Gauge Coupling Running in Minimal $SU(3) \times SU(2) \times U(1)$ Superstring Unification

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ABSTRACT

We study the evolution of the gauge coupling constants in string unification schemes in which the light spectrum below the compactification scale is exactly that of the minimal supersymmetric standard model. In the absence of string threshold corrections the predicted values $\sin^2 \theta_W = 0.218$ and $\alpha_s = 0.20$ are in gross conflict with experiment, but these corrections are generically important. One can express the string threshold corrections to $\sin^2 \theta_W$ and $\alpha_s$ in terms of certain modular weights of quark, lepton and Higgs superfields as well as the moduli of the string model. We find that in order to get agreement with the experimental measurements within the context of this minimal scheme, certain constraints on the modular weights of the quark, lepton and Higgs superfields should be obeyed. Our analysis indicates that this minimal string unification scheme is a rather constrained scenario.
Unification of coupling constants is a necessary phenomenon in string theory. Specifically, at tree level, the gauge couplings of a gauge group $G_a$ have simple relations [1] to the string coupling constant which is determined by the vacuum expectation value of the dilaton field: 

$$\frac{1}{g_a^2} = \frac{k_a}{g_{\text{string}}}$$

where $k_a$ is the level of the corresponding Kac-Moody algebra. At higher loop levels this relation holds only at the typical string scale which is of the order of the Planck mass $M_P$. Below this scale all couplings evolve according to their renormalization group equations in the same way as in standard GUT theories as first discussed in [2]. This allows a comparison of the coupling constants with the low energy data considering a specific string model. In addition, thresholds effects due the massive string excitations modify the above mentioned tree level relations.

Let us recall briefly the exact definition of the string mass scale. It is given in the $\overline{MS}$ scheme by [3]

$$M_{\text{string}}^2 = \frac{2}{\sqrt{27\pi\alpha'}} \frac{e^{(1-\gamma)}}{\sqrt{27\pi\alpha'}}$$

where $\gamma$ is the Euler constant and $\alpha' = \frac{16\pi}{g_{\text{string}}^2}M_P^2$. Numerically one finds [4]

$$M_{\text{string}} = 0.7 \times g_{\text{string}} \times 10^{18} \text{ GeV}. \quad (2)$$

(Note that this value differs from the one found in [3].) This mass scale has to be compared with low energy data using the field theory renormalization group equations and taking into account also the model-dependent stringy threshold corrections. A phenomenological very promising model is the minimal supersymmetric standard model with gauge group $G = SU(3) \times SU(2) \times U(1)$. The relevant evolution of the electro-weak and strong coupling constants was considered some time ago in [5], [6]. Recently this analysis was reconsidered [7] taking into account the up-dated low energy data. The results for $\sin^2 \theta_W$ and $\alpha_s$ are in very good agreement with data for a value of the unification mass $M_X \simeq 10^{16}$ GeV and a susy threshold close to the weak scale. On the other hand, as we show below, the large value for the string unification scale $M_{\text{string}}$ leads to rather embarrassing results for the couplings $\sin^2 \theta_W^0 = 0.218$ and $\alpha_s^0 = 0.20$. In this paper we discuss the question whether one can make consistent the unification scale of the minimal supersymmetric standard model with the relevant string unification
scale $M_{\text{string}}$. (String unification and threshold effects within the flipped $SU(5)$ model were considered in [8].) At the first sight, this seems very unlikely since $M_{\text{string}}$ is substantially larger than the minimal susy model scale $M_X$. However, one might hope that the effects of the string threshold contributions could make the separation of these two scales consistent. Although the threshold effects are rather small in usual grand unified models [9] it is not obvious that the same holds true in string unification since we have to remember that above the string scale an infinite number of massive states contribute to the threshold. This is obviously very different from field theory unification scenarios.

The structure of the paper is the following. First we will collect some formulas about one-loop gauge coupling constants with special emphasis on string threshold corrections and their relation to target space duality. Then we will apply these formulas to the case of the unification of the three physical coupling constants $g_1, g_2, g_3$. Our approach here will be mainly phenomenological. We will consider a possible situation in which

a) the massless particles with standard model gauge couplings are just those of the minimal supersymmetric standard model

b) there is no partial (field theoretical) unification scheme below the string scale.

This is in principle the simplest string unification scheme that one can think of and that is why we call it minimal string unification. Up to now no realistic string model with this characteristics has been built but the model search done up to now is extremely limited and by no means complete. We would like to answer the question whether such a minimal scenario can be made consistent with the measured values of the low energy coupling constants.

