Pointwise stability estimates for periodic traveling wave solutions of systems of viscous conservation laws

Soyeun Jung

May 3, 2014

Abstract

In the previous paper [J1], we established pointwise bounds for the Green function of the linearized equation associated with spatially periodic traveling waves $\bar{u}$ of a system of reaction diffusion equations, and also obtained pointwise nonlinear stability and behavior of $\bar{u}$ under small perturbations. In this paper, using periodic resolvent kernels and the Bloch-decomposition, we establish pointwise bounds for the Green function of the linearized equation associated with periodic standing waves $\bar{u}$ of a system of conservation laws. We also show pointwise nonlinear stability of $\bar{u}$ by estimating decay of modulated perturbation $v$ of $\bar{u}$ under small perturbation $|v_0| \leq E_0(1 + |x|)^{-\frac{3}{2}}$ for sufficiently small $E_0 > 0$.

1 Introduction

In this paper, we obtain pointwise bounds for the Green function of the linearized equations associated with spatially periodic traveling waves of systems of conservation laws extending previous work for reaction-diffusion systems in [J1], and using pointwise Green function bounds we establish the pointwise stability estimates for the periodic traveling waves. Compared with the previous work for reaction-diffusion systems, the main difference is that the Green function of the linearized operator with respect to the periodic traveling waves of conservation laws decays more slowly. This is because of the spectral structure of an eigenvalue $\lambda = 0$ of the linear operator (Lemma 1.3).

We consider systems of viscous conservation laws of form

\begin{equation}
(1.1) \quad u_t = u_{xx} + f(u)_x,
\end{equation}

where $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, $u \in U(\text{open}) \subset \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth.

The $L^p$ nonlinear stability of the periodic traveling waves of systems of conservation laws have been obtained by Johnson-Zumbrun in all dimensions ([JZ1] and [JZ3]). Here, following their basic approach, but a more detailed linear analysis, we establish the pointwise stability of the periodic traveling waves by deriving pointwise descriptions of localized modulated perturbations of $\bar{u}$.

*Indiana University, Bloomington, IN 47405; soyjung@indiana.edu
1 INTRODUCTION

1.1 Assumptions
We follow [JZ1] and [JZ3] in our assumptions. We assume the existence of an X-periodic traveling wave solution with boundary conditions \( \bar{u}(0) = \bar{u}(X) =: \bar{u}_0 \) of (1.1) of the form

\[
u(x, t) = \bar{u}(x - st),
\]

where \( s \) is the speed of the traveling wave. Plugging \( \bar{u}(x - st) \) into (1.1), we have

\[-s\bar{u}' = \bar{u}'' + f(\bar{u})'.\]

Integrating both sides, we obtain the profile equation

\[
-\bar{s} \bar{u} + q = \bar{u}' + f(\bar{u}),
\]

where \((\bar{u}_0, q, s, X) \equiv \text{constant. Without of loss of generality, we take } s=0, \text{ that is, } \bar{u}(x) \text{ is a periodic standing wave solution of (1.1). For the existence of periodic solutions of (1.2),}

we make the following assumptions ([JZ1], [JZ3], [S]):

(H1) The map \( H : \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}^n \) taking \( (X; w, q, s) \mapsto u(X; w, q, s) - w \) is full rank at \( (\bar{u}_0, 0, \bar{u}(0), 0, \bar{q}) \), where \( u(\cdot) \) is the solution operator of (1.2).

By the Implicit Function Theorem, the condition (H1) implies that the set of periodic solutions of (1.2) vicinity of \( \bar{u} \) form a smooth \((n+2)\)-dimensional manifold \( \{\bar{u}(x - \alpha - sa_t)\} \) with \( \alpha \in \mathbb{R} \) corresponding to translation and \( a \in \mathbb{R}^{n+1} \).

Linearizing (1.1) about a standing-waves solution \( \bar{u}(x) \) gives the second-order spectral problem

\[
\lambda v = Lv := v_{xx} + (df(\bar{u})v)_x = (\partial_x^2 + df(\bar{u})\partial_x + df(\bar{u})_x)v
\]

considered on the real Hilbert space \( L^2(\mathbb{R}) \). As coefficients of \( L \) are 1-periodic, Floquet theory implies that the \( L^2 \) spectrum is purely continuous and corresponds to the union of \( \lambda \) such that (1.3) admits a bounded eigenfunction of the form

\[
v(x) = e^{ix}w(x), \quad \xi \in \mathbb{R}
\]

where \( w(x + 1) = w(x) \), that is, the eigenvalues of the family of associated Floquet, or Bloch, operators

\[
L_\xi := e^{-i\xi x}Le^{i\xi x} = (\partial_x^2 + i\xi)^2 + df(\bar{u})(\partial_x + i\xi) + df(\bar{u})_x, \quad \text{for } \xi \in [-\pi, \pi),
\]

considered as acting on \( L^2 \) periodic functions on \([0, 1]\).

Recall that any function \( g \in L^2(\mathbb{R}) \) admits an inverse Bloch-Fourier representation

\[
g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} \tilde{g}(\xi, x) d\xi.
\]
where \( \hat{\gamma}(\xi, x) = \sum_{j \in \mathbb{Z}} e^{i2\pi j x} \hat{\gamma}(\xi + 2\pi j) \) is a 1-periodic functions of \( x \), and \( \hat{\gamma}(\cdot) \) denotes the Fourier transform of \( \gamma \) with respect to \( x \). Indeed, using the Fourier transform we have

\[
2\pi \gamma(x) = \int_{-\infty}^{\infty} e^{i\xi x} \hat{\gamma}(\xi) d\xi = \sum_{j \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{i(\xi + 2\pi j) x} \hat{\gamma}(\xi + 2\pi j) d\xi = \int_{-\pi}^{\pi} e^{i\xi x} \hat{\gamma}(\xi, x) d\xi.
\]

Since \( L(e^{i\xi x} f) = e^{i\xi x} (L_\xi f) \) for \( f \) periodic, the Bloch-Fourier transform diagonalizes the periodic-coefficient operator \( L \), yielding the inverse Bloch-Fourier transform representation

\[
e^{L_t} \gamma(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} e^{L_\xi t} \hat{\gamma}(\xi, x) d\xi.
\]

We now discuss the strong spectral stability conditions of the periodic traveling waves \( \tilde{u}(\cdot) \). By the translation invariant of (1.1), \( \tilde{u}'(x) \) is a 1-periodic function such that \( L_0 \tilde{u}' = 0 \). It follows that \( \lambda = 0 \) is an eigenvalue of the linear operator \( L_0 \). Moreover, the zero eigenspace of \( L_0 \) is at least \((n+1)\)-dimensional ([JZ1], [S]). Following [JZ1] and [OZ2], we assume along with (H1) the following strong spectral stability conditions:

\[
\text{(D1) } \sigma(L) \subset \{ \text{Re}\lambda < 0 \} \cup \{0\}.
\]

\[
\text{(D2) There exists a } \theta > 0 \text{ such that for all } \xi \in [-\pi, \pi] \text{ we have } \sigma(L_\xi) \subset \{ \text{Re}\lambda < -\theta|\xi|^2 \}.
\]

\[
\text{(D3) } \lambda = 0 \text{ is an eigenvalue of } L_0 \text{ of multiplicity exactly } n + 1.
\]

Conditions (D1)-(D3) correspond to “dissipativity” of the large-time behavior of the linearized system. By standard spectral perturbation theory and assumption (D3), there exist \( n + 1 \) smooth eigenvalues \( \lambda_j(\xi) \) analytic at \( \xi = 0 \) of \( L_\xi \) bifurcating from \( \lambda = 0 \) at \( \xi = 0 \) with

\[
\lambda_j(\xi) = -ia_j \xi - b_j \xi^2 + O(|\xi|^3),
\]

where \( a_j \) and \( b_j > 0 \) are real. Moreover, we make the further nondegeneracy hypothesis([JZ1], [OZ2]):

\[
\text{(H2) } a_j \text{ in (1.7) are distinct.}
\]

**Remark 1.1.** In (D3), \( \lambda = 0 \) does not need to be a semisimple eigenvalue of \( L_0 \). This is the main difficulty of systems of conservation laws compared with the previous work for reaction-diffusion system([J1]).

**Remark 1.2 ([J1]).** The condition (D3) may be readily verified by direct numerical Evans function analysis as described in [BJNRZ1, BJNRZ2].

### 1.2 First-order systems

Rewriting the eigenvalue equation (1.3) as a first-order system

\[
(1.8) \quad V' = A(\lambda, x)V,
\]
Remark 1.4. If \( \beta \) is a semisimple eigenvalue of \( L_0 \), the general right and left eigenvectors are genuine right and left eigenvectors. That is, we can simply say that there are right eigenfunctions \( q_j(\xi, x) \) and left eigenfunctions \( \tilde{q}_j(\xi, x) \) of the operator \( L_\xi \), respectively,
associated with the eigenvalue $\lambda_j(\xi)$ for each $j = 1, \ldots, n + 1$, analytic in $\xi$ for sufficiently small $|\xi|$, with the normalization condition $< \tilde{q}_j, q_k> = \delta^k_j$. In the semisimple case, the pointwise Green function $G(x, t; y)$ bound of the linearized operator $L$ is similar to that of the previous work (reaction-diffusion case, [J1]) with several modes of heat kernels, see [JZ3] and [OZ1] for the semisimple case.

1.4 Main results

With these preparations, we state here our two main results. In theorem 1.5, we determine pointwise estimates for the Green function $G(x, t; y)$ of (1.3) which is the linearization about standing-wave solutions $\tilde{u}$ of systems of conservation laws. In theorem 1.6, using pointwise bounds of $G$, we show pointwise stability estimates for $\tilde{u}$ by deriving the pointwise decay of the modulated perturbation of $\tilde{u}$ under the sufficiently small initial data.

**Theorem 1.5.** The Green function $G(x, t; y)$ for equation (1.3) satisfies the estimates:

$$G(x, t; y) = \tilde{u}'(x) \sum_{j=1}^{n+1} \sum_{l \neq n} \tilde{\beta}_{j,n}(0) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) \text{erf}(\frac{|x-y-a_j|}{\sqrt{t}})$$

$$+ \tilde{u}'(x) \sum_{j=1}^{n+1} \tilde{\beta}_{j,n}(0) \tilde{\beta}_{j,n}(0) \tilde{v}_n(0, y) \frac{1}{\sqrt{4\pi b_j t}} e^{-\frac{|x-y-a_j|^2}{4b_j t}}$$

$$+ O \left( \sum_{j=1}^{n+1} t^{-\frac{1}{2}} e^{-\frac{|x-y-a_j|^2}{M r^2}} \right),$$

$$G_y(x, t; y) = \tilde{u}'(x) \sum_{j=1}^{n+1} \tilde{\beta}_{j,n}(0) \left( \sum_{l \neq n} \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) + \tilde{\beta}_{j,n}(0) \tilde{v}_n'(0, y) \right) \frac{1}{\sqrt{4\pi b_j t}} e^{-\frac{|x-y-a_j|^2}{4b_j t}}$$

$$+ O \left( \sum_{j=1}^{n+1} t^{-1} e^{-\frac{|x-y-a_j|^2}{M r^2}} \right),$$

uniformly on $t \geq 0$, for some sufficiently large constant $M > 0$, where $\tilde{\beta}_{j,n}(0) = \lim_{\xi \to 0} \tilde{\beta}_{j,n}(\xi)$ and $\tilde{\beta}_{j,n}(0) = \lim_{\xi \to 0} \xi^{-1} \tilde{\beta}_{j,n}(\xi)$ for $\beta_{j,n}(\xi)$, $\tilde{\beta}_{j,n}(\xi)$, $v(\xi, x)$ and $\tilde{v}(\xi, x)$ defined in Lemma 1.3.

**Theorem 1.6.** Let $\tilde{u}$ be a periodic standing-wave solution of (1.1) and let $u := \tilde{u} - \tilde{u}$, where $\tilde{u}$ is any solution of (1.1) such that $|\tilde{u}(x, 0) - \tilde{u}(x, 0)| \leq E_0 (1 + |x|)^{-\frac{3}{2}}$, $E_0$ sufficiently small. Then for some $\varphi(\cdot, t) \in W^{2, \infty}$, we have the pointwise estimates

$$|\tilde{u}(x - \varphi(x, t), t) - \tilde{u}(x)| \leq C E_0 (\theta + \psi_1 + \psi_2),$$

where

$$\theta(x, t) := \sum_{j=1}^{n+1} (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-a_j|^2}{M r^2}},$$

$$\varphi(x, t) := \sum_{j=1}^{n+1} (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-a_j|^2}{M r^2}}.$$
\[ \psi_1(x, t) := \chi(x, t) \sum_{j=1}^{n+1} (1 + |x| + t)^{-\frac{1}{2}} (1 + |x - a_j t|)^{-\frac{1}{2}} \]

and

\[ \psi_2(x, t) := (1 - \chi(x, t))(1 + |x - a_1 t| + \sqrt{t})^{-\frac{3}{2}} + (1 - \chi(x, t))(1 + |x - a_{n+1} t| + \sqrt{t})^{-\frac{3}{2}}, \]

where \( \chi(x, t) = 1 \) for \( x \in [a_1 t, a_{n+1} t] \) and zero otherwise, and \( M' > 0 \) is a sufficiently large constant with \( M' > M \).

### 1.5 Discussion and open problems

\( L^p \) bounds on the Green function of \( L \) and \( L^p \) stability have been obtained by Johnson and Zumbrun. We emphasize again that it is the pointwise description that is the main new aspect here. Pointwise Green function bounds for systems of viscous conservation laws have been obtained by Oh and Zumbrun ([OZ1]) previously. However, this analysis was only for the nongeneric case in the conservation laws setting for in somewhat less detail which \( \lambda = 0 \) is a semisimple eigenvalue of \( L_0 \). As mentioned in Remark 1.4, if \( \lambda = 0 \) is a semisimple eigenvalue of \( L_0 \), we can easily define the existence of the right and left eigenfunctions \( q(\xi, x) \) and \( \tilde{q}(\xi, x) \) of \( L_\xi \) analytically for sufficiently small \( \xi \). In this case, we have the same Green function bounds as in the previous work for the reaction-diffusion case, only with several modes of heat kernels. However, for the generic case, noting first that the right and left eigenfunctions \( q(\xi, x) \) and \( \tilde{q}(\xi, x) \) of \( L_\xi \) have more complicated descriptions as in Lemma 1.3, the Green function decays more slowly (theorem 1.5) than in the previous work, [J1].

Similarly to the previous work for reaction diffusion-systems, the key to the pointwise nonlinear analysis is to subtract out the first two terms of \( G \) in Theorem 1.5 from the integral representation of modulated perturbations \( v(x, t) := \tilde{u}(x - \varphi(x, t), t) - \bar{u}(x) \) by defining \( \varphi(x, t) \) appropriately with an assumption \( \varphi(x, 0) = 0 \), that is, localized modulations (section 5.1). However, the pointwise nonlinear analysis with nonlocalized modulations \( h(x) := \varphi(x, 0), |\partial_x h(x)| \), treated at \( L^q \rightarrow L^p \) level in [JNRZ1, JNRZ2, JNRZ3], is an interesting direction for further investigation for both systems of reaction-diffusion and conservation laws. The main new ingredient compared to the localized case will be a detailed estimation of \( e^{ Lt}(\bar{u}' h_0) \) in terms of \( |\partial_x h_0| \). With further effort, we could also give a description of behavior for both localized and nonlocalized parts.

