On $j$-near closure operators induced from relations and its applications

Mohamed I. Abbas$^1$, W.S. Amer$^1$ and Mostafa K. El-Bably$^1$

Abstract: In this paper, for the first time, we introduce and study the new concepts near closure operators induced from binary relation. These operators represent a generalization to closure operators that are induced from binary relations. The suggested techniques will narrow the gap between topologists and those who are interested in applications with topology in their field. The fundamental properties of the suggested structures are obtained. Finally, we introduce some applications of the suggested structures in rough set theory. Moreover, we give a simple practical example to illustrate the importance of near concepts in rough set theory and we introduce comparisons between our approaches and other approaches.

Subjects: Science; Mathematics & Statistics; Advanced Mathematics; Pure Mathematics; Technology; Computer Science

Keywords: neighborhood spaces; near closure operators; topology and rough sets

1. Introduction
For a long time, abstract topological structures and concepts seem to be far from applications for most researchers in application fields such as engineers and computer scientists among others. In addition, it seems that there is a big gap between these structures and real-life applications. In order to reduce this gap, some proposals introduced some topological structures using binary relations

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In this paper, we introduced and studied the new concepts “near closure” operators induced from any binary relation. Moreover, we applied these concepts in rough set theory. We conclude that the intermingling of $j$-near concepts in the construction of some approximation space concepts will help in getting results with abundant logical statements, discovering hidden relationships among data and, moreover, probably help in producing accurate programs. Moreover, the $j$-near approximations can help in the discovery of hidden information in data that were collected from real-life applications since the boundary regions are decreased or canceled by increasing the lower and decreasing the upper approximations and this is very important in decision-making. We believe that using these techniques is easier in application fields and useful for applying many topological concepts in future studies.

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Relation concept is a simple mathematical model to which many real-life data can be connected. Relations are used in the construction of topological structures in many fields such as dynamics (Slapal, 2001), rough set theory and approximation space (Abd El-Monsef et al., 2014, 2015; Abo Khadra & El-Babby, 2008; Abd El-Monsef, Kozae, & El-Babby, 2015; Abo Khadra, Taher, & El-Babby, 2007; Allam, Bakeir, & El-Tabl, 2006; El-Babby, 2015, 2016; Galton, 2003; Yu & Zhan, 2014; Zhao, 2016).

Closure and interior operators are basic concepts for the study and investigations in general topological spaces. Now these two operators are widely used in computer science approximations, for example in rough set theory (Pawlak, 1982), where the topological structures is used as a knowledge base for extracting knowledge from available information. Accordingly, El-Babby (2016) introduced new methods to generate different closure operators from binary relation as a generalization to Galton definition (Galton, 2003), and Allam definition (Allam et al., 2006). The used technique is very useful since the generated operators have topological properties and hence, the topological concepts can be applied directly using the binary relation. In addition, El-Babby (2016) introduced new methods to generate different general topologies directly from any binary relation without using base or subbase. Hence, these techniques opened the way for more topological applications in several fields from real-life problems. In the present paper, for the first time, we introduce different types of “near closure operators” generated from any binary relation. These operators are a generalization to closure operators. Moreover, the properties of the suggested operators are introduced and sufficiently illustrated. In addition, we give the relationships among them and closure operators. The suggested structures are very useful since by using it many topological applications such as “separation axioms, connectedness, continuous functions and near concepts” can be applied in many fields such as “rough set theory, fuzzy set theory and soft set theory”.

Rough set theory, which was first formulated by Pawlak (1982) in the early 1980s, is an effective tool to deal with vagueness and incompleteness in information systems. The theory has been successfully applied to many fields, such as machine learning, knowledge acquisition, and decision analysis, etc. Rough set theory bears on the assumption that some elements of a universe may be indiscernible in view of the available information about the elements. Thus, the indiscernibility relation is the starting point of rough set theory. Such a relation was first described by equivalence relation in the way that two elements are related by the relation if and only if they are indiscernible from each other. Lower and upper approximations, which are the core concepts in rough set theory, are defined with respect to equivalence classes. However, the requirement of equivalence relation as the indiscernibility relation is too restrictive for many applications. In other words, many practical data-sets cannot be handled well by classical rough sets. To solve this problem, several authors have generalized the notion of approximation operators using arbitrary binary relations (Abd El-Monsef et al., 2014, 2015; Abo Khadra & El-Babby, 2008; Abo Khadra et al., 2007; D’eer, Restrepo, Cornelis, & Gómez, 2016; Restrepo, Cornelis, & Gómez, 2014; Wang, Wu, & Chen, 2008; Wang et al., 2013, 2014, 2015, 2016; Yao, 1996; Yao & Deng, 2014; Yu & Zhan, 2014; Zhao, 2016). El-Babby (2016), introduced a generalized definition for approximations using the closure operators generated from general relation as a generalization to classical rough set theory (Pawlak, 1982) and some of other generalization. In the present paper, we use the near closure operators to define generalized approximation operators as topological applications in rough set theory. In fact, we introduce generalizations to classical rough set theory using general binary relations without adding any restrictions.
These approximation operators are generalization to the approximations, which are introduced by El-Bably (2016), and the other generalizations. Moreover, they are more accurate than other approaches.

In the present paper, the concept of \( j \)-neighborhood space and \( j \)-closure operators that generated from a binary relation are introduced in Section 2. Section 3 is devoted to introduce and study the new concepts “near closure operators” which generated from a binary relation. In Section 4, we introduce a topological application in rough set theory. In fact, we introduce and study new generalization for rough approximations, so called “near approximations”. These new approximations are based on near closure operators that are induced from a general binary relation. Moreover, these approximations are more accurate than classical approximations, and the \( j \)-approximations that are given by El-Bably (2016). Comparisons between our approaches and the other approaches are introduced. Near concepts are provided to be easy tools to classify the sets and to help in measuring exactness and roughness of sets. Many proved results, examples and counter examples are provided. Simple practical example is introduced in Section 5 as a real application for our approaches in the network connectivity devices. This example illustrate the importance of our approaches in exactness of rough sets and give comparisons between our approaches and some of the others generalization (namely, Yao approach, Yao, 1996).

2. \( j \)-Neighborhood spaces and \( j \)-closure operators derived from binary relations

In this section, we present the main ideas about the \( j \)-neighborhood space and \( j \)-closure operators which are cited in Abd El-Monsef et al. (2014), El-Bably (2016).

