CHARACTERIZATION OF PPT STATES AND MEASURES OF ENTANGLEMENT

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Abstract. A detailed characterization of PPT states, both in the Heisenberg and in the Schrödinger picture, is given. Measures of entanglement are defined and discussed in details. Illustrative examples are provided.

1. Preliminaries

In this section we compile some basic facts on the theory of positive maps on C*-algebras. To begin with, let A and B be C*-algebras (with unit), $A_h = \{ a \in A; a = a^* \}$ - the set of all selfadjoint elements in A, $A^+ = \{ a \in A_h; a \geq 0 \}$ - the set of all positive elements in A, and $S(A)$ the set of all states on A, i.e. the set of all linear functionals $\varphi$ on A such that $\varphi(1) = 1$ and $\varphi(a) \geq 0$ for any $a \in A^+$. In particular

$$(A_h, A^+) \text{ is an ordered Banach space.}$$

We say that a linear map $\alpha : A \rightarrow B$ is positive if $\alpha(A^+) \subset B^+$.

The theory of positive maps on non-commutative algebras can be viewed as a jig-saw-puzzle with pieces whose exact form is not well known. On the other hand, as we address this paper to a readership interested in quantum mechanics and quantum information theory, in this section, we will focus our attention on some carefully selected basic concepts and fundamental results in order to facilitate access to main problems of that theory. Furthermore, the relations between the theory of positive maps and the entanglement problem will be indicated.

We begin with a very strong notion of positivity: the so called complete positivity (CP). Namely, a linear map $\tau : A \rightarrow B$ is CP iff

$$(1) \quad \tau_n : M_n(A) \rightarrow M_n(B); [a_{ij}] \mapsto [\tau(a_{ij})]$$

is positive for all $n$. Here, $M_n(A)$ stands for $n \times n$ matrices with entries in $A$.

To explain the basic motivation for that concept we need the following notion: an operator state of C*-algebra A on a Hilbert space $K$, is a CP map $\tau : A \rightarrow B(K)$, where $B(K)$ stands for the set of all bounded linear operators on $K$. Having this concept we can recall the Stinespring result, [37], which is a generalization of GNS construction and which was the starting point for a general interest in the concept of complete positivity.

**Theorem 1.** ([37]) *For an operator state $\tau$ there is a Hilbert space $H$, a *-representation ($\ast$-morphism) $\pi : A \rightarrow B(H)$ and a partial isometry $V : K \rightarrow H$ for which

$$(2) \quad \tau(a) = V^* \pi(a)V.$$

A nice and frequently used criterion for CP can be extracted from Takesaki book [42];
Criterion 2. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is CP if and only if

\[ \sum_{i,j=1}^{n} y_i^* \phi(x_i^* x_j) y_j \geq 0 \]

for every $x_1, \ldots, x_n \in \mathcal{A}$, $y_1, \ldots, y_n \in \mathcal{B}$, and every $n \in \mathbb{N}$.

Up to now we considered linear positive maps on an algebra without entering into the (possible) complexity of the underlying algebra. The situation changes when one is dealing with composed systems (for example in the framework of open system theory). Namely, there is a need to use the tensor product structure. At this point, it is worth citing Takesaki’s remark [42]: “...Unlike the finite dimensional case, the tensor product of infinite dimensional Banach spaces behaves mysteriously.” He had in mind “topological properties of Banach spaces”, i.e.: “cross norms in the tensor product are highly non-unique.”

But from the point of view of composed systems the situation is, even, more mysterious as finite dimensional cases are also obscure. To explain this point, let us consider positive maps defined on the tensor product of two $C^*$-algebras, $\tau : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A} \otimes \mathcal{B}$. But now the question of order is much more complicated. Namely, there are various cones determining the order structure in the tensor product of algebras (cf. [48])

\[ C_{inj} \equiv (\mathcal{A} \otimes \mathcal{B})^+ \supseteq \ldots \supseteq C_{\beta} \supseteq \ldots \supseteq C_{pro} \equiv \text{conv}(\mathcal{A}^+ \otimes \mathcal{B}^+) \]

and correspondingly in terms of states (cf [32])

\[ S(\mathcal{A} \otimes \mathcal{B}) \supseteq \ldots \supseteq S_{\beta} \supseteq \ldots \supseteq \text{conv}(S(\mathcal{A}) \otimes S(\mathcal{B})). \]

Here, $C_{inj}$ stands for the injective cone, $C_{\beta}$ for a tensor cone, while $C_{pro}$ for the projective cone. The tensor cone $C_\beta$ is defined by the property: the canonical bilinear mappings $\omega : \mathcal{A}_h \times \mathcal{B}_h \to (\mathcal{A}_h \otimes \mathcal{B}_h, C_\beta)$ and $\omega^* : \mathcal{A}_h^* \times \mathcal{B}_h^* \to (\mathcal{A}_h^* \otimes \mathcal{B}_h^*, C_\beta)$ are positive. The cones $C_{inj}, C_\beta, C_{pro}$ are different unless either $\mathcal{A}$, or $\mathcal{B}$, or both $\mathcal{A}$ and $\mathcal{B}$ are abelian (so a finite dimension does not help very much!). This feature is the origin of various positivity concepts for non-commutative composed systems and it was Stinespring who used the partial transposition (transposition tensored with identity map) for showing the difference between $C_\beta$ and $C_{inj}$ and $C_{pro}$ (see [48], also [21]). Clearly, in dual terms, the mentioned property corresponds to the fact that the set of separable states $\text{conv}(S(\mathcal{A}) \otimes S(\mathcal{B}))$ is different from the set of all states and that there are various special subsets of states if both subsystems are truly quantum.

In his pioneering work on Banach spaces, Grothendieck [19] observed the links between tensor products and mapping spaces. A nice example of such links was provided by Størmer [38]. To present this result we need a little preparation.

Let $\mathfrak{A}$ denote a norm closed self-adjoint subspace of bounded operators on a Hilbert space $\mathcal{K}$ containing the identity operator on $\mathcal{K}$. $\mathfrak{I}$ will denote the set of trace class operators on $\mathcal{B}(\mathcal{H})$. $x \to x^\tau$ denotes the transpose map of $\mathcal{B}(\mathcal{H})$ with respect to some orthonormal basis. The set of all linear bounded (positive) maps $\phi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{B}(\mathfrak{A}, \mathcal{B}(\mathcal{H}))$ ($\mathcal{B}(\mathfrak{A}, \mathcal{B}(\mathcal{H}))^+$ respectively). Finally, we denote by $\mathfrak{A} \otimes \mathfrak{I}$ the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{I}$ (algebraic tensor product of two vector spaces is defined as its $^*$-algebraic structure when the factor
spaces are \( ^\ast \)-algebras; so the topological questions are not considered) and denote by \( \mathcal{A} \otimes \mathcal{T} \) its Banach space closure under the projective norm defined by

\[
\| x \|_1 = \inf \{ \sum_{i=1}^{n} \| a_i \| \| b_i \| : x = \sum_{i=1}^{n} a_i \otimes b_i, \ a_i \in \mathcal{A}, \ b_i \in \mathcal{T} \},
\]

where \( \| \cdot \|_1 \) stands for the trace norm. Now, we are in a position to give (see [38])

**Lemma 3.** There is an isometric isomorphism \( \phi \rightarrow \tilde{\phi} \) between \( \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H})) \) and \( (\mathcal{A} \otimes \mathcal{T})^\ast \) given by

\[
(\tilde{\phi})(\sum_{i=1}^{n} a_i \otimes b_i) = \sum_{i=1}^{n} \text{Tr}(\phi(a_i)b_i^t),
\]

where \( \sum_{i=1}^{n} a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{T} \).

Furthermore, \( \phi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(\mathcal{H}))^+ \) if and only if \( \tilde{\phi} \) is positive on \( \mathcal{A}^+ \otimes \mathcal{T}^+ \).

To comment this result we make

**Remark 4.**

1. There is not any restriction on the dimension of Hilbert space.
   In other words, this result can be applied to true quantum systems.
2. In [39], Størmer showed that in the special case when \( \mathcal{A} = M_n(\mathbb{C}) \) and \( \mathcal{H} \) has dimension equal to \( n \), the above Lemma is a reformulation of Choi result [13], [15], (see also [23]).
3. One should note that the positivity of a functional is defined by the projective cone \( \mathcal{A}^+ \otimes \mathcal{T}^+ \).
4. A generalization of the Choi result (mentioned in 4.2) was also obtained by Belavkin and Staszewski [8].

We will also need the concept of co-CP maps. A map \( \phi \) is co-CP if its composition with the transposition is a CP map. To see that this is not a trivial condition it is enough to note that the transposition is not even a 2-positive (2-positivity means that the condition given in (11) is satisfied for \( n \) equal to 1 and 2 only). A larger class of positive maps is formed by decomposable maps. A map \( \phi \) is called decomposable if it can be written as a sum of CP and co-CP maps. Equivalently, if in (2) one replaces \( ^\ast \)-morphism by Jordan morphism (i.e. a linear map which preserves anticommutator) then the canonical form of a decomposable map is obtained.

Turning to states, it was mentioned that \( \text{conv}(\mathcal{S}(\mathcal{A}) \otimes \mathcal{S}(\mathcal{B})) \) are called separable states. The subset of states \( \mathcal{S}(\mathcal{A} \otimes \mathcal{B}) \setminus \text{conv}(\mathcal{S}(\mathcal{A}) \otimes \mathcal{S}(\mathcal{B})) \) is called the set of entangled states. We will be interested in the special subset of states:

\[
\mathcal{S}(\mathcal{A} \otimes \mathcal{B})_{\text{PPT}} \equiv \mathcal{S}_{\text{PPT}} = \{ \varphi \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B}); \varphi \circ (t \otimes \text{id}) \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B}) \}
\]

where \( t \) stands for transposition. Such states are called PPT states. It is worth observing that the condition in the definition of \( \mathcal{S}_{\text{PPT}} \) is non-trivial; namely the partial transposition does not need to be a positive map! Clearly

\[
\mathcal{S} \supseteq \mathcal{S}_{\text{PPT}} \supseteq \mathcal{S}_{\text{sep}}.
\]

Among entangled states, the states called maximally entangled are of special interest. They can be defined as those for which the state reduced to a subsystem is maximally chaotic (in the entropic sense). A nice example of such states is given by EPR (Einstein-Podolsky-Rosen) states.
The aim of this paper is to give a general characterization of PPT states. We shall present two approaches. The first one is based on the structure of positive maps and as the starting point we will take a modification of Lemma 3. The second approach employs the Hilbert space geometry. This equivalent description will offer rather strikingly very simple definitions of entanglement measures.

