Truncated Airy beam dynamics in wavelet-based representations

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Airy beams are special wave packets which appear to spontaneously accelerate without external potentials or applied forces. Since their physical realisation they have found many applications on various platforms, spanning from optics to plasma physics. Here we investigate the dynamics of truncated Airy beams from the perspective of the wavelet transform. We show that their main properties can be understood using a Madelung transformation in momentum-space combined with the wavelet analysis. We identify the modes responsible for the packet acceleration, and explain how it resharps into a Gaussian form. We propose a mode-selection scheme to obtain quasi-shape preserving and long-lived Airy wave packets. We extend our study to the nonlinear case and we show how the resulting bright solitons can be unambiguously detected using the wavelet transform.

Airy beams are wave packet solutions of the free Schrödinger equation which possess the spectacular properties of accelerating without the need of an external potential and without diffracting. They were first discovered by Berry and Balazs [1] as a non-physical solution of the Schrödinger equation and they were long considered to be mathematical curiosity.

About three decades later, physical (square-integrable) approximations of Airy beams were experimentally demonstrated [2],[4], in which the resulting Airy packets exhibited their special properties for a finite time. Truncated Airy beams (TABs), also called finite energy Airy beams, have since been observed in platforms other than optically-based ones, using e.g., electron beams [5] or surface plasmon polaritons [6]. These realizations set the ground for a wide range of applications such as optical particle manipulation [7], curved plasma channel generation [8], light-sheet microscopy [9], light bullet generation [10], or light propagation into turbulent environments [11]. Their ability to resist diffraction in strongly scattering media has been recently utilised to image dense yeast cells suspensions [12]. TABs have also been extensively studied in a nonlinear context, with e.g. the presence of a self-focusing nonlinearity (or attractive particle interactions), where the packet spontaneously splits between a weak accelerating remanent and an “off-shooting” bright soliton (BS) [13–19], bringing even richer physics and further potential applications.

In this letter we revisit the TAB dynamics from a new perspective. We employ the wavelet transform (WT), a spectral decomposition which provides broad insights into nontrivial wave packets dynamics [20]. In the context of Schrödinger physics, this technique is particularly suited to detect interference between different wave packets [21]. It has been recently applied to understand the intriguing phenomenon of wave packet self-interference in exciton-polaritons [22] or atomic condensates [23]. It was also employed to reveal the mechanism at the origin of the formation of nonlinear X-waves in systems which possess a hyperbolic dispersion [24]. The key advantage of the WT comes from the simultaneous representation of the complex field in both position and momentum at a given time, which allows one to study the dynamics of individual modes comprising the wave packet.

In the aforementioned cases, the intriguing wave dynamics arise due to the properties of the dispersion relation, such as its curvature or the presence of inflection points. The case of the TAB is different. The short time dynamics arise from the intrinsic phase engineering of the initial condition combined with the effect of the dispersion relation. At long times the latter becomes dominant, and the packet resharps into a diffusing Gaussian wave packet. To understand the resulting complex Airy phase dynamics, we use a Madelung decomposition of the wave function, not in real-space, but in momentum-space, which is here essential to interpret the results of the WT. We find that the accelerating fringes in the TAB density result from a dynamical self-interference of the packet’s internal modes. This allows us to derive, from the WT picture only, the parabolic trajectory of the peaks. Having understood which modes are responsible for the Airy beam’s unusual properties, we implement a method of dynamical mode filtering/amplification to obtain quasi-shape preserving and long-lived accelerating Airy-front packets. Finally, we consider Airy beams in the presence of attractive interactions and show how the WT can be applied to unambiguously detect the presence of BSs.