**Definition 2.1** (Abd El-Monsef et al., 2014). Let \( R \) be an arbitrary binary relation on a non-empty finite set \( X \). The \( j \)-neighborhood of \( x \in X \) (denoted by \( N_j(x) \), \( \forall j \in \{r, l, i, \langle l \rangle, \langle r \rangle, \langle i \rangle, \langle u \rangle, \langle w \rangle, \langle u \rangle \} \)) is defined by:

(i) \( r \)-neighborhood: \( N_r(x) = \{ y \in X | yRx \} \)

(ii) \( l \)-neighborhood: \( N_l(x) = \{ y \in X | yRx \} \)

(iii) \( r \)-neighborhood: \( N_r(x) = \{ y \in X | yRx \} \)

(iv) \( l \)-neighborhood: \( N_l(x) = \{ y \in X | yRx \} \)

(vi) \( u \)-neighborhood: \( N_u(x) = N_l(x) \cup N_r(x) \)

(vii) \( i \)-neighborhood: \( N_i(x) = N_l(x) \cup N_r(x) \)

(viii) \( \langle u \rangle \)-neighborhood: \( N_{\langle u \rangle}(x) = N_l(x) \cup N_r(x) \)

**Definition 2.2** (Abd El-Monsef et al., 2014). Let \( R \) be an arbitrary binary relation on a non-empty finite set \( X \) and \( f : X \to \wp(X) \) be a mapping which assigns for each \( x \) in \( X \) its \( N_j(x) \) in the power set of \( X \). \( f(X) \). The triple \( (X, R, f) \) is called a \( j \)-neighborhood space (in briefly, \( j \)-NS).

**Definition 2.3** (El-Bably, 2016). Let \( (X, R, f) \) be a \( j \)-NS, then we define the “\( j \)-closure operator”, generated by the relation \( R \), by the operator \( Cl_j f(x) : \wp(X) \to \wp(X) \) such that

\[
Cl_j(A) = A \cup \left\{ x \in X | N_j(x) \cap A \neq \emptyset \right\}, \quad \forall j \in \{r, l, i, \langle l \rangle, \langle r \rangle, \langle i \rangle, \langle u \rangle, \langle w \rangle, \langle u \rangle \}.
\]

The subset \( A \subseteq X \) is called “\( j \)-closed set” if \( A = Cl_j(A) \).

The class of all \( j \)-closed sets in \( U \) is: \( \Gamma_j = \left\{ A \subseteq X | A = Cl_j(A) \right\} \).

The complement of a \( j \)-closed set is called “\( j \)-open set”.
The following proposition gives the relationships among the above eight $j$-closure operators.

**Proposition 2.1** (El-Bably, 2016). Let $(X, R, F_j)$ be a $j$-NS, and $A \subseteq X$. Then

(i) $\text{Cl}_j(A) \subseteq \text{Cl}_l(A) \subseteq \text{Cl}_u(A)$.
(ii) $\text{Cl}_l(A) \subseteq \text{Cl}_i(A) \subseteq \text{Cl}_u(A)$.
(iii) $\text{Cl}_l(A) \subseteq \text{Cl}_r(A) \subseteq \text{Cl}_u(A)$.
(iv) $\text{Cl}_l(A) \subseteq \text{Cl}_m(A) \subseteq \text{Cl}_u(A)$.

**Proposition 2.2** (El-Bably, 2016). Let $(X, R, F_j)$ be a $j$-NS, then $\forall j \in \{r, l, r, l, u, i, (u), (i)\}$. The pair $(X, \text{Cl}_j)$ represents eight different closure spaces called “$j$-closure space”.

**Proposition 2.3** (El-Bably, 2016). Let $(X, R, F_j)$ be a $j$-NS. Then $\forall j \in \{r, l, i, u\}$, the $j$-closure space $(X, \text{Cl}_j)$ is a topological space.

**Remark 2.1** (El-Bably, 2016). Let $(X, \text{Cl}_j)$ be a $j$-closure space. Then $\forall j \in \{r, l, i, u\}$, $\text{Cl}_j$ need not be a topological closure in general.

**Proposition 2.4** (El-Bably, 2016). Consider $(X, R, F_j)$ be a $j$-NS. For each $j \in \{r, l, i\}$, if $R$ is a transitive relation, then the $j$-closure space $(X, \text{Cl}_j)$ is a topological space.

**Definition 2.4** (El-Bably, 2016). Let $(X, R, F_j)$ be a $j$-NS, then for any subset $A \subseteq X$, we define the “$j$-interior” operator of $A$, induced by the relation $R$, by the operator, $\text{int}_j \psi(X) \rightarrow \varphi(X)$ where

$\text{int}_j(A) = \{x \in A | N_j(x) \subseteq A \}, \ \forall j \in \{r, l, r, l, u, i, (u), (i)\}$

**Lemma 2.1** (El-Bably, 2016). Let $(X, R, F_j)$ be a $j$-NS. Then $\forall j \in \{r, l, r, l, u, i, (u), (i)\}$: $\text{int}_j(A) = \overline{\text{Cl}_j(A^c)}$, where $A^c$ indicates the complement of $A$.

**Lemma 2.2** (El-Bably, 2016). Let $(X, R, F_j)$ be a $j$-NS. Then $\forall j \in \{r, l, r, l, u, i, (u), (i)\}$: The interior operator $\text{int}_j(A)$ satisfies the following properties:

(i) $\text{int}_j(X) = X$.
(ii) $\text{int}_j(A) \subseteq A$, for each $A \subseteq X$.
(iii) $\text{int}_j(A \cap B) = \text{int}_j(A) \cap \text{int}_j(B)$, for each $A, B \subseteq X$.

3. $j$-Near closure operators induced from relations

Near (or nearly) open sets concept had been introduced in order to be new generalized sorts of open sets to topological spaces (see Abd El-Monsef, 1980; Andrijević, 1996; El-Atik, 1997; El-Bably, 2015; Levine, 1963; Mashhour et al., 1982; Njestad, 1965). In this section, for the first time, we introduce new types of closure operators so-called “near closure” operators, constructed from a binary relation, to be mathematical tools to generalize the $j$-closure operators. Properties of the suggested operators are investigated, and their connections are examined.

**Definition 3.1** Let $(X, R, F_j)$ be a $j$-NS. Then we define the $j$-near closure operators of any subset $A$ as follows: For each $j \in \{r, l, r, l, u, i, (u), (i)\}$ and $k \in \{p, s, l\}$, the near closure operator is $\text{Cl}_j^p \psi(X) \rightarrow \varphi(X)$ where

(i) The $j$-pre closure operator is defined by: $\text{Cl}_j^p(A) = A \cup \text{Cl}_j(\text{int}_j(A))$.
(ii) The $j$-semi closure operator is defined by: $\text{Cl}_j^s(A) = A \cup \text{int}_j(\text{Cl}_j(A))$.
(iii) The $j$-γ closure operator is defined by: $\text{Cl}_j^\gamma(A) = \text{Cl}_j^p(A) \cap \text{Cl}_j^s(A)$. 
Definition 3.2 Consider \( (X, R, T) \) be a \( j-\text{NS} \). The subset \( A \) is called \( j-\text{near closed} \) set if \( C^n_j(A) = A \). The family of all \( j-\text{near closed} \) sets of \( X \) is defined by:

\[
\Gamma^n_j = \{ A \subseteq X \mid C^n_j(A) = A \}.
\]

The complement of \( j-\text{near closed} \) set is called \( j-\text{near open} \) set and the family of all \( j-\text{near open} \) sets is defined by \( O^n_j = \{ A \subseteq X \mid \text{int}^n_j(A) = A \} \).

