ON SIGNED MEASURE VALUED SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS

BRUNO RÉMILLARD AND JEAN VAILLANCOURT

Abstract. We study existence, uniqueness and mass conservation of signed measure valued solutions of a class of stochastic evolution equations with respect to the Wiener sheet, including as particular cases the stochastic versions of the regularized two-dimensional Navier-Stokes equations in vorticity form introduced by Kotelenez.

1. Introduction

Measure-valued stochastic processes arise as mathematical descriptors for the limiting behaviour of many characteristic parameters used for modelling complex evolutions in the natural sciences. Genetic drift, bacterial spread, fluid dynamics, heat conduction and chemical reactions are but a few examples of challenging problems for which a good mathematical understanding can be reached by first building an appropriate interacting particle system and then looking at it through the lens of the associated (measure valued) empirical process. When properly rescaled, the processes give rise to measure valued limits solving stochastic evolution equations which are themselves of great interest. The study of the existence and uniqueness of measure valued solutions to stochastic evolution equations really started with [Dawson (1975)] and the subject has grown rapidly since then. For a nice review of measure valued processes, see [Dawson (1993)].

A broad class of such equations, the so-called stochastic McKean-Vlasov equations, e.g., [Dawson and Vaillancourt (1995)], are probability valued if their starting point is. This conservation of the initial mass turns out to be quite easy to show in this case, by way of a basic completeness argument which remains valid for many other examples in the literature. Caution must prevail, however, against a common misconception which recurs unfortunately too often, to the effect that an interacting particle system displaying no creation or annihilation of particles at any time, necessarily gives rise in its scaling limits to stochastic evolutions obeying this mass conservation property. While seemingly sensible, such a statement requires proof.

1991 Mathematics Subject Classification. Primary 76D06 60G57 60H15, Secondary 60H05 60J60.

Key words and phrases. Signed measure; stochastic evolution equation; McKean-Vlasov; Wiener sheet.

Partial funding in support of this work was provided by the Natural Sciences and Engineering Research Council of Canada, and by the Fonds québécois de la recherche sur la nature et les technologies.

Partial funding in support of this work was provided by the Natural Sciences and Engineering Research Council of Canada. To appear in Stochastic Processes and their Applications.
For a general treatment of the important and challenging family of stochastic Navier-Stokes equations in vorticity form, such as those appearing in Kotelenez (1995), it is necessary to look for solutions in the space of signed measures. Mainly because the space of signed measures is not complete for the usual metrics compatible with the topology of weak convergence, it is much harder to study existence, uniqueness and/or mass conservation of signed measure valued solutions of stochastic evolution equations.

In fact, because of the incompleteness, many of the results presented in the literature on signed measure valued solutions are either false or have only been provided with a false proof. This is the case for example in Marchioro and Pulvirenti (1982), Kotelenez (1995) and Amirdjanova (2000), where the space of signed measures was assumed to be complete. Other examples of incorrect proofs include Kotelenez (2010), Kotelenez and Seadler (2011, 2012). These articles will be discussed later.

In this paper, we study existence and uniqueness of signed measure valued solutions of a class of stochastic evolution equations with respect to the Wiener sheet, including as particular cases the stochastic versions of the regularized two-dimensional Navier-Stokes equations in vorticity form introduced by Kotelenez. These stochastic evolution equations can be seen as weak versions of equations of the form

\[ d\chi_t = L^*\chi_t - \nabla(\Gamma\chi_t) dW, \]

where \( L^* \) is the (formal) adjoint of a diffusion operator with coefficients depending on \( \chi \) and \( W \) is the Wiener sheet. Other weak versions of these equations appear for example in Amirdjanova and Xiong (2006). Also, additional references to similar equations are given in Section 5 where we discuss the relationship with two-dimensional regularized Navier-Stokes equations in vorticity form.

The problem is described in Section 2, where appropriate spaces of measures are defined. Next, in Section 3 using a particles representation, signed measure valued solutions are shown to exists when the initial signed measure has finite support. The existence is then shown to hold for general initial conditions. Uniqueness and mass conservation are shown to hold in Section 4 using fixed points arguments and duality. Finally, in Section 5 we revisit some of the results appearing in the literature concerning two-dimensional Navier-Stokes equations in vorticity form.

A commendable attempt at resolving all these issues can be found in Kurtz and Xiong (1999), where the authors show existence of solutions to a large class of nonlinear stochastic partial differential equations encompassing our own. Their techniques also allow them to prove uniqueness of solution, but only for those starting measures with square integrable densities with respect to Lebesgue measure. Our constructions yield the existence and uniqueness of solution for all starting signed measures.

2. Description of the problem

First, one needs to define correctly the spaces of measures and signed measures that will be used in the sequel. To this end, suppose that \( \rho \) is the metric on \( \mathbb{R}^d \) defined by

\[ \rho(x, y) = \min(1, \|x - y\|), \quad x, y \in \mathbb{R}^d, \]
where \( \| \cdot \| \) is the Euclidean norm.

Suppose that \( p \geq 1 \) is given. For any \( m \geq 0 \), let \( M(m) \) be the set of (non-negative) Borel measures \( \mu \) with \( \mu(\mathbb{R}^d) = m \). This space is equipped with the Wasserstein metric \( W_p \) defined by

\[
W_p(\mu, \nu) = \left[ \inf_{Q \in \mathcal{H}(\mu, \nu)} \int_{\mathbb{R}^{2d}} p^p(x, y)Q(dx, dy) \right]^{1/p},
\]

where \( \mathcal{H}(\mu, \nu) \) is the set of all joint representations of \((\mu, \nu)\), that is, the set of all Borel measures \( Q \) on \( \mathbb{R}^d \times \mathbb{R}^d \), so that for any Borel subset \( A \) of \( \mathbb{R}^d \), \( Q(A \times \mathbb{R}^d) = m\mu(A) \) and \( Q(\mathbb{R}^d \times A) = m\nu(A) \).

Further set \( M(m_1, m_2) = M(m_1) \times M(m_2) \) and consider the following “Wasserstein” metric \( \gamma_p \) on \( M(m_1, m_2) \):

\[
\gamma_p(\mu, \eta) = \gamma_p \{ (\mu^1, \mu^2), (\eta^1, \eta^2) \} = [W_p^p(\mu^1, \eta^1) + W_p^p(\mu^2, \eta^2)]^{1/p}.
\]

It follows easily from Minkowski’s inequality together with a lemma in [Dudley 1989, p.330] that for any \( p \geq 1 \), \( \gamma_p \) is a metric on \( M(m_1, m_2) \), since \( W_p \) is a metric on \( M(m) \).

Let \( M^{(f)}(m) \) be the subset of measures in \( M(m) \) concentrated on finite sets and \( M^{(f)}(m_1, m_2) = M^{(f)}(m_1) \times M^{(f)}(m_2) \).

Notice that since \( \rho \leq 1 \), for any \( \mu, \nu \in M(m) \), one has

\[
W_p^\rho(\mu, \nu) \leq W_1(\mu, \nu) \leq (m^2)^{p-1} W_p(\mu, \nu).
\]

It follows that for any \( \mu, \nu \in M(m_1, m_2) \),

\[
\gamma_p^\rho(\mu, \nu) \leq \gamma_1(\mu, \nu) \leq (2m^2)^{p-1} \gamma_p(\mu, \nu),
\]

where \( m = \max(m_1, m_2) \). It follows that \( \gamma_p \) and \( \gamma_2 \) generate the same topology. Therefore, from now on, we use \( p = 2 \).

The following lemma summarizes results in [Huber 1981] and [Dudley 1989] [Lemma 11.8.4] about spaces of measures endowed with metrics \( W_2 \) and \( \gamma_2 \).

**Lemma 2.1.** \((M(m), W_2)\) and \((M(m_1, m_2), \gamma_2)\) are Polish spaces. Their respective topology is that of weak convergence on \( M(m) \) and \( M(m_1, m_2) \). Moreover \( M^{(f)}(m) \) is dense in \( M(m) \) and \( M^{(f)}(m_1, m_2) \) is dense in \( M(m_1, m_2) \).

Throughout the rest of the paper, let \( m_1, m_2 \) be fixed.

Set \( M = M(m_1, m_2) \) and \( M^{(f)} = M^{(f)}(m_1, m_2) \). Further let \( \mathcal{M} \) be the space of \( M \)-valued random variables with metric \( \gamma \) defined by

\[
\gamma(\chi, \eta) = \left[ E\gamma_2^2(\chi, \eta) \right]^{1/2}, \quad \chi, \eta \in \mathcal{M},
\]

and denote by \( \mathcal{M}^{(f)} \) the subset of \( M^{(f)} \)-valued random variables.
Let \( C([0, T]; M) \) denote the space of continuous mappings from \([0, T]\) into \( M \) and let \( \mathbb{M} \) be the space of \( C([0, T]; M) \)-valued random variables with metric \( \gamma_{[0,T]} \) defined by

\[
\gamma_{[0,T]}(\chi, \eta) = \left[ E \sup_{0 \leq t \leq T} \frac{\gamma_2^2(\chi_t, \eta_t)}{\delta_n} \right]^{1/2}, \quad \chi, \eta \in \mathbb{M},
\]

and let \( \mathbb{M}^{(f)} \) be the subset of \( C([0, T]; M^{(f)}) \)-valued random variables.

It follows easily that \( (\mathcal{M}, \gamma) \) is a complete metric space, since \( (M, \gamma_2) \) is complete, by Lemma 2.1. Note that \( (\mathcal{M}, \gamma_{[0,T]}) \) is also complete and endowed with that metric, is complete, while it is not.

