ON THE INVERTIBILITY OF THE SUM OF OPERATORS

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Abstract. The primary purpose of this paper is to investigate the question of invertibility of the sum of operators. The setting is bounded and unbounded linear operators. Some interesting examples and consequences are given. As an illustrative point, we characterize invertibility for the class of normal operators. Also, we give a very short proof of the self-adjointness of a normal operator when the latter has a real spectrum.

1. Introduction

Let $H$ be a complex Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on $H$. An $A \in B(H)$ is called positive (symbolically $A \geq 0$) if

$$\langle Ax, x \rangle \geq 0 \quad \forall x \in H.$$ 

By a square root of $A \in B(H)$, we mean a $B \in B(H)$ such that $B^2 = A$. If $A \geq 0$, then there is one and only one $B \geq 0$ such that $B^2 = A$. This positive $B$ is denoted by $\sqrt{A}$.

Recall that any $T \in B(H)$ is expressible as $T = A + iB$ where $A, B \in B(H)$ are self-adjoint. Besides,

$$A = \text{Re}T = \frac{T + T^*}{2} \quad \text{and} \quad B = \text{Im}T = \frac{T - T^*}{2i}.$$ 

It is readily verifiable that $T$ is normal iff $AB = BA$.

We also recall some known results which will be called on below (these are standard facts, see [8] for proofs).

Theorem 1.1. Let $A, B \in B(H)$. Then

$$0 \leq A \leq B \iff \sqrt{A} \leq \sqrt{B}.$$ 

Lemma 1.2. Let $A, B \in B(H)$ be such that $AB = BA$ and $A, B \geq 0$. Then

$$\sqrt{A + B} \leq \sqrt{A} + \sqrt{B}.$$ 

Theorem 1.1 is known to hold for $\alpha \in (0, 1)$ instead of $\frac{1}{2}$ (the Heinz Inequality). Hence we may easily establish the analogue of Lemma 1.2 for $n$th roots.

Lemma 1.3. Let $A, B \in B(H)$ be such that $AB = BA$ and $A, B \geq 0$. If $n \in \mathbb{N}$, then $(A + B)^{\frac{1}{n}} \leq A^{\frac{1}{n}} + B^{\frac{1}{n}}$.

Let us say a few more words about the absolute value of an operator (that is, $|A| = \sqrt{A^*A}$ with $A \in B(H)$). It is well known that the properties of the absolute value of complex numbers cannot all just be carried over to $B(H)$ (even for self-adjoint operators). This applies for example to the multiplicativity property and to

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triangle inequalities. For counterexamples, readers may wish to consult [5]. See also [9] to see when these results hold. The similar question on unbounded operators may be found in the recent work [2]. Some results, however, do hold without any special assumption. One of them is the following simple result.

**Proposition 1.4.** Let $A, B \in B(H)$. Then

$$|A + B|^2 \leq 2|A|^2 + 2|B|^2.$$  

The following known result is also primordial.

**Proposition 1.5.** (see [5] for a new proof) Let $A, B \in B(H)$. If $0 \leq A \leq B$ and $A$ is invertible, then $B$ is invertible and $B^{-1} \leq A^{-1}$.

We digress a little bit to notice a simple proof of the positiveness of the spectrum of a positive operator using the previous proposition: If $\lambda < 0$, then $-\lambda I > 0$ and so $A - \lambda I \geq -\lambda I$ because $A \geq 0$. Hence $A - \lambda I$ is invertible as $-\lambda I$ is, i.e. $\lambda \notin \sigma(A)$.

A simple application of the Functional Calculus for self-adjoint operators is as follows.

**Example 1.6.** Let $A \in B(H)$ be such that $0 \leq A \leq I$. If $\alpha \in [0, 1]$, then $A^\alpha \geq A$.

From [6] we recall the following result.

