THE ANISOTROPY OF MHD ALFVÉNIC TURBULENCE

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ABSTRACT

We perform direct 3-dimensional numerical simulations for magnetohydrodynamic (MHD) turbulence in a periodic box of size $2\pi$ threaded by strong uniform magnetic fields. We use a pseudo-spectral code with hyperviscosity and hyperdiffusivity to solve the incompressible MHD equations. We analyze the structure of the eddies as a function of scale. A straightforward calculation of anisotropy in wavevector space shows that the anisotropy is scale-independent. We discuss why this is not the true scaling law and how the curvature of large-scale magnetic fields affects the power spectrum and leads to the wrong conclusion. When we correct for this effect, we find that the anisotropy of eddies depends on their size: smaller eddies are more elongated than larger ones along local magnetic field lines. The results are consistent with the scaling law $\tilde{k}_\parallel \sim \tilde{k}_\perp^{2/3}$ proposed by Goldreich and Sridhar (1995, 1997). Here $\tilde{k}_\parallel$ (and $\tilde{k}_\perp$) are wavenumbers measured relative to the local magnetic field direction. However, we see some systematic deviations which may be a sign of limitations to the model, or our inability to fully resolve the inertial range of turbulence in our simulations.

Subject headings: ISM:general-MHD-turbulence

1. INTRODUCTION

Many astrophysical plasmas, including the interstellar medium and the solar wind, often show magnetic fields whose energy density is greater than or equal to the local kinetic energy energy. In these plasmas the magnetic fields play a dominant dynamical role, mediated by magnetohydrodynamic (MHD) waves. In the incompressible limit, there are only two types of linear modes: shear Alfvén waves and pseudo Alfvén waves. While these two modes have different polarization directions, they have the same dispersion relation and propagate along the magnetic field lines at the Alfvén speed. Therefore, the nonlinear interactions of wave packets moving along the magnetic field lines at the Alfvén speed determine the dynamics of incompressible magnetized plasmas with a strong background field. In this paper, we study the anisotropy of the MHD turbulence in this regime. We will refer to this turbulence as incompressible Alfvénic turbulence.

Nonlinear processes and the corresponding energy spectrum of incompressible Alfvénic turbulence are still among the most controversial problems in MHD. Since the pioneering works of Iroshnikov (1963) and Kraichnan (1965), the Iroshnikov-Kraichnan (IK) theory has been widely accepted as a model for incompressible, highly conducting MHD turbulence. The IK theory predicts $E_M(k) \sim E_K(k) \sim k^{-3/2}$ from a Kolmogorov-like dimensional analysis. Here, $E_M(k)$ and $E_K(k)$ are the magnetic and kinetic energy spectra respectively. In this framework, two counter-traveling eddies (i.e. Alfvén wave packets) interact and transfer energy to smaller spatial scales only when they collide, as they move in opposite directions along the magnetic field lines. Since the duration of such a collision is shorter than the conventional eddy turnover time by a factor of $t_v(l)/t_A(l)$, this collisional process is inefficient and the spectral energy transfer time as a function of scale $l$ ($= t_{cas}(l)$) increases by the same factor compared to the eddy turnover time ($l/v_l$) in ordinary hydrodynamic turbulence. Here $t_v(l) = l/v_l$ and $t_A(l) = l/V_A$ are eddy turnover time and Alfvén time respectively, $V_A \equiv B/\sqrt{4\pi \rho}$, and $B$ is rms magnetic field strength. When the external field is strong, as assumed in IK
theory, this quantity is usually set to $B_0$, the strength of the uniform background field. If the spectral energy cascade rate

$$\epsilon \sim \frac{v_l^2}{t_{\text{cas}}(l)} \sim \frac{v_l^3 \tau_A(l)}{T v(l)}$$

(1)

is scale-independent and $E_M(k) \approx E_K(k)$, then we obtain the IK energy spectra.

The IK theory assumes an isotropic distribution of energy in $k$-space. However, many researchers have argued that anisotropy is an important characteristic in MHD turbulence (for example, Shebalin et al 1983, Montgomery and Matthaeus 1995). This anisotropy results from the resonant conditions for 3-wave interactions (or 4-wave interactions, when 3-wave interactions are null). The resonant conditions for the 3-wave interactions are

$$k_1 + k_2 = k_3,$$  

(2)

$$\omega_1 + \omega_2 = \omega_3,$$  

(3)

where $k$’s are wavevectors and $\omega$’s are wave frequencies. The first condition can be viewed as momentum conservation and the second as energy conservation. Alfvén waves satisfy the dispersion relation: $\omega = V_\text{A} |k||$, where $k||$ is the component of wavevector parallel to the background magnetic field. Since only opposite-traveling wave packets interact, $k_1$ and $k_2$ must have opposite signs. Then from equations (2) and (3), either $k_{||,1}$ or $k_{||,2}$ must be equal to 0. That is, zero frequency modes are essential for energy transfer. If $k_{||,2} = 0$, we have

$$k_{||,1} = k_{||,3},$$  

(4)

$$k_{||,2} = 0$$  

(5)

(Shebalin et al 1983). Therefore, in the wavevector space, 3-wave interactions make energy cascade in directions perpendicular to the mean magnetic field. Since the energy cascade is strictly perpendicular to the mean magnetic field, the actives modes in wavevector space have a slab-like geometry with a constant width. The implication is that the nonlinear cascade of energy works against isotropy in $k$ space. Furthermore, it is important to note that equations (2) and (3) are true only when wave amplitudes are constant. In reality, nonlinear interactions provide a natural broadening mechanism, following the uncertainty relation, $\Delta t \cdot \Delta \omega \sim 1$. In particular, if a wave has a frequency less than or comparable to the nonlinear interaction rate, it is effectively a zero frequency mode.

