Abstract

Random divisions of an interval arise in various context, including statistics, physics, and geometric analysis. For testing the uniformity of a random partition of the unit interval \([0, 1]\) into \(k\) disjoint subintervals of size \((S_k[1], \ldots, S_k[k])\), Greenwood (1946) suggested using the squared \(\ell_2\)-norm of this size vector as a test statistic, prompting a number of subsequent studies. Despite much progress on understanding its power and asymptotic properties, attempts to find its exact distribution have succeeded so far for only small values of \(k\). Here, we develop an efficient method to compute the distribution of the Greenwood statistic and more general spacing-statistics for an arbitrary value of \(k\). Specifically, we consider random divisions of \([1, 2, \ldots, n]\) into \(k\) subsets of consecutive integers and study \(\|S_{n,k}\|_{p,w}^p\), the \(p\)th power of the weighted \(\ell_p\)-norm of the subset size vector \(S_{n,k} = (S_{n,k}[1], \ldots, S_{n,k}[k])\) for arbitrary weights \(w = (w_1, \ldots, w_k)\). We present an exact and quickly computable formula for its moments, as well as a simple algorithm to accurately reconstruct a probability distribution using the moment sequence. We also study various scaling limits, one of which corresponds to the Greenwood statistic in the case of \(p = 2\) and \(w = (1, \ldots, 1)\), and this connection allows us to obtain information about regularity, monotonicity and local behavior of its distribution. Lastly, we devise a new family of non-parametric tests using \(\|S_{n,k}\|_{p,w}^p\) and demonstrate that they exhibit substantially improved power for a large class of alternatives, compared to existing popular methods such as the Kolmogorov-Smirnov, Cramér-von Mises, and Mann-Whitney/Wilcoxon rank-sum tests.
against a specific class of alternatives. Around the same time, Gardner (1952) fully described the case \( k = 3 \) by fruitfully interpreting the distribution function of the test statistic as the volume of intersection between a sphere and the simplex. Subsequently, Darling (1953) found a closed-form, but difficult to invert, characteristic function, while Weiss (1956) and Sethuraman and Rao (1970) investigated the role of Greenwood’s statistic in the context of the general goodness-of-fit test of whether a sample \( X_1, \ldots, X_n \) follows a given arbitrary distribution \( F \). There they proved that a test based on Greenwood’s statistic is both most powerful against symmetric linear alternatives, and enjoys the greatest asymptotic relative efficiency against a wide class of tests and alternative hypotheses. Given these favorable properties, but analytically intractable nature of Greenwood’s statistic, approximations and numerical schemes were devised by Burrows (1979), Currie (1981) and Stephens (1981) to tabulate scores for certain significance levels up to \( k = 20 \). Most recently, Schechtman and Zinn (2000) exploited the geometric interpretation of Gardner (1952) to derive an explicit rate function to characterize the large deviations of Greenwood’s statistic as \( k \to \infty \).

Closely connected to Greenwood’s statistic, and applicable in an equal variety of settings, is a discretized version of it: instead of sampling uniformly from the simplex \( \Delta^{k-1} \), the null distribution may be uniform over all integral points in the scaled simplex \( n \cdot \Delta^{k-1} \) for some \( n \in \mathbb{Z}^+ \). Tests and computations based on such measure occur in various contexts in computational biology (e.g., Palamara et al., 2018; Riley et al., 2007), physics (where it is known as the Bose-Einstein distribution) and theoretical statistics, where it emerges naturally as an urn model (e.g., Holst, 1979). Greenwood’s statistic has a natural analogue in this discretized scenario, where it is known as Dixon’s statistic after its proposed use by Dixon (1940) for performing non-parametric two-sample testing. However, although its asymptotic behavior has been well studied by Holst and Rao (1980), a description in the non-asymptotic regime as well as convergence rates have, to the best of our knowledge, remained elusive.

Here we fill this gap by studying a generalized family of Greenwood’s statistics (including Dixon’s statistic) for finite \( n \) and \( k \), for which we are able to exactly and efficiently compute its moments. Using this knowledge, we examine various scaling limits, proving CLT results as well as identifying novel limiting distributions. We then quantify the connection between Dixon’s and Greenwood’s statistic through precise convergence rates and monotonicity results, while using our understanding from the discrete setting to offer new insights into the moment sequence, smoothness and monotonicity of Greenwood’s statistic. Finally, we propose a simple and efficient algorithm that recovers an underlying continuous distribution from its first \( m \) moments up to \( O(m^{-1}) \) accuracy, and use it to devise a powerful hypothesis test of whether two data sets \( \{X_1, \ldots, X_{k-1}\} \) and \( \{Y_1, \ldots, Y_n\} \) were sampled from the same distribution. We demonstrate the test’s suitability for a large class of alternatives through extensive power studies, and compare it with the classical Kolmogorov-Smirnov, Cramér-von Mises and Mann-Whitney tests. We illustrate how the same principles employed in designing our two-sample test can be applied equally successfully to the settings of one-sample tests, and tests using paired data.

## 2 Preliminaries and notation

For a positive integer \( n \), we use \([n]\) to denote the set \( \{1, \ldots, n\} \). The two probability spaces underlying most of our discussion will consist of the \((k-1)\)-dimensional probability simplex

\[
\Delta^{k-1} = \left\{ (x_1, x_2, \ldots, x_k) \in [0,1]^k : \sum_{i=1}^k x_i = 1 \right\}
\]
together with the uniform measure \( \mu_{\Delta k^{-1}} = \sigma / \sigma(\Delta^{k-1}) = (k-1)! / \sqrt{k} \), where \( \sigma \) is surface measure in \( \mathbb{R}^k \), and its discretized version

\[
D_{n,k} = (n\Delta^{k-1}) \cap \mathbb{Z}^k = \left\{ (z_1, z_2, \ldots, z_k) \in \{0, \ldots, n\}^k : \sum_{i=1}^k z_i = n \right\}
\]

with its uniform measure \( \mu_{D_{n,k}} = |D_{n,k}|^{-1} = (n^{k-1})^{-1} \). In other words, \( \mu_{\Delta k^{-1}} \) is the law of a Dirichlet(1, 1, ..., 1) variable, while a \( k \)-part weak composition of \( n \) chosen uniformly at random is distributed according to \( \mu_{D_{n,k}} \). Occasionally we will refer to this latter distribution as a uniform configuration of \( n \) balls distributed over \( k \) bins.

To test the hypothesis of whether a sampled random variable \( X = (X_1, \ldots, X_k) \) in \( \mathbb{Z}^k \) or \( \mathbb{R}^k \) has distribution \( \mu_{\Delta k^{-1}} \) or \( \mu_{D_{n,k}} \), respectively, we are interested in comparing the distribution of its weighted \( \ell_p \)-norms

\[
\|X\|_{p,w}^p = \sum_{i=1}^k w_i X_i^p,
\]

for some fixed weight vector \( w = (w_1, \ldots, w_k) \in \mathbb{R}^k \), against its null distribution. That is, if \( S_k \sim \mu_{\Delta k^{-1}} \) and \( S_{n,k} \sim \mu_{D_{n,k}} \), we are interested in studying the distributions of \( \|S_k\|_{p,w}^p \) and \( \|S_{n,k}\|_{p,w}^p \) (to prevent confusion of powers and vector indices, we will denote entries of \( S_k \) and \( S_{n,k} \) by \( S_k[j] \) and \( S_{n,k}[j] \) for \( j \in [k] \)). For \( p = 2 \) and \( w = 1_k \), with \( 1_k := (1, \ldots, 1) \) being the all-ones vector of length \( k \), these are precisely Greenwood’s and Dixon’s statistics, respectively.

A primary reason why understanding these statistics is important is their application to non-parametric testing. If \( Z_1, \ldots, Z_N \) are independently sampled from the same continuous distribution \( F \), then the spacings (that is, the differences between consecutive order statistics) of \( 0, F(Z_1), \ldots, F(Z_N), 1 \) are distributed according to \( \mu_{\Delta N} \), so characterizing the distribution of Greenwood’s statistic allows to test if the sample is distributed according to \( F \). Similarly, if \( X_1, \ldots, X_{k-1}, Y_1, \ldots, Y_n \) are i.i.d. samples from the same continuous distribution \( G \), and we define the order statistics

\[
-\infty =: X(0) < X(1) < \cdots < X(k-1) < X(k) := \infty,
\]

and the number of \( Y_i \) sandwiched between every two consecutive order statistics

\[
S_{n,k}[j] := \# \{ i : X(i-1) \leq Y_i < X(i) \},
\]

for \( j \in [k] \), then \( (S_{n,k}[1], \ldots, S_{n,k}[k]) \) is distributed according to \( \mu_{D_{n,k}} \).

These two hypothesis tests will be our main application considered in Section 5. Before coming to those, in Section 3 we present our main theoretical results about \( \|S_{n,k}\|_{p,w}^p \) and \( \|S_k\|_{p,w}^p \). In particular, we will detail a simple and efficient way to compute their moment sequences exactly. Recovering the distribution of \( \|S_{n,k}\|_{p,w}^p \) and \( \|S_k\|_{p,w}^p \) from these moments seems analytically tractable, but can be done algorithmically in an efficient and exact manner. Section 4 is dedicated to describing such an algorithm.

3 Moments and scaling limits of generalized spacing-statistics

We start by investigating the discrete generalized spacing-statistics \( \|S_{n,k}\|_{p,w}^p \), from which their continuous analogues will follow. Our point of departure is the well-known Wilcoxon-Mann-Whitney \( U \) statistic (Mann and Whitney, 1947; Wilcoxon, 1945). It is easy to see that in the special case of \( p = 1 \) and \( w = (k-1)_\downarrow := (k-1, k-2, \ldots, 1, 0) \), we recover, up to an explicit constant depending
only on k, the U statistic. Indeed, for two samples \(X_1, \ldots, X_{k-1}\) and \(Y_1, \ldots, Y_n\), with \(R_1, \ldots, R_{k-1}\) the ranks of \(X_1, \ldots, X_{k-1}\) computed in the joint ensemble \(\{X_1, \ldots, X_{k-1}, Y_1, \ldots, Y_n\}\), their U statistic is given by \(U = \sum_{j=1}^{k-1} R_j\). In our notation introduced in (1), we then have \(R_j = j + \sum_{i=1}^{j} S_{n,k}[i]\), and consequently

\[
U = \sum_{j=1}^{k-1} R_j = \sum_{j=1}^{k-1} \left[ j + \sum_{i=1}^{j} S_{n,k}[i] \right] = \sum_{j=1}^{k-1} j + \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} S_{n,k}[i] = \binom{k-1}{2} + \sum_{i=1}^{k-1} (k-i)S_{n,k}[i] = \binom{k-1}{2} + \|S_{n,k}\|_{1,(k-1)↓},
\]

as desired. In order to both compute the exact distribution of \(U\) under the null hypothesis of all \(X_i, Y_i\) being generated i.i.d, as well as its asymptotic normality, Mann and Whitney exploited a recurrence relation which, in our language, consists of conditioning on the occupancy of the very last entry in \(S_{n,k}\). Conditional on the event \(\{S_{n,k}[k] > 0\}\) — that is, conditional on the last bin containing at least one ball — we may remove one such ball to find that \(\left(\|S_{n,k}\|_{1,(k-1)↓} \setminus \{S_{n,k}[k] > 0\}\right) \overset{d}{=} \|S_{n-1,k}\|_{1,(k-1)↓}\). Similarly, on \(\{S_{n,k}[k] = 0\}\) we may omit the very last bin to arrive at \(\left(\|S_{n,k}\|_{1,(k-1)↓} \setminus \{S_{n,k}[k] = 0\}\right) \overset{d}{=} \|S_{n-1,k}\|_{1,(k-2)↓} + n\). Combining these two observations, and writing \(q_{n,k}(x) = \mathbb{P}\left(\|S_{n,k}\|_{1,(k-1)↓} = x\right)\), yields the two-term recursion

\[
q_{n,k}(x) = \frac{n}{n+k-1} q_{n-1,k}(x) + \frac{k-1}{n+k-1} q_{n-1,k}(x-n),
\]

from which the whole probability mass function of \(\|S_{n,k}\|_{1,(k-1)↓}\) can be computed in \(O(n^2k^2)\) time. Moreover, this recursion directly translates into a recurrence of the moments of \(\|S_{n,k}\|_{1,(k-1)↓}\), which allowed Mann and Whitney to prove a central limit theorem as \(n, k \to \infty\), thus rendering \(U\) a versatile and quickly computable two-sample test statistic.

