Stress-Strength Reliability for \( P(T < X < Z) \) using Dagum Distribution

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Abstract. In this paper, the reliability formula of the stress-strength model is derived for probability \( P(T<X<Z) \) of a component having strength \( X \) falling between two stresses \( T \) and \( Z \), that these stresses and strength variables have Dagum distribution with unknown shape parameters and known common scale parameters. Three methods for estimating the Dagum parameters are discussed which are the Maximum Likelihood, Regression method and Method of Moment, and the comparison between these estimation method based on a simulation study by the mean square error criteria for each of the small, medium and large samples. The most important conclusion is that this comparison confirms that the performance of the maximum likelihood estimator works better for most of the experiments studied.

1. Introduction

In the reliability studies the stress-strength model describes the life of a component which has a random strength \( X \) and is subjected to random stress \( Y \). This idea arises in the classical stress-strength reliability, where the person is interested in estimating the probability \( P(X < Y) \) that can be interpreted as the probability of failure of the component, when the applied stress \( Y \) is greater than its strength \( X \) [5]. An important case is the estimation of \( R = P(T < X < Z) \) which represents the situation where the strength \( X \) should not only be greater than stress \( T \) but also be smaller than stress \( Z \). For example, there are many devices that do not work when the temperatures are high or when they are low. Similarly, person’s blood pressure should lie within two limits, systolic and diastolic [6]. The stress-strength model of \( P(T < X < Z) \) have wide applications in various subareas of engineering, psychology, genetics, clinical trials and so on [9]. Singh (1980) presented the minimum variance unbiased (MVU), Maximum Likelihood and empirical estimator of \( R = P(T < X < Z) \), where \( T,Z \) and \( X \) are mutually independent random variables and follows the normal distribution [11]. Dutta and Sriwastav (1986) deal with the estimation of \( R \) when \( T,Z \) and \( X \) are exponential random variables [3]. In 1988, Ivshin investigated the MLE and UMVUE of \( R \) when \( T,Z \) and \( X \) are either uniform or exponential random variable with the unknown location parameter [7]. In 2013, Hassan et al. focused on the estimate of \( R = P(T < X < Z) \), when \( T,Z \) and \( X \) have weibull distribution in presence of k outlies [6]. In 2019, Salman et al. focused on the estimate of \( R = P(T < X < Z) \), when \( T,Z \) and \( X \) are independent and that these stress and strength variables have Inverted Kumaraswamy distribution [5].
Dagum distribution was introduced by Dagum in (1977). This distribution was widely used in various fields such as, income and wealth data, meteorological data, reliability and survival analysis. The Dagum distribution (Dag) is also called the inverse Burr XII distribution [2],[4].

For any random variable X that follows Dagum distribution the cumulative density function (cdf) is given by:[2]

\[ F(x) = (1 + \lambda x^{-\delta})^{-\beta} ; x > 0; \beta, \lambda, \delta > 0 \]  \hspace{1cm} (1)

and the probability density function (pdf) :

\[ f(x) = \beta \lambda \delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1} ; x > 0; \beta, \lambda, \delta > 0 \]  \hspace{1cm} (2)

where \( \lambda \) is the scale parameter and the shape parameters are \( \beta, \delta \).

Since \( f(x) \) is probability density function, then we can rewrite equation (2) as:

\[ \int_{0}^{\infty} \lambda \delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1} dx = \frac{1}{\beta} \]  \hspace{1cm} (3)

The main aim of this paper is to obtain a mathematical formula for the probability of a component’s strength is between two stresses based on Dagum distribution in section 2. In order to find the estimators of the shape parameters \( (\beta, \beta_1, \beta_2) \) for the three random variables, three different estimation methods (Maximum Likelihood, Regression Method and Moment Method) are used and then the reliability parameter is estimated in section 3. A simulation study was conducted to compare the performance of the three different estimators of the reliability in section 4, based on twelve experiments of shape parameter values and at different sample sizes of (15) for small, (30) for medium and (90) for large sample sizes. The comparison is made by the Mean Square Error (MSE), and the conclusions are discussed in section 5.

