{R}$-linear independent paravectors in $A_n$ and using the comparison test, we obtain that the series (4) converges normally in $\mathbb{R}^n \setminus \mathbb{Z}^p$. ■

Returning to the series defined by (3)

$$
cot_{l,k}(x, u, v) = \sum_{m \in \mathbb{Z}^l} E_k(x + m e_1 + \cdots + m_l e_l, u, v)
$$

we observe that $Z_k\left(\frac{(x + m)u(x + m)}{||x + m||^2}, v\right)$ is a bounded function on a bounded domain in $\mathbb{R}^n$, because its first variable, $\frac{(x + m)u(x + m)}{||x + m||^2}$, is a reflection in the direction $\frac{x + m}{||x + m||}$ for each $m$, and hence is a linear transformation which is a continuous function. On the other hand, we would get with respect to the second variable, bounded homogeneous functions of degree $k$.

Consequently, applying Proposition 2 the series (3) is a uniformly convergent series and represents a kernel for the Rarita-Schwinger operators under translations by $m \in \mathbb{Z}^l$, with $1 \leq l \leq n - 2$.

Now, we want to define the $(n - 1)$-fold periodic cotangent. In order to do that, we decompose as in [K] the lattice $\mathbb{Z}^l$ into three parts: the origin $\{0\}$ and a positive and a negative parts. The last two parts are equal and disjoint:

$$
\Lambda_l = \{m_1 e_1 : m_1 \in \mathbb{N}\} \cup \{m_1 e_1 + m_2 e_2 : m_1, m_2 \in \mathbb{Z}, m_2 > 0\} \\
\cup \cdots \cup \{m_1 e_1 + \cdots + m_l e_l : m_1, \cdots, m_l \in \mathbb{Z}, m_l > 0\}
$$

$$
-A_l = (\mathbb{Z}^l \setminus \{0\}) \setminus \Lambda_l.
$$

For $l = n - 1$, we define

$$
cot_{n-1,k}(x, u, v) = E_k(x, u, v) + \sum_{m \in \Lambda_{n-1}} [E_k(x + m, u, v) + E_k(x - m, u, v)]. \quad (5)
$$

To show the uniform convergence of the above series, we need the following proposition which is a special case in [K].

**Proposition 3.** Let $\mathbb{Z}^{n-1}$ be the $(n - 1)$-dimensional lattice. Then the series

$$
q_0(x) + \sum_{m \in \mathbb{Z}^{n-1}\setminus\{0\}} (q_0(x + m) - q_0(m)), \quad (6)
$$
converges normally in \( \mathbb{R}^n \setminus \mathbb{Z}^{n-1} \).

\textbf{Proof.} Following the proof of Proposition \[2\] again we suppose that \( x \in \bar{B}(0, R) \) and consider only those lattice points with \( ||m|| > nR \geq ||x|| \). We consider the function \( q_0(x + m) - q_0(m) \) and expand it into a Taylor series in \( B(0, R') \) where \( 0 < R' < R \). So we have

\[
q_0(x + m) - q_0(m) = \sum_{\nu=1}^{\infty} \left( \sum_{l_1+\ldots+l_n=\nu} \frac{1}{n!} x_1^{l_1} \ldots x_n^{l_n} q_1(m) \right).
\]

Using similar arguments to those of proof of Proposition \[2\] we obtain

\[
||q_0(x + m) - q_0(m)|| \leq \sum_{\nu=1}^{\infty} \prod_{\gamma=1}^{n-2} (\nu + \gamma) \left( \frac{nR}{||m||} \right)^{\nu} \frac{1}{||m||^{n-1}} = \sum_{\nu=0}^{\infty} \prod_{\gamma=2}^{n-1} (\nu + \gamma) \left( \frac{nR}{||m||} \right)^{\nu} \frac{nR}{||m||^n}.
\]

