Rotating relativistic thin disks as sources of charged and magnetized Kerr-NUT spacetimes

Gonzalo García-Reyes

Universidad Tecnológica de Pereira, Departamento de Física,
A. A. 097, Pereira, Colombia

Guillermo A. González

Escuela de Física, Universidad Industrial de Santander,
A.A. 678, Bucaramanga, Colombia

Abstract

A family of models of counterrotating and rotating relativistic thin disks of infinite extension based on a charged and magnetized Kerr-NUT metric are constructed using the well-known “displace, cut and reflect” method extended to solutions of vacuum Einstein-Maxwell equations. The metric considered has as limiting cases a charged and magnetized Taub-NUT solution and the well-known Kerr-Newman solutions. We show that for Kerr-Newman fields the eigenvalues of the energy-momentum tensor of the disk are for all the values of the parameters real quantities so that these disks do not present heat flow in any case, whereas for charged and magnetized Kerr-NUT and Taub-NUT fields we find always regions with heat flow. We also find a general constraint over the counterrotating tangential velocities needed to cast the surface energy-momentum tensor of the disk as the superposition of two counterrotating charged dust fluids. We show that, in general, it is not possible to take the two counterrotating fluids as circulating along electrogeodesics nor take the two counterrotating tangential velocities as equal and opposite.

Key words: General relativity; Thin disks; Exact solutions; Einstein-Maxwell spacetimes

*e-mail: ggarcia@utp.edu.co
†e-mail: guillego@uis.edu.co
1 Introduction

Stationary or static axially symmetric exact solutions to the Einstein field equations representing relativistic thin disks are of great astrophysical importance since they can be used as models for certain galaxies, accretion disks, and the superposition of a black holes and a galaxy or an accretion disk as in the case of quasars. Disk sources for stationary axially symmetric spacetimes with electromagnetic fields, especially magnetic fields, are also of astrophysical importance in the study of neutron stars, white dwarfs and galaxy formation. In such situation one has to study the coupled Einstein-Maxwell equations.

Exact solutions which describe relativistic static thin disks without radial pressure were first studied by Bonnor and Sackfield [1], and Morgan and Morgan [2], and with radial pressure by Morgan and Morgan [3]. Also thin disks with radial tension were considered [4]. Several classes of exact solutions of the Einstein field equations corresponding to static thin disks with or without radial pressure have been obtained by different authors [5]–[12]. Rotating thin disks that can be considered as a source of a Kerr metric were presented by Bicák and Ledvinka [13], while rotating disks with heat flow were were studied by González and Letelier [14]. The nonlinear superposition of a disk and a black hole was first obtained by Lemos and Letelier [9]. Perfect fluid disks with halos were studied by Vogt and Letelier [15]. The stability of some general relativistic thin disks models using a first order perturbation of the energy-momentum tensor was investigated by Ujevic and Letelier [16].

Thin disks in presence of electromagnetic field have been discussed as sources for Kerr-Newman fields [17, 18], magnetostatic axisymmetric fields [19], conformastationary metrics [20], while models of electrovacuum static counterrotating dust disks were presented in [21]. Charged perfect fluid disks were also studied by Vogt and Letelier [22], and charged perfect fluid disks as sources of static and Taub-NUT-type spacetimes by García-Reyes and González [23, 24].

In all the above cases, the disks are obtained by an “inverse problem” approach, called by Synge the “g-method” [25]. The method works as follows: a solution of the vacuum Einstein equations is taken, such that there is a discontinuity in the derivatives of the metric tensor on the plane of the disk, and the energy-momentum tensor is obtained from the Einstein equations. The physical properties of the matter distribution are then studied by an analysis of the surface energy-momentum tensor so obtained. Another approach to generate disks is by solving the Einstein equations given a source (energy-momentum tensor). Essentially, they are obtained by solving a Riemann-Hilbert problem and are highly nontrivial [26]–[32]. A review of this kind of disks solutions to the Einstein-Maxwell equations was presented by Klein in [33].

Now, when the inverse problem approach is used for static spacetimes, the energy-momentum tensor is diagonal and its analysis is direct and, except for dust disks, the solutions obtained have anisotropic sources with azimuthal stress different from the radial stress. On the other hand, when the considered spacetime is stationary, the obtained energy-momentum tensor is non-diagonal and the analysis of its physical content is more involved and, in general, the obtained source is not only anisotropic but with nonzero heat flow. Due to this fact, there are very few works about of stationary disks and they are limited to disks without heat flow [13, 17, 24]. Indeed, only in one work disks with
heat flow has been considered, but only a partial analysis of the corresponding energy-momentum tensor was made [14].

The above disks can also be interpreted as made of two counterrotating streams of freely moving (charged) particles, i.e., which move along (electro-)geodesics, as was also indicated in [17]. This interpretation is obtained by means of the Counterrotating Model (CRM) in which the energy-momentum tensor of the source is expressed as the superposition of two counterrotating fluids. Now, even though this interpretation can be seen as merely theoretical, there are observational evidence of disks made of counterrotating matter as in the case of certain $S0$ and $Sa$ galaxies [35]–[38]. Has also been observed in some spiral galaxies as in NGC3626 the presence of counterrotating ionized (charged) gas disks [39]. Indeed recent investigations have shown that there is a large number of galaxies (the first was NGC4550 in Virgo) [38] which show counterrotating streams in the disk with up to 50% counterrotation. It is believe that the presence of counterrotating matter components in these galaxies is the consequence of the accretion or merger of galaxies. Thus these counterrotating streams may be the result from the capture by a massive early-type galaxy of a gas-rich dwarf galaxy that was orbiting in the opposite sense to the rotation of the main galaxy.

The purpose of the present paper is twofold. In first instance, we present a analysis of the energy-momentum tensor and the surface current density for electrovacuum stationary axially symmetric relativistic thin disks of infinite extension without radial stress and when there is heat flow. And, in the second place, we present a study of the counterrotating model (CRM) for these stationary thin disks. The paper is structured as follows. In Sec. II we present a summary of the procedure to obtain models of rotating thin disks with a purely azimuthal pressure and currents, using the well known “displace, cut and reflect” method extended to solutions of Einstein-Maxwell equations. In particular, we obtain expressions for the surface energy-momentum tensor and the surface current density of the disks.

In Sec. III the disks are interpreted in terms of the CRM. We find a general constraint over the counterrotating tangential velocities needed to cast the surface energy-momentum tensor of the disk as the superposition of two counterrotating charged fluids made of dust or pressureless matter (collisionless particles). We show that, in general, it is not possible to take the two counterrotating fluids as circulating along electro-geodesics nor take the two counterrotating tangential velocities as equal and opposite.

