Anomalous and dimensional scaling in anisotropic turbulence

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We present a numerical study of anisotropic statistical fluctuations in homogeneous turbulent flows. We give an argument to predict the dimensional scaling exponents, $\zeta(p) = (p + j)/3$, for the projections of $p$-th order structure function in the $j$-th sector of the rotational group. We show that measured exponents are anomalous, showing a clear deviation from the dimensional prediction. Dimensional scaling is subleading and it is recovered only after a random reshuffling of all velocity phases, in the stationary ensemble. This supports the idea that anomalous scaling is the result of a genuine inertial evolution, independent of large-scale behavior.

In recent years a huge amount of theoretical, numerical and experimental work has been done in order to study anisotropic turbulent fluctuations\textsuperscript{1--3,5}. Typical questions go from the theoretical point of calculating and measuring anomalous scaling exponents in anisotropic sectors\textsuperscript{4--7,11}, to the more applied problem of quantifying the rate of recovery of isotropy at scales small enough\textsuperscript{12}. Another important issue is the universality of anisotropic scaling exponents, i.e. whether they are an intrinsic characteristic of the Navier-Stokes nonlinear evolution or they are fixed by a dimensional matching with the external anisotropic forcing.

Important steps forward in the analysis of anisotropic fluctuations have recently been done in Kraichnan models, i.e. passive scalars/vectors advected by isotropic, Gaussian and white-in-time velocity fields\textsuperscript{13}, with a large-scale anisotropic forcing\textsuperscript{14}. In those models, anomalous scaling arises as the result of a non-trivial null-space structure for the advecting operator (zero modes). Also, correlation functions in different sectors of the rotational group show different scaling properties. Scaling exponents are universal: they do not depend on the actual value of the forcing and boundary conditions, and they are fully characterized by the order of the anisotropy. Non-universal effects are felt only in coefficients multiplying the power laws.

Similar problems, like the existence of scaling laws in anisotropic sectors and, if any, the values of the corresponding scaling exponents are at the forefront of experimental, numerical and theoretical research for real turbulent flows. Only few indirect experimental investigations of scaling in different sectors\textsuperscript{7,8} and direct decomposition in numerical simulations\textsuperscript{7,8,11} have been attempted up to now.

The question is still open, evidences of a clear improving of scaling laws by isolating the isotropic sector have been reported, supporting the idea that the undecomposed correlations are strongly affected by the superposition of isotropic and anisotropic fluctuations\textsuperscript{7,11}. On a theoretical ground, only recently it has been highlighted the potentiality of $SO(3)$ decomposition to quantify different degrees of anisotropies for any correlation function\textsuperscript{4}. On the basis of this analysis, preliminary experimental evidences of the existence of a scaling law also in sectors with total angular momentum $j = 2$ have been reported\textsuperscript{8}. The value of the exponent for the second order correlation function being close to the dimensional estimate $\zeta_d(2) = 4/3$\textsuperscript{7} (where, from now on, subscript $d$ denotes the dimensional value).

Typically, experimental investigations in real turbulent flows are flawed by the contemporary presence of anisotropies and strong non-homogeneities. The meaning of $SO(3)$ decomposition becomes opaque in presence of strong non-homogeneities and also the very existence of scaling laws cannot be given for granted\textsuperscript{18}.

To overcome such difficulties, some of us performed\textsuperscript{9} the numerical investigation of a “Random-Kolmogorov Flow” (RKF), a fully periodic Kolmogorov flow with random forcing phases, $\delta$-correlated in time.

