A TRIPLE COPRODUCT OF CURVES AND KNOTS

NOBORU ITO AND TAKESHI KOMATSUZAKI

Abstract. We suggest a triple coproduct $\Delta$ which decomposes pointed one-component curves on surfaces into three-component curves. Combined with intersection numbers $\nu$ on three component curves, $\Delta$ gives a stable equivalence invariant of one-component curves on surfaces. This study is motivated by relationship between the Milnor’s triple linking number and the affine index polynomial.

1. Introduction

Algebraic structures on curves on surfaces have been well studied. They have two natural products; one of them is of Goldman [6] and the other is of Andersen-Mattes-Reshetikhin [1, 2]. The former gives Lie bialgebra with Turaev cobracket [16]; for the latter, Cahn operation induces co-Jacobi and coskew symmetry identities [3].

In virtual knot theory, Kauffman [11], Folwaczny-Kauffman [5], Cheng-Gao [4], and Satoh-Taniguchi [13] independently introduce the affine index polynomial, aka writhe polynomial, where virtual knots are identified with stable equivalence classes of signed curves on surfaces (Turaev [15]). Before these works, Turaev [14] introduces the $u$-polynomial for virtual strings, and Henrich [7] defines a virtual knot polynomial that relates to the Goldman-Turaev Lie bialgebra.

For these coproducts, we seek a triple coproduct where two theories meet. Let $C^*$ be the set of stable homeomorphism classes of one-component pointed oriented curves on oriented surfaces; let $C^3$ be the set of stable homeomorphism classes of three-component oriented curves generically immersed on oriented surfaces. Either $C$ or a tuple $(C^{(1)}, C^{(2)}, C^{(3)})$ denotes a stable homeomorphism class of one-component pointed curves or three-component curves, respectively.

Figure 1. Smoothing. The label $L$ (resp. $R$) indicates “left” (resp. “right”).

Let $C$ be a curve with a base point. Traveling along the curve beginning at the base point in the direction of the orientation. An ordered pair $(a, b)$ of two crossings $a$ and $b$ are called parallel if and only if smoothing two crossings along the orientation (Figure 1) produces a three-component curve $C_{ab}^{(1)}, C_{ab}^{(2)}, C_{ab}^{(3)}$. For each parallel pair $a, b$, we define the components’ order $C_{ab}^{(1)}, C_{ab}^{(2)}, C_{ab}^{(3)}$ and words $w$ (Definition 3); we define a map $\Delta$ by

$$\Delta : C^* \to \mathbb{Z}[C^3] \otimes W_{LR}; C \mapsto \sum_{(a, b) \text{ parallel}} (C_{ab}^{(1)}, C_{ab}^{(2)}, C_{ab}^{(3)}) \otimes w.$$  

The map $\Delta$ does not keep stable equivalence, but $(\nu \otimes \mathrm{id}) \circ \Delta$ does; $\nu$ is an extension of the intersection number for three-component curves (Definition 5).

Theorem 1. Let $C$ be a stable homeomorphism class of a curve with a base point. Then $(\nu \otimes \mathrm{id}) \circ \Delta(C)$ is invariant under stable equivalence preserving the base point.

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2. Preliminaries

Definition 1 (pointed curves, stable homeomorphism). A curve is the image of a generic immersion of oriented circles into an oriented (closed) surface. Each self-intersection is called a crossing. A one-component curve with a base point, which is not an intersection, is called pointed. Two curves are stably homeomorphic if there is a homeomorphism of their regular neighborhoods in the ambient surfaces mapping the first curve onto the second one preserving the orientations of the curve and the surface. When the curves are pointed one-component curve, then we also require that the base point is preserved.

Remark 1. Adding handles to the ambient surface away from a neighborhood of a curve does not change a stable homeomorphism class.

Notation 1. Let \( \mathcal{C}_* \) be the set of stable homeomorphism classes of one-component pointed oriented curves on oriented surfaces. Let \( \mathcal{C}^3 \) be the set of stable homeomorphism classes of three-component oriented curves generically immersed on oriented surfaces.

Definition 2 (stable equivalence). Two pointed one-component curves stably equivalent if they are related by finite sequence of deformations \( \Omega_1 \), \( \Omega_2 \), \( \Omega_3 \), and \( \Omega_4 \) shown in Figure 2, up to stable homeomorphisms, away from the base point.

