On the Supersymmetric Index of the M-theory 5-brane and Little String Theory

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Abstract: We propose a six-dimensional framework to calculate the supersymmetric index of M-theory 5-branes wrapped on a six-manifold with product topology $M_4 \times T^2$, where $M_4$ is a holomorphic 4-cycle in a Calabi-Yau three-fold. This is obtained by zero-modes counting of the self-dual tensor contribution plus “little” string states and correctly reproduces the known results which can be obtained by shrinking or blowing the $T^2$ volume parameter. We also extract the geometric moduli space of the multi M5-brane system and infer the generic structure of the supersymmetric index for more general geometries.
1 Introduction

A proper definition of M-theory as a non perturbative framework for superstring theories is still an open problem. It is strongly shaded among the others by the lack of understanding the very structure of the world-volume theory of the M5-branes.

As far as this sector of the theory is concerned, its totally decoupled phase has been named little string theory [1–3]. Little string theory (for a review, see also [4,5] and references therein) is still anyway poorly understood. Its low energy limit is accepted to be a theory of (0, 2) self-dual tensor multiplets which lacks both locality and a Lagrangean formulation and can be therefore studied by now just with constructive methods. The UV complete theory has been conjectured [1] to be a closed string theory in 6 dimensions describing the boundary states of the membranes ending on the 5-branes. This strong characterization of M-theory has passed several consistency checks [1, 3, 4, 6]. Specifically, little string theory has been extensively studied on tori and orbifold K3, while much less is known about it in more general cases. Anyway it turns out [7] that the appropriate nature of the degrees of freedom which complete in the UV the interacting theory is stringy and this further enforces the above proposal.

The picture that is coming out and that we have in mind is the following. Suppose one is dealing with a bounce of M5-branes. In the low energy approximation, if they can be separated neatly one from each other, each brane hosts a self-dual tensor multiplet theory describing its effective degrees of freedom. They come as zero-modes of membranes ending on them. Suppose now that we take two M5-branes close-by. In this case, at some distance, the effective theory is expected to develop interacting terms corresponding to an analog of gauge symmetry enhancement $U(1) \times U(1) \rightarrow U(2)$, but it turns out that these kind of structure – i.e. a higher rank generalization of non-abelian gauge symmetry – does not exists in a local sense. Notice that this fact is natural because the boundary of the membranes stretched between the M5-branes are stringy objects while a local field theory description would describe particle vertices. This situation has to be compared with the other situation in which D-branes come together: in this case the boundary of the strings stretched between the D-branes is composed of point-like objects whose field theory description turns out to be given by the non abelian gauge degrees of freedom via the usual Chan-Paton construction. The outcome of this analysis (see the cited literature for more complete treatment) is that the effective interacting theory of M5-branes has to be described as a string theory in six dimensions. The aim of this paper is to check this picture with a supersymmetry index.
calculation. This means that we will start from a known result for this object (in a particular geometrical set-up) which has been already obtained by other methods and we will recalculate it from a six-dimensional string theory point of view. This will be done within an on-shell model. The off-shell formulation of the six dimensional string theory is out of reach in the present paper and an open problem.

In this paper, in fact, we use the analysis performed in [8] by proposing a six dimensional framework for the BPS state counting of the M5-brane which correctly reproduces the results for the supersymmetric index which have been obtained there in a dimensionally reduced framework. This interpretation gives also some hints on the structure of the moduli space of supersymmetry preserving solutions of the M5-brane world volume theory which we check to be compatible with non-perturbative superstring dualities results. In the framework that we propose, the complete counting of these configurations amounts of two sectors which are the set of fluxes of the self-dual tensor multiplet plus certain little string BPS saturated configurations.

