Abstract. We study the computational complexity of distance games, a class of combinatorial games played on graphs. A move consists of colouring an uncoloured vertex subject to it not being at certain distances determined by two sets, $D$ and $S$. $D$ is the set of forbidden distances for colouring vertices in different colors, while $S$ is the set of forbidden distances for the same colour. The last player to move wins. Well-known examples of distance games are NODE-KAYLES, SNORT, and COL, whose complexities were shown to be PSPACE-hard. We show that many more distance games are also PSPACE-hard.

Keywords. Combinatorial games · Distance games · Computational complexity

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1 Introduction

We begin by introducing distance games and defining combinatorial games needed in the remainder of the paper, then give an introduction to computational complexity and explain our proof strategy. At the end of the section we will give an overview of the organization of the paper.

1.1 Distance Games

Distance games were introduced by Huntemann and Nowakowski [8]. They are part of a larger class of combinatorial games called placement games studied in [3] and [6]. Distance games are played on a graph (board). Note that for our purposes all pieces only occupy one vertex. The distance between two pieces \(P_1\) and \(P_2\) is given by the graph distance between the two vertices that \(P_1\) and \(P_2\) occupy. In our diagrams, we will use B (blue) for a Left piece and R (red) for a Right piece.

**Definition 1** The distance game \(\text{GraphDistance}(D, S)\) is the combinatorial game in which

(i) The board is empty at the beginning of the game.
(ii) Two players, Left and Right, place pieces on empty vertices of the board so that:
   - A Left piece and a Right piece are not allowed to have distance \(d\) if \(d \in D\).
   - Two Left pieces or two Right pieces are not allowed to have distance \(s\) if \(s \in S\).
(iii) Pieces may not be moved or removed once placed.

The following are well-known combinatorial games of interest to us. They will be used in the reductions when considering the complexity of distance games.

**Definition 2** \(\text{Node-Kayles}\) is the impartial distance game \(\text{GraphDistance}([1], [1])\). Thus players play on a vertex which is not adjacent to any previously chosen one.

**Definition 3** \(\text{BiGraph-Node-Kayles}\) is a partizan version of \(\text{Node-Kayles}\) played on a bipartite graph. The bipartition of vertices \(V\) into \(V_L\) and \(V_R\) forces Left to choose vertices from \(V_L\) and Right to choose vertices from \(V_R\).

**Definition 4** \(\text{Snort}\) is the game \(\text{GraphDistance}([1], \emptyset)\), that is, adjacent vertices cannot be coloured with different colours. \(\text{Col}\) is the game \(\text{GraphDistance}(\emptyset, [1])\), that is, adjacent vertices cannot be coloured with the same colour.

More information about \(\text{Node-Kayles}\), \(\text{Snort}\), and \(\text{Col}\) can be found in [1].
1.2 Computational Complexity of Games

Computational complexity can be applied to combinatorial games to measure how hard it is to determine whether the next player has a winning strategy. For some games, such as Nim \(^2\), polynomial-time (P) algorithms exist. For other games, such as Chess, the best algorithms require an exponential amount of time (EXPTIME) in the worst case \(^7\).

PSPACE is the set of problems that can be solved using polynomial amount of storage with no restrictions on time. It is known that \( P \subseteq \text{PSPACE} \subseteq \text{EXPTIME} \), and \( P \neq \text{EXPTIME} \), but proper relationships from PSPACE to both P and EXPTIME are unknown \(^9\). Many decision problems are PSPACE-hard, meaning they are at least as hard as the most difficult problems in PSPACE. In other words, if decision problem \( K \) is PSPACE-hard, then:

- Currently, no known algorithm exists to solve \( K \) in polynomial time, and
- If a polynomial-time algorithm exists for \( K \), then every problem in PSPACE has a polynomial-time solving algorithm (P = PSPACE) \(^9\).

A decision problem is PSPACE-complete if it is both PSPACE-hard and in PSPACE.

