On analyticity up to the boundary for critical quasi-geostrophic equation

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Abstract. We study the Cauchy problem for the surface quasi-geostrophic equations with the critical dissipation in the two dimensional half space under the homogeneous Dirichlet boundary condition. We show the global existence, the uniqueness and the analyticity of solutions, and the real analyticity up to the boundary is obtained. We will show a natural way to estimate the nonlinear term for functions satisfying the Dirichlet boundary condition.

1. Introduction

We consider the critical surface quasi-geostrophic equations in the half space.

\[ \partial_t \theta + (u \cdot \nabla) \theta + \Lambda_D \theta = 0, \quad u = \nabla_\perp \Lambda_D^{-1} \theta, \quad t > 0, x \in \mathbb{R}^2_+, \quad (1.1) \]

\[ \theta(0, x) = \theta_0(x), \quad x \in \mathbb{R}^2_+, \quad (1.2) \]

where \( \mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 | x_n > 0\} \), \( \nabla_\perp := (-\partial_{x_2}, \partial_{x_1}) \), \( \Lambda_D \) is the square root of the Dirichlet Laplacian. The equations are known as an important model in geophysical fluid dynamics, which is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency (see [24, 26]). The purpose of this paper is to show the existence of global solutions for initial data in scaling critical Besov spaces, and the analyticity.

Let us recall existing results, where the space is the whole space. If we consider the fractional Laplacian of the order \( \alpha \), \((-\partial^2)^{\alpha/2}\), with \( 0 < \alpha \leq 2 \), instead of the square root of the Laplacian, then the case when \( \alpha < 1, \alpha = 1, \alpha > 1 \) are called sub-critical case, critical case, super-critical case, respectively. It is known that the global-in-time regularity is obtained for the sub-critical case and the critical case. The sub-critical case can be treated, by \( L^\infty \)-maximum principle, and the critical case is delicate. In the critical case, the regularity with small data was proved by Constantin, Cordoba and Wu [5] (see also Constantin and Wu [12]). The problem for large data case was solved by Caffarelli and Vasseur [2], Kiselev, Nazarov and Volberg [23]. As another approach, Constantin and Vicol [11] proved the global regularity by nonlinear maximum principles in the form of a nonlinear lower bound on the fractional Laplacian. On the other hand, in the super-critical case, the regularity only for small data is known (see [14]), and blow-up for smooth solutions is an open problem.

In bounded domains with smooth boundary, the equations was introduced by Constantin and Ignatova [6, 7]. Local existence was shown by Constantin and Nguyen [10], and global existence of weak solutions was proved by Constantin and Ignatova [7] (see...
also the paper by Constantin and Nguyen \cite{9} for the inviscid case). An interesting question here is how to understand the behavior of the solutions. A priori bounds of smooth solutions was obtained by Constantin and Ignatova \cite{6}, and interior Lipschitz continuity of weak solutions was studied by Ignatova \cite{15}. Recently, Constantin and Ignatova \cite{8} considered the quotient of the solution by the first eigen function to investigate near the boundary, and gave a condition to obtain the global regularity up to the boundary. Stokols and Vasseur \cite{27} constructed global-in-time weak solutions with Hölder regularity up to the boundary. We should note from the viewpoint of smooth solutions that regularity near the boundary is guaranteed for a short time to the best of our knowledge. As for the half space case, the possibility is pointed out in \cite{6} when the support of the initial data is away from the boundary.

In this paper, we consider the problem in the half space to show global-in-time regularity, and furthermore, analyticity in spacetime. Initial data in our theorem can have its support around the boundary. We will utilize the odd extension with respect to \( x_2 \) technically, but the reason of the half space is just for the sake of the simplicity. Our method is based on the one for the whole space, introducing Besov spaces associated with the Dirichlet Laplacian. Related idea to handle product estimate can be found in the paper \cite{20}, where the validity of bilinear estimates for functions with the Dirichlet boundary condition is discussed. We should also remark that our domain, the half space, is one of the simplest domains, and the odd reflection and the existing result in the whole space \( \mathbb{R}^2 \) imply the existence of solutions below formally, but the main subject here is the behavior of functions on the boundary. We suppose to establish a method applicable to more general domains in the future, and the purpose is to state theorems in intrinsic framework.

We state two theorems. The first theorem concerns with the integral equation with the small data, where it seems easier to explain the proof near boundary clearly. The second theorem studies the data belonging to the largest scaling critical Besov space, where the smooth functions exist dense. We here introduce a formal definition of Besov spaces, which is defined precisely in section 2. For every function \( f \in L^1(\mathbb{R}^2_+) + L^\infty(\mathbb{R}^2_+) \), let \( f_{\text{odd}} \) be defined by

\[
f_{\text{odd}}(x_1, x_2) := \begin{cases} f(x_1, x_2) & \text{if } x_2 > 0, \\ -f(x_1, -x_2) & \text{if } x_2 < 0, \end{cases}
\]

where this extension is justified as a locally integrable function at least. Norm of our Besov spaces can be understood through the spaces on the whole space by

\[
\|f\|_{B^{s}_{p,q}(\Lambda_D)} = \|f_{\text{odd}}\|_{B^{s}_{p,q}(\mathbb{R}^2)}; \quad \|f\|_{\dot{B}^{s}_{p,q}(\Lambda_D)} = \|f_{\text{odd}}\|_{\dot{B}^{s}_{p,q}(\mathbb{R}^2)}.
\]

We then have the following theorem.

**Theorem 1.1.** (Solutions of the integral equation with small data) Let \( \theta_0 \in \dot{B}_{\infty,1}^{0}(\Lambda_D) \) be sufficiently small. Then the integral equation

\[
\theta(t) = e^{-t\Lambda_D} \theta_0 - \int_0^t e^{-(t-\tau)\Lambda_D} \left( (u \cdot \nabla) \theta \right) d\tau, \quad u = \nabla_{\perp} \Lambda_D^{-1} \theta
\]

possesses a unique global solution \( \theta \) such that

\[
\theta \in C([0, \infty), \dot{B}_{\infty,1}^{0}(\Lambda_D)) \cap L^1(0, \infty; \dot{B}^1_{\infty,1}(\Lambda_D))
\]
Furthermore, there exists $C > 0$ such that for any $\alpha, \beta_1, \beta_2 \in \mathbb{N} \cup \{0\}$,
\[
t^{\alpha + \beta_1 + \beta_2} \| \partial_t^{\alpha} \partial_x^{\beta_1} \partial_x^{\beta_2} \theta(t) \|_{L^\infty(\mathbb{R}_+^2)} \leq C^{\alpha + \beta_1 + \beta_2} \alpha! \beta_1! \beta_2!,
\]
and in particular, $\theta(t)$ is real analytic in space and time if $t > 0$.

Next theorem concerns with the initial data in the space corresponding to the largest scaling critical Besov space $\dot{B}^0_{\infty, \infty}(\Lambda_D)$. We recall the the paper \[30\] by Wang-Zhang, who take the data in the space defined by the completion of $C^1_0(\mathbb{R}^2)$. We can generalize it to the Besov space $B^0_{\infty, \infty}(\mathbb{R}^2)$ with taking the completion with small high frequency (see \[19\]). Under this motivation, we have:

**Theorem 1.2. (Solutions in the largest scaling critical Besov space)** Let $\theta_0 \in B^0_{\infty, \infty}(\Lambda_D)$ be such that
\[
\lim_{j \to \infty} \| \phi_j(\Lambda_D) \theta_0 \|_{L^\infty} = 0, \quad \left(1 - \sum_{j \geq 0} \phi_j(\Lambda_D)\right) \theta_0 \in \dot{B}^0_{\infty, 1}(\Lambda_D).
\]

Then the problem (1.1) and (1.2) possess a unique solution $\theta$ such that
\[
\theta \in C([0, \infty), B^0_{\infty, \infty}(\Lambda_D)) \cap L^1_{loc}(0, \infty; \dot{B}^1_{\infty, \infty}(\Lambda_D)),
\]
\[
\left(1 - \sum_{j \geq 0} \phi_j(\Lambda_D)\right) \theta \in C([0, \infty), \dot{B}^0_{\infty, 1}(\Lambda_D)).
\]

Furthermore, $\theta(t)$ is real analytic in space and time if $t > 0$.

**Remark.** (i) If we replace the first condition of (1.3) with the smallness of the high spectral component
\[
\limsup_{j \to \infty} \| \phi_j(\Lambda_D) \theta_0 \|_{L^\infty} \leq \delta, \quad \delta \ll 1,
\]
then we can also construct a unique solution such that $\theta$ is weak-* continuous in $B^0_{\infty, \infty}(\Lambda_D)$ with respect to $t \geq 0$. We can regard the smallness as the possibility of only small shock discussed in \[11\].

