Wave packets of a harmonic oscillator with various
degrees of rigidity

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The time evolution of wave packets in a harmonic oscillator potential is
studied. Some new results for the most general case are obtained. A natural
number, called “degree of rigidity”, is introduced to describe qualitatively how
much the shape of a wave packet is changed with time. Two classes of wave
packets with an arbitrarily given degree of rigidity are presented.

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I. INTRODUCTION

In recent years, the time evolution of localized wave packets of various quantum-mechanical
systems has been widely discussed in the literature (see Ref. [1] for a review). As for wave
packets of the harmonic oscillator, the subject is as old as quantum mechanics itself. Though
some aspects of the Gaussian wave packets (displaced or squeezed ground state) are still being
discussed in the current literature [2, 3], the problem was essentially solved in the very early
years [4, 5, 6]. In the 1950’s, several authors studied more general displaced number states and
found that they keep their shape unchanged while their center oscillates like a classical particle
[7, 8, 9, 10]. Displaced and squeezed number states of special forms were also considered in
some of these papers [8, 9]; it turns out that their width is also oscillating, so the shapes of
such wave packets change with time apparently. Displaced and squeezed number states of the
most general form were studied in a recent paper [11]. This may represent the most general
case where the time dependence of the wave packets could be worked out explicitly.

The questions we are concerned with are: First, if the time dependence of the wave packet
cannot be obtained explicitly, what conclusions could be made in regard to its time evolution?
Second, if the shape of the wave packet changes with time, how can we describe the level of
changes for different cases? For the first question, it is known that the center of the wave
packet moves like a classical particle while its width pulsates. However, one can find wave

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packets whose width keeps unchanged but whose shape still changes with time. To describe the change of the shape in the general case we introduce general-order moments of $x$ about its center, defined as the mean value of $(x - \bar{x}_t)^K$ and denoted by $Q_K(t)$, where $\bar{x}_t$ is the mean value of $x$ (which represents the center of the wave packet) and $K$ a natural number. Differential equations for these moments are established for general $K$, and are solved for the most general initial conditions up to $K = 4$. They are essentially oscillating, but higher-order ones involve higher frequencies and thus their time dependence is more complicated. Wave packets could be found whose $Q_K(t)$ are time independent up to $K = 4$. Obviously the shape of such wave packets changes with time less than those that only keep their width [represented by $\sqrt{Q_2(t)}$] unchanged. At this stage the answer to the second question is becoming clear.

We introduce a natural number called the “degree of rigidity” for a wave packet: If $Q_K(t)$ is time independent for $K = 2, 3, \ldots, 2N$ [note that $Q_0(t) \equiv 1$ and $Q_1(t) \equiv 0$] but not for $K = 2N + 2$ (for $K = 2N + 1$ it may or may not be time independent), we say that the wave packet has degree of rigidity $N$. It is obvious that the larger $N$ is, the less the shape changes with time. The shape of a wave packet with a time-dependent width changes with time apparently and thus has no rigidity. One with a constant width but with all higher-order moments time dependent has the “ground” degree of rigidity 1. On the other hand, if $Q_K(t)$ is time independent for all $K$, then the degree of rigidity for such a wave packet is infinity. In other words, its shape is perfectly rigid. Typical examples of such wave packets are displaced number states. Two classes of wave packets with an arbitrarily given degree of rigidity are presented.

II. SOME GENERAL RESULTS

Consider localized wave packets of the harmonic oscillator in one dimension, whose time evolution is governed by the Hamiltonian

$$H = \frac{p^2}{2\mu} + \frac{1}{2} \mu \omega^2 x^2.$$  \hspace{1cm} (1)

The normalized initial wave function $\psi(x, 0)$ at $t = 0$ will be denoted by $\psi_0$ for convenience. Localization means that the mean value of $x^k p^l$ in $\psi_0$ is finite for any nonnegative $k$ and $l$. The wave function $\psi(x, t)$ at time $t$ will be denoted by $\psi_t$. The mean values of $x$ and $p$ at time $t$ are

$$\bar{x}_t = (\psi_t, x \psi_t), \quad \bar{p}_t = (\psi_t, p \psi_t).$$  \hspace{1cm} (2)

