CANCELLATION FOR 4-MANIFOLDS WITH VIRTUALLY ABELIAN FUNDAMENTAL GROUP

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Abstract. Suppose \(X\) and \(Y\) are compact connected topological 4-manifolds with fundamental group \(\pi\). For any \(r \geq 0\), \(Y\) is \(r\)-stably homeomorphic to \(X\) if \(Y \# r(S^2 \times S^2)\) is homeomorphic to \(X \# r(S^2 \times S^2)\). How close is stable homeomorphism to homeomorphism?

When the common fundamental group \(\pi\) is virtually abelian, we show that large \(r\) can be diminished to \(n + 2\), where \(\pi\) has a finite-index subgroup that is free-abelian of rank \(n\). In particular, if \(\pi\) is finite then \(n = 0\), hence \(X\) and \(Y\) are 2-stably homeomorphic, which is one \(S^2 \times S^2\) summand in excess of the cancellation theorem of Hambleton–Kreck [HK93].

The last section is a case-study investigation of the homeomorphism classification of closed manifolds in the tangential homotopy type of \(X = X_1 \# X_n\), where \(X_n\) are closed nonorientable topological 4-manifolds whose fundamental groups have order two [HKT94].

1. Introduction

Suppose \(X\) is a compact connected smooth 4-manifold, with fundamental group \(\pi\) and orientation character \(\omega: \pi \rightarrow \{\pm 1\}\). Our motivation herein is the Cappell–Shaneson stable surgery sequence [CS71, 3.1], whose construction involves certain stable diffeomorphisms. These explicit self-diffeomorphisms lead to a modified version of Wall realization \(\text{rel } \partial X:\)

\[
L^r_5(\mathbb{Z}[\pi^\omega]) \times S^r_{\text{DIFF}}(X) \longrightarrow \overline{S}^r_{\text{DIFF}}(X),
\]

where \(S\) is the simple smooth structure set and \(\overline{S}\) and is the stable structure set. Recall that the equivalence relation on these structure sets is smooth s-bordism of smooth manifold homotopy structures. The actual statement of [CS71, Theorem 3.1] is sharper in that the amount of stabilization, that is, the number of connected summands of \(S^2 \times S^2\), depends only on the rank of a representative of a given element of the odd-dimensional \(L\)-group.

In the case \(X\) is sufficiently large, in that it contains a two-sided incompressible smooth 3-submanifold \(\Sigma\), a periodicity argument using Cappell’s decomposition [Cap74, 7] shows that the restriction of the above action on \(S^r_{\text{DIFF}}(X)\) to the summand UNil_5 of \(L^r_5(\mathbb{Z}[\pi^\omega])\) is free. Therefore for each nonzero element of this exotic UNil-group, there exists a distinct, stable, smooth homotopy structure on \(X\), restricting to a diffeomorphism on \(\partial X\), which is not \(\mathbb{Z}[\pi_1(\Sigma)]\)-homology splittable along \(\Sigma\). If \(\Sigma\) is the 3-sphere, the TOF case is [Kha12]. Furthermore, when \(X\) is a connected sum of two copies of \(\mathbb{RP}^4\), see [JK06] and [BDK07].

For any \(r \geq 0\), denote the \(r\)-stabilization of \(X\) by

\[
X_r := X \# r(S^2 \times S^2).
\]

Acknowledgments. I would like to thank Jim Davis for having interested me in relating stabilization to non-splittably fake connected sums of 4-manifolds. Completed under his supervision, this long-delayed paper constitutes a chapter of the author’s thesis [Kha06]; note Proposition 2.2 was recently extended from virtually cyclic to virtually abelian groups.
2. On the topological classification of 4-manifolds

The main result (2.4) of this section is an upper bound on the number of $S^2 \times S^2$ connected summands sufficient for a stable homeomorphism, where the fundamental group of $X$ lies in a certain class of good groups. By using Freedman–Quinn surgery [FQ90, §11], if $X$ is also sufficiently large (2.3 for example), each nonzero element $\theta$ of the UNil-group and simple DIFF homotopy structure $(Y, h : Y \to X)$ pair to form a distinct TOP homotopy structure $(Y_\theta, h_\theta)$ that represents the DIFF homotopy structure $\theta \cdot (Y, h)$ obtained from (1).

2.1. Statement of results. For finite groups $\pi$, the theorem’s conclusion and the proof’s topology are similar to Hambleton–Kreck [HK93]. However, the algebra is quite different.

**Theorem 2.1.** Suppose $\pi$ is a good group (in the sense of [FQ90]) with orientation character $\omega : \pi \to \{\pm 1\}$. Consider $A := \mathbb{Z}[\pi^\omega]$, a group ring with involution: $\overline{x} = \omega(x)x^{-1}$. Select an involution-invariant subring $R$ of the commutative $\text{Center}(A)$. Its norm subring is

$$R_0 := \left\{ \sum x_i \pi \mid x_i \in R \right\}.$$

Suppose $A$ is a finitely generated $R_0$-module, $R_0$ is noetherian, and the dimension $d$ is finite:

$$d := \dim(\text{maxspec } R_0) < \infty.$$

Now suppose that $X$ is a compact connected TOP 4-manifold with

$$\pi(X), w_1 = (\pi, \omega)$$

and that it has the form

$$(X, \partial X) = (X_{-1}, \partial X) \# (S^2 \times S^2).$$

If $X_r$ is homeomorphic to $Y_r$, for some $r \geq 0$, then $X_d$ is homeomorphic to $Y_d$.

Here are the class of examples of good fundamental groups promised in the paper’s title.

**Proposition 2.2.** Suppose $\pi$ is a finitely generated, virtually abelian group, with any homomorphism $\omega : \pi \to \{\pm 1\}$. For some $R$, the pair $R \# \pi$, $R_0$ satisfies the above hypotheses: $\pi$ is good, $A$ is a finitely generated $R_0$-module, $R_0$ is noetherian, and $d$ is finite. Furthermore, $d = n + 1$, where $\pi$ contains a finite-index subgroup that is free-abelian of finite rank $n \geq 0$.

The author’s original motivations are infinite virtually cyclic groups of the second kind.

**Corollary 2.3.** Let $X$ be a compact connected TOP 4-manifold whose fundamental group is an amalgamated product $G_- \ast_F G_+$ with $F$ a finite common subgroup of $G_\pm$ of index two. If $Y$ is stably homeomorphic to $X$, then $Y \# 3(S^2 \times S^2)$ is homeomorphic to $X \# 3(S^2 \times S^2)$.