2. Entanglement mappings and PPT states

In this Section we present a modification of Belavkin-Ohya approach [6], [7], [33], and [5] to the characterization of entanglement. The basic concept of this approach is the entangling operator $H$. The aim of this section is to provide explicit formulas for both entangling operator $H$ and entanglement mapping $\phi^*$ as well as to give the first characterization of PPT states.

Let us consider a composed system $\sum$ consisting of two subsystems $\sum_1, \sum_2$. We assume that $\sum_1$ is defined by the pair ($\mathcal{H}, \mathcal{B}(\mathcal{H})$) while $\sum_2$ by the pair ($\mathcal{K}, \mathcal{B}(\mathcal{K})$) respectively, where $\mathcal{H}$ ($\mathcal{K}$) is a separable Hilbert space. Let $\omega$ be a normal compound state on $\sum$, i.e. $\omega$ is a normal state on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Thus

$$\omega(a \otimes b) = Tr_{\omega}(a \otimes b)$$

with $a \in \mathcal{B}(\mathcal{H}), b \in \mathcal{B}(\mathcal{K})$. $\rho_\omega \equiv \rho$ is a density matrix with the spectral resolution $\rho = \sum_i \lambda_i |e_i\rangle \langle e_i|$. Define a linear bounded operator $T_\zeta : \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{K}$ by

$$T_\zeta \eta = \zeta \otimes \eta$$

where $\zeta \in \mathcal{H}, \eta \in \mathcal{K}$.

Note that the adjoint operator $T_\zeta^* : \mathcal{H} \otimes \mathcal{K} \rightarrow \mathcal{K}$ is given by

$$T_\zeta^* \zeta' \otimes \eta' = (\zeta, \zeta') \eta'.$$

Now we wish, following B-O scheme, to define the operator

$$H : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{K} \otimes \mathcal{K}$$

by the formula:

$$H_\zeta = \sum \lambda_i^2 \left( J_{\mathcal{H} \otimes \mathcal{K}} \otimes T_{J_{\mathcal{H}}}^* \right) e_i \otimes e_i$$

where $J_{\mathcal{H} \otimes \mathcal{K}}$ is a complex conjugation defined by $J_{\mathcal{H} \otimes \mathcal{K}} f = J_{\mathcal{H}} (\sum_i (e_i, f) e_i) = \sum_i (e_i, f) e_i$ where $\{e_i\}$ is any CONS extending (if necessary) the orthogonal system $\{e_i\}$ determined by the spectral resolution of $\rho$. ($J_{\mathcal{H}}$ is defined analogously with the spectral resolution given by $H^*H$; using the explicit form of $H$, easy calculations show that the spectrum of $H^*H$ is discrete.) We wish to show

**Theorem 5.** The normal state $\omega$ can be represented as

$$\omega(a \otimes b) = Tr_{\omega} a^t H^* (1 \otimes b) H$$

where $a^t = J_{\mathcal{H}} a^* J_{\mathcal{H}}$. 

Proof.

\[
\text{Tra}^t H^* (1 \otimes b) H = \sum_k (h_k, a^t H^* (1 \otimes b) H h_k)
\]

\[
= \sum_k (h_k, J H a^* J H^* (1 \otimes b) H h_k)
\]

\[
= \sum_k (H J H a J H h_k, (1 \otimes b) H h_k)
\]

\[
= \sum_k \sum_{i,j} \left( \lambda_i^\frac{1}{2} (J H \otimes \mathcal{K} \otimes T_{J H J H h_k}^* e_i \otimes e_i, (1 \otimes b) \lambda_j^\frac{1}{2} (J H \otimes \mathcal{K} \otimes T_{J H h_k}^* e_j \otimes e_j) \right)
\]

\[
= \sum_{k,i,j} \lambda_i^\frac{1}{2} \lambda_j^\frac{1}{2} (e_i \otimes T_{a J H h_k}^* e_i, (1 \otimes b) e_j \otimes T_{J H h_k}^* e_j).
\]

Note that \(\{h_k\}\) can be chosen in such a way that it is CONS used in the definition of \(J_H\). So

\[
\text{Tra}^t H^* (1 \otimes b) H = \sum_{k,i,j} \lambda_i \left( T_{a h_k}^* e_i, b T_{h_k}^* e_i \right).
\]

Let \(\{v_m \otimes w_n\}\) be a CONS in \(\mathcal{H} \otimes \mathcal{K}\). Then

\[
\text{Tra}^t H^* (1 \otimes b) H
\]

\[
= \sum_{k,i,m,n,p,r} \lambda_i (e_i, v_m \otimes w_n) \left( T_{a h_k}^* v_m \otimes w_n, b T_{h_k}^* v_p \otimes w_r \right) (v_p \otimes w_r, e_i)
\]

\[
= \sum_{k,i,m,n,p,r} \lambda_i (e_i, v_m \otimes w_n) (v_p \otimes w_r, e_i) \overline{(a h_k, v_m)} (h_k, v_p) (w_n, b w_r).
\]

As \(\{v_m\}\) is a CONS in \(\mathcal{H}\) we can take \(\{v_m\} = \{h_m\}\). So that

\[
\text{Tra}^t H^* (1 \otimes b) H
\]

\[
= \sum_{k,i,m,n,p,r} \lambda_i (e_i, h_m \otimes w_n) (h_p \otimes w_r, e_i) \overline{(a h_k, h_m)} (h_k, h_p) (w_n, b w_r)
\]

\[
= \sum_{k,i,m,n,r} \lambda_i (e_i, h_m \otimes w_n) (h_k \otimes w_r, e_i) (h_m, a h_k) (w_n, b w_r)
\]

\[
= \sum_{k,i,m,n,r} \lambda_i (e_i, h_m \otimes w_n) (h_m \otimes w_n, (a \otimes b) h_k \otimes w_r) (h_k \otimes w_r, e_i)
\]

\[
= \sum_i \lambda_i (e_i, (a \otimes b) e_i) = Tr \rho (a \otimes b) = \omega (a \otimes b).
\]

As it was mentioned in the previous Section, in his fundamental paper on topological linear spaces, Grothendieck emphasized the importance of relating mapping space to tensor products. Another nice example of such relations is given by the following modification of Størmer’s result.
Lemma 6. (1) Let $\mathcal{B} [\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K})]_+$ stand for the set of all linear, bounded, normal (so weakly$^*$-continuous) maps from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$. There is an isomorphism $\psi \mapsto \Psi$ between $\mathcal{B} [\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K})]_+$ and $(\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))_+$ given by

$$\Psi \left( \sum a_i \otimes b_i \right) = \sum \text{Tr}_K \psi (a_i) b_i, \quad a_i \in \mathcal{B}(\mathcal{H}), \quad b_i \in \mathcal{B}(\mathcal{K}).$$

The isomorphism is isometric if $\Psi$ is considered on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$. Furthermore $\Psi$ is positive on $(\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))^+$ iff $\psi$ is complete positive.

(2) There is an isomorphism $\phi \mapsto \Phi$ between $\mathcal{B} [\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K})]_+$ and $(\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))_+$ given by

$$\Phi \left( \sum a_i \otimes b_i \right) = \sum \text{Tr}_K \phi (a_i) b_i, \quad a_i \in \mathcal{B}(\mathcal{H}), \quad b_i \in \mathcal{B}(\mathcal{K}).$$

The isomorphism is isometric if $\Phi$ is considered on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$. Furthermore $\Phi$ is positive on $(\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}))^+$ iff $\phi$ is complete co-positive.

Proof. A repetition of modified Størmer's and standard arguments (cf. [2]-[3] below).

We want to comment this lemma with

Remark 7. (1) Firstly, one should note the basic difference between Lemma 5 and Lemma 6. In Lemma 5 the order is defined by the projective cone while in Lemma 6, the order is defined by the injective cone.

(2) Secondly, as $\mathcal{B}(\mathcal{K})_+$ is isomorphic to the set of all trace class operators $\mathfrak{T} \equiv \mathfrak{T}_K$ on $\mathcal{K}$, $\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{K})_+$ can be considered as $\mathcal{B}(\mathcal{H}), \mathfrak{T}$.

(3) Thirdly, let $A$ and $B$ be $C^*$-algebras. Lemma 5 and Lemma 6 stem from the standard identification $\Psi \mapsto \psi$ of $(A \otimes B)^d$ with the set $\text{Hom}(A, \mathcal{B}(B^d))$ of linear maps from $A$ to $\mathcal{B}(B^d)$ where $\psi(a)(b) = \Psi (a \otimes b)$.

(4) Finally, Lemma 6 gains in interest if we realize that the operator $H$ defined in the first part of this section can be used for a definition of the entanglement mapping $\phi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})_+$.

Let us define

$$\phi (b) = (H^* (1 \otimes b) H)^\dagger = J_H H^* (1 \otimes b)^* H J_H.$$

Then

Proposition 8. The entanglement mapping

(i) $\phi^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})_+$ has the following explicit form

$$\phi^* (a) = \text{Tr}_{\mathcal{H} \otimes \mathcal{K}} Ha^d H^*$$

(ii) The state $\omega$ on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ can be written as

$$\omega (a \otimes b) = \text{Tr}_H a \phi (b) = \text{Tr}_K b \phi^* (a)$$

where $\phi$ was defined in (15).

