Quantum Space-time and Classical Gravity

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Abstract

A method has been recently proposed for defining an arbitrary number of differential calculi over a given noncommutative associative algebra. As an example a version of quantized space-time is considered here. It is found that there is a natural differential calculus using which the space-time is necessarily flat Minkowski space-time. Perturbations of this calculus are shown to give rise to non-trivial gravitational fields.
1 Motivation and Notation

Since the early days of quantum field theory physicists have been tempted to introduce some type of lattice structure on space-time to avoid the appearance of ultraviolet divergences, that is, to fix a natural value for an ultraviolet cut-off $\Lambda$. One of the disadvantages of these discrete structures is the fact that they destroy Lorentz invariance and they can be hardly considered in any respect as fundamental. It was Snyder (1947) who first had the idea of using non-commuting coordinates to mimic a discrete structure in a covariant way. Since then several attempts have been made to continue this initial effort. We refer to Madore & Mourad (1996) for a recent review with historical perspective. One typically introduces four hermitian generators $q^\mu$ of a noncommutative $\ast$-algebra $A$ which satisfy commutation relations of the form

$$[q^\mu, q^\nu] = i\mu_P^{-2}q^{\mu\nu}. \quad (1.1)$$

The problem lies then with the interpretation of the right-hand side. One can define a succession of elements $q^{\lambda_1\ldots\lambda_n}$ by the equations

$$[q^\lambda, q^{\mu\nu}] = i\mu_P^{-1}q^{\lambda\mu\nu}, \quad [q^\lambda, q^{\mu\nu\rho}] = i\mu_P^{-1}q^{\lambda\mu\nu\rho} \quad (1.2)$$

and so forth. The structure of the algebra $A$ is constrained by the value of these commutators. One possibility, considered by Snyder (1947), is to choose them so as to form a representation of the Lie algebra of the de Sitter group. A second possibility, considered by Dubois-Violette & Madore (Madore 1988, 1995) is to choose them to form a representation of the conformal algebra. Recently Doplicher, Fredenhagen & Roberts (1994, 1995) have argued that $q^{\mu\nu}$ should be chosen to lie in the center $Z(A)$ of $A$, that the $q$-tensors with more than 2 indices should vanish. We shall adopt this as a working hypothesis. There are 6 independent $q^{\mu\nu}$, which from (1.1) parameterize symplectic structures on space-time. In the commutative limit one obtains therefore a space of dimension greater than four except of course if the $q^{\mu\nu}$ are nilpotent. We shall not address here the question of the physical significance of the extra dimensions.

Let $T^{(0)}_{\mu\nu}$ be the bare energy-momentum tensor, including quantum corrections, of some field theory on space-time. Choose some separation of $T^{(0)}_{\mu\nu}$ into a divergent part $T^{(\Lambda)}_{\mu\nu}$ and a regular part $T^{(\text{Reg})}_{\mu\nu}$ which would remain finite if one were to let $\Lambda \to \infty$. Implicit in what follows is the assumption that the decomposition can be made so that the singular part is in some sense universal and independent of the particular (physically reasonable) field theory one starts with. We write then

$$T^{(0)}_{\mu\nu} = T^{(\Lambda)}_{\mu\nu} + T^{(\text{Reg})}_{\mu\nu}. \quad (1.3)$$

Denote by $\langle O \rangle_0$ the vacuum-expectation value of an operator $O$. Then in a quasi-classical approximation, considering the gravitational field as classical, one can write the Einstein field equations as

$$G_{\mu\nu} = -\mu_P^{-2}(\langle T^{(\Lambda)}_{\mu\nu} \rangle_0 + \langle T^{(\text{Reg})}_{\mu\nu} \rangle_0).$$

We shall be here interested in the divergent part of $T^{(0)}_{\mu\nu}$ and we shall neglect the regular term. The field equations become then

$$G_{\mu\nu} = -\mu_P^{-2}\langle T^{(\Lambda)}_{\mu\nu} \rangle_0. \quad (1.3)$$

This equation is quite unsatisfactory. One would like to replace it by an operator equation of the form

$$G_{\mu\nu} = -\mu_P^{-2}T^{(\Lambda)}_{\mu\nu} \quad (1.4)$$
such that

\[ G_{\mu\nu}^{(\infty)} = \lim_{\Lambda \to \infty} G_{\mu\nu} \]  

is non-vanishing in order to produce a gravitational field which acts as a regulator but such that

\[ (G_{\mu\nu}^{(\infty)})_0 = 0 \]  

so that the regularizing gravitational field is not classically observable. In any case it is reasonable to assume that a divergence gives rise to a gravitational field and so we have defined the left arrow of the diagram

\[ \text{Cut-off } \Lambda \iff \mu P \leftrightarrow \text{Cut-off } \mu P \]

\[ \Downarrow \]  

\[ \Downarrow \]  

\[ \Rightarrow A \]  

\[ (1.7) \]

It is an old idea, due to Pauli and developed by Deser (1957) and others (Isham et al. (1971), that perturbative ultraviolet divergences will one day be regularized by the gravitational field. The possibility which we would like to explore here is that the mechanism by which this works is through the introduction of noncommuting ‘coordinates’ such as the \( q^\lambda \). A hand-waving argument can be given (Madore & Mourad 1995) which allows one to think of the noncommutative structure of space-time as being due to quantum fluctuations of the light-cone in ordinary 4-dimensional space-time. This relies on the existence of quantum gravitational fluctuations. A purely classical argument based on the formation of black-holes has been given by Doplicher et al. (1995). In both cases the classical gravitational field \( g_{\mu\nu} \) is to be considered as regularizing the ultraviolet divergences through the introduction of the quantum structure of space-time.

The right arrow of the Diagram (1.7) has been discussed, for example, by Doplicher et al. The top arrow is a definition. We wish to discuss the implications which define the bottom arrow. For this we consider the following diagram:

\[ \text{Curvature} \iff \Omega^*(A) \]

\[ \Downarrow \]  

\[ \Downarrow \]  

\[ \Rightarrow A \]  

\[ (1.8) \]

We shall argue that it can in fact be used to define the bottom arrow, the same as in (1.7). The right arrow is a mathematical triviality; it gives a relation between a differential calculus over an algebra and the algebra itself. We shall argue that to a certain extent a differential calculus determines uniquely a curvature in the commutative limit. The uniqueness will allow us in certain cases to invert the top arrow. If we identify the curvature which we so obtain with that which we supposed was the origin of the structure of the algebra we can claim that curvature gives rise not only to a noncommutative algebra but also to an associated differential calculus. As a corollary we have defined the bottom arrow. Since the differential calculus is not unique we cannot claim that the curvature depends only on the right-hand side of (1.1). That is, although the non-vanishing Planck mass gives rise to commutation relations, the left-hand side of (1.3) does not depend only on \( \mu P \). Equation (1.3) can therefore not be considered as an equation for \( \Lambda \) in terms of \( \mu P \). Were this the case and were it possible to use some additional \textit{a priori} relation between \( \Lambda \) and \( \mu P \) then (1.3) would become an eigenvalue equation yielding the mass spectrum in units of \( \mu P \).

In Section 2 we give a general prescription of how one defines differential calculi based on a set of derivations and we determine the conditions which the derivations
must satisfy for the module of 1-forms to be a free module and for the module of
2-forms to be non-trivial. In Section 3 we associate a metric and a linear connection
to each differential calculus and we argue that the association is unique. In Section 4
we give the explicit calculations using a particularly simple set of derivations and we
find that the associated gravitational field is trivial. In Sections 5 and 6 we consider
differential calculi which are, in a sense which we shall make precise, small perturbations
of this basic differential calculus and we find that they lead to a rather simple but non-
trivial gravitational field. In Section 7 we consider a quotient algebra which could be
considered as the noncommutative version of de Sitter space. Finally in Section 8 we
discuss briefly the definition of ‘gauge invariance’.

