Structure-preserving algorithm for thermal conduction of local thermal nonequilibrium saturated porous medium

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Abstract. Based on the two-energy equation model and constitutive equations of local thermal nonequilibrium saturated porous media, one-dimensional thermal conduction equations of solid skeleton and pore fluid are established. Introducing orthogonal variables, a first order generalized multi-symplectic structure-preserving form of thermal conduction equations is derived as well as the errors of generalized multi-symplectic conservation law and local momentum. The temperature profiles of two phases are obtained and the effect of the heat exchange coefficient between two phases on the process from local thermal nonequilibrium to local thermal equilibrium is revealed numerically. According to the relative error between numerical solution and analytical solution derived from the separating variables method, it can be concluded that this structure-preserving scheme is a valid scheme with high accuracy. Numerical errors of generalized multi-symplectic conservation law and local momentum are presented for two typical types of the thermal conduction in one-dimensional saturated porous media. According to these numerical results, it can be concluded that this structure-preserving algorithm has long-time numerical stability and good conservation properties.

1. Introduction
In recent decades, the problems of heat transfer in porous media have been investigated widely in a variety of fields, such as composite material manufacturing processes[1,2], heat and moisture motion in porous media[3-5], heat and mass transfer process of biological tissues[6,7], and so on [8,9]. Simacek P and Advani S G[1,2] examined the temperature distribution during non-isothermal saturated flow through the porous media. Sajjadi M and Azaiez J[5] examined the effect the heat transfer coefficient on the melting process in the melting porous media. Zhu G M et al [7] pointed out that the model for an unsaturated porous medium can be used to describe the flow and heat transfer of sweat in the skin layer. In these fields, model of local thermal equilibrium is usually assumed in a representative element volume (REV). In order to reveal the heat exchange between two phases, the model of local thermal nonequilibrium in porous media has attracted researchers’ more attentions[10-12]. Nield D A and Kuznetsov A V[10] investigated the effect of local thermal nonequilibrium among the particle, fluid, and solid-matrix phases of a porous media saturated by a nanofluid analytically. Dehghan M et al [11] analytically studied the local thermal non-equilibrium effect in the developed region of forced convection in the saturated porous media at several boundary conditions. However, there are few researches on preserving-structure property in the thermal conduction problem.
As we known, structure-preserving algorithm has applied widely and its high precision and excellent stability are outstanding extremely[13-18]. Aydin A and Karasözen B[13] considered the multi-symplectic six-point scheme for the integration of coupled nonlinear Schrödinger equations. Wang Y Sh et al [14] discussed the concept of local structure-preserving algorithms for partial differential equations. Zhang S Y and Deng Z Ch[15] demonstrated the structure-preserving algorithms can preserve canonical character of system real solution. Mcdonald F et al [17] investigated the property of preserving travelling wave solutions for multi-symplectic discretizations. In view of the dissipative system, Hu W P et al [19-21] investigated the generalized multi-symplectic structure-preserving algorithm and indicated excellent long-time numerical behavior and good conservation property of this method in several various fields. Based on theory of porous media and two-energy equation model, thermal conduction equations of one-dimensional local thermal nonequilibrium saturated porous medium are established firstly in this paper. Secondly, introducing canonical variables, structure-preserving form for heat conduction equations are constructed. Then, a midpoint box discretization scheme is obtained. Finally, the thermal conduction process associated with heat exchange coefficient between two phases is simulated numerically. Meanwhile, the numerical errors of generalized multi-symplectic conservation law and local momentum are also investigated.

