ON A CONSTANT CURVATURE STATISTICAL MANIFOLD

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ABSTRACT. We will show that a statistical manifold \((M, g, \nabla)\) has a constant curvature if and only if it is a projectively flat conjugate symmetric manifold, that is, the affine connection \(\nabla\) is projectively flat and the curvatures satisfies \(R = R^\ast\), where \(R^\ast\) is the curvature of the dual connection \(\nabla^\ast\). Moreover, we will show that properly convex structures on a projectively flat compact manifold induces constant curvature \(-1\) statistical structures and vice versa.

INTRODUCTION

On a pseudo-Riemannian manifold \((M, g)\), consider a torsion-free affine connection \(\nabla\) on \((M, g)\) and a \((0, 3)\)-tensor field \(C\) defined by
\[
C(X, Y, Z) = (\nabla_X g)(Y, Z).
\]
This pair \((g, \nabla)\) is called a statistical structure if \(C\) is totally symmetric, and the tensor \(C\) will be called the cubic form or the cubic tensor. A pseudo-Riemannian manifold \((M, g)\) with a statistical structure \((g, \nabla)\) will be called the statistical manifold and it will be denoted by a triad \((M, g, \nabla)\). It is important to consider the dual torsion-free affine connection \(\nabla^\ast\) for a statistical manifold \((M, g, \nabla)\) defined by
\[
Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^\ast_X Z).
\]
Note that \(\nabla = \nabla^\ast\) if and only if \(\nabla\) is the Levi-Civita connection of the metric \(g\). Thus the statistical manifold is a natural generalization of a pseudo-Riemannian manifold with the Levi-Civita connection, which gives a trivial statistical structure. For the theory of statistical manifolds, we refer the readers to [2].

For a statistical manifold \((M, g, \nabla)\) let \(R\) denote the curvature tensor field of \(\nabla\), and it is said to be of constant curvature \(k\) if
\[
R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}
\]
holds for any vector fields \(X, Y\) and \(Z\) and some real constant \(k\), see [11]. The statistical manifold \((M, g, \nabla)\) is said to be conjugate symmetric if
\[
R = R^\ast
\]
holds, where \(R^\ast\) is the curvature of \(\nabla^\ast\), see [13]. Moreover, it is well known that projective flatness is a fundamental notion in projective differential geometry [5]. The projective...
flatness is defined on the equivalence classes of affine connections on a manifold $M$, see Definition 1.6. A torsion-free projective flat connection $\nabla$ on a manifold defines a projective structure and it has been called a $\mathbb{RP}^n$-structure.

In this paper, we will first show that a statistical manifold is constant curvature if and only if it is a projectively flat conjugate symmetric manifold (or a projectively flat conjugate Ricci symmetric manifold, that is, $\text{Ric} = \text{Ric}^\ast$), Theorem 2.6. Moreover, we will discuss a natural family of affine connections, the so-called $\alpha$-connections introduced in information geometry [2] and will show that almost all members of $\alpha$-connections are not constant curvature, Proposition 2.8.

Next we recall the basic definitions of properly convex $\mathbb{RP}^n$-structures, see [8, 12] for more details. For a $\mathbb{RP}^n$-structure (induced from a projectively flat connection $\nabla$) on a manifold $M$, the holonomy representation $\rho$ can be defined, which is a map from the fundamental group $\pi_1(M)$ to the projective group $\text{PSL}_{n+1}\mathbb{R}$. Accordingly the developing map follows, which is a local diffeomorphism $f$ from the universal cover $\tilde{M}$ taking values in $\mathbb{RP}^n$ such that it is $\rho$-equivariant, that is, for any $x \in \tilde{M}$ and any $\gamma \in \pi_1(M)$, $f(\gamma x) = \rho(\gamma)f(x)$ holds. Finally a $\mathbb{RP}^n$-structure is called convex if the developing map is a homeomorphism to a convex set in $\mathbb{RP}^n$ and it is called properly convex if this convex set is included in a compact convex set of an affine chart. It has been known that (properly) convex $\mathbb{RP}^n$-structures are fundamental and important geometric structures on a manifold.

We will then characterize the properly convex $\mathbb{RP}^n$-structures on a compact manifold $M$, see Theorem 3.1 that is, such structures can be constructed from constant curvature $-1$ statistical structures on $M$. This theorem is a reformulation of [12, Theorem 3.2.1], in terms of a constant statistical structure on $M$. In Appendix A, we will give a sketch of the proof of Theorem 3.1.

It is known that notion of statistical manifolds has also grown out of theory of affine immersions. The Codazzi equation of an affine immersion defines a statistical structure, and many of studies of statistical manifolds relate to it, for example, constant curvature statistical manifolds give affine spheres, see [11]. However, in this paper, we emphasize an intrinsic characterization of a constant curvature statistical manifold in terms of properties of the affine connection $\nabla$ and its curvature without any relation to affine immersions. As a result of it, we can see a direct connection between constant curvature statistical structures and properly convex $\mathbb{RP}^n$-structures on a manifold.

