Application of 1 D Finite Element Method in Combination with Laminar Solution Method for Pipe Network Analysis

O I Dudar¹, E S Dudar²

¹Applied Mathematics and Mechanics Faculty, Perm National Research Polytechnic University, 29, Komsomolsky Ave., Perm 614990, Russia
²Aerospace Faculty, Perm National Research Polytechnic University, 29, Komsomolsky Ave., Perm 614990, Russia

E-mail: olegdudar@yandex.ru

Abstract. The features of application of the 1D dimensional finite element method (FEM) in combination with the laminar solutions method (LSM) for the calculation of underground ventilating networks are considered. In this case the processes of heat and mass transfer change the properties of a fluid (binary vapour-air mix). Under the action of gravitational forces it leads to such phenomena as natural draft, local circulation, etc. The FEM relations considering the action of gravity, the mass conservation law, the dependence of vapour-air mix properties on the thermodynamic parameters are derived so that it allows one to model the mentioned phenomena. The analogy of the elastic and plastic rod deformation processes to the processes of laminar and turbulent flow in a pipe is described. Owing to this analogy, the guaranteed convergence of the elastic solutions method for the materials of plastic type means the guaranteed convergence of the LSM for any regime of a turbulent flow in a rough pipe. By means of numerical experiments the convergence rate of the FEM - LSM is investigated. This convergence rate appeared much higher than the convergence rate of the Cross – Andriyashev method. Data of other authors on the convergence rate comparison for the finite element method, the Newton method and the method of gradient are provided. These data allow one to conclude that the FEM in combination with the LSM is one of the most effective methods of calculation of hydraulic and ventilating networks. The FEM - LSM has been used for creation of the research application programme package “MineClimate” allowing to calculate the microclimate parameters in the underground ventilating networks.

1. Introduction

The networks of pipelines, canals, and vessels are widespread in engineering and nature. The most common problem in the analysis of such systems is the calculation of flow distribution in a network (e.g. water supply systems, ventilation networks). For a long time the Hardy Cross method (HCM) [1] and its Russian analogue the Andriyashev method [2] was the main method of calculating of pipeline networks. This method is intuitive, convenient for both manual calculation and computer implementation. However, computer calculations of large networks have shown that the Cross-Andriyashev method has slow convergence and does not always converge [3]. For this reason new and significantly more effective methods have appeared and found application: the Newton method [4,5], the finite element method (FEM) [6-8], the gradient method [9,10]. Reviews of the methods for calculating of hydraulic networks and comparison of their effectiveness are given in [3,8,11,12].
Earlier, the authors of this paper proposed their version of the one-dimensional FEM [13-16], which is presented here in the final form. The proposed version is primarily aimed at calculating of ventilation networks of mines and other underground structures. Such networks are characterized by heat and mass transfer processes that change the properties of a transferred substance (vapor-air mixture), which, in turn, leads to such phenomena as natural draft, local circulation on internal circuits, etc. In this case, mass flow rate should be used in calculations instead of volumetric flow rate. The action of gravity, the dependence of the substance properties on the temperature, pressure and water mass concentration should also be taken into account. Thus the possibility of involving of the fluid temperature and component mass concentration codes has to be provided in computer realization of the method. The foregoing distinguishes the considered version of the FEM from those presented in [6,8].

As well as in the papers [6-8], the proposed version of the FEM is based on mathematical identity of the equations describing deformation of a rod structure and a fluid flow in a network. The balance of forces in nodes, the compatibility of the deformations, and the stress-strain constitutive relation are considered for rod structures. For a network, the sum of flow rates is equal to zero in nodes, the node pressure is the same for all branches converging in a node and the flow-pressure drop relation is given either by the Hagen-Poiseuille formula [7], or by the Hazen-Williams formula [6] or by the Darcy formula [8]. The analogy of a rod elastic tension to a laminar flow in a pipe is well known [7]. In addition, the analogy of a rod plastic tension to a turbulent flow in a pipe was discussed in [15]. From this analogy it follows that the yield point corresponds to the critical Reynolds number, and the convex upward stress-strain curve of the plastic deformation corresponds to the convex upward mass flow-pressure drop curve under a turbulent flow in a pipe. This analogy allowed to extend the convergence conditions of the “elastic solutions method” considered in [17,18] and used as an iterative method in the analysis of plastic deformation of a rod, to the similar "laminar-solution method" (LSM) applied in this study and at root in the studies [6,8] for the analysis of a turbulent flow in a pipe.

2. FEM relations

The basic aspects of the FEM implementation are stated in numerous papers [7, 19]. Only a brief description of the method is considered in this article. According to the FEM a network must be divided into separate elements (branches) which are represented by the straight-line segments with the constant cross section. The topology of a network is set by the incident matrix coupling the indices of branches and nodes. The unknowns are the pressure values in network nodes.