**Proof.** Division of $G_\pm$ by the normal subgroup $F$ yields a short exact sequence of groups:

$$1 \longrightarrow F \longrightarrow G_- \ast_F G_+ \longrightarrow C_2 \ast C_2 \cong C_\infty \ast_{-1} C_2 \longrightarrow 1.$$

So $\pi_1(X)$ contains an infinite cyclic group of finite index (namely, twice the order of $F$). Now apply Proposition 2.2 with $n = 1$ ($d = 2$). Then apply Theorem 2.1 to $X \# (S^2 \times S^2)$. □

Given the full strength of the proposition, we generalize the above specialized corollary.

**Corollary 2.4.** Let $X$ be a compact connected TOP 4-manifold whose fundamental group is virtually abelian: say $\pi_1(X)$ contains a finite-index subgroup that is free-abelian of rank $n < \infty$. If $Y$ is stably homeomorphic to $X$, then $Y$ is $(n + 2)$-stably homeomorphic to $X$. □

More generally, can we reach the same conclusion if $\pi$ has a finite-index subgroup $\Gamma$ that is polycyclic of Hirsch length $n$? The example $\pi = \mathbb{Z}^2 \times\mathcal{Z}_{\{1\}}$ is not virtually abelian.
2.2. Definitions and lemmas. An exposition of the following concepts with applications is available in Bak’s book [Bak81]. We assume the reader knows the more standard notions.

**Definition 2.5 ([Bas73, I:4.1]).** A *unitary ring* $(A, \lambda, \Lambda)$ consists of a ring with involution $A$, an element

$$\lambda \in \text{Center}(A) \quad \text{satisfying} \quad \lambda \bar{\lambda} = 1,$$

and a *form parameter* $\Lambda$. This is an abelian subgroup of $A$ satisfying

$$\{ a + \lambda \overline{a} \mid a \in A \} \subseteq \Lambda \subseteq \{ a \in A \mid a - \lambda \overline{a} = 0 \}$$

and

$$ra \overline{a} \in \Lambda \quad \text{for all} \quad r \in A \text{ and } a \in \Lambda.$$

Here is a left-handed classical definition discussed in the equivalence after its reference.

**Definition 2.6 ([Bas73, I:4.4]).** We regard a *quadratic module* over a unitary ring $(A, \lambda, \Lambda)$ as a triple $(M, \langle \cdot, \cdot \rangle, \mu)$ consisting of a left $A$-module $M$, a bi-additive function

$$\langle \cdot, \cdot \rangle : M \times M \to A \quad \text{such that} \quad \langle ax, by \rangle = a \langle x, y \rangle \bar{\overline{y}} \quad \text{and} \quad \langle y, x \rangle = \lambda \langle x, y \rangle$$

(called a $\lambda$-*hermitian form*), and a function (called a $\Lambda$-*quadratic refinement*)

$$\mu : M \to A/\Lambda \quad \text{such that} \quad \mu(ax) = a \mu(x) \bar{\overline{y}} \quad \text{and} \quad \mu([x, y]) = \mu(x + y) - \mu(x) - \mu(y).$$

The following unitary automorphisms can be realized by diffeomorphisms [CS71, 1.5].

**Definition 2.7 ([Bas73, I:5.1]).** Let $(M, \langle \cdot, \cdot \rangle, \mu)$ be a quadratic module over a unitary ring $(A, \lambda, \Lambda)$. A *transvection* $\sigma_{u,v,\alpha}$ is an isometry of this structure defined by the formula

$$\sigma_{u,v,\alpha} : M \to M ; \quad x \mapsto x + \langle u, v \rangle u - \lambda \langle u, x \rangle v - \lambda \langle u, x \rangle au$$

where $u, v \in M$ and $a \in A$ are elements satisfying

$$\langle u, v \rangle = 0 \in A \quad \text{and} \quad \mu(u) = 0 \in A/\Lambda \quad \text{and} \quad \mu(v) = [a] \in A/\Lambda.$$

The following lemmas involve, for any finitely generated projective $A$-module $P = P^\ast$, a nonsingular $(+1)$-quadratic form over $A$ called the *hyperbolic construction*

$$\mathcal{H}(P) := (P \oplus P^\ast, \langle \cdot, \cdot \rangle, \mu) \quad \text{where} \quad \langle x + f, y + g \rangle := f(y) + g(x) \quad \text{and} \quad \mu(x + f) := [f(x)].$$

Topologically, $\mathcal{H}(A)$ is the equivariant intersection form of $S^2 \times S^2$ with coefficients in $A$.

**Lemma 2.8.** Consider a compact connected TOP 4-manifold $X$ with good fundamental group $\pi$ and orientation character $\omega : \pi \to \{\pm 1\}$. Define a ring with involution $A := \mathbb{Z}[\pi^\omega]$. Suppose that there is an orthogonal decomposition

$$K := \text{Ker} w_2(X) = V_0 \perp V_1$$

as quadratic submodules of the intersection form of $X$ over $A$, with a nonsingular restriction to $V_0$. Define a homology class and a free $A$-module

$$p_+ := \lbrack S_2^2 \times pt \rbrack$$

$$P_+ := A p_+.$$

Consider the summand

$$\mathcal{H}(P_+) = H_2(S_2^2 \times S_2^2; A)$$

of

$$H_2(X \# (S_2^2 \times S_2^2) \# (S_2^2 \times S_2^2) ; A).$$

Then for any transvection $\sigma_{p,v}$ on the quadratic module $K \perp \mathcal{H}(P_+)$ with $p \in V_0 \oplus P_+$ and $v \in K$, the stabilized isometry $\sigma_{p,v} \oplus 1_{H_2(S_2^2 \times S_2^2; A)}$ can be realized by a self-homeomorphism of $X \# 3(S_2^2 \times S_2^2)$ which restricts to the identity on $\partial X$. 

Remark 2.9. In the case that $\partial X$ is empty and $\pi_1(X)$ is finite, then Lemma 2.8 is exactly [HK93, Corollary 2.3]. Although it turns out that their proof works in our generality, we include a full exposition, providing details absent from Hambleton–Kreck [HK93].