In the following section, Sec. IV, a family of models of counterrotating and rotating relativistic thin disks based on a magnetized and charged Kerr-NUT metric is considered. This metric has as limiting cases a charged and magnetized Taub-NUT solution and the well known Kerr-Newman solutions. We show that for Kerr-Newman fields the eigenvalues of the energy-momentum tensor of the disk are for all the values of the parameters real quantities so that this disks do not present heat flow in any case, whereas for charged and magnetized Kerr-NUT and Taub-NUT fields we find always regions with heat flow. We then analyze the CRM for these disks and we study the angular velocities, surface energy densities and electric charge densities of both streams when the two fluids move along electrogeodesics and when they move with equal and opposite velocities. Also the stability against radial perturbation is analyzed in all the cases. Finally, in Sec. V, we summarize our main results.
2 Electrovacuum rotating relativistic thin disks

A sufficiently general metric for our purposes can be written as the Weyl-Lewis-Papapetrou line element \[34\],
\[
ds^2 = -e^{2\Psi}(dt + \mathcal{W}d\varphi)^2 + e^{-2\Psi}[r^2d\varphi^2 + e^{2\Lambda}(dr^2 + dz^2)],
\]
where we use for the coordinates the notation \((x^0, x^1, x^2, x^3) = (t, \varphi, r, z)\), and \(\Psi, \mathcal{W}, \text{ and } \Lambda\) are functions of \(r\) and \(z\) only. The vacuum Einstein-Maxwell equations, in geometric units in which \(8\pi G = c = \mu_0 = \varepsilon_0 = 1\), are given by
\[
G_{ab} = T_{ab},
\]
\[
F_{ab;b} = 0,
\]
with
\[
T_{ab} = F_{ac}F_b^c - \frac{1}{4}g_{ab}F_{cd}F^{cd},
\]
\[
F_{ab} = A_{b,a} - A_{a,b},
\]
where \(A_a = (A_t, A_\varphi, 0, 0)\) and the electromagnetic potentials \(A_t\) and \(A_\varphi\) are also functions of \(r\) and \(z\) only.

For the metric (1), the Einstein-Maxwell equations are equivalent to the system \[40\]
\[
\nabla \cdot [r^{-2}f(\nabla A_\varphi - \mathcal{W} \nabla A_t)] = 0,
\]
\[
\nabla \cdot [f^{-1}\nabla A_t + r^{-2}f\mathcal{W}(\nabla A_\varphi - \mathcal{W} \nabla A_t)] = 0,
\]
\[
f\nabla^2 f = \nabla \cdot \nabla f - r^{-2}f^4\nabla \mathcal{W} \cdot \nabla \mathcal{W} + f\nabla A_t \cdot \nabla A_t
\]
\[
+ r^{-2}f^3(\nabla A_\varphi - \mathcal{W} \nabla A_t) \cdot (\nabla A_\varphi - \mathcal{W} \nabla A_t),
\]
\[
\Lambda_x = r(\Psi_x^2 - \Psi_z^2) - \frac{1}{4r}(\mathcal{W}_x^2 - \mathcal{W}_z^2)e^{4\Psi} - \frac{1}{2r}(r^2e^{-2\Psi} - \mathcal{W}^2e^{2\Psi})(A_{t,x}^2 - A_{t,z}^2)
\]
\[
+ \frac{1}{2r}(A_{\varphi,x}^2 - A_{\varphi,z}^2)e^{2\Psi} - \frac{1}{r}\mathcal{W}(A_{\varphi,x}A_{t,x} - A_{\varphi,z}A_{t,z})e^{2\Psi},
\]
\[
\Lambda_z = 2r\Psi_x \Psi_z - \frac{1}{2r}\mathcal{W}_x\mathcal{W}_z e^{4\Psi} - \frac{1}{r}(r^2e^{-2\Psi} - \mathcal{W}^2e^{2\Psi})A_{t,x}A_{t,z}
\]
\[
+ \frac{1}{r}A_{\varphi,x}A_{\varphi,z}e^{2\Psi} - \frac{1}{r}\mathcal{W}(A_{\varphi,x}A_{t,z} + A_{\varphi,z}A_{t,x})e^{2\Psi},
\]
where \(\nabla\) is the standard differential operator in cylindrical coordinates and \(f = e^{2\Psi}\).

In order to obtain a solution of (2a) - (2b) representing a thin disk at \(z = 0\), we assume that the components of the metric tensor are continuous across the disk, but with first derivatives discontinuous on the plane \(z = 0\), with discontinuity functions
\[
b_{ab} = g_{ab,z}\big|_{z=0^+} - g_{ab,z}\big|_{z=0^-} = 2g_{ab,z}\big|_{z=0^+},
\]
Thus, by using the distributional approach \[41, 42, 43\] or the junction conditions on the extrinsic curvature of thin shells \[44, 45, 46\], the Einstein-Maxwell equations yield an energy-momentum tensor
\[T_{ab} = T_{elm,ab} + T_{mat,ab},\]
where
\[T_{mat,ab} = \frac{\mathcal{Q}_{ab}}{2} \delta(z),\]
and a current density
\[J_a = j_a \delta(z) = -2e^{2(\Psi - \Lambda)} A_{a,z} \delta(z),\]
where \(\delta(z)\) is the usual Dirac function with support on the disk. \(T_{elm,ab}\) is the electromagnetic tensor defined in Eq. (3a), \(j_a\) is the current density on the plane \(z = 0\), and
\[\mathcal{Q}_{ab} = \frac{1}{2} \left\{ b^a z \delta_b^z - b^z \delta^a_b + g^{az} b^z_b - g^{zz} b^a_b + b^c_c (g^{zz} \delta^a_b - g^{az} \delta^z_b) \right\}\]
is the distributional energy-momentum tensor. The “true” surface energy-momentum tensor (SEMT) of the disk, \(S_{ab}\), and the “true” surface current density, \(j_a\), can be obtained through the relations
\[S_{ab} = \int T_{mat,ab} \, ds_n = e^{\Lambda - \Psi} Q_{ab},\]
\[j_a = \int J_a \, ds_n = e^{\Lambda - \Psi} j_a,\]
where \(ds_n = \sqrt{g_{zz}} \, dz\) is the “physical measure” of length in the direction normal to the disk.