![Figure 2. A generating set of homotopy of curves: \( \Omega_1 \), \( \Omega_2 \), two patterns of \( \Omega_3 \) (from the left to the right, in the 1st line); \( \Omega_4 \) (the 2nd line).](image)

Note that the sum of (1) runs over the parallel pairs \((a, b)\) consisting of crossings \(a\) and \(b\). For a parallel pair \((a, b)\) of a curve \(C\), we consider smoothing them in order. We define the ordering of resulting three components and a word \(w_{ab}\) in \(W_{LR}\).

Definition 3 (Components’ order and words of \(L, R\)). Let \((a, b)\) be a parallel pair. If we smooth ‘a’ of \(C\), then we obtain two curves. We assign a label \(L\) (\(R\), resp.) to the component the left (right, resp.) of the smoothing. To define the order of them, let the component containing the base point be the first. Therefore, now the two components are specified by \((C_L, C_R)\) or \((C_R, C_L)\) in the order. Without loss of generality, suppose that it is \((C_L, C_R)\) in the following, i.e., \(C_L\) has the base point. When ‘b’ is in \(C_L\) or \(C_R\), the number of components increases by one, 

\[
(C_L, C_R) \rightarrow (C_L^*, C_L^1, C^R) \quad \text{(formar case)} \quad \text{or} \quad (C_L, C_R) \rightarrow (C_L, C_R^*, C_R^1) \quad \text{(latter case)},
\]

where \(C_L^*\) and \(C_L^1\) (resp. \(C_R^*\) and \(C_R^1\)) are newborn components from \(C_L\) (resp. \(C_R\)). The order “\(<\)” is (somewhat tricky) defined as follows. In the former case, ‘*’ means that it has the base point, and in the latter case, * (\(\), resp.) means that it is the left (right, resp.) component at the smoothing of \(D_R\). In the same way, listing all the patterns, we have

\[
(C_L, C_R) \rightarrow (C_L^*, C_L^1, C^R), (C_L, C_R^*, C_R^1), (C_R, C_L^*, C_L^1), (C_R^*, C_R^1, C_L).
\]

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1. The condition of “closed” is not essential, but is supposed here to avoid a detailed argument of boundaries.
2. Only in Section 4, since both curves and link diagrams appear, we call them separately.
**Definition 4** (sign $\varepsilon$ of a parallel pair $(a, b)$). Suppose that an application of the two successive smoothings at $a$ and $b$ to a curve $C$ yields three ordered components $\left(C_{ab}^{(1)}, C_{ab}^{(2)}, C_{ab}^{(3)}\right)$. Reading superscripts in a case of (2) implies a word $w$ in $W_{LR}$. We define 

$$w_{ab} := \varepsilon \cdot w$$

where $\varepsilon = -1$ if the first component is on the right to the second smoothing at $b$, otherwise $\varepsilon = 1$.

**Definition 5** (invariant $\nu$). Let $I(C, C')$ be the intersection number of $C$ and $C'$ in the same surface. Let $(C^{(i)}, C^{(j)}, C^{(k)})$ be an three component curve with an order $(i, j, k)$. For a 3-component curve $C = \left(C^{(1)}, C^{(2)}, C^{(3)}\right)$, let

$$\nu(C) = I(C^{(1)}, C^{(2)})I(C^{(2)}, C^{(3)}) + I(C^{(1)}, C^{(3)})I(C^{(2)}, C^{(3)}).$$

By definition, we have

**Proposition 1.** Let $C$ be a three-component curve on a surface. The integer-valued function $\nu(C)$ is invariant under stable equivalence.

**Example 1.** For a curve in Figure 4, $
u \left(C^{(1)}, C^{(2)}, C^{(3)}\right) = I(C^{(1)}, C^{(2)})I(C^{(2)}, C^{(3)}) = (-1) \cdot 1 = -1$.

**Figure 3.** An example $(D^L, D^R) \rightarrow (D^L, D^R, D^R)$ (upper line, $\varepsilon = 1$) and $(D^L, D^R) \rightarrow (D^R, D^R, D^L)$ (lower line, $\varepsilon = -1$). Enclosing letters correspond to $D^R, D^R$ and the three circled numbers indicate the order of components.

