Optimal decomposition of incoherent qubit channel

Swapan Rana¹ and Maciej Lewenstein¹,²

¹ ICFO—Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels, Barcelona, Spain
² ICREA, Lluis Companys, 23, E-08010 Barcelona, Spain

E-mail: swapanqic@gmail.com

Received 4 July 2018, revised 26 July 2018
Accepted for publication 8 August 2018
Published 14 September 2018

Abstract

We show that any incoherent qubit channel could be decomposed into four incoherent Kraus operators. The proof consists in showing existence of four incoherent Kraus operators by decomposing the corresponding Choi–Jamiołkowski–Sudarshan matrix. We mention some applications of this optimal decomposition. We also show that the Kraus rank and incoherent rank are different even for qubit channel.

Keywords: coherence, incoherent operation, Kraus operators

(Some figures may appear in colour only in the online journal)

1. Introduction

In resource theory of coherence [1], as introduced in [2], one first chooses a basis for the Hilbert space corresponding to the considered quantum system. Diagonal states are then regarded as free (called ‘incoherent’) states of the theory. The free operations are defined in terms of incoherent operators, which transform any free state to another free state. Precisely, a quantum operation is free, termed as incoherent operation (IO), if and only if it has a Kraus decomposition with all Kraus operators being incoherent. If additionally the transpositions of all such Kraus operators are also incoherent, the operation is called strictly incoherent (SIO) [3]. The SIOs, being more restrictive, neither create nor use coherence [4].

The free operations in entanglement theory—the local operations and classical communications (LOCC) [5], have clear physical restrictions that the (possibly spatially separated) parties are allowed to act only locally, albeit, hard to efficiently describe their mathematical decomposition in terms of Kraus operators. In contrast, the free operations in coherence theory—that is the IOs, are defined in terms of the Kraus decomposition. However, this poses a difficulty in implementing the IOs, as (besides the non-uniqueness of Kraus decomposition) the general way of implementing a quantum operation—adding an ancillary system, evolving
the combined system under a suitable unitary, and finally tracing off the ancillary system, is not applicable. Although an incoherent unitary and incoherent ancilla always yield an IO, these restrictions may not be necessary. Specifically, it is not known what is the necessary and sufficient condition for the unitary and ancilla to yield IOs. But, implementation of free operations are of immense importance as it dictates the whole structure of the theory starting from state conversion.

One way to understand the structure of IOs is to find their minimal parametrization, which necessarily involves finding the minimum number of Kraus operators. It will also allow to numerically simulate the IOs efficiently for small dimensions. To this aim, some bounds on the number of Kraus operators for the (S)IOs have been derived in [6]. However, optimality of these bounds are not known even for the simplest case of qubit IO. It was shown that every SIO can be decomposed into four Kraus operators (and this is optimal number), and every IO into five Kraus operators. A canonical representation of the Kraus operators for any incoherent qubit channel is given by [6] the set

\[ \{ (a_1, b_1), (0, 0), (a_2, b_2), (0, 0), (a_3, 0), (0, b_3), (a_4, 0), (0, 0), (a_5, 0) \} \],

(1)

where \( a_i \) can be chosen nonnegative, while \( b_i \in \mathbb{C} \). Moreover, it holds (for normalization) that \( \sum_{i=1}^{5} a_i^2 = \sum_{j=1}^{5} |b_j|^2 = 1 \) and \( a_1 b_1 + a_2 b_2 = 0 \). Similarly, a canonical (and optimal) representation of the Kraus operators for a qubit SIO is given by [6] the set

\[ \{ (a_1, 0), (0, b_1), (a_2, 0), (0, 0), (a_3, 0), (0, 0), (a_4, 0) \} \],

(2)

where \( a_i \geq 0, b_i \in \mathbb{C} \), and (for normalization) \( \sum_{i=1}^{4} a_i^2 = \sum_{j=1}^{4} |b_j|^2 = 1 \).

