QuAFL: Federated Averaging Can Be Both Asynchronous and Communication-Efficient

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Abstract

Federated Learning (FL) is an emerging paradigm to enable the large-scale distributed training of machine learning models, while still providing privacy guarantees. In this work, we jointly address two of the main practical challenges when scaling federated optimization to large node counts: the need for tight synchronization between the central authority and individual computing nodes, and the large communication cost of transmissions between the central server and clients. Specifically, we present a new variant of the classic federated averaging (FedAvg) algorithm, which supports both asynchronous communication and communication compression. We provide a new analysis technique showing that, in spite of these system relaxations, our algorithm essentially matches the best known bounds for FedAvg, under reasonable parameter settings. On the experimental side, we show that our algorithm ensures fast practical convergence for standard federated tasks.

1 Introduction

Federated learning (FL) \([20, 27, 22, 16]\) is a paradigm for large-scale distributed learning, in which multiple clients, orchestrated by a central authority, cooperate to jointly optimize a machine learning model given their local data. The key promise of federated learning is to enable joint model training over distributed client data, often located on end devices which are computationally-and communication-limited, without the data itself having to leave the client device.

The basic optimization algorithm underlying the learning process is known as federated averaging (FedAvg) \([27]\), and works roughly by having a central...
authority periodically communicate a shared model to all clients; then, the clients optimize this model locally based on their data, and communicate the resulting models to a central authority, which incorporates these models, often via some form of averaging, after which it initiates the next iteration. This algorithmic blueprint has been shown to be effective in practice \[27 22 3\], and has also motivated a rich line of research analyzing its convergence properties \[37 11 6 4\], as well as proposing improved variants \[32 17 24 10 23\].

Despite its popularity, scaling federated learning runs into a number of practical challenges \[22 16\]. One natural bottleneck is \textit{synchronization} between the server and the clients: as practical deployments may contain thousands of nodes, it is infeasible for the central server to orchestrate synchronous rounds among all participants. A simple mitigating approach is \textit{node sampling}, e.g. \[36 3 13\], by which the server only communicates with a subset of the nodes in a round; another, more general, approach is \textit{asynchronous communication}, e.g. \[42 30\], by which the server and the nodes may work with inconsistent versions of the shared model, avoiding the need for lock-step synchronization even between participants to the same round.

An orthogonal scalability barrier for federated learning is the \textit{high communication cost} of transmitting updates between the server and clients in every iteration \[22 16\]: even if the server samples clients in each round, the burden of repeatedly transmitting parameter-heavy updates may overwhelm the bandwidth of communication-limited clients. Several approaches have been proposed to address this bottleneck, by allowing the server and clients to apply communication-compression to the transmitted updates, e.g. \[15 14 24 41\].

It is reasonable to assume that both these bottlenecks would need to be mitigated in order to enable truly large-scale deployments: for instance, communication-reduction may not be as effective if the server has to wait for each of the clients to complete their local steps on a version of the model, before proceeding to the next round; yet, synchrony is assumed by most references with compressed communication. At the same time, removing the synchrony assumption completely would complicate the algorithm, and may lead to divergence, especially given that local data is commonly assumed to be heterogenous. Thus, it is interesting to ask under which conditions asynchrony and communication compression, on the one hand, and heterogenous local data, on the other, can be jointly supported in federated learning.

**Contributions.** In this paper, we take a step towards answering this question, and propose \textit{QuAFL}\(^1\) an extension of the FedAvg algorithm, specifically adapted to support both asynchronous communication and communication compression. Our analysis shows that, under reasonable parameter values, this new version of FedAvg essentially matches some of the best known bounds for FedAvg. Moreover, experimental results show that it also has good practical performance, confirming

\(^1\)The name stands for \textit{Qu}antized \textit{A}synchronous \textit{F}ederated \textit{L}earning, and can be pronounced as \textit{quaffle} \[34\], standing for the fact that information gets passed on as quickly as possible in our algorithm.
that the federated averaging approach can be made robust to both asynchrony and communication-compression without significant performance impact.

The general idea behind our algorithm is that we allow clients to perform their local steps independently of the round structure implemented by the server, and on a fully-local, inconsistent version of the model. Specifically, all clients receive a copy of the model when joining the computation, and start performing at most $K \geq 1$ optimization steps on it based on their local data. Independently, in each “logical round,” the server samples a set of $s$ clients uniformly at random, and sends them a compressed copy of its current model. Whenever receiving the server’s message, clients immediately respond with a compressed version of their current model, which may still be in the middle of the local optimization process, and therefore may not include recent server updates, nor the totality of the $K$ local optimization steps. Clients will carefully integrate the received server model into their next local iteration, while the server does the same with the models it receives from clients.

The key missing piece regards the quantization procedure. Here, it is tempting to apply a standard quantizer, e.g. [1, 18], similar to previous work. However, this runs into the issue that the quantization error may be too large, as it is proportional to the size of the generated model update at the client. In turn, resolving this in the context of our algorithm would require either assuming a second-moment bound on the maximum gradient update, e.g. [5], which would be unrealistic, or applying variance-reduction techniques [9], which may be complex to deploy in practice. We circumvent this issue differently, by leveraging the recent lattice-based quantizer of [8], which offers an order-optimal bias-variance trade-off, and has the property that the quantization error only depends on the difference between the quantized model and a carefully-chosen “reference point.” We instantiate this technique for the first time in the federated setting. To employ it successfully, we must overcome non-trivial challenges in defining the right reference points for encoding and decoding, as well as the fact that this procedure has a non-zero probability of error.

Our analysis technique relies on a new potential argument which shows that the discrepancy between the client and server models can still be bounded. This bound has dual usage: it is used both to control the “noise” at different optimization steps due to model inconsistency, but also to ensure that the local models are consistent enough to allow correct encoding and decoding via lattice quantization, with good probability. The technique is complex yet modular, and should allow further analysis of more complex algorithmic variants, such as controlled averaging [17] or adaptive optimization [32]. We obtain that our algorithm has surprisingly good convergence bounds, given the degree of relaxation in the system, and the fact that we consider local data to be heterogenous. Specifically, under reasonable parameter settings, we recover the best known bounds for the synchronous and full-communication FedAvg algorithm [17], showing that asynchrony and quantization can be supported without significant impact of convergence.
We validate our algorithm experimentally by simulating a standard FL setting, showing that the algorithm converges even under both quantized and asynchronous communication. Specifically, our experiments show that, in practice, QuAFL can compress updates by more than $3 \times$ without significant loss of convergence, and that the impact of asynchrony on convergence is relatively limited.

2 Related Work

The federated averaging (FedAvg) algorithm was introduced by McMahan et al. [27], and Stich [37] was among the first to consider its convergence rate, although in the homogeneous data setting. Here, we investigate whether one can jointly eliminate two of the main scalability bottlenecks of this algorithm, the synchrony between the server and client iterations, as well as the necessity of full-precision communication, with heterogeneous data distributions. Due to space constraints, we focus on prior work which seeks to mitigate these two constraints in the context of FL.

There has already been significant research into communication-compression for variants of federated averaging [31, 33, 15, 10, 2]. However, virtually all of this work considers synchronous iterations. Specifically, [33] introduced FedPAQ, a variant of FedAvg which supports quantized communication via standard compressors, and provides strong convergence bounds, under the strong assumption of i.i.d. client data. Jin et al. [15] examines the viability of a variant of the signSGD quantizer [35, 18] in the context of FedAvg, providing convergence guarantees; however, the rate guarantees have a polynomial dependence in the model dimension $d$, rendering them less practically meaningful. Haddadpour et al. [10] proposed FedCOM, a family of federated optimization algorithms which support communication-compression and come with convergence rates; yet, we note that, in order to prove convergence in the challenging heterogeneous-data setting, this reference requires non-trivial technical assumptions on the quantized gradients [10, Assumption 5]. Chen et al. [5] also considered update compression, but under convex losses, coupled with a rather strong second-moment bound assumption on the gradients. Finally, Jhunjhunwala et al. [14] examine adapting the degree of compression during the execution, proving convergence bounds for their scheme, under the non-standard i.i.d. data sampling assumption. We observe that each of these references requires at least one non-standard assumption for providing convergence guarantees for FedAvg under communication compression. By contrast, our analysis works for general (non-convex) losses, under a standard non-i.i.d. data distribution, without relying on second-moment bounds on the gradients.

A complementary approach to reducing communication cost in the federated setting has been to investigate optimizers with faster convergence, e.g. [28, 17], or adaptive optimizers [32, 39]. Moreover, recent work has shown that these approaches can be compatible with communication-compression [9, 24, 41]. Specifically, for non-convex losses, the MARINA method [9] offers theoretical
guarantees both in terms of convergence and bits transmitted. However, MARINA is structured in synchronous rounds; moreover, it periodically (with some probability) has clients compute full gradients and transmit uncompressed model updates, and requires fairly complex synchronization and variance-reduction to compensate for the extra noise due to quantization. Recent work by Tyurin and Richtarik [40] proposed a family of methods called DASHA, which combines the general structure of MARINA with Momentum Variance Reduction (MVR) methods [7], while relaxing the coupling between the server and the nodes and allowing fully-compressed updates. Specifically, the algorithm still uses a synchronous round structure, but nodes can probabilistically choose between sending full gradients or variance-reduced updates (thus reducing synchronization relative to MARINA, which required all nodes to send full gradients at the same round).

By contrast to this work, we focus on obtaining a practical algorithm with good convergence bounds: we always transmit compressed, low-precision messages, and consider a general notion of asynchronous communication, which allows the server and nodes to make progress independently, in non-blocking fashion. We focus on the classic, practical FedAvg algorithm, although our general algorithmic and analytic approach should generalize to more complex notions of local optimization.

Finally, we note that our approach borrows ideas from the analysis of decentralized variants of SGD [25, 38, 29, 19, 26], bringing them into the context of federated optimization. Specifically, we also rely on a global potential function, whose evolution we analyze across optimization steps, with various system relaxations. At the same time, there are very significant differences, from the fact that we need to define a novel potential function, adapted to FL, to the fact that we cannot rely on some of the stronger assumptions available in the decentralized setting, such as a second-moment bound on the gradients [26, 29].

3 The Algorithm

3.1 System Overview

We assume a distributed system with one coordinator node, and \( n \) client nodes, whose goal is to jointly minimize a \( d \)-dimensional, differentiable function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \). More precisely, we consider the classical empirical risk minimization (ERM) setting, in which a set of data samples are located at the \( n \) nodes. We assume that each agent \( i \) has a local function \( f_i \) associated to its own local fraction of the data, i.e. \( \forall x \in \mathbb{R}^d, f(x) = \sum_{i=1}^{n} f_i(x)/n \). In this setting, the goal would be to converge on a model \( x^* \) which minimizes the empirical loss over the sum of the loss functions, that is \( x^* = \arg\min_x f(x) = \arg\min_x (1/n) \sum_{i=1}^{n} f_i(x) \).

The clients will run a distributed variant of SGD, coordinated by the central node, which is described in detail in the next section. We will assume that each client \( i \) is able to obtain unbiased stochastic gradients \( \tilde{g}_i \) of its own local function \( f_i \), which are estimators with the property that \( \mathbb{E}[\tilde{g}_i(x)] = \nabla f_i(x) \).

More formally, these stochastic gradients can be computed by each agent by
sampling i.i.d. from its own local distribution, and computing the gradient at the chosen sample. Our analysis will consider the case where each client distribution is distinct, but that there is a bound on the maximum discrepancy between the gradients.

3.2 Algorithm Description

Algorithm 1 Pseudocode for QuAFL Algorithm.

% Initial models
\[ X_0 = X^1 = X^2 = ... = X^n = 0^d, \] number of local steps \( K \)
% Encoding (\( Enc(A) \)) and decoding (\( Dec(B, A) \)) functions, with common parametrization.

