Gerbes, $SU(2)$ WZW models and exotic smooth $\mathbb{R}^4$

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Abstract In this paper we bridge the gap between exotic smoothness structures on the Euclidean 4-space $\mathbb{R}^4$ and gerbes as well rational conformal field theories of WZW type. The first part of the paper describes the amazing relation between exotic $\mathbb{R}^4$ and non-cobordant codimension-1 foliations of the 3-sphere $S^3$ described by the elements of $H^3(S^3, \mathbb{R})$. This relation is used in the second part of the paper to relate the exotic $\mathbb{R}^4$ to $SU(2)$ WZW $\sigma$-models and its orientifolds via the integer cohomology classes $H^3(S^3, \mathbb{Z})$ as well as abelian gerbes. The full case of the real classes $H^3(S^3, \mathbb{R})$ is also discussed and can be related to Hitchin-Dirac structures. Finally we show the quantization of electric charge without magnetic monopoles but using small exotic $\mathbb{R}^4$.

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1. Introduction

One of the most important problems in modern physics is the unification of Einstein’s general relativity and quantum field theory into quantum gravity. Currently there are more or less two main approaches: string theory and loop quantum gravity. Both approaches have the modification of our view to concepts like space-time in common. If one formally considers the path integral
over space time geometries then one has to include the possibility of different smoothness structures for space time \cite{43}. Brans \cite{14,13,12} was the first who considered exotic smoothness as a possibility for space-time. He conjectured that exotic smoothness induces an additional gravitational field (Brans conjecture). The conjecture was established by Asselmeyer \cite{5} in the compact case and by Śladkowski \cite{46} in the non-compact case. But there is a big problem which prevents progress in the understanding of exotic smoothness especially for the $\mathbb{R}^4$; there is no known explicit coordinate representation. As the result no exotic smooth function on any such $\mathbb{R}^4$ is known even though there exist families of infinite continuum many different nondiffeomorphic smooth $\mathbb{R}^4$. Bizaca \cite{9} was able to construct an infinite coordinate patch by using Casson handles. But it seems hopeless to extract physical information from that approach.

In our paper we choose another way by considering a relative change of the smoothness structure. For that purpose we have to consider the technical tool of h-cobordism for 4-manifolds. Fortunately there is a structure theorem for such h-cobordisms reflecting the difference in the smoothness for two non-diffeomorphic but homeomorphic 4-manifolds: both manifolds differ by a contractable submanifold (with boundary) called an Akbulut cork\footnote{Named after Selman Akbulut who constructed the first exotic, contractable 4-manifold see \cite{1}.}. A careful analysis of that example leads us to the amazing relation between smoothness structures on 4-manifold and codimension-1 foliations of the boundary for the Akbulut cork. In case of the exotic $\mathbb{R}^4$, one can use the work of Bizaca to construct something like an Akbulut cork\footnote{The Akbulut cork of the $\mathbb{R}^4$ is not the 4-disk $D^4$ with boundary the 3-sphere $S^3$. We thank L. Taylor for communication of this error in a previous version of the paper.} inside of the $\mathbb{R}^4$. The embedding of this cork is described by an involution of the boundary, a homology 3-sphere. Thus we have to consider the codimension-1 foliations of the homology 3-sphere $\Sigma$ (up to foliated cobordism) which are related to the real cohomology $H^3(\Sigma, \mathbb{R})$ isomorphic to $H^3(S^3, \mathbb{R})$ also known as Godbillon-Vey invariant. By using the diffeomorphism $\Sigma \# S^3 = \Sigma$ there is a 3-sphere lying inside of the homology 3-sphere $\Sigma$. A codimension-1 foliation of the 3-sphere induces uniquely a foliation on the homology 3-sphere $\Sigma$.

The second amazing relation was found by using the isomorphism $S^3 = SU(2)$. By using the integer cohomology classes $H^3(S^3, \mathbb{Z})$ we are able to rewrite the whole model as $SU(2)$ WZW $\sigma$ models and its orientifolds. The levels $k$ of that conformal field theory label the exotic $\mathbb{R}^4$'s. But what about the real cohomology classes? Here we need the whole theory of gerbes and the deformations by gerbes as well the theory of generalized structures a la Hitchin.

General technique of our work is to trace the changes of various structures on $S^3$, like $S^1$-gerbes, the levels of $SU(2)$ WZW models or generalized geometries of Hitchin, as corresponding to changes of smoothness on 4-manifolds. Even though exotic manifolds are 4-dimensional, the structures we deal with are strongly related to string theory. This observation indicates that exotic $\mathbb{R}^4$'s may play a distinguished geometrical role for quantum gravity which extends the role of standard smooth $\mathbb{R}^4$ in the Riemannian geometry and classical gravity (see also \cite{6,38}). The further recognition of this extremely interesting point, as the role of exotic 4-smoothness in string theory and the relation to supersymmetry, we leave for a separate work, and this paper serves as crucial first step.
Another thing which should be commented here is the case of large exotic $\mathbb{R}^4$’s. This paper relates small exotic $\mathbb{R}^4$’s, i.e. those which are determined by an Akbulut cork, and 3-rd cohomologies of $S^3$. The interesting question would be to find an analogous characterization of large exotic $\mathbb{R}^4$’s, i.e. those which contain a compact set non-embeddable in standard $\mathbb{R}^4$. The techniques presented here seem to be crucial for that case too, though we have to take some further deformations of our test space, i.e. $S^3$, to detect correctly the changes in large smoothness on $\mathbb{R}^4$. The possible indications come from the categorical approach to gerbes and related generalized quotients of spaces, as well from geometries of Gualtieri-Hitchin again.

The paper is organized as follows. In the next section we present a short introduction into smoothness structures on manifolds and codimension-1 foliations. Then we will derive the relation between exotic $\mathbb{R}^4$ and codimension-1 foliations of $S^3$. The reader who is not interested in the details of that approach should keep in mind the following result:

*The exotic $\mathbb{R}^4$ is determined by the codimension-1 foliation with non-vanishing Godbillon-Vey class in $H^3(S^3, \mathbb{R}^3)$ of a 3-sphere seen as submanifold $S^3 \subset \mathbb{R}^4$.*

In Sect. 3 we will use this result to get a relation between exotic $\mathbb{R}^4$’s and the $SU(2)$ WZW $\sigma$ models. Then the exotic $\mathbb{R}^4$’s can differ by the level $k$ of that theory. By this model we are only able to cover the integer cohomology classes $H^3(S^3, \mathbb{Z})$. For the general case we make essentially usage of the deformed geometries of Hitchin where the case of integer classes corresponds now to the deformations by $S^1$-gerbes.

In the last section we will show that the appearance of small exotic $\mathbb{R}^4$ can be used to explain the quantization of electric charge but without magnetic monopoles.

### 2. Exotic smoothness and foliations

First we will give a short overview about differential or smooth structures on manifolds especially in dimension 4. We refer to the book [47] for further information relating smooth or differential structures on manifolds.

A manifold is described by charts $h_i$: homeomorphic maps from subsets of the manifold $M$ into the linear space $\mathbb{R}^n$ $$h_i: M \supset W_i \rightarrow U_i \subset \mathbb{R}^n.$$ These charts describe the local properties of the manifold captured by linear spaces. But the really interesting property is the transition map between these charts. Assume two charts $h_i: W_i \rightarrow U_i$ and $h_j: W_j \rightarrow U_j$. The overlapping origin $W_{ij} = W_i \cap W_j$ will be mapped into two (usually different) images $U_{ij} = h_i(W_{ij})$ and $U_{ji} = h_j(W_{ij})$. A *transition map* between two charts is a map between subsets of linear spaces: $$h_{ij}: U_{ij} \rightarrow U_{ji}, \ h_{ij}(x) = h_j\left(h_i^{-1}(x)\right).$$

Two charts $h_i$, $h_j$ are compatible if $U_{ij}$, $U_{ji}$ are open (possibly empty), and the transition maps $h_{ij}$, $h_{ji}$ (with $W_i \cap W_j \neq \emptyset$) are *diffeomorphisms*. A family of pairwise compatible charts that covers the whole manifold is an *atlas*, and two atlases are *equivalent* if their union is an atlas again.
2.1. Differential structures in dimension 4. A differential structure of the manifold \( M \) is an equivalence class of the atlases of the manifold \( M \). We call two atlases \( \mathcal{A}, \mathcal{A}' \) equivalent iff there are diffeomorphisms between the transition maps. As an important fact we will note that there is only one differential structure of any manifold of dimension smaller than four. For all manifolds larger than four dimensions there is only a finite number of possible differential structures \( \text{Diff}_{\text{dim}} M \). The following table lists the numbers of differential structures up to dimension 11.

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|----|----|
| \#Diff\(_n\) | 1 | 1 | 1 | \(\infty(?)\) | 1 | 1 | 28 | 2 | 8 | 6 | 992 |

In dimension four there is a countable number of differential structures on most compact four-manifolds and an uncountable number for most non-compact four-manifolds.

What we need for the following is a kind of “comparing the differential structures” i.e. an analysis of the map \( M \to N \) for two 4-manifolds with different differential structures but the same topological structure. This map can be also interpreted as a procedure to modify \( M \) in many steps to get \( N \). Then we will get a sequence of 4-manifolds starting with \( M \) and ending with \( N \). Such a 5-manifold is called a cobordism and if the start and end manifold are topological equivalent then it is called a h-cobordism.[3] Thus we have to study h-cobordisms between 4-manifolds to understand the differential structures. Now given two smooth, homeomorphic 4-manifolds \( M, N \) together with a smooth h-cobordism \( W \). If we can show that \( W \) is diffeomorphic to \( M \times [0, 1] \) then there is a diffeomorphism between \( M \) and \( N \), i.e. both manifolds have the same differential structure.

By the work of Freedman [25], one can only show that \( W \) is homeomorphic to \( M \times [0, 1] \) but there are many counterexamples for the smooth case, i.e. \( M \) and \( N \) carries different differential structures. In 1996, a bulk of mathematicians (see [22]) proved a structure theorem for h-cobordant 4-manifolds. Given two simply-connected, smooth, h-cobordant 4-manifolds \( M, N \) with different differential structure. Then there are submanifolds \( M_0 \subset M, N_0 \subset N \) which are contractable and having a common boundary \( \Sigma = \partial M_0 = \partial N_0 \) so that we have the decompositions \( M = M_0 \cup_{\Sigma} M_1, N = N_0 \cup_{\Sigma} N_1 \) with a diffeomorphism \( M_1 \to N_1 \) relative to the boundary \( \Sigma \). But that means that the “difference” between the differential structures is concentrated at \( M_0 \) and \( N_0 \). This “difference” in the differential structures can be localized in contractable pieces \( M_0, N_0 \). Furthermore it is known [25] that the boundary of \( M_0, N_0 \) is a homology 3-sphere \( \Sigma \).

To detect the non-triviality of the h-cobordism we have to understand where the non-triviality of the h-cobordism \( W \) comes from. The structure theorem states only that the h-subcobordism between \( M_0 \) and \( N_0 \) is the only non-trivial part of \( W \). An alternative way to describe this non-trivial subcobordism is give by the following procedure: Cut out \( M_0, N_0 \) and glue in \( M_0 \) with the identity map \( \text{id} : \partial M_0 \to \partial (M \setminus M_0) \) and \( N_0 \) with the map \( \theta : \partial N_0 \to \partial (N \setminus N_0) \). The map is called an involution and has the properties: \( \theta \neq \text{id} \) and \( \theta \circ \theta = \text{id} \). We denote the contractable pieces \( M_0, N_0 \) as Akbulut corks.

Firstly we remark that the most important point is the involution \( \theta \) of the boundary. Secondly we remark that the structure theorem above is valid for

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[3] The exact definition is: A h-cobordism is a 5-manifold \( W \) with boundary \( \partial W = M \cup N \) so that the embeddings \( M, N \to W \) induces homotopy-equivalences.
compact 4-manifolds. But we are also interested in the non-compact case. Then the concept of a h-cobordism has to be replaced by the so-called “engulfing”. In [22] it was remarked that the proof for the h-cobordism can be extended to the engulfing for non-compact manifolds. We will use implicitly that result in the following.

