Gravitational Wave Memory from the Relativistic Jet of Gamma-Ray Bursts

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(Received ; accepted )

Abstract

The gravitational wave (GW) memory from a radiating and decelerating point mass is studied in detail. It is found that for isotropic photon emission the memory generated from the photons is essentially the same with the memory from the point mass that radiated the photons so that it is anti-beamed. On the other hand, for anisotropic emission the memory from the photons may have a non-vanishing amplitude even if it is seen with small viewing angles. In the decelerating phases of gamma-ray burst (GRB) jets the kinetic energy of the jet is converted into the energy of gamma-ray photons. Then it would be possible to observe a variation in the GW memory associated with GRB jets on the timescale of the gamma-ray emission if the emission is partially anisotropic. Such an anisotropy in the gamma-ray emission has been suggested by the polarizations detected in recent observations of GRBs. The GW memory from GRB jets would provide clues to clarifying the geometry of the jets and the emission mechanism in GRBs. Thus it will be an interesting target for the next generation detectors of the GWs.

Key words: Physical Data and Processes: gravitational waves – stars: winds, outflows

1. Introduction

Gamma-ray bursts (GRBs) are considered to be relativistic jets with Lorentz factors $\gamma \sim 10^2 - 10^3$. The energy radiated in gamma-rays for a single burst amounts to $E_\gamma \sim 10^{50} - 10^{52}$ erg (Frail et al. 2001), which may be supplied by kinetic energy of the relativistic jets (e.g., Piran 1999). The origin of such energetic jets currently debated includes the coalescence of double neutron stars (NSs) in a binary as well as the collapsar, which is the collapse of a rotating massive star ending up with a system composed of a black hole (BH) and an accreting torus (Woosley 1993). It has long been considered that NS binaries and collapsing massive stars are the primary targets for the gravitational wave (GW) detection for missions such as LIGO, TAMA, Virgo, and LISA. In fact, some experiments have already begun to put constraints on the rates of such astrophysical phenomena (Akutsu et al. 2006; Harry et al. 2010). Soon after the discovery of the supernova-GRB association (Galama et al. 1998), a hydrodynamical simulation for the collapsar model was carried out by MacFadyen & Woosley (1999) and later it has been studied intensively by a number of numerical simulations (Nagataki 2009; Sekiguchi & Shibata 2011; Ott et al. 2011 to list only a few of the recent works). It is now established that the collapsar scenario is the most plausible model that we now know for long GRBs.

Then GRBs turn out to be major astrophysical sources of GWs. The mechanism of GW emission from GRBs is quite similar to that of the GWs from supernovae, which is mainly caused by asphericity in the matter motion in a bouncing core of collapsing massive stars and the anisotropic neutrino emission that emerges from the core (Müller & Janka 1997; Dimmelmeier et al. 2007). In the collapsar scenario of GRBs, the neutrinos are emitted from an accretion torus and the total energy of neutrinos are estimated to be $E_\nu \sim 10^{53} - 10^{54}$ erg. Such GW signals have been studied analytically (Hiramatsu et al. 2005; Suwa & Murase 2009) and by special relativistic magnetohydrodynamics simulations (Kotake et al. 2012). The GW from anisotropic neutrino emissions comes up as a burst with memory, that is, a sharp rise followed by a steady value of the metric perturbation (Braginsky & Thorne 1987). In fact the GW memory has been considered for a variety of sources such as relativistic fluids (Segalis & Ori 2001; Sago et al. 2004) and neutrinos (Hiramatsu et al. 2005; Suwa & Murase 2009) in the context of the GW from GRBs. The maximum amplitude of the GW memory produced by a relativistic point source with energy $E$ is given by

$$\Delta h = \frac{G}{c^4} \frac{4E}{d} \sim 1.1 \times 10^{-22} \left( \frac{E}{10^{51} \text{erg}} \right) \left( \frac{d}{1 \text{Mpc}} \right)^{-1}.$$  (1)

where $d$ is the distance to the source. Here we mean by a ‘point source’ a point mass or a single ray of photons/neutrinos, where the point mass is an approximation to treat a fluid element with a non-zero mass density.

Segalis & Ori (2001) studied the GW memory from a relativistic jet of GRBs. They derived the angular dependence of the GW amplitude for the case of a point mass and found that the amplitude becomes quite small in the direction of jet propagation (anti-beaming effect).
Another important finding is that the GW memory is polarized in the direction from the source to the line of sight on the transverse plane (in the TT gauge). Sago et al. (2004) analyzed the GW memory from the accelerating phases of the GRB jets. They find that the finite size of the opening half-angle \( \Delta \) of the jet leads to the reduction of the GW memory for viewing angles \( \theta_v \) smaller than \( \Delta \theta \), which means that the GW memory is hard to detect simultaneously with GRBs. The total energy of neutrinos is two orders of magnitude larger than in the case of relativistic jets of GRBs. However, the anisotropy in the neutrino emission is weak compared to GRB jets, resulting in the same consequence that it is not likely that we could observe the GW memory at the same time with GRB events (Hiramatsu et al. 2005; Suwa & Murase 2009).

