Abstract. Algebraic and combinatorial properties of a monomial ideal and its radical are compared.

1. Introduction

There are simple examples of Cohen-Macaulay ideals whose radical is not Cohen-Macaulay. The first such example is probably due to Hartshorne [5], who proved that in positive characteristic the toric ring $K[s^4, s^3t, st^3, t^4]$ is a set theoretic complete intersection. With CoCoA or other computer algebra systems many other examples, also in characteristic zero, can be constructed. The following example due Conca was computed with CoCoA: let $S = K[x_1, x_2, x_3, x_4, x_5]$ and $J = (x_2^2 - x_4x_5, x_1x_3 - x_3x_4, x_3x_4 - x_1x_5) \subset S$. Then $S/J$ is a 2-dimensional Cohen-Macaulay ring, $\sqrt{J} = (x_1x_3 - x_1x_5, x_3x_4 - x_1x_5, x_2^3 - x_4x_5, x_1^2x_2 - x_4x_5, x_1x_2x_4, x_2x_3^2 - x_2x_3x_5)$ and $S/\sqrt{J}$ is not Cohen-Macaulay. Indeed, the depth of $S/\sqrt{J}$ equals 1. On the other hand it is well-known that the Cohen-Macaulay property of a monomial ideal is inherited by its radical. The reason is that the radical of a monomial ideal is essentially obtained by polarization and localization. This observation, was communicated to the third author by David Eisenbud. Both operations, polarization and localization, preserve the Cohen-Macaulay property. An explicit proof of this fact can be found in [11]. The purpose of this paper is to exploit this idea and to show that many other nice properties are inherited by the radical of a monomial ideal.

2. The comparison

For the proof of the main result of this paper we need some preparation. We begin with the following extension [10, Theorem 1.1] of Hochster’s formula [11, Theorem 5.3.8] describing the local cohomology of a monomial ideal.

Let $K$ be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring and $I \subset S$ a monomial ideal. The unique minimal monomial system of generators of $I$ is denoted by $G(I)$. For $i = 1, \ldots, n$ we set

$$t_i = \max \{\nu_i(u) : u \in G(I)\},$$

where for a monomial $u \in S$, $u = x_1^{a_1} \cdots x_n^{a_n}$ we set $\nu_i(u) = a_i$ for $i = 1, \ldots, n$.

For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we set

$$G_a = \{i : 1 \leq i \leq n, \ a_i < 0\},$$

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and define the simplicial complex $\Delta_a(I)$ whose faces are the sets $L \setminus G_a$ with $G_a \subset L$, and such that $L$ satisfies the following condition: for all $u \in G(I)$ there exists $i \notin L$ such that $\nu_i(u) > a_i > 0$.

Notice that the inequality $a_i \geq 0$ in the definition of $\Delta_a(I)$ follows from the condition $i \notin L \supset G_a$. It is included only for the reader’s convenience.

With the notation introduced one has

**Theorem 2.1** (Takayama [10]). Let $I \subset S$ be a monomial ideal. Then the Hilbert series of the local cohomology modules of $S/I$ with respect to the $\mathbb{Z}^n$-grading is given by

$$\text{Hilb}(H^i_m(S/I), t) = \sum_{F \in \Delta} \sum_a \dim_K \bar{H}_{i-|F|-1}(\Delta_a(I); K) t^a$$

where $\Delta$ is the simplicial complex corresponding to the Stanley-Reisner ideal $\sqrt{I}$, and the second sum is taken over all $a \in \mathbb{Z}^n$ such that $a_i \leq t_i - 1$ for all $i$, and $G_a = F$.

As a first application of this theorem we have

**Corollary 2.2.** Let $I \subset S$ be a monomial ideal. Then

$$a(S/I) \leq \sum_{i=1}^n t_i - n,$$

where $a(S/I)$ is the $a$-invariant of $S/I$.

**Proof.** By Theorem 2.1 we know that $H^i_m(R)_a = 0$ for all $i$ and for all $a \in \mathbb{Z}^n$ such that $a_i > t_i - 1$ for some $i$. Thus in particular, if $d = \dim R$, then $H^d_m(R)_j = 0$ for $j \geq \sum_{i=1}^n t_i - n$. □

We say that $S/I$ has maximal $a$-invariant if the upper bound in Corollary 2.2 is attained, that is, if $a(S/I) = \sum_{i=1}^n t_i - n$.

For our main theorem the next corollary is important.

**Corollary 2.3.** Let $I \subset S$ be a monomial ideal. Then we have the following isomorphisms of $K$-vector spaces

$$H^i_m(S/I)_a \cong H^i_m(S/\sqrt{I})_a$$

for all $a \in \mathbb{Z}^n$ with $a_i \leq 0$ for $1 \leq i \leq n$.

