Quantum System Identification by Bayesian Analysis of Noisy Data: Beyond Hamiltonian Tomography

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We consider how to characterize the dynamics of a quantum system from a restricted set of initial states and measurements using Bayesian analysis. Previous work has shown that Hamiltonian systems can be well estimated from analysis of noisy data. Here we show how to generalize this approach to systems with moderate dephasing in the eigenbasis of the Hamiltonian. We illustrate the process for a range of three-level quantum systems. The results suggest that the Bayesian estimation of the frequencies and dephasing rates is generally highly accurate and the main source of errors are errors in the reconstructed Hamiltonian basis.

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I. INTRODUCTION

Recent advances in nanofabrication technology increasingly enable the construction of devices operating in the quantum regime. However, to utilize coherence effects for practical applications such as quantum information processing and communication tasks requires the ability to engineer their dynamics with high precision. Considerable progress in the area of laser technology and optimal control has shown that precise coherent manipulation of the dynamics is not infeasible for a variety of quantum systems, but such control requires accurate knowledge of the system’s dynamical behavior and response to external fields, which can be used to construct accurate models from which effective control designs can be engineered. The problem is particularly acute for manufactured systems, due to inevitable variations in the manufacturing processes, which ensure that the exact behavior of each device is unique and must be individually measured and characterized.

For the manufacture of large-scale practical devices, the design and operation of each device should be as simple as possible meaning that the physical resources available to initialize and measure the state of a system are usually restricted to a single basis set defined by static electrode geometry. In normal operation, any state can be produced from an initial fiducial one by applying a suitable unitary rotation. This also enables us to effectively perform measurements in an arbitrary basis, and given both these abilities, one can perform quantum process tomography [1, 2]. However, the problem of characterizing a device is not trivial since initially, if one does not yet know the control response of the system, one cannot generate the unitary rotations required in the first place, leading to a Catch-22 situation.

What is required is a method of bootstrapping the control and characterization process so that the system dynamics and response can be incrementally assessed until full control and process tomography is possible, and only using the in situ resources. Hence, we have developed techniques based upon the analysis of generalized coherent oscillation data from Rabi or Ramsey-type experiments. There are several approaches to the analysis of such experimental data including frequency-domain and time-domain analysis. In the regime of a single system transition, Fourier analysis is effective but in the presence of multiple signals, it ceases to an optimal estimator.

In previous work, we have shown how Bayesian signal analysis can be effective in determining accurate model parameters in generic two-qubit Hamiltonian systems where multiple frequencies are present. In this work, we extend the technique to systems with dephasing and use Bayesian signal analysis to reconstruct the underlying dynamics, which are now non-unitary. We apply this technique to three-level (qutrit) systems and analyze its performance for a range of dephasing rates and find that as long as coherent dynamics dominate, which would be the case for quantum information purposes, signal parameters can generally be reliably extracted and the system effectively reconstructed.

II. OPEN QUANTUM SYSTEMS

The evolution of a closed quantum system is governed by a time-dependent unitary operator $U(t)$ obeying the Schrodinger equation. The evolution of an open quantum system can be highly complicated but under certain conditions it can be described by a master equation