We start by introducing our method of analysis for a wave packet evolved with the one-dimensional Schrödinger equation, here written in momentum-space:
where the kinetic energy has the usual parabolic dispersion $E(k) = k^2/2$. We consider a truncated Airy function as the initial condition, which can be expressed in momentum-space as

$$
\psi_0(k) = \mathcal{F}_k[\text{Ai}(b|x|) \exp((a+ik_0)x)] = \frac{\exp\left(\frac{(a+i(k+k_0))^3}{3b^3}\right)}{2b\sqrt{2\pi}}.
$$

The parameter $b$ governs the width of the peak in position-space, and thus the spread in momentum-space. The parameter $a$ controls the exponential cut-off of the wave function density to ensure its square-integrability [25]. The last parameter $k_0$ specifies the momentum of the initial condition. The solution of Eq. (S1) is obtained by simple integration: $\psi(k,t) = \psi_0(k) \exp(-ik_0t)/2$. The real-space solution can then be found by inverse Fourier Transform as $\psi(x,t) = \mathcal{F}^{-1}_x[\psi(k,t)]$. The space-time dynamics for a TAB with a negative initial momentum is shown in Fig. 1(a), and density profiles at selected times in Fig. 1(b). Because of the negative initial “kick”, the centre of mass of the packet moves to the left (dashed-blue line) [27]. However, and as expected, the Airy peaks accelerate along a parabolic trajectory, initially moving to the left, until a reversing time $t_{rev}$ after which they continue to accelerate to the right. With zero initial momentum ($k_0 = 0$), the wave packet’s center of mass would instead remain at the origin ($x = 0$) and the peaks would always move to the right.

Other representations of the wave function can also be accessed through the Fourier Transform, such $\psi(k,E)$ (often referred as the far-field) or $\psi(x,E)$ [28]. Alternatively, the WT permits a representation of the wave function in both position ($x$) and momentum ($k$). The WT reads [20]

$$
\mathcal{W}(x,k) = \frac{1}{\sqrt{|k|}} \int_{-\infty}^{+\infty} \psi(x') G^*(x'-x)/k dx'.
$$

For this study we use Gabor wavelet family

$$
G(x) = \sqrt{\pi} \exp(iw_\varphi x) \exp(-x^2/2),
$$

which consists of Gaussian functions with an internal frequency $w_\varphi$. We apply the WT to the TAB and show its wavelet energy density $|\mathcal{W}(x,k)|^2$ at four selected times in Fig. 1(c-f). It shows two distinct branches that are initially separated at $t = 0$. However, the branches collapse onto each other at $t_{rev}$, before splitting and spreading at longer times. This peculiar distribution can be understood by analyzing the wave packet’s phase dynamics.

It is common to perform a Madelung decomposition of the complex real-space wave function into an amplitude and a phase term as $\psi(x,t) = \sqrt{N(x,t)} \exp(-i\phi(x,t))$, notably to perform a hydrodynamic analysis. In this picture, the gradient of the phase corresponds to the fluid velocity $v(x,t) = \partial_x \phi(x,t)$. Here, we perform the same decomposition, but in momentum-space with

$$
\psi(k,t) = \sqrt{N(k)} \exp(-i\phi(k,t)),
$$

where the amplitude is a time-independent Gaussian of width $\sigma_k = 3b^3/4a$, and the phase is

$$
\phi(k,t) = \frac{(k+k_0)^3 - 3a^2(k+k_0)}{3b^3} + \frac{1}{2}k^2t.
$$

What does the gradient of the phase (with respect to $k$) represent? For a more trivial initial condition without any applied phase, the $k$-dependent phase of the packet would be simply $\phi(k,t) = E(k)t$. As the derivative of $E(k)$ gives the group velocity dispersion [29], the gradient of the phase now represents a distance as $\partial_k \phi(k,t) =$...
\[ \partial_t E(k) = v(k) = kt = d(k, t). \]  Explicitly, the term \( d(k, t) \) gives the distance traveled by a given mode \( k \) after a time \( t \). The simple case of the WT for a Gaussian packet is discussed in the Supplemental Material [26].

With a TAB the modes propagate in a more complex fashion, with
\[ d(k, t) = \partial_k \phi(k, t) = \frac{(k + k_0)^2 - a^2}{b^4} + kt. \]  From Eq. (6) we can see that \( d(k, t) \) contains two terms: the first one arises from the TAB phase specifics, and the second from the dispersion \( E(k) \). The TAB’s dynamics arises from the interplay between these two phase terms, which govern the propagation of modes \( k \).

