Quasi-Normal Mode Expansion for Linearized Waves in Gravitational Systems

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Abstract

The quasinormal modes (QNM’s) of gravitational systems modeled by the Klein-Gordon equation with effective potentials are studied in analogy to the QNM’s of optical cavities. Conditions are given for the QNM’s to form a complete set, i.e., for the Green’s function to be expressible as a sum over QNM’s, answering a conjecture by Price and Husain [Phys. Rev. Lett. 68, 1973 (1992)]. In the cases where the QNM sum is divergent, procedures for regularization are given. The crucial condition for completeness is the existence of spatial discontinuities in the system, e.g., the discontinuity at the stellar surface in the model of Price and Husain.

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Radiation from optical cavities is often analysed in terms of the “modes” of the cavity [1]. Because the waves escape, causing the total energy in the cavity to decrease, these “modes” are quasi-normal modes (QNM’s). The observation of the QNM’s from the outside immediately gives information on the spatial structure of the cavity (but not the spatial structure of the source of radiation inside the cavity), e.g., in obvious notation, $\omega_j \sim j\pi c/L$, where $L$ is the length of a simple 1-dimensional optical cavity.

Similarly, gravitational radiations from relativistic systems can be described by QNM’s. Although “cavities” in gravitational systems, i.e., regions of spacetimes which reflect and scatter waves significantly, tend to be more leaky than optical cavities, we argued that QNM analysis and the optical analogy can still be very useful. Indeed the QNM’s of black holes [2] and relativistic stars [3] have been subjects of much study. In numerical simulations, it is often found [4,5] that the radiation observed in many black hole processes is dominated by the QNM’s of the hole. For processes as violent and non-linear as the head-on collision of two black holes, the domination of the radiation by QNM’s may seem surprising [5]. However, it is easy to understand with the optical analogy — the distant observer sees only the QNM’s of the optical cavity, but not the details of the radiation generation mechanism.

Gravitational QNM radiation may be observed by LIGO and VIRGO [6] in the next decade, and may therefore reveal spacetime structures of various gravitational systems, e.g., the strong field region around a black hole — in much the same way as the spectrum of a laser permits a distant observer to infer some characteristics of the cavity.

This exciting possibility calls for the study of the general properties of QNM’s of gravitational systems. For the interpretation and extraction of observational data, one would like to know, for example, how the QNM frequencies of a black hole are perturbed by a massive accretion disk around it.

A fundamental question, raised by Price and Husain [7], is whether QNM’s can form complete sets. This question is not only of theoretical interest, but also of technical importance, e.g., the application of the usual Rayleigh perturbation theory to calculate frequency shifts hinges on the existence of a complete set. The answer to this question is thought to
be in general negative, as the system giving rise to QNM’s is nonhermitian. Yet Price and Husain [7] have given a model of relativistic stellar oscillations that does have a complete set of QNM’s.

In this Letter, we consider this question in a general context, and give conditions for completeness to hold. The crucial condition is the existence of spatial discontinuities. (In the model of [7], this discontinuity corresponds to the stellar surface [8].) Such a condition is not surprising: a discrete sum of QNM’s can at best be complete over a finite interval; the discontinuities provide a natural demarcation of this interval, analogous to the boundaries of an optical cavity.

The propagation of waves in curved space is often modeled by the Klein-Gordon (KG) equation [9]

\[ D\phi(x, t) \equiv \left[ \partial_t^2 - \partial_x^2 + V(x) \right] \phi(x, t) = 0, \tag{1} \]

with the outgoing wave boundary condition at infinity. The potential \( V(x) \), assumed to be positive, bounded and vanishing at infinity, describes the scattering of wave by the background geometry. We show that the QNM’s of (1) form a complete set, in the sense that the Green’s function can be expressed as a sum over QNM’s (cf. (10) below), provided the following three conditions hold: (i) \( V(x) \) is everywhere finite, and vanishes sufficiently rapidly, as \( x \to \infty \), in a sense to be defined below; (ii) there are spatial discontinuities demarcating a finite interval; (iii) consideration is limited to certain domains of spacetime.