2 The differential calculi

Let \( A \) be any unital associative \( * \)-algebra. Of the many differential calculi which can
be constructed over \( A \) the largest is the differential envelope or universal differential
calculus \( (\Omega^*_u(A), d_u) \). Every other differential calculus can be considered as a quo
tient of it. For the definitions we refer, for example, to the book by Connes (1994). Let
\((\Omega^*(A), d)\) be another differential calculus over \( A \). Then there exists a unique
\( d_u \)-homomorphism \( \phi \) \( \Phi \)

\[
\begin{array}{c}
A \xrightarrow{d_u} \Omega^1_u(A) \xrightarrow{d_u} \Omega^2_u(A) \xrightarrow{d_u} \cdots \\
\downarrow \phi_1 \downarrow \phi_2 \downarrow \\
A \xrightarrow{d} \Omega^1(A) \xrightarrow{d} \Omega^2(A) \xrightarrow{d} \cdots
\end{array}
\]  

(2.1)

of \( \Omega^*_u(A) \) onto \( \Omega^*(A) \). It is given by

\[
\phi(d_u f) = df.
\]

(2.2)

The restriction \( \phi_p \) of \( \phi \) to each \( \Omega^p_u \) is defined by

\[
\phi_p(f_0 d_u f_1 \cdots d_u f_p) = f_0 df_1 \cdots df_p.
\]

Consider a given algebra \( A \) and suppose that we know how to construct an \( A \-
module \( \Omega^1(A) \) and an application

\[
\begin{array}{c}
A \xrightarrow{d} \Omega^1(A) \\
\end{array}
\]  

(2.3)

Then using (2.1) there is a method of constructing \( \Omega^p(A) \) for \( p \geq 2 \) as well as the
extension of the differential. Since we know \( \Omega^1_u(A) \) and \( \Omega^1(A) \) we can suppose that \( \phi_1 \)
is given. It is the map such that

\[
\Omega^1(A) = \Omega^1_u(A) / \text{Ker} \phi_1.
\]

We must construct \( \Omega^2(A) \). The largest consistent choice would be to set

\[
\Omega^2(A) = \Omega^2_u(A) / d_u \text{Ker} \phi_1
\]

(2.4)

where

\[
d_u \text{Ker} \phi_1 = d_u \text{Ker} \phi_1 + \Omega^1_u(A) \otimes \text{Ker} \phi_1 + \text{Ker} \phi_1 \otimes \Omega^1_u(A)
\]
is the bimodule generated by \( d_u \text{Ker} \phi_1 \). Since

\[
\Omega^2_u(A) = \Omega^1_u(A) \otimes_A \Omega^1_u(A)
\]
we find that $\Omega^2(A)$ can be written also as

$$\Omega^2(A) = \Omega^1(A) \otimes_A \Omega^1(A)/K$$  \hfill (2.5)

with

$$K = (\phi_1 \otimes \phi_1)(d_u \text{Ker} \phi_1) = (\phi_1 \otimes \phi_1)(d_u \text{Ker} \phi_1).$$  \hfill (2.6)

It can happen that $K = \Omega^1(A) \otimes_A \Omega^1(A)$, in which case $\Omega^2(A) = 0$.

Let $\pi$ be the projection

$$\Omega^1(A) \otimes_A \Omega^1(A) \rightarrow \Omega^2(A).$$  \hfill (2.7)

Then $\pi$ has a right inverse $\iota$, a map

$$\Omega^2(A) \rightarrow \Omega^1(A) \otimes_A \Omega^1(A)$$  \hfill (2.8)

such that $\pi \circ \iota = 1$. This will allow us to identify $\Omega^2(A)$ as a submodule of $\Omega^1(A) \otimes_A \Omega^1(A)$. The map $\phi_2$ is defined to be the projection of $\Omega^2_u(A)$ onto $\Omega^2(A)$ so defined. From the definition of $\pi$ one sees that $\phi_2$ is given by

$$\phi_2 = \pi \circ (\phi_1 \otimes \phi_1).$$  \hfill (2.9)

The wedge product of two elements $\xi$ and $\eta$ in $\Omega^1(A)$ is given by $\xi \eta = \pi(\xi \otimes \eta)$. Let $\xi_u$ be an inverse image of $\xi$ in $\Omega^1_u(A)$. Then the map $d$ from $\Omega^1(A)$ to $\Omega^2(A)$ can be written in terms of $d_u$ as

$$d(\phi_1(\xi_u)) = \phi_2(d_u \xi_u).$$  \hfill (2.10)

Equation (2.4) defines the largest set of 2-forms consistent with the constraints on $\Omega^1(A)$. The procedure can be continued by iteration to arbitrary order in $p$. Define for this the map $\psi_p$

$$\Omega^p_u(A) \xrightarrow{\psi_p} \bigotimes_{1}^{p} \Omega^1(A)$$  \hfill (2.11)

given by $\psi_p = \phi_1 \otimes \cdots \otimes \phi_1$ and define (Connes 1994)

$$\Omega^p(A) = \Omega^p_u(A)/(\text{Ker} \psi_p + d_u \text{Ker} \psi_{p-1}).$$  \hfill (2.12)

We have then by definition

$$\text{Ker} \phi_p = \text{Ker} \psi_p + d_u \text{Ker} \psi_{p-1}.$$  \hfill (2.13)

For example

$$\text{Ker} \psi_2 = \Omega^1_u(A) \otimes \text{Ker} \phi_1 + \text{Ker} \phi_1 \otimes \Omega^1_u(A)$$  \hfill (2.14)

and using (2.13) we find (2.4) as a particular case. Equation (2.12) can be rewritten as

$$\Omega^p(A) = \bigotimes_{1}^{p} \Omega^1(A)/\psi_p(d_u \text{Ker} \psi_{p-1})$$

or in the form

$$\Omega^p(A) = \bigotimes_{1}^{p} \Omega^1(A)/\mathcal{K}_p$$  \hfill (2.15)

with the $\mathcal{K}_p$ defined by the recurrence relations

$$\mathcal{K}_p = \mathcal{K}_{p-1} \otimes \Omega^1(A) + \Omega^1(A) \otimes \mathcal{K}_{p-1}, \quad \mathcal{K}_2 = \mathcal{K}. $$
In particular we find the expression
\[ \Omega^3(A) = \frac{\Omega^1(A) \otimes \Omega^1(A) \otimes \Omega^1(A)}{K \otimes \Omega^1(A) + \Omega^1(A) \otimes K} \] (2.16)
for the module of 3-forms.

To initiate the above construction we define the 1-forms using a set of derivations. We shall suppose that they are interior and exclude therefore the case where \( A \) is commutative. For each integer \( n \) let \( \lambda_i \) be a set of \( n \) linearly independent antihermitian elements of \( A \) and introduce the derivations \( e_i = \text{ad} \lambda_i \). In general the \( e_i \) do not form a Lie algebra but they do however satisfy commutation relations as a consequence of the commutation relations of \( A \). We shall suppose that if an element of \( A \) commutes with all of the \( \lambda_i \) then it is in the center \( Z(A) \) of \( A \):
\[ e_i f = 0 \Rightarrow f \in Z(A). \]

Define the map (2.3) by
\[ df(e_i) = e_i f = [\lambda_i, f]. \] (2.17)
We shall suppose that there exists a set of \( n \) elements \( \theta^i \) of \( \Omega^1(A) \) such that
\[ \theta^i(e_j) = \delta^i_j. \] (2.18)
In the examples which we consider we shall show that the \( \theta^i \) exist by explicit construction. We shall refer to the set of \( \theta^i \) as a frame or Stehbein. It commutes with the elements \( f \in A \),
\[ f \theta^i = \theta^i f. \] (2.19)

The \( A \)-bimodule \( \Omega^1(A) \) is generated by all elements of the form \( fdg \) or of the form \((df)g\). Because of the Leibniz rule these conditions are equivalent. By definition
\[ fdg(e_i) = fe_i g, \quad (dg)f(e_i) = (e_i g)f. \]
Using the frame we can write these as
\[ fdg = (f e_i g) \theta^i, \quad (dg)f = (e_i g)f \theta^i. \] (2.20)

