Node-wise monotone barrier coupling law for central pattern generation*

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Abstract

With the aim of providing new insights into the mechanisms behind the emergence of collective behaviors via nonlinear coupling, we study a node-wise monotone barrier coupling law, motivated by the synaptic coupling of neural central pattern generators. It is illustrated that this coupling imitates the desirable properties of neural central pattern generators. In particular, the coupling law 1) allows us to assign multiple central patterns on the circle and 2) allows for rapid switching between different patterns via simple ‘kicks’. In the end, we achieve full control by partitioning the state space by utilizing a barrier effect and assigning a unique steady-state behavior to each element of the resulting partition. We analyze the global behavior and study the viability of the design.

Keywords: Neural central pattern generators, formation control, nonlinear spaces, positivity, consensus

1. Introduction

Phenomena where multiple agents synchronize in some sense to result in the emergence of collective behaviors are frequently observed in nature, complex ecosystems, social networks, financial markets, and energy networks. It is expected that a detailed understanding of the various mechanisms behind such phenomena will provide new modes of acquiring, processing, disseminating, and reacting to information and data.

Indeed, studies of synchronous oscillations in linearly coupled non-identical nonlinear oscillators observed in power grids and biological systems [1] [2] [3] [4] have provided us with a general theory on the collective behavior of a heterogeneous network under linear coupling [5] [6]. The network design tool further developed by this general theory has provided control techniques relevant for the design of smart and resilient societies. Examples include the use of distributed optimization to solve economic dispatch problems, i.e. to minimize the total generation cost under the power supply-demand balance and individual generation capacity constraints [7], distributed state estimation for the design of sensor networks where each sensor node can autonomously collect, manage, and control the information [8] [9], and the distributed SLAM problem where the goal is to make a group of mobile robots simultaneously localize and map the environment [10].

Meanwhile, more general types of coupling, i.e. nonlinear and/or non-smooth couplings, and more general state spaces, e.g. nonlinear spaces such as Lie groups [11] [12] [13] [14] [15], the n-sphere [16], Grassmannians [17], and Stiefel manifolds [18], frequently appear in the modeling of physical, biological, or societal systems. Understanding the mechanism behind the enforcement of synchronization for heterogeneous agents and the emergence of collective behavior in this more general setting will provide additional advantageous features that are completely absent in linear models.

A particularly important example where we can see such features is that of the neural central pattern generators (CPGs) that produce diverse rhythms in networks for the purpose of collectively generating movements such as breathing, chewing, swallowing, walking, and heart beating [19] in animals. Understanding the mechanisms behind the control and regulation of CPGs may result in technological advances resulting in systems that can accommodate rapidly to sudden changes similar to the way that CPGs adapt in fractions of a second to respond to events, e.g. in choking prevention or predator escape [19]. Besides the ability to accommodate multiple central patterns in the network and rapidly switch between them, such systems also exhibit robustness with respect to individual variability. Indeed, in the study of a network of non-identical neurons interconnected via excitatory synaptic coupling (a particular type of CPGs), it was shown that the network is robust to heterogeneity and has the emergent behavior (central pattern) of synchronous spiking, even with the weak coupling strength and the impulsive nature of communication [20] [21].

However, the theory for understanding such systems is only in its infancy. For instance, the mechanisms of CPGs have been extensively investigated in robotics to build autonomous robots that adapt efficiently to changing environments, but a solid theoretical understanding of the robust and adaptive hardware of CPGs is still lacking [22].
and [21] is the only theoretical work that we are aware of on a network of heterogeneous neurons (using a conductance-based model) after nearly 30 years since the pioneering work of Somers and Kopell on a network of identical neurons [23]. This lack of understanding particularly applies to synchronization enforceability and associated emergent collective behavior. In particular, consensuasability, a prerequisite for synchronization enforceability, which answers the question of when does a network achieve consensus when there are no individual dynamics, has been studied for decades and only recently been understood under more general types of coupling [24].

The paper [21] argues that the key feature that provides these attractive properties of CPGs is the fast threshold modulation: a mechanism behind rapid synchronization initially discovered in [23], which can be illustrated by a barrier effect in phase models. Thus, to gain a deeper insight into such systems, in this paper we focus on the problem of formation control on the circle and study a node-wise monotone barrier coupling law. In the end, we illustrate that by only assuming the barrier effect, the network exhibits attractive properties of CPGs. In particular,

1. it allows us to assign multiple central patterns in the steady-state behavior of the network with possibly different formations and common angular frequencies,
2. it allows rapid switching between different central patterns via a simple ‘kick’ or sudden disturbance, and it brings robustness with respect to individual variability. For the considered node-wise coupling law, we then concentrate on the question of the viability of assigning one or multiple central patterns in the network, from an engineering viewpoint.

This paper is organized as follows. In Section 2, we motivate the relevance between coupling functions with barrier effect in phase models and the synaptic coupling in CPGs. Then, with a brief review of graph theory given in Section 3, we introduce the node-wise monotone barrier coupling law in Section 4, which is the subject of study in this paper. Given the basic convergence result in Section 5, we then specify our study to central patterns, where Sections 6 and 7 consider the analysis and design perspective, respectively. We conclude in Section 8. The proof for the convergence result is given in Appendix B which uses the graph theoretical lemma introduced in Appendix A.

**Notation:** \( \mathbb{R} \), \( \mathbb{Z} \), \( \mathbb{N} \), and \( \mathbb{N}_0 \) denote the set of real numbers, integers, positive integers, and non-negative integers, respectively. For vectors or matrices \( a \) and \( b \), \( \text{col}(a, b) := [a^T, b^T]^T \). For a set \( \mathcal{Z} \), its cardinality is denoted by \( |\mathcal{Z}| \). The function \( \text{sgn} : \mathbb{R} \to \mathbb{R} \) denotes the sign function defined as \( \text{sgn}(s) = s/|s| \) for non-zero \( s \), and \( \text{sgn}(s) = 0 \) for \( s = 0 \). In this paper, the modulo operation with respect to \( 2\pi \mod \pi \) results in a value in \([-\pi, \pi) \), and the modulo operation with respect to 1 \( \mod 1 \) results in a value in \([0, 1) \), for simplicity of notation.

2. Motivation on the barrier effect

**Fast threshold modulation** is a mechanism behind rapid and robust synchronization of non-identical neuronal spiking systems, e.g., Fitzhugh-Nagumo model, Morris-Lecar model, and Hodgkin-Huxley model, under weak synaptic coupling. In particular, it provides attractive properties of

- rapid convergence to a central pattern
- and robustness with respect to heterogeneity,

in the following way [21], at the singular limit, i.e., when there is a sufficient time scale separation (illustrated for the central pattern of synchronous spiking).

1. At the singular limit, the individual system is an oscillator having jumps, which can be described as a hybrid system, that has a jump set of a lower dimension (see [1] and its description for an example).
2. At the singular limit, this provides a rapid convergence to the neighborhood of the limit cycle given that the coupling is weak.
3. Then, at the singular limit, the synaptic coupling alters the jump set of an individual in a way that the network-wise jump set demonstrating a central pattern becomes an open set (see [2] and its description for an example)\(^2\)
4. This open set with non-zero volume is what provides rapid convergence and robustness with respect to heterogeneity.
5. In particular, by the creation of this open set, the phenomenon of synchronous spiking happens in a hierarchical way: one neuron spikes and this yields a spike of neighboring neurons, and so on to the entire network, at an instant in the singular limit.\(^3\)

For example, at the singular limit, the neuronal model in [21] can be illustrated by a one-dimensional hybrid system

\[
\begin{align*}
\dot{x} &= h(x) \in \mathbb{R}, \quad x > \hat{x}, \\
\dot{x} &= \hat{h}(\hat{x}), \\
x^+ &= \hat{x}^+ \in \mathbb{R}, \quad x = \hat{x},
\end{align*}
\]

where \( b(\cdot) < 0 \) dictates the state-dependent velocity of the individual oscillator, and \( \hat{x} \) and \( \hat{x}^+ \) are the position of the threshold (jump set) and the jumping point, respectively, such that \( \hat{x} < \hat{x}^+ \). It is intuitively clear that this is an oscillator repeating its trajectory from \( \hat{x}^+ \) to \( \hat{x} \).

Now, what a fast threshold modulation does is that given another system \( \hat{x} \), it widens a jump set for the synchronous jump from \( \{x\} \times \{\hat{x}\} \) to \( \{x\} \times [\hat{x}, \hat{x}^+] \cup [\hat{x}, \hat{x}] \times \{\hat{x}\} \) with some points \( \hat{x}, \hat{x}^+ \) such that \( \hat{x} < \hat{x}^+ \). In particular, at the singular limit, the neuronal network in [21] (with two neurons) can be illustrated by a hybrid system

\[
\begin{align*}
\dot{x} &= h(x), \\
\dot{x} &= \hat{h}(\hat{x}), \\
\text{col}(x^+, \hat{x}^+) &= \begin{cases} 
\text{col}(\hat{x}^+, \hat{x}), & \text{if } x = \hat{x}, \quad \hat{x} > \hat{x}, \\
\text{col}(\hat{x}, \hat{x}^+), & \text{if } x > \hat{x}, \quad \hat{x} = \hat{x}, \\
x^+ + (x, \hat{x}), & \text{if } x = \hat{x}, \quad \hat{x} \in [\hat{x}, \hat{x}^+] \\
\text{col}(x, \hat{x}) & \text{in } x \in [\hat{x}, \hat{x}], \quad \hat{x} = \hat{x}, 
\end{cases}
\end{align*}
\]

\(^2\)That its dimension equals the dimension of the entire hybrid system.
\(^3\)The mechanism is called fast threshold modulation, as the threshold (jump set) is altered rapidly.
with an appropriate network-wise jump map $X^+$. Note that by the creation of this open set, even when the neighboring neuron $\hat{x}$ is not at its jump set (threshold) $\hat{x}$, but only if sufficiently near to it: $\hat{x} \in [\hat{x}, \hat{\pi}]$, then the spike of the neuron $x$ modulates the jump set and triggers a spike of the neighboring neuron $\hat{x}$. This happens instantly in the singular limit. Therefore, even if the frequencies of individual oscillators are different (heterogeneity), we can obtain a synchronous spiking solution. We emphasize that this creation of an open set can happen even under weak coupling strength, and this is what provides rapid convergence. We refer to [21] for a more exhaustive illustration.

Now, although it is difficult to define a phase for individual systems in such a network, as the range of individual oscillation changes by the action of a neighborhood, if we were to model it in a phase model, then the action of a fast threshold modulation can only be realized by a barrier effect (whether it is discontinuous or continuous), as in the singular limit, the new open jump set introduces a region $[\hat{x}, \hat{\pi}]$, where a phase pulling of infinite power happens by the neuron $x$ at the threshold $\hat{x}$, and clearly demonstrates a boundary $\hat{\pi}$ that discriminates the behavior of neuron $\hat{x}$. For instance, if the phase that corresponds to position $\hat{x}$ is $\theta = \theta_0$, then as the phase difference $\hat{\theta} - \theta$ approaches the new threshold $\theta_{th}$, the effect of coupling becomes infinitely strong, where the phase $\theta_{th} + \theta_0$ corresponds to position $\hat{\pi}$.