% At the Server:
1: for \( t = 0 \) to \( T - 1 \) do
2: Server chooses the set \( S \) of clients uniformly at random.
3: for all clients \( i \in S \) do
4: Server sends \( Enc(X_t) \) to the client.
5: Server receives \( Enc(Y_i) \) from client \( i \).
6: \( Q(Y_i) \leftarrow Dec(X_t, Y_i) \) % Decodes client messages relative to \( X_t \)
7: end for
8: \( X_{t+1} = \frac{1}{|S|} X_t + \frac{1}{|S|} \sum_{i \in S} Q(Y_i) \)
9: end for

% At Client \( i \):
% Upon (asynchronous) contact from the server run \( \text{InteractWithServer} \)
% Local variables:
% \( X^i \) stores the base client model, following the last server interaction. Initially \( 0^d \).
% \( \tilde{h}_i \) accumulates local gradient steps since last server interaction, initially \( 0^d \).
1: function \( \text{InteractWithServer} \)
2: \( MSG_i \leftarrow Enc(X^i - \eta \tilde{h}_i) \) % Client \( i \) compresses its local progress since last contacted.
3: Client sends \( MSG_i \) to the server.
4: Client receives \( Enc(X_t) \) from the server, where \( t \) is the current server time.
5: \( Q(X_t) \leftarrow Dec(X^i, Enc(X_t)) \) % Client decodes the message using its old model as reference point.
6: % The client then updates its local model
7: \( X^i = \frac{1}{1+\tilde{h}_i} Q(X_t) + \frac{1}{1+\tilde{h}_i} (X^i - \eta \tilde{h}_i) \)
8: % Finally, it performs \( K \) new local steps on the updated \( X^i \), unless interrupted again.
9: \( \text{LocalUpdates}(X^i, K) \)
10: \( \text{Wait()} \)
11: end function

1: function \( \text{LocalUpdates}(X^i, K) \)
2: \( \tilde{h}_i = 0 \) % local gradient accumulator
3: for \( g = 0 \) to \( K - 1 \) do
4: \( \tilde{h}_i = \tilde{g}_i(X^i - \eta \sum_{q=0}^{g-1} \tilde{h}_i^q) \) % compute the \( q \)th local gradient
5: \( \tilde{h}_i = \tilde{h}_i + \tilde{h}_i^g \) % add it to the accumulator
6: end for
7: end function

Overview. Our algorithm starts from the standard pattern used by federated averaging (FedAvg), as well as by many other FL algorithms: computation and communication are organized in logical “rounds,” where in each round the server
transmits its current version of the model to either all, or a subset of clients. The clients should then take $K \geq 1$ local optimization steps on the received model, and transmit the result to the server, which would integrate these updates. Our algorithm will relax this pattern in two orthogonal ways, allowing for both quantized and asynchronous communication between server and clients.

**Quantized Communication.** The first relaxation is only allow for compressed communication of the server model and of the client updates, via quantization. For this, we employ a carefully-parametrized version of the lattice-based quantization technique of Davies et al. [8], whose analytical properties we describe in the analysis section. For practical purposes, this quantization technique presents an encoding function $Enc(A)$, which encodes an arbitrary input $A$ to a quantized representation. To “read” a message, a node must call the symmetric $Dec(A, B)$ function, which allows for the “decoding” of the input $B$ with respect to a reference point $A$, returning a quantized output. (Part of the challenge in our analysis will be to show that the reference points used in the algorithm will allow for correct decoding with high probability.) The computational complexity of encoding and decoding is $O(d \log d)$ for a vector of dimension $d$.

**Asynchronous Communication.** A key practical limitation of the FedAvg pattern is the fact that the server and its clients have to communicate in synchronous, lock-step fashion: thus, the server must wait for the results of computation at a round before it can move to the next round. In particular, this means that the server has to wait for the slowest client to complete its local steps before it can proceed. QuAFL relaxes this requirement by essentially allowing any contacted node $i$ to immediately return (a quantized version of) its current version of the model to the server upon being contacted, even though the client might still not have completed all its $K$ local optimization steps for the round. More precisely, the client always records its “base” model at the end of the last interaction with the server into parameter $X^i_t$, and sums up its gradient updates since the last interaction into the buffer $\tilde{h}^i$. Upon being contacted, the client simply sends its current progress $X^i_t - \eta \tilde{h}^i$ to the server, where $\eta$ is the learning rate, in quantized form. It then decodes the quantized model $X^t$ received for the server, using its old local model $X^i_t$ as the decoding key. Finally, the client updates $X^t$ to include the server’s new information via careful averaging. It is then ready to restart its local update loop, upon this new local model.

It is important to notice that the server interaction occurs *asynchronously*, and that it might occur either while the client is still performing local steps, or after the client has completed its $K$ local steps, and is idle, waiting for server contact. (See Figure 1.) In the former case, we assume that, upon being contacted, the client completes its pending local optimization step, and then immediately calls the server interaction function, without

![Figure 1: Illustration of server-client interaction. The client immediately replies with a quantized version of its local model, after which it restarts its local steps based on the updated model.](image-url)
performing additional steps.

Globally, the server contacts $s$ random agents in each logical round, sends them a quantized version of the global model $X_t$, then receives quantized versions of their progress, and then incorporates this into the global model which will be sent at the next round.

**Discussion.** The main practical advantage of this approach is that it prevents the server from having to wait for each of the contacted clients to complete their local optimization on the global model $X_t$. Obviously, we inherently assume that the frequency at which clients are contacted by the server will still allow them to make some progress on the local optimization problem in between two communications. This is a reasonable assumption for large client counts, which we will also revisit in our analysis and in the experimental validation.

Specifically, in the analysis we treat the number of local steps as a random variable which takes values in \( \{1, 2, \ldots, K\} \) and define its expected value to be $H$. Intuitively, a larger $H$ gives better convergence, since this means that nodes are able to perform larger number of local steps. We will show that, when $H = \Theta(K)$, convergence is asymptotically the same as in the case when nodes perform $K$ local steps without interruption from the server, similarly to the standard FedAvg algorithm.

In addition to the above, an important difference relative to the standard FedAvg pattern is that we perform a careful form of averaging between the server’s model and the client models. This is critical for convergence, as for instance simply adopting the client average at a step could lead to divergence.

## 4 Convergence Analysis

### 4.1 Analytical Assumptions

We begin by stating the assumptions we make in the theoretical analysis of our algorithm. Specifically, we assume the following for the global loss function $f$, the individual client losses $f_i$, and the stochastic gradients $\tilde{g}_i$:

1. **Uniform Lower Bound:** There exists $f_* \in \mathbb{R}$ such that $f(x) \geq f_*$ for all $x \in \mathbb{R}^d$.

2. **Smooth Gradients:** For any client $i$, the gradient $\nabla f_i(x)$ is $L$-Lipschitz continuous for some $L > 0$, i.e. for all $x, y \in \mathbb{R}^d$:
   \[
   \|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|. \tag{1}
   \]

3. **Bounded Variance:** For any client $i$, the variance of the stochastic gradients is bounded by some $\sigma^2 > 0$, i.e. for all $x \in \mathbb{R}^d$:
   \[
   \mathbb{E}\left\|\tilde{g}_i(x) - \nabla f_i(x)\right\|^2 \leq \sigma^2. \tag{2}
   \]

4. **Bounded Gradient Dissimilarity:** There exist constants $G^2 \geq 0$ and
\[ B^2 \geq 1, \text{ such that for all } x \in \mathbb{R}^d: \]
\[ \sum_{i=1}^{n} \frac{\| \nabla f_i(x) \|^2}{n} \leq G^2 + B^2 \| \nabla f(x) \|^2. \]  

(3)

All these assumptions are standard in the context of non-convex federated learning, e.g. [17, 15, 9]. Specifically, the first three assumptions are fairly universal in distributed stochastic optimization over non-convex losses, whereas the fourth is a standard way of encoding the fact that there must be a bound on the amount of divergence between the local distributions at the nodes in order to allow for joint optimization.

**Quantization Procedure.** We use a quantization function which is given by Lemma 23 in [8]. For completeness, we restate this result here:

**Lemma 4.1.** (Lattice Quantization) Fix parameters \( R \) and \( \gamma > 0 \). There exists a quantization procedure defined by an encoding function \( \text{Enc}_{R,\gamma} : \mathbb{R}^d \to \{0,1\}^* \) and a decoding function \( \text{Dec}_{R,\gamma} = \mathbb{R}^d \times \{0,1\}^* \to \mathbb{R}^d \) such that, for any vector \( x \in \mathbb{R}^d \) which we are trying to quantize, and any vector \( y \) which is used by decoding, which we call the decoding key, if \( \| x - y \| \leq R \gamma \) then with probability at least \( 1 - \log \log(\| x - y \| / \gamma) \) we have that

- (Unbiased decoding) \( \mathbb{E}[Q_{R,\gamma}(x)] = \mathbb{E}[\text{Dec}_{R,\gamma}(y, \text{Enc}_{R,\gamma}(x))] = x; \)
- (Error bound) \( \| Q_{R,\gamma}(x) - x \| \leq (R^2 + 7) \gamma; \)
- (Communication bound) \( O\left( d \log \left( \frac{R}{\gamma} \| x - y \| \right) \right) \) bits are needed to send \( \text{Enc}_{R,\gamma}(x) \).

Thus, our aim is to show that in our algorithm, local models of the clients stay close to the local model of the server, so that we can successfully apply the above lemma to our setting. With this in mind, we move to the main result.

### 4.2 Main Results

Let \( \mu_t = (X_t + \sum_{i=1}^{n} X_i^t)/(n+1) \) be the mean over all the node models in the system at a given \( t \). Our main result shows the following:

**Theorem 4.2.** Assume the total number of steps \( T \geq \Omega(n^3) \), the learning rate \( \eta = \frac{n+1}{\sqrt{T}} \), and quantization parameters \( R = 2 + T^2 \) and \( \gamma = \frac{n^2}{(2+T)^2} \left( \sigma^2 + 2KG^2 + f(\mu_0) - f^* \right) \). Let \( H \geq 1 \) be the expected number of local steps already performed by a client when interacting with the server. Then, with probability at least \( 1 - O(1/T) \) we have that Algorithm [\ref{alg:quantization}] converges at the following rate

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\| \nabla f(\mu_t) \|^2] \leq \frac{5(f(\mu_0) - f^*)}{s H \sqrt{T}} + \frac{8sKL(\sigma^2 + 2kg^2)}{H \sqrt{T}} + O \left( \frac{n^3sK^2L^2(\sigma^2 + 2KG^2)}{HT} \right)
\]

and uses \( O \left( d \log n + \log T \right) \) expected communication bits per step.
The result provides a trade-off between the convergence speed of the algorithm, the variance of the local distributions (given by $\sigma$ and $G$), the sampling set size $s$, and the average number of local steps $H$ performed by a node when contacted by the server. Intuitively, the first additive term appears to be asymptotically-optimal, as the server relies on progress from $s$ clients, each having taken $H$ local gradient steps on average, and the objective is general non-convex. The second term essentially contains a “total upper bound” on variance in its numerator, divided by the total number of expected gradient steps in the denominator. The third term contains similar “nuisance terms” to the second, with the addition of the $n^3$ factor, and also bounds the extra variance. Crucially, this larger term is divided by $T$, as opposed to $\sqrt{T}$; since $T$ is our asymptotic parameter, it is common to assume that this extra term becomes negligible as $T$ is large, e.g. [26].

Under this convention, the first two terms essentially recovers the convergence bound provided by Karimireddi et al. [17] for FedAvg if we assume that $H = \Theta(K)$ and that $s = \Theta(n)$; this reference also provided arguments showing that the dependency in $\sigma$ and $G$ should be inherent in the heterogeneous data setting that we consider [17, Section 3.2].

In practice, it should be reasonable to assume that $H = \Theta(K)$, that is, that on average each client $i$ will have completed its local steps on the old version of the model $X^i$ when being contacted: otherwise, the sampling frequency of the server is too high, and prevents clients from making progress on their local optimization, and the server should simply decrease it.

