ON THE MINIMUM SIZE OF SUBSET AND SUBSEQUENCE SUMS IN INTEGERS

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Abstract

Let $A$ be a sequence of $rk$ terms which is made up of $k$ distinct integers each appearing exactly $r$ times in $A$. The sum of all terms of a subsequence of $A$ is called a subsequence sum of $A$. For a nonnegative integer $\alpha \leq rk$, let $\Sigma_\alpha(A)$ be the set of all subsequence sums of $A$ that correspond to the subsequences of length $\alpha$ or more. When $r = 1$, we call the subsequence sums as subset sums and we write $\Sigma_\alpha(A)$ for $\Sigma_\alpha(A)$. In this article, using some simple combinatorial arguments, we establish optimal lower bounds for the size of $\Sigma_\alpha(A)$ and $\Sigma_\alpha(A)$. As special cases, we also obtain some already known results in this study.

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1. Introduction

Let $A$ be a set of $k$ integers. The sum of all elements of a subset of $A$ is called a subset sum of $A$. So, the subset sum of the empty set is 0. For a nonnegative integer $\alpha \leq k$, let

$$\Sigma_\alpha(A) := \left\{ \sum_{a \in A'} a : A' \subset A, |A'| \geq \alpha \right\}$$

and

$$\Sigma^\alpha(A) := \left\{ \sum_{a \in A'} a : A' \subset A, |A'| \leq k - \alpha \right\}.$$
That is, \( \Sigma_\alpha(A) \) is the set of subset sums corresponding to the subsets of \( A \) that are of the size at least \( \alpha \) and \( \Sigma^\alpha(A) \) is the set of subset sums corresponding to the subsets of \( A \) that are of the size at most \( k - \alpha \). So, \( \Sigma_\alpha(A) = \sum_{a \in A} a - \Sigma^\alpha(A) \). Therefore \( |\Sigma_\alpha(A)| = |\Sigma^\alpha(A)| \).

Now, we extend the above definitions for sequences of integers. Before we go for extension, we mention some notation that are used throughout the paper.

Let \( A = (a_1, a_2, \ldots, a_k, \ldots, a_k) \) be a sequence of \( rk \) terms, where \( a_1, a_2, \ldots, a_k \) are distinct integers each appearing exactly \( r \) times in \( A \). We denote this sequence by \( A = (a_1, a_2, \ldots, a_k)_r \). If \( A' \) is a subsequence of \( A \), then we write \( A' \subset A \). By \( x \in A \), we mean \( x \) is a term in \( A \). For the number of terms in a sequence \( A \), we use the notation \( |A| \). For an integer \( x \), we let \( x \cdot A \) be the sequence which is obtained from by multiplying each term of \( A \) by \( x \). For two nonempty sequences \( A, B \), by \( A \cap B \), we mean the sequence of all those terms that are in both \( A \) and \( B \). Furthermore, for integers \( a, b \) with \( b \geq a \), by \( [a, b]_r \), we mean the sequence \( (a, a + 1, \ldots, b)_r \).

Let \( A = (a_1, a_2, \ldots, a_k)_r \) be a sequence of integers with \( rk \) terms. The sum of all terms of a subsequence of \( A \) is called a subsequence sum of \( A \). For a nonnegative integer \( \alpha \leq rk \), let

\[
\Sigma_\alpha(A) := \left\{ \sum_{a \in A'} a : A' \subset A, \ |A'| \geq \alpha \right\}
\]

and

\[
\Sigma^\alpha(A) := \left\{ \sum_{a \in A'} a : A' \subset A, \ |A'| \leq rk - \alpha \right\}.
\]

That is, \( \Sigma_\alpha(A) \) is the set of subsequence sums corresponding to the subsequences of \( A \) that are of the size at least \( \alpha \) and \( \Sigma^\alpha(A) \) is the set of subsequence sums corresponding to the subsequences of \( A \) that are of the size at most \( rk - \alpha \). Then in the same line with the subset sums, we have \( |\Sigma_\alpha(A)| = |\Sigma^\alpha(A)| \) for all \( 0 \leq \alpha \leq rk \).

The set of subset sums \( \Sigma_\alpha(A) \) and \( \Sigma^\alpha(A) \) and the set of subsequence sums \( \Sigma_\alpha(A) \) and \( \Sigma^\alpha(A) \) may also be written as unions of sumsets:

For a finite set \( A \) of \( k \) integers and for positive integers \( h, r \), the \( h \)-fold sumset \( hA \) is the collection of all sums of \( h \) not-necessarily-distinct elements of \( A \), the \( h \)-fold restricted sumset \( h^rA \) is the collection of all sums of \( h \) distinct elements of \( A \), and the generalized sumset \( h^{(r)}A \) is the collection of all sums of \( h \) elements of \( A \) with at most \( r \) repetitions for each element (see [28]). Then \( \Sigma_\alpha(A) = \bigcup_{h=\alpha}^{k} h^rA, \Sigma^\alpha(A) = \bigcup_{h=0}^{k-\alpha} h^rA, \Sigma_\alpha(A) = \bigcup_{h=\alpha}^{k} h^{(r)}A, \) and \( \Sigma^\alpha(A) = \bigcup_{h=0}^{k-\alpha} h^{(r)}A, \) where \( A = (A)_r \) and \( 0^rA = 0^{(r)}A = \{0\} \).