In the present work, we will show that every qubit IO can be decomposed into four incoherent Kraus operators, thereby giving the optimal description of qubit IOs. This decomposition will be compared with Kraus rank and some applications will be mentioned later. Throughout this manuscript, the terms '(S)IO channel' and '(S)IO operations' will be used synonymously. We will also frequently use the fact that two sets of Kraus operators \( \{ K_i \} \) and \( \{ L_i \} \), each having exactly \( m \) members (possibly after adding zero operators to the set having less members) describe the same quantum channel if and only if [7, pp 372] there is an \( m \times m \) unitary matrix \( U = (u_{ij}) \) such that

\[ L_i = \sum_{j=1}^{m} u_{ij} K_j, \quad i = 1, 2, \ldots, m. \]

(3)

In what follows, we will first re-parametrize the canonical form of IO from equation (1) to

\[ \Lambda = \left\{ \begin{array}{c}
(r \alpha_1 \beta_1) , (0, 0), (\alpha_2, 0), (0, \beta_2), (\alpha_3, 0), (0, \beta_3), (\alpha_4, 0)
\end{array} \right\} .
\]

(4)

where \( r, \alpha_i \geq 0, \beta_i \in \mathbb{C} \), and for normalization,

\[ \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + (r^2 + 1) \alpha_4^2 = |\beta_1|^2 + |\beta_2|^2 + (r^2 + 1) |\beta_3|^2 = 1. \]

(5)

With this parametrization, we make the following observation.

**Observation 1.** If any of the \( \alpha_i \)’s or \( \beta_i \)’s is zero in equation (4), the number of Kraus operators reduces to four.
Proof of observation 1. If \( \alpha_4 = 0 \), there is nothing to prove. If \( \alpha_1 \beta_1 = 0 \), then \( \Lambda \) is a SIO and hence could be reduced to the canonical form in equation (2) with at most four Kraus operators. If \( \beta_2 = 0 \), then the operators \( \begin{pmatrix} \alpha_2 & 0 \\ 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} \alpha_4 & 0 \\ 0 & 0 \end{pmatrix} \) are scalar multiple of each other, hence one could be made zero via equation (3), thereby reducing the number of Kraus operators to four. Similarly, if \( \alpha_3 = 0 \), the three operators \( \begin{pmatrix} r \alpha_1 & 0 \\ 0 & \beta_3 \end{pmatrix} \), \( \begin{pmatrix} 0 & \beta_2 \\ 0 & 0 \end{pmatrix} \), and \( \begin{pmatrix} \alpha_4 & 0 \\ 0 & 0 \end{pmatrix} \) are linearly dependent, so one of them could be made zero. The remaining two cases of \( \alpha_2 = 0 \) and \( \beta_3 = 0 \) are similar, thus we consider the first one.

If \( \alpha_2 = 0 \), then consider the (unnormalized) unitary \( U \oplus V \oplus 1 \), with

\[
U = \begin{pmatrix} r \alpha_1 & \alpha_4 \\ -\alpha_4 & r \alpha_1 \end{pmatrix}, \quad V = \begin{pmatrix} r \beta_1^* & -\beta_2^* \\ \beta_2 & r \beta_1 \end{pmatrix},
\]

which transforms
\[
\left\{ \begin{pmatrix} r \alpha_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_4 \\ 0 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 0 \\ \alpha_1 \end{pmatrix}, \begin{pmatrix} 0 \\ -r \beta_1 \end{pmatrix} \right\} \text{ to } \left\{ \begin{pmatrix} * \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ * \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}
\]

respectively. But then the three operators \( \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \), \( \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \), and \( \begin{pmatrix} 0 & \beta_3 \\ \alpha_2 & 0 \end{pmatrix} \) are linearly dependent, and so one of those could be made zero. \( \square \)

2. Main result

Theorem 2. The optimal number of incoherent Kraus operators for an incoherent qubit channel is four. That is, every incoherent qubit channel could be decomposed into four incoherent Kraus operators and there are some which cannot be decomposed into three. Thus an incoherent qubit channel can be canonically represented as

\[
\Lambda = \left\{ \begin{pmatrix} r \alpha_1 & \beta_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \alpha_1 & -r \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix}, \begin{pmatrix} 0 & \beta_3 \\ \alpha_3 & 0 \end{pmatrix} \right\}, \tag{6}
\]

where \( r, \alpha_i \geq 0, \beta_i \in \mathbb{C} \), and for normalization,

\[
\alpha_1^2 + \alpha_2^2 + (r^2 + 1) \alpha_3^2 = |\beta_2|^2 + |\beta_3|^2 + (r^2 + 1) |\beta_1|^2 = 1. \tag{7}
\]

Since the four operators \( \{|i\langle j|, i,j = 1,2 \text{ are linearly independent} \) (in the vector space of 2-by-2 matrices over complex numbers), the channel \( \Lambda = \{|i\langle j|/\sqrt{2}\} \) is an (S)IO whose number of Kraus operators could not be reduced further. Thus to prove the optimality, it suffices to show that any qubit incoherent channel, which without loss of generality given in equation (4), could be decomposed into four incoherent Kraus operators.

