Robust Control Barrier Function for Systems Affected by a Class of Mismatched Disturbances

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Abstract: This paper proposes a robust exponential control barrier function (RECBF) for systems affected by a class of mismatched disturbances, which forces system states to remain in a given safety set expressed by constraint functions. We consider the case that given constraint functions have different relative degrees for control input and disturbances due to the property of mismatched disturbances, and we extend a concept of the nominal exponential control barrier function (EBCF) to such cases. As a main result, we show RECBF conditions to guarantee invariance of the given safety set and formulate a convex optimization based controller with the RECBF conditions. In particular, we combine a disturbance estimation using Gaussian process regression, which is one of the machine learning methods, with the controller to make use of good properties of RECBF conditions. This formulation enables us to realize robust disturbance compensation based on experimental data, and it can be easily applied to practical systems. We show the effectiveness of the proposed controller through a numerical simulation of a magnetic ball levitation system having model uncertainty.

Key Words: control barrier function, robust control, mismatched disturbances, Gaussian process regression.

1. Introduction

There are many constraints to be satisfied for safety and performance when we control practical systems. In particular, constraints for system states must be satisfied since violation of such constraints often makes the system unstable and poses a risk. Model predictive control (MPC) [1], which solves a constrained optimal control problem every control step, is well known as an example of controllers which can consider state constraints. Recently, a new method for calculating control input to satisfy given state constraints at a low calculation cost, a control barrier function (CBF), has been proposed. A CBF gives a set of control input to keep systems states inside a set satisfying given state constraints for all time. Since this set can be described by a linear inequality, we can calculate specific control input by solving convex optimization problems (e.g., linear programming, quadratic programming) having constraints based on CBF. The CBF based control has been applied to a lot of applications, such as bipedal walking robots [2] and multi-agent robot systems [3].

However, the nominal CBF based control is sensitive to unmodeled disturbances and model uncertainties, and its control performance cannot be guaranteed in the presence of such disturbances and uncertainties. In order to solve this problem, some researchers have studied a robust CBF which can attenuate the effects of disturbances. First, under the assumption that there exists an upper bound of a norm of disturbance terms, a robust CBF has been proposed in [4]. This method attenuates the disturbance effects by compensating the worst case disturbance obtained by the assumption. The controller always calculates conservative control input regardless of the actual disturbance in this method. Hence the other methods based on disturbance estimation have been proposed in [5],[6]. These methods estimate disturbances as random variables following a Gaussian distribution, and they construct a robust CBF by using the mean and covariance information of the disturbance model.

In previous studies, there is an assumption that given functions of state constraints have the same relative degrees for control input and disturbances, that is, an effect of disturbance appears in the same order time derivative of the given function as an effect of control input appears in. This assumption is not satisfied when there exist disturbances which cannot be canceled out by control input (this kind of disturbances is called as mismatched disturbances). Therefore, previous methods cannot deal with such cases, and it is worth considering to construct a new robust CBF coping with the mismatched disturbances.

The main contribution of this paper is to propose a new robust CBF for systems affected by a class of mismatched disturbances. We consider constraint functions having different relative degrees for control input and disturbances, especially the case that the relative degree for control input is larger than the relative degree for disturbances by one. This problem formulation can be applied to control problems of actual systems, e.g., a DC motor system affected by acceleration disturbances and a magnetic ball levitation system, which we will use for the numerical example in this paper. We give conditions to construct a robust CBF for such systems and formulate a robust controller combined with a disturbance estimation method as an optimization problem to make use of derived conditions. It is worth noting that the derived conditions do not require a perfect disturbance estimation to guarantee the invariance of safety sets. In order to utilize this good property, we use the Gaussian process regression (GPR) [7], which is one of the nonlinear regression methods to identify a disturbance model. Using the mean and covariance information of the GPR model, we can robustly compensate for the disturbance effects by taking the uncertainty of the identified model into consideration. This way
also enables us to identify a disturbance model from experimental data. Hence the proposed controller can be easily applied to practical systems.

The contents of this paper are as follows. In Section 2, we explain structures of systems and functions describing state constraint and their assumption considered in this paper. In Section 3, we first introduce a concept of CBF, especially exponential CBF (ECBF) which can be applied to functions having a high relative degree for control input [8]. Then, we extend it and propose a robust ECBF (RECBF) which is for systems affected by mismatched disturbances. In Section 4, we explain a concept of the GPR first, and then we construct an RECBF based controller using the disturbance model identified by the GPR. In Section 5, in order to verify the effectiveness, we apply the proposed controller to a magnetic ball levitation system model affected by a mismatched disturbance and show numerical results. Finally, Section 6 provides conclusions and future work.

Notation: The symbols \( \mathbb{R}, \mathbb{Z} \) denote the set of real numbers and the set of integers, and \( \mathbb{R}_+, \mathbb{Z}_+ \) denote the set of non-negative real numbers and integers, respectively. The notation \( \text{span}(g_{\text{vec}}) \) denotes the subspace spanned by the set of vectors \( g_{\text{vec}} \). For a differentiable function \( h(x) \) and vectors \( f(x) \), \( g(x) \) and an integer \( i \in \mathbb{Z}_+ \), the notations \( L^i_f h(x) \), \( L^i_g h(x) \), \( L^i_f L^i_g h(x) \) mean \( h(x) \), \( \frac{\partial^i}{\partial x^i} f(x) \), \( \frac{\partial^i}{\partial x^i} g(x) \), respectively.

2. Problem Formulation

In this paper, we consider systems which can be expressed by the following equation:

\[
\dot{x} = f(x) + g_1(x)u + g_2(x)d(x,t), \tag{1}
\]

where \( x \in \mathbb{R}^n \) is a state vector, \( u \in \mathbb{R}^m \) is a control input vector, and \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( g_1 : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) and \( g_2 : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are locally Lipschitz functions. The function \( d : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times m} \) is an unknown nonlinear function, but we assume that it is continuously differentiable. Furthermore, we assume that \( \text{span}(g_2(x)) \not\subseteq \text{span}(g_1(x)) \), which means that there exist disturbance effects which cannot be canceled out by control input \( u \) directly.

