The uniqueness of the energy momentum tensor in non-Abelian gauge field theories

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(Received 28 June 1988; accepted for publication 24 May 1989)

The uniqueness of the energy momentum tensor in non-Abelian gauge field theories is established under minimal hypothesis.

I. INTRODUCTION

In the general theory of relativity, the interaction of the gravitational field (characterized by a metric tensor $g_{ij}$) and a source-free gauge field (characterized by a curvature form $F_{ij}$; see Ref. 1 for definitions and notations) is assumed to be governed by the Einstein–Yang–Mills field equations

$$ R^{ij} - g^{ij} R = B_{ij} (F_{ik} F_{jk} - \frac{1}{4} g_{ik} g_{jk} F^2_{hh}), $$

(1)

$$ F^{ij}_{[IL]} = 0, $$

(2)

where $B_{ij}$ are the coefficients of a bilinear symmetric form in $LG$, the Lie algebra of the Lie group $G$, which are $Ad G$ invariant, i.e., $B_{ij} = Ad^*_a (a) Ad^*_b (a) B_{ab}$ for all $a \in G$. Besides, the covariant gauge derivative of $F_{ij}^a$ is defined as

$$ F_{[IL]} = F_{ij}^a - \Gamma^i_{jk} F_{jk}^a - \Gamma^j_{ik} F_{ik}^a + C^i_{jl} A^o_{ij} F^o_{[ij]}, $$

(3)

where $C^i_{jl}$ are the structure constants of the Lie group and $A^o_{ij}$ are the gauge potentials (see Ref. 1 or Ref. 2) related to the curvature form by

$$ F_{ij}^a = A^a_{[ij]} - A_{ij}^a + C_{ijk} A^b_{ij} A^o_{bj}. $$

(4)

It is easy to see that with these definitions, the following identity holds:

$$ F_{[IL]} + F_{[IL]} = 0. $$

(5)

Since the Einstein tensor given by the left-hand side of (1) is divergence-free, the same must be true for $T_{ij}$, the right-hand side of (1). This case because of the identity

$$ T_{ij}^{0} = B_{ij} F_{[IL]} A^o_{[ij]}, $$

(6)

and Eq. (2). For any $T^{ij}$ in the right-hand side it must be true that $T_{ij}^{0} = 0$, at least when (2) holds. In other words, it must be true that

$$ F_{[IL]} = 0 \Rightarrow T_{ij}^{0} = 0. $$

(7)

The uniqueness of the energy momentum tensor was established recently^1 under the restrictive hypothesis $T_{ij}^{0} = C_{ij} (g_{ij})^2 F_{[ij]}^2$. Clearly (7) is weaker and it is mandatory because of (1) and (2). In this paper we will prove that $T_{ij}^{0}$ is essentially the only solution to the following problem: to find all gauge invariant symmetric tensors $T_{ij}^{0} = T^{0} (g_{ij}, F_{[ij]}^a)$ such that (7) holds. Our result generalizes Ref. 3.

We want to point out that, due to the condition (7), one cannot generate energy momentum tensors by adding terms to the action.

II. CONSEQUENCES OF THE IMPLICATION (7)

We will work in a coordinate system for which $(g_{ij}) = \text{diag} (-1,1,1,1)$ and $g_{[IL]} = 0$ (which implies $\Gamma_{[ij]} = 0$).

Then (2) reads

$$ F_{[12]}^a = F_{[24]}^a + F_{[41]}^a + F_{[14]}^a. $$

(8)

It is easy to see that

$$ T_{ij}^{0} = \frac{\partial T^{0}}{\partial F_{[ij]}^a} F_{[ij]}^a = T_{[ik]}^{[ih]} F_{[ij]}^a, $$

(9)

because of the gauge invariance of $T^{0}$ and its tensorial character. Then $T_{ij}^{0} = 0$ is written out in full in the above mentioned coordinate system is