For the metric (1), the nonzero components of \(S_{ab}\) are
\[S^0_0 = \frac{e^{\Psi - \Lambda}}{r^2} \left[ 2r^2 (\Lambda, z - 2\Psi, z) - e^{4\Psi} \mathcal{W} \mathcal{W}_{,z} \right],\]
\[S^1_0 = -\frac{e^{\Psi - \Lambda}}{r^2} \left[ 4r^2 \mathcal{W} \Psi_{,z} + \mathcal{W}^2 e^{4\Psi} \mathcal{W}_{,z} \right],\]
\[S^1_1 = \frac{e^{\Psi - \Lambda}}{r^2} \left[ e^{4\Psi} \mathcal{W}_{,z} \right],\]
\[S^1_1 = \frac{e^{\Psi - \Lambda}}{r^2} \left[ 2r^2 \Lambda_{,z} + e^{4\Psi} \mathcal{W} \mathcal{W}_{,z} \right],\]
and the nonzero components of the surface current density \(j_a\) are
\[j_t = -2e^{\Psi - \Lambda} A_{t,z},\]
\[j_\varphi = -2e^{\Psi - \Lambda} A_{\varphi,z},\]
where all the quantities are evaluated at \(z = 0^+\).

These disks are essentially of infinite extension. Finite disks can be obtained introducing oblate spheroidal coordinates, which are naturally adapted to a disk source, and imposing appropriate boundary conditions. These solutions, in the vacuum and static case, correspond to the Morgan and Morgan solutions [2]. A more general class of solutions representing finite thin disks can be constructed using a method based on the use of conformal transformations and solving a boundary-value problem [3] [6] [4] [12] [23] [24].

Now, in order to analyze the matter content of the disks is necessary to compute the eigenvalues and eigenvectors of the energy-momentum tensor. The eigenvalue problem for the SEMT (7a) - (7d)
\[S_{ab} \xi^b = \lambda \xi^a,\]

has the solutions
\[
\lambda_\pm = \frac{1}{2} \left( T \pm \sqrt{D} \right),
\]
where
\[
T = S_0^0 + S_1^1, \quad D = (S_1^1 - S_0^0)^2 + 4 S_1^0 S_0^1,
\]
and \( \lambda_r = \lambda_z = 0 \). For the metric \((11)\)
\[
D = 4 e^{2(\Psi - \Lambda)} \left( 4 r^2 \Psi^2 \right)^{-\frac{1}{2}} \left( W_{,z} e^4 \Psi \right) = A^2 - B^2,
\]
where
\[
A = 4 \Psi_{,z} e^{\Psi - \Lambda}, \quad B = \frac{2}{r} W_{,z} e^{3 \Psi - \Lambda}.
\]
The corresponding eigenvectors are
\[
\xi_\pm = (\xi_\pm^0, \xi_\pm^1, 0, 0), \\
X^a = e^{\Psi - \Lambda} (0, 0, 1, 0), \\
Y^a = e^{\Psi - \Lambda} (0, 0, 1),
\]
with
\[
g(\xi_\pm, \xi_\pm) = 2 N_\pm e^{2 \Psi} \left( \frac{\xi_\pm^0}{S_0^0 - S_1^1 \pm \sqrt{D}} \right)^2,
\]
where
\[
N_\pm = \sqrt{D}(-\sqrt{D} \pm A).
\]
We only consider the case when \( D \geq 0 \), so that the two eigenvalues \( \lambda_\pm \) are real and different and the two eigenvectors are orthogonal, in such a way that one of them is timelike and the other is spacelike. Since \( |A| \geq \sqrt{D} \), from \((17)\) follows that when \( A > 0 \) the negative sign corresponds to the timelike eigenvector while the positive sign to the spacelike eigenvector. When \( A < 0 \) we have the opposite case. So the function \( \Psi_{,z} \) determines the sign of the norm.

Let \( V^a \) be the timelike eigenvector, \( V_a V^a = -1 \), and \( W^a \) the spacelike eigenvector, \( W_a W^a = 1 \). In terms of the orthonormal tetrad or comoving observer \( e_a^b = \{ V^b, W^b, X^b, Y^b \} \), the SEMT and the surface electric current density may be decomposed as
\[
S_{ab} = \epsilon V_a V_b + p_\varphi W_a W_b, \\
j_a = \delta^a V_a + j^a W_a,
\]
where
\[
\epsilon = -\lambda_\pm, \quad p_\varphi = \lambda_\mp,
\]
are, respectively, the surface energy density, the azimuthal pressure, and
\[
\hat{j}^0 = -V^a j_a, \quad \hat{j}^i = W^a j_a, \tag{20}
\]
are the surface electric charge density and the azimuthal current density of the disk measured by this observer. In (19) the sign is chosen according to which is the timelike eigenvector and which is the spacelike eigenvector. However, in order to satisfy the strong energy condition \( \rho = \epsilon + p_\phi \geq 0 \), where \( \rho \) is the effective Newtonian density, we must choose \( \xi^- \) as the timelike eigenvector and \( \xi^+ \) as the spacelike eigenvector. These condition characterizes a disk made of matter with the usual gravitational attractive property. Consequently \( \Psi_{\pm} \) must be taken positive. So we have
\[
\epsilon = -\lambda_-, \quad p_\phi = \lambda_+, \tag{21}
\]
and
\[
V^0 = \frac{\nu e^{-\psi}}{\sqrt{-2N_-}} (S_0^0 - S_1^1 - \sqrt{D}), \tag{22a}
\]
\[
V^1 = \frac{2\nu e^{-\psi}}{\sqrt{-2N_-}} S_1^0, \tag{22b}
\]
where \( \nu = \pm 1 \) so that the sign is chosen according to the causal character of the timelike eigenvector (observer’s four-velocity),
\[
W^0 = \frac{2}{\sqrt{2M}} S_1^0, \tag{23a}
\]
\[
W^1 = \frac{1}{\sqrt{2M}} (S_1^1 - S_0^0 + \sqrt{D}), \tag{23b}
\]
where
\[
M = \sqrt{D} \left\{ g_{11} \sqrt{D} + 2r\mathcal{W}B + (r^2 e^{-2\psi} + \mathcal{W}^2 e^{2\psi})A \right\}. \tag{24}
\]
When \( D < 0 \), the eigenvalues \( \lambda_\mp \) and the eigenvectors \( \xi_\pm \) are complex conjugates, \( \lambda_- = \bar{\lambda}_+ \), \( \xi_- = \bar{\xi}_+ \). In a comoving orthonormal tetrad \( e^a_b = \{V^b, W^b, X^b, Y^b\} \) the eigenvectors \( \xi^a_\pm \) can be expressed as \( \xi^a_\pm = V^a \pm iW^a \). So we can write the SEMT in the canonical form
\[
S_{ab} = \epsilon V_a V_b + q(V_a W_b + W_a V_b) + p_\phi W_a W_b, \tag{25}
\]
so that it can be interpreted as the energy-momentum tensor of a matter distribution with propagation of heat in the tangential direction. The energy density, the azimuthal pressure, and the heat flow function are respectively
\[
\epsilon = -\frac{T}{2}, \quad p_\phi = \frac{T}{2}, \quad q = \frac{\sqrt{-D}}{2}. \tag{26}
\]
3 Counterrotating charged dust disks

We now consider, based on Refs. [12] and [23], the possibility that the SEMT \( S^{ab} \) and the current density \( j^a \) can be written as the superposition of two counterrotating charged fluids that circulate in opposite directions; that is, we assume

\[
S^{ab} = S^{ab}_+ + S^{ab}_-, \tag{27a}
\]

\[
j^a = j^a_+ + j^a_-, \tag{27b}
\]

where the quantities on the right-hand side are, respectively, the SEMT and the current density of the prograde and retrograde counterrotating fluids.

