A NOTE ON CASTELNUOVO-MUMFORD REGULARITY AND
HILBERT COEFFICIENTS

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Abstract. New upper and lower bounds on the Castelnuovo-Mumford regularity
are given in terms of the Hilbert coefficients. Examples are provided to show that
these bounds are in some sense nearly sharp.

1. Introduction

Hilbert coefficients are basic invariants associated to primary ideals and the
Castelnuovo-Mumford regularity is one of the most important invariants measuring
the complexity of a graded algebra. It was shown in [2, Theorem 17.3.6], [9, Lemma
4] and [10, Theorem 2] that one can bound the Castelnuovo-Mumford regularity of
a graded algebra in terms of its Hilbert coefficients. These bounds are recursively
defined. In [5, Lemma 1.2], an explicit bound was given. However these bounds are
far from being sharp. In this note, under an additional assumption, we provide a
new upper bound (see Theorem 2.2). The main meaning of this work is not just to
provide another bound, but also to show that the new bound is nearly sharp in any
dimension.

The approach in [2] and [9] uses induction on the dimension and an idea of Mum-
ford; it works for graded modules, while our approach, like in the proofs of [4, Lemma
3.1] and [7, Theorem 9], uses an extended version of the Gotzmann’s regularity the-
orem in [1]. Therefore this new bound only holds for graded algebras.

Using a result in the Erratum of [5] we also provide a rough lower bound on
the Castelnuovo-Mumford regularity of a graded algebra in terms of its Hilbert

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coefficients.
coefficients, see Proposition 2.7. An example is given to show that this bound is also in some sense nearly sharp.

We also pose a question on bounding the Castelnuovo-Mumford regularity of the associated graded ring of a local ring in terms of fewer number of Hilbert coefficients and a question on the dependence of Hilbert coefficients.

2. Results

Let \( R = \oplus_{n \geq 0} R_n \) be a Noetherian standard graded ring over a local Artinian ring \((R_0, \mathfrak{m}_0)\). We always assume that \( R_0/\mathfrak{m}_0 \) is an infinite field. Let \( E \) be a finitely generated graded \( R \)-module of dimension \( d \). First let us recall some notation. For \( 0 \leq i \leq d \), put

\[
    a_i(E) = \sup \{ n \mid H^k_{R_n}(E)_{n} \neq 0 \},
\]

where \( R_+ = \oplus_{n > 0} R_n \). The Castelnuovo-Mumford regularity of \( E \) is defined by

\[
    \text{reg}(E) = \max \{ a_i(E) + i \mid 0 \leq i \leq d \},
\]

and the Castelnuovo-Mumford regularity of \( E \) at and above level \( p \), \( 0 \leq p \leq d \), is defined by

\[
    \text{reg}^p(E) = \max \{ a_i(E) + i \mid p \leq i \leq d \}.
\]

We denote the Hilbert function \( \ell_{R_0}(E_i) \) and the Hilbert polynomial of \( E \) by \( h_E(t) \) and \( p_E(t) \), respectively. Writing \( p_E(t) \) in the form:

\[
    p_E(t) = \sum_{i=0}^{d-1} (-1)^i e_i(E) \binom{t + d - 1 - i}{d - 1 - i},
\]

the numbers \( e_i(E) \) are called Hilbert coefficients of \( E \). For \( p \leq d - 1 \), let

\[
    \xi_p(E) = \max \{ e_0(E), |e_1(E)|, \ldots, |e_p(E)| \}.
\]

It was shown in [2, Theorem 17.3.6], [9, Lemma 4] and [10, Theorem 2] (see also [5, Lemma 1.2] for an explicit formula) that \( \text{reg}^1(E) \) can be bounded in terms of \( \xi_{d-1}(E) \), provided that \( E \) is generated in degrees at most 0. In the case of quotient rings of \( R \), the following main result provides a much better bound. Its proof uses an approach developed in [7, Section 3].

