CONCENTRATION OF GROUND STATE SOLUTIONS FOR QUASILINEAR SCHRODINGER SYSTEMS WITH CRITICAL EXPONENTS

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Abstract. This paper is concerned with the critical quasilinear Schrodinger systems in $\mathbb{R}^N$:

\[
\begin{align*}
-\Delta w + (\lambda a(x) + 1)w - (\Delta |w|^2)w &= p \frac{w}{|w|^p} |w|^{p-2} |w|^q + \frac{\alpha}{p+\beta} |w|^\alpha - 2 |w|^\beta \\
-\Delta z + (\lambda b(x) + 1)z - (\Delta |z|^2)z &= \frac{q}{p+q} |w|^p |z|^{q-2} z + \frac{\beta}{p+\beta} |w|^\alpha |z|^{\beta-2} z,
\end{align*}
\]

where $\lambda > 0$ is a parameter, $p, q > 2$, $\alpha, \beta > 2$, $2 \cdot (2^* - 1) < p + q < 2 \cdot 2^*$ and $\alpha + \beta = 2 \cdot 2^*$. By using variational method, we prove the existence of positive ground state solutions which localize near the set $\Omega = \text{int} \{ a^{-1}(0) \} \cap \text{int} \{ b^{-1}(0) \}$ for $\lambda$ large enough.

1. Introduction. Because of its deep applications in physics, in recent years, much attention has been devoted to the following quasilinear Schrodinger equation

\[
-\varepsilon^2 \Delta u + V(x)u - \varepsilon^2 k(\Delta |u|^2)u = g(x, u), \quad \text{in} \ \mathbb{R}^N,
\]

where $V(x)$ is a given potential function, $k$ is a real constant and $g$ is a real function. Equation (1.1) appears more naturally in mathematical physics and can model certain physical phenomena (see [4, 5, 6, 23]). Formally, this equation (1.1) has a variational structure, however we face the lack of a suitable working space such that the variational functional is smooth and has some compactness condition. Hence the standard critical point theory does not seem to apply directly. To overcome this difficulty, in the last decades, several approaches have been successfully employed in the literature to deal with the quasilinear problems, such as change of...
variable approach (see Liu, Wang and Wang [19] and Colin and Jeanjean [7]), Nehari manifold method (see Liu, Wang and Wang [20]), regularization approach (see Liu, Liu and Wang [21]), and so on. And a lot of results have been obtained by using these methods. First, we list some results obtained by change of variables. Bezerra do O, Miyagaki and Soares [3] showed the existence of a positive solution for the critical quasilinear equations. Wang and Zou [27], studied the concentration of positive bound states as $\varepsilon \to 0$. For the case $\varepsilon = 1$ and $V(x) = \lambda > 0$, under different conditions, Adachi, Shibata and Watanbe [1], Wang and Shen [25], proved that the solution of (1.1) would converge to a ground state solution of a semilinear Schrödinger equation as $\varepsilon \to 0$. Moreover, Zeng, Zhang and Zhou [30] studied the positive solutions of a quasilinear Schrödinger equation with Hardy potential and critical exponent. For more information about change of variable approach, we refer to [13, 24, 29, 8, 26, 14] and references therein. However, this method depends heavily on the special structure of the quasilinear term and cannot be generalized to treating more general quasilinear problems. In this case, Nehari manifold method and regularization approach, especially the latter, are robust to deal with general quasilinear problems. For example, Liu, Liu and Wang [16] obtained multiple sign-changing solutions by regularization approach. Liu, Wang and Guo [17], also by regularization approach, studied the multibump solutions. For more information about the regularization approach, we refer to [22, 15, 18].

On the other hand, in some practical physical problem, when several pulses with different frequencies are transmitted in a fiber, then we have to consider the mutual interaction of the different frequencies. Thus, from the viewpoint of mathematics, we are led to consider the quasilinear systems. For example, Guo and Li [9] proved the existence of a ground state positive solution for a class of quasilinear Schrödinger system with critical exponents by perturbation method. We also refer to [11, 10] and references therein for more related results on the quasilinear systems.

In this paper, we consider the following critical quasilinear Schrödinger systems in $\mathbb{R}^N$:

$$
\begin{align*}
-\Delta w + (\lambda a(x) + 1)w - (\Delta |w|^2)w &= \frac{p}{p+q} |w|^{p-2}w|z|^q + \frac{\alpha}{\alpha + \beta} |w|^{\alpha-2}w|z|^{\beta} \\
-\Delta z + (\lambda b(x) + 1)z - (\Delta |z|^2)z &= \frac{q}{p+q} |w|^p|z|^{q-2}z + \frac{\beta}{\alpha + \beta} |w|^p|z|^{\beta-2}z,
\end{align*}
$$

(1.2)

where $N \geq 3$, $\lambda > 0$ is a parameter, $p > 2$, $q > 2$, $\alpha > 2$, $\beta > 2$, $2 \cdot (2^* - 1) < p + q < 2 \cdot 2^*$, $\alpha + \beta = 2 \cdot 2^*$ and $2^*$ is the critical Sobolev exponent. We are interested in the existence of ground state solutions for $\lambda > 0$ and their concentration behavior as $\lambda \to \infty$.

Recently, Guo and Tang [12] studied the existence and the concentration behavior of the ground state solutions for systems (1.2) but with subcritical exponent. The aim of the present paper is to extend the results in [12] to the critical exponent case. Recall that the critical exponent for system (1.2) is $2 \cdot 2^*$. To do this, besides the difficulty caused by the lack of suitable working space, we have to overcome the loss of compactness caused by the critical exponent $2 \cdot 2^*$ and the whole space $\mathbb{R}^N$. We will follow the method used in Colin and Jeanjean [7] to deal with these difficulties. We first choose $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ as the working space, then introduce a limit functional of the original energy functional, combined the method of change of variables, we can prove the infimum of the original energy functional on the Nehari manifold can be achieved, which in turn lead to the results that we expect. We assume:
\( (A_1) \) \( a(x), b(x) \) are locally Hölder continuous in \( \mathbb{R}^N \) and \( a(x) \geq 0, b(x) \geq 0 \) for all \( x \in \mathbb{R}^N \). \( \Omega_a := \text{int} a^{-1}(0), \Omega_b := \text{int} b^{-1}(0), \Omega_a^\ast := a^{-1}(0), \) and \( \Omega_b^\ast := b^{-1}(0) \).

\( (A_2) \) \( \lim_{|x| \to \infty} a(x) = a_\infty < \infty, \lim_{|x| \to \infty} b(x) = b_\infty < \infty, \) and \( a(x) \leq a_\infty, b(x) \leq b_\infty \) and one of the last two inequalities is strict on a subset of positive measure in \( \mathbb{R}^N \).

\( (A_3) \) \( \Omega_a = \Omega_a^\ast \cup \Omega \) and \( \Omega_b = \Omega_b^\ast \cup \Omega \), where \( \text{dist}(\Omega, \Omega_a^\ast) > 0, \text{dist}(\Omega, \Omega_b^\ast) > 0 \) and \( \text{dist}(\Omega_a, \Omega_b) > 0 \). Moreover, \( \Omega_a, \Omega_b \) and \( \Omega \) are smooth connected domain.

**Remark 1.** Condition \( (A_3) \) implies that \( \Omega = \Omega_a \cap \Omega_b, \Omega_a = \Omega \setminus \Omega_b = \Omega_b \setminus \Omega \). Conditions \( (A_1)-(A_3) \) implies that the sets \( \Omega_a \) and \( \Omega_b \) consists of finite connected bounded domains in \( \mathbb{R}^N \). Moreover, any two connected components of \( \Omega_a \) and \( \Omega_b \) are either overlapped or isolated.

We denote by \( \| \cdot \| \) and \( | \cdot |_q \) the norms of \( H^1(\mathbb{R}^N) \) and \( L^q(\mathbb{R}^N) \) respectively, that is

\[
\|u\| = \left( \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx \right)^{\frac{1}{2}},
\]

and

\[
|u|_q = \left( \int_{\mathbb{R}^N} |u|^q \, dx \right)^{\frac{1}{q}}.
\]

Let \( X = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \) with norm \( \|(u, v)\| := (\|u\|^2 + \|v\|^2)^{\frac{1}{2}} \). The norm of \( L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \) is defined by

\[
|(u, v)|_q = \left( \int_{\mathbb{R}^N} |u|^q \, dx + \int_{\mathbb{R}^N} |v|^q \, dx \right)^{\frac{1}{q}}.
\]

We observe that the natural energy functional associated with problem (1.2) is given by:

\[
J_\lambda(w, z) = \frac{1}{2} \int_{\mathbb{R}^N} \left( 1 + 2w^2 \right) |\nabla w|^2 + A_\lambda(x) w^2 \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^N} \left( 1 + 2z^2 \right) |\nabla z|^2 + B_\lambda(x) z^2 \, dx \\
- \frac{1}{p+q} \int_{\mathbb{R}^N} |w|^p |z|^q \, dx - \frac{1}{2} \cdot \frac{1}{\alpha \beta} \int_{\mathbb{R}^N} |w|^\alpha |z|^\beta \, dx,
\]

where \( A_\lambda(x) = \lambda a(x) + 1 \) and \( B_\lambda(x) = \lambda b(x) + 1 \).

However, \( J_\lambda(w, z) \) is not well defined on \( X \). We follow the idea of [7] and make the following change of variable \( u = f^{-1}(w) \) and \( v = f^{-1}(z) \), where \( f \) is defined by

\[
f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}}, \quad \text{on} \ [0, +\infty),
\]

\[
f(t) = -f(-t), \quad \text{on} \ (-\infty, 0].
\]

After the change of variables, we obtain the following functional:

\[
\Phi_\lambda(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + A_\lambda f^2(u) + |\nabla v|^2 + B_\lambda f^2(v) \right) \, dx \\
- \frac{1}{p+q} \int_{\mathbb{R}^N} |f(u)|^p |f(v)|^q \, dx - \frac{1}{\alpha \beta} \int_{\mathbb{R}^N} |f(u)|^\alpha |f(v)|^\beta \, dx,
\]

which is well defined on \( X \) and is of class \( C^1 \). And for any \( (\phi, \psi) \in X \), we have

\[
\langle \Phi_\lambda(u, v), (\phi, \psi) \rangle = \int_{\mathbb{R}^N} \nabla u \nabla \phi \, dx + A_\lambda(x) f(u) f'(u) \phi \, dx + \int_{\mathbb{R}^N} \nabla v \nabla \psi \, dx + B_\lambda(x) f(v) f'(v) \psi \, dx
\]
Assume that

We also define the functional \( \Phi \)

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We also define the functional \( \Phi_{\lambda} \) on \( X \) by:

where \( A_{\lambda} = \lambda a_{\infty} + 1 \) and \( B_{\lambda} = \lambda b_{\infty} + 1 \), and the Nehari manifold \( N_{\lambda} \) by:

We say that \((u_{\lambda}, z_{\lambda}) = (f(u_{\lambda}), f(v_{\lambda}))\) is a least energy solution of \((1.2)\) if \((u_{\lambda}, v_{\lambda})\) is a critical point of \( \Phi_{\lambda} \) such that \( c_{\lambda} \) is achieved.