Let \( U^a_{\pm} = (U^0_{\pm}, U^1_{\pm}, 0, 0) = U^0_{\pm}(1, \omega_{\pm}, 0, 0) \) be the velocity vectors of the two counterrotating fluids, where \( \omega_{\pm} = U^1_{\pm}/U^0_{\pm} \) are the angular velocities of each stream. In order to do the decomposition (27a) and (27b) we project the velocity vectors onto the tetrad \( e^a_b \), using the relations [47]

\[
U^a_{\pm} = e^a_b U^b_{\pm}, \quad U^a_{\pm} = e^a_b U^b_{\pm}. \tag{28}
\]

In terms of the tetrad [15] we can write

\[
U^a_{\pm} = \frac{V^a + v^a W^a}{\sqrt{1 - v^2_{\pm}}}, \tag{29}
\]

so that

\[
V^a = \sqrt{1 - v^2 v^a U^a - \sqrt{1 - v^2 v^a U^a}}, \tag{30a}
\]

\[
W^a = \sqrt{1 - v^2 U^a - \sqrt{1 - v^2 U^a}}, \tag{30b}
\]

where \( v_{\pm} = U^1_{\pm}/U^0_{\pm} \) are the tangential velocities of the streams with respect to the tetrad.

Another quantity related with the counterrotating motion is the specific angular momentum of a particle rotating at a radius \( r \), defined as \( h_{\pm} = g_{\phi a} U^a_{\pm} \). This quantity can be used to analyze the stability of circular orbits of test particles against radial perturbations. The condition of stability,

\[
\frac{d(h^2)}{dr} > 0, \tag{31}
\]

is an extension of Rayleigh criteria of stability of a fluid in rest in a gravitational field [48]. For an analysis of the stability of a rotating fluid taking into account the collective behavior of the particles see for example Refs. [49] [16].
Substituting (30a) and (30b) in (25) we obtain

\[ S^{ab} = \frac{F(v_-, v_+)(1 - v_+^2)}{(v_+ - v_-)^2} \, U^a_+ U^b_+ \]
\[ + \frac{F(v_+, v_-)(1 - v_-^2)}{(v_+ - v_-)^2} \, U^a_- U^b_- \]
\[ - \frac{F(v_+, v_-)(1 - v_+^2)\hat{\tau}(1 - v_-^2)\hat{\tau}(U^a_+ U^b_- + U^a_- U^b_+)}{(v_+ - v_-)^2}, \]

where

\[ F(v_1, v_2) = \epsilon v_1 v_2 - q(v_1 + v_2) + p_\phi. \]  \hspace{1cm} (32)

Clearly, in order to cast the SEMT in the form (27a), the mixed term must be absent and therefore the counterrotating tangential velocities must satisfy the following constraint

\[ F(v_+, v_-) = 0, \] \hspace{1cm} (33)

where we assume that \(|v_\pm| \neq 1\).

Then, assuming a given choice for the tangential velocities in agreement with the above relation, we can write the SEMT as (27a) with

\[ S^{ab}_\pm = \epsilon_\pm \, U^a_\pm U^b_\pm, \] \hspace{1cm} (34)

so that we have two counterrotating dust fluids with energy densities, measured in the coordinates frames, given by

\[ \epsilon_\pm = \left[ \frac{1 - v_\pm^2}{v_\mp - v_\pm} \right] (\epsilon v_\mp - q). \] \hspace{1cm} (35)

Thus the SEMT \( S^{ab} \) can be written as the superposition of two counterrotating dust fluids if, and only if, the constraint (33) admits a solution such that \( v_+ \neq v_- \).

Similarly, substituting (30a) and (30b) in (18b) we can write the current density as (27b) with

\[ j^a_\pm = \sigma_\pm U^a_\pm, \] \hspace{1cm} (36)

where \( \sigma_\pm \) are the counterrotating electric charge densities, measured in the coordinates frames, which are given by

\[ \sigma_\pm = \left[ \frac{\sqrt{1 - v_\pm^2}}{v_\mp - v_\pm} \right] (j^1 - j^0 v_\mp). \] \hspace{1cm} (37)

Thus, we have a disk makes of two counterrotating charged dust fluids with energy densities given by (35), and electric charge densities given by (37).

As we can see from Eqs. (29), (35) and (37), all the main physical quantities associated with the CRM depend on the counterrotating tangential velocities \( v_\pm \). However, the constraint (33) does not
determine \( v_\pm \) uniquely so that we need to impose some additional requirement in order to obtain a complete determination of the tangential velocities leading to a well defined CRM.

A possibility, commonly assumed [17, 31], is to take the two counterrotating streams as circulating along electrogeodesics. Now, if the electrogeodesic equation admits solutions corresponding to circular orbits, we can write this equation as

\[
\frac{1}{2} \epsilon_{\pm g_{ab,r}} U^a_{\pm} U^b_{\pm} = -\sigma_{\pm} F_{ra} U^a_{\pm}.
\] (38)

In terms of \( \omega_\pm \) we obtain

\[
\frac{1}{2} \epsilon_{\pm} (U^0_{\pm})^2 (g_{11,r} \omega^2_\pm + 2g_{01,r} \omega_\pm + g_{00,r}) = -\sigma_{\pm} U^0_{\pm} (A_{t,r} + A_{\varphi,r} \omega_\pm).
\] (39)

From (27a), (27b), (34), and (36) we have

\[
\sigma_{\pm} U^0_{\pm} = \frac{j^1 - \omega_\pm j^0}{\omega_\pm - \omega_\mp},
\] (40a)

\[
\epsilon_{\pm} (U^0_{\pm})^2 = \frac{S^{01}_0 - \omega_\mp S^{00}_0}{\omega_\pm - \omega_\mp},
\] (40b)

\[
\omega_\mp = \frac{S^{11}_0 - \omega_\pm S^{01}_0}{S^{01}_0 - \omega_\pm S^{00}_0},
\] (40c)

and substituting (40a) and (40b) in (39) we find

\[
\frac{1}{2} (S^{01}_0 - \omega_\mp S^{00}_0)(g_{11,r} \omega^2_\pm + 2g_{01,r} \omega_\pm + g_{00,r}) = -(j^1 - \omega_\mp j^0)(A_{t,r} + A_{\varphi,r} \omega_\pm),
\] (41)

and using (40c) we obtain

\[
\frac{1}{2} [(S^{01})^2 - S^{00} S^{11}](g_{11,r} \omega^2_\pm + 2g_{01,r} \omega_\pm + g_{00,r}) = -[S^{01} j^1 - S^{11} j^0 + \omega_\pm (S^{01} j^0 - S^{00} j^1)](A_{t,r} + A_{\varphi,r} \omega_\pm).
\] (42)