**Convergence at the Server.** Finally, we show that not only convergence at the server, as opposed to the convergence of the mean of the local models as in Theorem 4.2. We get that:

**Corollary 4.3.** Assume the total number of steps $T \geq \Omega(n^4)$, the learning rate $\eta = \frac{n^4}{\sqrt{T}}$, and quantization parameters $R = 2 + T^{\frac{3}{2}}$ and $\gamma^2 = \frac{n^4}{(2T + 1)} \left( \sigma^2 + 2KG^2 + \frac{f(\mu_0) - f_\ast}{L} \right)$. Let $H \geq 1$ be the expected number of local steps already performed by a client when interacting with the server. Then, with probability at least $1 - O(\frac{1}{T})$ we have that Algorithm 1 converges at the following rate

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(X_t)\|^2] \leq \frac{15(f(X_0) - f_\ast)}{sH\sqrt{T}} + \frac{24sKL(\sigma^2 + 2KG^2)}{H\sqrt{T}} + O \left( \frac{n^4sKL^2(\sigma^2 + 2KG^2)}{HT} \right).
$$

This corollary yields a very similar bound to our main result, except for the larger dependency between $T$ and $n$, which is intuitively required due to the additional time required for the server to converge to a similar bound to the mean $\mu_t$. The third term may be significant for large number of nodes $n$; however, since it is divided by $T$ (as opposed to $\sqrt{T}$) it can be seen as negligible for moderate $n$ and large $T$. The fact that QuAFL can match some of the best known rates for FedAvg under some parameter settings may seem surprising, since our algorithm is asynchronous (in particular, nodes take steps on local, delayed versions of the server model) and also supports communication-compression.
4.3 Overview of the Analysis

The complete analysis is fairly complex, and is provided in full in the Supplementary material. Due to space constraints, we only provide an overview of the proofs, outlining the main intermediate results. The first step in the proof is bounding the deviation between the local models and their mean. For this, we introduce the potential function $\Phi_t = \|X_t - \mu_t\|^2 + \sum_{i=1}^{n} \|X_i^{i} - \mu_t\|^2$, and we use a load-balancing approach to show that this potential has the following supermartingale-type property:

**Lemma 4.4.** For any time step $t$ we have:

$$
E[\Phi_{t+1}] \leq \left(1 - \frac{1}{4n}\right) E[\Phi_t] + 8s\eta^2 \sum_{i=1}^{n} E[\|\tilde{h}_i\|^2] + 16n(R^2 + 7)^2 \gamma^2.
$$

The intuition behind this result is that potential $\Phi_t$ will stay well-concentrated around its mean, except for influences from the variance due to local steps (second term) or quantization (third term). With this in place, the next lemma allows us to track the evolution of the average of the local models, with respect to local step and quantization variance:

**Lemma 4.5.** For any step $t$

$$
E[\|\mu_{t+1} - \mu_t\|^2] \leq \frac{2s^2\eta^2}{n(n+1)^2} \sum_{i=1}^{n} E[\|\tilde{h}_i\|^2] + \frac{2}{(n+1)^2}(R^2 + 7)^2 \gamma^2.
$$

In both cases, the upper bound depends on the second moment of the nodes’ local progress $\sum_i E[\|\tilde{h}_i\|^2]$. (This is due to the fact that the server contacts $s$ clients, which are chosen uniformly at random.) Then, our main technical lemma uses properties (1), (2) and (3), to concentrate $\sum_i E[\|\tilde{h}_i(X_i^{i})\|^2]$ around the true gradient $E[\|\nabla f(\mu_t)\|^2]$, where the expectation is taken over the algorithm’s randomness. We show that:

**Lemma 4.6.** For any step $t$, we have that

$$
\sum_{i=1}^{n} E[\|\tilde{h}_i\|^2] \leq 2nK(\sigma^2 + 2KG^2) + 8L^2K^2 E[\Phi_t] + 4nK^2B^2 E[\|\nabla f(\mu_t)\|^2] - \sum_{t=0}^{T-1} E[\|\nabla f(\mu_t)\|^2].
$$

We can then combine Lemmas 4.4 and 4.6 to get an upper bound on the potential with respect to $E[\|\nabla f(\mu_t)\|^2]$. Summing over steps, we obtain the following:

**Lemma 4.7.**

$$
\sum_{t=0}^{T} E[\Phi_t] \leq 80Tn^2(R^2 + 7)^2 \gamma^2 + 80Tn^2sK\eta^2(\sigma^2 + 2KG^2) + 160B^2n^2sK^2\eta^2 T - \sum_{t=0}^{T-1} E[\|\nabla f(\mu_t)\|^2].
$$

Next, using the $L$-smoothness of the function $f$, implied by (1), we can show that

$$
E[f(\mu_{t+1})] \leq E[f(\mu_t)] + E[\nabla f(\mu_t), \mu_{t+1} - \mu_t] + \frac{L}{2} E[\|\mu_{t+1} - \mu_t\|^2]. \quad (4)
$$
Final argument. Using the above inequality, and given that $\mathbb{E}[\mu_{t+1} - \mu_t] = -\frac{\eta}{n+1} \sum_{i \in S} \tilde{h}_i(X_i)$, we observe that the sum $\sum_{i=1}^{n} \mathbb{E}(\nabla f(\mu_t), \mu_{t+1} - \mu_t)$ can be concentrated around $\mathbb{E}\|\nabla f(\mu_t)\|^2$, in similar fashion as in Lemma 4.6. Together with Lemma 4.5, this results in the following bound:

$$
\mathbb{E}[f(\mu_{t+1})] - \mathbb{E}[f(\mu_t)] \leq \frac{5\eta s K L^2 \mathbb{E}[\Phi_t]}{n(n+1)} + (\frac{4sL^2 \eta^3 K^3}{n+1} + \frac{2s^2 K^2 L}{(n+1)^2})(\sigma^2 + 2KG^2)
$$

$$
+ \left(\frac{R^2 + 7}{(n+1)^2}\right) \gamma^2 L + \left(\frac{-3\eta s H}{4(n+1)} + \frac{8B^2 L^2 \eta^3 s K^3}{n+1} + \frac{4B^2 s^2 K^2 L^2 \eta^2}{(n+1)^2}\right) \mathbb{E}\|\nabla f(\mu_t)\|^2.
$$

For $\eta = (n+1)/\sqrt{T}$, as stated in the Theorem, we can use Lemma 4.7 to cancel out the terms containing the potential $\Phi_t$ (after summing up the inequality over $T$ steps). Replacing these terms, and modulo some additional term wrangling, we obtain the claimed convergence bound.

Quantization Impact. Finally, we address the correctness of the quantization technique. We show that the quantization fails with negligible probability:

Lemma 4.8. Let $T \geq \Omega(n^3)$, then for quantization parameters $R = 2 + T^\frac{3}{2}$ and $\gamma^2 = \frac{s^2}{(n^2 + 1)^2}(\sigma^2 + 2KG^2 + \frac{f(\mu_0) - f_*}{L})$ we have that the probability of quantization never failing during the entire run of the Algorithm is at least $1 - O(T^\frac{1}{4})$.

Per Lemma 4.1, in order for the communication to fail with negligible probability, we need to show that whenever the server communicates with a client, the two norm of their local models is at most $R^d \gamma$. Hence, we need to use bound $\mathbb{E}[\Phi(t)]$. The only similar use of this technique was in [29]; however, the authors of this reference could benefit from assuming that the second-moment of the gradients was bounded. Since we make no such assumption here, we need to find a way to bound $\sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(\mu_t)\|^2$. Fortunately, our main result shows that the gradients are vanishing. This allows us to take the advantage of the convergence rate we prove and plug it back into Lemma 4.7.

Similarly, due to the Property 3, Lemma 4.1, the number of the bits used by our algorithm, in one communication between the server and a client, depends on the two norm of the distance between their local models. Thus, we can use the bound on $\sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(\mu_t)\|^2$ (as above) to show the following bound.

Lemma 4.9. Let $T \geq \Omega(n^3)$, then for quantization parameters $R = 2 + T^\frac{3}{2}$ and $\gamma^2 = \frac{s^2}{(n^2 + 1)^2}(\sigma^2 + 2KG^2 + \frac{f(\mu_0) - f_*}{L})$ we have that the expected number of bits used by Algorithm per step is $O(d \log(n) + \log(T))$. We note that the communication cost per step is also asymptotically optimal, modulo the multiplicative $\log n$ and additive $\log T$ terms. These terms occur because we must ensure that the quantization scheme may only fail with probability $1 - O(1/T)$. 

12
5 Experimental Results

Experimental Setup and Goals. We implemented our algorithm in Pytorch in order to train neural networks for image classification tasks, specifically residual CNNs [12] on the Fashion MNIST [43] and CIFAR [21] datasets. The model and dataset details are presented in the Supplementary Material. While a deployment-scale validation of our algorithm is beyond our present scope, we aim to validate our analysis relative to the impact of various hyper-parameters, such as the frequency of contact with the server, the number of local steps, and the number of quantization bits, on the algorithm characteristics.

We begin by defining the hyper-parameters which control the behavior of QuAFL and FedAvg. Then, we proceed by carefully describing the way in which we simulated each of the algorithms. Finally, we detail the datasets, tasks, and models used for our experiments.

5.1 Hyper-parameters

We first define our hyper-parameters; in the later sections, we will examine their impact on algorithm behavior through ablation studies.

\( n \): Number of the clients.

\( s \): Number of clients interacting with the server at each step.

\( K \): In QuAFL, this is the maximum number of allowed local steps by each client between two server calls. In FedAvg, this is the number of local steps performed by each client upon each server call.

\( b \): Number of bits used to send a coordinate after quantization.

\( swt \): Server waiting time, i.e. the amount of time that server waits between two consecutive calls.

\( sit \): Server interaction time, i.e. the amount of time that server needs to send and receive necessary data (excluding computation time).

5.2 Simulation

We attempt to simulate a realistic FL deployment scenario, as follows. We assume a server and \( n \) clients, each of which initially has a model copy. The training dataset is distributed among the clients so that each of them has access to \( 1/n \) of the training data. We track the performance of each algorithm by evaluating the server’s model, on an unseen validation dataset. We measure loss and accuracy of the model with respect to simulation time, server steps, and total local steps performed by clients. These setups so far were common between QuADL and FedAvg. In the following, we are going to describe their specifications and differences.

QuAFL: Upon each server call, the server chooses \( s \) clients uniformly at random. It then sends its model to those clients and asks for their current local
models. (Recall that clients send their model immediately to the server.)
Each of the clients will have taken a maximum of $K$ local steps by the time
it is contacted by the server. The server then replaces its model with a
carefully-computed average over the received models and its current model.
This process increases time on the server by the server interaction time ($sit$).
The server then waits for another interval of server waiting time ($swt$) to make its next call. The $s$ receiving clients replace their model with
the weighted average between their current model and the received server’s
model. Since each client performs local steps from its last interaction time
until the current server time, nodes are effectively executing asynchronously.
Moreover, note that communication is compressed, as all the models get
encoded in their source and decoded in their destination.

**Quantization:** To have a lightweight but efficient communication between
clients and the server, we use the lattice quantization algorithm of [8].
Using this method, we send $b$ instead of 32 bits for each scalar
dimension. Informally, each 32-bit number maps to one of the $2^b$
quantized levels and can be sent using $b$ bits only. The encoded
number can then be decoded to a sufficiently close number at the
destination, following the quantization protocol.

**FedAvg:** In the beginning of each round, server chooses $s$ clients randomly, and
sends its current model to them. Each of those clients receives the model,
uncompressed, and performs exactly $K$ local steps using this model as the
starting point, and then sends back the resulting model to the server. The
server then computes the average of the received models and adopts it as
its model. By this synchronous structure, in each round, the server must
wait for the slowest client to complete its local steps plus an extra $sit$ for
the communication time. After completing each round, the server starts
the next call immediately, that means $swt = 0$ in FedAvg.

**Timing Experiments.** We differentiate between two types of timing experi-
ments. **Uniform timing** experiments, presented in the paper body, assume all
clients take the same amount of time for a gradient step. However, in real-world
setups, different devices may require different amounts of time to perform a
single local step. This is one of the main disadvantages of synchronous federated
optimization algorithms. To demonstrate how this fact affects the experiments,
in our Non-uniform timing experiments we differentiate clients to be either fast
or slow. The length of each local step can be characterized as a memoryless
time event. Therefore, the length of each local step can be defined by a random
variable $X \sim \text{exponential}(\lambda)$. The parameter $\lambda$ is $1/2$ for fast clients and $1/8$
for slow clients; the expected runtime $E(X)$ would be 2 and 8, respectively. In
each timing experiment, we assumed only one fourth of clients to be slow.

5.3 Datasets and Models
We used Pytorch to manage the training process in our algorithm. We have
trained neural networks for image classification tasks on two well-known datasets
Figure 2: Impact of the maximum number of local steps $K \in \{5, 10, 20\}$ on the QuAFL algorithm / Fashion MNIST.

Figure 3: Impact of the number of interacting peers $s \in \{4, 8, 16\}$ on the convergence of the algorithm.

Figure 4: Impact of the server contact frequency (controlled via server timeout $swt$) on the convergence of the algorithm.

Figure 5: Convergence comparison relative to total number of rounds, between QuAFL, FedAvg, and the sequential baseline.

**Fashion MNIST, and CIFAR-10.** For all the datasets, we used the default train/test split of the dataset for our training/validation dataset. In the following, we describe the model architecture and the training hyper-parameters used to train on each of these datasets.

**Fashion MNIST:** Although this dataset has the same sample size and number of classes as MNIST, obtaining competitive performance on it requires a more complicated architecture. Therefore, we used a CNN model to train on this model and demonstrated the performance of our algorithm in a non-convex task. To optimize the models, we used Adam optimizer with constant $lr = 0.001$ and batch size 100.