In our way of estimating nonlinear interactions, we follow the strategy of [HZ]. Full details of the scattering part of the [HZ] argument given in the more restricted situation considered here help clarify that argument as well. With further effort, one should be able to derive a more detailed description in terms of ”nonlinear diffusion waves” as in [HRZ], by combining our argument with that of [JNRZ3].

### 2 The resolvent kernel

In this section, we develop a formula for the resolvent kernel on the whole line and the periodic boundary conditions on \([0,1]\) using solution operators and projections. Here, “
whole-line ” means the kernel of periodic-coefficient operator considered as acting on $L^2(\mathbb{R})$. For $\lambda$ in the resolvent set of $L$, we denote by $G_\lambda(x,y)$ the resolvent kernel defined by

$$(L - \lambda I)G_\lambda(\cdot,y) := \delta_y \cdot I,$$

$\delta_y$ denoting the Dirac delta distribution centered at $y$.

We already constructed the formula for $G_{\xi,\lambda}(\cdot,y)$ in the previous paper, [J1]. Here, we construct the formula of $(G_{\xi,\lambda} \partial_y G_{\xi,\lambda})(x,y)$. By [ZH](Lemma 4.3), we need to consider the adjoint operator $L^*_\xi$ of (1.10), and $z = G_{\xi,\lambda}(x,\cdot)$ satisfies

(2.1) 
$$z\lambda = zL^*_\xi = z'' - (zA_\xi)' - zC_\xi,$$

where $A_\xi = -2iI - df(\bar{u}) \in \mathbb{C}^{n \times n}$ and $C_\xi(x) = -df(\bar{u})_x - i\xi df(\bar{u}) + \xi^2 I \in \mathbb{C}^{n \times n}$. Rewriting (2.1) as a first-order system

(2.2) 
$$Z' = Z\tilde{A}_\xi(x,\lambda),$$

where

$$U = (z \ z'), \quad \tilde{A}_\xi = \begin{pmatrix} 0 & \lambda I - i\xi df(\bar{u}) + \xi^2 I \\ I & 2i\xi I + df(\bar{u}) \end{pmatrix},$$

similarly, denote by $\tilde{\mathcal{F}}^{x\rightarrow y}_\xi \in \mathbb{C}^{2n \times 2n}$ the solution operator of (2.2), defined by $\tilde{\mathcal{F}}^{x\rightarrow y}_\xi = I$, $\partial_y \tilde{\mathcal{F}}^{x\rightarrow y}_\xi = \tilde{\mathcal{F}}^{x\rightarrow y}_\xi \tilde{A}_\xi$.

In subsection 2.3, we give a simple example for construction of $(G_{\xi,\lambda} \partial_y G_{\xi,\lambda})(x,y)$.

### 2.1 The whole line case

We constructed $(G_{\xi,\lambda} \partial_y G_{\xi,\lambda})(x,y)$ in the previous paper [J1]. We state here again with $(G_{\xi,\lambda} \partial_y G_{\xi,\lambda})(x,y)$.

**Lemma 2.1.** For all $\xi \in [-\pi, \pi]$, the whole line kernel satisfies

$$
\begin{cases}
(G_{\xi,\lambda} \partial_y G_{\xi,\lambda})(x,y) = \begin{pmatrix} \mathcal{F}_\xi^{y\rightarrow x} \Pi_{\xi}^+(y) & 0 \\ 0 & I \end{pmatrix}, & x > y, \\
-\mathcal{F}_\xi^{y\rightarrow x} \Pi_{\xi}^-(y) & 0 \end{cases},
\quad x < y,
\end{cases}
$$

and

$$
\begin{cases}
(G_{\xi,\lambda} \partial_y G_{\xi,\lambda})(x,y) = \begin{pmatrix} 0 & I \\ \Pi_{\xi}^-(x) & \tilde{\mathcal{F}}^{x\rightarrow y}_\xi \end{pmatrix}, & x > y, \\
-\begin{pmatrix} 0 & I \\ \Pi_{\xi}^+(x) & \tilde{\mathcal{F}}^{x\rightarrow y}_\xi \end{pmatrix}, & x > y,
\end{cases}
$$

where $\Pi_{\xi}^\pm$ and $\tilde{\Pi}_{\xi}^\pm$ are projections onto the manifolds of solutions decaying as $x \to \pm\infty$ and $y \to \pm\infty$, respectively.
Proof. We must only check the jump condition \[ \left( G_{\xi,\lambda}, \frac{\partial y}{\partial x} G_{\xi,\lambda} \right) \big|_y = \begin{pmatrix} 0 \\ I \end{pmatrix} \] and \[ \left( G_{\xi,\lambda} \quad \frac{\partial y}{\partial x} G_{\xi,\lambda} \right) \big|_y = \begin{pmatrix} 0 \\ -I \end{pmatrix} \] which follows from \( \mathcal{F}_{\xi}^{y \rightarrow y} = I = \tilde{\mathcal{F}}_{\xi}^{y \rightarrow y} \) and \( \tilde{\Pi}_{\xi}^+ + \tilde{\Pi}_{\xi}^- = I = \tilde{\Pi}_{\xi}^+ + \tilde{\Pi}_{\xi}^- \), and the fact that \( G_{\xi,\lambda}(\cdot, y) \) and \( G_{\xi,\lambda}(x, \cdot) \) decay at \( \pm \infty \), which is clear by inspection.

2.2 The periodic case

Lemma 2.2. For all \( \xi \in [-\pi, \pi] \), the periodic kernel satisfies

\[
\left( G_{\xi,\lambda}, \frac{\partial y}{\partial x} G_{\xi,\lambda} \right)(x, y) = \begin{cases} 
\mathcal{F}_{\xi}^{y \rightarrow x} M_{\xi}^+(y) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x > y, \\
-\mathcal{F}_{\xi}^{y \rightarrow x} M_{\xi}^-(y) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x \leq y,
\end{cases}
\]

where \( M_{\xi}^+(y) = (I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1} \) and \( M_{\xi}^-(y) = -(I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1} \mathcal{F}_{\xi}^{y \rightarrow y+1} \),

\[
(G_{\xi,\lambda} \quad \frac{\partial y}{\partial x} G_{\xi,\lambda})(x, y) = \begin{cases} 
- \begin{pmatrix} 0 \\ I \end{pmatrix} \tilde{M}_{\xi}^+(x) \tilde{\mathcal{F}}_{\xi}^{x \rightarrow y}, & x > y, \\
\begin{pmatrix} 0 \\ I \end{pmatrix} \tilde{M}_{\xi}^-(x) \tilde{\mathcal{F}}_{\xi}^{x \rightarrow y}, & x < y,
\end{cases}
\]

where \( \tilde{M}_{\xi}^+(x) = (I - \tilde{\mathcal{F}}_{\xi}^{x \rightarrow x+1})^{-1} \) and \( \tilde{M}_{\xi}^-(x) = -\tilde{\mathcal{F}}_{\xi}^{x \rightarrow x+1} (I - \tilde{\mathcal{F}}_{\xi}^{x \rightarrow x+1})^{-1} \).

Proof. We must check the jump condition \[ \left( \frac{\partial y}{\partial x} G_{\xi,\lambda} \right) \big|_y = \begin{pmatrix} 0 \\ I \end{pmatrix} \] and \[ \left( G_{\xi,\lambda} \quad \frac{\partial y}{\partial x} G_{\xi,\lambda} \right) \big|_y = \begin{pmatrix} 0 \\ -I \end{pmatrix} \] which follows from \( \mathcal{F}_{\xi}^{y \rightarrow y} = I \) and \( M_{\xi}^+ + M_{\xi}^- = I \), and periodicity, \( \left( G_{\xi,\lambda} \quad \frac{\partial y}{\partial x} G_{\xi,\lambda} \right)(0, y) = (G_{\xi,\lambda}, \frac{\partial y}{\partial x} G_{\xi,\lambda})(1, y) \). By periodicity of the solution operator, \( \mathcal{F}_{\xi}^{0 \rightarrow y} \mathcal{F}_{\xi}^{y \rightarrow 1} = \mathcal{F}_{\xi}^{1 \rightarrow y+1} \mathcal{F}_{\xi}^{y \rightarrow y+1} \). By direct computation, we obtain \( \mathcal{F}_{\xi}^{y \rightarrow y}(I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1} = \mathcal{F}_{\xi}^{y \rightarrow 0}(I - \mathcal{F}_{\xi}^{y \rightarrow y+1})^{-1} \mathcal{F}_{\xi}^{y \rightarrow y+1} \) which gives us \( \left( G_{\xi,\lambda} \quad \frac{\partial y}{\partial x} G_{\xi,\lambda} \right)(0, y) = \left( G_{\xi,\lambda} \quad \frac{\partial y}{\partial x} G_{\xi,\lambda} \right)(1, y) \). Similarly we argue the periodicity for \( (G_{\xi,\lambda} \quad \frac{\partial y}{\partial x} G_{\xi,\lambda})(x, y) \).

2.3 Example

Consider the constant-coefficient scalar case

\[ u_t + au_x = u_{xx}, \quad a > 0 \quad \text{constant}. \]

This gives a eigenvalue equation for each \( \xi \in [-\pi, \pi] \),

\[ u'' - (a - i2\xi)u' - (\xi^2 + ia\xi)u = \lambda u \]
Rewriting as a first-order system

\[ U' = \mathcal{A}_\xi(x, \lambda)U, \]

where

\[ U = \begin{pmatrix} u \\ u' \end{pmatrix}, \quad \mathcal{A}_\xi = \begin{pmatrix} 0 & 1 \\ \lambda + \xi^2 + ia\xi & a - i2\xi \end{pmatrix}. \]

By a direct calculation we can find two eigenvalues of \( \mathcal{A}_\xi \),

\[ \mu_{\pm} = \frac{a - i2\xi \pm \sqrt{a^2 + 4\lambda}}{2}, \]

which are solutions of the characteristic equation

\[ \mu^2 - (a - i2\xi)\mu - \lambda - \xi^2 - ia\xi = 0. \]

Then for \( \text{Re}\lambda > 0 \), we can assume \( \text{Re}\mu_- < 0 \) and \( \text{Re}\mu_+ > 0 \).

To find \( (G_{\xi,\lambda} \quad \partial_y G_{\xi,\lambda}) (x, y) \), let's consider the equation for each \( \xi \in [-\pi, \pi] \),

\[ z'' + (a - i2\xi)z' - (\xi^2 + ia\xi)z = \lambda z \]

Rewriting as a first-order system

\[ Z' = Z\tilde{\mathcal{A}}_\xi(x, \lambda), \]

where

\[ Z = \begin{pmatrix} z \\ z' \end{pmatrix}, \quad \tilde{\mathcal{A}}_\xi = \begin{pmatrix} 0 & \lambda + \xi^2 + ia\xi \\ 1 & -a + i2\xi \end{pmatrix}. \]

Then the matrix \( z = G_{\xi,\lambda}(x, \cdot) \) satisfies (2.3) and \( (G_{\xi,\lambda}(x, \cdot) \quad \partial_y G_{\xi,\lambda}(x, \cdot)) \) satisfies (2.4). It is easily see that there are two eigenvalues of \( \tilde{\mathcal{A}}_\xi(x, \lambda) \)

\[ \tilde{\mu}_{\pm} = \frac{-a + i2\xi \pm \sqrt{a^2 + 4\lambda}}{2} = -\mu_{\mp}. \]

By the same calculation, we find \( G_{\xi,\lambda}(x, y) \) and \( \partial_y G_{\xi,\lambda}(x, y) \),

\[ G_{\xi,\lambda}(x, y) = \begin{cases} \frac{e^{\tilde{\mu}_- (y-x+1)}}{\tilde{\mu}_- - \tilde{\mu}_+ (1-e^{\mu_-})} - \frac{e^{\tilde{\mu}_+ (y-x+1)}}{(\tilde{\mu}_- - \tilde{\mu}_+) (1-e^{\mu_+})}, & x > y, \\ \frac{e^{\tilde{\mu}_- (y-x)}}{\tilde{\mu}_- - \tilde{\mu}_+ (1-e^{\mu_-})} - \frac{e^{\tilde{\mu}_+ (y-x)}}{(\tilde{\mu}_- - \tilde{\mu}_+) (1-e^{\mu_+})}, & x < y, \end{cases} \]

and

\[ \partial_y G_{\xi,\lambda}(x, y) = \begin{cases} \frac{\tilde{\mu}_- e^{\tilde{\mu}_- (y-x+1)}}{\tilde{\mu}_- - \tilde{\mu}_+ (1-e^{\mu_-})} - \frac{\tilde{\mu}_+ e^{\tilde{\mu}_+ (y-x+1)}}{(\tilde{\mu}_- - \tilde{\mu}_+) (1-e^{\mu_+})}, & x > y, \\ \frac{\tilde{\mu}_- e^{\tilde{\mu}_- (y-x)}}{\tilde{\mu}_- - \tilde{\mu}_+ (1-e^{\mu_-})} - \frac{\tilde{\mu}_+ e^{\tilde{\mu}_+ (y-x)}}{(\tilde{\mu}_- - \tilde{\mu}_+) (1-e^{\mu_+})}, & x < y. \end{cases} \]
3 **POINTWISE BOUNDS ON** \( G_{\xi,\lambda}(x,y) \) **AND** \( \partial_y G_{\xi,\lambda}(x,y) \) **FOR** \( |\lambda| > R, R \text{ sufficiently large} \)

The solution operator of (2.4) is

\[
\tilde{F}_\xi^{x,y} = e^{\tilde{\lambda}_i(y-x)} = \tilde{\Pi}_\xi^+(x)e^{\tilde{\lambda}_i(y-x)} + \tilde{\Pi}_\xi^-(x)e^{\tilde{\lambda}_i(y-x)},
\]

where

\[
\tilde{\Pi}_\xi^+ = \begin{pmatrix} \frac{-\mu_+}{\mu_- - \mu_+} & \frac{-\mu_-}{\mu_- - \mu_+} \\ \frac{1}{\mu_- - \mu_+} & \frac{1}{\mu_- - \mu_+} \end{pmatrix} \quad \text{and} \quad \tilde{\Pi}_\xi^- = \begin{pmatrix} \frac{\mu_+}{\mu_- - \mu_+} & \frac{\mu_-}{\mu_- - \mu_+} \\ \frac{-1}{\mu_- - \mu_+} & \frac{-1}{\mu_- - \mu_+} \end{pmatrix},
\]

which satisfies

\[
(G_{\xi,\lambda}, \partial_y G_{\xi,\lambda})(x,y) = \begin{cases} 
- \begin{pmatrix} 0 & I \end{pmatrix} \tilde{M}_\xi^-(x)\tilde{F}_\xi^{x,y}, & x > y, \\
\begin{pmatrix} 0 & I \end{pmatrix} \tilde{M}_\xi^+(x)\tilde{F}_\xi^{x,y}, & x < y,
\end{cases}
\]

where \( \tilde{M}_\xi^+(x) = (I - \tilde{F}_\xi^{x-x+1})^{-1} \) and \( \tilde{M}_\xi^-(x) = -\tilde{F}_\xi^{x-x+1}(I - \tilde{F}_\xi^{x-x+1})^{-1} \).