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1. Tensor analysis for a statistical manifold

In this section, we recall the basic facts about a statistical manifold, see for examples [2, 4, 17, 16, 3, 20].
1.1. **Preliminaries.** Let $\nabla$ be a torsion-free affine connection on a pseudo-Riemannian manifold $(M, g)$ and let $\nabla^*$ be the dual torsion-free affine connection in the sense of [0.2]. Then it is easy to see that

$$-C(X, Y, Z) = (\nabla^*_X g)(Y, Z)$$

holds and thus $(g, \nabla)$ is a statistical structure if and only if $(g, \nabla^*)$ is. For a statistical manifold $(M, g, \nabla)$, we define a tensor field $K$ of type $(1, 2)$ by

$$(1.1) \quad K(X, Y) = \nabla_X Y - \hat{\nabla}_X Y$$

and it will be called the difference tensor, where $\hat{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian metric $g$. Moreover, defining the $(1, 1)$-tensor $K_X$ by

$$K_X Y = K(X, Y),$$

we have

$$(1.2) \quad K_X = \nabla_X - \hat{\nabla}_X.$$

Since $\nabla$ and $\hat{\nabla}$ are torsion-free, thus $K(X, Y)$ is symmetric. Then from the compatibility of $g$, that is $\hat{\nabla} g = 0$, we have $(\nabla_X g)(Y, Z) = (K_X g)(Y, Z)$. We also compute

$$(K_X g)(Y, Z) = -g(K_X Y, Z) - g(Y, K_X Z).$$

From the definition of $C$ in [0.1] and the symmetry of $C$, we have

$$(1.3) \quad C(X, Y, Z) = -2g(K(X, Y), Z).$$

The relation $-C(X, Y, Z) = (\nabla^*_X g)(Y, Z)$ also implies

$$(1.4) \quad K_X = -\nabla^*_X + \hat{\nabla}_X.$$

Therefore the Levi-Civita connection $\hat{\nabla}$ is the mean of $\nabla$ and its dual $\nabla^*$:

$$\hat{\nabla}_X Y = \frac{1}{2}(\nabla_X Y + \nabla^*_X Y).$$

Moreover, from $[1.1]$ and $\hat{\nabla} g = 0$, it is easy to see that

$$(1.5) \quad (\hat{\nabla}_X C)(Y, Z, W) = -2g((\hat{\nabla}_X K)(Y, Z), W)$$

holds. We now characterize the total symmetry of the covariant derivative $\hat{\nabla} C$ of the cubic form $C$ as follows.

**Lemma 1.1** (Lemma 1 in [3]). For a statistical manifold $(M, g, \nabla)$, the followings are mutually equivalent:

1. $\nabla C$ is totally symmetric.
2. $\hat{\nabla} C$ is totally symmetric.
3. $\hat{\nabla} K$ is totally symmetric.

**Remark 1.2.** Lemma 1.1 has been proved in [3] for the statistical structure induced by Blaschke hypersurfaces of affine differential geometry. The proof can be easily generalized to any statistical manifold. In statistical manifolds point of view, Blaschke hypersurfaces correspond to the trace-free structures of difference tensors $K$, see [1.14].

Let $R$ denote the curvature tensor field of the connection $\nabla$, that is,

$$(1.6) \quad R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$
and let $R^*$ denote the curvature tensor field of its dual connection $\nabla^*$. Moreover, let $\hat{R}$ denote the curvature tensor field of the Levi-Civita connection $\hat{\nabla}$.

**Lemma 1.3** (Proposition 9.1 in [16]). For a statistical manifold $(M, g, \nabla)$ the following identities hold:

\begin{align}
R(X, Y) &= \hat{R}(X, Y) + (\hat{\nabla}_X K)_Y - (\hat{\nabla}_Y K)_X + [K_X, K_Y], \\
&= \hat{R}(X, Y) + (\nabla_X K)_Y - (\nabla_Y K)_X - [K_X, K_Y], \\
R^*(X, Y) &= \hat{R}(X, Y) - (\hat{\nabla}_X K)_Y + (\hat{\nabla}_Y K)_X + [K_X, K_Y], \\
&= \hat{R}(X, Y) - (\nabla_X K)_Y + (\nabla_Y K)_X + 3[K_X, K_Y].
\end{align}

Moreover, the following identities also hold:

\begin{align}
\frac{1}{2} R(X, Y) - \frac{1}{2} R^*(X, Y) &= (\hat{\nabla}_X K)_Y - (\hat{\nabla}_Y K)_X, \\
&= (\nabla_X K)_Y - (\nabla_Y K)_X - 2[K_X, K_Y],
\end{align}

\begin{align}
\frac{1}{2} R(X, Y) + \frac{1}{2} R^*(X, Y) &= \hat{R}(X, Y) + [K_X, K_Y].
\end{align}

We now assume that $M$ is orientable and take the pseudo-Riemannian volume form $\omega_g$ on $(M, g)$:

$$\omega_g = \sqrt{|\det g|} \, dx_1 \wedge \cdots \wedge dx_n.$$  

Then it is easy to see that the covariant derivative $\nabla_X$ of $\omega_g$ as

$$\nabla_X \omega_g = \frac{1}{2} \text{tr}_g (\nabla_X g)(\cdot, \cdot) \omega_g.$$  

From the symmetry of $C$ we see that $\text{tr}_g (\nabla_X g)(\cdot, \cdot) = \text{tr}_g (\nabla g)(\cdot, X)$, and the relation (1.3) and the self-adjointness of $K$ imply that

$$\frac{1}{2} \text{tr}_g (\nabla g)(\cdot, X) = -g(K(\cdot, \cdot), X) = -\text{tr} K_X,$$

and therefore we have

$$\nabla_X \omega_g = -\tau_g(X) \omega_g \quad \text{with} \quad \tau_g(X) = \text{tr} K_X.$$

The Ricci curvature tensor $\text{Ric}$ of $\nabla$ is defined by

$$\text{Ric}(Y, Z) = \text{tr} \{ X \mapsto R(X, Y)Z \}.$$  

Similarly, the Ricci curvature tensor $\text{Ric}^*$ (resp. $\hat{\text{Ric}}$) of $\nabla^*$ (resp. $\hat{\nabla}$) can be defined analogously.