$$
\begin{align*}
&\left(T_{[12]}^{[11]} + T_{[23]}^{[11]} + T_{[24]}^{[11]} + T_{[41]}^{[11]} \right) F_{[23]}^a + \left(T_{[12]}^{[12]} + T_{[23]}^{[12]} + T_{[24]}^{[12]} + T_{[41]}^{[12]} \right) F_{[24]}^a + \left(T_{[12]}^{[13]} + T_{[23]}^{[13]} + T_{[24]}^{[13]} + T_{[41]}^{[13]} \right) F_{[41]}^a + \left(T_{[12]}^{[14]} + T_{[23]}^{[14]} + T_{[24]}^{[14]} + T_{[41]}^{[14]} \right) F_{[14]}^a + \left(T_{[12]}^{[12]} + T_{[23]}^{[12]} + T_{[24]}^{[12]} + T_{[41]}^{[12]} \right) F_{[24]}^a + \\
&+ \left(T_{[12]}^{[13]} + T_{[23]}^{[13]} + T_{[24]}^{[13]} + T_{[41]}^{[13]} \right) F_{[41]}^a + \left(T_{[12]}^{[14]} + T_{[23]}^{[14]} + T_{[24]}^{[14]} + T_{[41]}^{[14]} \right) F_{[14]}^a + \left(T_{[12]}^{[12]} + T_{[23]}^{[12]} + T_{[24]}^{[12]} + T_{[41]}^{[12]} \right) F_{[24]}^a + \\
&+ \left(T_{[12]}^{[13]} + T_{[23]}^{[13]} + T_{[24]}^{[13]} + T_{[41]}^{[13]} \right) F_{[41]}^a + \left(T_{[12]}^{[14]} + T_{[23]}^{[14]} + T_{[24]}^{[14]} + T_{[41]}^{[14]} \right) F_{[14]}^a + \left(T_{[12]}^{[12]} + T_{[23]}^{[12]} + T_{[24]}^{[12]} + T_{[41]}^{[12]} \right) F_{[24]}^a + \\
&+ \left(T_{[12]}^{[13]} + T_{[23]}^{[13]} + T_{[24]}^{[13]} + T_{[41]}^{[13]} \right) F_{[41]}^a + \left(T_{[12]}^{[14]} + T_{[23]}^{[14]} + T_{[24]}^{[14]} + T_{[41]}^{[14]} \right) F_{[14]}^a + \left(T_{[12]}^{[12]} + T_{[23]}^{[12]} + T_{[24]}^{[12]} + T_{[41]}^{[12]} \right) F_{[24]}^a = 0.
\end{align*}
\]
Let us choose, for arbitrary but fixed $g_{ab}$, $F^n_{ab}$, the derivatives $F^n_{ab}$ such that (8) holds.

Taking account of (8) and (5), it is clear that the $F^n_{ab}$ appearing in (10) are arbitrary and independent. Then we deduce

$$T_{\beta}^{113} - T_{\beta}^{214} = T_{\beta}^{414} - T_{\beta}^{413} = T_{\beta}^{413} + T_{\beta}^{214} = T_{\beta}^{413} + T_{\beta}^{214} = 0$$

Taking $i = 1, 2, 3, 4$ in (11), a tedious but straightforward computation proves that

$$1212 = 1313 = 1414 = 2323 = 2424 = 3434 = 1234 = 1324 = 1423 = 2314 = 2413 = 3412 = 0$$
$$(12)$$

$$1232 = 1242 = -2212 = -2312 = -2312 = -3312 = 3112 = 4112 = 1112$$
$$(13)$$

$$-1223 = 1343 = -2213 = -2312 = -3312 = 3113 = 4113 = 1113$$
$$(14)$$

$$-1224 = -1334 = 2214 = -2412 = 3314 = -3413 = -4412 = 1114$$
$$(15)$$

$$-1231 = -1232 = 2232 = -2322 = 3232 = 3423 = -4423 = 1123$$
$$(16)$$

$$-1241 = -1422 = 2234 = -3324 = 3242 = 4424 = 1124$$
$$(17)$$

$$-1314 = 1413 = -2234 = 2324 = 3324 = 4344 = 1134$$
$$(18)$$

where, for the sake of simplicity, we have used the notation

$$T_{\beta}^{ijk} = ijk$$

for a fixed $\beta$.

### III. THE UNIQUENESS OF THE ENERGY MOMENTUM TENSOR

Let us denote, for fixed $\alpha$, $\beta$, and $\gamma$,

$$T^{ijklkrsm} = \frac{\partial^3 T^u}{\partial F^a_{\alpha} \partial F^b_{\beta} \partial F^c_{\gamma}}$$

We will prove that all these derivatives are zero. From (12)–(18) it is clear that it is enough to consider the cases $ijk = 1213, 1214, 1223, 1224, 1323, 1324, 1314$.

(i) The case $ijk = 1213$: It is clear that $T_{ijklkrsm}^{1213} = 0$ except perhaps for $(r,s) \neq (1,2) \neq (l,m)$ and $(r,s) \neq (3,4) \neq (l,m)$. Since $T^{ijk} = T^{ijk}$ for $i \neq j$ and $h \neq k$ as a consequence of (12)–(18), then in this case all the pairs commute, and so it is enough to consider $(r,s) \neq (1,3) \neq (l,m)$ and $(r,s) \neq (3,4)$, which leave us with the following cases:

(a) $(r,s,l,m) = (1,4,2,3)$,

(b) $(r,s,l,m) = (1,4,2,4),$

(c) $(r,s,l,m) = (2,3,2,4).$

In case (a) using (16), (15), and $T^{ijk}$ for $i \neq j$ and $h \neq k$, we have

$$T_{ijklkrsm}^{1213} = 0$$

(19)

In case (b) we have

$$T_{ijklkrsm}^{1213} = 0$$

(20)

Finally, in case (c) it is

$$T_{ijklkrsm}^{1213} = 0$$

(21)

where we have also used the equality of the cross derivatives. We conclude that

$$T_{ijklkrsm}^{1213} = 0$$

(22)

