Title
Nonparametric covariate-adjusted regression

Permalink
https://escholarship.org/uc/item/6808d9sf

Journal
Annals of Statistics, 44(5)

ISSN
0090-5364

Authors
Delaigle, A
Hall, P
Zhou, WX

Publication Date
2016-10-01

DOI
10.1214/16-AOS1442

Peer reviewed
NONPARAMETRIC COVARIATE-ADJUSTED REGRESSION

BY AURORE DELAIGLE¹,* , PETER HALL AND WEN-XIN ZHOU¹,* ,†

University of Melbourne* and Princeton University†

We consider nonparametric estimation of a regression curve when the data are observed with multiplicative distortion which depends on an observed confounding variable. We suggest several estimators, ranging from a relatively simple one that relies on restrictive assumptions usually made in the literature, to a sophisticated piecewise approach that involves reconstructing a smooth curve from an estimator of a constant multiple of its absolute value, and which can be applied in much more general scenarios. We show that, although our nonparametric estimators are constructed from predictors of the unobserved undistorted data, they have the same first-order asymptotic properties as the standard estimators that could be computed if the undistorted data were available. We illustrate the good numerical performance of our methods on both simulated and real datasets.

1. Introduction. We consider nonparametric estimation of a regression curve \( m(x) = E(Y | X = x) \) when \( X \) and \( Y \) are observed with multiplicative distortion induced by an observed confounder \( U \). Specifically, we observe \( \tilde{X} , \tilde{Y} \) and \( U \), where \( \tilde{Y} = \psi(U)Y \), \( \tilde{X} = \varphi(U)X \), \( \psi \) and \( \varphi \) are unknown functions and \( U \) is independent of \( X \) and \( Y \). This model is known as a covariate-adjusted regression model. It was introduced by Şentürk and Müller (2005a) to generalize an approach commonly employed in medical studies, where the effect of a confounder \( U \), for example,

**Tribute:** Peter was a wonderful man and an extraordinary researcher. The diversity of topics that he studied originated from his passion for science. He could get literally fascinated by all sorts of problems, ranging from the most applied biological or physical question, to the most theoretical puzzle in number theory. Faced with a new challenge (something he enjoyed particularly), he sought insights by first exploring its fundamental theoretical properties. This is how he managed to unravel the most surprising and important characteristics of problems, and, from there, suggest very innovative and creative statistical methods. His constant search for insight, and sheer tenacity as a researcher, has led him to develop some of the most difficult and most influential theory in modern statistics. Peter was one of the strongest statisticians of our time. He was a great source of inspiration and our daily interactions with him made research particularly exciting. We will always remember his passion, enthusiasm, optimism, and constant support. We feel fortunate for having had him as friend, mentor and colleague for so many years, but we miss him sorely.

Received May 2015; revised November 2015.

¹Supported by grants and fellowships from the Australian Research Council.

MSC2010 subject classifications. 62G05, 62G08, 62G20.

Key words and phrases. Discontinuities, local linear estimator, multiplicative distortion, Nadaraya–Watson estimator, nonparametric smoothing, predictors.
body mass index, is often removed by dividing by $U$. Motivated by the fibrinogen data on haemodialysis patients, where $\tilde{Y}$ was fibrinogen level, $\tilde{X}$ was serum transferrin level, and $U$ was body mass index, Şentürk and Müller (2005a) pointed that although it is often reasonable to assume that the effect of $U$ is multiplicative, it does not need to be proportional to $U$, and a more flexible model is obtained by allowing for distortions represented by the functions $\phi$ and $\psi$. More generally, this model is useful to describe the relationship between variables that are influenced by a confounding variable, and see if this relationship still exists once the effect of the confounder has been removed.

A number of authors have suggested estimators of the curve $m$ in various parametric settings. Linear regression models were considered by Şentürk and Müller (2005a, 2006) and Şentürk and Nguyen (2006), who generalized them to varying coefficient models [Şentürk (2006)] and generalized linear models [Şentürk and Müller (2009)]. A more general nonlinear regression model was suggested by Cui et al. (2009) and Zhang, Zhu and Liang (2012), and in Zhang et al. (2013), the authors considered a partially linear model, where the linear part is observed with multiplicative distortions.

In this work, we propose more flexible nonparametric estimators of the regression function $m$, which not only relax the parametric assumptions imposed in the existing literature, but also significantly weaken some of the strong assumptions on the curves $\phi$ and $\psi$ and on the distribution of the data made by previous authors. In particular, we propose estimators which, unlike in the previous studies, can be applied if $EX$ and $EY$ vanish, and even if the functions $\psi$ and $\phi$ are not strictly positive. Our procedures involve estimating the functions $\phi$ and $\psi$, deduce from there predictors of $X$ and $Y$, and construct nonparametric estimators of $m$ using those predictors. We show that, under the restrictive assumptions made in the existing literature, this is relatively straightforward to do, whereas under the much weaker assumptions we also consider, we need to use a sophisticated approach.

This paper is organized as follows. We describe the covariate-adjusted model and discuss the model restrictions in the existing literature in Section 2. We propose several nonparametric estimators in Section 3, ranging from the most basic estimators which can be applied under similar restrictions as those imposed in the existing literature, to the most sophisticated ones which rely on much milder assumptions. We derive theoretical properties of our estimators in Section 4, where we show that they have the same first-order asymptotic properties as the nonparametric estimators that could be computed if $X$ and $Y$ were observed directly. More surprisingly, in some particular cases, our new estimators can even achieve faster convergence rates than the standard estimators based on direct observations from $(X, Y)$. We discuss practical implementation of our methods in Section 5, where we also investigate their performance on simulated data, and apply them to analyze two real datasets studied in Şentürk and Müller (2005b) and Şentürk and Nguyen (2006). We discuss multivariate extensions in Section 6. Our proofs
are provided in Section 7 and in a supplementary file [Delaigle, Hall and Zhou (2016)].

2. Model and data. We observe independent and identically distributed (i.i.d.) triplets \( \{(\tilde{X}_i, \tilde{Y}_i, U_i)\}_{i=1}^n \) generated by the covariate-adjusted model of Şentürk and Müller (2005a), where

\[
Y = m(X) + \sigma(X)\varepsilon, \quad \tilde{Y} = \psi(U)Y, \quad \tilde{X} = \varphi(U)X,
\]

with \( m(x) = E(Y|X = x) \), an unknown regression curve that we wish to estimate nonparametrically, \( \sigma^2(x) = \text{var}(Y|X = x) \), an unknown variance function, and \( \varphi \) and \( \psi \), unknown smooth functions. The random variables \( U, X \) and \( \varepsilon \) are mutually independent, \( E(\varepsilon) = 0 \) and \( \text{var}(\varepsilon) = 1 \). We use \( f_X \) and \( f_U \) to denote the densities of \( X \) and \( U \), respectively. As in Şentürk and Müller (2005a), to make the problem identifiable, we assume that

\[
E\{\varphi(U)\} = E\{\psi(U)\} = 1.
\]

In other words, on average there is no distorting effect, which is similar to the standard condition imposed in the related classical measurement error problems [Carroll and Hall (1988), Fan and Truong (1993)], where one observes \( W = X + U \) with \( X \) and \( U \) independent, and the measurement error \( U \) is assumed to have zero mean.

As mentioned in the Introduction, several parametric estimators of \( m \) have been suggested in the literature. There, it is commonly assumed that

\[
(a) \quad \varphi(u), \psi(u) > 0 \quad \text{for all } u \in I_U,
(b) \quad E(X) \neq 0 \quad \text{and} \quad E(Y) \neq 0,
\]

where \( I_U = [u_L, u_R] \) denotes the compact support of \( U \). Without loss of generality, we assume that \( I_U = [0, 1] \) throughout the paper.

An approach used by some authors is based on constructing predictors of the \((X_i, Y_i)\)'s, which can be obtained from the data \((\tilde{X}_i, \tilde{Y}_i, U_i), i = 1, \ldots, n\), on noting that

\[
\varphi_0(U_i) \equiv E(\tilde{X}_i|U_i) = \varphi(U_i)E(X),
\]

\[
\psi_0(U_i) \equiv E(\tilde{Y}_i|U_i) = \psi(U_i)E(Y).
\]

Now, \( \varphi \) and \( \psi \) can easily be estimated nonparametrically, say by \( \hat{\varphi} \) and \( \hat{\psi} \), which motivates Cui et al.’s (2009) predictors \( \hat{Y}_i = \{\hat{\psi}(U_i)\}^{-1}\tilde{Y}_i \) and \( \hat{X}_i = \{\hat{\varphi}(U_i)\}^{-1}\tilde{X}_i \), and shows that (2.3) is needed by those authors to avoid dividing by zero. In the next section, we shall see that it is possible to construct consistent nonparametric estimators of \( m \), and that this can be done under much less restrictive conditions than (2.3).
3. Methodology.

3.1. Different methods under different conditions. The parametric methods developed in the literature crucially rely on assumption (2.3), and the examples considered there are always such that \( \phi, \psi, EX \) and \( EY \) are far from zero. We wish to construct nonparametric estimators of \( m \) that are consistent even if those assumptions do not hold. Let \( e_1 = (1, 0)^T \), and, for any pairs of random variables \((Q, R)\) and \((Q_i, R_i)\), let \( S_{Q,n}(x; K, h) = n^{-1} \sum_{i=1}^{n} K_h(Q_i - x)w(h^{-1}(Q_i - x))w(h^{-1}(Q_i - x))^T \in \mathbb{R}^{2 \times 2} \) and \( T_{Q,R,n}(x; K, h) = n^{-1} \sum_{i=1}^{n} R_i K_h(Q_i - x)w(h^{-1}(Q_i - x)) \), with \( w(s) = (1, s)^T \) and where \( K \) is a kernel function, \( h = h_n > 0 \) is a bandwidth and, for every \( t \in \mathbb{R} \), \( K_h(t) = h^{-1} K(t/h) \).

If the \((X_i, Y_i)\)'s were available, we could estimate \( m(x) \) nonparametrically by a standard local polynomial estimator constructed from the \((X_i, Y_i)\)'s, the two most popular versions of which are the Nadaraya–Watson and the local linear estimators, defined by

\[
\tilde{m}_{NW}(x) = \frac{\sum_{i=1}^{n} Y_i K_h(x - X_i)}{\sum_{i=1}^{n} K_h(x - X_i)},
\]

(3.1)

\[
\tilde{m}_{LL}(x) = e_1^T S_{X,n}^{-1}(x; K, h) T_{X,Y,n}(x; K, h),
\]

respectively. In our case, the \((X_i, Y_i)\)'s are not observed and these standard estimators cannot be computed. We develop new nonparametric estimators that can be computed from the \((\tilde{X}_i, \tilde{Y}_i, U_i)\)'s, and whose complexity depends on whether (2.3)(a) and (b) are satisfied or not. The simplest situation is the one where (2.3)(a) holds. There, we can estimate \( m \) by standard nonparametric estimators based on predictors of the \((X_i, Y_i)\)'s that are similar to, but less restrictive than, those used by Cui et al. (2009); see Section 3.2. The case where we do not assume (2.3)(a) requires more elaborate techniques: in Section 3.3, we suggest a method that can be used when (2.3)(b) is satisfied; we handle the most general case in Section 3.4, where we develop a sophisticated method which is valid regardless of whether (2.3)(a) and (b) hold or not. It involves computing estimators of unknown constant multiples of \(|\varphi|\) and \(|\psi|\), estimate the zeros of those functions, construct piecewise estimators of unknown constant multiples of \( \varphi \) and \( \psi \), estimate these constants and finally deduce estimators of \( \varphi \) and \( \psi \).

3.2. Basic method. We start by deriving simple nonparametric estimators of \( m \) that can be computed when (2.3)(a) holds, and which form the basis of the more sophisticated methods we introduce in the subsequent sections. The idea is similar to the one used in the parametric context by Cui et al. (2009): replace the unobserved \((X_i, Y_i)\)'s by predictors \((\tilde{X}_i, \tilde{Y}_i)\). Under (2.3), motivated by (2.4) and since \( EX = E\tilde{X} \) and \( EY = E\tilde{Y} \), Cui et al. (2009) take \( \tilde{Y}_i = (\hat{\psi}(U_i))^{-1} \tilde{Y}_i \) and \( \tilde{X}_i = (\hat{\varphi}(U_i))^{-1} \tilde{X}_i \), where \( \hat{\varphi} \) and \( \hat{\psi} \) denote Nadaraya–Watson estimators of \( \varphi_0 \) and \( \psi_0 \), divided by, respectively, \( E\tilde{X} = n^{-1} \sum_{i=1}^{n} \tilde{X}_i \) and \( E\tilde{Y} = n^{-1} \sum_{i=1}^{n} \tilde{Y}_i \).
It is because of this division that Cui et al. (2009) assume (2.3)(b), but the latter can be avoided and replaced by \( E|X|, E|Y| \neq 0 \) (which holds for all nondegenerate random variables), by better exploiting (2.3)(a). Specifically, under (2.3)(a), \(|\psi| = \psi, |\varphi| = \varphi, \) and

\[
\varphi_0^+ (U_i) \equiv E(|\tilde{X}_i||U_i) = \varphi(U_i)E|X|, \\
\psi_0^+ (U_i) \equiv E(|\tilde{Y}_i||U_i) = \psi(U_i)E|Y|.
\]

(3.2)

Motivated by this, we propose to estimate \( \psi \) and \( \varphi \) by

\[
\hat{\varphi}_{LL}(u) = \frac{\hat{\varphi}_{0,LL}(u)}{E|X|} \quad \text{and} \quad \hat{\psi}_{LL}(u) = \frac{\hat{\psi}_{0,LL}(u)}{E|Y|},
\]

where \( E|X| = n^{-1} \sum_{i=1}^n |\tilde{X}_i|, \) \( E|Y| = n^{-1} \sum_{i=1}^n |\tilde{Y}_i|, \) and where \( \hat{\varphi}_{0,LL}(u) = e_1^TS_{U,n}^{-1}(u; L, g_1)T_{U,|\tilde{X}_i,n}(u; L, g_1) \) and \( \hat{\psi}_{0,LL}(u) = e_1^TS_{U,n}^{-1}(u; L, g_2)T_{U,|\tilde{Y}_i,n}(u; L, g_2) \) are local linear estimators of \( \varphi_0^+ \) and \( \psi_0^+ \) computed with a kernel function \( L \) and bandwidths \( g_1 \) and \( g_2 \).