Using the Schrödinger equation it is easy to show that

$$\dot{\bar{x}}_t = \frac{\bar{p}_t}{\mu}, \quad \dot{\bar{p}}_t = -\mu \omega^2 \bar{x}_t.$$  \hspace{1cm} (3)

The solution of these equations is

$$\bar{x}_t = \bar{x}_0 \cos \omega t + \frac{\bar{p}_0}{\mu \omega} \sin \omega t, \quad \bar{p}_t = \bar{p}_0 \cos \omega t - \mu \omega \bar{x}_0 \sin \omega t.$$  \hspace{1cm} (4)

The solution means that the center of an arbitrary wave packet moves like a classical particle. This is a well known result.

To study the change with time of the shape of the wave packet we consider the mean value of $(x - \bar{x}_t)^K$ in $\psi_t$ where $K$ is a natural number. These mean values are denoted by $Q_K(t)$ in...
the following. If the $Q_K(t)$’s are time independent for all $K$, then the wave packet obviously keeps its shape unchanged while it is moving, and this is the case for displaced number states [8, 9, 10]. In order to examine their time dependence for a general wave packet, we define the real quantities

$$R_{kl}(t) = \frac{1}{2}(\psi_t, \{(x - \bar{x}_t)^k, (p - \bar{p}_t)^l\}\psi_t), \quad S_{kl}(t) = \frac{1}{2i}(\psi_t, \{(x - \bar{x}_t)^k, (p - \bar{p}_t)^l\}\psi_t),$$

(5)

where $k$ and $l$ are nonnegative integers, and $\{F, G\} = FG + GF$, $[F, G] = FG - GF$ for any operator $F$ and $G$. Obviously $R_{K0}(t) = Q_K(t)$. In the following we also denote $R_{0K}(t)$ by $P_K(t)$. It is straightforward to show that they satisfy the following equations

$$\dot{R}_{kl}(t) = \frac{k}{\mu}R_{k-1,l+1}(t) - l\mu\omega^2R_{k+1,l-1}(t) + \frac{\hbar}{2\mu}k(k-1)S_{k-2,l}(t) - \frac{\hbar\mu\omega^2}{2}l(l-1)S_{k,l-2}(t),$$

(6a)

$$\dot{S}_{kl}(t) = \frac{k}{\mu}S_{k-1,l+1}(t) - l\mu\omega^2S_{k+1,l-1}(t) - \frac{\hbar}{2\mu}k(k-1)R_{k-2,l}(t) + \frac{\hbar\mu\omega^2}{2}l(l-1)R_{k,l-2}(t).$$

(6b)

Note that by definition we have $S_{k0}(t) = S_{0k}(t) = 0$ for all $k$. We are mainly interested in $R_{kl}(t)$, especially $Q_K(t)$. It is easy to realize that the equations for the subset $\{R_{kl}(t)|k + l = K\}$ close among themselves provided that the lower-order subset $\{S_{kl}(t)|k + l = K - 2\}$ has been obtained. The equations for the subset $\{S_{kl}(t)|k + l = K\}$ also close among themselves, provided that the lower-order subset $\{R_{kl}(t)|k + l = K - 2\}$ is known. Now for $k = l = 0$ we have $R_{00}(t) = 1$ and $S_{00}(t) = 0$ by definition. Similarly for $k + l = 1$ we have $R_{10}(t) = R_{01}(t) = S_{10}(t) = S_{01}(t) = 0$. These enable us to solve the cases with $k + l \geq 2$.

