The Circular Chromatic Number of the Mycielski’s graph $M^t(K_n)$ *

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Abstract

As a natural generalization of chromatic number of a graph, the circular chromatic number of graphs (or the star chromatic number) was introduced by A.Vince in 1988. Let $M^t(G)$ denote the $t$th iterated Mycielski graph of $G$. It was conjectured by Chang, Huang and Zhu(Discrete mathematics,205(1999), 23-37) that for all $n \geq t + 2$, $\chi_c(M^t(K_n)) = \chi(M^t(K_n)) = n + t$. In 2004, D.D.F. Liu proved the conjecture when $t \geq 2$, $n \geq 2t - 1 + 2t - 2$. In this paper,we show that the result can be strengthened to the following: if $t \geq 4$, $n \geq \frac{11}{12}2^{t-1} + 2t + \frac{1}{3}$, then $\chi_c(M^t(K_n)) = \chi(M^t(K_n))$.

Keyword: circular chromatic number, complete graph, Mycielski graph

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1 Introduction

The circular chromatic number of a graph is a natural generalization of chromatic number $\chi(G)$ of a graph, introduced and studied by A.Vince in 1988, as the ‘star chromatic number’.

Definition 1[13] Let $k$ and $d$ be positive integers such that $k \geq 2d$, a $(k,d)$-coloring of a graph $G$ is a mapping $f : V(G) \rightarrow \mathbb{Z}_k = \{0,1,\ldots,k-1\}$ such that for any $uv \in E(G), d \leq |f(u) - f(v)| \leq k - d$. The circular chromatic number $\chi_c(G)$ of $G$ is defined as $\chi_c(G) = \inf\{\frac{k}{d} : \text{there exists a } (k,d)\text{-coloring of } G\}$.

Obviously, a $(k,1)$-coloring of a graph $G$ is just an ordinary $k$-coloring of $G$. The following properties of circular chromatic number can be found in [2,13,14]:

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\textbf{Property 1\textsuperscript{[13]}} \( \chi(G) - 1 < \chi_c(G) \leq \chi(G) \).

\textbf{Property 2\textsuperscript{[13]}} If \( H \) is a subgraph of \( G \), then \( \chi_c(H) \leq \chi_c(G) \).

\textbf{Property 3\textsuperscript{[2]}} If there is a homomorphism from \( G \) to \( H \), then \( \chi_c(G) \leq \chi_c(H) \).

\textbf{Property 4\textsuperscript{[14]}} For any graph \( G \), \( \chi_c(G) \) is a rational number. In fact, we have

\[ \chi_c(G) = \min \left\{ \frac{k}{d} : \text{there exists a} \ (k,d)\text{-coloring of} \ G, k \leq |V(G)|, d \leq \alpha(G) \right\} \]

where \( \alpha(G) \) is the independence number of \( G \).

The question of determining a graph \( G \) satisfying \( \chi_c(G) = \chi(G) \) was asked by Vince in \textsuperscript{[13]}. Unfortunately it is hard to determine whether a given graph \( G \) has \( \chi_c(G) = \chi(G) \) or not \textsuperscript{[5]}. In spite of this difficulty, many classes of graphs satisfying \( \chi_c(G) = \chi(G) \) have been found, and the Mycielski graph is one of the important classes\textsuperscript{[3,4,10,11,12]}.

\textbf{Definition 2} Let \( G \) be a simple graph with vertex set \( V(G) = \{x_1, x_2, \ldots, x_n\} \) and edge set \( E(G) \), the \textit{Mycielski graph} of \( G \), denoted by \( M(G) \), is the graph with vertex set

\[ V(M(G)) = \{x_1, x_2, \ldots, x_n\} \cup \{x'_1, x'_2, \ldots, x'_n\} \cup \{u\}, \]

and edge set

\[ E(M(G)) = E(G) \cup \{x'_ix_j : x_ix_j \in E(G), 1 \leq i, j \leq n\} \cup \{ux'_i : 1 \leq i \leq n\}. \]

For each \( x_i \in V(G), x'_i \) is called the \textit{twin} of \( x_i \) (\( x_i \) is also the \textit{twin} of \( x'_i \)), and the new vertex \( u \) is called the \textit{root} of \( M(G) \). Let \( M^0(G) = G \), and define the \( t \)-th Mycielski graph of \( G \) is \( M^t(G) = M(M^{t-1}(G)) \).