**Proposition 1.7.** Let $A, B \in B(H)$ be commuting. Then

$$\sigma(A + B) \subset \sigma(A) + \sigma(B)$$

where

$$\sigma(A) + \sigma(B) = \{\lambda + \mu : \lambda \in \sigma(A), \mu \in \sigma(B)\}.$$  

We call the result in the previous proposition the "subadditivity of the spectrum". There is also a "submultiplicativity of the spectrum", that is,

**Proposition 1.8.** (11) Let $A, B \in B(H)$ be commuting. Then

$$\sigma(AB) \subset \sigma(A)\sigma(B)$$

where

$$\sigma(A)\sigma(B) = \{\lambda\mu : \lambda \in \sigma(A), \mu \in \sigma(B)\}.$$  

In Proposition 2.20 we show that Proposition 1.8 implies Proposition 1.7 in the context of self-adjoint operators and that the backward implication also holds but for positive and invertible operators.

Recall also the following definition.

**Definition.** Let $T$ and $S$ be unbounded positive self-adjoint operators. We say that $S \geq T$ if $D(S^{1/2}) \subseteq D(T^{1/2})$ and $\|S^{1/2}x\| \geq \|T^{1/2}x\|$ for all $x \in D(S^{1/2})$.

The "natural but weak extension" is defined as in Definition 10.5 (Page 230) in [11]: If $S$ and $T$ are non-necessarily bounded symmetric operators, then $S \geq T$ if $D(S) \subseteq D(T)$ and

$$\langle Sx, x \rangle \geq \langle Tx, x \rangle \quad \forall x \in D(S).$$

Notice that Proposition 1.5 remains valid for unbounded operators. Indeed, as on Page 200 in [12], if $S$ and $T$ are self-adjoint, $T$ is boundedly invertible and $S \geq T \geq 0$, then $S$ is boundedly invertible and $S^{-1} \leq T^{-1}$.

Finally, we assume that readers are familiar with other basic notions and results of Operator Theory.
The sum of two invertible operators is not necessarily invertible even if strong conditions are imposed. For instance, if we take $A$ to be invertible and positive, then setting $B = -A$, we see that $AB = BA$ and that $B$ is invertible. But plainly $A + B$ is not invertible. Positivity must also be avoided as it may make some of the results evident. For instance, if $A, B \in B(H)$ are such that $A, B \geq 0$ and $A$ say is invertible, then obviously $A + B \geq (A)$ is invertible by Proposition 1.5. These two observations make the investigation of this question a little hopeless. However, the approach considered by Bikchentaev in [1] deserves to be investigated further. This is one aim of the paper. Another purpose is to treat some of these questions in an unbounded setting. Some interesting consequences, examples and counterexamples accompany our results.

2. Main Results

**Theorem 2.1.** Let $A, B \in B(H)$.

1. If $A + B$ is invertible, then so is $|A|^2 + |B|^2$.
2. If $A + B$ is invertible, then so are $|A| + |B|$ and $|A|^{2n} + |B|^{2n}$ ($n \in \mathbb{N}$) as well.
3. Assume here that $A, B \geq 0$ and let $\alpha, \beta \in \mathbb{C}$. Then $\alpha A + \beta B$ invertible $\implies A + B$ invertible.

**Remark.** Most of the previous results appeared in [1], but our proof is simpler.

**Proof.** (1): It is clear that $A + B$ invertible $\implies |A + B|^2$ invertible $\implies |A|^2 + |B|^2$ invertible by Propositions 1.4 & 1.5.

(2): By the first property, $|A|^2 + |B|^2$ is invertible from which we readily get that $|A|^4 + |B|^4$ is invertible and, by induction, we establish the invertibility of $|A|^{2n} + |B|^{2n}$.

The invertibility of $|A| + |B|$ is not hard to prove. WLOG we may assume that $||A|| \leq 1$ and $||B|| \leq 1$. Hence $A^*A \leq I$ and $B^*B \leq I$, and so $|A| \leq I$ and $|B| \leq I$ by Theorem 1.1. By the Functional Calculus, $|A|^2 \leq |A|$ and $|B|^2 \leq |B|$. This implies that $|A| + |B| \geq |A|^2 + |B|^2 \geq 0$.