The relationship between the mass flow rate and the nodal pressures for a standard element of a network is necessary for derivation of the system of equations with unknown nodal pressures. So it is required to set the relation of type [1]

\[ \{G\}^T = [k]^T \{p\} \]  

where \( \{G\}^T \) is the local vector of mass flows in nodes; \( \{p\} \) is the local vector of the nodal pressures; \( [k]^T \) is the local matrix of conductivity which form depends on the flow regime in an element.

For a laminar flow the dependence \( G(\Delta p) \) is given by the Hagen – Poiseuille equation [11]:

\[ \Delta p = \frac{8\nu_l l_e}{\pi r_e^2} G_i = -\frac{8\nu_l l_e}{\pi r_e^2} G_j \]  

where \( \Delta p = p_i - p_j \), \( l_e \) and \( r_e \) are the length and the hydraulic radius of a pipe considered as a branch; \( \nu_e \) is the kinematic viscosity of a fluid. The Hagen – Poiseuille law is written down for the network branch \( e \) with the nodes \( i \) and \( j \). The direction from \( i \) to \( j \) is accepted as positive. The flows from the nodes \( i \) and \( j \) (denoted as \( G_i \) and \( G_j \)) into the pipe element \( e \), are considered as positive. With the help of (2) we can write the next matrix relation:
where $k^e = \frac{\pi r_e^4}{8
u l^e}$; $G_0 = k^e \rho_e g \Delta h$ is the member, which accounts for the action of mass forces (gravity); $\rho_e$ is the density; $\Delta h = (h_i - h_j)$ is the height difference for vertical and inclined branches of a network.

Thus for a laminar regime the dependence $G(\Delta p)$ is linear and the conductivity matrix is defined by the equation:

$$[k]^e = k^e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(4)

For a turbulent regime the mass flow rate - pressure drop relation is given by the Darcy law [3]:

$$\Delta p = \lambda \frac{l_e}{4\rho_e \pi r_e^3} G^2$$

(5)

where the resistance coefficient $\lambda$ depends generally on the flow rate $G$ and the surface roughness.

As follows from the equation (5) the dependence $G(\Delta p)$ is not linear for a turbulent flow. To obtain the relation which is valid for any pipe and any liquid properties we rewrite (5) in a dimensionless form

$$Re = \frac{1}{\sqrt{\lambda}} \left( \frac{\Delta p^*}{\Delta p} \right)$$

(6)

where $Re$ is the Reynolds number; $\Delta p^* = \frac{16r_e^3}{\rho_l \nu_e^2} \Delta p$ is the dimensionless pressure drop.

The universal curves (6) for various relative roughness $\delta$ based on the Nikuradse’s experimental results [21] are shown in figure 1. In these experiments $\delta = k_e/r$, where $k_e$ is the height of roughness elements (sand grain), $r$ is the pipe radius. According to the equation (6) the coefficient $k^e$ in the conductivity matrix (4) for the case of a turbulent flow can be determined by the formula

$$k^e = \frac{G(\Delta p^*)}{\Delta p} = \frac{\pi r_e \nu_e \cdot Re(\Delta p^*)}{2} \frac{\Delta p}{\Delta p}$$

(7)

where $Re(\Delta p^*)$ is the Reynolds number value determined from the curve on figure 1 for given $\delta$. It should be noted that the density and the kinematic viscosity which are present in the finite element relation (3) are the functions of the thermodynamic parameters and are calculated through the values of these parameters determined at the midpoint of an element:

$$\rho_e = \rho_e(T_m, p_m, C_m), \nu_e = \nu_e(T_m, p_m, C_m)$$

where $T_m$, $p_m$ and $C_m$ are the temperature, pressure and concentration of water vapor at the midpoint.
Summarizing the equation (1) throughout all branches, we deduce the system for the entire network:

\[ [K] \{P\} = \{G\} \]  

(8)

where \( \{G\} = \sum \{G\}^e \) is the network nodal mass consumption vector; \( [K] = \sum [k]^e \) is the network conductivity matrix; \( \{P\} \) is the network nodal pressures vector.

As follows from the mass conservation law only the vector \( \{G\} \) components which correspond to the boundary nodes with the prescribed flow consumption will be nonzero. In the boundary nodes where the consumptions are unknown, the pressures must be introduced. In one boundary node the pressure must be introduced obligatorily, otherwise the conductivity matrix will be degenerate.

3. Convergence of the method

In general case the system of equations (8) is nonlinear because of the nonlinearity of the relationship \( Re = (\Delta p^*) \) (figure 1). The method of successive iteration is used for solving it. At first iteration the flow in each network branch is assumed to be laminar. According to (7) the relationship between \( G \) and \( \Delta p \) in each element (branch) remains linear ("laminar") at all subsequent iterations. The iterative procedure of the LSM leading to the satisfaction of the dependence \( G(\Delta p) \) (i.e. \( Re = (\Delta p^*) \)) (6) for any regime of a turbulent flow is illustrated by figure 2.

