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Gaussian representation of a class of Riesz probability distributions

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Abstract : The Wishart probability distribution on symmetric matrices has been initially defined by mean of the multivariate Gaussian distribution as an of the chi-square distribution. A more general definition is given using results for harmonic analysis. Recently a probability distribution on symmetric matrices called the Riesz distribution has been defined by its Laplace transform as a generalization of the Wishart distribution. The aim of the present paper is to show that some Riesz probability distributions which are not necessarily Wishart may also be presented by mean of the Gaussian distribution using Gaussian samples with missing data.

Keywords: Chi-square distribution, Gaussian distribution, Wishart probability distribution, Riesz probability distribution, Laplace transform.

AMS Classification : 60B11, 60B15, 60B20.

1 Introduction

Let \( V \) be the space of real symmetric \( r \times r \) matrices, \( \Omega \) be the cone of positive definite and \( \overline{\Omega} \) the cone of positive semi-definite elements of \( V \). We denote the identity matrix by \( I \), the trace of an element \( x \) of \( V \) by \( \text{tr}(x) \) and its determinant by \( \Delta(x) \). We equip \( V \) by the scalar product

\[
(\theta, x) = \text{tr}(\theta x).
\]

Denoting

\[
L_{\mu}(\theta) = \int_{V} e^{(\theta, x)} \mu(dx)
\]

the Laplace transform of a measure \( \mu \), it is well known (see Faraut and Korányi (1994)) that there exists a positive measure \( \mu_p \) on \( \overline{\Omega} \) such that the Laplace transform of \( \mu_p \) exists for \( -\theta \in \Omega \) and is equal to

\[
\int_{\overline{\Omega}} \exp(\text{tr}(\theta x)) \mu_p(dx) = \Delta^{-p}(-\theta),
\]

(1.1)

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if and only if $p$ is in the so called Gindikin set
\[
\Lambda = \left\{ \frac{1}{2}, 1, \ldots, \frac{r-1}{2} \right\} \cup \left( \frac{r-1}{2}, +\infty \right).
\] (1.2)

When $p > \frac{r-1}{2}$, the measure $\mu_p$ is absolutely continuous with respect to the Lebesgue measure and is given by
\[
\mu_p(dx) = \frac{1}{\Gamma_{\Omega}(p)} \Delta^{p-r+1/2}(x) \mathbf{1}_\Omega(x)dx,
\]
where
\[
\Gamma_{\Omega}(p) = (2\pi)^{-n/2} \prod_{k=1}^r \Gamma(p - k - 1/2),
\]
where $n$ is the dimension of $V$.

And when $p = \frac{j}{2}$ with $j$ integer, $1 \leq j \leq r - 1$, the measure $\mu_p$ is singular concentrated on the set of elements of rank $j$ of $\Omega$.

For $p \in \Lambda$ and $\sigma \in \Omega$, (1.1) implies that the measure $W_{p,\sigma}$ on $\Omega$ defined by
\[
W_{p,\sigma}(dx) = \Delta^{-p}(\sigma) \exp(-tr(x\sigma^{-1}))\mu_p(dx)
\] (1.3)
is a probability distribution. It is called the Wishart distribution with shape parameter $p$ and scale parameter $\sigma$. Its Laplace transform exists for $\theta \in \sigma^{-1} - \Omega$ and is equal to
\[
\int_{\Omega} \exp(tr(\theta x))W_{p,\sigma}(dx) = \Delta^{-p}(e - \sigma \theta).
\] (1.4)

When $p > \frac{r-1}{2}$, the Wishart distribution $W_{p,\sigma}$ is given by
\[
W_{p,\sigma}(dx) = \frac{\Delta^{-p}(\sigma)}{\Gamma_{\Omega}(p)} \exp(-tr(x\sigma^{-1}))\Delta^{p-r+1/2}(x) \mathbf{1}_\Omega(x)dx
\] (1.5)

Further details on the Wishart distribution on $\Omega$ may be obtained from Casalis and Letac (1996) or from Letac and Massam (1998). It is however important to mention that the Wishart probability distribution has initially been defined as a multivariate extension of the chi-square distribution. In fact, taking $U_1, \ldots, U_p$ random vectors in $\mathbb{R}^r$ with distribution $N(0, \sigma)$, the distribution of the matrix $X = \sum_{i=1}^p U_i^t U_i$ is a Wishart matrix.

This kind of Wishart probability distributions may be defined using the Gaussian random matrices. Let $u = \{u_1, \ldots, u_r\}$ be a random vector with distribution $N_r(0, \sigma)$. Suppose that we have a random sample of $u$ i.e., a sequence of independent and identically distributed random variables with distribution equal to the distribution of $u$, and consider the $r \times s$ Gaussian matrix
\[
U = \begin{pmatrix}
u_{1,1} & \ldots & u_{1,s} \\
\vdots & \ddots & \vdots \\
u_{r,1} & \ldots & u_{r,s}
\end{pmatrix}.
\]

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The distribution of $U$ is $N_{r,s}(0, \sigma)$ equal to
\[
\frac{1}{(2\pi)^{\frac{r^2}{2}} \Delta^2(\sigma)} e^{-\frac{1}{2} \langle u, \sigma^{-1} w \rangle},
\]
where $\langle u, \sigma^{-1} w \rangle = \text{tr}(\sigma^{-1} u^t u)$.

We have

**Proposition 1.1** If $U \sim N_{r,s}(0, \sigma)$, then the matrix $X = U^t U$ has the Wishart probability distribution $W_r(\frac{s^2}{2}, \sigma^{-1})$.

**Proof** The Laplace transform of $X$ estimated in $\theta \in (\frac{s^2}{2} - \Omega)$ is given by
\[
E(e^{\langle \theta, X \rangle}) = \frac{1}{(2\pi)^{\frac{r^2}{2}} \Delta^2(\sigma)} \int_{\mathbb{R}^r} e^{\langle \theta, a^t u \rangle - \frac{1}{2} \langle u, \sigma^{-1} w \rangle} du
\]
\[
= \frac{1}{(2\pi)^{\frac{r^2}{2}} \Delta^2(\sigma)} \int_{\mathbb{R}^r} e^{-\frac{1}{2} \langle u, (\sigma^{-1} - 2\theta) w \rangle} du
\]
\[
= \frac{1}{\Delta^2(\sigma)} \left( \frac{(\sigma^{-1} - \theta)^{-1}}{\Delta^2((\sigma^{-1})^{-1})} \right).
\]

It is then the Laplace transform of a $W_r(\frac{s^2}{2}, \sigma^{-1})$ distribution.

This distribution is absolutely continuous with respect to the Lebesgue measure if and only if $\frac{s^2}{2} > \frac{r^2}{2}$, that is if and only if the size $s$ of the sample is greater or equal to the dimension $r$ of the space of observations.

If we consider $p$ independent random matrices $U_1, ..., U_p$ with distribution $N_{r,s}(0, \sigma)$, then the distribution of the matrix $X = \sum_{i=1}^{p} U_i^t U_i \sim W_r(\frac{ps^2}{2}, \sigma^{-1})$.

Of course the class of Wishart distributions defined by mean of the Gaussian vectors or the Gaussian matrices doesn’t cover all the Wishart distributions, however the fact that this random matrix is expressed in terms of Gaussian random vectors allows to establish many important properties of this subclass of the class of Wishart probability distribution.

Using a theorem due to Gindikin which relays on the notion of generalized power, Hassairi and Lajmi [4] have introduced an important generalization of the Wishart distribution that they have called Riesz distribution. This distribution in its general form is defined by its Laplace transform. We will show that some among the Riesz distributions may be represented by the Gaussian distribution using samples of Gaussian vectors with missing data.

## 2 Riesz distributions

For $1 \leq i, j \leq r$, we define the matrix $\mu_{ij} = (\gamma_{kl})_{1 \leq k, l \leq r}$ such that $\gamma_{ij} = \frac{1}{\sqrt{2}}$ and the other entries are equal to 0. We also set
\[
e_i = \sqrt{2} \mu_{ii} \text{ for } 1 \leq i \leq r, \quad e_{ij} = (\mu_{ij} + \mu_{ji}), \text{ for } 1 \leq i < j \leq r.
\]
With these notations, an element $x$ of $V$ may be written

$$x = \sum_{i=1}^{r} x_i c_i + \sum_{i<j} x_{ij} e_{ij}.$$ 

In particular, for $1 \leq k \leq r$, we set $e_k = c_1 + \cdots + c_k$.

Now consider the map $P_k$ from $V$ into $V$ defined by

$$x = \sum_{i=1}^{r} x_i c_i + \sum_{i<j} x_{ij} e_{ij} \mapsto P_k(x) = \sum_{i=1}^{k} x_i c_i + \sum_{i<j \leq k} x_{ij} e_{ij}.$$ 

Then the determinant $\Delta^{(k)}(P_k(x))$ of the $k \times k$ bloc $P_k(x)$ is denoted by $\Delta_k(x)$, it is the principal minor of $x$ of order $k$. The generalized power of an element $x$ of $\Omega$ is then defined for $s = (s_1, \cdots, s_r) \in \mathbb{R}^r$, by

$$\Delta_s(x) = \Delta_1(x)^{s_1} \Delta_2(x)^{s_2} \cdots \Delta_r(x)^{s_r}.$$ 

Note that $\Delta_s(x) = \Delta^p(x)$ if $s = (p, \cdots, p)$ with $p \in \mathbb{R}$. It is also easy to see that $\Delta_{s+p'}(x) = \Delta_s(x) \Delta_{s'}(x)$. In particular, if $m \in \mathbb{R}$ and $s + m = (s_1 + m, \cdots, s_r + m)$, we have $\Delta_{s+m}(x) = \Delta_s(x) \Delta^m(x)$.

Now, we introduce the set $\Xi$ of elements $s$ of $\mathbb{R}^r$ defined as follows:

- For a real $u \geq 0$, we put $\varepsilon(u) = 0$ if $u = 0$ and $\varepsilon(u) = 1$ if $u > 0$.
- For $u_1, u_2, \cdots, u_r \geq 0$, we define $s_1 = u_1$ and $s_k = u_k + \frac{1}{2}(\varepsilon(u_1) + \cdots + \varepsilon(u_{k-1}))$, for $2 \leq k \leq r$.

We have the following result due to Gindikin [3], it is also proved in the monograph by Faraut and Korányi [2].

**Theorem 2.1** There exists a positive measure $R_s$ on $V$ such that the Laplace transform $L_{R_s}$ is defined on $-\Omega$ and is equal to $\Delta_s(-\theta^{-1})$ if and only if $s$ is in $\Xi$.

The measures $R_s$, defined in the previous theorem in terms of their Laplace transforms are divided into two classes according to the position of $s$ in $\Xi$. In the first class, the measures are absolutely continuous with respect to the Lebesgue measure on $\Omega$. More precisely, we have (see [4]) that when $s = (s_1, \cdots, s_r)$ in $\Xi$ is such that for all $i$, $s_i > \frac{i-1}{2}$,

$$R_s(dx) = \frac{1}{\Gamma_{\Omega}(s)} \Delta_{s-\frac{i-1}{2}}(x) \mathbf{1}_{\Omega}(x) dx$$

where $\Gamma_{\Omega}(s) = (2\pi)^{\frac{nr}{2}} \prod_{j=1}^{r} \Gamma(s_j - \frac{j-1}{2})$.

The second class corresponds to $s$ in $\Xi \setminus \bigcup_{i=1}^{r} \frac{[i-\frac{1}{2}}{2}, +\infty[$. In this case the Riesz measure $R_s$ is concentrated on the boundary $\partial\Omega$ of $\Omega$, it has a complicated form which is explicitly described in [5].

For $s = (s_1, \cdots, s_r)$ in $\Xi$ and $\sigma$ in $\Omega$, we define the Riesz distribution $R_r(s, \sigma)$ by

$$R_r(s, \sigma)(dx) = \frac{e^{-\langle \sigma, x \rangle}}{\Delta_s(\sigma^{-1})} R_s(dx).$$
The Laplace transform of the Riesz distribution is defined for \( \theta \) in \( \sigma - \Omega \) by

\[
L_{R_e(s,\sigma)}(\theta) = \frac{\Delta_s((\sigma - \theta)^{-1})}{\Delta_s(\sigma^{-1})}
\]

(2.6)

If \( s = (p, \cdots, p) \) such that \( p \in \Lambda = \{0, \frac{1}{2}, \cdots, \frac{r-1}{2}\} \cup \frac{r}{2}, +\infty\} \), then \( R_e(s,\sigma) \) is nothing but the Wishart distribution with shape parameter \( p \) and scale parameter \( \sigma \).

The definition of the absolutely continuous Riesz distribution on the cone of positive definite symmetric matrices \( \Omega \) relies on the notion of generalized power. It is defined for \( \sigma \) in \( \Omega \) and \( s = (s_1, \cdots, s_r) \in \mathbb{R}^r \) such that \( s_i > \frac{i-1}{2} \) by

\[
R(s, \sigma)(dx) = \frac{1}{\Gamma_{\Omega}(s) \Delta_s(\sigma^{-1})} e^{-\langle \sigma, x \rangle} \Delta_s - \frac{s+1}{2}(x) \mathbf{1}_\Omega(x) dx.
\]

3 Main result

We now show that some of the Riesz distributions have a representation in terms of the Gaussian distribution generalizing what accours in the case of the Wishart distribution. For simplicity, we take \( \sigma \) equal to the identity matrix.

Suppose that we have \( s_r \) independent observations of the vector \((u_1, ..., u_r)\) with Gaussian distribution \( \mathcal{N}(0, I_r) \), and that some data in the observations are missing. The sample is then partitioned according to the number of available components in the observation. More precisely, we suppose that there exist integers \( 0 < s_1 \leq s_2 \leq \cdots \leq s_r \) such that we have

- \( s_1 \) observations on \((u_1, ..., u_r)\) where all the components of the observation are available,
- \( (s_2 - s_1) \) observations on \((u_2, ..., u_r)\), the first component of the observation is missing,
- \( (s_{i+1} - s_i) \) observations on \((u_{i+1}, ..., u_r)\), the \( i \) first components of the observation are missing, \( 2 \leq i \leq r - 1 \).

The number \( s_i \) is in particular equal to the number of times the component \( u_i \) appears in the data.

Define

\[
\begin{align*}
p_1 &= \sup\{i \geq 1; s_i = s_1\}, \\
p_2 &= \sup\{i > p_1; s_i = s_{p_1+1}\}, \\
p_{l+1} &= \sup\{i > p_l; s_i = s_{p_l+1}\}, \\
p_k &= \inf\{i; s_i = s_r\} - 1,
\end{align*}
\]

so that we have

\[
\begin{align*}
s_1 &= \ldots = s_{p_1}, \\
s_{p_1} &\neq s_{p_1+1}, \\
s_{p_1+1} &= \ldots = s_{p_2}, \\
s_{p_2} &\neq s_{p_2+1} \\
&\quad \vdots \\
s_{p_{l+1}} &= \ldots = s_{p_{l+1}}, \\
&\quad \vdots \\
s_{p_k} &\neq s_r
\end{align*}
\]
$s_{p_k+1} = ... = s_r$.

We have that $p_{k+1} = r$.

Replacing the missing components by the theoretical mean that is by zero, we obtain the matrix

$$U = \begin{pmatrix}
    u_{1,1} & \ldots & u_{1,s_{p_1}} & 0 & \ldots & \ldots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    u_{p_1,1} & \ldots & u_{p_1,s_{p_1}} & 0 & \ldots & \ldots & 0 \\
    u_{p_1+1,1} & \ldots & u_{p_1+1,s_{p_1}} & u_{p_1+1,s_{p_2}} & 0 & \ldots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    u_{p_2,1} & \ldots & u_{p_2-1,s_{p_1}} & u_{p_2,s_{p_2}} & 0 & \ldots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    u_{p_k+1,1} & \ldots & \ldots & \ldots & \ldots & \ldots & u_{p_k+1,s_r} \\
    u_{r,1} & \ldots & \ldots & \ldots & \ldots & \ldots & u_{r,s_r}
\end{pmatrix}$$

This matrix $U$ consists of blocks defined recursively in the following way:

$$U^{(1)} = (u_{i,j})_{1 \leq i \leq p_1, 1 \leq j \leq s_{p_1}}$$

$$U^{(l+1)} = \begin{pmatrix}
    U_1^{(l+1)} & 0 \\
    U_2^{(l+1)} & U_0^{(l+1)}
\end{pmatrix}, \ 1 \leq l \leq k$$

with

$$U_1^{(l+1)} = U^{(l)}$$

$$U_2^{(l+1)} = (u_{i,j})_{p_1+1 \leq i \leq p_{l+1}, 1 \leq j \leq s_{p_1}}$$

$$U_0^{(l+1)} = (u_{i,j})_{p_1+1 \leq i \leq p_{l+1}, s_{p_1}+1 \leq j \leq s_{p_{l+1}}}$$

We have that $U^{(k+1)} = U$.

**Theorem 3.1** The distribution of $X = U^tU$ is $R(s, \frac{I}{2})$ with $s = (\frac{s_1}{2}, ..., \frac{s_r}{2})$.

**Proof** We use the following Peirse decomposition of an $r \times r$ matrix $\omega$.

For $1 \leq l \leq k$, we set

$$\omega^{(l)} = (\omega_{i,j})_{1 \leq i \leq p_1, 1 \leq j \leq p_l},$$

and we write $\omega^{(l+1)}$ as

$$\omega^{(l+1)} = \begin{pmatrix}
    \omega^{(l)} & t\omega_2^{(l+1)} \\
    \omega_2^{(l+1)} & \omega_0^{(l+1)}
\end{pmatrix}.$$  

From Proposition 1.1, we have that $U^{(1)} \cdot U^{(1)}$ has the $W_{p_1}(\frac{s_{p_1}}{2}, \frac{l_{p_1}}{2})$ distribution. On the other hand, since $s_1 = ... = s_{p_1}$, we have that

$$W_{p_1} \left( \frac{s_{p_1}}{2}, \frac{l_{p_1}}{2} \right) = R_{p_1} \left( \left( \frac{s_1}{2}, ..., \frac{s_{p_1}}{2} \right), \frac{l_{p_1}}{2} \right).$$
It suffices to show that if \( U^{(l)} \) is \( R \left( \frac{s_1}{2}, \ldots, \frac{s_p}{2} \right) \), then \( U^{(l+1)} \) is \( R(p_{l+1}) \left( \frac{s_1}{2}, \ldots, \frac{s_p}{2} \right) \). Again we use the Laplace transform. For \( \theta \in \left( \frac{t}{2} - \Omega \right) \) and with the decomposition (3.9), we have

\[
L_{U^{(l+1)}}(U^{(l+1)}(\theta^{(l+1)})) = E \left( \exp \left( \langle \theta^{(l+1)}, U^{(l+1)} \rangle \right) \right) = E \left( \exp \left( \langle \theta^{(l+1)}U^{(l)} \rangle \right) \right) \]

Concerning the second expectation, we have that

\[
E \left( \exp \left( \langle \theta^{(l+1)}U^{(l)} \rangle \right) \right) = \frac{\Delta^{sp(p_{l+1})-sp(l)}}{\Delta^{p_{l+1}-sp(l)}} \left( \frac{I_{p_{l+1}-p(l)}}{2} - \theta^{(l+1)} \right)^{-1}.
\]

We have that \( U_0^{(l+1)} \) is a \( (p_{l+1} - p(l)) \times (s_{p_{l+1}} - s_{p(l)}) \) Gaussian matrix with distribution \( N(0, I_{p_{l+1}-p(l)}) \).

According to Proposition 1.1, the matrix \( U_0^{(l+1)} \) has the Wishart distribution \( W_{\frac{sp(p_{l+1})-sp(l)}{2}, \frac{I_{p_{l+1}-p(l)}}{2}} \). Hence

\[
E \left( \exp \left( \langle \theta^{(l+1)}U_0^{(l+1)} \rangle \right) \right) = \frac{\Delta^{sp(p_{l+1})-sp(l)}}{\Delta^{p_{l+1}-sp(l)}} \left( \frac{I_{p_{l+1}-p(l)}}{2} - \theta^{(l+1)} \right)^{-1}.
\]

Concerning the second expectation, we have that

\[
\begin{align*}
\text{tr} \left( \langle \theta^{(l+1)}U^{(l+1)} \rangle \right) & = \text{tr} \left( \langle \theta^{(l+1)}U^{(l)} \rangle + \text{tr} \left( \langle \theta^{(l+1)} \rangle U^{(l)} \right) \right) \\
& = 2\langle U^{(l)} \rangle \langle \theta^{(l+1)} \rangle \end{align*}
\]

As \( U_{21}^{(l+1)} \) is a \( (p_{l+1} - p(l)) \times s_{p(l)} \) matrix with distribution \( N(0, I_{p_{l+1}-p(l)}) \), we obtain that

\[
E \left( \exp \left( 2\langle U^{(l)} \rangle \langle \theta^{(l+1)} \rangle + \langle \theta^{(l+1)} \rangle U^{(l)} \right) \right) = \Delta^{\frac{sp(l)}{2}} \left( \frac{I_{p_{l+1}-p(l)}}{2} - 2\theta^{(l+1)} \right)^{-1} \exp \left( 2\langle \theta^{(l+1)} \rangle U^{(l)} \right) \left( \frac{I_{p_{l+1}-p(l)}}{2} - 2\theta^{(l+1)} \right)^{-1} \theta^{(l+1)} U^{(l)} \right) \)

Now multiplying this by \( \exp \left( \text{tr} \left( \langle \theta^{(l+1)} \rangle U^{(l)} \right) \right) \), we need then to calculate the expectation

\[
E \left( \exp \left( \langle \theta^{(l+1)} \rangle U^{(l)} \right) \right) \left( \frac{I_{p_{l+1}-p(l)}}{2} - \theta^{(l+1)} \right)^{-1} \theta^{(l+1)} U^{(l)} \right) \)

As from the hypothesis, \( U^{(l)} \) is \( R \left( \frac{s_1}{2}, \ldots, \frac{s_p}{2} \right) \), using (2.6) this is equal to

\[
\frac{\Delta^{\frac{sp(l)}{2}}}{\Delta^{\frac{sp(l)}{2}}} \left( \frac{I_{p_{l+1}-p(l)}}{2} - \theta^{(l+1)} \right)^{-1} \theta^{(l+1)} U^{(l)} \right) \)

\[
\Delta^{\frac{sp(l)}{2}} \left( \frac{I_{p_{l+1}-p(l)}}{2} - \theta^{(l+1)} \right)^{-1} \theta^{(l+1)} U^{(l)} \right) \)
Finally, we obtain

\[ L_{U^{(l+1)}} I_{U^{(l+1)}} (\theta^{(l+1)}) = \frac{\Delta^{s_{p_{l+1}} - s_{p_l}}}{\Delta^{s_{p_{l+1}} - s_{p_l}}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

\[ \Delta^{s_{p_{l+1}} - s_{p_l}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

\[ \Delta \left( \frac{\ell_1}{2}, \ldots, \frac{\ell_{s_{p_{l+1}}}}{2} \right) \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

that is

\[ L_{U^{(l+1)}} I_{U^{(l+1)}} (\theta^{(l+1)}) = \frac{\Delta^{s_{p_{l+1}} - s_{p_l}}}{\Delta^{s_{p_{l+1}} - s_{p_l}}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

\[ \Delta^{s_{p_{l+1}} - s_{p_l}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

\[ \Delta \left( \frac{\ell_1}{2}, \ldots, \frac{\ell_{s_{p_{l+1}}}}{2} \right) \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

But \( s_{p_{l+1}} = \ldots = s_{p_{l+1}} \), then

\[ \Delta^{s_{p_{l+1}} - s_{p_l}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

\[ \Delta^{s_{p_{l+1}} - s_{p_l}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

\[ \Delta^{s_{p_{l+1}} - s_{p_l}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

and

\[ \Delta^{s_{p_{l+1}} - s_{p_l}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

\[ \Delta^{s_{p_{l+1}} - s_{p_l}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

As we have

\[ \Delta \left( \frac{\ell_1}{2}, \ldots, \frac{\ell_{s_{p_{l+1}}}}{2} \right) \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

\[ \Delta \left( \frac{\ell_1}{2}, \ldots, \frac{\ell_{s_{p_{l+1}}}}{2} \right) \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

it follows that

\[ L_{U^{(l+1)}} I_{U^{(l+1)}} (\theta^{(l+1)}) = \frac{\Delta^{s_{p_{l+1}} - s_{p_l}}}{\Delta^{s_{p_{l+1}} - s_{p_l}}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

\[ \Delta^{s_{p_{l+1}} - s_{p_l}} \left( \frac{I_{(p_{l+1})} - I_{(p_l)}}{2} - \theta_0^{(l+1)} \right)^{-1} \]

Therefore \( U^{(l+1)} I U^{(l+1)} \) is a Riesz random matrix with distribution

\[ R_{p_{l+1}} \left( \frac{S_1}{2}, \ldots, \frac{s_{p_{l+1}}}{2}, \frac{I_{(p_{l+1})}}{2} \right). \]
Given that $X = U^{(k+1)} \, t \, U^{(k+1)}$, the result follows.

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