The exterior gravitational potential of toroids

J.-M. Hure\textsuperscript{1,}\textsuperscript{*}, B. Basillais\textsuperscript{1}, V. Karas\textsuperscript{2}, A. Trova\textsuperscript{3}, and O. Semerák\textsuperscript{4}

\textsuperscript{1}Laboratoire d’Astrophysique de Bordeaux, Univ. Bordeaux, CNRS, B18N, allée Geoffroy Saint-Hilaire, 33615 Pessac, France
\textsuperscript{2}Astronomical Institute, Academy of Sciences, Boční I 1401, CZ-14100 Prague, Czech Republic
\textsuperscript{3}University of Bremen, Center of Applied Space Technology and Microgravity (ZARM), 28359 Bremen, Germany
\textsuperscript{4}Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, CZ-180 00 Prague, Czech Republic

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ABSTRACT

We perform a bivariate Taylor expansion of the axisymmetric Green function in order to determine the exterior potential of a static thin toroidal shell having a circular section, as given by the Laplace equation. This expansion, performed at the centre of the section, consists in an infinite series in the powers of the minor-to-major radius ratio $\epsilon$ of the shell. It is appropriate for a solid, homogeneous torus, as well as for inhomogeneous bodies (the case of a core stratification is considered). We show that the leading term is identical to the potential of a loop having the same main radius and the same mass — this “similarity” is shown to hold in the $O(\epsilon^2)$ order. The series converges very well, especially close to the surface of the toroid where the average relative precision is $\sim 10^{-3}$ for $\epsilon = 0.1$ at order zero, and as low as a few $10^{-6}$ at second order. The Laplace equation is satisfied exactly in every order, so no extra density is induced by truncation. The gravitational acceleration, important in dynamical studies, is reproduced with the same accuracy. The technique also applies to the magnetic potential and field generated by azimuthal currents as met in terrestrial and astrophysical plasmas.

Key words: Gravitation — Methods: analytical — Methods: numerical

1 INTRODUCTION

The derivation of reliable and compact expressions for the gravitational potential of massive toroids is a longstanding problem of dynamical astronomy, from planetary to galactic scales. This is essential not only to examine the motion of test-particles and fluids orbiting around, in the classical framework as well as in general relativity (Nieto 2005; Šubr & Karas 2005; Semerák & Suková 2010; Tresaco et al. 2011; Iorio 2012), but also to understand the conditions for the formation, evolution and stability of toroids themselves (Dyson 1893; Hachisu 1986; Chandrasekhar 1987; Tohline & Hachisu 1990; Woodward et al. 1992; Sterzer 1993; Christodoulou 1993; Hashimoto et al. 1993; Eriguchi & Mueller 1993; Nishida & Eriguchi 1994; Pickett et al. 1997; Horedt 2004; Lehmann et al. 2019). While it is relatively easy to deduce the mass density corresponding to a given potential (e.g. Binney & Tremaine 1987), the inverse procedure is very complicated by analytical means, and it is almost impossible to go beyond the classical series representations and to get closed forms (Clement 1974; Cohl et al. 2001; Petroff & Horatschek 2008). Fully numerical approaches may be preferred for their apparent simplicity, but the computing times are generally large, often prohibitive at high spatial resolution, especially for very inhomogeneous configurations and/or very extended systems like discs. The numerical accuracy of discretization schemes is mainly limited when treating thin sources (having less than three spatial dimensions) whose field typically suffers a certain irregularity at their position.

In axial symmetry, the Green function $G(r,r')$ of the Poisson equation involves the complete elliptic integral of the first kind $K$ whose argument (or modulus) gathers all the pertinent variables (Kellogg 1929; Durand 1953; Fukushima 2016). The presence of a special function is a real obstacle when it is to be convolved with any non-trivial mass density $\rho(r')$. One can overcome this difficulty by expanding $K$ over the modulus, but the “dual” nature of the series — different for large and for short separations — means piecewise approximations whose connection requires technical efforts. This is done for instance in Bannikova et al. (2011) who match together the internal and the external potentials of the solid (i.e. homogeneous) torus from a minimization procedure.

This article brings a new contribution to this general and challenging problem. It is inspired by Hure et al. (2019) who derived a reliable approximation for the interior po-

\textsuperscript{*} E-mail: jean-marc.hure@u-bordeaux.fr
tential of a toroidal shell of circular cross-section, based on a bivariate expansion performed at the pole (or focal ring) of the toroidal coordinates. At this singular point, all the partial derivatives of the Green function are exceptionally analytical. Unfortunately, the “pole” method does not apply outside the shell cavity because the line segment linking the focal ring to any exterior point crosses the shell where the Green function is basically singular. We generate accurate approximations for the exterior solution of the toroidal shell by expanding the axisymmetric Green function as a Taylor series before integrating over the source. As for the classical multipole expansion, the shell potential writes as an infinite series (e.g. Majic 2020), but our approach differs in that the origin of coordinates does not play a special role: the expansion is performed at the centre of the toroid section. Such an approach has been reported very recently by Kondratyev (2018) in the case of the solid torus. The author writes the external potential in the form \( \sum_n \phi_n e^{2\pi in\phi} \) (\( e \) is the minor-to-major radius ratio, i.e. the torus parameter). He then uses the second-order expression to set constraints on the masses of thin, virialized rings orbiting an asteroid.

In this article, we go beyond the hypothesis made in Bannikova et al. (2011) and in Kondratyev (2018) by considering inhomogenous systems too. In particular, we show that, when the toroid is radially stratified from the centre to the surface, only moments of the density need to be calculated. The method has an unexpected efficiency, not only at large distances, but also quite close to the surface of the toroid. The leading term has a correct behaviour at infinity as well as on the \( Z \)-axis, and it obeys the Laplace equation. As a matter of fact, these desirable properties are shared by all terms of the expansion. There is thus no spurious noise or extra density induced in space, whatever the truncation order. We treat orders 0 to 2 explicitly (a driver F90-program is appended). The resulting shell potential can be recast in the form of a “modified monopole” or in the form of an “equivalent loop”, which has been discussed in Stahler (1983), while proofs are found in Bannikova et al. (2011) and Kondratyev (2018). We show that the exterior potential of the solid torus, which is of more astrophysical relevance than the shell, is easily deduced, with all the properties observed for the shell maintained. The method also applies to the determination of the vector potential and magnetic field of electromagnetism for toroids carrying a purely azimuthal current (Trova et al. 2018).

The paper is organized as follows. In Sect. 2, the expression for the potential of a toroidal shell is given in its integral form. The axisymmetric Green function is expanded, in a bivariate manner, in Sect. 3. The leading term is calculated and compared with the potential of a monopole (i.e. a point mass) and of a circular loop. Its precision is checked against an “exact” numerical reference in Sect. 4. The 1st-order and 2nd-order approximations are treated then similarly in Sect. 5. In Sect. 6 we show how the exterior and interior solutions match together at the shell surface. The procedure leading to the \( n \)-th order term is detailed in Sect. 7. The case of a solid torus is treated in Sect. 8, while the case of core-stratified toroids is the aim of Sect. 9. The formula for the gravitational acceleration is derived in Sect. 10. In particular, we show that the vertical component that rules hydrostatic equilibrium differs by a factor of about 2 from Paczyński’s estimate valid for thin discs (Paczynski 1978). This result is suited to examining the stability of rings (e.g. Wisdom & Tremaine 1988). From the radial component, we deduce the circular velocity of test particles orbiting in the equatorial plane. This formula can be helpful in explaining the deviations to the Kepler’s law in massive systems (e.g. Guilloteau et al. 1999). Section 11 is devoted to the magnetic potential due to toroidal currents (the leading term is derived). Two general comments are found in Sect. 12. Conclusions and perspectives are found in the last section.