Goldreich and Sridhar (1995, 1997) showed that in the strong incompressible shear Alfvénic turbulence regime (i.e. $\tau_{NL}^{-1} \sim \omega$), these arguments lead to a new scaling law with a scale-dependent anisotropy. In this model smaller eddies are more elongated. Their arguments are based on the assumption of a critically balanced cascade, $k||V_A \sim k_\perp v_l$, where $k_\perp$ and $k||$ are wave numbers perpendicular and parallel to the external dc field. The argument given above for 3-wave interactions makes it clear that $k_\perp$ will tend to increase until it becomes important in the plasma dynamics. The assumption of strong nonlinearity implies that wave packets lose their identity after they travel one wavelength along the field lines. Consequently the eddy turnover time $((k_\perp v_l)^{-1})$ is actually the same as Alfvénic time $(k||V_A)^{-1}$. In this model, the cascade time, $t_{\text{cas}}(l)$ can be determined without ambiguity: $t_{\text{cas}} \approx (k_\perp v_l)^{-1} \approx (k||V_A)^{-1}$. Since the cascade time is comparable to the period of Alfvén wave, the 3-wave resonant condition can be violated according to the uncertainty relation $\Delta \omega \cdot \Delta t \sim V_A k|| \cdot t_{\text{cas}} \sim 1$. The quantity $k||$ is the width of the active region in wavevector space. Finally the assumption of a scale-independent cascade rate $\epsilon \sim v_l^2/t_{\text{cas}}(l) \sim E_{\text{waves}} (V_A/L$ gives

$$k|| \sim k_\perp^{2/3} L^{-1/3} \left(\frac{E_{\text{waves}}}{V_A^2}\right)^{1/3},$$  

(6)

$$v_l \sim V_A (k_\perp L)^{-1/3} \left(\frac{E_{\text{waves}}}{V_A}\right)^{1/3},$$  

(7)

where $E_{\text{waves}}$ is the wave energy per mass. These formulae assume that all scales, from $L$ on down, are within the inertial range of MHD turbulence. Here the typical $k||$ should be interpreted as the size of the range of parallel wavevectors, corresponding to a given $k_\perp$, that contain significant energy.

Matthaeus et al. (1998) recently tested this model numerically, and showed that the anisotropy of low frequency MHD turbulence scales linearly with the ratio of perturbed and total magnetic field strength $b/B$, a result which seems inconsistent with Goldreich and Sridhar’s model. To explain this scaling relation, they suggested that the region of Fourier space where the energy transfer takes place actively is given by

$$|k \cdot \frac{B_0}{\sqrt{4\pi \rho}} | \leq \frac{1}{\tau_{NL}},$$  

(8)

where $\tau_{NL}$ is the eddy turnover time of the energy containing length $L$. Consequently, “the region of the
that their results are in contradiction to our discussion in §1. "Vigorous" has a slab-like geometry with a constant tendency of reduced the Shebalin angles, \( \theta_{\mathbf{Q}} \), defined by \( \tan^2 \theta_{\mathbf{Q}} = (\sum k_\perp^2 |\mathbf{Q}(k, t)|^2 )/(\sum k_\parallel^2 |\mathbf{Q}(k, t)|^2 ) \), where \( \mathbf{Q} \) is vector potential \( \mathbf{A} \), magnetic field \( \mathbf{B} \), or current \( \mathbf{J} \), etc. Greater \( \theta_{\mathbf{Q}} \) means greater anisotropy. They found that \( \theta_A < \theta_B < \theta_J \). If the energy spectrum of \( \mathbf{B} \) scales as \( E_M(k) \propto k^{-8} \), then the spectra of vector potential and current scale as \( E_A(k) \propto k^{-8-2} \) and \( E_J(k) \propto k^{-8+2} \), respectively. The spectra of vector potential has the steepest slope among the three spectra. This means that the vector potential is least strongly dependent on small scales (and the current is most strongly dependent on small scales). Therefore they concluded that anisotropies are more pronounced at smaller scales\(^3\). They also found a similar ordering for velocity field (and vorticity, etc.).

On the other hand, Ghosh and Goldstein (1997) calculated the shear layer angles as a function of wave number. Bin. They found that the Hall-MHD simulations show increased anisotropy at small scales (i.e. greater Shebalin angles at smaller scales). However their standard MHD simulations do not show increased anisotropies at small scales. We refrain from comparing their work with ours because they used different physics (the Hall effect) and their simulations are 2-D dimensional. The simulations given in this paper are 3-D standard MHD simulations. We note that none of the previous papers quantitatively compared their results with particular theories of anisotropy.

