Spectrum generating algebras for few-body systems

Roelof Bijker
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A.P. 70-543, 04510 México, D.F., México
E-mail: bijker@nucleares.unam.mx

Abstract. I discuss an algebraic treatment of few-body systems based on spectrum generating algebras for the relative motion of the clusters. Particular attention is paid to the formalism for identical clusters including a discussion of the permutation symmetry and a study of special solutions which are shown to correspond to the harmonic oscillator, the deformed oscillator, the oblate symmetric top and the spherical top with tetrahedral symmetry.

1. Introduction
Algebraic models and spectrum generating algebras have played an important role in the study of both many-body and few-body systems. Generally speaking, in algebraic models energy eigenvalues and eigenvectors are obtained by diagonalizing a finite-dimensional matrix, rather than by solving a set of coupled differential equations in coordinate space. The benefits and merits of this type of models are illustrated beautifully by the interacting boson model (IBM), which has been very successful in the description of collective states in nuclei [1]. Its dynamical symmetries correspond to the quadrupole vibrator, the axially symmetric rotor and the γ-unstable rotor in a geometric description. In addition to these special solutions, the IBM can describe intermediate cases between any of them equally well. The first extension of the algebraic approach to few-body systems was the vibron model [2], which was introduced to describe vibrational and rotational excitations in diatomic molecules. The dynamical symmetries of the vibron model correspond to the (an)harmonic oscillator and the Morse oscillator. In spectroscopic studies these algebraic methods provide a powerful tool to study symmetries and selection rules, to classify the basis states, and to calculate matrix elements of physical observables.

The general procedure is to introduce a $U(\nu+1)$ spectrum generating algebra for a bound-state problem with $\nu$ degrees of freedom in which all states are assigned to the symmetric representation $[N]$ of $U(\nu+1)$. For the $\nu=5$ quadrupole degrees of freedom in collective nuclei this led to the introduction of the $U(6)$ interacting boson model [1]. Similarly, the $U(4)$ vibron model was proposed to describe the dynamics of the $\nu=3$ dipole degrees of freedom of the relative motion of two objects, e.g. two atoms in a diatomic molecule [2], two clusters in a nuclear cluster model [3], or a quark and antiquark in a meson [4]. An application to the three-body system involves the six degrees of freedom of two relative vectors which in the algebraic approach leads to a $U(7)$ spectrum generating algebra [5] as an extension of the vibron model. The $U(7)$ model was developed originally to describe the relative motion of the three constituent quarks in baryons [5], but has also found applications in molecular physics [6, 7] and nuclear physics ($^{12}\text{C}$ as a cluster of three $\alpha$ particles) [8].
The aim of this contribution is to introduce the so-called Algebraic Cluster Model (ACM) to the \(n\)-body system in which the relative motion of the clusters is described in terms of a \(U(3n-2)\) spectrum-generating algebra, to discuss the permutation symmetry for the case of identical objects, to study some special solutions and to establish the connection between algebraic interactions and geometric models. In the ACM, the full rotational-vibrational structure is present from the outset. There is no need for Born-Oppenheimer type approximations.

2. The Algebraic Cluster Model

In this section, I introduce the Algebraic Cluster Model for \(n\)-body systems. The eigenvalues and eigenfunctions are obtained by matrix diagonalization instead of by solving a set of differential equations. The ACM is an interacting boson model to describe the relative motion of the \(n\) clusters based on the spectrum generating algebra of \(U(\nu+1)\) with \(\nu = 3(n-1)\). As special cases the ACM contains the vibron model [2] for two-body problems \((n = 2)\) and the \(U(7)\) model [5, 7, 8] for three-body clusters \((n = 3)\).

First, I introduce relative Jacobi coordinates

\[
\tilde{\rho}_k = \frac{1}{\sqrt{k(k+1)}} \left( \sum_{i=1}^{k} \tilde{r}_i - k\tilde{r}_{k+1} \right), \quad k = 1, \ldots, n-1, \quad (1)
\]

and their conjugate momenta. Here \(\tilde{r}_i\) are the coordinates of the \(i\)-th cluster \((i = 1, \ldots, n)\). Instead of a formulation in terms of coordinates and momenta the method of bosonic quantization is used which consists of introducing a dipole boson with \(L^P = 1^-\) for each independent relative coordinate and an auxiliary scalar boson with \(L^P = 0^+\)

\[
s^\dagger, \quad b^\dagger_{k,m}, \quad (2)
\]

with \(k = 1, \ldots, n-1\) and \(m = -1, 0, 1\). The scalar boson does not represent an independent degree of freedom, but is added under the restriction that the Hamiltonian commutes with the number operator

\[
\hat{N} = s^\dagger s + \sum_k \sum_m b^\dagger_{k,m} b_{k,m}, \quad (3)
\]