Then we predict \( Y_i \) and \( X_i \) by taking

\[
\hat{Y}_i = \{\hat{\psi}_{LL}(U_i)\}^{-1}\tilde{Y}_i \quad \text{and} \quad \hat{X}_i = \{\hat{\varphi}_{LL}(U_i)\}^{-1}\tilde{X}_i.
\]

(3.4)

Finally, replacing \((X_i, Y_i)\) by \((\hat{X}_i, \hat{Y}_i)\) in (3.1), we obtain the following estimators of \( m(x) \):

\[
\hat{m}_{NW}(x) = \frac{\sum_{i=1}^n \hat{Y}_i K_h(x - \hat{X}_i)}{\sum_{i=1}^n K_h(x - \hat{X}_i)}, \\
\hat{m}_{LL}(x) = e_1^TS_{\tilde{X},n}^{-1}(x; K, h)T_{\tilde{X},\tilde{Y},n}(x; K, h).
\]

(3.5)

Remark 3.1. Using \( E(\tilde{Y}_i|X_i) = E(Y_i|X_i) = m(X_i) \), simpler estimators of \( m \) can also be defined by \( \hat{m}_{NW,0}(x) = \sum_{i=1}^n \hat{Y}_i K_h(x - \hat{X}_i)/K_h(x - \hat{X}_i) \) and \( \hat{m}_{LL,0}(x) = e_1^TS_{\tilde{X},n}^{-1}(x; K, h)T_{\tilde{X},\tilde{Y},n}(x; K, h) \). Since they require predicting only the \( X_i \)’s, these estimators seem more attractive than those in (3.5). However, it can be proved that their asymptotic “variance” is larger than that of the estimators in (3.5). Moreover, they cannot be adapted simply to the case where \( \varphi \) does not satisfy (2.3)(a); see Remark 3.3 in Section 3.3.

3.3. Refined procedure. As their parametric counterparts developed in the covariate-adjusted literature, the methods introduced in Section 3.2 can only be computed if (2.3)(a) holds. However, in practice, there is no reason why \( \varphi \) and \( \psi \) would always be positive, and even if they are, their estimators may vanish or get close to zero, which can cause numerical problems. In this section, we suggest a refined approach which can overcome these difficulties when (2.3)(b) holds. The more complex case where (2.3)(b) is violated will be dealt with in Section 3.4.
As in Section 3.2, to estimate \( m \), the first step is to construct predictors \( \hat{X}_i \) and \( \hat{Y}_i \), and thus estimators of \( \varphi \) and \( \psi \). Recall the notation in (2.4). Since we assume (2.3)(b) but not (2.3)(a), instead of (3.3) we take \( \hat{Y}_i = \{ \hat{\psi}_{LL}(U_i) \}^{-1}\hat{Y}_i \) and \( \hat{X}_i = \{ \hat{\varphi}_{LL}(U_i) \}^{-1}\hat{X}_i \), where \( \hat{\varphi}_{LL}(u) = \hat{\varphi}_{0,LL}(u)/\hat{E}X \) and \( \hat{\psi}_{LL}(u) = \hat{\psi}_{0,LL}(u)/\hat{E}Y \), and the local linear estimators

\[
\hat{\varphi}_{0,LL}(u) = e_1^TS_{U,n}^{-1}(u; L, g_1)T_{U,\hat{X},n}(u; L, g_1), \\
\hat{\psi}_{0,LL}(u) = e_1^TS_{U,n}^{-1}(u; L, g_2)T_{U,\hat{Y},n}(u; L, g_2)
\]

of \( \varphi_0 \) and \( \psi_0 \) computed with a kernel function \( L \) and bandwidths \( g_1 \) and \( g_2 \).

To derive consistent estimators of \( m \) without imposing (2.3)(a), recall that, for each \( i \), \( X_i \) and \( Y_i \) are independent of \( U_i \). As a consequence, for any subset \( S \subseteq \mathbb{R} \), we have \( E(Y_i|X_i = x, U_i \in S) = E(Y_i|X_i = x) \). In particular, if \( X_i \), \( Y_i \), \( \varphi \) and \( \psi \) were known, then letting \( \mathcal{C}_n(\rho_1, \rho_2) = \{ 1 \leq i \leq n : |\varphi(U_i)| \geq \rho_1, |\psi(U_i)| \geq \rho_2 \} \), with \( \rho_1, \rho_2 > 0 \) denoting two small numbers, the following modification of \( \hat{m}_{NW}(x) \) at (3.1) would be consistent:

\[
\hat{m}_{NW}(x; \rho_1, \rho_2) = \sum_{i \in \mathcal{C}_n(\rho_1, \rho_2)} Y_i K_h(x - X_i) / \sum_{i \in \mathcal{C}_n(\rho_1, \rho_2)} K_h(x - X_i),
\]

and a similar consistent version \( \hat{m}_{LL}(x; \rho_1, \rho_2) \) of \( \hat{m}_{LL}(x) \) at (3.1) could be constructed by replacing, in the definition of \( \hat{m}_{LL}(x) \), sums over all \( i \) by sums over \( i \in \mathcal{C}_n(\rho_1, \rho_2) \) as above. The advantage of this approach is that it enables us to exclude the data for which \( \hat{\psi}(U_i) \) or \( \varphi(U_i) \) are small, and thus it can be applied even if (2.3)(a) does not hold.

Motivated by this discussion, in the case that interests us, where \( X_i \), \( Y_i \), \( \varphi \) and \( \psi \) are unknown, we suggest estimating \( m \) as follows. First, let \( \hat{\mathcal{C}}_n(\rho_1, \rho_2) = \{ i = 1, \ldots, n : |\hat{\varphi}_{0,LL}(U_i)| \geq \rho_1, |\hat{\psi}_{0,LL}(U_i)| \geq \rho_2 \} \). (The choice of \( \rho_1 \) and \( \rho_2 \) will be discussed in Section 5.) We define a Nadaraya–Watson estimator of \( m(x) \), valid even if (2.3)(a) does not hold, by

\[
\hat{m}_{NW}(x; \rho_1, \rho_2) = \sum_{i \in \hat{\mathcal{C}}_n(\rho_1, \rho_2)} \hat{Y}_i K_h(x - \hat{X}_i) / \sum_{i \in \hat{\mathcal{C}}_n(\rho_1, \rho_2)} K_h(x - \hat{X}_i).
\]

Similarly, we define a local linear estimator \( \hat{m}_{LL}(x; \rho_1, \rho_2) \) in the same way as \( \hat{m}_{LL} \) in (3.5), replacing there, and in the definitions of \( S_{\hat{X},n}(x; K, h) \) and \( T_{\hat{X},\hat{Y},n}(x; K, h) \), the indices \( i = 1, \ldots, n \) by the indices \( i \in \hat{\mathcal{C}}_n(\rho_1, \rho_2) \).

**Remark 3.2.** While we shall prove in Section 4 that these estimators are consistent and have the same first-order asymptotic properties as their counterparts at (3.1) based on undistorted data, in practice performance can be further improved by excluding a small fraction (say 5%) of the observations corresponding to the \( U_i \)'s such that a kernel density estimator \( \hat{f}_U(U_i) \) of \( f_U(U_i) \) is the smallest. Indeed, we know from standard properties of kernel regression estimators that, at points \( u \)
where \( f_U(u) \) is small, \( \hat{\phi}(u) \) and \( \hat{\psi}(u) \) are more variable.] Doing this corresponds to enlarging the set \( S \) slightly, which does not affect consistency and convergence rates, again due to the fact that the \( U_i \)'s are independent of the \((X_i, Y_i)\)'s.

**Remark 3.3.** It is not possible to directly use this approach to modify the estimator discussed in Remark 3.1 for the case where \( \phi \) has zeros, because \( \hat{Y}_i \) and \( U_i \) are dependent. Particularly, we note that in general \( E(\hat{Y}_i|X_i = x, U_i \in S) \) and \( E(\hat{Y}_i|X_i = x) \) are not equal.

### 3.4. Elaborate procedure for the most general case.

Finally, we construct estimators of \( m \) that rely on neither part of (2.3). As before, we start by deriving predictors of the \((X_i, Y_i)\)'s. Constructing predictors \( \hat{X}_i \) (resp., \( \hat{Y}_i \)) without assuming (2.3) requires to derive an estimator of \( \phi \) (resp., \( \psi \)) without this assumption, which, unlike the methods used in the previous sections, turns out to be a challenging task. Our procedure is based on the fact that, from (2.1),

\[
\hat{\psi}(u) \equiv E(|\tilde{X}|U = u) = |\phi(u)||E|X| \quad \text{[resp., \( \psi^*(u) \equiv E(|\tilde{Y}|U = u) = |\psi(u)||E|Y| \]},
\]

which implies that we can estimate \( \phi^* \) (resp., \( \psi^* \)) by a standard local linear estimator \( \hat{\phi}_{LL}^* \) (resp., \( \hat{\psi}_{LL}^* \)) with kernel \( L \) and bandwidth \( g_1 \) (resp., \( g_2 \)) constructed from the \((U_i, |\tilde{X}_i|)\)'s [resp., the \((U_i, |\tilde{Y}_i|)\)'s]. In what follows, we explain how to deduce an estimator of \( \phi \) from \( \hat{\phi}_{LL}^* \). The same procedure can be applied to derive an estimator of \( \psi \) from \( \hat{\psi}_{LL}^* \).

Since \( \phi^* \) is proportional to \( |\phi| \), to extract an estimator of \( \phi \) from \( \hat{\phi}_{LL}^* \), we need to estimate the zeros of \( \phi \), say \( \tau_1, \ldots, \tau_M \) for some finite \( M \), at which \( \phi \) changes sign. To do this, we assume that, for each \( j \), \( \phi''(\tau_j) \neq 0 \). Then it is straightforward to see that the first derivative of \( \phi^* \) has jump discontinuities at the \( \tau_j \)'s. Moreover, the zeros of \( \phi \) coincide with those of \( \phi^* \), so that, at the \( \tau_j \)'s, \( \phi^* \) reaches its minimum value, 0. Therefore, the \( \tau_j \)'s can be estimated using procedures for detecting discontinuities in derivatives of a regression curve, such as those in Gijbels, Hall and Kneip (1999) and Gijbels and Goderniaux (2004), combined with the fact that the \( \hat{\tau}_j \)'s need to correspond to local minima of \( \hat{\phi}_{LL}^* \); see Section 5.2 for details of implementation. For \( j = 1, \ldots, M \), let \( \hat{\tau}_j \) denote the resulting estimator of \( \tau_j \), and let \( I_0 = (-\infty, \hat{\tau}_1), I_M = [\hat{\tau}_M, \infty) \) and for \( j = 1, \ldots, M - 1, I_j = [\hat{\tau}_j, \hat{\tau}_{j+1}) \).

Our next target is to construct an estimator of \( \phi \). Recall the notation \( \phi_{0}^+ = \phi \cdot E|X| \) in (3.2). Recalling that \( \phi \) changes sign at each \( \tau_j \), we can obtain a consistent estimator of either \( \phi_{0}^+ \) or \( -\phi_{0}^+ \) (we will see below how to distinguish these two cases) by taking \( \hat{\phi}_{\pm,0}^+(x) = \sum_{j=0}^{M} (-1)^j \hat{\phi}_{\pm,LL}^*(x) \cdot I(x \in I_j) \), where, for each \( j \), \( \hat{\phi}_{\pm,LL}^* \) denotes the local linear estimator of \( \phi^* \) constructed using only the \((U_i, |\tilde{X}_i|)\)'s for which \( U_i \in I_j \). Here, we use a different local estimator in each \( I_j \) because, under our assumptions, the first derivative of \( \phi^* = |\phi| \cdot E|X| \) is discontinuous at the \( \tau_j \)'s. It can be shown using standard kernel smoothing arguments that in this case the bias near the \( \tau_j \)'s is reduced by using this piecewise approach.
Our next step is to extract from $\hat{\phi}_+^{\pm}$ an estimator of $\varphi_0^+$ (recall that $\hat{\phi}_+^{\pm}$ is an estimator of $\varphi_0^+$ or $-\varphi_0^+$, but we cannot know of which one). To do this, recall that $E[\varphi(U)] = 1$, which implies that $E[\varphi_0^+(U)] > 0$. This fact motivates us to estimate $\varphi_0^+(x)$ by $\hat{\phi}_+^{\pm}(x) = \hat{\phi}_+^{\pm}(x)/\text{sign}(\sum_{i=1}^n \hat{\phi}_+^{\pm}(U_i))$. Since $\varphi_0^+(x) = \varphi(x)E[X|\cdot]$, once we have done this, to estimate $\varphi$ it remains to construct an estimator of $E[X]$.