The one-loop running gauge coupling constant of a (simple) gauge group $G_a$ is of the following form:

$$\frac{1}{g_a^2(\mu)} = \frac{k_a}{g_{\text{string}}^2} + \frac{b_a}{16\pi^2} \log \frac{M_a^2}{\mu^2} + \Delta_a. \quad (3)$$

Here $b_a = -3C(G_a) + \sum_{R_a} h_{R_a} T(R_a)$ is the $N = 1$ $\beta$-function coefficient ($h_{R_a}$ is the number of chiral matter fields in a representation $R_a$). $M_a$ is the renormalization point below which the effective field theory running of the coupling constant
begins. (As we will discuss below, $M_a$ will depend on the specific model and also on the considered gauge group.) $\Delta_a$ are the string threshold contributions [3] which arise due to the integration over the infinite number of massive string states, in particular momentum and winding states: $\Delta_a \propto \log \det \mathcal{M}$, where $\mathcal{M}$ is the mass matrix of the heavy modes.

In the following we would like to give a brief description of how one derives the expressions for the field-dependent stringy threshold corrections and for the renormalization scale $M_a$ which, in general, is also field dependent. Most directly, these quantities can be obtained by world-sheet string computations of string amplitudes involving external gauge fields and moduli as done [10], [11] for the case of (2,2) symmetric orbifold compactifications [12]. These computations are closely related to the calculation [13] of the target space free energies of compactified strings.

A second very useful approach to obtain information about the form of the string threshold corrections is the use of the target space duality symmetries [14] present in many known string compactifications. Here, the main idea is related to the observation [15] that in string compactifications the scale $M_a$ below which the effective field theory running of the gauge coupling constants starts becomes a moduli dependent quantity,

$$M_a^2 = (2R^2)^\alpha M_{\text{string}}^2. \quad (4)$$

$R$ is a background parameter denoting (in Planck units) the overall radius of the compact six-dimensional space, and the power $\alpha$ is a model- and gauge group dependent parameter. (In naive field theory compactifications one expects $\alpha = -1$. However, as we will discuss in the following, for orbifold compactifications $\alpha$ can also take different values.) Thus the running gauge coupling constant eq.(3) generically depends on the background radius. To be specific consider orbifold type of compactifications. Here the radius is related to the real part of a complex modulus field, $T = R^2 + iB$, ($B$ is an internal axion field) and the target space duality group is given [16] by the modular group $PSL(2, \mathbb{Z})$, acting on $T$ as $T \rightarrow \frac{aT + b}{cT + d}$ ($a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$). It follows that the effective action involving the $T$-field must be target space modular invariant and is given...
in terms of modular functions [17]. Now, requiring [18] the invariance of $g_a^2(\mu)$ under target space modular transformations enforces $\Delta_a$ to be a non-trivial $T$-dependent function. Specifically, as discussed in [18], [19] for the case $\alpha = -1$, target space modular invariance, together with the requirement of having no poles inside the fundamental region, implies

$$\Delta_a(T, \bar{T}) = \frac{\alpha b_a}{16\pi^2} \log |\eta(T)|^4,$$

where $\eta(T)$ is the Dedekind function. Notice that for large $T$ one recovers the linear behavior found in ref. [20].

The parameter $\alpha$ is intimately related to the modular weights of the charged matter fields which transform non-trivially under the gauge group $G_a$. To understand this, consider a standard supergravity, Yang Mills field theory [21] with massless gauge singlet chiral moduli fields $T_i$ and massless charged chiral matter fields $\phi_i^{R\alpha}$ ($\alpha = 1, \ldots, h_{R\alpha}$). The relevant part of the tree level supergravity Lagrangian is specified by the following Kähler potential at lowest order in $\phi_i^{R\alpha}$:

$$K(T_i, \phi_i^{R\alpha}) = K(T_i, \bar{T}_i) + K_{ij}^{R\alpha}(T_i, \bar{T}_i) \phi_i^{R\alpha} \bar{\phi}_j^{R\alpha}. \quad (6)$$

In the following we assume that the Kähler metric for the charged fields is proportional to the Kähler metric of the moduli, which was shown [23] to be true for (2,2) Calabi-Yau string compactifications [24], i.e. $K_{ij}^{R\alpha} \propto \partial T_i \partial \bar{T}_j K(T_i, \bar{T}_i)$. As discussed in [4], [25], [26], at the one loop level $\sigma$-model anomalies play a very important role for the determination of the renormalized gauge coupling constant. Specifically, one has to consider two types of triangle diagrams with two gauge bosons and several moduli fields as external legs and massless gauginos and charged (fermionic) matter fields circulating inside the loop: First the coupling of the moduli to the charged fields contains a part described by a composite Kähler connection, proportional to $K(T_i, \bar{T}_i)$, which couples to gauginos as well as to chiral matter fields $\phi_i^{R\alpha}$. It reflects the (tree level) invariance of the theory under Kähler transformations. Second, there is a coupling between the moduli