Proof. For $f, g \in \mathcal{K}$ and $h \in \mathcal{H}$

$$\text{Tr}_K \phi^* (a) \langle f | g \rangle = \langle g, \phi^* (a) f \rangle = \sum_i (e_i \otimes g, Ha^d H^* e_i \otimes f),$$

where $h \in \mathcal{H}$ is defined in the first part of this section can be used for a definition of the entanglement mapping $\phi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})_+$. We want to comment this lemma with

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where $\phi$ was defined in (15).

Proof. For $f, g \in \mathcal{K}$ and $h \in \mathcal{H}$

$$\text{Tr}_K \phi^* (a) \langle f | g \rangle = \langle g, \phi^* (a) f \rangle = \sum_i (e_i \otimes g, Ha^d H^* e_i \otimes f),$$
where as before \{e_i\} is a CONS in \(\mathcal{H} \otimes \mathcal{K}\). Note:

\[
(h, H^*e_i \otimes f) = (\lambda^\frac{1}{2} (J_{\mathcal{H}} \otimes e_i) (J_{\mathcal{H}} \otimes T_{J_{\mathcal{H}}}^* e_k \otimes e_i \otimes f)\]

\[
= \sum_k \lambda^\frac{1}{2} (e_k \otimes T_{J_{\mathcal{H}}}^* e_k \otimes e_i \otimes f)\]

\[
= \sum_{k,m,n} \lambda^\frac{1}{2} (e_k \otimes v_m \otimes w_n) (e_k \otimes (J_{\mathcal{H}} h,v_m) w_n, e_i \otimes f)\]

where \{v_m\} is a CONS in \(\mathcal{H}\) such that \(J_{\mathcal{H}}\) is defined w.r.t this basis, and \{w_n\} is a CONS in \(\mathcal{K}\),

\[
= \sum_{m,n} \lambda^\frac{1}{2} (e_i, v_m \otimes w_n) (J_{\mathcal{H}} h, v_m) (w_n, f)\]

In particular, putting \(h' = \lambda^\frac{1}{2} (a^t)^* H^*e_i \otimes g\) one has

\[
(h', v_m) = (H^*e_i \otimes g, a^t v_m)\]

\[
= \lambda^\frac{1}{2} (e_i, v_m \otimes f)\]

Hence

\[
Tr_{\mathcal{K}} \phi^* (a | f) \langle g |) = \sum_i ((a^t)^* H^*e_i \otimes g, H^*e_i \otimes f)\]

\[
= \sum_{i,m,n} ((a^t)^* H^*e_i \otimes g, v_m, H^*e_i \otimes w_n) (w_n, f)\]

\[
= \sum_{i,m,n} \lambda^\frac{1}{2} (J_{\mathcal{H}} a^t v_m \otimes g, e_i) \lambda^\frac{1}{2} (e_i, v_m \otimes w_n) (w_n, f)\]

\[
= \sum_{i,m} \lambda^\frac{1}{2} (J_{\mathcal{H}} a^t v_m \otimes g, e_i) (e_i, v_m \otimes f)\]

\[
= Tr_{\rho_\omega} \left( \sum_m |v_m \otimes f\rangle \langle J_{\mathcal{H}} a^t v_m \otimes g| \right)\]

\[
= Tr_{\rho_\omega} \left( \sum_m |v_m \otimes f\rangle \langle a^* v_m \otimes g| \right)\]

\[
= Tr_{\rho_\omega} (a \otimes | f\rangle \langle g |) = \omega (a \otimes | f\rangle \langle g |)\]

Thus

\[
Tr_{\mathcal{K}} b \phi^* (a) = \omega (a \otimes b)\]

The rest follows from Theorem 5.

□

Theorem 5, Lemma 6 and Proposition 8 lead to
Corollary 9. PPT states are completely characterized by entanglement mappings \( \phi^* \) which are both CP and co-CP.

This conclusion can be rephrased in the following way (cf [33]): Entanglement mapping \( \phi^* \) which is not CP will be called \( q \)-entanglement. The set of all \( q \)-entanglements will be denoted by \( \mathcal{E}_q \). Then PPT criterion can be formulated as:

Corollary 10. A state is PPT if and only if its associated entanglement mapping \( \phi^* \) is not in \( \mathcal{E}_q \).

3. Examples

To illustrate the strategy of B-O entanglement maps as well as to get better understanding of positive maps we present some examples.

Example 1: Let \( \omega : \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \rightarrow \mathbb{C} \) be a pure product state, i.e.

\[
\omega(a \otimes b) = \omega_{x \otimes y} (a \otimes b) = (x \otimes y, (a \otimes b) x \otimes y) = (x, ax) (y, by)
\]

where \( x \in \mathcal{H}, y \in \mathcal{K} \) and \( \|x\| = \|y\| = 1 \). Then

\[
H_\zeta = J_{\mathcal{H} \otimes \mathcal{K}} \otimes T_{j_\zeta} (x \otimes y) \otimes (x \otimes y) = J_{\mathcal{H} \otimes \mathcal{K}} (x \otimes y) \otimes (J_{\mathcal{H}} \zeta, x) y.
\]

For \( f \in \mathcal{H} \otimes \mathcal{K}, g \in \mathcal{K}, h \in \mathcal{H} \) we have

\[
(h, H^* f \otimes g) = (Hh, f \otimes g) = (J_{\mathcal{H} \otimes \mathcal{K}} (x \otimes y) \otimes (J_{\mathcal{H}} h, x) y, f \otimes g) = (J_{\mathcal{H} \otimes \mathcal{K}} (x \otimes y), f) (h, J_{\mathcal{H}} x) (y, g) = (h, (y, g) (J_{\mathcal{H} \otimes \mathcal{K}} (x \otimes y), f) J_{\mathcal{H}} x) = (h, (J_{\mathcal{H} \otimes \mathcal{K}} (x \otimes y) \otimes y, f \otimes g) J_{\mathcal{H}} x).
\]

so that

\[
H^* f \otimes g = (J_{\mathcal{H} \otimes \mathcal{K}} (x \otimes y) \otimes y, f \otimes g) J_{\mathcal{H}} x.
\]

Let \( v, z \in \mathcal{K} \) and \( \{e_i\} \) is a CONS in \( \mathcal{H} \otimes \mathcal{K} \) then, using the above calculation, we have

\[
(v, \phi^* (a) z) = (v, Tr_{\mathcal{H} \otimes \mathcal{K}} Ha^t H^* z) = \sum_i (e_i \otimes v, Ha^t H^* e_i \otimes z) = \sum_i (H^* e_i \otimes v, a^t H^* e_i \otimes z) = \sum_i ((J_{\mathcal{H} \otimes \mathcal{K}} (x \otimes y) \otimes y, e_i \otimes v) J_{\mathcal{H}} x, a^t (J_{\mathcal{H} \otimes \mathcal{K}} (x \otimes y) \otimes y, e_i \otimes z) J_{\mathcal{H}} x) = \sum_i (J_{\mathcal{H} \otimes \mathcal{K}} (x \otimes y) \otimes y, e_i \otimes z) (e_i \otimes v, J_{\mathcal{H} \otimes \mathcal{K}} (x \otimes y) \otimes y) (J_{\mathcal{H}} x, a^t J_{\mathcal{H}} x).
\]
Note that
\[
(J_{\mathcal{H}}x, a^i J_{\mathcal{H}}x) = (J_{\mathcal{H}}x, J_{\mathcal{H}}a^i x) = (a^i x, J_{\mathcal{H}}J_{\mathcal{H}}x) = (x, ax).
\]
Thus
\[
(v, \phi^*(a) z) = (J_{\mathcal{H}\otimes\mathcal{K}} (x \otimes y), J_{\mathcal{H}\otimes\mathcal{K}} (x \otimes y)) (y, z) (v, y) (x, ax)
\]
so that
\[
\phi^*(a) z = \|x \otimes y\|^2 (y, z) (x, ax) y = (x, ax) |y \rangle \langle y| \cdot z. \quad \text{(because of } \|x\| = 1 = \|y\|).)
\]
Put \(P_y = |y \rangle \langle y|\). Then, we have
\[
(18) \quad \phi^*(a) = (x, ax) P_y.
\]
To analyse CP and co-CP property let us observe that (we are applying Criterion 2):
\[
\forall \ w \in \mathcal{K}
\]
\[
\left( w, \sum_{i,j} b_i^* \phi^* (a_i^* a_j) b_j w \right) = \sum_{i,j} (x, a_i^* a_j x) (w, b_i^* y) (b_j^* y, w)
\]
\[
= \left( \sum_i \lambda_i a_i x, \sum_j \lambda_j a_j x \right) \geq 0,
\]
where \(\lambda_i = (w, b_i^* y)\). Also
\[
\left( w, \sum_{i,j} b_i^* \phi^* (a_i^* a_j) b_j w \right) = \sum_{i,j} (x, a_i^* a_j x) (w, b_i^* y) (b_j^* y, w)
\]
\[
= \left( \sum_j \lambda_j a_j x, \sum_i \lambda_i a_i x \right) \geq 0.
\]
So \(\phi^*\) is both CP and co-CP. This was expected because any pure separable state is a PPT state.

**Example 2: Separable states:** Let \(\omega = \sum_i \lambda_i \omega_{x_i \otimes y_i}\). One has
\[
\omega (a \otimes b) = \sum_i \lambda_i \omega_{x_i \otimes y_i} (a \otimes b)
\]
\[
= \sum_i \lambda_i Tr_{\mathcal{K}} b \phi_i^* (a)
\]
\[
= Tr_{\mathcal{K}} b \sum_i \lambda_i \phi_i^* (a)
\]
\[
= Tr_{\mathcal{K}} b \phi^*(a)
\]
where $\phi^* = \sum_i \lambda_i \phi_i^*$. But $\phi_i^*$ was described in Example 1 and is both CP and co-CP so $\phi^*$ also has this property. Clearly, the conclusion given at the end of Example 1 is also valid here.

