We study the reliability of extractions of $|V_{us}|$ based on flavor-breaking hadronic $\tau$ decay sum rules. The “(0,0) spectral weight”, proposed previously as a favorable candidate for this extraction, is shown to produce results having poor stability with respect to $s_0$, the upper limit on the relevant spectral integral, suggesting theoretical errors much larger than previously anticipated. We argue that this instability is due to the poor convergence of the integrated $D=2$ OPE series. Alternate weight choices designed to bring this convergence under better control are shown to produce significantly improved stability, and determinations of $|V_{us}|$ which are both mutually compatible, and consistent, within errors, with values obtained by other methods.

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I. BACKGROUND

Three-family unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) matrix implies

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 ,$$

with the $V_{ub}$ contribution playing a numerically negligible role. Analyses of $K_{\ell 3}$ incorporating recent updates to the $K_L$ lifetime, the $K^+\ell\nu$, $K_L\ell\nu$ and $K_s\ell\nu$ branching fractions, and the $K_{\ell 3}$ form factor slope parameters, together with strong isospin-breaking and long distance electromagnetic corrections computed in the framework of ChPT, lead to

$$f_+(0)|V_{us}| = 0.2173 \pm 0.0008 ,$$

which, with the Leutwyler-Roos estimate, $f_+(0) = 0.961 \pm 0.008$ (compatible within errors with recent quenched and unquenched lattice results), yields

$$|V_{us}| = 0.2261 \pm 0.0021 .$$
This result is in good agreement with expectations based on unitarity and the most recent update of the average of superallowed $0^+ \rightarrow 0^+$ nuclear $\beta$ decay \cite{11} and neutron decay \cite{12} results, $|V_{ud}| = 0.9738 \pm 0.0003 \pm 8$. The $\sim 2\sigma$ discrepancy observed when earlier $K$ decay results were employed thus appears finally to have been resolved. One should, however, bear in mind two recent developments relevant to $|V_{ud}|$: (i) a new measurement of the neutron lifetime, in strong disagreement with the previous world average \cite{13}, and (ii) a Penning trap measurement of the $Q$ value of the superallowed $^{46}V$ decay \cite{14} in significant disagreement with the average used as input in Ref. \cite{11}, and with the potential to raise doubts about current evaluations of structure-dependent isospin-breaking corrections \cite{15}. The potentially unsettled $|V_{ud}|$ situation makes alternate (non-$K_{\ell 3}$) determinations of $|V_{us}|$ of interest, both as a means of testing the Standard Model (SM) scenario for strangeness-changing interactions, and for reducing errors through averaging. Two such alternate methods have been proposed recently.

In the first, $|V_{us}/V_{ud}|$ is extracted using lattice results for $f_K/f_\pi$ in combination with experimental results for $\Gamma[K_{\mu 2}]/\Gamma[\pi_{\mu 2}]$ \cite{16}. With the recently updated MILC $n_f = 3$ unquenched lattice result, $f_K/f_\pi = 1.198^{+0.016}_{-0.006}$ \cite{17}, the first method yields

$$|V_{us}| = 0.2245^{+0.0011}_{-0.0031}, \quad (4)$$

compatible within errors with the $K_{\ell 3}$ determination.

The second of these proposals involves the analysis of flavor-breaking sum rules employing strange and non-strange hadronic $\tau$ decay data \cite{18}, and forms the subject of the rest of this paper. Existing results, based on the “(0,0) spectral weight” version of this analysis, will be discussed as part of the development below. The discussion to follow represents an update and extension of the preliminary results presented in Ref. \cite{19}.

II. $V_{us}$ FROM HADRONIC $\tau$ DECAY DATA

With $\Pi^{(J)}_{V/A;ij}$ the spin $J$ parts of the flavor $ij = ud, us$ vector/axial vector correlators, $\rho^{(J)}_{V/A;ij}$ the corresponding spectral functions, and $R_{V/A;ij} \equiv \Gamma[\tau^- \rightarrow \nu_\tau \text{hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e(\gamma)]$, the kinematics of hadronic $\tau$ decay imply \cite{20}

$$R_{V/A;ij} = 12\pi^2|V_{ij}|^2 S_{EW} \int_{m_\tau^2}^{m_\tau^2} \frac{ds}{m_\tau^2} (1 - y_\tau)^2 \left[(1 + 2y_\tau) \rho^{(0+1)}_{V/A;ij}(s) - 2y_\tau \rho^{(0)}_{V/A;ij}(s)\right] \quad (5)$$