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* If some of the matter fields become massive due to a trilinear superpotential one ends up with the same results about the form of the threshold corrections [22].
and the $\phi^R_i$'s by the composite curvature (holonomy) connection. It originates from the non-canonical kinetic energy $K^R_{ij}$ of the matter fields $\phi^R_i$ and shows the (tree level) invariance of the theory under general coordinate transformations on the complex moduli space. These two anomalous contributions lead, via supersymmetry, to the following one-loop modification of the gauge coupling constant [25],[4],[26]:

$$\frac{1}{g_a^2} = \frac{k_a}{g_{\text{string}}^2} - \frac{1}{16\pi^2} \left( C(G_a) - \sum_{R_a} h_{R_a} T(R_a) K(T_i, \bar{T}_i) \right) + 2 \sum_{R_a} T(R_a) \log \det K^R_{ij}(T_i, \bar{T}_i).$$  \hspace{1cm} (7)

Now assume that the string theory is invariant under target space duality transformations which are discrete reparametrizations of the moduli. (The simple $R \to 1/R$ duality symmetry in bosonic string compactification was shown [27] to be unbroken in each order of string perturbation.) These transformations do not leave invariant the Kähler potential $K(T_i, \bar{T}_i)$ and also $\log \det K_{ij}$. Thus eq.(7) is not invariant under duality transformations. It follows that the duality anomalies must be cancelled by adding new terms to the effective action. Specifically, there are two ways to cancel these anomalies. First [4],[26], one can perform a moduli dependent, but gauge group independent redefinition of the dilaton/axion field, the so-called $S$-field, such that $S + \bar{S}$ transforms non-trivially under duality transformations and cancels in this way some part or all of the duality non-invariance of eq.(7). This field redefinition of the $S$-field is analogous to Green-Schwarz mechanism [28] and leads to a mixing between the moduli and the $S$-field in the $S$-field Kähler potential. Second, the duality anomaly can be cancelled by adding to eq.(7) a term which describes the threshold contribution due to the massive string states. (Only the specific knowledge about the massive string spectrum can determine the exact coefficients for the Green-Schwarz and threshold terms whose combined variation cancels the total modular anomaly. However, as it will become clear in the following, the coefficient of Green-Schwarz term is irrelevant for the determination of the unification mass scales.) In analogy to the Dedekind function the threshold contributions are given in terms of automorphic functions of the target space duality group. Specifically, as described in [13], for general
(2,2) Calabi-Yau compactifications there exist two types of automorphic functions: the first one provides a duality invariant completion of $K(T_i, \bar{T}_i)$, where the second one is needed to cancel the duality anomaly coming from $\log \det K_{ij}$. These two types of automorphic functions can be, at least formally, constructed for all (2,2) Calabi-Yau compactifications [13].

In the following we restrict ourselves to symmetric (but not necessarily (2,2) symmetric) $\mathbb{Z}_N$ [12], [29] and $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds [30]. Every orbifold of this type has three complex planes corresponding to three two-dimensional subtori. For non-trivial examples each orbifold twist $\delta_m = (\delta_1, \delta_2, \delta_3)$ acts either simultaneously on two or all three planes. For simplicity we consider only the overall modulus $T = R^2 + iB$ where the target space duality transformations are given by the modular group $PSL(2, \mathbb{Z})$. The Kähler potential for this overall modulus looks like [31]

$$K(T, \bar{T}) = -3 \log(T + \bar{T}).$$

The Kähler metric of the matter fields has the following generic form [23]:

$$K_{ij}^{R_a} = \delta_{ij}(T + \bar{T})^{n_{R_a}}.$$  \hspace{1cm} (9)

Target space modular invariance then implies that the matter fields transform under $PSL(2, \mathbb{Z})$ as

$$\phi_i^{R_a} \rightarrow \phi_i^{R_a}(icT + d)^{n_{R_a}}.$$  \hspace{1cm} (10)

Thus we identify the integers $n_{R_a}$ as the modular weights of $\phi_i^{R_a}$. (For the $\mathbb{Z}_3$ orbifold see [32].) Specifically, for symmetric orbifold compactifications there are three different types of matter fields:

a) Untwisted matter with $n_R = -1$.

b) Twisted matter fields associated with an orbifold twist $\delta_m$. Here $n_R = -2$ if $\delta_m$ acts on all three planes. $n_R = -1$ if the twist acts only on two of the three planes. Thus the latter kind of twisted fields behave exactly like untwisted fields under modular transformations.

