Unique Stability Point in Social Storage Networks

Pramod Mane · Kapil Ahuja · Nagarajan Krishnamurthy

Abstract Social storage systems are a good alternative to existing data backup systems of local, centralized, and P2P backup. Till date, researchers have mostly focused on either building such systems by using existing underlying social networks (exogenously built) or on studying Quality of Service (QoS) related issues. In this paper, we look at two untouched aspects of social storage systems. One aspect involves modeling social storage as an endogenous social network, where agents themselves decide with whom they want to build data backup relation. This is more intuitive than exogenous social networks. The second aspect involves studying stability of social storage systems. The second aspect involves studying stability of social storage systems, which would help reduce maintenance cost of networks and further, help build efficient as well as contented networks.

We have a five fold contribution that covers the above two aspects. We, first, model the social storage system as a strategic network formation game. Second, we define the utility of each agent in the network under two different frameworks, one where the cost to add and maintain links is considered in the utility function and the other where budget constraints are considered. In the context of social storage and social cloud computing, these utility functions are the first of its kind, and we use them to define and analyze the social storage network game. Third, we define the concept of bilateral stability which refines the pairwise stability concept defined by Jackson and Wolinsky (1996), by requiring mutual consent for both addition and deletion of links, as compared to mutual consent just for link addition. Mutual consent for link deletion is especially important in the social storage setting. The notion of bilateral stability subsumes the bilateral equilibrium definition of Goyal and Vega-Redondo (2007). Fourth, we provide necessary and sufficient conditions for bilateral...
stability of social storage networks. Fifth, for symmetric social storage networks, we prove that there exists a unique neighborhood size, independent of the number of agents, where no pair of agents has any incentive to increase or decrease their neighborhood size. We call this neighborhood size as the stability point. Further, given the number of agents and other parameters, we discuss which bilaterally stable networks would evolve and also discuss which of these stable networks are efficient (that is, with maximum sum of utilities of all agents). We also discuss ways to build contented networks (where all agents achieve maximum possible utility).

**Keywords** Social Storage, Endogenous Network Formation, Bilateral Stability, Pairwise Stability, F2F Backup System, Peer-to-Peer System

1 Introduction

In this digital era, where personal data size is growing exponentially, data backup is not a new need. Data stored on an agent’s local machine is prone to loss due to disk-failure, malware, etc. Local backup, centralized on-line backup and decentralized (Peer-to-Peer) backup are some strategies available to agents. Each has its own merits and demerits. For example, maintaining data backup on a local external hard disk on a regular basis is cumbersome. As far as on-line backup systems are concerned, on the one hand, centralized on-line backup is not cost efficient, especially when the amount of data required to be backed up is huge. On the other hand, although Peer-to-Peer (P2P) backup systems are cost efficient, they require dealing with several issues like data availability, reliability and security.

In recent years, to cope up with the above issues in P2P storage systems, researchers have been focusing on on-line social network relationships. It is believed that social ties between agents (or players) will help to build backup systems that overcome aforementioned issues. This trend that takes real world social relationships (encoded in an on-line social network) into account for constructing a data backup system is emerging as a special case of P2P backup system, and tagged as Social Storage or Friend-to-Friend (F2F) Storage (Friendstore, FriendBox, BackupBuddy are a few examples).

Existing research on social storage is moving in two directions. One research trend has been focusing on various approaches to build the system, dealing with providing ways for agents to select their data-backup-partners explicitly. The other direction has been focusing on studying Quality of Service (QoS) related issues such as data availability, reliability, the cost associated with communication, data maintenance, data placement or scheduling polices, by taking online social relationships into account.

---

1 For technical details regarding social storage systems, we refer the reader to Li and Dabek (2006); Nguyen and Li (2007); Tran et al (2012); Gracia-Tinedo et al (2012b,c); Moreno-Martinez et al (2012), Sharma et al (2011); Gracia-Tinedo et al (2012a,c); Blackburn et al (2014).

2 http://www.buddybackup.com/
1.1 Social Storage Issues Addressed in this Paper

In this work, we address two other issues of social storage. First, major social storage studies (such as Gracia-Tinedo et al (2012a,c); Blackburn et al (2014)) have considered exogenous social networks (an underlying social network, for instance Facebook, Orkut, Venus, etc.) to construct a social storage system and to study QoS related issues. However, the approach of considering an exogenous social network to build a social storage system (or to do QoS analysis) fails to address various aspects. For example, the approach does not focus on participation benefits and costs. Rational (self-interested) behavior of agents involved in the data backup activity is not taken into consideration. Another issue with considering exogenous social networks as is for social storage is the assumption that an agent in the underlying network is involved in data backup activity with all its neighbors. However, it is possible that agents do not want to perform a data backup activity with their set of existing neighbors. The QoS analysis, which is based upon neighborhood size in the underlying network, is no longer valid. Thus, it is important to study when agents want to perform a data backup activity and when they want to discontinue it. Hence, in this paper, we model the social storage system as an endogenous network formation game.

Second, social storage systems may not be stable and even if stable, they may not be efficient. There is limited study on stability and efficiency of social storage systems. While proposing the idea of F2F backup systems, Li and Dabek (2006) argue that social ties between agents act as incentives for them to stay in the system, thereby resulting in a stable social storage system. In their context, a system is unstable when agents arrive and depart the social storage system randomly — lesser this randomness, more the stability of the system. In our case, a social storage system is stable when agents have no incentive to add new partners or delete existing partners. In the following subsection, we motivate this definition of stability in detail. Studying social storage systems as an endogenous network formation game, and then analyzing its stability may, on first glance, seem contradictory — since one cannot do anything from outside the (endogenous) system if the agents themselves do not form a stable or efficient network. In our case, the agents always form a stable network, but the network may not be efficient — that is, the sum of the utilities of all agents may not be the maximum possible (and in our case, as many agents as possible may not be “contented”). Contented networks (where all agents achieve maximum possible utility) are also efficient. Looking at both endogenous network formation and contentment is useful because, though the social storage system is built endogenously, an independent observer (say, an administrator or a regulator) can check whether the system is contented, and if not, can externally do a small perturbation to the network. In some scenarios we look at, the independent administrator may achieve contentment by just introducing a small number of dummy agents.

---

3 This is because social storage is in its infancy and an architectural prototype of social storage is in the development stage.

4 Although Sharma et al (2011) begin discussing about agents’ strategic behavior in a scenario where limited storage is available for the agents, this has just been touched upon and has not been looked at in detail.
1.2 Our Contribution

Both aspects of social storage systems that we address in this paper (that is, endoge-
nously evolved systems as well as stability, efficiency and contentment of such sys-
tems) are easily analyzed by using strategic network formation models. As mentioned
earlier, in this paper, we do not explicitly consider trust between pairs of agents. We
assume that all agents trust each other, and thus, anyone can form direct links with
anyone. In scenarios where agents do not necessarily trust all agents in the network,
our results extend to every clique (of mutually trusting agents) in the network. Fur-
ther, our results lay foundation for future work that may explicitly consider subsets
of trusted agents. Another point worth emphasizing here is that, in our analysis, each
agent only looks at its local neighborhood.

Strategic network formation literature is vast\(^5\), which includes approaches to model
network formation, its solution concepts, applications, and analysis. Stability and
equilibrium notions include the concept of pairwise stability (Jackson and Wolinsky,
1996), the concept of Nash equilibrium in networks (Bala and Goyal, 2000a), strong
and coalition-proof Nash equilibria (Dutta and Mutuswami, 1997), strong pairwise
stability (Jackson and van den Nouweland, 2005), pairwise stable Nash equilibrium
(Goyal and Joshi, 2006b), farsighted equilibrium (Dutta et al, 2005) and bilateral
equilibrium (Goyal and Vega-Redondo, 2007). Each of these solution concepts is rel-
evant in variants of the scenario we discuss in this paper.

In the endogenous network formation model of Jackson and Wolinsky (1996)
where rational decision makers build a network by interacting with each other, the
pairwise stability solution concept takes agents’ (or players’) mutual consent into
account while building a relationship (that is, adding a link in the network), but any
agent can decide not to maintain a relationship (that is, delete any of its existing links)
without consent of the agent at the other end of the link.

However, the social storage system discussed earlier impels us to focus on the
requirement of bilateral consent while deleting a link as well. For instance, let agents
\(i\) and \(j\) be backup partners. That is, \(i\) provides its storage space to \(j\) for the purpose
of storing \(j\)’s data, and vice versa. Now, let us assume that breaking a backup part-
nership without mutual consent is allowed. If agent \(i\) breaks the partnership without
consent of \(j\), then there is a threat that \(j\) will lose its data which is stored on \(i\)’s storage
resource. Hence, backup partnerships in social storage networks have to be viewed as
mutual contracts which cannot be broken unilaterally. We call this as “bilateral stabili-
ty”. This definition of bilateral stability also applies to other contexts where mutual

\(^5\) Dutta and Jackson (2003); Jackson (2005); Tennekes (2010); Goyal (2012); Surajit et al (2014) give
detailed surveys on network formation games and games on networks. A number of significant works,
including Myerson (1977); Aumann and Myerson (1988); Jackson and Wolinsky (1996); Dutta and Mu-
utswami (1997); Bala and Goyal (2000a,b); Bramoullé et al (2004); Dutta et al (2005); Jackson and van den
Nouweland (2005); Goyal and Joshi (2006b); Bloch and Jackson (2007); Bloch and Dutta (2009); Yann
et al (2009); Güth and Sarangi (2010); Billand et al (2011); Belhaj et al (2014); Bramoullé et al (2014)
have been done on network formation, stability, equilibrium and efficiency. Some interesting applications
of network games include Goyal and Moraga-González (2001); Goyal and Joshi (2003, 2006a); Furu-
sawa and Konishi (2007); Calvó-Armengol (2004); Belleflamme and Bloch (2004); Goyal and Moraga-
González (2001); Zafirulä et al (2006). A number of studies, Hummon (2000); Falk and Kosfeld (2003);
Goeree et al (2009), focus on agent based analysis of these models.
consent is required for deletion, for example, Service Level Agreements in the Cloud. Our definition of bilateral stability subsumes the concept of bilateral equilibrium proposed by Goyal and Vega-Redondo (2007). The set of all strategies that are bilaterally stable contains the set of all bilateral equilibrium strategies. A network which is bilaterally stable may contain agents who may be better off by deviating, where as a bilateral equilibrium network does not contain any such agent. Both definitions, however, allow only bilateral deviations (or pairwise addition as well as deletion with mutual consent). While Hummon (2000) also discuss mutual consent for deletion, they do not formally define or study the concept of stability with mutual consent for deletion too. They perform agent based simulation of the connection model proposed by Jackson and Wolinsky (1996) and discuss the simulation outputs.