\(i.e.\) the total number of bosons \(N = n_s + \sum_k n_k\) is conserved. The set of \((3n-2)^2\) bilinear products of creation and annihilation operators spans the Lie algebra of \(U(3n-2)\). All operators, such as the Hamiltonian and electromagnetic transition operators, are expanded into elements of this algebra. This procedure leads to a compact spectrum generating algebra of \(U(3n-2)\) whose model space is spanned by the symmetric irreducible representation \([N]\) which contains the oscillator shells with \(n_0 = \sum_k n_k = 0, 1, 2, \ldots, N\). The introduction of the scalar boson makes it possible to investigate the dynamics of three vector degrees of freedom including situations in which there is a mixing of oscillator shells.

For \(n\) identical objects, the Hamiltonian has to be invariant under the permutation group \(S_n\). The permutation symmetry of \(n\) identical objects is determined by the transposition \(P(12)\) and the cyclic permutation \(P(12\cdots n)\) [9]. All other permutations can be expressed in terms of these two elementary ones. The transformation properties under \(S_n\) of all operators in the model follow from those of the building blocks. The scalar boson, \(s^\dagger\), transform as the symmetric representation \([n]\), whereas the dipole Jacobi bosons, \(b^\dagger_k\) with \(k = 1, \ldots, n-1\) transform as the \(n-1\) components of the mixed symmetry representation \([n-1, 1]\).

Next, one can use the multiplication rules for \(S_n\) to construct physical operators with the appropriate symmetry properties. As a result, the most general one- and two-body
Hamiltonian that describes the relative motion of a system of $n$ identical clusters, is a scalar under the permutation group $S_n$, is rotationally invariant, conserves parity as well as the total number of bosons, is given by

$$H = \epsilon_0 s^L \tilde{s} - \epsilon_1 \sum_k b^L_k \cdot \tilde{b}_k + u_0 s^L \tilde{s} \tilde{s} - u_1 \sum_k s^L b^L_k \cdot \tilde{b}_k \tilde{s}$$

$$+ \nu_0 \left[ \sum_k b^L_k \cdot b^L_k \tilde{s} \tilde{s} + \text{h.c.} \right] + \sum_L \sum_{ijkl} v^{(L)}_{ijkl} [b^L_i \times b^L_j] [\tilde{b}^L_k \times \tilde{b}^L_l],$$

(4)

with $\tilde{b}_{k,m} = (-1)^{1-m} b_{k,-m}$ with $k = 1, \ldots, n-1$ and $\tilde{s} = s$. The invariance under the permutation symmetry imposes restrictions on the coefficients $v^{(L)}_{ijkl}$. By construction, the first five terms in Eq. (4) are invariant under the permutation group $S_n$.

In general, the eigenvalues and corresponding eigenvectors are obtained numerically by diagonalization in a coupled harmonic oscillator basis. By construction, the wave functions are characterized by the total number of bosons $N$, angular momentum and parity $L^P$, and their transformation property under $t$ the permutation group $S_n$. The relation between harmonic oscillators and permutation symmetry was studied by Kramer and Moshinsky [9]. In particular, it was shown that for the $n$-body problem, the reduction $U(3n-3) \supset U(3) \otimes U(n-1)$ with $U(3) \supset SO(3)$ and $U(n-1) \supset O(n-1) \supset S_n$ provides a basis for an explicit construction of states with good permutation symmetry. Here $U(3)$ denotes the symmetry group for the coupled harmonic oscillator and $U(n-1)$ is the symmetry group associated with the transformation of the indices of the $n-1$ relative Jacobi vectors. In the application to the ACM, it is preferred to generate the wave functions with good permutation symmetry $|\psi_t\rangle$ numerically by diagonalization of a $S_n$ invariant Hamiltonian. The permutation symmetry $t$ of a given wave function can then be determined from the transformation properties under $P(12)$ and $P(12 \cdots n)$ [9] with applications to systems of three and four identical clusters [5, 10].