3 **Pointwise bounds on** \( G_{\xi,\lambda}(x,y) \) **and** \( \partial_y G_{\xi,\lambda}(x,y) \) **for** \( |\lambda| > R, R \text{ sufficiently large} \)

In this section, we derive pointwise bounds on \( G_{\xi,\lambda}(x,y) \) and \( \partial_y G_{\xi,\lambda}(x,y) \). For the case \( |\lambda| \leq R \), since \( G_{\xi,\lambda}(x,y) \) is analytic in \( \lambda \), we have

\[
(3.1) \quad |G_{\xi,\lambda}(x,y)|, |\partial_y G_{\xi,\lambda}(x,y)| \leq C e^{-\theta|x-y|},
\]

for all \( x,y \), where \( C, \theta > 0 \) are constants (See [OZ1] and [ZH]). For \( |\lambda| > R, R \text{ sufficiently large} \), we use the direct construction of \( G_{\xi,\lambda}(x,y) \) in Section 2. However, the argument for this part is exactly same as our previous work [J1] for reaction-diffusion waves. Thus, we just state the pointwise bounds on \( G_{\xi,\lambda}(x,y) \) and \( \partial_y G_{\xi,\lambda}(x,y) \) for \( |\lambda| > R, R \text{ sufficiently large} \) without proof.

**Proposition 3.1.** ([J1]) For any \( |\xi| \leq \pi \) and any \( 0 \leq x, y \leq 1 \),

\[
|G_{\xi,\lambda}(x,y)| \leq C |\lambda|^{-1/2}|(e^{-\beta^{-1/2}|\lambda|^{1/2}|x-y|} + e^{-\beta^{-1/2}|\lambda|^{1/2}(1-|x-y|)})|
\]

\[
|\partial/\partial_x G_{\xi,\lambda}(x,y)| \leq C |e^{-\beta^{-1/2}|\lambda|^{1/2}|x-y|} + e^{-\beta^{-1/2}|\lambda|^{1/2}(1-|x-y|)})|
\]

provided \( |\lambda| \) is sufficiently large and \( C > 0 \), that is, \( |G_{\xi,\lambda}| \) is uniformly bounded as \( |\lambda| \to \infty \). Here, \( \beta^{-1/2} \approx \min_{\lambda \in \Omega \setminus \{\lambda > R\}} Re(\sqrt{\lambda}/|\lambda|) \).

4 **Pointwise bounds on** \( G \)

We now prove Theorem 1.5 which is pointwise bounds on the Green function \( G(x,t;y) \) of the linear operator \( L \) in (1.3). Let’s define the sector

\[
\Omega := \{ \lambda : Re(\lambda) \geq \theta_1 - \theta_2 |Im(\lambda)| \},
\]
where \( \theta_1, \theta_2 > 0 \) are small constants.

We first state the the standard spectral resolution (inverse Laplace transform) formula (see, [ZH, OZ1]). We use this formula to prove Theorem 1.5. This is the reason we constructed the resolvent kernels and their bounds in the previous sections.

**Proposition 4.1.** ([ZH]) The parabolic operator \( \partial_t - L \) has a Green function \( G(x, t; y) \) for each fixed \( y \) and \( (x, t) \neq (y, 0) \) given by

\[
G(x, t; y) = \frac{1}{2\pi i} \int_{\Gamma := \partial(\Omega \cup B(0, R))} e^{\lambda t} G_\lambda(x, y) d\lambda
\]

for \( R > 0 \) sufficiently large and \( \theta_1, \theta_2 > 0 \) sufficiently small.

**Proof of Theorem 1.5.** Case (i). \( \frac{|x - y|}{t} \) large. We first consider the case that \( |x - y|/t \geq S, S \) sufficiently large. For this case, as I mentioned in the previous paper [J1], it is hard to estimate \( G \) through \( |[G_\xi(x, t; y)]| \) directly, because of the problem of aliasing ([J1]). Instead we estimate \( |G_\lambda(x, y)| \) first and we estimate \( |G(x, t; y)| \) by (4.1). This is treated by exactly the same argument as in [ZH]. By [ZH], notice that

\[
|G_\lambda(x, y)| \leq C|\lambda|^{-1/2} e^{-|x-y|/2|\xi|},
\]

for all \( \lambda \in \Omega \setminus B(0, R) \) and \( R > 0 \) sufficiently large, and here, \( \beta^{-1/2} \sim \min_{\lambda \in \Omega \setminus \{|\lambda| > R\}} Re \sqrt{\lambda/|\lambda|} \).

Finally we have

\[
|G(x, t; y)| \leq C \left| \int_{\Gamma} e^{\lambda t} G_\lambda(x, y) d\lambda \right| \leq t^{-\frac{1}{2}} e^{-\eta t} e^{-\frac{|x-y|^2}{8t \epsilon}} \leq t^{-\frac{1}{2}} e^{-\eta t} e^{-\frac{|x-y-a_j|^2}{8t \epsilon}},
\]

for all \( a_j \), and for some \( \eta > 0 \) and \( M > 0 \) sufficiently large. (See [ZH] for details) Here, the last inequality is from that \( \frac{|x - y|}{t} \) large.

**Case (ii).** \( \frac{|x - y|}{t} < S \) bounded. To begin, notice that by standard spectral perturbation theory [K], the total eigenprojection \( P(\xi) \) onto the eigenspace of \( L_\xi \) associated with the eigenvalues \( \lambda(\xi) \) bifurcating from the \( (\xi, \lambda(\xi)) = (0, 0) \) state is well defined and analytic in \( \xi \) for \( \xi \) sufficiently small, since the discreteness of the spectrum of \( L_\xi \) implies that the eigenvalue \( \lambda(\xi) \) is separated at \( \xi = 0 \) from the remainder of the spectrum of \( L_0 \). By (D2), there exists an \( \epsilon > 0 \) such that \( Re \sigma(L_\xi) \leq -\theta|\xi|^2 \) for \( 0 < |\xi| < 2 \epsilon \). With this choice of \( \epsilon \), we first introduce a smooth cut off function \( \phi(\xi) \) such that

\[
\phi(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq \epsilon \\
0, & \text{if } |\xi| \geq 2\epsilon,
\end{cases}
\]

where \( \epsilon > 0 \) is a sufficiently small parameter. Now from the inverse Bloch-Fourier transform representation, we split the Green function

\[
G(x, t; y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} e^{L_\xi t} \delta_y(\xi, x) d\xi
\]
into its low-frequency part

\[ L = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} \phi(\xi) P(\xi)e^{L\xi t} \delta_y(\xi, x) d\xi \]

and high frequency part

\[ H = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} (1 - \phi(\xi) P(\xi)) e^{L\xi t} \delta_y(\xi, x) d\xi. \]

Let’s start by considering the second part \( H \). The proof of the high frequency part is similar to the previous work [J1]. Noting first that

\[ \delta_y(\xi, x) = \sum_{j \in \mathbb{Z}} e^{i2\pi j x} \delta_y(\xi + 2\pi j) = \sum_{j \in \mathbb{Z}} e^{i2\pi j x} e^{-i(\xi + 2\pi j)y} = e^{-i\xi y} \sum_{j \in \mathbb{Z}} e^{i2\pi j(x-y)} = e^{-i\xi y} [\delta_y(x)], \]

we have for \(|\xi| \geq 2\varepsilon\), \( \phi(\xi) = 0 \) and

\[
\begin{align*}
\int_{2\varepsilon \leq |\xi| \leq \pi} e^{i\xi x} (1 - \phi(\xi) P(\xi)) e^{L\xi t} \delta_y(\xi, x) d\xi &= \int_{2\varepsilon \leq |\xi| \leq \pi} e^{i\xi x} e^{L\xi t} \delta_y(\xi, x) d\xi \\
&= \int_{2\varepsilon \leq |\xi| \leq \pi} e^{i\xi(x-y)} e^{L\xi t} [\delta_y(x)] d\xi \\
&= \int_{2\varepsilon \leq |\xi| \leq \pi} e^{i\xi(x-y)} [G_\xi(x, t; y)] d\xi,
\end{align*}
\]

where the brackets \([\cdot]\) denote the periodic extensions of the given function onto the whole line. Assuming that \( \text{Re} \sigma(L_\xi) \leq -\eta < 0 \) for \(|\xi| \geq 2\varepsilon\), we have

\[ [G_\xi(x, t; y)] = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda t} [G_{\xi,\lambda}(x, y)] d\lambda, \]

here, we fix \( \Gamma_1 = \partial(\Omega \cup \{ \text{Re} \lambda \geq -\eta \}) \) independent of \( \xi \). Parameterizing \( \Gamma_1 \) by \( \text{Im} \lambda := k \), and applying the bounds of \( \sup_{|\xi| \leq \pi} |[G_{\xi,\lambda}(x, y)]| < O(|\lambda^{-\frac{1}{2}}|) \) for large \( |\lambda| \) in Proposition 3.1 and (3.1), we have

\[
\begin{align*}
|[G_\xi(x, t; y)]| &\leq C \int_{\Gamma_1} e^{\text{Re} \lambda t} |[G_{\xi,\lambda}(x, y)]| d\lambda \\
&\leq Ce^{-\eta t} \int_{0}^{\infty} k^{-\frac{1}{2}} e^{-\theta_3 k t} dk \\
&\leq Ct^{-\frac{1}{2}} e^{-\eta t} \\
&\leq Ct^{-\frac{1}{2}} e^{-\eta t} \sum_{j=1}^{n+1} e^{-\frac{|x-y-a_j|^2}{mt}},
\end{align*}
\]
for large $M > 0$. Here, the last inequality is from $|x-y-a_j| < S_1$ bounded for all $j$. Indeed, for large $M > 0$,

$$e^{-\frac{|x-y-a_j|^2}{M t}} = e^{-\left(\frac{|x-y-a_j|^2}{t}\right)\frac{t}{M}} \geq e^{-\frac{S_1^2}{M}} \geq e^{-\frac{\eta}{2}t},$$

and so,

$$\int_{2\varepsilon \leq |\xi| \leq \pi} e^{i \xi x} (1 - \phi(\xi) P(\xi)) e^{L_\xi t} \delta_y(\xi, x) d\xi \leq C \sup_{2\varepsilon \leq |\xi| \leq \pi} ||G_\xi(x, t; y)||$$

$$\leq C t^{-\frac{1}{2}} e^{-\frac{\eta}{2}t} \sum_{j=1}^{n+1} e^{-\frac{|x-y-a_j|^2}{M t}}.$$

(4.2)

For sufficiently small $|\xi|$, $I - \phi(\xi) P(\xi) = I - P(\xi) = Q(\xi)$, where $Q$ is the eigenprojection of $L_\xi$ associated with eigenvalues complementary to $\lambda_j(\xi)$ bifurcating from $\lambda = 0$ at $\xi = 0$, which have real parts strictly less than zero. So we can estimate for $|\xi| \leq \varepsilon$ in the same way as in (4.2). Combining these observations, we have the estimate

$$|B| \leq C t^{-\frac{1}{2}} e^{-\frac{\eta}{2}t} \sum_{j=1}^{n+1} e^{-\frac{|x-y-a_j|^2}{M t}},$$

for some $\eta > 0$ and sufficiently large $M > 0$.

We now consider the low-frequency part $\mathcal{L}$. By Lemma 1.3, we know that $\xi \beta_{j,n}(\xi)$ is analytic in $\xi$ for sufficiently small $|\xi|$. Letting $\tilde{\beta}_{j,n}(0) = \lim_{\xi \to 0} \xi \beta_{j,n}(\xi)$ and $\tilde{\lambda}_j(\xi) = -ia_j \xi - b_j \xi^2$
we have

\[
\mathcal{L} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi x} \phi(\xi) P(\xi) e^{t \xi^2} d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi(x-y)} \phi(\xi) \sum_{j=1}^{n+1} e^{\lambda_j(\xi) t} q_j(x, \xi) \hat{g}_j(y, \xi) d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi(x-y)} \phi(\xi) \sum_{j,k=1}^{n+1} e^{\lambda_j(\xi) t} \beta_{j,k}(x) v_k(x, \xi) \tilde{\beta}_{j,l}(\xi) \tilde{v}_l(y, \xi) d\xi
\]

\[
+ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi(x-y)} \phi(\xi) \sum_{j,l \neq k}^{n+1} e^{\lambda_j(\xi) t} \beta_{j,n}(0) v_n(0, x) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) \xi^{-1} d\xi
\]

\[
+ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi(x-y)} \phi(\xi) \sum_{j,l \neq k, l \neq n}^{n+1} e^{\lambda_j(\xi) t} \beta_{j,k}(x) v_k(x, \xi) \tilde{\beta}_{j,l}(\xi) \tilde{v}_l(y, \xi) d\xi
\]

\[
= I + II + III + IV + V
\]

We start with the estimate $I$.

\[
I = \sum_{j,l \neq n}^{n+1} \beta_{j,n}(0) v_n(0, x) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi(x-y)} \phi(\xi) e^{\lambda_j(\xi) t} \xi^{-1} d\xi
\]

\[
= \sum_{j,l \neq n}^{n+1} \tilde{\beta}_{j,n}(0) v_n(0, x) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) \frac{1}{2\pi} p.v. \int_{-\pi}^{\pi} e^{i\xi(x-y)} e^{\lambda_j(\xi) t} \xi^{-1} d\xi
\]

\[
+ O \left( \int_{|\xi| \geq \varepsilon} e^{i\xi(x-y)} e^{\lambda_j(\xi) t} \xi^{-1} d\xi \right)
\]

\[
= \sum_{j,l \neq n}^{n+1} \tilde{\beta}_{j,n}(0) v_n(0, x) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) \text{erf} \left( \frac{|x-y-a_j t|^2}{\sqrt{t}} \right) + O \left( \sum_{j=1}^{n+1} t^{-\frac{1}{2} - \frac{|x-y-a_j t|^2}{mt}} \right)
\]

See [J1] for the detail estimate of $O \left( \int_{|\xi| \geq \varepsilon} e^{i\xi(x-y)} e^{\lambda_j(\xi) t} \xi^{-1} d\xi \right)$. Setting $\tilde{\beta}_{j,n}(0) = \lim_{\xi \to 0} \xi^{-1} \tilde{\beta}_{j,n}(\xi)$,
we separate $II$ into two parts,

$$II = \frac{1}{2\pi} \sum_{j=1}^{n+1} \int_{-\pi}^{\pi} e^{i\xi(x-y)} \phi(\xi) e^{i\tilde{\lambda}_j(\xi)t} \tilde{\beta}_{j,n}(0) v_n(x,0) \tilde{v}_n(y,0) d\xi$$

$$+ \sum_{j=1}^{n+1} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} e^{i\tilde{\lambda}_j(\xi)t}$$

$$\times \left( e^{O(|\xi|^3)t} (\xi \beta_{j,n}(\xi)) v_n(x,\xi) (\xi^{-1} \tilde{\beta}_{j,n}(\xi)) \tilde{v}_n(y,\xi) - \tilde{\beta}_{j,n}(0) v_n(x,0) \tilde{\beta}_{j,n}(0) \tilde{v}_n(y,0) \right) d\xi$$

$$= \sum_{j=1}^{n+1} \beta_{j,n}(0) v_n(x,0) \tilde{\beta}_{j,n}(0) \tilde{v}_n(y,0) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(x-y)} e^{i\tilde{\lambda}_j(\xi)t} d\xi + O \left( \int_{|\xi| \geq \varepsilon} e^{i\xi(x-y)} e^{i\tilde{\lambda}_j(\xi)t} d\xi \right)$$

$$+ \sum_{j=1}^{n+1} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} e^{(-ia_j \xi - b_j \xi^2)t} e^{O(|\xi|^3)t - 1 + O(\xi)} d\xi$$

$$= \sum_{j=1}^{n+1} \beta_{j,n}(0) v_n(x,0) \tilde{\beta}_{j,n}(0) \tilde{v}_n(y,0) \frac{1}{\sqrt{4\pi b_j t}} e^{-\frac{|x-y-a_j t|^2}{4b_j t}} + O \left( t^{-\frac{1}{2}} e^{-\frac{|x-y-a_j t|^2}{4b_j t}} \right)$$