A ruleset \( Q \) can be shown to be PSPACE-hard with the help of another ruleset, say \( T \), already known to be PSPACE-hard. \( Q \) is PSPACE-hard if a function \( f \) exists where

- \( f : \text{positions}(T) \rightarrow \text{positions}(Q) \),
- \( f \) can be computed in polynomial time, and
- \( f \) preserves winnability (e.g. for \( t \in T \), Left has a winning move going next on \( f(t) \) exactly when Left has a winning move going next on \( t \)) \(^6\).

Such a function \( f \) is called a reduction (from \( T \) to \( Q \)). Finding reductions from PSPACE-hard games to new games is common practice for showing the PSPACE-hardness of these new games. Sometimes these reductions have a stronger property: each move from any position \( t \in T \) corresponds to exactly one move in \( f(t) \) (i.e. the game trees have exactly the same shape). Due to this injective homomorphism, we can refer to these reductions as play-for-play reductions. Readers interested further in the application of computational complexity to combinatorial games should reference \(^5\).

1.3 Reduction Strategy

In what follows, we will describe each reduction as a transformation of the graph \( G \) on which \( T \) is played to a graph \( G' \) on which \( Q \) is played via the insertion of subgraphs called gadgets. All reductions to be used will be play-for-play, as we will enforce the following two properties in all of our constructions:
– **Vertex condition**: None of the vertices that we add to convert \( G \) to \( G' \) can be playable. No vertices of the original graph \( G \) will be deleted.

– **Edge condition**: None of the additional edges will result in any restrictions on the play on any of the vertices \( v \in V \) from the original graph \( G \). That is, for any \( v \in V \), a Blue/Red piece can be played at \( v \) under ruleset \( T \) on \( G \) exactly when it can be played on \( v \) using ruleset \( Q \) on \( G' \).

We will use the fact that Node-Kayles, BiGraph-Node-Kayles, Snort, and Col are PSPACE-hard \([4,10]\) to create reductions showing that the games \( \text{GraphDistance}(D, S) \) are PSPACE-hard for many pairs \( D \) and \( S \).

### 1.4 Outline

We will derive results for the simplest case, namely \( D = \{1, 2\} \) in Section 2. In Section 3, we introduce the gadgets used in the remaining reductions. We then consider the more general case \( D = \{1, 2, \ldots, n\} \) with \( S \subseteq D \) and \( \max(S) < \max(D) \). In the next section, we study the case in which \( \max(D) = \max(S) \) and \( D \) or \( S \) equals \( \{1, 2, \ldots, n\} \). Finally, we look at the case \( S = \{1, 2, \ldots, k\} \) with \( D \subseteq S \) and \( \max(D) < \max(S) \). Since this reduction requires that Col is PSPACE-hard, a claim recently made in \([4]\), but not yet peer-reviewed, we give an alternate reduction for a subset of this case, namely when \( 1 < \max(D) < k < 2 \max(D) \). We conclude the paper with possible future work.

### 2 \( D = \{1, 2\} \) and either \( S = \emptyset \) or \( S = \{1\} \)

**Proposition 1** The games \( \text{GraphDistance}(\{1, 2\}, S) \) are PSPACE-hard for \( S = \emptyset \) and \( S = \{1\} \).

**Proof** Since BiGraph-Node-Kayles is PSPACE-hard, we will construct a reduction from a bipartite graph \( G = (V_L \cup V_R, E) \) on which BiGraph-Node-Kayles is played to a graph \( G' \) on which \( \text{GraphDistance}(\{1, 2\}, S) \) is played.

We first look at the reduction from \( G \) to \( G' \) when \( S = \emptyset \), which is illustrated in ?? Note that we do not show the edges connecting the sets \( V_L \) and \( V_R \) to better focus on the reduction which preserves \( G \) as a subgraph.