Here, we consider \( M^t(G) \) as a subgraph of \( M^j(G) = M^{j-i}(M^i(G)) \) if \( i \leq j \).

We know that for any nonempty graph \( G \), \( \chi(M(G)) = \chi(G) + 1 \). Unfortunately, no simple characterization of graphs with \( \chi_c(M(G)) = \chi(M(G)) \) have been found, though some graphs that satisfy this condition were studied in \textsuperscript{[4,8]}. An important class of graphs is the Mycielski graph of complete graph \( K_n \). The following conjecture was introduced by Chang \textsuperscript{[3]} and confirmed for the cases of \( t = 1, 2 \).

\textbf{Conjecture\textsuperscript{[3]}} If \( t \geq 1, n \geq t + 2 \), then \( \chi_c(M^t(K_n)) = \chi(M^t(K_n)) = n + t \).

For \( t \geq 2 \) and \( n \geq 2^{t-1} + 2t - 2 \), the conjecture is verified \textsuperscript{[10]}. Recently, Simonyi and Tardos proved the conjecture holds when \( t + n \) is even.

In this paper, we show that the result in \textsuperscript{[10]} can be strengthened to the following:

\textbf{Theorem 1} If \( t \geq 4, n \geq \frac{11}{17}2^{t-1} + 2t + \frac{3}{7} \), then \( \chi_c(M^t(K_n)) = \chi(M^t(K_n)) = n + t \).
2 Some definitions and lemmas

Let $k \geq 2d$, Fan introduced $(k, d)$-partition of $G$ [4]: a $(k, d)$-partition of $G$ is a partition $(X_0, X_1, \cdots, X_{k-1})$ of $V(G)$ such that $V(G) = \bigcup_{i=0}^{k-1} X_i$, and for each $j \in \mathbb{Z}_{k-1}$,

$$X_j \cup X_{j+1} \cup \cdots \cup X_{j+d-1}$$

is an independent set in $G$ (Here it is allowed that $X_i = \emptyset$). It is easy to see that a $(k, d)$-partition of $G$ is equivalent to a $(k, d)$-coloring of $G$ (Here we call $X_i$ color class of $i, 0 \leq i \leq n-1$). Thus, we have

$$\chi_c(G) = \min \left\{ \frac{k}{d} : \text{there exists a } (k, d) \text{-partition of } G \right\}.$$

Combining with the conclusions in [4] and [6], the following result holds:

**Lemma 1**[4,6] Let $\chi_c(G) = \frac{k}{d}$, $(k, d) = 1, (X_0, X_1, \cdots, X_{k-1})$ is a $(k, d)$-partition of $G$, then $\forall i \in \mathbb{Z}_k, X_i \neq \emptyset$, and $N(X_i) \cap X_{i+d} \neq \emptyset$.

To work with such complicated graphs as $M^i(G)$, we need to take a system to name the vertices of $M^i(G)$.

**Definition 3**[11] Suppose $x \in V(G)$, and $t$ is a positive integer,

1. As a vertex of $M^{i-1}(G)$, the twin of $x$ in $M^i(G)$ is called the $t$th twin of $x$, denoted by $x^t$, $x$ is called initial vertex of $M^i(G)$;

2. If $i$ and $j$ are positive integers, the $i$th twin of $x$ in $M^i(G)$ is $x^i$, the $j$th of $x^i$ in $M^j(M^i(G)) = M^{i+j}(G)$ is denoted by $(x^i)^j$, which can be simplified as $x^{ij}$ if there is no ambiguity.