2 POTENTIAL OF THE TOROIDAL SHELL

We consider the simplest possible toroidal shell, as depicted in Fig. 1. The major radius is \( R_c \) and the meridional section is circular, with centre \( C \) and minor radius \( b \equiv e R_c \leq R_c \),

\[
b \equiv e R_c \leq R_c, \quad (1)
\]

where \( e \in [0,1] \) denotes the shell parameter. We work in cylindrical coordinates \((R, Z)\), using the symmetry axis of the shell as the \( Z \)-axis, and \( xOy \) as the plane of symmetry. For this specific problem, the Green function of the Poisson equation (e.g. Kellogg 1929; Durand 1953) is

\[
G(R, Z; a, z) = -2 \sqrt{\frac{a}{R}} k \mathbf{K}(k), \quad (2)
\]

where

\[
\mathbf{K}(k) = \int_0^{\frac{\pi}{2}} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}} \quad (3)
\]

is the complete elliptic integral of the first kind,

\[
k = \frac{2\sqrt{a R}}{\Delta} \in [0,1] \quad (4)
\]

is its modulus, and

\[
\Delta^2 = (R + a)^2 + \zeta^2, \quad (5)
\]

where \( \zeta = Z - z \), and \((a, z)\) are the cylindrical coordinates of any point \( P \) belonging to the shell. Basically, (2) corresponds to the potential created by an infinitesimally thin circular ring with unit mass per unit length, radius \( a \) and

Figure 1. The infinitely thin, toroidal shell (main centre \( O \) and main radius \( R_c \)) with a circular meridional section (centre \( C \) and core radius \( b \)).
altitude $z$. This function is known to be logarithmically singular at the location of the ring (where $k \to 1$). The gravitational potential generated, at any point $P(\bar{r})$ of space, by such the axisymmetric shell is then given by the integral

$$
\Psi(\bar{r}) = -2G \int_0^{2\pi} \Sigma(\ell) \sqrt{\frac{R}{k}} K(k) d\ell,
$$

where $\Sigma$ is the local surface density, and $d\ell$ is the infinitesimal length along the shell section. In the case of a shell with a circular section of radius $R_c$, $a$ and $z$ are simply given by

$$
a = R_c + b \cos \theta, \quad z = b \sin \theta,
$$

where $\theta \in [0, 2\pi]$ is the angular position of any point $P'$ on the shell with respect to the equatorial plane (see Fig. 1). The infinitesimal length then takes its simplest form, namely $d\ell = bd\theta$. Other options are possible, but the subsequent calculations are much more complicated (see Sect. 12).

The surface density $\Sigma$ may be variable in local latitude $\theta$. However, even if it is independent of $\theta$, (6) cannot in general be integrated into a compact form, except on the $Z$-axis (Sácha & Semerák 2005). An example of a direct numerical estimate of $\Psi$ is given in Fig. 2 for $e = 0.1$. We use the trapezoidal rule as the quadrature scheme. We will use such a numerical potential, denoted $\Psi_{\text{rel}}$ in the following, as a “reference” against which we will compare our approximations. As shown in Huré et al. (2019), the potential inside the shell cavity is a quasi-linear function of the cylindrical radius $R$, and it is weakly sensitive to the $Z$-coordinate, especially when $e \ll 1$. We see that the potential outside the cavity has a more complex structure. It resembles the potential of a loop, as already pointed out (e.g. Wong 1973; Bannikova et al. 2011; Kondratyev 2018).

3 EXPANSION OF THE GREEN FUNCTION. ZERO-ORDER FORMULA

As quoted in the introduction, the elliptic integral $K$ may be expanded over $k$ at $k \to 0$, which means far away from the source or close to the $Z$-axis, and over $k' = \sqrt{1-k^2}$ at $k \to 1$, e.g. close to or even inside the source (see e.g. Abramowitz & Stegun 1964; Gradshteyn & Ryzhik 2007). However, such two series have to be matched somewhere (Bannikova et al. 2011). In the present paper, we propose a more synthetic approach which consists in expanding the axisymmetric Green function over $a$ and $z$, before integration over $\theta$ in (6). We expect to preserve the asymptotic behaviour of the potential at large distances. Let us remind that, for any “regular enough” function $f$ of two independent variables $x$ and $y$, the bivariate Taylor expansion at $(x_0, y_0)$ writes

$$
f(x, y) = f(x_0, y_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[ (x-x_0) \frac{\partial}{\partial x} + (y-y_0) \frac{\partial}{\partial y} \right]^n f(x', y') \bigg|_{x'=x_0, y'=y_0}.
$$

The expansion is performed in $x \equiv a$ and $y \equiv z$, at the centre $C$ of the shell, i.e. at $x_0 \equiv R_c$ and $y_0 \equiv 0$ (see Sect. 12 for the expansion at the focal ring). We see from (7) that this is valid for $e < 1$, which in astrophysical toroids (typically orbiting a massive central body) is safely satisfied.

In fact, it is not necessary to expand the whole Green function. In particular, the term $\sqrt{\Delta}$ is not problematic and it can be left aside. There are several options. For instance, if we expand $kK(k)$ or $K(k)$, the subsequent integration over $\theta$ will generate a new series of elliptic integrals (again, see Sect. 12). That is not a problem per se, but it complicates the calculations when the solid torus is considered. Seeing that the complication can be avoided when extracting the other factor $\sqrt{\Delta}$ contained in the modulus (4), we finally choose to expand

$$
\frac{K(k)}{\Delta} \equiv \kappa
$$

as

$$
\kappa = \kappa|_{a=R_c} + (a-R_c) \frac{\partial \kappa}{\partial a}|_{a=R_c} + \frac{1}{2} (a-R_c)^2 \frac{\partial^2 \kappa}{\partial a^2}|_{a=R_c} + \ldots
$$

where

$$
\Psi(\bar{r}) \approx -4G \Sigma_0 \kappa b \times 2\pi R_c S_{0,0} \equiv \Psi_0(\bar{r}),
$$

(11)

where

$$
k_0^2 = \frac{4 R_c R}{\Delta^3}.
$$
\[ J_{n,m} \quad S_{n,m} \quad V_{n,m} \]

| n + m | n | m | \( J_{n,m} \) | \( S_{n,m} \) | \( V_{n,m} \) |
|------|---|---|-------------|-------------|-------------|
| 0    | 0 | 0 | \( \frac{1}{\pi} B\left(\frac{1}{2}, \frac{1}{2}\right) \) | 1           | 1           |
| 1    | 1 | 0 | 0           | \( \frac{1}{4} e^{2} \) | \( \frac{1}{4} e^{2} \) |
|      | 0 | 1 | 0           | 0           | 0           |
| 2    | 2 | 0 | \( \frac{1}{\pi} B\left(\frac{3}{2}, \frac{1}{2}\right) \) | \( \frac{1}{4} e^{2} \) | \( \frac{1}{4} e^{2} \) |
|      | 1 | 1 | 0           | 0           | 0           |
|      | 0 | 2 | \( \frac{1}{\pi} B\left(\frac{3}{2}, \frac{3}{2}\right) \) | \( \frac{1}{2} e^{2} \) | \( \frac{1}{4} e^{2} \) |
| 3    | 3 | 0 | 0           | 0           | 0           |
|      | 2 | 1 | 0           | 0           | 0           |
|      | 1 | 2 | 0           | 0           | 0           |

Table 1. Expressions for \( J_{n,m} \) required when expanding the Green function up to second order. Also given are the surface factor \( S_{n,m} \) and the volume factor \( V_{n,m} \).

\[ \Delta_{0}^{3} = (R + R_{c})^{2} + Z^{2}, \]

\[ \kappa_{0} = \frac{K(k_{0})}{\Delta_{0}}, \]

and

\[ S_{0,0} = \frac{1}{2\pi \Sigma_{0} R_{c}} \int_{0}^{2\pi} \Sigma(\theta) a d\theta \]

called the “surface factor” in the following. In this paper, we will consider homogenous shells, so we set \( \Sigma = \text{const.} = \Sigma_{0} \).