The rest of this paper is divided into six sections. In Section 2, we formally describe our social storage model. We compute the utility of agents in a social storage network under two different frameworks, namely Multi-Objective Framework and Single-Objective Framework, in Section 3. The utility function includes the cost and benefit of an agent in a social storage network. As far as we know, in the social storage literature, this is the first attempt of its kind. In Section 4, we study endogenous social storage network formation by focusing on bilateral agreement for both link addition and deletion. To capture the notion of bilateral link addition and deletion, we redefine the pairwise stability solution concept introduced by (Jackson and Wolinsky, 1996), so that it is suitable to characterize our proposed social storage network formation game. We provide some necessary and sufficient conditions for pairwise stability of social storage networks in Section 5. Here, we first define the stability point (the ideal neighborhood size) such that no agent gains by deviating from the stability point. Then, we show that there exists a unique stability point in one type of social storage network, independent of the number of agents. We also show that there exist unique and non-unique pairwise stable storage networks under certain conditions, and given the number of agents and other parameters, we discuss which pairwise stable networks would evolve. Finally, we do analysis related to efficiency as well as contentment of networks, e.g. efficient networks are always stable in the context we study. We conclude the paper in Section 6.

2 Social Storage Network Model

**Definition 1** Social Storage \( g = (A, L) \) is a tuple of two sets \( A \) and \( L \). \( A \) is a set of agents (or players) and \( L \) is a set of edges (or links) connecting these agents.

A social storage \( g \) can be viewed as a social network constituted by a set of agents \( A \) to perform a data backup activity. The link \( \langle ij \rangle \in L \) represents the fact that agent \( i \) and \( j \) are neighbors of each other, and are involved in a data backup agreement \( a_{ij} \). This agreement indicates that both the agents share storage resources with each other so that they can store their data on each other’s provided storage space. The link \( \langle ij \rangle \) between agents \( i \) and \( j \) and the link \( \langle ji \rangle \) between agents \( j \) and \( i \) are identical. This implies that storage resource sharing and data backup activity are bidirectional and
Table 1 Notation Summary

| Symbol | Description |
|--------|-------------|
| $g$    | social storage network |
| $A$    | set of agents (or users or players) |
| $N$    | number of agents (i.e., $N = |A|$) |
| $L$    | set of edges or links |
| $(ij)$ | link between agents $i$ and $j$ |
| $a_{ij}$ | data backup agreement between agents $i$ and $j$ |
| $c$    | cost incurred by an agent to maintain a link |
| $b_i$  | worth (value) that agent $i$ has for its data |
| $s_i$  | amount of storage available with agent $i$ that it can contribute to other agents |
| $d_i$  | amount of data that agent $i$ wants to backup |
| $b_i$  | budget allocated by agent $i$ towards backup agreements |
| $\lambda$ | probability of failure of a disk (i.e., average disk failure rate) |
| $\eta_i(g)$ | neighborhood size of agent $i$ in $g$ (also denotes the set of neighbors of $i$) |
| $g + (ij)$ | new link $(ij)$ is added to $g$ |
| $g - (ij)$ | existing link $(ij)$ is deleted from $g$ |
| $\mathcal{G}(N)$ | the set of networks on $N$ agents |
| $g(A_k)$ | $k$th component of network $g$ |

A link establishment between a pair of agents reflect their commitment towards storage provision. A link $(ij)$ indicates that agent $i$ is a neighbor of agent $j$ and vice versa. We also refer to $i$ and $j$ as backup partners, and their mutual relationship as backup partnership.

A social storage network may be connected or may consist of two or more connected components. We say a network $g$ is connected if there exists a path between every pair of nodes $i$ and $j \in A$, or else the network $g$ is disconnected. A disconnected network $g$ can be divided into a number of sub-networks $g(A_1), g(A_2), \ldots, g(A_n)$, where $A_1 \cup A_2 \cup \ldots \cup A_n = A$, $A_i \cap A_j = \emptyset$ for $k \neq l$, such that a pair of nodes $i$ and $j$ is connected if and only if $i$ and $j$ are members of the same set $A_i$. These sub-networks are called as components of the network $g$. A complete network is one where every agent is connected to every other agent. A null (or empty) network is one where there are no links (that is, no agent is connected to any agent).

The set of agents with whom agent $i$ has links or direct connections is represented by $\eta_i(g)$. In other words, $\eta_i(g)$ is the neighborhood of agent $i$. One can look at $\eta_i(g)$ as the number of mutual data backup agreements in which agent $i$ is involved. We also use $\eta_i(g)$ for the neighborhood size of $i$, which will be clear from the context. Table 1 summarizes all notations used in this paper.

### 3 Utility of an Agent in a Social Storage Network

Data stored on local hard disk is in danger of getting lost or damaged due to local disk failure. Hence, to keep data safe, each data owner keeps multiple copies of the data on each backup partner’s hard disk. As hard drives are prone to failure, there is a chance that a data owner’s backup partner’s hard drive also fails. It is likely that each backup partner’s hard drive fails, so each data owner’s interest lies in recovering at least one copy of its data so that the value of the data is intact. It is not hard to observe that each agent’s chance of data recovery, given a particular disk failure rate, depends
on its neighborhood size. The more the number of neighbors, the higher the chance of data recovery.

Social storage systems use two types of techniques to backup data. The first is replication, and the second is erasure coding\(^6\) (Marcelo et al, 2008). Erasure coding is the data redundancy technique in which a data block is divided into \(x\) (coded) fragments such that the data block can be recovered from any subset \(y\) of \(x\), where \(y < x\). Replication is the data redundancy technique in which an agent maintains a single data copy on each partner’s storage device. In this paper, we consider the replication technique.

In the absence of costs to add and/or maintain links, the aim of each agent in a social storage network is to maximize his/her chance of data recovery, given that his/her local copy of data has been damaged or lost. However, every agent incurs a cost for each of its links. Keeping this in mind, we define the utility of each agent in the network under two frameworks. The utility of agent \(i\) in the network \(g\) is represented by a function \(u_i : \mathcal{G}(A) \rightarrow \mathbb{R}\), where \(\mathcal{G}\) is the set of all networks, \((g)\) an element of \(\mathcal{G}\). We first define the parameters required to write down the utility function. \(\lambda \in (0,1)\) is the average disk failure rate in the network. That is, at any point in time, the probability of failure of agent \(i\)’s disk is \(\lambda\). For data owner (agent) \(i\), the value of the local data that is to be backed up, is \(\beta_i\). Each agent incurs a cost \(c\) to maintain a link. That is, the total cost of adding and/or maintaining a link is \(2c\), and we assume that the agents connected by the link equally share this cost. This cost can be interpreted as the cost required for infrastructure, bandwidth, time, etc. There is no additional cost to add a new link. Each agent \(i\) also has allocated budget \(b_i\) for maintaining its links. Further, each agent \(i\) has a certain amount of local data \(d_i\) that the agent wants to store on storage devices of backup partners. Also, each agent \(i\) has a certain amount of storage space \(s_i\) available for sharing with other agents in the network. Using these notations, we now define the utility of an agent in the following two frameworks.

3.1 Multi-Objective Framework (MO-Framework)

In the first framework, there are two objective functions that each agent \(i\) tries to optimize. Firstly, each agent \(i\) wants to minimize the total cost associated with maintaining the links, i.e., \(c\eta_i(g)\). Secondly, each agent wants to maximize the expected value of backup data. Since the disk failure rate is \(\lambda\), and \(i\) has \(\eta_i(g)\) neighbors, the expected value of \(i\)’s backup data is \(\beta_i(1 - \lambda^{\eta_i(g)})\). Note that, as each agent is interested in “how many links to maintain”, we talk of the expected value of an agent’s backup data \textit{given} that the local copy of the agent’s data has been damaged or lost. For each agent \(i\), these two objective functions can be written as a single objective function as follows:

\[
[\alpha(\beta_i(1 - \lambda^{\eta_i(g)})]] - [(1 - \alpha)(c\eta_i(g))], \quad \text{where} \quad \alpha \in (0,1).
\]  

\(^6\) Weatherspoon and Kubiakowicz (2002) perform quantitative comparisons between these two techniques.
For elegance of results on stability, we let \( \alpha = 1/2 \). We drop the factor of \( 1/2 \) from (1), for all \( i \in \mathcal{A} \), and just consider the following utility function \( u_i(g) \), for all \( i \in \mathcal{A} \), for the given network \( g \):

\[
  u_i(g) = \beta_i(1 - \lambda \eta_i(g)) - c \eta_i(g).
\]

(2)

Each agent \( i \) wants to maximize \( u_i(g) \) over all possible values of \( \eta_i(g) \). The social optimization problem can formulated as

\[
  \max \sum_{i \in \mathcal{A}} (u_i(g))
\]

such that

\[
  \eta_i(g) = \sum_{i,j \in g} a_{ij} \quad \text{and} \quad s_i \geq \sum_{j \in \eta_i(g)} d_j a_{ij},
\]

where,

\[
  a_{ij} = \begin{cases} 
    1 & \text{if } i \text{ and } j \text{ have a backup agreement,} \\
    0 & \text{otherwise.}
  \end{cases}
\]

3.2 Single Objective Framework (SO-Framework)

In this framework, each agent \( i \) has only one objective (as compared to two in the previous framework). Each agent tries to maximize the expected value of backup data. The cost, \( c \eta_i(g) \), incurred by agent \( i \) to maintain links (which was the second objective function in the MO-Framework), appears in constraints here. This is because, in the SO-Framework, every agent \( i \) has an allocated budget, \( b_i \), towards backup agreements.

For the given network \( g \), utility \( u_i(g) \) of agent \( i \) is \( u_i(g) = \beta_i(1 - \lambda \eta_i(g)) \).

Each agent \( i \) wants to maximize \( u_i(g) \) over all possible values of \( \eta_i(g) \). The social optimization problem can formulated as

\[
  \max \sum_{i \in \mathcal{A}} (u_i(g))
\]

such that

\[
  \eta_i(g) = \sum_{i,j \in g} a_{ij},
\]

\[
  s_i \geq \sum_{j \in \eta_i(g)} d_j a_{ij}, \quad \text{and}
\]

\[
  b_i \geq c \eta_i(g),
\]
where,

\[ a_{ij} = \begin{cases} 
1 & \text{if } i \text{ and } j \text{ have a backup agreement,} \\
0 & \text{otherwise.} 
\end{cases} \]

*Remark 1* The utility function in the SO-Framework may be reduced to the CARA (Constant Absolute Risk Aversion) function. In the context of social storage, agents are risk averse as they do not want to “risk” losing their data, which is what the above utility function captures. This function may also be viewed as the Cumulative Distribution Function of an Exponential distribution, given than the disk failure rate is Poisson.

### 4 Network Formation Game (NFG) and Stability Under Constraints

At any given point in time, each agent plays a dual role: that of a data owner who wants to back up its data, and that of a backup partner who provides storage for each of its backup partners. Pairs of agents may add a new link (or continue to maintain the existing link) or delete the existing link (or continue to remain without a direct link). Note that, an agent neither adds a new link nor deletes an existing link without consent of the agent with whom it wants to perform an agreement or is involved in an agreement.

In the social storage context, mutual consent is a must for deleting links too. As discussed in Section 1, this assumption is practical but has not been focused upon in the network formation literature.

The structure of the network is defined by actions of the agents. Firstly, the network is updated when two agents \( i \) and \( j \) add a new link \( \langle ij \rangle \), and we denote this by \( g^+ \langle ij \rangle \). Secondly, the network is updated when a pair agents \( i \) and \( j \) delete an existing link \( \langle ij \rangle \), and we denote this by \( g^- \langle ij \rangle \). As agents explicitly decide with whom they want to perform backup partnerships and with whom they do not, this is a process of endogenous network formation (or partner selection).