Therefore we conclude that

\[
\omega_\pm = \frac{-T_2 \pm \sqrt{T_2^2 - T_1 T_3}}{T_1}
\] (43)

with

\[
T_1 = g_{11,r} + 2A_{\varphi,r} \frac{j^0 S^{01} - j^1 S^{00}}{S^{01} S^{01} - S^{00} S^{11}},
\] (44a)

\[
T_2 = g_{01,r} + A_{t,r} \frac{j^0 S^{01} - j^1 S^{00}}{S^{01} S^{01} - S^{00} S^{11}} + A_{\varphi,r} \frac{j^1 S^{01} - j^0 S^{11}}{S^{01} S^{01} - S^{00} S^{11}},
\] (44b)

\[
T_3 = g_{00,r} + 2A_{t,r} \frac{j^0 S^{01} - j^1 S^{00}}{S^{01} S^{01} - S^{00} S^{11}}.
\] (44c)
It is easy to see when \( D > 0 \) that electrogeodesic velocities satisfies the constraint (33). In the fact, in terms of \( \omega_\pm \) we get
\[
v_\pm = - \left[ \frac{W_0 + W_1 \omega_\pm}{V_0 + V_1 \omega_\pm} \right],
\]
and, by using (43), we have that
\[
v_+ v_- = \frac{T_1 W_0^2 - 2 T_2 W_0 W_1 + T_3 W_1^2}{T_1 V_0^2 - 2 T_2 V_0 V_1 + T_3 V_1^2},
\]
so that, using (18a), we get
\[
F(v_+, v_-) = \frac{32 e^{4(\Psi - \Lambda)} \Lambda_z (r^2 \Lambda_z \sqrt{D} + 4 r^2 \Lambda_z \Psi_s e^{\Psi - \Lambda} - \Psi_z e^{5\Psi - \Lambda})}{r^3 (A + \sqrt{D}) p_\phi (S_{00} S_{01} - S_{01} S_{11}) (T_1 V_0^2 - 2 T_2 V_0 V_1 + T_3 V_1^2)} \times \left[ \Lambda_{,z} - 2 r \Psi_{,z} \Psi_s + \frac{1}{2r} W_{,z} W_s e^{4\Psi} + \frac{1}{r} (r^2 e^{-2\Psi} - \Psi_s e^{2\Psi}) A_{t,r} A_{t,z} - \frac{1}{r} A_{\phi,r} A_{\phi,z} e^{2\Psi} + \frac{1}{r} W (A_{\phi,r} A_{t,z} + A_{\phi,z} A_{t,r}) e^{2\Psi} \right].
\]
Finally, using the Einstein-Maxwell equation (4f) follows immediately that \( F(v_+, v_-) \) vanishes and therefore the electrogeodesic velocities satisfy the constraint (33) and so, if the electrogeodesic equation admits solutions corresponding to circular orbits, we have a well defined CRM. When there is heat flow a proof as the previous one is not trivial and in this case we take the counterrotating hypothesis (34) and (36) as an ansatz.

Another possibility is to take the two counterrotating fluids not circulating along electrogeodesics but with equal and opposite tangential velocities,
\[
v_\pm = \pm v = \pm \sqrt{p_\phi/\epsilon}.
\]
This choice, that imply the existence of additional interactions between the two streams (e.g. collisions), leads to a complete determination of the velocity vectors. However, this can be made only when \( 0 \leq p_\phi/\epsilon \leq 1 \). So in precense of heat flow by Eq. (26) we have \( p_\phi/\epsilon = -1 \) which corresponds to an imaginary velocity and therefore these disks have an unphysical behavior in the regions where there is heat flow.

In the general case, the two counterrotating streams circulate with different velocities and we can write (33) as
\[
v_+ v_- = - \frac{p_\phi}{\epsilon}.
\]
However, this relation does not determine completely the tangential velocities, and therefore the CRM is undetermined. In summary, the counterrotating tangential velocities can be explicitly determined only if we assume some additional relationship between them, like the equal and opposite condition or the electro-geodesic condition. Now, can happen that the obtained solutions do not
satisfy any of these two conditions. That is, the counterrotating velocities are, in general, not completely determined by the constraint (33). Thus, the CRM is in general undetermined since the counterrotating energy densities and pressures cannot be explicitly written without a knowledge of the counterrotating tangential velocities.

4 Disks from a Charged and magnetized Kerr-NUT solution

As an example of the above presented formalism, we consider thin disk models obtained by means of the “displace, cut and reflect” method applied to the electromagnetic generalization of the Kerr-NUT metric, which can be written as

\[ \Psi = \frac{1}{2} \ln \left[ \frac{a_1^2 x^2 + b_1^2 y^2 - c^2}{u^2 + v^2} \right], \]  
\[ \Lambda = \frac{1}{2} \ln \left[ \frac{a_1^2 x^2 + b_1^2 y^2 - c^2}{a_1^2 (x^2 - y^2)} \right], \]  
\[ W = \frac{2kc}{a_1} \left\{ \frac{b_1 (1 - y^2)}{(a_1^2 x^2 + b_1^2 y^2 - c^2)} \left[ a_1 a_2 x + b_1 b_2 y + \frac{1}{2} c(1 + c^2) \right] + b_2 y \right\}, \]  
\[ A_t = \sqrt{2(c^2 - 1)} \left[ \frac{a_2 u + b_2 v}{u^2 + v^2} \right], \]  
\[ A_\phi = -k \sqrt{2(c^2 - 1)} \left\{ \frac{(a_2 u + b_2 v) \left[ -y(b_1 y + 2cb_2) + b_1 \right]}{u^2 + v^2} \right\} + b_2 y, \]

where

\[ u = a_1 x + c a_2, \quad v = b_1 y + c b_2, \]  
and \( a_1^2 + b_1^2 = a_2^2 + b_2^2 = c^2 \geq 1 \), being \( c \) the parameter that controls the electromagnetic field. \( x \) and \( y \) are the prolate spheroidal coordinates which are related with the Weyl coordinates by

\[ r^2 = k^2 (x^2 - 1)(1 - y^2), \quad z + z_0 = kxy, \]