**Figure 4.** A three component curve
Remark 2. We shall give a comment here for readers who are familiar with Gauss diagrams. The symbol $\bigcirc$ denotes the sum running over the possible pairs of signs; let $G_C$ be a Gauss diagram of a three-component curve $(C^{(1)}, C^{(2)}, C^{(3)})$. $\nu(C^{(1)}, C^{(2)}, C^{(3)}) = \langle \bigcirc \bigcirc \bigcirc, G_C \rangle$, which is none other than an analogue of Milnor’s triple linking number. For the terminologies including chord diagrams and chords, see [10].

Remark 3. If the reader is familiar with chord diagrams, a parallel pair is represented as $\{\}$ symbolically; then non-parallel case is $\{\}$.

Definition 6 (equivalence in $W_{LR}$). Let $W_{LR}$ be the quotient $\mathbb{Z}$-module generated by the set of words of length three in letters $L$ or $R$, so that $w = -w$ for each word $w$, where $\bar{w}$ is the word obtained by switching $L$ and $R$ in $w$.

Definition 7 (coproduct $\Delta$). Let $\mathbb{Z}[C^3]$ be the $\mathbb{Z}$-module generated by $C^3$. We define a map $\Delta$ by

$$\Delta : C_\ast \to \mathbb{Z}[C^3] \otimes W_{LR}; C \mapsto \sum_{(a,b): \text{parallel}} \left( C^{(1)}_{ab}, C^{(2)}_{ab}, C^{(3)}_{ab} \right) \otimes w_{ab}. $$

Then let $\Delta_{|ab}(C) = \left( C^{(1)}_{ab}, C^{(2)}_{ab}, C^{(3)}_{ab} \right) \otimes w_{ab}$; by definition, $\Delta(C) = \sum_{(a,b): \text{parallel}} \Delta_{|ab}(C)$.

3. Proof of Theorem 1

We are at the point to prove Theorem 1. In this section, we permit to use the symbol $\nu \Delta$ (resp. $\nu \Delta_{|ab}$) to indicate the composition $(\nu \otimes \text{id}) \circ \Delta$ (resp. $(\nu \otimes \text{id}) \circ \Delta_{|ab}$) if there is no confusion.

3.1. Invariance of $\Omega_{1a}$ and $\Omega_{1b}$. Suppose that the move $\Omega_{1a}$ (or $\Omega_{1b}$) on a curve $C'$ generates a single crossing $A$, and let $C$ be the resulting curve. Then we have

$$\nu \Delta(C) - \nu \Delta(C') = \sum_{(a,b): \text{parallel}} \nu \left( C^{(1)}_{ab}, C^{(2)}_{ab}, C^{(3)}_{ab} \right) \otimes w_{ab}. $$

For a parallel pair $(a,b)$ or $(a,A)$ in $C$, since there is a component $C^{(k)}_{ab}$ $(k = 1, 2, 3)$ with no crossings, we have

$$\nu \left( C^{(1)}_{ab}, C^{(2)}_{ab}, C^{(3)}_{ab} \right) = 0. $$

3.2. Invariance of $\Omega_{2b}$. Suppose that the move $\Omega_{2b}$ on a curve $C'$ generates two crossings $A$ and $B$, and let $C$ be the resulting curve. Firstly, Figure 5 implies the following.

Lemma 1.

$$\nu \left( C^{(1)}_{AB}, C^{(2)}_{AB}, C^{(3)}_{AB} \right) = \nu \left( C^{(1)}_{BA}, C^{(2)}_{BA}, C^{(3)}_{BA} \right) = 0. $$

\[3\text{Though [10] treats plane/spherical curves only, the extension to curves of surfaces is straightforward [9].}\]
There are two cases depending on the placement of $E$ (Case I, the first line and Case II, the second line), each of which produces three components $i$, $j$ and $k$. For each case, there are two possible orientations of a curve and four possible positions of the base point (the third and fourth lines).

Hence we have

$$\nu \Delta(C) - \nu \Delta(C') = \sum_{(a,b) : \text{parallel}} \nu \left( C^{(1)}_{ab}, C^{(2)}_{ab}, C^{(3)}_{ab} \right) \otimes w_{ab}. \tag{4}$$

We will check the cases such that $(a,b)$ : parallel and $|\{a,b\} \cap \{A,B\}| = 1$. Let $E$ be a crossing. We will consider the situations such that pairs $(E,A)$, $(A,E)$, $(E,B)$, or $(B,E)$ are parallel (Figure 6) where we note that $(a,b)$ indicates an ordered pair corresponding to the order of smoothings. It is elementary to list the cases. Although one can prove the invariance of $\Omega_{2b}$ by routine checks (Appendix A) of the cases using Figure 6, we will put here a constructive proof.