The commutation relations of the algebra constrain the relations between \( fdg \) and \((dg)f\) for all \( f \) and \( g \). Because of the commutation relations of the algebra or, equivalently, because of the kernel of \( \phi_1 \) in the quotient (2.4) the \( \theta^i \) satisfy in general commutation relations. With the identification \( \iota \) we have
\[ \pi(\theta^i \otimes \theta^j) = P^{ij}_{kl} \theta^k \otimes \theta^l, \quad P^{ij}_{kl} \in Z(A). \] (2.21)
Since \( \pi \) is a projection we have
\[ P^{ij}_{mn} P^{mn}_{kl} = P^{ij}_{kl} \] (2.22)
and the product \( \theta^i \theta^j \) satisfies
\[ \theta^i \theta^j = P^{ij}_{kl} \theta^k \theta^l. \] (2.23)
The module \( K \) is generated by the elements \((\delta^i_k \delta^j_l - P^{ij}_{kl}) \theta^k \otimes \theta^l\). In one important case which we shall consider the \( \theta^i \) anticommute. This corresponds to
\[ P^{ij}_{kl} = \frac{1}{2} (\delta^i_k \delta^j_l - \delta^i_l \delta^j_k). \] (2.24)
Define \( \theta = -\lambda_i \theta^i \). Then one sees that
\[
df = -[\theta, f]
\]
and it follows that as a bimodule \( \Omega^1(\mathcal{A}) \) is generated by one element. Under the condition (2.18) the \( \Omega^1(\mathcal{A}) \) is free of rank \( n \) as a left or right module. It can therefore by identified with the direct sum of \( n \) copies of \( \mathcal{A} \):
\[
\Omega^1(\mathcal{A}) = \bigoplus_1^n \mathcal{A}.
\]
(2.26)
In this representation \( \theta^i \) is given by the element of the direct sum with the unit in the \( i \)th position and zero elsewhere.

Any element \( \theta_u^i \) of \( \Omega_u^1(\mathcal{A}) \) can be written in the form
\[
\theta_u^i = \sum_\alpha f_\alpha^{(i)} \otimes g_\alpha^{(i)}
\]
with the \( f_\alpha^{(i)} \) and \( g_\alpha^{(i)} \) elements of \( \mathcal{A} \) such that
\[
\sum_\alpha f_\alpha^{(i)} g_\alpha^{(i)} = 0.
\]
(2.28)
Let \( \theta^i \) be the images of \( \theta_u^i \) in \( \Omega^1(\mathcal{A}) \). Then the condition (2.18) can be rewritten as
\[
\sum_\alpha f_\alpha^{(i)} \lambda_j g_\alpha^{(i)} = \delta_j^i.
\]
(2.29)
The assumption that \( \Omega^1(\mathcal{A}) \) is free then is equivalent to the assumption that these equations have a solution for \( f_\alpha^{(i)} \) and \( g_\alpha^{(i)} \).

The \( f_\alpha^{(i)} \) and \( g_\alpha^{(i)} \) can be used to give an explicit representation of \( K \). Introduce the notation
\[
\mathcal{A}[\theta^i_u, \mathcal{A}] = \{ \sum_\alpha f_\alpha[\theta^i_u, g_\alpha] \mid f_\alpha, g_\alpha \in \mathcal{A} \}.
\]
Then it is easily seen that
\[
\text{Ker} \phi_1 = \sum_i \mathcal{A}[\theta^i_u, \mathcal{A}].
\]
(2.30)
If we define then the elements \( \nu^i \) of \( \Omega^1(\mathcal{A}) \otimes \mathcal{A} \Omega^1(\mathcal{A}) \) by
\[
\nu^i = \sum_\alpha f_\alpha^{(i)} \theta \otimes \theta g_\alpha^{(i)}
\]
(2.31)
a short calculation yields the characterization
\[
\mathcal{K} = \sum_i \mathcal{A}[\nu^i, \mathcal{A}].
\]
(2.32)
If we define \( \lambda = (\lambda_1, \ldots, \lambda_n) \) then the condition (2.29) can be written in the form
\[
\mathcal{A}[\lambda, \mathcal{A}] = \bigoplus_1^n \mathcal{A}.
\]
(2.33)
More generally let \( \mathcal{B} \) be an \( \mathcal{A} \)-bimodule and \( \lambda \) an element of \( \mathcal{B} \). Define a map
\[
\mathcal{A} \xrightarrow{d} \mathcal{B}
\]
(2.34)
by \( df = [\lambda, f] \). Then we can set

\[
\Omega^1(\mathcal{A}) = \mathcal{A}[\lambda, \mathcal{A}] \subset \mathcal{B}.
\]  

(2.35)

The choice \( \mathcal{B} = \mathcal{A} \otimes \mathcal{A} \) with \( \lambda = 1 \otimes 1 \) yields the universal calculus. We see from this example that \( \lambda \) itself need not be an element of \( \Omega^1(\mathcal{A}) \).

From (2.31) we see that \( \pi(\nu^i) \) can be written as

\[
\pi(\nu^i) = \sum_{\alpha} f^{(i)}_{\alpha} \lambda^j \lambda^k P^{j k}_{l m} g^{(i)}_{\alpha} \theta^l \otimes \theta^m.
\]  

(2.36)

But from the identity \([\pi(\nu^i), f] = 0\) we see also that

\[
\pi(\nu^i) = \frac{1}{2} F^{i j k} \theta^j \theta^k
\]  

(2.37)

with \( F^{i j k} \) elements in \( \mathcal{Z}(\mathcal{A}) \) such that

\[
P^{j k} P^{i} = F^{i j k}.
\]  

(2.38)

Using (2.29) it follows that

\[
\sum_{\alpha} f^{(i)}_{\alpha} K_{j k} g^{(i)}_{\alpha} = 0
\]  

if we define \( K_{j k} \) by the equation

\[
2\lambda^l \lambda^m P^{l m}_{j k} - \lambda^i F^{i j k} - K_{j k} = 0.
\]  

(2.39)

The exterior derivative of \( \theta^i \) is given by

\[
d\theta^i = \sum_{\alpha} df^{(i)}_{\alpha} d g^{(i)}_{\alpha} = -[\theta, \theta^i] - \pi(\nu^i).
\]  

(2.40)

The bracket is a graded bracket. Multiplying both sides of this equation by \( \lambda_i \) we find the identity

\[
d\theta + \theta^2 = -\theta^2 + \pi(\nu^i \lambda_i).
\]  

Using (2.39) we find that this can be written in the form

\[
d\theta + \theta^2 = -\frac{1}{2} K_{i j} \theta^i \theta^j
\]  

(2.41)

and if we take the exterior derivative of (2.25) we see immediately that the coefficients \( K_{i j} \) must lie in \( \mathcal{Z}(\mathcal{A}) \).

The structure elements \( C^{i j k} \) are defined by the equation

\[
d\theta^i = -\frac{1}{2} C^{i j k} \theta^j \theta^k.
\]  

(2.42)

From (2.40) it follows that

\[
C^{i j k} = F^{i j k} - 2\lambda_l P^{(l i)}_{j k}.
\]  

(2.43)

We have started from an integer \( n \) and a set of \( \lambda_i \). The necessary and sufficient conditions for the existence of the basis \( \theta^i \) are expressed in the Equation (2.29). If \( \Omega^2(\mathcal{A}) \) is non-trivial there exists \( P^{(j i)}_{k l} \neq 0 \) in \( \mathcal{Z}(\mathcal{A}) \) such that (2.22) and (2.23) are
satisfied. Conversely we could have started from elements $P_{ijkl}$, $F_{jk}$, $K_{ij}$ in $Z(A)$ and looked for a solution $\lambda_i$ to the Equation (2.39). Define
\[ C^{ij}_{kl} = \delta^i_k \delta^j_l - 2P^{ij}_{kl}. \] (2.44)
Then from (2.22) we find that
\[ C^{ij}_{kl}C^{kl}_{mn} = \delta^i_m \delta^j_n. \]
From the associativity rule for the product in $\Omega^3(A)$ one finds that $C^{ij}_{kl}$ must satisfy a weak form of the Yang-Baxter equation.

