1. Introduction

Indirect exchange interactions among magnetic impurities embedded in conduction electrons is a rich and fascinating problem in solid state physics. The most familiar inter-impurity interaction mediated by the conduction electrons is the celebrated Ruderman–Kittel–Kasuya–Yosida (RKKY) interaction [1]. The discovery of the RKKY interaction allowed for the comprehension of magnetic order of a variety of magnetic materials [2]. This phenomena can be understood within the concept of perturbation theory: an electron scattered by a given magnetic impurity has its spin modified by a local exchange interaction—a Kondo-like coupling. This information is then transferred to a second impurity upon a second similar collision. The net effect is an effective indirect exchange coupling between the two impurities mediated by the conduction electrons [3–5]. In conventional systems, this resulting effective coupling exhibits an oscillatory behavior as a function of the distance \( x \) between the impurities, decaying as \( x^{-D} \), where \( D \) is the dimension of the system.

In the recent years we have witnessed a renewed interest in the indirect exchange interactions between magnetic impurities embedded in spin–orbit coupled conduction electrons [6–9], including topological insulators, Dirac [10] and Weyl [11] semimetals. In these materials, the spins of the conduction electrons and their momenta are coupled together. As a result, after being scattered by the one impurity, the spin of a given electron precesses while traveling towards the second impurity. This precession produces a more complex and reacher inter-impurity magnetic interaction [12, 13] such as twisted magnetic arrangement, a non-collinear exchange coupling known as Dzialoshinkii–Moryia interaction (DMI) [14, 15] and a Ising like coupling [16]. From a practical point of view, the Rashba spin–orbit coupling (SOC) opens up the possibility of controlling the inter-impurity magnetic interaction via external electric field with great potential application.
in spintronics [17, 18]. Particularly appealing, but hitherto less investigated, is the indirect the exchange interactions in 1D systems in the presence SOC. Since in 1D the electrons are forced to propagate along some particular direction, the spin-momentum locking induced by the SOC can drastically modify the scattering processes [19].

In a seminal paper published in 1990, Datta and Das proposed the idea of producing a highly spin-polarized current controlled via SOC by external electric field [20]. In their device, the spins of the electrons injected from a polarized source could be rotated by a tunable SOC while traveling towards a polarized drain. Likewise, it would interesting if one could use the SOC to control the indirect exchange interaction between two magnetic impurities embedded in a 1D conduction electron sea. The few studies addressing the RKKY interaction in one-dimensional systems with SOC, in general, employ a real space Green’s function [6]. It is known, however, that calculating the RKKY interaction in one-dimension is quite subtle [21]. This was first noticed by Kittel [22] and later discussed in detail by Yafet [23]. Yafet indeed showed that, depending on how the double integral is handled, it can lead to unphysical results. Moreover, Yafet observed that the problem arises because the Pauli’s exclusion principle is violated very severely. More recently, Rusin and Zawadzki [25] has examined the commonly used expression for the RKKY interaction [6] and noticed that there is an implicit change of variables in the calculation of the full indirect exchange interaction between two magnetic impurities in a 1D system. On the basis of the traditional second order perturbation theory, we obtained the known form of the inter-impurity couplings, which includes the usual RKKY, DM and Ising interaction terms. The effective couplings are calculated both numerically and analytically. Drastically different from the usual RKKY systems, we obtain additional oscillatory contributions to the effective couplings that do not decay with the distance. These unsuppressed terms vanish in the absence of SOC, in which case the traditional RKKY coupling is recovered. This feature is potentially important to spintronics as it can be used to control spin-spin interaction at longer distances as compared to the traditional RKKY couplings. Indeed, by employing a similar calculation we perform here, it was shown important enhancement in the magnetic coupling between two impurities in controlled Rashba spin–orbit interaction [24].

There are currently several modern 1D systems with SOC that are natural candidates for experimental investigation of this interesting physics [26–30].

2. Model and method

We consider two spin-1/2 magnetic impurities coupled to a quantum wire with spin–orbit interaction, as schematically shown in figure 1. We write the full Hamiltonian of the system

\[ H_{\text{wire}} = \sum_k \left( \varepsilon_k \delta_{k,k'} - \alpha_{R} \sigma_{z} \right) \vec{c}_k \cdot \vec{c}_{k'}, \]  

(1)

where \( \delta_{k,k'} \) denotes the scattering matrices due to the spin–orbit interaction.

The tildes on top of the spin operators above indicate that these operators are also written in the rotated basis. In equation (3) we have defined \( \vec{I}_\parallel = 2 \Re \left( \vec{I}_+ + \vec{I}_- \right) \) and \( \vec{I}_\perp = \vec{I}_+ - \vec{I}_- \), where

\[ I_{\Delta} = \frac{J}{N} \sum_{k k'} e^{-i(k-k')x} \left[ \vec{S}_I \cdot \vec{c}^{\dagger}_k \vec{c}^{\dagger}_{k'} - \vec{c}^{\dagger}_k \vec{c}^{\dagger}_{k'} \vec{c}_{k'} \right] + \vec{S}_I \cdot \vec{c}^{\dagger}_k \vec{c}^{\dagger}_{k'} \vec{c}_{k'} \right], \]  

(2)

Here, \( \alpha_{R} \) is the effective mass of the conduction electrons. Starting by diagonalizing the Hamiltonian (1), we follow the traditional second order perturbation theory approach. The resulting inter-impurity coupling is described by the effective Hamiltonian (see detail in appendix A)

\[ \tilde{H}_{\text{imp}} = I_{\parallel} \vec{S}_1 \cdot \vec{S}_2 + \left( I_{\perp} \vec{S}_1 \cdot \vec{S}_2 + I_{\perp} \vec{S}_1 \cdot \vec{S}_2 \right). \]

(3)

The tilde is on top of the spin operators above indicate that these operators are also written in the rotated basis. In equation (3) we have defined \( I_{\parallel} = 2 \Re \left( I_{+} + I_{-} \right) \) and \( I_{\perp} = I_{+} - I_{-} \), where

\[ I_{\Delta} = \frac{J}{N} \sum_{k k'} e^{-i(k-k')x} \left[ \vec{S}_I \cdot \vec{c}^{\dagger}_k \vec{c}^{\dagger}_{k'} - \vec{c}^{\dagger}_k \vec{c}^{\dagger}_{k'} \vec{c}_{k'} \right] + \vec{S}_I \cdot \vec{c}^{\dagger}_k \vec{c}^{\dagger}_{k'} \vec{c}_{k'} \right], \]  

(4)

Here, \( x = x_2 - x_1 \) is the distance between the impurities and \( k_{\Delta} = k_1 + k_{R} \), with \( k_{R} = m^* \alpha_{R} / \hbar^2 \) being the characteristic inverse of spin–orbit length. In the equation (4) we also have \( \delta, \nu \in \{+, - \} \) denoting the Rashba bands. The rather simple form of the Hamiltonian (3), written in the Rashba basis, hides very interesting physics. It can be seen that \( I_{\parallel} \neq I_{\perp} \), therefore, in the present form, the Hamiltonian (3) describes a highly anisotropic exchange interaction mediated by the conduction electrons. To highlight the physics buried in the equation (3) we transform it back to the original real spin basis, obtaining

\[ \tilde{H}_{\text{imp}} = I_{\text{RKKY}} \vec{S}_1 \cdot \vec{S}_2 + I_{\text{DM}} \left( \vec{S}_1 \times \vec{S}_2 \right) \cdot \vec{y} + I_{\text{Ising}} \vec{S}_1 \cdot \vec{S}_2. \]