**Example 3: A pure state.** Let $\omega$ be a pure state on $B(\mathcal{H} \otimes \mathcal{K})$. As any pure state on the factor $I$ is a vector state so there exists $x \in \mathcal{H} \otimes \mathcal{K}$ such that

$$\omega (a \otimes b) = (x, (a \otimes b) x).$$

Let $z \in \mathcal{H} \otimes \mathcal{K}$, $h$, $\zeta \in \mathcal{H}$, $g \in \mathcal{K}$ and $\{v_i\}$ be CONS in $\mathcal{H}$, $\{w_j\}$ be CONS in $\mathcal{K}$. Then

$$H \zeta = (J_{\mathcal{H} \otimes \mathcal{K}} \otimes T^*_{J_{\mathcal{H} \otimes \mathcal{K}}}) (x \otimes x) = J_{\mathcal{H} \otimes \mathcal{K}} x \otimes T^*_{J_{\mathcal{H} \otimes \mathcal{K}}} x.$$ 

Also

$$(h, H^* z \otimes y) = (H h, z \otimes y) = (J_{\mathcal{H} \otimes \mathcal{K}} x, z) (T^*_{J_{\mathcal{H} \otimes \mathcal{K}}} x, y)$$

where

$$T^*_{J_{\mathcal{H} \otimes \mathcal{K}}} x = \sum_{i,j} T^*_{J_{\mathcal{H} \otimes \mathcal{K}}} (v_i \otimes w_j, x) v_i \otimes w_j$$

Thus

$$(h, H^* z \otimes y) = \sum_{i,j} (J_{\mathcal{H} \otimes \mathcal{K}} x, z) (x, v_i \otimes w_j) (v_i, J_{\mathcal{H}} h) (w_j, y)$$

$$= \sum_{i,j} (J_{\mathcal{H} \otimes \mathcal{K}} x, z) (x, v_i \otimes w_j) (v_i \otimes w_j, J_{\mathcal{H}} \otimes y)$$

$$= (J_{\mathcal{H} \otimes \mathcal{K}} x, z) (J_{\mathcal{H}} h \otimes y, x).$$

(19)

Let \{e_i\} be CONS in $\mathcal{H} \otimes \mathcal{K}$ then, for $w, u \in \mathcal{K}$

$$w, \phi^* (a) u = (w, (Tr_{\mathcal{H} \otimes \mathcal{K}} H a^t H^*) u)$$

$$= \sum_n (e_n \otimes w, Ha^t H^* e_n \otimes u)$$

$$= \sum_n (H^* e_n \otimes w, a^t H^* e_n \otimes u).$$

Let us use (19), i.e. put $h = a^t H^* e_n \otimes u$, $z = e_n$, $y = w$. Then one gets

$$w, \phi^* (a) u$$

$$= \sum_n (e_n, J_{\mathcal{H} \otimes \mathcal{K}} x) \left( J_{\mathcal{H}} (a^t H^* e_n \otimes u) \otimes w, \sum_{i,j} (v_i \otimes w_j, x) v_i \otimes w_j \right)$$

$$= \sum_{n,i,j} (e_n, J_{\mathcal{H} \otimes \mathcal{K}} x) (v_i \otimes w_j, x) (J_{\mathcal{H}} (a^t H^* e_n \otimes u), v_i) (w, w_j)$$

$$= \sum_{n,i,j} (e_n, J_{\mathcal{H} \otimes \mathcal{K}} x) (x, v_i \otimes w_j) (w_j, w) ((a^t H^* e_n \otimes u), J_{\mathcal{H}} v_i)$$

$$= \sum_{n,i} (e_n, J_{\mathcal{H} \otimes \mathcal{K}} x) (v_i \otimes w, x) \left( (a^t)^* J_{\mathcal{H}} v_i, H^* e_n \otimes u \right)$$
Thus \( \phi(20) \)\n
Consequently \( x \) Assume that Again, using (19), i.e. putting \( h \) \( \phi(21) \)

This means that: \( \lambda \) \( \beta \)

Consequently

This means that:

Turning to the analysis of CP and co-CP we begin with co-CP property. To this end let \( \{ v_k \} \) be CONS in \( \mathcal{H} \) then

Thus \( \phi^* \) is a co-CP map.

Now, consider CP condition: Let \( \{ v_k \} \) be CONS in \( \mathcal{H} \) and \( \{ z_i \} \) be CONS in \( \mathcal{K} \). Assume that \( x \) is given by

where at least two elements of \( \{ \lambda_k \} \) are non-zero. In order to show the non-CP of \( \phi^* \) some preliminaries are necessary. We recall that \( M_n(\mathbb{C}) \) denotes the C\(^*\)-algebra of \( n \times n \) matrices with entries in \( \mathbb{C} \). Let \( \{ e_{ij} \} \) be the canonical basis for \( M_n(\mathbb{C}) \equiv M_n \), i.e. the \( n \times n \) matrices with a “1” in row \( i \), column \( j \), and zeros elsewhere. It is well known that every element \( y \) in \( \mathcal{A} \otimes M_n \) can be written

\( y = \sum a_{ij} \otimes e_{ij} \)
where the $a_{ij}$'s (being in $\mathcal{A}$) are unique. The map
\begin{equation}
\Theta : \mathcal{A} \otimes M_n \to M_n(\mathcal{A}) : \sum a_{ij} \otimes e_{ij} \mapsto \{a_{ij}\}
\end{equation}
is linear, multiplicative, $*$-preserving, and bijective. Therefore, it should be clear that the complete positivity of $\phi^*$ is equivalent to the positivity of operator $\sum_{i,j=1}^n e_{ij} \otimes \phi^*(a^*_ia_j)$, for any $n$.

Let $\{e_i\}$ be CONS in $\mathbb{C}^n$. One has
\[
\phi^*(a^*_ia_j) = \sum_{k,l} \lambda_k \lambda_l \langle \psi_k | \phi | \psi_l \rangle = \sum_{k,l} \lambda_k \lambda_l \langle \psi_k | \psi_l \rangle \langle e_i | \phi | e_j \rangle = \sum_{k,l} \lambda_k \lambda_l \langle e_i | \phi | e_j \rangle \langle z_k | e_j \rangle \langle e_i | \phi | z_l \rangle.
\]
Thus
\[
\sum_{i,j=1}^n |e_i\rangle \langle e_j| \otimes \phi^*(a^*_ia_j) = \sum_{i,j=1}^n \sum_{k,l} \lambda_k \lambda_l \langle e_i | \phi | e_j \rangle \langle z_k | e_j \rangle \langle e_i | \phi | z_l \rangle.
\]
Put $a_i = |y\rangle \langle v_i|$ ( $y \in \mathcal{H}$, $\|y\| = 1$) then
\[
\{\phi^*(|v_i\rangle \langle v_j|)\} \cong \sum_{i,j=1}^n \lambda_j \lambda_j |e_i\rangle \langle e_j| \otimes |z_j\rangle \langle z_i|.
\]
The positivity of $\{\phi^*(|v_i\rangle \langle v_j|)\}$ means the positivity of $(\Psi, (\sum \phi^*(|v_i\rangle \langle v_j|) \otimes e_{ij})\Psi)$ for any $\Psi \in \mathbb{C}^n \otimes \mathcal{K}$. Let us take $\Psi_{\pm}$ in the form
\[
\Psi_{\pm} = e_k \otimes z_l \pm e_l \otimes z_k.
\]
and assume that $k \neq l$. Then
\[
(\Psi_{\pm}, (\sum \phi^*(|v_i\rangle \langle v_j|) \otimes e_{ij})\Psi_{\pm}) = \pm 2Re \lambda_k \lambda_l \in \mathbb{R}.
\]
If $2Re \lambda_k \lambda_l$ is positive then
\[
(\Psi_{-}, (\sum \phi^*(|v_i\rangle \langle v_j|) \otimes e_{ij})\Psi_{-}) = -2Re \lambda_k \lambda_l < 0.
\]
Also if $2Re \lambda_k \lambda_l$ is negative then
\[
(\Psi_{+}, (\sum \phi^*(|v_i\rangle \langle v_j|) \otimes e_{ij})\Psi_{+}) = 2Re \lambda_k \lambda_l < 0.
\]
This means that $\phi^*$ is non-CP.

The above example can be readily generalized (cf Example 2). Namely, a normal state $\omega$ on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ can be written as
\[
\omega(a \otimes b) = Tr_{\mathcal{H} \otimes \mathcal{K}} \varrho a \otimes b,
\]
where $\varrho$ is the corresponding density matrix. But, the spectral representation of $\varrho$ implies
\[
\omega(a \otimes b) = Tr_{\mathcal{H} \otimes \mathcal{K}}(\sum_i \lambda_i P_{x_i}a \otimes b) = \sum_i \lambda_i \omega(x_i)(a \otimes b).
\]
Consequently, the entanglement mapping \( \phi^*_\omega \) associated with the state \( \omega \) would have the form
\[
\phi^*_\omega = \sum_i \lambda_i \phi^*_\omega x_i = \sum_i \text{Tr}_\mathcal{H}(a \otimes I)P_{x_i} = \text{Tr}_\mathcal{H}(a \otimes I)\rho,
\]
where the second equality follows from (21). Obviously, the analysis of CP and co-CP for \( \phi^*_\omega \) is, in general, much more complicated.

Finally, to get a better understanding of the difference between CP and co-CP given in Example 3, let us consider the very particular case of this example.

Example 4. Let in Example 3, \( \mathcal{H} \) and \( \mathcal{K} \) be three dimensional Hilbert spaces. Further, put in the place of \( x \) vectors giving maximally entangled pure states, i.e.
\[
x_1 = \frac{1}{\sqrt{3}}(e_1 \otimes f_2 - e_2 \otimes f_3 - e_3 \otimes f_1)
\]
and
\[
x_2 = \frac{1}{\sqrt{3}}(e_1 \otimes f_1 + e_2 \otimes f_2 + e_3 \otimes f_3)
\]
where \( \{e_i\}^3 \) \( \{f_i\}^3 \) is a CONS in \( \mathcal{H} \) (in \( \mathcal{K} \) respectively). Easy calculations, which are left to the reader, lead to the following maps
\[
[a_{ij}]_{i,j=1}^3 \mapsto \frac{1}{3} \begin{pmatrix} a_{33} & -a_{13} & a_{23} \\ -a_{31} & a_{11} & -a_{21} \\ a_{32} & -a_{12} & a_{22} \end{pmatrix}
\]
for \( x_1 \), and
\[
[a_{ij}]_{i,j=1}^3 \mapsto \frac{1}{3}([a_{ij}]_{i,j=1}^3)^t.
\]
for \( x_2 \). Here, \( ([a_{ij}]_{i,j=1}^3)^t \) stands for the transposed map. Clearly, transposition is not even 2-positive, so not CP. Now, non-CP observed of Example 3 should be well understood. The maps (24) will be useful in the last Section.