This paper is organized as follows. In the next section we will briefly review the results obtained in [8] for the supersymmetric index of certain M5-branes configurations in a specific geometrical setup. In the subsequent section we will explore a six-dimensional point of view about the M5-brane supersymmetric index which correctly reproduces the previous results by a combined argument coming from the analysis of the self-dual 2-form potential as counted in [12,13] and the little strings to which we referred above. In the next section we discuss some points about the multi 5-brane result, we completely determine the structure of the geometric moduli space of the multi five-brane bound states and infer the generic structure of the supersymmetric index for more general geometries. This is done by combining together the informations encoded in the supersymmetric index formula, superstring dualities and results about the structure of (0,2) theories. As an important by-product we check the appearance of extra massless string states corresponding to the interaction between M5-branes as they approach each others. A final section is dedicated to the discussion about open questions.

2 The M5-branes index on \(T^2 \times M_4\)

The geometric set-up that we refer to is the following [8,14]. We consider M-theory on \(W = Y_6 \times T^2 \times R^3\), where \(Y_6\) is a Calabi-Yau threefold of general holonomy. Let \(M_4\) be a supersymmetric simply connected four-cycle in \(Y_6\) which we take to be a representative
of a very ample divisor. Notice that $M_4$ is automatically equipped with a Kaehler form $\omega$ induced from $Y_6$ and is simply connected. We consider then $N$ M5-branes wrapped around $C = T^2 \times M_4$.

Very few is known about the full world-volume theory describing a bunch of parallel M5-branes, but from what we believe to be true in M-theory, we can extract already some informations about it. It can be shown that in this specific geometrical set-up the potential anomalies which tend to ruin gauge invariance of the world-volume theory are absent and that it is then meaningful to define a supersymmetric index for the above 5-branes bound states by extending the approach in $[8]$. As it is well known, the supersymmetric index is independent on smooth continue parameters and as a consequence we have that this counting of supersymmetry preserving states have to coincide in the large and small $T^2$ volume. In $[8]$ this calculation was performed and this equivalence was shown to be effective. In particular, in the large $T^2$ volume we calculated the supersymmetric index of the relevant two dimensional $\sigma$-model with target the moduli space of susy-preserving configurations of the corresponding twisted $\mathcal{N} = 4$ SYM theory on $M_4$. This was shown to consist of the Hilbert scheme of holomorphic coverings of $M_4$ in $Y_6$. On each stratum, characterized by the total rank of the covering $N$ and by the topological numbers of the associated spectral surface $\Sigma$, we calculate the supersymmetric index as

$$\mathcal{E} = \frac{(\text{Im}\tau)^{d/2}}{V_d} \text{Tr}_{RR} \left[ (-1)^F F_R^{\sigma/2} q^{L_0} \bar{q}^{L_0} \right],$$

where $d$ is the number of non-compact scalar bosons, $V_d$ their zero-mode volume and $(0, \sigma)$ are the supersymmetries of the model. By general arguments, $\mathcal{E}$ is a $(-d/2, -d/2) + (0, \sigma/2)$ modular form.

On the other side of the equivalence, our explicit calculation, done by making use of the lifting technique $[10, 23]$ (see also $[11]$), for the 4 dimensional gauge theory gave, for the generic irreducible sector relative to the partition $N = \sum_a n_a \cdot a$ and to a given irreducible holomorphic covering of $M_4$ in $Y_6$ of rank $a$,

$$\mathcal{E}_{n_a,a} = H_{n_a} \sum_{\varepsilon} \frac{\theta_{\Lambda^\Sigma_a + x}}{\eta_{\Lambda^\Sigma_a}}$$

where $H_n$ is the Hecke operator of order $n$, $\varepsilon$ is a label for the square-roots of the canonical line bundle (spin structures) on $M_4$ with respect to a given one as $\mathcal{O}_\varepsilon \otimes K^{1/2}$ with $\mathcal{O}_\varepsilon^2 = 1$, $x = [\mathcal{O}_\varepsilon^\otimes a + 1]$ shifts correspondingly the lattice of integer periods $\Lambda^\Sigma_a$ on $H^2(\Sigma_a, R)$. The
\( \theta \)-function on the lattice \( \Lambda \) is defined as

\[
\theta_\Lambda(q, \bar{q}) = \sum_{m \in \Lambda} q^{\frac{1}{4}(m, * m - m)} \bar{q}^{\frac{1}{4}(m, * m + m)}
\]  