**CIFAR-10:** To load this dataset, we used data augmentation and normalization. For this task, we trained ResNet20 models. Moreover, the SGD optimizer with constant $lr = ??$ is used to in the training process. The batch size 64/200 is used for training/validation.

### 5.4 Results on Fashion MNIST (FMNIST)

We begin by validating our earlier results, presented in the paper body, for the slightly more complex FMNIST dataset, and on a convolutional model.

In Figures 2 and 3 we examine the impact of the parameters $K$ and $s$, respectively, on the total number of interaction rounds at the server, to reach a certain training loss. As expected, we notice that higher $K$ and $s$ improve
the convergence behavior of the algorithm. In Figures 4 we examine the impact of the server waiting time on the convergence of the algorithm relative to the number of server rounds. Again, we notice that a higher server waiting time improves convergence, as it allows the server to take advantage of additional local steps performed at the clients, as predicted by our analysis. (Higher \( swt \) means higher average number of steps completed \( H \).)

Next, we examine the convergence, again in terms of number of optimization “rounds” at the server, between the sequential Baseline, FedAvg, and QuAFL. As expected, the Baseline is faster to converge than FedAvg, which in turn is faster than QuAFL in this measure. Specifically, the difference between QuAFL and the other algorithms comes because of the fact that, in our algorithm, nodes operate on old variants of the model at every step, which slows down convergence. Next, we examine convergence in terms of actual time, in the heterogeneous setting in which 25% of the clients are slow.

In Figure 6, we observe the validation accuracy ensured by various algorithms relative to the simulated execution time, whereas in Figure 7 we observe the training loss versus the same metric. (We assume that, in Baseline, a single node acts as both the client and the server, and that this node is slow, i.e. has higher per-step times.) We observe that, importantly, if time is taken into account rather than the number of server rounds, QuAFL can provide notable speedups in these metrics. This is specifically because of its asynchronous communication patterns, which allow it to complete rounds faster, without having to always wait for the slow nodes to complete their local computation. While this behaviour is simulated, we believe that this reflects the algorithm’s practical potential.

5.5 Results on CIFAR-10

We now present results for a standard image classification task on the CIFAR-10 dataset, using a ResNet20 model [12].

Figures 8 and 9 show the decrease in training loss versus the number of server steps (or rounds) for different values of \( K \) and \( s \) respectively. As our theory suggests, increasing \( K \) and \( s \) leads to an improvement in the convergence rate of the system. Figure 10 demonstrates the impact of the number of quantization bits \( b \), on the convergence behaviour of the algorithm. According to the definition of \( b \), increasing the number of quantization bits improves the communication
accuracy. Thus, as it can be seen in the graph, higher values of $b$ enhance the convergence relative to the number of server steps. Finally, Figure 11 shows the impact of the server interaction frequency, again controlled via the timeout parameter $s_{\text{swt}}$, on the algorithm’s convergence. It is apparent that a very high interaction frequency can slow the algorithm down, by not allowing it to take advantage of the clients’ local steps.

In Figures 12 and 13, we examine the validation accuracy and loss, respectively, ensured by various algorithms versus the simulated execution time. (As in the F-MNIST experiments, we assumed the Baseline to be a single slow node that performs an optimization step per round.) Again, the asynchronous nature of QuAFL provides a faster convergence rate than its synchronous counterparts; which can be clearly seen in the mentioned figures.
6 Conclusion

We have provided the first variant of FedAvg which incorporates both asynchronous and compressed communication, and have shown that this algorithm can still provide strong convergence guarantees, asymptotically matching some of the best known bounds for the (synchronous and non-compressed) version of FedAvg. Our analysis should be extensible to more complex federated optimizers, such as gradient tracking, e.g. [10], controlled averaging [17], or variance-reduced variants [9].

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A The Complete Analysis

A.1 Overview and Notation

Recall that $X_t$ denotes the model of the server at step $t$, and $X^i$ is the local model of client $i$ after its last interaction with the server. Also, $\tilde{h}_i$ is the sum of local gradient steps for model $X^i$ since its last interaction with the server.

For the convergence analysis, local steps of the clients that are not selected by the server don’t have any effect on the server or other clients. Therefore we do not need to assume that clients are doing their local steps asynchronous, and we can assume that all clients run their local gradient steps after the server contacts them. The only thing that we should consider is the randomness of the server selecting the clients, and the fact that the server can contact nodes before they have finished their $K$ steps. For this purpose, we assume that their number of steps is a random number $H^i_t$ with mean $H$.

To show the analysis in this setting, we introduce new notations that consider the server round. To this end, we use $X^i_t$ as the value of $X^i$ when the server is running its $t$th iteration, And $\tilde{h}_{i,t}$ for the sum of local steps at this time. We show each local step $q$ with a superscript. Formally, we have

$$\tilde{h}_{0,t}^i = 0,$$

and for $1 \leq q \leq H^i_t$ let:

$$\tilde{h}_{q,t}^i = \tilde{g}_i(X^i_t - \sum_{s=0}^{q-1} \eta \tilde{h}_{s,t}^i),$$

and

$$\tilde{h}_{i,t} = \sum_{q=0}^{H^i_t} \tilde{h}_{q,t}^i.$$

Further, for $1 \leq q \leq H^i_t$, let

$$h_{q,t}^i = E[\tilde{g}_i(X^i_t - \sum_{s=0}^{q-1} \eta \tilde{h}_{s,t}^i)] = \nabla f(X^i_t - \sum_{s=0}^{q-1} \eta \tilde{h}_{s,t}^i)$$

be the expected value of $\tilde{h}_{q,t}^i$ taken over the randomness of the stochastic gradient $\tilde{g}_i$. Also, we have:

$$h_{i,t} = \sum_{q=0}^{H^i_t} h_{q,t}^i.$$

A.2 Properties of Local Steps

Lemma A.1. For any agent $i$ and step $t$

$$E\|h_{i,t}^q\|^2 \leq \frac{\sigma^2}{K^2} + 8L^2E\|X^i_t - \mu_t\|^2 + 4E\|\nabla f_i(\mu_t)\|^2.$$
Proof.

\[ E\|h_{i,t}^q\|^2 \leq E\left\| \nabla f_i(X^i_t - \sum_{s=0}^{q-1} \eta \tilde{h}_{i,t}^s) - \nabla f_i(\mu_t) \right\|^2 + 2E\|\nabla f_i(\mu_t)\|^2 \]

\[ \leq 2E\left\| \nabla f_i(X^i_t - \sum_{s=0}^{q-1} \eta \tilde{h}_{i,t}^s) - \nabla f_i(\mu_t) \right\|^2 + 2E\|\nabla f_i(\mu_t)\|^2 \]

\[ \leq 4L^2E\|X^i_t - \mu_t\|^2 + 4\eta^2 L^2 q \sum_{s=0}^{q-1} E\|\tilde{h}_{i,t}^s\|^2 + 2E\|\nabla f_i(\mu_t)\|^2 \]

\[ \leq 4L^2E\|X^i_t - \mu_t\|^2 + 4\eta^2 L^2 q \sum_{s=0}^{q-1} E\|\tilde{h}_{i,t}^s\|^2 + 2E\|\nabla f_i(\mu_t)\|^2 \]

The rest of the proof is done by induction, and assuming \( \eta < \frac{1}{4LK^2} \). \( \square \)

Lemma 4.6. For any step \( t \), we have that

\[ \sum_{i=1}^{n} E\|\tilde{h}_{i,t}\|^2 \leq 2nK(\sigma^2 + 2KG^2) + 8L^2K^2E[\Phi_t] + 4nK^2B^2E\|\nabla f(\mu_t)\|^2. \]

Proof. Using Lemma A.1

\[ \sum_{i=1}^{n} E\|\tilde{h}_{i,t}\|^2 = \sum_{i=1}^{n} \sum_{h=1}^{K} Pr[H^i_t = h]E\| \sum_{q=1}^{h} \tilde{h}_{i,t}^q \|^2 \]

\[ \leq \sum_{i=1}^{n} \sum_{h=1}^{K} Pr[H^i_t = h]hE\| \sum_{q=1}^{h} \tilde{h}_{i,t}^q \|^2 \]

\[ \leq nK\sigma^2 + \sum_{i=1}^{n} \sum_{h=1}^{K} Pr[H^i_t = h]hE\| \sum_{q=1}^{h} \tilde{h}_{i,t}^q \|^2 \]

\[ \leq nK\sigma^2 + \sum_{i=1}^{n} K^2 \left( \frac{\sigma^2}{L^2} + 8L^2E\|X^i_t - \mu_t\|^2 + 4E\|\nabla f_i(\mu_t)\|^2 \right) \]

\[ \leq 2nK\sigma^2 + \sum_{i=1}^{n} K^2 \left( 8L^2E\|X^i_t - \mu_t\|^2 + 4E\|\nabla f_i(\mu_t)\|^2 \right) \]

\[ \leq 2nK\sigma^2 + 8L^2K^2E[\Phi_t] + 4nK^2G^2 + 4nK^2B^2E\|\nabla f(\mu_t)\|^2. \]

\( \square \)

Lemma A.2. For any local step \( 1 \leq q \), and agent \( 1 \leq i \leq n \) and step \( t \)

\[ E\|\nabla f_i(\mu_t) - h_{i,t}^q\|^2 \leq 4L^2\eta^2 q^2\sigma^2 + 4L^2E\|X^i_t - \mu_t\|^2 + 8L^2\eta^2 q^2E\|\nabla f_i(\mu_t)\|^2. \]

Proof.

\[ E\|\nabla f_i(\mu_t) - h_{i,t}^q\|^2 = E\|\nabla f_i(\mu_t) - \nabla f_i(X^i_t - \sum_{s=0}^{q-1} \eta \tilde{h}_{i,t}^s)\|^2 \]
\[ L^2 \mathbb{E}\|\mu_t - X^i_t\|^2 \leq 2L^2 \mathbb{E}\|X^i_t - \mu_t\|^2 + 2L^2 \eta^2 \mathbb{E} \left( \sum_{s=0}^{q-1} \eta h^s_{i,t} \right)^2 \]

\[ \leq 2L^2 \mathbb{E}\|X^i_t - \mu_t\|^2 + 2L^2 \eta^2 q \sum_{s=0}^{q-1} \mathbb{E}\|h^s_{i,t}\|^2 \]

Lemma A.3.

For any time step \( t \)

\[ \sum_{i=1}^{n} \mathbb{E}\langle \nabla f(\mu_t), -h_{i,t} \rangle = 4KL^2 \mathbb{E}[\Phi_t] + \left( -\frac{3Hn}{4} + 8B^2 L^2 \eta^2 K^3 n \right) \mathbb{E}\|\nabla f(\mu_t)\|^2 + 4nL^2 \eta^2 K^3 (\sigma^2 + 2G^2). \]

Proof.

\[ \sum_{i=1}^{n} \mathbb{E}\langle \nabla f(\mu_t), -h_{i,t} \rangle = \sum_{i=1}^{n} \sum_{h=1}^{K} \mathbb{P}[H_i^h = h] \mathbb{E}\langle \nabla f(\mu_t), -h_q^{i,t} \rangle \]

\[ = \sum_{i=1}^{n} \sum_{h=1}^{K} \mathbb{P}[H_i^h = h] \sum_{q=1}^{h} \left( \mathbb{E}\langle \nabla f(\mu_t), \nabla f_i(\mu_t) - h_q^{i,t} \rangle - \mathbb{E}\langle \nabla f(\mu_t), \nabla f_i(\mu_t) \rangle \right) \]

Using Young’s inequality we can upper bound \( \mathbb{E}\langle \nabla f(\mu_t), \nabla f_i(\mu_t) - h_q^{i,t} \rangle \) by

\[ \left( \mathbb{E}\|\nabla f(\mu_t)\|^2 \right)^{\frac{1}{4}} \left( \mathbb{E}\|\nabla f_i(\mu_t) - h_q^{i,t}\|^2 \right)^{\frac{3}{4}}. \]

Plugging this in the above inequality we get:

\[ \sum_{i=1}^{n} \mathbb{E}\langle \nabla f(\mu_t), -h_{i,t} \rangle \]

\[ \leq \sum_{i=1}^{n} \sum_{h=1}^{K} \mathbb{P}[H_i^h = h] \sum_{q=1}^{h} \left( \mathbb{E}\|\nabla f(\mu_t) - h_q^{i,t}\|^2 + \frac{\mathbb{E}\|\nabla f(\mu_t)\|^2}{4} - \mathbb{E}\langle \nabla f(\mu_t), \nabla f_i(\mu_t) \rangle \right) \]

and the last inequality comes from \( \eta < \frac{1}{4L^2} \).
Lemma A.2

\[ \sum_{i=1}^{n} \sum_{h=1}^{K} Pr[H_i^h = h] \sum_{q=1}^{h} \left( 4L^2\eta^2\sigma^2 + 4L^2E\|X_t^i - \mu_t\|^2 + 8L^2\eta^2E\|\nabla f_i(\mu_t)\|^2 \right. \\
+ \frac{\mathbb{E}\|\nabla f(\mu_t)\|^2}{4} - \mathbb{E}(\nabla f(\mu_t), \nabla f_i(\mu_t)) \right) \]

\[ \leq \sum_{i=1}^{n} \sum_{h=1}^{K} Pr[H_i^h = h] \left( 4L^2\eta^2\sigma^2 + 4L^2E\|X_t^i - \mu_t\|^2 + 8L^2\eta^2E\|\nabla f_i(\mu_t)\|^2 \right. \\
+ \frac{\mathbb{E}\|\nabla f(\mu_t)\|^2}{4} - \mathbb{E}(\nabla f(\mu_t), \nabla f_i(\mu_t)) \right) \]

\[ \leq 4KL^2\mathbb{E}[\Phi_t] + 4nL^2\eta^2K^3(\sigma^2 + 2G^2) + (8B^2nL^2\eta^2K^3 + \frac{Hn}{4} - Hn)\mathbb{E}\|\nabla f(\mu_t)\|^2 \]

Where in the last step we used that \( \mathbb{E}[H_i^h] = H \), and \( \sum_{i=1}^{n} \frac{f_i(x)}{n} = f(x) \), for any vector \( x \).