In the rest of the paper, we will relax this characteristic to arbitrary coupling law with barrier effect and consider a network of phase models. It will be shown that this is sufficient to recover the attractive properties of CPGs: rapid convergence and robustness. Moreover, unlike the specific case illustrated in this section only for synchronization, it will be illustrated that this phase network exhibits multiple central patterns, and we can even design them. Before starting this investigation, we briefly review the necessary graph theoretical tools in the next section.

3. Graph theoretical preliminaries

A (weighted directed) graph is a pair $G = (N, \mathcal{E})$ consisting of a finite nonempty set of nodes $N = \{1, 2, ..., N\}$ and an edge set of ordered pairs of nodes $\mathcal{E} \subseteq N \times N$, where $(i, i) \notin \mathcal{E}$ for all $i \in N$ (i.e., the graph does not contain self-loops). The set $N_i = \{j \in N | (j, i) \in \mathcal{E}\}$ denotes the neighbors of the node $i$. A tuple $(i_0, i_1, ..., i_l) \in N^{l+1}$ is called a path (of length $l$) from the node $i_0$ to the node $i_l$, if $i_k \in N_{i_{k+1}}$ for all $k = 0, ..., l - 1$. If $i_1, ..., i_l$ are distinct, then the path is called elementary. A loop is an elementary path with $i_0 = i_l$. A strongly connected graph $G$ is a graph for which any pair of distinct nodes $i$ and $j$ are connected via a path in $G$ from $j$ to $i$.

$G$ is called undirected, if $(i, j) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$. Given a graph $G = (N, \mathcal{E})$, let $N^G \subseteq N$ and

$$\mathcal{E}^G := \{(j, i) \in \mathcal{E} | i, j \in N^G\}.$$

Then the pair $G' = (N', \mathcal{E}')$ is called a subgraph of $G$. If $N' \supseteq N$, then $G'$ is a spanning subgraph. If a graph $G$ is connected, then there exists an agent $i$, called a root of the graph, from which information can propagate to all other agents $j$ along paths in $G$. A spanning subgraph $\mathcal{T}$ of $G$ obtained by removing all edges that do not belong to one of these paths is called a spanning tree of $G$. Note that any node of the strongly connected graph is a root. Note also that a strongly connected graph is connected, but not vice versa.

An independent strongly connected component (iSCC) of $G$ is an induced subgraph $G' = (N', \mathcal{E}|_{N'})$, $N' \subseteq N$, such that it is maximal subject to being strongly connected and satisfies $(j, i) \notin \mathcal{E}$ for any $j \in N' \setminus N''$ and $i \in N''$. For any digraph $G$, there exists a uniquely defined set of $M \geq 1$ number of iSCCs. $M = 1$ if and only if the graph is connected.

The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ of a graph is defined as $L := D - A$, where $A = [a_{ij}]$ is the adjacency matrix of the graph and $D$ is the diagonal matrix whose diagonal entries are determined such that each row sum of $L$ is zero. By its construction, it contains at least one eigenvalue of zero, whose corresponding eigenvector is $1^T \in \mathbb{R}^N$, and all the other eigenvalues have non-negative real parts. For directed graphs (digraphs), the zero eigenvalue is simple if and only if the corresponding graph is connected.

We like to stress that for any connected digraph $G$, the indices can be well assigned so that the Laplacian matrix associated with the graph can be written as

$$L = \begin{bmatrix} L_s & 0 \\ -A_{sf} & L_f + D_f \end{bmatrix} \in \mathbb{R}^{N \times N},$$

where $L_s \in \mathbb{R}^{S \times S}$ and $L_f \in \mathbb{R}^{(N - S) \times (N - S)}$ are the Laplacian matrices of the unique iSCC, $S$, and the subgraph induced by the rest of the agents, respectively. Since $L_s$ is the Laplacian matrix of a strongly connected graph, it is known that there exists a vector $\zeta := \text{col}(\zeta_1, ..., \zeta_S) \in \mathbb{R}^S$ which satisfies $\zeta^T L_s = 0$ and $\zeta_i > 0$ for all $i = 1, ..., S$. In particular, we have $\text{col}(\zeta, 0)^T L = 0$.

4. Node-wise monotone barrier coupling law

Motivated by Section 2, we consider the node-wise monotone barrier coupling law for a group of $N$ agents evolving on the circle $S^1$ as

$$\dot{\theta}_i = \omega_i + f_i (\nu_i + \phi_i), \quad \nu_i = \sum_{j \in N_i} \alpha_{ij} (\theta_j - \theta_i), \quad (3)$$

where $\theta_i \in S^1$ represents the phase of agent $i$, $N_i$ is a subset of $N$ whose elements are indices of the agents that send information to agent $i$ (hence $\mathcal{E} \equiv \{(j, i) : j \in N_i, i \in N\}$), $\omega_i \in \mathbb{R}$ is the ‘intrinsic’ frequency, and $f_i$ denotes a coupling function on the domain $(-\pi, \pi)$ extended to $\mathbb{R}$, $2\pi$-periodically. $\alpha_{ij} > 0$ is the interconnection weight and $\phi_i \in [-\pi, \pi]$ is the phase bias.

Let $\theta = (\theta_1, ..., \theta_N)$ denote an element of the $N$-torus $\mathbb{T}^N$. Note that as the coupling function $f_i$ will have a barrier effect, i.e., $f_i(s) \to \pm \infty$ as $s \to \pm \pi$, we want our trajectories to reside inside the set $\mathbb{T}^N = \{\theta \in \mathbb{T}^N : |\nu_i + \phi_i| \neq 2\pi, i \in N\} \subseteq \mathbb{T}^N$. Note also that for the network to be well-defined for $\theta \in \mathbb{T}^N$, we should have $\alpha_{ij} \in \mathbb{N}$; that is the network should be quantized, as any addition of $2\pi$ in the phase difference $\theta_j - \theta_i$ should not alter the coupling input $f_i(\nu_i + \phi_i)$.
We note here that in the rest of the paper, the $N$-torus $\mathbb{T}^N$ will be realized by $[-\pi, \pi)^N$, and in that regard, the partition of $\mathbb{T}^N$ consisting of a finite number of sets

$$\Theta_{\{n_i\}}^{\mathbb{T}^N} := \{ \theta \in [-\pi, \pi)^N | \nu_i + \phi_i - 2n_i\pi \in (-\pi, \pi), i \in \mathbb{N} \},$$

(4)

where each being associated with a sequence of integers $\{n_i\}_{N} \in \mathbb{Z}$ will take a critical role (as, e.g., in the following theorem) with its associated extension to $\mathbb{R}^N$ space:

$$\Theta_{\{n_i\}}^{\mathbb{R}^N} := \{ \theta \in \mathbb{R}^N | \nu_i + \phi_i - 2n_i\pi \in (-\pi, \pi), i \in \mathbb{N} \}.$$  

(5)

Note that for $\Theta_{\{n_i\}}^{\mathbb{T}^N}$ to be non-empty, we should have $n_i \in [-d_i, d_i]$, where $0 \leq d_i = \sum_{j \in \mathbb{N}} \alpha_{ij} \in \mathbb{Z}$, and thus, the number of sets is upper bounded by $\prod_{i \in \mathbb{N}} (2d_i + 1)$.

**Assumption 1.** The communication graph $\mathcal{G}$ is connected; $\mathcal{G}$ contains a spanning tree. The adjacency elements $\alpha_{ij}$, $(j, i) \in \mathcal{E}$ are positive integers.

**Assumption 2.** The coupling functions $f_i : (-\pi, \pi) \to \mathbb{R}$, $i \in \mathbb{N}$ are differentiable, strictly monotonically increasing, and have a barrier effect so that $f_i(s) \to \pm \infty$ as $s \to \pm \pi$.

**Theorem 1.** Under Assumptions 1 and 2, for any given $\{n_i\}_{N}$, each trajectory of $[\Theta_{\{n_i\}}^{\mathbb{T}^N}]$ starting from the extended space $\Theta_{\{n_i\}}^{\mathbb{T}^N}$ resides inside $\Theta_{\{n_i\}}^{\mathbb{R}^N}$ and converges to a central pattern $(\bar{\omega}, \{\Delta_{ij}\}_\mathcal{E})$ of phase-locking behavior determined by

- a common frequency $\bar{\omega} \in \mathbb{R}$
- and a formation $\{\Delta_{ij}\}_\mathcal{E} \in [-\pi, \pi)$: a phase difference given for each edge.

In particular, there exists a steady-state solution of $[\Theta_{\{n_i\}}^{\mathbb{T}^N}]$ corresponding to $(\bar{\omega}, \{\Delta_{ij}\}_\mathcal{E})$ that resides inside $\Theta_{\{n_i\}}^{\mathbb{R}^N}$. Finally, for each trajectory, the control input $f_i(\nu_i(t) + \phi_i)$, $i \in \mathcal{N}$ is bounded uniformly on $[0, \infty)$.

**Proof.** We show in Appendix B.1 that the set $\Theta_{\{n_i\}}^{\mathbb{R}^N}$ is positively invariant for the network $[\Theta_{\{n_i\}}^{\mathbb{T}^N}]$ extended to $\mathbb{R}^N$ and also that the control input is uniformly bounded. Then, the convergence to a central pattern and the existence of a corresponding steady-state solution that resides inside $\Theta_{\{n_i\}}^{\mathbb{R}^N}$ is shown in Appendix B.2. $\square$

In the following sections, we will further investigate

- the shape of the central pattern $(\bar{\omega}, \{\Delta_{ij}\}_\mathcal{E})$,
- the number of different central patterns $N^\mathcal{P}$,
- what central patterns can be assigned,
- and how to achieve them.

**Remark 1.** We neglect analysis of the convergence rate as it depends on the slope of our coupling functions $f_i(\cdot)$, which can be made arbitrarily large.

5. **Analysis of central patterns**

This subsection corresponds to the analysis part of the study of the considered node-wise monotone barrier coupling law. In particular, for a fixed given set of intrinsic frequencies $\{\omega_i\}_{\mathcal{N}}$, a set of coupling functions $f_i(\cdot)$, a set of interconnection weights $\{\alpha_{ij}\}_\mathcal{E} \in \mathbb{N}$, and a set of phase biases $\{\phi_i\}_{\mathcal{N}}$, we first specify

- the shape of the central pattern $(\bar{\omega}, \{\Delta_{ij}\}_\mathcal{E})$.