Finally, let \( SM(m_1, m_2) \) be the set of Borel signed measures with Hahn-Jordan decomposition \( (\mu^+, \mu^-) \in M(m_1, m_2) \). A natural distance on \( SM(m_1, m_2) \) is the one inherited from \( \gamma_2 \). More precisely, if \( \mu = (\mu^1, \mu^2) \) and \( \nu = (\nu^1, \nu^2) \) are respectively the Hahn-Jordan decompositions of \( \mu, \nu \in SM(m_1, m_2) \), then the distance between \( \mu \) and \( \nu \) is \( \gamma_2(\mu, \nu) \).

**Remark 2.2.** Many authors, because of Lemma 2.1, have stated that \( SM(m_1, m_2) \) endowed with that metric, is complete, while it is not.

To see that, simply take \( \mu_n = \delta_{1/n} - \delta_0 \). Then clearly its Hahn-Jordan decomposition is \( (\delta_{1/n} - \delta_0, \delta_0) \), so \( \mu_n \in SM(1,1) \). Moreover, it is obvious that \( \mu_n \) is a Cauchy sequence in \( SM(1,1) \) since \( \gamma_2((\delta_{1/n}, \delta_0), (\delta_{1/m}, \delta_0)) \to 0 \) as \( n, m \to \infty \). However, there is no measure \( \mu \in SM(1,1) \) with Hahn-Jordan decomposition \( (\mu_1, \mu_2) \) so that \( \gamma_2((\delta_{1/n}, \delta_0), (\mu_1, \mu_2)) \to 0 \) as \( n \to \infty \). For, the latter implies that \( \mu_1 = \mu_2 = \delta_0 \), so \( \mu = \delta_0 - \delta_0 = 0 \notin SM(1,1) \).

In order to preserve the mass in the limit, Kotelenez and Seadler (2011, 2012) introduce the following metric \( \lambda \) on the space of signed measures: If \( \mu \) and \( \nu \) have Hahn-Jordan decompositions \( (\mu^+, \mu^-) \) and \( (\nu^+, \nu^-) \), then the distance between \( \mu \) and \( \nu \), denoted by \( \lambda(\mu, \nu) \), is defined by

\[
(2.1) \quad \lambda(\mu, \nu) = \inf_{\eta \in M(m), m \geq 0} \max \left[ \gamma(\mu^+ - \nu^+ - \eta) + \gamma(\mu^- - \nu^- - \eta), \right. \]

\[
\gamma(\mu^+ - \nu^+ + \eta) + \gamma(\mu^- - \nu^- + \eta),
\]

where \( \gamma(\chi) = \sup_{f: \|f\| \leq 1} | < \chi, f > | \), for any finite signed measure \( \chi \), and \( \| \cdot \|_L \) is the so-called minimal Lipschitz constant defined by (A.1).

It is shown in Kotelenez and Seadler (2012, Theorem A.6) that with respect to \( \lambda \), a Cauchy sequence \( \mu_n \) converges to \( \mu \) if and only if there exists a sub-sequence \( \mu_{n_k} \) so that the Hahn-Jordan decompositions \( \mu_{n_k}^\pm \) converges to the Hahn-Jordan decomposition \( \mu^\pm \) of \( \mu \).

**Remark 2.3.** This new metric is not as useful as it seems since, in order to show that a sequence \( \mu_n \) converges to a limit \( \mu \), one needs to prove the convergence of the Hahn-Jordan decompositions to that limit, at least for a subsequence. In Kotelenez and Seadler (2011, 2012), from the convergence of the Hahn-Jordan decompositions \( \mu_n^+ \) and \( \mu_n^- \) to measures \( \mu_1 \) and \( \mu_2 \), they conclude that the sequence is a Cauchy sequence in \( \lambda \), which is true, but they also conclude that the sequence
converges in $\lambda$, which is not necessarily true. To see that, take the same example as before, i.e., $\mu_n = \delta_{1/n} - \delta_0$. Since $\mu_n \to \delta_0$, the sequence is Cauchy with respect to $\lambda$. However it is not a convergent sequence, according to Kotelnenez and Seadler (2012, Theorem A.6).

**Remark 2.4.** Before defining the stochastic evolution equations of interest here, we state some properties which are assumed to hold throughout the rest of this paper.

Suppose that $K(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded, differentiable with respect to $x$ and Lipschitz in both variables, i.e there exists a constant $C$ such that

$$\|K(x, y) - K(x', y')\| \leq C \left(\|x - x'\|^2 + \|y - y'\|^2\right)^{1/2}$$

holds for all $x, x', y, y' \in \mathbb{R}^d$.

Moreover assume that $\Gamma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is a continuous function that satisfies

$$\sum_{j=1}^{d} \sum_{l=1}^{d} \int \left\{|\Gamma_{jl}(r, p) - \Gamma_{jl}(q, p)|\right\}^2 dp \leq C_r^2 \|r - q\|^2. \tag{2.2}$$

for some positive constant $C_r$ and $G(x, y) = \int_{\mathbb{R}^d} \Gamma(x, p)\Gamma(y, p)^\top dp$ is such that

$$\max_{1 \leq i, j \leq d} \sup_{x, y} \left\{\frac{|G_{ij}(x, y)|}{(1 + |x|)(1 + |y|)} + \max_{1 \leq k, l \leq d} |\partial_{x_k} \partial_{y_l} G_{ij}(x, y)|\right\} < \infty.$$

Given is a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ which satisfies the usual conditions. All stochastic processes are assumed to live on $\Omega$ and to be $\mathbb{F}$-adapted, including all initial conditions in SDE’s and SPDE’s. Moreover, the processes are assumed to be $P \otimes \lambda$ measurable, where $\lambda$ is the Lebesgue measure on $[0, \infty)$.

Recall that a Wiener sheet $w$ is a stochastic process defined on $\mathbb{R}^d \times [0, \infty)$, such that for any Borel set $A$ with finite Lebesgue measure $\lambda(A)$, $t \mapsto w(A, t)$ is a continuous centered Gaussian process with covariance function $E\{w(A, s)w(B, t)\} = \min(s, t)\lambda(A \cap B)$, whenever $A$ and $B$ both have finite Lebesgue measures. That implies, in particular that $w(A \cup B, t) = w(A, t) + w(B, t)$ almost surely, whenever $A$ and $B$ both have finite Lebesgue measures and are disjoint.

The following stochastic evolution equation, with respect to a vector $w = (w_1, \ldots, w_d)$ of independent Wiener sheets, will be analyzed next. To be adapted for $w_t(r, t)$ means that $\int_A w_t(dp, t)$ is adapted for any Borel set $A \subset \mathbb{R}^d$ with $\lambda(A) < \infty$. The space of twice-differentiable real-valued functions with bounded second derivative on $\mathbb{R}^d$ is henceforth denoted by $C^2(\mathbb{R}^d)$.

**Definition 2.5.** A path $\chi = \chi^1 - \chi^2$, with $(\chi^1, \chi^2) \in M$, is called a solution of the stochastic evolution equation if

$$d < \chi_t, f > = < \chi_t, L(\chi_t)f > dt + \int < \chi_t, \nabla f(\cdot)^\top \Gamma(\cdot, p) > w(dp, dt), \tag{2.3}$$

holds for all $f \in C^2(\mathbb{R}^d)$, where we write, for any $x \in \mathbb{R}^d$,

$$U(x, \chi_t) = \int K(x, q)\chi_t(dq),$$
and

\[ L(\chi_t)f(x) = \sum_{j=1}^{d} \partial_{x_j}f(x)U_j(x, \chi_t) + \frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{d} \partial_{x_j} \partial_{x_k}f(x)G_{jk}(x, x), \]

with \( <\chi, f> \) being the integral of \( f \) with respect to \( \chi \).

The existence of such a solution is studied in the next section.

**Remark 2.6.** Note that even if \( \chi_t = \chi_t^1 - \chi_t^2 \), with \( (\chi_t^1, \chi_t^2) \in \mathcal{M} \) for all \( t \), it does not necessarily imply that \( (\chi_t^1, \chi_t^2) \) is the Hahn-Jordan decomposition of \( \chi_t \). That (desirable) property is hereafter called “mass conservation”. It will be studied in Section 4. In the setting of Kotelenez (1995), this is called conservation of vorticity.

### 3. Existence of Signed Measure Valued Solutions

The key argument in proving the existence of a solution to (2.3) is the construction of an appropriate particles system.

For instance, consider the following system of SODEs that describe the movement of \( N \) interacting particles \( r^i(t), \ldots, r^N(t) \):

\[
\begin{cases}
    dr^i(t) = \sum_{j=1}^{N} a_{ij} K(r^i(t), r^j(t))dt + \int \Gamma(r^i(t), p)w(dp, dt), \\
    r^i(0) = r_0^i, \quad i = 1, 2, \ldots, N,
\end{cases}
\]

where \( a_1, \ldots, a_N \) are fixed real numbers.

Existence and uniqueness of system of particles (3.1) was stated in [Kotelenez 1993] for particular cases of \( K \) and \( \Gamma \). Under the general Lipschitz conditions stated in Remark 2.4, one has the following result.

**Lemma 3.1.** For every \( r_0 \in \mathbb{R}^{dN} \), (3.1) has a unique \( \mathbb{F}_t \)-adapted solution \( r \in C([0, \infty); \mathbb{R}^{dN}) \) a.s., which is an \( \mathbb{R}^{dN} \)-valued Markov process.