2.2. Codimension-1 Foliations of the 3-sphere. During that section we will describe a connection between the differential structure on some exotic $\mathbb{R}^4$ and foliations of the 3-sphere $S^3$. In short, a foliation of a smooth manifold $M$ is an integrable subbundle $N \subset TM$ of the tangent bundle $TM$. The existence of codimension-1-foliations depends strongly on the compactness or non-compactness of the manifold. Every compact manifold admits a codimension-1-foliation if and only if the Euler characteristics vanish. In the following we will concentrate on the 3-sphere $S^3$ with vanishing Euler characteristics admitting codimension-1-foliations.

2.2.1. Definition of Foliation and its cobordism. A codimension $k$ foliation of an $n$-manifold $M^n$ (see the nice overview article [39]) is a geometric structure which is formally defined by an atlas $\{\phi_i : U_i \rightarrow M^n\}$, with $U_i \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$, such that the transition functions have the form $\phi_{ij}(x, y) = (f(x, y), g(y))$, $[x \in \mathbb{R}^{n-k}, y \in \mathbb{R}^k]$. Intuitively, a foliation is a pattern of $(n-k)$-dimensional stripes - i.e., submanifolds - on $M^n$, called the leaves of the foliation, which are locally well-behaved. The tangent space to the leaves of a foliation $\mathcal{F}$ forms a vector bundle over $M^n$, denoted $T\mathcal{F}$. The complementary bundle $\nu\mathcal{F} = TM^n/T\mathcal{F}$ is the normal bundle of $\mathcal{F}$. Such foliations are called regular in contrast to singular foliations or Haefliger structures. For the important case of a codimension-1 foliation we need an overall non-vanishing vector field or its dual, an one-form $\omega$. This one-form defines a foliation iff it is integrable, i.e.

$$d\omega \wedge \omega = 0$$

and the leaves are the solutions of the equation $\omega = \text{const}$.

Now we will discuss an important equivalence relation between foliations, cobordant foliations. Let $M_0$ and $M_1$ be two closed, oriented $m$-manifolds with codimension-$q$ foliations. Then these foliated manifolds are said to be foliated cobordant if there is a compact, oriented $(m+1)$-manifold with boundary $\partial W = M_0 \sqcup M_1$ and with a codimension-$q$ foliation transverse to the boundary and inducing the given foliation there. The resulting foliated cobordism classes form a group under disjoint union.

2.2.2. Non-cobordant foliations of $S^3$ detected by the Godbillon-Vey class. In [49], Thurston constructed a foliation of the 3-sphere $S^3$ depending on a polygon $P$ in the hyperbolic plane $\mathbb{H}^2$ so that two foliations are non-cobordant if the corresponding polygons have different areas. For later usage, we will present this construction now.

Consider the hyperbolic plane $\mathbb{H}^2$ and its unit tangent bundle $T_1\mathbb{H}^2$, i.e the tangent bundle $T\mathbb{H}^2$ where every vector in the fiber has norm 1. Thus the bundle

\footnote{In general, the differentiability of a foliation is very important. Here we consider the smooth case only.}
$T_1 \mathbb{H}^2$ is a $S^1$-bundle over $\mathbb{H}^2$. There is a foliation $\mathcal{F}$ of $T_1 \mathbb{H}^2$ invariant under the isometries of $\mathbb{H}^2$ which is induced by bundle structure and by a family of parallel geodesics on $\mathbb{H}^2$. The foliation $\mathcal{F}$ is transverse to the fibers of $T_1 \mathbb{H}^2$. Let $P$ be any convex polygon in $\mathbb{H}^2$. We will construct a foliation $\mathcal{F}_P$ of the three-sphere $S^3$ depending on $P$. Let the sides of $P$ be labeled $s_1, \ldots, s_k$ and let the angles have magnitudes $\alpha_1, \ldots, \alpha_k$. Let $Q$ be the closed region bounded by $P \cup P'$, where $P'$ is the reflection of $P$ through $s_1$. Let $Q_\epsilon$ be $Q$ minus an open $\epsilon$-disk about each vertex. If $\pi: T_1 \mathbb{H}^2 \to \mathbb{H}^2$ is the projection of the bundle $T_1 \mathbb{H}^2$, then $\pi^{-1}(Q_\epsilon)$ is a solid torus $Q \times S^1$(with edges) with foliation $\mathcal{F}_Q$ induced from $\mathcal{F}$.

For each $i$, there is an unique orientation-preserving isometry of $\mathbb{H}^2$, denoted $I_i$, which matches $s_i$ point-for-point with its reflected image $s'_i$. We glue the cylinder $\pi^{-1}(s_i \cap Q_\epsilon)$ to the cylinder $\pi^{-1}(s'_i \cap Q_\epsilon)$ by the differential $dI_i$ for each $i > 1$, to obtain a manifold $M = (S^2 \setminus \{k$ punctures$\}) \times S^1$, and a (glued) foliation $\mathcal{F}_P$, induced from $\mathcal{F}_Q$. To get a complete $S^3$, we have to glue-in $k$ solid tori for the $k S^1 \times$ punctures. Now we choose a linear foliation of the solid torus with slope $\alpha_k/\pi$ (Reeb foliation). Finally we obtain a smooth codimension-1 foliation $\mathcal{F}_P$ of the 3-sphere $S^3$ depending on the polygon $P$.

Now we consider two codimension-1 foliations $\mathcal{F}_1, \mathcal{F}_2$ depending on the convex polygons $P_1$ and $P_2$ in $\mathbb{H}^2$. As mentioned above, these foliations $\mathcal{F}_1, \mathcal{F}_2$ are defined by two one-forms $\omega_1$ and $\omega_2$ with $d\omega_a \wedge \omega_a = 0$ and $a = 0, 1$. Now we define the one-forms $\theta_a$ as the solution of the equation

$$d\omega_a = -\theta_a \wedge \omega_a$$

and consider the closed 3-form

$$\Gamma_{\mathcal{F}_a} = \theta_a \wedge d\theta_a$$

associated to the foliation $\mathcal{F}_a$. As discovered by Godbillon and Vey [28], $\Gamma_{\mathcal{F}}$ depends only on the foliation $\mathcal{F}$ and not on the realization via $\omega, \theta$. Thus $\Gamma_{\mathcal{F}}$, the Godbillon-Vey class, is an invariant of the foliation. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two cobordant foliations then $\Gamma_{\mathcal{F}_1} = \Gamma_{\mathcal{F}_2}$. In case of the polygon-dependent foliations $\mathcal{F}_1, \mathcal{F}_2$, Thurston [49] obtains

$$\Gamma_{\mathcal{F}_a} = \text{vol}(\pi^{-1}(Q)) = 4\pi \cdot \text{Area}(P_a)$$

and thus

- $\mathcal{F}_1$ is cobordant to $\mathcal{F}_2 \implies \text{Area}(P_1) = \text{Area}(P_2)$
- $\mathcal{F}_1$ and $\mathcal{F}_2$ are non-cobordant $\iff \text{Area}(P_1) \neq \text{Area}(P_2)$

We note that $\text{Area}(P) = (k - 2)\pi - \sum \alpha_k$. The Godbillon-Vey class is an element of the deRham cohomology $H^3(S^3, \mathbb{R})$ which will be used later to construct a relation to gerbes. Furthermore we remark that the classification is not complete. Thurston constructed only a surjective homomorphism from the group of cobordism classes of foliation of $S^3$ into the real numbers $\mathbb{R}$. We remark the close connection between the Godbillon-Vey class [11] and the Chern-Simons form if $\theta$ can be interpreted as connection of a suitable line bundle.

For later use we will also discuss the codimension-1 foliations of a homology 3-sphere $\Sigma$. Because of the diffeomorphism $\Sigma \# S^3 = \Sigma$, we can relate a foliation on $\Sigma$ to a foliation on $S^3$. By using the surgery along a knot or link, more is
true. One starts with a 3-sphere which is modified by using a knot or link to get every compact 3-manifold.

In the construction of foliation of the 3-sphere one glues in solid tori with a Reeb foliation. If we make a surgery along a knot by using one of these tori then we obtain an arbitrary 3-manifold which only depends on the used knot. Thus we choose a knot or link so that we will get $\Sigma$. Then the whole procedure produces a codimension-1 foliation on $\Sigma$ which is induced from the foliation on the 3-sphere.

2.3. The relation between smooth structures on 4-manifolds and foliations on 3-manifolds. Now we use the information about foliations of the $S^3$ and smooth structures on 4-manifolds to uncover a relation between both structures. The starting point is a simple problem: Given a 4-manifold $M$ with boundary $\Sigma = \partial M$ a 3-manifold. Suppose $M$ carries an exotic smooth structure. Is it possible, and if yes how, to detect the exoticness on the boundary? Every 3-manifold admits an unique smooth structure (up to diffeomorphisms). Thus if we want to give a positive answer to this question we have to look for another structure on 3-manifolds detecting the exoticness of the 4-manifold. Here we will show that non-cobordant foliations of $\Sigma$ detect exotic smoothness of $M$.

As starting point we consider the non-compact 4-manifold $\mathbb{R}^4$ with the boundary $S^3$ at infinity. In the subsection 2.1 we defined the Akbulut cork (the submanifolds $M_0, N_0$) now denoted by $A$ as a contractable pieces determing the smooth structure. One changes the smooth structure by cutting $A$ out and reglue it by using a non-trivial involution.

As remarked in [22], for non-compact manifolds like the $\mathbb{R}^4$ one can proof a similar theorem, the decomposition of proper h cobordisms, as for compact, closed 4-manifolds. Now we consider a particular exotic $\mathbb{R}^4$ constructed by Bizaca (see [9,10]). Bizaca used the Akbulut cork $A'$ of a compact 4-manifold $M$ (the so-called K3-surface) to construct a neighborhood $N(A')$ of the cork $A' \subset M$ in $M$ (see [31] for an explicit construction). The interior of that neighborhood $\text{int}(N(A'))$ is homeomorphic but not diffeomorphic to $\mathbb{R}^4$. Then Bizaca was able to proof that the corresponding $\mathbb{R}^4$ carries an exotic differential structure. The neighborhood $N(A)$ consists of two parts: the contractable manifold $A$ and the gluing map of $A$ into the K3 surface $M$. The gluing map can be understood as follows. Cut $A$ out from $M$ to get $M \setminus A$ and glue in $A$ via an involution $\tau : \partial A \to \partial A$ of the boundary. The gluing map can be understood as follows. Cut $A$ out from $M$ to get $M \setminus A$ and glue in $A$ via an involution $\tau : \partial A \to \partial A$ of the boundary. The boundary $\partial A$ of $A$ is a homology 3-sphere [24], i.e. we have to study the involution of the homology 3-sphere. But because of the diffeomorphism $\partial A = \partial A \# S^3$, an involution of $\partial A$ can be mainly described by an involution of the 3-sphere $S^3$. But Bizaca found another representation of this involution or gluing map of $\partial A$: it can be described by a Casson handle. Thus the gluing map of the Akbulut cork $A$ can be described by an involution. Especially using the diffeomorphism $\partial A = \partial A \# S^3$ this involution has the same properties as an involution of the 3-sphere.

Therefore we are faced with the problem to construct a non-trivial involution of the 3-sphere $S^3$. The following is known:

1. Every involution of the 3-sphere, i.e. a map $\tau : S^3 \to S^3$ with $\tau \circ \tau = \text{id}_{S^3}$, is orientation reversing.
2. The only fix-point-free involution $\tau$ is the trivial one $\tau = \pm \text{id}_{S^3}$.
3. The fix-point set $Fix(\tau)$ of the involution $\tau$ can be one points (north or south pole) or a 2-sphere $S^2$ (the equator). The involution $\tau$ with $Fix(\tau) = \{\star\}$ a point $\star$ is the trivial involution $\tau = \pm id_{S^3}$.

4. An involution $\tau$ with $Fix(\tau) = S^2$ defines an embedding $S^2 \to S^3$ and vice versa. The set of involutions with $Fix(\tau) = S^2$ has a dense subset $[8]$ given by the wild embeddings $S^2 \to S^3$ (Alexanders horned sphere $[4]$).