2. Reliability Formulation

The reliability formula of the stress-strength model that the probability of a component strength falling in between two stresses, is given by:[1]

\[ R = P(T < X < Z) \]

\[ = \int_{0}^{\infty} P(T < x, Z > x/X = x) dF_2(x) \]

\[ = \int_{0}^{\infty} H_T(x) \left( 1 - G_Z(x) \right) f(x) dx \]

\[ = \int_{0}^{\infty} H_T(x) f(x) dx - \int_{0}^{\infty} H_T(x) G_Z(x) f(x) dx \]  \hspace{1cm} (4)

Let T and Z be independent random stress variables with distribution functions \( H_T(t) \), \( G_Z(z) \) following \( \text{Dag}(\beta_1, \lambda, \delta) \) and \( \text{Dag}(\beta_2, \lambda, \delta) \) respectively. And let X be a random strength variable, assumed that X independent from T and Z following \( \text{Dag}(\beta, \lambda, \delta) \) with (cdf) \( F_i(x) \), then:

\[ H_T(t) = (1 + \lambda t^{-\delta})^{-\beta_1} t > 0; \beta_1, \lambda, \delta > 0 \]  \hspace{1cm} (5)

\[ G_Z(z) = (1 + \lambda z^{-\delta})^{-\beta_2} z > 0; \beta_2, \lambda, \delta > 0 \]  \hspace{1cm} (6)

Also \( f(x) = \beta \lambda \delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1} x > 0; \beta, \lambda, \delta > 0 \)

Now, from equation (4):

\[ R = \int_{0}^{\infty} (1 + \lambda x^{-\delta})^{-\beta_1} \beta \lambda \delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1} dx \]

\[ - \int_{0}^{\infty} (1 + \lambda x^{-\delta})^{-\beta_1} (1 + \lambda x^{-\delta})^{-\beta_2} \beta \lambda \delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta-1} dx \]

\[ R = \beta \int_{0}^{\infty} \lambda \delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta} dx - \beta \int_{0}^{\infty} \lambda \delta x^{-\delta-1}(1 + \lambda x^{-\delta})^{-\beta} dx \]

Similarly, from equation (3) we can get:
\[ R = \frac{\beta}{\beta + \beta_1} - \frac{\beta}{\beta + \beta_1 + \beta_2} = \frac{\beta \beta_2}{(\beta + \beta_3)(\beta + \beta_1 + \beta_2)} \]  

The behavior of reliability is illustrated in the following figures with different values of the three distribution shape parameters. The reliability behavior is illustrated in the following figures with different values of the three shape parameters of the distribution. The purpose of explaining this behavior is to know how much confidence we have in the component to work under certain conditions and to continue doing this work according to the special terms of the model.

Figure 1: The Reliability curve against parameter $\beta$

Figure (1-a,b,c) shows the change of reliability curve by the effect of different values of the strength parameter $\beta$ as a function of the parameters $(\beta, \beta_1, \beta_2)$, in three different cases for the values of the two parameters $\beta_1, \beta_2$, where reliability value increasing with the increasing in the value of the strength shape parameter $\beta$, then it gradually decreases.
Figure 2: The Reliability curve against parameter $\beta_1$

Figure (2-a,b,c) shows the effect of the stress parameter value $\beta_1$ on the change of the reliability curve as a function of the parameters $(\beta, \beta_1, \beta_2)$, in three different cases for the values of the two parameters $\beta, \beta_2$, where reliability value decreasing and its decrease is steep with increasing value of stress shape parameter $\beta_1$. 
Figure 3: The Reliability curve against parameter $\beta_2$

Figure (3-a,b,c) shows the effect of the stress parameter value $\beta_2$ on the change of the reliability curve as a function of the parameters $(\beta, \beta_1, \beta_2)$, in three different cases for the values of the two parameters $\beta, \beta_1$, where reliability value increasing with increasing value of stress shape parameter $\beta_2$, but it reaches a certain point and then begins to decrease.