Since

\[
||q_0(x + m) - q_0(m)|| \leq \frac{KR}{||m||^n} = \frac{KR}{||m||^{p+1}}, \ p = n - 1,
\]

where \( K \) is a positive real constant and applying the Eisenstein’s Lemma, the series \[6\] turns into a normally convergent series in \( \mathbb{R}^n \setminus \mathbb{Z}^{n-1} \). \( \blacksquare \)

\textbf{Remark 2.} Because \( \Lambda_{n-1} \) and \( -\Lambda_{n-1} \) belong to \( \mathbb{Z}^{n-1} \setminus \{0\} \), we have also the normal convergence of the series

\[
\sum_{m \in \Lambda_{n-1}} (q_0(x + m) - q_0(m)), \sum_{m \in -\Lambda_{n-1}} (q_0(x + m) - q_0(m))
\]

Now we consider the expression

\[
G(x + m)Z_k \left( \frac{(x + m)u(x + m)}{||x + m||^2}, v \right) + G(x - m)Z_k \left( \frac{(x - m)u(x - m)}{||x - m||^2}, v \right),
\]

where the function \( Z_k(\cdot, \cdot) \) has the hypotheses stated formerly. Rewriting the former expression as

\[
(G(x + m) - G(x - m))Z_k \left( \frac{(x + m)u(x + m)}{||x + m||^2}, v \right) + (G(x - m) - G(m))Z_k \left( \frac{(x - m)u(x - m)}{||x - m||^2}, v \right) + (G(x - m) + G(m))Z_k \left( \frac{(x - m)u(x - m)}{||x - m||^2}, v \right) + G(m)Z_k \left( \frac{(x + m)u(x + m)}{||x + m||^2}, v \right) - G(m)Z_k \left( \frac{(x - m)u(x - m)}{||x - m||^2}, v \right)
\]

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and taking the sum over $Z^{n-1}\setminus\{0\}$ we obtain

$$
\sum_{m \in Z^{n-1}\setminus\{0\}} (G(x+m)Z_k\left(\frac{(x+m)u(x+m)}{\|x+m\|^2}, v\right) + G(x-m)Z_k\left(\frac{(x-m)u(x-m)}{\|x-m\|^2}, v\right)) = \\
\sum_{m \in Z^{n-1}\setminus\{0\}} (G(x+m) + G(x-m))Z_k\left(\frac{(x+m)u(x+m)}{\|x+m\|^2}, v\right) + \\
\sum_{m \in Z^{n-1}\setminus\{0\}} (-G(x-m) - G(m))Z_k\left(\frac{(x-m)u(x-m)}{\|x-m\|^2}, v\right) + \\
\sum_{m \in Z^{n-1}\setminus\{0\}} (G(x-m) + G(m))Z_k\left(\frac{(x-m)u(x-m)}{\|x-m\|^2}, v\right) + \\
\sum_{m \in Z^{n-1}\setminus\{0\}} (G(m)Z_k\left(\frac{(x+m)u(x+m)}{\|x+m\|^2}, v\right) - G(m)Z_k\left(\frac{(x-m)u(x-m)}{\|x-m\|^2}, v\right)).
$$

The last sum in (7) vanishes because the terms $G(m)Z_k\left(\frac{(x+m)u(x+m)}{\|x+m\|^2}, v\right)$ for each $m \in \Lambda_{n-1}$ and $m \in -\Lambda_{n-1}$ cancel with the terms $G(m)Z_k\left(\frac{(x-m)u(x-m)}{\|x-m\|^2}, v\right)$ for each $m \in -\Lambda_{n-1}$ and $m \in \Lambda_{n-1}$, respectively. On the other side, since

$$||G(x+m) + G(x-m)|| \leq ||G(x+m) - G(m)|| + ||G(x+m) - G(m)||,$$

and observing that $Z_k(\cdot, \cdot)$ is bounded, we can apply Proposition 3 to obtain the normal convergence for the series (7). Consequently the series defined by (5) is uniformly convergent and it is a kernel for the Rarita-Schwinger type operator under translations by $m \in \Lambda_{n-1}$.

