Contribution of the $a_1$ meson to the axial nucleon-to-$\Delta$ transition form factors

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Abstract

We analyze the low-$Q^2$ behavior of the axial form factor $G_A(Q^2)$, the induced pseudoscalar form factor $G_P(Q^2)$, and the axial nucleon-to-$\Delta$ transition form factors $C_{5A}^A(Q^2)$ and $C_{6A}^A(Q^2)$. Building on the results of chiral perturbation theory, we first discuss $G_A(Q^2)$ in a chiral effective-Lagrangian model including the $a_1$ meson and determine the relevant coupling parameters from a fit to experimental data. With this information, the form factor $G_P(Q^2)$ can be predicted. For the determination of the transition form factor $C_{5A}^A(Q^2)$ we make use of an SU(6) spin-flavor quark-model relation to fix two coupling constants such that only one free parameter is left. Finally, the transition form factor $C_{6A}^A(Q^2)$ can be predicted in terms of $G_P(Q^2)$, the mean-square axial radius $\langle r_A^2 \rangle$, and the mean-square axial nucleon-to-$\Delta$ transition radius $\langle r_{AN\Delta}^2 \rangle$. 

I. INTRODUCTION

At the fundamental level, the electroweak form factors of hadrons originate from the dynamics of the constituents of quantum chromodynamics (QCD), namely, quarks and gluons. While a wealth of precision data exists for the electromagnetic form factors of the proton and, to a lesser extent, of the neutron (see, e.g., Refs. [1, 2] for a review), the nucleon form factors of the isovector axial-vector current, the axial form factor $G_A$ and, in particular, the induced pseudoscalar form factor $G_P$, are not as well known (see, e.g., Refs. [3, 4] for a review). A similar situation occurs in the case of the nucleon-to-$\Delta$ transition form factors. A considerable amount of data is available for the electromagnetic transition form factors (see, e.g., Refs. [5, 6] for a review), whereas very little is known about the axial nucleon-to-$\Delta$ transition form factors [7–11]. On the theoretical side, there have been various approaches to determining the nucleon-to-$\Delta$ transition form factors. Calculations have been performed in the framework of quark models [12–16], chiral effective field theory [17–19], lattice QCD [20–22], and light-cone QCD sum rules [23, 24]. Moreover, a substantial amount of work has been devoted to the question of how to parametrize and extract the form factors from experimental data [25–34].

In this article, based on the results of Ref. [35], we make use of a semi-phenomenological description of the nucleon axial form factor $G_A$ and the induced pseudoscalar form factor $G_P$ to predict, using certain model assumptions, two of the four axial nucleon-to-$\Delta$ transition form factors, namely, $C_A^5$ and $C_A^6$. We will assume that the exchange of the axial-vector meson $a_1(1260)$ provides a dominant contribution to the form factor $C_A^5$ at low values of $Q^2$. Such a scenario was already envisaged decades ago in Ref. [36], where the common use of a dipole form was questioned.

II. AXIAL-VECTOR CURRENT OPERATOR IN QCD

In terms of the up-quark and down-quark fields,

$$q(x) = \left( \begin{array}{c} u(x) \\ d(x) \end{array} \right),$$

the Cartesian components of the isovector axial-vector current operator are defined as

$$A_\mu^j(x) = \bar{q}(x) \gamma^\mu \gamma_5 \tau_j \frac{1}{2} q(x).$$  \hspace{1cm} (1)$$

In the isospin-symmetric limit, $m_u = m_d = \hat{m}$, the divergence of the isovector axial-vector current is given by

$$\partial_\mu A_\mu^j = i\hat{m} \bar{q} \gamma_5 \tau_j q \equiv \hat{m} P_j,$$

where $P_j$ is the $j$th component of the pseudoscalar quark density. After coupling external c-number axial-vector fields $a_{\mu j}(x)$ to the axial-vector current operators $A_\mu^j(x)$ [37],

$$\mathcal{L}_{\text{ext}} = \sum_{j=1}^{3} a_{\mu j}(x) A_\mu^j(x),$$  \hspace{1cm} (3)$$

the invariant amplitude for a transition from a hadronic state $|A(p_i)\rangle$ to $|B(p_f)\rangle$, induced by a plane-wave external field of the form $a_{\mu j}(x) = \epsilon_{\mu j}(q) e^{-iq\cdot x}$, is defined as (no summation
over $j$ implied)
\[
\mathcal{M} = i\epsilon_{\mu j}(q) \langle B(p_f)|A_{\mu}^{(1)}(0)|A(p_i)\rangle,
\]
where four-momentum conservation $p_f = p_i + q$ due to translational invariance is implied.

### III. PARAMETRIZATION OF THE NUCLEON-TO-NUCLEON AND NUCLEON-TO-$\Delta$ TRANSITIONS

The axial-vector current matrix element between nucleon states can be parametrized as \[35\]
\[
\langle N(p_f, s_f)|A_{\mu}^{(0)}(0)|N(p_i, s_i)\rangle = \bar{u}(p_f, s_f) \left[ \gamma^\mu \gamma_5 G_A(Q^2) + \frac{q^\mu}{2m_N} \gamma_5 G_P(Q^2) \right] \frac{\tau_j}{2} u(p_i, s_i),
\]
where $q = p_f - p_i$, $Q^2 = -q^2$, and $m_N$ is the nucleon mass. The Pauli matrix $\tau_j$ has to be evaluated between nucleon isospinors. At $Q^2 = 0$, the axial form factor reduces to the axial-vector coupling constant $g_A = 1.2723 \pm 0.0023$ \[38\]. At $Q^2 = m_\mu^2$, where $m_\mu$ is the muon mass, the induced pseudoscalar coupling constant is defined as
\[
g_p = \frac{m_\mu}{2m_N} G_P(m_\mu^2).
\]

Recently, the MuCap Collaboration obtained $g_\mu = 8.06 \pm 0.55$ \[39\], which is in very good agreement with the result of chiral perturbation theory \[40, 41\], $g_\mu = 8.26 \pm 0.23$ \[31\].