3. Estimation Method:
In this section, three different estimation methods are used to find the estimator of the Dagum unknown shape parameter; $\beta, \beta_1, \beta_2$ and the Reliability; $R$ of the stress-strength model. These methods are Maximum Likelihood, Regression Method and Method of Moment. These methods are used to arrive at the best reliability estimate.

3.1. Maximum likelihood estimator (ML)
The maximum likelihood method is the most widely used method for parameter estimation. There is no doubt that the success of the method stems from many desirable characteristics including consistency, asymptotic, efficiency, invariance and simply its intuitive appeal [2]. Let $x_1, x_2, ..., x_n$ be a random strength sample of size (n) from $\text{Dag}(\beta, \lambda, \delta)$ where $\beta$ is unknown parameter and $\lambda, \delta$ are known. Then the ML function is given by [4]:

$$L(x_1, x_2, ..., x_n; \beta, \lambda, \delta) = (\beta \lambda \delta)^n \prod_{i=1}^{n} x_i^{-\delta - 1} (1 + \lambda x_i^{-\delta})^{-\beta - 1}$$  \hspace{1cm} (8)

Then the natural logarithm function for equation (8) can be written as:

$$\ln L = n \ln(\beta \lambda \delta) - (\delta + 1) \sum_{i=1}^{n} \ln(x_i) - (\beta + 1) \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta})$$
$$= n \ln \beta + n \ln \lambda + n \ln \delta - \delta \sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta}) - \beta \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta})$$
$$- \beta \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta}) - \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta})$$  \hspace{1cm} (9)

By differentiate (9) with respect to the unknown parameter $\beta$ and equating the result to zero, we obtain:

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta})$$
$$\frac{n}{\beta} - \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta}) = 0$$
$$\frac{n}{\beta} = \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta})$$
\[ \hat{\beta}_{ML} = \frac{n}{\sum_{i=1}^{n} \ln(1+\lambda x_i^{-\delta})} \] (10)

In the same way, let \( t_1, t_2, \ldots, t_m \) and \( z_1, z_2, \ldots, z_m \) be a stress random samples of size (m) from \( \text{Dag}(\beta_1, \lambda, \delta) \) and \( \text{Dag}(\beta_2, \lambda, \delta) \), respectively and the ML estimator of the unknown parameters \( \beta_1, \beta_2 \) are [4]:

\[ \hat{\beta}_{1ML} = \frac{m}{\sum_{j=1}^{m} \ln(1+\lambda t_j^{-\delta})} \] (11)

\[ \hat{\beta}_{2ML} = \frac{m}{\sum_{k=1}^{m} \ln(1+\lambda z_k^{-\delta})} \] (12)

By substituting (10), (11) and (12) in equation (7), we can obtain the ML reliability estimator based on invariance property as:

\[ \hat{R}_{ML} = \frac{\hat{\beta}_{2ML} \hat{\beta}_{ML}}{(\hat{\beta}_{1ML} + \hat{\beta}_{ML})(\hat{\beta}_{1ML} + \hat{\beta}_{2ML} + \hat{\beta}_{ML})} \]

3.2. Regression Method (Rg)

Linear Regression is one of the important procedures that use auxiliary information to construct estimators with good efficiency. The standard regression equation [8]:

\[ z_i = a + bu_i + e_i \] (13)

Where \( z_i \) is dependent variable, \( u_i \) is independent variable and \( e_i \) is the error term and represent identically distributed random variable and \( a, b \) are called regression coefficients where \( a \) is the intercept and \( b \) is the slope [10].