### A.3 Upper Bounding Potential Functions

We proceed by proving the lemma 4.4 which upper bounds the expected change in potential:

**Lemma 4.4.** For any time step \( t \) we have:

\[ \mathbb{E}[\Phi_{t+1}] \leq \left( 1 - \frac{1}{4n} \right) \mathbb{E}[\Phi_t] + 8s\eta^2 \sum_{i=1}^{n} \mathbb{E}\|\tilde{h}_{i,t}\|^2 + 16n(R^2 + 7)^2\eta^2. \]

**Proof.** First we bound change in potential \( \Delta_t = \Phi_{t+1} - \Phi_t \) for some fixed time step \( t > 0 \).

For this, let \( \Delta_t^S \) be the change in potential when set \( S \) of agents wake up. For \( i \in S \) define \( S_i^t \) and \( S_t \) as follows:

\[ S_i^t = -\frac{s}{s+1}\eta\tilde{h}_{i,t} + \frac{Q(X_t) - X_t}{s+1} \]

\[ S_t = -\frac{1}{s+1}\eta \sum_{i \in S} \tilde{h}_{i,t} + \frac{1}{s+1} \sum_{i \in S} (Q(X_t^i - \eta\tilde{h}_{i,t}) - (X_t^i - \eta\tilde{h}_{i,t})) \]

We have that:

\[ X_{t+1}^i = \frac{sX_t^i + X_t}{s+1} + S_i^t \]

\[ X_{t+1} = \frac{\sum_{i \in S} X_t^i + X_t}{s+1} + S_t \]

\[ \mu_{t+1} = \mu_t + \frac{\sum_{j \in S} S_j^t + S_t}{n+1} \]

This gives us that for \( i \in S \):

\[ X_{t+1}^i - \mu_{t+1} = \frac{sX_t^i + X_t}{s+1} + S_i^t - \frac{\sum_{j \in S} S_j^t + S_t}{n+1} - \mu_t \]
\[ X_{t+1} - \mu_{t+1} = \frac{\sum_{i \in S} X^i_t + X_t}{s + 1} + \sum_{j \in S} S^j_t + S_t - \frac{\sum_{j \in S} S^j_t + S_t}{n + 1} - \mu_t \]

For \( k \not\in S \) we get that
\[ X^k_{t+1} - \mu_{t+1} = X^k_t - \frac{\sum_{j \in S} S^j_t + S_t}{n + 1} - \mu_t. \]

Hence:
\[
\Delta^S = \sum_{i \in S} \left( \left\| \frac{sX^i_t + X_t}{s + 1} + S^i_t - \frac{\sum_{j \in S} S^j_t + S_t}{n + 1} - \mu_t \right\|^2 - \left\| X^i_t - \mu_t \right\|^2 \right) \\
+ \left\| \sum_{i \in S} X^i_t + X_t - \sum_{j \in S} S^j_t + S_t \right\|^2 - \left\| X_t - \mu_t \right\|^2 \\
+ \sum_{k \not\in S} \left( \left\| X^k_t - \frac{\sum_{j \in S} S^j_t + S_t}{n + 1} - \mu_t \right\|^2 - \left\| X^k_t - \mu_t \right\|^2 \right)
\]
\[
= \sum_{i \in S} \left( \left\| \frac{sX^i_t + X_t}{s + 1} - \mu_t \right\|^2 + \left\| S^i_t + \frac{\sum_{j \in S} S^j_t + S_t}{n + 1} \right\|^2 \\
+ 2\left\langle \frac{sX^i_t + X_t}{s + 1} - \mu_t, S^i_t - \frac{\sum_{j \in S} S^j_t + S_t}{n + 1} \right\rangle - \left\| X^i_t - \mu_t \right\|^2 \right) \\
+ \left( \left\| \sum_{i \in S} X^i_t + X_t - \mu_t \right\|^2 + \left\| S_t - \frac{\sum_{j \in S} S^j_t + S_t}{n + 1} \right\|^2 \\
+ 2\left\langle \sum_{i \in S} X^i_t + X_t - \mu_t, S_t - \frac{\sum_{j \in S} S^j_t + S_t}{n + 1} \right\rangle - \left\| X_t - \mu_t \right\|^2 \right) \\
+ \sum_{k \not\in S} 2\left\langle X^k_t - \mu_t, -\frac{\sum_{j \in S} S^j_t + S_t}{n + 1} \right\rangle + \sum_{k \not\in S} \left\| \sum_{j \in S} S^j_t + S_t \right\|^2
\]

Observe that:
\[ \sum_{k=0}^n \left\langle X^k_t - \mu_t, -\frac{\sum_{j \in S} S^j_t + S_t}{n + 1} \right\rangle = 0. \]

After combining the above two equations, we get that:
\[
\Delta^S = \sum_{i \in S} \left( \left\| \frac{s(X^i_t - \mu_t) + (X_t - \mu_t)}{s + 1} \right\|^2 - \frac{s}{s + 1} \left\| X^i_t - \mu_t \right\|^2 - \frac{1}{s + 1} \left\| X_t - \mu_t \right\|^2 \right) \\
+ \left( \left\| \sum_{i \in S} \frac{(X^i_t - \mu_t) + (X_t - \mu_t)}{s + 1} \right\|^2 - \sum_{i \in S} \frac{1}{s + 1} \left\| X^i_t - \mu_t \right\|^2 - \frac{1}{s + 1} \left\| X_t - \mu_t \right\|^2 \right) \\
+ \sum_{i \in S} \left\| S^i_t - \frac{\sum_{j \in S} S^j_t + S_t}{n + 1} \right\|^2 + 2\left\langle \frac{sX^i_t + X_t}{s + 1} - \mu_t, S^i_t \right\rangle \\
+ \left\| S_t - \frac{\sum_{j \in S} S^j_t + S_t}{n + 1} \right\|^2 + 2\left\langle \sum_{i \in S} \frac{X^i_t + X_t}{n + 1} - \mu_t, S_t \right\rangle
\]
\[ + \sum_{k \notin S} \left\| \frac{\sum_{j \in S} S_i^j + S_k}{n + 1} \right\|^2 \]

By simplifying the above, we get:

\[
\Delta_i^S = \frac{-s}{(s + 1)^2} \sum_{i \in S} \|X_i^i - X_i\|^2 - \frac{1}{(s + 1)^2} \sum_{i \in S} \|X_i^i - X_i\|^2 - \frac{1}{(s + 1)^2} \sum_{i \neq j \in S} \|X_i^i - X_i^j\|^2 \\
+ \sum_{i \in S} \left\| S_i^i - \frac{\sum_{j \in S} S_i^j + S_i}{n + 1} \right\|^2 + \frac{2s}{s + 1} \sum_{i \in S} \langle X_i^i - \mu_t, S_i^i \rangle + \frac{2}{s + 1} \sum_{i \in S} \langle X_i - \mu_t, S_i \rangle \\
+ \left\| S_i - \frac{\sum_{j \in S} S_i^j + S_i}{n + 1} \right\|^2 + \frac{2}{s + 1} \sum_{i \in S} \langle X_i^i - \mu_t, S_i \rangle + \frac{2}{s + 1} \langle X_i - \mu_t, S_i \rangle \\
+ \sum_{k \notin S} \left\| \frac{\sum_{j \in S} S_i^j + S_k}{n + 1} \right\|^2 \\
\]

Let \( \alpha \) be a parameter we will fix later:

\[
\langle X_i^i - \mu_t, S_i^i \rangle \leq \alpha \|X_i^i - \mu_t\|^2 + \frac{\|S_i^i\|^2}{4\alpha} \\
\]

Finally, we get that

\[
\Delta_i^S \leq \frac{-1}{s + 1} \sum_{i \in S} \|X_i^i - X_i\|^2 + 2 \sum_{i \in S} \|S_i^i\|^2 + \frac{2s(s + 1)}{(n + 1)^2} \sum_{j \in S} \|S_i^j\|^2 + \frac{2s(s + 1)}{(n + 1)^2} \|S_i\|^2 \\
+ \sum_{i \in S} \frac{2s\alpha}{s + 1} \|X_i^i - \mu_t\|^2 + \sum_{i \in S} \frac{s\|S_i^i\|^2}{2\alpha(s + 1)} + \sum_{i \in S} \frac{2\alpha}{s + 1} \|X_i - \mu_t\|^2 + \sum_{i \in S} \frac{\|S_i\|^2}{2\alpha(s + 1)} \\
+ 2 \|S_i\|^2 + \frac{2(s + 1)}{(n + 1)^2} \sum_{j \in S} \|S_i^j\|^2 + \frac{2(s + 1)}{(n + 1)^2} \|S_i\|^2 + \sum_{i \in S} \frac{2\alpha}{s + 1} \|X_i^i - \mu_t\|^2 + \sum_{i \in S} \frac{\|S_i\|^2}{2\alpha(s + 1)} \\
+ \frac{2\alpha}{s + 1} \|X_i - \mu_t\|^2 + \frac{\|S_i\|^2}{2\alpha(s + 1)} + \sum_{i \in S} \frac{(n - s)(s + 1)}{(n + 1)^2} \|S_i^j\|^2 + \frac{(n - s)(s + 1)}{(n + 1)^2} \|S_i\|^2 \\
= \frac{-1}{s + 1} \sum_{i \in S} \|X_i^i - X_i\|^2 + (2 + \frac{2(s + 1)^2}{(n + 1)^2} + \frac{1}{2\alpha} + \frac{(n - s)(s + 1)}{(n + 1)^2}) \sum_{j \in S} \|S_i^j\|^2 + \\
(2 + \frac{2(s + 1)^2}{(n + 1)^2} + \frac{1}{2\alpha} + \frac{(n - s)(s + 1)}{(n + 1)^2}) \|S_i\|^2 + \sum_{i \in S} \frac{2\alpha}{s + 1} \|X_i^i - \mu_t\|^2 + \frac{\|S_i\|^2}{2\alpha} \|X_i - \mu_t\|^2 \\
\leq \frac{-1}{s + 1} \sum_{i \in S} \|X_i^i - X_i\|^2 + (4 + \frac{1}{2\alpha}) \sum_{i \in S} \|S_i^i\|^2 + \\
\frac{1}{2\alpha} \|S_i\|^2 + \sum_{i \in S} \frac{2\alpha}{s + 1} \|X_i^i - \mu_t\|^2 + \frac{\|S_i\|^2}{2\alpha} \|X_i - \mu_t\|^2}
\]
Using definitions of $S_t^i$ and $S_t$, Cauchy-Schwarz inequality and properties of quantization we get that

$$
\|S_t^i\|^2 \leq \frac{2s^2}{(s+1)^2} \eta^2 \|\tilde{h}_{i,t}\|^2 + \frac{2(R^2 + 7)^2 \gamma^2}{(s+1)^2}.
$$

$$
\|S_t\|^2 \leq \frac{2s}{(s+1)^2} \eta^2 \sum_{i \in S} \|\tilde{h}_{i,t}\|^2 + \frac{2s^2(R^2 + 7)^2 \gamma^2}{(s+1)^2}.
$$