**Theorem 2.1.** Let \( R_0 \) be an Artinian equicharacteristic local ring and \( I \subset R_+ \) a homogeneous ideal of \( R \) such that \( \dim R/I = d \geq 1 \). Then, for all \( 1 \leq p \leq d \), we have

\[
    \text{reg}^p(R/I) \leq (\xi_{d-p}(R/I) + 1)^{2^{d-p}} - 2.
\]

**Proof.** We may assume that \( I \neq 0 \). Let \( e_j := e_j(R/I) \) and \( \xi_j := \xi_j(R/I) \). By [1, Corollary 3.5(i)], the Hilbert polynomial can be uniquely written in the form

\[
    p_{R/I}(t) = \sum_{i=0}^{d-1} \binom{c_1 + t}{t} + \binom{c_2 + t - 1}{t - 1} + \cdots + \binom{c_s + t - s + 1}{t - s + 1},
\]

where \( c_1 \geq c_2 \geq \cdots \geq c_s \geq 0 \) are integers. For \( 0 \leq j \leq d - 1 \) set

\[
    B_j = B_j(R/I) = \sharp \{ i ; c_i \geq (d - 1) - j \}.
\]

Note that \( e_0 = B_0 \leq B_1 \leq \cdots \leq B_{d-1} = s \) and \( \xi_0 \leq \xi_1 \leq \cdots \leq \xi_{d-1} \). By [1, Corollary 3.5(ii)], \( \text{reg}^p(R/I) \leq B_{d-p} - 1 \). Hence, it suffices to show that

\[
    B_j \leq (\xi_j + 1)^{2^j} - 1,
\]
for all $0 \leq j \leq d - 1$. Since $B_0 = e_0 = \xi_0$, the inequality holds for $j = 0$. For $j \geq 1$, by [1, Proposition 3.9] we have
\[
B_j = (-1)^j e_j + \left(\frac{B_{j-1} + 1}{2}\right) - \left(\frac{B_{j-2} + 1}{3}\right) + \cdots + (-1)^{j-1}\left(\frac{B_0 + 1}{j + 1}\right).
\] (1)

For $j = 1$ it yields
\[
B_1 = -e_1 + \left(\frac{B_0 + 1}{2}\right) = -e_1 + \left(\frac{e_0 + 1}{2}\right) \leq \xi_1 + \left(\frac{\xi_1 + 1}{2}\right) < (\xi_1 + 1)^2 - 1.
\]

Let $j \geq 2$. By the induction assumption we may assume that
\[
B_{j-l} \leq (\xi_{j-l} + 1)^{2^{j-l}} - 1 \leq (\xi_j + 1)^{2^{j-l}} - 1,
\]
for all $1 \leq l \leq j$. Since $2^l \geq l + 1$, we have
\[
\left(\frac{B_{j-l} + 1}{l + 1}\right) \leq \left(\frac{B_{j-l} + 1}{l + 1}\right)^{l+1} \leq \left(\frac{B_{j-l} + 1}{l + 1}\right)^{l+1} \leq \left(\frac{(\xi_j + 1)^{2^l}}{(l + 1)!}\right).
\]

By (1) this implies
\[
B_j \leq |e_j| + \left(\frac{B_{j-1} + 1}{2}\right) + \left(\frac{B_{j-2} + 1}{4}\right) + \cdots \\
\leq \xi_j + (\xi_j + 1)^2 \left(\frac{1}{2^1} + \frac{1}{2^2} + \cdots\right) \\
< \xi_j + \frac{1}{12}(\xi_j + 1)^{2^l} < (\xi_j + 1)^{2^l} - 1.
\]

This completes the proof of the theorem. \qed

Let $I$ be an $m$-primary ideal of a $d$-dimensional Noetherian local ring $(A, m)$. Then, for $n \gg 0$, we can write
\[
H_{A/I}(n) := \ell(A/I^{n+1}) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{n + d - i}{d - i}.
\]
The integers $e_i(I)$ are called Hilbert coefficients of $I$. Let $G(I) = A/I \oplus I/I^2 \oplus \cdots$. Note that $e_i(I) = e_i(G(I))$ for $0 \leq i \leq d - 1$.