**Theorem 2.** Under Assumptions 1 and 2, every trajectory of $[\Theta_{\{n_i\}}^{\mathbb{T}^N}]$ starting from $\Theta_{\{n_i\}}^{\mathbb{R}^N}$ converges to the unique central pattern $(\bar{\omega}, \{\Delta_{ij}\}_\mathcal{E})$. Here, $\bar{\omega}$ is the unique solution of

$$F(\bar{\omega}) := \sum_{i \in \mathcal{S}} \zeta_i \left(f_i^{-1}(\bar{\omega} - \omega_i) + 2n_i\pi - \phi_i\right) = 0,$$

(6)

where $\mathcal{S}$ denotes the unique iSCC, $\zeta_i > 0$, $i \in \mathcal{S}$ are the components of the left eigenvector of $\mathcal{L}$ associated with the zero eigenvalue, and the inverse $f_i^{-1}$ is defined as a mapping from $\mathbb{R}$ into $(-\pi, \pi)$. $\{\Delta_{ij}\}_\mathcal{E} \equiv \Delta_i \mod 2\pi$, where $\text{col}(\Delta_1, \ldots, \Delta_N) \in \Theta_{\{n_i\}}^{\mathbb{R}^N}$ is the unique solution of

$$-\mathcal{L} \left[ \begin{array}{c} \Delta_1 \\ \vdots \\ \Delta_N \end{array} \right] = \left[ \begin{array}{c} f_1^{-1}(\bar{\omega} - \omega_1) + 2n_1\pi - \phi_1 \\ \vdots \\ f_N^{-1}(\bar{\omega} - \omega_N) + 2n_N\pi - \phi_N \end{array} \right],$$

(7)

such that $\Delta_1 = 0$.

**Proof.** Note first that by Theorem 1 for the given sequence $\{n_i\}_{\mathcal{N}}$, each trajectory starting from $\Theta_{\{n_i\}}^{\mathbb{R}^N}$ has a central pattern $(\bar{\omega}, \{\Delta_{ij}\}_\mathcal{E})$ of phase-locking behavior that the trajectory converges to. Being the central pattern, $(\bar{\omega}, \{\Delta_{ij}\}_\mathcal{E})$ should satisfy

$$\bar{\omega} = \omega_i + f_i \left( \sum_{j \in \mathcal{N}_i} \alpha_{ij} \Delta_{ij} + \phi_i \right), \quad i \in \mathcal{N}. $$

(8)

Now, by Theorem 1 there exists $\text{col}(\Delta_1, \ldots, \Delta_N) \in \Theta_{\{n_i\}}^{\mathbb{R}^N}$ such that $\Delta_{ij} = \Delta_j - \Delta_i \mod 2\pi$. Without loss of generality, we assume $\Delta_1 = 0$. This implies

$$\sum_{j \in \mathcal{N}_i} \alpha_{ij}(\Delta_j - \Delta_i) + \phi_i = f_i^{-1}(\bar{\omega} - \omega_i) + 2n_i\pi, \quad i \in \mathcal{N}. $$

(9)

Hence, by the identity

$$\sum_{i \in \mathcal{S}} \zeta_i \sum_{j \in \mathcal{N}_i} \alpha_{ij}(\Delta_j - \Delta_i) = 0,$$

we get $F(\bar{\omega}) = 0$, where $F$ is defined in (6).

Since the function $F : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing with respect to $\bar{\omega}$, and satisfies $\lim_{\bar{\omega} \to \pm \infty} F(\bar{\omega}) = \sum_{i \in \mathcal{S}} \zeta_i (\pm \pi + 2n_i\pi - \phi_i)$, where

$$-\sum_{i \in \mathcal{S}} \zeta_i \pi < \sum_{i \in \mathcal{S}} \zeta_i (2n_i\pi - \phi_i) < \sum_{i \in \mathcal{S}} \zeta_i \pi,$$

(10)

4Since $\Theta_{\{n_i\}}^{\mathbb{T}^N} \neq \emptyset$, there exists $\theta \in \Theta_{\{n_i\}}^{\mathbb{T}^N}$ such that $\nu_i + \phi_i - 2n_i\pi = \psi_i \in (-\pi, \pi), i \in \mathcal{N}$; hence, we get $\sum_{i \in \mathcal{S}} \zeta_i (2n_i\pi - \phi_i) = \sum_{i \in \mathcal{S}} \zeta_i (\nu_i - \psi_i) = -\sum_{i \in \mathcal{S}} \zeta_i \psi_i$. This implies (10).
we have the existence and the uniqueness of the solution \( \omega \) of (6). 

Now, (9) further implies (7) and this uniquely defines \( \text{col}(\Delta_1, \ldots, \Delta_N) \) such that \( \Delta_1 = 0 \) because the Laplacian matrix \( L \) has rank \( N - 1 \). \( \square \)

**Remark 2.** Here, we can see the resiliency of the generated patterns with respect to individual variability. In particular, \( f^{-1}(\cdot) \) is a sigmoidal function, and thus, equation (6) rejects outliers \( \omega_i \), as done by the median in statistics, which is the solution of the following equation

\[
\sum_{i=1}^{N} \text{sgn}(\bar{\omega} - \omega_i) = 0.
\]

In general, \( \bar{\omega} \) becomes more tolerant to variation in \( \omega_i \) that is far from \( \bar{\omega} \). This is also true for the formation, as can be seen in (7). The effect of variation in \( \omega_i \) that is far from \( \bar{\omega} \) (\( |f^{-1}(\bar{\omega} - \omega_i)| \)) is approximately \( \pi \) becomes negligible through \( f^{-1}(\cdot) \) due to the barrier effect.

From this characterization of the shape of the central pattern, we have the following conclusion.

**Theorem 3.** Under Assumptions 2 and 3 a network of agents on the circle communicating according to (3) introduces a partition of \( \mathbb{T}_\pi^N \) consisting of \( N \) number of sets \( P_j \subset \mathbb{T}_\pi^N \), \( j = 1, \ldots, N \). Every trajectory starting from \( P_j \) resides inside \( P_j \) and converges to a unique central pattern \( (\bar{\omega}, \{\Delta_j\}_j) \) of phase-locking behavior. Each set \( P_j \) has the following structure:

\[
P_j = \bigcup_{k=1}^{m_j} \Theta_{\{n^k_1\}_N}.
\]

**Proof.** Given that any initial point of a trajectory in \( \mathbb{T}_\pi^N \) is contained in one of the sets \( \Theta_{\{n^k_1\}_N} \subset \Theta_{\{n^k_i\}_N} \), associated with some \( \{n^k_i\}_N \), the convergence to a unique central pattern for any trajectory starting from each set \( \Theta_{\{n^k_i\}_N} \) is given by Theorem 2.

Note that for each set \( \Theta_{\{n^k_i\}_N} \), there exists a finite number of sequences \( \{n^k_i\}_N \), \( i = 1, \ldots, m_j \) such that any point in \( \mathbb{T}_\pi^N \) that corresponds to a set \( \Theta_{\{n^k_i\}_N} \) is contained in one of the sets \( \Theta_{\{n^k_i\}_N} \), \( k = 1, \ldots, m \) in the partition of \( \mathbb{T}_\pi^N \).

Now, for each unique central pattern, let us collect all the corresponding sets \( \Theta_{\{n^k_i\}_N} \), \( k = 1, \ldots, m \) for all positively invariant sets \( \Theta_{\{n^k_i\}_N} \). Let \( P_j \) be a representation of a union of a collection of sets \( \Theta_{\{n^k_i\}_N} \), \( k = 1, \ldots, m_j \). This is because, otherwise, there exist a sequence \( \{n^k_i\}_N \) and \( \theta^1, \theta^2 \in \Theta_{\{n^k_i\}_N} \) such that \( \theta^1 \in P_j \) and \( \theta^2 \in P_j \) with some \( j^1 \neq j^2 \). This yields a contradiction, as trajectories that start from \( \theta^1 \) and \( \theta^2 \) reside inside \( \Theta_{\{n^k_i\}_N} \) and converge to the same central pattern.

Now, we specify

- a collection of sequences \( \{n^k_i\}_N \), \( k = 1, \ldots, m_j \), that defines the set \( P_j \).

For that purpose, we define an equivalence relation for two sequences \( \{n^k_1\}_N \) and \( \{n^k_2\}_N \) if they have the same central pattern or equivalently that \( \Theta_{\{n^k_1\}_N} \) and \( \Theta_{\{n^k_2\}_N} \) are contained in the same set \( P_i \), and denote it as \( \{n^k_1\}_N \sim \{n^k_2\}_N \). We also denote the equivalence class as \( [\{n^k_1\}_N] \).

**Theorem 4.** Two sequences \( \{n^k_1\}_N \) and \( \{n^k_2\}_N \) are equivalent, i.e., \( \{n^k_1\}_N \sim \{n^k_2\}_N \), if and only if

- \( \sum_{i \in S} \zeta_i n^k_1 = \sum_{i \in S} \zeta_i n^k_2 \)
- and the following equation has a unique integer solution \( \delta_2, \ldots, \delta_N \). Or equivalently that the vector on the right-hand side is spanned by the columns of the Laplacian matrix with integer-valued weights.

\[
\text{L} = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\delta_2 & \cdots & \delta_N
\end{bmatrix}
\]

The second condition is equivalent to \( \Delta_1 = \Delta_2 \mod 2\pi \) for all \( i \in N \), where \( \{\Delta_1^i\}_N \) and \( \{\Delta_2^i\}_N \) is the corresponding solution of (7) for \( \{n^k_1\}_N \) and \( \{n^k_2\}_N \) respectively.

**Proof.** The first condition is a necessary and sufficient one for the unique solution of (6) to be equivalent for two sequences \( \{n^k_1\}_N \) and \( \{n^k_2\}_N \). This is because, for different \( \sum_{i \in S} \zeta_i n^k_1 \), we have different \( \bar{\omega} \), since \( \sum_{i \in S} \zeta_i f^{-1}(\bar{\omega} - \omega_i) \) is strictly increasing. The second condition is straightforward from (7). In particular, \( \Delta_1^i - \Delta_2^i = 2\delta_i \pi \) for \( i = 2, \ldots, N \). \( \square \)

**Remark 3.** According to Theorem 4, a strongly connected graph that has the following form of Laplacian matrix ensures that two sequences \( \{n^k_1\}_N \) and \( \{n^k_2\}_N \) are equivalent if and only if \( \sum_{i \in N} \zeta_i n^k_1 = \sum_{i \in N} \zeta_i n^k_2 \).

This includes the cases of directed ring graphs and undirected line graphs, which have one eigenvalue as the left eigenvector associated with the zero eigenvalue, resulting in \( N \) or \( N - 1 \) different central patterns (Remark 4).

Now, the number of different central patterns \( (\bar{\omega}, \{\Delta_j\}_j) \) of the network (3), \( N \), can be fully characterized by a graph theoretical interpretation as the number of different equivalence classes \( \left[ \{n^k_i\}_N \right] \) such that there exists \( \{n^k_i\}_N \in \left[ \{n^k_i\}_N \right] \) satisfying \( \Theta_{\{n^k_i\}_N} \neq \emptyset \). The following remark and corollary may aid in the analytical investigation to determine \( N \) or at least its upper bound (as in Remark 3). Meanwhile, such a number can be obtained by numerical computation.