(5)
Here, $I_{\text{RKKY}} = 2\Re I_\perp$, is the traditional RKKY interaction coupling renormalized by the SOC, $I_{\text{DM}} = -2\Im I_\perp$ is the Dzialoshinskii–Moriya interaction between the two impurities and $I_{\text{sing}} = I_\parallel - 2\Re I_\perp$ represents an Ising-like coupling. Again, for $\alpha_R = 0$ only the first term of (5) survives. In this case we left with one double integral, obtaining $I_{\text{sing}} = I_{\text{DM}} = 0$ and

$$I_{\text{RKKY}} = 4\Im \int_{-\infty}^{\infty} \frac{dk}{k} \int_{|k'| \geq |k|} \frac{dk'}{k'^2} \cos \left( \frac{(k-k')x}{k^2 - k'^2} \right),$$

with $I_0 = m^* \tau^2/2\pi^2 \hbar^2$. Performing the double integral (6) is known to be a delicate matter and have been discussed from way back [23]. Analytically, the integration can be performed if one extends the integral over $k'$ to the entire real axis. After this, the residue theorem can be employed. Apart from the singular point $k' = k = 0$ which can be accounted separately the contribution to the double integral added by including the interval $(-k, k)$ vanishes because the integrand is antisymmetric under exchange $k' \leftrightarrow k$. In the asymptotic limit $x \to \infty$, the final correct solution exhibits the usual form $\cos(2k_Fx)/x$.

In the presence of SO $\alpha_R \neq 0$, exact solutions for the integrals of equation (4) are, unfortunately, unavailable. In this case, even though we can subtract the contribution of the singularity from the integration over $k'$ within the entire real axis, the integrand is no longer antisymmetric. Therefore, by extending the integral over $k'$ to the interval $(−k_F, k_F)$, the extra contribution cannot be fully subtracted. As we will see below, great approximations for the integrals (4) can still be obtained in the limit $k_F \ll k_F$, in which case the asymmetry of the integrand is negligible. To carry out the calculations, we simplify the notation defining $a = 2k_F$ and $\tilde{a} = a/k_F$ and introducing the dimensionless momenta $q = k/k_F$ and $q' = k'/k_F$. Within these new variables, we can rewrite the equation (4) as

$$I_{\text{5F}} = I_0 \int_{-\infty}^{\infty} dq \int_{|q'| > |q|} \frac{e^{i(q-q')k_Fx} dq'}{(q^2 - q'^2)^2} + \delta \hat{a}(q - \delta v q')$$

(7)

Here, $q = 1 + \delta \tilde{a} = 2 = 1 + \delta k_F/k_F$. Following Yafet’s approach [23] we can write $I_{\text{5F}} = I_{\text{5}'\text{F}} - I_5', where

$$I_{\text{5}'\text{F}} = I_0 \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} \frac{e^{i(q-q')k_Fx} dq'}{(q^2 - q'^2)^2} + \delta \hat{a}(q - \delta v q')$$

(8)

in which the integral over $q'$ extends over the entire real axis, and $I_{\text{5}'\text{F}}$ corresponds to the undesirable singularities accounted within the extended limit of the integral over $q'$. The integration of (8) over $q'$ can be performed using Cauchy’s integral theorem. For instance, after a cumbersome integration over $q$ (see detail in appendix B) we obtain for $I_{\text{RKKY}}$ (without the corrections),

$$I_{\text{RKKY}} = \pi I_0 \left\{ \sin(\hat{a}k_F) \left[ C[(1 - \hat{a})2k_F] - C[(1 + \hat{a})2k_F] \right] + \cos(\hat{a}k_F) \left[ S[(1 - \hat{a})2k_F] - S[(1 + \hat{a})2k_F] \right] + \text{Si}(2k_F) - \text{Si}(-2k_F) - \ln \left| \frac{1 - \hat{a}}{1 + \hat{a}} \right| \sin(\hat{a}k_F) \right\}$$

(9)

In the equation above, $\text{Si}(x)$ and $\text{Ci}(x)$ are the known sine and cosine integral functions, respectively. To obtain the approximate expression we have to subtract the spurious contribution from the singularities. As an example, here we show in detail the calculation of the correction for $I_{\text{5}}'$ to the integral $I_{\text{5}}$ (see appendix B). Note that the unbalanced singularities occur when $q + q' = 0$ and $(q - q' - \tilde{a}) = 0$ simultaneously, from which we find $q' = -\tilde{a}/2 = -q$. We can evaluate the integral within an infinitesimal interval around this point as

$$I_{\text{5}}' = \pi I_0 e^{i\hat{a}2k_Fx} \int_{-\epsilon}^{\epsilon} dq' \int_{\frac{\pi}{2} - \epsilon}^{\frac{\pi}{2} + \epsilon} dq \frac{\ln |q - \frac{\pi}{2} + \epsilon| - \ln |q - \frac{\pi}{2} - \epsilon|}{q - \frac{\pi}{2}}.

(10)

with $\epsilon \to 0$. On the rhs of the equation (10) we have already used that at the singular point under analysis, $e^{i(q-q')k_Fx} = e^{i(\hat{a}+\tilde{a})k_Fx}/\sqrt{2} = e^{i\hat{a}k_Fx}$. Performing the integral over $q'$ we obtain

$$I_{\text{5}}' = \pi I_0 e^{i\hat{a}2k_Fx} \int_{-\epsilon}^{\epsilon} dq' \frac{\ln |q - \frac{\pi}{2} + \epsilon| - \ln |q - \frac{\pi}{2} - \epsilon|}{q - \frac{\pi}{2}}.

(11)

After a simple change of variable $y = q - \tilde{a}/2$ we can write $I_{\text{5}}' = \pi I_0 e^{i\hat{a}2k_Fx} \int_{-\epsilon}^{\epsilon} dy \frac{\ln |y + \frac{\pi}{2} + \epsilon| - \ln |y - \frac{\pi}{2} - \epsilon|}{y - \frac{\pi}{2}}$. The integral here can be written in terms of the Dilogarithm function $\text{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1 - t/t)$. With this and using $\text{Li}_2(1) = \pi^2/6 = -\text{Li}_2(-1)$ [35] we can write $I_{\text{5}}' = (\pi^2/2)\pi I_0 e^{i\hat{a}2k_Fx}$. Proceeding likewise, we obtain the correction $I_{\text{5}}' = (\pi^2/2)I_0 e^{i\hat{a}2k_Fx}$. Collecting all these terms, the correction for the RKKY coupling is given by $I_{\text{RKKY}} = 2\Re \left( I_{\text{5}}' + I_{\text{5}}'' \right) = 4\pi I_0 \left[ \frac{\pi}{2} \cos(\hat{a}k_F) \right]$. This is the quantity we must subtract from (9) to obtain the approximated result. This result generalizes the correction found in [25]. In the absence of SOC ($\tilde{a} = 0$) $I_{\text{RKKY}} = 2\pi^2 I_0$, which is exactly the correction discussed in [25]. The final expression for the RKKY coupling is then $I_{\text{RKKY}} = I_{\text{RKKY}} - I_{\text{DM}}$.