Introducing the spherical tensor notation \[42\],
\[
A_{\pm 1}^{(1)} = \mp \frac{1}{\sqrt{2}} (A_1^{\mu} \pm iA_2^{\mu}), \quad A^{(1)}_0 = A_3^{\mu},
\]
and using isospin symmetry, we express the matrix element of the spherical isospin components ($\alpha = +1, 0, -1$) between a nucleon state and a $\Delta$ state as
\[
\langle 3/2, \tau_\Delta|A^{(1)}_{j\alpha}(1/2, \tau)|1/2, \tau\rangle = (1/2, \tau; 1, \alpha|3/2, \tau_\Delta)\langle 3/2||A^{(1)}\rangle|1/2\rangle,
\]
where $\langle 3/2||A^{(1)}\rangle|1/2\rangle$ denotes the reduced matrix element and $(1/2, \tau; 1, \alpha|3/2, \tau_\Delta)$ is the relevant Clebsch-Gordan coefficient. For example, using $(1/2, 1/2; 1, 0|3/2, 1/2) = \sqrt{2}/3$, we obtain the reduced matrix element in terms of the $p$ to $\Delta^+$ transition as
\[
\langle 3/2||A^{(1)}\rangle|1/2\rangle = \frac{\sqrt{3}}{2} \langle \Delta^+|A^{(1)}_0|p\rangle.
\]

The Lorentz structure of the reduced matrix element may be written as
\[
\langle \Delta(p_f, s_f)||A^{(1)}(0)||N(p_i, s_i)\rangle = \bar{w}_\lambda(p_f, s_f) \Gamma^\lambda_A u(p_i, s_i).
\]

Here, the initial nucleon is described by the Dirac spinor $u(p_i, s_i)$ with real mass $m_N$ and $p_i^2 = m_N^2$, the final $\Delta(1232)$ is described via the Rarita-Schwinger vector-spinor $\bar{w}_\lambda(p_f, s_f)$.

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1 The result of older experiments has been somewhat under debate (see Table II of Ref. [4]) with a world average of $g_\mu = 10.5 \pm 1.8$ of all ordinary muon capture experiments.
with a complex mass $z_\Delta$ and $p_f^2 = z_\Delta^2$. In the following, it is always understood that the “tensor” $\Gamma_A^\mu$ is evaluated between on-shell spinors $u$ and $\bar{w}_\lambda$, satisfying
\begin{equation}
\bar{w}_\lambda(p_f, s_f) \bar{p}_i u(p_i, s_i) = m_N u(p_i, s_i),
\end{equation}
\begin{equation}
\bar{w}_\lambda(p_f, s_f) p_f = z_\Delta \bar{w}_\lambda(p_f, s_f), \quad \bar{w}_\lambda(p_f, s_f) \gamma^\lambda = 0, \quad \bar{w}_\lambda(p_f, s_f)p_f^\lambda = 0.
\end{equation}

The expressions for a stable $\Delta$ resonance are obtained via the replacement $z_\Delta \rightarrow m_\Delta$. The “tensor” $\Gamma_A^\mu$ contains a superposition of four Lorentz tensors \[25, 26\], which we choose to be \[19, 21\]
\begin{equation}
\Gamma_A^\mu = \frac{C_3^A(Q^2)}{m_N} (g^\mu \gamma^s - q^\mu \gamma^\lambda) + \frac{C_4^A(Q^2)}{m_N^2} (g^\lambda p_f \cdot q - q^\lambda p_f^\mu) + C_5^A(Q^2) g^\lambda + \frac{C_6^A(Q^2)}{m_N^2} q^\lambda q^\mu.
\end{equation}

Note that $p_f \cdot q = \frac{1}{2}(p_f + p_i + p_f - p_i) \cdot (p_f - p_i) = \frac{1}{2}(p_f^2 - p_i^2 + q^2) = \frac{1}{2}(z_\Delta^2 - m_\Delta^2 + q^2)$. In particular, $C_5^A$ and $C_6^A$ correspond to the axial nucleon form factor $G_A$ and the induced pseudoscalar form factor $G_P$, respectively.

### IV. AXIAL-VECTOR COUPLING CONSTANTS IN THE STATIC QUARK MODEL

Here, we recall an SU(6) spin-flavor quark-model relation, which will be applied in the subsequent calculations. In the static quark model, the operator $A_{x,3}$ is given by
\begin{equation}
A_{x,3} = \frac{1}{2} \sum_{i=1}^{3} \tau_3(i) \sigma_z(i).
\end{equation}

The axial-vector coupling constant is obtained as
\begin{equation}
\langle p, S_z = 1/2 | A_{x,3} | p, S_z = 1/2 \rangle = \frac{1}{2} g_A.
\end{equation}

Inserting the appropriate quark-model wave function,
\begin{equation}
| p, S_z = 1/2 \rangle = \frac{1}{\sqrt{18}} \left[ 2(u \uparrow u \uparrow d \downarrow + u \uparrow d \downarrow u \uparrow d \downarrow u \uparrow u \uparrow) \\
- (u \uparrow u \downarrow d \uparrow + u \downarrow u \uparrow d \uparrow u \uparrow d \uparrow u \downarrow) \\
+ d \uparrow u \uparrow u \downarrow + u \downarrow u \uparrow d \uparrow u \uparrow d \uparrow u \downarrow \right],
\end{equation}
one obtains
\begin{equation}
g_A = 2 \langle p, S_z = 1/2 | A_{x,3} | p, S_z = 1/2 \rangle = 3 \langle p, S_z = 1/2 | \tau_3(3) \sigma_z(3) | p, S_z = 1/2 \rangle = \frac{5}{3}.
\end{equation}

On the other hand, evaluating Eq. (5) for $\bar{p}_i = \bar{p}_f = \bar{0}$ and $S_{zi} = S_{zf} = 1/2$ yields
\begin{equation}
\bar{u}^{(1)}(\bar{0}) \gamma^3 \gamma_5 g_A u^{(1)}(\bar{0}) \times \frac{1}{2} = 2m_N \frac{g_A}{2} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix} = 2m_N \frac{g_A}{2}.
\end{equation}

\[2\] The explicit form of $\bar{w}_\lambda$ can be found in Ref. \[15\].
The factor $2m_N$ originates from our normalization of the Dirac spinors (see Appendix A). When comparing the expression of Eq. (16) to the quark-model result of Eq. (13), we have to discard this factor.

