On the critical boundary RSOS $\mathcal{M}(3,5)$ model

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February 17, 2016

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Abstract

We consider the critical non-unitary minimal model $\mathcal{M}(3,5)$ with integrable boundaries. We analyze the patterns of zeros of the eigenvalues of the transfer matrix and then determine the spectrum of the critical theory through the Thermodynamic Bethe Ansatz (TBA) equations. By solving the TBA functional equation satisfied by the transfer matrices of the associated $A_4$ RSOS lattice model of Forrester and Baxter in Regime III in the continuum scaling limit, we derive the integral TBA equations for all excitations in the $(r=1, s=1)$ sector then determine their corresponding energies. The excitations are classified in terms of $(m, n)$ systems.

keywords: $\mathcal{M}(3,5)$ model, conformal field theory, lattice models, Yang-Baxter integrability, non-unitary minimal models

1 Introduction

Solving 1+1 dimensional Quantum Field Theory (QFT) in finite volume to determine the energy spectrum and the field correlation functions is a complicated and non-trivial problem. The volume dependence is complicated and the energy cannot be calculated exactly even for the ground state. However, for a special category of models containing an infinite number of conservation law called integrable models, the solution is attainable. The bootstrap approach allows the determination of the masses of the particles and their scattering matrices. The Bethe-Yang (BY) equations for large volumes are used to determine an approximate spectrum using the infinite volume quantities [1, 2]. This finite size spectrum neglects small vacuum polarization contributions and contains polynomial corrections inversely proportional to the volume. Using the Thermodynamic Bethe Ansatz (TBA) method, we can exactly calculate the vacuum polarization effects for the ground state. For large Euclidean time, the partition function is dominated by the ground state contribution. As the roles of space and Euclidean time can be exchanged by an appropriate transformation, it is sufficient to evaluate the partition function in the large volume limit, where Bethe-Yang equations are accurate. Integral equations (TBA) can be derived for the pseudo energies by calculating the partition function in the saddle point approximation. The ground-state energy is obtained from the solutions of these nonlinear TBA equations [3, 4, 5, 6].

Extending the TBA method of exploiting the invariance properties of the partition function to all excited states is not generally an easy task. However, some information about certain excited
states can be obtained from the exact ground-state TBA equations \[7\] because these excited states and the ground-state are related by analytic continuation. This method using analytic continuation has not been successfully carried out to obtain TBA equations for the complete excitation spectrum of the non-unitary \( \mathcal{M}(3,5) \) model, and not even for the simpler non-unitary scaling Lee-Yang model \[8, 9\].

Solving the functional relations obtained from the Yang-Baxter integrable lattice regularization is a powerful and systematic way for obtaining the TBA integral equations for excited states \[10, 11, 12, 13\]. These functional equations take the form of \( T \) and \( Y \) fusion systems. The \( Y \)-system contains the pseudo energies and describes the conformal spectra at criticality. The functional relations can be recast into TBA integral equations for the full excitation spectrum using analytic and asymptotic properties. This method has been successfully applied for the tricritical Ising model \( \mathcal{M}(4;5) \) with conformal boundary conditions \[14\]. In \[15, 16\], the tricritical Ising model was described on the interval and its full spectrum was determined. The ground-state equation of the sine-Gordon theory and some excited states were derived in \[17, 18, 19\]. The lattice regularization approach has been systematically developed for the Lee-Yang theory \[20, 21, 22\]. On the other hand, the ground-state energy of the Lee-Yang model on the interval and the excited state TBA equations were derived in \[23, 24, 25\] and defect ground state TBA equations were obtained in \[26\].

The integrable structure of the Conformal Field Theory (CFT) is reflected in the functional form of the \( Y \)-system. In the lattice approach, it is obtained from the continuum scaling limit of an integrable lattice regularization of the theory. The lattice approach provides the asymptotic and analyticity properties and allows the complete classification of excited states of the theory.

Our aim is to start developing the next simplest non-unitary model, namely the critical lattice \( \mathcal{M}(3,5) \) model with integrable boundary conditions. In the present paper we study the critical TBA equations of the boundary model using a lattice approach. The paper is organized as follows: In Section 2, we introduce the conformal model as well as as the continuum scaling limit of the \( A_4 \) RSOS lattice model of Forrester-Baxter \[27, 28, 29, 30\] in Regime III with crossing parameter \( \lambda = \frac{2\pi}{5} \). We introduce the commuting double row transfer matrices with integrable boundaries. We show that the double row transfer matrix satisfies the universal functional relation in the form of a \( Y \)-system after an appropriate normalization. Section 3 analyzes the conformal spectra of the transfer matrices. We investigate the analytic structure of the transfer matrix eigenvalues, classify all excited states of the trigonometric theory in the \((m,n)\) system and plot representative zero configurations of an eigenvalue of the transfer matrix. In Section 4, we use the analytic information together with the functional relations to derive integral TBA equations for the finite volume spectrum of the \((1,1)\) boundary condition in the critical case, and calculate the finite-size energies. Lastly, we conclude with discussions in Section 6.

2 The \( \mathcal{M}(3,5) \) Lattice Model

The lattice \( \mathcal{M}(3,5) \) model under consideration is a Restricted Solid-on-Solid (RSOS) model defined on a square lattice with heights that live on an \( A_4 \) Dynkin diagram, with nearest neighbor heights differing by \( \pm 1 \). It belongs to the general \( A_L \) Forrester-Baxter models developed in \[29, 30, 31\].