Carrying out similar calculations we obtain the analytical results for all inter-impurity couplings,

$$I_{\text{RKKY}} = \pi I_0 \left\{ \sin(\hat{a}k_F) \left[ C[(1 - \hat{a})2k_F] - C[(1 + \hat{a})2k_F] \right] + \cos(\hat{a}k_F) \left[ S[(1 - \hat{a})2k_F] - S[(1 + \hat{a})2k_F] \right] + 2\text{Si}(2k_F) + \ln \left| \frac{1 - \hat{a}}{1 + \hat{a}} \right| \sin(\hat{a}k_F) \right\} + 4\pi I_0 \left[ \frac{\pi}{2} \cos(\hat{a}k_F) \right], \quad (12)

I_{\text{DM}} = -\pi I_0 \left\{ \cos(\hat{a}k_F) \left[ C[(1 + \hat{a})2k_F] - C[(1 - \hat{a})2k_F] \right] + \sin(\hat{a}k_F) \left[ S[(1 + \hat{a})2k_F] - S[(1 - \hat{a})2k_F] \right] + 2\text{Si}(2k_F) + \ln \left| \frac{1 - \hat{a}}{1 + \hat{a}} \right| \cos(\hat{a}k_F) \right\} + 4\pi I_0 \left[ \frac{\pi}{2} \sin(\hat{a}k_F) \right], \quad (13)

I_{\text{sing}} = 2\pi I_0 \left\{ \sin[(1 + \hat{a})2k_F] - \sin[(\hat{a} - 1)2k_F] \right\} - 4\pi I_0 \left( \frac{\pi}{2} \right) - I_{\text{RKKY}}. \quad (14)

As usual, the sine and cosine integral functions are respectively defined as $\text{Ci}(x) = \int_0^x (1/t) \cos(t) dt$ and $\text{Si}(x) = \int_0^x (1/t) \sin(t) dt$.
These rather complex expressions reduce to the known result $I_{\text{RKKY}} = 4\pi I_0[\text{Si}(2k_Fx) - \pi/2]$ in the absence of SOC ($\tilde{a} = 0$), that behaves as $\cos(2k_Fx)/x$ for large $x$ (with $I_{\text{DM}} = I_{\text{Ising}} = 0$). On the other hand, for $\tilde{a} \neq 0$ the leading terms for large $x$ are

$$I_{\text{RKKY}} = -\pi I_0 \ln \left| \frac{k_F - 2k_R}{k_F + 2k_R} \right| \sin(2k_Rx), \quad (15)$$

$$I_{\text{DM}} = -\pi I_0 \ln \left| \frac{k_F - 2k_R}{k_F + 2k_R} \right| \cos(2k_Rx), \quad (16)$$

$$I_{\text{Ising}} = \pi I_0 \ln \left| \frac{k_F - 2k_R}{k_F + 2k_R} \right| \sin(2k_Rx). \quad (17)$$

Here we have used $\tilde{a} = 2k_R/k_F$. These remarkable unsuppressed oscillatory terms summarize the main result of our work. These terms contrast with the decaying behavior of the usual RKKY interaction in the absence of SOC.

Before discussing these results, we compare the analytical results of equations (12)–(14) with the ones obtained by direct numerical integration of (7) for $\tilde{a} = 0.1$ ($k_R = 0.05k_F$). The results are shown in figure 2. Panels figures 2(a)–(c) correspond to the $I_{\text{RKKY}}$, $I_{\text{DM}}$ and $I_{\text{Ising}}$, respectively. Dashed red lines correspond to the analytical results of equations (12)–(14) while solid black lines refer to the numerical results obtained by direct integration of equation (7). The dash-dot blue lines show the asymptotic behavior of the coupling given by the equations (15)–(17). The extraordinary agreement between our numerical and analytical results shown in figure 2 confirms that we have indeed obtained very good approximate expressions for all couplings.

The striking features are the undamped slow oscillations in the coupling due to the SOC mentioned earlier. The fast oscillations along the slow oscillating line is the traditional behavior of the RKKY interaction and result from the polarization of the Fermi sea by one impurity and ‘felt’ by the other one. They are described by the $C_i$ and $S_i$ functions of equations (12)–(14) and have the traditional period $2\pi/k_F$. Note also in equations (12)–(14) the extra terms $\pm 2\tilde{a}k_Fx = \pm 2k_Rx$ in the arguments of the functions $C_i$ and $S_i$. They are responsible for the curious beating patterns observed in the fast oscillations. Physically, the beating patterns can be understood in the following way: the original RKKY interaction exhibits an oscillation with frequency of $2k_F$. In the absence of SOC, the spin up and down bands are degenerated leading to a single Fermi momenta $k_F$. Here, on the other hand, electrons move freely in different helical bands possessing Fermi momenta are slightly shifted as compared to each other. Since the real spin basis representation is a linear combination of each Rashba bands, the resulting spin polarization is a combination of two oscillating terms whose phase are slightly shifted. This renders the beating pattern observed along the distance between the impurities. These beating patterns are akin to what was found in [34] for the RKKY interaction in spin-polarized bands.

![Figure 2. RKKY (a) Dzialoshinkii–Moryia (b) and Ising (c) couplings as a function of the distance $x$ between the impurities. Solid black lines correspond to the results obtained by direct numerical integration of equation (4), dashed red lines correspond to the analytical results of equations (12)–(14) and dash-dot blue lines show the asymptotic behavior of the coupling given by the equations (15)–(17). 3. Discussions](image)
equations (15)–(17). When $k_F \rightarrow 2k_F$ the Fermi momenta matches precisely the distance (in the momentum space) between the two bands, providing a resonant forward scattering. In reality, similarly to all the traditional approach to RKKY interaction, our results are limited distances smaller than the coherent length of the material. For distances larger than this characteristic length, other scattering processes have to be taken into account in the conduction electron propagation.

Somewhat similar to our results was found by Simonin [35]. He has found a spin-spin correlation between two magnetic moments induced by spin that extends also to distances longer than those of the traditional RKKY interaction. Our results also resemble the persistent spin helix [36, 37] in which a ‘right’ combination of Rashba and Dresselhaus SOCs produces a long lived spin excitation in the system. This contrasts with the traditional scattering in the absence of the SOC, in which there is a $Q = 0$ scattering processes is allowed since the $\epsilon_F (k) = \epsilon_F (k)$. Previous studies usually employ a very attractive expression based on real space Green’s functions [6]. However, as thoroughly discussed by Valizadeh [34] the expression should be avoided in 1D systems. Essentially, the reason is because in the derivation of the equation (5) of [6] there is a change in the order of integration in double integral that should not be made in one-dimension. Here we circumvent this problem by directly performing the integrals (7) both analytically and numerically. See detailed discussion in appendix D.

To interpret our results, let us recall that the mechanism responsible for the decaying oscillations in the RKKY interaction results from the existence of a Fermi sea. Under the second order perturbation theory perspective, the propagating electrons with momentum $k_F$ suffer scatterings with the Fermi sea. In the absence of SOC, these scatterings occur independently of the spin orientation of the propagating electrons. In contrast, in the presence of SOC, spin and momentum are locked together. As a result, an scattering can only occur if the spin orientation is modified accordingly. In 1D, backward scattering, for instance, has to be accompanied by a spin flip. Therefore, some of scatterings allowed in the absence of SOC are prevented when spin and momentum are coupled together.

