SOLITARY WAVES IN AN INTENSE BEAM PROPAGATING THROUGH A SMOOTH FOCUSING FIELD

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Abstract

Based on the Vlasov-Maxwell equations describing the self-consistent nonlinear beam dynamics and collective processes, the evolution of an intense sheet beam propagating through a periodic focusing field has been studied. In an earlier paper [1] it has been shown that in the case of a beam with uniform phase space density the Vlasov-Maxwell equations can be replaced exactly by the macroscopic warm fluid-Maxwell equations with a triple adiabatic pressure law. In this paper we demonstrate that macroscopic warm fluid-Maxwell equations with a triple Vlasov-Maxwell equations can be replaced exactly by the hydrodynamic equations, and using the renormalization group (RG) technique [2, 3, 4, 5] a nonlinear Schrödinger equation for the slowly varying single-wave amplitude can be derived. The renormalized solution for the beam density describes the process of formation of periodic holes in intense particle beams.

1 INTRODUCTION

Of particular importance in modern accelerators and storage rings operating at high beam currents and charge densities are the effects of the intense self-fields produced by the beam space charge and current on determining detailed equilibrium, stability and transport properties. In general, a complete description of collective processes in intense charged particle beams is provided by the Vlasov-Maxwell equations for the self-consistent evolution of the beam distribution function and the electromagnetic fields. As shown in [1] in the case of a sheet beam with constant phase-space density the Vlasov-Maxwell equations are fully equivalent to a warm-fluid model with zero heat flow and triple-adiabatic equation-of-state.

In the present paper we demonstrate that starting from the hydrodynamic equations, and using the renormalization group (RG) technique [2, 3, 4, 5] a nonlinear Schrödinger equation for the slowly varying single-wave amplitude can be derived. The renormalized solution for the beam density describes the process of formation of periodic holes in intense particle beams.

2 THE HYDRODYNAMIC MODEL

We begin with the hydrodynamic model derived in [1]

\[
\frac{\partial \bar{\rho}}{\partial s} + \frac{\partial}{\partial x}(\bar{\rho} \bar{v}) = 0,
\]

\[
\frac{\partial \bar{v}}{\partial s} + \bar{v} \frac{\partial \bar{v}}{\partial x} + \bar{v}^2 \frac{\partial}{\partial x} \bar{\rho}^2 = -G(s) \bar{v} - \frac{\partial \bar{\psi}}{\partial x},
\]

\[
\frac{\partial^2 \bar{\psi}}{\partial x^2} = -2\pi K \bar{\rho}.
\]

Here \( \bar{\rho}(x; s) = \rho(x; s)/N \) and \( \bar{v}(x; s) \) are the normalized density and the current velocity, respectively, \( G(s + S) = G(s) \) is the periodic focusing lattice coefficient, \( \bar{v}^2 = 3P_0/2n_0^3 \) is the normalized thermal speed-squared, and \( P_0/n_0^3 = N^2/12A^2 \) is a constant coefficient [1], where \( N \) is the area density of sheet beam particles, and \( A \) is the constant phase-space density. Moreover, \( \bar{\psi}(x; s) \) is the renormalized self-field potential

\[
\bar{\psi}(x; s) = \frac{e_b \phi(x; s)}{m_b \gamma_b^0 \beta_c^0 c^2},
\]

where \( \phi(x; s) \) is the electrostatic (space-charge) potential, \( m_b \) and \( e_b \) are the rest mass and charge of a beam particle, and \( \beta_c \) and \( \gamma_b \) are the relative particle velocity and Lorentz factor, respectively. Finally, the quantity \( K \) is the normalized self-field perveance defined by

\[
K = \frac{2Ne_b^2}{m_b \gamma_b^3 \beta_c^2 c^2}.
\]

In what follows the analysis is restricted to the smooth focusing approximation

\[
G(s) = G = \text{const},
\]

and assume that there exist nontrivial stationary solutions to (2.1) in the interval \( x \in (-x^(-), x^(+)) \), and that the sheet beam density is zero (\( \bar{\rho} = 0 \)) outside of the interval. The change of variables

\[
\xi = x + x^(-), \quad \Psi = \bar{\psi} - Gx^(-)x
\]

enables us to rewrite (2.1) in the form

\[
\frac{\partial \bar{\rho}}{\partial \xi} + \frac{\partial}{\partial \xi}(\bar{\rho} \bar{v}) = 0,
\]

\[
\frac{\partial \bar{v}}{\partial \xi} + \bar{v} \frac{\partial \bar{v}}{\partial \xi} + \bar{v}^2 \frac{\partial}{\partial \xi} \bar{\rho}^2 = -G \xi - \frac{\partial \Psi}{\partial \xi},
\]

\[
\frac{\partial^2 \Psi}{\partial \xi^2} = -2\pi K \bar{\rho}.
\]