For the sake of completeness, the proof is given in Appendix B, completing some missing arguments in [Kotelenez 1993]. Note that by construction, if \( \pi \) is any permutation of \( \{1, \ldots, N\} \), the solution corresponding to \( a_{\pi} \) is \( r^{\pi} \).

**Lemma 3.2.** Suppose that \( (\nu^1, \nu^2) \in M^f \) is such that

\[
\nu^1 = \sum_{i=1}^{N} \max(a_i, 0)\delta_{r_0^i}, \quad \nu^2 = \sum_{i=1}^{N} \max(-a_i, 0)\delta_{r_0^i},
\]

for some \( r_0 \in \mathbb{R}^{dN} \) and \( a = (a_1, \ldots, a_N)^T \in \mathbb{R}^N \) with the property that

\[
\sum_{i=1}^{N} \max(a_i, 0) = m_1, \quad \sum_{i=1}^{N} \max(-a_i, 0) = m_2.
\]

Then, the mapping \( \Psi : (\nu^1, \nu^2) \in M^{(f)} \mapsto \Psi(\nu^1, \nu^2) = (\chi^1, \chi^2) \in M^{(f)} \), given for all \( t \geq 0 \) by

\[
\chi_t^1 = \sum_{i=1}^{N} \max(a_i, 0)\delta_{r^i(t)}, \quad \chi_t^2 = \sum_{i=1}^{N} \max(-a_i, 0)\delta_{r^i(t)},
\]
is well-defined, where \( r \) satisfies (3.1).

Moreover, \((\chi^1, \chi^2) \in \mathbb{M}^f\) gives rise to a solution of the stochastic evolution equation, i.e., the empirical signed measure \( \chi_t \) defined by

\[
\chi_t = \chi_t^1 - \chi_t^2 = \sum_{i=1}^N a_i \delta_{x_i(t)},
\]

satisfies (2.3).

**Proof.** First, let \( \pi \) be any permutation of \[1, \ldots, N\]. Then \((\chi^1, \chi^2) = (\nu^1, \nu^2)\) can also be written as

\[
\chi^1 = \pi \left( \max \{0, a_{\pi_i^0} \} \right) - \pi \left( \max \{-a_{\pi_i^0}, 0 \} \right),
\]

\[
\chi^2 = \pi \left( \max \{0, a_{\pi_i^0} \} \right) - \pi \left( \max \{-a_{\pi_i^0}, 0 \} \right).
\]

By Lemma 3.1 it follows that the unique solution \( q \) to

\[
\begin{align*}
\left\{ 
& dq^i(t) = \sum_{j=1}^N a_{\pi_j} K(q^j(t), q^i(t)) dt + \int \Gamma(q^j(t), p) w(dp, dt), \\
& q^i(0) = \pi_{\pi_i^0}, \quad i = 1, \ldots, N,
\end{align*}
\]

is \( q^i = r_{\pi_i} \). Hence \((\chi^1, \chi^2) = \Psi(\nu^1, \nu^2)\) is well-defined.

Let \( \chi = \chi^1 - \chi^2 \) and set \( U(x, \chi_t) = \int K(x, p) \chi_t(dp) \), \( x \in \mathbb{R}^d \). Note that for all \( i = 1, \ldots, N, \)

\[
r_i(t) = r_i^0 + \int_0^t U(r^i(s), \chi_s) ds + M^i(t),
\]

where the \( \mathbb{R}^d \)-valued martingale \( M^i \) has components

\[
M^i_j(t) = \sum_{l=1}^d \int_0^t \int \Gamma_{jl}(r^i(s), p) w_l(dp, ds)
\]

and quadratic covariation

\[
\langle M^i_j, M^i_k \rangle(t) = \sum_{l=1}^d \int_0^t \int \Gamma_{jl}(r^i(s), p) \Gamma_{kl}(r^i(s), p) dp ds,
\]

for any \( j, k = 1, \ldots, d, \)

Now, let \( f \in C^2_b(\mathbb{R}^d) \). Applying Itô’s formula to \( < \chi_t, f > = \sum_{j=1}^N a_j f(r^j(t)) \), we obtain

\[
\begin{align*}
\frac{d}{dt} < \chi_t, f > & = \sum_{i=1}^N a_i \partial_{x_i} f(r^i(t)) U_j(r^i(t), \chi_s) dt \\
& + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^d \sum_{k=1}^d a_i \partial_{x_j} \partial_{x_k} f(r^i(t)) G_{jk}(r^i(t)) dt \\
& + \sum_{i=1}^N \sum_{j=1}^d \sum_{k=1}^d a_i \partial_{x_j} f(r^i(t)) \int \Gamma_{jl}(r^i(t), p) w_l(dp, dt) \\
& = < \chi_t, \mathcal{L}(\chi) f > dt \\
& + \int < \chi_t, \nabla f(\cdot)^\top \Gamma(\cdot, p) > w(dp, dt),
\end{align*}
\]
which is exactly (2.3).

The following lemma will allow us to extend the solution of (2.3) for discrete initial conditions to one with arbitrary initial conditions in $M$.

**Lemma 3.3.** For any $T > 0$, there exist $c' = c'(T), c'' = c''(T) > 0$, independent of $x_0 = (\chi_0, \chi_0^2), \eta_0 = (\eta_0, \eta_0^2) \in M^{(f)}$, such that if $\chi = \Psi(x_0)$ and $\eta = \Psi(\eta_0)$, then

$$\gamma_{[0, T]}(\chi, \eta) \leq c' \gamma_2(x_0, \eta_0).$$

Moreover

$$E \sup_{0 \leq t \leq T} \sup_{\|\| \leq 1} \chi_t - \eta_t, f^2 \leq c'' \gamma_2(x_0, \eta_0).$$

The proof is relegated to Appendix B.

Using the previous result, it is now possible to extend the mapping $\Psi$ from $M^{(f)}$ to $M$. While the representation of the mapping $\Psi$ is explicit when $\nu \in M^{(f)}$, it is no longer the case when $\nu \in M \setminus M^{(f)}$.

**Theorem 3.4.** The map $\Psi := \chi = (\chi_0, \chi_0^2) \mapsto \chi = (\chi_1, \chi_2)$ from $M$ into $M^{(f)}$ extends uniquely to a map from $M$ into $M$. Moreover, for any $x_0, \eta_0 \in M$,

$$\gamma_{[0, T]}(\chi, \eta) \leq c' \gamma(x_0, \eta_0)$$

and $\chi = \chi_1 - \chi_2$ satisfies (2.3), that is $\chi$ is a solution of the stochastic evolution equation with initial condition $\chi_0$. The proof is relegated to Appendix B.

**Remark 3.5.** While the last theorem tells us that there is at least one solution to the weak stochastic evolution equation starting from some signed measure $\nu$, one cannot deduce yet that there is a unique solution, nor that the mass is conserved, that is, if $\chi_0 \in SM(m_1, m_2)$, then $\chi$ also belongs to $SM(m_1, m_2)$ for all $t \geq 0$. In order to prove these results, one needs to introduce another mapping.

4. **Fixed point representation and mass conservation**

The plan is the following. First, we start by defining a mapping that maintains, through time, the Hahn-Jordan decomposition of the initial signed measure. Then, we show that it has a unique fixed point and that the latter satisfies (2.3). Finally, we prove the uniqueness of the solution of (2.3), using a monotonicity condition.

**Definition 4.1.** Let $\nu = (\nu^1, \nu^2) \in M$ be given. Consider the operator $S = (S^1, S^2)$ acting on $M_\nu = \{\mu \in M; \mu_0 = \nu\}$, and defined by

$$(S\mu)_t^\tau = \nu^\tau \circ r^{-1}(t, \mu, \cdot), \quad \tau = 1, 2,$$

where for any $\mu \in M_\nu$ and any $x \in \mathbb{R}^d$, $r(t, \mu, x)$, with $\mu = \mu^1 - \mu^2$, is the unique solution of the following Itô equation:

$$\begin{align*}
\left\{ \begin{array}{l}
dr(t) = \int K(r(t), p)\mu_1(dp)dt + \int \Gamma(r(t), p)w(dp, dt), \\
r(0) = x.
\end{array} \right.
\end{align*}$$

(4.1)
Note that the measurability of the mapping $x \mapsto r(t,\mu,x)$ follows easily from the properties in Remark 2.4 ensuring that $(S\mu)^\tau_t$ is well-defined. In fact, from the proof of Lemma 4.2 below, the mapping $x \mapsto r(t,\mu,x)$ is a homeomorphism, thus it is measurable.

In other words, for all $t \geq 0$ and for any bounded and measurable $f$ on $\mathbb{R}^d$, $$< (S\mu)^\tau_t, f > = \int f(r(t,\mu,x))\nu^\tau(dx), \quad \tau = 1,2.$$ In particular, $$< (S\mu)_t, f > = \int f(r(t,\mu,x))\nu(dx).$$ The proof of existence and uniqueness of $r$ is similar to the proof of Lemma 3.1 so it is omitted.

The following lemma is essential and confirms that the Hahn-Jordan decomposition is preserved by the mapping $S$.

**Lemma 4.2.** If $\nu = (\nu^1,\nu^2)$ is the Hahn-Jordan decomposition of $\nu$, then $(S\mu)_t = ((S\mu)^1_t, (S\mu)^2_t)$ is the Hahn-Jordan decomposition of $(S\mu)^1_t - (S\mu)^2_t$.

