Modified Skellam, Poisson and Gaussian distributions in semi-open systems at charge-like conservation law

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Abstract

A modification of the Skellam and Poisson distributions is proposed for subsystems when the constraints imposed by the charge conservation law in the complete system are taken into account. Such distributions can be applied, for example, for an analysis of the fluctuations of baryon and net baryon numbers in certain pseudo-rapidity interval in $A + A$ and $p + p$ collisions with high multiplicities. The presented modified Skellam, Poisson and Gaussian distributions can be utilized also in various branches of science, when one studies the fluctuations of the two variables related to a subsystem, as well as the distribution of the difference of these variables, while the mentioned difference in the total system is fixed.

PACS numbers: 05.20, 13.85, 24.10.Pa, 24.60.Ky, 25.75.Dw, 25.75.Gz
I. INTRODUCTION

The famous Poisson distribution in statistics appears in 1837 [1]. More than one century later J. G. Skellam published the paper ”The frequency distribution of the difference between two Poisson variates belonging to different populations” [2]. Both the Poisson and Skellam distributions are widely utilized for an analysis of very different phenomena, in various branches of sciences. Despite our approach is quite general, to be concrete, this note we address to the actual and relatively new field of science - relativistic nucleus-nucleus and proton-proton collisions with high multiplicity at, particularly, CERN LHC and BNL RHIC. In these experiments an extremely high energy of colliding nucleons converts into multiparticle and multicomponent systems. The successful analysis of the particle production for different species within the statistical models [3–5], where a minimal number of parameters, such as the temperature and chemical potentials are almost enough to describe and predict the data, indicates a thermal nature of the final state formed in multiple elementary processes accompanied the collisions. Therefore the methods of statistical physics, that utilized micro-canonical, canonical and grand canonical ensembles to described the formed systems are widely used.

Typically, only a part of the system is acceptable for the direct data analysis at current LHC or RHIC detectors. It is clear, that in the case of full acceptance, the so-called 4π geometry, the net baryon number in the final state that is equal to initial one - number of nucleons in the two colliding nuclei or just value 2 in p + p collisions – is fixed and not fluctuated. In small subsystems of the total final system, the distributions of baryons and anti-baryons can be both approximated by the Poissonian ones, then difference between the corresponding particle numbers (the net baryon number) are distributed according to related Skellam function [2]. It was checked by the ALICE Collaboration [6] that such a situation really takes place in relatively small pseudo-rapidity interval in Pb + Pb collisions. In the case, when subsystem of baryons + anti-baryons, that is available for analysis, is not small and comparable to the total system - such a situation takes place in nucleus-nucleus collisions with relatively low energies (BES at RHIC, FAIR and NICA planning experiments) - the baryon charge conservation law will certainly deform both the Poisson and Skellam distributions for baryon/anti-baryon numbers and net baryon charge in the subsystem.

Despite the fact that in some publications [7–10], the baryon number conservation law in
such *semi-open* systems is discussed to some extent, the problem as whole is not solved up to now. In this note we start from generalization of the Skellam distribution; the corresponding modification of the Poisson one will follow from the former in straightforward way.

II. THE STATEMENT OF THE PROBLEM

Suppose that independent discrete values $n_i$ are distributed according to Poisson, $P(N_i, n_i)$, with the mean values $N_i$:

$$P(N_i, n_i) = \frac{N_i^{n_i}}{n_i!} \exp(-N_i)$$  \hspace{1cm} (1)

If there are several uncorrelated subsystems, then the sum of discrete values, $\sum n_i$, can be presented again as the Poisson distribution $P(\sum N_i, \sum n_i)$. Let us consider a system, consisting of particles of two species: say, baryons $(N, n)$ and anti-baryons $(\bar{N}, \bar{n})$. Since the distributions are of the Poissonian type and mutually independent, one can calculate distribution $p(k)$ of difference $k$ between particle numbers in both components, in our case, baryon minus antibaryon numbers $k = n - \bar{n}$: $p(k; N, \bar{N}) = \sum_{n} P(N, \bar{N} + k)P(N, \bar{N})$. The latter is expressed by the Skellam distribution [2]:

$$p(k; N, \bar{N}) = \exp(-N - \bar{N}) \left( \frac{N}{\bar{N}} \right)^{k/2} I_k(2\sqrt{N\bar{N}})$$  \hspace{1cm} (2)

