Gravitating monopoles and black holes in Einstein-Born-Infeld-Higgs model

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Abstract

We find static spherically symmetric monopoles in Einstein-Born-Infeld-Higgs model in 3 + 1 dimensions. The solutions exist only when a parameter $\alpha$ (related to the strength of Gravitational interaction) does not exceed certain critical value. We also discuss magnetically charged non Abelian black holes in this model. We analyse these solutions numerically.

1 Introduction

Some time ago monopoles in Einstein-Yang-Mills-Higgs(EYMH) model, for $SU(2)$ gauge group with Higgs field in adjoint representation, were studied as a generalization of the 't Hooft-Ployakov monopole to see the effect of gravity on it. In particular, it was found that solutions exist up to some critical value of a dimensionless parameter $\alpha$, characterising the strength of the gravitational interaction, above which there is no regular solution. The existance of these solutions were also proved analytically for the case of infinite Higgs mass. Also, non Abelian magnetically charged black hole solutions were shown to exist in this model for both finite as well as infinite value of the coupling constant for Higgs field. The Abelian black holes exists for $r_h \geq \alpha$ and non Abelian black holes exist in a limited region of the $(\alpha, r_h)$ plane.

Recently Born-Infeld theory has received wide publicity, especially in the context of string theory. Bogomol’nyi-Prasad-Sommerfield (BPS) saturated solutions were obtained in Abelian Higgs model as well as in $O(3)$ sigma model in 2 + 1 dimensions in presence of Born-Infeld term. Different models for domain wall, vortex and monopole solutions, containing the Born-Infeld Lagrangian were constructed in such a way that the self-dual equations are identical with the corresponding Yang-Mills-Higgs model. Recently non self-dual monopole solutions were found numerically in non Abelian Born-Infeld-Higgs theory.
In this paper we consider the Einstein-Born-Infeld-Higgs (EBIH) model and study the monopole and black hole solutions. The solutions are qualitatively similar to those of EYMH model. The black hole configurations have nonzero non Abelian field strength and hence they are called non Abelian black holes [13]. In Sec. II we consider the model and find the equations of motion for static spherically symmetric fields. In Sec III we find the asymptotic behaviours and discuss the numerical results. Finally we conclude the results in Sec. IV.

2 The Model

We consider the following Einstein-Born-Infeld-Higgs action for $SU(2)$ fields with the Higgs field in the adjoint representation

$$S = \int d^4 x \sqrt{-g} \left[ L_G + L_{BI} + L_H \right]$$  \hspace{1cm} (1)

with

$$L_G = \frac{1}{16\pi G} R,$$

$$L_H = -\frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \frac{e^2 g^2}{4} \left( \phi^a \phi^a - v^2 \right)^2$$

and the non Abelian Born-Infeld Lagrangian [14],

$$L_{BI} = \beta^2 \text{Str} \left( 1 - \sqrt{1 + \frac{\beta^2}{2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{8\beta^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2} \right)$$

where

$$D_\mu \phi^a = \partial_\mu \phi^a + e \epsilon^{abc} A^b_\mu \phi^c,$$

$$F_{\mu\nu} = F^a_{\mu\nu} t^a = \left( \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e \epsilon^{abc} A^b_\mu A^c_\nu \right) t^a$$

and the symmetric trace is defined as

$$\text{Str} \left( t_1, t_2, \ldots, t_n \right) = \frac{1}{n!} \sum \text{tr} \left( t_{i_1} t_{i_2} \ldots t_{i_n} \right).$$

Here the sum is over all permutations on the product of the $n$ generators $t_i$. Here we are interested in purely magnetic configurations, hence we have $F_{\mu\nu} \tilde{F}^{\mu\nu} = 0$. Expanding the square root in powers of $\frac{1}{\beta^2}$ and keeping up to order $\frac{1}{\beta^4}$ we have the Born-Infeld Lagrangian

$$L_{BI} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{96\beta^2} \left[ (F^a_{\mu\nu} F^{a\mu\nu})^2 + 2 F^a_{\mu\nu} F^a_{\rho\sigma} F^{b\mu\nu} F^{b\rho\sigma} \right] + O(\frac{1}{\beta^4}).$$
For static spherical symmetric solutions, the metric can be parametrized as

\[ ds^2 = -e^{2\nu(R)}dt^2 + e^{2\lambda(R)}dR^2 + r^2(R)(d\theta^2 + \sin^2 \theta d\varphi^2) \]  

