$p$-Summable Commutators in Bergman Spaces on Egg Domains

Mohammad Jabbari

Received: 8 March 2021 / Accepted: 11 October 2021 / Published online: 30 January 2022
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Abstract

The exact range of the positive real parameter $p$ is determined so that the multiplications by polynomial functions acting on Bergman spaces over generalized complex ellipsoids are essentially $p$-normal.

1 Introduction

Recall that an operator $T$ on a Hilbert space is (Schatten) $p$-summable, $0 < p < \infty$, if $(TT^*)^{p/2}$ is trace class [24][Chapter 11], [30,58]. (Equivalently, if the sequence of singular values $s_j = \inf\{|\|T - F\|| : \text{rank}(F) \leq j\}$ belongs to $l^p$.) A commuting tuple $(T_1, \ldots, T_m)$ of operators on a Hilbert space $\mathcal{H}$ is called $p$-essentially normal if all of the commutators $[T_j, T_k^*], j, k = 1, \ldots, m$ are $p$-summable. Alternatively, $p$-essential normality can be attributed to the Hilbert $\mathbb{C}[z_1, \ldots, z_m]$-module generated by $(T_1, \ldots, T_m)$, namely, $\mathcal{H}$ with the module action $P(z_1, \ldots, z_m) \cdot f, P \in \mathbb{C}[z_1, \ldots, z_m], f \in \mathcal{H}$ given by $P(T_1, \ldots, T_m) f$. In noncommutative geometry, $p$-essential normality is important in the study of differentiable structures [21–23,31] as well as the construction of the Chern-Connes character for noncommutative spaces [14,15].

This article is about the $p$-essential normality of the multiplications by the coordinate functions acting on the Bergman space $L^2_\alpha$ of square-integrable holomorphic functions over the egg domains of the form

$$\Omega_1 := \left\{ \sum_{j=1}^{m} |z_j|^{2p_j} < 1 \right\} \subseteq \mathbb{C}^m, \quad p_j > 0, \quad (1)$$

Communicated by Heinrich Begehr.

Mohammad Jabbari
mohammad.jabbari@cimat.mx

1 Centro de Investigacion en Matematicas, A.P. 402, Guanajuato, 36000 Gto, Mexico
or more generally of the form
\[
\Omega_2 := \left\{ \sum_{k=1}^{K} \left( \sum_{j=1}^{j_k} |z_{jk}|^{2p_{jk}} \right)^{a_k} \leq C^{j_1+\ldots+j_K}, \quad p_{jk}, a_k > 0. \right\}
\]

(When all \( p_{jk} \) equal 1, \( \Omega_2 \) is called a generalized complex ellipsoid in [37,38].) Note that \( \Omega_2 \) is always pseudoconvex (because it is a logarithmically convex, complete Reinhard domain [52, Theorem 3.28]), but, assuming all \( a_k \geq 1 \), it is strongly pseudoconvex with \( C^2 \) boundary exactly when all \( p_{jk}, a_k \) equal 1. (Compare [17].)

To put the main result of this paper in a proper context, a brief survey of the literature related to the \( p \)-essential normality of the usual Hilbert modules of analytic functions is given. In what follows, module actions are always given by multiplication with polynomial functions. First, note that the \( p \)-essential normality of, say, a Bergman module is closely related to the Schatten membership of the Hankel operators \( H_f := (I - P)M_f P \) as well as the commutators \( C_f := [M_f, P] \) associated to coordinate functions \( f \). (Here, \( M_f \) is the multiplication by \( f \), and \( P \) is the orthogonal projection in \( L^2 \) onto \( L^2_a \), the so-called Bergman projection.) More precisely, the formal identities
\[
[M_{z_j}, M_{z_k}^*] = C_{z_j} M_{z_k},
\]
show that the Bergman module is \( p \)-essentially normal if all commutators \( C_f \) associated to coordinate functions are \( p \)-summable, and that \( C_f \) is \( p \)-summable if and only if both \( H_f \) and \( H_f^* \) are so. (Obviously, \( H_f \) is zero for holomorphic functions \( f \).) Here is our survey:

- The Bergman module on the unit ball of \( \mathbb{C}^m \) is \( p \)-essentially normal exactly when \( p > 1/2 \) and \( m = 1 \), or \( p > m > 1 \) [2,4]. The \( m \)-shift (or Drury-Arveson) module \( H^2_m \) is \( p \)-essentially normal if and only if \( p > 0 \) and \( m = 1 \), or \( p > m > 1 \) [5].
- The Hardy module on a smoothly bounded, strongly pseudoconvex domain is \( p \)-essentially normal if \( p \) is strictly larger than the complex dimension of the domain [10, Theorem 13.1], [26, Proposition 1].
- The Schatten membership of the commutators \( C_f \), as well as the Hankel operators, are studied in:
  - [2–4, 34, 36, 45, 51, 54, 62–64, 67, 69] for the Bergman space on the unit ball.
  - [66] for the Bergman space on bounded symmetric domains.
  - [27, 28, 46–49, 53, 57, 68] for the Hardy spaces on the unit sphere.
  - [41, 42, 50] for the Hardy space on smoothly bounded, strongly pseudoconvex domains.
  - [33, 65] for the Segal–Bargmann (or Fock) space.
  - [8] for \( P \) being a Calderon-Zygmund operator associated to a homogeneous space.
• The essential normality of the Bergman module on a bounded, pseudoconvex domain is equivalent to the compactness of the \( \partial \)-Neumann operator \( N_1 \) on \((0,1)\)-forms with \( L^2 \) coefficients [13,29,55,56]. (The essential normality means that the \( \mathbb{C}^* \)-algebra generated by multiplications with coordinate functions is commutative modulo the ideal of compact operators.) Several sufficient conditions for this are given in the literature [11,12,32,44]. For example, strongly pseudoconvex domains, domains of finite type, and pseudoconvex domains with real analytic boundary have compact \( N_1 \). Other approaches to the essential normality of Bergman modules are given in [6,7,9,16,39,56,61].

The first main result of this article is:

**Theorem 1** The Bergman module on the domain \((1)\) is \( p \)-essentially normal if and only if

\[
p > \begin{cases} \frac{1}{2}, & m = 1, \\ \max \left\{ m, p_j(m - 1) : j = 1, \ldots, m \right\}, & m > 1. \end{cases}
\]

This result was first stated without proof in [35]. Our next result generalizes this:

**Theorem 2** The Bergman module on the domain \( \Omega_2 \) given in \((2)\) is \( p \)-essentially normal if and only if

\[
p > \begin{cases} \frac{1}{2}, & d = 1, \\ \max \left\{ d, q_1, \ldots, q_K \right\}, & d > 1, \end{cases}
\]

where

\[ d = j_1 + \cdots + j_K \]

is the complex dimension of \( \Omega_2 \), and for each \( k = 1, \ldots, K \),

\[
q_k = \begin{cases} \max \left\{ a_k p_{j_k}(d - j_k) : j = 1, \ldots, j_k \right\}, & j_k = 1, \\ \max \left\{ p_{j_k}(d - 1), a_k p_{j_k}(d - j_k) : j = 1, \ldots, j_k \right\}, & j_k > 1 = a_k, \\ \max \left\{ p_{j_k}(d - 1), a_k p_{j_k}(d - j_k), \frac{2a_k(d - j_k)}{1/p_{j_k} + 1/j_{l_k}} : j, l = 1, \ldots, j_k, j \neq l \right\}, & j_k > 1 \neq a_k. \end{cases}
\]

An interesting phenomenon appearing here (already observed in [40]; see also [35]) is that for weakly pseudoconvex domains, in contrast to strongly pseudoconvex ones, the cut-off values for the \( p \)-summability of the Bergman modules depend not only on the dimensions of the domains but also on their boundary geometry. In our case of the egg domains of type \((1)\) or \((2)\), this geometry is manifested in the maximum order of contact of the boundary with complex analytic curves (in the sense of D’Angelo [18,19]).

This paper is organized as follows: I gather several facts needed in the proof of the main results in Sect. 2. Theorems 1 and 2 are respectively proved in Sects. 3 and 4.
2 Preliminaries

This section provides some facts needed in the proof of Theorems 1 and 2.

Lemma 3 Given positive real numbers $a$, $b$, and real variable $x$, we have

\[
\frac{\Gamma(x + a)}{\Gamma(x + b)} x^{b-a} = 1 + \frac{(a-b)(a+b-1)}{2x} + \frac{(a-b)(a-b-1)}{24x^2} + O\left(\frac{1}{x^3}\right),
\]

\[
\frac{\Gamma(x + a)^2}{\Gamma(x + a + 2b)} = 1 - \frac{a^2}{x} + \frac{a^2(a^2 + 2a - 1)}{2x^2} + O\left(\frac{1}{x^3}\right),
\]

\[
\frac{\Gamma(x + a)\Gamma(x + 2a + b)}{\Gamma(x + a + b)\Gamma(x + 2a)} = 1 + \frac{ab}{x} + \frac{ab(ab - 3\alpha - b + 1)}{x^2} + O\left(\frac{1}{x^3}\right),
\]

\[
\frac{(x + a)^2}{x(x + 2a)} = 1 + \frac{a^2}{x^2} + O\left(\frac{1}{x^3}\right),
\]

\[
\frac{(x + a)(x + 2a + b)}{(x + a + b)(x + 2a)} = 1 - \frac{ab}{x^2} + O\left(\frac{1}{x^3}\right).
\]

as $x \to \infty$.

**Proof** The first formula is proved in [60] as well as [1, Appendix C]. The next two are immediate from it. 

Lemma 4 Let $\sum a_j$ and $\sum b_j$ be two series with nonnegative terms, and let $p > 0$. Then $\sum (a_j + b_j)^p$ converges if and only if both $\sum a_j^p$ and $\sum b_j^p$ converge.

**Proof** Immediate from

\[
\max\left\{a_j^p, b_j^p\right\} \leq (a_j + b_j)^p \leq (2 \max\{a_j, b_j\})^p \leq 2^p \left(a_j^p + b_j^p\right).
\]

The proof of Theorem 1 depends on the following fact about the convergence of higher zeta series. (Compare [43,59].) In what follows, $\mathbb{N}$ and $\mathbb{N}_+$ respectively denote the set of nonnegative and positive integers.