Here $I_k$ is the modified Bessel function of the first kind. The distribution is normalized:

$$\sum_{k=\pm\infty} p(k; N, \bar{N}) = 1$$  \hspace{1cm} (3)

The mean value $m$ is

$$m = \sum_{k=\pm\infty} kp(k; N, \bar{N}) = N - \bar{N}.$$  \hspace{1cm} (4)

Let us input constraint for this baryon-antibaryon system and associate it with the baryon number conservation in the isolated complete system. Namely, we consider the ensemble of the systems under the constraint that all the systems in the ensemble have exactly the same net baryon number $B$, for other observables the averaged values are fixed only. Such an ensemble can be realized, for instance, in very central nucleus-nucleus collisions. Then the total final multiplicity can fluctuate, but fluctuations of the net baryon number in the total system are completely suppressed. Let us divide the total system into the two $i$-subsystems,
FIG. 1. The total system. Any selected subsystem $i$, say “1”, is complemented by system “2” (and wise-verse) to a complete closed system.

$i = 1, 2$, with the mean numbers $\{N_1, \overline{N}_1\}$ and $\{N_2, \overline{N}_2\}$, Fig. 1. The fluctuations of particle numbers, baryon $n_i$ and anti-baryon $\overline{n}_i$, in the two subsystems are not independent anymore because the conservation law constraint, $n - \overline{n} = N - \overline{N} = B = const$ \quad ($n = \sum_i n_i$, $\overline{n} = \sum_i \overline{n}_i$), brings the condition for the two components $\{n_1, \overline{n}_1\}$ and $\{n_2, \overline{n}_2\}$, as well as for the differences $k_1 = n_1 - \overline{n}_1$ and $k_2 = n_2 - \overline{n}_2$, in both subsystems:

$$B = n_1 + n_2 - \overline{n}_1 - \overline{n}_2 = k_1 + k_2 = N_1 + N_2 - \overline{N}_1 - \overline{N}_2 \quad (5)$$

Our aim is to generalize the Skellam distribution (2) for the semi-open $i$-systems, $i = 1$ or 2. One of the possibilities to do this, is just to construct some function $F$ by multiplying the product of independent distributions by the Kronecker $\delta$-function, e.g., $F = P(N_1, n_1)P(N_2, n_2)P(\overline{N}_1, \overline{n}_1)P(\overline{N}_2, \overline{n}_2)\delta^B_{n_1-\overline{n}_1}$, or $F = P(\sum N_i, \sum n_i)P(\sum \overline{N}_i, \sum \overline{n}_i)\delta^B_{\overline{n}_1-\overline{n}_2}$, or $F = P(N_1, n_1)P(N_2, n_2)P(\sum \overline{N}_i, \sum \overline{n}_i)\delta^B_{n_1-\overline{n}_2}$, or half-sum of the last two expressions, etc. \footnote{e.g., in Ref. [7] the Kronecker-delta constraint is imposed for the product of the partition functions of initially independent subsystems.}

After that a Skellam-like distribution is presented for subsystem $i$ as

$$p(k_i; N_i, \overline{N}_i) = \sum_{n_j, \overline{n}_l} F(\{n_j, \overline{n}_l\})\delta^{k_i}_{n_j-\overline{n}_l} \quad (6)$$

The resulting distribution will depend on the choice of the initial form $F$. It cannot be derived unambiguously without a microscopic model/theory that includes mechanisms ensuring the charge conservation law in the total system.

Our goal is to find the simplest analytic approximation to this problem which is based on general constraints required for such semi-open systems. We will compare our results with the model based on the binomial distribution (see [9], [8]) with probabilities $q$ and $\overline{q}$ for baryon and anti-baryon to belong to certain subsystem, say “1”, when the total numbers $n$ and $\overline{n}$ in complete system are given. Let now these numbers $n$ and $\overline{n}$ fluctuate with the Poisson distributions having the mean values $D$ and $\overline{D}$ correspondingly. In such a model $F$
in Eq. (6) is

\[
\mathcal{F}(\{n_j, \bar{n}_l\}) = C \sum_{n,\bar{\pi}} \delta^{\bar{\pi}}_{n-\bar{\pi}} \delta^{\nu}_{n_1+n_2} \delta_{n_1+\pi_2} P(D, n) P(\bar{D}, \bar{n}) \\
\times q^{n_1} (1-q)^{n_2} \bar{q}^{\pi_1} (1-\bar{q})^{\pi_2} \frac{n!}{n_1! n_2!} \frac{\bar{n}!}{\pi_1! \pi_2!}
\]  