2. Thermal conduction model of local thermal nonequilibrium saturated porous medium

An incompressible elastic saturated porous medium $\phi$ is considered, which is completely filled with inviscid pore fluid $\phi$ and occupies the domain $V$ in the initial state. Based on the theory of porous media, each constituent is regarded as a continuum by using $u = u (X, t) , (X,t) \in V \times [0,+\infty)$ to represent its own motion. Energy balance equation is given as follows [12],

$$\rho' \dot{e} = - \sigma' \cdot D' - \rho' \dot{r} + \text{div} q' = \dot{e}' - \dot{p}' \cdot \dot{u}' , \quad i = s,f$$

(1)

where $\rho'$, $\sigma'$, $q'$ and $\dot{e}'$ are macroscopic quality density, macroscopic stress tensor, heat flux vector and specific internal energy of the constituent $\phi$, respectively. $D'$ is macroscopic strain rate tensor which satisfies $D' = (\text{grad} \dot{u} + \text{grad}^T \dot{u})/2$. $\dot{p}'$ and $\dot{e}'$ are the interaction force and heat exchange between two phases, which satisfy $\dot{p}' + \dot{p}' = 0$ and $\dot{e}' + \dot{e}' = 0$, respectively. $(\cdots)'$, denotes the material time derivative with the motion of constituent $\phi$. Assuming small deformation of solid phase and small velocity of pore fluid, neglecting thermal expansion of solid skeleton, additional thermal conduction and thermal convection caused by the relative motion between two constituents, the constitutive equations are expressed as follows (See reference 12).

$$\begin{cases}
\sigma' = -n' p / \mathbf{I} + \sigma' , \quad \sigma' = -n' p / \mathbf{I} + \sigma' , \\
\sigma' = 2 \mu' (E' \cdot I) I , \quad \sigma'^e = 0 , \\
\dot{p}' = p \text{grad} n' - S' w' , \quad \dot{e}' = \dot{p}' \cdot \dot{u} - e' (\theta' - \theta') , \\
q' = -k'' \text{grad} \theta , \quad q'^f = -k'' \text{grad} \theta' 
\end{cases}$$

(2)

where $\sigma'^e$ and $\sigma'^f$ are the effective stresses tensors of the solid skeleton and the porous fluid, respectively. $\mu'$ and $\lambda'$ are the macroscopic Lame constants of the solid skeleton. $k''$ is the ordinary thermal conduction coefficient of the constituent $\phi$.

**Figure 1.** Thermal conduction of local thermal nonequilibrium saturated porous media.

In view of one-dimensional saturated porous media shown as figure 1, substituting the constitutive equations into equation (1), two-energy equations are simplified as follows,
\[
\begin{aligned}
\rho' c' \partial_t \theta' - k'' \partial_x \theta' - e_o (\theta' - \theta') &= 0 \\
\rho' f' \partial_t \theta' - k'' \partial_x \theta' + e_o (\theta' - \theta') &= 0
\end{aligned}
\]  \tag{3}

In which the partial heat sources of two constituents is neglected (see Reference 12). In equation (3), introducing the following dimensionless constants and variables:

\[
\overline{\rho} = \rho' / (\rho' + \rho'), \overline{\rho} = \rho' / (\rho' + \rho'), z' = c' / (c' + c'), \overline{z} = c' / (c' + c'), \overline{k} = k'' / (k'' + k''), \overline{k} = k'' / (k'' + k''), \\
\overline{z}_o = H^2 e_o / (k'' + k''), \overline{\theta}' = \theta' / \Theta_o, \overline{\theta}' = \theta' / \Theta_o, \overline{x} = x / L, \overline{t} = t / T, \overline{H} = H^2 (\rho' + \rho') (c' + c') / (k'' + k''), \lambda = H / L,
\]

dimensionless thermal conduction equations can be obtained below,

\[
\begin{aligned}
\rho' c' \partial_t \theta' - k'' \lambda^2 \partial_x \theta' - e_o (\theta' - \theta') &= 0 \\
\rho' f' \partial_t \theta' - k'' \lambda^2 \partial_x \theta' + e_o (\theta' - \theta') &= 0
\end{aligned}
\]  \tag{4}

where a dimensional mark is still used to express a dimensionless quantity.

