Localization of the number of photons of ground states in nonrelativistic QED

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Abstract

One electron system minimally coupled to a quantized radiation field is considered. It is assumed that the quantized radiation field is massless, and no infrared cutoff is imposed. The Hamiltonian, $H$, of this system is defined as a self-adjoint operator acting on $L^2(\mathbb{R}^3) \otimes \mathcal{F} \cong L^2(\mathbb{R}^3; \mathcal{F})$, where $\mathcal{F}$ is the Boson Fock space over $L^2(\mathbb{R}^3 \times \{1, 2\})$. It is shown that the ground state, $\psi_g$, of $H$ belongs to $\bigcap_{k=1}^\infty D(1 \otimes N^k)$, where $N$ denotes the number operator of $\mathcal{F}$. Moreover it is shown that, for almost every electron position variable $x \in \mathbb{R}^3$ and for arbitrary $k \geq 0$, $\| (1 \otimes N^{k/2}) \psi_g(x) \|_{\mathcal{F}} \leq D_k e^{-\delta|x|^{m+1}}$ with some constants $m \geq 0$, $D_k > 0$, and $\delta > 0$ independent of $k$. In particular $\psi_g \in \bigcap_{k=1}^\infty D(e^{\beta|x|^{m+1}} \otimes N^k)$ for $0 < \beta < \delta/2$ is obtained.

1 Introduction

1.1 The Pauli-Fierz Hamiltonian

In this paper one spinless electron minimally coupled to a massless quantized radiation field is considered. It is the so-called Pauli-Fierz model of the nonrelativistic QED. The Hilbert space of state vectors of the system is given by

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F},$$

where $\mathcal{F}$ denotes the Boson Fock space defined by

$$\mathcal{F} = \bigoplus_{n=0}^\infty \left[ \otimes_n L^2(\mathbb{R}^3 \times \{1, 2\}) \right],$$

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where $\otimes_n^s L^2(\mathbb{R}^3 \times \{1,2\})$, $n \geq 1$, denotes the $n$-fold symmetric tensor product of $L^2(\mathbb{R}^3 \times \{1,2\})$ and $\otimes_n^0 L^2(\mathbb{R}^3 \times \{1,2\}) = \mathbb{C}$. The Fock vacuum $\Omega$ is defined by $\Omega = \{1,0,0,...\}$. Let

$$\mathcal{F}_0 = \{ \oplus_{n=0}^\infty \Psi^{(n)} \in \mathcal{F} | \Psi^{(n)} = 0 \text{ for } n \geq m \text{ with some } m \}. $$

For each $(k,j) \in \mathbb{R}^3 \times \{1,2\}$, the annihilation operator $a(k,j)$ is defined by, for $\Psi = \oplus_{n=0}^\infty \Psi^{(n)} \in \mathcal{F}_0$,

$$(a(k,j)\Psi^{(n)})(k_1,j_1,...,k_n,j_n) = \sqrt{n+1}\Psi^{(n+1)}(k,j,k_1,j_1,...,k_n,j_n).$$

The creation operator $a^*(k,j)$ is given by $a^*(k,j) = (a(k,j|\mathcal{F}_0)^*$. They satisfy the canonical commutation relations on $\mathcal{F}_0$

$$[a(k,j), a^*(k',j')] = \delta(k-k')\delta_{jj'},$$

$$[a(k,j), a(k',j')] = 0,$$

$$[a^*(k,j), a^*(k',j')] = 0.$$ 

The closed extensions of $a(k,j)$ and $a^*(k,j)$ are denoted by the same symbols respectively. The annihilation and creation operators smeared by $f \in L^2(\mathbb{R}^3)$ are formally written as

$$a^f(k,j) = \int a^*(k,j)f(k)dk, \quad a^f = a \text{ or } a^*,$$

and act as

$$(a(f,j)\Psi^{(n)} = \sqrt{n+1} \int f(k)\Psi^{(n+1)}(k,j,k_1,j_1,...,k_n,j_n)dk,$$

$$a^*(f,j)\Psi^{(n)} = \frac{1}{\sqrt{n}} \sum_{j_i=j} f(k)\Psi^{(n-1)}(k_1,j_1,...,j_i,j_i,...,k_n,j_n),$$

where $\sum_{j_i=j}$ denotes to sum up $j_i$ such that $j_i = j$, and $\widehat{X}$ means neglecting $X$. We work with the unit $\hbar = 1 = c$. The dispersion relation is given by

$$\omega(k) = |k|.$$ 

Then the free Hamiltonian $H_f$ of $\mathcal{F}$ is formally written as

$$H_f = \sum_{j=1,2} \int \omega(k)a^*(k,j)a(k,j)dk,$$
and acts as

$$(H_t \Psi)^{(n)}(k_1, j_1, \ldots, k_n, j_n) = \sum_{j=1}^{n} \omega(k_j) \Psi^{(n)}(k_1, j_1, \ldots, k_n, j_n), \quad n \geq 1,$$

$$(H_t \Psi)^{(0)} = 0$$

with the domain

$$D(H_t) = \left\{ \Psi = \bigoplus_{n=0}^{\infty} \Psi^{(n)} \bigg| \sum_{n=0}^{\infty} \| (H_t \Psi)^{(n)} \|_{L^2(\mathbb{R}^3 \times \{1, 2\})}^2 \right\}.$$ 

Since $H_t$ is essentially self-adjoint and nonnegative, we denote the self-adjoint extension of $H_t$ by the same symbol $H_t$. Under the identification

$$H \cong \int_{\mathbb{R}^3} F dx,$$

the quantized radiation field $A$ with a form factor $\varphi$ is given by the constant fiber direct integral

$$A = \int_{\mathbb{R}^3} A(x) dx,$$

where $A(x)$ is the operator acting on $F$ defined by

$$A(x) = \frac{1}{\sqrt{2}} \sum_{j=1,2} \int \frac{e(k, j)}{\sqrt{\omega(k)}} \left\{ a^*(k, j)e^{-ik \cdot x} \varphi(-k) + a(k, j)e^{ik \cdot x} \varphi(k) \right\} dk.$$ 

Here $\varphi$ denotes the Fourier transform of $\varphi$ and $e(k, j)$, $j = 1, 2$, are polarization vectors such that $(e(k, 1), e(k, 2), k/|k|)$ forms a right-handed system, i.e., $k \cdot e(k, j) = 0$, $e(k, j) \cdot e(k, j') = \delta_{jj'}$, and $e(k, 1) \times e(k, 2) = k/|k|$ for almost every $k \in \mathbb{R}^3$. We fix polarization vectors through this paper.

The decoupled Hamiltonian is given by

$$H_0 = H_p \otimes 1 + 1 \otimes H_t.$$ 

Here

$$H_p = \frac{1}{2} p^2 + V$$

denotes a particle Hamiltonian, where $p = (-i\nabla_{x_1}, -i\nabla_{x_2}, -i\nabla_{x_3})$ and $x = (x_1, x_2, x_3)$ are the momentum operator and its conjugate position operator in $L^2(\mathbb{R}^3)$, respectively, and $V : \mathbb{R}^3 \to \mathbb{R}$ an external potential. We are prepared to define the total Hamiltonian, $H$, of this system, which is given by the minimal coupling to $H_0$. I.e., we replace $p \otimes 1$ with $p \otimes 1 - eA$,

$$H = \frac{1}{2} (p \otimes 1 - eA)^2 + V \otimes 1 + 1 \otimes H_t,$$

where $e$ denotes the charge of an electron.
1.2 Assumptions on $V$ and fundamental facts

We give assumptions on external potentials. We say $V \in K_3$ (the three dimensional Kato class [23]) if and only if
\[
\lim_{\epsilon \to 0} \sup_{x \in \mathbb{R}^3} \int_{|x-y|<\epsilon} \frac{|V(y)|}{|x-y|} dy = 0,
\]
and $V \in K_3^{\text{loc}}$ if and only if $1_R V \in K_3$ for all $R \geq 0$, where
\[
1_R(x) = \begin{cases} 1, & |x| < R, \\ 0, & |x| \geq R. \end{cases}
\]

Let us define classes $K$ and $V_{\exp}$ as follows.

**Definition 1.1**

1. We say $V \in K$ if and only if $V = V_+ - V_-$ such that $V_+ \geq 0$, $V_+ \in K_3^{\text{loc}}$ and $V_- \in K_3$.

2. We say $V \in V_{\exp}$ if and only if $V = Z + W$ such that $\inf Z > -\infty$, $Z \in L_1^{\text{loc}}(\mathbb{R}^3)$, $W < 0$, and $W \in L^p(\mathbb{R}^3)$ for some $p > 3/2$.

For $V \in K$ a functional integral representation of $e^{-t(-\frac{1}{2}\Delta + V)}$ by means of the Wiener measure on $C([0, \infty); \mathbb{R}^3)$ is obtained. See e.g., [23]. For $V \in K \cap V_{\exp}$, using this functional integral representation, it can be proven that a ground state, $f_p$, of $-\frac{1}{2}\Delta + V$ decays exponentially, i.e.,
\[
|f_p(x)| \leq c_1 e^{-c_2|x|^3}
\]
for almost every $x \in \mathbb{R}^3$ with some positive constants $c_1, c_2, c_3$. Similar estimates are available to the Pauli-Fierz Hamiltonian $H$ with $V \in K \cap V_{\exp}$. See Proposition 1.5.

Furthermore we need to define class $V(m), m = 0, 1, 2, \ldots$ to estimate constant $c_3$ in (1.1) precisely.

**Definition 1.2** Suppose that $V = Z + W \in V_{\exp} \cap K$, where the decomposition $Z + W$ is that of the definition of $V_{\exp}$.

1. We say $V \in V(m), m \geq 1$, if and only if $Z(x) \geq \gamma|x|^{2m}$ for $x \notin O$ with a certain compact set $O$ and with some $\gamma > 0$.

2. We say $V \in V(0)$ if and only if $\lim \inf_{|x| \to \infty} Z(x) > \inf \sigma(H)$, where $\sigma(H)$ denotes the spectrum of $H$. 

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A physically reasonable example of $V$ is the Coulomb potential $-\frac{eZ}{4\pi|x|}$, where $Z > 0$ denotes the charge of a nucleus. Actually we see the following proposition.

**Proposition 1.3** Assume that

$$\int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} \frac{dk}{\omega(k)} < \frac{Z^2}{2(4\pi)^2}.$$ 

Then

$$-\frac{eZ}{4\pi|x|} \in V(0)$$

for all $e > 0$.

**Proof:** It is known that $-\frac{1}{|x|} \in K_3 \cap V_{\exp}$. Then we shall show $\inf \sigma(H) < 0$. Let $V = -eZ/(4\pi|x|)$ and $f$ be the ground state of $H_p = -\frac{1}{2}\Delta + V$, $H_p f = -E_0 f$, where

$$E_0 = \frac{e^2 Z^2}{2(4\pi)^2}.$$ 

Then we have

$$\inf \sigma(H) \leq (f \otimes \Omega, H f \otimes \Omega)_H = (f, H_p f)_{L^2(\mathbb{R}^3)} + \frac{e^2}{2}(f \otimes \Omega, A^2 f \otimes \Omega)_H$$

$$= -E_0 + \frac{e^2}{2} \sum_{\mu=1,2,3} \int_{\mathbb{R}^3} \left(1 - \frac{k^2}{|k|^2}\right) \frac{|\hat{\varphi}(k)|^2}{\omega(k)} dk = -\frac{e^2}{2} \left(\frac{Z^2}{(4\pi)^2} - 2 \int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} dk\right) < 0.$$ 

Thus the proposition follows. \qed

We introduce Hypothesis $\mathbb{H}_m$, $m = 0, 1, 2, ...$.

**Hypothesis** $\mathbb{H}_m$

1. $D(\Delta) \subset D(V)$ and there exists $0 \leq a < 1$ and $0 \leq b$ such that for $f \in D(\Delta)$,

$$\|V f\|_{L^2(\mathbb{R}^3)} \leq a\|\Delta f\|_{L^2(\mathbb{R}^3)} + b\|f\|_{L^2(\mathbb{R}^3)},$$

2. $\hat{\varphi}(-k) = \overline{\hat{\varphi}(k)}$, and $\hat{\varphi}/\omega$, $\sqrt{\omega \hat{\varphi}} \in L^2(\mathbb{R}^3)$,

3. $\inf \sigma_{\text{ess}}(H_p) - \inf \sigma(H_p) > 0$, where $\sigma(H_p)$ (resp. $\sigma_{\text{ess}}(H_p)$) denotes the spectrum (resp. essential spectrum) of $H_p$,

4. $V \in V(m).$
Proposition 1.4 We assume (1) and (2) of $\mathbb{H}_m$. Then for arbitrary $e \in \mathbb{R}$, $H$ is self-adjoint on $D(\Delta \otimes 1) \cap D(1 \otimes H_f)$ and bounded from below, moreover essentially self-adjoint on any core of $-\Delta \otimes 1 + 1 \otimes H_f$.

Proof: See [14, 15].

The number operator of $F$ is defined by

$$N = \sum_{j=1,2} \int a^*(k,j)a(k,j)dk.$$ 

The operator $N^k$, $k \geq 0$, acts as, for $\Psi = \oplus_{n=0}^{\infty} \Psi^{(n)}$,

$$(N^k \Psi)^{(n)} = n^k \Psi^{(n)}$$

with the domain

$$D(N^k) = \left\{ \Psi = \oplus_{n=0}^{\infty} \Psi^{(n)} \left| \sum_{n=0}^{\infty} n^{2k}\|\Psi^{(n)}\|_{L^2(\mathbb{R}^3 \times \{1,2\})}^2 < \infty \right. \right\}.$$ 

We give a remark on notations. We can identify $\mathcal{H}$ with the set of $\mathcal{F}$-valued $L^2$-functions on $\mathbb{R}^3$, i.e.,

$$\mathcal{H} \cong L^2(\mathbb{R}^3; \mathcal{F}). \quad (1.2)$$

Under this identification, $\Psi \in \mathcal{H}$ can be regarded as a vector in $L^2(\mathbb{R}^3; \mathcal{F})$. Namely for almost every $x \in \mathbb{R}^3$,

$$\Psi(x) \in \mathcal{F}.$$ 

We use identification (1.2) without notices in what follows. The following proposition is well known.

Proposition 1.5 Suppose $\mathbb{H}_m$. Then there exists $e_0 < \infty$ such that for all $|e| \leq e_0$, (i) $H$ has a ground state $\psi_g$, (ii) it is unique, (iii) $\| (1 \otimes N^{1/2}) \psi_g \|_\mathcal{H} < \infty$, (iv) $\| \psi_g(x) \|_\mathcal{F} \leq De^{-\delta|x|^{m+1}}$ for almost every $x \in \mathbb{R}^3$ with some constants $D > 0$ and $\delta > 0$.

Proof: See [5, 10] for (i) and (iii), [13] for (ii) and [16] for (iv).

Remark 1.6 It is not clear directly from Proposition 1.5 that $\psi_g \in D(e^{\delta|x|^{m+1}} \otimes N^{1/2})$.

See Corollary 1.11.
The condition
\[ I = \int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk < \infty \] (1.3)
is called the infrared cutoff condition. (1.3) is not assumed in Proposition 1.5. For suitable external potentials, \( e_0 = \infty \) is available in Proposition 1.5. This is established in [10]. In the case where \( \inf\text{ess}(H_p) - \inf \sigma(H_p) = 0 \), examples for \( H \) to have a ground state is investigated in [17, 19]. It is unknown, however, whether such a ground state decays in \( x \) exponentially or not. When electron includes spin, \( H \) has a twofold degenerate ground state for sufficiently small \( |e| \), which is shown in [18].