where $y_\tau = s/m_\tau^2$, $V_{ij}$ is the flavor $ij$ CKM matrix element, $S_{EW} = 1.0201 \pm 0.0003$ \cite{21} is a short-distance electroweak correction, and the superscript $(0 + 1)$ denotes the sum of $J = 0$ and $J = 1$ contributions. Eq. (5) is written in such a way that both terms on the RHS can be rewritten using the general finite energy sum rule (FESR) relation,

$$\int_{m_\tau^2}^{s_0} ds \, w(s) \rho(s) = -\frac{1}{2\pi i} \oint_{|s| = s_0} ds \, w(s) \Pi(s), \quad (6)$$

valid for any analytic weight $w(s)$ and any correlator $\Pi$ without kinematic singularities. Quantities $R^{(k,m)}_{V/A;ij}$, analogous to $R_{V/A;ij}$, are obtained by rescaling the experimental decay
distribution with the factor \((1 - y_f)\) before integrating. The corresponding FESR’s are referred to as the \(\langle k, m \rangle\) spectral weight sum rules. Similar FESR’s can be written down for general weights \(w(s)\), for \(s_0 < m_s^2\), and for the separate correlator combinations \(\Pi^{(0+1)}_{V/A;ij}(s)\) and \(s\Pi^{(0)}_{V/A;ij}(s)\). The corresponding spectral integrals, \(\int_{s_0}^s ds w(s)\rho^{(J)}_{V/A;ij}(s)\), will be denoted \(R^{w}_{ij}(s_0)\) in what follows. In FESR’s involving both the \(J = 0 + 1\) and \(J = 0\) combinations, the purely \(J = 0\) contribution will be referred to as “longitudinal”.

With this background, the \(\tau\)-based extraction of \(V_{us}\) works schematically as follows. Given experimental values for the spectral integrals \(R^{w}_{ij}(s_0)\), \(ij = ud, us\), corresponding to the same \(w(s)\) and same \(s_0\), the combination

\[
\delta R^w(s_0) = \frac{R^w_{ud}(s_0)}{|V_{ud}|^2} - \frac{R^w_{us}(s_0)}{|V_{us}|^2}
\]  

vanishes in the SU(3) flavor limit and hence has an OPE representation, \(\delta R^w_{OPE}(s_0)\), which begins at dimension \(D = 2\). Solving for \(|V_{us}|\), one has

\[
|V_{us}| = \sqrt{\frac{R^w_{us}(s_0)}{\left[ R^w_{ud}(s_0)/|V_{ud}|^2 \right] - \delta R^w_{OPE}(s_0)}}.
\]

At scales \(\sim 2 - 3\) GeV\(^2\), and for weights used in the literature, the dominant \(D = 2\) term in \(\delta R^w_{OPE}(s_0)\) is much smaller than the leading \(D = 0\) contribution and, as a consequence, similarly smaller than the separate \(ud, us\) spectral integrals (for physical \(m_s\), typically at the few to several percent level). The OPE uncertainty, \(\Delta(\delta R^w_{OPE}(s_0))\), thus produces a fractional \(|V_{us}|\) error \(\approx \Delta(\delta R^w_{OPE}(s_0))/2 R^w_{ud}(s_0)\), much smaller than the fractional uncertainty on \(\delta R^w_{OPE}(s_0)\) itself. High accuracy for \(|V_{us}|\) is thus obtainable with only modest accuracy for \(\delta R^w_{OPE}(s_0)\) provided experimental spectral integral errors can be kept under control.

At present, the absence of a \(V/A\) separation of the \(us\) spectral data means one must work with sum rules based on the observed \(V+A\) combination. This combination also reduces the fractional \(ud\) spectral integral errors. With present \(ud\) spectral data \([22, 23, 24]\), these errors are at the \(\sim 0.5\%\) level for weights used previously in the literature. The much smaller strange branching fraction leads to limited statistics and coarser binning for the \(us\) spectral distribution \([25, 26, 27]\). The \(K\) pole term is very accurately known, but errors are \(\sim 6 - 8\%\) in the \(K^*\) region and \(> 20 - 30\%\) above 1 GeV\(^2\). For weights used in the literature, the result is \(us\) spectral integrals with \(\sim 3 - 4\%\) uncertainties \([25, 27, 28]\). Experimental errors on \(|V_{us}|\) are thus at the \(\sim 1.5 - 2\%\) level, and dominated by uncertainties in the \(us\) sector. The situation should improve dramatically with the increase in statistics and improved \(K\) identification available from the B factory experiments.

