Research Article

Blow-Up Analysis for the Periodic Two-Component $\mu$-Hunter-Saxton System

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The two-component $\mu$-Hunter-Saxton system is considered in the spatially periodic setting. Firstly, a wave-breaking criterion is derived by employing the localization analysis of the transport equation theory. Secondly, several sufficient conditions of the blow-up solutions are established by using the classic method. The results obtained in this paper are new and different from those in previous works.

1. Introduction

In this article, we will consider the periodic two-component $\mu$-Hunter-Saxton system derived by Zuo [1]

$$
\begin{align*}
\mu_t - u_{txx} = -2\mu(u)u_x + 2u_{x}u_{xx} + uu_{xxx} + \rho \rho_x,
\end{align*}
$$

(1)

$$
\begin{align*}
\rho_t = (u\rho)_x + 2\gamma_2 \rho_x, \quad t > 0, \quad x \in \mathbb{R},
\end{align*}
$$

where $A(u) = \mu(u) - u_{xx}$, and it also can be viewed as a bivariational equation. Moreover, for $\gamma_i = 0$, $i = 1, 2$, system (1) has a Lax pair given by

$$
\begin{align*}
\psi_{xx} &= \lambda (A(u) - \lambda^2 \rho^2) \psi,
\end{align*}
$$

(3)

$$
\begin{align*}
\psi_t &= \left(u - \frac{1}{2\lambda}\right)\psi_x - \frac{1}{2}u_x \psi,
\end{align*}
$$

where $\lambda$ is a spectral parameter (see [1]).

In fact, system (1) is a generalization of the generalized Hunter-Saxton equation [2, 3]

$$
\begin{align*}
\mu_t - u_{txx} = -2\mu(u)u_x + 2u_{x}u_{xx} + uu_{xxx},
\end{align*}
$$

(4)

which describes the geodesic flow on $D'(\mathbb{S})$ with the right-invariant metric given at the identity by the inner product $\langle u, v \rangle = \mu(u)\mu(v) + \int_{\mathbb{S}} u_x v_x \, dx$ and models the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal with external magnetic nematic field and self-interaction. Here, the solution $u(t, x)$ denotes the director field of a nematic liquid crystal. It was observed in [2–4] that the $\mu$-Hunter-Saxton equation is formally integrable and has bi-Hamiltonian structure and infinite hierarchy of conservation laws. Further, the development of singularities in finite time and geometric descriptions of the system...
from nonstretching invariant curve flows in centroequiaffine geometries and pseudospherical surfaces and affine surfaces are described by Fu et al. [5].

For \( y_i = 0, \ i = 1, 2, \ \mu(u) = 0 \) and replacing \( t \) by \(-t\), system (1) reduces to periodic two-component Hunter-Saxton equation

\[
\begin{align*}
    u_{txx} + 2u_x u_{xx} + uu_{xxx} - \rho \rho_x = 0, \quad t > 0, \ x \in \mathbb{R}, \\
    \rho_i + (up)_x = 0, \quad t > 0, \ x \in \mathbb{R},
\end{align*}
\]

which is a generalization of the well-known Hunter-Saxton equation. It was also viewed as a particular case of the Gurevich-Zybin system [6] pertaining to nonlinear one-dimensional dynamics of dark matter as well as nonlinear ion-acoustic waves (cf. [7] and the references therein). The Hunter-Saxton system is formally integrable, possesses a bi-Hamiltonian structure, and admits peakon solutions [8]. The local well-posedness, global existence, blow-up phenomena, solitary wave solutions, and geometric properties of system (5) were recently discussed in literatures (see [9–17] and the references therein). It is worthwhile to mention that Moon and Liu [13] studied the Cauchy problem for the two-component Hunter-Saxton system in the periodic setting and gave some interesting results.

Recently, Liu and Yin [18, 19] investigated the Cauchy problem for system (1). In [18], the local well-posedness and several precise blow-up criteria for the system were obtained. Under the conditions \( \mu_0 = 0 \) and \( \mu_0 \neq 0 \), the sufficient conditions of blow-up solutions were presented. The global existence for strong solution for system (1) in the Sobolev space \( H^s(S) \times H^{s-1}(S) \) with \( s = 2 \) is also given [18], and in [19], existence of global weak solution is established for the periodic two-component \( \mu \)-Hunter-Saxton system. The objective of the present paper is to focus mainly on wave-breaking criterion and several sufficient conditions of blow-up solutions.

Motivated by the works in [13, 20], in the present paper, the localization analysis in the transport equation theory is employed to derive a new wave-breaking criterion of solutions for the system (1) in the Sobolev space \( H^s(S) \times H^{s-1}(S) \) with \( s \geq 2 \). It implies that the wave-breaking criterion is determined only by the slope of the component \( u \) of solution definitely. Inspired by the work in [21, 22], we use the Lyapunov function of \( \int_S u^2 \ dx \) to deduce several new blow-up results for the periodic two-component \( \mu \)-Hunter-Saxton system (1), which are different from the ones obtained in [18]. These results obtained in this paper are new and different from those in Liu and Yin’s work [18].

The rest of this paper is organized as follows. Section 2 states several properties for the periodic two-component \( \mu \)-Hunter-Saxton system and gives several lemmas. In Section 3, we employ the transport equation theory to prove a wave-breaking criterion in the Sobolev space \( H^s(S) \times H^{s-1}(S) \) with \( s \geq 2 \). Section 4 is devoted to the study of blow-up mechanism.