In BiGraph-Node-Kayles, \( V_L \) is restricted to only be playable by Left and \( V_R \) to only be playable by Right. Our goal is to create the graph \( G' \) via a play-by-play reduction so that playing \( \text{GraphDistance}(D, S) \) on \( G' \) is equivalent to playing BiGraph-Node-Kayles on \( G \). We describe the reduction as it relates to the vertices in \( V_L \). The first goal is to ensure that \( V_L \) cannot be played by Right, hence we connect all vertices in \( V_L \) to the terminal vertex of a path of length one, where the external vertex is coloured blue (labeled B) and the intermediate vertex \( (v_2) \) is uncoloured. Since the vertex labeled B is distance two from all vertices in \( V_L \), no vertex in \( V_L \) can be coloured red, and neither can \( v_2 \), as it is at distance one from B. We also need to ensure
that $v_2$ cannot be coloured blue. To do so we connect $v_2$ to a path on two vertices where the terminal vertex not connected to $v_2$ is coloured red (labeled R). The intermediate vertex $v_1$ is unplayable by both players as it is distance two from B and hence cannot be coloured red, and is adjacent to R and so cannot be coloured blue. Therefore the gadget consisting of these four vertices satisfies the vertex condition. We replicate this gadget on the right-hand side, switching the roles of R and B for $V_R$. We now turn to the edge condition, checking whether the additional edges create restrictions on vertices $v \in V_L$ for $G'$ that do not exist for $G$. Note that $v_2$ now connects every pair of vertices in $V_L$, but this does not create any new restrictions on these vertices because $S = \emptyset$, so the edge condition is satisfied.

We now turn to the case $S = \{1\}$. The additional constraint from the set $S$ allows us to simplify the reduction, as shown in ???. We again begin with a bipartite graph $G$ with bipartition $V_L$ and $V_R$ as described in the case $S = \emptyset$. We connect all vertices from $V_L$ to a path of length one with an uncoloured connected terminal vertex (labeled $v$) and an external vertex labeled B. Then $v$ and all vertices in $V_L$ cannot be coloured red as before. Also, $v$ cannot be coloured blue because it is distance one from the vertex labeled B. The gadget is replicated on the right-hand side of the bipartite graph with B replaced by R. This completes the reduction for $S = \{1\}$ as both the vertex and edge conditions are satisfied.

Now in both cases, let $V'$ be the union of all inserted vertices, $E'$ be the union of all inserted edges, and $G' = (V_L \cup V_R \cup V', E \cup E')$. Playing $\text{GraphDistance}(D, S)$ on $G'$ is now exactly the same as playing $\text{BiGraph-Node-Kayles}$ on $G$. The only playable vertices are those from $V_L \cup V_R$, and for all vertices $x$ that were played by one player in a previous turn, if $(x, y) \in E$, then $y$ may not be chosen in subsequent turns by the other player using either ruleset.

\[ \square \]
3 Construction of gadgets for the remaining reductions

For the remaining sets $S$ and $D$ to be considered in this paper, we will utilize a common construction for the various reductions. These reductions will not only modify the vertices of the graph $G$ as in the case $D = \{1, 2\}$, but the edges as well. As before, the concern is to make any vertex that is added into the graph unplayable by each of the two players in such a way that the vertices in the original graph $G$ from which we reduce are not affected. We will achieve this by creating a forbidden vertex gadget and a path gadget.

**Lemma 1** If $D$ or $S$ equals the set $\{1, 2, \ldots, r\}$ for some $r$, and the other is a subset, then we can create a forbidden vertex gadget $F(r)$ of size $r$ which creates a vertex $v$ such that $v$ is uncoloured, but neither player may choose to play at $v$, and the playability of any vertex connected to $v$ is not affected by the vertices in the gadget. Furthermore, all vertices in the gadget are either coloured, or uncoloured and unplayable.