(ii) We impose the second condition of (1.3) to justify $u = \nabla^\perp \Lambda_D^{-1} \theta \in X'_D$.

(iii) We will give a proof outline of global regularity based on the nonlinear maximum principle. Uniform bound in the Hölder spaces similar to Theorem 3.1 in \[4\] enables us to repeat the fixed point argument in a time interval of fixed length.

Let us give remarks for the proof of theorems. We apply a simple fixed point argument to the the proof of Theorem \[1.1\] as in \[16\] and \[19\]. To this end, we will derive bilinear estimates for $(u \cdot \nabla)\theta$, which is crucial to understand how to estimate near the boundary and contains the main idea of this paper. As for Theorem \[1.2\], we explain only proof outline, since the main idea near the boundary is same as in the proof of the first theorem and we can apply the proof in \[19\]. We also mention the linear estimates which was established for more general framework (see \[17\]).

We here focus on the discussion of the validity of the bilinear estimate for $(u \cdot \nabla)\theta$, how to obtain smoothness measured by the Dirichlet Laplacian up to the boundary. A standard argument would be to show
\[
\|(u \cdot \nabla)\theta\|_{H^s_D} \leq C(\|u\|_{\dot{H}^1_{p_1}} \|
abla \theta\|_{p_2} + \|u\|_{L^p_{p_3}} \|\nabla \theta\|_{H^s_{p_4}}), \quad s > 0, \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},
\]
and apply the boundedness of the Riesz transform. However, this method causes a problem from the boundary value of the functions in the right hand side because of $x_2$ derivative, which yields $\|\partial_{x_2}A_D^{-1}\theta\|_{H^s_{p_1}} \cdot \|\partial_{x_2}\theta\|_{H^s_{p_2}} = \infty$ for large $s$, in general. On the other hand, we investigate the boundary value of $(u \cdot \nabla)\theta$ itself which leads to a natural estimate. We can then have a estimate in Besov spaces (see Proposition 3.2).

\[
\| (u \cdot \nabla)\theta \|_{\dot{B}^1_{p,1}(\Lambda_D)} \leq C(\|\theta\|_{\dot{B}^1_{p,q}(\Lambda_D)} \|\theta\|_{\dot{B}^1_{q,1}(\Lambda_D)} + \|\theta\|_{\dot{B}^0_{p,1}(\Lambda_D)} \|\theta\|_{\dot{B}^0_{q,1}(\Lambda_D)}).
\]

In a word, the most important point is: If $\theta$ satisfies the Dirichlet boundary condition, then $(u \cdot \nabla)\theta$ does. It would be possible to apply this argument not only to the half space but also to more general domains with smooth boundary. One can also find how derivatives affect boundary value of functions with the Dirichlet and the Neumann Laplacian in the papers [13,20], where the validity of product estimate for $fg$ is discussed and we can not expect it when $s > 2+1/p$. Briefly speaking, $\partial_{x_1}$ maps from $\dot{B}^1_{p,q}(\Lambda_D)$ to $\dot{B}^0_{p,q}(\Lambda_D)$, but $\partial_{x_2}$ maps to $\dot{B}^0_{p,q}(\Lambda_D)$ to $\dot{B}^0_{p,q}(\Lambda_N)$, where $\Lambda_N$ is the square root of the Neumann Laplacian.

This paper is organized as follows. In section 2, we recall the definition of Besov spaces associated with the Dirichlet Laplacian, and several properties to study the boundary value of functions. We prove Theorem 1.1 in section 3, and explain idea of proof of Theorem 1.2 in section 4.

**Notations.** We denote by $-\Delta_D$ the Dirichlet Laplacian on $L^2(\Omega)$. We write $x = (x_1, x_2)$. For any function $f$ on $\mathbb{R}^2_+$, $f_{\text{odd}}$, $f_{\text{even}}$ are odd and even extenstions of $f$ with respect to $x_2$,

\[
f_{\text{odd}}(x) = \begin{cases} f(x) & \text{if } x_2 > 0, \\
-f(x_1, -x_2) & \text{if } x_2 < 0,
\end{cases}
f_{\text{even}}(x) = \begin{cases} f(x) & \text{if } x_2 > 0, \\
f(x_1, -x_2) & \text{if } x_2 < 0.
\end{cases}
\]

For multi-index $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{N} \cup \{0\})^2$, let $\partial^{\alpha}_{x_2} = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2}$ and $|\alpha| = \alpha_1 + \alpha_2$. We denote by $\Delta_{\mathbb{R}^2}$ the Laplacian on $\mathbb{R}^2$, $G_t = G_t(x)$ the Gauss kernel $G_t(x) = (4\pi t)^{-1}e^{-|x|^2/4t}$, and $e^{t\Delta_D}$ the semigroup generated by the Dirichlet Laplacian

\[
e^{t\Delta_D} f = \int_{\mathbb{R}^2_+} (G_t(x-y) - G_t(x+y)) f(y) \, dy.
\]

Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be the dyadic decomposition of the unity such that $\phi_j$ is a non-negative function in $C^\infty_0(\mathbb{R})$ and

\[
\text{supp} \, \phi_0 \subset [2^{-1}, 2], \quad \phi_j(\lambda) = \phi_0 \left( \frac{\lambda}{2^j} \right), \quad \sum_{j \in \mathbb{Z}} \phi_j(\lambda) = 1 \text{ for any } \lambda > 0.
\]

Let $\psi \in C^\infty_0(\mathbb{R})$ be a non-negative function such that

\[
\text{supp} \, \psi \subset (-\infty, 2], \quad \psi(\lambda) + \sum_{j=1}^{\infty} \phi_j(\lambda) = 1 \text{ for any } \lambda \geq 0.
\]

We write the Fourier transform and the inverse Fourier transform.

\[
\mathcal{F}[f](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) \, dx, \quad \mathcal{F}^{-1}[f](x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix \cdot \xi} f(\xi) \, d\xi.
\]
The convolution $f * g$ is defined by the standard integral on $\mathbb{R}^2$.
\[
f * g(x) = \int_{\mathbb{R}^2} f(x - y)g(y)dy.
\]
We use the following notations for norms of spaces in space and time as follows.
\[
\|f\|_{B^s_{p,q}(\Lambda_D)} = \|\psi(\Lambda_D)f\|_{L^p(\mathbb{R}^2_+^s)} + \left\| \left\{ 2^{sj}\|\phi_j(\Lambda_D)f\|_{L^p(\mathbb{R}^2_+^s)} \right\}_{j \in \mathbb{N}} \right\|_{\ell^1(\mathbb{N})},
\]
\[
\|f\|_{B^s_{p,q}(\Lambda_D)} = \left\| \left\{ 2^{sj}\|\phi_j(\Lambda_D)f\|_{L^p(\mathbb{R}^2_+^s)} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^1(\mathbb{Z})},
\]
\[
\|f\|_{L^r(0,\infty;X)} = \|f(t)\|_X \quad X = L^p(\mathbb{R}^2_+^s), B^s_{p,q}(\Lambda_D), \hat{B}^s_{p,q}(\Lambda_D),
\]
\[
\|f\|_{L^r(0,\infty;B^s_{p,q}(\Lambda_D))} = \left\| \left\{ 2^{sj}\|\phi_j(\Lambda_D)f\|_{L^r(0,\infty;L^p(\mathbb{R}^2_+^s))} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^1(\mathbb{Z})}.
\]

2. Preliminary

In this section, we recall the definition of the Besov spaces (see [22]), the boundedness of the spectral multipliers (see [21][25][28]), several lemmas to justify an argument by the odd or the even extention of $f$ and its derivatives.

We start by defining the Dirichlet Laplacian $-\Delta_D$, and spaces of test functions of non-homogeneous type, $\mathcal{X}_D$, and of homogeneous type $\mathcal{Z}_D$, and their duals.