## 2. Preliminaries

**Lemma 1** (see [20]). The following estimates hold.

(i) For \( s \geq 0 \),

\[
\begin{align*}
    \|f\|_{H^s(R)} \\
    \leq C \left( \|f\|_{H^s(R)} \|g\|_{L^\infty(R)} + \|f\|_{L^\infty(R)} \|g\|_{H^s(R)} \right).
\end{align*}
\]

(ii) For \( s > 0 \),

\[
\begin{align*}
    \|f\|_{H^s(R)} & \leq C \left( \|f\|_{H^s(R)} \|g\|_{L^\infty(R)} + \|f\|_{L^\infty(R)} \|\partial_x g\|_{H^s(R)} \right) \quad \text{if} \ s > 0,
\end{align*}
\]

where \( C \) are constants independent of \( f \) and \( g \).

**Lemma 2** (see [20]). Suppose that \( s > -d/2 \). Let \( v \) be a vector field such that \( Vv \) belongs to \( L^1([0, T]; H^{s-1}) \) if \( s > 1 + d/2 \) or to \( L^1([0, T]; H^s \cap L^\infty) \) otherwise. Suppose also that \( f_0 \in H^s \), \( F \in L^1([0, T]; H^s) \), and \( f \in L^\infty([0, T]; H^s) \cap C([0, T]; S') \) solves the \( d \)-dimensional linear transport equations

\[
\begin{align*}
    f_i + v \cdot \nabla f = F, \\
    f|_{t=0} = f_0.
\end{align*}
\]

Then \( f \in C([0, T]; H^s) \). More precisely, there exists a constant \( C \) depending only on \( s, p, \) and \( d \), such that the following statements hold:

(i) If \( s \neq 1 + d/2 \),

\[
\begin{align*}
    \|f\|_{H^s} & \leq \|f_0\|_{H^s} + C \int_0^T \|F(\tau)\|_{H^s} \ d\tau + C \int_0^T \|V(\tau)\|_{H^{s}} \|f(\tau)\|_{H^s} \ d\tau.
\end{align*}
\]

(ii) If \( f = v \), then, for all \( s > 0 \), the estimates (9) and (10) hold with \( V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{s-1}} \ d\tau \) else.

**Lemma 3** (see [20]). Let \( 0 < \sigma < 1 \). Suppose that \( f_0 \in H^s \), \( g \in L^1([0, T]; H^s) \), \( v, \partial_x v \in L^1([0, T]; L^\infty) \) and \( f \in L^\infty([0, T]; H^s) \cap C([0, T]; S') \) solves the \( 1 \)-dimensional linear transport equation

\[
\begin{align*}
    f_t + v \partial_x f = g, \\
    f|_{t=0} = f_0.
\end{align*}
\]

Then \( f \in C([0, T]; H^s) \). More precisely, there exists a constant \( C \) depending only on \( \sigma \), and such that the following statement holds:

\[
\begin{align*}
    \|f\|_{H^s} & \leq \|f_0\|_{H^s} + C \int_0^T \|g(\tau)\|_{H^s} \ d\tau + C \int_0^T \|V(\tau)\|_{H^s} \|f(\tau)\|_{H^s} \ d\tau.
\end{align*}
\]
Lemma 5, then there exists a maximal $S = (\text{see } [18])$.

Lemma 4, the initial-value problem (1) can be recast in the following:

$$u_t - (u + \gamma_1) u_x = A^{-1} \partial_x \left( 2 \mu_0 u + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2 \right),$$

$$t > 0, \, x \in \mathbb{R},$$

where $A = \mu - \partial_x^2$ is an isomorphism between $H^4$ and $H^{s-2}$ with the inverse $v = A^{-1} \omega$ given explicitly by

$$v(x) = \left( x^2 - \frac{x}{2} + \frac{13}{12} \right) \mu(\omega) + \left( x - \frac{1}{2} \right) \int_0^x \omega(s) ds dy$$

$$- \int_0^x \int_0^y \omega(s) ds dy + \int_0^1 \int_0^y \omega(r) dr dy.$$

Commuting $A^{-1}$ and $\partial_x$, we get

$$A^{-1} \partial_x \omega(x) = \left( x - \frac{1}{2} \right) \int_0^x \omega(s) ds dy$$

$$+ \int_0^1 \int_0^y \omega(r) dr dy,$$

and

$$A^{-1} \partial_x^2 \omega(x) = -\omega(x) + \int_0^1 \omega(x) dx.$$
Now, consider the initial-value problem for the Lagrangian flow map:

\[ \eta_t = u(t, -\eta) + 2\gamma, \quad t \in [0, T), \]

\[ \eta(0, x) = x, \quad x \in \mathbb{R}, \]  

(29)

where \( u \) denotes the first component of the solution \( z = (u, \rho) \) to system (1). Applying classical results from ordinary differential equations, one can obtain the result.