where \( 1 \leq x \leq \infty, 0 \leq y \leq 1 \), and \( k \) is an arbitrary constant. Note that we have displaced the origin of the \( z \) axis in \( z_0 \). This solution can be generated, in these coordinates, using the well-known complex potential formalism proposed by Ernst [40] using as seed solution the Kerr-NUT vacuum solution [34]. So when \( c = 1 \) this solution reduces to the Kerr-NUT vacuum solution. When \( b_1 = 0 \) we have a charged and magnetized Taub-NUT solution [24], and when \( b_2 = 0 \) we have the well known
Kerr-Newman solution. Let $\tilde{T} = kT$, $\tilde{D} = k^2D$, $\tilde{j}_t = kj_t$ be, therefore

$$
\tilde{T} = \frac{4ca_1}{(\tilde{x}^2 - \tilde{y}^2)^{3/2}(u^2 + v^2)^{3/2}} \left\{ -a_1a_2\tilde{y}\tilde{x}^4 + [-2c\tilde{y}^3 - 3b_1b_2\tilde{y}^2
\right. 
+c(1 - c^2)\tilde{y} + b_1b_2)\tilde{x}^3 + 3a_1a_2\tilde{y}(1 - \tilde{y}^2)\tilde{x}^2
\right. 
+\left[ -b_1b_2\tilde{y}^2 + c(1 - c^2)\tilde{y}^2 + 3b_1b_2\tilde{y} + 2c^2] \tilde{x}\tilde{y} + a_1a_2\tilde{y}^3 \right\},
$$

(53a)

$$
\tilde{D} = \frac{16c^2a_1^2}{(\tilde{x}^2 - \tilde{y}^2)(u^2 + v^2)^{3/2}} \left\{ b_1^2(2c^2\tilde{x}^2 - a_2^2)\tilde{y} + 2b_1b_2(c^3\tilde{x} + c\tilde{x} + 2a_1a_2)\tilde{x}\tilde{y}^3
\right. 
+ [c^2a_1^2\tilde{x}^4 + 2a_1a_2c(c^2 + 1)(\tilde{x}^2 + 1)\tilde{x} - (3c^2a_1^2 + 3c^2a_2^2 - 6a_1a_2 - c^6)
\right. 
-4c^4 - c^2)\tilde{x}^2 + c^2\tilde{x}^2]a^2\tilde{y}^2 + 2b_1b_2(2a_1a_2\tilde{x} + c^3 + c)\tilde{x}\tilde{y}^2 - b_2^2(2a_1^2\tilde{x} - c^2)\tilde{x}^2
\right\},
$$

(53b)

$$
\tilde{j}_t = \frac{2\sqrt{2}(c^2 - 1)a_1}{(\tilde{x}^2 - \tilde{y}^2)^{1/2}(u^2 + v^2)^{3/2}} \left\{ b_1b_2\tilde{x}\tilde{y}^4 + b_1^2(3a_1a_2\tilde{x}^2 + 2c^3\tilde{x} - a_1a_2)\tilde{y}^3
\right. 
- b_1b_2(3a_1^2\tilde{x}^2 - 3a_2^2 - c^2)\tilde{x}\tilde{y}^2 + [-a_2^2(a_1a_2\tilde{x} + 2c)\tilde{x}^3
\right. 
-a_1a_2(3a_2^2 + 2c^2 + c^4)\tilde{x}^2 - 2c^3(c^2 - 2a_1^2)\tilde{x} + c^4a_1a_2]\tilde{y}
\right. 
+b_1b_2(2a_1^2\tilde{x}^2 - c^4)\tilde{y}^2\right\},
$$

(53c)

$$
\tilde{j}_\nu = \frac{2\sqrt{2}(c^2 - 1)}{(\tilde{x}^2 - \tilde{y}^2)^{1/2}(u^2 + v^2)^{3/2}} \left\{ -a_1a_2b_1^2(\tilde{x}^2 - 1)\tilde{y}^5
+ b_1^2b_2(3a_1^2\tilde{x}^3 - 3a_2^2\tilde{x} + c^4\tilde{x} + c^2\tilde{x} + 2ca_1a_2)\tilde{y}^4
+ b_1[a_1^2(3a_1a_2\tilde{x} + 8c^3)\tilde{x}^3 - a_1a_2(4a_1^2 - 7c^4 - 3c^2)\tilde{x}^2
+ 2c(-4c^2a_1^2 + c^4 + 2a_1^2a_2^2 + c^6)\tilde{x} - a_1a_2(b_1^2 + c^4)]\tilde{y}^3
- b_2[a_1^4\tilde{x}^5 - 4a_1^2(c^4 - b_2)\tilde{x}^3 - 2c^3a_1a_2(2c^2 - 1)\tilde{x}^2
+( -4c^2a_1^2 + c^4 + 4c^4a_1^2 + 3a_1^4 - c^8)\tilde{x} + 2c^3a_1a_2]^2\tilde{y}^2
- b_1[3a_1^3a_2\tilde{x}^4 + 4ca_1^2(a_2^2 + c^2)\tilde{x}^3 - a_1a_2(3a_2^2 - 2c^2 - 7c^4)\tilde{x}^2
- 2c^3(2a_1^2 - c^4)\tilde{x} - c^4a_1a_2]\tilde{y}
+b_2\tilde{x}(a_1^2\tilde{x}^2 - c^4)(a_1^2\tilde{x}^2 + 2ca_1a_2 + b_1^2 + c^4)\right\}.
$$

(53d)

In the above expressions $\tilde{x}$ and $\tilde{y}$ are given by

$$
2\tilde{x} = \sqrt{\tilde{r}^2 + (\alpha + 1)^2} + \sqrt{\tilde{r}^2 + (\alpha - 1)^2},
$$

(54a)

$$
2\tilde{y} = \sqrt{\tilde{r}^2 + (\alpha + 1)^2} - \sqrt{\tilde{r}^2 + (\alpha - 1)^2},
$$

(54b)

where $\tilde{r} = r/k$ and $\alpha = z_0/k$, with $\alpha > 1$.