**Proof.** The conditions in Remark 2.4 ensure that the “local characteristics” of the semimartingale $B(x,t) + M(x,t)$, with $B(x,t) = \int_0^t K(x,p)\mu_+(dp)ds$ and $M(x,t) = \int_0^t \Gamma(x,p)w(dp,ds)$ satisfy the hypotheses of Kunita (1990, Theorem 4.5.1). It follows that $x \mapsto r(t,\mu,x)$ is a homeomorphism and therefore injective. The results then follow from Proposition A.1 in Appendix A.

The next result, similar to Lemma 3.3, is needed to show that $S$ has a unique fixed point in $M_\nu$.

**Lemma 4.3.** For any $T > 0$, there exist $c' = c'(T) > 0$, independent of $\nu \in M$, such that, if both $\mu$ and $\eta$ belong to $M_\nu$, then they satisfy

$$\gamma^2_{[0,T]}(S\mu, S\eta) \leq c' \int_0^T E\gamma^2_{[0,T]}(\mu_t, \eta_t)dt \leq C'T \gamma^2_{[0,T]}(\mu, \eta). \quad (4.2)$$

The proof is given in Appendix B.

One can now prove that $S$ has a unique fixed point.

**Theorem 4.4.** Let $\nu \in M$ be given. Then the mapping $S$ has a unique fixed point $\mu \in M_\nu$ given by $\mu = \Psi(\nu)$, and for any $\eta \in M_\nu$, $S^n\eta$ converges to $\mu$ as $n \to \infty$.

**Proof.** First, using Itô’s formula, it is easy to check that $\Psi(\nu)$ is a fixed point of $S$.

It follows from Lemma 4.3 that $S$ is continuous and that for any $\mu \in M_\nu$, the sequence $\mu_n = S^n\mu$ is Cauchy. Since the space $M_\nu$ is complete and the mapping $S$ is continuous, $\mu_n$ converges to $\mu \in M_\nu$, which must be a fixed point of $S$. Hence the set of fixed points of $S$ is not empty. Next, it follows easily from Lemma 4.3 that there are no more than one fixed point. For if $\mu$ and $\eta$ are fixed points, then

$$E \sup_{0 \leq t \leq T} \gamma^2_{[0,T]}(\mu_t, \eta_t) \leq c' \int_0^T E\gamma^2_{[0,T]}(\mu_t, \eta_t)dt,$$

so using Gronwall’s inequality, one may conclude that $\mu = \eta$. \qed

Suppose that for any given $\nu \in M$ and $\mu \in M_\nu$, the signed measure valued process $\chi = \chi^1 - \chi^2$, with $(\chi^1, \chi^2) \in M_\nu$, satisfies the weak (linear) evolution
equation
\[ < \chi_t, g > = < \nu, g > + \int_0^t < \chi_s, L(\mu_s) g > ds \]
(4.3)
\[ + \int_0^t \int < \chi_s, \nabla g(\cdot)\Gamma(\cdot, p) > w(dp, ds), \]
for any nice \( g \).

First note that such a solution exists. In fact, \( \chi = \chi^1 - \chi^2 \), with \( (\chi^1, \chi^2) = S\mu \), is a solution of (4.3), by a simple application of Itô's formula applied to \( g(r(t)) \) when \( g \) is sufficiently smooth.

Note also that equation (4.3) still makes sense when \( \chi \) is a trajectory in the space of Schwartz's tempered distributions, for any test function \( g \). Let \( H_p \) denote the Sobolev space of order \( p \), as defined in [25].

The proof of the next lemma is done in Appendix B. Before stating it, let \( L^*_\mu \) be the (Schwartz distributional) adjoint operator to \( L_\mu \) in the classical functional analytic sense, i.e., for \( \chi \in \mathcal{H}_{-q}, \phi \in H_q, < \chi, L_\mu \phi > = < L^*_\mu \chi, \phi > \). See, e.g., [15].

**Lemma 4.5.** Suppose that for all \( t \in [0, T] \) and any given \( p \geq 1 \), there exists \( q \geq p \) and a positive constant \( C_p \), so that for every solution \( \chi \in H_{-p} \) of (4.3),
\[ (4.4) \]
\[ E \{ < \chi, L^*_\mu \chi >_{-q} \} + \int_{\mathbb{R}^d} \| \nabla \Gamma(\cdot, x) \chi \|^2_{-q} dx \leq C_p \| \chi \|^2_{-q}. \]

Then (4.3) has at most one solution in trajectorial sense starting from \( \nu \in H_{-p} \).

As a consequence of the previous results, one obtains the uniqueness of the solution of the stochastic evolution equation (2.3) starting from \( \nu \in SM(m_1, m_2) \).

**Theorem 4.6.** Suppose that \( \nu \in SM(m_1, m_2) \) has Hahn-Jordan decomposition \( \nu = (\nu^1, \nu^2) \in M \). Then, under the conditions stated in Remark 2.4 and Lemma 4.5, the stochastic evolution equation (2.3), with initial condition \( \nu \), has a unique solution \( \chi \) which preserves the mass, that is, for every \( t \geq 0 \), there holds \( \chi_t \in SM(m_1, m_2) \) and its Hahn-Jordan decomposition is \( \chi_t \), where \( \chi = \Psi(\nu) \).

**Proof.** For every \( \nu \in M \), Theorem 3.4 yields the existence of a solution \( \mu = \mu^1 - \mu^2 \) to (2.3) starting from \( \nu \), where \( \mu = (\mu^1, \mu^2) = \Psi(\nu) \). In order to show uniqueness, it suffices to prove that \( \mu = S\mu \) invariably ensues, that is, any solution is a fixed point of \( S \), which we already know has a unique fixed point by Theorem 4.4.

First, it is easy to check that for any \( g \) in \( C^0_0(\mathbb{R}^d) \), \( (S\mu)^1 - (S\mu)^2 \) satisfies (4.3), for any given \( \mu \).

Now take \( \mathcal{M} \) to be any solution to (2.3). To show that \( (S\mu)^1 - (S\mu)^2 = \mu_t \), notice that \( \eta = (S\mu)^1 - (S\mu)^2 - \mu_t \) satisfies (4.3) with \( \nu \equiv 0 \). Applying Lemma 4.5, one may conclude that \( P(\eta_t = 0) \) holds for every \( t \geq 0 \). Hence \( \mu_t = (S\mu)^1 - (S\mu)^2 \) for every \( t \geq 0 \) with probability one.

It follows from Lemma 4.2 that \( (S\mu)_t \) is the Hahn-Jordan decomposition of \( (S\mu)^1 - (S\mu)^2 \). Hence, by Proposition A.2, we have \( \mu_t = (S\mu)_t \) since \( \mu_t = (S\mu)^1_t - (S\mu)^2_t \). Therefore, one may conclude that the stochastic evolution equation has a unique solution \( \mu \), with Hahn-Jordan decomposition \( \mu = S\mu \). This completes the proof. \[\Box\]
We can at this point discuss some of the more recent papers of Kotelenez and
his school, as requested by one of the referees, whom we thank for drawing them to
our attention. The main statement in Kotelenez (2010, Theorem 3.3) is similar to
our Theorem 4.6 except for the uniqueness. However the proof in Kotelenez (2010,
Theorem 3.3) is incomplete since the Hahn-Jordan decomposition (corresponding
to our Lemma 4.2) and stated in Kotelenez (2010, Lemma 3.1), is proven only for
discrete measures in Kotelenez (2010, Corollary 2.6).

Next, in Kotelenez and Seadler (2011), the main statement is Theorem 3.5,
which is our Theorem 3.4, with the additional claim that the Hahn-Jordan de-
composition is preserved. However the proof given there is incorrect since they do
not prove that the sequence of signed measures is convergent with respect to the
distance $\lambda$ defined by (2.1). It is an example of the kind of error we mentioned in
Remark 2.3.

Finally, in Kotelenez and Seadler (2012), the main statement is Theorem 3.5,
which is an extension of Kotelenez (2010, Theorem 3.3), with an additional claim
about the conservation of the mass of the Hahn-Jordan decomposition. The proof of
the latter is incomplete, since it is based on their Corollary A.7, whose proof is miss-
ing, claimed to be a direct consequence of Kotelenez and Seadler (2012, Theorem
A.6). Furthermore, the proof of the Hahn-Jordan decomposition in Kotelenez and Seadler
(2012, Theorem 3.5) is based on their Lemma 3.2, whose proof is circular with their
Corollary 3.3 upon which it implicitly relies. More precisely, their Lemma 3.2 is
a statement about the dominance of the distance $\lambda$ for the mapping $S$ with
respect to the initial measure. To be proven, one absolutely needs a result about the
Hahn-Jordan decomposition of $S\mu$, which is stated as a Corollary of Lemma 3.2.

5. TWO-DIMENSIONAL VORTICITY EQUATIONS AS SPECIAL CASES

The classical Navier-Stokes equations of fluid dynamics for the two dimensional
velocity field $v(t, x) = (v_1, v_2)(t, x) \in \mathbb{R}^2$ of an incompressible viscid planar fluid
submitted to a pressure field $p(t, x) \in \mathbb{R}^2$, with prescribed initial velocity $v_0$, are
given by

$$
\frac{\partial}{\partial t} v + (v \cdot \nabla) v + \nabla p - \nu \Delta v = 0, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^2,
$$

$$(\nabla \cdot v)(t, x) = \frac{\partial}{\partial x_1} v_1 + \frac{\partial}{\partial x_2} v_2 = 0, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^2,
$$

with initial and boundary conditions

$$
v(x, 0) = v_0(x), \quad x \in \mathbb{R}^2,
$$

$$
\lim_{|x| \to \infty} v(t, x) = 0, \quad t \in \mathbb{R}^1.
$$

Here the constant $\nu \geq 0$ denotes the kinematic viscosity coefficient and we write
$x = (x_1, x_2) \in \mathbb{R}^2$. $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ is the gradient and $\Delta = \nabla \cdot \nabla$ is the Laplace
operator.