As an application of the above theorem, we can give a much better bound for $\text{reg}(G(I))$ than the one in [5, Theorem 1.8]. We always assume that $A/m$ is infinite.

**Theorem 2.2.** Let $I$ be an $m$-primary ideal of a Noetherian local ring $(A, m)$ of dimension $d \geq 1$ such that $A/I$ is equicharacteristic, and let
\[
\xi_p(I) := \max\{e_0(I), |e_1(I)|, \ldots, |e_p(I)|\}.
\]
Then
\[
\text{reg}(G(I)) \leq (\xi_d(I) + 1)^{2^d} - 2.
\]
Moreover, if $\text{depth}(A) \geq 1$, then
\[
\text{reg}(G(I)) \leq (\xi_{d-1}(I) + 1)^{2^{d-1}} - 2.
\]

**Proof.** If $\text{depth}(A) \geq 1$, then by [6, Theorem 5.2], $\text{reg}(G(I)) = \text{reg}^1(G(I))$. Note that $\xi_{d-1}(G(I)) = \xi_{d-1}(I)$. Hence the second statement follows from Theorem 2.1.

In order to prove the first statement, let $x$ be an indeterminate of $\deg(x) = 1$ and let $S = G(I)[x]$ be a standard graded ring of dimension $d + 1$ over $A/m$. Then $h_S(n) = \sum_{i=0}^{d} \ell_A(I^i/I^{i+1}) = \ell(A/I^{n+1})$. Hence $e_i(I) = e_i(S)$ for all $i \leq d$ and $\xi_d(S) = \xi_d(I)$. Since $x$ is regular on $S$, $\text{reg}(G(I)) = \text{reg}^1(S)$. Now we can again apply Theorem 2.1 to $S$ in order to complete the proof. \qed

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As said in the introduction, bounds on \( \text{reg}^1(E) \) for any Noetherian graded \( R \)-module \( E \) in terms of Hilbert coefficients were given in [2] and [9]. A bound on \( \text{reg}(G_M(I)) \) is given in [5, Theorem 1.8], where \( I \) is an \( m \)-primary ideal of a Noetherian local ring \((A, m)\), \( M \) is a finite \( A \)-module and \( G_M(I) = \oplus_{n \geq 0} I^n M/I^{n+1} M \). The orders of these bounds are bigger than the one in the above two theorems. We don’t know if the bounds \( \text{reg}^p(E) \leq (\xi_{d-p}(E) + 1)^{2^d-p} - 2 \) and \( \text{reg}(G_M(I)) \leq (\xi_d(I, M) + 1)^{2^d} - 2 \) hold, where \( e_i(I, M) \) denotes the \( i \)-th Hilbert coefficients of \( I \) with respect to \( M \) and \( \xi_d(I, M) = \max\{e_0(I, M), |e_1(I, M)|, \ldots, |e_d(I, M)|\} \).

**Corollary 2.3.** With the assumptions in Theorem 2.2, we have

\[
(-1)^{i-1} e_i(I) < \frac{7}{12} (\xi_{i-1}(I) + 1)^{2^i} - e_0(I),
\]

for all \( 1 \leq i \leq d \).