Anticipating higher orders, let us define the whole series of definite integrals

\[ J_{n,m} = \frac{1}{2\pi} \int_{0}^{2\pi} \cos^{n} \theta \sin^{m} \theta d\theta, \]

where \( n \) and \( m \) are positive integers. They can all be written in terms of the complete Beta function \( B\left(\frac{n}{2}, \frac{m}{2}\right) \); see the Appendix A. We give \( J_{n,m} \) for the first few values of \( n \) and \( m \) in Tab. 1. We note in particular that \( J_{n,m} = 0 \) when either \( m \), or \( n \) or \( m + n \) is odd. We have \( S_{0,0} = J_{0,0} + \epsilon J_{1,0} = 1 \) and so the zero-order approximation for the potential of the shell is approximately given by

\[ \Psi_{0}(\vec{r}) = -8\pi G \Sigma_{0} b R_{c} \kappa_{0}. \]

3.1 Comparison with the potential of a point mass

It is easy to compare (17) to the potential of some simple sources, like the potential of a monopole (or point mass at the origin), which is of major interest in dynamical studies. Introducing the mass of the homogeneous shell \( M_{\text{shell}} = 4\pi \Sigma_{0} b R_{c} \), (17) writes

\[ \Psi_{0}(\vec{r}) = -\frac{GM_{\text{shell}}}{r} \times g_{0,0}, \]

where \( r = \sqrt{R^{2} + Z^{2}} \) is the spherical radius, and

\[ g_{0,0} = \frac{r}{\Delta_{0}^{3}} K(k_{0}). \]

We see that (17) differs from the monopole potential only by the quantity \( g_{0,0} \) which is a function of the position in space only. In the physical space, it also depends on \( R_{c} \), but not of \( e \), i.e. \( g_{0,0} \equiv g_{0,0}(\vec{r}; R_{c}) \). The meridional-plane contours of \( g_{0,0} \) are shown in Fig. 3 in the neighborhood of the shell. The contours are closed, except the \( g_{0,0} = 1 \) one. The region where \( g_{0,0} > 1 \) surrounds the shell section, while \( g_{0,0} < 1 \) concerns the central region near the \( Z \)-axis. This kind of map is helpful for dynamical studies since it indicates very well the families of bounded and unbounded trajectories of test particles moving with a constant angular momentum (on equatorial or inclined toroidal orbits and on purely meridional orbits).

3.2 Comparison with the potential of a circular loop

Let us also compare the zero-order shell potential (17) and that of a circular loop of radius \( R_{c} \) and mass \( M_{\text{loop}} = 2\pi \lambda R_{c} \), which writes

\[ \Psi_{\text{loop}}(\vec{r}) = -2\pi \lambda \sqrt{\frac{r}{R}} k_{0} K(k_{0}), \]

\[ = -\frac{GM_{\text{loop}}}{r} \frac{r}{\Delta_{0}^{3}} K(k_{0}). \]

We see that, at the zeroth order, \( \Psi_{0} = \Psi_{\text{loop}} \) at any point \( P \) in space, and for any value of \( e \), provided \( M_{\text{shell}} = M_{\text{loop}} \).

According to Kondratyev (2018) (see also below), there is no term in the expansion led by \( e \) and, more generally, by odd powers of \( e \) (this is not guaranteed as soon as \( \Sigma \) varies with \( \theta \)). This implies that \( \Psi_{\text{shell}} = \Psi_{\text{loop}} + O(e^{2}) \). We can thus conclude that (similarity theorem 1):

\[ a \text{ homogenous toroidal shell of main radius } R_{c} \text{ and circular section generates, at the first order in the } e-\text{parameter, the same exterior potential as a circular loop} \]
of same radius $R_c$ and same mass.

This result has a few important consequences. First, the approximation thus behaves correctly at infinity and on the $Z$-axis as well ($\lim_{R\to0} K(k_0) = \frac{2}{\pi}$ and $k_0/\sqrt{R}$ is finite at the $Z$-axis). Second, the gravitational acceleration inherits these properties, i.e. the similarity theorem also applies to $\vec{g} = -\nabla \Psi$ (see below). Third, the formula (17) does not generate any residual mass distribution in space. This is easily verifiable by calculating the Laplacian of $\kappa_0$ (see the Appendix B for the demonstration), i.e.

$$\nabla^2 \Psi_0 = 4\pi G\rho^{\text{ext}} = 0. \quad (21)$$

This property is also intrinsic to the interior solutions reported in Huré et al. (2019).

4 NUMERICAL TESTS. DOMAIN OF VALIDITY

Let us now compare the expression (17) to the numerical reference (see Sect. 2). We quantify the relative difference by

$$\epsilon = \log \left| \frac{\Psi - \Psi_{\text{ref}}}{\Psi_{\text{ref}}} \right|. \quad (22)$$

Figure 4 shows $\epsilon$ in the upper half-plane $Z > 0$ for $\epsilon = 0.1$. If we limit the statistics to the domain exterior to the shell, the average precision is of the order of $10^{-3}$ in the vicinity of the shell, and it is much lower in the far field. The deviation with respect to the reference never exceeds 1%. The error is maximal near the surface of the shell, which is not a surprise. On the other hand, the best approximation is achieved in a narrow domain going from the top of the shell to infinity along the line $Z \sim 0.7R$.

Our expansion is performed at the center C of the shell section. The expanded function thus has to be smooth along the line $CP'$, where $C$ is the center of the shell section and $P'$ is any point $P$ located at the shell surface. This, however, is not the case for $K(k)/\Delta$ which is singular for any point $P(R, Z)$ belonging to the line segment [CP']. Therefore, the formula (10) and subsequently the zero-order approximation is only valid outside the cavity, namely for

$$(R - R_c)^2 + Z^2 - b^2 > 0. \quad (23)$$

The comparison has been checked for different values of the shell parameter $\epsilon$. The results are plotted in Fig. 5 where values of $\epsilon$ have been averaged over values contained inside a squared box $[1-2e, 1+2e] \times [0, 4e]$ (in dimensionless units) encompassing the shell section (see Fig. 4); interior values are excluded. We see that the smaller the shell parameter, the better the approximation. The precision of the zero-order approximation remains better than 1% for shell parameter as large as about 0.3, which is remarkable.