c) Twisted moduli with $n_R = -3$ or $n_R = -2$ if the corresponding twist acts on all three planes or only on two planes respectively.
Then, using eq.(7), the one-loop contribution to the gauge coupling constant due to the anomalous triangle diagrams with massless charged fields has the following form [25],[4]:

$$\frac{1}{g_a^2} = \frac{k_a}{g_{\text{string}}^2} + \frac{1}{16\pi^2} b'_a \log(T + \bar{T}),$$

$$b'_a = 3C(G_a) - \sum_{R_a} h_{R_a} T(R_a)(3 + 2n_{R_a}) = -b_a - 2 \sum_{R_a} h_{R_a} T(R_a)(1 + n_{R_a}).$$

(11)

As discussed already, the modular anomaly of this contribution to $\frac{1}{g_a^2}$ from the massless fields can be cancelled by a universal Green-Schwarz term plus the threshold contribution from the massive orbifold excitations. The orbifold threshold contribution takes the following form (up to a small $T$-independent term [3]):

$$\Delta_a(T, \bar{T}) = \frac{1}{16\pi^2} (b'_a - k_a b_{\text{GS}}) \log |\eta(T)|^4.$$  \hspace{1cm} (12)

Here $b_{\text{GS}}$ is the universal coefficient of the Green-Schwarz term. Without going into any detail let us just state the main result concerning the coefficient $b'_a - k_a b_{\text{GS}}$ [10],[11]. The threshold contribution of the massive fields, i.e. $b'_a - k_a b_{\text{GS}}$, is non-vanishing if at least one of the three complex planes is not rotated by some of the orbifold twist $\delta_m$. Then, within this sector, the massive spectrum with $T$-dependent masses is $N = 2$ space-time supersymmetric and $b'_a - k_a b_{\text{GS}}$ is proportional to the $N = 2$ $\beta$-function coefficient. In this case $b'_a - k_a b_{\text{GS}}$ is in general non-zero for all gauge groups including the unbroken $E_8$ in the hidden sector. On the other hand, sectors corresponding to planes which are rotated by all twists $\delta_i$ lead to a massive $T$-dependent spectrum with $N = 4$ space-time supersymmetry and therefore do not contribute to the threshold corrections.

Let us insert the threshold contribution eq.(12) into the one-loop running coupling constant eq.(3):

$$\frac{1}{g_a^2(\mu)} = \frac{k_a}{g_{\text{string}}^2} + \frac{b_a}{16\pi^2} \log \frac{M_a^2}{\mu^2} + \frac{1}{16\pi^2} (b'_a - k_a b_{\text{GS}}) \log |\eta(T)|^4,$$

$$M_a^2 = (TR)^\alpha M_{\text{string}}^2, \quad \alpha = \frac{b'_a - k_a b_{\text{GS}}}{b_a},$$

(13)

where $T_R = T + \bar{T} = 2R^2$. This expression is explicitly target space modular invariant. Here we have absorbed the piece from the massless fields in eq.(11)
which is not cancelled by the Green-Schwarz term into the definition of the renormalization point \( M_a \) (the remainder is absorbed into \( \frac{1}{g_{\text{string}}} \) [4]) since it is a field theoretical, infrared effect and does not originate from the heavy string modes.

Now we are finally ready to discuss the unification of the gauge coupling constants. The unification mass scale \( M_X \) where two gauge group coupling constants become equal, i.e. \( \frac{1}{k_a g_a^2(M_X)} = \frac{1}{k_b g_b^2(M_X)} \), becomes using eq.(13)

\[
\frac{M_X}{M_{\text{string}}} = [T_R|\eta(T)|^4]\frac{b'_a k_a - b'_b k_b}{b_a k_b - b_b k_a}.
\]

Note that since we are interested only in the difference of two gauge couplings, the universal Green-Schwarz term is irrelevant for \( M_X \). Since the moduli-dependent function \( (T+\bar{T})|\eta(T)|^4 \) is smaller (bigger) than one if \( b'_a k_a - b'_b k_b \) is bigger (smaller) than zero. Comparing with the definition of \( b'_a \) in eq.(11) one recognizes that for \( k_a = k_b \) twisted states with \( n_R < -1 \) are necessarily required to have \( M_X < M_{\text{string}} \).

Let us now briefly discuss three known (2,2) orbifold examples \( (k = 1) \). First for the \( \mathbb{Z}_3 \) and \( \mathbb{Z}_7 \) orbifolds, each of the three planes is simultaneously rotated by all twists. Thus \( b'_a - b_{\text{GS}} = 0 \) for all gauge groups. It trivially follows that the renormalization point is given as \( M_a = M_{\text{string}} \). The radius independence of the renormalization point is due to the fact that the spectrum of the massive Kaluza-Klein and winding states is \( N = 4 \) supersymmetric and has therefore no effect in loop calculations. The absence of threshold corrections (in other words, only a universal gauge group independent piece contributes to the gauge coupling constant at one loop) also trivially implies that the unification scale \( M_X \), for example the unification point of \( E_8 \) and \( E_6 \), is given by \( M_{\text{string}} \).