An important problem in additive number theory is to find the optimal lower bound for \( |\Sigma_\alpha(A)| \) and \( |\Sigma_\alpha(A)| \). Such problems are very useful in some other combinatorial problems such as the zero-sum problems (see [17,21,24,36]). Nathanson [33] established the optimal lower bound for \( |\Sigma_1(A)| \) for sets of integers \( A \). Mistri and Pandey [29,30].
and Jiang and Li \cite{27} extended Nathanson’s results to $\Sigma_1(A)$ for sequences of integers $A$. Note that these subset and subsequence sums may also be studied in any abelian group (for earlier works, in case $\alpha = 0$ and $\alpha = 1$, see \cite{6, 8, 18, 21, 23, 25, 26, 35}). Recently, Balandraud \cite{7} proved the optimal lower bound for $|\Sigma_\alpha(A)|$ in the finite prime field $\mathbb{F}_p$, where $p$ is a prime number. Inspired by Balandraud’s work \cite{7}, in this paper we establish optimal lower bounds for $|\Sigma_\alpha(A)|$ and $|\Sigma_\alpha(A)|$ in the group of integers. Note that, in \cite{13}, we have already settled this problem when the set $A$ (or sequence $A$) contains nonnegative or nonpositive integers. So, in this paper we consider the sets (or sequences) which may contain both positive and negative integers.

A similar problem for sumsets also have been extensively studied in the past (see \cite{1–3, 14–16, 19–21, 33, 34} and the references therein). For some recent works along this line, one can also see \cite{4, 5, 9–12, 28, 31, 32}.

In Section 2, we prove optimal lower bounds for $|\Sigma_\alpha(A)|$ for finite sets of integers $A$. In Section 3, we extend the results of Section 2 to sequences of integers.

The following results are used to prove the results in this paper.

**Theorem 1.** \cite[Theorem 1.4]{34} Let $A$, $B$ be nonempty finite sets of integers. Set $A + B = \{a + b : a \in A, b \in B\}$. Then

$$|A + B| \geq |A| + |B| - 1.$$ 

This lower bound is optimal.

**Theorem 2.** \cite[Theorem 1.3]{34} Let $A$ be a nonempty finite set of integers and $h$ be a positive integer. Then

$$|hA| \geq h|A| - h + 1.$$ 

This lower bound is optimal.

**Theorem 3.** \cite[Theorem 4]{7} Let $A$ be a nonempty subset of $\mathbb{F}_p$ such that $A \cap (-A) = \emptyset$. Then for any integer $\alpha \in [0, |A|]$, we have

$$|\Sigma_\alpha(A)| \geq \min \left\{ p, \frac{|A||A| + 1}{2} - \frac{\alpha(\alpha + 1)}{2} + 1 \right\}.$$ 

This lower bound is optimal.

### 2. Subset sum

In Theorem \cite{4} and Corollary \cite{4} we prove optimal lower bound for $|\Sigma_\alpha(A)|$ under the assumptions $A \cap (-A) = \emptyset$ and $A \cap (-A) = \{0\}$, respectively. In Theorem \cite{5} and Corollary \cite{5} we prove optimal lower bound for $|\Sigma_\alpha(A)|$ for arbitrary finite sets of integers $A$. The bounds in Theorem \cite{5} and Corollary \cite{5} depends on the number of positive
elements and the number of negative elements in set \( A \). In Corollary 3, we prove lower bounds for \(|\Sigma_\alpha(A)|\), which holds for arbitrary finite sets of integers \( A \) and only depend on the total number of elements of \( A \) not the number of positive and negative elements of \( A \).

**Theorem 4.** Let \( A \) be a set of \( k \) integers such that \( A \cap (-A) = \emptyset \). For any integer \( \alpha \in [0, k] \), we have

\[
|\Sigma_\alpha(A)| \geq \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1. \tag{1}
\]

This lower bound is optimal.

**Proof.** Let \( p \) be a prime number such that \( p > \max\left\{2 \max^*(A), \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1\right\}\), where \( \max^*(A) = \max\{|a| : a \in A\} \). Now, the elements of \( A \) can be thought of residue classes modulo \( p \). Since \( p > 2 \max^*(A) \), any two elements of \( A \) are different modulo \( p \).

Furthermore \( A \cap (-A) = \emptyset \). Hence, by Theorem 3, we get

\[
|\Sigma_\alpha(A)| \geq \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1.
\]

Next, to verify that the lower bound in (1) is optimal, let \( A = [1, k] \). Then \( A \cap (-A) = \emptyset \) and

\[
\Sigma_\alpha(A) \subset [1 + 2 + \cdots + \alpha, 1 + 2 + \cdots + k] = \left[\frac{\alpha(\alpha+1)}{2}, \frac{k(k+1)}{2}\right].
\]

Therefore

\[
|\Sigma_\alpha(A)| \leq \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1.
\]

This together with (1) gives

\[
|\Sigma_\alpha(A)| = \frac{k(k+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1.
\]

Thus, the lower bound in (1) is optimal. This completes the proof of the theorem. \( \square \)

**Corollary 1.** Let \( A \) be a set of \( k \) integers such that \( A \cap (-A) = \{0\} \). For any integer \( \alpha \in [0, k] \), we have

\[
|\Sigma_\alpha(A)| \geq \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} + 1. \tag{2}
\]

This lower bound is optimal.

**Proof.** If \( A = \{0\} \), then \( \Sigma_\alpha(A) = \{0\} \). Therefore \(|\Sigma_\alpha(A)| = 1\), and (2) holds. So, let \( A \neq \{0\} \) and set \( A' = A \setminus \{0\} \). Then it is easy to see that \( \Sigma_0(A) = \Sigma_0(A') \) and \( \Sigma_\alpha(A) = \Sigma_{\alpha-1}(A') \) for \( \alpha \geq 1 \). Since \( A' \cap (-A') = \emptyset \), by Theorem 4 we get

\[
|\Sigma_0(A)| = |\Sigma_0(A')| \geq \frac{k(k-1)}{2} + 1
\]
and
\[ |\Sigma_\alpha(A)| = |\Sigma_{\alpha-1}(A')| \geq \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} + 1 \]
for \( \alpha \geq 1 \). Hence (2) is established.

Now, let \( A = [0, k-1] \). Then \( A \cap (-A) = \{0\} \) and \( \Sigma_\alpha(A) \subset \left[ \frac{\alpha(\alpha-1)}{2}, \frac{k(k-1)}{2} \right] \).
Therefore \( |\Sigma_\alpha(A)| \leq \frac{k(k-1)}{2} - \frac{\alpha(\alpha-1)}{2} + 1 \). This together with (2) gives that the lower bound in (2) is optimal. \( \square \)

**Remark 1.** Nathanson’s theorem [33, Theorem 3] is a particular case of Theorem 4 and Corollary 11 for \( \alpha = 1 \).

**Theorem 5.** Let \( n \) and \( p \) be positive integers and \( A \) be a set of \( n \) negative and \( p \) positive integers. Let \( \alpha \in [0, n+p] \) be an integer.

(i) If \( \alpha \leq n \) and \( \alpha \leq p \), then \( |\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} + 1 \).

(ii) If \( \alpha \leq n \) and \( \alpha > p \), then \( |\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} + 1 \).

(iii) If \( \alpha > n \) and \( \alpha \leq p \), then \( |\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} + 1 \).

(iv) If \( \alpha > n \) and \( \alpha > p \), then \( |\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} + 1 \).

These lower bounds are optimal.

**Proof.** Let \( A = A_n \cup A_p \), where \( A_n = \{b_1, \ldots, b_n\} \) and \( A_p = \{c_1, \ldots, c_p\} \) such that \( b_n < b_{n-1} < \cdots < b_1 < 0 < c_1 < c_2 < \cdots < c_p \).

(i) If \( \alpha \leq n \) and \( \alpha \leq p \), then

\[ (\Sigma_\alpha(A_n) + \Sigma_0(A_p)) \cup \left( \Sigma^1(\{b_1, \ldots, b_n\}) + \sum_{j=1}^p c_j \right) \subset \Sigma_\alpha(A) \]

with

\[ (\Sigma_\alpha(A_n) + \Sigma_0(A_p)) \cap \left( \Sigma^1(\{b_1, \ldots, b_n\}) + \sum_{j=1}^p c_j \right) = \emptyset. \]

Hence, by Theorem 1 and Theorem 4 we have

\[ |\Sigma_\alpha(A)| \geq |\Sigma_\alpha(A_n) + \Sigma_0(A_p)| + |\Sigma^1(\{b_1, \ldots, b_n\})| \]
\[ \geq |\Sigma_\alpha(A_n)| + |\Sigma_0(A_p)| + |\Sigma^1(\{b_1, \ldots, b_n\})| - 1 \]
\[ \geq \left( \frac{n(n+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1 \right) + \left( \frac{p(p+1)}{2} + 1 \right) + \frac{\alpha(\alpha+1)}{2} - 1 \]
\[ = \frac{n(n+1)}{2} + \frac{p(p+1)}{2} + 1. \]
(ii) If \( \alpha \leq n \) and \( \alpha > p \), then

\[
(\Sigma_\alpha(A_n) + \Sigma_0(A_p)) \cup \left( \Sigma_{\alpha-p}(\{b_1, \ldots, b_\alpha\}) + \sum_{j=1}^{p} c_j \right) \subset \Sigma_\alpha(A)
\]

with

\[
(\Sigma_\alpha(A_n) + \Sigma_0(A_p)) \cap \left( \Sigma_{\alpha-p}(\{b_1, \ldots, b_\alpha\}) + \sum_{j=1}^{p} c_j \right) = \left\{ \sum_{j=1}^{\alpha} b_j + \sum_{j=1}^{p} c_j \right\}.
\]

Hence, by Theorem 1 and Theorem 4, we have

\[
|\Sigma_\alpha(A)| \geq |
\Sigma_\alpha(A_n) + \Sigma_0(A_p)| + |\Sigma_{\alpha-p}(\{b_1, \ldots, b_\alpha\})| - 1
\geq \left( \frac{n(n+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1 \right) + \left( \frac{p(p+1)}{2} + 1 \right)
+ \left( \frac{\alpha(\alpha+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} + 1 \right) - 2
= \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} + 1.
\]

(iii) If \( \alpha > n \) and \( \alpha \leq p \), then by applying the result of (ii) for \(-A\), we obtain

\[
|\Sigma_\alpha(A)| = |\Sigma_\alpha(-A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} + 1.
\]

(iv) If \( \alpha > n \) and \( \alpha > p \), then

\[
\left( \sum_{j=1}^{n} b_j + \Sigma_{\alpha-n}(A_p) \right) \cup \left( \Sigma_{\alpha-p}(A_n) + \sum_{j=1}^{p} c_j \right) \subset \Sigma_\alpha(A)
\]

with

\[
\left( \sum_{j=1}^{n} b_j + \Sigma_{\alpha-n}(A_p) \right) \cap \left( \Sigma_{\alpha-p}(A_n) + \sum_{j=1}^{p} c_j \right) = \left\{ \sum_{j=1}^{n} b_j + \sum_{j=1}^{p} c_j \right\}.
\]

Hence, by Theorem 4 we get

\[
|\Sigma_\alpha(A)| \geq |\Sigma_{\alpha-n}(A_p)| + |\Sigma_{\alpha-p}(A_n)| - 1
\geq \left( \frac{p(p+1)}{2} - \frac{\alpha-n)(\alpha-n+1)}{2} + 1 \right)
+ \left( \frac{n(n+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} + 1 \right) - 1
= \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-n)(\alpha-n+1)}{2} - \frac{(\alpha-p)(\alpha-p+1)}{2} + 1.
\]
It can be easily verified that all the lower bounds mentioned in the theorem are optimal for \( A = [-n, p] \setminus \{0\}. \)

**Corollary 2.** Let \( n \) and \( p \) be positive integers and \( A \) be a set of \( n \) negative integers, \( p \) positive integers and zero. Let \( \alpha \in [0, n+p+1] \) be an integer.

(i) If \( \alpha \leq n \) and \( \alpha \leq p \), then \(|\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} + 1.\)

(ii) If \( \alpha \leq n \) and \( \alpha > p \), then \(|\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-p)(\alpha-p-1)}{2} + 1.\)

(iii) If \( \alpha > n \) and \( \alpha \leq p \), then \(|\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-n)(\alpha-n-1)}{2} + 1.\)

(iv) If \( \alpha > n \) and \( \alpha > p \), then \(|\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(\alpha-n)(\alpha-n-1) - (\alpha-p)(\alpha-p-1)}{2} + 1.\)

These lower bounds are optimal.

**Proof.** The lower bounds for \(|\Sigma_\alpha(A)|\) easily follows from Theorem 5 and the fact that \( \Sigma_0(A) = \Sigma_0(A') \) and \( \Sigma_\alpha(A) = \Sigma_{\alpha-1}(A') \) for \( \alpha \geq 1 \), where \( A' = A \setminus \{0\} \). Furthermore, the optimality of these bounds can be verified by taking \( A = [-n, p] \). \( \square \)

**Corollary 3.** Let \( k \geq 2 \) and \( A \) be a set of \( k \) integers. Let \( \alpha \in [0, k] \) be an integer. If \( 0 \notin A \), then

\[ |\Sigma_\alpha(A)| \geq \left\lfloor \frac{(k+1)^2}{4} \right\rfloor - \frac{\alpha(\alpha+1)}{2} + 1. \tag{3} \]

If \( 0 \in A \), then

\[ |\Sigma_\alpha(A)| \geq \left\lfloor \frac{k^2}{4} \right\rfloor - \frac{\alpha(\alpha-1)}{2} + 1. \tag{4} \]

**Proof.** **Case 1.** \((0 \notin A)\). If \( k = 2 \), then \( A = \{a_1, a_2\} \) for some integers \( a_1, a_2 \) with \( a_1 < a_2 \). Therefore \( \Sigma_1(A) = \{a_1, a_2, a_1 + a_2\} \subset \Sigma_0(A) \) and \( \Sigma_2(A) = \{a_1 + a_2\} \). Hence (3) holds for \( k = 2 \). So, assume that \( k \geq 3 \). As \( k(k+1)/2 > (k+1)^2/4 \) for \( k \geq 3 \), if \( |\Sigma_\alpha(A)| \geq k(k+1)/2 - \alpha(\alpha+1)/2 + 1 \), then we are done. So, let \( |\Sigma_\alpha(A)| < k(k+1)/2 - \alpha(\alpha+1)/2 + 1 \). Then, Theorem 4 implies that \( A \) contains both positive and negative integers. Let \( A_n \) and \( A_p \) be subsets of \( A \) that contain respectively, all negative and all positive integers of \( A \). Let also \(|A_n| = n\) and \(|A_p| = p\). Then \( n \geq 1 \) and \( p \geq 1 \). By Theorem 5, we have

\[ |\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1. \]
for all $\alpha \in [0, k]$. Since $k = n + p$, without loss of generality we may assume that $n \geq k/2$. Therefore

$$|\Sigma_\alpha(A)| \geq \frac{n(n+1)}{2} + \frac{(k-n)(k-n+1)}{2} - \frac{\alpha(\alpha+1)}{2} + 1$$

$$= \left( n - \frac{k}{2} \right)^2 + \frac{(k+1)^2 - 1}{4} - \frac{\alpha(\alpha+1)}{2} + 1$$

$$\geq \frac{(k+1)^2 - 1}{4} - \frac{\alpha(\alpha+1)}{2} + 1.$$ 

Hence

$$|\Sigma_\alpha(A)| \geq \left\lfloor \frac{(k+1)^2}{4} \right\rfloor - \frac{\alpha(\alpha+1)}{2} + 1.$$ 

Case 2. ($0 \in A$). Set $A' = A \setminus \{0\}$. Then $\Sigma_0(A) = \Sigma_0(A')$ and $\Sigma_\alpha(A) = \Sigma_{\alpha-1}(A')$ for $\alpha \geq 1$. Hence, by Case 1, we get

$$|\Sigma_0(A)| = |\Sigma_0(A')| \geq \left\lfloor \frac{k^2}{4} \right\rfloor + 1$$

and

$$|\Sigma_\alpha(A)| = |\Sigma_{\alpha-1}(A')| \geq \left\lfloor \frac{k^2}{4} \right\rfloor - \frac{\alpha(\alpha-1)}{2} + 1$$

for $\alpha \geq 1$. Hence

$$|\Sigma_\alpha(A)| \geq \left\lfloor \frac{k^2}{4} \right\rfloor - \frac{\alpha(\alpha-1)}{2} + 1$$

for all $\alpha \in [0, k]$. This completes the proof of the corollary.

**Remark 2.** Nathanson [33] have already proved this corollary for $\alpha = 1$. The purpose of this corollary is to prove a similar result for every $\alpha \in [0, k]$. Note that the lower bounds in Corollary 3 are not optimal for all $\alpha \in [0, k]$, except for $\alpha = 0$ and $\alpha = 1$.

**Remark 3.** The lower bounds in Corollary 3 can also be written in the following form:

If $0 \notin A$, then

$$|\Sigma_\alpha(A)| \geq \begin{cases} 
\frac{(k+1)^2 - \alpha(\alpha+1)}{2} + 1 & \text{if } k \equiv 1 \pmod{2} \\
\frac{(k+1)^2 - 1}{4} - \frac{\alpha(\alpha+1)}{2} + 1 & \text{if } k \equiv 0 \pmod{2}.
\end{cases}$$

If $0 \in A$, then

$$|\Sigma_\alpha(A)| \geq \begin{cases} 
\frac{k^2 - 1}{4} - \frac{\alpha(\alpha-1)}{2} + 1 & \text{if } k \equiv 1 \pmod{2} \\
\frac{k^2 - \alpha(\alpha-1)}{2} + 1 & \text{if } k \equiv 0 \pmod{2}.
\end{cases}$$

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3. Subsequence sum

In this section, we extend the results of the previous section from sets of integers to sequences of integers. In Theorem 6, we establish optimal lower bound for $|\Sigma_{\alpha}(A)|$ under the assumptions $A \cap (-A) = \emptyset$ and $A \cap (-A) = (0)_r$. In Theorem 7 and Corollary 4, we prove optimal lower bound for $|\Sigma_{\alpha}(A)|$ for arbitrary finite sequences of integers $A$. The bounds in Theorem 7 and Corollary 4 depends on the number of negative terms and the number of positive terms in sequence $A$. In Corollary 5, we prove lower bounds for $|\Sigma_{\alpha}(A)|$, which holds for arbitrary finite sequences of integers $A$ and only depend on the total number of terms of $A$ not the number of positive and negative terms of $A$.

If $\alpha = rk$ and $A = (a_1, \ldots, a_k)_r$, then $\Sigma_{\alpha}(A) = \{ra_1 + ra_2 + \cdots + ra_k\}$. Therefore $|\Sigma_{\alpha}(A)| = 1$. So, in the rest of this section, we assume that $\alpha < rk$.

**Theorem 6.** Let $k \geq 2$, $r \geq 1$, and $\alpha \in [0, rk - 1]$ be integers. Let $m \in [1, k]$ be an integer such that $(m - 1)r \leq \alpha < mr$. Let $A$ be a sequence of $rk$ terms which is made up of $k$ distinct integers each repeated exactly $r$ times. If $A \cap (-A) = \emptyset$, then

$$|\Sigma_{\alpha}(A)| \geq r \left( \frac{k(k + 1)}{2} - \frac{m(m + 1)}{2} \right) + m(mr - \alpha) + 1. \quad (5)$$

If $A \cap (-A) = (0)_r$, then

$$|\Sigma_{\alpha}(A)| \geq r \left( \frac{k(k - 1)}{2} - \frac{m(m - 1)}{2} \right) + (m - 1)(mr - \alpha) + 1. \quad (6)$$

These lower bounds are optimal.

**Proof.** Let $A$ be the set of all distinct terms of sequence $A$. Since $(m - 1)r \leq \alpha < mr$, we can write $\alpha$ as $\alpha = (m - 1)r + u$ for some integer $0 \leq u < r$. Then

$$(r - u)\Sigma_{m-1}(A) + u\Sigma_m(A) \subset \Sigma_{\alpha}(A),$$

where $(r - u)\Sigma_{m-1}(A)$ is the $(r - u)$-fold subset of $\Sigma_{m-1}(A)$ and $u\Sigma_m(A)$ is the $u$-fold subset of $\Sigma_m(A)$. So, by Theorem 1 and Theorem 2, we have

$$|\Sigma_{\alpha}(A)| \geq |(r - u)\Sigma_{m-1}(A)| + |u\Sigma_m(A)| - 1 \geq (r - u)|\Sigma_{m-1}(A)| + u|\Sigma_m(A)| - r + 1.$$  

If $A \cap (-A) = \emptyset$, then $A \cap (-A) = \emptyset$. Thus, by Theorem 4, we have

$$|\Sigma_{\alpha}(A)| \geq (r - u) \left( \frac{k(k + 1)}{2} - \frac{m(m - 1)}{2} + 1 \right) + u \left( \frac{k(k + 1)}{2} - \frac{m(m + 1)}{2} + 1 \right) - r + 1$$

$$= r \left( \frac{k(k + 1)}{2} - \frac{m(m + 1)}{2} \right) + m(r - u) + 1$$

$$= r \left( \frac{k(k + 1)}{2} - \frac{m(m + 1)}{2} \right) + m(mr - \alpha) + 1.$$
Similarly, if \( A \cap (-A) = (0)_r \), then \( A \cap (-A) = \{0\} \). Thus, by Corollary 1, we have

\[
|\Sigma_\alpha(A)| \geq (r - u) \left( \frac{k(k - 1)}{2} - \frac{(m - 1)(m - 2)}{2} + 1 \right) + u \left( \frac{k(k - 1)}{2} - \frac{m(m - 1)}{2} + 1 \right) - r + 1
\]

\[
= r \left( \frac{k(k - 1)}{2} - \frac{m(m - 1)}{2} \right) + (m - 1)(r - u) + 1
\]

\[
= r \left( \frac{k(k - 1)}{2} - \frac{m(m - 1)}{2} \right) + (m - 1)(mr - \alpha) + 1.
\]

Hence (5) and (6) are established.

Next, to verify that the lower bounds in (5) and (6) are optimal, let \( A = [1, k]_r \) and \( B = [0, k - 1]_r \). Then \( A \cap (-A) = \emptyset \) and \( B \cap (-B) = (0)_r \) with

\[
\Sigma_\alpha(A) \subset [r \cdot 1 + \cdots + r \cdot (m - 1) + (\alpha - (m - 1)r) \cdot m, r \cdot 1 + \cdots + r \cdot k]
\]

and

\[
\Sigma_\alpha(B) \subset [r \cdot 1 + \cdots + r \cdot (m - 2) + (\alpha - (m - 1)r) \cdot (m - 1), r \cdot 1 + \cdots + r \cdot (k - 1)].
\]

Therefore

\[
|\Sigma_\alpha(A)| \leq \frac{rk(k + 1)}{2} - \frac{rm(m + 1)}{2} + m(mr - \alpha) + 1
\]

and

\[
|\Sigma_\alpha(B)| \leq \frac{rk(k - 1)}{2} - \frac{rm(m - 1)}{2} + (m - 1)(mr - \alpha) + 1.
\]

These two inequalities together with (5) and (6) implies that the lower bounds in (5) and (6) are optimal. This completes the proof of the theorem. \( \square \)

**Remark 4.** Mistri and Pandey’s result [30, Theorem 1] is a particular case of Theorem 6, for \( \alpha = 1 \).

**Theorem 7.** Let \( k \geq 2, r \geq 1, \) and \( \alpha \in [0, rk - 1] \) be integers. Let \( m \in [1, k] \) be an integer such that \( (m - 1)r \leq \alpha < mr \). Let \( A \) be a sequence of \( rk \) terms which is made up of \( n \) negative integers and \( p \) positive integers each repeated exactly \( r \) times.

(i) If \( m \leq n \) and \( m \leq p \), then \( |\Sigma_\alpha(A)| \geq r \left( \frac{n(n + 1)}{2} + \frac{p(p + 1)}{2} \right) + 1 \).

(ii) If \( m \leq n \) and \( m > p \), then \( |\Sigma_\alpha(A)| \geq r \left( \frac{n(n + 1)}{2} + \frac{p(p + 1)}{2} - \frac{(m - p)(m - p + 1)}{2} \right) + (m - p)(mr - \alpha) + 1 \).

(iii) If \( m > n \) and \( m \leq p \), then \( |\Sigma_\alpha(A)| \geq r \left( \frac{n(n + 1)}{2} + \frac{p(p + 1)}{2} - \frac{(m - n)(m - n + 1)}{2} \right) + (m - n)(mr - \alpha) + 1 \).
(iv) If \( m > n \) and \( m > p \), then
\[
|\Sigma_\alpha(A)| \geq r \left( \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(m-n)(m-n+1)}{2} - \frac{(m-p)(m-p+1)}{2} \right) + (2m - n - p)(mr - \alpha) + 1.
\]

These lower bounds are optimal.

**Proof.** Let \( A_n \) and \( A_p \) be sets that contain respectively, all distinct negative terms and all distinct positive terms of \( A \). Then \( |A_n| = n \) and \( |A_p| = p \). Let also \( A_n = \{ b_1, b_2, \ldots, b_n \} \) and \( A_p = \{ c_1, c_2, \ldots, c_p \} \), where \( b_n < b_{n-1} < \cdots < b_1 < 0 < c_1 < c_2 < \cdots < c_p \).

(i) If \( m \leq n \) and \( m \leq p \), then
\[
r(\Sigma_m(A_n) + \Sigma_0(A_p)) \cup \left( \Sigma^1((b_1, \ldots, b_m)_r) + \sum_{j=1}^{p} rc_j \right) \subset \Sigma_\alpha(A)
\]
with
\[
r(\Sigma_m(A_n) + \Sigma_0(A_p)) \cap \left( \Sigma^1((b_1, \ldots, b_m)_r) + \sum_{j=1}^{p} rc_j \right) = \emptyset.
\]
Hence, by Theorem 1, Theorem 2, Theorem 4, and Theorem 6 we have
\[
|\Sigma_\alpha(A)| \geq \left| r(\Sigma_m(A_n) + \Sigma_0(A_p)) \right| + \left| \Sigma^1((b_1, \ldots, b_m)_r) \right|
\]
\[
\geq r|\Sigma_m(A_n)| + r|\Sigma_0(A_p)| + \left| \Sigma^1((b_1, \ldots, b_m)_r) \right| - 2r + 1
\]
\[
\geq r \left( \frac{n(n+1)}{2} - \frac{m(m+1)}{2} + 1 \right) + r \left( \frac{p(p+1)}{2} + 1 \right) + \frac{rm(m+1)}{2} - 2r + 1
\]
\[
= r \left( \frac{n(n+1)}{2} + \frac{p(p+1)}{2} \right) + 1.
\]

(ii) If \( m \leq n \) and \( m > p \), then
\[
r(\Sigma_m(A_n) + \Sigma_0(A_p)) \cup \left( \Sigma_{\alpha-pr}((b_1, \ldots, b_m)_r) + \sum_{j=1}^{p} rc_j \right) \subset \Sigma_\alpha(A)
\]
with
\[
r(\Sigma_m(A_n) + \Sigma_0(A_p)) \cap \left( \Sigma_{\alpha-pr}((b_1, \ldots, b_m)_r) + \sum_{j=1}^{p} rc_j \right) = \left\{ \sum_{j=1}^{m} rb_j + \sum_{j=1}^{p} rc_j \right\}.
\]
Hence, by Theorem 1, Theorem 2, Theorem 4, and Theorem 6, we have

\[ |\Sigma_\alpha(A)| \geq |\Sigma_{m}(A_n) + \Sigma_0(A_p)| + |\Sigma_{\alpha-pr}(b_1, \ldots, b_m)| - 1 \]

\[ \geq r|\Sigma_{m}(A_n)| + r|\Sigma_0(A_p)| + |\Sigma_{\alpha-pr}(b_1, \ldots, b_m)| - 2r + 1 \]

\[ \geq r\left(\frac{n(n+1)}{2} - \frac{m(m+1)}{2} + 1\right) + r\left(\frac{p(p+1)}{2} + 1\right) \]

\[ + r\left(\frac{m(m+1)}{2} - \frac{(m-p)(m-p+1)}{2}\right) + (m-p)(mr - \alpha) - 2r + 1 \]

\[ = r\left(\frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(m-p)(m-p+1)}{2}\right) + (m-p)(mr - \alpha) + 1. \]

(iii) If \( m > n \) and \( m \leq p \), then by applying the result of (ii) for \(-A\), we obtain

\[ |\Sigma_\alpha(A)| = |\Sigma_\alpha(-A)| \geq r\left(\frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(m-n)(m-n+1)}{2}\right) + (m-n)(mr - \alpha) + 1. \]

(iv) If \( m > n \) and \( m > p \), then

\[ \left(\sum_{j=1}^{n} rb_j + \Sigma_{\alpha-nr}(A_p)\right) \cup \left(\Sigma_{\alpha-pr}(A_n) + \sum_{j=1}^{p} rc_j\right) \subset \Sigma_\alpha(A) \]

with

\[ \left(\sum_{j=1}^{n} rb_j + \Sigma_{\alpha-nr}(A_p)\right) \cap \left(\Sigma_{\alpha-pr}(A_n) + \sum_{j=1}^{p} rc_j\right) = \left\{\sum_{j=1}^{n} rb_j + \sum_{j=1}^{p} rc_j\right\}. \]

Hence, by Theorem 6, we have

\[ |\Sigma_\alpha(A)| \geq |\Sigma_{\alpha-nr}(A_p)| + |\Sigma_{\alpha-pr}(A_n)| - 1 \]

\[ \geq r\left(\frac{p(p+1)}{2} - \frac{(m-n)(m-n+1)}{2}\right) + (m-n)(mr - \alpha) \]

\[ + r\left(\frac{n(n+1)}{2} - \frac{(m-p)(m-p+1)}{2}\right) + (m-p)(mr - \alpha) + 1 \]

\[ = r\left(\frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(m-n)(m-n+1)}{2} - \frac{(m-p)(m-p+1)}{2}\right) \]

\[ + (2m-n-p)(mr - \alpha) + 1. \]

Furthermore, the optimality of the lower bounds in (i)–(iv) can be verified by taking \( A = [-n, p] \setminus (0)_r \).

**Corollary 4.** Let \( k \geq 2, r \geq 1, \) and \( \alpha \in [0, rk - 1] \) be integers. Let \( m \in [1, k] \) be an integer such that \((m - 1)r \leq \alpha < mr\). Let \( A \) be a sequence of \( rk \) terms which is made up of \( n \) negative integers, \( p \) positive integers and zero, each repeated exactly \( r \) times.
(i) If \( m \leq n \) and \( m \leq p \), then \(|\Sigma_\alpha(A)| \geq r \left( \frac{n(n+1)}{2} + \frac{p(p+1)}{2} \right) + 1\).

(ii) If \( m \leq n \) and \( m > p \), then \(|\Sigma_\alpha(A)| \geq r \left( \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(m-p)(m-p-1)}{2} \right) + (m - p - 1)(mr - \alpha) + 1\).

(iii) If \( m > n \) and \( m \leq p \), then \(|\Sigma_\alpha(A)| \geq r \left( \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(m-n)(m-n-1)}{2} \right) + (m - n - 1)(mr - \alpha) + 1\).

(iv) If \( m > n \) and \( m > p \), then \(|\Sigma_\alpha(A)| \geq r \left( \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{(m-n)(m-n-1)}{2} - \frac{(m-p)(m-p-1)}{2} \right) + (2m - n - p - 2)(mr - \alpha) + 1\).

These lower bounds are optimal.

Proof. The lower bounds for \(|\Sigma_\alpha(A)|\) easily follows from Theorem 4 and the fact that \( \Sigma_\alpha(A) = \Sigma_0(A') \) for \( 0 \leq \alpha < r \) and \( \Sigma_\alpha(A) = \Sigma_{\alpha-r}(A') \) for \( r \leq \alpha < rk \), where \( A' = A \setminus \{0\} \). Furthermore, the optimality of these bounds can be verified by taking \( A = [-n, p]_r \).

Corollary 5. Let \( k \geq 3 \), \( r \geq 1 \), and \( \alpha \in [0, rk - 1] \) be integers. Let \( m \in [1, k] \) be an integer such that \( (m - 1)r \leq \alpha < mr \). Let \( A \) be a sequence of \( rk \) terms which is made up of \( k \) distinct integers each repeated exactly \( r \) times. If \( 0 \not\in A \), then

\[ |\Sigma_\alpha(A)| \geq \begin{cases} r \left( \frac{(k+1)^2}{4} - \frac{m(m+1)}{2} \right) + 1 & \text{if } k \equiv 1 \pmod{2} \\ r \left( \frac{(k+1)^2-1}{4} - \frac{m(m+1)}{2} \right) + 1 & \text{if } k \equiv 0 \pmod{2} \end{cases} \]

If \( 0 \in A \), then

\[ |\Sigma_\alpha(A)| \geq \begin{cases} r \left( \frac{k^2-1}{4} - \frac{m(m-1)}{2} \right) + 1 & \text{if } k \equiv 1 \pmod{2} \\ r \left( \frac{k^2}{4} - \frac{m(m-1)}{2} \right) + 1 & \text{if } k \equiv 0 \pmod{2} \end{cases} \]

Proof. Note that

\[ \frac{rk(k+1)}{2} \geq \begin{cases} r \left( \frac{(k+1)^2}{4} + 1 \right) & \text{if } k \equiv 1 \pmod{2} \\ r \left( \frac{(k+1)^2-1}{4} + 1 \right) & \text{if } k \equiv 0 \pmod{2} \end{cases} \]

and

\[ \frac{rk(k-1)}{2} + 1 \geq \begin{cases} r \left( \frac{k^2-1}{4} + 1 \right) & \text{if } k \equiv 1 \pmod{2} \\ r \left( \frac{k^2}{4} + 1 \right) & \text{if } k \equiv 0 \pmod{2} \end{cases} \]

for \( k \geq 3 \). If \( 0 \not\in A \) and \(|\Sigma_\alpha(A)| \geq r \left( \frac{k(k+1)}{2} - \frac{m(m+1)}{2} \right) + m(mr - \alpha) + 1 \), then we are done. So, let \(|\Sigma_\alpha(A)| < r \left( \frac{k(k+1)}{2} - \frac{m(m+1)}{2} \right) + m(mr - \alpha) + 1 \) when \( 0 \not\in A \). Then,
Theorem 6 implies that $\mathcal{A}$ contains both positive and negative integers. By similar arguments, when $0 \in \mathcal{A}$ also, we can assume that $\mathcal{A}$ contains both positive and negative integers. So, in both the cases $0 \in \mathcal{A}$ and $0 \notin \mathcal{A}$, we can assume that $\mathcal{A}$ contains both positive and negative integers. Let $A_n$ and $A_p$ be sets that contain respectively, all distinct negative terms and all distinct positive terms of sequence $\mathcal{A}$. Let also $|A_n| = n$ and $|A_p| = p$. Then $n \geq 1$ and $p \geq 1$.

Case 1. $(0 \notin \mathcal{A})$. By Theorem 7, we have

$$|\Sigma_\alpha(\mathcal{A})| \geq r \left( \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{m(m+1)}{2} \right) + 1$$

for all $\alpha \in [0, rk - 1]$. Therefore

$$|\Sigma_\alpha(\mathcal{A})| \geq r \left( \frac{n(n+1)}{2} + \frac{(k-n)(k-n+1)}{2} - \frac{m(m+1)}{2} \right) + 1$$

$$= r \left( \left( n - \frac{k}{2} \right)^2 + \frac{k^2 + 2k}{4} - \frac{m(m+1)}{2} \right) + 1.$$

Since $k = n + p$, without loss of generality we may assume that $n \geq \lceil k/2 \rceil$. If $k \equiv 1 \pmod{2}$, then $k = 2t + 1$ for some positive integer $t$. Hence

$$|\Sigma_\alpha(\mathcal{A})| \geq r \left( \left( n - t - \frac{1}{2} \right)^2 + \frac{k^2 + 2k}{4} - \frac{m(m+1)}{2} \right) + 1$$

$$= r \left( (n-t)(n-t-1) + \frac{(k+1)^2}{4} - \frac{m(m+1)}{2} \right) + 1$$

$$\geq r \left( \frac{(k+1)^2}{4} - \frac{m(m+1)}{2} \right) + 1.$$ 

If $k \equiv 0 \pmod{2}$, then $k = 2t$ for some positive integer $t$. Without loss of generality we may assume that $n \geq t$. Hence

$$|\Sigma_\alpha(\mathcal{A})| \geq r \left( (n-t)^2 + \frac{k^2 + 2k}{4} - \frac{m(m+1)}{2} \right) + 1$$

$$\geq r \left( \frac{(k+1)^2 - 1}{4} - \frac{m(m+1)}{2} \right) + 1.$$

Case 2. $(0 \in \mathcal{A})$. By Corollary 4, we have

$$|\Sigma_\alpha(\mathcal{A})| \geq r \left( \frac{n(n+1)}{2} + \frac{p(p+1)}{2} - \frac{m(m-1)}{2} \right) + 1$$

for all $\alpha \in [0, rk - 1]$. Therefore

$$|\Sigma_\alpha(\mathcal{A})| \geq r \left( \left( n - \frac{k-1}{2} \right)^2 + \frac{k^2 - 1}{4} - \frac{m(m-1)}{2} \right) + 1$$
Since \( k = n + p + 1 \), without loss of generality we may assume that \( n \geq \lceil (k - 1)/2 \rceil \). If \( k \equiv 1 \pmod{2} \), then \( k = 2t + 1 \) for some positive integer \( t \). Hence

\[
|\Sigma_\alpha(A)| \geq r \left( (n - t)^2 + \frac{k^2 - 1}{4} - \frac{m(m - 1)}{2} \right) + 1
\]

\[
\geq r \left( \frac{k^2 - 1}{4} - \frac{m(m - 1)}{2} \right) + 1.
\]

If \( k \equiv 0 \pmod{2} \), then \( k = 2t \) for some positive integer \( t \). Hence

\[
|\Sigma_\alpha(A)| \geq r \left( \left( n - t + \frac{1}{2} \right)^2 + \frac{k^2 - 1}{4} - \frac{m(m - 1)}{2} \right) + 1
\]

\[
= r \left( (n - t)(n - t + 1) + \frac{k^2 - 1}{4} - \frac{m(m - 1)}{2} \right) + 1
\]

\[
\geq r \left( \frac{k^2}{4} - \frac{m(m - 1)}{2} \right) + 1.
\]

This completes the proof of the corollary.

**Remark 5.** Mistri and Pandey [30] have already proved this corollary for \( \alpha = 1 \). The purpose of this corollary is to prove a similar result for every \( \alpha \in [0, rk - 1] \). Note that the lower bounds in Corollary 5 are not optimal for all \( \alpha \in [0, rk - 1] \), except for \( \alpha = 0 \) and \( \alpha = 1 \).

4. Open problems

1. Along this line, it is important to find the optimal lower bound for \( |\Sigma_{\alpha}(A)| \), for arbitrary finite sequence of integers \( A = (a_1, a_2, \ldots, \underbrace{a_1, \ldots, a_1}_{r_1 \text{ copies}}, a_2, \ldots, \underbrace{a_k, \ldots, a_k}_{r_k \text{ copies}}) \).

When the sequence \( A \) contains nonnegative or nonpositive integers, we already have the optimal lower bound for \( |\Sigma_{\alpha}(A)| \) (see [13]). So, the only case that remains to study is when the sequence \( A \) contains both positive and negative integers. Note that, in this paper we settled this problem in the special case \( r_i = r \) for all \( i = 1, 2, \ldots, k \).

2. It is also an important problem to study the structure of the sequence \( A \) for which the lower bound for \( |\Sigma_{\alpha}(A)| \) is optimal. When \( A \) contains nonnegative or nonpositive integers this problem has already been established (see [13]). So, it remains to solve this problem when the sequence \( A \) contains both positive and negative integers.
3. For a finite set $H$ of nonnegative integers and a finite set $A$ of $k$ integers, define the sumsets

$$HA := \bigcup_{h \in H} hA, \quad H \cdot A := \bigcup_{h \in H} h \cdot A \quad \text{and} \quad H^{(r)} A := \bigcup_{h \in H} h^{(r)} A.$$ 

Then $H \cdot A = \Sigma_{\alpha}(A)$ for $H = [\alpha, k]$, $H \cdot A = \Sigma_{\alpha}(A)$ for $H = [0, k - \alpha]$, $H^{(r)} A = \Sigma_{\alpha}(A)$ for $H = [\alpha, rk]$, and $H^{(r)} A = \Sigma_{\alpha}(A)$ for $H = [0, rk - \alpha]$, where $A = (A)_r$. Along the same line with the sumsets $hA$, $h \cdot A$, and $\Sigma_{\alpha}(A)$, the first author established optimal lower bounds for $|HA|$ and $|H \cdot A|$, when $A$ contains nonnegative or nonpositive integers (see [9]). The author also characterized the sets $H$ and $A$ for which the lower bounds are achieved [9]. It will be interesting to generalize such results to the sumset $H^{(r)} A$.

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