**Lemma 1.4** (Section 3 in [18]). For an orientable statistical manifold $(M, g, \nabla)$ with the 1-form $\tau_g$ in (1.11), the following identities hold:

$$\text{Ric}(Y, Z) = \hat{\text{Ric}}(Y, Z) + (\text{div} \hat{\nabla} K)(Y, Z) - (\hat{\nabla}_Y \tau_g)(Z) + \tau_g(K_Y Z) - g(K_Y, K_Z),$$

$$\text{Ric}^*(Y, Z) = \hat{\text{Ric}}^*(Y, Z) + (\text{div} \hat{\nabla} K)(Y, Z) - (\hat{\nabla}_Y \tau_g)(Z) - \tau_g(K_Y Z) + g(K_Y, K_Z),$$

$$\text{Ric}^*(Y, Z) = \hat{\text{Ric}}^*(Y, Z) - (\text{div} \hat{\nabla} K)(Y, Z) + (\hat{\nabla}_Y \tau_g)(Z) + 3\tau_g(K_Y Z) - 3g(K_Y, K_Z).$$
Moreover, the following identities also hold:

\[ \frac{1}{2} \text{Ric}(Y, Z) - \frac{1}{2} \text{Ric}^*(Y, Z) = (\text{div}^\nabla K)(Y, Z) - (\nabla_Y \tau_g)(Z), \]
\[ = (\text{div}^\nabla K)(Y, Z) - (\nabla_Y \tau_g)(Z) - 2\tau_g(K_Y Z) + 2g(K_Y, K_Z), \]
\[ (1.12) \]
\[ \frac{1}{2} \text{Ric}(Y, Z) + \frac{1}{2} \text{Ric}^*(Y, Z) = \widehat{\text{Ric}}(Y, Z) + \tau_g(K(Y, Z)) - g(K_Y, K_Z), \]
\[ (1.13) \]

From Lemma 1.4, it is clear that the Ricci curvature of a statistical manifold is not symmetric in general. We now recall that a torsion-free affine connection is called \textit{locally equiaffine} (resp. \textit{equiaffine}) if there exists a volume form \( \omega \) around any point on \( M \) (resp. a volume form \( \omega \) on \( M \)) such that \( \nabla \omega = 0 \). In particular if the volume form is the pseudo-Riemannian volume form \( \omega_g \), then \( \nabla \omega_g = 0 \) is equivalent to
\[ \text{tr} K_X = 0, \]
and the condition (1.14) will be called \textit{trace-free}.

\textbf{Corollary 1.5.} The Ricci curvature of the connection \( \nabla \) of a statistical manifold \( (M, g, \nabla) \) is symmetric if and only if it is locally equiaffine.

We next recall the projective equivalence of affine connections.

\textbf{Definition 1.6.} Two torsion-free locally equiaffine connections \( \nabla \) and \( \widehat{\nabla} \) on \( M \) are called \textit{projectively equivalent} if there exists a closed 1-form \( \rho \) such that
\[ \nabla_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X \]
holds. In particular when \( \widehat{\nabla} \) is flat, then \( \nabla \) is called \textit{projectively flat}. Moreover the \textit{projective curvature tensor} \( P \) is defined as
\[ P(X, Y)Z = R(X, Y)Z - \frac{1}{n - 1} \left\{ \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y \right\}. \]
\[ (1.15) \]

It is classically known that that the projectively flat connections has been characterized as follows:

\textbf{Theorem 1.7} (p.88 and p.96 in \[5\]). A torsion-free locally equiaffine connection \( \nabla \) on \( M \) is projectively flat if and only if the following condition holds:

\begin{enumerate}
  \item If \( \text{dim} M = 2 \), \( \nabla \text{Ric} \) is totally symmetric. In this case the projective curvature tensor \( P \) automatically vanishes.
  \item If \( \text{dim} M \geq 3 \), the projective curvature tensor \( P \) identically vanishes. In this case \( \nabla \text{Ric} \) is automatically totally symmetric.
\end{enumerate}

We also collect the basic identities for curvatures by a straight forward computation.

\textbf{Lemma 1.8.} The following identities hold:

\begin{enumerate}
  \item \( R(X, Y) = -R(Y, X) \) and \( R^*(X, Y) = -R^*(Y, X) \).
  \item \( g(R(X, Y)Z, W) = -g(Z, R^*(X, Y)W) \).
  \item \( R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \) (1st Bianchi identity),
  \item \( (\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0 \) (2nd Bianchi identity).
\end{enumerate}
2. Characterization of constant curvature statistical manifolds

In this section, we characterize constant curvature statistical manifolds in terms of various conjugate symmetries and the projective flatness of the dual connection.