Example 3.1 Consider \( (X, R, T) \) be a \( j-\text{NS} \), where \( X = \{ a, b, c, d \} \) and \( R = \{ (a, a), (a, b), (b, a), (b, b), (c, a), (c, b), (c, c), (c, d), (d, d) \} \). Then, we compute the \( j-\text{near open} \) (resp. \( j-\text{closed} \)) sets in the case of \( j = r \) (and the other cases similarly) as follows:

The \( r-\text{preopen} \) and \( r-\text{pre closed} \) sets are:

\[
O^n_j = \{ X, \emptyset, \{ a \}, \{ b \}, \{ d \}, \{ a, b \}, \{ a, d \}, \{ b, d \}, \{ a, b, d \}, \{ a, c, d \}, \{ b, c, d \} \},
\]

and

\[
\Gamma^n_j = \{ X, \emptyset, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \}, \{ a, c, d \}, \{ b, c, d \} \}.
\]

The \( r-\text{semi open} \) and \( r-\text{semi closed} \) sets are:

\[
O^n_j = \{ X, \emptyset, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \}, \{ a, b, d \}, \{ a, c, d \} \},
\]

and

\[
\Gamma^n_j = \{ X, \emptyset, \{ c \}, \{ d \}, \{ a, b \}, \{ a, c \}, \{ b, c \} \}.
\]

The \( r-\text{open} \) and \( r-\text{closed} \) sets are:

\[
O^n_j = \{ X, \emptyset, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \}, \{ a, b, d \}, \{ a, c, d \}, \{ b, c, d \} \},
\]

\[
\Gamma^n_j = \{ X, \emptyset, \{ a \}, \{ b \}, \{ c \}, \{ d \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \}, \{ a, c, d \}, \{ b, c, d \} \}.
\]

The following proposition gives the fundamental properties of the near closure operator \( C^n_j \):

Proposition 3.1 Let \( (X, R, T) \) be a \( j-\text{NS} \), and \( A, B \subseteq X \). Then

(i) \( C^n_j(\emptyset) = \emptyset \) and \( C^n_j(X) = X \).

(ii) \( A \subseteq C^n_j(A) \), for each \( A \subseteq X \).

(iii) If \( A \subseteq B \), then \( C^n_j(A) \subseteq C^n_j(B) \), for each \( A, B \subseteq X \).

(iv) \( C^n_j(A) \cup C^n_j(B) \subseteq C^n_j(A \cup B) \) and \( C^n_j(A \cap B) \subseteq C^n_j(A) \cap C^n_j(B) \).

(v) \( C^n_j(A) \subseteq C^n_j(C^n_j(A)) \), for each \( A \subseteq X \).

Proof (i) and (ii) By Definition 3.1, the proof is obvious.

(iii) Let \( A \subseteq B \), then \( \text{int}^n_j(A) \subseteq \text{int}^n_j(B) \) and thus \( C^n_j(A) = A \cup C^n_j(\text{int}^n_j(A)) \subseteq B \cup C^n_j(\text{int}^n_j(B)) = C^n_j(B) \).

(iv) and (v) Using (iii), the proof is obvious.

Remark 3.1 In the above proposition, the converse of the property (v) is not true in general as illustrated in the following example:
Example 3.2 Let \( (X, R, \mathcal{T}_j) \) be a \( j \)-\( \text{NS} \), where \( X = \{a, b, c, d\} \) and \( R = \{(a, a), (a, b), (b, c), (c, d), (d, a)\} \). Suppose that \( A = \{a\} \), then we get \( \text{Cl}_j(A) = \{a, d\} \) and thus \( \text{Cl}_j^1(A) = \{a, d\} \). This implies \( \text{Cl}_j^2(\text{Cl}_j(A)) = \{a, c, d\} \).

Lemma 3.1 (Galton, 2003). Consider \( (X, R, \mathcal{T}_j) \) be a \( j \)-\( \text{NS} \). If \( \text{Cl}_j \) is a topological closure, then \( (X, \text{Cl}_j) \) is a topological space.

Corollary 3.1 Consider \( (X, R, \mathcal{T}_j) \) be a \( j \)-\( \text{NS} \), and \( A \subseteq X \). If \( \text{Cl}_j \) is a topological closure, then the \( j \)-near closure (resp. \( j \)-near interior) of any subset \( A \) is the intersection of all \( j \)-near closed sets that contains \( A \) (resp. the union of all \( j \)-near open sets that is contained in \( A \)). That is, for each \( k \in \{p, s, \gamma\} \):

\[
\text{Cl}_j^k(A) = \bigcap \{H \in \mathcal{T}_j^k : A \subseteq H\}
\]

and

\[
\text{int}_j^k(A) = \bigcup \{G \in \mathcal{O}_j^k : G \subseteq A\}.
\]

Lemma 3.2 Consider \( (X, R, \mathcal{T}_j) \) be a \( j \)-\( \text{NS} \), and \( A, B \subseteq X \). For each \( j \in \{r, l, (r), (l), u, i, (u), (i)\} \) and \( k \in \{p, s, \gamma\} \): If \( \text{Cl}_j \) is a topological closure, then we can prove the following properties:

(i) \( \text{Cl}_j^k(A) = \text{Cl}_j^l(\text{Cl}_j^k(A)) \)

(ii) \( \text{Cl}_j^k(A \cup B) = \text{Cl}_j^k(A) \cup \text{Cl}_j^k(B) \).

Proof Obvious. \( \square \)

Corollary 3.1 Consider \( (X, R, \mathcal{T}_j) \) be a \( j \)-\( \text{NS} \), and \( A, B \subseteq X \). For each \( j \in \{r, l, (r), (l), u, i, (u), (i)\} \) and \( k \in \{p, s, \gamma\} \), the following properties hold in general:

(i) \( \text{Cl}_j^k(A) = \text{Cl}_j^l(\text{Cl}_j^k(A)) \).

(ii) \( \text{Cl}_j^k(A \cup B) = \text{Cl}_j^k(A) \cup \text{Cl}_j^k(B) \).

The following proposition illustrates the relationship between the \( j \)-closure operators and the \( j \)-near closure operators.

Proposition 3.2 Let \( (X, R, \mathcal{T}_j) \) be a \( j \)-\( \text{NS} \) and \( A \subseteq X \). Then \( \text{Cl}_j^k(A) \subseteq \text{Cl}_j(A) \).

Proof We will prove the proposition in case of \( k = p \) and the other cases similarly.

Let \( x \in \text{Cl}_j^p(A) \), then \( x \in A \cup \text{Cl}_j(\text{int}_j(A)) \) this implies \( x \in A \) or \( x \in \text{Cl}_j(\text{int}_j(A)) \). But \( A \subseteq \text{Cl}_j(A) \) and \( \text{int}_j(A) \subseteq A \). Thus, \( x \in \text{Cl}_j(A) \) or \( x \in \text{Cl}_j(\text{int}_j(A)) \subseteq \text{Cl}_j(A) \).