**Definition.** (i) Let $-\Delta_D$ be the Dirichlet Laplacian on $L^2(\mathbb{R}^2_+)$ defined by
\[
\begin{aligned}
D(-\Delta_D) &:= \{ f \in H^1_0(\mathbb{R}^2_+) \mid \Delta f \in L^2(\mathbb{R}^2_+) \}, \\
-\Delta_D f &:= -\Delta f = -\left( \frac{\partial^2}{\partial x_1^2} f + \frac{\partial^2}{\partial x_2^2} f \right), \quad f \in D(-\Delta_D).
\end{aligned}
\]
(ii) Let $\mathcal{X}_D = \mathcal{X}_D(\mathbb{R}^2_+)$ be a space of test functions of non-homogeneous type such that
\[
\mathcal{X}_D := \{ f \in L^1(\mathbb{R}^2_+) \cap L^2(\mathbb{R}^2_+) \mid p_m(f) < \infty \text{ for all } m \in \mathbb{N} \},
\]
where
\[
p_m(f) := \|f\|_{L^1} + \sup_{j \in \mathbb{N}} 2^{mj}\|\phi_j(\Lambda_D)f\|_{L^1}.
\]
(iii) Let $\mathcal{Z}_D = \mathcal{Z}_D(\mathbb{R}^2_+)$ be a space of test functions of homogeneous type such that
\[
\mathcal{Z}_D := \{ f \in \mathcal{X}_D \mid q_m(f) < \infty \text{ for all } m \in \mathbb{N} \},
\]
where
\[
q_m(f) := p_m(f) + \sup_{j \leq 0} 2^{mj}\|\phi_j(\Lambda_D)f\|_{L^1}.
\]
(iv) Let $\mathcal{X}_D^\prime$, $\mathcal{Z}_D^\prime$ be the topological duals of $\mathcal{X}_D$, $\mathcal{Z}_D$, respectively.

It was proved in [22] that the spaces $\mathcal{X}_D$, $\mathcal{Z}_D$ are Fréchet spaces, and can regard their duals $\mathcal{X}_D^\prime$, $\mathcal{Z}_D^\prime$ as distribution spaces of non-homogeneous type and homogeneous type, respectively, which are variants of the space of the tempered distributions and the quatient space by the polynomials. We define Besov spaces associated with the Dirichlet Laplacian on the half space as follows.

**Definition.** Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. 5
(i) (Non-homogeneous Besov space) \( B_{p,q}^s(\Lambda_D) \) is defined by
\[
B_{p,q}^s(\Lambda_D) = \{ f \in X'_D \mid \| f \|_{B_{p,q}^s(\Lambda_D)} < \infty \},
\]
where
\[
\| f \|_{B_{p,q}^s(\Lambda_D)} := \| \psi(\Lambda_D) f \|_{L^p(\mathbb{R}_+^2)} + \left\{ 2^{sj} \| \phi_j(\Lambda_D) f \|_{L^p(\mathbb{R}_+^2)} \right\}_{j \in \mathbb{N}}^{\ell_1(\mathbb{N})}.
\]

(ii) (Homogenous Besov space) \( \dot{B}_{p,q}^s(\Lambda_D) \) is defined by
\[
\dot{B}_{p,q}^s(\Lambda_D) := \{ f \in \mathcal{Z}'_D \mid \| f \|_{B_{p,q}^s(\Lambda_D)} < \infty \},
\]
where
\[
\| f \|_{\dot{B}_{p,q}^s(\Lambda_D)} := \left\{ 2^{sj} \| \phi_j(\Lambda_D) f \|_{L^p(\mathbb{R}_+^2)} \right\}_{j \in \mathbb{Z}}^{\ell_1(\mathbb{Z})}.
\]

It is proved in [22] that the Besov spaces \( B_{p,q}^s(\Lambda_D) \), \( \dot{B}_{p,q}^s(\Lambda_D) \) are Banach spaces and satisfy standard properties such as lift properties, embedding theorems of Sobolev type as well as the whole space case (see [29]). We here recall the uniform boundedness of the frequency restriction operator \( \phi_j(\Lambda_D) \) and some fundamental property of the Besov spaces for our purpose of this paper.

**Lemma 2.1.** ([21,25,28]) Let \( 1 \leq p \leq \infty \). Then
\[
\sup_{j \in \mathbb{Z}} \| \phi_j(\Lambda_D) \|_{L^p \to L^p} < \infty.
\]
In particular, we also have that for any \( \varphi \in C^\infty_0((0, \infty)) \), there exists \( C > 0 \) such that
\[
\| \varphi(2^{-j}\Lambda_D)\phi_j(\Lambda_D) f \|_{L^p} \leq C \| \phi_j(\Lambda_D) f \|_{L^p}
\]
for all \( j \in \mathbb{Z} \) and \( f \) with \( \phi_j(\Lambda_D) f \in L^p(\mathbb{R}_+^2) \).

**Lemma 2.2.** ([22]) (i) (Resolution of identity) For every \( f \in X'_D \), we have
\[
f = \psi(\Lambda_D) f + \sum_{j=1}^{\infty} \phi_j(\Lambda_D) f \quad \text{in } X'_D.
\]
For every \( f \in \mathcal{Z}'_D \), we have
\[
f = \sum_{j \in \mathbb{Z}} \phi_j(\Lambda_D) f \quad \text{in } \mathcal{Z}'_D.
\]

(ii) (A characterization of homogeneous spaces as a subspace of \( X'_D \)) Let \( s < 2/p \) or \( (s,p) = (2/p,1) \). Then \( \dot{B}_{p,q}^s(\Lambda) \) is equivalent to
\[
\left\{ f \in X'_D \mid \| f \|_{\dot{B}_{p,q}^s(\Lambda_D)} < \infty, \quad f = \sum_{j \in \mathbb{Z}} \phi_j(\Lambda_D) \text{ in } X'_D \right\}.
\]

Next, we show that functions that \( \dot{B}_{\infty,1}^0(\Lambda_D) \cap \dot{B}_{\infty,1}^m(\Lambda_D) \) is included in \( C^m(\mathbb{R}_+^2 \cup \partial \mathbb{R}_+^2) \) and the odd extension for \( x_2 \) is in \( C^m(\mathbb{R}^2) \).

**Lemma 2.3.** Let \( m = 0, 1, 2, \ldots \) and \( f \in \dot{B}_{\infty,1}^0(\Lambda_D) \cap \dot{B}_{\infty,1}^m(\Lambda_D) \). Then \( \partial_x^m f \) is in \( L^\infty(\mathbb{R}_+^2) \) and is extended to a continuous function in the closure of \( \mathbb{R}_+^2 \) provided that \( |\alpha| \leq m \). In particular, \( f_{\text{odd}} \) is regarded as a function belonging to \( C^m(\mathbb{R}^2) \).
Proof. Let \( f \in \dot{B}^0_{\infty,1}(\Lambda_D) \). We write \( f \) with the resolution of the identity and consider the estimate in \( L^\infty(\mathbb{R}_+^2) \).

\[
\|f\|_{L^\infty} \leq \sum_{j \in \mathbb{Z}} \|\phi_j(\Lambda_D)f\|_{L^\infty} = \|f\|_{\dot{B}^0_{\infty,1}(\Lambda_D)} < \infty.
\]

Next, we estimate the first derivatives of \( \phi_j(\Lambda_D)f \) with each \( j \in \mathbb{Z} \), by the derivative estimate for \( e^{i\Delta_D} \) and the uniform boundedness in Lemma 2.1.

\[
\|\nabla \phi_j(\Lambda_D)f\|_{L^\infty} = \|\nabla e^{t_j \Delta_D}(e^{-t_j \Delta_D}\phi_j(\Lambda_D))f\|_{L^\infty} \leq C 2^j \|f\|_{L^\infty} < \infty, \quad t_j := 2^{-2j},
\]

which implies the uniform continuity of \( \phi_j(\Lambda_D)f \) with respect to \( x \in \mathbb{R}_+^2 \). The continuity and the following convergence

\[
\left\| f - \sum_{|j| \leq N} \phi_j(\Lambda_D)f \right\|_{L^\infty} \leq \left\| f - \sum_{|j| \leq N} \phi_j(\Lambda_D)f \right\|_{\dot{B}^0_{\infty,1}(\Lambda_D)} \to 0 \quad \text{as} \quad N \to \infty
\]
yield the uniform continuity of \( f \), and we can then extend \( f \) as a continuous function up to the boundary \( \partial \mathbb{R}_+^2 \). We may then have \( f \in L^\infty(\mathbb{R}_+^2) \cap C(\mathbb{R}_+^2 \cup \partial \mathbb{R}_+^2) \).