A great deal of information about the solution $v$ can be gleaned from a scalar
parameter called the vorticity (or rotation) $\omega$ of the two dimensional flow, defined as

$$
\omega := \text{rot} \ v = \frac{\partial}{\partial x_1} v_2 - \frac{\partial}{\partial x_2} v_1.
$$
Using (5.2) and treating (5.1) formally, one obtains the corresponding (pressure invariant) two-dimensional vorticity equations

\[
\frac{\partial}{\partial t} \omega + (v \cdot \nabla) \omega - \nu \Delta \omega = 0, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^2, \\
(\nabla \cdot v)(t, x) = 0, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^2.
\]

Introducing the operator \(\nabla^\perp = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})\) yields, by virtue of \(\nabla \cdot v = 0\), the classical two dimensional Biot-Savard formula

\[
v(t, x) = \int (\nabla^\perp g)(x - y)\omega(t, y)dy,
\]

with \(g(r) = h(||r||)\), with \(h(s) = \frac{1}{2s} \ln s, s > 0\).

Making sense of (5.1), (5.3) and even (5.4) requires the precise identification of the spaces of values wherein lie \(v\) and \(\omega\). There is an extensive literature on the conditions of existence and uniqueness of solution for both equations (5.1) and (5.3), when the initial data is restricted to either a nice enough function space or a small subset of the space of all Borel signed measures. See Chorin and Marsden (1993), Ben-Artzi (2003) and Gallagher and Gallay (2005) for precise statements as well as some historical background. The existence of solutions for the rougher initial data selected arbitrarily within the space of signed measures, a legitimate requirement for the sake of good statistical modeling of particle behaviour at the microscopic level, is harder and was first accomplished rigorously for (5.3) by Cottet (1993), Ben-Artzi (2003) and Gallagher and Gallay (2005). In both cases, the solution is only shown to have continuous trajectories away from 0, a consequence of both the singularity at 0 of \(K\) and the use of the total variation norm to induce a manageable topology on the state space. To our knowledge mass conservation has never been proved rigorously for these equations.

**Remark 5.1.** Conditions in (5.2) cover the vorticity equations (5.3) only after the singularity at 0 of kernel \(g\) is removed through smoothing, which we explain next, using the same notation as in Marchioro and Pulvirenti (1982). Let \(g_\varepsilon(r) = h_{\varepsilon}(||r||), 0 < \varepsilon \leq 1\) where \(h_{\varepsilon} \in C^2_b(\mathbb{R})\) is selected so as to satisfy \(h_{\varepsilon}(s) = h(s)\) for \(s \leq \frac{1}{\varepsilon}\), \(h_{\varepsilon}'(0) = 0\), and for all \(s > 0, |h_{\varepsilon}'(s)| \leq |h'(s)|, |h_{\varepsilon}''(s)| \leq |h''(s)|\). Such a filter of smooth approximations is easy to build, e.g. see Leonard (1985). For \(r \neq 0\), set \(K_{\varepsilon}(r) = (\nabla^\perp g_{\varepsilon})(r), r \in \mathbb{R}^2\). It follows from the assumption \(h_{\varepsilon}'(0) = 0\) that \(K_{\varepsilon}(0) = 0\) makes \(K_{\varepsilon}\) continuous on \(\mathbb{R}^2\). Moreover, since \(h_{\varepsilon}'(0) = 0\) and \(h_{\varepsilon}''\) is bounded by \(C_{\varepsilon}\) (say), it follows that \(|K_{\varepsilon}(r) - K_{\varepsilon}(q)| \leq 2C_{\varepsilon}|r - q|\), that is \(K_{\varepsilon}\) is Lipschitz. Finally, note that the monotonicity condition (1.1) holds for these equations by Mikulevicius and Rozovskii (2005) [Proposition 2.12], provided that in addition, \(\Gamma\) and \(K_{\varepsilon}\) are bounded.

The regularized or smoothed vorticity equations are then given by (5.3) with \(v\) no longer being a solution to (5.1) but rather of the form (5.3) with \(\nabla^\perp g\) replaced by the approximating \(\nabla^\perp g_{\varepsilon}\). Since \(\nabla \cdot K_{\varepsilon} \equiv 0\), their weak form may be written as

\[
\begin{align*}
&d <\omega_t, f> = <\omega_t, L_{\varepsilon}(\omega_t)f> dt, \\
&U_{\varepsilon}(r, \omega_t) = \int K_{\varepsilon}(r - q)\omega_t(dq),
\end{align*}
\]

(5.5)
where, for any $x \in \mathbb{R}^2$,

$$L_\varepsilon(\omega_t) f(x) = \sum_{j=1}^{2} \partial_{x_j} f(x) (U_\varepsilon)_j(x, \omega_t) + \nu \Delta f(x),$$

with $\langle \omega, f \rangle$ is just the integral of $f$ with respect to the measure (or density) $\omega$.

The advantage of (5.5) over (5.1) or (5.3) is that it makes perfect sense for finite signed measures $\omega_t$ and offers a ready-made stochastic version under the guise of our (2.3).

This reformulation allows us to provide through Theorem 4.6 the first rigorous statements and proofs in the matters of existence, uniqueness, mass conservation and continuity at the origin for arbitrary initial data for both the regularized vorticity equations (5.5) and their stochastic counterparts analyzed in Marchioro and Pulvirenti (1982) and Kotelenez (1995), among others. It must be pointed out that not only did Marchioro and Pulvirenti claim to give explicit conditions for equations (5.5) to possess one and only one solution, they did so even when it was perturbed by independent Brownian motions. They also claimed to have proved mass conservation (see the statement of their Theorem 2.1) but, just as in their proofs of the two previous statements, the use of an incomplete state space mars their argument and the same difficulty affects the proof of their Theorem 3.1 in the stochastic case. Specifically, Marchioro and Pulvirenti (1982) defined a mapping $S$ from $\mathcal{S}(m_1, m_2)$ into the space of continuous functions $C([0,T]; \mathcal{S}(m_1, m_2))$ by setting

$$(S\mu)_t(A) = \int P_{\mu_t}(A|y) \nu(dy)$$

for all Borel sets $A \subset \mathbb{R}^2$, where $P_{\mu_t}(\cdot|y)$ are the transition probabilities of the diffusion process, solution of the stochastic differential equation

$$dx_t = U_\varepsilon(x_t, \mu_t)dt + \sigma dW_t, \quad x_0 = y,$$

which is a particular case of equation (1.1). They claimed that it is obvious that $(S\mu)_t \in \mathcal{S}(m_1, m_2)$ if $\nu \in \mathcal{S}(m_1, m_2)$; this is shown to be false in Appendix C. Next, the proof of their Theorem 2.1 relies on a fixed point theorem for operators on complete spaces in an essential way. Unfortunately, their statements are widely quoted and used in the literature.

In a similar fashion, Kotelenez (1995) claimed to be able to extend Marchioro and Pulvirenti’s treatment when the stochastic terms involve Brownian sheets as the driving random environment, a more realistic description from the physical standpoint since the energy conservation law of physics is then respected at the microscopic level. However several of his arguments involve repeated use of the completeness of spaces that are simply not complete. A further claim by Kotelenez to the effect that the conservation of total positive and negative vortices follows from his construction turns out to hinge on the (incorrect) completeness assumption already mentioned. For his stochastic version of the weak regularized vorticity equations (5.5), Kotelenez selected the kernel $\Gamma$ appearing in our equations (2.2) and (5.1) amongst a family of simple gaussian kernels with values inside the set of diagonal matrices. They are easily shown to satisfy our conditions (2.2).

For the sake of both papers and the ensuing literature, we provide in the appendix corrected statements and proofs of some of their claims.
We must finally draw the reader’s attention to a clever transformation introduced by [Jourdain (2000)] which enabled him to transfer the problem of manipulating sequences of signed measures to that of associated probability measures, thus obtaining existence and uniqueness of solutions to some viscous scalar conservation laws by a direct application of the propagation of chaos results of [Sznitman (1991)]. This idea allowed [Mélaréard (2001)] to provide a proof of uniqueness for the vorticity equation under some fairly unrestricted conditions on the initial measure.

Whether our methods extend to the original equation with the explosive kernel \( g \) above remains an open question.

### Appendix A. Auxiliary results

This appendix contains the proofs of the more technical results of this paper. By organizing the material of this paper in such a fashion, our hope is that the reader will get a better overview of the subject at hand without getting bogged down in small details that could hamper his understanding of the connexions between the various results and contributions.

#### Proposition A.1

If \( T \) is a measurable injection on \( \mathbb{R}^d \), then \( \mu \mapsto \mu \circ T \) preserves singularity, i.e., if \( \mu = (\mu^1, \mu^2) \) is the Hahn-Jordan decomposition of some given signed measure \( \mu \), then \( \mu \circ T = (\mu^1 \circ T, \mu^2 \circ T) \) is the Hahn-Jordan decomposition of \( \mu \circ T \).