There is need for a solution concept which is suitable for characterizing the storage network formation game. A strategic network formation game (NFG) is described as below. NFG consists of a set of agents \( A = \{1,2,\ldots,N\} \) who represent nodes in the network \( g \) — if \( i \) is an agent, we use \( i \in A \) and \( i \in g \) synonymously. In this setting, pairs of agents may form new links thereby increasing their expected value of backup data, by incurring higher costs to maintain links. Pairs of agents may also delete existing links, thereby reducing the costs incurred, but reducing the probability of retrieving the data too. The shape of the network is not only defined by each agent’s cost and benefit trade off, but also by limitation of resources available with the agents.

*Pairwise stability* (as a solution concept) introduced by Jackson and Wolinsky (1996) (see Definition 2) is an appropriate solution concept when agents require mutual consent while adding a link, but any agent can delete any of its existing link without consent.
Definition 2 (Jackson and Wolinsky, 1996) A network $g$ is pairwise stable if and only if

1. $u_i(g) \geq u_i(g - \langle ij \rangle)$ and $u_j(g) \geq u_j(g - \langle ij \rangle)$, for all $\langle ij \rangle \in g$, and
2. If $u_i(g + \langle ij \rangle) > u_i(g)$, then $u_j(g + \langle ij \rangle) < u_j(g)$, for all $\langle ij \rangle \notin g$.

We modify the pairwise stability concept introduced by Jackson and Wolinsky (1996) so as to ensure that deletion of links also happens with mutual consent. We call this modified pairwise stability as bilateral stability. Bilateral equilibrium (Goyal and Vega-Redondo, 2007) is another refinement of pairwise stability (Jackson and Wolinsky, 1996). Goyal and Vega-Redondo (2007) define strategies of agents as sets of links they would want to add, and define Bilateral Equilibrium as a strategy profile that is a Nash equilibrium (that is, no agent benefits by unilaterally deviating) and is pairwise stable (where both addition and deletion require mutual consent). The set of all bilaterally stable strategies (see Definition 3) is a superset of the set of all bilateral equilibrium strategies (Goyal and Vega-Redondo, 2007), as discussed earlier.

The modified definition of pairwise stability we use for social storage is given below.

Definition 3 A social storage network $g$ is bilaterally stable if and only if

1. for all $\langle ij \rangle \in g$, if $u_i(g - \langle ij \rangle) > u_i(g)$, then $u_j(g - \langle ij \rangle) < u_j(g)$, and
2. for all $\langle ij \rangle \notin g$, if $u_i(g + \langle ij \rangle) > u_i(g)$, then $u_j(g + \langle ij \rangle) < u_j(g)$.

Definition 3 is a network stability concept, whose first part states that no pair of agents with a link between them, wants to delete the link, and the second part states that no pair of agents has an incentive to add a new link. Note that neither link formation (addition) nor link deletion can happen without mutual consent. Our further discussions about social storage stability stands on Definition 3.

Now, we generalize Definition 3 so that it is suitable as a solution concept for the two frameworks discussed in the previous section.

For this, we first define remaining storage available with agent $i$ in a network $g$ as

$$RS_i = s_i - \sum_{j \in \eta_i(g)} d_j a_{ij},$$  \hspace{1cm} (3)$$

and remaining budget of agent $i$ in $g$ as

$$RB_i = b_i - \sum_{j \in \eta_i(g)} c a_{ij},$$ \hspace{1cm} (4)$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ have a backup agreement,} \\ 0 & \text{otherwise.} \end{cases}$$

For the MO-Framework, where we have storage constraints, the following modification of Definition 3 is appropriate.

Definition 4 A social storage network $g$ with storage constraints is bilaterally stable if and only if
1. for all $\langle ij \rangle \in g$, if $u_i(g - \langle ij \rangle) > u_i(g)$, then $u_j(g - \langle ij \rangle) < u_j(g)$, and
2. for all $\langle ij \rangle \not\in g$, if $[u_i(g + \langle ij \rangle) > u_i(g)$ and $RS_j \geq d_i]$, then $[u_j(g + \langle ij \rangle) < u_j(g) \text{ or } RS_i < d_j]$.

In the above definition, there is no change in the link deletion condition of Definition 3. However, while adding a link, an agent has to ensure that the other agent has sufficient storage to store its data (besides ensuring increase in its utility). We assume that the agents are rational and self-centered (and hence, it is up to agent $j$ to check whether agent $i$ has sufficient storage for agent $j$’s data or not).

Next, we adapt Definition 3 for the SO-Framework, where we have storage and budget constraints.

**Definition 5** A social storage network $g$ with storage and budget constraints is bilaterally stable if and only if

1. for all $\langle ij \rangle \in g$, if $u_i(g - \langle ij \rangle) > u_i(g)$, then $u_j(g - \langle ij \rangle) < u_j(g)$, and
2. for all $\langle ij \rangle \not\in g$, if $[u_i(g + \langle ij \rangle) > u_i(g)$ and $RS_j \geq d_i$ and $RB_i \geq c]$], then $[u_j(g + \langle ij \rangle) < u_j(g) \text{ or } RS_i < d_j \text{ or } RB_i < c]$.

As in the case of MO-Framework, there is no change in the link deletion condition of Definition 3. However, while adding a link, an agent has to ensure that the other agent has sufficient storage to store its data and agent itself has sufficient budget to form the link (besides ensuring increase in its utility). This is, again, based on the assumption that the agents are rational and self-centered.

### 5 Stability Point and Stable Networks

As discussed earlier, in this section we study stability aspects of a social storage network considering the utility of agents and network formation game defined in the previous two sections.

Free riding is a situation where an agent offers less storage space, but consumes more. To deal with free riding, many backup systems have used the concept of symmetric resource sharing (or equal resource trading). Internet Cooperative Backup System (Lillicbridge et al, 2003), PeerStore (Landers et al, 2004), Pastiche (Cox et al, 2002), are a few examples of P2P backup systems which use symmetric trading to mitigate free riding.

We term a social storage system with symmetric resource sharing as a symmetric social storage system. We consider symmetry in the worth of data, storage space available, amount of data to be shared, and budget in two different scenarios. These scenarios are discussed next.

**Definition 6** A symmetric value network (SVN) $g$ is a social storage network where the benefit (value) associated with backed-up data is the same for all agents in the network, i.e., $\beta_i = \beta_j$ (say $\beta$), for all $i, j \in A$, and hence, utility of each agent $i$ in the network is
\[ u_i(g) = \beta_i(1 - \lambda_i \eta_i(g)) - c \eta_i(g) \text{ for MO-Framework 3.1 and,} \\
\[ u_i(g) = \beta_i(1 - \lambda_i \eta_i(g)) \text{ for SO-Framework 3.2,} \]
where \( \beta_i, \lambda_i, c \in (0, 1) \).

**Definition 7** A symmetric resource network (SRN) \( g \) is a social storage network where all agents in \( g \) have an equal amount of (limited) storage space available to them, an equal amount of data that they want to backup, and have the same budget. That is, for all \( i, j \in g \), \( s_i = s_j \) (say \( s \)), \( d_i = d_j \) (say \( d \)), and \( b_i = b_j \) (say \( b \)).

In the next two subsections (Section 5.1 and Section 5.2), we first discuss the results for SVN under the MO-Framework, where each agent in the given network \( g \) has as much storage as is required for all other agents in \( g \). That is,

\[ s_i = \sum_{j \in g, j \neq i} d_j, \quad \text{for all } i \in g. \quad (6) \]

Note that \( s_i \) may be different from some other \( s_j \). For convenience, we shall call such networks as SVN with sufficient storage. The reason we do this is that it leads to the results of the realistic scenario, i.e., SV-SRN under the MO-Framework.

**Remark 2** An SV-SRN, \( g \) under the MO-Framework is a social storage network where the utility of each agent \( i \in g \) is \( u_i(g) = \beta_i(1 - \lambda_i \eta_i(g)) - c \eta_i(g) \), and for all agents \( i \in g \), \( \beta_i = \beta_j, s_i = s_j, d_i = d_j \).

Next, we work with SVN under the SO-Framework where each agent in the given network \( g \) has as much storage as is required for all other agents in \( g \), and each agent in the given network \( g \) has as much budget as is required by him/her to maintain backup-partnerships with every other agent in \( g \). That is,

\[ s_i \geq \sum_{j \in A, j \neq i} d_j, \quad \text{for all } i \in g, \quad \text{and} \quad b_i \geq c(N - 1), \quad \text{for all } i \in g \text{ where } N = |A|. \quad (7) \]

As in the SO-Framework, this leads to the scenario of SV-SRN under the SO-Framework. However, for SO-Framework, we present the results for SRN directly rather than SV-SRN. This is because SV-SRN is a subset of SRN and so, what holds for SRN does for SV-SRN as well.

**Remark 3** An SV-SRN, \( g \) under the SO-Framework is a social storage network where the utility of each agent \( i \in g \) is \( u_i(g) = \beta_i(1 - \lambda_i \eta_i(g)) \), and for all agents \( i \in g \), \( \beta_i = \beta_j, s_i = s_j, d_i = d_j, \) and \( b_i = b_j \).

For ease of exposition, from now onwards, whenever we talk of SVN networks, we will always assume sufficiency of every resource — that is, sufficient storage under MO-Framework, and sufficient storage and budget under SO-Framework. Whenever we talk of SRN or SV-SRN networks, we will not make these assumptions of sufficiency. These are summarised in Table 2.
We, first (in Section 5.1), apply the bilateral stability definitions discussed in the previous section to the two utility functions of Section 3. This gives us necessary and sufficient conditions for stability, in terms of the network parameters (that is, $c$, $\lambda$, etc.). This makes it easier to visualize a bilaterally stable network, and we use these conditions in Section 5.2 to derive the ideal neighborhood size for having a bilaterally stable network. As discussed earlier, we term it the stability point. We also prove uniqueness of such a point (except in one trivial case). In Section 5.3 we discuss bilaterally stable social storage networks, and in Section 5.4, we discuss efficient networks.

5.1 Conditions for Stability

In order to characterize symmetric (SVN, SRN or SV-SRN) social storage networks that are bilaterally stable, we first derive the deviation conditions that state when an agent has an incentive to add or delete a link, given the disk failure rate, value of backup data and the cost of each link. Using these conditions, we derive conditions for bilateral stability of SVN, SRN and SV-SRN networks, under the MO-Framework as well as the SO-Framework.

Table 2 Summary of Network Study under Different Frameworks with/ without Sufficient Resources.

| Network Type | Framework | Resource Availability            |
|--------------|-----------|----------------------------------|
| SVN          | MO-Framework | Sufficient Storage.              |
| SV-SRN       | MO-Framework | Limited Storage and Limited Budget. |
| SVN          | SO-Framework | Sufficient Storage and Sufficient Budget. |
| SRN          | SO-Framework | Limited Storage and Limited Budget. |

5.1.1 Conditions under MO-Framework

Lemma 1 In an SVN $g$, under the MO-Framework, for any agent $i \in g$, forming a partnership with another agent $j \in g$ is beneficial if and only if $c < B[\lambda^{n_i(g)} - \lambda^{n_i(g)+1}]$.

Proof Agent $i$’s utility in social storage $g$ is as given in Equation 5.