If $i_1, i_2, \cdots, i_n \in \mathbb{N}$, $x^{i_1i_2\cdots i_n}$ is defined recursively by $(x^{i_1i_2\cdots i_{n-1}})^{i_n}$. For any positive integers $i_1, i_2, \cdots, i_n$ such that $i_1 + i_2 + \cdots + i_n \leq t$, $x^{i_1i_2\cdots i_n}$ is called a derived vertex of $x$ in $M^i(G)$ (Here, the $i$th twin of $x$ is the derived vertex of $x$ too). The set of derived vertices of $x$ is denoted by $T(x)$:

$$T(x) = \{x^{i_1i_2\cdots i_n} : i_1 > 0, \cdots, i_n > 0, i_1 + i_2 + \cdots + i_n \leq t\}.$$

3. For $0 < i \leq t$, the new root in $M^i(G)$ formed from $M^{i-1}(G)$ to $M^i(G)$ is called the $i$th root of $G$, denoted by $u_i$. The set of roots and their derived vertices in $M^i(G)$ is denoted by $R(M^i(G))$:

$$R(M^i(G)) = \{u_i, u_i^{i_1i_2\cdots i_n} : 0 < i \leq t, i_1 > 0, \cdots, i_n > 0, i_1 + i_2 + \cdots + i_n \leq t - i\}.$$

Fig.1 shows $M^2(G)$.

For convenience, any vertex in $R(M^i(G))$ is called the root of $M^i(G)$.

**Definition 4** We call $u_i^{i_1i_2\cdots i_n}$ the $s$th root of $M^i(G)$ if $i + i_1 + i_2 + \cdots + i_n = s$, and denote $R_s(M^i(G))$ as the set of all $s$th roots of $M^i(G)$. 


By the definition of Mycielski graph, there is a bijection $h$ from $R(M^{t-1}(G))$ to $R_t(M^t(G)) - \{u_t\}$ as follows

$$h : R(M^{t-1}(G)) \rightarrow R_t(M^t(G)) - \{u_t\}.$$ 

For any $u \in R(M^{t-1}(G))$, $h(u)$ is the twin of $u$ in $M^t(G) = M(M^{t-1}(G))$. In other words, $h(u_{i_1i_2\cdots i_{n-1}}) = u_{i_1i_2\cdots i_n}$, $i_n \neq 0$, $i_1 + i_2 + \cdots + i_n = t$.

![Diagram](Fig.1)

We say $x \in V(G)$ is a universal vertex if for each $y \in V(G) - \{x\}$, $xy \in E(G)$. Concerning the circular chromatic number of $M^t(K_n)$, the following results holds:

**Lemma 2**[^6] Let $G$ be a graph with $n$ universal vertices, $n \geq 2$. If $\chi_c(M^t(G)) = \frac{k}{d}$, $(k, d) = 1$, then $(n - 3)(d - 1) \leq 2^t - 2$.

**Lemma 3**[^11] Let $G$ be a simple graph, $V(G) = \{x_1, x_2, \ldots, x_n\}$. The vertex set of $M(G)$ is $V(M(G)) = \{x_1, x_2, \ldots, x_n\} \cup \{x_1', x_2', \ldots, x_n'\} \cup \{u\}$. If $V(G)$ has $(k, d)$-partitions, then there is a $(k, d)$-partition of $V(G)$ such that

1. $u \in X_0$;
2. For some $i$, $d \leq i \leq k - d, \forall x \in V(G)$, we have $\{x, x'\} \subseteq X_i$ if $x \in X_i$.

**Definition 5**[^11] Let $(X_0, X_1, \ldots, X_{k-1})$ be a $(k, d)$-partition of $G$. For any $x \in V(G)$, there exist $j, 0 \leq j \leq k - 1$ such that $x \in X_j$. The following set

$$\delta(x) = X_{j-d+1} \cup X_{j-d+2} \cup \cdots \cup X_j \cup X_{j+1} \cup \cdots \cup X_{j+d-1}$$

is called the $d$-field of $x$ in partition $(X_0, X_1, \ldots, X_{k-1})$.

It is easy to see that for any $x \in V(G)$, $\delta(x) \cap N(x) = \emptyset$, where $N(x)$ is the adjacency set of $x$. In general, we use $C(x)$ to denote the color class containing $x$.