Accordingly \( \text{Cl}_j^p(A) \subseteq \text{Cl}_j(A) \). \( \square \)

The converse of the above proposition is not true in general as illustrated in the following example:

Example 3.3 Let \( (X, R, \mathcal{T}_j) \) be a \( j \)-\( \text{NS} \) where \( X = \{a, b, c, d\} \) and \( R = \{(a, a), (a, b), (b, c), (c, d), (d, a), (d, c)\} \). Suppose that \( A = \{b, c\} \) then \( \text{Cl}_j(A) = X \). But \( \text{Cl}_j^2(A) = \{a, b, c\} \).
Proposition 3.3 Let \((X, R, T_j)\) be a \(j\)-NS. Then the subset \(A\) is

(i) \(j\)-preopen set if \(A \subseteq \text{int}_j(\text{Cl}_j(A))\).
(ii) \(j\)-semi open set if \(A \subseteq \text{Cl}_j(\text{int}_j(A))\).
(iii) \(j\)-open set if \(A \subseteq \{\text{int}_j(\text{Cl}_j(A)) \cup \text{Cl}_j(\text{int}_j(A))\}\).

Proof We will prove (i) and the other statements similarly:

Let \(A\) is \(j\)-preopen set, then \(A^c\) is \(j\)-pre closed set. Then \(A^c = C_j^p(A^c)\) and this implies \(A^c = A^c \cup \text{Cl}_j(\text{int}_j(A^c))\). Thus, \(\text{Cl}_j(\text{int}_j(A^c)) \subseteq A^c\).

Accordingly, \(A \subseteq \text{int}_j(\text{Cl}_j(A))\). \(\square\)

Definition 3.3 Let \((X, R, T_j)\) be a \(j\)-NS. The \(j\)-near interior operators generated by the relation \(R\) is defined by the operator \(\text{int}_j^k: \phi(X) \rightarrow \phi(X)\), for each \(j \in \{r, l, (r), (l), u, i, \langle u\rangle, \langle i\rangle\}\) and \(k \in \{p, s, \gamma\}\), where:

(i) The \(j\)-pre interior operator is defined by: \(\text{int}_j^p(A) = A \cap \text{int}_j(\text{Cl}_j(A))\).
(ii) The \(j\)-interior operator is defined by: \(\text{int}_j^i(A) = A \cap \text{Cl}_j(\text{int}_j(A))\).
(iii) The \(j\)-near interior operator is defined by: \(\text{int}_j^k(A) = \text{int}_j^i(A) \cup \text{int}_j^p(A)\).

Lemma 3.3 Let \((X, R, T_j)\) be a \(j\)-NS and \(A \subseteq X\). Then \(\forall j \in \{r, l, (r), (l), u, i, \langle u\rangle, \langle i\rangle\}\) and \(k \in \{p, s, \gamma\}\), the \(j\)-near interior \(\text{int}_j^k\) and \(j\)-near closure \(\text{Cl}_j^k\) operator are dual operators which means that: \(\text{int}_j^k(A) = (C_j^p(A^c))^c\) and \(\text{Cl}_j^k(A) = (\text{int}_j^k(A))^c\).

Proof We will prove the lemma in the case of \(k = p\) and the other cases similarly.

\((C_j^p(A^c))^c = (A^c \cup \text{int}_j(\text{Cl}_j(A^c)))^c = A \cap \text{int}_j(\text{Cl}_j(A^c))^c\).

By duality of \(\text{int}_j\) and \(\text{Cl}_j\) we get \(C_j^p(A^c))^c = A \cap \text{Cl}_j^k(\text{int}_j(A)) = \text{int}_j^p(A)\). \(\square\)

The following proposition gives the fundamental properties of the \(j\)-near interior operator \(\text{int}_j^k\).

Proposition 3.4 Let \((X, R, T_j)\) be a \(j\)-NS and \(A, B \subseteq X\). Then

(i) \(\text{int}_j^k(\emptyset) = \emptyset\) and \(\text{int}_j^k(X) = X\).
(ii) \(\text{int}_j^k(A) \subseteq A\), for each \(A \subseteq X\).
(iii) \(\text{If } A \subseteq B\), then \(\text{int}_j^k(A) \subseteq \text{int}_j^k(B)\).
(iv) \(\text{int}_j^k(A) \cup \text{int}_j^k(B) \subseteq \text{int}_j^k(A \cup B)\) and \(\text{int}_j^k(A \cap B) \subseteq \text{int}_j^k(A) \cap \text{int}_j^k(B)\).
(v) \(\text{int}_j^k(A) \subseteq \text{int}_j^k\left(\text{int}_j^k(A)\right)\).

Proof By the duality of \(\text{int}_j^k\) and \(C_j^p\), the proof is obvious. \(\square\)

Corollary 3.2 Consider \((X, R, T_j)\) be a \(j\)-NS. The subset \(A \subseteq X\) is \(j\)-near open set if \(\text{int}_j^k(A) = A\).

Lemma 3.4 Consider \((X, R, T_j)\) be a \(j\)-NS, and \(A, B \subseteq X\). For each \(j \in \{r, l, (r), (l), u, i, \langle u\rangle, \langle i\rangle\}\) and \(k \in \{p, s, \gamma\}\), if \(\text{Cl}_j\) is a topological closure, then we can prove the following properties:

(i) \(\text{int}_j^k\left(\text{int}_j^k(A)\right) = \text{int}_j^k(A)\).
(ii) \(\text{int}_j^k(A \cap B) = \text{int}_j^k(A) \cap \text{int}_j^k(B)\).
Corollary 3.3 Consider \((X, R, r_p)\) be a \(j\)-NS, and \(A, B \subseteq X\). For each \(j \in \{r, l, u, l\} \) and \(k \in \{p, s, r\}\), the following properties are true in general:

(i) \(int^j_k(int^j_k(A)) = int^j_k(A)\).

(ii) \(int^j_k(A \cap B) = int^j_k(A) \cap int^j_k(B)\).

The following proposition illustrates the relationship between the \(j\)-interior and the \(j\)-near interior operators.

Proposition 3.5 Let \((X, R, r_p)\) be a \(j\)-NS and \(A \subseteq X\). Then \(int^j_k(A) \subseteq int^j_k(A)\).

Proof By the duality of \(int^j_k\) and \(Cl^j_k\), and from Proposition 3.2, the proof is obvious. \(\square\)

The following proposition gives the relations between different types of \(j\)-near operators.

Proposition 3.6 Let \((X, R, r_p)\) be a \(j\)-NS and \(A \subseteq X\). Then

(i) \(Cl^j_k(A) \subseteq Cl^j_k(A)\).

(ii) \(Cl^j_k(A) \subseteq Cl^j_k(A)\).

(iii) \(int^j_k(A) \subseteq int^j_k(A)\).

(iv) \(int^j_k(A) \subseteq int^j_k(A)\).