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**Figure 1.** The dependence of the Reynolds number \( Re \) on the dimensionless pressure difference \( \Delta p^* \):

- ■ – \( \delta = 0 \);
- ▲ – \( \delta = 1/60 \);
- ▼ – \( \delta = 1/30 \);
- ● – \( \delta = 1/15 \).

**Figure 2.** The iterative procedure of the LSM reducing a turbulent flow to the sequence of “laminar” flows.
In the rod theory the iterative method of reducing of the plastic deformation problem to the sequence of the elastic deformation problems is widely known [17,18]. The guaranteed convergence of such iteration procedure is proved for the rod system suffering tension-compression with the assumption that the stress-strain diagram $\sigma(\varepsilon)$ is a monotonically increasing function with decreasing rigidity [17,18]. In our case the analogue of $\sigma(\varepsilon)$ is the function $Re = (\Delta p \ast)$. From figure 1 we can see that this function is monotonically increasing with decreasing "conductivity" for any fixed relative roughness. This feature ensures the convergence of the FEM-LSM. As it was shown in [14] the dependence $G(\Delta p)$ for active elements (blowers) is also a monotonically increasing convex upward curve. Really the characteristic of a blower $h(Q)$ ($h$ is draught loss and $Q$ is delivery) is a convex upward but monotonously decreasing curve. The function $Q(h)$ is the same. Then $Q(\Delta p)$, and hence $G(\Delta p)$, are convex upward and monotonically increasing functions, since $\Delta p = p_i - p_f = -(p_f - p_i) = -h$ (the air moves from the blower inlet to the outlet diffuser with increasing pressure).

To compare the rate of convergence one and the same model problem (Figure 3) was solved by the Cross-Andriyashev method and the FEM-LSM. A numerical experiment showed that when the number of network branches equals 23, the convergence rate of the FEM-LSM is more than 4 times higher than the same of the Cross-Andriyashev method (table 1). Numerical experiments show that when the number of branches increases the advantage of the FEM-LSM also increases. So for the FEM-LSM with the increase of the number of branches almost by 6 times the number of iterations has increased only by 1.4 times (135 branches, 19 iterations), and for the Cross-Andriyashev method with the increase of the number of branches by 8 times the number of iterations has increased by more than 170 times (190 branches, 10431 iterations[11]).

![Figure 3. The model pipeline network.](image)

Figure 4 shows the curves of the convergence rate, namely, the dependence of the logarithm of the maximum relative error for the mass flow rate and for the pressure on the number of iterations. The curve rise on the 3rd and 4th iterations is explained by the features of the algorithm (the calculation of local resistances is involved). Then these dependences become close to linear.
Table 1. Comparison of the convergence rate in the calculation of the model network.

| Method                          | Maximum error, % | Number of iterations |
|---------------------------------|------------------|----------------------|
|                                 | pressure $\varepsilon_p$ | flow $\varepsilon_G$ |                      |
| Method of Cross–Andriyashev (23 branches) | 0,0002            | 0,001                | 60                   |
| FEM-LSM: (23 branches)          | 0,0005            | 0,00001              | 14                   |

Figure 4. The dependence of the logarithm of the relative error $\lg(\varepsilon)$ on the number of iterations $N$ in a numerical experiment to determine the convergence rate of the FEM-LSM: ■ - the maximum network error for the mass flow rate $\varepsilon_G$; ● – the maximum network error for the pressure $\varepsilon_p$.

If to compare the FEM-LSM with other high-speed methods (the Newton method and the gradient method) the results demonstrate [12] that all three methods have approximately the same rate of convergence (for the Newton method and the gradient method the convergence rate is slightly higher than for the FEM-LSM). However, for the Newton method oscillations take place, and for the FEM-LSM the grid was unsuccessfully numbered, as a result the band width of the conductivity matrix appeared to be the largest. It is known that the time of solving of simultaneous linear algebraic equations with a band matrix is proportional to a squared band width.

Based on the FEM-LSM the authors created the research application programme package "MineClimate" with the help of which a number of studies were performed. In particular the "MineClimate" was used to evaluate the influence of various parameters on the process of moisture condensation in an underground structure [22].

4. Conclusions
The features of the FEM-LSM application are shown and the finite-element relations are derived for the case of application of the method to the ventilation networks of underground structures.

The convergence of the FEM is considered.

The results of our own research and the studies of other authors on comparison of the convergence rate of the FEM-LSM and some other methods of network computation are represented.

Our investigations have shown:
- the FEM-LSM together with the gradient method and the Newton method, is one of the three most effective methods for calculating of hydraulic (ventilation) networks;
- the FEM-LSM as well as the gradient method has the guaranteed convergence.

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