Let \( f \in \dot{B}^0_{\infty,1}(\Lambda) \cap \dot{B}^m_{\infty,1}(\Lambda_D) \) for \( m \geq 0 \). Since we know \( e^{-t_j \Delta_D}\phi_j(\Lambda_D)f \in \dot{B}^0_{\infty,1}(\Lambda_D) \) can be extended to a continuous function on the closure of \( \mathbb{R}_+^2 \), we write \( \phi_j(\Lambda_D)f \) as a convolution of the Gauss kernel and an odd function for \( x_2 \).

\[
\phi_j(\Lambda_D)f = e^{t_j \Delta_D} e^{-t_j \Delta_D}\phi_j(\Lambda_D)f = \int_{\mathbb{R}^2} G_{t_j}(x - y) (e^{-t_j \Delta_D}\phi_j(\Lambda_D)f)_{\text{odd}}(y) \, dy,
\]

where \( t_j = 2^{-2j} \). It follows from Lemma 2.1 that for every multiindex \( \alpha \) with \( |\alpha| \leq m \)

\[
\|\partial^\alpha_x \phi_j(\Lambda_D)f\|_{L^\infty} \leq C \|\partial^\alpha_x G_{t_j}\|_{L^1} \|e^{-t_j \Delta_D}\phi_j(\Lambda_D)f\|_{L^\infty} \leq C 2^{|\alpha| j} \|e^{-t_j \Delta_D}\phi_j(\Lambda_D)f\|_{L^\infty} \leq C 2^{|\alpha| j} \|\phi_j(\Lambda_D)f\|_{L^\infty}.
\]

We have from the inequality above that

\[
\|\partial^\alpha_x f\|_{L^\infty} \leq \sum_{j \in \mathbb{Z}} \|\partial^\alpha_x \phi_j(\Lambda_D)f\|_{\dot{B}^0_{\infty,1}(\Lambda_D)} \leq C \|f\|_{\dot{B}^m_{\infty,1}(\Lambda_D)}.
\]

\( \partial^\alpha_x f \) is extended to a continuous function in the closure of \( \mathbb{R}_+^2 \) and \( f_{\text{odd}} \) is regarded as a function on \( \mathbb{R}^2 \). We can also see that \( \partial^\alpha_x f \) is uniformly continuous in a similar way to the previous case when \( m = 0 \).

Finally it is easy to see that the odd extension \( f_{\text{odd}} \) is regarded as a function in \( C^m(\mathbb{R}^2) \), since the last right hand side of (2.1) can be regarded as an odd function by taking \( x \) in the whole space.

Next lemma reveals that the spectral restriction operator \( \varphi(\Lambda_D) \) is written by using the Fourier transform and the odd extension, and relation between derivatives and smooth odd or even extension.

Lemma 2.4. (i) Let \( \varphi \in C^\infty_0((0, \infty)) \) and \( f \in L^1(\mathbb{R}_+^2) + L^\infty(\mathbb{R}_+^2) \). Then

\[
\varphi(\Lambda_D)f = \int_{\mathbb{R}_+^2} \left( \mathcal{F}^{-1} \left[ \varphi(|\xi|) \right] (x - y) - \mathcal{F}^{-1} \left[ \varphi(|\xi|) \right] (x_1 - y_1, x_2 + y_2) \right) f(y) \, dy = \mathcal{F}^{-1} \left[ \varphi(|\xi|) \right] * f_{\text{odd}}\bigg|_{\mathbb{R}_+^2}.
\]

\[\text{(2.2)}\]
(ii) Let \( f \in \hat{B}^m_{\infty,1}(\Lambda_D) \) for all \( m = 0,1,2,\ldots \). Then
\[
(\partial_{x_1}f)_{\text{odd}} = \partial_{x_1}f_{\text{odd}}, \quad (\partial_{x_2}f)_{\text{even}} = \partial_{x_2}(f_{\text{odd}}).
\]

**Proof.** We prove (i). Since the support of \( \varphi \) is away from the origin, there is \( \tilde{\varphi} \in C_0^\infty(0,\infty) \) such that
\[
\tilde{\varphi}(\lambda^2) = \varphi(\lambda),
\]
and let us consider \( \tilde{\varphi}(-\Delta_D) \) instead of \( \varphi(\Lambda_D) \), whose proof requires essentially same argument. Therefore we will show that
\[
\tilde{\varphi}(-\Delta_D)f = \mathcal{F}^{-1}[\tilde{\varphi}(|\xi|^2)] \ast f_{\text{odd}} \bigg|_{\mathbb{R}^2_+}. \tag{2.3}
\]
We can also suppose that there exists \( j_0 \in \mathbb{N} \) such that
\[
f = \sum_{|j| \leq j_0} \phi_j(\Lambda_D)f, \tag{2.4}
\]
\[
\tilde{\varphi}(-\Delta)f = \tilde{\varphi}(-\Delta) \sum_{|j| \leq j_0} \phi_j(\Lambda_D)f
\]
for sufficiently large \( j_0 \).

We recall the spectral multiplier theorem with the bound by \( H^s \) norm (see \([25, 28]\)), and it holds that
\[
\|\tilde{\varphi}(-\Delta_D)\|_{L^\infty \to L^\infty} \leq C\|\tilde{\varphi}\|_{H^l(\mathbb{R})} \text{ for } l > \frac{n}{2}. \tag{2.5}
\]

Let us fix \( l > n/2 \). Our strategy is to approximate \( \tilde{\varphi} \) by an polynomial, more precisely, we utilize an analytic function. For every \( \varepsilon > 0 \), there exists \( \tilde{\varphi}_\varepsilon \) such that the support of the Fourier transform of \( \tilde{\varphi}_\varepsilon \) is compact and
\[
\|\tilde{\varphi} - \tilde{\varphi}_\varepsilon\|_{H^l(\mathbb{R})} < \varepsilon.
\]

By the compactness of the Fourier support, there exists \( C > 0 \) such that
\[
\|\nabla^\alpha \tilde{\varphi}_\varepsilon\|_{L^\infty} \leq C|\alpha|\|\tilde{\varphi}_\varepsilon\|_{L^\infty},
\]
and the Taylor expansion of \( \tilde{\varphi}_\varepsilon \) has the convergence radius, infinity, for each point. We take \( \lambda_0 > 0 \) and write the Talor expansion
\[
\tilde{\varphi}_\varepsilon(\lambda) = \sum_{k=0}^\infty a_k(\lambda - \lambda_0)^k,
\]
where \( a_k (k = 0,1,2,\ldots) \) are real numbers and the convergence of the series is uniform on each bounded interval. It follows by the formula above that
\[
\tilde{\varphi}(-\Delta_D)f - \mathcal{F}^{-1}[\tilde{\varphi}(|\xi|^2)] \ast f_{\text{odd}} \bigg|_{\mathbb{R}^2_+}
\]
\[
=\tilde{\varphi}(-\Delta_D)f - \tilde{\varphi}_\varepsilon(-\Delta_D)f + \sum_{k=0}^\infty a_k(-\Delta_D - \lambda_0)^k f - \mathcal{F}^{-1}[\tilde{\varphi}(|\xi|^2)] \ast f_{\text{odd}} \bigg|_{\mathbb{R}^2_+}. \tag{2.6}
\]

On the first two terms, we have from the boundedness of the spectral multiplier \( 2.5 \) that
\[
\|\tilde{\varphi}(-\Delta_D)f - \tilde{\varphi}_\varepsilon(-\Delta_D)f\|_{L^\infty} \leq C\|\tilde{\varphi} - \tilde{\varphi}_\varepsilon\|_{H^l(\mathbb{R})}\|f\|_{L^\infty} \leq C\varepsilon\|f\|_{L^\infty}.
\]