(7)

where \(C\) is the normalization constant and

\[
q = \langle n_1 \rangle / \langle n \rangle = N_1/N, \quad \bar{q} = \langle \bar{n}_1 \rangle / \langle \bar{n} \rangle = \bar{N}_1/\bar{N}.
\]  

(8)

Only the numerical calculations are possible to provide summation of infinite series of hypergeometric functions in order to find from (7) the distribution over \(k_1 = n_1 - \bar{n}_1\) according to Eq. (6). Suddenly imposed constraint \(\delta^{\bar{\pi}}_{n-\bar{\pi}}\) dramatically changes the initial Poissonian-based “bare” distributions for \(n_j\) or \(\bar{n}_l\). Correspondingly, the mean values \(N = \langle n \rangle, \bar{N} = \langle \bar{n} \rangle\) are differ from the “bare” mean values \((D, \bar{D}\) in Eq. (7)), and their connection is expressed through the sum of hypergeometric functions. Nevertheless, to compare our results with results based, for certainty, on the binomial-Poissonian distribution (7) we provide such calculations.

III. GENERALIZATION OF THE SKELLAM DISTRIBUTION

In this note we propose the natural generalization of the Skellam and Poisson distributions that not deal with the “bare” distributions and corresponding “bare” mean values but are expressed analytically directly through experimentally observed particle numbers and its mean values for the system and subsystems. Pursuing this aim let us present the modified Skellam distribution for subsystem \(i\) in the following generalized form:

\[
\tilde{p}(k_i; N_i, \bar{N}_i) = e^{-(M_{i}+\bar{M}_{i})} \left( \frac{M_{i}}{\bar{M}_{i}} \right)^{(k_i-k_0^0)/2} I_{k_i-k_0^0}(2 \sqrt{M_i \bar{M}_i})
\]  

(9)

To fix the expressions for \(M_i, \bar{M}_i, k_0^0\) in the ansatz (9) let us list the conditions for the modified Skellam distribution:

1. The normalization condition (3) with substitutions \(p \rightarrow \tilde{p}, \ k \rightarrow k_i, \ \{N, \bar{N}\} \rightarrow \{N_i, \bar{N}_i\}\) must be satisfied for \(\tilde{p}(k_i; N_i, \bar{N}_i)\).

2. The equality (4) for the mean value must be satisfied under the above substitutions.
3. Because of the symmetry of the conservation law constraint (5) with respect to the mutual permutations $N_1 \leftrightarrow N_2, \overline{N}_1 \leftrightarrow \overline{N}_2$ and $k_1 \leftrightarrow k_2$, the analytical expressions for modified Skellam distributions for the two subsystems should transform to each other

$$\tilde{p}(k_1; N_1, \overline{N}_1) \leftrightarrow \tilde{p}(k_2; N_2, \overline{N}_2)$$

under these permutations.

4. If one of the subsystems (say, “1”) is much smaller than the other one, $N_1 + \overline{N}_1 \ll N_2, \overline{N}_2$, then fluctuations of the two components, baryon and anti-baryon, in the subsystem “1” are uncorrelated Poissonian ones (the second subsystem just plays the role of a “thermal bath”). So the distribution (9) tends to the Skellam one (2),

$$\tilde{p}(k_1; N_1, \overline{N}_1) \rightarrow p(k_1; N_1, \overline{N}_1), \quad N_1 + \overline{N}_1 \ll N_2, \overline{N}_2,$$  \hspace{1cm} (10)

see Fig. 2. The situation is wise-verse as for the permutation “1” ⇔ “2”.