The goal of this paper is to design controllers to keep the solution of the system (1) \( x(t) \) in the following set

\[
C = \{ x \in \mathbb{R}^n \mid c_0(x,t) \geq 0 \}, \tag{2}
\]

where \( c_0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is a continuously differentiable function. In particular, we consider the case that the solution \( x(t) \) starts from \( x(0) = x_0 \in C \). We call this set \( C \) defined by (2) a safety set in this paper. Before assuming the property of \( c_0 \), we note the following definitions of input relative degree (IRD) and disturbance relative degree (DRD) [9],[10].

Definition 1. (Input relative degree, disturbance relative degree) The input relative degree (IRD) of an output function \( h(x,t) \) and the system (1) is defined as an integer \( \rho \leq n \) at the point \( x_0 \), which implies that \( L^\rho h(x,t) = 0 \) for all time \( t \) and \( x \) in a neighborhood of \( x_0 \) and all \( k < \rho - 1 \), and \( L^\rho L^{\rho-1}_f h(x_0,t) \neq 0 \) for all \( t \). The disturbance relative degree (DRD) is also defined as an integer \( \nu \leq n \) satisfying that \( L^\nu h(x,t) = 0 \) for all \( t \) and \( x \) in a neighborhood of \( x_0 \) and all \( k < \nu - 1 \), and \( L^\nu L^{\nu-1}_f h(x_0,t) \neq 0 \) for all \( t \).

We assume that the function \( c_0 \) has uniform IRD \( r_0 \) and uniform DRD \( r_0 - 1 \) in this paper. From this assumption, the time derivative of \( c_0 \) can be written as

\[
c_0 = c_0(x,t), \tag{3}
\]

\[
\dot{c}_0 = L^1_f c_0(x,t) + \frac{\partial}{\partial t} c_0(x,t), \tag{4}
\]

\[
\vdots \]

\[
c_0^{(\nu-1)} = L^{\nu-1}_f c_0(x,t) + \frac{\partial^{\nu-1}}{\partial t^{\nu-1}} c_0(x,t), \tag{5}
\]

\[
c_0^{(\nu)} = L^{\nu}_f c_0(x,t) + L^\nu L^{\nu-1}_f c_0(x,t)u + \dot{\delta}_1(t) + \frac{\delta^{\nu}}{\partial t^{\nu}} c_0(x,t) + \frac{\partial}{\partial t} \frac{\partial^{\nu}}{\partial t^{\nu}} c_0(x,t), \tag{6}
\]

where \( \delta_1(t) := L^\nu L^{\nu-2}_f c_0(x,t)d(x,t) \) and \( \delta_2(t) := L^\nu L^{\nu-1}_f c_0(x,t)d(x,t) \).

Remark 1. As one of problems satisfying above condition, we can consider a state constraint \( c_0(x,t) \geq 0 \) for the following non-input-affine system of \( z \in \mathbb{R}^{n-m_u} \),

\[
z = f_c(z,v) + g_c(z)d_c(z,v,t), \tag{7}
\]

where \( z \in \mathbb{R}^{n-m_u} \) is a state vector, \( v \in \mathbb{R}^{m_u} \) is a control input vector, and \( f_c : \mathbb{R}^{n-m_u} \times \mathbb{R}^{m_u} \to \mathbb{R}^{n-m_u} \) is a locally Lipschitz function. Furthermore, \( g_c : \mathbb{R}^{n-m_u} \to \mathbb{R}^{n-m_u \times \mathbb{R}} \) is a locally Lipschitz function and \( d_c : \mathbb{R}^{n-m_u} \times \mathbb{R}^{m_u} \times \mathbb{R} \to \mathbb{R}^{n-m_u} \) is an unknown continuous differentiable nonlinear function. By considering an augmented state vector \( x = [z^T, v^T]^T \), the system (7) can be transformed into the input-affine system (1) as

\[
\frac{d}{dt} \begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} f_c(z,v) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{v} + \begin{bmatrix} g_c(z) \\ 0 \end{bmatrix} d_c(z,v,t). \tag{8}
\]

The condition \( \text{span}(g_c(z)) \not\subseteq \text{span}(f_c(z,v)) \) is satisfied in this case. Hence, there exists a function \( c_0(x,t) \) whose time derivatives have the same structure as (4), (5), (6).

3. Robust Control Barrier Function Conditions

In this section, we give conditions to keep the solution \( x(t) \) of the system (1) inside the safety set \( C \) defined by (2). The idea is based on an exponential control barrier function (ECBF) [8]. Therefore, we introduce the concept of ECBF for the nominal system (\( d(x,t) = 0 \)) first; then we give robust ECBF conditions for the disturbed system (\( d(x,t) \neq 0 \)).

3.1 Exponential Control Barrier Function Revisited

The ECBF is proposed to construct CBFs for high relative degree state constraints. It is noted that the following definition and theorem are slightly different from [8].

Definition 2. (Exponential control barrier function [8]) Consider the dynamical system (1) with \( d(x,t) = 0 \) and the safety set \( C = \{ x \in \mathbb{R}^n \mid c_0(x,t) \geq 0 \} \), where \( c_0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) has IRD \( r_0 \). Furthermore, consider the following family of output functions,

\[
y_0 = c_0(x,t), \tag{9}
\]

\[
y_i = y_{i-1} + p_i y_{i-1} \quad (i = 1, \ldots, n), \tag{10}
\]

where \( p_i \in \mathbb{R}_+ \) (\( i = 1, \ldots, n \)) are the designed parameters. The function \( c_0(x,t) \) is an exponential control barrier function (ECBF) if there exist the parameters \( p_i \) s.t.

\[
\sup \{ y_0(x,u,t) \} \geq 0 \forall X \in C, \tag{11}
\]

and \( c_0(x,t) \geq 0 \) when \( c_0(x_0,0) \geq 0 \).
Theorem 1. (ECBF conditions) Suppose that \( c_0(x_0, 0) \geq 0 \) and parameters \( p_i (i = 1, ..., r_b) \) are chosen s.t.

\[
p_i = \max \left( \epsilon - \frac{y_{i-1}(x_0, 0)}{y_{i-1}(x_0, 0)} \right) (i = 1, ..., r_b - 1), \tag{12}
\]

\[
p_b = \epsilon, \tag{13}
\]

where \( \epsilon \in \mathbb{R} \) is a constant value. Then, the following condition guarantees that \( c_0(x, t) \) is an ECFB for the system (1) with \( d(x, t) = 0 \):

\[
y_{o}(x, u, t) \geq 0. \tag{14}
\]

We prove Theorem 1 in a different way from [8]. We first introduce the definition of continuous-time positive linear systems and a lemma of positivity.