A second example, which is rather orthogonal to the previous case, is the symmetric \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold. (Also many four-dimensional heterotic strings obtained by the fermionic [33], [34] or by the covariant lattice [35] construction fall into his category.) Here each of the \( \mathbb{Z}_2 \) twists leaves invariant exactly one of the three orbifold planes. Then we obtain that \( b_{\text{GS}} = 0 \), i.e. there is no one-loop \( S-T \) mixing in the Kähler potential of this model. Furthermore, according to our general rules all twisted matter fields (27 of \( E_6 \)) have modular weight \( n_{27} = -1 \)
and behave like untwisted fields. It follows that $b'_a = -b_a$ ($a = E_8, E_6$). Thus we 
obtain $\alpha = -1$ and the radius dependence of the renormalization point agrees 
with the naive field theoretical expectation: $M_a = M_{\text{string}}T_R^{-1/2}$. The unification 
scale of $E_8$ and $E_6$ is given by $M_X = M_{\text{string}}/(T_R^{1/2}|\eta(T)|^2)$ and is therefore larger 
than the string scale for all values of the radius.

Now let us apply the above discussion to the case of the unification of the 
gauge coupling constants within the minimal string unification. We will make use 
of the threshold formulae of eqs.(13),(14) although they were originally derived 
for a general class of abelian $\mathbb{Z}_N$ and $\mathbb{Z}_N \times \mathbb{Z}_M$ $(2,2)$ orbifolds. In fact the gauge 
groups in these cases is always $E_6 \times E_8$ and not anything looking like the standard 
model group. However we would like to argue for the validity of these formulae 
in the presence of Wilson lines and for $(0,2)$ type of gauge embeddings because 
the structure of the untwisted moduli is exactly the same as in the corresponding 
$(2,2)$ orbifold. We will again only consider the string threshold effects dependence 
on the overall modulus $T$. Then the $T$-field Kähler potential and the Kähler 
metric of the matter fields are given by eqs.(8) and (9) respectively, and the low 
energy contribution to the gauge coupling constants is still described by eq.(11). 
Thus, using the requirement of target space modular invariance, the threshold 
formulae (13),(14) remain valid for generic symmetric orbifolds, and not only 
for their standard embeddings. (For example one can check that for $\mathbb{Z}_3$ $(0,2)$ 
orbitfolds the $b'$ coefficients of all gauge groups again exactly agree.) These type 
of models may in general yield strings with the gauge group of the standard 
model and appropriate matter fields as discussed e.g. in [29]. In reality the 
threshold effects will depend not only on the untwisted moduli but on other 
marginal deformations like the twisted moduli and even on extra charged scalars 
with flat potentials present in specific models. We believe that considering just 
the dependence on the overall (volume) modulus gives us an idea of the size 
and effects of the string threshold. Finally, as discussed above, we would expect 
to find similar results in more general (Calabi Yau) four dimensional strings in 
which the threshold effects have not been explicitly computed. The low energy 
anomaly arguments should be valid for an arbitrary string and similar formulae 
to those below should be found for those more general cases with the obvious 
replacements due to the different duality groups involved.
Let us first consider the joining of the $SU(2)$ and $SU(3)$ gauge coupling constants $g_2$ and $g_3$ at a field theory unification scale $M_X$. If such a unification takes place eq.(14) leads to the result ($k_2 = k_3 = 1$)

$$M_X^2 = M_{\text{string}}^2 \left(T_R \mid \eta(T) \mid^4 \right)^{\frac{b_i - b'_{i}}{2k_2 - 2k_3}}$$

(15)

for the unification scale of the $g_2$ and $g_3$ coupling constants. Recalling that one always has $T_R \mid \eta(T) \mid^4 \leq 1$, one concludes that $M_X$ may be smaller or bigger than $M_{\text{string}}$ depending on the relative sign of $(b'_2 - b'_3)$ versus $(b_2 - b_3)$. In general, the definition of $b'_i$ in eq.(11) shows that relative sign depends on the modular weights of the matter fields. In some cases (e.g. when all matter fields have modular weight $n = -1$) one has $b'_i = -b_i$ and $M_X$ is necessarily bigger than $M_{\text{string}}$. In these cases one cannot accommodate the difference between $M_X$ and $M_{\text{string}}$ we discussed above and the minimal string unification scheme is simply not viable. This is the case of any model based on the $Z_2 \times Z_2$ orbifold (or equivalent models constructed with free world-sheet fermions) since all matter fields have modular weight one.