Proposition 1.5 allows us to confirm the invertibility of $|A| + |B|$, as desired.

(3): Since $\alpha A + \beta B$ is invertible, by the previous property, so is $|\alpha||A| + |\beta||B|$ or merely $|\alpha|A + |\beta|B$ as $A, B \geq 0$. Since we can assume $|\alpha| \geq |\beta| > 0$, we infer that $|\alpha|(A + B) \geq |\alpha|A + |\beta|B$.

Consequently, $|\alpha|(A + B)$ or simply $A + B$ is invertible. \hfill $\Box$

**Corollary 2.2.** Let $A \in B(H)$ be invertible. Then $|A - B| + |B|$ is invertible for every $B \in B(H)$.

**Proof.** Since $A$ is invertible and $A = (A - B) + B$, it follows that $|A - B| + |B|$ too is invertible by the previous result. \hfill $\Box$

**Remark.** It is clear that if $T$ is invertible, then the self-adjoint $\text{Re}T + \text{Im}T$ need not be invertible. For instance:
Example 2.3. Let $A$ be self-adjoint and invertible and set $B = -A$. Then

$$A + iB = A - iA = (1 - i)A$$

is invertible while $A + B = 0$ is not.

Nonetheless, we have the following.

Corollary 2.4. Let $A \in B(H)$ be invertible. Then $|\text{Re} A| + |\text{Im} A|$ is invertible.

Proof. Just write $A = \text{Re} A + i\text{Im} A$, then apply Theorem 2.1. □

The next corollary appeared in [1].

Corollary 2.5. Let $A, B \in B(H)$ be such that $A + B$ is invertible. If $p, q \in (0, \infty)$, then $|A|^p + |B|^q$ is invertible.

Proof. WLOG, we assume that $|A| \leq I$ and $|B| \leq I$. We can always find an $n \in \mathbb{N}$ such that $p, q \leq 2^n$. Then by Example 1.6, we have $|A|^\frac{p}{2^n} \geq |A|$ and $|B|^\frac{q}{2^n} \geq |B|$.

Therefore,

$$|A|^p \geq |A|^{2^n} \quad \text{and} \quad |B|^q \geq |B|^{2^n}.$$ 

Thus,

$$|A|^p + |B|^q \geq |A|^{2^n} + |B|^{2^n}.$$ 

Since $|A|^{2^n} + |B|^{2^n}$ is already invertible (Theorem 2.1), we obtain the invertibility of $|A|^p + |B|^q$ from Proposition 1.5. □

Proposition 2.6. Let $A, B \in B(H)$ be such that $AB = BA$ and either $A$ or $B$ is normal. Then $|A| + |B|$ is invertible $\iff |A|^2 + |B|^2$ is invertible.

Proof. We already proved the implication "$\Rightarrow"$ in Theorem 2.1

Assume now that $|A|^2 + |B|^2$ is invertible. Since $A$ commutes with $B$, it follows by the Fuglede Theorem that $A^*B = BA^*$. Hence by known techniques, we may get the commutativity of $|A|^2$ and $|B|^2$. Therefore, from Lemma 1.2 we obtain

$$\sqrt{|A|^2 + |B|^2} \geq \sqrt{|A|^2} + |B|^2.$$ 

Hence

$$|A| + |B| \geq \sqrt{|A|^2 + |B|^2}.$$ 

Since $\sqrt{|A|^2 + |B|^2}$ is invertible, Proposition 1.5 gives the invertibility of $|A| + |B|$, as wished. □

Remark. The invertibility of $A^2 + B^2$ does not yield that of $A + B$ even in the event of the self-adjointness of $A$ and $B$. As a counterexample, just consider an invertible and self-adjoint $A$ such that $A = -B$.

Example 2.7. Let $A \in B(H)$. We know that $\cos^2 A + \sin^2 A = I$. It then follows that $|\cos A| + |\sin A|$ is invertible.

Example 2.8. Let $A$ be self-adjoint and invertible. Set $B = iA$. Then $A^2 + B^2 = 0$ is obviously not invertible.