Noting that $E[\varphi_0^+(U)] = E[\varphi(U)]E[X] = E[X]$, we can estimate $E[X]$ by $\hat{E}[X] = n^{-1} \sum_{i=1}^n \hat{\phi}_0^+(U_i) = |n^{-1} \sum_{i=1}^n \hat{\phi}_+^{\pm}(U_i)|$. Finally, we estimate $\varphi(x)$ by $\hat{\varphi}(x) = \hat{\phi}_0^+(x)/E[X]$. Then we can predict the $X_i$’s by taking $\hat{X}_i = (\hat{\varphi}(U_i))^{-1} \hat{X}_i$. We can proceed similarly to construct predictors $\hat{Y}_i$ of the $Y_i$’s. As in Section 3.3, since, to obtain these predictors, we divide by $\hat{\varphi}(U_i)$ and $\hat{\psi}(U_i)$, when constructing our estimator of $m$ we cannot use the $(\hat{X}_i, \hat{Y}_i)$’s for which $|\hat{\varphi}(U_i)|$ or $|\hat{\psi}(U_i)|$ is too small. Therefore, to estimate $m$ we use the estimators $\hat{m}_{NW}(x; \rho_1, \rho_2)$ and $\hat{m}_{LL}(x; \rho_1, \rho_2)$ defined in Section 3.3, but with the predictors $\hat{X}_i$ and $\hat{Y}_i$ constructed above.

4. Theoretical properties. We start by establishing theoretical properties of the estimators $\hat{m}_{NW}$ and $\hat{m}_{LL}$ from Section 3.2. While these estimators seem intuitively natural, because they are computed using variables obtained through nonparametric prediction, checking whether they are consistent, and deriving detailed asymptotic properties, are quite difficult. Recently, Mammen, Rothe and Schienle (2012) gave a deep account of nonparametric estimators computed from nonparametrically generated covariates, but our estimators do not fall into the class of settings they consider, not least because in our case, not only the covariate $X$, but also the dependent variable $Y$, are nonparametrically generated, which makes the problem even more complex than theirs. In addition to the basic model assumptions introduced in the first paragraph of Section 2, we make the following regularity assumptions:

(B1) $E[X] \neq 0$, $E[Y] \neq 0$ and $\inf_{u \in I_U} \varphi(u) > 0$, $\inf_{u \in I_U} \psi(u) > 0$.

(B2) $0 < \inf_{u \in I_U} f_U(u) \leq \sup_{u \in I_U} f_U(u) < \infty$; $f_U$, $\varphi$ and $\psi$ are twice differentiable, and their second derivatives are uniformly continuous and bounded.

(B3) (a) $f_X$ is continuous, $\sup_{x \in \mathbb{R}} f_X(x) < \infty$, and $E[\exp(c_1|X|)] < \infty$ for some constant $c_1 > 0$; (b) $m$ and $f_X$ are twice differentiable and their second derivatives are uniformly continuous and bounded; (c) $\sigma$ is continuous and bounded.

(B4) $E(\varepsilon) = 0$, $E(\varepsilon^2) = 1$ and $E[\exp(c_2|\varepsilon|)] < \infty$ for some $c_2 > 0$.

(B5) $K$ and $L$ are twice continuously differentiable, symmetric density functions, and are compactly supported on $[-1, 1]$. Moreover, $\int_0^1 t^2 L(t) dt > 2\{\int_0^1 t L(t) dt\}^2$.

(B6) The bandwidths $(h, g_1, g_2) = (h_n, g_{1n}, g_{2n})$ are such that $h \asymp n^{-\alpha_0}$ and $g_1 \asymp n^{-\beta_1}$ and $g_2 \asymp n^{-\beta_2}$ for some $0 < \alpha_0, \beta_1, \beta_2 < 1/3$. 


Condition (B1) is a relaxed version of assumption (2.3) often assumed in the covariate-adjusted regression literature. See, for example, Şentürk and Müller (2005a, 2006) and Cui et al. (2009). Condition (B2) includes standard regularity and smoothness assumptions for the asymptotic results of kernel-type nonparametric regression estimation. In (B3), we relax the conventional boundedness condition on the covariates used by Şentürk and Müller (2005a, 2006) and Mammen, Rothe and Schienle (2012), and assume instead that X has a finite exponential moment (e.g., this is satisfied if the distribution of X comes from the exponential family or is compactly supported). Condition (B4), which requires exponentially light tails of ε, is similar in spirit to Assumption 1(iv) in Mammen, Rothe and Schienle (2012). Like them, we need this technical assumption to employ an argument based on empirical processes. Condition (B5) is standard in the context of kernel regression, and is easy to satisfy since we can choose the kernels. Condition (B6) states the required range of magnitude of the bandwidths, and is easy to satisfy in practice.

The next two theorems establish uniform consistency and asymptotic normality of our estimators \( \hat{m}_{NW} \) and \( \hat{m}_{LL} \) defined in Section 3.2. Their proof can be found in Section 7 and in Section D in the supplementary file [Delaigle, Hall and Zhou (2016)].

**Theorem 4.1.** Assume that (2.2) and conditions (B1)–(B6) hold and let \([a, b] \subseteq \mathbb{I}_X \equiv \{x : f_X(x) > 0\}.

(i) If \( h \asymp g_1 \asymp g_2 \asymp (\log n)^{1/5}n^{-1/5} \), then \( \hat{m}_{NW} \) at (3.5) satisfies

\[
\max_{x \in [a, b]} |\hat{m}_{NW}(x) - m(x)| = O_P\{(\log n)^{2/5}n^{-2/5}\}.
\]

(ii) If \( \beta_1 \geq 1/5 \) and \( 0 < \alpha_0 < 1/2 - \beta_1 \), then for every \( x \in [a, b] \),

\[
\hat{m}_{NW}(x) - m(x) = \sqrt{V(x)} N(x) + B_0(x) + \tilde{B}(x) + R_0(x),
\]

where \( N(x) \xrightarrow{p} N(0, 1) \) as \( n \to \infty \), \( V(x) = \{nhf_X(x)\}^{-1} \sigma^2(x) f_X(x) \mu_{K,2}h^2/2 \), \( B_0(x) = \{m''(x)/f_X(x)m)\mu_{K,2}h^2/2 \), \( \tilde{B}(x) = \hat{B}_\psi(x) + \tilde{B}_\psi(x) \) with \( \hat{B}_\psi(x) = x m'(x)(\psi''(U)/\psi(U))_{\mu_{L,2}}g_1/2 \), \( \tilde{B}_\psi(x) = -m(x)(\psi''(U)/\psi(U))_{\mu_{L,2}}g_2/2 \), and the remainder \( R_0 \) is such that \( |R_0(x)| = o_P\{g_1^2 + g_2^2 + h^2 + (nh)^{-1/2}\} \).

**Theorem 4.2.** Assume that (2.2) and conditions (B1)–(B6) hold and let \([a, b] \subseteq \mathbb{I}_X \).

(i) If \( h \asymp g_1 \asymp g_2 \asymp (\log n)^{1/5}n^{-1/5} \), then \( \hat{m}_{LL} \) at (3.5) satisfies

\[
\max_{x \in [a, b]} |\hat{m}_{LL}(x) - m(x)| = O_P\{(\log n)^{2/5}n^{-2/5}\}.
\]

(ii) If \( \beta_1 \geq 1/5 \) and \( 0 < \alpha_0 < 1/2 - \beta_1 \), then for every \( x \in [a, b] \),

\[
\hat{m}_{LL}(x) - m(x) = \sqrt{V(x)} N(x) + B_1(x) + \tilde{B}(x) + R_1(x),
\]

where \( N(x) \xrightarrow{p} N(0, 1) \) as \( n \to \infty \), \( B_1(x) = \{m''(x)\mu_{K,2}h^2/2 \), \( V \) and \( \tilde{B} \) are as in part (ii) of Theorem 4.1, and \( R_1 \) is such that \( |R_1(x)| = o_P\{g_1^2 + g_2^2 + h^2 + (nh)^{-1/2}\} \).
We deduce from the theorems that, although they are constructed from distorted data, when computed with appropriate bandwidths, our estimators \( \hat{m}_{NW} \) and \( \hat{m}_{LL} \) defined in Section 3.2 have the same uniform convergence rates as the standard estimators in (3.1) used when the \((X_i, Y_i)\)'s are available. This contrasts with the errors-in-variables models studied by Fan and Truong (1993) and Delaigle, Fan and Carroll (2009), where convergence rates are significantly degraded by the measurement errors. The conclusions arising from the asymptotic distribution of our estimators are also interesting. Abusing terminology, we refer to \( V \) and \( B_0 + \tilde{B} \) (resp., \( B_1 + \tilde{B} \)) as the asymptotic variance and bias and of our estimator \( \hat{m}_{NW} \) (resp., \( \hat{m}_{LL} \)), and we call asymptotic mean squared error (AMSE) the sum of the asymptotic variance and squared bias. We use similar terminology for the standard estimators of \( m \).

We learn from part (ii) of both theorems that, if we choose \( g_1 \) and \( g_2 \) of order \( o(h) \), the asymptotic bias and variance of our estimators are identical to those of standard estimators, and there, as in the standard case, it is optimal to take \( h \propto n^{-1/5} \), so that AMSE \( \asymp n^{-4/5} \). Perhaps more surprisingly, in cases where \( B_0 \) (resp., \( B_1 \) for \( \hat{m}_{LL} \), \( \hat{B}_p \)) do not all have the same sign, it is possible to choose \( h \) and \( g_1 \) or \( g_2 \) an order of magnitude slightly larger than \( n^{-1/5} \) such that the asymptotic bias \( B_0 + \tilde{B} \) (resp., \( B_1 + \tilde{B} \)) vanishes and the AMSE our estimator is of order \( o(n^{-4/5}) \), thus smaller than the AMSE of the standard estimator (similar results can be established for the integrated AMSE). However, while it is theoretically interesting, we were not able to exploit this result in practice to make our estimator outperform the standard one, despite several attempts. In part, this is because to benefit from this result we need to choose the bandwidths in a very specialized way that requires estimating too many unknowns, and we found that the simpler bandwidths choice suggested in Section 5.2 almost always worked better.

Next, we develop theoretical properties of our estimator defined in Section 3.3. We start by rewriting \( \hat{c}_n(\rho_1, \rho_2) \) as \( \hat{c}_n(\rho_1, \rho_2) = \{1 \leq i \leq n : U_i \in \hat{L}_n(\rho_1, \rho_2)\} \), where \( \hat{L}_n(\rho_1, \rho_2) = \{u \in I_U : |\hat{\varphi}_{0,LL}(u)| \geq \rho_1, |\hat{\psi}_{0,LL}(u)| \geq \rho_2\} \). We can rewrite the estimator at (3.7) as

\[
\hat{m}_{NW}(x; \rho_1, \rho_2) = \frac{\sum_{i=1}^n \hat{Y}_i K_h(x - \hat{X}_i) I\{U_i \in \hat{L}_n(\rho_1, \rho_2)\}}{\sum_{i=1}^n K_h(x - \hat{X}_i) I\{U_i \in \hat{L}_n(\rho_1, \rho_2)\}}. 
\]

To emphasize the main idea while avoiding repetitive arguments, here we present the theoretical result only for this estimator, assuming that only \( \varphi \) may have zeros and, therefore, we take \( \rho_2 = 0 \) throughout this section. A straightforward adaptation of the arguments used to prove Theorem 4.3 below leads to similar results in the more general case where \( \varphi \) has zeros and \( \rho_2 > 0 \), and for the local linear estimator \( \hat{m}_{LL}(x; \rho_1, \rho_2) \).

When \( \rho_2 = 0 \), \( \hat{L}_n(\rho_1, \rho_2) \) depends only on \( \rho_1 \); to simplify notation, we rewrite it as \( \hat{L}_n(\rho_1) = \{u \in I_U : |\hat{\varphi}_{0,LL}(u)| \geq \rho_1\} \). Likewise, we rewrite \( \hat{m}_{NW}(x; \rho_1, 0) \) as \( \hat{m}_{NW}(x; \rho_1) \). Under certain regularity conditions on \( \varphi \), the random set \( \hat{L}_n(\rho_1) \)
is a consistent estimator of $\mathcal{L}(\rho_1) = \{u \in I_U : |\varphi_0(u)| \geq \rho_1\}$. Recalling that
$\varphi_0(u) = E(X)\varphi(u)$, this suggests taking $\rho_1$ to be some value between 0 and
$M_0 \equiv |E(X)| \max_{u \in I_U} |\varphi(u)|$. For $0 \leq t \leq M_0$, let $\partial \mathcal{L}(t) = \{u \in I_U : |\varphi_0(u)| = t\}$. We
will need the following assumptions:

(C1) $E(X),\ E(Y) \neq 0$ and $\inf_{u \in I_U} \psi(u) > 0$.

(C2) $\varphi$ is such that the set $\Theta = \{t \in (0, M_0) : \partial \mathcal{L}(t)$ consists of finitely many
points located in the interior of $I_U$ and $\min_{u \in \partial \mathcal{L}(t)} |\varphi'(u)| > 0\}$ is nonempty.

The next theorem establishes uniform consistency and asymptotic normality
of $\hat{m}_{NW}(x; \rho)$. See Section E in the supplementary file [Delaigle, Hall and Zhou
(2016)] for its proof.

**THEOREM 4.3.** Assume that (2.2), conditions (B2)–(B5), (C1) and (C2) hold
and that $\rho \in (0, M_0)$ in (3.7) is such that $\rho \in \Theta$. Let $[a, b] \subseteq I_X$.

(i) If $g_1 \asymp g_2 \asymp h \asymp (\log n)^{1/5} n^{-1/5}$, then $\hat{m}_{NW}(x; \rho) = \hat{m}_{NW}(x; \rho, 0)$ at
(3.7) satisfies $\max_{x \in [a, b]} |\hat{m}_{NW}(x; \rho) - m(x)| = O_p((\log n)^{2/5} n^{-2/5})$.