A number of points relevant to reducing OPE errors are outlined below. Note that use of the \(V+A\) sum rules has the added advantage of strongly suppressing duality violation at the scales considered \([24]\). Working with weights satisfying \(w(s = s_0) = 0\) further suppresses such contributions \([29, 30]\), as does working at scales \(s_0 > 2\) GeV\(^2\) \([31]\). A major, and irreducible, source of OPE uncertainty for “inclusive” sum rules (those involving both \(J = 0 + 1\) and \(J = 0\) contributions) is that produced by the bad behavior
of the integrated longitudinal \( D = 2 \) OPE series. This representation displays badly non-convergent behavior, order by order in \( \alpha_s \), even at the maximum scale, \( s_0 = m_s^2 \), allowed by kinematics \([32]\). Moreover, for the \((k, 0)\) spectral weights, those truncations of this series employed in the literature can be shown to strongly violate constraints associated with the positivity of the continuum (non-\( K \)-pole) part of \( \rho_{V+A;us}^{(0)}(s) \) \([33]\).

The impossibility of making sensible use of the longitudinal OPE representation necessitates working with sum rules based on the \( J = 0+1 \) combination. Since no complete \( J = 0 \) spin separation of the spectral data exists, a phenomenological subtraction of the longitudinal parts of the experimental decay distribution is necessary. This can be done with good accuracy because the (very accurately known) \( \pi \) and \( K \) pole terms dominate the subtraction, for a combination of chiral and kinematic reasons \([29, 34]\). Small continuum \( us \) longitudinal corrections have been constrained phenomenologically, by a sum rule analysis of the flavor \( us \) pseudoscalar channel \([35]\) and a coupled-channel dispersive analysis of the scalar channel which employs experimental \( K\pi \) phases, ChPT, and short-distance QCD constraints as input \([36]\). The contribution of the resulting phenomenological longitudinal continuum \( us \) model to the \( \tau \) strangeness branching fraction (corresponding to the \((0, 0)\) spectral weight, and \( s_0 = m_s^2 \)) is \(< 1\% \) of the total. The uncertainty in the bin-by-bin spectral weight, even for the \((0, 0)\) spectral weight. Dominance of the longitudinal \( us \) continuum by the \( K_0^*(1430) \) and \( K(1460) \) resonance contributions also means the impact of longitudinal subtraction uncertainties on the resulting \( J = 0 + 1 \) spectral integrals decreases rapidly with decreasing \( s_0 \) and is much reduced for weights (like the \((k > 0, 0)\) spectral weights) which fall off to zero faster at \( s = s_0 \) than does the \((0, 0)\) spectral weight.

For the remainder of this paper, we focus on the flavor-breaking combination

\[
\Delta \Pi(s) \equiv \Pi^{(0+1)}_{V+A;ud}(s) - \Pi^{(0+1)}_{V+A;us}(s) .
\]  

(9)

### A. OPE Input

The OPE representation of \( \Delta \Pi \) is known up to dimension \( D = 6 \).

The leading, \( D = 2 \), term is given by \([37]\)

\[
\left[ \Delta \Pi(Q^2) \right]_{D=2} = \frac{3}{2\pi^2} \frac{m_s}{Q^2} \left[ 1 + 2.333\bar{a} + 19.933\bar{a}^2 + 208.746\bar{a}^3 + (2378 \pm 200)\bar{a}^4 + \cdots \right]
\]  

(10)

where \( \bar{a} = \alpha_s(Q^2)/\pi \) and \( \bar{m}_s = m_s(Q^2) \), with \( m_s(Q^2) \) and \( \alpha_s(Q^2) \) the running strange quark mass and coupling in the \( \overline{MS} \) scheme. The \( O(\bar{a}^4) \) term \([37]\) was estimated using methods which provided an extremely accurate prediction for the \( O(\bar{a}^3) \) coefficient, and similarly reliable predictions of the \( n_f \)-dependent \( O(\bar{a}^3m_s^2) \) coefficients of the electromagnetic current correlator \([38]\), all in advance of their explicit calculation. For \( \bar{a} \) and \( \bar{m}_s \), we employ exact solutions corresponding to the 4-loop-truncated \( \beta \) and \( \gamma \) functions \([39]\), with initial conditions for \( m_s \) \([1]\) and \( \alpha_s \) \([23]\)

\[
m_s(2 \text{ GeV}) = 105 \pm 25 \text{ MeV}, \quad \alpha_s(m_s^2) = 0.334 \pm 0.022.
\]  

