A NOTE ON THE LUMER–PHILLIPS THEOREM FOR BI-CONTINUOUS SEMIGROUPS

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ABSTRACT. Given a Banach space $X$ and an additional coarser Hausdorff locally convex topology $\tau$ on $X$ we characterise the generators of $\tau$-bi-continuous semigroups in the spirit of the Lumer–Phillips theorem, i.e. by means of dissipativity w.r.t. a directed system of seminorms and a range condition.

1. Introduction

Characterising generators of strongly continuous semigroups on Banach spaces is a classical topic due to its relation to well-posedness of the corresponding abstract Cauchy problem [8, Chap. II, 6.7 Theorem, p. 150]. The two main generation theorems go back to Hille–Yosida [16, 34] and Feller–Miyadera–Phillips [14, 28, 29] for general semigroups and Lumer–Phillips [25] for contraction semigroups.

However, there are important examples of semigroups which are not strongly continuous for the Banach space norm, e.g. the Gauß–Weierstraß semigroup on $C_b(\mathbb{R}^d)$. To circumvent this issue the concept of bi-continuous semigroups which are strongly continuous only w.r.t. to a weaker Hausdorff locally convex topology $\tau$ has been introduced by Kühnemund [22]. Thus a natural question is to characterise the generators of bi-continuous semigroups in the spirit of the Hille–Yosida theorem and the Lumer–Phillips theorem. While a version of the Hille–Yosida theorem for bi-continuous semigroups was established directly at the beginning of the theory [23], a corresponding version of the Lumer–Phillips theorem for bi-continuous contraction semigroups was missing. Recently, in [5], Budde and Wegner introduced the notion of bi-dissipativity to characterise the generators of bi-continuous contraction semigroups. However, the result in [5] does not cover the key example of the Gauß–Weierstraß semigroup on $C_b(\mathbb{R}^d)$ (see [3, Example 3.9, p. 8]), see also Remark 3.25 below.

In this paper we make use of the notion of $\Gamma$-dissipativity, where $\Gamma$ is a directed set of seminorms generating a topology related to the mixed topology $\gamma := \gamma(\cdot, \tau)$ (see [33] and Definition 2.11 for the mixed topology), introduced in [1] to prove versions of the Lumer–Phillips theorem for bi-continuous (contraction) semigroups in Theorem 3.9 and Theorem 3.10. Note that also [3] used $\Gamma$-dissipativity to define bi-dissipativity; however, there $\Gamma$ is a fundamental system of seminorms generating the topology $\tau$. Working instead with the mixed topology yields a more natural concept which can also be applied to the Gauß–Weierstraß semigroup, see Remark 3.25.

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Let us mention that strongly continuous semigroups have also been considered in locally convex spaces, in particular including a Lumer–Phillips-type generation theorem, see [1].

Note that with these results we also answer a question on characterising generators of transition semigroups raised by Markus Kunze, see Problem 3 in [12, p. 4].

In Section 2 we review the notion of bi-continuous semigroups and their generators as well as the (sub-)mixed topology. We also recall further properties of locally convex spaces as well as operators which we make use of later on. In Section 3 we recall the notion of Γ-dissipativity and then prove the version of the Lumer–Phillips theorem, where we also comment on its relation to [5]. Further, we also provide some examples.

2. Notions and preliminaries

In this short section we recall some basic notions and results in the context of bi-continuous semigroups. For a vector space $X$ over the field $\mathbb{R}$ or $\mathbb{C}$ with a Hausdorff locally convex topology $\tau$ we denote by $(X, \tau)'$ the topological linear dual space and just write $X' = (X, \tau)'$ if $(X, \tau)$ is a Banach space. By $\Gamma_\tau$ we always denote a directed system of continuous seminorms that generates the Hausdorff locally convex topology $\tau$ on $X$. Further, for two Hausdorff locally convex spaces $(X, \tau)$ and $(Y, \sigma)$ we use the symbol $\mathcal{L}((X, \tau); (Y, \sigma))$ for the space of continuous linear operators from $(X, \tau)$ to $(Y, \sigma)$, and abbreviate $\mathcal{L}(X, \tau) = \mathcal{L}((X, \tau); (X, \tau))$. If $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ are Banach spaces, we denote by $\tau_{\| \cdot \|_X}$ and $\tau_{\| \cdot \|_Y}$ the corresponding topologies induced by the norms and just write $\mathcal{L}(X; Y) = \mathcal{L}((X, \tau_{\| \cdot \|_X}); (Y, \tau_{\| \cdot \|_Y}))$ with operator norm $\| \cdot \|_{\mathcal{L}(X; Y)}$, and $\mathcal{L}(X) = \mathcal{L}(X; X)$.

Let us recall the definition of the mixed topology, [33, Section 2.1], and the notion of a Saks space, [2, 1.3.2 Definition, p. 27–28], which will be important for the rest of the paper.

2.1. Definition ([18, Definition 2.2, p. 3], [3, Proposition 3.11 (a), p. 9]). Let $(X, \| \cdot \|)$ be a Banach space and $\tau$ a Hausdorff locally convex topology on $X$ that is coarser than the $\| \cdot \|$-topology $\tau_{\| \cdot \|}$. Then

(a) the mixed topology $\gamma = \gamma(\| \cdot \|, \tau)$ is the finest linear topology on $X$ that coincides with $\tau$ on $\| \cdot \|$-bounded sets and such that $\tau \subseteq \gamma \subseteq \tau_{\| \cdot \|};$

(b) a directed system of continuous seminorms $\Gamma_\tau$ that generates the topology $\tau$ is called norming if

$$\| x \| = \sup_{p \in \Gamma_\tau} p(x), \quad x \in X; \tag{1}$$

(c) the triple $(X, \| \cdot \|, \tau)$ is called a Saks space if there exists a norming directed system of continuous seminorms $\Gamma_\tau$ that generates the topology $\tau$.

The mixed topology is actually Hausdorff locally convex and the definition given above is equivalent to the one introduced by Wiweger [33, Section 2.1] due to [33, Lemmas 2.2.1, 2.2.2, p. 51].

2.2. Definition ([20, Definitions 2.2, 5.4, p. 2, 8]). Let $(X, \| \cdot \|, \tau)$ be a Saks space.

(a) We call $(X, \| \cdot \|, \tau)$ (sequentially) complete if $(X, \gamma)$ is (sequentially) complete.

(b) We call $(X, \| \cdot \|, \tau)$ semi-reflexive if $(X, \gamma)$ is semi-reflexive.

(c) We call $(X, \| \cdot \|, \tau)$ C-sequential if $(X, \gamma)$ is C-sequential, i.e. every convex sequentially open subset of $(X, \gamma)$ is already open (see [33, p. 273]).

2.3. Remark. If $(X, \| \cdot \|, \tau)$ is a sequentially complete Saks space, then $(X, \| \cdot \|, \gamma)$ is also a sequentially complete Saks space by [21, Lemma 5.5, p. 2680–2681] and
Remark 2.3 (c), p. 3. In particular, there exists a norming directed system of continuous seminorms \( \Gamma \gamma \) that generates \( \gamma \).

There is another kind of mixed topology (see [6, p. 41]) which becomes quite handy if one has to deal with the mixed topology because it is generated by a quite simple directed system of continuous seminorms and often coincides with the mixed topology.

2.4. Definition ([18, Definition 3.9, p. 9]). Let \((X, \| \cdot \|, \tau)\) be a Saks space and \( \Gamma \gamma \) a norming directed system of continuous seminorms that generates the topology \( \tau \). We set

\[ \mathcal{N} := \{ (p_n, a_n)_{n \in \mathbb{N}} \mid (p_n)_{n \in \mathbb{N}} \subseteq \Gamma \gamma, (a_n)_{n \in \mathbb{N}} \in \mathcal{C}_0, a_n \geq 0 \text{ for all } n \in \mathbb{N} \} \]

where \( \mathcal{C}_0 \) is the space of real null-sequences. For \((p_n, a_n)_{n \in \mathbb{N}} \in \mathcal{N}\) we define the seminorm

\[ \| x \|_{(p_n, a_n)_{n \in \mathbb{N}}} := \sup_{n \in \mathbb{N}} (p_n(x) a_n), \quad x \in X. \]

We denote by \( \gamma_s := \gamma_s(\| \cdot \|, \tau) \) the Hausdorff locally convex topology that is generated by the system of seminorms \( \{ \| x \|_{(p_n, a_n)_{n \in \mathbb{N}}} \}_{(p_n, a_n)_{n \in \mathbb{N}} \in \mathcal{N}} \) and call it the submixed topology.

Due to [6, I.1.10 Proposition, p. 9], [6, I.4.5 Proposition, p. 41–42] and [11, Lemma A.1.2, p. 72] we have the following observation.

2.5. Remark ([18, Remark 3.10, p. 9]). Let \((X, \| \cdot \|, \tau)\) be a Saks space, \( \Gamma \gamma \) a norming directed system of continuous seminorms that generates the topology \( \tau \), \( \gamma = \gamma(\| \cdot \|, \tau) \) the mixed and \( \gamma_s = \gamma_s(\| \cdot \|, \tau) \) the submixed topology.

(a) We have \( \tau \subseteq \gamma_s \subseteq \gamma \) and \( \gamma_s \) has the same convergent sequences as \( \gamma \).

(b) If

(i) for every \( x \in X, \varepsilon > 0 \) and \( p \in \Gamma \gamma \) there are \( y, z \in X \) such that \( x = y + z \),

\[ p(z) = 0 \text{ and } \| y \| \leq p(x) + \varepsilon, \]

or

(ii) the \( \| \cdot \|\)-unit ball \( B_{\| \cdot \|} = \{ x \in X \mid \| x \| \leq 1 \} \) is \( \tau \)-compact,

then \( \gamma = \gamma_s \) holds.