5 EXPANSION UP TO 2ND ORDER

Though already very good, the zero-order approximation can be improved by considering further terms in the expansion. For order one, we have to calculate

$$\int_0^{2\pi} \left[ (a - R_c) \frac{\partial \kappa}{\partial a} \bigg|_{a=R_c, z=0} + 2 \frac{\partial \kappa}{\partial \zeta} \bigg|_{a=R_c, z=0} \right] abd\theta, \quad (24)$$

where $a$ and $z$ are still given by (7). Because the derivatives are evaluated at $a = R_c$ and $z = 0$, they are not concerned by the integration over $\theta$ and can be carried out of the operator. There are two new surface factors to calculate, namely (we do not include $\Sigma$ in these definitions, since we assume it is
but this latter term vanishes (since $e_1$ and $J_1, J_0$ are zero). In the first order, the potential writes $\Psi \approx \Psi_0 + \Psi_1$, where $\Psi_0$ is given by (17) and

$$\Psi_1 = -8\pi G\Sigma_0 br_c^2 \left. \frac{\partial^2 \kappa}{\partial a^2} \right|_{a=R_c, z=0} \times S_{1,0},$$

(27)

where $S_{1,0} = e(J_{1,0} + eJ_{2,0})$. Note that the derivative $\frac{\partial^2 \kappa}{\partial a^2}$ is analytical (see the Appendix C). Since $J_{1,0} = 0$, the first-order correction depends on $e^2$. As $2\pi J_{2,0} = 2B(\frac{3}{2}, \frac{1}{2}) = \pi$, we have

$$S_{1,0} = \frac{1}{2} e^2.$$  

(28)

It is pertinent to account for the next term in the Taylor expansion which also contains a contribution varying as $e^2$. This term is

$$\frac{1}{2} \int_0^{2\pi} \left[ (a-R_c)^2 \left. \frac{\partial^2 \kappa}{\partial a^2} \right|_{a=R_c, z=0} + 2(a-R_c)z \left. \frac{\partial^2 \kappa}{\partial a \partial z} \right|_{a=R_c, z=0} \right] \times \left. \frac{\partial^2 \kappa}{\partial z^2} \right|_{z=0} \times S_{2,0} + \frac{1}{2} \left. \frac{\partial^2 \kappa}{\partial a^2} \right|_{z=0} \times S_{1,1} + \frac{1}{2} \left. \frac{\partial^2 \kappa}{\partial z^2} \right|_{a=R_c} \times S_{0,2},$$

(29)

and so the 2nd-order approximation is given by $\Psi \approx \Psi_0 + \Psi_1 + \Psi_2$ with

$$\Psi_2 = -8\pi G\Sigma_0 b r_c^3 \times \left[ \frac{1}{2} \left. \frac{\partial^2 \kappa}{\partial a^2} \right|_{z=0} S_{2,0} + 2 \left. \frac{\partial^2 \kappa}{\partial a \partial z} \right|_{z=0} S_{1,1} + \left. \frac{\partial^2 \kappa}{\partial z^2} \right|_{a=R_c} S_{0,2} \right],$$

(30)

where the derivatives are given in the Appendix C. The new surface factors are (again, $\Sigma$ is removed from these definitions)

$$S_{2,0} = \frac{1}{2\pi R_c^2} \int_0^{2\pi} (a-R_c)^2 a \, d\theta,$$  

(31)

$$S_{1,1} = \frac{1}{2\pi R_c^2} \int_0^{2\pi} (a-R_c)z a \, d\theta,$$  

(32)

and

$$S_{0,2} = \frac{1}{2\pi R_c^2} \int_0^{2\pi} z^2 a \, d\theta.$$  

(33)

We see that $S_{1,1} = 0$, again because of the odd power of $z$. The non-zero terms are $S_{2,0} = e^2(J_{2,0} + eJ_{3,0})$ and $S_{0,2} = e^2(J_{0,2} + eJ_{1,2})$, where $J_{1,0} = J_{1,2} = 0$, $2\pi J_{2,0} = 2B(\frac{3}{2}, \frac{1}{2}) = \pi$. $S_{2,0} = S_{0,2} = \frac{1}{2} e^2$. We thus see that it is necessary to include both orders 1 and 2 simultaneously in order to obtain a consistent $e^2$-approximation. This new approximation can be put in the form of a modified monopole, like we did for the zeroth-order expression. There is one specific correction factor $g_{n,m} \equiv g_{n,m}(\frac{r}{R_c}, e)$ for each non-zero surface factor $S_{n,m}$. We find

$$\Psi_1 = -\frac{GM_{\text{shell}}}{r} g_{1,0},$$

(34)

where

$$g_{1,0} = \frac{2}{\pi} \left. \frac{\partial \kappa}{\partial a} \right|_{a=R_c} z \, S_{1,0}(z=0),$$

(35)

and

$$\Psi_2 = -\frac{GM_{\text{shell}}}{r} \left( g_{2,0} + g_{0,2} \right),$$

(36)

where

$$g_{2,0} = \frac{2}{\pi} \left. \frac{\partial^2 \kappa}{\partial a^2} \right|_{a=R_c} R_c^2 S_{2,0}(z=0),$$

(37)

and

$$g_{0,2} = \frac{2}{\pi} \left. \frac{\partial^2 \kappa}{\partial z^2} \right|_{a=R_c} R_c^2 S_{0,2}(z=0).$$

(38)

Because of the circular section, $S_{2,0} = S_{0,2}$ which implies that the partial sum $\Psi_1 + \Psi_2$ can be rewritten in a very compact form. We actually find

$$g_{1,0} + \frac{1}{2} (g_{2,0} + g_{0,2}) = \frac{2}{\pi} \times \frac{e^2}{8k^2 \Delta_2^2} \times \left\{ \left[ \Delta_2^2 - 2R_c (R_c + R) \right] E(k) - k^2 \Delta_2^2 K(k) \right\},$$

(39)

which is to be multiplied by $-GM_{\text{shell}}/r$.

Figure 6 compares the second-order approximation obtained from (17), (27) and (30) with the reference values (see Sec. 2), as computed under the same conditions as in Fig. 4. We notice that the $e^2$-approximation reproduces the potential with almost 6-digits precision in the close vicinity of the shell. At larger distances, the expansion is extremely efficient (in the present example, the potential is known with more than 10 digits for $r/R_c \geq 5$ typically). The variation of the averaged precision as a function of the shell parameter $e$ is plotted in Fig. 5, in the same conditions as for the zeroth-order approximation.
Figure 6. The same legend and the same conditions as for Fig. 4, but for the 2nd-order approximation (i.e. $e^2$-approximation).

6 VALUES AT THE SURFACE. MATCH WITH THE INTERIOR SOLUTION

At the surface of the shell, we have $R = R_e + b \cos \theta$ and $Z = b \sin \theta$, or equivalently $(R - R_e)^2 + Z^2 - b^2 = 0$ with $R \in [R_e - b, R_e + b]$. If we introduce these expressions in (12), the modulus $k_0$ simplifies into

$$k_0^2 = \frac{4RR_e}{4RR_e + b^2}. \quad (40)$$

Using these values in (17), (27) and (30), we get the potential at the shell surface, which can be compared to the values obtained from the interior solution reported in Huré et al. (2019); see their equations (35), (36) and (39). The results are displayed in Fig. 7, again for $e = 0.1$. We see that the matching is very good at the second order (with at least 4 correct digits). At the zeroth order, the interior solution reduces to a constant potential throughout the toroidal cav-

7 GENERALIZATION TO NTH-ORDER EXPANSION

It is possible to include further terms in the Taylor series. Since the expansion writes formally

$$\Psi = \Psi_0 + \sum_{n=1}^{N} \Psi_n, \quad (42)$$

where the $n$th-order contribution $\Psi_n$ is made of $n + 1$ terms, namely

$$\Psi_n = -4G\Sigma_0 \times \frac{1}{n!} \sum_{m=0}^{\infty} \binom{n}{m} \frac{\partial^n \kappa}{\partial a^n \partial b^m} \bigg|_{a=R_e} \times 2\pi bR_e^{n+1} S_{n-m,m}, \quad (43)$$

where we have set (still in the homogeneous case)

$$S_{n,m} = \frac{1}{2\pi bR_e^{n+1}} \int_0^{2\pi} (a - R_e)^{n} b^m a d\theta. \quad (44)$$