If the link $(ij)$ is not present, then by adding $(ij)$ the structure of $g$ changes to $g + (ij)$, and the utility of agent $i$ in the new structure $g + (ij)$ will be

$$u_i(g + (ij)) = [\beta(1 - \lambda^{n_i(g)+1}) - c(n_i(g) + 1)].$$
Then, from Definition 3, adding a new link or backup partner is beneficial for any agent $i$ if and only if

$$u_i(g + (ij)) > u_i(g), \text{ if and only if}$$

$$[\beta(1 - \lambda \eta_i(g) - 1) - c(\eta_i(g) + 1)] > [\beta(1 - \lambda \eta_i(g)) - c(\eta_i(g))], \text{ if and only if}$$

$$c < \beta(\lambda \eta_i(g) - \lambda \eta_i(g) + 1).$$

□

Lemma 2 In an SVN $g$, under the MO-Framework, for any agent $i \in g$, breaking an existing partnership with another agent $j \in g$ is beneficial if and only if $c > \beta(\lambda \eta_i(g) - \lambda \eta_i(g)).$

Proof Agent $i$’s utility in social storage $g$ is as given in Equation 5.

If the link $(ij)$ is present, then by deleting $(ij)$ the structure of $g$ changes to $g - (ij)$ and the utility of agent $i$ in the new structure $g - (ij)$ will be

$$u_i(g - (ij)) = [\beta(1 - \lambda \eta_i(g) - 1) - c(\eta_i(g) - 1)].$$

Then, from Definition 3, deleting the existing link or backup partner is beneficial for agent $i$ if and only if

$$u_i(g - (ij)) > u_i(g), \text{ if and only if}$$

$$[\beta(1 - \lambda \eta_i(g) - 1) - c(\eta_i(g) - 1)] > [\beta(1 - \lambda \eta_i(g)) - c(\eta_i(g))], \text{ if and only if}$$

$$c > \beta(\lambda \eta_i(g) - \lambda \eta_i(g)).$$

□

Theorem 1 An SVN $g$, under the MO-Framework, $g$ is bilaterally stable if and only if

1. for all $(ij) \in g$, $\beta(\lambda \eta_i(g) - \lambda \eta_i(g)) < c \Rightarrow \beta(\lambda \eta_i(g) - \lambda \eta_i(g)) > c$, and
2. for all $(ij) \notin g$, $\beta(\lambda \eta_i(g) - \lambda \eta_i(g) + 1) > c \Rightarrow \beta(\lambda \eta_i(g) - \lambda \eta_i(g) + 1) < c$.

Proof Follows from Lemma 1, Lemma 2 and Definition 3 of bilateral stability. □

We state and prove the following for SV-SRN, under the MO-Framework.

Lemma 3 Let $g$ be an SV-SRN, under the MO-Framework. Then, in $g$, for an agent $i$, adding or deleting a link is beneficial if and only if

$$\begin{cases} c < \beta(\lambda \eta_i(g) - \lambda \eta_i(g) + 1) & \text{adding} \\ s - d \eta_i(g) \geq d & \text{or} \end{cases}$$

$$\begin{cases} c > \beta(\lambda \eta_i(g) - \lambda \eta_i(g) + 1) & \text{deleting} \\ \sum_{k \in \eta_i(g)} d_k \geq d_i. \end{cases}$$

Proof Agent $i$ has an incentive to add a link with agent $j$, if and only if

$$c < \beta(\lambda \eta_i(g) - \lambda \eta_i(g) + 1) \text{ (from Lemma 1), where } \eta_i(g) \text{ = neighborhood size of } i,$$

and the amount of storage available with agent $j \geq agent i$’s data size, if and only if

$$c < \beta(\lambda \eta_i(g) - \lambda \eta_i(g) + 1) \text{ and } s_j - \sum_{k \in \eta_j(g)} d_k \geq d_i.$$
where \( \eta_j(g) \) is the set of neighbors of \( j \), if and only if

\[
c < \beta [\lambda \eta_i(g) - \lambda \eta_i(g) + 1] \quad \text{and} \quad s - d \eta_j(g) \geq d, \quad (\text{as } s_k = s_l, d_k = d_l, \text{ for all } k, l \in g),
\]

where \( \eta_j(g) \) is the neighborhood size of \( j \).

To delete an existing link, agent \( i \) only looks at the cost for maintaining the link, and hence, from Lemma 2, agent \( i \) has an incentive to delete a link if and only if

\[
c > \beta [\lambda \eta_i(g) - \lambda \eta_i(g) + 1].
\]

\( \square \)

**Theorem 2** An SV-SRN \( g \), under the MO-Framework, is bilaterally stable if and only if

1. for all \( \langle ij \rangle \in g \), \( \beta [\lambda \eta_i(g) - \lambda \eta_i(g) + 1] < c \Rightarrow \beta [\lambda \eta_j(g) - \lambda \eta_j(g) + 1] > c \), and
2. for all \( \langle ij \rangle \notin g \), \( \beta [\lambda \eta_i(g) - \lambda \eta_i(g) + 1] > c \) and \( s - d \eta_j(g) \geq d \) \( \Rightarrow \)

\[
\beta [\lambda \eta_i(g) - \lambda \eta_i(g) + 1] < c \quad \text{or} \quad s - d \eta_j(g) < d.
\]

**Proof** Follows from Lemma 3, and Definition 4 of bilateral stability. \( \square \)

### 5.1.2 Conditions under SO-Framework

**Lemma 4** In an SVN, \( g \), under the SO-Framework, for any agent \( i \in g \), forming a partnership with another agent \( j \in g \) is always beneficial.

**Proof** Agent \( i \)'s utility in social storage \( g \) is as given in Equation 5.

If the link \( \langle ij \rangle \) is not present, then by adding \( \langle ij \rangle \) the structure of \( g \) changes to \( g + \langle ij \rangle \), and the utility of agent \( i \) in the new structure \( g + \langle ij \rangle \) will be

\[
u_i(g + \langle ij \rangle) = [\beta (1 - \lambda \eta_i(g) + 1)].
\]

Then, from Definition 3, adding a new link or backup partner is beneficial for agent \( i \) if and only if

\[
u_i(g + \langle ij \rangle) > u_i(g), \quad \text{if and only if}
\]

\[
[\beta (1 - \lambda \eta_i(g) + 1)] > [\beta (1 - \lambda \eta_i(g))], \quad \text{if and only if}
\]

\[
\lambda \eta_i(g) + 1 < \lambda \eta_i(g), \quad \text{if and only if}
\]

\[
\lambda < 1, \quad \text{which is always true}. \quad \square
\]

**Corollary 1** In an SVN, \( g \), under the SO-Framework, no agent benefits by deleting any existing partnership.

**Corollary 2** An SVN, \( g \), under the SO-Framework, is bilaterally stable if and only if each agent \( i \in g \) has backup partnerships with all agents \( j \in g \) with \( j \neq i \).

Now, we state and prove the following for SRN, under the SO-Framework.

**Lemma 5** In an SRN, \( g \), under the SO-Framework, for any agent \( i \in g \), forming a partnership with another agent \( j \in g \) is beneficial if and only if \( b - c \eta_i(g) \geq c \) and \( s - d \eta_j(g) \geq d \).
Proof In the SO-Framework, the utility of each agent \( i \in g \) increases with increase in its neighborhood size \( \eta_i(g) \).

Therefore, for any agent \( i \in g \), forming a partnership with another agent \( j \in g \) is beneficial if and only if agent \( i \)'s budget allows this link addition and agent \( j \) has free storage space for agent \( i \)'s data. (Refer Definition 5).

agent \( j \) has free storage space for \( i \)'s data, if and only if \( s - d \eta_j(g) \geq d \). (Similar to the proof of Lemma 3).

Similarly, agent \( i \)'s budget allows adding a link, if and only if \( b - c \eta_i(g) \geq c \).

\( \therefore \)

\textbf{Corollary 3} In an SRN, \( g \), under the SO-Framework, no agent benefits by deleting any existing partnership.

\textbf{Theorem 3} An SRN \( g \), under the SO-Framework is bilaterally stable if and only if for all \( (ij) \notin g \), \( b - c \eta_i(g) \geq c \) and \( s - d \eta_j(g) \geq d \) \( \Rightarrow \) \( b - c \eta_j(g) < c \) or \( s - d \eta_i(g) < d \).

\textbf{Proof} Follows from Lemma 5, and Definition 5.

\( \therefore \)

\textbf{Remark 4} Since in an SRN \( g \) under the SO-Framework, no agent benefits by deleting any existing partnership, link deletion does not appear in the bilateral stability conditions above.

We summarize the above results in Table 3.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Network Type} & \textbf{Framework} & \textbf{Condition(s) for Bilateral Stability} \\
\hline
SVN & MO-Framework & 1. For all \((ij) \in g\), \(\beta[\lambda \eta_i(g) - \lambda \eta_j(g)] < c \Rightarrow \beta[\lambda \eta_i(g) - \lambda \eta_j(g)] > c\), and \\
& & 2. For all \((ij) \notin g\), \\
& & \(\beta[\lambda \eta_i(g) - \lambda \eta_j(g)] > c \Rightarrow \beta[\lambda \eta_i(g) - \lambda \eta_j(g)] < c\). \\
SV-\text{SRN} & MO-Framework & 1. For all \((ij) \in g\), \(\beta[\lambda \eta_i(g) - \lambda \eta_j(g)] < c \Rightarrow \beta[\lambda \eta_i(g) - \lambda \eta_j(g)] > c\), and \\
& & 2. For all \((ij) \notin g\), \\
& & \(\beta[\lambda \eta_i(g) - \lambda \eta_j(g)] > c \Rightarrow \beta[\lambda \eta_i(g) - \lambda \eta_j(g)] < c\) or \(s - d \eta_i(g) < d\). \\
SVN & SO-Framework & Each agent \( i \in g \) has backup partnerships with all agents \( j \in g \) with \( j \neq i \). \\
SRN & SO-Framework & For all \((ij) \notin g\), \(b - c \eta_i(g) \geq c \) and \(s - d \eta_j(g) \geq d\) \( \Rightarrow \) \(b - c \eta_j(g) < c \) or \(s - d \eta_i(g) < d\). \\
\hline
\end{tabular}
\caption{Summary of stability condition for different network.}
\end{table}
5.2 Uniqueness of Stability Point

In this subsection, we prove uniqueness of the stability point (See Definition 8) of SVN, SRN and SV-SRN networks, under the MO- and SO-Frameworks. We also show that the stability point is independent of the number of agents for all cases under the MO-Framework and for all cases but one trivial case under the SO-Framework, the trivial case being SVN with sufficient storage and sufficient budget where it is easy to see that the complete network is the only stable network.

**Definition 8** Given a network $g$, we define the stability point $\hat{\eta}$ of $g$ as the neighborhood size (degree) such that no agent in $g$ has any incentive to increase its neighborhood size to more than $\hat{\eta}$ and to decrease it to less than $\hat{\eta}$.