**Proof.** Consider the multigraded Hilbert series of $H^i_m(S/I)$ and $H^i_m(S/\sqrt{I})$. Let $a \in \mathbb{Z}^n$ be such that $a_i \leq 0$ for all $1 \leq i \leq n$. Then by Theorem 2.1 we have

$$\dim_K H^i_m(S/I)_a = \dim_K \bar{H}_{i-|F|-1}(\Delta_a(I); K), \quad \text{and}$$

$$\dim_K H^i_m(S/\sqrt{I})_a = \dim_K \bar{H}_{i-|F|-1}(\Delta_a(\sqrt{I}); K).$$

For a monomial $u$ we set $\text{supp}(u) = \{i : x_i \text{ divides } u\}$. Now since for every $u \in G(I)$ there exists $v \in G(\sqrt{I})$ such that $\text{supp}(u) \supset \text{supp}(v)$, and since for every $v \in G(\sqrt{I})$ there exists $u \in G(I)$ such that $\text{supp}(v) = \text{supp}(u)$, it follows that $\Delta_a(I) = \Delta_a(\sqrt{I})$. Thus we have $\dim_K H^i_m(S/I)_a = \dim_K H^i_m(S/\sqrt{I})_a$. □
Let $M$ be a graded $S$-module. For the convenience of the reader we recall the following two concepts which generalize the Cohen-Macaulay property and non-pure shellability of simplicial complexes.

The following definition is due to Stanley [9, Section II, 3.9]:

**Definition 2.4.** Let $M$ be a finitely generated graded $S$-module. The module $M$ is sequentially Cohen-Macaulay if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$$

of $M$ by graded submodules of $M$ such that each quotient $M_i/M_{i-1}$ is CM, and $\dim M_1/M_0 < \dim M_2/M_1 < \ldots < \dim M_r/M_{r-1}$.

It is known (see for example [6, Corollary 1.7]) that if $M$ is sequentially Cohen-Macaulay, then the filtration given in the definition is uniquely determined. We call it the attached filtration of the sequentially Cohen-Macaulay module $M$.

The uniqueness of the filtration is seen as follows: suppose $\text{depth } M = t$, then $M_1$ is the image of the natural map $\text{Ext}_S^{n-t}(\text{Ext}_S^{n-t}(M, \omega_S), \omega_S) \to M$. Here $\omega_S = S(-n)$ is the canonical module of $S$. Then one notices that $M/M_1$ is again sequentially Cohen-Macaulay and uses induction on the length of the attached sequence.

In case $M$ is a cyclic module, say, $M = S/I$, with attached filtration $0 = M_0 \subset M_1 \subset M_2 \subset \ldots$, each of the the modules $M_i$ is an ideal in $S/I$, and hence is of the form $I_i/I$ for certain (uniquely determined) ideals $I_i \subset S$. Thus $S/I$ is sequentially Cohen-Macaulay, if and only of there exists a chain of graded ideals

$$I = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_r = S$$

such that each factor module $I_{i+1}/I_i$ is Cohen-Macaulay with

$$\dim I_{i+1}/I_i < \dim I_{i+2}/I_{i+1}$$

for $i = 0, \ldots, r - 2$. Moreover if this property is satisfied, then this chain of ideals is uniquely determined.

In the particular case that $I$ is a monomial ideal, the natural map

$$\text{Ext}_S^{n-t}(\text{Ext}_S^{n-t}(S/I, \omega_S), \omega_S) \to S/I$$

is a homomorphism of multigraded $S$-modules. This implies that the attached chain of ideals of the sequentially Cohen-Macaulay module $S/I$ is a chain of monomial ideals.

Now let us briefly describe the other concept which was introduced by Dress [4]:

**Definition 2.5.** Let $M$ be a finitely generated graded $S$-module. A filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$$

of $M$ by graded submodules of $M$ is called clean if for all $i = 1, \ldots, r$ there exists a minimal prime ideal $P_i$ of $M$ such that $M_i/M_{i-1} \cong S/P_i$. The module $M$ is called clean if it has a clean filtration.

Again, if $M = S/I$ is cyclic, then $S/I$ is clean if there exists a chain of ideals $I = I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_{r-1} \subset I_r = S$ such that $I_{i+1}/I_i \cong S/P_i$ with $P_i$ a minimal prime ideal of $I$. In other words, for all $i = 0, \ldots, r - 1$ there exists $f_{i+1} \in I_{i+1}$ such
that \( I_{i+1} = (I_i, f_{i+1}) \) and \( P_i = I_i : f_{i+1} \). In case \( I \) is a monomial ideal we require that all \( f_i \) are monomials.