Note that $\Psi_{n+1} \ll \Psi_n$ when $e \ll 1$, and the equality in (42) is obtained in the limit $N \to \infty$. In the form of the modified monopole representation, the $n$-order correction is

$$\Psi_n = -\frac{GM_{\text{shell}}}{r} \sum_{m=0}^{n} \Theta_{n,m}. \quad (45)$$
where

$$g_{n,m} = \frac{2}{\pi} \times \frac{1}{n!} \left( \frac{n}{m} \right) \frac{\partial^n \kappa}{\partial a^{n-m} \partial z^m} \bigg|_{a=R_0} R_0^2 S_{n-m-m}.$$  (46)

Since the two operators $\nabla_{R,Z}^2$ and $\partial^n / \partial a^{n-m} \partial z^m$ act on different spaces, we have

$$\nabla_{R,Z}^2 \left( \frac{\partial^n \kappa}{\partial a^{n-m} \partial z^m} \right) = \frac{\partial^n}{\partial a^{n-m} \partial z^m} \left( \nabla_{R,Z}^2 \kappa \right) = 0$$  (47)

for any pair $(n,m)$. This is expected because $\kappa$ is a harmonic function (see Sec. 4). We thus conclude that $\nabla_{R,Z}^2 \Psi_n = 0$, for any $n$, which means that each term of the expansion separately obeys the Laplace equation. Therefore, expanding the Green function over $a$ and $z$ induces no residual source mass in space, whatever the order of the truncation.

### 8 THE SOLID TORUS

The above solution for the shell can be employed to obtain the potential of a solid toroid. This is achieved by integrating (6) over $b$, while the surface density $\Sigma$ is changed for $\rho db$, $\rho$ being the mass density. The result is

$$\Psi(\vec{r}) = -4G \int_0^b \int_0^{2\pi} \rho(b', \theta) a b' d\theta db',$$  (48)

where $\kappa$ can be replaced by its Taylor expansion, namely (10). In the leading term, i.e. using just (17), we have

$$\Psi_0(\vec{r}) = -4G \rho_0 \times \pi R_0 b^2 \rho_0 V_{0,0},$$  (49)

where $\rho_0$ is some typical mass density, and $V_{0,0}$ is the “volume factor” defined in general by

$$V_{0,0} = \frac{1}{\pi \rho_0 R_0 b^2} \int_0^b b' dB \int_0^{2\pi} \rho(b', \theta) a d\theta,$$  (50)

where $b' = e' R_0 \leq b$. As quoted, the mass density $\rho$ may vary with both $\theta$ and $b'$ (see below). In the homogeneous case, we have $\rho = \text{const.}= \rho_0$, and so (50) becomes

$$V_{0,0} = \frac{2 e^2}{\rho_0 R_0^2} \int_0^e \rho(e') e' S_{0,0}(e') de' = \frac{2 e^2}{\pi^2} \int_0^e \rho(e') e' S_{0,0}(e') de',$$  (51)

where the dependence of the surface factor with the shell parameter $e$ has been-explicated. Introducing the total mass of the homogeneous torus $M_{\text{solid}} = 2\pi^2 \rho_0 b^2 R_a$, the zero-order formula can be written in the form

$$\Psi_0(\vec{r}) = -\frac{GM_{\text{solid}}}{r} g_{0,0},$$  (52)

where we have set

$$g_{0,0} = \frac{r}{\Delta_0} \frac{2}{\pi} K(h_0) V_{0,0}.  \quad (53)$$

Again, the difference from the point-mass potential is represented by the term $g_{0,0}$, while the deviation with respect to the potential of a massive loop (of radius $a = R_a$) is given by the volume factor $V_{0,0}$. For the zero-order approximation, we have $V_{0,0} = 1$, and so $\Psi_0 = \Psi_0^{\text{loop}} + O(e^2)$. We can thus conclude (similarity theorem 2):

a solid torus of main radius $R_c$ and circular section generates, at the first order in the $e$-parameter, the same exterior potential as a circular loop of radius $R_c$ and same mass.

The derivation of the $e^2$-term requires $V_{1,0}$, $V_{2,0}$ and $V_{0,2}$, which are listed in Tab. 2. These quantities happen to be equal (due to the circular section). As a consequence, the partial sum $\Psi_1 + \Psi_2$ resembles the formula derived in Kondratyev (2018). We finally find

$$\Psi_1 + \Psi_2 = e^2 \left( \frac{GM_0 R_0 b^2}{4k^2 \Delta_0^2} \right) \times \left[ \left( \Delta_0^2 - 2 R_c (R_c + R_0) \right) E(k) - \kappa^2 2 \Delta_0^2 \left( k^2 \Delta_0^2 \right) \right].$$  \quad (54)

We have compared (49) to a reference obtained by direct numerical integration of (48). As we have observed, the error map is the same as for the shell, which is expected since the onlydifference between the shell and the torus stands in the volume factor which is analytical. This remark holds for the $e^2$-approximation.

More terms in the expansion of $\kappa$ can be accounted for. The $n$-order contribution is

$$\Psi_n = -4G \frac{1}{ \pi} \sum_{m=0}^n \left( \frac{n}{m} \right) \frac{\partial^n \kappa}{\partial a^{n-m} \partial z^m} \bigg|_{a=R_c} R_c^2 S_{n-m-m},$$  \quad (55)

$$\times 2 \pi R_c^{n+1} \rho \int_0^b S_{n-m-m}(b') b' db',$n

where $S_{n-m-m}$ depends on $b'$ as indicated. If we set the volume factor $V_{n,m}$ to

$$V_{n,m} = \frac{1}{2 \pi R_c^{n+m+1}} \frac{2 e^2}{\pi^2} \int_0^b b' dB \int_0^{2\pi} (a - R_c) z^{m} a d\theta,$n

then $\Psi_n$ has the same form as (45) where $S_{n-m-m}$ is just to be replaced by $V_{n-m-m}$ and $M_{\text{shell}}$ by $M_{\text{solid}}$. It can be checked that $\Psi_n$ is harmonic.

1 We notice two differences between (54) and the formula (14) by Kondratyev (2018): the factor $R_0^2$ should be $R_0 R_a^2 \equiv R_c b^2$ (since the $\phi_2$ term is multiplied by $e^2$), and the factor 16 at the denominator should be 4.
9 INHOMOGENEOUS SYSTEMS

As a matter of fact, (41) works for inhomogenous systems. Actually, the expansion depends on α and z only through powers of \( \cos \theta \) and \( \sin \theta \). If \( \Sigma(\theta) \) is prescribed, the knowledge of any term \( \Psi_n \), just requires the calculation of the surface factors according to

\[
S_{n,m} = \frac{1}{2\pi \Sigma_0 R_c^{m+1}} \int_0^{2\pi} \Sigma(\theta)(a - R_c)^n z_m \, d\theta
\]

(57)

\[
= \frac{e^{m+n}}{2\pi \Sigma_0} \int_0^{2\pi} \Sigma(\theta) \cos^n \theta \sin^m \theta (1 + e \cos \theta) \, d\theta.
\]

We see that the \( S_{n,m} \)'s are combinations of moments of the surface density profile, which are analytical for a wide family of \( \Sigma(\theta) \)-profiles. In a similar way for the solid torus, if \( \rho \) depends both on \( \theta \) and on \( b' \leq b \), then the volume factors are calculated following

\[
V_{n,m} = \frac{1}{2\pi \rho_0 R_c^{m+1} b^2} \times \int_0^b b' db' \int_0^{2\pi} \rho(b', \theta)(a - R_c)^n z_m \, d\theta,
\]

