Leaderless and Multi-Leader Computation in Disconnected Anonymous Dynamic Networks

Giuseppe A. Di Luna∗ Giovanni Viglietta†

Abstract

We give a simple and complete characterization of which functions can be deterministically computed by anonymous processes in disconnected dynamic networks, depending on the number of leaders in the network. In addition, we provide efficient distributed algorithms for computing all such functions assuming minimal or no knowledge about the network. Each of our algorithms comes in two versions: one that terminates with the correct output and a faster one that stabilizes on the correct output without explicit termination. Notably, all of our algorithms have running times that scale linearly both with the number of processes and with a parameter of the network which we call dynamic disconnectivity. We also provide matching lower bounds, showing that all our algorithms are asymptotically optimal for any fixed number of leaders.

While most of the existing literature on anonymous dynamic networks relies on classical mass-distribution techniques, our work makes use of a recently introduced combinatorial structure called history tree. Among other contributions, our results establish a new state of the art on two popular fundamental problems for anonymous dynamic networks: leaderless Average Consensus (i.e., computing the mean value of input numbers distributed among the processes) and multi-leader Counting (i.e., determining the exact number of processes in the network).

∗DIAG, Sapienza University of Rome, g.a.diluna@gmail.com
†Japan Advanced Institute of Science and Technology (JAIST), viglietta@gmail.com
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1 Introduction

Dynamic networks. An increasingly prominent area of distributed computing focuses on algorithmic aspects of dynamic networks, motivated by novel technologies such as wireless sensors networks, software-defined networks, networks of smart devices, and other networks with a continuously changing topology [10, 35, 38]. Typically, a network is modeled by a set of \( n \) processes that communicate in synchronous rounds; at each round, the network’s topology changes unpredictably.

Disconnected networks. In the dynamic setting, a common assumption is that the network is 1-interval-connected, i.e., connected at all rounds [33, 41]. However, this is not a suitable model for many real systems, due to the very nature of dynamic entities (think of p2p networks of smart devices moving unpredictably) or due to transient communication failures, which may compromise the network’s connectivity. A weaker assumption is that the union of all the network’s links across any \( T \) consecutive rounds induces a connected graph on the processes [31, 43]. We say that such a network is \( T \)-interval-disconnected, and we call \( T \geq 1 \) its dynamic disconnectivity.

Anonymous processes. Several works have focused on processes with unique IDs, which allow for efficient algorithms for many different tasks [9, 32, 33, 34, 38, 41]. However, unique IDs may not be available due to operational limitations [41] or to protect user privacy: a famous example are COVID-19 tracking apps, where assigning temporary random IDs to users was not enough to eliminate privacy concerns [47]. Systems where processes are indistinguishable are called anonymous. The study of static anonymous networks has a long history, as well [7, 8, 12, 13, 14, 25, 46, 49].

Networks with and without leaders. It is known that several fundamental problems for anonymous networks (a notable example being the Counting problem, i.e., determining the total number of processes \( n \)) cannot be solved without additional “symmetry-breaking” assumptions. The most typical choice is the presence of a single distinguished process, called leader [2, 3, 4, 5, 22, 25, 26, 28, 37, 45, 50], or of a set of multiple leaders (and knowledge of their number) [27, 29, 30, 31].

However, this is often an unrealistic assumption: for example, in a mobile sensor network deployed by an aircraft, the leaders may be destroyed as a result of a bad landing; also, the leaders may malfunction during the system’s lifetime. This justifies the extensive existing literature on networks with no leaders [16, 17, 39, 40, 12, 48, 51]. In fact, a large portion of works on leaderless networks have focused on the Average Consensus problem, where the goal is to compute the mean of a list of numbers distributed among the processes [6, 15, 16, 24, 43, 44].

| Problem       | Reference | Leaders | Disconn. | Term. | Notes | Running time |
|---------------|-----------|---------|----------|-------|-------|--------------|
| Average Consensus | [43]     | \( \ell = 0 \) | ✓        | c-convergence, \( T \) unknown, upper bound on processes’ degrees known | \( O(Tn^3 \log(1/\epsilon)) \) |
|               | [16]     | \( \ell = 0 \) |          | c-convergence | \( O(n^4 \log(n/\epsilon)) \) |
|               | [15]     | \( \ell = 0 \) |          | randomized Monte Carlo | \( O(n) \) |
|               | [29]     | \( \ell \geq 1 \) | ✓        | \( \ell \) known | \( O(n^3 \log^3(n)/\ell) \) |
|               | this work | \( \ell = 0 \) | ✓        | \( T \) unknown | \( 2\ell n \) |
|               |          | \( \ell = 0 \) | ✓        | ✓ \( T \) and \( N \geq n \) known | \( T(n + N) \) |
| (Generalized) Counting | [27]     | \( \ell = 1 \) | ✓        |           | 2\( n - 2 \) |
|               |          | \( \ell = 1 \) | ✓        |           | 3\( n - 2 \) |
|               | [31]     | \( \ell \geq 1 \) | ✓        | \( \ell \) known | \( O(n^3 \log^3(n)/\ell) \) |
|               | [30]     | \( \ell \geq 1 \) | ✓        | ✓ \( O(\log n) \)-size messages, \( \ell \) and \( T \) known | \( O(n^{2\ell + 3}/\ell) \) |
|               | this work | \( \ell \geq 1 \) | ✓        | ✓ \( \ell \) known, \( T \) unknown | \( 2\ell n \) |
|               |          | \( \ell \geq 1 \) | ✓        | ✓ \( \ell \) and \( T \) known | \( (\ell^2 + \ell + 1)Tn \) |

Table 1: Comparing results for Average Consensus and Counting in anonymous dynamic networks. For algorithms that support disconnected networks, \( T \) indicates the dynamic disconnectivity.
1.1 Our Contributions

Summary. Focusing on anonymous dynamic networks, in this paper we completely elucidate the relationship between leaderless networks and networks with (multiple) leaders, as well as the impact of the dynamic disconnectivity $T$ on the efficiency of distributed algorithms. In fact, this is one of the few works that considers networks that are not necessarily connected at all times.

Computability. We give an exact characterization of which functions can be computed in anonymous dynamic networks with and without leaders, respectively. Namely, with at least one leader, all the so-called multi-aggregate functions are computable; with no leaders, only the scale-invariant multi-aggregate functions are computable (see Section 2 for definitions). Interestingly, computability is independent of the dynamic disconnectivity $T$. Our contribution considerably generalizes a recent result on the functions computable with exactly one leader and with $T = 1$ [23].

Complete problems. While computing the so-called Generalized Counting function $F_{GC}$ was already known to be a complete problem for the class of multi-aggregate functions [23], in this work we expand the picture by identifying a complete problem for the class of scale-invariant multi-aggregate functions, as well: the Concentration function $F_R$ (both $F_{GC}$ and $F_R$ are defined in Section 2). By “complete problem” we mean that computing such a function allows the immediate computation of any other function in the class with no overhead in terms of execution rounds.

Algorithms. We give efficient distributed algorithms for computing the Concentration function (Section 3) and the Generalized Counting function (Section 4). Since the two problems are complete, we automatically obtain efficient algorithms for computing all functions in the respective classes.

For each problem, we give two algorithms: a terminating version, where each process is required to commit on its output and never change it, and a stabilizing version, where processes are allowed to modify their outputs, provided that they eventually stabilize on the correct output.

Both our stabilizing algorithms run in $2Tn$ rounds, and do not require any knowledge of the dynamic disconnectivity $T$ or the number of processes $n$. Our terminating algorithm for leaderless networks runs in $T(n + N)$ rounds with knowledge of $T$ and an upper bound $N \geq n$; the terminating algorithm for $\ell \geq 1$ leaders runs in $(\ell^2 + \ell + 1)Tn$ rounds, with no knowledge of $n$. The latter running time is reasonable in most applications, as $\ell$ is typically very small compared to $n$.

Impact on fundamental problems. Since the mean is a scale-invariant aggregate function, our results also apply to the popular Average Consensus problem. Table 1 shows that our algorithms greatly improve upon the state of the art on this problem, as well as the (Generalized) Counting problem, in terms of (i) running time, (ii) assumptions on the network and the processes’ knowledge, and (iii) quality of the solution. For instance, previous leaderless algorithms only converge to an $\epsilon$-approximation of the mean or have a small chance of giving an incorrect output. A comprehensive discussion of related literature is found in Appendix D.

Negative results. Some of our algorithms assume processes to have a-priori knowledge of some parameters of the network; in Section 5 we show that all of these assumptions are necessary. We also provide lower bounds that asymptotically match our algorithms’ running times, assuming that the number of leaders $\ell$ is constant (which is a realistic assumption in most applications).

Multi-graphs. All of our results hold more generally if networks are modeled as multi-graphs, as opposed to the simple graphs traditionally encountered in nearly all of the literature. This is relevant in many applications: in radio communication, for instance, multiple links between processes naturally appear due to the multi-path propagation of radio waves.

Technique. Our approach departs radically from the mass-distribution techniques traditionally adopted by most previous works on anonymous dynamic networks; instead, we build upon history trees, a combinatorial structure recently introduced in [23] (see Section 2 for an introduction).
2 Definitions and Preliminaries

We will briefly give some preliminary definitions and results, and recall some properties of history trees from [23]. The interested reader may find a more rigorous discussion in Appendices A and B

Processes and networks. A dynamic network is modeled by an infinite sequence \( \mathcal{G} = (G_t)_{t \geq 1} \), where \( G_t = (V, E_t) \) is an undirected multigraph whose vertex set \( V = \{p_1, p_2, \ldots, p_n\} \) is a system of \( n \) anonymous processes and \( E_t \) is a multiset of edges representing links between processes.

Each process \( p_i \) starts with an input \( \lambda(p_i) \), which is assigned to it at round 0. It also has an internal state, which is initially determined by \( \lambda(p_i) \). At each round \( t \geq 1 \), every process composes a message (depending on its internal state) and broadcasts it to its neighbors in \( G_t \) through all its incident links. By the end of round \( t \), each process reads all messages coming from its neighbors and updates its internal state according to a local algorithm \( A \). Note that \( A \) is deterministic and is the same for all processes. The input of each process also includes a leader flag. The processes whose leader flag is set are called leaders (or supervisors). We will denote the number of leaders as \( \ell \).

Each process also returns an output at the end of each round, which is determined by its current internal state. A system is said to stabilize if the outputs of all its processes remain constant from a certain round onward; note that a process’ internal state may still change even when its output is constant. A process may also decide to explicitly terminate and no longer update its internal state. When all processes have terminated, the system is said to terminate, as well.

We say that \( A \) computes a function \( F \) if, whenever the processes are assigned inputs \( \lambda(p_1), \lambda(p_2), \ldots, \lambda(p_n) \) and all processes execute the local algorithm \( A \) at every round, the system eventually stabilizes with each process \( p_i \) giving the desired output \( F(p_i, \lambda) \). A stronger notion of computation requires the system to not only stabilize but also to explicitly terminate with the correct output.

Classes of functions. Let \( \mu = \{(z_1, m_1), (z_2, m_2), \ldots, (z_k, m_k)\} \) be the multiset of all processes’ inputs. That is, for all \( 1 \leq i \leq k \), there are exactly \( m_i \) processes \( p_{j_1}, p_{j_2}, \ldots, p_{j_{m_i}} \) whose input is \( z_i = \lambda(p_{j_1}) = \lambda(p_{j_2}) = \cdots = \lambda(p_{j_{m_i}}) \); note that \( n = \sum_{i=1}^{k} m_i \). A multi-aggregate function is defined as a function \( F \) of the form \( F(p_i, \lambda) = \psi(\lambda(p_i), \mu) \), i.e., such that the output of each process depends only on its own input and the multiset of all processes’ inputs.

The special multi-aggregate functions \( F_C(p_i, \lambda) = \mu \) and \( F_{GC}(p_i, \lambda) = \mu \) are called the Counting function and the Generalized Counting function, respectively. It is known that, if a system can compute the Generalized Counting function \( F_{GC} \), then it can compute any multi-aggregate function in the same number of rounds: thus, \( F_{GC} \) is complete for the class of multi-aggregate functions [23].

For any \( \alpha \in \mathbb{R}^+ \), we define \( \alpha \cdot \mu \) as \( \{(z_1, \alpha \cdot m_1), (z_2, \alpha \cdot m_2), \ldots, (z_k, \alpha \cdot m_k)\} \). We say that a multi-aggregate function \( F(p_i, \lambda) = \psi(\lambda(p_i), \mu) \) is scale-invariant if \( \psi(z, \mu) = \psi(z, \alpha \cdot \mu) \) for every positive integer \( \alpha \) and every input \( z \). That is, \( F \) depends only on the “concentration” of each input in the system, rather than on their actual multiplicities. Notable examples include statistical functions such as mean, variance, maximum, median, mode, etc. The problem of computing the mean of all input values is called Average Consensus [6, 15, 16, 17, 24, 29, 39, 40, 42, 43, 44, 48, 51].

The scale-invariant multi-aggregate function \( F_R(p_i, \lambda) = \frac{1}{n} \cdot \mu \) is called Concentration function, and is complete for the class of scale-invariant multi-aggregate functions, as shown below.

Proposition 2.1. If \( F_R \) can be computed (with termination), then all scale-invariant multi-aggregate functions can be computed (with termination) in the same number of rounds, as well.

Proof. Suppose that a process \( p_i \) has determined \( \frac{1}{n} \cdot \mu = \{(z_1, m_1/n), (z_2, m_2/n), \ldots, (z_k, m_k/n)\} \). Then it can immediately find the smallest integer \( d > 0 \) such that \( d \cdot (m_i/n) \) is an integer for all \( 1 \leq i \leq k \). Note that \( \frac{d}{n} \cdot \mu \) is a multiset. Hence, in the same round, \( p_i \) can compute any desired function \( \psi(\lambda(p_i), \frac{d}{n} \cdot \mu) \), and thus any scale-invariant multi-aggregate function, by definition. □
History trees. History trees were introduced in [23] as a tool of investigation for anonymous dynamic networks; an example is found in Figure 1. A history tree is a representation of a dynamic network given some inputs to the processes. It is an infinite graph whose nodes are partitioned into levels \( L_t \), with \( t \geq -1 \); each node in \( L_t \) represents a class of processes that are indistinguishable at the end of round \( t \) (with the exception of \( L_{-1} \), which contains a single node \( r \) representing all processes). The definition of distinguishability is inductive: at the end of round 0, two processes are distinguishable if and only if they have different inputs. At the end of round \( t \geq 1 \), two processes are distinguishable if and only if they were already distinguishable at round \( t - 1 \) or if they have received different multisets of messages at round \( t \).