5. An involution of a homology 3-sphere has the same possible fixpoint set as the involution $\tau$.

Thus the non-trivial involution can be detected by the wild embedding $S^2 \to S^3$. In the following we will construct such an embedding inside of the Casson handle to get a non-trivial involution.

Now we are faced with the problem to recognize the non-trivial involution in the Casson handle to finish our argumentation. To understand this problem we have to answer two questions: what is a Casson handle? and how it can be constructed? A Casson handle $CH$ is the trial to embed a disk $D^2$ into a 4-manifold. In most cases this trial fails and Casson $[20]$ looked for a substitute, which is now called a Casson handle. Freedman $[25]$ showed that every Casson handle $CH$ is homeomorphic to the open 2-handle $D^2 \times \mathbb{R}^2$ but in nearly all cases it is not diffeomorphic to the standard handle $[20][30]$. The Casson handle will be build by an iterated procedure. One starts with an immersed disk into some 4-manifold $M$, i.e. a map $D^2 \to M$ with injective differential. Every immersion $D^2 \to M$ is an embedding except on a countable set of points, the double points. One can kill one double point by immersing another disk into that place. These disks form the 1-stage of the Casson handle. By iteration one can produce the other stages. Finally one considers not the immersed disk but rather a tubular neighborhood $D^2 \times D^2$ of the immersed disk and as well for every stage. The union of all neighborhoods of all stages is the Casson handle $CH$. So, there are two input data to construct $CH$: the number of double points in every stage and the orientation $\pm$ of the double points. Thus we can visualize the Casson handle $CH$ by a tree: the root is the immersion $D^2 \to M$ with $k$ double points, the 1-stage forms the next level of the tree with $k$ vertices connected with the root by edges etc. The edges are evaluated by the orientation $\pm$. Every Casson handle can be represented by this infinite tree. The Casson handle $CH(R_+)$ represented by the link in Fig.1 is the simplest Casson handle represented by the simplest tree $R_+$: one vertex in each level connected by one edge with evaluation $+$. For the details of the construction we refer to the book $[31]$ and the original articles $[24][25]$. Now we consider the immersed disk (i.e. with self-intersections) in the Casson handle. The boundary of this disk is a knot which is also the building block of the Casson handle (see Fig.2). Now we need the following facts:

1. There is a surface with one boundary components having this knot as boundary.
2. This surface (see Fig. 3) consists of a tower (consisting of levels) of genus \( g \)-surfaces \( (g > 0) \) called a grope \( G \) (see the appendix of \[18\] for a readable overview).

3. This grope represents a perfect fundamental group \( \pi_1 \), i.e. a group identical to its commutator group \( \pi_1 = [\pi_1, \pi_1] \). Every element is a generator of the group or is represented by a commutator of two other elements.

4. The group is represented by a tree: the vertices are the genus of the surface and the edges are the orientation.

To every Casson handle one can construct a grope and vice versa \[23\]. The grope \( G \) represents the skeleton of the Casson handle. The closure of \( G \) in 3-dimensional Euclidean space \( \mathbb{E}^3 \) is what we call a closed grope \( G^+ \). The set \( G^+ \setminus G \) is a closed 0-dimensional set (a Cantor set in the case of the fundamental grope). Delete \( G^+ \setminus G \) from \( \mathbb{E}^3 \), and let \( N \) denote a regular neighborhood of \( G \) in \( \mathbb{E}^3 \setminus (G^+ \setminus G) \). Examine the set \( C = N \cup G^+ \), Then \( C \) is a compact subset of \( \mathbb{E}^3 \) which has \( G^+ \) as a strong deformation retract. The boundary \( \partial C \) is always a 2-sphere, and, in the case where \( G \) is the fundamental grope represented by the

\[\begin{align*}
\text{Figure 2.} & \text{ the boundary of an immersed disk (building block of Casson handle)} \\
\text{Figure 3.} & \text{ An example of a grope}
\end{align*}\]
simplest tree above, $C$ is precisely the Alexanders horned (wild) sphere (see [19] for a proof). Thus we have constructed the wild sphere.

Only one problem is open: the construction of the involution from the Casson handle. We showed above that we obtain a map with a fixed point set a wildly embedded sphere. Now it is enough to show that a second usage of the map above produces the identity map. But that part of the proof is already done: we use a result of Bing [8] that the sum of two horned spheres is the standard sphere. Thus the map must be the identity and that is the defining property of an involution. Thus Alexanders horned sphere [4] is the fix point set of an involution [8]. For the construction we used essentially the grope $G$. Lets take the simplest grope associated to the tree $R_+$ above. Every level of the grope is represented by a torus. The torus and all surfaces of genus $g > 1$ are represented by polygons in the hyperbolic space $\mathbb{H}^2$ with $2g$ vertices. By the uniformization theorem of 2-manifolds, the size of the polygon is given by the size of the surface (both are conformally invariant to each other). From that sequence of polygons we can construct a polygon with $k$ vertices with fixed size. Remember that the torus is uniquely characterized by the polygon with 4 vertices. Then we get a polygon $P_\infty$ as the disjoint union of these 4-vertices-polygons with decreasing size. Thus for every $\epsilon > 0$ we obtain a natural number $k$ so that the $k$-vertices-polygon has the size of the disjoint union $P_\infty$ of polygons up to an error of $\epsilon$. We choose the smallest possible number $k$ and take that polygon $P_k$ as a representative. By the procedure above we get a foliation of the 3-sphere from this polygon $P_k$. Now the circle of argumentation is complete.

For all readers getting lost in the unfamiliar constructions above, we present the main line of argumentation:

1. In Bizacas exotic $\mathbb{R}^4$ one starts with the neighborhood $N(A)$ of the Akbulut cork $A$ in the K3 surface $M$. The exotic $\mathbb{R}^4$ is the interior of $N(A)$.
2. This neighborhood $N(A)$ decomposes into $A$ and a Casson handle representing the non-trivial involution of the cork.
3. From the Casson handle we construct a grope containing Alexanders horned sphere.
4. Akbuluts construction gives a non-trivial involution, i.e. the double of that construction is the identity map.
5. From the grope we get a polygon in the hyperbolic space $\mathbb{H}^2$.
6. This polygon defines a codimension-1 foliation of the 3-sphere inside of the exotic $\mathbb{R}^4$ with an wildly embedded 2-sphere, Alexanders horned sphere. This foliation agrees with the corresponding foliation of the homology 3-sphere $\partial A$. This codimension-1 foliations of $\partial A$ is partly classified by the Godbillon-Vey class lying in $H^3(\partial A, \mathbb{R})$ which is isomorphic to $H^3(S^3, \mathbb{R})$.
7. Finally we get a relation between codimension-1 foliations of the 3-sphere and exotic $\mathbb{R}^4$.

This relation is very strict, i.e. if we change the Casson handle then we must change the polygon. But that changes the foliation and vice versa. Finally we obtain the result:

The exotic $\mathbb{R}^4$ (of Bizaca) is determined by the codimension-1 foliations with non-vanishing Godbillon-Vey class in $H^3(S^3, \mathbb{R})$ of a 3-sphere seen as submanifold $S^3 \subset \mathbb{R}^4$ lying at the boundary $\partial A$ of the Akbulut cork $A$. 
2.4. The conjecture: the failure of the smooth 4-dimensional Poincare conjecture.
In this section we will go a step further. We consider the 4-sphere and use the close relation between foliations and smooth structures. In case of the 4-sphere $S^4$ we need an exotic 4-disk $D^4$ which is the Akbulut cork for $S^4$. The boundary of $D^4$ is the 3-sphere $S^3$. Now we consider a polygon $P_k$ in $\mathbb{H}^2$ representing a codimension-1 foliation of the 3-sphere. A direct analogy of the construction above showed that we have a Casson handle attached to the $S^3$ which is related to the polygon $P_k$. But now we have a big difference to the non-compact exotic $\mathbb{R}^4$. The 4-sphere is compact and we have to consider an embedding of the Casson handle. This embedding leads to the cut-off of the Casson handle after $N$ levels (see [44] for the argument using PL methods). Then the polygon $P_k$ has also an integer size (in $\pi$ units). Thus the integer values in $H^3(S^3, \mathbb{Z})$ of the Godbillon-Vey invariant is represented by the foliation of the 3-sphere $S^3$ as part of the compact 4-sphere $S^4$. Thus we conjecture:

Conjecture: Every 3-sphere admitting a codimension-1 foliation with integer Godbillon-Vey invariant bounds an exotic 4-disk $D^4$.

But the existence of an exotic 4-disk implies the failure of the smooth Poincare conjecture\(^6\) in dimension 4. Or,

Conjecture: If we glue an exotic 4-disk to a standard 4-disk along its common boundary we obtain an exotic 4-spheres. Thus we obtain the failure of the smooth Poincare conjecture in dimension 4.

For the following, it is interesting to note by using the conjecture that countable infinite exotic structures of the 4-spheres are related to the elements in $H^3(S^3, \mathbb{Z})$ (canonically isomorphic to $H^4(S^4, \mathbb{Z})$). Unfortunately the conjecture cannot be checked by the lack of a suitable invariant.

3. Gerbes and exotic $\mathbb{R}^4$

In the second part of the paper we indicate the possible directions how to proceed with abstract topics like exotic $\mathbb{R}^4$’s or (conjectural) exotic 4-spheres, in order to extract some effective, computable and physically valid results. These results are motivated by physics and geometrical constructions and heavily rely on the relation of exotic smoothness on $\mathbb{R}^4$ with codimension-one foliations of $S^3$ and with third real cohomologies of the 3-sphere established in Sec. 2.3. In the following we will associate effective constructions in conformal field theory to these exotic $\mathbb{R}^4$’s. The emerging new relations can be especially important in string or field theory.

Thus, in next two sections we consider the levels of $SU(2)$ WZW $\sigma$ models as corresponding to exotic $\mathbb{R}^4$’s which are determined by foliations from even integral third cohomologies of $S^3$, and which have also CFT algebraic presentation. The case of WZW model with the target $S^3$ group manifold, is especially well recognized and solvable as 2-dimensional quantum field theory. This is also the case with the description of the orientifolds of the $SU(2)$ WZW theory, and D-branes in these.

In last section we extended the presentation over exotic $\mathbb{R}^4$’s which correspond to real third cohomologies of $S^3$. This is performed in the context of generalized
Hitchin’s structures. The relation to gerbes, in the case of integral cohomologies, and to 2-gerbes in the general case, are discussed as well. The presented approach indicates that exotic $\mathbb{R}^4$’s may play a fundamental role in the description of quantum gravity (string theory), similarly as standard $\mathbb{R}^4$ plays in classical gravity based on the Riemannian geometry.

3.1. WZW $\sigma$- models and $U(1)$- gerbes on $S^3$. From the analysis of Sec. 2.3 we know that to every orientation reversing involution $\tau : S^3 \to S^3$ corresponds some exotic $\mathbb{R}^4$ when this involution generates a regluing of the boundary of the Akbulut cork. More is known: different exotic $\mathbb{R}^4$’s correspond to different (integral, to begin with) cohomology classes $H^3(S^3, \mathbb{Z}) \subset H^3(S^3, \mathbb{R})$.

From the other hand, some orientation reversing involutions of $S^3$ are building blocks of $\mathbb{Z}_2$ orientifolds of the $SU(2)$ Wess-Zumino-Novikov-Witten (WZW) model, namely those involutions which leave the $S^2$ equator sphere fixed [17]. More precisely, the possible classes of orientifolds, correspond to the inequivalent orientation reversing $\mathbb{Z}_2$ isometries of $S^3$. There are two such possibilities:

1. $\mathbb{Z}_2$ isometries of $S^3$ fixing two points, i.e. the south and north poles;
2. $\mathbb{Z}_2$ isometries of $S^3$ fixing the equator two sphere.

The first case corresponds to two, so called, $O0$ orientifold’s planes while the second to the $O2$ orientifold plane [17]. The orientifolds with these planes were considered as targets for the WZW models. The invariance of the interaction term (WZ term) in the Lagrangian requires that the $\mathbb{Z}_2$ isometry reverses the orientation of $S^3$.