(58)

\[
= \frac{\pi \rho_0}{e^{m+n}} \times \int_0^1 x^{m+n+1} dx' \int_0^{2\pi} \rho(x', \theta) \cos^n \theta \sin^m \theta (1 + b x' \cos \theta) \, d\theta,
\]

where we have set \( b' = b x' \leq b \). We can go a little bit further in the analysis by considering the case where the two variables \( b' \) and \( \theta \) are separable, i.e. \( \rho(b', \theta) = f(b') \times g(\theta) \). This corresponds to toroids having a core stratification. For instance, if we assume the \( \theta \)-invariance and (with 2\( n > 1 \))

\[
\rho(b') = \rho_0 \left[ 1 - \left( \frac{b - b'}{b} \right)^{2n} \right],
\]

(59)

then the volume factor required at order zero (i.e. \( n = m = 0 \)) is

\[
V_{0,0} = \int_0^1 2x'(1 - x^{2n}) dx',
\]

\[
= \frac{\alpha}{1 + \alpha}.
\]

(60)

Note that \( V_{0,0} \rightarrow 1 \) as \( \alpha \rightarrow \infty \). Since \( M_{\text{total}} V_{0,0} \) is just the total mass \( M \) of the (inhomogeneous) core-stratified torus, we have \( \Psi = \Psi_{\text{loop}} + \mathcal{O}(e^2) \). We can conclude that (similarity theorem 3):

a core-stratified torus of main radius \( R_c \) and circular section generates, at the first order in the \( e \)-parameter, the same exterior potential as a circular loop of radius \( R_c \) and same mass.

Table 2 lists values of \( V_{n,m} \) corresponding to (59).

10 GRAVITATIONAL ACCELERATION

For a massive loop, the non-zero components of acceleration \( \ddot{g} = -\nabla \Psi \) are given by (Durand 1953; Huré 2005)

\[
g_{\tau \tau} = \frac{GM_{\text{loop}}}{2\pi R_c R} \sqrt{\frac{R_c}{R}}
\]

\[
\times k_0 \left[ E(k_0) - K(k_0) + \frac{(R_c - R) k_0^2 E(k_0)}{2R_c k_0^2} \right],
\]

and

\[
g_{z z} = -\frac{GM_{\text{loop}} Z}{4\pi R_c \sqrt{R_c \pi}} k_0^2 E(k_0).
\]

(61)

According to the similarity theorems to 3, the acceleration outside a shell, solid torus, or core-stratified toroid is the same as for a loop having the same mass \( M \), and deviations are \( \mathcal{O}(e^2) \). This result is very convenient for a study of the motion of orbiting test-particles. Several types of trajectories can be distinguished. Of particular interest are circular trajectories tied to the equatorial plane of the shell/torus and having \( R \notin [R_c - b, R_c + b] \). The orbital velocity \( v_{\text{o}}(R) \) is easily deduced from (61). We find

\[
v_{\text{o}}^2(R) = \frac{GM}{R + R_c} \frac{1}{\pi} \left[ K(k_0) + \frac{R + R_c}{R - R_c} E(k_0) \right],
\]

(63)

where \( k_0 = \frac{2\pi \sqrt{R_c R}}{R + R_c} \) follows from (12) where \( Z \) has been set to 0. Figure 8 displays (63) versus the radius. Note that \( v_{\text{o}}^2(R) \leq 0 \) for \( R \in [0, R_c - b] \), which means that orbits are in principle forbidden in this region, unless a massive central object is present. The Keplerian profile associated to a point mass at the origin is shown in comparison. For \( R > R_c + b \), the velocity is super-Keplerian. It is a decreasing function of the radius, the maximum value being reached at the outer radius \( R_c + b \) of the toroid. To the detriment of precision, we can replace the elliptic integrals by more standard functions when \( k_0 \rightarrow 1 \), which corresponds to particles orbiting very close to the inner/outer radius of the toroid. Within this limit, \( K(k_0) \sim \ln \left( \frac{b}{R_c - R_c} \right) \) and \( E(k_0) \sim 1 \).

Another interesting quantity is the vertical component of acceleration at the surface of thin/small rings. It is a fundamental ingredient that governs the hydrostatic equilibrium of astrophysical discs (e.g. Shakura & Sunyaev 1973; Pringle 1981). By setting \( E(k_0) \approx 1 \), which again corresponds to the vicinity of the toroid, we find

\[
g_z = -2\pi G \rho \frac{\sin \theta}{\sqrt{1 + e \cos \theta + \frac{a^2}{2}}}.
\]

(64)

where \( \theta \in [0, \pi] \) above the equatorial plane. This quantity is plotted in Fig. 9 for \( e = 0.1 \) as the torus parameter. It varies between 0 at the inner/outer edges to about \( -2\pi G \rho b \) at \( \theta = \frac{\pi}{2} \). It is interesting to see that Paczynski's approximation (Paczynski 1978), classically written as \( -4\pi G \rho Z \), overestimates the acceleration by a factor 2 in the middle of the toroid. This observation may e.g. be of importance in oscillation modes in planetary or other rings (Wisdom & Tremaine 1988; Lehmann et al. 2019). It also means that, in a geometrically thin discs where Paczynski's approximation is valid, half of the vertical acceleration comes from the local contribution of matter while the other half comes from the global or long-range distribution of matter (Trova et al. 2014).
11 MAGNETIC POTENTIAL AND FIELD FOR PURELY AZIMUTHAL CURRENTS (IN SURFACE AND VOLUME)

The method presented in this paper can also be applied to the determination of the vector potential $\vec{A}$ of electromagnetism. Toroidal currents are met in both terrestrial and astrophysical plasmas (Dini et al. 2009; Trova et al. 2018). The magnetic potential $\vec{A} = A_x \vec{e}_x + A_y \vec{e}_y + A_z \vec{e}_z$ of a toroidal shell carrying a purely azimuthal electric current $\sigma \vec{e}_z$ is obtained by summing over the contribution of individual current loops (Jackson 1998; Cohl et al. 2001), namely

$$A_x(r) = \int_0^{2\pi} \sigma(\theta) \sqrt{\frac{a}{R} (2 - k^2)} \frac{K(k) - 2E(k)}{k} \, d\theta,$$

(65)

where $I = b \oint \sigma(\theta) d\theta$ is the total current. Similarly as for the gravitational problem, we have to select some part of the Green function. A convenient choice appears to be

$$\frac{1}{\Delta} \left\{ \frac{2}{k^2} [K(k) - E(k)] - K(k) \right\} \equiv \kappa'.$$

(66)

By expanding $\kappa'$ over $a$ and $z$ at the center $C$ of the shell, i.e. at $a = R_c$ and $z = 0$, and integrating over the latitude $\theta$ (see Sec. 3), we get the leading term

$$A_\phi(r) = \frac{\mu_0 b}{\pi} \int_0^{2\pi} \sigma(\theta) d\theta \int_0^{2\pi} \sigma a d\theta$$

$$= 2\mu_0 \sigma a R_c \kappa'_0 S_{0,0},$$

(67)

where $\kappa'_0$ stands for $\kappa'$ evaluated at $C$, $\sigma =$const, is assumed, and $S_{0,0}$ is given by (15). A $\theta$-dependent surface density of current would lead to a different surface factor. Again, we notice that (67) formally differs from the expression for a current loop only by the term $S_{0,0}$, which is unity in the homogeneous case. We thus state (similarity theorem 5):

a toroidal shell of main radius $R_c$ and circular section carrying a uniform surface current generates, at the first order in the $e$-parameter, the same exterior magnetic potential as a circular loop of radius $R_c$ carrying the same current.