Each node in level \( L_0 \) has a label indicating the input of the processes it represents. There are also two types of edges connecting nodes in adjacent levels. The black edges induce an infinite tree rooted at node \( r \in L_{-1} \) which spans all nodes. The presence of a black edge \( \{v, v'\} \) with \( v \in L_t \) and \( v' \in L_{t+1} \), indicates that the child node \( v' \) represents a subset of the processes represented by the parent node \( v \). The red multi-edges represent communications between processes. The presence of a red edge \( \{v, v'\} \) with multiplicity \( m \), with \( v \in L_t \) and \( v' \in L_{t+1} \), indicates that, at round \( t + 1 \), each process represented by \( v' \) receives \( m \) (identical) messages from processes represented by \( v \).

As time progresses and processes exchange messages, they are able to locally construct finite portions of the history tree. In [23], it is shown that there is a local algorithm \( A^* \) that allows each

![Figure 1: The first rounds of a dynamic network with \( n = 9 \) processes and the corresponding levels of the history tree. The multiplicities of the red multi-edges of the history tree are explicitly indicated only when greater than 1. The letters A, B, C denote processes’ inputs; all other labels have been added for the reader’s convenience, and indicate classes of indistinguishable processes (non-trivial classes are also indicated by dashed blue lines). Note that the two processes in \( b_4 \) are still indistinguishable at the end of round 2, although they are linked to the distinguishable processes \( b_5 \) and \( b_6 \). This is because such processes were in the same class \( a_5 \) at round 1. The subgraph in the green blob is the view of the two processes in the class \( b_1 \).](image-url)
process to locally construct and update its own view of the history tree at every round (also see Appendix B.4). The view of a process \( p \) at round \( t \geq 0 \) is the subgraph of the history tree which is spanned by all the shortest paths (using black and red edges indifferently) from the root \( r \) to the node in \( L_t \) representing \( p \) (see Figure 1). As proved in [23, Theorem 3.1], the view of a process at round \( t \) contains all the information that the process may be able to use at that round. This justifies the convention that all processes always execute \( A^* \), constructing their local view of the history tree and broadcasting (a representation of) it at every round, regardless of their task. Then, they simply compute their task-dependent outputs as a function of their respective views.

We define the anonymity of a node \( v \) of the history tree as the number of processes that \( v \) represents, and we denote it as \( a(v) \). It follows that \( \sum_{v \in L_t} a(v) = n \) for all \( t \geq -1 \), and that the anonymity of a node is equal to the sum of the anonymities of its children. Naturally, a process is not aware of the anonymities of the nodes in its view of the history tree, unless it can somehow infer them from the view’s structure itself. In fact, computing the Generalized Counting function is equivalent to determining the anonymities of all the nodes in \( L_0 \). Similarly, computing the Concentration function corresponds to determining the value \( a(v)/n \) for all \( v \in L_0 \).

**Computation in disconnected networks.** Although the network at each individual round \( G_t \) may be disconnected, we assume the dynamic network to be \( T \)-interval-disconnected. That is, there is a dynamic disconnectivity parameter \( T \geq 1 \) such that the sum of any \( T \) consecutive \( G_t \)’s is a connected multigraph. Thus, for all \( i \geq 1 \), the multigraph \( (V, \bigcup_{t=i}^{i+T-1} E_t) \) is connected (we remark that a union of multisets adds up the multiplicities of equal elements). Note that a network is 1-interval-disconnected if and only if it is 1-interval-connected, as defined in [33].

**Observation 2.2.** Most functions (including the mean of the input values) are impossible to compute with termination unless the processes have some knowledge about \( T \).

*Proof.* We will assume that the function to be computed is the mean, although the same proof technique works for most non-trivial functions. Suppose the mean can be computed with termination in \( t \) rounds in some network \( G \) when all processes have input 0. Now consider the network \( G' \) obtained from \( G \) by adding a single process \( p \), which is kept disconnected from the rest for \( t \) rounds (hence \( T > t \)). Assign input 1 to \( p \) and input 0 to all other processes. Then, all processes in \( G' \) other than \( p \) still terminate in \( t \) rounds and output 0, which is not the mean of all input values.

**Proposition 2.3.** A function \( F \) can be computed (with termination) within \( f(n) \) rounds in any dynamic network with \( T = 1 \) if and only if \( F \) can be computed (with termination) within \( T \cdot f(n) \) rounds in any dynamic network with \( T \geq 1 \), assuming that \( T \) is known to all processes.

*Proof.* Let us subdivide time into blocks of \( T \) consecutive rounds. Each process collects and stores all messages it receives within a same block, and updates its state all at once at the end of the block. This reduces any \( T \)-interval-disconnected network \( G = ((V, E_t))_{t \geq 1} \) to a 1-interval-disconnected network \( G' = ((V, E'_t))_{t \geq 1} \), where \( E'_t = \bigcup_{i=(t-1)T+1}^{tT} E_i \). Thus, if \( F \) can be computed in within \( f(n) \) rounds in all 1-interval-disconnected networks (which include \( G' \)), then \( F \) can be computed within \( T f(n) \) rounds in the original network \( G \).

Conversely, consider a 1-interval-disconnected network \( G \), and construct a \( T \)-interval-disconnected network \( G' \) by inserting \( T - 1 \) empty rounds (i.e., rounds with no links at all) between any two consecutive rounds of \( G \). Since no information circulates during the empty rounds, if \( F \) cannot be computed within \( f(n) \) rounds in \( G \), then \( F \) cannot be computed within \( T f(n) \) rounds in \( G' \).

\[^{1}\text{Note that this argument is correct because we require algorithms to work for all multigraphs, as opposed to simple graphs only. Indeed, since a process \( p_i \) may receive multiple messages from the same process \( p_j \) within a same block, the resulting network \( G' \) may have multiple links between \( p_i \) and \( p_j \) in a same round, even if \( G \) does not.}\]
Listing 1: Constructing a system of equations in the anonymities of some nodes in a view.

```plaintext
# Input: a view \( \mathcal{V} \) with levels \( L_1, L_0, L_1, \ldots, L_h \)
# Output: \((t, S)\), where \( t \) is an integer and \( S \) is a system of linear equations

1. Assign \( s := 0 \)
2. For \( t := 0 \) to \( h \)
3.   If \( L_t \) contains a node with no children, return \((-1, \emptyset)\)
4.   If \( L_t \) contains a node with more than one child, assign \( s := t + 1 \)
5.   Else
6.      Let \( k = |L_s| = |L_t| \) and let \( u_i \) be the \( i \)th node in \( L_t \)
7.      Let \( P_i \) be the strand starting in \( L_s \) and ending in \( u_i \in L_t \)
8.      Let \( \mathcal{P} = \{P_1, P_2, \ldots, P_k\} \)
9.      Let \( G \) be the graph on \( \mathcal{P} \) whose edges are pairs of exposed strands
10.     If \( G \) is connected
11.        Let \( G' \subseteq G \) be any spanning tree of \( G \)
12.        Assign \( S := \emptyset \)
13.        For each edge \( \{P_i, P_j\} \) of \( G' \)
14.           Find any two exposed nodes \( v_1 \in P_i \) and \( v_2 \in P_j \)
15.           Let \((m_1, m_2)\) be the multiplicity of the exposed pair \((v_1, v_2)\)
16.           Add to \( S \) the equation \( m_1x_i = m_2x_j \)
17.     Return \((t, S)\)
```

3 Computation in Leaderless Networks

We will give a stabilizing and a terminating algorithm that efficiently compute the Concentration function \( F_R \) in all leaderless networks with finite dynamic disconnectivity \( T \). As a consequence, all scale-invariant multi-aggregate functions are efficiently computable as well, due to Proposition 2.1. Moreover, Proposition 5.1 states that no other functions are computable in leaderless networks, and Proposition 5.2 shows that our algorithms are asymptotically optimal.

3.1 Stabilizing Algorithm

We will use the procedure in Listing 1 as a subroutine in some of our algorithms. Its purpose is to construct a homogeneous system of \( k - 1 \) independent linear equations involving the anonymities of all the \( k \) nodes in a level of a process’ view. We will first give some definitions.

In (a view of) a history tree, if a node \( v \in L_t \) has exactly one child (i.e., there is exactly one node \( v' \in L_{t+1} \) such that \( \{v, v'\} \) is a black edge), we say that \( v \) is non-branching. Borrowing from [23], we say that two non-branching nodes \( v_1, v_2 \in L_t \), whose respective children are \( v_1', v_2' \in L_{t+1} \), are exposed with multiplicity \((m_1, m_2)\) if the red edges \( \{v_1', v_2\} \) and \( \{v_2', v_1\} \) are present with multiplicities \( m_1 \geq 1 \) and \( m_2 \geq 1 \), respectively. A strand is a path \((w_1, w_2, \ldots, w_k)\) in (a view of) a history tree consisting of non-branching nodes such that, for all \( 1 \leq i < k \), the node \( w_i \) is the parent of \( w_{i+1} \). We say that two strands \( P_1 \) and \( P_2 \) are exposed if there are two exposed nodes \( v_1 \in P_1 \) and \( v_2 \in P_2 \).

Intuitively, the procedure in Listing 1 searches for a long-enough sequence of levels in the given view \( \mathcal{V} \), say from \( L_s \) to \( L_t \), where all nodes are non-branching. That is, the nodes in \( L_s \cup L_{s+1} \cup \cdots \cup L_t \) can be partitioned into \( k = |L_s| = |L_t| \) strands. Then the procedure searches for pairs of exposed strands, each of which yields a linear equation involving the anonymities of some nodes of \( L_t \), until it obtains \( k - 1 \) linearly independent equations. Note that the search may fail (in which case Listing 1 returns \( t = -1 \)) or it may produce incorrect equations. The following lemma specifies a sufficient condition for Listing 1 to return a correct and non-trivial system of equations for some \( t \geq 0 \).
Lemma 3.1. Let \( \mathcal{V} \) be the view of a process in a \( T \)-interval-disconnected network of size \( n \) taken at round \( 2Tn \) or after. Then, [Listing 1] on input \( \mathcal{V} \) returns \( (t, S) \), where \( 0 \leq t \leq Tn \) and \( S \) is a homogeneous system of \( k - 1 \) independent linear equations in \( k = |L_t| \) variables \( x_1, x_2, \ldots, x_k \). Moreover, \( S \) is satisfied by assigning to \( x_i \) the anonymity of the \( i \)-th node of \( L_t \), for all \( 1 \leq i \leq k \).

Proof. It is well known that, if \( T = 1 \), information takes less than \( n \) rounds to travel from a process to any other process [33]. Thus, if \( T \geq 1 \), it takes less than \( Tn \) rounds (cf. Proposition 2.3). Since \( \mathcal{V} \) is a view taken at round \( 2Tn \) (or after), all levels of \( \mathcal{V} \) up to \( L_{Tn+1} \) are complete, i.e., all nodes in the first \( Tn + 1 \) levels of the network's history tree also appear in the view \( \mathcal{V} \).

Since the anonymity of the root of \( \mathcal{V} \) is \( n \), there must be less than \( n \) branching nodes in \( \mathcal{V} \). Therefore, the first \( Tn \) levels contain an interval of at least \( T \) consecutive levels, say from \( L_r \) to \( L_{r+T-1} \), where all nodes are non-branching and can be partitioned into \( |L_r| = |L_{r+T-1}| \) strands \( P_i \).

Note that a link between any round \( r' \) in the interval \([r + 1, r + T]\) determines a pair of exposed nodes in \( L_{r'-1} \). Thus, by definition of \( T \)-interval-disconnected network, the graph of exposed strands between \( L_r \) and \( L_{r+T-1} \) (constructed as \( G \) in Line 12) is connected. It follows that the execution of [Listing 1] terminates at Line 20 (as opposed to Line 6), at the latest when \( t = r + T - 1 \). Thus, the procedure returns a pair \((t, S)\) with \( 0 \leq t \leq r + T - 1 \leq Tn \).

Observe that \( S \) is homogeneous because it consists of homogeneous linear equations (cf. Line 19). Also, since the spanning tree \( G' \) constructed in Line 14 has \( k - 1 \) edges, \( S \) contains \( k - 1 \) equations.

We will prove that they are linearly independent by induction on \( k \). If \( k = 1 \), there is nothing to prove. Otherwise, let \( P_i \) be a leaf of \( G' \), and let \( \{P_i, P_j\} \) be its incident edge. Then, \( S \) contains an equation \( Q \) of the form \( m_1x_i = m_2x_j \) with \( m_1m_2 \neq 0 \). Let \( S' \) be the system obtained by removing \( Q \) from \( S \); equivalently, \( S' \) corresponds to the tree obtained by removing the leaf \( P_i \) from \( G' \). By the inductive hypothesis, no linear combination of equations in \( S' \) yields \( 0 = 0 \). On the other hand, if \( Q \) is involved in a linear combination with a non-zero coefficient, then the variable \( x_i \) cannot vanish, because it only appears in \( Q \). Therefore, the equations in \( S \) are independent.

It remains to prove that a solution to \( S \) is given by the anonymities of the nodes of \( L_t \). It was shown in [23, Lemma 4.1] that, if \( v_1 \) and \( v_2 \) are exposed in \( \mathcal{V} \), as well as in the history tree containing \( \mathcal{V} \), with multiplicity \( (m_1, m_2) \), then \( m_1a(v_1) = m_2a(v_2) \). To conclude our proof, it is sufficient to note that, since the nodes of a strand \( P_i \) are non-branching in \( \mathcal{V} \) as well as in the underlying history tree, they all have the same anonymity, which is the anonymity of the ending node \( w_i \in L_t \).

Theorem 3.2. There is an algorithm that computes \( F_R \) in all \( T \)-interval-disconnected anonymous networks with no leader and stabilizes in at most \( 2Tn \) rounds, assuming no knowledge of \( T \) or \( n \).

Proof. Our local algorithm is as follows. Run [Listing 1] on the process' view \( \mathcal{V} \), obtaining a pair \((t, S)\). If \( t = -1 \) or \( S \) is not a homogeneous system of \( k - 1 \) independent linear equations in \( k \) variables, output "Unknown". Otherwise, since the rank of the coefficient matrix of \( S \) is \( k - 1 \), the general solution to \( S \) has exactly one free parameter, due to the Rouché–Capelli theorem. Therefore, by Gaussian elimination, it is possible to express every variable \( x_i \) as a rational multiple of \( x_1 \), i.e., \( x_i = \alpha_i x_1 \) for some \( \alpha_i \in \mathbb{Q} \) (note that the coefficients of \( S \) are integers). Let \( L_t = \{w_1, w_2, \ldots, w_k\} \) and \( L_0 = \{v_1, v_2, \ldots, v_k\} \). For every node \( v_i \in L_0 \), define \( \beta_i \in \mathbb{Q}^+ \) as \( \beta_i = \sum_{j \in L_t} \alpha_j \), and let \( \beta = \sum \beta_i \). Then, output \((\operatorname{label}(v_1), \beta / \beta), \operatorname{label}(v_2), \beta_2 / \beta, \ldots, \operatorname{label}(v_k), \beta_k / \beta)\).