We point out that under our assumptions, for \( \lambda \) large enough, the following Dirichlet problem is a kind of "limit" problem:

Here are our main results:

**Theorem 1.1.** Assume that \((A_1), (A_2), (A_3)\) are satisfied, \( p > 2, q > 2, \alpha > 2, \beta > 2, 2 \cdot (2^* - 1) < p + q < 2 \cdot 2^*, \alpha + \beta = 2 \cdot 2^* \). Then for \( \lambda > 0 \), \( c_{\lambda} \) is achieved by a critical point \((u_{\lambda}, v_{\lambda})\) of \( \Phi_{\lambda} \) such that \((f(u_{\lambda}), f(v_{\lambda}))\) is a least energy solution of \((1.2)\). Furthermore, for any sequence \( \lambda_n \to +\infty \), \((u_{\lambda_n}, v_{\lambda_n})\) has a subsequence converging to \((u, v)\) such that \((f(u), f(v))\) is a least energy solution of \((1.5)\).

The paper is organized as follows: In section 2, we give some preliminaries. In section 3, behaviors of \((PS)\), sequence and some energy levels are studied. In section 4, the existence of positive ground state solution for \((1.2)\) is proved. Section 5 is devoted to the discussion of "limit problem". In section 6, the proof of Theorem 1.2 is given. Throughout of the paper, we will use the same \( C \) to denote different constants unless otherwise specified.

2. Preliminaries and estimates in \( X \). In this section, we give some properties about \( f \) and make some estimates about the norm in \( X = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \). The following properties were proved in [7] and [27].

**Lemma 2.1.** The function \( f(t) \) has the following properties:

1. \( f \) is uniquely defined, \( C^\infty \) and invertible.
2. \( |f(t)| \leq 1 \) for all \( t \in \mathbb{R} \).
3. \( |f(t)| \leq |t| \) for all \( t \in \mathbb{R} \).
(4) \( f(t)/2 \leq f'(t) \leq f(t) \) for all \( t \geq 0 \).
(5) \( |f(t)| \leq 2^{1/4}|t|^{1/2} \) for all \( t \in R \).
(6) There exists a positive constant \( C \) such that
\[
|f(t)| \geq \begin{cases} 
  C|t|, & |t| \leq 1, \\
  C|t|^{1/2}, & |t| \geq 1.
\end{cases}
\]
(7) \( |f(t)f'(t)| \leq 1/\sqrt{2} \) for all \( t \in R \).
(8) The function \( f^2(t) \) is strictly convex.
(9) \( f^2(\xi t) \leq C\xi^2 f^2(t) \) for \( \xi > 1 \) and \( t \in R \).
(10) The function \( f(t)f'(t)t^{-1} \) is decreasing for \( t > 0 \).

**Lemma 2.2.** There exists two constant \( C_1 > 0 \) independent of \( \lambda \), \( C_2 > 0 \) such that for any \( (u, v) \in X \),
\[
C_1 \min \left\{ \|u\|^{2(N+2)/(N+4)}, \|v\|^2 \right\} \leq \int_{R^N} (|\nabla u|^2 + A\lambda f^2(u))dx \leq C_2 \|u\|^2.
\]
\[
C_1 \min \left\{ \|v\|^{2(N+2)/(N+4)}, \|v\|^2 \right\} \leq \int_{R^N} (|\nabla v|^2 + B\lambda f^2(v))dx \leq C_2 \|v\|^2.
\]
\[
C_1 \min \left\{ \|(u, v)\|^{2(N+2)/(N+4)}, \|(u, v)\|^2 \right\} \leq \int_{R^N} (|\nabla u|^2 + A\lambda f^2(u) + |\nabla v|^2 + B\lambda f^2(v))dx \leq C_2 \|(u, v)\|^2.
\]

**Proof.** First, by (A2) and (3) of Lemma 2.1, there exists \( C_2 > 0 \), such that
\[
\int_{R^N} |\nabla u|^2 + A\lambda f^2(u)dx \leq C_2 \int_{R^N} |\nabla u|^2 + u^2 dx = C_2 \|u\|^2.
\]
On the other hand, by (6) of Lemma 2.1,
\[
|u|^2 = \int_{\{x:|u| \geq 1\}} u^2 dx + \int_{\{x:|u| \leq 1\}} u^2 dx 
\leq C \left( \int_{\{x:|u| \geq 1\}} f^4(u)dx + \int_{\{x:|u| \leq 1\}} f^2(u)dx \right)
\leq C \left( \|f(u)\|^4_{L^4} + \|f(u)\|^2_{L^2} \right)
\leq C \left( \|f(u)\|^4_{L^4} |\nabla u|^2_{L^2} + |f(u)|^2_{L^2} \right).
\]
Let \( a = \int_{R^N} |\nabla u|^2 + A\lambda f^2(u)dx \), then by the above inequality, we have
\[
\|u\|^2 \leq C(a + a^{\frac{N+4}{N+2}}).
\]
If \( a \geq 1 \), then we have \( \|u\|^2 \leq 2Ca^{\frac{N+4}{N+2}} \), thus \( a \geq \left( \frac{1}{2C} \right)^{\frac{N+4}{N+2}} \|u\|^{2(N+2)/(N+4)} \). If \( a \leq 1 \), then we have \( \|u\|^2 \leq 2Ca \), thus \( a \geq \frac{1}{2C} \|u\|^2 \). Therefore, there exists \( C_1 > 0 \) independent of \( \lambda \) such that
\[
\int_{R^N} (|\nabla u|^2 + A\lambda f^2(u))dx \geq C_1 \min \left\{ \|u\|^{2(N+2)/(N+4)}, \|u\|^2 \right\}.
\]
Thus,
\[
C_1 \min \left\{ \|u\|^{2(N+2)/(N+4)}, \|u\|^2 \right\} \leq \int_{R^N} (|\nabla u|^2 + A\lambda f^2(u))dx \leq C_2 \|u\|^2.
\]
Similarly, we have

\[ C_1 \min \left\{ \|v\|^{\frac{2(N+2)}{N+4}} \right\} \leq \int_{\mathbb{R}^N} (|\nabla v|^2 + B\lambda f^2(v)) \, dx \leq C_2 \|v\|^2. \]

From the above two estimates, we can easily get

\[ C_1 \min \left\{ \|(u, v)\|^{\frac{2(N+2)}{N+4}} \right\} \leq \int_{\mathbb{R}^N} (|\nabla u|^2 + A\lambda f^2(u) + |\nabla v|^2 + B\lambda f^2(v)) \, dx \leq C_2 \|(u, v)\|^2. \]

\[ \square \]

**Lemma 2.3.** The map : \((u, v) \mapsto (f(u), f(v))\) from \(X\) into \(L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)\) is continuous for \(2 \leq q \leq 2 \cdot 2^*\).

**Proof.** For \(u \in H^1(\mathbb{R}^N)\), we have

\[ |f(u)|_2 \leq |u|_2 \leq \|u\|, \]

and

\[ |f(u)|_{2^*} \leq C|u|^\frac{1}{2^*}, \leq C|\nabla u| \leq \|u\|^\frac{1}{2}. \]

Thanks to the inequality:

\[ |f(u)|_q \leq |f(u)|_{2^*}^{\frac{1}{q-1}} \quad \text{for } 0 < \theta < 1 \quad \text{and } q \in (2, 2 \cdot 2^*), \]

we obtain that \(f(v) \in L^q(\mathbb{R}^N)\) for \(q \in [2, 2 \cdot 2^*]\). Moreover,

\[ |f(u)|_q \leq \|u\|^{\frac{q}{2^*}}. \]

Therefore, there exists \(C > 0\) independent of \(\lambda\), such that

\[ |f(u)|_q \leq C \max \left\{ \|u\|, \|u\|^{\frac{1}{2}} \right\}. \quad (2.6) \]

Now we prove that if \(u_n \to u\) in \(H^1(\mathbb{R}^N)\), then \(f(u_n) \to f(u)\) in \(L^q(\mathbb{R}^N)\).

By (2.6), we have

\[ \|f(u_n) - u\|^2_q \leq C \max \left\{ \|u_n - u\|, \|u_n - u\|^2 \right\}, \]

so \(f(u_n) \to 0\) in \(L^q(\mathbb{R}^N)\), and \(u_n \to u\) a.e. in \(\mathbb{R}^N\).

On the other hand, using (8) and (9) of Lemma 2.1, we have

\[ |f^q(u_n)| = |f^2(u_n - u + v)|^{\frac{1}{q}} \leq |f^2(2(u_n - u)) + f^2(2(u))|^{\frac{1}{q}} \]

\[ \leq C(f^2(u_n - u) + f^2(v))^{\frac{1}{2}} \leq C(|f(u_n - u)| + |f(u)|)^q \]

so \(f(u_n) \to f(u)\) in \(L^q(\mathbb{R}^N)\).

Note that, if \((u_n, v_n) \to (u, v)\) in \(X\), then \(u_n \to u\) and \(v_n \to v\) in \(H^1(\mathbb{R}^N)\), hence \(f(u_n) \to f(u)\) and \(f(v_n) \to f(v)\) in \(L^q(\mathbb{R}^N)\). Then, as \(n \to \infty\), we have

\[ ||(f(u_n), f(v_n)) - (f(u), f(v))||_q^2 = ||f(u_n) - f(u)||_q^2 + ||f(v_n) - f(v)||_q^2 \to 0. \]

Moreover,

\[ ||(f(u), f(v))||_q^2 \leq \left( |f(u)|_q + |f(v)|_{L^q} \right)^2 \leq 2(|f(u)|_q^2 + |f(v)|_q^2) \leq C \max \left\{ \|u\|, \|v\|^{\frac{1}{2}} \right\} \]

\[ \leq C \max \left\{ \|u\| + \|v\|, (\|u\| + \|v\|)^2 \right\}. \]
\[ \leq C \max \{ \| (u, v) \|, \| (u, v) \|^2 \}. \]

So, we have
\[ |(f(u), f(v))|^2 \leq C \max \{ \| (u, v) \|, \| (u, v) \|^2 \} , \quad (2.7) \]
where \( C \) is independent of \( \lambda \).

3. **Behaviors of \((PS)_c\) sequence and some energy levels.** In this section, we study the boundedness of \((PS)_c\) sequence and give some energy levels about associated functional. Recall that we say \( \{ (u_n, v_n) \} \subset X \) is a Palais-Smale \(_c\) sequence ((\(PS)_c\) sequence in short) for \( \Phi_\lambda \) if \( \Phi_\lambda (u_n, v_n) \to c \) and \( \Phi_\lambda'(u_n, v_n) \to 0 \) in \( X^* \), the dual space of \( X \).

**Lemma 3.1.** Any of the \((PS)_c\) sequence \( \{ (u_n, v_n) \} \) for \( \Phi_\lambda \) is bounded, and
\[ \lim_{n \to \infty} \sup \| (u_n, v_n) \| \leq \max \left\{ \left( \frac{2(p+q)c}{(p+q-4)C_1} \right)^{\frac{N+4}{N}}, \left( \frac{2(p+q)c}{(p+q-4)C_1} \right) \right\} , \]
where \( C_1 \) is the number in Lemma 2.2.