In this paper, we examine the scaling law for Alfvénic MHD turbulence numerically and resolve the controversy concerning the anisotropic structure of the turbulence. Our results are consistent with Goldreich and Sridhar’s model. We describe our numerical methods in §2. In §3, we describe our results for anisotropy in wave vector space. In this section, we demonstrate that none of the scaling laws mentioned above agrees with our data, and explain why a straightforward evaluation of the distribution of spectral power does not correspond to a physically meaningful set of scaling laws. We attribute this to the systematic effects of large scale curvature of the magnetic fields. In §4, we determine the shape of individual eddies, avoiding the systematic error described above. We compare our results to Goldreich and Sridhar’s model and give our conclusions in §5.

2. NUMERICAL METHODS AND GENERAL RESULTS

We have employed a pseudospectral code to solve the incompressible MHD equations in a periodic box of size \( 2\pi \)

\[
\frac{\partial \mathbf{V}}{\partial t} = (\nabla \times \mathbf{V}) \times \mathbf{V} - V_{A0}^2 (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{V} + \mathbf{f} + \nabla P',
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \mathbf{B} \cdot \nabla \mathbf{V} - \mathbf{V} \cdot \nabla \mathbf{B} + \eta \nabla^2 \mathbf{B},
\]

where \( \mathbf{f} \) is random driving force and \( P' \equiv P + \mathbf{V} \cdot \nabla \mathbf{V} / 2 \). Other variables have their usual meaning. \( V_{A0} = B_0/(4\pi \rho)^{1/2} \) is the Alfvén velocity of the background field, which is set to be order unity in our simulations. In pseudo spectral methods, we calculate the temporal evolution of the equations (11) and (12) in Fourier

\[\left[ \begin{array}{c} k_\parallel \\ k_\perp \end{array} \right]_{\text{small } k_\perp} > \left[ \begin{array}{c} k_\parallel \\ k_\perp \end{array} \right]_{\text{large } k_\perp},\]

then we will have the ordering of Shebalin angles as observed by Shebalin et al. and Oughton et al. However, if \( k_\parallel \propto k_\perp \), then we do not expect any ordering among the angles. In §3, we will find that, no matter what the true functional form \( k_\parallel = k_\parallel(k_\perp) \) is, Fourier transformation smooths out the true relation and leads to a fake linear relationship between \( k_\parallel \) and \( k_\perp \). This suggests that their results are in contradiction to our discussion in §3. However, the apparent linear relationship between \( k_\parallel \) and \( k_\perp \) in §3 is not perfectly linear. Instead, we expect \( k_\parallel = c_1 k_\perp + c_2 \), where \( c_1 \) is a decreasing function of \( B_0 \) and \( c_2 \) depends on the \( k_\parallel \) of forcing terms (or initial values of \( k_\parallel \) for decaying turbulence). The presence of \( c_2 \) does not seem to be important in our Fig. 3 in §3. However, it does affect the calculation of Shebalin angles. That is, because of \( c_2 \), the ratio \( k_\parallel /k_\perp = c_1 /c_2 \) becomes a function of \( k_\perp \). At the largest energy containing eddy scale, \( k_\perp \sim c_2 \) and hence \( k_\parallel /k_\perp \sim c_1 + O(1) \sim O(1) \). But, at small scales, the ratio can be much smaller than unity. Therefore, because of \( c_2 \), we can obtain a scale-dependent anisotropy: \( (k_\parallel /k_\perp) \) at small \( k_\perp \) is greater than that at large \( k_\perp \). Note that this is a result of the initial conditions, rather than a true scaling relation. This will lead the ordering of the angles as observed by Shebalin et al. and Oughton et al. This interpretation is qualitatively consistent with their results. For example, the ratio \( k_\parallel /k_\perp = c_1 + c_2/k_\perp, \) therefore \( (\tan \theta_{\mathbf{Q}})^{-1} \), approaches to a constant value \( c_2 /k_\perp \) as \( B_0 \) becomes strong. Here \( k_\perp, \) is the wavenumber of the peak of the energy spectrum. It is also possible to explain the increased anisotropy at high magnetic Reynolds numbers. If the magnetic Reynolds number increases, then \( k_\perp \) increases and, therefore, the ratio decreases.

\(^3\)Hall MHD includes the Hall term, which is important at ion-cyclotron scales. In this paper, we consider only the standard MHD equations.
space. To obtain the Fourier components of nonlinear terms, we first calculate them in real space, and transform back into Fourier space. We use 21 forcing components with $2 \leq k \leq \sqrt{12}$. Each forcing component has correlation time of one. The peak of energy injection occurs at $k \approx 2.5$. The amplitudes of the forcing components are tuned to ensure $V \approx 1$. Since we expect the perturbed magnetic field strength $b$ to be comparable to or less than $V$ (i.e. $b \leq 1$), the resulting turbulence can be regarded as strong MHD turbulence (i.e. $V \sim B_0 \sim B$). The average helicity in these simulations is not zero. However, our results are insensitive to the value of helicity, possibly because the box size is only slightly bigger than the energy injection scale. We use an appropriate projection operator to calculate $\nabla P'$ term in Fourier space and also to enforce divergence-free condition ($\nabla \cdot V = \nabla \cdot B = 0$). We use up to $256^3$ collocation points. We use integration factor technique for kinetic and magnetic dissipation terms and leap-frog method for nonlinear terms. We eliminate $2\Delta t$ oscillation of the leap-frog method by using an appropriate average. At $t=0$, magnetic field has only uniform component and velocity has a support between $2 \leq k \leq 4$ in wavevector space.