Let us now be a bit more quantitative and try to answer the following question: what are the values for modular weights $n_\beta$ of quarks, leptons and Higgs as well as the corresponding values of $T_R$ which would allow for $\sin^2 \theta_W$ and $\alpha_s$ values in reasonable agreement with data? Making use of equation (14) one gets for the value of the electroweak angle $\theta_W$ after some standard algebra

$$\sin^2 \theta_W(\mu) = \frac{k_2}{k_1 + k_2} - \frac{k_1}{k_1 + k_2} \frac{\alpha_e(\mu)}{4\pi} \left( A \log \left( \frac{M_{\text{string}}^2}{\mu^2} \right) - A' \log(T_R \mid \eta(T) \mid^4) \right)$$

(16)

where $A$ is given by

$$A \equiv \frac{k_2}{k_1} b_1 - b_2$$

(17)

and $A'$ has the same expression after replacing $b_i \rightarrow b'_i$. The standard grand unification values of the Kac-Moody levels correspond to the choice $k_2 = k_3 = 1$ and $k_1 = 5/3$. Finally, $\alpha_e$ is the fine structure constant evaluated at a low energy scale $\mu$ (e.g. $\mu = M_Z$). In an analogous way one can compute the low energy
value of the strong interactions fine structure constant \( \alpha_s \)

\[
\frac{1}{\alpha_s(\mu)} = \frac{k_3}{(k_1 + k_2)} \left( \frac{1}{\alpha_e(\mu)} - \frac{1}{4\pi} B \log\left( \frac{M_{\text{string}}^2}{\mu^2} \right) - \frac{1}{4\pi} B' \log(T_R|\eta(T)|^4) \right)
\]

(18)

where

\[
B \equiv b_1 + b_2 - \frac{(k_1 + k_2)}{k_3} b_3
\]

(19)

and \( B' \) has the same expression after replacing \( b_i \rightarrow b'_i \). Let us now define

\[
\delta A \equiv A' + A = -2 \sum_\beta (n_\beta + 1) \left( \frac{k_2}{k_1} Y^2(\beta) - T_2(\beta) \right)
\]

(20)

where the sum runs over all the matter fields and \( n_\beta \) are the corresponding modular weights. \( Y(\beta) \) is the hypercharge of each field and \( T_2(\beta) \) the corresponding \( SU(2) \) quadratic Casimir \((T_2 = 1/2 \text{ for a doublet})\). Analogously let us define

\[
\delta B = B + B' = -2 \sum_\beta (n_\beta + 1) \left( Y^2(\beta) + T_2(\beta) - \frac{(k_1 + k_2)}{k_3} T_3(\beta) \right).
\]

(21)

We can now write equations (16) and (18) as follows \((k_2 = k_3 = 1, k_1 = 5/3)\)

\[
\sin^2 \theta_W = \frac{3}{8} - \frac{5\alpha_e}{32\pi} A \log\left( \frac{M_T^2}{\mu^2} \right) - \frac{5\alpha_e}{32\pi} \delta A \log(T_R|\eta(T)|^4),
\]

(22)

\[
\frac{1}{\alpha_s(\mu)} = \frac{3}{8\alpha_e} - \frac{3B}{32\pi} \log\left( \frac{M_T^2}{\mu^2} \right) - \frac{3\delta B}{32\pi} \log(T_R|\eta(T)|^4)
\]

(23)

where

\[
M_T^2 \equiv \frac{M_{\text{string}}^2}{T_R|\eta(T)|^4}.
\]

(24)

All the model dependence (through the modular weights) is contained in \( \delta A, \delta B \). Denoting by \( n^i_\beta \) the modular weight of the \( i-th \) generation field of type
\( \beta = Q, U, D, L, E \) one can explicitly evaluate that dependence and find

\[
\delta A = \frac{2}{5} \sum_{i=1}^{N_{\text{gen}}} (7n_Q^i + n_L^i - 4n_U^i - n_D^i - 3n_E^i) + \frac{2}{5} (2 + n_H + n_{\bar{H}}),
\]

(25)

\[
\delta B = 2 \sum_{i=1}^{N_{\text{gen}}} (n_Q^i + n_D^i - n_L^i - n_E^i) - 2 (2 + n_H + n_H)
\]

(26)

where \( N_{\text{gen}} \) is the number of generations and \( n_H, n_{\bar{H}} \) are the modular weights of the Higgs fields.