The theorem holds in the $O(e^2)$ order. It applies likewise to the magnetic field $\vec{B} = \nabla \times \vec{A}$. The $e^2$-approximations for the poloidal components $B_R$ and $B_Z$ of the shell are then given by

$$B_R = \frac{\mu_0 I}{2\pi R \Delta_0} \left[ \frac{R^2 + R_c^2 + Z^2}{(R - R_c)^2 + Z^2} E(k) - K(k) \right],$$

(68)

$$B_Z = \frac{\mu_0 I}{2\pi \Delta_0} \left[ \frac{R^2 - R_c^2 + Z^2}{(R - R_c)^2 + Z^2} E(k) + K(k) \right].$$

(69)

We can deduce the magnetic potential and field of a solid torus carrying a uniform current density $J = J_0 e_z$, following the procedure given in Sec. 8. The $e^2$-approximation for the vector potential is obtained from (67) where $S_{0,0}$ is to be replaced by $V_{0,0}$ (which is also unity in the present case). So we can state that (similarity theorem 5):

a toroid of main radius $R_c$ and circular section carrying a uniform volume current density generates, at the first order in the $e$-parameter., the same exterior magnetic potential (and the field) as a circular loop of radius $R_c$ carrying the same current.

The reader can verify that this theorem also works for a core-stratified current, as for the gravitational problem.

12 GENERAL COMMENTS

The paper resides on the expansion of $K(k)/\Delta$. Other options are possible as quoted before. If we expand $kK(k)$ instead of $K(k)/\Delta$ in the Green function, one can show that $g_{0,0}$ is changed for

$$g_{0,0} = \frac{r}{\Delta_0} \frac{2}{\pi} K(k) \times S_{0,0}.$$

(70)
Another important comment concerns the point where the expansion is performed. In Huré et al. (2019), the choice for the expansion at the focal ring \( a = R_0 \) was strategic: this is the only point in space which makes the modulus \( k \) of the elliptic integral constant all along the circular section of the shell. The motivation for choosing the centre of the circular section here (instead of the focal ring) is similar: the calculation of the integral in (6) is facilitated, in particular through the expression for \( d\ell = b dh \). If we use for instance the toroidal coordinates \((\eta, \zeta) \in [0, \infty] \times [-\pi, \pi] \), the integral over \( \theta \) in (6) can be converted into an integral over \( \zeta \). We have in this case

\[
a = R_0 \frac{\sinh \eta}{\cosh \eta - \cos \zeta}, \quad z = R_0 \frac{\sin \zeta}{\cosh \eta - \cos \zeta},
\]

where \( R_0 \) is the radius of the pole (or focal ring), and the line element is \( d\ell = \frac{R_0}{\cosh \eta - \cos \zeta} \). As a consequence, the potential writes

\[
\Psi_{(\vec r)} = -4GR_0^2 \int_{-\pi}^{\pi} \frac{K(k)}{\Delta} \sinh \eta \frac{\Sigma(\zeta) d\zeta}{(\cosh \eta - \cos \zeta)^2},
\]

where the modulus \( k \) and \( \Delta \) depend on \( \zeta \). The expansion of \( \kappa \) in \( x_0 = R_p \) (and \( y_0 = 0 \) still; see Sects. 3 and 7) generates, for the homogeneous shell, integrals of the form

\[
R_p^{n+m+2} \int_{-\pi}^{\pi} \frac{(\cosh \eta - \cos \zeta - \sinh \eta)^n}{(\cosh \eta - \cos \zeta)^{n+m+2}} \sin^m \zeta d\zeta.
\]

At order zero (i.e., for \( n = m = 0 \)), we find

\[\Psi_{(\vec r)} \approx -8GR_0^2 \Sigma_0 \kappa_0 S_{0,0},\]

where

\[S_{0,0} = 2 \sinh \eta_0 \int_{-\pi}^{\pi} \frac{d\zeta}{(\cosh \eta_0 - \cos \zeta)^2},\]

and so we recover (17). For higher terms, (80) have to be calculated analytically for all pairs \((n,m)\), and this manifestly requires more effort than for the \( J_{nm} \)'s.

13 CONCLUSION AND PERSPECTIVES

The exterior potential of a static thin toroidal shell, as given by the Laplace equation, is obtained from a double Taylor expansion of the axisymmetric Green function. Each term is then integrated over the source, as in the multipole theory. Here, the expansion is performed at the centre of the circular section instead of the origin of coordinates. The series converges very well and provides a solution which satisfies the Laplace equation in every order, so no “ghost” sources are induced by truncation. In practice, the efficiency of the method is remarkable, with already 3 correct digits at order zero for toroids having an axis ratio of 0.1. At order 2, this precision is almost doubled (to 6 digits), which should be sufficient for most applications.

At order 2 in the shell parameter (minor-to-major radius ratio), a shellular, solid or core-stratified toroid generates an exterior potential (and field) similar to that of a thin circular loop having same main radius and same mass. We meet the results by Bannikova et al. (2011) and Kondratyev (2018). A few similarity theorems, which all resemble the Gauss theorem, have been proposed. The approximations
for the exterior potential reported here together with the interior solutions reported in Huré et al. (2019) yield a complete description of the potential of a toroidal shell of circular section, at any point of space. It then becomes possible to deduce the interior solution for the solid torus since both interior and exterior shell solutions are required in this operation. Next, the energy for the formation of a solid torus becomes accessible. It would be worth to generalize the method to any kind of source shape, not limited to circular section, through specific prescriptions for $a(\theta)$ and $z(\theta)$, or $z(a)$. This would open exciting perspectives, in particular for oblate structures such as geometrically thin discs.

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APPENDIX A: INTEGRALS \( J_{n,m} \)

From Gradshteyn & Ryzhik (2007), we have

\[
\int_0^{\pi/2} \cos^n \theta \sin^m \theta d\theta = \frac{1}{2^n} \binom{n + m}{m} 
\]

(\( A1 \)),

where \( B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \) is the complete Beta function, and \( \Gamma(x) \) is the Gamma function. From this expression, we can easily deduce \( J_{n,0} \) (integral bounds 0 and 2\( \pi \)). We find

\[
J_{n,m} = \frac{1}{2} B \left( \frac{n + 1}{2}, \frac{m + 1}{2} \right) \left[ 1 + (-1)^n \right] \left[ 1 + (-1)^{n+m} \right].
\]

(\( A2 \))

It follows that \( J_{n,m} = 0 \) when \( n \) is even or when \( n + m \) is even (\( n \) and \( m \) have different parity). The expression for \( J_{n,m} \) are given in Tab. 1 for \( n \in \{0, 1, 2, 3, 4\} \) and \( m \in \{0, 1, 2, 3, 4\} \).

APPENDIX B: RESIDUAL MASS DENSITY

The residual density is found from the Poisson equation, i.e.

\[
\nabla^2 \left[ \frac{K(k)}{\sqrt{(a + R)^2 + z^2}} \right] = \frac{k_0 K(k_0)}{4R^2} + \left[ \frac{\partial^2 k}{\partial R^2} + \frac{\partial k}{\partial R} \right] \frac{\partial k_0}{k_0^2} + \left[ \frac{\partial k}{\partial R} \right] \left[ \frac{\partial k}{\partial z} \right]^2 \frac{(1 + k_0^2) E(k_0) - k_0^2 K(k_0)}{k_0^4},
\]

(\( B1 \))

where the partial derivatives of \( K(k) \) and \( E(k) \) with respect to the modulus \( k \) are found in mathematical textbooks (Gradshteyn & Ryzhik 2007). By expanding all the terms inside the curly brackets, this quantity is strictly zero provided \( a - R \neq 0 \) and \( Z \neq 0 \), which never occurs in free space.