**Lemma 7** (see [18]). Let \( u \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})) \), \( s \geq 2 \). Then (29) has a unique solution \( \eta \) in \( C^1([0, T) \times \mathbb{R}; \mathbb{R}) \). Moreover, the map \( \eta(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with

\[ \eta_t (t, x) = \exp \left( - \int_0^t u_x(s, -\eta(s, x)) \, ds \right) > 0, \]

(30)

\[ (t, x) \in [0, T) \times \mathbb{R}. \]

**Lemma 8** (see [18]). Let \( z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \), \( s \geq 2 \), and let \( T > 0 \) be the maximal existence time of the corresponding solution \( z = (u, \rho) \) to system (1). Then it has

\[ \rho(t, -\eta(t, x)) \eta_x(t, x) = \rho_0(-x), \]

(31)

\[ (t, x) \in [0, T) \times \mathbb{R}. \]

**Lemma 9** (see [18]). Assume that \( z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \), \( s \geq 2 \), and let \( T > 0 \) be the maximal existence time of corresponding solution \( z = (u, \rho) \) to system (1) with the initial data \( z_0 \). Then the corresponding solution blows up in finite time if and only if

\[ \limsup_{t \to T} \sup_{x \in \mathbb{S}} u_x(t, x) = +\infty \]

or

\[ \limsup_{t \to T} \| \rho_x \|_{L^\infty} = +\infty. \]

(32)

### 3. Wave-Breaking Criterion

**Theorem 10.** Let \( z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \) with \( s \geq 2 \), and \( z = (u, \rho) \) be the corresponding solution to (1). Assume that \( T > 0 \) is the maximal existence time. Then

\[ T < \infty \implies \int_0^T \| u_x \|_{L^\infty} \, d\tau = \infty. \]

(33)

**Proof.** We shall complete the proof of the theorem by an inductive argument with respect to the index \( s \). Let us first give upper bound for \( \| \rho \|_{L^\infty} \).

From (31), we derive

\[ \| \rho \|_{L^\infty} = \| \rho_0 \|_{L^\infty} e^{\int_0^t \| u_x(s, -\eta(s, x)) \| \, ds} \]

\[ \leq \| \rho_0 \|_{L^\infty} e^{\int_0^t \| u_x(s, x) \|_{L^\infty} \, ds}. \]

(34)

Next, we split four steps to finish the proof of Theorem 10.

**Step 1.** For \( s \in (2, 3) \), applying Lemma 3 to the second equation of system (23), we have

\[ \| \rho \|_{H^{s-2}([0, T])} \]

\[ \leq \| \rho_0 \|_{H^{s-2}} + C \int_0^t \| \rho u_x \|_{H^{s-2}} \, d\tau \]

\[ + C \int_0^t \| \rho \|_{H^{s-1}} \left( \| u + 2\gamma \|_{L^\infty} + \| \nabla_x u \|_{L^\infty} \right) \, d\tau. \]

(35)

Using (6), we get

\[ \| u_x \rho \|_{H^{s-2}([0, T])} \leq C \left( \| u_x \|_{H^{s-1}} \| \rho \|_{L^\infty} + \| u_x \|_{L^\infty} \| \rho \|_{H^{s-2}} \right) \]

\[ = C \left( \| u \|_{H^{s-1}} \| \rho \|_{L^\infty} + \| u_x \|_{L^\infty} \| \rho \|_{H^{s-2}} \right). \]

(36)

Therefore, it yields

\[ \| \rho \|_{H^{s-2}([0, T])} \leq \| \rho_0 \|_{H^{s-2}} + C \int_0^t \| u + \gamma \|_{H^{s-1}} \| \rho \|_{L^\infty} \, d\tau \]

\[ + C \int_0^t \| \rho \|_{H^{s-1}} \left( \| u \|_{L^\infty} + 1 + \| u_x \|_{L^\infty} \right) \, d\tau. \]

(37)

Differentiating once the second equation of system (23) with respect to \( x \), we have

\[ \partial_x \rho_x - (u + 2\gamma) \partial_x \rho_x = 2u_x \rho_x + u_{xx} \rho. \]

(38)

Using Lemma 3, we get

\[ \| \rho_x \|_{H^{s-2}([0, T])} \]

\[ \leq \| \rho_0 \|_{H^{s-2}} + C \int_0^t \| 2u_x \rho_x + \rho u_{xx} \|_{H^{s-2}} \, d\tau \]

\[ + C \int_0^t \| \rho \|_{H^{s-1}} \left( \| u \|_{L^\infty} + 1 + \| u_x \|_{L^\infty} \right) \, d\tau. \]

(39)

From (7), we have

\[ \| u_x \rho_x \|_{H^{s-2}} \leq C \left( \| u_x \|_{H^{s-1}} \| \rho \|_{L^\infty} + \| u_x \|_{L^\infty} \| \rho_x \|_{H^{s-2}} \right) \]

\[ = C \left( \| (u + \gamma) \rho_x \|_{H^{s-2}} + \| u_x \|_{L^\infty} \| \rho \|_{H^{s-2}} \right) \]

(40)

and

\[ \| \partial_x u_x \|_{H^{s-2}} \leq C \left( \| \rho \|_{H^{s-1}} \| u_x \|_{L^\infty} + \| \rho \|_{L^\infty} \| u_{xx} \|_{H^{s-2}} \right) \]

\[ = C \left( \| \rho \|_{H^{s-1}} \| u_x \|_{L^\infty} + \| \rho \|_{L^\infty} \left( \| u \|_{L^\infty} + \| \gamma \| \right) \right). \]

(41)