1.3 Localization of the number of bosons and infrared singularities for a linear coupling model

The Nelson Hamiltonian [22] describes a linear coupling between a nonrelativistic particle and a scalar quantum field with a form factor \( \varphi \). Let \( H_N = L^2(\mathbb{R}^3) \otimes \mathcal{F}_N \), where \( \mathcal{F}_N = \bigoplus_{n=0}^\infty [\otimes_n L^2(\mathbb{R}^3)] \). The Nelson Hamiltonian is defined as a self-adjoint operator acting in the Hilbert space \( \mathcal{H}_N \), which is given by

\[ H_N = H_p \otimes 1 + 1 \otimes H_f^N + g\phi, \]

where \( g \) denotes a coupling constant, \( H_f^N = \int \omega(k)a^*(k)a(k)dk \) is the free Hamiltonian in \( \mathcal{F}_N \), and under identification \( \mathcal{H}_N \cong \mathcal{F}_N \otimes \mathcal{F}_N \), \( \phi \) is defined by \( \phi = \int_{\mathbb{R}^3} \phi(x)dx \) with

\[ \phi(x) = \frac{1}{\sqrt{2}} \int \left\{ a^*(k)e^{-ikx} \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} + a(k)e^{ikx} \frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}} \right\} dk. \]

It has been established in [2, 4, 9, 25] that the Nelson Hamiltonian has the unique ground state, \( \psi_g^N \), under the condition

\[ I < \infty. \]

Let us denote the number operator of \( \mathcal{F}_N \) by the same symbol \( N \) as that of \( \mathcal{F} \). In [6] it has been proven that \( \psi_g^N \) decays superexponentially, i.e.,

\[ \| e^{+\beta(1 \otimes N)}\psi_g^N \|_{\mathcal{H}_N} < \infty \] (1.4)

for arbitrary \( \beta > 0 \). This kind of results has been obtained in [11, Section 3] and [24] for relativistic polaron models, and [26, Section 8] for spin-boson models. Moreover in [6] we see that

\[ \lim_{I \to \infty} \| (1 \otimes N^{1/2})\psi_g^N \|_{\mathcal{H}_N} = \infty. \] (1.5)
Actually in the infrared divergence case,

\[ I = \infty, \]  

(1.6)

it is shown in [20] that the Nelson Hamiltonian with some confining external potentials has no ground states in \( \mathcal{H}_N \). Then we have to take a non-Fock representation to investigate a ground state with (1.6). See [1, 3, 21] for details. That is to say, as the infrared cutoff is removed, the number of bosons of \( \psi_N^g \) diverges and the ground state disappears. A method to show (1.4) and (1.5) is based on a path integral representation of \((\psi_N^g, e^{+\beta (1 \otimes N)} \psi_N^g)_{\mathcal{H}_N}\). Precisely it can be shown that in the case \( I < \infty \) there exists a probability measure \( \mu \) on \( C(\mathbb{R}; \mathbb{R}^3) \) such that for arbitrary \( \beta > 0 \),

\[
(\psi_N^g, e^{+\beta (1 \otimes N)} \psi_N^g)_{\mathcal{H}_N} = \int_{C(\mathbb{R}; \mathbb{R}^3)} e^{-(g^2/2)(1-e^{+\beta})} \int_{0}^{\infty} ds \int_{-\infty}^{\infty} dt W(q_s - q_t, s - t) \mu(dq),
\]

(1.7)

where \((q_t)_{-\infty < t < \infty} \in C(\mathbb{R}; \mathbb{R}^3)\), and

\[
W(X, T) = \int_{\mathbb{R}^3} e^{-|T|\omega(k)} e^{ik \cdot X} \frac{|\hat{\phi}(k)|^2}{\omega(k)} dk.
\]

(1.8)

Note that the double integral \( \int_{-T}^{0} ds \int_{0}^{T} dt W(q_s - q_t, s - t) \) is estimated uniformly in path and \( T \) as

\[
\left| \int_{-T}^{0} ds \int_{0}^{T} dt W(q_s - q_t, s - t) \right| \leq I.
\]

(1.9)

This uniform bound is a core of the proof of identity (1.7).

1.4 The main theorems

In contrast to the Nelson Hamiltonian, for the Pauli-Fierz Hamiltonian, as is seen in Proposition 1.5, it is shown that the ground state, \( \psi_g \), exists and \( \|(1 \otimes N^{1/2}) \psi_g\|_{\mathcal{H}} < \infty \) even in the case \( I = \infty \). We may say that the infrared singularity for the Pauli-Fierz Hamiltonian is not so singular in comparison with the Nelson Hamiltonian, and one may expect that

\[
\|e^{+\beta (1 \otimes N)} \psi_g\|_{\mathcal{H}} < \infty
\]

(1.10)

holds for some \( \beta > 0 \) under \( I = \infty \). Unfortunately, however, we can not show (1.10), since the similar path integral method as the Nelson Hamiltonian is not available on account of the appearance of the so-called double stochastic integral ([13]) instead of
\[ f_0^\infty ds f_0^\infty dt W(q_s - q_t, s - t) \] in (1.7). The double stochastic integral is formally written as

\[ \sum_{\mu, \nu = 1, 2, 3} \int_{-\infty}^0 dq_{\mu, s} \int_{0}^\infty dq_{\nu, t} W_{\mu \nu}(q_s - q_t, s - t), \tag{1.11} \]

where \((q_s)_{-\infty < s < \infty} = (q_{1, s}, q_{2, s}, q_{3, s})_{-\infty < s < \infty} \in C(\mathbb{R}, \mathbb{R}^3)\) and

\[ W_{\mu \nu}(X, T) = \int_{\mathbb{R}^3} \left( \delta_{\mu \nu} - \frac{k_{\mu} k_{\nu}}{|k|^2} \right) e^{-|T|\omega(k)} e^{ik \cdot X} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} dk. \]

Actually we can not estimate (1.11) uniformly in path such as (1.9). Therefore we are not concerned here with (1.10). In place of this we will show the following theorems.

**Theorem 1.7** Assume \( \mathbb{H}_m \). Then \( \psi_g \in \bigcap_{k=1}^\infty D(1 \otimes N^{k/2}) \).

**Remark 1.8** Theorem 1.7 automatically follows if one assumes that photons have artificial positive mass, \( \nu \), i.e., \( \omega(k) = \sqrt{|k|^2 + \nu^2} \).

**Theorem 1.9** Assume \( \mathbb{H}_m \). Then for a fixed \( k \geq 0 \) there exist positive constants \( D_k \), and \( \delta \) independent of \( k \) such that

\[ \| (1 \otimes N^{k/2}) \psi_g(x) \|_F \leq D_k e^{-\delta |x|^{m+1}} \tag{1.12} \]

for almost every \( x \in \mathbb{R}^3 \).

**Remark 1.10** We do not assume \( \mathcal{I} < \infty \) in Theorems 1.7 and 1.9.

From Theorems 1.7 and 1.9 the following corollary is immediate.

**Corollary 1.11** Assume \( \mathbb{H}_m \). Then \( \psi_g \in \bigcap_{k=0}^\infty D(e^{2\beta |x|^{m+1}} \otimes 1) \) for \( \beta < \delta/2 \).

**Proof:** Since \( \psi_g \in D(e^{2\beta |x|^{m+1}} \otimes 1) \cap D(1 \otimes N^{k/2}) \) for all \( k \geq 0 \), the corollary follows from the fact that \( D(e^{2\beta |x|^{m+1}} \otimes 1) \cap D(1 \otimes N^k) \subset D(e^{\beta |x|^{m+1}} \otimes N^{k/2}) \). \( \square \)

### 1.5 Outline of proofs of the main theorems

For notational convenience, in the following we mostly omit the tensor notation \( \otimes \), e.g., we express as \( H_f \) for \( 1 \otimes H_f \), \( a^\sharp(k, j) \) for \( 1 \otimes a^\sharp(k, j) \), \( \Delta \) for \( \Delta \otimes 1 \), \( |x| \) for \( |x| \otimes 1 \), etc., and set

\[ k = (k, j) \in \mathbb{R}^3 \times \{1, 2\} \]
and
\[ \sum_{j_1, \ldots, j_n=1,2} \ldots dk_1 \ldots dk_n = \sum_{j_1, \ldots, j_n=1} \ldots \int_{j_1, \ldots, j_n} dk_1 \ldots dk_n. \]

The strategy of this paper is as follows. We check in Lemma 3.2 that
\[ \sum_{j_1, \ldots, j_n=1} \ldots dk_1 \ldots dk_n = \sum_{j_1, \ldots, j_n=1,2} \ldots \int_{j_1, \ldots, j_n} \Psi = \sum_{j_1, \ldots, j_n=1,2} \ldots \int_{j_1, \ldots, j_n} \]
if and only if
\[ \Psi \in D(N^{k/2}). \]

Thus in order to prove Theorem 1.7 it is enough to show that \( \psi_\gamma \in D(a(k_1) \ldots a(k_l)) \) for almost every \((k_1, \ldots, k_l) \in \mathbb{R}^l\), and
\[ \sum_{j_1, \ldots, j_n=1} \ldots dk_1 \ldots dk_n < \infty \]
holds for all \( l \geq 0 \). One subtlety to show 1.14 is that we do not assume \( \mathcal{I} < \infty \). Bach-Fröhlich-Sigal [5] proved 1.14 for \( l = 1 \). We extend it to \( l \geq 1 \).

To see (1.14) for all \( l \) we make a detour through the modified annihilation operator defined by
\[ b(k, j) = a(k, j) - i \frac{e}{\sqrt{2}}(x \cdot e(k, j)) \frac{e^{-ik \cdot x}}{\sqrt{\omega(k)}} \hat{\phi}(k). \]

For some \( \Psi \in \mathcal{H} \) we establish in Lemma 3.6 that
\[ \|a(k_1) \ldots a(k_n)\|_{\mathcal{H}} \leq \sum_{l=0}^n \sum_{(p_1, \ldots, p_l) \subset (1, \ldots, n)} \prod_{j=1}^l \frac{e^{i\hat{\phi}(p_j)}}{\sqrt{2\omega(p_j)}} \|b(k_1) \ldots b(p_j) \ldots b(k_n)\|_{\mathcal{H}} \|x^m \Psi\|_{\mathcal{H}}, \quad (1.15) \]

where \( \hat{\phi} \) means neglecting the term below, and \( \sum_{(p_1, \ldots, p_l) \subset (1, \ldots, n)} \) denotes to sum up all the combinations to choose \( l \) numbers from \( \{1, 2, \ldots, n\} \). In Lemma 3.7 we show that there exist constants \( c_k^{n,l} \) such that
\[ \sum_{l=0}^n \sum_{k=1}^{n-l} c_k^{n,l} \|N^{k/2} |x^{m+l}| \Psi\|_{\mathcal{H}}^2, \quad (1.16) \]

Combining (1.15) and (1.16), we see in Lemma 3.8 that
\[ \sum_{l=0}^n \sum_{k=1}^{n-l} c_k^{n,l} \|N^{k/2} |x^{m+l}| \Psi\|_{\mathcal{H}}^2, \quad (1.17) \]
with some constants \(d^n_l\), where

\[
\mathcal{R}_{n,m}(\Psi) = \sum_{l=0}^{n} \sum_{k=1}^{n-l} c^n_{k,l} \|N^{k/2}|x|^{m+l}\Psi\|_H^2.
\]

Furthermore if \(\psi_g \in D(N^{k/2})\) then we see that

\[
N^{k/2}\psi_g = e^{-tH}e^{tE}N^{k/2}\psi_g + e^{tE}[N^{k/2}, e^{-tH}]\psi_g,
\]

where

\[E = \inf \sigma(H).\]

Using this identity we show in Lemma 2.12 that if \(\psi_g \in D(N^{k/2})\) then for all \(l \geq 0\),

\[
|x|^l\psi_g \in D(N^{k/2}). \tag{1.18}
\]

Under these preparations we prove Theorem 1.7 by means of an induction. Let us assume that

\[
\psi_g \in D(N^{(n-1)/2}). \tag{1.19}
\]

Hence

\[
\sum \|a(k_1)...a(k_l)\psi_g\|_H^2 dk_1...dk_l < \infty, \quad l = 1,2,...,n-1. \tag{1.20}
\]

Then we see that by (1.19),

\[
\sum_{l=1}^{n} d^n_l \mathcal{R}_{n-l,l}(\psi_g) < \infty.
\]

Moreover by using pull through formula (2.14) we prove in Lemma 3.4 that

\[
\|b(k_1)...b(k_n)\psi_g\|_H \leq \sum_{p=1}^{n} \delta_1(k_p)\|b(k_1)...b(k_p)...b(k_n)|x|+1\psi_g\|_H
\]

\[+
\sum_{p=1}^{n} \sum_{q<p} \delta_2(k_p, k_q)\|b(k_1)...b(k_q)...b(k_p)...b(k_n)|x|\psi_g\|_H \tag{1.21}
\]

with

\[\delta_1 \in L^2(\mathbb{R}^3), \quad \delta_2 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3).\]

By (1.16), (1.18) and assumption (1.19), we show that

\[
\sum \|b(k_1)...b(k_n)\psi_g\|_H dk_1...dk_n < \infty.
\]
Hence by (1.17) we have
\[ \sum_{k_1, \ldots, k_n} \|a(k_1) \ldots a(k_n) \psi_g \|^2_H dk_1 \ldots dk_n \]
\[ \leq 2^n \left\{ \sum_{k_1, \ldots, k_n} \|b(k_1) \ldots b(k_n) \psi_g \|^2_H dk_1 \ldots dk_n + \sum_{l=1}^n d_l R_{n-l,l}(\psi_g) \right\} < \infty, \]
which implies, together with (1.20), that
\[ \psi_g \in D(N^{n/2}). \]

Since \( \psi_g \in D(N^{1/2}) \) is known, we obtain
\[ \psi_g \in \bigcap_{k=1}^{\infty} D(N^{k/2}). \]

This paper is organized as follows. In Section 2 we establish (1.21) by means of the pull-through formula. In Section 3 we give a proof of the main theorems. In Section 4 we show (1.18) by virtue of a functional integral representation.

## 2 Pull-through formula and exponential decay

### 2.1 Fundamental facts

Let \( T \) be an operator. We set
\[ C^\infty(T) = \bigcap_{k=1}^{\infty} D(T^k). \]

**Lemma 2.1** We have \( \psi_g \in C^\infty(\Delta) \cap C^\infty(D_\Omega). \)

*Proof:* By Proposition 1.4, \( D(H) = D(\Delta) \cap D(H_\Omega) \). Then \( \psi_g \in D(H) \), which implies \( \psi_g \in D(\Delta) \). By Proposition 1.5 (2) it holds that \( \psi_g \in C^\infty(\Delta) \). It is obtained in [8] that \( H_\Omega^l(\Delta) \) is bounded for all \( l \geq 0 \). Recall that \( E = \inf \sigma(H) \). Then it follows that for arbitrary \( l \geq 0 \),
\[ \|H^l \psi_g\| = \|H^l(\Delta - i)^{-l} (E - i)^l \psi_g\| \leq \|(E - i)^l\| \|H^l(\Delta - i)^{-l}\| \||\psi_g\| \).

Then \( \psi_g \in D(H_\Omega^l) \) for all \( l \geq 0 \). Thus the lemma follows. \( \square \)

Let
\[ \mathcal{F}_{\omega} = \mathcal{L}\{a^*(f_1,j_1) \ldots a^*(f_n,j_n) \Omega, \Omega | f_j \in C^\infty_0(\mathbb{R}^3), j = 1, \ldots, n, n = 0, 1, \ldots \}, \]
where $\mathcal{L}\{\ldots\}$ denotes the set of the finite linear sum of $\{\ldots\}$. We define

$$
\mathcal{D} = C^\infty(|x|) \cap C^\infty(H_t),
$$

and

$$
\mathcal{C} = C_0^\infty(\mathbb{R}^3) \otimes \mathcal{F}_\omega.
$$

**Lemma 2.2** Let $m \geq 0$ and $n \geq 0$. Then $(H_t + 1)^n + |x|^m$ is self-adjoint on $D((H_t + 1)^n) \cap D(|x|^m)$ and essentially self-adjoint on $\mathcal{C}$.

**Proof:** The self-adjointness is trivial. Since $C_0^\infty(\mathbb{R}^3)$ and $\mathcal{F}_\omega$ are the set of analytic vectors of $|x|^m$ and $(H_t + 1)^n$ respectively, $C_0^\infty(\mathbb{R}^3)$ and $\mathcal{F}_\omega$ are cores of $|x|^m$ and $(H_t + 1)^n$ respectively. Hence $\mathcal{C} = C_0^\infty(\mathbb{R}^3) \otimes \mathcal{F}_\omega$ is a core of $(H_t + 1)^n + |x|^m$. □

**Remark 2.3** Let $p, q \geq 0$. From Lemma 2.2 it follows that for $\Psi \in \mathcal{D} \subset D((H_t + 1)^p + |x|^q)$ there exists a sequence $\{\Psi_m\} \subset \mathcal{C}$ such that $\Psi_m \to \Psi$ and $((H_t + 1)^p + |x|^q)\Psi_m \to ((H_t + 1)^p + |x|^q)\Psi$ strongly as $m \to \infty$.

