Certain Results on Almost Contact Pseudo-Metric Manifolds

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Abstract. We study the geometry of almost contact pseudo-metric manifolds in terms of tensor fields $h := \frac{1}{2} L_\xi \varphi$ and $\ell := R(\cdot, \xi) \xi$, emphasizing analogies and differences with respect to the contact metric case. Certain identities involving $\xi$-sectional curvatures are obtained. We establish necessary and sufficient condition for a nondegenerate almost $CR$ structure $(\mathcal{H}(M), J, \theta)$ corresponding to almost contact pseudo-metric manifold $M$ to be $CR$ manifold. Finally, we prove that a contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is Sasakian if and only if the corresponding nondegenerate almost $CR$ structure $(\mathcal{H}(M), J)$ is integrable and $J$ is parallel along $\xi$ with respect to the Bott partial connection.

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1. Introduction

In 1969, Takahashi [20] initiated the study of contact structures associated with pseudo-Riemannian metrics. Afterwards, a number of authors studied such structures mainly focusing on a special case, namely Sasakian pseudo-metric manifolds. The case of contact Lorentzian structures $(\eta, g)$, where $\eta$ is a contact 1-form and $g$ a Lorentzian metric associated to it, has a particular relevance for physics and was considered in [12] and [4]. A systematic study of almost contact pseudo-metric manifolds was undertaken by Calvaruso and Perrone [7] in 2010, introducing all the technical apparatus which is needed for further investigations, and such manifolds have been extensively studied under several points of view in [16], [2], [9], [14], [15], [16], [22], [3], [10], and references cited therein.

The operators $h := \frac{1}{2} L_\xi \varphi$ and $\ell := R(\cdot, \xi) \xi$ play fundamental roles in the study of geometry of contact pseudo-metric manifolds. For contact metric
manifolds, Sharma [19] obtained the following beautiful results (Theorem 1.1 in [19]):

(a) a contact metric manifold is $K$-contact if and only if $h$ is a Codazzi tensor;

(b) a contact metric manifold is $K$-contact if and only if $\tau$, the tensor metrically equivalent to the strain tensor $\mathcal{L}_\xi g$ of $M$ along $\xi$, is a Codazzi tensor;

(c) the sectional curvatures of all plane sections containing $\xi$ vanish if and only if the tensor $\ell$ is parallel.

The proof of these results exploit, in an essential way, the fact that in the contact Riemannian case, the self-adjoint operator $h$ vanishes if $h^2 = 0$. But in the contact pseudo-metric case the condition $h^2 = 0$ does not necessarily imply that $h = 0$ (see [15]). So the corresponding results fail for general contact pseudo-metric structures.

Under these circumstances, becomes interesting to explore more the geometry of contact pseudo-metric manifolds. The paper is organized as follows. In section 2, we give the basics of almost contact pseudo-metric manifolds. In section 3, we study contact pseudo-metric manifold $M$ with $h$ satisfying Codazzi condition and we prove that $M$ is Sasakian pseudo-metric manifold if and only if the equation (2.10) is satisfied and $h$ is a Codazzi tensor. In Section 4, we investigate the Codazzi condition for the operator $\tau$, and we obtain a necessary and sufficient condition for $\tau$ to be a Codazzi tensor on contact pseudo-metric manifold. Moreover, if $\tau$ is a Codazzi tensor, then $h^2 = 0$ and the Ricci operator $Q$ satisfies $Q\xi = 2\varepsilon n \xi$, and we prove that $M$ is a Sasakian pseudo-metric manifold if and only if the equation (2.10) is satisfied and $\tau$ is a Codazzi tensor. In section 5, we obtain certain identities involving $\xi$-sectional curvatures of contact pseudo-metric manifolds. It is proved that the parallelism of the tensor $\ell$ together with the condition $\nabla_\xi h = 0$ on a contact pseudo-metric manifold implies that all $\xi$-sectional curvatures vanish. At the end, we investigate the nondegenerate almost $CR$ structure $(\mathcal{H}(M), J, \theta)$ corresponding to almost contact pseudo-metric manifold $M$, and establish a necessary and sufficient condition for an almost contact pseudo-metric manifold to be a $CR$ manifold. Finally, we show that a contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is Sasakian pseudo-metric if and only if the corresponding nondegenerate almost $CR$ structure $(\mathcal{H}(M), J)$ is integrable and $J$ is parallel along $\xi$ with respect to the Bott partial connection.

2. Preliminaries

In this section, we briefly recall some general definitions and basic properties of almost contact pseudo-metric manifolds. For more information and details, we recommend the reference [7].