(2)

and we consider the following ansatz for the gauge and scalar fields

\[ A^a_t(R) = 0 = A^a_\theta, \quad A^a_\varphi = e^a_{\varphi} W(R) - \frac{1}{e} \sin \theta, \]  

\[ A^a_\varphi = -e^a_{\varphi} W(R) - \frac{1}{e} \sin \theta, \]  

(3)

and

\[ \phi^a = e^a_\varphi vH(R). \]  

(4)

Putting the above ansatz in Eq.1, defining \( \alpha^2 = 4\pi Gv^2 \) and rescaling \( R \to R/ev, \beta \to \beta ev^2 \) and \( r(R) \to r(R)/ev \) we get the following expression for the Lagrangian

\[ \int dRe^{\nu+\lambda} \left[ \frac{1}{2} \left( 1 + e^{-2\lambda} \left( (r')^2 + \nu'(r^2') \right) \right) - \alpha^2 \left( e^{-2\lambda}V_1 - e^{-4\lambda}V_2 + V_3 \right) \right], \]  

(5)

where

\[ V_1 = (W')^2 + \frac{1}{2} r^2 (H')^2 - (W')^2 \frac{(W^2 - 1)^2}{6\beta^2 r^4}, \]  

(6)

\[ V_2 = (W')^4 / 3\beta^2 r^2, \]  

(7)

and

\[ V_3 = \frac{(W^2 - 1)^2}{2r^2} + W^2 H^2 + \frac{g^2 r^2}{4} (H^2 - 1)^2 - \frac{(W^2 - 1)^4}{8\beta^2 r^6}. \]  

(8)

Here the prime denotes differentiation with respect to \( R \). The dimensionless parameter \( \alpha \) can be expressed as the mass ratio

\[ \alpha = \sqrt{\frac{4\pi}{e}} \frac{M_W}{eM_{Pl}} \]  

(9)

with the gauge field mass \( M_W = ev \) and the Planck mass \( M_{Pl} = 1/\sqrt{G} \). Note that the Higgs mass \( M_H = \sqrt{2} gev \). In the limit of \( \beta \to \infty \) the above action reduces to that of the Einstein-Yang-Mills-Higgs model[1, 2]. For the case of \( \alpha = 0 \) we must have \( \nu(R) = 0 = \lambda(R) \) which corresponds to the flat space Born-Infeld-Higgs theory[12]. We now consider the gauge \( r(R) = R \), corresponding to the Schwarzschild-like coordinates and rename \( R = r \). We define \( A = e^{\nu+\lambda} \) and \( N = e^{-2\lambda} \). Varying the matter field Lagrangian with respect to the metric we find the energy-momentum tensor. Integrating the \( tt \) component of the energy-momentum we get the mass of the monopole equal to \( M/evG \) where

\[ M = \alpha^2 \int_0^\infty dr \left( NV_1 - N^2 V_2 + V_3 \right) \]  

(10)
Following 't Hooft the electromagnetic $U(1)$ field strength $F_{\mu\nu}$ can be defined as

$$ F_{\mu\nu} = \frac{\phi^a F_{\mu\nu}^a}{|\phi|} - \frac{1}{e|\phi|^3} \epsilon^{abc} \phi^a D_\mu \phi^b D_\nu \phi^c. $$

Then using the ansatz (3) the magnetic field

$$ B^i = \frac{1}{2} \epsilon^{ijk} F_{jk} $$

is equal to $e^i_1/er^2$ with a total flux $4\pi/e$ and unit magnetic charge.