Suppose $\{x_i, x_j\} \subseteq V(K_n), i \neq j$, by the definition of $M^t(K_n)$, it is obviously that $T(x_i) \subseteq N(x_j)$. Hence, $X_j \subseteq R(M^t(G))$ if $x_j \subseteq \delta(x_i) \cap \delta(x_{i+1})$, because $\forall v \in T(x_l), 1 \leq l \leq n, v \notin X_j$. 

[^6]: Lemma 2
[^11]: Definition 5
Definition 6  Let $t \geq 1, F_t^o$ is a digraph with vertex set

$$V(F_t^o) = R_t(M'(G)) - \{u_t\},$$

and arc set

$$A(F_t^o) = \{(u_{i_1 i_2 \cdots i_{n-1}}^t, u_{i_1 i_2 \cdots i_{n-1}-1}^t) : 1 \leq i \leq t - 1, i + \sum_{l=1}^{n} i_l = t, i_1 \geq 2, 1 \leq j \leq i_n - 1\}.$$
**Definition 7**  For $1 \leq i \leq t - 1$, $F(i) \sqcup F'(i)$ is a digraph with vertex set

$$V(F(i) \sqcup F'(i)) = V(F(i)) \cup V(F'(i)),$$

and arc set

$$A(F(i) \sqcup F'(i)) = A(F(i)) \cup A(F'(i)) \cup \{(u_i, v_i)\}.$$

**Lemma 6**  For $1 \leq i \leq t - 2$, $F(i) \cong F(i + 1) \sqcup F'(i + 1)$.

**proof**  Let $g$ be a mapping from $V(F(i))$ to $V(F(i + 1) \sqcup F'(i + 1))$, i.e.,

$$g : V(F(i)) \rightarrow V(F(i + 1) \sqcup F'(i + 1))$$

such that

$$g(x) = \begin{cases} 
  u_{i+1}, & x = u_i; \\
  u_{i+1}^{(i+1)_2 \ldots i_n}, & x = u_i^{i_2 \ldots i_n}, i_1 \geq 2; \\
  v_{i+1}, & x = u_1^i; \\
  v_{i+1}^{i_2 \ldots i_n}, & x = u_1^{i_2 \ldots i_n}.
\end{cases}$$

Then $(u_i, u_1^i) \in A(F(i)) \iff (u_{i+1}, v_{i+1}) \in A(F(i + 1) \sqcup F'(i + 1))$,

$$(u_i^{i_2 \ldots i_n-1}, u_1^{i_2 \ldots i_n}) \in A(F(i)) \iff (v_{i+1}^{i_2 \ldots i_n-1}, v_{i+1}^{i_2 \ldots i_n}) \in A(F(i + 1) \sqcup F'(i + 1))$$

and for $i_1 \geq 2$,

$$(u_i^{i_2 \ldots i_n-1}, u_1^{i_2 \ldots i_n}) \in A(F(i)) \iff (u_{i+1}^{(i+1)_2 \ldots i_n-1}, u_{i+1}^{(i+1)_2 \ldots i_n}) \in A(F(i + 1) \sqcup F'(i + 1)).$$

Hence $g$ is an isomorphism. \(\square\)

According to Lemma 6, it is straightforward to verify $|V(F(i))| = 2^{t-1-i}, 1 \leq i \leq t - 1$.

**Definition 8**  Let $D$ be a digraph, $\{u, v, w\} \subseteq V(D)$, $(u, v, w)$ is a directed triple of $D$ if there is a directed path $P(u, v, w)$ from $u$ to $w$, and contains $v$ as an inner vertex. Let $S \subseteq V(D)$, $S$ is a 3-cut set of $D$ if there is no directed triple of $D$ in $V(D) - S$.

**Lemma 7**  $t \geq 3$, $S$ is 3-cut set of $F_i$, then $|S| \geq 2^{t-3} - 1$, and there exists a 3-cut set such that $^t = ^t$ holds.

**proof**  If $t = 3$, then the length of the longest directed path of $F_3$ is 1. So the directed triple is $\emptyset$. Hence the proposition comes true.

Let $t \geq 4$, and suppose that $S$ is a 3-cut set of $F_i$, then $S \cap V(F(i)) = S_i$ is a 3-cut set of $F(i)$, and $S = \bigcup_{i=1}^{t-1} S_i$. Assume that $S_i$ is the smallest 3-cut set of $F(i)$, it is easy to see that we need to confirm $|S_i| = 2^{t-i-3}, 1 \leq i \leq t - 3$.  