A $(2n + 1)$-dimensional smooth connected manifold $M$ is said to be an almost contact manifold if there exists on $M$ a $(1, 1)$ tensor field $\varphi$, a vector
field $\xi$, and a 1-form $\eta$ such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0 \quad (2.1)$$

for all $X, Y \in TM$. It is known that the first relation along with any one of the remaining three relations in (2.1) imply the remaining two relations. Also, for an almost contact structure, the rank of $\varphi$ is $2n$. For more details, we refer to [4].

If an almost contact manifold is endowed with a pseudo-Riemannian metric $g$ such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y), \quad (2.2)$$

where $\varepsilon = \pm 1$, for all $X, Y \in TM$, then $(M, \varphi, \xi, \eta, g)$ is called an almost contact pseudo-metric manifold. The relation (2.2) is equivalent to

$$\eta(X) = \varepsilon g(X, \xi) \text{ along with } g(\varphi X, Y) = -g(X, \varphi Y). \quad (2.3)$$

In particular, in an almost contact pseudo-metric manifold, it follows that $g(\xi, \xi) = \varepsilon$ and so, the characteristic vector field $\xi$ is a unit vector field, which is either space-like or time-like, but cannot be light-like.

The fundamental 2-form of an almost contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y),$$

which satisfies $\eta \wedge \Phi^n \neq 0$. An almost contact pseudo-metric manifold is said to be a contact pseudo-metric manifold if $d\eta = \Phi$, where

$$d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y])).$$

The curvature operator $R$ is given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$ 

This sign convention of $R$ is opposite to the one used in [7, 9, 14, 15, 16]. The Ricci operator $Q$ is determined by

$$S(X, Y) = g(QX, Y).$$

In an almost contact pseudo-metric manifold $(M, \varphi, \xi, \eta, g)$ there always exists a special kind of local pseudo-orthonormal basis $\{e_i, \varphi e_i, \xi\}_{i=1}^n$, called a local $\varphi$-basis.

In a contact pseudo-metric manifold, the $(1,1)$ tensor $h = \frac{1}{2}L_\xi \varphi$ is self-adjoint and satisfies

$$h\xi = 0, \quad \varphi h + h \varphi = 0, \quad \text{tr}(h) = \text{tr}(\varphi h) = 0.$$
Further, one has the following formulas:

\[ \nabla_X \xi = -\varepsilon \varphi X - \varphi hX, \quad (2.4) \]

\[ (\mathcal{L}_\xi g)(X, Y) = 2g(h\varphi X, Y), \quad (2.5) \]

\[ (\nabla_\xi h)X = \varphi X - h^2 \varphi X + \varphi R(\xi, X)\xi, \quad (2.6) \]

\[ R(\xi, X)\xi - \varphi R(\xi, \varphi X)\xi = 2(h^2 + \varphi^2)X, \quad (2.7) \]

\[ \text{tr} \nabla \varphi = 2n \xi. \quad (2.8) \]

A contact pseudo-metric manifold \( M \) is said to be a \textit{K-contact pseudo-metric manifold} if \( \xi \) is a Killing vector field (or equivalently, \( h = 0 \)), and is said to be a \textit{Sasakian pseudo-metric manifold} if the almost complex structure \( J \) on the product manifold \( M \times \mathbb{R} \) defined by

\[ J \left( X, f \frac{d}{dt} \right) = \left( \varphi X - f\xi, \eta(X)\frac{d}{dt} \right), \]

is integrable, where \( X \in TM, t \) is the coordinate on \( \mathbb{R} \) and \( f \) is a \( C^\infty \) function on \( M \times \mathbb{R} \). It is well known that a contact pseudo-metric manifold \( M \) is a Sasakian pseudo-metric manifold if and only if

\[ R(X, Y)\xi = \eta(Y)X, \quad (2.10) \]

for all \( X, Y \in TM \). A Sasakian pseudo-metric manifold is always \( K \)-contact pseudo-metric. A 3-dimensional \( K \)-contact pseudo-metric manifold becomes a Sasakian pseudo-metric manifold, which may not be true in higher dimensions. Further on a Sasakian pseudo-metric manifold we have

\[ R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \]

In contact metric case, the condition \((2.10)\) implies that the manifold is Sasakian, which is not true in contact pseudo-metric case \[14\]. However, we have the following:

**Lemma 2.1.** \[14\] Let \( M \) be a \( K \)-contact pseudo-metric manifold. Then \( M \) is a Sasakian pseudo-metric manifold if and only if the curvature tensor \( R \) satisfies \((2.10)\).