3. Local conservation laws of the local thermal nonequilibrium saturated porous medium

Introducing canonical momenta: \( \partial_x \theta' = \phi, \partial_x \theta' = \phi \) and defining \( z = (\theta', \theta', \theta', \phi) \) as the state variable, equation (4) can be rewritten as the coupled first-order Hamilton partial differential equations,

\[
M \dot{z} + K \ddot{z} = \nabla_z S(z)
\]  \tag{5}

where \( S(z) = \lambda^2 (k'' \phi' + k'' \phi') / 2 + e_o [\theta' - \phi']^2 / 2 \) is the Hamiltonian function.

\[
M = \begin{pmatrix}
\rho' c' & 0 & 0 & 0 \\
0 & \rho' c' & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, K = \begin{pmatrix}
0 & 0 & -k'' \lambda^2 & 0 \\
0 & 0 & 0 & -k'' \lambda^2 \\
0 & k'' \lambda^2 & 0 & 0 \\
0 & k'' \lambda^2 & 0 & 0
\end{pmatrix}
\]  \tag{6}

Because \( K \) is skew-symmetric, local energy conservation law is satisfied exactly. \( M \) isn’t skew-symmetric, so the local momentum of this dissipative system isn’t preserved exactly. But if \( \rho' c' \) and \( \rho' f' \) are small enough with satisfying the inequality

\[
| \Delta | \leq o(\Delta t, \Delta x^2)
\]  \tag{7}

the system is generalized multi-symplectic, where \( o(\Delta t, \Delta x^2) \) is the difference truncation error. \( \Delta_i \) is expressed as

\[
| \Delta_i | = \max \left| \rho' c' d \theta' \Delta d (\partial_i \theta') + \rho' f' d \theta' \Delta d (\partial_i \theta') \right| \tag{8}
\]

\( \Delta_i \) is the error value of the discrete generalized multi-symplectic local conservation law. The error of generalized multi-symplectic local momentum can be expressed as follows,

\[
\Delta_p = \rho' c' [\partial_i (\partial_i \theta') - \phi'] / 2 + \rho' c' [\partial_i (\partial_i \theta') - \phi'] / 2 \tag{9}
\]

Referenced formula can consult reference 20. A centred box difference scheme is constructed below,

\[
M(z_{i+1/2}^{j+1/2} - z_{i+1/2}^{j-1/2}) / \Delta t + K(z_{i+1/2}^{j+1/2} - z_{i+1/2}^{j-1/2}) / \Delta x = \nabla_z S(z_{i+1/2}^{j+1/2})
\]  \tag{10}

where \( z_{i+1/2}^{j+1/2} = (z_{i+1}^{j+1} + z_{i+1}^{j+1}) / 2, z_{i+1/2}^{j-1/2} = (z_{i+1}^{j} + z_{i+1}^{j}) / 2, z_{i+1/2}^{j-1/2} = (z_{i+1}^{j} + z_{i+1}^{j}) / 4 \) and specific formulas are

\[
\begin{aligned}
\rho' c' & \theta_{i+1/2}^{j+1/2} - \theta_{i+1/2}^{j-1/2} - k'' \lambda^2 \phi_{i+1/2}^{j+1/2} = e_o (\theta_{i+1/2}^{j+1/2} - \theta_{i+1/2}^{j-1/2}), \\
\rho' c' & \theta_{i+1/2}^{j+1/2} - \theta_{i+1/2}^{j-1/2} - k'' \lambda^2 \phi_{i+1/2}^{j+1/2} = e_o (\theta_{i+1/2}^{j+1/2} - \theta_{i+1/2}^{j-1/2}), \\
k'' \lambda^2 & \phi_{i+1/2}^{j+1/2} - \phi_{i+1/2}^{j-1/2} = k'' \lambda^2 \theta_{i+1/2}^{j+1/2} - \theta_{i+1/2}^{j+1/2},
\end{aligned}
\]  \tag{11}