Let $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $j = 1, \ldots, n$, and $\Psi \in \mathcal{C}$. Then it is well known and easily proven that

$$
\sum_{j=1}^n \prod_{i=1}^n f_j(k_j)\|a(k_1)\ldots a(k_n)\Psi\|_{\mathcal{H}} dk_1 \ldots dk_n \leq \epsilon(f_1, \ldots, f_n)\|(H_t + 1)^{n/2}\Psi\|_{\mathcal{H}} \quad (2.1)
$$

with some constant $\epsilon(f_1, \ldots, f_n)$ independent of $\Psi$.

Let $A$ and $B$ be operators. We say $f \in D(AB)$ if $f \in D(B)$ and $Bf \in D(A)$.

**Lemma 2.4** Let $\Psi \in \mathcal{D}$. Then there exists $\mathcal{M}_\mathcal{D}(\Psi) \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero such that

$$
\Psi \in D(a(k_1)\ldots a(k_n)) \quad (2.2)
$$

and

$$
a(k_1)\ldots a(k_n)\Psi \in \mathcal{D} \quad (2.3)
$$

for $(k_1, \ldots, k_n) \notin \mathcal{M}_\mathcal{D}(\Psi)$. Moreover assume that $\{\Psi_m\} \subset \mathcal{C}$ satisfies that $\Psi_m \to \Psi$ and $(H_t + 1)^{n/2}\Psi_m \to (H_t + 1)^{n/2}\Psi$ strongly as $m \to \infty$. Then there exists a subsequence $\{m'\} \subset \{m\}$ and $\mathcal{M}_\mathcal{D}(\Psi, \{\Psi_m\}, \{m'\}) \subset \mathbb{R}^{3n}$ with the Lebesgue measure zero such that for $(k_1, \ldots, k_n) \notin \mathcal{M}_\mathcal{D}(\Psi, \{\Psi_m\}, \{m'\})$, (2.2) and (2.3) are valid and

$$
s - \lim_{m' \to \infty} a(k_1)\ldots a(k_n)\Psi_{m'} = a(k_1)\ldots a(k_n)\Psi.
$$
Proof: See Appendix A.

Lemma 2.5 The operator $|x|$ leaves $\mathcal{D}$ invariant.

Proof: Let $\Psi \in \mathcal{D}$. It is clear that $|x|\Psi \in C^\infty(|x|)$. We choose a sequence $\{\Psi_m\} \subset \mathcal{C}$ such that $\Psi_m \to \Psi$ and $((H_t + 1)^{2n} + |x|^2)\Psi_m \to ((H_t + 1)^{2n} + |x|^2)\Psi$ strongly as $m \to \infty$. In particular

$$|x|\Psi_m \to |x|\Psi$$

strongly as $m \to \infty$. $H^n_t|x|\Psi_m$ is well defined and it is obtained that

$$\|H^n_t|x|\Psi_m\|^2_{\mathcal{H}} \leq \|H^n_{2n}\Psi_m\|^2_{\mathcal{H}}\|\Psi_m\|^2_{\mathcal{H}} \leq \|(H_t + 1)^{2n} + |x|^2\|=\Psi_m\|^2_{\mathcal{H}}.$$

Then $H^n_t|x|\Psi_m$ converges strongly as $m \to \infty$. Since $H^n_t$ is closed, by (2.4) we have $|x|\Psi \in D(H^n_t)$. Here $n$ is arbitrary, hence $|x|\Psi \in C^\infty(H_t)$. The proof is complete. \square

Let

$$\beta(k) = \frac{e^{i\beta(k)}e^{-ix\cdot k}}{\sqrt{2\sqrt{\omega(k)}}}$$

and

$$b(k) = e^{i\beta(k)}a(k) - i\beta(k).$$

For simplicity we set $-ix \cdot \beta(k_j) = \theta_j$. Then

$$b(k_j) = a(k_j) + \theta_j.$$

Lemma 2.6 Let $\Psi \in \mathcal{C}$ and $f_j \in C^\infty_0(\mathbb{R}^3 \setminus \{0\})$, $j = 1, \ldots, n$. Then there exists a constant $c'(f_1, \ldots, f_n)$ independent of $\Psi$ such that

$$\sum |\prod_{j=1}^n f_j(k_j)||b(k_1)\cdots b(k_n)\Psi|_{\mathcal{H}}dk_1\cdots dk_n \leq c'(f_1, \ldots, f_n)||((H_t + 1)^{2n} + |x|^2)\Psi||_{\mathcal{H}}.$$  (2.5)

Proof: Since $[\theta_j, a(k)] = 0$ on $\mathcal{C}$, we have

$$b(k_1)\cdots b(k_n)\Psi = (a(k_1) + \theta_1)\cdots(a(k_n) + \theta_n)\Psi$$

$$= \sum_{p_1=0}^n \cdots \sum_{p_n=0}^n \theta_{p_1}\cdots\theta_{p_n}\hspace{1cm}a(k_1)p_1\cdots\cdots\cdots a(k_n)p_n.$$ 

Hence by (2.1),

$$\sum |\prod_{j=1}^n f_j(k_j)||b(k_1)\cdots b(k_n)\Psi|_{\mathcal{H}}dk_1\cdots dk_n$$
\[ \leq \sum_{l=0}^{n} \sum_{\{p_1, \ldots, p_l\} \subseteq \{1, \ldots, n\}} \left( \prod_{i=1}^{l} \frac{|e^{\hat{\varphi}(k)}|}{\sqrt{2\omega(k)}} f_{p_i}(k) \right) \times e(f_1, \ldots, \hat{f}_{p_1}, \ldots, \hat{f}_{p_l}, \ldots, f_n) \| (H_f + 1)^{(n-l)/2} |x|^l \Psi \|_H. \]

Since \( \| (H_f + 1)^{(n-l)/2} |x|^l \Psi \|_H \leq c_n \| ((H_f + 1)^n + |x|^{2n}) \Psi \|_H \) with some constant \( c_n \).

Thus (2.5) follows. \( \square \)

**Lemma 2.7** Let \( \Psi \in \mathcal{D} \). Then there exists \( \mathcal{N}_D(\Psi) \subset \mathbb{R}^{3n} \) with the Lebesgue measure zero such that
\[ \Psi \in D(b(k_1) \ldots b(k_n)) \quad (2.6) \]
and
\[ b(k_1) \ldots b(k_n) \Psi \in \mathcal{D} \quad (2.7) \]
for \( (k_1, \ldots, k_n) \notin \mathcal{N}_D(\Psi) \). Moreover assume that \( \{\Psi_m\} \subset \mathcal{C} \) satisfies that \( \Psi_m \to \Psi \) and \( ((H_f + 1)^n + |x|^{2n}) \Psi_m \to ((H_f + 1)^n + |x|^{2n}) \Psi \) strongly as \( m \to \infty \). Then there exists a subsequence \( \{m'\} \subset \{m\} \) and \( \mathcal{N}_D(\Psi, \{\Psi_m\}, \{m'\}) \subset \mathbb{R}^{3n} \) with the Lebesgue measure zero such that for \( (k_1, \ldots, k_n) \notin \mathcal{N}_D(\Psi, \{\Psi_m\}, \{m'\}) \), (2.6) and (2.7) are valid and
\[ s - \lim_{m' \to \infty} b(k_1) \ldots b(k_n) \Psi_{m'} = b(k_1) \ldots b(k_n) \Psi. \]

**Proof:** See Appendix A.

### 2.2 Pull-through formula

**Lemma 2.8** We have
\[ \mathcal{C} \subset D(Hb(k_1) \ldots b(k_n)) \cap D(b(k_1) \ldots b(k_n)H) \quad (2.8) \]
for all \( (k_1, \ldots, k_n) \in \mathbb{R}^{3n} \), and for \( \Psi \in \mathcal{C} \),
\[ [H, b(k_1) \ldots b(k_n)]\Psi = \mathcal{R}_0 \Psi + \mathcal{R}_1 \Psi + \mathcal{R}_2 \Psi. \]

Here
\[ \mathcal{R}_0 = \mathcal{R}_0(k_1, \ldots, k_n) = -\sum_{p=1}^{n} \omega(k_p) b(k_1) \ldots b(k_n), \]
\[ \mathcal{R}_1 = \mathcal{R}_1(k_1, \ldots, k_n) = \sum_{p=1}^{n} \varphi_1(k_p) b(k_1) \ldots \hat{b}(k_p) \ldots b(k_n), \]
\[ R_2 = R_2(k_1, \ldots, k_n) = \sum_{p=1}^{n} \sum_{q<p} \vartheta_2(k_p, k_q) b(k_1) \ldots b(k_q) \ldots b(k_p) \ldots b(k_n), \]

and

\[ \vartheta_1(k) = \vartheta_1(k, x, p) = \frac{i}{2} \left\{ (x \cdot \beta(k)) k \cdot (p - eA) + k \cdot (p - eA) (x \cdot \beta(k)) \right\} - i \omega(k) (x \cdot \beta(k)), \]

\[ \vartheta_2(k, k') = \vartheta_2(k, k', x) = (x \cdot \beta(k))(x \cdot \beta(k'))(k \cdot k'). \]

**Proof:** (2.8) is trivial. On \( C \) we have

\[ [H, b(k)] = -\omega(k) b(k) + \vartheta_1(k). \tag{2.9} \]

Moreover

\[ [b(k'), \vartheta_1(k)] = \vartheta_2(k, k'). \tag{2.10} \]

By (2.9) and (2.10) we have

\[
[H, b(k_1) \ldots b(k_n)] \Psi = \sum_{p=1}^{n} b(k_1) \ldots \{ -\omega(k_p) b(k_p) + \vartheta_1(k_p) \} \ldots b(k_n) \Psi
\]

\[ = - \sum_{p=1}^{n} \omega(k_p) b(k_1) \ldots b(k_n) \Psi + \sum_{p=1}^{n} \vartheta_1(k_p) b(k_1) \ldots b(k_p) \ldots b(k_n) \Psi
\]

\[ + \sum_{p=1}^{n} \sum_{q<p} \vartheta_2(k_p, k_q) b(k_1) \ldots b(k_q) \ldots b(k_p) \ldots b(k_n) \Psi. \]

The lemma follows. \( \square \)

\( \overline{B} \) denotes the closure of \( B \). We simply set \( \overline{R_1} = \overline{R_1|_C}. \)

**Lemma 2.9** Let \( \Psi \in D \cap D(\Delta) \). Then there exists \( N(\Psi) \subset \mathbb{R}^{3n} \) with the Lebesgue measure zero such that for \((k_1, \ldots, k_n) \notin N(\Psi),\)

\[ \Psi \in D(R_0(k_1, \ldots, k_n)) \cap D(\overline{R_1}(k_1, \ldots, k_n)) \cap D(R_2(k_1, \ldots, k_n)). \]

**Proof:** By Lemma 2.7, \( \Psi \in D(b(k_1) \ldots b(k_n)) \) and \( b(k_1) \ldots b(k_n) \Psi \in D \) for \((k_1, \ldots, k_n) \notin N_D(\Psi). \) Thus \( b(k_1) \ldots b(k_n) \Psi \in D(\sum_{p=1}^{n} \omega(k_p)) \cap D(\vartheta_2(k_p, k_q)), \) which implies that \( \Psi \in D(R_0(k_1, \ldots, k_n)) \cap D(R_2(k_1, \ldots, k_n)). \) Next we shall prove \( D(\overline{R_1}(k_1, \ldots, k_n)) \ni \Psi. \)

Simply we set \( K_n = ((H + 1)^n + |x|^{2n}). \) We have on \( C \)

\[ R_1 = \sum_{p=1}^{n} i x \cdot \beta(k_p)(k_p \cdot p) b(k_1) \ldots b(k_q) \ldots b(k_p) \ldots b(k_n) + \sum_{p=1}^{n} R_x(k_p) b(k_1) \ldots b(k_p) \ldots b(k_n), \]
where
\[ R_x(k_p) = (-ie)(x \cdot \beta(k_p))(k \cdot A) - \frac{i}{2}(i\beta(k_p) \cdot k_p + x \cdot \beta(k_p)|k_p|^2) - i\omega(k_p)(x \cdot \beta(k_p)). \]
It follows that for \( \Phi \in \mathcal{C} \),
\[ \sum \int \prod_{j=1}^n f_j(k_j) R_x(k_p)b(k_1) \ldots b(k_p) \Phi \|_{\mathcal{H}} dk_1 \ldots dk_n \leq c_1 \| K_{m_1} \Psi \|_{\mathcal{H}} \]
with some constants \( c_1 \) and \( m_1 \), and
\[ ix \cdot \beta(k_p)(k_p \cdot p)b(k_1) \ldots b(k_p) \Phi = ix \cdot \beta(k_p)b(k_1) \ldots b(k_p)(k_p \cdot p) \Phi \]
\[ + \sum_{q \neq p} R_x(k_p, k_q)b(k_1) \ldots b(k_q) \ldots b(k_p) \Phi, \quad (2.11) \]
where
\[ R_x(k_p, k_q) = ix \cdot \beta(k_p) (-k_q \beta(k_q) + ix \cdot \beta(k_q)(k_p \cdot k_q)). \]
The second term of (2.11) is estimated as
\[ \sum \int \prod_{j=1}^n f_j(k_j) \sum_{q \neq p} R_x(k_p, k_q)b(k_1) \ldots b(k_q) \ldots b(k_p) \Phi \|_{\mathcal{H}} dk_1 \ldots dk_n \]
\[ \leq c_2 \| K_{m_2} \Psi \|_{\mathcal{H}} \]
with some constants \( c_2 \) and \( m_2 \). By (2.5) the first term of (2.11) is estimated as
\[ \sum \int \prod_{j=1}^n f_j(k_j) (x \cdot \beta(k_p)) b(k_1) \ldots b(k_p) \Phi \|_{\mathcal{H}} dk_1 \ldots dk_n \]
\[ \leq \epsilon'(f_1, \ldots, f_p, \ldots, f_n) \int |f(k_p)| \frac{|\tilde{\phi}(k_p)|}{2^{1/2} \sqrt{\omega(k_p)}} \| K_{n-1} |x| (k_p \cdot p) \Phi \|_{\mathcal{H}} dk_p. \]
Let \( Q = K_{n-1} \). Note that
\[ \| Q |x|(k_p \cdot p) \Phi \|_{\mathcal{H}} = (|x|^2 Q^2 \Phi, (k_p \cdot p)^2 \Phi)_H + (\Phi, [(k_p \cdot p), Q^2|x|^2](k_p \cdot p) \Phi)_H. \]
Since \([(k_p \cdot p), |x|] = -i(k_p \cdot x)/|x| \), we have \([(k_p \cdot p), Q^2|x|^2] = k_p \cdot P \), where
\[ P = 2 \left\{ (H_l + 1)^n(-i)x \frac{|x|}{|x|} + (-i)x(|x| + 1)^{2n-3} + (-i)x \frac{|x|}{|x|} (|x| + 1)^{2n-2} \right\}. \]
Then
\[ \| Q |x|(k_p \cdot p) \Phi \|_{\mathcal{H}}^2 \leq |k_p|^2 \left( \| |x|^2 Q^2 \Phi \|_{\mathcal{H}} \| \Delta \Phi \|_{\mathcal{H}} + \| P \Phi \|_{\mathcal{H}} \| p \Phi \|_{\mathcal{H}} \right). \]
Hence
\[ \|Q|x|(k_p \cdot p)\Phi\|_H \leq |k_p| (c_3\|K_{m_3}\Psi\|_H + c'\|\Delta\Phi\|_H) \]
follows with some constants \( c_3, c' \) and \( m_3 \). Thus for \( \Phi \in C \)
\[ \sum_j \| \prod f_j(k_j)R_1(k_1, \ldots, k_n)\Phi \|_H dk_1 \ldots dk_n \leq c\|K_m\Phi\|_H + c'\|\Delta\Phi\|_H \quad (2.12) \]
follows with some constants \( c \) and \( m \). Set \( K = -\Delta + K_m = -\Delta + |x|^{2m} + (f + 1)^m \).
Then \( K \) is self-adjoint on \( D(-\Delta + |x|^{2m}) \cap D((f + 1)^m) \) and essentially self-adjoint on \( C \). Then for \( \Psi \in D \cap D(\Delta) \) there exists a sequence \( \{\Psi_l\} \subset C \) such that \( \Psi_l \to \Psi \) and \( K\Psi_l \to K\Psi \) strongly as \( l \to \infty \). By (2.12) it follows that
\[ \sum_j \| \prod f_j(k_j)R_1(k_1, \ldots, k_n)\Psi_l \|_H dk_1 \ldots dk_n \leq c\|K_m\Psi_l\|_H + c'\|\Delta\Psi_l\|_H. \]

Then there exist \( \mathcal{N}_D(\Psi)' \subset \mathbb{R}^{3n} \) with the Lebesgue measure zero and a subsequence \( \{l'\} \subset \{l\} \) such that \( \mathcal{R}_1(k_1, \ldots, k_n)\Psi_{l'} \) strongly converges as \( l' \to \infty \) for \( (k_1, \ldots, k_n) \notin \mathcal{N}_D(\Psi)' \). Then \( \Psi \in D(\overline{\mathcal{R}_1(k_1, \ldots, k_n)}) \) for \( (k_1, \ldots, k_n) \notin \mathcal{N}_D(\Psi)' \). Set
\[ \mathcal{N}(\Psi) = \mathcal{N}_D(\Psi) \cup \mathcal{N}_D(\Psi)' \]
We get the desired results. \( \Box \)

The following lemma is a variant of the pull-through formula.