**Proof.** Let \( A^1, A^2 \) be disjoint Borel sets such that \( \mu^1(A^1) = m_1, \mu^1(A^2) = 0, \mu^2(A^1) = 0 \) and \( \mu^2(A^2) = m_2 \). Set \( B^\tau = T(A^\tau), \tau = 1, 2 \). Then, for \( \tau = 1, 2 \),

\[
\mu^1 \circ T(B^\tau) = \mu^1 \{ T^{-1}(B^\tau) \} = \mu^1 (A^\tau)
\]

and

\[
\mu^2 \circ T(B^\tau) = \mu^2 \{ T^{-1}(B^\tau) \} = \mu^2 (A^\tau).
\]

Next, remark that \( B^1 \cap B^2 = \emptyset \) since \( T \) is an injection. Hence the result.

\[ \Box \]

#### Proposition A.2

Let \( \mu = (\mu^1, \mu^2), \nu = (\nu^1, \nu^2) \in M \) be such that \( \mu = \mu^1 - \mu^2 \) equals \( \nu = \nu^1 - \nu^2 \). If \( \nu \) is the Hahn-Jordan decomposition of \( \nu \), then \( \mu = \nu \).

**Proof.** Let \( P_1 \) and \( P_2 \) be the positive and negative sets of \( \nu \). Then \( \nu^1(P_1) = m_1, \nu^2(P_2) = m_2 \) and \( \nu^1(P_2) = 0 = \nu^2(P_1) \). It follows that

\[
\mu^1(P_1) - \mu^2(P_1) = \mu(P_1) = \nu(P_1) = \nu^1(P_1) - \nu^2(P_1) = m_1
\]

and

\[
\mu^1(P_2) - \mu^2(P_2) = \nu^1(P_2) - \nu^2(P_2) = -m_2.
\]

Since \( \mu^\tau(\mathbb{R}^d) = m_\tau \geq \mu^\tau(P_\tau) \) for \( \tau = 1, 2 \), it follows that \( \mu^1(P_1) = m_1, \mu^1(P_2) = 0, \mu^2(P_1) = 0 \) and \( \mu^2(P_2) = m_2 \). Therefore \( \mu = (\mu^1, \mu^2) \) is also the Hahn-Jordan decomposition of \( \mu = \nu \). Because the uniqueness of the Hahn-Jordan decomposition, one may conclude that \( \mu = \nu \).

\[ \Box \]

Recall that the minimal Lipschitz constant for a function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is defined by

\[
\| f \|_L = \sup_{r \neq q \in \mathbb{R}^d} \frac{|f(r) - f(q)|}{r(q, q)}.
\]
It follows from the Kantorovich-Rubinstein Theorem (Dudley, 1989, Theorem 11.8.2) that for any $\mu, \nu \in M(m),$

$$W_1(\mu, \nu) = m \sup_{\|f\|_L \leq 1} |<\mu - \nu, f>|.$$ 

If $\mu = (\mu^1, \mu^2), \nu = (\nu^1, \nu^2) \in M(m_1, m_2)$ then, by denoting $\mu = \mu^1 - \mu^2,$ $\nu = \nu^1 - \nu^2$ and $m = \max(m_1, m_2),$ note that

$$\text{(A.2)} \quad \sup_{\|f\|_L \leq 1} |<\mu - \nu, f>| \leq \gamma_1(\mu, \nu)$$

and

$$\text{(A.3)} \quad \sup_{\|f\|_L \leq 1} <\mu - \nu, f>^2 \leq 2m^2 \gamma_2^2(\mu, \nu).$$

Appendix B. Proof of the main results

Proof of Lemma 3.1. For any adapted and $P \otimes \lambda$ measurable stochastic process $q = (q^1, \ldots, q^N) \in C([0, T]; \mathbb{R}^{dN})$ set

$$Q_q(t) = \sum_{i=1}^{N} a_i \delta_{q^i(t)}$$

and define the mapping $q \mapsto F(q) = \hat{q},$ where, for all $i = 1, \ldots, d,$ and every $t \geq 0,$

$$\hat{q}^i(t) = \dot{r}^i(0) + \int_0^t \int K(q^i(s), p)Q_q(dp, s)ds + \int_0^t \int \Gamma(q^i(s), p)w(dp, ds).$$

One wants to show the existence and uniqueness by Picard’s iteration, using the same technique as in Ethier and Kurtz (1986).

To this end, for any adapted $C([0, T]; \mathbb{R}^{dN})$-valued random variables $q_1, q_2,$ one needs to estimate $\|\hat{q}^i_1(t) - \hat{q}^i_2(t)\|$ for all $i = 1, \ldots, N.$ First, set

$$A^i_1(t) = \int_0^t \int K(q^i_1(s), p)Q_{q_1}(dp, s)ds = \sum_{j=1}^{N} a_j \int_0^t K(q^j_1(s), q^i_1(s))ds,$$
\[ l = 1, 2. \text{ Then, for any } i = 1, \ldots, N, \]
\[ \| A_i^1(t) - A_i^2(t) \|^2 \]
\[ = \left\| \int_0^t \sum_{j=1}^N a_j \left\{ K(q_i^1(s), q_i^1(s)) - K(q_i^2(s), q_i^2(s)) \right\} ds \right\|^2 \]
\[ \leq T \int_0^t \left\| \sum_{j=1}^N a_j \left\{ K(q_i^1(s), q_i^1(s)) - K(q_i^2(s), q_i^2(s)) \right\} \right\|^2 ds \]
\[ \leq NT \sum_{j=1}^N a_j^2 \int_0^t \left\| K(q_i^1(s), q_i^1(s)) - K(q_i^2(s), q_i^2(s)) \right\|^2 ds \]
\[ \leq NT \| K \|^2 \int_0^t \left\| q_i^1(s) - q_i^2(s) \right\|^2 ds \]
\[ \leq 2NT \| K \|^2 \left( \sum_{j=1}^N a_j^2 \right) \int_0^t \left\| q_i(s) - q_i(s) \right\|^2 ds, \]
where \( \| r - q \|_N = \max_{1 \leq i \leq N} \| r^i - q^i \| \).

Next, using Doob’s inequality,
\[ E \sup_{0 \leq u \leq t} \left\| \int_0^u \Gamma(q_i^1(s), p)w(dp, ds) - \int_0^u \Gamma(q_i^2(s), p)w(dp, ds) \right\|^2 \]
\[ \leq 4 \sum_{j=1}^d E \left\{ \left( \int_0^t \int \sum_{l=1}^d \left\{ \Gamma_{jl}(q_i^1(s), p) - \Gamma_{jl}(q_i^2(s), p) \right\} w_l(dp, ds) \right)^2 \right\} \]
\[ = 4 \sum_{j=1}^d \sum_{l=1}^d E \int_0^t \int \left\{ \Gamma_{jl}(q_i^1(s), p) - \Gamma_{jl}(q_i^2(s), p) \right\}^2 dpds \]
\[ \leq 4C_T^2 \int_0^t E \| q_i^1(s) - q_i^2(s) \|^2 ds, \]
where (2.2) was used in the last chain of inequalities. Hence, setting \( C = 8C_T^2 + 4NT\| K \|^2 \left( \sum_{j=1}^N a_j^2 \right) \), one obtains
\[ (B.1) \quad E \sup_{0 \leq s \leq t} \| q_1(s) - q_2(s) \|^2_N \leq C \int_0^t E \| q_1(s) - q_2(s) \|^2_N ds. \]
Therefore, if \( X_0(t) \equiv r(0) \) and \( X_{k+1} = F(X_k) \), one obtains, for all \( t \in [0, T] \),
\[ E \sup_{0 \leq s \leq t} \| X_1(s) - X_0(s) \|^2_N \leq 2t^2 \max_{1 \leq i \leq N} \left\| \int K(r^i(0), p)Q_0(dp) \right\|^2 \]
\[ + 8t \max_{1 \leq i \leq N} \sum_{j=1}^d \sum_{l=1}^d \int \Gamma_{jl}^2(r^i(0), p) dp \]
\[ \leq C' t, \]
for some constant $C_1$. Next, using the last equality together with \(B.1\), one obtains
\[
E \sup_{s \in [0, t]} \|X_{k+1}(s) - X_k(s)\|_N^2 \leq \frac{C_1 (Ct)^{k+1}}{C(k+1)!}, \quad k \geq 0.
\]
It follows from Borel-Cantelli Theorem that with probability one,
\[
\|X_{k+1}(s) - X_k(s)\|_N^2 \leq 2^{-(k+1)}
\]
for all large $k$. Since $C([0, t]; \mathbb{R}^{dN})$ is complete under the sup norm, it follows that $X_k$ converges almost surely to $X \in C([0, t]; \mathbb{R}^{dN})$. Since $F$ is a continuous mapping, one has $F(X) = X$ so $X$ is a solution. Finally, if $X$ and $Y$ are two solutions, i.e. $F(X) = X$ and $F(Y) = Y$, \(B.1\) yields
\[
E \sup_{s \in [0, t]} \|X(s) - Y(s)\|_N^2 \leq C \int_0^t E \sup_{u \in [0, s]} \|X(u) - Y(u)\|_N^2 du
\]
so Gronwall’s inequality entails that $E \sup_{s \in [0, t]} \|X(s) - Y(s)\|_N^2 = 0$, proving uniqueness.

**Proof of Lemma 3.1** Consider the following two $\mathbb{R}^d$-valued Itô equations with deterministic initial conditions $x_0, y_0 \in \mathbb{R}^d$:
\[
\begin{align*}
\frac{dr(t)}{dt} &= \int K(r(t), p)\chi_t(dp)dt + \int \Gamma(r(t), p)w(dp, dt), \\
\quad r(0) &= x_0; \\
\frac{dq(t)}{dt} &= \int K(q(t), p)\eta(p)dt + \int \Gamma(q(t), p)w(dp, dt), \\
\quad q(0) &= y_0.
\end{align*}
\]
When $x_0 = x^i(0)$, then $r(t, x_0) = x^i(t)$, using uniqueness property in Lemma 3.1. Similarly, $q(t, y_0) = y^j(t)$, if $y_0 = y^j(0)$.