When $b_2 = 0$ we have

$$
\tilde{D}_{KN} = \frac{16c^4a_1^2\tilde{y}^2\{[a(\tilde{x}^2 + 1) + \tilde{x}(1 + c^2)]^2 - b_1^2\tilde{r}^2\}}{\tilde{x}^2 - \tilde{y}^2)(a\tilde{x} + c^2)^2 + b_1^2\tilde{y}^2]^3}.
$$

(55)

In order to analyze the behavior of $\tilde{D}_{KN}$, is enough to consider the expression

$$
\tilde{D}_0 = [a(\tilde{x}^2 + 1) + \tilde{x}(1 + c^2)]^2 - b_2^2\tilde{r}^2,
$$

(56)
that can be written as
\[
\tilde{D}_0 = a(1 + c^2)R_+ [\alpha(\alpha - 1) + 2 + \tilde{r}] + a(1 + c^2)R_- [\alpha(\alpha + 1) + 2 + \tilde{r}]
\]
\[
+ \frac{1}{2} R_+ R_- [(c^2 + 1)^2 + a^2(\tilde{r}^2 + \alpha^2 + 3)] + \frac{1}{2} [(c^2 + 1)^2 + 5a^2]
\]
\[
+ \frac{1}{2} \tilde{r}^2 [c^4 + 1 + a^2(\tilde{r}^2 + 2\alpha^2 + 6)] + \frac{1}{2} \alpha^2 [(c^2 + 1)^2 + a^2(\alpha^2 + 2)],
\]

(57)

where \( R_\pm = \sqrt{\tilde{r}^2 + (\alpha \pm 1)^2} \). Since \( \alpha(\alpha \mp 1) + 2 > 0 \) for any \( \alpha \), from (57) follows that \( D \) always is a positive quantity for Kerr-Newman fields and therefore the eigenvalues of the energy-momentum tensor are always real quantities. So we conclude that these disks can be interpreted, for all the values of parameters, as a matter distribution with currents and purely azimuthal pressure and without heat flow [18].

In order to study the behavior of \( D \) when \( b_2 \neq 0 \) and of the other physical quantities associated with the disks, we shall perform a graphical analysis of them for charged and magnetized Kerr-NUT disks with \( b_1 = 0.2, b_2 = 0.9 \), charged and magnetized Taub-NUT disks with \( b_2 = 0.9 \), and Kerr-Newman disks with \( b_1 = 0.2, \) for \( \alpha = 2 \) and \( c = 1.0, 1.5, 2.0, 2.5, 3.0 \), as functions of \( \tilde{r} \). In Fig. [a] we plot \( \tilde{D} \) for charged and magnetized Kerr-NUT disks and in Fig. [b] we plot \( \tilde{D} \) for charged and magnetized Taub-NUT disks. We found that \( \tilde{D} \) takes negative values after certain value of \( \tilde{r} = \tilde{r}_0 \). Therefore these disks always have heat flow beginning at the root \( \tilde{r}_0 \). We also computed this function for other values of the parameters and, in all the cases, we found the same behavior.

In Figs. [c] - [h] we show the energy density \( \tilde{\epsilon} \) and the azimuthal pressure \( \tilde{p}_\phi \). We see that the energy density is in all cases always a positive quantity in agreement with the weak energy condition, whereas the pressure becomes negative for a value of \( \tilde{r} > \tilde{r}_0 \) in the case of charged and magnetized Kerr-NUT and Taub-NUT disks. We also see that the presence of electromagnetic field decreases the energy density at the central region of the disk and later increases it. The heat flow function \( \tilde{q} = kq \) is also represented in Figs. [i] and [j] for charged and magnetized Kerr-NUT and Taub-NUT disks.

The electric charge density \( \tilde{j}_t \) and the azimuthal current density \( \tilde{j}_\phi \), measured in the coordinates frame, are represented in Fig. [2]. We observe that the electric charge density has a similar behavior to the energy density which is consistent with the fact that the mass is more concentrated in the disks center. We also computed this functions for other values of the parameters and, in all the cases, we found the same behavior.

Now we will interpret the matter in the disks as two counterrotating streams of charged dust particles (CRM). First we consider the two streams of particles moving on electrogeodesics for charged and magnetized Kerr-NUT disks with \( b_1 = 0.2, b_2 = 0.9 \), and Kerr-Newman disks with \( b_1 = 2, \) for \( \alpha = 1.2, \) and \( c = 1.0, 1.1, 1.3, 1.4 \), as functions of \( \tilde{r} \). For charged and magnetized Taub-NUT fields we found a similar behavior to the charged and magnetized Kerr-NUT fields.

In Figs. [a] - [d] we plot the angular velocities curves \( \omega_\pm \). We see that these velocities are real and for charged and magnetized Kerr-NUT field \( \omega_+ \) presents a nonsmooth behavior at certain value of \( \tilde{r} \). In Figs. [e] - [h] we have plotted the energy densities \( \tilde{\epsilon}_\pm \). We see that for charged
and magnetized Kerr-NUT disks $\tilde{e}_-$ is everywhere positive whereas $\tilde{e}_+$ becomes negative after certain value of $\tilde{r}$ in violation of the weak energy condition. Therefore for charged and magnetized Kerr-NUT fields one finds that only the central region of the these disks presents a physically reasonable behavior. Thus these disks only model the inner portions of galaxies or accretion disks. However, since the surface energy density decreases rapidly one can to define a cut off radius at the point where the energy density changes of sign and, in principle, we consider these disks as finite so that we can use them to model the whole galaxy or accretion disk. By contrast for Kerr-Newman disks the counterrotating energy densities are always positive quantities.

In Figs. 3(i) - 3(l) we plotted the electric charge densities $\tilde{\sigma}_\pm$. We also find that these quantities have a similar behavior to $\tilde{e}_\pm$. In Figs. 3(m) - 3(p) we have drawn the specific angular momenta $h^2_\pm$ for the same values of the parameters. We see that for charged and magnetized Kerr-NUT disks there is a strong change in the slope at certain values of $\tilde{r}$, which means that there is a strong instability there. We also find regions with negative slope where the CRM is also unstable. However, these disks models are stable in the central region and away of the center of the disks. Meanwhile for Kerr-Newman disks we find that the presence of electromagnetic field can also make unstable these disks against radial perturbations. Thus the CRM cannot apply for $c = 3.0$ (bottom curves). We also computed this functions for other values of $b_1$ and $b_2$ and, in all the cases, we found the same behavior.

Second we consider the case when the two fluids move with equal and opposite tangential velocities (non-electrogeodesic motion) for charged and magnetized Kerr-NUT disks with $\alpha = 2$, $b_1 = b_2 = 0.1$ and $c = 1.0, 1.5, 2.0, 2.5$. In Fig. 4(a) we plot the tangential velocity curves of the counterrotating streams, $v^2$. We see that the quantity $v^2$ is also always less than the light velocity, but after of certain value of $r$ becomes negative. Therefore these disks are also well behaved only in the central region of the disk. In Figs. 4(b) - 4(d) we have plotted the energy densities $\tilde{e}_\pm$ and the electric charge densities $\sigma_\pm$. We see that this quantities have a similar behavior to the previous cases. In Figs. 4(e) and 4(f) we have drawn the specific angular momenta $h^2_+$ and $h^2_-$ for the same values of the parameters. We find that these disks models are stable only in the central region of the disks. In the case of charged and magnetized Taub-NUT disks we also find that the physical quantities have a similar behavior to the charged and magnetized Kerr-NUT disks, and in the case of Kerr-Newman disks to the previous ones.