Proof. In view of observation 1, we can assume each \( \alpha_i, \beta_i \) to be non-zero in the channel (4). The Choi–Jamiołkowski–Sudarshan (CJS) matrix for this channel is
\[
M = \frac{1}{2} \begin{pmatrix}
    r^2 \alpha_1^2 + \alpha_2^2 + \alpha_3^2 & r \alpha_1 \beta_1^* & 0 & \alpha_2 \beta_2^*
    \\
    r \alpha_1 \beta_1 & |\beta_1|^2 + |\beta_3|^2 & \alpha_3 \beta_3 & 0
    \\
    0 & \alpha_3 \beta_3 & \alpha_1^2 + \alpha_3^2 & -r \alpha_1 \beta_1^*
    \\
    \alpha_2 \beta_2^* & 0 & -r \alpha_1 \beta_1 & r^2 |\beta_1|^2 + |\beta_2|^2
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
    a & e & 0 & g
    \\
    e^* & b & f & 0
    \\
    0 & f^* & c & -e
    \\
    g^* & 0 & -e^* & d
\end{pmatrix}.
\]  

Assuming \( M \geq 0 \), we want to show that \( M \) admits a decomposition of the form 
\[
2M = \begin{pmatrix}
    A & \cdot & \cdot & g
    \\
    \cdot & \cdot & \cdot & \cdot
    \\
    \cdot & \cdot & \cdot & \cdot
    \\
    g^* & \cdot & \cdot & |\beta|^2/A
\end{pmatrix} + \begin{pmatrix}
    \cdot & \cdot & \cdot & \cdot
    \\
    B & f & \cdot & \cdot
    \\
    \cdot & \cdot & \cdot & \cdot
    \\
    \cdot & \cdot & \cdot & \cdot
\end{pmatrix} + \begin{pmatrix}
    a - A & e & \cdot & \cdot
    \\
    e^* & b - B & \cdot & \cdot
    \\
    \cdot & \cdot & \cdot & \cdot
    \\
    c - |\beta|^2/B & -e & \cdot & \cdot
\end{pmatrix} + \begin{pmatrix}
    \cdot & \cdot & \cdot & \cdot
    \\
    \cdot & \cdot & \cdot & \cdot
    \\
    \cdot & \cdot & \cdot & \cdot
    \\
    -e^* & d - |\beta|^2/A & \cdot & \cdot
\end{pmatrix}
\]  

for some \( A, B > 0 \). If, additionally, each of the two diagonal blocks of the last matrix is positive semidefinite (PSD) with rank one, then the matrix \( M \) is decomposed into sum of four rank one PSD matrices which are CJS matrix of the four incoherent Kraus operators of the form 
\[
\begin{pmatrix}
    \ast & 0 & 0 & 0
    \\
    0 & \ast & 0 & 0
    \\
    0 & 0 & \ast & 0
    \\
    0 & 0 & 0 & \ast
\end{pmatrix}
\]
respectively. These four CJS matrices will uniquely determine the parameters \( r, \alpha_i, \beta_i \) in the canonical form given in equation \( (6) \). Therefore, to make the number of (incoherent) Kraus operators four, it suffices to show that the last matrix is PSD and it has rank two. This is the case if and only if 
\[
(a - A)(b - B) = \begin{pmatrix}
    c - |\beta|^2/B
    \\
    \cdot
\end{pmatrix} \begin{pmatrix}
    d - |\beta|^2/A
    \\
    \cdot
\end{pmatrix} = |\beta|^2,
\]  

and 
\[
A \in \begin{pmatrix}
    |\beta|^2/B
    \\
    d
\end{pmatrix},\quad B \in \begin{pmatrix}
    |\beta|^2/A
    \\
    c
\end{pmatrix}.
\]  

Since this is an equation involving only positive variables, we remove the absolute signs for brevity, thereby \( \beta_i \) should be understood as \( |\beta_i| \) henceforth. (Indeed the condition \( M \geq 0 \) remains valid if we assume all of \( a, b, c, d, e, f, g \) to be positive\(^3\).