(11)
The forms of the \( D = 4 \) and \( D = 6 \) contributions are well known, and may be found in Ref. [21]. The dominant \( D = 4 \) contribution is that proportional to the RG-invariant strange quark condensate. This is evaluated using ChPT quark mass ratios \([41]\), GMOR for the light quark condensate, and the conventional estimate \( r_c \equiv \langle m_s \bar{c} \rangle / \langle m_s \bar{s} \rangle = 0.8 \pm 0.2 \) for the ratio of the two condensates. \( D = 6 \) contributions are evaluated using the vacuum saturation approximation (VSA) and assigned an error of \( \pm 500\% \). \( D > 6 \) terms are assumed negligible. This assumption can be tested for self-consistency since, \( \alpha \) unsuppressed by any additional factors of \( s \) with respect to \( s_0 \), determinations of the first involves the difference of the truncated correlator and Adler function versions of \( \Delta \Pi \), or its Adler function \( \Delta \), having such large higher order coefficients. The second source of error is the truncation of the \( \Delta \Pi \) and \( \Delta \) contributions potentially enhanced through large \( \alpha_s \) scale determinations of the series, the second the stability with respect to \( s_0 \). The dominant \( s \) contributions scale as \( 1/\delta s \), while \( s \) contributions scale as \( 1/\delta s^2 \). For weights, \( w(y) = \sum_s c_m y^m \) (with \( y = s/s_0 \)), integrated \( D = 2m + 2 \) OPE contributions scale as \( 1/s_0^{6m} \). Neglected, but non-negligible, \( D > 6 \) contributions will thus show up as unphysical instabilities in the output of a given sum rule (in this case, \(|V_{us}|\)) with respect to \( s_0 \). Since, when \( c_m \neq 0 \), the \( D = 2m + 2 \) contribution to \( \delta R^w_{\text{OPE}}(s_0) \) is unsuppressed by any additional factors of \( \alpha_s \), an \( s_0 \)-stability test is particularly important for weights, \( w(y) \), having \( D > 6 \) OPE contributions potentially enhanced through large values for one or more of the \( c_m \) with \( m > 2 \). The \( (2, 0) \), \( (3, 0) \) and \( (4, 0) \) \( J = 0 + 1 \) spectral weights, \( w^{(2,0)}(y) = 1 - 2 y - 2 y^2 + 8 y^3 - 7 y^4 + 2 y^5 \), \( w^{(3,0)}(y) = 1 - 3 y + 10 y^3 - 15 y^4 + 9 y^5 - 2 y^6 \), and \( w^{(4,0)}(y) = 1 - 4 y + 3 y^2 + 10 y^3 - 25 y^4 + 24 y^5 - 11 y^6 + 2 y^7 \), are examples of weights having such large higher order coefficients.

The \( D = 2 \) OPE integrals are evaluated using the CIPT prescription [41], in which the RG-improved expression for \( \Delta \Pi \), or its Adler function \( \Delta D(Q^2) = -Q^2 d \Pi(Q^2)/dQ^2 \), is used point-by-point along the integration contour. To all orders, the two versions of the \( D = 2 \) integral are necessarily equal, being related by a partial integration. With both \( \Delta \Pi \) and \( \Delta D \) truncated at the same order, however, they differ by terms of higher order in \( \bar{a} \). Our central values employ the \( O(\bar{a}^4) \) truncated RG-improved correlator version (for arguments in favor of this choice, see Ref. [37]).

For the scales employed in this study, both the magnitude and error of \( \delta R^w_{\text{OPE}}(s_0) \) are dominated by the \( D = 2 \) contribution. The \( D = 2 \) error has two important sources. The first is an overall scale uncertainty, associated with the error on the input strange quark mass, \( \bar{m}_s(2 \text{ GeV}) \). This uncertainty is \( \sim 50\% \) for the PDG04 input, Eq. (11), but should be reduced considerably by ongoing progress in unquenched lattice simulations. The second source of error is the truncation of the \( D = 2 \) series.

The \( D = 2 \) truncation error is potentially significant because the series in Eq. (10) is slowly converging near the spacelike point on the FESR contour. In fact, with high-scale determinations of \( \alpha_s(M_Z) \) [1] corresponding to an \( n_f = 3 \) coupling \( \bar{a}(Q^2 = m^2) \simeq 0.10 - 0.11 \), the last three terms in Eq. (10) are actually slowly increasing with order at the spacelike point throughout the whole of the kinematically allowed region. Convergence of the integrated series will thus typically be slow for weights which emphasize this part of the contour. The \( (k, 0) \) spectral weights, which involve \( 2 + k \) powers of \( 1 - y \), fall more and more into this category as \( k \) is increased, since, on the contour \( y = e^{i \theta} \), \( |1 - y|^{2+k} \propto \sin^{2+k} (\theta/2) \) is more and more peaked in the spacelike direction.