Noting first that $\xi \beta_{j,n}(\xi) v_n(x,\xi) \tilde{\beta}_{j,l}(\xi) \tilde{v}_l(y,\xi)$ is analytic in $\xi$, we have

$$III = \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} \sum_{j,l \neq n}^{n+1} \left( e^{i\lambda_j(\xi)t} \xi \beta_{j,n}(\xi) v_n(x,\xi) \tilde{\beta}_{j,l}(\xi) \tilde{v}_l(y,\xi) - e^{i\tilde{\lambda}_j(\xi)t} \beta_{j,n}(0) v_n(x,0) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0,0) \right) \xi^{-1} d\xi$$

$$= \sum_{j,l \neq n}^{n+1} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} e^{(-ia_j \xi - b_j \xi^2)t} \left( e^{O(|\xi|^3)} - 1 + O(\xi) \right) \xi^{-1} d\xi$$

Similarly to [J1], viewing this as complex contour integral in complex variable $\xi$, define

$$\alpha_j := \frac{x - y - a_j}{2b_j t}$$

which is bounded because $|x - y|/t$ is bounded. Setting

$$\tilde{\alpha} := \min\{\varepsilon, \alpha_j\},$$
we have

\[(4.3)\]

\[|III| = \left| \sum_{j=1}^{n+1} \int_{-\varepsilon}^{\varepsilon} e^{i(\xi+y-a_j) t} e^{-b_j \xi^2 t} \left( e^{O(\xi^3)t} - 1 + O(\xi) \right) \xi^{-1} d\xi \right| \]

\[= \left| \sum_{j=1}^{n+1} \int_{-\varepsilon}^{\varepsilon} e^{i(\xi+i\alpha)(y-a_j) t} e^{-b_j (\xi+i\alpha)^2 t} \left( e^{O((\xi+i\alpha)^3)t} - 1 + O(\xi + i\alpha) \right) (\xi + i\alpha)^{-1} d\xi \right| \]

\[+ \sum_{j=1}^{n+1} \int_{0}^{\tilde{\alpha}} e^{i(\varepsilon+i\alpha)(y-a_j) t} e^{-b_j (\varepsilon+i\alpha)^2 t} \left( e^{O((\varepsilon+i\alpha)^3)t} - 1 + O(\varepsilon + i\alpha) \right) (\varepsilon + i\alpha)^{-1} dz \]

\[\leq C \left| \sum_{j=1}^{n+1} e^{-b_j \varepsilon^2 t} \int_{-\varepsilon}^{\varepsilon} e^{-b_j \xi^2 t} (O(|\xi|^3 t) + O(\xi)) |\xi + i\alpha|^{-1} d\xi \right| \]

\[+ C \left| \sum_{j=1}^{n+1} e^{-b_j \varepsilon^2 t} \int_{0}^{\tilde{\alpha}} e^{-b_j \xi^2 t} (O(|\xi|^3 t) + O(\varepsilon)) |\xi + iz|^{-1} dz \right| \]

\[\leq C \left| \sum_{j=1}^{n+1} e^{-b_j \varepsilon^2 t} \int_{-\varepsilon}^{\varepsilon} e^{-b_j \xi^2 t} (O(\xi) + O(\tilde{\alpha})) |\xi + i\alpha|^{-1} d\xi \right| \]

\[+ C \left| \sum_{j=1}^{n+1} e^{-b_j \varepsilon^2 t} \int_{0}^{\tilde{\alpha}} e^{-b_j \xi^2 t} (O(\varepsilon) + O(z)) |\varepsilon + iz|^{-1} dz \right| \]

\[\leq C \left| \sum_{j=1}^{n+1} e^{-b_j \varepsilon^2 t} \int_{-\varepsilon}^{\varepsilon} e^{-b_j \xi^2 t} d\xi \right| + C \left| \sum_{j=1}^{n+1} e^{-b_j \varepsilon^2 t} \int_{0}^{\tilde{\alpha}} e^{-b_j \xi^2 t} dz \right| \]

\[\leq O \left( \sum_{j=1}^{n+1} t^{-1} e^{-\frac{|x-y-a_j|^2}{Mt}} \right).\]

By Lemma 1.3, noting first that

\[\beta_{j,k}(\xi)v_k(x,\xi)\tilde{\beta}_{i,j}(\xi)\tilde{v}_i(y,\xi) = O(1), \quad \text{for} \quad l \neq n, k \neq n,\]

\[\beta_{j,k}(\xi)v_k(x,\xi)(\xi^{-1}\beta_{j,n}(\xi))\tilde{v}_n(y,\xi) = O(1), \quad \text{for} \quad k \neq n\]

and

\[(\xi\beta_{j,n}(\xi)v_n(x,\xi)(\xi^{-1}\beta_{j,n}(\xi))\tilde{v}_n(y,\xi) = O(1),\]

we have

\[IV = \sum_{j,l \neq n, k \neq n}^{n+1} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} e^{\lambda_j(\xi)t} O(1) d\xi \]

\[= \sum_{j,l \neq n, k \neq n}^{n+1} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} e^{(-ia_j \xi - b_k^2)t} e^{O(\xi^3)t} d\xi \]
and

\[ V = \sum_{j=1}^{n+1} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} e^{\lambda_j(\xi)t} O(\xi) d\xi, \]

\[ = \sum_{j=1}^{n+1} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} e^{(-ia_j \xi - b_j \xi^2)t} e^{O(\xi^3)t} O(\xi) d\xi. \]

Similarly to (4.3), we have \( IV = V = O \left( \sum_{j=1}^{n+1} t^{-\frac{1}{2}} e^{-\frac{1}{4} \frac{|x-y-a_jt|^2}{M^2}} \right) \).

We now consider the estimate of \( G_y(x, t; y) \). By Lemma 1.3, recalling \( \tilde{v}_l(0, y) \) is constant for all \( l \neq n \), we have

\[ \partial_y I = \sum_{j,l \neq n}^{n+1} \tilde{\beta}_{j,n}(0) v_n(0, x) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi(x-y)} \phi(\xi) e^{\lambda_j(\xi)t} \xi^{-1}(-i\xi) d\xi \]

\[ = \sum_{j,l \neq n}^{n+1} \tilde{\beta}_{j,n}(0) v_n(0, x) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) \frac{1}{\sqrt{4\pi\beta_j t}} e^{-\frac{|x-y-a_jt|^2}{4\beta_j t}} + O \left( \int_{|\xi| \geq \varepsilon} e^{i\xi(x-y)} \lambda_j(\xi)t d\xi \right) \]

\[ = \sum_{j,l \neq n}^{n+1} \tilde{\beta}_{j,n}(0) v_n(0, x) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) \frac{1}{\sqrt{4\pi\beta_j t}} e^{-\frac{|x-y-a_jt|^2}{4\beta_j t}} + O \left( t^{-1} e^{-\frac{|x-y-a_jt|^2}{M^2}} \right), \]

and

\[ \partial_y II = \sum_{j=1}^{n+1} \tilde{\beta}_{j,n}(0) \tilde{v}_n'(y, 0) \tilde{\beta}_{j,n}(0) \tilde{v}_n'(y, 0) \frac{1}{\sqrt{4\pi\beta_j t}} e^{-\frac{|x-y-a_jt|^2}{4\beta_j t}} + O \left( t^{-1} e^{-\frac{|x-y-a_jt|^2}{M^2}} \right). \]

Since \( \tilde{v}_l(0, y) \) is constant for all \( l \neq n \), \( \partial_y \tilde{v}_l(\xi, y) = O(|\xi|) \) for all \( l \neq n \), and so we have

\[ \partial_y III = \sum_{j,l \neq n}^{n+1} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} e^{(-ia_j \xi - b_j \xi^2)t} \left( e^{O(\xi^3)} - 1 + O(\xi) \right) O(1) d\xi \]

\[ + \sum_{j,l \neq n}^{n+1} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} e^{(-ia_j \xi - b_j \xi^2)t} e^{O(\xi^3)t} O(\xi) d\xi \]

\[ = O \left( \sum_{j=1}^{n+1} t^{-1} e^{-\frac{|x-y-a_jt|^2}{M^2}} \right). \]
Similarly, \( \partial_y(\beta_{j,k}(\xi)v_k(x,\xi)\beta_{j,l}(\xi)\tilde{v}_l(y,\xi)) = O(|\xi|) \) for all \( l \neq n \), so we have

\[
\partial_y IV = \sum_{j,l \neq n, k \neq n}^{n+1} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} e^{i\xi(x-y)} e^{-ia_j\xi - b_j\xi^2} t e^{O(\xi^3)t} O(\xi) d\xi
\]

\[= O\left(\sum_{j=1}^{n+1} t^{-1} e^{-\frac{|x-y-a_jt|}{Mt}}\right).
\]

Since \( \partial_y(\beta_{j,k}(\xi)v_k(x,\xi)(\xi^{-1}\beta_{j,n})(\xi)\tilde{v}_n(y,\xi)) = O(1) \), we have

\[
\partial_y V = IV = O\left(\sum_{j=1}^{n+1} t^{-1} e^{-\frac{|x-y-a_jt|}{Mt}}\right).
\]

5 Pointwise description of perturbations of \( \tilde{u} \)

In this section we describe the pointwise bound of perturbations of (1.1). Let \( \tilde{u}(x, t) \) be a solution of systems of conservation laws (1.1) and let \( \bar{u}(x) \) be a periodic stationary solution on \([0, 1]\). We now define perturbations

\[
(5.1) \quad u(x, t) = \tilde{u}(x, t) - \bar{u}(x) \quad \text{and} \quad v(x, t) = \tilde{u}(x - \varphi(x, t), t) - \bar{u}(x),
\]

for some unknown functions \( \varphi(x, t) : \mathbb{R}^2 \to \mathbb{R} \) to be determined later with \( \varphi(x, 0) = 0 \).

In this section, using the pointwise estimate of the linear operator \( L \) in Theorem 1.5, we establish a pointwise description of perturbations \( v \) for a initial condition \( v_0 = v(x, 0) = u(x, 0) \):

\[|v_0(x)| \leq E_0 (1 + |x|)^{-\frac{3}{2}} \quad \text{and} \quad |v_0(x)|_{H^2} \leq E_0,
\]

where \( E_0 > 0 \) sufficiently small.

Recalling Theorem 1.5, the Green function \( G(x, t; y) \) of the linear equation \( u_t = Lu \).
satisfies the estimates:

\[
G(x, t; y) = \bar{u}'(x) \sum_{j=1}^{n+1} \sum_{l \neq n} \tilde{\beta}_{j,n}(0) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) \text{erf} \left( \frac{|x - y - a_j t|^2}{\sqrt{t}} \right) \\
+ \bar{u}'(x) \sum_{j=1}^{n+1} \tilde{\beta}_{j,n}(0) \tilde{\gamma}_{j,n}(0) \tilde{v}_n(0, y) \frac{1}{\sqrt{4\pi b_j t}} e^{-\frac{|x - y - a_j t|^2}{4b_j t}} \\
+ O \left( \sum_{j=1}^{n+1} t^{-\frac{1}{2}} e^{-\frac{|x - y - a_j t|^2}{M t}} \right),
\]

\[
G_y(x, t; y) = \bar{u}'(x) \sum_{j=1}^{n+1} \tilde{\beta}_{j,n}(0) \left( \sum_{l \neq n} \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) + \tilde{\gamma}_{j,n}(0) \tilde{v}_n(0, y) \right) \frac{1}{\sqrt{4\pi b_j t}} e^{-\frac{|x - y - a_j t|^2}{4b_j t}} \\
+ O \left( \sum_{j=1}^{n+1} t^{-\frac{1}{2}} e^{-\frac{|x - y - a_j t|^2}{M t}} \right),
\]

uniformly on \( t \geq 0 \), for some sufficiently large constant \( M > 0 \), where \( \bar{\beta}_{j,n}(0) = \lim_{\xi \to 0} \xi \beta_{j,n}(\xi) \) and \( \tilde{\beta}_{j,n}(0) = \lim_{\xi \to 0} \xi^{-1} \tilde{\beta}_{j,n}(\xi) \) for \( \beta_{j,n}(\xi) \), \( \tilde{\beta}_{j,n}(\xi) \), \( v(\xi, x) \) and \( \tilde{v}(\xi, x) \) defined in Lemma 1.3.

First off, let \( \chi(t) \) be a smooth cut off function defined for \( t \geq 0 \) such that \( \chi(t) = 0 \) for \( 0 \leq t \leq 1 \) and \( \chi(t) = 1 \) for \( t \geq 2 \) and define

\[
E(x, t; y) := \bar{u}'(x)e(x, t; y),
\]

where

\[
e(x, t; y) = \sum_{j=1}^{n+1} \sum_{l \neq n} \beta_{j,n}(0) \tilde{\beta}_{j,l}(0) \tilde{v}_l(0, y) \text{erf} \left( \frac{|x - y - a_j t|^2}{\sqrt{t}} \right) \chi(t) \\
+ \sum_{j=1}^{n+1} \tilde{\beta}_{j,n}(0) \tilde{\gamma}_{j,n}(0) \tilde{v}_n(0, y) \frac{1}{\sqrt{4\pi b_j t}} e^{-\frac{|x - y - a_j t|^2}{4b_j t}} \chi(t)
\]

Now we set

\[
\tilde{G}(x, t; y) = G(x, t; y) - E(x, t; y).
\]

so that

\[
|\tilde{G}(x, t; y)| \leq Ct^{-\frac{1}{2}} \sum_{j=1}^{n+1} e^{-\frac{|x - y - a_j t|^2}{M t}} \quad \text{and} \quad |\tilde{G}_y(x, t; y)| \leq Ct^{-1} \sum_{j=1}^{n+1} e^{-\frac{|x - y - a_j t|^2}{M t}}.
\]

To establish a pointwise description of perturbations \( v \), we first start with the nonlinear perturbation equation of \( v \) ([JZ1]).
Lemma 5.1 (Nonlinear perturbation equations, [JZ1]). For $v$ defined in (5.1), we have

$$(5.3) \quad (\partial_t - L)v = - (\partial_t - L)\bar{u}'(x)\varphi + Q_x + R_x + (\partial_x^2 + \partial_t)S,$$

where

$$(5.4) \quad Q := f(v(x,t) + \bar{u}(x)) - f(\bar{u}(x)) - df(\bar{u}(x))v = O(|v|^2),$$

$$(5.5) \quad R := -v\psi_t - v\psi_{xx} + (\bar{u}_x + v_x)\frac{\varphi_x^2}{1 - \varphi_x},$$

$$(5.6) \quad S := v\varphi_x = O(|v||\varphi_x|),$$

Proof. Direct computation; see [JZ1].