**Proof.** Keep the notation in the proof of the previous two theorems. Then by (1), we have

\[
(-1)^{i-1} e_i = \left( \frac{B_{i-1}(S) + 1}{2} \right) - \left( \frac{B_{i-2}(S) + 1}{3} \right) + \cdots + (-1)^{i-1} \left( \frac{B_0(S) + 1}{i+1} \right) - B_i(S).
\]

In the proof of Theorem 2.1 we have shown that

\[
B_j(S) \leq (\xi_j(S) + 1)^{2^j} - 1 \leq (\xi_{i-1}(S) + 1)^{2^j} - 1 = (\xi_{i-1} + 1)^{2^j} - 1,
\]

for all \( j \leq i - 1 \). Since \( B_i(S) \geq B_0(S) = e_0 \), the last part of computation in the proof of Theorem 2.1 shows that

\[
(-1)^{i-1} e_i < \frac{7}{12} (\xi_{i-1} + 1)^{2^i} - e_0.
\]

\( \square \)

It is well-known that \( e_1(I) \leq \binom{e_0(I)}{2} \). Without any assumption on the local ring \( A \), one cannot bound \( |e_i(I)| \) in terms of \( \xi_{i-1}(I) \), see [5, Example 2.7]. On the other hand, when \( A \) is a Cohen-Macaulay, both \( e_1(I) \) and \( e_2(I) \) are non-negative and \( e_2(I) \leq \binom{e_1(I)}{2} \) (see [8]). Moreover, in this case, it was first proved in [9] that \( \text{reg}(G(I)) \) and all \( |e_i(I)| \), where \( 1 \leq i \leq d \), are bounded in terms of \( e_0(I) \) (see [11] for an improvement). Generalizing this fact, it was shown in [5, Theorem 2.4] that \( \text{reg}(G(I)) \) and all \( |e_i(I)| \) are bounded in terms of \( \xi_{d-t}(I) \), where \( t = \text{depth} A \) and \( d - t + 1 \leq i \leq d \). However, the bounds in [5] are too large. Therefore, in view of Theorem 2.2 and Corollary 2.3, we would like to ask

**Question 2.4.** Let \( I \) be an \( m \)-primary ideal of a Noetherian local ring \((A, m)\) of dimension \( d \geq 1 \) and depth \( t \) such that \( A/I \) is equicharacteristic. Do the following inequalities hold

(i) \( \text{reg}(G(I)) < (\xi_{d-t}(I) + 1)^{2^d} \), and

(ii) \( |e_i(I)| < (\xi_{d-t}(I) + 1)^{2^i} \) for all \( d - t + 1 \leq i \leq d \)?

Note, by Theorem 2.2, that (i) holds for \( t \leq 1 \). The above question is of interest even in the case of a Cohen-Macaulay ring \( A \) and \( I = m \). The following example shows that the bounds in Theorem 2.1 and Theorem 2.2 are almost optimal.
Example 2.5. Let \( f_1, \ldots, f_e \) be a regular sequence of homogeneous polynomials of degree \( \delta \geq \max\{e, 36\} \) in \( R = K[x_1, \ldots, x_n] \) such that \( 2d + 1 < c \), where \( d := n - c \geq 3 \). Let \( a = (f_1, \ldots, f_c) \subset R \). Then \( e_0 := e_0(R/a) = \delta^c \) and \( \text{reg}(R/a) = c\delta - c \). By [5, Proposition A in Corrigendum] we have

\[
|e_i(R/a)| \leq e_0(c\delta)^i < e_0\delta^d < c_0^{1+\varepsilon}/\delta \quad \text{for all} \quad 1 \leq i \leq d - 1,
\]

where \( 0 < \varepsilon \leq 1 \) and \( \varepsilon \to 0 \) if \( c/d \to \infty \). Hence \( \xi_{d-1}(R/a) < c_0^{1+\varepsilon}/\delta \).