Proof Straightforward. \(\square\)

Remark 3.2 Suppose that \((X, R, r_p)\) be a \(j\)-NS, and \(A \subseteq X\). Thus, the family of all \(j\)-pre closed (resp. the \(j\)-preopen) sets and the family of all \(j\)-semi closed (resp. \(j\)-semi open) sets are not comparable. Accordingly \(Cl^j_k\) (resp. \(int^j_k\)) operators and \(Cl^j_k\) (resp. \(int^j_k\)) operators are not comparable as the following example illustrates.

Example 3.4 Let \((X, R, r_p)\) be a \(j\)-NS, where \(X = \{a, b, c, d\}\) and \(R = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, b), (c, c), (c, d), (d, d)\}\). Then we can get:

The family of all \(r\)-pre closed sets of \(X\) is:

\[\Gamma_p^r = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}.\]

The family of all \(r\)-pre open sets of \(X\) is:

\[O_p^r = \{X, \emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}.\]

The family of all \(r\)-semi closed sets of \(X\) is:

\[\Gamma_s^r = \{X, \emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}\}.\]

The family of all \(r\)-semi open sets of \(X\) is:

\[O_s^r = \{X, \emptyset, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}.\]

Also, \(Cl^j_k(\{a\}) = \{a\}\) and \(Cl^j_k(\{a, b\}) = \{a, b, c\}\). But \(Cl^j_k(\{a\}) = \{a, b\}\) and \(Cl^j_k(\{a, b\}) = \{a, b\}\). Also \(int^j_k(\{b, c, d\}) = \{b, c, d\}\) and \(int^j_k(\{c, d\}) = \{d\}\). But \(int^j_k(\{b, c, d\}) = \{c, d\}\) and \(int^j_k(\{c, d\}) = \{c, d\}\).
4. Some applications of near closure operators in rough set theory

El-Bably (2016) has introduced some new generalized approximation operators of rough sets as a topological application of j-closure operators. In the present section, we introduce, for the first time, new generalized definitions for approximations (so called j-near approximation operators) as a generalization and modification for j-approximations. In fact, using j-near closure (resp. j-near interior) operators for each $j \in \{ (r), (l), (u), (i) \}$, new generalized definitions for rough approximation operators are obtained as a generalization to Pawlak rough approximations (Pawlak, 1982). In addition, comparisons between our generalization and some of other generalizations are provided.

Definition 4.1 (El-Bably, 2016). Let $(X, R, F_j)$ be a j-NS, and $A \subseteq X$. The j-lower and j-upper approximations of $A$ for each $j \in \{ (r), (l), (u), (i) \}$ are defined, respectively, by

$$L_j^r(A) = \left\{ x \in A | N_j(x) \subseteq A \right\} = \text{int}_j(A)$$

and

$$U_j^r(A) = A \bigcup \left\{ x \in X | N_j(x) \cap A \neq \emptyset \right\} = \text{Cl}_j(A).$$

Definition 4.2 (El-Bably, 2016). Let $(X, R, F_j)$ be a j-NS, and $A \subseteq X$. The j-boundary, j-positive and j-negative regions of $A$ are defined, respectively,

$$B_j(A) = U_j^r(A) - L_j^r(A), \text{POS}_j(A) = L_j^r(A) \text{ and } \text{NEG}_j(A) = X - U_j^r(A).$$

Definition 4.3 (El-Bably, 2016). Let $(X, R, F_j)$ be a j-NS, and $A \subseteq U$. Thus, the subset $A$ is called “j-exact” set if $L_j^r(A) = U_j^r(A) = A$. Otherwise, $A$ is called “j-rough set”.

Definition 4.4 (El-Bably, 2016). Let $(X, R, F_j)$ be a j-NS, and $A \subseteq X$. Thus, the j-accuracy of the approximations of $A$ is defined as follows:

$$\delta_j(A) = \frac{|L_j^r(A)|}{|U_j^r(A)|} \text{ where } |L_j^r(A)| \neq 0 \text{ and } |A| \text{ denotes the cardinality of } A.$$ 

Obviously, $0 \leq \delta_j(A) \leq 1$ and if $\delta_j(A) = 1$, then $A$ is j-exact set. Otherwise, it is j-rough.

The following definitions give the new approximation operators so called “j-near approximations” as a generalization to Pawlak approximations (Pawlak, 1982), j-approximations (El-Bably, 2016) and the other generalizations such as (Abd El-Monsef et al., 2014, 2015; Abo Khadra & El-Bably, 2008; Abo Khadra et al., 2007; D’eer et al., 2016; Martin, 2000; Restrepo et al., 2014; Tripathy & Mitra, 2010; Wang et al., 2008, 2013, 2014, 2015, 2016; Yao, 1996; Yao & Deng, 2014; Yu & Zhan, 2014; Zhao, 2016).

Definition 4.6 Let $(X, R, F_j)$ be a j-NS, and $A \subseteq X$. Then we define the j-near approximations of any subset $A$ as follows: For each $j \in \{ (r), (l), (u), (i) \}$:

i) The j-pre lower and j-pre upper approximation of $A$ is defined, respectively, by

$$L_j^p(A) = A \bigcap L_j(U_j^r(A)) = \text{int}_j(A)$$

and

$$U_j^p(A) = A \bigcup U_j(L_j^r(A)) = \text{Cl}_j(A).$$

ii) The j-semi lower and j-semi upper approximation of $A$ is defined, respectively, by

$$L_j^s(A) = A \bigcap U_j(L_j^r(A)) = \text{int}_j(A)$$
and
\[ \mathcal{U}^j_r(A) = A \cup \mathcal{L}_j(\mathcal{U}^j_r(A)) = \mathcal{C}_j^r(A). \]

(iii) The \( \gamma_j \)-lower and \( \gamma_j \)-upper approximation of \( A \) is defined, respectively, by
\[ \mathcal{L}_j^\gamma(A) = \mathcal{L}_j^j(A) \bigcup \mathcal{L}_j^r(A) = \text{int}_j^\gamma(A) \]
and
\[ \mathcal{U}_j^\gamma(A) = \mathcal{U}_j^j(A) \bigcap \mathcal{U}_j^r(A) = \mathcal{C}_j^\gamma(A). \]

From the above definition, we can notice that the new approximations “near approximations” are more accurate than the approximations “\( j \)-approaches” that are given by El-Bably (2016). It is clear that the \( j \)-near lower approximations are bigger than the \( j \)-lower approximations and also the \( j \)-near upper approximations are smaller than the \( j \)-approaches. Thus, the \( j \)-near approximations are very useful in exactness of sets as illustrated in the following example and definitions.

**Example 4.1** Consider \( (X, R, \mathcal{P}_j) \) be a \( j \)-NS, where \( X = \{a, b, c, d\} \) and \( R = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, b), (c, b), (c, d), (d, d)\} \). Suppose that \( A = \{a, d\}, B = \{a, b, c\} \) and \( C = \{b, c, d\} \). Then, we compute the \( j \)-approaches and the \( j \)-near approximations of the subsets \( A \) and \( B \), in the case of \( j = r \)(and the other cases similarly), as follows:

From Table 1, we noted that the subsets \( A, B, \) and \( C \) are rough in \( j \)-approaches but it may be exact in \( j \)-near approaches (such as the bold sets).