\(^3\) Notice that for the Hermitian matrix
\[
X = \begin{pmatrix}
    a & e & 0 & g
    \\
    e^* & b & f & 0
    \\
    0 & f^* & c & -e
    \\
    h^* & 0 & -e^* & d
\end{pmatrix},
\]
\( X \geq 0 \) implies \( \tilde{X} \geq 0 \), where \( \tilde{X} \) is the matrix obtained from \( X \) by keeping the same diagonals and replacing the off-diagonal elements \( e, f, -c, g \) by \( |e|, |f|, -|c|, |g| \) respectively. This is due to the fact that in the condition for PSDness of \( X \), phases appear only in the determinant condition, which reads
\[
0 \leq \det(X) = (ad - |g|^2)(bc - |f|^2) - |e|^2(ab + cd - |e|^2) + 28 |e^2f^g|^2
\]
\[
\leq (ad - |g|^2)(bc - |f|^2) - |e|^2(ab + cd - |e|^2) + 2|e|^2f\|g| = \det(\tilde{X}).
\]
The caveat is that \( X \) and \( \tilde{X} \) could be entirely different matrix (e.g. with different rank). So, in our problem, we could not use the decomposition of \( 2M \) to claim the same about \( 2M \).
We will first parametrize $A, B$, using the first equation of equation (10a). To this end,

\[
\frac{A(a - A)}{Ad - g^2} = \frac{Bc - f^2}{B(b - B)} : = k > 0.
\]

(11a)

\[
\Rightarrow A = \frac{(a - kd) + \sqrt{(a - kd)^2 + 4kg^2}}{2},
\]

(11b)

\[
B = \frac{(k c - b) + \sqrt{(k c - b)^2 + 4kf^2}}{2k}.
\]

(11c)

Note that

\[
a - A = \frac{(a + kd) - \sqrt{(a + kd)^2 - 4k(ad - g^2)}}{2},
\]

(12a)

\[
b - B = \frac{(c + kb) - \sqrt{(c + kb)^2 - 4k(bc - f^2)}}{2k},
\]

(12b)

where $ad - g^2 > 0$ and $bc - f^2 > 0$, whereby $0 < A < a$ and $0 < B < b$ irrespective of the value of $k > 0$. This, together with equation (11a) shows that equation (10b) is satisfied for any $k > 0$.

The final task is to show that there is a positive solution for the equation $(a - A)(b - B) = e^2$ in $k$. Substituting the values from equation (12), this gives

\[
\frac{(a + kd) - \sqrt{(a + kd)^2 - 4k(ad - g^2)}}{2} \left[\frac{(c + kb) - \sqrt{(c + kb)^2 - 4k(bc - f^2)}}{2k}\right] = 4ke^2
\]

\[
\Rightarrow (a - A) = \frac{(a + kd) + \sqrt{(a + kd)^2 - 4k(ad - g^2)}}{2}.
\]

(13a)

where

\[
\alpha = e^2d_2d_4,
\]

(13b)

\[
\beta = \Delta \left[ \Delta + e^4(4f + ab + cd + 2fg) \right] + 2e^4(abfg + adf^2 + bcg^2 + cdfg),
\]

(13c)

\[
\gamma = e^2d_1d_3,
\]

(13d)

\[
d_1 := a(bc - f^2) - ce^2, \quad d_2 := b(ad - g^2) - de^2,
\]

(13e)

\[
d_3 := a(cd - e^2) - cg^2, \quad d_4 := b(cd - e^2) - df^2,
\]

(13f)

\[
\Delta := (ad - g^2)(bc - f^2) - e^2(ab + cd - e^2 + 2fg).
\]

(13g)