2.1. Conjugate symmetries. We first define various conjugate symmetries for a statistical manifold.

**Definition 2.1.** Let \((M, g, \nabla)\) be a statistical manifold, and let \(R\) and \(R^*\) denote curvatures of the connection \(\nabla\) and its dual connection \(\nabla^*\), respectively. Moreover, let \(\text{Ric}\) and \(\text{Ric}^*\) denote the Ricci curvatures of \(\nabla\) and \(\nabla^*\), respectively. Then \((M, g, \nabla)\) will be respectively called a self-dual, a conjugate symmetric or a conjugate Ricci-symmetric if

\[
\nabla = \nabla^*, \quad R = R^* \quad \text{or} \quad \text{Ric} = \text{Ric}^*
\]

holds.

Note that notion of “self-dual” and “conjugate symmetric” had been introduced in \([1]\) and \([13]\), respectively. By (0.2), a self-dual manifold is nothing but \(\nabla\) is the Levi-Civita connection of \(g\). Moreover, by (0.1) and (1.2), the conditions \(\nabla = \nabla^*, \ C = 0 \) and \(K = 0\) are mutually equivalent. From now on we always assume that a manifold \(M\) is orientable. Then we have the following.

**Lemma 2.2.** The condition \(\nabla = \nabla^*\) (resp. \(R = R^*\)) of a statistical manifold \((M, g, \nabla)\) implies \(R = R^*\) (resp. \(\text{Ric} = \text{Ric}^*\)). Moreover, \(\text{Ric} = \text{Ric}^*\) implies \(\text{Ric}(X, Y) = \text{Ric}(Y, X)\).

**Proof.** The first statements follow immediately from the definition. We now assume \(\text{Ric} = \text{Ric}^*\). From the formulas in Lemma 1.4, we have

\[
0 = \text{Ric}(Y, Z) - \text{Ric}^*(Y, Z) = 2(\text{div} \nabla^* K)(Y, Z) - 2(\nabla^* \tau_g)(Z).
\]

In particular, \((\nabla^* \tau_g)(Z)\) is symmetric with respect to \(Y\) and \(Z\), and thus \(d\tau_g(Y, Z) = (\nabla^* \tau_g)(Z) - (\nabla^* \tau_g)(Y) = 0\) follows. From the formulas in Lemma 1.4 and symmetries of \(\nabla \tau_g(Y, Z)\) and \((\text{div} \nabla^* K)(Y, Z) + \tau_g(K(Y, Z)) - g(K_Y, K_Z)\) with respect to \(Y\) and \(Z\), it is easy to see that

\[
\text{Ric}(Y, Z) - \text{Ric}(Z, Y) = -(\nabla^* \tau_g)(Z) + (\nabla^* \tau_g)(Y) = -d\tau_g(Y, Z).
\]

This completes the proof.

From (2.2), we have the following corollary.

**Corollary 2.3.** The followings are mutually equivalent:

1. \(\text{Ric} = \text{Ric}^*\).
2. \((\text{div} \nabla^* K)(Y, Z) = (\nabla^* \tau_g)(Z)\).

We next characterize the conjugate symmetry of the curvature \(R\).

**Proposition 2.4.** The followings are mutually equivalent:

1. \(R = R^*\).
(2) $\nabla C$ is totally symmetric.
(3) $\hat{\nabla} C$ is totally symmetric.
(4) $\hat{\nabla} K$ is totally symmetric.

Proof. The equivalences of (2), (3) and (4) follow from Lemma 1.1. The equivalence of (1) and (4) follows from the formula in (1.9). □

2.2. Characterization of constant curvature statistical manifolds. The notion of constant curvature statistical manifold was introduced in [11], see (0.3). Then the following lemma (Schur's lemma) can be used for the characterization of a constant curvature statistical manifold.

Lemma 2.5. Let $S$ be the $(1,1)$-tensor $S$ defined by $\text{Ric}(X, Y) = (n - 1)g(SX, Y)$. Assume that $\text{Ric}$ and $\nabla \text{Ric}$ are totally symmetric. Then if there exists a smooth function $\lambda$ such $SY = \lambda Y$ for any vector field $Y$, then $\lambda$ is constant.

Proof. Using the definition of $\nabla_X \text{Ric}$ and the symmetry of $\text{Ric}$ we compute

$$(\nabla_X \text{Ric})(Y, Z) = X \text{Ric}(Y, Z) - \text{Ric}(\nabla_X Y, Z) - \text{Ric}(Y, \nabla_X Z),$$

$$= (n - 1)\left\{Xg(SY, Z) - g(SY, \nabla_X Z) - g(SZ, \nabla_X Y)\right\},$$

$$= (n - 1)\left\{g(\nabla_X^*(\lambda Y), Z) - g(\lambda Z, \nabla_X Y)\right\}.$$

By using the total symmetry of $\nabla \text{Ric}$ we compute

$$0 = \frac{1}{n - 1}(\nabla_X \text{Ric})(Y, Z) - \frac{1}{n - 1}(\nabla_Y \text{Ric})(X, Z),$$

$$= g(\nabla_X^*(\lambda Y), Z) - g(\lambda Z, \nabla_X Y) - g(\nabla_Y^*(\lambda X), Z) + g(\lambda Z, \nabla_Y X),$$

$$= g((X\lambda)Y - (Y\lambda)X, Z).$$

Thus we obtain $(X\lambda)Y - (Y\lambda)X = 0$ and $X\lambda = 0$ follows, that is, $\lambda$ is constant. □

We now characterize a constant curvature statistical manifold.