$$\dot{\rho}(t) = -i[H, \rho(t)] + L_D \rho(t),$$

where $[A, B] = AB - BA$ is the usual matrix commutator, $L_D$ is a super-operator describing the interaction with the environment, and $\rho$ is a unit-trace positive operator $\rho$ on $H$ representing the state of the system. This
form of master equation is generally applicable to systems interacting with a memory-less (Markovian) reservoir such as an effectively infinite bath, where it can be shown that the superoperator $L_D(\rho)$ takes the form $L_D(\rho) = \sum_k D[V_k] \rho$, where $V_k$ are operators on $H$ and the superoperators $D[V_k]$ are defined by
\begin{equation}
D[V_k] \rho = V_k \rho V_k^\dagger - \frac{1}{2}(V_k^\dagger V_k \rho + \rho V_k^\dagger V_k).
\end{equation}
Under certain conditions we can make further simplifying assumptions. For example, dissipative effects in open systems weakly coupled to an environment are often dominated by a certain types of decoherence such as pure phase relaxation or population relaxation processes such as the spontaneous emission of photons or phonons. These types of processes can be described by relatively simple master equations. In the case of pure dephasing the dissipation superoperator is often determined by a single Hermitian operator $V$. In this case, it is easy to show that the master equation simplifies
\begin{equation}
\dot{\rho}(t) = -i[H, \rho(t)] - \frac{1}{2}[V, [V, \rho(t)]].
\end{equation}
Even with these simplifying assumptions on the open system dynamics we see that full system identification now requires the identification two generally independent Hermitian operators $H$ and $V$, which in general means the identification of $2(N^2 - 1)$ real parameters. Fortunately, dephasing often acts in the eigenbasis of the Hamiltonian, in which case $H$ and $V$ commute and are simultaneously diagonalizable, i.e., there exists a basis $\{ |e_\nu \rangle \}$ such that
\begin{equation}
H = \sum_{\nu=1}^N \lambda_\nu |e_\nu \rangle \langle e_\nu |, \quad V = \sum_{\nu=1}^N \gamma_\nu |e_\nu \rangle \langle e_\nu |
\end{equation}
where $\lambda_\nu$ and $\gamma_\nu$ are real, and in this case the identification problem reduces to finding a joint eigenbasis $\{ |e_\nu \rangle \}$ and the corresponding eigenvalues $\lambda_\nu$ and $\gamma_\nu$ of $H$ and $V$, respectively. This simplifies the problem. If $\hat{H} = \text{diag}(\lambda_\nu)$, $\hat{V} = \text{diag}(\gamma_\nu)$ and $\tilde{\rho}$ is the representation of the state in a joint eigenbasis of $\hat{H}$ and $\hat{V}$ then it is easy to see that the master equation (1) gives
\begin{equation}
\frac{d}{dt} \tilde{\rho}_{\mu\nu}(t) = -i(\omega_{\mu\nu} - i \Gamma_{\mu\nu}) \tilde{\rho}_{\mu\nu}
\end{equation}
where $\omega_{\mu\nu} = \lambda_\mu - \lambda_\nu$ and $\Gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu - \gamma_\nu)^2$, i.e., we have
\begin{equation}
\tilde{\rho}_{\mu\nu}(t) = e^{-it(\omega_{\mu\nu} - i \Gamma_{\mu\nu})} \tilde{\rho}_{\mu\nu}(0),
\end{equation}
and if $W$ is the unitary basis transformation that maps the measurement basis to the joint eigenbasis of $H$ and $V$, then the evolution of the density operator $\rho$ with respect to the measurement basis is given by $\rho(t) = W^\dagger \tilde{\rho}(t) W$. Thus the evolution is determined by the transition frequencies $\omega_{\mu\nu}$, dephasing rates $\Gamma_{\mu\nu}$ and the relation between the system and measurement basis $W$, which are to be determined.

![FIG. 1: Simulated measurement traces of a Qutrit. Ideal signal trace with no projection noise (top) and with 1000 repetition samples per time point (bottom). The signal consists of three damped sinusoids and represents the probability of measuring the system to be in the computational state $|0\rangle$ if it was original initialized in $|0\rangle$. At long times, noise dominates the signal which leads to an optimal total sampling time.]

III. EXPERIMENTAL IDENTIFICATION PROTOCOL

As in previous work [3, 4, 5, 6, 7, 8, 9] we assume that we can prepare and measure the system in a fixed set of (orthonormal) computational basis states $\{|1\rangle, |2\rangle, \ldots, |N\rangle\}$, where $N$ is the Hilbert space dimension. No other measurements or resources such as non-basis states are assumed to be available initially. The basic protocol is to prepare the system in a computational state, let it evolve for a period of time, then measure the probabilities that the system ends up in one of the computational basis states, repeating it for different times and all computational basis states. The experimental data thus consists of $N^2$ time traces, $p_{k\ell}(t)$, with $k, \ell = 1, 2, 3$, which represents the probability that the system, initially in state $|k\rangle$ is measured in state $|\ell\rangle$ after evolving under the system Hamiltonian for time $t$.