The $O2$ orientifold plane corresponds to particular involutions of $S^3$ which were considered in Sec. 2.3 point 4, in context of smooth structures. Given such an involution we have some embedding $S^2 \hookrightarrow S^3$, namely the equator $S^2$ in the WZW case, hence a smooth structure on $R^4$ corresponds to it. This is in fact the standard smooth structure, since the regular embedding $S^2 \hookrightarrow S^3$ means that the equator 2-sphere has „two simply-connected sides-complements” in $S^3$ or, the grope $G$ is derived from Casson handle which is smoothly an ordinary 2-handle.

We claim that the structure of $\mathbb{Z}_2$ orientifolds of $SU(2)$ WZW $\sigma$- models is sensitive to the existence and change of exotic $\mathbb{R}^4$’s. More precisely, the levels, $k$, of the $\mathbb{Z}_2$ orientifold of $SU(2)$ WZW $\sigma$- model with D-branes wrapping $S^2$-conjugacy class, correspond to different small exotic $\mathbb{R}^4$’s. Namely to those which are determined by foliations of $S^3$, i.e. classes from $H^3(S^3, \mathbb{Z})$. These cohomologies refer directly to the levels $k$ of the WZW model through $k[H] \in H^3(S^3, \mathbb{Z})$ where $[H]$ is the generator of the 3-rd cohomologies. (see the Appendix A). Thus, from that point of view the smooth structures are distinguished by the levels of the $\sigma$ - model. However, even though the levels correspond to third integral cohomologies of 3-sphere, and these correspond to different smooth structures on $\mathbb{R}^4$, to be sure that these two, the levels and the structures, are correlated more essentially, one should try to recover some smooth structure from the quantum regime of the WZW model. Again, the results of Sec. 2.3 are crucial: a smooth structure on $\mathbb{R}^4$ is determined by the embedding of $S^2$ into $S^3$. Thus, constructing the embedding $S^2 \hookrightarrow S^3$ we determine (an exotic) smooth structure for $\mathbb{R}^4$. 
As we have already observed standard smoothness on $\mathbb{R}^4$ is determined by the regular embedding $S^2 \hookrightarrow S^3$ as the equator 2-sphere. This is the involution of $S^3$ fixing the equator $S^2$. The 2-sphere $S^2$ can be seen as $O2$ orientifold plane in the $\mathbb{Z}_2$ orientifold of $SU(2)$ WZW $\sigma$ model. (see the Appendix A).

The semi-classical limit of the WZW model corresponds to taking the level $k$ of the theory to infinity. In that limit $S^3$ becomes effectively flat. The important thing is that the geometry of orientifold planes is recovered as $S^2$ from quantum regime of the model. This kind of ,,shape computation” relies on considering the scattering processes of localized packets of gravitons (closed string states) with branes and orientifold planes, which are represented in the conformal field theory (CFT) by the so called boundary and cross-cap states [15]. Then we note that generic models for 2-dimensional CFT are WZW theories, and the so-called boundary 2-dimensional CFT are WZW orientifolds with branes. We have placed the elements of the shape computation in the Appendix B. The relation with CFT is the topic of next section.

The quantum shape computation in the Appendix C shows that the geometry of the orientifold plane is $S^2$ when $k$, the level of the model, is very large. Hence indeed this semi-classical limit of the model pinpoints the standard smoothness of $S^3$ as corresponding to the involution fixing the equator orientifold plane $S^2$.

Now, let us consider finite levels $k$ of the WZW model corresponding to integral cohomology classes $k[H] \in H^3(S^3, \mathbb{Z})$, $k \in \mathbb{Z}$.

The $k$ - level of $\mathbb{Z}_2$ orientifold of the WZW model with the $S^3$ target, corresponds to the pair: the involution $\tau: S^3 \to S^3$ fixing the equator $S^2$ together with the integral class $k[H] \in H^3(S^3, \mathbb{Z})$.

From the other hand, every $k[H] \in H^3(S^3, \mathbb{Z})$ determines different codimension-1 foliation of $S^3$ which, in turn, corresponds to different wild embedding $S^2 \to S^3$. Hence the effect of the level $k$ of the orientifold WZW model is correlated with the different embeddings $S^2 \to S^3$. From Sec. 2.3 it follows that different classes $H^3(S^3, \mathbb{Z}) \subset H^3(S^3, \mathbb{R})$ determine different smooth structures on $\mathbb{R}^4$. Moreover, when $k \to \infty$ the standard smooth $\mathbb{R}^4$ is fixed. We take that as indication for a correlation between $k$ levels and smooth exotic structures on $\mathbb{R}^4$.

We can summarize the above discussion as follows

If the family of small exotic $\mathbb{R}^4$’s is determined by integral cohomology classes from $H^3(S^3, \mathbb{Z})$, the change of smoothness on $\mathbb{R}^4$ is correlated with the change of the level $k$ of $\mathbb{Z}_2$ orientifold of WZW $SU(2)$ model defined on $S^3$, lying at the boundary of the Akbulut cork.

In that way, we have in principle a computational tool, enabling one to represent the changes of the smoothness of $\mathbb{R}^4$ simply by shifting the levels of the orientifold WZW model. Note that, at this stage we are dealing only with those smooth $\mathbb{R}^4$’s which correspond to integral third cohomologies of $S^3$. The more general case of real third cohomologies is discussed in Sec. 3.3.

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7 In general, the so called integral conjugacy classes are orientifold planes for a consistent quantum WZW $SU(2)$ $\sigma$ model.
Let us summarize the up to now discussion in the table below

| SU(2) WZW $\mathbb{Z}_2$ orientifolds | smooth (exotic) $\mathbb{R}^4$ |
|--------------------------------------|-------------------------------|
| $k \to \infty$: involution of $S^3$ fixing equator $S^2$ | standard $\mathbb{R}^4$ |
| $k$-level SU(2) WZW $\mathbb{Z}_2$ orientifolds | $\tau_k$-involution of $S^3$ fixing (wild) $S^2$ |
| $k[H] \in H^3(S^4, \mathbb{Z})$ | $k[H] \in H^3(S^4, \mathbb{Z})$ |

The conjecture from Sec. 2.4 allows for the following formulation of the correspondence:

Provided that every 3-sphere admitting a codimension-1 foliation with integer Godbillon-Vey invariant, bounds an exotic 4-disk $D^4$, then different smooth exotic $S^4$’s are in $1 \div 1$ correspondence with the levels $k$ of $\mathbb{Z}_2$ orientifolds of SU(2) WZW $\sigma$ model defined on $S^3$, lying at the boundary of the Akbulut cork in $S^4$.

Again, some computations performed in 2-dimensional boundary CFT should be relevant for grasping effects of changing the smoothness of $S^4$, unless the smooth Poincare conjecture in dimension 4 is true.

In the remaining part of this section we will focus on some constructions referring to gerbes. Abelian gerbes and gerbe bundles on manifolds are geometrical objects interpreting third integral cohomologies on manifolds, similarly as isomorphism classes of linear complex line bundles are classified by $H^2(M, \mathbb{Z})$. From that point of view abelian gerbes are important for the constructions in this paper. There exists vast literature in mathematics and physics devoted to gerbes and bundle gerbes. Gerbes were first considered by Giraud [27]. The classic reference is the Brylinski book [16]. We are interested here in abelian gerbes and when forgetting about the questions of uniqueness of the choices made, we can, following Hitchin [34], make use of the simplified working definition of a gerbe. Hence we do not refer to categorical constructions (sheaves of categories, see e.g. [42]) which, from the other hand, are essential for correct recognition of gerbes. In that way we can state easily the relation to 3-rd integral cohomologies on $S^3$. Only in the end of the paper we comment on categorically defined 2-gerbes in context of smooth exotic structures.

Abelian, or $S^1$, gerbes are best understood in terms of cocycles and corresponding transition objects, similarly as $S^1$ principal bundle is specified by a cocycle $g_{\alpha\beta} : U_\alpha \cap U_\beta \to S^1$ which is the Čech cocycle from $\check{C}^1(M, C^\infty(S^1))$ where $U_{\alpha,\beta}$ are elements of a good cover of a manifold $M$. However, to define $S^1$ gerbe we need to compare data on each triple intersections of elements of a good cover. Hence, let $g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to S^1$ be the cocycle in $\check{C}^2(M, C^\infty(S^1))$ with

$$g_{\alpha\beta\gamma} = g_{\beta\gamma\alpha} = g_{\gamma\alpha\beta}$$  \hspace{1cm} (2)

and on each fourth intersection $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$ they should satisfy the cocycle condition

$$\delta g = g_{\beta\gamma\delta}g_{\alpha\gamma\delta}^{-1}g_{\alpha\beta\delta}g_{\alpha\beta\gamma}^{-1} = 1$$  \hspace{1cm} (3)

The above data define the $S^1$-gerbe.

The cocycles from $\check{C}^2(M, C^\infty(S^1))$ give rise to the cohomology group $H^2(M, C^\infty(S^1))$ which classifies gerbes as defined above. However, we have the canonical exact sequence of sheaves on $M$
0 \rightarrow \mathbb{Z} \rightarrow C^\infty(\mathbb{R}) \rightarrow C^\infty(S^1) \rightarrow 1 \quad (4)

where the third morphism is given by $e^{2\pi ix}$. However, $C^\infty(\mathbb{R})$ is fine, hence

$$H^2(M,C^\infty(S^1)) = H^3(M,\mathbb{Z}) \quad (5)$$

We see that gerbes are classified by third integral cohomologies on $M$ similarly as line bundles are classified topologically by Chern classes. The elements of $H^3(M,\mathbb{Z})$ are called the Dixmier-Douady classes of the gerbe local data. It is worth noticing that gerbes are neither manifolds nor bundles. These can be considered as generalization of both: sheaves (bundle gerbes) and vector bundles (cocycle description).

A trivialization of a $S^1$-gerbe is given by functions

$$f_{\alpha\beta} = f^{-1}_{\beta\alpha} : U_\alpha \rightarrow U_\beta \quad (6)$$

such that

$$g_{\alpha\beta\gamma} = f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} \quad (7)$$

which is a representation of a cocycle by the functions. The difference of two trivializations is thus given by $h_{\alpha\beta} = f_{\alpha\beta}/f_{\alpha\beta}'$ which means $h_{\alpha\beta} = h_{\beta\alpha}^{-1}$ and

$$h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} = 1 \quad (8)$$

and this is exactly a cocycle for some line bundle on $M$. We say that the transition (generalized) functions of an abelian gerbe are line bundles. One can iterate this construction and define higher ,,gerbes'' with the transition generalized functions given by lower rank gerbes.

A connection on a gerbe is specified by 1-forms $A_{\alpha\beta}$ and 2-forms $B_\alpha$ satisfying the following two conditions

$$iA_{\alpha\beta} + iA_{\beta\gamma} + iA_{\gamma\alpha} = g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma} \quad (9)$$

$$B_\beta - B_\alpha = dA_{\alpha\beta} \quad (10)$$

This implies that there exists a globally defined 3-form $H$, with integral 3-rd de-Rham cohomologies corresponding to $[H/2\pi]$ and defined by local 2-forms $B_\alpha$

$$H|_{U_\alpha} = dB_\alpha \quad (11)$$

This 3-form $H$ is the curvature of a gerbe with connection as above. The local data defining a gerbe exists whenever the 3-form $H$ has its 3-periods in $2\pi\mathbb{Z}$, hence $[H/2\pi]$ is integral.

The trivialization $f_{\alpha\beta}$ is the element of $\tilde{C}^1(M,C^\infty(S^1))$ which implies that when a gerbe is trivial then $g_{\alpha\beta\gamma}$ satisfies

$$[g_{\alpha\beta\gamma}] = 0 \quad (12)$$

The connection on a gerbe is flat when $H|_{U_\alpha} = dB_\alpha = 0$. In that case we have $B_\alpha = da_\alpha$ for a suitably chosen cover.
In the case $M = S^3$ again the integral third de-Rham cohomologies are important, this time for classifying $S^1$-gerbes on $S^3$ hence these are Dixmier-Douady classes for local data of such gerbes.