Definition 3. (Continuous-time positive linear system [11]) A linear system \((A, b, c)^T\) is said to be positive if and only if its state and output are non-negative for every non-negative initial state and for every non-negative input.

Lemma 1. (Conditions for positivity of continuous-time linear system [11, Thm. 2]) A continuous-time linear system \((A, b, c)^T\) is positive if and only if the matrix \( A \) is a Metzler matrix, that is, its non-diagonal elements are non-negative \( [a_{ij} \geq 0 \ \forall (i, j), i \neq j] \) and \( b \geq 0, c^T \geq 0^T \).

Proof of Theorem 1. The following continuous-time linear system can be constructed by using the new coordinates (9), (10):

\[
\frac{d}{dt} \eta_t = A \eta_t + B y_{o}(x, u, t), \tag{15}
\]

\[
c_0(x, t) = C \eta_t(x, t) \tag{16}
\]

where

\[
A \eta := \begin{bmatrix}
-p_1 & 1 & \cdots & 0 & 0 \\
0 & -p_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -p_{n-1} & 1 \\
0 & 0 & \cdots & 0 & -p_n
\end{bmatrix}, \quad B \eta := \begin{bmatrix}
y_{0}(x, t) \\
y_{1}(x, t) \\
\vdots \\
y_{o-2}(x, t) \\
y_{o-1}(x, t)
\end{bmatrix}, \quad C \eta := \begin{bmatrix}
y_{0}(x, t) \\
y_{1}(x, t) \\
\vdots \\
y_{o-2}(x, t) \\
y_{o-1}(x, t)
\end{bmatrix}.
\]

The matrix \( A \eta \) is a Metzler matrix which does not have negative value in non-diagonal elements, and the initial values of \( \eta(x_0, 0) \) are non-negative due to (10) and the assumption (12) and \( c_0(x_0, 0) \geq 0 \). Then, according to Lemma 1, the condition (14) makes the system described by (15), (16) satisfy the conditions of continuous-time positive linear systems. Hence the output \( c_0(x, t) \) is non-negative for all time and satisfies the following inequality for all time:

\[
c_0(x, t) \geq C e^{\lambda t} \eta_0(x_0, 0) \geq 0. \tag{18}
\]

Moreover, the lower bound \( C e^{\lambda t} \eta_0(x_0, 0) \) goes to zero exponentially since the matrix \( A \) is a stable matrix from the assumption \( p_i \geq \epsilon \). This completes the proof.

Remark 2. The family of output functions (9), (10) can be rewritten as

\[
y_i = \left( \prod_{j=0}^{i} \left( \frac{d}{dt} + p_j \right) \right) c_0(x, t) (i = 0, ..., r_b). \tag{19}
\]

The vector \( \eta(x, t) \) corresponds to \( \eta_c := [c_0(x, t), ..., c_0^{(r_b-1)}(x, t)]^T \) one-to-one through (19), and (14) is equivalent to (38) in [8]. Hence Definition 2 and Theorem 1 are essentially the same as [8], but it is worth mentioning that this different formulation and the way to prove Theorem 1 are very useful to consider a robust exponential control barrier function explained in the next subsection.

3.2 Robust ECFB Conditions for the Disturbed Cases

In the case that the system is affected by disturbance, i.e., \( d(x, t) \neq 0 \), the output functions \( y_{o-1} \) and \( y_0 \) defined by (19) include disturbance terms \( \delta_1 \) and \( \delta_2 \). We cannot observe the true value of the disturbance \( d(x, t) \) in many cases; hence we must obtain the estimated value \( \hat{d}(x, t) \) in some way and use it to design the controller.

We consider the following family of output functions instead of (9), (10):

\[
y_i(x, t) := \left( \prod_{j=0}^{i} \left( \frac{d}{dt} + p_j \right) \right) c_0(x, t) (i = 0, ..., r_b - 2), \tag{20}
\]

\[
y_{o-1}(x, \hat{\lambda}), \quad y_{o-1}(x, \hat{\lambda}) := y_{o-1}^e + p_{n-1} y_{o-1} + \hat{\delta}_1, \tag{21}
\]

\[
y_{o-1}(x, \hat{\alpha}), \quad y_{o-1}(x, \hat{\alpha}) := y_{o-1}^e + p_{n-1} y_{o-1} + p_{n-1} y_{o-1} \tag{22}
\]

where \( \hat{\delta}_1, \hat{\delta}_2 \) are estimated values of \( \delta_1, \delta_2 \) respectively, and \( \hat{\lambda} := [\hat{\delta}_1, \hat{\delta}_2, \hat{\alpha}]^T \). Furthermore, \( y_{o-1}^e \) and \( y_{o-1}^m \) are the nominal terms without \( \hat{\delta}_1, \hat{\delta}_2 \), which can be expressed as

\[
y_{o-1}^e := y_{o-1} - \delta_1
\]

\[
L_0 \frac{\partial}{\partial y_0} c_0(x, t) + \delta_1 + \frac{\partial^2 c_0}{\partial y_0^2} c_0(x, t) + \cdots
\]

\[
+ \prod_{j=1}^{n-2} p_{j} \left( \frac{\partial}{\partial y_{o-j}} c_0(x, t) - \delta_j \right)
\]

\[
L_0 \frac{\partial}{\partial y_0} c_0(x, t) + \cdots + \prod_{j=1}^{n-2} p_{j} \left( \frac{\partial}{\partial y_{o-j}} c_0(x, t) \right), \tag{23}
\]

\[
y_{o-1}^m := y_{o-1} - \sum_{j=1}^{n-2} p_{j} \hat{\delta}_j - \delta_2
\]

\[
L_0 \frac{\partial}{\partial y_0} c_0(x, t) + L_0 \frac{\partial}{\partial y_0} c_0(x, t) u + \cdots + \prod_{j=1}^{n-2} p_{j} \hat{\delta}_j c_0(x, t) \tag{24}
\]
Now we define a robust ECBF for the system (1) with \( d(x,t) \neq 0 \) by using these new family of output functions (20), (21), (22).