The Boltzmann weights of the general \( A_L \) Forrester-Baxter models are given by

\[
W\left( \begin{array}{cc}
  a \pm 1 & a \\
  a & a \mp 1
\end{array} \right) = \frac{s((a \pm 1) \lambda)}{s(a \lambda)} \\
W\left( \begin{array}{cc}
  a & a \pm 1 \\
  a \mp 1 & a
\end{array} \right) = \frac{g_{a \pm 1}}{g_{a \mp 1}} \frac{s((a \pm 1) \lambda)}{s(a \lambda)} \frac{s(u)}{s(\lambda)}
\]

\[1\]
\[ W \left( \begin{array}{cc} a & a+1 \\ a+1 & a \end{array} \right) = \frac{s(a\lambda \pm u)}{s(a\lambda)} \]

where \( a = 1, \ldots, L \), while \( u \) is the spectral parameter and \( s(u) = \vartheta_1(u, p) \) for the massive theory with

\[ \vartheta_1(u, q) = 2q^{\frac{1}{4}} \sin u \prod_{n=1}^{\infty} \left( 1 - 2q^{2n} \cos 2u + q^{4n} \right) \left( 1 - q^{2n} \right) \]  \( \text{(2)} \)

\( \vartheta_1 \) is the elliptic theta function \([32]\) where \( q \) is the elliptic nome related to the a temperature-like quantity \( T = q^2 \) corresponding to the massive bulk perturbation of the model. At criticality, \( s(u) = \sin(u) \) and corresponds to the conformal massless model.

The crossing parameter \( \lambda \) is given by

\[ \lambda = \frac{(p' - p)\pi}{pp'} \]  \( \text{(3)} \)

where \( p' = L + 1 \) and \( p, p' \) are coprime integers with \( p < p' \).

These local face weights satisfy the Yang-Baxter equation which ensures the integrability of the model. The gauge factors \( g_a \) are arbitrary and can be all taken to be equal to 1. Unitary models with \( p' = p + 1 \) have positive Boltzmann weights while the non-unitary models with \( p' \neq p + 1 \) may have negative Boltzmann weights.

The critical Forrester-Baxter models in Regime III in the continuum scaling limit

\[ \text{Regime III:} \quad 0 < u < \lambda, \quad 0 < q < 1 \]  \( \text{(4)} \)

correspond to the minimal models \( \mathcal{M}(p, p') \) whose central charge is

\[ c = 1 - \frac{6(p-p')^2}{pp'} \]  \( \text{(5)} \)

Here we consider the \( \mathcal{M}(3, 5) \) model having \( \lambda = \frac{2\pi}{5} \) and \( c = -\frac{3}{5} \).

A minimal \( \mathcal{M}(p, p') \) model has \( \frac{(p-1)(p'-1)}{2} \) scaling fields which result in four independent scaling fields for the \( \mathcal{M}(3, 5) \) model. As generally prescribed in \([33]\), we can determine the scaling fields, scaling dimensions and fusion rules. Those fields and their symbols are given in Table 1 below:

| \( (r, s) \) | equivalent \( (r, s) \) | Dimension \( h_{r,s} \) | Symbol |
|---|---|---|---|
| \( (1, 1) \) | \( (2, 4) \) | 0 | \( I \) |
| \( (2, 1) \) | \( (1, 4) \) | \( \frac{4}{5} \) | \( \sigma'' \) |
| \( (2, 2) \) | \( (1, 3) \) | \( \frac{1}{5} \) | \( \sigma' \) |
| \( (2, 3) \) | \( (1, 2) \) | \( -\frac{1}{20} \) | \( \sigma \) |

The fusion rules of those fields can be obtained using the general fusion relation

\[ \phi_{(r,s)} \times \phi_{(m,n)} = \sum_{k=1}^{k_{max}} \sum_{l=1}^{l_{max}} \phi(k,l) \quad k = 1 + |r - m| \quad l = 1 + |s - n| \quad k + r + m = 1 \ mod \ 2 \quad l + s + n = 1 \ mod \ 2 \]  \( \text{(6)} \)
where
\[ k_{\text{max}} = \min(r + m - 1, 2p' - 1 - r - m) \]
\[ l_{\text{max}} = \min(s + n - 1, 2p - 1 - s - n) \] (7)
and \( k \) and \( l \) are incremented by 2. We summarize the fusion rules of the fields of \( M(3, 5) \) here:
\[
\begin{cases}
\sigma \times \sigma &= I + \sigma' \\
\sigma \times \sigma' &= \sigma + \sigma' \\
\sigma \times \sigma'' &= \sigma' \\
\sigma' \times \sigma' &= I + \sigma' \\
\sigma' \times \sigma'' &= \sigma \\
\sigma'' \times \sigma'' &= I
\end{cases}
\] (8)

Minimal models have the following fractional decompositions
\[
\frac{p'}{p} = \nu_0 + 1 + \frac{1}{\nu_1 + \frac{1}{\nu_2 + \ldots + \frac{1}{\nu_n + 2}}} \quad \text{if } 2 < 2p < p'
\] (9)
and
\[
\frac{p'}{p'} = \nu_0 + 1 + \frac{1}{\nu_1 + \frac{1}{\nu_2 + \ldots + \frac{1}{\nu_n + 2}}} \quad \text{if } 2p > p'
\] (10)

where the parameters satisfy \( \nu_0 > 0 \) and \( \nu_j \geq 1 \) for \( j = 1, 2, \ldots, n \). and the number of particles in the theory is given by
\[ t = \sum_{j=0}^{n} \nu_j \] (11)

In this particular model, with \( 2p > p' \), we obtain \( \nu_0 = 1 \) and all other \( \nu_{n \neq 0} = 0 \). Thus \( t = 1 \) and this model has one type of particles. This is in direct analogy with its dual \( M(2, 5) \) Lee Yang model which only has one type of particles and same values of \( \nu_n \).