Let $Q_0^\tau$ be joint representations of $(\chi_0^\tau, \eta_0^\tau)$, $\tau = 1, 2$. The following expressions define joint representations $Q_t^\tau$ of $(\chi_t^\tau, \eta_t^\tau)$ for every $t \geq 0$ and $\tau = 1, 2$: for $f \in C_b(\mathbb{R}^{2d})$ set
\[
\int \int f(y, z)Q_t^\tau(dy, dz) = \int \int f(r(t, y), q(t, z))Q_0^\tau(dy, dz).
\]
To see that $Q_t^\tau$ is indeed a representation for $(\chi_t^\tau, \eta_t^\tau)$, remark that
\[
\int \int f(y)Q_t^\tau(dy, dz) = \int \int f(r(t, y))Q_0^\tau(dy, dz)
\]
\[
= m_2 \int f(r(t, y))\chi_0^\tau(dy)
\]
\[
= m_2 \sum_{i:a_i \geq 0} a_if(r(t, x^i(0))
\]
\[
= m_2 \sum_{i:a_i \geq 0} a_if(x^i(t))
\]
\[
= m_2 \int f(y)\chi_t^\tau(dy).
\]
Similarly,
\[
\int f(y)Q_t^\tau_2(dy, dz) = m_2 \int f(y)\chi_t^2(dy)
\]
and
\[
\int f(z)Q_t^\tau_2(dy, dz) = m_1 \int f(z)\eta_t^\tau_2(dz), \quad \tau \in \{1, 2\}.
\]
It follows that if \( \chi = (\chi^1, \chi^2) \) and \( \eta = (\eta^1, \eta^2) \), then
\[
\gamma_2^2(\chi_t, \eta_t) \leq \int \rho^2(r(t, y), q(t, z))Q^1_0(dy, dz) + \int \rho^2(r(t, y), q(t, z))Q^2_0(dy, dz).
\]

Also,
\[
\|r(t) - r(0) - q(t) + q(0)\| \geq \rho(r(t) - q(t), r(0) - q(0)) \geq \rho(r(t), q(t)) - \rho(r(0), q(0)).
\]

Next we calculate \( \|r(t) - r(0) - q(t) + q(0)\|^2 \). First, set \( \chi = \chi^1 - \chi^2 \) and \( \eta = \eta^1 - \eta^2 \).

Proceeding as in Lemma 3.1, one has
\[
E \sup_{0 \leq t \leq T} \left\| \int_0^t \Gamma(r(s), p)w(dp, ds) - \int_0^t \Gamma(q(s), p)w(dp, ds) \right\|^2 \leq 2c \int_0^T E\rho^2(r(s), q(s))ds
\]
and
\[
\left\| \int_0^t K(r(s), p)\chi_s(dp)ds - \int_0^t K(q(s), p)\eta_s(dp)ds \right\|^2 \\
\leq 2 \left\| \int_0^t (K(r(s), p) - K(q(s), p))\chi_s(dp)ds \right\|^2 \\
+ 2 \left\| \int_0^t K(q(s), p)(\chi_s(dp) - \eta_s(dp))ds \right\|^2.
\]

Recall the notation \( m = \max(m_1, m_2) \). Since \( \|K\|_L \leq C \), one has that
\[
\left\| \int_0^t (K(r(s), p) - K(q(s), p))\chi_s(dp)ds \right\|^2 \\
\leq 4T(mC)^2 \int_0^t \rho^2(r(s), q(s))ds.
\]

In addition, it follows from (A.3) that
\[
\left\| \int_0^t K(q(s), p)(\chi_s(dp) - \eta_s(dp))ds \right\|^2 \\
\leq 2T(mC)^2 \int_0^t \gamma_2^2(\chi_s, \eta_s)ds.
\]
Proof of Theorem 3.4. To this end, let \( \chi \) satisfy the evolution equation (2.3). Using Gronwall’s inequality, we have

\[
E \sup_{0 \leq t \leq T} \rho^2(r(t), q(t)) \leq 2E \sup_{0 \leq t \leq T} \|r(t) - r(0) - q(t) + q(0)\|^2 + 2\rho^2(y, z) + C_1 \int_0^T E \sup_{0 \leq s \leq T} \rho^2(r(s), q(s))ds
\]

where \( C_1 = 16T(c + (2mC)^2) \).

By Gronwall’s inequality, we have

\[
E \sup_{0 \leq t \leq T} \rho^2(r(t), q(t)) \leq 2e^{C_1T} \rho^2(y, z) + e^{C_1T} \int_0^T \gamma^2_2(\chi_s, \eta_s)ds.
\]

Integrating the last inequality with respect to \( Q^0_0 + Q^2_0 \), one obtains

\[
E \sup_{0 \leq s \leq t} \gamma^2_2(\chi_t, \eta_t) \leq E \sup_{0 \leq t \leq T} \rho^2(r(t), q(t))\{Q^0_0 + Q^2_0\}\{dy, dz\}
\]

\[
\leq 2e^{C_1T} \int_0^T \rho^2(y, z)\{Q^0_0 + Q^2_0\}\{dy, dz\} + me^{C_1T} \int_0^T \gamma^2_2(\chi_s, \eta_s)ds.
\]

Taking the infimum over all \( Q^0_0 \in H(\chi_0^0, \eta_0^0) \), \( \tau = 1, 2 \), one gets

\[
E \sup_{0 \leq t \leq T} \gamma^2_2(\chi_t, \eta_t) \leq 2e^{C_1T} \gamma^2_2(\chi_0, \eta_0) + me^{C_1T} \int_0^T \gamma^2_2(\chi_s, \eta_s)ds.
\]

Using Gronwall’s inequality again yields (3.3). Finally, (3.4) is obtained by combining (3.3) and (3.3).

Proof of Theorem 3.4. To this end, let \( \chi_0 \in \mathcal{M} \) be given. Since \( \mathcal{M}^{(f)} \) is dense in \( \mathcal{M} \), there exists a sequence \( \chi_{0,n} \in \mathcal{M}^{(f)} \) so that \( \gamma(\chi_0, \chi_{0,n}) \to 0 \), as \( n \to \infty \). Using Lemma 3.3 it follows that \( \chi_n = \Psi(\chi_{0,n}) \) is a Cauchy sequence in \( \mathcal{M} \). Thus there exists \( \chi \in \mathcal{M} \) so that \( \gamma_{[0,T]}(\chi, \chi_n) \to 0 \). Since the limit does not depend on the sequence, the mapping is well-defined.

It also follows that for any \( \chi_0, \eta_0 \in \mathcal{M} \), the corresponding paths \( \chi, \eta \in \mathcal{M}_{[0,T]} \) satisfy

\[
\gamma_{[0,T]}(\chi, \eta) \leq c'\gamma(\chi_0, \eta_0).
\]

Next one will show that this extension gives a weak solution of the stochastic evolution equation (2.3).

Using (3.3) and the last inequality, one can conclude that for any \( f \) such that \( \|f\|_L \leq 1 \), one has

\[
E \sup_{0 \leq t \leq T} \sup_{\|f\|_L \leq 1} < \chi_t - \eta_t, f \|^2 \leq c''\gamma(\chi_0, \eta_0),
\]

where \( \chi = \chi_1 - \chi_2 \) and \( \eta = \eta_1 - \eta_2 \). In particular,

\[
\lim_{n \to \infty} E \sup_{0 \leq t \leq T} \sup_{\|f\|_L \leq 1} < \chi_t - \chi_{t,n}, f \|^2 = 0.
\]
The next step is to show that for any $f \in C_b^2(\mathbb{R}^d)$ and $\chi_0 \in M$, $\chi$ satisfies (2.3). Note that by the choice of $f$, both $f$ and $\nabla f$ are bounded, so the right-hand side of the stochastic evolution equation is defined for $< \chi_t, f >$. Moreover, since $\|f\|_L < \infty$ and $\|\nabla f\|_L < \infty$, it follows that
\[
\lim_{n \to \infty} E \sup_{0 \leq t \leq T} < \chi_t - \chi_{t,n}, f >^2 = 0.
\]
Similarly,
\[
E \sup_{0 \leq t \leq T} \sup_p |U(p, \chi_t) - U(p, \chi_{t,n})|^2 \to 0
\]
and one can prove that all right-hand side terms of (2.3) tends to zero, as $n$ tends to infinity. This completes the proof.

Proof of Lemma 4.3. For any $\mu \in M_{\nu}$, $\mu = \mu^1 - \mu^2$, and any $x \in \mathbb{R}^d$, let $r(t, \mu, x)$ be the unique solution of the following Itô equation:
\[
\begin{cases}
    dr(t) = \int K(r(t), p)\mu(t)(dp)dt + \int \Gamma(r(t), p)w(dp, dt), \\
    r(0) = x.
\end{cases}
\]
The proof of existence and uniqueness is similar to that of Lemma 3.1.