The correctness and stabilization time of the above algorithm directly follow from Lemma 3.1. Specifically, at any round \( \geq 2Tn \), the system \( S \) is satisfied by the anonymities of the nodes in \( L_t \). Thus, \( a(v_i) = \alpha_i a(v_1) \) for all \( v_i \in L_0 \), and therefore \( \beta_i / \beta = a(v_i) / n \). We conclude that, for any input assignment \( \lambda \), the algorithm stabilizes on the correct output \( \frac{1}{n} \cdot \mu_\lambda \) within \( 2Tn \) rounds.

\[ \text{If } v_1 \text{ and } v_2 \text{ are exposed in } \mathcal{V} \text{ but not in the underlying history tree, then they have some children not in } \mathcal{V}, \text{ and therefore the equation may not hold.} \]
3.2 Terminating Algorithm

We will now give a certificate of correctness that can be used to turn the stabilizing algorithm of [Theorem 3.2] into a terminating algorithm. The certificate relies on a-priori knowledge of the dynamic disconnectivity T and an upper bound N on the size of the network n; these assumptions are justified by Observation 2.2 and Proposition 5.3 respectively.

**Theorem 3.3.** There is an algorithm that computes \( F_R \) in all \( T \)-interval-disconnected anonymous networks with no leader and terminates in at most \( T(n+N) \) rounds, assuming that T and an upper bound \( N \geq n \) are known to all processes.

*Proof.* Our terminating algorithm is as follows. Run [Listing 1] on the process' view \( V \), obtaining a pair \( (t, S) \), and then do the same computations in the algorithm in [Theorem 3.2] If \( t \neq -1 \) and the current round \( t' \) satisfies \( t' \geq t + TN \), then the output is correct, and the process terminates.

The above termination condition relies on the fact that information takes less than \( Tn \leq TN \) rounds to travel from a process to any other process, and therefore at round \( t' \geq t + TN \) all the nodes of the history tree up to level \( t \) are in the view \( V \). As shown in [Lemma 3.1] this is enough to guarantee that the system \( S \) is indeed satisfied by the anonymities of the nodes in level \( L_t \). Also, [Lemma 3.1] ensures that \( t \leq Tn \), which yields a total running time of at most \( T(n+N) \) rounds. \( \Box \)

4 Computation in Networks with Leaders

We will give a stabilizing and a terminating algorithm that efficiently compute the Generalized Counting function \( F_{GC} \) in all networks with \( \ell \geq 1 \) leaders and finite dynamic disconnectivity \( T \). Therefore, all multi-aggregate functions are efficiently computable as well, due to [23, Theorem 2.1]. Moreover, Proposition 5.4 states that no other functions are computable in networks with leaders, and Proposition 5.6 shows that our algorithms are asymptotically optimal for any fixed \( \ell \geq 1 \).

4.1 Stabilizing Algorithm

We will once again make use of the subroutine in [Listing 1] this time assuming that the number of leaders \( \ell \geq 1 \) is known to all processes. This assumption is justified by Proposition 5.5.

**Theorem 4.1.** There is an algorithm that computes \( F_{GC} \) in all \( T \)-interval-disconnected anonymous networks with \( \ell \geq 1 \) leaders and stabilizes in at most \( 2TN \) rounds, assuming that \( \ell \) is known to all processes, but assuming no knowledge of \( T \) or \( n \).

*Proof.* The algorithm proceeds as in [Theorem 3.2] When the fractions \( \beta_1, \beta_2, \ldots, \beta_{k'} \) have been computed, as well as their sum \( \beta \), we perform the following additional steps. Let \( L_0 = \{v_1, v_2, \ldots, v_{k'}\} \) and let \( \{v_{j_1}, v_{j_2}, \ldots, v_{j_l}\} \subseteq L_0 \) be the set of nodes in \( L_0 \) representing leader processes, i.e., such that label\( (v_{j_i}) \) has the leader flag set for all \( 1 \leq i \leq l \). Compute \( \beta' = \sum_{i=1}^{l} \beta_{j_i} \) and \( \gamma_i = \ell \beta_i / \beta' \) for all \( 1 \leq i \leq k' \), and return \( \{(\text{label}(v_1), \gamma_1), (\text{label}(v_2), \gamma_2), \ldots, (\text{label}(v_{k'}), \gamma_{k'}\} \).

The correctness follows from the fact that, as shown in [Theorem 3.2] at any round \( 2TN \) we have \( \beta_i / \beta = a(v_i)/n \) for all \( 1 \leq i \leq k' \). Adding up these equations for all \( i \in \{j_1, j_2, \ldots, j_l\} \), we obtain \( \beta' / \beta = \ell / n \), and therefore \( n = \ell \beta / \beta' \). We conclude that

\[
\gamma_i = \frac{\ell \beta_i}{\beta'} = \frac{\ell \beta \beta_i}{\beta'^2} = \frac{n \beta_i}{\beta} = a(v_i).
\]

Thus, within \( 2TN \) rounds, the algorithm stably outputs the anonymities of all nodes in \( L_0 \). As observed in [Section 2] this is equivalent to computing the Generalized Counting function \( F_{GC} \). \( \Box \)
4.2 Terminating Algorithm

Giving an efficient certificate of correctness for the (Generalized) Counting problem with multiple leaders is a highly non-trivial task, for which we require a radically different approach.

**The subroutine ApproxCount.** We will first introduce the subroutine ApproxCount, whose formal description and proof of correctness are found in Appendix C. The purpose of ApproxCount is to compute an approximation \( n' \) of the total number of processes \( n \) (or report various types of failure). It takes as input a view \( \mathcal{V} \) of a process, the number of leaders \( \ell \), and two integer parameters \( s \) and \( x \), representing the index of a level of \( \mathcal{V} \) and the anonymity of a node in \( L_s \), respectively.

The procedure ApproxCount\((\mathcal{V}, s, x, \ell)\) begins by assuming that \( x \) is the anonymity of the first leader node \( \tau \in L_s \). Then it scans the levels of \( \mathcal{V} \) starting from \( L_s \), inferring the anonymities of other nodes based on known anonymities. As soon as it has obtained enough information, it stops and returns \((n', t)\), where \( L_t \) is the level scanned thus far. If the information gathered satisfies certain criteria, then \( n' \) is an approximation of \( n \). Otherwise, \( n' \) is an error code, as detailed below.

If \( L_s \) contains no leader nodes, the procedure returns the error code \( n' = -1 \). If, before gathering enough information on \( n \), the procedure encounters a leader node with more than one child in \( \mathcal{V} \), it returns the error code \( n' = -2 \). Finally, if it determines that the sum of anonymities of the leader nodes is not \( \ell \) (possibly because the assumption \( x = a(\tau) \) was incorrect), it returns the error code \( n' = -3 \). A proof of the following technical lemma is found in Appendix C.

**Lemma 4.2.** Let \( \text{ApproxCount}(\mathcal{V}, s, x, \ell) \) return \((n', t)\), and let \( L_{t'} \) be the last level of \( \mathcal{V} \). Then:

(i) If \( t' \geq s + (\ell + 2)n - 1 \), then \( s \leq t \leq s + (\ell + 1)n - 1 \).

(ii) If \( x = a(\tau) \) and \( t' \geq t + n \), then \( n' \neq -3 \).

(iii) If \( x \geq a(\tau) \) and \( n' > 0 \) and \( t' \geq t + n' \), then \( n' = n \).

Our terminating algorithm assumes that all processes know the number of leaders \( \ell \geq 1 \) and the dynamic disconnectivity \( T \). Again, this is justified by Proposition 5.5 and Observation 2.2.

**Theorem 4.3.** There is an algorithm that computes \( F_{GC} \) in all \( T \)-interval-disconnected anonymous networks with \( \ell \geq 1 \) leaders and terminates in at most \((\ell^2 + \ell + 1)Tn\) rounds, assuming that \( \ell \) and \( T \) are known to all processes, but assuming no knowledge of \( n \).

**Proof.** Due to Proposition 2.3, since \( T \) is known and appears as a factor in the claimed running time, we can assume that \( T = 1 \) without loss of generality. Also, note that determining \( n \) is enough to compute \( F_{GC} \). Indeed, if a process determines \( n \) at round \( t' \), it can wait until round \( \max\{t', 2Tn\} \) and run the algorithm in **Theorem 4.1** which is guaranteed to give the correct output by that time.

In order to determine \( n \) assuming that \( T = 1 \), we run the algorithm in **Listing 2** with input \((\mathcal{V}, \ell)\). We will prove that this algorithm returns a positive integer (as opposed to “Unknown”) within \((\ell^2 + \ell + 1)n\) rounds, and the returned number is indeed the correct size of the system \( n \).

**Algorithm description.** The algorithm goes through at most \( \ell \) phases (Line 5). The first phase starts by calling \( \text{ApproxCount} \) with \( s = 0 \) and \( x = \ell \), i.e., the maximum possible value for a leader node’s anonymity (Line 8). Let \((n', t)\) be the returned value. If \( n' > 0 \), then this approximation of \( n \) is stored in the variable \( n^* \), and the algorithm proceeds with the next phase (Lines 10–12). If \( n' = -1 \), then no leader nodes were found, implying that the process’ view is still missing some relevant nodes. In this case, the algorithm immediately returns “Unknown” (Line 13). If \( n' = -2 \), then a leader node with multiple children was found. As this is an undesirable event, the algorithm proceeds to the next phase (Line 14). If \( n' = -3 \), then \( x \) may not be the correct anonymity of the
Listing 2: Solving the Counting problem with $\ell$ leaders.

```
# Input: a view $\mathcal{V}$ and a positive integer $\ell$
# Output: either an integer $n$ or "Unknown"

Assign $n^* := -1$ and $s := 0$
Repeat $\ell$ times
  For $x = \ell$ downto 1
    Assign $(n',t) := \text{ApproxCount}(\mathcal{V}, s, x, \ell)$ # see Listing 3
    Assign $t^* := \max\{t^*, t\}$
    If $n' > 0$
      Assign $n^* := \max\{n^*, n'\}$
      Break out of the for loop
    Else if $n' = -1$, return "Unknown"
    Else if $n' = -2$, break out of the for loop
  Assign $s := t^* + 1$
Let $L_\ell$ be the last level of $\mathcal{V}$
If $n^* > 0$ and $t^* \geq s - 1 + n^*$, return $n^*$
Else return "Unknown"
```

leader node $\tau$ (see the description of $\text{ApproxCount}$), and therefore the algorithm calls $\text{ApproxCount}$ again with $s = 0$ and $x = \ell - 1$. If the same happens, then $x = \ell - 2$ is tried, and so on. After all possible assignments down to $x = 1$ have failed, the algorithm just proceeds to the next phase.

The second phase starts from the first level $L_{\ell+1}$ that was not visited in the first phase (Line 15), but it is otherwise identical to the first phase. The algorithm proceeds in this fashion, always keeping the largest approximation of $n$ in the variable $n^*$ (Line 11) until the $\ell$th phase is done. At this point, a correctness check is performed: the algorithm takes the last level $L_{\ell^*}$ visited thus far; if $n^* > 0$ (i.e., if at least one meaningful approximation was found) and the current round $t'$ satisfies $t' \geq t^* + n^*$, then $n^*$ is accepted as correct; otherwise "Unknown" is returned (Lines 16–18).

**Correctness and running time.** We will prove that, if the output of Listing 2 is not "Unknown", then it is indeed the number of processes, i.e., $n^* = n$. Recall statement (iii) of Lemma 4.2 if $x \geq a(\tau)$ and $n' > 0$ and $t' \geq t + n'$, then $n' = n$. Observe that, if the condition in Line 17 is satisfied, then $t' \geq t + n'$ must hold every time a call to $\text{ApproxCount}$ returns an approximation value $n' > 0$, because $t^* \geq t$ and $n^* \geq n'$. Also, because the loop at Lines 7–14 starts with $x = \ell$, which is the largest possible anonymity of a leader node, statements (ii) and (iii) of Lemma 4.2 combined imply that, as long as $n' = -3$, the inequality $x \geq a(\tau)$ is always satisfied. Thus, whenever $n' > 0$, it must be $n' = n$. We conclude that, if the condition in Line 17 is satisfied, then indeed $n^* = n$.

It remains to prove that Listing 2 actually gives an output other than “Unknown” within the claimed number of rounds; it suffices to show that it does so if it is executed at round $t' = (\ell^2 + \ell + 1)n$. It is known that all nodes in the first $t' - n = \ell(\ell + 1)n$ levels of the history tree are contained in the view $\mathcal{V}$ at round $t'$ (cf. Corollary 4.3]). Also, it is straightforward to prove by induction that the assumption of statement (i) of Lemma 4.2 holds every time $\text{ApproxCount}$ is invoked in any of the $\ell$ phases. Indeed, at each phase, $s$ increases by at most $(\ell + 1)n$; in other words, each phase involves an interval of at most $(\ell + 1)n$ levels. It follows that all levels ever scanned by $\text{ApproxCount}$ are completely contained in $\mathcal{V}$. Therefore, at round $t'$, none of the calls to $\text{ApproxCount}$ can return the error code $n' = -1$. Also, since these $\ell$ intervals are disjoint (due to Line 15) and at most $\ell - 1$ leader nodes in $\mathcal{V}$ have more than one child (because the number of leader nodes at any level is at most $\ell$), there is at least one interval $L$ where $\text{ApproxCount}$ does not return the error code $n' = -2$. 

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Let us consider the phase where \texttt{ApproxCount} is executed in the interval $\mathcal{L}$. Let $L_s$ be the first level of $\mathcal{L}$, and let $\tau$ be the first leader node in $L_s$. Since the loop at Lines 7–14 repeatedly calls \texttt{ApproxCount} trying all possible options $\ell \geq x \geq 1$ for $a(\tau)$ until $n' \neq -3$, statement (ii) in Lemma 4.2 implies that at least one of these calls is bound to return $n' > 0$ (i.e., when $x$ takes the correct value $a(\tau)$). Thus, in the end, $n' > 0$, and the condition in Line 17 is satisfied; hence, Listing 2 does not return “Unknown” if executed at round $t' = (\ell^2 + \ell + 1)n$. 