**Remark 4.** Unlike the number of different central patterns, \( N \), which is complicated to find, the number of different \( \sum_{i \in S} \zeta_i n^k_1 =: n_S \) is straightforward. In particular, the left eigenvector of \( \text{L} \) associated with the zero eigenvalue can always be taken as an integer vector where the common denominator of the components is 1. Then, the number of different \( n_S \) is simply upper bounded by

\[5\text{This is because, } \text{L} - 0I \text{ is integer-valued, and thus, Gaussian elimination will produce rational eigenvectors.} \]

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\[ \sum_{i \in S} \xi_i =: \zeta_S \text{ or } \zeta_S - 1. \text{ This is because, we have from } \nu_i + \phi_i - 2n_i \pi \in (-\pi, \pi) \text{ that} \]
\[ \sum_{i \in S} \xi_i \phi_i - 2n_i \pi \in (-\zeta_S \pi, \zeta_S \pi), \]

hence for each \( \phi_S = \sum_{i \in S} \xi_i \phi_i \), we get
\[ n_S \in \left( -\frac{\zeta_S}{2} + \frac{\phi_S}{2\pi}, \frac{\zeta_S}{2} + \frac{\phi_S}{2\pi} \right). \]

**Corollary 4.1.** \( \{n^1_i\}_{N} \sim \{n^2_i\}_{N} \) if and only if

- \( \{n^1_i\}_{S} \sim \{n^2_i\}_{S} \) for the graph \( (S, \mathcal{E}|_{S}) \), which is strongly connected
- and there exist integers \( \delta_i, i \in N \setminus S \) such that
\[ \sum_{j \in N_i} \alpha_{ij}(\delta_j - \delta_i) = n^1_i - n^2_i, \quad \forall i \in N \setminus S. \]

In other words, if \( \{n^1_i\}_{S} \sim \{n^2_i\}_{S} \), then for any set of integers \( \{\delta_i\}_{N \setminus S} \), we have
\[ \{n^1_i\}_{N} \in [\{n^2_i\}_{N}], \]
where \( n^1_i = n^2_i + \sum_{j \in N_i} \alpha_{ij}(\delta_j - \delta_i), i \in N \setminus S. \) In particular, if all the followers \( f_i \in N \setminus S \) have only one neighbor, \( |N| = 1 \), and \( \alpha_{ij} = 1, j \in N_i \), then \( \{n^1_i\}_{N} \sim \{n^2_i\}_{N} \) if and only if \( \{n^1_i\}_{S} \sim \{n^2_i\}_{S} \).

**Proof.** The first claim directly follows from Theorem 4.1. The second claim follows because under this additional assumption, for any \( \{n^1_i\}_{N \setminus S} \) and \( \{n^2_i\}_{N \setminus S} \), there exist integers \( \delta_i, i \in N \setminus S \) such that
\[ \delta_j - \delta_i = n^1_i - n^2_i, \quad \forall i \in N \setminus S, \quad j \in N_i. \]
This is because there is no loop in graph \((N \setminus S, \mathcal{E}|_{N \setminus S})\). Or one can simply consider \( S \) as a single node, which makes the entire graph a spanning tree. \( \square \)

Before concluding this section, we note that, ultimately, by only assuming the barrier effect in our coupling functions, we observe the desirable properties of neural CPGs in the network on the circle:

- The network exhibits \( N^P \) number of different central patterns.
- A simple ‘kick’ (e.g., an impulsive input that instantly shifts the state of the network) that pushes the steady-state solution of \( P_j \) outside the boundary of \( P_j \) rapidly switches the central pattern (Remark 4).
- The number of different central patterns and the steady-state solution is robust to model uncertainties, noises, and disturbances, given that the barrier effect is consistent (Remark 5).

This provides numerous advantages in the problem of formation control on the circle:

- When considering a single formation in the network on the circle, global convergence property is hard to achieve in general (unless the control is stochastic), and to the best of our knowledge, most of the works either apply only to particular interconnection topologies \([25, 12, 17]\), use auxiliary variables \([26, 12, 17]\), use global information \([27]\), or can lead to unnecessarily slow convergence rates \([25]\). This is because the required convexity is not retained globally in nonlinear spaces. In other words, in general, there are multiple steady-state behaviors or even chaotic ones. From an engineering viewpoint, this issue can be resolved, if we have control over the multiple central patterns and their associated domain of attraction. Barrier coupling laws partition the state space into finite regions, where for each partition there exists a unique steady-state behavior.
- For instance, in a situation where a fleet of drones move in a balanced formation until they encounter obstacles, e.g., a scenario in which they have to pass between two buildings, and this impulsive event is detected by some of the drones in the formation. Then, this event can be made to trigger a ‘kick,’ which could alternate the formation of the network, for example, to a line, so that they can be safely guided through a narrow passage.

To best utilize these advantages, in the next section, we seek the viability of assigning multiple patterns, from the engineering viewpoint.

**Remark 5.** If we cut the barrier function at a finite region, that is \( \bar{f}_i : (-\pi, \pi) \rightarrow (M_i^-, M_i^+) \) with some \( M_i^- < M_i^+ \) but \( f_i \) is still strictly increasing and satisfies \( \lim_{s \to -\pi} f_i(s) = M_i^{-} \), then since the monotonicity is preserved, the behavior of the network will be either converging to some central pattern and achieving phase-locking behavior or moving to the discontinuous boundary. In particular, if we let \( \delta > 0 \)
\[ \bar{f}_i(s) \begin{cases} 
\in (M_i^-, M_i^+ + \delta), & \text{if } s \in (-\pi, s_i^-), \\
\in (M_i^+ - \delta, M_i^+), & \text{if } s \in (s_i^+, \pi), \\
= f_i(s), & \text{if } s \in [s_i^-, s_i^+].
\end{cases} \]
where \( s_i^-, s_i^+ \in (-\pi, \pi) \) are such that \( f_i(s_i^-) = M_i^- + \delta \) and \( f_i(s_i^+) = M_i^+ - \delta \) so that the resulting function is still strictly increasing, then there are no new central patterns generated when \( \delta \) is sufficiently small. Therefore, for the partition, where the central pattern \( \bar{w}, \{\Delta_i\}_{E} \) corresponding to the coupling functions \( f_i, i \in N \) is outside the saturated region, i.e., which has \( i \in N \) such that \( \bar{w} - \omega_i \notin (M_i^-, M_i^+) \), the trajectory starting from that partition moves to the boundary. Then, depending on the vector field on the opposite side, it either moves to another region associated with another sequence \( \{n_i\}_N \) in a finite time or stays on the boundary. In other words, if we set \( M_i^- \) and \( M_i^+ \) for each \( i \in N \) such that \( \bar{w} - \omega_i \in (M_i^-, M_i^+) \) for all central patterns \( \bar{w}, (\omega, \{\Delta_i\}_{E}) \), then the original behavior will be mostly maintained, while some might converge to the boundary and stay there. \footnote{This becomes clear if we observe the dynamics of \( x_i := \nu_i + \phi_i \).}
6. Design of central patterns

This subsection corresponds to the synthesis part of the study of the considered node-wise monotone barrier coupling law. In particular, for a given fixed connected digraph \( G = (\mathcal{N}, E) \), a set of intrinsic frequencies \( \{\omega_i\}_{i \in \mathcal{N}} \), and a central pattern \((\bar{\omega}, \{\Delta_{ij}\}_{ij})\), we seek the viability of assigning the given central pattern to the network under several scenarios that are governed by the choice of freedom we have for the design parameters:

- a set of interconnection weights \( \{\alpha_{ij}\} \in \mathbb{N}_0 \) (we allow \( \alpha_{ij} = 0 \) for design purposes),
- a set of phase biases \( \{\phi_i\}_{i \in \mathcal{N}} \),
- and a set of coupling functions \( \{f_i(\cdot)\}_{i \in \mathcal{N}} \).

Moreover, among the viable solutions that we can get, we further consider the problem of providing some of the additional desired characteristics such as:

- utilizing a minimal number of edges,
- minimizing the number of alternative central patterns,
- or assigning other given central patterns.

In so doing, we illustrate our findings with examples.

We begin with the following theorem on the viability of assigning one central pattern.

**Theorem 5.**

1. It is almost impossible to assign the given central pattern \((\bar{\omega}, \{\Delta_{ij}\})\) if we only have the freedom to choose the set of interconnection weights \( \{\alpha_{ij}\} \).

2. If we have freedom of choice for either the set of coupling functions \( \{f_i\}_{i \in \mathcal{N}} \) or the set of phase biases \( \{\phi_i\}_{i \in \mathcal{N}} \), then we could assign the given central pattern \((\bar{\omega}, \{\Delta_{ij}\})\).

3. If our coupling functions are the scaled version of a single fixed function \( f \), i.e., \( f_i(\cdot) = g_i \tilde{f}(\cdot) \) with positive coefficients \( g_i \) \footnote{The coefficients must be positive to ensure monotonicity.}, then we can assign the given central pattern \((\bar{\omega}, \{\Delta_{ij}\})\) if and only if

\[
\text{sgn} \left( \Delta \left( \sum_{j \in \mathcal{N}_i} \alpha_{ij} \Delta_{ij} + \phi_i \right) \right) = \text{sgn} (\bar{\omega} - \omega), \quad i \in \mathcal{N}. \tag{12}
\]

**Proof.** The first claim follows from the fact that

\[
\begin{bmatrix}
  f_1 \left( \sum_{j \in \mathcal{N}_i} \alpha_{ij} \Delta_{ij} + \phi_i \right) \\
  \vdots \\
  f_N \left( \sum_{j \in \mathcal{N}_i} \alpha_{ij} \Delta_{ij} + \phi_N \right)
\end{bmatrix}
\]

forms only a measure zero set in \( \mathbb{R}^N \), as it is parametrized by a set of integers \( \{\alpha_{ij}\} \), hence is countable.

The second claim follows from the fact that there always exist \( \{f_i\}_{i \in \mathcal{N}} \) or \( \{\phi_i\}_{i \in \mathcal{N}} \) such that

\[
f_i \left( \sum_{j \in \mathcal{N}_i} \alpha_{ij} \Delta_{ij} + \phi_i \right) = \bar{\omega} - \omega_i, \quad i \in \mathcal{N},
\]

as \( f_i, i \in \mathcal{N} \), are barrier functions. In particular, if we have freedom of choice for the set of coupling functions \( \{f_i\}_{i \in \mathcal{N}} \), then we choose any \( f_i \) such that it satisfies assumption \footnote{The coefficients must be positive to ensure monotonicity.} \( f_i(\theta_i) = \bar{\omega} - \omega_i \), where \( \theta_i = \sum_{j \in \mathcal{N}_i} \alpha_{ij} \Delta_{ij} + \phi_i \). Otherwise, if we have freedom of choice for the set of phase biases \( \{\phi_i\}_{i \in \mathcal{N}} \), then we choose \( \phi_i \in [-\pi, \pi) \) as

\[
\phi_i = f_i^{-1}(\bar{\omega} - \omega_i) - \sum_{j \in \mathcal{N}_i} \alpha_{ij} \Delta_{ij} \mod 2\pi.
\]

The final claim also follows from the fact that \( f_i, i \in \mathcal{N} \), are barrier functions. In particular, we choose \( g_i \), as

\[
g_i = \frac{\bar{\omega} - \omega_i}{f_i \left( \sum_{j \in \mathcal{N}_i} \alpha_{ij} \Delta_{ij} + \phi_i \right)}.
\]

Coupling functions that have a prototypical shape, i.e., \( f_i(\cdot) = g_i \tilde{f}(\cdot) \), are reminiscent of the physiology of CPGs, where the central patterns are designed by the maximal conductances \( g_i \) of the synaptic coupling \( g_i \tilde{f}(\cdot) \). From the engineering viewpoint, it simplifies the choice of the coupling functions to an algebraic problem.