Let \( x_1, x_2, \ldots, x_n \) be a random sample of size (n) from \( \text{Dag}(\beta, \lambda, \delta) \). Then the Rg estimators of the unknown parameter \( \beta \), can be obtained by taking the natural logarithm to equation (1):

\[ \ln(F(x_i)) = -\beta \ln(1 + \lambda x_i^{-\delta}) \] (14)

Substituted plotting position \( P_i \) instead of \( F(x_i) \) in equation (14), we get:

\[ \ln(P_i) = -\beta \ln(1 + \lambda x_i^{-\delta}) \] (15)

Where \( P_i = \frac{i}{n+1} \), \( i = 1, 2, \ldots, n \)

By the comparison between equation (15) and equation (13), we can get:

\[ z_i = \ln(P_i), \ a = 0, \ b = \beta, \ u_i = -\ln(1 + \lambda x_i^{-\delta}) \] (16)

Where \( b \) can be estimated by minimizing the summation of squared error with respect to \( b \), then we get:

\[ \hat{b} = \frac{n \sum_{i=1}^{n} x_i \left[ \sum_{i=1}^{n} x_i \sum_{i=1}^{n} u_i \right]}{n \sum_{i=1}^{n} [u_i]^2 - \left[ \sum_{i=1}^{n} u_i \right]^2} \] (17)

By substitution (16) in (17), the Rg estimator for the unknown parameter \( \beta \), \( \hat{\beta}_{Rg} \) is:

\[ \hat{\beta}_{Rg} = \frac{-n \sum_{i=1}^{n} \ln(P_i) \ln(1 + \lambda x_i^{-\delta}) + \sum_{i=1}^{n} \ln(P_i) \sum_{j=1}^{n} \ln(1 + \lambda x_j^{-\delta})}{n \sum_{i=1}^{n} [\ln(1 + \lambda x_i^{-\delta})]^2 - \left[ \sum_{i=1}^{n} \ln(1 + \lambda x_i^{-\delta}) \right]^2} \] (18)

In the same step above, we will estimate the unknown parameters \( \beta_1 \) and \( \beta_2 \) as follows:

\[ \hat{\beta}_{1Rg} = \frac{-m \sum_{j=1}^{m} \ln(P_j) \ln(1 + \lambda t_j^{-\delta}) + \sum_{j=1}^{m} \ln(P_j) \sum_{j=1}^{m} \ln(1 + \lambda t_j^{-\delta})}{m \sum_{j=1}^{m} [\ln(1 + \lambda t_j^{-\delta})]^2 - \left[ \sum_{j=1}^{m} \ln(1 + \lambda t_j^{-\delta}) \right]^2} \] (19)

\[ \hat{\beta}_{2Rg} = \frac{-m \sum_{k=1}^{m} \ln(P_k) \ln(1 + \lambda z_k^{-\delta}) + \sum_{k=1}^{m} \ln(P_k) \sum_{k=1}^{m} \ln(1 + \lambda z_k^{-\delta})}{m \sum_{k=1}^{m} [\ln(1 + \lambda z_k^{-\delta})]^2 - \left[ \sum_{k=1}^{m} \ln(1 + \lambda z_k^{-\delta}) \right]^2} \] (20)

Where \( P_j = \frac{j}{m+1} \), \( j = 1, 2, \ldots, m \) and \( P_k = \frac{k}{m+1} \), \( k = 1, 2, \ldots, m \)

By substitution equations (18),(19) and (20) in equation (7), we obtain the Rg estimator for reliability approximately as:
\[ \hat{R} = \frac{\beta_{2Rg}}{\beta_{1Rg} + \beta_{1Rg}(\beta_{1Rg} + \beta_{2Rg} + \beta_{Rg})} \]