**Proof** Consider the gadget $F(r)$ shown in Fig. 2, which is connected to vertices $a$ and $b$. We now prove that any uncoloured vertex in the gadget is unplayable by either player. Since the gadget is symmetric, we assume without loss of generality that $D = \{1, 2, \ldots, r\}$. For any $r$, the paths from the vertices labeled $R$ and $B$, respectively, to vertex $v$ are of length $r$, so each of these cannot be coloured blue and red, respectively. This means the vertices common to both paths cannot be coloured with either red or blue, and in addition, any vertex connected to $v$ is not affected by the vertices labeled $R$ and $B$ as their distance is at least $r + 1$. For the upper portions of the two paths we now need to ensure that these vertices also cannot be coloured with the other colour. When $r$ is even, the shortest path from $R$ to $B$ using the dashed edge has length $2\left\lceil \frac{r-1}{2} \right\rceil + 1 = r + 1$, so each of those vertices is within distance $r$ of the $R$ and $B$ vertices and cannot be coloured in either colour. When $r$ is odd, then

```latex
\begin{center}
\begin{tikzpicture}
\node (B) at (0,0) {$B$};
\node (V) at (0,-1) {$V$};
\node (R) at (0,-2) {$R$};
\node (V_L) at (-1,-2) {$V_L$};
\node (V_R) at (1,-2) {$V_R$};
\node (V_L) at (-1,-2) {$V_L$};
\node (V_R) at (1,-2) {$V_R$};
\draw (B) -- (V);
\draw (V) -- (R);
\draw (V_L) -- (B);
\draw (V_R) -- (B);
\end{tikzpicture}
\end{center}

**Fig. 2** The reduction from BiGraph-Node-Kayles to GraphDistance($\{1, 2\}, \{1\}$)
the shortest path from R to B has length $2\lceil \frac{r-1}{2} \rceil + 2 = r + 1$. Overall, all the unlabeled vertices in $F(r)$ cannot be played by either player, as stated.

Note that the construction above allows us to insert subgraphs into the graph $G$ that consist of vertices that are either labeled or unlabeled but unplayable, and none of the labeled vertices affects the playability of any vertices in the original graph. We will use a path of an appropriate length made up of forbidden vertex gadgets in the various reductions, where the size of the forbidden vertex gadget is equal to the maximal element in the distance set that consists of consecutive integers. We refer to a path consisting of $t$ forbidden vertices $F(r)$ as $FP(t, r)$, as shown in ?? . Such a path will be used to replace an edge between two vertices of the original graph, as shown in ?? . We will refer to this operation as edge replacement. Note that inserting either one of these gadgets into the graph $G$ automatically satisfies the vertex condition of the play-for-play reduction by ?? .

$$FP(t, r) = F(r)$$
Fig. 5 The edge replacement operation for edges \((x, y)\) in the original graph \(G\) for the reduction to graph \(G'\) for \(\text{GraphDistance}(D, S)\), where \(t\) and \(r\) depend on the particular reduction.

We are now ready to prove that \(\text{GraphDistance}(D, S)\) is PSPACE-hard for more general sets \(D\) and \(S\).

4 \(D = \{1, 2, \ldots, n\}, \ S \subset D, \ \max(S) < n\)

As in ??, we will first consider the case \(S = \emptyset\). These distance games are generalizations of SNORT.

**Proposition 2** \(\text{ENSNORT}(n) := \text{GraphDistance}(\{1, 2, \ldots, n\}, \emptyset)\) is PSPACE-hard.