5.1 Integral representation and $\varphi$-evolution scheme

We now recall the nonlinear iteration scheme of [JZ1]. Setting

$$N(x,t) = (Q_x + R_x + (\partial_x^2 + \partial_t)S)(x,t),$$

and applying Duhamel’s principle to (5.3), we obtain the integral representation of $v$

$$v(x,t) = -\bar{u}'(x)\varphi(x,t) + \int_{-\infty}^{\infty} G(x,t;y)v_0(y)dy + \int_{0}^{t} \int_{-\infty}^{\infty} G(x,t-s;y)N(y,s)dyds.$$

for the nonlinear perturbation $v$. Defining $\varphi$ implicitly by

$$\varphi(x,t) := \int_{-\infty}^{\infty} e(x,t;y)v_0(y)dy + \int_{0}^{t} \int_{-\infty}^{\infty} e(x,t-s;y)N(y,s)dyds$$

to subtract out $E(x,t;y)$ from $G(x,t;y)$, we have the new integral representation of $v$

$$(5.7) \quad v(x,t) = \int_{-\infty}^{\infty} \tilde{G}(x,t;y)v_0(y)dy + \int_{0}^{t} \int_{-\infty}^{\infty} \tilde{G}(x,t-s;y)N(y,s)dyds.$$

Differentiating and using $e(x,t;y) = 0$ for $0 < t \leq 1$ we obtain

$$(5.8) \quad \partial_t^{k} \partial_x^m \varphi(x,t) = \int_{-\infty}^{\infty} \partial_t^{k} \partial_x^m e(x,t;y)v_0(y)dy + \int_{0}^{t} \int_{-\infty}^{\infty} \partial_t^{k} \partial_x^m e(x,t-s;y)N(y,s)dyds.$$

Together, (5.7) and (5.8) form a complete system in $(v, \partial_t^k \varphi, \partial_x^m \varphi), 0 \leq k \leq 1, 0 \leq m \leq 2,$

that is, $v$ and derivatives of $\varphi$, from solutions of which we may afterward recover the shift function $\varphi$ by integration in $x$, completing the description of $\bar{u}$. 
5.2 Pointwise description of $v$ for initial perturbations $|v_0(x)| \leq E_0(1 + |x|)^{-\frac{3}{2}}$ with $|v_0(x)|_{H^2} \leq E_0$, sufficiently small $E_0 > 0$

In this section, we prove Theorem 1.6. We start with $L^p$ estimates of $v$, $u$ and $\varphi$ which are proved in [JZ1]. We state the main theorem of [JZ1] describing the $L^p$ stability of periodic standing waves of (1.1) in dimension $d = 1$. We use the following Theorem 5.2 when we derive pointwise estimates of the nonlinear terms of $v$ in (5.7), and this is the reason why we need $H^2$ condition in our initial perturbations.

**Theorem 5.2 (Nonlinear stability, [JZ1]).** Let $v(x,t)$ and $u(x,t)$ be defined as in (5.1) and $|u_0(x)| = |v_0(x)|_{L^1 \cap H^2(\mathbb{R})} \leq E_0$, for sufficiently small $E_0 > 0$. Then for all $t \geq 0$ and $p \geq 1$ we have the estimates

$$|v(\cdot,t)|_{L^p(\mathbb{R})}(t) \leq CE_0(1 + t)^{-\frac{1}{p}(1 - \frac{1}{d})},$$

$$|u(\cdot,t)|_{L^p(\mathbb{R})}(t), \quad |\varphi(\cdot,t)|_{L^p(\mathbb{R})}(t) \leq CE_0(1 + t)^{-\frac{1}{p}(1 - \frac{1}{d}) + \frac{1}{2}},$$

$$|v(\cdot,t)|_{H^2(\mathbb{R})}(t), \quad |(\varphi_t, \varphi_x)(\cdot,t)|_{H^2(\mathbb{R})}(t) \leq CE_0(1 + t)^{-\frac{1}{2}}.$$

**Proof.** See [JZ1] for the proof. \hfill $\square$

To prove Theorem 1.6, we first prove the following lemma. We follow the strategy of [HZ]. We give here details of [HZ] to help clarify that argument.

**Lemma 5.3.** Suppose that the initial perturbation $v_0$ satisfies $|v_0(x)| \leq E_0(1 + |x|)^{-\frac{3}{2}}$ and $|v_0(x)|_{H^2} \leq E_0$, for sufficiently small $E_0 > 0$. For $v$, $\varphi_t$, $\varphi_x$ and $\varphi_{xx}$ defined in the integral system (5.7) - (5.8), define

$$(5.9) \quad \zeta(t) := \sup_{0 \leq s \leq t, x \in \mathbb{R}} |(v, \varphi_t, \varphi_x, \varphi_{xx})(x,s)|(\theta + \psi_1 + \psi_2)^{-1},$$

where

$$\theta(x,t) := \sum_{j=1}^{n+1} (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-a_jt|^2}{M't}},$$

$$\psi_1(x,t) := \chi(x,t) \sum_{j=1}^{n+1} (1 + |x| + t)^{-\frac{1}{2}} (1 + |x - a_jt|)^{-\frac{1}{2}},$$

and

$$\psi_2(x,t) := (1 - \chi(x,t))(1 + |x - a_1t| + \sqrt{t})^{-\frac{a}{2}} + (1 - \chi(x,t))(1 + |x - a_{n+1}t| + \sqrt{t})^{-\frac{3}{2}},$$

where $\chi(x,t) = 1$ for $x \in [a_1t, a_{n+1}t]$ and zero otherwise, and $M' > M$. Then, for all $t \geq 0$ for which $\zeta(t)$ defined in (5.9) is finite,

$$(5.10) \quad \zeta(t) \leq C(E_0 + \zeta^2(t))$$

for some constant $C > 0$. 
Proof. It is enough to estimate \( v \),
\[
|v(x, t)| \leq CE_0 + \zeta^2(t)(\theta + \psi_1 + \psi_2). \tag{5.11}
\]
We can prove similarly for \((\varphi_t, \varphi_x, \varphi_{xx})\) because
\[
|\partial_t^k \partial_x^m e(x; t; y)| \lesssim |\hat{G}(x; t; y)| \quad \text{and} \quad |\partial_y(\partial_x^k \partial_x^m e(x; t; y))| \lesssim |\partial_y \hat{G}(x; t; y)|,
\]
for \(0 \leq k \leq 1\) and \(0 \leq m \leq 2\). Notice first that by Theorem 5.2, we have \(|v_x|_{\infty} \leq |v|_{H^2} \leq CE_0(1 + t)^{-4} \leq C\). Then by (5.4)-(5.6) and (5.9), for all \(y \in \mathbb{R}\) and \(0 \leq s \leq t\),
\[
|(Q, R, S)(y, s)| \leq C|(v, \varphi_s, \varphi_y, \varphi_{yy})(y, s)|^2C \leq \zeta^2(t)(\theta + \psi_1 + \psi_2)^2,
\]
and hence by (5.7) and integration by parts, we have
\[
|v(x, t)| \leq \int_{-\infty}^{\infty} |\hat{G}(x, t; y)||v_0(y)|dy + \int_0^t \int_{-\infty}^{\infty} |\hat{G}_y(x, t - s; y)||(v, \varphi_s, \varphi_y, \varphi_{yy})(y, s)|^2dyds
\leq \int_{-\infty}^{\infty} |\hat{G}(x, t; y)||v_0(y)|dy + \zeta^2(t) \int_0^t \int_{-\infty}^{\infty} |\hat{G}_y(x, t - s; y)||\theta + \psi_1 + \psi_2|^2dyds.
\]
To argue (5.11), we need to prove following estimates:
\[
\int_{-\infty}^{\infty} |\hat{G}(x, t; y)||v_0(y)|dy \leq CE_0(\theta + \psi_1 + \psi_2)(x, t), \tag{5.12}
\]
\[
\int_0^t \int_{-\infty}^{\infty} |\hat{G}_y(x, t - s; y)||\theta(y, s)|^2dyds \leq C(\theta + \psi_1 + \psi_2)(x, t), \tag{5.13}
\]
\[
\int_0^t \int_{-\infty}^{\infty} |\hat{G}_y(x, t - s; y)||\psi_1(y, s)|^2dyds \leq C(\theta + \psi_1 + \psi_2)(x, t), \tag{5.14}
\]
\[
\int_0^t \int_{-\infty}^{\infty} |\hat{G}_y(x, t - s; y)||\psi_2(y, s)|^2dyds \leq C(\theta + \psi_1 + \psi_2)(x, t). \tag{5.15}
\]

**Proof of the estimate (5.12).** We start with the linear estimate of \( v \),
\[
\int_{-\infty}^{\infty} |\hat{G}(x, t; y)||v_0(y)|dy \leq CE_0(\theta + \psi_1 + \psi_2)(x, t).
\]
By [J1] and [HZ], we have
\[
\int_{-\infty}^{\infty} |\hat{G}(x, t; y)||v_0(y)|dy \leq CE_0 \int_{-\infty}^{\infty} t^{-\frac{3}{2}} \sum_{j=1}^{n+1} e^{-\frac{|x-y-a_j t|^2}{4t}}(1 + |y|)^{-\frac{3}{2}}dy
\leq CE_0 \sum_{j=1}^{n+1} \left[(1 + |x - a_j t| + \sqrt{t})^{-\frac{3}{2}} + (1 + t)^{-\frac{3}{2}}e^{-\frac{|x-a_j t|^2}{4t}}\right].
\]
We consider several cases. Here we assume $a_1 < a_2 < \cdots < a_{n+1}$.

**case1.** $x \leq a_1 t$ or $x \geq a_{n+1} t$. For any $j = 1, \cdots, n+1$, $|x - a_j t| = |x - a_j| t$ for $x \leq a_1 t$ and $|x - a_{n+1} t| = |x - a_j| t$, for $x \geq a_{n+1} t$. Thus

$$
(1 + |x - a_j t| + \sqrt{t})^{-\frac{3}{2}} \leq (1 + |x - a_1 t| + \sqrt{t})^{-\frac{3}{2}}, \quad \text{for } x \leq a_1 t, \\
$$

and

$$
(1 + |x - a_j t| + \sqrt{t})^{-\frac{3}{2}} \leq (1 + |x - a_{n+1} t| + \sqrt{t})^{-\frac{3}{2}}, \quad \text{for } x \geq a_{n+1} t.
$$

**case2.** $x \in [a_1 t, a_{n+1} t]$, and $x$ and $a_j$ have opposite signs. In this case, $|x - a_j t| \leq C(|x| + t)$ because of no cancellation. So

$$
(1 + |x - a_j t| + \sqrt{t})^{-\frac{3}{2}} = (1 + |x - a_j t|)\frac{1}{2}(1 + |x - a_j t|)^{-\frac{3}{2}} \leq C(1 + |x| + t)^{-\frac{3}{2}}(1 + |x - a_j t|)^{-\frac{3}{2}}
$$

**case3.** $x \in [a_1 t, a_{n+1} t]$, and $x$ and $a_j$ have same signs. If $x \in [\frac{a_1}{2} t, 2a_j t]$, then $t^{-\frac{1}{2}} \leq C(|x| + t)^{-\frac{3}{2}}$, so

$$
(1 + |x - a_j t| + \sqrt{t})^{-\frac{3}{2}} \leq C(1 + |x - a_j t|)\frac{1}{2}(1 + \sqrt{t})^{-1} \leq C(1 + |x - a_j t|)^{-\frac{1}{2}}(1 + |x - a_j t|)^{-\frac{3}{2}} \leq C(1 + |x - a_j t|)^{-\frac{1}{2}}(1 + |x|)^{-\frac{1}{2}}.
$$

If $x \notin [\frac{a_1}{2} t, 2a_j t]$, there can be only limited cancellation between $x$ and $a_j t$, and so $|x - a_j t| \leq C(|x| + t)$, that is,

$$
(1 + |x - a_j t| + \sqrt{t})^{-\frac{3}{2}} \leq C(1 + |x - a_j t| + \sqrt{t})^{-\frac{3}{2}}(1 + |x - a_j t|)^{-\frac{3}{2}} \leq C(1 + |x - a_j t| + \sqrt{t})^{-\frac{3}{2}}(1 + |x| + t)^{-\frac{1}{2}}.
$$

**Proof of the estimate (5.13).** We now estimate the first nonlinear term of $v$,

$$
I = \int_0^t \int_{-\infty}^\infty |\tilde{G}_y(x, t - s; y)| \theta(y, s) dy ds \leq C(\theta + \psi_1 + \psi_2)(x, t).
$$
By (5.2),

\[
I \leq \int_0^t \int_{-\infty}^{\infty} (t - s)^{-1} \sum_{j=1}^{n+1} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}} \theta^2(y, s) dy ds
\]

\[
\leq C \int_0^t \int_{-\infty}^{\infty} (t - s)^{-1} (1 + s)^{-1} \sum_{j,k=1}^{n+1} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}} e^{-\frac{2|y-a_k|s|^2}{M's}} dy ds
\]

\[
\leq C \int_0^t \int_{-\infty}^{\infty} (t - s)^{-1} (1 + s)^{-1} \sum_{j,k=1}^{n+1} e^{-\frac{|x-y-a_j(t-s)|^2}{N(t-s)}} e^{-\frac{|y-a_k|s|^2}{N's}} dy ds,
\]

for sufficiently large \(N > 0\) with \(\frac{M}{2} < N < M'\). Noting first that for any \(j = k\),

\[
\int_{-\infty}^{\infty} e^{-\frac{|x-y-a_j(t-s)|^2}{N(t-s)}} e^{-\frac{|y-a_k|s|^2}{N(t-s)}} dy \leq C(1+t)^{-\frac{1}{2}}(t-s)^{\frac{1}{2}}(1+s)^{\frac{1}{2}} e^{-\frac{|x-a_jt|^2}{N(1+t)}},
\]

we have

\[
I \leq C \sum_{j=1}^{n+1} (1+t)^{-\frac{1}{2}} e^{-\frac{|x-a_jt|^2}{M'(1+t)}} \int_0^t (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{1}{2}} ds \leq C \sum_{j=1}^{n+1} (1+t)^{-\frac{1}{2}} e^{-\frac{|x-a_jt|^2}{M'(1+t)}}.
\]

We now assume \(j \neq k\). Noting first that for \(j \neq k\),

\[
\int_{-\infty}^{\infty} e^{-\frac{|x-y-a_j(t-s)|^2}{N(t-s)}} e^{-\frac{|y-a_k|s|^2}{N(t-s)}} dy \leq C(1+t)^{-\frac{1}{2}}(t-s)^{\frac{1}{2}}(1+s)^{\frac{1}{2}} e^{-\frac{|x-a_j(t-s)-a_k|^2}{N(1+t)}},
\]

we have

\[
(5.16) \quad I \leq C \sum_{j \neq k} (1+t)^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{1}{2}} e^{-\frac{|x-a_j(t-s)-a_k|^2}{N(1+t)}} ds = I'.
\]

To estimate the right hand side \(I'\) of (5.16), we consider 6 cases only with assumption \(x \leq 0\). The case \(x \geq 0\) is entirely symmetric.

