Correction to “Knotted Hamiltonian cycles in spatial embeddings of complete graphs”

Joel Foisy

Abstract. We state and prove a correct version of a theorem presented in [1].

Professor Masakazu Teragaito has recently pointed out that Theorem 3.3 of [1] is incorrect as stated. The fact that \( \mu_f(G, \Gamma; 6) = 3 \) is independent of embedding of \( K_8 \) does not necessarily imply that there are at least 3 knotted Hamiltonian cycles. For example, there could be exactly one knotted Hamiltonian cycle with \( a_2(K) = 3 \).

Professor Kouki Taniyama has further pointed out that Lemma 2, in [2], has a gap in its proof. He has proposed a rigorous proof of a weaker version Lemma 2. In this short paper, we will state and prove this weaker version of Lemma 2 in [2], and then apply it to obtain a weaker version of Theorem 3.3 of [1]. For definitions of terms, see [2] and [1]. Here is the modified version of Shimabara’s Lemma 2 that we will prove:

**Lemma 0.1.** Let \( \Gamma \) be a set of cycles in an undirected graph \( G \). The invariant \( \mu_f(G, \Gamma; n) \) does not depend on the spatial embedding \( f \) of \( G \) if the following two conditions hold:

1. For any edges \( A, B, E \) such that \( A \) is adjacent to \( B \),
   \( \nu_1(\Gamma; A, B, E) \equiv 0 \pmod{n} \).
2. For any pairs of non-adjacent edges \( (A, B) \) and \( (E, F) \),
   \( \nu_2(\Gamma; A, B; E, F) \equiv 0 \pmod{2n} \).

The difference between this new lemma and the original comes in the second condition, where the equivalence is \( \text{mod} \ 2n \), not \( n \). For the proof of Lemma 2, case 2, on p. 410 of [2], the definition of linking number used does not work. For the version of linking number used, \( \zeta(A, B) \) depends on the order of \( A \) and \( B \), whereas the equality \( \sum_{E,F \gamma \in \Gamma_1} \epsilon(c) \zeta(f_\gamma(E), f_\gamma(F)) = \sum_{E,F} (n_3 - n_4) \zeta(f(E), f(F)) \) implicitly uses the assumption that \( \zeta(A, B) = \zeta(B, A) \).

A proof of Lemma 0.1 is possible if one uses a different (but equivalent) version of \( \zeta \) and linking number. For \( A \) and \( B \) two disjoint oriented arcs or circles in \( \mathbb{R}^3 \), define \( \zeta'(A, B) = (1/2) \sum c \epsilon(c) \), with the summation being over all crossings.
between A and B. If A and B are circles, then $\zeta'(A, B)$ gives the linking number of A and B, $lk(A, B)$. It then follows immediately that $\zeta'(A, B) = \zeta'(B, A)$. The proof of Lemma 0.1 proceeds as in the proof of Lemma 2 in [2], but now ends with:

$$\delta(u) = \sum_{E, F} \sum_{\gamma \in \Gamma} \epsilon(c) \zeta(f_\gamma(E), f_\gamma(F)) = \sum_{E, F} (n_3 - n_4) \zeta'(f(E), f(F)).$$

It then follows, since each $\zeta'(f(E), f(F))$ is either an integer or an integer divided by 2, that, $\delta(u)$ is congruent to 0 mod n if $|n_3 - n_4| \equiv 0 \mod 2n$.

Now, in [1], it was shown that $\nu_2 \equiv 0 \mod 6$. It was also shown that there exists an embedding of $K_8$ with exactly 21 knotted Hamiltonian cycles, each with Arf invariant 1. One can also verify that each of these knotted cycles is a trefoil, with $a_2 = 1$. This embedding, together with Lemma 0.1, implies that $\mu_f(K_8, \Gamma, 3) \equiv 0$ for every spatial embedding of $K_8$. By Theorem 2.2 of [1], there is at least one Hamiltonian cycle with Arf invariant 1, in every spatial embedding of $K_8$.

We thus have the following corrected version of Theorem 3.3 from [1].

**Theorem 0.2.** Given an embedding of $K_8$, at least one of the following must occur in that embedding:

1. At least 3 knotted Hamiltonian cycles.
2. Exactly 2 knotted Hamiltonian cycles $C_1$ and $C_2$, with $a_2(C_1) \equiv 1 \mod 3$ and $a_2(C_2) \equiv 2 \mod 3$, or $a_2(C_1) \equiv 0 \mod 3$ and $a_2(C_2) \equiv 0 \mod 3$.

   Either $C_1$ or $C_2$ has non-zero Arf invariant.
3. Exactly 1 knotted Hamiltonian cycle, $C$ with $a_2(C) \equiv 1 \mod 2$ and $a_2(C) \equiv 0 \mod 3$. (Equivalently: $a_2(C) \equiv 3 \mod 6$.)

It thus remains an open question to determine if 1 is the best lower bound for the minimum number of knotted Hamiltonian cycles in every spatial embedding of $K_8$.

**Acknowledgments** The author would like to thank Professors Masakazu Teragaito, Kouki Taniyama and Ryo Nikkuni for valuable comments and suggestions.

**References**

[1] P. Blain, G. Bowlin, J. Hendricks, J. LaCombe, J. Foisy, *Knotted Hamiltonian cycles in spatial embeddings of complete graphs*, New York Journal of Mathematics, 13 (2007), 11-16.

[2] M. Shimabara *Knots in Certain Spatial Graphs* Tokyo J. Math. Vol. 11, No 2, 1988.

Department of Mathematics, SUNY Potsdam, Potsdam, NY 13676
foisyjs@potsdam.edu