**Proof** \(\text{ENSNORT}(n)\) is a generalization of \(\text{SNORT}\), which is PSPACE-hard [10], so we will use a reduction from \(\text{SNORT}\) to prove the result. Let \(G = (V, E)\) be any graph. We will construct a graph \(G'\) such that a position of \(\text{SNORT}\) played on \(G\) maps to a position of \(\text{ENSNORT}(n)\) played on \(G'\). Since colouring a vertex in \(\text{ENSNORT}(n)\) affects vertices up to distance \(n\) from the coloured vertex, we need to create a reduction that allows us to increase the distance between the vertices in \(G\) in such a way that any vertex that is inserted is not playable by either player. This can be achieved by performing an appropriate edge replacement, namely replacing each edge in \(G\) with a forbidden path \(FP(n - 1, n)\). Let \(V'\) be the union of all vertices in the respective forbidden path gadgets, \(E'\) be the union of all edges in the forbidden path gadgets along with the edges connecting them to vertices in \(V\), and \(G' = (V \cup V', E')\). Playing \(\text{ENSNORT}(n)\) on \(G'\) is now exactly the same as playing \(\text{SNORT}\) on \(G\). The only playable vertices are those already in \(V\) and for all vertices \(x\) that were played by one player in a previous turn, if \((x, y) \in E\), then \(y\) may not be chosen in subsequent turns by the other player using either ruleset, as the distance of \(x\) and \(y\) in \(G'\) is \(n\). \(\square\)

We now consider more general sets \(S\), namely \(S \subset D\) with \(k = \max(S) < n = \max(D)\).

**Corollary 1** \(\text{GraphDistance}(\{1, 2, 3, \ldots, n\}, S)\) is PSPACE-hard when \(n > \max(S)\) and \(S \subset D\).

**Proof** The same reduction as in the case \(S = \emptyset\) works because we only used properties of \(D\) in the construction of the forbidden vertices \(F(n)\) and paths \(FP(n - 1, n)\). As long as \(\max(S) < n\), colouring restrictions from the set \(S\) do not impact any of the uncoloured vertices from \(V\) in \(G'\), as all vertices from \(V\) are now at distance \(n\) from any other vertex from \(V\). \(\square\)
Note that if $\max(S) = n$, then if a vertex $x$ is coloured in one colour, then the vertex $y$ such that $(x, y) \in V$ would now be uncolourable in either colour, not just the opposite colour. This is why we need a separate reduction in this case, one where the original game played on $G$ has the feature that a vertex adjacent to a coloured vertex in $V$ cannot be coloured in either colour. This suggests a reduction from NODE-KAYLES.

5 $D$ or $S$ equals $\{1, 2, \ldots, m\}$ and $m = \max(D) = \max(S)$

In this section we consider distance games in which the maximum distance not playable by the same and opposite player are identical.

**Proposition 3** GraphDistance$(D, S)$ is PSPACE-hard when either $D$ or $S$ equals $\{1, 2, \ldots, m\}$ and the other set is a subset of $\{1, 2, \ldots, m\}$ with $m = \max(D) = \max(S)$.

**Proof** We reduce from NODE-KAYLES, which is PSPACE-hard [10]. Let $G = (V, E)$ be any graph. We will construct a graph $G'$ such that a position of NODE-KAYLES played on $G$ maps to a position of GraphDistance$(D, S)$ played on $G'$. We start with the extreme case where $D = S = \{1, 2, \ldots, m\}$. Here all vertices at distances less than or equal to $m$ are unplayable by either player, so we replace each edge $(x, y) \in E$ by the path gadget $FP(m - 1, m)$ connected on one side to $x$ and on the other side to $y$. Now let $V'$ be the union of all vertices in the inserted path gadgets, $E'$ be the union of all edges in the added path gadgets along with the edges connecting them to vertices in $V$, and $G' = (V \cup V', E')$.

Playing GraphDistance$(D, S)$ on $G'$ is now exactly the same as playing NODE-KAYLES on $G$. The only playable vertices are those from $V$. Also, for vertices $x$ that have been chosen in previous turns, if $(x, y) \in V$, then $y$ may not be chosen in subsequent turns using either ruleset. Indeed, in GraphDistance$(D, S)$, each such vertex $x$ has distance $m$ from any previously played vertex $x$.