Lemma 5 Suppose positive integer $m$ and real numbers $b, a_1, \ldots, a_m$. Then: (a) The series

\[
A := \sum_{i \in \mathbb{N}_+^m} \frac{i_1^{a_1} \cdots i_m^{a_m}}{(i_1 + \cdots + i_m)^b}
\]

converges if and only if

\[
b > \max\left\{\sum_{j \in J} (a_j + 1) : \emptyset \neq J \subseteq \{1, \ldots, m\}\right\}. \quad (3)
\]

Specially, if all $a_j$ are $\geq -1$, then $A$ converges if and only if

\[
b > a_1 + \cdots + a_m + m.
\]
(b) Given positive integer \( k \) and real number \( a \), the series

\[
B := \sum_{i \in \mathbb{N}^{m+k}} i_1^{a_1} \cdots i_m^{a_m} (i_{m+1} + \cdots + i_{m+k})^a \quad (i_1 + \cdots + i_{m+k})^b
\]

converges if and only if the series

\[
\sum_{i \in \mathbb{N}^{m+1}} i_1^{a_1} \cdots i_m^{a_m} i_{m+1}^{a+k-1} \quad (i_1 + \cdots + i_{m+1})^b
\]

converges, if and only if

\[
b > \max \left\{ \sum_{j \in J} (a_j + 1), a + k, a + k + \sum_{j \in J} (a_j + 1) : \emptyset \neq J \subseteq \{1, \ldots, m\} \right\}.
\]

**Proof** (a) We prove by induction on \( m \) that \( A < \infty \) if and only if (3) holds, if and only if

\[
A' := \sum_{i \in \mathbb{N}^{m-1}_+} i_1^{a_1} \cdots i_m^{a_m} \log(i_1 + \cdots + i_m) \quad (i_1 + \cdots + i_m)^b < \infty.
\]

The case \( m = 1 \) is clear, so assume \( m > 1 \). We split the proof into two cases. Case I: All \( a_j \) are nonnegative. In this case, \( A \) is dominated by \( \sum (i_1 + \cdots + i_m)^{-c} \), where \( c = b - \sum a_j \). This latter series equals \( \sum_{N \in \mathbb{N}^+} N^{-c} \mu \), where \( \mu = \binom{N-1}{m-1} \) is the number of tuples \( (i_1, \ldots, i_m) \) of positive integers that sum up to \( N \). Since \( \mu \) grows like \( N^{m-1} \) when \( N \to \infty \), it follows that \( A \) converges if \( c > m \), namely if \( b > \sum a_j + m \). Conversely, we show that \( A \) diverges when \( b = \sum a_j + m \). Since \( b > a_m \), the general summand of \( A \) is decreasing with respect to \( i_m \) provided that \( i_m \geq r (i_1 + \cdots + i_{m-1}) \), where \( r = a_m / (b - a_m) > 0 \). By the integral test, it suffices to show that the series

\[
A_1 := \sum_{i \in \mathbb{N}^{m-1}_+} i_1^{a_1} \cdots i_{m-1}^{a_{m-1}} \int_{r(i_1+\cdots+i_{m-1})+1}^\infty \frac{x^{a_m}}{(i_1 + \cdots + i_{m-1} + x)^b} dx
\]

diverges. After the change of variables \( x = (i_1 + \cdots + i_{m-1}) t \),

\[
A_1 = \sum_{i \in \mathbb{N}^{m-1}_+} i_1^{a_1} \cdots i_{m-1}^{a_{m-1}} \int_{r+1}^{\infty} \frac{t^{a_m}}{(t+1)^b} dt \geq \sum_{i \in \mathbb{N}^{m-1}_+} i_1^{a_1} \cdots i_{m-1}^{a_{m-1}} \int_{r+1}^{\infty} \frac{t^{a_m}}{(t+1)^b} dt.
\]
By the induction hypothesis, $A_1$ diverges. So far, we have shown that $A < \infty$ if and only if (3) holds. If (3) holds, choosing $\epsilon > 0$ such that (3) still holds when $b$ is replaced by $b - \epsilon$, then $A'$ converges because it is dominated by the convergent series

$$\sum_{i \in \mathbb{N}^m} i_1^{a_1} \cdots i_m^{a_m} (i_1 + \cdots + i_m)^{b - \epsilon}.$$ 

Clearly, the convergence of $A'$ implies the convergence of $A$.

Case II: At least one $a_j$ is negative. Without loss of generality suppose $a_m < 0$.

First assume that $A$ converges. Then, by the integral test,

$$A_2 := \sum_{i \in \mathbb{N}^{m-1}} i_1^{a_1} \cdots i_{m-1}^{a_{m-1}} \int_1^{\infty} \frac{x^{a_m} dx}{(i_1 + \cdots + i_{m-1} + x)^b} < \infty.$$ 

After the change of variables $x = (i_1 + \cdots + i_{m-1})t$,

$$A_2 = \sum_{i \in \mathbb{N}^{m-1}} i_1^{a_1} \cdots i_{m-1}^{a_{m-1}} \int_{\frac{1}{i_1 + \cdots + i_{m-1}}}^{\infty} \frac{t^{a_m} dt}{(t + 1)^b}.$$ 

Since

$$A_2 \geq \sum_{i \in \mathbb{N}^{m-1}} i_1^{a_1} \cdots i_{m-1}^{a_{m-1}} \int_{\frac{1}{i_1 + \cdots + i_{m-1}}}^{\infty} \frac{t^{a_m} dt}{(t + 1)^b},$$

it follows that both

$$A_3 := \sum_{i \in \mathbb{N}^{m-1}} i_1^{a_1} \cdots i_{m-1}^{a_{m-1}}$$

and

$I := \int_{1}^{\infty} \frac{t^{a_m} dt}{(t + 1)^b}$

converge. The integral $I$ converges exactly when

$$b > a_m + 1. \quad (4)$$

By the induction hypothesis and Case I, $A_3$ converges exactly when

$$b > \max \left\{ a_m + 1 + \sum_{j \in J} (a_j + 1) : J \subseteq \{1, \ldots, m-1\} \right\}. \quad (5)$$

Therefore, the convergence of $A_2$ implies the convergence of

$$A_4 := \sum_{i \in \mathbb{N}^{m-1}} i_1^{a_1} \cdots i_{m-1}^{a_{m-1}} \int_{\frac{1}{i_1 + \cdots + i_{m-1}}}^{1} \frac{t^{a_m} dt}{(t + 1)^b}.$$
Since \((t + 1)^{-b}\) is bounded between \(2^{-b}\) and 1 when \((i_1 + \cdots + i_{m-1})^{-1} \leq t \leq 1\), it follows that the convergence of \(A_4\) is equivalent to the convergence of

\[
A_5 := \sum \frac{i_1^{a_1} \cdots i_{m-1}^{a_{m-1}}}{(i_1 + \cdots + i_{m-1})^{b-a_{m-1}}} \int_1^{1/(i_1 + \cdots + i_{m-1})} t^{a_m} \, dt.
\]

According to whether \(a_m < -1\), \(a_m = -1\), or \(a_m > -1\), the convergence of \(A_5\) is respectively equivalent to the convergence of the series:

\[
\sum \frac{i_1^{a_1} \cdots i_{m-1}^{a_{m-1}}}{(i_1 + \cdots + i_{m-1})^b},
\]

\[
\sum \frac{i_1^{a_1} \cdots i_{m-1}^{a_{m-1}} \log(i_1 + \cdots + i_{m-1})}{(i_1 + \cdots + i_{m-1})^b},
\]

or

\[
\sum \frac{i_1^{a_1} \cdots i_{m-1}^{a_{m-1}}}{(i_1 + \cdots + i_{m-1})^{b-a_{m-1}}}.
\]

By the induction hypothesis and Case I, it follows that the convergence of \(A_2\) implies ((4) and (5)) together with

\[
b > \max \left\{ \sum_{j \in J} (a_j + 1) : J \subseteq \{1, \ldots, m - 1\} \right\}
\]

whenever \(a_m < -1\).

Similarly, ((4) and (5) and (6)) is logically equivalent to (3). Conversely, assume that (3) holds. Reversing the arguments above shows that \(A_2\) converges. Then, by the integral test,

\[
A_6 := \sum_{i \in \mathbb{N}^{m-1}, i_m > 1} \frac{i_1^{a_1} \cdots i_m^{a_m}}{(i_1 + \cdots + i_{m-1})^b} < \infty.
\]

Note that

\[
A = A_6 + \sum_{i \in \mathbb{N}^{m-1}} \frac{i_1^{a_1} \cdots i_{m-1}^{a_{m-1}}}{(i_1 + \cdots + i_{m-1} + 1)^b} \leq A_6 + \sum_{i \in \mathbb{N}^{m-1}} \frac{i_1^{a_1} \cdots i_{m-1}^{a_{m-1}}}{C^b(i_1 + \cdots + i_{m-1})^b},
\]

where \(C\) equals 1 if \(b > 0\), and equals 2 otherwise. Since the second series converges by the induction hypothesis and Case I, it follows that \(A < \infty\). We have shown that \(A < \infty\) if and only if (3) holds. Similar to Case I, one can show that (3) is equivalent to \(A' < \infty\).
(b) Note that
\[ B = \sum_{i \in \mathbb{N}_+^{m+1}, N \in \mathbb{N}_+} \frac{i_1^{a_1} \cdots i_m^{a_m} N^\mu \mu}{(i_1 + \cdots + i_m + N)^b}, \]
where \( \mu = \binom{N-1}{k-1} \) is the number of tuples \((i_{m+1}, \ldots, i_{m+k})\) of positive integers that sum up to \( N \). To study the convergence, one can replace \( \mu \) by \( N^{k-1} \), and the result follows from (a).

During the proof of Theorem 2, we will need to know exactly when each of the following higher zeta series (beside the ones appearing in Lemma 5) converge:

\[
\sum_{i_1,j_1} \cdots \sum_{i_m,j_m} \frac{i_1^{a_1} \cdots i_m^{a_m}}{(i_1 + \cdots + i_m)^b},
\]

\[
\sum_{i_1,j_1} \cdots \sum_{i_m,j_m} \frac{i_1^{a_1} \cdots i_m^{a_m}}{(i_1 + \cdots + i_m)^b},
\]

\[
\sum_{i_1,j_1} \cdots \sum_{i_m,j_m} \frac{i_1^{a_1} \cdots i_m^{a_m} \cdot \text{something}}{(i_1 + \cdots + i_m)^b},
\]

Here, all parameters \( b, a_1, a_2, \ldots \) are real numbers. The following two lemmas solve this problem. Their proof is modeled on the arguments in [59, Lemma 2].

**Lemma 6** Let \( L \) be a finite collection of linear functionals \( l = l(i_1, \ldots, i_m) : \mathbb{N}_+^m \to \mathbb{R} \) with nonnegative coefficients \( \partial l / \partial i_j, j = 1, \ldots, m \). Suppose that to each \( l \in L \), there associates a real number \( a \). Let \( b \) be a real number. If the infinite series

\[
S := \sum_{i \in \mathbb{N}_+^m} \frac{\prod_{l \in L} l(i_1, \ldots, i_m)^a}{(i_1 + \cdots + i_m)^b}
\]

converges, then

\[
b > \max \left\{ |J| + \sum_{J \sim l, l \in L} a : \emptyset \neq J \subseteq \{1, \ldots, m\} \right\},
\]

where \( |J| \) is the cardinality of \( J \), and the notation \( J \sim l \) indicates that \( \partial l / \partial i_j \neq 0 \) for some \( j \in J \). The same is true if one \( l(i_1, \ldots, i_m) \) is replaced by \( | - i_1 + i_2 + \cdots + i_m | \).