Unfortunately, the recurrence relation (2) lacks robustness with respect to varying either \(p\) or \(w = (w_1, \ldots, w_k)\): for \(p > 1\) and generic \(w\), removing a ball from any bin changes \(\|S_{n,k}\|_{p,w}\) by an amount depending on the total number of balls in that bin, and removing a bin may result in weights that are in no way related to the original weights. This can be fixed by conditioning not only on the vacancy of \(S_{n,k}[k]\), but its precise occupancy. Defining \(q_{n,k}^{p,w}(x) := \mathbb{P}\left(\|S_{n,k}\|_{p,w} = x\right)\) and \(w_{-k} := (w_1, \ldots, w_{k-1})\), we have

\[
q_{n,k}^{p,w}(x) = \sum_{j=0}^{n} \mathbb{P}(S_{n,k}[k] = j) q_{n-j,k-1}^{p,w-k}(x - w_k j^p) = \binom{n+k-1}{k-1}^{-1} \sum_{j=0}^{n} \binom{n-j+k-2}{k-2} q_{n-j,k-1}^{p,w-k}(x - w_k j^p),
\]

with initial conditions

\[
q_{n,k}^{p,w}(x) = \begin{cases} 0, & \text{if } x \notin \{x_{\min}, x_{\max}\} \text{ or } n < 0, \\ 1_{x=x_1,n^p}, & \text{if } k = 1, \\ 1_{x=0}, & \text{if } n = 0, \\ \end{cases}
\]
where \( x_{\min} = \min_{s,n,k \in D_{n,k}} \|S_{n,k}\|_{p,w}^p \) and \( x_{\max} = \|w\|_{\infty}^p \) bound the range of \( \|S_{n,k}\|_{p,w}^p \). However, solving (3) and (4) generically requires \( O \left( n^{p+1} k \|w\|_{\infty} \right) \) time, thus rendering it unfeasible to compute for even modestly large \( n \) and \( p \). To circumvent this problem, in what follows we will instead use a moment-based approach.

### 3.1 Exact moments of \( \|S_{n,k}\|_{p,w}^p \) and \( \|S_{k}\|_{p,w}^p \)

The following theorem, together with Proposition 6 in Section 4, demonstrates how the aforementioned computational intractability can be circumvented by lifting (3) to the moments of \( \|S_{n,k}\|_{p,w}^p \):

**Theorem 1.** Let \( G(x, y) = \sum_{m=0}^{\infty} \frac{\text{Li}_{-m}(x) y^m}{m!} \), where \( \text{Li}_s(x) = \sum_{j=1}^{\infty} j^{-s} x^j \) is the polylogarithm function. Denoting by \([x^n y^m]P(x, y)\) the \((n,m)\)th coefficient of a power series \( P \) in \( x \) and \( y \), we have

\[
\mathbb{E} \left( \|S_{n,k}\|_{p,w}^p \right)^m = \frac{m!}{(n+k-1)!} \left[ x^n y^m \right] \prod_{i=1}^{k} G(x, w_i y).
\]

In particular, the first \( m \) moments of \( \|S_{i,j}\|_{p,w}^p \) for \((i, j) \in \{0, \ldots, n\} \times \{1, \ldots, k\} \) can be computed in \( O \left( n m \cdot (\log nm) \cdot (\log k) \right) \) time.

**Proof.** We first expand the left-hand side of (5) to find

\[
\mathbb{E} \left( \|S_{n,k}\|_{p,w}^p \right)^m = \sum_{\sigma \in D_{n,k}} \mathbb{P}(S_{n,k} = \sigma) \left( \sum_{j=1}^{k} w_j \sigma_j^p \right)^m
\]

\[
= \left( \frac{n + k - 1}{k - 1} \right)^{-1} \sum_{\sigma \in D_{n,k}} \sum_{\eta \in D_{m,k}} \left( \sum_{j=1}^{m} \prod_{j=1}^{k} \frac{(w_j \sigma_j^p \eta_j)}{\eta_j!} \right),
\]

so it remains to show that \( A_{n,k,m,w} = [x^n y^m] \prod_{j=1}^{k} G(x, w_j y) \). By definition of \( \text{Li}_s(x) \), we have for every fixed \( \eta \in D_{m,k} \)

\[
\sum_{\sigma \in D_{n,k}} \prod_{j=1}^{k} w_j^{\eta_j} \sigma_j^{p \eta_j} = [x^n] \prod_{j=1}^{k} \frac{\text{Li}_{-p \eta_j}(x)}{\eta_j!} w_j^{\eta_j},
\]

and so

\[
A_{n,k,m,w} = [x^n] \left\{ [y^m] \prod_{j=1}^{k} \left( \sum_{i=0}^{\infty} \frac{\text{Li}_{-p \eta_j}(x)}{i!} (w_j y)^i \right) \right\}
\]

\[
= [x^n y^m] \prod_{j=1}^{k} G(x, w_j y),
\]

as desired. The \( O \left( n m \cdot (\log nm) \cdot (\log k) \right) \) runtime is now a direct consequence of computing the Cauchy product of \( k \) bivariate degree-\((n, m)\) polynomials using the Fast Fourier Transform. \( \square \)
The above theorem will be our main tool for devising an efficient, powerful, and general two-sample test in Section 5, if the two samples are modestly sized. In the case where one sample is significantly larger than the other, the following result, paired with PROPOSITION 6 below, yields an even more efficient testing procedure.

**Proposition 1.** For fixed $k$, $S_{n,k}/n$ converges in distribution to $S_k \sim \mu_{\Delta^{k-1}}$. In particular, we have

$$\lim_{n \to \infty} \mathbb{P}\left( \frac{S_{n,k}}{n} \in E \right) = \mathbb{P}(S_k \in E)$$

for any Lebesgue-measurable set $E \subset \Delta^{k-1}$. Moreover, by Dynkin’s $\pi$-$\lambda$ theorem, we may without loss of generality assume $E$ to be a box with rational vertex points inside $\Delta^{k-1}$ (see e.g. Theorem 1.1 in Kallenberg, 2006), which will allow us to count the number of lattice points in $E$ as $n \to \infty$.

To wit, let $L(E,n)$ be the cardinality of the set $nE \cap \mathbb{Z}^k$, then Ehrhart theory informs us that (cf. Ehrhart, 1967)

$$L(E,n) = \text{Vol}_\Lambda(E) \cdot n^{k-1} + O(n^{k-2}),$$

where $\text{Vol}_\Lambda(E) = \lambda_{k-1}(E)/d(\Lambda)$ is the $(k-1)$-dimensional Lebesgue volume of $E$, normalized by the co-volume of the lattice $\Lambda = \mathbb{Z}^k \cap H$ induced by $\mathbb{Z}^k$ on the hyperplane $H = \{x \in \mathbb{R}^k : \sum_{j=1}^k x_j = 0\}$.

But since the fundamental region of $\Lambda$ is the parallelepiped formed by $\{e_1 - e_j\}_{j \in \{2,\ldots,k\}}$, where $e_j \in \mathbb{R}^k$ is the $j$th standard basis vector, the co-volume can be computed to be

$$d(\Lambda)^2 = \det V^T V = \det \left( I_{k-1} + 1_{k-1}1_{k-1}^T \right),$$

with the columns of $V$ being given by $\{e_1 - e_j\}_{j \in \{2,\ldots,k\}}$. It is straightforward to check that $V^T V$ has eigenvalue 1 of multiplicity $k-2$ (with the associated eigenspace spanned by $\{e_1 - e_j\}_{j \in \{2,\ldots,k-1\}}$), and eigenvalue $k$ of multiplicity 1 (with eigenvector $1_{k-1} = (1,\ldots,1)$), and therefore

$$d(\Lambda) = \sqrt{k}. \quad (13)$$

Since $\sqrt{k}/(k-1)!$ is precisely the $(k-1)$-dimensional volume of $\Delta^{k-1}$, we finally arrive at

$$\mathbb{P}\left( \frac{S_{n,k}}{n} \in E \right) = \frac{L(E,n)}{\lambda_{k-1}(E)} \xrightarrow{n \to \infty} \frac{(k-1)! \sqrt{k}}{\lambda_{k-1}(E)} = \mathbb{P}(S_k \in E), \quad (14)$$

which proves the first part of our proposition. The result in (9) is now a direct consequence of the continuous mapping theorem.\]

We note in particular that for $p = 2$ and $w = 1_k = (1,\ldots,1)$, the limiting random variable in (9) is precisely Greenwood’s original test statistic. It is in this sense that we consider $\|S_{n,k}\|_{p,w}$ generalized Greenwood statistics. In order for this connection between two-sample testing and tests of uniformity to be of any use, a clear understanding of both the limiting distributions, as well as the convergence rates is necessary. We begin with the former, for which reasoning akin to THEOREM 1 is available:
Theorem 2. Let \( Q_p(x) = \sum_{m=0}^{\infty} (pm)! x^m / m! \). Then,

\[
\mathbb{E} \left( \|S_k\|^{p}_{p,u} \right)^m = \frac{(k-1)!m!}{(pm+k-1)!} [x^m] \prod_{j=1}^{k} Q_p(w_j x).
\]

(15)

In particular, the first \( m \) moments of \( \|S_j\|^{p}_{p,u} \) for \( j \in \{1,\ldots,k\} \) can be computed in \( O(m \cdot (\log m) \cdot (\log k)) \) time.

Proof. As in (6), we expand the left-hand side of (15) to obtain

\[
\mathbb{E} \left( \|S_k\|^{p}_{p,u} \right)^m = \int_{\Delta^{k-1}} (\|x\|^{p}_{p,u})^m d\mu_{\Delta^{k-1}}(x)
\]

\[
= \sum_{\eta \in D_{m,k}} \binom{m}{\eta_1, \ldots, \eta_k} \int_{\Delta^{k-1}} \prod_{j=1}^{k} \left( \eta_j ! \right) \int_{\Delta^{k-1}} \prod_{i=1}^{k} x_i^{\eta_i} d\sigma(x)
\]

\[
= \frac{(k-1)!m!}{\sqrt{k}} \sum_{\eta \in D_{m,k}} \binom{k-1}{\eta_1, \ldots, \eta_k} \int_{\Delta^{k-1}} \prod_{j=1}^{k} \left( \eta_j ! \right) \int_{\Delta^{k-1}} \prod_{i=1}^{k} x_i^{\eta_i} d\sigma(x)
\]

(16)

\[
= \frac{(k-1)!m!}{(pm+k-1)!} \sum_{\eta \in D_{m,k}} \binom{k-1}{\eta_1, \ldots, \eta_k} \int_{\Delta^{k-1}} \prod_{j=1}^{k} \left( \eta_j ! \right) \int_{\Delta^{k-1}} \prod_{i=1}^{k} x_i^{\eta_i} d\sigma(x)
\]

(17)

(18)

where \( \sigma(dx) \) is (unnormalized) surface measure on \( \Delta^{k-1} \), \( \Pi \Delta^{k-1} \) the projection of \( \Delta^{k-1} \) on the hyperplane spanned by the first \( k-1 \) coordinate axes, and (17) follows from recognizing the integral in (16) as the partition function of a Dirichlet variable with parameters \( (p\eta_1, \ldots, p\eta_k) \). We identify (18) as (15), and thus complete the first part of the proof. The second part now follows as in THEOREM 1 from computing (18) using the Fast Fourier Transform.

Remark 1. The generating function \( Q_p(x) \) in THEOREM 2 belongs to a class of well-known special function, which is not overly surprising given the occurrence of \( \mu_{\Delta^{k-1}} \) in various applications in physics (where it is known as the Bose-Einstein distribution). More precisely, \( Q_p(x) \) can be expressed as the generalized hypergeometric series

\[
Q_p(x) = {}_pF_0 \left[ \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p} \right] (p^2 x).
\]

In particular, for \( p = 2 \) (i.e., including the Greenwood statistic \( \|S_k\|_{2,1_k}^2 \)), we have \( Q_2(x) = 2F_0 \left[ 1, \frac{1}{2} \right] (4x) = \frac{1}{\sqrt{2}} D \left( \frac{1}{2\sqrt{2}} \right) \), where Dawson’s integral

\[
D(x) = e^{-x^2} \int_0^x e^{t^2} dt
\]

(19)

is interpreted through its asymptotic expansion (cf. Abramowitz and Stegun, 1965, formula 7.1.23).
Together with Proposition 6 to be discussed later, Theorem 2 clarifies the distributional properties of the continuous approximations of \( \|S_{n,k}\|_{p,w}^p \) for large \( n \) (while also satisfactorily answering Greenwood’s question of describing the distributional properties of \( \|S_k\|_{2,1_k}^2 \), cf. Section 3.2). The following proposition guarantees the quality of these approximations:

**Proposition 2.** Let \( F_{n,k}^{p,w}, F_k^{p,w} : [0, 1] \to [0, 1] \) be the cumulative distribution functions of \( \|S_{n,k}\|_{p,w}^p \) and \( \|S_k\|_{p,w}^p \), respectively. Then we have

\[
\|F_{n,k}^{p,w} - F_k^{p,w}\|_\infty = O(n^{-1}),
\]

for every fixed \( k \geq 2 \).