\section{Negative Results}

\subsection{Leaderless Networks}

\textbf{Proposition 5.1.} No function other than the scale-invariant multi-aggregate functions can be computed with no leader, even when restricted to simple connected static networks.

\textit{Proof.} Let $m_1, m_2, \ldots, m_k$ be integers greater than 2 with $\gcd(m_1, m_2, \ldots, m_k) = 1$, and let $B$ be the complete $k$-partite graph with partite sets $V_1, V_2, \ldots, V_k$ of sizes $m_1, m_2, \ldots, m_k$, respectively. For any positive integer $\alpha$, construct the static network $G_\alpha$ consisting of $\alpha$ disjoint copies of $B$, augmented with $k$ cycles $C_1, C_2, \ldots, C_k$ such that, for each $1 \leq i \leq k$, the cycle $C_i$ spans all the $\alpha m_i$ processes in the $\alpha$ copies of $V_i$. Clearly, $G_\alpha$ is a simple connected static network.

Let the function $\lambda_\alpha$ assign input $z_i$ to all processes in the $\alpha$ copies of $V_i$ in $G_\alpha$, and let $\lambda = \lambda_1$. As a result, $\mu_{\lambda_\alpha} = \{(z_1, \alpha m_1), (z_2, \alpha m_2), \ldots, (z_k, \alpha m_k)\} = \alpha \cdot \mu_\lambda$ for all $\alpha \geq 1$. Moreover, all the networks $G_\alpha$ have isomorphic history trees. This is because, at every round, each process in any of the copies of $V_i$ receives exactly two messages from other processes in copies of $V_i$ and exactly $m_j$ messages from processes in copies of $V_j$, for all $j \neq i$. Thus, it can be proved by induction that all processes in the copies of $V_i$ have isomorphic views, regardless of $\alpha$.

Due to the fundamental theorem of history trees [23, Theorem 3.1], all the processes with input $z_i$ must give the same output $\psi(z_i, \mu_\lambda) = \psi(z_i, \mu_{\lambda_\alpha}) = \psi(z_i, \alpha \cdot \mu_\lambda)$, regardless of $\alpha$. Hence, by definition, only scale-invariant multi-aggregate functions can be computed in these networks. 

\textbf{Proposition 5.2.} No algorithm can solve the Average Consensus problem in a $T$-interval-disconnected leaderless network in less than $2Tn - O(T)$ rounds (with or without termination).

\textit{Proof.} According to [23, Theorem 5.2], the number of processes $n$ in a network with $\ell = 1$ and $T = 1$ cannot be determined in less than $2n - O(1)$ rounds (with or without termination). We can reduce this problem to Average Consensus with $\ell = 0$ and $T = 1$ as follows. In any given network with $\ell = T = 1$, assign input 1 to the leader and clear its leader flag; assign input 0 to all other processes. If the processes can compute the mean input value, 1/n, they can invert it to obtain $n$ in the same number of rounds. It follows that Average Consensus with $\ell = 0$ and $T = 1$ cannot be solved in less than $2n - O(1)$ rounds; this immediately generalizes to an arbitrary $T$ by Proposition 2.3.

\textbf{Proposition 5.3.} No algorithm can solve the leaderless Average Consensus problem with explicit termination if nothing is known about the size of the network, even when restricted to simple connected static networks.

\textit{Proof.} Assume for a contradiction that there is such an algorithm $\mathcal{A}$. Let $\mathcal{G}$ be a static network consisting of three processes forming a cycle, and assign input 0 to all of them. If the processes execute $\mathcal{A}$, they eventually output the mean value 0 and terminate, say in $t$ rounds.

Now construct a static network $\mathcal{G}'$ consisting of a cycle of $2t + 2$ processes $p_1, p_2, \ldots, p_{2t+2}$; assign input 1 to $p_1$ and input 0 to all other processes. It is easy to see that, from round 0 to round $t$, the view of the process $p_{t+1}$ is isomorphic to the view of any process in $\mathcal{G}$. Therefore, if $p_{t+1}$ executes $\mathcal{A}$, it terminates in $t$ rounds with the incorrect output 0. Thus, $\mathcal{A}$ is incorrect.
5.2 Networks with Leaders

Proposition 5.4. No function other than the multi-aggregate functions can be computed (with or without termination), even when restricted to simple connected static networks with a known number of leaders.

Proof. It is sufficient to construct a static network where all processes with the same input are indistinguishable at every round. Such is, for example, the complete graph $K_n$, where the output of a process can only depend on its input and the multiset of all processes’ inputs [23, Theorem 5.1]. This is the definition of a multi-aggregate function, and thus no other functions can be computed.

Proposition 5.5. No algorithm can compute the Counting function $F_C$ (with or without termination) with no knowledge about $\ell$, even when restricted to simple connected static networks with a known and arbitrarily small ratio $\ell/n$.

Proof. Let us fix a positive integer $k$; we will construct an infinite class of networks whose ratio $\ell/n$ is $1/k$ as follows. For every $i \geq 3$, let $G_i$ be the static network consisting of a cycle of $n_i = k \cdot i$ processes of which $\ell_i = i$ are leaders, such that the leaders are evenly spaced among the non-leaders. Assume that all processes get the same input (apart from their leader flags). Then, at any round, all the leaders in all of these networks have isomorphic views, which are independent of $i$. It follows that, if nothing is known about $\ell_i$ (other than the ratio $\ell_i/n_i$, which is fixed), all the leaders in all the networks $G_i$ always give equal outputs. Since the number of processes $n_i$ depends on $i$, it follows that at most one of these networks can stabilize on the correct output $n_i$.

Proposition 5.6. For any $\ell \geq 1$, no algorithm can compute the Counting function $F_C$ (with or without termination) in all simple $T$-interval-disconnected networks with $\ell$ leaders in less than $T(2n - \ell) - O(T)$ rounds.

Proof. It was shown in [23, Theorem 5.2] that there is a family of simple 1-interval-disconnected networks $G_n$, with $n \geq 1$, with the following properties. $G_n$ has $\ell = 1$ leader and $n$ processes in total; moreover, up to round $2n - O(1)$, the leaders of $G_n$ and $G_{n+1}$ have isomorphic views.

Let us fix $\ell \geq 1$, and let us construct $G'_n$, for $n \geq \ell$, by attaching a chain of $\ell - 1$ additional leaders $p_1, p_2, \ldots, p_{\ell - 1}$ to the single leader $p_\ell$ of $G_{n-\ell+1}$ at every round. Note that $G'_n$ has $n$ processes in total and a stable subpath $(p_1, p_2, \ldots, p_\ell)$ which is attached to the rest of the network via $p_\ell$.

It is straightforward to see that the process $p_1$ in $G'_n$ and the process $p_\ell$ in $G'_{n+1}$, which correspond to the leaders of $G_{n-\ell+1}$ and $G_{n-\ell+2}$ respectively, have isomorphic views up to round $2(n - \ell) - O(1)$. Since the view of $p_1$ is completely determined by the view of $p_\ell$, and it takes $\ell - 1$ rounds for any information to travel from $p_\ell$ to $p_1$, we conclude that the process $p_1$ in $G'_n$ and the process $p_1$ in $G'_{n+1}$ have isomorphic views up to round $2n - \ell - O(1)$.

Thus, up to that round, the two processes must give an equal output, implying that they cannot both output the number of processes in their respective networks. It follows that the Counting function with $\ell \geq 1$ leaders and $T = 1$ cannot be computed in less than $2n - \ell - O(1)$ rounds, which generalizes to an arbitrary $T$ by Proposition 2.3.

6 Conclusions

We have shown that anonymous processes in disconnected dynamic networks can compute all the multi-aggregate functions and no other functions, provided that the network contains a known number of leaders $\ell \geq 1$. If there are no leaders or the number of leaders is unknown, the class of computable functions reduces to the scale-invariant multi-aggregate functions. We have also identified...
the functions $F_{GC}$ and $F_R$ as the complete problems for each class. Notably, the network’s dynamic disconnectivity $T$ does not affect the computability of functions, but only makes computation slower.

Moreover, we gave efficient stabilizing and terminating algorithms for computing all the aforementioned functions. Some of our algorithms make assumptions on the processes’ a-priori knowledge about the network; we proved that such assumptions are actually necessary. All our algorithms have optimal linear running times in terms of $T$ and the size of the network $n$.

In one case, there is still a small gap in terms of the number of leaders $\ell$. Namely, for terminating computation with $\ell \geq 1$ leaders, we have a lower bound of $T(2n - \ell) - O(T)$ rounds (Proposition 5.6) and an upper bound of $(\ell^2 + \ell + 1)Tn$ rounds (Theorem 4.3). Although these bounds asymptotically match if the number of leaders $\ell$ is constant (which is a realistic assumption in most applications), optimizing them with respect to $\ell$ is left as an open problem.

Our algorithms require processes to send each other explicit representations of their history trees, which have cubic size in the worst case \cite{23}. It would be interesting to develop algorithms that only send messages of logarithmic size, possibly with a trade-off in terms of running time.
APPENDIX

A Formal Model Definition and Basic Results

A multiset on an underlying set $X$ is a function $\mu: X \rightarrow \mathbb{N}$. The non-negative integer $\mu(x)$ is the multiplicity of $x \in X$, and specifies how many copies of each element of $X$ are in the multiset. The set of all multisets on the underlying set $X$ is denoted as $\mathcal{M}_X$.

A.1 Model of Computation

A dynamic network is an infinite sequence $\mathcal{G} = (G_i)_{i \geq 1}$, where $G_i = (V, E_i)$ is an undirected multigraph, i.e., $E_i$ is a multiset of unordered pairs of elements of $V$. In this context, the set $V = \{1, 2, \ldots, n\}$ is called system, and its $n \geq 1$ elements are the processes. The elements of the multiset $E_i$ are called links; note that we allow any number of “parallel links” between two processes.

The standard model of computation for systems of processes specifies the following computation parameters:

- three sets $\mathcal{I}$, $\mathcal{O}$, $\mathcal{S}$, representing the possible inputs, outputs, and internal states for a process, respectively;
- A partition of $\mathcal{S}$ into two subsets $\mathcal{S}_T$ and $\mathcal{S}_N$, representing the terminal and non-terminal states, respectively;
- an input map $\iota: \mathcal{I} \rightarrow \mathcal{S}$, where $\iota(x)$ represents the initial state of a process whose input is $x$;
- an output map $\omega: \mathcal{S}_T \rightarrow \mathcal{O}$, where $\omega(s)$ represents the output of a process in a terminal state $s$;
- a function $A: \mathcal{S}_N \times \mathcal{M}_\mathcal{S} \rightarrow \mathcal{S}$, representing a deterministic algorithm for local computations. The algorithm takes as input the (non-terminal) state of a process $p$, as well as the multiset of states of all processes linked to $p$, and it outputs the new state for $p$.

Given the above parameters and a dynamic network $\mathcal{G} = (G_i)_{i \geq 1}$, the computation proceeds as follows. The system $V$ is assigned an input in the form of a function $\lambda: V \rightarrow \mathcal{I}$. Time is discretized into units called rounds $r_1, r_2, r_3$, etc.; the multigraph $G_i$ describes the links that are present at round $r_i$, and is called the topology of the network at round $r_i$.

Each process in the system updates its own state at the end of every round. We denote by $\sigma_i(p) \in \mathcal{S}$ the state of process $p \in V$ after round $r_i$, for $i \geq 1$. Moreover, we define the initial state of $p$ as $\sigma_0(p) = \iota(\lambda(p))$ (we could say that each process is assigned its initial state at “round $r_0$”).

During round $r_i$, each process $p \in V$ broadcasts its state to all its neighbors in $G_i$; then, $p$ receives the multiset $\mathcal{M}_i(p)$ of states of all its neighbors (one state per incident link in $E_i$)[3] Finally, if $p$’s state is non-terminal, it computes its next state based on its current state and $\mathcal{M}_i(p)$. In

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[3] In the dynamic networks literature, $G_i$ is typically assumed to be a simple graph, as at most one link between the same two processes is allowed. However, our results hold more generally for multigraphs.

[4] In a slightly different model, a process does not necessarily broadcast its entire state, but a message, which in turn is computed as a function of the internal state.
formulas, if $\sigma_{i-1}(p) \in S_T$, then $\sigma_i(p) = \sigma_{i-1}(p)$. Otherwise, $\sigma_i(p) = A(\sigma_{i-1}(p), M_i(p))$, where $M_i(p)$ is defined as
\[
M_i(p): S \rightarrow \mathbb{N} \text{ such that } S \ni s \mapsto \sum_{q \in V_{i-1}(q) = s} E_i(\{p,q\})^5
\]
Note that processes are anonymous: they are initially identical and indistinguishable except for their input, and they all update their internal states by executing the same deterministic algorithm $A$.

Once all processes in $V$ are in a terminal state, the system’s output is defined as the function $\zeta: V \rightarrow O$ such that $\zeta(p) = \omega(\sigma_t(p))$. If this happens for the first time after round $r$, we say that the execution terminates in $t$ rounds.

Given an input set $I$ and an output set $O$, a problem on $(I, O)$ is a sequence of functions $P = (\pi_n)_{n \geq 1}$, where $\pi_n$ maps every function $\lambda: \{1,2,\ldots,n\} \rightarrow I$ to a function $\zeta: \{1,2,\ldots,n\} \rightarrow O$. Essentially, a problem prescribes a relationship between inputs and outputs: whenever a system of $n$ processes is assigned a certain input $\lambda$, it must eventually terminate with the output $\zeta = \pi_n(\lambda)$.

We denote the set of all problems on $(I, O)$ as $\mathcal{P}(I, O)$.

We say that a problem $P = (\pi_n)_{n \geq 1} \in \mathcal{P}(I, O)$ is solveable in $f(n)$ rounds if there exists computation parameters (i.e., an algorithm $A$, as well as a set of states $S = S_N \cup S_T$, an input map $\iota$, and an output map $\omega$) such that, whenever a system $V = \{1,2,\ldots,n\}$ of $n$ processes is given an input $\lambda: V \rightarrow I$ and carries out its computation on a dynamic network $G$ as described above, it terminates in at most $f(n)$ rounds and outputs $\pi_n(\lambda)$, regardless of the topology of $G$. Note that the algorithm $A$ must be the same for every $n$, i.e., it is uniform; in other words, the system is unaware of its own size.

A weaker notion of solvability involves stabilization instead of termination. Here, the processes are only required to output $\pi_n(\lambda)$ starting at round $f(n)$ and at all subsequent rounds, without necessarily reaching a terminal state.

