Non-commutative Kerr Black Hole

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Abstract

We investigate the behavior of a non-commutative Kerr black hole, inspired by a recent proposal of Aschieri et al. about a deformation of the metric for canonically deformed spaces with constant deformation parameter. It is shown that non-commutativity modifies the Hawking temperature and the efficiency of the Penrose process of energy extraction from a black hole.

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I. INTRODUCTION

In general relativity the problems concerning the singularities in gravitational collapse are at the base of acceptability of the Einstein theory.

In 1795 Laplace, relying upon Newtonian gravity, found that a very dense and massive object would appear black, because light would not be able to escape from it. Then in 1915 Einstein developed the theory of general relativity which predicts the possible existence of such dark objects, black holes, caused by singularities, which are objects with infinite curvature resulting from an infinite density, so that everything nearby is drawn into the black hole.

In the first half of the twentieth century, in particular in 1916, Schwarzschild obtained the first solution for the Einstein equation of general relativity, which described the space-time around a static spherically symmetric massive object, that does not have an angular momentum or charge. This was called, since then, the Schwarzschild metric.

Later, in 1963, Kerr discovered another solution: the Kerr metric, which describes the space-time outside a massive axi-symmetric rotating object. A rotating black hole has rotation in addition to the static black hole.

These two solutions describe the static and rotating black holes respectively. Thus, black hole solutions of the Einstein equations are characterized by three parameters, i.e., mass $M$, angular momentum $J$, and charge $Q$ by the no-hair, or uniqueness, theorem [1].

The description of a rotating black hole uses two of the three parameters: mass and angular momentum.

These three properties are conserved during the collapse of the star, because of global conservation laws. All other properties of the collapsed star to the black hole are lost during the collapse (no Hair theorem). Then there are four laws, derived from standard laws of physics, which describe the thermodynamics of a black hole [2]. In particular the temperature $T = \kappa/2\pi$ is the Hawking temperature of the black hole and is defined by [2, 3, 4, 5, 6, 7]:

$$T \equiv -\left( \frac{1}{4\pi\sqrt{-G_{00}G_{rr}}} \frac{dG_{00}}{dr} \right)_{r=r_H}$$  \hspace{1cm} (1.1)

The Kerr metric describes the space-time around a rotating or spinning singularity (black hole) without charge and time-independent, axi-symmetric gravitational field of a collapsed object that has retained its angular momentum. Stellar black holes are caused by the collapse
of stars. A star is a very massive, rotating but charge-less object. Because charges of opposite sign cancel each other, stars are neutral. Hence, the space-time around a stellar black hole is described by the Kerr metric. Both metrics are able to describe black holes that are caused by curvature singularities; they share a coordinate singularity at their event horizon; in both metrics two observers experience a time-dilation and curvature, and for both of them the space-time is empty (with the exception of the one massive object) and asymptotically flat. Different coordinate systems are indeed used for the Kerr metric, and a coordinate system commonly used is the Boyer-Lindquist which exploits spherical coordinates. These are easy to work with and some features are easily noticed, but it has a coordinate singularity at the event horizon. Thus, using different coordinate systems, different properties or features of a rotating black hole can be described.

In the second half of the twentieth century, the black hole is no longer a theoretical construction, but it could be a real physical object because astronomers observed very small objects that emitted jets of particles with very high energy. They proposed black holes as the sources of these jets. In fact massive stars undergoing a gravitational collapse in the final state, are expected to become a black hole. Moreover, a rotating star will collapse in a spinning or rotating black hole. Thus, the Kerr metric is a good candidate to describe the space-time around the final state of a very massive star.

Rotating black holes, discovered by Kerr as exact solutions to general relativity equations, are of great astrophysical interest because their emissions provide a method for identifying and studying black holes.

There are several motivations for studying black holes and the metrics that describe the space-time around them; one is that they form objects which probably have to be explained in terms of quantum gravity: large mass at small size. Secondly, because stellar mass black holes give information about the last stage in stellar evolution, thirdly these black holes are important in cosmology since they might be seeds of galaxy formation. However, it is unclear whether they are properly described by the Kerr metric. Super-massive black holes can give information about the very early universe era.

Hence there are three general types of black holes: stellar black holes, primordial black holes and super-massive black holes, distinguished by their mass and size.