The idea of the proof of Proposition 2.6 leads to the following generalization.
Proposition 2.9. Let $A, B \in B(H)$ be such that $AB = BA$ and either $A$ or $B$ is normal. Then

$$|A| + |B| \text{ is invertible} \iff |A|^n + |B|^n \text{ is invertible}$$

where $n \in \mathbb{N}$.

The following example shows that both the real and imaginary parts of an invertible operator may be not invertible.

Example 2.10. Consider the multiplication operators

$$Af(x) = \cos xf(x) \text{ and } Bf(x) = \sin xf(x)$$

both defined on $L^2(\mathbb{R})$. Then $A$ and $B$ are not invertible. However,

$$Tf(x) = (A + iB)f(x) = (\cos x + i\sin x)f(x) = e^{ix}f(x)$$

is invertible (even unitary!).

It is fairly easy to see that a normal $T = A + iB$ is invertible iff $A^2 + B^2$ is invertible (see e.g. [5]). With this observation, we may state the following interesting characterization of invertibility for the class of normal operators.

Proposition 2.11. Let $T = A + iB$ be normal in $B(H)$. Then

$$T \text{ is invertible} \iff |A| + |B| \text{ is invertible.}$$

In particular, if $\lambda = \alpha + i\beta$, then

$$\lambda \in \sigma(T) \iff |A - \alpha I| + |B - \beta I| \text{ is not invertible.}$$

Remark. (cf. [2]) Another way of establishing the previous result is as follows. By [9], we know that if $T \in B(H)$ is normal, then $|T| \leq |\text{Re}T| + |\text{Im}T|$. Hence the invertibility of $T$ entails that $|\text{Re}T| + |\text{Im}T|$. Conversely, if $T$ is normal (in fact hyponormality suffices here), then $|\text{Im}T| \leq |T|$ and $|\text{Re}T| \leq |T|$ and so

$$|\text{Im}T| + |\text{Re}T| \leq 2|T|.$$ 

Therefore, the invertibility of $|\text{Im}T| + |\text{Re}T|$ implies that of $T$.

The following related version to Proposition 1.7 does not make use of the Gelfand Transform.

Proposition 2.12. Let $S, T \in B(H)$ be normal and such that $ST = TS$. Then

$$\sigma(S + T) \subset \sigma(\text{Re}S + \text{Re}T) + i\sigma(\text{Im}S + \text{Im}T).$$

Proof. Write $S = A + iB$ and $T = C + iD$. Since $ST = TS$ and $S$ and $T$ are normal, $S + T$ is normal (see e.g. [10]). Hence, if we let $\lambda = \alpha + i\beta \in \mathbb{C}$, then

$$S + T - \lambda I = (A + C - \alpha I) + i(B + D - \beta I)$$

becomes normal. So, if $\lambda \in \sigma(S + T)$, then Proposition 2.11 says that $|A + C - \alpha I| + |B + D - \beta I|$ is not invertible. If either $|A + C - \alpha I|$ or $|B + D - \beta I|$ is invertible, then clearly $|A + C - \alpha I| + |B + D - \beta I|$ would be invertible! Therefore, both $|A + C - \alpha I|$ and $|B + D - \beta I|$ are not invertible, i.e. $A + C - \alpha I$ and $B + D - \beta I$ are not invertible. In other language, $\alpha \in \sigma(A + C)$ and $\beta \in \sigma(B + D)$. Accordingly, $\lambda = \alpha + i\beta \in \sigma(A + C) + i\sigma(B + D)$, as wished. \qed

Corollary 2.13. Let $T = A + iB$ be normal in $B(H)$. Then

$$\sigma(T) \subset \sigma(A) + i\sigma(B).$$
As another consequence, we have a new and shorter proof of a well known result.