### 2.1 Transfer matrices

The transfer matrices are constructed from the local face weights. They form commuting families \([D(u), D(v)] = 0\) since the local face weights satisfy the Yang-Baxter equations. This model satisfies the same functional relation satisfied by the tricritical hard squares, hard hexagon models and the Lee-Yang model, with spectral parameter \( \lambda = \frac{2\pi}{5} \) instead of \( \lambda = \frac{\pi}{5} \) and \( \frac{3\pi}{5} \) in the other models [13, 31, 32, 36, 21, 22]. The new crossing parameter leads to similar analyticity properties with the Lee-Yang model with a single analyticity strip, but quite different from the other two aforementioned models having two analyticity strips.

The double row transfer matrices satisfy the functional relation given by
\[ D(u)D(u + \lambda) = 1 + YD(u + 3\lambda) \] (12)

where \( Y \) appearing in \( \text{(12)} \) is the \( \mathbb{Z}_2 \) height reversal symmetry.

The conformal spectrum of energies \( E_n \) of the \( M(3, 5) \) model can be obtained from the logarithm of the double row transfer matrix eigenvalue through finite size corrections [37]. In the boundary case, those finite size corrections are given by
2.1 Transfer matrices

\[-\log T(u) = N f_{\text{bulk}}(u) + f_{\text{boundary}}(u, \xi) + \frac{2\pi}{N} E_n \sin \vartheta\]

where \(T(u)\) are the eigenvalues of \(D(u)\) and \(N\) is the number of face weights (\(N\) is even in the boundary case) and

\[\vartheta = \frac{\pi u}{\lambda} = \frac{5u}{2}\] (13)

is the anisotropy angle.

The bulk free energy and the boundary free energy are given by \(f_{\text{bulk}}\) and \(f_{\text{boundary}}\) respectively. Using the inversion relation methods one can calculate those free energies [38, 39, 40].

2.1.1 Boundary weights

The integrability of this model in presence of a boundary requires commuting row transfer matrices and triangle boundary conditions that satisfy the left and right boundary Yang Baxter equations [41]. In this model, we label the conformal boundary conditions by the Kac labels \((r, s)\) where \(1 \leq r \leq 2\) and \(1 \leq s \leq 4\). However, due to height reversal symmetry it is sufficient to determine the triangle weights corresponding to independent \((r, s)\) Kac labels shown in Table 1. These conformal boundaries can be expressed in terms of integrable boundary conditions in several weights due to gauge transformations. In fact, as can be proved in the solution of the boundary Yang-Baxter equations, the \((1, 1)\) triangle boundary weights are arbitrary and here they are given by

\[K_L \left( \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \right| u \right) = \frac{s(2\lambda)}{s(\lambda)}, \quad K_R \left( \begin{array}{c} 2 \\ 1 \\ 1 \end{array} \right| u \right) = 1\] (14)

The other integrable boundary conditions can be constructed by the repeated action of a seam on the integrable \((1, 1)\) boundary [42]. The non-zero left and right boundary weights are explicitly calculated as

\[K_L \left( \begin{array}{c} 2 \\ 2 \\ 1 \end{array} \right| u, \xi_L \right) = W \left( \begin{array}{c} 2 \\ 1 \\ 2 \end{array} \right| u + \xi \right) W \left( \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \right| \lambda - u + \xi \right) K_L \left( \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \right| u \right)
\]

\[= \frac{s(u - 2\lambda + \xi_L) s(u - 2\lambda - \xi_L)}{s(\lambda)^2}\] (15)

\[K_L \left( \begin{array}{c} 2 \\ 2 \\ 3 \end{array} \right| u, \xi_L \right) = W \left( \begin{array}{c} 2 \\ 1 \\ 3 \end{array} \right| u + \xi \right) W \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right| \lambda - u + \xi \right) K_L \left( \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \right| u \right)
\]

\[= \frac{s(3\lambda) s(u + \xi_L) s(u - \xi_L)}{s(\lambda)^3}\] (16)

in short notation, we can express this non-zero left boundary weight by

\[K_L \left( \begin{array}{c} 2 \\ 2 \\ a \end{array} \right| u, \xi_L \right) = \frac{s(a\lambda) s(u + \xi_L + (a - 3)\lambda) s(u - \xi_L + (a - 3)\lambda)}{s(\lambda)^3} \quad a = 1, 3\] (17)

and similarly the non-zero right boundary weight by

\[K_R \left( \begin{array}{c} a \\ 2 \\ 2 \end{array} \right| u, \xi_R \right) = W \left( \begin{array}{c} a \\ 2 \\ 1 \end{array} \right| u + \xi_R \right) W \left( \begin{array}{c} 2 \\ a \\ 1 \end{array} \right| \lambda - u + \xi_R \right) K_R \left( \begin{array}{c} 2 \\ 1 \\ 1 \end{array} \right| u \right)\]
\[
\frac{s(u + \xi_R + (2 - a)\lambda) s(u - \xi_R + (2 - a)\lambda)}{s(\lambda) s(2\lambda)} \quad a = 1, 3
\]  

(18)

Varying the imaginary parts of \( \xi_L \) and \( \xi_R \), one can obtain different \((r, s)\) conformal boundary conditions in this theory. The fact that the boundary weights satisfy the left and right boundary Yang-Baxter equations ensures the integrability of the model in presence of those boundaries.