### 3. The Codazzi condition for \( h \)

A self-adjoint tensor \( A \) of type \((1,1)\) on a pseudo-Riemannian manifold is known to be a Codazzi tensor if

\[ (\nabla_X A)Y = (\nabla_Y A)X \quad (3.1) \]

for all \( X, Y \in TM \). Now, we prove the following:

**Theorem 3.1.** Let \( M \) be a contact pseudo-metric manifold. Then the following statements are true:

(a) If \( h \) is a Codazzi tensor, then \( h^2 = 0 \).

(b) \( M \) is a Sasakian pseudo-metric manifold if and only if \( M \) satisfies \((2.10)\) and \( h \) is a Codazzi tensor.
Proof. (a). Suppose that $h$ is a Codazzi tensor, that is,
\[(\nabla_X h)Y = (\nabla_Y h)X, \quad X, Y \in TM.\]
For $Y = \xi$, using (2.4) in the above equation, we obtain
\[(\nabla_\xi h)X = -\varepsilon \varphi hX - h^2 \varphi X.\]
In view of (2.6), the above equation turns into
\[\varphi R(\xi, X)\xi = -\varepsilon \varphi hX - \varphi X.\] (3.2)
Operating $\varphi$ on both sides of (3.2), it follows that
\[R(\xi, X)\xi = \varphi^2 X - \varepsilon hX.\] (3.3)
Making use of (3.3) in (2.7), shows that $h^2 = 0$.

(b). If $M$ is a Sasakian pseudo-metric manifold, then $h = 0$ and $M$ satisfies (2.10); and the result is trivial. Conversely, suppose that (2.10) is true and $h$ is a Codazzi tensor. From (2.10), we obtain that
\[R(\xi, X)\xi = \varphi^2 X, \quad X \in TM.\] (3.4)
Equations (3.3) and (3.4) imply that $h = 0$, that is, $M$ is a $K$-contact pseudo-metric manifold. Thus, the result follows from Lemma 2.1. □

Remark 3.2. In a contact Riemannian manifold, if $h$ is a Codazzi tensor, then $h = 0$, that is, the manifold becomes $K$-contact manifold [19]. In the Riemannian case, as $h^2 = 0$ implies $h = 0$, Theorem 3.1 (a) holds in a stronger form, that is, $M$ is $K$-contact if and only if $h$ is a Codazzi tensor. But, in the case of $M$ being contact pseudo-metric, the condition $h^2 = 0$ does not imply that $h = 0$, because $h$ may not be diagonalizable (see [15]). Note that the result (b) of Theorem 3.1 is stronger than the Lemma 2.1 which was proved in [14].

In a contact Lorentzian manifold, just like the case of contact metric manifold, the condition $h^2 = 0$ implies $h = 0$ (see [6]). Hence, we immediately have the following

Corollary 3.3. Let $M$ be a contact Lorentzian manifold. If $h$ is a Codazzi tensor, then $h = 0$, that is, $M$ is $K$-contact Lorentzian manifold.

4. The Codazzi condition for $\tau$

We denote by $\tau$, the tensor metrically equivalent to the strain tensor $\mathcal{L}_\xi g$ along $\xi$, that is,
\[g(\tau X, Y) = (\mathcal{L}_\xi g)(X, Y)\]
for all $X, Y \in TM$. As pointed out in the introduction, in a contact metric manifold, if $\tau$ satisfies the Codazzi condition, then $h = 0$, that is, the manifold is a $K$-contact manifold. This fact need not be true in the case of contact pseudo-metric manifolds. So, it is quite interesting to study contact pseudo-metric manifolds, which satisfy the Codazzi condition for $\tau$. Now we prove the following:
Lemma 4.1. In a contact pseudo-metric manifold, $\tau$ is a Codazzi tensor if and only if the curvature tensor $R$ satisfies

$$R(\xi, X)Y = \varepsilon(\nabla_X \varphi)Y.$$  \hspace{1cm} (4.1)

Proof. Treating $\nabla \xi$ as a tensor of type $(1,1)$, that is $\nabla \xi : X \mapsto \nabla_X \xi$, one can see that

$$R(X, Y)\xi = (\nabla_X \nabla \xi)Y - (\nabla_Y \nabla \xi)X,$$

which together with (2.4) gives

$$R(X, Y)\xi = -\varepsilon(\nabla_X \varphi)Y - (\nabla_X \varphi h)Y + \varepsilon(\nabla_Y \varphi)X + (\nabla_Y \varphi h)Y. \hspace{1cm} (4.2)$$