Theorem 2.6. For a statistical manifold $(M, g, \nabla)$, the followings are mutually equivalent:

1. It has a constant curvature.
2. It is a projectively flat conjugate symmetric manifold.
3. It is a projectively flat conjugate Ricci symmetric manifold.

Proof. (1) $\Rightarrow$ (2): From the formula (2) in Lemma 1.8 and the constancy of the curvature, that is,

$$R(X, Y)Z = k\left\{g(Y, Z)X - g(X, Z)Y\right\}$$

for some constant $k$, we have

$$g(R^*(X, Y)Z, W) = -g(Z, R(X, Y)W),$$

$$= -g(k\left\{g(Y, W)X - g(X, W)Y\right\}, Z),$$

$$= g(k\left\{g(Y, Z)X - g(X, Z)Y\right\}, W),$$

$$= g(R(X, Y)Z, W).$$
Since $W$ is arbitrary, thus the conjugate symmetry of the curvature $R$ follows.

The Ricci curvature for a constant curvature statistical manifold is easily computed as
\[(2.4)\quad Ric(Y, Z) = (n-1)kg(Y, Z),\]
and therefore the projective curvature $P$ in $(1.15)$ vanishes. Thus for $n \geq 3$, $\nabla$ is projectively flat, see Theorem $1.7$. Since $R = R^*$, $\nabla^*$ is also projectively flat.

For $n = 2$, the projective flatness of $\nabla$ is equivalent to the total symmetry of $\nabla^* Ric$, however from $(2.4)$ it is equivalent to the total symmetry of $\nabla^* g$, which is of course true by the definition of the statistical manifold $(M, g, \nabla)$. Then the projective flatness of $\nabla^*$ follows from the duality.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1): We define a $(1,1)$-tensor $S$ by $Ric(X,Y) = (n-1)g(SX,Y)$, then from the symmetry of $Ric$, $g(SX,Y) = g(X,SY)$ follows. Moreover, the projective flatness of $\nabla$ implies that the projective curvature tensor $P$ vanishes, that is,
\[(2.5)\quad R(X,Y)Z = g(SY,Z)X - g(SX,Z)Y.\]
By using the formula (2) in Lemma $1.8$ and the equation above, we compute
\[
g(R^*(X,Y)Z, W) = -g(Z, R(X,Y)W),
= -g(Z, g(SY, W)X - g(SX, W)Y),
= g(g(Z, Y)SX - g(Z, X)SY, W).\]
Since $W$ is an arbitrary, thus we have
\[(2.6)\quad R^*(X,Y)Z = g(Y, Z)SX - g(X, Z)SY.\]
Taking the trace with respect to $X$ for the above equation, we have
\[(2.7)\quad Ric^*(Y, Z) = g(Y, Z) \text{tr} S - g(SY, Z).\]
Then the conjugate symmetry of $Ric$ implies that $ng(Y, SZ) = g(Y, \text{tr} S \cdot Z)$, thus $S = (\text{tr} S)\text{id}$ follows, and it is equivalent to
\[(2.8)\quad SY = \lambda Y,\]
where $\lambda = \text{tr} S/n$. The projective flatness of $\nabla$ also implies that $\nabla^* Ric$ is totally symmetric. Thus Lemma $2.5$ implies that $\lambda$ is constant and thus $R$ is of constant curvature. □

Remark 2.7. It should be remarked that (1) $\Rightarrow$ (2) part of Theorem 2.6 has been proved in [19, Theorem 2], [15, Theorem 3.3] or [4, Theorem 9.7.2], however, (3) (or (2)) $\Rightarrow$ (1) part has not been proved in those references. Moreover, the equivalence of (2) and (3) have been noted by anonymous referee of the paper [10].

2.3. Conjugate symmetric $\alpha$-connections. For a statistical manifold $(M, g, \nabla)$, it is natural to consider a family of statistical structure $(g, \nabla^\alpha)$ as follows:
\[(2.9)\quad \nabla^\alpha = \hat{\nabla} + \alpha K,\]
where $\alpha$ is a real constant. Note that $\nabla^1 = \nabla$, and from $(1.4)$ we have $\nabla^{-1} = \nabla^*$. Moreover,
\[
\nabla^{-\alpha} = (\nabla^*)^\alpha
\]
The family of connections \( \{ \nabla^\alpha \}_{\alpha \in \mathbb{R}} \) has been called the \( \alpha \)-connections, see [2]. It is easy to see that \((g, \nabla^\alpha)\) is a statistical structure for any \( \alpha \in \mathbb{R} \), and if \( \nabla \) (resp. \( R \) or \( \text{Ric} \)) is conjugate symmetric, then \( \nabla^\alpha \) (resp. \( R^\alpha \) or \( \text{Ric}^\alpha \)) is conjugate symmetric for any \( \alpha \in \mathbb{R} \).

The following proposition is a slight generalization of Proposition 4.1 in [18].

