Quadrupole Moments of Rapidly Rotating Compact Objects in Dilatonic Einstein-Gauss-Bonnet Theory

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We consider rapidly rotating black holes and neutron stars in dilatonic Einstein-Gauss-Bonnet (EGBd) theory and determine their quadrupole moments, which receive a contribution from the dilaton. The quadrupole moment of EGBd black holes can be considerably larger than the Kerr value. For neutron stars, the universality property of the $I$-$Q$ relation between the scaled moment of inertia and the scaled quadrupole moment appears to extend to EGBd theory.

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I. INTRODUCTION

With numerous high precision telescopes and satellite missions operating already or becoming operational within the next decade Einstein’s theory of general relativity will be tested with increasing accuracy, in particular, in the strong gravity regime (see, e.g., the reviews [1–5]).

From a theoretical point of view general relativity is expected to be superseded by a new theory of gravity, which on the one hand will not be suffering from those pathological singularities present in many solutions of General Relativity (GR), and, on the other hand, will be compatible with quantum theory.

String theory, as a leading candidate for a unified theory of the fundamental interactions and a quantum theory of gravity, for instance, predicts the presence of higher curvature corrections to Einstein gravity as well as the existence of further fields. The low energy effective theory deriving from heterotic string theory contains as basic ingredients a Gauss-Bonnet term and a dilaton field [6, 7].

The effects of the presence of such terms on the properties of compact astrophysical objects can be strong, as has been shown for black holes [8–12] and neutron stars [13], which have been studied in a truncated version of this theory, named dilatonic Einstein-Gauss-Bonnet (EGBd) theory. Moreover, EGBd theory allows for wormhole solutions without the necessity of introducing exotic fields [14, 15].

In this letter we focus on rapidly rotating compact objects in EGBd theory and compare their properties with those of their Einstein gravity counterparts. In particular, we here determine the quadrupole moments of rapidly rotating black holes and neutron stars.

Black holes in EGBd theory can slightly exceed the Kerr bound, $J/M^2 = 1$, given by the scale invariant ratio of the angular momentum $J$ and the square of the mass $M$ in appropriate units [10]. The innermost-stable-circular-orbits (ISCOs) can differ from the respective Kerr values by a few percent in the case of slow rotation [9] and up to 10% for rapid rotation [10]. Similarly, the orbital frequencies exhibit the largest deviations from the Kerr frequencies for rapid rotation, which can be as large as 60%. Such large effects might be observable for astrophysical black holes.

For neutron stars, on the other hand, recent calculations showed, that for slowly rotating neutron stars universal relations hold between the scaled moment of inertia $I$ and the Love number, and the scaled quadrupole moment $Q$ in Einstein gravity [16–18]. Subsequently, the $I$-$Q$ relation was generalized to the case of rapidly rotating neutron stars, where for fixed rotation parameter $j = J/M^2$ again only little dependence on the equation of state was seen [19], after appropriately scaled quantities were considered [20].

In the following we briefly recall EGBd theory and obtain the quadrupole moment for compact objects in this theory. We then discuss the quadrupole moments of rapidly rotating black holes and show that they may deviate from their Kerr counterparts by 20% and more. Subsequently, we present rapidly rotating neutron stars in EGBd theory and discuss their $I$-$Q$ relation. We conclude, that just like in Einstein gravity the properly scaled dimensionless quantities exhibit little dependence on the equation of state.

II. EINSTEIN-GAUSS-BONNET-DILATON THEORY

As motivated by the low-energy heterotic string theory [6, 7] we consider the following low-energy effective action [8, 10, 21, 22]

$$S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} (\partial_{\mu}\phi)^2 + \alpha e^{-\gamma\phi} R_{\text{GB}}^2 \right],$$

(1)
where $G$ is Newton’s constant, $c$ is the speed of light, $\phi$ denotes the dilaton field with coupling constant $\gamma$, $\alpha$ is a numerical coefficient given in terms of the Regge slope parameter, and $R_{\text{GB}} = R_{\mu
u\rho\sigma}R^{\mu
u\rho\sigma} - 4R_{\mu
u}R^{\mu
u} + R^2$ is the Gauss-Bonnet (GB) term. We have chosen the dilaton coupling constant according to its string theory value, $\gamma = 1$.