**Definition 4.** (Robust exponential control barrier function for the disturbed system (1)) Consider the dynamical system (1) and the safety set \( C = \{ x \in \mathbb{R}^n \mid c_0(x,t) \geq 0 \} \), where \( c_0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) has IRD \( r_0 \) and DRD \( r_0 - 1 \), and the family of output functions defined by (20), (21), (22). The function \( c_0(x,t) \) is a robust exponential control barrier function (RECBF) if there exist parameters \( p_1 \) and the disturbance estimation vector \( \tilde{\Delta} \) s.t.

\[
\sup_u \{ \tilde{y}_0(x,u,\tilde{\Delta},t) \} \geq 0 \quad \forall \ x \in C,
\]

and \( c_0(x,t) \geq 0 \) when \( c_0(x_0,0) \geq 0 \).

By introducing the disturbance estimation errors \( \delta_1 := \delta_1 - \tilde{\delta}_1, \delta_2 := \delta_2 - \tilde{\delta}_2 \), the RECBF conditions can be given by the following theorem.

**Theorem 2.** (RECBF conditions) Suppose that \( c_0(x_0,0) \geq 0 \) and parameters \( p_1, (i = 1, \ldots, r_0) \) are chosen s.t.

\[
p_i = \max \left( \epsilon, \frac{\tilde{y}_2(x_0,0)}{\tilde{y}_1(x_0,0)} \right) \quad (i = 1, \ldots, r_0 - 2),
\]

\[
p_{r_0-1} = \max \left( \epsilon, -\frac{\tilde{y}_2(x_0,0) + \delta_1(0)}{\tilde{y}_{r_0-2}(x_0,0)} \right),
\]

\[
p_{r_0} = \epsilon,
\]

where \( \epsilon \in \mathbb{R}_+ \) is a constant value. Then, the following conditions guarantee that \( c_0(x,t) \) is an RECBF for the system (1):

\[
\tilde{y}_n(x,u,\tilde{\Delta},t) \geq 0,
\]

\[
\delta_1 \leq 0,
\]

\[
\delta_2 \leq 0.
\]

**Proof of Theorem 2.** We make a proof in the same fashion as the proof of Theorem 1. By using (23),(24), the output functions (21) and (22) can be rewritten as follows:

\[
\tilde{y}_{n-1} = \tilde{y}_{n-2} + p_{n-1} \tilde{y}_{n-2} + \delta_1 + \tilde{\delta}_1,
\]

\[
\tilde{y}_{n} = \tilde{y}_{n-2} + p_{n-1} \tilde{y}_{n-2} + \delta_1 + \tilde{\delta}_1,
\]

\[
\tilde{y}_{n} = \tilde{y}_{n-2} + p_{n-1} \tilde{y}_{n-2} + \delta_1 + \tilde{\delta}_1 + \sum_{j=1}^{n-2} p_j \tilde{\delta}_1 + \delta_2 + \tilde{\delta}_2,
\]

\[
\tilde{y}_{n} = \tilde{y}_{n-2} + p_{n-1} \tilde{y}_{n-2} + \delta_1 + \tilde{\delta}_1 + \sum_{j=1}^{n-2} p_j \tilde{\delta}_1
\]

\[
+ \delta_2 + p_{n_0} \tilde{\delta}_0 + \tilde{y}_{n-1} + \sum_{j=1}^{n-1} p_j \tilde{\delta}_1 + \delta_2.
\]

Hence we can consider the following continuous time linear system having a state vector \( \tilde{\eta}_t := [\tilde{y}_0, \ldots, \tilde{y}_{n-1}]^T \),

\[
\frac{d}{dt} \tilde{\eta}_t = A \tilde{\eta}_t + \tilde{B}_t \left[ \begin{array}{c} \tilde{y}_0(x,u,\tilde{\Delta},t) \\ -\tilde{\delta}_1 \\ -\tilde{\delta}_2 \end{array} \right],
\]

\[
c_0(x,t) = C \tilde{\eta}_t,
\]

where \( A, C \) are matrices defined by (17), and

\[
B_t := \left[ \begin{array}{cccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \sum_{j=1}^{n_0} p_j 1
\]

From the assumption \( c_0(x_0,0) \geq 0 \) and (26), (27), the initial values of \( \tilde{\eta}_t(x_0,t) \) are non-negative. Hence the conditions (29), (30), (31) make the system (34), (35) satisfy the conditions of continuous time positive linear systems from Lemma 1. Then, \( c_0(x,t) \) is non-negative for all time and satisfies the following inequality,

\[
c_0(x,t) \geq C_1 e^{\alpha_1 \tilde{\eta}_t(x_0,0)} \geq 0,
\]

and its lower bound converges to zero exponentially since the matrix \( A_t \) is a stable matrix due to \( p_1 \geq \epsilon \). This completes the proof.

**Remark 3.** It is important to note that the term \( \tilde{\delta}_1 \) does not appear in (32), (33). It means that we do not have to consider the effect of \( \delta_1 \). Furthermore, the condition (29) can be described as a linear inequality in the vectors \( u, \tilde{\Delta} \) due to the structure of (22), (23), (24).

**Remark 4.** We need to estimate the disturbance term \( \delta_1 \) and \( \delta_2 \) in order to construct controllers by using RECBF conditions, but it is important to note that the estimation values \( \delta_1, \delta_2 \) do not have to be same as the true values \( \delta_1, \delta_2 \) as long as they satisfy the conditions (30), (31). We show an effective way to determine the disturbance estimation values which utilizes this property in the next section.