**Proof.** As in the proof of Proposition 1, let \( \Lambda = \mathbb{Z}^k \cap H \) where \( H = \{ x \in \mathbb{R}^k : \sum_{j=1}^k x_j = 0 \} \). Denoting by

\[
E^t = \{ x \in \Delta^{k-1} : \|x\|_{p,w}^p \leq t \} = \{ \|S_k\|_{p,w}^p \leq t \}
\]

the \( t \)-level set of \( F_k^{p,w} \), we observe that since the fundamental domain of \( \Lambda \) has diameter \( \|(-1, -1, \ldots, -1)\|_2 = \sqrt{k(k-1)} \), the number \( L(E^t, n) \) of lattice points in \( nE^t \) is bounded by

\[
\left( n - \sqrt{k(k-1)} \right)^{k-1} \text{Vol}_\Lambda (E^t) \leq d(\Lambda) L(E^t, n)
\]

\[
\leq \left( n + \sqrt{k(k-1)} \right)^{k-1} \text{Vol}_\Lambda (E^t).
\]

Thus, in particular,

\[
\mu_{D_{n,k}} \left( nE^t \right) - \mu_{\Delta^{k-1}} \left( E^t \right) = \frac{L(E^t, n)}{n^{k-1}} - (k-1)! \text{Vol}_\Lambda (E^t)
\]

\[
\leq (k-1)! \text{Vol}_\Lambda (E^t) \left[ \left( 1 + \frac{k}{n} \right)^{k-1} - 1 \right]
\]

\[
\leq \sqrt{k} \sum_{j=1}^{k-1} \binom{k-1}{j} \left( \frac{k}{n} \right)^j,
\]

(23)

where using \( \text{Vol}_\Lambda (E^1) = \sqrt{k} / (k-1)! \) as an upper bound for \( \text{Vol}_\Lambda (E^t) \) turns (23) independent of \( t \). Similarly, a uniform lower bound is given by

\[
\mu_{D_{n,k}} \left( nE^t \right) - \mu_{\Delta^{k-1}} \left( E^t \right) \geq (k-1)! \text{Vol}_\Lambda (E^t) \left[ \left( 1 - \frac{2k}{n+k-1} \right)^{k-1} - 1 \right]
\]

\[
\geq \sqrt{k} \sum_{j=1}^{k-1} \binom{k-1}{j} \left( \frac{-2k}{n+k-1} \right)^j.
\]

(24)

Combining (23) and (24) gives (20) as desired. \( \square \)

The results presented so far cover a wide range of scenarios encountered in practice when performing two-sample tests. Before considering other limiting situations, we first examine the insight that Theorem 2 provides into the distributional properties of the generalized Greenwood statistics \( \|S_k\|_{p,w}^p \), with particular emphasis on the Greenwood statistic \( \|S_k\|_{2,1_k}^2 \) itself.
3.2 Analysis of $\|S_k\|_{p,w}^p$ and the right tail probability

While providing an efficient means of computing large $n$ limits of any generalized Greenwood statistics $\|S_k\|_{p,w}^p$, THEOREM 2 (together with PROPOSITION 6) also satisfactorily answers Greenwood’s question of describing the distribution of $\|S_k\|_{2,1_k}^2$, which had remained open since Greenwood (1946). In particular, using the formulas given in (15) and PROPOSITION 6, the approximate z-score tabulations of Burrows (1979); Currie (1981); Stephens (1981) can be extended to arbitrary $k$ and arbitrary accuracy at reasonable runtime. In practice, this should be useful both for moderate and large $k$, for even though a central limit theorem (in $k$) exists for $\|S_k\|_{2,1_k}^2$ (Moran, 1947), its convergence rate is unfeasibly slow. Apart from computational improvements, THEOREM 2 also sheds light on analytic properties of $\|S_k\|_{p,w}^p$ and $\|S_k\|_{2,1_k}^2$ more specifically.

We begin by controlling the decay of moments, which in turn will inform us about tail behavior near $x_0 = 1$.

**Proposition 3.** For $p \geq 2$ and $k \geq 2$, and fixed weights $w_i \in [0,1]$, for all $i \in [k]$, we have

$$\lim_{m \to \infty} \left( \frac{\mathbb{E}\|S_k\|_{p,w}^p}{m^{k-1}} \right)^m = \frac{(k-1)!}{p^{k-1}} \cdot W_w,$$

(25)

where $W_w = |\{1 \leq i \leq k : w_i = 1\}|$ is the number of weights taking value 1. In particular, the Greenwood statistic satisfies

$$\lim_{m \to \infty} \left( \frac{\mathbb{E}\|S_k\|_{2,1}^2}{m^{k-1}} \right)^m = \frac{k!}{2^{k-1}}.$$

(26)

**Proof.** We first rewrite (17) as

$$\mathbb{E}\left(\|S_k\|_{p,w}^p\right)^m = \frac{1}{\binom{pm+k-1}{k-1}} \sum_{\eta \in D_{m,k}} \binom{m}{\eta} \binom{pm}{\eta p} \prod_{j=1}^k w_j^{\eta_j} = \frac{1}{\binom{pm+k-1}{k-1}} s_m^w,$$

(27)

which has leading order $O(m^{-(k-1)})$, if we can show that $s_m^w$ is $\Omega(1)$. To do so, we proceed by induction on $k$, the length of $w$, proving that in fact $\lim_{m \to \infty} s_m^w = W_w$. It is straightforward to check that for $\eta \in \{2, \ldots, m-1\}$, $(\binom{m}{\eta})/\binom{pm}{\eta p}$ is bounded above by $(\binom{m}{2})/(\binom{2m}{2\eta})$, and thus for the base case $k = 2$ we have

$$s_{m}^{(w_1,w_2)} = \sum_{\eta=0}^{m} \binom{m}{\eta} \binom{pm}{\eta p} w_1^{\eta} w_2^{m-\eta} \leq w_1^m + w_2^m + \binom{m}{1} \binom{pm}{p} + (m-2) \binom{m}{2} \binom{pm}{2p} \to_{m \to \infty} 1_{w_1=1} + 1_{w_2=1} = W_{(w_1,w_2)},$$

(28)

as desired. For the inductive step, we condition on the first entry of $\eta$ to obtain

$$s_{m}^{(w_1,\ldots,w_k)} = \sum_{\ell=0}^{m} \binom{m}{\ell} \binom{pm}{\ell p} w_1^{\ell} \sum_{\eta \in D_{m-\ell,k-1}} \binom{m-\ell}{\eta_1,\ldots,\eta_{k-1}} \binom{pm-\ell}{\eta_1,\ldots,\eta_{k-1} p} \prod_{j=1}^{k-1} w_j^{\eta_j+1} = s_{m}^{(w_2,\ldots,w_k)} + w_1^m + O\left(m^{-1}\right) \to_{m \to \infty} W_{(w_2,\ldots,w_k)} + 1_{w_1=1} = W_{(w_1,\ldots,w_k)},$$

(29)

where we used the inductive hypothesis on $s_{m}^{(w_1,\ldots,w_k)}$, and as in (28), bounded summands corresponding to $\ell \in \{2, \ldots, m-1\}$ by $(\binom{m}{2})/(\binom{pm}{2p})$. (25) and (26) now follow from taking the limit as $m \to \infty$ in (27).
The above result is useful primarily for describing the right tail of \( \|S_k\|_{p,w}^p \). Despite the obvious utility of such a result to hypothesis testing, such a description has not been available so far, even for \( \|S_k\|_{2,1_k}^2 \).

**Corollary 1.** For \( w = (w_1, \ldots, w_k) \) such that \( w_w \geq 1 \), the density \( f_k^{p,w} \) of \( \|S_k\|_{p,w}^p \) is analytic at \( x_0 = 1 \), and its first non-zero term in the Taylor expansion is \((k-1)W_w/2^{k-2}(1-x)^{k-2}\). That is, for \( x \) close to \( x_0 = 1 \), we have

\[
f_k^{p,w}(x) = \frac{(k-1)W_w}{2^{k-2}}(1-x)^{k-2} + O\left((1-x)^{k-1}\right).
\]

In particular, Greenwood’s statistic satisfies

\[
f_k^{2,1_k}(x) = \frac{k}{2k-2}(1-x)^{k-2} + O\left((1-x)^{k-1}\right).
\]

**Proof.** Let \( f_k^{p,w}(x) = \sum_{j=0}^{\infty} c_j (1-x)^j \) be the Taylor expansion of \( f_k^{p,w} \) around \( x_0 = 1 \). We first notice that for any \( r \geq 0 \),

\[
\int_0^1 x^m (1-x)^r \, dx = \frac{1}{m+r+1} \cdot \frac{1}{r^{m+r}}.
\]

and hence, using the fact that \( f_k^{p,w} \) is bounded,

\[
E\left(\|S_{n,k}\|_{p,w}^p\right)^m = \int_0^1 x^m f_k^{p,w}(x) \, dx + O\left(e^{-m}\right)
= \sum_{j=0}^{\infty} c_j \int_0^1 x^m (1-x)^j \, dx + O\left(e^{-m}\right)
= \sum_{j=0}^{\infty} c_j \frac{1}{m+j+1} \left(\frac{1}{j}\right)^{m+j} + O\left(e^{-m}\right).
\]

Identifying the \((k-2)\)nd term with \((25)\) immediately yields \((30)\).

While **Corollary 1** is phrased so as to clarify the decay properties of the right tail, its proof readily allows characterization of all the coefficients beyond the \((k-2)\)nd one in the Taylor expansion of \( f_k^{p,w} \) around \( x_0 = 1 \). For instance, it is straightforward to compute \( c_{k-1} = \frac{k}{2} (k+2)/2^k \) and \( c_k = (k+1)(k+6)(k^2+7k+16)/2^{k+4} \) by hand, and more generally, \( c_r \) for arbitrary \( r \in \mathbb{N} \) can be efficiently computed in \( O\left((\log \log k + [r \log r]^{2})\right) \) time. See **Appendix A** for the detailed algorithm. This distributional understanding of the right tail complements an explicit description of \( f_k^{p,w} \) on the far left worked out by Moran (1953) for the Greenwood statistic, valid for \( x \in [0,1/(k-1)] \). The reason for this rather narrow understanding near 0, and a guarantee that the right tail fares much better, is provided by the following lemma.

**Lemma 1.** For \( k,p > 1 \), the density \( f_k^{p,1_k}(x) \) of \( \|S_k\|_{p,1_k}^p \) is analytic on the intervals \( \left\{ \left(\frac{1}{j^{p-1}}, \frac{1}{(j-1)^{p-1}}\right) \right\}_{j\in\{2,\ldots,k\}} \). In particular, the Taylor expansion of \( f_k^{p,1_k}(x) \) around \( x_0 = 1 \) has radius of convergence \( 1/2^{p-1} \).

**Proof.** As we increase the radius \( r \) of an \( \ell_p \) ball centered at the origin, the ball will intersect the \( j < k \) dimensional faces of \( \Delta^{k-1} \) for the first time at \( r_j^p = \|\frac{1}{j^{p-1}}1_j\|_p^p = \frac{1}{2^{p-1}} \), which can be seen from projecting the \( \ell_p \) ball onto the \( j \)-dimensional coordinate-hyperplanes. These are the only points where \( f_k^{p,1_k} \) is not smooth, and therefore \( f_k^{p,1_k} \) must be analytic on \( \left\{ \left(\frac{1}{j^{p-1}}, \frac{1}{(j-1)^{p-1}}\right) \right\} \) for \( j \in \{2,\ldots,k\} \). \( \square \)
In light of Lemma 1, it is clear that the narrow applicability of the left-tail formulas in Moran (1953) is due to the quickly decreasing volume $\lambda_{k-1}(\Delta^{k-1})$ in $k$: the only regime in which the intersection of an $\ell_2$ and an $\ell_1$ ball is straightforward to compute is when this intersection is empty (i.e., $\|S_k\|_{2,1_k}^2 \leq 1/k$) or restricted to the $k-1$-dimensional face of $\Delta^{k-1}$ (in which case this computation reduces to a calculation of the volume of spherical caps). However, the volumes of these regimes are exhausted quickly, highlighting the importance of radius-independent descriptions like Theorem 2 (in combination with Proposition 6). For the particular case of (one-sided) hypothesis testing, the following monotonicity result guarantees that small $k$ approximations provide conservative estimates to large $k$ instances.

**Proposition 4.** For $p > 1$, the c.d.f. $F_{k,p,1,k}^p$ of $\|S_k\|^{p}_{p,1_k}$ is increasing in $k$. That is, $F_{k,p,1,k'}(x) \geq F_{k,p,1,k}(x)$ for all $x \in [0,1]$ and $k' > k$.