Using
\[ |\Delta^+, S_z = 1/2\rangle = \frac{1}{3}(u \uparrow u \uparrow d \downarrow + u \uparrow d \downarrow u \uparrow + d \downarrow u \uparrow u \uparrow \]
\[ + u \uparrow u \downarrow d \uparrow + u \downarrow u \uparrow d \uparrow + u \uparrow d \downarrow + u \downarrow d \uparrow + u \downarrow d \downarrow + u \uparrow u \downarrow + d \uparrow u \downarrow + d \uparrow u \downarrow + d \downarrow u \uparrow u \uparrow \]
(17)

together with Eq. (14), one obtains for the nucleon-to-\(\Delta\) axial-vector transition
\[ \langle \Delta^+, S_z = 1/2 | A_{z3} | p, S_z = 1/2 \rangle = \frac{3}{2} \langle \Delta^+, S_z = 1/2 | r_3(3) \sigma_z(3) | p, S_z = 1/2 \rangle \]
\[ = \frac{2}{3} \sqrt{2} = \frac{5}{3} \sqrt{2} = \frac{2}{5} \sqrt{2} g_A. \]
(18)

V. CONNECTION TO CHIRAL EFFECTIVE FIELD THEORY

At lowest order in the quark-mass and momentum expansion, the relevant interaction Lagrangian for nucleons reads [46]
\[ \mathcal{L}_{\text{int}} = \frac{g_A}{2} \bar{\Psi} \gamma^\mu \gamma_5 u_\mu \Psi, \]
(19)
where $g_A$ is the chiral limit of the axial-vector coupling constant and
\[ \Psi = \begin{pmatrix} p \\ n \end{pmatrix} \]
(20)
denotes the nucleon field with two four-component Dirac fields for the proton and the neutron. The so-called chiral vielbein $u_\mu$ (see Chapt. 4 of Ref. [47] for a detailed discussion),
\[ u_\mu = i[u^\dagger (\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger], \]
involves the external fields $r_\mu$ and $l_\mu$ as well as pions. The latter are contained in the unimodular, unitary, $(2 \times 2)$ matrix $u$:
\[ u(x) = \exp \left( i \frac{\Phi(x)}{2F} \right), \]
\[ \Phi(x) = \sum_{i=1}^{3} \tau_i \phi_i(x) = \begin{pmatrix} \pi^0(x) \sqrt{2} \pi^+(x) \\ \sqrt{2} \pi^-(x) - \pi^0(x) \end{pmatrix}, \]
(21)
where $F$ denotes the pion-decay constant in the chiral limit: $F_\pi = F [1 + O(\hat{m})] = 92.2$ MeV.

Making use of the replacement $r_\mu = -l_\mu = a_\mu$ and keeping only the leading term of the expansion $u = 1 + \cdots$, the chiral vielbein reduces to
\[ u_\mu \rightarrow u^\dagger (i\partial_\mu + a_\mu)u - u(i\partial_\mu - a_\mu)u^\dagger \rightarrow 2a_\mu. \]
where

\[ a_\mu = \sum_{j=1}^{3} a_{\mu j} \tau_j / 2. \]

The relevant interaction Lagrangian is then given by

\[ \mathcal{L}_{\text{int}} = \sum_{j=1}^{3} a_{\mu j} \frac{g_A}{2} \bar{\Psi} \gamma^\mu \gamma_5 \tau_j \Psi. \]  

(22)

The invariant amplitude for \[ a_{\mu j}(x) = \epsilon^\mu_{\mu j}(q) e^{-iq \cdot x}, \] with \( j \) fixed, reads

\[ \mathcal{M} = i \epsilon_{\mu j}(q) g_A \bar{u}(p_f) \gamma^\mu \gamma_5 \tau_j / 2 u(p_i). \]

A comparison with Eqs. (4) and (5) yields

\[ G_A(Q^2) = g_A. \]  

(23)

At lowest order, there is no \( Q^2 \) dependence and \( G_A(Q^2) \) reduces to the axial-vector coupling constant in the chiral limit.

For the nucleon-to-\( \Delta \) transition the lowest-order Lagrangian is given by [see Eq. (4.200) of Ref. [47] with \( \tilde{z} = -1 \)]

\[ \mathcal{L}_{\pi N \Delta}^{(1)} = g \Psi_\lambda,i \xi_{3ij}^g (g^{\lambda \mu} - \gamma^\lambda \gamma^\mu) u_{\mu j} \Psi + \text{H.c.} \rightarrow g \bar{\Psi}_\lambda,i \xi_{3ij}^g (g^{\lambda \mu} - \gamma^\lambda \gamma^\mu) a_{\mu j} \Psi + \text{H.c.,} \]  

(24)

where \( \Psi_\lambda,i \) denotes a vector-spinor isovector-isospinor field and \( \xi_{3ij}^g \) is a matrix representation of the projection operator for the isospin-\( 3/2 \) component of the fields. Considering \( j = 3 \) and making use of Eq. (4.184) of Ref. [47],

\[ \bar{\Psi}_\lambda,i \xi_{3ij}^g = \sqrt{\frac{2}{3}} (\bar{\Delta}_i^+ - \bar{\Delta}_j^0), \]

we obtain

\[ \mathcal{L}_{\text{int}} = \sqrt{\frac{2}{3}} g (\bar{\Delta}_i^+ - \bar{\Delta}_j^0) (g^{\lambda \mu} - \gamma^\lambda \gamma^\mu) a_{\mu j} \left( \frac{p}{n} \right) + \text{H.c.} \]

Using Eq. (11), the invariant amplitude of \( p \rightarrow \Delta^+ \) reads

\[ \mathcal{M} = i \sqrt{\frac{2}{3}} g \bar{w}_\lambda(p_f, s_f) g^{\lambda \mu} u(p_i, s_i) \epsilon_{\mu 3}(q). \]

The reduced matrix element [see Eq. (8)] is obtained by multiplying by \( \sqrt{3/2} \) and crossing out the factors \( i \) and \( \epsilon_{\mu 3}(q) \):

\[ \bar{w}_\lambda(p_f, s_f) \Gamma A^{\lambda \mu} u(p_i, s_i) = g \bar{w}_\lambda(p_f, s_f) g^{\lambda \mu} u(p_i, s_i). \]  

(25)

A comparison with Eq. (12) then yields the analogue of Eq. (23), namely,

\[ C_A^5(Q^2) = g. \]  

(26)
Finally, how is this related to the static quark model? For this purpose, we consider

$$\langle \Delta^+(0), S_z = 1/2 | A_{z,3}(0) | p(0), S_z = 1/2 \rangle = -\sqrt{\frac{2}{3}} g\bar{w}_3(0, S_z = 1/2) u(0, S_z = 1/2),$$

where we made use of $g^{33} = -1$. Since $\epsilon_{3,3} = -1$ and $\epsilon_{1,3} + i\epsilon_{2,3} = 0$, we obtain in terms of the appropriate Clebsch-Gordan coefficients,

$$w_3(0, S_z = 1/2) = \sqrt{\frac{2}{3}} \epsilon_{3,3} u(0, S_z = 1/2) + \frac{1}{\sqrt{3}} \left(-\frac{1}{\sqrt{2}} (\epsilon_{1,3} + i\epsilon_{2,3}) \right) u(0, S_z = -1/2)$$

$$= -\sqrt{\frac{2}{3}} u(0, S_z = 1/2).$$

Putting the pieces together, the matrix element is given by

$$\langle \Delta^+(0), S_z = 1/2 | A_{z,3}(0) | p(0), S_z = 1/2 \rangle = -\sqrt{\frac{2}{3}} g(-1)\sqrt{\frac{2}{3}} \bar{u}(0, S_z = 1/2) u(0, S_z = 1/2)$$

$$= g\sqrt{2} m_\Delta \sqrt{2} m_N. \quad (27)$$

Again, when we compare this to Eq. (18) for the static quark model, we have to cross out the normalization factors $\sqrt{2} m_\Delta$ and $\sqrt{2} m_N$. In combination with Eq. (23) we obtain

$$\frac{2}{3} g = \frac{2}{5} \sqrt{2} g_A,$$

or

$$g = \frac{3}{5} \sqrt{2} g_A. \quad (28)$$

In Table I we collect the numerical values of the masses and coupling constants which are taken as fixed in the subsequent calculations.