Dress [4] shows that a Stanley-Reisner ideal \( I_\Delta \) is clean if and only if the simplicial complex \( \Delta \) is non-pure shellable in the sense of Björner and Wachs [3].

In the proof of our main theorem we use polarization, as indicated in the introduction. Let \( I = (u_1, \ldots, u_m) \) with \( u_i = x_1^{a_{1i}} \cdots x_n^{a_{ni}} \). We fix some number \( i \) with \( 1 \leq i \leq n \), introduce a new variable \( y \), and set \( v_k = x_1^{a_{1i}} \cdots x_{i-1}^{a_{(i-1)i}} y x_{i+1}^{a_{i(i+1)}} \cdots x_n^{a_{ni}} \) if \( a_{ki} > 1 \), and \( v_k = u_k \) otherwise. We call \( J = (v_1, \ldots, v_m) \) the 1-step polarization of \( I \) with respect to the variable \( x_i \). The element \( y - x_i \) is regular on \( S[y]/J \) and \( (S[y]/J)/(y - x_i)(S[y]/J) \cong S/I \), see [1] Lemma 4.2.16.

Let as above \( t_i = \max \{ \nu_i(u_j) : j = 1, \ldots, m \} \), and set \( t = \sum_{i=1}^n t_i - n \). Then it is clear that if we apply \( t \) suitable 1-step polarizations, we end up with a squarefree monomial ideal \( I^p \), which is called the complete polarization of \( I \).

Now we are ready to present the main result of this section.

**Theorem 2.6.** Let \( K \) be a field, \( S = K[x_1, \ldots, x_n] \) the polynomial ring over \( K \), and \( I \subset S \) a monomial ideal. Suppose that \( S/I \) satisfies one of the following properties: \( S/I \) is (i) Cohen-Macaulay, (ii) Gorenstein, (iii) sequentially Cohen-Macaulay, (iv) generalized Cohen-Macaulay, (v) Buchsbaum, (vi) clean, or (vii) level and has maximal a-invariant. Then \( S/\sqrt{T} \) satisfies the corresponding property.

**Proof.** We first use the trick, mentioned in the introduction, to show that the Betti-numbers \( \beta_i(I) \) of \( I \) do not increase when passing to \( \sqrt{T} \).

We denote by \( I^p \) the complete polarization of \( I \). Let \( T \) be the polynomial ring in the variables that are needed to polarize \( I \). Then \( I^p \) is a squarefree monomial ideal in \( T \) with \( \beta_i(I^p) = \beta_i(I) \) for all \( i \). It is easy to see that if we localize at the multiplicative set \( N \) generated by the new variables which are needed to polarize \( I \), one obtains \( I^p T_N = (\sqrt{T}) T_N \). Since localization is an exact functor, the localized free resolution will be a possibly non-minimal free resolution of \( (\sqrt{T}) T_N \). Since the extension \( S \to T_N \) is flat, the desired inequality follows.

Proof of (i) and (ii): The inequality \( \beta_i(\sqrt{T}) \leq \beta_i(I) \) implies that \( \text{depth} S/\sqrt{T} \geq \text{depth} S/I \). On the other hand, \( \dim S/I = \dim S/\sqrt{T} \). This implies that \( S/\sqrt{T} \) is Cohen-Macaulay, if \( S/I \) is so.

Suppose now that \( S/I \) is Gorenstein. Then \( \beta_q(S/I) = 1 \) where \( q \) is the codimension of \( I \), see [1] Theorem 3.3.7 and Corollary 3.3.9]. Therefore, \( \beta_q(S/\sqrt{T}) \leq 1 \). Since \( I \) and \( \sqrt{T} \) have the same codimension, we see that \( \beta_q(S/\sqrt{T}) > 0 \), and hence \( \beta_q(S/\sqrt{T}) = 1 \). Again using [1] Theorem 3.3.7 and Corollary 3.3.9] we conclude that \( S/\sqrt{T} \) is Gorenstein. This fact follows also from [2] Corollary 3.4.]

Proof of (iii): Since \( S/I \) is sequentially Cohen-Macaulay there exists a chain of monomial ideals

\[
I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_k = S
\]
such that \( I_{j+1}/I_j \) is Cohen-Macaulay for all \( j = 0, \ldots, k-1 \) and such that \( \dim I_1/I_0 < \dim I_2/I_1 < \ldots < \dim I_k/I_{k-1} \).
Suppose \( x_1^a \) with \( a > 1 \) divides a generator of \( I \). Then we apply a 1-step polarization for \( x_1 \) to all the ideals \( I_i \), and obtain a chain of ideals \( J = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_k = \tilde{S} \) where \( \tilde{S} = \bar{S} \) where \( \bar{S} = S[y] \). It follows that \( y - x_1 \) is \( \tilde{S}/J_i \)-regular and \( (\tilde{S}/J_i)/(y - x_1)(\tilde{S}/J_i) \cong S/I_i \) for all \( i \). Therefore \( y - x_1 \) is \( J_{i+1}/J_i \)-regular, and \( (J_{i+1}/J_i)/(y - x_1)(J_{i+1}/J_i) \cong I_{i+1}/I_i \). Thus \( J \) is sequentially Cohen-Macaulay.