**Corollary 2.14.** Let \( A \in B(H) \) be self-adjoint. Then \( \sigma(A) \subseteq \mathbb{R} \).

**Proof.** Let \( \lambda \notin \mathbb{R} \), i.e. \( \lambda = \alpha + i\beta \) (\( \alpha, \beta \in \mathbb{R} \)). Since \( A - \alpha I \) is self-adjoint, it follows that \( A - \alpha I - i\beta I \) is normal. By the invertibility of \(|\beta|I\), it follows that \(|A - \alpha I| + |\beta I|\) (by Proposition 1.5). By Proposition 2.11, this means that \( A - \lambda I \) is invertible, that is, \( \lambda \notin \sigma(A) \). □

The following result appeared in [9].

**Proposition 2.15.** Let \( A, B \in B(H) \) be such that \( AB = BA \). If \( A \) is normal and \( B \) is hyponormal, then the following inequality holds:
\[
||A| - |B|| \leq |A - B|.
\]

As a consequence of the previous result, we get a very short proof concerning the spectrum of unitary operators.

**Corollary 2.16.** Let \( A \in B(H) \) be unitary. Then \( \sigma(A) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \).

**Proof.** We have \( |A| = I \) and so \( |1 - |\lambda||I = |I - |\lambda||I \leq |A - \lambda I| \). Thus, if \( |\lambda| \neq 1 \), then \( \lambda \notin \sigma(A) \). □

It is known that a normal operator having a real spectrum is self-adjoint. The proof is very simple if we know the very complicated Spectral Theorem for normal operators. It would be interesting to prove this result along the lines of the proof of Corollary 2.14. Notice also that this can very easily be established if the imaginary part of \( T \) is a scalar operator. A new proof in the general case has not been obtained yet. Nonetheless, as an application of Proposition 2.17, we have the following short proof (which seems to have escaped notice) of this result.

**Proposition 2.17.** Let \( T = A + iB \) be normal in \( B(H) \) and such that \( \sigma(T) \subseteq \mathbb{R} \). Then \( T \) is self-adjoint.

**Proof.** Recall that \( A \) and \( B \) are self-adjoint. The normality of \( T \) is equivalent to \( AB = BA \). Hence \( TA = AT \). Since \( T - A = iB \), we have by Proposition 1.7
\[
 i\sigma(B) = \sigma(iB) = \sigma(T - A) \subseteq \sigma(T) + \sigma(-A) \subseteq \mathbb{R}.
\]
Thus, necessarily \( \sigma(B) = \{0\} \). Accordingly, the Spectral Radius Theorem gives us \( B = 0 \) and so \( T = A \), i.e. \( T \) is self-adjoint. □

What is also interesting is that since Proposition 2.17 holds in the context of Banach algebras (see Theorem 11.23 in [10]), Proposition 2.17 becomes valid in the context of \( C^* \)-algebras. The proof is identical and so it is omitted.

**Proposition 2.18.** Let \( A \) be a \( C^* \)-algebra and let \( a \in A \) be a normal element. If \( \sigma(a) \subseteq \mathbb{R} \), then \( a \) is hermitian.

We also have the following related result.

**Proposition 2.19.** Let \( A \) be a \( C^* \)-algebra and let \( a \in A \) be a normal element. If \( \sigma(a) \) is purely imaginary, then \( a \) is skew-hermitian (i.e. \( a^* = -a \)).

**Proof.** Write \( a = x + iy \) where \( x \) and \( y \) are commuting hermitian elements of \( A \). Write \( x = a - iy \) and proceed as above to force \( x = 0 \). □

The last result about the spectrum is the following.
Proposition 2.20. The "subadditivity of the spectrum" is equivalent to the "sub-multiplicativity of the spectrum" in the context of bounded positive, commuting and invertible operators.