**Definition 4.7** Let \( (X, R, \mathcal{P}_j) \) be a \( j \)-NS, and \( A \subseteq X \). Then we define the \( j \)-near boundary and \( j \)-near accuracy of the \( j \)-near approximations of \( A \), respectively, as follows: For each \( j \in \{\{r\}, \{l\}, \{u\}, \{i\}\} \) and \( k \in \{p, s, \gamma\} \),

\[ \mathcal{B}_j^k(A) = \mathcal{U}_j^k(A) - \mathcal{L}_j^k(A) \]
and
\[ \delta_j^k(A) = \frac{|\mathcal{L}_j^k(A)|}{|\mathcal{U}_j^k(A)|}, \text{where}|\mathcal{U}_j^k(A)| \neq 0. \]

Obviously \( 0 \leq \delta_j^k(A) \leq 1 \).

**Definition 4.8** Let \( (X, R, \mathcal{P}_j) \) be a \( j \)-NS, and \( A \subseteq X \). Then \( \forall j \in \{\{r\}, \{l\}, \{u\}, \{i\}\} \) and \( k \in \{p, s, \gamma\} \), the subset \( A \) is called “\( j \)-near definable (briefly, \( k_j \)-exact) set” if \( \mathcal{L}_j^k(A) = \mathcal{U}_j^k(A) = A \). Otherwise, it is called \( j \)-near rough (briefly, \( k_j \)-rough).

It is clear that \( A \) is \( k_j \)-exact if \( \delta_j^k(A) = 1 \) and \( B_j^k(A) = \emptyset \). Otherwise, it is \( k_j \)-rough.

**Table 1. Comparison between the \( j \)-approaches and the \( j \)-near approximations**

| A       | \( \{a, d\} \)  | \( \{a, b, c\} \) | \( \{b, c, d\} \) |
|---------|-----------------|-----------------|-----------------|
| \( \mathcal{L}_j(A) \) | \( \{a, d\} \)  | \( \{a, b, c\} \) | \( \{b, c, d\} \) |
| \( \mathcal{U}_j^r(A) \) | \( \{a, d\} \)  | \( \{a, b, c\} \) | \( \{b, c, d\} \) |
| \( \mathcal{L}_j^p(A) \) | \( \{a, d\} \)  | \( \{a, b, c\} \) | \( \{b, c, d\} \) |
| \( \mathcal{L}_j^s(A) \) | \( \{a, d\} \)  | \( \{a, b, c\} \) | \( \{b, c, d\} \) |
| \( \mathcal{L}_j^\gamma(A) \) | \( \{a, d\} \)  | \( \{a, b, c\} \) | \( \{b, c, d\} \) |
| \( \mathcal{U}_j^\gamma(A) \) | \( \{a, d\} \)  | \( \{a, b, c\} \) | \( \{b, c, d\} \) |
The following proposition introduces the fundamental properties of the \( j \)-near approximations.

**Proposition 4.1** Let \( (X, R, \mathcal{P}) \) be a \( j \)-NS, and \( A, B \subseteq X \). Then \( \forall j \in \{r, l, u, i\}, k \in \{p, s, r\} \):

(i) \( \mathcal{L}_j^k(A) \subseteq A \subseteq \mathcal{L}_j^k(A) \).

(ii) \( \mathcal{L}_j^k(X) = \mathcal{L}_j^k(X) = X \) and \( \mathcal{L}_j^k(\emptyset) = \mathcal{L}_j^k(\emptyset) = \emptyset \).

(iii) If \( A \subseteq B \) then \( \mathcal{L}_j^k(A) \subseteq \mathcal{L}_j^k(B) \).

(iv) If \( A \subseteq B \) then \( \mathcal{L}_j^k(A) \subseteq \mathcal{L}_j^k(B) \).

(v) \( \mathcal{L}_j^k(A \cap B) \subseteq \mathcal{L}_j^k(A) \cap \mathcal{L}_j^k(B) \).

(vi) \( \mathcal{L}_j^k(A \cap B) \subseteq \mathcal{L}_j^k(A) \cap \mathcal{L}_j^k(B) \).

(vii) \( \mathcal{L}_j^k(A \cup B) \supseteq \mathcal{L}_j^k(A) \cup \mathcal{L}_j^k(B) \).

(viii) \( \mathcal{L}_j^k(A \cup B) \supseteq \mathcal{L}_j^k(A) \cup \mathcal{L}_j^k(B) \).

(ix) \( \mathcal{L}_j^k(A) = \left[ \mathcal{L}_j^k(A^c) \right]^c \), where \( A^c \) is the complement of \( A \).

(x) \( \mathcal{L}_j^k(A) = \left[ \mathcal{L}_j^k(A^c) \right]^c \), where \( A^c \) is the complement of \( A \).

(xi) \( \mathcal{L}_j^k(A^c) = \mathcal{L}_j^k(A) \).

(xii) \( \mathcal{L}_j^k(A^c) = \mathcal{L}_j^k(A) \).

**Proof** By the properties of the \( j \)-near interior and the \( j \)-near closure, the proof is obvious. \( \square \)

The following results introduce the relationships between the \( j \)-approximations and the \( j \)-near approximations. Moreover, they show that the near approximations are more accurate than other approaches.

**Theorem 4.1** Let \( (X, R, \mathcal{P}) \) be a \( j \)-NS, and \( A \subseteq X \). Then \( \forall j \in \{r, l, u, i\}, k \in \{p, s, r\} \):

\( \mathcal{L}_j^k(A) \subseteq \mathcal{L}_j^k(A) \subseteq \mathcal{L}_j^k(A) \subseteq \mathcal{L}_j^k(A) \)

**Proof** We will prove the proposition in case of \( k = p \) and the other cases similarly:

Let \( x \in \mathcal{L}_j^k(A) \), then \( x \in A \) such that \( N_j(x) \subseteq A \). Thus \( x \in A \) such that \( N_j(x) \subseteq \mathcal{L}_j^k(A) \) and this implies \( x \in \mathcal{L}_j^k(A) \). By duality, we get \( \mathcal{L}_j^k(A) \subseteq \mathcal{L}_j^k(A) \). \( \square \)

**Corollary 4.1** Let \( (X, R, \mathcal{P}) \) be a \( j \)-NS, and \( A \subseteq X \). Then \( \forall j \in \{r, l, u, i\}, k \in \{p, s, r\} \):

(i) \( \mathcal{N}_j^k(A) \subseteq \mathcal{B}_j^k(A) \).

(ii) \( \delta_j(A) \leq \delta_j^k(A) \).

**Remark 4.2** The main goals of the following examples are:

(i) The converse of the above results is not true in general.