**Lemma 2.10** For \( (k_1, \ldots, k_n) \notin \mathcal{N}(\psi_g) \), the following (1), (2) and (3) hold;

1. \( \psi_g \in D(b(k_1)b(k_n)) \cap D(\mathcal{R}_0) \cap D(\overline{\mathcal{R}_1}) \cap D(\mathcal{R}_2) \),
2. \( b(k_1)b(k_n)\psi_g \in D(H) \),
3. \( \left( H - E + \sum_{p=1}^n \omega(k_p) \right) b(k_1)b(k_n)\psi_g = \overline{\mathcal{R}_1}\psi_g + \mathcal{R}_2\psi_g \quad (2.13) \)

In particular it follows that for \( (k_1, \ldots, k_n) \notin \mathcal{N}(\psi_g) \) and \( (k_1, \ldots, k_n) \neq (0, \ldots, 0) \),
\[ b(k_1)b(k_n)\psi_g = \left( H - E + \sum_{p=1}^n \omega(k_p) \right)^{-1} \left( \overline{\mathcal{R}_1}\psi_g + \mathcal{R}_2\psi_g \right). \quad (2.14) \]
Proof: Note that \( \psi_g \in D \cap D(\Delta) = C^\infty(|x|) \cap C^\infty(H_t) \cap D(\Delta) \). Then (1) follows from Lemma 2.9. Since \( C \) is a core of \( H \), we have \( \phi_m \in D \) such that \( \phi_m \to \psi_g \) and \( H\phi_m \to H\psi_g = E\psi_g \) strongly as \( m \to \infty \). Then we have for \( \phi \in C \)

\[
(H\phi, b(k_1) \ldots b(k_n) \phi_m)_\mathcal{H} = \sum_{j=0,1,2} (\phi, R_j \phi_m)_\mathcal{H} + (\phi, b(k_1) \ldots b(k_n) H\phi_m)_\mathcal{H}.
\]

It follows that

\[
\lim_{m \to \infty} (H\phi, b(k_1) \ldots b(k_n) \phi_m)_\mathcal{H} = \lim_{n \to \infty} (b^*(k_n) \ldots b^*(k_1) H\phi, \phi_m)_\mathcal{H}
= (b^*(k_n) \ldots b^*(k_1) H\phi, \psi_g)_\mathcal{H} = (H\phi, b(k_1) \ldots b(k_n) \psi_g)_\mathcal{H}.
\]

\[
\lim_{m \to \infty} (\phi, R_j \phi_m)_\mathcal{H} = \lim_{m \to \infty} (R_j^* \phi, \psi_g)_\mathcal{H} = (R_j^* \phi, \psi_g)_\mathcal{H} = (\phi, R_j \psi_g)_\mathcal{H},
\]

and

\[
\lim_{m \to \infty} (\phi, b(k_1) \ldots b(k_n) H\phi_m)_\mathcal{H} = \lim_{n \to \infty} (b^*(k_n) \ldots b^*(k_1) \phi, H\phi_m)_\mathcal{H}
= (b^*(k_n) \ldots b^*(k_1) \phi, E\psi_g)_\mathcal{H} = (\phi, b(k_1) \ldots b(k_n) \psi_g)_\mathcal{H}.
\]

Hence

\[
(H\phi, b(k_1) \ldots b(k_n) \psi_g)_\mathcal{H} = \sum_{j=0,1,2} (\phi, R_j \psi_g)_\mathcal{H} + E(\phi, b(k_1) \ldots b(k_n) \psi_g)_\mathcal{H}.
\]

Then \( b(k_1) \ldots b(k_n) \psi_g \in D(H) \) and we have

\[
Hb(k_1) \ldots b(k_n) \psi_g = \sum_{j=0,1,2} R_j \psi_g + Eb(k_1) \ldots b(k_n) \psi_g.
\]

Note that \( R_0 \psi_g = R_0 \psi_g \) and \( R_2 \psi_g = R_2 \psi_g \). Then (2.13) follows. \( \square \)

2.3 Exponential decay of \( N^{k/2} \psi_g \)

**Lemma 2.11** Suppose that \( \psi_g \in D(N^{k/2}) \). Then there exist positive constants \( D_k \), and \( \delta \) independent of \( k \) such that

\[
\|N^{k/2} \psi_g(x)\|_F \leq D_k e^{-\delta |x|^{m+1}}
\]

for almost every \( x \in \mathbb{R}^3 \). In particular \( N^{k/2} \psi_g \in D(e^{\delta |x|^{m+1}}) \).

The proof of Lemma 2.11 is based on a functional integral representation of \( e^{-tH} \). Essential ingredients of the proof have been obtained in [14]. The proof is, however, long and complicated. Then we move it to Appendix B.
Lemma 2.12  Suppose that \( \psi_g \in D(N^{k/2}) \). Then \( |x|^{l} \psi_g \in D(N^{k/2}) \) for all \( l \geq 0 \).

Proof: This lemma is immediately follows from Lemma 2.11 and the following fundamental lemma. \( \square \)

Lemma 2.13  Let \( \mathcal{K} \) be a Hilbert space, and \( A \) and \( B \) self-adjoint operators such that \([e^{-itsA}, e^{-isB}] = 0\) for \( s, t \in \mathbb{R} \). Suppose that \( \phi \in D(A) \cap D(B) \) and \( A\phi \in D(B) \). Then \( B\phi \in D(A) \) with \( AB\phi = BA\phi \).

Proof: It follows that \( t^{-1}(e^{-itA} - 1)e^{-isB}\phi = t^{-1}e^{-isB}(e^{-itA} - 1)\phi \). Take \( t \to 0 \) on the both sides. Then it follows that \( e^{-isB}\phi \in D(A) \) with \( Ae^{-isB}\phi = e^{-isB}A\phi \). From this identity we have \( s^{-1}A(e^{-isB} - 1)\phi = s^{-1}(e^{-isB} - 1)A\phi \). Take \( s \to 0 \) on the both sides. Since \( A \) is closed and assumption \( A\phi \in D(B) \), we see that \( B\phi \in D(A) \) and \( AB\phi = BA\phi \). \( \square \)

Proof of Lemma 2.12

In Lemma 2.13 we set \( \mathcal{K} = \mathcal{H} \), \( A = N^{k/2} \) and \( B = |x|^l \). Since \( \psi_g \in D(N^{k/2}) \cap D(|x|^l) \) and \( N^{k/2}\psi_g \in D(|x|^l) \) by Lemma 2.11, the lemma follows. \( \square \)

3  Proof of the main theorems

Lemma 3.1  The following statements are equivalent.

(1) \( \Psi \in D(a(k_1)...a(k_n)) \) for almost every \( (k_1, ..., k_n) \in \mathbb{R}^{3n} \) and
\( \sum_{k_1}^{f} \|a(k_1)...a(k_n)\Psi\|_{H}dk_{1}...dk_{n} < \infty \). \hspace{1cm} (3.1)

(2) \( \Psi \in D(\prod_{j=1}^{n}(N - j + 1)^{1/2}) \).

Moreover if (1) or (2) is satisfied, then it holds that
\( \sum_{k_1}^{f} \|a(k_1)...a(k_n)\Psi\|_{H}dk_{1}...dk_{n} = \prod_{j=1}^{n}(N - j + 1)^{1/2}\Psi\|_{H}^{2} \).

Proof: We prove \( (1) \implies (2) \). We identify \( \mathcal{H} \) as
\( \mathcal{H} \cong \bigoplus_{n=0}^{\infty} \mathcal{H}_{n} \). \hspace{1cm} (3.2)
where
\[ \mathcal{H}_n = L^2(\mathbb{R}^3 \times (\mathbb{R}^3 \times \{1, 2\})^n_{sym}) \]

We note that
\[ \left( \prod_{j=1}^{n} (N - j + 1)^{1/2} \Psi \right)^{(k)} = \begin{cases} 0, & k = 0, 1, \ldots, n - 1, \\ \sqrt{k(k - 1) \ldots (k - n + 1)} \Psi^{(k)}, & k \geq n. \end{cases} \]

Define \( \Psi_p = \oplus_{m=0}^{\infty} \Psi_p^{(m)} \in \mathcal{H} \) by
\[
\Psi_p^{(m)} = \begin{cases} \Psi^{(m)}, & m \leq p, \\ 0, & m > p. \end{cases} \tag{3.3}
\]

By the definition of \( a(k) \) we have
\[
(a(k_1) \ldots a(k_n) \Psi_p)^{(l)}(x, k_1', \ldots, k_l') = \sqrt{l + 1} \sqrt{l + 2} \ldots \sqrt{l + n} \Psi_p^{(l+n)}(x, k_1, \ldots, k_n, k_1', \ldots, k_l').
\]

Then
\[
\|a(k_1) \ldots a(k_n) \Psi_p\|_{\mathcal{H}}^2 = \sum_{l=0}^{\infty} (l + 1)(l + 2) \ldots (l + n) \sum_{k} \|\Psi_p^{(l+n)}(\cdot, k_1, \ldots, k_n, k_1', \ldots, k_l')\|_{L^2(\mathbb{R}^3)}^2 dk_1' \ldots dk_l'.
\]

By (1) we see that
\[
\lim_{p \to \infty} \|a(k_1) \ldots a(k_n) \Psi_p\|_{\mathcal{H}}^2 = \lim_{p \to \infty} \sum_{k} \|a(k_1) \ldots a(k_n) \Psi^{(k)}\|_{\mathcal{H}_k}^2.
\]

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\[ = \|a(k_1) \ldots a(k_n) \Psi\|^2_H \]  

(3.6)

for almost every \((k_1, \ldots, k_n) \in \mathbb{R}^{3n}\), and

\[ \|a(k_1) \ldots a(k_n) \Psi\|^2_H \in L^1(\mathbb{R}^{3n}). \]

Thus the Lebesgue dominated convergence theorem yields that

\[ \lim_{p \to \infty} \sum_{j \in \mathbb{Z}} \|a(k_1) \ldots a(k_n) \Psi_p\|^2_H dk_1 \ldots dk_n < \infty. \]

Then from (3.5), it follows that

\[ \lim_{p \to \infty} \sum_{j \in \mathbb{Z}} \|a(k_1) \ldots a(k_n) \Psi\|^2_H dk_1 \ldots dk_n < \infty. \]

Thus (2) follows.

We prove (2) \(\Rightarrow\) (1). By (3.5) and (2) we see that

\[ \lim_{p \to \infty} \sum_{j \in \mathbb{Z}} \|a(k_1) \ldots a(k_n) \Psi_p\|^2_H dk_1 \ldots dk_n < \infty, \]

and by (3.4), \(\|a(k_1) \ldots a(k_n) \Psi\|^2_H\) is increasing in \(p\). Then we have by the Lebesgue monotone convergence theorem,

\[ \lim_{p \to \infty} \sum_{j \in \mathbb{Z}} \|a(k_1) \ldots a(k_n) \Psi_p\|^2_H dk_1 \ldots dk_n = \sum_{j \in \mathbb{Z}} \lim_{p \to \infty} \|a(k_1) \ldots a(k_n) \Psi_p\|^2_H dk_1 \ldots dk_n < \infty. \]

(3.7)

Then (1) follows from (3.6).

\[ \square \]

**Lemma 3.2** The following statements are equivalent.

(1) \(\Psi \in D(a(k_1) \ldots a(k_n))\) for almost every \((k_1, \ldots, k_n) \in \mathbb{R}^{3n}\) and

\[ \sum_{j \in \mathbb{Z}} \|a(k_1) \ldots a(k_n) \Psi\|^2_H dk_1 \ldots dk_n < \infty \]

for \(n = 1, 2, \ldots, k\).

(2) \(\Psi \in D(N^{k/2})\).

**Proof:** By Lemma 3.1, it is enough to show that

\[ \bigcap_{k=1}^{n} D\left(\prod_{j=1}^{k} (N - j + 1)^{1/2}\right) = D(N^{k/2}). \]
Assume that
\[ \Psi \in \bigcap_{k=1}^{n} D(\prod_{j=1}^{k} (N^n - j + 1)^{1/2}). \] (3.8)

Let \( \Psi_p \) be defined by (3.3). Let \( W_n = \prod_{j=1}^{n} (N - j + 1) \). For example
\( N = W_1, N^2 = W_2 + W_1, N^3 = W_3 + 3W_2 + W_1, N^4 = W_4 + 6W_3 + 7W_2 + W_1 \), etc. One can inductively see that there exist constants \( a_j, j = 1, \ldots, k \), such that on \( \mathcal{F}_0 \),
\[ N^k = \sum_{j=1}^{n} a_j W_j. \]

Then it follows that
\[ \| N^{k/2} \Psi_p \|_{\mathcal{H}}^2 = a_1 \| W_1^{1/2} \Psi_p \|_{\mathcal{H}}^2 + a_2 \| W_2^{1/2} \Psi_p \|_{\mathcal{H}}^2 + \cdots + a_k \| W_k^{1/2} \Psi_p \|_{\mathcal{H}}^2. \] (3.9)

As \( n \to \infty \), from (3.8) it follows that the right hand side of (3.9) converges to
\[ a_1 \| W_1^{1/2} \Psi \|_{\mathcal{H}}^2 + a_2 \| W_2^{1/2} \Psi \|_{\mathcal{H}}^2 + \cdots + a_k \| W_k^{1/2} \Psi \|_{\mathcal{H}}^2. \]

Since
\[ \| N^{k/2} \Psi \|_{\mathcal{H}}^2 = \lim_{p \to \infty} \sum_{k=0}^{p} \| (N^{k/2} \Psi)^{(k)} \|_{\mathcal{H}}^2 = \lim_{p \to \infty} \| N^{k/2} \Psi_p \|_{\mathcal{H}}^2 < \infty, \]
\( \Psi \in D(N^{k/2}) \) follows. Then
\[ \bigcap_{k=1}^{n} D(\prod_{j=1}^{k} (N^n - j + 1)^{1/2}) \subset D(N^{k/2}). \] (3.10)

Next we assume that
\[ \Psi \in D(N^{k/2}). \]

Note that
\[ \Psi \in \bigcap_{l=1}^{k} D(N^{l/2}). \]