Since the complete polarization \( I_i^p \) of the ideals \( I_i \) for \( i = 1, \ldots, k \), is obtained by a sequence of 1-step polarizations, it follows that \( I^p \) is sequentially Cohen-Macaulay. As \( I_i^p/J_{i+1}^p \) is Cohen-Macaulay, we conclude as in the proof of (i) that \( \sqrt{I_{i+1}}/\sqrt{I_i} \) is Cohen-Macaulay of the same dimension as \( I_{i+1}/I_i \). This shows that \( \sqrt{T} \) is sequentially Cohen-Macaulay.

Proof of (iv) and (v): Assuming that \( S/I \) is generalized Cohen-Macaulay or Buchsbaum, one has that \( S/I \) is equidimensional and that \( H^i_m(S/I)_j = 0 \) for all \( i < \dim S/I \), and all but finitely many \( j \). Since \( I \) and \( \sqrt{T} \) have the same minimal prime ideals, it follows that \( \sqrt{T} \) is again equidimensional.

Let \( Z^n_a \) be the set of all \( a \in \mathbb{Z}^n \) such that \( a_i \leq 0 \) for \( i = 1, \ldots, n \). By Corollary 2.3, \( H^i_m(S/I)_a = H^i_m(S/\sqrt{T})_a \) for all \( a \in Z^n_a \). Moreover, by Hochster’s formula, \( H^i_m(S/\sqrt{T})_a = 0 \) for all \( a \notin Z^n_a \). Therefore, \( \dim_K H^i_m(S/\sqrt{T})_j \leq \dim_K H^i_m(S/I)_j \) for all \( j \leq 0 \) and \( H^i_m(S/I)_j = 0 \) for \( j > 0 \). It is known [8] that a squarefree monomial ideal is Buchsbaum if and only if it is generalized Cohen-Macaulay. Thus (iv) and (v) follow.

Proof of (vi): Assuming that \( S/I \) is clean, there exists a chain of monomial ideals \( I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_{r-1} \subset I_r = S \) such that \( I_{i+1}/I_i \cong S/P_i \) with \( P_i \) a minimal prime ideal of \( I \). We claim that \( \sqrt{I_{i+1}}/\sqrt{I_i} = S/P_i \), if \( \sqrt{I_{i+1}} \neq \sqrt{I_i} \). This then implies that \( S/\sqrt{T} \) is clean, since the prime ideals \( P_i \) are also minimal prime ideals of \( \sqrt{T} \).

In order to prove this claim we introduce some notation: let \( u = x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} \) and \( v = x_1^{b_1}x_2^{b_2} \cdots x_n^{b_n} \) be two monomials. Then we set

\[
\begin{align*}
  u : v &= \prod_{i=1}^{n} x_i^{\max\{a_i-b_i,0\}}, & u_{\text{red}} &= \prod_{a_i>0} x_i.
\end{align*}
\]

We then have

\[
(u : v)_{\text{red}} = (u_{\text{red}} : v_{\text{red}}) \prod_{a_i>b_i>0} x_i.
\]

Note that if \( I \) is a monomial ideal with monomial generators \( u_1, \ldots, u_m \), then

\[
\sqrt{T} = ((u_1)_{\text{red}}, \ldots, (u_m)_{\text{red}}) \quad \text{and} \quad I : v = (u_1 : v, \ldots, u_m : v).
\]

Back to the proof of our claim, our assumption implies that for all \( i = 0, \ldots, r-1 \) there exists a monomial \( v_{i+1} \in I_{i+1} \) such that \( I_{i+1} = (I_i, v_{i+1}) \) and \( P_i = I_i : v_{i+1} \). Suppose \( P_i = (x_{i_1}, \ldots, x_{i_s}) \). Then \( P_i \) is a prime ideal if and only if

(a) for all \( j = 1, \ldots, s \) there exists \( u \in I_i \) such that \( u : v_{i+1} = x_{i_j} \), and
(b) for all monomial generators \( w \in I_i \) there exists an integer \( j \) with \( 1 \leq j \leq s \) such that \( x_{ij}|w : v_{i+1} \).

We need to show that \( P_i = \sqrt{T_i} : (v_{i+1})_{\text{red}} \), if \( (v_{i+1})_{\text{red}} \not\in \sqrt{T_i} \), and prove this by checking (a) and (b) for the pair \( \sqrt{T_i} \) and \( (v_{i+1})_{\text{red}} \).