As only the second term in Eq. (6) is time-dependent, at long times, the mode displacement essentially obeys the dispersion relation. Indeed, by comparing the two contributions in Eq. (6), one can essentially retrieve straightforward wave packet diffusion in real-space when
\[ t > \frac{k + 2k_0}{b^2} - \frac{a^2 - k_0^2}{b^4} \]  There is an interesting point to make from Eq. (7): for modes with a larger momentum, the more time it takes for the effect of the dispersion to dominate over the initial phase arrangement of the packet.

The mode displacement \( d(k, t) \) [Eq. (6)] is plotted in Fig. 1(c–f) as dashed-orange/purple lines on top of the wavelet energy density, and show excellent agreement [20]. For comparison, the mode displacement obtained from the dispersion relation alone is shown as dashed-green lines. One can see how, at long times, the mode displacement of the TAB becomes essentially the one obtained from the dispersion.

From the WT analysis, the presence of fringes in the real-space density can now be understood as self-interference of the wave packet. Indeed, \( d(k, t) \) is here a multi-valued function (see Fig. 1(c–f)), which leads to a self-interference when the wavelet energy density spreads over its extremum, i.e. where it becomes multi-valued. For a given position \( x \), two \( k \) modes can have support in the wave function and overlap in real-space, resulting in interference. This effect was first identified for condensed-matter systems possessing a non-parabolic dispersion relation, where the extrema correspond to inflection points of the dispersion [22, 23].

The trajectory of the branch’s extremum point, with coordinates \( \{d(k_{ext}), k_{ext}\} \) in \( x-k \) phase-space, can be determined from the expression of \( d(k, t) \). First, solving \( \partial_t d(k, t) = 0 \) for \( k \), one obtains \( k_{ext} = \frac{1}{2} (2k_0 - b^2 t) \). One can then substitute \( k_{ext} \) back into Eq. (6), which gives:
\[ d(k_{ext}) = \frac{a^2}{b^3} + k_0 + \frac{b^2 t^2}{4}. \]  The point \( \{d(k_{ext}), k_{ext}\} \) is shown in Fig. 1(c–f) as a red dot and its trajectory as a solid red line. As \( d(k_{ext}) \) corresponds to the largest mode displacement, it gives the trajectory of the TAB’s front wave in real-space, and it is indeed parabolic. It is shown as a dashed-red line in Fig. 1(a). One can also obtain the reversal time for the acceleration by solving \( \partial_t d(k_{ext}) = 0 \) for \( t \), which gives
\[ t_{rev} = -2k_0/b^3. \]

In the systems with a non-parabolic dispersion mentioned earlier [22, 23], the value \( k_{ext} \) is time-independent, i.e. the self-interference always occurs around the same value of momentum. What makes the TAB special is the fact that both the coordinates of the point \( \{d(k_{ext}), k_{ext}\} \) around which the self-interference occurs, are time-dependent. This explains why the TAB’s density fringes vanish at long times. One can observe from Fig. 1(c–f) that the wavelet energy density distribution along \( k \) is roughly constant in time (it mostly spreads along \( x \)). However, the point \( \{d(k_{ext}), k_{ext}\} \) linearly shifts to large momenta as \( k_{ext} \propto t \). At long times, there is then less and less signal available to participate into the self-interference effect, which explains why the wave packet fringes inevitably disappear.

From the wavelet spectra in Fig. 1 it can be seen that, at long times, the energy density around \( \{d(k_{ext}), k_{ext}\} \) is small. The majority of the signal comes from modes participating in the reshaping of the packet into a Gaussian. Therefore we consider damping out those modes, and enhancing those contributing to the self-interference, and hence to the accelerating peaks. We set up a dynamical high-pass filter to damp all the modes below a certain...
momentum close to \( k_{\text{ext}} \). This translates as a momentum and time-dependent loss term for the Schrödinger equation. Similarly, we set a dynamical amplification for the remaining modes with a gain term, in order to limit the decay of the normalisation. We can now rewrite Eq. (S1) as an open-dissipative Schrödinger equation