**Proposition 2.8.** Let \((M, g, \nabla)\) be a constant curvature statistical manifold, and let \( \nabla^\alpha \) the \( \alpha \)-connection in (2.9) for \( \alpha \in \mathbb{R} \). Assume that the pseudo-Riemannian metric \( g \) is not constant curvature. Then the statistical manifold \((M, g, \nabla^\alpha)\) does not have constant curvature except \( \alpha = \pm 1 \), that is the cases for \( \nabla^1 = \nabla \) and \( \nabla^{-1} = \nabla^* \).

**Proof.** From Theorem 2.6, the constant curvature property of \((M, g, \nabla)\) is equivalent to the conjugate symmetry of \( R \) and the projective flatness of \( \nabla^* \). By the conjugate symmetry of \( R \), we have

\[
R(X, Y) = \hat{R}(X, Y) + [K_X, K_Y].
\]

Moreover, the projective flatness of \( \nabla^* \) can be written as

\[
R(X, Y)Z = R^*(X, Y)Z = \frac{1}{n-1} \{ \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y \}.
\]

Then since \( \nabla^\alpha = \hat{\nabla} + \alpha K \), we have

\[
R^\alpha(X, Y) = \hat{R}(X, Y) + \alpha^2 [K_X, K_Y] = R(X, Y) + (\alpha^2 - 1)[K_X, K_Y].
\]

It follows that for \( \alpha \neq \pm 1 \)

\[
\hat{R}(X, Y) = \frac{1}{1 - \alpha^2} (R^\alpha(X, Y) - \alpha^2 R(X, Y))
\]

holds. Then the claim follows. \( \Box \)

**Remark 2.9.** The assumption of non-constant curvature of the pseudo-Riemannian metric \( g \) in Proposition 2.8 is a mild restriction. In fact, many Fisher metrics determined from statistics are not constant curvature, for examples, the multivariate normal distributions and the gamma distributions etc, see [13].

We finally give an example of flat statistical manifolds.

**Example 2.10** (Multivariate normal distributions). The density function \( p \) of the multivariate normal distribution is given by

\[
p(x, \mu, \Sigma) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\},
\]

where \( \mu \in \mathbb{R}^n \) is the mean and \( \Sigma \) is the covariance, which takes values in the set of degree \( n \) positive definite matrices. Therefore the set of density functions is parameterized by

\[
\mathcal{N} = \text{Pos}(n, \mathbb{R}) \times \mathbb{R}^n.
\]

Since the affine group \( \text{Aff}(n, \mathbb{R}) = \text{GL}_n \mathbb{R} \ltimes \mathbb{R}^n \) acts transitively on \( \mathcal{N} \), it can be represented by a homogeneous space \( \text{Aff}(n, \mathbb{R})/\text{Stab} \), where \( \text{Stab} \) denotes the stabilizer at a base point,
see [7], thus $\mathcal{N}$ has a manifold structure. We then define the Fisher metric and the cubic tensor of $\mathcal{N}$ as

$$g^F(X, Y) = E[(X \log p)(Y \log p)],$$
$$C(X, Y, Z) = E[(X \log p)(Y \log p)(Z \log p)],$$
where $X, Y, Z$ are tangent vectors of $\mathcal{N}$, and

$$E[f] = \int_{\mathbb{R}^n} f(x)p(x, \mu, \Sigma) \, dx$$

for an integrable function $f$ on $\mathbb{R}^n$, see [2]. For any real constant $\alpha$, we define an affine connection $\nabla^\alpha$ by

$$g^F(\nabla^\alpha_X Y, Z) = g^F(\nabla^g_F X Y, Z) - \frac{\alpha}{2} C(X, Y, Z),$$

where $\nabla^g_F$ denotes the Levi-Civita connection of the Riemannian metric $g^F$. The affine connection $\nabla^\alpha$ has been called the the Amari-Chentsov $\alpha$-connection for the space of normal distributions.

Then a straightforward computation shows that all Christoffel symbols $\Gamma_{ijk}$ are zero for $\alpha = 1$ and by the duality, the flatness of $\nabla^{\pm 1}$ follows.

For $n = 1$, the Fisher metric $g^F$ can be computed as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2s^2} \end{pmatrix},$$

where $\Sigma = s > 0$. It is easy to see that $g^F$ gives a constant curvature hyperbolic metric, thus the Riemannian curvature $\hat{R}$ is constant $-1/2$. Applying the formula in (2.12), we have the constant curvature $-1 - \alpha^2)/2$ for $R^\alpha$.

On the other hand for $n \geq 2$, the metric connection $\nabla^g_F$ does not have constant curvature, thus by Proposition 2.8, the curvature of $R^\alpha$ is not constant except $\alpha = \pm 1$. However since $R^{\pm 1} = 0$, $R^\alpha$ satisfies $R^\alpha = R^{-\alpha}$ for any $\alpha \in \mathbb{R}$.

Remark 2.11.

(1) The dual flat structures on a statistical manifold is particularly important for information geometry, see [2].

(2) It is known that the Fisher metric $g^F$ and the Amari-Chentsov $\alpha$-connection $\nabla^\alpha$ of $\mathcal{N}$ is invariant under a Lie group action of $\text{Aff}(n, \mathbb{R})$ and moreover, a subgroup of $\text{Aff}(n, \mathbb{R})$ acts simply and transively on $\mathcal{N}$. Thus $(\mathcal{N}, g^F, \nabla^\alpha)$ becomes a statistical Lie group, see Definition 1.4.1 in [6].