Proof. Let $A, B \in B(H)$ be self-adjoint and such that $AB = BA$. Assume that Proposition 1.8 holds. Let $\lambda \in \sigma(A + B)$. Then the Spectral Mapping Theorem yields

$$e^{\lambda} \in \sigma(e^{A+B}) = \sigma(e^A)e^B \subset \sigma(e^A)\sigma(e^B),$$

that is, $e^{\alpha}e^{\beta}$ for $\alpha \in \sigma(A)$ and $\beta \in \sigma(B)$. Hence

$$\lambda = \ln(e^{\alpha}e^{\beta}) = \alpha + \beta \in \sigma(A) + \sigma(B).$$

Now, suppose that Proposition 1.7 holds. Suppose also here that $A$ and $B$ are positive and invertible. Let $\lambda \in \sigma(AB)$. Since $AB$ is positive and invertible, by the Spectral Mapping Theorem we get

$$\ln \lambda \in \sigma(\ln(AB)) = \sigma(\ln A + \ln B) \subset \sigma(\ln A) + \sigma(\ln B) = \ln(\sigma(A)) + \ln(\sigma(B)),$$

i.e. $\ln \lambda = \ln \alpha + \ln \beta$ with $\alpha \in \sigma(A)$ and $\beta \in \sigma(B)$. Thus, $\lambda = \alpha \beta \in \sigma(A)\sigma(B)$, as required. \(\square\)

Let’s go back to invertibility.

Proposition 2.21. Let $A, B \in B(H)$ be such that $AB \geq 0$ and $AB$ is invertible. Then $|A^*|^2 + |B|^2$ is invertible.

Proof. Let $x \in H$. Then

$$\langle ABx, x \rangle = |\langle ABx, x \rangle| = |\langle Bx, A^*x \rangle| \leq \|A^*x\|\|Bx\| \leq 2(\|A^*x\|^2 + \|Bx\|^2) = 2(|\langle AA^*x, x \rangle + \langle B^*Bx, x \rangle|) = 2(|\langle AA^* + B^*B \rangle x, x |),$$

that is, $AB \leq 2(AA^* + B^*B) = 2(|A^*|^2 + |B|^2)$. Proposition 1.5 allows us to establish the invertibility of $|A^*|^2 + |B|^2$, as required. \(\square\)

Remark. Notice that the result is obvious if either $A$ or $B$ is invertible. In order to keep the result non-trivial we need also to avoid $\ker A = \ker A^*$ and $\ker B = \ker B^*$ (see [4]). This remark applies to the unbounded case as well (treated in Theorem 2.24 below).

Remark. The fact that we have assumed the invertibility of $AB$ is essential as seen by the following example.

Example 2.22. Let $A$ be the positive operator defined by

$$Af(x) = xf(x), \ f \in L^2(0, 1).$$

Setting $B = A$, we see that $ABf(x) = x^2f(x)$ is positive. However,

$$|A^*|^2 + |B|^2f(x) = (A^2 + B^2)f(x) = 2x^2f(x)$$

is not invertible.

As a consequence of the foregoing proposition, we have the following.

Corollary 2.23. Let $A \in B(H)$ be right (resp. left) invertible with $B \in B(H)$ being its right (resp. left) inverse. Then $|A^*|^2 + |B|^2$ (resp. $|A|^2 + |B^*|^2$) is always invertible.
Proof: Since $A$ is right invertible, for some $B \in B(H)$, we have $AB = I$. Since the latter is positive and invertible, the result follows immediately. The case of left-invertibility is identical. \qed

We can generalize Proposition 2.21 to unbounded operators.

**Theorem 2.24.** Let $A$ and $B$ be two operators such that $A$ is closed and $B \in B(H)$.

1. If $BA$ is positive (i.e. $\langle BAx, x \rangle \geq 0$ for all $x \in D(BA)$) and invertible, then $|A|^2 + |B^*|^2$ is invertible. Besides,

$$\langle (|A|^2 + |B^*|^2)^{-1}x, x \rangle \leq \langle (BA)^{-1}x, x \rangle \forall x \in H.$$ 

2. If $AB$ is densely defined, positive and invertible, then $|A^*|^2 + |B|^2$ is invertible. Moreover,

$$\langle (|A^*|^2 + |B|^2)^{-1}x, x \rangle \leq \langle (AB)^{-1}x, x \rangle \forall x \in H.$$ 