Moreover, the discrete errors of generalized multi-symplectic conservation law and local momentum in step \( j \)th can be obtained, respectively,
$|\Delta| = \max_{i,j} \left| \rho' c' d\theta'_j / \Delta t \left[ (\theta''_j^n - \theta''_j) / \Delta t + \rho' c' d\theta'_j / \Delta t \right] \right| \leq o(\Delta t, \Delta x)$ (12)

$|\Delta, c| = \max_{i,j} \left| 1/2 \rho' c' \left[ \int_0^1 (\theta''_j^n - \theta''_j) / \Delta t - \theta''_j / \Delta t \right] + \int_0^1 (\theta''_j^n - \theta''_j) / \Delta t \right| \leq o(\Delta t, \Delta x)$ (13)

where $o(\Delta t, \Delta x^2) \approx \Delta t^2 + \Delta x^2$.

4. Numerical experiment

In order to simulate the thermal conduction process of one-dimensional local thermal nonequilibrium saturated porous media, the following two cases are considered.

4.1. Case 1

$\theta'(x,0) = -0.01x^2 + 0.02x, \quad \theta'(x,0) = 0, \quad x \in [0,1]$ (14)

$\theta'(x,t) = \frac{1}{\Delta t} \left[ \int_0^1 (\theta''_j^n - \theta''_j) / \Delta t - \theta''_j / \Delta t \right] + \int_0^1 (\theta''_j^n - \theta''_j) / \Delta t \right| \leq o(\Delta t, \Delta x)$ (15)

Letting $c'=0.3, \quad \rho'=0.35, \quad k'=0.4, \quad c'=0.7, \quad \rho'=0.65, \quad k''=0.6$ and $L=0.0001$, the numerical solution of equation can be obtained by scheme (11), where the time-step and space-step are $\Delta t=0.03$ and $\Delta x=0.1$, respectively. The following analytical solution of equation (4) is obtained by separation of variables.

$\theta'(x,t) = \sum_{n=1}^{\infty} \left[ \rho' c' \xi / \Delta t + \zeta [k'' \chi^2 (2n-1)^2 \pi^2 / 4 + e_\alpha] \right] \sin[(2n-1)\pi x / 2]$ (16)

$\theta'(x,t) = \sum_{n=1}^{\infty} \left[ \rho' c' \xi / \Delta t + \zeta [k'' \chi^2 (2n-1)^2 \pi^2 / 4 + e_\alpha] \right] \sin[(2n-1)\pi x / 2]$ (17)

where $\xi = C_{i,n} \exp(\beta_{i,n} t) + C_{i,n} \exp(\beta_{i,n} t), \quad \zeta = C_{i,n} \exp(\beta_{i,n} t) + C_{i,n} \exp(\beta_{i,n} t), \quad C_{i,n} = -C_{i,n} , C_{i,n} = 32 (2n-1)^2 \pi^2 / 4 + e_\alpha$}

According to table 1 and table 2, the relative errors between solid phase and pore fluid are both small. Therefore, the thermal conduction process is simulated effectively by the scheme (11) and the high precision of this method is also reflected well.

**Table 1.** Relative error between analytical solution and numerical solution of solid phase temperature with time and location (i.e. x) when $\epsilon_0=1.0$.

| time | x=0.1 relative error (%) | x=0.2 relative error (%) | x=0.4 relative error (%) | x=0.7 relative error (%) | x=0.9 relative error (%) |
|------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 0.45 | 4.03                     | 2.90                     | 3.11                     | 3.21                     | 3.20                     |
| 0.9  | 1.30                     | 0.39                     | 0.06                     | 0.03                     | 0.03                     |
| 1.2  | 1.58                     | 0.48                     | 0.08                     | 5.52×10^-4               | 6.65×10^-3               |

**Table 2.** Relative error between analytical solution and numerical solution of pore fluid temperature with time and location (i.e. x) when $\epsilon_0=1.0$.