Next, we plug this in the previous inequality:

$$
\Delta_t^S \leq \frac{-1}{s+1} \sum_{i \in S} \|X_i - X_t\|^2 + \sum_{i \in S} 2\alpha \|X_i^t - \mu_t\|^2 + 2\alpha \|X_t - \mu_t\|^2
$$

$$
+ (4 + \frac{1}{2\alpha}) \frac{2s^2 + 2s}{(s+1)^2} \eta^2 \|	ilde{h}_{i,t}\|^2 + \frac{2s^2 + 2s(R^2 + 7)^2 \gamma^2}{(s+1)^2}
$$

$$
\leq \frac{-1}{s+1} \sum_{i \in S} \|X_i - X_t\|^2 + \sum_{i \in S} 2\alpha \|X_i^t - \mu_t\|^2 + 2\alpha \|X_t - \mu_t\|^2
$$

$$
+ (4 + \frac{1}{2\alpha}) (\eta^2 \sum_{i \in S} \|\tilde{h}_{i,t}\|^2 + 2(R^2 + 7)^2 \gamma^2)
$$

Next, we calculate probability of choosing the set $S$ and upper bound $\Delta_t$ in expectation, for this we define $E_t$ as expectation conditioned on the entire history up to and including step $t$

$$
E_t[\Delta_t] = \sum_{S} \frac{1}{\binom{n}{S}} E_t[\Delta_t^S]
$$

$$
\leq \sum_{S} \frac{1}{\binom{n}{S}} \left( \frac{-1}{s+1} \sum_{i \in S} \|X_i - X_t\|^2 + \sum_{i \in S} 2\alpha \|X_i^t - \mu_t\|^2 + 2\alpha \|X_t - \mu_t\|^2
$$

$$
+ (4 + \frac{1}{2\alpha}) (\eta^2 \sum_{i \in S} \|\tilde{h}_{i,t}\|^2 + 2(R^2 + 7)^2 \gamma^2) \right)
$$

$$
= \frac{(-s-1)}{(s+1)} \frac{\binom{n}{S}}{\binom{n}{k}} \sum_{i} \|X_i - X_t\|^2 + \sum_{i} \frac{2\alpha (n-1) n}{\binom{n}{k}} \|X_i^t - \mu_t\|^2 + 2\alpha \|X_t - \mu_t\|^2
$$

$$
+ (4 + \frac{1}{2\alpha}) (\eta^2 \frac{\binom{n-1}{k}}{\binom{n}{k}} \sum_{i} \|\tilde{h}_{i,t}\|^2 + 2(R^2 + 7)^2 \gamma^2)
$$

$$
\leq - \sum_{i} \frac{s \|X_i - \mu_t\|^2}{(s+1)n} + \sum_{i} \frac{2\alpha s_n}{n} \|X_i^t - \mu_t\|^2 + 2\alpha \|X_t - \mu_t\|^2
$$

$$
+ (8 + \frac{1}{\alpha}) (R^2 + 7)^2 \gamma^2 + \sum_{i} \frac{s}{n} (4 + \frac{1}{2\alpha}) \eta^2 \|\tilde{h}_{i,t}\|^2
$$

$$
\leq \frac{(-s)}{(s+1)n} + 2\alpha \Phi_t + (8 + \frac{1}{\alpha}) (R^2 + 7)^2 \gamma^2 + \sum_{i} \frac{s}{n} (4 + \frac{1}{2\alpha}) \eta^2 \|\tilde{h}_{i,t}\|^2
$$

28
we also use upper bound which come from the properties of quantization).

\[ \sum \Phi \]

For any time step we have:

\[ \sum \Phi \leq 0 \]

Next we remove the conditioning, and use the definitions of \( \Delta \) and \( S_t \) (for \( S_t \) we also use upper bound which come from the properties of quantization).

\[ \sum \Phi \leq (1 - \frac{1}{5m})E[\Phi_t] + 16n(R^2 + 7)^2\gamma^2 + 8s\eta^2E_h^2 \]

Proof. By combining Lemma 4.4 and 4.6 we have:

\[ \sum \Phi \leq (1 - \frac{1}{5m})E[\Phi_t] + 16n(R^2 + 7)^2\gamma^2 + 8s\eta^2E_h^2 \]

Lemma A.4. For any time step \( t \) we have:

\[ \sum \Phi \leq (1 - \frac{1}{5m})E[\Phi_t] + 16n(R^2 + 7)^2\gamma^2 + 8s\eta^2E_h^2 \]

Proof. By combining Lemma 4.4 and 4.6 we have:

\[ \sum \Phi \leq (1 - \frac{1}{5m})E[\Phi_t] + 16n(R^2 + 7)^2\gamma^2 + 8s\eta^2E_h^2 \]

Lemma A.5. For the sum of potential functions in all \( T \) steps we have:

\[ \sum_{t=0}^{T} \sum \Phi \leq 80Tn^2(R^2 + 7)^2\gamma^2 + 80Tn^2s\eta^2(\sigma^2 + 2KG^2) + 160B^2n^2sK\eta^2 \]

Proof.

\[ \sum_{t=0}^{T} \sum \Phi \leq \sum_{t=0}^{T-1} \left( (1 - \frac{1}{5m})E[\Phi_t] + 16n(R^2 + 7)^2\gamma^2 + 8s\eta^2E_h^2 \right) \]

\[ \leq (1 - \frac{1}{5m})\sum_{t=0}^{T-1} E[\Phi_t] + 16Tn(R^2 + 7)^2\gamma^2 + 16Tn\sum \Phi + 32B^2n^2sK\eta^2 \]

\[ \sum \Phi \leq (1 - \frac{1}{5m})\sum_{t=0}^{T-1} E[\Phi_t] + 16Tn(R^2 + 7)^2\gamma^2 + 16Tn\sum \Phi + 32B^2n^2sK\eta^2 \]
\[ \sum_{t=0}^{T} \mathbb{E}[\Phi_t] \leq 5n(16Tn(R^2 + 7)^2\gamma^2 + 16TnsK\eta^2(\sigma^2 + 2KG^2) \]
\[ + 32B^2nsK^2\eta^2 \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(\mu_t)\|^2 \]
\[ = 80Tn^2(R^2 + 7)^2\gamma^2 + 80Tn^2sK\eta^2(\sigma^2 + 2KG^2) \]
\[ + 160B^2n^2sK^2\eta^2 \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(\mu_t)\|^2 \]

\[ \square \]

**Lemma 4.5.** For any step \( t \)
\[ \mathbb{E}\|\mu_{t+1} - \mu_t\|^2 \leq \frac{2s^2\eta^2}{n(n+1)^2} \sum_i \mathbb{E}\|\tilde{h}_{i,t}\|^2 + \frac{2}{(n+1)^2}(R^2 + 7)^2\gamma^2. \]

**Proof.**
\[ \mathbb{E}\|\mu_{t+1} - \mu_t\|^2 \leq \sum_S \frac{1}{\binom{n}{s}(n+1)^2} \left( \mathbb{E}\| - \eta \sum_{i \in S} \tilde{h}_{i,t} + \frac{Q(X_i) - X_i}{s+1} \right) \]
\[ + \frac{1}{s+1} \sum_{i \in S} (Q(X_i^t - \eta\tilde{h}_{i,t}) - (X_i^t - \eta\tilde{h}_{i,t})) \]
\[ \leq \sum_S \frac{1}{\binom{n}{s}(n+1)^2} \left( 2s\eta^2 \sum_{i \in S} \mathbb{E}\|\tilde{h}_{i,t}\|^2 + \frac{2}{s+1} \mathbb{E}\|Q(X_i) - X_i\|^2 \right) \]
\[ + \frac{2}{s+1} \sum_{i \in S} \mathbb{E}\|Q(X_i^t - \eta\tilde{h}_{i,t}) - (X_i^t - \eta\tilde{h}_{i,t})\|^2 \]
\[ \leq \sum_S \frac{1}{\binom{n}{s}(n+1)^2} \left( 2s\eta^2 \sum_{i \in S} \mathbb{E}\|\tilde{h}_{i,t}\|^2 + 2(R^2 + 7)^2\gamma^2 \right) \]
\[ = \sum_s \frac{2s\eta^2 \binom{n-1}{s}}{(n+1)^2} \mathbb{E}\|\tilde{h}_{i,t}\|^2 + \frac{2}{(n+1)^2}(R^2 + 7)^2\gamma^2 \]
\[ = \sum_s \frac{2s\eta^2}{n(n+1)^2} \mathbb{E}\|\tilde{h}_{i,t}\|^2 + \frac{2}{(n+1)^2}(R^2 + 7)^2\gamma^2. \]

\[ \square \]

By plugging Lemma 4.6 in the above upper bound we get that:

**Lemma A.6.** For any step \( t \)
\[ \mathbb{E}\|\mu_{t+1} - \mu_t\|^2 \leq \frac{4s^2K\eta^2(\sigma^2 + 2KG^2)}{(n+1)^2} + \frac{16s^2L^2K^2\eta^2\mathbb{E}[\Phi_t]}{n(n+1)^2} \]
\[ + \frac{8B^2s^2K^2\eta^2\mathbb{E}\|\nabla f(\mu_t)\|^2}{(n+1)^2} + \frac{2(R^2 + 7)^2\gamma^2}{(n+1)^2}. \]
Proof.

\[ \mathbb{E}\|\mu_{t+1} - \mu_t\|^2 \leq \sum_i \frac{2s^2\eta^2}{n(n+1)^2} \mathbb{E}\|\tilde{h}_{i,t}\|^2 + \frac{2}{(n+1)^2}(R^2 + 7)^2\gamma^2 \]

\[ \leq \sum_i \frac{2s^2\eta^2}{n(n+1)^2} \left( 2nK(\sigma^2 + 2K^2) + 8L^2K^2\mathbb{E}[\Phi_t] + 4nK^2B^2\mathbb{E}\|\nabla f(\mu_t)\|^2 \right) \]

\[ + \frac{2(R^2 + 7)^2\gamma^2}{(n+1)^2} \]

\[ = \frac{4s^2K\eta^2}{(n+1)^2}(\sigma^2 + 2K^2) + \frac{16s^2L^2K^2\eta^2}{n(n+1)^2}\mathbb{E}[\Phi_t] \]

\[ + \frac{8B^2s^2K^2\eta^2}{(n+1)^2}\mathbb{E}\|\nabla f(\mu_t)\|^2 + \frac{2(R^2 + 7)^2\gamma^2}{(n+1)^2} \]

A.4 Convergence

Theorem A.7. For learning rate \( \eta = \frac{n+1}{\sqrt{T}} \), Algorithm \Box converges at rate:

\[ \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\|\nabla f(\mu_t)\|^2 \leq \frac{2(f(\mu_0) - f_\star)}{sH\sqrt{T}} + \frac{800nKL^2(R^2 + 7)^2\gamma^2}{H} + \frac{8sK\sigma^2 + 2K^2}{H\sqrt{T}} + \frac{2(R^2 + 7)^2\gamma^2L\sqrt{T}}{(n+1)^2sH} \]

Proof. Let \( \mathbb{E}_t \) denote expectation conditioned on the entire history up to and including step \( t \). By \( L \)-smoothness we have that

\[ \mathbb{E}_t[f(\mu_{t+1})] \leq f(\mu_t) + \mathbb{E}_t(\nabla f(\mu_t), \mu_{t+1} - \mu_t) + L\frac{1}{2}\mathbb{E}_t\|\mu_{t+1} - \mu_t\|^2. \]  

(5)

First we look at \( \mathbb{E}_t(\nabla f(\mu_t), \mu_{t+1} - \mu_t) = \langle \nabla f(\mu_t), \mathbb{E}_t[\mu_{t+1} - \mu_t] \rangle \). If set \( S \) is chosen at step \( t + 1 \), We have that

\[ \mu_{t+1} - \mu_t = \frac{1}{n+1} \left( -\eta \sum_{i \in S} \tilde{h}_{i,t} + \frac{Q(X_t)}{s+1} - X_t + \frac{1}{s+1} \sum_{i \in S} (Q(X^j_t - \eta \tilde{h}_{i,t}) - X^j_t - \eta \tilde{h}_{i,t}) \right) \]

Thus, in this case:

\[ \mathbb{E}_t[\mu_{t+1} - \mu_t] = -\frac{\eta}{n+1} \sum_{i \in S} \tilde{h}_{i,t}. \]