5. When one of the subsystems (say, “1”) vanishes, $N_1, \overline{N}_1 \rightarrow 0$, the subsystem “2” occupies, in fact, the total system, $N_2 - \overline{N}_2 = B$, and then, according to the net baryon charge conservation law,

$$\tilde{p}(k_2; N_2, \overline{N}_2) \rightarrow \delta_{k_2}^B, \quad N_2 \rightarrow N, \overline{N}_2 \rightarrow \overline{N}.$$  \hspace{1cm} (11)

The situation must be, of course, wise-verse when one permutes the systems, “1” ⇔ “2”.

6. One more restricting condition appears if the total system has only one-component, e.g. when $\overline{N}_i \rightarrow 0$ for both $i = 1, 2$, see Fig. 3.
Then the fluctuations in \( k_i = n_i - \pi_i = n_i \) inside the selected \( i \)-subsystem arise only because of the fluctuations of the \textit{baryons} between the subsystems “1” and “2”. It is obviously, that when \( N_1 = N_2 = N/2 \), the relative fluctuations, \( \sigma_i/N_i \), in any single subsystem will be twice suppressed as compare to the Skellam \( N_i \rightarrow 0 \) Poisson one in an independent subsystems. The fluctuation in the single \( i \)-subsystem is extended to the entire system \( N \): it enforces the same fluctuation (with opposite sign) in the other subsystem because of the charge conservation law. The dispersion of fluctuations related to (9) is defined, similar as in the Skellam case, by \( \sigma = \sqrt{M_1 + M_2} \) (see below, Eq. (17)) and in independent Poisson subsystems, when \( M_i = N/2 \), is \( \sigma_{\text{ind}} = \sqrt{N} \). So, to get \( \sigma = \sigma_{\text{ind}}/2 \) when the conservation law constraint is imposed, one should put \( M_i = N/8 \). On the other hand, if \( N_i \rightarrow N \), it must be: \( \tilde{p} \rightarrow \delta_{k_i = n_i} \).

As we will find, these six obvious conditions are enough to define a simplest expression for the modified Skellam distribution (9). In what follows we will refer to these conditions as (C1), (C2), (C3), (C4), (C5), (C6).

For the compactness of the subsequent presentation in an analysis of the items (C1)-(C5), we temporary rename the notations: \( M_i, \overline{M}_i \rightarrow G_i, \overline{G}_i \). The normalization condition (C1) are satisfied when \( G_i, \overline{G}_i \geq 0 \) and \( k_i^0 \) is integer. This can be seen immediately when one changes the summation variable: \( k_i \rightarrow q_i = k_i - k_i^0 \). In what follows, using the concrete expressions for \( k_i^0 \), we always imply the nearest integer numbers to the corresponding values. With the same changing of the summation variable, one can find that condition (C2) brings us the expression for \( k_i^0 \):

\[
k_i^0 = N_i + \overline{G}_i - \overline{N}_i - G_i \tag{12}
\]

To guarantee condition (C3) – the invariant form of the modified distribution under permutation “1” \( \leftrightarrow \) “2” – let us transform \( \tilde{p}(k_1; N_1, \overline{N}_1) \) into \( \tilde{p}(k_2; N_2, \overline{N}_2) \) by means of Eqs. (5) and (12). Then one has \( k_1 - k_1^0 = -(k_2 - k_2^0) + G_1 - \overline{G}_1 + G_2 - \overline{G}_2 \). Putting \( G_1 + G_2 = \overline{G}_1 + \overline{G}_2 \) one gets \( k_1 - k_1^0 = -(k_2 - k_2^0) \) and also transforms the exponent \( e^{-G_1 - \overline{G}_1} \) into \( e^{-G_2 - \overline{G}_2} \) in (9). Then, accounting for Bessel function property \( I_n(z) = I_{-n}(z) \) and negative sign before the expression \( (k_2 - k_2^0) \) after transformation of \( k_1 - k_1^0 \), one gets the final result to fulfill the condition (C3):