For $x, y \in \mathbb{R}^n \setminus \mathbb{Z}^l$, $1 \leq l \leq n-1$, the functions $\cot_{l,k}(x-y, u, v)$ induce functions

$$
\cot'_{l,k}(x', y', u, v) = \cot_{l,k}(\pi_l^{-1}(x') - \pi_l^{-1}(y'), u, v).
$$

These functions are defined on $(C_l \times C_l) \setminus diag(C_l \times C_l)$ for each fixed $u, v \in \mathbb{R}^n$, where $diag(C_l \times C_l) = \{(x', x') : x' \in C_l\}$ and they are $l$-cylindrical Rarita-Schwinger functions, i.e, the equation $R_{k,l}^{C_l} \cot'_{l,k}(x', y', u, v) = 0$ is satisfied. Furthermore for each $l$ they represent a kernel for the operator $R_{k,l}^{C_l}$.

4 Some integral formulas on cylinders

**Definition 1.** For any $Cl_n$-valued polynomials $P, Q$, the inner product $(P(u), Q(u))_u$ with respect to $u$ is given by

$$(P(u), Q(u))_u = \int_{S^{n-1}} P(u)Q(u) dS(u),$$

where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. 

9
See [DLRV].
For any $p_k \in \mathcal{M}_k$, one obtains

$$p_k(u) = (Z_k(u,v), p_k(v))_v = \int_{\mathbb{S}^{n-1}} Z_k(u,v)p_k(v)dS(v).$$

See [BDS].

**Theorem 1.** [DLRV] (Stokes' Theorem for $R_k$) Let $\Omega$ and $\Omega'$ be domains in $\mathbb{R}^n$ and suppose the closure of $\Omega$ lies in $\Omega'$. Further suppose the closure of $\Omega$ is compact and $\partial \Omega$ is piecewise smooth. Then for $f, g \in C^1(\Omega', \mathcal{M}_k)$, we have

$$\int_{\Omega} [(g(x,u)R_k, f(x,u))_u + (g(x,u), R_kf(x,u))_u]dx^n = \int_{\partial\Omega} (g(x,u), P_k d\sigma_x f(x,u))_u,$$

where $dx^n = dx_1 \wedge \cdots \wedge dx_n$, $d\sigma_x = \sum_{j=1}^n (-1)^{j-1} e_j d\hat{x}_j$, and $d\hat{x}_j = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \cdots \wedge dx_n$.

**Theorem 2.** [DLRV] (Borel-Pompeiu Theorem) Let $\Omega'$ and $\Omega$ be as in Theorem 1. Then for $f \in C^1(\Omega', \mathcal{M}_k)$

$$f(y,v) = \int_{\partial\Omega} (E_k(x - y, u, v), P_k d\sigma_x f(x,u))_u - \int_{\Omega} (E_k(x - y, u, v), R_k f(x,u))_u dx^n.$$

Now by Stokes' Theorem and Borel-Pompeiu Theorem for the Rarita-Schwinger operator $R_k$ in $\mathbb{R}^n$, we may easily obtain:

**Theorem 3.** Let $V$ be a bounded domain in $\mathbb{R}^n$ and $\overline{V}$ be the closure of $V$. For each $x \in \overline{V}$, the shifted lattice $x + \mathbb{Z}^l$ intersected with $V$ only contains the points $x$. Suppose that the boundary of $V$, $\partial V$, is piecewise smooth and $\overline{V}$ is compact. Further suppose $f(x,u) : \overline{V} \times \mathbb{R}^n \to \text{Cl}_u$ is a monogenic homogeneous polynomial of degree $k$ in $u$ and with respect to $x$ is $C^1$. Then for $1 \leq l \leq n - 1$ and each $y \in V$,

$$f(y,v) = \int_{\partial V} (\cot_{l,k}(x - y, u, v), P_k d\sigma_x f(x,u))_u - \int_{V} (\cot_{l,k}(x - y, u, v), R_k f(x,u))_u dx^n.$$