On the other hand, if $\tau$ is a Codazzi tensor, then from (2.5) we have

$$(\nabla_X h \varphi)Y = (\nabla_Y h \varphi)X.$$ 

Thus, (4.2) shows that $\tau$ is a Codazzi tensor if and only if

$$R(X, Y)\xi = -\varepsilon\{g(\nabla_Y \varphi)X - (\nabla_X \varphi)Y\}.$$ \hspace{1cm} (4.3)

Now if $\tau$ is a Codazzi tensor, then by using Bianchi identity and (4.3), we get

$$R(\xi, X, Y, Z) = \varepsilon\{g(X, (\nabla_X \varphi)Y) - g(Y, (\nabla_X \varphi)Z) + g(Z, (\nabla_X \varphi)Y)$$

$$- g(X, (\nabla_Z \varphi)Y)\}$$

$$= -2\varepsilon g((\nabla_X \varphi)Z, Y) + R(Z, Y, \xi, X),$$

and so

$$R(\xi, X, Y, Z) = -\varepsilon g((\nabla_X \varphi)Z, Y),$$

which gives (4.1).

Conversely, if (4.1) is true, then from Bianchi identity we have

$$R(X, Y, \xi, Z) = R(\xi, Z, X, Y) = -R(Z, X, \xi, Y) - R(X, \xi, Z, Y)$$

$$= -R(\xi, Y, Z, X) + R(\xi, X, Z, Y)$$

$$= -\varepsilon\{g((\nabla_Y \varphi)Z, X) - g((\nabla_X \varphi)Z, Y)\},$$

which leads to (4.3), and hence $\tau$ is a Codazzi tensor. \hspace{1cm} □

Theorem 4.2. Let $M$ be a contact pseudo-metric manifold. Then the following statements are true.

(i) If $\tau$ is a Codazzi tensor, then $h^2 = 0$ and the Ricci operator $Q$ satisfies

$$Q \xi = 2\varepsilon n \xi.$$ \hspace{1cm} (4.4)

(ii) $M$ is Sasakian if and only if $M$ satisfies (2.10) and $\tau$ is a Codazzi tensor.

Proof. (i). If $\tau$ is a Codazzi tensor, then (4.1) gives

$$R(\xi, X)\xi = \varepsilon(\nabla_X \varphi)\xi = \varphi^2 X - \varepsilon hX,$$

where we used (2.4). This implies

$$\varphi R(\xi, \varphi X)\xi = -\varphi^2 X - \varepsilon hX,$$
and so
\[ R(\xi, X)\xi - \varphi R(\xi, \varphi X)\xi = 2\varphi^2 X. \] (4.5)
Comparing (2.7) and (4.5), we obtain \( h^2 = 0 \).

Now, if \( \{e_i\}_{i=1}^{2n+1} \) is any local pseudo-orthonormal basis, then considering (4.1) we get
\[
S(X, \xi) = \sum_{i=1}^{2n+1} \varepsilon_i R(e_i, X, \xi, e_i) = \varepsilon \sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{e_i}\varphi)e_i, X)
= \varepsilon g(\text{tr}(\nabla\varphi), X),
\]
which by using (2.8) we have (4.4).

(ii). Suppose that \( M \) is a Sasakian pseudo-metric manifold, then \( M \) satisfies (2.10) and \( h = 0 \).

Conversely, suppose that \( M \) satisfies (2.10) and \( \tau \) is a Codazzi tensor. Then (4.1) shows that
\[
g((\nabla_X\varphi)Y, Z) = \varepsilon R(\xi, X, Y, Z) = -\varepsilon R(Z, Y, \xi, X)
= -\varepsilon \{\eta(Y)g(Z, X) - \varepsilon g(Z, \xi)g(X, Y)\}
\]
which gives (2.9). Hence \( M \) becomes a Sasakian pseudo-metric manifold. \( \square \)

**Corollary 4.3.** Let \( M \) be a contact Lorentzian manifold. If \( \tau \) is a Codazzi tensor, then \( h = 0 \), that is, \( M \) is \( K \)-contact Lorentzian manifold.

5. \( \xi \)-Sectional Curvatures

The \( \xi \)-sectional curvature \( K(\xi, X) \) of a contact pseudo-metric manifold is defined by
\[
K(\xi, X) = \varepsilon \varepsilon_X g(R(\xi, X)X, \xi),
\]
where \( X \) is a unit vector field such that \( X \in \text{Ker} \eta \) and \( g(X, X) = \varepsilon_X = \pm 1 \).