To construct rotating compact objects we employ the Lewis-Papapetrou line element \cite{23} for a stationary, axially symmetric spacetime with two Killing vector fields $\xi_1 = \partial_t$, $\eta = \partial_\varphi$. In terms of the spherical coordinates $r$ and $\theta$, the isotropic metric reads \cite{24}

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2e^{2\nu_0}dt^2 + e^{2(\nu_1-\nu_0)}\left(e^{2\nu_2}\left[dr^2 + r^2d\theta^2\right] + r^2\sin^2\theta(d\varphi - \omega dt)^2\right),$$

where $\nu_0$, $\nu_1$, $\nu_2$ and $\omega$ are functions of $r$ and $\theta$ only.

In the asymptotic region the metric and dilaton functions possess the expansion

$$\begin{align*}
\nu_0 &= -\frac{M}{r} + \frac{D_1M}{3r^3} - \frac{M_2}{r^3}P_2(\cos\theta) + \mathcal{O}(r^{-4}), \\
\nu_1 &= \frac{D_1}{r^2} + \mathcal{O}(r^{-3}), \\
\nu_2 &= -\frac{4M^2 + 16D_1 + q^2}{8r^2}\sin^2\theta + \mathcal{O}(r^{-3}), \\
\omega &= \frac{2J}{r^3} + \mathcal{O}(r^{-4}), \\
\phi &= \frac{q}{r} + \mathcal{O}(r^{-2}),
\end{align*}$$

where $P_2(\cos\theta)$ is a Legendre polynomial, $M$ is the mass, $J$ is the angular momentum, $q$ is the dilaton charge, and $D_1$ and $M_2$ are further expansion constants.

To extract the quadrupole moment of the compact solutions, we follow Geroch and Hansen \cite{25, 26} (see also \cite{27, 28}) and obtain

$$Q = -M_2 + \frac{4}{3} \left[\frac{1}{4} + \frac{D_1}{M^2} + \frac{q^2}{16M^2}\right] M^3. $$

We note, that there is no explicit contribution from the GB term, since this term decays sufficiently fast. The dilaton field enters only via the Coulomb-like term $q/r$, which coincides with the analogous term in Einstein-Maxwell theory to lowest order in the expansion. Thus the expressions for the quadrupole moment are completely analogous. In the vacuum limit, when the dilaton charge vanishes, the quadrupole moment $Q$ reduces to the expression in \cite{29} (up to an overall sign), with $b = D_1/M^2$. On the other hand, in the static limit the solution is spherically symmetric. In this case $M_2 = 0$ and $4M^2 + 16D_1 + q^2 = 0$, and the quadrupole moment vanishes.

### III. Rotating Black Holes

The domain of existence of the EGBd black hole solutions \cite{38} resides within the boundaries formed by (i) the Kerr black holes, (ii) the critical black holes, where a radicant vanishes \cite{8, 10}, and (iii) the set of extremal black holes with $j > 1$, which have temperature $T = 0$, a regular metric, but a dilaton field that diverges on the horizon at the poles \cite{10}. In Fig. 1a the shaded area represents the domain of existence, with (i) the lower boundary, (ii) the upper boundary, and (iii) the right lower boundary in the inset.

In Fig. 1b we present the scaled quadrupole moment $\hat{Q} = Q/MJ^2$ of the rotating black hole solutions versus the scaled angular momentum $\hat{j} = J/M^2$ for several values of the dimensionless horizon angular velocity $\Omega_H\alpha^{1/2}$. The curves are obtained, by fixing the value of $\Omega_H$ and the value of $\alpha$ and by varying the size of the black hole.