4. Conclusions

We have investigated the exchange interaction between two magnetic impurities mediated by conduction electrons in a one-dimensional system with spin orbit [6]. The apparent difference between our results and the those from [6] arises from the fact that the expression used in the later cannot be straightforwardly applied in the 1D systems [34], specially in the presence of SOC Here, we avoid the problem by performing explicitly the integral resulting from the second order perturbation theory. Our work extends the expression for the 1D indirect exchange interactions to the case in which SOC is present. This is not only important because it is fundamentally distinct from the usual case in the absence of the SOC but also may be useful for practical application where long-distance couplings are relevant. Magnetic impurities in materials such as GaAs/AlGaAs [26] or InAs [27] spin–orbit coupled quantum wires are examples of potential candidates for experimental verification of our predictions.

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Appendix A. Effective inter-impurities Hamiltonian

To derive the effective inter-impurities Hamiltonian we follow the traditional approach use to obtain the usual RKKY. We assume

$$H_{\text{wire}} = \sum_k \left( \varepsilon_k \delta_{k'k} - \alpha_R k \sigma_x \right) c_{k'} ^\dagger c_k', \quad (A.1)$$

as the unperturbed Hamiltonian that includes the spin–orbit interaction. The perturbation

$$H_i = \frac{J}{N} \sum_{kk'} e^{-i(k-k')\alpha} \left[ S^\alpha_i (c_{k'} ^\dagger c_{k'} - c_{k+} ^\dagger c_{k+}) + S^\alpha_i c_{k+} ^\dagger c_{k+} + S^\alpha_i c_{k-} ^\dagger c_{k-} \right], \quad (A.2)$$

accounts for the impurities. To apply the second order perturbation theory we diagonalize the Hamiltonian (A.1). This is achieved by defining the new operators by the transformation

$$\begin{pmatrix} c_{k+} \\ c_{k-} \end{pmatrix} = U_k \begin{pmatrix} c_{k} \\ c_{k} \end{pmatrix}, \quad (A.3)$$

where

$$U_k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad (A.4)$$

is a unitary matrix. The transformation above corresponds to a momentum-dependent rotation in the spin space. In the new base $H_{\text{wire}}$ acquires the diagonal form

$$H_{\text{wire}} = \sum_{kh} \varepsilon_{kh} c_{kh} ^\dagger c_{kh}, \quad (A.5)$$

in which $h = +,-$ is the helical quantum number and $\varepsilon_{kh} = \hbar k^2 / 2m^* + \hbar \alpha_R k$ are the eigenvalues of $H_{\text{wire}}$. The eigenstates are then defined as $|k, h\rangle$ such that $H_{\text{wire}} |k, h\rangle = \varepsilon_{kh} |k, h\rangle$. 

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For simplicity, here we assume impurities have spin 1/2 so that the spin operators can be easily written in terms of fermion operators as $\hat{S}^x = (\hat{d}_k^\dagger \hat{d}_{-k})/2$, $\hat{S}^y = \hat{d}_k^\dagger \hat{d}_{-k}$, and $\hat{S}^z = \hat{d}_k^\dagger \hat{d}_{-k}$, where $\hat{d}_k^\dagger (\hat{d}_k)$ corresponds to the creation (annihilation) spin-1/2 fermion operator. This is very useful because we can now perform the same rotation (A.3) for these fermion operators, after which we can rewrite (A.2) as

$$H_i = \frac{J}{N} \sum_{kk'} e^{-i(k-k')x} \left[ \hat{S}^+_i \hat{c}_{k'}^+ \hat{c}_k + \hat{c}_{k'}^- \hat{c}_k \right] + \hat{S}^z_i \hat{c}_{k'}^\dagger \hat{c}_k$$

(A.6)

Here, the $\hat{S}^z_i$ emphasizes that the impurity spin operators are written on the rotated spin basis. Having the eigenstates and eigenenergies of the unperturbed Hamiltonian, the prescription to obtain the RKKY coupling is to compute the correction to the total energies up to the second order perturbation theory. To account for the degrees of freedom of the impurities, an eigenstate of $H_{\text{wire}}$ can be written as $|k, h\rangle$, where $h$ is the helical quantum number.

The textbook expression for the second order energy correction can be written as

$$H_{\text{eff}} = \sum_{kk'} \sum_{k,k'} \frac{\langle k,h|[H_1 + H_2]|k',h'\rangle \langle k',h'|[H_1 + H_2]|k,h\rangle}{\varepsilon_{kh} - \varepsilon_{k'h'}}$$

where $H_i$ is given by (A.6). In the equation (A.7) we assume that we are at temperature $T = 0$, in which case, the bands are fully occupied up to the Fermi level while fully empty above it. The exchange energy is only due the mixed terms of (A.7), we thus drop the self-interaction terms and write

$$H_{\text{eff}} = \sum_{kk'} \sum_{k,k'} \frac{\langle k,h|[H_1 + H_2]|k',h'\rangle \langle k',h'|[H_1 + H_2]|k,h\rangle}{\varepsilon_{kh} - \varepsilon_{k'h'}}$$

(A.8)

The non-vanish contributions of (A.8) can be calculated applying the creator and annihilator operators on the state $|k,h\rangle$. For example $\hat{S}^x \hat{c}_{k'}^\dagger \hat{c}_k |k, +\rangle = \hat{S}^x |k, +\rangle$, $\hat{S}^z \hat{c}_{k'}^\dagger \hat{c}_k |k, +\rangle = \hat{S}^z |k, +\rangle$, $\hat{S}^+ \hat{c}_{k'}^\dagger \hat{c}_k |k, -\rangle = \hat{S}^+ |k, -\rangle$. Using these relations we obtain

$$\langle k, +|H_1|k', +\rangle = \frac{J}{N} \sum_{kk'} e^{i(k-k')x} \hat{S}^+_i$$

(A.9)

$$\langle k, -|H_1|k', -\rangle = -\frac{J}{N} \sum_{kk'} e^{i(k-k')x} \hat{S}^+_i$$

(A.10)

$$\langle k, -|H_1|k', +\rangle = \frac{J}{N} \sum_{kk'} e^{i(k-k')x} \hat{S}^+_i$$

(A.11)

$$\langle k, +|H_1|k', -\rangle = \frac{J}{N} \sum_{kk'} e^{i(k-k')x} \hat{S}^+_i$$

(A.12)

Here we have used the orthogonality relation $\langle k, h|k', h'\rangle = \delta_{kk'} \delta_{hh'}$. Carrying out the calculation for $\langle k, h|H_2|k', h'\rangle$ we obtain similar results. Unlike the usual case of absence of SOC, in which the energy is equal for both spin components, here the energies $\varepsilon_{kh}$ depend of the helical number. Using $\varepsilon_{kh} = \hbar^2 k^2/2m^* + \hbar v_F k$, the energy differences that appears in the denominator of the four non-vanishing terms of (A.8) are

$$\varepsilon_{k+} - \varepsilon_{k'} = \frac{\hbar^2}{2m^*} (k^2 - k'^2) + \delta \alpha_R (k - k')$$