**Proof** We argue by induction on \( m \). Since the case \( m = 1 \) is clear, assume \( m > 1 \). Note that (7) can be written as

\[
b > \max\{b_1, b_1 + b_{12}, b_2\},
\]
where

\[b_1 := 1 + \sum_{\{1\} \sim I} a,\]

\[b_{12} := \max \left\{ |J| + \sum_{J \sim I, \{1\} \sim I} a : \emptyset \neq J \subseteq \{2, \ldots, m\} \right\},\]

\[b_2 := \max \left\{ |J| + \sum_{J \sim I} a : \emptyset \neq J \subseteq \{2, \ldots, m\} \right\}.\]

Define the function

\[\chi : \mathbb{R} \to \mathbb{R}, \quad \chi(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 2, & \text{if } x < 0. \end{cases}\]  

(9)

The convergence of \(S\) implies the convergence of

\[\sum_{i_1 \geq 1 = i_2 = \cdots = i_m} \prod l(i_1, \ldots, i_m)^a \over (i_1 + \cdots + i_m)^b\]  

(10)

as well as

\[\sum_{i_1 \geq i_2 + \cdots + i_m} \prod l(i_1, \ldots, i_m)^a \over (i_1 + \cdots + i_m)^b\]  

(11)

The convergence of the series (10) implies \(b > b_1\). Note that on the region \(i_1 \geq i_2 + \cdots + i_m\), two quantities \(i_1\) and \(i_1 + \cdots + i_m\) are comparable in the sense that

\[\frac{i_1 + \cdots + i_m}{2} \leq i_1 \leq i_1 + \cdots + i_m.\]  

(12)

Since \(b > b_1\), this shows that the series (11) dominates

\[\sum_{i_1 \geq i_2 + \cdots + i_m} \prod l \left( \frac{i_1 + \cdots + i_m}{\chi(-a)}, i_2, \ldots, i_m \right)^a \over (i_1 + \cdots + i_m)^b l_i \geq \sum_{i_1 \geq i_2 + \cdots + i_m} \prod_{\{1\} \sim I} l^a \over (i_1 + \cdots + i_m)^{b-b_1+1} \]

\[\geq \sum_{i_2, \ldots, i_m \geq 1} \int_{i_2 + \cdots + i_m}^{\infty} \prod_{\{1\} \sim I} l^a \over (i_1 + \cdots + i_m)^{b-b_1+1} di_1 \cong \sum_{i_2, \ldots, i_m \geq 1} \prod_{\{1\} \sim I} l^a \over (i_2 + \cdots + i_m)^{b-b_1+1}.\]

Here, \(A \gtrsim B\) means that \(A\) is greater than \(B\) multiplied by a positive constant depending only on the parameters \(a, b\); and, \(A \cong B\) means \(A \gtrsim B \cong A\). Therefore, by the induction hypothesis, \(b - b_1 > b_{12}\). We have shown that part of (7) which contains the contribution from \(1 \in J\). Similar arguments (namely, summing over each of the regions \(i_j \geq \sum_{k \neq j} i_k, j = 2, \ldots, m\) instead of \(i_1 \geq \sum_{k=2}^m i_k\)) imply \(b > b_2\).
Finally, assume that one \( I(i_1, \ldots, i_m) \) is of the form \(| - i_1 + i_2 + \cdots + i_m |\). The proof above works after making the following modifications:

1 In the proof of \( b > b_1 + b_{12} \), replace the region \( i_1 \geq i_2 + \cdots + i_m \) with \( i_1 \geq 3(i_2 + \cdots + i_m) \), and the comparability estimate (12) with

\[
\frac{i_1 + 3(i_2 + \cdots + i_m)}{2} \leq i_1 \leq i_1 + 2(i_2 + \cdots + i_m).
\]

2 In the proof of \( b > b_2 \), for each \( j = 2, \ldots, m \), replace the region \( i_j \geq \sum_{k \neq j} i_k \) with \( i_j \geq 3 \sum_{k \neq j} i_k \), and the comparability estimate (12) with

\[
\frac{i_j + 3 \sum_{k \neq j} i_k}{2} \leq i_j \leq i_j + 2 \sum_{k \neq j} i_k.
\]

The proof is complete. \( \square \)

Lemma 7 The converse of Lemma 6 holds if \( S \) is any of the following series:

(a) \[
A := \sum_{i \in \mathbb{N}^4_+} \frac{i_1^{a_1} i_2^{a_2} i_3^{a_3} (i_1 + i_2)^{a_{12}}}{(i_1 + i_2 + i_3)^b}.
\]

(b) \[
B := \sum_{i \in \mathbb{N}^4_+} \frac{i_1^{a_1} i_2^{a_2} i_3^{a_3} i_4^{a_4} (i_1 + i_2)^{a_{12}}}{(i_1 + i_2 + i_3 + i_4)^b}.
\]

(c) \[
C := \sum_{i \in \mathbb{N}^4_+} \frac{i_1^{a_1} i_2^{a_2} i_3^{a_3} i_4^{a_4} (i_1 + i_2 + i_3)^{a_{123}}}{(i_1 + i_2 + i_3 + i_4)^b}.
\]

(d) \[
D := \sum_{i \in \mathbb{N}^5_+} \frac{i_1^{a_1} i_2^{a_2} i_3^{a_3} i_4^{a_4} i_5^{a_5} (i_1 + i_2 + i_3 + i_4)^{a_{1234}}}{(i_1 + i_2 + i_3 + i_4 + i_5)^b}, \quad a_{1234} > 0.
\]

(e) \[
E := \sum_{i \in \mathbb{N}^4_+} \frac{i_1^{a_1} i_2^{a_2} i_3^{a_3} i_4^{a_4} (i_1 + i_2)^{a_{12}} (i_3 + i_4)^{a_{34}}}{(i_1 + i_2 + i_3 + i_4)^b}, \quad a_1 + a_{12} > -1.
\]

(f) \[
F := \sum_{i \in \mathbb{N}^5_+} \frac{i_1^{a_1} i_2^{a_2} i_3^{a_3} i_4^{a_4} i_5^{a_5} (i_1 + i_2)^{a_{12}} (i_3 + i_4)^{a_{34}}}{(i_1 + i_2 + i_3 + i_4 + i_5)^b}, \quad a_1 + a_{12} > -1.
\]
Proof (a) We should show that $A$ converges assuming
\[ b > \max\{b_1, b_2, b_1 + b_2\}, \tag{13} \]
where
\[ b_1 := a_3 + 1, \]
\[ b_2 := \max\{a_1 + a_{12} + 1, a_2 + a_{12} + 1, a_1 + a_2 + a_{12} + 2\}. \]

Since
\[ A \leq A_1 + A'_2 + A''_2, \]
\[ A_1 := \sum_{i_3 \geq i_1 + i_2} \frac{i_1^{a_1} i_2^{a_2} i_3^{a_3} (i_1 + i_2)^{a_{12}}}{(i_1 + i_2 + i_3)^b}, \]
\[ A'_2 := \sum_{1 \leq i_3 \leq i_1 + i_2} \frac{i_1^{a_1} i_2^{a_2} i_3^{a_3} (i_1 + i_2)^{a_{12}}}{(i_1 + i_2 + i_3)^b}, \]
\[ A''_2 := \sum_{1 < i_3 \leq i_1 + i_2} \frac{i_1^{a_1} i_2^{a_2} i_3^{a_3} (i_1 + i_2)^{a_{12}}}{(i_1 + i_2 + i_3)^b}, \]
it suffices to show that $A_1, A'_2$ and $A''_2$ converge. Since $b > b_1$,
\[
A_1 \leq \sum_{i_3 \geq i_1 + i_2} \frac{i_1^{a_1} i_2^{a_2} (i_1 + i_2)^{a_{12}} (i_1 + i_2 + i_3)^{a_3}}{(i_1 + i_2 + i_3)^b} \leq \sum_{i_1, i_2 \geq 1} \int_{i_1 + i_2 - 1}^{\infty} \frac{i_1^{a_1} i_2^{a_2} (i_1 + i_2)^{a_{12}}}{(i_1 + i_2 + i_3)^b} d_i_3
\]
\[
\lesssim \sum \frac{i_1^{a_1} i_2^{a_2}}{(i_1 + i_2)^{b - b_1 - a_{12}}}. \]

Therefore, $A_1 < \infty$ according to Lemma 5.(a) and because $b > b_1 + b_2$. We have $A'_2 < \infty$ because $b > b_2$. The series $A''_2$ is dominated by
\[
\sum_{1 < i_3 \leq i_1 + i_2} \frac{i_1^{a_1} i_2^{a_2} (i_1 + i_2)^{a_{12}} (i_1 + i_2 + i_3)^{a_3}}{(\chi(b)(i_1 + i_2))^b} \lesssim \sum_{i_1, i_2 \geq 1} \int_{i_1}^{i_1 + i_2 + 1} \frac{i_1^{a_1} i_2^{a_2} i_3^{a_3}}{(i_1 + i_2)^{b - a_{12}} d_i_3}. \]

Therefore, if $a_3 > -1$, then $A''_2$ is dominated by
\[
\sum \frac{i_1^{a_1} i_2^{a_2}}{(i_1 + i_2)^{b - a_{12} - b_1}}, \]
and this latter series converges according to Lemma 5.(a) and because $b > b_1 + b_2$. If $a_3 < -1$, then $A''_2$ is dominated by
\[
\sum \frac{i_1^{a_1} i_2^{a_2}}{(i_1 + i_2)^{b - a_{12}}}, \]
and this latter series converges according to Lemma 5.(a) and because $b > b_2$. If $a_3 = -1$, then $A_p'$ is dominated by

$$
\sum \frac{i_1^a i_2^b \log(i_1 + i_2)}{(i_1 + i_2)^{b-a_12}} \lesssim \sum \frac{i_1^a i_2^b}{(i_1 + i_2)^{b-a_{12} - \varepsilon}},
$$

where we have chosen $\varepsilon > 0$ small enough such that (13) holds for $b$ replaced by $b - \varepsilon$. This latter series is finite according to Lemma 5.(a) and because $b - \varepsilon > b_2$.

(b, c) The series $B$ and $C$ are treated similar to $A$, now summing over regions $i_4 \geq i_1 + i_2 + i_3$, $1 = i_4 \leq i_1 + i_2 + i_3$, and $1 < i_4 \leq i_1 + i_2 + i_3$.

(d) Since $a_{1234} > 0$, the series $D$ is dominated by

$$
\sum \frac{i_1^a i_2^b i_3^a i_4^b i_1 + i_2 + i_3 + i_4)^{b-a_{1234}}}{(i_1 + i_2 + i_3 + i_4)^{b-a_{1234}}},
$$

and we are reduced to (c).