The submixed topology \( \gamma_s \) was originally introduced in [33, Theorem 3.1.1, p. 62] where a proof of Remark 2.5 (b) can be found as well.

2.6. Remark. Let \((X, \| \cdot \|, \tau)\) be a Saks space and \( \Gamma \gamma \) a norming directed system of continuous seminorms that generates the topology \( \tau \). Then there is a norming directed system of continuous seminorms \( \Gamma \gamma \) that generates the submixed topology \( \gamma_s = \gamma_s(\| \cdot \|, \tau) \). Indeed, we set

\[ \mathcal{N}_1 := \{ (p_n, a_n)_{n \in \mathbb{N}} \mid (p_n)_{n \in \mathbb{N}} \subseteq \Gamma \gamma, (a_n)_{n \in \mathbb{N}} \in \mathcal{C}_0, 0 \leq a_n \leq 1 \text{ for all } n \in \mathbb{N} \} \]

and \( \Gamma \gamma := \{ \| x \|_{(p_n, a_n)_{n \in \mathbb{N}}} \mid (p_n, a_n)_{n \in \mathbb{N}} \in \mathcal{N}_1 \} \). Let \((p_n)_{n \in \mathbb{N}} \subseteq \Gamma \gamma \) and \((a_n)_{n \in \mathbb{N}} \in \mathcal{C}_0\) with \( a_n \geq 0 \) for all \( n \in \mathbb{N} \). Then \( C := \sup_{n \in \mathbb{N}} a_n < \infty \) and w.l.o.g. \( C > 0 \). We have

\[ \| x \|_{(p_n, a_n)_{n \in \mathbb{N}}} = \sup_{n \in \mathbb{N}} p_n(x) a_n \leq C \sup_{n \in \mathbb{N}} p_n(x) a_n/C = C \| x \|_{(p_n, a_n)_{n \in \mathbb{N}}} \]

for all \( x \in X \). In combination with \( \mathcal{N}_1 \subseteq \mathcal{N} \) this shows that \( \Gamma \gamma \) generates \( \gamma_s \). Furthermore, for every \( p \in \Gamma \gamma \) we have with \( p_n := p \) for all \( n \in \mathbb{N} \) that \( p(x) \leq \| x \|_{(p_n, 1/n)_{n \in \mathbb{N}}} \) for all \( x \in X \). Together with the norming property of \( \Gamma \gamma \) this implies

\[ \| x \| = \sup_{p \in \Gamma \gamma} p(x) \leq \sup_{(p_n, a_n)_{n \in \mathbb{N}} \in \mathcal{N}_1} \| x \|_{(p_n, a_n)_{n \in \mathbb{N}}} \leq \sup_{(a_n)_{n \in \mathbb{N}} \in \mathcal{C}_0, 0 \leq a_n \leq 1} \| x \|_{(a_n)_{n \in \mathbb{N}}} \leq \| x \| \]

for all \( x \in X \). Hence \( \Gamma \gamma \) is norming.
Recall that for a Banach space $(X, \| \cdot \|)$ and a Hausdorff locally convex topology $\tau$ on $X$ the triple $(X, \| \cdot \|, \tau)$ is called bi-admissible space if $\tau$ is coarser than $\tau_{1\| \cdot \|}$. $\tau$ is sequentially complete on the $\| \cdot \|$-closed unit ball (or equivalently on $\| \cdot \|$-bounded sets), and $(X, \tau')$ is norming for $X$; cf. [3, Assumption 2.1, p. 3].

2.7. Lemma. Let $(X, \| \cdot \|)$ be a Banach space and $\tau$ a Hausdorff locally convex topology on $X$ that is coarser than the $\| \cdot \|$-topology $\tau_{1\| \cdot \|}$. Then the following are equivalent:

(i) $(X, \| \cdot \|, \tau)$ is a sequentially complete Saks space.

(ii) $(X, \| \cdot \|, \tau)$ is a bi-admissible space.

Proof. By [33, Corollary 2.3.2, p. 55] a Saks space $(X, \| \cdot \|, \tau)$ is sequentially complete if and only if $(X, \tau)$ is sequentially complete on $\| \cdot \|$-bounded sets, meaning that every $\| \cdot \|$-bounded $\tau$-Cauchy sequence converges in $X$. Combined with [18, Remark 2.3 (c), p. 3] it follows that a triple $(X, \| \cdot \|, \tau)$ fulfills [23, Assumptions 1, p. 206], which provides a bi-admissible space, if and only if it is a sequentially complete Saks space.

Let us recall the notion of a bi-continuous semigroup.

2.8. Definition ([23, Definition 3, p. 207]). Let $(X, \| \cdot \|, \tau)$ be a sequentially complete Saks space. A family $(T(t))_{t \geq 0}$ in $\mathcal{L}(X)$ is called $\tau$-bi-continuous semigroup if

(i) $(T(t))_{t \geq 0}$ is a semigroup, i.e. $T(t+s) = T(t)T(s)$ and $T(0) = \text{id}$ for all $t, s \geq 0$,

(ii) $(T(t))_{t \geq 0}$ is $\tau$-strongly continuous, i.e. the map $T_x : [0, \infty) \to (X, \tau)$, $T_x(t) := T(t)x$, is continuous for all $x \in X$,

(iii) $(T(t))_{t \geq 0}$ is exponentially bounded (of type $\omega$), i.e. there exists $M \geq 1$, $\omega \in \mathbb{R}$ such that $\| T(t) \|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$,

(iv) $(T(t))_{t \geq 0}$ is locally bi-equicontinuous, i.e. for every sequence $(x_n)_{n \in \mathbb{N}}$ in $X$, $x \in X$ with $\sup_{n \in \mathbb{N}} \| x_n \| < \infty$ and $\tau$- $\lim_{n \to \infty} x_n = x$ it holds that

$$\tau\lim_{n \to \infty} T(t)(x_n - x) = 0$$

locally uniformly for all $t \in [0, \infty)$. 

2.9. Remark. Let $(X, \| \cdot \|, \tau)$ be a Saks space.

(a) A sequence in $X$ is $\gamma$-convergent if and only if it is $\| \cdot \|$-bounded and $\tau$-convergent by [1, L.1.10 Proposition, p. 9].

(b) Let $(X, \| \cdot \|, \tau)$ be sequentially complete. A semigroup of linear operators $(T(t))_{t \geq 0}$ from $X$ to $X$ is $\gamma$-strongly continuous and locally sequentially $\gamma$-equicontinuous (i.e. for all $\gamma$-null sequences $(x_n)_{n \in \mathbb{N}}$ in $X$, $t_0 > 0$ and $p \in \Gamma_\gamma$ we have $\lim_{n \to \infty} \sup_{t \in [0, t_0]} p(T(t)x_n) = 0$) if and only if it is a $\tau$-bi-continuous semigroup on $X$. This follows directly from part (a) and [18, Remark 2.6 (b), p. 5], and remains true if $\gamma$ is replaced by any other Hausdorff locally convex topology on $X$ that has the same convergent sequences as $\gamma$ (cf. [11, Proposition A.1.3, p. 73] for $\gamma$ replaced by $\gamma_a$).

We already observed in Remark 2.9 (b) that $\tau$-bi-continuous semigroups are locally sequentially $\gamma$-equicontinuous. Under some mild conditions on the Saks space $(X, \| \cdot \|, \tau)$ they are even quasi-$\gamma$-equicontinuous. Let us recall what that means.

2.10. Definition. Let $(X, v)$ be a Hausdorff locally convex space and $\Gamma_v$ a directed system of continuous seminorms that generates $v$. A family $(T(t))_{t \in I}$ of linear maps...
from $X$ to $X$ is called $v$-equicontinuous if
\[ \forall p \in \Gamma_v \exists \tilde{p} \in \Gamma_v, \ C \geq 0 \ \forall t \in I, \ x \in X : \ p(T(t)x) \leq C\tilde{p}(x). \]
The family $(T(t))_{t \geq 0}$ is called locally $v$-equicontinuous if $(T(t))_{t \in [0,t_0]}$ is $v$-equicontinuous for all $t_0 \geq 0$. The family $(T(t))_{t \geq 0}$ is called quasi-$v$-equicontinuous if there is $\alpha \in \mathbb{R}$ such that $(e^{-\alpha t}T(t))_{t \geq 0}$ is $v$-equicontinuous. Note that one often drops the $v$ if the topology is clear.

2.11. Remark. Let $(X, \cdot, \| \cdot \|, \tau)$ be a sequentially complete Saks space. Due to Remark 2.9 (b) a $\gamma$-strongly continuous, locally $\gamma$-equicontinuous semigroup of linear operators $(T(t))_{t \geq 0}$ from $X$ to $X$ is a $\tau$-bi-continuous semigroup on $X$. The converse is not true in general by [13, Example 4.1, p. 101]. Even more is true, namely, that every $\tau$-bi-continuous semigroup on $X$ is quasi-$\gamma$-equicontinuous if $(X, \cdot, \| \cdot \|, \tau)$ is C-sequential.

There is another related notion to equicontinuity on Saks spaces.

2.12. Definition ([13, Definitions 3.4, 3.5, p. 6, 7]). Let $(X, \cdot, \| \cdot \|, \tau)$ be a Saks space and $\Gamma_\tau$ a directed system of continuous seminorms generating the topology $\tau$. A family of linear maps $(T(t))_{t \in I}$ from $X$ to $X$ is called $(\| \cdot \|, \tau)$-tight if
\[ \forall \varepsilon > 0, \ p \in \Gamma_\tau \exists \tilde{p} \in \Gamma_\tau, \ C \geq 0 \ \forall t \in I, \ x \in X : \ p(T(t)x) \leq C\tilde{p}(x) + \varepsilon \|x\|. \]
The family $(T(t))_{t \in I}$ is called $(\| \cdot \|, \tau)$-equitight if $(T(t))_{t \in [0,t_0]}$ is $(\| \cdot \|, \tau)$-tight for all $t_0 \geq 0$. The family $(T(t))_{t \geq 0}$ is called quasi-$(\| \cdot \|, \tau)$-equitight if there is $\alpha \in \mathbb{R}$ such that $(e^{-\alpha t}T(t))_{t \geq 0}$ is $(\| \cdot \|, \tau)$-equitight.