**Proof.** When $m = 0$, this is the Borel-Pompeiu Formula given by Theorem 2

$$f(y,v) = \int_{\partial V} (E_k(x - y, u, v), P_k d\sigma_x f(x,u))_u - \int_{V} (E_k(x - y, u, v), R_k f(x,u))_u dx^n. \quad (8)$$

When $m \neq 0$, for each $m \in \mathbb{Z}^l$, by the hypothesis that the shifted lattice $x + \mathbb{Z}^l$ intersected with $V$ only contains the points $x$, we have $x + m \notin \overline{V}$ but $y \in \overline{V}$, so there
is no singularity in $E_k(x - y + m, u, v) = E_k(x + m - y, u, v)$. Hence by Stokes’ Theorem we obtain
\[
\int_{\partial V} (E_k(x - y + m, u, v), P_k d\sigma_x f(x, u))_u - \int_V (E_k(x - y + m, u, v), R_k f(x, u))_u dx^n
\]
\[
= \int_V (E_k(x - y + m, u, v), R_k f(x, u))_u dx^n = 0. \tag{9}
\]
Now for all $m \in \mathbb{Z}^l$, by adding the equations (8) and (9), we obtain
\[
f(y, v) = \int_{\partial V'} \left( \sum_{m \in \mathbb{Z}^l} E_k(x - y + m, u, v), P_k d\sigma_x f(x, u) \right)_u
\]
\[
- \int_V \left( \sum_{m \in \mathbb{Z}^l} E_k(x - y + m, u, v), R_k f(x, u) \right)_u dx^n. \tag*{\blacksquare}
\]

The case we are most interested in here is the one that $V$ is $l$–fold periodic and $f$ is $l$–fold periodic. If we use the projection map $\pi_l$ at this moment, we obtain the Borel-Pompeiu Theorem on the cylinder $C_l$.

**Theorem 4.** (Borel-Pompeiu Theorem for $R_k^C_l$) Suppose $V'$ is a domain in $C_l$ with compact closure and smooth boundary. Suppose $f(x, v)$ is defined as in Theorem 3. Then for $1 \leq l \leq n - 1$ and each $y' \in V'$,
\[
f'(y', v) = \int_{\partial V'} \left( \cot_{t, k}(x', y', u, v), P_k d\sigma_x f'(x', u) \right)_u
\]
\[
- \int_{V'} \left( \cot_{t, k}(x', y', u, v), R_k^C f'(x', v) \right)_v d\mu(x'),
\]
where $x' = \pi_l(x), d\sigma_{x'} = \partial_x \pi_l d\sigma_x, \partial_x \pi_l$ is the derivative of $\pi_l$ at $x, \mu$ is the projection of Lebesgue measure on $\mathbb{R}^n$ onto $C_l, d\sigma_x = n(x) d\sigma(x), n(x)$ is the unit exterior normal vector at $x = \pi_l^{-1}(x')$ and $d\sigma(x)$ is the surface measure element.

**Theorem 5.** (Cauchy integral formula) Suppose the hypotheses as in Theorem 4 and $f'(x', u)$ is annihilated by the operator $R_k^C_l$, then for $1 \leq l \leq n - 1$ and each $y' \in V'$, we have
\[
f'(y', u) = \int_{\partial V'} \left( \cot_{t, k}(x', y', u, v), P_k d\sigma_x f'(x', u) \right)_u.
\]
If the function given in the Borel-Pompeiu Theorem has its support with respect to $x'$ in $V' \subset C_l$, then by the same theorem, we have the following:

**Theorem 6.** \[
\int_{V'} -\left( \cot_{t, k}(x', y', u, v), R_k^C_l \psi(x', u) \right)_u dx^n = \psi(y', v), \text{ for } \psi \in C^\infty(C_l).
\]