Since only trivial problems can be solved if no restrictions are made on the topology of the dynamic network $G$ (think of a dynamic network with no links at all), we assume that the dynamic network be $T$-disconnected. That is, if $G = (G_t)_{t \geq 1} = ((V, E_t))_{t \geq 1}$, we assume that there is a $T \geq 1$ such that, for all $i \geq 1$, the multigraph $(V, E_i \cup E_{i+1} \cup E_{i+2} \cup \cdots \cup E_{i+T-1})$ is connected.

Another assumption that is often made about the system is the presence of a known number $\ell \geq 1$ of leaders or supervisors among the processes. That is, the input set $I$ is of the form $I = \{L, N\} \times \mathcal{I}$, and an input assignment $\lambda: V \rightarrow \mathcal{I}$ is valid if and only if there are exactly $\ell$ processes, the leaders, whose input is of the form $(L, x)$ for some $x \in \mathcal{I}$ (thus, all non-leader processes have an input of the form $(N, x)$).

### A.2 Multi-Aggregation Problems

In this section we will define an important class of problems in $\mathcal{P}(I, O)$, the multi-aggregation problems. As shown in Section 4, these are precisely the problems that can be solved in $T$-disconnected anonymous dynamic networks with a known positive number of leaders.

Given a system $V$ and an input assignment $\lambda: V \rightarrow \mathcal{I}$, we define the inventory of $\lambda$ as the function $\mu_\lambda: \mathcal{I} \rightarrow \mathbb{N}$ which counts the processes that are assigned each given input. Formally, $\mu_\lambda$ is the multiset on $\mathcal{I}$ such that $\mu_\lambda(x) = |\lambda^{-1}(x)|$ for all $x \in \mathcal{I}$.

A problem $P \in \mathcal{P}(I, O)$ is said to be a multi-aggregation problem if the output to be computed by each process only depends on the processes’ own input and the inventory of all processes’ inputs.

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5Recall that $G_i$ is a multigraph, and therefore $E_i$ is a multiset, i.e., a function that maps every possible edge to its multiplicity (if an edge is absent, its multiplicity is 0).
Formally, \( P = (\pi_n)_{n \geq 1} \) is a multi-aggregation problem if there is a function \( \psi : I \times M_I \to O \), called signature function, such that every input assignment \( \lambda : \{1, 2, \ldots, n\} \to I \) is mapped by \( \pi_n \) to the output \( \zeta : \{1, 2, \ldots, n\} \to O \) where \( \zeta(p) = \psi(\lambda(p), \mu_\lambda) \).

### A.3 Counting Problem

If the signature function \( \psi(x, \mu) \) of a multi-aggregation problem \( P \in \mathcal{P}(I, O) \) does not depend on the argument \( x \), then \( P \) is simply said to be an aggregation problem. Thus, all aggregation problems are consensus problems, where all processes must terminate with the same output. Notable examples of aggregation problems in \( \mathcal{P}(\mathbb{R}, \mathbb{R}) \) include computing statistical functions on the input values, such as sum, average, maximum, median, mode, variance, etc.

The Counting Problem \( C_I \) is the aggregation problem in \( \mathcal{P}(I, M_I) \) whose signature function is \( \psi(x, \mu) = \mu \). That is, all processes must determine the inventory of their input assignment \( \lambda \), i.e., count the number of processes that have each input. If all inputs are the same (or, in the presence of a leader, if \( I = \{L, N\} \)), the Counting Problem reduces to determining the total number of processes in the system, \( n \).

The Counting Problem is complete for the class of multi-aggregation problems, in the sense that solving it efficiently implies solving any other multi-aggregation problem efficiently (actually, with no overhead at all). The following theorem makes this observation precise.

**Theorem A.1.** For every input set \( I \) and output set \( O \), if the Counting Problem \( C_I \) is solvable in \( f(n) \) rounds, then every multi-aggregation problem in \( \mathcal{P}(I, O) \) is solvable in \( f(n) \) rounds, as well.

**Proof.** The proof is essentially the same whether the requirement is termination or stabilization. For brevity, we only discuss termination.

Let \( A \) be an algorithm that solves \( C_I \) in \( f(n) \) rounds with the computation parameters \( \mathcal{S} = \mathcal{S}_N \cup \mathcal{S}_T \), \( \tau \), \( \omega \) as defined in Appendix A.1. We will show how to solve \( P \) by modifying \( A \) and the above parameters.

The idea is that each process should execute \( A \) while remembering its own input \( x \). Once it has computed the inventory \( \mu_\lambda \) of the system’s input assignment, it immediately uses \( x \) and \( \mu_\lambda \) to compute the signature function of \( P \).

We define the set of internal states for \( P \) as \( \mathcal{S}' = \mathcal{S} \times I \), with non-terminal states \( \mathcal{S}'_N = \mathcal{S}_N \times I \). The new input map is \( \iota'((x) = ((\iota(x), x) \in \mathcal{S}' \) for all \( x \in I \). The algorithm \( A' : \mathcal{S}'_N \times M_{\mathcal{S}'} \to \mathcal{S}' \) is defined as \( A'((s, x), \mu) = (A(s, \mu'), x) \), where \( \mu' \) is defined as

\[
\mu' : \mathcal{S} \to \mathbb{N} \quad \text{such that} \quad \mathcal{S} \ni s \mapsto \sum_{x \in I} \mu((s, x)).
\]

Finally, we define the output map as \( \omega'(s, x) = \psi(x, \omega(s)) \), where \( \psi : I \times M_I \to O \) is the signature function of \( P \).

Since \( A \) solves the Counting Problem, when a system \( V \) of \( n \) processes is assigned an input \( \lambda \) and executes \( A \) on a network \( G \), within \( f(n) \) rounds each process \( p \in V \) reaches a terminal state \( s \in \mathcal{S}_T \) such that \( \omega(s) = \mu_\lambda \). Therefore, if the system executes \( A' \) on the same input and in the same network, the process \( p \) reaches the terminal state \( (s, \lambda(p)) \in \mathcal{S}' \) in the same number of rounds. Thus, \( p \) gives the output \( \omega'((s, \lambda(p))) = \psi(\lambda(p), \omega(s)) = \psi(\lambda(p), \mu_\lambda) \), as required by the problem \( P \).

We remark that **Theorem A.1** makes no assumption on the dynamic network’s topology or the presence of a leader.
B Formal Definition of History Trees and Basic Properties

B.1 Abstract Structure of a History Tree

A history tree on an input set $I$ is a quintuplet $H = (H, u, B, R, l)$ such that:

- $H$ is a countably infinite set of nodes, with $u \in H$;
- $(H, B)$ is an infinite undirected tree rooted at $u$, where every node has at least one child;
- $(H, R)$ is an infinite undirected multigraph, where all edges have finite multiplicities;
- $l: H \setminus \{u\} \rightarrow I$ is a function that assigns a label $l(h) \in I$ to each node $h \in H$ other than $u$.

The elements of $B$ are called black edges, and $(H, B)$ is the black tree of $H$. Similarly, $(H, R)$ is the red multigraph of $H$, and its edges (with positive multiplicity) are called red edges.

The depth of $h \in H$ is the distance between the nodes $u$ and $h$ as measured in the black tree. For all $i \geq -1$, we define the $i$th level $L_i \subseteq H$ as the set of nodes that have depth $i + 1$. Thus, $L_{-1} = \{u\}$.

The nodes of a history tree inherit their “parent-child-sibling” relationships from the black tree: for example, $u$ is the parent of all the nodes in $L_0$ because it is connected to all of them by black edges; thus, every two nodes in $L_0$ are siblings, etc.

A descending path in a history tree is a sequence of $k \geq 1$ nodes $(h_1, h_2, \ldots, h_k)$ such that, for all $1 \leq i < k$, the unordered pair $\{h_i, h_{i+1}\}$ is either a black or a red edge and, if $h_i \in L_j$, then $h_{i+1} \in L_{j'}$ with $j' > j$. Equivalently, we say that $(h_k, h_{k-1}, \ldots, h_1)$ is an ascending path. The node $h_1$ is the upper endpoint of the path, and $h_k$ is the lower endpoint. If $k = 1$, then $h_1$ is both the upper and the lower endpoint.

Let $h$ be a node in a history tree $H = (H, u, B, R, l)$, and let $H_h \subset H$ be the set of all upper endpoints of ascending paths whose lower endpoint is $h$. The view of $h$ in $H$, denoted as $\nu(h)$, is the (finite) subtree of $H$ induced by $H_h$. Namely, $\nu(h) = (H_h, B_h, R_h, l_h)$, where $B_h \subset B$ is the set of black edges whose endpoints are both in $H_h$, $R_h$ is the restriction of $R$ to the unordered pairs of nodes in $H_h$, and $l_h$ is the restriction of $l$ to $H_h$. The node $h$ is said to be the viewpoint of the view $\nu(h)$; note that the viewpoint is the unique deepest node in the view. We will denote by $\mathcal{V}_I$ the set of all views of all history trees on the input set $I$.

B.2 History Tree of a Dynamic Network

Given a dynamic network $G = (G_i = (V, E_i))_{i \geq 1}$ and an input assignment $\lambda: V \rightarrow I$, we will show how to construct a history tree $H = (H, u, B, R, l)$ with labels in $I$ that is naturally associated with $G$ and $\lambda$.

At the same time, for all $i \geq -1$, we will construct a representation function $\rho_i: V \rightarrow L_i \subset H$. Intuitively, if $h \in L_i$, the preimage $\rho_i^{-1}(h)$ is a set of processes that, based on their “history”, are necessarily “indistinguishable” at the end of round $r_i$ (we will also give a meaning to this sentence when $i = -1$ and $i = 0$). The anonymity of a node $h \in L_i \subset H$ is the number of processes that $h$ represents, and is given by the function $\alpha: H \rightarrow \mathbb{N}^+$, where $\alpha(h) = |\rho_i^{-1}(h)|$.

In this paradigm, the label $l(h)$ of a node $h \neq u$ is the input $\lambda(p)$ that each process $p$ represented by $h$ has received at the beginning of the execution (all such processes must have received the same input, or else they would have different histories, and they would not be necessarily indistinguishable).

\footnote{Note that, in Figure 1, the root node $u$ has been renamed $r$ to match the notation of Section 2.}
The black tree keeps track of the progressive “disambiguation” of processes: if two processes are represented by the same node $h \in L_{i-1}$, they have had the same history up to round $r_{i-1}$. However, if they receive different multisets of messages at round $r_i$, they are no longer necessarily indistinguishable, and will therefore be represented by two different nodes in $L_i$, each of which is a child of $h$ in $\mathcal{H}$.

Red edges represent “observations”: if, at round $r_i$, each of the processes represented by node $h \in L_i$ has received messages from processes represented by node $h' \in L_{i-1}$ through a total of $m$ links in $G_i$, then $R$ contains the red edge $\{h, h'\}$ with multiplicity $m$.

We inductively construct the history tree $\mathcal{H}$ and the representation functions $\rho_i$ level by level. First we define $\rho_{-1}(p) = u$ for all $p \in V$, where $u$ is the root of $\mathcal{H}$. The intuitive meaning is that, before processes are assigned inputs (i.e., at “round $r_{-1}$”), they are all indistinguishable, because the network is anonymous. Thus, the anonymity of the root node $u$ is $|V| = n$.

The level $L_0$ of $\mathcal{H}$ represents the system at the end of “round $r_0$”, i.e., after every process $p \in V$ has been assigned its input $\lambda(p) \in I$ and has acquired an initial state $i'(\lambda(p))$. At this point, processes with the same input are necessarily indistinguishable. Thus, for every input $x \in \lambda(V)$, there is a node $h_x \in L_0$ with label $l(h_x) = x$. Accordingly, for every process $p \in \lambda^{-1}(x)$, we define $\rho_0(p) = h_x$.

In order to inductively construct the level $L_i$ of $\mathcal{H}$ for $i \geq 1$, we define the concept of observation multiset $o_i(p) \in \mathcal{M}_{L_{i-1}}$ of a process $p \in V$ at round $r_i$. This corresponds to the multiset of “necessarily indistinguishable” messages received by $p$ at round $r_i$, and its underlying set is $L_{i-1}$, i.e., the collection of equivalence classes of processes that are necessarily indistinguishable after round $r_{i-1}$. The definition of $o_i(p)$ is similar to the definition of $M_i(p)$ in Appendix A.1, as these are two closely related concepts:

$$o_i(p) \colon L_{i-1} \to \mathbb{N} \text{ such that } \begin{array}{l} L_{i-1} \ni h \mapsto \sum_{q \in \rho_{i-1}^{-1}(h)} E_i(\{p, q\}). \end{array}$$

Now, the children of $h \in L_{i-1}$ in $L_i$ are constructed as follows. Define the equivalence relation $\sim_h$ on the set of processes $V_h = \rho_{i-1}^{-1}(h)$ such that $p \sim_h q$ if and only if $o_i(p) = o_i(q)$. Let $W_1, W_2, \ldots, W_k$ be the equivalence classes of $\sim_h$, with $W_1 \cup W_2 \cup \cdots \cup W_k = V_h$. The node $h$ has exactly $k$ children $h_1, h_2, \ldots, h_k$ in $L_i$, one for each equivalence class of $\sim_h$. Thus, for every $1 \leq j \leq k$ and every process $p \in W_j$, we define $\rho_i(p) = h_j$. Also, since $W_j \subseteq V_h$ and a process’ input never changes, we set $l(h_j) = l(h)$. The red edges connecting $h_j$ with nodes in $L_{i-1}$ match the observation multiset of the processes in $W_j$. That is, if the node $h' \in L_{i-1}$ has multiplicity $m$ in $o_i(p)$, where $p \in W_j$, then the red edge $\{h_j, h'\}$ has multiplicity $m$ in $R$.

### B.3 Basic Properties of History Trees

The following properties of history trees are easily derived from the definitions in Appendix B.2.

**Observation B.1.**

- No two nodes in $L_0$ have the same label, and each node in $H \setminus (L_{i-1} \cup L_0)$ has the same label as its parent.
- Edges only connect nodes in adjacent levels; if $\{h, h'\}$ is a black or red edge with $h \in L_i$, then $h' \in L_{i-1} \cup L_{i+1}$. 

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• If \( h \in L_i \) with \( i \geq -1 \) and \( h_1, h_2, \ldots, h_k \in L_{i+1} \) are the children of \( h \), then
  \[
  \bigcup_{j=1}^{k} \rho_{i+1}^{-1}(h_j) = \rho_{i}^{-1}(h), \quad \text{and therefore} \quad \sum_{j=1}^{k} \alpha(h_j) = \alpha(h).
  \]

We will now give a concrete meaning to the idea that the processes represented by a node of a history tree are “necessarily indistinguishable”. That is, if a node is in the \( i \)th level, then all the processes it represents must have the same state at the end of round \( r_i \), regardless of the deterministic algorithm being executed (cf. Corollary B.3).