The canonical $S^1$-bundle gerbe on $S^3$ was first constructed in [26] and later [37, 41]. The canonical $U(1)$ - gerbe $G$ on $S^3$corresponds to the 3-form $H = \frac{1}{12\pi} \text{tr}(g^{-1} dg)^3$. Other gerbes on $S^3$, correspond to the curvatures $kH$, $k \in \mathbb{Z}$, and are determined by the tensor powers $G^k$. Given $kH$, $k \in \mathbb{Z}$ one has unique gerbe $G^k$ up to stable isomorphism, since $H^2(SU(2), U(1)) = \{1\}$.

Thus, we can represent uniquely (up to stable isomorphisms of gerbes) the action of elements of $H^3(S^3, \mathbb{Z})$ by the deformed action of the corresponding gerbes. The elements of $H^3(S^3, \mathbb{Z})$ determine some exotic $\mathbb{R}^4$’s as stated above. The connection is through generalized geometries of Hitchin (Dirac’s structures) and is referred to in Sec. 3.3.

3.2. CFT spectra and small exotic $\mathbb{R}^4$’s . Every $SU(2)$ WZW $\mathbb{Z}_2$ orientifold is a model of rational 2-dimensional CFT, hence the techniques of CFT can be used in principle in the realm of distinguishing different exotic $\mathbb{R}^4$’s, at least those corresponding to integral cohomologies $H^3(S^3, \mathbb{Z})$. Thus the change of some small exotic $\mathbb{R}^4$ to another can be described as the algebraic task coded in the spectra of 2-dimensional boundary CFT theory.

WZW models are especially important in string theory. For example $SU(2)$ WZW $\mathbb{Z}_2$ orientifolds describe naturally the limiting geometry of the stack of NS 5-branes [21]. In general, strings on group manifolds of a simple and simply connected group $G$ are described by the WZW model. Its action is a functional on fields $g : \Sigma \to G$ taking values in $G$ and there is one integer, the coupling constant $k$, which is the ‘level’ of the model. One can think of $k$ as controlling the size of the background. Large values of $k$ correspond to a large volume of the group manifold.

In Sec. 3.1 we discussed the relation between different classes of exotic smooth structures on $\mathbb{R}^4$ and the levels $k$ of the $\mathbb{Z}_2$ orientifold $SU(2)$ WZW model. Thus, a way to relate exoticness with the CFT data is by considering the level $k \mathbb{Z}_2$ orientifolds of $SU(2)$ WZW model with $O2$ planes. The boundary states of the theory correspond to D-branes whereas the cross-caps to the involution of $S^3$ reversing the orientation, hence to $O2$ planes. In the large $k$ limit the fixed point set of the involution is recognized geometrically from the quantum amplitudes as $S^2 \to S^3$. Quantum amplitudes of the states appearing at the level $k$ of this model, should say something about the exoticness of smooth $\mathbb{R}^4$’s (or $S^4$ when the smooth Poincare hypothesis fails in dimension 4) where $S^3$ is the boundary of the corresponding Akbulut cork. The smooth structures are determined by the class $k[H] \in H^3(S^3, \mathbb{Z})$. The amplitudes are expressed in terms of the characters of the representations of the chiral algebra $C_A$ which is among the defining data for any CFT (see the Appendix [B]). In the case of $SU(2)$ at the level $k$ these are given in terms of the characters of the representations of $SU(2)_k$.

Given a general cross-cap state in rational CFT

$$|C> = \sum_j \frac{P_{j0}}{\sqrt{S_{j0}}} |C, j>$$

(13)
where $P$ is the matrix appearing in the Appendices formula $\text{[01]}$ and $\text{[02]}$, $S$ is given in $\text{[01]}$, $|C, j \gg$ is the cross-cap Ishibashi state. The equator cross-cap state is given in $\text{[61]}$ and the boundary Cardy state $|k 4 >$ representing the D-brane wrapping of the equator is

$$|k 4 > = \left( \frac{2}{k+2} \right)^{\frac{k}{2}} \sum_{j=0}^{k/2} (-1)^j \left( \frac{2j+1}{k+2} \frac{\pi}{k} \right) \frac{\sin \left( \frac{(2j+1)(2j+1)}{2(k+2)} \pi \right)}{\sin \left( \frac{2j+1}{2(k+2)} \pi \right)} |B, j \gg \quad (14)$$

Given the conjugacy class $J$, let the boundary Cardy state for the D-brane wrapping this class, be $|J >$. The M"obius strip amplitude then reads

$$M^J = \sum_{l=0}^{k/2} \frac{1}{\sqrt{k+2}} (-1)^l \sin \frac{(2l+1)(2l+1)}{k+2} \sin \frac{2l+1}{2(k+2)} \chi_l$$

Thus, based on the shape of the amplitudes containing cross-cap states, and the fact that Cardy states for D-branes in $SU(2)$ exist only for even $k$, a general, conjectural by now, statement is in order:

**Exotic smooth $\mathbb{R}^4$’s corresponding to third even degree cohomology classes, $k = 2s$, from $H^3(S^3, \mathbb{Z})$, are distinguished by invariants which are functions of the characters of at most spin $s$ representations of $SU(2)$.

This kind of truncation of symmetries on 3-sphere in $\mathbb{R}^4$ (lying at the boundary of the Akbulut cork) could be relevant for the construction of invariants of exotic $\mathbb{R}^4$ with few symmetries $\text{[48, 46]}$.

The explicit construction of the invariants, i.e. from a given exotic $\mathbb{R}^4$ one constructs the invariant function of the characters as above, is missing now. The construction should establish the correspondence between wildly embedded 2-spheres in $S^3$ and the quantum geometry of D-branes wrapping the conjugacy classes. The geometry of the world-volumes of the branes in $SU(2)$ WZW model for finite $k$ was recognized algebraically as fuzzy and non-associative 2-spheres which refer to the truncation to finite $J$ of the algebra of spherical functions on $S^3$. In order to determine the smooth invariant by our construction, this quantum algebra should be related somehow to wild embeddings of $S^2$. A possible impact can be derived from some categorical constructions, as gerbes, or implementing generalized structures of Hitchin and others. This last issue is discussed in the next section. Such invariants should be also helpful in working on the smooth Poincare conjecture in dimension 4.

### 3.3. Generalized (complex) structures of Hitchin

In the previous subsections we have presented a way how to distinguish algebraically small exotic $\mathbb{R}^4$’s, at least those which correspond to even integral third cohomologies from $H^3(S^3, \mathbb{Z})$ where $S^3$corresponds to the homology 3-spheres bounding the Akbulut cork. The third cohomology class of the target determines $k$ and the exotic small $\mathbb{R}^4$. The CFT data describe the level $k$ of the WZW model. Composing this with the conjecture from Section 2.4 the CFT data describe different smoothness of $S^4$ too.

In this section we are looking for tools enabling distinguishing between other exotic $\mathbb{R}^4$’s which correspond not only to integral classes from $H^3(S^3, \mathbb{R})$. The description via CFT and orientifolds of WZW $\sigma$- models in this general case
is not possible at all, since a 2-dimensional CFT corresponding to non-integral cohomology classes is ill defined. The idea to look for the effects of various exotic $R^4$’s in the geometries of the boundary $S^3$ was proposed by Asselmeyer and Brans [47] and we used it in Sec. 2.3. This is due to the fact that $S^3$ has an unique smooth structure and the change of smooth structures on $\mathbb{R}^4$ cannot affect the smoothness of $S^3$. However, the geometries, foliations or some $K$-theoretic and categorical ingredients in dimension three, can vary.

In this section we try to establish a similar correspondence between some others structures defined on the boundary $S^3$ whose variation reflects the change of smoothness on $\mathbb{R}^4$. These are generalized geometries and complex structures introduced by Hitchin [34].

The idea of generalized structures is based on the substituting the tangent space $TM$ of a manifold $M$ rather by the sum $TM \oplus T^*M$ of the tangent and cotangent bundles such that the spin structure for such generalized ,,tangent” bundle now becomes the bundle of all forms $\wedge^*M$ on $M$.

Our interest in generalized Hitchin’s structures is due to the following correspondences

A. Continuum many distinct small exotic smooth structures on $\mathbb{R}^4$ correspond to the $H$-deformed classes of generalized Hitchin’s geometries on $S^3$, where $[H] \in H^3(S^3, \mathbb{R})$ and $S^3$ lies at the boundary of the Akbulut cork.

B. For integral $[H]$ these deformations are geometrically described by $S^1$ gerbes.

C. For non-integral $[H]$ the geometrical description requires $S^1$ (fully abelian) categorical 2-gerbes.

This is based on usual exotic 4-smoothness from one side, and generalized extended ,,smoothness” of Hitchin’s structures in dimension three, from the other. As we observed, the relation 4-smoothness/3-smoothness is not possible due to the uniqueness of the smooth structure on $S^3$. The class of generalized $H$-deformed Hitchin’s structure on $S^3$ could be well referred to as $H$-twisted Courant bracket on $TS^3 \oplus T^*S^3$. The class of structures is integrable with respect to this bracket. In what follows we will try to explain main points of this correspondence. The excellent reference for generalized geometries and complex structures is [32].

For a given smooth manifold $M$, the Courant bracket $[,]$ is defined on smooth sections of $TM \oplus T^*M$, as

$$[X + \xi, Y + \eta] = [X,Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)$$ (16)

where $X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M)$, $\mathcal{L}_X$ is the Lie derivative in the direction of the field $X$, $i_X \eta$ is the inner product of a 1-form $\eta$ and a vector field $X$. On the RHS of (16) $[,]$ is the Lie bracket on fields. This is not misleading since on fields Courant bracket reduces to Lie bracket, i.e. $\pi([X,Y]) = [\pi(X), \pi(Y)]$ where $\pi : TM \oplus T^*M \to TM$. It follows that the bracket is skew symmetric and on 1-forms vanishes. However, the Courant bracket is not a Lie bracket, since the first does not fulfill the Jacobi identity. The expression measuring the failure of the identity is the following Jacobiator:

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8 Generalized complex structures require even manifolds dimension hence should be considered on $S^3 \times \mathbb{R}$. 

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\[ \text{Jac}(X, Y, Z) = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] \]  

(17)

The Jacobiator can be expressed as the derivative of a quantity which is Nijenhuis operator, and it holds

\[ \text{Jac}(X, Y, Z) = d\text{Nij}(X, Y, Z) \]  

(18)

\[ \text{Nij}(X, Y, Z) = \frac{1}{3} \left( <[X, Y], Z> + <[Y, Z], X> + <[Z, X], Y> \right) \]  

(19)

where $<,>$ is the inner product on $TM \oplus T^*M$. This is a natural product given by

\[ <X + \xi, Y + \eta> = \frac{1}{2} (\xi(Y) + \eta(X)) \]  

(20)

This product is symmetric and has the signature $(n, n)$, where $n = \dim(M)$, thus the non-compact orthogonal group reads $O(TM \oplus T^*M) = O(n, n)$. A subbundle $L < TM \oplus T^*M$ is involutive iff it is closed under the Courant bracket defined on its smooth sections, and is isotropic when $<X, Y> = 0$ for $X, Y$ smooth sections of $L$. In the case that $\dim(L) = n$, hence is maximal, we call such an isotropic subbundle a maximal isotropic subbundle. The following property characterizes this subbundles ([32], Proposition 3.27):

If $L$ is a maximal isotropic subbundle of $TM \oplus T^*M$ then the following are equivalent:

- $L$ is involutive
- $\text{Nij}_L = 0$
- $\text{Jac}_L = 0$.

A Dirac structure on $TM \oplus T^*M$ is a maximal isotropic and involutive subbundle $L < TM \oplus T^*M$. As follows from the above properties, the involutiveness of a Dirac structure means $\text{Nij}_L = 0$.

The advantage of using the Dirac structures is its generality in contrast to structures used in Poisson geometry, complex structures, foliated or symplectic geometries. This has a great unifying power. The $H$- deformed Dirac structures (to be discussed below) include also generalized complex structures which are well defined on some manifolds without any complex or symplectic structures. Moreover, this kind of geometry became extremely important in string theory (flux compactification, mirror symmetry, branes in YM manifolds) and related WZW models. This $H$- deformed Dirac structures are also important for the recognition of exotic smoothness in topological trivial case of $\mathbb{R}^4$. The main idea behind this recognition is the suitable modification of Lie product of fields on smooth manifolds\(^9\). The modification is given firstly by the Courant bracket on $TM \oplus T^*M$ and then by the $H$-deformation of it.