(e) Assume that the condition (7) holds for the series $E$. If $a_{12} \geq 0$, then, by Lemma 4, it suffices to show that both of the following series converge:

$$
\sum \frac{i_1^a i_2^b i_3^a i_4^b (i_1 + i_2 + i_3 + i_4)^{a_{12}}}{(i_1 + i_2 + i_3 + i_4)^{b}},
$$

$$
\sum \frac{i_1^a i_2^b i_3^a i_4^b (i_1 + i_2 + i_3 + i_4)^{a_{12}}}{(i_1 + i_2 + i_3 + i_4)^{b}}.
$$

Both of these series converge by (b). Now, assume $a_{12} < 0$. Since

$$
E \leq E_1 + E_1' + E_1'',
$$

$$
E_1 := \sum_{i_1 \geq i_2 + i_3 + i_4} \frac{i_1^a i_2^b i_3^a i_4^b (i_1 + i_2 + i_3 + i_4)^{a_{12}}}{(i_1 + i_2 + i_3 + i_4)^{b}},
$$

$$
E_1' := \sum_{1 = i_1 \leq i_2 + i_3 + i_4} \frac{i_1^a i_2^b i_3^a i_4^b (i_1 + i_2 + i_3 + i_4)^{a_{12}}}{(i_1 + i_2 + i_3 + i_4)^{b}},
$$

$$
E_1'' := \sum_{1 < i_1 \leq i_2 + i_3 + i_4} \frac{i_1^a i_2^b i_3^a i_4^b (i_1 + i_2 + i_3 + i_4)^{a_{12}}}{(i_1 + i_2 + i_3 + i_4)^{b}},
$$

...
it suffices to show that $E_1$, $E_2'$ and $E_2''$ converge. That $E_1$ and $E_2'$ converge can be proved by similar arguments given in (a) for the convergence of $A_1$ and $A_2'$, respectively. Since $a_{12} < 0$ and $a_1 + a_{12} > -1$, $E_2''$ is dominated by

$$
\sum_{1 < i_1 \leq i_2 + i_3 + i_4} \left\langle \left( i_1 + i_2 + 1 \right)^a (i_3 + i_4)^b \right\rangle \lesssim \sum_{1 < i_1 \leq i_2 + i_3 + i_4} \left\langle \left( i_1 + i_2 + 1 \right)^a (i_3 + i_4)^b \right\rangle \chi(b)(i_2 + i_3 + i_4)^b \lesssim \sum_{i_2, i_3, i_4 \geq 1} \left\langle \left( i_2 + i_3 + i_4 \right)^b - a_1 - a_{12} - 1 \right\rangle.
$$

The last series converges by (b).

(f) Arguing as in (a), namely summing over regions $i_5 \geq i_1 + i_2 + i_3 + i_4, 1 = i_5 \leq i_1 + i_2 + i_3 + i_4$, and $1 < i_5 \leq i_1 + i_2 + i_3 + i_4$, reduces us to (e). □

**Remark 8** The author guesses that the converse of Lemma 6 is true in general, but he has not yet been able to prove this. Lemma 7 is enough to prove Theorem 2.

## 3 Proof of Theorem 1

This section proves Theorem 1 about the $p$-essential normality of the Bergman module $L^2_a(\Omega_1)$ over the domain $\Omega_1$ given in (1). Since $\Omega_1$ is a complete Reinhardt domain, polynomials are dense in $L^2_a(\Omega_1)$ with respect to the topology of uniform convergence on compact subsets [52, Page 47]. Then a standard shrinking argument ([68][Page 43], [25, Page 11]) shows that the normalized monomials

$$
b_i := \frac{z^i}{\sqrt{\omega_1(i)}}, \quad i \in \mathbb{N}^m,
$$

where

$$
\omega_1(i) := \|z^i\|^2_{L^2_a(\Omega_1)},
$$

constitute an orthonormal basis for the Hilbert space $L^2_a(\Omega_1)$. An explicit formula for the norm of monomials is given by:

**Proposition 9** Given multi-index $i \in \mathbb{N}^m$, we have

$$
\omega_1(i) = \frac{\pi^m}{\prod p_j} \frac{B \left( \frac{i+1}{p} \right)}{\left\lfloor \frac{i+1}{p} \right\rfloor}.
$$
where \( \frac{i+1}{p} = \left( \frac{i_1+1}{p_1}, \ldots, \frac{i_m+1}{p_m} \right), \left\| \frac{i+1}{p} \right\| = \sum_{j=1}^{m} \frac{i_j+1}{p_j}, \) and

\[
B \left( \frac{i+1}{p} \right) = \frac{\prod_{j=1}^{m} \Gamma \left( \frac{i_j+1}{p_j} \right)}{\Gamma \left( \sum_{j=1}^{m} \frac{i_j+1}{p_j} \right)}
\]

is the multi-variable Beta function.

Proof \cite{[20]} or \cite{[35]}. \qed

Theorem 1 follows immediately from:

Proposition 10 For each coordinate function \( f = z_j \), let \( M_f : L^2_\alpha(\Omega_1) \to L^2_\alpha(\Omega_1) \) be the multiplication by \( f \). Then:

(a) Given \( i \in \mathbb{N}^m \), we have

\[
\left[ M_{z_1}, M^*_z \right] (b_i) = \lambda b_i,
\]

\[
\sqrt{\left[ M_{z_2}, M^*_z \right] \left[ M_{z_2}, M^*_z \right]^*} (b_i) = \mu b_i,
\]

where

\[
\lambda = \frac{\omega(i_1, i_2, \ldots, i_m)}{\omega(i_1 - 1, i_2, \ldots, i_m)} \delta(i_1) - \frac{\omega(i_1 + 1, i_2, \ldots, i_m)}{\omega(i_1, i_2, \ldots, i_m)} \delta(i_1),
\]

\[
\mu = \left| \sqrt{\frac{\omega(i_1, \ldots, i_m) \omega(i_1 + 1, i_2 - 1, i_3, \ldots, i_m) \omega(i_1, i_2, i_3, \ldots, i_m) \omega(i_1, i_1 + 1, i_2 - 1, i_3, \ldots, i_m)}{\omega(i_1 - 1, i_2, \ldots, i_m) \omega(i_1 - 1, i_2, \ldots, i_m) \omega(i_1, i_1 + 1, i_2 - 1, i_3, \ldots, i_m)} \delta(i_2)} \right|
\]

and

\[
\delta(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{otherwise}. \end{cases}
\]

(b) \( \left[ M_{z_1}, M^*_z \right] \) is \( p \)-summable if and only if

\[
p > \begin{cases} \frac{1}{2}, & \text{if } \dim \Omega_1 = 1, \\ \max \{ \dim \Omega_1, p_1(\dim \Omega_1 - 1) \}, & \text{if } \dim \Omega_1 > 1, \end{cases}
\]

where \( \dim \Omega_1 = m \) is the complex dimension of \( \Omega_1 \).

(c) Assume \( \dim \Omega_1 > 1 \). Then \( \left[ M_{z_2}, M^*_z \right] \) is \( p \)-summable if and only if \( p > \dim \Omega_1 \).

Proof (a) Straightforward computations.
(b) We can write
\[
\lambda = \lambda'(\lambda'' - 1),
\]
which, by Lemma 11,
\[
\lambda' = \frac{\Gamma \left( \frac{i_1+2}{p_1} \right) \Gamma \left( \frac{i_1+1}{p_1} + N \right) \frac{i_1+1}{p_1} + N}{\Gamma \left( \frac{i_1+1}{p_1} \right) \Gamma \left( \frac{i_1+2}{p_1} + N \right) \frac{i_1+2}{p_1} + N},
\]
\[
\lambda'' = \frac{\Gamma \left( \frac{i_1+1}{p_1} \right)^2 \Gamma \left( \frac{i_1}{p_1} + N \right) \Gamma \left( \frac{i_1+2}{p_1} + N \right) \left( \frac{i_1+1}{p_1} + N \right)^2}{\Gamma \left( \frac{i_1}{p_1} \right) \Gamma \left( \frac{i_1+2}{p_1} \right) \Gamma \left( \frac{i_1+1}{p_1} + N \right)^2 \left( \frac{i_1+1}{p_1} + N \right)^2} \delta(i_1),
\]
where
\[
N := \sum_{j=2}^{m} \frac{i_j + 1}{p_j}.
\]

Formula (14) shows that \( [M_{z_1}, M_{z_1}^*] \) is \( p \)-summable if and only if the infinite series \( \sum_{i \in \mathbb{N}} |\lambda|^p \) converges. In the following, we derive the asymptotic formula for \( \lambda \) when at least one of \( i_1, N \) tends to infinity.

First, assume \( m > 1 \). By Lemma 3,
\[
\lambda' \approx \begin{cases} 
\frac{\frac{1}{p_1}}{i_1} / (N + i_1) \frac{1}{p_1}, & \text{if } i_1 > 0, \\
N - \frac{1}{p_1}, & \text{if } i_1 = 0,
\end{cases}
\]
and
\[
\lambda'' - 1 \approx \begin{cases} 
\frac{\frac{1}{p_1}}{i_1} - \frac{1}{i_1+N} - \frac{1}{(i_1+N)^2} \approx \frac{N}{i_1(i_1+N)}, & \text{if } i_1 > 0, \\
1, & \text{if } i_1 = 0.
\end{cases}
\]

Here, \( f(x) \approx g(x) \) means that as \( x \) grows large, the dominant term in the asymptotic expansion of \( f(x) \) is \( g(x) \), namely \( f(x) = C g(x) + O(x^{-k}) \) for some \( k \in \mathbb{N}_0 \) and \( C > 0 \). Note that to obtain the asymptotic formula (16) (respectively, (17)), we have used Lemma 3 up to \( O(x^{-1}) \) (respectively, \( O(x^{-2}) \)). We have shown that
\[
\lambda \approx \begin{cases} 
\frac{1}{p_1} - \frac{1}{N+1} / (N + i_1) \frac{1}{p_1} + 1, & \text{if } i_1 > 0, \\
N - \frac{1}{p_1}, & \text{if } i_1 = 0.
\end{cases}
\]

Therefore, \( \sum |\lambda|^p \) converges if and only if both of the following series converge:
\[
\sum i_1 \left( \frac{1}{p_1} - 1 \right)^p N^p / (N + i_1) \left( \frac{1}{p_1} + 1 \right)^p, \quad \sum N^{\frac{p}{p_1}}.
\]
According to Lemma 5. (b), the first series converges exactly when \( p > \max \{(p_1 - 1)m, m\} \), and the second converges exactly when \( p > p_1(m - 1) \).

Next, assume \( m = 1 \). Then \( N = 0 \), and by Lemma 3,
\[
\lambda = \frac{i_1 + 1}{i_1 + 2} \left( \frac{i_1(i_1 + 2)}{(i_1 + 1)^2} - 1 \right) \approx \frac{1}{i_1^2}.
\]

Therefore, \( \sum |\lambda|^p \) converges exactly when \( p > 1/2 \).

The whole analysis proves (b).