On the third and fourth terms, we note that
\[
-\Delta_D f \in L^\infty \quad \text{if and only if} \quad \lim_{t \to 0} \frac{e^{t\Delta_D f} - f}{t} \text{ in } L^\infty(\mathbb{R}^2_+) \text{ exists},
\]
\[
\|\nabla^\alpha (\tilde{\varphi}_\varepsilon(-\Delta_D)f)\|_{L^\infty} \leq C|\alpha|\|\tilde{\varphi}_\varepsilon(-\Delta_D)f\|_{L^\infty}
\]
and the Taylor expansion of \( \tilde{\varphi}_\varepsilon(-\Delta_D)f \) has the convergence radius, infinity, for each point.
and we know $-\Delta_D f \in L^\infty(\mathbb{R}^2_+)$ for $f \in L^\infty(\mathbb{R}^2_+)$ since (2.4). We also have that
\[
\lim_{t \to 0} \frac{e^{t\Delta_D} f - f}{t} = \lim_{t \to 0} \frac{e^{t\Delta_{\mathbb{R}^2}} f_{\text{odd}}|_{\mathbb{R}^2_+} - f}{t} = -\Delta f_{\text{odd}}|_{\mathbb{R}^2_+} \text{ in } L^\infty(\mathbb{R}^2_+) \text{ exists,}
\]
where $\Delta_{\mathbb{R}^2}$ is the Laplacian on $\mathbb{R}^2$. This implies that
\[
\sum_{k=0}^\infty a_k (-\Delta_D - \lambda_0)^k f = \sum_{k=0}^\infty a_k (-\Delta_{\mathbb{R}^2} - \lambda_0)^k f_{\text{odd}}|_{\mathbb{R}^2_+} = \mathcal{F}^{-1}[\varphi_\epsilon(\xi)] * f_{\text{odd}}|_{\mathbb{R}^2_+}.
\]
This equality and the boundedness of the Fourier multiplier implies that
\[
\left\| \sum_{k=0}^\infty a_k (-\Delta_D - \lambda_0)^k f - \mathcal{F}^{-1}[\varphi(\xi)] * f_{\text{odd}}|_{\mathbb{R}^2_+} \right\|_{L^\infty} \leq C\|\varphi_\epsilon - \varphi\|_{H^1(\mathbb{R})} \|f\|_{L^\infty} \leq C\epsilon \|f\|_{L^\infty}.
\]
Since $\epsilon > 0$ is arbitrary, we obtain (2.3) by (2.6) and the two inequalities above. The second statement (ii) follows from the equality (2.2) and a symmetric property of the evenness and the oddness with respect to $x_2$ taking $x$ in the whole space $\mathbb{R}^2$. In fact, we can write
\[
\partial_{x_k} f = \sum_{j \in \mathbb{Z}} \partial_{x_k} \phi_j(\Lambda_D) f = \sum_{j \in \mathbb{Z}} \partial_{x_k} \mathcal{F}^{-1}[\phi_j(\xi)] * f_{\text{odd}}|_{\mathbb{R}^2_+}, \quad k = 1, 2,
\]
and notice the radially symmetricity $\mathcal{F}^{-1}[\phi_j(\xi)](x) = \mathcal{F}^{-1}[\phi_j(\xi)](|x|)$. For $x_1$ derivative, it is easy check that
\[
\partial_{x_1} \mathcal{F}^{-1}[\phi_j(\xi)] * f_{\text{odd}} = \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|} \left(\partial_{|x-y|} \mathcal{F}^{-1}[\phi_j(\xi)](x - y) f_{\text{odd}}(y) dy, \quad x \in \mathbb{R}^2
\]
is odd for $x_2$, which proves that
\[
(\partial_{x_1} f)_{\text{odd}} = \sum_{j \in \mathbb{Z}} \partial_{x_1} \mathcal{F}^{-1}[\phi_j(\xi)] * f_{\text{odd}} = \partial_{x_1} f_{\text{odd}} \text{ in } \mathbb{R}^2.
\]
For $x_2$ derivative, we see that
\[
\partial_{x_2} \mathcal{F}^{-1}[\phi_j(\xi)] * f_{\text{odd}} = \int_{\mathbb{R}^2} \frac{x_2 - y_2}{|x - y|} \left(\partial_{|x-y|} \mathcal{F}^{-1}[\phi_j(\xi)](x - y) f_{\text{odd}}(y) dy, \quad x \in \mathbb{R}^2
\]
is even for $x_2$, and hence,
\[
(\partial_{x_2} f)_{\text{even}} = \sum_{j \in \mathbb{Z}} \partial_{x_2} \mathcal{F}^{-1}[\phi_j(\xi)] * f_{\text{odd}} = \partial_{x_2} f_{\text{odd}} \text{ in } \mathbb{R}^2.
\]
\]

3. Proof of Theorem 1.1

We prepare two propositions: The first proposition is about the linear estimate and the second proposition studies bilinear estimates of $(u \cdot \nabla)\theta$. We then prove Theorem 1.1

**Proposition 3.1.** ([17,19]) Let $s \in \mathbb{R}$, $1 \leq q, q_1, q_2 \leq \infty$.

(i) **(Smoothing estimates)** If $s > 0$ and $f \in \dot{B}^0_{\infty,q}(\Lambda_D)$, then
\[
\|e^{-t\Lambda_D} f\|_{\dot{B}^s_{\infty,q}(\Lambda_D)} \leq C t^{-s} \|f\|_{\dot{B}^s_{\infty,q}(\Lambda_D)}.
\]

(ii) **(Maximal regularity)** If $f \in \dot{B}^0_{\infty,q}(\Lambda_D)$, then
\[
\|e^{-t\Lambda} f\|_{\dot{B}^0_{\infty,q}(\Lambda_D)} + \|e^{-t\Lambda} f\|_{L^1(0,T;\dot{B}^s_{\infty,q}(\Lambda_D))} \leq C \|f\|_{\dot{B}^s_{\infty,q}(\Lambda_D)}.
\]
If \( f \in L^1(0, \infty; \dot{B}^0_{\infty,1}) \), then
\[
\left\| \int_0^t e^{-(t-\tau)\Lambda_D} f(\tau) \, d\tau \right\|_{L^1(0, \infty; \dot{B}^0_{\infty,1}(\Lambda_D) \cap L^1(0, \infty; \dot{B}^1_{\infty,1}(\Lambda_D)))} \leq \| f \|_{L^1(0, \infty; \dot{B}^0_{\infty,1}(\Lambda))}. \tag{3.3}
\]
(ii) There exists a positive constant \( C \) such that
\[
\| t^{\beta_1+\beta_2} \partial_t^{\beta_1} \Lambda_D^{\beta_2} e^{-t\Lambda_D} f \|_{\dot{B}^0_{\infty,q}(\Lambda_D)} + \| t^{\beta_1+\beta_2} \partial_t^{\beta_1} \Lambda_D^{\beta_2} e^{-t\Lambda_D} f \|_{L^1(0, \infty; \dot{B}^1_{\infty,q}(\Lambda_D))} \leq C^{\beta_1+\beta_2} \beta_1! \beta_2! \| f \|_{\dot{B}^0_{\infty,q}(\Lambda_D)},
\tag{3.4}
\]
for all \( f \in \dot{B}^0_{\infty,q}(\Lambda_D) \) and \( \beta_1, \beta_2 \in \mathbb{N} \cup \{0\} \).

Remark. The proof of (i) and (ii) can be found in [17]. The inequality (ii) can be proved in a similar way to the proof of Proposition 3.2 in [19]. Corresponding estimate to (3.4) has the constant \((\beta_1 + \beta_2)!\), but it should have been \(\beta_1!\beta_2!\) for the proof of the analyticity. It is possible to modify the proof to obtain \(\beta_1!\beta_2!\), by estimating the derivative orders of \(t, x\) separately.

**Proposition 3.2.** Let \( s \geq 0 \). Then for every \( f \in \dot{B}^0_{\infty,1}(\Lambda_D) \cap \dot{B}^s_{\infty,1}(\Lambda_D) \) and \( g \in \dot{B}^1_{\infty,1}(\Lambda_D) \cap \dot{B}^{s+1}_{\infty,1}(\Lambda_D) \)
\[
\left\| \left( \nabla \Lambda_D^{-1} f \cdot \nabla \right) g \right\|_{\dot{B}^s_{\infty,1}(\Lambda_D)} \leq C^{s+1} \left\{ \begin{array}{ll}
\| f \|_{\dot{B}^s_{\infty,1}(\Lambda_D)} \| g \|_{\dot{B}^1_{\infty,1}(\Lambda_D)} & \text{if } s = 0, \\
\| f \|_{\dot{B}^s_{\infty,1}(\Lambda_D)} \| g \|_{\dot{B}^1_{\infty,1}(\Lambda_D)} + \| f \|_{\dot{B}^s_{\infty,1}(\Lambda_D)} \| g \|_{\dot{B}^{s+1}_{\infty,1}(\Lambda_D)} & \text{if } s > 0.
\end{array} \right. \tag{3.5}
\]

Remark. In the whole space case, the inequality (3.5) is proved based by Bony paraproduct formula [4],
\[
\left( \nabla \Lambda_D^{-1} \right)^{-1/2} f \cdot \nabla \right) g = \left( \sum_{k \geq l+3} + \sum_{l \geq k+3} + \sum_{|k-l| \leq 2} \right) \left( \nabla \Lambda_D^{-1} \right)^{-1/2} f_k \cdot \nabla \right) g_l,
\]
where \( f_k = \mathcal{F}^{-1}[\phi_j(|\xi|)] * f \) and \( g_l = \mathcal{F}^{-1}[\phi_j(|\xi|)] * g \). In fact, the first two terms are estimated by uniformity of the spectral multiplier. The third term needs an additional argument by the divergence free property from \( \nabla \) and
\[
\left( \nabla \Lambda_D^{-1} \right)^{-1/2} f_k \cdot \nabla \right) g_l = \nabla \cdot \left( \nabla \Lambda_D^{-1} \right)^{-1/2} f_k \right) g_l.
\]
One can prove the dependence for the constant \( C^{s+1} \) in (3.5) with respect to the regularity number \( s \geq 0 \) by estimating the paraproduct formula carefully, and we left it to the reader. We admit the estimate in the whole space case in the proof of Proposition 3.2 below.