We will first prove a fundamental result: the internal state \( \sigma_i(p) \) of any process \( p \in V \) at the end of any round \( r_i \), with \( i \geq 0 \), can be inferred from the process’ history \( \xi_i(p) \), which is defined as the view of the node of \( L_i \) representing \( p \), i.e., \( \xi_i(p) = \nu(\rho_i(p)) \).

**Theorem B.2.** Given any set of computation parameters (as defined in Appendix A.1), there exists a function \( F : \mathcal{I} \rightarrow S \) such that, for every history tree \( H \) associated with a dynamic network \( G \) and an input assignment \( \lambda \) (as defined in Appendix B.2), and for every process \( p \in V \) and every \( i \geq 0 \), \( F(\xi_i(p)) = \sigma_i(p) \).

**Proof.** Let \( h = \rho_i(p) \), and let \( \xi_i(p) = \nu(h) = (H_h, B_h, R_h, l_h) \) be the view of \( h \) in the history tree \( H = (H, u, B, R, l) \). We will define \( F \) (in terms of the computation parameters \( \iota : \mathcal{I} \rightarrow S \) and \( A : \mathcal{M}_S \rightarrow S \)) such that, for every history tree \( H \), \( \xi_i(p) = F(\nu(h)) = \sigma_i(p) \).

Note that \( F \) can identify \( h \) as the deepest node of \( \xi_i(p) \). Also, it can compute \( i \) as the depth of \( h \) in the black tree \((H_h, B_h)\) minus 1. Hence, \( h \) and \( i \) do not have to be explicitly provided as arguments to \( F \).

The construction of \( F \) is done by induction on \( i \). If \( i = 0 \), the view \( \xi_0(p) = \nu(h) \) contains the node \( h \in L_0 \), whose label \( l_h(h) = l(h) \) is the input of \( p \), i.e., \( \lambda(p) \) (cf. the construction of \( L_0 \) in Appendix B.2). Therefore, we define \( F(\xi_0(p)) = \iota(l(h)) \); indeed, \( F(\xi_0(p)) = \iota(\lambda(p)) = \sigma_0(p) \) (cf. the definition of \( \sigma_0(p) \) in Appendix A.1).

Now let \( i \geq 1 \), and assume that \( F(\xi_{i-1}(p)) \) has been defined for every \( q \in V \). We will show how, given \( \xi_i(p) = \nu(h) \), the function \( F \) can compute \( \sigma_i(p) \).

By definition of view, it immediately follows that the view of any node \( h' \in H_h \) is contained in the view of \( h \). That is, all nodes and all black and red edges of \( \nu(h') \) are contained in \( \nu(h) \), and the two views also agree on the labels. Thus, given the history \( \xi_i(p) = \nu(h) \), the function \( F \) can infer the view of any node \( h' \in H_h \) by taking all the ascending paths in \( \nu(h) \) with lower endpoint \( h' \). In particular, if \( \{h, h'\} \) is a black or red edge in \( \nu(h) \), then \( h' \in L_{i-1} \), and therefore \( F \) can determine the state \( \sigma_{i-1}(q) \) of any process \( q \in \rho_{i-1}^{-1}(h') \), by the inductive hypothesis.

Observe that the only black edge \( \{h, h''\} \in B_h \) incident to \( h \) connects it with its parent \( h'' \in L_{i-1} \), and \( p \in \rho_{i-1}^{-1}(h'') \), due to Observation B.1. Thus, \( F \) can identify the parent of \( h \) and determine \( \sigma_{i-1}(p) \). Similarly, the observation multiset \( o_i(p) \) can be inferred by taking the multiplicities of all red edges of the form \( \{h, h'''\} \in R_h \) (cf. the construction of the red edges in Appendix B.2). Again, for each such \( h''' \in L_{i-1} \), it is possible to determine the state \( \sigma_{i-1}(q) \) of any process \( q \in \rho_{i-1}^{-1}(h''') \). The multiset of these states (with the multiplicities inherited from \( o_i(p) \)) is precisely \( M_i(p) \), i.e., the multiset of states that \( p \) receives at round \( r_i \) (cf. the definition of \( M_i(p) \) in Appendix A.1).

We conclude that \( F \) can compute \( \sigma_i(p) \) by first determining \( \sigma_{i-1}(p) \) and \( M_i(p) \), and then computing \( A(\sigma_{i-1}(p), M_i(p)) \) (cf. the definition of \( \sigma_i(p) \) in Appendix A.1).

**Corollary B.3.** During a computation in an anonymous dynamic network, at the end of round \( r_i \), with \( i \geq 0 \), all processes represented by the same node (in the \( i \)th level) of the history tree have the same state.
We will now describe the algorithm $A^*$ mentioned in Section 2. This algorithm takes as input a process $p$’s history at the end of the previous round $\xi_{i-1}(p)$, as well as the multiset of histories of neighboring processes $M_i(p)$, and constructs the new history $\xi_i(p)$ by merging $\xi_{i-1}(p)$ with all the histories in $M_i(p)$.

Let us give a preliminary definition. A homomorphism from a view $\nu(a) = (H_a, B_a, R_a, l_a) \in V_I$ to a view $\nu(b) = (H_b, B_b, R_b, l_b) \in V_I$ is a function $\varphi : H_a \rightarrow H_b$ that “preserves structure”. That is, for all $\{h, h’\} \in B_a$, we have $\varphi(h), \varphi(h’) \in B_b$; for all $h, h’ \in H_a$, we have $R_a(\{h, h’\}) = R_b(\{\varphi(h), \varphi(h’)\})$; for all $h \in H_a$, we have $l_a(h) = l_b(\varphi(h))$.

The new history $\xi_i(p)$ can be constructed from $\xi_{i-1}(p)$ and $M_i(p)$ as follows. The first step is to identify the viewpoint of $\xi_{i-1}(p)$, which is the (unique) deepest node $h$ in the view; note that $h = \rho_i(p)$. The second step is to “extend” $\xi_{i-1}(p)$ by adding a new node $h'$, with the same label as $h$, and the new black edge $\{h, h’\}$. Let $Z_0 \in V_I$ be the resulting view; note that $h'$ is the (unique) child of $h$ in $Z_0$. Eventually, the node $h'$ will be the viewpoint of $\xi_i(p)$.

Let $\xi_{i-1}(q_1), \xi_{i-1}(q_2), \ldots, \xi_{i-1}(q_k)$ be the histories with positive multiplicity in $M_i(p)$ (note that they must all be histories of processes at round $r_{i-1}$: such are the messages received by $p$ at round $r_i$). The next phase of the algorithm is to construct a sequence of views $Z_0, Z_1, Z_2, \ldots, Z_k \in V_I$ such that $Z_j$ is the smallest view in $V_I$ that contains both $\xi_{i-1}(q_j)$, for all $1 \leq j \leq k$.

In practice, the algorithm constructs $Z_j$ by starting from $Z_{j-1}$ and gradually adding the “missing nodes” from $\xi_{i-1}(q_j)$, at the same time constructing an (injective) homomorphism $\varphi_j$ from $\xi_{i-1}(q_j)$ to $Z_j$. First, the root of $\xi_{i-1}(q_j)$ is mapped by $\varphi_j$ to the root of $Z_{j-1}$. Then, the algorithm scans all the nodes of $\xi_{i-1}(q_j)$ level by level (i.e., doing a breadth-first traversal along the black edges). Let $v$ be a node of $\xi_{i-1}(q_j)$ encountered during the traversal, and let $v'$ be its parent in $\xi_{i-1}(q_j)$. The algorithm attempts to match $v$ with a child $w$ of the node $w' = \varphi_j(v')$ in $Z_{j-1}$. The label of $w$ should be the same as the label of $v$ and, for every red edge $\{v, v''\}$ in $\xi_{i-1}(q_j)$, connecting $v$ with a previous-level node $v''$, the edge $\{w, \varphi_j(v'')\}$ should also appear in $Z_{j-1}$ with the same multiplicity. If a node $w$ with these properties does not exist, the algorithm creates one, and then sets $\varphi_j(v) = w$. When the traversal is over, the resulting structure is $Z_j$, by definition.

Note that the final structure $Z_k$ coincides with $\xi_i(p)$, except for some missing red edges: these are the red edges incident to the viewpoint $h'$ which represent the messages received by $p$ at round $r_i$. Thus, for every $1 \leq j \leq k$, the viewpoint $h_j$ of $\xi_{i-1}(q_j)$ is found (as the unique deepest node in $\xi_{i-1}(q_j)$) and the red edge $\{h', \varphi_j(h_j)\}$ is added to $Z_k$, with the same multiplicity as $\xi_{i-1}(q_j)$ in $M_i(p)$. The resulting history is $\xi_i(p)$.

### C The Subroutine $\text{ApproxCount}$ and Its Correctness

In this section we define the subroutine $\text{ApproxCount}(V, s, x, \ell)$ introduced in Section 4.2 and invoked in Listing 2. It is an adaptation and generalization of the algorithm in Section 4.2 to the case where there is a strand of leader nodes in the view $V$ starting at the first leader node $\tau$ in level $L_i$, where the anonymity $a(\tau)$ is an unknown number not greater than $\ell$ (as opposed to $a(\tau) = 1$, which is assumed in Section 4.2). The algorithm starts by assuming that $a(\tau)$ is the given parameter $x$, and then it makes deductions on the anonymities of other nodes, until it is able to make an estimate $n' > 0$ on the total number of processes, or report failure in the form of an error code $n' \in \{-1, -2, -3\}$. In particular, since the algorithm requires the existence of a long-enough strand of descendants of $\tau$,
it reports failure if some descendants of $\tau$ (in the relevant levels of $\mathcal{V}$) have more than one child. We will now revisit and modify [23 Section 4.2] in order to formally state our new subroutine and prove its correctness and running time. We will conclude this section with a proof of Lemma 4.2.

We remark that \textit{ApproxCount} assumes that the network is 1-interval-(dis)connected, as this is sufficient for the main result of Section 4.2 to hold for any $T$-interval-connected network (see the proof of Theorem 4.3).

\textbf{Discrepancy} $\delta$. Suppose that \textit{ApproxCount} is invoked with arguments $\mathcal{V}, s, x, \ell$, where $1 \leq x \leq \ell$, and let $\tau$ be the first leader node in level $L_s$ of $\mathcal{V}$ (if $\tau$ does not exist, the procedure immediately returns the error code $n' = -1$). We define the \textit{discrepancy} $\delta$ as the ratio $x/a(\tau)$. Clearly, $\delta \leq \ell$. Note that, since $a(\tau)$ is not a-priori known by the process executing \textit{ApproxCount}, then neither is $\delta$.

\textbf{Conditional anonymity.} \textit{ApproxCount} starts by assuming that the anonymity of $\tau$ is $x$, and makes deductions on other anonymities based on this assumption. Thus, we will distinguish between the actual anonymity of a node $a(v)$ and the \textit{conditional anonymity} $a'(v) = \delta a(v)$ that \textit{ApproxCount} may compute under the initial assumption that $a'(\tau) = x = \delta a(\tau)$.

\textbf{Guessing conditional anonymities.} Let $u$ be a node of a history tree, and assume that the conditional anonymities of all its children $u_1, u_2, \ldots, u_k$ have been computed: such a node $u$ is called a \textit{guesser}. If $v$ is not a child of $u$ and the red edge $\{v, u\}$ is present with multiplicity $m \geq 1$,
we say that \( v \) is guessable by \( u \). In this case, we can make a guess \( g(v) \) on the conditional anonymity \( a'(v) \):

\[
g(v) = \left[ \frac{a'(u_1) \cdot m_1 + a'(u_2) \cdot m_2 + \cdots + a'(u_k) \cdot m_k}{m} \right],
\]

where \( m_i \) is the multiplicity of the red edge \( \{u_i, v'\} \) for all \( 1 \leq i \leq k \), and \( v' \) is the parent of \( v \) (possibly, \( m_i = 0 \)). Although a guess may be inaccurate, it never underestimates the conditional anonymity:

**Lemma C.1.** If \( v \) is guessable, then \( g(v) \geq a'(v) \). Moreover, if \( v \) has no siblings, \( g(v) = a'(v) \).

**Proof.** Let \( u, v' \in L_t \) and let \( P_1 \) and \( P_2 \) be the sets of processes represented by \( u \) and \( v' \), respectively. By counting the links between \( P_1 \) and \( P_2 \) in \( G_{t+1} \) in two ways, we have

\[
\sum_i a(u_i) m_i = \sum_i a(v_i) m'_i,
\]

where the two sums range over all children of \( u \) and \( v' \), respectively (note that \( v = v_j \) for some \( j \)), and \( m'_i \) is the multiplicity of the red edge \( \{v_i, u\} \) (so, \( m = m'_j \)). Our lemma now easily follows from the above equation and from the definition of conditional anonymity. \( \square \)

**Heavy nodes.** As the algorithm in [23], also our subroutine ApproxCount assigns guesses in a well-spread fashion, i.e., in such a way that at most one node per level is assigned a guess.

Suppose now that a node \( v \) has been assigned a guess. We define its weight \( w(v) \) as the number of nodes in the subtree hanging from \( v \) that have been assigned a guess (this includes \( v \) itself). Recall that subtrees are determined by black edges only. We say that \( v \) is heavy if \( w(v) \geq g(v) \).

**Lemma C.2.** In a well-spread assignment of guesses, if \( w(v) > a'(v) \), then some descendants of \( v \) are heavy (the descendants of \( v \) are the nodes in the subtree hanging from \( v \) other than \( v \) itself).

**Proof.** Our proof is by well-founded induction on \( w(v) \). Assume for a contradiction that no descendants of \( v \) are heavy. Let \( v_1, v_2, \ldots, v_k \) be the “immediate” descendants of \( v \) that have been assigned guesses. That is, for all \( 1 \leq i \leq k \), no internal nodes of the black path with endpoints \( v \) and \( v_i \) have been assigned guesses (observe that \( k \geq 1 \) because, by assumption, \( w(v) > 1 \)).

By the basic properties of history trees, \( a(v) \geq \sum_i a(v_i) \), and therefore \( a'(v) \geq \sum_i a'(v_i) \). Also, the induction hypothesis implies that \( w(v_i) \leq a'(v_i) \) for all \( 1 \leq i \leq k \), or else one of the \( v_i \)’s would have a heavy descendant. Therefore,

\[
w(v) - 1 = \sum_i w(v_i) \leq \sum_i a'(v_i) \leq a'(v) \leq w(v) - 1.
\]

It follows that \( w(v_i) = a'(v_i) \) and \( a'(v) = \sum_i a'(v_i) \). Hence, \( a(v) = \sum_i a(v_i) \).