At first, tight operators $T \in \mathcal{L}(X)$ as well as families of equitight operators $(T(t))_{t \in I}$ in $\mathcal{L}(X)$ for $t_0 \geq 0$ appeared in [11, Definitions 1.2.20, 1.2.21, p. 12] under the name local. In the setting of $\tau$-bi-continuous semigroups $(T(t))_{t \geq 0}$ the notion of tightness is used in [9, Definition 1.1, p. 668], meaning that $(T(t))_{t \in [0,t_0]}$ is equitight (or local) for all $t_0 \geq 0$. Local equitightness plays an important role in perturbation results for bi-continuous semigroups, see e.g. [9, Theorem 1.2, p. 669], [12, Theorems 2.4, 3.2, p. 92, 94–95], [12, Remark 4.1, p. 101], [2, Theorem 5, p. 8], [3, Theorem 3.3, p. 582], and the corrections regarding 2 in [18, Remark 3.8, p. 8–9].

Due to [18, Proposition 3.16, p. 12–13] $(\| \cdot \|, \tau)$-equitightness of a family of linear maps $(T(t))_{t \in I}$ from $X$ to $X$ implies $\gamma$-equicontinuity. If $(X, \| \cdot \|, \tau)$ is a sequentially complete C-sequential Saks space and $\gamma = \gamma_\tau$, then any $\tau$-bi-continuous semigroup $(T(t))_{t \geq 0}$ on $X$ is quasi-$\gamma$-equicontinuous and quasi-$(\| \cdot \|, \tau)$-equitight by [18, Theorem 3.17, p. 13], and both properties are equivalent by [18, Proposition 3.16, p. 12–13].

We close this section by recalling the definition of the generator of a $\tau$-bi-continuous semigroup and two of its properties which we will need.

2.13. Definition ([11, Definition 1.2.6, p. 7]). Let $(X, \cdot, \| \cdot \|, \tau)$ be a sequentially complete Saks space and $(T(t))_{t \geq 0}$ a $\tau$-bi-continuous semigroup on $X$. The generator $(A, D(A))$ is defined by
\[ D(A) := \{ x \in X \mid \tau \lim_{t \to 0_+} \frac{T(t)x - x}{t} \text{ exists in } X \text{ and } \sup_{t \in (0,1]} \frac{|T(t)x - x|}{t} < \infty \}; \]
\[ Ax := \tau \lim_{t \to 0_+} \frac{T(t)x - x}{t}, \quad x \in D(A). \]

2.14. Proposition ([23, Corollary 13, p. 215]). Let $(X, \cdot, \| \cdot \|, \tau)$ be a sequentially complete Saks space and $(T(t))_{t \geq 0}$ a $\tau$-bi-continuous semigroup on $X$ with generator $(A, D(A))$. Then the following assertions hold:
The generator \((A, D(A))\) is bi-closed, i.e. whenever \((x_n)_{n \in \mathbb{N}}\) is a sequence in \(D(A)\) such that \(\tau\)-\lim_{n \to \infty} x_n = x\) and \(\tau\)-\lim_{n \to \infty} Ax_n = y\) for some \(x, y \in X\) and both sequences are \(\| \cdot \|\)-bounded, then \(x \in D(A)\) and \(Ax = y\).

The generator \((A, D(A))\) is bi-densely defined, i.e. for each \(x \in X\) there exists a \(\| \cdot \|\)-bounded sequence \((x_n)_{n \in \mathbb{N}}\) in \(D(A)\) such that \(\tau\)-\lim_{n \to \infty} x_n = x\).

3. Lumer–Phillips for bi-continuous semigroups

First, we recall the relevant notions from [1] concerning dissipative linear operators on Hausdorff locally convex spaces. We write in short that \((A, D(A))\) is a linear operator on a Hausdorff locally convex space \(X\) if \(A : D(A) \subseteq X \to X\) is a linear operator.

3.1. Definition ([1], Definitions 3.1, 3.5, p. 923). Let \((X, v)\) be a Hausdorff locally convex space and \((A, D(A))\) a linear operator on \(X\).

\(a\) \((A, D(A))\) is called \(v\)-closed if for each net \((x_i)_{i \in I} \subseteq D(A)\) satisfying \(x_i \to x\) and \(Ax_i \to y\) w.r.t. \(v\) for some \(x, y \in X\), we have \(x \in D(A)\) and \(Ax = y\).

\(b\) A linear operator \((B, D(B))\) on \(X\) is called an extension of \((A, D(A))\) if \(D(A) \subseteq D(B)\) and \(B|_{D(A)} = A\). The operator \((A, D(A))\) is called \(v\)-closable if it admits a \(v\)-closed extension. The smallest \(v\)-closed extension of an \(v\)-closable operator \((A, D(A))\) is called the \(v\)-closure of \((A, D(A))\) and denoted by \((\overline{A}, D(\overline{A}))\).

\(c\) \((A, D(A))\) is called \(\langle\text{sequentially}\rangle\) \(v\)-densely defined if \(D(A)\) is \(\langle\text{sequentially}\rangle\) \(v\)-dense in \(X\).

\(d\) Let \((A, D(A))\) be \(\langle\text{v}\rangle\)-densely defined. The \(v\)-dual operator \((A', D(A'))\) of \((A, D(A))\) on \((X, v)\) is defined by setting
\[
D(A') = \{ x' \in (X, v)' \mid \exists y' \in (X, v)' \forall x \in D(A) : \{ Ax, x' \} = \{ x, y' \} \}
\]
and \(A'x' = y'\) for \(x' \in D(A')\).

We have the following relation to the notions of a \(\tau\)-closed resp. \(\tau\)-densely defined operator.

3.2. Remark. Let \((X, \| \cdot \|, \tau)\) be a Saks space and \((A, D(A))\) a linear operator. Due to Remark 2.14(a) \((A, D(A))\) is \(\tau\)-closed (see Proposition 2.14(a)) if and only if it is \(\langle\text{sequentially}\rangle\) \(\tau\)-closed (which is defined analogously to \(\tau\)-closedness but with nets replaced by sequences). Again by Remark 2.14(a) \((A, D(A))\) is \(\langle\text{bi-closed}\rangle\) defined (see Proposition 2.14(b)) if and only if it is \(\langle\text{sequentially}\rangle\) \(\tau\)-closed defined. Moreover, if \((A, D(A))\) is \(\langle\text{bi-closed}\rangle\) defined, then it is obviously \(\tau\)-closed defined, too (as every sequence is a net).

3.3. Definition ([1], p. 922]). Let \((X, v)\) be a Hausdorff locally convex space and \((A, D(A))\) a linear operator on \(X\). If \(\lambda \in \mathbb{C}\) is such that \(\lambda - A = \lambda \text{id} - A : D(A) \to X\) is injective, then the linear operator \((\lambda - A)^{-1}\) exists and is defined on the domain \(\text{Ran}(\lambda - A) = \{ (\lambda - A)x \mid x \in D(A) \}\), i.e. the range of \(\lambda - A\). The resolvent set of \(A\) is defined by
\[
\rho_v(A) = \{ \lambda \in \mathbb{C} \mid \lambda - A \text{ is bijective and } (\lambda - A)^{-1} \in \mathcal{L}(X, v) \}.
\]
If \(v = \tau\) for a Banach space \((X, \| \cdot \|, \tau)\), we just write \(R(\lambda, A) := (\lambda - A)^{-1}\) and \(p(A) := \rho_v(A)\).

3.4. Definition ([1], Definition 3.9, p. 925]). Let \((X, v)\) be a Hausdorff locally convex space and \(\Gamma_v\) a directed system of continuous seminorms that generates \(v\). A linear operator \((A, D(A))\) on \(X\) is called \(\Gamma_v\)-dissipative if
\[
\forall \lambda > 0, x \in D(A), p \in \Gamma_v : p((\lambda - A)x) \geq \lambda p(x).
\]
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It is important to note that in contrast to equicontinuity or equitightness the notion of dissipativity depends on the selection of the directed system of continuous seminorms that generates the topology \( v \) by [1, Remark 3.10, p. 925–926].

3.5. **Remark.** Let \( (X, \| \cdot \|, \tau) \) be a Saks space, \( v \) a Hausdorff locally convex topology on \( X \) and \( (A, D(A)) \) a \( \Gamma_v \)-dissipative operator on \( X \). If \( \Gamma_v \) is norming, then it follows from the \( \Gamma_v \)-dissipativity and \( (1) \) that

\[
\forall \lambda > 0, x \in D(A) : \| (\lambda - A)x \| \geq \lambda \| x \|.
\]

Thus \( (A, D(A)) \) is also a dissipative operator on the Banach space \( (X, \| \cdot \|) \) in the sense of [8, Chap. II, 3.13 Definition, p. 82] (cf. [5, Remark 3.3 (i), p. 5] for \( v = \tau \)). We also denote such kind of dissipativity on a Banach space \( (X, \| \cdot \|) \) by \( \| \cdot \|_\text{dissipativity} \).

In [3] another notion of dissipativity on Saks spaces was introduced.