Let \( j \) be an integer with \( 1 \leq j \leq s \). Then there exists \( u \in I_i \) such that \( u : v_{i+1} = x_{ij} \). Suppose \( u = \prod_{k=1}^{n} x_k^{a_k} \) and \( v_{i+1} = \prod_{k=1}^{n} x_k^{b_k} \), then (1) implies that \( x_{ij} = (u : v_{i+1})_{\text{red}} = (u_{\text{red}} : (v_{i+1})_{\text{red}})w \) where \( w = \prod_{k, a_k > b_k} x_k \). Suppose \( x_{ij} \) divides \( w \), then \( u_{\text{red}} : (v_{i+1})_{\text{red}} = 1 \). This implies that \( (v_{i+1})_{\text{red}} \in \sqrt{T_i} \), a contradiction. Therefore \( u_{\text{red}} : (v_{i+1})_{\text{red}} = x_{ij} \), and this proves (a). The argument also shows that \( b_{ij} = 0 \) for \( j = 1, \ldots, s \).

For the proof of (b), let \( w \in I_i \) be a monomial generator. Then there exists an integer \( j \) with \( 1 \leq j \leq s \) such that \( x_{ij}|w : v_{i+1} \). It follows that \( x_{ij} \) divides \( (w : v_{i+1})_{\text{red}} \).

Let \( w = \prod_{k=1}^{n} x_k^{c_k} \). Then (1) implies that \( x_{ij} \) divides \( (w_{\text{red}} : (v_{i+1})_{\text{red}}) \prod_{k, c_k > b_k} x_k \). However, \( b_{ij} = 0 \), as we have seen in the proof of (a). Therefore, \( x_{ij} \) divides \( (w_{\text{red}} : (v_{i+1})_{\text{red}}) \). Since \( \sqrt{T_i} \) is generated by the monomials \( w_{\text{red}} \) where the monomials \( w \) are the generators of \( I_i \), condition (b) follows.

Proof of (vii): By assumption \( S/I \) is level. This means that \( S/I \) is Cohen-Macaulay and that all generators of the canonical module \( \omega_{S/I} \) of \( S/I \) have the same degree, say \( g \). In this situation the \( a \)-invariant \( a(S/I) \) of \( S/I \) is just \(-g \), see [11 Section 3.6]. Suppose \( d = \dim S/I \); then \( I \) has a graded minimal free resolution \( F \) of length \( q = n - d - 1 \) with \( F_q = S^b(-c) \). Since \( \omega_{S/I} \) may be represented as the cokernel of \( F^*_q \rightarrow F^*_q \), which is dual of the map \( F_q \rightarrow F_{q-1} \) with respect to \( S(-n) \), it follows that \( a(S/I) = c - n \).

For \( i = 1, \ldots, n \) we set again
\[
t_i = \max\{v_i(u) : u \in G(I)\}.
\]

By Corollary 222 one has the upper bound \( a(S/I) \leq \sum_{i=1}^{n} t_i - n \). Since we assume that \( S/I \) has maximal \( a \)-invariant, the upper bound is reached. Let \( I^p \subset T \) the complete polarization of \( I \). This polarization requires precisely \( t = \sum_{i=1}^{n} t_i - n \) 1-step polarizations. It follows that \( S/I \) is obtained from \( T/I^p \) as a residue class ring modulo a regular sequence of linear forms of length \( t \). From the above description of the \( a \)-invariant we now conclude that \( a(T/I^p) = a(S/I) - t = 0 \). Let \( G \) be the multigraded minimal free resolution of the squarefree monomial ideal \( I^p \). Since \( \operatorname{proj} \dim I^p = \operatorname{proj} \dim I = q \), and since \( a(T/I^p) = 0 \), we see that \( G_q = T(-m)^b \), where \( m = n + t = \dim T \). This implies that \( G_q \) as a multigraded module is isomorphic to \( T(-e)^b \) where \( e = (1, 1, \ldots, 1) \).

For \( i = 1, \ldots, m \) let \( e_i \) be the \( i \)-th canonical basis vector of \( \mathbb{Z}^m \). Then \( e = \sum_{i=1}^{m} e_i \), and we may assume that \( \deg x_i = e_i \) for \( i = 1, \ldots, n \), while the new variables have the multidegrees \( e_i \) with \( i = n + 1, \ldots, m \). We define a new multigrading on \( T \) and \( T/I^p \): for an element \( f \) of multidegree \( a \) we set \( \deg f = \pi(a) \), where \( \pi : \mathbb{Z}^m \rightarrow \mathbb{Z}^n \) is the projection onto the first \( n \) components of \( \mathbb{Z}^m \).