We will use two different monitors of the convergence of the integrated \( D = 2 \) series. The first involves the difference of the truncated correlator and Adler function versions of the series, the second the stability with respect to \( s_0 \) of the sum rule output.

For a series with good convergence, the correlator and Adler function versions of the truncated sum should be in good agreement and, moreover, show improved agreement
with increasing truncation order. We define \( r_k^w(s_0) \) to be the fractional change in the relevant integrated order-\( k \)-truncated \( D = 2 \) sum produced by shifting from the correlator to corresponding Adler function version. Large values of \( |r_k^w(s_0)| \) and/or an increase of \( |r_k^w(s_0)| \) with \( k \) then signal slow convergence of the \( D = 2 \) series \([42]\). We will take twice the sum, in quadrature, of the last included term and the difference between the truncated correlator and Adler function versions of the sum as our estimate for the \( D = 2 \) truncation error. The resulting estimate is considerably more conservative than those used previously in the literature.

Regarding \( s_0 \) stability, a truncated series well-converged at \( s_0 = m_\tau^2 \) should remain so for some range of \( s_0 < m_\tau^2 \). Since the exact \( \delta R^w(s_0) \) would produce a \( |V_{us}| \) independent of \( s_0 \), a well-converged truncation of the integrated \( D = 2 \) OPE series should produce \( |V_{us}| \) values stable over some interval of \( s_0 \). If, however, neglected higher order terms are actually important at \( s_0 = m_\tau^2 \), they will be even more so at lower scales, making the accuracy of the truncated expression even worse at those scales, and producing an unphysical \( s_0 \) dependence to the extracted \( |V_{us}| \) \([43]\). The absence of a stability window in \( s_0 \) thus implies the unreliability of the truncated integrated \( D = 2 \) series and/or the importance of neglected, but non-negligible, higher dimension terms.

B. Data Input

For the \( ud \) data we use the ALEPH spectral distribution and covariance matrix \([23]\), with overall normalization corrected for the small changes in the \( e, \mu \), and strangeness branching fractions subsequent to the original ALEPH publication.

For the \( us \) data, we have, unfortunately, been unable to obtain the covariance matrix from the OPAL collaboration. The OPAL publication \([27]\) quotes correlated \( us \) spectral integral errors only for a range of the \((k, m)\) spectral weights, and only for \( s_0 = m_\tau^2 \). This information is insufficient to allow the errors resulting from other choices for either \( s_0 \) or \( w(y) \) to be inferred, precluding implementation of the crucial \( s_0 \)-stability test, even for the \((k, 0)\) spectral weights. We have thus chosen to work with the somewhat older ALEPH data \([25]\), whose covariance matrix is publicly available. The two data sets differ mainly in the values of a small number of the strange branching fractions, a particularly important difference being that for \( \tau^- \to K^-\pi^+\pi^-\nu_\tau \). To take into account the changes in the branching fraction values, we follow the strategy adopted in Ref. \([44]\). In this approach, the ALEPH distribution for each mode is rescaled by the ratio of the current world average to the ALEPH 1999 branching fraction value. The resulting rescaled mode-by-mode distributions are then recombined to form the modified total \( us \) V+A spectral distribution \([13]\). This scheme should be reliable for modes whose rescalings are close to 1, but is less clearly so for those, like \( \tau^- \to K^-\pi^+\pi^-\nu_\tau \), where this is not the case.

The \( us \) \( K \) pole spectral integral contribution is fixed by the \( \Gamma(K_{\mu 2}) \)-based SM prediction. This is done because (i) the SM prediction is compatible with the observed \( \tau \to K\nu_\tau \) branching fraction, but \( \sim 6 \) times more precise, and (ii) Eq. \([8]\) already pre-supposes the validity of the SM mechanism for hadronic \( \tau \) decays.