(A.13)

$$\varepsilon_{k-} - \varepsilon_{k'} = \frac{\hbar^2}{2m^*} (k^2 - k'^2) - \delta \alpha_R (k - k')$$

(A.14)

$$\varepsilon_{k+} - \varepsilon_{k'} = \frac{\hbar^2}{2m^*} (k^2 - k'^2) + \delta \alpha_R (k + k')$$

(A.15)

$$\varepsilon_{k-} - \varepsilon_{k'} = \frac{\hbar^2}{2m^*} (k^2 - k'^2) - \delta \alpha_R (k + k')$$

(A.16)

Replacing the results of the equations (A.9)–(A.12) and equations (A.13)–(A.16) into equation (A.8) we obtain

$$H_{\text{imp}} = (I_{++} + I_{+-} + I_{-+} + I_{--}) \hat{S}_i \hat{S}_j + (I_{++} + I_{--}) \hat{S}_i \hat{S}_j$$

(A.17)

with

$$I_{\delta \nu} = \frac{J}{N^2} \sum_{kk'} \sum_{k,k'} e^{i(k-k')x}$$

in which $x = x_2 - x_1$ is the distance between the impurities. The effective Hamiltonian (A.17) can be written in a more compact form

$$H_{\text{imp}} = I_1 \hat{S}_i \hat{S}_j + I_{++} \hat{S}_i \hat{S}_j + I_{--} \hat{S}_i \hat{S}_j$$

(A.19)

where we have defined $I_1 = 2Re \left( I_{++} + I_{--} \right)$ e $I_2 = \langle I_{++} + I_{--} \rangle$. We now transform the summations into integrals using the usual prescription $(1/N) \sum_k \rightarrow (1/2\pi) \int dk$ in the limit $N \rightarrow \infty$, so that the equation (A.18) can now be written as

$$I_{\delta \nu} = \int_{-\kappa}^{\kappa} \frac{dk}{2\pi} \int_{k' > k} \frac{dk'}{2\pi} e^{i(k-k')x}$$

(A.20)

Here we also used the fact that, because of the SOC, the bands $+ -$ have different Fermi momenta, namely $k_\delta = k_\delta + \delta k_Q$ (for $\delta = +, -$). In the helical basis, the Hamiltonian (A.19) has the form of a anisotropic Heisenberg Hamiltonian. Although simple, it hides the physics we want to study here. We can rewrite the impurity operators on the reals spin basis, on which we have

$$\hat{S}_i \hat{S}_j = S_1 \cdot S_2 + i(S_1 \times S_2) \cdot \hat{y} - S_1 \hat{S}_2$$

(A.21)

$$\hat{S}_i \hat{S}_j = S_1 \cdot S_2 - i(S_1 \times S_2) \cdot \hat{y} - S_1 \hat{S}_2$$

(A.22)
Thus, in the real spin space, the exchange Hamiltonian is given by
\[ H_{\text{imp}} = \text{RKKY} \mathbf{S}_1 \cdot \mathbf{S}_2 + I_{\text{DM}} ([\mathbf{S}_1 \times \mathbf{S}_2] \cdot \mathbf{y}) + I_{\text{sing}} \mathbf{S}_1 \cdot \mathbf{S}_2, \]
(2.4)
where, \( I_{\text{RKKY}} = 2 \Re \left[ I_1 \right], \) \( I_{\text{DM}} = -2 \Im \left[ I_1 \right]\) and \( I_{\text{sing}} = I_2 - 2 \Re \left[ I_1 \right]\) are the known RKKY, Dzialoshinskii–Moryia, and the Ising couplings.

### Appendix B. Analytical calculation of the couplings

We now focus on the calculation of the couplings \( I_{\text{RKKY}}, \) \( I_{\text{DM}} \) and \( I_{\text{sing}}. \) This requires performing the integrals (A.20). To simplify the notation we define the dimensionless variables \( q = k/k_F, \) \( q' = k'/k_F, \) together with \( a = 2k_R, \) with \( k_R = m\omega_B/\hbar^2, \) and \( \tilde{a} = a/k_F. \) With these definitions the equation (A.20) acquires the form
\[ I_{\delta q} = I_0 \int_{q_{\min}}^{q_{\max}} dq \int_{q_{\min}}^{q_{\max}} dq' \frac{e^{i(q-q')k_Fx}}{(q^2 - q'^2) + \tilde{a}(q - \tilde{q}q')}, \]
(2.1)
where \( I_0 = J^2m/2\pi^2\hbar^2, \) and \( q_{\min} = 1 + \tilde{a} \tilde{a}/2. \) An important point here that should be highlighted is that the order of the integrations above should not be changed as discussed by Yafet [23]. Later Valizadeh [34] revisited the problem and noted that results of the integrals (A.20) do not obey the Fubini’s condition [38, 39], leading to different results depending on the order in which the integrations are performed. Here we keep the order of integrations as it is in equation (A.20), avoiding the aforementioned problem. To perform the integral over \( q \) using the residues theorem we need to extend it to the entire real axis. With this we can write
\[ I_{\delta q} = I_0 \int_{q_{\min}}^{q_{\max}} dq \int_{-\infty}^{\infty} dq' \frac{e^{i(q-q')k_Fx}}{(q^2 - q'^2) + \tilde{a}(q - \tilde{q}q')}, \]
(B.2)
This deformation of the integral limits introduce undesirable contributions. If we are able to account for these extra contributions separately, we can subtract them from the final results to obtain the correct expression. In the absence of SOC, the integrand of (B.2) is antisymmetric under the exchange \( q \leftrightarrow q', \) thus the extra contributions added to the results are solely those coming from correspond to the singularities occurring at \( q = q'. \) However, in the presence of the SOC \( (\tilde{a} \neq 0) \) the integrand is no longer antisymmetric. Therefore, there are contributions other than those arising from the singularities. Here we assume that the only relevant additional contributions are those arising from the singularities of (B.2). Thus, within this approximation, we can write
\[ I_{\delta q} = I_{\delta q}^\text{corr} - I_{\delta q}^\text{sing}, \]
where \( I_{\delta q}^\text{sing} \) corresponds to the undesirable singularities. We first integrate over \( q' \) and then over \( q. \) The equation (B.2) can be written as
\[ I_{\delta q} = I_0 \int_{q_{\min}}^{q_{\max}} dq I_{\delta q}^\text{corr}, \]
(3.3)
where we have defined
\[ I_{\delta q}^\text{corr} = \frac{\mathcal{P}}{2\pi - \tilde{a}} \int dq' \frac{e^{i(q-q')k_Fx}}{q^2 - q'^2 + \tilde{a}(q - q')}, \]
(4.4)
in the above \( \mathcal{P}[\cdots] \) denote the Cauchy principal value. Let us start with by calculating \( I_{\delta q}^\text{corr} \) that has the form
\[ I_{\delta q}^\text{corr} = \frac{2\pi}{2\tilde{a} - \tilde{a}} \sin[(q - \tilde{a}/2)k_Fx]e^{i(q+\tilde{a}/2)k_Fx}. \]
(5.5)
Noticing from (B.2) that we can obtain \( I_{\delta q}^\text{corr} \) by doing \( \tilde{a} \rightarrow -\tilde{a} \) in the equation (B.6). Therefore we immediately obtain
\[ I_{\delta q}^\text{corr} = \frac{2\pi}{2\tilde{a} - \tilde{a}} \sin[q - \tilde{a}/2]k_Fx]e^{i(q-\tilde{a}/2)k_Fx}. \]
(B.7)
Proceeding in a similar way for the other two integrals we obtain
\[ I_{\delta q}^\text{corr} = \frac{2\pi}{2\tilde{a} + \tilde{a}} \sin[(q + \tilde{a}/2)k_Fx]e^{i(q+\tilde{a}/2)k_Fx} \]
and
\[ I_{\delta q}^\text{corr} = \frac{2\pi}{2\tilde{a} - \tilde{a}} \sin[(q - \tilde{a}/2)k_Fx]e^{i(q-\tilde{a}/2)k_Fx}. \]
(B.10)
Collecting the results (B.6)–(B.9) and grouping them properly, we obtain
\[ I_{\text{RKKY}}' = 2\Re \left( I_{\delta q}^\text{corr} + I_{\delta q}^\text{corr} \right) = 4\pi I_0 \left[ \int_{q_{\min}}^{q_{\max}} dq \cos[(q - \tilde{a}/2)k_Fx] \sin[(q + \tilde{a}/2)k_Fx] \right] \]
\[ + \int_{q_{\min}}^{q_{\max}} dq \sin[(q + \tilde{a}/2)k_Fx] \sin[(q - \tilde{a}/2)k_Fx] \]
(B.11)
and
\[ I_{\text{DM}}' = -2\Im \left( I_{\delta q}^\text{corr} + I_{\delta q}^\text{corr} \right) \]
\[ = -4\pi I_0 \left[ \int_{q_{\min}}^{q_{\max}} dq \sin[(q - \tilde{a}/2)k_Fx] \sin[(q - \tilde{a}/2)k_Fx] \right] \]
\[ + \int_{q_{\min}}^{q_{\max}} dq \sin[(q + \tilde{a}/2)k_Fx] \sin[(q - \tilde{a}/2)k_Fx] \]
(B.12)
The indices \( 'r' \) denote the uncorrected results, i.e. before subtracting the extra contribution. The six integrals appearing in the expressions (B.10)–(B.12) above are rather complicated.
but can still be computed analytically. After a tiresome work, apart from additive constants, we obtain the expressions for the undefined integrals