When we include dephasing in the Hamiltonian eigenbasis, it can be shown that the observable probabilities are
\begin{equation}
p_{k\ell}(t) = c_{k\ell}(t) + \frac{1}{2} \sum_{\nu > \mu} a_{k\ell, \mu\nu} e^{-i\Gamma_{\mu\nu} t} \cos(\omega_{\mu\nu} t) + b_{k\ell, \mu\nu} e^{-i\Gamma_{\mu\nu} t} \sin(\omega_{\mu\nu} t),
\end{equation}
where the coefficients are
\begin{align}
a_{k\ell, \mu\nu} &= s_{k\ell, \mu\nu} s_{k\ell, \mu} \cos(\Delta_{k\ell, \mu\nu}), \\
b_{k\ell, \mu\nu} &= s_{k\ell, \mu\nu} s_{k\ell, \mu} \sin(\Delta_{k\ell, \mu\nu}), \\
c_{k\ell} &= \sum_{\nu} s_{k\ell, \nu}^2.
\end{align}
Here $s_{k\ell, \nu}$ and $\delta_{k\ell, \nu}$ are the amplitude and phase of the complex number $\langle |\ell\rangle |\xi_{\nu} \rangle \langle \xi_{\nu} |k \rangle$ and $\Delta_{k\ell, \mu\nu} = \delta_{k\ell, \nu} - \delta_{k\ell, \mu}$ is the phase difference.
IV. BAYESIAN PARAMETER ESTIMATION

This shows that the identification problem for dephasing that acts in the system’s natural basis is similar to the Hamiltonian identification problem except that we also have to determine the dephasing rates $\Gamma_{\mu\nu}$. From the measurement results obtained from time traces like in Fig. 1 we must extract signal frequencies $\omega_{\mu\nu}$ and damping rates $\Gamma_{\mu\nu}$ as well as the amplitudes $c_{k\ell}$, $a_{k\ell;\mu\nu}$ and $b_{k\ell;\mu\nu}$ in order to be able to perform reconstruction of the system dynamics.

We can do this again by Bayesian estimation, maximizing the likelihood that a particular process generated the observed signal \[1\]. For convenience we label the transition frequencies of the system $\omega_m$, assuming $\omega_{m+1} > \omega_m > 0$, and the corresponding dephasing rates $\Gamma_m$, and define the vectors $\omega = (\omega_m)$, $\Gamma = (\Gamma_m)$, $a_{k\ell} = (a_{k\ell;m})$ and $b_{k\ell} = (b_{k\ell;m})$ where $k, \ell$ range from 1 to $N$ and $m$ from 1 to the number of transition frequencies $M$. According to Eq. \[7\] the traces should be linear combinations

$$p_{k\ell}(t) = \sum_{m=1}^{3} a_{k\ell;m} g_{2m-1}(t) + b_{k\ell;m} g_{2m}(t) + c_{k\ell} \quad (10)$$

of the $m_b = 2M + 1$ basis functions

$$g_{2m-1}(t) = e^{-t\Gamma_m} \cos(\omega_m t), \quad g_{2m}(t) = e^{-t\Gamma_m} \sin(\omega_m t), \quad g_{2M+1}(t) = 1, \quad (11a, 11b, 11c)$$

or in the case where $H$ is real-symmetric, the $m_b = M + 1$ basis functions

$$g_m(t) = e^{-t\Gamma_m} \cos(\omega_m t), \quad g_{M+1}(t) = 1, \quad (12a, 12b)$$

and our objective is to find parameters $\Gamma_m$, $\omega_m$, $a_{k\ell;m}$, $b_{k\ell;m}$ and $c_{k\ell}$ that maximize the likelihood of the measured data

$$L(a_{k\ell}, b_{k\ell}, c_{k\ell}, \omega, \sigma) = \prod_{k,\ell=1}^{N} \sigma_{k\ell}^{-N} \exp \left[ -\frac{\|p_{k\ell} - d_{k\ell}\|^2}{2\sigma_{k\ell}^2} \right]. \quad (13)$$

We can eliminate the explicit dependence on the linear coefficients $a_{k\ell}$, $b_{k\ell}$, $c_{k\ell}$ and the noise variances $\sigma_{k\ell}$ by integration over suitable priors to obtain an explicit expression for the probability of a particular model given the observed data $d_{k\ell}$ that depends only on the $M$ transition frequencies $\omega_m$ and corresponding dephasing rates $\Gamma_m$. Following standard Bayesian analysis \[10\] we obtain

$$P(\omega, \Gamma | d) \propto \prod_{k,\ell=1}^{N} \left[ 1 - \frac{m_b \langle h_{k\ell}^2 \rangle}{N \langle d_{k\ell}^2 \rangle} \right]^{(m_b-N)/2}, \quad (14)$$

where the averages are defined by

$$\langle d_{k\ell}^2 \rangle = \frac{1}{N} \sum_{n=1}^{N} d_{k\ell;n}^2, \quad (15a)$$

$$\langle h_{k\ell}^2 \rangle = \frac{1}{m_B} \sum_{m=1}^{m_B} h_{k\ell;m}^2. \quad (15b)$$