For the more general case where $S \neq D$, the same construction works as any vertex inserted through the path gadget is unplayable as long as one of the two sets equals $\{1, 2, \ldots, m\}$ by ???. The conclusion follows as in the case $S = D$ since the only relevant distances for play are the maximal distances. Any play restriction on $G'$ that would arise from a distance $i < m$ in either $D$ or $S$ would be on one of the forbidden vertices, which are already unplayable.

The final cases to be considered are those where the maximal unplayable distance for the same player is larger than the largest unplayable distance for the opposite player.
6 \( S = \{1, 2, \ldots, k\}, D \subset S, \max(D) < k \)

This case is very similar to the one treated in ??, with the roles of \( S \) and \( D \) interchanged. In ??, we used a reduction from \textsc{Snort} to \textsc{EnSnort}(\( n \)). Since the counterpart to \textsc{Snort} is \textsc{Col} (see ??), we would like to reduce from \textsc{Col}.

A recent preprint [4] asserts that \textsc{Col} is \textsc{PSPACE}-hard.

**Proposition 4** If \textsc{Col} is \textsc{PSPACE}-hard, then \( \text{EnCol}(k):=\text{GraphDistance}(\emptyset, \{1, 2, \ldots, k\}) \) is \textsc{PSPACE}-hard.

**Proof** Let \( G = (V, E) \) be any graph on which we play \textsc{Col}. Create the graph \( G' \) by performing an edge replacement where the edge \((x, y) \in E\) is replaced by the path gadget \( FP(k-1, k) \). If we let \( V', E', \) and \( G' \) be defined as before, then vertices \( x \) and \( y \) from \( G \) are now at distance \( k \) in \( G' \), and playing \textsc{Col} on \( G \) is the same as playing \text{EnCol}(k) on \( G' \).

\( \square \)

**Corollary 2** If \textsc{Col} is \textsc{PSPACE}-hard, then \( \text{GraphDistance}(D, \{1, 2, 3, \ldots, k\}) \) is \textsc{PSPACE}-hard when \( D \subset S \) and \( \max(D) < k \).

**Proof** Follows as in ?? as any impact of the set \( D \) in \( G' \) is only on forbidden vertices, which are unplayable already.

Since the result on \textsc{Col} being \textsc{PSPACE}-hard has not yet undergone peer-review, we offer an alternative proof for some of the cases based on a reduction from \textsc{BiGraph-Node-Kayles} which is reminiscent of the reduction in the case \( D = \{1, 2\} \), bringing us full circle.

7 \( S = \{1, 2, \ldots, k\}, D \subseteq \{1, 2, \ldots, n\}, n < k < 2n \)

**Proposition 5** \( \text{GraphDistance}(D, S) \) is \textsc{PSPACE}-hard when \( D \subseteq \{1, 2, \ldots, n\}, n = \max(D), \) and \( S = \{1, 2, \ldots, k\} \) with \( 1 < n < k < 2n \).

**Proof** In the proof, whenever we refer to \( \text{GraphDistance}(D, S) \), we assume that \( D \subseteq \{1, 2, \ldots, n\}, n = \max(D), \) and \( S = \{1, 2, \ldots, k\} \) with \( 1 < n < k < 2n \). We once more reduce from \textsc{BiGraph-Node-Kayles}. Let \( G = (V_L \cup V_R, E) \) be a bipartite graph. We will construct a graph \( G' \) such that a position of \textsc{BiGraph-Node-Kayles} played on \( G \) maps to a position of the particular games \( \text{GraphDistance}(D, S) \) played on \( G' \). As in the reduction from \textsc{Col}, we need to increase the distance between the vertices of \( G \). For this to work, we need to perform the edge replacement \( FP(n-1, k) \) for each edge in \( G \) as shown in ??, which reflects the influences of both \( S \) and \( D \).

We need forbidden vertex gadgets of size \( k \) as the set \( S \) is the one that consists of consecutive integers, which is what is needed to make all the vertices unplayable (see ??). On the other hand, we need to increase the distance between vertices in \( V_L \) and \( V_R \) to \( n \), which is why we use \( n-1 \) forbidden vertex gadgets in the path gadget. The requirement that \( k < 2n \) ensures that
we do not restrict play on a vertex \( x \in V_L \) that is at distance two from a vertex \( z \in V_L \), and which is now at distance \( 2n \) in \( G' \) (see ??).