(ii) The \( j \)-near approximations has accuracy measures more accurate than other measures such as Pawlak (1982), Yao (1996) and other measures. So, the use of the \( j \)-near concepts in rough set context is very useful for removing the vagueness of rough sets and thus it is very interesting in decision-making.

**Example 4.2** Let \( (X, R, \mathcal{P}) \) be a \( j \)-NS, where \( X = \{a, b, c, d\} \) and \( R = \{(a,a), (a,b), (b,a), (b,b), (c,a), (c,b), (c,c), (c,d), (d,d)\} \). Thus, we get

\( N_j(A) = \{a, b\} = N_j(B), N_j(c) = X, N_j(d) = \{d\} \).
Abbas et al., Cogent Mathematics (2016), 3: 1247505
http://dx.doi.org/10.1080/23311835.2016.1247505

Table 2 gives comparisons between the \(j\)-accuracy of \(j\)-approximations and the \(j\)-near accuracy of \(j\)-near approximations of the all subsets of \(X\), in case of \(j = \langle r \rangle\):

From Table 2, we notice that:

(i) The accuracy of the \(j\)-near approximations is more accurate than the accuracy of \(j\)-approximations, for example the shaded sets in Table 2 has accuracy measure 100% in each \(j\)-near approximations (\(p\), \(s\) and \(\gamma\)-approximations).

(ii) Using \(\gamma\)-method in constructing the approximations of sets is more accurate than others types, since for any subset \(A \subseteq X\), \(\delta_{\gamma}(A)\) \(\leq\) \(\delta_{p}(A)\) and \(\delta_{\gamma}(A)\) \(\leq\) \(\delta_{s}(A)\), \(\forall k \in \{p, s\}\). Thus, these approaches will helps to extract and discovery the hidden information in data that were collected from real-life applications.

(iii) Every \(\langle r \rangle\)-exact set is \(\langle r \rangle\)-near exact, but the converse is not true. For example, all shaded sets in Table 2.

**Remark 4.3** The following result is very interesting because it proves that the \(j\)-near approaches are more accurate than the classical \(j\)-approaches (El-Bably, 2016). Moreover, it illustrates the importance of \(j\)-near concepts in exactness of sets.

**Proposition 4.2** Consider \((X, R, T)\) be a \(j\)-NS, and \(A \subseteq X\). For each \(j \in \{\langle r \rangle, \langle l \rangle, \langle u \rangle\}\) and \(k \in \{p, s, \gamma\}\), if \(A\) is \(j\)-exact set, then it is \(k\)-exact.

**Proof** If \(A\) is \(j\)-exact set, then \(B_j(A) = \emptyset\). Thus, by Corollary 4.1, \(B_j^k(A) = \emptyset\) and accordingly \(A\) is \(k\)-exact.

**Remark 4.4** The converse of the above proposition is not true in general as Example 4.2 illustrated. The main goal of the following results is to introduce the relationships among different types of \(j\)-near approximations, \(j\)-near boundary, \(j\)-near accuracy, and \(j\)-near exactness, respectively. Moreover, it gives the best approaches.
Proposition 4.3 Let \((X, R, \tau_j)\) be a \(j\)-NS, and \(A \subseteq X\). Then \(\forall j \in \{(r), (l), (u), (i)\}\):

The following statements are true in general.

(i) \(L^j(A) \subseteq L^j_\tau(A)\).
(ii) \(L^j_\tau(A) \subseteq L^j_\tau(A)\).
(iii) \(U^j(A) \subseteq U^j_\tau(A)\).
(iv) \(U^j_\tau(A) \subseteq U^j_\tau(A)\).

Proof Obvious. \(\Box\)

Corollary 4.2 Let \((X, R, \tau_j)\) be a \(j\)-NS, and \(A \subseteq X\). Then \(\forall j \in \{(r), (l), (u), (i)\}\):

The following statements are true in general.

(i) \(\delta^j(A) \leq \delta^j_\tau(A)\).
(ii) \(\delta^j_\tau(A) \leq \delta^j(A)\).
(iii) \(R^j_\tau(A) \subseteq R^j_\tau(A)\).
(iv) \(R^j_\tau(A) \subseteq R^j_\tau(A)\).

Corollary 4.3 Let \((X, R, \tau_j)\) be a \(j\)-NS, and \(A \subseteq X\). Then \(\forall j \in \{(r), (l), (u), (i)\}\):

The following statements are true in general.

(i) \(A\) is \(p_j\)-exact \(\Rightarrow\) \(A\) is \(x_j\)-exact.
(ii) \(A\) is \(s_j\)-exact \(\Rightarrow\) \(A\) is \(y_j\)-exact.

Remark 4.5

(i) For each \(\forall j \in \{(r), (l), (u), (i)\}\): \(\delta^j_\tau(A) = \max\left(\delta^j_\tau(A), \delta^j(A)\right)\), where \(\max\) represents the maximum value of the two quantities. Accordingly, we can say that the using \(\tau_j\) in constructing the approximations of sets is more accurate than others types, since for any subset \(A \subseteq X\), \(\delta_j(A) \leq \delta_j^\tau(A)\) and \(\delta_j^\tau(A) \leq \delta_j(A)\), \(\forall k \in \{p, s\}\). Thus, these approaches will helps to extract and discover the hidden information in data that were collected from real-life applications.

(ii) The converse of the above results is not true in general as illustrated by Examples 4.1 and 4.2.

5. Practical example

In this section, an applied example (Martin, 2000) of the current method in the network connectivity devices is introduced. Moreover, the introduced example gives comparisons between our approaches and the other approaches such as “El-Bably approaches” (El-Bably, 2016) and “Yao approaches” (Yao, 1996).

Networks deploy many different hardware components known as devices. They are connected as resources for network users. We are already familiar with PCs, printers, and other office equipment. Other equipment that we may not be familiar with are network interface card, hubs, switches, repeaters, bridges, routers, and gate-ways (Martin, 2000).

Example 5.1 Let \(X = \{x_1, x_2, x_3, x_4\}\) be a set of four network connectivity devices and \(A = \{A_1, A_2, A_3\}\) be the attributes of network connectivity devices, where \(x_1\) is a hub, \(x_2\) is a switch, \(x_3\) is a bridge and \(x_4\) is a router, and the set of attributes is:
$A_1 = \text{Connection} = \{a_1, b_1, c_1, d_1, e_1, f_1, g_1\}$,

where $a_1 = \text{Connect all computers on each side of the network}$, $b_1 = \text{Connect two segments of the same LAN}$, $c_1 = \text{Connect two local area networks (LANs)}$, $d_1 = \text{Connect two similar network}$, $e_1 = \text{Connect two dissimilar network}$, $f_1 = \text{Divide a busy network into two segments}$, and $g_1 = \text{Select the best path to route a message based on the destination address and origin}$.

$A_2 = \text{Read or cannot read the addresses} = \{a_2, b_2, c_2, d_2\}$,

where $a_2 = \text{Read the addresses of all computers on each side of the network}$, $b_2 = \text{Cannot read the addresses of any computer in the network}$, $c_2 = \text{Read the addresses of bridges on the network}$, and $d_2 = \text{Read the addresses of other routers on the network}$.