(c) Note that an operator \( T \) on a Hilbert space is \( p \)-summable if and only if \( \sqrt{T^*T} \) is so. Formula (15) shows that \( \left[ M_{z_2}, M_{z_1}^* \right] \) is \( p \)-summable if and only if the infinite series \( \sum_{i \in \mathbb{N}^m} \mu_i^p \) converges. Since the region \( i_2 = 0 \) has no effect on the summability of this series, we assume \( i_2 > 0 \) from now on. We can write
\[
\mu = \mu' |\mu'' - 1|,
\]
\[
\mu' = \mu_1' \mu_2',
\]
which, by Lemma 11,
\[
\mu_1' = \sqrt{\frac{\Gamma \left( \frac{i_1+1}{p_1} \right) \Gamma \left( \frac{i_2+1}{p_2} \right) \Gamma \left( \frac{i_1+1}{p_1} + \frac{i_2+1}{p_2} + M \right) \Gamma \left( \frac{i_1+2}{p_1} + \frac{i_2+1}{p_2} + M \right)}{\Gamma \left( \frac{i_1+1}{p_1} \right) \Gamma \left( \frac{i_2+1}{p_2} \right) \Gamma \left( \frac{i_1+2}{p_1} + \frac{i_2+1}{p_2} + M \right)^2}},
\]
\[
\mu_2' = \sqrt{\frac{\left( \frac{i_1+1}{p_1} + \frac{i_2+1}{p_2} + M \right) \left( \frac{i_1+2}{p_1} + \frac{i_2+1}{p_2} + M \right)}{\left( \frac{i_1+2}{p_1} + \frac{i_2+1}{p_2} + M \right)^2}},
\]
\[
\mu'' = \frac{\Gamma \left( \frac{i_1+1}{p_1} + \frac{i_2+1}{p_2} + M \right) \Gamma \left( \frac{i_1+2}{p_1} + \frac{i_2+1}{p_2} + M \right) \Gamma \left( \frac{i_1+1}{p_1} + \frac{i_2+1}{p_2} + M \right) \Gamma \left( \frac{i_1+2}{p_1} + \frac{i_2+1}{p_2} + M \right)}{\Gamma \left( \frac{i_1+1}{p_1} + \frac{i_2+1}{p_2} + M \right) \Gamma \left( \frac{i_1+2}{p_1} + \frac{i_2+1}{p_2} + M \right) \Gamma \left( \frac{i_1+1}{p_1} + \frac{i_2+1}{p_2} + M \right) \Gamma \left( \frac{i_1+2}{p_1} + \frac{i_2+1}{p_2} + M \right)},
\]
where
\[
M := \sum_{j=3}^{m} \frac{i_j + 1}{p_j}.
\]

By Lemma 3,
\[
\mu' \approx i_1^{\frac{1}{p_1}} i_2^{\frac{1}{p_2}} / (i_1 + i_2 + M)^{\frac{1}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right),
\]
and
\[ \mu'' - 1 \approx \frac{1}{i_1 + i_2 + M} - \frac{1}{(i_1 + i_2 + M)^2} \approx \frac{1}{i_1 + i_2 + M}, \]
hence
\[ \mu \approx i_1^\frac{1}{p_1} i_2^\frac{1}{p_2} / ((i_1 + i_2 + M)^{\frac{1}{p_1} + \frac{1}{p_2}} + 1). \]
Therefore, \( \sum \mu^p < \infty \) if and only if
\[ \sum i_1^\frac{1}{p_1} i_2^\frac{1}{p_2} / ((i_1 + i_2 + M)^{\frac{1}{p_1} + \frac{1}{p_2}}) < \infty. \]

According to Lemma 5.(b), this happens exactly when \( p > m \). \( \square \)

### 4 Proof of Theorem 2

This section proves Theorem 2 about the \( p \)-essential normality of the Bergman module \( L^2_a(\Omega_2) \) over the domain \( \Omega_2 \) given in (2). For notational simplicity, we work on
\[ \Omega_2 := \left\{ \left( \sum_{j=1}^m |z_j|^{2p_j} \right)^a + \left( \sum_{k=1}^n |w_k|^{2q_k} \right)^b + \left( \sum_{l=1}^o |u_l|^{2r_l} \right)^c + \cdots < 1 \right\} \subseteq \mathbb{C}^{m+n+o+\cdots}, \quad (18) \]
instead of (2). Similar to the discussions in Sect. 3, the normalized monomials
\[ b_{\alpha, \beta, \ldots} := \frac{z^\alpha w^\beta \cdots}{\sqrt{\omega_2(\alpha, \beta, \ldots)}}, \quad (\alpha, \beta, \ldots) \in \mathbb{N}^{m+n+\cdots}, \quad (19) \]
where
\[ \omega_2(\alpha, \beta, \ldots) := \| z^\alpha w^\beta \cdots \|^2_{L^2_a(\Omega_2)} \]
constitute an orthonormal basis for the Hilbert space \( L^2_a(\Omega_2) \). An explicit formula for the norm of monomials is given by:

**Lemma 11** Given multi-indices \( \alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n, \ldots, \) we have
\[ \omega_2(\alpha, \beta, \ldots) = \frac{\pi^{m+n+\cdots}}{\prod p_j \prod q_k \cdots ab \cdots} B \left( \frac{\alpha + 1}{ap} \right) B \left( \frac{\beta + 1}{aq} \right) \cdots B \left( \frac{\alpha + 1}{ap} \right) + \frac{\beta + 1}{aq} + \cdots. \]

**Proof** [20] or [35]. \( \square \)

Theorem 2 follows immediately from:
Proposition 12  For each coordinate function $f = z_j, w_k, \ldots$, let $M_f : L^2_a(\Omega_2) \to L^2_a(\Omega_2)$ be the multiplication by $f$. Then:

(a) Given $(\alpha, \beta, \ldots) \in \mathbb{N}^{m+n+\ldots}$, we have

$$[M_{z_1}, M^*_{z_1}](b_{\alpha, \beta, \ldots}) = \lambda b_{\alpha, \beta, \ldots}, \quad (20)$$

$$\sqrt{[M_{z_2}, M^*_{z_1}][M_{z_1}, M^*_{z_1}]}(b_{\alpha, \beta, \ldots}) = \mu b_{\alpha, \beta, \ldots}, \quad (21)$$

$$\sqrt{[M_{z_2}, M^*_{w_1}][M_{z_1}, M^*_{w_1}]}(b_{\alpha, \beta, \ldots}) = \nu b_{\alpha, \beta, \ldots}, \quad (22)$$

where

$$\lambda = \frac{\omega(\alpha, \beta, \ldots)}{\omega(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta, \ldots)} \delta(\alpha_1) - \frac{\omega(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta, \ldots)}{\omega(\alpha, \beta, \ldots)} \delta(\alpha_1) - \frac{\omega(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta, \ldots)}{\omega(\alpha, \beta, \ldots)} \delta(\alpha),$$

$$\mu = \sqrt{\frac{\omega(\alpha, \beta, \ldots) \omega(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta, \ldots)}{\omega(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta, \ldots)^2}} \delta(\alpha_2),$$

$$\nu = \sqrt{\frac{\omega(\alpha, \beta, \ldots) \omega(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n, \gamma, \ldots)}{\omega(\alpha, \beta_1 - 1, \beta_2, \ldots, \beta_n, \gamma, \ldots)^2}} - \frac{\omega(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n, \gamma, \ldots)}{\omega(\alpha, \beta, \ldots) \omega(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1 - 1, \beta_2, \ldots, \beta_n, \gamma, \ldots)^2} \delta(\beta_1),$$

and

$$\delta(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{otherwise}. \end{cases}$$

(b) $[M_{z_1}, M^*_{z_1}]$ is $p$-summable if and only if

$$p > \begin{cases} \frac{1}{2}, & \text{if } \dim \Omega_2 = 1, \\ \max \{ \dim \Omega_2, ap_1(\dim \Omega_2 - m) \}, & \text{if } \dim \Omega_2 > 1, m = 1, \\ \max \{ \dim \Omega_2, p_1(\dim \Omega_2 - 1), ap_1(\dim \Omega_2 - m) \}, & \text{if } \dim \Omega_2 > 1, m > 1. \end{cases}$$

(c) Assume $m > 1$. $[M_{z_1}, M^*_{z_1}]$ is $p$-summable if and only if

$$p > \max \{ \dim \Omega_2, \frac{2p(\dim \Omega_2 - m)}{1/p_1 + 1/p_2} \}, \quad \text{if } a = 1,$$

$$p > \max \{ \dim \Omega_2, \frac{2p(\dim \Omega_2 - m)}{1/p_1 + 1/p_2} \}, \quad \text{if } a \neq 1.$$ 

(d) $[M_{z_1}, M^*_{w_1}]$ is $p$-summable if and only if $p > \dim \Omega_2$.

Proof (a) Straightforward computations.
(b) We can write

$$\lambda = \lambda'(\lambda'' - 1),$$

$$\lambda'' = \lambda_1 \lambda_2 \delta(\alpha_1),$$
which, by Lemma 11,

\[
\lambda' = \frac{\Gamma\left(\frac{a_1+2}{p_1}\right) \Gamma\left(\frac{a_1+1}{p_1} + A\right) \Gamma\left(\frac{a_1+1}{ap_1} + \frac{A}{a} + L\right) \Gamma\left(\frac{a_1+2}{ap_1} + \frac{A}{a} + L\right)}{\Gamma\left(\frac{a_1+1}{p_1}\right) \Gamma\left(\frac{a_1+2}{p_1} + A\right) \Gamma\left(\frac{a_1+2}{ap_1} + \frac{A}{a} + L\right) \Gamma\left(\frac{a_1+1}{ap_1} + \frac{A}{a} + L\right)},
\]

\[
\lambda'_{1} = \frac{\Gamma\left(\frac{a_1+1}{p_1}\right)^2 \Gamma\left(\frac{a_1+1}{p_1} + A\right) \Gamma\left(\frac{a_1+2}{p_1} + A\right) \Gamma\left(\frac{a_1+2}{ap_1} + \frac{A}{a} + L\right)}{\Gamma\left(\frac{a_1+1}{p_1}\right)^2 \Gamma\left(\frac{a_1+2}{p_1} + A\right)^2 \Gamma\left(\frac{a_1+1}{ap_1} + \frac{A}{a} + L\right) \Gamma\left(\frac{a_1+2}{ap_1} + \frac{A}{a} + L\right)},
\]

\[
\lambda'_{2} = \frac{\Gamma\left(\frac{a_1+1}{ap_1} + \frac{A}{a} + L\right) \Gamma\left(\frac{a_1+2}{ap_1} + \frac{A}{a} + L\right) \Gamma\left(\frac{a_1+1}{ap_1} + \frac{A}{a} + L\right)^2 \Gamma\left(\frac{a_1+2}{ap_1} + \frac{A}{a} + L\right)^2}{\Gamma\left(\frac{a_1+1}{ap_1} + \frac{A}{a} + L\right)^2 \Gamma\left(\frac{a_1+2}{ap_1} + \frac{A}{a} + L\right)^2}.
\]

where

\[
A = \sum_{j=2}^{m} \frac{\alpha_j + 1}{p_j},
\]

\[
L = \sum_{k=1}^{n} \frac{\beta_k + 1}{bq_k} + \sum_{l=1}^{o} \frac{\gamma_l + 1}{cr_l} + \cdots.
\]

Formula (20) shows that \([M_{z_1}, M_{z_1}^*]\) is \(p\)-summable if and only if the infinite series \(\sum |\lambda|^p\) converges. In the following, we derive the asymptotic formula for \(\lambda\) when at least one of \(\alpha_1, A, L\) tends to infinity.