Smoothing at crossings $A$ and $B$ corresponds to exchanging left $(L)$- and right $(R)$-labels of components connected by the 2-gon of $\Omega_{2b}$. Hence the term given by smoothing $(A,E)$ and that of smoothing $(B,E)$ are canceled out using $w = -\pi$. For terms given by the smoothing $(E,A)$ and that of $(E,B)$, the difference of the two terms is described by exchanging signs $\varepsilon$ of Definition 4 or orders of the second and third components, which implies the cancellation among the two terms.

3.3. Invariance of $\widetilde{\Omega_{3a}}$. Let $A$, $B$, and $C$ be vertices of a triangle of $\widetilde{\Omega_{3a}}$ as in Figure 2. Let $D_r$
and $D_\ell$ be the curves as in Figure 7. For each $\bullet = \ell$ or $r$, let

$$\sum \ast_k(D_\bullet) = \sum \nu(C_{ab}^{(1)}, C_{ab}^{(2)}, C_{ab}^{(3)}) \otimes w_{ab}.$$  

3.3.1. **Pair** $(a, b)$ including exactly **zero** or **one** element in $\{A, B, C\}$. By definition, $\sum \ast_0(D_\ell) = \sum \ast_0(D_r)$. By Figure 8 and using the homotopy invariance $\nu$ under $\Delta_2b$ (Section 3.2), $\sum \ast_1(D_\ell) = \sum \ast_1(D_r)$.

![Figure 8](image)

**Figure 8.** A triangle which will be applied by $\Delta_3a$ after a crossing is smoothened.

3.3.2. **Pair** $(a, b)$ including exactly **two** elements in $\{A, B, C\}$. Seeing $D_\ell$ (resp. $D_r$) in Figure 7, any pair of two crossings in $\{A, B, C\}$ is parallel (resp. not parallel). Hence it is enough to prove the following.

**Lemma 2.**

$$\nu|_{AB}(D_\ell) + \nu|_{BA}(D_\ell) + \nu|_{BC}(D_\ell) + \nu|_{CB}(D_\ell) + \nu|_{AC}(D_\ell) + \nu|_{CA}(D_\ell) = 0.$$  

**Proof of Lemma 2.** We have two cases depending on the orientation of the curve (Cases I, II).

**Case I** (Figure 9).

![Figure 9](image)

**Figure 9.** Case I

$$\nu|_{AB}(D_\ell) = \nu(C_{AB}^{(1)}, C_{AB}^{(2)}, C_{AB}^{(3)}) \otimes LRR, \nu|_{BA}(D_\ell) = \nu(C_{BA}^{(1)}, C_{BA}^{(2)}, C_{BA}^{(3)}) \otimes RRL,$$

$$\nu|_{AC}(D_\ell) = \nu(C_{AC}^{(1)}, C_{AC}^{(2)}, C_{AC}^{(3)}) \otimes LRR, \nu|_{CA}(D_\ell) = \nu(C_{CA}^{(1)}, C_{CA}^{(2)}, C_{CA}^{(3)}) \otimes RRL,$$

$$\nu|_{BC}(D_\ell) = \nu(C_{BC}^{(1)}, C_{BC}^{(2)}, C_{BC}^{(3)}) \otimes (-RRL), \nu|_{CB}(D_\ell) = \nu(C_{CB}^{(1)}, C_{CB}^{(2)}, C_{CB}^{(3)}) \otimes (-RRL).$$

Since $\nu$ is homotopy invariant and $\nu(C^{(1)}, C^{(2)}, C^{(3)}) = -\nu(C^{(1)}, C^{(2)}, C^{(3)})$, the formula (6) holds.