(2.3)

and is a modular form of weight \((b_-/2, b_+/2)\). \( \eta(q) \) is the Dedekind \( \eta \)-function and \( \chi = 2 + b_2 \Sigma_a \) is the Euler number of the spectral surface \( \Sigma_a \). In particular (2.2) for the single 5-brane reads

\[
\mathcal{E} = \frac{\theta_\Lambda(q, \bar{q})}{\eta^\chi}
\]  

(2.4)

which is a modular form of weight \((-b_-/2, b_+/2) + (-\chi/2, 0) = (-1 - b_+/2, b_+/2)\). All this agrees with the result obtained by the explicit evaluation of the supersymmetric index (2.1). In particular, we have \( \sigma = 2b_+ + 2 = 4b^{(2,0)} + 4 \) right fermions and \( d = 3 + 2b^{(2,0)} = 2 + b_+ \) non-compact scalar bosons by dimensional reduction giving \( \mathcal{E} \) to be a modular form of total weight \((-1 - b_+/2, b_+/2)\) as we just obtained.

3 The single M5-brane case

3.1 The low energy contribution

Let us start with the single M5-brane case. The bosonic spectrum of the low energy world-volume theory of this 5-brane is given by a 2-form \( V \) with self-dual curvature and five real bosons taking values in the normal bundle \( N_C \) induced by the structure of the embedding as \( T_W|_C = T_C \oplus N_C \). Passing to the holomorphic part and to the determinants and using the properties of \( Y_6 \), it follows that the five transverse bosons are respectively, three non-compact real scalars \( \phi_i \) and one complex section \( \Phi \) of \( K_{M_4} = \Lambda^{-2}T^{(1,0)}_{M_4} \), which is the canonical line bundle of \( M_4 \).

The (partially) twisted chiral \((0, 2)\) supersymmetry completes the spectrum. It is given by a doublet of complex anti-commuting fields which are \((2, 0)\)-forms in six dimensions and a doublet of complex anti-commuting fields which are scalars in six dimensions. Notice that these fermionic content reduces to the two relevant fermionic spectra in the large and small \( T^2 \) volume limits respectively.

In principle there would be two multiplicative contributions to the supersymmetric index. One given by zero modes counting and another given by the one loop determinants. In our case, anyhow, we are dealing with a (partially) twisted version of the \((0, 2)\) supersymmetric theory which we expect to be of a topological type (see for example \[15\] for the link between
twisting and topological six dimensional QFTs). This suggests that the oscillatory contribution to the index is 1 and we will take this point of view. Actually, by adapting the results in [17] to the relevant quadratic lagrangean of the type considered in [15], it can be checked to happen if $M_4$ is a $T^4$-orbifold [1]. Therefore, in the subsequent analysis we will consider the zero-modes contribution.

The contribution to the partition function coming from a single self-dual tensor can be analyzed following the results of [12, 13]. It consists of a $\theta$-function of the lattice of the self-dual harmonic three forms. The $\theta$-function is not completely specified because of the possible inequivalent choices of its characteristics $[\alpha \beta]$. Notice that the technique developed in [13] does not in fact single out a particular value for the characteristic as an ambiguity in the choice of the relevant holomorphic factor. We will find that, in the case at hand, a simple choice is automatically made by the requirement of reconstructing the supersymmetric index that we have reviewed in the previous section.

The relevant $\theta$-function (as calculated in [13]) is

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (Z^0|0) = \sum_k e^{i\pi(k+\alpha)Z^0(k+\alpha)+2(k+\alpha)\beta},$$

where $Z^0$ is a period matrix of the relevant six-manifold cohomology that we specify in the following. Let $\{E^{(6)}, \tilde{E}^{(6)}\}$ be a symplectic basis of harmonic 3-forms on the six-manifold at hand such that, in matrix notation,

$$\int E^{(6)} E^{(6)} = 0, \quad \int \tilde{E}^{(6)} E^{(6)} = 1, \quad \int \tilde{E}^{(6)} \tilde{E}^{(6)} = 0.$$

We can expand $\tilde{E}^{(6)} = X^0 E^{(6)} + Y^{0*} E^{(6)}$, where $*$ is the Hodge operator. Then $Z^0$ is defined as $Z^0 = X^0 + iY^0$.