3.3. Method of Moment (MOM)

The Method of Moment was introduced by Pearson in (1894). It was one of the first methods used to estimate a population parameter [8]. To derived the method of moment estimators of parameters of Dagum, let \( x_1, x_2, \ldots, x_n \) be a strength random sample of size (n) from \( \text{Dag}(\beta, \lambda, \delta) \) and let \( t_1, t_2, \ldots, t_m \) and \( z_1, z_2, \ldots, z_m \) be a stress random samples of size (m) from \( \text{Dag}(\beta, \lambda, \delta) \) and \( \text{Dag}(\beta, \lambda, \delta) \), respectively. Then their population means when \( \delta > 1 \) are given by:[2]

\[
\begin{align*}
E(x) &= \beta \frac{1}{\lambda} B \left( 1 - \frac{1}{\delta}, \beta + \frac{1}{\delta} \right) \\
E(t) &= \beta_1 \frac{1}{\lambda} B \left( 1 - \frac{1}{\delta}, \beta_1 + \frac{1}{\delta} \right) \\
E(z) &= \beta_2 \frac{1}{\lambda} B \left( 1 - \frac{1}{\delta}, \beta_2 + \frac{1}{\delta} \right)
\end{align*}
\]

According to the method of moment, equating the samples mean with the corresponding populations mean, then the moment estimators of \( \beta, \beta_1, \beta_2 \) are:

\[
\begin{align*}
\hat{\beta}_{\text{MOM}} &= \frac{\bar{x}}{\lambda^3 B \left( 1 - \frac{1}{\delta}, \beta_0 + \frac{1}{\delta} \right)} \quad \text{(21)} \\
\hat{\beta}_1 &= \frac{\bar{t}}{\lambda^3 B \left( 1 - \frac{1}{\delta}, \beta_1 + \frac{1}{\delta} \right)} \quad \text{(22)} \\
\hat{\beta}_2 &= \frac{\bar{z}}{\lambda^3 B \left( 1 - \frac{1}{\delta}, \beta_2 + \frac{1}{\delta} \right)} \quad \text{(23)}
\end{align*}
\]

By substitution equation (21), (22) and (23) in equation (7), we can obtain the approximate estimator of R as bellow:

\[
\hat{R}_{\text{MOM}} = \frac{\hat{\beta}_{2\text{MOM}}}{(\hat{\beta}_{1\text{MOM}} + \hat{\beta}_{\text{MOM}})(\hat{\beta}_{1\text{MOM}} + \hat{\beta}_{2\text{MOM}} + \hat{\beta}_{\text{MOM}})}
\]

4. Simulation study

In this section, a simulation study is used to determine the best estimate of the reliability with unknown parameters of the Dagum distribution, and to performance the three different estimates from the maximum likelihood, regression method and the method of moment; where the estimators of regression was used as initial value, are evaluated by using the mean square error criteria (MSE), with different sample sizes (15,30,90) and \( (\lambda = 2, 4, 0.5; \delta = 2, 1.2, 5) \), for four different experiments in each case of the parameters value \( \lambda \) and \( \delta \).

For the twelve different experiments, a simulation study is conducted by using MATLAB 2020 to compare the performance of the reliability estimators by the following steps:

Step1: Generating the random values of the random variables by the inverse function according to the following formula: \( x = \left[ (F(x))^{-1/\beta} - 1 \right]/\lambda \)^{-1/\delta}

Step2: Calculate the mean by the equation: \( \text{Mean} = \frac{\sum_{i=1}^{N} \hat{R}_i}{N} \)

Step3: The comparison of estimation methods is done by using the mean square error criteria: \( \text{MSE} = \frac{1}{N} \sum_{i=1}^{N} (\hat{R}_i - R)^2 \), where \( N \) the number of replication in each experiment is 1000.

The results are recorded in the following tables from 1 to 3. The comparison of these estimator’s performance based on the MSE values, has been observed as:
1- In table (1) and for experiments (1), (2), (3) and (4), the value of MSE decreases with increasing sample sizes for each of MLE, Rg and MOM. The best value of MSE is MLE, followed by Rg, MOM.

2- In table (2) and for experiments (5), (6), (7) and (8), the value of MSE decreases with increasing sample sizes for each of MLE, Rg and MOM. The best value of MSE is MLE, followed by Rg, MOM.