**Case II** (Figure 10). (The proof is essentially the same as that of Case I.)

For the first and second Reidemeister moves, we use the symbol $C$ to indicate a curve, whereas we use $D$ because the crossing $C$ appears.
Figure 10. Case II

\[
\nu \Delta|_{AB}(D_t) = \nu \left( C_{AB}^{(1)}, C_{AB}^{(2)}, C_{AB}^{(3)} \right) \otimes \text{RLL}, \nu \Delta|_{BA}(D_t) = \nu \left( C_{BA}^{(1)}, C_{BA}^{(2)}, C_{BA}^{(3)} \right) \otimes (-\text{LLR}),
\]

\[
\nu \Delta|_{AC}(D_t) = \nu \left( C_{AC}^{(1)}, C_{AC}^{(2)}, C_{AC}^{(3)} \right) \otimes \text{RLL}, \nu \Delta|_{CA}(D_t) = \nu \left( C_{CA}^{(1)}, C_{CA}^{(2)}, C_{CA}^{(3)} \right) \otimes (-\text{LLR}), \text{ and}
\]

\[
\nu \Delta|_{BC}(D_t) = \nu \left( C_{BC}^{(1)}, C_{BC}^{(2)}, C_{BC}^{(3)} \right) \otimes \text{LLR}, \nu \Delta|_{CB}(D_t) = \nu \left( C_{CB}^{(1)}, C_{CB}^{(2)}, C_{CB}^{(3)} \right) \otimes \text{LLR}.
\]

Hence the formula (6) holds. (End of Proof of Lemma 2) □

Summerized above,

\[
\nu \Delta(D_t) = \sum *_{0}(D_t) + \sum *_{1}(D_t) + \sum *_{2}(D_t)
\]

Lemma 2 \[= \sum *_{0}(D_t) + \sum *_{1}(D_t) \]

Section 3.3.1 \[= \sum *_{0}(D_t) + \sum *_{1}(D_t) \]

\[= \nu \Delta(D_r). \]

4. The relationship with the affine index polynomial

Throughout this section, to avoid confusion, if self-intersections are of (virtual) knot diagrams, we call them crossings; if self-intersections are of diagrams of curves on surfaces, we call them double points.

A knot diagram on a surface corresponds to a curve with over/under informations of double points; further, stable equivalence classes of curves on surfaces are called virtual strings, and its element is often presented by a flat virtual knot diagram with a base point.

We recall a construction of the affine index polynomial [8]. Let \( \gamma_i \) and \( \bar{\gamma}_i \) be components of two-component virtual link diagrams by smoothing \( i \)th crossing, where \( \gamma_i \) (resp. \( \bar{\gamma}_i \)) is the component \( R \) (resp. \( L \)) appearing in Figure 1. Let \( I(\gamma_i, \bar{\gamma}_i) \) be the intersection number of two curves \( \gamma_i \) and \( \bar{\gamma}_i \).

Theorem 2 (The affine index polynomial [11, 11, 13]). For a virtual knot \( K \), let \( D \) be an \( n \)-crossing knot diagram, \( c_1, c_2, \ldots, c_n \) crossings, and \( \text{sign}(c_i) \) the local writhe. The Laurent polynomial

\[
W_D(t) = \sum_{i=1}^{n} \text{sign}(c_i)(t^{I(\gamma_i, \bar{\gamma}_i)} - 1)
\]

is an invariant of \( K \).

\[\gamma_i \cdot \bar{\gamma}_i \] indicates the intersection number in [8].
Looking the polynomial $W_D(t)$, we see that

$$\frac{d}{dt}W_D(t)|_{t=1} = \sum_i \text{sign}(c_i) I(\gamma_i, \bar{\gamma}_i).$$

In order to compare with Theorem 2 easily, we use the symbol $\left(C_i^{(1)}, C_i^{(2)}\right)$ to indicate the pair of two curves $(\gamma_i, \bar{\gamma}_i)$. The intersection number is also represented by a bilinear function, called a Gauss diagram formula, $(\bigcirc \bigcirc, G_C)$ of a Gauss diagram $G_C$ of a curve $C$ on surface. Hence

$$\left\langle \bigcirc \bigcirc, \left(C_i^{(1)}, C_i^{(2)}\right) \right\rangle = I(\gamma_i, \bar{\gamma}_i).$$

In summary, we have a comparison.

| Invariants | $\frac{d}{dt}W_D(t)|_{t=1}$ | $(\nu \otimes \text{id}) \circ \Delta$ |
|------------|-----------------------------|-----------------------------------|
| Transit objects | $(C_i^{(1)}, C_i^{(2)})$ | $(\nu \otimes \text{id}) \circ \Delta$ |
| Smoothing | a single crossing | two crossings |
| Corresponding Gauss diagrams | $\bigcirc \bigcirc$ | $\bigcirc \bigcirc \bigcirc \bigcirc$ (order restricted) |
| Corresponding intersection number | $I\left(C_i^{(1)}, C_i^{(2)}\right)$ | $\nu$ |
| Sum | $\sum_i \text{sign}(c_i) \cdot (\text{term})$ | $\sum_{i,j;\parallel} (\text{term} \otimes (\text{word})$ |