It is well known that a contact metric manifold is \( K \)-contact if and only if all \( \xi \)-sectional curvatures are equal to \( +1 \) (see [5]). The corresponding result in pseudo-Riemannian case need not be true. In fact, we have the following:

**Theorem 5.1.** If \( M \) is a \( K \)-contact pseudo-metric manifold, then all \( \xi \)-sectional curvatures are equal to \( \varepsilon \).

**Proof.** If \( M \) is a \( K \)-contact pseudo-metric manifold, then \( h = 0 \). So (2.6) becomes
\[
\varphi R(\xi, X)\xi = -\varphi X,
\]
which upon applying \( \varphi \) gives
\[
R(\xi, X)\xi = -X
\]
for \( X \in \text{Ker} \eta \). Thus
\[
K(\xi, X) = \varepsilon \varepsilon_X g(R(\xi, X)X, \xi) = \varepsilon \varepsilon_X g(X, X) = \varepsilon.
\]
\( \square \)
Remark 5.2. The converse of above result is not true in general. In fact, a contact pseudo-metric manifold $M$, which satisfies (2.10), has $\xi$-sectional curvatures equal to $\epsilon$. But we already know that the condition (2.10) does not necessarily imply that $M$ is a $K$-contact pseudo-metric manifold.

Now we prove the following:

**Theorem 5.3.** On a contact pseudo-metric manifold $M$, the $\xi$-sectional curvatures satisfy
\[
K(\xi, X) = \epsilon \{1 - \epsilon_X g(h^2 X, X) - \epsilon_X g((\nabla_\xi h)X, \varphi X)\},
\]
(5.1)
\[
K(\xi, X) = K(\xi, \varphi X) - 2\epsilon \epsilon_X g((\nabla_\xi h)X, \varphi X)
\]
(5.2)
for any unit vector $X \in \text{Ker} \eta$.

**Proof.** Using (2.4), we have
\[
K(\xi, X) = -\epsilon \epsilon_X R(\xi, X, \xi, X)
\]
\[
= -\epsilon \epsilon_X g(-\varphi(\nabla_\xi h)X - X + h^2 X, X)
\]
\[
= \epsilon \{\epsilon_X g(\varphi(\nabla_\xi h)X, X) + \epsilon_X^2 - \epsilon \epsilon_X g(h^2 X, X)\},
\]
which gives (5.1).

Now, plugging $X$ by $\varphi X$ in (5.1) keeping $h\varphi = -\varphi h$ and $\nabla_\xi \varphi = 0$ in mind, we obtain
\[
K(\xi, \varphi X) = \epsilon \{1 - \epsilon_X g(h^2 X, X) + \epsilon_X g((\nabla_\xi h)X, \varphi X)\}. \tag{5.3}
\]
Now, from (5.1) and (5.3), we get (5.2).

**Theorem 5.4.** Let $M$ be a contact pseudo-metric manifold with $\nabla_\xi h = 0$. Then $h^2 = 0$ if and only if all $\xi$-sectional curvatures are equal to $\epsilon$.

**Proof.** Taking the inner product of the unit vector field $X \in \text{Ker} \eta$ with (2.7) yields the following formula for sectional curvatures:
\[
K(\xi, X) + K(\xi, \varphi X) = 2\epsilon \{1 - \epsilon_X g(h^2 X, X)\}. \tag{5.4}
\]
Now, since $\nabla_\xi h = 0$, (5.2) yields
\[
K(\xi, X) = K(\xi, \varphi X) \tag{5.5}
\]
for any unit vector $X \in \text{Ker} \eta$. From (5.4) and (5.5) we see
\[
K(\xi, X) = \epsilon \text{ if and only if } g(h^2 X, X) = 0.
\]
This concludes the proof.

**Corollary 5.5.** A contact Lorentzian manifold is a $K$-contact Lorentzian manifold if and only if all $\xi$-sectional curvatures are equal to $-1$.

As we discussed in introduction, due to the fact that $h^2 = 0$ does not imply $h = 0$ in a contact pseudo-metric manifold, the parallel condition of $\ell$ does not imply that $\xi$-sectional curvatures vanish. However, we have the following:
Theorem 5.6. If $M$ is a contact pseudo-metric manifold with $\nabla_\xi h = 0$ and $\nabla \ell = 0$, then all $\xi$-sectional curvatures vanish.