In our case the world volume is in the product form $\Sigma \times M_4$, where $\Sigma = T^2$, and therefore, being $M_4$ simply connected, we have

$$H^3(\Sigma \times M_4) = H^1(\Sigma) \otimes H^2(M_4) \quad (3.5)$$

This means that we can expand $\{E^{(6)}, \tilde{E}^{(6)}\}$ in terms of a symplectic basis $\{[a], [b]\}$ for $H_1(\Sigma)$, where

$$\int_\Sigma [a][a] = 0, \quad \int_\Sigma [a][b] = 1, \quad \int_\Sigma [b][b] = 0,$$

1Notice that this result should extend to a generic simply connected Kaehler $M_4$ by an extension of the methods worked out in [13, 17].
and an orthonormal basis \( \{ e^{(4)} \} \) for \( H_2(M_4) \), i.e. \( \int_{M_4} * e^{(4)} e^{(4)} = 1 \). In terms of the previous objects we have 

\[
E^{(6)} = e^{(4)} \otimes [b] \quad \text{and} \quad \tilde{E}^{(6)} = Q e^{(4)} \otimes [a]
\]

where \( Q \) is the intersection matrix on \( M_4 \) given by \( Q = \int_{M_4} e^{(4)} e^{(4)} \). We calculate \( *E^{(6)} = -Q e^{(4)} \otimes *[b] \), where \(*[b] \) is in the two dimensional sense. By recalling the relation

\[
[a] = -\Omega^{(1)} [b] - \Omega^{(2)} *[b]
\]

which holds on every Riemann surface [16] with period matrix \( \Omega = \Omega^{(1)} + i\Omega^{(2)} \) and the property \( Q^2 = 1 \), we get

\[
\tilde{E}^{(6)} = \left( -Q \otimes \Omega^{(1)} \right) E^{(6)} + \left( 1 \otimes \Omega^{(2)} \right) * E^{(6)}.
\]

Comparing with the general definition we finally read

\[
Z^0 = -Q \otimes \Omega^{(1)} + i1 \otimes \Omega^{(2)}.
\]

In our specific case, since \( \Sigma = T^2 \) and \( \Omega = \tau = \tau^{(1)} + i\tau^{(2)} \) is the modulus of the torus, we have simply that

\[
Z^0 = -\tau^{(1)} Q + i\tau^{(2)} 1.
\]

Now we calculate the relevant \( \theta \)-function from the zero modes of the self-dual form in six dimension choosing the zero characteristic candidate

\[
\Theta(Z^0) = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (Z^0 | 0) = \sum_k e^{i\pi k Z^0 k}
\]

Defining \( m = k \cdot e^{(4)} \), we calculate \( k Z^0 k = -\tau^{(1)} (m, m) + i\tau^{(2)} (m, *m) \) and we rewrite

\[
\Theta(Z^0) = \sum_{m \in \Lambda} q^{1/4 (m, *m-m)} q^{-1/4 (m, *m+m)}
\]

where \( q = e^{2i\pi \tau} \), which is equal to (2.3) as it has been calculated from the reduced dimensional perspectives.

Let us notice that the choice of the null characteristics candidate coincides with that of [17] where it has been shown, in the contest of calculating the self-dual tensor partition function on \( T^6 \), to be the only possible contribution leading to a fully modular invariant result.
3.2 The Little Strings contribution

To count the full spectrum of the theory a second sector is still lacking. In fact, the 5-brane theory is completed in the UV by the little string theory which has BPS saturates strings which eventually have to be kept into account in the calculation of the complete supersymmetric index. Even if a full off-shell model for this six-dimensional string theory is not available at the moment, we will propose an on-shell simple calculation scheme for the supersymmetric index.