5.2.1 Stability Point under MO-Framework

**Lemma 6** In an SVN $g$, under the MO-Framework, for an agent $i \in g$, increasing neighborhood size is not beneficial if and only if

$$\eta_i(g) \leq \frac{|\ln \frac{c}{|\ln \lambda|}}{|\ln \lambda|}, \text{ for all } i \in g.$$

**Proof** From Lemma 1, adding a link for agent $i$ is beneficial if and only if

$$\beta [\eta_i(g) - \lambda \eta_i(g) + 1] > c,$$

and only if

$$\eta_i(g) \ln \lambda > \ln \left( \frac{c}{|\ln (1 - \lambda)|} \right),$$

Therefore, for agent $i$, increasing neighborhood size is not beneficial if and only if

$$\eta_i(g) \geq \frac{|\ln \frac{c}{|\ln \lambda|}}{|\ln \lambda|}.$$

**Lemma 7** In an SVN $g$, under the MO-Framework, for an agent $i \in g$, decreasing neighborhood size is not beneficial if and only if

$$\eta_i(g) \leq \frac{|\ln \frac{c \beta}{|\ln \lambda|}}{|\ln \lambda|}, \text{ for all } i \in g.$$

**Proof** From Lemma 2, deleting a link for agent $i$ is beneficial if and only if

$$c > \beta [\eta_i(g) - \lambda \eta_i(g)]$$

and only if

$$\ln \left( \frac{c}{|\ln (1 - \lambda)|} \right) > \eta_i(g) \ln \lambda,$$

Therefore, decreasing neighborhood size is not beneficial for agents $i$ if and only if

$$\frac{|\ln \frac{c \beta}{|\ln \lambda|}}{|\ln \lambda|} \geq \eta_i(g).$$

**Theorem 4** Let $|\ln \frac{c \beta}{|\ln \lambda|}|$ be a non-integer. (Note that for most values of $c, \beta, \lambda$, this is true). Let $g$ be an SVN network under the MO-Framework. Then, the stability point $\hat{\eta}$ of $g$ is unique and is given by
\[ \hat{\eta} = \left\lceil \frac{\left| \ln \left( \frac{c^\beta (1 - \lambda)}{\lambda} \right) \right|}{\ln \lambda} \right\rceil = \left\lceil \frac{\left| \ln \left( \frac{c^\beta (1 - \lambda)}{\lambda} \right) \right|}{\ln \lambda} \right\rceil. \]

**Proof** Let \( L = \frac{\left| \ln \left( \frac{c^\beta (1 - \lambda)}{\lambda} \right) \right|}{\ln \lambda} \) and \( U = \frac{\left| \ln \left( \frac{c^\beta (1 - \lambda)}{\lambda} \right) \right|}{\ln \lambda} \). From Lemma 6, Lemma 7 and Definition 8, \( L \) and \( U \) are, respectively, the lower and upper bounds of \( \hat{\eta} \).

\[ U = \frac{\left| \ln \left( \frac{c^\beta (1 - \lambda)}{\lambda} \right) \right|}{\ln \lambda} + \frac{1}{\ln \lambda} = L + 1. \]

It is easy to see that if \( L \) is not an integer (and hence, \( U \) is not an integer), the stability point \( \hat{\eta} \) is the unique positive integer between \( L \) and \( U \). \( \square \)

The following example shows that in a network, if an agent has neighborhood size \( \hat{\eta} \), then it is not beneficial for him/her to add any more neighbors or break any of the existing backup partnerships.

**Example 1** Consider the networks \( g \) and \( s \) (see Fig. 1), consisting of six agents \( (a, b, c, d, e, f) \) and seven agents \( (a, b, c, d, e, f, g) \), respectively. In both the networks \( g \) and \( s \), let the cost \( c \) for maintaining a link be 0.0055, benefit \( \beta \) associated with data be 0.6, and disk failure rate \( \lambda \) be 0.2. Here, \[ \left\lceil \frac{\left| \ln \left( \frac{c^\beta (1 - \lambda)}{\lambda} \right) \right|}{\ln \lambda} \right\rceil = 2.72 \] and \[ \left\lfloor \frac{\left| \ln \left( \frac{c^\beta (1 - \lambda)}{\lambda} \right) \right|}{\ln \lambda} \right\rfloor = 3.72, \] and hence, \( \hat{\eta} = 3 \). This means that in both the networks \( g \) and \( s \), for any agent, maintaining more than three neighbors and having less than three is not beneficial. In network \( g \), all agents have three neighbors, and in network \( s \), all agents except agent \( g \) have three neighbors each. Despite the fact that agent \( g \) in the network \( s \) has an incentive to add one more link, both the networks are bilaterally stable. Let us look at each of these networks (see Fig. 1(a) and 1(b)) in detail.

![Fig. 1](image-url)  
Stable SVN Networks under the MO-Framework with Sufficient Storage

1. Network \( g \) (Fig. 1(a)) is bilaterally stable.
   Without loss of generality (owing to symmetry), assume that agents \( a \) and \( d \) decide to form a link in \( g \). Then agent \( a \)'s and agent \( d \)'s neighborhood size increases by one and become four each. Adding this link is not beneficial to agent \( a \) as \( c = 0.0055 \not< \beta \left( \lambda^{(a)} - \lambda^{(a)+1} \right) = 0.0038 \), and is, similarly, not beneficial to agent \( d \) either. Observe that this is true for all possible links \( \langle ae \rangle, \langle bc \rangle, \langle bf \rangle, \langle cf \rangle, \langle de \rangle \), which are not part of the network.
Now, assume that agents $a$ and $b$ want to delete the existing link $(ab)$. Deleting this link is not beneficial for the either of the agents as $c = 0.0055 < 0.0192 = \beta[\lambda^{n_2}(0)^{-1} - \lambda^{n_1}(0)]$. This is true for all existing pairs $(ac), (be), (af), (bd), (cd), (ce), (df), (ef)$.

As no pair of agents has any incentive to add a new link or delete an existing link, $g$ is bilaterally stable.

2. Network $s$ (Fig. 1(b)) is bilaterally stable.

As seen in the case of network $g$ above, no pair of agents want to delete an existing link. Each agent (except $g$) has a neighborhood size of three and has no incentive to delete any existing link, as seen above. Assume agent $g$ wants to break the relationship with agent $e$. Breaking the relationship $(ge)$ is not beneficial for agent $g$ either (as $c = 0.0055 < 0.0192, \beta[\lambda^{n_2}(0)^{-1} - \lambda^{n_1}(0)] = 0.096$). The same is true for $(gf)$ too.

Now, no agent (except agent $g$) has an incentive to add a new link. Assume $g$ wants to add a link with agent $a$. By adding link $(ag)$, agent $g$ benefits as $c = 0.0055 < 0.0192 = \beta[\lambda^{n_2}(0)^{-1} - \lambda^{n_1}(0)]$, but agent $a$ does not benefit as $c = 0.0055 < 0.0192, \beta[\lambda^{n_2}(0)^{-1} - \lambda^{n_1}(0)] = 0.0038$. As adding $(ag)$ requires mutual consent of $g$ and $a$, the link will not be added as $a$ would lose by adding this (or any additional) link, and hence will not consent to adding $(ag)$.

As no pair of agents has any incentive to add a new link or delete an existing link, $s$ is bilaterally stable. □

We have seen stability point for SVN under the MO-Framework. Next, we derive stability point for SV-SRN network under the MO-Framework. Here, Definition 4 is relevant, and for simplicity we assume that $\frac{n}{2}$ is an integer.

**Theorem 5** Let $g$ be an SV-SRN, under the MO-Framework.

Then, $\tilde{n} = \min\{\tilde{n}, \frac{n}{2}\}$, is the stability point of $g$.

**Proof** As seen in the previous subsection, if all agents have sufficient storage, then from Lemma 1 and Lemma 2, it follows that an agent has no incentive to add or delete a link if and only if $c < \beta[\lambda^{n_2}(0) - \lambda^{n_1}(0)+1]$ and $c > \beta[\lambda^{n_2}(0)^{-1} - \lambda^{n_1}(0)]$, respectively. Then from Theorem 4, $\tilde{n}$ is the stability point.

Now let us assume that each agent has a total amount of storage, $s$, available for sharing, and that s/he has an amount of data, $d$, that s/he wants to backup. Then, $\frac{n}{2}$ defines the maximum neighborhood size of each agent in the network.

Therefore, $\min\{\tilde{n}, \frac{n}{2}\}$ is the stability point, given $c, \lambda, \beta, s, d$.

Alternatively, we may also use the bound $\frac{n}{2}$ in Lemmas 1, 2 and Theorem 4. □

Henceforth, for the sake of uniformity, we shall use $\tilde{n}$ (and not $\tilde{n}$) for the stability point of SV-SRN under the MO-Framework too.

### 5.2.2 Stability Point under SO-Framework

The following case (Theorem 6) is the only case where the stability point depends on $N$. 

Theorem 6 In an SVN $g$, under the SO-Framework, $\hat{\eta} = N - 1$, is the stability point, where $N$ is the number of agents.

Proof Follows from Lemma 4.

Except the above (Theorem 6), in all other scenarios (including the following), the stability point is independent of $N$. In the following, for simplicity, we assume that $\frac{d}{s}$ and $\frac{b}{c}$ are integers.

Theorem 7 In an SRN $g$, under the SO-Framework, $\hat{\eta} = \min\{\frac{d}{s}, \frac{b}{c}\}$, for all $i \in s$, is the stability point, where no agent has incentive to add or delete a link.

Proof A constructive proof follows from Lemma 5. Alternatively, it is clear that it is beneficial for each agent to add as many links as possible. The degree of agent $i$ in $g$, $\eta_i(g)$, is limited only by its storage space $s$ and budget $b$. That is,

$$s \geq d\eta_i(g) \text{ and } b \geq c\eta_i(g).$$

The theorem follows as the above is true for all $i \in g$.

Example 2 Let us consider the networks $g$ (see Fig. 2(a)) and $s$ (see Fig. 2(b)), each consisting of six agents, and network $t$ (see Fig. 2(c)) consisting of seven agents. Assume that, in $g$ and $t$, $s = 60 \text{ TB}, d = 20 \text{ TB}, b = 0.5$, and $c = 0.1$. Assume that, in network $s$, $s = 60 \text{ TB}, d = 10 \text{ TB}, b = 0.4$, and $c = 0.1$.

![Fig. 2 Stable SRN Networks under the SO-Framework](image)

Note that in networks $g$ and $t$, although the budget constraints permit agents to maintain five neighbors each, storage limitations do not permit agents to maintain more than three neighbors each, and hence, in $g$ and $t$ which are bilaterally stable networks, each agent would maintain exactly three neighbors each. For instance, in network $g$, it is not possible for agents $a$ and $d$ to form the link $\langle ad \rangle$ because of the unavailability of enough storage space for backing up the data of neighbors. This is true for all other pairs $\langle ae \rangle, \langle be \rangle, \langle bf \rangle, \langle cf \rangle, \langle de \rangle$ too. Now, if agents $a$ and $b$ decide to delete the link $\langle ab \rangle$, then both the agents’ utility will decrease. This is true for all other existing links $\langle ac \rangle, \langle af \rangle, \langle bd \rangle, \langle be \rangle, \langle cd \rangle, \langle ce \rangle, \langle df \rangle, \langle ef \rangle$ too. Thus network $g$ is bilaterally stable.

Similarly, network $t$ is also bilaterally stable. Note that, though agent $g$ has $20 \text{ TB}$ free storage space, no new link with $g$ can be formed. For example, $\langle ag \rangle$ will not be
added as a has no space available to store g’s data. Recall that adding a link means both agents forming the link store a copy of their data on each other’s storage space.