2.1.2 Double row transfer matrix

The face and triangle boundary weights defined before are used to construct a family of commuting double row transfer matrices \( D(u) \). For a lattice of width \( N \), transfer matrix \( D(u) \) is given by

\[
D(u)^b_a = \sum_{c_0, \ldots, c_N} K_L \left( \begin{array}{c} r \\ c_0 \end{array} \right) A \left( \begin{array}{c} b_1 \\ c_0 \\ c_1 \\ b_2 \\ c_2 \\ \ldots \\ b_{N-1} \\ c_{N-1} \\ c_N \end{array} \right) \left( \begin{array}{c} \lambda - u \\ \lambda - u \\ \ldots \\ \lambda - u \end{array} \right) W(\lambda - u) W(\lambda - u) \ldots W(\lambda - u) 
\]

(19)

This matrix satisfies periodicity \( D(u + \pi) = D(u) \), commutativity \([D(u), D(v)] = 0\) and the crossing symmetry property \( D(u) = D(\lambda - u) \). In general, \( D(u) \) is not symmetric or normal, but it can be diagonalized because \( D(u) = GD(u) = D(u)^T \) is symmetric where the diagonal matrix \( G \) is expressed by

\[
G^b_a = \prod_{j=1}^{N-1} G_{a_j, a_{j+1}} \delta(a_j, b_j) \quad \text{with} \quad G(a,b) = \begin{cases} \frac{s(\lambda)}{s(2\lambda)} & b = 1, 4 \\ 1 & \text{otherwise} \end{cases}
\]

(20)

The normalized transfer matrix is defined by

\[
D(u) = S_b(u) \frac{s^2(2u - \lambda)}{(2u + \lambda)(2u - 3\lambda)} \left( \frac{s(\lambda) s(u + 2\lambda)}{(s(u + \lambda)(s(u + 3\lambda))} \right)^N T(u)
\]

(21)

In the following analysis we limit our discussion to the \((1, 1)\) left and right boundary weights corresponding to the \((r, s) = (1, 1)\) boundary. The boundary row transfer matrix reduces to

\[
D(u)^b_a = \sum_{c_1, \ldots, c_{N-1}} K_L \left( \begin{array}{c} r \\ c_0 \end{array} \right) A \left( \begin{array}{c} b_1 \\ c_0 \\ c_1 \\ b_2 \\ c_2 \\ \ldots \\ b_{N-1} \\ c_{N-1} \\ c_N \end{array} \right) \left( \begin{array}{c} \lambda - u \\ \lambda - u \\ \ldots \\ \lambda - u \end{array} \right) W(\lambda - u) W(\lambda - u) \ldots W(\lambda - u) 
\]

(22)

and

\[
S_b = 1 \quad \text{for} \quad (r, s) = (1, 1)
\]

Restricting the analysis to the \( Y = +1 \) eigenspace, the eigenvalues of the normalized double row transfer matrix \( T(u) \) satisfies the universal \( Y \)-system independent of the boundary conditions \([41]\), hence satisfies the functional equation

\[
t(u)t(u + \lambda) = 1 + t(u + 3\lambda)
\]
3 Classification of states

In this section, we analyze the complex zero distributions of the eigenvalues of the double row transfer matrix, without exploring their corresponding RSOS paths related to the one-dimensional configurational sums of Baxter’s Corner Transfer Matrices (CTMs) [43, 44, 45, 46], as our aim is to finally solve the TBA equations of the model. We consider the behavior of finite excitations above the ground state.

3.1 (m,n) systems, zero patterns, RSOS paths and characters

In the critical $\mathcal{M}(3,5)$ lattice model with $\lambda = \frac{2\pi}{5}$, the face weights and the triangle boundary weights are expressed in terms of the trigonometric functions $s(u) = \sin(u)$. This model corresponds to the conformal field theory model with central charge $c = -\frac{3}{5}$. Its Virasoro algebra has four irreducible modules with characters

$$
\chi_h(q) = q^{-\frac{nh}{2}} \sum_{n=0}^{\infty} \dim(V^h_n) q^n, \quad h = 0, \frac{1}{5}, \frac{3}{4}, -\frac{1}{20} \quad (23)
$$

where $n = E$ is the $L_0$ eigenvalue or the energy of the given state. The eigenvalues are characterized by the location and the pattern of the zeros in the complex $u$-plane. The entries of the unrenormalized transfer matrix are Laurent polynomials in the variables $z = e^{iu}$ and $z^{-1} = e^{-iu}$ of finite degree determined by $N$. The transfer matrices are commuting families with a common set of $u$-independent eigenvectors. It follows that the eigenvalues are also Laurent polynomials of the same degree. Numerical diagonalization gives those polynomials and numerical factorization gives their zeros. As a result, the eigenvalues are characterized by the location and the pattern of the zeros in the complex $u$-plane. We analyze those patterns in terms of the $(m,n)$ systems. In this paper we analyze the boundary case with $(r,s) = (1,1)$ boundary.