3. Geometric analysis

The geometric properties of the algebraic Hamiltonian of Eq. (4) can be studied by using a coherent state as a variational wave function for the $n$-body system

$$|N; \tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1}\rangle = \frac{1}{\sqrt{N!}} (b^L_{\nu})^N |0\rangle,$$

(5)

with

$$b^L_{\nu} = \sqrt{1 - \sum_k \tilde{\alpha}_k \cdot \tilde{\alpha}_k^* s^L + \sum_k \tilde{\alpha}_k \cdot \tilde{b}_k^L \tilde{s}^L}.$$

(6)

The condensate boson $b^L_{\nu}$ is parametrized in terms of $3(n-1)$ complex variables corresponding to the $n-1$ relative Jacobi coordinates and their conjugate momenta. For the geometric analysis of the ACM Hamiltonian it is convenient to use spherical rather than cartesian coordinates and momenta. We transform the variables $\tilde{\alpha}_k$ to intrinsic coordinates $(q_k, \theta_k, \phi_k)$ and their conjugate momenta $[11, 12]$

$$\alpha_{k,\mu} = \frac{1}{\sqrt{2}} \sum_\nu D^{(1)}_{\mu \nu}(\phi_k, \theta_k, 0) \beta_{k, \nu},$$

(7)

with

$$\begin{pmatrix} \beta_{k,1} \\ \beta_{k,0} \\ \beta_{k,-1} \end{pmatrix} = \begin{pmatrix} [-p_{\phi_k} \sin \theta_k - i p_{\theta_k}] / q_k \sqrt{2} \\ q_k + i p_k \\ [-p_{\phi_k} \sin \theta_k + i p_{\theta_k}] / q_k \sqrt{2} \end{pmatrix},$$

(8)
for \( k = 1, \ldots, n - 1 \). The classical limit of the ACM Hamiltonian is then given by the coherent state expectation value of the normal ordered Hamiltonian divided by the number of bosons \( N \)
\[
H_{\text{cl}} = \frac{1}{N} \langle N; \vec{\alpha}_1, \ldots, \vec{\alpha}_{n-1} | : H : | N; \vec{\alpha}_1, \ldots, \vec{\alpha}_{n-1} \rangle .
\] (9)

4. Special solutions

The Algebraic Cluster Model has a rich algebraic structure, which includes both continuous and discrete symmetries. It is of general interest to study limiting cases of the Hamiltonian of Eq. (4), in which the energy spectra can be obtained in closed form. We first consider two dynamical symmetries of the \( S_n \) invariant ACM Hamiltonian. These dynamical symmetries can be studied in general for the \( n \)-body problem for any number of clusters and are shown to correspond to the \( 3(n-1) \)-dimensional spherical and deformed oscillators.

In addition, there are some other cases of \( S_n \) invariant Hamiltonians for which approximate solutions can be obtained in a semiclassical mean-field analysis. For the case of three-body clusters I discuss the oblate symmetric top and for four-body clusters the spherical top with tetrahedral symmetry.

4.1. Harmonic oscillator

For \( v_0 = 0 \) in Eq. (4), there is no coupling between different harmonic oscillator shells. The oscillator is harmonic if all terms, except \( \epsilon_0 \) and \( \epsilon_1 \), are set to zero; otherwise it is anharmonic. This dynamical symmetry corresponds to the reduction
\[
\begin{aligned}
&U(3n - 2) \supset U(3n - 3) \supset SO(3n - 3) \\
&N, n_b, \sigma
\end{aligned}
\] (10)

We consider the one-body Hamiltonian
\[
H_1 = \epsilon \sum_{km} b_{k,m}^{\dagger} b_{k,m} ,
\] (11)
whose eigenvalues are those of a \( 3(n - 1) \)-dimensional spherical oscillator
\[
E_1 = \epsilon n_b .
\] (12)
The label \( n_b \) represents the total number of oscillator quanta \( n_b = \sum_k n_k = 0, 1, \ldots, N \). The energy levels are grouped into oscillator shells characterized by \( n_b \) and parity \( P = (-1)^{n_b} \). The levels belonging to an oscillator shell are further classified by the symmetric irreducible representation \( \sigma \) of \( SO(3n - 3) \) with \( \sigma = n_b, n_b - 2, \ldots, 1 \) or 0 for \( n_b \) odd or even (see Fig. 1). This special case is called the \( U(3n - 3) \) limit.