(ii) If $\beta_1 \geq 1/5$ and $0 < \alpha_0 < 1/2 - \beta_1$, then for every $x \in [a, b]$,

$$\hat{m}_{NW}(x; \rho) - m(x) = \sqrt{V(x; \rho)} N(x) + B_0(x) + \hat{B}(x; \rho) + R_2(x; \rho),$$

where $N(x) \xrightarrow{d} N(0, 1)$ as $n \to \infty$, $V(x; \rho) = V(x)/P\{U \in \mathcal{L}(\rho)\}$, $V, B_0$ are
as in part (ii) of Theorem 4.1. $\hat{B}(x; \rho) = \beta m'(x)E[\varphi''(U)I\{U \in \mathcal{L}(\rho)\}/\varphi(U)] \times
\mu_L 2g_1^2/2 - m(x)E[\psi''(U)I\{U \in \mathcal{L}(\rho)\}/\psi(U)]\mu_L 2g_2^2/2$, and $R_2$ is such that
$|R_2(x; \rho)| = o_P\{g_1^2 + g_2^2 + h^2 + (nh)^{-1/2}\}$.

We deduce from the theorem that our estimator defined in Section 3.3 has
the same uniform convergence rate as the standard Nadaraya–Watson estimator
in (3.1), used when the data $(X_i, Y_i)$ are available. Moreover, as long as we choose
g_1$ and $g_2$ of order $o(h)$, the asymptotic “bias” and “variance” of our estimator
from Section 3.3 are equal to those of the standard Nadaraya–Watson estimator,
where $i \in \{1 \leq j \leq n : U_j \in \mathcal{L}_n(\rho)\}$. As we already indicated below Theorems 4.1 and 4.2, in theory in some cases it is possible to choose the band-
widths in such a way that the AMSE of our estimator tends to zero faster than
that of the standard estimator, but it seems very hard to find a way to exploit
this in practice. Similar results can be established for the local linear estimator
$\hat{m}_{LL}(x; \rho_1, \rho_2)$.

Establishing theoretical results for the more general procedure described in Sec-
tion 3.4 is particularly challenging. Recall that this method combines a change
point detection algorithm and the ridge-parameter based method introduced in
Section 3.3. The complex nature of this approach implies that deriving its theo-
retical properties rigorously requires long and tedious arguments. Since our pa-
er is already very long, and even the proofs for our simpler methods are fairly
tedious, we leave such rigorous derivations for future work. However, our preliminary calculations already indicate that the procedure from Section 3.4 should have asymptotic properties similar to those described in Theorem 4.3. In particular, these calculations indicate that estimating the $\tau_j$’s and the sign of $\varphi$ and/or $\psi$ has no first-order asymptotic effect on the properties of our estimators of $m$.

5. Numerical results.

5.1. Which method to use. The approach in Section 3.4 can be applied in essentially all cases, but since the methods from Sections 3.2 and 3.3 are simpler, the user might prefer to use these if all parts of (2.3) hold. While (2.3) can be verified by standard tests of hypothesis applied to the observed data (see Remark 5.1 below), when these conditions are needed, it is because the techniques employed involve dividing by estimators of $\psi$, $\varphi$, $EX$ or $EY$. Therefore, in practice, to avoid numerical issues, we suggest using the method from Section 3.3, and to use instead the method from Section 3.4 if the absolute values of estimators of $EX$ or $EY$ are small, the extent of which depends on the magnitude of other quantities involved and the precision of the software employed. This is generally rather easy to determine by examining the data, but if unsure the user can just apply the method of Section 3.4, which is valid in the most general case.

We note too that one does not necessarily need to predict the $X_i$’s and the $Y_i$’s with the same method. For example, if one is confident that $EX$ is far from zero, but is not sure about $EY$, then the $X_i$’s could be predicted using the approach from Section 3.3, and the predictors of the $Y_i$’s could be obtained from the approach suggested in Section 3.4.

REMARK 5.1. The assumption at (2.3) can be tested in several ways. For example, since $EX\tilde{X} = EX$, we can first test the sign of $EX$ by a standard test of hypothesis for the mean applied to the data $\tilde{X}_1, \ldots, \tilde{X}_n$, and then test the sign of the function $\varphi_0 = \varphi \cdot EX$ at (2.4) using, for example, tests such as those in Dümbgen and Spokoiny (2001), Chetverikov (2012) and Lee, Song and Whang (2013), applied to the observed data.

5.2. Details of implementation. As in the case where the $(X_i, Y_i)$’s are available, in practice we recommend using the local linear versions of our estimators, and in this section we suggest ways of choosing the parameters required to compute them. Similar ideas can be used for the Nadaraya–Watson estimators. We know from Section 4 that, while we have to choose $h$ with care, we have more flexibility for the bandwidths $g_1$ and $g_2$, which can take a large range of values. If we take $h$ to be of the standard size for nonparametric regression, and $g_1 = o(h)$ and $g_2 = o(h)$, then our estimators have the same first-order asymptotic properties as the estimators at (3.1).
Motivated by this, for the estimators in Section 3.2, we take \( g_1 = n^{-0.1}g_{1,\text{PI}} \), \( g_2 = n^{-0.1}g_{2,\text{PI}} \) and \( h = h_{\text{PI}} \), where the subscript \( \text{PI} \) means that we use a standard plug-in bandwidth for local linear estimators [Ruppert, Sheather and Wand (1995)] constructed based on, respectively, the data \((U_i, |\tilde{X}_i|), (U_i, |\tilde{Y}_i|)\) and \((\tilde{X}_i, \tilde{Y}_i)\). For the estimators in Sections 3.3 and 3.4, we take \( g_1 = n^{-1/10}g_{1,\text{PI}} \) and \( g_2 = n^{-1/10}g_{2,\text{PI}} \), where \( g_{1,\text{PI}} \) and \( g_{2,\text{PI}} \) denote standard plug-in bandwidths for local linear estimators constructed based on, respectively, the data \((U_i, \tilde{X}_i)\) and \((U_i, \tilde{Y}_i)\). Then, in Section 3.3, we choose \( \rho_1 = \max(0.1, \rho_1^\ast) \) and \( \rho_2 = \max(0.1, \rho_2^\ast) \), where \( \rho_1^\ast \) (resp., \( \rho_2^\ast \)) denotes the square root of an estimator of the asymptotic “mean squared error” of \( \hat{\phi}_{\text{LL}} \) (resp., \( \hat{\psi}_{\text{LL}} \)), integrated over the set of \( x \)-values where \( |\hat{\phi}_{\text{LL}}(x)| \) (resp., \( |\hat{\psi}_{\text{LL}}(x)|\)) take its smallest values; see Appendix A in the supplementary file [Delaigle, Hall and Zhou (2016)] for details. We do the same for the method from Section 3.4, except that we use the estimators \( \hat{\phi} \) and \( \hat{\psi} \) of \( \phi \) and \( \psi \) derived there. Finally, we take \( h = h_{\text{PI}} \), a standard plug-in bandwidth for local linear estimators computed from the data \((\tilde{X}_i, \tilde{Y}_i), i \in \hat{C}_n(\rho_1, \rho_2)\).

The estimators from Section 3.4 also require to estimate the zeros \( \tau_1, \ldots, \tau_M \) at which \( \phi \) changes sign, and the same is required for \( \psi \) if the method in that section is used to compute predictors of the \( Y_i \)’s. We proceed as follows. First, since the \( \tau_j \)’s all correspond to a local minimum of \( \phi^\ast \), we find all the points at which \( \hat{\phi}_{\text{LL}} \) has local minima. Then, among those points we keep only those which are close to the discontinuity points of the derivative \( \phi^\ast \) detected by the method of Gijbels and Goderniaux (2004). Here, we define “close” by less than \( 2h \) away, where \( h \) is the bandwidth in Section 2.2.1 of Gijbels and Goderniaux (2004). Finally, to slightly improve numerical performance, we implement Remark 3.2 and remove the data corresponding to the 5% smallest \( \hat{f}_U(U_i) \)’s.

5.3. Simulations. We applied our methods to a variety of simulated examples, ranging from the simplest ones in which \( \psi > 0 \) and \( \phi > 0 \), where we can use the method from Section 3.2, to more complex ones in which \( EX = 0 \) and both \( \psi \) and \( \phi \) oscillate between positive and negative values, where we need to use the sophisticated approach suggested in Section 3.4.

We generated data \((\tilde{X}_i, \tilde{Y}_i, U_i), i = 1, \ldots, n\), from model (2.1) for \( n = 100, 200, 500 \) and 1000, and considered various combinations of \( m, \phi, \psi \) and \( \sigma \), and various distributions of \( X_i \) and \( U_i \). We took \( \varepsilon_i \sim N(0,1) \), and considered shifted versions of three regression curves \( m \), denoted by \( m_1, m_2 \) and \( m_3 \) and defined as \( m_1(x) = \sin\{\pi(x - 1)/2\}/\{1 + 2(x - 1)^2\}\{\text{sign}(x - 1) + 1\}, m_2(x) = x^2\phi_{0,1}(x), \) and \( m_3(x) = 2x + \phi_{0.5,0.1}(x) \), where \( \phi_{\mu,\theta} \) denotes the density of a \( N(\mu, \theta^2) \). In all cases below, the generic constant const. was chosen so that \( E\{\phi(U)\} = E\{\psi(U)\} = 1 \).

First, we considered models where the local linear estimators from Sections 3.2 to 3.4 could all be applied: (i.a) \( m = m_1, X_i \sim N(1, 1.5^2), \sigma(x) = 0.3 \); (ii.a) \( m = m_2, X_i \sim N(1, 1.5^2), \sigma(x) = 0.05 \); (iii.a) \( m = m_3, X_i \sim N(0.5, 0.75^2), \sigma(x) = \).

Finally, we considered models where the local linear estimators from Sections 3.2 to 3.4 could all be applied: (i.a) \( m = m_1, X_i \sim N(1, 1.5^2), \sigma(x) = 0.3 \); (ii.a) \( m = m_2, X_i \sim N(1, 1.5^2), \sigma(x) = 0.05 \); (iii.a) \( m = m_3, X_i \sim N(0.5, 0.75^2), \sigma(x) = \).
0.55; (i.b) \( m(\cdot) = m_1(\cdot - 1) + 2, X_i \sim N(2, 1.5^2), \sigma(x) = 0.3; (i.i.b) m(\cdot) = m_2(\cdot - 1), X_i \sim N(2, 1.5^2), \sigma(x) = 0.05; (i.i.i.b) m(\cdot) = m_3(\cdot - 1), X_i \sim N(1.5, 0.75^2), \sigma(x) = 0.55. \) Each time we took \( U_i \sim \beta(2, 5), \psi(u) = \text{const.}(u + 0.5)^2 \) and \( \varphi(u) = \text{const.}(u + 0.25)^2. \)

Next, we considered models (i.c)–(iii.c) and (i.d)–(iii.d), where we took \( m, X_i \) and \( \sigma \) as in models (i.a)–(iii.a) and (i.b)–(iii.b), respectively, but took \( U_i \sim \beta(3, 5) \) and \( \varphi(\cdot) = \psi(\cdot) = \text{const.} m_1(5 \cdot -2). \) Here, \( \varphi \) and \( \psi \) have zeros and change signs, so that the method from Section 3.2 cannot be applied. Finally, in our last models, \( \varphi \) and \( \psi \) change signs and have several zeros and \( E(X_i) = 0, \) so that we can apply only the method from Section 3.4: (iv.a) \( m(\cdot) = m_1(\cdot + 1), X_i \sim N(0, 1.5^2), \sigma(x) = 0.3; (v.a) m(\cdot) = m_2(\cdot + 1), X_i \sim N(0, 1.5^2), \sigma(x) = 0.05; (vi.a) m(\cdot) = m_3(\cdot + 0.5), X_i \sim N(0, 0.75^2), \sigma(x) = 0.55; (iv.b) m(\cdot) = m_1(\cdot), X_i \sim \chi^2(4)\,/2, \sigma(x) = 0.3; (v.b) m(\cdot) = m_2(\cdot), X_i \sim \chi^2(4)\,/2, \sigma(x) = 0.05; (vi.b) m(\cdot) = m_3(\cdot), X_i \sim \chi^2(4)\,/3.5, \sigma(x) = 0.55. \) Each time we took \( U_i \sim \beta(3, 5) \) and \( \varphi(\cdot) = \psi(\cdot) = \text{const.} m_1(5 \cdot -2). \) Heteroscedastic versions of these models gave similar results; see Appendix B in the supplementary file [Delaigle, Hall and Zhou (2016)].

We compared each of our estimators with the ideal estimator \( \hat{m}_\text{LL} \) at (3.1) computed from the \((X_i, Y_i)\)’s, which are not available in real data applications but are available when we simulate data, and with the inconsistent naive estimator \( \hat{m}_\text{LL, naive} \), which is the standard local linear estimator computed from the contaminated \((\tilde{X}_i, \tilde{Y}_i)\)’s. For each \( n \) and each model, we generated 1000 samples and constructed each estimator for each sample. Let \( \hat{m} \) denote any one of the estimators considered below. To summarize the performance of \( \hat{m} \), we computed, for each sample, the integrated squared error \( \text{ISE} = \int_a^b [\hat{m}(x) - m(x)]^2 \, dx \), where, in each case, \( a \) and \( b \) were the quantiles 0.025 and 0.975 of the distribution of \( X \).

In Tables 1 to 4 in Appendix B in the supplementary file [Delaigle, Hall and Zhou (2016)], for each method we report the first, second and third quartiles of the resulting 1000 ISEs. See Appendix B for a detailed discussion of the simulation results. In summary, we found that, as expected, when \( \varphi, \psi, EX \) and \( EY \) were different from zero, but \( EX \) and/or \( EY \) were relatively close to zero, the estimator that worked best was the one from Section 3.2, but the most complex estimator from Section 3.4 worked well. When \( EX \) and \( EY \) were far from zero, all three estimators worked well, with the simplest one from Section 3.2 giving the best results and the one from Section 3.4 working the worst. When \( \varphi \) and/or \( \psi \) had zeros, the estimator from Section 3.2 could not be applied, and when \( EX \) and \( EY \) were close to zero, the best results were obtained with the estimator from Section 3.4, whereas when \( EX \) and \( EY \) were far from zero, the estimator from Section 3.3 worked best. Finally, we found that our approach also performed well when the errors were heteroscedastic.