4. Tomita’s scheme for partial transposition (see [25], [26])

Let \( \mathcal{H} \) be a (separable) Hilbert space. Using an invertible density matrix \( \rho \) we can define a faithful state \( \omega \) on \( \mathcal{B}(\mathcal{H}) \) as \( \omega(a) = \text{Tr} \rho a \) for \( a \in \mathcal{B}(\mathcal{H}) \). Let us consider the GNS triple \( (\mathcal{H}_\pi, \pi, \Omega) \) associated with \( (\mathcal{B}(\mathcal{H}), \omega) \). Such triple is given by:

- GNS Hilbert space: \( \mathcal{H}_\pi = \{a\Omega : a \in \mathcal{B}(\mathcal{H})\} \) with \( (a,b) = \text{Tr} a^*b \) for \( a,b \in \mathcal{B}(\mathcal{H}) \).
- cyclic vector: \( \Omega = \rho^{1/2} \).
- representation: \( \pi(a) \Omega = a\Omega \).

In the considered GNS representation, the modular conjugation \( J_m \) is just the hermitian involution \( J_m \rho^{1/2} = \rho^{1/2} a^* \), and the modular operator \( \Delta \) is equal to the map \( \rho \cdot \rho^{-1} \). However, some remarks are necessary here. As we have assumed that \( \mathcal{H} \) is a separable Hilbert space then \( \rho^{-1} \) is, in general, an unbounded operator. Hence, the domain of \( \Delta \) should be described. To this end we note that: i) \( \{A\rho^{1/2} : A \in \mathcal{B}(\mathcal{H})\} \) is a dense subset in the set of all Hilbert-Schmidt operators \( \mathcal{F}_{HS}(\mathcal{H}) \) on the Hilbert space \( \mathcal{H} \), ii) \( \alpha_t(\sigma) = \rho^t \sigma \rho^{-t} \) is an one parameter group of automorphisms
on $\mathcal{F}_{HS}(\mathcal{H})$. So, there exists (cf. [11]) the set of entire analytic elements $\mathcal{F}^0_{HS}(\mathcal{H})$ of $\alpha_t(\cdot)$. Thus $\Delta \sigma = \alpha_t(\sigma)|_{t=-i} = \rho \sigma \rho^{-1}$ is well defined for $\sigma \in \mathcal{F}^0_{HS}(\mathcal{H})$. In particular, the polar decomposition of Tomita’s operator (cf. [11]) is also well defined

$$S\Omega = A^*\Omega = J_m \Delta^{1/2} A \Omega$$

Note, that $\{ A\Omega : A \in \mathcal{B}(\mathcal{H}) \} \subseteq D(\Delta^{1/2})$, $D(\cdot)$ stands for the domain. In order to discuss the transposition on $\pi (\mathcal{B}(\mathcal{H}))$ we introduce the following two conjugations: $J_c$ on $\mathcal{H}$ and $J$ on $\mathcal{H}_\pi$. Thanks to the faithfulness of $\omega$ the eigenvectors $\{ e_i \}$ of $\rho$ form an orthogonal basis in $\mathcal{H}$. Hence we can define

$$J_c x = \sum_i (e_i, x) e_i$$

for every $x \in \mathcal{H}$. Due to the fact that $\{ E_{ij} = |e_i \rangle \langle e_j| \}$ form an orthogonal basis in $\mathcal{H}_\pi$ we can also define a conjugation $J$ on $\mathcal{H}_\pi$

$$J a \Omega = \sum_i (E_{ij}, a\Omega) E_{ij}$$

with $J \Omega = \Omega$.

Following the construction presented in [25] and [26] let us define a transposition on $\mathcal{B}(\mathcal{H})$ as the map $a \in \mathcal{B}(\mathcal{H}) \mapsto a^t \equiv J_c a^* J_c$. By $\tau_0$ we will denote the map induced on $\mathcal{H}_\pi$ by the transposition, i.e.

$$\tau_0 a \Omega = a^t \Omega.$$ 

Here are the main properties of $\tau_0$:

**Proposition 11.** (cf. [25]) (1) Let $a \in \mathcal{B}(\mathcal{H})$ and $\xi \in \mathcal{H}_\pi$. Then

$$a^t \xi = J a^* J \xi.$$ 

(2) The map $\tau_0$ has a polar decomposition, i.e.

$$\tau_0 = U \Delta^{1/2}$$

where $U$ is an unitary operator on $\mathcal{H}_\pi$ defined by $U = \sum_{ij} |E_{ij}\rangle \langle E_{ji}|$.

In the above setting we can introduce the natural cone $\mathcal{P}$ (cf. [3], [17]) associated with $(\pi (\mathcal{B}(\mathcal{H})), \Omega)$:

$$\mathcal{P} = \{ \Delta^{1/4} a\Omega : a \geq 0, a \in \pi (\mathcal{B}(\mathcal{H})) \}.$$

The relationship between the Tomita-Takesaki scheme and transposition has the following form:

**Proposition 12.** (see [25]) Let $\xi \mapsto \omega_\xi$ be the homeomorphism between the natural cone $\mathcal{P}$ and the set of normal states on $\pi (\mathcal{B}(\mathcal{H}))$, such that

$$\omega_\xi (a) = (\xi, a\xi), \ a \in \mathcal{B}(\mathcal{H}).$$

For every state $\omega$ define $\omega^t (a) = \omega (a^t)$. If $\xi \in \mathcal{P}$ then the unique vector in $\mathcal{P}$ mapped into the state $\omega^t_\xi$ by the homeomorphism described above, is equal to $U \xi$, i.e.

$$\omega^t_\xi (a) = (U \xi, aU \xi), \ a \in \mathcal{B}(\mathcal{H}).$$
5. PPT states, a Hilbert space approach

Let \( \mathcal{H}_A \) and \( \mathcal{H}_B \) be finite dimensional Hilbert spaces. We want to emphasize that finite dimensionality of Hilbert spaces is assumed only in the proof of Theorem 13. More precisely, due to technical questions concerning the domain of modular operator \( \Delta \) we were able to prove this Theorem only for finite dimensional case (see [25]). On the other hand, we emphasize that the description of compound system based on Tomita’s approach is very general. It relies on the construction of tensor product of standard forms of von Neumann algebras and this description can be done in very general way (so infinite dimensional case is included, cf [18]).

Again let us consider a composite system \( A + B \). Suppose that the subsystem \( A \) is described by \( \mathcal{A} = \mathcal{B} (\mathcal{H}_A) \) and is equipped with a faithful state \( \omega_A \) given by an invertible density matrix \( \rho_A \) as \( \omega_A(a) = Tr \rho_A a \). Similarly, let \( \mathcal{B} = \mathcal{B} (\mathcal{H}_B) \) define the subsystem \( B \), \( \rho_B \) be an invertible density matrix in \( \mathcal{B} (\mathcal{H}_B) \) and \( \omega_B \) be a state on \( \mathcal{B} \) such that \( \omega_B(b) = Tr \rho_B b \) for \( b \in \mathcal{B} \). By \((\mathcal{K}, \pi, \Omega), (\mathcal{K}_A, \pi_A, \Omega_A)\) and \((\mathcal{K}_B, \pi_B, \Omega_B)\) we denote the GNS representations of \((\mathcal{A} \otimes \mathcal{B}, \omega_{A \otimes \omega_B}), (\mathcal{A}, \omega_A)\) and \((\mathcal{B}, \omega_B)\) respectively. Then the triple \((\mathcal{K}, \pi, \Omega)\) can be given by the following identifications (cf [18], [27]):

\[
\mathcal{K} = \mathcal{K}_A \otimes \mathcal{K}_B, \quad \pi = \pi_A \otimes \pi_B, \quad \Omega = \Omega_A \otimes \Omega_B.
\]

With these identifications we have

\[
J_n = J_A \otimes J_B, \quad \Delta = \Delta_A \otimes \Delta_B
\]

where \( J_n, J_A, J_B \) are modular conjugations and \( \Delta, \Delta_A, \Delta_B \) are modular operators for \((\pi (\mathcal{A} \otimes \mathcal{B})^{\prime\prime}, \Omega), (\pi_A (\mathcal{A})^{\prime\prime}, \Omega_A), (\pi_B (\mathcal{B})^{\prime\prime}, \Omega_B)\) respectively. Due to the finite dimensionality of the corresponding Hilbert spaces, just to simplify our notation, we will identify \( \pi_A (\mathcal{A})^{\prime\prime} \) and \( \pi_A (\mathcal{A}) \), etc. Moreover we will also write \( a \Omega_A \) and \( b \Omega_B \) instead of \( \pi_A(a) \Omega_A \) and \( \pi_B(b) \Omega_B \) for \( a \in \mathcal{A}, b \in \mathcal{B} \) when no confusion can arise. Furthermore we denote the finite dimension of \( \mathcal{H}_B \) by \( n \). Thus \( \mathcal{B} (\mathcal{H}_B) \equiv \mathcal{B} (\mathbb{C}^n) \equiv M_n (\mathbb{C}) \). To put some emphasis on the dimensionality of the ”reference” subsystem \( B \), we denote by \( \mathcal{P}_n \), the natural cone for \((M_n^\pi (\mathcal{A}), \omega_A \otimes \omega_0)\), where \( \pi (\mathcal{A} \otimes M_n (\mathbb{C})) \) is denoted by \( M_n^\pi (\mathcal{A}) \) and \( \omega_0 \) is a faithful state on \( M_n (\mathbb{C}) \).