\[
i \partial_t \psi(k, t) = \left[\frac{k^2}{2} + i(\tau(k, t) - \gamma(k, t))\right] \psi(k, t), \tag{9}\]

where \( \tau \) and \( \gamma \) are the gain and loss terms, respectively. Using the same parameters as before for the TAB (with \( k_0 = 0 \)), we solve Eq. (9) for specific \( \gamma \) and \( \tau \) [20]. The newly Airy-engineered wave packet is shown in Fig. 2(a,b) and does not display any Gaussian reshaping. Instead, it is an essentially shape-preserving and accelerating front wave. The normalisation is approximately constant over the considered time interval due to the amplification of the high \( k \) modes. The effect of the high-pass filter is shown in Fig. 2(c,d). The signal overlapping the right branch is then progressively damped, following the drift of \( d(k_{\text{ext}}) \), while the remaining signal is amplified.

Finally, it is interesting to consider the effects of introducing an attractive interaction on the dynamics of Airy beams. Equation (S1) can now be written as a 1D Gross-Pitaevskii equation:

\[
i \partial_t \psi(x, t) = -\frac{1}{2} \partial_x^2 + g(\psi(x, t))^2 \psi(x, t), \tag{10}\]

where \( g < 0 \) is the interaction strength. We solve Eq. (10) for the same parameters as before, with \( k_0 = -1 \) and an attractive interaction \( g = -0.5 \). The wave packet dynamics is shown in Fig. 3(a,b). The same parabolic trajectory of the front is observed, along with a bright peak propagating at constant velocity. This peak was previously identified as an “off-shooting” BS [13–19]. This can be confirmed by looking at the far-field \( |\psi(k, E)|^2 \), plotted in Fig. 3(c). It follows the usual noninteracting parabolic dispersion, but also displays a clear linear dispersion, tangential to the parabola at the point \( k = k_0 \), which is an explicit signature of a BS [31, 32]. The BS dispersion can be obtained by performing a Taylor expansion of the main branch around \( k_0 \):

\[E_{\text{BS}}(k) = k k_0 - \frac{k_0^2}{2}. \tag{11}\]

The BS velocity can be determined from the slope of the linear dispersion as \( v_{\text{BS}} = \partial_k E_{\text{BS}}(k) = k_0 \). In the wavelet analysis, this corresponds to a single mode displacement:

\[d_{\text{BS}}(k, t) = \partial_k E_{\text{BS}}(k) t = k_0 t, \tag{12}\]

which appears as a vertical line in the \( x-k \) representation, as shown in Fig. 3(d). As the BS arises from a non-diffusing mode, i.e. a linear dispersion, its wavelet energy density remains localised around the point \( \{x = k_0 t, k = k_0\} \). This illustrates how the WT can be used to detect the signature of a BS, complementary to the usual Fourier techniques.

In conclusion, we have shown that the TAB’s properties can be fully understood from a careful phase dynamics analysis using the WT and a Madelung decomposition. We have identified the key self-accelerating property of the TAB arising from the transient self-interference of the wave packet. We have then engineered a long-lived and quasi shape-preserving Airy wave front using a dynamical mode filtering/amplification. This is reminiscent of earlier theoretical work showing that an Airy beam can be decomposed as two traveling Hankel waves, only one of which exhibits the parabolic caustic property of the beam [33]. As Airy beams are known to be part of a wider family of wave packets with similar properties (Bessel, Mathieu, Weber beams, etc) our method of WT-based analysis could be usefully applied to further understand other classes of wave packets.

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[25] (1), one can compute the total normalisation from Eq. (2) as $N_{\text{Tot}} = \int_{-\infty}^{\infty} |\psi_0|^2 dx = \exp\left(\frac{a^2}{\delta^2}\right)/8\sqrt{2}\pi ab$ and this result is finite.
[26] (1), see Supplemental Material at [URL will be inserted by publisher] for a wavelet-based analysis of the diffusion of a Gaussian wave packet, further discussing of the Airy-engineered wave front, and three movies providing time-animated versions of Figures 1, 2 and 3.
In this Supplemental Material we apply the wavelet analysis to the simple case of a diffusing Gaussian wave packet with a parabolic dispersion relation. We also comment on the choice of the damping/amplification functions used to dynamically engineer the truncated Airy beam. Finally, we describe the content of the Supplementary Videos. Equations and figures from the main text are here quoted with numbers, whereas those from the Supplementary Material are prefixed by “S”.