Proof. Applying by $\phi$ on both sides of (2.6) and using $\nabla_\xi h = 0$, it follows that
\[
\ell X = -h^2 X + X - \eta(X)\xi,
\]
for any $X \in TM$. Now, in view of $(\nabla_X \ell)\xi = 0$ and (5.6), we have
\[
\epsilon h^2 \varphi X - h^3 \varphi X - \epsilon \varphi X + h \varphi X = 0.
\]
If $X \in \text{Ker} \eta$ is a unit vector field, then taking the inner product of $\phi X$ with (5.7) leads to
\[
\epsilon g(h^2 X, X) + g(h^3 X, X) - \epsilon g(X, X) - g(hX, X) = 0.
\]
Now replacing $X$ by $\phi X$ in (5.7) and then taking inner product of $X$ with the resulting equation gives
\[
- \epsilon g(h^2 X, X) + g(h^3 X, X) + \epsilon g(X, X) - g(hX, X) = 0.
\]
Now subtracting (5.8) from (5.9) yields
\[
g(h^2 X, X) = g(X, X) = \epsilon X
\]
for any unit vector $X \in \text{Ker} \eta$. Using (5.10) and $\nabla_\xi h = 0$ in (5.1) we conclude that $K(\xi, X) = 0$. \hfill \Box

6. Almost CR Structures

First, we recall few notions of almost CR structures (see [11, 15, 17]). Let $M$ be a $(2n + 1)$-dimensional (connected) differentiable manifold. Let $\mathcal{H}(M)$ be a smooth real subbundle of rank $2n$ of the tangent bundle $TM$ (also called Levi distribution), and $J : \mathcal{H}(M) \to \mathcal{H}(M)$ be a smooth bundle isomorphism such that $J^2 = -I$. Then the pair $(\mathcal{H}(M), J)$ is called an almost CR structure on $M$. An almost CR structure is called a CR structure if it is integrable, that is, the following two conditions are satisfied
\[
[JX, Y] + [X, JX] \in \mathcal{H}(M), \quad (6.1)
\]
\[
J([X, Y]) = [JX, JY] - [X, JY] \quad (6.2)
\]
for all $X, Y \in \mathcal{H}(M)$.

On an almost CR manifold $(M, \mathcal{H}(M), J)$, we define a 1-form $\theta$ such that $\text{Ker} \theta = \mathcal{H}(M)$, and such a differential 1-form $\theta$ is called a pseudo-Hermitian structure on $M$. Then on $\mathcal{H}(M)$, the Levi form $L_\theta$ is defined by
\[
L_\theta(X, Y) = d\theta(X, JY)
\]
for all $X, Y \in \mathcal{H}(M)$. Furthermore, we define a $(0, 2)$-tensor field on $\mathcal{H}(M)$ by
\[
\alpha(X, Y) = (\nabla_X \theta)(JY) + (\nabla_{JX} \theta)(Y)
\]
for all $X, Y \in \mathcal{H}(M)$.

Then we have the following:
Proposition 6.1. For an almost CR structure \((\mathcal{H}(M), J, \theta)\), the following statements are equivalent:

(i) \(L_\theta\) is Hermitian, that is, \(L_\theta(JX, JY) = L_\theta(X, Y)\);
(ii) \(L_\theta\) is symmetric, that is, \(L_\theta(X, Y) = L_\theta(Y, X)\);
(iii) \([JX, Y] + [X, JY] \in \mathcal{H}(M)\);
(iv) \(\alpha\) is symmetric, that is, \(\alpha(X, Y) = \alpha(Y, X)\).

Proof. It is immediate that (i) \(\Leftrightarrow\) (ii) and (ii) \(\Leftrightarrow\) (iii) follows from the fact that \(d\theta(X, Y) = \frac{-1}{2}\theta([X, Y])\) for all \(X, Y \in \mathcal{H}(M)\). On the other hand, as in general
\[
d\theta(X, Y) = \frac{1}{2}((\nabla_X \theta)Y - (\nabla_Y \theta)X),
\]
the condition (ii) is equivalent to
\[
(\nabla_X \theta)(JY) + (\nabla_{JX} \theta)Y = (\nabla_Y \theta)(JX) + (\nabla_{JY} \theta)X,
\]
and so (ii) \(\Leftrightarrow\) (iv). \(\square\)

An almost pseudo-Hermitian CR structure \((\mathcal{H}(M), J, \theta)\) is said to be nondegenerate if the Levi form \(L_\theta\) is a nondegenerate Hermitian form, and so the 1-form \(\theta\) is a contact form.