In differential geometry, Lie bracket of smooth vector fields on a smooth manifold $M$ is invariant under diffeomorphisms, and there are no other symmetries of the tangent bundle preserving the Lie bracket. More precisely, let $(f, F)$ be a

\[^9\] Such a modification was suggested to one of the authors by Robert Gompf some time ago.
pair of diffeomorphisms \( f : M \to M \) and \( F : TM \to TM \) and \( F \) is linear on each fiber, \( \pi \) be the canonical projection \( \pi : TM \to M \). Suppose that \( F \) preserves the Lie bracket \([\cdot,\cdot]\), i.e. \( F([X,Y]) = [F(X),F(Y)] \) for vector fields \( X, Y \) on \( M \), and suppose the naturality of \((f,F)\) i.e. \( \pi \circ F = f \circ \pi \), then \( F \) has to be equal \( df \).

In the case of our extended ,,tangent space'', which is \( TM \oplus T^*M \), the Courant bracket and the inner product are diffeomorphisms invariant. However, there exists another symmetry extending the diffeomorphisms which is \( B \)-field transformation. Let us see how this works. Given a two-form \( B \) on \( M \) one can think of it as the map \( TM \to T^*M \) by contracting \( B \) with \( X, X \to i_X B \). The transformation of \( TM \oplus T^*M \) given by \( e^B : X + \xi \to X + \xi + i_X B \) has the properties (see [32], Propositions 3.23, 3.24)

- The map \( e^B \) is an automorphism of the Courant bracket if and only if \( B \) is closed, i.e. \( dB = 0 \),
- The \( e^B \) extension of diffeomorphisms are the only allowed symmetries of the Courant bracket.

which means, that for a pair \((f,F)\) which is the (orthogonal) automorphism of \( TM \oplus T^*M \) and for \( F \) preserving the Courant bracket \([\cdot,\cdot] \), i.e. \( F([A,B]) = [F(A), F(B)] \) for all sections \( A,B \in C^\infty(TM \to T^*M) \), \( F \) has to be a composition of a diffeomorphism of \( M \) and a \( B \)-field transform. This means that the group of orthogonal Courant automorphisms of \( TM \oplus T^*M \) is the semidirect product of \( \text{Diff}(M) \) and \( \Omega^3_{\text{closed}}(M) \).

Given a Courant bracket on \( TM \oplus T^*M \) we are able to define various involutive structures with respect to it. The most important fact is the possibility to deform the Courant bracket on \( TM \oplus T^*M \) by a real closed \( 3 \)-form \( H \) on \( M \). For any real \( 3 \)-form \( H \) one has the twisted Courant bracket on \( TM \oplus T^*M \) defined as

\[
[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta] + iY i_X H
\]

(21)

where \([\cdot,\cdot]\) on the RHS is the non-twisted Courant bracket. This can be also restated as the splitting condition in non-trivial twisted Courant algebroid defined later.

This deformed bracket allows for defining various involutive and (maximal) isotropic structures with respect to \([\cdot,\cdot]_H \), which is again an analog for the integrability of distributions on manifolds. These structures correspond to new \( H \)-twisted geometries which are different for the previously considered Dirac structures in case of the untwisted Courant bracket.

In particular, the \( B \)-field transform of \([\cdot,\cdot]_H \) is the symmetry of the bracket if and only if \( dB = 0 \), since it holds

\[
[e^B(C), e^B(D)]_H = e^B[C, D]_{H + dB}, \quad \forall C, D \in C^\infty(TM \oplus T^*M)
\]

(22)

Then the tangent bundle \( TM \) is not involutive with respect to \([\cdot,\cdot]_H \) for non-zero \( H \). In general a subbundle \( L \) is involutive with respect to \([\cdot,\cdot]_H \) if and only if \( e^{-B}L \) is for \([\cdot,\cdot]_{H + dB} \).

The correspondence \( A \)

Continuum many distinct small exotic smooth structures on \( \mathbb{R}^4 \) correspond \( 1 \div 1 \) to the \( H \)-deformed classes of generalized Hitchin’s geometries on \( S^3 \), where \( [H] \in \)
$H^3(S^3,\mathbb{R})$ and $S^3$ lies at the boundary of the Akbulut cork. can be established by taking $H \in H^3(S^3,\mathbb{R})$ as deforming the Courant bracket on $TS^3 \oplus T^*S^3$ and recalling that exotic small $\mathbb{R}^4$’s correspond to the foliations of $S^3$ distinguished by 3-rd real cohomologies ($S^3$ is the boundary of the Akbulut cork). However, the choice of the specific generalized Hitchin’s geometry (from the class of $H$- deformed and integrable Dirac structures) to yield some exotic smooth $\mathbb{R}^4$, can not be answered uniquely now. Let us mention the possibility to take generalized complex structures on $S^3 \times \mathbb{R}$ by using the isomorphism $H^3(S^3 \times \mathbb{R},\mathbb{R}) \simeq H^3(S^3,\mathbb{R})$ and considering specific embedding of the collar into the neighborhood $N(A) \subset \mathbb{R}^3$ of the Akbulut cork $A$.

Besides, there is a close connection between foliations and generalized structures, since every foliation determines integrable involutive distributions. Moreover, in case of the Hopf surface, i.e. $S^3 \times S^1$, it is known that there is no generalized Courant complex structure, but a H-twisted generalized complex structure (GCS) which is also helpful. Moreover, ([33], Example 4.1) GCS on $S^3 \times S^1$ is integrable with respect to the generator of $H^3(S^3,\mathbb{Z})$, which could correspond to the case of compactified $S^3 \times \mathbb{R}$. The analysis of this interesting issues will be presented elsewhere.

In the following we will comment on the relation to gerbes. Then we are able to understand a way how $TM \oplus T^*M$ appears from the broader perspective of the extensions of bundles. For that purpose one defined the Courant algebroid $E$ as an extension of real vector bundles given by the sequence

$$0 \to T^*M \to \pi_* E \to \pi TM \to 0$$

(23)

On $E$ a non-degenerate symmetric bilinear form $<\cdot,\cdot>$ is given, such that $<\pi^*\xi,a>=\xi(\pi(a))$ where $\xi$ is smooth covector field on $M$ and $a \in C^\infty(E)$. A bilinear Courant bracket $[,]$ on $C^\infty(E)$ can be defined such that

$$- [a,[b,c]]-[a,b],[c]+[b,[a,c]]=0 \quad \text{(Jacobi identity)}$$

$$- [a,f b]=f[a,b]+(\pi(a))f b \quad \text{(Leibniz rule)}$$

$$- \pi(a)<b,c>=<a,[b,c]>+<b,[a,c]> \quad \text{(Invariance of bilinear form)}$$

$$- [a,a]=\pi d<a,a>$$

The sequence (23) defines a splitting of $E$. Each splitting determines a closed 3-form $H \in \Omega^3(M)$, given by

$$(i_X i_Y H)(Z)=<[s(X),s(Y)],s(Z)>$$

(24)

where $s : TM \to E$ is the splitting derived from the sequence. The cohomology class $[H]/2\pi \in H^3(M,\mathbb{R})$ is independent of the choice of splitting, and coincides with the image of the Dixmier-Douady class of the gerbe in real cohomology [35]. The Dixmier-Duady class classifies the bundle gerbes as done in Section 3.1.

The condition (0) in Sec. 3.1 for the $S^1$-gerbe with connection can be interpreted that $dA_{\alpha \beta}$ is a cocycle. Using this fact, one can glue local $(TM \oplus T^*M)\alpha$ with another $(TM \oplus T^*M)\beta$ by the automorphism

$$\left( \begin{array}{cc} 1 & 0 \\ dA_{\alpha \beta} & 1 \end{array} \right)$$

and the action of $dA_{\alpha \beta}$ on TM is defined by $X \to i_X dA_{\alpha \beta}$.

The second condition for connection data of a gerbe i.e. (10) in Sec. 3.1 defines exactly a splitting of the Courant algebroid as in (24). In that way we have the proposition 3.47 in [82], i.e.
If \([H/2\pi] \in H^3(M, \mathbb{Z})\) then the twisted Courant bracket \([\cdot, \cdot]_H\) on \(TM \oplus T^*M\) can be obtained from a \(S^1\) gerbe with connection.

In fact, when \([H/2\pi]\) is integral, trivializations \(f_{\alpha\beta}\) of a flat gerbe with connection are symmetries of \([\cdot, \cdot]_H\) (B-field transforms) since \(dB = 0\) and \((22)\). The difference of two such trivializations is a line bundle with connection as we saw in Sec. 3.1 and these line bundles play the role of gauge transformations (integral B-fields) \([32]\).

Now the relation with gerbes stated in our correspondence B: For integral \([H]\) these deformations are geometrically described by \(S^1\) gerbes is established. A general result for the family of small smooth \(\mathbb{R}^4\)'s can be re-stated as

The change of exotic smooth structure on \(\mathbb{R}^4\) results in the change of a generalized Dirac structures on \(S^3\), lying at the boundary of the Akbulut cork in \(\mathbb{R}^4\), and composing with the conjecture from Sec. 2.4 we have a geometrical interpretation

The change of smoothness on \(S^4\) corresponds to the deformation of a generalized Dirac structure by \(S^1\)-gerbes on \(S^3\), lying at the boundary of the Akbulut cork for \(S^4\).

Regarding the correspondence C:

For non-integral \([H]\) the geometrical description requires \(S^1\) (fully abelian) categorical 2-gerbes

we note that non-integral third cohomologies can be geometrically interpreted by gerbes. But then one needs a 2-categorical extension of these gerbes, namely fully abelian 2-gerbes \([2]\). In this work we omitted the usage of a categorical language and kept the presentation down to earth from that point of view. The connection of exotic smoothness in dimension four and (higher) category theory constructions will be presented in a separate work.

4. An application: charge quantization without magnetic monopoles

A concrete physical example is the quantization of electric charge. The discussion of Dirac’s magnetic monopole shows that the quantization condition for electrical charge follows from the existence of Dirac’s magnetic monopoles. However, the condition for the magnetic monopoles are expressed in terms of abelian gerbes and that gives non-vanishing of the third integral cohomologies \(H^3(S^3, \mathbb{Z})\) \([16]\), Chapter 7). Thus, we have as a consequence of our correspondence between exotic 4-smoothness and gerbes:

The quantization condition for electric charge in space time can be seen as a consequence of certain non-standard 4-smoothness appearing in space time.

In the following we will present the details to understand this consequence.

4.1. Dirac’s magnetic monopoles and \(S^1\)-gerbes. When magnetic field \(B\) is defined over the whole euclidean space \(\mathbb{R}^3\), there exists a globally defined potential vector, and any two potential vectors differ by the gradient of some function. In terms of the connection on the line bundle \(L\), one can trivialize the bundle, and the connection \(\nabla\) on \(L\) is given by

\[
\nabla = d + \frac{e}{\hbar c}A
\]  

(25)
Dirac considered a magnetic field $B$ defined on $\mathbb{R}^3 \setminus \{0\}$ which has a singularity at the origin. This singularity corresponds to the existence of a magnetic monopole localized at the origin. The magnetic monopole has the strength $\mu$

$$\mu = \frac{1}{4\pi} \int \int_{\Sigma} \overrightarrow{B} \times d\sigma$$

(26)

which is the flux of $\overrightarrow{B}$ through the 2-sphere $\Sigma$ up to the constant, and the integral does not depend on the choice of the $\Sigma$ centered at the origin. This is so, because $\text{div}(\overrightarrow{B}) = 0$.

Equivalently $\mu$ can be expressed in terms of the curvature 2-form $R$, of a connection on $L$, as

$$\mu = -i \frac{\hbar}{4\pi e} \int_{\Sigma} R$$

(27)

Now, the integral $\int_{\Sigma} R$ has values which are integer multiplicities of $2\pi i$ on a complex line bundle $L$. Thus, the following quantization condition for the strength of a magnetic monopole, follows

$$\mu = \frac{e}{2\hbar} \cdot n, \ n \in \mathbb{Z}$$

(28)

This is based on the fact that the cohomology class of $\frac{\mathcal{R}}{2\pi i}$ is integral and the magnetic field $\overrightarrow{B}$ is proportional to the curvature of some line bundle with connection on $\mathbb{R}^3 \setminus \{0\}$. We see, that (28) is the same as $\frac{e}{2\hbar} \mu \cdot e = n$ and this means that electric charge is to be quantized.