Next, as in the case of \( D = \{1, 2\} \) in ??, we need to add a subgraph for each vertex \( v \in V_L \cup V_R \) to ensure that Left can only play on \( V_L \) and Right can only play on \( V_R \). To each vertex in \( V_L \), we attach a path gadget \( FP(k−1, k) \), and then connect all the gadgets to a single terminal vertex coloured R. Let’s look at the effect of this reduction on the vertices in \( V_L \). Since \( \max(S) = k \), the vertices from \( V_L \) cannot be coloured red. On the other hand, since \( \max(D) = n < k \), the vertices in \( V_L \) can be coloured in blue, exactly what we need to ensure. We use the same reduction on the right side of the graph, except now the terminal vertex is labeled B. This creates a play-for-play reduction as both the vertex and edge conditions are satisfied. Playing \( \text{GraphDistance}(D, S) \) on \( G' = (V_L \cup V_R \cup V', E') \) is exactly the same as playing \( \text{BiGraph-Node-Kayles} \) on \( G \). The only playable vertices for Left are those from \( V_L \) and for Right those from \( V_R \). Also, for vertices \( x \) that have been chosen in previous turns, if \( (x, y) \in V_L \cup V_R \), then \( y \) may not be chosen in subsequent turns using either ruleset.

We now illustrate with an example why the condition \( k < 2n \) is needed in the statement of ??.

**Example 1** Let \( S = \{1, 2, 3, 4\} \) and \( D = \{1, 2\} \), so \( k = 4 \) and \( n = 2 \). Suppose we have a bipartite graph with a subgraph consisting of vertices \( x, y, \) and \( z \) as shown in ??.

The reduction described in ?? requires that each edge of the bipartite graph be replaced by a single forbidden vertex gadget of size four, increasing the distance between vertices in \( V_L \) and \( V_R \) to two as required by \( D \). However, vertices \( x \) and \( z \) are only distance \( 2n = 4 \) apart, and hence if Left moves on \( x \) it eliminates \( z \) as an option.

Therefore, if \( k \geq 2n \) the reduction used in ?? does not work.
8 Conclusion and Future Work

To summarize, we used various play-for-play reductions from known PSPACE-hard games to show that $\text{GraphDistance}(D, S)$ is PSPACE-hard when $D$ or $S$ is $\{1, 2, \ldots, n\}$ and the other is a subset. The games we were reducing from were in most cases the natural choices based on the properties of the distance sets $D$ and $S$. When $S$ is larger than $D$, we provided a proof for a subcase based on a known PSPACE-hard game, BiGraph-Node-Kayles, while the more general case relies on the verification that Col is PSPACE-hard.

Our study leads to a number of open questions. At the heart of the reductions was the forbidden vertex gadget. To obtain a play-for-play reduction required that the larger of the distance sets consists of consecutive integers. This leads to the following question:

**Open Problem 1** Is $\text{GraphDistance}(D, S)$ PSPACE-hard for cases not covered by our results?

Further, the edge replacement operation we have used in our various reductions results in a planar graph $G'$ when starting from a planar graph $G$. Thus if planar Snort, planar Col, or planar Node-Kayles are shown to be PSPACE-hard, then our constructions show that the corresponding planar $\text{GraphDistance}(D, S)$ games are also PSPACE-hard. Thus we are interested in the following problem:

**Open Problem 2** Are planar Snort, planar Col, or planar Node-Kayles PSPACE-hard?

Note that this is not an immediate result of our propositions, as a game on a more specialized (potentially simpler) graph may be easier to solve, and therefore might not be PSPACE-hard even though the game played on a general graph is PSPACE-hard.
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