Duff: A Dataset-Distance-Based Utility Function Family for the Exponential Mechanism

Jennifer Gillenwater  
Google Research  
111 8th Ave, New York, NY  
jengi@google.com

Andrés Muñoz Medina  
Google Research  
111 8th Ave, New York, NY  
ammedina@google.com

Abstract

We propose and analyze a general-purpose dataset-distance-based utility function family (Duff) for differential privacy’s exponential mechanism. Given a particular dataset and a statistic (e.g., median, mode), this function family assigns utility to a possible output \( o \) based on the number of individuals whose data would have to be added to or removed from the dataset in order for the statistic to take on value \( o \). We show that the exponential mechanism based on Duff often offers provably higher fidelity to the statistic’s true value compared to existing differential privacy mechanisms based on smooth sensitivity. In particular, Duff is an affirmative answer to the open question of whether it is possible to have a noise distribution whose variance is proportional to smooth sensitivity and whose tails decay at a faster-than-polynomial rate. We conclude our paper with an empirical evaluation of the practical advantages of Duff for the task of computing medians.

1 Introduction

The age of big data has brought with it an unprecedented ease-of-access to large amounts of information. From predicting traffic in cities to explaining voting patterns across the country, readily available large datasets have made it possible for researchers to analyze and model the behavior of people at a very fine granularity. Unrestricted access to this data, however, puts the privacy of individuals at risk. Indeed, many datasets used today contain very sensitive information about individuals, such as one’s political affiliation or home location. For this reason, analysts and machine learning practitioners must ensure that the output of their research (models or aggregated statistics) provide only information about the population as a whole and do not leak personal details. One might feel, intuitively, that certain outputs are inherently “safe”. For example, statistics aggregated over a sufficiently large number of individuals, such as the median of a dataset that contains data from thousands of people. But in fact it has been proven [9], and later verified in practice [1], that even with such aggregated statistics one can discover very personal information.

A formal guarantee, called differential privacy (DP) [11], has become the standard way to ensure that individual information is protected. Examples of its use in industry and by government entities can be found in Google [14], Apple [18], and the census bureau [1]. In order to protect user information, a DP mechanism usually perturbs the output of an aggregation with noise. The amount of noise depends on the amount of privacy we want to provide. There is an inherent trade-off here: more privacy implies that the perturbed value will likely be further from the true value. Proximity to the truth is known as the utility of the mechanism. The goal of DP research is to maximize utility while preserving a desired level of privacy. This process has lead to the creation of a plethora of mechanisms whose privacy-utility trade-offs are often difficult to compare. Moreover, even for the problem of releasing simple statistics such as the median, there is a lack of thorough empirical evaluation. This has made adoption of DP by practitioners difficult.
The goal of this paper is to help simplify practitioners’ work by providing utility guarantees for DP’s exponential mechanism [16] when it is used with a general-purpose dataset-distance-based utility function family, or Duff. Duff is general enough to be useful for many tasks, but also specific enough that we can prove meaningful theorems about its privacy-utility trade-off. The paper is organized as follows: We begin by introducing the basic definitions of DP and discussing previous work. In Section 4 we introduce Duff and provide an analysis of its utility. This analysis shows a previously unknown connection between the exponential mechanism and smooth sensitivity [17]. We also instantiate Duff for several statistics. In Section 5 we present a general sampling algorithm for an exponential mechanism using Duff. Finally, in Section 6 we empirically evaluate on the task of computing medians.

2 Definitions

In this section we present some common concepts of DP. We will denote by $\mathcal{S}$ a universe of datasets. A dataset $S \in \mathcal{S}$ is a collection of information about individuals. We assume that each individual contributes no more than one value to a given $S$.

**Definition 1.** Datasets $S, S' \in \mathcal{S}$ are neighbors if $S$ can be obtained from $S'$ by adding or removing a single element. We denote the neighbors of a dataset $S$ by $\mathcal{N}(S)$.

**Definition 2.** Let $M$ denote a (possibly randomized) function mapping a dataset $S$ to an output in $[a, b]$. We say $M$ is an $(\epsilon, \delta)$-DP mechanism if, for any two neighboring datasets $S$ and $S'$, and any set of outcomes $A \subseteq [a, b]$, it holds that: $\Pr(M(S) \in A) \leq e^{\epsilon} \Pr(M(S') \in A) + \delta$.

Given some target statistic of a dataset, $T : \mathcal{S} \rightarrow [a, b]$ (e.g., median, mode), one common mechanism $M$ that is often used to release a DP version of $T$ is $M(S) = T(S) + \text{noise}$. The exact distribution and scale of the noise required to satisfy the DP definition has been the subject of substantial research. Much of this research has focused on understanding the so-called sensitivity of $T$.

**Definition 3.** Given a $T : \mathcal{S} \rightarrow [a, b]$ and an $S \in \mathcal{S}$, the local sensitivity of $T$ at $S$ is:

$$LS(T, S) = \max_{S' \in \mathcal{N}(S)} |T(S) - T(S')|.$$  

**Definition 4.** Given a $T : \mathcal{S} \rightarrow [a, b]$, the (global) sensitivity of $T$ is: $GS(T) = \max_{S \in \mathcal{S}} LS(T, S)$.

Adding noise to $T$ with variance proportional to $GS(T)$ is one way of constructing a mechanism $M$ that is DP.

**Proposition 1 (Laplace mechanism).** Let $T$ be a function with sensitivity $GS(T)$. Then the mechanism $M$ that releases $M(S) = T(S) + \frac{GS(T)}{\epsilon} Z$, where $Z \sim \text{Lap}(0, 1)$, is $(\epsilon, 0)$-DP.