**Proof.** We let $B_k^p(r)$ be the $\ell_p$ ball of radius $r$ in $\mathbb{R}^k$, and recall that

$$\mu_{\Delta^{k-1}} \left\{ \|S_k\|^{p}_{p,1_k} \leq r^p \right\} = \frac{\lambda_{k-1}(B_k^p(r) \cap \Delta^{k-1})}{\lambda_{k-1}(\Delta^{k-1})}. \tag{34}$$

From Lemma 1 it is clear that the proposition is true for $x \leq \frac{1}{(k-1)^{p-1}}$. In order to relate $\lambda_{k-1}(B_k^p(r) \cap \Delta^{k-1})$ to $\lambda_{k-2}(B_{k-1}^p(r) \cap \Delta^{k-2})$ for $x > \frac{1}{(k-1)^{p-1}}$, we define $K_i(r)$ for $i \in \{1, \ldots, k\}$ to be the cone of apex $\frac{1}{k}1_k$ and base formed by the intersection of $B_k^p(r)$ with the $i^{th}$ $(k-2)$-dimensional face of $\Delta^{k-1}$ (for some fixed enumeration of the $k$ $(k-2)$-dimensional faces). Since $\bigcup_{i=1}^k K_i(r) \subset B_k^p(r) \cap \Delta^{k-1}$, it follows that

$$\lambda_{k-1}(B_k^p(r) \cap \Delta^{k-1}) > \sum_{i=1}^k \lambda_{k-1}(K_i) \tag{35}$$

$$= \frac{k}{k-1} \cdot \frac{1}{\sqrt{k(k-1)}} \lambda_{k-2}(B_{k-1}^p(r) \cap \Delta^{k-2}) \tag{36}$$

where by slight abuse of notation we used $\lambda_{k-2}(B_{k-1}^p(r) \cap \Delta^{k-1})$ for the $(k-2)$-dimensional volume of $B_k^p(r)$ intersected with the $(k-2)$-dimensional faces of $\Delta^{k-1}$. The fact that $k \cdot \lambda_{k-2}(B_{k-1}^p(r) \cap \Delta^{k-1}) = \lambda_{k-2}(B_{k-1}^p(r) \cap \Delta^{k-1})$ follows from the same projection argument as used in Lemma 1. \hfill $\square$

### 3.3 Alternative scaling limits

Our treatment up to now has primarily focused on the large $n$ limit of $\|S_{n,k}\|^{p}_{p,w}$ while keeping $k$ fixed, for which we saw a rich limiting distribution with fruitful connections to the previously studied Greenwood statistic emerge. However, in the setting of two-sample testing, it may very well happen that $k$ and $n$ are of comparable order, in which case similar behavior cannot be expected to govern the distribution of $\|S_{n,k}\|^{p}_{p,w}$. What happens in these cases is much simpler, as the following proposition demonstrates.

**Proposition 5.** Assume $n, k \to \infty$, and define $\mu_{n,k,p} = k^{-1}\mathbb{E}\|S_{n,k}\|_{p,1_k}$, $\sigma_{n,k,p} = k^{-1}\text{Var}\|S_{n,k}\|_{p,1_k}$. If $0 < W_{\min} \leq w_i \leq W_{\max}$ for all $i \in \{1, \ldots, k\}$, then

$$Z_{n,k,p,w} = \frac{\|S_{n,k}\|^{p}_{p,w} - \mu_{n,k,p} \left( \sum_{j=1}^k w_j \right)}{\sigma_{n,k,p} \left( \sum_{j=1}^k w_j^2 \right)^{1/2}} \frac{d}{d} \begin{cases} \mathcal{N}(0,1), & \text{if } k \frac{n}{n} \to \alpha \leq 1, \\ 0, & \text{if } k = o(n). \end{cases} \tag{37}$$
Moreover, in the case of $k \to \infty$ while $n$ remains fixed, if for every $k$, $\{w_i\}_{i \in [k]}$ is the discretization $w_i = w(i/k)$ of some function $w : [0, 1] \to [W_{\text{min}}, W_{\text{max}}]$ continuous (Lebesgue) almost everywhere, then

$$\|S_{n,k}\|_{p,w}^p \xrightarrow{d} \sum_{j=1}^n w(U_j),$$

where $\{U_j\}_{j \in [n]}$ are i.i.d. Uniform ([0, 1]).

The proof of this central limit theorem is by the method of moments, where explicit combinatorial expressions like (27) and known large deviations allow for precise quantification of the decorrelation in $S_{n,k}$. The full details are presented in the Appendix B. We point out here that in the two-sample setting, constraining $k$ to grow at most linearly in $n$ is no restriction, as the roles of balls and bins turn out to be easily exchanged. Before elaborating on the application to two-sample testing and making this statement precise, we give an efficient algorithm to reconstruct a distribution from its moments.

4 Reconstruction of a distribution from its moment sequence

Reconstructing a probability measure from its moments is a task that has received attention both in theoretical settings (see e.g. Akhiezer and Kemmer, 1965), where existence and uniqueness questions are addressed, and applied statistical problems, where existence and uniqueness are typically taken for granted, and efficient algorithms for computing the distribution in question are sought.

For a discrete distribution of $n$ atoms, the latter can be done by solving an $n \times n$ Vandermonde system, which Björck and Pereyra (1970) showed is solvable in $O(n^2)$ time. However, in our setting $\|S_{n,k}\|_{p,w}$ generically has $O(n^{k-1})$ atoms, whose precise location within $\{x_{\text{min}}, \ldots, \|w\|_{\infty}n^p\}$ is typically unknown. That is, solving the moment problem for $\|S_{n,k}\|_{p,w}$ exactly via its associated Vandermonde system requires $O(\min\{\|w\|_{\infty}n^{2p}, n^{2(k-1)}\})$ operations, which already for small values of $p$ or $k$ becomes prohibitively large. Moreover, aside from this computational intractability in our discrete setting, it is clear that such direct approach is unfeasible to conduct in the corresponding infinite-dimensional scenario required for $\|S_k\|_{p,w}$, and hence new algorithms are needed.

A commonly proposed alternative consists of forfeiting the demand for an exact recovery and focus on approximate reconstruction instead, attempting to trade off accuracy for accelerated runtimes. Examples of such approximation schemes include maximum entropy based algorithms (see e.g. Mead and Papanicolaou, 1984) as well as various applications of the method of moments. While the latter relies to a large extent on strong parametric assumptions, which are not available for our generalized Greenwood statistics, the former is primarily useful for density estimations with only a few explicit (or estimated) moments. THEOREM 1 and THEOREM 2, however, allow us access to a vast number of moments quickly, suggesting that a maximum entropy ansatz could waste valuable information. To remedy this situation, we recall a fact that is mentioned in Feller (2008) (p. 227, Theorem 2), but that to the best of our knowledge has not found widespread use in applied statistics.

**Fact 1.** Let $X \in [0, 1]$ be a (not necessarily continuous) random variable with cumulative distribution function $F$ and moments $\mu_m = \mathbb{E}X^m$, then at every continuity point $x$ of $F$, we have

$$\lim_{n \to \infty} \hat{F}_n(x) = F(x),$$

where

$$\hat{F}_n := \sum_{j=0}^{n-1} \mathbb{1}_{\frac{j}{n} \leq x} \binom{n}{j} (-1)^{n-j} \left( \delta^{n-j} \mu \right)_j,$$

(39)
with \( \delta : \mathbb{R}^N \rightarrow \mathbb{R}^N \) being the difference operator \( \delta : (a_j)_{j \in \mathbb{N}} \mapsto (a_{j+1} - a_j)_{j \in \mathbb{N}} \).

Part of the reason for the modest popularity of FACT 1 in statistical estimation problems may be the presence of generally large alternating summands, causing possibly uncontrollable instabilities if the moments \( \mu_m \) are not known exactly. In our situation, this instability does not present any limitations, since THEOREM 1 and THEOREM 2 allow computation of \( \mu_m = \mathbb{E}\|S_{n,k}\|_p^p \) (or \( \mu_m = \mathbb{E}\|S_k\|_p^p \) \( \mu_m \), respectively) to arbitrary precision, thus rendering (39) a promising candidate for reconstructing the distributions in question. It remains to clarify its convergence speed:

**Proposition 6.** Let \( X \in [0,1] \) be a random variable which is either (i) absolutely continuous with respect to \( \lambda_1 \), with density \( f \in C^1([0,1]) \), or (ii) discrete with support \( \text{supp} \ X = \{x_0, \ldots, x_N\} \). Then for any resolution \( \varepsilon_n \to 0, \varepsilon_n > \frac{1}{n^{1/2}}, \) there exists \( n_0(f, \varepsilon) \in \mathbb{N} \), so that for all \( n \geq n_0, \)

\[
\sup_{x \in [0,1]} \left| \hat{F}_n(x) - F(x) \right| \leq \frac{\|f\|_\infty + 2\|f'\|_\infty + 2}{n + 1}, \tag{i}
\]

\[
\sup_{x \in [0,1]\setminus\text{supp}\ X} \left| \hat{F}_n(x) - F(x) \right| \leq 2e^{-2n\varepsilon_n^2} + (|\text{supp} \ X| - 2) e^{-2nh^2}, \tag{ii}
\]

where \( \hat{F}_n(x) \) is defined in (39), \( \text{supp} \ X = \{x \in [0,1] : d(x, \text{supp} \ X) < \varepsilon \} \) is the \( \varepsilon \)-fattening, and \( h = \min_{i,j} |x_i - x_j| \) is the mesh of \( \text{supp} \ X \).

**Proof.** We first tackle (i) by recalling from Feller (2008) that the summation in (39) is nothing but

\[
\mathbb{E}B_{n,x}(X) = \mathbb{E} \sum_{j=0}^{n-1} \frac{1}{n^j} \binom{n}{j} X^k (1 - X)^{n-k}, \tag{40}
\]

where \( B_{n,x} \) is the degree \( n \) approximation of \( 1_{[0,x]} \) by Bernstein polynomials (see Bernstein (1912)).

To compute its approximation error, we choose a threshold \( \varepsilon_n \to 0 \) and investigate

\[
F(x) - \mathbb{E}B_{n,x}(X) = \mathbb{E} \left( 1_{[0,x]}(X) - B_{n,x}(X) \right)
= \int_{[0,1]\setminus\{x\}} \underbrace{\left( 1_{[0,x]}(y) - B_{n,x}(y) \right) f(y) \ dy}_{A_{n,x}}
+ \int_{\{x\}} \underbrace{\left( 1_{[0,x]}(y) - B_{n,x}(y) \right) f(y) \ dy}_{A'_{n,x}}, \tag{41}
\]

in which we treat the term \( A_{n,x} \) first: Interpreting \( B_{n,x}(y) \) as \( \mathbb{P}(S_{n,y} \leq nx) \), where \( S_{n,y} \sim \text{Binomial}(n,y) \), we see that by standard large deviation estimates and Pinsker’s inequality

\[
|A_{n,x}| \leq (x - \varepsilon_n) \|f\|_\infty e^{-nD_{KL}(x|x - \varepsilon_n)}
+ \|f\|_\infty (1 - x + \varepsilon_n) e^{-nD_{KL}(x|x + \varepsilon_n)} \leq \|f\|_\infty e^{-2n\varepsilon_n^2}, \tag{42}
\]

where \( D_{KL}(p \| q) \) is the Kullback-Leibler divergence (or the relative entropy) between a Bernoulli\((p)\) and Bernoulli\((q)\) distribution. To control \( A'_{n,x} \) then, we Taylor expand \( f \) to rewrite the integral in
where \( \min_{y \in (x, r]} f'(y) \leq M_n \leq \max_{y \in (x, r]} f'(y) \). In particular, since we assumed \( f \in C^1 ([0, 1]) \) and \( \varepsilon_n \to 0 \), there must exist a \( n_0' \) so that \( f'(x) - 1 \leq M_n \leq f'(x) + 1 \) for all \( n \geq n_0' \). So it remains to control \( A''_{n,x} \) and \( A'''_{n,x} \), which can be done in a manner similar to (42):

\[
|A''_{n,x}| \leq \int_{[0,1]} \left( \mathbb{1}_{[0,x]}(y) - B_{n,x}(y) \right) y \, dy + e^{-2n \varepsilon_n^2} = \frac{x}{n+1} + e^{-2n \varepsilon_n^2},
\]

(44)

\[
|A'''_{n,x}| \leq \int_{[0,1]} \left( \mathbb{1}_{[0,x]}(y) - B_{n,x}(y) \right) y \, dy + e^{-2n \varepsilon_n^2} = \frac{3nt(x-1) + 2(x^2-1)}{2(n+1)(n+2)} + e^{-2n \varepsilon_n^2} \leq \frac{1}{n+1} + e^{-2n \varepsilon_n^2},
\]

(45)

provided \( n \geq 4 \). Finally, combining (41)-(45), we obtain

\[
\left| \hat{F}_n(x) - F(x) \right| \leq \frac{\|f\|_\infty + 2\|f'\|_\infty + 2}{n+1} + 2 (\|f\|_\infty + \|f'\|_\infty) e^{-2n \varepsilon_n^2},
\]

(46)

independently of \( x \). Choosing \( \varepsilon_n \geq n^{-\frac{1}{2}+\delta} \) and \( n_0 \) so large that the first term dominates the second yields (i). (ii) follows in a very similar manner by observing that for \( n \) such that \( \varepsilon_n < h \), any \( x \in [0, 1] \setminus \text{supp} X \) satisfies

\[
\left| \mathbb{1}_{[0,x]}(y) - B_{n,x}(y) \right| \leq \begin{cases} e^{-2n \varepsilon_n^2}, & \text{if } y \in \{y\min(x), y\max(x)\}, \\ e^{-2nh^2}, & \text{for all other } y \in \text{supp} X, \end{cases}
\]

(47)

where \( y\min(x) = \min \{y' \in \text{supp} X : y' > x\} \) and \( y\max(x) = \max \{y' \in \text{supp} X : y' < x\} \) are the two atoms of \( X \) left and right of \( x \). Therefore,

\[
|F(x) - \mathbb{E}B_{n,x}(X)| \leq \sum_{y \in \text{supp} X} \mathbb{P} (X = y) \left| \mathbb{1}_{[0,x]}(y) - B_{n,x}(y) \right| \\
\leq 2 e^{-2n \varepsilon_n^2} + (|\text{supp} X| - 2) e^{-2nh^2},
\]

(48)

which is (ii).