There are several ways in which black holes could be observed in an indirect way: X-ray, spectral shift, gravitational lensing. The only direct way to observe black holes is via
gravitational waves. So far, no black hole has been directly observed.

In astrophysical contexts $Q$ is negligible, because the electric charge is shortened out by the surrounding plasma [9]. Thus, the variation in the observational properties of black holes is due to external parameters, such as the angle between the black hole spin vector and the line of sight, the gas accretion flow geometry and black hole spin $j \equiv J/M^2 = a/M$.

The Kerr metric written in the Boyer-Lindquist coordinates reads as

$$ds^2 = -(1 - \frac{ar}{\Sigma})dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left\{ (r^2 + h^2) \sin^2 \theta + \frac{a r h \sin^4 \theta}{\Sigma} \right\} d\phi^2 + 2 \frac{a r h \sin^2 \theta}{\Sigma} dtd\phi,$$

(1.2)

where $2M$ (the mass term), $a$ (the angular momentum) are two constants and $\Sigma$, $\Delta$ are given by

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + h^2 - 2Mr.$$  

(1.3)

On the other hand, from the point of view of current developments in field theory, the axisymmetry of the Kerr metric makes it interesting to study how non-commutativity would affect it. Following Ref. [? ], we assume that non-commutativity of space-time can be encoded in the commutator of operators corresponding to space-time coordinates, i.e. (the integer $D$ below is even)

$$\comm{x^\mu, x^\nu} = i \Lambda^{\mu\nu}, \quad \mu, \nu = 1, 2, ..., D,$$

(1.4)

where the antisymmetric matrix $\Lambda^{\mu\nu}$ is taken to have block-diagonal form

$$\Lambda^{\mu\nu} = \text{diag} \left( \Lambda_1, ..., \Lambda_{D/2} \right),$$

(1.5)

with

$$\Lambda_i = \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \forall i = 1, 2, ..., D/2,$$

(1.6)

the parameter $\lambda$ having dimension of length squared and being constant. The authors of Ref. [? ] solve the Einstein equations with mass density of a static, spherically symmetric, smeared particle-like gravitational source as (hereafter we work in $G = c = \hbar = 1$ units)

$$\rho_\lambda(r) = \frac{M}{(4\pi\lambda)^2} e^{-\frac{r^2}{4\lambda}}.$$  

(1.7)

We here use a different non-commutative prescription [10] to analyze the modification of Schwarzschild and Kerr metric.
We shall now introduce the plan of the paper. In Section 2 we apply the prescription of Aschieri et al. [10] to the Kerr metric, and we study the modifications on temperature. In Section 3 we compare different prescriptions existing in the literature when applied to the Schwarzschild case and discuss their implications on the process of energy extraction from a black hole. In Section 4 concluding remarks are presented. Relevant details are given in the appendix.

II. NON-COMMUTATIVE PRESCRIPTION OF ASCHIERI. EXPANSION IN $\Lambda$ OF THE AXISYMMETRIC METRIC

Following [10] the basic quantity is the deformed tetrad $E^a_\mu$ to all orders in the non-commutativity tensor $\Lambda$, denoted by $\theta$ in [10]. On the other hand, in general relativity, the tetrad for the Kerr metric takes the form [11]

$$
(e^a_\mu) = \begin{pmatrix}
    i\sqrt{1 - \frac{2Mr}{\Sigma}} & 0 & 0 & -i\frac{a2Mr \sin^2 \theta}{\sqrt{\Sigma(r^2 + a^2 \cos^2 \theta - 2Mr)}} \\
    0 & \frac{\Sigma}{\Delta} & 0 & 0 \\
    0 & 0 & \sqrt{\Sigma} \\
    0 & 0 & 0 & r\sqrt{(r^2 + a^2) \sin^2 \theta + \frac{2Ma^2 r \sin^4 \theta}{r^2 \Sigma} + \frac{(2M)^2 a^2 \sin^4 \theta}{(2.1)}}
\end{pmatrix}.
$$