Finally, the Figs. 1 - 4 show that the two fluids are continuous in $r$ which implies to have two particles in counterrotating movement in the same point in spacetime. So this model could be possible when the distance between streams (or between the counterrotating particles) were very small in comparing with the length $r$ so that we can consider, in principle, the fluids continuous like is the case of counterrotating gas disks present in disk galaxies.

5 Discussion

We presented a detailed analysis of the energy-momentum tensor and the surface current density for electrovacuum stationary axially symmetric relativistic thin disks without radial stress and when there is heat flow. The surface energy-momentum tensor and the surface current density were ex-
pressed in terms of the comoving tetrad and explicit expressions were obtained for the kinematical and dynamical variables that characterize the disks. That is, we obtained expressions for the velocity vector of the disks, as well for the energy density, azimuthal pressure, electric charge density and azimuthal current density.

We also presented in this paper the stationary generalization of the counterrotating model (CRM) for electrovacuum thin disks that had been previously only analyzed for the static case in [21, 23]. We considered both counter rotation with equal and opposite velocities and counter rotation along electrogeodesics. A general constraint over the counterrotating tangential velocities was obtained, needed to cast the surface energy-momentum tensor of the disk in such a way that can be interpreted as the superposition of two counterrotating dust fluids. The constraint obtained is the generalization of the obtained for the vacuum case in [14], for disks without radial pressure and with heat flow, where we only consider counterrotating fluids circulating along geodesics. We also found that, in general, there is not possible to take the two counterrotating tangential velocities as equal and opposite neither take the two counterrotating fluids as circulating along geodesics.

A family of models of counterrotating and rotating relativistic thin disks based on a charged and magnetized Kerr-NUT metric are constructed where we obtain some disks with a CRM well behaved. This solution has as limiting cases a charged and magnetized Taub-NUT solution and the well known Kerr-Newman solutions. For charged and magnetized Kerr-NUT and Taub-NUT fields we find that these disks always present heat flow, whereas for Kerr-Newman fields we shows that these disks can be interpreted, for all the values of parameters, as a matter distribution with currents and purely azimuthal pressure and without heat flow. Finally, the inclusion of radial pressure to these models is being considered.

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Figure 1: The discriminant $\tilde{D}$ (a) for charged and magnetized Kerr-NUT disks with $b_1 = 0.2$, $b_2 = 0.9$ and (b) for charged and magnetized Taub-NUT disks with $b_2 = 0.9$, the energy density $\tilde{\epsilon}$ (c) for charged and magnetized Kerr-NUT disks with $b_1 = 0.2$, $b_2 = 0.9$, (d) for charged and magnetized Taub-NUT disks with $b_2 = 0.9$ and (e) for Kerr-Newman disks with $b_1 = 0.2$, the azimuthal pressure $\tilde{p}_\varphi$ (f) for charged and magnetized Kerr-NUT disks with $b_1 = 0.2$, $b_2 = 0.9$, (g) for charged and magnetized Taub-NUT disks with $b_2 = 0.9$ and (h) for Kerr-Newman disks with $b_1 = 0.2$, the heat flow function $\tilde{q} = kq$ (i) for charged and magnetized Kerr-NUT disks with $b_1 = 0.2$, $b_2 = 0.9$ and (j) for charged and magnetized Taub-NUT disks with $b_2 = 0.9$, and $\alpha = 2.0$, $c = 1.0$ (top curves), 1.5, 2.0, 2.5, and 3.0 (bottom curves), as functions of $\tilde{r}$. 
Figure 2: The electric charge density $\tilde{j}_t$ (a) for charged and magnetized Kerr-NUT disks with $b_1 = 0.2$, $b_2 = 0.9$, (b) for charged and magnetized Taub-NUT disks with $b_2 = 0.9$, and $\alpha = 2.0$, $c = 1.0$ (axis $\tilde{r}$), 1.5 (top curves), 2.0, 2.5, and 3.0 (bottom curves), and (c) for Kerr-Newman disks with $b_1 = 0.2$, and $\alpha = 2.0$, $c = 1.0$ (axis $\tilde{r}$), 2.0 (top curve), 1.5, 2.5, and 3.0 (bottom curve), and the azimuthal current density $j_\varphi$ (d) for charged and magnetized Kerr-NUT disks with $b_1 = 0.2$, $b_2 = 0.9$, (e) for charged and magnetized Taub-NUT disks with $b_2 = 0.9$, and (f) for Kerr-Newman disks with $b_1 = 0.2$, and $\alpha = 2.0$, $c = 1.0$ (axis $\tilde{r}$), 1.5 (top curves), 2.0, 2.5, and 3.0 (bottom curves), as functions of $\tilde{r}$. 
Figure 3: We plot for electrogeodesic charged and magnetized Kerr-NUT fields with $\alpha = 1.2, b_1 = 0.2, b_2 = 0.9$, and electrogeodesic Kerr-Newman fields with $\alpha = 1.2, b_1 = 0.2$, the angular velocities $\omega_{\pm}$ $(a), (b)$ for charged and magnetized Kerr-NUT disks and $(c), (d)$ for Kerr-Newman disks, the energy densities $\tilde{\epsilon}_{\pm}$ $(e), (f)$ for charged and magnetized Kerr-NUT disks and $(g), (h)$ for Kerr-Newman disks, with $c = 1.0$ (top curves), 1.1, 1.3, 1.4 (bottom curves), the electric charge densities $\tilde{\sigma}_{\pm}$ $(i), (j)$ for charged and magnetized Kerr-NUT disks and $(k), (l)$ for Kerr-Newman disks, with $c = 1.0$ (axis $\tilde{r}$), 1.1 (bottom curves), 1.3, 1.4 (top curves), the specific angular momenta $h^2_{\pm}$ $(m), (n)$ for charged and magnetized Kerr-NUT disks and $(o), (p)$ for Kerr-Newman disks, with $c = 1.0$ (top curves), 1.1, 1.3, 1.4, 3.0 (bottom curves).
Figure 4: We plot for non-electrogeodesic charged and magnetized Kerr-NUT field with $\alpha = 2$, $b_1 = b_2 = 0.1$, (a) the tangential velocity $v^2$, (b) the energy densities $\tilde{\epsilon}_\pm$ for $c = 1.0$ (top curves), 1.5, 2.0, 2.5 (bottom curves), (c), (d) the electric charge densities $\tilde{\sigma}_\pm$ for $c = 1.0$ (axis $\tilde{r}$), 1.5 (bottom curves), 2.0, 2.5 (top curves), (e) and (f) the angular momentum specific $h^2_\pm$ for $c = 1.0$ (top curves), 1.5, 2.0, 2.5 (bottom curves), as functions of $\tilde{r}$. 