The case $k + l = 2$ is of essential interest, because it involves the width of the wave packet. By definition we have $S_{20}(t) = S_{02}(t) = 0$, and $S_{11}(t) = \hbar/2$. These will be useful when the case $k + l = 4$ is considered. For the moment we are interested in the remaining three. It is not difficult to find that

$$Q_2(t) = \frac{\mu^2\omega^2Q_2(0) + P_2(0)}{2\mu^2\omega^2} + \frac{\mu^2\omega^2Q_2(0) - P_2(0)}{2\mu^2\omega^2} \cos 2\omega t + \frac{R_{11}(0)}{\mu\omega} \sin 2\omega t,$$

(7a)

$$P_2(t) = \frac{\mu^2\omega^2Q_2(0) + P_2(0)}{2} - \frac{\mu^2\omega^2Q_2(0) - P_2(0)}{2} \cos 2\omega t - \mu\omega R_{11}(0) \sin 2\omega t,$$

(7b)

$$R_{11}(t) = R_{11}(0) \cos 2\omega t - \frac{\mu^2\omega^2Q_2(0) - P_2(0)}{2\mu\omega} \sin 2\omega t.$$  

(7c)

The width of the wave packet at time $t$ is characterized by the quantity $\Delta x = \sqrt{Q_2(t)}$, and that in the momentum space by $\Delta p = \sqrt{P_2(t)}$. Thus the width of the wave packet oscillates with frequency $2\omega$, just like that for a squeezed ground state. It can be shown that both $Q_2(t)$ and $P_2(t)$ are positive as they should be. The uncertainty product $\Delta x\Delta p$ is also oscillating. These conclusions (for a general wave packet) have been discussed in some different way previously [8, 9], so we will not go into more details. A relation that seems not to be emphasized in the literature is

$$\mu^2\omega^2(\Delta x)^2 + (\Delta p)^2 = \mu^2\omega^2(\Delta_0 x)^2 + (\Delta_0 p)^2.$$

(8)

According to this result, $\Delta x$ reaches its minimum when $\Delta p$ reaches its maximum and vice versa.
When \(k + l = 3\), all the \(S_{kl}(t)\) can be found to be zero. The equations for the \(R_{kl}(t)\) can be solved without much difficulty. They are all linear combinations of \(\{\sin \omega t, \cos \omega t, \sin 3\omega t, \cos 3\omega t\}\). Since the results are lengthy and not important for further discussions we will not write them down. We just point out that they all vanish if the corresponding initial values \(R_{kl}(0)\) are all zero.

When \(k + l = 4\), it can be found that \(S_{40}(t) = S_{04}(t) = 0, S_{31}(t) = 3\hbar Q_2(t)/2, S_{13}(t) = 3\hbar P_2(t)/2, S_{22}(t) = 2\hbar R_{11}(t)\). However, the results for the \(R_{kl}(t)\) are rather lengthy, they are linear combinations of \(\{1, \sin 2\omega t, \cos 2\omega t, \sin 4\omega t, \cos 4\omega t\}\). We only write down one of them here:

\[
Q_4(t) = \frac{3\mu^4 \omega^4 Q_4(0) + 3P_4(0) + 6\mu^2 \omega^2 R_{22}(0) + 3\hbar^2 \mu^2 \omega^2}{8\mu^4 \omega^4}
+ \frac{\mu^4 \omega^4 Q_4(0) - P_4(0)}{2\mu^4 \omega^4} \cos 2\omega t + \frac{R_{13}(0) + \mu^2 \omega^2 R_{31}(0)}{\mu^3 \omega^3} \sin 2\omega t
+ \frac{\mu^4 \omega^4 Q_4(0) + P_4(0) - 6\mu^2 \omega^2 R_{22}(0) - 3\hbar^2 \mu^2 \omega^2}{8\mu^4 \omega^4} \cos 4\omega t
- \frac{R_{13}(0) - \mu^2 \omega^2 R_{31}(0)}{2\mu^3 \omega^3} \sin 4\omega t.
\]

We see that higher-order moments about the center have more complicated time dependence, but they are essentially oscillating and hence are periodic functions. This is also true for still higher ones. Indeed, \(\bar{x}_t, \bar{p}_t\) and \(\psi_t\) are all periodic, and so are \(R_{kl}(t)\) and \(S_{kl}(t)\).

For larger values of \(k + l\), the solutions are more difficult to find. We will not proceed further in this respect.