In this Letter, we present a more extended analysis of the same data set, but focusing on new evidences that anisotropic scaling exponents are indeed universal and anomalous, i.e. they do not follow simple dimensional scaling. In order to do this, we also give a clear phenomenological background able to predict the dimensional scaling in anisotropic sectors. The Letter is organized as follows. First, we present a simple dimensional argument for all anisotropic sectors of structure functions of any order. With respect to this dimensional prediction, we show that anisotropic exponents are indeed anomalous. Moreover, we show that by performing a random reshuffling of the velocity phases (in the stationary ensemble of anisotropic configurations), the leading anomalous scaling is filtered out and the sub-leading dimensional prediction is recovered. This is both a test of our dimensional prediction and a clean indicator that forced velocity correlations are dominated by inertial terms; in this sense one may refer to them as the “equivalent” of the zero-modes responsible for anomalous scaling and universality in linear hydrodynamical problems\textsuperscript{13}. These findings lead to conclude that anisotropic fluctuations in turbulence are anomalous and universal.
We recall few details on the numerical simulations \([\ddagger]\). The RKF is fully periodic; the large-scale anisotropic random forcing points in one direction, \(\hat{z}\), has a spatial dependency only from the \(x\) coordinate and it is different from zero at the two wavenumbers: \(k_1 = (1, 0, 0), k_2 = (2, 0, 0)\). Namely, \(f_i(k_{1,2}) = \delta_i,3f_{(1,2)} \exp(\theta_{(1,2)})\), where \(f_{(1,2)}\) are fixed amplitudes and \(\theta_{(1,2)}\) are independent random phases, \(\delta\)-correlated in time. Random phases gives an homogeneous statistics, without destroying the high anisotropy introduced by the chosen forced wavenumber. We simulated the RKF at resolution 256\(^3\) and collected up to 70 eddy turn over times.

Anisotropy is studied by means of \(SO(3)\) decomposition of longitudinal structure functions:

\[
S_p(R) = \left\langle \left[ (v(x + R) - v(x)) \cdot \hat{R} \right]^p \right\rangle, \tag{1}
\]

where we have kept only the dependency on \(R\) and neglected the small non-homogeneous fluctuations. We expect that the undecomposed structure functions are not the real “scaling” bricks of the theory. Theoretical and numerical analysis showed \([\ddagger\ddagger]\) that one must first decompose the structure functions onto irreducible representations of the rotational group and then study the scaling behavior of the projections. In practice, being the longitudinal structure functions scalar objects, their decomposition reduces to the projections on the spherical harmonics:

\[
S_p(R) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} S_p^{jm}(|R|) Y_{jm}(\hat{R}). \tag{2}
\]

As usual, we use indexes \((j, m)\) to label, respectively, the total angular momentum and its projection along a reference axis, say \(\hat{z}\). The whole physical information is hidden in the functions \(S_p^{jm}(R)\). In particular, the main question we want to address here concerns their scaling properties:

\[
S_p^{jm}(|R|) \sim A_{jm}|R|^{\zeta_p^{(j)}}. \tag{3}
\]

First, we need an estimate for the “dimensional” values of the exponents \(\zeta_p^{(j)}\) in all sectors. Our argument is based on the idea that large-scale energy pumping and/or boundary conditions are such as to enforce a large-scale, anisotropic, driving velocity field \(U\). Dimensional predictions for intermediate (small) scales anisotropic fluctuations may then be obtained by studying the influence of the large-scale \(U\) on the inertial range. Let us therefore evaluate the weight of anisotropic contributions as it comes out from a balance between inertial advection of the small scales and the “shear effect”, induced by the instantaneous large scale velocity configuration. Decomposing the velocity field in a small scale component, \(u\), and a large scale, strongly anisotropic component, \(U\), one finds the following equation for the time evolution of \(u\):

\[
\partial_t u_i + u_k \partial_k u_i + U_k \partial_k u_i + u_k \partial_k U_i = -\partial_i \nu + \Delta u_i. \tag{4}
\]

The major effect of the large-scale field is given by the instantaneous shear \(S_{ik} = \partial_i U_k\) which acts as an anisotropic forcing term on small scales. A simple dimensional reasoning can be done as follows. Let us first consider the equation of motion for two point quantities \(\langle u_i(x')u_i(x)\rangle\) in the stationary regime; we may balance inertial terms and shear-induced terms as follows:

\[
\langle u_i(x')u_k(x)\partial_k u_i(x)\rangle \sim \langle S_{ik}(x)u_i(x')u_k(x)\rangle, \tag{5}
\]