**Proposition 1.** The necessary and sufficient condition \([12]\) for viability can always be satisfied

- If we have freedom of choice for the phase biases.
- If we only have freedom of choice for the interconnection weights, then \([12]\) can be satisfied when
  - \( \bar{\omega} \neq \omega_i \)
  - and there exists \( j \in \mathcal{N}_i \) such that \( \Delta_{ij}/(2\pi) \) is an irrational number.
- If, in addition, our interconnection weights are to take a prototypical ratio, i.e., \( \alpha_{ij} = \beta_i \bar{\alpha}_{ij} \) with positive integers \( \beta_i \), then \([12]\) can be satisfied when
  - \( \bar{\omega} \neq \omega_i \)
  - and \( \Delta_{ij}/(2\pi) \) is an irrational number, where \( \Delta := \sum_{j \in \mathcal{N}_i} \bar{\alpha}_{ij} \Delta_{ij} \).

**Proof.** The first item is trivial, as, for any given \( \sum_{j \in \mathcal{N}_i} \alpha_{ij} \Delta_{ij} \), we can always choose \( \phi_i \in [-\pi, \pi) \) so that \( f \left( \sum_{j \in \mathcal{N}_i} \alpha_{ij} \Delta_{ij} + \phi_i \right) \) becomes positive (or negative).

For the second and the third items, we first prove the following claim:

Let \( \delta \in (0, 1) \) be an irrational number. Then, for any \( \epsilon > 0 \), there exists \( n_\epsilon \in \mathbb{N} \) such that \( n_\epsilon \delta < \epsilon \mod 1 \).

**Proof of the claim:** The proof is by contradiction. Suppose that

\[
\epsilon^* := \inf_{n \in \mathbb{N}} n \delta \mod 1 > 0.
\]

Then, there exists \( n^* \in \mathbb{N} \) such that

\[
\epsilon^* \in \left[ \frac{1}{n^* + 1}, \frac{1}{n^* + 1} \right].
\]

By the definition of \( \epsilon^* \), for

\[
\tilde{\epsilon} := \frac{1 - n^* \epsilon^*}{n^* + 1} > 0,
\]
there exists \( \tilde{n}_\epsilon \in \mathbb{N} \) such that

\[
\epsilon^* \leq \tilde{n}_\epsilon \delta < \epsilon^* + \tilde{\epsilon} \mod 1.
\]

Now, if we let

\[
\tilde{\delta}_\epsilon := \tilde{n}_\epsilon \delta \mod 1.
\]

Then, this implies that \( \epsilon^* \leq \tilde{\delta}_\epsilon < \epsilon^* + \tilde{\epsilon} \), hence

\[
n^* \tilde{\delta}_\epsilon < n^*(\epsilon^* + \tilde{\epsilon}) = 1 - \tilde{\epsilon} < 1
\]

and

\[
(n^* + 1) \tilde{\delta}_\epsilon \geq (n^* + 1)\epsilon^* \geq 1.
\]

Since \( \delta \) is an irrational number, we have \((n^* + 1)\tilde{\delta}_\epsilon > 1\), and \( \tilde{\epsilon} \) further implies

\[
0 < (n^* + 1)\tilde{\delta}_\epsilon - 1 < \tilde{\delta}_\epsilon - \tilde{\epsilon} < \epsilon^*.
\]

In other words, we have

\[
(n^* + 1)\tilde{n}_\epsilon \delta < \epsilon^* \mod 1,
\]

which is a contradiction. This completes the proof of the claim.

Now, for the second item, under this sufficient condition on the existence of \( j \in \mathbb{N}_i \), we can simply let \( \alpha_{ik} = 0 \), \( j \neq k \in \mathbb{N}_i \) and seek for a non-negative integer \( \alpha_{ij} \) such that

\[
\text{sgn}(\tilde{f}(\alpha_{ij}\Delta_{ij} + \phi_i)) = \text{sgn}(\omega - \omega_i).
\]

Such an integer \( \alpha_{ij} \) always exists, because \( \Delta_{ij}/(2\pi) \) is an irrational number. In particular, without loss of generality assume that \( \tilde{\omega} - \omega_i \) is positive and let \( \tilde{\theta}_0 \in (-\pi, \pi) \) be such that \( \tilde{f}(\tilde{\theta}_0) = 0 \). Then, for \( \epsilon := (\pi - \tilde{\theta}_0)/(4\pi) \) and \( \delta := \Delta_{ij}/(2\pi) \), our claim ensures the existence of \( n\in \mathbb{N} \) such that \( n\epsilon\Delta_{ij}/(2\pi) < (\pi - \tilde{\theta}_0)/(4\pi) \mod 1 \). Therefore, if there exists \( \alpha_{ij} \in \mathbb{N}_0 \) such that

\[
\alpha_{ij}\Delta_{ij} + \phi_i \in (-\pi, \tilde{\theta}_0) \mod 2\pi,
\]

then this implies that there exists \( n \in \mathbb{N} \) such that

\[
(\alpha_{ij} + nn\epsilon)\Delta_{ij} + \phi_i \in (\tilde{\theta}_0, \pi) \mod 2\pi,
\]

hence \( \tilde{f}(\alpha_{ij} + nn\epsilon)\Delta_{ij} + \phi_i \) also becomes positive.

The third item follows similarly, except that we consider \( \Delta_i \) instead of \( \Delta_{ij} \):

\[
\text{sgn}(\tilde{f}(\tilde{\delta}_i\Delta_i + \phi_i)) = \text{sgn}(\omega - \omega_i).
\]

\[\square\]

Note that given a formation \( \{\Delta_{ij}\}_\mathcal{E} \), we can perform an arbitrarily small perturbation so that the above irrational number condition is satisfied, hence we can achieve our design goal with arbitrary precision. In particular, in the former case, we can even select an arbitrary edge to satisfy that \( \Delta_{ij}/(2\pi) \) is an irrational number. This is because, any formation \( \{\Delta_{ij}\}_\mathcal{E} \) can be generated by a sequence of phases \( \{\Delta_i\}_\mathcal{N} \) as \( \Delta_{ij} = \Delta_j - \Delta_i \mod 2\pi \), and thus, with any irrational number \( \delta \), if we perturb \( \{\Delta_i\}_\mathcal{N} \) as \( \Delta_i = \Delta_i + (\epsilon \cdot i \cdot \delta)2\pi \), \( i \in \mathcal{N} \), then for almost all sufficiently small rational numbers \( \epsilon \), \( \Delta_{ij}/(2\pi) \) becomes irrational.

Based on this viability analysis, in the following subsections, we further provide guidelines for achieving the additional desired characteristics, illustrated and further discussed by examples.

### 6.1. Design guideline for utilizing a minimal number of edges

Among all of the possible choices we could take for assigning a central pattern, we provide a design guideline that maximizes the number of interconnection weights that we can set to zero, given a digraph \( \mathcal{G} = (\mathcal{N}, \mathcal{E}) \).

If we have freedom of choice for either the set of coupling functions or the set of phase biases, then according to Theorem 3, we can simply choose our interconnection weights so that the reduced subgraph (governed by positive weights) still contains a spanning tree (Assumption 1), and maximizes the number of interconnection weights that are zero. In particular, \( \mathcal{N} - 1 \) edges is sufficient. For the design, one could pick any node from the unique iSCC of the original graph, and then take any spanning tree that connects to it.

On the other hand, if we consider the scaled version of coupling functions as in Theorem 3 and only have freedom of choice for the interconnection weights, then (to assign the given central pattern) we must include at least one edge \( (j, i) \) for each \( i \in \mathcal{N} \), such that \( \Delta_{ij}/(2\pi) \) is an irrational number (Proposition 1). So, for each \( i \in \mathcal{N} \), we need at least one positive interconnection weight \( \alpha_{ij} \). Now, if we denote such a set of at most \( N \) edges as \( \mathcal{E}^* \subset \mathcal{E} \), then an associated least communication subgraph, which contains \( \mathcal{E}^* \) (hence satisfies (12)) and satisfies Assumption 1, can be found as follows.

1. Consider the reduced graph \( \mathcal{G}_\mathcal{S}^* = (\mathcal{S}, \mathcal{E}_\mathcal{S}^*) \). Add a minimum number of edges so that the new reduced graph \( (\mathcal{S}, \mathcal{E}_\mathcal{S}) \) contains a spanning tree. This can be done as follows.

   (a) The reduced graph \( \mathcal{G}_\mathcal{S}^* \) consists of its isSCCs and followers. Consider each iSCC and its followers as a single node (a follower can be included in multiple nodes), and define an edge from one node to another if there is an edge in the original edge set \( \mathcal{E} \) from any agent inside one node to any agent in the iSCC of another node.

   (b) This new graph is strongly connected. So, take any spanning tree of it and add one corresponding edge from the original edge set \( \mathcal{E} \).

2. Then, include all other edges in \( \mathcal{E}^\prime \), and add a minimum number of edges so that the new subgraph \( (\mathcal{N}, \mathcal{E}_\mathcal{N}) \) contains a spanning tree. This can be done as follows.

   (a) The graph obtained by including all other edges in \( \mathcal{E}^\prime \) consists of its isSCCs and followers. By construction, there exists unique iSCC included in \( \mathcal{S} \).

   (b) Now, consider the iSCC included in \( \mathcal{S} \) and its followers as a single node and consider all other isSCCs each as a single node, and define a graph according to the edge set \( \mathcal{E} \).

   (c) Then, this new graph contains a spanning tree which has its root node as the node that corresponds to the iSCC included in \( \mathcal{S} \). Take this spanning tree and add one corresponding edge from the original edge set \( \mathcal{E} \).
Example 1. Let’s take an example for the above procedure. For this purpose, let us consider a graph \((N, E)\) with \(N = \{1, \ldots, 8\}\) and \(E\) with \((j, i) \in E\) represented by \(j \rightarrow i\) in Figure 1. If the formation \(\{\Delta_{ij}\}_E\) is governed by 
\[
\Delta_1 = 0, \quad \Delta_2 = -1/50, \quad \Delta_3 = \pi/4, \quad \Delta_4 = \pi/2, \\
\Delta_5 = 3\pi/4, \quad \Delta_6 = \pi + 1/100, \quad \Delta_7 = 5\pi/4 + 1/100, \\
\Delta_8 = 3\pi/2, \quad \Delta_9 = 7\pi/4,
\]
then the set \(E'\) and the set of edges \((j, i) \in E\) such that \(\Delta_{ij}/(2\pi)\) is an irrational number are obtained as in Figure 2. The procedure described above is illustrated in Figure 3.