\[
\int dq \frac{\sin((2q + \tilde{a})k_Fx)}{2q + \tilde{a}} = \frac{1}{2} \text{Si}[(2q + \tilde{a})k_Fx], \quad \text{(B.13)}
\]

\[
\int dq \frac{\sin((2q - \tilde{a})k_Fx)}{2q - \tilde{a}} = \frac{1}{2} \text{Si}[(2q - \tilde{a})k_Fx], \quad \text{(B.14)}
\]

\[
\int dq \frac{\cos((q - \tilde{a}/2)k_Fx) \sin((q + \tilde{a}/2)k_Fx)}{2q + \tilde{a}} = \frac{1}{4} \left\{ -\sin(\tilde{ak}_Fx) \text{Ci}[(2q + \tilde{a})k_Fx] + \cos(\tilde{ak}_Fx) \text{Si}[(2q + \tilde{a})k_Fx] + \sin(\tilde{ak}_Fx) \ln[2(2q + \tilde{a})] \right\}, \quad \text{(B.15)}
\]

\[
\int dq \frac{\cos((q + \tilde{a}/2)k_Fx) \sin((q - \tilde{a}/2)k_Fx)}{2q - \tilde{a}} = \frac{1}{4} \left\{ -\cos(\tilde{ak}_Fx) \text{Ci}[(2q - \tilde{a})k_Fx] - \sin(\tilde{ak}_Fx) \text{Si}[(2q - \tilde{a})k_Fx] - \sin(\tilde{ak}_Fx) \ln[2(2q - \tilde{a})] \right\}, \quad \text{(B.16)}
\]

and

\[
\int dq \frac{\sin((q + \tilde{a}/2)k_Fx) \sin((q - \tilde{a}/2)k_Fx)}{2q - \tilde{a}} = \frac{1}{4} \left\{ -\cos(\tilde{ak}_Fx) \text{Ci}[(2q - \tilde{a})k_Fx] - \sin(\tilde{ak}_Fx) \text{Si}[(\tilde{a} - 2q)k_Fx] + \cos(\tilde{ak}_Fx) \ln[2(\tilde{a} - 2q)] \right\}. \quad \text{(B.18)}
\]

In the above we use the usual definitions

\[
\text{Ci}(x) = \int_{0}^{x} \frac{\cos(t)}{t} dt \quad \text{(B.19)}
\]

\[
\text{Si}(x) = \int_{0}^{x} \frac{\sin(t)}{t} dt. \quad \text{(B.20)}
\]

After imposing the proper limits to the results \text{(B.13)}–\text{(B.18)} and some algebraic manipulations we can write

\[
I_{\text{RKKY}} = \pi I_0 \left\{ \sin(\tilde{ak}_Fx) \left[ \text{Ci}[(1 - \tilde{a})2k_Fx] - \text{Ci}[(1 + \tilde{a})2k_Fx] \right] + \cos(\tilde{ak}_Fx) \left[ \text{Si}[(1 + \tilde{a})2k_Fx] - \text{Si}[(\tilde{a} - 1)2k_Fx] \right] + 2\text{Si}(2k_Fx) \right\} \ln \left( \frac{1 - \tilde{a}}{1 + \tilde{a}} \right) \sin(\tilde{ak}_Fx), \quad \text{(B.21)}
\]

\[
I_{\text{SM}} = -\pi I_0 \left\{ \cos(\tilde{ak}_Fx) \left[ \text{Ci}[(1 + \tilde{a})2k_Fx] - \text{Ci}[(1 - \tilde{a})2k_Fx] \right] + \sin(\tilde{ak}_Fx) \left[ \text{Si}[(1 + \tilde{a})2k_Fx] - \text{Si}[(\tilde{a} - 1)2k_Fx] \right] + 2\text{Si}(2k_Fx) \right\} \ln \left( \frac{1 - \tilde{a}}{1 + \tilde{a}} \right) \cos(\tilde{ak}_Fx), \quad \text{(B.22)}
\]

\[
I_{\text{kang}} = 2\pi I_0 \left[ \text{Si}[(1 + \tilde{a})2k_Fx] - \text{Si}[(\tilde{a} - 1)2k_Fx] \right] - I_{\text{RKKY}}. \quad \text{(B.23)}
\]

To obtain the final results we still need to compute the contribution from the singularities of the integrals \text{(B.2)}.