**$(m,n)$ systems and zero patterns.** The single relevant analyticity strip in the complex $u$-plane is the full periodicity strip

$$\frac{\pi}{5} < \text{Re} \ u < \frac{6\pi}{5} \quad (24)$$

In the boundary case, the transfer matrix is symmetric under complex conjugation so it is enough to study the eigenvalue zero distributions on the upper half plane. The zeros form strings and the excitations are described by the string content in the analyticity strip. Here we notice the occurrence of four different kinds of strings which we assign as “1-strings”, “short 2-strings”, “long 2-strings” and “real 2-strings”. Figure 1 below gives an example on this string content for a prototype configuration of zeros.

A 1-string $u_j = \frac{7\pi}{10} + iv_j$ whose real part is $\frac{7\pi}{10}$ lies in the middle of the analyticity strip. It appears here in the $(r,s) = (1,1)$ boundary on a fixed location for all eigenvalues. It appears in some of the sectors/boundaries of the model, while it doesn’t exist in other sectors. Each short 2-string has a pair of zeros whose real parts are at $\frac{3\pi}{5}$ and $\frac{4\pi}{5}$, with equal imaginary parts, thus $u_j = \frac{3\pi}{5} + iy_j$. The long 2-string lies at $u_j = \frac{\pi}{5} + iy_j$ with equal imaginary parts and with real parts $\frac{\pi}{5}$ and $\frac{6\pi}{5}$. The zeroes of a long 2-string lie at the edges of the analyticity strip and due to periodicity. In fact, those 2 zeroes are equivalent and correspond to a single zero. The
Figure 1: A typical configuration of zeros of an eigenvalue of the transfer matrix. The "long 2-string" is in green, the "short 2-string" in red, the "1-string" occurs at the center of the strip furthest from the real axis and the "real 2-string" occurs on the real axis.

reason for this naming follows from the general classification of RSOS models with more than one analyticity strips. Finally, a real 2-string consists of a pair of zeros $u_j = \frac{\pi}{5}, \frac{9\pi}{10}$ lying on the real axis with zero imaginary parts. Due to symmetries, the values of these real parts are exact for finite $N$.

The string contents are described by $(m, n)$ systems \[47, 48\]. For this model in the $(1, 1)$ sector, we have:

$$2m + n = \frac{N}{2} - 2$$

where $m$ is the number of short 2-strings, $n$ is the number of long 2-strings and $N$ is even.

In this sector, we always have a single 1-string furthest from the real axis, and a real 2-string on the real axis. The 1-string contributes to one zero, and similarly does the real 2-string due to the symmetry of the upper and the lower half planes. In addition, each short 2-string contributes two zeroes, while each long 2-string contributes only one zero due to periodicity. Hence, the $(m, n)$ system expresses the conservation of the $2N$ zeroes in the periodicity strip.

Note that the appearance of short 2-strings expresses excited states, and the absence of short 2-strings occurs in the ground state, where only long 2-strings appear. For finite excitations, $m$ is finite while $n \to N$ as $N \to \infty$.

An excitation with string content $(m, n)$ is labeled by a unique set of quantum numbers $I$

$$I = (I_1, I_2, ..., I_m)$$

where the integers $I_j \geq 0$ equal the number of long 2-strings whose imaginary parts are greater than that of the given short 2-string $y_j$. The long 2-strings and short 2-strings labeled by $j = 1$ are nearest to the real axis. Those quantum numbers $I_j$ satisfy the equation

$$n \geq I_1 \geq I_2 \geq ... \geq I_m \geq 0$$

For the example given above in Figure 1, we simply have $I_1 = 3$. No other quantum numbers $I_j$ exist as $m = 1$. 
3.2 Continuum scaling limit

For any string content \((m, n)\), the lowest excitation occurs when all of the short 2-strings are further away from the real axis than all of the long 2-strings. This is equivalent to say that all \(I_j = 0\). Bringing the location of a short 2-string closer to the real axis below a long 2-string increases its quantum number by one and increases the energy.

3.2 Continuum scaling limit

In the continuum scaling limit, where the even \(N \to \infty\), the spacing of the zeroes becomes denser. We find that the imaginary parts of the furthest zeros from the real axis grow as \(\frac{2}{9} \log N\), hence the spacing between zeros tends to 0 as \(\frac{2 \log N}{N}\). Finite energy states for large \(N\) have zero patterns as depicted in Figure 2. We denote the imaginary part of the 1-string by \(\alpha\) and those of the short 2-strings by \(\beta_j\). The number of short 2-strings is finite. \(\alpha, \beta_j\), and the imaginary parts of the long 2-strings furthest from the real axis scale as \(\frac{2}{9} \log N\) in the continuum scaling limit.

![Figure 2: A typical zero configuration for an eigenvalue in the (1,1) sector for large value of N.](image)

4 Critical TBA Equations

4.1 Critical TBA

The critical TBA equations of this model are derived by solving the functional relation

\[
t(u)t(u + \lambda) = 1 + t(u + 3\lambda)
\]

while using the analytic structure of the function \(t(u)\) which is factorized according to the large volume behavior as

\[
t(u) = f(u)g(u)l(u)
\]

where \(\log f(u)\) is of order \(N\), \(\log g(u)\) of order 1 and \(\log l(u)\) is on the order \(\frac{1}{N}\). The leading order term satisfies the relation

\[
f(u)f(u + \lambda) = 1
\]
and contains the order $N$ zeros and poles of the normalization factor. The function $g(u)$ satisfies a
similar relation

$$g(u)g(u + \lambda) = 1$$

and accounts for the order 1 boundary dependent zeroes and poles. The remaining finite size
function $l(u)$ is derived from an appropriate integral equation.