Where we used unbiasedness of quantization and stochastic gradients. We would like to note that even though we do condition on the entire history up to and including step \( t \) and this includes conditioning on \( X^j_t \), the algorithm has not yet used \( \tilde{h}_{i,t} \) (it does not count towards computation of \( \mu_t \)), thus we can safely use all properties of stochastic gradients. Hence, we can proceed by taking into the
account that each set of agents $S$ is chosen as initiator with probability $\frac{1}{|S|}$:

$$E_t[\mu_{t+1} - \mu_t] = \sum_S \frac{1}{|S|} \sum_i \sum_{i \in S} - \frac{\eta}{n+1} \sum_i h_{i,t} = -\frac{sn\eta}{n(n+1)} \sum_i h_{i,t}.$$ 

and subsequently

$$E_t(\nabla f(\mu_t), \mu_{t+1} - \mu_t) = \sum_{i=1}^n \frac{sn\eta}{n(n+1)} E_t(\nabla f(\mu_t), -h_{i,t}).$$

Hence, we can rewrite (5) as:

$$E_t[f(\mu_{t+1})] \leq f(\mu_t) + \sum_{i=1}^n \frac{sn\eta}{n(n+1)} E_t(\nabla f(\mu_t), -h_{i,t}) + \frac{L}{2} E_t[\mu_{t+1} - \mu_t]^2.$$ 

Next, we remove the conditioning

$$E[(\mu_{t+1})] = E[E_t[f(\mu_{t+1})]] \leq E[f(\mu_t)] + \sum_{i=1}^n \frac{sn\eta}{n(n+1)} \mathbb{E}(\nabla f(\mu_t), -h_{i,t})$$

This allows us to use Lemmas A.6 and A.3

$$E[f(\mu_{t+1})] - E[f(\mu_t)] \leq \frac{sn\eta}{n(n+1)} \left( 4KL^2 \mathbb{E}[\Phi] + \left( -\frac{3Hn}{4} + 8B^2L^2\eta^2K^3n \right) \mathbb{E}\|\nabla f(\mu_t)\|^2 
+ 4nL^2\eta^2K^3(\sigma^2 + 2G^2) \right)$$

$$+ \frac{L}{2} \left( \frac{4s^2K\eta^2(\sigma^2 + 2KG^2)}{(n+1)^2} + \frac{16s^2L^2K^2\eta^2\mathbb{E}[\Phi]}{n(n+1)^2} 
+ \frac{8B^2s^2K^2\eta^2\mathbb{E}\|\nabla f(\mu_t)\|^2}{(n+1)^2} + \frac{2(R^2 + 7)\gamma^2}{(n+1)^2} \right)$$

$$= \left( \frac{4\eta sKL^2}{n(n+1)} + \frac{8s^2KL^2\eta^2}{n(n+1)^2} \right) \mathbb{E}[\Phi]$$

$$+ \left( \frac{4sL^2\eta^3K^3}{n+1} + \frac{2s^2K\eta^2L}{(n+1)^2} \right) (\sigma^2 + 2KG^2) + \left( \frac{(R^2 + 7)\gamma^2L}{(n+1)^2} \right) \mathbb{E}\|\nabla f(\mu_t)\|^2$$

By simplifying the above inequality we get:

$$E[f(\mu_{t+1})] - E[f(\mu_t)] \leq \frac{5\eta sKL^2}{n(n+1)} + \left( \frac{4sL^2\eta^3K^3}{n+1} + \frac{2s^2K\eta^2L}{(n+1)^2} \right) (\sigma^2 + 2KG^2)$$

$$+ \left( \frac{(R^2 + 7)\gamma^2L}{(n+1)^2} \right) \mathbb{E}\|\nabla f(\mu_t)\|^2$$

$$+ \left( \frac{8B^2L^2\eta^3sK^3}{n+1} \right) + \left( \frac{-3\eta sH}{4(n+1)} + \frac{4B^2s^2K^2L\eta^2}{(n+1)^2} \right) \mathbb{E}\|\nabla f(\mu_t)\|^2$$

32
by summing the above inequality for $t = 0$ to $t = T - 1$, we get that
\[
\mathbb{E}[f(\mu_T)] - f(\mu_0) \leq \frac{5\eta s KL^2}{n(n+1)} \sum_{t=0}^{T-1} \mathbb{E}[\Phi_t] + \left(\frac{4sL^2\eta^3K^3}{n+1} + \frac{2s^2Kn^2L}{(n+1)^2}\right)(\sigma^2 + 2KG^2)
\]
\[
+ \left(\frac{-3\eta s H}{4(n+1)} + \frac{8B^2L^2\eta^3sK^3}{n+1} + \frac{4B^2s^2K^2L\eta^2}{(n+1)^2}\right) \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]
\]
\[
+ \frac{(R^2 + 7)^2\gamma^2LT}{(n+1)^2}
\]
Further, we use Lemma [A.5]
\[
\mathbb{E}[f(\mu_T)] - f(\mu_0) \leq \frac{5\eta s KL^2}{n(n+1)} \left(80Tn^2(R^2 + 7)^2\gamma^2 + 80Tn^2sK\eta^2(\sigma^2 + 2KG^2)
\]
\[
+ \frac{160B^2n^2sK^2\eta^2}{n+1} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]
\]
\[
+ \left(\frac{4sL^2\eta^3K^3}{n+1} + \frac{2s^2Kn^2L}{(n+1)^2}\right)(\sigma^2 + 2KG^2) + \frac{(R^2 + 7)^2\gamma^2LT}{(n+1)^2} + \frac{8B^2L^2\eta^3sK^3}{n+1} + \frac{4B^2s^2K^2L\eta^2}{(n+1)^2} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]
\]
\[
\leq \frac{400\eta s sK^2L2(R^2 + 7)^2\gamma^2}{n+1} + \frac{404Tn^2sK^2\eta^2(\sigma^2 + 2KG^2)}{n+1} 
\]
\[
+ \frac{3s^2Kn^2LT}{(n+1)^2} + \frac{(R^2 + 7)^2\gamma^2LT}{(n+1)^2} + \frac{8B^2L^2\eta^3sK^3}{n+1} + \frac{4B^2s^2K^2L\eta^2}{(n+1)^2} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]
\]
by assuming $\eta < \frac{1}{100B\sqrt{nsv^2L}}$ we get:
\[
\mathbb{E}[f(\mu_T)] - f(\mu_0) \leq \frac{400\eta s sK^2L2(R^2 + 7)^2\gamma^2}{n+1} 
\]
\[
+ \frac{3s^2Kn^2LT}{(n+1)^2} + \frac{404Tn^2sK^2\eta^2(\sigma^2 + 2KG^2)}{n+1} + \frac{(R^2 + 7)^2\gamma^2LT}{(n+1)^2} + \frac{8B^2L^2\eta^3sK^3}{n+1} + \frac{4B^2s^2K^2L\eta^2}{(n+1)^2} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]
\]
Next, we regroup terms, multiply both sides by $\frac{2(n+1)}{\eta sHnT^2}$ and use the fact that $f(\mu_T) \geq f_*$
\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2] \leq \frac{2(n+1)(f(\mu_0) - f_*)}{sH\eta T} + \frac{800nKL^2(R^2 + 7)^2\gamma^2}{H} + 
\]
33
Proof.

Lemma A.8. For quantization parameters $(R^2 + 7)^2 \gamma^2 = \frac{(n+1)^2}{T} (\sigma^2 + 2KG^2 + f(\mu_0)^2 - f_*)$ we have:

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2] \leq \frac{2(f(\mu_0) - f_*)}{sH \sqrt{T}} + \frac{808nKL^2(n+1)^2}{HT}(\sigma^2 + 2KG^2) + \frac{80n(n+1)^2KL(f(\mu_0) - f_*)}{HT}
$$

Finally, we set $\eta = \frac{n+1}{\sqrt{T}}$:

$$
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2] \leq \frac{2(f(\mu_0) - f_*)}{sH \sqrt{T}} + \frac{808nKL^2(R^2 + 7)^2}{HT}(\sigma^2 + 2KG^2) + \frac{6sKL(\sigma^2 + 2KG^2)}{H \sqrt{T}}
$$

$$
+ \frac{808n(n+1)^2KL^2}{HT}(\sigma^2 + 2KG^2) + \frac{2(R^2 + 7)^2\gamma^2L}{(n+1)sH} \tag{6}
$$

$$
+ \frac{80n(n+1)^2KL(f(\mu_0) - f_*)}{HT} \tag{7}
$$

\[\square\]

\[\square\]

34
Lemma A.9. We have:

$$5s \sum_{t=0}^{T-1} \mathbb{E}[\Phi_t] + 3\eta^2 \sum_{t=0}^{T-1} \sum_i \mathbb{E}[\tilde{h}_{i,t}]^2 \leq 1000Tn^3s(R^2 + 7)^2\gamma^2$$

$$+ 10000B^2n^2s^2K^2LT(R^2 + 7)^2\gamma^2$$

Proof.

$$5s \sum_{t=0}^{T-1} \mathbb{E}[\Phi_t] + 3\eta^2 \sum_{t=0}^{T-1} \sum_i \mathbb{E}[\tilde{h}_{i,t}]^2$$

$$\leq 5s \sum_{t=0}^{T-1} \mathbb{E}[\Phi_t] + 3n^2 \sum_{t=0}^{T-1} (2nK(\sigma^2 + 2KG^2) + 8L^2K^2\mathbb{E}[\Phi_t] + 4nK^2B^2\mathbb{E}[\|\nabla f(\mu_t)\|^2])$$

$$\leq 5s \sum_{t=0}^{T-1} \mathbb{E}[\Phi_t] + 6nT\eta^2K(\sigma^2 + 2KG^2) + 24n^2L^2K^2 \sum_{t=0}^{T-1} \mathbb{E}[\Phi_t]$$

$$+ 12nB^2\eta^2K^2 \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]$$

$$\leq 6s \sum_{t=0}^{T-1} \mathbb{E}[\Phi_t] + 6nT\eta^2K(\sigma^2 + 2KG^2) + 12B^2n^2\eta^2K^2 \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]$$

$$\leq 6s(80Tn^2(R^2 + 7)^2\gamma^2 + 80Tn^2sK\eta^2(\sigma^2 + 2KG^2))$$

$$+ 160B^2n^2sK^2\eta^2 \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]$$

$$+ 6nT\eta^2K(\sigma^2 + 2KG^2) + 12B^2n^2\eta^2K^2 \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]$$

$$\leq 480Tn^2s(R^2 + 7)^2\gamma^2 + (480Tn^2sK^2\eta^2 + 6nT\eta^2K)(\sigma^2 + 2KG^2)$$

$$+ (960n^2s^2K^2B^2\eta^2 + 12B^2n^2\eta^2K^2) \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]$$

$$\leq 480Tn^2s(R^2 + 7)^2\gamma^2 + 486Tn^2sK^2\eta^2(\sigma^2 + 2KG^2)$$

$$+ 1000B^2n^2s^2K^2\eta^2 \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2]$$

$$\leq 480Tn^2s(R^2 + 7)^2\gamma^2 + 486Tn^2s^2K^2\eta^2(\sigma^2 + 2KG^2)$$

$$+ 1000B^2n^2s^2K^2\eta^2 \left(\frac{2(n + 1)(f(\mu_0) - f_*)}{sH\eta} + \frac{800TnKL^2(R^2 + 7)^2\gamma^2 + 800TnKL^2(R^2 + 7)^2\gamma^2}{H} + \frac{6TsK\eta L}{H(n + 1)} + \frac{80sTsK^2L^2\eta^2}{H}\right)(\sigma^2 + 2KG^2) + \frac{2T(R^2 + 7)^2\gamma^2L}{(n + 1)sH\eta}$$

$$\leq 480Tn^2s(R^2 + 7)^2\gamma^2 + 486Tn^2s^2K^2\eta^2(\sigma^2 + 2KG^2)$$
and $\gamma$ never failing during the entire run of the Algorithm 1 is at least

$$\Pr[5s\Phi_t + 3\gamma^2 \sum_i \|h_{i,t}\|^2 \leq T^{30}\gamma^2] \leq \frac{\mathbb{E}[5s\Phi_t + 3\gamma^2 \sum_i \|h_{i,t}\|^2|\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_t]}{T^{30}\gamma^2}$$

Proof. Let $\mathcal{L}_t$ be the event that quantization does not fail during step $t$. Our goal is to show that $\Pr[\bigcup_{t=1}^T \mathcal{L}_t] \geq 1 - O\left(\frac{1}{T}\right)$. In order to do this, we first prove that $\Pr[\neg \mathcal{L}_{t+1}|\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_t] \leq O\left(\frac{1}{T^2}\right)$ (O is with respect to $T$ here).