| TABLE I. Masses and coupling constants. |
|----------------------------------------|
| Pion mass                              | $M_\pi = 139.57$ MeV                    |
| Nucleon mass                           | $m_N = 938.92$ MeV                      |
| $a_1$ mass                             | $M_{a_1} = 1260$ MeV                    |
| Pion-decay constant                    | $F_\pi = 92.2$ MeV                      |
| Axial-vector coupling constant         | $g_A = 1.2723$                         |
| Pion-nucleon coupling constant         | $g_{\pi N}^2/(4\pi) = 13.69$            |
VI. INCLUSION OF THE $a_1$ AXIAL-VECTOR MESON

The vector mesons $\rho$ and $\omega$ play an important role in the description of the electromagnetic form factors of the nucleon in chiral effective field theory [48–50]. Similarly, the $a_1$ axial-vector meson leads to an improved description of the axial form factor $G_A$ [35]. Moreover, in the $\gamma^*N \rightarrow \Delta$ transition, the contribution of the $\rho$ meson is needed to obtain a good description of the experimental data [51]. Even though we have practically no data for the axial $N\Delta$ transition, we expect that the $a_1$ meson plays a similar role as in the nucleon case. For that reason, we discuss the relevant Lagrangians and calculate their contribution to the form factors.

A. Nucleon

The Lagrangian for the interaction of the $a_1$ meson with the building block $f_{-\mu\nu}$ is given by [see Eq. (52) of Ref. [52]]

$$-\frac{1}{4} f_A \langle A^{\mu\nu} f_{-\mu\nu} \rangle,$$

(29)

where $\langle \ldots \rangle$ denotes $\text{Tr}(\ldots)$ and

$$A^{\mu\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu,$$

$$\nabla^\mu A^\nu = \partial^\mu A^\nu + [\Gamma^\mu, A^\nu],$$

$$\Gamma^\mu = \frac{1}{2} [u^\dagger (\partial^\mu - ir^\mu) u + u (\partial^\mu - il^\mu) u^\dagger],$$

$$f_{-\mu\nu} = uf_{L\mu\nu} u^\dagger - u^\dagger f_{R\mu\nu} u,$$

$$f_{L\mu\nu} = \partial_{l\mu} l^\nu - \partial_{l\nu} l^\mu - i[l_{\mu}, l_{\nu}],$$

$$f_{R\mu\nu} = \partial_{r\mu} r^\nu - \partial_{r\nu} r^\mu - i[r_{\mu}, r_{\nu}].$$

In comparison with Ref. [52], we omit the roof sign, i.e., we write $A^\mu$ instead of $\hat{A}^\mu$. Moreover, we introduce an additional factor $1/\sqrt{2}$, because our normalization of the field matrix is

$$A_\mu = \sum_{i=1}^{3} A_{\mu i} \tau_i,$$

whereas Ecker et al. use [see Eq. (3.4) of Ref. [53]]

$$A_\mu = \frac{1}{\sqrt{2}} \sum_{i=1}^{3} A_{\mu i} \tau_i.$$

The replacement

$$r^\mu \rightarrow a^\mu, \quad l^\mu \rightarrow -a^\mu, \quad f_{-\mu\nu} \rightarrow -2(\partial^\mu a^\nu - \partial^\nu a^\mu),$$

results in the interaction Lagrangian

$$\frac{1}{2} f_A \langle A^{\mu\nu}(\partial_\mu a_\nu - \partial_\nu a_\mu) \rangle = \frac{f_A}{2} (\partial^\mu A^\nu_i - \partial^\nu A^\mu_i)(\partial_\mu a_{\nu i} - \partial_\nu a_{\mu i}).$$
The invariant amplitude for the coupling of an incoming external axial source with four-momentum $q$, polarization vector $\epsilon$, and isospin component 3 to an outgoing $a_1$ meson with four-momentum $q$, polarization vector $\epsilon_A$, and isospin component 3 reads

$$\mathcal{M} = i\frac{f_A}{2}(iq^\mu \epsilon_A^{\mu*} - iq^\nu \epsilon_A^{\nu*})(-iq_{\nu}\epsilon_{\nu} + iq_{\mu}\epsilon_{\mu}) = if_A\epsilon^{*}_{\mu}(q^2g^{\nu\mu} - q^\nu q^\mu)\epsilon_{\mu}. \quad (30)$$

The lowest-order Lagrangian for the interaction of the $a_1$ meson with the nucleon is given by [see Eq. (20) of Ref. [35]]

$$\mathcal{L}_{a_1N} = \frac{g_{a_1N}}{2}\bar{\Psi}\gamma^\mu\gamma_5 A_\mu \Psi. \quad (31)$$

The corresponding Feynman rule for the absorption of an $a_1$ meson with isospin index $i$ reads

$$ig_{a_1N}^2 \gamma^\mu\gamma_5 \tau_i.$$
In comparison to Eqs. (46) and (47) of Ref. [35], we obtain the opposite sign.

In practice, the loop diagrams play no role in the one-loop calculation of the axial form factor \(G_A\). The low-\(Q^2\) behavior is encoded in the two constants \(g_A\) and \(\langle r_A^2 \rangle\) which chiral symmetry does not predict:

\[
G_A^{\text{linear}}(Q^2) = g_A \left( 1 - \frac{1}{6} \langle r_A^2 \rangle Q^2 \right). \tag{34}
\]

Experimental data are commonly analyzed in terms of the dipole parametrization,

\[
G_A^{\text{dipole}}(Q^2) = \frac{g_A}{\left( 1 + \frac{Q^2}{M_A^2} \right)^2}, \tag{35}
\]

where the parameter \(M_A\) is referred to as the axial mass. The weighted average extracted from (quasi)elastic neutrino and antineutrino scattering experiments is \(M_A = (1.026 \pm 0.021)\) GeV [3] corresponding to a mean-square axial radius \(\langle r_A^2 \rangle = (0.444 \pm 0.018)\) fm\(^2\). A subsequent re-analysis of quasielastic data on deuterium has reported \(M_A = (1.016 \pm 0.026)\) GeV [\(\langle r_A^2 \rangle = (0.453 \pm 0.023)\) fm\(^2\)] [24]. The results of more recent experiments on (quasi)elastic neutrino and antineutrino scattering experiments are summarized in Table II.