The components $h_{k\ell;m}$ are essentially the orthogonal projections of the data onto a set of orthonormal basis vectors $H_m(t_n)$

$$h_{k\ell;m} = \sum_{n=1}^{N} H_m(t_n) d_{k\ell;n} \quad (16)$$
derived from the (non-orthogonal) basis functions $g_m(t)$ defined above, evaluated at the respective sample times $t_n$, via

$$H_m(t_n) = \frac{1}{\sqrt{\alpha_m}} \sum_{m'=1}^{m_n} \epsilon_{m'm} g_{m'}(t_n), \quad (17)$$

where $\epsilon_{m'm}$ is an $m_b \times m_b$ matrix whose columns $e_m$ are the normalized eigenvectors $-G e_m = \alpha_m e_m$ of the $m_b \times m_b$ matrix $G = (G_{m_1 m_2})$ with

$$G_{m_1 m_2} = \sum_{n=1}^{N} g_{m_1}(t_n) g_{m_2}(t_n). \quad (18)$$

Thus, the parameter estimation problem for a system with decoherence acting in the Hamiltonian basis is similar to that for a Hamiltonian system, except that the sine and cosine basis functions for the Bayesian analysis must be modified to damped sinusoids with unknown damping rates.

The objective is to find the frequencies $\omega$ and damping rates $\Gamma$ that maximize $P(\omega, \Gamma | d_{kl})$, or equivalently, the log-likelihood function

$$\log_{10} P(\omega, \Gamma | d_{kl}) = \frac{m_b - N}{2} \sum_{k,\ell=1}^{N} \log_{10} \left[ 1 - \frac{m_b \langle \omega^2 \rangle}{N \langle d_{kl}^2 \rangle} \right]. \quad (19)$$

Given a solution $\omega$ and $\Gamma$ that maximizes this log-likelihood, it can be shown that the corresponding optimal coefficients in the general case [11] are

$$a_{k\ell} = \langle x_{k\ell};1, x_{k\ell};3, \ldots, x_{k\ell;m_B-2} \rangle, \quad (20a)$$

$$b_{k\ell} = \langle x_{k\ell};2, x_{k\ell};4, \ldots, x_{k\ell;m_B-1} \rangle, \quad (20b)$$

$$c_{k\ell} = \langle x_{k\ell};m_B \rangle, \quad (20c)$$

where $\langle x_{k\ell};m \rangle$ is shorthand notation for the expectation values $E(x_{k\ell};m | \omega, \Gamma, d_{kl})$ of the linear coefficients of the basis functions, given the optimal frequencies $\omega$ and damping rates $\Gamma$ and the data $d_{kl}$. Similarly, in the special case [12]

$$a_{k\ell} = \langle x_{k\ell};1, x_{k\ell};3, \ldots, x_{k\ell;m_B-1} \rangle, \quad (21a)$$

$$c_{k\ell} = \langle x_{k\ell};m_B \rangle. \quad (21b)$$

Since the log-likelihood function is sharply peaked with generally many local extrema, finding the global optimum using gradient-type optimization algorithms starting with a completely random guess for $\omega$ and $\Gamma$ is inefficient. A global optimization such as pattern search or evolutionary algorithms might circumvent this problem, but neither proved either very effective in our case, especially for higher-dimensional search spaces. Alternatively, starting with a somewhat reasonable initial guess, especially for the frequencies, a standard quasi-Newton optimization method with cubic line search [11] [12] [13] [14] proved generally very effective in finding the global maximum.

To obtain an initial estimate for the frequencies we used the sum of the power spectra of the signals. Although the peaks in the power spectrum are not optimal frequency estimators when there are multiple frequencies and the exact peak locations can be difficult to ascertain even for systems with only three frequencies, as Fig. 2 shows, rough estimates of the peak locations usually seem to provide a reasonable initial guess for the gradient-based likelihood optimization routine. In principle the damping rates could be estimated from the peaks widths as well but these estimates can be tricky, especially for overlapping and minor peaks, hence we chose multiple runs with random initial guesses for the damping rates $\Gamma$ and selected the run with the highest final likelihood (“global” maximum).