As is clear from (2.1) this is just the square root of the Kerr metric. Hence, bearing in mind that [10]

$$
G_{\mu\nu} = \frac{1}{2} \left( E^a_\mu \star E^b_\nu + E^a_\nu \star E^b_\mu \right) \eta_{ab},
$$

$$
G^{(0)}_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu},
$$

where the non-commutativity plays a role both in the deformed tetrad $E^a_\mu$ and in the star-product, one finds up to second order (the contribution is only due to the $\Lambda^{23}$ component, because the metric depends on $(r, \theta)$)

$$
G_{\mu\nu} = g_{\mu\nu} - \frac{1}{8} \Lambda^{a_1b_1} \Lambda^{a_2b_2} (\partial_{a_1} \partial_{a_2} e^a_\mu)(\partial_{b_1} \partial_{b_2} e^b_\nu) \eta_{ab} + \ldots
$$

$$
= g_{\mu\nu} - \frac{1}{8} (\Lambda^{23})^2 \left[ (\partial_2 \partial_2 e^a_\mu)(\partial_3 \partial_3 e^b_\nu) + (\partial_3 \partial_3 e^a_\mu)(\partial_2 \partial_2 e^b_\nu) - 2(\partial_2 \partial_3 e^a_\mu)(\partial_2 \partial_3 e^b_\nu) \right] \eta_{ab} + \ldots
$$

(2.3)
There is no contribution to first order of $\Lambda^{23}$ and for all odd orders in $\Lambda^{23}$, because by definition $G_{\mu\nu}$ is real while the first order in $\Lambda^{23}$ in the $*$-product is purely imaginary. The 00-component $G_{00}$ is

$$G_{00} = \frac{\sum_{n=0}^{7} F_{2n}(r, \theta) \cos^{2n}(\theta)}{4 \left(r^2 + a^2 \cos^2(\theta)\right)^4 \left(r \left(-2M + r + a^2 \cos^2(\theta)\right)\right)^2}, \quad (2.4)$$

where the expression of $F_{2n}(r, \theta)$ is in appendix. The rr-component $G_{rr}$ is the following:

$$G_{rr} = \frac{(\sum_{n=1}^{3} G_{2n}(r, \theta) \cos^{2n}(\theta))}{8 \left(a^2 - 2Mr + r^2\right)^3 \left(r^2 + a^2 \cos^2(\theta)\right)^2}, \quad (2.5)$$

where the expression of $G_{2n}(r, \theta)$ is in appendix.

Now we consider the event horizon with equation $G_{rr}^{-1} = 0$ and the stationary surface with equation $G_{00} = 0$ for the Kerr metric and in the limit of Schwarzschild metric.

In the ordinary case we have

Kerr metric :
\begin{align*}
r &= M + \sqrt{M^2 - a^2 \cos^2 \theta} \quad \text{(stationary limit or ergosphere)} \\
r &= M + \sqrt{M^2 - a^2} \quad \text{(event horizon)}
\end{align*}

Schwarzschild metric :
\begin{align*}
r &= 2M \quad \text{(stationary limit or ergosphere)}, \\
r &= 2M \quad \text{(event horizon)} \quad (2.6)
\end{align*}

In the Schwarzschild case the event horizon is not affected and the stationary limit due to $a = 0$, in fact the presence of $\Lambda^{23}$, induces a non spherically symmetric metric, hence it is plausible to study the modifications induced to metric with axial symmetric if one has only one component of $\Lambda^{23}$. In fact in the Kerr case we have to solve an equation of twelfth degree in $r$ for $G_{00} = 0$, depending on $\Lambda^{23}$, to obtain the stationary limit, while in correspondence of $G_{rr} = 0$ we have the same event horizon as in the ordinary case.

Let us now consider for example the black hole temperature $T$ defined in Eq. (1.1). This is the Hawking temperature of the Kerr background, and in general relativity it becomes equal to

Standard $T = -\left(\frac{1}{4\pi \sqrt{-G_{00}G_{rr}}} \frac{2 \left(M r^2 - a^2 M \cos^2(\theta)\right)}{\left(r^2 + a^2 \cos^2(\theta)\right)^2}\right)_{r=r_H}, \quad (2.7)$

while in the non-commutative case it reads as

N.C. $T = -\left(\frac{1}{4\pi \sqrt{-G_{00}G_{rr}}} \frac{\sum_{n=1}^{7} L_{2n}(r, \theta) \cos^{2n}(\theta)}{4 \left(2Mr - r^2 - a^2 \cos^2(\theta)\right)^3 \left(r^2 + a^2 \cos^2(\theta)\right)^5}\right)_{r=r_H}. \quad (2.8)$
The plots of the numerator of $G_{00}$ in the standard and in the non-commutative case show below where the zeros lie:

**FIG. 1:** The $G_{00}$ component in the standard case.

**FIG. 2:** The $G_{00}$ component in the non-commutative case.

**FIG. 3:** The $G_{00}$ component in the standard (full line) and non-commutative case (dashed line). They overlap only at high values of $r$, which are not shown here.
III. MODIFIED EFFICIENCY OF THE PENROSE PROCESS

In this section we consider the process of energy extraction from a black hole. In this process proposed by Penrose (1969), a particle falling onto a black hole splits up into two fragments at some \( r > r_+ \) where \( V < 0 \), and energy can be extracted from a black hole with an ergosphere. We study the energetics of a Kerr-Newman black hole by the Penrose process using charged particles in the NC case, in particular we analyze negative-energy states. It turns out that the presence of electromagnetic field gives conditions for energy extraction. In \( \Delta \) there is an additive term \( Q \) that is the charge on the black hole

\[
\Delta = r^2 + h^2 - ar + Q^2. \tag{3.1}
\]

In this space-time there exists an electromagnetic field due to the presence of charge \( Q \), which is obtained from the vector potential \( A_i = (-Qr/\rho^2, 0, 0, aqr \sin^2 \theta/\rho^2) \), hence the rotation of the black hole gives rise to a magnetic dipole potential in addition to the usual electrostatic potential. In the approximation when the metric and the electromagnetic field are both static and axisymmetric one has two integrals of motion \( E \) and \( L \), i.e., the energy and the \( \phi \)-component of the angular momentum per unit of rest mass of the particle, and if the particle has \( p_\theta = 0 \) (\( p_i \) is 4-momentum of the particle) in the equatorial plane, will stay in the plane for all time, i.e. \( p_\theta = 0 \) all through the motion. Henceforth one can consider motion in the equatorial plane and set \( \rho_\theta = 0 \) with the restriction of motion in the equatorial, then the effective potential for radial motion could be obtained by putting \( p_r = p_\theta = 0 \) in the following Equation of [13]

\[
E = -eA_t - g_{t\phi}/g_{\phi\phi}(L - eA_\phi) + \left( \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}/g_{\phi\phi} \right) [(L - eA_\phi)^2 + g_{\phi\phi}(g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 + \mu^2)]^{1/2}.
\]

Now we substitute the components of the metric in our case to obtain the modifications induced by NC with respect to the ordinary case [13].

If one of the fragments has negative energy (relative to infinity), it will be absorbed by the black hole while the other fragment will come out, by conservation of energy, with energy greater than the parent particle. This is known as the mechanism of energy extraction from the black hole. In fact for a test particle of 4-momentum \( p^a = mu^a \), the energy \( E = -p^a\xi_a \) need not be positive in the ergosphere, hence one can extract energy from a black hole by absorbing a particle with negative energy [12].
In the case of the Kerr-Newman black hole, the extracted energy can be given by the rotational and/or the electromagnetic energy (Christodoulou 1970). One has to consider the conservation equations for the 4-momenta of the particles, and one can follow the recipe for energy extraction given in [13]. At the point of split, we assume that the 4-momentum is conserved, i.e., $P_1 = P_2 + P_3$ where $p_i$ $(i = 1, 2, 3)$ denotes the 4-momentum of the ith particle. The above relation stands for the following three relations:

$$
E_1 = \mu_2 E_2 + \mu_3 E_3, \\
l_1 = \mu_2 l_2 + \mu_3 l_3, \\
\dot{r}_1 = \mu_2 \dot{r}_2 + \mu_3 \dot{r}_3,
$$

(3.2)

where we have set $\mu_1 = 1$. The other conservation relation follows from the conservation of charge:

$$
\lambda_1 = \mu_2 \lambda_2 + \mu_3 \lambda_3,
$$

(3.3)

where the quantities $\mu_i, l_i, \lambda_i, E_i, r_i$ refer to the i-th particle. These relations contain in all eleven parameters, seven of which are freely specificable. The choice of these parameters will be constrained by the requirements that particle 1 should reach the point of split where $V < 0$ for some suitable $\lambda$, such that particle 2 can have $E_2 < 0$ and particle 3 has a runaway orbit.

The most important question in the black hole energetics is the efficiency of the energy extraction process, for example from a supermassive black hole, that is one of the many important parameters of any model in Active Galactic Nuclei. It is therefore very pertinent to examine how efficient the Penrose process is.