A. Gaussian wave packet

We begin with the one-dimensional Schrödinger equation, written in momentum-space

\[ i\partial_t \psi(k, t) = E(k) \psi(k, t) , \tag{S1} \]

where the kinetic energy has the parabolic dispersion \( E(k) = k^2/2 \). Taking a Gaussian wave packet as the initial condition for Eq. (S1), which can be either written in real or momentum space as \( \psi_0(k) = F_k[\exp(-x^2/2\sigma_x^2)] \simeq \exp(-k^2/2\sigma_k^2) \), with \( \sigma_x = 1/\sigma_k \), the full solution in momentum-space is

\[ \psi(k, t) = \exp\left(-\frac{i k^2 t}{2}\right) \exp\left(-\frac{k^2}{2\sigma_k^2}\right) . \tag{S2} \]

In Fig. S1(a) we show an example of the well-known freely diffusing Gaussian wave packet, initialized with \( \sigma_x = 1 \), as seen in the density profiles in Fig. S1(b). The dashed-blue line here shows the position of the packet’s center of mass.

We apply the WT to the diffusing Gaussian packet and show the wavelet energy density \( |W(x, k)|^2 \) at selected times in Fig. S1(c–f). The wavelet energy density is initially tightly distributed around \( x = 0 \) where the packet stands, and then spreads as two branches.

As in the main text, we decompose the complex momentum-space wave function into an amplitude term and a phase term as \( \psi(k, t) = \sqrt{N(k)} \exp(-i\phi(k, t)) \), with the amplitude being

\[ \sqrt{N(k)} = \exp\left(-\frac{k^2}{2\sigma_k^2}\right) , \tag{S3} \]

and the phase

\[ \phi(k, t) = E(k)t = k^2t/2 . \tag{S4} \]

We now computed the gradient of the phase (with respect
to $k$):
\[
d(k, t) = \partial_k \phi(k, t) = \partial_k E(k) t = v(k)t = kt.
\] (S5)

As the $k$-dependent velocity is obtained by taking the derivative of the dispersion relation, the gradient of the phase in momentum-space represents a distance $d(k, t)$ of propagation of a given mode $k$ at a time $t$. This distance travelled for each mode of the wave packet is superimposed on the wavelet energy density shown in Fig. S1(c–f). We note that this result would be qualitatively comparable to other non-Gaussian wave packets evolved on the same parabolic dispersion, as long as they do not initially contain any complex phase relationships.

**B. Dynamical mode filtering/amplification**

In the main text we demonstrated a self-sustaining accelerating Airy beam by implementing time-dependent gain and loss. Here we provide further details. We define the momentum and time dependent loss and gain terms as:

\[
\gamma(k, t) = \Theta[-k + \frac{1}{2} b^3 t], \quad (S6)
\]
\[
\tau(k, t) = \beta \Theta[k - \frac{1}{2} b^3 t], \quad (S7)
\]

where $\Theta$ is a Heavyside step function and $\beta$ a constant chosen to control the amplification of the remaining modes. The “boundary” in $k$-space between damping and amplification here follows the position of $d(k_{\text{ext}})$ which is a linear function of time.

This is the simplest way to dynamically damp and amplify a desired range of modes. This is sufficient to prevent the total population varying by more than a factor of two over the time interval we considered. This procedure could be further optimised using more complex functions for $\tau(k, t)$ and $\gamma(k, t)$.

**C. Supplementary videos**

Four videos are provided showing animations of Figs. 1–3 of the main text, and Fig. S1 of the Supplementary Material. In Supplementary Videos S1 and S2, corresponding to Fig. 1 and 2, we report the dynamical position of the point $\{d(k_{\text{ext}}), k_{\text{ext}}\}$. The displacement $d(k_{\text{ext}})$ corresponds to the position of the main peak in the real-space density, and it is indicated a solid vertical red line.