While the Laplace mechanism is simple, the amount of noise added can be more than is strictly necessary. This is because $GS(T)$ is a max over all possible datasets. It might seem like an easy fix is to just add noise proportional to $LS(S)$. However, this is not DP, because $LS(S)$ is itself a sensitive quantity; see Section 2.1 of [17] for a detailed example. To bridge the gap between $LS$ and $GS$, [17] introduced the notion of smooth sensitivity, which depends on the distance between datasets.

**Definition 5.** Datasets $S, S'$ are at distance $k$, denoted $d(S, S') = k$, if the existence of a sequence $S = S_0, \ldots, S_m = S'$ such that $S_i \in \mathcal{N}(S_{i-1})$ implies $m \geq k$.

For completeness, the appendix contains a proof that $d$ is a true distance function (a metric) over $\mathcal{S}$.

**Definition 6.** Given a function $T : \mathcal{S} \rightarrow [a, b]$, $\beta > 0$ and a dataset $S \in \mathcal{S}$, the $\beta$-smooth sensitivity of $T$ at $S$ is: $SS_\beta(T, S) = \max_{k \geq 0} \max_{S' : d(S, S') = k} e^{-\beta k} LS(T, S')$.

The smooth sensitivity is a low-sensitivity estimate of the the local sensitivity. If elements of $\mathcal{N}(S)$ have similar local sensitivity, then $SS_\beta \approx LS$. On the other hand, if there are datasets near $S$ with local sensitivity close to $GS$ then $SS_\beta \approx GS$. Formally (see [17]), for all $\beta > 0$:

$$LS(T, S) \leq SS_\beta(T, S) \leq GS(T).$$  

(1)

One crucial property of $SS_\beta$ is that the Laplace mechanism remains DP if $GS$ is replaced by $SS_\beta$.

**Proposition 2.** [7, Theorem 34] Fix some $\epsilon, \delta > 0$. Let $\alpha > 0$ and $\beta > 0$ satisfy $\epsilon \geq \alpha + (e^\beta - 1) \log(1/\delta) - \beta$. Then the mechanism that returns $T(S) + \frac{SS_\beta(T, S)}{\alpha} Z$, where $Z \sim \text{Lap}(0, 1)$, is $(\epsilon, \delta)$-DP.
It is also possible to achieve \((\epsilon, 0)\)-DP with a smooth sensitivity mechanism. However, the noise that must be added in this case has polynomial rather than exponential tail decay.

**Proposition 3** (Combining Lemmas 2.6 and 2.7 from [17]). Fix some \(\epsilon > 0\) and some \(\gamma > 1\). Define the density \(h(z) \propto \frac{1}{1 + |z|^\gamma}\). Let \(\alpha = \beta = \frac{\epsilon}{\pi (\gamma + 1)}\). Then the mechanism that returns \(T(S) + \frac{55u(T,S)}{\alpha}Z\), where \(Z \sim h(z)\), is \((\epsilon, 0)\)-DP.

Smooth sensitivity is particularly useful for estimating medians or more generally, quantiles. Indeed, it is not hard to see that the global sensitivity of the median, for a dataset where elements are in \([a, b]\), is simply \(b - a\). Adding noise of that scale would overwhelm any signal in the output. On the other hand, for a *usual* dataset, changing a few elements won’t change the median much. Therefore, we would expect smooth sensitivity to be much smaller than global sensitivity.

We now turn our attention to another popular DP mechanism and the main focus of this paper: the exponential mechanism (EM). This mechanism defines a distribution using a *utility function*.

**Definition 7.** [16, Definition 2] Given a utility function \(u : [a, b] \times S \rightarrow \mathbb{R}\), the **exponential mechanism** outputs \(x \in [a, b]\) with probability proportional to \(\exp\left(\frac{u(x,S)}{\Delta_u}\right)\), where \(\Delta_u\) is the sensitivity of \(u\): \(\Delta_u = \max_{S \in \mathcal{V}} \max_{S' \in N(S)} \max_{x \in [a, b]} |u(x, S) - u(x, S')|\).

The EM is always \((\epsilon, 0)\)-DP. Moreover, it is known to have the following utility guarantee, whose proof is included in the appendix for completeness.

**Proposition 4** ([16]). Let \(S\) be a dataset and \(u : [a, b] \times S \rightarrow \mathbb{R}\) be a utility function with sensitivity \(\Delta_u\). Let \(X\) be a random variable sampled according to the EM and let \(\lambda\) denote the Lebesgue measure. If \(OPT = \max_x u(x, S)\) and \(H_t = \{x : u(x, S) > OPT - t\}\) then:

\[
P(u(X, S) < OPT - t) \leq \frac{(b - a)}{\lambda(H_{t/2})} e^{-\frac{\epsilon t}{\Delta_u}}.
\]

In words: the probability of an output with utility \(t\) or more below \(OPT\) is exponentially small in \(t\). While the above guarantee is useful as a general-purpose bound, it is often hard to interpret. In Section 4 we present the first analysis expressing the utility guarantees of the exponential mechanism in terms of the more intuitive notion of smooth sensitivity. This connection:

* helps us theoretically compare two seemingly unrelated mechanisms, EM and the smooth-sensitivity-based mechanism of Proposition 2, and
* shows that it is possible to define an \((\epsilon, 0)\)-DP mechanism with faster-than-polynomial noise decay whose variance is also as small as the smooth sensitivity.

This might seem like a contradiction to [17], which states that *admissible* distributions [17, Definition 2.4] require polynomial tails. However, their definition of “admissible” does not encompass all possible DP mechanisms, so it does not actually imply the non-existence of a DP mechanism with faster-than-polynomial decay.