The quantities $d_1, d_2, d_3, d_4$ are the third-ordered principal minors of $2M$ and can be shown strictly positive by either directly evaluating those minors of $2M$ from equation (8), or substituting the values of $a, b, c, d, e, f, g$ in term of $a_i, \beta_i$, the latter showing the strict positivity of $\Delta$,.
\[ \Delta = \alpha_1^2 \alpha_2^2 \beta_1^2 \left( \beta_1^2 + \beta_2^2 \right) + \alpha_2^2 \beta_1^2 \left( \beta_2^2 + \beta_1^2 r^2 \right) + r^2 \left( \alpha_2 \alpha_3 \beta_1^2 - \alpha_1^2 \beta_1 \beta_3 \right)^2 > 0. \]

Therefore, \( \alpha > 0, \beta > 0, \gamma > 0. \) Noticing that the discriminant \(^4\)

\[ D := \beta^2 - 4 \alpha \gamma = \Delta \left( \Delta + 4 e^2 fg \right) \left[ (ad - g^2) (bc - f^2) - e^4 \right]^2 > 0, \]

it follows that the two roots of the quadratic equation (13a) are positive (and distinct). \(\square\)

### 3. IO versus SIO

A general (without any restriction like incoherentness) single qubit channel can optimally be decomposed into four Kraus operators. So, incoherentness makes no difference for qubit channels in terms of optimal Kraus decomposition. Note, however, that it no way means that a general channel could be decomposed into incoherent Kraus operators. A 2-by-2 unitary with non-vanishing entries (even in any single column) is a simple example of a channel which is not incoherent. This is also true for any finite dimension. A \(d\)-dimensional incoherent unitary is a general permutation (that is, the entries are arbitrary phases, \(e^{i \phi}\), not necessarily 1). Thus, all IO unitaries (channels with Kraus rank one) are necessarily SIO. For non-unitary channels, this is not true already at qubit level.

**Proposition 3.** There are (qubit-) incoherent channels which are not strictly incoherent.

**Proof.** As an example, consider the following incoherent channel

\[
\Lambda = \left\{ \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \right\}. \tag{14}
\]

If it were SIO, there must be a unitary converting those four Kraus operators via equation (3) to the standard SIO form given in equation (2). Comparing the last two operators, the fourth and third (unnormalized) row of the unitary must be \((1, -1, -1, 1)x\) and \((1, 1, 1, -1)y\) for some nonzero \(x, y \in \mathbb{C}\). The orthogonality of those two row vectors with the first row implies that the first row must be \((a, b, -b, a)\) with \(a, b \in \mathbb{C}\). However, the resulting first Kraus operator is then

\[
\begin{pmatrix}
\frac{a-b}{2} \\
\frac{a+b}{2}
\end{pmatrix}. \tag{a, b, -b, a}
\]

In order it to be an SIO, we must have \(a = 0 = b\), an impossibility as a unitary cannot have a zero row. \(\square\)

The four Kraus operators in equation (14) are linearly independent. Hence the channel cannot be reduced further to have three Kraus operators.

In the qubit case, it is easy to characterize all IO with two Kraus operators which are also SIO.

**Proposition 4.** All qubit IO with two Kraus operators are essentially SIO, except the following canonical one

\(^4\)The inequality \((ad - g^2) (bc - f^2) - e^4 > 0\) could be shown either by substituting \(a, b, c, d\), etc in terms of \(\alpha_i, \beta_i\), or by writing the left hand side as \(\Delta + e^2 [(ab - e^2) + (cd - e^2) + 2fg] \).
\[
\begin{pmatrix}
\cos \theta & \sin \theta e^{i\phi} \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
\sin \theta & -\cos \theta e^{i\phi}
\end{pmatrix}, \quad 0 < \theta < \frac{\pi}{2}, \phi \in \mathbb{R}.
\]  \quad (15)

**Proof.** There are only three IOs to verify:
\[
\left\{ \begin{pmatrix} \ast & \ast \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \ast & \ast \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 & 0 \\ \ast & \ast \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \ast & \ast \end{pmatrix} \right\}, \left\{ \begin{pmatrix} \ast & \ast \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \ast & \ast \end{pmatrix} \right\},
\] where $\ast$ denotes an arbitrary non-zero complex number (subject to the restriction of forming a channel). The first one could be parametrized as
\[
\left\{ \begin{pmatrix} \cos \theta & \sin \theta e^{i\phi} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \sin \theta & -\cos \theta e^{i\phi} \\ 0 & 0 \end{pmatrix} \right\}, \quad 0 < \theta < \frac{\pi}{2}, \phi \in \mathbb{R},
\] and hence the unitary
\[
U = \begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{pmatrix}
\] transforms it to an SIO. Similarly, the second one is also an SIO. For the last one, however, one verifies that no 4-by-4 unitary could transform it to an SIO given in equation (2). \qed