We use hyperviscosity and hyperdiffusivity for dissipation terms to maximize the inertial range. The only exception is Run 256P-$B_01$, where we use physical viscosity and diffusivity. The power of hyperviscosity is set to 8, so that the dissipation term in the above equation is replaced with

$$-\nu_8(\nabla^2)^8V,$$

where we set the value of $\nu_8$ from the condition $\nu_8(N/2)^{2h}\Delta t \approx 0.5$ (see Borue and Orszag 1996). Here $\Delta t$ is the time step and $N$ is the number of grids in each direction. We use exactly same expression for the magnetic dissipation term. We list parameters used for the simulations in Table 1. We use the notation 256X-Y, where X = H or P refers to hyperviscosity or physical viscosity; Y = $B_01$ or $B_0.5$ refers to the strength of the external magnetic fields.

Fig. 1 shows the evolution of $V^2$ and $B^2$ from Runs 256H-$B_01$ and REF2. Results of 256H-$B_01$ are plotted as solid lines, and results of REF2 are plotted as dotted lines. Both runs have similar results for the overlapped time, which means that the values of $V^2$ and $B^2$ are not sensitive to the spatial resolution. The magnetic energy density grows fast at the beginning of the simulations, as a result of the stretching of external field lines by turbulent motion. Since the external field is strong, magnetic forces soon become strong enough to balance the stretching effect. This balance happens at $t \sim 1$ and, after a transient stage ($1 \leq t \leq 5$), the turbulence reaches the saturation stage ($t \geq 5$). In this case the saturation stage begins quite early, since the external uniform magnetic field is strong and the magnetic diffusivity is effectively 0. In general, when the external field is weaker and/or magnetic diffusivity is larger, the saturation stage occurs later. At the saturation stage, $B^2 \sim 1.45$ and $V^2 \sim 0.6$. Since $b^2 = B^2 - B_0^2 (\approx 1) \approx 0.45$, there is a rough energy equipartition between $b$ and $V$. Note that, since $V \sim B$, the condition for strong MHD turbulence is met.

3. FOURIER ANALYSIS: A FALSE SCALING LAW?

3.1. Fourier Analysis

In this subsection, we investigate the spectral structure of Alfvénic turbulence. It is important to note that we lose some information about individual eddies when we perform a global transformation, such as the Fourier transformation. In particular, the scaling law given in this subsection may not be true for individual eddies (See §3.2 for details).

We plot the 1-dimensional energy spectra $E_K(k)$ and $E_M(k)$ in Fig. 2. Both spectra peak at the same $k$, which is also the scale of the energy injection. This reflects that there is a rough energy equipartition between $b$ and $V$ at the largest energy containing scale. In general, $E_K(k)$ always peaks at the energy injection scale. However, $E_M(k)$ peaks at a larger $k$ than $E_K(k)$ when the external field is weak. The inertial range exhibits of two different scaling ranges: a small $k$ range where the slopes are steep and a large $k$ range where the slopes are mild. We believe that the former reflects a true inertial range, whereas the latter is a result of bottleneck effect. The $1/k$ bottleneck effect is a common feature in numerical hydrodynamic simulations with hyperviscosity (see Borue and Orszag 1996, 1995). Interestingly enough, the slope of the bottleneck is actually steeper than $-1$. This might mean that the bottleneck effect is less serious in MHD case. The kinetic and magnetic energy spectra have slightly different powers in the inertial range. Although the power indexes of both spectra are close to $-5/3$ in the (true) inertial range, the data are also compatible with IK theory, where the power index is $-3/2$.

We plot normalized 3-dimensional energy spectra of Run 256H-$B_01$ in Fig. 3. In the figure, we plot contours of same

$$E_3(k_{\perp}, k_{||})/E_3(k_{\perp}, 0)$$

in $(k_{\perp}, k_{||})$ plane. As the energy cascades in the directions perpendicular to the mean field, this ratio drops as we move away from $k_{||}$ axis. The figure implies most of the energy is confined in a region around $k_{\perp}$ axis. The thickness of the region depends on the
strength of the external magnetic fields (Fig. 4). We use the contours of \(E_3(k_\perp, k_\parallel)/E_3(k_\perp, 0) = 0.5\) to measure the thickness. The angle \(\Theta_{0.5}\) is the angle formed by the contours and the horizontal axis. We can see that \(\tan \Theta_{0.5} (= k_\parallel/k_\perp)\) is proportional to \(b/B_0\). This result confirms the scaling relation found by Matthaeus et al. (1998). Both velocity and magnetic fields have very similar structures. From the fact that \(k_\parallel < k_\perp\) in the active region (see Fig. 3), one can conclude that eddies do have anisotropic structure: eddies are stretched along the direction of mean field. Apparently, Fig. 3 suggests that \(k_\parallel \propto k_\perp\) and, hence, that anisotropy is scale-independent. No theory mentioned in \$1\) agrees with our result. However, it is not clear from the figure whether or not the true anisotropy is a function of scale: although contours show linear relationship between \(k_\parallel\) and \(k_\perp\), the relationship doesn’t mean that all individual eddies have the same major axis to minor axis ratio. As explained in next subsection, the rotation of eddies by large-scale waves in the magnetic fields can distort the actual scaling relation and lead to the linear relationship shown in the figure.