Lemma 2.10. Suppose $X$ and $p$ satisfy the hypotheses of Lemma 2.8. If $p$ is unimodular in $V_0 \oplus P_+$, then the summand $X_1 = X \# (S^2 \times S^2)$ of $X_2$ can be topologically re-split so that $S^2 \times \text{pt}$ represents $p$. 

Proof. Since $V_0 \perp \mathcal{H}(P_+)$ is nonsingular, there exists an element $q \in V_0 \perp \mathcal{H}(P_+)$ such that $(p, q)$ is a hyperbolic pair. Since $p, q \in \text{Ker} \ w_2(X)$ and $w_2$ is the sole obstruction to framing the normal bundle in the universal cover, each homology class is represented by a canonical regular homotopy class of framed immersion

$$\alpha, \beta : S^2 \times \mathbb{R}^2 \to X_1$$

with transverse double-points. Since the self-intersection number of $\alpha$ vanishes, all its double-points pair to yield framed immersed Whitney discs; consider each disc separately:

$$W : D^2 \times \mathbb{R}^2 \to X_1.$$ 

Upon performing finger-moves to regularly homotope $W$, assume that one component of $\alpha (S^2 \times 0) \setminus W(D^2 \times \mathbb{R}^2)$ is a framed embedded disc

$$V : D^2 \times \mathbb{R}^2 \to X_1$$

and, by an arbitrarily small regular homotopy of $\beta$, that $\beta|S^2 \times 0$ is transverse to $W|\text{int} D^2 \times 0$ with algebraic intersection number 1 in $\mathbb{Z}[\pi_1(X_1)]$. Hence $W$ is a framed properly immersed disc in

$$\overline{X}_1 := X_1 \setminus \text{Im} V.$$

So, since $\pi_1(X_1) \cong \pi_1(X)$ is a good group, by Freedman’s disc theorem [FQ90, 5.1A], there exists a framed properly TOP embedded disc

$$W' : D^2 \times \mathbb{R}^2 \to \overline{X}_1$$

such that

$$W' = W \text{ on } \partial D^2 \times \mathbb{R}^2 \quad \text{and} \quad \text{Im } W' \subset \text{Im } W.$$

Therefore, by performing a Whitney move along $W'$, we obtain that $\alpha$ is regularly homotopic to a framed immersion with one fewer pair of self-intersection points. Thus $\alpha$ is regularly homotopic to a framed TOP embedding $\alpha'$. A similar argument, allowing an arbitrarily small regular homotopy of $\alpha'$, shows that $\beta$ is regularly homotopic to a framed TOP embedding $\beta'$ transverse to $\alpha'$, with a single intersection point

$$\alpha'(x_0 \times 0) = \beta'(y_0 \times 0)$$

such that the open disc

$$\Delta := \beta'(y_0 \times \mathbb{R}^2) \subset \alpha'(S^2 \times 0).$$

Define a closed disc

$$\Delta' := S^2 \setminus (\alpha')^{-1}(\Delta).$$

Surgery on $X_1$ along $\beta'$ yields a compact connected TOP 4-manifold $X'$. Hence $X_1$ is recovered by surgery on $X'$ along the framed embedded circle

$$\gamma : S^1 \times \mathbb{R}^2 \approx \text{nbhd}_{S^2}(\partial \Delta') \times \mathbb{R}^2 \to \alpha' X_1 \setminus \text{Im } \beta' \subset X'.$$
But the circle $\gamma$ is trivial in $X'$, since it extends via $\alpha'$ to a framed embedding of the disc $\Delta'$ in $X'$. Therefore we obtain a TOP re-splitting of the connected sum

$$X_1 \cong X' \# (S^2 \times S^2)$$

so that $S^2 \times pt$ of the right-hand side represents the image of $p$. \hfill \Box

The next algebraic lemma decomposes certain transvections so that the pieces fit into the previous topological lemma.

**Lemma 2.11.** Suppose $(A, \Lambda, \Lambda)$ is a unitary ring such that: the additive monoid of $A$ is generated by a subset $S$ of the unit group $(A^\times, \cdot)$. Let $K = V_0 \perp V_1$ be a quadratic module over $(A, \Lambda, \Lambda)$ with a nonsingular restriction to $V_0$, and let $P_{\pm}$ be free left $A$-modules of rank one. Then any stabilized transvection

$$\sigma_{p,a,v} \otimes 1_{\mathcal{H}(P_{\pm})} \quad \text{on} \quad K \perp \mathcal{H}(P_+) \perp \mathcal{H}(P_-)$$

with $p \in V_0 \oplus P_+$ and $v \in K$ is a composite of transvections $\sigma_{p,0,v_j}$ with unimodular $p_i \in V_0 \oplus P_+$ and isotropic $v_j \in K \oplus \mathcal{H}(P_-)$.

**Proof.** Using a symplectic basis $\{p_+, q_\pm\}$ of each hyperbolic plane $\mathcal{H}(P_{\pm})$, define elements of $K \oplus \mathcal{H}(P_+ \oplus P_-)$:

$$v_0 := v + p_- - aq_- \quad v_1 := -p_- \quad v_2 := aq_-$$

Then

$$v = 2 \sum_{i=0}^2 v_i.$$  

Observe that each $v_i \in K \oplus \mathcal{H}(P_-)$ is isotropic with $\langle v_i, p \rangle = 0$. So transvections $\sigma_{p,0,v_j}$ are defined. Note, by Definition 2.7, for all $x \in K \oplus \mathcal{H}(P_+ \oplus P_-)$, that

$$(\sigma_{p,0,v_i} \circ \sigma_{p,0,v_j} \circ \sigma_{p,0,v_0})(x) = x + \sum_i \langle v_i, x \rangle p - \sum_i \overline{\langle p, x \rangle} v_i - \overline{\langle p, x \rangle} \sum_{i<j} \langle v_j, v_i \rangle p$$

$$= x + \langle v, x \rangle p - \overline{\langle p, x \rangle} v - \overline{\langle p, x \rangle} ap$$

$$= \sigma_{p,a,v}(x) \otimes 1_{\mathcal{H}(P_{\pm})}.$$  

Therefore it suffices to consider the case that $v \in K \oplus \mathcal{H}(P_-)$ is isotropic. Write

$$p = p' \oplus p'' \in V_0 \oplus P_+.$$  

Define a unimodular element

$$p_0 := p' \oplus 1p_+.$$  

Note, since $P_+$ has rank one and by hypothesis, there exist $n \in \mathbb{Z}_{>0}$ and unimodular elements $p_1, \ldots, p_n \in S P_+ \subseteq P_+$ such that

$$p - p_0 = p'' - 1p_+ = \sum_{i=1}^n p_i.$$  

For each $1 \leq i \leq n$, write

$$p_i := s_ip_+ \quad \text{for some} \quad s_i \in S.$$
Observe for all $1 \leq i, j \leq n$ that

$$\langle v, p_i \rangle = 0$$
$$\mu(p_i) = s_i \mu(p_+ \) = 0$$
$$\langle p_i, p_j \rangle = s_i \langle p_+, p_+ \rangle s_j = 0.$$  