It is seen that there exist constants \( b^n_l, l = 1, \ldots, n \), such that
\[ \| \prod_{j=1}^{n} (N - j + 1)^{1/2} \Psi_p \|^{2} = (\Psi_p, \prod_{j=1}^{n} (N - j + 1) \Psi_p) \]
\[ = (\Psi_p, N(N - 1)(N - 2) \cdots (N - n + 1) \Psi_p) \leq \sum_{l=1}^{n} b^n_l \| N^{l/2} \Psi_p \|^{2}. \] (3.11)
Take $p \to \infty$ on the both sides above. Then the right hand side of (3.11) converges to
\[
\sum_{l=1}^{n} b_{n}^{l} \| N^{l/2} \Psi \|^{2}.
\]
Hence
\[
\| \prod_{j=1}^{n} (N - j + 1)^{1/2} \Psi \|^{2}_{\mathcal{H}} = \lim_{p \to \infty} \sum_{k=0}^{p} \| \left( \prod_{j=1}^{n} (N - j + 1)^{1/2} \Psi \right)^{(k)} \|^{2}_{\mathcal{H}_{k}}
\]
\[
= \lim_{p \to \infty} \| \prod_{j=1}^{n} (N - j + 1)^{1/2} \Psi_{p} \|^{2}_{\mathcal{H}} < \infty.
\]
Thus $\Psi \in \cap_{k=1}^{n} D(\prod_{j=1}^{n} (N - j + 1)^{1/2})$. We obtain
\[
\bigcap_{k=1}^{n} D(\prod_{j=1}^{n} (N - j + 1)^{1/2}) \supset D(N^{k/2}). \tag{3.12}
\]
The lemma follows from (3.10) and (3.12). \hfill \box

We set
\[
\mathcal{R}_{\omega} = \mathcal{R}_{\omega}(k_{1}, \ldots, k_{n}) = \left( H - E + \sum_{p=1}^{n} \omega(k_{p}) \right)^{-1}.
\]

Lemma 3.3 There exist $\delta_{1}(\cdot) \in L^{2}(\mathbb{R}^{3})$ and $\delta_{2}(\cdot, \cdot) \in L^{2}(\mathbb{R}^{3} \times \mathbb{R}^{3})$ such that for $\Psi \in \mathcal{D}$,
\[
\| \mathcal{R}_{\omega} \varphi_{1}(k_{q}) \Psi \|_{\mathcal{H}} \leq \delta_{1}(k_{q}) \| (|x| + 1) \Psi \|_{\mathcal{H}}, \tag{3.13}
\]
and
\[
\| \mathcal{R}_{\omega} \varphi_{2}(k_{q}, k_{p}) \Psi \|_{\mathcal{H}} \leq \delta_{2}(k_{q}, k_{p}) \| |x|^{2} \Psi \|_{\mathcal{H}}. \tag{3.14}
\]

Proof: By the closed graph theorem there exists a constant $C$ such that
\[
\| (-\Delta + H_{I}) \Psi \|_{\mathcal{H}} \leq C \| (H + 1) \Psi \|_{\mathcal{H}}.
\]
First we shall prove that $\mathcal{R}_{\omega}(p \cdot k_{q})$ and $\mathcal{R}_{\omega}(A \cdot k_{q})$ are bounded with
\[
\| \mathcal{R}_{\omega}(p \cdot k_{q}) \| \leq c_{1}(k_{q}) \tag{3.15}
\]
and
\[
\| \mathcal{R}_{\omega}(A \cdot k_{q}) \| \leq c_{2}(k_{q}), \tag{3.16}
\]
where $c_{1}(k_{q}) = \sqrt{(|k_{q}| + |1 + E|)C}$ and $c_{2}(k_{q}) = \sqrt{2} (c_{1}(k_{q}) + 1) (2 \| \varphi/\omega \| + \| \varphi/\sqrt{\omega} \|)$. Let $\Psi \in \mathcal{C}$. Since
\[
\| (p \cdot k_{q}) \Psi \|^{2}_{\mathcal{H}} \leq |k_{q}|^{2} \| (\Psi, C(H + 1) \Psi) \| \leq |k_{q}|^{2} C \left\{ \| (H - E)^{1/2} \Psi \|^{2}_{\mathcal{H}} + |1 + E| \| \Psi \|^{2}_{\mathcal{H}} \right\},
\]
we see that
\[ \| (p \cdot k_q) R_\omega \Psi \|^2_{L^2} \leq C |k_q| \| \Psi \|^2_{L^2} + C |1 + E \| \Psi \|^2_{L^2}. \]
Thus (3.15) follows. Note that
\[ \| a(f, j) \Psi \|_\mathcal{H} \leq \| f / \sqrt{\omega} \| \| H_1^{1/2} \Psi \|_F, \]
and
\[ \| a^*(f, j) \Psi \|_F \leq \| f / \sqrt{\omega} \| \| H_1^{1/2} \Psi \|_F + \| f \| \| \Psi \|_F. \]
Since
\[ \| (A \cdot k_q) \Psi \|_\mathcal{H} \leq \sqrt{2} |k_q| \left( 2 \| \tilde{\varphi} / \omega \| + \| \tilde{\varphi} / \sqrt{\omega} \| \right) \left( \| H_1^{1/2} \Psi \|_\mathcal{H} + \| \Psi \|_\mathcal{H} \right) \]
and
\[ \| H_1^{1/2} \Psi \|_\mathcal{H}^2 \leq C (\Psi, (H + 1) \Psi)_\mathcal{H} \leq C \| (H - E)^{1/2} \Psi \|^2_{\mathcal{H}} + C |1 + E \| \| \Psi \|^2_{\mathcal{H}}, \]
we have
\[ |k_q|^2 \| H_1^{1/2} R_\omega \Psi \|^2_{L^2} \leq C |k_q| \| \Psi \|^2_{L^2} + C |1 + E \| \| \Psi \|^2_{L^2}. \]
Hence
\[ \| (A \cdot k_q) R_\omega \Psi \|_\mathcal{H} \leq \sqrt{2} \left( 2 \| \tilde{\varphi} / \omega \| + \| \tilde{\varphi} / \sqrt{\omega} \| \right) \left( \| H_1^{1/2} \Psi \|_\mathcal{H} + |k_q| \| R_\omega \Psi \|_\mathcal{H} \right) \]
\[ \leq \sqrt{2} \left\{ \sqrt{(|k_q| + |1 + E|)C + 1} \right\} \left( 2 \| \tilde{\varphi} / \omega \| + \| \tilde{\varphi} / \sqrt{\omega} \| \right) \| \Psi \|_\mathcal{H}. \]
Thus (3.16) follows. We have on \( \mathcal{C} \)
\[ \Phi_1(k) = i(p - eA) \cdot k(x \cdot \beta(k)) + \frac{i}{2} \left( i \beta(k) \cdot k + x \cdot \beta(k) |k|^2 \right) - i \omega(k)(x \cdot \beta(k)). \]
Then by (3.15) and (3.16) we have for \( \Psi \in \mathcal{C}, \)
\[ \| R_\omega i(p - eA) \cdot k_p, (x \cdot \beta(k_p)) \Psi \|_\mathcal{H} \leq (c_1(k_p) + |e|c_2(k_p)) \frac{|e| \| \tilde{\varphi}(k_p) \|}{\sqrt{2} \omega(k_p)} \| x \| \| \Psi \|_{L^2}, \]
\[ \| R_\omega (-i \omega(k_p)(x \cdot \beta(k_p))) \Psi \|_\mathcal{H} \leq \frac{|e|}{\sqrt{2} \omega(k_p)} \| \tilde{\varphi}(k_p) \| \| x \| \| \Psi \|_{L^2}, \]
and
\[ \| R_\omega \frac{i}{2} \left( i \beta(k_p) \cdot k_p + x \cdot \beta(k_p) |k_p|^2 \right) \Psi \|_\mathcal{H} \leq \frac{1}{2} \frac{|e|}{\sqrt{2} \omega(k_p)} \left( \| \tilde{\varphi}(k_p) \| \| \Psi \|_{L^2} + \| \tilde{\varphi}(k_p) \| \| x \| \| \Psi \|_{L^2} \right). \]
Since $\|\hat{\varphi}\| < \infty$, $\|\sqrt{\omega}\hat{\varphi}\| < \infty$ and $\|\hat{\varphi}/\omega\| < \infty$, (3.13) follows for $\Psi \in \mathcal{C}$. By a limiting argument it can be extended for $\Psi \in \mathcal{D}$. (3.14) is rather easier than (3.13). We have for $\Psi \in \mathcal{C}$,

$$
\|R_\omega \varphi_2(k_p, k_q)\psi\|_{\mathcal{H}} \leq \frac{c^2}{2} \sqrt{\omega(k_p)\omega(k_q)|\hat{\varphi}(k_q)\hat{\varphi}(k_p)||x|^2\Psi\|_{\mathcal{H}}.
$$

Thus the lemma follows from a limiting argument and $\|\sqrt{\omega}\hat{\varphi}\| < \infty$. \hfill \Box

From Lemma 3.3 the next lemma immediately follows.

**Lemma 3.4** For almost every $(k_1, \ldots, k_n) \in \mathbb{R}^{3n}$ it follows that

$$
\psi_g \in D(b(k_1) \ldots b(k_n)) \cap \bigcap_{p=1}^n D(b(k_1) \ldots b(k_p)(|x| + 1)) \cap \bigcap_{q \leq p} D(b(k_1) \ldots b(k_q) \ldots b(k_p)(|x|^2))
$$

and

$$
\|b(k_1) \ldots b(k_n)\psi_g\|_{\mathcal{H}} \leq \sum_{p=1}^n \delta_1(k_p)\|b(k_1) \ldots b(k_p) \ldots b(k_n)(|x| + 1)\psi_g\|_{\mathcal{H}}
$$

$$
+ \sum_{p=1}^n \sum_{q < p} \delta_2(k_p, k_q)\|b(k_1) \ldots b(k_q) \ldots b(k_p) \ldots b(k_n)|x|^2\psi_g\|_{\mathcal{H}}.
$$

**Proof:** Note that for $(k_1, \ldots, k_n) \notin \mathcal{N}(\psi_g)$ and $(k_1, \ldots, k_n) \neq (0, \ldots, 0),

$$
b(k_1) \ldots b(k_n)\psi_g = R_\omega(k_1, \ldots, k_n)R_1(k_1, \ldots, k_n)\psi_g + R_\omega(k_1, \ldots, k_n)R_2(k_1, \ldots, k_n)\psi_g.
$$

Let $\Psi \in \mathcal{C}$ and $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $j = 1, \ldots, n$. It is obtained that

$$
\sum_{j=1}^n \prod_{j=1}^n f_j(k_j)\|R_\omega(k_1, \ldots, k_n)R_1(k_1, \ldots, k_n)\psi\|_{\mathcal{H}dk_1 \ldots dk_n}
$$

$$
\leq \sum_{p=1}^n \prod_{j=1}^n f_j(k_j)\|\delta_1(k_p)\|b(k_1) \ldots b(k_p) \ldots b(k_n)(|x| + 1)\psi\|_{\mathcal{H}dk_1 \ldots dk_n}.
$$

(3.18)

Similarly we obtain that

$$
\sum_{j=1}^n \prod_{j=1}^n f_j(k_j)\|R_\omega(k_1, \ldots, k_n)R_2(k_1, \ldots, k_n)\psi\|_{\mathcal{H}dk_1 \ldots dk_n}
$$

$$
\leq \sum_{p=1}^n \prod_{j=1}^n f_j(k_j)\|\delta_2(k_p, k_q)\|b(k_1) \ldots b(k_q) \ldots b(k_p) \ldots b(k_n)|x|^2\psi\|_{\mathcal{H}dk_1 \ldots dk_n}.
$$

(3.19)
We choose a sequence \( \{ \Psi_m \} \subset C \) such that \( \Psi_m \to \psi_g \) and \((H_1 + 1)^K + |x|^{2K} \Psi_m \to (H_1 + 1)^K + |x|^{2K} \psi_g\) strongly as \( m \to \infty \) for sufficiently large \( K \). Note that \(|x|^j \Psi_m \to |x|^j \Psi\) and \(((H_1 + 1)^n + |x|^{2n}) |x|^j \Psi_m \to ((H_1 + 1)^n + |x|^{2n}) |x|^j \Psi\) strongly as \( m \to \infty \) for \( j = 1, 2 \), since \( K \) is sufficiently large. By Lemma 2.7 there exists a subsequence \( \{ m' \} \subset \{ m \} \) such that for almost every \( (k_1, \ldots, k_n) \in \mathbb{R}^n \), (3.17) follows and

\[
\begin{align*}
&b(k_1)\ldots b(k_n) \Psi_{m'} \to b(k_1)\ldots b(k_n) \psi_g, \\
&\quad b(k_1)\ldots \hat{b}(k_p)\ldots b(k_n) (|x| + 1) \Psi_{m'} \to b(k_1)\ldots \hat{b}(k_p)\ldots b(k_n) (|x| + 1) \psi_g,
\end{align*}
\]

and

\[
\begin{align*}
&b(k_1)\ldots \hat{b}(k_q)\ldots b(k_p)\ldots b(k_n) |x|^2 \Psi_{m'} \to b(k_1)\ldots \hat{b}(k_q)\ldots b(k_p)\ldots b(k_n) |x|^2 \psi_g
\end{align*}
\]

strongly as \( m' \to \infty \). Moreover

\[
\sum_{j=1}^{n} | \prod_{j=1}^{n} f_j(k_j) \delta_1(k_p) ||b(k_1)\ldots \hat{b}(k_p)\ldots b(k_n) (|x| + 1) \Psi_{m'} ||_{H} dk_1 \ldots dk_n
\]

\[
\leq \left( \int \delta_1(k_p) |f_p(k_p)| dk_p \right) e' \left( f_1, \ldots, \hat{f}_p, \ldots, f_n \right) \|((H_1 + 1)^{n-1} + |x|^{2n-2})(|x| + 1) \Psi_{m'} \|_{H}
\]

and

\[
\sum_{j=1}^{n} | \prod_{j=1}^{n} f_j(k_j) \delta_2(k_p, k_q) ||b(k_1)\ldots \hat{b}(k_q)\ldots b(k_p)\ldots b(k_n) |x|^2 \Psi_{m'} ||_{H} dk_1 \ldots dk_n
\]

\[
\leq \left( \int \delta_2(k_p, k_q) |f_p(k_p)| f_q(k_q) dk_p dk_q \right)
\]

\[
\times e' \left( f_1, \ldots, \hat{f}_p, \ldots, \hat{f}_q, \ldots, f_n \right) \|((H_1 + 1)^{n-2} + |x|^{2n-2}) |x|^2 \Psi_{m'} \|_{H}.
\]

Then we have by (3.18) and (3.19)

\[
\sum_{j=1}^{n} | \prod_{j=1}^{n} f_j(k_j) ||b(k_1)\ldots b(k_n) \Psi_{m'} ||_{H} dk_1 \ldots dk_n
\]

\[
\leq \sum_{j=1}^{n} \sum_{p=1}^{n} | \prod_{j=1}^{n} f_j(k_j) \delta_1(k_p) ||b(k_1)\ldots \hat{b}(k_p)\ldots b(k_n) (|x| + 1) \Psi_{m'} ||_{H} dk_1 \ldots dk_n
\]

\[
+ \sum_{p=1}^{n} \sum_{q<p} \sum_{j=1}^{n} | \prod_{j=1}^{n} f_j(k_j) \delta_2(k_p, k_q) ||b(k_1)\ldots \hat{b}(k_q)\ldots b(k_p)\ldots b(k_n) |x|^2 \Psi_{m'} ||_{H} dk_1 \ldots dk_n
\]

\[
\leq C' \|((H_1 + 1)^K + |x|^{2K}) \Psi_{m'} \|_{H} \leq C'' \|((H_1 + 1)^K + |x|^{2K}) \psi_g \|_{H}
\]

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with some constant $C$ and $C'$. Thus by the Lebesgue dominated convergence theorem
and (3.20), (3.21) and (3.22), we have

$$\sum_{j=1}^{n} \prod_{j=1}^{n} f_j(k_j) \|b(k_1)\ldots b(k_n)\psi_g\|_\mathcal{H} dk_1\ldots dk_n$$

$$\leq \sum_{p=1}^{n} \sum_{j=1}^{n} f_j(k_j) \delta_1(k_p) \|b(k_1)\ldots b(k_p)\|_\mathcal{H} dk_1\ldots dk_n$$

$$+ \sum_{p=1}^{n} \sum_{q<p}^{n} f_j(k_j) \delta_2(k_q, k_p) \|b(k_1)\ldots b(k_p)\|_\mathcal{H} dk_1\ldots dk_n$$

Since $f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, $j = 1,\ldots, n$, are arbitrary, the lemma follows. \qed