\[
G_1 = \overline{G}_2, \ G_2 = \overline{G}_1 \tag{13}
\]
Now let us present \( G_i, \overline{G}_i \) in the form \( G_i = \alpha_i N_i \) and \( \overline{G}_i = \overline{\alpha}_i \overline{N}_i \). The limits described by the condition (C4) in the situation when, say subsystem \(^{\prime\prime}N_1 + \overline{N}_1\) is much smaller than both of components, \( N_2 \) and \( \overline{N}_2 \), of the subsystem \(^{\prime\prime}2\), require that \( \alpha_1 \rightarrow 1 \) and \( \overline{\alpha}_1 \rightarrow 1 \) in this case. Then the modified Skellam distribution tends to standard one according to Eq. (10). Similarly for the second system. To satisfy the condition (C5) when system \(^{\prime\prime}2\) tends to be the total system, and so \( N_1 \rightarrow 0 \) and \( \overline{N}_1 \rightarrow 0 \), one must to put \( G_2 \rightarrow 0 \) and \( \overline{G}_2 \rightarrow 0 \). Then \( k_2 = k^0_2 = N_2 - \overline{N}_2 \) and the Bessel function in Eq. (9) is zero at all orders except zero, when it is unity. So the equation (11) and condition (C5) are satisfied.

To guaranty the symmetries (13) and above discussed limiting values, one has to put \( \alpha_1 = N_2/(N_1 + \overline{N}_2), \overline{\alpha}_2 = N_1/(N_1 + \overline{N}_2) \) and \( \alpha_2 = \overline{N}_1/(N_2 + \overline{N}_1), \overline{\alpha}_1 = N_2/(N_2 + \overline{N}_1) \). Finally

\[
G_1 = \overline{G}_2 = \frac{N_1 \overline{N}_2}{N_1 + \overline{N}_2}, \quad G_2 = \overline{G}_1 = \frac{N_2 \overline{N}_1}{N_2 + \overline{N}_1}
\]

(14)

Note, a common function \( Q(N_i, \overline{N}_i) \), which is symmetric under permutation \(^{1\prime\prime} \leftrightarrow ^{2\prime\prime}\) and vanish in the limits discussed in (C4),(C5), can be add to \( G_i, \overline{G}_i \) without violation of all the properties discussed in (C1)-(C5) and values for \( k^0_i \) (12). To fix it let us take into account the condition (C6). Then the simplest function \( Q \) that guaranties all the requirements is:

\[
Q = \left| k^0_1 k^0_2 \right| /2(N + \overline{N}).
\]

(15)

So, finally

\[
M_i = G_i + Q, \quad \overline{M}_i = \overline{G}_i + Q.
\]

(16)

The modified Skellam distribution (9) with the expressions (16) for \( M_i, \overline{M}_i \) and (12) for \( k^0_i \) generalizes the original distribution (2) for semi-open (sub)systems \( N_i, \overline{N}_i \) with the constraint for the total system \( N - \overline{N} = B = \text{const} \). The generalization is satisfied the obvious and necessary physical conditions (C1)-(C6).

In Fig. 4 we present, just for illustration, the comparison between modified Skellam distribution (2), Skellam-like binomial-based distribution (7), (6) and just Skellam distributions (2) at the same average values \( N_i, \overline{N}_i \). The probabilities (8) are \( q = 0.6, \overline{q} = 0.5 \), and \( B = 80 \). One can see that the uncorrected for charge conservation law Skellam distribution is much wider than the ones accounting for this restriction.

Let us find for the semi-open \( i \)-subsystem the variance \( \sigma^2 \), skewness \( S \) – the measure of lack of symmetry of the probability distribution, and excess kurtosis \( K \) – the measure of the
FIG. 4. A comparison of the modified Skellam distribution (9) with the Skellam-like one based on the Poissonian-binomial distribution (7), (6), and also with the original Skellam distribution.

"tailedness". The calculations with modified Skellam distribution (9), (12), (16) accounting for the corresponding mean values $m_i = N_i - \overline{N}_i$, see (C2), give the results

$$\sigma_i^2 = \sum_{k=-\infty}^{\infty} (k - m_i)^2 \tilde{p}(k; N_i, \overline{N}_i) = (M_i + \overline{M}_i) \quad (17)$$

$$S_i = \sum_{k=-\infty}^{\infty} \left( \frac{k - m_i}{\sigma} \right)^3 \tilde{p}(k; N_i, \overline{N}_i) = \frac{M_i - \overline{M}_i}{(M_i + \overline{M}_i)^{3/2}} \quad (18)$$

$$K_i = \sum_{k=-\infty}^{\infty} \left( \frac{k - m_i}{\sigma} \right)^4 \tilde{p}(k; N_i, \overline{N}_i) - 3 = \frac{1}{M_i + \overline{M}_i} \quad (19)$$