Let \( b = \text{lex}(I) \) be the lex-segment ideal of \( a \), that is the ideal of \( R \) generated by all first \( h_a(m) \) monomials in \( R \) of degree \( m \) with respect to the lexicographic order, where \( m \) runs through all positive integers. This ideal has the same Hilbert function as \( a \). Hence \( p_{R/b}(t) = p_{R/a}(t) \), which implies \( \xi_{d-1}(R/b) = \xi_{d-1}(R/a) < c_0^{1+\varepsilon}/\delta \leq (c_0/4)^{1+\varepsilon} \). Since \( R/a \) is a Cohen-Macaulay ring, by [7, Proposition 12], we have

\[
\text{reg}^1(R/b) = \text{reg}(R/b) \geq \frac{c_0^{2d-1}}{9^{2d-2}} - 1 = \frac{9(c_0/3)^{2d-1} - 1}{9(\xi_{d-1}(R/b))^{(1+\varepsilon)2d-2}},
\]

where \( 0 < \varepsilon = 2/(1+\varepsilon) - 1 < 1 \) and \( \varepsilon \to 1 \) when \( c/d \to \infty \). This shows that the bound in Theorem 2.1 (in the case \( p = 1 \)) is almost optimal in the sense that there is no constant \( c < 1 \) such that \( \text{reg}^1(R/I) \leq c(\xi_{d-1}(R/I))^{2d-1} \) for all \( I \) and \( R \). Note that an upper bound on \( \text{reg}(R/b) \) (in terms of \( \delta \)) in this example was first given in [4, Lemma 3.1].

For an example in the local case, let \( S = K[[x_1, \ldots, x_n]] \) and \( A = S/b, I = m. \) Then \( G(m) \cong R/b. \) It was observed in the proof of Theorem 2.2, that \( e_d(m) = e_d((R/b)[x]) = e_d((R/a)[x]), \) where \( x \) is an indeterminate. Again by [5, Proposition A in Corrigendum] we have \( e_d(m) \leq e_0(\text{reg}(R/I) + 1)^d < c_0^{1+\varepsilon}/\delta. \) So, in this case, we still have \( \xi_d(m) < c_0^{1+\varepsilon}/\delta, \) and

\[
\text{reg}(G(m)) = \text{reg}(R/b) > 9(\xi_d(m))^{(1+\varepsilon)2d-2}.
\]

We can also give very rough lower bounds for \( \text{reg}(R) \) in terms of Hilbert coefficients.

Proposition 2.6. Let \( R_0 \) be an Artinian local ring and \( I \subset R_+ \) a homogeneous ideal of a standard graded algebra \( R = R_0[x_1, \ldots, x_n] \) in \( n \) indeterminates such that \( \dim R/I = d \geq 1 \). Let \( c = n - d \). Then we have

\[
e_0(R/I) \leq \ell(R_0) \left( \frac{\text{reg}(R/I) + c}{c} \right).
\]

The equality holds if and only if \( R/I \) is a Cohen-Macaulay ring and its Hilbert-Poincaré series \( HP_{R/I}(z) := \sum_{n \geq 0} h_{R/I}(n)z^n \) is equal to

\[
\sum_{i=0}^{a} \ell(R_0) \frac{(c+i-1)!}{(1-y)^d} z^i,
\]

for some \( a \geq 0. \)

Proof. Put \( a := \text{reg}(R/I). \) Without loss of generality, we may assume that \( y_1 := x_{c+1}, \ldots, y_d := x_n \) form a filter-regular sequence of \( R/I, \) that is, all modules

\[
0 :_{R/(I, y_1, \ldots, y_d)} R y_j + 1, \quad 0 \leq j \leq d, \quad \text{are of finite length. Since} \quad \text{reg}(R/(I, y_1, \ldots, y_d)R) \leq
\]

\[
\sum_{i=0}^{a} \ell(R_0) \frac{(c+i-1)!}{(1-y)^d} z^i,
\]

for some \( a \geq 0. \)
reg\((R/I) = a\) (see, e.g., [2, Proposition 18.3.11]) and \(R/(I, y_1, ..., y_d)R\) is an epimorphic image of \(R_0[x_1, ..., x_c]\), we have

\[
e_0(R/I) \leq B := \ell_{R_0}(R/(I, y_1, ..., y_d)R) \leq \sum_{i=0}^{a} \ell_{R_0}(R_0[x_1, ..., x_c]_i) = \ell(R_0)\binom{a + c}{c}.
\]