### 4. Gaussian Process Regression Synthesis

In this section, we construct a controller to keep the solution \( x_t \) of the system (1) inside the safety set (2) based on Theorem 2. In particular, in order to use the good property of the RECBF condition described in Remark 4, we identify the disturbance model by Gaussian Process Regression (GPR) [7] and determine the estimated values based on it.

#### 4.1 Gaussian Process Regression

GPR is one of the nonlinear regression methods [7]. A latent function is represented by a Gaussian process in which all data hold a joint Gaussian distribution in GPR. When \( N \) pairs of training data are obtained, i.e., input data \( X = [x_1, x_2, \ldots, x_N] \in \mathbb{R}^n \times \mathbb{R}^n \) and normalized output data \( Y = [y_1, y_2, \ldots, y_N]^T \in \mathbb{R}^N \), the predictive distribution of output \( y \), corresponding to a new input \( x \), can be derived as

\[
p(y|x, X, Y, \sigma_n^2) \sim \mathcal{N}(y|\mu, \sigma_n^2),
\]

\[
\mu_x = k(x, X)k(X, X) + \sigma_n^2 I^{-1}X
\]

\[
\sigma_n^2 = k(x, x) - k(x, X)(k(X, X) + \sigma_n^2 I)^{-1}k(X, x)
\]

where \( k(\cdot, \cdot) \) is a certain kernel function and \( \sigma_n^2 \) is the variance of the observation noise. The matrices \( k(X, X), k(x, X) \) are kernel matrices defined by

\[
k(X, X) = \begin{bmatrix} k(x_1, x_1) & \ldots & k(x_1, x_N) \\ \vdots & \ddots & \vdots \\ k(x_N, x_1) & \ldots & k(x_N, x_N) \end{bmatrix}
\]

\[
k(x, X) = k(x, x)^T = \begin{bmatrix} k(x_1, x_1) & \ldots & k(x_1, x_N) \end{bmatrix}
\]
The characteristic of GPR is determined by the kernel function \( k(\cdot, \cdot) \). In this paper, we use an automated relevance determination (ARD) squared exponential kernel defined as

\[
k(x, x') = \sigma_f^2 \exp \left( -\frac{1}{2}(x - x')^T \Lambda^{-1}(x - x') \right),
\]

where \( \Lambda = \text{diag}(\ell_1^2, \ell_2^2, \ldots, \ell_D^2) \) is a diagonal matrix of the characteristic length-scales that relate between input \( x \) and \( x' \), and \( \sigma_f^2 \) is the variance of the latent function. All parameters \( \theta = [\ell_1^2, \ell_2^2, \ldots, \ell_D^2, \sigma_f^2, \sigma_n^2] \) are called hyperparameters, and an optimal parameters \( \theta^* \) can be obtained by solving the following optimization problem:

\[
\theta^* = \arg \max_{\theta} \log p(Y|X, \theta),
\]

\[
\log p(Y|X, \theta) = -\frac{1}{2}Y^T (k(X, X) + \sigma_n^2 I)^{-1} Y - \frac{1}{2} \log |k(X, X) + \sigma_n^2 I|.
\]

The above GPR model (38), (39), (40) is defined for functions which have a single output and multi inputs. When we identify the GPR model of vector functions from the training data \( X = [x_1, x_2, \ldots, x_N] \in \mathbb{R}^{N \times d} \) and \( Y_{vec} = [y_1, y_2, \ldots, y_N]^T \in \mathbb{R}^{N \times n} \), their each element must be identified independently.

Therefore, the GPR model of vector functions which expresses a relationship between \( Y_{vec} \) and \( x \), has the following structure:

\[
p(Y_{vec} | x, X, Y_{vec}, \Sigma) \sim \mathcal{N}(Y_{vec}, \Sigma_{vec}),
\]

\[
\mu_{vec} = [\mu_1, \mu_2, \ldots, \mu_m]^T,
\]

\[
\Sigma_{vec} = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2),
\]

where \( \Sigma : = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2) \) and \( \sigma_{vec}, \mu_{vec}, \sigma_{vec} \) corresponds to the ith element of \( y_{vec}, x_{vec} \), respectively.

### 4.2 Disturbance Model and Robust Estimation Conditions

From the assumption of the disturbance \( d(x, t) \), we identify the following disturbance model from \( N \) sets of experiment data \( X_D := [x_D^{(1)}, \ldots, x_D^{(N)}], Y_D := [d^{(1)}, \ldots, d^{(N)}]^T \) by using the GPR,

\[
d(x_D) \sim \mathcal{N}(\mu_D(x_D), \Sigma_d(x_D)),
\]

where \( x_D := [x^{(1)}, x^{(2)}] \in \mathbb{R}^{n \times 1} \). By using the property of linear transformation of random variables following a Gaussian distribution, the model of \( \delta_1, \delta_2 \) can also be expressed as random variables with Gaussian distributions:

\[
\delta_1 \sim \mathcal{N}(\mu_{\delta_1}(x_D), \sigma_{\delta_1}^2(x_D)),
\]

\[
\delta_2 \sim \mathcal{N}(\mu_{\delta_2}(x_D), \sigma_{\delta_2}^2(x_D)),
\]

(50)

where

\[
\mu_{\delta_1}(x_D) := L_{\delta_1}L_{\theta_1}^T c_0(x_D)\mu_d(x_D),
\]

\[
\mu_{\delta_2}(x_D) := L_{\delta_2}L_{\theta_2}^T c_0(x_D)\mu_d(x_D),
\]

\[
\sigma_{\delta_1}^2(x_D) := L_{\delta_1}L_{\theta_1}^T c_0(x_D)\Sigma_d(x_D)L_{\delta_1}L_{\theta_1}^T c_0(x_D),
\]

\[
\sigma_{\delta_2}^2(x_D) := L_{\delta_2}L_{\theta_2}^T c_0(x_D)\Sigma_d(x_D)L_{\delta_2}L_{\theta_2}^T c_0(x_D)^T.
\]

Note that \( \mu_{\delta_1}(x_D), \sigma_{\delta_1}^2(x_D) (i = 1, 2) \) are all scalar functions.