For a discussion of theoretical studies abandoning the dipole form in their analyses, see, e.g., Refs. [60–64]. The weighted average extracted from charged pion electroproduction experiments is \(M_A = (1.069 \pm 0.016)\) GeV [3] resulting in \(\langle r_A^2 \rangle = (0.409 \pm 0.012)\) fm\(^2\).

### Table II. Axial masses and mean-square axial radii obtained from recent (quasi)elastic neutrino and antineutrino scattering experiments.

| Experiment | \(M_A\) [GeV] | \(\langle r_A^2 \rangle\) [fm\(^2\)] |
|------------|---------------|----------------------------------|
| K2K [55]   | 1.20 ± 0.12   | 0.32 ± 0.06                     |
| NOMAD [56] | 1.05 ± 0.06   | 0.42 ± 0.05                     |
| MiniBooNE [57] | 1.35 ± 0.17   | 0.26 ± 0.06                     |
| MINERvA [58] | 0.99          | 0.48                            |
| MINOS [59] | 1.23\(^{+0.13}_{-0.09}\)(fit)\(^{+0.12}_{-0.15}\)(syst) | 0.31\(^{+0.07}_{-0.05}\)(fit)\(^{+0.06}_{-0.08}\)(syst) |

Including the \(a_1\) meson, the axial form factor may be written as

\[
G_A(Q^2) = g_A + c_1 Q^2 + c_2 \frac{Q^2}{M_{a_1}^2 + Q^2}
= g_A \left[ 1 + \tilde{c}_1 Q^2 - \tilde{c}_2 \frac{(Q^2)^2}{M_{a_1}^2 (M_{a_1}^2 + Q^2)} \right], \tag{36}
\]

where \(g_A \tilde{c}_1 = c_1 + c_2 / M_{a_1}^2\) and \(g_A \tilde{c}_2 = c_2 = f_A g_{a_1 N}\). Introducing the normalized axial form factor as

\[
F_A(Q^2) = \frac{G_A(Q^2)}{G_A(0)}, \tag{38}
\]
the parametrization of $F_A(Q^2)$ contains two parameters, namely, $\tilde{c}_1$ and $\tilde{c}_2$, which can be determined from a fit to experimental data. Expanding the normalized axial form factor as

$$F_A(Q^2) = 1 - \frac{1}{6} \langle r_A^2 \rangle Q^2 + \frac{1}{120} \langle r_A^4 \rangle (Q^2)^2 + \cdots,$$

(39)

for the parametrization including the $a_1$ meson, Eq. (37), we can identify the mean-square and mean-quartic axial radii as

$$\langle r_A^2 \rangle = -6\tilde{c}_1, \quad \langle r_A^4 \rangle = -120 \frac{\tilde{c}_2}{M_{a_1}^4},$$

(40)

respectively. On the other hand, for the dipole parametrization, Eq. (35), one obtains

$$\langle r_A^2 \rangle = \frac{12}{M_A^2}, \quad \langle r_A^4 \rangle = \frac{360}{M_A^4}.$$

(41)

Figure 2 shows the results of fitting the dipole parametrization to experimental data extracted from pion electroproduction experiments [3]. The fits are performed for different values of the maximal squared momentum transfer, $Q_{\text{max}}^2$, and the corresponding axial masses, mean-square axial radii, and mean-quartic axial radii are summarized in Table III. In their common domain, the curves associated with $Q_{\text{max}}^2 = 0.6$ GeV$^2$ and $Q_{\text{max}}^2 = 1$ GeV$^2$ are hardly distinguishable in Fig. 2 because the difference between the fitted axial masses is very small.

**TABLE III.** Comparison of the axial masses, mean-square axial radii, and mean-quartic axial radii obtained from the dipole expression of the form factor $F_A$ fitted to different ranges of momentum transfer.

| $Q_{\text{max}}^2$ [GeV$^2$] | $M_A$ [GeV] | $\langle r_A^2 \rangle$ [fm$^2$] | $\langle r_A^4 \rangle$ [fm$^4$] | $\chi^2_{\text{red}}$ |
|-----------------------------|-------------|-------------------------------|-------------------------------|-----------------|
| 0.24                       | 1.057 ± 0.027 | 0.418 ± 0.021 | 0.437 ± 0.045 | 2.87 |
| 0.6                        | 1.084 ± 0.020 | 0.398 ± 0.015 | 0.395 ± 0.029 | 3.21 |
| 1.0                        | 1.082 ± 0.019 | 0.399 ± 0.014 | 0.398 ± 0.028 | 2.97 |

Figure 3 shows the corresponding fits using the parametrization of Eq. (37) including the $a_1$ meson ($a_1$ fits for short). The respective parameters $\tilde{c}_1$ and $\tilde{c}_2$, mean-square axial radii, and mean-quartic axial radii are summarized in Table IV. When comparing the $a_1$ fit to the dipole fit, one should keep in mind that Eq. (37) represents a model for the low-$Q^2$ behavior of the axial form factor with a restricted domain of validity. The fits of Fig. 3 share the common feature that $F_A$, when extrapolated beyond $Q_{\text{max}}^2$, very soon starts to rise again

3 We would like to thank U.-G. Meißner for providing the data in the form of a table.

4 Strictly speaking, because of a loop correction to the threshold electric dipole amplitude $E_{0+}$, the mean-square axial radius extracted from pion electroproduction has to be modified by an amount

$$\frac{3}{64 F_{\pi}^2} \left( \frac{12}{\pi^2} - 1 \right) = 0.0456 \text{fm}^2,$$

such that the true axial radius is slightly larger [3, 65]. This is consistent with the observation that the average for $M_A$ extracted from charged pion electroproduction experiments is larger than the value from (quasi)elastic neutrino and antineutrino scattering experiments.
and diverges as $Q^2 \to \infty$. This is, of course, an unphysical feature, originating from the linear term proportional to $c_1$ in Eq. (36). Moreover, the $a_1$ contribution asymptotically does not fall off as $1/(Q^2)^2$ as predicted by perturbative QCD [66]. Guided by the fact that the dipole fit and the $a_1$ fit produce very similar results for $Q^2_{\text{max}} = 0.6$ GeV$^2$, we will assume that this value provides a reasonable upper limit for the range of applicability of the $a_1$ model. According to Eqs. (36) and (40), the mean-square axial radius obtains a contribution from both the low-energy constant (LEC) $c_1$ and the $a_1$-pole diagram (see Fig. 1). For the values of $\tilde{c}_1$ and $\tilde{c}_2$ of Table IV, the $a_1$ contribution to $\langle r_A^2 \rangle$ is larger than the total result, implying a negative contribution from the LEC $c_1$. To be specific, for $Q^2_{\text{max}} = 0.6$ GeV$^2$ we obtain $\langle r_A^2 \rangle_{\text{LEC}} + \langle r_A^2 \rangle_{a_1} = (-0.366 + 0.781) \text{fm}^2 = 0.415 \text{fm}^2$.