(ii) The case $ijk = 1214$: It is easy to see that $T_{ijklkrsm}^{1214} = 0$ except perhaps for $(r,s) \neq (1,2) \neq (l,m)$ and $(r,s) \neq (3,4) \neq (l,m)$, and $(r,s) \neq (2,3) \neq (l,m)$, which leaves us only with the case $T_{ijklkrsm}^{1214}$, and this is zero because of (20). Then

$$T_{ijklkrsm}^{1214} = 0$$

(23)

(iii) The case $ijk = 1223$: As in case (ii) $T_{ijklkrsm}^{1223} = 0$ except perhaps for $(r,s) \neq (1,2), (2,3), (3,4), (1,4) \neq (l,m)$, and $(r,s) \neq (l,m)$. This leaves us only with the case $T_{ijklkrsm}^{1223}$ which is zero by (21). Then

$$T_{ijklkrsm}^{1223} = 0$$

(24)

(iv) The case $ijk = 1224$: As before, it is enough to consider the cases $(r,s) \neq (1,2), (2,3), (3,4), (1,4) \neq (l,m)$, and $(r,s) \neq (l,m)$. Then there is only the case $T_{ijklkrsm}^{1224} = 0$. Then

$$T_{ijklkrsm}^{1224} = 0$$

(25)

(v) The case $ijk = 1323$: It is enough to consider $(r,s) \neq (1,3), (2,3), (2,4), (1,4) \neq (l,m)$, and $(r,s) \neq (l,m)$. Then

$$T_{ijklkrsm}^{1323} = 0$$

(26)

(vi) The case $ijk = 1314$: It is enough to consider $(r,s) \neq (1,3), (1,4), (2,4), (2,3), (l,m)$, and $(r,s) \neq (l,m)$. But then $T_{ijklkrsm}^{1314} = 0$, and so

$$T_{ijklkrsm}^{1314} = 0$$

(27)

From (22)–(27) we conclude that

$$T_{ijklkrsm}^{ijk} = 0$$

for all $i,j,h,k,r,s,l,m$, and so $T^u$ is a polynomial in $F^n_{ab}$ of degree not greater than two.

Consequently

$$T^u = A_{x,y}^{ijk}(g_{lm}) F^r_{hk} F^s_{rs} + B_x^{ijk}(g_{lm}) F^r_{hk} + C^u(g_{lm}).$$

(28)
The tensorial concomitants of $g_{lm}$ were recently found for any valence of the tensor. Taking account of the fact that we are dealing with all coordinate systems, and not merely with those belonging to an oriented atlas, then it follows that

$$T^{ij} = (d_{a}F_{abk}F_{b}^{a} + \lambda)g^{ij}$$

$$+ \frac{1}{2}a_{a}(F_{a}^{ij}F_{bij} + F_{a}^{ij}F_{bij}),$$

(29)

where $d_{a}$, $\lambda$, and $a_{a}$ are real numbers and $a_{a} = a_{b}a$, $d_{a} = d_{b}a$. Then

$$T^{ij} = (d_{a}F_{abk}F_{b}^{a} + \lambda)g^{ij} + a_{a}F_{a}^{ij}F_{bij}. $$

(30)

Assuming $F_{a}^{ij} = 0$, it follows that $T^{ij} = 0$, and so, using the identity (5) to change indices, we have

$$[2d_{a}g^{ij}F_{abk} + \frac{1}{2}a_{a}g^{ij}F_{abk}]F_{b}^{a}g^{ij} = 0.$$  

(31)

It is easy to see that if $S^{ij}_{abk}$ is the term within brackets in (31) then, because of (12)–(18), we have

$$S^{ij}_{ab}g^{ij} = 0.$$  

(32)

From (32) and the definition of $S^{ij}_{abk}$ it follows that

$$2(4d_{a} + a_{a})F_{abk} = 0.$$  

(33)

Differentiating (33) with respect to $F_{abk}$ we obtain

$$(4d_{a} + a_{a})(g^{ik}g^{jk} - g^{ik}g^{jk}) = 0.$$  

(34)

Multiplying (34) by $g_{ab}g_{ck}$, we deduce

$$d_{a} = - \frac{1}{4}a_{a},$$

(35)

and so, replacing (33) in (30), it follows that

$$T^{ij} = a_{a}(F_{aij}F_{bij} - \frac{1}{4}g^{ij}F_{abk}F_{b}^{k}) + \lambda g^{ij}. $$

(36)

It follows easily from the gauge invariance of $T^{ij}$ that the $a_{a}$ are Ad $G$ invariant.

In summary, we have proved the following.

**Theorem:** If $T^{ij} = T^{ij}(g_{ab}, F_{abk})$ is a gauge invariant tensor whose divergence vanishes when the divergence of $F_{aij}$ is zero, and if $T^{ij} = T_{ij}$, then

$$T^{ij} = T^{ij} + \lambda g^{ij},$$

where $T^{ij}$ is the usual energy momentum tensor.

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