The amplitude of the GW memory expected for the relativistic jet of GRBs is much smaller than that expected for neutrinos from GRBs. However, a possible detection of the GWs characteristic for the jets would support the general view that GRBs are associated with relativistic jets and might resolve the issues on the physical mechanism of the gamma-ray emission. Sago et al. (2004) investigated the GW memory from relativistic jets based on the ‘unified model’ of GRBs, in which a bunch of multiple sub-jets launched from a central engine are assumed to be seen as GRBs, X-ray flushes (XRFs), and X-ray rich GRBs, depending on the viewing angle (Yamazaki et al. 2004). They also pointed out that GRB jets may be GW sources in deci herz frequency bands, reflecting the characteristic time scale of the central engine’s activity, and thus will be suitable targets for DECIGO and BBO (Seto, Kawamura, & Nakamura 2001; Sago et al. 2004).

It is thought that GRB phenomena detected by GW memory will be seen from off-axis with \( \theta_v > \gamma^{-1} \) because of the anti-beaming effect. Thus, such phenomena should actually be observed as XRFs rather than GRBs, or completely be missed, in electromagnetic observations. In the GRB phenomena, the relativistic jet will lose its kinetic energy and will be decelerated because of the gamma-ray emission. Then part of the memory carried initially by the jet will be transformed into the memory produced by electromagnetic radiation. Although the anti-beaming effect does not exist for the memory generated by a single ray of photons, photons are emitted toward a small but finite solid angle with a typical opening half-angle of \( \sim \gamma^{-1} \) so that the phase cancellation of the amplitude still occurs. However, if the emission is partially anisotropic, we might be able to see a variation in the GW memory from GRB jets for a moderate range of viewing angles \( \theta_v \sim \gamma^{-1} \sim \Delta \theta \) on the time scale of gamma-ray emission. In fact, the polarizations detected in recent observations of GRBs suggest that the gamma-ray emission is partially anisotropic (Steele et al. 2009; Yonetoku et al. 2011).

In this paper, we will present the results of a detailed study on the change of the GW memory in the decelerating relativistic shocks of GRBs. The paper is organized as follows. In §2 we briefly describe the formulation for calculating the gravitational waveform and focus on the change of GW memory generated by a radiating and decelerating point mass. In §3, we apply the result of §2 to realistic cases corresponding to GRB phenomena. The finiteness of the opening half-angle of the jet is now taken into consideration. Then our analyses will be applied to the internal shock model of GRBs. We also present calculated waveforms for a specific model that has appeared in the previous literature. Finally, the summary and discussion including the detectability are given in §4.

2. Gravitational Wave Memory from a Radiating Point Mass

We assume a flat background space-time with the Minkowski metric \( \eta_{\mu\nu} = (-1, +1, +1, +1) \) and restrict ourselves to dealing with the small perturbation \( h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \) of the metric \( g_{\mu\nu} \). We set \( c = G = 1 \) hereafter. Then the linearized Einstein equation reads

\[
\left( -\frac{\partial^2}{\partial t^2} + \Delta \right) h_{\mu\nu} = -16\pi T_{\mu\nu}
\]

(2)

under the Lorentz gauge condition \( h^\mu\nu = 0 \), where \( h_{\mu\nu} \) is defined as

\[
h_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h_\lambda^\lambda,
\]

(3)

and \( T_{\mu\nu} \) is the energy-momentum tensor. For a point mass \( m \) with a world-line \( x^\mu(\tau) \) parametrized by the proper time \( \tau \), \( T^{\mu\nu}(x) \) is given by

\[
T^{\mu\nu}(x) = \int n^\mu(\tau) u^\nu(\tau) \delta(4)(x - x(\tau)) d\tau.
\]

(4)

where \( u^\mu(\tau) \equiv \frac{dx^\mu}{d\tau} = \gamma(1, \beta n) \) and \( \beta, n \) are the velocity, the unit vector proportional to the velocity of the point mass, respectively.

Following the notation in Segalis & Ori (2001), the retarded solution of equation (2) is written as

\[
h^{\mu\nu}(x) = 4m \frac{u^\mu(\tau_r) u^\nu(\tau_r)}{-u_\lambda(\tau_r) \cdot (x - x(\tau_r)) \lambda},
\]

(5)

where \( \tau_r \) is the retarded time defined by the conditions

\[
(x - x(\tau_r))^\mu \cdot (x - x(\tau_r))_\mu = 0, \quad x^0 - x^0(\tau_r) > 0
\]

The transverse-traceless (TT) part of \( h_{ij} \) defined by \( h_{ij}^{TT} = P_{ik} h_{kj} - \frac{1}{2} P_{ij} h_{kl} P_{kl} \), where \( P_{ij} = \delta_{ij} - n_i n_j \) and \( n_i \) is \( i \)-th component of the unit vector from the source to the observer, has been derived to be

\[
h_+ \equiv h_{22}^{TT} = -h_{yy}^{TT} = \frac{2\gamma m \beta^2}{d} \frac{\sin^2 \theta}{1 - \beta \cos \theta} \cos 2\phi,
\]

\[
h_\times \equiv h_{21}^{TT} = h_{12}^{TT} = \frac{2\gamma m \beta^2}{d} \frac{\sin^2 \theta}{1 - \beta \cos \theta} \sin 2\phi,
\]

(7)

in Segalis & Ori (2001). Here we choose \( z \) axis to be the line of sight. \( d \) is the distance to the source and \( \theta, \phi \) are the polar and azimuthal angle in spherical coordinates, respectively, that indicate the direction of the jet. It should be noted that \( \beta \) and \( \gamma \) are evaluated at the retarded time. It is found that the \( \theta \) dependent part of \( h_+ \) and \( h_\times \), which we denote \( h(\theta) \), has asymptotic forms.
\[
\begin{align*}
\theta(\theta) &= \frac{\sin^2 \theta}{1 - \beta \cos \theta} \simeq \frac{2 \theta^2}{\gamma^2 + \theta^2} \\
&\text{for } \gamma \gg 1 \text{ and } \theta \ll 1 \\
\theta(\theta) &\approx 1 + \cos \theta \\
&\text{for } \theta \text{ large compared to } \gamma^{-1}. 
\end{align*}
\]

This angular dependence shows that the GW memory is diminished in the forward direction within \( \theta < \gamma^{-1} \), which is called as “anti-beaming” (Segalis & Ori 2001; Sago et al. 2004).