We remark that the proof works equally well for distributions that have both an absolutely continuous and a singular part (with respect to \( \lambda_1 \)), in which case the continuous component presents the bottleneck, resulting in an \( O \left( n^{-1} \right) \) bound like in (i). For purely discrete measures
however, we notice that by setting \( \varepsilon_n = \varepsilon < h/2 \) in (ii), we can reconstruct \( F(x) \) for \( x \in \text{supp}_X \) up to exponentially decreasing error (in the number of moments) by computing \( \hat{F}(x + 2\varepsilon) \). Moreover, the bounds (i)-(ii) present worst case errors that are achieved at \( x \) for which \( f(x), |f'(x)| \) are large or atoms of \( X \) densely packed, respectively. Away from these bottlenecks, and in particular in the tails of \( \|S_{n,k}\|_{p,w} \) and \( \|S_k\|_{p,w} \), these guarantees should improve significantly. Lastly, we may replace each use of Bernstein polynomials throughout the entire analysis with any other expedient polynomial approximation of \( 1_{[0,x]} \) in order to impose desired properties on the reconstructed density. If, e.g., one-sided reconstructions are preferable (for instance, in order to give rise to conservative hypothesis tests in Section 5), then resorting to appropriate one-sided polynomial approximations (the optimal of which is worked out in Bustamante et al., 2012) will enforce this preference. To summarize our situation then:

1. We can approximate the distribution of \( \|S_k\|_{p,w} \) on \([a_{p,w}, 1]\) (where \( a_{2,k} = \frac{1}{2} \) and \( a_{p,w} < 1 \) generally) by computing its exact Taylor expansion of order \( r \) around \( x_0 = 1 \) in \( O\left( \frac{1}{p} \log \frac{1}{p} \log k + \left[ \frac{1}{r} \log r \right]^2 \right) \) time (see Proposition 3 and Appendix).

2. On \([0, a_{p,w}]\), we can achieve a uniform approximation error of \( \varepsilon \) in \( O\left( \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \log k \right) \) (see Proposition 3 and Proposition 6 (ii)). Moreover, in the special case that \( p = 2 \) and \( w = 1_k \), exact formulas for any \( x \in \left[0, \frac{1}{k-1}\right] \) are available by Lemma 1 and its preceding remarks.

3. For \( n \) large, we can approximate the distribution of \( \|S_{n,k}\|_{p,w} \) by that of \( \|S_k\|_{p,w} \) and using the two bullet points above. The additional error incurred is of order \( O\left( n^{-1} \right) \) by Proposition 2.

4. For any \( n \) and \( k \), Proposition 6 and the remark following its proof allow an \( \varepsilon \)-approximation of the distribution of \( \|S_{n,k}\|_{p,w} \) in \( O\left( \log \frac{1}{\varepsilon} \right) \) time.

Observations 1, 3 and 4 in particular render the generalized Greenwood statistics as promising statistics for hypothesis testing.

5 Application to non-parametric hypothesis tests

Having developed a thorough distributional understanding of both Greenwood’s and Dixon’s statistics as well as their various generalizations, we are now in a position to illuminate their role in the hypothesis tests that motivated them. We begin by carrying out the original test of uniformity proposed by Greenwood (1946). We demonstrate its power in comparison to other commonly used test statistics, and describe its implications for the wider class of one-sample tests. Our second application is then devoted to clarifying completely the two-sample test described in Dixon (1940) by replacing low-order approximations and lifting limiting sample size constraints that were assumed therein; in addition to illustrating how the flexibility that comes with our family of generalized test statistics can substantially improve power.

5.1 Tests of uniformity and one-sample tests

Recall from Section 1 that the null hypothesis to be queried in Greenwood (1946) concerned determining the distributional family of \( k \) sample times \( T_1, \ldots, T_k \). Namely,

\[ H_0 : \{T_j\}_{j \in [k]} \overset{i.i.d.}{\sim} \mathcal{E} (\lambda), \quad \text{for some } \lambda \in \mathbb{R}_+, \]
where $\mathscr{E}(\lambda)$ denotes the exponential distribution of rate $\lambda$. It is of particular interest to be able to detect alternatives consisting of point-processes whose inter-arrival times show either under- or overdispersion with respect to this homogeneous Poisson($\lambda$)-process; that is, it is desirable to maximize power against

$$H_1 : \{T_j\}_{j \in [k]} \overset{i.i.d.}{\sim} X, \quad \text{where } c_Y^2 = \frac{\text{Var}X}{(EX)^2} \neq 1. \quad (50)$$

Through normalizing by $\sum_j T_j$, Greenwood noticed that this decision problem is tantamount to the task of distinguishing the null hypothesis

$$H_0 : \left( T_1, T_1 + T_2, \ldots, \sum_{j=1}^{k-1} T_j \right) / \left( \sum_{j=1}^k T_j \right) \sim (U_{(1)}, \ldots, U_{(k-1)}), \quad (51)$$

where $\{U_j\}_{j \in [k-1]} \overset{i.i.d.}{\sim} \text{Uniform}([0,1])$, and $U_{(j)}$ is the $j^{th}$ order statistic, from a class of alternatives where points in $[0,1]$ tend to, intuitively, be overly equi-spaced (corresponding to $c_Y^2 < 1$) or overly clustered (resulting from $c_Y^2 > 1$). This, in turn, is translated to the level of spacings as

$$H_0 : \left( T_1, \ldots, T_k \right) / \left( \sum_{j=1}^k T_j \right) \sim \mu_{\Delta^{k-1}}, \quad (52)$$

with spacings in the alternative class exhibiting either smaller ($c_Y^2 < 1$) or larger ($c_Y^2 > 1$) variances than under the null. It is this last formulation (52) that motivated Greenwood to introduce his eponymous statistic $(\sum_{j=1}^k T_j^2) / (\sum_{j=1}^k T_j)^2$, whose law under the null is simply that of $\|S_k\|_2^2,_{1,k}$ in our notation above. Greenwood successfully treated the case $k = 2$, but was unable to extend his results to larger sample sizes. **Theorem 2 and Proposition 6** fill this gap by allowing us to compute $p$-values efficiently and accurately. Indeed, our algorithm proceeds fast enough to run large scale power studies for an extensive range of $k$ (computing the $p$-value of a sample of 10,000 points takes roughly 5 seconds on an ordinary laptop), all of which return results that qualitatively resemble those depicted in Figure 1: both the absolute power of Greenwood’s test, as well as its performance relative to three other popular test of uniformity (Pearson’s $\chi^2$, Kolmogorov-Smirnov, Cramer-von-Mises), are uniformly high (and particularly pronounced in the case of underdispersed data), rendering it a suitable hypothesis test to decide (49) against (50).

This performance is especially encouraging in light of the role that tests of uniformity play in the larger context of one-sample testing, where one is given a sample $Z_1, \ldots, Z_k$ of size $k$, and wants to ascertain whether these $k$ samples all arose in an $i.i.d.$ fashion from the same continuous distribution $F$. To ask whether $\{Z_i\}_{i \in [k]}$ are $i.i.d.$ $F$ however, is the same as to ask whether $\{F(Z_i)\}_{i \in [k]}$ are distributed $i.i.d.$ as $U \sim \text{Uniform}([0,1])$, which is nothing but the test of uniformity we just conducted. Naturally, the classes of alternatives likely encountered in this new setting will most often differ from those in our previous considerations. However, part of the benefit of having substantial analytical control on the entire *family* of generalized Greenwood statistics $\{\|S_{n,k}\|_p,\omega\}_{\rho_{\in \mathbb{N}}, \omega \in \mathbb{R}^k}$ is the ability to accommodate various, even strongly disparate, alternatives by means of adjusting $p$ and $\omega$. As this is best illustrated in the framework of two-sample tests, and readily reduced to one-sample tests from there, we will not delineate the details here, but rather develop them in the following subsection on two-sample tests, while being careful to point out any particular adjustments that may be necessary for application to one-sample tests.
attractive solution to this problem: it allows the user to optimize any quantity of interest (like
presenting themselves, it is often unclear how an appropriate test statistic is to be chosen. Having
other words, given various classes of alternatives $A$ standing of their relative power against each such family of alternatives has remained elusive. In
Von Mises, 2013)) or specific (e.g. the Mann-Whitney test (Mann and Whitney, 1947)) families of
Smirnov test (Kolmogorov, 1933; Smirnov, 1948) or the Cramer-von-Mises test (Cramér, 1928;
An array of test statistics devised to be sensitive against either arbitrary (e.g. the Kolmogorov-
Consequently, to probe
0.0 0.2 0.4 0.6 0.8 1.0
ROC curve for $ℋ_1: \sigma^2/µ^2 = 1/50$
0.0 0.2 0.4 0.6 0.8 1.0
ROC curve for $ℋ_1: \sigma^2/µ^2 = 10$
0.0 0.2 0.4 0.6 0.8 1.0
Testing uniformity ($ℋ_0: σ/µ = 1$) at significance $α = 0.05$

Figure 1: One-sample test results. ROC and power curves of tests based on the Greenwood
statistic $\|S_k\|_{2,1k}$ on under- and overdispersed data for $k = 10$, compared to three other commonly
used tests of uniformity (Pearson’s $χ^2$, Kolmogorov-Smirnov, Cramer-von-Mises): each experiment
consists of 1000 independently drawn samples from the null ($μ_{Δk-1}$) and alternative (Erlang and
Hyperexponential, either as individual classes as in the top two panels, or mixed into one class
and presented at equal probability as in the bottom left panel) distributions matching the stated
coefficients of variation; $α$ denotes the type I error, while $β$ is the type II error (and $1 − β$ is power).

5.2 Two-sample tests

One-sample tests are in a concrete sense (namely, that of (38) in Proposition 5) large sample-size
limits of two-sample tests: Instead of judging whether the generating distribution of one given
sample $\{Z_i\}_{i∈[k]}$ matches a suspected given continuous $F$, the task is to decide whether two drawn
samples $\{X_i\}_{i∈[k-1]}$ and $\{Y_j\}_{j∈[n]}$ have identical generating mechanisms. That is, assuming that
$\{X_i\}_{i∈[k-1]}$ and $\{Y_j\}_{j∈[n]}$ are generated $i.i.d.$ from $F$ and $G$, respectively, the null hypothesis to be
tested is

$$ℋ_0 : F = G,$$

which indeed in the $n → ∞$ limit (where $G$ becomes fully known) reduces to the one-sample setting.
The equivalent of the uniformizing transformation $\{Z_i\}_{i∈[k]} → \{F(Z_i)\}_{i∈[k]}$ in the one-sample setting
is now given by the discrete uniformization $\{X_i\}_{i∈[k-1]} → \{S_{n,k}[i]\}_{i∈[k-1]}$, $\{Y_j\}_{j∈[n]} → \{S_{n,k}[j]\}_{j∈[k]}$, where
$S_{n,k}[j]$ is as defined in (1). It is straightforward to verify that $S_{n,k} = (S_{n,k}[1], S_{n,k}[2], \ldots, S_{n,k}[k])$
is distributed as Multinomial($n, α$) with $α ∼$ Dirichlet($1, \ldots, 1$), and that this law is precisely $μ_{D_{n,k}}$.
Consequently, to probe $ℋ_0$ is really to probe whether $S_{n,k}$ is distributed according to $μ_{D_{n,k}}$ or not.
An array of test statistics devised to be sensitive against either arbitrary (e.g. the Kolmogorov-
Smirnov test (Kolmogorov, 1933; Smirnov, 1948) or the Cramer-von-Mises test (Cramér, 1928;
Von Mises, 2013)) or specific (e.g. the Mann-Whitney test (Mann and Whitney, 1947)) families of
alternatives have surfaced over the last century, yet, to the best of our knowledge, a clear understand-
ing of their relative power against each such family of alternatives has remained elusive. In
other words, given various classes of alternatives $A_1, \ldots, A_d$ that a practitioner might deem likely
to present themselves, it is often unclear how an appropriate test statistic is to be chosen. Having
access to fast numerical evaluations of $\|S_{n,k}\|_{ρ,w}$ and their laws for arbitrary $p$ and $w$ offers one
attractive solution to this problem: it allows the user to optimize any quantity of interest (like
power) over this family of generalized Greenwood statistics quickly. To wit, assuming for now a sufficiently well-behaved class of alternatives \( \mathcal{A} \), i.e. \( H_1 : G \in \mathcal{A} \), and denoting by \( \mathbb{H}_{n,k}^{p,w} \) the law of \( \|S_{n,k}\|_{p,w} \) on \( \mathbb{R} \) induced by discretely uniformizing \( \{Y_j\}_{j \in [\alpha]} \sim i.i.d. H \in \mathcal{A} \) through \( \{X_i\}_{i \in [\alpha]} \), the (two-sided) power at significance threshold \( \alpha \) is computed as

\[
\min_{H \in \mathcal{A}} \{1 - \beta_{p,w}^\alpha (H)\} = \min_{H \in \mathcal{A}} \left\{1 - \mathbb{H}_{n,k}^{p,w} (\{z_{p,w}^-(\alpha), z_{p,w}^+(\alpha)\})\right\}, \tag{54}
\]

where \( z_{p,w}^{\pm}(\alpha) \) are the \( \frac{\alpha}{2} \) and \( (1-\alpha) \) quantiles of \( \|S_{n,k}\|_{p,w} \) under \( \mu_{D_{n,k}} \) (i.e. \( \mu_{D_{n,k}} (\{z_{p,w}^-(\alpha), z_{p,w}^+(\alpha)\}) = 1-\alpha \)). \( z_{p,w}^{\pm}(\alpha) \) are efficiently computed from the moments of \( \mu_{D_{n,k}} \), so in principle, if \( \mathcal{A} \) is tractable enough (relative to \( F \)) to allow for an explicit characterization of \( \mathbb{H}_{n,k}^{p,w} \) for every \( H \in \mathcal{A} \), (54) is amenable to fast numerical optimization. Alas, in practice we can hardly expect \( \|S_{n,k}\|_{p,w} \) under \( H \) to be as accessible as under the null, so computing (54) to arbitrarily high accuracy will likely prove unfeasible. Nevertheless, reasonable approximations to (54) are often available under mild assumptions on \( \mathcal{A} \). The following two examples illustrate the process of performing such approximate optimization, its impact on statistical power, as well as how to extend this selection process to composite hypotheses.