We also can introduce a Cauchy transform for the Rarita-Schwinger operator $R_k^C_l$:
**Definition 2.** For a subdomain $V'$ of cylinder $C_l$ and a function $f' : V' \to Cl_n$, the Cauchy transform of $f'$ is formally defined to be

\[(Tf')(y', v) = -\int_{V'} (\cot_{l,k}(x', y', u, v), f'(x', u))_u \, dx^m, \quad y' \in V'.\]

Consequently, one may obtain a right inverse for $R^C_l$.

**Theorem 7.** \[R^C_l \int_{V'} (\cot_{l,k}(x', y', u, v), \psi(x', u))_u \, dx^m = \psi(y', v), \text{ for } \psi \in C^\infty(C_l).\]

## 5 Conformally inequivalent spinor bundles on $C_l$

We shall now show a construction of a number of conformally inequivalent spinor bundles on $C_l$. In the previous sections the spinor bundle over $C_l$ is chosen to be the trivial one $C_l \times Cl_n$, however we can construct $2^l$ spinor bundles on $C_l$.

Different spin structures on a spin manifold $M$ are detected by the number of different homomorphisms from the fundamental group $\Pi_1(M)$ to the group $\mathbb{Z}_2 = \{0,1\}$. In our case we have $\Pi_1(C_l) = \mathbb{Z}^l$. Because there are two homomorphisms of $\mathbb{Z}$ to $\mathbb{Z}_2$, we have $2^l$ distinct spin structures on $C_l$, see [MP],[KR1].

The following construction is for some of spinor bundles over $C_l$ but all the others are constructed similarly. First let $p$ be an integer in the set $\{1,\ldots,l\}$ and consider the lattice $\mathbb{Z}^p := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_p$. We also consider the lattice $\mathbb{Z}^{l-p} := \mathbb{Z}e_{p+1} + \cdots + \mathbb{Z}e_l$. In this case $\mathbb{Z}^l = \{m + n : m \in \mathbb{Z}^p \text{ and } n \in \mathbb{Z}^{l-p}\}$. Suppose that $m = m_1 e_1 + \cdots + m_p e_p$.

Let us make the identification $(x, X)$ with $(x + m + n, (-1)^{m_1 + \cdots + m_p} X)$ where $x \in \mathbb{R}^n$ and $X \in Cl_n$. This identification gives rise to a spinor bundle $E^p$ over $C_l$.

The Rarita-Schwinger operator over $\mathbb{R}^n$ induces a Rarita-Schwinger operator acting on sections of the bundles $E^p$ over $C_l$. We will denote this operator by $R^p_{k,p}$. The projection $\pi_l$ maps $U \subset \mathbb{R}^n$ to a domain $U' \subset C_l$. Now if $f : U \times \mathbb{R}^n \to Cl_n$ is a $l$–fold periodic function then the projection $\pi_l$ induces a well defined function $f'_p : U' \times \mathbb{R}^n \to E^p$, where $f'_p(x' + m + n, u) = (-1)^{m_1 + \cdots + m_p} f(\pi_l^{-1}(x') + m + n, u)$ for each $x' + m + n = \pi_l(x) + m + n \in U'$. If $R^p_{k,p}(f'_p) = 0$ then $f'_p$ is called an $E^p$ left Rarita-Schwinger section. Moreover, any $E^p$ left Rarita-Schwinger section $f'_p : U' \times \mathbb{R}^n \to E^p$ lifts to a $l$–fold periodic function $f : U \times \mathbb{R}^n \to Cl_n$, where $U = \pi_l^{-1}(U')$, and $R^C_l f = 0$.