### III. SIMPLIFIED RESULTS FOR SPECIAL CASES

The results obtained in Sec. II are rather complicated. In the following sections we will discuss the results for special cases with the initial condition

\[
\psi_0 = \psi(x, 0) = \varphi(x - x_0)e^{ip_0 x/\hbar},
\]

where \(x_0\) and \(p_0\) are real constants, and \(\varphi(x)\) has definite parity, namely

\[
\varphi(-x) = \pm \varphi(x).
\]

In this initial state it is not difficult to show that

\[
\bar{x}_0 = x_0, \quad \bar{p}_0 = p_0,
\]

and

\[
R_{kl}(0) = 0, \quad S_{kl}(0) = 0, \quad k + l = 1, 3, 5, \ldots.
\]

The latter could be derived from the useful relations

\[
(p_0, (x - x_0)^k(p - p_0)^l \psi_0) = (\varphi(x), x^k p^l \varphi(x)),
(p_0, (p - p_0)^l(x - x_0)^k \psi_0) = (\varphi(x), p^l x^k \varphi(x)),
\]

\(k, l = 0, 1, 2, \ldots\).
which are easy to show.

The shape of the initial wave packet, $|\psi_0|^2$, is obviously symmetric about the center $x_0$. This symmetry will be kept at later times. To prove this it is sufficient to show that $Q_{2K-1}(t) = 0$ for all natural numbers $K$. Indeed, we will show that

$$R_{kl}(t) = 0, \quad S_{kl}(t) = 0, \quad k + l = 1, 3, 5, \ldots.$$  \hspace{1cm} (14)

For $k + l = 1$ it is true as given in Sec. II. Now suppose that it is true for $k + l = 2K - 1$, and consider the case with $k + l = 2K + 1$. It is easy to realize that the equations for the subset $\{R_{kl}(t)|k + l = 2K + 1\}$ close among themselves and are all homogeneous because all of the $S_{kl}(t)$ in the subset $\{S_{kl}(t)|k + l = 2K - 1\}$ vanish (according to the assumption). Since the initial conditions are all homogeneous too, the solutions are obviously $R_{kl}(t) = 0, S_{kl}(t) = 0$, for $k + l = 2K + 1$ as well. The case with $k + l = 3$ has been explicitly calculated, and the result is consistent with this general conclusion, as pointed out in Sec. II.

If $\varphi(x)$ is real, we have another useful consequence:

$$R_{kl}(0) = 0, \quad k, l = 1, 3, 5, \ldots.$$  \hspace{1cm} (15)

As a result the solutions $R_{kl}(t)$ with $k + l = 2K$ are also simplified. For example, the sine terms in Eqs. (7a), (7b), and (9) all vanish. However, that $\varphi(x)$ is real is just a sufficient condition, not a necessary one for Eq. (15).

Now if $\varphi(x)$ in the initial state satisfies the condition

$$(\varphi(x), (xp + px)\varphi(x)) = 0, \quad (\varphi(x), \mu^2\omega^2x^2\varphi(x)) = (\varphi(x), p^2\varphi(x)),$$  \hspace{1cm} (16)

we have [cf. Eq. (13)]

$$R_{11}(0) = 0, \quad \mu^2\omega^2Q_2(0) = P_2(0),$$

and from Eq. (7) we obtain

$$Q_2(t) = Q_2(0), \quad P_2(t) = P_2(0), \quad R_{11}(t) = 0.$$  

That is, the width of the wave packet keeps unchanged. The second condition in Eq. (16) means that the kinetic energy and the potential energy have the same mean value in $\varphi(x)$. It is easy to realize that all the number states $\varphi_n(x)$ satisfy the conditions (10b) and (16). However, there exist many other functions that satisfy these conditions. For example,

$$\varphi^{even}(x) = \sum_{i=1}^{\infty} a_i \varphi_{2n_i}(x), \quad 0 \leq n_1 < n_2 < \ldots, \quad \sum_{i=1}^{\infty} |a_i|^2 = 1 \hspace{1cm} (17a)$$