**Example 1** (Detecting heteroskedasticity). Assume without loss of generality that \( \mathbb{E}X = 0, \text{Var}X = 1 \), and that we would like to test against alternatives of the form \( G = F \circ (y \mapsto y/\sigma) \), i.e. \( Y = \sigma X \), for any constant \( \sigma \in \mathbb{R}_+ \). It is straightforward to see that as \( \sigma \to \infty \), \( S_{n,k} \) will be concentrated mostly on its far ends \( S_{n,k}[1] \) and \( S_{n,k}[k] \) weighted by \( F(0) \), i.e., \( S_{n,k} \overset{d}{\to} S_{n,k}^\infty := N_\infty 1\{1\} + (n-N_\infty)1\{k\} = (N_\infty, 0, \ldots, 0, n-N_\infty) \), where \( N_\infty \sim \text{Binomial} (n, F(0)) \) and \( \mathbb{P} (S_{n,k} \neq S_{n,k}^\infty) = O (\sigma^{-1}) \). Likewise, the limiting law of \( S_{n,k} \) as \( \sigma \to 0 \) is quickly verified to be that of a \( S_{n,k}^0 := n 1\{N_0\} \) variable, where \( N_0 \sim \text{Binomial} (n, F(0)) \) as before, again with \( \mathbb{P} (S_{n,k} \neq S_{n,k}^0) = O (\sigma) \). It is therefore plausible to assume that most mass of \( \|S_{n,k}\|_{p,w} \) is tightly concentrated around \( \|S_{n,k}^\infty\|_{p,w} = (w_1 N_\infty^p + w_k (n-N_\infty)^p) \) and \( \|S_{n,k}^0\|_{p,w} = n^p 1\{N_0\} \), respectively, and that thus (54) is appreciably large whenever

\[
\mu_{D_{n,k}} (\|S_{n,k}\|_{p,w} \notin \|S_{n,k}^0\|_{p,w}, \|S_{n,k}^\infty\|_{p,w}) \\
= \sum_{i=0}^{n} \sum_{j=0}^{n} \mathbb{P} (N_\infty = i) \mathbb{P} (N_0 = j) \\
\times \mu_{D_{n,k}} (\|S_{n,k}\|_{p,w} \notin \|S_{n,k}^0\|_{p,w}, \|S_{n,k}^\infty\|_{p,w} \mid N_\infty = i, N_0 = j) \lesssim \alpha, \tag{55}
\]

where by slight abuse of notation we use \([a,b] \) to denote the interval \([\min\{a,b\}, \max\{a,b\}] \). This motivates solving the approximate, yet computationally tractable, optimization problem of finding

\[
\arg\min_{p,w} \mu_{D_{n,k}} (\|S_{n,k}\|_{p,w} \notin \|S_{n,k}^0\|_{p,w}, \|S_{n,k}^\infty\|_{p,w}), \tag{56}
\]

for any given \( F(0) \), instead of optimizing (54) directly. To verify empirically that any such minimizers do indeed give rise to a powerful test of heteroskedasticity, we ran large scale simulations for various \( F \) and \( G \) in the assumed family of distributions. As Figure 2A illustrates, where performing the optimization in (56) yielded parameter choices of \( p = 1 \) and \( w = \frac{1}{10} (10, 2, 1, 0, 0, 0, 1, 2, 10) \), our new two-sample test based on the generalized spacing-statistics \( \|S_{n,k}\|_{p,w} \) compares favorably with other non-parametric tests (Mann-Whitney, Kolmogorov-Smirnov, and Cramer-von-Mises) commonly used for such tasks.
Although the alternatives in Example 1 are composite, the laws they induce on $D_{n,k}$ are all tightly clustered around two universal ones, thereby effectively reducing the decision task to a semi-simple hypothesis test. The extension to truly composite settings is standard:

**Example 2** (Sensing location and scale). We enrich the class of alternatives in Example 1 by location shifts, i.e. we consider $G$ of the form $G = F \circ (x \mapsto (x - \mu)/\sigma)$ for $\mu \in \mathbb{R}, \sigma \in \mathbb{R}_+$. The laws $G^{\mu,\sigma}$ induced on $D_{n,k}$ now exhibit infinitely many accumulation points, barring any simple optimization of the kind we performed before. Indeed, even if we had the capacity to identify maximizers of (54) explicitly, they likely would not deliver satisfactory power, for we have

$$\max_{p,w} \min_{H \in \mathcal{A}} \left(1 - \beta_{p,w}^\alpha(H)\right) \leq \max_{p,w} \min_{H \in \mathcal{A}_\mu \cup \mathcal{A}_\sigma} \left(1 - \beta_{p,w}^\alpha(H)\right),$$

(57)

where $\mathcal{A}_\sigma = \{G : G = F \circ (x \mapsto x/\sigma)$ for some $\sigma \in \mathbb{R}_+\} \subset \mathcal{A}$ and $\mathcal{A}_\mu = \{G : G = F \circ (x \mapsto x - \mu)$ for some $\mu \in \mathbb{R}\} \subset \mathcal{A}$ are the pure location shifts and pure scales, respectively. Computations as in Example 1 successfully yield powerful parameter choices against $\mathcal{A}_\mu$ and $\mathcal{A}_\sigma$ individually, yet fail to do so for $\mathcal{A}_\mu$ and $\mathcal{A}_\sigma$ jointly, producing values of $1 - \beta$ not exceeding $\approx 0.6$. To mend this shortcoming, we can capitalize on the successful optimizers $(p_\mu, w_\mu)$

Figure 2: Two-sample test results. The null hypothesis tested is $F = G$ using samples $X_1, \ldots, X_{k-1} \overset{iid}{\sim} F = \text{Normal}(0, 1)$ and $Y_1, \ldots, Y_n \overset{iid}{\sim} G = \text{Normal}(\mu, \sigma^2)$, for $k = 10$ and $n = 30$ (different parameter choices led to minor qualitative changes only). The experimental setup is similar to that of Figure 1. **A.** ROC and power curves for detecting heteroskedasticity. Our new test based on the generalized spacing-statistics $\|S_{n,k}\|_{p,w}$ exhibits substantially improved performance over the other non-parametric two-sample tests. **B.** Power against joint variations in location and scale. $G = \text{Normal}(\mu, \sigma^2)$, with $\mu \in \{-2, -1, 0, 1, 2\}$ and $\sigma^2 \in \{1, 2, 3, 4, 5\}$. Colors of bubble indicate the test statistic used—generalized spacing-statistics $\|S_{n,k}\|_{p,w}$ (red), Mann-Whitney (black), Kolmogorov-Smirnov (dark gray), Cramer-von-Mises (light gray)—while its radius indicates power $1 - \beta$. The column for $\mu = 0$ corresponds to the results illustrated in the bottom panel of **A**.
in the former scenario, while exception of a single randomly chosen outcome is depicted in the Figure 2B. Since the choices of \(m\) intersected with \(\Delta\) for \(p\) other than 1 overdispersion (times are again drawn from an underdispersed distribution \(H\)). In our present case, however, we investigate an alternative hypothesis where we sought to distinguish Markovian arrival times from over- or underdispersed alternatives. (Spiked spacing model and higher scales in mind. As a consequence, our test based on \(\|S_{n,k}\|_{p,\mu,\mathbf{w}}^p\) and \(\|S_{n,k}\|_{p,\sigma,\mathbf{w}_\sigma}^p\) boasts power comparable to the Mann-Whitney test when sensing location shifts, while extending sensitivity to heteroskedastic alternatives as well. Notably, this comparative advantage in detecting scale changes persists even against tests like Kolmogorov-Smirnov and Cramér-von-Mises whose design is not centered around mean shifts.

Examples 1 and 2 provide manifestations of (54) in two concrete two-sample instances, for which the optimal choice of \(p\) happened to be \(p = 1\). As mentioned before, these optimization tools are easily adapted to the one-sample setting, which often requires mere replacement of \(S_{n,k}\) with \(S_k\). The following example illustrates such adaptation in practice, while also supplying a family of circumstances in which choices of \(p\) greater than 2 lead to greater power.

**Example 3** (Spiked spacing model and higher \(p\)-norms). We revisit the one-sample test in (49), where we sought to distinguish Markovian arrival times from over- or underdispersed alternatives. In our present case, however, we investigate an alternative hypothesis \(H_1\) consisting of a distribution \(G_k\) of \(T_1, \ldots, T_k\) that is both over- and underdispersed in the following sense: Under \(G_k\), arrival times are again drawn iid from an underdispersed distribution \(H_\cdot\) (that is, \(c_1^2(H_\cdot) < 1\)), with the exception of a single randomly chosen \(T_K\) (i.e., \(K \sim \text{Uniform}([k])\)) whose law \(H_+\) now exhibits overdispersion (\(c_1^2(H_+) > 1\)). In other words, \(G_k\) mixes \(k - 1\) underdispersed arrivals with 1 uniformly chosen overdispersed spacing. We will call this overdispersed \(T_K\) the spiked or outlier arrival time, and refer to the just described model of \(G_k\) as the spiked spacing model. Though the subsequent analysis is phrased in terms of this spiked spacing model, much of its reasoning pertains to similar outlier or correlation models of this kind as well.

To design a test capable of reliably detecting this spiked spacing model, we first observe that the symmetry in \(T_1, \ldots, T_k\) (induced by the uniform choice of \(K\)) suggests little benefit of choices for \(w\) other than \(1_k\), leaving \(p\) as the sole parameter to optimize in (54). To choose among the candidates for \(p\) then, it is useful to clarify and compare the geometry that various \(\ell_p\) balls give rise to when intersected with \(\Delta^{k-1}\): as the 2-dimensional illustrations of Figure 3A demonstrate, the growth of the (normalized) intersection volume \(V_k^p(s) = \mu_{\Delta^{k-1}}(\|S_k\|_{p,1_k}^p \leq \|s\|_{p,1_k}^p)\) depends noticeably on the precise location of our observation \(s\). If \(s\) localizes exactly along any of the line segments \(L_+ = \left\{\frac{1}{k}1_k, e_i\right\}_{i \in [k]}\), where \(e_i\) is the \(i\)th standard basis vector, then \(V_k^p(s) \subset V_k^p(s)\) whenever \(p < q\), while \(V_k^p(s) \supset V_k^q(s)\) in case \(s\) falls precisely on any of the line segments \(L_- = \left\{\frac{1}{k}1_k, m_i\right\}_{i \in [k]}\), where \(m_i = \frac{1}{k}(1_k - e_i)\) is the midpoint of the \((k-2)\)-dimensional face opposite of vertex \(e_i\). Since \(p\)-values are nothing but \(1 - V_k^p(s)\), it follows that tests based on \(\|S_k\|_{\infty,1_k}^2\) should be most powerful in the former scenario, while \(\|S_k\|_{2,1_k}^2\)-based tests shine in the latter scenario, with intermediate localizations giving rise to optimal \(p^*\) between 2 and \(\infty\). In our spiked spacing model at hand, the
A. Illustration of tail probabilities on $\Delta^2$ in the cases of $p = 2$ and $p = \infty$, and the samples (denoted by solid dots) giving rise to them. While a fixed sample near line segments in $L_+$ (purple, dashed lines in top panel) produces smaller sub-level sets in $\ell_2$ than $\ell_\infty$ (thereby increasing the $p$-value of said sample, which corresponds to 1 minus the shaded area), this trend reverses for observations near line segments in $L_-$ (orange, dashed lines in bottom panel). B. ROC and power curves. The spiked spacing model largely concentrates around $L_+$ in $\Delta^{k-1}$, with the degree of this concentration increasing with spike size. As a consequence, $p$-norms of samples generated under such alternative tend to separate more markedly for larger $p$, which in turn affords increases in power of $\|S_k\|_{p,1_k}$ when $p > 2$. The experimental design and choice of under- and overdispersed distributions match those of Figure 1; in particular, $k = 10$.