The $tt$ and $rr$ components of Einstein’s equations are

$$ \frac{1}{2} (1 - (rN)'') = \alpha^2 \left( NV_1 - N^2 V_2 + V_3 \right) $$
$$ \frac{A'}{A} = \frac{2\alpha^2}{r}(V_1 - 2NV_2). $$

The equations for the matter fields are

$$ (ANV_4)' = AW \left( \frac{2}{r^2}(W^2 - 1) + 2H^2 - \frac{(W^2 - 1)^3}{\beta^2 r^6} - \frac{2N(W')^2}{3\beta^2 r^4}(W^2 - 1) \right) $$
$$ (ANr^2H')' = AH \left( 2W^2 + g^2 r^2(H^2 - 1) \right) $$

with

$$ V_4 = 2W' - \frac{W'}{3\beta^2 r^4}(W^2 - 1)^2 - \frac{4N}{3\beta^2 r^2}(W')^3 $$

It is easy to see that $A$ can be eliminated from the matter field equations using Eq.(12). Hence we have to solve three differential equations Eqs. (11),(13) and (14) for the three fields $N, W$ and $H$.

## 3 Solutions

### 3.1 Monopoles

For finite $g$, demanding the solutions to be regular and the monopole mass to be finite gives the following behaviour near the origin

$$ H = ar + O(r^3), $$
$$ W = 1 - br^2 + O(r^4), $$
$$ N = 1 - cr^2 + O(r^4), $$

where $a$ and $b$ are free parameters and $c$ is given by

$$ c = \alpha^2 \left( a^2 + 4b^2 + \frac{g^2}{6} - \frac{20b^4}{3\beta^2} \right). $$
In general, with these initial conditions \( N \) can be zero at some finite \( r \) where the solutions become singular. In order to avoid this singularity we have to adjust the parameters \( a \) and \( b \) suitably.

For \( r \to \infty \) we require the solutions to be asymptotically flat. Hence we impose

\[
N = 1 - \frac{2M}{r} \tag{19}
\]

Then for finite mass configuration we have the following expressions for the gauge and the Higgs fields

\[
W = Cr^M e^{-r} \left( 1 + O\left(\frac{1}{r}\right) \right) \tag{20}
\]

\[
H = \begin{cases} 
1 - Br^{-\sqrt{2}gM-\frac{1}{2}}e^{-\sqrt{2}gr}, & \text{for } 0 < g \leq \sqrt{2} \\
1 - \frac{C^2}{g^2}r^{-2M-2e^{-2r}}, & \text{for } g = 0 \text{ and } g > \sqrt{2}.
\end{cases} \tag{21}
\]

Note that the fields have similar kind of asymptotic behaviour in the EYMH model\[3\]. We have solved the equations of motion numerically with the boundary conditions given by Eqs.(16-21). For \( \alpha = 0, \ g = 0 \) and \( \beta \to \infty \) they corresponds to the exact Prasad-Sommerfield solution\[16\]. For nonzero \( \alpha, g \) and finite \( \beta \) the qualitative behaviour of the solutions are similar to the corresponding solutions of EYMH model. For large \( r \) these solutions converges to their asymptotic values given as in Eqs.(19-21). For a fixed value of \( g \) and \( \beta \) we solved the equations increasing the value of \( \alpha \). For small value of \( \alpha \) the solutions are very close to flat space solution. As \( \alpha \) is increased the minimum of the metric function \( N \) was found to be decreasing. The solutions cease to exist for \( \alpha \) greater then certain critical value \( \alpha_{\text{max}} \). For \( g = 0 \) and \( \beta = 3 \) we find \( \alpha_{\text{max}} \sim 2 \). The profile for the fields for different values of \( \alpha \) with \( g = 0 \) and \( \beta = 3 \) are given in Figs.1,2 and 3. The profile for the fields for \( g = 1, \alpha = 1.0 \) and \( \beta = 3 \) are given in Fig. 4. We find numerically the mass \( M = 0.7865 \) of the monopole for \( g = 1, \alpha = 1.0 \) and \( \beta = 3 \).