As above, let \( N \) be the multiplicative set generated by the \( t \) new variables which are needed to polarize \( I \). Then \( I^pT_N = \sqrt{T_N} \), and localization with respect to \( N \) preserves the new multigrading since \( \deg f = 0 \) for all \( f \in N \). Therefore \( G_N \) is,
with respect to the new grading, a multigraded free $T_N$-resolution of $\sqrt{TT}$ with $(G_q)_N = T_N(-1, \ldots, -1)^b$ and $(-1, \ldots, -1) \in \mathbb{Z}^n$.

Let $\mathbb{H}$ be the multigraded minimal free $S$-resolution of $\sqrt{T}$. Then $\mathbb{H}T_N$ is the minimal multigraded free $T_N$-resolution of $\sqrt{TT}$. A comparison with the (possibly non-minimal) graded free $T_N$-resolution $G_{T_N}$ shows that $H_q$ is a direct summand of copies of $S(-1, \ldots, -1)$. Since $S/I$ and $S/\sqrt{T}$ are Cohen-Macaulay of the same dimension, we see that $q = \text{proj dim } I = \text{proj dim } \sqrt{T}$. Therefore all summands in the last step of the resolution $\mathbb{H}$ of $S/\sqrt{T}$ have the same shift. This show that $S/\sqrt{T}$ is level. □

**Remark 2.7.** In Theorem 2.6(i) (or (iv)), it suffices to require that $I$ is an arbitrary homogeneous (generalized) Cohen-Macaulay ideal whose radical $\sqrt{T}$ is a monomial ideal, i.e. we do not need to require that $I$ itself is a monomial ideal.

Indeed it is enough to prove that there is a surjective homomorphism $H^i_m(S/I) \rightarrow H^i_m(S/\sqrt{T})$ for all $i$. The natural surjective map $S/I \rightarrow S/\sqrt{T}$ induce for all $i$ commutative diagrams

$$
\begin{array}{ccc}
\text{Ext}^i(S/\sqrt{T}, S) & \longrightarrow & \text{Ext}^i(S/I, S) \\
\downarrow & & \downarrow \\
H^i_{\sqrt{T}}(S) & \longrightarrow & H^i_I(S).
\end{array}
$$

Since $H^i_{\sqrt{T}}(S) \cong H^i_I(S)$ and since $\text{Ext}^i(S/\sqrt{T}, S) \longrightarrow H^i_{\sqrt{T}}(S)$ is an essential extension (see [12]), it follows that $\text{Ext}^i(S/\sqrt{T}, S) \longrightarrow \text{Ext}^i_I(S/I, S)$ is injective for all $i$. Hence the desired conclusion follows by local duality.

On the other hand, as for the Gorenstein property, we must assume that $I$ is a monomial ideal. For example, $I = (xy + yz, xz)$ is a complete intersection, hence, a Gorenstein ideal, while $\sqrt{T} = (xy, yz, xz)$ is not Gorenstein.

### 3. The inverse problem

The results of the previous section indicate the following question: for a subset $F \subseteq [n]$, let $P_F$ be the prime ideal generated by the $x_i$ with $i \in F$. The minimal prime ideals of a squarefree $I$ are all of this form, and since $I$ is a radical ideal it is the intersection of its minimal prime ideals, say, $I = \bigcap_{i=1}^r P_{F_i}$ with $F_i \subseteq [n]$.

Suppose $I$ is Cohen-Macaulay. For which exponents $a_{ij}$ is the ideal

$$J = \bigcap_{i=1}^r (x_j^{a_{ij}} : j \in F_i)$$

again Cohen-Macaulay?

Of course if we raise the $x_i$ uniformly to some power, say $x_i$ is replaced by $x_i^{a_i}$ everywhere in the intersection, then the resulting ideal $J$ is the image of the flat map $S \rightarrow S$ with $x_i \mapsto x_i^{a_i}$ for all $i$. Thus in this case $J$ will be Cohen-Macaulay, if $I$ is so. On the other hand, if we allow arbitrary exponents, the question seems to be quite delicate, and we do not know a general answer. However, if we require that for all choices of exponents the resulting ideal is again Cohen-Macaulay, a complete answer is possible.
We need a definition to state the next result. Let \( L \) be a monomial ideal. Lyubeznik \[7\] defines the size of \( L \) as follows: let \( L = \bigcap_{j=1}^{r} Q_j \) be an irredundant primary decomposition of \( L \), where the \( Q_j \) are monomial ideals. Let \( h \) be the height of \( \sum_{j=1}^{r} Q_j \), and denote by \( v \) the minimum number \( t \) such that there exist \( j_1, \ldots, j_t \) with \( \sqrt{\sum_{i=1}^{t} Q_{j_i}} = \sqrt{\sum_{j=1}^{r} Q_j} \). Then size \( L = v + (n - h) - 1 \).

