An average theorem for tuples of \( k \)-free numbers in arithmetic progressions

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1 - Introduction

The Barban-Davenport-Halberstam (BDH) Theorem and its refinement due to Hooley and Montgomery (HM) are important theorems in analytic number theory since they suggest what one believes to be the correct order of magnitude for the error term in the Prime Number Theorem for Arithmetic Progressions. The question of additive patterns in prime numbers is also a central problem, but a theorem of BDH type is out of reach - indeed it is not even known, at the time of writing, that there are infinitely many primes \( p \) such that \( p + 2 \) is also prime, never mind the Hardy-Littlewood Conjecture.

Let \( k \geq 2 \). If for a given \( n \) there is no prime \( p \) for which \( p^k | n \) then \( n \) is said to be \( k \)-free. An asymptotic formula of similar shape to that in the BDH-HM Theorem is known for the \( k \)-free numbers, the current state of knowledge attained and summarised by Vaughan in [10]. Crucially, the corresponding question on additive patterns in the \( k \)-frees is accessible. For given non-negative integers \( 0 \leq h_1 < \cdots < h_r \) we call \( n \) a \( k \)-free \( r \)-tuple associated to \( h := (h_1, \ldots, h_r) \) if the \( n + h_i \) are all \( k \)-free, and write \( \mathcal{R} = \mathcal{R}(h) \) for the set of all such \( n \). The asymptotic count

\[
\sum_{\substack{n \leq x \\ n \in \mathcal{R}}} 1 = \mathcal{O}_x + O \left( x^{2/(k+1)+\epsilon} \right),
\]

for some \( \mathcal{O} = \mathcal{O}_x > 0 \), is easily established (see [5]) and restricting to arithmetic progressions isn’t too much harder. Indeed in [2] twins in arithmetic progressions were investigated and it was shown easily that

\[
\sum_{\substack{n \leq x \\ n \equiv q \pmod{a}}} \mu_k(n)\mu_k(n+1) = \eta(q, a)x + O \left( x^{2/(k+1)+\epsilon} \right),
\]

for some \( \eta(q, a) > 0 \), and that

\[
V(x, Q) := \sum_{q \leq Q} \left| \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \mu_k(n)\mu_k(n+1) - \eta(q, a)x \right|^2 \ll Q^2 \left( \frac{x}{Q} \right)^{2/k+\epsilon} + x^{4/(k+1)+\epsilon};
\]
and in [4] the method of Vaughan (that in [10]) is followed to show that
\[ V(x, Q) \ll Q^2 \left( \frac{x}{Q} \right)^{1/k+\epsilon} + x^{1+2/k} \log Q + x^{3/2+1/2k+\epsilon}; \]
these results are important because the same results for primes are out of reach.

As far as we can see, however, there is no recorded asymptotic form for this variance of \( k \)-free twins. In this paper we achieve this, indeed for general tuples.

**Theorem.** Fix natural numbers \( k \geq 2 \) and \( r \geq 1 \), denote by \( K \) the set of \( k \)-free numbers, fix non-negative integers \( 0 \leq h_1 < \cdots < h_r \), and let
\[ \mathcal{R} = \{ n \in \mathbb{N} | n + h_i \in K, \ i = 1, \ldots, r \} \]  
be the set of \( k \)-free \( r \)-tuples associated to \( h := (h_1, \ldots, h_r) \). Let for \( q, a \in \mathbb{N} \) and \( x, Q \geq 1 \)
\[ \eta(q, a) = \sum_{d_1, \ldots, d_r=1}^{\infty} \frac{\mu(d_1) \cdots \mu(d_r)}{[q, d_1 \cdots d_r]} \]  
\[ E_x(q, a) = \sum_{n \leq x, n \equiv a (q)} 1 - \eta(q, a)x \]  
and
\[ V(x, Q) = \sum_{q \leq Q} \sum_{a=1}^{q} E_x(q, a)^2. \]  
Take \( \epsilon = \epsilon(r) \) to be any number in \([1/2, 1]\) for which we know
\[ \int_{\pm \infty} \frac{|\zeta(\sigma + it)|^r dt}{(1 + |t|)^{3/2}} \]
converges absolutely for all \( \sigma \geq \epsilon \). For each prime \( p \) write \( R_p \) for the number of different residues represented by the \( h_1, \ldots, h_r \) modulo \( p^k \). If always \( R_p < p^k \) then for \( 1 \leq Q \leq x \) and \( \epsilon > 0 \)
\[ V(x, Q) = Q^2 \left( \frac{x}{Q} \right)^{1/k} P \left( \log(x/Q) \right) + O_{k, r, h, \epsilon} \left( Q^2 \left( \frac{x}{Q} \right)^{\epsilon(r)/k+\epsilon} + x^{1+2/(k+1)+\epsilon} \right) \]
where \( P = P(r, k, h) \) is a polynomial of degree at most \( r - 1 \).

As already mentioned the only theorems in this direction are upper bound result for twins of squarefree numbers. In this case we can take \( \epsilon = 1/2 \) since it is contained in classical results that for \( \sigma \geq 1/2 \)
\[ \int_T^{2T} |\zeta(\sigma + it)|^2 dt \ll_T T \log T \]
and so our theorem then says
\[ V(x, Q) = x^{1/2} Q^{3/2} P \left( \log(x/Q) \right) + O_\epsilon \left( x^{1/4} Q^{7/4} + x^{5/3+\epsilon} \right) \]
for some linear function \( P \). Of course if the \( h_i \) cover a complete residue system modulo some \( p^k \) then there are no \( k \)-free \( r \)-tuples.
In [10] the evaluation of the variance of $k$-free numbers is translated into a binary additive problem in $k$-free numbers which can be tackled with the circle method (following the general method of [9]). Aside from the last stage of the proof we use the method laid out there. The main difficulty when comparing with [10] is that the Gauss sum associated to $k$-free $r$-tuples is less accessible than that associated to $k$-frees; we use the methods of [1] to get hold of this object, although a direct argument is also possible.

The paper is structured as follows: In Section 2 we collect the elementary facts about the distribution of $k$-free $r$-tuples in arithmetic progressions; in Section 3 we discuss the Gauss sum; in Section 4 we do most of the circle method work; in Section 5 we obtain the necessary results for the application of Perron’s formula to the quantity remaining after the circle method work; and in Section 6 we carry out the main argument, using the results of the previous sections.

Throughout we consider $k \geq 2$, $r \geq 1$ and $0 \leq h_1 < \cdots < h_r$ as fixed and write $h = (h_1, ..., h_r)$. The implied constants in the $O$ symbol will always be understood to be dependent on $k, r, h$ and $\epsilon$, and $\epsilon$ may be taken to be arbitrarily small at each of its occurrences. Often (but always with explicit mention) we will write statements such as

$$f(X) \ll g(X)$$

where the $\ll$ contains terms up to $X^\epsilon$ - here we mean

$$f(X) \ll X^\epsilon g(X).$$

Whenever $s, \sigma$ and $t$ appear in the same context we will always mean a complex number $s$ with real and imaginary parts $\sigma$ and $t$. We will write statements that involve $r$-tuples using vectors and mean that that statement is to hold for each vector component. For example, $\nu \equiv d \ (q)$ would mean $\nu \equiv d, (q_i)$ for each $i = 1, ..., R$, where the $R, d_i, q_i$ would be clear from context. A sum $\sum q$ will mean that the summation variables are restricted to numbers coprime to $q$. The $R$-fold divisor estimate $d_R(n) \ll n^\epsilon$ is well known, as is the (General) Chinese Remainder Theorem which says

$$n \equiv a \mod (q)$$

has exactly one solution modulo $[q_1, ..., q_R]$ if $(q_i, q_j) | a_i - a_j$ and has no solutions otherwise. We will use both these facts frequently but often forget to mention where they come from. A coprimality condition may often disappear from one line to the next with the introduction of the Möbius function; here we are using

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

## 2 - Counting $k$-free numbers in arithmetic progressions

Counting $k$-free numbers amounts to counting solutions of congruences modulo $k$-th powers. The precision we need for $r$-tuples is contained in [5] but we reproduce the proof since we need a slightly different result to the one stated there.

**Lemma 2.1.** (i) For any $R, D \in \mathbb{N}$ with $D$ being $k$-free we have for $Z > 0$

$$\sum_{d_R \leq Z} 1 \ll_R Z^{r} \left(\frac{Z}{D}\right)^{1/k}.$$
(ii) For any $R \in \mathbb{N}$, any distinct $a_1, \ldots, a_R \in \mathbb{N}_0$, any $\delta \in [0, 1/3]$ and any $Y > 0$
\[ \sum_{d_1, \ldots, d_R > Y, \ (d_i^k, d_j^k) | a_i - a_j} [d_1, \ldots, d_R]^{k(\delta - 1)} \ll_{R,a} Y^{1 + k(\delta - 1) + \varepsilon}. \]

(iii) For $t \geq 1$, $R, d_1, \ldots, d_R \in \mathbb{N}$ and distinct $a_1, \ldots, a_R \in \mathbb{N}_0$ denote by $\mathcal{N}_{d,a}(t)$ the number of solutions $n \leq t$ to the system $n \equiv -a \ (d^k)$. Then for $Y > 0$ we have
\[ \sum_{d_1, \ldots, d_R > Y} \mathcal{N}_{d,a}(t) \ll_{R,a} t^{(kY^{1-k} + l^{2/(k+1)})}. \] (4)

Proof. (i) We have
\[ \sum_{[d_1^k, \ldots, d_n^k, D] \leq Z} 1 = \sum_{[n^k, D] \leq Z} \sum_{[d_1, \ldots, d_n] = n} 1 \ll_R Z^\varepsilon \sum_{[n^k, D] \leq Z} 1 = Z^\varepsilon \sum_{l | D} \sum_{n^k \leq z/l | D} 1. \]

Write $l_0 = \prod_{p | l} p$. Then the inner sum above is
\[ \sum_{n^k \leq z/l | D, (n^k, l) = 1} 1 \leq \left( \frac{Z^\varepsilon}{D} \right)^{1/k} \frac{l^{1/k}}{l_0} \leq \left( \frac{Z}{D} \right)^{1/k} \]
since $D$ and therefore $l$ is $k$-free, and the claim follows for $D \leq Z$. If $D > Z$ the LHS of the sum in question is zero.

(ii) It is straightforward to establish with induction that for any $d_1, \ldots, d_n \in \mathbb{N}$
\[ [d_1, \ldots, d_n] \geq \prod_{i \neq j} \frac{d_i}{(d_i, d_j)} \]

Therefore for any $d_1, \ldots, d_R$ with $(d_i^k, d_j^k) | a_i - a_j$
\[ \frac{1}{[d_1^k, \ldots, d_R^k]} \ll_{R,a} \frac{1}{d_1^k \cdots d_R^k} \]
and so (since $k(\delta - 1) < -1$)
\[ \sum_{d_1, \ldots, d_R > Y, \ (d_i^k, d_j^k) | a_i - a_j} [d_1, \ldots, d_R]^{k(\delta - 1)} \ll_{R,a} \sum_{d_1, \ldots, d_R > Y} (d_1 \cdots d_R)^{k(\delta - 1)} \ll \sum_{n > Y} n^{k(\delta - 1) + \varepsilon} \ll Y^{1 + k(\delta - 1) + \varepsilon}. \]

(iii) We prove the claim by induction on $R$. Suppose $t$ is larger than all the $a_1, \ldots, a_R$ since otherwise the LHS of the sum in question is anyway. We have
\[ \sum_{d > Y} \mathcal{N}_{d,a}(t) = \sum_{d > Y} \sum_{n \leq t, \ (a_{n+1}, a_n | d^k)} 1 \leq \sum_{Y < d \leq (t+a)^{1/k}} \left( \frac{t}{dk} + 1 \right) \ll_a t Y^{1-k} + t^{1/k} \] (5)
which is (stronger than) the result for \( R = 1 \) so suppose now the result holds for some \( R \). Let \( Z > 0 \) be a parameter. We have

\[
\sum_{d_1 \cdots d_{R+1} > Y \atop d_1 \cdots d_R > Z} \mathcal{N}_{d_1, \ldots, d_{R+1}; a_1, \ldots, a_{R+1}}(t)
\]

\[
\leq \sum_{d_1 \cdots d_R > Z} \sum_{a_1 \leq \Delta} \left( \sum_{n=\min(a_1, a_{R+1})+1}^{d_{R+1}} 1 \right)
\]

\[
\ll (t + a_{R+1})^\varepsilon \sum_{d_1 \cdots d_R > Z} \mathcal{N}_{d_1, \ldots, d_R; a_1, \ldots, a_R}(t)
\]

\[
\ll R, a \ t^\varepsilon \left( tZ^{1-k+\varepsilon} + t^{2/(k+1)} \right)
\]

by assumption, and since the argument would obviously be the same if we had the summation condition \( d_1 \cdots d_{R+1}/d_i > Z \) for some \( 1 \leq i \leq R \) instead of \( i = R+1 \) we deduce

\[
\sum_{d_1 \cdots d_{R+1} > Y \atop d_1 \cdots d_{R+1}/d_i > Z \atop \text{for some } i} \mathcal{N}_{d_1, \ldots, d_{R+1}; a_1, \ldots, a_{R+1}}(t)
\]

\[
\ll R, a \ t^\varepsilon \left( tZ^{1-k+\varepsilon} + t^{2/(k+1)} \right)
\]

On the other hand if always \( d_1 \cdots d_{R+1}/d_i \leq Z \) then we must have \( d_1 \cdots d_{R+1} \leq Z^{1+1/R} \) so that

\[
\sum_{d_1 \cdots d_{R+1} > Y \atop d_1 \cdots d_{R+1}/d_i \leq Z} \mathcal{N}_{d_1, \ldots, d_{R+1}; a_1, \ldots, a_{R+1}}(t)
\]

\[
\leq \sum_{Y < d_1 \cdots d_{R+1} \leq Z^{1+1/R}} \mathcal{N}_{d_1, \ldots, d_{R+1}; a_1, \ldots, a_{R+1}}(t)
\]

\[
\leq \sum_{Y < d_1 \cdots d_{R+1} \leq Z^{1+1/R} \atop (a_1, \ldots, a_R \in \mathbb{N})} \left( \frac{t}{[d_1, \ldots, d_{R+1}]} + 1 \right)
\]

\[
\ll R, a \left( tY^{1-k+\varepsilon} + Z^{1+1/R+\varepsilon} \right)
\]

from part (ii). Together (6) and (7) imply, assuming \( Z \leq t \),

\[
\sum_{d_1 \cdots d_{R+1} > Y} \mathcal{N}_{d_1, \ldots, d_{R+1}; a_1, \ldots, a_{R+1}}(t) \ll R, a \ t^\varepsilon \left( tY^{1-k+\varepsilon} + Z^{1+1/R} + tZ^{1-k} + t^{2/(k+1)} \right)
\]

\[
\ll t^\varepsilon \left( tY^{1-k+\varepsilon} + t(R+1)/(Rk+1) + t^{2/(k+1)} \right)
\]

having chosen \( Z = t^{R/(Rk+1)} \). The second term being less than the third, this is the result for \( R+1 \).

\[ \boxed{ } \]

**Lemma 2.2.** Let \( \mathcal{R} \) be as in (1), let \( \eta(q, a) \) and \( E_t(q, a) \) be as in (2), and let \( \theta = 1/k \) and \( \Delta = 2/(k+1) \).

(i) For \( t \geq 1 \) and \( q, a \in \mathbb{N} \)

\[
\sum_{a \leq t \atop a \equiv 0 \mod R} 1 = \eta(q, a)t + O(t^{\Delta+\varepsilon})
\]
(ii) For $t, \gamma \geq 1$

\[
\sum_{q \leq \gamma} \sum_{\nu = 1}^{q} |E_t(q, \nu)|^2 \ll \gamma^{2 - \vartheta} t^{2\vartheta} + t^{2\Delta} + \gamma t^\Delta, \\
\sum_{q \leq \gamma} \sum_{\nu = 1}^{q} |E_t(q, \nu)|^2 \ll \gamma^{1 - \vartheta} t^{2\vartheta} + t^{2\Delta}, \\
\sum_{q \leq \gamma} \frac{1}{q} \sum_{\nu = 1}^{q} |E_t(q, \nu)|^2 \ll \gamma^{1 - \vartheta} t^{2\vartheta} + t^{2\Delta} + \gamma t^\Delta \\
\text{and} \quad \sum_{q \leq \gamma} \frac{1}{q^2 - \theta} \sum_{\nu = 1}^{q} |E_t(q, \nu)|^2 \ll t^{2\Delta};
\]

here the $\ll$ symbol may include terms of size $t^\epsilon, \gamma^\epsilon$.

Proof. Recall that $h = (h_1, ..., h_r)$ is fixed from the start. A sum $\Sigma^*$ over variables $d_1, ..., d_r$ will mean that for all $i, j$ we have $(d_i^*, d_j^*) | h_i - h_j$. For natural numbers $d_1, ..., d_r$ write $d^* = |d_1, ..., d_r|$. For given $q, a, d_1, ..., d_r \in \mathbb{N}$ and $t > 0$ write $N_{d,h}^q(t)$ for the number of solutions $n \leq t$ to the system of congruences $n \equiv -h \pmod{(d^k)}$ and $n \equiv a \pmod{(q)}$, and write $N_{d,h}^q(t) = N_{d,h}^{q,1}$. As in Lemma 2.1 Since it is well-known that for any $N \in \mathbb{N}$

\[
\sum_{d \mid N} \mu(d) = \begin{cases} 1 & \text{if } N \text{ is k-free} \\
0 & \text{if not}
\end{cases}
\]

we see from (8) that

\[
\sum_{n \leq \gamma} 1 = \sum_{n \leq \gamma} \mu(d)N_{d,h}^q(t); \quad (8)
\]

here we obviously write $\mu(d) = \mu(d_1) \cdots \mu(d_r)$.

(i) From (8) we have for a parameter $Y \leq t^{1/k}$ to be chosen

\[
\sum_{n \leq \gamma} 1 = \sum_{d_1 \cdots d_r \leq Y} \mu(d)N_{d,h}^q(t) + \mathcal{O}\left(\sum_{d_1 \cdots d_r > Y} N_{d,h}^q(t)\right)
\]

\[
= \sum_{d_1 \cdots d_r \leq Y} \mu(d)N_{d,h}^q(t) + \mathcal{O}\left(t (tY^{1-k+\epsilon} + t^\Delta)\right) \quad (9)
\]

from Lemma 2.1 (iii). The main term here is

\[
\sum_{d_1 \cdots d_r \leq Y} \mu(d) \left(\frac{t}{[q, d^*k]} + \mathcal{O}(1)\right)
\]

\[
= t \sum_{d_1 \cdots d_r \leq Y} \frac{\mu(d)}{[q, d^*k]} + \mathcal{O}\left(t \sum_{d_1 \cdots d_r > Y} \frac{1}{d^*k} + \sum_{d_1 \cdots d_r \leq Y} 1\right)
\]

\[
= t \gamma(q, a) + \mathcal{O}\left(t(Y^{1-k+\epsilon} + Y^{1+\epsilon})\right)
\]

\footnote{see the notation explained in the introduction}
from \[2\] and Lemma 2.1 (ii), so (9) becomes

\[
\sum_{n \leq t \atop n \equiv \nu(q)} 1 = t \eta(q, \nu) + O\left( t Y^{1-k+\epsilon} + Y^{1+\epsilon} + t^{\epsilon} \left( t Y^{1-k+\epsilon} + t^\Delta \right) \right)
\]

which gives (i) on choosing \(Y = t^{1/k}\).

(ii) From (8) we have for a parameter \(Y \leq t^{1/k}\) to be chosen

\[
\sum_{n \leq t \atop \nu(n) = \nu(q)} 1 = \sum_{d_1, \ldots, d_r \leq Y} \mu(d) \mathcal{N}^{a_{n_1}^{d_1}} \mathcal{N}^{a_{n_2}^{d_2}} \cdots \mathcal{N}^{a_{n_r}^{d_r}} (t) + O \left( t \sum_{d_1, \ldots, d_r \leq Y} \frac{1}{|q, d^{k}|} + \sum_{d_1 \cdot \ldots \cdot d_r \leq Y} 1 \right).
\]

The main term here is

\[
\sum_{d_1 \cdot \ldots \cdot d_r \leq Y} \mu(d) \left( \frac{t}{|q, d^{k}|} + O(1) \right) = t \sum_{d_1, \ldots, d_r \leq Y} \frac{\mu(d)}{|q, d^{k}|} + O \left( t \sum_{d_1 \cdot \ldots \cdot d_r \leq Y} \frac{1}{|q, d^{k}|} + \sum_{d_1 \cdot \ldots \cdot d_r \leq Y} 1 \right)
\]

from \[2\], therefore (10) implies

\[
\sum_{n \leq t \atop \nu(n) = \nu(q)} 1 - t \eta(q, \nu) \ll \sum_{d_1 \cdot \ldots \cdot d_r \leq Y} \left( \sum_{n \leq t \atop \nu(n) = \nu(q)} 1 + \frac{t}{|q, d^{k}|} \right) + Y^{1+\epsilon}
\]

\[
=: \mathcal{T}_Y(q, \nu) + Y^{1+\epsilon}.
\]

In general for general positive functions \(f, g\) and \(S := \sum_d \{f(d) + g(d)\}\) it is easy to establish that \(S^2 \ll \sum_{d, d'} \{f(d) f(d') + g(d) g(d')\}\). Therefore

\[
\sum_{\nu=1}^{q} |\mathcal{T}_Y(q, \nu)|^2 \leq \sum_{d_1 \cdot \ldots \cdot d_r \leq Y} \left( \sum_{n \leq t \atop \nu(n) = \nu(q)} 1 + \frac{t^2}{|q, d^{k}|^2 |q, d^{k'}|^2} \right).
\]

and the congruence conditions in the \(\nu\) sum amount to one congruence modulo

\[
[(q, d_1^{k_1}), \ldots, (q, d_r^{k_r}), (q, d_1^{k_1'}), \ldots, (q, d_r^{k_r'})] = \frac{(q, d^{k_1}) (q, d^{k_r}) (q, d^{k_1'}) (q, d^{k_r'})}{(q, d^{k_1}, d^{k_r})}.
\]
so that the whole \( \nu \) sum is

\[
\sum_{\nu=1}^{q} \left| T_Y(q, \nu) \right|^2 \leq \sum_{\nu=1}^{q} \left( 1 + \frac{t^2 q(q, d^{*k}, d'^{*k})}{qd^{*k}d'^{*k}} \right).
\]

and therefore

\[
\sum_{\nu=1}^{q} \left| T_Y(q, \nu) \right|^2 \leq \sum_{\nu=1}^{q} \left( 1 + \frac{t^2 q(q, d^{*k}, d'^{*k})}{qd^{*k}d'^{*k}} \right).
\]

Since for any \( N \in \mathbb{N} \)

\[
\sum_{q \leq \gamma} \frac{(q, N)}{q} \ll N^\epsilon (\log \gamma + 1) \ll N^\epsilon \gamma^\epsilon
\]

we see, on separating the terms with \( n = n' \) since for these no divisor estimate is applicable, that

\[
\sum_{q \leq \gamma} \sum_{\nu=1}^{q} \left| T_Y(q, \nu) \right|^2 \ll \sum_{d_1 \cdots d_{t'} > Y} \left( t^\epsilon \sum_{n,m \leq \gamma} 1 + \sum_{n,m \leq \gamma} 1 + \frac{t^2 q(q, d^{*k}, d'^{*k})}{qd^{*k}d'^{*k}} \right)
\]

\[
\ll t^\epsilon \left( \sum_{d_1 \cdots d_{t'} > Y} \mathcal{N}_{d,h}(t) + \gamma \mathcal{N}_{d,h}(t) \right) + t^2 \gamma^{\epsilon} \left( \sum_{d_1 \cdots d_{t'} > Y} d^{*k(\epsilon-1)} \right)^2
\]

\[
\ll t^\epsilon \left( t^2 Y^{2-2k+2\epsilon} + t^2 \Delta + \gamma (t Y^{1-k+\epsilon} + t^{\Delta}) \right) + t^2 \gamma^{\epsilon} Y^{2-2k+4\epsilon}
\]

(12)

from Lemma 2.1 (iii) and (ii); in the second term in the second line we summed first over \( d' \) and used \( \sum_{d_{t'} > h} \ll t^\epsilon \), valid for large \( t \); if \( t \) is not large the first claim to be proven is clear since the obvious bound \( \eta(q, \nu) \ll q^{-1} \) means the LHS is then

\[
\sum_{q \leq \gamma} \sum_{\nu=1}^{q} 1 + t^2 q^{2\epsilon-2} \ll \gamma.
\]

Putting (12) in (11) we get, assuming \( Y \leq t \),

\[
\sum_{q \leq \gamma} \sum_{\nu=1}^{q} \left| 1 - t \eta(q, \nu) \right|^2 \ll t^\epsilon \left( t^2 Y^{2-2k+2\epsilon} + t^2 \Delta + \gamma (t Y^{1-k+\epsilon} + t^{\Delta}) \right) + t^2 \gamma^{\epsilon} Y^{2-2k+4\epsilon} + Y^{2+2\epsilon} \sum_{q \leq \gamma} \sum_{\nu=1}^{q} 1
\]

\[
\ll t^2 Y^{2-2k} + t^2 \Delta + \gamma (t Y^{1-k} + t^{\Delta}) + Y^2 \gamma^2
\]
the $t', \gamma'$ terms going into the $\ll$ symbol again. Choosing $Y = (t/\gamma)^{1/k}$ gives the first claim and the others follow from partial summation. □

3 - Gauss sums

In this section we collect from [1] the results needed to study the Gauss sum associated to $k$-free $r$-tuples. The letter $S$ will always denote a general sequence whilst, as in the statement of our theorem, $K$ denotes the $k$-free numbers and $R$ the $k$-free $r$-tuples.

If a sequence $S$ satisfies for fixed $q$ and $a$

$$\sum_{n \leq x \atop n \in S} 1 = xf_S(q, a) + E_S(x; q, a)$$

for some $f(q, a)$ and some $E_S(x; q, a) = o(x)$, $x \to \infty$;

we say that $S$ satisfies Criterion D. We define the density of $S$ as $\rho_S = f(1, 1)$ and if this is non-zero we define

$$g_S(q, a) = \frac{f_S(q, a)}{\rho_S}.$$  

The Gauss sum of $S$ is defined as

$$G_S(q, a) = \sum_{\nu=1}^{q} e\left(\frac{a\nu}{q}\right) g_S(q, a).$$

These definitions are all on page 92 of [1]. From Lemma 2.3 of [1] (page 101) we can consider the Gauss sum as a function on $\mathbb{Q}/\mathbb{Z}$ and write

$$G_S(a/q) = G_S(q, a).$$

The Gauss sum is crucial to the exponential sum approximation in the circle method application later; indeed sorting the $n$ into arithmetic progressions modulo $q$ we see that for any $S$ with non-zero density we have for $t > 0$

$$\sum_{n \leq t \atop n \in S} e\left(\frac{an}{q}\right) = \rho_S G_S(a/q) t + \sum_{\nu=1}^{q} e\left(\frac{a\nu}{q}\right) E_S(t; q, \nu).$$

Let $E_t(q, a)$ be as in (2). Of course Lemma 2.2 says that $E_t(q, a) = o(t)$ for fixed $q$ and $a$ so we must have $E_R(t; q, a) = E_t(q, a)$. Therefore the above says for any $t > 0$

$$\sum_{n \leq t \atop n \in R} e\left(\frac{an}{q}\right) = \rho_R G_R(a/q) t + \sum_{\nu=1}^{q} e\left(\frac{a\nu}{q}\right) E_t(q, \nu), \quad \text{if } \rho_R > 0. \ (13)$$

For $h \in \mathbb{N}$ define the $h$-shift of a sequence $S$ as the sequence

$$\{n \in \mathbb{N}|n + h \in S\},$$
which also obviously satisfies Criterion D, and write $G_S^h$ for its Gauss sum. From (2.18) and (2.19) of [1] (page 108) we have for any $q, a \in \mathbb{N}$

$$G_S^h(a/q) = e\left(-\frac{a h}{q}\right) G_S(a/q).$$  \tag{14}

The rest of this section is concerned with evaluating the Gauss sum of the $k$-free $r$-tuples. The underlying principle is that the Gauss sum associated to an intersection of sequences can be expressed in terms of the Gauss sums of the individual sequences via a convolution. Since

$$\mathcal{R} = \bigcap_{i=1}^r K_{h_i}$$  \tag{15}

and since the Gauss sum for the $k$-free numbers, and so from (14) also the Gauss sum of their shifts, is accessible, we can therefore handle the Gauss sum of the $r$-tuples.

**Lemma 3.1.** For prime $p$ and $a, l \in \mathbb{N}$ with $p \nmid a$

$$G_K(a/p^l) = \frac{-1}{p^l - 1} \left\{ \begin{array}{ll}
1 & \text{if } l \leq k \\
0 & \text{if } l > k.
\end{array} \right.$$

*Proof.* The $k$-free numbers are $\{ s \in \mathbb{N} \mid$ for all primes $p$, we have $p^k \nmid s \}$ so this is Lemma 5.3 of [1] (page 128).

As in definition (4.20) of [1] (page 125) define, for a prime $p$, the $p$-local Gauss sum $G_S^p : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{C}$ of $S$ through

$$G_S^p(a/q) = \left\{ \begin{array}{ll}
G_S(a/q) & \text{if } a/q \text{ in lowest form has denominator a non-negative power of } p \\
0 & \text{if not.}
\end{array} \right.$$

As on page 118 of [1] define, for two sequences $S$ and $S'$ satisfying Criterion D, the convolution of $G_S$ and $G_{S'}$ as the function $G_S \ast G_{S'} : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{C}$ given through

$$(G_S \ast G_{S'})(a/q) = \sum_{b/r \in \mathbb{Q}/\mathbb{Z}} G_{S'}(b/r) G_S(a/q - b/r);$$

this is absolutely convergent by Lemma 1.1 of [1] (page 92) so that in particular it is commutative, and for shifts of $k$-free numbers it must also be associative since in that case all the summations are finite summations, in view of (14) and Lemma 3.1.

**Lemma 3.2.** Take $R \in \mathbb{N}$. For any $H_1, \ldots, H_R \in \mathbb{N}$ write $G_i$ for the Gauss sum of the $k$-frees shifted by $H_i$. Then for any prime $p$ and any $a, l \in \mathbb{N}$ with $p \nmid a$ we have, if the $H_i$ are distinct modulo $p^k$, $$(G_1^p \ast \cdots \ast G_R^p)(a/p^l) = \frac{-1/p^k}{(1 - 1/p^k)^R} \sum_{n=1}^R e\left(-\frac{a H_n}{p^l}\right) \left\{ \begin{array}{ll}
1 & \text{if } l \leq k \\
0 & \text{if } l > k.
\end{array} \right.$$ and

$$(G_1^p \ast \cdots \ast G_R^p)(0) = \frac{1 - R/p^k}{(1 - 1/p^k)^R}. $$
Proof. We prove the first claim only, the proof of the second being essentially no different. The result is clearly valid for \( R = 1 \) in view of (14) and Lemma 3.1, so suppose the result is true for some \( R \in \mathbb{N} \) and take arbitrary distinct \( H_1, ..., H_{R+1} \in \mathbb{N} \). Write \( G \) for the Gauss sum of the \( k \)-frees. For \( b, L \in \mathbb{N} \) we have from (14) \( G_{R+1}(b/p^L) = e\left(-\frac{bH_{R+1}}{p^L}\right) G(b/p^L) \)

so that, writing \( q = p^l \),

\[
(G_1 \cdots G_{R+1})(a/q) = \sum_{b=1}^{p^l} e\left(-\frac{bH_{R+1}}{p^k}\right) G(b/p^k) (G_1 \cdots G_R)(a/q - b/p^k),
\]

the terms with \( L > k \) vanishing in view of Lemma 3.1. We now use the inductive hypothesis and (continue using) Lemma 3.1; we also drop the \( p \) superscripts and write \( g_n = (1 - 1/p^k)^n \). Since clearly \( G(0) = 1 \) the term \( b = p^k \) contributes

\[
G(0) (G_1 \cdots G_R)(a/q) = -\frac{1}{g_R} \sum_{n=1}^{R} e\left(-\frac{aH_n}{q}\right),
\]

the term \( b = ap^k/q \) contributes

\[
G(a/q) (G_1 \cdots G_R)(0) = -\frac{1}{g_R} \cdot \frac{1 - R/p^k}{g_1} e\left(-\frac{aH_{R+1}}{q}\right),
\]

and the remaining terms contribute (both \( G \) factors having non-trivial arguments)

\[
-\frac{1}{g_1} \cdot \frac{1}{g_R} \sum_{n=1}^{R} e\left(-\frac{aH_n}{q}\right) \sum_{\substack{b=1 \in \mathbb{N} \setminus \{p^k, \ldots, p^R\} \cap \mathbb{N} \\cap \mathbb{N}\}} e\left(-\frac{b(H_{R+1} - H_n)}{p^k}\right) = \frac{1}{g_1 + R} \sum_{n=1}^{R} e\left(-\frac{aH_n}{q}\right) (0 - e\left(-\frac{a(H_{R+1} - H_n)}{q}\right)),
\]

so the whole sum is

\[
\left(\frac{-1}{g_R} - \frac{1}{g_1 + R} \sum_{n=1}^{R} e\left(-\frac{aH_n}{q}\right)\right) + e\left(-\frac{aH_{R+1}}{q}\right) \left(\frac{-1}{g_1} \cdot \frac{1 - R/p^k}{g_R} - \frac{1}{g_1 + R} \sum_{n=1}^{R} 1\right)
\]

\[
= -\frac{g_1}{g_1 + R} - \frac{1}{g_1 + R} \sum_{n=1}^{R} e\left(-\frac{aH_n}{q}\right) - \frac{1}{g_1 + R} \sum_{n=1}^{R} e\left(-\frac{aH_{R+1}}{q}\right) \left(\frac{1 - R/p^k}{g_R} - \frac{R/p^k}{g_1 + R} \sum_{n=1}^{R} 1\right)
\]

\[
= -\frac{1}{g_1 + R} \sum_{n=1}^{R+1} e\left(-\frac{aH_n}{q}\right)
\]

and first claim follows. \( \square \)
We now introduce the concept of quasi-multiplicativity and introduce functions \( G \) and \( H \) which will be present throughout the paper, being essentially our exponential sum approximation for the \( k \)-free \( r \)-tuples.

**Definition 3.3.** Take a function \( f : \mathbb{N}^2 \rightarrow \mathbb{C} \) for which \( f(q, \cdot) \) has period \( q \) for each \( q \in \mathbb{N} \). If for any \( \omega \in \mathbb{N} \), any pairwise coprime \( q_1, ..., q_\omega \in \mathbb{N} \), and any \( a_1, ..., a_\omega \in \mathbb{N} \) we have

\[
f(q_1 \cdots q_\omega, a_1q/q_1 + \cdots + a_\omega q/q_\omega) = f(q_1, a_1) \cdots f(q_\omega, a_\omega)
\]

where \( q = q_1 \cdots q_\omega \), then we say that \( f \) is quasi-multiplicative; through induction this holds if and only if it holds for \( \omega = 2 \). We now take \( q, a \in \mathbb{N} \) with \( (a, q) = 1 \) and look at the value of \( f(q, a) \) if \( f \) is quasi-multiplicative. Write \( q = q_1 \cdots q_\omega \) for the prime factorisation of \( q \) and define \( a_\omega \) through \( a \equiv a_\omega \mod (q) \). Then

\[
a \equiv a_\omega q/q_1q/q_1 + \cdots + q/q_\omega \mod (q),
\]

where \( q/q_i \) is inverse to \( q/q_i \mod (q_i) \). Therefore

\[
f(q, a) = f(q_1, a_1 q/q_1) \cdots f(q_\omega, a_\omega q/q_\omega)
\]

so that specifying the value of a quasi-multiplicative function \( f(q, a) \) at prime powers \( q \) and all \( a \) with \( (a, q) = 1 \) (and saying \( f(1, 1) = 1 \)) is enough to determine \( f \) for all \( q \in \mathbb{N} \) and all \( a \) with \( (a, q) = 1 \). Recall that the \( h_1, ..., h_r \) from our theorem are fixed from the outset. For any prime \( p \) denote by \( H_1, ..., H_R \) the \( R = R_p \) different residues represented modulo \( p^k \) by the \( h_1, ..., h_r \). For any prime \( p \) and any \( a, l \in \mathbb{N} \) with \( p \nmid a \) define

\[
G(p^l) = \left\{ \begin{array}{ll}
-1/p^k & \text{if } l \leq k \\
1 - R/p^k & \text{if } l > k
\end{array} \right.
\]

and

\[
H(p^l, a) = \sum_{n=1}^{R} e\left( -aH_n/p^l \right) \left\{ \begin{array}{ll}
1 & \text{if } l \leq k \\
0 & \text{if } l > k
\end{array} \right.
\]

Define \( G(q) \) and \( H(q, a) \) for all \( q, a \in \mathbb{N} \) with \( (q, a) = 1 \) by extending multiplicatively and quasi-multiplicatively; note that \( G \) is well-defined in view of the assumption in our theorem. \[ \Box \]

For large \( p \) we have \( R_p = r \) so \( 1 - R_p/p^k \geq 1/2 \) and so

\[
|G(p^l)| \leq \left\{ \begin{array}{ll}
2/p^k & \text{for large } p \\
0 & \text{for all } p \text{ and } t > k,
\end{array} \right.
\]

which we will use later, therefore for all \( p \)

\[
|G(p^l)| \ll \left\{ \begin{array}{ll}
1/p^k & \text{always, in particular for } t \leq k, \\
0 & \text{for } t > k.
\end{array} \right.
\]

We deduce \( G(q) \ll 1/q \) for prime powers \( q \) and so for general \( q \)

\[
G(q) \ll q^{-1};
\]

also note \( |H(q, a)| \leq R \ll 1 \) holds for prime powers \( q \) and for \( a \in \mathbb{N} \) with \( p \nmid a \) so for general \( q \) and \( a \) with \( (q, a) = 1 \)

\[
H(q, a) \ll q^e.
\]
If a sequence $S$ satisfying Criterion D has quasi-multiplicative Gauss sum $g_S$ we say that $S$ satisfies Criterion C; see page 93 of [1]; and we look at the intersection of such sequences. In the paragraph containing equation (4.21) of [1] (page 125) we have two sequences $U$ and $V$ satisfying Criterion C with Gauss sums $u$ and $v$. Shortly after $w$ is defined as the Gauss sum of the intersection $W := U \cap V$ and then for any prime $p$

$$w_p = \frac{u_p \ast v_p}{(u_p \ast v_p)(0)}$$

according to (4.22) of [1], so long as $\rho_W > 0$. It follows for given $R \in \mathbb{N}$ that, if we have given sequences $U_1, \ldots, U_R$ satisfying Criterion C with Gauss sums $u_1$ and if $w_R$ denotes the Gauss sum of the intersection $W := S_1 \cap \cdots \cap S_R$, then for any prime $p$ and any $a/q \in \mathbb{Q}/\mathbb{Z}$ we have

$$w_R^p(a/q) = \frac{(u_1^p \ast \cdots \ast u_R^p)(a/q)}{(u_1^p \ast \cdots \ast u_R^p)(0)}$$

so long as $\rho_W > 0$; moreover according to (1.4), (4.7) and Lemma 4.3 of [1] (pages 92, 119 and 120) we have

$$u_i \ast u_i = (u_i \ast u_i)(0) \cdot u_i$$

and therefore we may drop repeated $p$-local Gauss sums from the above quotient of convolutions.

From (14) and Lemma 3.1 we have for any $h \in \mathbb{N}$, any prime $p$, any $l \geq 0$, and any $a \in \mathbb{N}$ with $p \not| a$

$$G_{K_h}(a/p^l) = \begin{cases} e(aH_Hn/p^l)G_{K_H}(a/p^l) & \text{if } l \leq k \\ 0 & \text{if } l > k \end{cases} = G_{K_H}(a/p^l)$$

for any $H \in \mathbb{N}$ with $H \equiv h \mod(p^k)$. From (15) this discussion implies that for any prime $p$, any $l \geq 0$, and any $a \in \mathbb{N}$ with $p \not| a$

$$G_R^p(a/p^l) = \left(\frac{G_{K_{H_1}}^p \ast \cdots \ast G_{K_{H_R}}^p}{G_{K_{H_1}}^p \ast \cdots \ast G_{K_{H_R}}^p}(0)\right)(a/p^l)$$

where the $H_1, \ldots, H_R$ are the $R = R_p$ different residues represented modulo $p^k$ by the $h_1, \ldots, h_R$, and so from Lemma 3.2 and Definition 3.3

$$G_R(p^l, a) = \frac{-1/p^k}{1 - R/p^k} \sum_{n=1}^{R} e\left(-\frac{aH_n}{q}\right) \begin{cases} 1 & \text{if } l \leq k \\ 0 & \text{if } l > k \end{cases} = G(p^l)H(p^l, a)$$

(19)

for any prime $p$, any $l \geq 0$, and any $a \in \mathbb{N}$ with $p \not| a$; not to forget is that this is all subject to $\rho_R > 0$. Moreover by Lemma 2.9 and Theorem 4.6 of [1] (pages 110 and 125) it follows from (15) that $R$ satisfies Criterion C and therefore, from Lemma 2.6 of [1] (page 106), that $G_R$ is quasi-multiplicative. We deduce from Definition 3.3 that

$$G_R(q, a) = G(q)H(q, a)$$

holds for $q, a \in \mathbb{N}$ with $(q, a) = 1$ and so for general $q, a \in \mathbb{N}$

$$G_R(q, a) = G\left(\frac{q}{(q, a)}\right)H\left(\frac{q}{(q, a)}, \frac{a}{(q, a)}\right), \quad \text{if } \rho_R > 0.$$ 

(20)

We finish this section by establishing some easy properties of $G$ and $H$. 

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Lemma 3.4. Define $G$ as in Definition 3.3 and write $\theta = 1/k$. For any $Z \geq 1$

\[ \sum_{Z < q \leq 2Z} |G(q)| \ll Z^{\theta - 1+\epsilon}. \]

This implies in particular

\[ \sum_{q \leq Z} q|G(q)| \ll Z^{\theta+\epsilon}, \quad \sum_{q \leq Z} q^2|G(q)|^2 \ll Z^{2\theta+\epsilon}, \quad \sum_{q > Z} |G(q)| \ll Z^{\theta-1+\epsilon}, \quad \sum_{q > Z} |G(q)|^2 \ll Z^{\theta-2+\epsilon}, \quad \sum_{q > Z} q^{1+\epsilon}|G(q)|^2 \ll Z^{\theta-1+\epsilon} \]

and

\[ \sum_{q > Z} q^2|G(q)|^2 \ll Z^{\theta}. \]

Proof. By (16) we have

\[ p^{t(1-1/k)}|G(p^t)| \leq \begin{cases} 2/p & \text{for large } p \text{ and } 1 \leq t \leq k \\ 0 & \text{for all } p \text{ and } t > k \end{cases} \]

therefore by multiplicativity

\[ \sum_{q \leq Z} q^{1-1/k}|G(q)| \leq \prod_{p \leq 2Z} \left( 1 + \sum_{t \geq 1} p^{t(1-1/k)}|G(p^t)| \right) \]

\[ \leq \prod_{p \leq 2Z} \left( 1 + \sum_{t \leq k} p^{t(1-1/k)}|G(p^t)| \right) \prod_{p \leq 2Z} (1 + 2k/p) \]

\[ \ll \prod_{p \leq 2Z} (1 + 1/p)^{2k} \ll (\log Z)^{2k} + 1 \ll Z^\epsilon \]

by one of Merten’s formulas. Consequently

\[ \sum_{Z < q \leq 2Z} |G(q)| \ll Z^{1/k-1} \sum_{q \leq 2Z} q^{1-1/k}|G(q)| \ll Z^{1/k-1+\epsilon}. \]

The first “in particular” claim follows from the main claim after partial summation and then a dyadic split. The second then follows from (17) and the first. The third follows from the main claim and a dyadic split. The fourth follows from the third and (17). The fifth follows from the fourth and partial summation. The sixth follows from

\[ \sum_{Z < q \leq 2Z} q^{2-\theta}|G(q)|^2 \ll Z^\epsilon \]

and a dyadic split, and this in turn follows from the main claim with partial summation. \qed

Lemma 3.5. Define $H$ as in Definition 3.3. Define for $q, n \in \mathbb{N}$

\[ \Phi_q(n) = \sum_{a=1}^q |H(q,a)|^2 e\left(\frac{an}{q}\right), \quad \Phi_q^*(n) = \sum_{a=1}^q \overline{H(q,a)} e\left(\frac{an}{q}\right) \quad \text{and} \quad \Phi(q) = \Phi_q(0). \]
(i) Both $\Phi_q(n)$ and $\Phi_q^*(n)$ are, for each $n$, multiplicative in $q$. If a function $F(q, d)$ defined for $q \in \mathbb{N}$ and $d|q$ satisfies for all $(q, q') = 1$ and $d|q, d'|q'$

$$F(qq', dd') = F(q, d)F(q', d')$$

then the sum

$$\sum_{A=1}^q F\left(q, (q, A)\right)\Phi_q(A)$$

is multiplicative in $q$.

(ii) For $q$ a power of a prime and for $d|q$

$$\sum_{A=1}^{q/d} \Phi_q(-Ad) = \Phi(q)\mu(q/d).$$

(iii) For any $q \in \mathbb{N}$

$$\sum_{A=1}^q |\Phi_q(A)| \ll q^{1+\epsilon}$$

and the same claim holds with $\Phi_q(A)$ replaced by $\Phi_q^*(A)$.

(iv) Let $\eta$ be as in our theorem, $G$ as in Definition [3.3] and define

$$\rho = \prod_p \left(1 - \frac{R_p}{p^k}\right).$$

Then for any $q \in \mathbb{N}$ we have

$$\eta(q, a)^2 = \rho^2 \sum_{d|q} \Phi(d)G(d)^2.$$

Proof. For comparison with [10] think of $\Phi_q(n)$ as Ramanujan’s sum and see Lemma 2.4 of that paper.

(i) All these claims are simple consequences of the fact that $H$ is quasi-multiplicative.

(ii) Write $p$ for the prime in question and suppose $q/p^k$ since otherwise $H(q, a) = 0$ so that the claim is trivial. We have for any $N \in \mathbb{N}_0$

\begin{equation}
\Phi_q(N) = \sum_{n, n'=1}^{R_p} \sum_{a=1}^q e\left(-\frac{a(H_n - H_{n'})}{q} - N\right)
\end{equation}

so that

$$\sum_{A=1}^{q/d} \Phi_q(-Ad) = \sum_{n, n'=1}^{R_p} \sum_{a=1}^q e\left(-\frac{a(H_n - H_{n'})}{q} - N\right) \sum_{A=1}^{q/d} e\left(-\frac{-aAd}{q}\right)$$

$$= \mu(q/d) \sum_{n, n'=1}^{R_p} \sum_{a=1}^q e\left(-\frac{a(H_n - H_{n'})}{q}\right)$$

$$= \mu(q/d)\Phi_q(0)$$
from (21).

(iii) As in (i) the sum in question is multiplicative so it is enough to prove the bound for \( q \)
power of a prime \( p \) and as in (ii) it is enough to prove it for \( q | p^k \). In that case (21) implies

\[
\Phi_q(N) = \sum_{n, n' = 1}^{R_p} c_q(-H_n + H_{n'} + N)
\]

so that, since for given \( M \) there are only \( \ll r \) many \( (n, n', N) \in \{1, ..., R_p\}^2 \times \{1, ..., q\} \) such that \(-H_n + H_{n'} + N \equiv M \) modulo \( q \),

\[
\sum_{N=1}^{q} |\Phi_q(N)| \ll \sum_{M=1}^{q} |c_q(M)| \ll q^{1+\epsilon}
\]

by a standard bound for Ramanujan’s sum and the proof is similar for \( \Phi_q^*(n) \).

(iv) For \( P \) a power of a prime \( p \) and \( A \in \mathbb{N} \) we have

\[
H(P, A) = \sum_{n=1}^{R_p} e\left(\frac{AH_n}{P}\right)
\]

(22)

where \( H_1, ..., H_{R_p} \) are the distinct residues represented by \( h_1, ..., h_r \) modulo \( p^k \). Therefore for \( d \mid q \) with \( q \) a power of a prime \( p \) we have

\[
\sum_{b=1}^{d} H(d, b) e\left(\frac{-ab}{d}\right) = \sum_{n=1}^{R_p} c_d(H_n - a) \quad (23)
\]

By orthogonality and (20)

\[
g_R(q, a) = \frac{1}{q} \sum_{b=1}^{q} G_R(q, b) e\left(\frac{ab}{q}\right) = \frac{1}{q} \sum_{d \mid q} G(d) \sum_{b=1}^{d} H(d, b) e\left(\frac{ab}{d}\right) = \sum_{n=1}^{R_p} \sum_{d \mid q} G(d) c_d(H_n - a)
\]

so that

\[
\sum_{a=1}^{q} |g_S(q, a)|^2 = \sum_{n, n' = 1}^{R_p} \sum_{d, d'|q} G(d) G(d') \sum_{a=1}^{q} c_d(H_n - a) c_{d'}(a - H_{n'})
\]

\[
= q \sum_{n, n' = 1}^{R_p} \sum_{d, d'|q} G(d) G(d') \sum_{a=1}^{[d, d']} c_d(H_n - a) c_{d'}(a - H_{n'}).
\]

(24)
Then
\[ \sum_{a=1}^{[d,d']} c_d(H_n - a)c_{d'}(a - H_{n'}) \]
\[ = \sum_{A=1}^{d} \sum_{A'=1}^{d'} e \left( \frac{AH_n}{d} - \frac{A'H_{n'}}{d'} \right) \sum_{a=1}^{[d,d']} e \left( \frac{a - A[d,d']/d + A'[d,d']/d'}{[d,d']} \right) \]
\[ = \sum_{A=1}^{d} \sum_{A'=1}^{d'} e \left( \frac{AH_n}{d} - \frac{A'H_{n'}}{d'} \right) \]
but the only (prime power) \( d, d' \) which can satisfy these summation conditions are those with \( d = d' \), in which case the \( A, A' \) sum becomes
\[ \sum_{A=1}^{d} e \left( \frac{A(H_n - H_{n'})}{d} \right) = c_d(H_n - H_{n'}) \]
and so
\[ \frac{1}{[d,d']} \sum_{a=1}^{[d,d']} c_d(H_n - a)c_{d'}(H_{n'} - a) = \begin{cases} c_d(H_n - H_{n'}) & \text{if } d = d' \\ 0 & \text{if not.} \end{cases} \]

Therefore (24) says
\[ \sum_{a=1}^{q} |g_R(q,a)|^2 = q \sum_{n,n'=1}^{R_p} \sum_{d|q} (G(d))^2 c_d(H_n - H_{n'}) \]
\[ = q \sum_{d|q} (G(d))^2 \sum_{a=1}^{q} - e \left( \frac{aH_n}{d} \right) \]
\[ = q \sum_{d|q} \Phi(d)(G(d))^2. \] (25)

from (22). This holds initially only for \( q \) a prime power, but the LHS is multiplicative since \( g_R(q,a) \) is quasi-multiplicative (as in part (i)) and the RHS is multiplicative from part (i), so (25) holds in fact for general \( q \). Since obviously Lemma 2.2 says \( \eta(q,a) = p_R g_R(q,a) \) we deduce from (25)
\[ \sum_{a=1}^{q} |\eta(q,a)|^2 = |p_R|^2 q \sum_{d|q} \Phi(d)(G(d))^2. \] (26)

From (23)
\[ \sum_{a=1}^{q} |H(q,a)|^2 = \sum_{n,n'=1}^{R_p} c_q(H_n - H_{n'}) \]
on prime powers, so that from Definition (3.3)
\[ \sum_{t=0}^{p^k} \sum_{a=1}^{p^t} |H(p^t, a)|^2 = 1 + \frac{1/p^{2k}}{(1 - R/p^k)^2} \sum_{n,n'=1}^{R_p} \sum_{1 \leq t \leq k} c_{p^t}(H_n - H_{n'}). \] (27)
But it is easy to establish that for any $D \not| N$

$$\sum_{d|D} c_d(N) = 0$$

and therefore

$$\sum_{n, n' = 1 \atop q

\sum \phi(q) = -R(R - 1) + R(p^k - 1) = R(p^k - R)$$

which we put in (27) to see that

$$\sum_{t \geq 0} G(p^t)^2 \sum_{a=1}^{p^t} |H(p^t, a)|^2 = 1 + \frac{R(p^k - R)/p^{2k}}{(1 - R/p^k)^2} = \frac{1}{1 - R/p^k}$$

and so from the multiplicativity of $G$ and from part (i)

$$\sum_{q=1}^{\infty} |G(q)|^2 \sum_{a=1}^{q'} |H(q, a)|^2 = \prod_p \frac{1}{1 - R/p^k} = \rho^{-1}$$

and therefore from (20)

$$\sum_{q=1}^{\infty} \sum_{a=1}^{q'} |G(q, a)|^2 = \rho^{-1}.$$ 

But from page 92 of [1] (more precisely from (1.4), (E) and the following paragraph) the LHS is $\rho R^2$, so that in fact $\rho R = \rho$, and the result follows from (26).

In the last lemma we showed $\rho R = \rho \neq 0$, where $\rho$ is as given in that lemma. From (13) and (20) we conclude for any $t > 0$ and any $q, a \in \mathbb{N}$ with $(q, a) = 1$

$$\sum_{n \in \mathbb{R} \atop q|n} e\left(\frac{an}{q}\right) = \rho G(q) H(q, a) t + \sum_{\nu=1}^{q} e\left(\frac{a\nu}{q}\right) E_t(q, \nu)$$

which will be our exponential sum approximation in the circle method application.

4 - The circle method application

In this section we carry out most of the circle method work.

Let $\gamma > 0$ be a parameter. Consider the set of all irreducible fractions in $[0, 1]$ with denominator not exceeding $\gamma$: the Farey fractions. If $a'/q' < a/q$ are consecutive Farey fractions in lowest form, define their median as $

\frac{a + a'}{q + q'}.$

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Since this lies in \((a'/q', a/q)\) we may partition some \(\mathfrak{F}\) unit interval into disjoint intervals each containing a Farey point \(a/q\) in lowest form and extending to the median of \(a/q\) with its neighbouring Farey points. Denoting each interval by \(\mathfrak{F}(a/q)\), the Farey arc at \(a/q\), we see that

\[
\int_{\mathfrak{F}} f(t) dt = \sum_{q \leq \gamma} \sum_{a=1}^{q'} \int_{\mathfrak{F}(a/q)} f(t) dt
\]

for any continuous function \(f : \mathbb{R} \rightarrow \mathbb{C}\). Denote by \(\mathfrak{U}(a/q)\) the interval of unit length centered at \(a/q\). It can be shown that

\[
\left( \frac{a}{q} - \frac{1}{2q'q}, \frac{a}{q} + \frac{1}{2q'q} \right) \subseteq \mathfrak{F}(a/q) \subseteq \left( \frac{a}{q} - \frac{1}{q'q}, \frac{a}{q} + \frac{1}{q'q} \right) \subseteq \mathfrak{U}(a/q);
\]

for a discussion of these matters, see Sections 3.1 and 3.8 of [3].

**Lemma 4.1.** Let \(x, \gamma \geq 1\) and \(Q > \sqrt{x}\). Let \(\Delta, \Delta, G, H\) and \(\rho\) be as in Lemma 3.2, Definition 3.3 and Lemma 3.5. As explained above, denote by \(\mathfrak{F}(a/q)\) the Farey arc at \(a/q\) in the Farey dissection of order \(\gamma\), where \((a,q) = 1\), and by \(\mathfrak{U}(a/q)\) the unit interval centered at \(a/q\). Define for \(t > 0\) and \(q, a \in \mathbb{N}\)

\[
\Delta_t(q,a) = \sum_{\nu=1}^{q} e\left(\frac{aw}{q}\right) E_t(q,\nu)
\]

where \(E_t(q,\nu)\) is as in Lemma 3.2. For \(\alpha \in \mathbb{R}\) define

\[
f(\alpha) = \sum_{n \leq x} e(n\alpha), \quad F(\alpha) = \sum_{n \leq x} e(n\alpha\nu) \quad \text{and} \quad I(\alpha) = \int_1^x e(\alpha t) dt.
\]

For \(\alpha \in \mathfrak{F}(a/q)\) write \(\beta = \alpha - a/q\) and define

\[
J(\alpha) = -2\pi i \beta \int_1^x e(\beta t) \Delta_t(q,a) dt \quad \text{and} \quad \hat{J}(\alpha) = e(x\beta) \Delta_t(q,a) + J(\alpha).
\]

For \(2\sqrt{x} \leq \gamma \leq x^{3/4}\) we have

(A) \quad for \(\alpha \in \mathfrak{F}(a/q)\), \quad \(f(\alpha) = \rho G(q) H(q,a) I(\beta) + \hat{J}(\alpha)\)

(B) \quad for any \(\beta \in \mathbb{R}\) and \(q \leq x\), \quad \(\sum_{a=1}^{q} |H(q,a)|^2 F(-a/q - \beta) \ll x\)

(C) \quad \(\sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q'} \int_{\mathfrak{F}(a/q)} |F(-\alpha)| \cdot |\hat{J}(\alpha)|^2 d\alpha \ll x^{1+\theta} + \frac{x^{1+2\Delta}}{\gamma}\)

(D) \quad \(\sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q} \int_{\mathfrak{U}(a/q)} |F(-\alpha)| \cdot |I(\beta)| \hat{J}(\alpha) d\alpha \ll x^{1+\Delta + \gamma x^{1/2+\theta}}\)

(E) \quad \(\sum_{2\sqrt{x} < q \leq \gamma} \sum_{a=1}^{q'} \int_{\mathfrak{F}(a/q)} |F(-\alpha)| \cdot |f(\alpha)|^2 d\alpha \ll x\gamma\)

(F) \quad \(\sum_{a=1}^{q'} \int_{\mathfrak{U}(a/q) \setminus \mathfrak{F}(a/q)} |F(-\alpha)| \cdot |I(\alpha - a/q)|^2 d\alpha \ll q^2 \gamma^2\).
Here the $\ll$ symbol is allowed to contain terms of size $x^\varepsilon$.

Proof. This is essentially all contained in [10]. We use a specific notation just for this proof: we will write $f(x) \ll g(x)$ to mean $f(x) \ll x^\varepsilon g(x)$. We are basically telling the reader to ignore logs and epsilons. Write $\lambda = 1/q\gamma$ so that (30) reads

$$\left( \frac{a}{q} - \frac{\lambda}{2}, \frac{a}{q} + \frac{\lambda}{2} \right) \subseteq \mathfrak{A}(a/q) \subseteq \left( \frac{a}{q} - \lambda, \frac{a}{q} + \lambda \right) \subseteq \mathfrak{U}(a/q).$$

(31)

For $\alpha \in \mathfrak{A}(a/q)$ (and so assuming $q \leq \gamma$) we have from (31) that $|\beta| \leq \lambda \leq 1/2\sqrt{x}$ so from display (2.7), Lemma 2.9, (the second part of) Lemma 2.11 and Lemma 2.12 of [10] we have

$$F(\alpha) \ll \frac{x}{q(1 + x|\beta|)} + \sqrt{x}q \ll \frac{x}{q} + q, \quad \text{for } \alpha \in \mathfrak{A}(a/q),$$

(32)

so that from (31)

$$\int_{\mathfrak{A}(a/q)} |F(\alpha)|d\alpha \ll \frac{1}{q} \int_{1 - \lambda x}^{x} \frac{xd\beta}{1 + x|\beta|} + \lambda(\sqrt{x} + q) \ll \frac{1}{q},$$

(33)

and

$$|F(\alpha)| \cdot |\beta| \ll \frac{1}{q} + \left( \sqrt{x} + q \right)|\beta| \ll \frac{1}{q}, \quad \text{for } \alpha \in \mathfrak{A}(a/q).$$

(34)

From displays (2.7), (2.9), (2.11), Lemma 2.9 and (taking $q = 1$ in the first part of) Lemma 2.11 of [10]

$$F(\alpha) \ll \sum_{u \leq \sqrt{x}} \frac{x}{u + x|u\alpha|} \quad \text{for } \alpha \in \mathbb{R},$$

(35)

Simply integrating shows

$$I(\beta) \ll \frac{x}{2 + x|\beta|}.$$  

(36)

Now we prove the claims of the lemma.

(A) Write $f_t(\alpha) = \sum_{n \leq t, n \in \mathbb{R}} e(n\alpha)$ so that (28) reads

$$f_t(a/q) = \rho G(q)H(q,a)t + \Delta_t(q,a)$$

for $t > 0$, so that partial summation to

$$f(\alpha) = \sum_{n \leq \frac{a}{q}} e\left( \frac{an}{q} + \beta n \right)$$

gives

$$f(\alpha) = e(x\beta)f_x(a/q) - 2\pi i\beta \int_{1}^{x} e(\beta t)f_t(\alpha)dt$$

$$= \rho G(q)H(q,a) \left( xe(x\beta) - 2\pi i\beta \int_{1}^{x} te(\beta t)dt \right) + \quad e(x\beta)\Delta_x(q,a) - 2\pi i\beta \int_{1}^{x} e(\beta t)\Delta_t(q,a)dt$$

3see the notation explained in the introduction

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which gives the result after an integration by parts.

(B) Take $\Phi_q(n)$ as in Lemma 3.5. By part (iii) of that lemma

$$
\sum_{a=1}^{q'} |H(q,a)|^2 F(-a/q - \beta) = \sum_{u,v \leq x} \Phi_q(-uv)e(-uv\beta) \\
\ll x^\epsilon (x/q + 1) \sum_{n=1}^{q} |\Phi_q(n)| \\
\ll x^\epsilon (x/q + 1) q^{1+\epsilon}.
$$

(C) From (34) and (31) we have for $\alpha \in F(a/q)$

$$
|F(\alpha)| \cdot |J(\alpha)|^2 \ll |F(\alpha)| \cdot |\beta|^2 \left| \int_1^{x} \Delta_t(q,a)e(\beta t) dt \right|^2 \ll \frac{1}{q^2 \gamma} \left| \int_1^{x} \Delta_t(q,a)e(\beta t) dt \right|^2
$$

and therefore from (31)

$$
\sum_{a=1}^{q'} \int_{\mathfrak{g}(a/q)} |F(\alpha)| \cdot |J(\alpha)|^2 d\alpha \\
\ll \frac{1}{q^2 \gamma} \sum_{a=1}^{q} \int_{\pm \lambda} \left| \int_1^{x} \Delta_t(q,a)e(\beta t) dt \right|^2 d\beta \\
= \frac{1}{q^2 \gamma} \int_1^{x} \left( \sum_{a=1}^{q} \Delta_t(q,a) \Delta_{t'}(q,a) \right) \left( \int_{\pm \lambda} e(\beta(t-t')) d\beta \right) dt dt'.
$$

We have ($E_t$ is defined in Lemma 2.2)

$$
\sum_{a=1}^{q} \Delta_t(q,a) \Delta_{t'}(q,a) = q \sum_{\nu=1}^{q} E_t(q,\nu) E_{t'}(q,\nu)
$$

and the second factor in the double integral above is

$$
\ll \min \left( \frac{1}{|t-t'|}, \lambda \right)
$$

so

$$
\sum_{q \leq \gamma} \sum_{a=1}^{q'} \int_{\mathfrak{g}(a/q)} |F(\alpha)| \cdot |J(\alpha)|^2 d\alpha \\
\ll \frac{1}{\gamma} \int_1^{x} \int_1^{x} \left( \sum_{q \leq \gamma} \frac{1}{q} \sum_{\nu=1}^{q} |E_t(q,\nu) E_{t'}(q,\nu)| \min \left( \frac{1}{|t-t'|}, 1 \right) \right) dt dt' \\
=: \frac{V(x,\gamma)}{\gamma}.
$$
Applying twice the Cauchy-Schwarz inequality we see that
\[ V(x, \gamma) \leq \int_1^x \left( \sum_{q \leq \gamma} \frac{1}{q} \sum_{\nu = 1}^q |E_t(q, \nu)|^2 \right) \int_1^x \min \left( \frac{1}{|t|}, 1 \right) dt \]
\[ \ll x \cdot \max_{1 \leq t \leq x} \left( \sum_{q \leq \gamma} \frac{1}{q} \sum_{\nu = 1}^q |E_t(q, \nu)|^2 \right) \tag{39} \]
therefore from (38)
\[ \sum_{q \leq \gamma} \sum_{a = 1}^\gamma \int_{\mathfrak{g}(a/q)} |F(\alpha)| \cdot |J(\alpha)|^2 d\alpha \ll x \cdot \max_{1 \leq t \leq x} \left( \sum_{q \leq \gamma} \frac{1}{q} \sum_{\nu = 1}^q |E_t(q, \nu)|^2 \right) \tag{40} \]
From (33) and (37)
\[ \sum_{q \leq 2\sqrt{x}} \sum_{a = 1}^\gamma \int_{\mathfrak{g}(a/q)} |\Delta_x(q, a)|^2 d\alpha \ll \sum_{q \leq 2\sqrt{x}} |E_x(q, \nu)|^2 \]
therefore
\[ \sum_{q \leq 2\sqrt{x}} \sum_{a = 1}^\gamma |\Delta_x(q, a)|^2 \int_{\mathfrak{g}(a/q)} |F(\alpha)| d\alpha \ll \max_{1 \leq t \leq x} \left( \sum_{q \leq 2\sqrt{x}} \sum_{\nu = 1}^q |E_t(q, \nu)|^2 \right) \]
which with (40) says
\[ \sum_{q \leq 2\sqrt{x}} \sum_{a = 1}^\gamma \int_{\mathfrak{g}(a/q)} |F(\alpha)| \cdot |J(\alpha)|^2 d\alpha \]
\[ \ll \max_{1 \leq t \leq x} \left( \sum_{q \leq 2\sqrt{x}} \sum_{\nu = 1}^q |E_t(q, \nu)|^2 + \frac{x}{\gamma} \sum_{q \leq \gamma} \frac{1}{q} \sum_{\nu = 1}^q |E_t(q, \nu)|^2 \right) \]
\[ \ll x^{1+\theta} + x^{2\Delta} + x^{1/2+\Delta} + \frac{x}{\gamma} \left( \gamma^{-1} + \gamma^2 + \gamma^{2\Delta} \right) \]
from Lemma 2.2 (ii). The second term is bounded by the fifth (since \( \gamma \leq x \)), and the third and fourth are bounded by the first (the third since \( \Delta \leq 1/2 \) unless \( k = 2 \), in which case \( 1/2 + \Delta \leq 3/2 = 1 + \theta \), and the fourth since \( \gamma \geq \sqrt{x} \)).

(D) For \( \alpha \in \mathfrak{g}(a/q) \) write
\[ C(\alpha) = G(q)H(q, a)I(\beta) \]
For \( \alpha \in \mathfrak{g}(a/q) \setminus (a/q - \lambda/2, a/q + \lambda/2) \) we have \( |\beta| \gg \lambda \) so from (34) and (36)
\[ F(-\alpha)l|I(\beta)|^2 \ll \frac{1}{q|\beta|^2} \left( \frac{x}{1+|x|} \right)^2 \ll \frac{1}{q\lambda^2} \]
so that from (34)
\[ \int_{\mathfrak{g}(a/q) \setminus (a/q - \lambda/2, a/q + \lambda/2)} |F(-\alpha)| \cdot |I(\beta)|^2 d\alpha \ll \frac{1}{q\lambda^2} = q^{-2} \]
and therefore by (18)

$$\sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q} \int_{\mathfrak{F}(a/q) \setminus (a/q - \lambda/2, a/q + \lambda/2)} |F(-\alpha)| \cdot |C(\alpha)|^2 d\alpha \leq \gamma^2 \sum_{q \leq 2\sqrt{x}} q^2 |G(q)|^2$$

$$\ll \gamma^2 x^{\theta/2}$$

from Lemma 3.4. Therefore the Cauchy-Schwarz Inequality and part (C) imply

$$\left( \sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q} \int_{\mathfrak{F}(a/q) \setminus (a/q - \lambda/2, a/q + \lambda/2)} |F(-\alpha)| \cdot |C(\alpha)| \cdot |\hat{J}(\alpha)| d\alpha \right)^2$$

$$\leq \left( \sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q} \int_{\mathfrak{F}(a/q) \setminus (a/q - \lambda/2, a/q + \lambda/2)} |F(-\alpha)| \cdot |C(\alpha)|^2 d\alpha \right)$$

$$\times \left( \sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q} \int_{\mathfrak{F}(a/q)} |F(-\alpha)| \cdot |\hat{J}(\alpha)|^2 d\alpha \right)$$

$$\ll \gamma^2 x^{\theta/2} \left( x^{1+\theta} + \frac{x^{1+2\Delta}}{\gamma} \right)$$

so that, since $\gamma \leq x^{3/4}$,

$$\sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q} \int_{\mathfrak{F}(a/q) \setminus (a/q - \lambda/2, a/q + \lambda/2)} |F(-\alpha)| \cdot |C(\alpha)| \cdot |\hat{J}(\alpha)| d\alpha$$

$$\ll \gamma x^{1/2+3\theta/4} + \gamma^{1/2} x^{1/2+\Delta+\theta/4}$$

$$\ll \gamma x^{1/2+\theta} + x^{1+\Delta}.$$  \hspace{1cm} (41)

For $\alpha \in \mathfrak{F}(a/q)$ (so assuming $(a, q) = 1$ and $q \leq \gamma$) and $u \leq \sqrt{x}$ we have from (31)

$$|u\beta| \leq \frac{1}{2q}$$

so if $q \nmid u$ then

$$||ua|| \geq \left| \frac{ua}{q} \right| - ||u\beta|| \gg \left| \frac{ua}{q} \right|$$

and therefore using the standard bound for a linear exponential sum

$$H_q(\alpha) := \sum_{v < \sqrt{x}} \left( \sum_{v \leq u \leq \sqrt{x}} + \sum_{\sqrt{x} < u < Q, x/v} \right) e(\alpha uv)$$

$$\ll \sum_{v < \sqrt{x}} \frac{1}{||ua/q||}$$

$$\ll (\sqrt{x}/q + 1) \sum_{u=1}^{q-1} \frac{1}{||ua/q||}$$

$$\ll \log q \left( \sqrt{x} + q \right) \ll \gamma \log q$$  \hspace{1cm} (42)
so that, breaking the $u$ summation in the definition of $F$ at $\sqrt{x}$ and then swapping sums in the second part,

$$F(\alpha) = \sum_{u \leq \sqrt{x}} \sum_{v \leq \sqrt{x}/u} e(\alpha uv) + \sum_{u \leq \sqrt{x}} \sum_{Q \leq x} e(\alpha uv)$$

$$= \sum_{u \leq \sqrt{x}} \sum_{v \leq \sqrt{x}/u} e(\beta uv) + \sum_{u \leq \sqrt{x}} \sum_{Q \leq x} e(\beta uv) + H_q(\alpha)$$

$$=: K_q(\beta) + O(\gamma \log q) \quad (43)$$

whenever $\alpha \in \mathfrak{S}(a/q)$. Therefore by (31) and (18)

$$\sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q'} H(q,a) \int_{\mathfrak{S}(a/q)} F(-\alpha)I(\beta)J(\alpha) d\alpha$$

$$= \int_{\pm \lambda/2} K_q(-\beta)I(-\beta) \left( \sum_{a=1}^{q'} H(q,a) \int_{\mathfrak{S}(a/q)} J(\alpha) \cdot I(\beta) J(\alpha) d\alpha \right)$$

$$=: \int_{\pm \lambda/2} K_q(-\beta)A_q(\beta) d\beta + O \left( q' \gamma \sum_{a=1}^{q'} \int_{\mathfrak{S}(a/q)} |B(\alpha)| d\alpha \right)$$

so that

$$\sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q'} F(-\alpha)C(\alpha) \int_{\mathfrak{S}(a/q)} J(\alpha) d\alpha$$

$$\ll \sum_{q \leq 2\sqrt{x}} |G(q)| \int_{\pm \lambda/2} K_q(-\beta)A_q(\beta) d\beta + \gamma \sum_{q \leq 2\sqrt{x}} |G(q)| \sum_{a=1}^{q'} \int_{\mathfrak{S}(a/q)} |B(\alpha)| d\alpha$$

and therefore from (31)

$$\sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q'} G(q) H(q,a) \int_{\mathfrak{S}(a/q)} F(-\alpha)I(\beta)J(\alpha) d\alpha$$

$$= \sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q'} F(-\alpha)C(\alpha)J(\alpha) d\alpha$$

$$+ O \left( \sum_{q \leq 2\sqrt{x}} \sum_{a=1}^{q'} \int_{\mathfrak{S}(a/q)\setminus(a/q-\lambda/2,a/q+\lambda/2)} |F(-\alpha)| \cdot |C(\alpha)| \cdot |J(\alpha)| d\alpha \right)$$

$$\ll \sum_{q \leq \gamma} |G(q)| \int_{\pm \lambda/2} K_q(-\beta)A_q(\beta) d\beta$$

$$+ \gamma \sum_{q \leq \gamma} |G(q)| \sum_{a=1}^{q'} \int_{\mathfrak{S}(a/q)} |B(\alpha)| d\alpha + \gamma x^{1/2+\theta} + x^{1+\Delta}. \quad (44)$$

Recall the definition of $E_\ell(q,\nu)$ from Lemma 2.2 and of $H(q,a)$ from Lemma 3.3 and let $\Phi^*(n)$
be as in Lemma 3.5. From Lemma 2.2 (i) and then Lemma 3.5 (iii) we have for \( q, t \leq x \)

\[
\sum_{a=1}^{q'} \Delta_x(q, a) H(q, a) = \sum_{\nu=1}^{q} \Phi^*_q(\nu) \left( \sum_{a \in \mathbb{Z} \setminus \nu} 1 - t\eta(q, v) \right)
\]

\[
\prec t^\Delta \sum_{\nu=1}^{q} |\Phi^*_q(\nu)|
\]

\[
\prec q^\Delta t^\Delta
\]

so that for any \( \beta \in \mathbb{R} \)

\[
\sum_{a=1}^{q'} \frac{H(q, a)}{q} J(a/q + \beta) \prec \sum_{a=1}^{q'} \Delta_x(q, a) H(q, a) + |\beta| \int_{1}^{x} \sum_{a=1}^{q'} \Delta_t(q, a) H(q, a) \, dt
\]

\[
\prec q^\Delta (1 + |\beta|x)
\]

and therefore from (36)

\[
A_q(\beta) \sim q^{x^{1+\Delta}}.
\] (45)

Therefore from (43), (31) and (33)

\[
\int_{\pm \lambda/2} K_q(-\beta) A_q(\beta) d\beta \ll q^{x^{1+\Delta}} \int_{\pm 1/2 \gamma} \left( |F(-a/q - \beta)| + |\gamma| \right) d\beta \ll q^{x^{1+\Delta}} \left( \frac{1}{q} + \gamma \lambda \right) \ll x^{1+\Delta}
\]

and so from (17)

\[
\sum_{q \leq \gamma} |G(q)| \left| \int_{\pm \lambda/2} K_q(-\beta) A_q(\beta) d\beta \right| \ll x^{1+\Delta}.
\] (46)

We have from (31)

\[
\sum_{a=1}^{q'} \int_{\gamma(a/q)} |J(\alpha)|^2 d\alpha
\]

\[
\leq \int_{1}^{x} \int_{1}^{x} \left( \sum_{a=1}^{q} \Delta_t(q, a) \Delta_{t'}(q, a) \right) \cdot \left| \int_{\pm \lambda} |\beta|^2 e\left( \beta(t - t') \right) d\beta \right| dt' dt.
\]

The first factor is from (37)

\[
\sum_{\nu=1}^{q} E_t(q, \nu) E_{t'}(q, \nu)
\]

and the second factor is

\[
\ll \min \left( \frac{\lambda^2}{1 - \nu^2}, \lambda^3 \right)
\]
\[
\sum_{q \leq \gamma} \frac{1}{q^{1-\theta}} \sum_{a=1}^{q'} \int_{\mathfrak{H}(a/q)} |J(\alpha)|^2 d\alpha \\
\ll \int_1^x \int_1^x \left( \sum_{q \leq \gamma} q^{\theta} \sum_{\nu=1}^{q} |E_t(q,\nu) E_{t'}(q,\nu)| \min \left( \frac{1}{t-t'},1 \right) \right) dt \, dt' \\
=: \quad U(x, \gamma). \quad (47)
\]

As in (39) we have
\[
U(x, \gamma) \leq \int_1^x \left( \sum_{q \leq \gamma} q^{\theta} \sum_{\nu=1}^{q} |E_t(q,\nu)|^2 \int_1^x \min \left( \frac{1}{t-t'},1 \right) \right) dt
\ll \int_1^x \frac{1}{\gamma^2} \max_{1 \leq t \leq x} \left( \sum_{q \leq \gamma} q^{\theta-2} \sum_{\nu=1}^{q} |E_t(q,\nu)|^2 \right)
\]
so that (47) says
\[
\sum_{q \leq \gamma} \frac{1}{q^{1-\theta}} \sum_{a=1}^{q'} \int_{\mathfrak{H}(a/q)} |J(\alpha)|^2 d\alpha \\
\ll \frac{1}{\gamma^2} \max_{1 \leq t \leq x} \left( \sum_{q \leq \gamma} q^{\theta-2} \sum_{\nu=1}^{q} |E_t(q,\nu)|^2 \right).
\quad (48)
\]

From (31) and (37)
\[
\sum_{q \leq \gamma} \frac{1}{q^{1-\theta}} \sum_{a=1}^{q'} \int_{\mathfrak{H}(a/q)} |\Delta_t(a/q)|^2 d\alpha \\
\ll q^{\theta} \sum_{a=1}^{q} |E_t(q,\nu)|^2 = \frac{1}{\gamma} \sum_{q \leq \gamma} q^{\theta-2} \sum_{\nu=1}^{q} |E_t(q,\nu)|^2
\]
so that
\[
\sum_{q \leq \gamma} \frac{1}{q^{1-\theta}} \sum_{a=1}^{q'} \int_{\mathfrak{H}(a/q)} |\Delta_t(a/q)|^2 d\alpha \\
\ll \frac{1}{\gamma} \max_{1 \leq t \leq x} \left( \sum_{q \leq \gamma} q^{\theta-2} \sum_{\nu=1}^{q} |E_t(q,\nu)|^2 \right)
\]
and therefore from (48) and Lemma 2.2 (ii)
\[
\sum_{q \leq \gamma} \frac{1}{q^{1-\theta}} \sum_{a=1}^{q'} \int_{\mathfrak{H}(a/q)} |\hat{J}(\alpha)|^2 d\alpha \\
\ll \frac{1}{\gamma} \max_{1 \leq t \leq x} \left( \sum_{q \leq \gamma} q^{\theta-2} \sum_{\nu=1}^{q} |E_t(q,\nu)|^2 \right)
\ll \frac{1}{\gamma} \left( \gamma^{1-\theta} x^{2\theta} + x^{2\Delta} + \gamma^{2\theta} x^{\Delta} + \frac{x^{1+2\Delta}}{\gamma} \right)
\ll \frac{x^{2\theta} + x^{1+2\Delta}}{\gamma^2}, \quad (49)
\]
the second term being less than the fourth in the penultimate line (since \(x \geq \gamma\)), and the third less than the first (since \(2\theta \geq \Delta\) and \(1 - \theta \geq \theta\)). From orthogonality
\[
\int_{\pm \lambda} |I(\beta)|^2 \ll x
\]

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so from Lemma 3.4
\[
\sum_{q \leq \gamma} q^{1-\theta} |G(q)|^2 \sum_{a=1}^{q} |I(\beta)|^2 d\beta \ll x. \tag{50}
\]

From the Cauchy-Schwarz inequality and then (49) and (50)
\[
\sum_{q \leq \gamma} |G(q)| \sum_{a=1}^{q} \int_{B(q)} |B_q(\alpha)| d\alpha
\leq \left( \sum_{q \leq \gamma} q^{1-\theta} |G(q)|^2 \sum_{a=1}^{q} |I(\beta)|^2 d\beta \right)^{1/2} \left( \sum_{q \leq \gamma} q^{1-\theta} \sum_{a=1}^{q} \int_{B(q)} |J(\alpha)|^2 d\alpha \right)^{1/2}
\ll x^{1/2+\theta} + \frac{x^{1+\Delta}}{\gamma}. \tag{51}
\]

From this, (44) and (46) we deduce
\[
\sum_{2^{\sqrt{x}} < q \leq \gamma} \sum_{a=1}^{q} \int_{B(a/q)} \left| F(-\alpha) \right| |J(\alpha)| d\alpha \ll \gamma \left( x^{1/2+\theta} + \frac{x^{1+\Delta}}{\gamma} \right). \tag{52}
\]

(E) For \( q \gg \sqrt{x} \) and \( \alpha \in B(a/q) \) (so assuming \( q \leq \gamma \)) we have from (32)
\[
F(\alpha) \ll \gamma
\]
therefore (since the collection of all Farey arcs gives some interval of unit length)
\[
\sum_{2^{\sqrt{x}} < q \leq \gamma} \sum_{a=1}^{q} \int_{B(a/q)} \left| F(-\alpha) \right| |f(\alpha)|^2 d\alpha \ll \gamma \int_{0}^{1} |f(\alpha)|^2 d\alpha \ll x\gamma \tag{52}
\]
by orthogonality.

(F) Write
\[
\mathcal{V} = \sum_{a=1}^{q} \sum_{u \leq \sqrt{x}} \int_{u/2q \gamma < |\beta| \leq u/2} \frac{xd\beta}{|\beta|^2 (u+x||ua/q+u\beta||)}
\]
so that (55) and the bound \( I(\beta) \ll 1/|\beta| \) (from (50)) says
\[
\sum_{a=1}^{q} \int_{1/2q \gamma < |\beta| \leq 1/2} \left| F(-a/q - \beta) \right| |I(\beta)|^2 d\beta \ll \mathcal{V}. \tag{53}
\]
We have
\[
\int_{1/2q \gamma < |\beta| \leq 1/2} \frac{xd\beta}{\beta^2 (u+x||ua/q+u\beta||)} = u \int_{u/2q \gamma < |t| \leq u/2} \frac{xdt}{t^2 (u+x||ua/q+t||)}
\]
and the part of the integral with \( t \geq 1/2 \) is
\[
\leq \sum_{0 < |j| \leq u/2} \int_{-1/2}^{1/2} \frac{xdt}{(j+t)^2 (u+x||ua/q+j+t||)}
\ll \left( \int_{|ua/q+t| \leq 1/2} + \int_{1/4 \leq |ua/q+t| \leq 1/2} \right) \frac{xdt}{u+x||ua/q+t||}
\ll 1 + \log x \ll x^5
\]
so that the whole integral in the definition of \( V \) is
\[
\int_{u/2q^2 < |t| \leq 1/2} u \frac{xdt}{t^2(u + x||u/q + t||)} + O(x^\varepsilon u)
\]
and therefore
\[
V = \sum_{u \leq \sqrt{x}} \int_{u/2q^2 < |t| \leq 1/2} \frac{1}{t^2} \left( \sum_{a=1}^{q'} \frac{x}{u + x||u/q + t||} \right) dt + O \left( x^\varepsilon \sum_{a=1}^{q'} \sum_{u \leq \sqrt{x}} u \right)
\]
\[
= \sum_{u \leq \sqrt{x}} \int_{u/2q^2 < |t| \leq 1/2} \frac{1}{t^2} \left( \sum_{a=1}^{q'} \frac{x}{u + x||u/q + t||} \right) dt + O \left( x^{1+\varepsilon} q \right).
\]
Write \( q' = q/(q, u) \) and \( u' = u/(q, u) \). The inner sum is
\[
\leq (q, u) \sum_{a=1}^{q'} \frac{x}{u + x||u/a/q' + t||}
\]
\[
= (q, u) \sum_{a=1}^{q'} \frac{x}{u + x||a/q' + t||}
\]
\[
\ll (q, u) \sum_{|a/q' + t| \leq 1/2} \frac{x}{u + x||a/q' + t||} + (q, u) q' \log q'
\]
so that
\[
V \ll \sum_{u \leq \sqrt{x}} u(q, u) \int_{u/2q^2 < |t| \leq 1/2} \frac{1}{t^2} \left( \sum_{a=1}^{q'} \frac{x}{u + x||a/q' + t||} \right) dt + q q \sum_{u \leq \sqrt{x}} \int_{u/q^2 < |t| \leq 1/2} \frac{dt}{t^2} + xq
\]
\[
\ll \sum_{u \leq \sqrt{x}} u(q, u) \int_{u/2q^2 < |t| \leq 1/2} \left( \sum_{a=1}^{q'} \frac{x}{t^2(u + x||a/q' + t||)} \right) dt + q^2 \gamma \sqrt{x}
\]
In the sum we have \( t \gg a/q' \) so that the whole integral is for \( q \leq x \)
\[
\ll q^2 \int_{u/2q^2 < |t| \leq 1/2} \left( \sum_{a=1}^{q'} \frac{x}{a^2(u + x||a/q' + t||)} \right) dt \ll q^2
\]
and therefore
\[
V \ll \sum_{u \leq \sqrt{x}} u(q, u) q'^2 + q^2 \gamma \sqrt{x} \ll q^2 \gamma \sqrt{x}
\]
so that from (38)
\[
\sum_{a=1}^{q'} \int_{|\lambda/2| < |\beta| \leq 1/2} |F(-a/q - \beta)| \cdot |I(\beta)|^2 d\beta \ll q^2 \gamma^2
\]
or in other words
\[
\sum_{a=1}^{q} \int_{(-1/2,1/2) \setminus \mathcal{X}} |F(-a/q - \beta)| \cdot |I(\beta)|^2 d\alpha \ll q^2 \gamma^2
\]
for any subset \( \left( \frac{\lambda}{2}, \frac{\lambda}{2} \right) \subseteq \mathcal{X} \subseteq (-1/2,1/2) \).

The result now follows from (31). \( \square \)

5 - Evaluation of a character sum

In the last stage of the proof we will be left with a quantity which we have chosen to analyse with Perron’s formula. The main difficulty will be evaluating

\[
\sum_{n \leq X} \chi(n) n^{-s}
\]
where \( \chi \) is a Dirichlet character and \( s = it \) for \( t \in \mathbb{R} \).

As in Chapter 9 of [6] we make the convention that a primitive character may be principal (and so necessarily of modulus one).

**Lemma 5.1.** For any \( M \in \mathbb{N}, Q > 0, t_0 \in \mathbb{R}, T \geq 1, \) and any primitive character \( \chi \) modulo \( M \),

\[
\int_1^T \frac{L(iv + it_0, \chi) Q^{iv} dv}{v} \ll \sqrt{MT} \left( 1 + \sqrt{|t_0|} \right).
\]

Here the \( \ll \) may contain four terms up to \( M^r, T^s, |t_0|^r \).

**Proof.** Throughout we allow the \( M^r, T^s, |t_0|^r \) terms to go into the \( \ll, O \) symbols - we are basically telling the reader to ignore logs and epsilons.

We first suppose \( \chi \) is non-principal. Take parameters \( Z_0 > Z \geq 2 \). Summing by parts and applying the Polya-Vinogradov Inequality (Theorem 9.18 of [6]) we have for any \( t \in \mathbb{R} \)

\[
\sum_{Z < n \leq Z_0} \frac{\chi(n)}{n^{1-it}} = \frac{1}{Z^{1-it}} \sum_{Z < n \leq Z_0} \chi(n) + (1 - it) \int_Z^{Z_0} \frac{1}{y^{1-it}} \left( \sum_{Z < n \leq y} \chi(n) \right) dy \ll \frac{(1 + |t|) \sqrt{Z}}{Z}
\]

so that letting \( Z_0 \to \infty \)

\[
L(1-it, \chi) = \sum_{n \leq Z} \frac{\chi(n)}{n^{1-it}} + O \left( \frac{\log Z}{1 + |t|} \right)
\]

so long as

\[
Z > (1 + |t|)^2 \sqrt{M};
\]

\[
\text{as explained in the introduction.}
\]
on the other hand Theorem 4.11 of [8] says that if \( Z > 1 + |t| \) then

\[
\zeta(1 - it) = \sum_{n \leq Z} \frac{1}{n^{1-it}} + O \left( \frac{\log Z}{1 + |t|} \right)
\]

so that (54) subject to (55) remains true also in the case of principal \( \chi \) (that is, \( M = 1 \)). For any \( \kappa \) there is some \( A_\kappa \) for which

\[
\sin \left( \frac{\pi (it + \kappa)}{2} \right) = A_\kappa e^{\pi|t|/2} \left( 1 + O \left( \frac{1}{1 + |t|} \right) \right)
\]

and by standard formulas for the Gamma function there is some \( B \) for which

\[
\Gamma(1 - it) = B |t|^{1/2-it} e^{-\pi|t|/2} \left( 1 + O \left( \frac{1}{1 + |t|} \right) \right)
\]

so that with (54) we have for any \( t \in \mathbb{R} \)

\[
L(1 - it, \chi) \Gamma(1 - it) \sin \left( \frac{\pi (it + \kappa)}{2} \right) = \left( \sum_{n \leq Z} \frac{\chi(n)}{n^{1-it}} + O \left( \frac{\log Z}{1 + |t|} \right) \right) \left( BA_\kappa |t|^{1/2-it} e^{it} + O \left( \frac{1}{(1 + |t|)^{1/2}} \right) \right)
\]

so long as (55) holds. Let \( \kappa \) and \( \epsilon(\chi) \) be given respectively as in (10.15) and (10.17) of [6]; from the comments immediately following (10.17) we have \( \epsilon(\chi) \ll 1 \). Therefore Corollary 10.9 of [6] and the last equality say that for some \( C_\chi \ll 1 \) we have for any \( t \in \mathbb{R} \)

\[
L(it, \chi) = \pi^{-1} \epsilon(\chi) \sqrt{M} \left( \frac{2\pi}{M} \right)^it L(1 - it, \chi) \Gamma(1 - it) \sin \left( \frac{\pi (it + \kappa)}{2} \right)
\]

\[
= \left( \sum_{n \leq Z} \frac{\chi(n)}{n^{1-it}} + O \left( \frac{\log Z}{1 + |t|} \right) \right) \left( \sqrt{M} \log Z \right)
\]

so long as (55) holds. Therefore for any \( 1 \leq v \leq T \) and so long as

\[
Z > (1 + T + |t_0|)^2 \sqrt{M}
\]

we have

\[
L \left( i(v + t_0), \chi \right) Q^{iv} = \frac{C_\chi \sqrt{M|v + t_0|}}{v} \sum_{n \leq Z} \frac{\chi(n)e^{f(v)}}{n} + O \left( \frac{\sqrt{M} \log Z}{v} \right),
\]

where

\[
f(v) = f_{n,M,Q,t_0}(v) = \frac{(v + t_0)(\log(2\pi/M) - \log |v + t_0| + 1 + \log n) + v \log Q}{2\pi};
\]

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note that \( f \) is twice differentiable for \( v + t_0 \neq 0 \) and there we have
\[
f''(v) = \pm \frac{1}{2\pi|v + t_0|}.
\]
Therefore
\[
\int_1^T \frac{L(iv + it_0, \chi)Q^v}{v} dv
\]
\[
= C \chi \sqrt{M} \int_1^T \frac{\sqrt{|v + t_0|}}{v} \left( \sum_{n \leq Z} \frac{\overline{n}e(f(v))}{n} \right) dv + \mathcal{O} \left( \sqrt{M \log Z} \int_1^T \frac{dv}{v} \right)
\]
\[
\ll \sqrt{M} \sum_{n \leq Z} \frac{1}{n} \left| \int_1^T G(v) e(f(v)) \, dv \right| + \sqrt{M \log Z}
\]
subject to (56), where
\[
G(v) = \frac{\sqrt{|v + t_0|}}{v}.
\]
We now bound the integral in (58). Take \( R \geq 1 \). For \( v \in (R, 2R) \) we have \( |v + t_0| \ll R + |t_0| \) so from (57)
\[
v \in (R, 2R) \implies \left\{ \begin{array}{l}
G(v) \ll \frac{\sqrt{R + |t_0|}}{R} \quad \text{always} \\
|f''(v)| \gg \frac{1}{R} \quad \text{if } v \neq -t_0.
\end{array} \right.
\]
and we now consider the various scenarios for the sizes of \( R \) and \( t_0 \). Suppose first that \( R \) is large and \( -t_0 \in (R + 1, 2R - 1) \). Then the above bounds become
\[
v \in (R, 2R) \setminus \{t_0\} \implies |f''(v)| \gg \frac{1}{R} \quad \text{and} \quad G(v) \ll \frac{1}{\sqrt{R}}
\]
so that from Lemma 4.5 of [8]
\[
\int_R^{2R} G(v)e(f(v)) \, dv = \left( \int_R^{-t_0-1} + \int_{-t_0-1}^{-t_0+1} + \int_{-t_0+1}^{2R} \right) G(v)e(f(v)) \, dv
\]
\[
\ll 1
\]
having bounded the second integral crudely with (59). If \( -t_0 \not\in (R + 1, 2R - 1) \) then \( v + t_0 \neq 0 \) for \( v \in (R + 1, 2R - 1) \) so the above bounds and the same lemma imply
\[
\int_R^{2R} G(v)e(f(v)) \, dv = \left( \int_{R}^{R+1} + \int_{R+1}^{2R-1} + \int_{2R-1}^{2R} \right) G(v)e(f(v)) \, dv
\]
\[
\ll \frac{\sqrt{R + |t_0|}}{R} + \frac{\sqrt{R + |t_0|}}{R} + \frac{1}{\sqrt{R + |t_0|}} + \frac{\sqrt{R + |t_0|}}{R}
\]
\[
\ll 1 + \sqrt{|t_0|}
\]
having bounded the first and third integrals crudely with (59). If \( R \) is not large then
\[
\int_R^{2R} G(v)e(f(v)) \, dv \ll 1 + \sqrt{|t_0|}
\]
(60)
is clear from (59) so we conclude that (60) holds for all $R \geq 1$ and subject to no constraints on $t_0$. Consequently
\[
\int_1^T G(v)e(f(v)) \, dv \ll 1 + \sqrt{|t_0|}
\]
so (58) implies
\[
\int_1^T L(iv + it_0, \chi)Q^s \, dv \ll \sqrt{M} \left(1 + \sqrt{|t_0|}\right) \sum_{n \leq Z} \frac{1}{n} + \sqrt{M} \log Z
\]
\[
\ll \sqrt{M} \log Z \left(1 + \sqrt{|t_0|}\right)
\]
which proves the lemma if we set for example $Z = 1 + (1 + T + |t_0|)^2 \sqrt{M}$ in accordance with (56).

Suppose $X \geq 1$ and $m \in \mathbb{N}$. For $w \in \mathbb{C}$ with $\Im w \ll X \mathcal{O}(1)$ and $\Re w \geq 0$ and for a primitive character $\chi^*$ modulo $m$, it is well known that
\[
L(w, \chi^*) \ll X^\varepsilon \sqrt{m(1 + |\Im w|)}.
\]

\[\text{(61)}\]

**Lemma 5.2.** For $q, d \in \mathbb{N}$ with $d | q$ and $s \in \mathbb{C}$ define
\[
U_s(q, d) = \frac{1}{q} \sum_{D | d} D^s \phi(q/D).
\]

Then for any $d, M \in \mathbb{N}$, $t \in \mathbb{R}$, $x, Q \geq 1$, and any Dirichlet character $\chi \mod M$, we have for $d, M, |t|, Q \leq x \mathcal{O}(1)$
\[
\sum_{u \leq Q} \frac{\chi(u/(u, d))}{(u/(u, d))^s} = \frac{U_s(dM, d)Q^{1-s}}{1-s} + \mathcal{O} \left(x^\varepsilon \sqrt{M(1 + |t|)}\right)
\]
where $s = it$, and where the main term is present if and only if $\chi$ is principal.

Moreover, the result remains true if $s$ is assumed to be in the region \{ $s \in \mathbb{C} | \sigma \geq 0$ and $|s| \leq 1/2$ \}.

**Proof.** Throughout we write $s = it$ and for $w \in \mathbb{C}$ always $w = u + iv$, for real $u, v$. As in the last proof we allow the $\ll, \mathcal{O}$ symbols to contain terms up to $x^\varepsilon$ (and therefore also $d^\varepsilon, M^\varepsilon, |t|^\varepsilon, Q^\varepsilon$).

Let $\chi^*$ be a primitive character of modulus $m$ say, with $m \leq x \mathcal{O}(1)$. Since $\chi^*$ is principal if and only if $m = 1$ we may define for any $X > 0$
\[
R_{\chi^*}(X) = \begin{cases} 
\frac{X}{1-s} & \text{if } m = 1 \\
0 & \text{if } \chi^* \text{ is not principal}
\end{cases}
\]

Write $A$ for the implied constant in the hypothesis and take parameters $2 \leq X, T \leq x^{A+2}$ with $T$ so large that
\[
T > |t| \tag{62}
\]
and
\[
T > X^2. \tag{63}
\]
Perron’s formula (Theorem 2 in Part II, Section 2 of [7]) implies for \( \kappa > 1 \)
\[
\sum_{n \leq X} \frac{\chi^*(n)}{n^s} = \frac{1}{2\pi i} \int_{\kappa \pm iT} \frac{L(w + s, \chi^*) X^w dw}{w} + \mathcal{O}\left( X^\kappa \sum_{n=1}^\infty \frac{1}{n^s (1 + T |\log(X/n)|)} \right)
=: I(X, T) + \mathcal{O}\left( E(X, T) \right). 
\tag{64}
\]
If \( m = 1 \) then \( L(w + s, \chi^*) = \zeta(w + s) \) and if \( \chi^* \) is non-principal then \( L(w + s, \chi^*) \) is holomorphic for \( u > 0 \), so by the Residue Theorem and (62)
\[
I(X, T) = R_{\chi^*}(X) = \frac{1}{2\pi i} \left( \int_{\kappa+iT}^{iT} + \int_{-\kappa-iT}^{-iT} \right) L(w + s, \chi^*) X^w dw \tag{65}
\]
where \( \mathcal{L} \) is the vertical line from \( iT \) to \(-iT\) except for a half circle \( \mathcal{C} \) from \( \delta i \) to \(-\delta i \) to the right of 0, where \( \delta = 1/\log X \). From (61), we have
\[
\int_{\kappa+iT}^{iT} \frac{L(w + s, \chi^*) X^w dw}{w} \ll X^\kappa \int_0^\kappa \frac{|L(u + iT + it)| du}{|u + iT|} \ll x^\kappa X^\kappa \sqrt{m(1 + T + |t|)} / T
\]
and similarly for the other horizontal integral in (65). For the vertical integral Lemma 5.1 and (61) imply
\[
\int_{\mathcal{L}} \frac{L(w + s, \chi^*) X^w dw}{w} \ll \left| \int_1^T \frac{L(iv + it, \chi^*) X^w dv}{v} \right| + \left( \int_{\mathcal{C}} + \int_{|v| \leq 1} \right) \frac{|L(w + s, \chi^*)| \cdot |X^w| \cdot dw}{|w|}
\ll x^\kappa \sqrt{m(1 + |t|)} + X^\kappa \sqrt{m(1 + |t|)} \left( \int_{\mathcal{C}} + \int_{|v| \leq 1} \right) \frac{dw}{|w|}
\ll x^\kappa \sqrt{m(1 + |t|)} + X^{1/\log X} \sqrt{m(1 + |t|)} \cdot |\log \delta|
\ll x^\kappa \sqrt{m(1 + |t|)}.
\]
Using these bounds for the integrals in (65) and inserting the result into (64) we get
\[
\sum_{n \leq X} \frac{\chi^*(n)}{n^s} = R_{\chi^*}(X) + \mathcal{O} \left( x^\kappa \left( X^\kappa \sqrt{m(1 + T + |t|)} / T + \sqrt{m(1 + |t|)} + E(X, T) \right) \right).
\tag{66}
\]
In general for \( Z > -1 \)
\[
|\log(1 + Z)| \geq \frac{|Z|}{1 + |Z|}.
\]
For \( X/2 \leq n \leq 3X/2 \) we have \( (n - X)/X > -1 \) so that
\[
|\log(X/n)| = \left| \log \left( 1 + \frac{n - X}{X} \right) \right| \geq \frac{|n - X|}{n} \geq \left| \frac{n - X}{n} \right| / n
\]
and therefore
\[
\sum_{X/2 \leq n \leq 3X/2} \frac{1}{n^s |\log(X/n)|} \leq X^{-\kappa} \left( 1 + 2 \sum_{n \leq X} \frac{1}{n} \right) \ll X^{1-\kappa}. 
\tag{67}
\]
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If \( n \) is not in this range then \( |\log(X/n)| \gg 1 \) so from (67)

\[
X^\kappa \sum_{n=1}^{\infty} \frac{1}{n^\kappa (1 + T|\log(X/n)|)} \ll X^\kappa \left( \frac{\zeta(\kappa)}{T} + \frac{X^{1-\kappa}}{T} \right) \\
\ll \frac{1}{T} X^\kappa \frac{X^\kappa}{\kappa - 1} + X \\
\ll \frac{X^\kappa}{T}
\]

if we set \( \kappa = 1 + 1/\log X \). Therefore \( E(X, T) \ll X/T \) which we put in (66) to get

\[
\sum_{n \leq X} \frac{\chi^*(n)}{n^s} = R_{\chi^*}(X) + O \left( x^s \left( \frac{X\sqrt{m(1 + T + |t|)}}{T} + \sqrt{m(1 + |t|)} + \frac{X}{T} \right) \right) \\
= R_{\chi^*}(X) + O \left( x^s \sqrt{m(1 + |t|)} \right)
\]

from (63). The equality obviously still valid if \( 0 \leq X \leq 2 \) we conclude that for any \( 0 < X \leq x^{A+2} \)

\[
\sum_{n \leq X} \frac{1}{n^s} = \frac{X^{1-s}}{1-s} + O \left( x^s \sqrt{(1 + |t|)} \right) \\
\tag{68}
\]

and

\[
\sum_{n \leq X} \frac{\chi^*(n)}{n^s} \ll x^s \sqrt{m(1 + |t|)} \\
\tag{69}
\]

if \( \chi^* \) is non-principal.

If \( \chi \) is non-principal then there is an \( m \) with \( m|M \) and non-principal primitive character \( \chi^* \mod m \) for which

\[
\chi(n) = \begin{cases} 
\chi^*(n) & \text{if } (n, M) = 1 \\
0 & \text{if not}
\end{cases}
\]

so that

\[
\sum_{u \leq Q} \frac{\chi(u/(u, d))}{(u/(u, d))^s} = \sum_{D|d} \sum_{\gcd(u/D, (u/d)/D) = 1} \frac{\chi(u)}{u^s} \\
= \sum_{D|d} \sum_{u \leq Q/D} \frac{\chi^*(u)}{u^s} \\
= \sum_{D|d} \sum_{\Delta|dM} \Delta^s \sum_{u \leq Q/D\Delta} \frac{\chi^*(u)}{u^s} \\
\ll \sum_{D, \Delta|dM} \left| \sum_{u \leq Q/D\Delta} \frac{\chi^*(u)}{u^s} \right| \\
\ll x^s \sqrt{m(1 + |t|)}
\]

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from (69), which proves the lemma for \( \chi \) non-principal. If \( \chi \) is principal then we use (68) to deduce
\[
\sum_{u \leq Q} \frac{\chi(u/(u, d))}{(u/(u, d))} = \sum_{u \leq Q} \frac{1}{(u/(u, d))}
\]
\[
= \sum_{D|d} \sum_{\Delta|dM/D} \frac{\phi(d)}{\Delta} \sum_{u \leq Q/\Delta} \frac{1}{u^s}
\]
\[
= Q^{1-s} \sum_{D|d} \frac{1}{D^{1-s}} \sum_{\Delta|dM/D} \frac{\phi(dM/D)}{dM/D} + O \left( x^\epsilon \sqrt{1+|\epsilon|} \right)
\]
which proves the lemma for \( \chi \) principal. The last claim is an easy consequence of partial summation and the Polya-Vinogradov Inequality.

6 - Proof of theorem

Let 1 \( \leq Q \leq x \) be given. If \( Q \leq \sqrt{x} \) then (3) and the first claim of Lemma 2.2 (ii) imply
\[
V(x, Q) \ll x^\epsilon \left( x^{1+\eta} + x^{2\Delta} + x^{1/2 + \Delta} \right) \ll x^{\Delta + \epsilon}
\]
which is our theorem, so we assume
\[
Q > \sqrt{x}.
\]
(70)

Since for \((d_j^k, d_j^e)|h_i - h_j \) (as in the proof of part (iii) of Lemma 2.1)
\[
\sum_{d_1, \ldots, d_r = 1 \atop d_i^k - d_j^k | h_i - h_j} \frac{1}{|d_1^k \cdots d_r^e|} \ll \frac{1}{d_1^k \cdots d_r^e}
\]
we have
\[
\sum_{d_1, \ldots, d_r = 1 \atop d_i^k - d_j^k | h_i - h_j} \frac{|\mu(d_1) \cdots \mu(d_r)|}{|q, [d_1^k, \ldots, d_r^e]|} \ll \frac{1}{q} \sum_{d_1, \ldots, d_r} \frac{(q, d_1^k \cdots d_r^e)}{d_1^k \cdots d_r^e} \ll \frac{1}{q} \sum_{n=1}^{\infty} \frac{(q, n^k)n^{s-k} \ll q^{s-1}}{n^{s-k}}.
\]
(71)

From (3) and (2)
\[
V(x, Q) = \sum_{q \leq Q} \sum_{a=1}^{q} \sum_{q n \leq x} 1 - 2x \sum_{q \leq Q} \sum_{a=1}^{q} \eta(q, a) \sum_{n \geq \frac{x}{Qa}} 1 + x^2 \sum_{q \leq Q} \sum_{a=1}^{q} \eta(q, a)^2
\]
\[
= S_1(x, Q) - 2xS_2(x, Q) + x^2 \sum_{q \leq Q} W(q).
\]
(72)

For \( d_1, \ldots, d_r \in \mathbb{N} \) write \( d^* = [d_1, \ldots, d_r] \). Denote by \( V \) the unique solution modulo \([q, d_1, \ldots, (q, d_r)] = (q, d^*) \to n \equiv -h ((q, d)) \). From (2) we have for a new parameter \( X > 0 \)
\[
\sum_{a=1}^{q} \eta(q, a) \sum_{n \leq X} \eta(q, n) = \sum_{n \leq X} \mu(d_1) \cdots \mu(d_r) \frac{1}{[q, d^*]} \sum_{d_1, \ldots, d_r = 1 \atop d_i^k - d_j^k | h_i - h_j} \frac{1}{n^{s-k}}.
\]
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From Lemma 2.2 (i) the inner sum here is
\[ A_{q,d} X + \mathcal{O} \left( X^{\Delta+\epsilon} \right) \]
for some \( A_{q,d} \) and therefore from (71)
\begin{align*}
  \sum_{a=1}^{q} \eta(q,a) \sum_{n \in X \atop n \equiv a(q)} 1 &= X \sum_{d_1=1}^{\infty} \cdots \sum_{d_r=1}^{\infty} \frac{A_{q,d} \cdot \mu(d_1) \cdots \mu(d_r)}{[q,d^k]} + \mathcal{O} \left( X^{\Delta+\epsilon} \sum_{d_1=1}^{\infty} \cdots \sum_{d_r=1}^{\infty} \frac{|\mu(d_1) \cdots \mu(d_r)|}{[q,d^k]} \right) \\
  &=: \quad XB_q + \mathcal{O} \left( X^{\Delta+\epsilon} q^{-1} \right). \tag{73}
\end{align*}

On the other hand Lemma 2.2 (i) says
\[ \sum_{n \leq X \atop n \equiv a(q)} 1 = X \eta(q,a) + o(X) \]
so from (72)
\[ \sum_{a=1}^{q} \eta(q,a) \sum_{n \leq X \atop n \equiv a(q)} 1 = X \sum_{a=1}^{q} \eta(q,a)^2 = XW(q) + o_q(X) \]
and so from (73)
\[ XB_q + \mathcal{O} \left( X^{\Delta+\epsilon} q^{-1} \right) = XW(q) + o_q(X). \]

Therefore we must have
\[ B_q = W(q) \]
and setting \( X = x \) in (73) we deduce
\[ \sum_{a=1}^{q} \eta(q,a) \sum_{n \leq x \atop n \equiv a(q)} 1 = xW(q) + \mathcal{O} \left( x^{\Delta+\epsilon} q^{-1} \right) \]
so that, from (72),
\[ S_2(x,Q) = \sum_{q \leq Q} \left( xW(q) + \mathcal{O} \left( x^{\Delta+\epsilon} q^{-1} \right) \right) = x \sum_{q \leq Q} W(q) + \mathcal{O} \left( x^{\Delta+\epsilon} \right). \tag{74} \]

Let \( \rho \) be as in Lemma 3.5. From Definition 3.3 we have \( G(1)H(1,1) = 1 \) so from (25) and Lemma 2.2 (i)
\[ \sum_{n \leq x} 1 = \rho x + \mathcal{O} \left( x^{\Delta+\epsilon} \right) \]
so that
\[ \sum_{q \leq Q} \sum_{n \leq x} 1 = \rho xQ + \mathcal{O} \left( x^{1+\Delta+\epsilon} \right) \]

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and therefore from \((72)\)

\[
S_4(x, Q) = \sum_{q \leq Q} \sum_{\substack{n, m \leq x \atop n \equiv m \pmod{q}}} 1
\]

\[
= 2 \sum_{q \leq Q} \sum_{\substack{n, m \leq x \atop n \equiv m \pmod{q}}} 1 + \sum_{q \leq Q} \sum_{n \in \mathbb{R}} 1
\]

\[
=: 2S_4(x, Q) + \rho xQ + O(x^{1+\Delta + \epsilon}).
\]

Putting this and \((74)\) in \((72)\) gives

\[
V(x, Q) = 2S_4(x, Q) + \rho xQ - x^2 \sum_{q \leq Q} W(q) + O(x^{1+\Delta + \epsilon}) \tag{75}
\]

and now our task is to study \(S_4(x, Q)\) using the circle method.

Let \(F, f\) be as in Lemma 4.1. Writing the congruence condition in \(S_4(x, Q)\) out explicitly and using orthogonality we have

\[
S_4(x, Q) = \sum_{q \leq Q} \sum_{l \leq x/q} \sum_{\substack{n, m \leq x \atop n \equiv m \pmod{q}}} 1 = \int_{\Delta} F(-\alpha)|f(\alpha)|^2 d\alpha \tag{76}
\]

for any unit interval \(\Delta\). As in the comments preceding Lemma 4.1 denote by \(\mathcal{F}(a/q)\) the Farey arc at \(a/q\) in the Farey dissection of order \(\gamma\), where \((a, q) = 1\). Then \((76)\) and \((29)\) imply

\[
S_4(x, Q) = \sum_{q \leq 2\sqrt{x}} \sum_{l = 1}^q \int_{\mathcal{F}(a/q)} F(-\alpha)|f(\alpha)|^2 d\alpha
\]

\[
+ \mathcal{O}\left(1 + \sum_{2\sqrt{x} < q \leq \gamma} \sum_{a = 1}^q \int_{\mathcal{F}(a/q)} |F(-\alpha)| \cdot |f(\alpha)|^2 d\alpha \right)
\]

\[
=: M(\gamma) + \mathcal{O}\left(1 + E_4(\gamma)\right). \tag{77}
\]

Let \(\theta, \Delta, G, H, \rho, I\) and \(\mathcal{J}\) be as in Lemma 4.1 and as in that lemma write \(\alpha = a/q + \beta\) whenever \(\alpha \in \mathcal{F}(a/q)\). Suppose \(2\sqrt{x} \leq \gamma \leq x^{3/4}\). From part (A) of that lemma

\[
|f(\alpha)|^2 = |\rho G(q)H(q, a)I(\beta)|^2 + 2\Re \left(\rho G(q)H(q, a)I(\beta)\mathcal{J}(\alpha)\right) + |\mathcal{J}(\alpha)|^2
\]
so that from parts (D) and (C) we have

\[
M(\gamma) = |\rho|^2 \sum_{q \leq 2 \sqrt{x}} |G(q)|^2 \sum_{a=1}^{q} |H(q,a)|^2 \int_{\delta(a/q)} F(-\alpha)|I(\beta)|^2 d\alpha
\]

\[
+ 2\text{Re} \left( \sum_{q \leq 2 \sqrt{x}} \sum_{a=1}^{q} G(q) \sum_{a=1}^{q} H(q,a) \int_{\delta(a/q)} F(-\alpha)\bar{I}(\beta)J(\alpha) d\alpha \right)
\]

\[
+ \sum_{q \leq 2 \sqrt{x}} \sum_{a=1}^{q} \int_{\delta(a/q)} F(-\alpha)\bar{J}(\alpha)|^2 d\alpha
\]

\[
= \rho^2 \sum_{q \leq 2 \sqrt{x}} |G(q)|^2 \sum_{a=1}^{q} |H(q,a)|^2 \int_{\delta(a/q)} F(-\alpha)|I(\alpha-a/q)|^2 d\alpha + O \left( x^\epsilon \left( x^{1+\Delta} + \gamma x^{1/2+\theta} + \frac{x^{1+2\Delta}}{\gamma} \right) \right)
\]

\[
=: M^*(\gamma) + O \left( x^\epsilon \left( x^{1+\Delta} + \gamma x^{1/2+\theta} + \frac{x^{1+2\Delta}}{\gamma} \right) \right). 
\]  (78)

Take a parameter \(1 \leq Z \leq 2 \sqrt{x}\). From Lemma 3.4

\[
\sum_{q \leq Z} q^2 |G(q)|^2 \ll Z^{\theta+\epsilon}
\]

so from (15) and Lemma 4.1 (F)

\[
\sum_{q \leq Z} |G(q)|^2 \sum_{a=1}^{q} |H(q,a)|^2 \int_{\delta(a/q)} F(-\alpha)|I(\alpha-a/q)|^2 d\alpha \ll x^\epsilon Z^\theta
\]  (79)

where \(\delta(a/q)\) denotes the unit interval centered at \(a/q\). On the other hand Lemma 4.1 (B) and then orthogonality gives

\[
\sum_{a=1}^{q} |H(q,a)|^2 \int_{X} F(-\alpha)|I(\alpha-a/q)|^2 d\alpha \ll x^{1+\epsilon} \int_{-1/2}^{1/2} |I(\beta)|^2 d\beta \ll x^{2+\epsilon}
\]

for any \(X \subseteq [a/q - 1/2, a/q + 1/2]\), and from Lemma 3.4

\[
\sum_{q > Z} |G(q)|^2 \ll Z^{\theta-2+\epsilon},
\]

therefore

\[
\sum_{q > Z} |G(q)|^2 \sum_{a=1}^{q} |H(q,a)|^2 \int_{X} F(-\alpha)|I(\alpha-a/q)|^2 d\alpha \ll x^{2+\epsilon} Z^{\theta-2}. 
\]  (80)
From (79) and (80)

\[
M^*(\gamma) = \rho^2 \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{q} |H(q,a)|^2 \int_{I(a/q)} F(-\alpha) |I(\alpha - a/q)|^2 d\alpha
\]

\[
+ O \left( \sum_{q \leq Z} |G(q)|^2 \sum_{a=1}^{q} |H(q,a)|^2 \int_{I(a/q) \setminus \beta(a/q)} |F(-\alpha)| \cdot |I(\alpha - a/q)|^2 d\alpha \right)
\]

\[
+ \sum_{q > Z} |G(q)|^2 \left| \sum_{a=1}^{q} |H(q,a)|^2 \int_{I(a/q) \setminus \beta(a/q)} F(-\alpha) |I(\alpha - a/q)|^2 d\alpha \right|
\]

\[
+ \sum_{q > Z} |G(q)|^2 \left| \sum_{a=1}^{q} |H(q,a)|^2 \int_{I(a/q)} F(-\alpha) |I(\alpha - a/q)|^2 d\alpha \right|
\]

\[
= \rho^2 \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{q} |H(q,a)|^2 \int_{I(a/q)} F(-\alpha) |I(\alpha - a/q)|^2 d\alpha + O \left( x^\epsilon (\gamma^2 Z^\theta + x^2 Z^{\theta - 2}) \right)
\]

\[
= \rho^2 \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{q} |H(q,a)|^2 \int_{-1/2}^{1/2} F(-a/q - \beta) |I(\beta)|^2 d\beta + O \left( \gamma^{2-\theta} x^{\theta + \epsilon} \right) \quad (81)
\]

on choosing \( Z = x/\gamma \). Since it is straightforward to establish that

\[
\int_{-1/2}^{1/2} |I(\beta)|^2 e(-\beta n) d\beta = x - n + O(1)
\]

we have (\( F \) is defined in Lemma 4.1).

\[
\int_{-1/2}^{1/2} F(-a/q - \beta) |I(\beta)|^2 d\beta = \sum_{\frac{nu}{w+u} \leq Q} e(-auv/q)(x - uv) + O(x^{1+\epsilon})
\]

so we deduce from (81), (18) and (17)

\[
M^*(\gamma) = \rho^2 \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{q} |H(q,a)|^2 \left( \sum_{\frac{nu}{w+u} \leq Q} e(-auv/q)(x - uv) + O(x^{1+\epsilon}) \right) + O(\gamma^{2-\theta} x^{\theta + \epsilon})
\]

\[
= \rho^2 \sum_{q \leq x} |G(q)|^2 \sum_{\frac{nu}{w+u} \leq Q} \left( x - uv \right) \Phi_q(-uv) + O \left( x^\epsilon \left( x + \gamma^{2-\theta} x^\theta \right) \right),
\]

where \( \Phi_q(n) \) is as in Lemma 3.5 so from (77) and (78)

\[
S_4(x, Q) = \rho^2 \sum_{q \leq x} |G(q)|^2 \sum_{\frac{nu}{w+u} \leq Q} \left( x - uv \right) \Phi_q(-uv)
\]

\[
+ O \left( x^\epsilon \left( x^{1+\Delta} + \gamma x^{1/2+\theta} + \frac{x^{1+2\Delta}}{\gamma} + E_x(\gamma) + \gamma^{2-\theta} x^\theta \right) \right) \quad (82)
\]

Recall that \( \theta = 1/k \) and \( \Delta = 2/(k + 1) \). If \( k > 2 \) we set \( \gamma = 2\sqrt{x} \) so that \( E_x(\gamma) = 0 \) and \( \gamma \geq x^\Delta \).
to deduce that
\[ S_4(x, Q) = \rho^2 \sum_{q \leq x} |G(q)|^2 \sum_{\substack{uv \leq x \\mod q \leq Q \\mod (\mathcal{D})}} \Phi_q(-uv) \big( x - uv \big) + O \left( x^{1+\Delta} \right) \]
\[ =: \rho^2 J(x, Q) + O \left( x^{1+\Delta + \epsilon} \right). \tag{83} \]
If \( k = 2 \) we set \( \gamma = x^{2/3} \) and deduce from Lemma \( \text{(E)} \) that the error term in \( \text{(82)} \) is up to an \( x^\epsilon \) bound
\[ \ll x^{5/3} + \gamma x + \frac{\gamma^{7/3}}{\gamma} + x \gamma + \gamma^{3/2} x^{1/2} \ll x^{5/3} = x^{1+\Delta} \]
to conclude that \( \text{(83)} \) holds for all \( k \geq 2 \).

This finishes our circle method work and it remains to evaluate \( J(x, Q) \). We use the periodicity of \( \Phi_q(n) \) modulo \( q \) and apply Perron’s formula to evaluate precisely the remaining quantity.

We make the convention that whenever we have the letter \( \mathcal{D} \) appearing in a context involving natural numbers \( q, a \) we mean \( \mathcal{D} = (q, a) \). For any \( u \in \mathbb{N} \) we then write \( u' = u/(u, \mathcal{D}) \).

Sorting the \( uv \) according to the residue \( a \mod q \) we have
\[ J(x, Q) = \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{\varphi(q')} \Phi_q(-a) \sum_{\substack{uv \leq x \\mod q \leq Q \\mod (\mathcal{D})}} \big( x - uv \big) \]
\[ = \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{\varphi(q')} \Phi_q(-a) \sum_{\substack{uv \leq x \\mod q \leq Q \\mod (\mathcal{D})}} \big( x - uv \big) \]
\[ =: \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{\varphi(q')} \Phi_q(-a) U(q, a). \tag{84} \]

For \( n \in \mathbb{N} \) with \( (n, q') = 1 \) denote by \( \mathfrak{m} \) the inverse of \( n \) modulo \( q' \). We have
\[ U(q, a) = \sum_{u \leq Q} \sum_{\substack{v \leq x/u \\mod \mathcal{D}/(\mathcal{D}, u) \\mod \mathcal{D}/(\mathcal{D}, u') \\mod \mathcal{D}/(\mathcal{D}, u')'}} \big( x - uv \big) \]
\[ = \sum_{u \leq Q} \mathcal{D} u' \sum_{(u', q') = 1} \sum_{v \leq x/\mathcal{D} u'} \big( x/\mathcal{D} u' - v \big) \]
\[ =: \sum_{(u', q') = 1} \mathcal{D} u' V_{q,a}(u). \tag{85} \]

Through the orthogonality of Dirichlet characters and a Perron formula (taking \( w = 1 \) in (11) of Section 2, Part II in [7], page 134, the relevant quantities being defined at the start of that section) we have
\[ V_{q,a}(u) = \frac{1}{\varphi(q')} \sum_{\chi} \overline{\chi(u'a')} \sum_{v \leq x/\mathcal{D} u'} \chi(v) \big( x/\mathcal{D} u' - v \big) \]
\[ = \frac{1}{\varphi(q')} \sum_{\chi} \overline{\chi(u'a')} \int_{2-i\infty}^{2+i\infty} L(s, \chi) \left( \frac{x}{\mathcal{D} u'} \right)^{s+1} ds; \tag{86} \]
here and in what follows the sum \( \Sigma \chi \) runs over the Dirichlet characters modulo \( q' \) and for \( s \in \mathbb{C} \) we always write \( s = \sigma + it \) for real numbers \( \sigma, t \). Denote by \( \mathcal{L} \) the contour from \(-ix^6 \) to \( ix^6 \) which is a vertical line except for a small detour to the right of 0. Define

\[
\delta(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_0 \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}
\]

The parts of the above integral with \( |t| \geq x^6 \) contribute to the integral

\[
\ll x^3 \int_{|t| \geq x^6} \frac{|L(2 + it, \chi)|}{t^2} dt \ll x^3 \int_{|t| \geq x^6} \frac{dt}{t^2} \ll \frac{1}{x^3}
\]

so pulling the remaining part of the integral to the left, and so picking up a simple pole at \( s = 1 \) if \( \chi = \chi_0 \), we see that the integral in (86) is

\[
\int_{\mathcal{L}} \frac{L(s, \chi)}{s(s + 1)} \left( \frac{x}{Du'} \right)^{s+1} ds + \frac{\text{Res}_{s=1} L(s, \chi_0)}{2} \left( \frac{x}{Du'} \right)^2 \delta(\chi) + O \left( \frac{1}{x^3} \right)
\]

so that for \((u', q') = 1\)

\[
\mathcal{V}_{q, a}(u) = \frac{1}{\phi(q')} \sum_{\chi} \chi(u') \chi(a') \int_{\mathcal{L}} \frac{L(s, \chi)}{s(s + 1)} \left( \frac{x}{Du'} \right)^{s+1} ds + \frac{1}{2q'} \left( \frac{x}{Du'} \right)^2 + O \left( \frac{1}{x^4 \phi(q')} \right)
\]

and so from (85) for \( q \leq x \)

\[
\mathcal{U}(q, a) = \frac{1}{\phi(q')} \int_{\mathcal{L}} \frac{x^{s+1}}{s(s + 1)D^u} \left( \sum_{\chi} \chi(u') L(s, \chi) \sum_{u \leq Q} \chi(u') \right) ds + \frac{x^2}{2q'} \sum_{u \leq Q} \frac{1}{Du'} + O \left( \frac{1}{x} \sum_{u \leq Q} Du' \right)
\]

\[
=: \mathcal{I}(q, a) + \frac{x^2}{2q'} \sum_{u \leq Q : (u/D, q/D) = 1} \frac{(u/D)}{u} + O(1). \tag{87}
\]

We have

\[
L(s, \chi_0) = \zeta(s) \prod_{p | q'} (1 - p^{-s}) =: \zeta(s)\omega_s(q'); \tag{88}
\]

define for \( d | q \)

\[
\theta_s(q, d) = \frac{\omega_s(q/d)U_s(q, d)}{d^s \phi(q/d)} \tag{89}
\]

where \( U_s(q, d) \) is as in Lemma 5.2. For \( q', |t| \leq x \) we have the standard estimate

\[
L(s, \chi) \ll x' \sqrt{q'(1 + |t|)}, \quad 0 \leq \sigma \leq 1,
\]
so that with Lemma 5.2 we see that the term in the brackets in $I(q, a)$ is for $q \leq x$

\[
\sum_{q \leq x} \Phi_q(-a) I(q, a) = Q^2 \int_{\mathcal{L}} \frac{\zeta(s)\theta_s(q, D)}{(s+1)(1-s)} \left( \frac{x}{Q} \right)^{s+1} ds + O\left(x^{1+\epsilon} |x|^{s+1} \right) \]

and so

\[
I(q, a) = Q^2 \int_{\mathcal{L}} \frac{\zeta(s)\theta_s(q, D)}{(s+1)(1-s)} \left( \frac{x}{Q} \right)^{s+1} ds + O\left(x^{1+\epsilon} \right)
\]

so from Lemma 3.3 (iii) we have for $q \leq x$

\[
\sum_{a=1}^{q} \Phi_q(-a) I(q, a) = -Q^2 \int_{\mathcal{L}} \frac{\zeta(s)(x/Q)^{s+1}}{(s+1)s(s+1)} \left( \sum_{a=1}^{q} \theta_s(q, D) \Phi_q(-a) \right) ds + O\left(x^{1+\epsilon} q^2 \right)
\]

and therefore from Lemma 3.4 (and since 17) says $|G(q)| \ll$

\[
\sum_{q \leq x} |G(q)|^2 \sum_{q \leq x} \Phi_q(-a) I(q, a)
\]

\[
= -Q^2 \int_{\mathcal{L}} \frac{\zeta(s)(x/Q)^{s+1}}{(s+1)s(s+1)} \left( \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{q} \theta_s(q, D) \Phi_q(-a) \right) ds + O\left(x^{1+\epsilon} \sum_{q \leq x} |G(q)|^2 \right)
\]

\[
= -Q^2 \int_{\mathcal{L}} \frac{\zeta(s)(x/Q)^{s+1}}{(s+1)s(s+1)} \left( \sum_{q \leq x} |G(q)|^2 \Delta_s(q) \right) ds + O\left(x^{1+\theta+\epsilon} \right).
\]

For $\sigma \geq 0$ and $d|q$ it is clear that $U_s(q, d) \ll |d|^\sigma q^\epsilon$ and so from Lemma 3.3 (iii) we have

\[
\Delta_s(q) \ll q^{1+\epsilon}
\]

and therefore from Lemma 3.4

\[
\sum_{q \geq x} |G(q)|^2 \Delta_s(q) \ll x^{\theta-1+\epsilon}
\]

so we can add in these terms to 41 at the cost of an error of size

\[
\ll x^{\theta-1+\epsilon} Q^2 \int_{\mathcal{L}} \frac{\zeta(s)(x/Q)^{s+1}}{(s+1)s(s+1)} ds \ll x^{\theta+\epsilon} Q \int_{|t|} \frac{|t|^{1/2} dt}{1+|t|^2} \ll x^{1+\theta+\epsilon}
\]
Recall from Lemma 5.2 that 

$$\sum_{q \leq x} |G(q)|^2 \Phi_q(-a) \zeta(q,a) = -Q^2 \int \frac{\zeta(s)(x/Q)^{s+1}}{(s-1)s(s+1)} \left( \sum_{q=1}^{\infty} |G(q)|^2 \Delta_s(q) \right) ds + O \left(x^{1+\theta+\epsilon}\right)$$

$$= -Q^2 \int \frac{\zeta(s)(x/Q)^{s+1}G(s)ds}{(s-1)s(s+1)} + O \left(x^{1+\theta+\epsilon}\right)$$

$$= -Q^2 \mathcal{V}(x/Q) + O \left(x^{1+\theta+\epsilon}\right), \quad (92)$$

where $G(s)$ converges absolutely (at least) for $\sigma \geq 0$. The last equality with (84) and (87) implies

$$\mathcal{J}(x, Q) = -Q^2 \mathcal{V}(x/Q) + \frac{x^2}{2} \sum_{q \leq x} |G(q)|^2 \Phi_q(-a) \sum_{a \leq q} \frac{(u,D)}{u}$$

$$+ O \left(x^{1+\theta+\epsilon} + x^{1+\theta+\epsilon} \sum_{q \leq x} |G(q)|^2 \sum_{a=1}^{q} |\Phi_q(-a)|\right)$$

$$= -Q^2 \mathcal{V}(x/Q) + \frac{x^2 \mathcal{V}(Q)}{2} + O \left(x^{1+\theta+\epsilon}\right), \quad (93)$$

with Lemma 5.3 (iii) and Lemma 5.4. Our application of Perron’s formula is complete and now our task is now to evaluate $\mathcal{G}(s)$. The main point is we can write down an analytic continuation for this thanks to the explicit expressions for the Gauss sum in Section 3. Recall the assumption $G_{\sigma}$ of our theorem: we always have $R_p < p^k$, where $R_p$ is the number of distinct residue classes represented by the $h_1, \ldots, h_r$.

Recall from Lemma 5.2 that

$$U_s(q,d) = \frac{1}{q} \sum_{D | d} D^s \phi(q/D) = \frac{F^*_s(q,d)}{q}. \quad (94)$$

It is straightforward to establish that $U_s(q,d)$ satisfies $U_s(qq', dd') = U_s(q,d)U_s(q', d')$ for $(q, q') = 1$ and $d | q$ so from (89) the same must be true of $\theta_s(q, d)$, and therefore Lemma 5.3 (i) says that $\Delta_s(q)$ is multiplicative (this is defined in (91)), so from Definition 5.3 we have for $\sigma \geq 0$

$$\mathcal{G}(s) = \prod_p \left( \sum_{t \geq 0} |G(q)|^2 \Delta_s(q) \right) = \prod_p \left( 1 + \frac{1}{p^{2k}(1 - R_p/p^k)^2} \sum_{1 \neq q | p^k} \Delta_s(q) \right). \quad (95)$$

Define $\rho$ as in Lemma 5.3 (iv), namely

$$\rho = \prod_p \left( 1 - \frac{R_p}{p^k} \right);$$

then from (93) for $\sigma \geq 0$

$$\rho^2 \mathcal{G}(s) = \prod_p \left( 1 - \frac{R_p}{p^k} \right)^2 \frac{1}{p^{2k}} \sum_{1 \neq q | p^k} \Delta_s(q). \quad (96)$$
From (89) and (94) we have

\[ \theta_s(q, (q, a)) = \frac{\omega_s(q/D) F_s^*(q, D)}{q D^s \phi(q/D)} \]

so from (91)

\[ \Delta_s(q) = \frac{1}{q} \sum_{d|q} \omega_s(q/d) F_s^*(q, d) \sum_{a=1}^{q/d} \Phi_s(-ad). \]  

(97)

Define \( H(q, a) \) and \( \Phi(q) \) as in Lemma 3.5 and take a prime \( p \). Denote the different residues represented by \( h_1, ..., h_r \) modulo \( p^k \) by \( H_1, ..., H_R \). For \( q \mid p^k \)

\[ H(q, a) = \sum_{n=1}^{R_p} c \left( \frac{a H_n}{q} \right) \]

so that

\[ \Phi(q) = \sum_{n,n'=1}^{R_p} c_q (H_n - H_{n'}) \]

so from Lemma 3.5 (ii)

\[ \sum_{a=1}^{q/d} \Phi_s(-ad) = \mu(q/d) \sum_{n,n'=1}^{R_p} c_q (H_n - H_{n'}) \]

and therefore from (97)

\[ \Delta_s(q) = \frac{1}{q} \left( \sum_{d|q} \mu(q/d) \omega_s(q/d) F_s^*(q, d) \frac{d^s \phi(q/d)}{d^s \phi(q)} \right) \left( \sum_{n,n'=1}^{R_p} c_q (H_n - H_{n'}) \right) \]

\[ =: P_s(q) \sum_{n,n'=1}^{R_p} c_q (H_n - H_{n'}). \]  

(98)

Simple calculations show

\[ F_s^*(q, q) = q^s + \frac{\phi(q) (q^{s-1} - 1)}{p^{s-1} - 1} \]

and

\[ F_s^*(q, q/p) = F_s^*(q, q) - q^s \]

so that

\[ P_s(q) = \frac{F_s^*(q, q)}{q^s} - \frac{(1-p^{-s}) F_s^*(q, q/p)}{(q/p)^s \phi(p)} \]

\[ = \left( q^s + \frac{\phi(q) (q^{s-1} - 1)}{p^{s-1} - 1} \right) \left( \frac{1 - p^{-s}}{(q/p)^s \phi(p)} \right) + q^s (1 - p^{-s}) \]

\[ = q^{1-s} \]

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so from (98)

$$\Delta_s(q) = q^{-s} \sum_{n,n'=1}^{R_p} c_q(H_n - H_{n'})$$

for $q|p^k$. The sum here is

$$\sum_{n,n'=1}^{R_p} c_q(H_n - H_{n'}) + R_p \phi(q) =: \Delta^*(q) + R_p \phi(q)$$  \hspace{1cm} (99)

so

$$\Delta_s(q) = q^{-s} \left( \Delta^*(q) + R_p \phi(q) \right)$$

and therefore, writing $X = p$ and $Y = p^{-s}$,

$$\sum_{1 \neq q|p^k} \Delta_s(q) = \sum_{1 \neq q|p^k} q^{-s} \Delta^*(q) + R_p \left( \sum_{1 \neq q|p^{k-1}} q^{-s} \phi(q) + p^{-sk} \phi(p^k) \right)$$

$$= \sum_{1 \neq q|p^k} q^{-s} \Delta^*(q) + R_p \left( \frac{(1 - 1/p)(p^k(1-s) - p^{-1-s})}{p^{1-s} - 1} \right) + R_p p^{(1-s)}$$

$$= \sum_{1 \leq t \leq k} Y^t \Delta^*(X^t) + R_p \left( \frac{(1 - 1/X)((XY)^k - XY)}{XY - 1} \right) + R_p (XY)^k$$

$$=: P_1(X,Y) + R_p (XY)^k$$  \hspace{1cm} (100)

so from (98) for $\sigma \geq 0$

$$\rho^2 G(s) = \prod_{p} \left( 1 - 2R_p \frac{X^k}{X^{2k}} + R_p^2 \frac{X^{2k}}{X^{4k}} + R_p \left( \frac{Y}{X} \right)^k \right)$$

$$= \prod_{p} \left( (1 + (Y/X)^k)^{R_p} + 1 + R_p \left( \frac{Y}{X} \right)^k - (1 + (Y/X)^k)^{R_p} - 2R_p \frac{Y^k}{X^{2k}} + R_p^2 \frac{Y^2}{X^{4k}} + \frac{P_1(X,Y)}{X^{2k}} \right)$$

$$= : \prod_{p} \left( (1 + (Y/X)^k)^{R_p} + P_2(X,Y) + \frac{P_1(X,Y)}{X^{2k}} \right);$$  \hspace{1cm} (101)

in particular

$$\rho^2 G(0) = \prod_{p} \left( 1 - 2R_p \frac{X^k}{X^{2k}} + R_p^2 \frac{X^{2k}}{X^{4k}} + \frac{P_1(X,0)}{X^{2k}} + \frac{R_p}{X^k} \right).$$  \hspace{1cm} (102)

For $q$ a power of $p$ and $n \in \mathbb{N}$ with $q \nmid n$ we have $c_q(n) \ll q/p$, so for $q|p^k$ all the summands in $\Delta^*(q)$ are $\ll p^{k-1}$. Therefore

$$\sum_{1 \leq t \leq k} Y^t \Delta^*(X^t) \ll R_p^2 X^{k-1} \sum_{1 \leq t \leq k} |Y|^t \ll X^{k-1} \left( |Y| + |Y|^k \right).$$  \hspace{1cm} (103)

Fix $0 < \delta < 1/2k$. If $\sigma \in [-1, 1/2]$ then $|Y|/X \leq 1$. Moreover $5X|Y|/7 \geq 5\sqrt{2}/7 > 1$ so with
\[ P_1(X,Y) \ll X^{k-1} \left( |Y| + |Y|^k \right) + X^k |Y|^k + X^k |Y|^k \]
\[ \ll X^k + X^{2k-1} \left( |Y| \right)^k \]
\[ \ll \frac{X^{2k}}{X^{1+5k}}, \quad \text{for } \sigma \geq -1 + \delta. \quad \text{(104)} \]

From the Binomial Theorem (and even if \( R_p = 1 \))
\[ (1 + (Y/X)^k)^{R_p} = 1 + R_p \left( \frac{Y}{X} \right)^k + O \left( \left( \frac{Y}{X} \right)^{2k} + \cdots + \left( \frac{Y}{X} \right)^{R_p k} \right) \]
\[ = 1 + R_p \left( \frac{Y}{X} \right)^k + O \left( \frac{1}{p^{1+2k \delta}} \right), \quad \text{for } \sigma \geq -1 + \frac{1}{2k} + \delta, \quad \text{(105)} \]

so
\[ P_2(X,Y) \ll \frac{1}{p^{1+2k \delta}}, \quad \text{for } \sigma \geq -1 + \frac{1}{2k} + \delta, \]

so that with (104) we have
\[ P_2(X,Y) + \frac{P_1(X,Y)}{X^{2k}} \ll \frac{1}{p^{1+\delta}}, \quad \text{for } \sigma \geq -1 + \frac{1}{2k} + \delta. \quad \text{(106)} \]

For \( \sigma \geq -1 + \delta \) we have \(|(Y/X)^k| \leq 1/p^{3k} \) so (105) says \((1 + (Y/X)^k)^{R_p} \gg 1 \) so we deduce from (101) and (106) that for \( \sigma \geq 0 \)
\[ \rho^2 \mathcal{G}(s) = \prod_p \left( 1 + (Y/X)^k \right)^{R_p} \prod_p \left( 1 + \frac{P_2(X,Y) + P_1(X,Y)/X^{2k}}{(1 + (Y/X)^k)^{R_p}} \right) \]
\[ = \prod_p \left( 1 + (Y/X)^k \right)^{R_p} \prod_{p^{\leq h_v}} \left( 1 + (Y/X)^k \right)^{R_p} \prod_p \left( 1 + \frac{P_2(X,Y) + P_1(X,Y)/X^{2k}}{(1 + (Y/X)^k)^{R_p}} \right) \]
\[ =: \zeta(sk + k)^{R_p} / \zeta(2sk + 2k)^{R_p} \]

where the third product in the second line is absolutely convergent and uniformly bounded for \( \sigma \geq -1 + 1/2k + \delta \), so that \( \mathcal{F}(s) \) is holomorphic and uniformly bounded for \( \sigma \geq -1 + 1/2k + \delta \).

We now have our analytic extension for \( \mathcal{G}(s) \) in place. With this extension and in view of the clear fact
\[ \zeta(s) \zeta(sk + k)^{R_p} / \zeta(2sk + 2k)^{R_p} \ll |t|^{3/2 + \epsilon} \log^\tau |t|, \quad \text{for } \sigma \geq -1 + 1/k \text{ and } |t| \geq 1, \]
we see that the integral in \( V(y) \) (see (122)) is certainly absolutely convergent for \( \sigma \geq -1 + 1/k + \delta \).
and so we may pull it to the left, picking up a simple pole at \( s = 0 \), to deduce

\[
\rho^2 \mathcal{V}(y) = -\rho^2 \zeta(0) G(0)y + \int_{(-1+1/k,\infty)} \frac{\zeta(s)(sk+k)^r F(s)y^{s+1}ds}{(s-1)s(s+1)\zeta(2sk+2k)^r} =: \frac{\rho^2 G(0)y}{2} + \frac{1}{k} \int_{(1+k\delta)} \frac{\zeta(-1+s/k)\zeta(s)r F(-1+s/k)y^{s/k}ds}{(-2+s/k)(-1+s/k)(s/k)\zeta(2s)^r}.
\]

Since for \( 1/2 \leq \sigma \leq 3/2 \) and \( |t| \geq 1 \) we have the standard bounds \( \zeta(-1+s/k) \ll |t|^{3/2-\sigma/k+\epsilon} \) and \( \zeta(2s) \gg 1/(\log|t|)^{2} \) we see that

\[
f(s) \ll \frac{(1 + |t|^{3/2-\sigma/k+\epsilon})|\zeta(s)|^r}{(1 + |t|)^{3/2}}, \quad \text{for} \quad \frac{1}{2} \leq \sigma \leq \frac{3}{2}.
\]

By the definition of \( c \) the integral above therefore converges absolutely for \( \sigma \geq c \), and we may move the line of integration to \( \sigma = c \), picking up a pole at \( s = 1 \), to deduce

\[
\rho^2 \mathcal{V}(y) = \frac{\rho^2 G(0)y}{2} + \text{Res}_{s=1} \left( f(s)y^{s/k} \right) + \int_{(c)} f(s)y^{s/k}ds =: \frac{\rho^2 G(0)y}{2} + \text{Res}_{s=1} \left( f(s)y^{s/k} \right) + \mathcal{O}_c \left( y^{c/k} \right).
\] (107)

Since the pole of \( f \) is of order \( r \) a standard formula from complex analysis tells us that

\[
\text{Res}_{s=1} \left( f(s)y^{s/k} \right) = \frac{1}{(r-1)!} \sum_{i+j=r-1} \left( \frac{d}{ds} \right)^i \left( y^{s/k} \right) \bigg|_{s=1} \left( \text{Res}_{s=1} \left( (s-1)^{-i-1}f(s) \right) \right) = \frac{y^{1/k}}{(r-1)!} \sum_{i+j=r-1} \left( \frac{\log y}{k} \right)^j \left( \text{Res}_{s=1} \left( (s-1)^{-i-1}f(s) \right) \right) = \frac{-y^{1/k}P(\log y)}{2},
\]

where \( P = P_r \) is a polynomial of degree at most \( r - 1 \), so from (107)

\[
-2\rho^2 \mathcal{V}(y) = -\rho^2 G(0)y + y^{1/k}P(\log y) + \mathcal{O} \left( y^{c/k} \right). \tag{108}
\]

It is straightforward to establish that for \( q \nmid n \)

\[
\sum_{d|q} c_d(n) = 0
\]

so from (99)

\[
\sum_{1 \leq t \leq k} \Delta^*(p^t) = \sum_{n,n' \leq X} \sum_{d|p^k \atop d \neq 1} c_d(H_n - H_{n'}) = -R_p(R_p - 1)
\]

so from (100)

\[
P_l(X,0) = \sum_{1 \leq t \leq k} \Delta^*(p^t) + R_p \left( \frac{1 - 1/X}{X - X^{k-1}} - X^{k-1} \right) = -R_p^2
\]

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and so from (102)
\[ \rho^2 G(0) = \prod_p \left( 1 - \frac{R_p}{X^p} \right) = \rho. \] (109)

From (93)
\[ W(Q) = \sum_{u \leq Q} \frac{1}{u} \sum_{q \leq x} \frac{|G(q)|^2}{q} \sum_{a=1}^q (u, D) \Phi_q(-a) \]
\[ = \sum_{u \leq Q} \frac{1}{u} \sum_{q \leq x} \frac{|G(q)|^2 (u, q)}{q} \sum_{a=1}^q \Phi_q(-a) \]
\[ =: \sum_{u \leq Q} \frac{1}{u} \sum_{q \leq x} \frac{|G(q)|^2 f_u(q)}{q}. \] (110)

From Lemma 3.5(ii) we see that \( f_u(q) \) is multiplicative and for prime powers \( q \) Lemma 3.5(ii) implies
\[ f_u(q) = \sum_{d|q} \Phi_q(-ad) \]
\[ = \Phi(q) \sum_{d|q} \mu(q/d) \]
\[ =: (u, q) \Phi(q) F_u(q) \]
so that for general \( q \)
\[ f_u(q) = (u, q) \Phi(q) \left( \prod_{p^\alpha ||q} F_u(p^\alpha) \right). \]

If \( p^\beta | u \) then the summation condition in \( F_u(p^\beta) \) is impossible unless \( d = q \) and so \( F_u(p^\beta) = 1 \). If \( p^\beta \not| u \) then the condition holds for \( d = q \) and \( d = q/p \) so \( F_u(p^\beta) = 0 \). Therefore
\[ f_u(q) = \begin{cases} q \Phi(q) & \text{if } q|u \\ 0 & \text{if not} \end{cases} \]
and so from (110)
\[ W(Q) = \sum_{u \leq Q} \frac{1}{u} \sum_{q|u} \Phi(q)|G(q)|^2 = \frac{1}{\rho^2} \sum_{u \leq Q} W(u) \] (111)
from Lemma 3.5(iv) and (72). Our theorem now follows from (75), (83), (93), (103), (109) and (111).

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