The cohomologies involved here are $H^2(\mathbb{R}^3 \setminus \{0\})$. We extend, following [16], Chap. 7, the forms and cohomologies over whole 3-space, including the origin. To this end let us consider generalized 3-forms on $\mathbb{R}^3$ which are supported at the origin, i.e. the relative cohomology group $H^3(\mathbb{R}^3, \mathbb{R}^3 \setminus \{0\}) =: H^3_0(\mathbb{R}^3)$. Given the exact sequence

$$H^2(\mathbb{R}^3) = 0 \rightarrow H^2(\mathbb{R}^3 \setminus \{0\}) \rightarrow H^3_0(\mathbb{R}^3) \rightarrow H^3(\mathbb{R}^3) = 0$$

(29)

we have the isomorphism

$$H^2(\mathbb{R}^3 \setminus \{0\}) = H^3_0(\mathbb{R}^3)$$

(30)

We saw in (28) that a topological analog of the monopole is the element of the 2-nd cohomology $H^2(\mathbb{R}^3 \setminus \{0\}, \mathbb{Z})$. Thus, the extension of the description of a monopole, located at the origin, over entire $\mathbb{R}^3$, gives the topological analog of the monopole as an element of $H^3_0(\mathbb{R}^3)$. A monopole is moving now inside the 3-space, and from the canonical isomorphisms

$$H^3_0(\mathbb{R}^3) \simeq H^3_0(\mathbb{R}^3 \cup \{\infty\}) \simeq H^3(S^3)$$

(31)

the topological counterpart of a monopole is the element of $H^3(S^3, \mathbb{Z})$. Conversely, given the topological realization of some element from $H^3(S^3, \mathbb{Z})$ in space time, defines some line bundle with connection on $\mathbb{R}^3 \setminus \{0\}$, which is equivalent to the existence of a Dirac monopole in space time and consequently the electric charge is quantized.
The whole discussion can be extended to the relativistic theory in $\mathbb{R}^4$ as well (see the introduction of [11]). Then we consider the Coulomb potential of a point particle of charge $q$ in $0 \in \mathbb{R}^3$ as the connection one-form of a line bundle

$$A = -q \cdot \frac{1}{r} \cdot dt$$

over $\mathbb{R}^4 \setminus \mathbb{R}_t$ with $r^2 = x^2 + y^2 + z^2 \neq 0$. The curvature is given by the exact 2-form

$$F = dA$$

fulfilling the first Maxwell equation $dF = 0$. Now we consider the 2-form $*F$

$$*F = \frac{q}{4\pi} \cdot \frac{x \, dy \wedge dz - y \, dx \wedge dz + z \, dx \wedge dy}{r^3}$$

as element of $H^2(\mathbb{R}^4 \setminus \mathbb{R}_t)$ thus fulfilling the second Maxwell equation $d*F = 0$. A simple argument showed the isomorphism

$$H^* (\mathbb{R}^4 \setminus \mathbb{R}_t) \cong H^* (\mathbb{R}^3 \setminus \{0\})$$

and the integral along an embedded $S^2$ surrounding the point 0 gives

$$\int_{S^2} *F = q$$

the charge of the particle. Together with the isomorphism [30] we obtain the relation between a relativistic particle of charge $q$ in $\mathbb{R}^4$ and elements of $H^3(S^3)$.

The elements of $H^3(S^3, \mathbb{Z})$ have well-defined topological realizations, similarly as the elements of $H^2(\mathbb{R}^3 \setminus \{0\}, \mathbb{Z})$ corresponds to line bundles with connection. Namely, $h \in H^3(S^3, \mathbb{Z})$ corresponds to $S^1$-gerbe $G_h$ on $S^3$. However, the elements of $H^3(S^3, \mathbb{Z})$ have yet another realizations in space time.

4.2. $S^1$-gerbes on $S^3$ and exotic smooth $R^4$’s. In Subsect. 2.3 it was shown that third real de-Rham cohomology classes of $S^3$ correspond to the isomorphy classes of codimension-1 foliations of $S^3$. These last correspond to different isotopy classes of exotic smooth $R^4$’s where $S^3$ lies at the boundary of the Akbulut cork in $R^4$ [17]. For integral 3-rd cohomologies of $S^3$ we have the correspondence between $S^1$-gerbes on $S^3$ and some exotic smooth $R^4$’s. The correspondence means, in particular, that some exotic smooth small structures on some open region in $\mathbb{R}^4$ serves as the space time realization of the integral 3-rd cohomology class of $S^3$. In other words, non-trivial 3-rd cohomology class of $S^3$ is realized in space time by exotic smooth 4-structure in some region.

From the other hand, given a class from $H^3(S^3, \mathbb{Z})$, which is the same as a class in $H^2(\mathbb{R}^3 \setminus \{0\})$, means that one has a line bundle with connection on $\mathbb{R}^3 \setminus \{0\}$, which could be equivalent to the action of a monopole existing somewhere in space time. However, if a monopole exists, electric charge is quantized. To be sure that exoticness of a region of space time can give the same effect as the existence of a monopole, we will make some additional suppositions.

Namely, suppose that magnetic field propagates over a region, with smooth exotic structure, in 4-space time with the Minkowski metric, such that
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- the exotic smooth $\mathbb{R}^4$ corresponds to the class $[h] \in H^3(S^3, \mathbb{Z})$ and
- the strength of the magnetic field is proportional to the curvature of the line bundle on $\mathbb{R}^3\setminus\{0\}$ which corresponds to this $[h] \in H^2(\mathbb{R}^3 \setminus \{0\})$,

then this smooth $\mathbb{R}^4$ acts as a source for the magnetic field in $\mathbb{R}^4$, i.e. magnetic monopole, and electric charge is quantized, since given a class $[h] \in H^2(\mathbb{R}^3 \setminus \{0\})$ we always find a monopole solution fulfilling the requirements. Such a monopole gives the quantization of electric charge and is the source for magnetic field.

Some large smooth exotic $\mathbb{R}^4$’s can act as the external sources of gravitational field in space time. This follows from the Brans conjecture which states that exotic smooth structures on 4-manifolds (compact and non-compact) can serve as external sources for gravitational field [12]. The Brans conjecture was proved by Asselmeyer [5] in the compact case, and by Śladkowski [46] in the non-compact case.

In that way, we yielded an essential extension of the Brans conjecture for magnetic fields:

Some small, exotic smooth structures on $\mathbb{R}^4$ can act as sources of magnetic field, i.e. monopoles, in space time. Electric charge in space time has to be quantized, provided some region has this small exotic smoothness.

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A. Branes and orientifold planes in $SU(2)$ WZW $\sigma$- models

WZW models, which are interacting field theories, allows also a Lagrangian description. Free string theory on a group manifold $G$ (a Lie group) can either be described by the abstract CFT (built on an affine extension $\tilde{g}$ of the algebra $g$ of $G$) or by the Wess-Zumino-Witten action given by

$$ S = \frac{-k}{16\pi}\int_{\Sigma} d^2x \text{Tr}(g^{-1}\partial_\mu g)^2 + \frac{k}{24\pi}\int_{D} d^3y \text{Tr}(g^{-1}dg)^3 $$

(32)

where $g$ is a field that takes values in the group $G$ and is defined on the string worldsheet $\Sigma$, hence $g$ describes an embedding $\Sigma \hookrightarrow G$. The second integral is topological; $D$ is a three-manifold whose boundary is $\Sigma$ and $k$ is an integer level of the theory. The WZW theory possesses so called chiral currents

$$ J = g^{-1}\partial g, \quad \overline{J} = -\overline{\partial}gg^{-1} $$

(33)

where $\partial = \partial_\tau + \partial_t$, $\overline{\partial} = \partial_\tau - \partial_t$ and $\tau, t$ are coordinates on $\Sigma$. The modes of these currents span two commuting copies of the Lie algebra $g$.

Let $Z_N$ be the center of $G$ - a compact, simple and semisimple Lie group, and $\gamma$ be a generator of $Z_N$. Let $P$ be the worldsheet parity transformation of $\Sigma$ interchanging the chiral coordinates. Then, $R_n$ defined on the coordinates as

$$ R_n : g \rightarrow \gamma^n g^{-1}, \quad n = 0, ..., N - 1 $$

when composed with $P$, interchanges the chiral currents and can be considered as a global invariance of this theory. Let us denote this composition as $P_n = P \circ R_n$. Gauging out this symmetry, gives orientifold fixed planes, i.e. points of the target space $G$ that are fixed under $P_n$.

$P$ alone is not a symmetry of the theory, hence in WZW models there does not exist ,,maximal space-time filling orientifold planes”. There are only lower dimensional planes which contain points $g \in G$ determined from the condition

$$ g = \gamma^n g^{-1}. $$

(34)

When $g$ is the solution of (34), every point of the conjugacy class $C_g = \{hg^{-1}h^{-1} : h \in G\}$ is either. One requires an involution on the target group manifold, such that the induced action on the currents is to exchange the left and right moving current algebra. The fixed point set is then interpreted geometrically as the location of the orientifold. Thus, the existence of geometric fixed points orientifolds affects the algebra spanned by currents.

In the case of our interest, i.e. $G = SU(2) \simeq S^3$, these data have the following meaning. The center of $SU(2)$ consists of the two elements $\{1, -1\} \simeq \mathbb{Z}_2$, hence there are two possible involutions generating orientifolds. The fixed point set of the inversion $g \rightarrow g^{-1}$ consists of 1 poles of $S^3$; the fixed point set of $g \rightarrow -g^{-1}$ consists of the conjugacy class of the group element and this is the $S^2$ conjugacy class located at the equator of $S^3$. This agrees with our previous discussion.

Under the action of $P_n$ the gluing condition on currents (33), i.e. $J = -\overline{J}$ can be written as

$$ g^{-1}\partial_t g - \partial_t gg^{-1} = g^{-1}\partial_\tau g + \partial_\tau gg^{-1} $$

(35)
Let us denote the adjoint action of $G$ on its Lie algebra as $Ad(g)y = ggy^{-1}$. Then, equation (35) gives

$$(1 - Ad(g))g^{-1} \partial_t g = (1 + Ad(g))g^{-1} \partial_x g$$

(36)

In the case of closed string theory one easily sees that this means that so called D-branes coincide with the conjugation classes of some group elements of $S^3$. On a conjugacy class $C_g$ we get

$$g^{-1} \partial_t g = \frac{1 + Ad(g)}{1 - Ad(g)} g^{-1} \partial_x g$$

(37)

Thus, on $C_g$ we get globally defined two-form $B$ given by

$$B = \frac{k}{8\pi} Tr(g^{-1}dg \frac{1 + Ad(g)}{1 - Ad(g)} g^{-1}dg)$$

(38)

whose curvature 3-form $H = dB$ is given (on $C_g$) by

$$dB = -\frac{k}{12\pi} Tr(dgg^{-1})^3$$

(39)

Now, the so called D-branes in WZW models are specified by a choice of sub-manifolds $M \subset G$ such that:

1. the WZ form derived from the topological WZ term in the Lagrangian, as in $WZ = -\frac{k}{8\pi} Tr(dgg^{-1})^3$, is exact, when on $M$, and
2. there exists 2-form $B$ on $M$ such that $dB = WZ|_M$.

The WZ terms of the $SU(2)$ WZW models at level $k \in \mathbb{Z}$ are determined by 2-forms $kB$, $k \in \mathbb{Z}$, with curvature 3-forms $kH$.