![Figure 1: Illustration of the graphs \((N, E)\) and \((N', E')\), and the process 1) and 2) in obtaining the least communication subgraph in Example 1](image)

![Figure 2: Illustration of the graphs \((N', E')\) and \((N', E'')\), and the process 1) and 2) in obtaining the least communication subgraph in Example 1](image)

6.2. Design guideline for minimizing the number of alternative central patterns

Note that according to Corollary 11, when we choose our non-negative interconnection weights to be such that the reduced subgraph governed by positive weights is a spanning tree and those positive weights are unity, then our central pattern becomes unique and we have almost global convergence. This is because all of the followers have only one neighbor and \(\alpha_{ij} = 1, j \in N_i\), hence \(\{n_i\}_N \sim \{n_i^\pi\}_N\) if and only if \(\{n_i^\pi\}_S \sim \{n_i^\pi\}_S\). Note that \(S\) is a singleton \(\{\}\), hence \(\{\} \sim \{\}\) if and only if \(n^\pi_i = n^\pi_j\). Since \(\nu_i \equiv 0\), any admissible \(\{\} \sim \{\} \neq \emptyset\) gives \(n_i = 0\); the number of different central patterns is one.

In this manner, if we have freedom of choice for either the set of coupling functions or the set of phase biases, then as in Section 6.1, we can simply choose our interconnection weights so that the reduced subgraph is a spanning tree. Then, no alternative central pattern exists.

On the other hand, if our coupling functions have a prototypical shape as in Theorem 5 and only have freedom of choice for the interconnection weights, then in principle, \(\alpha_{ij}\) becomes a large integer and so the number of alternative central patterns becomes large. Meanwhile, if we have a large number of neighbors for each agent, and the formation is uniformly distributed, then we have a better chance of decreasing the number, as there likely be an edge \((j, i)\) such that \(\text{sgn}(f(\Delta_{ij} + \alpha_{ij})) = \text{sgn}(\omega - \omega_i)\). But, in general, finding a set of interconnection weights that gives a minimal number of alternative central patterns under the restriction of our coupling functions is a hard problem. The best we could do is to reduce the number of neighbors and reduce the interconnection weights, as, in general, the equivalence relation specified in Theorem 4 is complicated, and the fact that \(\alpha_{ij}\) is an integer gives an additional restriction via the equality (11) and it is most likely that different \(\{n_i\}_N\) are not equivalent. This is to reduce the number of admissible \(\{n_i\}_N \neq \emptyset\), in particular, its upper bound \(\prod_{i \in N} (2d_i + 1)\).

Remark 6. According to Remark 5, if we have chosen our interconnection weights, coupling functions, and phase biases, then we can simply cut our coupling functions at a finite region \((M^-_i, M^+_i)\), to satisfy \(\omega - \omega_i \in (M^-_i, M^+_i)\), \(i \in N\) only for the desired central pattern. This increases the chance to yield almost global convergence, even under the restriction \(f_i = g_i f\). The trajectory might converge to the boundary and stay, but we can always give a kick to make it converge to the desired central pattern.

6.3. Further discussions on Example 2

In this subsection, we follow Example 1. But, instead we consider an infinitesimal perturbation on the given formation, so that we can choose the set \(E'\) of edges \((j, i) \in E\) such that \(\Delta_{ij}/(2\pi)\) is an irrational number, which yields the graph represented in Figure 2 a spanning tree with one additional edge (as discussed at the end of Section 6.1). This is achieved for the formation \(\{\Delta_{ij}\}_E\) governed by 
\[
\begin{align*}
\Delta_1 &= 0, \quad \Delta_2 = -1/50, \quad \Delta_3 = \pi/4, \quad \Delta_4 = \pi/2, \\
\Delta_5 &= 3\pi/4, \quad \Delta_6 = \pi + 1/50, \quad \Delta_7 = 5\pi/4 + 1/100, \\
\Delta_8 &= 3\pi/2 + 1/50, \quad \Delta_9 = 7\pi/4 + 1/100.
\end{align*}
\]

Meanwhile, note that as illustrated after Proposition 1, we can introduce an infinitesimal small perturbation in the formation to select \(E'\) as whatever we want. In this sense, we can make the number of positive interconnection weights \(N - 1\) or \(N\), by choosing \(E'\) such that \((N', E')\) contains a spanning tree. In particular, we can simply make \((N', E')\) to be a spanning tree, if there exists \(i \in S\) such that 
\[
\text{sgn}(f(\phi_i)) = \text{sgn}(\omega - \omega_i),
\]
or if not, then make \((N', E')\) to be a spanning tree with an additional edge \((j, i)\) for the root node \(i\). Such an attempt for the situation illustrated in Example 1 can be found in Section 6.3.
Note that the only difference is $\Delta_6$, $\Delta_8$, and $\Delta_9$, and the difference is smaller than $1/50$.

$$\phi_{ij} = \phi_i - \phi_j$$

The corresponding simulation result with different initial conditions is given in Figure 4 (Remark 4), and they result in three different equivalence classes \{1\} \{0,1\} \{0,0\}, where \{0,1\} corresponds to the objective formation. The graph represents the phase differences $\theta_i(t)-\theta_j(t)$, $i \neq j$.

On the contrary, if we preserve the restriction that the phase biases $\{\phi_i\}_N$ are fixed as (14), but, as discussed in Remark 6, instead saturate the prototypical barrier function so that now $f(s) : (-\pi, \pi) \rightarrow (-200 - \delta, 40 + \delta]$ with sufficiently small $\delta \succ 0$, then we again obtain almost global convergence. The saturation region is chosen so that $f(\alpha_{ij}\Delta_{ij} + \phi_i) \in [-200, 40]$ for all $i \in \mathcal{N}$. Note that, in this case, we must have $g_1 = 1/\tan(\pi/2 - 1/40)$ and $g_2 = 1/\tan(\pi/2 - 1/200)$. This is illustrated in Figure 4(b) with the same initial condition that resulted in a different central pattern in Figure 3. This happens because, for the alternative central patterns, $\bar{\omega}$ does not sat-
isfy $\bar{\omega} - \omega_i \in (-g_i(200 + \delta), g_i(40 + \delta)), \ i \in \mathcal{N}$, as this is equivalent to $\bar{\omega} \in (1 - \eta_i, 1 + \eta_i)$ with some small $\eta_i > 0$, while they have their common frequency bigger than 2 for $\omega \in (0, 0)$ and smaller than 0 for $\omega = 1$. Therefore, the trajectory starting from that corresponding partition travels to the discontinuous boundary, which in this case, results in a transition to the partition that corresponds to our desired central pattern.

6.4. Assigning multiple central patterns

The following theorem characterizes the viability of assigning multiple central patterns $\bar{\omega}^k, (\Delta^k)\mathcal{E}$, $k = 1, \ldots, n^P$ for a fixed given connected digraph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ and a set of intrinsic frequencies $\{\omega_i\}_\mathcal{N}$.

**Theorem 6.** For a set of interconnection weights $\{\alpha_{ij}\}\mathcal{E}$ and a set of phase biases $\{\phi_i\}_\mathcal{N}$, if we have freedom of choice for the set of coupling functions $\{f_i\}_\mathcal{N}$, then we could assign the given multiple central patterns $\bar{\omega}^k, (\Delta^k)\mathcal{E}$, $k = 1, \ldots, n^P$ ($n^P < N^P$) if and only if

- $\Theta^k_{i} := \sum_{j \in \mathcal{N}} \alpha_{ij} \Delta^k_{ij} + \phi_i, k = 1, \ldots, n^P$ if and only if

$$\bar{\omega}^{1}_{i} \leq \bar{\omega}^{2}_{i} \leq \cdots \leq \bar{\omega}^{n^P}_{i},$$

then we have, with equality being preserved:

$$\Theta^{1}_{i} \leq \Theta^{2}_{i} \leq \cdots \leq \Theta^{n^P}_{i}, \ i \in \mathcal{N}.$$

**Hence,** if, in addition, we have freedom of choice for the set of phase biases $\{\phi_i\}_\mathcal{N}$, then, it is always possible when $n^P = 2$.

**Proof.** We can simply choose any $f_i$ such that it satisfies Assumption [2] and $f_i(\Theta^k_i) = \bar{\omega}^k - \omega_i$ for all $k = 1, \ldots, n^P$, where $\Theta^k_i = \sum_{j \in \mathcal{N}} \alpha_{ij} \Delta^k_{ij} + \phi_i, k = 1, \ldots, n^P$.

- Given a digraph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, a set of intrinsic frequencies $\{\omega_i\}_\mathcal{N}$, and multiple desired central patterns $\bar{\omega}^k, (\Delta^k)\mathcal{E}$, $k = 1, \ldots, n^P$, it might be possible to select a set of interconnection weights $\{\alpha_{ij}\}_\mathcal{E}$ so that the above necessary and sufficient condition is fulfilled. However, in such cases, due to the choice of interconnection weights (which in general satisfies $\alpha_{ij} > 1$) the number of alternative central patterns increases (at least the number of $\{n_i\}_\mathcal{N}$ such that $\Theta(n_i) \neq 0$).

- Theorem [3] and the corresponding advantages in formation control is illustrated in the following examples.

**Example 2.** For digraph $\mathcal{G} = \{(1, 2), (1, 2), (2, 1)\}$, if $\alpha_{12} = 1$ and $\alpha_{21} = 1$, $\phi_1 = \pi/2$ and $\phi_2 = \pi/2$, then we have two different central patterns. If $\omega_1 = 0, \omega_2 = 1, \omega_3 = 2$, then $\bar{\omega}_2 = \pi/2$. Thus, we can set the alternate central pattern to be near the boundary, for instance, as $\bar{\omega}' = 2$ and $\Delta_2 = 2\pi/3$, since $\Theta_1 = -\pi/2, \Theta_2 = \pi - \epsilon$ and $\Theta_2 = -\pi/2, \Theta_2 = \epsilon$. A suitable set of coupling functions is

$$f_i(s) = \frac{\tan(\frac{s}{2}) + \tan(\frac{\pi}{2})}{\tan(\frac{s}{2}) + \tan(\frac{\pi}{2})} + 1,$$

$$f_2(s) = \frac{\tan(\frac{s}{2}) + \tan(\frac{\pi}{2})}{\tan(\frac{s}{2}) + \tan(\frac{\pi}{2})} - 1.$$

Then, since only the alternative central pattern is near the boundary, with a persistent small kick (e.g., a train of impulsive inputs) on $\Theta_1$ or $\Theta_2$, we have almost global convergence to the desired central pattern $\bar{\omega} = 1$ and $\Delta_2 = \pi$.

This is illustrated in Figure 5.