If \(e_0(R/I) = \ell(R_0)\binom{a+c}{c}\), then \(e_0(R/I) = B\) and \(\ell_{R_0}([R/(I, y_1, ..., y_d)R]_i) = \binom{c+i-1}{i}\) for all \(0 \leq i \leq a\). This implies that \(R/I\) is a Cohen-Macaulay ring and (using, e.g., [3, Remark 4.1.11])

\[
HP_{R/I}(z) = \frac{\sum_{i=0}^{a} \ell_{R_0}([R/(I, y_1, ..., y_d)R]_i)z^i}{(1-z)^d} = \frac{\sum_{i=0}^{a} \ell(R_0)\binom{c+i-1}{i}z^i}{(1-z)^d}.
\]

Conversely, if \(R/I\) is a Cohen-Macaulay ring, then \(\text{reg}(R/(I, y_1, ..., y_d)R) = \text{reg}(R/I) = a\), and if also \(HP_{R/I}(z) = \sum_{i=0}^{a} \ell(R_0)\binom{c+i-1}{i}z^i\), then from this Hilbert-Poincaré series we can compute the Hilbert coefficient (see, e.g., [3, Proposition 4.1.9]):

\[
e_0(R/I) = \ell(R_0)\sum_{i=0}^{a} \binom{c+i-1}{i} = \binom{a+c}{c}.
\]

\(\square\)

The lower bound in Proposition 2.6 is attained by the ideal \(I = (x_1, ..., x_c)^{a+1}, a \geq 0\). It is also attained by the so-called Stanley-Reisner ideal of a cyclic polytope \(C(n, d)\), i.e., the intersection of all possible monomial ideals generated by \(c\) variables in \(R = K[x_1, ..., x_n]\), where \(c = n - d \geq 1\) (see, e.g., [3, Subsection 5.2]). In this case, \(e_0(R/I) = \binom{n}{c}\) and \(\text{reg}(R/I) = d\).

Below is a lower bound using other Hilbert coefficients. Using the ideal \(I = (x_1, ..., x_c)^{a+1}, a \geq 0\), one can easily see that the bound is nearly sharp.

**Proposition 2.7.** Let \(R_0\) be an Artinian local ring and \(I \subset R_+\) a homogeneous ideal of a standard graded algebra \(R = R_0[x_1, ..., x_n]\) in \(n\) indeterminates such that \(\dim R/I = d \geq 1\). Let \(c = n - d\). Then we have

\[
\text{reg}(R/I) \geq \max\{\sqrt[\ell(R_0)-1]{|e_i(R/I)|/\ell(R_0)}, \sqrt[\ell(R_0)-1]{|e_i(R/I)|/\ell(R_0)} (c+1)/2 : 1 \leq i \leq d-1\}.
\]

**Proof.** Keep the notation in the proof of the previous proposition. Then it was shown that \(B \leq \ell(R_0)\binom{a+c}{c}\). This implies \(e_0(R/I) \leq B \leq \ell(R_0)\sqrt[a+c][\ell(R_0)-1]/2\), and so

\[
a \geq \sqrt[\ell(R_0)-1]{|e_i(R/I)|/\ell(R_0)} - (c+1)/2.
\]

Since \(\binom{a+c}{c} \leq (a+1)^c\), we also have \(B \leq \ell(R_0)(a+1)^c\). Hence, by [5, Proposition A in Corrigendum], for all \(1 \leq i < d\) we get

\[
|e_i(R/I)| \leq B(a+1)^i \leq \ell(R_0)(a+1)^c(a+1)^i = \ell(R_0)(a+1)^{c+i},
\]

which yields the statement of the proposition. \(\square\)
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