6
Let $S_i = \{u_i^{i_1i_2\cdots i_n}: i_1 + i_2 + \cdots + i_n + i \leq t - 3\}$, we will prove $S_i$ is the smallest 3-cut set of $F(i)$ by induction on $i$.

For $i = t - 3$, $S_{t-3} = \{u_{t-3}\}$ is the smallest 3-cut set of $F(t - 3)$, the proposition holds. For $i, 1 \leq i \leq t - 4$, hypothesize the conclusion holds for $i + 1$. It means that $S_{i+1} = \{u_{i+1}^{i_1i_2\cdots i_n}: i_1 + i_2 + \cdots + i_n + (i + 1) \leq t - 3\}$ is the smallest 3-cut set of $F(i + 1)$.

In the case of $F(i)$, by Lemma 6, $F(i) \cong F(i + 1) \cup F'(i + 1)$. Since $F(i + 1) \cong F'(i + 1)$, as the image of $S_{i+1}$, $S_{i+1}'$ is the smallest 3-cut set of $F'(i + 1)$. So $S_{i+1} \cup S_{i+1}'$ is a 3-cut set of $F(i + 1) \cup F'(i + 1)$. By the mapping $g$ in Lemma 6, the preimage of $S_{i+1} \cup S_{i+1}'$ is $\{u_i^{i_1i_2\cdots i_n}, u_{i+1}^{i_1i_2\cdots i_n}: i + i_1 + i_2 + \cdots + i_n + i \leq t - 3\} = S_i$, which is a 3-cut set of $F(i)$. $|S_i| = 2^{t-i-3}$ follows by induction.

If $S_i$ is not the smallest 3-cut set of $F(i)$, then there is a 3-cut set of $F(i)$, denoted as $T$, such that $|T| = |g(T)| < |S_i| = 2^{t-i-3}$. Since $g(T)$ is a 3-cut set of $F(i + 1) \cup F'(i + 1)$, and $g(T) = (g(T) \cap V(F(i + 1))) \cup (g(T) \cap V(F'(i + 1)))$, assume $|(g(T) \cap V(F(i + 1)))| \leq 2^{t-(i+1)-3} - 1 < |S_{i+1}|$, contradicting to $S_{i+1}$ is the smallest 3-cut set of $F(i + 1)$.

Hence $S = \bigcup_{i=0}^{k-1} S_i$ is the smallest 3-cut set of $F_i$ satisfying $|S| = 2^{t-3} - 1$. \hfill \Box

**Corollary 1** Let $t \geq 4, U \subseteq V(F_t^\circ)$, if $|U| > 3 \cdot 2^{t-3}$, then there exists a directed triple in $U$.

**proof** Assume to the contrary that there is no directed triple in $U$, then $R_t(M^t(G)) - \{u_i\} - U$ is 3-cut set, and $|R_t(M^t(G)) - \{u_i\} - U| < 2^{t-1} - 1 - 3 \cdot 2^{t-3} = 2^{t-3} - 1$, contradicts to $F_t^\circ \cong F_t$ and lemma 7. \hfill \Box

### 3 Main result

It is straightforward to verify the following lemma by definition of $M^t(G)$.

**Lemma 8** Let $v_1, v_2$ be two $t$th roots of $M^t(K_n)$, and there is a directed path $P(v_1, v_2)$ from $v_1$ to $v_2$ in $F_t^\circ$, then $N(\{v_2, h^{-1}(v_2)\}) \cap T(V(K_n)) \subseteq N(\{v_1, h^{-1}(v_1)\}), N(v_2) \cap T(V(K_n)) \subseteq N(v_1)$.

**Lemma 9** Let $t \geq 3$, $\chi_c(M^t(K_n)) = \frac{k}{2}$, and a $(k,2)$-coloring satisfy Lemma 3. If there exists a set of $3$ $t$th roots $\{v_1, v_2, v_3\}$ such that $C(v_i) \subseteq \delta(v_i) \cap R(M^t(K_n)) \subseteq \{v_i, h^{-1}(v_i)\}, 1 \leq i \leq 3$, then $(v_1, v_2, v_3)$ is not the directed triple of $F_t^\circ$.