Fig. 5 shows \(t_{phase}\) as a function of wavenumber \(k\). This time scale is defined as the average correlation time \(\Delta t\) such that Fourier components at \(k = k\) have a phase shift of 60° with respect to the original phase. We plot the result for zero frequency modes (i.e. \(k_\parallel = 0\) modes). We see that \(t_{phase} \sim k^{-1}\) (that is, \(k^{-1}\)). How can we interpret this result? If a turbulent structure, characterized by a wavenumber \(k\), moves a distance \(l\) then the phase is shifted by roughly \(kl\), even if the eddy is unaffected by the motion. This implies that a large scale velocity \(V\) will change the phase at a rate \(kV\), so that \(t_{phase} \propto k^{-1}\). This implies that our calculation of \(t_{phase}\) is dominated by large scale motions rather than by the local cascade of energy, as long as the nonlinear cascade rate is proportional to \(k\) to some power less than one, which is generally the case. This is an example of how large-scale fluctuations can complicate attempts to find physically meaningful scaling relations.

3.2. Rotation Effect

In the previous subsection, we showed that anisotropy appears to be scale-independent. In this subsection, we will show that this apparent scaling relation is an artifact caused by the Fourier transformation. That is, we will demonstrate that large-scale modes in the magnetic field can distort the actual scaling law at smaller scales when we perform a Fourier transformation. In this manner, a straight-
forward evaluation of anisotropy in Fourier space is strongly contaminated by the curvature of the large-scale magnetic fields and does not reflect the actual local anisotropy when anisotropy is more pronounced at smaller scales. Consequently, figures 3 and 4 are compatible with any scaling law that predicts that smaller eddies are more elongated.

Fig. 3 implies that eddies have anisotropic shapes: on average, eddies are stretched along the direction of \( \mathbf{B}_0 \). However, not all eddies are aligned along the large scale field. The elongation of an eddy is determined by its interaction with the local magnetic field, not the background field. Since the large-scale magnetic field lines wander with respect to \( \mathbf{B}_0 \), all smaller scale eddies have similar angular distributions around \( \mathbf{B}_0 \). We illustrate this effect in Fig. 6.

In this way, we can explain the results of Goldreich and Sridhar (1995) and Matthaeus et al. (1998) simultaneously. Suppose that eddies are oriented according to the local field lines (Fig. 6a). Goldreich and Sridhar’s result implies smaller eddies are relatively more elongated. When we perform a Fourier transformation and measure the ratio of \( k_{\parallel} / k_{\perp} \), what we actually measure is not the ratio of the minor axis to the major axis of individual eddies. Instead, because the direction of the local magnetic field varies according to location and Fourier transformation is none other than a (weighted) averaging process, we actually measure the ratio averaged over all possible orientation of the eddies (Fig. 6b). The Fourier transformation sees that \( L_1 \) (and \( l_1 \)) is the major axis and \( L_2 \) (and \( l_2 \)) is the minor axis of an eddy. The ratio of \( L_1 / L_2 \) (\( = l_1 / l_2 \)) is determined by the degree of the wandering of the large scale magnetic field lines with respect to \( \mathbf{B}_0 \) and, therefore, the ratio is nearly constant for all eddies, regardless of their sizes and shapes. In fact, the ratio will depend on the tangent of the angle between \( \mathbf{B}_0 \) and \( \mathbf{B} \). Since \( k_{\perp} \propto 1/L_2 \) and \( k_{\parallel} \propto 1/L_1 \), the measured value of the ratio \( k_{\parallel} / k_{\perp} \) is nearly scale-independent. We expect the angle \( \theta \) (\( \equiv 90^\circ - \theta_w \)) between \( \mathbf{B}_0 \) and \( \mathbf{B} \) to be

\[
\tan \theta = b / B_0. \tag{13}
\]

Therefore, we have

\[
\sin \theta = b / B = \cos \theta_w, \tag{14}
\]

which is exactly the scaling relation found by Matthaeus et al.

4. MEASURING EDDY SHAPES

In this section, we analyze the shape of eddies in real space. As explained in the previous section, we assume that elongated eddies are aligned in the direction of the local magnetic fields. If we want to visualize the shape of eddies, we need to first identify
the direction of the local magnetic fields. It is important to note that, although we use the term 'local magnetic fields' for simplicity, the fields are different from $\mathbf{B}(\mathbf{r})$. Let us consider an eddy of size $l$. The 'local magnetic field' of the eddy must act as the 'large-scale magnetic field' for the eddy. Therefore, the 'local magnetic fields' for eddies of size $l$, must be smoothly varying vector fields whose characteristic length of variation is $> l$. Note also that another eddy of size $l'$ ($\neq l$) at the same location can have a slightly different direction for the 'local magnetic field.' The 'local magnetic fields' are functions of position ($\mathbf{r}$) and eddy size ($l$). Then, how can we define the direction of the local magnetic fields? We implement 2 independent numerical algorithms to calculate the direction of the local magnetic fields.