In network a, although storage constraints permit agents to maintain six neighbors each, budget constraints do not allow agents to maintain more than four neighbors each. Hence, a is bilaterally stable.

Table 4 summarizes the stability points of each type of network under both frameworks.

| Network Type | Framework | Stability Point |
|--------------|-----------|----------------|
| SVN          | MO-Framework | $\hat{\eta} = \left\lceil \frac{\ln(\beta(1-\lambda))}{\ln \lambda} \right\rceil \left\lfloor \frac{\ln \lambda}{\ln \lambda} \right\rfloor$ |
| SV-SRN       | MO-Framework | $n = \min\{\hat{\eta}, \frac{4}{7}\}$ |
| SVN          | SO-Framework | $\hat{\eta} = N - 1$ |
| SRN          | SO-Framework | $\hat{\eta} = \min\{\frac{4}{7}, \frac{4}{7}\}$ |

5.3 Stable Networks

In the previous subsection, we proved that, given $N$ agents, we have a unique stability point $\hat{\eta}$ for symmetric social storage networks. In this subsection, we first discuss conditions on $N$ and $\hat{\eta}$ for connected networks to be bilaterally stable (in Section 5.3.1). Second, we look at networks that are comprised of multiple connected components, and discuss conditions on $N$, $\hat{\eta}$ as well as number of agents in individual components that lead to a bilaterally stable network (in Section 5.3.2). Finally, we discuss uniqueness of resulting bilaterally stable networks (in Section 5.3.3).

Henceforth, whenever we say $g$ is a symmetric social storage network, $g$ may be any of the networks SVN, SRN or SV-SRN, under either framework (MO- or SO-Framework), consists $N$ agents, with the unique stability point $\hat{\eta}$ corresponding to that network type and framework.

Up to this point, we have not explicitly discussed the process of network formation, whether the network evolves from the null (empty) network by addition of links (with mutual consent between pairs of agents adding links) or from the complete network by deletion of links (with mutual consent between pairs of agents deleting links). This is because all our results above are independent of any process or protocol for network formation. However, the following results depend on the where we start the network formation from. Note that starting from the complete network is also a possibility in the case of Social Storage Systems, as one can think of a default setting of free sharing (mutual sharing of storage space with all agents in the network) while signing up for a system.
5.3.1 Connected Networks

We start our discussion with the following remark, which is easy to see, owing to results in the previous subsection.

**Remark 5** Each agent *tries* to maximize his/her utility by achieving neighborhood size \( \hat{\eta} \).

It is easy to see that the utility (or part of the utility) of each agent in terms of the expected value of data backup increases with the increase in neighborhood size. However, agents incur a cost for adding or maintaining each link and by Theorem 4, no agent can have a neighborhood size greater than \( \hat{\eta} \) in a stable network. Hence, each agent tries to increase their neighborhood size up to \( \hat{\eta} \). We illustrate this with the following example.

**Example 3** This example demonstrates how agents form a stable network when initially all agents are isolated or when initially all are connected.

![Networks](image)

Consider networks \( g \) (see Fig. 3(a)) and \( s \) (see Fig. 3(b)). Let \( \hat{\eta} = 3 \). The networks \( g \) and \( s \) are both bilaterally stable, where \( g \) evolves when agents start from the empty network and add links and \( s \) evolves when agents start from the complete network and delete links.

In \( g \), although agent \( e \) has an incentive to add another link, no other agent (who does not have a link with \( e \)) would consent to adding a new link with \( e \) as they (that is, agents \( a, b, c \) and \( d \)) have already reached their stability point \( \hat{\eta} \) (that is, their neighborhood size is \( \hat{\eta} = 3 \)) and hence, have no incentive to add or delete any link.

In \( s \), although agent \( f \) has an incentive to delete a link, no other agent (who has a link with \( f \)) would consent to deleting their link with \( f \) as they (that is, agents \( g, h, i \) and \( j \)) have already reached their stability point \( \hat{\eta} \) (that is, their neighborhood size is \( \hat{\eta} = 3 \)) and hence, have no incentive to add or delete any link. \( \Box \)

In Propositions 1 and 2 below, we provide results that would be useful for an independent observer in checking that for a pairwise stable symmetric social storage network, how many agents have maximized their utility. Thus, as discussed in the introduction, such an observer (say, an administrator or regulator) can externally perturb the system so that all agents achieve maximum utility.

**Proposition 1** Let \( N \) and \( \hat{\eta} \) be (positive) odd integers, with \( \hat{\eta} < N \). Then:
1. Any symmetric social storage network $g$ with $N$ agents and stability point $\hat{\eta}$ consists of at least one agent who has an incentive to either add or delete a link.

2. There exists a connected, bilaterally stable, symmetric social storage network with exactly $N-1$ agents who have no incentive to add or delete any link.

**Proof** Let $g$ be bilaterally stable, and let $\ell$ be the number of links in $g$.

If $\hat{\eta} < N$, then $\ell$ does not exceed the maximum number of links $g$ can possibly have, that is, $\frac{N \times (N-1)}{2}$.

As the utility of each agent is maximum when its neighborhood size is $\hat{\eta}$, total number of links $\tilde{\ell} = \frac{N \times \hat{\eta}}{2}$ will be attained if possible. However, $\tilde{\ell}$ is not an integer, as both $N$ and $\hat{\eta}$ are odd.

This implies that, not all $N$ agents have a neighborhood size of $\hat{\eta}$ at stability. This proves (1).

Now, $N-1$ agents having $\hat{\eta}$ neighbors and the $N^{th}$ agent having $\hat{\eta}-1$ or $\hat{\eta}+1$ neighbors are, however, possible. Let $g$ be such a network with exactly $N-1$ agents who have no incentive to add or delete any link. These $N-1$ agents have neighborhood size $\hat{\eta}$. None of these $N-1$ agents will consent to add or delete any link (among themselves, or with the $N^{th}$ agent). Thus, the symmetric social storage network $g$ is bilaterally stable. If $g$ is connected, we are done. Otherwise, all non-trivial components (that is, components with 2 or more agents in each) of $g$ can be connected as follows, without changing the neighborhood sizes of any of the agents. Let $\langle i_1j_1 \rangle$ and $\langle i_2j_2 \rangle$ be links in two different (non-trivial) components, say $g(k_1)$ and $g(k_2)$ of $g$. Deleting both these links, and replacing them with $\langle i_1j_2 \rangle$ and $\langle i_2j_1 \rangle$ connects $g(k_1)$ and $g(k_2)$, without changing the neighborhood sizes of any of the agents. As neighborhood sizes of all the agents remain the same, the resulting graph is bilaterally stable too. Now, if $i$ is an isolated agent and $\langle ij \rangle$ is a link in $g$, delete $\langle ij \rangle$, and add $\langle ik \rangle$ instead. In this case, clearly, $i$ continues to be the only agent with an incentive to either add or delete a link. This proves (2).

**Remark 6** In the proof of Proposition 1, on the one hand, when the network evolves from the null network, $\hat{\eta} - 1$ neighbors for the $N^{th}$ agent is as beneficial as possible, and the total number of links will, hence, be $\ell = \frac{[(N-1)\hat{\eta}+(\hat{\eta}-1)]}{2}$.

On the other hand, when the network evolves from the complete network, $\hat{\eta} + 1$ neighbors for the $N^{th}$ agent is as beneficial as possible, and the total number of links will, hence, be $\ell = \frac{[(N-1)\hat{\eta}+(\hat{\eta}+1)]}{2}$. This number also does not exceed the maximum possible number of links, as $\hat{\eta} \leq N - 2$ (because $\hat{\eta} < N$, and both $\hat{\eta}$ and $N$ are odd).

The following example shows a bilaterally stable network which is not connected.

**Example 4** Let $N = 15$ and $\hat{\eta} = 3$. Consider the network $g$ on $N$ agents (see Fig. 4) that consists of three components, $g(k_1)$, $g(k_2)$ and $g(k_3)$. Though $g$ consists of three agents, $a$, $f$, and $k$, who have an incentive to delete a link each, $g$ is bilaterally stable. This is because the agents, $a$, $f$, and $k$, are in three different components, in each of which all other agents have neighborhood size $= \hat{\eta}$.

**Proposition 2** Let at least one of $N$ and $\hat{\eta}$ be even, and let $\hat{\eta} < N$. Then, there exists a connected bilaterally stable symmetric social storage network $g$ where no agent has incentives to add or delete any link.
5.3.2 Multiple Connected Components

We, now, discuss results on stability of symmetric storage networks with 2 or more components. Examples of scenarios where this might be useful include companies under the same umbrella group, where the social storage networks of each of these companies may be viewed as a component of a larger network which may be monitored or analysed by an independent observer (administrator or regulator, as discussed in the previous section). The results of this section (Proposition 3, Corollary 3, and subsequent claims) will help the independent observer check the stability of the system as a whole and, if required, externally perturb the system so that all agents achieve maximum utility.

Claim 1 Suppose \( g \) is a symmetric social storage network with 2 or more components. If \( g \) is bilaterally stable, then there is at most one component with less than or equal to \( \hat{\eta} \) agents.

Proposition 3 Let \( g \) be a symmetric social storage network which has evolved from the null network. Let \( \hat{\eta} \) be odd. Suppose \( g \) consists of \( \kappa \) connected components, \( \kappa \geq 2 \). Suppose at least two of the components, say \( g(\kappa_1) \) and \( g(\kappa_2) \), each have either \( \leq \hat{\eta} \) agents or an odd number of agents more than \( \hat{\eta} \). Then \( g \) is not bilaterally stable.

Proof From Proposition 1 and Claim 1, one agent in \( g(\kappa_1) \) (say \( p_{\kappa_1} \)) and one agent in \( g(\kappa_2) \) (say \( p_{\kappa_2} \)) each have neighborhood size less than \( \hat{\eta} \). Hence, both \( p_{\kappa_1} \) and \( p_{\kappa_2} \) have incentive to add a link, and gain by forming \( \langle p_{\kappa_1} p_{\kappa_2} \rangle \not\in g \). Thus, the network \( g \) is not bilaterally stable.

Example 5 Consider network \( g \) (see Fig. 5) consisting of 6 agents \((a, b, c, d, e, f)\), with two components \( g(\kappa_1) \) and \( g(\kappa_2) \). Let \( \hat{\eta} = 3 \). Note that, in Fig. 5(a), each agent in components \( g(\kappa_1) \) and \( g(\kappa_2) \) has neighborhood size 2. Though each component is complete, as \( \hat{\eta} = 3 \), each agent in either of the components can increase their neighborhood size by one by connecting to agents from the other component, and hence, \( g \) is not bilaterally stable. In Fig. 5(b), agents \( a \) and \( e \), \( c \) and \( g \), and \( d \) and \( f \) form links, resulting in the bilaterally stable network \( g' \) in Fig. 5(c).

Now, if \( \hat{\eta} = 2 \), then the network \( g \) (see Fig. 5(a)) is bilaterally stable though it consists of two components \( g(\kappa_1) \) and \( g(\kappa_2) \).
Corollary 4 Let $g$ be a symmetric social storage network which has evolved from the null network and which consists of $\kappa$ components, $\kappa \geq 2$. Let $\hat{\eta}$ be odd, and let $N > \hat{\eta}$. If $g$ is bilaterally stable, then at least $\kappa - 1$ components must consist of an even number of agents greater than $\hat{\eta}$.