The energy of the states can be calculated from the finite size corrections of the double row
transfer matrix eigenvalues

$$\log t(u) = \log f(u) + \log g(u) - \frac{i}{N} \left( e^{-\frac{5u}{2}E^+} - e^{\frac{5u}{2}E^-} \right)$$

where the finite-size conformal energies are given in their exponential form as

$$E = e^{-\frac{5u}{2}E^+} - e^{\frac{5u}{2}E^-}$$

The first term dominates in the $u \to +i\infty$ limit, while the second dominates in the $u \to -i\infty$
Using equation (27), we find that

$$t(u)t(u + \frac{2\pi}{5}) = 1 + t(u + \frac{9\pi}{5})$$

Exploiting the periodicity of the transfer matrix of $t(u) = t(u + \pi)$ and after an appropriate
shift in the variables we obtain the functional relation

$$t(u - \frac{\pi}{5})t(u + \frac{\pi}{5}) = 1 + t(u)$$

The normalization introduces zeros of order $N$ (even) at $\frac{\pi}{5}$ and $\frac{6\pi}{5}$, and poles of order $N$ at $\frac{3\pi}{5}$ and $\frac{4\pi}{5}$. They should be normalized by the function $f(u)$ whose solution compatible with the
analytic structure is

$$f(u) = \left( \frac{-\sin(\frac{5u}{2} - \frac{\pi}{4}) \sin(\frac{5u}{2} + \frac{\pi}{4})}{\cos(\frac{5u}{2} - \frac{\pi}{4}) \cos(\frac{5u}{2} + \frac{\pi}{4})} \right)^N = \left( \frac{\cos(\frac{5u}{2} + \frac{\pi}{4}) + \cos \frac{\pi}{4}}{\cos(\frac{5u}{2} + \frac{\pi}{4}) - \cos \frac{\pi}{4}} \right)^N$$

This function satisfies the relation

$$f(u - \frac{\pi}{5})f(u + \frac{\pi}{5}) = 1$$

In addition to the relation $f(u)f(u + \lambda) = 1$ stated above.

In the thermodynamic limit, the imaginary part of the outermost string from the real $u$ axis
goes to infinity as $\frac{3}{2} \log \kappa N$ with

$$\kappa = 4 \sin \frac{\pi}{4} = 2\sqrt{2}$$

Consequently, we define a real variable $x$ as a vertical coordinate along the center of the analyticity
strip as:

$$u = \frac{7\pi}{10} + \frac{2ix}{5}$$

In this new coordinate $x$, the functional relation becomes

$$t(x - \frac{\pi}{2})t(x + \frac{\pi}{2}) = 1 + t(x)$$
4.1 Critical TBA

And upon this change of variable, \( f(u) \) becomes

\[
f(x) = \left( \frac{\cosh x + \cos \frac{\pi}{4}}{\cosh x - \cos \frac{\pi}{4}} \right)^N
\]

and satisfies the functional relation

\[
f(x - i\frac{\pi}{2})f(x + i\frac{\pi}{2}) = 1
\] (38)

The boundary normalization also introduces a double zero at \( u = \lambda^2 = \frac{\pi}{5} \) and poles at \( u = -\lambda^2 + \pi = \frac{2\pi}{5} \) and at \( u = \frac{3\pi}{5} = \frac{2\pi}{5} \). Due to the presence of the argument \( 2u \), the periodicity of order 1 functions is \( \frac{\pi}{2} \). Thus, we have a double zero at \( u = \frac{7\pi}{10} \), and poles at \( u = \frac{3\pi}{10} \) and \( u = \frac{11\pi}{10} \). Finally, the presence of a real 2-string indicates a couple of zeroes at \( \frac{\pi}{2} \) and \( \frac{9\pi}{10} \).

To account for those zeros and poles, \( g(u) \) is defined as

\[
g(u) = \frac{\tan^2(\frac{5u}{4} - \frac{\pi}{4})\tan^2(\frac{5u}{4} - \frac{7\pi}{8})\tan(\frac{5u}{4} - \frac{5\pi}{8})\tan(\frac{5u}{4} - \frac{9\pi}{8})}{\tan(\frac{5u}{4} - \frac{3\pi}{4})}\]

Transforming into the \( x \) variable we obtain

\[
g(x) = \frac{\tan^2(\frac{i\pi}{2} + \frac{5\pi}{8})\tan^2(\frac{i\pi}{2} + \frac{\pi}{4})\tan(\frac{i\pi}{2} - \frac{\pi}{4})}{\tan^2(\frac{i\pi}{2} + \frac{7\pi}{8})\tan(\frac{i\pi}{2} + \frac{3\pi}{8})}
\] (40)

The function \( g(u) \) satisfies the functional relations \( g(u - \frac{\pi}{5})g(u + \frac{\pi}{5}) = 1 \) which is equivalent to

\[
g(x - i\frac{\pi}{2})g(x + i\frac{\pi}{2}) = 1
\] (41)

The order one \( g \) term appears in the lattice boundary TBA but it doesn’t explicitly contribute to the energy. Due to the symmetry with respect to the real \( u \)-axis obtained by complex conjugation, the variables are scaled around \( \log \kappa N \), and disappear in the scaling limit.

**Quantum states**

To find the quantum states and the corresponding critical TBAs, we need to solve the functional relation (37). To do this we need to ensure that \( l(x) \) is analytic and non-zero in the analyticity strip, and that its logarithm has constant asymptotic behavior as \( x \to \pm \infty \). This is done by characterizing the eigenvalues of the transfer matrix by their patterns of zeros in the analyticity strip \( \frac{\pi}{5} < u < \frac{6\pi}{5} \). The long 2-strings occur at the boundaries of the analyticity strip, and they become dense in the thermodynamic limit \( N \to \infty \), consequently they define the boundaries of the analyticity strip at \( \frac{\pi}{5} \) and \( \frac{5\pi}{5} \).