There exist three determinations of the branching fraction \( B[\tau^- \to K^-\pi^+\pi^-\nu_\tau] \), the
original ALEPH result, $0.214 \pm 0.037 \pm 0.029\%$, and the more recent CLEO and OPAL results, $0.384 \pm 0.014 \pm 0.038\%$ and $0.415 \pm 0.059 \pm 0.031\%$, respectively. While the CLEO and OPAL results are in good agreement, the agreement with ALEPH is less compelling. The OPAL $u\bar{s}$ spectral integral results employed the three-fold average, $0.330 \pm 0.028\%$, in setting the overall normalization of the $K^-\pi^+\pi^-$ spectral distribution. Results based on this normalization are denoted ‘ACO’ below. To stress the sensitivity to apparently minor changes in the branching fraction values, as well as the importance of improved precision, we also present results (denoted ‘CO’ in what follows) corresponding to the alternate rescaling, produced by the average, $0.40\%$, of the OPAL and CLEO central values. The ‘ACO’/‘CO’ branching fraction difference, though only $0.07\%$, represents more than 2% of the $\sim 3\%$ total strangeness branching fraction and hence has the potential to shift $|V_{us}|$ by as much as $\sim 1\%$ ($\sim 0.0020$).

III. ANALYSIS AND RESULTS

In this section we first discuss the existing $(0,0)$ spectral weight analysis, then present alternate determinations, based on weights with improved $s_0$-stability for $|V_{us}|$.

A. The $(0,0)$ Spectral Weight Determination of $V_{us}$

The $(0,0)$ spectral weight has been proposed in the literature as a particularly favorable case for the $|V_{us}|$ analysis \cite{18}. A key potential advantage is the very close cancellation between the weighted $ud$ and $us$ spectral integrals. This manifests itself in the OPE representation in suppressed values for the integrated $D = 2$ OPE series and hence similarly suppressed values for the $m_s$-induced $D = 2$ scale uncertainty. This scale uncertainty is the dominant component of the estimated theoretical error in Ref. \cite{18}, being a factor of $\sim 2$ larger than the estimated $D = 2$ truncation uncertainty, and much larger than any of the other contribution. The combined theory error, $\pm 0.0009$ \cite{18}, is swamped by the current $\pm 0.0033$ experimental error, but, if reliable, would make the $(0,0)$ analysis a very favorable one for use in determining $|V_{us}|$ once the much improved $us$ spectral data from BABAR and BELLE becomes available. Unfortunately, as we will see below, the theoretical uncertainty on $\delta R_{OPE}^{(0,0)}(m^2_s)$ almost certainly significantly exceeds the estimate of Refs. \cite{18}. Indeed, we will argue that the convergence of the integrated $D = 2$ $(0,0)$ spectral weight series is sufficiently bad that the $(0,0)$ analysis is, in fact, an unfavorable one for the extraction of $|V_{us}|$. Fortunately, alternatives exist with significantly improved convergence behavior, which allow one to take advantage of the general approach proposed in Refs. \cite{18}. We return to these in the next subsection, after elaborating on the problematic features of the $(0,0)$ analysis.

The $(0,0)$-spectral-weight-based results of Refs. \cite{18} were obtained using the $O(\bar{a}^3)$-truncated Adler function version of the integrated $D = 2$ OPE series. The truncation error was estimated by combining the magnitude of the last ($O(\bar{a}^3)$) included term in quadrature with a measure of the residual scale dependence, the latter obtained by chang-
TABLE I: Values of $r^w_k(s_0)$, as defined in the text, for various weight choices $w(y)$ and $s_0 = m^2$.

| Weight $w_j(y)$ | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ |
|----------------|---------|---------|---------|---------|
| $w^{(0,0)}_{j=0+1}$ | -0.01  | 0.06    | 0.20    | 0.67    |
| $\hat{w}_{10}$   | -0.11  | -0.07   | -0.05   | -0.03   |
| $w_{20}$         | -0.11  | -0.08   | -0.05   | -0.03   |
| $w_{10}$         | -0.10  | -0.06   | -0.03   | -0.01   |

TABLE II: $|V_{\text{us}}|$ as a function of $s_0$ for various FESR weight choices and the ACO and CO treatments of the $\text{us}$ data. $s_0$ is given in units of GeV$^2$.

| $s_0$ | $w^{(0,0)}_{\text{ACO}}$ | $\hat{w}_{\text{ACO}}$ | $w_{20}$ | $w_{10}$ | $w^{(0,0)}_{\text{CO}}$ | $\hat{w}_{\text{CO}}$ | $w_{20}$ | $w_{10}$ |
|-------|----------------|----------------|---------|---------|----------------|----------------|---------|---------|
| 2.35  | 0.2149         | 0.2220         | 0.2243  | 0.2201  | 0.2172         | 0.2236         | 0.2255  | 0.2218  |
| 2.55  | 0.2167         | 0.2218         | 0.2235  | 0.2203  | 0.2192         | 0.2236         | 0.2250  | 0.2223  |
| 2.75  | 0.2181         | 0.2218         | 0.2230  | 0.2207  | 0.2207         | 0.2239         | 0.2246  | 0.2229  |
| 2.95  | 0.2193         | 0.2220         | 0.2227  | 0.2211  | 0.2219         | 0.2243         | 0.2245  | 0.2235  |
| 3.15  | 0.2202         | 0.2223         | 0.2226  | 0.2216  | 0.2228         | 0.2246         | 0.2246  | 0.2241  |

Using the CIPT scale choice $\mu^2 = Q^2$ to $\mu^2 = \xi Q^2$, with $0.75 < \xi < 1.5$. The resulting truncation uncertainty is $+10\%$ to $-20\%$ of the $O(\bar{a}^3)$-truncated Adler function sum. The results of Ref. [37], however, show that the truncated sum is decreased by 47% if evaluated instead using the $O(\bar{a}^4)$ integrated correlator form. This shift is larger by a factor of nearly 2.5 than the truncation error estimate of Refs. [18].

Further evidence for the slow convergence of the integrated $D = 2$ $(0,0)$ series is provided by the values of $r^w_k(m^2)$, given in Table I. The values are not small in general, and grow rapidly with increasing $k$. This increase raises serious doubts about any truncation error estimate based on features of the $O(\bar{a}^4)$-truncated sum, even one, like ours, which is more numerically conservative than that of Refs. [18].

A final illustration of the unreliability of the convergence of the integrated $D = 2$ $(0,0)$ spectral weight series is provided by the $s_0$-stability results, shown in columns 2 (ACO case) and 6 (CO case) of Table II. No stability window for $|V_{\text{us}}|$ is evident in either case, as expected given the indications for poor convergence already discussed above. The level of instability is much larger than the total theory error estimated in Refs. [18], even if we restrict our attention to the upper half of the interval displayed in the table.

**B. Alternate Weight Choices**

From Refs. [34, 37] it is clear that the integrated $J = 0 + 1$, $D = 2$ OPE series for the $(k,0)$ spectral weights display rather unfavorable convergence behavior, making such weights problematic for use in extracting $|V_{\text{us}}|$. Although $(k,0)$ spectral weights occur...
frequently in treatments of hadronic $\tau$ decay data, one should bear in mind that one of the primary reasons for their introduction, namely the possibility of using them in inclusive analyses, is entirely negated by the necessity of avoiding inclusive analyses, which follows from the extremely bad behavior of the integrated longitudinal OPE representation.

In Ref. [34], the possibility of constructing weights more suitable for use in $J = 0 + 1$ non-inclusive sum rules was investigated. These weights were chosen to (i) emphasize contributions from regions of the contour showing improved convergence for the $D = 2$ $\Delta\Pi$ series, (ii) suppress contributions from the region of the spectrum above $\sim 1 \text{ GeV}^2$ where $us$ spectral errors are large, and (iii) control the size of higher order coefficients which might otherwise enhance $D > 6$ contributions. Three such weights, $w_{10}$, $\tilde{w}_{10}$, and $w_{20}$ [46], were constructed, all having profiles on the timelike axis intermediate between those of the $J = 0 + 1$, $(0, 0)$ and $(2, 0)$ spectral weights, and hence similarly intermediate relative $us$ spectral integral errors. The much improved $D = 2$ convergence, compared to that of the $(k, 0)$ spectral weights, is shown explicitly in Ref. [34]. Further evidence for this improvement is contained in Table I, which shows good agreement, improving with increasing truncation order, between the correlator and Adler function versions of the truncated $D = 2$ sums. The contrast to the $(0, 0)$ spectral weight case is striking.

As $s_0$ is decreased, terms in the integrated $D = 2$ series of $O(\bar{a}^k)$, with $k > 4$, grow in size relative to the leading $0^\text{th}$ order term. This growth is least rapid for $w_{20}$ and most rapid for $w_{10}$. Though the coefficients multiplying these terms in $\Delta\Pi$ are not known, this nonetheless indicates that stability for the improved $D = 2$ convergence will be best for $w_{20}$ and worst for $w_{10}$. The $r_k^{w_{10}}(s_0)$ values in fact display a cross-over in sign and increase in magnitude with increasing $k \leq 4$ below $s_0 \sim 2.55 \text{ GeV}^2$, signalling probable deteriorating convergence. We thus base our final results on the highest available scale, $s_0 = m_{\tau}^2$, and favor the $\tilde{w}_{10}$ and $w_{20}$ analyses over that based on $w_{10}$.

C. Results

Much improved $s_0$ stability is observed for the weights with improved $D = 2$ convergence, particularly $w_{20}$ and $\tilde{w}_{10}$. Contributions to the errors on $|V_{us}|$ for the various weights are given in Table III [47]. Sources contributing < 0.0003 theoretical uncertainty for all weights considered are not listed explicitly but are included in the total theoretical error. Since the results for our favored weights, $w_{20}$ and $\tilde{w}_{10}$, are in excellent agreement, and the combined errors are minimized for the latter, we take as our final determination the $\tilde{w}_{10}$ results. Displaying the larger of the asymmetric theory errors, the ACO (CO) $us$ data treatments yield the following results, both compatible, within errors, with those of Eqs. (3) and (4):

$$|V_{us}| = 0.2223 (0.2246) \pm 0.0032_{exp} \pm 0.0038_{th}.$$  \hspace{1cm} (12)
TABLE III: Contributions to the error on $|V_{us}|$, at $s_0 = m_{\tau}^2$, for various FESR weight choices.

| Weight       | $us$ data | $ud$ data | $m_s$-scale | $\delta \alpha_s$ | $\delta r_c$ | $D = 2$ truncation | Theory (total) |
|--------------|-----------|-----------|-------------|-------------------|--------------|-------------------|---------------|
| $w^{(0,0)}$  | ±0.0040   | ±0.0006   | +0.0006     | +0.0006           | ±0.0000      | ±0.0020           | ±0.0022       |
| $\hat{w}_{10}$ | ±0.0031   | ±0.0006   | −0.0004     | −0.0007           | ±0.0001      | ±0.0009           | ±0.0038       |
| $w_{20}$     | ±0.0028   | ±0.0007   | +0.0036     | +0.0001           | ±0.0005      | ±0.0014           | ±0.0054       |
| $w_{10}$     | ±0.0033   | ±0.0006   | −0.0027     | −0.0002           | ±0.0003      | ±0.0004           | ±0.0028       |

### IV. CONCLUSIONS

We have shown that the values of $|V_{us}|$ extracted using the $(0, 0)$ spectral weight sum rule display a sizeable instability with respect to $s_0$. This instability, combined with the results of Ref. [37], strongly suggests that the true $D = 2$ truncation uncertainty is much larger than previously estimated. Given the level of instability, even our much more conservative estimate, shown in row 1 of Table III, seems far from being overly conservative. We see no plausible way of obtaining a reliable, but more restrictive, estimate of this uncertainty. The $D = 2$ truncation error thus represents a sizeable, and irreducible, limitation on the accuracy of the $(0, 0)$ spectral weight determination of $|V_{us}|$.

In contrast, for the weights $\hat{w}_{10}$, $w_{20}$ and $w_{10}$ (whose integrated $D = 2$ series, by design, display improved convergence), good consistency, and much improved $s_0$-stability, is found. At present the theoretical errors on $|V_{us}|$ for these weights, shown in Table III, are dominated by the $m_s$-scale uncertainty. Near-term improvements in unquenched lattice simulations should significantly reduce this error. A determination of $m_s(2 \text{ GeV})$ to $\pm 5$ MeV, for example, would reduce the combined $\hat{w}_{10}$ theory error to $\pm 0.0013$, bringing a sub-1% determination of $|V_{us}|$ easily within reach with the improved B factory data. Note that $\hat{w}_{10}$ would be favored over $w_{20}$ because of its smaller truncation error.

It is also possible to construct weights having, simultaneously, improved $D = 2$ convergence and reduced $m_s$-scale sensitivity. We have generated a number of such weights, but find they typically weight the region of the spectrum above 1 GeV$^2$ more strongly than do those weights discussed above. As a result, with current $us$ data, they produce very large experimental errors, and are not presently useful. We will report on these weights elsewhere, once the improved $us$ B factory data has become available and meaningful stability tests can be performed.
Acknowledgments

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42 A slowly converging series may, despite its slow convergence, display neither of these prob-
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and Adler function versions, are similar in size [37]. The problem is, however, not exposed
by the behavior of the series of $r_k(4,0)(m^2)$ values ($−0.17, −0.12, −0.07$ and $−0.06$ for
$k = 1, · · · , 4$), though it does show up in the poor $s_0$-stability of the extracted $|V_{us}|$ results
(see, for example, the results quoted in Ref. [19]).
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and the additional information required to make this rescaling analysis possible.
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polynomial coefficients are available from the current authors, on request.
47 Our error for the (0, 0) spectral weight is based on the ALEPH data and hence somewhat
larger than that quoted in the second of Refs. [18], which was based on the OPAL data.