**proof** Suppose on the contrary that $(v_1, v_2, v_3)$ is a directed triple of $F_t^\circ$, and $C(v_i) \subseteq \{v_i, h^{-1}(v_i)\}, v_i \in X_{s_i}, 1 \leq i \leq 3$. It is sufficient to consider two cases as follows:

Case 1. $h^{-1}(v_3) \notin C(v_3)$. According to Lemma 3, Lemma 8 and the assumption of this lemma, we get $h^{-1}(v_3) \in \delta(u_i), N(v_3) \cap \delta(v_2) = \emptyset$. Hence, we can dye $v_3$ color $s_2$, and obtain a new $(k,2)$-coloring $(Y_0, Y_1, \cdots, Y_{k-1})$ such that $Y_{s_2} = X_{s_2} \cup \{v_3\}, Y_{s_3} = \emptyset$ and $Y_i = X_i, i \neq s_2, s_3$, a
contradiction to Lemma 1.

Case 2. $h^{-1}(v_3) \in C(v_3)$. Suppose that $h^{-1}(v_2) \in C(v_2)$, by Lemma 8 and the assumption of this lemma, we get $N(v_3, h^{-1}(v_3)) \cap \delta(v_2, h^{-1}(v_2)) = \emptyset$. Hence, we can dye $v_3, h^{-1}(v_3)$ color $s_2$ and obtain a new $(k, 2)$-coloring $(Y_0, Y_1, \ldots, Y_{k-1})$ such that $Y_{s_2} = \emptyset$, a contradiction. So $h^{-1}(v_2) \notin C(v_2)$, then we can dye $v_2$ color of $v_1$ and also obtain a new $(k, 2)$-coloring $(Y_0, Y_1, \ldots, Y_{k-1})$ such that $Y_{s_2} = \emptyset$, a contradiction. □

**Lemma 10** Let $G$ be a simple graph with $V(G) = \{x_1, x_2, \cdots, x_n\}$. The vertex set of $M(G)$ is $V(M(G)) = \{x_1, x_2, \cdots, x_n\} \cup \{x_1', x_2', \cdots, x_n'\} \cup \{u\}$. If $M(G)$ has $(k, d)$-partition, then there is a $(k, d)$-partition of $M(G)$ such that

1. $X_0 = \{u\}$;
2. For some $i, d \leq i \leq k - d, \forall x \in V(G)$, we have $\{x, x'\} \subseteq X_i$ if $x \in X_i$.

**proof** For any vertex $v \in V(M(G)) - \{u\}, v'$ is the twin of $v$.

By Lemma 3, there exists a $(k, d)$-partition satisfying 2 and $u \in X_0$. If $x \neq u, x \in X_0$, and $x' \in X_i, d \leq i \leq k - d, \text{then } x \in V(G)$. We can dye $x$ color $i$. Otherwise, there exists $y \in \delta(x')$ such that $x'y \notin E(M(G)), xy \in E(M(G))$. So $y \notin V(G) \cup \{u\}$, $x'y' \in E(M(G))$. Hence $y$ can’t be dyed any color, this is impossible. Hence $x \notin X_0, X_0 = \{u\}$ and conclusion 1 holds as well. □

**Theorem 1** Let $t \geq 4, n \geq \frac{11}{12}2^{t-1} + 2t + \frac{1}{4}$, then $\chi_c(M^t(K_n)) = \chi(M^t(K_n)) = n + t$.

**proof** Suppose that $\chi_c(M^t(K_n)) = \frac{1}{2}d, d \neq 1$, by Lemma 2

\[
\begin{align*}
    d - 1 & \leq \frac{2t - 2}{n - 3} \leq \frac{2t - 2}{\frac{11}{12}2^{t-1} + 2t - \frac{8}{3}} \leq 3 + \frac{2t - 2 - 3\left(\frac{11}{12}2^{t-1} + 2t - \frac{8}{3}\right)}{\frac{11}{12}2^{t-1} + 2t - \frac{8}{3}} \\
    & = 3 - 3\frac{2t - 3}{\frac{11}{12}2^{t-1} + 2t - \frac{8}{3}} < 3
\end{align*}
\]

holds for $t \geq 3$. So $d \leq 3$.

