ON DIRICHLET FORMS AND SEMI-DIRICHLET FORMS

REMARKS ON THE BOOK
“SEMIDI-RICHLET FORMS AND MARKOV PROCESSES”
BY YOICHI OSHIMA

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Abstract. One aim of this note is to give an overview of some developments in the area of Dirichlet forms. A second aim is to review the new book “Semi-Dirichlet forms and Markov processes” by Yoichi Oshima. The book [Osh13] appeared last year, but first versions were written as lecture notes 25 years ago. We first give a rather short and light introduction into the field of Dirichlet forms with a special emphasis on the subjects presented in the book under consideration. After a small account on the history of Dirichlet forms we comment on the book by Oshima against the background of related works.

1. DIRICHLET FORMS AND SEMI-DIRICHLET FORMS

In short, the theory of Dirichlet forms is an advancement and an abstraction of potential theory. The theory of Dirichlet forms has created a lot of interesting and interrelated research since 1970. Dirichlet forms are related to probability theory, Riemannian geometry, pseudo-differential operators and mathematical physics. The strength and the beauty of this theory are that it provides a framework that connects spectral theory, functional analysis and stochastic processes in a natural way. Given some Hilbert space $L^2(X, m)$ of square-integrable functions on some topological space $X$ with measure $m$, a Dirichlet form is a pair $(\mathcal{E}, \mathcal{F})$ of a bilinear form $(u, v) \mapsto \mathcal{E}(u, v)$ for $u$ and $v$ from some domain $\mathcal{F} \subset L^2(X, m)$. The domain $\mathcal{F}$ itself, historically, is called Dirichlet space.

Before discussing further requirements and examples, let us explain the main characteristics. A Dirichlet form is called symmetric if $\mathcal{E}(u, v) = \mathcal{E}(v, u)$ for all $u, v$. Whether $X$ is a locally compact separable metric space like a subset of $\mathbb{R}^d$ or whether $X$ is a more general infinite-dimensional state space has significant consequences for the whole theory. The book [Osh13] focuses on lower-bounded semi-Dirichlet forms which are defined on locally compact separable metric spaces $X$.

Let us look at examples which are basic and were in the mind of those who founded the theory of Dirichlet forms. Assume $D \subset \mathbb{R}^d$ is open. We denote by $H^1(D)$ the space of all elements $u \in L^2(D)$ whose distributional derivatives $\nabla u$ are elements of $L^2(D)$. $H^1(D)$ is a Banach space with respect to the norm $\|u\|_{H^1(D)}^2 = \|u\|_{L^2(D)}^2 + \|\nabla u\|_{L^2(D)}^2$. $H_0^1(D)$

2010 Mathematics Subject Classification. 31-XX, 60-XX.

Key words and phrases. Dirichlet forms, potential theory, Markov processes.

The author would like to thank several participants of the Oberwolfach workshop “Dirichlet Form Theory and its Applications” in 2014 for helpful hints and discussions.
denotes the closure of $C_c^\infty(D)$ with respect to $\|u\|_{H^1(D)}$. Set $\mathcal{E}^{(1)}(u, v) = \int_D \nabla u \nabla v \, d\lambda$, $\mathcal{F}^{(1)} = H^1(D)$, $\mathcal{F}^{(2)} = H^1_0(D)$, where $\lambda$ denotes the Lebesgue measure. Then $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ and $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ are symmetric Dirichlet forms on $L^2(D, \lambda)$. For $\alpha \geq 0$ and $u, v \in \mathcal{F}$ let us denote $\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha_0(u, v)$ and $\mathcal{E}_\alpha(u) = \mathcal{E}_\alpha(u, u)$. The following conditions/properties turn out to be important. Assume $\alpha \geq 0$.

\begin{itemize}
  \item[(E.1)] $\mathcal{E}_\alpha(u) \geq 0$ for all $u \in \mathcal{F}$.
  \item[(E.2)] For some $K \geq 1$ and for all $u, v \in \mathcal{F}$ \(|\mathcal{E}(u, v)| \leq K \sqrt{\mathcal{E}_\alpha(u)} \sqrt{\mathcal{E}_\alpha(v)}\).
  \item[(E.3)] For all $\beta > \alpha$ the domain $\mathcal{F}$ is a Hilbert space with respect to the scalar product $\mathcal{E}_\beta(u, v) + \mathcal{E}_\beta(v, u)$.
  \item[(E.4)] For all $u \in \mathcal{F}$ the function $u^+ \wedge 1$ belongs to $\mathcal{F}$ and $\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$.
\end{itemize}