In particular, for propagation from a source point \( y \) to an observation point \( x \) in a time \( t \), the QNM’s give a complete description for (a) \( y \) inside the interval, \( x \) outside the interval, and \( t > t_p(x, y) \), for a certain \( t_p \); or (b) both \( y \) and \( x \) inside the interval, and \( t > 0 \) (with the retarded Green’s function being zero for \( t \leq 0 \)). Case (b) is analogous to the completeness of normal modes in hermitian systems defined on a finite interval.

The above conditions allow dispersion, backscatter, as well as differences in the damping times, the absence of which has been conjectured to be important [4]. The results are valid to all orders in the rate of leakage. (Lowest order results would be trivial, since the system
becomes hermitian in that limit.) These results extend our previous work on the wave equation [10].

The spatial coordinate \(x\) in (1) often represents a radial variable, so first consider a half-line problem \((x \geq 0)\) with \(\phi(x = 0, t) = 0\). The QNM’s are eigen-solutions to the time-independent KG equation

\[
[-\omega^2 - \partial_x^2 + V(x)]\tilde{\phi}(x) = 0 ,
\]

with the boundary conditions \(\tilde{\phi}(0) = 0\) and the outgoing wave condition at infinity. The eigenfunctions and eigenvalues are denoted as \(\tilde{\phi}(x) = f_j(x)\) and \(\omega = \omega_j\).

The retarded Green’s function for the system is defined by \(DG(x, y; t) = \delta(x - y)\delta(t)\) with \(G = 0\) for \(t \leq 0\), and the boundary conditions are (i) \(G = 0\) for either \(x = 0\) or \(y = 0\), and (ii) the outgoing wave condition as either \(x \to \infty\) or \(y \to \infty\). In the frequency domain, and henceforth choosing \(y \leq x\), \(\tilde{G}(x, y; \omega) = f(\omega, y)g(\omega, x)/W(\omega)\), where \(f\) and \(g\) are solutions to (2) with the boundary conditions \(f(\omega, x = 0) = 0; f'(\omega, x = 0) = 1\) [11], and \(g(\omega, x) \to \exp(i\omega x)\) as \(x \to \infty\), where \(' = d/dx\).

The Wronskian \(W(\omega) = gf' - fg'\) is independent of \(x\), and its zeros are the QNM frequencies \(\omega_j\), with \(f(\omega_j, x) \propto g(\omega_j, x) \propto f_j(x)\). For simple zeros [12], the residues of \(\tilde{G}(x, y; \omega)\) are \(K_j = f(\omega_j, y)g(\omega_j, x)/[\partial W(\omega = \omega_j)/\partial \omega]\). Multiple zeros can be handled readily. To analyse the denominator \(\partial W/\partial \omega\), start with the defining equation for \(f(\omega_j, x)\) and \(g(\omega, x)\). The usual manipulations lead to

\[
\int_0^X dx f(\omega_j, x)g(\omega, x)
= [g(\omega, x)f'(\omega_j, x) - g'(\omega, x)f(\omega_j, x)]_0^X / (\omega^2 - \omega_j^2) ,
\]

where the integral is taken along any contour from \(x = 0\) to \(x = X\). Use the outgoing wave condition to evaluate \(f\) and \(g\) at the upper limit, and take \(\omega \to \omega_j\) by l’Hospital’s rule. This then gives

\[
\int_0^X dx f(\omega_j, x)g(\omega_j, x)
= -[\partial W/\partial \omega + ifg]_{\omega=\omega_j,x=X} / (2\omega_j) ,
\]

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for $X \to \infty$. Since $f$ and $g$ are proportional at these poles, a generalized norm of the QNM’s can be defined as

$$
\ll f_j | f_j \gg \equiv \lim_{X \to \infty} \int_0^X dx \ f_j(x)^2 + i f_j(X)^2/(2\omega_j) ,
$$

(5)

with which the residue of $\tilde{G}$ at $\omega_j$ can be expressed as

$$
K_j = -f_j(x)f_j(y)/[2\omega_j \ll f_j | f_j \gg] .
$$

(6)

The generalized norm, introduced in other contexts [1,13,14], involves $f^2$ rather than $|f|^2$, and is therefore complex. Although each term on the right of (5) does not have a limit as $X \to \infty$, the limit exists for the sum.