**Proof.** Suppose that \( \{ (u_n, v_n) \} \) is a \((PS)_c\) sequence of \( \Phi_\lambda \), we have
\[ \Phi_\lambda (u_n, v_n) \to c, \quad \Phi_\lambda'(u_n, v_n) \to 0. \]
Taking \( \phi_n = \frac{f(u_n)}{f'(u_n)}, \psi_n = \frac{f(v_n)}{f'(v_n)} \), then \( \| \phi_n \| \leq C \| u_n \| \) and \( \| \psi_n \| \leq C \| v_n \| \), thus
\[ c + o(1) + o(1) \| (u_n, v_n) \| = \Phi_\lambda (u_n, v_n) - \frac{1}{p+q} (\Phi_\lambda'(u_n, v_n) \cdot f(u_n) - f'(u_n)) \]
\[ = \int_{\mathbb{R}^N} \left( \frac{1}{2} - \frac{1}{p+q} \right) \left( 1 + \frac{2f^2(u_n)}{1+2f^2(u_n)} \right) |\nabla u_n|^2 dx + \left( \frac{1}{2} - \frac{1}{p+q} \right) \int_{\mathbb{R}^N} A_\lambda f^2(u_n) dx \]
\[ + \int_{\mathbb{R}^N} \frac{1}{2} - \frac{1}{p+q} \left( 1 + \frac{2f^2(v_n)}{1+2f^2(v_n)} \right) |\nabla v_n|^2 dx \]
\[ + \left( \frac{1}{2} - \frac{1}{p+q} \right) \int_{\mathbb{R}^N} B_\lambda f^2(v_n) dx \]
\[ \geq \int_{\mathbb{R}^N} \left( \frac{1}{2} - \frac{2}{p+q} \right) |\nabla u_n|^2 dx + \left( \frac{1}{2} - \frac{1}{p+q} \right) \int_{\mathbb{R}^N} A_\lambda f^2(u_n) dx \]
\[ + \int_{\mathbb{R}^N} \left( \frac{1}{2} - \frac{2}{p+q} \right) |\nabla v_n|^2 dx + \left( \frac{1}{2} - \frac{1}{p+q} \right) \int_{\mathbb{R}^N} B_\lambda f^2(v_n) dx \]
\[ \geq \left( \frac{1}{2} - \frac{2}{p+q} \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 + A_\lambda f^2(u_n) + |\nabla v_n|^2 + B_\lambda f^2(v_n) dx \]
It follows that
\[ \left( \frac{1}{2} - \frac{2}{p+q} \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 + A_\lambda f^2(u_n) + |\nabla v_n|^2 + B_\lambda f^2(v_n) dx \leq c + o(1) + o(1) \| (u_n, v_n) \| , \]
by Lemma 2.2 , we obtain
\[ \min \left\{ \left( \frac{2(p+q)c}{(p+q-4)C_1} \| (u_n, v_n) \|^2 \right), \left( \frac{2(p+q)c}{(p+q-4)C_1} \right) \right\} \leq \left( \frac{2(p+q)c}{(p+q-4)C_1} \right)^{\frac{N+4}{N}} + o(1) + o(1) \| (u_n, v_n) \| , \]
thus \( \{ (u_n, v_n) \} \) is bounded in \( X \), and
\[ \lim_{n \to \infty} \sup \| (u_n, v_n) \| \leq \max \left\{ \left( \frac{2(p+q)c}{(p+q-4)C_1} \right)^{\frac{N+4}{N}}, \left( \frac{2(p+q)c}{(p+q-4)C_1} \right) \right\} , \]
\[ \square \]
Lemma 3.2. There exists $0 < \sigma < 1$ which is independent of $\lambda$ such that 
\[ \| (u, v) \| > \sigma \quad \text{and} \quad \Phi_\lambda (u, v) \geq \frac{p + q - 4}{2(p + q)} C_1 \sigma^2 \quad \text{for all} \quad (u, v) \in N_\lambda \setminus \{(0, 0)\} . \]

Proof. For any $(u, v) \in N_\lambda \setminus \{(0, 0)\}$, without loss of of generality, we may assume that $\| (u, v) \| \leq 1$ ( otherwise the conclusion is also true up to a constant ). Then, by (4) of Lemma 2.1, we have
\[
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + A_\lambda f^2(u) + |\nabla v|^2 + B_\lambda f^2(v) dx \\
\leq \int_{\mathbb{R}^N} |\nabla u|^2 + A_\lambda f(u)f'(u)u + |\nabla v|^2 + B_\lambda f(v)f'(v)v dx \\
= \frac{p}{p + q} \int_{\mathbb{R}^N} |f(u)|^{p - 2} f(u)f'(u)|f(v)|^q u dx \\
+ \frac{q}{p + q} \int_{\mathbb{R}^N} |f(u)|^p |f(v)|^{q - 2} f(v)f'(v)v dx \\
+ \frac{\alpha}{\alpha + \beta} \int_{\mathbb{R}^N} |f(u)|^{\alpha - 2} f(u)f'(u)|f(v)|^\beta u dx \\
+ \frac{\beta}{\alpha + \beta} \int_{\mathbb{R}^N} |f(u)|^{\alpha} |f(v)|^{\beta - 2} f(v)f'(v)v dx \\
\leq \int_{\mathbb{R}^N} |f(u)|^p |f(v)|^q dx + \int_{\mathbb{R}^N} |f(u)|^\alpha |f(v)|^\beta dx .
\]
From Lemma 2.2 and (2.7), we have
\[
\frac{C_1}{2} \min \left\{ \| (u, v) \|^{\frac{2(N + 2)}{N + 2}} , \| (u, v) \|^2 \right\} \\
\leq |f(u)|^p \| f(v) \|^q \| f(v) \|^{\alpha - 2} + \| f(u) \|^\alpha |f(v)|^\beta \| f(v) \|^{\alpha + \beta} \\
\leq (|f(u), f(v)|)^{p + q} + (f(u), f(v))^{\alpha + \beta} \\
\leq C(\max \{ \| (u, v) \|, \| (u, v) \|^2 \})^{\frac{N + 2}{N + 4}} + C(\max \{ \| (u, v) \|, \| (u, v) \|^2 \})^{\frac{N + 2}{N + 4}}.
\]
Since $\| (u, v) \| \leq 1$, one can easily deduce the first desired result. On the other hand, by (4) of Lemma 2.1, we have
\[
\int_{\mathbb{R}^N} |\nabla u|^2 + A_\lambda f(u)f'(u)u + |\nabla v|^2 + B_\lambda f(v)f'(v)v dx \\
= \frac{p}{p + q} \int_{\mathbb{R}^N} |f(u)|^{p - 2} f(u)f'(u)|f(v)|^q u dx \\
+ \frac{q}{p + q} \int_{\mathbb{R}^N} |f(u)|^p |f(v)|^{q - 2} f(v)f'(v)v dx \\
+ \frac{\alpha}{\alpha + \beta} \int_{\mathbb{R}^N} |f(u)|^{\alpha - 2} f(u)f'(u)|f(v)|^\beta u dx \\
+ \frac{\beta}{\alpha + \beta} \int_{\mathbb{R}^N} |f(u)|^{\alpha} |f(v)|^{\beta - 2} f(v)f'(v)v dx \\
\geq \frac{1}{2} \int_{\mathbb{R}^N} |f(u)|^p |f(v)|^q dx + \frac{1}{2} \int_{\mathbb{R}^N} |f(u)|^\alpha |f(v)|^\beta dx .
\]
Thus,
\[
\Phi_\lambda (u, v) \geq (\frac{1}{2} - \frac{2}{p + q}) \int_{\mathbb{R}^N} (|\nabla v|^2 + A_\lambda f^2(v)) + |\nabla v|^2 + B_\lambda f^2(v) dx
\]
above equality is increasing about \( t \).

From (10) of Lemma 2.1 and Lemma 3.1 of [12], we know that the right side of the

\[
\begin{align*}
\geq \left( \frac{1}{2} - \frac{2}{p+q} \right) C_1 \min \left\{ \| (u, v) \|^{\frac{2(N+2)}{N+4}}, \| (u, v) \|^{2} \right\} \\
\geq \frac{p+q-4}{2(p+q)} C_1 \sigma^2.
\end{align*}
\]

This completes the proof of the lemma. \( \square \)

Let

\[
c_\lambda = \inf_{(u, v) \in N_\lambda} \Phi_\lambda(u, v), \quad c^* = \inf_{(u, v) \in X \setminus \{(0, 0)\}} \sup_{t \geq 0} \Phi_\lambda(t(u, v)), \quad c^{**} = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} \Phi_\lambda(\gamma(t)),
\]

where

\[
\Gamma = \{ \gamma(t) \in C([0, 1], X) : \gamma(0) = 0, \Phi_\lambda(\gamma(1)) < 0 \}.
\]

Define

\[
E = \{ (u, v) \in X : \mu \{ x : |u| > 0, |v| > 0 \} > 0 \}.
\]

where \( \mu \) denotes the Lebesgue measure on \( \mathbb{R}^N \).

**Lemma 3.3.** For any \((u, v) \in E\), there exists a unique \( t_{(u,v)} > 0 \) such that \( t_{(u,v)}(u, v) \in N_\lambda \) and \( \Phi_\lambda(t_{(u,v)}(u, v)) = \sup_{t \geq 0} \Phi_\lambda(t(u, v)) \).

**Proof.** Let \( g(t) = \Phi_\lambda(t(u, v)) \) for \( t > 0 \), then \( g'(t) = \langle \Phi'(t(u, v)), (u, v) \rangle = 0 \) if and only if \( t(u, v) \in N_\lambda \), which can be written as:

\[
\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 = \frac{p}{p+q} \int_{\mathbb{R}^N} \frac{|f(tu)|^p f(tu) f'(tu)|f(tu)|^q u}{t} dx \\
+ \frac{q}{p+q} \int_{\mathbb{R}^N} \frac{|f(tu)|^p f(tu)|f'(tu)v|^2}{t} dx \\
+ \frac{\alpha}{\alpha + \beta} \int_{\mathbb{R}^N} \frac{|f(tu)|^{\alpha - 2} f(tu) f'(tu)|f(tu)|^u}{t} dx \\
+ \frac{\beta}{\alpha + \beta} \int_{\mathbb{R}^N} \frac{|f(tu)|^{\beta - 2} f(tu) f'(tu)v|^2}{t} dx \\
- \int_{\mathbb{R}^N} A \lambda f(tu) f'(tu) u dx - \int_{\mathbb{R}^N} B \lambda f(tu) f'(tu)v dx.
\]

From (10) of Lemma 2.1 and Lemma 3.1 of [12], we know that the right side of the
above equality is increasing about \( t \), if there exists \( t_{(u,v)} \) that makes it hold, then \( t_{(u,v)} \) must be unique, now we turn to the existence of \( t_{(u,v)} \).