Now we turn to the case of photons. Since there is no rest frame for a zero-mass particle, the world-line is a null geodesic, being parametrized as \( x^\mu(x^0) \) by the coordinate time \( x^0 \). Then the energy momentum tensor is expressed similarly as

\[
T^{\mu\nu}(x) = E n^\mu(x^0) n^\nu(x^0) \delta^{(3)}(x - x(x^0))
\]

\[
= \int E n^\nu(t) n^\mu(t) \delta^{(4)}(x - x(t)) dt
\]

where \( x^0(t) = t, E \) is the energy of the photon, \( n^\mu = (1, n) \) is the unit four-vector that is proportional to the photon’s four momentum.

Comparing equations (4) and (10), we notice that the GW memory from photons (equation 11) has the same angular dependence with the memory from a point mass (jet) moves in the \( z' \)-direction and the \( z \)-axis corresponds to the line of sight. The viewing angle is indicated by \( \theta_v \).

coordinates, the following relations hold among \( \theta, \phi \) and \( \theta', \phi' \).

\[
\begin{align*}
\sin \theta \cos \phi &= -\sin \theta_v \cos \theta' + \cos \theta_v \sin \theta' \cos \phi' \\
\sin \theta \sin \phi &= \sin \theta_v \sin \phi' \\
\cos \theta &= \cos \theta_v \cos \phi' + \sin \theta_v \sin \theta' \cos \phi' 
\end{align*}
\]

Then, the GW memory should be averaged over the angular distribution of photons as follows.

\[
\begin{align*}
\theta_v &\equiv h_+(\theta_v) + ih_x(\theta_v) = \frac{E}{d} < 2(1 + \cos \theta) e^{2i\phi} > \\
&= \frac{2E}{d} \int \int \sin \theta' d\theta' d\phi' f(\theta', \phi') (1 + \cos \theta) e^{2i\phi} 
\end{align*}
\]

where \( \theta \) and \( \phi \) are related to \( \theta' \) and \( \phi' \) by equation (12), \( E \) is the total energy of the photons, and \( f(\theta', \phi') \) is the distribution function. The fraction of the energy of photons emitted into a solid angle \( d\Omega' = \sin \theta' d\theta' d\phi' \) is given by \( f(\theta', \phi') d\Omega' \).

Figure 2 shows a schematic view of the cone of emitted photons.

We evaluate the GW memory from photons for several cases of different angular distributions of the photon emission. First we assume an isotropic emission in the rest frame of the point mass for simplicity. Some GRBs have shown relatively high degrees of polarization in their early optical emission (e.g., Steele et al. 2009) and in gamma-ray emission itself (e.g., Yonetoku et al. 2011). Such high degrees of polarizations may suggest the existence of ordered magnetic fields in the relativistic shock. If the gamma-rays are caused by synchrotron radiation
2.2. Case of Maximally Anisotropic Emission

If the photon emission is isotropic in the rest frame of the point mass, the angular distribution function is given by

\[ f(\theta', \phi')d\Omega' = \frac{1}{4\pi} \frac{1}{\gamma^4(1 - \beta \cos \theta')^2} \delta(\theta' - \phi_0)d\Omega', \tag{14} \]

and \( \beta, \gamma \) is the velocity and the Lorentz factor of the point mass, respectively (Rybicki & Lightman 1979). For \( \gamma \gg 1 \) the emission is beamed within small angles of order \( \theta' \sim \gamma^{-1} \) so that the amplitude depends only on \( \gamma \theta_v \).

Figure 3 shows two polarization components \( h_+ \) and \( h_\times \) of the GW memory from photons for isotropic emission generated by a point mass. The normalized amplitude \( h/(E/d) \) is shown as a function of \( \gamma \theta_v \) (solid line). The \( h_\times \) component vanishes owing to the axisymmetry. We also plotted the memory expected from a point mass with the same energy \( \gamma m = E \) (dotted line). Since \( h_+ \) takes almost the same value with the memory from the point mass, it is hard to distinguish them in Figure 3.

2.2. Case of Maximally Anisotropic Emission

Next we consider the second case in which the angular distribution is given by

\[ f(\theta', \phi')d\Omega' = \frac{1}{2} \frac{1}{\gamma^4(1 - \beta \cos \theta')^2} \delta(\theta' - \phi_0)d\Omega', \tag{15} \]

where \( \beta \) and \( \gamma \) is the velocity and the Lorentz factor of the point mass that radiates the photons.

For \( \theta_v = 0 \) the GW memory from photons with the angular distribution of equation (15) can be evaluated analytically as

\[ h(0) = \frac{2E(1 + \beta)}{d} e^{2i\phi_0}, \tag{16} \]

For large \( \gamma \sim 10^2 - 10^3 \) the amplitude of the GW memory takes a maximum value and the polarization is simply determined by the angle \( \phi_0 \).