Next, recall that the operator $S$, acting on $\mu \in M_{\nu}$, is defined by
\[
(S\mu)_{\tau} = \nu^\tau \circ r(t, \mu, \cdot), \quad \tau = 1, 2.
\]
For $\tau = 1, 2$, let $Q^\tau$ be joint representations for $(\nu^\tau, \nu^\tau)$ and define, for any $\mu, \eta \in M_{\nu}$, the joint representations $Q^\tau$ for $(S\mu)_{\tau}$ and $(S\eta)_{\tau}$ as follows: for every $f \in C_b(\mathbb{R}^d)$
\[
\int \int f(y, z)Q^\tau(dy, dz) = \int \int f(r(t, \mu, y), r(t, \eta, z))Q^\tau(dy, dz).
\]
To see that $Q^\tau_{\tau}$ is indeed a representation for $((S\mu)_{\tau}, (S\eta)_{\tau})$, $\tau = 1, 2$, remark that
\[
\int \int f(y)Q^\tau_{\tau}(dy, dz) = \int \int f(r(t, \mu, y))Q^\tau_{\tau}(dy, dz) = m_2 \int f(r(t, \mu, y))\nu^\tau(dy) = m_2 < (S\mu)_{\tau}, f >.
\]
Similarly, for $\tau = 1, 2$,
\[
\int \int f(z)Q^\tau_{\tau}(dy, dz) = \int \int f(r(t, \eta, z))Q^\tau_{\tau}(dy, dz) = m_1 \int f(r(t, \eta, z))\nu^\tau(dz) = m_1 < (S\eta)_{\tau}, f >.
\]
It follows that
\[
\gamma^2_{\tau}((S\mu)_{\tau}, (S\eta)_{\tau}) \leq \int \rho^2(r(t, \mu, y), r(t, \eta, z))Q_{\tau}(dy, dz) + \int \rho^2(r(t, \mu, y), r(t, \eta, z))Q_{\tau}(dy, dz).
\]
The rest of the proof is almost identical to the proof of Lemma 3.3 so it is omitted. □

**Proof of Lemma 4.5.** The proof is classical. Suppose there are two solutions, and write \( \chi \) for their difference. It is easy to check that for any \( \phi \in H_q \),

\[
< \chi_t, \phi > = \int_0^t < \chi_s, L(\mu_s)\phi > ds + \int_0^t \int < \chi_s, \nabla \phi(\cdot) \top \Gamma(\cdot, p) > w(dp, ds).
\]

Therefore applying Ito’s formula to \( e^{-tC_p} < \chi_t, \phi >^2 \) and taking expectations, one ends up with

\[
e^{-2tC_p} E \{ < \chi_t, \phi >^2 \} = 2E \left\{ \int_0^t e^{-2sC_p} < \chi_s, \phi > < \chi_s, L(\mu_s)\phi > ds \right\} - 2C_p E \left\{ \int_0^t e^{-2sC_p} < \chi_s, \phi >^2 ds \right\} + E \left\{ \int \int e^{-2sC_p} \| \nabla \phi(\cdot, x) \chi_s \|^2 dx ds \right\}.
\]

Summing over a complete orthonormal system of \( H_q \), one gets

\[
e^{-tC_p} E \{ \| \chi_t \|_{-q}^2 \} = 2 \int_0^t e^{-2sC_p} E \{ < \chi_s, L(\mu_s)\chi_s >_{-q} \} ds - 2C_p E \left\{ \int_0^t e^{-2sC_p} \| \chi_s \|_{-q}^2 ds \right\} + E \left\{ \int \int e^{-2sC_p} \| \nabla^* \Gamma(\cdot, x) \chi_s \|^2 dx ds \right\} \leq 0,
\]

using the monotonicity condition (4.4). Hence the result. □

**Appendix C. Disproof of Marchioro and Pulvirenti claim**

Denote by \( x(t, y) \) the solution of the stochastic differential equation

\[
dx_t = U_t(x_t, \mu_t)dt + \sigma dW_t, \quad x_0 = y.
\]

Recall that their mapping \( S \) is defined by

\[
(S\mu)_t(A) = \int P_t(A|y)\nu(dy)
\]

for all Borel sets \( A \subset \mathbb{R}^2 \), where \( P_t(\cdot|y) \) are the transition probabilities of the diffusion process \( x(t, y) \).

Suppose that \( \nu \in SM(m_1, m_2) \) and \( (S\mu)_t \in SM(m_1, m_2) \), as claimed in Marchioro and Pulvirenti (1982). It follows from Proposition A.2 that the Hahn-Jordan decomposition of \( (S\mu)_t \) is

\[
((S\mu)_t^1, (S\mu)_t^2) = \left( \int P_t(\cdot|y)\nu^1(dy), \int P_t(\cdot|y)\nu^2(dy) \right),
\]

if \( (\nu^1, \nu^2) \) is the Hahn-Jordan decomposition of \( \nu \). Therefore, there exist disjoint sets \( A^1 \) and \( A^2 \) so that \( (S\mu)_t^1(A^\tau) = m_\tau, \tau = 1, 2, (S\mu)_t^1(A^2) = 0 \) and \( (S\mu)_t^2(A^1) = 0 \).
It follows that there exist Borel sets $N^1$ and $N^2$ so that $\nu^\tau(N^\tau) = m_\tau$ and $P_t(A^\tau|y) = 1$ for all $y \in N^\tau$, $\tau = 1, 2$. In addition $P_t(A^1|y) = 0$ for all $y \in N^2$, and $P_t(A^2|y) = 0$ for all $y \in N^1$. Hence, $N^1$ and $N^2$ are disjoint, showing that $\nu^1(N^2) = 0 = \nu^2(N^1)$.

Because for any $z \in \mathbb{R}^2$, $P_t(\cdot|z)$ has a positive density with respect to Lebesgue measure (since it can transformed into a Wiener process with respect to an equivalent measure using Girsanov’s formula), it follows that both $A^1$ and $A^2$ would have zero Lebesgue measure, contradicting the equations $P_t(A^\tau|y) = 1$ for all $y \in N^\tau$, $\tau = 1, 2$.

Therefore, $(S\mu)_t \not\in SM(m_1, m_2)$.

References

Amirdjanova, A. (2000). *Topics in stochastic fluid dynamics: A vorticity approach*. PhD thesis, University of North Carolina at Chapel Hill.

Amirdjanova, A. and Xiong, J. (2006). Large deviation principle for a stochastic Navier-Stokes equation in its vorticity form for a two-dimensional incompressible flow. *Discrete and Continuous Dynamical Systems (Series B)*, 6(4):651–666.

Ben-Artzi, M. (2003). Planar Navier-Stokes equations: vorticity approach. In *Handbook of mathematical fluid dynamics, Vol. II*, pages 143–167. North-Holland, Amsterdam.

Chorin, A. J. and Marsden, J. E. (1993). *A mathematical introduction to fluid mechanics*, volume 4 of *Texts in Applied Mathematics*. Springer-Verlag, New York, third edition.

Cottet, G.-H. (1986). Équations de Navier-Stokes dans le plan avec tourbillon initial mesure. *C. R. Acad. Sci. Paris Sér. I Math.* , 303(4):105–108.

Dawson, D. and Vaillancourt, J. (1995). Stochastic McKean-Vlasov equations. *Nonlinear Diff. Eq. Appl.*, 2:199–229.

Dawson, D. A. (1975). Stochastic evolution equations and related measure processes. *J. Multivariate Anal.*, 5:1–52.

Dawson, D. A. (1993). Measure-valued Markov processes. In *École d’Été de Probabilités de Saint-Flour XXI—1991*, volume 1541 of *Lecture Notes in Math.*, pages 1–260. Springer, Berlin.

Dudley, R. M. (1989). *Real Analysis and Probability*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA.

Ethier, S. N. and Kurtz, T. G. (1986). *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York. Characterization and convergence.

Gallagher, I. and Gallay, T. (2005). Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity. *Math. Ann.*, 332(2):287–327.

Huber, P. J. (1981). *Robust statistics*. John Wiley & Sons Inc., New York. Wiley Series in Probability and Mathematical Statistics.

Jourdain, B. (2000). Diffusion processes associated with nonlinear evolution equations for signed measures. *Methodol. Comput. Appl. Probab.*, 2(1):69–91.

Kotelenez, P. (1995). A stochastic Navier-Stokes equation for the vorticity of a two-dimensional fluid. *Ann. Appl. Probab.*, 5:1126–1160.

Kotelenez, P. M. (2010). Stochastic flows and signed measure valued stochastic partial differential equations. *Theory Stoch. Process.*, 16(2):86–105.
Kotelenez, P. M. and Seadler, B. T. (2011). Conservation of total vorticity for a 2D stochastic Navier Stokes equation. *Adv. Math. Phys.*, pages Art. ID 862186, 14.

Kotelenez, P. M. and Seadler, B. T. (2012). On the Hahn-Jordan decomposition for signed measure valued stochastic partial differential equations. *Stoch. Dyn.*, 12(1):1150009, 23.

Kunita, H. (1990). *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.

Kurtz, T. G. and Xiong, J. (1999). Particle representations for a class of nonlinear SPDEs. *Stochastic Process. Appl.*, 83(1):103–126.

Leonard, A. (1985). Computing three-dimensional incompressible flows with vortex filaments. *Annual Review of Fluid Mechanics*, 17:523–559.

Marchioro, C. and Pulvirenti, M. (1982). Hydrodynamics in two dimensions and vortex theory. *Comm. Math. Phys.*, 84:483–503.

Méléard, S. (2001). Monte-Carlo approximations for 2d Navier-Stokes equations with measure initial data. *Probab. Theory Related Fields*, 121(3):367–388.

Mikulevicius, R. and Rozovskii, B. L. (2005). Global $L_2$-solutions of stochastic Navier-Stokes equations. *Ann. Probab.*, 33(1):137–176.

Rudin, W. (1973). *Functional analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Compagny.

Sznitman, A.-S. (1991). Topics in propagation of chaos. In *École d’Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin.