Perturbative Analysis of the Two-Body Problem in 
\((2 + 1)-AdS\) Gravity

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**Abstract**

We derive a perturbative scheme to treat the interaction between point sources and \(AdS\)-gravity. The interacting problem is equivalent to the search of a polydromic mapping \(X^A = X^A(x^\mu)\), endowed with \(O(2, 2)\) monodromies, between the physical coordinate system and a Minkowskian 4-dimensional coordinate system, which is however constrained to live on a hypersurface. The physical motion of point sources is therefore mapped to a geodesic motion on this hypersurface. We impose an instantaneous gauge which induces a set of equations defining such a polydromic mapping. Their consistency leads naturally to the Einstein equations in the same gauge. We explore the restriction of the monodromy group to \(O(2, 1)\), and we obtain the solution of the fields perturbatively in the cosmological constant.
1 Introduction

The scattering of particles in gravity is one of the typical problems of physics, from the times relativity theory has been discovered [1]-[2]-[3]-[4]. Our scheme of solution [5]-[6]-[7]-[8] is based on introducing the constants of motion of the problem by allowing for a polydromic mapping $X^A = X^A(x^\mu)$ of the flat metric, whose polydromy automatically builds up the delta singularities of the particles’ energy-momentum tensors.

Thanks to this mapping the physical motion can be trivialized to an elementary motion, which in the case of pure gravity reduces to a free motion on a plane, while, as it has been shown in [9], in the case of $AdS$ gravity reduces to a geodesic motion of a hyper-surface immersed in a flat 4-dimensional Minkowskian space-time with signature (+ + - - ). In the last scheme we can unify not only particle-like singularities, moving as time-like geodesics, but also $BTZ$ black-hole singularities [10]-[11]-[12]-[13], moving as space-like geodesics.

In this article we restrict such a mapping problem in a particular gauge, the instantaneous gauge defined by the equations $g_{zz} = 0$ and $K = 0$, which has allowed to compute, in the case of pure (2 + 1)-gravity [6], the fields in terms of analytic and anti-analytic functions. Our aim is to show how this scheme can be extended to the case of (2 + 1) $AdS$-gravity [14]-[15], although we are not able to solve completely for the fields. Our method at least gives a framework in which it is possible to obtain perturbatively an unique answer.

Such a polydromic mapping is constrained by the gauge choice to satisfy a complete set of equations governing this immersion in the flat 4-dimensional space-time. This system of equations is related to the gauge choice and is valid only for those regions of space-time consistent with the conformal gauge. An obvious counterexample is the case of the region of the black hole internal to the horizon, whose metric is too complex to satisfy such a choice.

The first-order formalism is simpler than the second-order formalism, as it allows to avoid distributions, related to the singularities and it instead introduces polydromies which take into account globally the presence of the particles.

In this sense sources are viewed as singularities of the fields once that the monodromy conditions are imposed around them. The particles’ scattering is then reduced to a composition of monodromies, that, in the case of $O(2, 2)$ monodromy group, produces two invariant masses [9], determining the solution at great distances. In the particular case of zero angular momentum, these two invariant masses coincide between them and the monodromy group is restricted to $O(2, 1)$ (see also [16]). In such a case it is useful to introduce a parameterization that reduces the cut to a Möbius transformation. However we are not able, as in the gravity case, to find a parameterization of the fields in terms of analytic functions, and here
the problem seems quite involved, being the equations for the $X^A$ coordinates non-analytic.

It is however possible to solve for the cuts with non-analytic functions, since the non-analytic part can be perturbatively represented as a finite number of terms, obtained as products of analytic and anti-analytic functions. It remains an arbitrariness of analytic functions in the naive integration that is removed imposing that the whole solution satisfies the monodromies. We obtain therefore, order by order in perturbation theory, a field satisfying all the requirements and depending from the constants of motion, which describe the particles’ motion. We finally discuss the limits of such research to the study of black holes scattering.

2 Instantaneous gauge in the second order formalism

The splitting ADM of space-time in terms of space-like surfaces can be derived from the Einstein-Hilbert action with negative cosmological constant (with modulus $\Lambda$) rewriting the scalar curvature $R^{(3)}$ in its spatial part $R^{(2)}$, intrinsic to the surfaces $\Sigma(t)$, and an extrinsic part, coming from the embedding, as follows ($8\pi G = 1$)

$$S = -\frac{1}{2} \int \sqrt{|g|} R^{(3)} + \Lambda \int \sqrt{|g|} = -\frac{1}{2} \int \sqrt{|g|} (R^{(2)} + (Tr K)^2 - Tr (K^2)) + \Lambda \int \sqrt{|g|}. \quad (2.1)$$

where the equivalence is true apart from a surface term. Here we have introduced the extrinsic curvature tensor $K_{ij}$, a second fundamental form of the surface $\Sigma(t)$, given by

$$K_{ij} = \frac{1}{2} \sqrt{|g|} \left( \nabla_i g_{0j} + \nabla_j g_{0i} - \partial_0 g_{ij} \right), \quad (2.2)$$

with the notation that the spatial indices are raised or lowered with the spatial part of the metric $g_{ij}$ and its inverse.

Let us choose as parameterization for the metric, once that it is constrained to be in conformal gauge $g_{zz} = 0$:

$$g_{00} = \alpha^2 - e^{2\phi} \beta \overline{\beta}, \quad g_{0z} = \frac{1}{2} \alpha e^{2\phi}, \quad g_{zz} = -\frac{1}{2} e^{2\phi}. \quad (2.3)$$

Let us compute the trace of the extrinsic curvature tensor and pose it equal to zero, in such a way that all the dependences on time derivatives are avoided in the action:

$$K(z, \overline{z}, t) = K_{z\overline{z}} = \frac{1}{2\alpha} (\partial_z g_{0\overline{z}} + \partial_{\overline{z}} g_{0z} - \partial_0 g_{z\overline{z}}) = 0. \quad (2.4)$$

The other part of the tensor $K_{ij}$ can be shown to be a meromorphic analytic function, as we shall see in a moment from the equations of motion

$$K_{zz} = N(z) = \frac{e^{2\phi}}{2\alpha} \partial_z \overline{\beta}. \quad (2.5)$$
The condition of instantaneous gauge (see also [6]) is therefore defined by

\[ g_{zz} = K = 0. \]  

(2.6)

Combining the conditions above, we obtain a new action without temporal derivatives, showing that the propagation of the fields \( \alpha, \beta, \phi \) can be made instantaneous. Adding the matter coupling (i.e. a set of \( N \) particles), the equations of AdS-gravity have as sources both the cosmological constant that can be thought as the mean effect of an uniform distribution of matter and the energy-momentum tensor of the external particles:

\[
\nabla^2 \phi + \frac{e^{2\phi}}{\alpha^2} \partial_z \beta \partial_z \beta = \nabla^2 \phi + 4N\bar{N}e^{-2\phi} = \Lambda e^{2\phi} - |g|e^{-2\phi}T_{00}
\]

\[
\partial_z \left( \frac{e^{2\phi}}{2\alpha} \partial_z \beta \right) = \partial_z N(z) = -(4\alpha)^{-1} |g| (T^{0z} - \beta T^{00})
\]

\[
\nabla^2 \alpha - 2\frac{e^{2\phi}}{\alpha^2} \partial_z \beta \partial_z \beta = \nabla^2 \alpha - 8N\bar{N}e^{-2\phi} = 2\Lambda \alpha e^{2\phi} + \alpha^{-1} |g| (T^{z\bar{z}} - \beta T^{z\bar{z}} - \bar{\beta} T^{0z} + \beta \bar{\beta} T^{00}),
\]

(2.7)

where \( \nabla^2 \equiv 4\partial_z \partial_z \) denotes the Laplacian, and the energy-momentum tensor for the particles is given by

\[
T^\mu{}^\nu = \frac{1}{\sqrt{|g|}} \sum_{(i)} m_i \left( \frac{dt}{ds_i} \right) \dot{\xi}_i^\mu \dot{\xi}_i^\nu \delta^2(x - \xi_i(t_i)), \quad (i = 1, 2, ..., N).
\]

(2.8)

Avoiding time derivatives allows to treat the particle motion as an external independent datum, from which the metric has a parametric dependence, while only by imposing both the monodromies and the boundary conditions one obtains constraints on the geodesic motion of the particles.

With the introduction of the conformal gauge \( g_{zz} = 0 \) the Einstein equations for the corresponding components of the Ricci tensor \( R_{\mu\nu} \) are missing in eq. (2.7), and should be added as a constraint, i.e.

\[
R_{zz} = T_{zz}.
\]

(2.9)

These two constraints, together with the gauge condition \( K = 0 \), are indeed equivalent to the covariant conservation of the energy-momentum tensor, which in turn implies the geodesic equations, as shown in ref. [6].

3 Comparison with the first-order formalism

The introduction of the cosmological constant makes useless searching for a simplification of the equations of motion in the dreibein formalism [14], while it is fruitful to start from the
typical construction of space-times with constant curvature [10]-[11], that are obtained from a 4-dimensional flat metric with a constraint between the coordinates:

\[ ds^2 = dX^A dX^B \eta_{AB} \quad X^A \partial_0 X_A = \frac{1}{\Lambda}. \]  

(3.1)

This formalism can also be considered a first-order formalism, defining a new dreibein \( E^A_\mu = \partial_\mu X^A \), in which the Lorentz index depends on four coordinates.

From the constraint (3.1) we can easily deduce the following properties:

\[ X^A \partial_z X_A = 0 \quad X^A \partial_0 X_A = 0. \]  

(3.2)

If we suddenly impose the conformal gauge, i.e.

\[ g_{zz} = \partial_z X^A \cdot \partial_z X_A = 0 \quad g_{z\tau} = \partial_z X^A \cdot \partial_\tau X_A = -\frac{1}{2} e^{2\phi}, \]  

(3.3)

many other properties of the \( X^A \)-mapping are shown to be valid.

An intrinsic local frame of the four-dimensional space-time is given by these 4 four-vectors \( X^A, \partial_z X^A, \partial_\tau X^A \), and \( V^A \), defined as

\[ V^A = 2i\sqrt{\Lambda} e^{-2\phi} \epsilon^{ABCD} X_B \partial_z X_C \partial_\tau X_D \]  

(3.4)

that is orthogonal to the first three ones, and is defined to have norm equal to unity. Instead \( \partial_z X^A \) and \( \partial_\tau X^A \) are two null vectors that have a non-vanishing scalar product between them. We can easily check the following property:

\[ \partial_0 X^A = -\beta \partial_z X^A - \beta \partial_\tau X^A + \alpha V^A. \]  

(3.5)

Therefore once that the mapping problem \( X^A \) is solved at fixed time, we can also determine the field \( \alpha \) and \( \beta \), while the knowledge of the field \( \phi \) doesn’t require to compute any time derivative on the mapping.

In the second order formalism we have chosen an instantaneous gauge, which is the same that we have use to solve the N-body problem in pure \((2 + 1)\)-gravity, and it is defined by the conditions in eq. (2.6). These gauge conditions combined together imply that:

\[ \partial_z \partial_\tau X^A = \frac{\Lambda}{2} e^{2\phi} X^A. \]  

(3.6)

In particular, the \( K = 0 \) condition implies that additional terms proportional to \( V^A \) are absent in the r.h.s. of eq. (3.6).

Let us compute \( \partial^2_z X^A \). We can develop it in the basis of four-vectors

\[ \partial^2_z X^A = c_0 X^A + c_1 \partial_z X^A + c_2 \partial_\tau X^A + c_3 V^A. \]  

(3.7)
It is easy to see that $c_0 = 0$ and $c_2 = 0$ are consequences of the eqs. (3.2) and (3.3). The scalar product with $\partial_\tau X^A$ implies that $c_1 = 2\partial_\tau \phi$, while $c_3$ can be constrained since it satisfies the analyticity condition:

$$\partial_\tau c_3 = 0,$$

therefore $c_3 = c_3(z)$ is a monodromic analytic function. The only candidate for such a function is the meromorphic function $N(z)$, previously identified as the $K_{zz}$ component of the extrinsic curvature tensor; in fact, starting from the definition (2.5) of $N(z)$ we can show that $c_3 = N(z)$. Therefore we find the equation:

$$\partial_\tau^2 X^A = 2\partial_\tau \phi \partial_\tau X^A + N(z)V^A. \quad (3.9)$$

Now let us compute $\partial_\tau V^A$. From the definition of $V^A$ we can suddenly notice that $\partial_\tau V^A$ is orthogonal both to $V^A$ and $X^A$. We can therefore pose

$$\partial_\tau V^A = a_1 \partial_\tau X^A + a_2 \partial_\tau X^A. \quad (3.10)$$

Introducing eq. (3.9) in the computation of (3.10) we can directly find $\partial_\tau V^A$, and obtain that:

$$\partial_\tau V^A = 2e^{-2\phi} N(z) \partial_\tau X^A. \quad (3.11)$$

In summary, since we work in an instantaneous gauge, the equations at fixed time of the immersion $X^A = X^A(x^\mu)$ are given by:

$$\partial_\tau \partial_\tau X^A = \Lambda e^{2\phi} X^A$$

$$\partial_\tau^2 X^A = 2\partial_\tau \phi \partial_\tau X^A + N(z)V^A$$

$$\partial_\tau V^A = 2e^{-2\phi} N(z) \partial_\tau X^A. \quad (3.12)$$

The consistency of these equations leads to the fields equations (2.7) in the second-order formalism.

Let us also notice the following property, i.e., that, being $X^A$ and $V^A$ two vectors of constant norm, their dynamics can be described by a sigma model $O(2,2)$, as a consequence of the equations (3.12)

$$\partial_\tau \partial_\tau X^A = -\Lambda(\partial_\tau X^B \partial_\tau X_B)X^A$$

$$\partial_\tau \partial_\tau V^A = -(\partial_\tau V^B \partial_\tau V_B)V^A. \quad (3.13)$$

Both eqs. (3.12) and (3.13) are automatically covariant with respect to the $O(2,2)$ cuts, that can be decomposed as products of $SU(1,1)$ cuts, as shown in ref. [9]. In general, the one-particle cut can be identified as ($X^A = (X^t, X^z, X^t, X^z)$)

$$\begin{pmatrix} X^t & X^z \\ X^z & X^t \end{pmatrix} \to \begin{pmatrix} A_1 & B_1 \\ B_1 & A_1 \end{pmatrix} \begin{pmatrix} X^t & X^z \\ X^z & X^t \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ B_2 & A_2 \end{pmatrix},$$

(3.14)
where the entries $A_{i}$ and $B_{i}$ of the monodromy matrices are given by:

\[
\begin{align*}
A_{1} &= \cos \pi \mu - i \chi (\lambda_{1} - \lambda_{2}) \sin \pi \mu \\
B_{1} &= -ie^{-i(\alpha + \beta)} \sinh (\lambda_{1} - \lambda_{2}) \sin \pi \mu \\
A_{2} &= \cos \pi \mu + i \chi (\lambda_{1} + \lambda_{2}) \sin \pi \mu \\
B_{2} &= -ie^{i(\alpha - \beta)} \sinh (\lambda_{1} + \lambda_{2}) \sin \pi \mu.
\end{align*}
\]

(3.15)

In general the solution of the two-body problem can be obtained in global terms. The result of the composition of two monodromies, in the case of particles, is of course of the type:

\[
\left( \begin{array}{cc} X' & X'z \\ X & X' \end{array} \right) = M_{L} \left( \begin{array}{cc} X & X' \\ X^z & X' \end{array} \right) M_{R},
\]

(3.16)

where to the $M_{L}, M_{R}$ matrices it is possible to associate the corresponding invariant masses $M_{L}, M_{R}$:

\[
\begin{align*}
\cos(\pi M_{L}) &= \cos(\pi \mu_{1}) \cos(\pi \mu_{2}) - \frac{P_{1}^{L} \cdot P_{2}^{L}}{m_{1}m_{2}} \sin(\pi \mu_{1}) \sin(\pi \mu_{2}) \\
\cos(\pi M_{R}) &= \cos(\pi \mu_{1}) \cos(\pi \mu_{2}) - \frac{P_{1}^{R} \cdot P_{2}^{R}}{m_{1}m_{2}} \sin(\pi \mu_{1}) \sin(\pi \mu_{2})
\end{align*}
\]

(3.17)

and we have defined the following vectors, constants of motion:

\[
\begin{align*}
P_{1}^{L} &= m_{1} \gamma_{1}^{L}(1, v_{1}^{L}) & \gamma_{1}^{L} &= \cosh (\lambda_{1} - \lambda_{2}) & \gamma_{1}^{L} v_{1}^{L} &= e^{-i(\alpha + \beta)} \sinh (\lambda_{1} - \lambda_{2}) \\
P_{1}^{R} &= m_{1} \gamma_{1}^{R}(1, v_{1}^{R}) & \gamma_{1}^{R} &= \cosh (\lambda_{1} + \lambda_{2}) & \gamma_{1}^{R} v_{1}^{R} &= e^{-i(\alpha - \beta)} \sinh (\lambda_{1} + \lambda_{2}).
\end{align*}
\]

(3.18)

For generic values of the constants of motions, the left invariant mass $M_{L}$ will be different from the right invariant mass $M_{R}$ and therefore the composed system has spin, other than invariant mass.

In the simplified case in which $A_{1} = A_{2} = A$, $B_{1} = B_{2} = B$, that correspond to null angular momentum, it is useful to parameterize the $X^{A}$ transformation in the following way, that makes explicit the reduction of monodromies from the generic case of $O(2, 2)$ to the particular case of $O(2, 1)$ ( $X^{A} = (X^{0}, X^{1}, X^{z}, X^{z})$ )

\[
X^{A} = \frac{1}{\sqrt{A}} \left( \begin{array}{c} 1 + Z \overline{Z} \cos T, \sin T \\ 2Z \overline{Z} \cos T, \frac{2Z}{1 - Z \overline{Z}} \cos T \end{array} \right),
\]

(3.19)
in which the only polydromic variable is $Z$, transforming as $O(2, 1)$ around the particles

$$Z \rightarrow \frac{AZ + B}{BZ + A}. \quad (3.20)$$

Exploring the consequences of eqs. (3.12) we find that there are two eqs. with a Laplacian for the $T$ and $Z$ variables:

$$\begin{align*}
\partial_z \partial_T &= 2\sin T \cos T \left( \frac{\partial_z Z \partial_z Z + \partial_z Z \partial_z Z}{(1 - ZZ)^2} \right), \\
\partial_z \partial_Z &= \frac{2Z \partial_z Z \partial_z Z}{(1 - ZZ)} = \frac{2\sin T \sqrt{\partial_z Z \partial_z Z}}{(1 - ZZ)} \left( \sqrt{\partial_z Z \partial_z Z} + \sqrt{\partial_z Z \partial_z Z} \right). \quad (3.21)
\end{align*}$$

The first one is integrated by the following first integral,

$$\partial_z T = \frac{2\sqrt{\partial_z Z \partial_z Z}}{1 - ZZ} \cos T, \quad (3.22)$$

that is also obtained developing the condition of null norm $\partial_Z X^4 \cdot \partial_Z X_A = 0$.

The second one is integrated solving the eqs. (3.12) in terms of $N(z)$:

$$N(z) = \frac{1}{\sqrt{\Lambda}} \left( \sqrt{\partial_z Z \partial_z Z} - \sqrt{\partial_z Z \partial_z Z} \right) e^{2\phi} \partial_z (e^{-2\phi} \partial_z \sin T). \quad (3.23)$$

Using these first integrals we can express the conformal factor $e^{2\phi}$ in such a way that makes explicit its invariance under $O(2, 1)$:

$$e^{2\phi} = \frac{4}{\Lambda} \left( \frac{\sqrt{\partial_z Z \partial_z Z} - \sqrt{\partial_z Z \partial_z Z}}{(1 - ZZ)} \right)^2 \cos^2 T. \quad (3.24)$$

This method appears to be more involved than the second-order formalism, however the latter one would require to analyze in detail all the various contributions, of distribution type, while introducing the cuts allows to avoid to write down mathematically ill-defined equations.

### 4 Perturbative expansion in $\Lambda$ for two-body scattering

Let us firstly recall the single body metric, which is defined in the radial gauge as

$$ds^2 = ((1 - \mu)^2 + \Lambda r^2)dt^2 - \frac{dr^2}{(1 - \mu)^2 + \Lambda r^2} - r^2 d\theta^2 \quad (4.1)$$
This metric can be expressed as a polydromic mapping

\[ X^t = X^0 + iX^1 = \frac{1}{\sqrt{\Lambda}} \left( 1 + \frac{\Lambda r^2}{(1 - \mu)^2} e^{i \sqrt{\Lambda} (1 - \mu)t} \right) \]

\[ X^z = X^2 + iX^3 = \frac{r e^{(1 - \mu) \theta}}{(1 - \mu)}. \]  

(4.2)

Introducing the spatial conformal gauge \( g_{zz} = 0 \), which is obtained with a radial coordinate transformation

\[ r \rightarrow \frac{(1 - \mu) r (1 - \mu)}{1 - \frac{\Lambda^2}{4} r^2 (1 - \mu)}, \]  

(4.3)

the \( X^A \)-mapping becomes

\[ X^t = \frac{1 + \frac{\Lambda r^2 (1 - \mu)^2}{4} e^{i \sqrt{\Lambda} (1 - \mu)t}}{\sqrt{\Lambda} \left( 1 - \frac{\Lambda}{4} r^2 (1 - \mu) \right)} \]

\[ X^z = \frac{z^{1 - \mu}}{1 - \frac{\Lambda}{4} r^2 (1 - \mu)}. \]  

(4.4)

In conformal gauge it appears a physical limit on the radial coordinate, implying that the solution is defined on a disk instead on the whole \( z \)-plane, and it is given by \( r^{1 - \mu} \leq \frac{2}{\sqrt{\Lambda}} \).

Before starting the study of particles’ scattering, let us analyze in detail the one-body problem. Firstly, we can say that \( N(z) = 0 \), implying by consistency with eq. (3.23) that

\[ e^{-2\phi} \partial_z \sin T = c(t) \zeta. \]  

(4.5)

From the one-body problem solution (4.4), the following expressions for \( T \) and \( Z \) have been obtained in [9]

\[ Z = \sqrt{1 - a^2} \frac{\sqrt{\Lambda}}{4} z^{1 - \mu} \left[ 1 + \frac{4r^{2(1 - \mu)}}{\Lambda} \left( 1 - \sqrt{1 - \frac{\Lambda}{2} \left( \frac{1 + a^2}{1 - a^2} \right) r^{2(1 - \mu)} + \frac{\Lambda^2}{16} r^{4(1 - \mu)} \right) \right] \sim \]

\[ \sim \frac{\sqrt{\Lambda}}{2} \frac{z^{1 - \mu}}{\sqrt{1 - a^2}} \left[ 1 + \frac{\Lambda}{4} \frac{a^2}{1 - a^2} r^{2(1 - \mu)} + O(\Lambda^3) \right] \]

\[ \sin T = a \frac{1 + \frac{\Lambda}{4} r^{2(1 - \mu)}}{1 - \frac{\Lambda}{4} r^{2(1 - \mu)}} \quad a = \sin(\sqrt{\Lambda} (1 - \mu)t). \]  

(4.6)

Therefore we can fix the time-dependent constant of eq. (4.5) as

\[ c(t) = \frac{\Lambda}{2(1 - \mu)} a(t). \]  

(4.7)

This solution is not valid with the choice of the parameterization (3.19) in the whole disk, but only in a subset defined by the condition \( |Z| \leq 1 \), that in the case of a single body
is also a disk in the physical coordinates defined by

\[ r^{2(1-\mu)} \leq \frac{4 \left( 1 - a(t) \right)}{\Lambda 1 + a(t)} \quad \text{for} \quad a(t) > 0 \]

\[ \leq \frac{4 \left( 1 + a(t) \right)}{\Lambda 1 - a(t)} \quad \text{for} \quad a(t) < 0. \]  

(4.8)

At the limiting value, the \( X^A \) mapping reduces to \( \Theta = (1 - \mu)\theta \). The continuation of \( Z \)-field outside the physical region meets another region, disconnected from the physical one, in which the condition \( |Z| \leq 1 \) is valid. In this specular universe there is also an image of the source. In the two-body case this image will remain an isolated singularity, whose cut will be identified with the composition of the two-body cut.

Let us notice that this limit on the coordinates is not perturbative in \( \Lambda \) and that at each order of the perturbative expansion this limit can be considered \( \infty \). Only after re-summing the perturbative series, we can understand what is precisely the domain of the exact solution, from which it must be continued with another choice of the parameterization, instead of eq. (3.19).

With respect to the geodesic motion of a test particle situated in \( \xi(t) \), under the influence of a mass source situated in the origin of the coordinates, the knowledge of this patch is enough to give the complete solution, if the \( Z \)-value corresponding to \( \xi(t) \) has modulus less than unity:

\[
Z(\xi) = \frac{t h(\sqrt{\Lambda}\lambda)}{\sqrt{1 - a^2(t)}} \left[ (Z(\xi) + \frac{1}{Z(\xi)}) - \sqrt{(Z(\xi) + \frac{1}{Z(\xi)})^2 - 4(1 - a^2(t))} \right].
\]

(4.9)

The resulting motion is limited on a line, similar to the harmonic motion, and the test particle can meet the source after a finite time \( t_0 \).

To introduce the two-body problem, it is useful to give the explicit development of eq. (4.6) at the first orders:

\[
T = \sqrt{\Lambda} (T^{(0)} + \Lambda T^{(1)} + O(\Lambda^2))
\]

\[
= \sqrt{\Lambda} (1 - \mu) t \left( 1 + \frac{\Lambda}{2} r^{2(1-\mu)} + O(\Lambda^2) \right)
\]

\[
Z = \frac{\sqrt{\Lambda}}{2} (Z^{(0)} + \Lambda Z^{(1)} + \Lambda^2 Z^{(2)} + O(\Lambda^3))
\]

\[
= \frac{\sqrt{\Lambda}}{2} z^{1-\mu} \left( 1 + \frac{\Lambda}{2} (1 - \mu)^2 t^2 + \frac{5\Lambda^2}{24} (1 - \mu)^4 t^4 + \frac{\Lambda^2}{4} (1 - \mu)^2 r^{2(1-\mu)} \right). \]  

(4.10)
If we compare the exact non-perturbative limit (4.8) with what is given by the first order approximation, i.e.

$$|Z^{(0)}| = 1 \leftrightarrow r^{2(1-\mu)} \sim \frac{4}{\Lambda}$$

we note that the correct value is almost obtained, and that the following perturbative terms give only relatively small corrections.

To describe the scattering of point sources the monodromies must be non-abelian between them, in such a way that their fixed points are distinct for each particle:

$$Z \rightarrow \frac{a^1 Z + b^1}{b^1 Z + a^1} \quad Z \rightarrow \frac{a^2 Z + b^2}{b^2 Z + a^2} .$$

(4.12)

With the coefficients of the transformations defined as

$$a^i = \cos \pi \mu^i - i \gamma^i \sin \pi \mu^i,$$

$$b^i = -i \gamma^i \nabla^i \sin \pi \mu^i,$$

(4.13)

the corresponding fixed points are defined by

$$Z^{(0)}_i = -\frac{\gamma^i \nabla^i}{1 + \gamma^i} .$$

(4.14)

Let us do the case of head-on collision, choosing the coefficients given in eq. (4.13) in the following way

$$a_1 = \cos \pi \mu_1 - i \gamma_1 \sin \pi \mu_1 \quad a_2 = \cos \pi \mu_2 - i \gamma_2 \sin \pi \mu_2,$$

$$b_1 = -i \gamma_1 |v_1| \sin \pi \mu_1 \quad b_2 = i \gamma_2 |v_2| \sin \pi \mu_2 .$$

(4.15)

We can parameterize the coefficients only in terms of rapidities, without extra phases, i.e.

$$\gamma_1 = \text{ch}(2\sqrt{\Lambda} \lambda_1) \quad \gamma_2 = \text{ch}(2\sqrt{\Lambda} \lambda_2) \quad \gamma_1 |v_1| = \text{sh}(2\sqrt{\Lambda} \lambda_1) \quad \gamma_2 |v_2| = \text{sh}(2\sqrt{\Lambda} \lambda_2) .$$

(4.16)

Developing the coefficients $a_k$ e $b_k$ in powers of the perturbative parameter, the cosmological constant $\Lambda$,

$$a^k = a^k_0 + \Lambda a^k_1 + \Lambda^2 a^k_2 + O(\Lambda^3)$$

$$b^k = \sqrt{\Lambda} (b^k_0 + \Lambda b^k_1 + \Lambda^2 b^k_2 + O(\Lambda^3)) \quad k = 1, 2$$

(4.17)

we find at the lowest orders in $\Lambda$

$$a_1 = e^{-i\pi \mu_1} - 2i\Lambda \sin \pi \mu_1 \lambda_1^2 + O(\Lambda^2) \quad a_2 = e^{-i\pi \mu_2} - 2i\Lambda \sin \pi \mu_2 \lambda_2^2 + O(\Lambda^2)$$

$$b_1 = -2i\sqrt{\Lambda} \left( \lambda_1 + \frac{2\lambda_1^3}{3} + O(\Lambda^2) \right) \sin \pi \mu_1 \quad b_2 = +2i\sqrt{\Lambda} \left( \lambda_2 + \frac{2\lambda_2^3}{3} + O(\Lambda^2) \right) \sin \pi \mu_2 .$$

(4.18)
The system is defined non-perturbatively by an invariant mass $\mathcal{M}$ characterizing the system at great distances as a one-body metric. Speaking of limit of great distances is a little misleading since the metric in conformal gauge is valid only on a finite size region, however in a perturbative expansion, the limit coming from re-summing the perturbative series is not visible.

The invariant mass defined by

$$
\cos \pi \mathcal{M} = \cos \pi \mu_1 \cos \pi \mu_2 - \text{ch}(\sqrt{\Lambda}(\lambda_1 + \lambda_2))\sin \pi \mu_1 \sin \pi \mu_2, \quad (4.19)
$$

is a real number included between 0 and 1, implying some non-perturbative limits on the values of the constants of motion:

$$
\text{ch}(\sqrt{\Lambda}(\lambda_1 + \lambda_2)) \leq \frac{1 + \cos \pi \mu_1 \cos \pi \mu_2}{\sin \pi \mu_1 \sin \pi \mu_2}. \quad (4.20)
$$

In the computation of the perturbative fields it is useful to verify that the perturbative terms can be recombined to give $\mathcal{M}$, confirming that the complete solution, which we are not able to give explicitly, is deeply dependent on it. In fact, as in the one-body case there is another singularity, characterizing the $Z$-field at infinity in the unphysical region, whose mass must be identified with $\mathcal{M}$, as we shall see later on.

Before analyzing the two-body problem, we must notice that the cosmological constant produces by itself a sort of background field, which is obtained in the massless limit $\mu \to 0$:

$$
Z = \sqrt{1 - a^2} \frac{\sqrt{\Lambda}}{4} \left[ 1 + \frac{4r^{-2}}{\Lambda} \left( 1 - \sqrt{1 - \frac{\Lambda}{2} \left( 1 + a^2 \right) r^2 + \frac{\Lambda^2}{16} r^4} \right) \right],
$$

$$
\sin T = a \frac{1 + \frac{\Lambda}{4} r^2}{1 - \frac{\Lambda}{4} r^2}, \quad a = \sin(\sqrt{\Lambda} t). \quad (4.21)
$$

Therefore at each order the development of the background metric gives rise to a contribution, which is divergent at infinity. This is made smoother by the presence of the sources, that at infinity are seen as a unique central body with total mass given by the invariant mass $\mathcal{M}$. Since we have to fix the boundary conditions we choose to require that the analytic functions that naturally arise from the integration are regular around the particles and have a behavior at infinity that is no more divergent than a one-body metric with the mass given by the invariant mass $\mathcal{M}$:

$$
T \xrightarrow{r \to \infty} \sqrt{\Lambda}(1 - \mathcal{M}) t \left( 1 + \frac{\Lambda}{2} r^2 (1 - \mathcal{M}) + O(\Lambda^2) \right),
$$

$$
Z \xrightarrow{r \to \infty} \frac{\sqrt{\Lambda}}{2} \frac{z^{1-\mathcal{M}}}{\sqrt{1 - a^2(t)}} \left( 1 + \frac{\Lambda^2}{4} (1 - \mathcal{M})^2 r^2 z^{2(1-\mathcal{M})} \right), \quad (4.22)
$$

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where \( \tilde{a}(t) = \sin[(1 - \mathcal{M})t] \). It can be necessary to adjust the solution asymptotically with a boost of \( Z \), ( \( a_\infty = ch(2\sqrt{\Lambda}\lambda_\infty) \), \( b_\infty = sh(2\sqrt{\Lambda}\lambda_\infty) \) ), implying in practice a redefinition of \( Z \) with respect to the static case (4.22):

\[
Z \to \frac{a_\infty Z + b_\infty}{b_\infty Z + a_\infty}.
\]

(4.23)

In (4.15) we have supposed that at the lowest order the cut is purely a rotation,

\[
\begin{align*}
a^k_0 & = e^{-i\pi\mu_k} \\
b^k_0 & = 2i(-)^k\lambda_k sin\pi\mu_k & k = 1, 2,
\end{align*}
\]

(4.24)

then the monodromy conditions reduce at the same order to

\[
Z^{(0)} \to e^{-2i\pi\mu_k} Z^{(0)} + (-)^k 4i\lambda_k e^{-i\pi\mu_k} sin\pi\mu_k,
\]

(4.25)

from which we recover the static solution defined in the rescaled variable \( \zeta = (z - \xi_2)/(\xi_1 - \xi_2) \) by

\[
Z^{(0)} = k_2 \int_0^\zeta d\zeta \zeta^{-\mu_1}(1 - \zeta)^{-\mu_2} + k_1.
\]

(4.26)

Eq. (4.25) determines \( k_1 \) and \( k_2 \)

\[
\begin{align*}
k_1 & = -2\lambda_1 \\
k_2 & = (1 - \mu_1 - \mu_2)\xi_{12}^{1-\mu_1-\mu_2} \frac{2(\lambda_1 + \lambda_2)}{B(1 - \mu_1, 1 - \mu_2)} + O(\Lambda).
\end{align*}
\]

(4.27)

The fact that it is possible to relate the physical distance \( \xi_{12} \) in terms of the bosonic distance \( Z^{(0)}(1) - Z^{(0)}(0) \), is connected with the requirement that the mapping \( Z^{(0)} \) reduces to the identity mapping \( z \) in the massless limit.

In practice, from the one body case, we can deduce what is the approximate figure of the patch on which the solution is valid, defined by the equation

\[
|Z^{(0)}|^2 = \frac{4}{\Lambda}.
\]

(4.28)

If we insist too much on the validity of eq. (4.28), we find some perturbative limit on the constant of motion \( \lambda_i \leq \frac{4}{\sqrt{\Lambda}} \), which really doesn’t exist, and therefore we deduce that this approximate equation is valid only for small values of \( \lambda_i \). However more subtle limits on the constants of motion appear if we require that the asymptotical behavior is related to a particle, and not to a tachyon, as shown in eq. (4.20).
Let us rewrite the zero order solution in terms of an hypergeometric function:

\[
Z^{(0)} = k_2 \int_0^\zeta d\zeta \zeta^{-\mu_1} (1 - \zeta)^{-\mu_2} + k_1
\]

\[
= k_2 \frac{\zeta^{1-\mu_1}}{1 - \mu_1} F(\mu_2, 1 - \mu_1, 2 - \mu_1; \zeta) + k_1 \zeta^{\zeta \to \infty} \zeta^{1-\mu_1-\mu_2}.
\] (4.29)

The analytic part of the solution for \(Z\) can be directly generalized at all orders representing it as a ratio of hyper-geometric functions

\[
\tilde{Z} = \coth(\sqrt{\Lambda}(\lambda_1 + \lambda_2)) \frac{\tilde{F} \left( \begin{array}{c} \mathcal{M} - \mu_1 + \mu_2 \\ 2 \\ \end{array}, 1 + \frac{-\mathcal{M} - \mu_1 + \mu_2}{2}, 2 - \mu_1; \zeta \right)}{\tilde{F} \left( -1 + \frac{\mathcal{M} + \mu_1 + \mu_2}{2}, -\frac{-\mathcal{M} + \mu_1 + \mu_2}{2}, \mu_1; \zeta \right)},
\] (4.30)

where the symbol \(\tilde{F}\) denotes a novel normalization of the hyper-geometric function (see also [6]):

\[
\tilde{F}(a, b, c; z) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b, c; z).
\] (4.31)

More generally, the solution will be defined by a Lorentz transformation in order to produce the right fixed points of the \(Z\)-monodromy

\[
Z = \frac{\tilde{Z} + Z^{(0)}}{1 + Z^{(0)}\tilde{Z}}.
\] (4.32)

which, in the special case of head-on collision, become

\[
Z^{(0)} = -\frac{\gamma_1 V_1}{1 + \gamma_1} = -th(\sqrt{\Lambda} \lambda_1) \quad Z(1) = \frac{\gamma_2 V_2}{1 + \gamma_2} = th(\sqrt{\Lambda} \lambda_2) \quad \tilde{Z}(1) = th(\sqrt{\Lambda}(\lambda_1 + \lambda_2)).
\] (4.33)

It is easy to verify that the first order solution (eq. (4.29)) is obtained with the following limit on the invariant mass

\[
\Lambda \to 0 \equiv \mathcal{M} \to \mu_1 + \mu_2 + 2\Lambda(\lambda_1 + \lambda_2)^2 \frac{\sin\pi\mu_1\sin\pi\mu_2}{\pi\sin(\mu_1 + \mu_2)} + O(\Lambda^2)
\] (4.34)

once that the particular values of head-on collision are substituted. From the definition of the hyper-geometric function it is clear that there is another singularity at infinity, specular to the ones situated in \(\zeta = 0\) and \(\zeta = 1\), whose cut is related to the invariant mass \(\mathcal{M}\).

The scale \(\xi_{12}\) appear to be undetermined in the exact analytic solution. As in the lowest order in \(\Lambda\), corresponding to pure gravity, there is no way to determine it unless we require that the solution respects a given asymptotic behavior, as we have discussed before deriving eqs. (3.21) and (3.22).
To give the contribution to the geodesic equations, it is useful to make an asymptotic development of the $Z$ solution in the limit $\zeta \to \infty$:

$$\tilde{Z} \xrightarrow{\zeta \to \infty} \frac{a_1 + a_2 \xi^{1-M}}{a_3 + a_4 \xi^{1-M}}, \quad (4.35)$$

with the coefficients defined as:

$$a_1 = \coth(\sqrt{\Lambda}\,(\lambda_1 + \lambda_2)) \frac{\Gamma(b)\Gamma(a-b)}{\Gamma(c-b)} a_2 = \coth(\sqrt{\Lambda}\,(\lambda_1 + \lambda_2)) \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(c-a)}$$

$$a_3 = \frac{\Gamma(b-c+1)\Gamma(a-b)}{\Gamma(1-b)} a_4 = \frac{\Gamma(a-c+1)\Gamma(b-a)}{\Gamma(1-a)}$$

$$a = \frac{\mathcal{M} - \mu_1 + \mu_2}{2} b = 1 + \frac{-\mathcal{M} - \mu_1 + \mu_2}{2} c = 2 - \mu_1. \quad (4.36)$$