For \( \theta_v \neq 0 \) we calculated the amplitudes averaged with equation (15) numerically. Figure 4 shows the two components of polarization \( h_+ \) and \( h_\times \) calculated for \( \phi_0 = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \pi \) as a function of \( \gamma \theta_v \). Figure 5 shows the same amplitudes for \( \phi_0 = 0, \frac{\pi}{2}, \frac{3\pi}{2}, \pi \). The amplitudes are normalized by \( E/d \) again. The GW memory for a point mass with \( \gamma m = E \) is also shown with a dotted line for comparison. As \( \theta_v \) increases, \( h_+ \) becomes a dominant component, approaching to the memory for a point mass asymptotically.

2.3. Case of Synchrotron Emission with Ordered Magnetic Fields

Here we consider synchrotron emission from electrons as a more physically-motivated model. We assume that an ordered magnetic field \( \mathbf{B} = (B \cos \phi_B, B \sin \phi_B, 0) \) is present (Figure 6), the emitted photons have a simple power-law energy spectrum \( P_\nu \propto \nu^{-a} \), and that the electrons have an isotropic pitch angle distribution (Granot & Königl 2003). The photons emitted in the direction \( \mathbf{n} = (\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta') \) have a pitch angle \( \zeta \) such that \( \cos \zeta = \frac{\mathbf{B} \cdot \mathbf{n}_e}{B} \), where \( \mathbf{n}_e \) represents the direction of the photons in the comoving frame,

\[ \mathbf{n}_e = (\sin \theta_e \cos \phi', \sin \theta_e \sin \phi', \cos \theta_e), \tag{17} \]
Fig. 4. GW memory from photons with anisotropic angular distribution of equation (15) normalized by $E/d$ (case of maximally anisotropic emission). Two polarization components $h_+$ and $h_\times$ (solid lines) are shown for $\phi_0 = 0$ (top left), $\frac{\pi}{8}$ (top right), $\frac{\pi}{4}$ (bottom left), and $\frac{3}{8}\pi$ (bottom right) as a function of $\gamma \theta_\nu$. The amplitude from a point mass with the same energy $\gamma m = E$ is shown for comparison (dotted line).
Fig. 5. The same with Fig. 4 but for $\phi_0 = \frac{\pi}{2}$ (top left), $\frac{5}{6}\pi$ (top right), $\frac{3}{4}\pi$ (bottom left), and $\frac{7}{8}\pi$ (bottom right). GW memory from photons with anisotropic angular distribution of equation (15) normalized by $E/d$ (case of maximally anisotropic emission). The amplitude from a point mass with the same energy $\gamma m = E$ is shown for comparison (dotted line).
and \( \theta_c \) is given by \( \cos \theta_c = \frac{\cos \theta' - \beta}{1 - \beta \cos \theta'} \), which reduces to \( \cos \theta_c \approx \frac{1 - (\gamma \theta')^2}{1 + (\gamma \theta')^2} \) for \( \gamma > 1 \) and \( \theta' \ll 1 \). We note that the magnetic field in the comoving frame is given by \( \gamma B \) as a result of Lorentz transformation. A simple calculation yields

\[
\cos \xi = \sin \theta_c \cos \phi' \cos \phi_B + \sin \theta_c \sin \phi' \sin \phi_B \\
= \sin \theta_c \cos (\phi' - \phi_B)
\]

(18)

Since the synchrotron power is proportional to \((\sin \xi)^{a+1}\) (Rybicki & Lightman 1979), the angular distribution is given by

\[
f(\theta', \phi')d\Omega' = K(\sin \xi)^{a+1} \frac{1}{\gamma^4(1 - \beta \cos \theta')} \sin \theta_c d\Omega'
\]

(19)

where \( K \) is a normalization constant. We choose \( a = 1 \) from a range of values considered in the literature (Granot & Königl 2003; Lazzati 2006; Toma et al. 2009). For \( a = 1 \) the distribution function has a simple form and \( K = \frac{3}{8\pi} \). The GW memory depends on \( \phi_B \) as well as \( \theta_c \), so that we denote the amplitude as \( h(\theta_c, \phi_B) \) in this case. For \( \theta_c = 0 \) the integration in equation (13) can be done analytically, where we find

\[
h(0, \phi_B) = -\frac{1}{4} \cdot \frac{2E(1 + \beta)}{d} e^{2i\phi_B}.
\]

(20)

We see that \( \phi_0 \) dependence of \( h(0) \) in equation (16) is reproduced by replacing \( \phi_B \) with \( \phi_0 - \frac{\pi}{2} \) in equation (20).

For \( \theta_c \neq 0 \) we calculated the amplitudes averaged with equation (19) and \( a = 1 \) numerically.

Figure 7 shows the normalized amplitudes \( h(\theta_c, \phi_B) = h_+(\theta_c, \phi_B) + ih_\times(\theta_c, \phi_B) \) calculated for \( \phi_B = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \) as a function of \( \gamma \theta_c \). Figure 8 shows the same for \( \phi_B = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \). The GW memory for a point mass with \( \gamma m = E \) is also shown with a dotted line. The overall behavior is quite similar to the case of maximally anisotropic emission except that the amplitude for \( \theta_c = 0 \) is reduced to one fourth of the maximum amplitude.