Note, that the above equalities are exact, strictly speaking, when $k_i^0$ are integer numbers. Nevertheless, one can utilize the above expressions as the corresponding analytical continuations. Note, if $i$-system is a fairly small part of the total system (each of its components), then $M_i \to N_i$, $\overline{M}_i \to \overline{N}_i$ and all the moments of the modified Skellam distribution are coincided with the known ones. If the subsystem, say "2", tends to the total system: $N_2 \to N$, $\overline{N}_2 \to \overline{N}$, then $M_1$ and $M_2$ go to zero, so the dispersion $\sigma$ and all the other central moments, describing the fluctuations of the net baryon number, tends to zero.

In the special case $q = \bar{q}$ in (8) the ratio of $\sigma_i^2/\sigma_{i,\text{Skellam}}^2$ coincides (deviations not exceed 0.5%) with the result [6] for this ratio, $1 - q$, obtained in the binomial-based model [7], (6). In Fig. 5 we demonstrate the ratios of variances $\sigma_{\text{bin}}^2$, obtained from Eqs. (6), (7) for the binomial-Poissonian distribution, and our result (17) for $\sigma^2$, in the cases when $q = \bar{q}$ and also when $q = \bar{q} + \frac{1}{2}$. Both situations are considered in the two limits: when the total net baryon numbers in the total system are $B = 1$ and $B = 150$. The maximal deviation between the models is reached 7% and achieved at small $B$ when $q = \bar{q} + \frac{1}{2}$. 
FIG. 5. The ratios of the variance in the binomial-Poisson distribution (7), (9) to the one (17) in the modified Skellam distribution (9). The red and black lines are related to the total net charge $B = 1$ and $B = 150$ correspondingly at $q = q_1^2$; blue and green lines - to $B = 150$ and $B = 1$ correspondingly at $q = q + 1/2$.

IV. THE MODIFIED POISSON AND GAUSSIAN DISTRIBUTION

The analogy of the Poisson distribution $\tilde{P}_{1+2}$ for semi-open, say, “1”-subsystem containing only single component, say, baryons $N_1$, is described by Eq. (9) with $\overline{N}_1 = 0$. See Fig. 6. So that $\tilde{P}_{1+2}(n) = \tilde{p}(k_1 = n_1 - \overline{n}_1 = n_1 \equiv n; N_1, N_2, \overline{N}_2, \overline{N}_1 = 0)$.

Let us consider the case when the subsystem “2” has also only one component, namely, $\overline{N}_2 \neq 0$, so the only subsystems $N_1, \overline{N}_2 \neq 0$ compose the total system, see Fig. 7. Then the modified Poisson distribution is $\tilde{P}(n) = \tilde{p}(k_1 = n_1 - \overline{n}_1 = n_1 \equiv n; N_1, N_2, \overline{N}_2, \overline{N}_1 = 0, N_2 = 0)$. A significant analytical simplification in this case can be achieved if one neglects the term $Q$ (it is, at least, one order of the value $(1/8)$ less than $M_1$) in Eq. (16) for such a reduced system. Later we shall check such an approximation, see Figs. 8, 9. Then $M_i = G_i$. Denoting the mean numbers of baryons $N_1 = N$ and antibaryons $\overline{N}_2 = \overline{N}$, one has $M_2 \approx G_2 = 0$, $M_1 \approx G_1 \equiv M = \frac{NN}{N + \overline{N}}$, $k_1^0 \equiv k_0 = N - M$, and using the passage to the zeroth argument in
Bessel function in Eq. (9), one can write the modified Poisson distribution, $\tilde{P}$, accounting for the charge number conservation law in the system such as in Fig. 7:

$$\tilde{P}(n; N) = \frac{e^{-M} M^{(n-k_0)} (n-k_0)!}{N + N}, \text{where } M = \frac{N N}{N + N}, k_0 = N - M. \quad (20)$$

As this was marked earlier, $k_0$ is a nearest integer number of the corresponding value. At $\alpha \equiv \frac{N}{N + N} \to 1$ the Eq. (20) coincides with the Poisson distribution (1), $\tilde{P}(n; N) = P(n; N)$, at $\alpha \ll 1$, $\tilde{P} \to \delta^{N-B}$. In Fig. 8 we demonstrate comparison of modified Poisson distribution $\tilde{P}(n; N)$ (20) with some other distributions at the same mean values $N, N$ and $B = 1$. One can see that the results in our simple approximation (20) and the binomial-Poissonian model (7), (6) are fairly close, and at the same time the standard Poisson distribution (1) is much wider because does not take into account the charge conservation law.