Therefore, we have

\[ \| \rho_x \|_{H^{s-2}([0, T])} \]

\[ \leq \| \rho_0 \|_{H^{s-2}} + C \int_0^t \| u + \gamma \|_{H^{s-1}} \| \rho \|_{L^\infty} \, d\tau \]

\[ + C \int_0^t \| \rho \|_{H^{s-1}} \left( \| u \|_{L^\infty} + 1 + \| u_x \|_{L^\infty} \right) \, d\tau. \]

(42)
From (37) and (42), it has
\[
\|\rho\|_{H^{r-1}(S)} \leq \|\rho_0\|_{H^{r-1}} + C \int_0^t \|u + y_1\|_{H^r} \|\rho\|_{L^\infty} \, d\tau 
\]
(43)
\[+ C \int_0^t \|\rho\|_{H^{r-1}} (\|u\|_{L^\infty} + 1 + \|u_x\|_{L^\infty}) \, d\tau. \]

On the other hand, the first equation of system (23) is equivalent to
\[
\partial_t (u - y_1) + (-u - y_1) \partial_x (u - y_1) 
= -A^{-1} \partial_x \left( 2\mu_0 (u + y_1) + \frac{1}{2} (u + y_1)^2 + \frac{1}{2} \rho^2 \right).
\]
(44)

Therefore, using Lemma 2, we get from (44)
\[
\|u + y_1\|_{H^r(S)} \leq C \int_0^t \|A^{-1} \partial_x \left( 2\mu_0 (u + y_1) 
+ \frac{1}{2} (u + y_1)^2 + \frac{1}{2} \rho^2 \right) \|_{H^r} \, d\tau + \|u_0 + y_1\|_{H^r} 
+ C \int_0^t \|u + y_1\|_{H^r} \|\partial_x (u)\|_{L^\infty} \, d\tau.
\]
(45)

Note that
\[
\|A^{-1} \partial_x f\|_{H^r(S)} \leq \|A^{-1} \partial_x f\|_{L^2(S)} + \|A^{-2} \partial_x f\|_{H^{r-1}(S)} 
\leq \|g_x \ast f\|_{L^2(S)} 
+ \|f\|_{L^2(S)} + 2 \|f\|_{H^{r-1}(S)} 
\leq C \|f\|_{H^{r-1}(S)}
\]
where we have used (27), (28), and Young’s inequality. Using (6) and (46), one has
\[
\|A^{-1} \partial_x \left( 2\mu_0 (u + y_1) + \frac{1}{2} (u + y_1)^2 + \frac{1}{2} \rho^2 \right) \|_{H^r} 
\leq C \|2\mu_0 (u + y_1) + \frac{1}{2} (u + y_1)^2 + \frac{1}{2} \rho^2 \|_{H^r} 
\leq C \left( \|u + y_1\|_{H^r} + \left( \|u + y_1\|_{L^\infty} + 2 \|f\|_{H^{r-1}(S)} \right) \|\rho\|_{H^{r-1}} + \|\rho\|_{L^\infty} \right).
\]
(47)

Thus, we reach
\[
\|u + y_1\|_{H^r(S)} \leq \|u_0 + y_1\|_{H^r} 
+ C \int_0^t \|u + y_1\|_{H^r} (1 + \|u_x\|_{L^\infty}) \, d\tau 
+ C \int_0^t \|\rho\|_{H^{r-1}} \|\rho\|_{L^\infty} \, d\tau.
\]
(48)

which together with (43) reaches
\[
\|u + y_1\|_{H^r(S)} + \|\rho\|_{H^{r-1}(S)} 
\leq \|u_0 + y_1\|_{H^r} + \|\rho_0\|_{H^{r-1}} 
+ C \int_0^t (\|u + y_1\|_{H^r} + \|\rho\|_{H^{r-1}}) 
\times (\|u_x\|_{L^\infty} + 1 + \|u\|_{L^\infty} + \|\rho\|_{L^\infty}) \, d\tau.
\]
(49)

Using Gronwall’s inequality, we have
\[
\|u(t) + y_1\|_{H^r(S)} + \|\rho(t)\|_{H^{r-1}(S)} 
\leq (\|u_0 + y_1\|_{H^r} + \|\rho_0\|_{H^{r-1}}) \exp \left( \int_0^t \|u_x\|_{L^\infty} + 1 + \|u\|_{L^\infty} + \|\rho\|_{L^\infty} \, d\tau \right).
\]
(50)

From (22) and (34), we get
\[
\|u(t) + y_1\|_{H^r(S)} + \|\rho(t)\|_{H^{r-1}(S)} 
\leq (\|u_0 + y_1\|_{H^r} + \|\rho_0\|_{H^{r-1}}) \exp \left( \int_0^t \|u_x\|_{L^\infty} \, d\tau \right).
\]
(51)

where \(C_1 = C_1(\mu_0, \mu_1, \|\rho_0\|_{L^\infty})\).

Hence, if the maximal existence time \(T < \infty\) satisfies
\[
\int_0^t \|u_x\|_{L^\infty} \, d\tau < \infty,
\]
we obtain (51) that
\[
\limsup_{t \to T^-} \left( \|u(t) + y_1\|_{H^r(S)} + \|\rho(t)\|_{H^{r-1}(S)} \right) < \infty,
\]
(52)

which contradicts the assumption on the maximal existence time \(T < \infty\). It completes the Theorem 10 for \(s \in (2, 3)\).