(3) More generally for a given Lie group $G$ with a left-invariant metric $g$, one can construct a left-invariant affine connection $\nabla$ on $G$ such that the curvature of $\nabla$ is constant, that is, $(G, g, \nabla)$ is a statistical Lie group with constant curvature, as follows: A left-invariant connection $\nabla$ on $G$ is given by values of Christoffel symbols (which are constant) at a base point and by Theorem 2.6 Then constancy of the curvature of $\nabla$ is characterized by $R = R^*$ and the projective flatness of $\nabla$. Moreover $R = R^*$ is equivalent to the total symmetry of $\hat{\nabla} C$ by Proposition 2.4. Thus constancy of the curvature can be determined by algebraic equations among Christoffel symbols. It is not difficult to see that there are many solutions in general.
3. Projectively flat structures and constant curvature statistical structures

In this section we will apply the characterization of constant curvature statistical structures to properly convex $\mathbb{RP}^n$-structures on a compact Riemannian manifold.

In [12], Labourie characterized properly convex $\mathbb{RP}^n$-structures on a compact manifold $M$ by a certain condition, which he called the Condition (E). To explain it, let $g$ be a Riemannian metric on $M$, and let $L = \mathbb{R} \times M$ the trivial bundle over $M$. The Condition (E) is defined as follows:

Condition (E) $\iff$ \begin{align*}
1. & \ \nabla \text{ preserves a volume.} \\
2. & \ \nabla^g \text{ is flat.}
\end{align*}

Here $\nabla$ is a torsion-free affine connection on $M$, and $\nabla^g$ is a connection on $TM \oplus L$ given by

$$\nabla^g_X \left( \begin{array}{c} Z \\ \lambda \end{array} \right) = \left( \begin{array}{c} \nabla_X Z + \lambda X \\ L_X \lambda + g(Z, X) \end{array} \right).$$

It is immediately that $\nabla^g$ is flat if and only if the following conditions hold:

$$\begin{align*}
(3.1) & \ \begin{cases}
\nabla_X g(Y, Z) - \nabla_Y g(X, Z) = 0, \\
R(X, Y)Z - g(X, Z)Y + g(Y, Z)X = 0.
\end{cases}
\end{align*}$$

Then the properly convex $\mathbb{RP}^n$-structures on a compact manifold have been characterized by Condition (E), see Theorem 3.2.1 in [12], which is the following Theorem 3.1.

From the discussion in Introduction, the first condition in (3.1) is just equivalent to that $(\nabla, g)$ defines a statistical structure, see (0.1), and the second condition is equivalent to the constancy of the curvature $R$ with $k = -1$, see (0.3). Moreover, since constancy of the curvature implies symmetry of the Ricci curvature, and by Corollary 1.5 we have automatically the condition (1) in Condition (E). Therefore, Theorem 3.2.1 in [12] can be reformulated as follows:

**Theorem 3.1.** Let $(M, g, \nabla)$ be a compact constant curvature $-1$ statistical manifold such that the metric $g$ is positive definite. Then $\nabla$ is projectively flat and defines a properly convex structure on $M$. Conversely, every properly convex projectively flat structure on $M$ is obtained in this way.

We will give a sketch of the proof in Appendix A.

**Remark 3.2.** The projective flatness of the connection $\nabla$ for a constant curvature statistical manifold $(M, g, \nabla)$ in Theorem 3.1 immediately follows from Theorem 2.6. Moreover, the dual connection $\nabla^*$ of $\nabla$ also defines a properly convex $\mathbb{RP}^n$-structure. This induces a certain duality on the moduli space of properly convex $\mathbb{RP}^n$-structures, see the discussion below for the case of surfaces.

Moreover, Labourie has characterized convex $\mathbb{RP}^2$-structures on a compact surface $\Sigma$ in terms of pairs of a complex structure and a holomorphic cubic differential, [12, Theorem 1.0.2], which is a consequence of Theorems 4.2.1 and 4.1.1. of loc. cit. In the proof of Theorem 4.1.1, on page 1070, Condition (E) can be translated to four conditions

(1) $A(X)$ is symmetric and trace free.
(2) $A(X)Y = A(Y)X$.
(3) $d^\nabla A = 0$.
(4) $R^g(X, Y)Z + [A(X), A(Y)]Z + g(Y, Z)X - g(X, Z)Y = 0$,

where $g$ is a metric on $\Sigma$, $\nabla$ is the Levi-Civita connection and $A(X)Y$ is the difference tensor. In terms of our terminology, $A = K$ and $\nabla = \hat{\nabla}$. The first condition is that the trace-free structure of $K$ in (1.14) and the third condition is the total symmetry of $\hat{\nabla}K$, which follows from the constant curvature property of $\nabla$, see Proposition 2.3. The fourth condition the first formula Lemma 1.3 under constancy of the curvature $R$. Therefore, they can be naturally interpreted by notion on a statistical manifold.