**Case 1.** \(x \leq 0\) and \(0 \leq a_j < a_k\). In this case, we can rewrite

\[
(5.17) \quad x - a_j(t-s) - a_k s = (x - a_j t) + (a_j - a_k) s.
\]

Here, \(x - a_j t\) and \((a_j - a_k) s\) are both negative and there is no cancellation, so we have

\[
I \leq C \sum_{j=1}^{n+1} (1+t)^{-\frac{1}{2}} e^{-\frac{|x-a_jt|^2}{M'(1+t)}} \int_0^t (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{1}{2}} ds
\]

\[
\leq C \sum_{j=1}^{n+1} (1+t)^{-\frac{1}{2}} e^{-\frac{|x-a_jt|^2}{M'(1+t)}}.
\]
Case 2. $x \leq 0$ and $0 \leq a_k < a_j$. This is exactly same as the case 1 with rewriting
\begin{equation}
(5.19) \quad x - a_j(t - s) - a_k s = (x - a_k t) - (a_j - a_k)(t - s).
\end{equation}

Case 3. $x \leq 0$ and $a_k < 0 \leq a_j$. In this case, we consider two subcases $|x| \geq |a_k| t$ and $|x| \leq |a_k| t$. For $|x| \geq |a_k| t$, $x - a_k t$ and $-(a_j - a_k)(t - s)$ are both negative and no cancellation occurs in (5.19), so we have the same estimate as (5.18). In the event $|x| \leq |a_k| t$, we integrate $I'$ separately $[0, t/2]$ and $[t/2, t]$. For $s \in [0, t/2]$, since $x - a_j t$ is negative and $(a_j - a_k)s$ is positive in (5.17), cancellation occurs. In this case, we use the following balance estimate:
\begin{align}
(1 + s)^{-\frac{1}{2}} & e^{-\frac{|x - a_j(t - s) - a_k s|^2}{N(1+t)}} \\
& \leq C \left[ (1 + |x - a_j t|)^{-\frac{1}{2}} e^{-\frac{|x - a_j(t - s) - a_k s|^2}{N(1+t)}} + (1 + s)^{-\frac{1}{2}} e^{-\frac{|x - a_j|^2}{M'(1+t)}} \right].
\end{align}

We can easily prove this by considering two cases $(a_j - a_k)s \geq C|x - a_j t|$ and $(a_j - a_k)s \leq C|x - a_j t|$ for some constant $C > 0$ in the relation (5.17). So we have,
\begin{align}
I' & \leq C(1 + t)^{-\frac{1}{2}} (1 + |x - a_j t|)^{-\frac{1}{2}} \int_{0}^{t/2} (t - s)^{-\frac{1}{2}} e^{-\frac{|x - a_j(t - s) - a_k s|^2}{N(1+t)}} ds \\
& \quad + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x - a_j|^2}{M'(1+t)}} \int_{0}^{t/2} (t - s)^{-\frac{1}{2}} (1 + s)^{-\frac{1}{2}} ds \\
& \leq C \left[ (1 + |x| + t)^{-\frac{1}{2}} (1 + |x - a_j t|)^{-\frac{1}{2}} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x - a_j|^2}{M'(1+t)}} \right].
\end{align}

Here, the last inequality is from $|x| \leq |a_k| t$. In the case $s \in [t/2, t]$, we start with rewriting (5.19). Since $x - a_k t$ and $-(a_j - a_k)(t - s)$ have opposite signs in (5.19), we argue similarly the balance estimate for $(t - s)^{-\frac{1}{2}} e^{-\frac{|x - a_j(t - s) - a_k s|^2}{N(1+t)}}$. Thus we have
\begin{align}
I' & \leq C(1 + t)^{-\frac{1}{2}} (1 + |x - a_k t|)^{-\frac{1}{2}} \int_{t/2}^{t} (1 + s)^{-\frac{1}{2}} e^{-\frac{|x - a_j(t - s) - a_k s|^2}{N(1+t)}} ds \\
& \quad + C(1 + t)^{-\frac{1}{2}} e^{-\frac{|x - a_k|^2}{M'(1+t)}} \int_{t/2}^{t} (t - s)^{-\frac{1}{2}} (1 + s)^{-\frac{1}{2}} ds \\
& \leq C \left[ (1 + |x| + t)^{-\frac{1}{2}} (1 + |x - a_k t|)^{-\frac{1}{2}} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x - a_k|^2}{M'(1+t)}} \right].
\end{align}
Case 4. \( x \leq 0 \) and \( a_j < 0 \leq a_k \). This is exactly same as the case 3 by considering 
\(|x| \geq |a_j|t \) and \(|x| \leq |a_j|t \).

Case 5. \( x \leq 0 \) and \( a_k < a_j < 0 \). In this case, we consider 3 subcases, \(|x| \geq |a_k|t \), 
\(|x| \leq |a_j|t \) and \(|a_j|t \leq |x| \leq |a_k|t \). For \(|x| \geq |a_k|t \) and \(|x| \leq |a_j|t \), we use (5.19) and (5.17) 
respectively because the expression \( x - a_j(t - s) - a_k s \) has no cancellation. In the event 
that \(|a_j|t \leq |x| \leq |a_k|t \), we use the balance estimate for \( s \in [0, t/2] \) and \( s \in [t/2, t] \) similarly 
to (5.20) and (5.21), respectively.

Case 6. \( x \leq 0 \) and \( a_j < a_k < 0 \). This is exactly same as the case 5 by considering 
\(|x| \geq |a_j|t \), \(|x| \leq |a_k|t \) and \(|a_k|t \leq |x| \leq |a_j|t \).

\[ \square \]

Proof of the estimate (5.14). We now estimate the second nonlinear term of \( v \),

\[
II = \int_0^t \int_{-\infty}^{\infty} |\tilde{G}_y(x, t - s; y)||\psi_1(y, s)|^2dyds \leq C E_0(\theta + \psi_1 + \psi_2)(x, t).
\]

Notice first that

\[
II \leq \int_0^t \int_{-\infty}^{\infty} (t - s)^{-1} \sum_{j=1}^{n+1} e^{-\frac{|x - y - a_j(t - s)|^2}{M(t - s)}} |\psi_1(y, s)|^2dyds
\]

\[
\leq \int_0^t \int_{-\infty}^{\infty} (t - s)^{-1} \sum_{j=1}^{n+1} e^{-\frac{|x - y - a_j(t - s)|^2}{M(t - s)}} \left[ \chi(y, s) \sum_{k=1}^{n+1} (1 + |y| + s)^{-\frac{1}{2}} (1 + |y - a_k s|)^{-\frac{1}{2}} \right]^2dyds.
\]

It is enough to estimate

\[
II' = \sum_{j,k=1}^{n+1} \int_0^t \int_{a_j s}^{a_{j+1} s} (t - s)^{-1} e^{-\frac{|x - y - a_j(t - s)|^2}{M(t - s)}} (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-1}dyds.
\]

We estimate \( II' \) by considering three parts: \( x < a_1 t \), \( x > a_{n+1} t \) and \( x \in [a_1 t, a_{n+1} t] \).

For \( x < a_1 t \), we use \( x - y - a_j(t - s) = (x - a_1 t) - (y - a_1 s) - (a_j - a_1)(t - s) \) for 
\( y \in [a_1 s, a_j s] \) and \( x - y - a_j(t - s) = (x - a_1 t) - (y - a_j s) - (a_j - a_1) t \) for \( y \in [a_j s, a_{j+1} s] \).
so that they have no cancellation. Thus, we have

\[(5.22)\]
\[
\int_0^t \int_{a_1 s}^{a_0 + 1 s} (t - s)^{-1} e^{-\frac{|x - y - a_j(t - s)|^2}{2M(t - s)}} (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-1} dy ds \\
\leq e^{-\frac{|x - a_1|^2}{2M(t)}} \int_0^t \int_{a_1 s}^{a_0 + 1 s} (t - s)^{-1} e^{-\frac{|x - y - a_j(t - s)|^2}{2M(t - s)}} (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-1} dy ds \\
+ e^{-\frac{|x - a_1|^2}{2M(t)}} \int_0^t \int_{a_1 s}^{a_0 + 1 s} (t - s)^{-1} e^{-\frac{|x - y - a_j(t - s)|^2}{2M(t - s)}} (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-1} dy ds \\
\leq e^{-\frac{|x - a_1|^2}{2M(t)}} \int_0^t \int_{a_1 s}^{a_0 + 1 s} (t - s)^{-1} e^{-\frac{|x - y - a_j(t - s)|^2}{2M(t - s)}} (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-1} dy ds \\
\leq (1 + t)^{-\frac{1}{2}} e^{-\frac{|x - a_1|^2}{2M(t)}}.
\]

To argue the final inequality, let’s show the following estimate with an assumption \(a_1 \leq 0\),

\[(5.23)\]
\[
\int_0^t \int_{a_1 s}^0 (t - s)^{-1} (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-\frac{1}{2}} e^{-\frac{|x - y - a_j(t - s)|^2}{2M(t - s)}} dy ds \leq (1 + t)^{-\frac{1}{2}}.
\]

If \(a_k > 0\), then

\[
\int_0^t \int_{a_1 s}^0 (t - s)^{-1} (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-\frac{1}{2}} e^{-\frac{|x - y - a_j(t - s)|^2}{2M(t - s)}} dy ds \\
\leq \int_0^t \int_{a_1 s}^{t/2} (t - s)^{-1} (1 + |y| + s)^{-\frac{1}{2}} e^{-\frac{|x - y - a_j(t - s)|^2}{2M(t - s)}} dy ds \\
\leq C \int_0^{t/2} (t - s)^{-\frac{1}{2}} (1 + s)^{-\frac{1}{2}} \int_{a_1 s}^0 (t - s)^{-\frac{1}{2}} e^{-\frac{|x - y - a_j(t - s)|^2}{2M(t - s)}} dy ds \\
\leq (1 + t)^{-\frac{1}{2}}.
\]

If \(a_k \leq 0\), then

\[
\int_0^t \int_{a_1 s}^0 (t - s)^{-1} (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-\frac{1}{2}} e^{-\frac{|x - y - a_j(t - s)|^2}{2M(t - s)}} dy ds \\
\leq C \int_0^{t/2} (t - s)^{-1} (1 + s)^{-1} \int_{a_1 s}^0 (1 + |y - a_k s|)^{-\frac{1}{2}} dy ds \\
+ \int_{t/2}^t (t - s)^{-\frac{1}{2}} (1 + s)^{-\frac{1}{2}} \int_{0}^{t/2} (t - s)^{-\frac{1}{2}} e^{-\frac{|x - y - a_j(t - s)|^2}{2M(t - s)}} dy ds \\
\leq C \int_0^{t/2} (t - s)^{-1} (1 + s)^{-\frac{1}{2}} ds + \int_{t/2}^t (t - s)^{-\frac{1}{2}} (1 + s)^{-1} ds \\
\leq C(1 + t)^{-\frac{1}{2}}.
\]
Similarly, we can prove
\[
\int_0^t \int_0^{a_{n+1}s} (t-s)^{-1}(1+|y|+s)^{-1}(1+|y-a_ks|)^{-\frac{1}{2}} e^{-\frac{|y-a_j(t-s)|^2}{2M(t-s)}} \ dyds \leq C(1+t)^{-\frac{1}{2}},
\]
with an assumption \(a_{n+1} \geq 0\).

For \(x > a_{n+1}t\), we use \(x - y - a_j(t-s) = (x - a_{n+1}t) - (y - a_j) + (a_{n+1} - a_j)t\) for \(y \in [a_1s, a_j]\) and \(x - y - a_j(t-s) = (x - a_{n+1}t) - (y - a_{n+1}s) - (a_j - a_{n+1})(t-s)\) for \(y \in [a_j, a_{n+1}s]\). Then we argue similarly to estimate (5.22).

We now assume \(x \in [a_1t, a_{n+1}t]\) with \(a_1 < 0\) and \(a_{n+1} > 0\). We estimate \(\Pi'\) into two parts,
\[
\Pi'_N = \int_0^t \int_{a_1s}^{a_{n+1}s} \quad \text{and} \quad \Pi'_P = \int_0^t \int_0^{a_{n+1}s} \ .
\]
This is why we can assume \(a_1 \leq 0\) and \(a_{n+1} \geq 0\). For \(j = k\) which is a simple case, we first notice that
\[
(1+|y-a_j|)^{-\frac{1}{2}}e^{-\frac{|y-a_j(t-s)|^2}{M(t-s)}} \leq (1+|x-a_j|)^{-\frac{1}{2}}e^{-\frac{|x-a_j|^2}{M(t-s)}} + (1+|y-a_j|)^{-\frac{1}{2}}e^{-\frac{|y-a_j(t-s)|^2}{bM(t-s)}},
\]
for some constant \(b > 0\). Then we have
\[
\Pi'_N \leq C(1+|x-a_j|)^{-\frac{1}{2}} \int_0^t \int_{a_1s}^{a_{n+1}s} (t-s)^{-1}(1+|y|+s)^{-1}(1+|y-a_j|)^{-\frac{1}{2}}e^{-\frac{|y-a_j(t-s)|^2}{M(t-s)}} \ dyds
+ Ce^{-\frac{|x-a_j|^2}{M(t-s)}} \int_0^t \int_{a_1s}^{a_{n+1}s} (t-s)^{-1}(1+|y|+s)^{-1}(1+|y-a_j|)^{-1}e^{-\frac{|y-a_j(t-s)|^2}{bM(t-s)}} \ dyds
\leq C \left[(1+|x-a_j|)^{-\frac{1}{2}} + e^{-\frac{|x-a_j|^2}{M(t-s)}}\right] 
\times \int_0^t \int_{a_1s}^{a_{n+1}s} (t-s)^{-1}(1+|y|+s)^{-1}(1+|y-a_j|)^{-\frac{1}{2}}e^{-\frac{|y-a_j(t-s)|^2}{bM(t-s)}} \ dyds
\leq C \left[(1+|x-a_j|)^{-\frac{1}{2}} + e^{-\frac{|x-a_j|^2}{M(t-s)}}\right](1+t)^{-\frac{1}{2}}.
\]
Here, we already proved the last inequality in (5.23). Similarly, we estimate \(\Pi'_P\).

Let’s estimate \(\Pi'\) for \(j \neq k\) with \(a_k < 0\) for the case of \(a_k > 0\), we can estimate \(\Pi'_P\) similarly to \(\Pi'_N\) in the case \(a_k < 0\) and estimate \(\Pi'_N\) similarly to \(\Pi'_P\) in the case \(a_k < 0\). It is easy to estimate \(\Pi'_P\) while we have to consider several cases again for \(\Pi'_N\). For \(\Pi'_P\), since \(a_k < 0\) and \(y \geq 0\), we say that \(1+|y-a_k| \sim 1+|y|+s\), and so we have
\[
\Pi'_P \leq \sum_{j=1}^{n+1} \int_0^t \int_0^{a_{n+1}s} (t-s)^{-1}e^{-\frac{|y-a_j(t-s)|^2}{M(t-s)}}(1+|y|+s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}}dyds.
\]
Noting first that

\[(1 + |y| + s)^{-\frac{1}{2}} e^{-\frac{|x - a_j(t-s)|^2}{M(t-s)}} \leq (1 + |x - a_j(t-s)| + s)^{-\frac{1}{2}} e^{-\frac{|x - a_j(t-s)|^2}{M(t-s)}} + (1 + |y| + s)^{-\frac{1}{2}} e^{-\frac{|x - a_j(t-s)|^2}{M'(t-s)}} e^{-\frac{|x - y - a_j(t-s)|^2}{bM(t-s)}},\]

we have

\[II_p' \leq \int_0^t \int_0^{a_{n+1}s} (t-s)^{-1}(1 + |x - a_j(t-s)| + s)^{-\frac{1}{2}} (1 + s)^{-\frac{3}{2}} e^{-\frac{|x - a_j(t-s)|^2}{M(t-s)}} dyds + \int_0^t \int_0^{a_{n+1}s} (t-s)^{-1}(1 + |y| + s)^{-\frac{1}{2}} (1 + s)^{-\frac{3}{2}} e^{-\frac{|x - a_j(t-s)|^2}{M'(t-s)}} dyds,
\]

\[= A + B.
\]

For \(B\),

\[B \leq \int_0^{t/2} (t-s)^{-1}(1 + s)^{-1} e^{-\frac{|x - a_j(t-s)|^2}{M(t-s)}} ds + \int_0^t (t-s)^{-\frac{1}{2}} (1 + s)^{-\frac{3}{2}} e^{-\frac{|x - a_j(t-s)|^2}{M'(t-s)}} ds \leq (1 + t)^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (1 + s)^{-\frac{3}{2}} e^{-\frac{|x - a_j(t-s)|^2}{M'(t-s)}} ds \leq C1',\]

which is estimated in the proof of (5.13). For \(A\), noting first that

\[(1 + |x - a_j(t-s)| + s)^{-\frac{1}{2}} \leq C(1 + |x - a_j(t-s)| + s)^{-\frac{1}{2}} + C(1 + |x - a_j(t-s)| + |x - a_j(t)|)^{-\frac{1}{2}} \leq C(1 + |x - a_j(t)|)^{-\frac{1}{2}},\]

we have

\[A \leq (1 + |x - a_j(t)|)^{-\frac{3}{2}} \int_0^t \int_0^{a_{n+1}s} (t-s)^{-1}(1 + s)^{-\frac{3}{2}} e^{-\frac{|x - y - a_j(t-s)|^2}{bM(t-s)}} dyds \leq (1 + |x - a_j(t)|)^{-\frac{3}{2}} \int_0^t (t-s)^{-\frac{3}{2}} (1 + s)^{-\frac{3}{2}} ds \leq C(1 + t)^{-\frac{3}{2}} (1 + |x - a_j(t)|)^{-\frac{3}{2}}.
\]