Figure 3: Analysis of spiked spacing model (described in detail in Example 3). A. Illustration of tail probabilities on $\Delta^2$ in the cases of $p = 2$ and $p = \infty$, and the samples (denoted by solid dots) giving rise to them. While a fixed sample near line segments in $L_+$ (purple, dashed lines in top panel) produces smaller sub-level sets in $\ell_2$ than $\ell_\infty$ (thereby increasing the $p$-value of said sample, which corresponds to 1 minus the shaded area), this trend reverses for observations near line segments in $L_-$ (orange, dashed lines in bottom panel). B. ROC and power curves. The spiked spacing model largely concentrates around $L_+$ in $\Delta^{k-1}$, with the degree of this concentration increasing with spike size. As a consequence, $p$-norms of samples generated under such alternative tend to separate more markedly for larger $p$, which in turn affords increases in power of $\|S_k\|_{p,1_k}$ when $p > 2$. The experimental design and choice of under- and overdispersed distributions match those of Figure 1; in particular, $k = 10$.

We close this section with a few remarks on the scope and availability of our proposed hypothesis tests:

1. Even though our entire discussion is phrased around continuous null and alternative distributions $F$ and $G$, the extension to discrete variables is straightforward: it merely requires recourse to a source of independent noise to randomly break ties when forming $S_n,k$.

2. Due to their widespread use, our primary focus lies on applications of generalized Greenwood statistics $\|S_n,k\|_{p,w}$ to unpaired one- and two-sample test. However, they can naturally be deployed in any other goodness-of-fit context in which null distributions effectively reduce to $\mu_{D_n,k}$ or $\mu_{\Delta^{k-1}}$, e.g. paired two-sample tests.
3. (54) and its derived optimization problems are stated so as to incorporate rare events (under \( H_0 \)) in both the left and right tail of \( \| S_{n,k} \|_{p,w}^{p} \). Of course, a one-sided hypothesis test can be enforced by only considering one such tail.

4. The significance threshold adjustment \( \alpha/2 \) in (58) when considering the ensemble of two generalized Greenwood statistics is exact only if their individual rejection regions are disjoint; in all other circumstances it is conservative. To extract additional power, it is possible to apply the same tools we developed throughout this paper to compute the joint moments \( \mathbb{E}(\| S_{n,k} \|_{p,w}^{p} : \| S_{n,k} \|_{q,v}^{q})^{m} \), and recover from those the joint distribution \( \mathbb{P}(\| S_{n,k} \|_{p,w}^{p} \leq s, \| S_{n,k} \|_{q,v}^{q} \leq t) \), which would allow for more refined adjustment of significance thresholds.

5. An implementation of both the one- and two-sample test in Mathematica together with precomputed parameter configurations optimal against shifts in location, scale, skewness and kurtosis (as well as combinations thereof) is available at [https://github.com/songlab-cal/mochis](https://github.com/songlab-cal/mochis).

6 Conclusions

Since early on, Greenwood’s statistic and its relatives were theorized to be powerful candidates for a variety of goodness-of-fit tasks, yet proving them to be such, either rigorously or empirically, has, due to a lack of distributional understanding, largely remained open. Here we contribute to such distributional understanding by embedding Greenwood’s statistic into a larger family of laws, the generalized Greenwood statistics \( \| S_{n,k} \|_{p,w}^{p} \), whose distributional properties are more amenable to analysis. In particular, we were able to obtain explicit, efficiently computable, expressions for their associated moment sequences, and glean both qualitative (e.g. convergence, regularity and monotonicity results) as well as quantitative (convergence rates, tail behaviour, CLT) insights from them. By providing an algorithmic procedure to recover a given distribution to arbitrary accuracy from its truncated moment sequence, we are able to quickly compute quantiles and \( p \)-values, which in turn enables accurate and adaptive hypothesis tests based on said generalized Greenwood statistics. As a consequence, we were in a position to empirically verify the gains in power in two such goodness-of-fit settings, namely one- and two-sample tests, compared to conventional non-parametric test statistics widely used for these tasks.
Appendix

A Computing Taylor coefficients

**Proposition 7** (Taylor coefficients). Let \( f^{p,w}_k \) be the density (with respect to Lebesgue measure) of \( \|S_k\|_{p,w} \), then on \([1/2,1]\), we have

\[
f^{p,w}_k = \sum_{j=k-2}^{\infty} c^w_j (1 - x)^j,
\]

where \( c^w_r \) can be computed in \( O \left( \frac{p}{p} \log \frac{p}{p} \log k + [r \log r]^2 \right) \) time.

**Proof.** We recall from (27) that \( \mu_m = E(\|S_k\|_{p,w})^{pm} \) can be written as

\[
\mu_m = \left( \begin{array}{c} \frac{1}{(pm+k-1)} \sum_{\eta \in D_{m,k}} \frac{m!}{\prod_{j=1}^{k} w_j^{\eta_j}} \prod_{j=1}^{k} w_j^{\eta_j} \end{array} \right) = s^w_{m} \left( \begin{array}{c} \frac{1}{(pm+k-1)} \end{array} \right),
\]

where \( s^w_{m} = \sum_{j=0}^{\infty} \sigma^w_j (m) \cdot m^{-j} \) with \( \sigma^w_j (m) \) remaining constant \( \sigma^w_j \) past some threshold \( m^w_j \). By **Lemma 1** and the geometric interpretation of \( \|S_k\|_{p,w} \), \( f^{p,w}_k \) is analytic on \([1/2,1]\), and hence we also have

\[
\mu_m = \int_0^1 x^m f^{p,w}_k (x) \, dx = \sum_{j=0}^{\infty} c^w_j \int_0^1 x^m (1 - x)^j \, dx + O \left( e^{-m} \right)
\]

\[
= \sum_{j=0}^{\infty} c^w_j \left[ (m + j + 1) \binom{m + j}{j} \right]^{-1} + O \left( e^{-m} \right),
\]

which suggests that by matching coefficients in (60) and (61) we should be able to translate between \( \sigma^w_j \) and \( c^w_j \). For this to be helpful, we need to understand \( \sigma^w_j \):

**Lemma 2** (\( \sigma^w_j \) recursion). Defining \( b^j_r = \binom{m-r}{j} / \binom{m}{j} \) and employing notation as in (3), we have

\[
\sigma^w_r = \sum_{j=0}^{r'} \left( \sigma^w_{r-k} \cdot \sigma^w_{r-k} \cdot \mathbb{1}_{w_k=1} \cdot b^j_r \right),
\]

with initial condition \( \sigma^w_{r,1,2} = \sum_{j=0}^{r'} b^j_r \left( \mathbb{1}_{w_2=1} w_1^{j} + \mathbb{1}_{w_1=1} w_2^{j} \right) \), where \( r' = \lfloor r / (p-1) \rfloor \). In particular, we can compute \( \sigma^w_r \) in \( O \left( r' \log r' \log k + [r \log r]^2 \right) \) time.
Proof of Lemma 2. Slightly abusing notation, we have

\[
\sigma^w_r = [m^{-r}] s^w_m = [m^{-r}] \sum_{\eta \in \mathcal{D}_{m,k}} \left( \frac{m}{p(m-\omega)} \right) \prod_{j=1}^{k} w^\eta_j
\]

\[
= [m^{-r}] \sum_{\omega=0}^{m} \left( \frac{m}{p(m-\omega)} \right) w^\omega_k \cdot s^{w-k}_{m-\omega}
\]

\[
= [m^{-r}] \sum_{\omega=0}^{m-1} \left( \frac{m}{p(m-\omega)} \right) w^\omega_k \cdot s^{w-k}_{m-\omega}
\]

\[
= [m^{-r}] \sum_{\omega=0}^{m-1} w^{\omega-k} \sum_{j=0}^{\infty} b^\omega_j m^{\omega-j} + [m^{-r}] \sum_{\omega=0}^{m-1} w^{\omega-k} s^{w-k}_{m-\omega} \sum_{j=0}^{\infty} b^\omega_j m^{\omega-j}
\]

\[
= \sum_{j=0}^{r'} \sigma^{w-k}_{r-j} \cdot \sigma^{w-k}_{j} + s^{w-k}_{1} \cdot b^\omega_j
\]

as desired. To see that (62) can be computed in \( O \left( r' \log r' \log k + [r \log r]^2 \right) \) time, we notice that calculation of \( s^{w-k}_r \) is \( O \left( r' \log r' \log k \right) \) by the same reasoning as in Proposition 2, and \( b^\omega_j \), written as,

\[
b^\omega_j = [m^{-r}] \left( \frac{m}{p(j)} \right) = (pj - 1)! \cdot \left( \frac{p^{m-1}}{p(m-j) + 1} \right) \prod_{\ell=p(m-j)+1}^{p^{m-1}} \frac{1}{\ell^{1 - 1/p}} \]

\[
= (pj - 1)! \cdot \sum_{j=0}^{\infty} x^j
\]

where \( R(x) = \sum_{j=0}^{\infty} x^j \) is again a convolution of \( (p - 1) \cdot r' = r \) polynomials and hence computable in \( O \left( [r \log r]^2 \right) \).

With a proper understanding of \( \sigma^w_j \) at hand from Lemma 2, we may rewrite (60) as

\[
\mu_m = \sum_{j=0}^{\infty} \left( \sum_{\omega=0}^{j} a^\omega \cdot \sigma^{w-\omega}_j \right) m^{-j} + O \left( e^{-m} \right),
\]

where \( a^\omega = [m-\omega] \left( \frac{p^{m+k-1}}{k-1} \right) \). Similarly, expanding (61) yields

\[
\mu_m = \sum_{j=0}^{\infty} \left( \sum_{\omega=0}^{j-1} a^\omega \cdot c^\omega \right) m^{-j} + O \left( e^{-m} \right),
\]

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where $d_j = [m^{-j}] [m + \omega + 1](m+\omega)]^{-1}$. Consequently, matching the $r$th coefficients in (64) and (65) allows to solve for $c_r^w$:

$$
c_r^w = \frac{1}{r!} \left[ \sum_{j=0}^{r+1} a_j \cdot \sigma_{r+1-j} - \sum_{j=k-2}^{r-1} d_j c_j^w \right], \quad (66)
$$

where in the choice of summation indices we have used the fact that $a_j = 0$ for $j \in \{0, \ldots, k-2\}$ and $c_j^w = 0$ for $j \in \{0, \ldots, k-3\}$ by PROPOSITION 4. Now $\{d_{r+1}^{\omega}, \ldots, d_{r+1}^{\omega-1}\}$ can be found in $O\left( r (\log r)^2 \right)$ time, and given $a, b$ and $d$, the recursion is solved in $O\left( r^2 \right)$ steps, amounting to a total complexity of $O\left( r (\log r)^2 + r^2 + r^r \log r^r \log k + [r \log r]^2 \right) = O\left( r^r \log r^r \log k + [r \log r]^2 \right)$. \qed

B Alternative scaling limits

Proof of PROPOSITION 5. We will show that the moments of $Z_{n,k,p,w}$ converge to the respective limiting moments of a $\mathcal{N}(0,1)$ variable, which together with Theorem 30.2 of Billingsley (1995) implies the desired result. Hence we investigate

$$
\mathbb{E}\left( Z_{n,k,p,w}^m \right) = \sigma_{n,k,p,w}^{-m/2} \mathbb{E}\left( \|S_{n,k}\|_{p,w}^m - \mu_{n,k,p,w} \right)^m \\
= \sigma_{n,k,p,w}^{-m/2} \mathbb{E}\left( \sum_{j=1}^k w_j \left[ \langle S_{n,k}\|j] \right]^p - \mu_{n,k,p} \right)^m \\
= \sigma_{n,k,p,w}^{-m/2} \sum_{t=1}^m \sum_{a \in D_{m,t}^{\omega}} C_a \sum_{i_1,\ldots,i_t \text{ distinct}} \mathbb{E}\left[ (w_{i_1} X_{i_1})^{a_1} \cdots (w_{i_t} X_{i_t})^{a_t} \right], \quad (67)
$$

where $C_a$ is a combinatorial factor to be determined later, and $D_{m,t}^{\omega}$ is the set of ordered (strong) compositions of $m$ into $t$ parts (which is in bijection with the set of partitions of $m$ into $t$ parts).

We will argue that for case 1 of (37), all summands in (67) vanish in the limit of $n \to \infty$, while only the $t = m/2$, $a_1 = a_2 = \cdots = a_t = 2$ term survives in the $k = \alpha n$ regime. We begin by determining the asymptotics of $\mu_{n,k,p}$.

Lemma 3 (Asymptotics of $\mu_{n,k,p}$). We have

$$
\mu_{n,k,p} = \frac{n \cdot q_p(n,k)}{\langle k \rangle_p^p} = O\left( \frac{n^p}{k^p} \right), \quad (68)
$$

where $q_p(n,k) = O(n^{p-1})$ is a polynomial in $n$ and $k$, and $\langle k \rangle_p = k \cdot (k+1) \cdots (k+p-1)$ is the rising factorial.