**Proof.** (1): The first step is to show that $BA \leq 2|A|^2 + 2|B^*|^2$. Observe that

$$D(|A|^2 + |B^*|^2) = D(A^*A + BB^*) = D(A^*A) \subset D(A) = D(BA).$$

Now, let $x \in D(A^*A)$. As in the bounded case, we may prove

$$\langle BAx, x \rangle = \langle BAx, x \rangle = \langle Ax, A^*x \rangle \leq \|Ax\|\|B^*x\| \leq 2(\|A^2 + |B^*|^2\| x, x).$$

This means that $BA \leq 2|A|^2 + 2|B^*|^2$. Since $BA$ is positive, it is (only) symmetric. Since it is invertible, it follows that $BA$ is actually self-adjoint (and positive). Thus, by Lemma 10.10 in [11], "≤" becomes "≤", that is, we have established the desired inequality $BA \leq 2|A|^2 + 2|B^*|^2$. Since $BA$ is positive, invertible and $BA$ and $|A|^2 + |B^*|^2$ are self-adjoint, it follows that $|A|^2 + |B^*|^2$ is invertible by the unbounded version of Proposition [13] (recalled also in the introduction), as wished.

(2): The idea is similar to the previous case. As $AB$ is symmetric and invertible, then it is self-adjoint (and positive). Hence $B^*A^* \subset (AB)^*$ and so $D(A^*) = D(B^*A^*) \subset D(AB)$. The main point is to show that

$$AB \leq 2|A^*|^2 + 2|B|^2.$$ 

Clearly,

$$D(|A^*|^2 + |B|^2) = D(|A^*|^2) = D(AA^*) \subset D(A^*) \subset D(AB).$$

Now, let $x \in D(AA^*)$. Then

$$\langle ABx, x \rangle = \langle Bx, A^*x \rangle \leq \|Bx\|\|A^*x\| \leq 2\|Bx\|^2 + \|A^*x\|^2 = 2\langle Bx, Bx \rangle + 2\langle A^*x, A^*x \rangle = 2(\|B^2 + |A|^2\| x, x).$$

As above, $AB \leq 2|B|^2 + 2|A|^2$ becomes $AB \leq 2|B|^2 + 2|A|^2$. Thus, the invertibility of $|B|^2 + |A|^2$ follows from that of $AB$, as wished. \qed

Let’s give an explicit and non-trivial application of the previous result.

**Example 2.25.** Let $A$ be defined by $Af(x) = f’(x)$ on the domain

$$D(A) = \{ f \in L^2(0,1) : f’ \in L^2(0,1) \}.$$ 

Then $A$ is densely defined and closed (but it is not normal, see e.g. [7]). Also, $A^*f(x) = -f’(x)$ on $D(A^*) = \{ f \in L^2(0,1) : f’ \in L^2(0,1), f(0) = f(1) = 0 \}$ so that $|A^*|^2 = AA^*f(x) = -f''(x)$ with

$$D(AA^*) = \{ f \in L^2(0,1) : f’’ \in L^2(0,1), f(0) = f(1) = 0 \}.$$
Let $V$ be the Volterra operator defined on $L^2(0,1)$, i.e.

$$(Vf)(x) = \int_0^x f(t)dt, \ f \in L^2(0,1).$$

Then (see e.g. §) $|V|^2 f = V^*V f = \sum_{n=1}^\infty \lambda_n (f, f_n)f_n$ with $(f_n)$ being the eigenvectors corresponding to the distinct eigenvalues $\lambda_n$ of $V^*V$ (and $f \in L^2(0,1)$).

Now, neither $A$ nor $V$ is invertible. However, $A$ is right invertible for $D(AV) = L^2(0,1)$ and $AVf(x) = f(x)$ for $f \in L^2(0,1)$. This means that $AV$ is positive and invertible. Therefore, the unbounded operator

$$-\frac{d^2}{dx^2} + |V|^2$$

is invertible on the domain $D(AA^*)$ given above.

Next, we pass to the invertibility of finite sums.