Clearly, the system (2.4) possesses a stationary solution

\[
\bar{\rho}_0 = \frac{G}{2\pi K}, \quad \bar{v}_0 \equiv 0, \quad \Psi_0 = -\frac{G\bar{\rho}_0^2}{2} + \text{const}.
\]

Here, the uniform density \( \bar{\rho}_0 \) is normalized according to

\[
x^(-) + x^(+)= \frac{1}{\bar{\rho}_0} = \frac{2\pi K}{G}.
\]
3 Renormalization Group Reduction of the Hydrodynamic Equations

Following the basic idea of the RG method, we represent the solution to equations (2.1) in the form of a standard perturbation expansion \([6]\) in a formal small parameter \(\epsilon\) as

\[
q = q_0 + \sum_{k=1}^{\infty} \epsilon^k q_k, \quad v = \sum_{k=1}^{\infty} \epsilon^k v_k, \quad (3.1)
\]

\[
\Psi = - \frac{G\xi^2}{2} + \sum_{k=1}^{\infty} \epsilon^k \Psi_k.
\]

Before proceeding with explicit calculations order by order, we note that in all orders the perturbation equations acquire the general form

\[
\frac{\partial \varrho_n}{\partial s} + \varrho_0 \frac{\partial v_n}{\partial \xi} = \alpha_n, \quad (3.2)
\]

\[
\frac{\partial v_n}{\partial s} + 2\varrho_0 v_2 \frac{\partial \varrho_n}{\partial \xi} = -\frac{\partial \Psi_n}{\partial \xi} + \beta_n,
\]

\[
\frac{\partial^2 \Psi_n}{\partial \xi^2} = -2\pi K \varrho_n,
\]

where the functions \(\alpha_n(\xi; s)\) and \(\beta_n(\xi; s)\) involve contributions from previous orders and are considered known. Eliminating \(v_n\) and \(\Psi_n\), it is possible to obtain a single equation for \(\varrho_n\) alone, i.e.,

\[
\frac{\partial^2 \varrho_n}{\partial s^2} - 2\varrho_0 v_2 \frac{\partial^2 \varrho_n}{\partial \xi^2} + G \varrho_n = \varrho_0 \frac{\partial \alpha_n}{\partial s} - \varrho_0 \frac{\partial \beta_n}{\partial \xi}. \quad (3.3)
\]

It is evident that in first order \(\alpha_1 = \beta_1 = 0\). Imposing the condition

\[
\int_0^{1/\varrho_0} d\xi \varrho_1(\xi; s) = 0, \quad (3.4)
\]

which means that linear perturbation to the uniform stationary density \(\varrho_0\) should average to zero and not affect the normalization properties on the interval \((0, x^(-) + x^(+))\), we obtain the first-order solution

\[
\varrho_1(\xi; s) = \sum_{m \neq 0} A_m e^{i\chi_m(\xi; s)}, \quad \chi_m(\xi; s) = \omega_m s + m\sigma\xi. \quad (3.5)
\]

Here, \(A_m\) are constant complex wave amplitudes, and the following conventions and notations

\[
\omega_{-m} = -\omega_m, \quad \sigma = \frac{G}{K}, \quad A_{-m} = A_m^*. \quad (3.6)
\]

have been introduced. Moreover, the discrete mode frequencies \(\omega_m\) are determined from the dispersion relation

\[
\omega_m^2 = G + \frac{\nu_2^2\sigma^4}{2\pi^2 m^2}. \quad (3.7)
\]

In addition, the first-order solution for the current velocity and for the self-field potential can be expressed as

\[
v_1(\xi; s) = -\frac{1}{\varrho_0\sigma} \sum_{m \neq 0} \frac{\omega_m}{m} A_m e^{i\chi_m(\xi; s)}, \quad (3.8)
\]

\[
\Psi_1(\xi; s) = \frac{2\pi K}{\sigma} \sum_{m \neq 0} \frac{A_m}{m^2} e^{i\chi_m(\xi; s)}. \quad (3.9)
\]