**Proof.** We may assume that \( f, g \in \dot{B}^s_{\infty,1}(\Lambda_D) \) for all \( s \in \mathbb{R} \), since the intersection of \( \dot{B}^s_{\infty,1}(\Lambda_D) \) with all \( s \) is dense in \( \dot{B}^s_{\infty,1}(\Lambda_D) \) for each \( s \in \mathbb{R} \). We write
\[
\left( \nabla \Lambda_D^{-1} f \cdot \nabla \right) g = -\partial_{x_1} \left( (\partial_{x_2} \Lambda_D^{-1} f) g \right) + \partial_{x_2} \left( (\partial_{x_1} \Lambda_D^{-1} f) g \right), \tag{3.6}
\]
and handle the two terms separately, with Lemma 2.3 (ii). We need the odd extension of the product above to take the norm of \( \dot{B}^s_{\infty,1}(\Lambda_D) \). For the first term, we write by
Lemma 2.2 (i) and \( f \in \dot{B}^s_{\infty,1}(\Lambda_D) \) for all \( s \in \mathbb{R} \)

\[
\Lambda_D^{-1} f = \sum_{j \in \mathbb{Z}} \mathcal{F}^{-1} \left[ |\xi|^{-1} \phi_j(1) \right] \hat{f}_{\text{odd}} \bigg|_{\mathbb{R}^2_+} = (-\Delta_{\mathbb{R}^2})^{-1/2} \hat{f}_{\text{odd}} \bigg|_{\mathbb{R}^2_+} \quad \text{in } \mathcal{A}',
\]

and

\[
\partial_{x_2} \Lambda_D^{-1} f = \partial_{x_2} \left( (-\Delta_{\mathbb{R}^2})^{-1/2} \hat{f}_{\text{odd}} \right) \bigg|_{\mathbb{R}^2_+},
\]

and notice that \( \partial_{x_2} \left( (-\Delta_{\mathbb{R}^2})^{-1/2} \hat{f}_{\text{odd}} \right) \) is even for \( x_2 \). We then write the odd extension of the above product

\[
\left( \partial_{x_1} \left( \left( \partial_{x_2} \Lambda_D^{-1} f \right) g \right) \right)_{\text{odd}} = \partial_{x_1} \left( \left( \partial_{x_2} \left( (-\Delta_{\mathbb{R}^2})^{-1/2} \hat{f}_{\text{odd}} \right) \right) g \right)_{\text{odd}}
\]

Similarly, we write the second term

\[
\left( \partial_{x_2} \left( \left( \partial_{x_1} \Lambda_D^{-1} f \right) g \right) \right)_{\text{odd}} = \partial_{x_2} \left( \left( \partial_{x_1} \left( (-\Delta_{\mathbb{R}^2})^{-1/2} \hat{f}_{\text{odd}} \right) \right) g \right)_{\text{odd}}
\]

It follows from the above two equalities and the bilinear estimates in Besov spaces on the whole space \( \mathbb{R}^2 \) that

\[
\| \left( \nabla \Lambda_D^{-1} f \cdot \nabla \right) g \|_{\dot{B}^0_{\infty,1}(\Lambda_D)} = \sum_{j \in \mathbb{Z}} \left\| \mathcal{F}^{-1} \left[ |\phi_j(1) \right] \right\| \left( \left( \nabla \left( (-\Delta_{\mathbb{R}^2})^{-1/2} \hat{f}_{\text{odd}} \right) \right) g \right)_{\text{odd}} \bigg|_{L^\infty(\mathbb{R}^2)} \leq C \| \hat{f}_{\text{odd}} \|_{\dot{B}^0_{\infty,1}(\mathbb{R}^2)} \| g \|_{\dot{B}^1_{\infty,1}(\mathbb{R}^2)}
\]

If \( s > 0 \), it holds that

\[
\| \left( \nabla \Lambda_D^{-1} f \cdot \nabla \right) g \|_{\dot{B}^0_{\infty,1}(\Lambda_D)} \leq C C_0^s \left( \| f \|_{\dot{B}^0_{\infty,1}(\Lambda_D)} \| g \|_{\dot{B}^1_{\infty,1}(\Lambda_D)} + \| f \|_{\dot{B}^0_{\infty,1}(\Lambda_D)} \| g \|_{\dot{B}^{1+s}_{\infty,1}(\Lambda_D)} \right).
\]

Proof of Theorem 1.1 For the sake of the simplicity, we write

\[
L^\infty \dot{B}^0_{\infty,1} := L^\infty(0, \infty; \dot{B}^0_{\infty,1}(\Lambda_D)), \quad L^1 \dot{B}^1_{\infty,1} := L^1(0, \infty; \dot{B}^1_{\infty,1}(\Lambda_D)).
\]

Let \( \Psi \) be the right hand side of the integral equation,

\[
\Psi(\theta) := e^{-t \Lambda_D} \theta_0 - \int_0^t e^{-(t-\tau) \Lambda_D} (u \cdot \nabla) \theta \ d\tau, \quad u := \nabla \Lambda_D^{-1} \theta.
\]

Step 1. (Existence and space analyticity) Let a complete metric space \( X_\infty \) be defined by

\[
X_\infty := \{ \theta \in C([0, \infty), \dot{B}^0_{\infty,1}(\Lambda_D)) \mid \| \theta \|_X \leq \| u_0 \|_{\dot{B}^0_{\infty,1}(\Lambda_D)} \},
\]

where

\[
\| \theta \|_X := \sup_{\beta=0,1,2,\ldots} \frac{\| t^\beta \Lambda_D^\beta \theta \|_{L^\infty \dot{B}^0_{\infty,1} \cap L^1 \dot{B}^1_{\infty,1}}}{C_0^{2\beta+1} \beta!}.
\]
for some large constant $C_0$, with the metric

$$d(θ, \tilde{θ}) := \|θ - \tilde{θ}\|_{L^∞(0,∞; \dot{B}^{0}_{∞,1}(Λ_D)) ∩ L^1(0,∞; \dot{B}^{1}_{∞,1}(Λ_D))}.$$  

The main point is to prove that

$$\|Ψ(θ)\|_X \leq C\|θ_0\|_{\dot{B}^{0}_{∞,1}(Λ_D)} + C\|θ\|^2_X, \text{ for } θ ∈ X_∞$$  

(3.7)

which implies $Ψ(θ) ∈ X_∞$ provided that $u_0$ is small in $\dot{B}^{0}_{∞,1}(Λ_D)$. Hereafter, we estimate, supposing the smallness.

By (3.4), the linear part is estimated by

$$\|e^{-tΛ_D}θ_0\|_X \leq C \left( \sup_{β=0,1,2,\ldots} \frac{C^{β} β!}{C^{2} β!} \right) \|θ_0\|_{\dot{B}^{0}_{∞,1}(Λ_D)},$$

and we have the finiteness of the supremum in the right hand side for large $C_0$. We turn to estimate the nonlinear part, dividing the interval in half.

For the first-half time integral, it follows that by the smoothing effect (3.1)

$$\left\| t^{β_2} Λ_D^{β_2} \int_0^{t/2} e^{-(t-τ)Λ_D} \left( (u \cdot \nabla)θ \right) dτ \right\|_{\dot{B}^{0}_{∞,1}(Λ_D)} \leq C^{β_2} β_2! \int_0^{t/2} (t-τ)^{-β_1} \| (u \cdot \nabla)θ \|_{\dot{B}^{0}_{∞,1}} dτ \leq C^{β_2} β_2! \int_0^{t/2} \| (u \cdot \nabla)θ \|_{\dot{B}^{0}_{∞,1}} dτ,$$

and by the smoothing effect (3.1) and maximal regularity (3.3)

$$\left\| t^{β_2} Λ_D^{β_2} \int_0^{t/2} e^{-(t-τ)Λ_D} \left( (u \cdot \nabla)θ \right) dτ \right\|_{L^1(0,∞; \dot{B}^{1}_{∞,1}(Λ_D))} \leq C^{β_2} β_2! \left\| t^{β_2} \int_0^{t/2} \left( \frac{t-τ}{2} \right)^{-β_1} e^{-\frac{t-τ}{2} Λ_D} (u \cdot \nabla)θ \right\|_{L^1(0,∞)} \leq C^{β_2} β_2! \int_0^{∞} \| (u \cdot \nabla)θ \|_{\dot{B}^{0}_{∞,1}} dτ.$$