The efficiency $\eta$ is indeed defined as

$$
\eta = \frac{\text{gain in energy}}{\text{input energy}} = \mu_3 \frac{E_3}{E_1} - 1,
$$

(3.4)

and it can be calculated in the presence or absence of charge, and/or electromagnetic interactions. It is known that $\eta_{\text{max}} \sim 20.7\%$ for the pure extreme Kerr case, in absence of electromagnetic fields [14]. However, there is no upper limit on $\eta$ in presence of electromagnetic field [15] and in the non-commutative case, as will be shown.

We take the limit as $r$ approaches the split point $r_+$, in the absence of electromagnetic field in the following equation of [13]:

$$
\mu_3 E_3 = \chi E = \left( \frac{\Omega_1 - \Omega_+}{\Omega_+ - \Omega_-} \right) \left( \frac{g_{tt} + g_{t\phi}\Omega_+}{g_{tt} + g_{t\phi}\Omega_1} \right),
$$

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where
\[
\Omega_1 = \frac{-g_{t\phi}(1 + g_{tt}) + ((g^2_{t\phi} - g_{tt}g_{\phi\phi})(1 + g_{tt}))^{1/2}}{(g^2_{t\phi} + g_{\phi\phi})}, \quad \Omega_{\pm} = (-g_{t\phi} \pm \sqrt{(g^2_{t\phi} - g_{tt}g_{\phi\phi})})/g_{\phi\phi},
\]
and one obtains the following modification:
\[
\mu_3 E_3 = [(1 + G_{00})^{1/2} + 1]/2, \quad (3.5)
\]
where a purely geometric factor occurs, and in the presence of electromagnetic field there is an additive term depending on the electromagnetic field \[15\]. For the extreme Kerr-Newman black hole, \(a = M,\) and from Eq. (3.5) we obtain for small \(\Lambda_{23}\) to first non-vanishing order in \(\Lambda_{23}^2\) (all terms of odd order in \(\Lambda_{23}\) being identically zero)
\[
\mu_3 E_3 = \frac{\sum_{n=0}^{7} F_{2n}(r, \theta) \cos^{2n}(\theta)}{4 (r^2 + a^2 \cos^2(\theta))^4} (r (-2 M + r) + a^2 \cos^2(\theta)) \Lambda_{23}^2 M^2 (r^5 (-3 M + 2 r) \cos^2(\theta)
+ 2 a^2 (5 M - 2 r) r^3 \cos^4(\theta) + a^4 (M - 6 r) r \cos^6(\theta)
- (r^2 + a^2 \cos^2(\theta))^2 (r (-3 M + 2 r) + 2 a^2 \cos^2(\theta)) \sin^2(\theta)), \quad (3.6)
\]
which imply the following efficiency \(\eta:\)
\[
\eta \sim [(2 - Q^2)^{1/2} - 1]/2 + (\Lambda_{23}^2)[(2 - Q^2)^{-1/2}]\quad (3.8)
\]
Thus, in the absence of charge, i.e. when \(Q = 0,\)
\[
\eta \sim [\sqrt{2} - 1]/2 + \left[\frac{1}{2\sqrt{2}}\right] (\Lambda_{23}^2) + O((\Lambda_{23}^4)). \quad (3.9)
\]
In presence of non-commutativity, the efficiency of a given energy-extraction event depends on the competition between two factors in equation (3.8), i.e., the standard geometric term, and the non-commutative one. It increases the maximum efficiency of the Penrose process in the absence of electromagnetic interaction, i.e. when the particles are uncharged. If the maximum is set at 0.207 we have a bound on \(\Lambda_{23}\), it must be very small, and this agrees with our hypothesis.
However, since the geometry allows for the existence of negative energy states only up to the static limit, if the split point is close to $r_s$ then the efficiency is approximately given by

$$\eta \sim -m_3 \lambda_3 A_t + (\Lambda^{23})^2 \frac{1}{2\sqrt{2}}. \quad (3.10)$$

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IV. CONCLUDING REMARKS

Several mechanisms have been proposed to account for the power engine for active galactic nuclei, X-ray binaries, and quasars. One of the most suggestive is that suggested in general relativity, the Penrose mechanism, that allows for the existence of black holes and predicts the energy extraction from a rotating black hole.