In all cases, our estimators performed considerably better than the naive estimator, but were of course outperformed by the oracle estimator. As expected, the
performance of our estimators improved as sample size increased. In all our simulation settings, the estimator from Section 3.4 gave reasonable results. However, if $\varphi$ and $\psi$ were far from zero, we got better results by using the simplest estimator from Section 3.2, and if $EX$ and $EY$ were far from zero, we got better results using the estimator from Section 3.3.

To illustrate these results graphically, we present a few figures that are representative of the conclusions of our simulations. For each estimator $\hat{m}$ presented in the figures, we show the three estimated curves corresponding to the first three quartiles of the 1000 ISEs defined above. In Figure 1, using example (i.a), we illustrate the fact that, when all three methods can be applied, they often give similar results. Figure 2 shows estimated curves for examples (ii.c) and (ii.d). We can see that, in case (ii.c), where $EX$ is close to zero, the estimator $\hat{m}_{\text{LL},\text{naive}}$ from Section 3.4 worked better than the one from Section 3.3, but that the reverse is true in case (ii.d), where $EX$ and $EY$ are both far from zero. In that figure, we also depict the naive estimator $\hat{m}_{\text{LL},\text{naive}}$, which performed very poorly. Finally, in Figure 3, we use example (vi.a) to demonstrate the improvement that our estimator $\hat{m}_{\text{LL},\text{naive}}$ from Section 3.4 benefits from as the sample size $n$ increases. Here, too, the naive estimator performed very poorly, even for $n$ large.

5.4. Real data illustrations. We applied our new method to the Boston house-price dataset described in Harrison and Rubinfeld (1978), available at https://archive.ics.uci.edu/ml/datasets/Housing, and which contains information about houses and their owners at 506 locations around Boston. As in Şentürk and Müller (2005b), we are interested in the relationship between the median price (in USD 1000s) of houses, $\tilde{Y}$, and per capita crime rate by town, $\tilde{X}$, with the confounding effect of the proportion of population of lower educational status, $U$, removed. Şentürk and Müller’s (2005b), whose interest was in the correlation between $\tilde{X}$ and $\tilde{Y}$, concluded that this correlation alters dramatically after adjusting for the confounding effect of lower educational status. On the left panel of Figure 4, we depict the covariate-adjusted regression curve obtained using the local
FIG. 2. Naive estimator $\hat{m}_{\text{LL, naive}}$ (left), $\hat{m}_{\text{LL}}(\cdots; \rho_1, \rho_2)$ from Section 3.3 (center), and estimator $\hat{m}_{\text{LL}}(\cdots; \rho_1, \rho_2)$ from Section 3.4 (right) for three samples coming from model (ii.c) (top) and model (ii.d) (bottom) with $n = 500$, and corresponding to the 1st, 2nd and 3rd quartiles of the ISEs. The continuous line depicts the true $m$.

FIG. 3. $\hat{m}_{\text{LL}}(\cdots; \rho_1, \rho_2)$ from Section 3.4 (first row) and naive estimator $\hat{m}_{\text{LL, naive}}$ (second row) for three samples coming from model (vi.a) with $n = 100$ (left), $n = 200$ (centre) and $n = 500$ (right), and corresponding to the 1st, 2nd and 3rd quartiles of the ISEs. The continuous line depicts the true $m$. 
linear estimator $\hat{m}_{\text{LL}}(\cdot; \rho_1, \rho_2)$ from Section 3.3, the estimator $\hat{m}_{\text{LL}}(\cdot; \rho_1, \rho_2)$ from Section 3.4, and the naive regression estimator $\hat{m}_{\text{LL, naive}}$ obtained by regressing $\tilde{Y}$ on $\tilde{X}$ after removing a few outliers. In this example, the estimator from Section 3.2 was identical to the one from Section 3.3. We can see that $\hat{m}_{\text{LL, naive}}$ indicates a pronounced relationship between house price and crime rate (as crime rate increases, house price decreases), but once we adjust for the effect of lower educational status, the regression curve obtained by both versions of our estimator is almost flat, indicating a weak relationship between the adjusted $X$ and $Y$.

Next, we applied our procedure to the diabetes dataset used by Schorling et al. (1997) and Willems et al. (1997), available at http://biostat.mc.vanderbilt.edu/DataSets, which represents a subset of 403 individuals taken from a larger cohort of 1046 subjects who participated in a study for African Americans about obesity, diabetes and related factors in central Virginia. As in Şentürk and Nguyen (2006), our goal was to examine the relationship between glycosolated hemoglobin level $\tilde{Y}$, a biomarker for diabetes, and diastolic blood pressure $\tilde{X}$, adjusting for the effect of body mass index, $U$, which was found to be a confounder for both variables. As in Şentürk and Nguyen (2006), we removed a few outliers before our analysis. As in the previous example, $\bar{X}$ and $\bar{Y}$ were far from zero, so that we used the estimator $\hat{m}_{\text{LL}}(\cdot; \rho_1, \rho_2)$ from Section 3.3, which we compared with the naive estimator $\hat{m}_{\text{LL, naive}}$. Here, too, the estimator from Section 3.2 was identical to the one from Section 3.3. We also computed the estimator from $\hat{m}_{\text{LL}}(\cdot; \rho_1, \rho_2)$ from Section 3.4. These estimators, depicted on the right panel of Figure 4, show that after adjusting for body mass index, the relationship between glycosolated hemoglobin level and diastolic blood pressure is noticeably less pronounced. We should highlight that, in this example, the data were rather sparse for diastolic blood pressure greater than 100, and the few patients for which $\tilde{X}$ was greater than 100 had a rather low value of $\tilde{Y}$, whence the decreasing shape on the right-hand side of the graph, which may just be an artifact of the sparseness of the data in that area.

Another interesting application of our method is to the baseline data collected from studies A and B of the Modification of Diet in Renal Disease Study [Levey
et al. (1994)]. The nonlinear relationship between the baseline unadjusted glomerular filtration rate (GFR) and serum creatinine (SCr) is of particular interest. Taking body surface area (BSA) as the confounder, Cui et al. (2009) used a parametric nonlinear model of the form \( m(x) = \beta_1 \exp(-\beta_2 - \beta_3 x^2) + \beta_4 \) to study the relationship between GFR and SCr after correcting for the distorting effect of BSA. Because this dataset is not publicly available, we shall not compare the proposed nonparametric method with that of Cui et al. (2009) in this paper.

6. Generalizations to the multivariate case. Our approach can be generalized to the \( d \)-variate case, \( d \geq 1 \), where we observe data distributed like a vector \((U, \tilde{X}^T, \tilde{Y})\), with \( \tilde{X} \in \mathbb{R}^d \) a distorted version of \( X \in \mathbb{R}^d \). Reflecting the fact that some components of \( \tilde{X} \) may not all be distorted, we write \( d = d_1 + d_2 \), with \( d_1 \geq 0 \) and \( d_2 \geq 1 \), and let \( X = (X_1^T, X_2^T)^T \) and \( \tilde{X} = (\tilde{X}_1^T, \tilde{X}_2^T)^T \), where \( X_1 = (X_1, \ldots, X_{d_1})^T \) and \( \tilde{X}_2 = (\tilde{X}_{d_1+1}, \ldots, \tilde{X}_d)^T \) is a distorted version of \( X_2 = (X_{d_1+1}, \ldots, X_d)^T \), and where we use the convention that \( X = X_2 \) if \( d_1 = 0 \). In this notation, the data \( \{(U_i, \tilde{Y}_i, X_{1i}^T, \tilde{X}_{2i}^T)\}_{i=1}^n \) we observe are generated by the model

\[
Y = m(X) + \varepsilon \sigma(X), \\
\tilde{Y} = \psi(U)Y, \tilde{X}_{d_1+r} = \varphi_r(U)X_{d_1+r}, \quad r = 1, \ldots, d_2,
\]

where \( m(x) = E(Y|X = x) \) is a curve we wish to estimate, the random variables \( X, U \) and \( \varepsilon \) are mutually independent, \( E(\varepsilon) = 0 \) and \( \text{var}(\varepsilon) = 1 \). As in (2.2), we assume that \( E(\psi(U)) = 1, E(\varphi_r(U)) = 1 \), for \( r = 1, \ldots, d_2 \).

The procedures from Section 3.2 to 3.4 can each be generalized to the multivariate setting, but for space constraint here we show only how to generalize the approach from Section 3.2. The same ideas can be applied for the methods from Sections 3.3 and 3.4. To construct a nonparametric version of the estimator from Section 3.2, we first construct predictors \( \hat{Y}_i \) and \( \hat{X}_{i, d_1+1}, \ldots, \hat{X}_{id} \) as in equation (3.4), and let

\[
\hat{X}_i = (X_{i1}, \ldots, X_{id_1}, \hat{X}_{i, d_1+1}, \ldots, \hat{X}_{id})^T.
\]

Next, we use a standard multivariate local linear regression estimator applied to the data \( (\hat{X}_i^T, \hat{Y}_i) \). That is, we define [see Fan and Gijbels (1996)] \( \hat{m}_{LLL}(x) = \hat{\alpha}_0 \), where

\[
(\hat{\alpha}_0, \hat{\alpha}_1) = \arg \min_{\alpha_0, \alpha_1 \in \mathbb{R}^d} \sum_{i=1}^n (\hat{Y}_i - \alpha_0 - \alpha_1^T (\hat{X}_i - x))^2 K_h(\hat{X}_i - x),
\]

with \( K_h(x) = \prod_{r=1}^d h_r^{-1} K(x_r/h_r) \) a \( d \)-dimensional product kernel, \( K \) a univariate kernel, and \( h = (h_1, \ldots, h_d)^T \) a vector of bandwidths.

It is well known that fully nonparametric estimators suffer from the curse of dimensionality, which means that as \( d \) increases, such estimators can only work reasonably well if the sample size is very large. To overcome this problem, a common approach is to restrict the regression model so that only univariate curves have to be fitted. A popular example is the additive model [Hastie and Tibshirani
which assumes that $m(\mathbf{X}) = m_0 + \sum_{j=1}^{d} m_j(X_j)$. In our context, the additive covariate-adjusted regression model can be written as

\[
\begin{aligned}
Y &= m_0 + \sum_{j=1}^{d} m_j(X_j) + \varepsilon \sigma(X), \\
\tilde{Y} &= \psi(U)Y, \tilde{X}_{d_1+r} = \varphi_r(U)X_{d_1+r}, \quad r = 1, \ldots, d_2,
\end{aligned}
\]

where $m_1, \ldots, m_d$ are unknown univariate functions satisfying $E\{m_j(X_j)\} = 0$ for $j = 1, \ldots, d$ and $m_0$ is an unknown parameter.

In the standard setting where the $(X_i^T, Y_i)$'s are directly observed, there are several ways to fit the additive model; see Horowitz (2014) for an overview of estimation and inference for nonparametric additive models. The simplest approach is to adapt to our setting the iterative backfitting algorithm of Buja, Hastie and Tibshirani (1989), as follows. First, let \( \hat{m}_0 = n^{-1} \sum_{i=1}^{n} \tilde{Y}_i \) and \( \hat{m}_j \equiv 0 \) for $j = 1, \ldots, d$. For $j = 1, \ldots, d$, update \( \hat{m}_j \) by taking it equal to a local linear regression estimator using the data \( \{(\hat{X}_{ij}, \tilde{Y}_i - \hat{m}_0 - \sum_{k \neq j} \hat{m}_j(\hat{X}_{ik}))\}_{i=1}^{n} \). Iterate until the estimates \( \hat{m}_j \) stabilize. (Here, \( \hat{X}_{ij} = X_{ij} \) if $j \leq d_1$.) Alternatively, instead of taking \( \hat{m}_j = 0 \) as initial estimators, we could start with a linear approximation of the model in (6.2). See Appendix C in the supplementary file [Delaigle, Hall and Zhou (2016)] for details. We could also apply similar transformations to other existing methods for fitting additive models, such as the approach suggested by Horowitz and Mammen (2004). The main theoretical challenge is a delicate analysis on how the presence of generated response and predictors affects the first-order asymptotic properties of the final estimators. However, deriving such results requires much more work than can possibly done in this paper, and so we leave this problem for future research. The method proposed in this section can be applied to creatinine data, which was analyzed by Şentürk and Müller (2006). In this study, serum creatinine level is taken as the response and the two predictors include cholesterol level and serum albumin level. The confounder variable $U$ is taken to be body mass index defined as weight/height. The readers can find more details about this dataset in Şentürk and Müller (2006).

7. Proof of Theorem 4.1. We start by introducing basic notations. For a kernel function $K$, we write $\mu_{K, \ell} = \int u^\ell K(u) du$ for nonnegative integers $\ell$. For any set $S$, we denote its complement by $S^c$ and its cardinality by $\#S$. Throughout, we let const denote a finite positive constant independent of $n$, which may take different values at each occurrence. We also use the following notation: $\mu_0 = E(X)$, $m_0 = E(Y)$, $\mu_0^+ = E|X|$, $m_0^+ = E|Y|$ and

\[
\begin{align}
\varphi_0 &= \mu_0 \varphi, \quad \psi_0 = m_0 \psi, \quad \varphi_0^+ = \mu_0^+ \varphi, \quad \psi_0^+ = m_0^+ \psi.
\end{align}
\]

We proceed with the proof of Theorem 4.1. For $u \in I_U = [0, 1]$, write

\[
\begin{aligned}
w_0(u) &\equiv 1, \quad \hat{w}_X(u) = \mu_0^+ \varphi(u)/\hat{\psi}_{0,LL}^+(u), \\
\hat{w}_Y(u) &\equiv \hat{m}_0^+ \psi(u)/\hat{\psi}_{0,LL}^+(u),
\end{aligned}
\]
where $\hat{\mu}_0^+ = \hat{E}[X] = n^{-1}\sum_{i=1}^n |\hat{X}_i|$, $\hat{m}_0^+ = \hat{E}[Y] = n^{-1}\sum_{i=1}^n |\hat{Y}_i|$ and $\hat{\varphi}_{0,\text{LL}}^+$ and $\hat{\psi}_{0,\text{LL}}^+$ are local linear estimators of $\varphi_0^+$ and $\psi_0^+$ defined below (3.3).