Firstly, by Lemma 2.2 and (2.7), when \( t \) is small enough, we have

\[
g(t) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla tu|^2 + A \lambda f^2(tu) + |\nabla v|^2 + B \lambda f^2(tv)) dx \\
- \frac{1}{p+q} \int_{\mathbb{R}^N} |f(tu)|^p |f(tu)|^q dx - \frac{\alpha}{\alpha + \beta} \int_{\mathbb{R}^N} |f(tu)|^\alpha |f(tu)|^\beta dx, \\
\geq C \min \left\{ \| (tu, tv) \|^\frac{2(N+2)}{N+4}, \| (tu, tv) \|^{2} \right\} - \frac{1}{p+q} |(f(tu), f(tv))|^p q \\
- \frac{1}{\alpha + \beta} |(f(tu), f(tv))|^q \frac{\alpha + \beta}{\alpha + \beta} \\
\geq C \min \left\{ \| (tu, tv) \|^\frac{2(N+2)}{N+4}, \| (tu, tv) \|^{2} \right\} \\
- C(\max \left\{ \| (tu, tv) \|, \| (tu, tv) \|^{2} \right\})^{\frac{p+q}{p+q}} - C(\max \left\{ \| (tu, tv) \|, \| (tu, tv) \|^{2} \right\})^{\frac{p+q}{p+q}} \\
= C t^2 \| (u, v) \|^2 - C t^{\frac{p+q}{p+q}} \| (u, v) \|^{\frac{p+q}{p+q}} - C t^2 \| (u, v) \|^{2}.
\]
> 0.

Then, by Lemma 2.2 and (6) of Lemma 2.1, we get

\[
g(t) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla tu|^2 + A_\lambda f^2(tu) + |\nabla tv|^2 + B_\lambda f^2(tv))\,dx
- \frac{1}{p + q} \int_{\mathbb{R}^N} |f(tu)|^p |f(tv)|^q\,dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |f(tu)|^\alpha |f(tv)|^\beta\,dx
\leq C\|tu, tv\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} |f(tu)|^\alpha |f(tv)|^\beta\,dx
\leq C\|tu, tv\|^2 - \frac{1}{2} \int_{\{x : |tu|, |tv| \geq 1\}} |f(tu)|^\alpha |f(tv)|^\beta\,dx
\leq C\|tu, tv\|^2 - Ct^{2*} \int_{\{x : |tu|, |tv| \geq 1\}} |u|^\frac{\alpha}{2} |v|^\frac{\beta}{2}\,dx.
\]

Since \((u, v) \in E\), there exist \(t_0 > 0\) such that \(\mu\{x : |u| \geq \frac{1}{t_0}, |v| \geq \frac{1}{t_0}\} > 0\), thus

\[
\int_{\{x : |tu|, |tv| \geq 1\}} |u|^\frac{\alpha}{2} |v|^\frac{\beta}{2}\,dx > 0.
\]

When \(t\) is large enough, we have

\[
g(t) \leq Ct^{2*}\|u, v\|^2 - Ct^{2*} \int_{\{x : |tu|, |tv| \geq 1\}} |u|^\frac{\alpha}{2} |v|^\frac{\beta}{2}\,dx < 0.
\]

Hence, there exists a unique \(t_{(u,v)} > 0\) such that \(t_{(u,v)}(u, v) \in \mathcal{N}_{\lambda}\). Moreover, \(g'(t) > 0\) in \((0, t_{(u,v)})\) and \(g'(t) < 0\) in \((t_{(u,v)}, \infty)\), we get \(\Phi_{\lambda}(t_{(u,v)}(u, v)) = \sup_{t \geq 0} \Phi_{\lambda}(t(u, v))\).

\[\Box\]

**Remark 2.** From the proof of Lemma 3.3, we know that when \(\|u, v\| \neq 0\) is small enough, \(\Phi_{\lambda}(u, v) > 0\). Moreover, \(\Phi_{\lambda}(u, v)\) has mountain pass geometry properties.

**Lemma 3.4.** \(c_\lambda = c^* = c^{**}\).

**Proof.** To prove the lemma, we first prove

\[
c^* = \inf_{(u, v) \in \mathcal{X} \setminus \{0, 0\}} \sup_{t \geq 0} \Phi_{\lambda}(t(u, v)) = \inf_{(u, v) \in E} \sup_{t \geq 0} \Phi_{\lambda}(t(u, v)).
\]

In fact, if \((u, v) \in X \setminus E\) and \((u, v) \neq (0, 0)\), then

\[
\Phi_{\lambda}(t(u, v)) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla tu|^2 + A_\lambda f^2(tu) + |\nabla tv|^2 + B_\lambda f^2(tv))\,dx
\geq C \min \left\{\|tu, tv\|^{\frac{2(N+2)}{N+4}}, \|tu, tv\|^2\right\} \to \infty \quad \text{as } t \to \infty.
\]

Thus,

\[
c^* = \inf_{(u, v) \in \mathcal{X} \setminus \{0, 0\}} \sup_{t \geq 0} \Phi_{\lambda}(t(u, v)) = \inf_{(u, v) \in E} \sup_{t \geq 0} \Phi_{\lambda}(t(u, v)).
\]

We divide the proof into the following three steps.

**Step1.** \(c^* = c_{\lambda}\). By Lemma 3.3,

\[
c^* = \inf_{(u, v) \in \mathcal{X} \setminus \{0, 0\}} \sup_{t \geq 0} \Phi_{\lambda}(t(u, v)) = \inf_{(u, v) \in E} \sup_{t \geq 0} \Phi_{\lambda}(t(u, v))
= \inf_{(u, v) \in E} \Phi_{\lambda}(t_{(u,v)}(u, v)) = \inf_{(u, v) \in \mathcal{N}_{\lambda}} \Phi_{\lambda}(u, v) = c_\lambda.
\]
Step2. \( c^* \geq c^{**} \). From Lemma 3.3, for \((u, v) \in E\), there exists \( t_0 \) large enough, such that \( \Phi_\lambda(t_0(u, v)) < 0 \). Define \( \gamma_0(t) = tt_0(u, v) \) for \( t \in [0, 1] \), then \( \gamma_0(t) \in \Gamma \), thus
\[
c^{**} = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} \Phi_\lambda(\gamma(t)) \leq \sup_{t \in [0, 1]} \Phi_\lambda(\gamma(t)) \leq \sup_{t \geq 0} \Phi_\lambda(t(u, v)).
\]
we get \( c^* \geq c^{**} \).

Step3. \( c^{**} \geq c_\lambda \). The manifold \( N_\lambda \) separates \( X \) into two components. By Remark 2, the component contains the origin also contains a small ball around the origin. Moreover \( \Phi_\lambda(u, v) \geq 0 \) in this component, since \( g'(t) = (\Phi_\lambda'(t(u, v)), (u, v)) \geq 0 \) for all \( 0 \leq t \leq t(u, v) \). Thus every \( \gamma \in \Gamma \) has to cross \( N_\lambda \) and \( c^{**} \geq c_\lambda \). \( \square \)

The following lemma can be found in [2].

**Lemma 3.5.** Let \( \Omega \) be a domain of \( \mathbb{R}^N (N \geq 3) \) (not necessarily bounded). Define
\[
S_{a,b}(\Omega) := \inf_{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \setminus \{(0,0)\}} \left\{ \int_\Omega |\nabla u|^2 + |\nabla v|^2 dx - \frac{\int_\Omega |u|^a |v|^b dx}{\frac{1}{a} + \frac{1}{b}} \right\}.
\]
where \( a > 1, b > 1 \) and \( a + b \leq 2^* \), then
\[
S_{a,b}(\Omega) = ((\frac{a}{b})^{\frac{1}{a+b}} + (\frac{1}{a})^{\frac{1}{a+b}}) S_{a+b}(\Omega).
\]
Especially, when \( a + b = 2^* \), we have \( S_{a+b}(\Omega) = S_{2^*}(\Omega) = S_{2^*}(\mathbb{R}^N) = S \), where \( S \) is the Sobolev constant, and \( S_{a,b}(\Omega) \) is independent of the domain \( \Omega \), so we denote \( S_{a,b} \) instead of \( S_{a,b}(\Omega) \).

**Lemma 3.6.** \( c_\lambda < \frac{1}{2N} (S_{2^*,2} \frac{N}{2})^\frac{N}{2} \).

**Proof.** We use the similar arguments used in [13] and [10]. Let
\[
Y = \left\{ (u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \left| \int_{\mathbb{R}^N} |\nabla u|^2 < \infty, \int_{\mathbb{R}^N} |\nabla v|^2 < \infty \right. \right\}.
\]
Define the functional \( J_\lambda : Y \to \mathbb{R} \) by:
\[
J_\lambda(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} (1 + 2u^2)|\nabla u|^2 + A_\lambda(x) u^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} (1 + 2v^2)|\nabla v|^2 + B_\lambda(x) v^2 \, dx
- \frac{1}{p+q} \int_{\mathbb{R}^N} |u|^p |v|^q \, dx - \frac{1}{2} \cdot 2^* \int_{\mathbb{R}^N} |u|^a |v|^b \, dx.
\]
Then \( (f^{-1}(u), f^{-1}(v)) \in X \) and \( \Phi_\lambda(f^{-1}(u), f^{-1}(v)) = J_\lambda(u, v) \).

It suffices to prove that there is \( (0,0) \neq (w_1, w_2) \in Y \) such that
\[
\sup_{\varepsilon \geq 0} J_\lambda(t(w_1, w_2)) < \frac{1}{2N} (S_{2^*,2} \frac{N}{2})^\frac{N}{2}.
\]
In fact, since \( \Phi_\lambda(f^{-1}(t w_1), f^{-1}(t w_2)) = J_\lambda(t w_1, t w_2) \to -\infty \), as \( t \to \infty \), then there exists \( t^* > 0 \) such that \( \Phi_\lambda(f^{-1}(t^* w_1), f^{-1}(t^* w_2)) < 0 \). Define \( \gamma^*(t) = (f^{-1}(t^* w_1), f^{-1}(t^* w_2)) \), by Lemma 3.4, we have \( c_\lambda = c^{**} \leq \sup_{t \geq 0} J_\lambda(t w_1, t w_2) < \frac{1}{2N} (S_{2^*,2} \frac{N}{2})^\frac{N}{2} \). For given \( \varepsilon > 0 \), we consider the function \( U_\varepsilon(x) = \frac{1}{(\varepsilon + |x|^2)^\frac{N-2}{2}} \), which satisfies \( \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \, dx = \int_{\mathbb{R}^N} |U_\varepsilon|^2 \, dx = S^\frac{N}{2} \). Let \( \varphi \in C_0^\infty(\mathbb{R}^N, [0, 1]) \) such
that $\varphi = 1$ if $|x| < 1$ and $\varphi = 0$ if $|x| \geq 2$. Considering the following function $Z_\varepsilon(x) := \varphi(x)U_\varepsilon(x)\frac{1}{\varepsilon}$. Then by direct computation, we get

$$\|\nabla Z_\varepsilon\|_{L^2}^2 = S_\varepsilon^+ + O(\varepsilon^{-\frac{N-2}{2}}),$$

$$\|\nabla Z_\varepsilon\|_{L^2}^2 = O(\varepsilon^{-\frac{N-2}{2}}|\ln \varepsilon|),$$

$$\|Z_\varepsilon\|_{L^2}^2 = S_\varepsilon^+ + O(\varepsilon^{-\frac{N}{2}}),$$

$$\|Z_\varepsilon\|_{L^r} \approx \begin{cases} \varepsilon^{\frac{r(N-2)}{2}} N, & r \in (1, 2^*), \\ \varepsilon^\frac{N}{2} \ln \varepsilon, & r = 2^*, \\ \varepsilon^\frac{N}{2} \cdot \varepsilon^{-(N-2)} \frac{1}{N}, & r \in (2^*, 22^*). \end{cases}$$