In the \((1, 1)\) sector, a single 1-string appears at the center of the strip furthest out from the real axis, with symmetry in the upper and lower parts of the \( u \)-plane. Its position always occurs at

\[
u_0 = \frac{7\pi}{10} + i\alpha
\] (42)

The short 2-strings correspond to finite excitations above the ground state and their real parts can occur at \( \frac{3\pi}{5} \) and \( \frac{4\pi}{5} \), and they are expressed as

\[
u_j = \begin{cases} 
\frac{3\pi}{5} + i\beta_j \\
\frac{4\pi}{5} + i\beta_j
\end{cases}
\] (43)
In the thermodynamic limit, with \( N \to \infty \), those zeros in the scaling regions furthest from the real axis approach infinity in the upper and lower half planes as

\[
\begin{align*}
\alpha &= \frac{2}{5} \left( \pm \log \kappa N + \tilde{\alpha}^\mp \right) \\
\beta &= \frac{2}{5} \left( \pm \log \kappa N + \tilde{\beta}^\pm_j \right)
\end{align*}
\]

Transforming into the \( x = \frac{5}{2i} (u - \frac{7\pi}{10}) \) variable, we find that the locations of the zeros of the 1-strings occur at:

\[
x_0^\pm = \frac{5\alpha}{2} = \pm \log \kappa N + \tilde{\alpha}^\mp
\]

while the zeros of the short 2-strings occur at:

\[
\begin{cases}
(x_j^\pm + \frac{4\pi}{5}, x_j^\pm - \frac{4\pi}{5}) \\
x_j^\pm = \pm \log \kappa N + \tilde{\beta}^\pm_j
\end{cases}
\]

The remaining task is to convert the functional equation into an integral TBA equation that can be solved by Fourier transforms in the continuum scaling limit. To satisfy ANZC functions that are free of zeros and poles in the strip containing \( \text{Im} x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \), appropriate functions are introduced to remove the 1-string and the short 2-string zeros.

\[
\sigma_0 = \tan \left( \frac{5u}{4} + \frac{\pi}{8} \right)
\]

removes the zero of the one string while

\[
\sigma_1 = -\tan \left( \frac{5u}{4} \right) \tan \left( \frac{5u}{4} + \frac{\pi}{4} \right) = \frac{\cos \left( \frac{5u}{4} + \frac{\pi}{4} \right) - \cos \frac{\pi}{4}}{\cos \left( \frac{5u}{4} + \frac{\pi}{4} \right) + \cos \frac{\pi}{4}}
\]

removes the two zeros of the short two strings.

In the \( x \) variable those functions are given by

\[
\begin{cases}
\sigma_0 = \tanh \frac{x}{2} \\
\sigma_1 = \cosh \frac{x}{2} - \cos \frac{x}{4}
\end{cases}
\]

Those functions satisfy the relations

\[
\sigma_0(x - \frac{i\pi}{2}) \sigma_0(x + \frac{i\pi}{2}) = 1 \quad ; \quad \sigma_1(x - \frac{i\pi}{2}) \sigma_1(x + \frac{i\pi}{2}) = 1
\]

Consequently, the appropriate parametrization of the normalized transfer matrix eigenvalue is

\[
t(x) = f(x)g(x) \prod_{\pm} \sigma_0(x - x_0^\pm) \prod_{j=1}^M \sigma_1(x - x_j^\pm)l(x)
\]

where \( M \) is the number of short 2-strings in the strip for a given excited state.

Using the functional relation (37), and exploiting the properties of the functions \( f(x) \) and \( g(x) \) given in (39) and (41), we obtain the equality

\[
l(x - \frac{\pi}{2})l(x + \frac{\pi}{2}) = 1 + t(x)
\]
4.1 Critical TBA

With our construction of all necessary functions, \( l(x) \) is analytic and non-zero in the analyticity strip, and its logarithm has constant asymptotic (ANZC) as \( x \to \pm \infty \). Taking the logarithm on both sides and solving the equations using Fourier transforms of the derivatives \( \left[ \log l(x) \right]' \) we obtain that:

\[
\log l(x) = -\varphi \star \log [1 + t(x)]
\]

where the convolution \( \star \) is defined by

\[
(f \star g)(x) = (g \star f)(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x-y)g(y)dy
\]

and the function \( \varphi \) and its transform \( \hat{\varphi} \) are given by

\[
\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \hat{\varphi}(k)e^{ikx}
\]

and

\[
\hat{\varphi}(k) = -\frac{1}{e^{k\pi/2} + e^{-k\pi/2}}
\]

Consequently, and following the procedure of [21], an explicit expression of \( \varphi(x) \) can be obtained as:

\[
\varphi(x) = -\frac{1}{2\pi \cosh x}
\]

In general, the kernel \( \varphi(x) \) is related to the two-particle \( S \)-matrix of the corresponding continuum model, but this \( S \)-matrix is not explicitly determined yet.