Hence, we also have

$$\langle v, p_0 \rangle = 0$$
$$\mu(p_0) = 0.$$  

Then transvections $\sigma_{p_0,v}$ are defined and commute, so note

$$\sigma_{p_0,v} = \prod_{i=0}^{n} \sigma_{p_i,0,v}.$$  

□

Proof of Lemma 2.8. Define a homology class and a free $A$-module

$$p_- := [S^2 \times pt]$$
$$P_- := Ap_-.$$  

Consider the $A$-module decomposition

$$H_2(X_2; A) = H_2(X; A) \oplus \mathcal{H}(P_+) \oplus \mathcal{H}(P_-).$$  

Observe that the unitary ring

$$(A, \lambda, \Lambda) = (\mathbb{Z}[\pi^\alpha], +1, \{a - \overline{a} | a \in A\})$$  

satisfies the hypothesis of Lemma 2.11 with the multiplicative subset

$$S = \pi \cup -\pi.$$  

Therefore the stabilized transvection

$$\sigma_{p,a,v} \oplus 1_{\mathcal{H}(P_-)}$$  

is a composite of transvections $\sigma_{p_0,v}$ with unimodular $p_i \in V_0 \oplus P_+$ and isotropic $v_j \in K \oplus \mathcal{H}(P_-)$. Then by Lemma 2.10, for each $i$, a TOP re-splitting

$$f_i : X_1 \approx X'\#(S^2 \times S^2)$$  

of the connected sum can be chosen so that $S^2 \times pt$ represents $p_i$. So by the Cappell–Shaneson realization theorem [CS71, 1.5]¹, for each $i$ and $j$, the pullback under $(f_i)$, of the stabilized transvection

$$\sigma_{p_0,v} \oplus 1_{H_2(S^2 \times S^2; A)} = \sigma_{p_0,0,v}$$  

is an isometry induced by a self-diffeomorphism of

$$(X'\#(S^2 \times S^2)) \#(S^2 \times S^2).$$  

Hence, by conjugation with the homeomorphism $f_i$, the above isometry is induced by a self-homeomorphism of

$$X_2 = X_1 \#(S^2 \times S^2).$$  

Thus the stabilized transvection

$$(\sigma_{p,a,v} \oplus 1_{\mathcal{H}(P_-)} \oplus 1_{H_2(S^2 \times S^2; A)})$$  

is induced by the stabilized composite self-homeomorphism of

$$X_3 = X_2 \#(S^2 \times S^2).$$  

¹Their theorem realizes any transvection of the form $\sigma_{p,a,v}$ by a diffeomorphism of the 1-stabilization.
2.3. Proof of the main theorem. Now we modify the induction of [HK93, Proof B]; our result will be one $S^2 \times S^2$ connected summand less efficient than Hambleton–Kreck [HK93] in the case that $\pi$ is finite. The main algebraic technique is a theorem of Bass [Bas73, IV:3.4] on the transitivity of a certain subgroup of isometries on the set of hyperbolic planes. We refer the reader to [Bas73, §IV:3] for the terminology used in our proof. The main topological technique is a certain clutching construction of an $s$-cobordism.

Proof of Theorem 2.1. We may assume $r \geq d + 1$. Let
\[ f : X \# r(S^2 \times S^2) \to Y \# r(S^2 \times S^2) \]
be a homeomorphism. We show that
\[ X := X \# (r - 1)(S^2 \times S^2) \]
is homeomorphic to
\[ Y := Y \# (r - 1)(S^2 \times S^2), \]
thus the result follows by backwards induction on $r$.

Consider Definition 2.15 and [Bas73, Hypotheses IV:3.1]. By our hypothesis and Lemma 2.16, the minimal form parameter $\Lambda := \{ a - \pi | a \in A \}$ makes $(A, \Lambda)$ a quasi-finite unitary $(R, +1)$-algebra. Note, since $X = X_{r-1}(S') \times S^2$ by hypothesis, that the rank $r + 1$ free $A$-module summand
\[ P := H_2((S')^2 \times pt \sqcup r(S^2 \times pt); A) \]
of
\[ \text{Ker } w_2(X \# (S^2 \times S^2)) \]
satisfies [Bas73, Case IV:3.2(a)]. Then, by [Bas73, Theorem IV:3.4], the subgroup $G$ of the group $U(H(P))$ of unitary automorphisms defined by
\[ G := \langle H(E(P)), EU(H(P)) \rangle \]
acts transitively on the set of hyperbolic pairs in $H(P)$. So, by [Bas73, Corollary IV:3.5] applied to the quadratic module
\[ V := \text{Ker } w_2(X_{r-1}), \]
the subgroup $G_1$ of $U(V \perp H(P))$ defined by
\[ G_1 := \left\{ 1_V \perp G, EU(H(P), P; V), EU(H(P), V, V) \right\} \]
acts transitively on the set of hyperbolic pairs in $V \perp H(P)$. Let
\[ (p_0, q_0) \quad \text{and} \quad (p_0', q_0') \]
be the standard basis of the summand $H_2(S^2 \times S^2; A)$ of
\[ H_2(X \# (S^2 \times S^2); A) \quad \text{and} \quad H_2(Y \# (S^2 \times S^2); A). \]
Therefore there exists an isometry $\varphi \in G_1$ of
\[ V \perp H(P) = \text{Ker } w_2(X \# (S^2 \times S^2)) \]
such that
\[ \varphi(p_0, q_0) = (f^-1)(p_0', q_0'). \]
Lemma 2.12. The isometry
\[ \varphi \oplus 1_{\mathcal{H}(3(S^2 \times S^2); A)} \]
is induced by a self-homeomorphism \( g \) of
\[ \mathcal{X} \# 4(S^2 \times S^2). \]

Then the homeomorphism
\[ h := (f \# 1_{3(S^2 \times S^2)}) \circ g : \mathcal{X} \# 4(S^2 \times S^2) \to \mathcal{Y} \# 4(S^2 \times S^2) \]
satisfies the equation
\[ h_*(p_i, q_i) = (p_i', q_i') \quad \text{for all} \quad 0 \leq i \leq 3. \]

Here the hyperbolic pairs
\[ \{ (p_i, q_i) \}_{i=1}^3 \quad \text{and} \quad \{ (p_i', q_i') \}_{i=1}^3 \]
in the last three \( S^2 \times S^2 \) summands are defined similarly to \((p_0, q_0)\) and \((p_0', q_0')\).