Let $w_1 = \alpha^\frac{1}{4} Z_\varepsilon, w_2 = \beta^\frac{1}{4} Z_\varepsilon$, then $(w_1, w_2) \in Y$ and there exist $T_1, T_2 > 0$ such that

$$\sup_{t \geq 0} J_\lambda(t(w_1, w_2))$$

$$= \sup_{t_1 \leq t \leq t_2} J_\lambda(t(w_1, w_2))$$

$$= \sup_{t_1 \leq t \leq t_2} \left\{ \frac{(\alpha + \beta)t^4}{4} \int_{\mathbb{R}^N} |\nabla (Z_\varepsilon)|^2 dx + \frac{\beta t^4}{4} \int_{\mathbb{R}^N} |\nabla (Z_\varepsilon)|^2 dx \\ - \frac{t^{2^*} \alpha^\frac{\beta}{2} \beta^\frac{\alpha}{2}}{2 \cdot 2^*} \int_{\mathbb{R}^N} (Z_\varepsilon)^{2^{*}} dx \right\} + C \int_{\mathbb{R}^N} (Z_\varepsilon)^{2^*} + C \int_{\mathbb{R}^N} (Z_\varepsilon)^{p+q} dx$$

$$\leq \sup_{t_1 \leq t \leq t_2} \left\{ \frac{(\alpha + \beta)t^4}{4} (S_\varepsilon^+ + O(\varepsilon^{-\frac{N-2}{2}})) - \frac{t^{2^*} \alpha^\frac{\beta}{2} \beta^\frac{\alpha}{2}}{2 \cdot 2^*} (S_\varepsilon^+ + O(\varepsilon^{-\frac{N}{2}})) \right\}$$

$$+ O(\varepsilon^{-\frac{N-2}{2}} \ln \varepsilon) + O(\varepsilon^{-\frac{N-2}{4}}) - C\varepsilon^{-\frac{N}{2}} - \frac{(p+q)(N-2)}{N}$$

$$\leq \sup_{t_1 \leq t \leq t_2} \left\{ \frac{(\alpha + \beta)t^4}{4} - \frac{t^{2^*} \alpha^\frac{\beta}{2} \beta^\frac{\alpha}{2}}{2 \cdot 2^*} \right\} S_\varepsilon^+ + O(\varepsilon^{-\frac{N-2}{2}} \ln \varepsilon) - C\varepsilon^{-\frac{N}{2}} - \frac{(p+q)(N-2)}{N}$$

$$= \frac{1}{2N} \left( \frac{\alpha + \beta}{\alpha^\frac{\beta}{2} \beta^\frac{\alpha}{2}} \right) S_\varepsilon^+ + O(\varepsilon^{-\frac{N-2}{4}} \ln \varepsilon) - C\varepsilon^{-\frac{N}{2}} - \frac{(p+q)(N-2)}{N}$$

$$= \frac{1}{2N} \left( \frac{\alpha/\beta}{\alpha^\frac{\beta}{2} \beta^\frac{\alpha}{2}} \right) S_\varepsilon^+ + O(\varepsilon^{-\frac{N-2}{4}} \ln \varepsilon) - C\varepsilon^{-\frac{N}{2}} - \frac{(p+q)(N-2)}{N}$$

$$= \frac{1}{2N} \left( \frac{\beta/\alpha}{\alpha^\frac{\beta}{2} \beta^\frac{\alpha}{2}} \right) S_\varepsilon^+ + O(\varepsilon^{-\frac{N-2}{4}} \ln \varepsilon) - C\varepsilon^{-\frac{N}{2}} - \frac{(p+q)(N-2)}{N}$$

for $\varepsilon > 0$ small enough. This completes the proof.

\textbf{Remark 3.} It is easy to know that the related results from Lemma 2.2 to Lemma 3.6 also hold for $N_{\lambda \infty}$ and $\Phi_{\lambda \infty}$. 

\begin{flushright} \Box \end{flushright}
4. the existence of positive ground state solution for (1.2). In this section, we prove the existence of positive ground state solution for (1.2).

**Lemma 4.1.** $c_{\infty} := \inf_{N \lambda \in \Phi_{\lambda \infty}(u,v)}$ is achieved by some $(u_1, v_1) \in X$ and $u_1, v_1$ are positive in $\mathbb{R}^N$.

**Proof.** The proof is almost the same with Lemma 4.2, we omit it. \hfill $\Box$

**Lemma 4.2.** $c_\lambda := \inf_{N \lambda} \Phi_{\lambda}(u,v)$ is achieved by some $(u, v) \in X$.

**Proof.** By Lemma 3.4 and Remark 2, there exists a sequence $(u_n, v_n)$ which is a $(PS)_{c_\lambda}$ sequence of $\Phi_{\lambda}$. By Lemma 3.1, we know that $(u_n, v_n)$ is bounded in $X$. Hence, there exists $(u, v) \in X$ such that (up to a subsequence)

\begin{align*}
u_n & \to u \quad \text{in } H^1, \quad v_n \to v \quad \text{in } H^1, \\
u_n & \to u \quad \text{a.e. in } \mathbb{R}^N, v_n \to v \quad \text{a.e. in } \mathbb{R}^N, \\
f(u_n) & \to f(u) \quad \text{in } L^q, \quad 2 \leq q \leq 2 \cdot 2^*, \quad f(v_n) \to f(v) \quad \text{in } L^q, \quad 2 \leq q \leq 2 \cdot 2^*.
\end{align*}

A standard argument shows that $\Phi'_{\lambda}(u, v) = 0$.

Now we prove $(u, v) \neq (0, 0)$.

Since $(u_n, v_n)$ is a $(PS)_{c_\lambda}$ sequence of $\Phi_{\lambda}$, we have

\begin{align*}
o(1) &= \langle \Phi'_{\lambda}(u_n, v_n), \left(\frac{f(u_n)}{f'(u_n)}, \frac{f(v_n)}{f'(v_n)}\right) \rangle \\
&= \int_{\mathbb{R}^N} (1 + \frac{2f^2(u_n)}{1 + 2f^2(u_n)})|\nabla u_n|^2 + A_\lambda f^2(u_n) \\
&\quad + (1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)})|\nabla v_n|^2 + B_\lambda f^2(v_n)dx \\
&\quad - \int_{\mathbb{R}^N} |f(u_n)|^p|f(v_n)|^qdx - \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta dx.
\end{align*}

Assume

\begin{align*}
\int_{\mathbb{R}^N} \frac{2f^2(u_n)}{1 + 2f^2(u_n)} |\nabla u_n|^2 + A_\lambda f^2(u_n) + (1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}) |\nabla v_n|^2 + B_\lambda f^2(v_n)dx & \to b, \\
(4.12)
\end{align*}

and

\begin{align*}
\int_{\mathbb{R}^N} |f(u_n)|^p|f(v_n)|^qdx + \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta dx & \to b. \\
(4.13)
\end{align*}

If $b = 0$, by (4.12) and Lemma 2.2, we have $(u_n, v_n) \to (0, 0)$ in $X$, then $\Phi_{\lambda}(u_n, v_n) \to 0$, which contradicts $c_{\lambda} > 0$. Thus, $b \neq 0$. If $\int_{\mathbb{R}^N} |f(u_n)|^p|f(v_n)|^qdx \to 0$, it follows from (4.13) that

\begin{align*}
\int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta dx & \to b. \\
(4.14)
\end{align*}

By the definition of $S_{\alpha/2, \beta/2}$, we have

\begin{align*}
S_{\alpha/2, \beta/2} & \leq \frac{\int_{\mathbb{R}^N} |\nabla f^2(u_n)|^2 + |\nabla f^2(v_n)|^2dx}{(\int_{\mathbb{R}^N} |f^2(u_n)|^\frac{2}{2} |f^2(v_n)|^\frac{2}{2} dx)^\frac{2}{2}} \\
& \leq \frac{\int_{\mathbb{R}^N} (1 + \frac{2f^2(u_n)}{1 + 2f^2(u_n)}) |\nabla u_n|^2 + A_\lambda f^2(u_n)}{(\int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta dx)^\frac{2}{2}} \\
&\quad + \frac{\int_{\mathbb{R}^N} (1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}) |\nabla v_n|^2 + B_\lambda f^2(v_n)dx}{(\int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta dx)^\frac{2}{2}}.
\end{align*}
Using (4.12) and (4.14), we have $S_{\alpha/2,\beta/2} \leq b^{\frac{\alpha}{\beta}}$. It follows that

$$\frac{1}{2N}(S_{\alpha/2,\beta/2})^{\frac{2}{N}} > c_{\lambda} = \lim_{n \to \infty} \Phi_{\lambda}(u_n, v_n)$$

$$= \lim_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + A_{\lambda} f^2(u_n) + |\nabla v_n|^2 + B_{\lambda} f^2(v_n)) dx$$

$$- \frac{1}{22^*} \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta dx$$

$$\geq \lim_{n \to \infty} \frac{1}{4} \int_{\mathbb{R}^N} ((1 + \frac{2 f^2(u_n)}{1 + 2 f^2(u_n)}) |\nabla u_n|^2 + A_{\lambda} f^2(u_n) + (1 + \frac{2 f^2(v_n)}{1 + 2 f^2(v_n)}) |\nabla v_n|^2$$

$$+ B_{\lambda} f^2(v_n)) dx - \frac{1}{22^*} \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta dx$$

$$= (\frac{1}{4} - \frac{1}{22^*}) b \geq \frac{1}{2N}(S_{\alpha/2,\beta/2})^{\frac{2}{N}},$$

which is a contradiction.

Therefore, $\int_{\mathbb{R}^N} |f(u_n)|^p |f(v_n)|^q dx \to b_1 > 0$, as $n \to \infty$. By Lions lemma, there exists $(y_n) \subset \mathbb{R}^N, \rho > 0, \tau_1 > 0, \tau_2 > 0$ such that

$$\limsup_{n \to \infty} \int_{B_{\rho}(y_n)} |u_n|^2 dx \geq \tau_1, \quad \limsup_{n \to \infty} \int_{B_{\rho}(y_n)} |v_n|^2 dx \geq \tau_2. \quad (4.15)$$

In fact, for $(u_n)$, there exists $(y^1_n) \subset \mathbb{R}^N, \rho_1 > 0, \tau_1 > 0$ such that

$$\limsup_{n \to \infty} \int_{B_{\rho_1}(y^1_n)} |u_n|^2 dx \geq \tau_1.$$

For $(v_n)$, there exists $(y^2_n) \subset \mathbb{R}^N, \rho_2 > 0, \tau_2 > 0$ such that

$$\limsup_{n \to \infty} \int_{B_{\rho_2}(y^2_n)} |v_n|^2 dx \geq \tau_2.$$

Let $(y_n) = (y^1_n) \cup (y^2_n), \rho = \max \{\rho_1, \rho_2\}$, then we have (4.15).