Quite generally let $V$ be an $A$-module and $\pi$ a module morphism
\[ V \otimes_A V \xrightarrow{\pi} V \otimes_A V \] (2.45)
with $\pi^2 = \pi$. This is the algebraic generalization of the product given by (2.7). The equivalent generalization of the product in the 3-forms is a module morphism
\[ V \otimes_A V \otimes_A V \xrightarrow{\pi'} \otimes_A V \otimes_A V \] (2.46)
with $\pi'^2 = \pi'$. One has then
\[ \text{Ker} \, \pi' = \text{Ker} \, \pi \otimes V + V \otimes \text{Ker} \, \pi. \] (2.47)
For the product to be non-trivial we must require that $\pi' \neq 0$. Since we have
\[ \pi'(\text{Ker} \, \pi \otimes V) = 0, \quad \pi'(V \otimes \text{Ker} \, \pi) = 0, \]
there must exist two morphisms $\phi'$, $\phi''$ of $V \otimes_A V \otimes_A V$ into itself such that
\[ \pi' = \phi' \circ \pi_{12}, \quad \pi' = \phi'' \circ \pi_{23}. \]
We have here used the standard convention of setting $\pi_{12} = \pi \otimes 1$ and $\pi_{23} = 1 \otimes \pi$. The associativity rule becomes the compatibility condition
\[ \phi' \circ \pi_{12} = \phi'' \circ \pi_{23}. \] (2.48)
In particular if $C^{ij}_{kl}$ satisfies the Yang-Baxter condition
\[ C_{23} \circ C_{12} \circ C_{23} = C_{12} \circ C_{23} \circ C_{12} \]
then one can set
\[ \phi' = -\frac{1}{3}(1 - 4\pi_{12} \circ \pi_{23}), \quad \phi'' = -\frac{1}{3}(1 - 4\pi_{23} \circ \pi_{12}). \] (2.49)
However more general solutions to (2.48) do exist.

The $F_{jk}$ must satisfy a set of modified Jacobi identities. If we choose the $\lambda_i$ so that the $e_i$ are a basis of the Lie algebra of all derivations of a matrix algebra (Dubois-Violette et al. 1989) then one can choose $P^{ij}_{kl} = \delta^i_k \delta^j_l$. If a smaller Lie algebra (Madore 1995) is chosen then $P^{ij}_{kl}$ is given by (2.24), $K_{ij} = 0$ and the $F_{jk}$ are equal to the structure constants of the Lie algebra. An example with $K_{ij} \neq 0$ is to be found in Dubois-Violette et al. (1996b). Examples with $F_{jk} = 0$ and $K_{ij} = 0$ are given in Dimakis & Madore (1996). If $P^{ij}_{kl}$ is given by (2.24) with a plus instead of a minus sign, $n$ is even and $F_{ij} = 0$ then a solution to (2.39) is given by Dirac matrices. If also $K_{ij} = 0$ then a solution is given by ‘super-coordinates’. In these two cases the 1-forms $\theta^i$ commute.
3 The linear connections

The definition of a connection as a covariant derivative was given an algebraic form in the Tata lectures by Koszul (1960) and generalized to noncommutative geometry by Karoubi (1981) and Connes (1986, 1994). We shall use here the expressions ‘connection’ and ‘covariant derivative’ synonymously. A ‘bimodule connection’ is a connection on a general bimodule $\mathcal{M}$, which satisfies a left and right Leibniz rule. In the particular case where $\mathcal{M}$ is the module of 1-forms we shall speak of a ‘linear connection’.

Let $A$ be an arbitrary algebra and $(\Omega^*(A), d)$ a differential calculus over $A$. One defines a left $A$-connection on a left $A$-module $\mathcal{H}$ as a covariant derivative

$$\mathcal{H} \xrightarrow{D} \Omega^1(A) \otimes_A \mathcal{H}$$

which satisfies the left Leibniz rule

$$D(f \psi) = df \otimes \psi + f D\psi$$

for arbitrary $f \in A$. This map has a natural extension

$$\Omega^*(A) \otimes_A \mathcal{H} \xrightarrow{D} \Omega^*(A) \otimes_A \mathcal{H}$$

given, for $\psi \in \mathcal{H}$ and $\alpha \in \Omega^n(A)$, by

$$D(\alpha \psi) = d\alpha \otimes \psi + (-1)^n \alpha D\psi.$$

The operator $D^2$ is necessarily left-linear.

A covariant derivative on the module $\Omega^1(A)$ must satisfy (3.2). But $\Omega^1(A)$ has also a natural structure as a right $A$-module and one must be able to write a corresponding right Leibniz rule in order to construct a bilinear curvature. Quite generally let $\mathcal{M}$ be an arbitrary bimodule. Consider a covariant derivative

$$\mathcal{M} \xrightarrow{D} \Omega^1(A) \otimes_A \mathcal{M}$$

which satisfies both a left and a right Leibniz rule. In order to define a right Leibniz rule which is consistent with the left one, it was proposed by Mourad (1995), by Dubois-Violette & Michor (1996) and by Dubois-Violette & Masson (1996) to introduce a generalized permutation

$$\mathcal{M} \otimes_A \Omega^1(A) \xrightarrow{\sigma} \Omega^1(A) \otimes_A \mathcal{M}.$$ 

The right Leibniz rule is given then as

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f$$

for arbitrary $f \in A$ and $\xi \in \mathcal{M}$. The purpose of the map $\sigma$ is to bring the differential to the left while respecting the order of the factors. It is necessarily bilinear. Consider the case

$$\mathcal{M} = \Omega^1(A)$$

and let $\pi$ be the projector defined by (2.7). It was shown by Mourad (1995) and by Dubois-Violette et al. (1995) that a necessary as well as sufficient condition for torsion to be right-linear is that $\sigma$ satisfy the consistency condition

$$\pi \circ (\sigma + 1) = 0.$$
We define a bimodule $\mathcal{A}$-connection to be the couple $(D, \sigma)$. Using the fact that $\pi$ is a projection one sees that the most general solution to the constraint (3.6) is given by

$$1 + \sigma = (1 - \pi) \circ \tau$$

where $\tau$ is an arbitrary map

$$\Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \xrightarrow{\tau} \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}).$$

If we choose $\tau = 2$ then we find $\sigma = 1 - 2\pi$ and $\sigma^2 = 1$. The eigenvalues of $\sigma$ are then equal to ±1.

There is at the moment no general consensus of the correct definition of the curvature of a bimodule connection. The problem is that the operator $D^2$ need not in general be right-linear. Nevertheless in the particular cases of interest to us here, with a module of 1-forms which is free and has a special basis such that (2.19) is satisfied, the ordinary definition of curvature is quite satisfactory. We refer to Dubois-Violette $et$ $al.$ (1996b) or to Dimakis (1996) for recent discussions.