Step 2. For \(s \in [2, 5/2]\), applying Lemma 2 to the second equation of system (23), we get
\[
\|\rho\|_{H^{r-1}(S)} \leq \|\rho_0\|_{H^{r-1}} + C \int_0^t \|u_x\|_{H^{r-1}} \|\rho\|_{L^\infty} \, d\tau 
+ C \int_0^t \|\rho\|_{H^{r-1}} \|\partial_x (u + y_1)\|_{L^\infty} \, d\tau.
\]
(53)

From (6), we get
\[
\|\rho u_x\|_{H^{r-1}(S)} \leq C \left( \|u_x\|_{H^{r-1}} \|\rho\|_{L^\infty} + \|\rho\|_{H^{r-1}} \|u_x\|_{L^\infty} \right).
\]
(54)

which ensures that
\[
\|\rho\|_{H^{r-1}(S)} \leq \|\rho_0\|_{H^{r-1}} + C \int_0^t (\|u\|_{H^{r-1}} \|\rho\|_{L^\infty} + \|\partial_x (u + y_1)\|_{L^\infty} \, d\tau 
+ C \int_0^t \|\rho\|_{H^{r-1}} (\|u_x\|_{L^\infty} + \|\partial_x (u + y_1)\|_{L^{2s} \cap H^{s}}) \, d\tau,
\]
(55)

which together with (48) gives rise to
\[
\|u(t) + y_1\|_{H^r} + \|\rho(t)\|_{H^{r-1}} 
\leq \|u_0 + y_1\|_{H^r} + \|\rho_0\|_{H^{r-1}} 
+ C \int_0^t (\|u + y_1\|_{H^r} + \|\rho(t)\|_{H^{r-1}}) 
\times (\|u_x\|_{L^\infty} + 1 + \|u\|_{L^\infty} + \|\rho\|_{L^\infty}) \, d\tau.
\]
(56)
Using Lemma 2, we get from (60) that
\[ \|u(t) + \gamma_1\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \leq C \int_0^t (\|u + \gamma_1\|_{H^s} + \|\rho\|_{H^{s-1}}) \, dt, \]
where \( \epsilon \in (0, 1/2) \) and we used the fact that \( H^{1/2+\epsilon} \hookrightarrow L^\infty \cap H^{1/2} \).

Therefore, using Step 1 and arguing by induction assumption, we get that
\[ \|u + \gamma_1\|_{H^{s+2\epsilon, s}} + \|\rho\|_{L^\infty} \]
is uniformly bounded. We obtain from (57) that
\[ \limsup_{t \to T} (\|u(t) + \gamma_1\|_{H^s} + \|\rho(t)\|_{H^{s-1}}) < \infty, \]
which contradicts the assumption on the maximal existence time \( T < \infty \). It completes the Theorem 10 for \( s = 2, 5/2 \).

**Step 3.** For \( s = k \in \mathbb{N}, k \geq 3 \), differentiating \( k - 2 \) times the second equation of system (23) with respect to \( x \), we obtain
\[
\begin{align*}
\partial_x^k \partial_x^2 \rho - (u + 2\gamma_1) \partial_x (\partial_x^k \rho) & + \sum_{l_1, l_2 = 3, l_3, l_4 = 0} C_{l_1, l_2, l_3, l_4} \partial_x^{l_3+1} u \partial_x^{l_4+1} \rho + \rho \partial_x (\partial_x^k u) = 0.
\end{align*}
\]

Using Lemma 2, we get from (60) that
\[
\begin{align*}
\|\partial_x^k u\|_{H^{1/2}, L^\infty} & \leq \|\partial_x^k \rho_0\|_{H^s} + C \int_0^t (\|\partial_x^k \rho\|_{H^s}) \, dt \\
& + C \int_0^t \sum_{l_1, l_2 = 3, l_3, l_4 = 0} C_{l_1, l_2, l_3, l_4} \partial_x^{l_3+1} u \partial_x^{l_4+1} \rho + \rho \partial_x \partial_x^{k-1} u \|_{H^s} \, dt.
\end{align*}
\]
Since \( H^1 \) is an algebra, we have
\[
\begin{align*}
\|\partial_x^{k-1} u\|_{H^s} & \leq C \|\rho\|_{H^s} \|\partial_x^{k-1} u\|_{H^s} \\
& \leq C \|\rho\|_{H^s} \|u + \gamma_1\|_{H^s}.
\end{align*}
\]
and
\[
\begin{align*}
\|\sum_{l_1, l_2 = 3, l_3, l_4 = 0} C_{l_1, l_2, l_3, l_4} \partial_x^{l_3+1} u \partial_x^{l_4+1} \rho\|_{H^s} & \leq C \|\rho\|_{H^{s-1}} \|u + \gamma_1\|_{H^{s-1}}.
\end{align*}
\]
Therefore, we have
\[
\begin{align*}
\|\partial_x^{k-2} \rho\|_{H^s} & \leq \|\partial_x^{k-2} \rho_0\|_{H^s} \\
& + C \int_0^t (\|u + \gamma_1\|_{H^s} + \|\rho\|_{H^{s-1}}) \, dt \\
& \times (\|u + \gamma_1\|_{H^{s-1}} + \|\rho\|_{H^s}) \, dt.
\end{align*}
\]
Applying Lemma 3 to the second equation of system (23) yields
\[
\begin{align*}
\|\rho\|_{H^s} & \leq \|\rho_0\|_{H^s} + C \int_0^t (\|u + \gamma_1\|_{H^s} + \|\rho\|_{H^{s-1}}) \, dt \\
& \times (\|u + \gamma_1\|_{H^{s-1}} + \|\rho\|_{H^s}) \, dt,
\end{align*}
\]
where we used Lemma 1. From (64) and (65), it yields
\[
\begin{align*}
\|\rho\|_{H^{s-1}} & \leq \|\rho_0\|_{H^{s-1}} + C \int_0^t (\|u + \gamma_1\|_{H^{s-1}} + \|\rho\|_{H^{s-1}}) \, dt \\
& \times (\|u + \gamma_1\|_{H^{s-1}} + \|\rho\|_{H^{s-1}} + 1) \, dt,
\end{align*}
\]
where we used the Gagliardo-Nirenberg inequality \( \|\rho\|_{H^{s-1}} \leq C(\|\rho\|_{H^s} + \|\partial_x^k \rho\|_{H^s}) \) for \( \sigma \in (0, 1) \).