Now we consider the variation in the GW memory as a result of photon emission from a point mass. If a point mass \( m \) with a Lorentz factor \( \gamma \) (velocity \( \beta \)) radiates photons with power \( P \) during a short time interval \( \Delta t \), the change in the total GW memory \( \Delta h = \Delta h_+ + i\Delta h_\times \) is given by

\[
\Delta h = \frac{2\gamma' m}{d} \frac{\beta^2 \sin^2 \theta_v}{1 - \beta \cos \theta_v} + \frac{P \Delta t}{d} < 2(1 + \cos \theta)e^{2i\phi} > \]

\[
\left( -\frac{2\gamma m}{d} \frac{\beta^2 \sin^2 \theta_v}{1 - \beta \cos \theta_v} \right)\right)
\]

(21)

where \( \gamma'(\beta') \) is the Lorentz factor (velocity) of the point mass after time \( \Delta t \). Since the conservation of energy implies \( \gamma' = \gamma - P \Delta t \), we find that the following relation holds to first order in \( \Delta t \),

\[
\Delta h = \frac{2\gamma m}{d} \frac{\beta^2 \sin^2 \theta_v}{1 - \beta \cos \theta_v} + \frac{P \Delta t}{d} \left( 2(1 + \cos \theta)e^{2i\phi} - \frac{2\beta^2 \sin^2 \theta_v}{1 - \beta \cos \theta_v} \right)
\]

\[
\left( -\frac{2\gamma m}{d} \frac{\beta^2 \sin^2 \theta_v}{1 - \beta \cos \theta_v} \right)\right)
\]

(22)

where we assume \( \gamma, \gamma' \gg 1 \) and retain the terms in leading orders of \( \gamma^{-2} \). In equation (22) the first term is the variation in the memory of the point mass due to broadening of the hole of anti-beaming, while the second term is the change in the memory ascribed to the emission of photons. The size of the terms in the parenthesis of equation (22) can be easily read from figures 3, 4, 5, 7, and 8 as the difference between \( h_+ \) (solid line) and the memory from a point mass (dotted line) as well as the value of \( h_\times \). For the case of isotropic emission, the second term is negligibly small compared to the first term (Figure 3). Thus the memory decreases monotonically as the point mass slows down, even if the contribution from photons is added. However, for the case of anisotropic emission above considered (equation 15 and 19), the second term in equation (22) makes a dominant contribution to the change in the memory compared to the first term for small \( \gamma \theta_v \). For large \( \gamma \theta_v \), the first term in equation (22) is smaller than the second term in the parenthesis by a factor of \( \gamma^2 \) and thus it is negligible again. As a consequence, for \( \gamma \theta_v < 1 \) and even for \( \theta_v = 0 \) the GW memory varies by a significant fraction of its maximum value on the conversion of energy from a point mass into photons.
Fig. 7. GW memory from photons with anisotropic angular distribution of equation (19) normalized by $E/d$ (case of synchrotron emission with ordered magnetic fields). Two polarization components $h_+$ and $h_x$ (solid lines) are shown for $\phi_0 = 0$ (top left), $\frac{\pi}{8}$ (top right), $\frac{\pi}{4}$ (bottom left), and $\frac{3}{8}\pi$ (bottom right) as a function of $\gamma \theta_v$. The memory from a point mass with the same energy $\gamma m = E$ is shown for comparison (dotted line).
Fig. 8. The same with Fig. 7 but for $\phi_B = \frac{\pi}{2}$ (top left), $\frac{5}{8}\pi$ (top right), $\frac{3}{4}\pi$ (bottom left), and $\frac{7}{8}\pi$ (bottom right). GW memory from photons with anisotropic angular distribution of equation (19) normalized by $E/d$ (case of synchrotron emission with ordered magnetic fields). The amplitude from a point mass with the same energy $\gamma m = E$ is shown for comparison (dotted line).
3. Gravitational Wave Memory from the Decelerating Phase of GRB Jets

Here we estimate the overall behavior of time variation in the GW memory from GRB jets by applying our results in §2 to specific models of GRB phenomena. It has been shown that the temporal structure of GRBs are naturally explained by the internal shock model, in which gamma-rays are radiated in an inhomogeneous relativistic wind possibly generated by a variable central engine (Piran 1999). Such a wind has been modeled by multiple shells moving in procession with various Lorentz factors. It is assumed that if a rapid shell catches up a slower shell ahead the two shells would collide and merge converting a fraction of kinetic energy into thermal energy. The thermal energy released will be radiated on the spot. We follow the formulation by Kobayashi et al. (1997) to model the internal shock. Here we use the internal shock model only to reproduce the time variability of typical GRB light curves.

We assume that a rapid shell with mass \( m_r \) and Lorentz factor \( \gamma_r \) collides with a slow shell with mass \( m_s \) and Lorentz factor \( \gamma_s \) to form a merged shell with mass \( m_r + m_s \) and Lorentz factor \( \gamma_m \). The conservation of energy and momentum leads to

\[
m_r \gamma_r + m_s \gamma_s = (m_r + m_s + \epsilon) \gamma_m,
\]

\[
m_r \sqrt{\gamma_r^2 - 1} + m_s \sqrt{\gamma_s^2 - 1} = (m_r + m_s + \epsilon) \sqrt{\gamma_m^2 - 1},
\]

where \( \epsilon \) is the internal energy being released in the rest frame of the merged shell. For large \( \gamma_r, \gamma_s \gg 1 \), \( \gamma_m \) is approximately given by

\[
\gamma_m \simeq \sqrt{\frac{m_r \gamma_r + m_s \gamma_s}{m_r \gamma_r + m_s \gamma_s}},
\]

and then the energy converted into radiation \( E \) is estimated to be

\[
E = \gamma_m \epsilon = m_r (\gamma_r - \gamma_m) + m_s (\gamma_s - \gamma_m).
\]