First, assume \(m > 1\). By Lemma 3,

\[
\lambda' \approx \begin{cases} \frac{1}{\alpha_1} (\alpha_1 + A)^{-\frac{1}{p_1}} \left(1 - \frac{1}{\alpha}\right) / (\alpha_1 + A + L)^\frac{1}{\alpha p_1}, & \text{if } \alpha_1 > 0, \\ A^{-\frac{1}{p_1}} \left(1 - \frac{1}{\alpha}\right) / (A + L)^\frac{1}{\alpha p_1}, & \text{if } \alpha_1 = 0, \end{cases}
\]

and

\[
\lambda'' - 1 \approx -\frac{1}{\alpha_1} + \frac{1}{\alpha_1 + A} - \frac{a^{-2}}{\alpha_1 + A} + \frac{a^{-2}}{\alpha_1 + A + L} \approx \frac{A}{\alpha_1 (\alpha_1 + A)} + \frac{a^{-2}L}{(\alpha_1 + A)(\alpha_1 + A + L)},
\]

if \(\alpha_1 > 0\), and \(\lambda'' - 1 = -1\) if \(\alpha_1 = 0\). As before, \(f(x) \approx g(x)\) means that as \(x\) grows large, the dominant term in the asymptotic expansion of \(f(x)\) is \(g(x)\), namely \(f(x) = C g(x) + O(x^{-k})\) for some \(k \in \mathbb{N}_+\) and \(C > 0\). Note that to obtain the asymptotic formula (23) (respectively, (24)), we have used Lemma 3 up to \(O(x^{-1})\) (respectively, \(O(x^{-2})\)). We have shown that

\[
\lambda \approx \begin{cases} \frac{\alpha_1^{-1}}{p_1} (\alpha_1 + A)^{-\frac{1}{p_1}} (1 - \frac{1}{\alpha})^{-1} A + \frac{\alpha_1^{-1}}{p_1} (\alpha_1 + A)^{-\frac{1}{p_1}} (1 - \frac{1}{\alpha})^{-1} L, & \text{if } \alpha_1 > 0, \\ A^{-\frac{1}{p_1}} (\alpha_1 + A + L)^{-\frac{1}{\alpha p_1}}, & \text{if } \alpha_1 = 0, \end{cases}
\]
According to Lemma 4, \( \sum |\lambda|^p \) converges if and only if all of the following series converge:

\[
\sum \alpha_1 \left( \frac{1}{p_1} \right)^{(\alpha_1 - \frac{1}{a}) + 1} (\frac{1}{p_1} (1 - \frac{1}{a}) + 1)^p A^p / (\alpha_1 + A + L)^{\frac{p}{p_1}}, \tag{25}
\]

\[
\sum \alpha_1 \left( \frac{p}{p_1} \right)^{(\alpha_1 + A - \frac{1}{a}) + 1} L^p / (\alpha_1 + A + L)^{\frac{p}{p_1}}, \tag{26}
\]

\[
\sum A \left( \frac{p}{p_1} \right)^{1 - \frac{1}{a}} / (A + L)^{\frac{p}{p_1}}. \tag{27}
\]

Similar arguments as in the proof of Lemma 5.(b) show that the convergence of the series (25) is equivalent to the convergence of

\[
\sum_{i \in \mathbb{N}^3} \frac{(\frac{1}{i_1} - 1)^p}{(i_1 + i_2 - \frac{1}{a}) + 1} \left( \frac{1}{i_1} (1 - \frac{1}{a}) + 1 \right)^{p_1 - p + m - 2} \sum_{i_3} \frac{1}{i_3} \left( \frac{p}{p_1} \right)^{1 - \frac{1}{a}} / (A + L)^{\frac{p}{p_1}},
\]

where

\[ m' := \dim \Omega_2 - m = n + o + \cdots. \]

According to Lemma 7.(a), this latter series converges exactly when

\[ p > \max \{m + m', p_1 (m + m' - 1), ap_1 m'\}. \]

Likewise, the series (26) and (27) converge exactly when

\[ p > \max \{m + m', ap_1 m'\} \quad \text{and} \quad p > \max \{p_1 (m + m' - 1), ap_1 m'\}, \]

respectively.

Next, assume \( m = 1 \). Then, \( A = 0 \) and

\[
\lambda' = \frac{\Gamma \left( \frac{1}{i_1} + \frac{1}{i_1 + 1} \right)}{\Gamma \left( \frac{1}{i_1 + 1} \right)} \frac{\alpha_1 + 1}{\alpha_1} \frac{a_1 + 1}{a_1} + L,
\]

\[
\lambda'' = \left\{ \begin{array}{ll}
\frac{1}{\Gamma \left( \frac{1}{a_1 \alpha_1} \right)} \Gamma \left( \frac{a_1 + 1}{a_1 \alpha_1} \right) \Gamma \left( \frac{a_1 + 1}{a_1 \alpha_1} + L \right) \Gamma \left( a_1 + 1 + L \right), & \text{if } a_1 > 0, m' > 0, \\
\frac{1}{\Gamma \left( \frac{1}{a_1 \alpha_1} \right)} \Gamma \left( \frac{1}{a_1 \alpha_1} + L \right) \left( \frac{a_1 + 1}{a_1 \alpha_1} \right)^2 \left( \frac{a_1 + 1}{a_1 \alpha_1} + L \right)^2, & \text{if } a_1 > 0, m' = 0, \\
0, & \text{if } a_1 = 0.
\end{array} \right.
\]
(Note that $L = 0$ if $m' = 0$.) By Lemma 3,

$$
\lambda' \approx \begin{cases} 
\frac{1}{\alpha_1^{\alpha_1}} / (\alpha_1 + L) \frac{1}{\alpha_1^{\alpha_1}}, & \text{if } \alpha_1 > 0, \\
L \frac{1}{\alpha_1^{\alpha_1}}, & \text{if } \alpha_1 = 0,
\end{cases}
$$

$$
\lambda'' - 1 \approx \begin{cases} 
\frac{1}{\alpha_1} - \frac{1}{\alpha_1 + L} - \frac{1}{(\alpha_1 + L)^2} \approx \frac{L}{\alpha_1 (\alpha_1 + L)}, & \text{if } \alpha_1 > 0, m' > 0, \\
1/\alpha_1^2, & \text{if } \alpha_1 > 0, m' = 0, \\
L - \frac{1}{\alpha_1}, & \text{if } \alpha_1 = 0.
\end{cases}
$$

We have shown that

$$
\lambda \approx \begin{cases} 
\frac{1}{\alpha_1^{\alpha_1}} L / (\alpha_1 + L) \frac{1}{\alpha_1^{\alpha_1 + 1}}, & \text{if } \alpha_1 > 0, m' > 0, \\
1/\alpha_1^2, & \text{if } \alpha_1 > 0, m' = 0, \\
L - \frac{1}{\alpha_1}, & \text{if } \alpha_1 = 0.
\end{cases}
$$

According to Lemma 5.(b), the series $\sum |\lambda|^p$ converges exactly when

$$
p > \begin{cases} 
\max \{ap_1 m', m' + 1\}, & \text{if } m' > 0, \\
\frac{1}{2}, & \text{if } m' = 0.
\end{cases}
$$

The whole analysis proves (b).

(c) Formula (21) shows that $[M_{z_2}, M_{z_1}^a]$ is $p$-summable if and only if the infinite series $\sum \mu^p$ converges. Since the region $\alpha_2 = 0$ has no effect on the summability of this series, we assume $\alpha_2 > 0$. We can write

$$
\mu = \mu' |\mu'' - 1|,
$$

$$
\mu' = \mu_1 \mu_2^2 \mu_3^4,
$$

$$
\mu'' = \mu_1^{''} \mu_2^2 \mu_3^4,
$$

which, by Lemma 11,

$$
\mu_1' = \sqrt{\frac{\Gamma\left(\frac{\alpha_1 + 2}{p_1}\right) \Gamma\left(\frac{\alpha_2 + 1}{p_2}\right) \Gamma\left(\frac{\alpha_1 + 1}{p_1} + \frac{\alpha_2 + 1}{p_2} + \mathcal{A}\right) \Gamma\left(\frac{\alpha_1 + 2}{p_1} + \frac{\alpha_2}{p_2} + \mathcal{A}\right)} {\Gamma\left(\frac{\alpha_1 + 1}{p_1}\right) \Gamma\left(\frac{\alpha_2}{p_2}\right) \Gamma\left(\frac{\alpha_1 + 1}{p_1} + \frac{\alpha_2}{p_2} + \mathcal{A}\right)^2}},
$$

$$
\mu_2' = \sqrt{\frac{\Gamma\left(\frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + \frac{\mathcal{A}}{a}\right)^2} {\Gamma\left(\frac{\alpha_1 + 1}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + \frac{\mathcal{A}}{a}\right) \Gamma\left(\frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2}{ap_2} + \frac{\mathcal{A}}{a}\right)}},
$$

where $\mathcal{A}$ is a parameter related to the problem.
\[
\begin{align*}
\mu_3' &= \Gamma \left( \frac{\alpha_1 + 1}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + \frac{4}{a} + L \right) \Gamma \left( \frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2 + 2}{ap_2} + \frac{4}{a} + L \right) \\
&\quad \times \Gamma \left( \frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + \frac{4}{a} + L \right)^2, \\
\mu_4' &= \left( \frac{\alpha_1 + 1}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + \frac{4}{a} + L \right) \left( \frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2 + 2}{ap_2} + \frac{4}{a} + L \right) \\
&\quad \times \left( \frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + \frac{4}{a} + L \right)^2,
\end{align*}
\]

\[
\begin{align*}
\mu_1'' &= \frac{\Gamma \left( \frac{\alpha_1 + 1}{p_1} + \frac{\alpha_2}{p_2} + A \right) \Gamma \left( \frac{\alpha_1 + 2}{p_1} + \frac{\alpha_2 + 1}{p_2} + A \right)}{\Gamma \left( \frac{\alpha_1 + 1}{p_1} + \frac{\alpha_2 + 1}{p_2} + A \right) \Gamma \left( \frac{\alpha_1 + 2}{p_1} + \frac{\alpha_2}{p_2} + A \right)}, \\
\mu_2'' &= \frac{\Gamma \left( \frac{\alpha_1 + 1}{ap_1} + \frac{\alpha_2}{ap_2} + A \right) \Gamma \left( \frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + A \right)}{\Gamma \left( \frac{\alpha_1 + 1}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + A \right) \Gamma \left( \frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2}{ap_2} + A \right)}, \\
\mu_3'' &= \frac{\Gamma \left( \frac{\alpha_1 + 1}{ap_1} + \frac{\alpha_2}{ap_2} + A \right) \Gamma \left( \frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + A \right)}{\Gamma \left( \frac{\alpha_1 + 1}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + A \right) \Gamma \left( \frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2}{ap_2} + A \right)}, \\
\mu_4'' &= \frac{\left( \frac{\alpha_1 + 1}{ap_1} + \frac{\alpha_2}{ap_2} + A \right) \left( \frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + A \right)}{\left( \frac{\alpha_1 + 1}{ap_1} + \frac{\alpha_2 + 1}{ap_2} + A \right) \left( \frac{\alpha_1 + 2}{ap_1} + \frac{\alpha_2}{ap_2} + A \right)},
\end{align*}
\]

where

\[
\begin{align*}
A &= \sum_{j=3}^m \frac{i_j + 1}{p_j}, \\
L &= \sum_{k=1}^n \frac{\beta_k + 1}{bq_k} + \sum_{l=1}^o \frac{\gamma_l + 1}{cr_l} + \cdots.
\end{align*}
\]