The classical limit of the Hamiltonian in Eq. (11) is given by
\[
H_{1,\text{cl}} = \epsilon \sum_k \vec{\alpha}_k \cdot \vec{\alpha}_k^* = \frac{1}{2} \sum_k \left( p_k^2 + q_k^2 + \frac{L_k^2}{q_k^2} \right) ,
\] (13)
where \( L_k^2 \) is the angular momentum in polar coordinates of the \( k \)-th oscillator
\[
L_k^2 = p_{\theta_k}^2 + \frac{p_{\phi_k}^2}{\sin^2 \theta_k} .
\] (14)
A change of variables of the coordinates $q_k$ to hyperspherical coordinates in index space, the hyperradius $q$ and the angles $\chi_1, \ldots, \chi_{n-2}$

\begin{align*}
q_1 &= q \sin \chi_1 \cdots \sin \chi_{n-3} \sin \chi_{n-2}, \\
q_2 &= q \sin \chi_1 \cdots \sin \chi_{n-3} \cos \chi_{n-2}, \\
& \vdots \\
q_{n-1} &= q \cos \chi_1 ,
\end{align*}

(15)

and their conjugate momenta, $p$ and $p_{\chi_k}$, reduces the classical limit to

$$
H_{1,cl} = \frac{1}{2} \left( p^2 + q^2 + \frac{\Lambda_{n-1}^2}{q^2} \right).
$$

(17)

Here $\Lambda_{n-1}^2$ is the generalized angular momentum for rotations in $3(n-1)$ dimensions

\begin{align*}
\Lambda_{n-k}^2 &= p_{\chi_k}^2 + \frac{L_{n-k}^2 \cos^2 \chi_k}{\chi_k} + \frac{1}{\sin^2 \chi_k} \Lambda_{n-k-1}^2 , \quad k = 1, \ldots, n-2 , \\
\Lambda_1^2 &= L_1^2 .
\end{align*}

(18)

Eq. (17) confirms the interpretation of the $U(3n-3)$ limit of the ACM in terms of a $3(n-1)$-dimensional spherical oscillator.
4.2. Deformed oscillator

For the (an)harmonic oscillator, the number of oscillator quanta \( n_b \) is a good quantum number. However, when \( v_0 \neq 0 \) in Eq. (4), the oscillator shells with \( \Delta n_b = \pm 2 \) are mixed, and the eigenfunctions are spread over many different oscillator shells. A dynamical symmetry that involves the mixing between oscillator shells, is provided by the reduction

\[
\left| \begin{array}{c}
U(3n-2) \ni SO(3n-2) \ni SO(3n-3) \ni \sigma \\
N \ni \omega \\
\end{array} \right| .
\]

We consider now a dipole-dipole interaction which can be rewritten as the difference between the Casimir operators of \( SO(3n-2) \) and \( SO(3n-3) \)

\[
H_2 = \kappa \left[ \hat{N}(\hat{N} + 3n - 4) - \hat{C}_{2SO(3n-2)} + \hat{C}_{2SO(3n-3)} \right] \\
= \kappa \left[ (s^\dagger s^\dagger - \sum_k b_k^\dagger b_k^\dagger) (\text{h.c.}) + \hat{C}_{2SO(3n-3)} \right],
\]

where \( \hat{N} \) is the number operator of Eq. (3). The energy spectrum in this case, called the \( SO(3n-2) \) limit, is given by the eigenvalues of the Casimir operators as

\[
E_2 = \kappa \left[ (N - \omega)(N + \omega + 3n - 4) + \sigma(\sigma + 3n - 5) \right] .
\]

The label \( \omega = N, N-2, \ldots, 1 \) or 0 for \( N \) odd or even, respectively, characterizes the symmetric representations of \( SO(3n-2) \), and \( \sigma = 0, 1, \ldots, \omega \) those of \( SO(3n-3) \).