**Lemma 3.5** Let $\Psi \in \mathcal{D}$. Then for almost every $(k_1,\ldots, k_n) \in \mathbb{R}^{3n}$ it follows that

$$\Psi \in D(b(k_1)\ldots b(k_n)) \bigcap_{l=0}^{\infty} \bigcap_{(p_1,\ldots, p_l) \subset \{1,\ldots, n\}} D(a(k_1)\ldots a(k_{p_1})\ldots a(k_{p_l})\ldots a(k_n)|x|^l)$$

and

$$\|b(k_1)\ldots b(k_n)\psi\|_\mathcal{H}$$

$$\leq \sum_{l=0}^{n} \sum_{(p_1,\ldots, p_l) \subset \{1,\ldots, n\}} \prod_{j=1}^{l} \frac{c_2(k_{p_j})}{\sqrt{2\omega(k_{p_j})}} \|a(k_1)\ldots a(k_{p_1})\ldots a(k_{p_l})\ldots a(k_n)|x|^l\psi\|_\mathcal{H}. \quad (3.23)$$

**Proof:** Take a sequence $\{\Psi_m\} \subset \mathcal{C}$ such that $\Psi_m \to \Psi$ and $(H_1^K + |x|^{2K} + 1)\Psi_m \to (H_1^K + |x|^{2K} + 1)\Psi$ strongly as $m \to \infty$ for sufficiently large $K$. (3.23) is valid for $\Psi$ replaced by $\Psi_m$, since

$$b(k_1)\ldots b(k_n)\Psi_m = (a(k_1) + \theta_1)\ldots (a(k_n) + \theta_n)\Psi_m$$

$$= \sum_{l=0}^{n} \sum_{(p_1,\ldots, p_l) \subset \{1,\ldots, n\}} \theta_{p_1}\ldots \theta_{p_l} a(k_1)\ldots a(k_{p_1})\ldots a(k_{p_l})\ldots a(k_n)\Psi_m.$$ 

Note that $|x|^l\Psi_m \to |x|^l\Psi$ and $((H_1 + 1)^n + |x|^{2n})|x|^l\Psi_m \to ((H_1 + 1)^n + |x|^{2n})|x|^l\Psi$ strongly as $m \to \infty$, since $K$ is sufficiently large. By Lemmas 2.4 and 2.7 there exists a subsequence $\{m'\} \subset \{m\}$ such that for almost every $(k_1,\ldots, k_n) \in \mathbb{R}^{3n}$,

$$b(k_1)\ldots b(k_n)\Psi_{m'} \to b(k_1)\ldots b(k_n)\Psi$$

and

$$a(k_1)\ldots a(k_{p_1})\ldots a(k_{p_l})\ldots a(k_n)|x|^l\Psi_{m'} \to a(k_1)\ldots a(k_{p_1})\ldots a(k_{p_l})\ldots a(k_n)|x|^l\Psi.$$ 

Thus the proof is complete. \qed

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Lemma 3.6 Let $\Psi \in \mathcal{D}$. Then for almost every $(k_1, \ldots, k_n) \in \mathbb{R}^n$,
\[
\Psi \in D(a(k_1) \ldots a(k_n)) \cap \{x_{l=0}^n \cap \{(p_1, \ldots, p_l) \in \{1, \ldots, n\} \mid D(b(k_1) \ldots b(k_{p_1}) \ldots b(k_{p_l}) \ldots b(k_n)|x|^l]\}
\]
and
\[
\|a(k_1) \ldots a(k_n)\| \leq \sum_{l=0}^n \sum_{\{(p_1, \ldots, p_l) \in \{1, \ldots, n\}\}} \prod_{j=1}^l \left| e_{\mathcal{F}}(k_{p_j}) \right| \|b(k_1) \ldots b(k_{p_1}) \ldots b(k_{p_l}) \ldots b(k_n)|x|^l\|_\mathcal{H}.
\]

Proof: Note $b(k_1) \ldots b(k_n) = (a(k_1) - \theta_1) \ldots (a(k_n) - \theta_n)$. The lemma is proven in the similar way as Lemma 3.5. \(\square\)

Lemma 3.7 Suppose that $|x|^z \Psi \in D(N^{n/2}) \cap \mathcal{D}$ for $z = m, m+1, \ldots, m+n$. Then there exist constants $c_k^{n,l}$ such that
\[
\mathcal{P} \int \|b(k_1) \ldots b(k_n)|x|^m\|_\mathcal{H}^2 dk_1 \ldots dk_n \leq \sum_{l=0}^n \sum_{k=1}^{n-l} c_k^{n,l} \|N^{k/2}|x|^{m+l}\|_\mathcal{H}^2.
\] (3.24)

Proof: We have by Lemma 3.5 and $|\sum_{j=1}^N x_j|^2 \leq N \sum_{j=1}^N x_j^2$,
\[
\mathcal{P} \int \|b(k_1) \ldots b(k_n)|x|^m\|_\mathcal{H}^2 dk_1 \ldots dk_n \leq 2^n \sum_{l=0}^n \sum_{\{(p_1, \ldots, p_l) \in \{1, \ldots, n\}\}} \left( \left\| \frac{e_{\mathcal{F}}}{\sqrt{2\omega}} \right\| \right)^l \times
\]
\[
\times \mathcal{P} \int \|a(k_1) \ldots a(k_{p_1}) \ldots a(k_{p_l}) \ldots a(k_n)|x|^{m+l}\|_\mathcal{H}^2 dk_1 \ldots dk_{p_1} \ldots dk_{p_l} \ldots dk_n.
\]
By the assumption it follows that $|x|^{m+l}\Psi \in D(N^{n/2})$. Thus we see that
\[
\mathcal{P} \int \|a(k_1) \ldots a(k_{p_1}) \ldots a(k_{p_l}) \ldots a(k_n)|x|^{m+l}\|_\mathcal{H}^2 dk_1 \ldots dk_{p_1} \ldots dk_{p_l} \ldots dk_n
\]
\[
= \prod_{j=1}^{n-l} (N - j + 1)^{1/2} |x|^{m+l}\Psi \|_\mathcal{H}^2 \leq \sum_{l=0}^{n-l} a_{k}^{n,l} \|N^{k/2}|x|^{m+l}\|_\mathcal{H}^2
\]
with some constants $a_{k}^{n,l}$. Then
\[
\mathcal{P} \int \|b(k_1) \ldots b(k_n)|x|^m\|_\mathcal{H}^2 dk_1 \ldots dk_n
\]
\[
\leq \sum_{l=0}^n \sum_{\{(p_1, \ldots, p_l) \in \{1, \ldots, n\}\}} \left( \left\| \frac{e_{\mathcal{F}}}{\sqrt{2\omega}} \right\| \right)^l \sum_{k=1}^{n-l} a_{k}^{n,l} \|N^{k/2}|x|^{m+l}\|_\mathcal{H}^2.
\]
Hence we conclude (3.24). \(\square\)

We set the right hand side of (3.24) by $\mathcal{R}_{n,m}(\Psi)$, i.e.,
\[
\mathcal{R}_{n,m}(\Psi) = \sum_{l=0}^n \sum_{k=1}^{n-l} c_k^{n,l} \|N^{k/2}|x|^{m+l}\|_\mathcal{H}^2.
\]
Lemma 3.8 Let $\Psi \in D$. Then there exist constants $d^n_l$ such that

$$\sum \|a(k_1)...a(k_n)\Psi\|_{H}^2 dk_1...dk_n$$

$$\leq 2^n \left\{ \sum \|b(k_1)...b(k_n)\Psi\|_{H}^2 dk_1...dk_n + \sum_{l=1}^{n} d^n_l \mathcal{R}_{n-l,l}(\Psi) \right\}.$$  

Proof: We have by Lemma 3.6,

$$\sum \|a(k_1)...a(k_n)\Psi\|_{H}^2 dk_1...dk_n \leq 2^n \sum_{l=0}^{n} \sum_{\{p_1,...,p_l\} \subset \{1,...,n\}} \left( \left\| \frac{e^\phi}{\sqrt{2\omega}} \right\|^2 \right)^l \times$$

$$\times \int \|b(k_1)...b(k_{p_1})...b(k_{p_l})b(k)\|_{H}^2 dk_1...dk_{p_1}...dk_{p_l}...dk_n. \quad (3.25)$$

The term with $l = 0$ in (3.25) is just $\sum \|b(k_1)...b(k_n)\Psi\|_{H}^2 dk_1...dk_n$. The lemma follows from Lemma 3.7. \hfill $\Box$

Proof of Theorem 1.7

We prove the theorem by means of an induction. It is known that

$$\psi_g \in D(N^{1/2}).$$

Suppose that

$$\psi_g \in D(N^{(n-1)/2}).$$

Then by Lemma 3.2,

$$\sum \|a(k_1)...a(k_l)\psi_g\|_{H}^2 dk_1...dk_n < \infty, \quad l = 1, 2, ..., n-1, \quad (3.26)$$

and by Lemma 2.12,

$$\|N^{l/2}|x|^m \psi_g\|_H < \infty \quad (3.27)$$

follows for all $m \geq 0$ and $l \leq n-1$. By Lemma 3.8

$$\sum \|a(k_1)...a(k_n)\psi_g\|_{H}^2 dk_1...dk_n$$

$$\leq 2^n \left\{ \sum \|b(k_1)...b(k_n)\psi_g\|_{H}^2 dk_1...dk_n + \sum_{l=1}^{n} d^n_l \mathcal{R}_{n-l,l}(\Psi) \right\}.$$ 

By (3.27) we see that

$$\mathcal{R}_{n-l,l}(\Psi) < \infty.$$
From Lemma 3.4 it follows that
\[
\sum_{k_1=1}^{n} \sum_{k_n=1}^{n} \| b(k_1) \cdots b(k_n) \psi_g \|_{H}^2 dk_1 \cdots dk_n \\
\leq \delta_1 \sum_{p=1}^{n} \sum_{k_p=1}^{n} \| b(k_1) \cdots \hat{b}(k_p) \cdots b(k_n) (|x| + 1) \psi_g \|_{H}^2 dk_1 \cdots \hat{dk}_p \cdots dk_n \\
+ \delta_2 \sum_{p=1}^{n} \sum_{q=p+1}^{n} \sum_{k_q=1}^{n} \sum_{k_p=1}^{n} \| b(k_1) \cdots \hat{b}(k_q) \cdots \hat{b}(k_p) \cdots b(k_n) |x|^2 \psi_g \|_{H}^2 dk_1 \cdots \hat{dk}_q \cdots \hat{dk}_p \cdots dk_n, \tag{3.28}
\]
where \( \delta_1 = \int \delta_1(k) dk \) and \( \delta_2 = \int \delta_2(k, k') dk dk' \). Then the right hand side of (3.28) is finite by Lemma 3.7. Hence
\[
\sum_{k_1=1}^{n} \sum_{k_n=1}^{n} \| a(k_1) \cdots a(k_n) \psi_g \|_{H}^2 dk_1 \cdots dk_n < \infty
\]
follows, which implies, together with (3.26), that
\[
\psi_g \in D(N^{n/2})
\]
by Lemma 3.2. Thus the theorem follows. \( \square \)

**Proof of Theorem 1.9**

This follows from Theorem 1.7 and Lemma 2.11. \( \square \)

4 Appendix

4.1 Appendix A

**Lemma 4.1** Let \( \Psi \in D(H_t^{n/2}) \). Then there exists \( \mathcal{M}(\Psi) \subset \mathbb{R}^{3n} \) with the Lebesgue measure zero such that
\[
\Psi \in D(a(k_1) \cdots a(k_n)) \tag{4.1}
\]
for \( (k_1, \ldots, k_n) \not\in \mathcal{M}(\Psi) \). Moreover assume that \( \{\Psi_m\} \subset \mathcal{C} \) satisfies that \( \Psi_m \to \Psi \) and \( (H_t + 1)^{n/2} \Psi_m \to (H_t + 1)^{n/2} \Psi \) strongly as \( m \to \infty \). Then there exists a subsequence \( \{m'\} \subset \{m\} \) and \( \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\}) \subset \mathbb{R}^{3n} \) with the Lebesgue measure zero such that (4.1) follows and
\[
\lim_{m' \to \infty} a(k_1) \cdots a(k_n) \Psi_{m'} = a(k_1) \cdots a(k_n) \Psi
\]
for \( (k_1, \ldots, k_n) \not\in \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\}) \).
Proof: We fix a sequence \( \{ \Psi_m \} \). The lemma is proven inductively. Note that
\[
\| (H_l + 1)^p \Psi \| \leq \| (H_l + 1)^q \Psi \| \quad (4.2)
\]
for \( p \leq q \). By (2.1) we see that
\[
\sum_j |f_1(k_1)||a(k_1)\Psi_m\|_\mathcal{H}dk_1 \leq \epsilon(f_1)(H_l + 1)^{1/2}\Psi_m\|_\mathcal{H} \quad (4.3)
\]
for arbitrary \( f_1 \in C^\infty_0(\mathbb{R}^3 \setminus \{0\}) \). The right hand side of (4.3) converges as \( m \to \infty \) by (4.2). Then the left hand side of (4.3) is a Cauchy sequence. Then there exist \( N_1(\Psi) \subset \mathbb{R}^3 \) with the Lebesgue measure zero and a subsequence \( \{m_1\} \subset \{m\} \) such that \( a(k)\Psi_{m_1}\) converges strongly as \( m_1 \to \infty \) for \( k_1 \not\in N_1(\Psi) \). Since \( a(k_1) \) is closed, it follows that for \( k_1 \not\in N_1(\Psi) \), \( \Psi \in D(a(k_1)) \) and
\[
s - \lim_{m_1 \to \infty} a(k_1)\Psi_{m_1} = a(k_1)\Psi.
\]
For \( \Psi_{m_1} \) we have by (2.1)
\[
\sum_j |f_1(k_1)f_2(k_2)||a(k_1)a(k_2)\Psi_{m_1}\|_\mathcal{H}dk_1dk_2 \leq \epsilon(f_1,f_2)(H_l + 1)^{1/2}\Psi_{m_1}\|_\mathcal{H}
\]
for arbitrary \( f_1, f_2 \in C^\infty_0(\mathbb{R}^3 \setminus \{0\}) \). Then we also see that there exist \( N_2(\Psi) \subset \mathbb{R}^3 \times \mathbb{R}^3 \) with the Lebesgue measure zero and a subsequence \( \{m_2\} \subset \{m_1\} \) such that \( a(k_1)a(k_2)\Psi_{m_2}\) converges strongly as \( m_2 \to \infty \) for \( (k_1, k_2) \not\in N_2(\Psi) \). Since \( a(k_2)\Psi_{m_2} \to a(k_2)\Psi \) strongly as \( m_2 \to \infty \) for \( k_2 \not\in N_1(\Psi) \) and \( a(k_1) \) is closed, we see that for \( (k_1, k_2) \not\in N_2(\Psi) \cup [\mathbb{R}^3 \times N_1(\Psi)] \), \( a(k_2)\Psi \in D(a(k_1)) \) and
\[
s - \lim_{m_2 \to \infty} a(k_1)a(k_2)\Psi_{m_2} = a(k_1)a(k_2)\Psi.
\]
Repeating this procedure we see that there exist subsets \( N_j(\Psi) \subset \mathbb{R}^{3j} \), \( j = 1, \ldots, n \), with the Lebesgue measure zero and subsequences \( \{m_n\} \subset \{m_{n-1}\} \subset \ldots \subset \{m\} \) such that for \( (k_1, \ldots, k_n) \not\in N_n(\Psi) \), \( a(k_1)\ldots a(k_n)\Psi_{m_n} \) converges strongly as \( m_n \to \infty \) and \( a(k_2)\ldots a(k_n)\Psi_{m_n} \to a(k_2)\ldots a(k_n)\Psi \) strongly as \( m_n \to \infty \) for \( (k_2, \ldots, k_n) \not\in N_{n-1}(\Psi) \cup [\mathbb{R}^3 \times N_{n-2}(\Psi)] \cup \ldots \cup [\mathbb{R}^{3(n-2)} \times N_1(\Psi)] \). Let
\[
\mathcal{M}(\Psi, \{\Psi_m\}, \{m'\}) = N_n(\Psi) \cup [\mathbb{R}^3 \times N_{n-1}(\Psi)] \cup \ldots \cup [\mathbb{R}^{3(n-1)} \times N_1(\Psi)]
\]
Since \( a(k_1) \) is closed, we see that for \( (k_1, \ldots, k_n) \not\in \mathcal{M}_p(\Psi, \{\Psi_m\}, \{m'\}) \),
\[
a(k_2)\ldots a(k_n)\Psi \in D(a(k_1))
\]
and
\[ s - \lim_{\substack{m \to \infty}} a(k_1) \ldots a(k_n) \Psi_{m} = a(k_1) \ldots a(k_n) \Psi. \]