Given the extracted signal parameters we have to solve two further inverse problems: (i) reconstructing the level structure from the frequencies and (ii) constructing the matrix $W$ that relates the Hamiltonian basis to the computational basis. The former usually involves analyzing the relationships between the frequencies as illustrated in [9]. In general this is be tricky but for a qutrit system, analysis is essentially trivial. The basis reconstruction requires solving further optimization problems to find the coefficients $s_{k\ell,\mu}$ such that Eqs (8) are satisfied given the estimates for the parameters $a_{k\ell,\mu}, b_{k\ell,\mu}, c_{k\ell}$ and $\Delta_{k\ell,\mu\nu}$ derived in the previous step. Due to finite sampling and noise, the inversion may not be exact, hence we recast it as a constrained optimization problem and solve it as described in [9].

Our previous analysis [9] also shows that we can only identify a single generic Hamiltonian up to equivalence

$$\hat{H} \simeq D^\dagger \hat{H} D + \lambda \mathbf{1}, \quad (22)$$

where $D = \text{diag}(1, e^{i\delta_1}, \ldots, e^{i\delta_N})$ is a diagonal unitary matrix, in the basis of the measurement. However, if the off-diagonal elements in the Hamiltonian are known to be real and positive, for instance, then the Hamiltonian will be uniquely determined up to a global energy level shift $\lambda \mathbf{1}$, at least in the generic case, since we have $|H_{k\ell}| = |\tilde{H}_{k\ell}|$ for $k \neq \ell$. For a quantum control situation, the system dynamics can be controlled and hence different Hamiltonians can be applied, and in the case subsequent Hamiltonians can be fully determined up to the gauge fixed by the initial Hamiltonian. By varying control parameters and tracking the change in the system dynamics, a dynamical control model can be built of the system.

V. RESULTS

We randomly generated 100 real-symmetric qutrit Hamiltonians and dephasing operators with different spectral properties and the geometric average of the system $Q$-factors ranging from 12 to 72. From these we generated various data traces corresponding to the stroboscopic sampling described in section [11]. We considered
three cases, the zero noise case \((N_{\infty} = \text{infinite samples per point})\), fixed finite sampling with \(N_c = N_{1000} = 1000\) experimental repetitions per time point, and an adaptive sampling strategy \(N_{\text{var}}\) which varies the number of samples per point to reach an estimated target signal to noise ratio of \(\langle p_k(t) \rangle \geq 10/\sqrt{N_c}\) for all \(k, \ell\) and \(t\) with an upper limit of \(N_c \leq 10,000\) for each data point. We then applied our parameter estimation and reconstruction algorithms to the resulting data traces. A range of dephasing rates was studied to see the effect on the reconstruction of the Hamiltonian part of the dynamics. For the purposes of control, accurate determination of the Hamiltonian is much more important than a precise determination of the dephasing rate, usually it suffices to know that they are below certain limits.

Table I shows the median errors for various cases. Comparing the dephasing/no dephasing cases, the errors are similar in the absence of projection noise \((N_{\infty})\). The frequency \(\omega\) estimation is slightly more accurate but estimation of the signal amplitudes \(a_{k\ell,\mu\nu}\) is slightly less accurate since the basis functions depend on \(\Gamma\), hence errors in both \(\omega\) and \(\Gamma\) contribute to errors in the coefficients \(a_{k\ell,\mu\nu}\). For reduced signal to noise, dephasing decreases the maximum likelihood and increases frequency, basis and reconstructed Hamiltonian errors with a marked increase in median of the amplitude errors. Adaptive sampling overall increases the accuracy of the parameter estimation step and the reconstructed Hamiltonian for both Hamiltonian and dephasing systems but the improvement is more pronounced for dephasing systems. This may be due to adaptive sampling being more beneficial for small signal amplitudes i.e., decaying signals. This suggests the use of adaptive sampling to increase the signal to noise ratio for samples at increasing times. Alternatively, the sample data can be weighted to give precedence to earlier samples. Further exploration of these methods will be the subject of future study.