These conditions define what is called a lower-bounded semi-Dirichlet form in [Osh13, Section 1.1]. It makes sense that the author calls such forms Dirichlet forms in [Osh13] but it might be confusing for the reader of this review. The by now classical definition given in [Fuk80, FOT11] requires $\mathcal{E}$ to be symmetric, conditions (E.1), (E.3) to hold with $\alpha = 0$ and for all $u \in \mathcal{F}$ the condition $\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leq \mathcal{E}(u, u)$ which is stronger than (E.4). The notion of a nonsymmetric Dirichlet form is slightly more tricky. Again, one requires (E.1), (E.3) to hold with $\alpha = 0$. In this case one observes that (E.4) is equivalent to $\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$. A nonsymmetric Dirichlet form now additionally requires

\begin{itemize}
  \item[(E.4')] For all $u \in \mathcal{F}$ the function $u^+ \wedge 1$ belongs to $\mathcal{F}$ and $\mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0$.
\end{itemize}

The above examples obviously are very special cases with additional features. First, they are symmetric forms. Second, we can choose $\alpha = 0$ and $K = 1$ by the Cauchy-Bunyakovsky-Schwarz inequality. (E.4) is called Markov property because it relates to the Markov property of the related stochastic process. Note that $\mathcal{E}^{(1)}(u, u^+ \wedge 1) \geq \mathcal{E}^{(i)}(u^+ \wedge 1)$ trivially for $i = 1, 2$.

When relating these Dirichlet forms to extensions of $(-\Delta, C_c^\infty(D))$, readers might find [Str96] quite informative despite the fact that the author does not hide his personal view and his preference for another approach.

Let us provide an example which makes use of the flexibility of the above conditions. Define $\mathcal{E}^{(3)}(u, v) = \mathcal{E}^{(1)}(u, v) + \sum_{i=1}^d \int_D b_i \frac{\partial u}{\partial x_i} v \, d\lambda$ for some functions $b_1, \ldots, b_d : D \rightarrow \mathbb{R}$ which either have bounded absolute values in $D$ or (for $d \geq 3$) have the property that $\|b\|_{L^4(D)}$ is finite and $\text{div } \mathbf{b}$ is bounded from above. Here and below, we write $\mathbf{b}$ for the vector $(\mathbf{b}_1, \ldots, \mathbf{b}_d)^T$. The assumptions on $\mathbf{b}$ are tailored for conditions (E.1) and (E.2). This time, $\alpha \geq 0$ and $K \geq 1$ are chosen in dependence of $\mathbf{b}$. Thus the term lower bounded makes sense because of (E.1). The tuple $(\mathcal{E}^{(3)}, H^1_0(D))$ is a lower bounded semi-Dirichlet form. In general, it is not a nonsymmetric Dirichlet form in the above sense. Note that our previous examples all were local forms in the sense that $\mathcal{E}(u, v) = 0$ if $u, v \in \mathcal{F}$ have disjoint supports. There is a whole universe of nonlocal symmetric/nonsymmetric Dirichlet/semi-Dirichlet forms.

There is a natural link between closed bilinear forms, semi-groups and resolvent operators. Assume $(\mathcal{E}, \mathcal{F})$ satisfies (E.1)-(E.3). Then there exist strongly continuous semigroups $(T_t, (\widehat{T}_t))$ on $L^2(X, m)$ such that $\|T_t\| \leq e^{\alpha t}$, $\|\widehat{T}_t\| \leq e^{\alpha t}$, $(T_t f, g) = (f, \widehat{T}_t g)$ and for the
resolvents \(G_\beta, \hat{G}_\beta\) given by \(G_\beta f = \int_0^\infty e^{-\beta t}T_tf \, dt\) and analogously for \(\hat{G}_\beta\): \(E_\beta(G_\beta f, u) = (f, u) = E_\beta(u, \hat{G}_\beta f)\) for all \(f \in L^2(X, m)\), \(u \in \mathcal{F}\). The term semi-Dirichlet form relates to the fact that, different from \((T_t)\), the dual semi-group is not Markov in general.

What we have explained so far, holds true in infinite dimensions, too. This changes, when it comes to the important concept of regularity. A lower bounded semi-Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \(L^2(X, m)\) is called regular if \(C_c(X) \cap \mathcal{F}\) is (a) dense in \(\mathcal{F}\) with respect to the norm induced by \(\mathcal{E}_1(\cdot)\) and (b) dense in \(C_c(X)\) with respect to the supremum norm. The concept of regularity needs to be changed significantly when working with infinite dimensional state spaces, which we comment on below. The major achievement of the theory of Dirichlet forms is that there is a correspondence between Hunt processes (= quasi-left strong Markov processes) and regular Dirichlet forms if \(X\) is a locally compact separable metric space. More precisely, there exists a Hunt process whose resolvent \(R_\alpha f\) is a quasi-continuous modification \(G_\alpha f\) for any \(f \in L^\infty(X; m)\) and \(\alpha > 0\). Note that \(R_\alpha f(x) = \int f(y) R_\alpha(x, dy)\) where \(R_\alpha(x, E) = \int_0^\infty e^{-\alpha t}p_t(x, E) \, dt\) and \(p_t\) is the transition function of the Hunt process. It is possible to give a complete characterization of all symmetric and nonsymmetric Dirichlet forms satisfying the sector condition in terms of right processes, see [MR92].