If $(u, v) = (0, 0)$, we have the following two Claims.

Claim1: $\Phi_{\lambda,\infty}(u_n, v_n) \to c_{\lambda}, \quad \Phi_{\lambda,\infty}'(u_n, v_n) \to 0$.

In fact, for $R > 0$,

$$\Phi_{\lambda,\infty}(u_n, v_n) - \Phi_{\lambda}(u_n, v_n) = \frac{1}{2} \int_{\mathbb{R}^N} (A_{\lambda,\infty} - A_{\lambda}) f^2(u_n) + (B_{\lambda,\infty} - B_{\lambda}) f^2(v_n) dx.$$

For any $(\phi, \psi) \in X$, we have

$$\langle \Phi_{\lambda,\infty}'(u_n, v_n), (\phi, \psi) \rangle$$

$$= \int_{\mathbb{R}^N} (A_{\lambda,\infty} - A_{\lambda}) f(u_n)f'(u_n)\phi + (B_{\lambda,\infty} - B_{\lambda}) f(v_n)f'(v_n)\psi dx,$$

thus

$$||\Phi_{\lambda,\infty}'(u_n, v_n) - \Phi_{\lambda}'(u_n, v_n)||$$

$$= \sup_{||\phi,\psi|| \leq 1} \langle \Phi_{\lambda,\infty}'(u_n, v_n) - \Phi_{\lambda}'(u_n, v_n), (\phi, \psi) \rangle$$

$$\leq C \int_{\mathbb{R}^N} |A_{\lambda,\infty} - A_{\lambda}| f^2(u_n) + |B_{\lambda,\infty} - B_{\lambda}| f^2(v_n) dx \frac{2}{N},$$
When \( R \) large enough, by (A2) and \( u_n \to 0, v_n \to 0 \) in \( H^1 \), we have Claim 1.

**Claim 2:** Let \( \tilde{u}_n(x) = u_n(x-y_n), \tilde{v}_n(x) = v_n(x-y_n) \), then there exists \((\tilde{u}, \tilde{v}) \in X\) such that \((\tilde{u}, \tilde{v}) \) is a critical point of \( \Phi_{\lambda_\infty} \) and \( (\tilde{u}, \tilde{v}) \neq (0,0)\).

By Claim 1, it is easy to prove that
\[
\Phi_{\lambda_\infty}(\tilde{u}_n, \tilde{v}_n) \to c_\lambda, \quad \Phi'_{\lambda_\infty}(\tilde{u}_n, \tilde{v}_n) \to 0.
\]
Moreover, \( \|(\tilde{u}_n, \tilde{v}_n)\| \leq C \), then there exist \((\tilde{u}, \tilde{v}) \in X\) such that \( \tilde{u}_n \to \tilde{u}, \tilde{v}_n \to \tilde{v} \) in \( H^1 \) and \( \tilde{u}_n \to \tilde{u}, \tilde{v}_n \to \tilde{v} \) a.e in \( \mathbb{R}^N \). By (4.15), we get \((\tilde{u}, \tilde{v}) \neq (0,0)\). Using a standard argument, we have \( \Phi'_{\lambda_\infty}(\tilde{u}, \tilde{v}) = 0 \) and \((\tilde{u}, \tilde{v}) \in N_{\lambda_\infty}\).

From Claim 2, we get
\[
c_\lambda = \lim_{n \to \infty} \left[ \Phi_{\lambda_\infty}(\tilde{u}_n, \tilde{v}_n) - \frac{1}{2} \Phi'_{\lambda_\infty}(\tilde{u}_n, \tilde{v}_n), (\tilde{u}_n, \tilde{v}_n) \right] \\
\geq \int_{\mathbb{R}^N} \left[ \frac{1}{2} A_{\lambda_\infty}(f^2(\tilde{u}) - f(\tilde{u})'f(\tilde{v})\tilde{u}) + \frac{1}{2} B_{\lambda_\infty}(f^2(\tilde{v}) - f(\tilde{v})'f(\tilde{v})\tilde{v}) \right] dx \\
+ \int_{\mathbb{R}^N} \frac{p}{2(p+q)} |f(\tilde{u})|^p f(\tilde{u})'f(\tilde{v})|f(\tilde{v})|^q\tilde{u} - \frac{1}{2(p+q)} |f(\tilde{u})|^p |f(\tilde{v})|^q dx \\
+ \int_{\mathbb{R}^N} \frac{q}{2(p+q)} |f(\tilde{u})|^p |f(\tilde{v})|^q f(\tilde{v})' f(\tilde{v})\tilde{v} - \frac{1}{2(p+q)} |f(\tilde{u})|^p |f(\tilde{v})|^q dx \\
+ \int_{\mathbb{R}^N} \frac{\alpha}{2(\alpha + \beta)} |f(\tilde{u})|^\alpha f(\tilde{v})' f(\tilde{v})\tilde{v} - \frac{1}{2(\alpha + \beta)} |f(\tilde{u})|^\alpha |f(\tilde{v})|^\beta dx \\
+ \int_{\mathbb{R}^N} \frac{\beta}{2(\alpha + \beta)} |f(\tilde{u})|^\alpha |f(\tilde{v})|^\beta f(\tilde{v})' f(\tilde{v})\tilde{v} - \frac{1}{2(\alpha + \beta)} |f(\tilde{u})|^\alpha |f(\tilde{v})|^\beta dx \\
= \Phi_{\lambda_\infty}(\tilde{u}, \tilde{v}) - \frac{1}{2} \Phi'_{\lambda_\infty}(\tilde{u}, \tilde{v}), (\tilde{u}, \tilde{v}) \\
= \Phi_{\lambda_\infty}(\tilde{u}, \tilde{v}).
\]
By Lemma 4.1, we know that \( \Phi_{\lambda_\infty} \) has least energy solution \((u_1, v_1)\) and \( u_1, v_1 \) are positive, then \( c_\lambda \geq \Phi_{\lambda_\infty}(\tilde{u}, \tilde{v}) \geq \Phi_{\lambda_\infty}(u_1, v_1) \). Let \( T \) is large enough, such that \( \Phi_{\lambda_\infty}(T(u_1, v_1)) < 0 \). Define \( \gamma(t) = tT(u_1, v_1), t \in [0,1] \), then we have \( \gamma(t) \in C([0,1], X) \) and \( \gamma(0) = 0 \) and \( \Phi_{\lambda_\infty}(\gamma(1)) = \Phi_{\lambda_\infty}(T(u_1, v_1)) < 0 \), and max \( \Phi_{\lambda_\infty}(\gamma(t)) = \Phi_{\lambda_\infty}(u_1, v_1) \). By Lemma 3.4, there exists \( t_0 \in (0,1) \) such that
\[
c_\lambda \leq \max_{t \in [0,1]} \Phi_{\lambda_\infty}(\gamma(t)) = \Phi_{\lambda_\infty}(\gamma(t_0)).
\]
By (A2),
\[
\Phi_{\lambda}(\gamma(t_0)) < \Phi_{\lambda_\infty}(\gamma(t_0)) \leq \max_{t \in [0,1]} \Phi_{\lambda_\infty}(\gamma(t)) = \Phi_{\lambda_\infty}(u_1, v_1) \leq c_\lambda,
\]
which is a contradiction. Thus \((u, v) \neq (0,0)\), which implies \((u, v) \in N_\lambda\).

At last, we prove \( \Phi_{\lambda}(u, v) = c_\lambda \). If \( \Phi_{\lambda}(u, v) > c_\lambda \), then we have
\[
c_\lambda = \lim_{n \to \infty} \left[ \Phi_{\lambda}(u_n, v_n) - \frac{1}{2} \Phi'_{\lambda}(u_n, v_n), (u_n, v_n) \right] \\
= \lim_{n \to \infty} \int_{\mathbb{R}^N} \left[ \frac{1}{2} A_{\lambda}(f^2(u_n) - f(u_n) f'(u_n) u_n) + \frac{1}{2} B_{\lambda}(f^2(v_n) - f(v_n) f'(v_n) v_n) \right] dx \\
+ \int_{\mathbb{R}^N} \frac{p}{2(p+q)} |f(u_n)|^p f(u_n) f'(u_n) |f(v_n)|^q u_n - \frac{1}{2(p+q)} |f(u_n)|^p |f(v_n)|^q dx \\
+ \int_{\mathbb{R}^N} \frac{q}{2(p+q)} |f(u_n)|^p |f(v_n)|^q f(v_n) f'(v_n) v_n - \frac{1}{2(p+q)} |f(u_n)|^p |f(v_n)|^q dx
\]
loss of generality, we can assume

\begin{align*}
\text{Proof.} & \quad \text{Let } \Phi \text{ be a constant, } (|u|, |v|) \in N_{\lambda}, \text{ and } \Phi_{\lambda}(|u|, |v|) = c_{\lambda}, \text{ according to the proof of Theorem 4.3 of } [28], \text{ we can prove that } \Phi_{\lambda}^\prime(|u|, |v|) = 0, \text{ without loss of generality, we can assume } u \geq 0 \text{ and } v \geq 0, \text{ by using strong maximum principle, we get } u > 0 \text{ and } v > 0 \text{ in } \mathbb{R}^N. \\
\text{Lemma 4.3. Let } M > 0 \text{ be a constant, } \lambda_n \to \infty \text{ as } n \to \infty. \text{ If } (u_n, v_n) \in N_{\lambda_n}, \text{ and } u_n \to 0 \text{ or } v_n \to 0 \text{ in } H^1(\mathbb{R}^N), \text{ then for } 4 < p + q < 2 \cdot 2^*, \text{ we have}
\end{align*}
Using Hölder inequality and (2.7), there exists $0 < \theta < 1$ such that
\[
\left(\int_{|x| \geq R_0} |f(v_n)|^{p+q} dx\right)^{\frac{1}{p+q}} \leq \left(\int_{|x| \geq R_0} |f(v_n)|^2 dx\right)^{\frac{\theta}{2}} \left(\int_{|x| \geq R_0} |f(v_n)|^{2\gamma} dx\right)^{\frac{1-\theta}{2}} 
\leq C\left(\int_{|x| \geq R_0} |f(v_n)|^2 dx\right)^{\frac{\theta}{2}}.
\]
Thus,
\[
\int_{\mathbb{R}^N} |f(v_n)|^{p+q} dx = \int_{|x| \leq R_0} |f(v_n)|^{p+q} dx + \int_{|x| \geq R_0} |f(v_n)|^{p+q} dx
\leq \int_{|x| \leq R_0} |f(v_n)|^{p+q} dx + C\left(\frac{1}{\lambda_n}\right)^{\frac{q(p+q)}{2}}.
\]
Without loss of generality, we assume $v_n \to 0$ in $H^1(\mathbb{R}^N)$. From the above inequality, we have $f(v_n) \to 0$ in $L^{p+q}(\mathbb{R}^N)$ as $n \to \infty$. Thus, for $4 < p + q < 2 \cdot 2^*$, we get
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |f(u_n)|^p |f(v_n)|^q dx = 0.
\]

5. **Discussion of the “limit problem”**. In this section, we discuss the existence of ground state for the “limit problem” and make some comparison among the least energy.