In order to characterize the set of PPT states we need the notion of the ”transposed cone” \( \mathcal{P}_{\tau} = (I \otimes U) \mathcal{P}_n \), where \( \tau \) is the transposition on \( M_n (\mathbb{C}) \) and \( U \) is the unitary operator given in Proposition 11 with the eigenvectors of density matrix \( \rho_0 \) corresponding to \( \omega_0 \).

Then the construction of \( \mathcal{P}_n \) and \( \mathcal{P}_{\tau}^n \) may be realized as follows:

\[
\mathcal{P}_n = \left\{ \Delta^{1/4} [a_{ij}] \Omega : [a_{ij}] \in M_n^\pi (\mathcal{A})^+ \right\},
\]

\[
\mathcal{P}_{\tau}^n = \left\{ \Delta^{1/4} [a_{ij}] \Omega : [a_{ij}] \in M_n^\pi (\mathcal{A})^+ \right\}.
\]

Consequently, we arrived to

**Theorem 13. (see [25])** In the finite dimensional case

\[
\mathcal{P}_{\tau}^n \cap \mathcal{P}_n = \left\{ \Delta^{1/4} [a_{ij}] \Omega : [a_{ij}] \geq 0, [a_{ji}] \geq 0 \right\}.
\]

**Corollary 14.** (1) There is one to one correspondence between the set of PPT states and \( \mathcal{P}_{\tau}^n \cap \mathcal{P}_n \).

(2) There is one to one correspondence between the set of separable states and \( \mathcal{P}_A \otimes \mathcal{P}_B \) (cf [27]).
Remark 15. The correspondence given in Corollary [14] holds for a general case. Thus, the above characterization is applicable to a true quantum system.

We wish to close this Section with the following remark. Also, here, in the Hilbert space approach we met many “cones”; \( \mathcal{P}_A, \mathcal{P}_n, \mathcal{P}_A \otimes \mathcal{P}_M \). All these cones, as we have seen, play the crucial role in the description of important classes of states: all states, PPT states, and separable states respectively. This should be considered as another manifestation of “mysterious behavior” of tensor products (see Section 1).

6. Equivalence between two types of characterization of PPT states

In this Section, we wish to discuss the relation between the Hilbert space description of PPT states and B-O characterization. Firstly, we note that Tomita’s approach leads to the following representation of the compound state \( \omega \):

\[
\omega \left( \sum_i a_i \otimes b_i \right) = \sum_i (\xi, a_i \otimes b_i \xi) = \sum_i \varphi_{\xi,a_i}(b_i)
\]

where \( \varphi_{\xi,a_i}(b_i) \equiv (\xi, (a_i \otimes b_i) \xi) \) and \( \xi \in \mathcal{P}_n \). We have used here the well known result from Tomita-Takesaki theory saying that for any normal state \( \omega \) on a von Neumann algebra with cycling and separating vector \( \Omega \) there is unique vector \( \xi \) in the natural cone \( \mathcal{P}_n \) such that \( \omega(a) = (\xi, a\xi) \).

Let us observe that for \( a \in A, a \geq 0 \), and any \( b \in B \)

\[
\omega(a \otimes b^t) = (\xi, a \otimes b^t \xi)
\]

\[
= Tr_{K_A}|\xi> <\xi|a^{1/2} \otimes 1 \cdot a^{1/2} \otimes 1 \cdot b^t
\]

\[
= Tr_{K_A}Tr_{K_B}a^{1/2} \otimes 1 \cdot |\xi> <\xi| \cdot a^{1/2} \otimes 1 \cdot b^t
\]

\[
= Tr_{K_B}(Tr_{K_A}a^{1/2} \otimes 1 \cdot |\xi> <\xi| \cdot a^{1/2} \otimes 1 \cdot b^t)
\]

\[
= Tr_{K_B}(\rho_{A\xi}b^t) = (\chi_{\rho,\xi}, b^t \chi_{\rho,\xi})
\]

\[
= (U_{A\rho,\xi}, bU_{A\rho,\xi}) = (\chi_{\rho,\xi}, UbU_{A\rho,\xi})
\]

\[
= Tr_{K_B}U_{A\rho,\xi}UbU = \omega(a \otimes UbU)
\]

where \( \chi_{\rho,\xi} \) is Tomita’s representation of \( Tr_{K_B}(\rho_{A\xi}) \) and we have used the notation given in Section 5, Proposition 12 and that the fact the partial trace \( Tr_{K_A}(\cdot) \) is well defined conditional expectation.

As \( \omega(a \otimes b) \) is linear in \( a \), and any \( a \) can be written as a sum of four positive elements (Jordan decomposition) the previous result can be extended to

\[
(31) \quad \omega(a \otimes b^t) = \omega(a \otimes UbU)
\]

for any \( a \in A \) and \( b \in B \).

Now we are in position to compare the strategy given by Lemma 6 and B-O approach with the Hilbert space description of PPT states. Firstly we note (cf Lemma 5) that maps \( \varphi_{\xi} (\cdot) \) can be considered as

\[
(32) \quad \mathcal{B}(K_A) \ni a \mapsto \varphi_{\xi,a} (\cdot) \in \mathcal{B}(K_B)
\]
Secondly, note that the positivity used in Lemma 6(2) implies
\[
0 \leq \omega \left( \sum_{i,j} a_i^* a_j \otimes b_i^* b_j \right)
\]
\[
= \sum_{i,j} \varphi_{\xi, a_i^* a_j} (b_i^* b_j) = \sum_{i,j} (\xi, (a_i^* a_j \otimes b_i^* b_j) \xi).
\]

By using the same vector $\xi \in \mathcal{P}_n$ let us define $\omega^\tau \in (A \otimes B)_+$
\[
\omega^\tau \left( \sum_i a_i \otimes b_i \right) = \sum_i \varphi_{\xi, a_i} (b_i)
\]
where $\varphi_{\xi, a_i} (b_i) \equiv (\xi, (a_i \otimes b_i^*) \xi)$. The positivity used in Lemma 6(1) implies
\[
\omega^\tau \left( \sum_{i,j} a_i^* a_j \otimes b_i^* b_j \right) = \sum_{i,j} \varphi^\tau_{\xi, a_i^* a_j} (b_i^* b_j)
\]
\[
= \sum_{i,j} \left( \xi, (a_i^* a_j \otimes b_i^* b_j)^t \xi \right)
\]
\[
= \sum_{i,j} \left( \xi, (a_i^* a_j \otimes b_j^* (b_i^*)^t ) \xi \right)
\]
\[
= \sum_{i,j} \left( I \otimes U \xi, (a_i^* a_j \otimes b_j (b_i^*)^t) I \otimes U \xi \right) \geq 0,
\]
where in the last equality we have used (31). Hence, CP and co-CP of entangling mapping is equivalent to $\xi \in \mathcal{P}_n^+ \cap \mathcal{P}_n$. Consequently, we conclude that

**Theorem 16.** The description of PPT states by $\mathcal{P}_n^+ \cap \mathcal{P}_n$ can be recognized as the dual description of PPT states by $\mathcal{E}/\mathcal{E}_q$.

In the base of the above equivalence of two types of description of PPT states we may discuss the effectiveness of such characterizations from different points of view. This will be the topic of next Sections. We will start with an analysis of decomposable maps (cf [28]).

### 7. On Decomposable Maps

In [40] Størmer gave the following characterization of decomposable maps:

**Theorem 17.** ([40]) Let $\phi : A \rightarrow B(\mathcal{H})$ be a positive map. A map $\phi$ is decomposable if and only if for all $n \in \mathbb{N}$ whenever $[x_{ij}]$ and $[x_{ji}]$ belong to $M_n(A)^+$ then $[\phi(x_{ij})] \in M_n(B(\mathcal{H}))^+$.

As our aim is to discuss effectiveness of description of PPT states given in Section 5, we again assume finite dimensionality of Hilbert space $\mathcal{H}$. Further, recall (see Criterion [23]) that the positivity of the matrix $[\phi(x_{ij})]$ (with operator entries!) is equivalent to

\[
(33) \quad \sum_{ij} y_i^* \phi(x_{ij}) y_j \geq 0
\]
where \( \{ y_i \} \) are arbitrary elements of \( B(\mathcal{H}) \). Furthermore, any positive matrix \([x_{ij}]\) can be written as (cf \[12\])

\[
[x_{ij}] = \sum_k [(v_i^{(k)})^* v_j^{(k)}]
\]

Hence, applying condition \[33\] to matrices of the form \([a_i^*a_j]\) with the choice of \( y_i \) such that all \( y_i = 0 \) except for \( i_0 \) and \( j_0 \), then changing the numeration in such way that \( y_{i_0} = y_1 \) and \( y_{j_0} = y_2 \) we arrive to study the positivity of the following matrix

\[
\begin{pmatrix}
  a_1^*a_1 & a_1^*a_2 \\
  a_2^*a_1 & a_2^*a_2
\end{pmatrix} \geq 0
\]

and its transposition. On the other hand, block matrix techniques leads to necessary and sufficient conditions for positivity of such matrices. Namely, let \( A, B, C \) be \( d \times d \) matrices. Then

**Lemma 18.** (see \[50\])

\[
\begin{bmatrix}
  A & B \\
  B^* & C
\end{bmatrix} \geq 0
\]

if and only if \( A \geq 0 \), \( C \geq 0 \) and there exists a contraction \( W \) such that \( B = A^*WC^* \).