**Example 3.** Another example is given for the three agents that constitute a directed ring: $\mathcal{E} = \{(2, 1), (3, 2), (1, 3)\}$. From Remark [4] we notice that if $\phi_2 = -\pi$, then the number of different $n_2$ is two, and by the structure of the Laplacian matrix, if $\alpha_{12} = 1$, for all $(j, i) \in \mathcal{E}$, then all the sequences associated with each $n_2$ are equivalent (Remark [5]). Therefore, the number of different central patterns becomes two. We assign for this network a uniformly distributed central pattern with two different permutations by letting

$$\bar{\omega}_1 = -1, \ \Delta_1 = 0, \ \Delta_2 = 2\pi/3, \ \Delta_3 = 4\pi/3,$$

$$\bar{\omega}_2 = 1, \ \Delta_1 = 0, \ \Delta_2 = 4\pi/3, \ \Delta_3 = 2\pi/3.$$

To satisfy the necessary and sufficient condition in Theorem [4] and to satisfy $\phi_2 = \sum_{i=1}^{3} \phi_i = -\pi$, we utilize

$$\phi_1 = -\frac{3\pi}{4}, \ \phi_2 = -\frac{3\pi}{4}, \ \phi_3 = \frac{\pi}{2}.$$

Then, for any intrinsic frequency $\omega_1$, we can assign both central patterns simultaneously. Here, we take $\omega_1 = -2, \ \omega_2 = 0, \ \omega_3 = 2$. A suitable set of coupling functions is

$$f_1(s) = 2 \frac{\tan(\frac{s}{2}) + \tan(\frac{\pi}{2})}{\tan(\frac{s}{2}) + \tan(\frac{\pi}{2})} + 1,$$

$$f_2(s) = 2 \frac{\tan(\frac{s}{2}) + \tan(\frac{\pi}{2})}{\tan(\frac{s}{2}) + \tan(\frac{\pi}{2})} - 1,$$

$$f_3(s) = 2 \frac{\tan(\frac{s}{2}) + \tan(\frac{\pi}{2})}{\tan(\frac{s}{2}) + \tan(\frac{\pi}{2})} - 1.$$

This is illustrated in Figure 7.

**Example 4.** Our final example concerns an arbitrary odd number $N > 1$ with star shaped graph $\mathcal{G} = \{(1, \ldots, N),\{(1, 2), (1, N), (2, 1)\}\}$. If $\alpha_{ij} = 1$ for all $(j, i) \in \mathcal{E}$, $\phi_1 = (N - 1)\pi/N$ and $\phi_2 = 0$, then we have two different central patterns. In particular, if $\omega_i = 0$
for all $i \in \mathcal{N}$, $\phi_i = 0$, $i = 2, \ldots, (N + 1)/2$ and $\phi_i = (N - 1)\pi/N$, $i = (N + 3)/2, \ldots, N$, then we can assign two central patterns (with completely opposite behaviors), where one represents perfect balanced formation, i.e., $\omega = 0$, $\Delta_i = 2(i - 1)\pi/N$ for all $i = 1, \ldots, N$, and the other represents perfect synchronisation, i.e., $\omega' = 1$, $\Delta_i' = 0$ for all $i = 1, \ldots, N$. This is because, $0 = \omega < \omega' = 1$ and $-(N - 1)\pi/N = \Theta_i < \Theta_i' = (N - 1)\pi/N$, $-2(i - 1)\pi/N = \Theta_i < \Theta_i' = 0$, $i = 2, \ldots, (N + 1)/2$, and $-(2i - N - 1)\pi/N = \Theta_i < \Theta_i' = (N - 1)\pi/N$, $i = (N + 3)/2, \ldots, N$. We can utilize coupling functions in the form of $f_i(s) = a_i \tan(s/2) + b_i$, where $a_i$ and $b_i$ form the unique solution of the linear equation
\[
\begin{bmatrix}
\tan\left(-\frac{N-1}{2N}\pi\right) & 1 \\
\tan\left(-\frac{N}{2N}\pi\right) & 1
\end{bmatrix}
\begin{bmatrix}
a_1 \\
b_1
\end{bmatrix}
= \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
For $i = 2, \ldots, (N+1)/2$, $a_i$ and $b_i$ form the unique solution of the linear equation
\[
\begin{bmatrix}
\tan\left(-\frac{i-1}{2N}\pi\right) & 1 \\
\tan(0) & 1
\end{bmatrix}
\begin{bmatrix}
a_i \\
b_i
\end{bmatrix}
= \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
For $i = (N+3)/2, \ldots, N$, $a_i$ and $b_i$ form the unique solution of the linear equation
\[
\begin{bmatrix}
\tan\left(-\frac{2N-1}{2N}\pi\right) & 1 \\
\tan\left(-\frac{2N-3}{2N}\pi\right) & 1
\end{bmatrix}
\begin{bmatrix}
a_i \\
b_i
\end{bmatrix}
= \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
This is illustrated in Figure 7 for the case when $N = 9$.

7. Conclusion

By introducing the node-wise monotone barrier coupling law, we proposed a tool to simultaneously assign multiple central patterns on the circle, where a transition between different patterns can happen via a simple 'kick.' This shows one of the benefits of using a nonlinear coupling, especially in the emergence of collective behaviors, and is motivated by CPGs. We characterized the shape of the generated central patterns, identified the number of different central patterns, analyzed the viability of assigning desired patterns, and provided design guidelines.

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Appendix A. Essential graph theoretical lemma

The following lemma is essential in the proof of Theorem 1 which is given in Appendix B.

Lemma 1. Under Assumption 1, for any \( \mathcal{I} \subseteq \mathcal{N} \) such that \( \mathcal{S} \setminus \mathcal{I} \neq \emptyset \), where \( \mathcal{S} \) denotes the unique iSCC, there exists \( \zeta_i > 0 \), for \( i \in \mathcal{I} \), such that for any vector \( \chi = [\chi_i] \in \mathbb{R}^N \),

\[
\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{N}} \alpha_{ij} (\chi_j - \chi_i) = \sum_{j \in \mathcal{N} \setminus \mathcal{I}} \beta_j \chi_j - \sum_{i \in \mathcal{R}} \gamma_i \chi_i \quad (A.1)
\]

with some coefficients \( \beta_j, \gamma_i \geq 0 \). Moreover, there exists \( j^* \in \mathcal{N} \setminus \mathcal{I} \) and \( i^* \in \mathcal{I} \) such that \( \beta_{j^*} > 0 \) and \( \gamma_i > 0 \).

Proof. Consider a subgraph \( \mathcal{G}_\mathcal{I} = (\mathcal{I}, \mathcal{E}_\mathcal{I}) \). Then, there exists iSCCs denoted as \( \mathcal{S}^p \subset \mathcal{I} \), \( p \in \mathcal{M} := \{1, \ldots, M\} \), and the rest, denoted as \( \mathcal{R} \subset \mathcal{I} \). We will first assume that Lemma 1 holds for the index set \( \mathcal{R} \); that is, \( \zeta_i > 0 \) is given for \( i \in \mathcal{R} \) such that for any vector \( \chi = [\chi_i] \in \mathbb{R}^N \),

\[
\sum_{i \in \mathcal{R}} \sum_{j \in \mathcal{N}} \alpha_{ij} (\chi_j - \chi_i) = \sum_{j \in \mathcal{N} \setminus \mathcal{I}} \beta_j \chi_j - \sum_{i \in \mathcal{R}} \gamma_i \chi_i,
\]

then we will show that for each \( p \in \mathcal{M} \), we can find \( \zeta_i > 0 \), \( i \in \mathcal{S}^p \) such that for any vector \( \chi = [\chi_i] \in \mathbb{R}^N \),

\[
\sum_{i \in \mathcal{S}^p} \sum_{j \in \mathcal{N}} \alpha_{ij} (\chi_j - \chi_i) = \sum_{j \in \mathcal{N} \setminus \mathcal{I}} \beta_j \chi_j - \sum_{i \in \mathcal{S}^p} \gamma_i \chi_i \quad \text{and} \quad \gamma_i > \beta_i, \quad \forall i \in \mathcal{S}^p. \quad (A.2)
\]

This will complete the proof. By repeating this argument (i.e., by replacing the role of \( \mathcal{I} \) with \( \mathcal{R} \)), we arrive after finitely many steps at the stage of assuming that Lemma 1 holds for the index set \( \mathcal{R} \), but \( \mathcal{R} \) is empty; that is the subgraph \( \mathcal{G}_\mathcal{I} \) only consists of iSCCs, so that the assumption holds trivially.

Let us carry out the above described proof steps. For this purpose, note that by Assumption 1 and the fact that \( \mathcal{S} \setminus \mathcal{I} \neq \emptyset \), for each \( p \in \mathcal{M} \), there exist \( j \in \mathcal{N} \setminus \mathcal{I} \) and \( i \in \mathcal{S}^p \) such that \( (j, i) \in \mathcal{E} \). Now, let \( \mathcal{L}^p \) denote the Laplacian matrix of the subgraph \( \mathcal{G}^p = (\mathcal{S}^p, \mathcal{E}^p) \). Then, (A.2) for only those corresponding to \( \mathcal{S}^p \) can be represented as

\[
(\xi^p)^T (\mathcal{L}^p + \mathcal{D}^p) = (\gamma^p)^T > (\beta^p)^T
\]

with some diagonal matrix \( \mathcal{D}^p \geq 0 \) such that \( \mathcal{D}^p \neq 0 \) by the existence of \( (j, i) \in \mathcal{E} \), \( j \in \mathcal{N} \setminus \mathcal{I}, i \in \mathcal{S}^p \). So, first let \( \xi_1^p \) be the left eigenvector of \( \mathcal{L}^p \) associated with the zero eigenvalue. Then, we have \( \xi_1^p > 0 \) and

\[
(\xi_1^p)^T (\mathcal{L}^p + \mathcal{D}^p) = (\xi_1^p)^T \mathcal{D}^p \geq 0.
\]

Now, let \( \mathcal{N}_1^p \subset \mathcal{S}^p \) be the set of all indices \( i \in \mathcal{S}^p \) such that the element of \( (\xi_1^p)^T \mathcal{D}^p \) corresponding to agent \( i \) is positive. Clearly, \( \mathcal{N}_1^p \) is not empty. Now, we can find sufficiently small \( \epsilon_1 > 0 \) such that\(^8\)

\[
\epsilon_1^p := \epsilon_1 \mathbf{1}_{\mathcal{N}_1^p} > 0
\]

\(^8\)This is because, otherwise \( \mathcal{S}^p \) becomes the unique iSCC \( \mathcal{S} \) of the entire graph, but this is not possible, since \( \mathcal{S} \setminus \mathcal{I} \neq \emptyset \).
satisfies
\[(\xi^2)^T(\mathcal{L}^p + D^p) = -\varepsilon_1 T_{i,j}(\mathcal{L}^p + D^p) + (\xi^2)^T D^p \geq 0\]
and that \(N_1^2 \subseteq N_2^2\). In particular,
\[N_2^2 = N_1^2 \cup \{j \in S^p \mid (j, i) \in \mathcal{E}\}\]

Therefore, by repeating this argument, due to the fact that \(G^p\) is strongly connected, we arrive after finitely many steps at the situation where \(N_2^p = S^p\). This implies that
\[(\xi^2)^T(\mathcal{L}^p + D^p) > 0\]

Therefore, we can find sufficiently large \(K > 0\) such that \(\zeta^p := K \xi^2 > 0\) satisfies
\[(\zeta^p)^T(\mathcal{L}^p + D^p) > (\beta^p)^T\]
and we conclude (A.2) as desired.