In the first method, the local magnetic fields are calculated by

$$\mathbf{B}_l = (\mathbf{B}(\mathbf{r}_1) + \mathbf{B}(\mathbf{r}_2))/2$$  \hspace{1cm} (15)

and the second order structure functions for $v$ and $b$ are given by

$$\begin{align*}
F_2^v(R, z) &= \langle |\mathbf{V}(\mathbf{r}_1) - \mathbf{V}(\mathbf{r}_2)|^2 \rangle, \\
F_2^b(R, z) &= \langle |\mathbf{b}(\mathbf{r}_1) - \mathbf{b}(\mathbf{r}_2)|^2 \rangle,
\end{align*}$$

where $R = |\mathbf{z} \times (\mathbf{r}_2 - \mathbf{r}_1)|$, $z = \hat{\mathbf{z}} \cdot (\mathbf{r}_2 - \mathbf{r}_1)$ and $\hat{\mathbf{z}} = \mathbf{B}_l/|\mathbf{B}_l|$. That is, $R$ and $z$ are coordinates in a cylindrical coordinate system in which the $z$-axis is parallel to $\mathbf{B}_l$ (Fig. 7). $\mathbf{B}(\mathbf{r})$ and $\mathbf{b}(\mathbf{r})$ are the total and perturbed magnetic fields at a point $\mathbf{r}$. As usual, brackets denotes a spatial average.

In Fig. 8 we plot the second order structure functions in $z$-$R$ plane. The horizon axis ($z$-axis) is parallel to $\mathbf{B}_l$. The contours reflect the shapes of the eddies. Eddies are clearly elongated along the local field lines, and smaller eddies are more elongated. The velocity and magnetic fields show different structure at large scales. However, their small scale structure is quite similar.

In Fig. 9, we plot R-intercepts and z-intercepts of the contours. The R-intercept and z-intercept of a given contour can be regarded as a measure of $\hat{k}_l$ and $\hat{k}_\perp$ for the corresponding eddy scale$^5$. Fitting the results for velocity fields gives

$$v : \hat{k}_l \sim \begin{cases} \hat{k}_l^{0.69}, & (256H - B_00.5) \\ \hat{k}_l^{0.70}, & (256H - B_01) \\ \hat{k}_l^{0.73}, & (256P - B_01) \end{cases}$$  \hspace{1cm} (16)

On the other hand, for the magnetic fields we find

$$b : \hat{k}_l \sim \begin{cases} \hat{k}_l^{0.64}, & (256H - B_00.5) \\ \hat{k}_l^{0.50}, & (256H - B_01) \\ \hat{k}_l^{0.53}, & (256P - B_01) \end{cases}$$  \hspace{1cm} (17)

The velocity fields show good agreement with the scaling relation, $\hat{k}_l \sim \hat{k}_l^{2/3}$, proposed by Goldreich and Sridhar (1995). The power indices are insensitive to strength of $B_0$ or the form of viscosity (and diffusivity). However, the magnetic field shows different scaling behavior. When the external field is moderately strong ($256H - B_00.5$), the power index is very close to $2/3$. On the other hand, for stronger external fields ($256H - B_01$ and $256P - B_01$), the power indices are smaller than $2/3$. The results are insensitive to the choice of viscosity (and diffusivity). It is not clear whether the existence of a separate scaling law for the magnetic field represents a physical effect not included in Goldreich and Sridhar’s model, e.g. the first signs of small scale intermittency in the magnetic field distribution, or merely the failure of the numerical models used here to fully resolve the inertial range of strong MHD turbulence.

In the second method, we employ a completely different approach. We obtain the local large scale magnetic fields by filtering out the small scale magnetic fields:

$$\mathbf{B}_\sigma(\mathbf{r}) = \sum_{\mathbf{r}'} \mathbf{B}(\mathbf{r}') \phi(|\mathbf{r} - \mathbf{r}'|),$$  \hspace{1cm} (18)

where $\phi(r) \propto \exp(-r^2/\sigma_r^2)$ is a gaussian function. To determine the shape of small scale eddies (i.e. eddies smaller than the filter size, $\sim \sigma_r$), we consider the following quantities:

$$\begin{align*}
\mathbf{v}_\sigma(\mathbf{r}) &= \mathbf{V}(\mathbf{r}) - \mathbf{V}_\sigma(\mathbf{r}), \\
\mathbf{b}_\sigma(\mathbf{r}) &= \mathbf{B}(\mathbf{r}) - \mathbf{B}_\sigma(\mathbf{r}),
\end{align*}$$

where $\mathbf{V}_\sigma$ and $\mathbf{B}_\sigma$ are filtered fields (cf. eq. (20)). The fields $\mathbf{v}_\sigma(\mathbf{r})$ and $\mathbf{b}_\sigma(\mathbf{r})$ represent small scale fluctuation of velocity and magnetic fields, the shape of which can be regarded as an adequate approximation of small scale eddies. From these two fields we calculate the structure functions

$$\begin{align*}
F_2^v(R, z) &= |\mathbf{v}_\sigma(\mathbf{r}_2) - \mathbf{v}_\sigma(\mathbf{r}_1)|, \\
F_2^b(R, z) &= |\mathbf{b}_\sigma(\mathbf{r}_2) - \mathbf{b}_\sigma(\mathbf{r}_1)|,
\end{align*}$$

where $R$ and $z$ are similarly defined as in the first method with $\hat{\mathbf{z}} \parallel \mathbf{B}_\sigma(\mathbf{r}_1)$ (Fig. 10).