Next write $G$ in terms of $\tilde{G}$ by a Fourier integral, and distort the contour to a large semicircle in the lower half $\omega$ plane. One then sees that

$$
G(x, y; t) = \frac{i}{2} \sum_j f_j(x)f_j(y)e^{-i\omega_j t} + I_c + I_s .
$$

(7)

The sum comes from the zeros of $W(\omega)$, $I_c$ is the integral along a semicircle at infinity, and $I_s$ comes from any singularities of $f$ and $g$ in the lower half $\omega$ plane. It remains to show that (i) $f$ and $g$ are regular in $\omega$, in which case $I_s = 0$; and (ii) $I_c$ vanishes if there is a discontinuity in $V(x)$ at some $x = a > 0$.

The first step is straightforward, by appealing to well-known results in the quantum theory of scattering [15]. Since $f$ and $g$ satisfy the Schroedinger equation with energy $\omega^2$, they are analytic functions of $\omega$ if the potential is bounded and “has no tail” [15], in the sense that $\int_{-\infty}^\infty dx \ x \ |V(x)| < \infty$, and $\int_{-\infty}^\infty dx \ x \ e^{\alpha x} \ |V(x)| < \infty$ for any $\alpha > 0$. Our work on potentials that have a tail, e.g., inverse power laws, will be reported elsewhere.

Secondly, on the large semicircle $\tilde{G}$ can be estimated by the WKB approximation. As $|\omega| \to \infty$ WKB approximation is valid except at points where the potential has discontinuities, which can be handled by connecting the WKB solutions across them. Let $\tilde{\phi}(x) \equiv \exp[iS(x)]$ be a solution of (2); then $S(x) \approx \pm \int k(x)dx$, where $k(x) = [\omega^2 - V(x)]^{1/2}$. $\tilde{G}$ can then be obtained in terms of $\phi(x)$ and $k(x)$.
Consider a potential with a step discontinuity at \( x = a \). The reflection coefficient is
\[
R = \frac{S'(a^-) - S'(a^+)}{[S'(a^-)* + S'(a^+)]} \sim \omega^{-2}
\] at high frequencies. For simplicity we give the argument only for Case b: \( 0 < y \leq x < a \). It is straightforward to show that
\[
\tilde{G}(x, y; \omega) \approx \frac{\sin[I(0, y)][e^{-iI(x, a)} + Re^{iI(x, a)}]}{\sqrt{k(x)k(y)}[e^{-iI(0, a)} + Re^{iI(0, a)}]}
\] (8)

where \( I(u, v) = \int_u^v k(x)dx \approx \omega(v - u) \). Now on the semicircle \( \omega = \omega_R + i\omega_I = Ce^{i\theta}, \pi < \theta < 2\pi \), as \( \omega_I = C\sin \theta \to -\infty \), both the numerator and the denominator of \( \tilde{G} \) are dominated by the term proportional to \( R \), and
\[
\tilde{G}(x, y; \omega)e^{-i\omega t} \approx e^{-i\omega(t+2a)}/\omega .
\] (9)

As \( C \to \infty \), this vanishes for \( t > 0 \). This conclusion remains valid if \( V(x) \) has a discontinuity only in its \( p \)-th derivative \( (p = 0, 1, 2, \ldots) \), since the reflection coefficient \( R \) would vary as \( \omega^{-(p+2)} \) [16]. Thus we have proved (apart from a technical hitch mentioned below) that for a discontinuous potential, the QNM’s are complete for \( t > 0 \). For Case a, i.e. \( 0 < y < a < x \), a similar estimate gives, in place of (9), \( e^{-i\omega(t-2a)}/(R\omega) \). This vanishes and consequently completeness holds only if \( t > t_p(x, y) = \max(x+y-2a, 0) \).

For problems on a full line \(-\infty < x < -\infty \) (as in the case of Schwarzschild hole), \( f \) is still defined as the solution satisfying the left boundary condition, which is now a unit outgoing wave as \( x \to -\infty \); all the arguments remain the same. Generalization to multiple discontinuities is likewise straightforward. In this case, completeness in Case b holds in the interval between the leftmost and the rightmost discontinuities.