\subsection*{B.1. Contribution from the singularities}

To compute the contributions from the singularities we use the same method applied to the traditional RKKY problem in 1D \cite{23, 25}. Let us start with the integral

\[
I_{++} = I_0 \int_{-q'}^{q} dq \int_{|q'| > q} dq' \frac{e^{i(q-q')k_Fx}}{q' - q + \tilde{a}}. \quad \text{(B.24)}
\]

The singularities of this integral occur when \((q - q') = 0\) and \((q + q') = 0\) or \(q' = -\tilde{a}/2\) and \(q = -\tilde{a}/2\). In the following we calculate the integral above around \(q = q' = \tilde{a}/2\). At this point we have \(e^{i(q-q')k_Fx} = e^{i(-\tilde{a}+\tilde{a})k_Fx/2} = 1\). Therefore,

\[
I_{++} = \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dq \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dq' \frac{dq'}{q' - q + \tilde{a}}. \quad \text{(B.25)}
\]

with \(\epsilon \to 0^+\). The integral over \(q'\) variable can be calculated analytically using

\[
\int \frac{dx}{(y-x)(y+x+a)} = \ln(-a - x - y) - \ln(y - x) + \text{constant}. \quad \text{(B.26)}
\]

Imposing the limits, after some algebraic manipulation we obtain

\[
I_{++} = \int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dq \ln\left| \left( q + \tilde{a}/2 \right) + \epsilon \right| - \ln |q + \tilde{a}/2 - \epsilon| \left| q + \tilde{a}/2 \right|. \quad \text{(B.27)}
\]

Performing the variable change \(x = q + \tilde{a}/2\) the above integral becomes

\[
I_{++} = \int_{-\epsilon}^{\epsilon} dx \left[ \ln |x| \right] - \ln |x - \epsilon| \left| x \right|. \quad \text{(B.28)}
\]

This expression can be written as

\[
I_{++} = \int_{-1}^{1} dx \left[ \ln \left| 1 + u \right| - \ln \left| 1 - u \right| \right] = 2\text{Li}_2(1) - 2\text{Li}_2(-1) = \frac{\pi^2}{12}. \quad \text{(B.29)}
\]
where
\[ \text{Li}_2(x) = -\int_0^1 \frac{\ln(1-t)}{t} dt \] (B.30)
is the Dilogarithmic function. In the last line passage in (B.29) we have used [33] \( \text{Li}_2(1) = \pi^2/6 \) and \( \text{Li}_2(-1) = -\pi^2/12 \).
Likewise, we can show that
\[ I_{-+} = \frac{\pi^2}{2}. \] (B.31)
The others two integrals render slightly different results. Let us look at the correction for
\[ I_{++} = I_0 \int_{q+}^{q+} dq \int_{-\infty}^{\infty} dq' \frac{e^{i(q-q')kx}}{q^2 - q'^2 + \tilde{\alpha}(q + q')} \] (B.32)
Here the contribution are accounted when \( (q + q') = 0 \) and \( (q - q' + \tilde{\alpha}) = 0 \), from which we extract \( q' = \tilde{\alpha}/2 \) and \( q = -\tilde{\alpha}/2 \). At this point, \( e^{i(q-q')kx} = e^{i(-\tilde{\alpha} - \tilde{\alpha})kx/2} = e^{-\tilde{\alpha}kx} \), so that
\[ I_{++} = e^{-\tilde{\alpha}kx} \int_{-\tilde{\alpha}}^{\tilde{\alpha} + \epsilon} dq \int_{-\tilde{\alpha}}^{\tilde{\alpha} + \epsilon} dq' \frac{q + q' + \tilde{\alpha}}{q^2 - q'^2 + \tilde{\alpha}(q + q')} . \] (B.33)
Using the indefinite integral
\[ \int \frac{dx}{(y+x)(y-x+a)} = \ln|y+x| - \ln\left(-a + x - y\right) + \text{constant}, \] (B.34)
we obtain
\[ I_{++} = e^{-\tilde{\alpha}kx} \times \int_{-\tilde{\alpha}}^{\tilde{\alpha} + \epsilon} dq \ln\left|\left(q + \frac{\tilde{\alpha}}{2}\right) + \epsilon\right| - \ln\left|\left(q + \frac{\tilde{\alpha}}{2}\right) - \epsilon\right| \] (B.35)
Apart from the prefactor \( e^{-\tilde{\alpha}kx} \), this is the same as in (B.27), therefore,
\[ I_{++} = \frac{\pi^2}{2} e^{-\tilde{\alpha}kx} . \] (B.36)
The last correction, for \( I_{--} \), can be obtained using same argument of changing \( \tilde{\alpha} \to -\tilde{\alpha} \) in (B.36), leading to
\[ I_{--} = \frac{\pi^2}{2} e^{\tilde{\alpha}kx} . \] (B.37)
Collecting the results of (B.29), (B.31), (B.36) and (B.37) we obtain the corrections for the couplings
\[ I_{\text{KKY}}^{\text{Re}} = 2 \text{Re} \left( I_{++}^{\text{Re}} + I_{--}^{\text{Re}} \right) = 4\pi I_0 \left[ \frac{\pi}{2} \cos(\tilde{\alpha}kx) \right], \] (B.38)
\[ I_{\text{DM}}^{\text{Im}} = -2 \text{Im} \left( I_{++}^{\text{Re}} + I_{--}^{\text{Re}} \right) = -4\pi I_0 \left[ \frac{\pi}{2} \sin(\tilde{\alpha}kx) \right], \] (B.39)
\[ I_{\text{sing}}^{\text{Re}} = 2 \text{Re} \left( I_{++}^{\text{Re}} + I_{--}^{\text{Re}} \right) - I_{\text{KKY}}^{\text{Re}} = 4\pi I_0 \left[ \frac{\pi}{2} \right] - I_{\text{KKY}}^{\text{Re}} . \] (B.40)
We now subtract the results of the equation (B.38) from those of equation (B.21) to obtain our final analytical results for the indirect coupling
\[ I_{\text{KKY}} = \pi I_0 \left\{ \sin(\tilde{\alpha}kx) \left[ \text{Ci} \left( (1 - \tilde{\alpha})2kx \right) - \text{Ci} \left( (1 + \tilde{\alpha})2kx \right) \right] + \cos(\tilde{\alpha}kx) \left[ \text{Si} \left( (1 + \tilde{\alpha})2kx \right) - \text{Si} \left( (1 - \tilde{\alpha})2kx \right) \right] + 2\text{Si}(2kx) \right\} - \ln \left| \frac{1 - \tilde{\alpha}}{1 + \tilde{\alpha}} \right| \sin(\tilde{\alpha}kx) \right\} - 4\pi I_0 \left[ \frac{\pi}{2} \cos(\tilde{\alpha}kx) \right], \] (B.41)
\[ I_{\text{DM}} = -\pi I_0 \left\{ \cos(\tilde{\alpha}kx) \left[ \text{Ci} \left( (1 + \tilde{\alpha})2kx \right) - \text{Ci} \left( (1 - \tilde{\alpha})2kx \right) \right] + \sin(\tilde{\alpha}kx) \left[ \text{Si} \left[ (1 + \tilde{\alpha})2kx \right] - \text{Si} \left[ (1 - \tilde{\alpha})2kx \right] \right] + 2\text{Si}(2kx) \right\} + \ln \left| \frac{1 - \tilde{\alpha}}{1 + \tilde{\alpha}} \right| \cos(\tilde{\alpha}kx) \right\} + 4\pi I_0 \left[ \frac{\pi}{2} \sin(\tilde{\alpha}kx) \right], \] (B.42)
\[ I_{\text{sing}} = 2\pi I_0 \left[ \text{Si} \left[ (1 + \tilde{\alpha})2kx \right] - \text{Si} \left[ (1 - \tilde{\alpha})2kx \right] \right] - 4\pi I_0 \left[ \frac{\pi}{2} \right] - I_{\text{KKY}} . \] (B.43)
Notice that if we take \( \tilde{\alpha} = 0 \) the usual result \( I_{\text{KKY}} = 4\pi I_0 \left[ \text{Si}(2kx) - \pi/2 \right] \) and \( I_{\text{DM}} = I_{\text{sing}} = 0 \) is recovered, as expected. Interestingly, however, the asymptotic behavior of these expressions are
\[ I_{\text{KKY}} = -\pi I_0 \ln \left| \frac{k_F - 2k_R}{k_F + 2k_R} \right| \sin(2kRx), \] (B.44)
\[ I_{\text{DM}} = -\pi I_0 \ln \left| \frac{k_F - 2k_R}{k_F + 2k_R} \right| \cos(2kRx), \] (B.45)
\[ I_{\text{sing}} = -\pi I_0 \ln \left| \frac{k_F - 2k_R}{k_F + 2k_R} \right| \sin(2kRx). \] (B.46)
Where we use \( \lim_{x \to \infty} \text{Ci}(x) = 0 \), and \( \lim_{x \to \infty} \text{Si}(x) = \pi/2 \). These unsuppressed oscillations appearing in these asymptotic expressions is the principal result of our work.