In the presence of non-commutativity the total energy of a particle has a contribution from the non-commutative parameter in the efficiency. In this case the extraction of energy from a non-commutative Kerr black hole can become more efficient, although we expect only a tiny gain since the approximate formula for $\eta$ results from a perturbative expansion with a ‘small’ value of $\Lambda^{23}$. Our result could apply to the general behavior of negative-energy states and energy extraction process.

We have considered a Kerr black hole immersed in a non-commutative background (cf. ref. [16]) which is perturbative, that is, which does not appreciably alter the geometrical background though it would affect significantly the motion of particles. With this assumption the background geometry has been taken as the one described by the Kerr metric perturbatively modified by non-commutativity.

(i) One could perform a detailed study of black-hole thermodynamics in the non-commutative background (cf. ref. [17, 18]) pertaining to the applicability of the Penrose process in relativistic astrophysics.

(ii) The non-commutativity could be the seed to accelerate the fragments to relativistic speeds in the rare occurrence, not only if the black hole is immersed in a magnetic field [19].
APPENDIX A: COEFFICIENTS $F_i, G_i, L_i$

The coefficients $F_i(r, \theta)$ of Eq. (2.4) are

$$F_0(r, \theta) = -3 a^2 \Lambda M^3 r^5 + 2 a^2 \Lambda M^2 r^6 + 32 M^3 r^9 - 48 M^2 r^{10} + 24 M r^{11} - 4 r^{12}, \quad (A1)$$

$$F_2(r, \theta) = -6 a^4 \Lambda M^3 r^3 + 6 a^4 \Lambda M^2 r^4 + 3 a^2 \Lambda M^3 r^5 - 2 a^2 \Lambda M^2 r^6 + 96 a^2 M^3 r^7 - 192 a^2 M^2 r^8 + 8 M (15 a^2 - 4 M^2) r^9 + 24 (-a^2 + 2 M^2) r^{10} - 24 M r^{11} + 4 r^{12}, \quad (A2)$$

$$F_4(r, \theta) = -3 a^6 \Lambda M^3 r + 6 a^6 \Lambda M^2 r^2 - 10 a^4 \Lambda M^3 r^3 + 4 a^4 \Lambda M^2 r^4 + 3 a^4 M^3 (32 a^2 - \Lambda) r^5 + 2 a^2 M^2 (-144 a^2 + \Lambda) r^6 + 48 a^2 M (5 a^2 - 2 M^2) r^7 + 12 a^2 (-5 a^2 + 16 M^2) r^8 - 120 a^2 M r^9 + 24 a^2 r^{10}, \quad (A3)$$

$$F_6(r, \theta) = 2 a^8 \Lambda M^2 - a^6 \Lambda M^3 r + 6 a^6 \Lambda M^2 r^2 + 2 a^4 M^3 (16 a^2 + 5 \Lambda) r^3 + 4 a^4 M^2 (-48 a^2 - \Lambda) r^4 + 48 a^4 M (5 a^2 - 2 M^2) r^5 + 16 a^4 (-5 a^2 + 18 M^2) r^6 - 240 a^4 M r^7 + 60 a^4 r^8, \quad (A4)$$

$$F_8(r, \theta) = a^6 \Lambda M^3 r + 6 a^6 M^2 (-8 a^2 - \Lambda) r^2 + 8 a^6 M (15 a^2 - 4 M^2) r^3 + 12 a^6 (-5 a^2 + 16 M^2) r^4 - 240 a^6 M r^5 + 80 a^6 r^6, \quad (A5)$$

$$F_{10}(r, \theta) = 24 a^{10} M r - 24 a^{10} r^2 + 48 a^8 M^2 r^2 - 120 a^8 M r^3 + 60 a^8 r^4, \quad (A6)$$

$$F_{12}(r, \theta) = -4 a^{12} - 24 a^{10} M r + 24 a^{10} r^2, \quad (A7)$$

$$F_{14}(r, \theta) = 4 a^{12}. \quad (A8)$$

The coefficients $G_i(r, \theta)$ of (2.5) are

$$G_0(r, \theta) = 2 a^4 \Lambda M r^3 + (-3 a^4 \Lambda - a^2 \Lambda M^2) r^4 + 2 a^2 \Lambda M r^5 + 8 a^4 r^6 - 32 a^2 M r^7 + 16 (a^2 + 2 M^2) r^8 - 32 M r^9 + 8 r^{10}, \quad (A9)$$