In Fig. 11, we plot the results of the second method. We can clearly observe the flattening effect: When the filter size is large, eddies are less

$^5$We interpret $\hat{k}_l$ as the inverse of the major axis of eddies and $\hat{k}_\perp$ as that of the minor axis. Since the major axis is assumed to be parallel to the local magnetic fields, $\hat{k}_l$ is the parallel wavenumber with respect to the local magnetic field direction. On the other hand, in $\S 3.1$, $k_{\parallel}$ is parallel to $\mathbf{B}_0$. Therefore, $\hat{k}_l$ and $k_{\parallel}$ in this section have different meaning from $k_{\parallel}$ and $k_{\perp}$ in $\S 3.1$. 

Fig. 7.— Method 1. We adopt a cylindrical coordinate system in which z-axis is parallel to $B_l = (B(r_1) + B(r_2))/2$.

Fig. 8.— 256H-$B_01$. Visualization of eddies using method 1. Note that horizontal axis is z-axis (see Fig. 7 for definition). Unit is grid spacing. Smaller eddies are more elongated.

Fig. 9.— R-intercept (semi-minor axis; $\sim 1/k_\perp$) versus z-intercept (semi-major axis; $\sim 1/k_\parallel$) from Fig. 8. In 256H-$B_00.5$, both velocity and magnetic fields follow the relation $k_\parallel \sim k_\perp^{2/3}$. In 256H-$B_01$ and 256P-$B_01$, velocity fields follow the same scaling relation. However, magnetic fields scale slightly differently.

Fig. 10.— Method 2. We adopt a cylindrical coordinate system in which z-axis is parallel to large-scale fields $B_\sigma$.

Fig. 11.— 256H-$B_01$. Visualization of eddies using method 2. Note that horizontal axis is z-axis (see Fig. 10 for definition). Unit is grid spacing. Only magnetic fields are shown. Smaller eddies are more flattened.
anisotropic. When filter size is small, eddies show highly anisotropic structure. In the figure, we plot only the results for magnetic fields. Velocity fields show similar trends. The thick contours represent \( F^b(R, z) = < |b_\parallel|^2 >^{1/2} \). The values next to the thick contours are \(< |b_\parallel|^2 >^{1/2} \).

This second method is useful for visualizing small scale eddies, but may not be as useful for quantitative analysis. The difficulty comes from the fact that there is no well defined eddy scale associated with filtered fields \( \mathbf{v}_\alpha(\mathbf{r}) \) and \( b_\alpha(\mathbf{r}) \).

In this paper, we will not pursue quantitative analysis using the method 2. Instead, we just wish to point out that both methods describe the same scaling law: smaller eddies are relatively more stretched along the local magnetic field lines. In particular, the results from the first method are consistent with the proposed scaling law, \( k_\parallel \sim k_\perp^{1/3} \) proposed by Goldreich and Sridhar (1995).

5. DISCUSSION AND CONCLUSIONS

Here we rederive the scaling law \( k_\parallel \sim k_\perp^{2/3} \) in the framework of 3-wave interactions. Except for the use of the uncertainty principle, the work in this section is independent of Goldreich and Sridhar’s derivation. As noted by Goldreich and Sridhar (1995), 3-wave interactions are an adequate proxy for wave-wave interactions of all orders in a strong MHD turbulence. As long as we assume the locality of interactions, it is pointless to distinguish \( k_\parallel \) and \( k_\perp \) from \( \tilde{k}_\parallel \) and \( \tilde{k}_\perp \). Hence, for simplicity, we use \( k_\parallel \) and \( k_\perp \) instead of \( \tilde{k}_\parallel \) and \( \tilde{k}_\perp \) during the derivation.

Suppose the 3-dimensional energy spectrum is given by

\[
E_3(k_\perp, k_\parallel) \equiv |\tilde{V}(k)|^2 \propto k_\perp^{-2\alpha} f, \tag{25}
\]

where \( \tilde{V}(k) \) is the amplitude of the mode whose wavevector is \( \mathbf{k} \) and \( f(u) \) is a positive, symmetric function of \( u \) (cf. equation (7) of Goldreich and Sridhar (1995)) which describes the power distribution as a function of eddy shape. If the width (or, thickness) of the energy spectrum in the direction of \( k_\parallel \) is \( k_\perp^\beta \), then we can write

\[
E_3(k_\perp, k_\parallel) \propto k_\perp^{-2\alpha} f(k_\parallel/k_\perp^\beta). \tag{26}
\]

If the width is caused by the uncertainty principle \( (\Delta t \cdot \Delta \omega \approx 1 \) with \( \Delta t \propto t_{\text{cas}}(l) \) and \( \Delta \omega \propto k_\parallel \)), then

\[
t_{\text{cas}}(l) \propto k_\perp^{-\beta}. \tag{27}
\]

Suppose the energy cascade rate \( \epsilon \sim v_l^2/t_{\text{cas}}(l) \), where \( l = 2\pi/k_\perp \), is scale-independent. Because \( v_l^2 \sim k_\perp^{-2\alpha} f(k_\parallel/k_\perp^\beta) \sim k_\perp E(k_\perp) \), \( E(k_\perp) = 1\)-dimensional spectrum) and \( t_{\text{cas}}(l) \sim k_\perp^\beta \), we have

\[
k_\perp^{-2\alpha+2+2\beta} = \text{const}. \tag{28}
\]