### 3.2 Black holes

Apart from the regular monopoles, magnetically charged black holes can also exist in this model. Black hole arises when the field \( N \) vanishes for some finite \( r = r_h \). Demanding the solutions to be regular near horizon \( r_h \) we find the following behaviour of the fields

\[
N(r_h + \rho) = N'_h \rho + O(\rho^2), \tag{22}
\]

\[
H(r_h + \rho) = H'_h + H_h' \rho + O(\rho^2), \tag{23}
\]

\[
W(r_h + \rho) = W'_h + W_h' \rho + O(\rho^2) \tag{24}
\]

with

\[
N'_h = \frac{1}{r_h} \left[ 1 - \alpha^2 \left\{ \frac{(W_h^2 - 1)^2}{r_h^2} + 2W_h^2 H_h^2 + \frac{g^2 r_h^2}{2} (H_h^2 - 1)^2 - \frac{(W_h^2 - 1)^4}{4 \beta^2 r_h^2} \right\} \right] \tag{25}
\]
\[ H'_h = \frac{H_h}{N'_h r^2_h} \left\{ 2W'^2_h + g^2 r^4_h (H^2_h - 1) \right\} \quad (26) \]

\[ W'_h = \frac{W_h r^2_h (W'^2_h - 1)}{N'_h \left[ 1 - \frac{(W'^2_h - 1)^2}{6 \beta^2 r^4_h} \right]} \quad (27) \]

Here \( r_h, W_h(\equiv W(r_h)) \) and \( H_h(\equiv W(r_h)) \) are arbitrary. For \( r \to \infty \) the behaviour of the fields is same as the regular monopole solution as given by Eqs.(19-21). The black hole has unit magnetic charge with nontrivial gauge field strength. We found numerical solutions to the non Abelian black hole for different \( r_h \). For a fixed value of \( r_h \) we find the solutions for \( r > r_h \) adjusting the parameters \( W_h \) and \( H_h \). For \( r_h \) close to zero the solutions approach the regular monopole solutions. The profile for the fields are given in Fig.5. We found the mass of the black hole equals to be 0.6796 for \( \alpha = 1.0, g = 0 \) and \( \beta = 3 \).

## 4 Conclusion

In this paper we have investigated the effect of gravity on the Born-Infeld-Higgs monopole. We found that solutions exist only up to some critical value \( \alpha_{max} \) of the parameter \( \alpha \). In the limit \( \beta \to \infty \) these solutions reduces to those of EYMH monopoles. We also found numerically magnetically charged non Abelian black hole solutions in this model. It would be interesting to prove analytically the existence of these solutions for finite value of the parameters. Recently dyons and dyonic black holes were found in EYMH model numerically\[17\] and the existence of critical value for \( \alpha \) was also proved analytically \[18\]. It may be possible to generalize these solutions to find dyons and dyonic black holes in EBIH model. We hope to report on this issue in future.

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Figure 1: Plot for the metric function $N$ as a function of $r$ for $g = 0, \beta = 3$ for different values of $\alpha$. Curve I is for $\alpha = 0.7$, curve II for $\alpha = 1.6$ and curve III for $\alpha = 1.9$.

Figure 2: Plot for the gauge field $W$ as a function of $r$ for $g = 0, \beta = 3$ for different values of $\alpha$. Curve I for $\alpha = 0.7$, curve II for $\alpha = 1.6$ and curve III for $\alpha = 1.9$. 
Figure 3: Plot for the Higgs field $H$ as a function of $r$ for $g = 0$, $\beta = 3$ for different $\alpha$. Curve I is for $\alpha = .7$, curve II for $\alpha = 1.6$ and curve III for $\alpha = 1.9$.

Figure 4: Plot for the fields $N$, $W$ and $H$ as function of $r$ for $g = 0.1$, $\beta = 3$ and $\alpha = 1$. 
Figure 5: Black hole solutions for $g = 0$, $\beta = 3$, $\alpha = 1$, $r_h = 0.3$, $H_h = 0.057271$ and $W_h = 0.980997$. 