It is easy to see from equations (16) and (18) that both \( \delta A \) and \( \delta B \) have to be positive in order to have any chance to obtain the correct values for \( \sin^2 \theta_W \) and \( \alpha_s \). In the minimal supersymmetric standard model one has \( b_3 = -3, b_2 = m_H \) and \( b_1 = 10 + m_H \), where \( m_H \) is the number of pairs of Higgs doublets \((m_H = 1 \) in the minimal case). For the standard unification \( k_i \) values one then finds \( A = 6 - \frac{2}{5}m_H = 28/5 \) and \( B = 18 + 2m_H = 20 \). As already explained, the requirement \( M_X < M_{\text{string}} \) implies \( A'/A > 0 \) and also \( B'/B > 0 \) (remember \( \log(T_R|\eta(T)|^4) \) is negative). Then one has the conditions

\[
\delta A > A = \frac{28}{5},
\]

(27)

\[
\delta B > B = 20
\]

(28)

in the minimal model. Notice that these conditions are violated explicitly in the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifolds (in which case \( \delta A = \delta B = 0 \)) and also in \( \mathbb{Z}_3 \) and \( \mathbb{Z}_7 \). In the latter cases one has \( \delta A = A \) and \( \delta B = B \) since \( A' = B' = 0 \). (Parenthetically, these latter equations can be used combined with eqs.(25) and (26) in order to get constraints on the number of \( SU(2) \)-doublets and \( SU(3) \)-triplets coming from untwisted, twisted and twisted moduli sectors in specific \( \mathbb{Z}_3 \) and \( \mathbb{Z}_7 \) \((0,2)\) orbifolds).

Another point to remark is that if there are \( SU(5) \)-type boundary conditions for the matter kinetic terms (and, hence, for the modular weights) one has \( n_U = n_Q = n_E \) and \( n_D = n_E \). In this case \( \delta A = \frac{2}{5}(2 + n_H + n_{\bar{H}}) \) and \( \delta B = -2(2 + \)
\( n_H + n_{\bar{H}} \) = \(-5\delta A \) and both quantities cannot be simultaneously positive. Thus \( SU(5) \)-like boundary conditions for the modular weights cannot accommodate the values of the measured low energy couplings in the context of minimal string unification. On the other hand there is no reason why those boundary conditions should hold since we are assuming that the gauge group is \( SU(3) \times SU(2) \times U(1) \) up to the string scale. Furthermore, other unification schemes e.g. inside semisimple groups like \( SU(4) \times SU(2) \times SU(2) \) or \( SU(3) \times SU(3) \times SU(3) \) do not lead to those boundary conditions. All of these unification schemes are consistent with \( k_1 = 5/3 \). Incidentally, let us recall that for the standard values \( k_2 = k_3 = 1 \) and \( k_1 = 5/3 \) the low energy symmetry is enlarged to \( SU(5) \) only in the case we insist on the absence of massive fractionally charged states \[36\]. We do not insist on that, we just assume the minimal low energy susy particle content but nothing specific about the massive sector.

In principle, if the conditions (27) and (28) are met there may exist a value of \( T_R \) such that one can accommodate the measured low energy values of coupling constants. In the absence of string threshold effects (i.e. for \( \delta A = A \) and \( \delta B = B \)) one finds from equations (16),(18) \( \sin^2 \theta_W(M_Z) = 0.218 \) and \( \alpha_s^0(M_Z) = 0.205 \). The effect of non-vanishing threshold effects in the minimal scenario we are discussing is displayed in figures 1 and 2. The first shows the value of \( \sin^2 \theta_W(M_Z) \) as a function of \( \text{Re}T \equiv T_R/2 \) for different values of \( \delta A \). A similar plot for \( \alpha_s(M_Z) \) is shown in figure 2. The shaded areas correspond to the experimental results. The bounds in eqs.(27) and (28) are apparent in the figures. One also observes that one can get results within the experimental constraints for sufficiently large values of \( \delta A, \delta B \) and \( T_R \). In fact one can eliminate the explicit dependence on \( T_R \) by combining equations (22) and (23). In this way one finds a linear equation relating \( \delta A \) and \( \delta B \):

\[
(\delta B - B) = \gamma (\delta A - A),
\]

\[
\gamma = \frac{5}{3} \alpha_e \left( \frac{1}{\alpha_s^0} - \frac{1}{\alpha_s(\mu)} \right) \left( \frac{\sin^2 \theta_W^0}{\sin^2 \theta_W(\mu)} \right).
\]

If \( \delta A/A = \delta B/B \) the string corrections may be entirely contained in a change in the scale in the original field theoretical analysis, and all three coupling
constants meet (at the one loop level) at the same energy scale. This corresponds to \( \gamma = B/A = 25/7 \). Allowing for the experimental errors in \( \alpha_s(\mu) \) and \( \sin^2 \theta_W(\mu) \) one more generally finds \( 2.2 \leq \gamma \leq 4.0 \). One can then search for values for the modular weights \( n_\beta \) of the standard model particles compatible with eqs. (27),(28),(29). Assuming generation independence for the \( n_\beta \) as well 

\[-3 \leq n_\beta \leq -1\]

one finds, interestingly enough, a unique answer for the matter fields:

\[ n_Q = n_D = -1 \quad ; \quad n_U = -2 \quad ; \quad n_L = n_E = -3 \]  