$$G_2(r, \theta) = 2 a^6 \Lambda M r - a^4 (3 a^2 \Lambda + 2 \Lambda M^2) r^2 + a^2 (24 a^4 + 5 a^2 \Lambda + 2 \Lambda M^2) r^4 + 4 a^2 M (-24 a^2 - \Lambda) r^5 + 12 a^2 (4 a^2 + 8 M^2) r^6 - 96 a^2 M r^7 + 24 a^2 r^8, \quad (A10)$$
\[ G_4(r, \theta) = -(a^8 \Lambda) - a^6 \Lambda M^2 + 2a^6 \Lambda M r + 4a^4(6a^4 + a^2 \Lambda - \Lambda M^2) r^2 \]
\[ + 2a^4 M(-48a^2 + \Lambda) r^3 + 2a^4(24a^2 - \Lambda + 48M^2) r^4 \]
\[ - 96a^4 M r^5 + 24a^4 r^6, \quad (A11) \]
\[ G_6(r, \theta) = 8a^{10} + a^8 \Lambda - 2a^6 \Lambda M^2 - 2a^6(16a^2 M + \Lambda M) r + a^6(16a^2 - \Lambda + 32M^2) r^2 \]
\[ - 32a^6 M r^3 + 8a^6 r^4. \quad (A12) \]

Last, the coefficients \( L_i(r, \theta) \) of (2.8) are
\[ L_0(r, \theta) = -30a^2 \Lambda M^4 r^7 + 37a^2 \Lambda M^3 r^8 - 12a^2 \Lambda M^2 r^9 + 64M^4 r^{11} \]
\[ - 96M^3 r^{12} + 48M^2 r^{13} - 8M r^{14}, \quad (A13) \]
\[ L_2(r, \theta) = -66a^4 \Lambda M^4 r^5 + 116a^4 \Lambda M^3 r^6 + 12a^2 \Lambda M^2 (-4a^2 \Lambda + 5 \Lambda M^2) r^7 \]
\[ - 74a^2 \Lambda M^3 r^8 + 8a^2 M^2 (3\Lambda + 16M^2) r^9 - 288a^2 M^3 r^{10} \]
\[ + 192a^2 M^2 r^{11} - 40a^2 M r^{12}, \quad (A14) \]
\[ L_4(r, \theta) = -42a^6 \Lambda M^4 r^3 + 126a^6 \Lambda M^3 r^4 - 4a^4 M^2 (18a^2 \Lambda + 23 \Lambda M^2) r^5 + 32a^4 \Lambda M^3 r^6 \]
\[ + 16a^4 \Lambda M^2 r^7 - 192a^4 M^3 r^8 + 240a^4 M^2 r^9 - 72a^4 M r^{10}, \quad (A15) \]
\[ L_6(r, \theta) = -6a^8 \Lambda M^4 r + 52a^8 \Lambda M^3 r^2 + 4a^6 M^2 (-12a^2 \Lambda + 11, \Lambda M^2) r^3 + 40a^6 \Lambda M^3 r^4 \]
\[ - 16a^6 M^4 (\Lambda + 2) r^5 + 192a^6 M^3 r^6 - 40a^6 M r^8, \quad (A16) \]
\[ L_8(r, \theta) = 5a^{10} \Lambda M^3 + 4a^8 \Lambda M^2 (-3a^2 + M^2) r - 72a^8 \Lambda M^3 r^2 + 16a^8 M^2 (\Lambda - 4M^2) r^3 \]
\[ + 288a^8 M^3 r^4 - 240a^8 M^2 r^5 + 40a^8 M r^6, \quad (A17) \]
\[ L_{10}(r, \theta) = -6a^{10} \Lambda M^3 + 24a^{10} \Lambda M^2 r + 96a^{10} M^3 r^2 - 192a^{10} M^2 r^3 + 72a^{10} M r^4, \quad (A18) \]
\[ L_{12}(r, \theta) = -48a^{12} M^2 r + 40a^{12} M r^2, \quad (A19) \]
\[ L_{14}(r, \theta) = 8a^{14} M. \quad (A20) \]

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[20] It has been pointed out to us by D. Vassilevich that the very definition of Hawking temperature in the non-commutative case is an open problem, so that our $T$ is more clearly related to the surface gravity.