Assume, if necessary, that \( a_1 \) and \( a_2 \) have inverses, otherwise \( a_i^{-1} \) is understood to be generalized inverse of \( a_i \). Then, application of Lemma \[18\] to the Störmer condition leads to the following question: When \( |a_1|^{-1}a_2^*a_1|a_2|^{-1} \) is a contraction? But an operator \( T \in B(\mathcal{H}) \) is a contraction if and only if \( ||T|| \leq 1 \) what is equivalent to \( ||Tx||^2 \leq ||x||^2 \). This can be written as

\[
(x, T^*Tx) \leq (x, x)
\]

what is equivalent to

\[
T^*T \leq 1
\]

Consequently, \[38\] and Zhan’s lemma \[18\] give (see also \[1\] and \[16\])

\[
a_1^*a_2|a_1|^{-2}a_2^*a_1 \leq |a_2|^2
\]

Hence

\[
\forall f \quad (f, a_1^*a_2(a_1^*a_1)^{-1}a_2^*a_1f) \leq (f, a_2^*a_2f)
\]

So, putting \( f = a_1^{-1}g \) one gets

\[
\forall g \quad ||(a_1^{-1})a_2^*g|| \leq ||a_2a_1^{-1}g||
\]

This means hyponormality of operators \((a_2a_1^{-1})^*\) (cf. \[20\], and \[36\]). But, as considered operators are defined on a finite dimensional Hilbert space, in particular, they are completely continuous. Therefore, hyponormality of \((a_2a_1^{-1})^*\) implies normality (see \[2\], \[9\], and \[36\]).

Consequently, \( a_2a_1^{-1} \) is a normal operator. This means that there is a unitary operator \( U \) (equivalently unitary matrix as finite dimensions are assumed) such that

\[
Ua_2a_1^{-1}U^* = \text{diag}(\lambda_i)
\]
where $\lambda_i \in \mathbb{C}$. This can be rewritten as
\begin{equation}
a_2 a_1^{-1} = \sum_i \lambda_i Q_i
\end{equation}
where $\lambda_i \in \mathbb{C}$ and $\{Q_i\}$ is the resolution of identity. Hence, putting $Q_i \equiv |e_i><e_i|$ where $\{e_i\}$ is a CONS in the Hilbert space $\mathcal{H}$ on which operators $\{a_i\}$ act and defining rank one operators $|f><g| \equiv (g,z)|f>$, one gets
\begin{equation}
a_2 = \sum_i \lambda_i |e_i><a_i^* e_i|
\end{equation}

Thus we proved:

**Proposition 19.** For any matrix \[
\begin{pmatrix}
a_1^* a_1 & a_1^* a_2 \\
a_2^* a_1 & a_2^* a_2
\end{pmatrix}
\] satisfying the Størmer condition, $a_2$ is of the form (44).

**Remark 20.** Using the Ando-Choi inequality (see [1], [10]) one gets analogous formula for $a_1$ in terms of $a_2$.

As a next step we note that (44) and Størmer condition lead to the following form of the matrix \[
\begin{pmatrix}
a_1^* a_1 & a_1^* a_2 \\
a_2^* a_1 & a_2^* a_2
\end{pmatrix}
\] :
\begin{equation}
\begin{pmatrix}
a_1^* a_1 & a_1^* a_2 \\
a_2^* a_1 & a_2^* a_2
\end{pmatrix} = \sum_i \left( \begin{array}{c}
1 \\
\lambda_i/|\lambda_i|^2
\end{array} \right) \begin{pmatrix}
|a_i^* e_i><a_i^* e_i| & 0 \\
0 & |a_i^* e_i><a_i^* e_i|
\end{pmatrix}
\end{equation}

To rewrite the above equality in a more compact form, let us denote the norm of the vector $|a_i^* e_i>$ by $\alpha_i$ and the normalized vector $\frac{1}{\alpha_i}|a_i^* e_i>$ by $\phi_i$. Then
\begin{equation}
\begin{pmatrix}
a_1^* a_1 & a_1^* a_2 \\
a_2^* a_1 & a_2^* a_2
\end{pmatrix} = \sum_i \alpha_i^2 \left( \begin{array}{c}
1 \\
\lambda_i/|\lambda_i|^2
\end{array} \right) \begin{pmatrix}
|\phi_i><\phi_i| & 0 \\
0 & |\phi_i><\phi_i|
\end{pmatrix}
\end{equation}
or symbolically
\begin{equation}
\begin{pmatrix}
a_1^* a_1 & a_1^* a_2 \\
a_2^* a_1 & a_2^* a_2
\end{pmatrix} = \sum_i \alpha_i^2 \cdot \Lambda_i \cdot R_i
\end{equation}

where $\Lambda_i$ are “matrix” coefficients while $R_i$ are “matrix” projectors (not mutually orthogonal!). This leads to:

**Corollary 21.** (47) implies “separability” for $[a_i^* a_j]$ satisfying the Størmer condition. Namely, using the identification $M_2(\mathcal{B}(\mathcal{H})) \cong M_2(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H})$ (cf discussion concerning equations (22) and (23)) and noting that $(1 + |\lambda_i|^2)^{-\frac{1}{2}} \Lambda_i \equiv P_i$ is a projector one can write
\begin{equation}
\begin{pmatrix}
a_1^* a_1 & a_1^* a_2 \\
a_2^* a_1 & a_2^* a_2
\end{pmatrix} = \sum_i \alpha_i^2 (1 + |\lambda_i|^2) (P_i \otimes 1) (1 \otimes |\phi_i><\phi_i|).
\end{equation}

Hence
\begin{equation}
\begin{pmatrix}
a_1^* a_1 & a_1^* a_2 \\
a_2^* a_1 & a_2^* a_2
\end{pmatrix} \in M_2(\mathbb{C})^+ \otimes \mathcal{B}(\mathcal{H})^+.
\end{equation}

Therefore, it is important to realize that non-triviality of Størner condition follows from the fact that when a positive matrix $[x_{ij}] = \sum_k [(v_k^{(i)})^* v_k^{(j)}]$ satisfies the Størner condition some of its summands $[(v_k^{(i)})^* v_k^{(j)}]$ may not.
We end this Section with

**Remark 22.** Formula (46) can serve as a part of recipe for producing PPT states and some non-decomposable maps on matrix algebras (see next Section).

### 8. Effectiveness of the description of PPT states

Now we are able to discuss the question of effectiveness of the construction of $\mathcal{P}_n \cap \mathcal{P}^\tau_n$. In other words we are interested in the following question: Can one provide a canonical form for a vector in $\mathcal{P}_n \cap \mathcal{P}^\tau_n$?

We begin with the remark that the structure of $\mathcal{P}_n \cap \mathcal{P}^\tau_n$ given by Theorem 13 reflects the Størmer characterization of decomposable maps (see Theorem 17). Hence the posed problem seems to be equivalent to the question whether the given characterization of decomposable maps is an effective one in the sense that we wish to know the canonical form of matrices $[a_{ij}]$ such that $[a_{ij}] \geq 0$ and $[a_{ji}] \geq 0$.

The important point to note here is the Tomiyama characterization of positive transpositions (see [43]). Let $\mathcal{A}$ be a $C^*$-algebra. The transposition $\tau$ on the set of matrices $[a_{ij}]$ with $a_{ij} \in \mathcal{A}$ is a positive map if and only if $\mathcal{A}$ is abelian. This result suggests that the condition $f \in \mathcal{P}_n \cap \mathcal{P}^\tau_n$ reflects a kind of “local commutativity”.

Let us elaborate briefly this point. Firstly, we note : $\Lambda_i$ (see formula 47) is a matrix with complex entries and the transposition on such matrices is a positive map. Secondly, $R_i$ is a matrix with operator entries but this matrix is diagonal. Thus, for transposition, $R_i$ is a fixed point. Furthermore, $\Lambda_i$ commutes with $R_i$. We emphasize that all these remarks stem from (46), (47) - so this is a “local” property as we singled out two indices only. Nevertheless we can conclude : any summand of a positive matrix $[x_{ij}]$ in (34) satisfying the Størmer condition has “local-commutativity” which guarantees the nice behavior (positivity) of the transposition. But not every summand in (34) has this property (see Corollary 21)!

Finally, we are able to discuss the question of effectiveness of the description of PPT states. To this end we recall that Theorem 13 says: PPT states are characterized (uniquely) by vectors of the form $[a_{ij}]\Omega = \sum_k [(a^{(k)}_i)^*a^{(k)}_j]\Omega$ with $[a_{ij}] \geq 0$, where the last equality follows from (34). However, we would like to note here the important point (cf the discussion following (47)) : some summands $[(a^{(k)}_i)^*a^{(k)}_j]\Omega$ may not be in $\mathcal{P}_n \cap \mathcal{P}^\tau_n$. Consequently, some vectors in the subcone $\mathcal{P}_n \cap \mathcal{P}^\tau_n$ which represent non-trivial (that is non-separable) PPT states can be obtained as a convex hull of vectors in such way that some summand(s) is (are) not necessarily in this subcone. This can be expected as $\mathcal{P}_n \cap \mathcal{P}^\tau_n$ is a convex set which could be “far” from being a simplex. In other words, a convex decomposition of a vector in $\mathcal{P}_n \cap \mathcal{P}^\tau_n$ is far from being the unique one. Concluding, the presented arguments suggest that the universally effective prescription for a vector representing PPT state is not available. However, the above discussion provides some recipe for construction of concrete vectors in $\mathcal{P}_n \cap \mathcal{P}^\tau_n$.

### 9. Measures of entanglement.

In [29] and [30] using the $C^*$-algebraic approach to Quantum theory, we have introduced the degree of quantum correlations. The basic idea is to describe how a given quantum system is close to the “classical” world. We wish to repeat this idea but now in the context of Hilbert spaces (cf [31]). For that purpose we will employ the geometry of Hilbert spaces.
Definition 23. Let $\xi$ be a vector in the natural cone $\mathcal{P}$ corresponding to a normal state of a composite system $A + B$ (cf Proposition 12). Then

(1) Degree of entanglement (or quantum correlations) is given by:

$$D_e(\xi) = \inf_{\eta} \{||\xi - \eta||; \eta \in \mathcal{P}_A \otimes \mathcal{P}_B\}$$

(2) Degree of genuine entanglement (or genuine quantum correlations) is defined as

$$D_{ge}(\xi) = \inf_{\eta} \{||\xi - \eta||; \eta \in \mathcal{P}_n \cap \mathcal{P}^\tau_n\}$$

We will briefly discuss the geometric idea behind this definitions. The key to the argument is the concept of convexity (in Hilbert spaces). Namely, we observe

(1) $\mathcal{P} \supset \mathcal{P}_A \otimes \mathcal{P}_B$ is a convex subset,
(2) $\mathcal{P} \supset \mathcal{P}_n \cap \mathcal{P}^\tau_n$ is a convex subset,
(3) The theory of Hilbert spaces says: $\exists! \xi_0 \in \mathcal{P}_A \otimes \mathcal{P}_B$, such that $D_e(\xi) = ||\xi - \xi_0||$,
(4) Analogously, $\exists! \eta_0 \in \mathcal{P} \cap \mathcal{P}^\tau$, such that $D_{ge}(\xi) = ||\xi - \eta_0||$.