Once we obtain the above disturbance model, we can use the mean functions \( \mu_{\delta_1}, \mu_{\delta_2} \) as the disturbance estimation values \( \delta_1, \delta_2 \). However, they often become bad estimations since it is impossible to achieve \( \mu_d(x_D) \approx d(x_D) \) under few training data. Hence we calculate \( \delta_1, \delta_2 \) by using not only the mean functions but also the covariance functions to satisfy the estimation conditions (30), (31) as much as possible. Introducing a parameter \( k_d \in \mathbb{R}_+ \) which defines confidence intervals of Gaussian distributions (\( k_d = 3 \) corresponds to the 99.7% confidence interval), and using the information of the models (49), (50), we can define bounds for the left hand sides of the conditions (30), (31):

\[
\hat{\delta}_1 - \mu_{\delta_1} - k_d\sigma_{\delta_1} \leq \delta_1 \leq \hat{\delta}_1 - \mu_{\delta_1} + k_d\sigma_{\delta_1},
\]

\[
\hat{\delta}_2 - \mu_{\delta_2} - k_d\sigma_{\delta_2} \leq \delta_2 \leq \hat{\delta}_2 - \mu_{\delta_2} + k_d\sigma_{\delta_2}.
\]

From these inequalities, robust estimation conditions considering the uncertainty of the identified disturbance model (48) to satisfy (30), (31) can be defined as

\[
\hat{\delta}_1 - \mu_{\delta_1} + k_d\sigma_{\delta_1} \leq 0,
\]

\[
\hat{\delta}_2 - \mu_{\delta_2} + k_d\sigma_{\delta_2} \leq 0.
\]

Then, under the assumption that the actual disturbance values \( \delta_1 \) and \( \delta_2 \) are inside in the \( k_d \) confidence intervals, the estimated values \( \hat{\delta}_1, \hat{\delta}_2 \) based on (54), (55) satisfy (30), (31). If we choose enough big \( k_d \), this assumption can be satisfied as long as the covariance of the identified GPR model is not zero.

**Remark 5.** GPR models overestimate the covariance of an output big when given inputs are not close to training data. Thus the final terms of the left hand sides of (54) and (55) change depending on the uncertainty of the GPR model. This property enables the estimation conditions (30), (31) to be satisfied even if there are not a lot of training data to identify the GPR model.

**Remark 6.** If each element of the disturbance \( d(x_D) \) has a bounded reproducing kernel Hilbert space (RKHS) norm on a compact set \( \mathcal{D} \subset \mathbb{R}^n \times \mathbb{R} \), i.e., \( \|d(x_D)\|_2 \leq \gamma_j \) \( (j = 1, \ldots, m_d) \), the estimation error is stochastically bounded \([12]\) as

\[
\mathbb{P} \left\{ \|\mu_d(x_D) - d(x_D)\| \leq \|\beta'(\Sigma_d(x_D))\|_2 \right\} \geq 1 - \beta_p,
\]

where \( x_D \in \mathcal{D}, \nu_p \in (0, 1), \beta, \gamma \in \mathbb{R}^n \) and

\[
\beta_p = \sqrt{2\sigma_f^2 + 300\gamma_j^2 \log \left( \frac{N + 1}{1 - \nu_p} \right)^3},
\]

\[
\gamma_j = \max_{x_D \in \mathcal{D}} \max_{1 \leq l \leq m_d} \frac{1}{2} \log \|I + \sigma_f^2 K_n(x_D, x_D')\|
\]

\[
x_{D}, x_{D}' \in \{x_D^{(1)}, \ldots, x_D^{(N)}\}.
\]

The vector \( \beta \) corresponds to the parameter which determines the confidence parameter, and it can be determined by choosing the probability parameter \( \nu_p \). Therefore, we can use (56) to systematically design the parameter \( k_d \) when the upper bound of the RKHS norm of the disturbance \( K_n \) is known.

### 4.3 Quadratic Programming Based Controller

We construct a controller as an optimization problem based on the RECBF condition (29) and the robust estimation conditions (54), (55). The other constraints (e.g., input constraints) can also be considered directly in this formulation.

Before constructing the optimization problem, we transform the robust estimation condition for \( \delta_1 \) (54) into an additional ECBF because the RECBF condition requires not only \( \delta_1 \) but also \( \delta_2 \). The additional ECBF \( c_1 \) and its ECBF condition are defined as follows:
\[ c_1 = -\dot{\delta}_1 + F_{\delta_1} \geq 0, \]  
\[ -\dot{\delta}_1 + \ddot{\delta}_1 + \gamma_1 c_1(x) \geq 0, \]

where \( F_{\delta_1} := \mu_{\delta_1} - k_{\delta_1} \sigma_{\delta_1} \), and \( \gamma_1 \in \mathbb{R}_+ \) is an ECBF parameter. If we choose \( \dot{\delta}_1(0) \) satisfying (58), \( \dot{\delta}_1 \) satisfying (59) guarantees that (54) is satisfied for all time. Thus we first calculate \( \dot{\delta}_1 \) and obtain \( \dot{\delta}_1 \) by using the following equation in the controller:

\[ \dot{\delta}_1(t) = \dot{\delta}_1(0) + \int_0^t \dot{\delta}_1(\tau)d\tau. \]

It is worth mentioning that \( F_{\delta_1} \) is a function of \( x, \dot{x} \) and the analytic form of \( F_{\delta_1} \) can be obtained by calculating the gradient of the GPR model, but it is better to approximate \( F_{\delta_1} \) by using some kind of numerical differentiation methods (e.g., difference approximation, exact differentiator [13]). In many cases that there are few training data, GPR models cannot estimate the gradient of the latent function with enough accuracy. Furthermore, the other linear constraints \( \delta_{\text{ref}} \) can decrease the computational cost. For such reasons, we use the difference approximation to estimate \( F_{\delta_1} \) in a numerical example which we show in the later section.