which allows for a dimensional estimate of the anisotropic components of the LHS in terms of the RHS shear intensity and of the \(\langle uu\rangle\) isotropic part. Similarly for three point quantities we have (neglecting tensorial details): \(\langle uu'u'\rangle \sim \langle S uu u\rangle\) which can be easily generalized to any order velocity correlations. The shear term is a large-scale “slow” quantity and therefore, as far as scaling properties are concerned, we may safely estimate: \(\langle S_{ik}(x)u_i(x')u_k(x)\rangle \sim D_{ik} \langle u_i(x')u_k(x)\rangle\). Here the matrix \(D_{ik}\) is associated to the combined probability to have a given shear and a given small scale velocity configuration. Clearly the \(D_{ik}\) tensor brings angular momenta only up to \(j = 2\). One may therefore argue, by using simple composition of angular momenta, the following dimensional matching \([\ddagger\ddagger]\):

\[
S_p^j(R) \sim R^j S \cdot S_{p-2}^{j-2}(R), \tag{6}
\]

where on top of each term we have written the total angular momentum of that contribution. \(S_p^j(R)\) is the projection on the \(j\)-th sector of the \(p\)-th order correlation function at scale \(R\) (see equation \([\ddagger]\)), and \(S\) is the intensity of the shear term, \(D_{ik}\), in the \(j = 2\) sector.
FIG. 2. Comparison between the dimensional estimate, \( \zeta_d^j(p) = (p + j)/3 \), (straight lines), the measured exponents, \( \zeta^j(p) \), and the exponents, \( \zeta_d^j(p) \), obtained after random dephasing (\( \times \)), for \( p = 2, 4, 6 \). Top: sector \( j = 6 \), bottom: sector \( j = 4 \).

For instance, the leading behaviour of the \( j = 2 \) anisotropic sector of the 3-th order correlation function in the LHS of (3) is given by the coupling between the \( j = 2 \) components of \( D_k \) and the \( j = 0 \) sector of the 2-th order velocity correlation in the RHS of (3):

\[
S_2^3(R) \sim RS \cdot S_0^2(R) \sim R^{d_j^2(3)}.
\]

By using the same argument and considering that now we know the scaling of \( j = 0 \) and \( j = 2 \) sectors of the third order correlation, we may estimate the scaling exponents of the fourth order correlation for \( j = 2, 4 \). From equation (3), we have the dimensional matching in the \( j = 2 \) sector:

\[
S_2^2(R) \sim RS \cdot S_0^0(R) \sim R^{d_j^4(3)} \quad \text{and in the } j = 4 \text{ sector:}
\]

\[
S_4^4(R) \sim RS \cdot S_0^0(R) \sim R^{d_j^4(4)}.
\]

The procedure is easily extended to higher orders:

\[
\begin{align*}
\zeta_d^{j=2}(p) &= \zeta_d^{j=0}(p-1) + 1 = (p+2)/3, \quad p > 2; \quad (7) \\
\zeta_d^{j=4}(p) &= \zeta_d^{j=2}(p-1) + 1 = (p+4)/3, \quad p > 3; \quad (8) \\
\zeta_d^{j=6}(p) &= \zeta_d^{j=4}(p-1) + 1 = (p+6)/3, \quad p > 4; \quad (9)
\end{align*}
\]

which can be summarized as

\[
\zeta_d^j(p) = \frac{(p+j)}{3},
\]

where intermittency effects in the isotropic sector have been neglected for simplicity. In this way, giving as input only the isotropic exponents, \( \zeta_d^{j=0}(p) \), we are able to predict the scaling exponents up to \( j = 2 \) for the third order structure functions, to \( j = 4 \) for the fourth order, to \( j = 6 \) for the fifth order and so on. We may do a little better by giving a prediction also for anisotropic fluctuations of second order correlation functions. This cannot be simply obtained by using the equations of motion, because the first one involving velocity correlations at different spatial locations, i.e. inertial range quantities, is that for \( \partial_t \langle u_i(x)u_j(x') \rangle \), which fixes a constraint only for the third order correlation function [3].