### 3 Related work

Nissim et al. [17] introduced the notion of smooth sensitivity and applied it to reduce the noise required to estimate statistics such as medians. They proposed \((\epsilon, 0)\)-DP mechanisms with polynomially decaying tails and \((\epsilon, \delta)\)-DP mechanisms with exponentially decaying tails. These mechanisms were tightened by [7], who introduced light-tailed distributions with variance proportional to the smooth sensitivity. The privacy guarantees of [7], however, are under the relaxed model of *concentrated* DP. Thus, to the best of our knowledge it remains an open question whether there is a truly \((\epsilon, 0)\)-DP mechanism with variance proportional to smooth sensitivity that also has faster-than-polynomial tail decay. In this work we answer the question affirmatively with Duff.

In addition to smooth sensitivity, another common approach to tailoring noise to a dataset is the propose-test-release method [12]. This method tests (in a private way) whether the sensitivity of a function on a particular dataset is below a given value \(\tau\). If the test passes, then the mechanism outputs the true function value perturbed by noise with variance proportional to \(\tau\). If the test fails, then null (no output) is returned. Because there is always a probability of having a false positive in the test, this mechanism can never achieve \((\epsilon, 0)\)-DP. Moreover, the performance of this mechanism depends
heavily on the type of test used on the first step. Since this dependence is not easily parametrized, designing the best test for each function is a non-trivial task.

Finally, our work builds upon the well-studied exponential mechanism (EM) of [16]. Traditionally, the EM has been used for output spaces where generalizations of the Laplace distribution are not readily available. For instance, the EM has been exploited when the output space consists of: databases [6], eigenvectors [2], and infinite-dimensional vectors [4]. In contrast, we propose applying the EM to outputs, such as medians, which have commonly been made private using a mechanism involving a Laplace distribution. To do so, we introduce a dataset-distance-based utility function family, or Duff, which allows us to a) easily control the EM’s sensitivity, and b) provide a data-dependent analysis of the EM. Before this work, the utility guarantees for the EM were either dataset-independent [16], specific to a particular task [2, 4], or conditioned on the data being drawn from a smooth distribution [19].

4 Duff: dataset-distance-based utility function family

As described in the previous sections, the EM is a popular tool for releasing private information. However, the particular choice of utility function is crucial to its performance. For example, consider the task of estimating the median of a dataset \( S = \{y_1, \ldots, y_n\} \), with \( y_i \in [a, b] \). We will assume the median index \( m \) to be \((n + 1) / 2\) if \( n \) is odd and \( n / 2 \) if \( n \) is even. One could use either of the following utility functions:

\[
  u_1(x, S) = -|x - T(S)| \quad \text{or} \quad u_2(x, S) = -\left| \sum_{y \in S} 1(y < x) - \sum_{y \in S} 1(y > x) \right|.
\]

Both functions achieve their maximum value at the true median, \( x = T(S) \). However, if we consider their ranges and sensitivities, it becomes clear that \( u_2 \) will provide much better median estimates:

- \( u_1 \): Range is \([- (b - a), 0]\). Sensitivity is \( \Delta_{u_1} = b - a \); in the extreme case where there is only a single point in the dataset, \( S = \{b\} \), the median can shift from the max value \( b \) all the way to the min value \( a \) with the addition of a single new point: \( S' = \{a, b\} \).
- \( u_2 \): Range is \([-n, 0]\). Sensitivity is \( \Delta_{u_2} = 1 \); if a point is added to (or removed from) \( S \), then an indicator function is added to (or removed from) each summation in \( u_2 \), and only one of these two indicators will ever be active for a given \( x \).

The range of \( u_1 \) is equal to its sensitivity. In contrast, \( u_2 \)'s range is much larger than its sensitivity. This means that the corresponding EM density is much smaller at quantiles far from the median.

**Definition 8.** For dataset \( S \) and statistic \( T \), Duff is \( u_d(x, S) = -\min_{S' : T(S') = x} d(S, S') \).

Intuition: This utility function considers all datasets that have statistic value \( x \), and finds one that is closest to the actual dataset \( S \). The \( d \) value is then a measure of how difficult it is to go from \( T(S) \) to \( x \), and so its negation is the utility of output \( x \) for the dataset \( S \). One of the important features of \( Duff \) is that the sensitivity of this function is always 1. (See Lemma 1 in the appendix.)

4.1 Instantiations

We now instantiate \textit{Duff} for a few statistics to illustrate that it often reduces to a simple step function.

**Modes.** Consider a dataset \( S \) with values from a finite set \( \mathcal{Y} \). For every \( y \in S \), let \( n_y \) denote the frequency of that element in \( S \), and let \( n_{max} = \max_{y \in S} n_y \) denote the frequency of the mode. Then it is not hard to see that \textit{Duff} reduces to the following simple function: \( u_d(x, S) = n_x - n_{max} \).

**Medians.** Let \( S = \{y_1, \ldots, y_n\} \) with each \( y_i \in [a, b] \), and assume \( y_i < y_{i+1} \). We will again consider the median index \( m \) to be \((n + 1) / 2\) if \( n \) is odd and \( n / 2 \) if \( n \) is even. For convenience, we also define \( y_0 = a \) and \( y_{n+1} = b \). Then we can show that \( u_d \) reduces to the step function given below.

- For \( 1 \leq k \leq m \) and \( y_{m-k} \leq x \leq y_{m-k+1} \), we have \( u_d(x, S) = -2k + 1(n \text{ odd}) \).
We now analyze how similar the output of the EM based on our utility function $u$ where

$$\lambda \in \mathbb{R}$$

is continuous and is the product of a decreasing and a strictly decreasing function, which makes it defined as the inverse of the function $\beta$. Proofs of all theorems, corollaries, etc. can be found in the appendix.