At the time of collision the forward and reverse shocks arise. The Lorentz factors of the forward and reverse shocks, \( \gamma_{fs} \) and \( \gamma_{rs} \), are given by

\[
\gamma_{fs} \simeq \gamma_m \sqrt{1 + \frac{2 \gamma_m}{\gamma_r}} / \left( 2 + \frac{\gamma_m}{\gamma_r} \right),
\]

\[
\gamma_{rs} \simeq \gamma_m \sqrt{1 + \frac{2 \gamma_m}{\gamma_s}} / \left( 2 + \frac{\gamma_m}{\gamma_s} \right),
\]

respectively (Sari & Piran 1995). The time in which gamma-rays are emitted is approximately given by the time in which the reverse shock traverses the rapid shell,

\[
\Delta t = \frac{l_r}{\beta_r - \beta_{rs}},
\]

where \( l_r \) is the width of the rapid shell and the \( \beta_{rs} \) is the velocity of the reverse shock. We note that the emitting region moves at a speed of \( \beta_m \) and that the observed timescale is reduced by a factor of \( 1 - \beta_m \cos \theta_v \), which is \(~1/2\gamma_m^2\) for \( \theta_v = 0 \).

The width of the merged shell \( l_m \) is simply given by

\[
l_m = l_r \frac{\beta_{fs} - \beta_m}{\beta_{fs} - \beta_s} + l_s \frac{\beta_m - \beta_{rs}}{\beta_r - \beta_{rs}}.
\]

In calculating the GW memory, we assume that for each collision the energy of photons \( E \) (equation 25) is radiated in time \( \Delta t \) (equation 26) so that the emission power is given by \( P = E/\Delta t \). Applying equation (21) in the last section, the change in the GW memory for a single collision is evaluated as

\[
\Delta h(\theta_v, \phi_B) = \Delta h_{\text{photons}}(\theta_v, \phi_B) + \Delta h_{\text{jet}}(\theta_v),
\]

where

\[
\Delta h_{\text{photons}}(\theta_v, \phi_B) = m_r (\gamma_r - \gamma_m) + m_s (\gamma_s - \gamma_m) < 2(1 + \cos \theta_v) e^{2i\phi} >
\]

for photons’ contribution and

\[
\Delta h_{\text{jet}}(\theta_v) = \frac{2 m_r \sin^2 \theta_v}{d} \left( \frac{\gamma_m \beta_m^2}{1 - \beta_m \cos \theta_v} - \frac{\gamma_r \beta_r^2}{1 - \beta_r \cos \theta_v} \right)
\]

\[
+ \frac{2 m_s \sin^2 \theta_v}{d} \left( \frac{\gamma_m \beta_m^2}{1 - \beta_m \cos \theta_v} - \frac{\gamma_s \beta_s^2}{1 - \beta_s \cos \theta_v} \right)
\]

for the fluid contribution in the shells. We assume that this change in the memory occurs during time \( \Delta t \) at a constant rate. We study the case of synchrotron emission with ordered magnetic fields for \( \Delta h_{\text{photons}} \). We always have a real and negative \( \Delta h_{\text{jet}} \), while \( \Delta h_{\text{photons}} \) may be complex reflecting the non-axisymmetric distribution of the photon emission.

Now we describe the method for simulating the collision of the multiple shells in the internal shock. We take \( N \) shells with the same mass \( m \) and width \( l \) (i.e., the same density) being placed with an equal separation \( L \) at time \( t = 0 \). To each shell a Lorentz factor, uniformly distributed between \( \gamma_{\text{min}} \) and \( \gamma_{\text{max}} \), is assigned at the outset. Given the Lorentz factor \( \gamma_i \), the velocity \( \beta_i \), and the position \( x_i = -(L + l) i \) for \( i \)-th shell \( (i = 1, \cdots, N) \) as the initial conditions, we follow the time evolution of the position of all the shells until the first collision occurs. At the collision two shells are merged and again the movement of the shells is followed till the next collision. The same procedure is repeated until either all shells will be merged into a single large shell or we get the configuration where the shells are marching with decreasing velocities from the head to the end. We used parameters \( N = 100, \gamma_{\text{min}} = 10^2, \gamma_{\text{max}} = 10^3, \) and \( L = 1, l = 0.1 \). The overall evolution of the GW memory with time and the gamma-ray light curve are determined by the ratio \( L/l, N, \gamma_{\text{min}}, \gamma_{\text{max}} \) and \( \theta_v \).

Since the GW memory is polarized in the direction from the source to the line of sight on the transverse plane, we...
need to account for the finiteness of the conical opening half-angle of the GRB jets to estimate the net memory. The typical size of the opening half-angle has been estimated to be $\Delta \theta \sim 0.1$ rad, which is much larger than the beaming angle $\gamma^{-1} < 0.01$ (Frail et al. 2001). If we consider the memory from an axisymmetric jet that is seen nearly head-on, not only the memory from the central part of the jet vanishes but also the memory added up from the edge parts turns out to be canceled because of the symmetric distribution of polarization angles. Then the GW memory from the acceleration phase of the GRB jets can be observed only if the jet is seen off-axis $\theta_v > \Delta \theta$ (Sago et al. 2004). We note that this conclusion depends on the geometry of the jet. If the jet is uniform within its opening angles, the GW memory from the jet still has about half of its maximum values at $\theta_v = \Delta \theta$.