3.6. **Remark.** Let \( (X, \| \cdot \|, \tau) \) be a sequentially complete Saks space. In [5, Definition 3.2, p. 5] a linear operator \( (A, D(A)) \) on \( X \) is called bi-dissipative if there exists a norming directed system of continuous seminorms \( \Gamma_v \) that generates \( \tau \) such that \( (A, D(A)) \) is \( \Gamma_v \)-dissipative. It is then shown in the proof of [5, Theorem 3.15, p. 11] that a bi-dissipative operator \( (A, D(A)) \) is also \( (\| (p_n, a_n)_{n\in\mathbb{N}} \|_{(p_n, a_n)_{n\in\mathbb{N}}})_{(p_n, a_n)_{n\in\mathbb{N}} \in \mathcal{N}} \)-dissipative since

\[
\| (\lambda - A)x \|_{(p_n, a_n)_{n\in\mathbb{N}}} = \sup_{n \in \mathbb{N}} p_n((\lambda - A)x)a_n \geq \sup_{n \in \mathbb{N}} \lambda p_n(x)a_n = \lambda \| x \|_{(p_n, a_n)_{n\in\mathbb{N}}}
\]

for all \( \lambda > 0, x \in D(A) \) and \( (p_n, a_n)_{n\in\mathbb{N}} \in \mathcal{N} \), where \( (\| (p_n, a_n)_{n\in\mathbb{N}} \|_{(p_n, a_n)_{n\in\mathbb{N}}})_{(p_n, a_n)_{n\in\mathbb{N}} \in \mathcal{N}} \) is the system of seminorms that generates the submixed topology \( \gamma_{\mathcal{N}} \) from Definition [2.4]. We observe that this also implies that \( (A, D(A)) \) is \( \Gamma_{\gamma_{\mathcal{N}}} \)-dissipative w.r.t the norming directed system of continuous seminorms \( \Gamma_{\gamma_{\mathcal{N}}} \) that generates the submixed topology \( \gamma_{\mathcal{N}} \) from Remark [2.6].

3.7. **Proposition.** Let \( (X, \| \cdot \|, \tau) \) be a sequentially complete Saks space, \( v \) a Hausdorff locally convex topology on \( X \), \( (A, D(A)) \) a \( \Gamma_v \)-dissipative operator on \( X \). Then the following assertions hold:

(a) \( \lambda - A \) is injective for all \( \lambda > 0 \). Moreover, we have

\[
\forall \lambda > 0, x \in \text{Ran}(\lambda - A), p \in \Gamma_v : p((\lambda - A)^{-1}x) \leq \frac{1}{\lambda} p(x).
\]

(b) If \( \text{Ran}(\lambda - A) \) is (sequentially) \( v \)-closed for some \( \lambda > 0 \), then \( (A, D(A)) \) is (sequentially) \( v \)-closed. If \( v = \gamma \), then the converse even holds for all \( \lambda > 0 \).

(c) Let \( \Gamma_v \) be norming. Then \( \lambda - A \) is surjective for some \( \lambda > 0 \) if and only if it is surjective for all \( \lambda > 0 \). In such a case, \( (0, \infty) \subseteq \rho(A) \).

(d) Let \( v = \tau \). Then \( \lambda - A \) is surjective for some \( \lambda > 0 \) if and only if it is surjective for all \( \lambda > 0 \). In such a case, \( (0, \infty) \subseteq \rho_{\gamma}(A) \).

**Proof.** Parts (a), (b), and (d) are just [1, Proposition 3.11, p. 927] in combination with the sequential completeness of \( (X, \gamma) \). Part (c) is a consequence of [8, Chap. II, 3.14 Proposition (ii), p. 82] and Remark [3.7].

In the case \( v = \tau \) parts (a) and (c) of Proposition [3.7] are [5, Proposition 3.4, p. 6].

3.8. **Definition.** Let \( (X, \| \cdot \|) \) be a Banach space. We call a semigroup of linear operators \( (T(t))_{t \geq 0} \) from \( X \) to \( X \) a contraction semigroup if \( \| T(t) \|_{\mathcal{L}(X)} \leq 1 \) for all \( t \geq 0 \).
3.9. Theorem. Let \((X, \| \cdot \|, \tau)\) be a sequentially complete Saks space, \(v\) a Hausdorff locally convex topology on \(X\) with \(\tau \subseteq v \subseteq \tau_{\| \cdot \|}\) such that \(\gamma\)-convergent sequences are \(v\)-convergent, \((A, D(A))\) a bi-densely defined, \(\Gamma_v\)-dissipative operator on \(X\) and \(\Gamma_v\) norming. Then the following assertions are equivalent:

(a) \((A, D(A))\) generates a \(\tau\)-bi-continuous contraction semigroup on \(X\).

(b) \(\lambda - A\) is surjective for some \(\lambda > 0\).

Proof. We use the Hille–Yosida theorem for bi-continuous semigroups to prove both implications (see [23, Theorem 16, p. 217] and [3, Theorem 5.6, p. 340]).

(a)\(\Rightarrow\)(b): Let \((A, D(A))\) generate a \(\tau\)-bi-continuous contraction semigroup on \(X\). Due to [23, Theorem 16, p. 217] with \(\omega = 0\) we obtain that \((0, \infty) \subseteq \rho(A)\), in particular, that \(\lambda - A\) is surjective for all \(\lambda > 0\).

(b)\(\Rightarrow\)(a): Let \(\lambda - A\) be surjective for some \(\lambda > 0\). Due to [4, Theorem 5.6, p. 340] we only need to prove that

\[
\begin{align*}
(i) & \quad (0, \infty) \subseteq \rho(A), \\
(ii) & \quad \|R(\lambda, A)^n\| \leq \frac{1}{\lambda^n} \quad \text{for all } \lambda > 0, \quad n \in \mathbb{N}, \quad \text{and each } \| \cdot \|\text{-bounded } \tau\text{-null sequence } (x_m)_m \text{ in } X \text{ one has that } \\
(iii) & \quad \{\lambda A^n \mid \lambda > 0, \lambda \in \mathbb{N}\} \text{ is bi-equicontinuous for each } \lambda > 0, \quad \text{i.e. for each } \lambda > 0 \text{ and each } \| \cdot \|\text{-bounded } \tau\text{-null sequence } (x_m)_m \text{ in } X \text{ one has that } \tau\text{-lim}_{m \to \infty} (\lambda A^n)x_m = 0 \text{ uniformly for all } n \in \mathbb{N} \text{ and all } \lambda > 0.
\end{align*}
\]

Since \((A, D(A))\) is \(\Gamma_v\)-dissipative, we get that \(\lambda - A\) is bijective for all \(\lambda > 0\) and \(\rho(A) \subseteq (0, \infty)\) by Proposition 3.7 (a) and (c). From Remark 2.9 and \(\Gamma_v\), being norming we deduce that \(|R(\lambda, A)x| \leq \frac{1}{\lambda^n} \|x\| \quad \text{for all } \lambda > 0 \text{ and } x \in \text{Ran}(\lambda - A) = X\), yielding \(|R(\lambda, A)^n\| \leq \frac{1}{\lambda^n} \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda > 0\). Let \(\Gamma_\tau\) be a directed system of continuous seminorms that generates the topology \(\tau\) and \(q \in \Gamma_\tau\). Thanks to (2) we know that \(p((\lambda - A)^{-1}x) \leq \frac{1}{\lambda^n} p(x) \) for all \(p \in \Gamma_v\), \(\lambda > 0\) and \(x \in \text{Ran}(\lambda - A) = X\).

As \(\tau \subseteq v\), there are \(p \in \Gamma_v\) and \(C \geq 0\) such that for each \(\alpha > 0\) we have

\[
q((\lambda - \alpha)^n R(\lambda, A)^n x) \leq C(\lambda - \alpha)^n p((\lambda - A)^{-n} x) \leq C \left(1 - \frac{\alpha}{\lambda}\right)^n p(x) \leq Cp(x)
\]

for all \(x \in X\), \(n \in \mathbb{N}\) and \(\lambda > \alpha\). Since \(\| \cdot \|\text{-bounded } \tau\text{-null sequences are exactly the } \gamma\text{-null-sequences by Remark 2.9 (a) and } \gamma\text{-convergent sequences are assumed to be } v\text{-convergent, this inequality implies that } \{\lambda A^n \mid \lambda > 0, \lambda \in \mathbb{N}\} \text{ is bi-equicontinuous for all } \lambda > 0\). This finishes the proof. \(\square\)

In the case \(v = \tau\) we know that \(\gamma\)-convergent sequences are \(\tau\)-convergent and thus Theorem 3.9 is [3, Theorem 3.6, p. 6] (without the superfluous assumption that \(A\) should be norm-closed) in this case. Another possible choice is \(v = \gamma_s\) since the submixed topology \(\gamma_s\) has the same convergent sequences as \(\gamma\) by Remark 2.5 (a). However, we are mostly interested in the choice \(v = \gamma\). Our second generation result involves complete Saks spaces.

3.10. Theorem. Let \((X, \| \cdot \|, \tau)\) be a complete Saks space and \((A, D(A))\) a \(\gamma\)-densely defined, \(\Gamma_v\)-dissipative operator. Assume that \(\text{Ran}(\lambda - A)\) is \(\gamma\)-dense in \(X\) for some \(\lambda > 0\). Then the following assertions hold:

(a) The \(\gamma\)-closure \((\overline{A}, D(A))\) generates a \(\gamma\)-strongly continuous, \(\gamma\)-equicontinuous semigroup \((T(t))_{t \geq 0}\) on \(X\).

(b) If \(\Gamma_\gamma\) is norming, then \((T(t))_{t \geq 0}\) is a contraction semigroup.

(c) If \(\Gamma_\gamma\) is norming and \(\gamma = \gamma_s\), then \((T(t))_{t \geq 0}\) is \((\| \cdot \|, \tau)\)-equitight.