**Means.** Let $S = \{y_1, \ldots, y_n\}$ with each $y_i \in [a, b]$. Our target statistic is $\mu \eqdef \frac{1}{n} \sum_{y \in S} y$. Consider how this statistic changes when points are added to or removed from $S$:

- Adding a single new element $y$: This changes the mean to a value of $\mu'(y) \eqdef \frac{n}{n+1} \cdot \frac{y}{n+1}$. Thus, any mean $x$ in the range $\mu'(a) \leq x \leq \mu'(b)$ has utility $u_d(x, S) = -1$ (except for the true mean value, which has $u_d(\mu, S) = 0$).
- Removing a single element $y_i$: This changes the mean to a value of $\mu \eqdef \frac{n}{n-1} - \frac{y}{n-1}$. But since the value of each $y_i$ is fixed, the measure of the output space that we can reach with removals alone is zero. Thus, removals are really only meaningful when combined with additions.

To efficiently determine the minimum combination of additions and removals that covers the remaining regions of the output space $[a, b]$ requires a dynamic program. To see that the combinations that it must consider are bounded, notice that any mean value is achievable with $n+1$ changes to the original data—remove all $n$ original data points and add a point at value $x$ to get that value as the mean. Thus, ultimately we end up with a step function that has no more than $n+1$ levels.

### 4.2 Utility bounds

We now analyze how similar the output of the EM based on our utility function $u_d$ will be to the true value of a target statistic $T$. Our guarantees will be stated in terms of the function $\beta^*_T, S : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, defined as the inverse of the function $\beta \mapsto \frac{SS_{T, S}}{\beta}$. The inverse $\beta^*_T, S$ is well-defined since $SS_{T, S}$ is continuous and is the product of a decreasing and a strictly decreasing function, which makes it strictly decreasing. When clear from context, we will remove the dependency on $T$ and $S$ from the function $\beta^*$. Proofs of all theorems, corollaries, etc. can be found in the appendix.

**Theorem 1.** Let $x \in [a, b]$ denote the output of the EM with utility function $u_d$. Let $\lambda$ denote the Lebesgue measure and $\gamma = \frac{1}{2\pi(\sqrt{2} - 1)}$. If $H_t = \{x \mid u_d(x, S) \geq -t\}$, then:

$$P(\lvert x - T(S) \rvert > t) \leq \frac{2 \exp\left(-\frac{\pi t^2}{2}\right)(b - a)}{\lambda(H_t)}. \quad (2)$$

As a corollary of the previous theorem we obtain the following high-probability bound.

**Corollary 1.** Let $H_t$ be as in Theorem 1 and fix $\eta > 0$. Assume $\lambda(H_t) \geq Ct$ for some constant $C > 0$. Let $\beta_{exp} = \frac{\epsilon}{4W(\frac{\epsilon}{C\eta})}$, where $W$ is the main branch of the Lambert function$^1$. Then with probability at least $1 - \eta$:

$$\lvert x - T(S) \rvert < 4(e - 1) \frac{SS_{exp} \beta_{exp} W\left(\frac{\epsilon(b - a)}{C\eta}\right)}{\epsilon} \quad (3)$$

Our corollary depends on the assumption that $\lambda(H_t) \geq Ct$. A simple setting where we can estimate the value of $C$ is the following: Let $S$ consist of points evenly-spaced on $[a, b]$ and consider the problem of estimating the median. Using the Duff introduced in Section 4.1 we observe that $u_d(x, S) > -t$ for $\lvert x - T(S) \rvert < \frac{t(b - a)}{n}$. In this case $C = \frac{2(b - a)}{n}$. In general, it is hard to estimate $C$, but we believe for many typical datasets and common statistics it will be $O(\frac{1}{n})$.

We now compare our bound with the one provided in the original smooth sensitivity paper [17]. We use the $(\epsilon, 0)$-DP mechanism from in Proposition 3 with $\gamma = 2$ (yielding the Cauchy distribution).

$^1$The Lambert function is the inverse of the function $x \mapsto xe^x$
Therefore, sampling from the EM is equivalent to sampling an index $k$ included in camera-ready), in which they considered the numerical instability issues of estimating medians via the reservoir sampling algorithm (of [13] and depends on the fact that the minimum of a collection of exponential random variables also follows an exponential distribution. The proof that Algorithm 1 is correct hinges on Proposition 5, which is proven in the appendix.

5 Sampling algorithms

One of the main disadvantages of the EM is that, depending on the shape of the utility function, sampling may not be straightforward or may be too costly in practice [2, 5]. In this section we take advantage of the fact that Duff takes values in $N_0$ to provide a sampling algorithm for it.

Let $A_k = v_d^{-1}(-k, S) \subset [a, b]$ represent the portion of the output space that has utility $-k$. Then let $K \in N_0$ be a random variable with the following probability mass function:

$$p_k := P(K = k) \propto \lambda(A_k)e^{-\frac{\lambda(A_k)}{2}}.$$  

(5)

Further, let $Z \in [a, b]$ denote a random variable defined by the following conditional density:

$$P(Z = z \mid K = k) = \frac{1}{M(A_k)}1(z \in A_k).$$

We claim that the distribution of $Z$ is the one induced by the EM. To verify this, let $z \in [a, b]$ and let $k_0$ be such that $z \in A_{k_0}$. Then:

$$P(Z = z) = \sum_{k \geq 0} P(Z = z \mid K = k)P(K = k) = P(Z = z \mid K = k_0)P(K = k_0)$$

$$\propto e^{-\frac{\lambda(A_{k_0})}{2}} = e^{-\frac{\gamma_d(z, S)}{2}}.$$

Therefore, sampling from the EM is equivalent to sampling an index $k$ proportional to $p_k$ and then sampling uniformly an element from $A_k$. Sampling uniformly from $A_k$ is typically trivial. For example, for the medians estimation task discussed in Section 4.1, each $A_k$ is a single continuous interval on the real line. So we focus here on how to sample $k$ proportional to $p_k$. In order to solve this problem we need to address two issues.