By Lemma 3,

\[
\begin{align*}
\mu' \approx \alpha_1^{\frac{1}{p_1}} \alpha_2^{\frac{1}{p_2}} (\alpha_1 + \alpha_2 + A)^{-\frac{1}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right) \left( 1 - \frac{1}{a} \right) \\
&\quad \times (\alpha_1 + \alpha_2 + A + L)^{\frac{1}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right),
\end{align*}
\]

\[
\begin{align*}
\mu'' - 1 &\approx \frac{1}{\alpha_1 + \alpha_2 + A} - \frac{a^{-2}}{\alpha_1 + \alpha_2 + A} + \frac{a^{-2}}{\alpha_1 + \alpha_2 + A + L} - \frac{a^{-2}}{(\alpha_1 + \alpha_2 + A + L)^2} \\
&\approx \frac{a^2 - 1}{\alpha_1 + \alpha_2 + A} + \frac{1}{\alpha_1 + \alpha_2 + A + L}.
\end{align*}
\]
Hence,

\[
\mu \approx \begin{cases} 
\alpha_1^{-p} \alpha_2^{-p} (\alpha_1 + \alpha_2 + A)^{-\frac{p}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{(1 - \frac{1}{p})^{-1}} |\alpha_1 + \alpha_2 + A - tL|, & \text{if } a < 1, \\
\left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{p} & \text{if } a = 1, \\
\frac{1}{\alpha_1^{1/p_1} \alpha_2^{1/p_2}} (\alpha_1 + \alpha_2 + A + L)^{-\frac{p}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{(1 - \frac{1}{p})^{-1}} & \text{if } a > 1,
\end{cases}
\]

where

\[ t := a^{-2} - 1 > 0. \]

Therefore, \( \sum \mu^p \) converges if and only if the following series converges:

\[
\sum \alpha_1^{-p} \alpha_2^{-p} (\alpha_1 + \alpha_2 + A)^{-\frac{p}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{(1 - \frac{1}{p})^{-1}} |\alpha_1 + \alpha_2 + A - tL|^{p}, \quad \text{if } a < 1,
\]
\[
\left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{p} \frac{1}{\alpha_1^{1/p_1} \alpha_2^{1/p_2}} (\alpha_1 + \alpha_2 + A + L)^{-\frac{p}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{(1 - \frac{1}{p})^{-1}} \frac{1}{p} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{p}, \quad \text{if } a = 1,
\]
\[
\frac{1}{\alpha_1^{1/p_1} \alpha_2^{1/p_2}} (\alpha_1 + \alpha_2 + A)^{-\frac{p}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{(1 - \frac{1}{p})^{-1}} \frac{1}{p} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{p}, \quad \text{if } a > 1.
\]

Similar arguments as in the proof of Lemma 5.(b) show that the convergence of this series is equivalent to the convergence of

\[
\sum_{i \in \mathbb{N}^4} \frac{p}{i_1} \frac{p}{i_2} (i_1 + i_2 + i_3 + i_4)^{-\frac{p}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{(1 - \frac{1}{p})^{-1}} |i_1 + i_2 + i_3 + i_4|^{p} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{p}, \quad \text{if } a < 1,
\]
\[
\sum_{i \in \mathbb{N}^4} \frac{p}{i_1} \frac{p}{i_2} (i_1 + i_2 + i_3 + i_4)^{-\frac{p}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{(1 - \frac{1}{p})^{-1}} \frac{1}{p} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{p}, \quad \text{if } a = 1,
\]
\[
\sum_{i \in \mathbb{N}^4} \frac{p}{i_1} \frac{p}{i_2} (i_1 + i_2 + i_3 + i_4)^{-\frac{p}{2}} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{(1 - \frac{1}{p})^{-1}} \frac{1}{p} \left( \frac{1}{p_1} + \frac{1}{p_2} \right)^{p}, \quad \text{if } a > 1.
\]

According to Lemmas 5 and 7.(c, d), this latter series converges exactly when

\[
p > \begin{cases} 
\max \left\{ \dim \Omega_2 \right\}, & \text{if } a = 1, \\
\max \left\{ \dim \Omega_2, \frac{2a \left( \dim \Omega_2 - m \right)}{p_1 + p_2} \right\}, & \text{if } a \neq 1.
\end{cases}
\]

(d) Formula (22) shows that \( \left[ M_{z_1}, M_{w_1}^* \right] \) is \( p \)-summable if and only if the infinite series \( \sum v^p \) converges. Since the region \( \beta_1 = 0 \) has no effect on the summability of this series, we assume \( \beta_1 > 0 \). We can write

\[ v = v' |v'' - 1|, \]
where \( \nu' = \nu_1' \nu_2' \nu_3' \),

\[ \nu'' = \nu''_1 \nu''_2 \]

which, by Lemma 11,

\[
\nu' = \frac{\Gamma\left(\frac{\alpha_1+2}{p_1}\right) \Gamma\left(\frac{\beta_1+1}{q_1}\right)}{\Gamma\left(\frac{\alpha_1+1}{p_1} + A\right) \Gamma\left(\frac{\beta_1+1}{q_1} + B\right)} \frac{\Gamma\left(\frac{\alpha_1+1}{p_1} + A\right) \Gamma\left(\frac{\beta_1+1}{q_1} + B\right)}{\Gamma\left(\frac{\alpha_1+2}{p_1}\right) \Gamma\left(\frac{\beta_1+1}{q_1}\right)} \frac{\Gamma\left(\frac{\alpha_1+2}{p_1} + \frac{\beta_1+1}{b_{q_1}} + \frac{A}{a} + \frac{B}{b} + \mathcal{L}\right)}{\Gamma\left(\frac{\alpha_1+2}{p_1} + \frac{\beta_1+1}{b_{q_1}} + \frac{A}{a} + \frac{B}{b} + \mathcal{L}\right)^2},
\]

\[
\nu''_1 = \frac{\Gamma\left(\frac{\alpha_1+1}{p_1} + \frac{\beta_1+1}{b_{q_1}} + \frac{A}{a} + \frac{B}{b} + \mathcal{L}\right)}{\Gamma\left(\frac{\alpha_1+1}{p_1} + \frac{\beta_1+1}{b_{q_1}} + \frac{A}{a} + \frac{B}{b} + \mathcal{L}\right)} \frac{\Gamma\left(\frac{\alpha_1+2}{p_1} + \frac{\beta_1+1}{b_{q_1}} + \frac{A}{a} + \frac{B}{b} + \mathcal{L}\right)}{\Gamma\left(\frac{\alpha_1+2}{p_1} + \frac{\beta_1+1}{b_{q_1}} + \frac{A}{a} + \frac{B}{b} + \mathcal{L}\right)^2},
\]

\[
\nu''_2 = \frac{\Gamma\left(\frac{\alpha_1+1}{p_1} + \frac{\beta_1+1}{b_{q_1}} + \frac{A}{a} + \frac{B}{b} + \mathcal{L}\right)}{\Gamma\left(\frac{\alpha_1+1}{p_1} + \frac{\beta_1+1}{b_{q_1}} + \frac{A}{a} + \frac{B}{b} + \mathcal{L}\right)} \frac{\Gamma\left(\frac{\alpha_1+2}{p_1} + \frac{\beta_1+1}{b_{q_1}} + \frac{A}{a} + \frac{B}{b} + \mathcal{L}\right)}{\Gamma\left(\frac{\alpha_1+2}{p_1} + \frac{\beta_1+1}{b_{q_1}} + \frac{A}{a} + \frac{B}{b} + \mathcal{L}\right)^2},
\]

where

\[
A = \sum_{j=2}^{m} \frac{\alpha_j + 1}{p_j},
\]

\[
B = \sum_{k=2}^{n} \frac{\beta_k + 1}{q_k},
\]

\[
\mathcal{L} = \sum_{l=1}^{o} \frac{\gamma_l + 1}{cr_l} + \ldots.
\]
By Lemma 3,
\[
\nu' \approx \tilde{\alpha}_1^{\frac{1}{p_1}} \beta_1^{\frac{1}{q_1}} (\alpha_1 + A) - \frac{1}{2 p_1} \left( 1 - \frac{1}{a} \right) (\beta_1 + B) - \frac{1}{2 q_1} \left( 1 - \frac{1}{b} \right) / (\alpha_1 + \beta_1 + A + B + \mathcal{L})^{\frac{1}{2}} \left( \frac{1}{p_1} + \frac{1}{q_1} \right).
\]

\[
\nu'' - 1 \approx \frac{1}{\alpha_1 + \beta_1 + A + B + \mathcal{L}} - \frac{1}{(\alpha_1 + \beta_1 + A + B + \mathcal{L})^2} \approx \frac{1}{\alpha_1 + \beta_1 + A + B + \mathcal{L}},
\]
so
\[
\nu \approx \tilde{\alpha}_1^{\frac{1}{p_1}} \beta_1^{\frac{1}{q_1}} (\alpha_1 + A) - \frac{1}{2 p_1} \left( 1 - \frac{1}{a} \right) (\beta_1 + B) - \frac{1}{2 q_1} \left( 1 - \frac{1}{b} \right) / (\alpha_1 + \beta_1 + A + B + \mathcal{L})^{\frac{1}{2}} \left( \frac{1}{p_1} + \frac{1}{q_1} \right)^{1/2}.
\]

Therefore, \( \sum \nu^p \) converges if and only if the following series converges:
\[
\sum \alpha_1^{\frac{p}{p_1}} \beta_1^{\frac{p}{q_1}} (\alpha_1 + A) - \tilde{\alpha}_1^{\frac{p}{2 p_1}} \left( 1 - \frac{1}{a} \right) (\beta_1 + B) - \tilde{\beta}_1^{\frac{p}{2 q_1}} \left( 1 - \frac{1}{b} \right) / (\alpha_1 + \beta_1 + A + B + \mathcal{L})^p \left( \frac{1}{p_1} + \frac{1}{q_1} \right)^p.
\]

Similar arguments as in the proof of Lemma 5.(b) show that the convergence of this series is equivalent to
\[
\sum_{i \in \mathbb{N}^5} i_1^{\frac{p}{p_1}} i_3^{\frac{p}{2 p_1}} (i_1 + i_2) - \tilde{\alpha}_1^{\frac{p}{2 p_1}} \left( 1 - \frac{1}{a} \right) (i_3 + i_4) - \tilde{\beta}_1^{\frac{p}{2 q_1}} \left( 1 - \frac{1}{b} \right) i_2^{p-2} i_4^{p-2} / i_5^{p-1} \left( \frac{1}{p_1} + \frac{1}{q_1} \right)^p.
\]

According to Lemma 7.(f), this series converges exactly when \( p > \dim \Omega_2 \).

\[\square\]

**Acknowledgements** The author would like to thank Richard Rochberg and Xiang Tang for reading a preliminary version of this paper and making helpful suggestions.