Remark 7 In Proposition 3 and Corollary 4, if we consider networks which have evolved from the complete network, then Example 4 itself acts as a counter example. If $\hat{\eta}$ is even, we apply Proposition 2 to each component having more than $\hat{\eta}$ agents to see that each of these components is bilaterally stable. Now, there can be at most one component with $\leq \hat{\eta}$ agents (refer Claim 1), and if there is such a component, $g$ is bilaterally stable if and only if that component is complete.

Claim 2 Suppose $g$ is a symmetric social storage network. If $\hat{\eta} = 1$, and if $g$ has evolved from the null network, then $g$ is bilaterally stable if and only if $g$ consists of a set of $N - 1$ connected pairs of agents plus one isolated agent if $N$ is odd, and a set of $N$ connected pairs of agents if $N$ is even.

Remark 8 In Claim 2, if $g$ has evolved from the complete network (by mutual deletion of links), then networks consisting of star components \(^7\), are also bilaterally stable as per Definitions 3, 4, and 5.

Remark 9 In Claim 2, if $g$ is a given network, then in addition to the star components as discussed in Remark 8, $g$ may also consist of (at most) one isolated agent and continue to be bilaterally stable.

It is interesting to note that in any star network, given that $\hat{\eta} = 1$, though the universal agent has incentive to delete a link (or links), no other (pendant) agent will consent to deletion. However, if we start from the null network, we have the following observation.

Claim 3 Suppose $g$ has evolved from the null network. Then, if $g$ is bilaterally stable, $g$ can never contain a star network as component.

\(^7\) An $N$ agent star network consists of a single universal agent and $N - 1$ pendant agents. A universal agent is one who is adjacent to other $N - 1$ pendant agents. A pendant agent is one who is adjacent to only the universal agent. A star component is a component which is a star (sub-)network.
5.3.3 Unique Bilaterally Stable Networks

In the previous subsections, we have seen results on the existence of a bilaterally stable social storage networks. In this subsection, we look at conditions under which a unique bilaterally stable social storage network exists. Whenever a unique bilaterally stable network exists, the agents themselves endogenously form this network. Any independent observer or regulator knows precisely which network would form (or has formed).

**Claim 4** If $N = \eta + 1$ or $\eta \geq N$, then there exists a unique symmetric social storage network $g$ that is bilaterally stable, namely the complete network on $N$ agents.

**Claim 5** If $N > \eta + 1$, then there are always two or more unique (with respect to degree sequence$^8$) bilaterally stable networks.

![Fig. 6 Bilaterally Stable Networks with $N = 6$ agents, $\eta = 3$.](image)

**Example 6** Let $\eta = 3$ and $N = 6$. Then, there are four networks $n_1$ (see Fig. 6(a)), $n_2$ (see Fig. 6(b)), $n_3$ (see Fig. 6(c)) and $n_4$ (see Fig. 6(d)) which are bilaterally stable.

If we look at specific protocols of network formation, then we get further uniqueness results. For example, in Claim 2, starting from the null network (or any network where no agent has more than 1 neighbor), the resulting bilaterally stable network is unique (if we do not label the agents).

5.4 Efficient Social Storage Networks

In this subsection, we look at efficient social storage networks, that is social storage networks where as many agents as possible achieve maximum utility, and contented social storage networks, that is social storage networks where all agents achieve maximum utility. As discussed earlier, an observer who observes or monitors or regulates

---

$^8$ Two networks are unique with respect to degree sequence if the sorted sequence of degrees (neighborhood sizes) in one is different from that of the other. Note that, both sequences are sorted in the ascending order (or both in the descending order).
the network may externally perturb the system so as to reach an efficient or a contented network.

**Definition 9** A social storage network $g$ is efficient with respect to utility profile $(u_1, \ldots, u_N)$ if $\sum_i u_i(g) \geq \sum_i u_i(g'), \forall g' \in \mathcal{G}(N)$.

We have seen in Section 5.2 that there exists a unique stability point, $\hat{\eta}$, (for each network type, under the given framework) such that, no agent gains by adding more neighbors than $\hat{\eta}$, and severing existing relationships resulting in a neighborhood size of less than $\hat{\eta}$. An efficient social storage network is, hence, one in which maximum possible number of agents have $\hat{\eta}$ neighbors.

**Remark 10** An efficient social storage network is bilaterally stable.

Note that any network or any stable network in which maximum possible number of agents have $\hat{\eta}$ neighbors is not necessarily efficient, as per Definition 10. We, now, discuss an example to highlight the fact that not all stable networks are efficient.

![Network Structures](image)

**Fig. 7** Network Structure and Social Welfare

**Example 7** Suppose there are six agents, $a, b, c, d, e$ and $f$, in a social storage network with stability point $\hat{\eta} = 3$. Assume that, starting from the null network, these agents add links (that is, build mutual data backup partnerships). Different network structures may emerge, for example Fig. 7(a), Fig. 7(b), and Fig. 7(c)).

In network $g_1$ (see Fig. 7(a)), agent $e$’s expected value of data backup is less than that of the rest of the agents. In $g_2$ (see Fig. 7(b)), all agents achieve the same (and maximum) expected value of data backup, and in $g_3$ (see Fig. 7(c)), agents $a, b, c, d$ obtain a higher expected value of data backup than agents $e$ and $f$. $g_2$ is efficient, whereas $g_1$ and $g_3$ are not (though they are bilaterally stable).

We, now, discuss contented networks.

**Definition 10** A social storage network $g$ is contented with respect to utility profile $(u_1, \ldots, u_N)$ if, for each $i \in A$, $u_i = \max_{\eta_i(g)} (\beta_i(1 - \lambda^{\eta_i(g)}) - c_i(\eta_i(g)))$, under the MO-Framework, and $u_i = \max_{\eta_i(g)} (\beta_i(1 - \lambda^{\eta_i(g)}))$, under the SO-Framework.
Remark 11 A contented social storage network is bilaterally stable.

It is easy to see that not all stable networks are contented. In Example 3, though both $g$ and $s$ are stable, neither of these networks are contented. Consider $g$. An independent observer could just add a storage device, $p$, to the network, which leads to a contented network as explained below. This storage device acts as a dummy agent, not trying to maximize its utility, and always agreeing to add or delete any link with any agent. Hence, when we talk of a contented network, we are not considering any dummy agent as a part of the network. In $g$, agent $e$ has not achieved the maximum possible utility (as $\hat{\eta} = 3$, but $e$ has 2 neighbors) while all other agents have. By allowing $e$ to store a copy of its data on storage device $p$, $e$ also obtains the maximum possible utility. This network is, now, a contented network (where the utility of the dummy agent $p$ is not considered). We are, in fact, talking about a hybrid model — hybrid between a centralised storage system and a decentralised one.

Next, we relate contented networks and efficient networks.

Proposition 4 Let $g$ be a symmetric social storage network with $N$ agents and stability point $\hat{\eta}$. Then:

1. If at least one of $N$ and $\hat{\eta}$ is/ are even, then, $g$ is efficient if and only if $g$ is contented.
2. Suppose $N$ and $\hat{\eta}$ are odd. Then, an efficient network does exist but there exists no contented network.

Proof 1 follows from Proposition 2, since, if at least one of $N$ and $\hat{\eta}$ is/ are even, then, $g$ is efficient if and only if $g$ is $\hat{\eta}$-regular.\footnote{An $r$-regular network is one where each agent has exactly $r$ neighbors.}

2 follows from Proposition 1. $\Box$

Remark 12 Not all efficient networks are contented.

In Example 3, neither $g$ nor $s$ are contented. However, (at least) one of them is efficient. The following Propositions help identify which of them is/ are efficient, under the MO- as well as SO-Frameworks, for SVN, SRN and SV-SRN networks, as the case may be (Refer Table 2).

Proposition 5 Let $g$ be an SVN or SV-SRN under the MO-Framework, with $N$ agents and stability point $\hat{\eta}$. Suppose both $N$ and $\hat{\eta}$ are odd. Then $g$ is efficient if and only if $g$ has $N - 1$ agents with neighborhood size $\hat{\eta}$ and one of the following holds:

1. $c < \frac{\beta \lambda \hat{\eta}}{2} \left( \frac{1}{\lambda} - \lambda \right)$ and $g$ has one agent with neighborhood size $\hat{\eta} + 1$.
2. $c > \frac{\beta \lambda \hat{\eta}}{2} \left( \frac{1}{\lambda} - \lambda \right)$ and $g$ has one agent with neighborhood size $\hat{\eta} - 1$.
3. $c = \frac{\beta \lambda \hat{\eta}}{2} \left( \frac{1}{\lambda} - \lambda \right)$ and $g$ has one agent with neighborhood size either $\hat{\eta} + 1$ or $\hat{\eta} - 1$.\footnote{An $r$-regular network is one where each agent has exactly $r$ neighbors.}
Proof For each $i \in A$, its utility is $u_i(g) = \beta_i (1 - \lambda \eta_i(g)) - c \eta_i(g)$. (Refer Equation 2). As the network is SVN or SV-SRN, $\beta_i = \beta$, for all $i$.

$$\max_{\eta_i(g)} u_i(g) = \max_{\eta_i(g)} (\beta (1 - \lambda \eta_i(g)) - c \eta_i(g)) = \beta (1 - \lambda \hat{\eta}) - c \hat{\eta}$$

Let $g_1$ be the network where $N - 1$ agents have $\hat{\eta}$ neighbors and the other agent has $\hat{\eta} - 1$ neighbors. Let $g_2$ be the network where $N - 1$ agents have $\hat{\eta}$ neighbors and the other agent has $\hat{\eta} + 1$ neighbors.

$$u_i(g_1) = \beta (1 - \lambda (\hat{\eta} - 1)) - c (\hat{\eta} - 1)$$

$$u_i(g_2) = \beta (1 - \lambda (\hat{\eta} + 1)) - c (\hat{\eta} + 1)$$

From Proposition 1 and Definition 10, it is easy to see that either $g_1$ or $g_2$ (or both) is (are) efficient. That is, $\max(\sum_i u_i(g))$ is either $u_i(g_1)$ or $u_i(g_2)$. If $u_i(g_1) < u_i(g_2)$, we get result 1, if $u_i(g_1) > u_i(g_2)$, we get result 2 and if $u_i(g_1) = u_i(g_2)$, both $g_1$ and $g_2$ are efficient, leading to result 3.

Proposition 6 An SVN under the SO-Framework is efficient if and only if it is a complete network.

Proof Follows from Theorem 6. □

Proposition 7 For an SRN under the SO-Framework, $s/d$ and $b/c$ act as constraints for the maximum number of links possible.

Let $g$ be an SRN under the SO-Framework, with $N$ agents and stability point $\hat{\eta}$. Then:

1. If at least one of $N$ and $\hat{\eta}$ is/ are even, then $g$ is efficient if and only if $g$ is $\hat{\eta}$-regular.
2. If both $N$ and $\hat{\eta}$ are odd, then $g$ is efficient if and only if $g$ has $N - 1$ agents with $\hat{\eta}$ neighbors and the other agent with $\hat{\eta} - 1$ neighbors.

Proof From Theorem 7, $\hat{\eta} = \min\{\frac{s}{d}, \frac{b}{c}\}$.