However, there is one technical hitch. The QNM sum is in fact a divergent series, but converges to the correct answer when regularised in a standard way [17]. The simplest regularization is to first invoke symmetry and only keep modes with \( \text{Re} \ \omega_j > 0 \) (hereafter denoted as \( j > 0 \)) in the sum in (7); secondly replace all times \( t \) by \( t - i\tau \), with \( \tau \to 0^+ \). This implies the use of a regulating factor \( I_j(\tau) = \exp(-\omega_j\tau) \) multiplying the contribution of the \( j \)-th QNM. The need for regularization can be seen as follows. The proof that the integrand vanishes on the large semicircle as \( C \to \infty \) is not valid near the real axis, in particular,
where $\text{Im} \, \omega = O(\log C)$. In this part, a factor such as $\exp(\text{Im} \, \omega \sigma)$ ($\sigma$ real and positive, e.g. $\sigma = t - x - y + 2a$) behaves like a power of $C$ rather than exponentially, and it is necessary to examine the integrand more carefully [18]. If $R \sim \omega^{-q}$, the powers of $\omega$ in the integrand for $G$ are inadequate to control the contribution along the semicircle if $q \geq 1$ [19]. For the KG equation with $V(x)$ discontinuous in its $p$-th derivative, $q = p + 2$, so regularization is always necessary. For the wave equation with a discontinuity in the $p$-th derivative of the refractive index $n(x)$, $q = p$, so regularization is not necessary for $p = 0$. The difference by two powers can be understood with the transformation relating the wave equation to the KG equation [10].

For large $p$, the divergence is more severe, the regulated sum converges more slowly to the correct result as $\tau \to 0$, and the use of a small finite $\tau$ becomes increasingly inaccurate. Thus, in practice, the QNM sum is only useful for relatively strong discontinuities, e.g. $p = 0, 1$.

To summarize, we have proved, under the conditions stated, that the Green’s function $G$ is expressible, in a certain domain of $x, y, t$, as the regulated QNM sum

$$G = \lim_{\tau \to 0} \text{Re} \left[ i \sum_{j>0} f_j(x)f_j(y) e^{-i\omega_j t} I_j(\tau) \right] \quad \omega_j \ll f_j |f_j|.$$  \hspace{1cm} (10)

Considering $\dot{G}$ as $t \to 0^+$,

$$\delta(x-y) = \dot{G}(x,y; t = 0^+)$$

$$= \lim_{\tau \to 0} \text{Re} \left[ \sum_{j>0} f_j(x)f_j(y) I_j(\tau) \right] \quad \ll f_j |f_j| \quad \text{valid for } x, y \in [0, a].$$  \hspace{1cm} (11)

We now focus on (11), noting that such a resolution of the identity will be useful for a variety of problems similar to those of hermitian systems. The QNM representation of $G$ for $t > 0$ will be further discussed elsewhere in the context of initial value problems. As an example, consider a potential defined on the full line, with $V(x) = V_1 > 0$ for $0 \leq x \leq a$, and zero otherwise. This potential has discontinuities and no tail, thus satisfying the conditions necessary for our results. The QNM’s of this system are easily found. We consider (11),
but as an approximate equality for some small nonzero regularization parameter \( \tau \), and in a distribution sense, e.g.

\[
\text{Re} \sum_{j>0} f_j(x) \int_{y_1}^{y_2} dy f_j(y) I_j(\tau) / \ll f_j \mid f_j \gg \\
\simeq \theta(x - y_1) - \theta(x - y_2) ,
\]

(12)

for \( x, y_1, y_2 \in [0,a] \). Figure 1 shows that for small \( \tau \), the partial sum indeed converges close to the right hand side. Without the regulator, the sum does not converge.

This example further illustrates why the series needs to be regulated. Suppose that for a full-line problem, the reflection coefficients on both sides behave as \( R(\omega) \sim \omega^{-q} \) at high frequencies \( (q = 2 \text{ for the present situation}) \). By setting the denominator in (8) to zero [20], it is straightforward to show that \( \text{Im} \omega_j a \sim -q \log |j| \). In the region \( 0 < x, y < a \), a typical term in the product \( f_j(x)f_j(y) \) goes as

\[
|f_j(x)f_j(y)| / \ll f_j \mid f_j \gg \sim (|j|\pi)^{q\Lambda/a} ,
\]

(13)

where \( \Lambda = \max(|x + y - a|, |x - y|) \). Since the maximum value of \( \Lambda \) is \( a \), without the regulator the worst behavior of the summand in (10) is \( |j|^{q-1} \). The sum would not converge [19] without a regulating factor if \( q \geq 1 \). The culprit is \( \text{Im} \omega_j a \sim -q \log |j| \), from which one can say unequivocally that the need for regulating the sum is an intrinsic property of KG open systems.

We have shown that for potentials that have no tail but which contain spatial discontinuities, the QNM sum is complete, if suitably regulated. This result settles a question in the literature [7]. The generic need for regularization, in particular, distinguishes the KG equation from the wave equation. When there is a tail, there will be scattering of the wave from asymptotic regions of space, leading to power-law type long time behavior, which will be discussed elsewhere.

In so far as the Green’s function \( G \) provides the solution to all the dynamics, the QNM expansion of \( G \) will lead to a variety of physical applications, much as the normal mode expansion of hermitian systems.
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[8] Price and Husain considered the wave equation with \( n(z) \) assuming different constant values inside and outside the star. Other models which contain discontinuities have also been studied. The Regge-Wheeler potential with a step truncation has been studied and reported by R.H. Price in the Gregynog meeting, Aug. 1993. H.P. Nollert has used step potentials to model the Regge-Wheeler potential (preprint, Universitat Tubingen, 1994).

[9] For linearized waves (including KG scalar, electromagnetic and gravitational waves) in static spherical symmetric spacetimes, the propagation is exactly described by (1).
Our work on the wave equation \([n^2(z)\partial_z^2 - \partial_t^2] \psi(z, t) = 0\) has been reported in Ref. 1. The wave equation is related to the KG equation (1) by a transformation \(dx/dz = n(z), \psi = n^{-1/2}\phi\), with the potential related to \(n(z)\) by \(V = (2n^3)^{-1}(d^2 n/dz^2) - (3/4n^4)(dn/dz)^2\). However, there are two important differences: (a) A continuous \(V(x)\) implies a continuous \(n(z)\) but not the other way around, (b) the need to regulate the QNM sum in the KG case is generic, see below.

We assume that here \(f'\) is nonzero at the origin. Otherwise, if \(f \sim x^{l+1}\) as \(x \to 0\) then we require \(\lim_{x \to 0} x^{-(l+1)} f(\omega, x) = 1\).

Generalization to higher order poles is straightforward. For high order poles, the time dependence acquires an extra prefactor going like \(t^n\); see e.g. J.S. Bell and C.J. Goebel, Phys. Rev. B138, 1198 (1965).

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If there are discontinuities of any order, a connection formula has to be used, as the WKB series is not valid at those points.

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Note that Jordan’s lemma states that an integral \(\int \omega e^{-i\omega t} f(\omega)\) along a large semicircle in the lower half plane vanishes for \(t > 0\), provided that \(|f(\omega)| \to 0\) at infinity. The necessary bound on \(f\) arises similarly from the need to bound the part with \(|\omega_I| = O(\log \omega_R)\).

To be specific, in this Letter we consider pointwise convergence for \(G\) (which is a well defined quantity) and convergence in the distribution sense for the representation of
\( \delta(x - y) \), which comes from \( \dot{G} \). Going from \( G \) to \( \dot{G} \) costs a factor \( \omega \), but going from the pointwise to the distribution sense effectively gains a factor \( \omega^{-1} \). This happens because asymptotically the typical dependence is \( e^{i\omega y} \), so integration over \( y \) gives a factor \( \omega^{-1} \).

Thus the pointwise validity of (10) and the validity of (11) in a distribution sense require the same conditions.

[20] Note that (8) has to be slightly modified for this full line case.
FIGURES

FIG. 1. Smooth envelope forming upper bound of $\log_{10} E$ vs $J/10^3$, where $E$ is the difference between the right hand side of (12) and the partial sum on the left hand side, up to $J$ terms. The parameters are $V_1 = 10^2$, $a = 1$, $y_1 = 0$, $y_2 = 0.5$ and $x = 0.25$, $\tau = 5 \times 10^{-3}$ (curve 1); $x = 0.25$, $\tau = 5 \times 10^{-4}$ (curve 2); $x = 0.75$, $\tau = 5 \times 10^{-3}$ (curve 3); $x = 0.75$, $\tau = 5 \times 10^{-4}$ (curve 4).