The eigenstates are ordered in bands labeled by \( \omega \). Although the size of the model space, and hence the total number of states, is the same as for the harmonic oscillator, the ordering and classification of the states is different. For example, in the \( U(3n-3) \) limit all states are vibrational, whereas the \( SO(3n-2) \) limit gives rise to a rotational-vibrational spectrum, characterized by a series of rotational bands which are labeled by \( \omega \), or equivalently by the vibrational quantum number \( v = (N - \omega)/2 \)

\[
E_2 = 4\kappa N v \left( 1 - 2v - 3n + 4 \right) + \kappa \sigma(\sigma + 3n - 5) .
\]

The classical limit of the \( SO(3n-2) \) limit is given by

\[
H_{2,\text{cl}} = \kappa(N-1) \left[ q^2 p^2 + (1-q^2)^2 + \Lambda_n^2 \right].
\]

Here I have used the hyperspherical variables of Eq. (16). Just as for the harmonic oscillator, the potential only depends on the hyperradius \( q \). In the limit of small oscillations around the minimum of the potential energy surface \( q = 1 + \Delta q \) one finds to leading order in \( N \) a one-dimensional harmonic oscillator in the hyperradius \( q \)

\[
H_{2,\text{cl}} \rightarrow \kappa N \left( p^2 + 4(\Delta q)^2 \right) ,
\]

with frequency \( 4\kappa N \). To leading order in \( N \), the frequency coincides with that of Eq. (22). This analysis shows the explicit connection between the \( SO(3n-2) \) dynamical symmetry and the deformed (or displaced) oscillator.
4.3. Three-body clusters: oblate top

The spectrum generating algebra of three identical clusters is given by $U(7)$. The potential energy surfaces of the $U(6)$ and $SO(7)$ limits (spherical and deformed oscillators) only depend on the hyperspherical radius $q$. The corresponding equilibrium shapes are characterized by $q_0 = 0$ and $q_0 = 1$, respectively. Another interesting case is provided by the Hamiltonian

$$H_3 = \xi_1 (R^2 s_1^3 s_1^1 - b_1^1 \cdot b_1^1 - b_2^1 \cdot b_2^1) (R^2 s_1 s_1 - b_1 \cdot b_1 - b_2 \cdot b_2) + \xi_2 \left[ (b_1^1 \cdot b_1^1 - b_2^1 \cdot b_2^1) (b_1 \cdot b_1 - b_2 \cdot b_2) + 4 (b_1^1 \cdot b_2^1) (b_2 \cdot b_1) \right] + \kappa \vec{L} \cdot \vec{L}. \quad (25)$$

where $\vec{L}$ denotes the angular momentum in coordinate space. For $R^2 = 0$, this Hamiltonian has $U(7) \supset U(6)$ symmetry and corresponds to a spherical vibrator, whereas for $R^2 = 1$ and $\xi_2 = 0$ it has $U(7) \supset SO(7)$ symmetry and corresponds to a deformed oscillator. The general case with $R^2 \neq 0$ and $\xi_1, \xi_2 > 0$ does not correspond to a dynamical symmetry, and hence its energy spectrum cannot be obtained in closed analytic form. In this case, the energy eigenvalues and eigenvectors are calculated numerically by diagonalizing the Hamiltonian in an appropriate basis. However, an approximate energy formula can still be derived in a semiclassical mean-field analysis. The general expression of the classical limit of the Hamiltonian of Eq. (25) has a complicated structure. We first study the potential energy surface which is obtained by setting all momenta equal to zero. Its equilibrium configuration corresponds to coordinates that have equal length ($q_{1,0} = q_{2,0} = q_0/\sqrt{2}$)

$$q_0 = \sqrt{2R^2/(1 + R^2)}, \quad \chi_{1,0} = \pi/4, \quad (26)$$

and are perpendicular

$$\zeta_{12,0} = \pi/4, \quad (27)$$

where $2\zeta_{12}$ denotes the relative angle between $\vec{a}_1$ and $\vec{a}_2$. Geometrically, the three clusters are located at the vertices of an equilateral triangle.

In the limit of small oscillations around the equilibrium shape the intrinsic degrees of freedom decouple and become harmonic. The resulting energy spectrum is that of the vibrational excitations of an oblate symmetric top [8]

$$E_{3,\text{vib}} = \omega_1 (v_1 + \frac{1}{2}) + \omega_2 (v_2 + 1), \quad (28)$$

with frequencies

$$\omega_1 = 4NR^2 \xi_1, \quad \omega_2 = \frac{4NR^2}{1 + R^2} \xi_2, \quad (29)$$

in agreement with the results obtained in a normal mode analysis [5]. Here $v_1$ represents the vibrational quantum number for a symmetric stretching $A$ vibration, and $v_2 = v_{2a} + v_{2b}$ for a degenerate $E$ vibration. The rotational spectrum is given by

$$E_{3,\text{rot}} = \kappa_1 L(L + 1). \quad (30)$$

In Fig. 2 I show the structure of the spectrum of the oblate top according to the approximate energy formula of Eqs. (28) and (30). The energy spectrum consists of a series of rotational bands labeled by $(v_1, v_2)$. It is assumed that the spin of the identical clusters is zero, as is relevant...
Figure 2. Schematic spectrum of an oblate symmetric top. The rotational bands are labeled by \((v_1, v_2)\) (bottom). All states are symmetric under \(S_3\).