Since for monomial ideals the operations of forming sums and taking radicals can be exchanged, the numbers \( v \) and \( h \), and hence the size of \( L \) depends only on the associated prime ideals of \( L \).

We shall need the following result of Lyubeznik \[7, Proposition 2\]:

**Lemma 3.1.** Let \( L \) be a monomial ideal in \( S \). Then \( \text{depth} \ S/L \geq \text{size} \ L \).

Now we can state the main result of this section.

**Theorem 3.2.** Let \( I \subset S = K[x_1, \ldots, x_n] \) be a Cohen-Macaulay squarefree monomial ideal, and write

\[
I = \bigcap_{i=1}^{r} P_{F_i},
\]

where the sets \( F_i \subset [n] \) are pairwise distinct, and all have the same cardinality \( c \). For \( i = 1, \ldots, r \) and \( j = 1, \ldots, c \) we choose integers \( a_{ij} \geq 1 \), and set

\[
Q_{F_i} = (x_j^{a_{ij}} : j \in F_i) \quad \text{for} \quad i = 1, \ldots, r.
\]

Then the following conditions are equivalent:

(a) for all choices of the integers \( a_{ij} \) the ideal

\[
J = \bigcap_{i=1}^{r} Q_{F_i}
\]

is Cohen-Macaulay;

(b) for each subset \( A \subset [r] \), the ideal \( I_A = \bigcap_{i \in A} P_{F_i} \) is Cohen-Macaulay;

(c) height \( P_{F_i} + P_{F_j} = c + 1 \) for all \( i \neq j \);

(d) for \( r \geq 2 \) either \( \left| \bigcup_{i=1}^{r} F_i \right| = c + 1 \), or \( \left| \bigcap_{i=1}^{r} F_i \right| = c - 1 \);

(e) after a suitable permutation of the elements of \([n]\) we either have

\[
F_i = \{1, \ldots, i-1, i+1, \ldots, c, c+1\} \quad \text{for} \quad i = 1, \ldots, r,
\]

or

\[
F_i = \{1, \ldots, c-1, c-1+i\} \quad \text{for} \quad i = 1, \ldots, r;
\]

(f) size \( I = \dim S/I \);

(g) \( S/L \) is Cohen-Macaulay for any monomial ideal \( L \) such that \( \text{Ass} \ L = \text{Ass} \ I \).

**Proof.** (a) \( \Rightarrow \) (b): Let \( Q_{F_i} = (x_j^{2^i} : j \in F_i) \) if \( i \in A \), and \( Q_{F_i} = P_{F_i} \) if \( i \notin A \). By assumption, \( J = \bigcap_{i=1}^{r} Q_{F_i} \) is Cohen-Macaulay. Hence the complete polarization \( J^p \) of \( J \) is again Cohen-Macaulay. We have \( J^p = \bigcap_{i=1}^{r} Q_{F_i}^p \) with \( Q_{F_i}^p = (x_j y_j : j \in F_i) \) if
$i \in A$, and $Q^p_{F_i} = P_{F_i}$ if $i \notin A$. Let $N$ be the multiplicative set generated by all the variables $x_i$. Then $J^p_N$ is Cohen-Macaulay, and hence

$$J^p_N = \bigcap_{i \in A} (y_j : j \in F_i).$$

This shows that $I_A = \bigcap_{i \in A} P_{F_i}$ is Cohen-Macaulay.

(b) $\Rightarrow$ (c): Consider the exact sequence

$$0 \rightarrow S/(P_{F_i} \cap P_{F_j}) \rightarrow S/P_{F_i} \oplus S/P_{F_j} \rightarrow S/(P_{F_i} + P_{F_j}) \rightarrow 0.$$  

The rings $S/P_i$ and $S/P_j$ are Cohen-Macaulay of dimension $n - c$, while $S/(P_{F_i} + P_{F_j})$ is Cohen-Macaulay of dimension $n - d$ where $d$ is the height of $P_{F_i} + P_{F_j}$. The exact sequence yields that $S/(P_{F_i} \cap P_{F_j})$ is Cohen-Macaulay if and only if $d = c + 1$.

Since by assumption $S/P_{F_i} \cap P_{F_j}$ is Cohen-Macaulay for all $i \neq j$, the assertion follows.

(c) $\Rightarrow$ (d): We must show: given a collection of subsets $F_1, \ldots, F_r \subset [n]$ with

(i) $|F_i| = c$ for all $i$;
(ii) $|F_i \cup F_j| = c + 1$ for all $i \neq j$.