If $d = 3$, there exists a $(3n + 3t - i, 3)$-coloring of $M^t(K_n), i = 1 \text{ or } 2$. According to box principle, there are at least $5n - (3n + 3t - i) = 2n - 3t + i$ color classes in the intersection sets of $d$-fields of different initial vertices. By Lemma 10, there are at most 3 such color classes in $\delta(u_t)$. So $V(M^t(K_n)) - \delta(u_t)$ have at least $2n - 3t + i - 3$ such color classes in which contains at least one vertex in $R_t(M^t(G)) - \{u_t\}$ respectively. Thus $2n - 3t + i - 3 \leq 2^{t-1} - 1$, i.e.,

\[
n \leq 2^{t-2} + \frac{3}{2}t + \frac{2 - i}{2} \leq 2^{t-2} + \frac{3}{2}t + \frac{1}{2}.
\]

If $t \geq 4, \text{then }

\[
\frac{11}{12}2^{t-1} + 2t + \frac{1}{3} - (2^{t-2} + \frac{3}{2}t + \frac{1}{2}) = \frac{5}{6}2^{t-2} + \frac{1}{2}t - \frac{1}{6} > 0.
\]
Hence of classes in the intersection set of $t$ and has at least one such color class in $\delta$, a contradiction.

Now we discuss such color classes in $V(M^t(K_n)) - \delta(u_t)$. It is easy to see that the number of these color classes is at least $n - 2t$. Suppose that the number of these color classes that contain exactly 1 $t$th root is $y$, the number of those which contain at least 2 $t$th roots is $x$ and the number of $t$th roots which is different from $\{u_t\}$ and not in such intersection sets is $z$, then we have

$$x + y \geq n - 2t, \quad 2x + y + z \leq 2^{t-1} - 1.$$ 

Hence

$$y \geq 2n - 2^{t-1} - 4t + 1 + z, \quad z \leq 2^{t-1} + 2t - n - 1.$$ 

Since $d = 2$, the color classes in the intersection set of $d$-field of different initial vertices are mutually disjoint. Hence, in the set of $y$ classes which contain exactly 1 $t$th root, the number of classes whose $d$-fields contain other $t$th roots is at most $2z$, furthermore there are at most 2 classes whose $d$-fields contain $r$th roots, $1 \leq r \leq t - 1$, $X_2$ and $X_{2n+2t-3}$. Hence, there are at least $y - 2z - 2$ color classes whose $d$-fields contain exactly 1 $t$th root.

By Corollary 1, if $y - 2z - 2 > 3 \cdot 2^{t-3}$, then there exists a directed triple of $F^t_i$ in the set of $t$th roots that contained in these $y - 2z - 2$ color classes, contradict to Lemma 9. Hence, $y - 2z - 2 \leq 3 \cdot 2^{t-3}$, we have

$$(2n - 2^{t-1} - 4t + 1 + z) - 2z - 2 \leq y - 2z - 2 \leq 3 \cdot 2^{t-3}$$

$$\implies 2n - 2^{t-1} - 4t - 1 - z \leq 3 \cdot 2^{t-3}$$

$$\implies 2n - 2^{t-1} - 4t - 1 - (2^{t-1} + 2t - 1 - n) \leq 3 \cdot 2^{t-3}$$

$$\implies 3n \leq 11 \cdot 2^{t-3} + 6t$$

$$\implies n \leq \frac{11}{12} \cdot 2^{t-1} + 2t,$$

contradict to $n \geq \frac{11}{12} 2^{t-1} + 2t + \frac{1}{3}$. So $d \neq 2$.

In other words, if $n \geq \frac{11}{12} 2^{t-1} + 2t + \frac{1}{3}$, then $d = 1$, $\chi_c(M^t(K_n)) = \chi(M^t(K_n)) = n + t$. \hfill \square
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