Using (48) implies that
\[
\begin{align*}
\|u(t) + \gamma_1\|_{H^{s-1}} + \|\rho(t)\|_{H^{s-1}} & \leq C (\|u_0 + \gamma_1\|_{H^s} \\
& + \|\rho_0\|_{H^{s-1}}) + C \int_0^t (\|u + \gamma_1\|_{H^{s-1}} + \|\rho(t)\|_{H^{s-1}}) \\
& \times (\|u + \gamma_1\|_{H^{s-1}} + \|\rho\|_{H^{s-1}} + 1) \, dt.
\end{align*}
\]
Using Gronwall’s inequality, we get
\[
\begin{align*}
\|u(t) + \gamma_1\|_{H^{s-1}} + \|\rho(t)\|_{H^{s-1}} & \leq C (\|u_0 + \gamma_1\|_{H^s} \\
& + \|\rho_0\|_{H^{s-1}}) e^{C \int_0^t (\|u + \gamma_1\|_{H^s} + \|\rho(t)\|_{H^{s-1}}) \, dt}.
\end{align*}
\]
Therefore, if the maximal existence time \( T < \infty \) satisfies
\[
\int_0^T \|u_\xi\|_{L^\infty} \, dt < \infty,
\]
using Step 2, we get that
\[
\|u + \gamma_1\|_{H^{s-1}} + \|\rho\|_{H^s}
\]
is uniformly bounded by the induction assumption. From (66), we get
\[
\limsup_{t \to T} (\|u(t) + \gamma_1\|_{H^s} + \|\rho(t)\|_{H^{s-1}}) < \infty,
\]
which contradicts the assumption that \( T < \infty \) is the maximal existence time. This completes the proof of Theorem 10 for \( s = k \in \mathbb{N} \) and \( k \geq 3 \).
Step 4. For $s \in (k, k+1)$, $k \in \mathbb{N}$ and $k \geq 3$. Differentiating $k-1$ times the second equation of system (23) with respect to $x$, we obtain
\[
\begin{align*}
\partial_s \partial^{-k+1} \rho - (u + 2\gamma_s) \partial_x \partial^{-k+1} \rho 
&+ \sum_{l_1+l_2=k-2,l_1,l_2 \geq 0} C_{l_1,l_2} \partial_x \partial^{-k+1} u \partial_x \partial^{-k+1} \rho + \rho \partial_x \partial^{-k+1} u \partial_x \partial^{-k+1} \rho 
&= 0.
\end{align*}
\]
Applying Lemma 3 to (71) and noting $s-k \in (0,1)$, we get
\[
\begin{align*}
\|\partial^{-k+1} \rho\|_{L^p} &\leq C \int_0^t \left( \sum_{l_1+l_2=k-2,l_1,l_2 \geq 0} C_{l_1,l_2} \partial_x \partial^{-k+1} u \partial_x \partial^{-k+1} \rho 
&+ \rho \partial_x \partial^{-k+1} u \partial_x \partial^{-k+1} \rho 
&+ \|\partial_x \partial^{-k+1} u\|_{L^\infty} + 1 \right) d\tau + \|\partial^{-k+1} \rho\|_{L^p} .
\end{align*}
\]
For each $\varepsilon \in (0,1/2)$, using (7) and the fact that $H^{1/2+\varepsilon}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we have
\[
\begin{align*}
\|\rho \partial^{-k+1} u\|_{L^p} &\leq C \left( \|\rho\|_{H^{k-3/2+\varepsilon}} \right) \left( \|\partial^{-k+1} u\|_{L^\infty} + \|\partial_x \partial^{-k+1} u\|_{L^\infty} \right) 
&\leq C \left( \|\rho\|_{H^{k-3/2+\varepsilon}} \right) \left( \|\partial^{-k+1} u\|_{L^\infty} + \|\partial_x \partial^{-k+1} u\|_{L^\infty} \right)
\end{align*}
\]
and
\[
\begin{align*}
\sum_{l_1+l_2=k-2,l_1,l_2 \geq 0} C_{l_1,l_2} \partial_x \partial^{-k+1} u \partial_x \partial^{-k+1} \rho 
&\leq C \sum_{l_1+l_2=k-2,l_1,l_2 \geq 0} C_{l_1,l_2} \left( \|\partial^{-k+1} u\|_{L^\infty} \|\partial_x \partial^{-k+1} \rho\|_{L^\infty} \right) 
&\leq C \left( \|\rho\|_{H^{k-3/2+\varepsilon}} \right) \left( \|\partial^{-k+1} u\|_{L^\infty} + \|\partial_x \partial^{-k+1} u\|_{L^\infty} \right),
\end{align*}
\]
where we used Lemma 1.