Given the relation between a given Dirichlet form and the corresponding Hunt process many properties of the form can be studied by investigating the process and vice versa. A fundamental result in the theory of regular symmetric Dirichlet forms is the formula of Beurling–Deny which provides a unique representation. Together with the results of Le Jan it leads to the following beautiful description which we give in a simple setting. Assume \(X = D \subset \mathbb{R}^d\) is a domain and \((\mathcal{E}, C_c^\infty(D))\) is a closable symmetric bilinear form on \(L^2(D, m)\) satisfying the Markov property \((\mathcal{E}.4)\). Then \(\mathcal{E}\) can be expressed uniquely by

\[
\mathcal{E}(u, v) = \sum_{i,j=1}^d \int_D \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \nu_{ij}(dx) + \int_{D \times D} (u(y) - u(x))(v(y) - v(x)) J(dx \, dy) + \int_D u(x)v(x)\kappa(dx),
\]

where \(\nu_{ij}, J\) and \(k\) are nondegenerate positive Radon measures satisfying, among other properties: \(\sum_{i,j=1}^d \xi_i \xi_j \nu_{ij}(K) \geq 0\) and \(\int_{K \times K} |x-y|^2 J(dx \, dy) < +\infty\) for any compact \(K \subset \mathbb{R}^d\). From the probabilistic point of view, a major result in the theory of Dirichlet forms is the Fukushima decomposition of an additive functional into a martingale additive functional and an additive functional of zero energy. This decomposition is similar to the semi-martingale decomposition for Markov processes and leads to results which, in the simplest cases, can be obtained by the Itô formula. We do not elaborate on this important result here.

The development of Dirichlet forms and corresponding Markov processes on infinite-dimensional state spaces is motivated by questions arising in quantum field theory and interacting particle systems. The main mathematical challenge is to find a substitute for the notion of regular Dirichlet forms. To this end, the notion of quasi-regularity
for Dirichlet forms is introduced. Leaving technicalities aside, a Dirichlet form is quasi-regular if and only if it is quasi-homeomorphic to a regular Dirichlet form on a locally compact metric space. It can be shown that a Dirichlet form is associated with a nice Markov process if and only if it is quasi-regular. The quasi-homeomorphism allows the results developed for regular Dirichlet forms to be applied to quasi-regular Dirichlet forms on general infinite-dimensional state spaces. Note that the term “quasi” relates to exceptional sets which are defined with respect to capacity. These exceptional sets appear naturally and are abundant in the theory of Dirichlet forms. One task is to overcome them using regularity theory.

2. A SMALL AND INCOMPLETE ACCOUNT ON THE HISTORY OF DIRICHLET FORMS

The articles [BD59, BD58, Den70] study the domains of Dirichlet forms as Hilbert spaces and show that the concept of Dirichlet space captures a lot of the classical potential theory. Usually, these studies are regarded as the birth of the analytic side of Dirichlet form theory. The correspondence with a strong Markov process is provided in [Fuk71]. This review is not the right place to list all articles that contributed to this theory. We restrict ourselves to monographs and to those articles which are related to the focus of [Osh13].

The books [Sil74, Fuk75, Sil76, Fuk80] lay out the foundations of symmetric Dirichlet forms and corresponding Markov processes. It is important to note that nonsymmetric forms and related stochastic calculus were studied already at the very beginning of the theory, e.g. in [Kun70, Bli71b, Bli71a, Anc75, Anc76, CM76, LJ77, Sil77, Sil77, LJ78, Pac78, Sil78, LJ82, Kim87]. Note that these works benefit from the corresponding theory for elliptic differential operators in divergence form worked out in [Sta65]. A standard reference for nonsymmetric forms is the monograph [MR92] which appeared roughly at the same time as [BH91] and the first edition of [FOT11] which is an extension of [Fuk80] and nowadays is the main reference for the field. Note that the main emphasis of [MR92] is on the development of the theory of Dirichlet forms on general state spaces. As mentioned above, the motivation to relax the assumptions (locally compactness) on the state space is closely connected to mathematical physics. First studies in this direction include [AHK77a, AHK77b]. Since the book under review does not add results in this direction we do not provide more references on this important development and refer the interested reader to the discussion in [AR90, BH91, MR92, AMR93, Sta99, Tru00, Alb03, AMR14]. Note that the exposition in [Sta99] goes beyond the scope of other books and covers truly nonsymmetric (without sector condition) and rather general time-dependent Dirichlet forms for infinite-dimensional state spaces.