for the description of the \(^{12}\)C nucleus as a cluster of three \(\alpha\) particles. As a consequence, all states in Fig. 2 are symmetric under \(S_3\). For this case, the bands with \((v_1, 0)\) can have angular momenta and parity \(L^P = 0^+, 2^+, 3^-, 4^\pm, \ldots\), whereas the angular momentum content of the doubly degenerate vibrations \((v_1, 1)\) is given by \(L^P = 1^-, 2^\mp, 3^\pm, \ldots\), in agreement with Ref. [13].

For the schematic Hamiltonian of Eq. (25), the angular momentum states belonging to different \(K\)-bands are degenerate [8].

4.4. Four-body clusters: spherical top

As a last example, I discuss the spherical top with tetrahedral symmetry as a special case of the ACM for four identical clusters whose spectrum generating algebra is given by \(U(10)\). Let us consider a \(S_4\) invariant Hamiltonian of the form

\[
H_4 = \xi_1 (R^2 s^+ s^- - b^+_1 \cdot b^+_1 - b^+_2 \cdot b^+_2 - b^+_3 \cdot b^+_3) \text{(h.c.)}
\]

\[
+ \xi_2 \left[ (-2\sqrt{2} b^+_1 \cdot b^+_3 + 2b^+_1 \cdot b^+_2) \text{(h.c.)} \right]
\]

\[
+ \xi_3 \left[ (2b^+_1 \cdot b^+_3 + 2\sqrt{2} b^+_1 \cdot b^+_2) \text{(h.c.)} \right]
\]

\[
+ \kappa_1 \vec{L} \cdot \vec{L} + \kappa_2 (\vec{L} \cdot \vec{L} - \vec{T} \cdot \vec{T})^2,
\]  

(31)

where \(\vec{L}\) denotes the angular momentum in coordinate space and \(\vec{T}\) the angular momentum in index space.

For \(R^2 = 0\), this Hamiltonian has \(U(10) \supset U(9)\) symmetry and corresponds to a spherical vibrator, whereas for \(R^2 = 1\) and \(\xi_2 = \xi_3 = 0\) it has \(U(10) \supset SO(10)\) symmetry and corresponds to a deformed oscillator. The general case with \(R^2 \neq 0\) and \(\xi_1, \xi_2, \xi_3 > 0\) does not correspond to a dynamical symmetry, and hence its energy spectrum cannot be obtained in closed analytic form.
In this case, the energy eigenvalues and eigenvectors are calculated numerically by diagonalizing the Hamiltonian in an appropriate basis. However, an approximate energy formula can still be derived in a semiclassical mean-field analysis. The general expression of the classical limit of the Hamiltonian of Eq. (31) has a complicated structure. We first study the potential energy surface which is obtained by setting all momenta equal to zero. Its equilibrium configuration corresponds to coordinates that have equal length ($q_{1,0} = q_{2,0} = q_{3,0} = q_0 / \sqrt{3}$)

$$q_0 = \sqrt{2R^2/(1 + R^2)}, \quad \chi_{1,0} = \arctan \sqrt{2}, \quad \chi_{2,0} = \pi/4,$$

and are mutually perpendicular

$$\zeta_{12,0} = \zeta_{23,0} = \zeta_{31,0} = \pi/4,$$

where $2\zeta_{ij}$ denotes the relative angle between $\vec{\alpha}_i$ and $\vec{\alpha}_j$. Geometrically, the four clusters are located at the vertices of an regular tetrahedron.