Similar effects of the finite opening half-angle exist for the change in the GW memory from the decelerating phase of the GRB jets. For isotropic gamma-ray emission in the rest frame of the jet, the GW memory from photons is found to be nearly equal with that from the point mass that emits the photons (§2). Thus, the memory decreases monotonically owing to the decline in the Lorentz factor and the resultant expansion of the anti-beaming hole. In this case the GW memory might be observable only if $\theta_v > \Delta \theta$ as in the case of the acceleration phase.

Recent successes in the detection of the polarization in the prompt emission of GRBs (Yonetoku et al. 2011) suggests the presence of ordered magnetic fields and/or an anisotropic photosphere in the emission region. The coherence length scale of the magnetic fields is not well understood for any GRBs and the origin of such ordered magnetic fields is still a debated issue. A variety of different magnetic field configurations and viewing angles have been considered to explain a large degree of polarization (Granot & Königl 2003; Lazzati 2006; Toma et al. 2009).

If the angular scale of coherent magnetic field is small compared to the beaming angle $\gamma^{-1}$ or the magnetic fields are randomly oriented, there is no preferred azimuthal angles into which the gamma-ray emission is focused so that the emission ought to be essentially isotropic. Then the change in the GW memory on each collision should be quite small so that we would not observe a variation in the memory from the decelerating phases of GRB jets.

Instead if the shell contains a number of patches of locally ordered magnetic fields we would have a non-vanishing variation in the memory for some viewing angles. The case in which we would expect the largest change in GW memory is when an ordered transverse magnetic field exists in the relativistic shell. We consider a uniform jet with an opening half angle $\Delta \theta$ illustrated in Figure 9. The line of sight (z axis) is located at a point tilted from $z'$ axis by an angle $\theta_v$ toward $x'$ direction. We assume that the magnetic field has only a transverse component and that the field line is parallel to the meridional plane at $\phi' = \alpha$. Then the unit vector $\hat{B} = B/|B|$ at a point $(\theta', \phi')$ is expressed as

$$B = \frac{1}{\sqrt{\sin^2 \theta' \cos^2 (\phi' - \alpha) + \cos^2 \theta'}} \left( \cos \theta' \cos \alpha, \cos \theta' \sin \alpha, -\sin \theta' \cos (\phi' - \alpha) \right).$$

(31)

We study the case of $\alpha = \pi/2$, in which the highest degree of anisotropy is expected for GW memory. In order to use $\Delta h(\xi, \phi_B)$ calculated for the synchrotron model (Equation 28), we define $\phi_B$ such that $\cos \phi_B = \hat{B} \cdot \hat{t}$ where $\hat{t}$ is a unit vector that is tangent to the great circle connecting the points $(\theta', \phi')$ and $(\theta_v, 0)$ at $(\theta', \phi')$,

$$\hat{t} = \frac{\hat{p} - \hat{q} \cos \xi}{\sin \xi},$$

(32)

where

$$\hat{p} = (\sin \theta_v, 0, \cos \theta_v),$$

$$\hat{q} = (\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta'),$$

(33)

and $\cos \xi = \hat{p} \cdot \hat{q}$. The angle $\xi$ is equal to $\theta$, the polar angle in $xyz$ frame. The GW memory is calculated by summing up additive contributions to the amplitude from all points in the shell as follows.

$$h = \int \frac{\sin \theta' d\theta' d\phi'}{\Delta \Omega} \Delta h(\xi, \phi_B) e^{2i\phi},$$

(34)

where $\Delta \Omega = 2\pi(1 - \cos \Delta \theta)$ and the integration is performed over the angles inside the jet. We note that equation (12) relates $(\theta, \phi)$ to $(\theta', \phi')$.

Figure 10 shows the GW memory $h = \sqrt{|h_+|^2 + |h_x|^2}$ calculated for the case of synchrotron emission with ordered magnetic fields with $\Delta \theta = 0.1$ and $\alpha = \pi/2$. We show two plots for cases with different viewing angles,
The amplitudes of GW memory $h = \sqrt{|h_\times|^2 + |h_\times|^2}$ normalized by $E_\gamma/d$, where $E_\gamma$ is the total energy radiated in gamma-rays, are shown for the case of synchrotron emission with ordered magnetic fields with $\Delta \theta = 0.1$ and $\alpha = \pi/2$. The two panels are GW memory for different viewing angles $\theta_v = 0$ (left) and $\theta_v = 0.001$ (right). The time $t$ is arbitrarily scaled. The gamma-ray pulses, also plotted with an arbitrary scale, correspond to the collision events, which are modeled by internal shocks with $N = 100$ relativistic shells of Lorentz factors $\gamma = 10^2 - 10^3$.

$\theta_v = 0$ (left) and $\theta_v = 0.001$ (right). The time is arbitrarily scaled. The time which passed till the memory reaches the plateau corresponds to the duration of the gamma-ray emission of the GRB. The amplitude is normalized by $E_\gamma/d$, where $E_\gamma$ is the total energy radiated in gammarays. The value of $h_{\text{jet}}$ is shifted so that the final level of $h_{\text{jet}}$ turns out to be zero. The gamma-ray pulses, also plotted with an arbitrary scale, correspond to the collision events. We see that the memory from photons emitted at each collision is added up, leading to a monotonic rise in net memory with time. For $\theta_v = 0$ the memory from the jet is suppressed and unobservable throughout. We ignored the broadening of the gamma-ray light curves that should appear because of the finite opening angle of the jet. The broadening of the GW memory at each collision should also be smeared. However, even if such smearing is taken into account, the waveform, which is made from cumulative contributions, would not change its shape so much.