Let \((M, \mathcal{H}(M), J, \theta)\) be a nondegenerate pseudo-Hermitian almost CR manifold. We extend the complex structure \(J\) to an endomorphism \(\varphi\) of the tangent bundle \(TM\) in such a way that \(\theta = J\) on \(\mathcal{H}(M)\) and \(\varphi \xi = 0\), where \(\xi\) is the Reeb vector field of \(\theta\). Then the Webster metric \(g_\theta\), which is a pseudo-Riemannian metric, is defined by
\[
g_\theta(X, Y) = L_\theta(X, Y), \quad g_\theta(X, \xi) = 0, \quad g_\theta(\xi, \xi) = \varepsilon
\]
for all \(X, Y \in \mathcal{H}(M)\). In this case, \((\varphi, \xi, \eta = -\theta, g = g_\theta)\) defines a contact pseudo-metric structure on \(M\). Conversely, if \((\varphi, \xi, \eta, g)\) is a contact pseudo-metric structure, then \((\mathcal{H}(M), J, \theta)\), where \(\mathcal{H}(M) = \text{Ker}\eta\), \(\theta = -\eta\), and \(J = \varphi|_{\mathcal{H}(M)}\), defines a nondegenerate almost CR structure on \(M\). Thus, we have:

Proposition 6.2 ([15]). The notion of nondegenerate almost CR structure \((\mathcal{H}(M), J, \theta)\) is equivalent to the notion of contact pseudo-metric structure \((\varphi, \xi, \eta, g)\).

Now, we prove the following:

Theorem 6.3. The nondegenerate almost CR structure \((\mathcal{H}(M), J, \theta)\) corresponding to almost contact pseudo-metric manifold \(M\) is a CR manifold if and only if
\[
(\nabla_X J)Y - (\nabla_{JX} J)Y = \alpha(X, Y)\xi
\]
for all \(X, Y \in \mathcal{H}(M)\).
Proof. Applying $J$ to (6.2) gives

$$(\nabla Y J)X - (\nabla X J)Y = JJX Y - J(\nabla JY J)X$$

for all $X, Y \in \mathcal{H}(M)$. Since $J(\nabla JX J)Y = -(\nabla JY J)X$, the above equation becomes

$$(\nabla Y J)X - (\nabla X J)Y = (\nabla JY J)X - (\nabla JX J)Y$$

(6.4)

for all $X, Y \in \mathcal{H}(M)$. If we define a $(0,3)$-tensor field $A$ on $\mathcal{H}(M)$ as

$$A(X, Y, Z) = g((\nabla JX J)Y - (\nabla JY J)X, Z)$$

(6.5)

for all $X, Y \in \mathcal{H}(M)$, then from (6.4) one obtain

$$A(X, Y, Z) = A(Y, X, Z).$$

(6.6)

Next, a simple computation shows that

$$A(X, Y, Z) + A(X, Z, Y) =$$

$$= g((\nabla JX J)Y - (\nabla X J)Y, Z) + g((\nabla JX J)Z - (\nabla X J)Z, Y)$$

$$= -g((\nabla JX J)Z, JY) + g((\nabla JX J)JZ, Y)$$

$$= -g((\nabla JX JZ, JY) - g((\nabla JX JZ, J^2 Y)$$

$$+ g((\nabla JX J^2 Z, Y) - g(J(\nabla JX JZ), Y)$$

$$= 0,$$

where the skew-symmetry of $J$ and $\nabla J$ are used. This together with (6.6) gives the following:

$$A(X, Y, Z) = -A(X, Z, Y) = -A(Z, X, Y) = A(Z, Y, X)$$

$$= A(Y, Z, X) = -A(Y, X, Z) = -A(X, Y, Z).$$

Hence it follows that $A = 0$, and so (6.5) implies

$$((\nabla JX J)Y - (\nabla X J)Y) = \gamma(X, Y) \xi$$

(6.7)

for all $X, Y \in \mathcal{H}(M)$, for certain $(0,2)$-tensor field $\gamma$ on $\mathcal{H}(M)$. It remains to show that $\gamma = \alpha$. From (6.7), it follows that

$$\gamma(X, Y) = \varepsilon g((\nabla JX J)Y - (\nabla X J)Y, \xi)$$

$$= \varepsilon\{-g((\nabla JX J)\xi, JY) + g((\nabla X J)\xi, Y)}$$

$$= \varepsilon\{g(\nabla JX \xi, Y) - g(J\nabla X \xi, Y)}$$

$$= (\nabla JX \theta)Y + (\nabla X \theta)JY$$

$$= \alpha(X, Y).$$

Conversely, suppose that (6.3) holds true. Then projecting (6.3) onto $\xi$, it follows that $\alpha$ is symmetric and is equivalent to (6.1). The symmetry of $\alpha$ together with (6.3) gives (6.4), which yields