This development can be put in the form of a Lorentz transformation with respect to the static case

$$\tilde{Z} = \frac{a_\infty Z + b_\infty}{b_\infty Z + a_\infty} \quad Z \xrightarrow{\zeta \to \infty} \frac{\sqrt{\Lambda}}{2} \frac{z^{1-M}}{\sqrt{1-\tilde{a}(t)\,a_2}}, \quad (4.37)$$

where we have taken into account the fact that the asymptotic metric contains terms dependent explicitly on time ($\tilde{a}(t) = \sin(\sqrt{\Lambda}\,(1-\mathcal{M})t)$). The solution for the scale $\xi_{12}$ becomes then:

$$\xi_{12}^{1-M} = 2\sqrt{1-\tilde{a}^2(t)} \frac{a_2}{\sqrt{\Lambda}} a_3 \quad (4.38)$$

as the following identity $a_1 a_2 = a_3 a_4$ is valid.

At the next order $\Lambda^2$ we find that the field time dependence comes directly from the equations of motion and not only from the boundary conditions. To compute its contribution we must first know $T^{(1)}$.

From the asymptotic behavior (eqs. (4.22) and (4.23)) we deduce that:

$$T^{(0)} = (1 - \mathcal{M}t). \quad (4.39)$$

Instead, the equation of motion for $T^{(1)}$ can be integrated giving

$$\partial_z T^{(1)} = \frac{T^{(0)}}{2} \partial_z \tilde{Z}^{(0)} (\tilde{Z}^{(0)} + h(\zeta)). \quad (4.40)$$

In the limit $\mu_i \to 0$ we impose that the contribution of the extra function $h(\zeta)$ vanishes, so to recover the background field (4.21). Let us define $h(\zeta)$ in such a way that $\partial_z T^{(1)}$ is a meromorphic function, which therefore requires that the combination $\tilde{Z}^{(0)} + h(\zeta)$ transforms as $\partial_z \tilde{Z}^{(0)}$ under all monodromies, from which we deduce the monodromy properties of $h(\zeta)$ as follows:

$$h(\zeta) \to e^{2\pi i \mu_k} h(\zeta) + 4i(-)^k \lambda_k e^{i\pi \mu_k} \sin \pi \mu_k, \quad (4.41)$$
Adding the condition that \( h(\zeta) \rightarrow 0 \) in the limit of small masses, \( \mu_i \rightarrow 0 \), we obtain the solution, perturbative in the mass parameters,

\[
h(\zeta) \simeq -2\lambda_1 \mu_1 \log \zeta + 2\lambda_2 \mu_2 \log (1 - \zeta).
\] (4.42)

This behavior, divergent around the particles, is still acceptable, because the logarithmic divergence is cancelled by \( z \)-integration and \( T^{(1)} \) is again well defined around the particles.

Let us also notice that in the case of one particle \( h(\zeta) \) is constrained by

\[
h(\zeta) \mu_2 \rightarrow 0 - \rightarrow 2 \lambda_1 (1 - C \mu_1)
\] (4.43)

hence the solution for \( T^{(1)} \) must be for one particle with rapidity \( \lambda_1 \neq 0 \),

\[
T^{(1)} = \frac{T^{(0)}}{2} \left[ (Z^{(0)} + 2\lambda_1)(Z^{(0)} + 2\lambda_1) - 2\lambda_1 C(1 - \mu_1)(z + \overline{z}) \right] + T^{(1)}(t).
\] (4.44)

We can generalize eq. (4.42) to the case of any masses choosing \( h(\zeta) \) as

\[
h(\zeta) = A_1 \int_0^\zeta \! d\zeta \zeta_{\mu_1-1} (1 - \zeta)^{\mu_2} + A_2 \int_0^\zeta \! d\zeta \zeta_{\mu_1} (1 - \zeta)^{\mu_2-1}.
\] (4.45)

The monodromy conditions (4.41) are satisfied by the following positions:

\[
A_1 = -\frac{2\lambda_1}{B(\mu_1, 1 + \mu_2)} \quad A_2 = -\frac{2\lambda_2}{B(1 + \mu_1, \mu_2)},
\] (4.46)

that in the small mass limit reproduces the solution (4.42).

We have added the second line to make explicit the property that \( T^{(1)} \) is automatically monodromic around both particles, without need to introduce logarithmic terms to adjust an eventual translation monodromy, which fortunately is absent.

Starting from the knowledge of \( T^{(1)} \) we can deduce, using the first integral (3.18) constraints for the solution of the \( Z^{(2)} \) field,

\[
\partial_\zeta Z^{(2)} = \frac{(\partial_\zeta T^{(1)})^2}{\partial_\zeta Z^{(0)}} = \frac{1}{4} (T^{(0)})^2 (Z^{(0)} + \overline{h}(\zeta))^2 \partial_\zeta Z^{(0)}.
\] (4.48)
This first integral is automatically solution of the perturbative expansion of eq. (3.17), which remains also non linear in the unknown $\partial_\tau Z^{(2)}$ after the development in $\Lambda$:

$$\partial_\tau \partial_\tau Z^{(2)} = T^{(0)} \partial_\tau Z^{(0)} \sqrt{\partial_\tau \bar{Z}^{(0)} \partial_\tau Z^{(2)}}. \quad (4.49)$$

The other first integral, which is relative to $N(z)$, gives at the lowest order in $\Lambda$:

$$N(z) = \Lambda \frac{T^{(0)}}{2} k_2 \left( \frac{A_1}{\zeta} + \frac{A_2}{1 - \zeta} \right) \quad (4.50)$$

and we check that is a meromorphic function with simple poles.

The solution for $Z^{(2)}$ can be decomposed in an analytic part and a non-analytic one, satisfying eq. (4.48):

$$Z^{(2)} = Z_a^{(2)}(z) + Z_n^{(2)}(z, \bar{z}) \quad (4.51)$$

While the analytic part $Z_a^{(2)}(z)$ is determined by the development of the general solution (4.30), the non-analytic part must satisfy only the homogeneous part of the monodromies

$$Z_n^{(2)}(z, \bar{z}) \xrightarrow{k} e^{-2\pi \mu_k} Z_n^{(2)}(z, \bar{z}). \quad (4.52)$$

When we integrate $\partial_\tau Z^{(2)}$ we have at disposition an arbitrary polydromic function $f(\zeta)$, to be fixed in order to make $Z_n^{(2)}(z, \bar{z})$ again covariant under the rule (4.52).