Therefore, from (72), (73), and (74), we get
\[
\begin{align*}
\|\partial^{-k+1} \rho\|_{L^p} &\leq C \int_0^t \left( \|u + \gamma_1\|_{H^p} + \|\rho\|_{H^{p-1}} \right) d\tau 
&\cdot \left( \|u + \gamma_1\|_{H^{p-1/2+\varepsilon}} + \|\rho\|_{H^{p-3/2+\varepsilon}} + 1 \right) d\tau + \|\partial^{-k+1} \rho\|_{L^p} ,
\end{align*}
\]
which together with (48) and (37) (where $s - 2$ is replaced by $s-k$) gives rise to
\[
\begin{align*}
\|u(t) + \gamma_1\|_{H^p} + \|\rho(t)\|_{H^{p-1}} &\leq C \left( \|u_0 + \gamma_1\|_{H^p} 
&+ \|\rho_0\|_{H^{p-1}} \right) + C \int_0^t \left( \|u + \gamma_1\|_{H^p} + \|\rho(t)\|_{H^{p-1}} \right) 
&\times \left( \|u + \gamma_1\|_{H^{p-1/2+\varepsilon}} + \|\rho\|_{H^{p-3/2+\varepsilon}} + 1 \right) d\tau,
\end{align*}
\]
from which we have
\[
\begin{align*}
\|u(t) + \gamma_1\|_{H^p} + \|\rho(t)\|_{H^{p-1}} 
&\leq C \left( \|u_0 + \gamma_1\|_{H^p} + \|\rho_0\|_{H^{p-1}} \right) \left( e^{C \int_0^t (t + \|u + \gamma_1\|_{H^p} + \|\rho(t)\|_{H^{p-1}}) d\tau} \right)
&\cdot e^{C (t + \|u + \gamma_1\|_{H^p} + \|\rho(t)\|_{H^{p-1}}) d\tau} .
\end{align*}
\]
Noting that $k - 1/2 + e < k$, $k - 3/2 + e < k - 1$ and $k \geq 3$. Therefore, if the maximal existence time $T < \infty$ satisfies $\int_0^T \|u_\tau\|_{L^\infty} d\tau < \infty$, using Step 3, we get that
\[
\begin{align*}
\|u + \gamma_1\|_{H^{p-1/2+\varepsilon}} + \|\rho\|_{H^{p-3/2+\varepsilon}} 
&\leq C \left( \|u_0 + \gamma_1\|_{H^p} + \|\rho_0\|_{H^{p-1}} \right) \left( e^{C \int_0^t (t + \|u + \gamma_1\|_{H^p} + \|\rho(t)\|_{H^{p-1}}) d\tau} \right)
&\cdot e^{C (t + \|u + \gamma_1\|_{H^p} + \|\rho(t)\|_{H^{p-1}}) d\tau} .
\end{align*}
\]
which contradicts the assumption that $T < \infty$ is the maximal existence time. This completes the proof of Theorem 10 for $s = k \in \mathbb{N}$ and $k \geq 3$.

Therefore, from Steps 1–4, we complete the proof of Theorem 10.}

\section{Wave-Breaking Phenomena}

In this section, we will present several blow-up results by using the Lyapunov function of $\int_0^3 u_3^2 \, dx$ (see [21]). Let $z_0 = (u_0, \rho_0) \in H^4(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s \geq 2$ and let $T$ be the maximal existence time of the corresponding solution to system (1) with the initial data $z_0$.

Now, we give the first blow-up result.

\textbf{Theorem 11.} Let $z_0 = (u_0, \rho_0) \in H^4(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s \geq 2$ and $N$ be a sufficient large positive real number and let $T$ be the maximal existence time of the corresponding solution to system (1) with the initial data $z_0$ and $\int_{\mathbb{S}} \rho_0(x) = 0$.

\textbf{(i)} If $\rho_0^2 < 2(\mu_2^2 - N^2)$, then the corresponding solution to system (1) must blow up in finite time $T$ with
\[
0 \leq T \leq \inf_{a \in I} \left( \frac{6}{(1 - 6\alpha \|u_0\|)} \left( \int_{\mathbb{S}} u_0^3 dx \right)^{1/3} \right),
\]
where $I = (|\mu_0|/\pi^2 (\mu_2^2 - N^2), 1/6(\|u_0\|)$, such that
\[
\lim_{t \to T^-} \sup_{x \in \mathbb{S}} u_n (t, x) = +\infty.
\]

\textbf{(ii)} If $\int_{\mathbb{S}} u_0^3 dx > \sqrt{(3|\mu_0|/2\pi^2 \alpha + 3N^2)|\mu_4^4/(1 - 6\alpha |\mu_0|)}$ then the corresponding solution to system (1) must blow up in finite time $T$ with
\[
0 < T < \frac{\mu_4^2}{(1 - 6\alpha |\mu_0|)} \ln \left( \int_{\mathbb{S}} u_0^3 dx + K \right),
\]
where $0 < \alpha < 1/6|\mu_0|$ and $K = (3|\mu_0|/2\pi^2 \alpha + 3N^2)|\mu_4^4/(1 - 6\alpha |\mu_0|)$. 