By considering the series

$$\cot_{l,k}(x, u, v) = \sum_{m \in \mathbb{Z}^l} E_k(x + m, u, v), \quad \text{for } 1 \leq l \leq n - 2$$

which converge normally on $\mathbb{R}^n \setminus \mathbb{Z}^l$, we can obtain the kernel (see section 3)

$$\cot_{l,k}(x, y, u, v) = \sum_{m \in \mathbb{Z}^l} E_k(x - y + m, u, v), \quad \text{for } 1 \leq l \leq n - 2.$$
Applying the projection map \( \pi_l : \mathbb{R}^n \to C_l \) to these kernels induce kernels 

\[
\cot_{l,k}(x', y', u, v)
\]

defined on \((C_l \times C_l) \setminus \text{diag}(C_l \times C_l)\), where \(\text{diag}(C_l \times C_l) = \{(x', x') : x' \in C_l\}\). We can adapt these functions as follows. For \(1 \leq l \leq n - 2\) we define

\[
\cot_{l,k,p}(x, u, v) = \sum_{m \in \mathbb{Z}^{p}, n \in \mathbb{Z}^{n - 1 - p}} (-1)^{m_1 + \cdots + m_p} E_k(x + m + n, u, v)
\]

These are well defined functions on \(\mathbb{R}^n \setminus \mathbb{Z}^l\). Therefore we obtain from these functions the cotangent kernels

\[
\cot_{l,k,p}(x, y, u, v) = \sum_{m \in \mathbb{Z}^{p}, n \in \mathbb{Z}^{n - 1 - p}} (-1)^{m_1 + \cdots + m_p} E_k(x - y + m + n, u, v).
\]

Again applying the projection map \(\pi_l\) these kernels give rise to the kernels

\[
\cot'_{l,k,p}(x', y', u, v).
\]

In the case \(l = n - 1\), by considering the series

\[
\cot_{n-1,k}(x, u, v) = E_k(x, u, v) + \sum_{m \in \mathbb{Z}^{n - 1}} [E_k(x + m, u, v) + E_k(x - m, u, v)]
\]

we obtain the kernel

\[
\cot_{n-1,k}(x, y, u, v) = E_k(x - y, u, v) + \sum_{m \in \mathbb{Z}^{n - 1}} [E_k(x - y + m, u, v) + E_k(x - y - m, u, v)]
\]

which in turn using the projection map induces kernels \(\cot'_{n-1,k}(x', y', u, v)\). Defining

\[
\cot_{n-1,k,p}(x, u, v) = E_k(x + m + n, u, v) + \sum_{m \in \mathbb{Z}^{p}, n \in \mathbb{Z}^{n - 1 - p}} (-1)^{m_1 + \cdots + m_p} [E_k(x + m + n, u, v) + E_k(x - m - n, u, v)]
\]

we obtain the cotangent kernels

\[
\cot_{n-1,k,p}(x, y, u, v) = E_k(x - y + m + n, u, v) + \sum_{m \in \mathbb{Z}^{p}, n \in \mathbb{Z}^{n - 1 - p}} (-1)^{m_1 + \cdots + m_p} [E_k(x - y + m + n, u, v) + E_k(x - y - m - n, u, v)]
\]

and by \(\pi_l\) the kernels \(\cot'_{n-1,k,p}(x', y', u, v)\).
Consider the fundamental solution of the remaining Rarita-Schwinger operator $Q_k$ in $\mathbb{R}^n$, see [LR]:

$$H_k(x, u, v) := \frac{-1}{\omega_n c_k} \frac{x}{\|x\|^n} Z_{k-1}(\frac{ux}{\|x\|^2}, v)v,$$

where $c_k = \frac{n-2}{n-2+2k}$.