Then either $|\bigcup_{i=1}^r F_i| = c + 1$, or $|\bigcap_{i=1}^r F_i| = c - 1$.

Suppose this is not the case. Then, since $|F_i \cap F_j| = c - 1$ and $|F_i \cup F_j| = c + 1$, there exist integers $i$ and $j$ such that $F_i \cap F_j \neq F_i$, and $F_j \neq F_i \cup F_j$. The conditions (i) and (ii) then imply that there exists an element $x \in F_i \cap F_j$ such that $F_i \cup F_j \setminus \{x\} = F_i$, and an element $y \in F_j \setminus (F_i \cup F_j)$ such that $F_j = \{y\} \cup (F_i \cap F_j)$. It follows that $F_i \cup F_j = (F_i \cup F_j) \cup \{y\}$. This contradicts (ii).

(d) $\Rightarrow$ (e): Assume that $|\bigcup_{i=1}^r F_i| = c + 1$. After a suitable permutation of the elements of $[n]$ we may assume that $\bigcup_{i=1}^r F_i = \{1, \ldots, c + 1\}$. Since $|F_i| = c$, there exists $j_i \in \{1, \ldots, c + 1\}$ such that $F_i = \{1, \ldots, c + 1\} \setminus \{j_i\}$. Since the sets $F_i$ are pairwise distinct it follows that $j_i \neq j_k$ for $i \neq k$. Thus after applying again suitable permutation we may assume that $j_i = i$ for $i = 1, \ldots, r$.

The second statement follows similarly.

(e) $\Rightarrow$ (f): In the first case, $v = 2$ and $h = (c + 1)$, while in the second case, $v = r$ and $h = c - 1 + r$. Thus in both cases size $I = n - c = \dim S/I$.

(f) $\Rightarrow$ (g): By Lemma 3.1 and the remark preceding the lemma, we have

$$\text{depth } S/L \geq \text{size } L = \text{size } I = \dim S/I = \dim S/L.$$ 

Hence $S/L$ is Cohen-Macaulay.

Finally the implication (g) $\Rightarrow$ (a) is trivial. \hfill \square

**Corollary 3.3.** With notation as above, the following conditions are equivalent:

(a) $J$ is a Gorenstein ideal for all choices of the integers $a_{ij}$;
(b) $r = 1$ or $c = 1$.

**Proof.** If $r = 1$ or $c = 1$, then $J$ is complete intersection for all choices of the integers $a_{ij}$. Thus (b) implies (a).

Conversely suppose condition (b) is not satisfied. We assume that $c > 1$, and have to show that $r = 1$. By Theorem 3.2 we have $|\bigcap_{i=1}^r F_i| = c - 1$ or $|\bigcup_{i=1}^r F_i| = c + 1$. 


In the first case we may assume that \( F_i = \{1, \ldots, c - 1, i + c - 1\} \) for \( i = 1, \ldots, r \). Assume \( r > 1 \), and let \( Q_{F_1} = (x_1^2, x_2, \ldots, x_c) \) and \( Q_{F_i} = P_{F_i} \) for \( i \geq 2 \). Then \( J = \bigcap_{i=1}^{r} Q_{F_i} = (x_1^2, x_1 x_2, \prod_{i=0}^{r-1} x_{c+i}) \) is not Gorenstein, a contradiction.

In the second case suppose that \( r \geq 3 \). With the same argument as in the proof of Theorem 3.2 it follows that \( I_A = \bigcap_{i \in A} P_{F_i} \) is a Gorenstein ideal for all subsets \( A \subset [r] \). Therefore \( P_{F_1} \cap P_{F_2} \cap P_{F_3} \) is Gorenstein. We may assume that \( F_1 = \{1, 2, \ldots, c\}, F_2 = \{2, 3, \ldots, c + 1\} \) and \( F_3 = \{1, 3, 4, \ldots, c + 1\} \). Then \( P_{F_1} \cap P_{F_2} \cap P_{F_3} = (x_1 x_2, x_1 x_{c+1}, x_2 x_{c+1}, x_3, \ldots, x_c) \) is not Gorenstein, a contradiction.

On the other hand, if \( r = 2 \), then \( |\bigcap_{i=1}^{r} F_i| = c - 1 \), and we are again in the first case. Thus we must have that \( r = 1 \). □

**Remark 3.4.** From a viewpoint of Stanley-Reisner rings, the ideal \( I \) in the first case of condition (e) in Theorem 3.2 corresponds to an iterated cone of a 0-dimensional simplicial complex. In this case it is known that \( S/I \) itself is Gorenstein if the corresponding 0-dimensional simplicial complex consists of at most 2 points, see [9, Theorem 5.1(e)]. The corollary also follows from this fact.
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