APPENDIX C: PARTIAL DERIVATIVES

There are different ways to calculate the partial derivatives of \( \kappa \) with respect to \( a \) and \( z \). We find convenient to rewrite \( K(k) \) as the definite integral over the azimuth, i.e. (3). The denominator is then expanded and rearranged so that the n-order derivative with respect to \( a \) and \( z \) writes

\[
\frac{\partial^n \kappa}{\partial a^{n-m} \partial z^m} = \frac{\partial^n}{\partial a^{n-m} \partial z^m} \int_0^{\tilde{\phi}} \frac{d\phi}{\sqrt{\Delta^2 - 4aR \sin^2 \phi}} = \int_0^{\tilde{\phi}} d\phi \frac{\partial^n}{\partial a^{n-m} \partial z^m} \left\{ [a + R \cos(2\phi)]^2 + [R \sin(2\phi)]^2 + \zeta^2 \right\}^{-1/2}.
\]

(\( C1 \))

Denoting \( D = [a + R \cos(2\phi)]^2 + [R \sin(2\phi)]^2 + \zeta^2 \), we have

\[
\frac{\partial D^{-1/2}}{\partial a} = -[a + R \cos(2\phi)] D^{-3/2}
\]

(\( C2 \))

and

\[
\frac{\partial^2 D^{-1/2}}{\partial a^2} = -D^{-3/2} + 3[a + R \cos(2\phi)]^2 D^{-5/2}.
\]

(\( C3 \))

It follows that

\[
\frac{\partial \kappa}{\partial a} = \int_0^{\tilde{\phi}} \frac{\partial D^{-1/2}}{\partial a} d\phi = -(a + R) \Delta^{-3} \frac{E(k)}{k^2} + 2R \Delta^{-3} \frac{E(k) - k^2 K(k)}{k^2 k^2}
\]

(\( C4 \))

and

\[
\frac{\partial^2 \kappa}{\partial a^2} = \int_0^{\tilde{\phi}} \frac{\partial^2 D^{-1/2}}{\partial a^2} d\phi
\]

(\( C5 \))

\[
= -\Delta^{-3} \frac{E(k)}{k^2} + 3 \Delta^{-3} \frac{E(k)}{k^2} - 3 \zeta^2 \Delta^{-5} \frac{2(1 + k^2) E(k) - k^2 K(k)}{3k^4} - 12R^2 \Delta^{-5} \frac{(1 + k^2) E(k) - 2k^2 K(k)}{3k^2 k^2}
\]

\[
= 2 \Delta^{-3} \frac{E(k)}{k^2} - \zeta^2 \Delta^{-5} \frac{2(1 + k^2) E(k) - k^2 K(k)}{k^4} - 4R^2 \Delta^{-5} \frac{(1 + k^2) E(k) - 2k^2 K(k)}{k^4 k^2}
\]

For the \( z \)-derivatives, we have

\[
\frac{\partial D^{-1/2}}{\partial z} = \zeta D^{-3/2}, \quad \frac{\partial^2 D^{-1/2}}{\partial z^2} = -D^{-3/2} + 3 \zeta^2 D^{-5/2}
\]

(\( C6 \))
and, consequently,
\[
\frac{\partial \kappa}{\partial z} = \zeta \int_0^{\pi} \frac{\partial D^{-1/2}}{\partial z} d\phi = \zeta \Delta^{-3} \frac{E(k)}{k'^2}
\]
(C7)

and
\[
\frac{\partial^2 \kappa}{\partial z^2} = \int_0^{\pi} \frac{\partial^2 D^{-1/2}}{\partial z^2} d\phi = -\Delta^{-3} \frac{E(k)}{k'^2} + \zeta^2 \Delta^{-5} \frac{2(1 + k'^2)E(k) - k'^2 K(k)}{k'^4}
\]
(C8)

APPENDIX D: F90 PROGRAM FOR THE EXTERIOR POTENTIAL

Program F90drivercode3

! "The exterior gravitational potential of toroids"
! Hure, Basillais, Karas, Trova & Semerak (2019), MNRAS
! gfortran F90drivercode3.f90; ./a.out
! not optimized
Implicit None

Integer, Parameter :: AP = Kind(1.00D+00)
Real(Kind = AP), Parameter :: PI = ATAN(1._AP)*4

Real(KIND = AP) :: B, RC, MASS, E ! core radius, main radius and mass of the shell, and axis ratio
Real(KIND = AP) :: KMOD, KMOD2, KPRIM, KPRIM2 ! moduli
Real(KIND = AP) :: R, Z, PSI ! cylindrical coordinates and potential value where it is estimated
Real(KIND = AP) :: Z2, VAL, DELTA, DELTA2, DELTA3, DELTA4, DELTA5 ! misc

Real(KIND = AP) :: KMOD, KPRIM4, S10, S20, S32, S2D32, S2C2D52, S2C2D52, S2C2D52 ! misc
Real(KIND = AP) :: ELLIPTIC, ELLIPTIC2 ! complete elliptic integrals

! input parameters (properties of the shell)
B=0.1_AP
RC=1._AP
E=B/RC
MASS=B*RC*PI**2*4
print*,"Mass of the shell",MASS

! ? values of R and Z where the potential is requested (must be outside the cavity!)
R=RC*2
Z=RC*2
Z2=Z**2

If ((R-RC)**2+Z2-B**2<0._AP) Then
! approximation not valid inside the shell
PSI=0._AP
Else

DELTA2=(R+RC)**2+Z2
KMOD2=RC*R*4/DELTA2
DELTA=SQRT(DELTA2)
KPRIM2=((R-RC)**2+Z2)/DELTA2
! values of K(k) and E(k) to be set here!
! ELLIPTIC=
! ELLIPTIC=
! misc
DELTA3=DELTA2*DELTA
DELTA5=DELTA3*DELTA2
KMOD4=KMOD2**2
KPRIM4=KPRIM2**2
! surface factors
S0=1._AP
S10=E**2/2
S20=E**2/2
S20=S20
! coefficients
D32=ELLIPTIC*KPRIM2
S2D32=(ELLIPTIC-KPRIM2*ELLIPTIC)/KPRIM2/KMOD2
D52=((1._AP+KPRIM2)*ELLIPTIC*2-KPRIM2*ELLIPTIC)/KMOD4/3
S2C2D52=((1._AP+KPRIM2)/KPRIM2*ELLIPTIC-ELLIPTIC*2)/KMOD4/3
! order 0
The gravitational potential of toroids

\[ \text{Val} = \text{Elliptick} / \Delta \times S_0 \]

! order 1
\[ \text{Val} = \text{Val} + \text{RC} \times (-R + \text{RC}) \times D_{32} / \Delta \times \text{S}_{2D32} / \Delta \times R \times 2 \times S_{10} \]

! order 2
\[ \text{Val} = \text{Val} + \text{RC}^2 \times ((-D_{32} / \Delta \times 3 - D_{52} \times Z_2 / \Delta \times 3 - S_{2C2B52} / \Delta \times R \times 12) \times S_{20} \]
\[ + ((-D_{32} / \Delta \times 3 - D_{52} \times Z_2 / \Delta \times 3) \times S_{02}) / 2 \]
\[ \text{Psi} = -\text{Val} \times B \times \text{RC} \times \pi \times 8 \]
\[ \text{Print} *, "Potential value (2nd-order)", \text{Psi}, \text{Psi} / \text{Mass} \times \text{RC} \]

Endif

End Program F90drivercode3