**Appendix C. Analytical versus numerical results**

Despite the complexities involved in obtaining the analytical results, numerically it is rather straightforward. Basically, we need to calculate the integrals (A.20) numerically. In fact, here we simply perform these integrals using a numerical subroutine built in Julia programming language [40]. To get convergence, as usual we add an infinitesimal imaginary to the denominator of (A.20) so that the integrals we indeed solve numerically are
\[ I_{\delta\nu} = f^2 \int_{|k'| > k_x} \frac{dk'}{2\pi} \int_{|k'| > k_x} \frac{dk''}{2\pi} \frac{1}{(k''^2 - k'^2)^2} + \delta k_R (k - \delta v k') + \eta, \] (C.1)
with \( \eta = 0^+ \). The expression above is exactly the same we obtain when we used scattering theory to obtain the indirect interaction via the Lippmann–Schwinger equation [41], having in mind that we need to account for the Fermi sea and the Pauli’s exclusion principle. Having calculated the integrals numerically, we obtain the indirect coupling using the expressions just using the expressions for \( I_{\text{KKY}}, I_{\text{DM}} \) and \( I_{\text{sing}} \) obtained in the end of section (appendix A). The analytical (dashed red line) and the numerical (solid black line) results
and transform the summation
\[ H_{\text{eff}} = \sum_{kk'} \sum_{\sigma} \sum_{\sigma'} \langle k, \sigma | \hat{H}_{1} | k', \sigma' \rangle \langle k', \sigma' | \hat{H}_{2} | k, \sigma \rangle + \text{H.c.}. \]  
\hfill (D.1)

Defining the retarded Green’s function
\[ \hat{G}_{k', k} (\varepsilon_{kh}) = \frac{|k', h|}{\varepsilon_{kh} - \varepsilon_{k' h}'}, \]  
\hfill (D.2)
in which \( \varepsilon_{kh} \rightarrow \varepsilon_{kh} + 0^+ \), the equation (D.1) can be written as
\[ H_{\text{eff}} = \sum_{kk'} \sum_{\sigma} \sum_{\sigma'} \langle k, \sigma | \hat{H}_{1} \hat{G}_{k', k} (\varepsilon_{kh}) | k', \sigma' \rangle \langle k', \sigma' | \hat{H}_{2} | k, \sigma \rangle + \text{H.c.}. \]  
\hfill (D.3)

Let us now introduce two closure relations in the position space, \( \sum_{x, \sigma} |x, \sigma \rangle \langle x, \sigma| \), to obtain
\[ H_{\text{eff}} = \sum_{kk'} \sum_{\sigma} \sum_{\sigma'} \sum_{\sigma''} \langle k, \sigma | \hat{H}_{1} | x, \sigma \rangle \langle x, \sigma | \hat{G}_{k', k} | \sigma' \rangle \langle \sigma' | \hat{H}_{2} | k, \sigma \rangle + \text{H.c.}. \]  
\hfill (D.4)

In the real position and spin space, the Hamiltonian \( H_i \) of equation (A.6) acquires the form
\[ H_i = J \sum_{hh'} S_{ij}^a (c_{i h}^\dagger c_{j h'} - c_{i h}^\dagger c_{j h'}) + S_{ij}^s c_{i h}^\dagger c_{j h'} + S_{ij}^s c_{i h}^\dagger c_{j h'} \delta(x - x_i) \delta(x - x_i). \]  
\hfill (D.5)

Inserting this into equation (D.4), after some straightforward algebraic manipulations we obtain
\[ H_{\text{eff}} = J \sum_{hh'} \sum_{\sigma} \langle h | \langle h' | (S_{1} \cdot s) | h' \rangle (S_{2} \cdot s) | h \rangle \times \sum_{k} \sum_{k'} G_{k', k} (x_1, x_2, \varepsilon_{kh}) \langle k | x_1 \rangle \langle x_2 | k \rangle + \text{H.c.}. \]  
\hfill (D.6)

Here we have defined the scalar retarded Green’s function
\[ G_{k', k} (x_1, x_2, \varepsilon_{kh}) = \frac{\langle x_1 | k' \rangle \langle k' | x_2 \rangle}{\varepsilon_{kh} - \varepsilon_{k' h'}}. \]  
\hfill (D.7)

We now use \( \langle x_i | k \rangle = \frac{1}{N} \sum_c \delta(k - c) \) and transform the summation into integrals we obtain
\[ H_{\text{eff}} = NJJ' \sum_{hh'} \sum_{\sigma} \sum_{\sigma'} \langle h | \langle h' | (S_{1} \cdot s) | h' \rangle (S_{2} \cdot s) | h \rangle \times \int_{\text{occ}} \frac{dk}{2\pi} e^{ik(x_1 - x_2)} \int_{\text{empty}} \frac{dk'}{2\pi} G_{k', k} (x_1, x_2, \varepsilon_{kh}) + \text{H.c.}. \]  
\hfill (D.8)

As discussed in detail by Valizadeh [34], further simplification of equation (D.8) towards the a similar expression as equation (5) of [6] requires changing the order of the integrals over \( k \) and \( k' \), which may lead to spurious result in 1D case. Moreover, the integral over \( k' \) cannot be extended from \( -\infty \) to \( \infty \), since the extra contribution to the double integral does not vanish in the presence of spin–orbit coupling.

**Appendix F. Effective Hamiltonian in terms of Green’s function**

In this section we present a derivation of an expression for the effective Inter-impurity Hamiltonian in terms of Green’s function in the position space for the 1D system in the presence of the spin–orbit interaction.

Let us start with equation (A.8) that can be written as

\[ H_{\text{eff}} = \sum_{kk'} \sum_{\sigma} \sum_{\sigma'} \langle k, \sigma | \hat{H}_{1} | k', \sigma \rangle \langle k', \sigma | \hat{H}_{2} | k, \sigma \rangle + \text{H.c.}. \]  
\hfill (D.1)

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