We now estimate \(II_N'\). To estimate this part, we agree several cases. We try here only the case \(x < 0\) and \(a_k < 0 < a_j\). We can agree similarly other cases. Using \(x - y - a_j(t-s) = (x - a_j(t-s) - a_k s) - (y - a_k s)\), we have
5  POINTWISE DESCRIPTION OF PERTURBATIONS OF $\bar{U}$

$$(1 + |y - a_k s|)^{-\frac{1}{2}} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}}$$

$$\leq C(1 + |x - a_j(t - s) - a_k s|)^{-\frac{1}{2}} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}}$$

$$+ (1 + |y - a_k s|)^{-\frac{1}{2}} e^{-\frac{|x-y-a_j(t-s)|^2}{N(t-s)}} e^{-\frac{|x-y-a_j(t-s)|^2}{bM(t-s)}},$$

for some constant $b > 0$. Thus,

$$II_N' \leq \int_0^t \int_{a_1 s}^0 (t-s)^{-1} (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-\frac{1}{2}}$$

$$\times (1 + |x - a_j(t - s) - a_k s|)^{-\frac{1}{2}} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}} dyds$$

$$+ \int_0^t \int_{a_1 s}^0 (t-s)^{-1} (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-1}$$

$$\times e^{-\frac{|x-a_j(t-s)-a_k s|^2}{N(t-s)}} e^{-\frac{|x-y-a_j(t-s)|^2}{bM(t-s)}} dyds$$

$$= A + B.$$

For $B$,

$$B \leq \int_0^{t/2} (t-s)^{-1} e^{-\frac{|x-a_j(t-s)-a_k s|^2}{N(t-s)}} \int_{a_1 s}^0 (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-1} dyds$$

$$+ \int_{t/2}^t (t-s)^{-1} e^{-\frac{|x-a_j(t-s)-a_k s|^2}{M(t-s)}} \int_{a_1 s}^0 (1 + |y| + s)^{-1} e^{-\frac{|x-y-a_j(t-s)|^2}{bM(t-s)}} dyds$$

$$\leq C \int_0^{t/2} (t-s)^{-1} e^{-\frac{|x-a_j(t-s)-a_k s|^2}{N(t-s)}} (1 + s)^{-1} \ln(1 + s) ds$$

$$+ \int_{t/2}^t (t-s)^{-\frac{1}{2}} e^{-\frac{|x-a_j(t-s)-a_k s|^2}{M(t-s)}} (1 + s)^{-1} ds$$

$$\leq C(1 + t)^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} (1 + s)^{-\frac{1}{2}} e^{-\frac{|x-a_j(t-s)-a_k s|^2}{M(t-s)}} ds$$

$$\leq C(\theta + \psi_1 + \psi_2)(x,t).$$

Here, the last inequality is proved in $I'$. To estimate $A$, we separate $A$ into two parts $|x| \geq |a_k|t$ and $|x| \leq |a_k|t$. For $|x| \geq |a_k|t$, using

$$x - a_j(t - s) - a_k s = (x - a_k t) - (a_j - a_k)(t - s),$$
for which there is no cancellation, we have

\[ A \leq (1 + \|x - a_k t\|)^{-\frac{1}{2}} \int_0^t (t - s)^{-1} \int_{a_k s}^t (1 + |y| + s)^{-1} (1 + |y - a_k s|)^{-\frac{1}{2}} e^{-\frac{|x - y - a_j (t-s)|^2}{M(t-s)}} dyds \]

\[ \leq (1 + \|x - a_k t\|)^{-\frac{1}{2}} \left[ \int_0^{t/2} (t - s)^{-1} (1 + s)^{-\frac{1}{2}} (1 + s)^{-\frac{1}{2}} ds + \int_{t/2}^t (t - s)^{-1} (1 + s)^{-1} (1 - s)^{-\frac{3}{2}} ds \right] \]

\[ \leq (1 + t)^{-\frac{1}{2}} (1 + \|x - a_k t\|)^{-\frac{1}{4}}. \]

For \(|x| \leq \|a_k t\|\), we divide again the analysis into the cases \(s \in [0, t/2]\) and \(s \in [t/2, t]\). In the case \(s \in [0, t/2]\), using

\[ x - a_j (t - s) - a_k s = (x - a_j t) + (a_j - a_k) s, \]

we have

\[ (1 + \|x - a_j (t - s) - a_k s\|)^{-\frac{1}{2}} (1 + |y| + s)^{-\frac{1}{2}} \]

\[ \leq C[(1 + \|x - a_j t\|)^{-\frac{1}{2}} (1 + |y| + s)^{-\frac{1}{2}} + (1 + \|x - a_j (t - s) - a_k s\|)^{-\frac{1}{2}} (1 + |y| + |x - a_j t|)^{-\frac{1}{2}}]. \]

Thus, we consider \(A\) into two terms \(A'\) and \(A''\). For \(A'\),

\[ A' \leq C(1 + \|x - a_j t\|)^{-\frac{1}{2}} \int_0^{t/2} (t - s)^{-1} \int_{a_k s}^t (1 + |y - a_k s|)^{-\frac{1}{2}} e^{-\frac{|x - y - a_j (t-s)|^2}{M(t-s)}} (1 + |y| + s)^{-1} dyds \]

\[ \leq C(1 + \|x - a_j t\|)^{-\frac{1}{2}} \int_0^{t/2} (t - s)^{-1} (1 + s)^{-1} \int_{a_k s}^t (1 + |y - a_k s|)^{-\frac{1}{2}} e^{-\frac{|x - y - a_j (t-s)|^2}{M(t-s)}} dyds \]

\[ \leq C(1 + \|x - a_j t\|)^{-\frac{1}{2}} \int_0^{t/2} (t - s)^{-1} (1 + s)^{-1} (1 + s)^{-\frac{3}{2}} ds \]

\[ \leq C(1 + t)^{-\frac{1}{2}} (1 + \|x - a_j t\|)^{-\frac{1}{4}}. \]

For \(A''\),

\[ A'' \leq C(1 + \|x - a_j t\|)^{-\frac{1}{2}} \int_0^t (t - s)^{-1} (1 + s)^{-\frac{1}{2}} (1 + \|x - a_j (t - s) - a_k s\|)^{-\frac{1}{2}} (1 + s)^{-\frac{3}{2}} ds \]

\[ \leq C(1 + t)^{-1} (1 + \|x - a_j t\|)^{-\frac{1}{2}} \int_0^{t/2} (1 + |x - a_j (t - s) - a_k s|)^{-\frac{1}{2}} ds \]

\[ \leq C(1 + t)^{-1} (1 + \|x - a_j t\|)^{-\frac{1}{2}} (1 + t)^{\frac{1}{2}} \]

\[ \leq C(1 + t)^{-\frac{3}{4}} (1 + \|x - a_j t\|)^{-\frac{1}{4}}. \]
Here, the inequalities are from $|x| \leq |a_k|t$ and $(|x| + t) \sim |x - a_j|t$ because of $x < 0$ and $a_j > 0$.

In the case $s \in [t/2, t]$, using

$$x - a_j(t-s) - a_k s = (x - a_k t) - (a_j - a_k)(t-s),$$

we have

$$\begin{align*}
(t-s)^{-\frac{1}{2}}(1 + |x - a_j(t-s) - a_k s|)^{-\frac{1}{2}} &
\leq C[|x - a_k t|^{-\frac{1}{2}}(1 + |x - a_j(t-s) - a_k s|)^{-\frac{1}{2}} + (t-s)^{-\frac{1}{2}}(1 + |x - a_k t|)^{-\frac{1}{2}}].
\end{align*}$$

Thus,

$$A \leq C(1 + t)^{-\frac{1}{2}} e^{-\frac{|x-a_k t|^2}{M(t-s)}} + A' + A''.$$

Here, since $|A| \leq C(1 + t)^{-\frac{1}{2}}$, for $|x - a_k t| \leq C \sqrt{t}$, we get the first term. So to estimate $A'$, we assume $|x - a_k t| \geq C \sqrt{t}$ and $t > 1$. If $t \leq 1$, then $|x - a_k t| \geq C \sqrt{t} \geq Ct \geq C(t-s)$ which is a contraction to the expression (5.24). Then, we have

$$A' \leq |x - a_k t|^{-\frac{1}{2}} \int_{t/2}^{t} (1 + s)^{-1} (1 + |x - a_j(t-s) - a_k s|) \cdot \frac{1}{2} \int_{a_k s}^{0} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}} dy ds$$

$$\leq (1 + t)^{-1} (1 + |x - a_k t|)^{-\frac{1}{2}} \int_{t/2}^{t} (1 + |x - a_j(t-s) - a_k s|)^{-\frac{3}{2}} ds$$

$$\leq (1 + t)^{-\frac{1}{2}} (1 + |x - a_k t|)^{-\frac{1}{2}}$$

and

$$A'' \leq (1 + |x - a_k t|)^{-\frac{1}{2}} \int_{t/2}^{t} (1 + s)^{-1} (t-s)^{-\frac{1}{2}} \int_{a_k s}^{0} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}} dy ds$$

$$\leq C(1 + t)^{-\frac{1}{2}} (1 + |x - a_k t|)^{-\frac{1}{2}}.$$

**Remark 5.4.** We argue other cases very similarly. However, for some cases, we need to separate $A$ into three parts, not just two parts. For example, in the case of $a_k < a_j < 0$, we need to consider $A$ by $|x| \geq |a_k|t$, $|x| \leq |a_j|t$ and $|a_j|t \leq |x| \leq |a_k|t.$
Proof of the estimate (5.15). We now estimate the third nonlinear term of $v$,

$$III = \int_0^t \int_{-\infty}^{\infty} |\hat{G}_y(x, t - s; y)||\psi_2(y, s)|^2 \, dy \, ds \leq CE_0(\theta + \psi_1 + \psi_2)(x, t).$$

Notice that

$$III \leq \int_0^t \int_{-\infty}^{\infty} (t - s)^{-1} \sum_{j=1}^{n+1} e^{-\frac{|x - y - a_j(t - s)|^2}{M(t - s)}}$$

$$\times \left[ (1 - \chi(y, s))(1 + |y - a_1s| + \sqrt{s})^{-\frac{3}{2}} + (1 - \chi(y, s))(1 + |y - a_{n+1}s| + \sqrt{s})^{-\frac{3}{2}} \right]^2 \, dy \, ds.$$

We here estimate

$$III' = \int_0^t \int_{-\infty}^{a_1s} (t - s)^{-1} e^{-\frac{|x - y - a_j(t - s)|^2}{M(t - s)}} (1 + |y - a_1s| + \sqrt{s})^{-3} \, dy \, ds.$$

We can argue the other terms similarly. Using

$$x - y - a_j(t - s) = (x - a_j(t - s) - a_1s) - (y - a_1s),$$

we have

$$(1 + |y - a_1s| + \sqrt{s})^{-\frac{3}{2}} e^{-\frac{|x - y - a_j(t - s)|^2}{M(t - s)}}$$

$$\leq (1 + |(x - a_j(t - s) - a_1s)| + \sqrt{s})^{-\frac{3}{2}} e^{-\frac{|x - y - a_j(t - s)|^2}{M(t - s)}}$$

$$+ (1 + |y - a_1s| + \sqrt{s})^{-\frac{3}{2}} e^{-\frac{|x - a_j(t - s) - a_1s|^2}{N(t - s)}} e^{-\frac{|x - y - a_j(t - s)|^2}{bM(t - s)}},$$

for some constant $b > 0$. Thus,

(5.26)

$$III' \leq \int_0^t \int_{-\infty}^{a_1s} (t - s)^{-1} (1 + |y - a_1s| + \sqrt{s})^{-\frac{3}{2}} (1 + |(x - a_j(t - s) - a_1s)| + \sqrt{s})^{-\frac{3}{2}} e^{-\frac{|x - y - a_j(t - s)|^2}{M(t - s)}} \, dy \, ds$$

$$+ \int_0^t \int_{-\infty}^{a_1s} (t - s)^{-1} (1 + |y - a_1s| + \sqrt{s})^{-3} e^{-\frac{|x - a_j(t - s) - a_1s|^2}{N(t - s)}} e^{-\frac{|x - y - a_j(t - s)|^2}{bM(t - s)}} \, dy \, ds.$$
The second term of (5.26) is estimated by $I'$ because

\[
\int_0^t \int_{-\infty}^{a_1s} (t-s)^{-1} (1 + |y-a_1s| + \sqrt{s})^{-\frac{3}{2}} e^{-\frac{|x-y-a_j(t-s)-a_1s|^2}{M(t-s)}} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}} \, dy \, ds
\]

\[
\leq \int_0^{t/2} (t-s)^{-1} e^{-\frac{|x-a_j(t-s)-a_1s|^2}{N(t-s)}} \int_{-\infty}^{a_1s} (1 + |y-a_1s| + \sqrt{s})^{-\frac{3}{2}} \, dy \, ds
\]

\[
+ \int_{t/2}^t (t-s)^{-1} (1 + \sqrt{s})^{-\frac{3}{2}} e^{-\frac{|x-a_j(t-s)-a_1s|^2}{N(t-s)}} \int_{-\infty}^{a_1s} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}} \, dy \, ds
\]

\[
\leq C(1+t)^{-\frac{1}{2}} \int_0^{t/2} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-a_j(t-s)-a_1s|^2}{N(t-s)}} (1 + s)^{-1} \, ds
\]

\[
+ C(1+t)^{-\frac{1}{2}} \int_{t/2}^t (t-s)^{-1} (1 + s)^{-\frac{1}{2}} e^{-\frac{|x-a_j(t-s)-a_1s|^2}{N(t-s)}} (t - s)^{\frac{1}{2}} \, ds
\]

\[
\leq C I'.
\]

We now prove the first term of $III'$:

\[
III' = \int_0^t \int_{-\infty}^{a_1s} (t-s)^{-1} (1 + |y-a_1s| + \sqrt{s})^{-\frac{3}{2}}
\]

\[
\times (1 + |(x-a_j(t-s)-a_1s)| + \sqrt{s})^{-\frac{3}{2}} e^{-\frac{|x-y-a_j(t-s)-a_1s|^2}{M(t-s)}} \, dy \, ds.
\]

For $x < a_1 t$, using

\[
x - a_j(t-s) - a_1s = (x - a_1 t) - (a_j - a_1)(t-s)
\]

for which there is no cancellation, we have

\[
III' \leq \int_0^t \int_{-\infty}^{a_1s} (t-s)^{-1} (1 + |y-a_1s| + \sqrt{s})^{-\frac{3}{2}} (1 + |x-a_1t| + \sqrt{s})^{-\frac{3}{2}} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}} \, dy \, ds
\]

\[
\leq C(1+t)^{-1} (1 + |x-a_1t|)^{-\frac{3}{2}} \int_0^{t/2} (1 + |y-a_1s| + \sqrt{s})^{-\frac{3}{2}} \, dy \, ds
\]

\[
+ (1 + |x-a_1t| + \sqrt{t})^{-\frac{3}{2}} (1 + \sqrt{t})^{-\frac{3}{2}} \int_{t/2}^t \int_{-\infty}^{a_1s} (t-s)^{-1} e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}} \, dy \, ds
\]

\[
\leq C(1+t)^{-1} (1 + |x-a_1t|)^{-\frac{3}{2}} \int_0^{t/2} (1 + s)^{-\frac{1}{2}} \, ds
\]

\[
+ (1 + |x-a_1t| + \sqrt{t})^{-\frac{3}{2}} (1 + \sqrt{t})^{-\frac{3}{2}} \int_{t/2}^t (t-s)^{-\frac{1}{2}} \, ds
\]

\[
\leq C(1+t)^{-\frac{1}{2}} (1 + |x-a_1t|)^{-\frac{3}{2}} + C(1+t)^{-\frac{1}{2}} (1 + |x-a_1t| + \sqrt{t})^{-\frac{3}{2}}
\]

\[
\leq C(1 + |x-a_1t| + \sqrt{t})^{-\frac{3}{2}} + (1+t)^{-\frac{1}{2}} e^{-\frac{|x-a_1t|^2}{M(t)}}.
\]
For the last inequality, we consider two cases \(|x - a_1 t| \leq \sqrt{t}\) and \(|x - a_1 t| \geq \sqrt{t}\). Since 

\(|III''| \leq (1 + t)^{-\frac{1}{2}}\), we have \((1 + t)^{-\frac{1}{2}} e^{-\frac{|x-a_1 t|^2}{M^4 t}}\), for \(|x - a_1 t| \leq \sqrt{t}\). For \(|x - a_1 t| \geq \sqrt{t}\), 

\((1 + |x - a_1 t|)^{-\frac{3}{2}} \leq C(1 + |x - a_1 t| + \sqrt{t})^{-\frac{3}{2}}\).