We see from (39) that indeed the conjugacy classes $C_g$ correspond to D-branes in the $SU(2)$ WZW $\sigma$-models. To determine which conjugacy classes can be considered as D-branes, this is the choice of $B$-field, i.e. local 2-form $B$, and the topological WZ term in the Lagrangian. In the $SU(2)$ case we have $S^2$ conjugacy classes. Taking two 3-balls bounded by such a $S^2$, one finds that there are $k - 1$ conjugacy classes which can be D-branes. This is due to the difference with the phase of the boundary state of the world-sheet theory. To every D-brane in the target corresponds some boundary state in the world-sheet CFT theory. The difference in the phase of this boundary state should be $\Delta(\phi) = 2\pi j$, $j = 1, 2, ..., k - 1$ [3]. The corresponding conjugacy classes, hence D-branes, are 2-spheres passing through the points

$$\left( e^{i\pi j/k} 0 \right)$$

(40)

under the usual identification of matrix from $SU(2)$ with points of $S^3$. 
B. Elements of boundary CFT

Let us consider spherical (fully symmetric) D-brane wrapping the conjugacy \( S^2 \) class in \( S^3 = SU(2) \) the target for WZW model. Given two such branes one can calculate the spectrum of open strings ending on these branes. In the case of \( \mathbb{Z}_2 \) orientifold theory branes wrapping \( S^2 \) equator plane become geometrically the projective \( \mathbb{R}P^2 \) spaces. Thus, the cross-caps states correspond to them. The calculation of the spectra is performed in 2-dimensional CFT. Let us denote by \(|B_a>\) and \(|C_P>\) the boundary and cross-cap states, where \( a \) is the boundary condition and \( P = \tau\Omega \) a parity symmetry, i.e. \( \tau \) is an internal transformation and \( \Omega \) is the space inversion. Let \( \mathcal{H}_{a,b} \) be the Hilbert space of states on a segment with the boundary conditions \( a, b \) on the left and right ends, then the cylinder amplitude is given by

\[
\text{Tr}_{\mathcal{H}_{a,b}} e^{-\beta H_c(L)} = g < B_1 | e^{-LH_c(\beta)} | B_2 >_g
\]  

(41)

the Klein bottle by

\[
\text{Tr}_{\mathcal{H}_{P(a)}} e^{-\beta H_c(L)} = P_2 < B_a | e^{-LH_c(2\beta)} | C_P >
\]  

(42)

and the Moebius strip by

\[
\text{Tr}_{\mathcal{H}_{a,P(a)}} e^{-\beta H_c(L)} = P_2 < B_a | e^{-LH_c(2\beta)} | C_P >
\]  

(43)

where \( H_c(L) \) is the Hamiltonian of the system on a circle of circumference \( L \), \( H_0(L) \) the Hamiltonian on a segment of length \( L \), \( P(a) \) is the image by \( P \) of the boundary conditions \( a \). In the case of involutive parity symmetries one has \( P^2 = 1 \). The CFT theory is defined by specifying a chiral algebra of currents \( C_A = \{ W^r \} \) with its set of representations \( \{ \mathcal{H}_i \} \), thus the Hilbert space of states is given as \( \mathcal{H} = \bigoplus_i \mathcal{H}_i \otimes \mathcal{H}_\bar{i} \), and \( \bar{i} \) is the conjugate of \( i \). The spin \( s_r \) of the current \( W^r \) is defined by the anti-unitary operator \( U : \mathcal{H}_i \to \mathcal{H}_{\bar{i}} \) such that

\[
U W^r U^{-1} = (-1)^r W^r, \text{ for every } r.
\]

In terms of the characters \( \chi_i \) of the representation \( i \) and Ishibashi boundary, \(|B, i>\), and Ishibashi cross-caps \(|C, i>\) states, the corresponding to (41-43) amplitudes read

\[
\langle B, i | e^{2\pi i H_c} | B, j > = \delta_{i,j} \chi_i(2\tau)
\]  

(44)

\[
\langle C, i | e^{2\pi i H_c} | C, j > = \delta_{i,j} \chi_i(2\tau)
\]  

(45)

\[
\langle B, i | e^{2\pi i H_c} | C, j > = \delta_{i,j} \tilde{\chi}_i(2\tau)
\]  

(46)

where \( \tau \) is complex, \( H_c = L_0 + \tilde{L}_0 - c/12 \), \( c \) the central charge, and \( \tilde{\chi}_i(\tau) = e^{-\pi i (h_i - \frac{c}{12})} \chi_i(\tau + \frac{1}{2}) \). \( L_0, \tilde{L}_0 \) are the 0-generators in \( W^r \) and \( \tilde{W}^r \) correspondingly; \( h_i \) are defined by the relation between Ishibashi states

\[
|C, i > = e^{\pi i (L_0 - h_i)} |B, i>
\]  

(47)

General boundary and cross-caps states are the combinations of these Ishibashi states.
\[ |B_a > = \sum_i n_{ai}|B, i > \quad (48) \]
\[ |C_P > = \sum_i \gamma_{Pi}|C, i > \quad (49) \]

The characters that appear in the partition functions are the characters of the chiral algebra \( \mathcal{A} \) and they form a representation of the modular group, generated by \( T : \tau \to \tau + 1 \) and \( S : \tau \to -1/\tau \).

\[
\begin{align*}
\chi_j(\tau + 1) &= \sum_i T_{ij}\chi_i(\tau) \\
\chi_j(-1/\tau) &= \sum_i S_{ij}(\tau)\chi_i \\
\hat{\chi}_j(-1/4\tau) &= \sum_i P_{ij}(\tau)\hat{\chi}_i
\end{align*}
\]

where \( P = \sqrt{TST^2S\sqrt{T}} \) and \( \sqrt{T_{ij}} = \delta_{ij}e^{\pi i(h_i - \frac{c}{24})} \).

The spectrum of open strings stretching between the branes that are associated with the labels \( I \) and \( J \) is encoded in the associated partition functions

\[ Z_{IJ}(q) = \sum_J \{N_{IJ}\}^J \chi_j(q) \quad (53) \]

The operator product expansion for two primary fields, \( \psi^{IJ}_j(u) \), is of the form

\[ \psi^{LM}_i(u_1)\psi^{MN}_j(u_2) = \sum_k (u_1 - u_2)^{h_i + h_j - h_k}F_{Mk}^{i j L N}U_{ijk}\psi^{LN}_k(u_2) + ... \quad (54) \]

The one-point function in general boundary CFT of the boundary field \( \phi_{jj}' \) is given by (p.42, 43 [45]):

\[ <\phi_{jj'}(z, \bar{z})> = \frac{S_{jj'}}{\sqrt{S_{0j}}} \frac{U_{jj'}}{|z - \bar{z}|^{2h_j}} \quad (55) \]

where \( U_{jj'} \) is the unitary intertwinner between the actions of the zero-modes on two spaces of ground states.

In the case of SU(2) WZW model, in the closed string sector, the one-point function is given by (formula (4.20), p. 56, [45])

\[ <\phi^{ab}_{jj}(z, \bar{z})> = \left( \frac{2}{k+2} \right)^{\frac{1}{2}} \sin \frac{\pi(2j+1)(2J+1)}{k+2} \frac{\delta^{ab}}{|z - \bar{z}|^{2h_j}} \quad (56) \]

The open string sector, in the case of strings ending on the same \( S^2 \) - brane, gives the following OPE rule (formula (4.25), p. 58, [45])

\[ \psi^a_i(u_1)\psi^b_j(u_2) = \sum_{k,c} \delta_{i12}^{h_k - h_i} \left\{ \begin{array}{ccc} i & j & k \\ J & J & J \end{array} \right\} \left\{ \begin{array}{ccc} i & j & k \\ a & b & c \end{array} \right\} \psi^c_k(u_2) \quad (57) \]
where \( \{ ijk \}_{III} \) are 6-\( j \) Glebsh-Gordon coefficients and \( [ijk]_{abc} \) are analogous coefficients respecting the truncation of the affine algebra and correspond to the fusion matrix \([15]\).

C. The result of shape computation in CFT \([15]\)

The shape of the orientifold \( S^2 \) equator plane as emerging from the quantum regime of WZW orientifold can be determined by scattering massless closed string states. The geometrical result will be obtained in the limit \( k \to \infty \) and following \([15]\) we use the CFT description.

Scattering amplitude between cross-cap states and massless closed strings are computed as overlaps of the cross-cap states with closed string ground states. In the case of \( SU(2) \) the closed string ground states are \( |j,m,m_1> \), with no descendants. Let us consider the space of functions \( \mathcal{F}(G) \) on a group manifold \( G \). It is known from Peter-Weyl theorem that \( \mathcal{F}(G) \) is isomorphic to a direct sum of tensor products of irreducible representations. Matrix elements of these representations form a complete orthogonal basis of the space \( \mathcal{F}(G) \). In the case of \( SU(2) \) and the Haar measure on it, one yields the following orthonormal basis of functions on \( S^3 \):

\[
<j|m|R(g)|m_1> = \sqrt{2j+1}D^j_{mm_1}(g)
\]  

(58)

The localized graviton state is then

\[
|g> = \sum_j e^{-j^2} \sqrt{2j+1} |j,m,m_1>
\]

(59)

where we have inserted the exponent factor to leave only modes of the closed string with low values of \( j \), which makes the states well localized in order to probe classical geometry. The best approximation (\( \delta \)-shape) is in the limit \( k \to \infty \).

Parametrizing \( SU(2) \) by the angles \( \theta, \phi, \psi \), we take \( \psi \) as the label of the conjugacy classes. \( \theta \) and \( \psi \) give the parametrization in the \( S^2_\psi \) class and \( g_\psi \in S^3 \) means \( g \in S^3_\psi \subset S^3 \).

We want to find the overlaps of the cross-cap state \( |Ceg> \) corresponding to the equator orientifold plane and the localized closed string packet \([59]\). The CFT analysis ensures us that the cross-cap state is the combination of the cross-cap Ishibashi states which are invariant with respect to \( \Omega : g \to -g^{-1} \) leaving only even \( j \) states.

The characters of \( SU(2)_k \) are labeled by \( j = 0, 1/2, 1,...k/2 \). The matrix \( P \) for \( SU(2)_k \) where \( k \) is even, reads

\[
P_{jl} = \frac{2}{\sqrt{k+2}} \sin \frac{\pi(2j+1)(2l+1)}{2(k+2)}, \ j+l \in Z
\]

(60)

The simple current of the theory corresponds to the representation \( k/2 \) and the current group is \( \{0,k/2\} \simeq \mathbb{Z}_2 \), thus the cross-cap state corresponding to the equator of \( G \) is given as
\[|C_{eq} > = \sqrt{\frac{2}{k+2}} \sum_{j=0}^{k/2} (-1)^j \cot^{1/2} \left( \frac{(2j + 1)\pi}{2(k + 2)} \right)|C, j \gg \]  

(61)

where \(|C, j \gg\) are cross-cap Ishibashi states. The above equation follows from the following formula for the general cross-cap state as the combination of the Ishibashi states [7]

\[|C > = \sum_j \frac{P_{j0}}{\sqrt{S_{j0}}} |C, j \gg \]  

(62)

where the matrix \(P = T^\dagger S T S T^\dagger\) and \(T, S\) are the matrices corresponding to the modular transformations. In the presence of the simple current \(L\), as is our case where \(L = k/2\), we have

\[|C_L > = \sum_j \frac{P_{jL}}{\sqrt{S_{j0}}} |C, j \gg \]  

(63)

Thus the computation of the amplitude gives

\[< C_{eq}|g_\psi > \simeq \sum_{j=0}^{k/2} \sum_m (-1)^j e^{-j^2 \cot^{1/2} \left( \frac{2j+1}{2(k+2)} \pi \right) (2j + 1)^{1/2} D_{mn}^{ij} (g_\psi)} \]  

(64)

and using the relation of the matrix elements of \(D_{mn}^{ij}\) with characters of \(SU(2)\), i.e. \(\sum_m D_{mn}^{ij} = \frac{\sin(2j+1)\psi}{\sin \psi}\), and taking \(j^2 << k\) limit, one has

\[< C_{eq}|g_\psi > \simeq \sum_j (-1)^j \frac{\sin(2j+1)\psi}{\sin \psi} \simeq \delta(\psi - \frac{\pi}{2}) \]  

(65)

This last shows that the geometrical location of the orbifold plane is indeed at \(S^2\) given by \(\psi = \pi/2\).

Communicated by name