$$-[JX, Y] - [X, JY] = J[JX, JY] - J[X, Y],$$

for all $X, Y \in \mathcal{H}(M)$, and so satisfies the equation (6.2). □
Let \((M, \varphi, \xi, \eta, g)\) be an almost contact pseudo-metric manifold with \((\mathcal{H}(M), J)\) as the corresponding almost CR structure. For \(Y \in TM\), we denote \(Y|_{\mathcal{H}(M)}\) to the orthogonal projection on \(\mathcal{H}(M)\). Then, the Bott partial connection \(\hat{\nabla}\) on \(\mathcal{H}(M)\) (along \(\xi\)) is the map \(\hat{\nabla} : S(\xi) \times \mathcal{H}(M) \to \mathcal{H}(M)\) defined by

\[
\hat{\nabla}_\xi X := ((\mathcal{L}_\xi X)|_{\mathcal{H}(M)} = [\xi, X]|_{\mathcal{H}(M)}
\]

for any \(X \in \mathcal{H}(M)\) (see, [18, p. 18]), where \(S(\xi)\) is the 1-dimensional linear subspace of \(TM\) generated by \(\xi\).

**Theorem 6.4.** Let \((M, \varphi, \xi, \eta, g)\) be an almost contact pseudo-metric manifold, and \(\xi\) a geodesic vector field. Then \(h = 0\) if and only if \(\hat{\nabla}_\xi J = 0\).

**Proof.** As \(h\xi = 0\), we may observe that, \(h = 0\) if and only if \(hX = 0\) for any \(X \in \mathcal{H}(M)\).

Now using \(\nabla_\xi \xi = 0\), for any \(X \in \mathcal{H}(M)\), we have

\[
\eta([\xi, X]) = \varepsilon g(\xi, \nabla_\xi X - \nabla_X \xi) = 0,
\]

which means \(\mathcal{L}_\xi X \in \mathcal{H}(M)\). Thus, we get

\[
2hX = \mathcal{L}_\xi (\varphi X) - \varphi(\mathcal{L}_\xi X) = \hat{\nabla}_\xi (\varphi X) - \varphi(\hat{\nabla}_\xi X)
\]

\[
= \hat{\nabla}_\xi (JX) - J(\hat{\nabla}_\xi X) = (\hat{\nabla}_\xi J)X
\]

for any \(X \in \mathcal{H}(M)\), completing the proof. \(\square\)

For contact pseudo-metric manifold, the structure vector field is geodesic. So we have the following:

**Corollary 6.5.** A contact pseudo-metric manifold is \(K\)-contact if and only if \(\hat{\nabla}_\xi J = 0\).

**Theorem 6.6.** A contact pseudo-metric manifold \((M, \varphi, \xi, \eta, g)\) is Sasakian if and only if the corresponding nondegenerate almost CR structure \((\mathcal{H}(M), J)\) is integrable and \(\hat{\nabla}_\xi J = 0\).

**Proof.** First we observe that, following the same proof given in [21] for the Riemannian case, the integrable condition (that is, (6.1) and (6.2)) of the corresponding CR structure \((\mathcal{H}(M), J)\) is equivalent to

\[
(\nabla_X \varphi)Y = -\{(\nabla_X \eta)\varphi Y\} \xi - \eta(X)\varphi(\nabla_X \xi),
\]

where

\[
(\nabla_X \eta)\varphi Y = -g(X, Y) + \varepsilon \eta(X)\eta(Y) - \varepsilon g(hX, Y),
\]

and

\[
\varphi(\nabla_X \xi) = \varepsilon X - \varepsilon \eta(X)\xi + hX.
\]

Thus, (6.8) becomes

\[
(\nabla_X \varphi)Y = g(X + \varepsilon hX, Y)\xi - \varepsilon \eta(Y)(X + \varepsilon hX).
\]

If the contact pseudo-metric manifold \((M, \varphi, \xi, \eta, g)\) is Sasakian, then (6.9) satisfies with \(h = 0\), and so corresponding nondegenerate almost CR structure \((\mathcal{H}(M), J)\) is integrable and \(\hat{\nabla}_\xi J = 0\).
Conversely, as $\nabla_{\xi}J = 0$ implies $h = 0$, equation (6.9) reduces to (2.9), and so the structure is Sasakian.

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