Noting the model at (2.1), and hence by (3.4) and (7.2),

(7.3) \[ \hat{X}_i = X_i \hat{w}_X(U_i), \quad \hat{Y}_i = Y_i \hat{w}_Y(U_i). \]

Substituting the expressions in (7.3) into (3.5) gives

\begin{align*}
\hat{m}_{\text{NW}}(x) - m(x) &= \left\{ n \hat{f}_X(x) \right\}^{-1} \sum_{i=1}^n K_h(x - \hat{X}_i) \{ \hat{w}_Y(U_i) - w_0(U_i) \} Y_i \\
&\quad + \left\{ n \hat{f}_X(x) \right\}^{-1} \sum_{i=1}^n K_h(x - \hat{X}_i) \{ m(X_i) - m(x) \} \\
&\quad + \left\{ n \hat{f}_X(x) \right\}^{-1} \sum_{i=1}^n K_h(x - \hat{X}_i) \sigma(X_i) \varepsilon_i \\
&= \hat{\Pi}_{01}(x) + \hat{\Pi}_{02}(x) + \hat{\Pi}_{03}(x),
\end{align*}

where

(7.5) \[ \hat{f}_X(x) = n^{-1} \sum_{i=1}^n K_h(x - \hat{X}_i). \]

**Proof of (1).** We start by establishing uniform bounds for $\hat{w}_X$ and $\hat{w}_Y$ which will be useful throughout the proof. Recalling that the $U_i$’s are supported on $I_U = [0,1]$, for $Z = Y$ or $Z = X$ we use the notation $\sup_{u \in [0,1]} |\hat{w}_Z(u) - w_0(u)|$. To derive our bounds, note that under conditions (B1)–(B6), for $\ell = 0, 1, 2$, we have [Hansen (2008), Masry (1996)]

\begin{align*}
&\sup_{u \in [0,1]} \left| \hat{\varphi}_{0,\text{LL}}^{+(\ell)}(u) - \varphi_0^{+(\ell)}(u) \right| = O_P \{ \delta_{\ell n}(g_1) \}, \\
&\sup_{u \in [0,1]} \left| \hat{\psi}_{0,\text{LL}}^{+(\ell)}(u) - \psi_0^{+(\ell)}(u) \right| = O_P \{ \delta_{\ell n}(g_2) \},
\end{align*}

(7.6) where, for all $t > 0$,

(7.7) \[ \delta_{\ell n}(t) \equiv t^2 + (n^{2\ell+1})^{-1/2}(\log n)^{1/2}. \]

In particular, for $g_1 = g_{1n} \asymp n^{-\beta_1}$ and $g_2 = g_{2n} \asymp n^{-\beta_2}$, we have $\delta_{0n}(g_1) = O(n^{-\lambda_1} / \sqrt{\log n})$ and $\delta_{0n}(g_2) = O(n^{-\lambda_2} / \sqrt{\log n})$, where $\lambda_v \equiv \min(2\beta_v, 1/2 - \beta_v/2) \in (0, 2/5]$, for $v = 1, 2$.

Now, using (7.1) and (7.2), we can write

\begin{align*}
\hat{w}_X(u) - w_0(u) &= \frac{\hat{\mu}_0^+ \varphi(u) - \hat{\varphi}_{0,\text{LL}}^+(u)}{\hat{\varphi}_{0,\text{LL}}^+(u)} \\
&= \frac{(\hat{\mu}_0^+ \varphi(u) - \mu_0^+ \varphi(u))}{\hat{\varphi}_{0,\text{LL}}^+(u)} + \frac{\varphi_0^+(u) - \hat{\varphi}_{0,\text{LL}}^+(u)}{\hat{\varphi}_{0,\text{LL}}^+(u)},
\end{align*}

(7.8)
Moreover, we also have
\[
\hat{\varphi} + \varphi = \hat{\varphi} - \varphi
\]
Later in our proof, it will also be useful to use the fact that (7.9)
\[
\alpha
\]
available. For this estimator, it is well known [see, e.g., Theorem 6 in Hansen (2008)] that max \( x \in [a, b] \) that \( f_X(x) > 0 \), we obtain
\[
\{ \hat{f}_X(x) \}^{-1} = \{ f_X(x) + \hat{f}_X(x) - f_X(x) \}^{-1}
\]
uniformly over \( x \in [a, b] \), and that max \( x \in [a, b] \) that \( \hat{f}_X(x) \) is O(1).

Next, we prove (7.11). For this, note that for any \( C > 0 \), we can write
\[
P \left\{ \max_{x \in [a, b]} | \hat{f}_X(x) - f_X(x) | > C h^{-1} \delta_0 n(g_1) \right\}
\leq P \left\{ \max_{x \in [a, b]} | \hat{f}_X(x) - f_X(x) | > C h^{-1} \delta_0 n(g_1), A_n \right\} + P(A_n^c),
\]
where \( A_n \) is an event that we shall define below, and which is such that \( P(A_n) \to 1 \) as \( n \to \infty \). Therefore, to prove (7.11), it suffices to handle the first term on the right.
side of the inequality above. Toward this end, first, comparing the definitions (7.5) and (7.10) we see that for each \( x \in \mathbb{R} \),

\[
|\hat{f}_X(x) - \hat{f}_X(x)|
\]

(7.13)

\[
\leq \| K' \|_\infty \| \hat{w}_X - w_0 \|_\infty \frac{1}{nh^2} \sum_{i=1}^{n} |X_i| I(|X_i - x| \leq h \text{ or } |\hat{X}_i - x| \leq h).
\]

To further bound the right-hand side of (7.13), we shall show that \( \hat{X}_i \) and \( X_i \) are uniformly close [see (7.16) below] as long as the estimation error of \( \hat{w}_X \) is well-controlled. To see this, for \( \lambda \geq 0 \), define the event

\[
E_n(\lambda) = \{ \| \hat{w}_X - w_0 \|_\infty \leq \lambda \log n \}.
\]

(7.14)

By (7.9), we have \( P \{ E_n(\lambda) \} \rightarrow 1 \) as \( n \rightarrow \infty \) provided that \( \lambda < \lambda_1 \). Moreover, define events

\[
E_{1n}(\lambda) = \left\{ \max_{1 \leq i \leq n} |X_i| \leq \lambda \log n \right\}
\]

(7.15)

\[
E_{2n}(\lambda) = \left\{ \max_{1 \leq i \leq n} |\varepsilon_i| \leq \lambda \log n \right\}.
\]

In the proof of Lemma F.1 in the supplementary file [Delaigle, Hall and Zhou (2016)], we shall show that for every given \( c > 0 \), there exist a constant \( C_1 > 0 \) such that

\[
P \{ E_{1n}(C_1) \} \geq 1 - \text{const}.n^{-c}.
\]

Let \( \alpha \in (\alpha_0, \lambda_1) \) be a constant, such that under condition (B6), \( n^{-\alpha} = o(h) \) and \( P \{ E_n(\alpha) \} \rightarrow 0 \) as \( n \rightarrow \infty \). On the event \( E_n(\alpha) \cap E_{1n}(C_1) \), we have

\[
\max_{1 \leq i \leq n} |\hat{X}_i - X_i| \leq \| \hat{w}_X - w_0 \|_\infty \max_{1 \leq i \leq n} |X_i| \leq C_1 n^{-\alpha} \log n,
\]

(7.16)

such that for every \( x \in [a, b] \), \( |X_i - x| \leq |\hat{X}_i - x| + C_1 n^{-\alpha} \log n \). Therefore, on the event \( E_n(\alpha) \cap E_{1n}(C_1) \) with \( n \) sufficiently large,

\[
I(|\hat{X}_i - x| \leq h) \leq I(|X_i - x| \leq 2h).
\]

(7.17)

It follows from (7.13) and (7.17) that, on \( E_n(\alpha) \cap E_{1n}(C_1) \) with \( n \) large enough,

\[
\max_{x \in [a, b]} |\hat{f}_X(x) - \hat{f}_X(x)|
\]

\[
\leq \| K' \|_\infty \| \hat{w}_X - w_0 \|_\infty (nh^2)^{-1} \max_{x \in [a, b]} \sum_{i=1}^{n} |X_i| I(|X_i - x| \leq 2h)
\]

(7.18)

\[
\leq \| K' \|_\infty \| \hat{w}_X - w_0 \|_\infty (nh^2)^{-1} \max_{x \in [a, b]} (|x| + 2h) \sum_{i=1}^{n} I(|X_i - x| \leq 2h)
\]

\[
\leq \text{const.} \| \hat{w}_X - w_0 \|_\infty h^{-2} \max_{x \in [a, b]} \{ \hat{f}_X(x + 2h) - \hat{f}_X(x - 2h) \},
\]
where \( \hat{F}_X(x) = n^{-1} \sum_{i=1}^{n} I(X_i \leq x) \) denotes the empirical distribution function. To further bound the right-hand side of (7.18), we let \( F_X \) be the distribution function of \( X \) and then apply the Dvoretzky–Kiefer–Wolfowitz inequality [Massart (1990)] to obtain that \( P(\sqrt{n}\|\hat{F}_X - F_X\|_\infty > y) \leq 2 \exp(-2y^2) \) for all \( y > 0 \), where \( \|\hat{F}_X - F_X\|_\infty = \sup_{x \in \mathbb{R}} |\hat{F}_X(x) - F_X(x)| \). For \( \lambda > 0 \), define the event

(7.19) \[ \mathcal{E}_{on}(\lambda) = \{ \sqrt{n}\|\hat{F}_X - F_X\|_\infty \leq (\lambda \log n)^{1/2} \}, \]

such that \( P(\mathcal{E}_{on}(1/2)) \geq 1 - 2n^{-1} \). Under condition (B3), we deduce that on the \( \mathcal{E}_{on}(1/2) \) with \( n \) sufficiently large,

\[
\begin{align*}
\max_{x \in [a,b]} \{ \hat{F}_X(x + 2h) - \hat{F}_X(x - 2h) \} & \leq \max_{x \in [a,b]} \{ F_X(x + 2h) - F_X(x - 2h) \} + \{ 2(\log n)/n \}^{1/2} \\
& \leq 4\|f_X\|_\infty h + \{ 2(\log n)/n \}^{1/2} \leq \text{const.} \ h.
\end{align*}
\]

Substituting this into (7.18) and taking \( \mathcal{A}_n = \mathcal{E}_n(\alpha) \cap \mathcal{E}_{on}(1/2) \cap \mathcal{E}_{1n}(C_1) \) imply that for all sufficiently large \( n \),

(7.21) \[ P \left\{ \max_{x \in [a,b]} |\hat{f}_X(x) - \hat{f}_X(x)| > Ch^{-1}\delta_{0n}(g_1), \mathcal{A}_n \right\} \]

\[ \leq P \{ \|\hat{w}_X - w_0\|_\infty > \text{const.} \ \delta_{0n}(g_1), \mathcal{A}_n \} \]

\[ \leq P \{ \|\hat{w}_X - w_0\|_\infty > \text{const.} \ \delta_{0n}(g_1) \}, \]

and that \( P(\mathcal{A}_n^c) \to 0 \) as \( n \to \infty \). Together, (7.9) and (7.21) prove (7.11).

Next, we study \( \hat{\Pi}_{01}(x) \). For this, we first write \( \hat{f}_X(x)\hat{\Pi}_{01}(x) \) as

\[
\begin{align*}
n^{-1} \sum_{i=1}^{n} K_h(x - X_i)(\hat{w}_Y - w_0)(U_i)Y_i & + n^{-1} \sum_{i=1}^{n} \{ K_h(x - \hat{X}_i) - K_h(x - X_i) \}(\hat{w}_Y - w_0)(U_i)Y_i \\
& \equiv J_1(x) + J_2(x).
\end{align*}
\]

Applying Lemma F.4 with \( g_2 \asymp n^{-\beta_2} \) to \( J_1(x) \) implies

(7.23) \[ \max_{x \in [a,b]} |J_1(x)| = O_p(g_2^2) = O_p(n^{-2\beta_2}). \]

For \( J_2(x) \), note that \( J_2(x) \leq \|K'\|_\infty \|\hat{w}_X - w_0\|_\infty \|\hat{w}_Y - w_0\|_\infty (nh^2)^{-1} \times \sum_{i=1}^{n} |X_i| \left( |m(X_i)| + \sigma(X_i)|\epsilon_i| I(|X_i - x| \leq h) \right) \). The argument leading to (7.11) can be used to prove that \( \max_{x \in [a,b]} n^{-1} \sum_{i=1}^{n} |X_i m(X_i)| \times I(|X_i - x| \leq h) = O_p(1) \) and the same bound holds if the \( m(X_i)'s \)
are replaced by the $\sigma(X_i)$’s. Moreover, similar to (F.9) in the proof of Lemma F.1, it can be proved that

$$\max_{1 \leq i \leq n} |e_i| = O_P(\log n).$$

(7.24)

This, together with (7.9) and the two displays before (7.24), yields

$$\max_{x \in [a,b]} |J_2(x)| = O_P\{h^{-1}\delta_0n(g_1)\delta_0n(g_2)\log n\} = o_P\{g_2^2 + (ng_2)^{-1}/2\}.$$

(7.25)

Here, the last step follows from condition (B6) and the assumption that $\alpha_0 < 2\beta_1$.