This general formalism can be applied in particular to the differential calculi which we have constructed in the previous section. Since $\Omega^1(\mathcal{A})$ is a free module the maps $\sigma$ and $\tau$ can be defined by their action on the basis elements:

$$\sigma(\theta^i \otimes \theta^j) = S^{ij}_{kl} \theta^k \otimes \theta^l, \quad \tau(\theta^i \otimes \theta^j) = T^{ij}_{kl} \theta^k \otimes \theta^l.$$  (3.9)

By the sequence of identities

$$f S^{ij}_{kl} \theta^k \otimes \theta^l = \sigma(f \theta^i \otimes \theta^j) = \sigma(\theta^i \otimes \theta^j f) = S^{ij}_{kl} f \theta^k \otimes \theta^l$$

and the corresponding ones for $T^{ij}_{kl}$ we conclude that the coefficients $S^{ij}_{kl}$ and $T^{ij}_{kl}$ must lie in $\mathcal{Z}(\mathcal{A})$. From (3.7) the most general form for $S^{ij}_{kl}$ is

$$S^{ij}_{kl} = (\delta^i_m \delta^j_n - \delta^i_l \delta^j_m) T^{mn}_{kl} - \delta^i_k \delta^j_l.$$  (3.11)

Since $\Omega^1(\mathcal{A})$ is a free module a covariant derivative can be defined by its action on the basis elements:

$$D\theta^i = -\omega^j_{jk} \theta^j \otimes \theta^k.$$  (3.12)

The coefficients here are elements of the algebra. The torsion 2-form is defined as usual as

$$\Theta^i = d\theta^i - \pi \circ D\theta^i.$$  

There is a natural covariant derivative $D_{(0)}$ (Dubois-Violette $et$ $al.$ 1996b) given by

$$D_{(0)}\theta^i = -\theta \otimes \theta^i + \sigma(\theta^i \otimes \theta) - \nu^i,$$  (3.13)

which is torsion-free by (2.40). The corresponding coefficients are given by

$$\omega_{(0)}^i{}_{jk} = \lambda^i_l (S^{il}_{jk} - \delta^i_j \delta^i_k) + \frac{1}{2} F^i_{jk}.$$  (3.14)

The most general $D$ is of the form

$$D = D_{(0)} + \chi$$  (3.15)

where $\chi$ is an arbitrary bimodule morphism

$$\Omega^1(\mathcal{A}) \xrightarrow{\chi} \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}).$$  (3.16)
If we write
\[ \chi(\theta) = -\chi_{jk}^i \theta^j \otimes \theta^k \] (3.17)
then by an argument similar to (3.10) we conclude that
\[ \chi_{jk}^i \in Z(A). \] (3.18)

In general a covariant derivative is torsion-free provided the condition
\[ \omega^i_{jk} - \omega^i_{lm} S^m_{jk} = C^i_{jk} \] (3.19)
is satisfied. The covariant derivative (3.15) is torsion free if and only if
\[ \pi \circ \chi = 0. \] (3.20)

One can define a metric by the condition
\[ g(\theta^i \otimes \theta^j) = g^{ij} \] (3.21)
where the coefficients \( g^{ij} \) are elements of the algebra. To be well defined on all elements of the tensor product \( \Omega^1(A) \otimes_A \Omega^1(A) \) the metric must be bilinear and by the sequence of identities
\[ fg^{ij} = g(f \theta^i \otimes \theta^j) = g(\theta^i \otimes \theta^j f) = g^{ij} f \] (3.22)
one concludes that the coefficients must lie in \( Z(A) \). This restriction plays an important role in the unicity argument which allows us to invert the top arrow of Diagram (1.8). In the commutative limit the \( g^{ij} \) cannot be functions of the coordinates. In ordinary geometry an equivalence class of moving frames determines a metric and all equivalence classes correspond to the same differential calculus, the ordinary de Rham differential calculus. The choice of differential calculus does not fix the metric. In the noncommutative case on the other hand, as we have defined it, each differential calculus determines a Stehbein and thereby a metric. In the commutative limit all of the noncommutative differential calculi are either singular, if \( n \) is not equal to the classical dimension of the manifold, or have a common limit. The moving frame however and the associated metric remain however as a shadow of the noncommutative structure.

The covariant derivative (3.12) is compatible with the metric (Dubois-Violette et al. 1995) if and only if
\[ \omega^i_{jk} + \omega^i_{lm} S^m_{jm} = 0. \] (3.23)
When \( F^i_{jk} = 0 \) the condition that (3.14) be metric compatible can be written as
\[ S^{im}_{ln} g^{np} S^{kj}_{mp} = g^{ik} \delta^j_l. \] (3.24)
The metric we have chosen is not symmetric with respect to \( \sigma \). That is
\[ g^{ij} \neq S^{ij}_{kl} g^{kl} \]
in general. If one wishes to find a metric symmetric in the above sense then one must consider (3.24) as an equation for \( S \) and the metric and add the additional equation
\[ g^{ij} = S^{ij}_{kl} g^{kl}. \] (3.25)
The system (3.24), (3.25), if it has a solution, would yield a symmetric metric with a compatible connection.

Since we are primarily interested in the first-order effects in the commutative limit we can identify the curvature with the operator \( D^2 \). We set as usual then
\[ D^2 \theta^i = -\frac{1}{2} R^i_{jkl} \theta^j \theta^k \otimes \theta^l. \] (3.26)
Since \( D^2 \) is not necessarily right-linear as an operator the last of the equivalent of the sequence of identities (3.10) is not valid and we cannot conclude that the coefficients \( R^i_{jkl} \) necessarily lie in the center of the algebra.
4 The basic calculus

In this section we shall return to the ‘space-time’ algebra defined in Section 1 and we shall suppose, with Doplicher et al., that the \( q^{\mu\nu} \) lie in the center of the algebra. This permits us to suppose further that the matrix \( q^{\mu\nu} \) has an inverse \( q_{\lambda\mu}^{-1} \):

\[
q_{\lambda\mu}^{-1}q^{\mu\nu} = \delta^\nu_\lambda.
\]

We shall use this inverse to lower the indices of the generators \( q^\mu \):

\[
\tilde{q}_\lambda = \mu^2 P q_{\lambda\mu}^{-1} q^\mu.
\]

A natural choice of \( n \) is \( n = 4 \) and a natural choice of \( \lambda^\mu \) is given by

\[
\lambda^\mu = -i \tilde{q}_\mu.
\] (4.1)

The associated derivations defined in Section 2 satisfy then

\[
e^\mu q^\lambda = \delta^\lambda_\mu
\] (4.2)

and it follows that

\[
[e^\mu, e^\nu] = 0.
\] (4.3)

From (4.2) it follows that

\[
\theta^\lambda = dq^\lambda, \quad \theta = i\tilde{q}_{\lambda}dq^\lambda
\] (4.4)

from which we deduce that

\[
P^{\mu\nu}_{\rho\sigma} = \frac{1}{2}(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma), \quad F^\lambda_{\mu\nu} = 0, \quad K_{\mu\nu} = i\mu^2 \tilde{q}_{\mu\nu}^{-1}.
\] (4.5)

One can interpret \( \theta \) as a connection on a trivial bundle with the unitary elements of the algebra as structural group (Dubois-Violette et al. 1989, 1990). We see from the above formula for \( K_{\mu\nu} \) that \( q^{\mu\nu} \) is related to the corresponding curvature. This is the noncommutative analogue of the classical result of mechanics which interprets the symplectic 2-form as the curvature of a line bundle.

From the commutation relations (2.19) one finds that the \( \theta^\lambda \) anticommute. A possible form for \( \sigma \) is given therefore by

\[
S^{\mu\nu}_{\rho\sigma} = \delta^\nu_\rho \delta^\mu_\sigma.
\] (4.6)

From (3.14) we see that in this case the coefficients of the connection necessarily lie in the center of the algebra. From (3.6) we see that the most general \( S^{\mu\nu}_{\rho\sigma} \) must satisfy the constraint

\[
S^{\mu\nu}_{[\rho\sigma]} + \delta^\mu_\rho \delta^\nu_\sigma = 0.
\] (4.7)

The most general \( \sigma \) is defined by a solution to the Equations (3.24) and (4.7). If we restrict the \( g_{\lambda\mu} \) to be the components of the Minkowski metric then the unique solution is given by (4.6). The Minkowski metric is then symmetric also with respect to \( \sigma \). From (3.19) and (3.23) we see that if we require that the torsion vanish then we have

\[
\omega^\lambda_{\mu\nu} = 0.
\] (4.8)

The space-time is therefore a noncommutative version of Minkowski space and the right-hand side of Equation (1.3) must vanish.
It is of interest to notice that Equation (4.2) defines a derivation of the algebra whatever the form of the matrix \( q^{\mu \nu} \). The derivation is inner if the matrix is invertible; otherwise it is outer. Let \( \theta^\lambda \) be a set of Grassmann variables. Define
\[
q^\lambda = x^\lambda + \mu_P^{-1} \theta^\lambda, \quad q^{\mu \nu} = -2i \theta^\mu \theta^\nu.
\] (4.9)

Then (1.1) is satisfied and \( q^{\mu \nu} \in \mathcal{Z}(\mathcal{A}) \). In this case the matrix \( q^{\mu \nu} \) is not invertible; it is in fact nilpotent.