These inequalities above and the bilinear estimate (3.5) imply that

$$\left\| t^{β_2} Λ_D^{β_2} \int_0^{t/2} e^{-(t-τ)Λ_D} \left( (u \cdot \nabla)θ \right) dτ \right\|_{L^∞(0,∞; \dot{B}^{0}_{∞,1}(Λ_D)) ∩ L^1(0,∞; \dot{B}^{1}_{∞,1}(Λ_D))} \leq C^{β_2} β_2! \left\| θ \right\|_{\dot{B}^{0}_{∞,1}(Λ_D)} \left\| θ \right\|_{\dot{B}^{1}_{∞,1}(Λ_D)} dτ \leq C^{β_2} β_2! \left\| θ \right\|^2_X.$$
As for the second-half time integral with \( \beta_1 = 0 \), by maximal regularity (3.3) and the bilinear estimate (3.5) give that
\[
\left\| \beta^2 \Lambda_D^2 \int_{t/2}^t e^{-(t-\tau)\Lambda_D} (u \cdot \nabla) \theta \right\|_{L^\infty \dot{B}^{0}_{\infty,1} \cap L^1 \dot{B}^{1}_{\infty,1}} \\
\leq C \int_0^{\infty} (2s)^{\beta_2} \left\| \Lambda_D^2 (u \cdot \nabla) \theta \right\|_{\dot{B}^{0}_{\infty,1}(\Lambda_D)} d\tau \\
\leq C^{\beta_2} \int_0^{\infty} s^{\beta_2} \left( \| \theta \|_{\dot{B}^{2\beta_2}_{\infty,1}(\Lambda_D)} \| \theta \|_{\dot{B}^{3\beta_2}_{\infty,1}(\Lambda_D)} + \| \theta \|_{\dot{B}^{0}_{\infty,1}(\Lambda_D)} \| \theta \|_{\dot{B}^{2\beta_2+1}_{\infty,1}(\Lambda_D)} \right) d\tau \\
\leq C^{\beta_2} \beta_2! \cdot \| \theta \|^2_X
\]
Therefore, we obtain (3.7) for large \( C_0 \), and we can also prove that for \( \theta, \tilde{\theta} \in X_\infty \)
\[
d(\theta, \tilde{\theta}) \leq C(\| \theta \|_X + \| \tilde{\theta} \|_X) d(\theta, \tilde{\theta}) \leq \frac{1}{2} d(\theta, \tilde{\theta}),
\]
provided that \( \| u_0 \|_{\dot{B}^{\beta}_{\infty,1}(\Lambda_D)} \) is sufficiently small. The fixed point argument yields the existence of the solution, and the analyticity for \( x \) holds by the estimate of derivatives by factorials.

Step 2. (With time analyticity) By the result of Step 1, we have \( \Lambda^\beta \theta(t) \in \dot{B}^{0}_{\infty,1}(\Lambda_D) \) for all \( \beta = 1, 2, \cdots \) if \( t > 0 \). Then we know the solution becomes smooth, and let us assume the initial data \( \theta_0 \in \dot{B}^{s}_{\infty,1}(\Lambda_D) \) for all \( s \geq 0 \). Our argument is to construct a solution for smooth data.

Let a complete metric space \( Y_\infty \) be defined by
\[
Y_\infty := \{ \theta \in C([0, \infty), \dot{B}^{0}_{\infty,1}(\Lambda_D)) \mid \| \theta \|_Y \leq \| u_0 \|_{\dot{B}^{\beta}_{\infty,1}(\Lambda_D)} \},
\]
where
\[
\| \theta \|_Y := \sup_{\beta_1, \beta_2 = 0, 1, 2, \cdots} \frac{(1 + \beta_1 + \beta_2)^4 \| t^{\beta_1 + \beta_2} \partial_t^{\beta_1} \Lambda_D^{\beta_2} \theta \|_{L^\infty \dot{B}^{0}_{\infty,1} \cap L^1 \dot{B}^{1}_{\infty,1}}}{C_0^{3\beta_1 + 2\beta_2 - 1} \beta_1! \beta_2!},
\]
for some large constant \( C_0 \), with the metric
\[
d(\theta, \tilde{\theta}) := \| \theta - \tilde{\theta} \|_{L^\infty(0, \infty; \dot{B}^{0}_{\infty,1}(\Lambda_D)) \cap L^1(0, \infty; \dot{B}^{1}_{\infty,1}(\Lambda_D))}.
\]
Let us focus on the following estimate.
\[
\| \Psi(\theta) \|_Y \leq C \| \theta \|_{\dot{B}^{\beta}_{\infty,1}(\Lambda_D)} + C \| \theta \|_Y, \text{ for } \theta \in Y_\infty
\]
To this end, we will apply an induction argument for time derivatives. For \( \beta = 0, 1, 2, \cdots \) we write
\[
\| \theta \|_{Y_{\leq \beta}} := \sup_{0 \leq \beta_1, \beta_2 = 0, 1, 2, \cdots} \frac{(1 + \beta_1 + \beta_2)^4 \| t^{\beta_1 + \beta_2} \partial_t^{\beta_1} \Lambda_D^{\beta_2} \theta \|_{L^\infty \dot{B}^{0}_{\infty,1} \cap L^1 \dot{B}^{1}_{\infty,1}}}{C_0^{3\beta_1 + 2\beta_2 + 1} \beta_1! \beta_2!}.
\]
When \( \beta = 0 \), the estimate is essentially proved in Step 1, modifying \( C_0 \) larger to lead to a boundedness with an additional factor \( (1 + \beta_1 + \beta_2)^4 \).

Let \( \beta \geq 1 \) and assume that
\[
\| \Phi(\theta) \|_{Y_{\leq \beta - 1}} \leq \| u_0 \|_{\dot{B}^{\beta}_{\infty,1}}.
\]
Therefore we obtain that
\[ \partial_t (t^{\beta} \partial_t^\beta \Psi(t)) + \Lambda_D (t^{\beta} \partial_t^\beta \Psi(t)) - \beta t^{\beta-1} \partial_t^\beta \Psi(t) + t^{\beta} \partial_t^\beta (u \cdot \nabla) \theta = 0, \]
the initial data of \( t^{\beta} \partial_t^\beta \Psi(t) \) is zero because of our smooth \( \theta_0 \), and we have the integral equation
\[ t^{\beta} \partial_t^\beta \Psi(t)(t) = \int_0^t e^{-(t-\tau)\Lambda_D} \left( \beta \tau^{\beta-1} \partial_t^\beta \Phi(t)(\tau) - \tau^{\beta} \partial_t^\beta ((u \cdot \nabla) \theta) \right) d\tau. \]

Maximum regularity (3.3) implies that
\[
\| t^{\beta} \partial_t^\beta \Psi(t)(t) \|_{L^\infty \mathcal{B}^0_{\infty,1} \cap L^1 \mathcal{B}^1_{\infty,1}} \leq C \int_0^\infty \left( \beta \tau^{\beta-1} \| \partial_t^\beta \Phi(t) \|_{\mathcal{B}^0_{\infty,1}} + \tau^{\beta} \| \partial_t^\beta (u \cdot \nabla) \theta \|_{\mathcal{B}^0_{\infty,1}} \right) d\tau. \tag{3.9}
\]
For the first integrand, we write the equation \( \partial_t \Phi(t) = -\Lambda_D \Phi(t) - (u \cdot \nabla) \theta \) and apply the assumption of the induction to the first term in the right hand side, and the Leibniz rule, the bilinear estimate (3.5) for the second term. We then have that
\[
C \int_0^\infty \beta \tau^{\beta-1} \| \partial_t^\beta \Phi(t) \|_{\mathcal{B}^0_{\infty,1}} d\tau \leq C \beta \int_0^t \tau^{\beta-1} \| \partial_t^\beta \Psi(t) \|_{\mathcal{B}^1_{\infty,1}} d\tau + C \beta \int_0^\infty \tau^{\beta-1} \sum_{\gamma=0}^{\beta-1} \frac{(\beta-1)!}{(\beta-1-\gamma)!\gamma!} \| \partial_t^{\beta-1-\gamma} \theta \|_{\mathcal{B}^0_{\infty,1}} \| \partial_t^\gamma \theta \|_{\mathcal{B}^0_{\infty,1}} d\tau
\]
\[
\leq C \beta \cdot \frac{C_0^{3(\beta-1)+2+1}(\beta_1 - 1)!}{(1+\beta)^4} \| u_0 \|_{\mathcal{B}^0_{\infty,1}} + C \sum_{\gamma=0}^{\beta} \frac{C_0^{3(\beta-1)+2+1}}{(1+\beta-1-\gamma)^4(1+\gamma)^4} \| u_0 \|^2_{\mathcal{B}^0_{\infty,1}}
\]
\[
\leq C \left( \frac{1}{C_0} + \| u_0 \|_{\mathcal{B}^0_{\infty,1}} \right) \frac{C_0^{3+1+\beta!}}{(1+\beta)^4} \| u_0 \|_{\mathcal{B}^0_{\infty,1}}.
\]
As for the second term of the right hand side of (3.9), a similar argument to the second estimate above implies that
\[ C \int_0^\infty \tau^{\beta} \| \partial_t^\beta (u \cdot \nabla) \theta \|_{\mathcal{B}^0_{\infty,1}} d\tau \leq C \frac{C_0^{3+1+\beta!}}{(1+\beta)^4} \cdot C_0 \| u_0 \|^2_{\mathcal{B}^0_{\infty,1}}. \]

Therefore we obtain that
\[ \| \Phi(t) \|_{Y_{\infty,\beta}} \leq \| u_0 \|_{\mathcal{B}^0_{\infty,1}}. \]
This together with (3.8) allows us to apply the fixed point argument and we have the solution analytic in space and time.