For fixed $\alpha$ the range of the scaled area $\hat{a} = A/M^2$ for EGBd black holes is largest in the static case and decreases to zero for the maximal value of $\hat{j}$ \cite{10}. This is reflected in the range of the scaled quadrupole moment $\hat{Q}$, which is likewise largest in the limit of slow rotation, and decreases with increasing $\hat{j}$. Thus for slow rotation deviations from the Kerr value of up to 20% and more are seen. Moreover, the solutions with $j > 1$, not present in General Relativity (GR), always have $\hat{Q} > 1$.

In Fig. 1b we exhibit the scaled moment of inertia $\hat{I} = J/(\Omega_HM^3)$ versus the scaled quadrupole moment $\hat{Q}$ for fixed values of $\hat{j}$. For $\hat{j} \leq 1$ the families of solutions with fixed $\hat{j}$ start at the respective Kerr values, $I_{Kerr} = 2(1 + \sqrt{1 - j^2})$ and $Q_{Kerr} = 1$, indicated by the dots on the axis. They end at the boundary (ii) given
FIG. 1: (a) The scaled quadrupole moment $\hat{Q} = QM/J^2$ is shown versus the scaled angular momentum $j = J/M^2$ for fixed values of the dimensionless horizon angular velocity $\Omega_H^{1/2}$. The shaded area represents the domain of existence of the EGBd black holes. (b) The scaled moment of inertia $\hat{I} = J/(\Omega_H M^3)$ is shown versus the scaled quadrupole moment $\hat{Q}$ for fixed values of $j$. The dots indicate the corresponding values of the Kerr black holes. The dotted curve corresponds to the boundary of the domain of existence given by the critical solutions. The straight dotted lines represent the perturbative results of [12].

by the critical solutions and represented by the dotted curve. The inset again shows the region $j > 1$. The figure also contains the perturbative results derived in [12] for small $\alpha$ and small $j$.

Clearly, the $\hat{I}-\hat{Q}$ relation for black holes depends on the value of the scaled angular momentum $j$. However, for black holes there is no dependence of this relation on the coupling parameter $\alpha$. This is a consequence of the invariance of $\hat{I}$, $\hat{Q}$, $j$ and the EGBd equations under the scaling transformation $r \rightarrow \lambda r$, $\omega \rightarrow \omega/\lambda$, and $\alpha \rightarrow \lambda^2 \alpha$.

The absolute value of the scaled dilaton charge $q/M$ decreases monotonically with decreasing scaled quadrupole moment $\hat{Q}$. In the limit $\hat{Q} \rightarrow 1$ the scaled dilaton charge $q/M$ vanishes. Therefore the scaled moment of inertia $\hat{I}$ approaches $\hat{I}_{Kerr}$ in this limit. Its dependence on the value of the scaled angular momentum $j$ is only weak.

IV. ROTATING NEUTRON STARS

Rapidly rotating neutron stars have been studied extensively in GR (see, e.g., [30, 31]). Here we address rapidly rotating neutron stars in EGBd theory [10]. Considering rigid rotation, the four-velocity has the form $U^\mu = (u, 0, 0, \Omega u)$ with angular velocity $\Omega$. The normalization condition $U^\mu U_\mu = -1$ then yields

$$u^2 = \frac{e^{-2\nu_0}}{1 - (\Omega - \omega)^2 r^2 \sin^2 \theta e^{2\nu_1 - 4\nu_0}}.$$ (9)

The differential equations for the pressure $P$ and the energy density $\epsilon$ are obtained from the constraints $\nabla_\mu T^{\mu\nu} = 0$,

$$\frac{\partial_r P}{\epsilon + P} = \frac{\partial_r u}{u}, \quad \frac{\partial_\theta P}{\epsilon + P} = \frac{\partial_\theta u}{u}. $$ (10)