Now, we show the resultant optimization problem to calculate the control input \( u \) and the disturbance estimation values \( \dot{\delta}_1, \dot{\delta}_2 \), which is expressed as the following robust ECBF based quadratic programming (RECBF-QP):

\[
\begin{align*}
\min_{u, \dot{\delta}_1, \dot{\delta}_2} & \quad \frac{1}{2} (u - u_{\text{ref}})^T (u - u_{\text{ref}}) \\
\text{s.t.} \quad & \frac{\dot{\delta}_1}{\dot{x}_1} (x, u, t) \geq 0, \\
& -\dot{\delta}_1 + \dot{\delta}_2 + \gamma_1 c_1(x) \geq 0, \\
& \dot{\delta}_2 - \mu_{\delta_2} + k_{\delta_2} \sigma_{\delta_2} \leq 0,
\end{align*}
\]

where \( u_{\text{ref}} \in \mathbb{R}^m \) is a reference control input vector given by another controller. This formulation enables us to design the reference controller \( u_{\text{ref}} \) independently. It is worth mentioning that the set of three constraints used in (61) realizes only one RECBF. Hence, if there exist several RECBFs to be considered, additional constraint sets must be added to the above optimization problem. Furthermore, the other linear constraints can also be added to (61) as additional constraints.

5. Numerical Example

To validate the effectiveness of the proposed method, we apply the proposed controller to a two-dimensional nonlinear system model having the same structure of magnetic levitation systems affected by a mismatched disturbance [14] in this section.

5.1 System Model

Dynamic equation of the two dimensional nonlinear system can be described by the following input non-affine system:

\[ \begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= g_0 - \frac{k_0}{(l_0 + x_1)^2} u^2 + d(x, u),
\end{align*} \]

where \( x := \begin{bmatrix} x_1, x_2 \end{bmatrix} \in \mathbb{R}^2 \) is a two dimensional state vector, and \( u \in \mathbb{R} \) is a control input. Furthermore, \( g_0, k_0, l_0 \in \mathbb{R}_+ \) are nominal parameters of the system model. The function \( d(x, u) \) is a disturbance function, and we assume that it is generated by a model uncertainty of the parameter \( k_0 \).
Finally, we construct the RECBF-QP controller using (69), (70), (71) to calculate the control input and the disturbance estimation values \( \hat{\delta}_{1_1}, \hat{\delta}_{1_2}, \hat{\delta}_{1_3}, \hat{\delta}_{2_1}, \hat{\delta}_{2_2}, \hat{\delta}_{2_3} \). In order to compare results with and without RECBF-QP, we give the following nominal output zeroing controller for an output function \( h(x) = x - x_{id} \) as a reference controller \( u_{ref} \):

\[
\hat{u}_{ref} = (L_x L_2 h(x))^{-1} \left( -L_2^2 h(x) + v_{ref}(x) \right),
\]

\[
v_{ref} = -k_1 h(x) - k_2 L_x h(x) - k_3 L_2^2 h(x),
\]

where \( k_1, k_2, k_3 \in \mathbb{R}_+ \) are control gain parameters. Furthermore, there is no disturbance effect in third-order differentiations of RECBFs \( \hat{c} \) because \( L_x L_2^2 \hat{c} = L_x L_2 \hat{c} = L_x \hat{c} = 0 \) are satisfied for all time due to the structure of RECBFs and the system (65). Hence we use the equality constraints \( \hat{\delta}_2 = 0 \) instead of the estimation condition (55) in the RECBF-QP.

5.3 Disturbance Model Identification

From the assumption that the disturbance \( d(x) \) is generated by the model uncertainty, we identify the following model by using GPR:

\[
d(x) \sim N(\mu_d(x), \sigma_d^2(x)).
\]

In order to identify this model, we simulate the system (62), (63) with the nominal output zeroing controller (72) and obtain data \( x_k[k] (k = 0, ..., N) \) sampled at a constant sampling period \( \Delta t \). The disturbance values \( d[k] \) are estimated by using obtained data \( x_k[k] \) and the following approximation:

\[
d[k] \approx \frac{x_3[k + 1] - x_3[k - 1]}{2\Delta t} - g_0 + \frac{k_0}{(l_0 + x_3[k])^2} u[k]^2.
\]

We used the central difference approximation to approximate the time derivative of \( x_3[k] \), but other ways having higher accuracy can also be used to obtain more accurate data. In particular, we give the following equation as the desired trajectory \( x_{id}(t) \),

\[
x_{id}(t) = b_1 + b_2 \sin(t),
\]

where \( b_1, b_2 \in \mathbb{R} \) are constant parameters. Finally, we obtained data by simulating the closed loop systems with two different desired trajectories having parameters \( b_1 = [0.4, 0.6], b_2 = 0.1 \) and identified the disturbance model (74) by using \( N_d \) set of the training data extracted from all data.

5.4 Simulation Results

To validate the effectiveness of the proposed controller, we demonstrate two controllers through numerical simulations:

- Controller A: The nominal output zeroing controller (72);
- Controller B: RECBF-QP with the nominal output zeroing controller (61).

These two controllers are used at a constant period \( \Delta t = 0.001 \) in each simulation. Model parameters of the system and control parameters are summarized in Table 1. In particular, we assume that there exists 10% deviation between the nominal and actual values of the model parameter \( k_o \).