At order $\mathcal{O}(p^3)$ in chiral perturbation theory, the low-$Q^2$ behavior of the induced pseudoscalar form factor $G_P(Q^2)$ can entirely be written in terms of known physical quantities [35, 40],

$$G_P(Q^2) = \frac{4m_N F_\pi g_{\pi N}}{M_\pi^2 + Q^2} - \frac{2}{3} m_N^2 g_A \langle r_A^2 \rangle,$$

(42)

where $g_{\pi N}$ denotes the pion-nucleon coupling constant with $g_{\pi N}^2/(4\pi) = 13.69 \pm 0.19$ [67]. Using Eq. (33), the relevant expression including the $a_1$ meson reads

$$G_P(Q^2) = 4 m_N F_\pi g_{\pi N} \frac{M_\pi^2 + Q^2}{M_\pi^2 + Q^2} - 2 m_N^2 g_A \langle r_A^2 \rangle - 4 m_N^2 g_A \tilde{c}_2 \frac{Q^2}{M_{a_1}^2 (M_{a_1}^2 + Q^2)},$$

(43)

Note that the dipole form shows this behavior.
where the mean-square axial radius is given in Eq. (40). In Fig. 4, we compare the results for \( G_P(Q^2) \) including the \( a_1 \) contribution (solid line) and without the \( a_1 \) contribution (dashed line). Clearly, at low \( Q^2 \), the form factor is dominated by the pion-pole contribution and a deviation due to the \( a_1 \) meson is only seen for larger values of \( Q^2 \), where the form factor is small.

**TABLE IV.** Comparison of the parameters \( \tilde{c}_1 \) and \( \tilde{c}_2 \), mean-square axial radii, and mean-quartic axial radii obtained from the expression Eq. (36) of the form factor \( F_A \) fitted to different ranges of momentum transfer.

| \( Q^2_{\text{max}} \) [GeV\(^2\)] | \( \tilde{c}_1 \) [GeV\(^{-2}\)] | \( \tilde{c}_2 \) | \( \langle r_A^2 \rangle \) [fm\(^2\)] | \( \langle r_A^4 \rangle \) [fm\(^4\)] | \( \chi^2_{\text{red}} \) |
|-------------------------------|-----------------|-------------|-----------------|-----------------|-------------|
| 0.24                          | \(-2.44 \pm 0.32\) | \(-14.8 \pm 4.4\) | 0.570 \( \pm 0.075\) | 1.068 \( \pm 0.318\) | 2.08        |
| 0.6                           | \(-1.78 \pm 0.09\) | \(-5.31 \pm 0.69\) | 0.416 \( \pm 0.021\) | 0.383 \( \pm 0.050\) | 2.68        |
| 1.0                           | \(-1.61 \pm 0.07\) | \(-3.84 \pm 0.39\) | 0.376 \( \pm 0.016\) | 0.277 \( \pm 0.028\) | 3.27        |

FIG. 3. (Color online) \( F_A(Q^2) = G_A(Q^2)/G_A(0) \) fitted to different ranges of momentum transfer \( Q^2 \) using the parametrization of Eq. (37) including the \( a_1 \) meson. The (black) solid line corresponds to a fit up to and including \( Q_{\text{max}}^2 = 0.24 \) GeV\(^2\), the long-dashed (red) line up to and including \( Q_{\text{max}}^2 = 0.6 \) GeV\(^2\), and the short-dashed (green) line up to and including \( Q_{\text{max}}^2 = 1 \) GeV\(^2\), respectively. The corresponding parameters are given in Table IV.
B. Nucleon-to-Δ transition

In order to discuss the $a_1$-meson contribution to $C_5^A(Q^2)$ and $C_6^A(Q^2)$, we need the coupling of the $a_1$ meson to the $N\Delta$ system. We model this interaction in analogy to the coupling of the external axial-vector field $a_{\mu j}(x)$ [see Eqs. (22) and (24)]. For the neutral $a_1$ meson we obtain

$$\mathcal{L}_{a_1,N\Delta} = g_{a_1,N\Delta} \sqrt{\frac{2}{3}} \left( \bar{\Delta}_+ \Delta_0^0 \right) (g^{\lambda\mu} - \gamma^\lambda \gamma^\mu) A_{\mu3} \left( \frac{p}{n} \right) + \text{H.c.}$$

In particular, since the external axial-vector field $a_{\mu j}(x)$ and the field $A_{\mu j}(x)$ carry the same quantum numbers, it is natural to assume the same SU(6) relation for the coupling constants $g_{a_1N}$ and $g_{a_1N\Delta}$ as for $g_A$ and $g$ [see Eq. (28)],

$$g_{a_1,N\Delta} = \frac{3}{5} \sqrt{2} g_{a_1N}.$$  \hspace{1cm} (45)

The contribution of the $a_1$ meson to $C_5^A(Q^2)$ is obtained from Eq. (26) by the replacement

$$g \rightarrow f_A g_{a_1,N\Delta} \frac{Q^2}{M_{a_1}^2 + Q^2}.$$  \hspace{1cm} (46)

As in the nucleon case, the loop contributions to the low-$Q^2$ behavior of $C_5^A(Q^2)$ are small and we can write

$$C_5^A(Q^2) = g_{AN\Delta} + c_3 Q^2 + c_4 \frac{Q^2}{M_{a_1}^2 + Q^2},$$  \hspace{1cm} (46)
where $c_4 = f_A g_{a_1 N \Delta}$. Extracting $C_5^A(0) = g_{AN \Delta}$, we get

$$C_5^A(Q^2) = g_{AN \Delta} \left[1 + \tilde{c}_3 Q^2 - \frac{(Q^2)^2}{M_{a_1}^2 (M_{a_1}^2 + Q^2)}\right],$$

(47)

where $g_{AN \Delta} \tilde{c}_3 = c_3 + c_4/M_{a_1}^2$ and $\tilde{c}_4 = c_4/g_{AN \Delta}$. By analogy with Eq. (40), we find for the mean-square and mean-quartic axial transition radii

$$\langle r_{AN \Delta}^2 \rangle = -6 \tilde{c}_3, \quad \langle r_{AN \Delta}^4 \rangle = -120 \frac{\tilde{c}_4}{M_{a_1}^2}.$$  

(48)

At this point, we make use of the quark-model relation of Eq. (45) between the coupling constants $g_{a_1 N \Delta}$ and $g_{a_1 N}$ to reexpress $\tilde{c}_4$ as

$$\tilde{c}_4 = \frac{f_A g_{a_1 N \Delta}}{g_{AN \Delta}} = \frac{3}{5} \sqrt{2} \frac{f_A g_{a_1 N}}{g_{AN \Delta}} = \frac{3}{5} \sqrt{2} \frac{g_A}{g_{AN \Delta}} \tilde{c}_2.$$  

Applying, in addition, to $g_{AN \Delta}$ and $g_A$ the quark-model relation of Eq. (28), we obtain the simple result

$$\tilde{c}_4 = \tilde{c}_2.$$  

(49)

With these assumptions, the form factor $C_5^A(Q^2)$ contains only one single free parameter $\tilde{c}_3$ (or $c_3$). In order to show the dependence on this parameter, as a starting point we make use of the assumption

$$C_5^A(Q^2) = g_{AN \Delta} F_A(Q^2) = \frac{3}{5} \sqrt{2} G_A(Q^2),$$  

(50)

i.e., $\tilde{c}_3 = \tilde{c}_1$, and then vary the LEC $\tilde{c}_3$. Figure 5 shows a comparison between $G_A(Q^2)$ and $C_5^A(Q^2)$. The parameters for $G_A(Q^2)$ [(black) long-dashed line] are taken from the fit with $Q_{\text{max}}^2 = 0.6 \text{ GeV}^2$ (second row of Table IV). The (black) solid line corresponds to Eq. (50) for $C_5^A(Q^2)$, the (blue) short-dashed line and the (red) dashed line correspond to a decrease and an increase of the mean-square axial transition radius by 5 %, respectively.

By analogy with Eq. (42), the low-$Q^2$ behavior of $C_6^A(Q^2)$ without the $a_1$ meson can be written as (see Appendix B)

$$C_6^A(Q^2) = \frac{m_N F_\pi g_{\pi N \Delta}}{M_\pi^2 + Q^2} + m_N^2 C_5^A(0),$$  

(51)

where $g_{\pi N \Delta} = G_{\pi N \Delta}(-M_\pi^2)$ is the pion-nucleon-$\Delta$ coupling constant. Including the $a_1$ meson, we obtain

$$C_6^A(Q^2) = \frac{m_N F_\pi g_{\pi N \Delta}}{M_\pi^2 + Q^2} + m_N^2 C_5^A(0) - m_N^2 g_{AN \Delta} \tilde{c}_4 \frac{Q^2}{M_{a_1}^2 (M_{a_1}^2 + Q^2)}.$$  

(52)

In terms of the lowest-order Lagrangian, Eq. (24), and the lowest-order prediction $g_{AN \Delta} = g$, Eq. (26), the pion-nucleon-$\Delta$ coupling constant satisfies the generalization of the Goldberger-Treiman relation

$$g_{\pi N \Delta} = \frac{m_N}{F_\pi} g_{AN \Delta}.$$  

(53)

Using the values of Table I, the Goldberger-Treiman discrepancy at the nucleon level, $\Delta = 1 - m_N g_A/(F_\pi g_{\pi N})$, amounts to $\Delta = 1.2 \%$.  

6 Using the values of Table I, the Goldberger-Treiman discrepancy at the nucleon level, $\Delta = 1 - m_N g_A/(F_\pi g_{\pi N})$, amounts to $\Delta = 1.2 \%$.
Using \( \tilde{c}_2 = \tilde{c}_4 \) of Eq. \((49)\) and the quark-model relation \( g_{AN\Delta} = 3\sqrt{2}g_A/5 \), we obtain the following prediction,

\[
C_6^A(Q^2) = \frac{5}{6\sqrt{2}} G_P(Q^2) + m_N^2 \left( C_5^{A'}(0) - \frac{5}{3} \sqrt{2} G_A'(0) \right) = \frac{5}{6\sqrt{2}} G_P(Q^2) - \frac{1}{6} m_N^2 \left( \langle r_{AN\Delta}^2 \rangle - \frac{5}{3} \sqrt{2} \langle r_A^2 \rangle \right),
\]

where \( G_P(Q^2) \) is the induced pseudoscalar form factor of Eq. \((43)\). At this stage, we assume that \( G_A'(0) = c_1 \) and \( C_5^{A'}(0) = c_3 \) are independent. Figure 6 shows a comparison between \( C_6^A(Q^2) \) without and including the \( a_1 \) meson. In each case we make use of the quark-model estimate \( g_{\pi N\Delta} = 3\sqrt{2}g_{\pi N}/5 \) as obtained from the respective Goldberger-Treiman relations. The (black) long-dashed line corresponds to Eq. \((51)\) with \( C_5^{A'}(0) = g_{AN\Delta} \tilde{c}_1 \) and \( \tilde{c}_1 = -1.78 \) GeV\(^{-2} \). For the result that includes the \( a_1 \) we assume, in addition, \( \tilde{c}_4 = \tilde{c}_2 = -5.31 \) (second row of Table IV). The (black) solid line corresponds to Eq. \((52)\) for \( C_6^A(Q^2) \), the (blue) short-dashed line and the (red) dashed line correspond to a decrease and an increase of the mean-square axial transition radius by 10 %, respectively.

**VII. SUMMARY AND CONCLUSIONS**

We analyzed the low-\( Q^2 \) behavior of the axial form factor \( G_A(Q^2) \), the induced pseudoscalar form factor \( G_P(Q^2) \), and the axial nucleon-to-\( \Delta \) transition form factors \( C_5^A(Q^2) \).
and \( C_{6}^{A}(Q^{2}) \). To this end we made use of a chiral effective Lagrangian for the interaction of the \( a_{1} \) meson with an external axial current, the nucleon, and the \( \Delta \). Within this approach, the axial form factor \( G_{A}(Q^{2}) \) is described in terms of three parameters [see Eq. (36)]. We investigated the parameters by fitting the model to empirical data, choosing different values of the maximal squared momentum transfer (see Table IV). We compared the results with the commonly used dipole parametrization (see Figs. 2 and 3). Extending a relation known from chiral perturbation theory, we made a prediction for the induced pseudoscalar form factor \( G_{P}(Q^{2}) \). For the determination of the transition form factor \( C_{6}^{A}(Q^{2}) \) we drew on an SU(6) spin-flavor quark-model relation to fix \( g_{AN\Delta} \) and \( g_{a_{1}N\Delta} \) in terms of \( g_{A} \) and \( g_{a_{1}N} \), respectively. With this assumption, the result for \( C_{6}^{A}(Q^{2}) \) depends only on a single parameter \( \tilde{c}_{3} \), which is related to the mean-square axial transition radius (see Fig. 5). Finally, the transition form factor \( C_{5}^{A}(Q^{2}) \) was predicted in terms of \( G_{P}(Q^{2}) \), and the derivatives \( G_{A}'(0) \) and \( C_{5}^{A'}(0) \). We emphasize that the predictions at hand represent a model of the relevant form factors at low \( Q^{2} \). To be specific, we expect \( Q^{2} = 0.6 \text{ GeV}^{2} \) to be a reasonable upper limit for the applicability of the model.

The purpose of the present investigation was to identify the \( a_{1} \) meson as an important messenger particle in the context of axial-vector current transitions. The use of SU(6) spin-flavor quark-model relations has to be regarded as a first attempt to restrict the number of free parameters. Clearly, merging the \( a_{1} \)-meson contribution with the inclusion of pion loops within a consistent power counting is a desirable next step. However, as far as the predictive power is concerned, one has to keep in mind that the chiral effective field theory calculation will essentially contain the same number of free parameters, i.e., LECs.
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Appendix A: Conventions for Dirac spinors

For the normalization of spinors and states, we follow Appendix A of Ref. [46]. We only include the relations which are necessary for our calculation.

\[ \langle \vec{p}', r | \vec{p}, s \rangle = \frac{2}{2E(\vec{p}) (2\pi)^3} \delta^3(\vec{p}' - \vec{p}) \delta_{rs}, \]

\[ E(\vec{p}) = \sqrt{m^2 + \vec{p}^2}, \]

\[ |N(\vec{p}, s)\rangle = b_s^\dagger(\vec{p}) |0\rangle, \]

\[ \{ b_r(\vec{p}''), b_s^\dagger(\vec{p}) \} = \frac{2}{2E(\vec{p}) (2\pi)^3} \delta^3(\vec{p}' - \vec{p}) \delta_{rs}, \]

\[ \Psi(x) = \sum_{r=1}^{2} \int \frac{d^3p}{2E(\vec{p}) (2\pi)^3} \left( b_r(\vec{p}) u^{(r)}(\vec{p}) e^{-ip \cdot x} + d_r^\dagger(\vec{p}) v^{(r)}(\vec{p}) e^{ip \cdot x} \right), \]

\[ p^0 = E(\vec{p}), \]

\[ u^{(r)}(\vec{p}) = \sqrt{E(\vec{p}) + m} \left( \begin{array}{c} \chi_r \cr \frac{\vec{p} \cdot \vec{p}}{E(\vec{p}) + m} \chi_r \end{array} \right), \]

\[ \chi_1 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad \chi_2 = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \]

\[ \bar{u}^{(r)}(\vec{p}) u^{(s)}(\vec{p}) = 2m \delta_{rs}, \]

\[ \langle 0 | \Psi(x) | N(\vec{p}, s) \rangle = u^{(s)}(\vec{p}) e^{-ip \cdot x}. \]

Appendix B: Low-\(Q^2\) expansion of \(C_6^A(Q^2)\)

We define the pion-nucleon-\(\Delta\) form factor \(G_{\pi\Delta}(Q^2)\) in terms of the reduced matrix element

\[ \langle \Delta(p_f, s_f) | \hat{m} P(1) | N(p_i, s_i) \rangle = \frac{M_\pi^2 F_\pi}{M_\pi^2 + Q^2} G_{\pi\Delta}(Q^2) i \vec{w}_\lambda(p_f, s_f) \frac{q^\lambda}{m_N} u(p_i, s_i). \]  

Using the parametrization of Eq. (12), the equation for the divergence of the axial-vector current, Eq. (2), results in

\[ C_5^A(Q^2) - \frac{Q^2}{m_N^2} C_6^A(Q^2) = \frac{M_\pi^2 F_\pi}{M_\pi^2 + Q^2} G_{\pi\Delta}(Q^2) \frac{q^\lambda}{m_N} u(p_i, s_i). \]  

Truncating the expansion of the form factors \(C_5^A(Q^2)\) and \(G_{\pi\Delta}(Q^2)\) after the linear order in \(Q^2\),

\[ C_5^A(Q^2) = C_5^A(0) + Q^2 C_5^A'(0), \]

\[ G_{\pi\Delta}(Q^2) = G_{\pi\Delta}(0) + Q^2 G_{\pi\Delta}'(0), \]
and using

\[ g_{\pi N\Delta} = G_{\pi N\Delta}(-M_{\pi}^2) = G_{\pi N\Delta}(0) - M_{\pi}^2G_{\pi N\Delta}'(0), \]

we obtain

\[
C_6^A(Q^2) = \frac{m_N^2}{Q^2} \left[ C_5^A(Q^2) - \frac{M_{\pi}^2 F_{\pi}}{2} \frac{G_{\pi N\Delta}(Q^2)}{m_N} \right]
\]

\[
= \frac{m_N^2}{Q^2} \frac{1}{M_{\pi}^2 + Q^2} \left[ (M_{\pi}^2 + Q^2)(C_5^A(0) + Q^2 C_5^A'(0)) - \frac{M_{\pi}^2 F_{\pi}}{m_N} (G_{\pi N\Delta}(0) + Q^2 G_{\pi N\Delta}'(0)) \right]
\]

\[
= \frac{m_N^2}{Q^2(Q^2 + M_{\pi}^2)} \left[ M_{\pi}^2 C_5^A(0) + M_{\pi}^2 Q^2 C_5^A'(0) + Q^2 C_5^A(0) + (Q^2)^2 C_5^A'(0) \right.
\]

\[
- \frac{M_{\pi}^2 F_{\pi}}{m_N} G_{\pi N\Delta}(0) - \frac{M_{\pi}^2 F_{\pi}}{m_N} Q^2 G_{\pi N\Delta}'(0) \right]
\]

\[
= \frac{m_N^2}{M_{\pi}^2 + Q^2} \left[ C_5^A(0) - M_{\pi}^2 F_{\pi} \frac{G_{\pi N\Delta}(0)}{m_N} + (M_{\pi}^2 + Q^2 C_5^A'(0) \right]
\]

\[
= \frac{m_N^2}{M_{\pi}^2 + Q^2} \left[ \frac{F_{\pi} G_{\pi N\Delta}(0)}{m_N} - M_{\pi}^2 \frac{F_{\pi}}{m_N} G_{\pi N\Delta}(0) + (M_{\pi}^2 + Q^2 C_5^A'(0) \right]
\]

\[
= \frac{m_N^2 F_{\pi} g_{\pi N\Delta}}{M_{\pi}^2 + Q^2} + m_N^2 C_5^A'(0).
\]
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