Thus the proof is complete. \(\square\)

We define \(\text{ad}_A^l(B)\) by \(\text{ad}^0_A(B) = B\) and \(\text{ad}_A^l(B) = [A, \text{ad}_A^{l-1}(B)]\). Note that on \(\mathcal{F}_\omega\)
\[
[H^l, a(k_1) \ldots a(k_n)] = \sum_{l=1}^p \left( \begin{array}{c} p \\ l \end{array} \right) \text{ad}_{H^l}^l(a(k_1) \ldots a(k_n)) H^{p-l},
\]
\[
\text{ad}_{H^l}^l(a(k_1) \ldots a(k_n)) = \sum_{p_1=0}^{l-p_1} \sum_{p_2=0}^{l-p_2} \cdots \sum_{p_n=0}^{l-p_n} \left( \begin{array}{c} l \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \right) \left( l - p_1 \right) \left( l - p_2 \right) \cdots \left( l - \sum_{i=1}^{n-1} p_i \right)
\]
\[
\times \text{ad}_{H^l}^{p_1}(a(k_1)) \text{ad}_{H^l}^{p_2}(a(k_2)) \cdots \text{ad}_{H^l}^{p_n}(a(k_n)),
\]
and
\[
\text{ad}_{H^l}^l(a(k)) = (-1)^p \omega(k)^p a(k).
\]

Hence we have
\[
[H^p, a(k_1) \ldots a(k_n)] = \sum_{l=1}^p \left( \begin{array}{c} p \\ l \end{array} \right) \sum_{p_1=0}^{l-p_1} \sum_{p_2=0}^{l-p_2} \cdots \sum_{p_n=0}^{l-p_n} \left( \begin{array}{c} l \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{array} \right) \left( l - p_1 \right) \left( l - p_2 \right) \cdots \left( l - \sum_{i=1}^{n-1} p_i \right)
\]
\[
\times (-1)^l \omega(k_1)^{p_1} \omega(k_2)^{p_2} \ldots \omega(k_n)^{p_n} a(k_1) \ldots a(k_n).
\] (4.4)

**Lemma 4.2** Let \(\Psi \in C^\infty(H_t)\). Then there exists \(\mathcal{M}_\infty(\Psi) \subset \mathbb{R}^{3n}\) with the Lebesgue measure zero such that for \((k_1, \ldots, k_n) \notin \mathcal{M}_\infty(\Psi)\),
\[
\Psi \in D(a(k_1) \ldots a(k_n))
\]
and
\[
a(k_1) \ldots a(k_n) \Psi \in C^\infty(H_t).
\]

**Proof:** Let \(\{\Psi_m\} \subset \mathcal{C}\) be such that \(\Psi_m \to \Psi\) and \((H_t + 1)^q \Psi_m \to (H_t + 1)^q \Psi\) strongly as \(m \to \infty\) for \(q = (n/2) + p\). In particular, \((H_t + 1)^{n/2} \Psi_m \to (H_t + 1)^{n/2} \Psi\) strongly as \(m \to \infty\). By Lemma 4.1, there exists a subsequence \(\{m'\} \subset \{m\}\) such that for \((k_1, \ldots, k_n) \notin \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\})\),
\[
\Psi \in D(a(k_1) \ldots a(k_n))
\]
and
\[
\lim_{m' \to \infty} a(k_1) \ldots a(k_n) \Psi_{m'} = a(k_1) \ldots a(k_n) \Psi.
\] (4.5)
We reset \( m' \) as \( m \). By (4.4), for \( f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}) \), \( j = 1, \ldots, n \),

\[
\sum_j \left| \prod f_j(k_j) \right| ||H_t^p a(k_1) \ldots a(k_n) \Psi_m||_H dk_1 \ldots dk_n \leq \sum_j \left| \prod f_j(k_j) \right| ||a(k_1) \ldots a(k_n) H_t^p \Psi_m||_H dk \\
+ \sum_{l=1}^p \frac{p^l}{l} \sum_{p_1=0} \sum_{p_2=0} \cdots \sum_{p_{n-1}=0} \frac{1}{p} \left( \frac{l-1}{p_1} \right) \left( \frac{l-1}{p_2} \right) \cdots \left( \frac{l-1}{p_{n-1}} \right) \times \sum_j \left| \prod f_j(k_j) \omega(k_j)^{p_j} \right| ||a(k_1) \ldots a(k_n) H_t^{n-l} \Psi_m||_H dk_1 \ldots dk_n \\
\leq \varepsilon(f_1, \ldots, f_n) ||(H_t + 1)^{n/2} H_t^p \Psi_m||_H \\
+ \sum_{l=1}^p \frac{p^l}{l} \sum_{p_1=0} \sum_{p_2=0} \cdots \sum_{p_{n-1}=0} \frac{1}{p} \left( \frac{l-1}{p_1} \right) \left( \frac{l-1}{p_2} \right) \cdots \left( \frac{l-1}{p_{n-1}} \right) \times \varepsilon(\omega^{p_1} f_1, \ldots, \omega^{p_n} f_n) ||(H_t + 1)^{n/2} H_t^{(p-l)} \Psi_m||_H \\
\leq C ||(H_t + 1)^{(n/2) + p} \Psi_m||_H 
\tag{4.6}
\]

with some constant \( C \). The right hand side of (4.6) converges strongly as \( m \to \infty \). Since \( f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}) \), \( j = 1, \ldots, n \), are arbitrary, there exist \( N_p(\Psi) \subset \mathbb{R}^{3n} \) with the Lebesgue measure zero and a subsequence \( \{m'\} \subset \{m\} \) such that \( H_t^p a(k_1) \ldots a(k_n) \Psi_{m'} \) strongly converges as \( m' \to \infty \) for \( (k_1, \ldots, k_n) \notin N_p(\Psi) \). Since \( H_t^p \) is closed, we obtain by (4.5) that

\[ a(k_1) \ldots a(k_n) \Psi \in D(H_t^p) \]

for \( (k_1, \ldots, k_n) \notin \Omega_p = N_p(\Psi) \cup \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\}) \). Define

\[ \mathcal{M}_\infty(\Psi) = \bigcup_p \Omega_p. \]

Then it follows that \( a(k_1) \ldots a(k_n) \Psi \in C^\infty(H_t) \) for \( (k_1, \ldots, k_n) \notin \mathcal{M}_\infty \). \( \square \)

**Proof of Lemma 2.4**

Let \( \{\Psi_m\} \subset \mathcal{C} \) be such that \( \Psi_m \to \Psi \) and \( ((H_t + 1)^{n/2} + |x|^{2p}) \Psi_m \to ((H_t + 1)^{n/2} + |x|^{2p}) \Psi \) strongly as \( m \to \infty \). From Lemma 4.2 it follows that for \( (k_1, \ldots, k_n) \notin \mathcal{M}_\infty(\Psi) \),

\[ a(k_1) \ldots a(k_n) \Psi \in C^\infty(H_t) \]

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and from Lemma 4.1

\[ s - \lim_{m' \to \infty} a(k_1) \ldots a(k_n) \Psi_{m'} = a(k_1) \ldots a(k_n) \Psi \]  

(4.7)

with some subsequence \( \{m'\} \) for \((k_1, \ldots, k_n) \not\in \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\}) \). We reset \( m' \) as \( m \). Let \( f_j \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}) \), \( j = 1, \ldots, n \). Since \( |x|^p, a(k_1) \ldots a(k_n) \Psi_m = 0 \), we have

\[
\left( \sum_{j=1}^{n} | \prod_{j=1}^{n} f_j(k_j)| |x|^p a(k_1) \ldots a(k_n) \Psi_m \right)_{H} dk_1 \ldots dk_n \leq \epsilon(f_1, \ldots, f_n)^2 \left\| |x|^p \Psi \right\|_{H} \leq \epsilon(f_1, \ldots, f_n)^2 \left\| (H_1 + 1) + |x|^p \right\|_{H} \Psi_m. 
\]

Since the right hand side converges as \( m \to \infty \), there exist \( \mathbf{N}_p(\Psi)' \subset \mathbb{R}^{3n} \) with the Lebesgue measure zero and a subsequence \( \{m'\} \) such that \( |x|^p a(k_1) \ldots a(k_n) \Psi_{m'} \) strongly converge as \( m' \to \infty \) for \((k_1, \ldots, k_n) \in \mathbf{N}_p(\Psi)' \). Since \( |x|^p \) is closed and by (4.7),

\[ a(k_1) \ldots a(k_n) \Psi \in D(|x|^p) \]

follows for \((k_1, \ldots, k_n) \not\in \Omega'_p = \mathbf{N}_p(\Psi)' \cup \mathcal{M}(\Psi, \{\Psi_m\}, \{m'\}) \). Then for \((k_1, \ldots, k_n) \not\in \cup_p \Omega'_p, \]

\[ a(k_1) \ldots a(k_n) \Psi \in C^\infty(|x|). \]

Let

\[ \mathcal{M}_{D}(\Psi, \{\Psi_m\}, \{m'\}) = \mathcal{M}_\infty(\Psi) \bigcup [\cup_p \Omega'_p]. \]

Then the lemma follows. \( \square \)

**Proof of Lemma 2.6**

Applying (2.5) instead of (2.1), we can show the lemma in the similar way as Lemmas 4.1, 4.2 and 2.4. \( \square \)

### 4.2 Appendix B

In this section we prove Lemma 2.11. In [14] we proved that \( e^{-tH} \) maps \( D(N^{k/2}) \) into itself for the case when \( V = 0 \). We extend this result for some nonzero potential \( V \).

We see that if \( \psi_g \in D(N^{k/2}) \) then the identity

\[ N^{k/2} \psi_g = e^{-tH} e^{tE} N^{k/2} \psi_g + e^{tE} [N^{k/2}, e^{-tH}] \psi_g \]  

(4.8)

is well defined. Using (4.8) we shall prove that \( \|N^{k/2} \psi_g(x)\|_\mathcal{F} \) decays exponentially. To see it we prepare some probabilistic notations.
It is known that there exist a probability space \((Q, \mu)\) and Gaussian random variables \((\phi(f), f \in \oplus^3 L^2_{\text{real}}(\mathbb{R}^3))\) such that
\[
\int_Q \phi(f) \phi(g) d\mu = \frac{1}{2} \sum_{\mu, \nu=1,2,3} \int_{\mathbb{R}^3} \left( \delta_{\mu} - \frac{k_\mu k_\nu}{|k|^2} \right) \tilde{f}_\mu(k) \tilde{g}_\nu(k) dk.
\]
For a general \(f \in \oplus^3 L^2(\mathbb{R}^3)\), we set \(\phi(f) = \phi(\Re f) + i\phi(\Im f)\). It is also known that there exists a unitary operator implementing \(1 \cong \Omega, L^2(Q) \cong \mathcal{F}\) and \(\phi(\oplus_{\nu=1}^3 \delta_{\mu \nu} \lambda(\cdot - x)) \cong \Lambda_\mu(x)\), where \(\lambda\) is the inverse Fourier transform of \(\tilde{\lambda} = \tilde{\phi} / \sqrt{\omega}\).

The free Hamiltonian in \(L^2(Q)\) corresponding to \(H_t\) in \(\mathcal{F}\) is denoted by \(\tilde{H}_t\). To have a functional integral representation of \(e^{-i\tilde{H}_t}\), we go through another probability space \((Q_0, \nu_0)\) and Gaussian random variables \((\phi_0(f), f \in \oplus^3 L^2_{\text{real}}(\mathbb{R}^4))\) such that
\[
\int_{Q_0} \phi_0(f) \phi_0(g) d\nu_0 = \frac{1}{2} \sum_{\mu, \nu=1,2,3} \int_{\mathbb{R}^4} \left( \delta_{\mu} - \frac{k_\mu k_\nu}{|k|^2} \right) \tilde{f}_\mu(k, k_0) \tilde{g}_\nu(k, k_0) dk dk_0.
\]
Here \(\phi_0(f)\) is also extended to \(f \in \oplus^3 L^2(\mathbb{R}^4)\) such as \(\phi(f)\). Let \(j_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^4)\) be the isometry defined by
\[
\tilde{j}_t f(k, k_0) = \frac{e^{-itk_0}}{\sqrt{\pi}} \sqrt{\omega(k) / (\omega(k)^2 + |k_0|^2)} \tilde{f}(k)
\]
and \(J_t : L^2(Q) \to L^2(Q_0)\) by
\[
J_t \phi(f_1) \ldots \phi(f_n) := \phi_0([\oplus^3 j_t] f_1) \ldots \phi_0([\oplus^3 j_t] f_n),
\]
\[J_1 1 = 1.\]
Here \(\hat{X}\): denotes the Wick power of \(X\) inductively defined by
\[
\hat{x}(f) := \phi_x(f),
\]
\[
\hat{x}(f) \hat{x}(f_1) \ldots \hat{x}(f_n) := \phi_x(f) \hat{x}(f_1) \ldots \hat{x}(f_n);
\]
\[
- \sum_{j=1}^n (\phi_x(f_j) \hat{x}(f))_{L^2(Q_0)} \hat{x}(f_1) \ldots \hat{x}(f_j) \ldots \hat{x}(f_n),
\]
where \(Q_x = Q, Q_0\) and \(\phi_x = \phi, \phi_0\). Then \(J_t\) can be extended to an isometry and \(J^*_t J_s = e^{-|t-s|H_t}\) follows for \(t, s \in \mathbb{R}\). We identify \(\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}\) with \(L^2(\mathbb{R}^3; L^2(Q))\). Under this identification \(\Psi \in \mathcal{H}\) can be regarded as \(L^2(Q)\)-valued \(L^2\)-function on \(\mathbb{R}^3\),
i.e., $\Psi(x) \in L^2(Q)$ for almost every $x \in \mathbb{R}^3$. In [14, Lemma 4.9] and [12] we established that

$$
\left( e^{-tH}\Psi \right)(x) = \mathbb{E}^Q_x \left( e^{-\int_0^t V(X_s)ds} \mathcal{J}_t\Psi(X_t) \right)
$$

for almost every $x \in \mathbb{R}^3$. Here $(X_t)_{t \geq 0} = (X_{1,t}, X_{2,t}, X_{3,t})_{t \geq 0} \in C([0, \infty); \mathbb{R}^3)$ denotes an $\mathbb{R}^3$-valued continuous path, $\mathbb{E}^Q_x$ an $L^2(Q)$-valued expectation value with respect to the wiener measure $P_x$ on $C([0, \infty); \mathbb{R}^3)$ with $P_x(X_0 = x) = 1$, and

$$
\mathcal{J}_t = \mathcal{J}_t(x, X.) : L^2(Q) \rightarrow L^2(Q)
$$

is given by

$$
\mathcal{J}_t = J^*_0 e^{-i\phi_0(K(x, X_0))} J_t,
$$

where $K(x, X.)$ is a $\oplus^3 L^2(\mathbb{R}^4)$-valued stochastic integral defined by

$$
K = K(x, X.) = \oplus_{\mu=1,2,3} \int_0^t j_{s\lambda}(\cdot - X_s) dX_{\mu,s}.
$$

Let $N$ and $N_0$ be the number operators in $L^2(Q)$ and $L^2(Q_0)$, respectively. Note that

$$
J_t N = N_0 J_t
$$
on a dense domain. The expectation value with respect to $P_x$ is denoted by $\mathbb{E}_x$. We show a fundamental inequality.

**Lemma 4.3** Let $\xi = \xi(x, X.) = \|K(x, X.)\|_{\oplus^3 L^2(\mathbb{R}^4)}$. Then, for all $m \geq 0$,

$$
\mathbb{E}_x \left( \xi^{2m} \right) \leq \frac{3(2m)!}{2m} t^{m-1} \mathbb{E}_x \left( \int_0^t \| j_{s\lambda}(\cdot - X_s) \|_{L^2(\mathbb{R}^4)}^{2m} ds \right) = \frac{3(2m)!}{2m} t^m \| \hat{\phi}/\sqrt{\omega} \|_{2m}^{2m}. \quad (4.9)
$$

In particular $\sup_{x \in \mathbb{R}^3} \mathbb{E}_x (\xi^{2m}) < \infty$.