For \(x > a_{n+1} t\), using 

\[x - a_j (t - s) - a_1 s = (x - a_{n+1} t) - (a_j - a_{n+1})(t - s)\]

for which there is no cancellation, we estimate \(III''\) similarly to \(x < a_1 t\). Thus, for \(x > a_{n+1} t\), we have 

\[III'' \leq C(1 + |x - a_{n+1} t| + \sqrt{t})^{-\frac{3}{2}} + (1 + t)^{-\frac{1}{2}} e^{-\frac{|x-a_{n+1} t|^2}{M^4 t}}\].

Now we consider the last part \(a_1 t < x < a_{n+1} t\). We prove only the case \(x < 0\) and \(a_j \leq 0\). We can prove other cases similarly. In this part, we need to consider two cases.

case1. \(|x| \leq |a_j| t\). Here, we have no cancellation in 

\[x - a_j (t - s) - a_1 s = (x - a_j t) + (a_j - a_1) s\].

Thus, we argue similarly to the event \(x < a_1 t\).

case2. \(|x| \geq |a_j| t\). In this case, we consider \(s \in [0, t/2]\) and \(s \in [t/2, t]\) similarly to the proof of (5.14). For \(s \in [0, t/2]\), noting first that 

\[x - a_j (t - s) - a_1 s = (x - a_j t) + (a_j - a_1) s\],

we have 

\[
(1 + |x - a_j (t - s) - a_1 s| + \sqrt{s})^{-\frac{3}{2}} \\
\leq (1 + |x - a_j t| + \sqrt{s})^{-\frac{3}{2}} + (1 + |x - a_j (t - s) - a_1 s| + |x - a_j t| + \sqrt{s})^{-\frac{3}{2}}.
\]
Thus,

\[
III'' \leq \int_0^{t/2} \int_{-\infty}^{a_1s} (t-s)^{-1}(1+|y-a_1s|+\sqrt{s})^{-\frac{3}{2}}(1+|x-a_2t|+\sqrt{s})^{-\frac{3}{2}}e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}}dyds \\
+ \int_0^{t/2} \int_{-\infty}^{a_1s} (t-s)^{-1}(1+|y-a_1s|+\sqrt{s})^{-\frac{3}{2}}
\times (1+|x-a_j(t-s)-a_1s|+|x-a_2t|^{\frac{3}{2}}+\sqrt{s})^{-\frac{3}{2}}e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}}dyds \\
\leq C \int_0^{t/2} (t-s)^{-1}(1+|x-a_2t|+\sqrt{s})^{-\frac{3}{2}} \int_{-\infty}^{a_1s} (1+|y-a_1s|+\sqrt{s})^{-\frac{3}{2}}dyds \\
+ \int_0^{t/2} (t-s)^{-1}(1+|x-a_2t|^{\frac{3}{2}}+\sqrt{s})^{-\frac{3}{2}} \int_{-\infty}^{a_1s} (1+|y-a_1s|+\sqrt{s})^{-\frac{3}{2}}dyds \\
\leq C(1+|x-a_2t|)^{-\frac{1}{2}} \int_0^{t/2} (t-s)^{-1}(1+\sqrt{s})^{-1}(1+\sqrt{s})^{-\frac{3}{2}}ds \\
+ C(1+|x-a_2t|^{\frac{3}{2}})^{-1} \int_0^{t/2} (t-s)^{-1}(1+\sqrt{s})^{-\frac{3}{2}}(1+\sqrt{s})^{-\frac{3}{2}}ds \\
\leq C(1+t)^{-\frac{1}{2}}(1+|x-a_2t|)^{-\frac{1}{2}} \\
\leq C(1+t+|x|)^{-\frac{1}{2}}(1+|x-a_2t|)^{-\frac{1}{2}}.
\]

For \( s \in [t/2, t] \), noting first that \( x-a_j(t-s)-a_1s = (x-a_1t) - (a_j-a_1)(t-s) \), we have

\[
(t-s)^{-\frac{1}{2}}(1+|x-a_j(t-s)-a_1s|+\sqrt{s})^{-\frac{3}{2}} \\
\leq |x-a_1t|^{-\frac{1}{2}}(1+|x-a_j(t-s)-a_1s|+\sqrt{s})^{-\frac{3}{2}} + (t-s)^{-\frac{1}{2}}(1+|x-a_1t|+\sqrt{s})^{-\frac{3}{2}}.
\]

Thus,

\[
III'' \leq C(1+t)^{-\frac{1}{2}}e^{-\frac{|x-a_j(t-s)|^2}{M}} \\
+ \int_{t/2}^{t} \int_{-\infty}^{a_1s} (t-s)^{-\frac{1}{2}}(1+|y-a_1s|+\sqrt{s})^{-\frac{3}{2}}|x-a_1t|^{-\frac{1}{2}}
\times (1+|x-a_j(t-s)-a_1s|+\sqrt{s})^{-\frac{3}{2}}e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}}dyds \\
+ \int_{t/2}^{t} \int_{-\infty}^{a_1s} (t-s)^{-1}(1+|y-a_1s|+\sqrt{s})^{-\frac{3}{2}}(1+|x-a_1t|+\sqrt{s})^{-\frac{3}{2}}e^{-\frac{|x-y-a_j(t-s)|^2}{M(t-s)}}dyds \\
\leq C(1+t)^{-\frac{1}{2}}e^{-\frac{|x-a_j(t-s)|^2}{M}} + A + B.
\]
By the same argument as (5.25), we assume \(|x - a_1t| \geq C > 0\) for \(A\). Finally, we have

\[
A \leq (1 + |x - a_1t|)^{-\frac{1}{2}} \int_{t/2}^{t} \int_{-\infty}^{a_1s} (t-s)^{-\frac{1}{2}} (1 + |y - a_1s| + \sqrt{s})^{-\frac{3}{2}} \times (1 + |x - a_j(t-s) - a_1s| + \sqrt{s})^{-\frac{1}{2}} e^{-\frac{|x-y-a_j(t-s)|^2}{\mathcal{M}(t-s)}} dy ds \\
\leq (1 + |x - a_1t|)^{-\frac{1}{2}} \int_{t/2}^{t} (1 + \sqrt{s})^{-\frac{3}{2}} (1 + |x - a_1t| + \sqrt{s})^{-\frac{1}{2}} dy ds \\
\times \int_{-\infty}^{a_1s} (t-s)^{-\frac{1}{2}} e^{-\frac{|x-y-a_j(t-s)|^2}{\mathcal{M}(t-s)}} dy ds \\
\leq (1 + \sqrt{t})^{-\frac{3}{2}} (1 + |x - a_1t|)^{-\frac{1}{2}} \int_{t/2}^{t} (1 + |x - a_1(t-s) - a_1s| + \sqrt{s})^{-\frac{3}{2}} ds \\
\leq C (1 + t)^{-\frac{1}{2}} (1 + |x - a_1t|)^{-\frac{1}{2}}
\]

and

\[
B \leq \int_{t/2}^{t} \int_{-\infty}^{a_1s} (t-s)^{-1} (1 + |y - a_1s| + \sqrt{s})^{-\frac{3}{2}} (1 + |x - a_1t| + \sqrt{s})^{-\frac{3}{2}} e^{-\frac{|x-y-a_j(t-s)|^2}{\mathcal{M}(t-s)}} dy ds \\
\leq (1 + |x - a_1t| + \sqrt{t})^{-\frac{3}{2}} \int_{t/2}^{t} (t-s)^{-\frac{1}{2}} (1 + \sqrt{s})^{-\frac{3}{2}} ds \\
\leq (1 + |x - a_1t| + \sqrt{t})^{-\frac{3}{2}} \\
\leq (1 + t)^{-\frac{1}{2}} (1 + |x - a_1t|)^{-\frac{1}{2}}.
\]

Now we complete the proof of (5.12) - (5.15), which implies that we have (5.10).

Proof of theorem (1.6). By Lemma 5.3, we have \(\zeta(t) \leq CE_0 + \zeta^2(t)\) for all \(t \geq 0\) for which \(\zeta(t)\) defined in (5.9) is finite. Since \(\zeta(t)\) is continuous so long as it remains finite, it follows by continuous induction that \(\zeta(t) \leq 2CE_0\) for all \(t \geq 0\) provided \(E_0 \leq \frac{1}{4}\) and (as holds without loss of generality) \(C \geq 1\). Thus, recalling \(\zeta(t) := \sup_{0 \leq s \leq t,x \in \mathbb{R}} |v_{x,t} + \varphi_x \varphi_{xx}(x,s)(\theta + \psi_1 + \psi_2)^{-1}|\), we have

\[
|v(x,t)| \leq CE_0(\theta + \psi_1 + \psi_2),
\]

for all \(t \leq 0\) and all \(x \in \mathbb{R}\).

Acknowledgement. This project was completed while studying within the PhD program at Indiana University, Bloomington. Thanks to my thesis advisor Kevin Zumbrun for suggesting the problem and for helpful discussions.
References

[BJNRZ1] B. Barker, M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto-Sivashinsky equation, preprint (2012)

[BJNRZ2] B. Barker, M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, Efficient numerical evaluation of the periodic Evans function of Gardner and spectral stability of periodic viscous roll waves, in preparation

[CKTR] B.I. Cohen, J.A. Krommes, W.M. Tang, and M.N. Rosenbluth, Non-linear saturation of the dissipative trapped-ion mode by mode coupling Nucl. Fusion 16 (1976) 971

[DSSS] Arjen Doelman, Björn Sandstede, Arnd Scheel, and Guido Schneider. The dynamics of modulated wave trains. Mem. Amer. Math. Soc. 199 (2009) 934:viii+105

[F] A. Friedman, Partial differential equations of parabolic type, Prentice-Hall, Englewood Cliffs, NY (1964), Reprint Ed. (1983).

[FST] U. Frisch, Z.S. She, and O. Thual, Viscoelastic behaviour of cellular solutions to the Kuramoto–Sivashinsky model J. Fluid Mech. 168 (1987) 221–240

[G1] R. Gardner, On the structure of the spectra of periodic traveling waves, J. Math. Pures Appl. 72 (1993), 415-439.

[G2] R.A. Gardner, Spectral analysis of long wavelength periodic waves and applications, J. Reine Angew. Math. 491 (1997), 149–181.

[HZ] P. Howard and K. Zumbrun, Stability of undercompressive viscous shock waves, J. Differential Equations, 225 (2006), no. 1, 308–360.

[HRZ] P. Howard, M. Raoofi, and K. Zumbrun, sharp pointwise bounds for perturbed viscous shock waves, J. Hyperbolic Differential Equations, 3 (2006), no. 2, 1–77.

[J1] S. Jung, Pointwise asymptotic behavior of modulated periodic reaction-diffusion waves, J. Differential Equations, 253 (2012), no. 6, 1807-1861.

[JNRZ1] M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, Nonlocalized modulation of periodic reaction diffusion waves: Nonlinear stability, to appear, Arch. Ration. Mech. Anal.

[JNRZ2] M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, Nonlocalized modulation of periodic reaction diffusion waves: The Whitham equation, to appear, Arch. Ration. Mech. Anal.
REFERENCES

[JNRZ3] M. Johnson, P. Noble, M. Rodrigues, and K. Zumbrun, Behavior or periodic solutions of viscous conservation laws under localized and nonlocalized perturbations, manuscript/private communication.

[JZ1] M. Johnson and K. Zumbrun, Nonlinear stability of periodic traveling waves of viscous conservation laws in the generic case, J. Diff. Eq. 249 (2010) no. 5, 1213-1240.

[JZ2] M. Johnson and K. Zumbrun, Nonlinear stability of spatially-periodic traveling-wave solutions of systems of reaction diffusion equations, to appear, J. Math. Pure et Appl.

[JZ3] M. Johnson and K. Zumbrun, Nonlinear Stability of Periodic Traveling-Wave Solutions of Viscous Conservation Laws in Dimensions One and Two, SIAM J. Math. Anal. 10 (2011), no. 1, 189-211.

[JZN] M. Johnson, K. Zumbrun, and P. Noble, Nonlinear stability of viscous roll waves, to appear, SIAM J. Math. Anal.

[K] T. Kato, Perturbation theory for linear operators, Springer–Verlag, Berlin Heidelberg (1985).

[MaZ] C. Mascia and K. Zumbrun Pointwise green’s function bounds for shock profiles of systems with real viscosity, Arch. Ration. Mech. Anal. 169 (2003), no. 3, 177–263.

[OZ1] M. Oh and K. Zumbrun, Stability of periodic solutions of conservation laws with viscosity: Pointwise bounds on the Green function, Arch. Ration. Mech. Anal. 166 (2003), no. 2, 167–196.

[OZ2] M. Oh and K. Zumbrun, Stability and asymptotic behavior of traveling-wave solutions of viscous conservation laws in several dimensions, Arch. Ration. Mech. Anal. 196 (2010) 1-20.

[RZ] M. Raoofi and K. Zumbrun, Stability of undercompressive viscous shock profiles of hyperbolic-parabolic systems, J. Differential Equations, (2009) 1539–1567.

[S] D. Serre, Spectral stability of periodic solutions of viscous conservation laws: Large wavelength analysis, Comm. Partial Differential Equations 30 (1-3) (2005) 259-282.

[S1] G. Schneider, Nonlinear diffusive stability of spatially periodic solutions– abstract theorem and higher space dimensions, Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997), 159–167, Tohoku Math. Publ., 8, Tohoku Univ., Sendai, 1998.
[S2] G. Schneider, *Diffusive stability of spatial periodic solutions of the Swift-Hohenberg equation*, (English. English summary) Comm. Math. Phys. 178 (1996), no. 3, 679–702.

[ZH] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves*, Indiana Math. J. 47 (1998), 741–871.