The behavior of gravitational waveforms shown in Figure 10 can be understood as follows. As we changed contributions, would not change its shape so much. As is found in §2, $\Delta h(\theta_v, \phi_B)$ takes a non-zero value of order $E/d$ only for $\gamma \theta_v$ less than $\sim 1$ (Figures 7 and 8). Thus, we use equation (35) in (34) and change variables of integration into $\theta_v, \phi$ to obtain a rough estimate

$$h \sim \frac{E_\gamma}{d} (\gamma \Delta \theta)^{-2},$$

where $E_\gamma$ is the energy radiated as photons. This is valid both for a single collision and for the sum of radiated energy for all the collisions. For $\Delta \theta = 0.1$ and $\gamma \sim 3 \times 10^2$ we have $h \sim 10^{-3} E_\gamma/d$. This is a change of the GW memory. The initial amplitude of the GW memory is carried by the uniform jet and its approximate value is given by

$$h_0 \sim \frac{4 E_{\text{jet}}}{d} \frac{\theta_v^2}{\Delta \theta^2 + \theta_v^2},$$

where $E_{\text{jet}}$ is the kinetic energy of the relativistic jet. We have confirmed that this approximate expression agrees with the exact value (Sago et al. 2004; Hiramatsu et al. 2005) to an accuracy of 10 percent for $\gamma = 10^2 - 10^3$ and a wide range of $\Delta \theta$. If we assume $E_{\text{jet}} = E_\gamma$, we obtain $h_0 \sim 4 \times 10^{-3} E_\gamma/d$ for $\Delta \theta = 0.1$ and $\theta_v = 0.001$, being in a qualitatively good agreement with the results shown in Figure 10.

As is seen in Figures 7 and 8, the change in the GW memory is seen only within small angles of $\gamma^{-1}$ from the line of sight. As long as the small solid angles around the line of sight (a shaded circle in Figure 9) is included in the jet with an opening half angle $\Delta \theta$, which is equivalent to $\theta_v < \Delta \theta - 1/\gamma$, we would observe almost the same variation in the GW memory regardless of the position of the line of sight. This implies that an ordered magnetic field is required only within a small solid angle from which gamma-rays we observe are radiated.
The GW memory would also provide clues to discriminating different models of GRBs and related GRB phenomena. Based on the unified model of GRBs the relativistic jet is composed of a number of sub-jets launched from a central engine (Yamazaki et al. 2004). In this case each sub-jet would be threaded with ordered magnetic fields that might be oriented in various directions. Then the polarization angles of the GW memory from each sub-jet may be different, which leads to a sharp fluctuation of the memory with a relatively small amplitude. The unified model also predicts that relativistic jets seen with large viewing angles are observed as XRFs. For such cases with moderate viewing angles the GW memory should vary owing to the broadening of the anti-beaming hole. Whether the memory has such features in its time variation could help us to test GRB models.

The GW memory from decelerating GRB jets is expected to have characteristic frequencies $0.1 - 10$ Hz, corresponding to the duration of GRBs, $T \sim 0.1 - 10$ sec, so that it is a suitable target for DECIGO and BBO (Seto, Kawamura, & Nakamura 2001; Sago et al. 2004). Figure 11 shows the characteristic amplitude $h_c(f) = f \langle \hat{h}(f) \rangle$, where $2 \pi i f h_c(f) = \int_{-\infty}^{\infty} \hat{h}(t) e^{-2 \pi i f t} dt$, from the decelerating phase of a GRB jet with $E_\gamma = 10^{51}$ erg and a duration of 1 sec. The noise amplitude $h_n(f) = [5S_b(f)]^{1/2}$, where $S_b(f)$ is the spectral density of the strain noise in the detector at frequency $f$, is also shown for Advanced LIGO, LISA, and DECIGO/BBO. We use the same formula for $h_n(f)$ or $S_b(f)$ with the one used in Sago et al. (2004) and Suwa & Murase (2009). The flat spectrum at around $f = 1 \times 10^3$ Hz is a direct consequence of the time variability of the gamma-ray emission. A GRB with $E_\gamma = 10^{52} - 10^{53}$ erg and $\Delta \theta = 0.1$ at $d = 10$ kpc (Galactic Center) is likely to provide an amplitude at a level close to the sensitivity of Advanced LIGO detector at around 100 Hz (Flanagan & Hughes 1998). Thus the GW from GRB jets may be an interesting target for Einstein Telescope (ET) (ET-webpage) as well as for Advanced LIGO.

Unfortunately, the local GRB event rate has been estimated to be relatively small as $\sim 1$ Gpc$^{-3}$ yr$^{-1}$ (Matsubayashi et al. 2005; Wanderman & Piran 2010). This translates into an event rate $\sim 10^{-9}$ yr$^{-1}$ (within $d = 1$ Mpc) or $\sim 10^{-6}$ yr$^{-1}$ (within $d = 10$ Mpc) as a rate of GRBs with detectable GW memory. The possibility of detecting the GW memory from on-axis viewing angles, that is, almost simultaneously with gamma-rays from GRBs, makes it easier to extract the memory component embedded from the detector’s signals. Thus the GW memory from GRB jets will be one of the interesting targets for the next generation detectors such as KAGRA (Somiya et al. 2012), DECIGO/BBO, and ET.

The authors would like to thank the anonymous referee for valuable comments and suggestions, which greatly improved the manuscript. K.I. is grateful to Drs. Takashi Nakamura, Kunihito Ioka, and Kenji Toma for stimulating discussion.
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