**Proof:** See [14, Theorem 4.6].

**Lemma 4.4** For each $(x, X.) \in \mathbb{R}^3 \times C([0, \infty); \mathbb{R}^3)$ and $\Psi \in D(N^{k/2})$,

$$
\| [N^{k/2}, \mathcal{J}_t(x, X.)] \Psi \|_{L^2(Q)} \leq P_k(\xi) \| (N + 1)^{k/2} \Psi \|_{L^2(Q)},
$$

with some polynomial $P_k(\cdot)$.
Proof: Note that for each \((x,X), \mathcal{J}_t = \mathcal{J}_t(x,X)\) maps \(D(N^{k/2})\) into itself. We have

\[
\begin{aligned}
[N^{k/2}, \mathcal{J}_t] \Psi &= J_0^* e^{-ie\phi_0(K)} [e^{ie\phi_0(K)} N_0^{k/2} e^{-ie\phi_0(K)} - N_0^{k/2}] J_t \Psi \\
&= J_0^* e^{-ie\phi_0(K)} \left\{ \left( N_0 - e\phi'_0(K) + \frac{e^2}{2} \xi \right)^{k/2} - N_0^{k/2} \right\} J_t \Psi \\
&= -J_t N^{k/2} \Psi + J_0^* e^{-ie\phi_0(K)} \left\{ \left( N_0 - e\phi'_0(K) + \frac{e^2}{2} \xi \right)^{k/2} \right\} J_t \Psi,
\end{aligned}
\]

where \(\phi'_0(K) = i[N_0, \phi_0(K)]\). We see that

\[
\|J_t N^{k/2} \Psi\|_{L^2(Q)} \leq \|N^{k/2} \Psi\|_{L^2(Q)}.
\]

Note that

\[
\|\phi_0(K) \Psi\| \leq \sqrt{2} \xi \|(N_0 + 1)^{1/2} \Psi\|.
\]

Then it is obtained that

\[
\| \left( N_0 - e\phi'_0(K) + \frac{e^2}{2} \xi \right)^k J_t \Psi\|_{L^2(Q)} \leq R_k(\xi) \|(N + 1)^k \Psi\|_{L^2(Q)}
\]

with some polynomial \(R_k(\cdot)\). Then

\[
\|[N^{k/2}, \mathcal{J}_t] \Psi\|_{L^2(Q)} \leq R_k(\xi) \|(N + 1)^{k/2} \Psi\|_{L^2(Q)} + \|N^{k/2} \Psi\|_{L^2(Q)}
\]

\[
\leq (R_k(\xi) + 1) \|(N + 1)^{k/2} \Psi\|_{L^2(Q)}.
\]

Thus the proof is complete.

Proposition 4.5 Let \(1 \leq p \leq \infty\) and \(a \geq 0\). Then there exists a constant \(c_p = c_p(a)\) such that

\[
\sup_{x \in \mathbb{R}^3} \left| \mathbb{E}_x \left( e^{-a \int_0^t V(X_s) ds} f(X_t) \right) \right| \leq c_p \|f\|_{L^p(\mathbb{R}^3)}.
\]

Proof: See [23, Theorem B.1.1].

Lemma 4.6 We see that \(e^{-tH}\) maps \(D(N^{k/2})\) into itself.
Proof: Let \( \Phi, \Psi \in D(N^{k/2}) \). We have
\[
(N^{k/2}\Phi, e^{-tH}\Psi)_{\mathcal{H}} = \int \left( (N^{k/2}\Phi)(x), \mathbb{E}_x^Q \left( e^{-\int_0^t V(X_s)ds} \mathcal{J}_t\Psi(X_t) \right) \right)_{L^2(Q)} dx
\]
\[
= \int \mathbb{E}_x \left\{ (N^{k/2}\Phi(x), \mathcal{J}_t\Psi(X_t))_{L^2(Q)} e^{-\int_0^t V(X_s)ds} \right\} dx.
\]
Then
\[
(N^{k/2}\Phi, e^{-tH}\Psi)_{\mathcal{H}} = \int \mathbb{E}_x \left\{ \left( \Phi(x), \mathcal{J}_tN^{k/2}\Psi(X_t) \right)_{L^2(Q)} e^{-\int_0^t V(X_s)ds} \right\} dx
\]
\[
+ \int \mathbb{E}_x \left\{ \left( \Phi(x), [N^{k/2}, \mathcal{J}_t]\Psi(X_t) \right)_{L^2(Q)} e^{-\int_0^t V(X_s)ds} \right\} dx.
\]
Hence we have by Lemma 4.4
\[
|(N^{k/2}\Phi, e^{-tH}\Psi)_{\mathcal{H}}| \leq \int \mathbb{E}_x \left( e^{-\int_0^t V(X_s)ds} \| \Phi(x) \|_{L^2(Q)} \| N^{k/2}\Psi(X_t) \|_{L^2(Q)} \right) dx \tag{4.11}
\]
\[
+ \int \mathbb{E}_x \left( P_k(\xi)e^{-\int_0^t V(X_s)ds} \| \Phi(x) \|_{L^2(Q)} \| (N + 1)^{k/2}\Psi(X_t) \|_{L^2(Q)} \right) dx. \tag{4.12}
\]
The first term (4.11) is estimated as
\[
(4.11) = \left( \| \Phi(\cdot) \|_{L^2(Q)}, e^{-tH_p} \| N^{k/2}\Psi(\cdot) \|_{L^2(Q)} \right)_{L^2(\mathbb{R}^3)} \leq e^{-tE_p} \| \Phi \|_{\mathcal{H}} \| N^{k/2}\Psi \|_{\mathcal{H}},
\]
where \( E_p = \inf \sigma(H_p) \). The second term (4.12) is estimated as
\[
(4.12) \leq \int \| \Phi(x) \|_{L^2(Q)} \times
\]
\[
\times \left( \mathbb{E}_x P_k(\xi)^2 e^{-2\int_0^t V(X_s)ds} \right)^{1/2} \left( \mathbb{E}_x \| (N + 1)^{k/2}\Psi(X_t) \|_{L^2(Q)}^2 \right)^{1/2} dx
\]
\[
\leq \int \| \Phi(x) \|_{L^2(Q)} \left( \mathbb{E}_x P_k(\xi)^4 \right)^{1/4} \times
\]
\[
\times \left( \mathbb{E}_x e^{-4\int_0^t V(X_s)ds} \right)^{1/4} \left( \mathbb{E}_x \| (N + 1)^{k/2}\Psi(X_t) \|_{L^2(Q)}^2 \right)^{1/2} dx.
\]
By Lemma 4.3 we have
\[
\theta = \sup_{x \in \mathbb{R}^3} \left( \mathbb{E}_x P_k(\xi)^4 \right)^{1/4} < \infty,
\]
and by (4.10),
\[
\eta = \sup_{x \in \mathbb{R}^3} \left( \mathbb{E}_x e^{-4\int_0^t V(X_s)ds} \right)^{1/4} < \infty.
\]
Then we have
\[
(4.12) \leq \theta \eta \int \| \Phi(x) \|_{L^2(Q)} \left( \mathbb{E}_x \| (N + 1)^{k/2}\Psi(X_t) \|_{L^2(Q)}^2 \right)^{1/2} dx
\]
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Thus we conclude that
\[ |(N^{k/2} \Phi, e^{-tH} \Psi)_H| \leq \| \Phi \|_H \left( e^{-tE_0} \| N^{k/2} \Psi \|_H + \theta \eta \| (N + 1)^{k/2} \Phi \|_H \right). \]
This implies that \( e^{-tH} \Psi \in D(N^{k/2}) \).

**Lemma 4.7** Assume that \( \psi_g \in D(N^{k/2}) \). Then \( \sup_{x \in \mathbb{R}^3} \| N^{k/2} \psi_g(x) \|_{L^2(Q)} < \infty \).

**Proof:** By Lemma 4.6 the identity \( N^{k/2} \psi_g = e^{tE} e^{-tH} N^{k/2} \psi_g + e^{tE} [N^{k/2}, e^{-tH}] \psi_g \) is well defined, and we obtained that
\[ N^{k/2} \psi_g(x) = e^{tE} \mathbb{E}_x \left( e^{-\int_0^t V(x_s) ds} \mathcal{J}_t N^{k/2} \psi_g(X_t) \right) + e^{tE} \mathbb{E}_x \left( e^{-\int_0^t V(x_s) ds} [N^{k/2}, \mathcal{J}_t] \psi_g(X_t) \right) \]
for almost every \( x \in \mathbb{R}^3 \). We see that by Lemma 4.4
\[ \| N^{k/2} \psi_g(x) \|_{L^2(Q)} \leq e^{tE} \mathbb{E}_x \left( e^{-\int_0^t V(x_s) ds} \| N^{k/2} \psi_g(X_t) \|_{L^2(Q)} \right) + e^{tE} \mathbb{E}_x \left( e^{-\int_0^t V(x_s) ds} P_k(\xi) \| (N + 1)^{k/2} \psi_g(X_t) \|_{L^2(Q)} \right). \]
By (4.10) it is obtained that
\[ \sup_{x \in \mathbb{R}^3} (4.13) < \infty. \] (4.15)
(4.14) is estimated as
\[ (4.14) \leq \left( \mathbb{E}_x P_k(\xi)^2 \right)^{1/2} \left( \mathbb{E}_x e^{-\int_0^t V(x_s) ds} \| (N + 1)^{k/2} \psi_g(X_t) \|_{L^2(Q)}^2 \right)^{1/2}. \]
By (4.10) we yield that
\[ \sup_{x \in \mathbb{R}^3} \mathbb{E}_x \left( e^{-\int_0^t V(x_s) ds} \| (N + 1)^{k/2} \psi_g(X_t) \|_{L^2(Q)}^2 \right) < \infty, \]
and by Lemma 4.3, \( \sup_{x \in \mathbb{R}^3} \mathbb{E}_x (P_k(\xi)^2) < \infty \). Hence
\[ \sup_{x \in \mathbb{R}^3} (4.14) < \infty. \] (4.16)
Thus the lemma follows from (4.15) and (4.16). \( \square \)
Proof of Lemma 2.11

It is enough to prove the lemma for sufficiently large $|x|$ by Lemma 4.7. Set $\theta = \sup_{x \in \mathbb{R}^3} \|(N + 1)^{k/2} \psi_g(x)\|_{L^2(Q)} < \infty$. We have by (4.13) and (4.14) for almost every $x \in \mathbb{R}^3$

$$\|N^{k/2} \psi_g(x)\|_{L^2(Q)} \leq E_x \left( e^{-\int_0^t V(X_s)ds} (1 + P_k(\xi)) \right) e^{tE} \theta$$

$$\leq \left\{ E_x \left( (1 + P_k(\xi))^2 \right) \right\}^{1/2} \left( E_x e^{-2 \int_0^t V(X_s)ds} \right)^{1/2} e^{tE} \theta.$$  

By (4.9) we have

$$E_x \left( (1 + P_k(\xi))^2 \right) \leq Q_k(t),$$

where $Q_k$ is some polynomial of the same degree as $P_k$. Then we have

$$\|N^{k/2} \psi_g(x)\|_{L^2(Q)} \leq \theta Q_k(t) e^{tE} E_x \left( e^{-2 \int_0^t V(X_s)ds} \right).$$

Here $t$ is arbitrary. Take $t = t(x) = |x|^{1-m}$. Then by [7] we see that there exist positive constants $D$ and $\delta$ such that for sufficiently large $|x|$, 

$$e^{t(x)E} E_x \left( e^{-2 \int_0^{t(x)} V(X_s)ds} \right) \leq De^{-\delta|x|^{m+1}}.$$  

In the case of $m \geq 1$ it is trivial that $Q_k(t(x)) \leq \theta'$ with some constant $\theta'$ independent of $x$. Hence

$$\|N^{k/2} \psi_g(x)\|_{L^2(Q)} \leq \theta' De^{-\delta|x|^{m+1}}$$

follows for sufficiently large $|x|$. Thus the lemma follows for $m \geq 1$. In the case of $m = 0$, we see that $\|N^{k/2} \psi_g(x)\|_{L^2(Q)} \leq \theta Q_k(|x|) De^{-\delta|x|}$, and hence

$$\|N^{k/2} \psi_g(x)\|_{L^2(Q)} \leq \theta D' e^{-\delta'|x|}$$

follows for $\delta' < \delta$ with some constant $D'$ for sufficiently large $|x|$. The lemma is complete. \hfill $\Box$

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References

[1] A. Arai, Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation, Rev. Math. Phys. 13 (2001), 1075–1094.
[2] A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized
spin-boson model, J. Funct. Anal. 151 (1997), 455–503.
[3] A. Arai, M. Hirokawa, and F. Hiroshima, On the absence of eigenvectors of Hamiltonians in
a class of massless quantum field models without infrared cutoff, J. Funct. Anal. 168 (1999),
470–497.
[4] V. Bach, J. Fröhlich, I. M. Sigal, Quantum electrodynamics of confined nonrelativistic particles,
Adv. Math. 137 (1998), 299–395.
[5] V. Bach, J. Fröhlich, I. M. Sigal, Spectral analysis for systems of atoms and molecules coupled
to the quantized radiation field, Commun. Math. Phys. 207 (1999), 249–290.
[6] V. Betz, F. Hiroshima, J. Lörinczi, R. A. Minlos and H. Spohn, Gibbs measure associated with
particle-field system, Rev. Math. Phys., 14 (2002), 173–198.
[7] R. Carmona, Pointwise bounds for Schrödinger operators, Commun. Math. Phys. 62 (1978),
97–106.
[8] J. Fröhlich, M. Griesemer and B. Schlein, Asymptotic electromagnetic fields in a mode of
quantum-mechanical matter interacting with the quantum radiation field, Adv. in Math. 164
(2001), 349–398.
[9] C. Gérard, On the existence of ground states for massless Pauli-Fierz Hamiltonians, Ann. Henri
Poincaré 1 (2000), 443–459.
[10] M. Griesemer, E. Lieb and M. Loss, Ground states in non-relativistic quantum electrodynamics,
Invent. Math. 145 (2001), 557–595.
[11] L. Gross, The relativistic Polaron without cutoffs, Commun. Math. Phys. 31 (1973), 25–73.
[12] F. Hiroshima, Functional integral representation of a model in quantum electrodynamics, Rev.
Math. Phys. 9 (1997), 489–530.
[13] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics I, J. Math.
Phys. 40 (1999), 6209–6222, II, J. Math. Phys. 41 (2000), 661–674.
[14] F. Hiroshima, Essential self-adjointness of translation-invariant quantum field models for arbi-
trary coupling constants, Commun. Math. Phys. 211 (2000), 585–613.
[15] F. Hiroshima, Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling
constants, Ann. Henri Poincaré, 3 (2002), 171–201.
[16] F. Hiroshima, Analysis of ground states of atoms interacting with a quantized radiation fields,
to be published in Int. J. Mod. Phys. B.
[17] F. Hiroshima and H. Spohn, Enhanced binding through coupling to a quantum field, Ann. Henri
Poincaré 2 (2001), 1159–1187.
[18] F. Hiroshima and H. Spohn, Ground state degeneracy of the Pauli-Fierz model with spin, Adv.
Theor. Math. Phys. 5 (2001), 1091–1104.
[19] C. Hainzl, V. Vougalter and S. A. Vugalter, Enhanced binding in non-relativistic QED, mp-arc
01-455, preprint, 2001.
[20] J. Lörinczi, R. A. Minlos and H. Spohn, The infrared behaviour in Nelson’s model of a quantum
particle coupled to a massless scalar field, Ann. Henri Poincaré 3 (2001), 1–28.
[21] J. Lörinczi, R. A. Minlos and H. Spohn, Infrared regular representation of the three dimensional
massless Nelson model, Lett. Math. Phys. 59 (2002), 189–198.
[22] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* 5 (1964), 1190–1197.

[23] B. Simon, Schrödinger semigroups, *Bull. Amer. Math. Soc.* 7 (1982), 447–526. *J. Funct. Anal.* 32 (1979), 97–101.

[24] A. Sloan, The polaron without cutoffs in two space dimensions, *J. Math. Phys.* 15 (1974), 190–201.

[25] H. Spohn, Ground state of quantum particle coupled to a scalar boson field, *Lett. Math. Phys.* 44 (1998), 9–16.

[26] H. Spohn, Ground state(s) of the spin-boson Hamiltonian, *Commun. Math. Phys.* 123 (1989), 277–304.