| time | x=0.1 relative error (%) | x=0.2 relative error (%) | x=0.4 relative error (%) | x=0.7 relative error (%) | x=0.9 relative error (%) |
|------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| 0.45 | 0.06                     | 1.02                     | 0.84                     | 0.74                     | 0.75                     |
| 0.9  | 1.18                     | 0.40                     | 0.10                     | 1.97×10^-3               | 0.01                     |
| 1.2  | 1.47                     | 0.46                     | 0.08                     | 3.55×10^-3               | 7.54×10^-3               |

**Table 3.** Temperatures of solid phase and pore fluid at x=0.4 and x=0.9 when $\epsilon_0=1.0$.

| time | solid phase temperature | pore fluid temperature | solid phase temperature | pore fluid temperature |
|------|--------------------------|------------------------|--------------------------|------------------------|
| x=0.4| x=0.4                    | x=0.9                  | x=0.4                    | x=0.9                  |
From figure 2, the time from local thermal nonequilibrium to local thermal equilibrium decreases with the value of $e_\Theta$ increasing. When $t=0.54$, the system with $e_\Theta=1.0$ has been already close to local thermal equilibrium state. The reason is that larger value of $e_\Theta$ leads to more adequate heat transfer between two phases. In figure 3, the effect of heat exchange constant $e_\Theta$ on the time from local thermal nonequilibrium to local thermal equilibrium is also indicated exactly. Table 3 gives the specific values of temperatures of solid phase and pore fluid when the time is close to local thermal equilibrium state. Because the temperature of pore fluid is lower than the temperature of solid phase in the initial state, temperature of pore fluid increases until reach the local thermal equilibrium state. Then with conducting heat to the location with lower temperature, the local thermal equilibrium temperature decreases, so the moment with maximum temperature of pore fluid is the time reaching local thermal equilibrium state. When $e_\Theta=1.0$, the moments reaching the local thermal equilibrium state are $t=0.84$ and $t=0.87$ at $x=0.4$ and $x=0.9$, respectively. But the relative error between these two moments is small (3.45%), so it can be considered that different spatial locations reach the local thermal equilibrium state at the same time ($t=0.87$). In the same way, above conclusion can also be obtained when $e_\Theta=0.5$. Moreover, the temperatures of local thermal equilibrium are independent on the value of $e_\Theta$ at both spatial positions. It is attributed to the independence of local thermal equilibrium on the heat exchange coefficient between two phases.
According to a series of small relative errors in table 4 and table 5, the temperatures of local thermal equilibrium are 0.001199 and 0.001856 at \( x=0.4 \) and \( x=0.9 \), respectively.

4.2. Case2

\[
\theta'(x,0) = -0.01x^2 + 0.01x, \quad \theta'(x,0) = 0, \quad x \in [0,1]
\]

\[
\theta'(0) = \theta'(1) = 0, \quad \theta'(x) = 0, \quad x \in [0,1], \quad t \geq 0
\]

In case 2, letting \( \Delta t=0.01 \) and \( \Delta x=0.1 \), the numerical solutions are obtained by scheme (11).

5. Conclusions

Based on multi-symplectic idea, a generalized multi-symplectic structure-preserving method is proposed to solve the thermal conduction equations in local thermal nonequilibrium saturated porous media. Then, a centred box difference scheme is constructed, which satisfies the condition of the generalized multi-symplectic. In addition, the errors of the generalized multi-symplectic conservation law and local momentum are also obtained. Two typical types of the initial-boundary value problems are surveyed. Comparing the numerical solutions with the analytical solutions, it is concluded that the multi-symplectic structure-preserving scheme constructed in this paper can solve the thermal conduction equations accurately. Moreover, the time from local thermal nonequilibrium to local thermal equilibrium decreases with the value of \( e_\Theta \) increasing, while the temperature of local thermal...
equilibrium is independent on the value of \( e_\Theta \). Finally, according to the numerical errors of the generalized multi-symplectic conservation law and local momentum with two typical cases, the excellent local structure-preserving property and long-time numerical stability of this generalized multi-symplectic structure-preserving algorithm are presented well. These conclusions may provide some useful information and a new numerical method for the temperature analysis of porous medium.

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