Lemma 2.13. The manifold triad \((W; \mathcal{X}, \mathcal{Y})\) is a compact \( \text{TOP} \) \( s \)-cobordism rel \( \partial \mathcal{X} \):
\[ W^5 := \mathcal{X} \times [0, 1] \sqcup 4(S^2 \times D^3) \bigcup_h \mathcal{Y} \times [0, 1] \sqcup 4(S^2 \times D^3). \]

Therefore, since \( \pi_1(\mathcal{X}) \cong \pi_1(\mathcal{X}) \) is a good group, by the \( \text{TOP} \) \( s \)-cobordism theorem [FQ90, 7.1A], \( \mathcal{X} \) is homeomorphic to \( \mathcal{Y} \). This proves the theorem by induction on \( r \). \( \square \)

Remark 2.14. The reason for restriction to the \( A \)-submodule
\[ K = \text{Ker} \, w_2(\mathcal{X}(S^2 \times S^2)) \]
is two-fold. Geometrically [CS71, p504], a unique quadratic refinement of the intersection form exists on \( K \), hence \( K \) is maximal. Also, the inverse image of \((p_0', q_0')\) under the isometry \( f \) is guaranteed to be a hyperbolic pair in \( K \), hence \( K \) is simultaneously minimal.

2.4. Remaining lemmas and proofs.

Definition 2.15 ([Bas73, IV:1.3]). An \( R_0 \)-algebra \( A \) is quasi-finite if, for each maximal ideal \( m \in \text{maxspec}(R_0) \), the following containment holds:
\[ mA_m \subseteq \text{rad} \, A_m \]
and that the following ring is left artinian:
\[ A[m] := A_m / \text{rad} \, A_m. \]

Here
\[ A_m := (R_0)_m \otimes_{R_0} A \]
is the localization of \( A \) at \( m \), and \( \text{rad} \, A_m \) is its Jacobson radical. The pair \((A, \Lambda)\) is a quasi-finite unitary \((R, \lambda)\)-algebra if \((A, \lambda, \Lambda)\) is a unitary ring, \( A \) is an \( R \)-algebra with involution, and \( A \) is a quasi-finite \( R_0 \)-algebra. Here \( R_0 \) is the subring of \( R \) generated by norms:
\[ R_0 = \left\{ \sum_i r_i \mathcal{I} \; \big| \; r_i \in R \right\}. \]

Lemma 2.16. Suppose \( A \) is an algebra over a ring \( R_0 \) such that \( A \) is a finitely generated left \( R_0 \)-module. Then \( A \) is a quasi-finite \( R_0 \)-algebra.
Proof. Let \( m \in \text{maxspec}(R_0) \). By [Bas68, Corollary III:2.5] to Nakayama’s lemma,
\[
A_m \cdot m = A_m \cdot \text{rad}(R_0)_m \subseteq \text{rad} A_m.
\]
Then
\[
A[m] = (A_m/mA_m)/((\text{rad} A_m)/mA_m)
\]
and is a finitely generated module over the field
\[
(R_0)_m/m(R_0)_m.
\]
by hypothesis. Therefore \( A[m] \) is left artinian, hence \( A \) is quasi-finite. □

The existence of the realization \( g \) is proven algebraically; refer to [Bas73, §II:3].

Proof of Lemma 2.12. Consider Lemma 2.8 applied to
\[
\mathcal{X}\#(S^2 \times S^2) \quad V_0 = \mathcal{H}(P) \quad V_1 = V.
\]
It suffices to show that the group \( G_1 \) is generated by a subset of the transvections \( \sigma_{p,a,v} \)
with \( p \in \mathcal{H}(P) \) and \( v \in V \oplus \mathcal{H}(P) \).

By [Bas73, Cases II:3.10(1–2)], the group \( EU(\mathcal{H}(P)) \)
is generated by all transvections \( \sigma_{u,a,v} \) with \( u \in P \) or \( u \in \overline{P}, v \in P \). By
[HK93, Definition 1.4], the group \( EU(\mathcal{H}(P), P; V) \)
is generated by all \( \sigma_{u,a,v} \) with \( u \in \overline{P}, v \in V \). In any case, \( p \in \mathcal{H}(P) \) and \( v \in V \oplus \mathcal{H}(P) \). □

The assertion is essentially that \( (W; \mathcal{X}, \mathcal{Y}) \) is a \( h \)-cobordism with zero Whitehead torsion.

Proof of Lemma 2.13. By the Seifert–vanKampen theorem, we have a pushout diagram
\[
\begin{array}{ccc}
\pi_1(\mathcal{X}\times [0,1]\natural 4(S^2 \times D^3)) & \xrightarrow{\partial} & \pi_1(\overline{\mathcal{Y}}\times [0,1]\natural 4(S^2 \times D^3)) \\
\downarrow & & \downarrow \\
\pi_1(\mathcal{X}\times [0,1]\natural 4(S^2 \times D^3)) & \xrightarrow{\partial} & \pi_1(W).
\end{array}
\]
So the maps induced by the inclusion \( \mathcal{X} \sqcup \overline{\mathcal{Y}} \to W \) are isomorphisms:
\[
i_* : \pi_1(\mathcal{X} \times 0) \to \pi_1(W)
\]
\[
j_* : \pi_1(\overline{\mathcal{Y}} \times 0) \to \pi_1(W).
\]
Denote \( \pi \) as the common fundamental group using these identifications.