First, the domain of $K$ could technically be infinite. In practice however, the domain of $K$ is often finite. This is the case for the mode, median, and mean estimation tasks described in Section 4.1. In all three of those cases the set of values Duff can take is a linear function of the number of elements in the dataset. So we focus here on developing an algorithm for the case where Duff only takes on a finite number $N$ of values.

The other issue we have to deal with is numeric instability—the probabilities $p_k$ are numerically equal to 0 even for moderate values of $k$. Thus, a naïve sampling of $K$ could ignore the tail values of the distribution. To address this issue we use a racing algorithm\(^2\) that samples $N$ random variables and then chooses the smallest one. The algorithm is inspired by the reservoir sampling algorithm of [13] and depends on the fact that the minimum of a collection of exponential random variables also follows an exponential distribution. The proof that Algorithm 1 is correct hinges on Proposition 5, which is proven in the appendix.

\(^2\)This algorithm is from unpublished work by Anonymous (name redacted for double-blind review, will be included in camera-ready), in which they considered the numerical instability issues of estimating medians via the EM.
Proposition 5. Let \( Z_k = \log \log \frac{1}{U_k} - \log p_k \) and \( K = \arg \min Z_k \), where \( U_k \) are independent and uniformly distributed over \([0, 1]\). Then \( P(K = k) \propto p_k \).

Algorithm 1 Numerically stable sampling

| Input: | Weights \( p_k = \lambda(A_k)e^{-\frac{k\epsilon}{2}} \) for \( k = 0, \ldots, N - 1 \) |
|:------|:------------------------------------------------------------------|
| Output: | Random variable \( K \) s.t. \( P(K = k) \propto p_k \) |
|        | Sample \( N \) uniform r.v.s \( U_0, \ldots, U_{N-1} \) |
|        | Let \( Z_k = \log \log \frac{1}{U_k} - \log \lambda(A_k) + \frac{\epsilon_2}{2} \) |
|        | Return \( \arg \min Z_k \) |

6 Experiments

In this section we empirically demonstrate the practical advantages of **Duff** for the task of computing medians. Code to reproduce all experiments is provided in the supplementary material.

As baselines we use three mechanisms which scale noise according to the smooth sensitivity.

- **SMOOTHSENS**: \((\epsilon, 0)\)-DP method of Proposition 3, with \( \gamma \) set to 2 (giving the Cauchy distribution).
- **SMOOTHSENS\_\_\_****: \((\epsilon, \delta)\)-DP method of Proposition 2. The parameters \( \alpha \) and \( \beta \) are chosen to minimize \( \frac{SS_\alpha}{\alpha} \) under the constraints of Proposition 2.
- **LAPLACE\_\_\_**: Recently proposed by [7], this method releases \( T(S) + \frac{SS_\alpha(T,S)}{\alpha} Xe^\sigma Y \) where \( X \sim \text{Lap}(0, 1) \) and \( Y \sim \mathcal{N}(0, \sigma) \). Parameters \( \alpha, \beta, \) and \( \sigma \) are chosen optimally based on [7]. This mechanism yields \( \frac{\epsilon^2}{2} \)-concentrated DP. We use Lemma 9 from [7] to map to an \((\epsilon, \delta)\)-DP guarantee.

We also ran preliminary experiments comparing with the propose-test-release mechanism of [3], but it was not a strong competitor; even when allowed a high failure rate (e.g., a return value of “no result” for 50% of queries), its error on the remaining results was substantially higher than that of **Duff**.

6.1 Synthetic data

We fix the dataset size at \(|S| = 1000\), and generate data from three distributions: 1) \( \mathcal{N}(0, 1) \), the zero-mean, unit-variance Normal, 2) \( U(0, 1) \), uniform on \([0, 1]\), and 3) \( B(0.5, 0.5) \), the bimodal Beta. After computing the true median, \( T(S) \), we truncate the data to a reasonable range, \([a, b]\), effectively imposing limits on the magnitude of user contributions. In the case of \( \mathcal{N}(0, 1) \) we set \([a, b] = [-10, 10] \), and otherwise we set \([a, b] = [0, 1] \). We note that the exact values of \([a, b]\) are not a significant factor in the performance of **Duff**, but if we loosen these limits then it dramatically increases the average error of **SMOOTHSENS** and **LAPLACE\_\_\_**. This is because in many settings a substantial fraction of these methods’ probability mass before clipping to \([a, b]\) lies outside of this range. Hence, our tight setting of \([a, b]\) bounds is actually an advantage that competing methods might not have in a more realistic setting.

We generate 100 datasets of each type, and call each privacy mechanism 100 times per dataset. We compute the average difference between the true median and the mechanism’s estimate, \( |T(S) - M(S)| \), and average this value for each dataset. In Figure 1 we plot the average of these average errors, with error bars representing the standard deviation across datasets. (Note that the plots have a log scale on the y-axis, which accounts for the error bars looking non-symmetric.)