and

$$\varphi^{odd}(x) = \sum_{i=1}^{\infty} b_i \varphi_{2n_i+1}(x), \quad 0 \leq n_1 < n_2 < \ldots, \quad \sum_{i=1}^{\infty} |b_i|^2 = 1 \hspace{1cm} (17b)$$

all have definite parity. And as long as $n_{i+1} - n_i \geq 2$ for all $i$, they also satisfy the conditions in Eq. (16).
However, the shapes of such wave packets change with time, though their width keeps unchanged. The reason is that $Q_4(t)$, and in general the higher-order moments, are still oscillating. The conditions for $Q_4(t)$, $P_4(t)$ etc. to be time independent are

$$R_{13}(0) = R_{31}(0) = 0, \quad \mu^4 \omega^4 Q_4(0) = P_4(0), \quad 2\mu^2 \omega^2 Q_4(0) - 6R_{22}(0) = 3\hbar^2.$$  \hspace{1cm} (18)$$

If $\varphi(x) = \varphi_n(x)$, a number state, then it can be shown that these conditions are satisfied, as expected. However, there exist many other functions that satisfy these conditions. In fact, the two classes of functions given in Eq. (17) do if $n_{i+1} - n_i \geq 3$ for all $i$. In this case Eq. (16) is of course satisfied too. Since now $Q_4(t)$ is also time independent, the shapes of the wave packets changes with time less than the case where only the width is kept unchanged. In other words, their shapes are more rigid.

In general, the higher-order moments, say $Q_6(t)$, of the above wave packets will still be time dependent. It will become more and more difficult to discuss the problem in the above manner. Since we are not going to find the most general results for these higher-order moments, we will proceed in a different way in the next section.

**IV. WAVE PACKETS WITH VARIOUS DEGREES OF RIGIDITY**

In the last section we have found wave packets whose width keeps unchanged with time, and ones whose $Q_K(t)$ up to $K = 4$ all keep unchanged with time. Obviously the shape of the latter changes with time less than the former. Now we introduce a natural number called the “degree of rigidity” for a wave packet: If $Q_K(t)$ is time independent for $K = 2, 3, \ldots, 2N$ [note that $Q_0(t) \equiv 1$ and $Q_1(t) \equiv 0$] but not for $K = 2N + 2$ (the situation for $K = 2N + 1$ is not important in this definition), we say that the wave packet has degree of rigidity $N$. Thus the two cases mentioned above have degrees of rigidity 1 and 2, respectively. It is obvious that the larger $N$ is, the less the shape changes with time. If the width of a wave packet changes with time, then its shape changes with time apparently and thus has no rigidity. On the other hand, if $Q_K(t)$ is time independent for all $K$, then the degree of rigidity for such a wave packet is infinity. In other words, its shape is perfectly rigid. Typical examples of such wave packets are displaced number states.

In order to find wave packets with a given degree of rigidity, we consider the quantity

$$W_{kl}(t) = (\psi_t, (x - \bar{x}_t)^k(p - \bar{p}_t)^l\psi_t) = R_{kl}(t) + iS_{kl}(t).$$  \hspace{1cm} (19)$$

Because $\psi_t = e^{-i\hat{H}t/\hbar}\psi_0$, we have

$$W_{kl}(t) = (\psi_0, (x_t - \bar{x}_t)^k(p_t - \bar{p}_t)^l\psi_0),$$  \hspace{1cm} (20)$$

where

$$x_t = e^{i\hat{H}t/\hbar}xe^{-i\hat{H}t/\hbar} = x \cos \omega t + \frac{p}{\mu \omega} \sin \omega t,$$  \hspace{1cm} (21a)$$

$$p_t = e^{i\hat{H}t/\hbar}pe^{-i\hat{H}t/\hbar} = p \cos \omega t - \mu \omega x \sin \omega t.$$  \hspace{1cm} (21b)$$
Now we confine ourselves to initial states of the form $|\psi_0\rangle$. On account of Eqs. (4) and (11), we have

$$W_{kl}(t) = \langle \psi_0, [(x - x_0) \cos \omega t + (\mu \omega)^{-1}(p - p_0) \sin \omega t]^k[(p - p_0) \cos \omega t - \mu \omega (x - x_0) \sin \omega t]^l \psi_0 \rangle. \tag{22}$$