**Lemma 2.26.** (cf. §) Let $(A_k)_{k=1,\ldots,n}$ be in $B(H)$ and let $(a_k)_{k=1,\ldots,n}$ be in $\mathbb{C}$. Then

$$\left\| \sum_{k=1}^n a_k A_k x \right\|^2 \leq \sum_{k=1}^n |a_k|^2 \left( \sum_{k=1}^n A_k^* A_k x, x \right)$$

for all $x \in H$.

**Proof.** Clearly,

$$\left\| \sum_{k=1}^n a_k A_k x \right\| \leq \sum_{k=1}^n \|a_k A_k x\| \leq \left( \sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^n \|A_k x\|^2 \right)^{\frac{1}{2}}$$

for all $x \in H$. \hfill $\square$

**Corollary 2.27.** Let $(A_k)_{k=1,\ldots,n}$ be in $B(H)$ and let $(a_k)_{k=1,\ldots,n}$ be in $\mathbb{C}$. If $\sum_{k=1}^n a_k A_k$ is invertible, then so is $\sum_{k=1}^n |A_k|^2$.

**Proof.** As $\sum_{k=1}^n a_k A_k$ is invertible, it is bounded below, i.e., for some $\alpha > 0$ and all $x \in H$, we have $\|\sum_{k=1}^n a_k A_k x\| \geq \alpha \|x\|$. By Lemma 2.20 and by the self-adjointness of $\sum_{k=1}^n A_k^* A_k$, it follows that $\sum_{k=1}^n |A_k|^2$ is invertible (given that the $a_k$ cannot all vanish simultaneously!). \hfill $\square$

**Theorem 2.28.** Let $(A_k)_{k=1,\ldots,n}$ and let $(B_k)_{k=1,\ldots,n}$ be in $B(H)$. If $\sum_{k=1}^n A_k B_k$ is positive and invertible, then $\sum_{k=1}^n |A_k|^2 + \sum_{k=1}^n |B_k|^2$ too is invertible.

**Proof.** Let $x \in H$. Then

$$\left\langle \sum_{k=1}^n A_k B_k x, x \right\rangle = \left\langle \sum_{k=1}^n A_k B_k x, x \right\rangle = \left\langle \sum_{k=1}^n (A_k B_k x, x) \right\rangle$$

$$= \sum_{k=1}^n \langle B_k x, A_k^* x \rangle \leq \sum_{k=1}^n |\langle B_k x, A_k^* x \rangle| \leq \sum_{k=1}^n \|B_k x\| \|A_k^* x\|$$

$$\leq \sum_{k=1}^n \|B_k x\|^2 \sum_{k=1}^n \|A_k^* x\|^2 = \sum_{k=1}^n \langle B_k^* B_k x, x \rangle \sum_{k=1}^n \langle A_k^* A_k x, x \rangle$$

$$= \sum_{k=1}^n \|B_k x\|^2 \sum_{k=1}^n \|A_k x\|^2.$$
\[
\sqrt{\left( \sum_{k=1}^{n} B_k^* B_k x, x \right)} \leq \frac{1}{2} \left( \sum_{k=1}^{n} |B_k|^2 x, x \right) + \frac{1}{2} \left( \sum_{k=1}^{n} |A_k|^2 x, x \right). 
\]

Since \( \sum_{k=1}^{n} A_k B_k \) is positive and invertible, Proposition \ref{prop:invertibility} gives the invertibility of \( \sum_{k=1}^{n} |A_k|^2 + \sum_{k=1}^{n} |B_k|^2 \), as needed. \( \blacksquare \)

**Corollary 2.29.** Let \( (A_k)_{k=1}^{n} \) be in \( B(H) \). If \( \sum_{k=1}^{n} A_k^* A_k \) (or \( \sum_{k=1}^{n} A_k A_k^* \)) is invertible, then so is \( \sum_{k=1}^{n} |A_k|^2 + \sum_{k=1}^{n} |B_k|^2 \).

**Proof.** First, remember that \( A A_k^* \geq 0 \) for every \( k \). Hence clearly \( \sum_{k=1}^{n} A_k^* A_k \geq 0 \). Now apply Theorem \ref{thm:invertibility} to get the desired result. \( \blacksquare \)

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