Observe that the nontrivial boundary map \( \partial_3 \) of the cellular chain complex
\[
C_*(j; \mathbb{Z}[\pi]) : 0 \to \bigoplus_{0 \leq k < 4} \mathbb{Z}[\pi] \cdot (S^2 \times D^3) \xrightarrow{h \cdot \alpha} \bigoplus_{0 \leq k < 4} \mathbb{Z}[\pi] \cdot (D^2 \times S^2) \to 0
\]
is obtained as follows. First, attach thickened 2-cells to kill 4 copies of the trivial circle in $\overline{Y}$. Then, onto the resultant manifold

$$\overline{Y} \# 4(S^2 \times S^2),$$

attach thickened 3-cells to kill certain belt 2-spheres, which are the images under $h$ of the normal 2-spheres to the 4 copies of the trivial circle in $\overline{X}$. Hence, as morphisms of based left $\mathbb{Z}[\pi]-modules, the boundary map

$$\partial_3 = h_{\#} \circ \partial$$

is canonically identified with the morphism

$$h_* = 1 : H_2(4(S^2 \times S^2); \mathbb{Z}[\pi]) \longrightarrow H_2(4(S^2 \times S^2); \mathbb{Z}[\pi])$$

on homology induced by the attaching map $h$. This last equality holds by the construction of $h$, since

$$h_*(p_i, q_i) = (p'_i, q'_i) \quad \text{for all} \quad 0 \leq i < 4.$$ 

So the inclusion $f : \overline{Y} \to W$ has torsion

$$\tau(C_*(j; \mathbb{Z}[\pi])) = [h_\#] = [h_*] = [1] = 0 \in Wh(\pi).$$

A similar argument using $h^{-1}$ shows that the inclusion $i : \overline{X} \to W$ has zero torsion in $Wh(\pi)$. Therefore $(W; \overline{X}, \overline{Y})$ is a compact TOP s-cobordism rel $\partial X$. \hfill $\Box$

The final proof of this section employs the theory of commutative rings and subrings (including invariant theory), as well as language from algebraic geometry (spec and maxspec).

**Proof of Proposition 2.2.** Since $\pi$ is virtually polycyclic, it is a good group [FQ90, 5.1A]. Since $\pi$ is virtually abelian, by intersection with finitely many conjugates of a finite-index abelian subgroup, we find an exact sequence of groups with $\Gamma$ normal abelian and $G$ finite:

$$1 \longrightarrow \Gamma \longrightarrow \pi \longrightarrow G \longrightarrow 1.$$ 

This induces an action $G \curvearrowright \Gamma$. Consider these rings with involution and norm subring $R_0$: 

\begin{align*}
A & := \mathbb{Z}[\pi^G] \quad \text{where} \quad \forall g \in \pi : g = \omega(g)g^{-1} \\
A_0 & := \mathbb{Z}[\Gamma^G] \quad \text{which is a commutative ring} \\
R & := (A_0)^G = \{x \in A_0 \mid \forall g \in G : gx = x\} \\
R_0 & := \left\{ \sum_i x_i \overline{x}_i \mid x_i \in R \right\}.
\end{align*}

Note $\mathbb{Z}[\Gamma^G] \subset R \subset \text{Center}(A)$. Since $\pi$ is finitely generated, by Schreier’s lemma, so is $\Gamma$. So, by enlarging $G$ as needed, we may assume $\Gamma$ is a free-abelian group of a finite rank $n$.

First, we show that $A$ is a finitely generated $R_0$-module. Since $\Gamma$ has only finitely many right cosets in $\pi$, the group ring $A$ is a finitely generated $A_0$-module. Since $G$ is finite and $A_0$ is a finitely generated commutative ring and $\mathbb{Z}$ is a noetherian ring, by Bourbaki [Bou98, §V.1: Theorem 9.2], $A_0$ is a finitely generated $R$-module and $R$ is a finitely generated ring. By Bass [Bas73, Intro IV:1.1], the commutative ring $R$ is integral over its norm subring $R_0$. So, since $R$ is a finitely generated integral $R_0$-algebra, it follows that $R$ is a finitely generated $R_0$-module [Eis95, Corollary 4.5]. Therefore, $A$ is a finitely generated $R_0$-module.

Second, we show that $R_0$ is a noetherian ring. It follows from Hilbert’s basis theorem [Eis95, Corollary 1.3] that the finitely generated commutative $\mathbb{Z}$-algebra $A_0$ is noetherian. So $R_0$ is too, by Eakin’s theorem [Eis95, A3.7a], since $A_0$ is a finitely generated $R_0$-module.

Third, we show that the irreducible-dimension of the Zariski topology on $\Psi := \text{spec}(R_0)$ is $n + 1$. Here, by **irreducible-dimension** of a topological space, we mean the supremum of the lengths of proper chains of closed irreducible subsets, where **reducible** means being
the union of two nonempty closed proper subsets \([\text{Har77}, \S I:1]\). Krull dimension of a ring equals irreducible-dimension of its spec \([\text{Har77}, \S II:3.2.7]\); in particular \(\dim(\Psi) = \dim(R_0)\).

Since \(A_0\) is a finitely generated \(R_0\)-module, \(A_0\) is integral over \(R_0\) by \([\text{Eis95}, \text{Corollary 4.5}]\). So \(\dim(R_0) = \dim(A_0)\), by the Cohen–Seidenberg theorems \([\text{Eis95}, 4.15, 4.18; \text{Axiom D3}]\). Note \(\dim(A_0) = n + 1\) since \(\dim(\mathbb{Z}) = 1\), by \([\text{Eis95}, \text{Exercise 10.1}]\). Thus \(\dim(\Psi) = n + 1\).

Last, we show the topological space \(\Psi = \text{spec}(R_0)\) and its subspace \(\mathfrak{m} := \text{maxspec}(R_0)\) have equal irreducible-dimensions. Since \(R\) is a finitely generated commutative ring and \(R\) is a finitely generated \(R_0\)-module, by the Artin–Tate lemma \([\text{Eis95}, \text{Exercise 4.32}]\), also \(R_0\) is a finitely generated ring. Then \(R_0\) is a Jacobson ring, by the generalized Nullstellensatz \([\text{Eis95}, \text{Theorem 4.19}]\), since \(\mathbb{Z}\) is Jacobson. So we obtain an isomorphism of posets:

\[
\text{ClosedSets}(\Psi) \rightarrow \text{ClosedSets}(\mathfrak{m}); \quad C \mapsto C \cap \mathfrak{m} \quad \text{with inverse } \quad D \mapsto \text{closure}_\Psi(D).
\]