Let \( v_d \) be the deepest of the \( v_i \)’s, which is unique, since the assignment of guesses is well spread. Note that \( v_d \) has no siblings at all, otherwise we would have \( a(v) > \sum_i a(v_i) \). Due to Lemma C.1, we conclude that \( g(v_d) = a'(v_d) = w(v_d) \), and so \( v_d \) is heavy. \( \square \)

**Correct guesses.** We say that a node \( v \) has a correct guess if \( v \) has been assigned a guess and \( g(v) = a'(v) \). The next lemma gives a criterion to determine if a guess is correct.

**Lemma C.3.** In a well-spread assignment of guesses, if a node \( v \) is heavy and no descendant of \( v \) is heavy, then \( v \) has a correct guess.
Proof. Because \( v \) is heavy, \( g(v) \leq w(v) \). Since \( v \) has no heavy descendants, Lemma C.2 implies \( w(v) \leq a'(v) \). Also, by Lemma C.1 \( a'(v) \leq g(v) \). We conclude that

\[
g(v) \leq w(v) \leq a'(v) \leq g(v),
\]

and therefore \( g(v) = a'(v) \). \( \square \)

When the criterion in Lemma C.3 applies to a node \( v \), we say that \( v \) has been \emph{counted}. So, counted nodes are nodes that have been assigned a guess, which was then confirmed to be the correct conditional anonymity.

\textbf{Cuts and isles.} Fix a view \( \mathcal{V} \) of a history tree \( \mathcal{H} \). A set of nodes \( C \) in \( \mathcal{V} \) is said to be a \emph{cut} for a node \( v \not\in C \) of \( \mathcal{V} \) if two conditions hold: (i) for every leaf \( v' \) of \( \mathcal{V} \) that lies in the subtree hanging from \( v \), the black path from \( v \) to \( v' \) contains a node of \( C \), and (ii) no proper subset of \( C \) satisfies condition (i). A cut for the root \( \tau \) whose nodes are all counted is said to be a \emph{counting cut}.

Let \( s \) be a counted node in \( \mathcal{V} \), and let \( F \) be a cut for \( v \) whose nodes are all counted. Then, the set of nodes spanned by the black paths from \( s \) to the nodes of \( F \) is called \emph{isle}; \( s \) is the \emph{root} of the isle, while each node in \( F \) is a \emph{leaf} of the isle. The nodes in an isle other than the root and the leaves are called \emph{internal}. An isle is said to be \emph{trivial} if it has no internal nodes.

If \( s \) is an isle’s root and \( F \) is its set of leaves, we have \( a(s) \geq \sum_{v \in F} a(v) \), because \( s \) may have some descendants in the history tree \( \mathcal{H} \) that do not appear in the view \( \mathcal{V} \). This is equivalent to \( a'(s) \geq \sum_{v \in F} a'(v) \). If equality holds, then the isle is said to be \emph{complete}; in this case, we can easily compute the conditional anonymities of all the internal nodes by adding them up starting from the nodes in \( F \) and working our way up to \( s \).

\textbf{Overview of ApproxCount.} Our subroutine \textbf{ApproxCount} is found in Listing 3. It repeatedly assigns guesses to nodes based on known conditional anonymities (starting from \( \tau \) and its descendants). Eventually some nodes become heavy, and the criterion in Lemma C.3 causes the deepest of them to become counted. In turn, counted nodes eventually form isles; the internal nodes of complete isles are marked as counted, which gives rise to more guessers, and so on. In the end, if a counting cut is created, the algorithm checks whether the conditional anonymities of the leader nodes in the cut add up to \( \ell \).

\textbf{Algorithmic details of ApproxCount.} The algorithm \textbf{ApproxCount} uses flags to mark nodes as “guessed” or “counted”; initially, no node is marked. Thanks to these flags, we can check if a node \( u \in \mathcal{V} \) is a guesser: let \( u_1, u_2, \ldots, u_k \) be the children of \( u \) that are also in \( \mathcal{V} \) (recall that a view does not contain all nodes of a history tree); \( u \) is a \emph{guesser} if and only if it is marked as counted, all the \( u_i \)'s are marked as counted, and \( a'(u) = \sum_i a'(u_i) \) (which implies \( a(u) = \sum_i a(u_i) \)).

\textbf{ApproxCount} will ensure that nodes marked as guessed are well-spread at all times; if a level of \( \mathcal{V} \) contains a guessed node, it is said to be \emph{locked}. A level \( L_t \) is \emph{guessable} if it is not locked and has a non-counted node \( v \) that is guessable, i.e., there is a guesser \( u \) in \( L_{t-1} \) and the red edge \( \{v, u\} \) is present in \( \mathcal{V} \) with positive multiplicity.

The algorithm starts by assigning a conditional anonymity \( a'(\tau) = x \) to the first leader node \( \tau \in L_s \). (If no leader node exists in \( L_s \), it immediately returns the error code \(-1\), Line 5.) It also finds the longest strand \( P_\tau \) starting in \( \tau \), assigns the same conditional anonymity \( x \) to all of its nodes, and marks them as counted. Then, as long as there are guessable levels and no counting cut has been found yet, it keeps assigning guesses to non-counted nodes (Line 12). When a guess is made on a node \( v \), some nodes in the path from \( v \) to its ancestor in \( L_s \) may become heavy; if so, the algorithm marks the deepest heavy node \( v' \) as counted (Lines 13–16). Furthermore, if the newly counted node \( v' \) is the root or a leaf of a complete isle \( I \), then the conditional anonymities of all the

25
internal nodes of $I$ are determined, and such nodes are marked as counted; this also unlocks their levels if such nodes were marked as guessed (Lines 17–20).

In the end, the algorithm performs a “reality check” and possibly returns an estimate $n'$ of $n$, as follows. If it has not found a counting cut, it returns the error code $-2$ and the depth of the last node in $P_r$ (Lines 21–23). Otherwise, if a counting cut $C$ has been found, let $L_t$ be the level of its deepest node. The algorithm computes $n'$ (respectively, $\ell'$) as the sum of the conditional anonymities of all nodes (respectively, all leader nodes) in $C$. If $\ell' = \ell$, then $\text{ApproxCount}$ returns $n'$ and the index $t$ (Line 29). Otherwise, it returns the error code $-3$ and the index $t$ (Line 30).

Invariants. We can easily show by induction that $\text{ApproxCount}$ maintains some invariants, i.e., conditions that are satisfied whenever Line 11 is reached. Namely, (i) the nodes marked as guessed are well spread, (ii) there are no heavy nodes (i.e., as soon as a heavy nodes are created, they immediately become non-heavy due to one of them becoming counted), and (iii) all complete isles are trivial (i.e., as soon as a complete isle is created, it is immediately reduced to a set of trivial islands).

Correctness and running time. We will now study the correctness and running time of $\text{ApproxCount}$, culminating with a proof of Lemma 4.2

Lemma C.4. Whenever Line 11 is reached, at most $\delta n$ levels are locked.

Proof. We will prove that, if the subtree hanging from a node $v$ of $V$ contains more than $a'(v)$ guessed nodes, then it contains a guessed node $v'$ such that $w(v') > a'(v')$. The proof is by well-founded induction based on the subtree relation in $V$. If $v$ is guessed, then we can take $v' = v$. Otherwise, by the pigeonhole principle, $v$ has at least one child $u$ whose hanging subtree contains more than $a'(u)$ guessed nodes. Thus, $v'$ is found in this subtree by the induction hypothesis.

Assume for a contradiction that at least $\delta n$ levels of $V$ are locked; hence, $V$ contains more than $\delta n$ guessed nodes. Since the conditional anonymity of the root $r$ of $V$ is $\delta n$, by the above paragraph we know that $V$ contains a guessed node $v'$ such that $w(v') > a'(v')$. Since the algorithm’s invariant (i) holds, we can apply Lemma C.2 to $v'$, which implies that there exist heavy nodes. In turn, this contradicts invariant (ii). We conclude that at most $\delta n$ levels are locked. \hfill $\square$

Lemma C.5. Assume that all the levels of the history tree up to $L_{s+d}$ are entirely contained in the view $V$, and $L_{s+d}$ contains a node in $P_r$. Then, Whenever Line 11 is reached, there are at most $n - 2$ levels in the range from $L_{s+1}$ to $L_{s+d}$ that lack a guessable non-counted node.

Proof. Observe that there are no counting cuts, or Line 11 would not be reachable.

Initially, $\text{ApproxCount}$ makes every node in $P_r$ (except the last one) into a guesser (Line 9). Hence, all levels between $L_s$ and $L_{s+d-1}$ must have a non-empty set of guessers at all times. Consider any level $L_i$ with $s < i \leq s + d$ such that all the guessable nodes in $L_i$ are already counted. Let $S$ be the set of guessers in $L_{i-1}$; note that not all nodes in $L_{i-1}$ are guessers, or else they would give rise to a counting cut. Since the network is 1-interval-connected, there is a red edge $\{u, v\}$ (with positive multiplicity) such that $u \in S$ and the parent of $v$ is not in $S$. By definition, the node $v$ is guessable; therefore, it is counted. Also, since the parent of $v$ is not a guesser, $v$ must have a non-counted parent or a non-counted sibling; note that such a non-counted node is in $V$.

We have proved that every level between $L_{s+1}$ and $L_{s+d}$ lacking a guessable non-counted node contains a counted node $v$ having a parent or a sibling that is not counted: we call such a node $v$ a bad node. To conclude the proof, it suffices to show that there are at most $n - 2$ bad nodes up to level $L_{s+d}$.

We will prove by induction that, if a subtree $W$ of $V$ contains the root $r$, no counting cuts, and no non-trivial isles, then $W$ contains at most $f - 1$ bad nodes, where $f$ is the number of leaves of
which is a counted node. Since there can be no counted nodes after level \( L \), if \( s = \tau(t) \), then \( t \) is the level of the deepest node in a counting cut, and all of them are completely contained in \( V \). Therefore, \( W' \) has exactly one less bad node than \( W \) and at least one less leaf; the induction hypothesis now implies that \( W \) contains at most \( f - 1 \) bad nodes.

Observe that the subtree \( V' \) of \( V \) formed by all levels up to \( L_{s+d} \) satisfies all of the above conditions, as it contains the root \( r \) and has no counting cuts, because a counting cut for \( V' \) would be a counting cut for \( V \), as well. Also, invariant (iii) ensures that \( V' \) contains no non-trivial complete isles. However, since the levels up to \( L_{s+d} \) are contained in \( V' \), all isles in \( V' \) are complete, and thus must be trivial. We conclude that, if \( V' \) has \( f \) leaves not in \( P_T \), it contains at most \( f - 1 \) bad nodes.

Since the leaves of \( V \) not in \( P_T \) induce a partition of the at most \( n - 1 \) processes not represented by \( \tau \), we have \( f \leq n - 1 \), implying that the number of bad nodes up to \( L_{s+d} \) is at most \( n - 2 \).

We are now ready to prove Lemma 4.2.

**Lemma C.6.** Let \( \text{ApproxCount}(V, s, x, \ell) \) return \((n', t')\), and let \( L_{t'} \) be the last level of \( V \). Then:

(i) If \( t' \geq s + (\ell + 2)n - 1 \), then \( s \leq t \leq s + (\ell + 1)n - 1 \).

(ii) If \( x = a(\tau) \) and \( t' \geq t + n \), then \( n' \neq -3 \).

(iii) If \( x \geq a(\tau) \) and \( n' > 0 \) and \( t' \geq t + n' \), then \( n' = n \).

**Proof.** If \( L_{t'} \) is the last level of \( V \), then \( t' \) is the current round. By [23] Corollary 4.3], all levels of the history tree up to \( L_{t' - n} \) are completely contained in \( V \).

Let us prove statement (i). If \( \text{ApproxCount} \) returns the error code \(-1\), then \( t = s \), and our claim holds. Otherwise, let \( L_{t''} \) be the level of the last node in \( P_T \). It is immediate to prove by induction on the number of counted nodes in \( V \) that there can never be any guesser after level \( L_{t''} \). Indeed, this is true at the beginning, because the initial guessers are all in \( P_T \) (Line 9). After that, if there are no guessers after level \( L_{t''} \), then no subsequent guesses can be performed on nodes after level \( L_{t'' + 1} \), and thus no node after level \( L_{t'' + 1} \) can become heavy and then counted. Since a guesser is a counted node with counted children, it follows that there will never be any guesser after level \( L_{t''} \).

Let us first assume that \( t'' \leq s + (\ell + 1)n - 2 \). If \( \text{ApproxCount} \) returns the error code \(-2\), then \( t = t'' \), and our claim holds. In all other cases, \( L_t \) is the level of the deepest node in a counting cut, which is a counted node. Since there can be no counted nodes after level \( L_{t'' + 1} \), we have once again \( t \leq s + (\ell + 1)n - 1 \), as desired.

Assume now that \( t'' \geq s + (\ell + 1)n - 1 \). Since

\[
t' - n \geq s + (\ell + 1)n - 1 \geq s + (\delta + 1) - 1,
\]

all levels of the history tree from \( L_s \) to \( L_{s+(\delta+1)n-1} \) are completely contained in \( V \), and all of them are spanned by \( P_T \). Thus, [Lemma C.5] applies with \( d = (\delta + 1)n - 1 \). Due to Lemmas C.4 and C.5, as long as a counting cut has not been created, at least one level in the interval from \( L_{s+1} \) to \( L_{s+(\delta+1)n-1} \) is guessable. Indeed, there are \((\delta + 1)n - 1\) levels in this interval, of which at most
$\delta n$ are locked and at most $n - 2$ lack a guessable non-counted node. Hence, a counting cut $C$ is eventually found, and its deepest node is in $L_t$, with

$$t \leq s + (\delta + 1)n - 1 \leq s + (\ell + 1)n - 1.$$  

As this is the $t$ returned by $\text{ApproxCount}$, our claim is proved.

Let us prove statement (ii). Assume that $x = a(\tau)$ (which implies that $\delta = 1$) and $\text{ApproxCount}$ returns $(n', t)$ with $t' \geq t + n$. Then, there are no missing nodes in the levels of $\mathcal{V}$ between $L_s$ and $L_t$. Assume for a contradiction that $\text{ApproxCount}$ returns the error code $-3$. This implies that a counting cut $C$ was found between $L_s$ and $L_t$. Hence, the nodes in $C$ encompass all $n$ processes; moreover, since $\delta = 1$, all conditional anonymities are equal to the actual anonymities. It follows that $n = n'$ and $\ell = \ell'$, where $n'$ (respectively, $\ell'$) is the sum of the conditional anonymities of all nodes (respectively, all leader nodes) in $C$. Thus, the “reality check” of Line 29 succeeds, and $\text{ApproxCount}$ must return $n' = n > 0$, a contradiction. Therefore, the error code $-3$ cannot be returned, as claimed.