Proof. (a) Due to [1, Theorem 3.14, p. 929] \((\overline{A}, D(A))\) generates a \(\gamma\)-equicontinuous, \(\gamma\)-strongly continuous semigroup \((T(t))_{t \geq 0}\) on \(X\).
(b) By Proposition 3.13, p. 929 the operator $\overline{A}$, $D(\overline{A})$ is also $\gamma_s$-dissipative and $\lambda - \overline{A}$ is surjective for all $\lambda > 0$. As a consequence of part (a) and Remark 2.9 (b) $\lambda \overline{A}$ is also a $\tau$-bi-continuous semigroup on $X$. By p. 5 the generator $\overline{A}$ of $(\overline{A})_{t\geq 0}$ as a $\gamma_s$-strongly continuous, $\gamma_s$-equicontinuous semigroup (see p. 922) and the generator of $\overline{A}$ as a $\tau$-bi-continuous semigroup (see Definition 2.13) coincide. Thus $\overline{A}$ is bi-densely defined by Proposition 2.14 (b). Hence we get that $\overline{A}$ is a contraction semigroup by Theorem 3.9 with $\nu = \gamma_s$ and the norming property of $\gamma_s$.

(c) It follows from part (b) that $\|T(t)\|_{\mathcal{L}(X)} \leq 1$ for all $t \geq 0$. Further, $(\overline{A})_{t\geq 0}$ is $\gamma_s$-equicontinuous by part (a). In combination with $\gamma = \gamma_s$ we derive that $(\overline{A})_{t\geq 0}$ is $(\mathcal{L}(X), \mathcal{L}_\gamma)$-equiuniform by Proposition 3.16, p. 12–13].

Let us compare Theorem 3.11 with one of the main theorems of [5], namely, [5, Theorem 3.15, p. 11]. We note that the topology that is called mixed topology (and denoted by $\gamma$ there in [5, p. 10]) is actually the submixed topology $\gamma_s$. With this observation at hand let us phrase [5, Theorem 3.15, p. 11] in our terminology.

3.11. Theorem ([5, Theorem 3.15, p. 11]). Let $(X, \|\cdot\|, \tau)$ be a sequentially complete Saks space such that $(X, \gamma_s)$ is complete, and $(A, D(A))$ a bi-densely defined, bi-dissipative operator. Assume that $\text{Ran}(\lambda - A)$ is bi-dense, i.e. sequentially $\gamma_s$-dense, in $X$ for some $\lambda > 0$. Then the $\gamma_s$-closure $(\overline{\lambda}, D(\overline{\lambda}))$ generates a $\tau$-bi-continuous contraction semigroup on $X$.

3.12. Remark. (a) First, it is actually shown in the proof of Theorem 3.11 that $(\overline{\lambda}, D(\overline{\lambda}))$ generates a $\gamma_s$-strongly continuous, $\gamma_s$-equicontinuous semigroup on $X$, in particular, a $\tau$-bi-continuous semigroup by Remark 2.9 and Remark 2.5 (a). However, the proof that the generated semigroup is contractive is missing. In this proof it is used that a bi-dissipative operator is $(\|\cdot\|_{(p_0, a_0), \mathcal{A}}(\rho_0, a_0), \mathcal{A}_\gamma)$-dissipative as well (see Remark 3.6). In order to prove that the generated semigroup is also contractive, the only available tool in [5] is [5, Theorem 3.6, p. 6]. However, to apply the latter theorem one has to show that $(\overline{\lambda}, D(\overline{\lambda}))$ is also bi-dissipative. Due to Proposition 3.13, p. 929 we only know that $(\overline{\lambda}, D(\overline{\lambda}))$ is $(\|\cdot\|_{(p_0, a_0), \mathcal{A}}(\rho_0, a_0), \mathcal{A}_\gamma)$-dissipative (and $\lambda - \overline{A}$ is surjective for all $\lambda > 0$). To circumvent this obstacle, we relaxed [5, Theorem 3.6, p. 6] to Theorem 3.9 where one has several possible choices for the topology $\nu$, not only $\nu = \tau$ as in [5, Theorem 3.6, p. 6]. Using Remark 3.6 and Proposition 3.13, p. 929, we see that $(\overline{\lambda}, D(\overline{\lambda}))$ is $\Gamma_s$-dissipative with the norming directed system of continuous seminorms $\Gamma_s$ from Remark 2.6. Now, it is possible to apply Theorem 3.9 with $\nu = \gamma_s$ to conclude that the generated semigroup is contractive. This closes the gap in the proof of Theorem 3.11.

(b) There is no nice characterisation (known to us) of the completeness of $(X, \gamma_s)$, that is assumed in Theorem 3.11. However, there is a nice characterisation of the completeness of the Saks space $(X, \|\cdot\|, \tau)$. By definition the Saks space is complete if and only if $(X, \gamma)$ is complete. The space $(X, \gamma)$ is complete if and only if $B_{\|\cdot\|} = \{x \in X \mid \|x\| \leq 1\}$ is $\tau$-complete by [5, I.1.14, Proposition, p. 11]. But, since $\gamma_s$ is in general a weaker topology than $\gamma$ by Remark 2.5 (a), the completeness of $(X, \gamma_s)$ does in general not imply the completeness of $(X, \gamma)$. Let us suppose that $\gamma = \gamma_s$. Then Theorem 3.11 is covered by Theorem 3.10 (a) and (b). Further, we point out that in comparison to Theorem 3.11 we weakened the assumptions from $(A, D(A))$ being a bi-densely defined, bi-dissipative operator and $\text{Ran}(\lambda - A)$ being $\nu$-dense for some $\lambda > 0$ to
(A, D(A)) being a $\gamma$-densely defined, $\Gamma_\gamma$-dissipative operator and $\text{Ran}(\lambda - A)$ being $\gamma$-dense for some $\lambda > 0$ (see Remark 3.2 in Theorem 3.10).

Let us take a closer look at the completeness assumption on the Saks space $(X, \| \cdot \|, \tau)$ in Theorem 3.10 which is actually fulfilled for many important examples, and its characterisation in Remark 3.12 (b). Especially, $(X, \| \cdot \|, \tau)$ is complete, thus $(X, \| \cdot \|, \tau)$ as well, if $B_1$ is $\tau$-compact, which is condition (ii) of Remark 2.5 (b) and also a sufficient condition for $\gamma = \gamma_\tau$. We recall the following observations from Examples 2.4, 3.11, p. 4–5, 10, Remark 3.20 (a), p. 15, Example 4.12, p. 24–25] and Corollary 3.23, p. 17], and add a proof of the completeness of the Saks spaces considered in Remark 3.13 (c), (d) and (f) below.

3.13. Remark. (a) Let $\Omega$ be a Hausdorff $k_2$-space and recall that a completely regular space $\Omega$ is called $k_2$-space if any map $f: \Omega \to \mathbb{R}$ whose restriction to each compact $K \subset \Omega$ is continuous, is already continuous on $\Omega$ (see p. 487]). Further, let $C_0(\Omega)$ be the space of bounded continuous functions on $\Omega$, and $\| \cdot \|_\infty$ the sup-norm as well as $\tau_{co}$ the compact-open topology, i.e. the topology of uniform convergence on compact subsets of $\Omega$. Then $(C_0(\Omega), \| \cdot \|_\infty, \tau_{co})$ is a complete Saks space and $\gamma(\| \cdot \|_\infty, \tau_{co}) = \gamma_\tau(\| \cdot \|_\infty, \tau_{co})$.

Let $V$ denote the set of all non-negative bounded functions $\nu$ on $\Omega$ that vanish at infinity, i.e. for every $\varepsilon > 0$ the set \{ $x \in \Omega \mid \nu(x) \geq \varepsilon$ \} is compact. Let $\beta_0$ be the Hausdorff locally convex topology on $C_0(\Omega)$ that is induced by the seminorms

$$ ||f||_\nu := \sup_{x \in \Omega} |f(x)|\nu(x), \quad f \in C_0(\Omega), $$

for $\nu \in V$. Then we have $\gamma(\| \cdot \|_\infty, \tau_{co}) = \beta_0$. If $\Omega$ is locally compact, then $V$ may be replaced by the functions in $C_0(\Omega)$ that are non-negative where $C_0(\Omega)$ is the space of real-valued continuous functions on $\Omega$ that vanish at infinity.

If $\Omega$ is a hemi-compact Hausdorff $k_2$-space or a Polish space, then we even have

$$ \gamma(\| \cdot \|_\infty, \tau_{co}) = \beta_0 = \mu(C_0(\Omega), M_1(\Omega)) $$

where $M_1(\Omega) = (C_0(\Omega), \beta_0)'$ is the space of bounded Radon measures and $\mu(C_0(\Omega), M_1(\Omega))$ the Mackey-topology of the dual pair $(C_0(\Omega), M_1(\Omega))$.

(b) Let $(X, \| \cdot \|)$ be a Banach space and $\sigma^* := \sigma(X', X)$ the weak*-topology. Then condition (ii) of Remark 2.5 (b) is fulfilled, $(X', \| \cdot \|_{X'}, \sigma^*)$ is a complete Saks space and $\gamma(\| \cdot \|_{X'}, \sigma^*) = \gamma_s(\| \cdot \|_{X'}, \sigma^*) = \tau_s(X', X)$ where $\tau_s(X', X)$ is the topology of uniform convergence on compact subsets of $X$.