FIG. 8. A comparision of the modified Poisson distribution (20) with the Poisson-like one based on binomial-Poissonian distribution (7), (6), and also with the standard Poisson distribution. The net baryon number $B = 1$.

The modified Poisson distribution is normalized as easy to check. The mean value $m = \langle n \rangle = N$. In the approximation (20) the variance $\sigma^2$, skewness $S$ and kurtosis $K$ are defined by the formulas (17)-(19) with $M_1 = M$ and $M_2 = 0$. In Fig. 9 we present the ratio of variances obtained in the modified Poisson function $\tilde{P}$ (20) and in the corresponding binomial-Poissonian distribution (7) with probabilities $q = 1$ and $\bar{q} = 0$, for a wide interval of net baryon charge $B$. We see a good agreement within 1% between these two models. As for the Poisson-like distribution, $\tilde{P}_{1+2}$, the deviation for the corresponding results can reach 14% at relatively small $B$.

The transition to the correspondent Gaussian distribution is straightforward by means of the Stirling approximation and standard procedure: $x = n = M(1 + \delta)$; $N, M \gg 1$, $\delta \ll 1$. Then one get from (20):

$$\tilde{P}(n; N) \to \frac{1}{\sqrt{2\pi M}} e^{\frac{(x-N)^2}{2M}} \quad (21)$$
FIG. 9. The ratios of variance of the binomial-based distribution \( (7), (9) \) to the one obtained with the modified Poisson distribution \( (20) \) - blue points, and to the Poisson-like one \( \tilde{P}_{1+2} \) (see Fig. 6) - red points, as the function of net baryon number \( B \).

The limits are obvious: if \( N \ll \bar{N} \), then \( M \to N \) in \( (21) \), if \( N \gg \bar{N} \) (\( N \to B \)), then \( \tilde{P}(n;N) \to \delta(x-N) \).

V. SUMMARY

A simple analytical generalization of the Skellam distribution for an arbitrary two-component subsystem accounting for a charge-like conservation law in the total system is proposed. It is compared with the numerically evaluated binomial-Poissonian model, a very good agreement with previously found in such a model\(^2\) the variation of the net baryon charge is observed. The results coincide within less than 0.5\%. The same concerns the case when the number of baryons is much more than anti-baryons, and vice versa. Being consider in full region of baryon and antibaryon probabilities to belong the subsystem, the deviation in results of these two models do not exceed 15\%. The extremely simple approximations for the Poisson and the corresponding Gaussian distribution for considered type of systems are obtained based on the modified Skellam distribution. The presented formulas for the modified Poisson distribution are in a good agreement with numerical calculations in the binomial-Poissonian model, and so can be considered as a good analytic approximations for the later.

It is worthy noting that despite the closeness of the results, the analytic approach proposed in the note is fully independent and the simplest among the possible models generalizing the Skellam and Poisson distributions for semi-open subsystems under the total charge-like conservation constraint. The analytic expressions for variation, skewness and kurtosis generated by the modified distributions are presented. The work is planning to

\(^2\) for the particular case of equal probabilities to find baryon and anti-baryon in the selected subsystem.
apply for an analysis of different baryon & anti-baryon observables in pp and AA collisions at high and intermediate energies.

ACKNOWLEDGMENTS

Author is grateful to P. Braun-Munzinger and A. Rustamov for initializing discussions. The research was carried out within the scope of the EUREA: European Ultra Relativistic Energies Agreement (European Research Network ”Heavy ions at ultrarelativistic energies”) and support by NASU, Agreement F-2018. The work is partially supported by NAS of Ukraine Targeted research program “Fundamental research on high-energy physics and nuclear physics (international cooperation)”.

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