3- In table (3) and for experiments (10) and (11), the value of MSE decreases with increasing sample sizes for each of MLE, Rg and MOM. The best value of MSE is MLE, followed by Rg, MOM, but in the case of experiments (9) and (12), the best value of MSE was for the MOM, followed by MLE, Rg.

Thus, the estimators of the maximum likelihood method give better performance than those of the regression and moment methods through the small values of the MSE for most of the experiments, except for experiments (9) and (12) in which the moment method is the best.

Table 1: Estimate for Reliability when $\gamma = 2, \delta = 2$

| Exp.1: $\beta = 1.2, \beta_1 = 0.6, \beta_2 = 0.6$ | Exp.2: $\beta = 0.9, \beta_1 = 0.3, \beta_2 = 0.9$ |
|---|---|
| $R = 0.1667$ | $R = 0.3214$ |
| n | m | MLE | Rg | MOM | Best | MLE | Rg | MOM | Best |
|---|---|---|---|---|---|---|---|---|---|
| 15 | 30 | Mean 0.1640 0.1667 0.1680 MSE 0.0019 0.0035 0.0042 | MLE | 0.3165 | 0.3110 | 0.2904 |
| | | Mean 0.1674 0.1678 0.1685 MSE 0.0009 0.0020 0.0023 | MLE | 0.3188 | 0.3163 | 0.2897 |
| | | Mean 0.1676 0.1672 0.1720 MSE 0.0003 0.0008 0.0013 | MLE | 0.3206 | 0.3193 | 0.2916 |
| 15 | 30 | Mean 0.1678 0.1701 0.1732 MSE 0.0018 0.0037 0.0039 | MLE | 0.3199 | 0.3181 | 0.2962 |
| | | Mean 0.1634 0.1578 0.1669 MSE 0.0003 0.0009 0.0010 | MLE | 0.3152 | 0.3046 | 0.2895 |
| | | Mean 0.1655 0.1625 0.1678 MSE 2.9340e-04 8.4170e-04 8.8600e-04 | MLE | 0.0007 | 0.0018 | 0.0030 |
| | | Mean 0.0848 0.0858 0.0977 MSE 0.0007 0.0015 0.0020 | MLE | 0.0028 | 0.0053 | 0.0063 |
| | | Mean 0.0830 0.0840 0.0958 MSE 0.0004 0.0008 0.0011 | MLE | 0.0013 | 0.0032 | 0.0034 |
| | | Mean 0.0835 0.0833 0.0962 MSE 1.1620e-04 3.2390e-04 5.4940e-04 | MLE | 0.0005 | 0.0011 | 0.0013 |
| | | Mean 0.0859 0.0891 0.0981 MSE 0.0007 0.0015 0.0019 | MLE | 0.2301 | 0.2342 | 0.2258 |
| | | Mean 0.0829 0.0826 0.0951 MSE 1.2100e-04 3.2970e-04 5.2130e-04 | MLE | 0.0025 | 0.0051 | 0.0049 |
| | | Mean 0.0828 0.0826 0.0968 MSE 1.2180e-04 3.0800e-04 7.0930e-04 | MLE | 0.0006 | 0.0015 | 0.0016 |