Notice also that in this case the center \( \mathcal{Z}(\mathcal{A}) \) is nontrivial and in fact it is possible to impose that it be a smooth 6-dimensional manifold with the \( q^{\mu \nu} \) as coordinates. We can, with Doplicher et al., impose the conditions
\[
q_\mu q^{\mu \nu} = 0, \quad (4.10)
\]
as well as
\[
\epsilon_{\mu \nu \rho \sigma} q^{\mu \nu} q^{\rho \sigma} = 12,
\]
which is equivalent to
\[
q^{-1}_\mu = -\frac{1}{3} \epsilon_{\mu \nu \rho \sigma} q^{\rho \sigma}. \quad (4.11)
\]
The normalization is chosen for later convenience. The manifold can then be reduced to 4 dimensions. In the limit \( \mu_P \to 0 \) we have in fact a structure which can be regarded as a 4-dimensional manifold with a non-commutative extension à la Kaluza-Klein similar to the structures which are mentioned, for example, in Madore & Mourad (1996). In the limit \( \mu_P \to \infty \) the structure can be considered to be that of an ordinary space-time with an extra 4- or 6-dimensional factor in which the Poisson structure defined by the commutator (1.1) takes its values (Dubois-Violette et al. 1996a). An element of \( \mathcal{Z}(\mathcal{A}) \) can in no way correspond to a function on space-time in the commutative limit. Also if \( f \) is an element of \( \mathcal{A} \) such that \( \epsilon_{\mu} f = 0 \) then we can only conclude that \( f \) is an arbitrary function of the \( q^{\mu \nu} \); we cannot conclude that it is proportional to the identity. We regard the non-trivial center as something which is to be eventually eliminated for example by choosing \( q^{\mu \nu} \) not to lie in the center. To simplify the calculations we shall suppose that the matrix of coefficients \( g^{\mu \nu} \) of the metric is symmetric in the ordinary sense of the word we shall impose the condition that it be equal to the matrix of components of the ordinary Minkowski metric.

5 Variations of the calculus

We shall now check the stability of the result (4.8) under the perturbation of the differential calculus. Introduce four arbitrary ‘small’ elements \( f^\lambda \) of \( \mathcal{A} \) and define \[
\tilde{f}_\lambda = \mu_P^2 q^{-1}_\lambda f^\mu. \quad (5.1)
\]
Then the elements
\[
\lambda'_\mu = -i(\tilde{q}_\mu + \tilde{f}_\mu) \quad (5.2)
\]
are ‘near’ to (4.1). In general, unless the condition (2.39) is satisfied, \( \Omega^2(\mathcal{A}) = 0 \) and the curvature will vanish. Impose the condition (2.39) and let \( P^{\rho \sigma}_{(1)} \), \( P^{\lambda}_{(1)} \mu \nu \) and \( K^{(1)} \mu \nu \) be the first-order perturbations respectively of the coefficients. By simple dimensional arguments one can argue that \( P^{\mu \nu}_{(1) \rho \sigma} \) must vanish. In fact it must tend to zero when the Planck mass tends to infinity but on the other hand it is without dimension and therefore cannot depend on the Planck mass. Therefore we set
\[
P^{\mu \nu}_{(1) \rho \sigma} = 0. \quad (5.3)
\]
Using (4.5) we find that the linearization of (2.39) yields the equation

$$[\tilde{q}_\mu, \tilde{f}_\nu] - [\tilde{q}_\nu, \tilde{f}_\mu] = iF^\lambda_{(1)\mu\nu}\tilde{q}_\lambda - K_{(1)\mu\nu}. \quad (5.4)$$

Let $k_\mu$ be an arbitrary ‘small’ 4-vector with the dimension of mass. Then a solution is given by

$$\tilde{f}_\lambda = k_\mu q^\mu \tilde{q}_\lambda \quad (5.5)$$

and

$$F^\lambda_{(1)\mu\nu} = k_\mu \delta^\lambda_\nu + 2q^{-1}_\mu q_\alpha k_\sigma, \quad K_{(1)\mu\nu} = 0. \quad (5.6)$$

The corresponding frame is given by

$$\theta^{\lambda\mu} = (1 - k_\rho q^\rho) dq^\lambda + q^{\lambda\rho} k_\rho q_{\mu\sigma} dq^\sigma. \quad (5.7)$$

It will generate a new differential calculus $\Omega^*(\mathcal{A})$ which will be in general different from $\Omega^*(\mathcal{A})$. Using (2.42) or (2.43) we find that

$$C^{\lambda}_{(1)\mu\nu} = F^\lambda_{(1)\mu\nu}. \quad (5.8)$$

The frame (5.7) explicitly breaks Lorentz invariance through the vector $k_\lambda$. This is quite natural since there is now present a gravitational field in the commutative limit. What is less desirable is the dependence on the extra ‘coordinates’ $q_{\mu\nu}$. This can be eliminated by introducing a Lorentz-invariant probability measure on the space of $q^{\mu\nu}$ with

$$\langle q^{\mu\nu} q^{\rho\sigma} \rangle = \delta^{\mu\nu}\delta^{\rho\sigma}. \quad (5.9)$$

The normalization in (4.11) was chosen so that this equation is compatible with the condition $\langle 1 \rangle = 1$. We have then

$$\langle \theta^{\lambda\mu} \rangle = dq^\lambda - q^\lambda k_\mu dq^\mu \quad (5.10)$$

and

$$\langle F^\lambda_{(1)\mu\nu} \rangle = -k_\mu \delta^\lambda_\nu. \quad (5.11)$$

The particular, simple form of (5.10) and (5.11) is due to the choice of normalization of the measure on the space of $q^{\mu\nu}$.

Let $D'$ be a covariant derivative and define the coefficients $\omega^{\lambda\mu\nu}$ by the equation

$$D' \theta^{\lambda\mu} = -\omega^{\lambda\mu\nu} \theta^{\mu\nu} \quad (5.12)$$

equivalent to (3.12). Because of (4.8) we have $\omega^{\lambda\mu\nu} = \omega^{\lambda}_{(1)\mu\nu}$ and we can write to lowest order

$$D' \theta^{\lambda\mu} = -\omega^{\lambda}_{(1)\mu\nu} \theta^{\mu\nu}. \quad (5.12)$$

To extend the covariant derivative to the entire module of 1-forms we shall need the expression for the perturbed value $\sigma'$ of $\sigma$. We define the coefficients $S^{\mu\nu\rho\sigma}$ by the equation

$$\sigma' (\theta^{\mu\nu} \otimes \theta^{\rho\sigma}) = S^{\mu\nu\rho\sigma} \theta^{\rho\sigma} \otimes \theta^{\rho\sigma} \quad (5.13)$$

equivalent to (3.12). We expand

$$S^{\mu\nu\rho\sigma} = \delta^{\nu}_\rho \delta^{\mu}_\sigma + S^{\mu\nu}_{(1)\rho\sigma}. \quad (5.13)$$

Each choice of $S^{\mu\nu}_{(1)\rho\sigma}$ corresponds to a definite choice of covariant derivative. As above, by simple dimensional arguments one can argue that

$$S^{\mu\nu}_{(1)\rho\sigma} = 0. \quad (5.14)$$
Therefore from (3.14) and a proper choice of $\chi$ so that (3.23) is satisfied we find that the torsion-free metric connection is given by

$$\omega^{\lambda}_{(1)\mu\nu} = \frac{1}{2}(F^{\lambda}_{(1)\mu\nu} - F^{\lambda}_{(1)\nu\mu} + F_{(1)\mu\nu}\lambda).$$