We also note that the linking number $\text{lk}(L)$ is $\left\langle \frac{1}{2}(\bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc \bigcirc), G_L \right\rangle$ [12]. Analogously, Milnor’s triple linking number $\mu(L)$ is a linear combination of $\bigcirc \bigcirc \bigcirc \bigcirc$, $\bigcirc \bigcirc \bigcirc \bigcirc$, and $\bigcirc \bigcirc \bigcirc \bigcirc$. Thus it is natural to extend Theorem 1 on this curve structure to knots. We will treat it elsewhere.

5. Computation of the invariant $(\nu \otimes \text{id}) \circ \Delta$

The author TK gives an infinitely many nontrivial examples Proposition 2, each of which is distinct from any of the others up to stable equivalence.

**Proposition 2.** There exists infinitely many pointed curves $\{C_i\}$ on surfaces shown in Figure 11 such that if $i \neq j$, $C_i$ and $C_j$ are not stably equivalent.

![Figure 11](image1.png)

**Figure 11.** From the left to the right, $C_1, C_2, C_3, C_4, \ldots$.

**Proof.** Since $C_n$ is a curve on a surface, it is represented as a virtual string diagram. For $C_n$, let $i$ and $j$ be crossings as follows. We proceed along $C_n$ according to the orientation starting from the base point, and we assign labels $1, \ldots, 2n$ to crossings in the encountering order (i.e. from the left to the right in each of the figures of Figure 11).
Figure 12. The curve $C_4$ and cases (a)–(e) correspond to the terms of $\Delta(C_4)$. 
Recalling that $\Delta(C_n) = \sum_{i,j,\text{parallel}} \left( C_{ij}^{(1)} \otimes C_{ij}^{(2)} \otimes C_{ij}^{(3)} \right) \otimes w$ where

(a) $\sum_{i,j} \left( \left( C_{ij}^{(1)} C_{ij}^{(2)} C_{ij}^{(3)} \right) \otimes RLL + \left( C_{ij}^{(1)} C_{ij}^{(2)} C_{ij}^{(3)} \right) \otimes (RRL) \right)$ $(1 \leq i \leq n - 1, j = i + 1)$,

(b) $\sum_{i,j} \left( \left( C_{ij}^{(1)} C_{ij}^{(2)} C_{ij}^{(3)} \right) \otimes RLL + \left( C_{ij}^{(1)} C_{ij}^{(2)} C_{ij}^{(3)} \right) \otimes (RRL) \right)$ $(1 \leq i \leq n - 2, i + 2 \leq j \leq n)$,

(c) $\sum_{i,j} \left( \left( C_{ij}^{(1)} C_{ij}^{(2)} C_{ij}^{(3)} \right) \otimes LRR + \left( C_{ij}^{(1)} C_{ij}^{(2)} C_{ij}^{(3)} \right) \otimes LLR \right) (n + 1 \leq i \leq 2n - 1, j = i + 1)$,

(d) $\sum_{i,j} \left( \left( C_{ij}^{(1)} C_{ij}^{(2)} C_{ij}^{(3)} \right) \otimes LRR + \left( C_{ij}^{(1)} C_{ij}^{(2)} C_{ij}^{(3)} \right) \otimes LLR \right)$ $(n + 1 \leq i \leq 2n - 2, i + 2 \leq j \leq 2n)$, and

(e) $\sum_{i,j} \left( \left( C_{ij}^{(1)} C_{ij}^{(2)} C_{ij}^{(3)} \right) \otimes RLL + \left( C_{ij}^{(1)} C_{ij}^{(2)} C_{ij}^{(3)} \right) \otimes (LLR) \right)$ $(1 \leq i \leq n - 1, i + n + 1 \leq j \leq 2n)$.

Then taking care of the order of the components, $(\nu \otimes \text{id})\Delta(C_n) = (\nu \otimes \text{id})\Delta((a) + (b) + (e))$. For the $n = 4$ case, see Figures [12].