In obtaining the second-order perturbation equation \([3.3]\), we note that

\[
\alpha_2 = -\frac{\partial}{\partial \xi}(\varrho_1 v_1), \quad \beta_2 = -\frac{1}{2} \frac{\partial}{\partial \xi} \left(v_1^2 + 2v_1^2 q_1^2\right). \quad (3.10)
\]

Thus the second-order solution for the density \(\rho_2(\xi; s)\) is found to be

\[
\rho_2(\xi; s) = -\sum_{m, k \neq 0} \alpha_{mk} A_mA_k e^{i[\chi_m(\xi; s) + \chi_k(\xi; s)]}, \quad (3.11)
\]

where

\[
\alpha_{mk} = \frac{m + k}{D_{mk}} \left[\frac{\omega_m + \omega_k}{k\varrho_0} + \frac{m + k}{2\varrho_0} \left(\frac{v_1^2\sigma^4}{2\pi^2} + \frac{\omega_m\omega_k}{m} \right)\right], \quad (3.12)
\]

\[
D_{mk} = -(\omega_m + \omega_k)^2 + \frac{v_1^2\sigma^4}{2\pi^2} (m + k)^2 + G. \quad (3.13)
\]

Having determined \(\rho_2\), the second-order current velocity \(v_2(\xi; s)\) can be found in a straightforward manner. The result is

\[
v_2(\xi; s) = \frac{1}{\varrho_0\sigma} \sum_{m, k \neq 0} \beta_{mk} A_mA_k e^{i[\chi_m(\xi; s) + \chi_k(\xi; s)]}, \quad (3.14)
\]

where

\[
\beta_{mk} = \frac{\omega_k}{k\varrho_0} + \frac{\omega_m + \omega_k}{m + k} \alpha_{mk}, \quad \beta_{m, m, m} = 0. \quad (3.15)
\]

In third order, the functions \(\alpha_3\) and \(\beta_3\) entering the right-hand-side of equation \([3.3]\) can be calculated utilizing the already determined quantities from the first and second orders, according to

\[
\alpha_3 = -\frac{\partial}{\partial \xi}(\varrho_1 v_2 + \vartheta_2 v_1), \quad (3.16)
\]

\[
\beta_3 = -\frac{\partial}{\partial \xi} \left(v_1 v_2 + 2v_1^2 q_1 q_2\right). \quad (3.17)
\]

It is important to note that the right-hand-side of equation \([3.3]\) for \(\rho_3\) contains terms which yield oscillating terms with constant amplitudes to the solution for \(\rho_2\). Apart from these, there is a resonant term (proportional to \(e^{i\chi_m(\xi; s)}\)) leading to a secular contribution. To determine the renormalization group reduction of the hydrodynamic equations, we select this particular resonant third-order term on the
right-hand-side of equation (3.3). The latter can be written as

\[
\left( \frac{\partial \alpha_3}{\partial s} - \theta_0 \frac{\partial \beta_3}{\partial \xi} \right)_{res} = \sum_{m,k \neq 0} \Gamma_{mk} A_m |A_k|^2 e^{i \chi_m(\xi; s)}, \tag{3.18}
\]

where

\[
\Gamma_{mk} = \frac{m}{\theta_0} \left[ \omega_m \left( \beta_{mk} + \frac{\omega_k \psi_{mk}}{k} \right) + \frac{m \omega_k \beta_{mk}}{k} \right]. \tag{3.19}
\]

Some straightforward algebra yields the solution for \( \varrho_3(\xi; s) \) to equation (3.3) in the form

\[
\varrho_3(\xi; s) = \sum_{m \neq 0} P_m(\xi; s) e^{i \chi_m(\xi; s)} + \ldots, \tag{3.20}
\]

where the dots stand for non-secular oscillating terms. Moreover, the amplitude \( P_m(\xi; s) \) is secular and satisfies the equation

\[
\hat{L}_m(\xi; s) P_m(\xi; s) = \sum_{k \neq 0} \Gamma_{mk} A_m |A_k|^2, \tag{3.21}
\]

where the operator \( \hat{L}_m \) is defined by

\[
\hat{L}_m = \frac{\partial^2}{\partial s^2} + 2i \left( \omega_m \frac{\partial}{\partial s} - \frac{\psi^2 \sigma^3}{2m} \frac{\partial}{\partial \xi} \right) - \frac{\psi^2 \sigma^2}{2m} \frac{\partial^2}{\partial \xi^2}. \tag{3.22}
\]