Together, (7.12), (7.22), (7.23) and (7.25) imply

$$\max_{x \in [a,b]} \left| \hat{\Pi}_{01}(x) \right| = O_P\{g_2\} + o_P\{(ng_2)^{-1/2}\}.$$

(7.26)

For $\hat{\Pi}_{02}(x)$, we write $K_h(x - \hat{X}_i)$ in $\hat{f}_X(x)\hat{\Pi}_{02}(x)$ as $K_h(x - \hat{X}_i) - K_h(x - X_i) + K_h(x - X_i)$. A similar argument to what we used to study (7.13) gives

$$\max_{x \in [a,b]} \left| \hat{\Pi}_{02}(x) \right| = O_P\{\delta_{n,0}(g_1)\}.$$

(7.28)

where $\Pi_{02}(x) = \{n\hat{f}_X(x)\}^{-1}\sum_{i=1}^n K_h(x - X_i)\{m(X_i) - m(x)\}.$

Next, we write $\hat{f}_X(x)\Pi_{02}(x) = n^{-1}\sum_{i=1}^n K_h(x - X_i)\{m(X_i) - m(x)\}$ as

$$n^{-1}\sum_{i=1}^n \{g_{n,i}(x) - E_{g_{n,i}}(x)\} + n^{-1}\sum_{i=1}^n E_{g_{n,i}}(x)$$

(7.29)

$$\equiv R_n(x) + n^{-1}\sum_{i=1}^n E_{g_{n,i}}(x),$$

where $g_{n,i}(x) = K_h(x - X_i)\{m(X_i) - m(x)\}$. To bound $\max_{x \in [a,b]} |R_n(x)|$, we create a grid using $N$ points of the form $x_j = a + j\delta$ with $\delta = (b - a)/N$ for some $N \geq 1$ to be determined below (7.32). Since $g'_{n,i}(x) = h^{-1}K'_h(x - X_i)\{m(X_i) - m(x)\}$
\[ m(x) - m'(x)K_h(x - X_i) \text{, by the mean value theorem we have, for every } x, y \in \mathbb{R}, \quad |g_{n,i}(x) - g_{n,i}(y)| \leq (\|K\|_\infty + \|K'\|_\infty)\|m'\|_\infty h^{-1} |x - y|. \] Therefore,

\[ \text{(7.30)} \quad \max_{x \in [a,b]} |R_n(x)| \leq \max_{1 \leq j \leq N} |R_n(x_j)| + 2\|K\|_\infty + \|K'\|_\infty \|m'\|_\infty \delta h^{-1}. \]

For each \( x \in \mathbb{R} \) fixed, \( g_{n,i}(x), \ldots, g_{n,n}(x) \) are independent random variables satisfying \( |g_{n,i}(x)| \leq \|K\|_\infty \|m\|_\infty \) and \( E\{g_{n,i}(x)\}^2 = h^{-1} \int K^2(t)(m(x - ht) - m(x))^2 f_X(x - ht) dt \leq \|m'\|^2 f_X(\infty) h \int t^2 K^2(t) dt \). Hence, by Bernstein's inequality and Boole's inequality, for every \( y \geq 0 \),

\[ P\left\{ \max_{1 \leq j \leq N} |R_n(x_j)| \geq y \right\} \]

\[ \leq \sum_{j=1}^N P\left[ \left| n^{-1} \sum_{i=1}^n \{g_{n,i}(x_j) - E\{g_{n,i}(x_j)\}\} \geq y \right| \right] \]

\[ \leq 2N \exp\left\{ -\frac{ny^2}{2(c_K^2 \|m'\|_\infty^2 \|f_X\|_\infty h + \|K\|_\infty \|m'\|_\infty^2 \|f_X\|_\infty) / 3} \right\}, \]

where \( c_K = \{ \int t^2 K^2(t) dt \}^{1/2} \). For every \( \lambda > 0 \), define the event

\[ \text{(7.32)} \quad C_n(\lambda) = \left\{ \max_{1 \leq j \leq N} |R_n(x_j)| \leq c_K \|m'\|_\infty \|f_X\|_\infty^{1/2} \sqrt{\frac{h\lambda}{n} + \|K\|_\infty \|m'\|_\infty \frac{\lambda}{n}} \right\}. \]

such that in view of (7.31), \( P\{C_n(\lambda)^c\} \leq 2N \exp(-\tau \lambda) \) for some absolute constant \( \tau > 0 \). By taking \( N = n \) and \( \lambda = 2\tau^{-1} \log n \), it follows from (7.30) and (7.32) that

\[ \text{(7.33)} \quad \max_{x \in [a,b]} |R_n(x)| = O_P\{h^{1/2}(n / \log n)^{-1/2} + n^{-1} \log n + (nh)^{-1}\}. \]

For the second term on the right-hand side of (7.29), standard arguments show that, under conditions (B3) and (B5),

\[ \text{(7.34)} \quad E\{g_{n,i}(x)\} = \left\{ m''(x) f_X(x) / 2 + m'(x) f_X'(x) \right\} \mu_K, 2h^2 + o(h^2) \]

uniformly in \( x \in [a, b] \). Consequently, combining (7.12), (7.28), (7.33) and (7.34), we get

\[ \text{(7.35)} \quad \max_{x \in [a,b]} |\hat{\Pi}_{02}(x)| = O_P(h^2). \]

For the last term \( \hat{\Pi}_{03}(x) \) in (7.4), we need to control the stochastic error

\[ \text{(7.36)} \quad \Delta_{n,\infty} \equiv \max_{x \in [a,b]} \left| n^{-1} \sum_{i=1}^n \left( K_h(x - \hat{X}_i) \sigma(X_i) \varepsilon_i \right) \right| \]

for \( \hat{X}_i = X_i \hat{w}_X(U_i) \) as in (7.3). To this end, we shall use a lattice argument by making a finite approximation of the compact interval \([a, b]\) using a sequence \( \{x_j\}_{j=1}^N \) of equidistant points \( x_j = a + j\delta \) for \( \delta = (b - a) / N \), and then discretize \( \Delta_{n,\infty} \) to
define $\Delta_{n,N} \equiv \max_{1 \leq j \leq N} |n^{-1} \sum_{i=1}^{n} K_h(x_j - \hat{X}_i)\sigma(X_i)\varepsilon_i|$. Here, $N$ is a positive integer that will be determined after (7.44).

Instead of dealing with $\Delta_{n,\infty}$ directly, we shall prove that $\Delta_{n,N}$ provides a fine approximation to $\Delta_{n,\infty}$, at least with high probability, and then restrict attention to $\Delta_{n,N}$. By definition of $\Delta_{n,N}$, we have $|\Delta_{n,\infty} - \Delta_{n,N}| \leq \|\sigma\|_{\infty} \|K\|_{\infty} \delta h^{-2} \max_{1 \leq i \leq n} |\varepsilon_i|$. Together with (7.24), this leads to

$$|\Delta_{n,\infty} - \Delta_{n,N}| = O_P(N^{-1} h^{-2} \log n).$$

For $\Delta_{n,N}$, shortly we shall prove by taking $N = n$ that

$$\Delta_{n,N} = O_P\{nh/\log n\}^{-1/2},$$

which together with (7.37) leads to

$$\Delta_{n,\infty} = O_P\{nh/\log n\}^{-1/2} + (nh^2)^{-1} \log n$$

(7.39)

where the last step relies on the identity $(nh^2)^{-1} \log n = (nh/\log n)^{-1/2} (nh^3/\log n)^{-1/2}$ and condition (B6). Combining (7.12) and (7.39) yields

$$\max_{x \in [a,b]} |\hat{f}_{103}(x)| \leq \max_{x \in [a,b]} \{\hat{f}_{\tilde{X}}(x)\}^{-1} \Delta_{n,\infty}$$

(7.40)

$$= O_P\{nh/\log n\}^{-1/2}.$$  

Together, (7.26), (7.35) and (7.40) complete the proof of (4.1).

Next, we prove (7.38). For $\lambda > 0$, let $V_1n(x) = \{\sum_{i=1}^{n} K_h^2(x - \hat{X}_i)\sigma^2(X_i)\}^{1/2}$, $V_2n(x) = \max_{1 \leq i \leq n} K_h(x - \hat{X}_i)\sigma(X_i)$ and define the event

$$D_n(N,\lambda)$$

(7.41)

$$= \{|\Delta_{n,N}| \leq \max_{1 \leq j \leq N} V_1n(x_j) \sqrt{\lambda}/n + \max_{1 \leq k \leq N} V_2n(x_j) \lambda/n\}.$$  

To deal with $V_1n(x)$, as in the proof of (7.21), put $A_n = \mathcal{E}_n(\alpha) \cap \mathcal{E}_1n(C_1) \cap \mathcal{E}_0n(1/2)$ with $\alpha \in (\alpha_0, \lambda_1)$ such that $P(A_n^c) \to 0$ as $n \to \infty$, where $\mathcal{E}_n(\alpha)$, $\mathcal{E}_1n(C_1)$ and $\mathcal{E}_0n(1/2)$ are as in (7.14), (7.15) and (7.19), respectively. On the event $A_n$ with $n$ sufficiently large, it follows from (7.17) and (7.20) that

$$\max_{1 \leq j \leq N} V_1n(x_j)$$

(7.42)

$$\leq \max_{x \in [a,b]} V_1n(x) \leq \|\sigma\|_{\infty} \|K\|_{\infty} \max_{x \in [a,b]} \sqrt{\sum_{i=1}^{n} \{I(|X_i - x| \leq 2h)\}}$$

$$\leq \text{const.} \|\sigma\|_{\infty} \|K\|_{\infty} (n/h)^{1/2}.$$
It is easy to see that \( \max_{x \in [a, b]} V_{2n}(x) \leq \| \sigma \|_{\infty} \| K \|_{\infty} h^{-1} \). This, combined with (7.41) and (7.42) yields, on the event \( D_n(N, \lambda) \cap A_n \) with \( n \) large enough,

\[
\Delta_{n,N} \leq \text{const.} \| \sigma \|_{\infty} \| K \|_{\infty} \left\{ \sqrt{\lambda/(nh)} + \lambda/(nh) \right\}.
\]

Next, we show that for properly chosen \( N \) and \( \lambda \), \( P\{D_n(N, \lambda)^c\} \to 0 \) as \( n \to \infty \). Observe that \( \hat{w}_X \) defined in (7.2) is a measurable function of \( \{(X_i, U_i)\}_{i=1}^n \), and thus is independent of \( \{\varepsilon_i\}_{i=1}^n \). Conditional on \( \{(X_i, U_i)\}_{i=1}^n \), taking \( a = (a_1, \ldots, a_n)^T = (K_h(x - \hat{X}_i)\sigma(X_i), \ldots, K_h(x - \hat{X}_n)\sigma(X_n))^T \) in Lemma F.2 and using Boole’s inequality, we obtain that for every \( \lambda \geq 0 \), \( P[D_{n,N} > \max_{1 \leq j \leq N} V_{1n}(x_j)\sqrt{\lambda}/n + \max_{1 \leq k \leq N} V_{2n}(x_j)\lambda/n |\{(X_i, U_i)\}_{i=1}^n] \leq 2N \exp(-c\lambda) \) where \( c > 0 \) is a constant independent of \( n \) and \( N \). Taking expectations on both sides of the inequality gives that for every \( \lambda \geq 0 \), \( P\{D_n(N, \lambda)^c\} \leq 2N \exp(-c\lambda) \).

Taking \( N = n \) and \( \lambda = 2c^{-1} \log n \), we get

\[
P\{D_n(n, \lambda)^c\} \leq 2n^{-1}.
\]

Combining (7.43) with \( N = n, \lambda = 2c^{-1} \log n \), (7.44) and the fact that \( P(A_n^c) \to 0 \) proves (7.38) as claimed. \( \square \)

**Proof of (II).** To prove the asymptotic normality, we need to use a more refined argument. In what follows, \( x \in [a, b] \) is fixed and we deal with the sum in (7.4) over each \( \hat{\Pi}_{0j}(x) \) separately.