![Fig. 3: Hamiltonian vs Basis Reconstruction Error S for various samplings shows a strong correlation and suggests that the total error in the Hamiltonian is dominated by errors in the basis reconstruction step that comes about from the separate optimization of each basis function from the amplitude estimation, which may not lead to orthogonal data vectors.](image)

Dephasing leads to a reduction in signal at long times which can lead us to fitting noise. For strong dephasing, this leads to reduced accuracy in the estimation of the frequencies, and hence increased errors in the other parameters. The spread in the Fourier peaks can also lead to problems for closely spaced frequencies. This in itself is not a problem per se for the Bayesian parameter estimation step [9], except that it can lead to inaccurate initial search parameters coming from peak detection in the power spectrum. This can be obviated somewhat by trying different initial parameters assuming that either of the two remaining peaks were doublets and using the most likely result.

For systems of interest for quantum information processing, the dephasing rates should be sufficiently low so that the damping of the Rabi-type oscillations do not impact the scheme greatly. For very small dephasing rates, however, it can be a problem if the algorithm overestimates the dephasing rates which means that the basis functions used are not suitable, and this is reflected in errors of the estimated amplitudes. For such systems, it is a simple enough matter to test models which are purely Hamiltonian to see which gives the larger likelihood.

One factor which limits the reconstruction is that we may obtain a set of \(N \times N\) matrices \(P_{\mu} = (s_{k\ell,\mu\nu})\), which ideally should be projectors onto orthogonal eigenspaces, but may not always form an orthogonal set of projectors. We can quantify this basis error by

\[
S = \max_{\mu\nu} |\text{Tr}(P_{\mu}^\dagger P_{\mu}) - \delta_{\mu\nu}|. \tag{23}
\]

Fig. 3 shows that there is a strong correlation between \(S\) and the (relative) error in the final reconstructed Hamil-
tonian. Thus, we can use $S$ to choose the best reconstructed Hamiltonian from multiple optimization runs and as a rough indication of the likely accuracy of the reconstructed Hamiltonian. The data also suggests that there is little direct correlation between the likelihood and errors in the parameter estimation step and the final Hamiltonian error, suggesting that the final error in the Hamiltonian is dominated by errors in the basis reconstruction step. The reconstruction step obviously depends on the parameter estimates obtained in the first step, and poor estimates for the parameters will generally result in large Hamiltonian errors, but in some cases the basis reconstruction produces poor results even when the individual errors in the estimated parameters are small. It should be possible to improve the reconstruction step by solving the $N^2$ optimization problems for the $s_{k\ell \mu}$ simultaneously rather than independently and enforcing orthonormality constraints for the basis vectors, but doing so would require solving a rather more complicated optimization problem with several nontrivial constraints.

VI. DISCUSSION

Other researchers have also begun to address the problem of system characterization with limited resources. For example, Leghtas et al. \cite{15} also consider estimating parameters of three-level quantum systems using weak continuous population measurements. However, in their case it is assumed that most of the system is already known including the transition frequencies and the precise structure of the Hamiltonian, and there is no intrinsic decoherence. They consider extracting only two real parameters of the system, the dipole transition strengths between levels 1-2 and 2-3, which simplifies the problem enormously.

Burgarth et al. \cite{16,17} also consider Hamiltonian characterization with restricted resources for Heisenberg spin chains where only a small subset of spins are individually addressable. The form and structure of the Hamiltonian is known \textit{a priori} to be of a particular class, and only the coupling strengths and anisotropy of the system Hamiltonian are to be determined. The sign of the couplings is also known beforehand. Characterization is achieved in this case by preparing different initial states of the first spin, letting the system evolve and then performing quantum state tomography on the accessible spins. If we consider a system of three spins, the first excitation subspace acts as a qutrit. Our protocol could be applied to this problem with some modifications. Our scheme does not require state tomography, only the determination of position of the up-spin, and there is no requirement to know the network topology. It would be interesting to explore Bayesian analysis of the response of such systems for Hamiltonian characterization, and especially the role of topology in identifiability, and whether it is possible to relax the requirement for addressability of all spins.

In summary, we have shown that our current two-step procedure of Bayesian parameter estimation followed by a reconstruction via optimization works in the presence of dephasing on three-level systems. However, we find that the reconstruction step is a weak point of our current implementation. It may be possible to eliminate the parameter estimation step and directly apply Bayesian maximum likelihood estimation upon the dynamical system parameters. This would have the advantage of always giving admissible solutions at all steps. Another direction which should be explored is adaptive sampling, not only varying experimental repetitions per data point, but also using non-uniform time-domain sampling for better frequency discrimination.

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