(c) Let $(X, \| \cdot \|)$ be a Banach space and $\mu^* := \mu(X', X)$ the dual Mackey-topology. Then $(X', \| \cdot \|_{X'}, \mu^*)$ is a complete Saks space, where the completeness follows from [17, p. 74], and $\gamma(\| \cdot \|_{X'}, \mu^*) = \mu^*$. If $X$ is a $\beta^*$-space, i.e. every $\sigma(X, X')$-convergent sequence is $\| \cdot \|_{X'}$-convergent (see [10, p. 253]), then condition (ii) of Remark 2.5 (b) is fulfilled and $\gamma(\| \cdot \|_{X'}, \mu^*) = \gamma_s(\| \cdot \|_{X'}, \mu^*)$.

(d) Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be Banach spaces, and $\tau_{\text{Sot}}$ the strong operator topology on $\mathcal{L}(X; Y)$. Then $(\mathcal{L}(X; Y), \| \cdot \|_{\mathcal{L}(X; Y), \tau_{\text{Sot}}})$ is a Saks space. Let $(T_i)_{i \in I}$ be a $\tau_{\text{Sot}}$-Cauchy net in $B_{\| \mathcal{L}(X; Y) \|}(T \in \mathcal{L}(X; Y) \mid \|T\|_{\mathcal{L}(X; Y)} \leq 1)$. Then for each $x \in X$ the net $(T_ix)_{i \in I}$ is $\| \cdot \|_Y$-convergent to some $Tx \in Y$ with $\|Tx\|_Y \leq \|x\|_X$ in the Banach space $(Y, \| \cdot \|_Y)$. Thus the map $T: x \mapsto Tx$ belongs to $\mathcal{L}(X; Y)$ with $\|T\|_{\mathcal{L}(X; Y)} \leq 1$ and $(T_i)_{i \in I}$ is $\tau_{\text{Sot}}$-convergent to $T$. Hence $B_{\| \mathcal{L}(X; Y) \|}$ is $\tau_{\text{Sot}}$-complete and so $(\mathcal{L}(X; Y), \| \cdot \|_{\mathcal{L}(X; Y), \tau_{\text{Sot}}})$ is complete.

If $Y$ is in addition finite-dimensional, then condition (ii) of Remark 2.5 (b) is fulfilled and $\gamma(\| \cdot \|_{\mathcal{L}(X; Y), \tau_{\text{Sot}}}) = \gamma_s(\| \cdot \|_{\mathcal{L}(X; Y), \tau_{\text{Sot}}})$. 

Remark 3.13 (a).

Let $\beta$ be the symmetric strong operator topology, i.e. the Hausdorff locally convex topology on $\mathcal{L}(H)$ generated by the directed system of seminorms

$$p_N(R) = \max\{\sup_{x \in N} |Rx|_H, \sup_{x \in N} |R^*x|_H\}, \quad R \in \mathcal{L}(H),$$

for finite $N \subset H$ where $R^*$ is the adjoint of $R$. We denote by $\beta_{\text{sort}}$, the mixed topology $\gamma(\cdot, \mathcal{L}(H), \tau_{\text{sort}})$. Then the triple $(\mathcal{L}(H), \| \cdot \|_{\mathcal{L}(H)}, \tau_{\text{sort}})$ is a complete Saks space and $\beta_{\text{sort}} \cong \mu(\mathcal{L}(H), \mathcal{N}(H))$.

(f) Let $\Omega$ be a completely regular Hausdorff space, $M_\gamma$ the space of bounded Radon measures on $\Omega$, and $\| \cdot \|_{M_\gamma(\Omega)}$ the total variation norm on $M_\gamma(\Omega)$. Then $(M_\gamma(\Omega), \| \cdot \|_{M_\gamma(\Omega)}, \sigma(M_\gamma(\Omega), C_b(\Omega)))$ is a complete Saks space where the completeness follows from $R_{\| \cdot \|_{M_\gamma(\Omega)}}$ being $\sigma(M_\gamma(\Omega), C_b(\Omega))$-compact by [13, Corollary 3.23 (a), p. 17], i.e. condition (ii) of Remark 2.5 (b) is fulfilled. Furthermore, we have

$$\beta_0' = \gamma(\cdot, \| \cdot \|_{M_\gamma(\Omega)}, \sigma(M_\gamma(\Omega), C_b(\Omega))) = \tau(M_\gamma(\Omega), (C_b(\Omega), \| \cdot \|_{\infty})).$$

Let us consider a toy example for an application of Theorem 3.10, namely, the multiplication operator on $C_b(\Omega)$, which we will revisit for other generation results.

3.14 Example. Let $\Omega$ be a Hausdorff $k_2$-space and $q: \Omega \to \mathbb{C}$ be continuous with $C := \sup_{x \in \Omega} \Re q(x) < \infty$. We define the multiplication operator $(M_q, D(M_q))$ by setting

$$D(M_q) := \{ f \in C_b(\Omega) \mid qf \in C_b(\Omega) \}$$

and $M_q := qf$ for $f \in D(M_q)$. By solving the equation $(\lambda - q)f = g$ we can compute the resolvent $R(\lambda, M_q)$ of $M_q$ explicitly by

$$R(\lambda, M_q)f = \frac{1}{\lambda - q}f, \quad f \in C_b(\Omega),$$

for all $\lambda \in (C \setminus \Omega) = \rho(M_q)$, which shows that $\lambda - M_q$ is surjective, i.e. $\text{Ran}(\lambda - M_q) = C_b(\Omega)$, for all $\lambda \in C \setminus \Omega$. Suppose that $C \leq 0$. Then $(0, \infty) \in \rho(M_q)$ and $\text{Ran}(\lambda - M_q) = C_b(\Omega)$ for all $\lambda > 0$. Furthermore, we have for all $\lambda > 0$, $f \in C_b(\Omega)$ and $\nu \in \mathcal{V}$ from Remark 3.13 (a) that

$$|R(\lambda, M_q)f|_\nu = \sup_{x \in \Omega} \left| \frac{1}{\lambda - q(x)} |f(x)| \nu(x) - \frac{1}{\lambda} |f(x)| \nu(x) \right| \leq \sup_{x \in \Omega} \left| \frac{1}{\lambda} |f(x)| \nu(x) \right|.$$

Therefore $(M_q, D(M_q))$ is $\beta_0$-dissipative for the directed system of seminorms $\Gamma_{\beta_0} = (\| \cdot \|_{\nu})_{\nu \in \mathcal{V}}$ that generates the mixed topology $\beta_0 = \gamma(\| \cdot \|_{\infty}, \tau_{\text{co}})$. Moreover, due to Proposition 3.7 (b) and $\text{Ran}(\lambda - M_q) = C_b(\Omega)$ for all $\lambda > 0$ the operator $(M_q, D(M_q))$ is $\beta_0$-closed and thus generates a $\beta_0$-strongly continuous, $\beta_0$-equicontinuous semigroup $(T(t))_{t \geq 0}$ on $C_b(\Omega)$ by Theorem 3.10 (a) and Remark 3.13 (a). Choosing $V_1 = \{ \nu \in \mathcal{V} \mid \| \nu \| = 1 \}$ instead of $\mathcal{V}$, we get a norming directed system of continuous seminorms that generates $\beta_0$ for which $(M_q, D(M_q))$ is dissipative, too. Hence $(T(t))_{t \geq 0}$ is also a $(\| \cdot \|_{\infty}, \tau_{\text{co}})$-equitight contraction semigroup by Theorem 3.10 (b) and (c) since $\beta_0 = \gamma(\| \cdot \|_{\infty}, \tau_{\text{co}}) = \gamma_0(\| \cdot \|_{\infty}, \tau_{\text{co}})$ by Remark 3.13 (a).

Our next generation result involves the $\gamma$-dual operator. Let us recall some observations from [13, Remark 4.5, p. 9]. Let $(X, \| \cdot \|, \tau)$ be a sequentially complete Saks space. Then $X_\gamma' = (X, \gamma)'$ is a closed linear subspace of $X'$, in particular...
a Banach space, by [3, 1.1.18 Proposition, p. 15], and we denote by \( \| \cdot \|_{X'} \) the restriction of \( \| \cdot \|_{X} \) to \( X' \). We note that \((X, \gamma)\) is a Mazur space, i.e. \( X' \) coincides with the space of linear \( \gamma \)-sequentially continuous functionals on \( X \) (see [32, p. 50]), if and only if 
\[
X'_{\gamma} = \{ x' \in X' \mid x' \text{ } \gamma \text{-sequentially continuous on } \| \cdot \|_{X} \text{-bounded sets} \} = X^\circ
\]
by [3, 1.1.10 Proposition, p. 9]. The space \( X^\circ \) was introduced in [13 Proposition 2.1, p. 314] in the context of dual semigroups of bi-continuous semigroups.

3.15. Corollary. Let \((X, \| \cdot \|, \tau)\) be a complete Saks space. Let both \((A, D(A))\) and its \( \gamma \)-dual operator \((A', D(A'))\) be \( \Gamma_\gamma \)-dissipative and \( \| \cdot \|_{X'_{\gamma}} \)-dissipative operators, respectively. Then the following assertions hold:
(a) The \( \gamma \)-closure \((\overline{A}, D(\overline{A}))\) generates a \( \gamma \)-equicontinuous, \( \gamma \)-strongly continuous semigroup \((T(t))_{t \geq 0} \) on \( X \).
(b) If \( \Gamma_\gamma \) is norming, then \((T(t))_{t \geq 0} \) is a contraction semigroup.
(c) If \( \Gamma_\gamma \) is norming and \( \gamma = \gamma_\tau \), then \((T(t))_{t \geq 0} \) is \((\| \cdot \|, \tau)\)-equitight.

Proof. By [3, 1.1.18 Proposition (i), p. 15] we have \((X'_{\gamma}, \tau_\gamma) = (X'_{\gamma}, \| \cdot \|_{X'})\) where \( \tau_\gamma \) denotes the topology of uniform convergence on \( \gamma \)-bounded sets. Due to [3, Corollary 3.17, p. 931] we get that \((\overline{A}, D(\overline{A}))\) generates a \( \gamma \)-strongly continuous, \( \gamma \)-equicontinuous semigroup on \( X \). Parts (b) and (c) follow as in Theorem 3.10. \( \square \)

3.16. Example. Let \( \Omega = \mathbb{N} \) be equipped with the metric induced by the absolute value. Then \( \Omega \) is a Polish space, in particular, a Hausdorff \( k_\mathbb{R} \)-space. Moreover, \( C_0(\mathbb{N}) = \ell^\infty \) and \( M_1(\mathbb{N}) = \ell^1 \) (see e.g. [6, p. 477]). It follows from Remark 3.13 (a) that \( \beta_0 = \mu(\ell^\infty, \ell^1) \) and so 
\[
(\ell^\infty, \beta_0)' = (\ell^\infty, \mu(\ell^\infty, \ell^1))' = \ell^1.
\]
Let \( q: \mathbb{N} \to \mathbb{C} \) be a function with \( C := \sup_{n \in \mathbb{N}} \text{Re} q(n) \leq 0 \). Again, we consider the multiplication operator \( M_q \) from Example 3.14, i.e. 
\[
D(M_q) = \{ f \in \ell^\infty \mid qf \in \ell^\infty \}
\]
and \( M_q = qf \) for \( f \in D(M_q) \). We already know that \((M_q, D(M_q))\) is \( \Gamma_{\beta_0} \)-dissipative with \( \Gamma_{\beta_0} \) from Example 3.14. Furthermore, we have for the \( \beta_0 \)-dual operator \((M_q', D(M_q'))\) that 
\[
D(M_q') = \{ f \in \ell^1 \mid qf \in \ell^1 \}
\]
and \( M_q' = qf \) for \( f \in D(M_q') \). For all \( \lambda > 0 \) and \( f \in D(M_q') \) we get 
\[
|\lambda - M_q'f|_{\ell^1} = \sum_{n=1}^{\infty} |(\lambda - q(n))f_n| \geq C_0 \sum_{n=1}^{\infty} |(\lambda - \text{Re} q(n))f_n| \geq \lambda \sum_{n=1}^{\infty} |f_n| = \lambda \|f\|_{\ell^1},
\]
meaning that \((M_q', D(M_q'))\) is \( \ell^1 \)-dissipative. Thus we may apply Corollary 3.15 (a) to deduce that \((M_q, D(M_q))\) generates a \( \mu(\ell^\infty, \ell^1) \)-strongly continuous, \( \mu(\ell^\infty, \ell^1) \)-equicontinuous semigroup on \( \ell^\infty \).

Instead of Remark 3.13 (a) we may also use Remark 3.13 (c) in Example 3.16 since \( \ell^1 \) is a Schur space by [10, Theorem 5.36, p. 252].

Next, we would like to transfer [3, Theorem 3.18, p. 931] to the setting of Saks spaces \((X, \| \cdot \|, \tau)\). However, looking at the assumptions of [3, Theorem 3.18, p. 931], we see that this requires \((X, \gamma)\) to be reflexive. Since reflexive spaces are barrelled, this requirement implies that \( \tau = \gamma = \tau_{\ell^1} \) by [3, 1.1.15 Proposition, p. 12] and so we are in an uninteresting situation from the perspective of \( \tau \)-bi-continuous semigroups. But if we could relax the assumption to \((X, \gamma)\) being semi-reflexive, then there are non-trivial (i.e. not Banach) Saks spaces. This is actually possible by the following observation.
3.17. **Remark.** [11, Theorem 3.18, p. 931] is stated for reflexive Hausdorff locally convex spaces \((X, v)\). However, a closer look at its proof reveals that it is actually valid for semi-reflexive \((X, v)\) because the only part where reflexivity comes into play is that it implies that a \(v\)-bounded set \(B \subset X\) is relatively \(\sigma(X, (X, v)')\)-compact; see [11, p. 931, l. 9–10 from below]. But the latter assertion is equivalent to semi-reflexivity by [26, Proposition 23.18, p. 270].

3.18. **Theorem.** Let \((X, \| \cdot \|, \tau)\) be a complete, semi-reflexive Saks space, \((A, D(A))\) a \(\gamma\)-densely defined, \(\Gamma_\gamma\)-dissipative operator and \(\text{Ran}(\lambda - A) = X\) for some \(\lambda > 0\). Then the following assertions hold:

(a) \((A, D(A))\) generates a \(\gamma\)-equicontinuous, \(\gamma\)-strongly continuous semigroup \((T(t))_{t \geq 0}\) on \(X\).

(b) If \(\Gamma_\gamma\) is norming, then \((T(t))_{t \geq 0}\) is a contraction semigroup.

(c) If \(\Gamma_\gamma\) is norming and \(\gamma = \gamma_s\), then \((T(t))_{t \geq 0}\) is \((\| \cdot \|, \tau)\)-equitight.

**Proof.** Part (a) follows from [11, Theorem 3.18, p. 931] (noting that (ii) in Remark 2.5 (b) again and also a sufficient condition for 3.10. □

Let \((X, \| \cdot \|, \tau)\) be a Saks space. By definition the Saks space is semi-reflexive if and only if \((X, \gamma)\) is semi-reflexive. The space \((X, \gamma)\) is semi-reflexive if and only if \(B_{\gamma} = \{ x \in X \mid \| x \| \leq 1 \}\) is \(\sigma(X, (X, \gamma)')\)-compact by [11, I.1.21 Corollary, p. 16]. Due to [11, I.1.20 Proposition, p. 16], \(B_{\gamma}\) is \(\sigma(X, (X, \gamma)')\)-compact if and only if it is \(\sigma(X, (X, \gamma)')\)-compact. Further, \((X, \gamma)\) is a semi-Montel space, thus semi-reflexive, if and only if \(\gamma = \gamma_s\) is \(\tau\)-compact by [11, I.1.13 Proposition, p. 11] which is condition (ii) in Remark 2.5 (b) again and also a sufficient condition for \(\gamma = \gamma_s\). Therefore we have by Remark 3.13 the following observations where we only have to add an additional argument in parts (a), (c) and (e) of Remark 3.19 below.

3.19. **Remark.**

(a) Let \(\Omega\) be a discrete space. Then \((C_b(\Omega), \| \cdot \|_\infty, \tau_{co})\) is a complete, semi-reflexive Saks space by [11, II.1.24 Remark 4], p. 88–89.

(b) Let \((X, \| \cdot \|)\) be a Banach space. Then \((X', \| \cdot \|_{X'}, \sigma\ast)\) is a complete, semi-reflexive Saks space.

(c) Let \((X, \| \cdot \|)\) be a Banach space. Then \((X', \| \cdot \|_{X'}, \mu\ast)\) is a complete, semi-reflexive Saks space where the semi-reflexivity follows from \((X', \mu\ast)'' = X'\) by the Mackey–Arens theorem.

(d) Let \((X, \| \cdot \|_{X})\) and \((Y, \| \cdot \|_{Y})\) be Banach spaces and \(Y\) finite-dimensional. Then \((L(X; Y), \| \cdot \|_{L(X; Y), \tau_{sot}})\) is a complete, semi-reflexive Saks space.

(e) Let \(H\) be a separable Hilbert space. Then \((L(H), \| \cdot \|_{L(H), \tau_{sot}})\) is a complete, semi-reflexive Saks space where the semi-reflexivity follows from \((L(H), \beta_{sot})'' = N(H)' = L(H)\).

(f) Let \(\Omega\) be a completely regular Hausdorff space. Then we have that the triple \((M_1(\Omega), \| \cdot \|_{M_1(\Omega)}, \sigma(M_1(\Omega), C_b(\Omega)))\) is a complete, semi-reflexive Saks space.

3.20. **Example.** Due to Example 3.16 and Remark 3.19 \((\ell^\infty, \| \cdot \|_\infty, \tau_{co})\) is a complete, semi-reflexive Saks space. Therefore we may also apply Theorem 3.18 (a) to prove that the multiplication operator \((M_n, D(M_n))\) with \(\sup_{n \in \mathbb{N}} \text{Re} q(n) \leq 0\) generates a \(\mu(\ell^\infty, \ell^1)\)-strongly continuous, \(\mu(\ell^\infty, \ell^1)\)-equicontinuous semigroup on \(\ell^\infty\) (we already checked in Example 3.14 that the other assumptions of Theorem 3.18 are satisfied).

We close this section with a characterisation of the bi-continuous semigroups with dissipative generators. First, we start with a refinement of Theorem 3.9. In the case \(v = \tau\) this was already done in [3, Proposition 3.11, p. 9] whose prove needs
Proposition. Let \((X, \| \cdot \|, \tau)\) be a sequentially complete Saks space, \(v\) a Hausdorff locally convex topology on \(X\) with \(\tau \subseteq v \subseteq \tau|_1\) such that \(\gamma\)-convergent sequences are \(v\)-convergent, and \((A, D(A))\) bi-densely defined. Then the following assertions are equivalent:

(a) \((A, D(A))\) generates a \(\tau\)-bi-continuous contraction semigroup \((T(t))_{t \geq 0}\) on \(X\) and there exists a norming directed system of continuous seminorms \(\Gamma_v\) that generates \(v\) such that \(p(T(t)x) \leq p(x)\) for all \(t \geq 0, p \in \Gamma_v\) and \(x \in X\).

(b) \(\lambda - A\) is surjective for some \(\lambda > 0\) and \((A, D(A))\) is a \(\Gamma_v\)-dissipative operator on \(X\) for some norming directed system of continuous seminorms \(\Gamma_v\) that generates \(v\).

Proof. (a)\(\Rightarrow\)(b): First, we show that \((A, D(A))\) is \(\Gamma_v\)-dissipative. We note that \((0, \infty) \subseteq \rho(A)\) and

\[ R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x dt \]

for all \(\lambda > 0\) and \(x \in X\) by [23, Theorem 12, p. 215] and [23, Definition 9, p. 213] where the integral is an improper \(\tau\)-Riemann integral. The sequence of Riemann sums that approximate the integral on the right-hand side w.r.t. \(\tau\) are \(\| \cdot \|\)-bounded for each \(\lambda > 0\) and \(x \in X\). Due to Remark 2.9 (a) this means that this sequence of Riemann sums is actually \(\gamma\)-convergent and thus \(v\)-convergent by assumption. Therefore we have for all \(\lambda > 0, p \in \Gamma_v\) and \(x \in X\) that

\[ p(R(\lambda, A)x) \leq \int_0^\infty e^{-\lambda t} p(T(t)x) dt \leq \int_0^\infty e^{-\lambda t} p(x) dt = \frac{1}{\lambda} p(x) \]

where we used that \(p\) is \(v\)-continuous for the first inequality. Hence \((A, D(A))\) is \(\Gamma_v\)-dissipative. In combination with Theorem 3.9 this yields that \(\lambda - A\) is surjective for some \(\lambda > 0\).

(b)\(\Rightarrow\)(a): Due to Theorem 3.9 \((A, D(A))\) generates a \(\tau\)-bi-continuous contraction semigroup on \(X\), and \((0, \infty) \subseteq \rho(A)\) by Proposition 3.17 (c). Furthermore, we have by the Post–Widder inversion formula [23, Corollary 2.10, p. 47] that

\[ T(t)x = \tau- \lim_{n \to \infty} \left( \frac{n}{t} R \left( \frac{n}{t}, A \right) \right)^n x \]

for all \(t > 0\) and \(x \in X\). As a consequence of Remark 3.5 the \(\gamma\)-convergent sequence \(\left( \left( \frac{n}{t} R \left( \frac{n}{t}, A \right) \right)^n x \right)_{n \in \mathbb{N}}\) is \(\| \cdot \|\)-bounded for each \(t > 0\) and \(x \in X\), thus \(\gamma\)-convergent by Remark 2.9 (a) and so \(v\)-convergent by assumption. We deduce from (2) that for all \(t > 0, p \in \Gamma_v\) and \(x \in X\) it holds that

\[ p(T(t)x) = \lim_{n \to \infty} \left( \frac{n}{t} \right)^n p \left( \frac{n}{t}, A \right)^n x \leq p(x) \]

where we used that \(p\) is \(v\)-continuous for the first equality. Further, for \(t = 0\) we have \(p(T(t)x) = p(x)\). We conclude that statement (a) holds.

The assumptions of Proposition 3.21 (a) are up to rescaling fulfilled for any \(\tau\)-bi-continuous semigroup on \(X\) if it is \(v\)-equicontinuous for \(v = \tau\) or \(\gamma_s\) or \(\gamma\) (the assumption on \(v\)-equicontinuity may fail if \(v = \tau\) by Remark 3.20).

Remark. Let \((X, \| \cdot \|, \tau)\) be a sequentially complete Saks space and \(v = \tau, \gamma_s\) or \(\gamma\). Then there exists a norming directed system of continuous seminorms that generates the topology \(v\) by Definition 2.24 (c) for \(v = \tau\), by Remark 2.6 for \(v = \gamma_s\) and by Remark 2.3 for \(v = \gamma\). Further, let \((T(t))_{t \geq 0}\) be a \(\tau\)-bi-continuous,
v-equicontinuous semigroup on X and \( \omega \in \mathbb{R} \) be its type (see Definition 2.23 (iii)). By modifying the proof of [3, Remark 3.12, p. 9–10] one can show that for the \( \tau \)-bi-continuous, \( v \)-equicontinuous contraction semigroup \( (e^{-\omega t}T(t))_{t \geq 0} \) on X there exists a norming directed system of continuous seminorms \( \Gamma_v \) that generates \( v \) such that \( p(e^{-\omega t}T(t)x) \leq p(x) \) for all \( t \geq 0 \), \( p \in \Gamma_v \), and \( x \in X \).

In addition, we have the following characterisation in the case of complete, C-sequential Saks spaces \((X, \| \cdot \|, \tau)\) and \( v = \gamma \).

3.23. Proposition. Let \((X, \| \cdot \|, \tau)\) be a complete, C-sequential Saks space and \((A, D(A))\) the generator of a \( \tau \)-bi-continuous semigroup \((T(t))_{t \geq 0}\) on X. Then the following assertions are equivalent:

(a) \((T(t))_{t \geq 0}\) is \( \gamma \)-equicontinuous.

(b) There is a directed system of continuous seminorms \( \Gamma_{\gamma} \) that generates the mixed topology \( \gamma \) such that \((A, D(A))\) is \( \Gamma_{\gamma} \)-dissipative.

Proof. By Remark 2.11 we know that \((T(t))_{t \geq 0}\) is quasi-\( \gamma \)-equicontinuous if \((X, \| \cdot \|, \tau)\) is a sequentially complete C-sequential Saks space. Hence the equivalence of the assertions (a) and (b) follows from [1, Propositions 4.2, 4.4, p. 933, 935]. \( \square \)

The condition that \((X, \| \cdot \|, \tau)\) is a complete, C-sequential Saks space is quite often fulfilled, e.g. for the examples from Remark 3.14 under some minor constraints by [18, Remark 3.19, p. 14], [18, Remark 3.20 (c), p. 15], [18, Example 4.12, p. 24–25] and [18, Corollary 3.23 (b), p. 17].

3.24. Remark. (a) Let \( \Omega \) be a hemi-compact Hausdorff \( k_{\mathbb{R}} \)-space or a Polish space. Then \((C_b(\Omega), \| \cdot \|_{\infty}, \tau_{co})\) is a complete, C-sequential Saks space.

(b) Let \((X, \| \cdot \|)\) be a separable Banach space. Then \((X', \| \cdot \|_{X'}, \sigma^*)\) is a complete, C-sequential Saks space.

(c) Let \((X, \| \cdot \|)\) be an SWCG space (see [30, p. 387]), or a sequentially \( \sigma(X, X') \)-complete space with an almost shrinking basis (see [17, p. 75]). Then the triple \((X', \| \cdot \|_{X'}, \mu^*)\) is a complete, C-sequential Saks space.

(d) Let \((X, \| \cdot \|_{X})\) be a separable Banach space and \((Y, \| \cdot \|_{Y})\) a Banach space. Then \((\mathcal{L}(X, Y), \| \cdot \|_{\mathcal{L}(X, Y), \tau_{sot}})\) is a complete, C-sequential Saks space.

(e) Let \( H \) be a separable Hilbert space. Then \((\mathcal{L}(H), \| \cdot \|_{\mathcal{L}(H), \tau_{sot}})\) is a complete, C-sequential Saks space.

(f) Let \( \Omega \) be a Polish space. Then \((M_k(\Omega), \| \cdot \|_{M_k(\Omega)}, \sigma(M_k(\Omega), C_b(\Omega)))\) is a complete, C-sequential Saks space.

3.25. Remark. Let \((X, \| \cdot \|, \tau)\) be a complete, C-sequential Saks space. Due to Remark 2.11 assertion (a) of Proposition 3.23 always holds up to rescaling, and thus for any \( \tau \)-bi-continuous semigroup \((T(t))_{t \geq 0}\) on X there is a rescaling such that the generator of the rescaled semigroup is \( \Gamma_{\gamma} \)-dissipative for some system of continuous seminorms \( \Gamma_{\gamma} \) that generates the mixed topology \( \gamma \).

On the other hand, there are important examples of \( \tau \)-bi-continuous semigroups that are not quasi-\( \tau \)-equicontinuous. For instance, the Gauß–Weierstraß semigroup on the complete, C-sequential Saks space \((C_b(\mathbb{R}^d), \| \cdot \|_{\infty}, \tau_{co})\) is \( \tau_{co} \)-bi-continuous but not locally \( \tau_{co} \)-equicontinuous by [23, Examples 6 (a), p. 209–210]. Since there is some \( \lambda > 0 \) such that \( \lambda - A \) is surjective for the generator \((A, D(A))\) of the Gauß–Weierstraß semigroup by [23, Lemma 7, Proposition 8, Theorem 12, p. 211–212, 215], it follows from Proposition 3.21 with \( v = \tau_{co} \) that its generator \((A, D(A))\) is not bi-dissipative (even after rescaling, cf. [4, Example 3.9, p. 8]). Another example of a \( \tau_{co} \)-bi-continuous semigroup which has no bi-dissipative generator (even after rescaling) is the left translation semigroup on the complete, C-sequential Saks space \((C_b(\mathbb{R}), \| \cdot \|_{\infty}, \tau_{co})\) which is \( \tau_{co} \)-bi-continuous, even locally \( \tau_{co} \)-equicontinuous but
not quasi-$\tau_{cc}$-equicontinuous by [23, Examples 6 (b), p. 209–210] and [24, Example 3.2, p. 549].

Nevertheless, the Gauß–Weierstraß semigroup and the left translation semigroup are quasi-$\beta_0$-equicontinuous by Remark 2.11 and Remark 3.24 (a), and thus both have (after rescaling) a $\Gamma_{\beta_0}$-dissipative generator for some system of continuous seminorms $\Gamma_{\beta_0}$ that generates the mixed topology $\beta_0 = \gamma(\|\cdot\|_\infty, \tau_{cc})$. This underlines that in the framework of $\tau$-bi-continuous semigroups the concept of a bi-dissipative operator resp. generator is not the correct choice whereas the concept of a $\Gamma_{\gamma}$-dissipative operator resp. generator is the more reasonable one.

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