We observe that **Duff** has the smallest average error in all experiments. It far outperforms **SMOOTHSENS** in the \( \delta = 0 \) setting; for example, for the \( \mathcal{N}(0, 1) \) data, the errors of **Duff** are a factor of 187 times smaller for \( \epsilon = 0.1 \) and 34 times smaller for \( \epsilon = 2 \). More remarkably, even if we allow **SMOOTHSENS** a non-zero \( \delta \) value, it still does not reach the performance of **Duff**. **Duff** has errors a factor of 130 (\( \epsilon = 0.1 \)) to 4 (\( \epsilon = 2 \)) times smaller than those of **SMOOTHSENS** with the reasonable \( \delta \) of \( 1/|S| = 0.001 \). Comparing to the more recent **LAPLACE\_\_\_** method, we still see that **Duff** has errors a factor of 4 (\( \epsilon = 0.1 \)) to 15 (\( \epsilon = 2 \)) times smaller.

Runtimes of **Duff** and **SMOOTHSENS** with \( \delta = 0 \) are similar, and the big-O cost of all methods is dominated by the required \( O(n \log n) \) step of sorting the data. However, the optimal parameter search
for **SMOOTHSENS** and **LAPLACE LN** with $\delta > 0$ requires substantial extra time. In practice, it took 15 to 20 times longer to run these methods.

### 6.2 Real data

![Histograms of vertebral measurements](image)

Figure 2: From left to right, the plots are histograms of: 1) the actual data, 2) median estimates from **LAPLACE LN** with $\delta = 1/n$, and 3) median estimates from the EM with **Duff**.

As a test of our method on real data, we use the Vertebral Column Dataset from the UCI Repository [10, 8]. Each row in this dataset contains six measurements related to a patient’s vertebral column, and each patient is classified as either normal (NO) or abnormal (AB). There are $n_{NO} = 100$ normal patients and $n_{AB} = 210$ abnormal patients. The leftmost plot in Figure 2 shows a normalized histogram of the first measurement for each class, with class medians denoted by a vertical bar. Fixing $\epsilon = 0.5$ and setting the $[a,b]$ range to the actual range observed when the data from both classes is combined, we ran 1000 trials of DP median estimation. The center and rightmost plots in Figure 2 show normalized histograms of the resulting estimates from **LAPLACE LN** with $\delta = 1/n$ and **Duff**, respectively. Notice that the **LAPLACE LN** histograms for the two classes overlap significantly. In contrast, the **Duff** histograms are much tighter, providing more useful estimates of the class medians. Similar plots for the other five vertebral measurements can be found in the appendix.

### 7 Conclusion

In this work we proposed a general-purpose distance-based utility function family, **Duff**, for the exponential mechanism. We proved that **Duff** is an affirmative answer to the open question of whether it is possible to have differential privacy with a noise distribution whose variance is proportional to smooth sensitivity and whose tails decay at a faster-than-polynomial rate. This is the first time a connection between the EM and smooth sensitivity has been studied and we believe there are very interesting research questions that can expand this connection. We also provide the first ever comparison of multiple median algorithms, demonstrating the advantages of **Duff**. An interesting open question is whether we can instantiate **Duff** on other tasks for which smooth sensitivity constitutes the state of the art. For instance, the task of privately calculating the number of triangles in a graph. Please see below for a broader impact statement.
Broader impact

Differential privacy has become a standard method for anonymization. This work presents a method that allows organizations to release private statistics such as the median with much higher fidelity to the true statistic value while at the same time providing greater user privacy protection. Notably, in contrast to the existing state-of-the-art [7], our method can produce more accurate results while using no \( \delta \). Since the implications of non-zero \( \delta \) are not well-understood [15], this may be especially important for giving private user information the best protection.

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A  Additional proofs

A.1  Distance properties

Proposition 6. The function \( d: \mathbb{S} \times \mathbb{S} \to \mathbb{R}_+ \) of Definition 5 is a metric.

Proof. The fact that \( d \) is positive and symmetric is trivial. Therefore, we focus on proving that it satisfies the triangle inequality. Let \( S, S' \) and \( S'' \) be three data sets. Let \( d(S, S'') = k_1 \) and \( d(S'', S') = k_2 \). Now let \( S = S_0, \ldots, S_{k_1} = S'' \) and \( S'' = S_{k_1}, \ldots, S_{k_1+k_2} = S' \) be the sequence of datasets defining \( d(S, S'') \) and \( d(S'', S') \). Since \( S_0, \ldots, S_{k_1+k_2} \) defines a sequence of neighbors starting at \( S \) and ending at \( S' \), by definition we must have \( d(S, S') \leq k_1 + k_2 = d(S, S'') + d(S'', S') \). \( \square \)

A.2  Sensitivity of Duff

Lemma 1. Given any statistic \( T \), let \( u_d \) be defined as

\[
  u_d(x, S) = - \min_{S': T(S') = x} d(S, S').
\]

Then \( \Delta_{u_d} \), the sensitivity of \( u_d \), is less than 1.

Proof. Let \( S'' \in \mathcal{N}(S) \). We will show that \( u_d(x, S) \geq u_d(x, S'') - 1 \), and then the inequality \( u_d(x, S') \geq u_d(x, S) - 1 \) will follow simply by symmetry on the choice of \( S'' \) and \( S \). The combination of these two implies \( u_d(x, S'') - 1 \leq u_d(x, S) \leq u_d(x, S'') + 1 \), which yields the statement of the lemma.

Let \( S_1 \in \arg \min_{S': T(S') = x} d(S'', S') \). Starting from the definition of \( u_d \):

\[
  u_d(x, S) = - \min_{S': T(S') = x} d(S, S') \\
  \geq -d(S, S_1) \\
  \geq -d(S, S'') - d(S'', S_1) \\
  = -1 + u_d(x, S'').
\]

The first inequality follows from the fact that \( S_1 \) is a feasible dataset for the optimization problem. The second inequality follows from the fact that \( S_1 \) is a minimizer.