In the limit of small oscillations around the equilibrium shape the intrinsic degrees of freedom decouple and become harmonic. The resulting energy spectrum is that of the vibrational excitations of a spherical top with tetrahedral symmetry

$$E_{4,\text{vib}} = \omega_1(v_1 + \frac{1}{2}) + \omega_2(v_2 + 1) + \omega_3(v_3 + \frac{3}{2}),$$

with frequencies

$$\omega_1 = 4NR^2\xi_1, \quad \omega_2 = \frac{8NR^2}{1 + R^2}\xi_2, \quad \omega_3 = \frac{8NR^2}{1 + R^2}\xi_3.$$

in agreement with the results obtained in a normal mode analysis. Here $v_1$ represents the vibrational quantum number for a symmetric stretching $A$ vibration, $v_2 = v_{2a} + v_{2b}$ denotes a doubly degenerate $E$ vibration, and $v_3 = v_{3a} + v_{3b} + v_{3c}$ a three-fold degenerate $F$ vibration.

Whereas the angular momentum $L$ is an exact symmetry of $H_4$ of Eq. (31), the angular momentum in index space $I$ in general does not commute with the Hamiltonian. Only if $\xi_2 = \xi_3$ in Eq. (31), does $I$ become a good quantum number. The rotational excitations of the ground state vibrational band of $H_4$ with $(v_1, v_2, v_3) = (0, 0, 0)$ are characterized by $L = I$. This property is a consequence from the fact that the operator $\vec{L} \cdot \vec{L} - I \cdot I$ annihilates the coherent (or intrinsic) state corresponding to the rigid equilibrium shape of Eqs. (32,33). Therefore, the rotational energies of the ground state band are given by

$$E_{4,\text{rot}} = \kappa_1 L(L + 1).$$

In Fig. 3 I show the structure of the spectrum of a spherical top with tetrahedral symmetry according to the approximate energy formula of Eqs. (34) and (36). The energy spectrum consists of a series of rotational bands labeled by $(v_1, v_2, v_3)$. It is assumed that the spin of the identical clusters is zero, as is relevant for the description of the $^{16}$O nucleus as a cluster of four $\alpha$ particles. As a consequence, all states in Fig. 3 are symmetric under $S_4$. For this case, the bands with $(v_1, 0, 0)$ can have angular momenta and parity $L^P = 0^+, 3^-, 4^+, 6^-, \ldots$, whereas the angular momentum content of the doubly degenerate $E$ vibration $(v_1, 1, 0)$ is given by $L^P = 2^+, 4^+, 5^-, \ldots$, and for the triply degenerate $F$ vibration $(v_1, 0, 1)$ by $L^P = 1^-, 2^+, 3^-, 4^+, \ldots$, in agreement with Ref. [13].
5. Summary and conclusions

In this contribution, I discussed an algebraic treatment of few-body systems in terms of the Algebraic Cluster Model which is based on the algebraic quantization of the relative Jacobi variables. The ensuing $U(3n - 2)$ spectrum generating algebra incorporates all vibrational and rotational degrees of freedom from the beginning. It was shown that the permutation symmetry of identical clusters can be taken into account in an exact manner.

I reviewed four special solutions of $S_n$ invariant ACM Hamiltonians for which the energy eigenvalue problem can be solved in closed analytic form. Two cases can be solved exactly. They correspond to dynamical symmetries, called the $U(3n - 2) \supset U(3n - 3)$ and $U(3n - 2) \supset SO(3n - 2)$ limits, which were interpreted as the spherical and deformed oscillator, respectively. In addition, there are some other cases of $S_n$ invariant Hamiltonians for which approximate solutions can be obtained in a semiclassical mean-field analysis. For the case of three-body clusters I discussed the oblate symmetric top (equilateral triangle) and for four-body clusters the spherical top with tetrahedral symmetry (regular tetrahedron).

In this contribution, I focussed on the symmetry structure of the ACM. For two- and three-body clusters interesting applications were found both in molecular, nuclear and hadronic physics [2]-[8]. Future work includes the derivation of electromagnetic transition rates and form factors as well as the analysis of possible applications of the ACM for four-body systems in molecular physics ($X_4$ molecules), nuclear physics ($^{16}$O as a cluster of four $\alpha$ particles), and hadronic physics ($q^4 - \bar{q}$ multiquark configurations).

Finally, it is important to note that the ACM provides a general framework to study the full rotational and vibrational structure of many-body systems which is not restricted to the case of identical particles discussed in this contribution. It can be applied to other situations as well, such as nonidentical particles [7] and/or other geometric configurations.
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