Figures 2 and 3 show the trajectories of the state \( x_1(t) \) and the error \( e(t) \) in the two cases, respectively. While the controller could not control the state \( x_1(t) \) appropriately and \( x_1(t) \) went outside the designed funnel-like set \( F_\psi \), the controller B could keep the state \( x_1(t) \) inside the funnel-like set \( F_\psi \). The effectiveness of the controller B can also be confirmed from Fig. 4 showing the trajectories of three RECBFs, \( c_{0a}, c_{0b}, c_{0c} \), since all RECBFs do not have negative values for all time. Finally, Fig. 5

| Table 1  | Model and control parameters used in simulations. |
|----------|--------------------------------------------------|
| \( g_0 = 9.81 \) | \( k_0 = 1 \) | \( \Delta k_0 = -0.1 \) | \( l_0 = 0.02 \) |
| \( a_{eq} = 0.005 \) | \( a_{eq} = 0.8 \) | \( a_{eq} = 1 \) | \( x_{id} = 0.1 \) |
| \( b_1 = 0.5 \) | \( b_2 = 0.15 \) | \( k_1 = 6 \) | \( k_2 = 11 \) |
| \( k_3 = 6 \) | \( \epsilon = 3 \) | \( k_4 = 3 \) | \( \gamma_1 = 10 \) |
| \( \Delta t = 0.001 \) | \( N_d = 200 \) |
shows the estimation error of $\delta_{t_a}, \delta_{t_b}, \delta_{t_c}$. According to this figure, it can be seen that all estimation errors are negative values for all time; hence we can confirm that the performance of disturbance estimation is enough to satisfy the estimation condition (30), (31) in this case.

6. Conclusion and Future Work

In this paper, we discussed a robust exponential control barrier function (RECBF) for systems affected by a class of mismatched disturbances. We considered the case that given constraint functions have different relative degrees for control input and disturbances due to the property of mismatched disturbances, and we constructed RECBF conditions to keep system states inside safety sets in the presence of such disturbances. In addition, we showed that the derived RECBF condition can be connected to a disturbance estimation based on Gaussian process regression (GPR). We finally formulated an optimization based controller using RECBF and disturbance estimation.

As future work, we will first extend the proposed RECBF conditions to more general cases. In this paper, we dealt with only the case that given constraints have IRD $r_3$ and DRD $r_2 - 1$, that is, the difference between IRD and DRD is only one. However, it is possible to derive the RECBF conditions for other cases by using the same idea which we used to derive Theorem 2. Secondly, we will consider an efficient way to obtain data to well identify disturbance models satisfying the estimation conditions (30), (31). Although a large confidence interval parameter $k_d$ can satisfy the disturbance estimation conditions, too large $k_d$ makes control input too conservative and not efficient. Hence it is important to obtain an accurate disturbance model so that we do not have to use large $k_d$. We guess that a good way to obtain training data for disturbances can be constructed based on the bound of the estimation error (56). Finally, we will introduce an on-line GPR method [17] into our method. Since our proposed method identifies the disturbance model only once at the beginning of the control process, it cannot guarantee the control performance for systems which have parameters and disturbances changing with time. The on-line GPR method can update the model by using new data which is given every sampling period; therefore, we expect that introducing the on-line GPR method gives adaptability for such time-varying systems to our method.

Acknowledgments

This work was supported by JSPS KAKENHI Grant Number JP19J13314.

References

[1] C.E. Garcia, D.M. Prett, and M. Morari: Model predictive control: Theory and practice a survey, *Automatica*, Vol. 25, No. 3, pp. 335–348, 2004.

[2] S. Hsu, X. Xu, A.D. Ames, and J.W. Grizzle: Control barrier function based quadratic programs with application to bipedal walking, *Proc. American Control Conference 2015*, pp. 4542–4548, 2015.

[3] L. Wang, A.D. Ames, and M. Egerstedt: Safety barrier certificates for collision-free multirobot systems, *IEEE Transactions on Robotics*, Vol. 33, No. 3, pp. 661–674, 2017.

[4] Q. Nguyen and K. Sreenath: Optimal robust control for constrained nonlinear hybrid systems with application to bipedal locomotion, *Proc. American Control Conference 2016*, pp. 4807–4813, 2016.

[5] R. Takano and M. Yamakita: Robust constrained stabilization control using control Lyapunov and control barrier function in the presence of measurement noises, *Proc. IEEE Conference on Control Technology and Applications 2018*, pp. 300–305, 2018.

[6] L. Wang, E.A. Theodorou, and M. Egerstedt: Safe learning of quadrotor dynamics using barrier certificates, *Proc. IEEE International Conference on Robotics and Automation 2018*, pp. 2460–2465, 2018.

[7] C.E. Rasmussen and C.K.I. Williams: *Gaussian Process for Machine Learning*, MIT Press, 2006.

[8] Q. Nguyen and K. Sreenath: Exponential control barrier functions for enforcing high relative-degree safety-critical constraints, *Proc. American Control Conference 2016*, pp. 322–328, 2016.

[9] A. Isidori: *Nonlinear Control Systems: An Introduction*, 3rd edition, Springer-Verlag, New York, 1995.

[10] J. Yang, W.H. Cheng, S. Li, and X. Chen: Static disturbance-to-output decoupling for nonlinear systems with arbitrary disturbance relative degree, *International Journal of Robust and Nonlinear Control*, Vol. 23, No. 5, pp. 562–577, 2013.

[11] L. Farina and S. Rinaldi: *Positive Linear Systems Theory and Applications*, John Wiley & Sons, 2000.

[12] T. Beckers, D. Kulić, and S. Hirche: Stable Gaussian process based tracking control of Euler-Lagrange systems, *Automatica*, Vol. 103, No. 1, pp. 390–397, 2019.

[13] A. Levant: Globally convergent fast exact differentiator with variable gains, *Proc. European Control Conference 2014*, pp. 2925–2930, 2014.

[14] T. Binazadeh and M.A. Rahgoshay: Robust output tracking of a class of non-affine systems, *Systems Science & Control Engineering*, Vol. 5, No. 1, pp. 426–433, 2017.

[15] A. Ilichmann, E.P. Ryan, and P. Townsend: Tracking with prescribed transient behavior for nonlinear systems of known relative degree, *SIAM Journal on Control and Optimization*, Vol. 46, No. 1, pp. 210–230, 2007.

[16] C.M. Hackl, C. Endisch, and D. Schröder: Funnel-control in robotics: An introduction, *Proc. 2008 16th Mediterranean Conference on Control and Automation*, pp. 913–919, 2008.

[17] M.F. Huber: Recursive Gaussian process: On-line regression and learning, *Pattern Recognition Letters*, Vol. 45, No. 1, pp. 85–91, 2014.