By using Theorem 4.2.1 of [12], it has been shown that there is one-to-one correspondence between a set of triads $(\nabla, \omega, J)$ ($\omega$ is a volume form and $J$ is a complex structure on $\Sigma$) which satisfies Condition (H), see the equation (13) in page 1084, and

$$\text{Rep}_H(\pi_1(\Sigma), \text{PSL}_3\mathbb{R}),$$

where $\text{Rep}_H$ denotes the holonomy representation. In Section 7.1, using Condition (H), a duality in the space of representations has been discussed. It has been shown that a triad $(\nabla, \omega, J)$ satisfies Condition (H) if and only if $(-J\nabla J, \omega, J)$ satisfies it, thus the above correspondence defines a duality on the space of representations. In the statistical structure point view, $-J\nabla J$ is nothing but the dual connection $\nabla^*$ of $\nabla$. In fact for a non-degenerate metric $g$ on $\Sigma$ such that $g$ induces the volume $\omega$, $Xg(Y, Z) = Xg(Z, Y) = Xg(JZ, JY)$ because $g$ is compatible with the complex structure $J$. A straightforward computation shows by using (0.2) that

$$g(\{\nabla_X + J\nabla^*_X J\} Y, Z) + g(Y, J \{\nabla_X + J\nabla^*_X J\} JZ) = 0$$

holds. Thus $\nabla^* = -J\nabla J$ follows, and the dual constant curvature statistical structure naturally induces a duality on the moduli space of representations.

Remark 3.3. In Theorem 4 of [14], Loftin has characterized properly convex structures in terms of affine sphere structures for a given $\mathbb{RP}^n$-manifold. It is evident that an affine sphere structure gives a statistical structure with constant curvature. On the one hand, Theorem 3.1 in this paper has shown that a compact constant $-1$ statistical manifold $M$ is a $\mathbb{RP}^n$-manifold.

Appendix A. A sketch of the proof of Theorem 3.1

As explained in Section 3 property convex $\mathbb{RP}^n$-structures on a compact manifold $M$ have been characterized by Condition (E), and we have rephrased it in terms of constant curvature $-1$ statistical manifold structure on $M$, and we have obtained Theorem 3.1. In this appendix, we will briefly give a proof of this theorem along the proof of Theorem 3.2.1 in [12].

Let us prove the necessary part. Let $\tilde{M}$ be the universal cover of $M$ and consider the bundle $T\tilde{M} \oplus L$, where $L = \mathbb{R} \times \tilde{M}$. Then it is evident that by the flatness of $\nabla^g$ (which is equivalent to the constant curvature $-1$ statistical structure), $T\tilde{M} \oplus L$ is isomorphic to the trivial bundle $\mathbb{R}^{n+1} \times \tilde{M}$. Let us take the projection $p$ from $T\tilde{M} \oplus L \cong \mathbb{R}^{n+1} \times \tilde{M}$ to $\mathbb{R}^{n+1}$. Moreover, let us take the canonical section of $T\tilde{M} \oplus L$ by $u_0 : m \rightarrow (0, 1)$, and define
\[ \phi = p \circ u_0. \] 

Then by the construction, \( \phi \) is a \( \rho \)-equivariant mapping from \( \widetilde{M} \) to \( \mathbb{R}^{n+1} \), where \( \rho \) is the holonomy representation of the flat connection \( \nabla^g \). Then Labourie has proved that \( \phi(M) \) is a locally convex proper hypersurface in a sequence of Propositions, 3.2.2 (immersion), 3.2.3 (strictly locally convex and radial) and 3.2.4 (proper) in [12], respectively. Moreover, the geodesic \( \gamma(t) \) with respect to \( \nabla \) gives a sub-bundle

\[ P = \mathbb{R}(\gamma) \oplus \mathbb{R} \subset TM \oplus L, \]

which is parallel along \( \gamma(t) \). Then \( \phi(\gamma(t)) \) is the projective line defined by \( P \), and \( \nabla \) and \( \nabla^g \) define the same projective flat structure. Therefore the connection \( \nabla \) gives a properly convex \( \mathbb{R}P^n \)-structure on \( M \).

Let us prove the sufficient part. An important step has been proved by Vinberg [21] and see Lemma 3.1.1 in [12]: For a given properly convex \( \mathbb{R}P^n \)-structure on a manifold \( M \) induced by the pair \((f, \rho)\), there exists a proper \( \rho \)-equivariant immersion \( \tilde{f} \) from a universal cover \( \widetilde{M} \) into \( \mathbb{R}^{n+1} \) such that the image is strictly convex and radial and \( \pi \circ \tilde{f} = f \), where \( \pi \) is the projection \( \mathbb{R}^{n+1} \setminus \{0\} \) to \( \mathbb{R}P^n \). Note that a hypersurface is strictly convex means that it does not contain any segment and a hypersurface is radial means that if the vector pointing from the origin points inward. Let \( \Sigma = \tilde{f}(\widetilde{M}) \) be the locally strictly convex hypersurface in \( \mathbb{R}^{n+1} \). Since \( \Sigma \) is radial, thus

\[ T\mathbb{R}^{n+1}|\Sigma = T\Sigma \oplus \mathbb{R}N. \]

The standard flat connection \( \nabla^0 \) on \( \mathbb{R}^{n+1} \) then induces a volume preserving connection \( \nabla \) given as

\[ \nabla^0_X(Z + \lambda N) = \nabla_XZ + \lambda \nabla^0_XN + (L_X\lambda).N + g(Z, X).N \]

for vector fields \( X, Z \) on \( \Sigma \). Since \( \Sigma \) is strictly locally convex and radial, \( \nabla^0_XN = X \) and \( g(X, X) > 0 \). Now the pair \((\nabla, g)\) clearly gives a constant curvature \(-1\) statistical manifold structure on \( M \).

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