A.3  Proof of Proposition 4

Proposition 4 (116). Let \( S \) be a dataset and \( u: [a, b] \times \mathbb{S} \to \mathbb{R} \) be a utility function with sensitivity \( \Delta_u \). Let \( X \) be a random variable sampled according to the EM and let \( \lambda \) denote the Lebesgue measure. If \( \text{OPT} = \max_x u(x, S) \) and \( H_t = \{ x: u(x, S) > \text{OPT} - t \} \) then:

\[
P(u(X, S) < \text{OPT} - t) \leq \frac{(b - a)}{\lambda(H_{t/2})} e^{-\frac{t}{\Delta_u}}.
\]

Proof. Let \( H_t^c = \{ x: u(x, S) \leq \text{OPT} - t \} \), then

\[
P(u(x, S) \leq \text{OPT} - t) = P(H_t^c) \leq \frac{P(H_t^c)}{P(H_{t/2})} \\
= \frac{\int_{H_t^c} e^{\frac{\text{OPT}}{\Delta_u}} dx}{\int_{H_{t/2}} e^{\frac{\text{OPT}}{\Delta_u}} dx} \\
\leq \frac{e^{-\frac{t}{\Delta_u}} \int_{H_t} dx}{e^{-\frac{(\text{OPT} - t)}{\Delta_u}} \int_{H_{t/2}} dx} \\
\leq \frac{e^{-\frac{t}{\Delta_u}} (b - a)}{\lambda(H_{t/2})}.
\]
A.4 Proof of Theorem 1

Theorem 1. Let \( x \in [a, b] \) denote the output of the EM with utility function \( u_d \). Let \( \lambda \) denote the Lebesgue measure and \( \gamma = \frac{1}{2\beta^\gamma t}/(e^{\gamma}-1) \). If \( H_t = \{ x \mid u_d(x, S) \geq -t \} \), then:

\[
P(|x - T(S)| > t) \leq \frac{2 \exp \left( -\beta \right) (b - a)}{\lambda (H_t)}.
\]

(2)

Proof. We begin by bounding the difference between \( x \) and \( T(S) \) in terms of local sensitivity. Let \( S^* \) denote a dataset achieving the minimum in the definition of \( u_d(x, S) \). Let \( K \) be a random variable given by \( d(S, S^*) \). By definition of \( d \), there exists a sequence of neighboring datasets \( S = S_0, \ldots, S_K = S^* \). Therefore:

\[
|x - T(S)| = |T(S) - T(S^*)| = |T(S) - T(S^*) + \sum_{i=1}^{K-1} T(S_i) - \sum_{i=1}^{K-1} T(S_i)|
\]

Rearranging summands we get:

\[
\left| \sum_{i=0}^{K-1} (T(S_i) - T(S_{i+1})) \right| \leq \sum_{i=0}^{K-1} |T(S_i) - T(S_{i+1})| \leq \sum_{i=0}^{K-1} \text{LS}(T, S_i).
\]

where the first inequality follows from the triangle inequality and the second from the definition of local sensitivity (Definition 3).

We also know from Definition 6 that, for any \( \beta > 0 \) and \( i \in \mathbb{N} \): \( \text{LS}(T, S_i) \leq e^{\beta_i} \Sigma_{\beta}(T, S) \). Thus, conditioned on \( K\beta \leq 1 \), we have:

\[
|x - T(S)| \leq \Sigma_{\beta}(T, S) \sum_{i=0}^{K-1} e^{\beta_i} = \Sigma_{\beta}(T, S) \frac{e^{K\beta} - 1}{e^{\beta} - 1}
\]

\[
\leq \frac{(e^{K\beta} - 1) \Sigma_{\beta}(T, S)}{\beta} \leq \Sigma_{\beta}(T, S)(K\beta e + (1 - K\beta) - 1)
\]

\[
= K(e - 1) \Sigma_{\beta}(T, S) = d(S, S^*)(e - 1) \Sigma_{\beta}(T, S) = -u_d(x, S)(e - 1) \Sigma_{\beta}(T, S),
\]

where the first equality is an application of the geometric summation formula, and the third inequality follows from convexity of the function \( x \mapsto e^x \) and the condition that \( K\beta \leq 1 \). We can now bound the error of our mechanism.

\[
P(|x - T(S)| > t) \leq P(|x - T(S)| > t \land K\beta \leq 1) + P(K\beta > 1)
\]

\[
\leq P (u_d(x, S)(e - 1) \Sigma_{\beta}(T, S) < -t) + P(u_d(x, S) < -1)
\]

\[
= P \left( u_d(x, S) < \frac{-t}{(e - 1) \Sigma_{\beta}(T, S)} \right) + P \left( u_d(x, S) < -\frac{1}{\beta} \right).
\]

(6)

Notice that the above inequality holds for every value of \( \beta \) and that the summands introduce a trade-off. The larger \( \beta \) is, the smaller \( \Sigma_{\beta} \) is, and therefore the smaller the probability of the first event. On the other hand, a larger value of \( \beta \) makes the probability of the second term larger. Letting \( \beta = \beta^* \left( \frac{1}{e^{\gamma}-1} \right) \) makes both events equally likely. Then Equation 6 becomes:

\[
P(|x - T(S)| > t) \leq 2P \left( u_d(x, S) < -\frac{1}{\beta^* t/(e^{\gamma}-1)} \right).
\]

The result now follows from Proposition 4 and the fact that the sensitivity of \( u_d \) is 1: \( \Delta_{u_d} = 1 \).