Now we construct functions

$$Cot_{l,k}(x, u, v) = \sum_{m \in \mathbb{Z}^l} H_k(x + m e_1 + \cdots + m e_l, u, v)$$

$$= \sum_{m \in \mathbb{Z}^l} uG(x + m)Z_{k-1}(\frac{(x + m)u(x + m)}{\|x + m\|^2}, v)v, \quad 1 \leq l \leq n - 2.$$

These functions are defined on the $l$-fold periodic domain $\mathbb{R}^n/\mathbb{Z}^l$ for fixed $u$ and $v$ in $\mathbb{R}^n$ and are $C_l$-valued. We can observe that they are $l$-fold periodic functions. Using the similar arguments in Section 3, we may easily obtain that the series $Cot_{l,k}(x, u, v)$ is normally convergent over $\mathbb{R}^n \setminus \mathbb{Z}^l$.

For $l = n - 1$, we define

$$Cot_{n-1,k}(x, u, v) = H_k(x, u, v) + \sum_{m \in \mathbb{Z}^{n-1}} [H_k(x + m, u, v) + H_k(x - m, u, v)].$$

We can establish that the previous series converges uniformly following the proof in Section 3 for the case $l = n - 1$.

For $x, y \in \mathbb{R}^n \setminus \mathbb{Z}^l$, $1 \leq l \leq n - 1$, the functions $Cot_{l,k}(x - y, u, v)$ induce functions

$$Cot_{l,k}'(x', y', u, v) = Cot_{l,k}(\pi_i^{-1}(x') - \pi_i^{-1}(y'), u, v).$$

These functions are defined on $(C_l \times C_l) \setminus \text{diag}(C_l \times C_l)$ for each fixed $u, v \in \mathbb{R}^n$, where $\text{diag}(C_l \times C_l) = \{(x', x') : x' \in C_l\}$ and they are $l$-cylindrical remaining Rarita-Schwinger functions. Consequently, $Q_k^{C_l}Cot_{l,k}'(x', y', u, v) = 0$. Furthermore for each $l$ they represent a kernel for the operator $Q_k^{C_l}$.

Now we will establish some integral formulas associated with the remaining Rarita-Schwinger operators on cylinders.

**Theorem 8.** [LR] (Stokes’ Theorem for $Q_k$ operators) Let $\Omega'$ and $\Omega$ be domains in $\mathbb{R}^n$ and suppose the closure of $\Omega$ lies in $\Omega'$. Further suppose the closure of $\Omega$ is compact and
the boundary of $\Omega$, $\partial \Omega$, is piecewise smooth. Then for $f, g \in C^1(\Omega', M_{k-1})$, we have
\[
\int_{\Omega} [(g(x, u)uQ_{k,r})_u, u f(x, u)] u + \int_{\Omega} (g(x, u)u, Q_k u f(x, u)] u \, dx^n
\]
\[
= \int_{\partial \Omega} (g(x, u)u, (I - P_k)d\sigma_x u f(x, u)] u
\]
\[
= \int_{\partial \Omega} (g(x, u)u d\sigma_x (I - P_{k,r}), u f(x, u)] u ,
\]
Where $Q_{k,r}$ is the right remaining Rarita-Schwinger operator.

**Theorem 9.** [LR](Borel-Pompeiu Theorem for $Q_k$ operators) Let $\Omega'$ and $\Omega$ be as in the previous Theorem. Then for $f \in C^1(\Omega', M_{k-1})$ and $y \in \Omega$, we obtain
\[
uf(y, u) = \int_{\Omega} (H_k(x - y, u, v), Q_k u f(x, v)]_u, dx^n
\]
\[
- \int_{\partial \Omega} (H_k(x - y, u, v), (I - P_k) d\sigma_x u f(x, v)]_u ,
\]
Applying Stokes' Theorem and Borel-Pompeiu Theorem for the $Q_k$ operator in $\mathbb{R}^n$, we may have:

**Theorem 10.** Let $V$ be a bounded domain in $\mathbb{R}^n$ and $\nabla$ be the closure of $V$. For each $x \in \bar{V}$, the shifted lattice $x + \mathbb{Z}^l$ intersected with $V$ only contains the points $x$. Suppose that the boundary of $V$, $\partial V$, is piecewise smooth and $\nabla$ is compact. Further suppose $g(x, u) : \bar{V} \times \mathbb{R}^n \to Cl_n$ is a monogenic homogeneous polynomial of degree $k - 1$ in $u$ and with respect to $x$ is $C^1$. Then for $1 \leq l \leq n - 1$ and each $y \in V$,
\[
vg(y, v) = \int_{\partial V} \int_{V} (Cot_{l,k}(x - y, u, v), Q_k u f(x, u)]_u
\]
\[
- (Cot_{l,k}(x - y, u, v), (I - P_k) d\sigma_x u g(x, u)]_u, dx^n.
\]
Now using the projection map $\pi_l$, we obtain the Borel-Pompeiu Theorem for the $Q_k$ operators on the cylinder $C_l$.

**Theorem 11.** (Borel-Pompeiu Theorem for $Q_k^{C_l}$) Suppose $V'$ is a domain in $C_l$ with compact closure and smooth boundary. Suppose $g(x, u)$ is defined as in Theorem 10. Then for $1 \leq l \leq n - 1$ and each $y' \in V'$,
\[
vg'(y', v) = \int_{\partial V'} (Cot_{l,k}(x', y', u, v), (I - P_k) d\sigma_x u g'(x', u)]_u
\]
\[
- \int_{V'} (Cot_{l,k}(x', y', u, v), Q_k^{C_l} u g'(x', u)]_u, d\mu(x'),
\]
where $x' = \pi_l(x), d\sigma'_x = \partial x \pi_l d\sigma_x, \partial x \pi_l$ is the derivative of $\pi_l$ at $x$, $\mu$ is the projection of Lebesgue measure on $\mathbb{R}^n$ onto $C_l$, $d\sigma_x = n(x) d\sigma(x)$, $n(x)$ is the unit exterior normal vector at $x = \pi_l^{-1}(x')$ and $d\sigma(x)$ is the surface measure element.
Theorem 12. (Cauchy integral formula for $Q_{\mathcal{C}l}^{C_1}$) Suppose the hypotheses as in Theorem 10 and $u'g'(x', u)$ is annihilated by the operator $Q_{\mathcal{C}l}^{C_1}$, then for $1 \leq l \leq n-1$ and each $y' \in V'$, we have

$$v'g'(y', v) = \int_{\partial V'} (\text{Cot}_{l,k}(x', y', u, v), (I - P_k)\sigma_x u'g'(x', u))_u.$$  

If the function given in Borel-Pompeiu Theorem has its support with respect to $x'$ in $V' \subset C_l$, then we have the following:

Theorem 13. \[ \int_{V'} -(\text{Cot}_{l,k}(x', y', u, v), Q_{\mathcal{C}l}^C u\psi(x', u))_u dx^m = v\psi(y', v), \]

for $\psi \in C^\infty(C_l \times \mathbb{R}^n)$.

We may introduce a Cauchy transform for the remaining Rarita-Schwinger operator $Q_{\mathcal{C}l}^C$:

Definition 3. For a subdomain $V'$ of cylinder $C_l$ and a function $g' : V' \times \mathbb{R}^n \rightarrow \mathcal{C}l_n$, which is monogenic in $u$ with degree $k-1$, the Cauchy transform of $f'$ is formally defined to be

$$(Tv'g')(y', v) = -\int_{V'} (\text{Cot}_{l,k}(x', y', u, v), u'g'(x', u))_u dx^m, y' \in V'.$$

Consequently, one may obtain:

Theorem 14. \[ Q_{\mathcal{C}l}^C \int_{V'} (\text{Cot}_{l,k}(x', y', u, v), u\psi(x', u))_u dx^m = v\psi(y', v), \]

for $\psi \in C^\infty(C_l \times \mathbb{R}^n)$.

Similarly, we can carry on the theory of the remaining Rarita-Schwinger operators to the setting of conformally spinor bundles over the cylinders.

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