Consider the following quasilinear Schrödinger equation in a bounded smooth domain $\Omega \subset \mathbb{R}^N (N \geq 3)$:
\[
\begin{aligned}
-\Delta w + w - (\Delta |w|^2)w &= \frac{p}{p+q} |w|^{p-2}w|z|^q + \frac{\alpha}{\alpha + \beta} |w|^{\alpha-2} w|z|^\beta \quad \text{in } \Omega \\
-\Delta z + z - (\Delta |z|^2)z &= \frac{q}{p+q} |w|^{p}z^{q-2}z + \frac{\beta}{\alpha + \beta} |w|^{\alpha}z^{\beta-2}z \quad \text{in } \Omega \\
w &= 0, \quad z = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Let $Z := H^1_0(\Omega) \times H^1_0(\Omega)$ with norm $\| (u, v) \|_Z^2 = \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla v|^2 dx$. We do the same change of variables and define the functional $\Phi_\Omega$ on $Z$ by:
\[
\Phi_\Omega(u, v) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + f^2(u) + |\nabla v|^2 + f^2(v)) dx - \frac{1}{p+q} \int_{\Omega} |f(u)|^p |f(v)|^q dx
\]
\[
- \frac{1}{\alpha + \beta} \int_{\Omega} |f(u)|^\alpha |f(v)|^\beta dx,
\]
then $\Phi_\Omega(u, v)$ is well defined on $Z$ and is of class $C^1$.

We define the Nehari manifold $N_\Omega$ by
\[
N_\Omega := \{(u, v) \in Z \setminus \{(0, 0)\} : \langle \Phi_\Omega(u, v), (u, v) \rangle = 0\}.
\]

Let $c_{\Omega} = \inf_{N_\Omega} \Phi_\Omega(u, v)$, $c_{\Omega}^* = \inf_{(u, v) \in N_\Omega} \sup_{t \geq 0} \Phi_\Omega(t(u, v))$, $c_{\Omega}^{**} = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} \Phi_\Omega(\gamma(t))$, where
\[
\Gamma = \{ \gamma(t) \in C([0, 1], Z) : \gamma(0) = 0, \Phi_\Omega(\gamma(1)) < 0 \}
\]
We say that $(f(u), f(v))$ is a least energy solution of (5.16) if $(u, v)$ is a critical point of $\Phi_\Omega$ such that $c_{\Omega}$ is achieved. By using the similar arguments used in the proof of those Lemmas in Section 2, we have the following Lemmas.
Lemma 5.1. There exist $C_1 > 0, C_2 > 0$ such that for any $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, it holds
\[ C_1 \|(u, v)\|_2^2 \leq \int_{\Omega} |\nabla u|^2 dx + f^2(u) + |\nabla v|^2 dx + f^2(v) dx \leq C_2 \|(u, v)\|_2^2. \]

Lemma 5.2. $c_\Omega = c_\Omega^* = c_\Omega^{**} < \frac{1}{2N} (S_{a/2, \beta/2})^{\frac{\gamma}{\alpha}}$.

Lemma 5.3. Problem (5.16) has a ground solution.

Proof. Similar to Lemma 4.2, there exits a sequence \{$(u_n, v_n)$\} which is a $(PS)_{c_\Omega}$ sequence of $\Phi_\Omega$ (up to a subsequence) such that
\[
\begin{align*}
&u_n \to u \quad \text{in} \quad H_0^1(\Omega), \quad v_n \to v \quad \text{in} \quad H_0^1(\Omega), \\
&u_n \to u \quad \text{a.e. in} \quad \Omega, \quad v_n \to v \quad \text{a.e. in} \quad \Omega, \\
&f(u_n) \to f(u) \quad \text{in} \quad L^q, \quad 2 \leq q < 2 \cdot 2^*, \\
&f(v_n) \to f(v) \quad \text{in} \quad L^q, \quad 2 \leq q < 2 \cdot 2^*, \\
&f(u_n) \to f(u) \quad \text{in} \quad L^q, \quad q = 2 \cdot 2^*, \\
&f(v_n) \to f(v) \quad \text{in} \quad L^q, \quad q = 2 \cdot 2^*.
\end{align*}
\]

A standard argument shows that $\Phi'_\lambda(u, v) = 0$.

Now we prove $(u, v) \neq (0, 0)$.

If not, since \{$(u_n, v_n)$\} is a $(PS)_{c_\Omega}$ sequence of $\Phi_\Omega$, we have
\[
o(1) = \langle \Phi'_\lambda(u_n, v_n), (f(u_n), f(v_n)) \rangle = \int_{\Omega} (1 + 2f^2(u_n)) |\nabla u_n|^2 + f^2(u_n) + (1 + 2f^2(v_n)) |\nabla v_n|^2 + f^2(v_n) dx - \int_{\Omega} |f(u_n)|^\alpha |f(v_n)|^\beta.
\]
Assume
\[
\int_{\Omega} (1 + 2f^2(u_n)) |\nabla u_n|^2 + f^2(u_n) + (1 + 2f^2(v_n)) |\nabla v_n|^2 + f^2(v_n) dx \to b,
\]
and
\[
\int_{\Omega} |f(u_n)|^\alpha |f(v_n)|^\beta dx + \int_{\Omega} |f(u_n)|^\alpha |f(v_n)|^\beta dx \to b.
\]
If $b = 0$, by (5.18) and Lemma 5.1, we have $(u_n, v_n) \to 0$ in $Z$, then $\Phi_\Omega(u_n, v_n) \to 0$, which contradicts $c_\Omega > 0$.

If $b \neq 0$, from (5.19), we get
\[
\int_{\Omega} |f(u_n)|^\alpha |f(v_n)|^\beta dx \to b.
\]

By the definition of $S_{a/2, \beta/2}$,
\[
S_{a/2, \beta/2} \leq \frac{\int_{\Omega} |\nabla f^2(u_n)|^2 + |\nabla f^2(v_n)|^2 dx}{(\int_{\Omega} f^2(u_n))^2} \leq \frac{\int_{\Omega} (1 + 2f^2(u_n)) |\nabla u_n|^2 + f^2(u_n) + (1 + 2f^2(v_n)) |\nabla v_n|^2 + f^2(v_n) dx}{(\int_{\Omega} f(u_n))^2}.
\]
Using (5.18) and (5.20), we have $S_{\alpha/2,\beta/2} \leq b^{\frac{\alpha}{2}}$. It follows that

$$
\frac{1}{2N}(S_{\alpha/2,\beta/2})^{\frac{\alpha}{2}} > c_{\Omega} = \lim_{n \to \infty} \Phi_{\Omega}(u_n, v_n)
$$

$$
= \lim_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + f'(u_n))|\nabla v_n|^2 + f'(v_n)) dx - \frac{1}{22} \int_{\Omega} |f(u_n)|^\alpha |f(v_n)|^\beta dx
$$

$$
\geq \lim_{n \to \infty} \frac{1}{4} \int_{\Omega} \left((1 + \frac{2f^2(u_n)}{1 + 2f^2(u_n)})|\nabla u_n|^2 + f^2(u_n)ight)
$$

$$
+ (1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)})|\nabla v_n|^2 + f^2(v_n)) dx - \frac{1}{22} \int_{\Omega} |f(u_n)|^\alpha |f(v_n)|^\beta dx
$$

$$
= \left(\frac{1}{4} - \frac{1}{22}\right)b \geq \frac{1}{2N}(S_{\alpha/2,\beta/2})^{\frac{\alpha}{2}},
$$

which is a contradiction. We can also prove that $\Phi_{\Omega}(u, v) = c_{\Omega}$ as the way of Lemma 4.2.

**Lemma 5.4.** $c_{\lambda} \leq c_{\Omega}$.

**Proof.** **Claim:** For any $(u, v) \in N_{\Omega}$, we have $(u, v) \in N_{\lambda}$ and $\Phi_{\Omega}(u, v) = \Phi_{\lambda}(u, v)$.

Since $(u, v) \in N_{\Omega}$, we have

$$
\int_{\Omega} |\nabla u|^2 + f(u) f'(u) u + |\nabla v|^2 + f(v) f'(v) v dx
$$

$$
= \frac{p}{p+q} \int_{\Omega} |f(u)|^p f'(u) f(v) |f(v)|^q dx + \frac{q}{p+q} \int_{\Omega} |f(u)|^p f(v) |f'(v)|^q dx
$$

$$
+ \frac{\alpha}{\alpha+\beta} \int_{\Omega} |f(u)|^\alpha f'(u) |f(v)|^\beta dx + \frac{\beta}{\alpha+\beta} \int_{\Omega} |f(u)|^\alpha f(v) |f'(v)|^\beta dx.
$$

By $A_{\lambda} = B_{\lambda} = 1$ in $\Omega$ and $u = v = 0$ in $\mathbb{R}^N \setminus \Omega$, the above equality can be written as

$$
\int_{\mathbb{R}^N} |\nabla u|^2 + A_{\lambda} f(u) f'(u) u + |\nabla v|^2 + B_{\lambda} f(v) f'(v) v dx
$$

$$
= \frac{p}{p+q} \int_{\mathbb{R}^N} |f(u)|^p f'(u) f(v) |f(v)|^q dx + \frac{q}{p+q} \int_{\mathbb{R}^N} |f(u)|^p f(v) |f'(v)|^q dx
$$

$$
+ \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} |f(u)|^\alpha f'(u) |f(v)|^\beta dx + \frac{\beta}{\alpha+\beta} \int_{\mathbb{R}^N} |f(u)|^\alpha f(v) |f'(v)|^\beta dx.
$$

thus $(u, v) \in N_{\lambda}$. On the other hand,

$$
\Phi_{\Omega}(u, v)
$$

$$
= \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + f^2(u) + |\nabla v|^2 + f^2(v)) dx - \frac{1}{p+q} \int_{\Omega} |f(u)|^p f(v)|^q dx
$$

$$
- \frac{1}{\alpha+\beta} \int_{\Omega} |f(u)|^\alpha f(v)|^\beta dx
$$

$$
= \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + A_{\lambda} f^2(u) + |\nabla v|^2 + B_{\lambda} f^2(v)) dx - \frac{1}{p+q} \int_{\Omega} |f(u)|^p f(v)|^q dx
$$

$$
- \frac{1}{\alpha+\beta} \int_{\Omega} |f(u)|^\alpha f(v)|^\beta dx
$$
By Lemma 4.2, suppose that 

\[ \Phi \lambda (u, v). \]

Therefore \( c_\lambda \leq c_\Omega. \)

6. **Proof of the main Theorem.** In this section, we give a proof of our main results, that is, the least energy solution of (1.2) converging to the least energy solution of (1.5).