□

Corollary 1. Let \( H_t \) be as in Theorem 1 and fix \( \eta > 0 \). Assume \( \lambda(H_t) \geq C t \) for some constant \( C > 0 \). Let \( \beta_{\exp} = \frac{1}{4W \left( \frac{\epsilon \gamma}{\epsilon \gamma - 1} \right)} \), where \( W \) is the main branch of the Lambert function\(^1\). Then with probability at least \( 1 - \eta \):

\[
|x - T(S)| \leq 4(e - 1) \Sigma_{\beta_{\exp}} \left( \frac{e(b - a)}{C\eta} \right).
\]

(3)

\(^1\)The Lambert function is the inverse of the function \( x \mapsto xe^x \).
Proof. Using the fact that $\lambda(H_t/2) \geq C_t/2$ we can bound the probability in Theorem 1:

$$P(|x - T(S)| > t) \leq \frac{2 \exp \left(-\frac{\epsilon_t}{2}\right) (b - a)}{C_t\gamma}.$$ 

Setting the righthand side of the above inequality to $\eta$ and rearranging terms yields:

$$\frac{\epsilon_t}{2} \exp \left(\frac{\epsilon_t}{2}\right) = \frac{\epsilon (b - a)}{C_t\eta}.$$ 

Applying the Lambert function, we have:

$$\frac{\epsilon_t}{2} = W \left(\frac{\epsilon (b - a)}{C_t\eta}\right).$$

Expanding the definition of $\gamma$, this implies:

$$\beta^\ast \left(\frac{t}{e - 1}\right) = \frac{\epsilon}{4W \left(\frac{\epsilon (b - a)}{C_t\eta}\right)}.$$ 

Since $\beta^\ast$ is the inverse function of $\beta \mapsto SS_{\beta}$, we arrive at the following expression for $t$:

$$t = \frac{(e - 1)SS_{\beta^{\ast}}}{{\beta^{\ast}}_{exp}}.$$ 

A.5 Utility guarantee for the smooth sensitivity mechanism

Proposition 7. Let $\gamma > 0$ and let $M$ be a mechanism that releases $x = T(S) + \gamma Z$ where $Z$ is sampled from density $h(z) \propto \frac{1}{1 + |z|^2}$. Then with probability $1 - \eta$:

$$|x - T(S)| < \gamma \tan \left(\frac{\pi}{2} (1 - \eta)\right).$$

Proof. We begin by calculating the normalization constant of $h$:

$$\int_{-\infty}^{\infty} h(z)\,dz = 2 \int_{0}^{\infty} \frac{1}{1 + z^2}\,dz = \pi.$$ 

We can now measure the error of the mechanism:

$$P(|x - T(S)| > t) = P \left(|Z| > \frac{t}{\gamma}\right)$$

$$= \frac{2}{\pi} \int_{t/\gamma}^{\infty} \frac{1}{1 + z^2}\,dz$$

$$= 1 - \frac{2}{\pi} \arctan \left(\frac{t}{\gamma}\right)$$

Setting the righthand side to $\eta$ and solving for $t$ yields:

$$t = \gamma \tan \left(\frac{\pi}{2} (1 - \eta)\right)$$

A.6 Proof of Proposition 5

Proposition 5. Let $Z_k = \log \log \frac{1}{U_k} - \log p_k$ and $K = \arg\min Z_k$, where $U_k$ are independent and uniformly distributed over $[0, 1]$. Then $P(K = k) \propto p_k.$
Proof. Let \( Z'_k = \frac{\log \frac{1}{U_k}}{p_k} \). From elementary statistics, we know \( Z'_k \) is distributed exponential with parameter \( p_k \). That is, \( P(Z'_k > z) = e^{-p_k z} \). Now, from Lemma 2 we know that \( \min_{j \neq k}(Z'_j) \) is distributed exponential with parameter \( Q = \sum_{j \neq k} p_j \). Thus, using Lemma 3 we have:

\[
P(K = k) = P(Z_k \leq \min_j Z_j) = P(Z'_k \leq \min_j Z'_j)
\]

\[
= \frac{p_k}{p_k + Q} = \frac{p_k}{\sum_j p_j}.
\]

\[\square\]

**Lemma 2.** Let \( Z_1, Z_2 \) be two independent random variables from exponential distributions with parameters \( \lambda_1 \) and \( \lambda_2 \) respectively. Then \( \min(Z_1, Z_2) \) is an exponential random variable with parameter \( \lambda_1 + \lambda_2 \).

Proof. By definition we have:

\[
P(\min(Z_1, Z_2) \geq z) = P(Z_1 > z \text{ and } Z_2 > z)
\]

\[
= e^{-\lambda_1 z} e^{-\lambda_2 z} = e^{-(\lambda_1 + \lambda_2)z}.
\]

\[\square\]

**Lemma 3.** Let \( Z_1, Z_2 \) be two independent random variables from exponential distributions with parameters \( \lambda_1 \) and \( \lambda_2 \) respectively. Then:

\[
P(Z_1 \leq Z_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}
\]

Proof.

\[
P(Z_1 \leq Z_2) = \int_0^\infty \int_z^\infty \lambda_1 \lambda_2 e^{-\lambda_1 z_1} e^{-\lambda_2 z_2} dz_2 dz_1
\]

\[
= \int_0^\infty -\lambda_1 e^{-(\lambda_1 + \lambda_2)z_1} dz_1
\]

\[
= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\]

\[\square\]

**B Additional plots for real data experiments**
Figure 3: From top to bottom, the plots are histograms of: 1) the actual data, 2) median estimates from LAPLACELN with $\delta = 1/n$, and 3) median estimates from the EM with Duff.