We need need to lower bound probability that:

$$\forall i \in S : \|X_i - X^*\|^2 \leq (R^d\gamma)^2 \tag{8}$$

$$\|X_i - (X^* - \eta \tilde{h}_{i,t})\|^2 \leq (R^d\gamma)^2 \tag{9}$$

$$\|X_i - X^*\|^2 = O\left(\frac{\gamma^2(poly(T))^2}{R^2}\right) \tag{10}$$

$$\|X_i - (X^* - \eta \tilde{h}_{i,t})\|^2 = O\left(\frac{\gamma^2(poly(T))^2}{R^2}\right) \tag{11}$$

We would like to point out that these conditions are necessary for decoding to succeed, we ignore encoding since it will be counted when someone will try to decode it. Since, $R = 2 + T^{\frac{2}{3}}$ this means that $(R^d\gamma)^2 \geq 2T^3 \geq T^{30}$, for large enough $T$. Hence, it is suffices to upper bound the probability that $\sum_{i \in S} \|X_i - X^*\|^2 + \sum_{i \in S} \|X_i - (X^* - \eta \tilde{h}_{i,t})\|^2 \geq T^{30}\gamma^2$. To prove this, we have:

$$\sum_{i \in S} \|X_i - X^*\|^2 + \sum_{i \in S} \|X_i - (X^* - \eta \tilde{h}_{i,t})\|^2 \leq \sum_{i \in S} (5\|X_i - \mu_i\|^2 + 5\|\mu_i - X^*\|^2 + 3\eta^2 \|\tilde{h}_{i,t}\|^2) \leq 5s\Phi_t + 3\gamma^2 \sum_i \|\tilde{h}_{i,t}\|^2$$

Now, we use Markov’s inequality, and Lemma $\ref{lem:4.8}$
\[ \leq \frac{1000Tn^3s(R^2 + 7)^2\gamma^2 + 10000B^2n^3s^2K^2L(T^2 + 7)^2\gamma^2}{T^{30}\gamma^2} \leq O\left(\frac{1}{T^2}\right) \]

Thus, the failure probability due to the models not being close enough for quantization to be applied is at most \( O\left(\frac{1}{T^2}\right) \). Conditioned on the event that \( \|X_t - X_t^i\| \) and \( \|X_t - (X_t^i - \eta \tilde{h}_{i,t})\| \) are upper bounded by \( T^{15}\gamma \) (This is what we actually lower bounded the probability for using Markov), we get that the probability of quantization algorithm failing is at most

\[ \sum_{i \in S} \log \log(\frac{1}{\gamma} \|X_t - X_t^i\|) \cdot O(R^{-d}) \]

\[ + \sum_{i \in S} \log \log(\frac{1}{\gamma} \|X_t - (X_t^i - \eta \tilde{h}_{i,t})\|) \cdot O(R^{-d}) \]

\[ \leq O\left(\frac{s \log \log T}{T^3}\right) \leq O\left(\frac{1}{T^2}\right). \]

By the law of total probability (to remove conditioning) and the union bound we get that the total probability of failure, either due to not being able to apply quantization or by failure of quantization algorithm itself is at most \( O\left(\frac{1}{T^2}\right) \). Finally we use chain rule to get that

\[ Pr[\cup_{t=1}^T \mathcal{L}_t] = \prod_{t=1}^T Pr[\mathcal{L}_t | \cup_{s=0}^{t-1} \mathcal{L}_s] = \prod_{t=1}^T \left(1 - Pr[\neg \mathcal{L}_t | \cup_{s=0}^{t-1} \mathcal{L}_s]\right) \geq 1 - \sum_{t=1}^T Pr[\neg \mathcal{L}_t | \cup_{s=0}^{t-1} \mathcal{L}_s] \geq 1 - O\left(\frac{1}{T}\right). \]

\[ \square \]

Lemma 4.9: Let \( T \geq O(n^3) \), then for quantization parameters \( R = 2 + T^{\frac{2}{3}} \) and \( \gamma^2 = \frac{n^3}{(R^2 + 7)^2}(\sigma^2 + 2KG^2 + \frac{\mu_0}{T^{15}}L_\gamma) \) we have that the expected number of bits used by Algorithm 1 per step is \( O(d \log(n) + \log(T)) \).

Proof. At step \( t + 1 \), by Corollary 4.3 we know that the total number of bits used is at most

\[ \sum_{i \in S} O\left(d \log\left(\frac{R}{\gamma} \|X_t^i - X_t\|\right)\right) + O\left(d \log\left(\frac{R}{\gamma} \|X_t - (X_t^i - \eta \tilde{h}_{i,t})\|\right)\right) \]

By taking the randomness of agent interaction at step \( t + 1 \) into the account, we get that the expected number of bits used is at most:

\[ \sum_{i} \frac{1}{n} \sum_{s \in S} \left(O\left(d \log\left(\frac{R}{\gamma} \|X_t^i - X_t\|\right)\right) + O\left(d \log\left(\frac{R}{\gamma} \|X_t - (X_t^i - \eta \tilde{h}_{i,t})\|\right)\right) \right) \]

\[ = \sum_{i} \frac{s}{n} \left(O\left(d \log\left(\frac{R^2}{\gamma^2} \|X_t^i - X_t\|^2\right)\right) + O\left(d \log\left(\frac{R^2}{\gamma^2} \|X_t - (X_t^i - \eta \tilde{h}_{i,t})\|^2\right)\right) \right) \]

\[ \leq \sum_{i} \frac{s}{n} \left(O\left(d \log\left(\frac{R^2}{\gamma^2} \|X_t^i - X_t\|^2\right)\right) + O\left(d \log\left(\frac{R^2}{\gamma^2} \|X_t - (X_t^i - \eta \tilde{h}_{i,t})\|^2\right)\right) \right) \]
and uses when interacting with the server. Then, with probability at least \( H \)

\[
\text{Let have that Algorithm 1 converges at the following rate}
\]

\[
\text{Theorem 4.2.}
\]

\[
\text{Proof. The proof simply follows from combining Lemmas A.8, 4.8 and 4.9}
\]

\[
\text{For the convergence of the server, we have:}
\]

\[
\text{Lemma A.10.}
\]

\[
\text{So the expected number of bits per communication in all rounds is at most:}
\]

\[
\frac{1}{sT} \sum_{t=0}^{T-1} s \left( O \left( d \log \left( \frac{R^2}{\gamma^2} \sum_i \frac{1}{n} \left( \|X_i^t - X_t\|^2 + \|X_i^t - \bar{X}_i^t\|^2 \right) \right) \right) \]

\[
\leq s \left( O \left( d \log \left( \frac{R^2}{\gamma^2} \Phi_t + \frac{\eta^2}{n} \sum_i |\bar{h}_{i,t}|^2 \right) \right) \]

\[
\text{Next, By Jensen inequality and Lemma A.9 We get that the expected number of bits used is at most,}
\]

\[
O \left( d \mathbb{E} \left[ \log \left( \frac{R^2}{\gamma^2} \sum_{t=0}^{T-1} \Phi_t + \frac{1}{T} \sum_{t=0}^{T-1} \frac{\eta^2}{n} \sum_i |\bar{h}_{i,t}|^2 \right) \right] \right)
\]

\[
\leq O \left( d \log \left( \frac{R^2}{\gamma^2} \left( \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\Phi_t] + \frac{1}{T} \sum_{t=0}^{T-1} \frac{\eta^2}{n} \sum_i \mathbb{E}[|\bar{h}_{i,t}|^2] \right) \right) \]

\[
\leq O \left( d \log \left( \frac{R^2}{\gamma^2} \left( \frac{1}{T} \sum_{t=0}^{T-1} \frac{\eta^2}{n} \sum_i \mathbb{E}[|\bar{h}_{i,t}|^2] \right) \right) \]

\[
\leq O \left( d \log (R^2(1000n^3s(R^2 + 7)^2\gamma^2 + 10000B^2n^3s^2K^2L(R^2 + 7)^2\gamma^2)) \right)
\]

\[
\leq O \left( d \log (R^2(1000n^3s(R^2 + 7)^2 + 10000B^2n^3s^2K^2L(R^2 + 7)^2)) \right) = O(d \log (n) + \log (T))
\]

\[
\square
\]

\[
\text{Theorem 4.2. Assume the total number of steps } T \geq \Omega(n^3), \text{ the learning rate } 
\]

\[
\eta = \frac{n+1}{\sqrt{T}} \text{, and quantization parameters } R = 2 + T^{\frac{3}{2}} \text{ and } \gamma^2 = \frac{\eta^2}{(n^2+1)^2} \left( \sigma^2 + 2KG^2 + \frac{f(\mu_0) - f_\star}{L} \right).
\]

\[
\text{Let } H \geq 1 \text{ be the expected number of local steps already performed by a client}
\]

\[
\text{when interacting with the server. Then, with probability at least } 1 - O(\frac{1}{T}) \text{ we have that Algorithm 1 converges at the following rate}
\]

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(\mu_t)\|^2] \leq \frac{5(f(\mu_0) - f_\star)}{sH \sqrt{T}} + \frac{8sKL(\sigma^2 + 2KG^2)}{H \sqrt{T}} + O \left( \frac{n^3sK^2L(\sigma^2 + 2KG^2)}{HT} \right)
\]

\[
\text{and uses } O \left( d \log n + \log T \right) \text{ expected communication bits per step.}
\]

\[
\text{Proof. The proof simply follows from combining Lemmas A.8, 4.8 and 4.9} \quad \square
\]

\[
\text{Lemma A.10. For the convergence of the server, we have:}
\]

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla f(X_t)\|^2] \leq \frac{15(f(\mu_0) - f_\star)}{sH \sqrt{T}} + \frac{24sKL(\sigma^2 + 2KG^2)}{H \sqrt{T}}
\]

\[
38
\]
Proof.

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \| \nabla f(X_t) \|^2 \leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \| \nabla f(X_t) - \nabla f(\mu_t) \|^2 \\
\leq \frac{2}{T} \sum_{t=0}^{T-1} \mathbb{E} \| \nabla f(X_t) - \nabla f(\mu_t) \|^2 + \frac{2}{T} \sum_{t=0}^{T-1} \| \nabla f(\mu_t) \|^2 \\
\leq \frac{2L^2}{T} \sum_{t=0}^{T-1} \mathbb{E} \| X_t - \mu_t \|^2 + \frac{2}{T} \sum_{t=0}^{T-1} \| \nabla f(\mu_t) \|^2 \\
\leq 2L^2 (80n^2(R^2 + 7)^2\gamma^2 + 80n^2sK\eta^2(\sigma^2 + 2KG^2) + 160B^2n^2sK^2\eta^2 \left( \frac{T}{T} \sum_{t=0}^{T-1} \mathbb{E} \| \nabla f(\mu_t) \|^2 \right) + \frac{2}{T} \sum_{t=0}^{T-1} \| \nabla f(\mu_t) \|^2 \\
\leq 160n^2L^2(R^2 + 7)^2\gamma^2 + 160n^2sK^2\eta^2(\sigma^2 + 2KG^2) + \frac{15(f(\mu_0) - f_*)}{sH\sqrt{T}} \\
+ \frac{24sKL(\sigma^2 + 2KG^2)}{HT} + \frac{4824n(n + 1)^2sK^2L^2(\sigma^2 + 2KG^2)}{HT} \\
+ \frac{2400n(n + 1)^2KL(f(\mu_0) - f_*)}{HT} \\
\leq 160n^2L^2\eta^2(\sigma^2 + 2KG^2 + \frac{f(\mu_0) - f_*}{L}) + 160n^2sK^2\eta^2(\sigma^2 + 2KG^2) + \frac{15(f(\mu_0) - f_*)}{sH\sqrt{T}} \\
+ \frac{24sKL(\sigma^2 + 2KG^2)}{HT} + \frac{4824n(n + 1)^2sK^2L^2(\sigma^2 + 2KG^2)}{HT} \\
+ \frac{2400n(n + 1)^2KL(f(\mu_0) - f_*)}{HT} \\
\leq \frac{15(f(\mu_0) - f_*)}{sH\sqrt{T}} + \frac{24sKL(\sigma^2 + 2KG^2)}{HT} \\
+ \frac{320n^2(n + 1)^2sKL^2}{T} + \frac{4824n(n + 1)^2sK^2L^2}{HT}(\sigma^2 + 2KG^2) \\
+ \frac{160n^2(n + 1)^2L}{T} + \frac{2400n(n + 1)^2KL}{HT}(f(\mu_0) - f_*)
\]

\[ \square \]
Finally, the proof of Corollary 4.3 follows from combining Lemmas A.10, 4.8 and 4.9.