Note that the channel \{K_1, K_2\} in equation (15) trivially extends to higher dimension: the IO \{K_1 \oplus 0, K_2 \oplus 0, |2\rangle \langle 2|, \ldots, |d\rangle \langle d|\} is not an SIO.

### 4. Kraus rank versus IO rank versus SIO rank

In analogy with Kraus rank, we say the minimum number of (S)IO Kraus operators of an (S) IO channel its (S)IO rank. Since SIO \subseteq IO, it follows that for an SIO,
\[
\text{Kraus rank} \leq \text{IO rank} \leq \text{SIO rank},
\] where the inequalities are expected to be strict for higher dimension. In qubit case, however, the two terminal ranks are same.

**Proposition 5.** For all qubit SIO, the SIO rank equals to Kraus rank. This is not necessarily true in higher dimension.

**Proof.** The CJS matrix for a qubit SIO, which without loss of generality given by equation (2), can be decomposed as
\[
2M = \begin{pmatrix}
a_3^2 & \ldots & \ldots \\
\ldots & \ldots & \ldots \\
\ldots & \ldots & a_4^2
\end{pmatrix} + \begin{pmatrix}
a_1^2 & \ldots & a_1^* b_1^* \\
\ldots & \ldots & \ldots \\
\ldots & \ldots & a_2^2
\end{pmatrix} + \begin{pmatrix}
|b_2|^2 & a_2 b_2 \\
\ldots & \ldots \\
|b_1|^2 & a_1 b_1
\end{pmatrix}.
\]  \quad (16)

This shows that the Kraus rank of the SIO is four, unless $b_1 b_2 = 0$ or some of the Kraus operators vanish entirely. By the channel (normalization) condition, $b_1, b_2$ cannot vanish both. If $b_1 = 0$, the two operators \(\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}\) and \(\begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix}\) are scalar multiple of each other and so one of...
them could be made zero, thereby diminishing the SIO rank to match with Kraus rank. Similar reasoning applies to the case $b_2 = 0$. If exactly $k$ number of Kraus operators vanish, both the Kraus rank and SIO rank also diminish by $k$. Thus the two ranks are always same.

To show that this is not necessarily true in higher dimensions, consider the (unnormalized) qutrit SIO consisting of the six 3-by-3 permutation matrices as Kraus operators. The CJS matrix has rank five thereby Kraus rank is 5. However, there is no 6-by-6 unitary $U$ which could reduce the SIO rank from 6. To prove this latter claim, we first note that $a = (1, -1, 1, 1, 1, -1)$ must be an unnormalized row of $U$ (so that one of the transformed Kraus operators vanishes). Then we verify that among the resulting Kraus operators, no two can have the same form of any of the six-permutations; for example, if two have the same form as the identity permutation, then the two corresponding rows of $U$ must be $b = (x_1, x_2, x_2, -x_2, -x_2, x_2)$ and $c = (y_1, y_2, y_2, -y_2, -y_2, y_2)$. However, $a, b, c$ has to be mutually orthogonal, forcing one of $b, c$ to be zero, an impossibility. All the other cases of repeating one form can be discarded by similar arguments. So, the transformed non-vanishing five Kraus operators must be of the form of six permutations, in particular two of them must have the form of $\{1, 2\}$, or $\{2, 3\}$, or $\{3, 1\}$. But, like the previous cases, all these lead to a zero row and hence it is impossible to reduce the SIO rank.