Restoring \( t(x) \) we obtain the critical TBA equations on the lattice for the \((1,1)\) boundary condition as

\[
\log t(x) = \log f(x) + \log g(x) + \sum_{\pm} \log \sigma_0(x - x_0^\pm) + \sum_{j=1}^{M} \log \sigma_1(x - x_j^\pm) - \varphi \star \log [1 + t(x)]
\]

The parameters of the excited state \( x_i = \{ x_0^\pm, x_j^\pm \} \) are determined self-consistently from the fact that they are zeros of the transfer matrix:

\[
t(x) \bigg|_{x=x_\pm} = -1
\]

In the continuum scaling limit with \( N \to \infty \), \( f(x) \) has nontrivial behavior in the two scaling regions \( x \sim \pm \log \kappa N \). Then, two scaling functions are introduced as

\[
e^{\mp}(x) = \lim_{N \to \infty} t(x \pm \log \kappa N)
\]

The behavior of \( f(x) \) in the scaling regions is important, and in this scaling limit we obtain that
It is interesting to observe that \( g(x) \) scales to 1 around \( \log \kappa N \), hence it has no contribution to the subsequent TBA equations, and no explicit contribution to the energy.

This leads to the massless boundary TBA equations

\[
\epsilon^\mp (x) = e^\mp x + \sum_{\pm} \log \sigma_0(x - \tilde{\alpha}^\mp) + \sum_{j=1}^{M} \log \sigma_1(x - \tilde{\beta}_j^\mp) - \varphi \star \log \left(1 + e^x\right)
\]

The location of the zeros \( \tilde{\alpha}^\mp \) and \( \tilde{\beta}_j^\mp \) were defined in equations (14) and (15).

The lowest energy state of this sector, or what we may call the ground state of the \((1,1)\) sector has no short strings that represent excitations, hence the term \( \sigma_1(x) \to 1 \), and does not appear in the equations so the corresponding massless boundary TBA equation for the lowest energy state in this sector is given by

\[
\epsilon^\mp (x) = e^\mp x + \sum_{\pm} \log \sigma_0(x - \tilde{\alpha}^\mp) - \varphi \star \log \left(1 + e^x\right)
\]

### 4.2 Energies

The finite size energies of excited states can be determined from \( \log l(x) \) and equations (48). The energy formula is

\[
\frac{1}{N} (e^x E^+ + e^{-x} E^-) = \sum_{\pm} \log \sigma_0(x - x_0^\pm) + \sum_{j=1}^{M} \log \sigma_1(x - x_j^\pm) - \log l(x)
\]

\[
= \sum_{\pm} \log \sigma_0(x - x_0^\pm) + \sum_{j=1}^{M} \log \sigma_1(x - x_j^\pm) + \int_{-\infty}^{\infty} \frac{dy}{2\pi} \varphi(x-y) \log \left(1 + e^{x(y)}\right)
\]

With appropriate scaling in the infinite regions as \( x \sim \pm \log \kappa N \), we find that the limit

\[
\lim_{N \to \infty} \kappa N \varphi(x - \log \kappa N) = -e^x
\]

and this allows to determine \( E^+ \) and \( E^- \) as:

\[
E^\pm = \sum_i e^{\pm \tilde{\gamma}_i^\pm} - \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{\mp y} \log \left(1 + e^{\pm y}\right)
\]

where \( \tilde{\gamma}_i^\pm \) is either \( \tilde{\alpha}_i^\pm \) or \( \tilde{\beta}_j^\pm \), where \( i \) runs over \( \{\pm, j = 1, ..., M\} \).

### 5 Conclusion

We analyzed a nontrivial relativistic integrable theory, namely the boundary \( \mathcal{M}(3;5) \) model, from the lattice point of view, in the \((r = 1, s = 1)\) sector. This is a nontrivial non-unitary minimal model, dual to the Lee-Yang model. The \( A_4 \) Restricted Solid-on-Solid (RSOS) Forrester-Baxter model with trigonometric weights was solved in the continuum scaling limit. We described the patterns of zeros.
of the corresponding double row transfer matrix eigenvalues. Those zeros are directly related to the
RSOS paths on the lattice. Complementing previous work done in the $\mathcal{M}(2,5)$ Lee-Yang model
solved before [21, 22], we adopted a similar approach to analyze this model. However, the boundary
conformal field theory (BCFT) model is not fully solved, which prohibits direct comparison of the
Virasoro states with configurational paths, as well as the corresponding TBA equations on the
continuum side of the theory.

For the critical theory with integrable boundary, the transfer matrix satisfies the same universal
$Y$ system as [19]. The TBA equations describe the finite-size scaling spectra of the $\mathcal{M}(3,5)$ model
in the continuum scaling limit. The other sectors of this boundary case are similar in their patterns
of zeros with the only difference is that some of them would contain a fixed zero at the center of
the analytic strip, while other sectors would not.

The lattice description of the integrable scattering theory is used in order to determine spectrum
of the model. It may also allow further to explore interesting and relevant physical quantities
such as vacuum expectation values and form factors. These quantities can be obtained from the
bootstrap approaches and they are available for the dual Lee-Yang [8, 50] but were not solved yet
for the $\mathcal{M}(3,5)$ model. Future work should study the critical and massive $\mathcal{M}(3,5)$ models using the
bootstrap methods and determine its scattering, reflection and transmission matrices in different
geometries, and its vacuum expectation values as well as its form factors. It should also explore
other non-unitary models using both the RSOS lattice models and the bootstrap approach.

Acknowledgments

This work was performed during work at the Beirut Arab University (BAU). I would like to thank
the Physics department at BAU for its support towards concluding this work. The advices of Maya
Jalloul on Latex code and figures were also beneficiary to finish the paper in its current form.

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