A second field of current interest which is not touched by the book under consideration is the connection between geometry and Dirichlet forms. Again, we decide not to list many articles but rather give only a few hints where to find more information. The proceedings volume [JKM+98] might be a good start because it contains several related articles. On the one hand, Dirichlet forms provide a tool to define a Laplacian on general state spaces. On the other hand, they provide a framework for results which are robust with respect to geometric quantities. The geometric significance of Harnack inequalities and heat kernel bounds for Dirichlet forms are studied in [Stu94, BM95, Stu95, Stu96], see also the
references therein. In typical situations, the Gaussian short time off-diagonal behavior of the heat kernel is a function of the intrinsic distance. This holds true for general strongly local Dirichlet forms [HR03, AH05]. Aronson-type bounds have been studied in metric measure spaces and on fractals using the theory of Dirichlet forms. The works [Bar98, Kig01, BBK06, Gri10, BBKT10, BGK12, Kig12, GT12, KZ12, KSZ14, GH14, AB14] contain several important results. The recent book [BGL14] is a good source for the relation of functional inequalities and local Dirichlet forms in a general context. Note that the aforementioned contributions mainly concentrate on local Dirichlet forms. Similar studies for nonlocal Dirichlet forms and their relation to geometry seem to be much more subtle. Let us also mention that Dirichlet forms can be applied to other areas like discrete groups or random media. Two chapters of [VSCC92] address discrete groups and estimates of the decay of convolution powers of probability measures on these groups. For applications to random media see [Kum14].

Last, let us mention the monograph [CF12]. It provides a 100-pages summary of the theory of symmetric regular and quasi-regular Dirichlet forms which can be used as a first read. Moreover, it treats new developments in more specialized fields, i.e., trace processes, boundary theory and reflected Dirichlet spaces for regular symmetric Dirichlet forms.

As explained above, nonsymmetric Dirichlet forms were studied right on from the beginning of research activities around Dirichlet forms. The lectures of Y. Oshima at Friedrich-Alexander-Universität Erlangen-Nürnberg in 1988 and 1994 contributed significantly to this development. It is not clear to the author of this review when the notion of a semi-Dirichlet form was used for the first time. Regularity resp. quasi-regularity of semi-Dirichlet forms are studied in [MOR95, AMR95]. Several examples of semi-Dirichlet forms can be found in [RS95, FU12, Uem14]. Chapter 7 of [BB04] contains several results on quasi-regular semi-Dirichlet forms. Recently, the Fukushima decomposition and the Beurling-Deny formulae have been studied for semi-Dirichlet forms [HM04, MMS12, CMS12, HMS10, MS12]. See also the very recent survey [MSLF14] and which the author is thankful for having been provided with.

3. Remarks on the new book by Yoichi Oshima

The new book [Osh13] by Y. Oshima extends the theory of Dirichlet forms on locally compact separable metric spaces. After having set up the analytic theory of lower bounded semi-Dirichlet forms in the first two chapters, the relation to Hunt processes is studied in Chapter 3. Chapters 4 and 5 are devoted to additive functionals and decompositions. Thus the book can be viewed as a natural extension of [FOT11]. The book is carefully written and has got a nice layout. It certainly will serve as a standard reference for its respective field. Because of the development of the theory of Dirichlet forms during the past 25 years, it is natural that there are by now other monographs. The standard reference for symmetric Dirichlet forms on locally compact separable metric spaces remains [FOT11]. Symmetric Dirichlet forms in a more general framework are presented nicely also in [CF12]. If interested in symmetric or nonsymmetric Dirichlet forms satisfying the sector condition on more general state spaces, then [MR92] is the first choice. A special feature of [Osh13] is the treatment of time-dependent Dirichlet forms in Chapter 6.
which is an advancement of parabolic potential theory. These questions are also covered in [Sta99] and even in much greater generality. However, the presentation in [Osh13] is easier accessible and sufficient for many purposes. Altogether, the new book [Osh13] is a welcome addition to the literature. The accuracy of the presentation and the similarity of the style to the one of [FOT11] will be appealing to many researchers. As mentioned above, the main source for the book are the lectures by the author on nonsymmetric Dirichlet forms from 1988 and 1994 in Erlangen. Although the corresponding lecture notes are not available to the public, they have been circulated with the author’s permission among interested colleagues. In this way, the book certainly has had an impact on the theory long before its publication.

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