The little string theory model for the world volume theory of the M5-brane is built by identifying the boundary states of the membranes ending of the 5-branes with closed strings configurations whose target space is the 5-brane world volume itself. These strings are naturally coupled to the self-dual tensor as the Poincare’ dual of their world-sheet acts as a source for it. This in fact guarantees the gauge invariance of the effective action for the 5-brane/membrane system. To calculate the contribution to the supersymmetric index we can proceed as follows. As it is given by a trace on the string Hilbert space, it corresponds to a one-loop string path integral. Moreover, as it is usual in these index calculations, the semiclassical approximation is exact. Now, since $M_4$ is simply connected, the only contributions to this path integral can arise from string worldsheets wrapping the $T^2$ target itself. Therefore the configuration space of $n$ of these string worldsheets will be given by the symmetric product $(M_4^n/S_n)$ whose points parametrize the transverse positions. Notice that here we are assuming that since there exists only one kind of membranes ending on the M5-brane, there is a single type of BPS strings to be counted. Now, since the supersymmetric index calculated the Euler characteristics of the configuration space, we claim that the full contribution from these string BPS configurations is given by

$$q^{-\chi M_4/24} \sum_n q^n \chi (M_4^n/S_n) = q^{-\chi M_4/24} \prod_{n>0} \frac{(1-q^n)^{b_{odd}}}{(1-q^n)^{b_{even}}} = \eta(q)^{-\chi M_4}$$

where we fixed a global multiplicative factor $q^{-\chi M_4/24}$ because of modularity requirements and we used well known results from [18].

We can compare this result with a natural generalization of the construction done in [1] for toroidal and K3 compactifications to our case. In fact, although a covariant quantization scheme does not exists for six dimensional superstring theory, one can consider its light cone quantization [2] as a temptative good definition. This can be done in our case since

\[2\] As far as the anomaly cancellation problem in a covariant quantization scheme it is concerned, it is likely to be solved as indicated in [2].
the world volume is in the product form $T^2 \times M_4$ by placing the light-cone coordinates along the $T^2$. Generalizing the construction in [1] the twisted superalgebra can be obtained from the fermionic zero-modes and the brane supercharges whose anti-commutation relations can be given with a central charge matrix modeled on the extended intersection matrix $Z = H \oplus Q$, where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Here the two factors corresponds to two-form fluxes and momentum/winding degrees of freedom. Therefore we can switch on a set of charges in the lattice $\Gamma_{b_+, b_- + 1} = \Gamma_{1,1} \oplus \Lambda_{b_+, b_-}$ and, due to the one to one correspondence between charges and fluxes we can calculate the contribution from the BPS strings. In fact, since each BPS saturated string corresponds to a chiral scalar bosonic mode, these contribute with a factor of $\frac{1}{\eta(\tau)}$ for each possible flux that we can turn on, which is the dimension of the above lattice that is $b_+ + b_- + 2 = \chi_{M_4}$. Therefore we get a total further multiplicative contribution of

$$\left( \frac{1}{\eta(\tau)} \right)^{\chi_{M_4}},$$

which is exactly the same contribution that we have found before.

Multiplying this last factor with that coming from the low energy degrees of freedom the total supersymmetric index is given by

$$\frac{\theta_\Lambda}{\eta^N}$$

as given by the dimensionally reduced calculations (2.4).

4 The multiple M5-brane case

As far as the multi M5-brane case is concerned, we do not have still a direct six-dimensional way to perform a precise counting similar to the one that we have done for the single M5-brane case since it is still not known very much about the relevant world-volume theory. Anyhow, we have some constructive arguments which explain the structure of the multi 5-brane index calculation that we review in Section 2.

It is natural to read from these results [8] that the moduli space of susy preserving $N$ M5-branes wrapped on a supersymmetric Kaehler six manifold is given by the space of rank $N$ holomorphic coverings of the manifold itself plus some (non local) analog of the gauge bundle on them.