These equations have to be supplemented by an equation of state (EOS), $\epsilon = \epsilon(P)$ (or $P = P(\epsilon)$). For a polytropic EOS, $P = P_0 \Theta^{N+1}$, $\epsilon = NP + \rho_0 \Theta^N$, with central pressure $P_0$ and central density $\rho_0$. Substitution in Eq. (10) yields $\Theta = c_0 u - \rho_0/P_0(N+1)$, where $c_0$ is an integration constant, which we express as $c_0 = \rho_0/\sigma P_0(N+1)$ to obtain the more convenient expression

$$\Theta = \frac{\rho_0}{\sigma P_0(N+1)} (u - \sigma).$$ (11)

The boundary of the star is defined by $P = P_b = 0$, implying $\Theta_b = 0$ and consequently $u_b = \sigma$. Outside the star $\Theta = 0$. 

In Fig. 2, we exhibit the quadrupole moment $Q$ in units of $M_\odot \cdot \text{km}^2$ versus the angular velocity $\Omega$ in Hz for a family of neutron stars for a fixed scaled angular momentum of $j = 0.4$, the GB couplings $\alpha = 0, 1$ and 2, and two well-known equations of state.

The first EOS corresponds to a polytropic EOS with $N = 0.7463$, taken from (32) and denoted by DI-II, which is widely used in neutron star physics and also employed for neutron stars in scalar-tensor theory (33-34), while the second EOS corresponds to an approximation (39) to the FPS EOS (35). Clearly, the two equations of state give rise to rather different quadrupole moments, whereas the dependence on the GB coupling is only moderate, while increasing with increasing $\alpha$.

To address the $\hat{I} - \hat{Q}$ relation for neutron stars we exhibit in Fig. 2b $\hat{I}$ versus $\hat{Q}$ for a fixed value of the scaled angular momentum, $j = 0.4$. Again, we employ the polytropic EOS DI-II (32) and the analytical fit to the FPS EOS (35).

Comparison shows, that for a given value of $\alpha$ and a given value of $j$ the dependence on the EOS is only weak, but increasing with increasing $\alpha$. Moreover, the $\alpha$-dependence is the stronger the smaller $j$. We did not go beyond $\alpha = 2$, since the strongest current observational upper bound on $\alpha$, obtained from low-mass X-ray binaries, is slightly less than this value (in the units employed) (12, 36).

\section{V. CONCLUSION}

Following the procedure of Geroch and Hansen (25, 26) we have derived the spin induced quadrupole moment for rapidly rotating compact objects in EGBd theory. The scaled quadrupole moment of black holes can deviate by 20\% and more from the corresponding Kerr value for a given value of the scaled angular momentum $j$. Their $\hat{I} - \hat{Q}$ relation has no explicit dependence on the Gauss-Bonnet coupling $\alpha$.

For neutron stars the presence of the matter breaks the scale invariance of the $\hat{I} - \hat{Q}$ relation. We now observe a dependence on $\alpha$, which is however weak in its allowed range. While the unscaled quantities depend quite significantly on the equation of state, the scaled quantities satisfy to good approximation universal relations, which depend only on the scaled angular momentum $j$, in EGBd theory like in GR.

We are currently extending the calculations to quark stars, to see whether the universal relations hold also in this case (16) for EGBd theory. Unlike the case of Chern-Simons theory also studied in (16), however, the $\hat{I} - \hat{Q}$ relations do not differ significantly for EGBd theory and GR. Here other observables including ISCOs and orbital periods seem more promising handles to test the viability of EGBd theory.

Finally, another type of compact object can be obtained in EGBd theory, namely wormholes, which may possibly correspond to black hole foils (37). We expect that the static EGBd wormholes (14, 15) can be generalized to rapidly rotating wormholes, while retaining their linear stability with respect to changes of the size of their throat. It will be interesting to extract their quadrupole moments and compare them with the quadrupole moments of the EGBd black holes.
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[39] The analytical fit to the FPS EOS is approximated by a fit to a polytropic EOS.