(31)

and a constraint \( n_H + n_{\bar{H}} = -5, -4 \). For \( n_H + n_{\bar{H}} = -5 \) one obtains \( \delta A = 42/5, \delta B = 30 \), and the three coupling constants meet at a scale \( M_X \sim 2 \times 10^{16} \text{GeV} \) provided that \( \text{Re}T \) is of order \( \text{Re}T \sim 16 \) (see the two figures). For \( n_H + n_{\bar{H}} = -4 \) one has \( \delta A = 44/5, \delta B = 28 \). Now the three couplings only meet approximately (within the experimental errors of \( \sin^2 \theta_W \) and \( \alpha_s \)) for similar values of \( \text{Re}T \). Thus we see that, in principle, a situation with a compactification scale below the string scale may work within a minimal \( SU(3) \times SU(2) \times U(1) \) string provided, e.g., the above modular weights are possible. Notice that (twisted) moduli fields are not necessarily gauge singlets (e.g. the \( SU(2) \) doublets in the \( Z_4 \) orbifold). (Allowing for non-standard modular weights, i.e. \( n \leq -3 \), the minimal unification scenario would be possible for smaller values of \( \text{Re}T \), and in particular for \( \text{Re}T \sim 1 \).)

We have just shown that the minimal string unification scenario is in principle compatible with the measured low energy coupling constants for i) sufficiently large \( \text{Re}T \) and ii) restricted choices of standard particles modular weights. The question now is whether these two conditions are easy to meet. Concerning the first condition, we need to have an idea of the non-perturbative string dynamics which trigger the compactification process and fixes the value of \( \text{Re}T \). In the context of duality-invariant effective actions, recent analysis \[18],[19], [37] shows that the preferred values of \( \text{Re}T \) are of order one. This is expected since a duality-invariant potential will typically have its minima not very far away from the self-dual point. Thus large values of \( \text{Re}T \) are not expected within this philosophy. However a deeper understanding about non-perturbative string effects is definitely needed to give a final answer to this question. Concerning the second condition, it would be interesting to investigate \( Z_N \) and \( Z_M \times Z_N \) orbifold models
to see whether there are choices of modular weights leading to the appropriate \( \delta A, \delta B \). It is certainly intriguing the degree of uniqueness in the possible choices of modular weights (eq.(31)) leading to adequate results and this point deserves further study. Notice, however that one may relax the condition of generation independence which led to eq.(31). In addition one can consider the separate contribution of the three orbifold planes in terms of the three untwisted moduli \( T_i \). In summary, we believe that the analysis presented in this paper shows that minimal string unification is a possible but rather constrained scenario.

If no string model with the characteristics of the above minimal unification scenario is found, it may still be possible to explain the success of the unification of the three \( g_1, g_2, g_3 \) coupling constants in the context of strings provided one of the following possible alternatives is realized: i) There is an intermediate grand unification scale \( M_X \sim 10^{16} \) GeV at which a GUT simple group like \( SU(5) \) or \( SO(10) \) is realized; ii) There is instead some semi – simple group like \( SU(4) \times SU(2) \times SU(2), SU(3)^3 \) or non semi – simple group like \( SU(5) \times U(1) \) beyond that scale. Both of these possibilities has its shortcomings. The first requires that the Kac-Moody level of the gauge groups \( SU(5) \) or \( SO(10) \) be bigger than one and the construction of higher level models is both complicated and phenomenologically problematic [38]. Alternative ii) has the problem that there is further relative running of the coupling constants in the region \( M_X - M_{\text{string}} \) which will typically spoil the predictions of the minimal susy model. In any other possible alternative it would be difficult to understand why the couplings tend to join around \( 10^{16} \) GeV, it would just be a mere coincidence. For example, it is possible to consider extensions of the minimal particle content in such a way that the low-energy gauge couplings directly meet around \( 10^{18} \) GeV. This possibility was already considered in ref. [39]. In any case it is clear that the present precision of the measurement of low energy gauge couplings has reached a level which is sufficient to test some fine details of string models.

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Figure Captions

**Figure 1:** $\sin^2 \theta_W(M_Z)$ as a function of the compactification radius squared $\text{Re}T = R^2$ for different values of $\delta A$. The shaded area corresponds to the experimentally allowed range.

**Figure 2:** $\alpha_s(M_Z)$ as a function of $\text{Re}T$ for different values of $\delta B$. 