Proof of Lemma 3. Let us first rewrite $\mu_{n,k,p}$ into a form more amenable to extract asymptotics:

$$
\mu_{n,k,p} = k^{-1} \mathbb{E}\|S_{n,k}\|_{p,1}^p = k^{-1} \sum_{j=1}^k \mathbb{E} \langle S_{n,k}\|j] \rangle^p = \mathbb{E} \langle S_{n,k}\|1] \rangle^p \\
= \frac{1}{(n+k-1)} \sum_{j=0}^n j^p \binom{n-j+k-2}{k-2}, \quad (69)
$$
Inductive step: Using the binomial recursion whose RHS sum we claim can in general form be expressed as

\[
\sum_{j=0}^{n} j^p \binom{n-j+\ell}{\ell} = \sum_{j=\ell}^{n+\ell} (n-j+\ell)^p \binom{j}{\ell} = \binom{n+\ell-1}{\ell} \frac{(n+\ell)(n+\ell+1)}{\ell+1} q_p(n, \ell), \tag{70}
\]

for some polynomial \( q_p(n, \ell) = O(n^{p-1}) \) if \( p \geq 1 \) and \( q_0(n, \ell) = n^{-1} \). We prove (70) by induction on \( p \).

**Base case:** When \( p = 0 \), (70) simply becomes the hockey stick identity

\[
\sum_{j=\ell}^{n+\ell} \binom{j}{\ell} = \binom{n+\ell+1}{\ell+1} = \binom{n+\ell-1}{\ell} \frac{(n+\ell)(n+\ell+1)}{\ell+1} \cdot n^{-1}
\]

\[
= \binom{n+\ell-1}{\ell} \frac{(n+\ell)(n+\ell+1)}{\ell+1} q_0(n, \ell). \tag{71}
\]

**Inductive step:** Using the binomial recursion \( \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \), we find for \( p \geq 1 \)

\[
\sum_{j=\ell}^{n+\ell} (n-j+\ell)^p \binom{j}{\ell} = \sum_{j=\ell}^{n+\ell} (n-j+\ell)^p \binom{j}{\ell} - \sum_{j=\ell}^{n+\ell} (n-j+\ell)^p \binom{j}{\ell+1}
\]

\[
= \sum_{j=\ell+1}^{n+\ell} (n+\ell+1-j)^p \binom{j}{\ell+1} - \sum_{j=\ell+1}^{n+\ell} \sum_{i=0}^{p-1} \binom{p}{i} (n+\ell+1-j)^i (-1)^{p-i} \binom{j}{\ell+1}
\]

\[
= \sum_{i=1}^{p-1} \binom{p}{i} (-1)^{p-i-1} \sum_{j=\ell+1}^{n+\ell+1} (n+\ell+1-j)^i \binom{j}{\ell+1}
\]

\[
+ (-1)^{p-1} \sum_{j=\ell+1}^{n+\ell} \binom{j}{\ell+1}
\]

\[
= \left[ \sum_{i=1}^{p-1} \binom{p}{i} (-1)^{p-i-1} \binom{n+\ell+1}{\ell+1} \frac{(n+\ell+1)(n+\ell+2)}{(\ell+2)_{i+1}} q_{i}(n, \ell+1) \right]
\]

\[
+ (-1)^{p-1} \left[ \binom{n+\ell+2}{\ell+2} - \binom{n+\ell+1}{\ell+1} \right]
\]

\[
= \binom{n+\ell-1}{\ell} \frac{(n+\ell)(n+\ell+1)}{\ell+1} \sum_{i=0}^{p-1} \binom{p}{i} (-1)^{p-i-1} (n+\ell+2)(\ell+3+i)_{p-1} q_{i}(n, \ell+1)
\]

\[
+ \sum_{i=1}^{p-1} \binom{p}{i} (-1)^{p-i-1} (n+\ell+2)(\ell+3+i)_{p-1} q_{i}(n, \ell+1)
\]
where the second equality follows from binomial expansion of \((n - j + \ell)^p = (n + \ell + 1 - j - 1)^p\), while the fourth equality follows from applying the inductive hypothesis. Note that \(q'_{p}(n, \ell)\) is a sum of polynomials in \(O(n^{p-1})\), and hence it itself is a polynomial in \(O(n^{p-1})\).

To finish the proof of the lemma, it suffice to observe from (68) and (70) with \(\ell = k - 2\) that

\[
\mu_{n,k,p} = \frac{(n+k-3)}{(n+k-1)} \frac{(n+k-2)(n+k-1)}{\langle k-1 \rangle_{p+1}} q'_{p}(n,k-2) = \frac{n \cdot q_{p}(n,k)}{\langle k \rangle_{p}},
\]

where \(q_{p}(x,y) = q'_{p}(x,y - 2)\).

With the dependence of \(\mu_{n,k,p}\) on \(n\) and \(k\) in hand, we can elucidate the asymptotics of the summands in (67):

**Lemma 4** (Asymptotics of summands). For \(\ell \leq t \leq m\) and \(a \in D_{m,t}^{\leq}\) such that \(a_1 = \cdots = a_{\ell} = 1\) and \(a_j \geq 2\) for all \(j \in \{\ell + 1, \ldots, t\}\), we have

\[
\sigma_{n,k,p,w}^{-m} \sum_{i_1, \ldots, i_t \text{ distinct}} \mathbb{E} \left[ \prod_{j=1}^{\ell} w_{i_j} X_{i_j} \prod_{j=\ell+1}^{t} (w_{i_j} X_{i_j})^{a_j} \right] = O \left[ \left( \frac{k}{n} \right)^{p(m-t)} \frac{k^{t-m/2}}{n^t} \right]. \tag{74}
\]

**Proof.** Three cumbersome applications of (70) to the computation of the variance show that \(\sigma_{n,p,k,w}^{-m} = O \left( \left( \frac{n}{k} \right)^p \sqrt{k} \right)\), which takes care of the denominator on the LHS of (74). To treat the enumerator, we use exchangeability of the \(X_i\) as well as the compact support of the weights \(w_j\) to upper bound the magnitude of the sum on the LHS by the magnitude of

\[
k^{t} W_{\text{max}}^{m} \cdot \mathbb{E} \left[ \prod_{j=1}^{\ell} X_{j} \prod_{j=\ell+1}^{t} X_{j}^{a_j} \right] = k^{t} W_{\text{max}}^{m} \mathbb{E} \left[ \mathbb{E} [X_1 | X_2, \ldots, X_t] \prod_{j=2}^{t} X_{j} \prod_{j=\ell+1}^{t} X_{j}^{a_j} \right]. \tag{75}
\]

For fixed \(n\) and growing \(k\), we expect the bin occupations to decorrelate, and hence the conditional expectation on the RHS of (75) to vanish. Indeed, referring once more to (70) and writing \(X_{2}^t := \sum_{j=2}^{t} X_j\), we have

\[
\mathbb{E} [X_1 | X_2, \ldots, X_t] = \mathbb{E} [X_1 | X_2^t] = \mu_{n-X_{2}^t,k-t+1,p} - \mu_{n,k,p}
\]

\[
= \frac{n - X_{2}^t}{\langle k - t + 1 \rangle_p} q_{p} \left( n - X_{2}^t, k - t + 1 \right) - \frac{n}{\langle k \rangle_p} q_{p} \left( n, k \right)
\]

\[
= O \left( \frac{n^{p-1}}{k^p} \right), \tag{76}
\]

\(\square\)
as long as $X_2^t = \sum_{j=2}^t S_{n,k}[j] = o(n)$. Whence the magnitude of (74) is bounded above by

$$k^t W_{\max}^m \mathbb{E} \left\{ O\left(\frac{n^{\beta-1}}{k^p}\right) \prod_{j=2}^t X_j \prod_{j=\ell+1}^t X_j^{a_j} \right\}$$

$$\leq k^t W_{\max}^m O\left(\frac{n^{\beta-1}}{k^p}\right) \mathbb{E} \left\{ \prod_{j=2}^t X_j \prod_{j=\ell+1}^t X_j^{a_j} \right\} + k^t W_{\max}^m \mathbb{P} [X_2^t \neq o(n)] n^m$$

$$= k^t W_{\max}^m O\left(\frac{n^{\beta-1}}{k^p}\right) \mathbb{E} \left\{ \prod_{j=2}^t X_j \prod_{j=\ell+1}^t X_j^{a_j} \right\} + 2n^m k^t W_{\max}^m \mathbb{P} [X_2^t \neq o(n)]. \quad (77)$$

Repeating the same reasoning used in (75) and (77) $\ell$ times on $A$ yields an upper bound for the magnitude of (74) of

$$k^t W_{\max}^m O\left(\frac{n^{\ell(p-1)}}{k^{\ell p}}\right) \mathbb{E} \left\{ \prod_{j=\ell+1}^t X_j^{a_j} \right\} + 2(\ell - 1) k^t n^m \mathbb{P} [S_{n,k}[1] \neq o(n)]. \quad (78)$$

We will argue below in \textsc{Lemma 5} that bin sizes concentrate around their means $n/k$, which implies that $C$ is exponentially small in $k$, while $B$ scales like $(n/k)^{p(t-\ell)}$. Combining these with the $O(\sqrt{k}(n/k)^p)$ scaling of $\sigma_{n,k,p,w}$ gives the final upper bound

$$O \left[ k^t \frac{1}{k^{m/2}} \left( \frac{k}{n} \right)^{pm} \left( \frac{k}{n} \right)^{t-p\ell} \frac{1}{k^p} \left( \frac{k}{n} \right)^{p(t-t)} \right], \quad (79)$$

which simplifies to (74) as desired. \hfill \Box

To substantiate the claims about $B$ and $C$ in the proof of \textsc{Lemma 4}, we establish the following lemma:

\textbf{Lemma 5} (Large deviations for bin sizes). \textit{If } $k = O(n^\beta)$, \textit{then for all } $0 < \varepsilon < \beta$ \textit{there exists } $C_\varepsilon > 0$ \textit{so that}

$$\mathbb{P} [X_j \geq n^{1-\varepsilon}] < C e^{-n^{\beta-\varepsilon}}. \quad (80)$$

\textit{Proof.} We can compute explicitly

$$\mathbb{P} [S_{n,k}[1] \geq n^{1-\varepsilon}] = \frac{(n-n^{1-\varepsilon}+k-1)}{(n+k-1)} \left[ \prod_{j=n-n^{1-\varepsilon}+1}^n \left( 1 + \frac{k-1}{j} \right) \right]^{-1}$$

$$\leq C \exp \left\{ -(k-1) \log \left( \frac{1}{1-n^{1-\varepsilon}} \right) + O \left( \frac{k^2}{n^{1+\varepsilon}} \right) \right\}$$

$$\leq C \exp \left\{ -n^{\beta-\varepsilon} + O \left( n^{\beta-2\varepsilon} + n^{2\beta-1-\varepsilon} \right) \right\}, \quad (81)$$

which is dominated by the $n^{\beta-\varepsilon}$ term as long as $\varepsilon < \beta$. \hfill \Box

At last, we are now in shape to establish Proposition 5 from (74), for we see that
1. if \( t < m/2 \), the RHS is of \( O\left(\frac{(k)}{n}^{m/2} k^{t-m/2}\right) \) and hence vanishes as \( n \to \infty \);

2. if \( t > m/2 \), then since \( \ell \geq 2t - m \) (because \( \ell + 2(t - \ell) \leq m \)), the RHS is of \( O \left( (k/n^2)^{t-m/2} \right) \) and vanishes as \( n \to \infty \);

3. if \( t = m/2 \) and \( k = o(n) \), we obtain asymptotics of \( O \left( \frac{(k^n)}{n^{2m/2}} \right) \), which vanishes once again as \( n \to \infty \).

Hence, the only terms in (67) contributing to the limiting moments are those for which \( t = m/2 \) (and consequently \( \ell = 0, a_j = 2 \) for all \( j \in \{1, \ldots, t\} \), \( C_0 \ldots C_{n-2} = (m-1)!! \) and \( m \) must be even), when \( k = \Theta(n) \). That is,

\[
\mathbb{E} \left( Z_{n, m, p, w}^{m} \right) = (m-1)!! \cdot \mathbb{E} \prod_{j=1}^{m/2} \left[ \frac{(S_{n,k}[j])^p - \mu_{n,kp}}{\sigma_{n,k,p,w}^{-1/2}} \right] ^2,
\]

which by decorrelation computations very similar to (76) is readily seen to converge to \((m-1)!!\). These are precisely the normal moments, which proves the first half (37) of Proposition 5. To show the second half (38), we sidestep individual moment computations and resort directly to Lévy’s continuity theorem. To wit, we have

\[
E e^{-t\|S_{n,k}\|_{\infty,p,w}} = P \left( \|S_{n,k}\|_{\infty} \leq 1 \right) \cdot E \left[ e^{-t\|S_{n,k}\|_{p,w}} \right] + O \left( k^{-1} \right)
\]

\[
= \frac{1}{k(k-1) \cdots (k-n+1)} \sum_{s_1=1}^{k} \sum_{s_2=1}^{s_1} \cdots \sum_{s_n=1}^{s_{n-1}} e^{-t\sum_{j=1}^{n} w(s_j)} + O \left( k^{-1} \right),
\]

which, since \( w \) is bounded and continuous almost everywhere (and hence Riemann integrable) converges, as \( k \to \infty \), to

\[
\left( \int_0^1 e^{-tw(x)} \, dx \right)^n = E e^{-t\sum_{j=1}^{n} w(U_j)}
\]

as desired.

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