This can be checked by the following arguments. Let us restrict to the case in which the base six manifold admits a free $S^1$ action and a $S^1$ fibration. These are the cases

3See an extended discussion about this point in section 4 of [8].
where the known formulations for the self-dual two form \([19, 20]\) can be formulated without entering further problems and therefore where the \((0, 2)\) little string theory admits a better defined low energy limit (moreover there are no problems with anomaly \([12]\) since the Euler characteristic vanishes automatically). We compactify further one of the flat transverse directions on a circle that we take to be the M-theory eleven coordinate and map the M5-branes to NS5-branes in IIA. Under these circumstances, the \((0, 2)\) theory can be mapped by fiber-wise T-duality to the \((1, 1)\) little string theory which represents the decoupled limit of the NS5-branes of IIB. By the S self-duality of type IIB we map this system to an equivalent system of D5-branes in type IIB. In fact, D5-brane bound states are described, in the low energy approximation, by susy-preserving configurations of the dimensional reduction of the \(\mathcal{N} = 1\) U(N) SYM10 to the susy-cycle \(\mathcal{C}\) on which they are wrapped on. In particular this means that the spectrum of the transverse bosonic fields define an holomorphic covering of rank \(N\) in the total space of the normal bundle of \(\mathcal{C}\) in the ambient ten dimensional manifold (which is the ambient manifold itself). We naturally expect that the UV completition of the S-dual \((1, 1)\) theory still contains this BPS geometric moduli space naturally and therefore also its fiber-wise T-dual \((0, 2)\) little string theory that we started from.

From this perspective one could try to speculate about the possible structure of the analog of the gauge bundle structure. In the case of a single M5-brane, it is given by a higher dimensional analog of the Abelian gauge bundle structure realized within the 1-form valued Cech cohomology \([4]\). As it seems by now, there does not exist any straightforward non-Abelian analog for it (at least in a local framework generalization \([6]\)). A possible way out is given anyhow by the pullback procedure once the covering map from the base cycle to the multi 5-brane world-volume is given. Notice that the pull back map fails exactly at the branching locus of the covering, where in the typical D-brane cases the full non-Abelian structure of the underlying vector bundle becomes crucial, and therefore we can’t refer to any well defined local geometrical structure on the base cycle to be understood to be the pull back of a gerbe on the covering cycle. We conclude therefore that a non-local completition of the pull back map should be accounted for by the UV non-local structure of the theory.

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4That is because in both the cases non zero well defined vector fields are required to exist on the manifold to implement the self-duality condition everywhere.

5This means that the two form potential undergoes a gauge like transformation from patch to patch and that the patching forms satisfy certain consistency conditions.

6Notice the strong resemblance with the no-go theorem in \([21]\) about local interactions between chiral forms.
In the case in which the world volume manifold is in the product form $T^2 \times M_4$, with $M_4$ Kaehler and simply connected, the formula (2.2) applies to the supersymmetric index. The $T^2$ holomorphic self-covering is unbranched and the little strings contribution is explicitly exposed and we interpret it also to encode the above mentioned effect. In fact it enters in the form $(\frac{1}{\eta})^{\chi_{\Sigma}}$, where $\Sigma_a$ is a rank $a$ holomorphic covering of $M_4$ (spectral manifold). The explicit dependence on the branching locus $B_a$ appears just because

$$\chi_{\Sigma_a} = a \cdot \chi_{M_4} - \chi_{B_a}.$$ 

We can compare positively all these arguments with the approach developed in [7] where BPS stringy representations of the $(0, 2)$ algebra are obtained and the natural role of string degrees of freedom in encoding the very structure of the interacting M5-brane world-volume theory is enlighten.

Let us notice moreover also the following consequence of the construction given in the previous section, which appears once we compare it with the geometrical moduli space which we calculated in [8]. Once we consider this moduli space from the point of view of the little string theory BPS states, we find these strings to wrap along the $T^2$ and join/split in points located at the branching locus of the $M_4$ covering. Notice that this behavior is typical of the matrix strings [22, 23] and is coherent with the possibility of generalizing the approach in [6] to our case. Unfortunately, the problem of the formulation of Matrix Theory on curved manifold is still far from being understood, but it turns out that these two similar structures likely correspond each other under the M-theory electric/magnetic duality which exchanges membranes and 5-branes.