□

Figure 1. Numerical simulation of achievable region for single-qubit IO. The red colored area (each dot represents a state) shows the projection of the achievable region in the $x$-$z$ plane for initial Bloch vector $(0.5,0,0.5)^T$ [blue dot], while the blue thick curve is the exact analytical boundary of the achievable region. In this figure we have simulated $10^6$ random IO channels, given by the canonical form in equation (6), in Mathematica: $r = \text{RandomReal}\{\{1, 2\}\}$, $a = \text{Normalize}[\text{RandomReal}\{\{\}, 3\}]$, $\beta = \text{Normalize}[\text{RandomComplex}\{-2-2i, 2+2i\}, 3]$, and replacing $\alpha_1, \beta_1$ by $\alpha[1]/\sqrt{1+r^2}$, $\beta[1]/\sqrt{1+r^2}$ respectively.
For IO channel, the two ranks can differ already at qubit level.

**Proposition 6.** There are (qubit-) incoherent channels with Kraus rank < IO rank.

**Proof.** As an example, consider the following incoherent channel
\[
\Lambda = \left\{ \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.
\tag{17}
\]
This channel has Kraus rank 3, but IO rank 4.

\[\square\]

5. Discussion and conclusion

Our primary aim in this work was to find a minimal description of qubit incoherent operations. We have achieved this, as detailed in the main result section, by showing that any qubit IO could be decomposed into four incoherent Kraus operators. This shows that at most eight real parameters are needed to simulate any qubit IO.

As an application of this optimal decomposition, we have numerically simulated the set of reachable states, from all the given initial qubit states considered in [6] by all possible qubit IOs. The results are good approximations of the exact analytic achievable regions obtained there. We have depicted one such simulation in figure 1. Evidently, the figure is quite suggestive as many points reach the exact analytic boundary.

Regarding state conversion under various models of free operations (see e.g. [1, table II]) in coherence theory, it is known that all hierarchies collapse for qubit state conversion [6, 8]. The achievable regions remain same even for probabilistic IO and SIO [9]. However, the optimal decomposition serves much higher purpose. For example, the results of qubit transformations readily become insufficient to describe the possible output two qubit states when IOs are applied to one of the two input qubits. In contrast, the optimal parametrization allows to simulate the output states when IOs are applied to one (or more) part of a multipartite systems, which is the main theme in distributed scenario [10].

To conclude, we have shown that every IO can be decomposed into at most four incoherent Kraus operators and there are some requiring exactly four. The problem of finding minimum number of (strictly) incoherent Kraus operators for (S)IO beyond qubit systems remains open.

**Acknowledgments**

We thank Alexander Streltsov and Preeti Parashar for helpful discussions. We acknowledge financial support from ERC grant OSYRIS (ERC-2013-AdG grant no. 339106), EU grant QUIC (H2020-PETPROACT-2014 grant no. 641122), the European Social Fund, the Spanish MINECO grant FISICATEAMO (FIS2016-79508-P), the Severo Ochoa Programme (SEV-2015-0522), MINECO CLUSTER (ICFO15-EE-3785), the Generalitat de Catalunya (2014 SGR 874 and CERCA/Program), the Fundació Privada Cellex, and the National Science Centre, Poland-Symfonia (grant no. 2016/20/W/ST4/00314).

**ORCID iDs**

Swapan Rana 🅰️ https://orcid.org/0000-0002-6604-997X
References

[1] Streltsov A, Adesso G and Plenio M B 2017 Rev. Mod. Phys. 89 041003
[2] Baumgratz T, Cramer M and Plenio M B 2014 Phys. Rev. Lett. 113 140401
[3] Winter A and Yang D 2016 Phys. Rev. Lett. 116 120404
[4] Yadin B, Ma J, Girolami D, Gu M and Vedral V 2016 Phys. Rev. X 6 041028
[5] Chitambar E, Leung D, Mančinska L, Ozols M and Winter A 2014 Commun. Math. Phys. 328 303
[6] Streltsov A, Rana S, Boes P and Eisert J 2017 Phys. Rev. Lett. 119 140402
[7] Nielsen M A and Chuang I L 2010 Quantum Computation and Quantum Information 10th edn (Cambridge: Cambridge University Press)
[8] Chitambar E and Gour G 2016 Phys. Rev. Lett. 117 030401
[9] Theurer T, Streltsov A and Plenio M B 2018 (arXiv: 1804.09467)
[10] Streltsov A, Rana S, Bera M N and Lewenstein M 2017 Phys. Rev. X 7 011024