Appendix B. Proof of Theorem \[\text{[1]}\]

Appendix B.1. \(\Theta^p_{(n),s}\) is positively invariant

The proof is by contradiction. Suppose that there is a particular solution of (3) such that \(\theta \in \Theta^p_{(n),s}\) holds only for a finite time interval \([0, T]\) and is violated at \(t = T\). This implies that there is a time sequence \(\{\tau_k\}\) such that \(\tau_k\) is strictly increasing and \(\lim_{k \to \infty} \tau_k = T\), and
\[\mathcal{I}_+\{\tau_k\} := \left\{ i \in \mathcal{N} : \lim_{k \to \infty} \nu_i(\tau_k) + \phi_i = (2n_i + 1)\pi \right\}
\]
is non-empty, or
\[\mathcal{I}_-\{\tau_k\} := \left\{ i \in \mathcal{N} : \lim_{k \to \infty} \nu_i(\tau_k) + \phi_i = (2n_i - 1)\pi \right\}
\]
is non-empty,

where \(n_i \in \mathcal{I}\) is such that \(\nu_i(0) + \phi_i = ((2n_i - 1)\pi, (2n_i + 1)\pi)\).\[\text{[10]}\]

Let us first assume that \(\mathcal{I}_+\{\tau_k\}\) is non-empty. We will first show that a contradiction occurs if \(S \subseteq \mathcal{I}_+\{\tau_k\}\), where \(S\) denotes the unique SCC. If \(S \setminus \mathcal{I}_+\{\tau_k\} \neq \emptyset\), we will then show that it is possible to construct another time sequence \(\{\tilde{\tau}_k\}\) (based on \(\{\tau_k\}\)), such that
\[|\mathcal{I}_+\{\tilde{\tau}_k\}| < |\mathcal{I}_+\{\tau_k\}| \tag{B.1}\]

where the notation \(| \cdot |\) implies the cardinality of the set. By repeating the argument (i.e., by replacing the role of \(\{\tau_k\}\) with \(\{\tilde{\tau}_k\}\)), we arrive after finitely many steps at the index set \(\mathcal{I}_+\{\tilde{\tau}_k\}\) such that \(S \subseteq \mathcal{I}_+\{\tilde{\tau}_k\}\), which yields a contradiction. This means that there is no such sequence \(\{\tau_k\}\) that makes \(\mathcal{I}_+\{\tau_k\}\) non-empty. Similarly, it can be seen that there is no sequence that makes \(\mathcal{I}_-\{\tau_k\}\) non-empty. Therefore, we conclude there is no such finite time \(T\), and thus \(\theta(t) \in \Theta^p_{(n),s}\) for all \(t \geq 0\).

Let us carry out the steps of the proof described above. For convenience, we write \(\mathcal{I}\) instead of \(\mathcal{I}_+\{\tau_k\}\) in the following. Note that, by the definition of \(\mathcal{I}\), for each \(\eta > 0\), there exists a sufficiently large \(k^* \in \mathbb{N}\) such that \(\nu_i(\tau_k) + \phi_i > (2n_i + 1)\pi - \eta\) for all \(k \geq k^*\). Hence, if \(S \subseteq \mathcal{I}\), then for any given set of positive coefficients \(\zeta_i, i \in \mathcal{S}\), there is \(k\) such that\[\text{[11]}\]
\[\sum_{i \in S} \zeta_i \nu_i(t) = \sum_{j \in \mathcal{I}} \alpha_{ij} \cdot (\theta_j(\tau_k) - \theta_i(\tau_k)) \neq 0\]

However, this is violated if \(\zeta_i, i \in \mathcal{S}\) corresponds to the elements of the left eigenvector of the Laplacian matrix associated with the zero eigenvalue, because then
\[\sum_{i \in S} \zeta_i \sum_{j \in \mathcal{I}} \alpha_{ij} (\theta_j(t) - \theta_i(t)) \equiv 0, \quad \forall t \in [0, T].\]

Hence we have shown that \(S \subseteq \mathcal{I}\) is not possible and we continue the proof for the case that \(S \setminus \mathcal{I} \neq \emptyset\). For this purpose, let
\[W(t) := \sum_{i \in \mathcal{I}} \zeta_i \nu_i(t) = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \alpha_{ij} \cdot (\theta_j(t) - \theta_i(t)),\]

where \(\zeta_i > 0, i \in \mathcal{I}\) is given by Lemma \[\text{[1]}\] considering the index set \(\mathcal{I} \subseteq \mathcal{N}\). Note that \(W(t)\) is continuously differentiable, \(W(t) < \sum_{i \in \mathcal{I}} \zeta_i ((2n_i + 1)\pi - \phi_i) =: \overline{W}_\mathcal{I}\) on \([0, T]\), and \(\lim_{k \to \infty} W(\tau_k) = \overline{W}_\mathcal{I}\). Let us now consider a strictly decreasing sequence \(\{\varepsilon_q\}\) of positive numbers such that \(\lim_{q \to \infty} \varepsilon_q = 0\) and that \(W(0) < \overline{W}_\mathcal{I} - \varepsilon_0\). Choose a subsequence \(\{\kappa_q\}_{q \in \mathbb{N}}\) of \(\{\tau_k\}\) such that
\[W(\kappa_q) \geq \overline{W}_\mathcal{I} - \frac{\epsilon_q}{2}, \quad \forall q \in \mathbb{N}. \tag{B.2}\]

Based on this subsequence, we now construct a sequence \(\{s_q\}_{q \in \mathbb{N}}\) such that (see Figure \[\text{B.8}\])
\[s_q := \max \left\{ s \in [0, \kappa_q] \mid W(s) = \overline{W}_\mathcal{I} - \varepsilon_q \right\}. \tag{B.3}\]

![Figure B.8: Illustration of the choice of the sequence \(\{s_q\}\) based on \(\{\tau_k\}\).](image)

By \[\text{B.2}\] and \[\text{B.3}\], the sequence \(\{s_q\}\) is strictly increasing and \(\lim_{q \to \infty} s_q = T\). Moreover, since \(\lim_{q \to \infty} W(s_q) = \overline{W}_\mathcal{I}\),
\[\lim_{q \to \infty} \nu_i(s_q) + \phi_i = (2n_i + 1)\pi, \quad \forall i \in \mathcal{I}. \tag{B.4}\]

In addition, from \[\text{B.2}\] and \[\text{B.3}\], it follows that \(s_q < \tau_{\kappa_q}\) and
\[W(s_q) \geq 0, \quad \forall q \in \mathbb{N}. \tag{B.5}\]

\[\text{[11]}\] This is because, if \(\sum_{i \in \mathcal{S}} \zeta_i ((2n_i + 1)\pi - \phi_i) = 0\), then the sum converges to zero but is negative for all finite \(k\), and otherwise if \(\sum_{i \in \mathcal{S}} \zeta_i ((2n_i + 1)\pi - \phi_i) \neq 0\), then there exists sufficiently small \(\eta\) such that the sum is non-zero for all \(k \geq k^*\).
On the other hand, if we compute  the estimate of the trajectory:
\[
\tilde{W}(t) = \sum_{i \in I} \zeta_i \sum_{j \in \mathcal{N}} \alpha_{ij} (\omega_j - \omega_i) + \sum_{i \in \mathcal{I}} \zeta_i \sum_{j \in \mathcal{N}} \alpha_{ij} (f_j(t) - f_i(t)),
\]
where \( f_k(t) := f_k(\nu_k(t) + \phi_k), \) \( k \in \mathcal{N}, \) for simplicity. We denote the first sum by \( M_0. \) Invoking Lemma 1 for the index set \( \mathcal{I}, \) we therefore have that \( (\beta_j, \gamma_i \geq 0) \)
\[
\tilde{W}(t) \leq M_0 + \sum_{j \in \mathcal{N} \setminus \mathcal{I}} \beta_j f_j(t) - \sum_{i \in \mathcal{I}} \gamma_i f_i(t). \tag{B.6}
\]
Let \( \mathcal{J} := \mathcal{N} \setminus \mathcal{I} \) (which is non-empty). Then, (B.5) and (B.6) yield
\[
\sum_{j \in \mathcal{J}} \beta_j f_j(s_q) \geq \sum_{i \in \mathcal{I}} \gamma_i f_i(s_q) - M_0 =: M_q.
\]
Again by Lemma 1, at least one \( \beta_j (\gamma_i), \) where \( j \in \mathcal{J} (i \in \mathcal{I}), \) is positive. Thus, it follows from (B.4) that \( M_q \to \infty \) as \( q \to \infty. \) Since
\[
\sum_{j \in \mathcal{J}} \beta_j f_j(s_q) \leq \bar{\beta} \sum_{j \in \mathcal{J}} \max \{ f_j(s_q), 0 \}
\]
where \( \bar{\beta} := \max_{j \in \mathcal{J}} \beta_j > 0, \) we have
\[
\sum_{j \in \mathcal{J}} \max \{ f_j(s_q), 0 \} \geq \frac{M_q}{\bar{\beta}}. \tag{B.7}
\]
Therefore, for each sufficiently large \( q, \) there is an index \( j_k \in \mathcal{J} \) such that \( f_{j_k}(s_q) \geq M_q/(|\mathcal{J}| \bar{\beta}). \) Since \( \mathcal{J} \) is a finite set, there is a subsequence \( \{ \tilde{\tau}_k \} = \{ s_{n_k} \} \) such that \( j^* = j_{n_k} \in \mathcal{J}; \) hence
\[
f_{j^*}(\nu_j(\tilde{\tau}_k) + \phi_{j^*}) \to \infty
\]
i.e., \( \nu_j(\tilde{\tau}_k) + \phi_{j^*} \to (2n_{j^*} + 1)\pi. \) Consequently,
\[
\mathcal{I}_+((\tau_k)) \subseteq \mathcal{I}_+(\{ s_q \}) \subseteq \mathcal{I}_+((\tilde{\tau}_k)).
\]
By construction, \( j^* \in \mathcal{I}_+(\{ \tilde{\tau}_k \}) \setminus \mathcal{I}_+(\{ \tau_k \}) \) and we can conclude (B.3) as desired.

Noting that the above proof holds even if we let \( T = \infty, \) so that there is no time sequence \( \{ \tau_k \} \) such that \( \tau_k \) is strictly increasing and \( \lim_{k \to \infty} \tau_k = \infty \) which makes \( \mathcal{I}_+(\{ \tau_k \}) (\mathcal{I}_-(\{ \tau_k \}) \) non-empty, it follows that the control input \( f_{j^*}(\nu_j(t) + \phi_{j^*}) \) is bounded uniformly on \([0, T) \) \( = [0, \infty), \)

Appendix B. Convergence to a unique central pattern

Given the forward invariance in \( \Theta_{\{\nu_i\}}^R, \) the convergence of the trajectory to an integral curve corresponding to a central pattern of phase locking behavior can be shown simply by monotonicity. In particular, the linearization of (4) can be found as
\[
\dot{\theta}_i = f'_i(\nu_i + \phi_i) \sum_{j \in \mathcal{N}_i} \alpha_{ij} (\delta \theta_j - \delta \theta_i), \quad i \in \mathcal{N}
\]
and is strictly monotonic with respect to the positive orthonormal inside the positively invariant set \( \Theta_{\{\nu_i\}}^R \) due to Assumptions 1 and 2. The Perron-Frobenius vector of the system on \( \Theta_{\{\nu_i\}}^R \) is \( 1_N \) as this is the eigenvector of
\[
-\text{diag}(f'_1(\nu_1 + \phi_1), \ldots, f'_N(\nu_N + \phi_N)) \mathcal{L}
\]