Here to unify the terms, when we apply $\nu$ in each case (a)–(e) consisting of two terms, the order of the corresponding components is always adjusted to the components’ order $1, 2, 3$ by applying a sign $\pm$ (if necessary). Also note that the sign-contribution of $\nu$ is indicated just before $\Sigma$.

$$(\nu \otimes \text{id})(a) = \sum_{i=1}^{n-1} i \otimes (RLL + RRL),$$

$$(\nu \otimes \text{id})(b) = - \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} i(j - i) \otimes (-RLL - RRL),$$

$$(\nu \otimes \text{id})(e) = \sum_{i=1}^{n-1} \sum_{j=i+n+1}^{2n} i(2n + 1 - j) \otimes (RLL - LLR).$$

Then the coefficient $S_n$ of $RLL + RRL$ of $(\nu \otimes \text{id})\Delta(C_n)$ is as follows (Appendix [B] for the details):

$$S_n = \frac{(n - 1)n}{2} + \frac{(n - 2)(n - 1)n(n + 5)}{24} + \frac{(n - 1)n(n + 1)(n + 2)}{24}.$$ 

Then

$$S_n - S_{n-1} = n - 1 + \frac{1}{6}(n - 2)(n - 1)(n + 3) + \frac{1}{6}(n - 1)n(n + 1) > 0 \quad (n > 1).$$

In summary, $S_n > S_{n-1}$ $(n > 1)$ and $S_n > 0$ $(n > 1)$, which implies the claim of Proposition 2.

**APPENDIX A. INVARIANCE OF $\Omega_{2n}$ BY DIRECT COMPUTATION**

Let $A$, $B$, and $E$ be crossings such that $A$ and $B$ are generated by the move $\widehat{\Omega_{2n}}$. Let $(a, b)$ be an ordered pair corresponding to the order of smoothings. We list keys of case distinctions as follows:

(1) There are two positions of $E$ for an unoriented curve (Cases I, II).
(2) For each of cases I and II, there are two orientations of curves.
(3) For each oriented curve, there are four possible positions of the base points.
(4) Considering smoothed curved (Components $i, j, k$ as in Figure [6]), case distinctions of the positions of the base point are essentially three since the problem is which component has the base point, i.e. Case $\gamma$ is merged into Case $\delta$. 

□
Table 1. Each case corresponding to each term in RHS of (4).

| Case | base point / orientation | \((a, b)\) | order of \((i, j, k)\) | word in \(W_{LR}\) |
|------|--------------------------|------------|---------------------|-------------------|
| Case I | \(\alpha^\uparrow\) | \((E, A)\) | \((2, 3, 1)\) | \(RLL\) |
|       | \((A, E)\) | \((3, 2, 1)\) | \(-RRL\) |
|       | \((E, B)\) | \((3, 2, 1)\) | \(RLL\) |
|       | \((B, E)\) | \((3, 2, 1)\) | \(-LLR\) |
|       | \((E, A)\) | \((3, 2, 1)\) | \(LLR\) |
|       | \((A, E)\) | \((3, 2, 1)\) | \(LRR\) |
|       | \((E, B)\) | \((2, 3, 1)\) | \(LRR\) |
|       | \((B, E)\) | \((3, 2, 1)\) | \(RRL\) |
|       | \((E, A)\) | \((1, 2, 3)\) | \(LLR\) |
|       | \((A, E)\) | \((1, 2, 3)\) | \(LRR\) |
|       | \((E, B)\) | \((1, 2, 3)\) | \(-LLR\) |
|       | \((B, E)\) | \((1, 2, 3)\) | \(RLL\) |
|       | \((E, A)\) | \((1, 2, 3)\) | \(-RRL\) |
|       | \((A, E)\) | \((1, 3, 2)\) | \(RLL\) |
|       | \((E, B)\) | \((1, 2, 3)\) | \(RRL\) |
|       | \((B, E)\) | \((1, 3, 2)\) | \(LRR\) |
|       | \((E, A)\) | \((2, 1, 3)\) | \(-LLR\) |
|       | \((A, E)\) | \((3, 1, 2)\) | \(RRL\) |
|       | \((E, B)\) | \((2, 1, 3)\) | \(LRR\) |
|       | \((B, E)\) | \((3, 1, 2)\) | \(LRR\) |
|       | \((E, A)\) | \((2, 1, 3)\) | \(-RRL\) |
|       | \((A, E)\) | \((3, 1, 2)\) | \(-LRR\) |
|       | \((E, B)\) | \((3, 1, 2)\) | \(-RRL\) |
|       | \((B, E)\) | \((3, 1, 2)\) | \(-RRL\) |