(5.15)

The curvature of the covariant derivative defined by this expression is essentially constant as far as ‘space-time’ is concerned. From this point of view it is not particularly interesting. Using (5.11) we find that

$$\langle \omega^{\lambda}_{(1)\mu\nu} \rangle = \frac{1}{2}(k^{\lambda} g_{\mu\nu} - k_{\mu} \delta^{\lambda}_{\nu}).$$

(5.16)

To lowest order we find the expression

$$R^{\rho\nu\rho\sigma} = \omega^{\rho\mu}_{(1)} \omega^{\tau}_{(1)\sigma\nu} - \omega^{\rho\mu}_{(1)} \omega^{\tau}_{(1)\sigma\rho} - \omega^{\mu}_{(1)} \omega^{\tau}_{(1)\nu\rho} C^{\rho}_{(1)\rho\sigma}$$

(5.17)

for the components of the curvature. It is not of particular interest to give an expression for the expectation value $\langle R^{\rho\nu\rho\sigma} \rangle$ since it will depend critically on the probability measure. However the expectation value of the Einstein tensor must be of the form

$$\langle G_{\mu\nu} \rangle = a k_{\mu} k_{\nu} + b k^{2} g_{\mu\nu}$$

(5.18)

where $a$ and $b$ are dimensionless constants which depend on the details of the probability measure. According to the logic of the diagram (1.8) this expression is to be substituted for the left-hand side of Equation (1.3). At the present preliminary stage of the understanding of the relation between the differential calculus and the resulting curvature it is premature to consider this equation further.

### 6 Variations of the algebra

Another way to obtain a non-vanishing gravitational field is to vary the structure of the algebra $\mathcal{A}$. We introduce 6 ‘small’ elements $q^{\mu\nu}_{(1)}$ of $\mathcal{A}$ and define

$$q^{\mu\nu} = q^{\mu\nu}_{(1)} + q^{\mu\nu}_{(1)}.$$  

(6.1)

We have then

$$q^{-1}_{\mu\nu} = q_{\mu\nu}^{-1} + q^{-1}_{\mu\rho} q^{-1}_{\rho\sigma} q^{\rho\sigma}_{(1)}.$$  

(6.2)

Since we are here primarily interested in the effect of varying the structure of the algebra we keep the basic calculus and set

$$\lambda'_{\mu} = \lambda_{\mu}.$$  

(6.3)

From (1.2) we find

$$[q^{\lambda}, q^{\mu\nu}] = [q^{\lambda}, q^{\mu\nu}_{(1)}] = i\mu^{-1} q^{\lambda\mu\nu}.$$  

The simplest generalization of the basic algebra is obtained by supposing that $q^{\lambda\mu\nu}_{(1)}$ lies in the center of $\mathcal{A}$. This is the extended model of Doplicher et al. If we impose this condition we can choose

$$q^{\mu\nu}_{(1)} = -\mu^{-1} q^{\lambda\mu\nu}_{(1)} q_{\lambda}.$$  

(6.4)

We have then

$$e^{\prime}_{\mu} q^{\nu} = e_{\mu} q^{\nu} = \delta^{\nu}_{\mu} + \mu^{-1} q^{-1}_{\mu\rho} q^{\rho\sigma}_{(1)} q_{\sigma}.$$  

(6.5)

Using (4.5) we find that the linearization of (2.39) yields now the solution

$$F^{\lambda}_{(1)\rho\sigma} = \mu_{P} q^{-1}_{\rho\mu} q^{-1}_{\sigma\nu} (q^{\lambda\mu\nu}_{(1)} - q^{\mu\nu\lambda}_{(1)} + q^{\nu\lambda\mu}_{(1)}).$$  

(6.6)

The corresponding frame is given by

$$\theta^{\nu\lambda} = dq^{\lambda} - \mu^{-1} q^{-1}_{\mu\rho} q_{\sigma} dq^{\rho}.$$  

(6.7)
7 de Sitter space

To define a noncommutative version of a space which is not topologically trivial we shall use elementary techniques from classical geometry as well as from the quantum mechanics of constrained hamiltonian systems. A non-trivial topological manifold $V$ can be defined by its imbedding in a flat space of sufficiently high dimension. The algebra $\mathcal{B} = \mathcal{C}(V)$ of continuous functions on $V$ can be identified with the algebra $\mathcal{A}$ of all continuous functions on the enveloping space modulo the ideal $\mathcal{I}$ of continuous functions which vanish on $V$: $\mathcal{B} = \mathcal{A}/\mathcal{I}$. As a first attempt to define a noncommutative version of de Sitter space we set $\mathcal{A}$ equal to a $\ast$-algebra generated by a set of elements $q^i$, $0 \leq i \leq 4$, which satisfy commutation relations similar to those given by (1.1):

$$[q^i, q^j] = i\mu_p^{-2}q^{ij}. \quad (7.1)$$

We introduce a Minkowski-signature metric with components $g_{ij}$ and we define $\mathcal{I}$ to be the 2-sided ideal generated by the element

$$c_1 = g_{ij}q^i q^j - r^2 \quad (7.2)$$

where $r^2$ is a real constant. Two problems present themselves immediately. The $q^{ij}$ cannot be invertible; it would otherwise define a symplectic form on an odd-dimensional manifold. Using the commutation relations (7.1) one sees also that the 2-sided ideal generated by the element $c_1$ is the entire algebra and therefore $\mathcal{B} = 0$. We saw something similar to this in Section 2 where we noticed that the 2-sided ideal generated by the element $\theta$ is the entire algebra of forms. This second problem is connected with the fact that we are trying to define the manifold $V$ exactly as a submanifold in spite of the fact that its ‘points’ are fuzzy and only defined to within the uncertainty $\mu_p^{-1}$. As a solution to the first problem we add another dimension and we define $V$ as a submanifold of a 6-dimensional space. We let then $0 \leq i \leq 5$ and we add a constraint

$$c_2 = q^5. \quad (7.3)$$

To circumvent the second problem we follow the example furnished by the quantization of constrained hamiltonian systems. The two constraints are of second class and Dirac (1964) has shown how in this case one can introduce a new bracket with respect to which they commute with each other and with the observables. One can similarly modify the commutation relations (7.1) so that $c_1$ commutes with the generators of $\mathcal{A}$ and thus with $c_2$. We introduce the components $r^{ij}$ of an antisymmetric tensor and we set

$$[q^i, q^j]' = i\mu_p^{-2}(q^{ij} - r^{ij}). \quad (7.4)$$

From the condition $[q^i, c_2]' = 0$ we find that

$$r^{i5} = q^{i5}. \quad (7.5)$$

From the condition $[q^i, c_1]' = 0$ we find the equation

$$q_j r^{ij} + r^{ij} q_j = 2q^{ij} q_j \quad (7.6)$$

for $r^{ij}$. Suppose that the element $q^{i5} q_i$ of $\mathcal{A}$ is invertible. The appropriate solution is given then by

$$r^{ij} = \frac{1}{2} q^i [q^j q^{i5} q_k, (q^{i5} q_l)^{-1}]_+. \quad (7.7)$$
which can also be written in the form

\[ r_{ij} = q^{[ik} q^{j]5} q_k (q^{15} q_l)^{-1} - \frac{i}{2} \mu_p^{-2} q^{[ik} q^{j]5} q_k q_l q_m (q^{m5} q_m)^{-2}. \]  

(7.8)

We define \( I \) to be the 2-sided ideal of \( A \) generated by the two elements \( c_1 \) and \( c_2 \). By modifying the original structure of the algebra we have obtained a non-trivial quotient algebra \( B \). The modification of the algebra is not 'small' in the sense of the preceding section. We shall consider \( B \) as a possible noncommutative equivalent of de Sitter space. We refer to Masson (1996) for some examples of quotient algebras defined without modification of the bracket.