We can now construct the perturbative solution for \( \varrho \) up to third order in the small parameter \( \epsilon \). Confining attention to the constant stationary density \( \theta_0 \) and the fundamental modes (first harmonic in the phase \( \chi_m \)), we obtain

\[
\varrho(\xi; s) = \theta_0 + \epsilon \sum_{m \neq 0} \left[ A_m + \epsilon^2 P_m(\xi; s) \right] e^{i \chi_m(\xi; s)}. \tag{3.23}
\]

Following the basic philosophy of the RG method, we introduce the intermediate coordinate \( X \) and “time” \( S \) and transform equation (3.23) to

\[
\varrho(\xi; s) = \theta_0 + \epsilon \sum_{m \neq 0} \{ A_m(X; S) + e^{2} [P_m(\xi; s) - P_m(X; S)] \} e^{i \chi_m(\xi; s)}. \tag{3.24}
\]

Note that the transition from equation (3.23) to equation (3.24) can always be performed by enforcing the constant amplitude \( A_m \) to be dependent on \( X \) and \( S \), which is in fact the procedure for renormalizing the standard perturbation result. Since the general solution for \( \varrho(\xi; s) \) should not depend on \( X \) and \( S \), by applying the operator \( \hat{L}_m(X; S) \) [which is the same as that in equation (3.22), but with \( \xi \rightarrow X \) and \( s \rightarrow S \)] on both sides of equation (3.24), we obtain

\[
\hat{L}_m(X; S) A_m(X; S) = \sum_{k \neq 0} \Gamma_{mk} A_m(X; S) |A_k(X; S)|^2, \tag{3.25}
\]

where we have dropped the formal parameter \( \epsilon \) on the right-hand-side. Since the above equation should hold true for any choice of \( X \) and \( S \), we can set \( X = \xi \) and \( S = s \). Thus, we obtain the so-called proto RG equation [3, 4, 5]

\[
\hat{L}_m(\xi; s) A_m(\xi; s) = \sum_{k \neq 0} \Gamma_{mk} A_m(\xi; s) |A_k(\xi; s)|^2. \tag{3.26}
\]

Introducing the new variable

\[
\zeta_m = \frac{\psi^2 \sigma^3 m}{2m^2} s + \omega_m \xi, \tag{3.27}
\]

and neglecting the second order derivatives \( \partial^2/\partial s^2 \) and \( \partial^2/\partial \xi \partial \zeta_m \), we finally arrive at the RG equation for the \( m \)-th mode amplitude

\[
2i \omega_m \frac{\partial A_m}{\partial s} - \frac{\psi^2 \sigma^2 G}{2m^2} \frac{\partial^2 A_m}{\partial \zeta_m^2} = -\sum_{k \neq 0} \Gamma_{mk} A_m |A_k|^2. \tag{3.28}
\]

4 THE NONLINEAR SCHRÖDINGER EQUATION FOR A SINGLE MODE

Equation (3.28) represents a system of coupled nonlinear Schrödinger equations for the mode amplitudes. Neglecting the contribution from modes with \( k \neq m \), for a single mode amplitude \( A_m \), we obtain the equation

\[
2i \omega_m \frac{\partial A_m}{\partial s} - \frac{\psi^2 \sigma^2 G}{2m^2} \frac{\partial^2 A_m}{\partial \zeta_m^2} = -\Gamma_m |A_m|^2 A_m, \tag{4.1}
\]

where

\[
\Gamma_m = -\Gamma_{mm} = \frac{2}{3G \theta_0^2} \left( 16 \omega_m^4 - 11 \omega_m^2 G^2 \right). \tag{4.2}
\]

It is easy to verify that \( \Gamma_m \) is always positive. In nonlinear optics equation (4.1) is known to describe the formation and evolution of the so-called dark solitons [6]. In the case of charged particle beams these correspond to the formation of holes or caviatons in the beam. Since the renormalized solution for the beam density \( \varrho(\xi; s) \) can be expressed as

\[
\varrho(\xi; s) = \theta_0 + \sum_{m \neq 0} A_m(\xi; s) e^{i \chi_m(\xi; s)}, \tag{4.3}
\]

these holes have periodic structure in space \( \xi \) and “time” \( s \).

5 CONCLUDING REMARKS

Based on the renormalization group method, a system of coupled nonlinear Schrödinger equations has been derived for the slowly varying amplitudes of interacting beam-density waves. Under the approximation of an isolated
wave neglecting the effect of the rest of the waves, this system reduces to a single nonlinear Schrödinger equation with repulsive nonlinearity. The latter describes the formation and evolution of holes in intense charged particle beams.

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