Therefore,

\[
1 - \alpha + \beta = 0. \tag{29}
\]

Now, let’s pick up a mode at a wavevector \( \mathbf{p} \) and consider nonlinear interactions with other wave modes. First, the strength of the interaction with another mode at \( \mathbf{q} \) is \( \propto p|\tilde{V}(\mathbf{p})||\tilde{V}(\mathbf{q})| \). This comes from the \((\nabla \times \mathbf{V}) \times \mathbf{V}\) term in equation (9) Hereafter we assume \( p \equiv |\mathbf{p}| \approx p_\perp \). Second, the number of interactions is \( \propto p^2 p^{\beta} \). This is the number of modes near \( \mathbf{p} \). Here we use locality of 3-wave interactions. If the interactions are random, the net change of amplitude per unit time will be the strength of the interaction times the square root of the number of interactions, or

\[
|\Delta \tilde{V}(\mathbf{p})| \propto p|\tilde{V}(\mathbf{p})||\tilde{V}(\mathbf{q})| \cdot (p^2 p^{\beta})^{1/2}. \tag{30}
\]

Therefore, we have

\[
t_{\text{cas}} \propto |\tilde{V}(\mathbf{p})|/|\Delta \tilde{V}(\mathbf{p})| \propto p^{-\alpha} p^{-\beta}/2, \tag{31}
\]

where we assumed \( p \propto q \). Equating this with \( t_{\text{cas}} \propto p^{-\beta} \), we can write

\[
\alpha - 2 = -\beta/2. \tag{32}
\]

From equations (29) and (32), we have

\[
\alpha = 5/3, \beta = 2/3, \tag{33}
\]

which is just the result of Goldreich and Sridhar (1995):

\[
k_\parallel \propto k_\perp^{2/3}. \tag{34}
\]

As a consequence, the 3-D energy spectrum becomes

\[
E_3(k_\perp, k_\parallel) \propto k_\perp^{-10/3} f(k_\parallel/k_\perp^{2/3}) \tag{35}
\]

and the corresponding 1-D spectrum is given by

\[
E(k) \propto k^{-5/3}. \tag{36}
\]

In summary, we have shown that the anisotropy of Alfvénic turbulence depends on the spatial scales of eddies. In particular, our results confirm the claim by Goldreich and Sridhar (1995, 1997) that smaller eddies are relatively more elongated along the direction of the local magnetic field lines than larger ones. Quantitative measurements of the anisotropy using the velocity fields show good agreement with their proposed scaling law, \( k_\parallel \sim k_\perp^{2/3} \) as long we interpret these wavenumbers as referring to the local magnetic field direction. However, when the external magnetic...
field is very strong, magnetic fields scale somewhat differently, showing a slightly more rapid increase in anisotropy at smaller scales.

It is important to note that the correct scaling laws depend on comparing the eddy shape to the local magnetic field direction.

As a final note, we wish to stress that our results are not in agreement with the IK theory. The IK theory is based on the assumption of isotropy in wavenumber space, which may be true when the external magnetic field is very weak or zero. However, even in these cases, the turbulence is globally isotropic, but locally very anisotropic. In this paper, we showed that eddies do show anisotropy and that the anisotropy is scale-dependent when there is a strong large scale field (which should apply to very small scales within any MHD turbulence cascade). On the other hand, our results are consistent with Goldreich and Sridhar’s theory of strong MHD turbulence. More precisely, if we consider the ratio of hydrodynamic to Alfvénic rates, that is \( (kv_k/\tilde{k}_\parallel V_A) \), we find from equations (18) and (19) that

\[
\frac{kv_k}{\tilde{k}_\parallel V_A} \propto \tilde{k}_\perp^{0.3-0.5} v_k, \tag{37}
\]

where \( k \approx \tilde{k}_\perp \). From Fig. 2 we see that for the inertial range this implies a ratio which is either constant or increasing with wavenumber. A constant ratio is predicted by Goldreich and Sridhar’s model. The IK model predicts a slow decline.

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Table 1
PARAMETERS

| Run   | $N^3$   | $\nu$           | $\eta$           | $B_0$ |
|-------|---------|-----------------|------------------|-------|
| REF1  | $144^3$ | $3.20 \times 10^{-28}$ | $3.20 \times 10^{-28}$ | 0.5   |
| REF2  | $144^3$ | $3.20 \times 10^{-28}$ | $3.20 \times 10^{-28}$ | 1     |
| REF3  | $144^3$ | $3.20 \times 10^{-28}$ | $3.20 \times 10^{-28}$ | 2     |
| REF4  | $144^3$ | $3.20 \times 10^{-28}$ | $3.20 \times 10^{-28}$ | 3     |
| 256H-$B_00.5$ | $256^3$ | $6.42 \times 10^{-32}$ | $6.42 \times 10^{-32}$ | 0.5   |
| 256H-$B_01$  | $256^3$ | $6.42 \times 10^{-32}$ | $6.42 \times 10^{-32}$ | 1     |
| 256P-$B_01$  | $256^3$ | 0.001           | 0.001            | 1     |

For $256^3$ grids, we use the notation 256X-Y, where X = H or P refers to hyper- or physical viscosity; Y = $B_01$ or $B_00.5$ refers to the strength of the external magnetic fields.