We define \( \tilde{q}_i \) and \( \lambda_i \) as in Section 4. The associated derivations \( e_i \) satisfy then

\[ e_i q^j = \delta^j_i - \frac{1}{2} [q_i q^{j5} - \delta^5_i q^{jk} q_l, (q^{15} q_l)^{-1}]_+, \]  

and they no longer commute. They do not even close to form a Lie algebra:

\[ [e_i, e_j] = i \mu_p^{-2} q^{ik} q^{j5} r^{kl}. \]  

(7.10)

From the relations

\[ e_i c_1 = 0, \quad e_i c_2 = 0 \]  

(7.11)

it follows that the \( e_i \) can be considered as derivations of \( B \).

If we set

\[ f_i^{jk} = \delta_i^{[j} q^{k]5} + \delta^5_i q^{jk} \]

then in the commutative limit we can write

\[ e_i = (q^{15} q_l)^{-1} f_i^{jk} q_k \partial_j. \]  

(7.12)

Since \( f_i^{jk} \) is antisymmetric in the last 2 indices and since \( f_i^{j5} = 0 \) the \( e_i \) are vector fields on the de Sitter space. There are 6 of them and they satisfy 2 relations:

\[ q_i q^{j5} e_j = 0, \quad q^{j5} e_i = 0. \]  

(7.13)

We define a differential calculus as in Section 4. It follows from (7.11) and the construction of the differential that

\[ dc_1 = 0, \quad dc_2 = 0. \]  

(7.14)

The differential calculus can be considered then as a calculus \( \Omega^*(B) \) over \( B \). It tends as it must in the commutative limit to the algebra of de Rham forms over de Sitter space. Classical de Sitter space is parallelizable but it is not obvious that \( \Omega^1(B) \) is free as a left or right module. Because of the relations (7.13) the \( \theta^i \) dual to the \( e_i \) cannot be constructed. Because of the factor \( (q^{15} q_l)^{-1} \) in the expression (7.7) the equation (2.39) has no solution for the \( \lambda_i \) which we have chosen.

8 Gauge invariance

In (5.15) we calculated the connection associated to a perturbation of the basic differential calculus. One might have expected from the origin (5.2) of the perturbation that it would resemble rather a coordinate transformation and that the perturbed connection would vanish. The fact that this is not the case is due to the existence of the extra
‘coordinates’ $q^{\mu\nu}$ which have been used to raise and lower indices. Our formalism is in fact analogous to that which can be used to describe a manifold $V$ which is defined by its imbedding in a flat space of higher dimension so we mention this briefly. Let $x^a$ be the coordinates of the imbedding space and $y^a$ local coordinates of $V$. Then $V$ is defined locally by functions of the form $x^a(y^a)$. A variation of $V$ to a surface $V'$ is given by functions of the form $x^a(y^a) = x^a(y^a) + h^a(x^b)$. A variation of the coordinates of the imbedding space can be written in the form $x^a = x^a + h^a(x^b)$. The most general variation of the metric on $V$ is obtained by a variation $g^\prime_{ab} = g_{ab} + \partial(a h_b)$ of the components of the flat metric. In the commutative limit one can consider ($q^\lambda, q^{\mu\nu}$) as the ‘coordinates’ of the imbedding space and the conditions $q^{\mu\nu} = 0$ the equations which define $V$.

Let $V$ be a smooth manifold and $\phi$ a smooth map of $V$ into itself. Then $\phi$ induces an automorphism $\phi^*$ of the algebra $\mathcal{C}(V)$ of smooth functions given by $\phi^* f = f \circ \phi$ and thereby a map $\phi_*$ of the derivations. If $X$ is a derivation of $\mathcal{C}(V)$ then so is $\phi_* X = \phi^{-1} X \circ \phi^*$. The noncommutative equivalent of $\phi$ is an automorphism of the algebra $\mathcal{A}$. Consider the inner automorphism which acts on the generators by the transformation

$$q^\lambda \mapsto \text{ad} U^{-1} q^\lambda, \quad q^{\mu\nu} \mapsto \text{ad} U^{-1} q^{\mu\nu}. \quad (8.1)$$

Then it is obvious that $\lambda_\mu \mapsto \text{ad} U^{-1} \lambda_\mu$ and a solution to Equation (3.39) is transformed into another solution with the same values of $P_{\rho\sigma}^\mu, F_\mu^{\lambda\nu}$ and $K_{\mu\nu}$. A derivation $X$ is transformed into $X' = \text{ad} U^{-1} \circ X \circ \text{ad} U$ from which we deduce that

$$\theta^\alpha \mapsto \theta'^\alpha = \text{ad} U \circ \theta^\alpha \circ \text{ad} U^{-1}. \quad (8.2)$$

If the geometry of $V$ is described (locally) by a moving frame $\theta^\alpha$ then a change of moving frame is a map

$$\theta^\alpha \mapsto \theta'^{\alpha} = \Lambda^\alpha_\beta \theta^\beta \quad (8.3)$$

with $\Lambda^\alpha_\beta$ smooth functions of $V$. If the metric is to be left invariant then there are restrictions on the $\Lambda^\alpha_\beta$:

$$\Lambda^\alpha_\beta \Lambda^\beta_\delta g^{\gamma \delta} = g^{\alpha \beta}. \quad (8.4)$$

In the noncommutative case one could define a change of Stehbein using the same formula (8.3) but with $\Lambda^\alpha_\beta$ elements of the algebra $\mathcal{A}$. If $\Lambda^\alpha_\beta \in \mathcal{Z}(\mathcal{A})$ then the new Stehbein has the same status as the old; it is dual to a set of derivations. Otherwise the change is purely formal. From the left-linearity of $D^2$ we conclude that

$$D^2(\Lambda^\alpha_\beta \theta^\beta) = \Lambda^\alpha_\beta D^2 \theta^\beta$$

and so from (3.26) we find that

$$\Lambda^\alpha_\beta R^{\lambda}_{\quad \beta \gamma \delta} = R^\alpha_{\lambda \mu \nu} \Lambda^\lambda_\beta \Lambda^\mu_\gamma \Lambda^\nu_\delta. \quad (8.4)$$

If we multiply this equation on both sides by $g^{\beta \gamma}$ and define the ‘Ricci tensor’ as the map $\theta^\alpha \mapsto R^\alpha_{\beta \delta} \theta^\delta$ defined by $R^\alpha_{\beta \delta} = g^{\beta \gamma} R^\alpha_{\gamma \beta \delta}$ then we find the condition

$$\Lambda^\alpha_\beta R^{\lambda}_{\quad \beta \delta} = R^\alpha_{\nu \lambda} \Lambda^\nu_\delta. \quad (8.5)$$

The order of the factors is here important. There is no reason for the trace $R^\alpha_{\alpha \delta}$ of $R^\alpha_{\beta \delta}$ to be ‘invariant’ except if $\Lambda^\alpha_\beta \in \mathcal{Z}(\mathcal{A})$. To define the analogue of the Einstein-Hilbert action one would also have to introduce a trace on the algebra such that

$$\text{Tr}(R^\alpha_{\alpha \delta}) \rightarrow \int_V R^\alpha_{\alpha}$$
in the commutative limit. There is no obvious way in which this can be done. We are therefore unable at the moment to propose a satisfactory definition of an action and indeed we are not in a position to argue that there is even a valid action principle. A discussion of this point has been made by Connes and coworkers in a series of articles (Kalau & Walze 1995, Ackermann & Tolksdorf 1996, Chamseddine & Connes 1996) but the definition which these authors propose is valid only on the noncommutative generalizations of compact spaces with euclidean-signature metrics. Cyclic homology groups have been proposed (Connes, 1986) as the appropriate generalization to noncommutative geometry of topological invariants; the appropriate definition of other, non-topological, invariants in not clear.

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