Table 2: Estimate for Reliability when $\delta = 1.2$
| n  | m  | MLE         | Rg         | MOM         | Best | MLE        | Rg        | MOM       | Best |
|----|----|-------------|------------|-------------|------|------------|----------|-----------|------|
| 15 | 15 | Mean        | 0.1653     | 0.1651      | 0.1682 | 0.3110     | 0.3076   | 0.2956    |
|    |    | MSE         | 0.0017     | 0.0035      | 0.0137 | 0.0037     | 0.0064   | 0.0235    | MLE  |
| 30 | 30 | Mean        | 0.1653     | 0.1646      | 0.1660 | 0.3166     | 0.3136   | 0.3029    |
|    |    | MSE         | 0.0009     | 0.0019      | 0.0108 | 0.0017     | 0.0038   | 0.0187    | MLE  |
| 90 | 90 | Mean        | 0.1656     | 0.1661      | 0.1621 | 0.3221     | 0.3214   | 0.3113    |
|    |    | MSE         | 0.0003     | 0.0007      | 0.0069 | 0.0006     | 0.0017   | 0.0143    | MLE  |
| 30 | 15 | Mean        | 0.1673     | 0.1697      | 0.1710 | 0.3157     | 0.3115   | 0.3052    |
|    |    | MSE         | 0.0018     | 0.0038      | 0.0134 | 0.0038     | 0.0071   | 0.0257    | MLE  |
| 15 | 90 | Mean        | 0.1635     | 0.1587      | 0.1673 | 0.3158     | 0.3057   | 0.3032    |
|    |    | MSE         | 0.0004     | 0.0009      | 0.0069 | 0.0009     | 0.0026   | 0.0134    | MLE  |
| 30 | 90 | Mean        | 0.1657     | 0.1630      | 0.1696 | 0.3196     | 0.3148   | 0.3022    |
|    |    | MSE         | 0.0003     | 0.0008      | 0.0076 | 0.0007     | 0.0017   | 0.0134    | MLE  |

Table 3: Estimate for Reliability when $\lambda = 0.5, \delta = 5$

| n  | m  | MLE         | Rg         | MOM         | Best | MLE        | Rg         | MOM       | Best |
|----|----|-------------|------------|-------------|------|------------|-------------|-----------|------|
| 15 | 15 | Mean        | 0.1643     | 0.1644      | 0.1684 | 0.3166     | 0.3121    | 0.2750    |
|    |    | MSE         | 0.0018     | 0.0033      | 0.0011 | 0.0036     | 0.0066    | 0.0041    | MLE  |
| 30 | 30 | Mean        | 0.1657     | 0.1680      | 0.1702 | 0.3181     | 0.3146    | 0.2733    |
|    |    | MSE         | 0.0008     | 0.0019      | 0.0005 | 0.0018     | 0.0039    | 0.0033    | MLE  |
| 90 | 90 | Mean        | 0.1673     | 0.1674      | 0.1712 | 0.3203     | 0.3186    | 0.2725    |
|    |    | MSE         | 2.8460e-04 | 7.8490e-04  | 2.2200e-04 | 0.0006 | 0.0017    | 0.0028    | MLE  |
| 30 | 15 | Mean        | 0.1664     | 0.1660      | 0.1692 | 0.3229     | 0.3204    | 0.2797    |
|    |    | MSE         | 0.0018     | 0.0034      | 0.0011 | 0.0036     | 0.0069    | 0.0038    | MLE  |
| 15 | 90 | Mean        | 0.1636     | 0.1591      | 0.1681 | 0.3158     | 0.3059    | 0.2693    |
|    |    | MSE         | 3.2260e-04 | 8.5930e-04  | 1.8590e-04 | 0.0008 | 0.0022    | 0.0031    | MLE  |
| 30 | 90 | Mean        | 0.1652     | 0.1625      | 0.1693 | 0.3184     | 0.3142    | 0.2718    |
|    |    | MSE         | 3.2020e-04 | 8.0850e-04  | 2.0100e-04 | 0.0007 | 0.0019    | 0.0028    | MLE  |

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5. Conclusion
In this paper, three methods are used to estimate reliability $P(T < X < Z)$ as each of $T$, $Z$ and $X$ follow the Dagum distribution with different parameters, and these methods are the maximum likelihood, regression method and method of moment. A simulation study was conducted, and through the results that appeared in the twelve experiments, it was found that the estimators of the maximum likelihood method give better performance than those of the regression and moment methods through the small values of the MSE for most of the experiments, except for experiments (9) and (12) in which the moment method is the best. Thus, the simulation results confirm that the performance of the maximum likelihood estimator is much better than the estimators of the regression and moment methods for most experiments and for various sample sizes.

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