A way out is to ask the second order anisotropic fluctuations to be analytic in the shear intensity, \( S \), consistently with what one finds for higher order structure functions by the above dimensional estimate. With this assumption, we recover for \( j = 2 \) Lumley prediction [7], \( \zeta_d^{j=2}(2) = 4/3 \) by simply writing the first two terms dimensionally consistent with an expansion in the shear intensity: \( \langle uu \rangle \sim (\varepsilon R)^{3/3} + SR^{1/3} + \ldots \) where the first corresponds to the isotropic scaling, while the second captures anisotropies up to \( j = 2 \) (higher \( j \)-sectors could be captured by adding other terms in the expansions). By using this argument, we may now remove the limit of validity of the dimensional prediction, and extend it to all \( p \) values.

We now come to our numerical results for the \( SO(3) \) decomposition of longitudinal structure functions. In Figure 3, we present for the 4-th order longitudinal structure function, an overview of all sectors \( (j, m) \) which have a signal-to-noise ratio high enough to ensure stable results. Sectors with odd \( j \) are absent due to the parity symmetry of our observable. We measured anisotropic fluctuations up to \( j = 6 \). Scaling exponents can be measured in almost all sectors except for \( j = 2 \) where an annoying oscillation in the sign of \( S_2^{j=4}(R) \) prevents us from giving a quantitative statement.

We notice, as it is also summarized in Figure 3, that all measured exponents show a clear departure from the dimensional prediction. For example we measure in the \( j = 4 \) sectors the values: \( \zeta^4(2) = 1.65(5), \zeta^4(4) = 2.20(5), \zeta^4(6) = 2.55(10) \), and in the \( j = 6 \) sector: \( \zeta^6(2) = 3.2(2), \zeta^6(4) = 3.1(2), \zeta^6(6) = 3.3(2) \). This is a first clear sign that anisotropic scaling exponents are intermittent.

The importance of being anomalous does not stand on the exact values of the exponents, but on the connection between anomalous scaling and universality. Indeed, if correlation functions in the inertial range are not given by a dimensional matching with the large-scale shear, it means that they are fixed only by the inertial part of the Navier-Stokes evolution. In other words, they should enjoy strong universality properties with respect to changes of the large-scale physics, similarly to what happens to "zero-modes" responsible for anomalous scaling and universality in linear hydrodynamical problems [10].

Such a statement can even be tested in a different way.

We have taken the stationary configurations of the RKF and randomly re-shuffled all velocity phases: \( \hat{u}_i(k) \rightarrow P_{il}(k) \hat{u}_i(k) \exp(i \theta_i(k)) \), where \( P_{il}(k) \) is the incompressibility projector and \( \theta_i(k) = -\theta_i(-k) \). In this way we expect to filter out the dominant intermittent fluctuations coming from the inertial evolution, or at least those intermittent contributions connected to nontrivial phase organization. The rationale of the above statement comes from the observation that anomalous scaling, in linearly advected hydrodynamical models, is connected to the existence of statistically preserved structures with highly complex geometrical properties [20].
We imagine that once canceled the anomalous fluctuations, the sub-dominant fluctuations, due to the dimensional balancing with the forcing-shear terms, should show up. Still, it is worth to remark, the statistics of the velocity field stays non-gaussian.

In Figure 3 we show the results for the decomposition of 4-th order structure functions (after phase randomization) in the $j = 4$ anisotropic sector. As it can be seen, scaling properties change significantly going from the anomalous value (before randomization) to the dimensional predictions (after randomization). This happens for all sectors and moments we have measured, as it is summarized in Figure 2, with the notable exception of the second order structure function where phase randomization has almost no effect. An interesting fact which can have two explanations. Phases randomization is not enough to completely filter out intermittency, especially for two points quantities which should be less sensible to phase correlation. Or, as noticed before, because second order correlation function is not constrained by any equation of motion, dimensional scaling may never exist for it even not as a sub-leading contribution. This is an important point which certainly deserves further numerical, experimental tests.

In conclusions we have presented a dimensional argument able to predict, by means of a matching between inertial correlations and shear-induced inertial terms, scaling exponents for all structure functions in any anisotropic sector. We have shown by a direct numerical simulation that anisotropic scaling exponents deviate from the previous dimensional prediction, showing anomalous values. When performing a random re-shuffling of all velocity phases, the dimensional scaling comes out as a sub-leading contribution. Everything points toward the conclusion that anisotropic fluctuations are anomalous and universal.

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