As a consequence of the above observations, it is therefore natural to conjecture that the little string theory supersymmetric index (and moreover also a wider set of correlation functions) can be built from $\theta$-functions of the period matrix of the covering manifolds by combining them both via their characteristics (see the structure of (2.2) where the shifted lattice corresponds to non zero characteristic for the relevant $\theta$-function) and via the covering structure itself as far as the UV completion with the little string contributions is concerned. In particular one expects that the above ratio works for the calculation of correlators of self-dual strings in the form of surface operators in the low energy approximation.

We expect then that, for any given rank $N$ covering $C$ of the world-volume manifold on which the $N$ 5-branes are wrapped, the contribution to the supersymmetric index is of the
form
\[ \theta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \left( \frac{Z_0^0|0}{N(Z_0^0)} \right) \]  \hspace{1cm} (4.6)\]

where \( N(Z_0^0) \) is a modular form built from the periods of the covering six manifold which generalizes the little strings contribution to the generic case while the non-zero characteristic \( \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] \) encodes the twist induced by inequivalent spin structures [25–27] analogous to the one which appears in (2.2). These structures and the possible rank \( N \) holomorphic coverings has then to be summed up.

We expect also that a structure analogous to (4.6) applies also to the set of amplitudes relative to a proper supersymmetric version of the surface operators considered in [13, 24] where a non zero argument of the \( \theta \)-function encodes the surface periods.

5 Conclusions and open questions

In this note we have proposed a six dimensional framework for the evaluation of the super-symmetric index of the M-theory five-brane which correctly reproduces the results in [8] and proved the explicit need of taking into account the full spectrum of the little string theory to reproduce a precise counting in the form of BPS saturated string states.

It would be very interesting of course to check how the methods that we developed in this note extend to other possible M5-brane geometries and to continue the analysis of the Little String theory to try to better understand the several unclear points which are left over. Possible interesting configurations which naturally generalize the one we have studied here could be M-theory on \( R^3 \times Y \), with \( Y = CY_4, K3 \times K3 \) and the 5-branes wrapped on a six holomorphic cycle in \( Y \).

The question related to how one should exactly build (at least a set of) correlation functions in little string theory by using these \( \theta \)-function building blocks remains open and needs a much more accurate analysis. For an example in a close-by perspective see [25]. In particular one expects that the above ratio works for the calculation of correlators of self-dual strings in the form of surface operators.

Another important issue which is raised by taking seriously the Little String Theory hypothesis is the following. Since it is a superstring theory in 6 dimensions, it is non critical and this means that one should take into account also the Liouville sector which does not

\footnote{As an alternative, one might try to add to the theory a non geometrical sector to reach the Virasoro...}
decouple from the $\sigma$-model. Notwithstanding it has been extensively studied, the theory of the Liouville (super-)field still lacks a full solution and this problem, which in critical perturbative superstring theory was token for avoided, seems to come back into the game again. Let us notice here just the following coincidence. The near horizon geometry of the M5-brane in eleven dimensional super-gravity is $AdS_7 \times S^4$. Suppose we study the coupled theory of the 5-brane as a membrane theory on this fixed background. Then the longitudinal radial membrane field spans the radial direction in the $AdS_7$. The boundary value of this field will presumably play a crucial role in a decoupling procedure and it should be linked to the Liouville field which appears in the six dimensional string theory in a way similar to the original point of view of [28]. An apparently unrelated question concerns the claim that the low energy limit of little string theory does not contain gravitational degrees of freedom. This issue could also be related to the specific nature of the full six dimensional non-critical string theory.

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central charge saturation, but these degrees of freedom should not contribute both to the low energy effective theory and neither to the supersymmetric index.
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