| Case II | \(\alpha^\uparrow\) | \((E, A)\) | \((3, 2, 1)\) | \(LRR\) |
|         | \((A, E)\) | \((3, 2, 1)\) | \(LRR\) |
|         | \((E, B)\) | \((2, 3, 1)\) | \(RRL\) |
|         | \((B, E)\) | \((3, 2, 1)\) | \(-LRR\) |
|         | \((E, A)\) | \((2, 3, 1)\) | \(LLR\) |
|         | \((A, E)\) | \((3, 2, 1)\) | \(-RRL\) |
|         | \((E, B)\) | \((3, 2, 1)\) | \(RRL\) |
|         | \((B, E)\) | \((3, 2, 1)\) | \(LRR\) |
|         | \((E, A)\) | \((1, 2, 3)\) | \(-RRL\) |
|         | \((A, E)\) | \((1, 3, 2)\) | \(RRL\) |
|         | \((E, B)\) | \((1, 3, 2)\) | \(RRL\) |
|         | \((B, E)\) | \((1, 3, 2)\) | \(LRR\) |
|         | \((E, A)\) | \((2, 1, 3)\) | \(-RRL\) |
|         | \((A, E)\) | \((2, 1, 3)\) | \(-RRL\) |
|         | \((E, B)\) | \((2, 1, 3)\) | \(-RRL\) |
|         | \((B, E)\) | \((2, 1, 3)\) | \(-RRL\) |
|         | \((E, A)\) | \((2, 1, 3)\) | \(-RRL\) |
|         | \((A, E)\) | \((3, 1, 2)\) | \(RRL\) |
|         | \((E, B)\) | \((2, 1, 3)\) | \(LLR\) |
|         | \((B, E)\) | \((3, 1, 2)\) | \(LLR\) |
Note that $\nu$ has the antisymmetry with respect to the order of the second and third components. Note also that $\overline{w} = -w$ in $W_{LR}$. By using Table 1 it is elementary to check the right hand side of (4) is zero.

**Appendix B. Elementary sums**

\[
\sum_{i=1}^{n} \sum_{j=i+1}^{n} i(j-i) = \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} i(j-i) = \sum_{i=1}^{n-2} \left( i \cdot \frac{n(n+1)}{2} - i \cdot \frac{(i+1)(i+2)}{2} - i^2(n-i-1) \right) \\
= \frac{(n-2)(n-1)}{2} \cdot \frac{n(n+1)}{2} - \frac{(n-2)(n-1)n(n+1)}{8} \\
- \frac{(n-2)(n-1)^2(2n-3)}{6} + \frac{(n-2)^2(n-1)^2}{4} \\
= \frac{(n-2)(n-1)}{24} \left( 3n(n+1) - 4(n-1)(2n-3) + 6(n-2)(n-1) \right) \\
= \frac{(n-2)(n-1)n(n+5)}{24}.
\]

\[
\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (2n+1-j) = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} i(k) = \sum_{i=1}^{n-1} i \cdot \frac{(n-i)(n-i+1)}{2} = \sum_{i=1}^{n-1} (n-i) \cdot \frac{i(i+1)}{2} \\
= \frac{1}{2} \sum_{i=1}^{n-1} (n-i) \cdot (i^2 + i) = \frac{1}{2} \sum_{i=1}^{n-1} \left( n(i^2 + i) - (i^3 + i^2) \right) \\
= \frac{1}{2} \sum_{i=1}^{n-1} ((n-1)i^2 + i^3) + \frac{1}{2} \sum_{i=1}^{n-1} ni \\
= \frac{(n-1)^2n(2n-1)}{12} - \frac{(n-1)^2n^2}{8} + \frac{(n-1)n^2}{4} \\
= \frac{(n-1)n}{24} \cdot \left( (n-1)(4n-2) - (n-1)(3n) + 6n \right) \\
= \frac{(n-1)n(n+1)(n+2)}{24}.
\]

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National Institute of Technology, Ibaraki College, 312-8508, Japan

Current Address: Faculty of Engineering, Shinshu University, 4-17-1, Wakasato, Nagano, 380-8553, Japan

Email address: nito@shinshu-u.ac.jp

National Institute of Technology, Ibaraki College, 312-8508, Japan

Current Address: Department of Applied Physics, School of Engineering, The University of Tokyo, 113-8566, Japan

Email address: st18082tk@gm.ibaraki-ct.ac.jp