$A_3 = \text{Cost} = \{a_3, b_3, c_3, d_3\}$,

where $a_3 = 100 \text{ L.E.}$, $b_3 = 300 \text{ L.E.}$, $c_3 = 500 \text{ L.E.}$ and $d_3 = 800 \text{ L.E.}$.

Consider Table 3:

Let $R$ be a general relation as follows:

$x_1 Rx_2 \iff A(x_1) \subseteq A(x_2), \ x_1, x_2 \in X$.

Thus, we compute the approximations of first attribute $A_1$ and the others similarly:

For the first attribute $A_1$, we get

$R_{A_1} = \{(x_1, x_1), (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_2), (x_2, x_3), (x_2, x_4), (x_3, x_3), (x_3, x_4), (x_4, x_4)\}$.

Thus, using Definition 2.1, the $i$-neighborhoods of all elements of $X$ are:

$N_i(x_1) = N_i(x_2) = \{x_1, x_2\}$, $N_i(x_3) = \{x_3\}$, $N_i(x_4) = \{x_4\}$.

Yao (1996), introduced the lower and upper approximation of any subset as follows:

$\text{Ap}(A) = \{x \in X : xR \subseteq A\}$ and $\text{Ap}(A) = \{x \in X : xR \cap A \neq \emptyset\}$.

The boundary and the accuracy of Yao approximations are:

$B(A) = \text{Ap}(A) - \text{Ap}(A)$ and $\delta(A) = \frac{\text{Ap}(A)}{\text{Ap}(A)}$.

Also, using Definition 4.1 and Definition 4.6, we compute the $j$-approximations (resp. $j$-accuracy) and $j$-near approximations “$k_j$-approximations” (resp. $j$-near accuracy) in the case of $j = i, k = p$ for some subsets of $X$, and the other cases similarly, as follows:

| Table 3. Attributes network connectivity devices |
|----------------|----------------|----------------|----------------|
| $X/A$ | $A_1$ | $A_2$ | $A_3$ |
| $(x_1)$ | $(a_1, b_1)$ | $(b_1)$ | $(a_1)$ |
| $(x_2)$ | $(a_2, c_2)$ | $(c_2)$ | $(c_2)$ |
| $(x_3)$ | $(a_3, b_3, c_3, d_3, f_3)$ | $(c_3)$ | $(c_3)$ |
| $(x_4)$ | $(a_4, b_4, c_4, d_4, e_4, g_4)$ | $(d_4, c_4, b_4)$ | $(d_4)$ |
From Tables 4 and 5, we notice that:

(i) The accuracy of the j-near approximations is more accurate than the accuracy of j-approxi-
mations and Yao approaches.

(ii) Using j-near approaches in constructing the approximations of sets is more accurate than others types, because the current approximation decreases the boundary region by increasing the lower approximation and decreasing the upper approximation. Thus, these approaches will help to extract and discover the hidden information in data that were collected from real-life applications which are very useful in decision-making.

(iii) Every exact set in Yao approaches (Yao, 1996) is exact in j-near approaches (El-Bably, 2016) and also is exact in j-near approaches, but the converse is not true.

| A                  | Yao approaches (Yao, 1996) | j-Approaches (El-Bably, 2016) | j-Near approaches |
|--------------------|-----------------------------|--------------------------------|------------------|
|                    | Apr(A)                      | L^j(A)                         | L^j_1(A)         | L^j_2(A)         |
| (x_1)              | ∅                           | (x_1, x_j)                     | {x_1}            | {x_1}            |
| (x_2)              | ∅                           | (x_2, x_j)                     | {x_2}            | {x_2}            |
| (x_1, x_2)         | (x_1)                       | (x_1, x_2)                     | (x_1, x_2)       | (x_1, x_2)       |
| (x_1, x_3)         | (x_1)                       | (x_1, x_3)                     | (x_1, x_3)       | (x_1, x_3)       |
| (x_2, x_3)         | (x_2)                       | (x_2, x_3)                     | (x_2, x_3)       | (x_2, x_3)       |
| (x_1, x_2, x_3)    | (x_1, x_2)                  | (x_1, x_2, x_3)                | (x_1, x_2, x_3)  | (x_1, x_2, x_3)  |

| A                  | Yao approaches (Yao, 1996) | j-Approaches (El-Bably, 2016) | j-Near approaches |
|--------------------|-----------------------------|--------------------------------|------------------|
|                    | B(A)                        | δ(B(A))                        | δ_1(A)           | δ_2(A)           | δ_3(A)           |
| (x_1)              | {x_1, x_j}                  | 0                              | 0                | ∅                | 1                |
| (x_2)              | {x_1, x_j}                  | 0                              | 0                | ∅                | 1                |
| (x_1, x_2)         | {x_1, x_2}                  | 1/3                            | 1                | ∅                | 1                |
| (x_1, x_3)         | {x_1, x_3}                  | 1/3                            | 1/3              | ∅                | 1                |
| (x_2, x_3)         | {x_2, x_3}                  | 1/3                            | 1/3              | ∅                | 1                |
| (x_1, x_2, x_3)    | {x_1, x_2, x_3}             | 1/3                            | 1/3              | ∅                | 1                |
6. Conclusion and future work

In short, topology is a branch of mathematics, whose concepts are fundamental not only to all branches of mathematics, but also in real-life applications. Closure spaces “or closure operator induced by relations” are the generalization to topological spaces. Closure operators which are constructed from binary relation play an important role in real life. In this paper, we have introduced and studied the new concepts “near closure” operators induced from any binary relation. The suggested operators can be considered as a generalization for -closure operators (El-Bably, 2016), Allam definition (Allam et al., 2006), and Galton definition (Galton, 2003). Moreover, we study the relationships between rough sets and topological space. Also, we obtained the lower and the upper approximation operators from the interior and the closure operators (respectively) and vice versa. Finally, we have introduced a practical example of our structures in the network connectivity devices. Comparisons between our approaches and some of the other generalization (namely, Yao approach, Yao, 1996) are obtained. Thus, we can say that the introduced structures open the way for more topological applications in many fields. Therefore, we choose this line aiming to fill the gap between topology and application.

We conclude that the intermingling of topology and j-near concepts in the construction of some approximation space concepts will help in getting results with abundant logical statements, discovering hidden relationships among data and, moreover, probably help in producing accurate programs. In the present paper, we put the starting point for topological applications in rough set theory. The relevance of the rough set theory is that several methods for solving problems in machine learning and decision-making have been developed based on it (also in other type of problems). So, the following points will be studied in the future:

- The division of the boundary region can be applied to enlarge the range of rough membership. Consequently, it adds new notions in rough membership relations, rough membership functions, rough equality, rough inclusion, and rough power set.
- Apply the near concepts in many fields of rough set theory such as “reduction, data mining, decision-making and granular computing”.

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