This correspondence is worked out by Grothendieck \([\text{Gro66, \IV.10: Proposition 1.2(c')}]; \text{D\'e\text{finitions} 1.3, 3.1, 4.1; Corollaire 4.6}]\). Hence \(\dim(\mathfrak{m}) = \dim(\Psi)\). Thus \(d = n + 1\). \(\square\)

3. Manifolds in the tangential homotopy type of \(\mathbb{RP}^4 \# \mathbb{RP}^4\)

Given a tangential homotopy equivalence to a certain \(\text{TOP} 4\)-manifold, the main goal of this section is to uniformly quantify the amount of topological stabilization sufficient for smoothing and for splitting along a two-sided 3-sphere. In particular, we sharpen a result of Jahren–Kwasik \([\text{JK06, Theorem 1(f)}]\) on connected sum of real projective 4-spaces \((3.5)\). Let \(X\) be a compact connected \(\text{DIFF} 4\)-manifold, and write

\[
(\pi, \omega) := (\pi_1(X), w_1(X)).
\]

Suppose \(\pi\) is good \([\text{FQ90}]\). Let \(\theta \in L^1_c(\mathbb{Z}[\pi^c])\); represent it by a simple unitary automorphism of the orthogonal sum of \(r\) copies of the hyperbolic plane for some \(r \geq 0\). Recall \([\text{FQ90, \S I}11]\) that there exists a unique homeomorphism class

\[
(X_\theta, h_\theta) \in S^r_{\text{TOP}}(X)
\]

as follows. It consists of a compact \(\text{TOP} 4\)-manifold \(X_\theta\) and a simple homotopy equivalence \(h_\theta : X_\theta \rightarrow X\) that restricts to a homeomorphism \(h : \partial X_\theta \rightarrow \partial X\) on the boundary, such that there exists a normal bordism rel \(\partial X\) from \(h_\theta\) to \(1_X\) with surgery obstruction \(\theta\). Such a homotopy equivalence is called tangential; equivalently, a homotopy equivalence \(h : M \rightarrow X\) of \(\text{TOP}\) manifolds is tangential if the pullback microbundle \(h^*(\tau_X)\) is isomorphic to \(\tau_M\).

**Theorem 3.1.** The following \(r\)-stabilization admits a \(\text{DIFF}\) structure:

\[
X_\theta \# r(S^2 \times S^2).
\]

Furthermore, there exists a \(\text{TOP}\) normal bordism between \(h_\theta\) and \(1_X\) with surgery obstruction \(\theta \in L^1_c(\mathbb{Z}[\pi^c])\), such that it consists of exactly \(2r\) many 2-handles and \(2r\) many 3-handles. In particular \(X_\theta\) is \(2r\)-stably homeomorphic to \(X\).

**Proof.** The existence and uniqueness of \((X_\theta, h_\theta)\) follow from \([\text{FQ90, Theorems 11.3A, 11.1A, 7.1A}]\). But by \([\text{CS71, Theorem 3.1}]\), there exists a \(\text{DIFF} s\)-bordism class of \((X_\alpha, h_\alpha)\) uniquely determined as follows. Given a rank \(r\) representative \(\alpha\) of the isometry class \(\theta\), this pair \((X_\alpha, h_\alpha)\) consists of a compact \(\text{DIFF} 4\)-manifold \(X_\alpha\) and a simple homotopy equivalence \(h_\alpha\) that restricts to a diffeomorphism on the boundary:

\[
\begin{align*}
h_\alpha & : (X_\alpha, \partial X_\alpha) \rightarrow (X_\alpha, \partial X) \\
X_r & := X \# r(S^2 \times S^2).
\end{align*}
\]
It is obtained from a DIFF normal bordism \((W_\alpha, H_\alpha)\) rel \(\partial X\) from \(h_\alpha\) to \(1_X\), with of surgery obstruction \(\theta\), constructed with exactly \(r\) 2-handles and \(r\) 3-handles, and clutched along a diffeomorphism which induces the simple unitary automorphism \(\alpha\) on the surgery kernel

\[
K_2(W_\alpha) = \mathcal{H} \left( \bigoplus_r \mathbb{Z}[\pi] \right).
\]

This is rather the consequence, and not the construction itself, of Wall realization [Wal99, 6.5] in high odd dimensions.

By uniqueness in the simple TOP structure set, the simple homotopy equivalences \(h_\theta\#^1_r(\mathbb{S}^2 \times \mathbb{S}^2)\) and \(h_\alpha\) are \(s\)-bordant. Hence they differ by pre-composition with a homeomorphism, by the \(s\)-cobordism theorem [FQ90, Thm. 7.1A]. In particular, the domain \(X_\theta\#^r(\mathbb{S}^2 \times \mathbb{S}^2)\) is homeomorphic to \(X_\alpha\), inheriting its DIFF structure. Therefore, post-composition of \(H_\alpha\) with the collapse map \(X_r \rightarrow X\) yields a normal bordism between the simple homotopy equivalences \(h_\theta\) and \(1_X\), obtained by attaching \(r + r\) 2- and 3-handles.

Next, we recall Hambleton–Kreck–Teichner classification of the homeomorphism types and simple homotopy types of closed 4-manifolds with fundamental group \(\mathbb{C} - \mathbb{2}\). Then, we shall give a partial classification of the simple homotopy types and stable homeomorphism types of their connected sums, which have fundamental group \(\mathbb{D} - \mathbb{1}\), \(\mathbb{C} - \mathbb{2}\). The star operation \(*\) [FQ90, §10.4] flips the Kirby–Siebenmann invariant of some 4-manifolds.

**Theorem 3.2** ([HKT94, Theorem 3]). Every closed nonorientable topological 4-manifold with fundamental group order two is homeomorphic to exactly one manifold in the following list of so-called \(w_2\)-types.

1. The connected sum of \(*\mathbb{CP}^2\) with \(\mathbb{RP}^4\) or its star. The connected sum of \(k \geq 1\) copies of \(*\mathbb{CP}^2\) with \(\mathbb{RP}^4\) or \(\mathbb{RP}^2 \times \mathbb{S}^2\) or their stars.
2. The connected sum of \(k \geq 0\) copies of \(\mathbb{S}^2 \times \mathbb{S}^2\) with \(\mathbb{RP}^2 \times \mathbb{S}^2\) or its star.
3. The connected sum of \(k \geq 0\) copies of \(\mathbb{S}^2 \times \mathbb{S}^2\) with \(S\) \((\gamma^1 \oplus \gamma^1 \oplus \varepsilon^1)\) or \#_{\mathbb{S}^1\mathbb{RP}^4}\) or their stars, for unique \(1 \leq r \leq 4\).