**Proof of Theorem 1.2.** By Lemma 4.2, suppose that \( \{ (u_n, v_n) \} \subset N_\lambda \) is a sequence such that \( \Phi_\lambda (u_n, v_n) = c_\lambda, \Phi_\lambda' (u_n, v_n) = 0 \), by Lemma 3.1 and Lemma 5.4, there exists \( C > 0 \), such that \( \| (u_n, v_n) \| \leq C \). Thus, there exists \( (u, v) \in X \), such that \( u_n \rightarrow u, \ v_n \rightarrow v \) in \( H^1 (\mathbb{R}^N) \), \( f(u_n) \rightarrow f(u), \ f(v_n) \rightarrow f(v) \) in \( L^q_{\text{loc}} (\mathbb{R}^N) \), for \( 2 \leq q < 2 \cdot 2^* \), \( u_n \rightarrow u, \ v_n \rightarrow v \) a.e. in \( \mathbb{R}^N \).

We claim that \( u|_{\Omega^c} = 0 \) and \( v|_{\Omega^c} = 0 \), where \( \Omega^c = \{ x \in \mathbb{R}^N \setminus \Omega \} \) and \( \Delta^c = \{ x \in \mathbb{R}^N \setminus \Omega_b \} \). Indeed, it is sufficient to prove \( f(u)|_{\Omega^c} = 0 \) and \( f(v)|_{\Omega^c} = 0 \). If not, we have \( f(u)|_{\Omega^c} \neq 0 \) or \( f(v)|_{\Omega^c} \neq 0 \). Without loss of generality, we assume \( f(u)|_{\Omega^c} \neq 0 \). Then there exists a compact subset \( F \subset \Omega^c \) with \( \text{dist} \{ F, \partial \Omega_b \} > 0 \) such that \( f(u)|_F \neq 0 \) and

\[ \int_F f^2(u_n) dx \rightarrow \int_F f^2(u) > 0. \]

Moreover, there exists \( \epsilon_0 > 0 \) such that \( a(x) \geq \epsilon_0 \) for any \( x \in F \).

By the choice of \( \{ (u_n, v_n) \} \), we have

\[
0 = \langle \Phi_\lambda' (u_n, v_n), \ (f(u_n), f(v_n)) \rangle \\
= \int_{\mathbb{R}^N} (1 + \frac{2f^2(u_n)}{1 + 2f^2(u_n)}) |\nabla u_n|^2 + A_\lambda f^2(u_n) \\
+ (1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}) |\nabla v_n|^2 + B_\lambda f^2(v_n) dx \\
- \int_{\mathbb{R}^N} |f(u_n)|^p |f(v_n)|^q dx - \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta,
\]

hence,

\[
\Phi_\lambda (u_n, v_n) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + A_\lambda f^2(u_n) + |\nabla v_n|^2 + B_\lambda f^2(v_n) dx \\
- \frac{1}{p + q} \int_{\mathbb{R}^N} |f(u_n)|^p |f(v_n)|^q dx - \frac{1}{\alpha + \beta} \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta dx \\
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + A_\lambda f^2(u_n) + |\nabla v_n|^2 + B_\lambda f^2(v_n) dx \\
- \frac{1}{p + q} \int_{\mathbb{R}^N} |f(u_n)|^p |f(v_n)|^q dx + \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta dx \\
\geq \left( \frac{1}{2} - \frac{2}{p + q} \right) \int_{\mathbb{R}^N} A_\lambda f^2(u_n) dx \\
\geq \left( \frac{1}{2} - \frac{2}{p + q} \right) \int_F (\lambda_n \epsilon_0 + 1) f^2(u_n) dx.
\]
\[ \rightarrow +\infty \text{ as } n \rightarrow \infty. \]

This contradiction shows that \( f(u)|_{\Omega_n} = 0 \) and so does \( u \) and \( u \in H^1_0(\Omega_n) \). Similarly, \( v \in H^1_0(\Omega_n) \).

For any \( \phi \in C^\infty_c(\Omega_n) \), we have

\[ 0 = \langle \Phi_{\lambda_n}'(u_n, v_n), (\phi, 0) \rangle \]

\[ = \int_{\mathbb{R}^N} \nabla u_n \nabla \phi dx + A_{\lambda_n}(x)f(u_n)f'(u_n)\phi dx \]

\[ - \frac{p}{p + q} \int_{\mathbb{R}^N} |f(u_n)|^{p - 2} f(u_n)f'(u_n)|f(v_n)|^q \phi dx \]

\[ - \frac{\alpha}{\alpha + \beta} \int_{\mathbb{R}^N} |f(u_n)|^{\alpha - 2} f(u_n)f'(u_n)|f(v_n)|^\beta \phi dx, \]

\[ = \int_{\Omega_n} \nabla u_n \nabla \phi dx + f(u_n)f'(u_n)\phi dx \]

\[ - \frac{p}{p + q} \int_{\Omega_n} |f(u_n)|^{p - 2} f(u_n)f'(u_n)|f(v_n)|^q \phi dx \]

\[ - \frac{\alpha}{\alpha + \beta} \int_{\Omega_n} |f(u_n)|^{\alpha - 2} f(u_n)f'(u_n)|f(v_n)|^\beta \phi dx. \]

Let \( n \rightarrow \infty \), then we have

\[ \int_{\Omega_n} \nabla u \nabla \phi dx + f(u)f'(u)\phi dx - \frac{p}{p + q} \int_{\Omega_n} |f(u)|^{p - 2} f(u)f'(u)|f(v)|^q \phi dx \]

\[ - \frac{\alpha}{\alpha + \beta} \int_{\Omega_n} |f(u)|^{\alpha - 2} f(u)f'(u)|f(v)|^\beta \phi dx = 0. \]

Noting that \( v \in H^1_0(\Omega_n) \), we have

\[ \int_{\Omega_n} \nabla u \nabla \phi dx + f(u)f'(u)\phi dx = 0, \]

then we have

\[ -\Delta u + f(u)f'(u) = 0 \quad \text{in} \quad \Omega_n. \]

As \( u \in H^1_0(\Omega_n) \) and \( u \) is nonnegative, by maximum theorem, we have \( u \equiv 0 \) in \( H^1_0(\Omega_n) \). Thus, \( u \in H^1_0(\Omega) \). Similarly, \( v \in H^1_0(\Omega) \).

Now we show \( (u, v) \neq (0, 0) \). If not, \( u_n \rightarrow 0, v_n \rightarrow 0 \) in \( H^1(\mathbb{R}^N) \), by the choice of \( \{(u_n, v_n)\} \), we have

\[ 0 = \langle \Phi_{\lambda_n}'(u_n, v_n), (\frac{f(u_n)}{f'(u_n)}, \frac{f(v_n)}{f'(v_n)}) \rangle \]

\[ = \int_{\mathbb{R}^N} (1 + \frac{2f^2(u_n)}{1 + 2f^2(u_n)})|\nabla u_n|^2 + A_{\lambda_n}f^2(u_n) \]

\[ + \frac{2f^2(u_n)}{1 + 2f^2(v_n)}|\nabla v_n|^2 + B_{\lambda_n}f^2(v_n)dx \]

\[ - \int_{\mathbb{R}^N} |f(u_n)|^p|f(v_n)|^q dx - \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta. \]
Assume
\[
\int_{\mathbb{R}^N} \left( 1 + \frac{2f^2(u_n)}{1 + 2f^2(v_n)} \right) |\nabla u_n|^2 + \lambda_n f^2(u_n) \\
+ \left( 1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)} \right) |\nabla v_n|^2 + B \lambda_n f^2(v_n) \, dx \to b, \quad (6.21)
\]
and
\[
\int_{\mathbb{R}^N} |f(u_n)|^p |f(v_n)|^q \, dx + \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta \to b, \quad (6.22)
\]
In (6.22), by Lemma 4.3, we have
\[
\int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta \to b. \quad (6.23)
\]
If \( b \neq 0 \), by the definition of \( S_{\alpha/2, \beta/2} \), we have
\[
S_{\alpha/2, \beta/2} \leq \frac{\int_{\mathbb{R}^N} |\nabla f^2(u_n)|^2 + |\nabla f^2(v_n)|^2 \, dx}{\left( \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta \, dx \right)^{\frac{1}{N}}}
\]
\[
\leq \frac{\int_{\mathbb{R}^N} \left( 1 + \frac{2f^2(u_n)}{1 + 2f^2(v_n)} \right) |\nabla u_n|^2 + \lambda_n f^2(u_n) \, dx}{\left( \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta \, dx \right)^{\frac{1}{N}}}
\]
\[
+ \frac{\int_{\mathbb{R}^N} \left( 1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)} \right) |\nabla v_n|^2 + B \lambda_n f^2(v_n) \, dx}{\left( \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta \, dx \right)^{\frac{1}{N}}}.
\]
Using (6.21) and (6.23), we have \( S_{\alpha/2, \beta/2} \leq b^{\frac{1}{N}} \). It follows that
\[
\frac{1}{2N} (S_{\alpha/2, \beta/2})^{\frac{N}{2}} \geq c_\Omega \geq \lim \inf_{n \to \infty} c_{\lambda_n} = \lim \inf_{n \to \infty} \Phi_{\lambda_n}(u_n, v_n)
\]
\[
= \lim \inf_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u_n|^2 + \lambda_n f^2(u_n) + |\nabla v_n|^2 + B \lambda_n f^2(v_n) \right) \, dx
\]
\[
- \frac{1}{2N} \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta \, dx
\]
\[
\geq \lim \inf_{n \to \infty} \frac{1}{4} \int_{\mathbb{R}^N} \left( \left( 1 + \frac{2f^2(u_n)}{1 + 2f^2(v_n)} \right) |\nabla u_n|^2 + \lambda_n f^2(u_n) \right)
\]
\[
+ \left( 1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)} \right) |\nabla v_n|^2 + B \lambda_n f^2(v_n) \, dx
\]
\[
- \frac{1}{22} \int_{\mathbb{R}^N} |f(u_n)|^\alpha |f(v_n)|^\beta \, dx
\]
\[
= \left( \frac{1}{4} - \frac{1}{22} \right) b \geq \frac{1}{2N} (S_{\alpha/2, \beta/2})^{\frac{N}{2}},
\]
which is a contradiction.

If \( b = 0 \), by Lemma 2.2, we have \( \|(u_n, v_n)\| \to 0 \). Noting the choice of \((u_n, v_n)\) and Lemma 3.2, then there is a contradiction. Therefore, \((u, v) \neq (0, 0)\). By standard argument, we have \( \Phi_\Omega(u, v) = 0 \). As in the proof of the last statement of Lemma 4.2, we can show \( \Phi_\Omega(u, v) = c_\Omega \). Thus we complete the proof of the Theorem 1.2. \( \square \)

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