As the budget $b$ and the storage space available $s$ act as constraints, no agent can have more than $\hat{\eta}$ neighbors. Therefore, part 1 follows from Proposition 1. (We do not have the possibility of one agent having $\hat{\eta} + 1$ neighbors as we had in Proposition 5).

Part 2 follows from Proposition 2. □

6 Conclusion and Future Work

In this paper, we have expanded on two untouched aspects of social storage systems, namely, endogenous network formation and bilateral stability of such networks. We have formalized social storage networks as a network formation game where each agent tries to maximize his/her utility. We considered two frameworks for utility of agents in the network. We assumed that the cost to maintain a link is shared equally among the agents on either side of the link. Looking at asymmetric cost sharing, for example centrally-sponsored star networks, is one direction for future work. We
modified the pairwise stability definition of Jackson and Wolinsky (1996) to include mutual consent for link deletion too (as required for social storage networks), and also to include storage and budget constraints.

After defining bilateral stability as a modification of pairwise stability, we analysed bilateral stability of symmetric social storage networks. Our stability analysis involved restudying conditions of stability under the new definition of pairwise stability (that is, bilateral stability), derivation of a unique stability point (which is a neighborhood size where no agent has any incentive to add or delete a link), and some necessary and sufficient conditions for symmetric social storage networks to be bilaterally stable. We also showed that ideally all agents in a network want to achieve the stability point but a network can be bilaterally stable even when this stability point is not reached for one agent. We discussed which bilaterally stable networks would evolve.

In all our discussions on bilaterally stable networks as well as efficient networks, we have assumed that any pair of agents can potentially form a link. In scenarios where agents do not necessarily trust all agents in the network, our results on bilateral stability extend to every clique (of mutually trusting agents) in the network. If not all agents trust each other, we may use an extension of Hall’s marriage theorem (Hall, 1935) to aid independent observers determine whether it is possible to form an efficient network or not.

Note that, if we had used the concept of Pairwise Nash Stability as defined by Goyal and Joshi (2006b) and had applied the mutual consent requirement for deletion too, we would have the same results we have obtained in this paper. This is because the mutual consent requirement for addition and deletion overrides the requirement for Nash equilibrium. We are currently working on modifying the definition of Pairwise Nash Stability as defined by Goyal and Joshi (2006b) to have mutual consent only for link deletion, and Nash equilibrium for link addition. Looking at strong and coalition-proof Nash equilibria (Dutta and Mutuswami, 1997), strong pairwise stability (Jackson and van den Nouweland, 2005), and farsighted equilibrium (Dutta et al, 2005), are also future research directions.

In our current work, we have not focused on the heterogeneous behavior of agents in social storage settings. Although incorporating complex and heterogeneous behavior of agents into the model is closer to real world scenarios, it makes it difficult to deal with the model and as well as predict its outcome. Meier et al (2014) propose a social range matrix, which is a novel approach to deal with heterogeneous behavior of agents in the network. In particular, social range matrices capture three scenarios: anarchy, monarchy and coalitions. In anarchy, each agent is selfish. In monarchy, agents only care about one agent in the network. In the coalitions scenario, agents support each other within the same coalition but act selfishly or maliciously towards agents in other coalitions. In this work, they propose a network creation game for capturing the effect of social range matrix, and further explore how the social range matrix affects equilibria in a network game. In our view, the social range matrix is a useful tool to capture a complex behavior of agents in the network, especially when heterogeneous agents themselves decide the network structure. Investigating the applicability of social range matrix for the frameworks 3.1 and 3.2 is part of our future work.
References

Aumann RJ, Myerson RB (1988) Endogenous formation of links between players and of coalitions: an application of the shapley value. In: Roth AE (ed) The Shapley value, Cambridge University Press, pp 175–192

Bala V, Goyal S (2000a) A noncooperative model of network formation. Econometrica 68(5):1181–1229

Bala V, Goyal S (2000b) A strategic analysis of network reliability. Review of Economic Design 5(3):205–228

Belhaj M, Bramoullé Y, Frédérici (2014) Network games under strategic complementarities. Games and Economic Behavior

Belleflamme P, Bloch F (2004) Market sharing agreements and collusive networks*. International Economic Review 45(2):387–411

Billand P, Bravard C, Sarangi S (2011) Strict nash networks and partner heterogeneity. International Journal of Game Theory 40(3):515–525

Blackburn XZJ, Kourtellis N, Skvoretz J, Iamnitchi A (2014) The power of indirect ties in friend-to-friend storage systems. pp 1–5, DOI 10.1109/P2P.2014.6934314

Bloch F, Dutta B (2009) Communication networks with endogenous link strength. Games and Economic Behavior 66(1):39 – 56

Bloch F, Jackson MO (2007) The formation of networks with transfers among players. Journal of Economic Theory 133(1):83 – 110

Bramoullé Y, López-Pintado D, Goyal S, Vega-Redondo F (2004) Network formation and anti-coordination games. International Journal of Game Theory 33(1):1–19

Bramoullé Y, Kranton R, D’Amours M (2014) Strategic interaction and networks. American Economic Review 104(3):898–930

Calvó-Armengol A (2004) Job contact networks. Journal of Economic Theory 115(1):191 – 206

Cox LP, Murray CD, Noble BD (2002) Pastiche: Making backup cheap and easy. SIGOPS Oper Syst Rev 36(SI):285–298

Dutta B, Jackson MO (2003) Networks and Groups: Models of Strategic Formation, Springer Berlin Heidelberg, chap On the Formation of Networks and Groups, pp 1–15

Dutta B, Mutuswami S (1997) Stable networks. Journal of Economic Theory 76(2):322 – 344

Dutta B, Ghosal S, Ray D (2005) Farsighted network formation. Journal of Economic Theory 122(2):143 – 164

Falk A, Kosfeld M (2003) It’s all about connections: Evidence on network formation. Discussion papers 777, University of Zurich, IZA, URL http://www.econstor.eu/bitstream/10419/21462/1/dp777.pdf

Furusawa T, Konishi H (2007) Free trade networks. Journal of International Economics 72(2):310 – 335

Gilles RP, Sarangi S (2010) Network formation under mutual consent and costly communication. Mathematical Social Sciences 60(3):181–185

Goeree JK, Riedl A, Ule A (2009) In search of stars: Network formation among heterogeneous agents. Games and Economic Behavior 67(2):445 – 466
Goyal S (2012) Social networks on the web. In: Peitz M, Waldfogel J (eds) The
Oxford Handbook of the Digital Economy, Oxford University Press, pp 434–459
Goyal S, Joshi S (2003) Networks of collaboration in oligopoly. Games and Eco-
nomic Behavior 43(1):57 – 85
Goyal S, Joshi S (2006a) Bilateralism and free trade*. International Economic Re-
view 47(3):749–778
Goyal S, Joshi S (2006b) Unequal connections. International Journal of Game Theory
34(3):319–349
Goyal S, Moraga-González JL (2001) R&d networks. The RAND Journal of Eco-
nomics 32(4):686–707
Goyal S, Vega-Redondo F (2007) Structural holes in social networks. Journal of Eco-
nomic Theory 137(1):460 – 492
Gracia-Tinedo R, Artigas MS, López PG (2012a) Analysis of data availability in f2f
storage systems: When correlations matter. pp 225–236, DOI 10.1109/P2P.2012.
6335803
Gracia-Tinedo R, Sánchez-Artigas M, Garcia-Lopez P (2012b) F2box: Cloudifying
f2f storage systems with high availability correlation. pp 123–130, DOI 10.1109/
CLOUD.2012.22
Gracia-Tinedo R, Sánchez-Artigas M, Moreno-Martínez A, Garcia-Lopez P (2012c)
Friendbox: A hybrid f2f personal storage application. pp 131–138, DOI 10.1109/
CLOUD.2012.20
Hall P (1935) On representatives of subsets. J London Math Soc 10(1):26–30
Hummon NP (2000) Utility and dynamic social networks. Social Networks 22(3):221
– 249
Jackson MO (2005) A survey of network formation models: Stability and efficiency.
In: Demange G, Wooders M (eds) Group Formation in Economics, Cambridge
University Press, pp 11–57
Jackson MO, van den Nouweland A (2005) Strongly stable networks. Games and
Economic Behavior 51(2):420 – 444, special Issue in Honor of Richard D. McK-
elvey
Jackson MO, Wolinsky A (1996) A strategic model of social and economic networks.
Journal of Economic Theory 71(1):44 – 74
Landers M, Zhang H, Tan KL (2004) Peerstore: better performance by relaxing in
peer-to-peer backup. pp 72–79, DOI 10.1109/PTP.2004.1334933
Li J, Dabek F (2006) F2f: Reliable storage in open networks. URL http://
iptps06.cs.ucsb.edu/
Lillibridge M, Elnikety S, Birrell A, Burrows M, Isard M (2003) A cooperative inter-
net backup scheme. In: Proceedings of the Annual Conference on USENIX Annual
Technical Conference, USENIX Association, Berkeley, CA, USA, ATEC ’03, pp
3–3, URL http://dl.acm.org/citation.cfm?id=1247340.1247343
Marcelo, Cirne W, Brasilheiro F, Guerrero D (2008) On the impact of the data
redundancy strategy on the recoverability of friend-to-friend backup systems.
In: 26th Brazilian Symposium on Computer Networks and Distributed Sys-
tems, URL http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.
1.1.333.1732&rep=rep1&type=pdf
Meier D, Pignolet YA, Schmid S, Wattenhofer R (2014) On the windfall and price of friendship: Inoculation strategies on social networks. Computer Networks 62:221–236

Moreno-Martínez A, Gracia-Tinedo R, Sánchez-Artigas M, García-Lopez P (2012) Friendbox: A cloudified f2f storage application. pp 75–76, DOI 10.1109/P2P.2012.635818

Myerson RB (1977) Graphs and cooperation in games. Mathematics of Operations Research 2(3):225–229

Nguyen TD, Li J (2007) Blockparty: Cooperative ofﬁsite backup among friends. URL http://www.news.cs.nyu.edu/~trandinh/publications/blockparty_poster.pdf

Sharma R, Datta A, DeH’Amico M, Michiardi P (2011) An empirical study of availability in friend-to-friend storage systems. pp 348–351, DOI 10.1109/P2P.2011.6038754

Surajit B, Loyimee G, Sudipta S (2014) A survey of player-based and link-based allocation rules for network games. Studies in Microeconomics 2(1):5–26

Tennekes M (2010) Network formation games. Maastricht University, URL http://digitalarchive.maastrichtuniversity.nl/fedora/get/guid:6cb420d8-aeb4-4122-a813-cad7717f9ac6/ASSET1

Tran N, Chiang F, Li J (2012) Efﬁcient cooperative backup with decentralized trust management. ACM Transactions on Storage 8(3):8:1–8:25

Weatherspoon H, Kubiatowicz JD (2002) Peer-to-Peer Systems, Berlin Heidelberg: Springer, chap Erasure Coding Vs. Replication: A Quantitative Comparison, pp 328–337

Yann B, Habiba D, Bernard F (2009) Identiﬁcation of peer effects through social networks. Journal of Econometrics 150(1):41–55

Zirulia L, et al (2006) Industry proﬁt maximizing r and d networks. Economics Bulletin 12(1):1–6