Note that for any function $f(x)$ we have $(p - p_0)[e^{in\omega x/h}f(x)] = e^{in\omega x/h}pf(x)$, and that the inner product is an integral over $x$, the above equation can be simplified as

$$W_{kl}(t) = \langle \varphi(x), [x \cos \omega t + (\mu \omega)^{-1}p \sin \omega t]^k(p \cos \omega t - \mu \omega x \sin \omega t)^l \varphi(x) \rangle. \tag{23}$$

Using Eqs. (21) again this becomes

$$W_{kl}(t) = \langle \varphi_l(x), x^k p^l \varphi_l(x) \rangle, \tag{24}$$

where

$$\varphi_l(x) = e^{-iHt/\hbar} \varphi(x). \tag{25}$$

If $\varphi(x) = \varphi_n(x)$, a number state, then $\varphi_l(x) = e^{-iE_n t/\hbar} \varphi_n(x)$, and

$$W_{kl}(t) = \langle \varphi_n(x), x^k p^l \varphi_n(x) \rangle = W_{kl}(0), \quad k, l = 0, 1, 2, \ldots. \tag{26}$$

Especially, $Q_K(t) = W_{K0}(t) = Q_K(0)$ for all $K$, so that the shape of a displaced number state is perfectly rigid. This is a well-known conclusion. Here we obtain the conclusion in a different and also simple way.

Consider the function $\varphi^\text{even}(x)$ given in Eq. (17a), we have

$$\varphi_t^\text{even}(x) = \sum_{i=1}^\infty a_i \exp(-iE_{2n_i} t/\hbar) \varphi_{2n_i}(x). \tag{27}$$

Assuming that $n_{i+1} - n_i \geq N + 1$. Because $x^k p^l \varphi_{2n_i}(x)$ is a linear combination of $\{ \varphi_{2n_i+k+l}(x), \varphi_{2n_i+k+l-2}(x), \ldots, \varphi_{2n_i-(k+l)}(x) \}$, we have for $k + l \leq 2N$ that

$$W_{kl}(t) = \langle \varphi_t^\text{even}(x), x^k p^l \varphi_t^\text{even}(x) \rangle = \sum_{i=1}^\infty |a_i|^2 \langle \varphi_{2n_i}(x), x^k p^l \varphi_{2n_i}(x) \rangle = W_{kl}(0). \tag{28}$$

Especially

$$Q_K(t) = Q_K(0), \quad K = 2, 3, \ldots, 2N. \tag{29}$$

The same result can be obtained for the function $\varphi^\text{odd}(x)$ given in Eq. (17b). [Remember that $Q_K(t) = 0$ for all odd $K$ as proved in Sec. III.] Thus, if $n_{i+1} - n_i \geq N + 1$, the two classes of functions given in Eq. (17) lead to wave packets [whose initial wave functions are given by Eq. (10a)] that have degree of rigidity not less than $N$. If some of the differences $n_{i+1} - n_i$ equals $N + 1$, they will in general have degree of rigidity $N$. Otherwise the degree of rigidity will be greater than $N$. For large $N$, the shapes of these wave packets are almost unchanged with time.
V. SUMMARY

In this paper we studied the time evolution of a general wave packet in the harmonic oscillator potential by examining the time dependence of the various moments $Q_K(t)$ of $x$ about the center of the wave packet. The differential equations for these objects are derived for general $K$, and are solved for the most general initial conditions up to $K = 4$. Wave packets with constant width, and ones whose $Q_K(t)$ up to $K = 4$ are all constants, are discussed. These include displaced number states as special cases. In general the shapes of these wave packets still change with time because the higher order moments are oscillating. A natural number $N$, called the degree of rigidity, is introduced to describe qualitatively how much the shape is changed with time. The larger $N$ is, the less the shape is changed with time. Displaced number states are perfectly rigid and have $N = \infty$. Two classes of wave packets with an arbitrarily given degree of rigidity are given explicitly.

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