**Data Availability Statement** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**References**

1. Andrews, G., Askey, R., Roy, R.: Special Functions. Cambridge University Press, Cambridge (1999)
2. Arazy, J., Fisher, S., Peetre, J.: Hankel operators on weighted Bergman spaces. Am. J. Math. 110(6), 989–1053 (1988)
3. Arazy, J., Fisher, S., Peetre, J.: Hankel operators on planar domains. Constr. Approx. 6(2), 113–138 (1990)
4. Arazy, J., Fisher, S., Janson, S., Peetre, J.: Membership of Hankel operators on the ball in unitary ideals, J. Lond. Math. Soc. (2) 43(3), 485–508 (1991)
5. Arveson, W.: Subalgebras of C*-algebras III: multivariable operator theory. Acta Math. 181, 159–228 (1998)
6. Axler, S.: The Bergman space, the Bloch space, and commutators of multiplication operators. Duke Math. J. 53(2), 315–332 (1986)
7. Beatrous, F., Li, S.-Y.: On the boundedness and compactness of operators of Hankel type. J. Funct. Anal. 111, 279–350 (1993)
8. Beatrous, F., Li, S.-Y.: Trace ideal criteria for operators of Hankel type. Illinois J. Math. 39, 723–754 (1995)
9. Békollé, D., Berger, C., Coburn, L., Zhu, K.: BMO in the Bergman metric on bounded symmetric domains. J. Funct. Anal. 93, 310–350 (1990)
10. Boutet de Monvel, L., Guillemin, V.: The spectral theory of Toeplitz operators. Ann. Math. Stud., Vol. 99. Princeton University Press, Princeton (1981)
11. Catlin, D.: Global regularity of the Neumann problem. Proc. Symp. Pure Math. 41, 39–49 (1984)
12. Catlin, D.: Boundary invariants of pseudoconvex domains. Ann. Math. 120, 529–586 (1984)
13. Catlin, D., D’Angelo, J.: Positivity conditions for bihomogeneous polynomials. Math. Res. Lett. 4, 555–567 (1997)
14. Connes, A.: Noncommutative differential geometry. Inst. Hautes Études Sci. Publ. Math. 62, 257–360 (1985)
15. Connes, A.: Noncommutative Geometry. Academic Press Inc, San Diego, CA (1994)
16. Curto, R., Muhly, P.: C*-algebras of multiplication operators on Bergman spaces. J. Funct. Anal. 64, 315–329 (1985)
17. D’Angelo, J.: A note on the Bergman kernel. Duke Math. J. 45(2), 259–265 (1978)
18. D’Angelo, J.: Real hypersurfaces, orders of contact, and applications. Ann. Math. (2) 115(3), 615–637 (1982)
19. D’Angelo, J.: Several Complex Variables and the Geometry of Real Hypersurfaces. CRC Press, Boca Raton, FL (1993)
20. D’Angelo, J.: An explicit computation of the Bergman kernel function. J. Geom. Anal. 4, 23–34 (1994)
21. Douglas, R.: On the smoothness of elements of Ext, Topics in modern operator theory Timisoara/Herculane, 1980, pp. 63–69, Operator Theory: Adv. Appl., 2, Birkhauser, Basel-Boston, Mass., 1981
22. Douglas, R.: A New Kind of Index Theorem, Analysis, Geometry and Topology of Elliptic Operators, pp. 369–382. World Sci. Publ., Hackensack, NJ (2006)
23. Douglas, R., Voiculescu, D.: On the smoothness of sphere extensions. J. Oper. Theory 61(1), 103–111 (1981)
24. Dunford, N., Schwartz, J.: Linear Operators. Interscience Publishers Inc, New York (1963)
25. Duren, P.: Bergman Spaces. American Mathematical Society, Providence (2004)
26. Engliš, M., Eschmeier, J.: Geometric Arveson–Douglas conjecture. Adv. Math. 274, 606–630 (2015)
27. Fang, Q., Xia, J.: Schatten class membership of Hankel operators on the unit sphere. J. Funct. Anal. 257(10), 3082–3134 (2009)
28. Feldman, M., Rochberg, R.: Singular value estimates for commutators and Hankel operators on the unit ball and the Heisenberg group, Analysis and partial differential equations, 121–159, Lecture Notes in Pure and Appl. Math., 122. Dekker, New York (1990)
29. Fu, S., Straube, E.: Compactness in the Neumann problem, Complex Analysis and Geometry (J. McNeal, ed.), Ohio State Math. Res. Inst. Publ. 9 (2001), 141–160
30. Gohberg, I., Krein, M.: Introduction to the Theory of Linear Nonsselfadjoint Operators in Hilbert Space. Amer. Math. Soc, Providence, RI (1969)
31. Gong, G.: Smooth extensions for finite CW complexes. Trans. Am. Math. Soc. 342(1), 343–358 (1994)
32. Henkin, G., Iordan, A.: Compactness of the Neumann operator for hyperconvex domains with non-smooth B-regular boundary. Math. Ann. 307(1), 151–168 (1997)
33. Isralowitz, J.: Schatten p class Hankel operators on the Segal–Bargmann space $H^2(n, d\mu)$ for $0 < p < 1$, J. Oper. Theory 55(1), 145–160 (2011)
34. Isralowitz, J.: Schatten p class commutators on the weighted Bergman space $L^2_{\alpha}(n, dv_\gamma)$ for $2n/(n + 1 + \gamma) < p < \infty$, Indiana Univ. Math. J. 62 (2013), 201–233
35. Jabbari, M., Tang, X.: An index theorem for quotients of Bergman spaces on egg domains, in preparation
36. Janson, S.: Hankel operators between weighted Bergman spaces. Ark. Mat. 26(2), 205–219 (1988)
37. Jarnicki, M., Pflug, P.: First Steps in Several Complex Variables: Reinhardt Domains. European Mathematical Society, Zürich (2008)
38. Kodama, A., Krantz, S., Ma, D.: A characterization of generalized complex ellipsoids in $\mathbb{C}^n$ and related results. Indiana Univ. Math. J. 41(1), 173–195 (1992)
39. Krantz, S., Li, S.-Y.: Boundedness and compactness of integral operators on spaces of homogeneous type and applications. II. J. Math. Anal. Appl. 258(2), 642–657 (2001)
40. Krantz, S., Li, S.-Y., Rochberg, R.: The effect of boundary geometry on Hankel operators belonging to the trace ideals of Bergman spaces. Integral Equ. Oper. Theory 28(2), 196–213 (1997)
41. Li, H.: Schatten class Hankel operators on the Bergman spaces of strongly pseudoconvex domains. Proc. Am. Math. Soc. 119(4), 1211–1221 (1993)
42. Li, H., Luecking, D.: Schatten class of Hankel and Toeplitz operators on the Bergman space of strongly pseudoconvex domains, Multivariable operator theory (Seattle, WA, 1993), 237–257, Contemp. Math., 185, Amer. Math. Soc., Providence, RI (1995)
43. Matsumoto, K.: On the analytic continuation of various multiple zeta-functions. In: Bennett, M.A. et al. (Eds.), Number Theory for the Millennium II, Proc. Millennial Conference on Number Theory, A K Peters, Wellesley (2002)  
44. McNeal, J.: A sufficient condition for compactness of the Neumann operator. J. Funct. Anal. 195(1), 190–205 (2002)
45. Pau, J.: Characterization of Schatten-class Hankel operators on weighted Bergman spaces. Duke Math. J. 165(14), 2771–2791 (2016)
46. Peller, V.: Hankel operators of class $S_p$ and their applications (rational approximation, Gaussian processes, the problem of majorization of operators), Mat. Sbornik 41.: 538–581. English translation: Math. USSR Sbornik 41(1982), 443–479 (1980)
47. Peller, V.: Vectorial Hankel operators, commutators and related operators of the Schatten-von Neumann class $\gamma_p$. Integral Equ. Oper. Theory 5(2), 244–272 (1982)
48. Peller, V.: A description of Hankel operators of class $S_p$ for, an investigation of the rate of rational approximation, and other applications, Mat. Sbornik 122 (1983), no. 4, 481–510. English translation: Math. USSR Sbornik 50 (1985), 465–494
49. Peller, V.: Hankel Operators and Their Applications. Springer, New York (2003)
50. Peloso, M.: Hankel operators on weighted Bergman spaces on strongly pseudoconvex domains. Illinois J. Math. 38(2), 223–249 (1994)
51. Raimondo, R.: Schatten-von Neumann Hankel operators on the Bergman space of planar domains. Integral Equ. Oper. Theory 62(2), 219–232 (2008)
52. Range, M.: Holomorphic Functions and Integral Representations in Several Complex Variables. Springer Verlag, New York (1986)
53. Rochberg, R.: Trace ideal criteria for Hankel operators and commutators. Indiana Univ. Math. J. 31(6), 913–925 (1982)
54. Rochberg, R., Semmes, S.: Nearly weakly orthonormal sequences, singular value estimates, and Calderon–Zygmund operators. J. Funct. Anal. 86(2), 237–306 (1989)
55. Salinas, N.: The formalism and the C*-algebra of the Bergman $n$-tuple. J. Oper. Theory 22(2), 325–343 (1989)
56. Salinas, N., Sheu, A., Upmeier, H.: Toeplitz operators on pseudoconvex domains and foliation C*-algebras. Ann. Math. (2) 130(3), 531–565 (1989)
57. Semmes, S.: Trace ideal criteria for Hankel operators, and applications to Besov spaces. Integral Equ. Oper. Theory 7, 241–281 (1984)
58. Simon, B.: Trace Ideals and Their Applications, 2nd edn. American Mathematical Society, Providence, RI (2005)
59. Tornheim, L.: Harmonic double series. Am. J. Math. 72, 303–314 (1950)
60. Tricomi, F., Erdélyi, A.: The asymptotic expansion of a ratio of gamma functions. Pacific J. Math. 1, 133–142 (1951)
61. Upmeier, H.: Toeplitz operators and index theory in several complex variables. Birkhäuser Verlag, Basel (1996)
62. Wallstén, R.: Hankel operators between weighted Bergman spaces in the ball. Ark. Mat. 28(1), 183–192 (1990)
63. Xia, J.: On the Schatten class membership of Hankel operators on the unit ball. Illinois J. Math. 46, 913–928 (2002)
64. Xia, J.: Bergman commutators and norm ideals. J. Funct. Anal. 263(4), 988–1039 (2012)
65. Xia, J., Zheng, D.: Standard deviation and Schatten class Hankel operators on the Segal–Bargmann space. Indiana Univ. Math. J. 53(5), 1381–1399 (2004)
66. Zheng, D.: Schatten class Hankel operators on the Bergman space. Integral Equ. Oper. Theory 13(3), 442–459 (1990)
67. Zhu, K.: Schatten class Hankel operators on the Bergman space of the unit ball. Am. J. Math. 113, 147–167 (1991)
68. Zhu, K.: Spaces of Holomorphic Functions in the Unit Ball. Springer, New York (2005)
69. Zhu, K.: Operator Theory in Function Spaces, 2nd edn. American Mathematical Society, Providence, RI (2007)

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