A possible solution is

$$Z_n^{(2)}(z, \bar{z}) = \frac{T^{(0)} k_2}{4} \left[ (Z^{(0)} + 2\lambda_1)^2 (\bar{Z}^{(0)} + 2\lambda_1) + 2(Z^{(0)} + 2\lambda_1) \int_{\xi_1}^{\bar{\zeta}} d\zeta \partial_\tau \bar{Z}^{(0)} (\bar{h}(\zeta) - 2\lambda_1) + \right.$$  

$$\left. + \int_{\xi_1}^{\bar{\zeta}} d\zeta \partial_\tau \bar{Z}^{(0)} (\bar{h}(\zeta) - 2\lambda_1)^2 + f(\zeta) \right] =$$

$$= \frac{T^{(0)} k_2}{4} \left[ (Z^{(0)} - 2\lambda_2)^2 (\bar{Z}^{(0)} - 2\lambda_2) + 2(Z^{(0)} - 2\lambda_2) \int_{\xi_2}^{\zeta} d\zeta \partial_\tau \bar{Z}^{(0)} (\bar{h}(\zeta) + 2\lambda_2) + \right.$$  

$$\left. + \int_{\xi_2}^{\zeta} d\zeta \partial_\tau \bar{Z}^{(0)} (\bar{h}(\zeta) + 2\lambda_2)^2 + g(\zeta) \right], \quad (4.53)$$

in which the first expression is automatically covariant under the first particle if

$$f(\zeta) \xrightarrow{1} e^{-2\pi \mu_1} f(\zeta), \quad (4.54)$$

while the second expression is covariant around the second particle if

$$g(\zeta) \xrightarrow{2} e^{-2\pi \mu_2} g(\zeta). \quad (4.55)$$
Developing these two formulas we find that

\[ f(\zeta) = g(\zeta) - 2(\lambda_1 + \lambda_2)(Z^{(0)})^2 + 2C_1 Z^{(0)} + C_2 \]
\[ = g(\zeta) - 2(\lambda_1 + \lambda_2)(Z^{(0)} - 2\lambda_2)^2 + 2\bar{C}_1(Z^{(0)} - 2\lambda_2) + \bar{C}_2 \]
\[ \bar{C}_2 = C_2 + 4\lambda_2 C_1 + 8\lambda_2^2(\lambda_1 + \lambda_2), \]  
(4.56)

where to know \( C_1 \) and \( C_2 \) we must solve the following integrals

\[ C_1 = \int_1^{\xi_2} d\zeta \partial_2 Z^{(0)}(\zeta), \quad C_2 = \int_1^{\xi_2} d\zeta \partial_2 Z^{(0)} h^2(\zeta). \]  
(4.57)

The simplest solution is in fact

\[ f_0(z) = \Delta_1 \int_1^{\xi_1} dw \partial_w Z^{(0)}(w) \int_1^{w} dv \partial_v Z^{(0)}(v) \int_0^{\zeta=(v-\xi_2)/\xi_1} d\zeta \zeta^\mu_1 (1 - \zeta)^{\mu_2 - 1} + \Delta_2(Z^{(0)} + \lambda_1) \]
\[ \Delta_1 = -\frac{2(\lambda_1 + \lambda_2)}{B(1 + \mu_1, \mu_2)} \]
\[ \Delta_2 = \frac{\bar{C}_2}{(\lambda_1 + \lambda_2)} + \frac{2}{B(1 + \mu_1, \mu_2)} \int_1^{\xi_1} dw \partial_w Z^{(0)}(w) \int_1^{w} dv \partial_v Z^{(0)}(v) \int_0^{\zeta=(v-\xi_2)/\xi_1} d\zeta \zeta^\mu_1 (1 - \zeta)^{\mu_2 - 1}. \]  
(4.58)

To match the background metric (4.21) we can always add to the particular solution \( f_0(z) \) terms of the type:

\[ f(z) = f_0(z) + (A + Bw + Cw^2)(1 - \zeta)^{\mu_2}. \]  
(4.59)

At the level of geodesic equations we notice that the condition (4.33) is solved for an arbitrary scale \( \xi_1 \) and that it doesn’t give rise to new constraints, while the criterium of reproducing a certain asymptotic behavior is already satisfied by the first order of eq. (4.38), i.e. by eq. (4.27). If we continue perturbation theory it is possible that other constraints result from the requirement that the residue of the simple poles of \( N(z) \) has the following form, valid at all orders:

\[ N(z) \sim \Lambda(\xi_{12})^{1-M} \left( \frac{A_1(t)}{\zeta} + \frac{A_2(t)}{1-\zeta} \right) \]  
(4.60)

with the invariant mass \( M \) that replaces the sum of masses.

We cannot forget that we are treating a particular case of scattering at zero angular momentum, in which the particles have to collide after a finite time. This case is typically ill-defined from the distributional point of view, as in the solution products of distributions are expected to appear at a certain time, while the geodesic limit is still well defined. A verification of the consistency of this solution can be made with a perturbative computation at non-vanishing angular momentum, in which case it is no more possible to reduce the general \( O(2,2) \) monodromies to \( O(2,1) \).
5 Conclusions

We have deduced the general equations for the immersion $X^A$ that governs the scattering of point sources coupled to $AdS$-gravity. We have found the complete solution for the analytic part of the fields and a partial one for the non-analytic part. The choice of conformal gauge allows to study the scattering problem with instantaneous propagation of the fields avoiding the difficulties connected with the retarded potentials. This gauge is globally defined for the scattering of particles, but it is not for what concerns the scattering of black holes, where its validity is reduced to the region external to the horizons.

The scattering of particles is governed by the composition of monodromies, which gives rise to two invariant masses $\mathbb{R}$; someone may object that these are not relevant for the solution of the fields, since in conformal gauge there is a physical limit on the values of spatial coordinates and it is not possible to see the particles as a unique body at great distance from them. However such a limit is non-perturbative with respect to the cosmological constant and at a perturbative level the fields do have infinite extension, and the leading contribution at great distances of each perturbative order must be dominated by these two invariant masses, which in our particular case of head-on collision coincide. The physical scale is produced only re-summing the perturbative series. At a non-perturbative level, the two invariant masses are still present, since in the non-physical region of the $Z$ field it appears a specular image of the $N$-body system with an unique singularity, defined by the two invariant masses. Therefore, at least mathematically, this extra singularity is important to give the parameterization of the solution.

It would be interesting to continue our study to include the scattering of $BTZ$ black holes, where the equations (3.12) still remain valid outside their horizons. A similar problem has already an analogy with the case of the scattering of spinning particles in $(2 + 1)$-gravity $\mathbb{R}$, in which the problem of closed time-like curves produces, in conformal gauge, some $CTC$ horizons around the particles. Work is in progress in this direction.

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