Let us prove statement (iii). Assume that $x \geq a(\tau)$ (hence $\delta \geq 1$) and $\text{ApproxCount}$ returns $(n', t)$ with $n' > 0$ and $t' \geq t + n'$. Since $n' > 0$, a counting cut $C$ was found whose deepest node is in $L_t$, and $n'$ is the sum of the conditional anonymities of all nodes in $C$. Let $S_C$ be the set of processes represented by the nodes of $C$; note that $n' \geq |S_C|$, because $\delta \geq 1$. We will prove that $S_C$ includes all processes in the system. Assume the contrary; [23] Lemma 4.2 implies that, since $t' \geq t + n' \geq t + |S_C|$, there is a node $z \in L_t$ representing some process not in $S_C$. Thus, the black path from $z$ to the root $r$ does not contain any node of $C$, contradicting the fact that $C$ is a counting cut whose deepest node is in $L_t$. Therefore, $|S_C| = n$, i.e., the nodes in $C$ represent all processes in the system. Since $\text{ApproxCount}$ returns $n' > 0$, the “reality check” of Line 29, $\ell' = \ell$, succeeds. However, $\ell'$ is the sum of the conditional anonymities of all leader nodes in $C$, and hence $\ell' = \delta \ell$, implying that $\delta = 1$. Thus, $n' = \delta n = n$, as claimed.

D  Survey of Related Work

We examine related work on Counting and Average Consensus by first discussing the case of dynamic networks with unique IDs, then the case of static anonymous networks, and finally the case of dynamic anonymous networks.

D.1 Dynamic Networks with IDs

The problem of counting the size of a dynamic network has been first studied by the peer-to-peer systems community [36]. In this case having an exact count of the network at a given time is impossible, as processes may join or leave in an unrestricted way. Therefore, their algorithms mainly focus on providing estimates on the network size with some guarantees. The most related is the work that introduced 1-interval-connected networks [33]. They show a counting algorithm that terminates in at most $n + 1$ rounds when messages are unrestricted and in $O(n^2)$ rounds when the message size is $O(\log n)$ bits. The techniques used heavily rely on the presence of unique IDs and cannot be extended to our settings.

D.2 Anonymous Static Networks

The study of computability on anonymous networks has been pioneered by Angluin in [1] and it has been a fruitful research topics for the last 30 years [1, 7, 12, 13, 14, 25, 46, 49]. A key concept
in anonymous networks is the symmetry of the system; informally, it is the indistinguishability of processes that have the same view of the network. As an example, in an anonymous static ring topology, all processes will have the exact same view of the system, and such a view does not change between rings of different size. Therefore, non-trivial computations including counting are impossible on rings, and some symmetry-breaking assumption is needed (such as a leader [25]). The situation changes if we consider topologies that are asymmetric. As an example, on a wheel graph the central process has a view that is unique, and this allows for the election of a leader and the possibility, among other tasks, of counting the size of the network.

Several tools have been developed to characterize what can be computed on a given network topology (examples are views [49] or fibrations [8]). Unfortunately, these techniques are usable only in the static case and are not defined for highly dynamic systems like the ones studied in our work. Regarding the counting problem in anonymous static networks with a leader, [37] gives a counting algorithm that terminates in at most $2n$ rounds.

### D.3 Counting in Anonymous Interval-Connected Networks

The papers that studied counting in anonymous dynamic networks can be divided into two periods. A first series of works [11, 19, 20, 37] gave solutions for the counting problem assuming some initial knowledge on the possible degree of a processes. As a matter of fact [37] conjectured that some kind of knowledge was necessary to have a terminating counting algorithm. A second series of works [18, 23, 26, 28, 29, 31, 30] has first shown that counting was possible without such knowledge, and then has proposed increasingly faster solutions, culminating with the linear time asymptotically optimal solution of [23]. We remark that all these papers assume that a leader (or multiple leaders in [29]) is present. This assumption is needed to deterministically break the system’s symmetry.

**Counting with knowledge on the degrees.** Counting in interval-connected anonymous networks was first studied in [37], where it is observed that a leader is necessary to solve counting in static (and therefore also dynamic) anonymous networks (this result can be derived from previous works on static networks such as [8, 49]). The paper does not give a counting algorithm but it gives an algorithm that is able to compute an upper bound on the network size. Specifically, [37] proposes an algorithm that, using an upper bound $d$ on the maximum degree that each process will ever have in the network, calculates an upper bound $U$ on the size of the network; this upper bound may be exponential in the actual network size ($U \leq d^n$).

Assuming the knowledge of an upper bound on the degree, [19] given a counting algorithm that computes $n$. Such an algorithm is really costly in terms of rounds; it has been shown in [11] to be doubly exponential in the network size. The algorithm proposes a mass distribution approach akin to local averaging [38].

An experimental evaluation of the algorithm in [19] can be found in [21]. The result of [19] has been improved in [11], where, again assuming knowledge of an upper bound $d$ on the maximum degree of a process, an algorithm is given that terminates in $O\left(\frac{n(2d)^n+1\log n}{\log d}\right)$ rounds. A later paper [20] has shown that counting is possible when each process knows its degree before starting the round (for example, by means of an oracle). In this case, no prior global upper bound on the degree of processes is needed. [20] only show that the algorithm eventually terminates but does not bound the termination time.

We remark that all the above works assume some knowledge on the dynamic network, as an upper bound on the possible degrees, or as a local oracle. Moreover, all of these works give exponential-time algorithms.
Counting without knowledge on the degrees. The first work proposing an algorithm that
does not require any knowledge of the network was [18]. The paper proposed an exponential-time
algorithm that terminates in $O(n^4)$ rounds. Moreover, it also gives an asymptotically optimal
algorithm for a particular category of networks (called persistent-distance). In this type of network,
a process never changes its distance from the leader.

This result was improved in [26, 28], which presented a polynomial-time counting algorithm. The
paper proposes Methodical Counting, an algorithm that counts in $O(n^5 \log^2(n))$ rounds. Similar
to [19, 20], the paper uses a mass-distribution process that is coupled with a refined analysis of
collapse convergence and clever techniques to detect termination. The paper also notes that, using the
same algorithm, all algebraic and boolean functions that depend on the initial number of processes
in a given state can be computed. The authors of [26, 28] extended their result to work in networks
where $\ell \geq 1$ leaders are present (with $\ell$ known in advance) in [31]. They create an algorithm
that terminates in $O\left(\frac{n^{4+\epsilon}}{\ell^3} \log^3(n)\right)$ rounds, for any $\epsilon > 0$. In particular, when $\ell = 1$, this results
improves the running time of [26, 28].

Finally, in [29], they show a counting algorithm parameterized by the isoperimetric number of
the dynamic network. The technique used is similar to [26, 28], and it uses the knowledge of the
isoperimetric number to shorten the termination time. Specifically, for adversarial graphs (i.e., with
non-random topology) with $\ell$ leaders ($\ell$ is assumed to be known in advance), they give an algorithm
terminating in $O\left(\frac{n^{3+\epsilon}}{i_{\text{min}}} \log^2(n)\right)$ rounds, where $i_{\text{min}}$ is a known lower bound on the isoperimetric
number of the network. This improves the work in [31], but only in graphs where $i_{\text{min}}$ is $\omega(1/\sqrt{n})$.
The authors also study various types of graphs with stochastic dynamism; we remark that in this
case they always obtain superlinear results, as well. The best case is that of Erdős–Rényi, graphs
where their algorithm terminates in $O\left(\frac{n^{1+\epsilon}}{p_{\text{min}}} \log^5(n)\right)$ rounds; here $p_{\text{min}}$ is the smallest among the
probabilities of creating an arc on all rounds. Specifically, if $p_{\text{min}} = O(1/n)$, their algorithm is at
least cubic.

A recent breakthrough has been shown in [23], which proposed the novel technique of history
trees. A history tree is a combinatorial structure that models the entire evolution of an anonymous
dynamic graph. By developing a theory of history trees for dynamic networks with a unique leader,
the authors have shown a terminating solution for the Generalized Counting problem in $3n - 3$ rounds. The authors have shown that the Generalized Counting problem is complete for the class of problems solvable in general dynamic networks, proving that computing in anonymous dynamic
networks with a unique leader is linear. The authors have also given a stabilizing, non-terminating,
algorithm for Generalized Counting that stabilizes in roughly $2n$ rounds, providing an almost
matching lower bound (we will discuss lower bounds below). We remark that the results of [23] do
not apply to the leaderless or the multi-leader cases.

All the above works assume the dynamic network to be connected at each round.

The only works that studied counting in disconnected networks is the recent pre-print [30]. The
paper proposes an algorithm that solves the Generalized Counting in $\tilde{O}\left((R + \frac{n}{\ell})^{2T(1+\epsilon)}\right)$ rounds, where $T$ is the dynamic disconnectivity, $\ell$ is the number of leaders, $i_{\text{min}}$ is the isoperimetric number of
the network, and $R$ is the cumulative bit length of all inputs. In unknown networks, assuming
inputs of small size, the number of rounds is roughly $\tilde{O}\left(\frac{n^{2T(1+\epsilon)+3}}{\ell}\right)$. Interestingly, the algorithm
requires messages of size $O(\log n)$ bits. We highlight that the algorithm has a running time of

\footnote{In the Generalized Counting problem, each process starts with a certain input, and the problem is solved when each process has computed the multiset of these inputs.}

\footnote{A function is $\tilde{O}(f(n))$ if it is $O(f(n)g(n))$ for some polylogarithmic function $g(n)$.}
roughly $\tilde{O}\left(\frac{n^5}{\tau}\right)$ in dynamic networks that are connected at each round, and the complexity grows exponentially with the dynamic disconnectivity $T$.

Summarizing, to the best of our knowledge, there exists no worst-case cubic-time algorithm for the Generalized Counting with multiple leader and no algorithm that scales linearly with the dynamic disconnectivity and the size of the network.

**Lower bounds on counting.** From [33], a trivial lower bound of $n - 1$ rounds can be derived, as counting obviously requires information from each process to be spread in the network. The first non-trivial lower bound for general dynamic networks has been given in the recent work [28]: any algorithm that stabilizes on the correct count requires $2n - 6$ rounds. We remark that such a lower bound also holds for terminating algorithms. Another interesting lower bound is in [18], which shows a specific category of anonymous dynamic networks with constant temporal diameter (the time needed to spread information from a process to all others is at most 3 rounds), but where counting requires $\Omega(\log n)$ rounds.

**D.4 Average Consensus**

In the Average Consensus problem, each process $v_i$ starts with an input value $x_i(0)$, and the goal is to compute the average of these initial values. This problem has been studied for decades in the communities of distributed control and distributed computing [6, 15, 16, 17, 29, 39, 42, 44, 48, 51]. In the following, we give an overview that places our result in the current body of knowledge. A more detailed picture can be found in the surveys [24, 40, 43]. We can divide current papers between the ones that give convergent solutions to Average Consensus and the ones that give finite-time solutions.

**D.5 Convergent Average Consensus**

In convergent Average Consensus algorithms, the consensus is not reached in finite time, but each process has a local value that asymptotically converges to the average. A prototypical family of solutions [6, 48] is based on the so-called convex combination algorithms, where each process updates its local value $x_i(r)$ at every round $r$ as follows:

$$x_i(r) = \sum_{\forall v_j \in N(r, v_i) \cup \{v_i\}} a_{ij}(r) \cdot x_j(r - 1).$$

The value $a_{ij}(r)$ is taken from a weight matrix that models a dynamic graph. We remark that convex combination algorithms do not need unique IDs, and thus work in anonymous networks.

The $\epsilon$-convergence of an algorithm is defined as the time it takes to be sure that the maximum discrepancy between the local value of a process and the mean is at most $\epsilon$ times the initial discrepancy. That is, if the mean is $m = \sum_{i \in V} x_i(0)/|V|$, the following should hold:

$$\max_i \{|x_i(r) - m|\} \leq \epsilon.$$  

The local averaging approach has been studied in depth, and several upper and lower bounds for $\epsilon$-convergence are known for both static and dynamic networks [43, 44]. The procedure $\epsilon$-converges in $O(Tn^3 \log(\frac{1}{\epsilon}))$ rounds if, at every round, the weight matrix $A(r)$ such that $(A(r))_{ij} = a_{ij}(r)$ is doubly stochastic (i.e., the sum of the values on rows and columns is 1) and the dynamic graph is $T$-connected [39]. In a dynamic network, it is possible to have doubly stochastic weight matrices when an upper bound on the processes’ degrees is known [17]. Without such knowledge, if the
dynamic graph is always connected and stable (i.e., it changes every two rounds), then it is possible to implement a Metropolis weights strategy that converges in $O\left(n^2 \log\left(\frac{n}{\epsilon}\right)\right)$ rounds [39]. Unfortunately, these results do not apply to our setting, as we do not assume previous knowledge, and our dynamic graphs are not guaranteed to be stable. The only paper that assumes a similar setting to ours is [16], but it restricts the dynamic graph to be 1-connected. The paper shows an algorithm that uses MaxMetropolis weights and converges in $O\left(n^4 \log\left(\frac{n}{\epsilon}\right)\right)$ rounds.

We remark that all the above works just converge to the average of the inputs, and they do not stabilize to such an average in finite time.

### D.6 Finite-Time Average Consensus

The algorithms in the second class solve the finite-time Average Consensus; in this case, the value stabilizes to the actual average in a finite number of rounds. The majority of the literature on finite-time Average Consensus has considered static networks [24, 42, 51]. In such a setting, algorithms that stabilize in a linear number of rounds are known [51]. Few works considered anonymous dynamic networks; [39] describes an algorithm that stabilizes in $O(n^2)$ rounds and requires the network to change every three rounds, while in [13] a randomized Monte Carlo linear algorithm is given. An interesting take is given in [29], which investigates terminating Average Consensus algorithms for adversarial dynamic graphs and random dynamic graphs (i.e., Watts–Strogatz, Barabási–Albert, RGG, and Erdős–Rényi–Gilbert graphs). The algorithm for the adversarial case has a time complexity of $O\left(\frac{n^2}{\ell} \log^3(n)\right)$ rounds, where $\ell$ is the (known) number of leaders in the system. For random dynamic graphs, the complexity of the algorithm changes according to the model, but all the algorithms presented are super-linear and terminate require the knowledge of the number of leaders to terminate.

To the best of our knowledge, there is no deterministic solution to Average Consensus that stabilizes, or even converges, in a linear number of rounds in unknown dynamic networks.

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