The important point to note here is that we used the well known property of convex subsets in a Hilbert space: a closed convex subset $W$ in a Hilbert space $\mathcal{H}$ contains the unique vector with the smallest norm. This ensures the existence of vectors $\xi_0$ and $\eta_0$ introduced in 3. and 4. respectively.

It is expected that any well defined entanglement measure $D(\cdot)$ should, at least, satisfy the following requirements (see [22], [34], [44], [46], and [47]):

(1) $D(\xi) \geq 0$,
(2) $D(\xi) = 0$ if $\xi$ is not entangled,
(3) $D(U_A \otimes U_B \xi) = D(\xi)$ where $U_A$ ($U_B$) are unitary operators representing local symmetry for subsystem $A$ ($B$ respectively),
(4) convexity, i.e. $\sum_i \alpha_i D(\xi_i) \geq D(\sum_i \alpha_i \xi_i)$,
(5) continuity.

Clearly, $D_e$ satisfies all above listed requirements, while for $D_{ge}$ 1-2 and 4-5 hold. Note that the failure of 3 for $D_{ge}$ is not surprising. Namely, recall that for any automorphism on a von Neumann algebra in the standard form there exists the unique unitary operator on the Hilbert space which leaves the natural cone globally invariant (again this is a result of Tomita-Takesaki theory, see also Section 4). Thus unitary operators appearing in 3 can describe a local symmetry. On the other hand it is hard to expect that the set of PPT states has such general symmetry.

Another desirable property of degree (measure) of entanglement would be the monotonicity with respect to arbitrary nonselective operations (see [45]). Here, nonselective operations are understood as CP maps on the set of observables. If such a map $\psi$ leaves the selected vector state $\omega_\Omega$ invariant (in the GNS construction $\Omega$ is interpreted either as the vacuum (field theoretic interpretation) or as equilibrium (statistical interpretation); so this assumption is natural) then, by the generalized Schwarz-Kadison inequality, $\psi$ induces the contraction $\hat{\psi}$ on the GNS space. Consequently such condition in our framework is obviously satisfied provided that $\hat{\psi}$ leaves $\mathcal{P}_1 \otimes \mathcal{P}_2$ globally invariant.
We end this review of properties of entanglement measures with the remark that the idea of measuring entanglement of vectors in terms of their distance to separable vectors appeared in the papers cited in this Section (see also [4]). BUT our approach is carried out in a very different setting and our concept of degree of entanglement stems from the definition given in [30]. In particular, Ozawa arguments on Hilbert-Schmidt distance are not applicable here (cf [34]).

To illustrate our measures of entanglement we present the example which could be considered as a continuation of Example 4 and it is based on a modification of Kadison-Ringrose arguments (cf [24]) with Tomita-Takesaki theory (cf [11]):

**Example 5** Let \( \{e_1, e_2, e_3\} \) (\( \{f_1, f_2, f_3\} \)) be an orthonormal basis in the three dimensional Hilbert space \( \mathcal{H} \) (\( \mathcal{K} \)) respectively. By \( P \) we denote the following rank one orthogonal projector

\[
P = \frac{1}{3}|e_1 \otimes f_1 + e_2 \otimes f_2 + e_3 \otimes f_3| \equiv |x_2 \rangle \rangle x_2| \in B(\mathcal{H} \otimes \mathcal{K})^+
\]

Let \( S \) be an operator of the form

\[
S = \sum_{i=1}^{k<\infty} a_i \otimes b_i
\]

where \( a_i \in B(\mathcal{H})^+ \) and \( b_i \in B(\mathcal{K})^+ \). It can be shown (see [24]) that

\[
||P - S|| \geq \frac{1}{6}
\]

where \( || \cdot || \) stands for the operator norm. Any separable state on \( B(\mathcal{H}) \otimes B(\mathcal{K}) \) can be expressed in the form

\[
\rho_0 = \sum_{i=1}^{l<\infty} \omega_{z_i} \otimes \omega_{y_i}
\]

where \( z_1, \ldots, z_l \in \mathcal{H}, \ y_1, \ldots, y_l \in \mathcal{K} \), and the vector state \( \omega_z \) is defined as \( \omega_z(a) \equiv (z, az) \). Then, again, following Kadison-Ringrose exercise one can show that

\[
||\omega_{x_2} - \rho_0|| \geq \frac{1}{6}
\]

On the other hand (see [11]), if \( \xi \) (a vector \( \eta \)) \( \in \mathcal{P} \) defines the normal positive form \( \omega_\xi \) (\( \omega_\eta \) respectively) then one has

\[
||\xi - \eta||^2 \leq ||\omega_\xi - \omega_\eta|| \leq ||\xi - \eta|| ||\xi + \eta||
\]

Consequently

\[
D_c(\rho^\perp P \rho^\perp) \geq \frac{1}{12}.
\]

Obviously, \( \rho \equiv \rho_{\mathcal{H}} \otimes \rho_{\mathcal{K}} \), etc (cf Sections 4 and 5). We end this example with a remark that the same arguments applied to \( x_2' = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2) \) in 2D case lead to

\[
D_c(\rho^\perp P \rho^\perp) \geq \frac{1}{8}.
\]
Concluding this Section, for an entangled (non-PPT state) we are able to find the best approximation among separable states (PPT states, respectively). Moreover, this approach offers a classification of entanglement (genuine entanglement, respectively).

10. Final remarks

In Section 1, we emphasized that we should deal with many cones. In other words, many types of “positivity” should be taken into account. Furthermore, Lemmas 3 and 6 employ various “orders”: (plain) positivity for the first lemma and CP for the second. On the other hand, we recall that there are many examples on non-CP maps. Consequently, there appears natural question how to understand the difference between Lemmas 3 and 6 in this context. To clarify these subtleties, in this Section we will argue that following the scheme offered by Lemma 3 one can use (very) non-CP maps to describe states on \((A \otimes B, A^+ \otimes B^+)\) which could have very strong correlations (cf [35]). To make our presentation as simple as possible we restrict ourselves to 3 dimensional case. We recall that 3 dimensional models are the simplest cases with non-decomposable maps (see [13], [49], and [14]). We begin with recalling Cho, Kye and Lee [12] results. They studied the following family of maps \(C^3 \rightarrow C^3\)

\[
\phi[a, b, c](x) = \psi[a, b, c](x) - x
\]

\[
\psi[a, b, c](x_{ij}) = \begin{pmatrix}
ax_{11} + bx_{22} + cx_{33} & 0 & 0 \\
0 & ax_{22} + bx_{33} + cx_{11} & 0 \\
0 & 0 & ax_{33} + bx_{11} + cx_{22}
\end{pmatrix}
\]

The properties of these maps are collected in the following Theorem (see [12])

**Theorem 24.**

1. \(\phi[2, 0, \mu]\) for \(\mu \geq 1\) are indecomposable
2. \(\phi[2, 0, 1]\) is atom
3. \(\phi[a, b, c]\) is positive if and only if \(a \geq 1\), \(a + b + c \geq 3\), and \(bc \geq (2 - a)^2\) if \(1 \leq a \leq 2\)
4. \(\phi[a, b, c]\) is CP if and only if \(a \geq 3\)
5. \(\phi[a, b, c]\) is decomposable if and only if \(a \geq 1\), \(bc \geq (\frac{2a}{a+1})^2\) if \(1 \leq a \leq 3\)

For some choices of the parameters \(a, b, c\) one can arrive to maps very similar to that given by (24) (see Section 3). But, as Theorem 24 is saying, there are many concrete very non-CP maps. We can use them and Lemma 3 to produce very “quantum” functionals on compound systems - note that Lemma 6 always deals with CP maps and states (normalized positive functionals with respect to the cone \((A \otimes B)^+\)) on \(C^*\)-algebraic tensor product. Therefore, following Lemma 3 we will define states \(\omega\) by

\[
\omega(a \otimes b) = Tr\phi(a)b^d
\]

where \(\phi\) is any map described by Theorem 24. Any \(a, b \in B(C^3)\) can be written as

\[
a = \sum a_{ij}E_{ij}, \quad b = \sum b_{kl}F_{kl}
\]
where $a_{ij}, b_{kl} \in \mathbb{C}$ while $E_{ij}, F_{kl}$ are basis in $\mathcal{B}(\mathbb{C}^3)$. Hence

$$ (58) \quad \sum_{ijkl} a_{ij} b_{kl} \omega(E_{ij} \otimes F_{kl}) = Tr\phi[abc](a)b^{T} = \sum_{ijkl} a_{ij} b_{kl} Tr\phi[abc](E_{ij}) F_{lk} $$

But

$$ (59) \quad \phi[abc](E_{ij}) = \psi[abc](E_{ij}) - E_{ij} $$

Let $i \neq j$. Then

$$ (60) \quad \phi[abc](E_{ij}) = -E_{ij} $$

To consider the case $i = j$ define

$$ (61) \quad f_{k}(l) = a_{kl} $$

where

$$ (62) \quad [a_{kl}] = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} $$

Then

$$ (63) \quad \phi[abc](E_{ii}) = -E_{ii} + \begin{pmatrix} f_{1}(i) & 0 & 0 \\ 0 & f_{2}(i) & 0 \\ 0 & 0 & f_{3}(i) \end{pmatrix} $$

Taking $a = 2$, $b = 0$, and $c > 1$, equations (68), (60), and (62) give very “quantum” functionals, positive on the projective cone (so admitting negative values on $C_{inj} \setminus C_{pro}$) and showing another difference between Lemmas 3 and 6. Moreover, this shows how powerful “machinery” was proposed in Section 2 as well as gives another explanation of question studied in [35].

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