We explain the terms in the above theorem. Firstly,

\[
R \rightarrow \gamma^1 \rightarrow \mathbb{RP}^2
\]
denotes the canonical line bundle, and

\[
\varepsilon^1 := R \times \mathbb{RP}^2
\]
denotes the trivial line bundle. Secondly,

\[
\mathbb{S}^2 \rightarrow S(\gamma^1 \oplus \gamma^1 \oplus \varepsilon^1) \rightarrow \mathbb{RP}^2
\]

is the sphere bundle of the Whitney sum. Finally, the circular sum

\[
M \#_{\mathbb{S}^1} N := M \setminus E \bigcup_{\partial E} N \setminus E
\]

is defined by codimension zero embeddings of \(E\) in \(M\) and \(N\) that are not null-homotopic, where \(E\) is the nontrivial bundle:

\[
D^3 \rightarrow E \rightarrow S^1.
\]
Corollary 3.3 ([HKT94, Corollary 1]). Let \( M \) and \( M' \) be closed nonorientable topological 4-manifolds with fundamental group of order two. Then \( M \) and \( M' \) are (simple) homotopy equivalent if and only if

1. \( M \) and \( M' \) have the same \( w_2 \)-type,
2. \( M \) and \( M' \) have the same Euler characteristic, and
3. \( M \) and \( M' \) have the same Stiefel–Whitney number: \( w^1_3(M) \equiv w^1_3(M') \mod 2; \)
4. \( M \) and \( M' \) have \( \pm \) the same Brown–Arf invariant \( \mod 8 \), in case of \( w_2 \)-type III.

The following theorem is the main focus of this section. The pieces \( M \) and \( M' \) are classified by Hambleton–Kreck–Teichner [HKT94], and the \( UNil \)-group is computed by Connolly–Davis [CD04]. Since \( Z \) is a regular coherent ring, by Waldhausen’s vanishing theorem [Wald78, Theorems 1, 2, 4], \( \tilde{\text{Nil}}_0(\mathbb{Z}; \mathbb{Z}^+, \mathbb{Z}^-) = 0 \). Hence \( \text{UNil}_0^5 = \text{UNil}_0^5 \) [Cap74].

Theorem 3.4. Let \( M \) and \( M' \) be closed nonorientable topological 4-manifolds with fundamental group of order two. Write \( X = M \# M' \), and denote \( S \) as the 3-sphere defining the connected sum. Let \( \theta \in \text{UNil}_0^5(\mathbb{Z}; \mathbb{Z}^+, \mathbb{Z}^-) \).

1. There exists a unique homeomorphism class \((X_\theta, h_\theta)\), consisting of a closed \( \text{TOP} \) 4-manifold \( X_\theta \) and a tangential homotopy equivalence \( h_\theta : X_\theta \to X \), such that it has splitting obstruction

\[
\text{split}_L(h_\theta; S) = \theta.
\]

The function which assigns \( \theta \) to such a \((X_\theta, h_\theta)\) is a bijection.

2. Furthermore,

\[
X_\theta \# 3(S^2 \times \mathbb{S}^2) \text{ is homeomorphic to } X' \# 3(S^2 \times S^2).
\]

It admits a \( \text{DIFF} \) structure if and only if \( X \) does. There exists a \( \text{TOP} \) normal bordism between \( h_\theta \) and \( 1_X \), with surgery obstruction \( \theta \in L^5_2(D^-_\infty) \), such that it is composed of exactly six 2-handles and six 3-handles.

Proof. Recall that the forgetful map

\[
L^5_2(D^-_\infty) \to L^5_2(D^-_\infty)
\]

is an isomorphism, since the Whitehead group \( \text{Wh}(D_\infty) \) vanishes. Then the existence and uniqueness of \((X_\theta, h_\theta)\) and its handle description follow from Theorem 3.1, using \( r = d + 1 = 3 \) from Proposition 2.2 and Proof 2.1. By [Cap74, Theorem 6], the following composite function is the identity on \( \text{UNil}_0^5(\mathbb{Z}; \mathbb{Z}^+, \mathbb{Z}^-) \):

\[
\theta \mapsto (X_\theta, h_\theta) \mapsto \text{split}_L(h_\theta; S).
\]

In order to show that the other composite is the identity, note that two tangential homotopy equivalences \((X_\theta, h_\theta)\) and \((X'_\theta, h'_\theta)\) with the same splitting obstruction \( \theta \) must be homeomorphic, by freeness of the \( \text{UNil}_0^5 \) action on the structure set \( S^6_\text{TOP}(X) \). Finally, since the 4-manifolds \( X_\theta \) and \( X \) are 6-stably homeomorphic via the \( \text{TOP} \) normal bordism between \( h_\theta \) and \( 1_X \), we conclude that they are in fact 3-stably homeomorphic by Corollary 2.3. □

The six 2-handles are needed for map data and only three are needed to relate domains.

Corollary 3.5. The above theorem is true for \( X = \mathbb{RP}^4 \# \mathbb{RP}^4 \), with \( \mathbb{RP}^4 \) of \( w_2 \)-type III. □

Remark 3.6. We comment on a specific aspect of the topology of \( X \). Every homotopy automorphism of \( \mathbb{RP}^4 \# \mathbb{RP}^4 \) is homotopic to a homeomorphism [JK06, Lemma 1]. Then any automorphism of the group \( D_\infty \) can be realized [JK06, Claim]. The homeomorphism classes of closed topological 4-manifolds \( X' \) in the (not necessarily tangential) homotopy
type of $X$ has been computed in [BDK07, Theorem 2]. The classification involves the study [BDK07, Theorem 1] of the effect of transposition of the bimodules $\mathbb{Z}^+$ and $\mathbb{Z}^-$ in the abelian group $\text{UNil}_h^3(\mathbb{Z}; \mathbb{Z}^-, \mathbb{Z}^-)$. As promised in the introduction, Corollary 3.5 provides a uniform upper bound on the number of $S^2 \times S^2$ connected-summands sufficient for [JK06, Theorem 1(f)], and on the number of 2- and 3-handles sufficient for [JK06, Proof 1(f)].

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