We can also prove the uniqueness analogously to the paper [16] by introducing odd extension with respect to \( x_2 \), where the uniqueness in \( C([0, \infty), \mathcal{B}^{0}_{\infty,1}(\Lambda_D)) \cap L^1(0, \infty; \mathcal{B}^1_{\infty,1}(\Lambda_D)) \) is proved without smallness. It follows from the uniqueness that the solutions in Step 1 and Step 2 coincide. Overall, we complete the proof of Theorem 1.1.

\[ \square \]

4. Proof outline of Theorem 1.2

In this section, we explain proof outline, since the argument is similar to the whole space case, once we introduce odd extension of the equations. The essence of how to
introduce the odd extension is found in section 2. We write the odd extension of the equations.
\[
\partial_t \theta_{\text{odd}} + (-\Delta_D)^{1/2} \theta_{\text{odd}} + (u \cdot \nabla)\theta_{\text{odd}} = 0.
\]
The odd extension of the nonlinear term becomes
\[
\left( (u \cdot \nabla)\theta \right)_{\text{odd}} = (\nabla^\perp (-\Delta_{\mathbb{R}^2})^{-1/2} \theta_{\text{odd}} \cdot \nabla) \theta_{\text{odd}}.
\]
Therefore, we have the equation in the whole space, and it is possible to apply the argument in the paper [19, 30] with a certain modification for the low spectral component due to the second condition of (1.3) for the boundedness of the Riesz transform. We then obtain a local-in-time unique solution in the whole space, analytic in space time for some \(T_0 > 0\). The restriction to the half space gives the solution such that
\[
\|\theta\|_{L^1(0, T_0; \dot{B}^0_{\infty, \infty}(\Lambda_D))} < 1,
\]
and that we can extend the existence time as far as such kind of smallness holds for the linear solution. We fix a time \(t_0 \in (0, T_0]\) and consider the data \(\theta(t_0)\). We here utilize the uniform bounds in H"older spaces (see Theorem 3.1 in [3]), and this implies that if \(0 < a < 1, T_0 > 0\) for the local existence is taken such that
\[
\|\theta\|_{L^1(0, T_0; \dot{B}^0_{\infty, 1}(\Lambda_D))} \le M_{t_0}, \quad \text{for all } t \ge t_0, \quad (4.1)
\]
as far as the solution exists and is smooth. If we consider \(\theta(t_0)\) as a data, we have on a short time interval \([0, \delta]\) that
\[
\|e^{-t\Lambda_D} \theta(t)\|_{L^1(0, \delta; \dot{B}^1_{\infty, \infty}(\Lambda_D))} \le \|e^{-t\Lambda_D} \theta(t)\|_{L^1(0, \delta; \dot{B}^1_{\infty, 1}(\Lambda_D))} \le C\delta^a \|\theta(t)\|_{\dot{B}^0_{\infty, \infty}(\Lambda_D)} \le C\delta^a M_{t_0}.
\]
Let us take \(\delta\) sufficiently small, and we see that it is possible to extend the existence time of the solution longer than \([0, t_0]\), which is \([0, t_0 + \delta]\). Since the bound (4.1) is independent of \(t\), it is possible to repeat this procedure and we have the existence time \([0, t_0 + n\delta]\) for all \(n = 1, 2, \cdots\), which proves the global-in-time regularity of the solution. \(\square\)

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**Conflict of Interest.** The author declares that he has no conflict of interest.
References

[1] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. (4) 14 (1981), no. 2, 209–246 (French).

[2] L. A. Caffarelli and A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Ann. of Math. (2) 171 (2010), no. 3, 1903–1930.

[3] P. Constantin, *Energy Spectrum of Quasi-geostrophic Turbulence*, Phys. Rev. Lett. 89 (2002), 184501.

[4] ———, *Nonlocal nonlinear advection-diffusion equations*, Chin. Ann. Math. Ser. B 38 (2017), no. 1, 281–292.

[5] P. Constantin, D. Córdoba, and J. Wu, *On the critical dissipative quasi-geostrophic equation*, Indiana Univ. Math. J. 50 (2001), no. Special Issue, 97–107.

[6] P. Constantin and M. Ignatova, *Critical SQG in bounded domains*, Ann. PDE 2 (2016), no. 2, Art. 8, 42.

[7] ———, *Remarks on the fractional Laplacian with Dirichlet boundary conditions and applications*, Int. Math. Res. Not. IMRN 6 (2017), 1653–1673.

[8] ———, *Estimates near the boundary for critical SQG*, Ann. PDE 6 (2020), no. 1, Paper No. 3, 30.

[9] P. Constantin and H. Q. Nguyen, *Global weak solutions for SQG in bounded domains*, Comm. Pure Appl. Math. 71 (2018), no. 11, 2323–2333.

[10] ———, *Local and global strong solutions for SQG in bounded domains*, Phys. D 376/377 (2018), 195–203.

[11] P. Constantin and V. Vicol, *Nonlinear maximum principles for dissipative linear nonlocal operators and applications*, Geom. Funct. Anal. 22 (2012), no. 5, 1289–1321.

[12] P. Constantin and J. Wu, *Behavior of solutions of 2D quasi-geostrophic equations*, SIAM J. Math. Anal. 30 (1999), no. 5, 937–948.

[13] A. Córdoba and D. Córdoba, *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys. 249 (2004), no. 3, 511–528.

[14] M. Coti Zelati and V. Vicol, *On the global regularity for the supercritical SQG equation*, Indiana Univ. Math. J. 65 (2016), no. 2, 535–552.

[15] M. Ignatova, *Construction of solutions of the critical SQG equation in bounded domains*, Adv. Math. 351 (2019), 1000–1023.

[16] T. Iwabuchi, *Global solutions for the critical Burgers equation in the Besov spaces and the large time behavior*, Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015), no. 3, 687–713.

[17] ———, *The semigroup generated by the Dirichlet Laplacian of fractional order*, Anal. PDE 11 (2018), no. 3, 683–703.

[18] ———, *Derivatives on function spaces generated by the Dirichlet Laplacian and the Neumann Laplacian in one dimension*, Commun. Math. Anal. 21 (2018), no. 1, 1–8.

[19] ———, *Analyticity and large time behavior for the Burgers equation and the quasi-geostrophic equation, the both with the critical dissipation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 37 (2020), no. 4, 855–876.

[20] ———, *The Leibniz rule for the Dirichlet and the Neumann Laplacian*, preprint, [arXiv:1905.02854v2](https://arxiv.org/abs/1905.02854).

[21] T. Iwabuchi, T. Matsuyama, and K. Taniguchi, *Boundedness of spectral multipliers for Schrödinger operators on open sets*, Rev. Mat. Iberoam. 34 (2018), no. 3, 1277–1322.

[22] ———, *Besov spaces on open sets*, Bull. Sci. Math. 152 (2019), 93–149.

[23] A. Kiselev, F. Nazarov, and A. Volberg, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation*, Invent. Math. 167 (2007), no. 3, 445–453.

[24] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*, Translated from the Russian by J. B. Sykes and W. H. Reid. Course of Theoretical Physics, Vol. 6, Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959.

[25] E. M. Ouhabaz, *Analysis of heat equations on domains*, London Mathematical Society Monographs Series, vol. 31, Princeton University Press, Princeton, NJ, 2005.

[26] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag New York, 1979.

[27] L. F. Stokols and A. F. Vasseur, *Hölder regularity up to the boundary for critical SQG on bounded domains*, Arch. Ration. Mech. Anal. 236 (2020), no. 3, 1543–1591.

[28] X. Thinh Duong, E. M. Ouhabaz, and A. Sikora, *Plancherel-type estimates and sharp spectral multipliers*, J. Funct. Anal. 196 (2002), no. 2, 443–485.
[29] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.

[30] H. Wang and Z. Zhang, *A frequency localized maximum principle applied to the 2D quasi-geostrophic equation*, Comm. Math. Phys. **301** (2011), no. 1, 105–129.