First, for \( \hat{\Pi}_{01}(x) \), recall in (7.22) that \( \hat{f}_\hat{X}(x)\hat{\Pi}_{01}(x) = J_1(x) + J_2(x) \). By (7.25) and condition (B6), \( |J_2(x)| = o_P(h^{-1}\delta_{0n}(g_1)\delta_{0n}(g_2) \log n) = o_P(g_1^2 + g_2^2) \). For \( J_1(x) \), Lemma F.4 with \( g_{2n} \propto n^{-\beta_2} \) implies \( J_1(x) = -\frac{1}{2} m(x) f_\hat{X}(x) E[\psi''(U)/\psi(U)] \mu_{L,2} g_2^2 + o_P(g_2^2) \). The last two displays and (7.12) imply

\[
\hat{\Pi}_{01}(x) = -m(x) E[\psi''(U)/\psi(U)] \mu_{L,2} g_2^2/2 + o_P(g_1^2 + g_2^2).
\]

For \( \hat{\Pi}_{02}(x) \), by a first-order Taylor’s expansion we obtain

\[
\hat{f}_\hat{X}(x)\hat{\Pi}_{02}(x) = n^{-1} \sum_{i=1}^n K_h(x - \hat{X}_i)\{m(X_i) - m(x)\}
\]

\[
= n^{-1} \sum_{i=1}^n K_h(x - X_i)\{m(X_i) - m(x)\} + (nh^2)^{-1} \sum_{i=1}^n \frac{K'(x - X_i)}{h}
\]

\[
\times (w_0 - \hat{w}_X)(U_i)X_i\{m(X_i) - m(x)\} + (2nh^3)^{-1} \sum_{i=1}^n K''(\xi_n)
\]

\[
\times (w_0 - \hat{w}_X)^2(U_i)X_i^2\{m(X_i) - m(x)\} I(|\hat{X}_i - x| \leq h)
\]

\[
\equiv I_1(x) + I_2(x) + I_3(x),
\]

where \( \xi_n \) is a random variable that lies between \( (x - X_i)/h \) and \( (x - \hat{X}_i)/h \).
A standard argument shows that \( I_1(x) = O_P[h^2 + (nh)^{-1/2}] \). Together with (7.12), this yields
\[
\{ \hat{f}_X(x) \}^{-1} I_1(x) = \{ f_X(x) \}^{-1} I_1(x) + O_P\left[h^{-1}\delta_{n,0}(g_1)\right] = \{ f_X(x) \}^{-1} I_1(x) + o_P\left[h^2 + (nh)^{-1/2}\right].
\] (7.47)

For \( I_2(x) \), it follows from (F.12) in Lemma F.3 and (7.12) that \( \{ \hat{f}_X(x) \}^{-1} I_2(x) = x m'(x) E\left[\varphi''(U)/\varphi(U)\right] \mu_{L,2} g_1^2/2 + o_P(g_1^2) \). For \( I_3(x) \), a similar argument to that leading to (7.11) yields \( \max_{x \in [a, b]} |I_3(x)| = O_P\left(h^{-1}\delta_{n,0}(g_1)\right) \) and hence, \( \{ \hat{f}_X(x) \}^{-1} I_3(x) = o_P\left((nh)^{-1/2}\right) \). Combining with (7.46), we get
\[
\hat{f}_{02}(x) = \{ f_X(x) \}^{-1} I_1(x) + x m'(x) E\left[\varphi''(U)/\varphi(U)\right] \mu_{L,2} g_1^2/2 + o_P\left(g_1^2 + h^2 + (nh)^{-1/2}\right)
\] (7.48)

for \( I_1(x) \) as in (7.46). Finally, for the stochastic error term \( \hat{f}_{03}(x) \), we shall use an argument similar to that employed in Mammen, Rothe and Schienle (2012) based on empirical process theory. Write \( \beta_1 = (1 + \xi_0)/5 \) for some \( \xi_0 \geq 0 \). First, we argue that the estimator \( \hat{w}_X \) falls within a “nice” function space, the complexity of which can be measured via covering numbers. Let \( M \) be the set of functions \( [0, 1] \rightarrow \mathbb{R} \) whose derivatives up to order two exist and are uniformly bounded in order by \( (ng_1^2/\log n)^{-1/2} \approx n^{5/3}(\log n)^{1/2} \). Since \( \beta_1 \geq 1/5 \), we have \( \lambda_1 = \min(2\beta_1, 1/2 - \beta_1/2) = 1/2 - \beta_1/2 \). For some \( \alpha \in (\alpha_0, 1/2 - \beta_1/2) \) to be specified in the paragraph after (7.53), we define the following set of functions:
\[
N_{0n} = \{ w \in M_{0n} : \|w - w_0\|_\infty \leq n^{-\alpha} \}.
\] (7.49)

By (7.6), using the same argument that we used to derive (7.9), we have \( P(\hat{w}_X \in N_{0n}) \rightarrow 1 \) as \( n \rightarrow \infty \).

Note that \( \hat{f}_X(x) \hat{f}_{03}(x) \) in (7.4) can be written as \( n^{-1} \sum_{i=1}^n \{ K_h(x - \hat{X}_i) - K_h(x - X_i)\} \sigma(X_i) \epsilon_i + n^{-1} \sum_{i=1}^n K_h(x - X_i) \sigma(X_i) \epsilon_i \). For the first term, by Lemma F.1 we have, for any \( \kappa_1 \in (0, 1/2 + 3\alpha/4 - 3\alpha_0/2 - \xi_0/8) \), \( n^{-1} \sum_{i=1}^n \{ K_h(x - \hat{X}_i) - K_h(x - X_i)\} \sigma(X_i) \epsilon_i = O_P(n^{-\kappa_1}) \). On the other hand, it is straightforward to show that \( n^{-1} \sum_{i=1}^n K_h(x - X_i) \sigma(X_i) \epsilon_i = O_P\left((nh)^{-1/2}\right) = O_P\left(n^{-1/2 + \alpha_0/2}\right) \). Combining this and (7.12), we get
\[
\hat{f}_{03}(x) = \{nf_X(x)\}^{-1} \sum_{i=1}^n K_h(x - X_i) \sigma(X_i) \epsilon_i
\] + \( O_P\left(n^{-\kappa_1} + n^{-1 + \beta_1/2 + 3\alpha_0/2} (\log n)^{1/2}\right) \)
\[
= \{nf_X(x)\}^{-1} \sum_{i=1}^n K_h(x - X_i) \sigma(X_i) \epsilon_i + O_P\left(n^{-\kappa_1}\right).
\] (7.50)
Assembling (7.45), (7.48) and (7.50), we obtain that for any \( \alpha \in (\alpha_0, 1/2 - \beta_1/2) \) and \( \kappa_1 \in (0, 1/2 + 3\alpha/4 - 3\alpha_0/2 - \xi_0/8) \),

\[
\hat{m}_{NW}(x) - m(x) = \tilde{B}(x) + [f_X(x)]^{-1} I_1(x) + \sqrt{V(x)} N(x) + o_P(n^{-\kappa_1} + g_1^2 + g_2^2 + h^2),
\]

(7.51)

where \( \tilde{B}(x) \) and \( I_1(x) \) are as in part (ii) of Theorem 4.1 and (7.46), respectively, and \( N(x) \equiv \{V(x)^{-1/2}[nf_X(x)]^{-1}\sum_{i=1}^n K_h(x - X_i)\sigma(X_i)\varepsilon_i \} \) for \( V(x) \) is as in part (ii) of Theorem 4.1. Further, for \( I_1(x) = n^{-1}\sum_{i=1}^n K_h(x - X_i)[m(X_i) - m(x)] \), proceeding as in (7.29) we derive that

\[
\{f_X(x)^{-1}\}^{-1} I_1(x) = B_0(x) + o_P\{h^2 + (nh)^{-1/2}\}
\]

(7.52)

for \( B_0(x) \) as in part (ii) of Theorem 4.1. For the third addend on the right-hand side of (7.51), Lyapounov’s central limit theorem combined with Slutsky’s theorem yield

\[
N(x) \overset{D}{\longrightarrow} N(0, 1), \quad \text{as } n \to \infty.
\]

(7.53)

In particular, for \( h = h_n \asymp n^{-\alpha_0} \) with \( \alpha_0 \in (0, 1/2 - \beta_1) \), by taking \( \alpha \) and \( \kappa_1 \) in such a way that \( \frac{3}{4}\alpha_0 < \alpha < \frac{1}{2} - \frac{1}{2}\beta_1 \) and \( \frac{1}{2} - \frac{1}{2}\alpha_0 < \kappa_1 < \frac{1}{2} + \frac{3}{4}\alpha - \frac{3}{2}\alpha_0 - \frac{1}{8}\xi_0 \), we have \( n^{-\kappa_1} = o((nh)^{-1/2}) \). This, together with (7.51)–(7.53) proves (4.1). □

Acknowledgments. We thank three referees and an Associate Editor for their helpful comments which led to an improved version of the manuscript.

SUPPLEMENTARY MATERIAL

Supplement to “Nonparametric covariate-adjusted regression” (DOI: 10.1214/16-AOS1442SUPP; .pdf). This supplemental material contains more details for the implementation of the proposed estimators, additional simulation results as well as additional proofs omitted in the main text.

REFERENCES

BUJA, A., HASTIE, T. and TIBSHIRANI, R. (1989). Linear smoothers and additive models. Ann. Statist. 17 453–555. MR0994249

CARROLL, R. J. and HALL, P. (1988). Optimal rates of convergence for deconvolving a density. J. Amer. Statist. Assoc. 83 1184–1186. MR0997599

CHETVERIKOV, D. (2012). Adaptive test of conditional moment inequalities. Available at arXiv:1201.0167.

CUI, X., GUO, W., LIN, L. and ZHU, L. (2009). Covariate-adjusted nonlinear regression. Ann. Statist. 37 1839–1870. MR2533473

DELAIGLE, A., FAN, J. and CARROLL, R. J. (2009). A design-adaptive local polynomial estimator for the errors-in-variables problem. J. Amer. Statist. Assoc. 104 348–359. MR2504382

DELAIGLE, A., HALL, P. and ZHOU, W.-X. (2016). Supplement to “Nonparametric covariate-adjusted regression.” DOI:10.1214/16-AOS1442SUPP.
DÜMBGEN, L. and SPOKOINY, V. G. (2001). Multiscale testing of qualitative hypotheses. *Ann. Statist.* 29 124–152. MR1833961

FAN, J. and GIJBEELS, I. (1996). *Local Polynomial Modelling and Its Applications*. Monographs on Statistics and Applied Probability 66. Chapman & Hall, London. MR1383587

FAN, J. and TRUONG, Y. K. (1993). Nonparametric regression with errors in variables. *Ann. Statist.* 21 1900–1925. MR1245773

GIJBEELS, I. and GODERNIAUX, A.-C. (2004). Data-driven discontinuity detection in derivatives of a regression function. *Comm. Statist. Theory Methods* 33 851–871. MR2042771

GIJBEELS, I., HALL, P. and KNEIP, A. (1999). On the estimation of jump points in smooth curves. *Ann. Inst. Statist. Math.* 51 231–251. MR1707773

HANSEN, B. E. (2008). Uniform convergence rates for kernel estimation with dependent data. *Econometric Theory* 24 726–748. MR2409261

HARRISON, D. and RUBINFELD, D. L. (1978). Hedonic prices and the demand for clean air. *J. Environ. Econ. Manag.* 5 81–102.

HASTIE, T. J. and TIBSHIRANI, R. J. (1990). *Generalized Additive Models*. Monographs on Statistics and Applied Probability 43. Chapman & Hall, London. MR1082147

HOROWITZ, J. L. (2014). Nonparametric additive models. In *The Oxford Handbook of Applied Nonparametric and Semiparametric Econometrics and Statistics* (J. Racine, L. Su and A. Ullah, eds.) 129–148. Oxford Univ. Press, Oxford. MR3306924

HOROWITZ, J. L. and MAMMEN, E. (2004). Nonparametric estimation of an additive model with a link function. *Ann. Statist.* 32 2412–2443. MR2153990

LEE, S., SONG, K. and WHANG, Y. (2013). Testing functional inequalities. *J. Econometrics* 172 14–32. MR2997127

LEVEY, A. S., ADLER, S., BECK, G. J. et al. (1994). The effects of dietary protein restriction and blood pressure control on the progression of renal disease. *N. Engl. J. Med.* 330 877–884.

MAMMEN, E., ROTHE, C. and SCHIENLE, M. (2012). Nonparametric regression with nonparametrically generated covariates. *Ann. Statist.* 40 1132–1170. MR2985946

MASRY, E. (1996). Multivariate local polynomial regression for time series: Uniform strong consistency and rates. *J. Time Series Anal.* 17 571–599. MR1424907

MASSART, P. (1990). The tight constant in the Dvoretzky–Kiefer–Wolfowitz inequality. *Ann. Probab.* 18 1269–1283. MR1062069

RUPPERT, D., SHEATHER, S. J. and WAND, M. P. (1995). An effective bandwidth selector for local least squares regression. *J. Amer. Statist. Assoc.* 90 1257–1270. MR1379468

SCHORLING, J. B., ROACH, J., SIEGEL, M., BATURKA, N., HUNT, D. E., GUTERBOCK, T. M. and STEWART, H. L. (1997). A trial of church-based smoking cessation interventions for rural African Americans. *Preventive Med.* 26 92–101.

ŞENTÜRK, D. (2006). Covariate-adjusted varying coefficient models. *Biostatistics* 7 235–251.

ŞENTÜRK, D. and MÜLLER, H. (2005a). Covariate-adjusted regression. *Biometrika* 92 75–89. MR2158611

ŞENTÜRK, D. and MÜLLER, H. (2005b). Covariate adjusted correlation analysis via varying coefficient models. *Scand. J. Stat.* 32 365–383. MR2204625

ŞENTÜRK, D. and MÜLLER, H. (2006). Inference for covariate adjusted regression via varying coefficient models. *Ann. Statist.* 34 654–679. MR2281880

ŞENTÜRK, D. and MÜLLER, H. (2009). Covariate-adjusted generalized linear models. *Biometrika* 96 357–370. MR2507148

ŞENTÜRK, D. and NGUYEN, D. V. (2006). Estimation in covariate-adjusted regression. *Comput. Statist. Data Anal.* 50 3294–3310. MR2239669

WILLEMS, J. P., SAUNDERS, J. T., HUNT, D. E. and SCHORLING, J. B. (1997). Prevalence of coronary heart disease risk factors among rural blacks: A community-based study. *South. Med. J.* 90 814–820.
Zhang, J., Yu, Y., Zhu, L. and Liang, H. (2013). Partial linear single index models with distortion measurement errors. *Ann. Inst. Statist. Math.* **65** 237–267. MR3011622

Zhang, J., Zhu, L. and Liang, H. (2012). Nonlinear models with measurement errors subject to single-indexed distortion. *J. Multivariate Anal.* **112** 1–23. MR2957282

A. Delaigle

**Australian Research Council Centre of Excellence for Mathematical and Statistical Frontiers (ACEMS)**

**School of Mathematics and Statistics**

**University of Melbourne**

**Parkville, Victoria 3010**

**Australia**

E-MAIL: A.Delaigle@ms.unimelb.edu.au

W.-X. Zhou

**Australian Research Council Centre of Excellence for Mathematical and Statistical Frontiers (ACEMS)**

**School of Mathematics and Statistics**

**University of Melbourne**

**Parkville, Victoria 3010**

**Australia**

AND

**Department of Operations Research and Financial Engineering**

**Princeton University**

**Princeton, New Jersey 08544**

**USA**

E-MAIL: wenxinz@princeton.edu