Proof. If the statement is not true, then from Lemma 9 it implies that there exists a large enough real number $N > 0$, such that $u_i(t, x) < N$ and $\|\rho_i(t, x)\|_2 \leq N$ for all $t \in [0, T)$. From the assumption $\int_S \rho_0(x) dx = 0$, we know
\[
\int_S \rho(t, x) dx = \int_S \rho_0(x) dx = 0. \tag{83}
\]
It then follows that, for any $t \in [0, T)$, $\rho(t, x)$ has a zero point $\xi$. Thus we have
\[
\rho(t, x) = \int_\xi^x \rho_x(t, s) ds, \quad x \in [\xi, \xi + 1], \tag{84}
\]
which results in
\[
|\rho(t, x)| = \left| \int_\xi^x \rho_x(t, s) ds \right| \leq N. \tag{85}
\]
(i) Differentiating the first equation of system (23) with respect to $x$ yields
\[
u_{xx} - (u + \gamma_1) u_{xx} = A^{-1} \partial_x^2 \left( 2\mu_0 u + \frac{1}{2} u_{x}^2 + \frac{1}{2} \rho^2 \right). \tag{86}
\]
Using (26), we get
\[
u_x = \frac{1}{2} u_x^2 + (u + \gamma_1) u_{xx} - 2\mu_0 u + \frac{1}{2} \rho^2 + \frac{1}{2} \mu_1^2. \tag{87}
\]
Multiplying (87) by $3u_x^2$ and integrating on $S$ with respect to $x$, we obtain for any $t \in [0, T]$ that
\[
\frac{d}{dt} \int_S u_x^3 dx = \int_S 3u_x^2 u_{xx} dx
\]
\[
= \frac{3}{2} \int_S u_x^4 dx + \int_S 3u_x^2 (u + \gamma_1) u_{xx} dx
\]
\[
- 6\mu_0 \int_S u_x^2 u_{xx} + 6\mu_0^2 \int_S u_x^2 dx
\]
\[
- \frac{3}{2} \int_S \rho^2 u_x^2 dx + \frac{3}{2} \mu_1^2 \int_S u_x^2 dx
\]
\[
= \frac{1}{2} \int_S u_x^4 dx - 6\mu_0 \int_S (u - \mu_0) u_x^2 dx
\]
\[
- \frac{3}{2} \int_S \rho^2 u_x^2 dx + \frac{3}{2} \mu_1^2 \int_S u_x^2 dx. \tag{88}
\]
On the other hand, it follows from Lemma 6 for any $\alpha > 0$ that
\[
\mu_0 \int_S (u - \mu_0) u_x^2 dx \leq \left| \mu_0 \right| \left| \frac{1}{2} \int_S u_x^4 dx + \frac{1}{2\alpha} \mu_0 \int_S (u - \mu_0)^2 dx \right|
\]
\[
\leq \left| \mu_0 \right| \left[ \int_S (u - \mu_0)^2 dx \right]^{1/2} \left( \int_S u_x^4 dx \right)^{1/2}
\]
\[
\leq \left| \mu_0 \right| \frac{\alpha}{2} \int_S u_x^4 dx + \frac{1}{2\alpha} \left| \mu_0 \right| \int_S (u - \mu_0)^2 dx
\]
\[
\leq \frac{\alpha}{2} \left| \mu_0 \right| \int_S u_x^4 dx + \frac{1}{8\pi^2} \left| \mu_0 \right| \int_S u_x^2 dx. \tag{89}
\]
Therefore, we deduce that
\[
\frac{d}{dt} \int_S u_x^3 dx \geq \left( \frac{1}{2} - 3\alpha \left| \mu_0 \right| \right) \int_S u_x^4 dx
\]
\[
+ \left( \frac{3}{2} \mu_1^2 - \frac{3}{4\pi^2} \left| \mu_0 \right| \right) \int_S u_x^2 dx
\]
\[
- \frac{3}{2} \int_S \rho^2 u_x^2 dx \tag{90}
\]
\[
\geq \left( \frac{1}{2} - 3\alpha \left| \mu_0 \right| \right) \int_S u_x^4 dx
\]
\[
+ \left( \frac{3}{2} \mu_1^2 - \frac{3}{4\pi^2} \left| \mu_0 \right| - \frac{3}{2} \left| N \right| \right) \int_S u_x^2 dx,
\]
where we have used Lemma 8 and (19). From the assumption of the theorem, we know that $\left| \mu_0 \right| / [\pi^2 (\mu_1^2 - N^2)] < 1/(6\left| \mu_0 \right|)$. Let $\alpha > 0$ satisfy
\[
0 \leq \frac{\left| \mu_0 \right|}{\pi^2 (\mu_1^2 - N^2)} < \alpha < \frac{1}{6 \left| \mu_0 \right|.} \tag{91}
\]
It implies that $1/2 - 3\alpha \left| \mu_0 \right| > 0$ and $(3/2)\mu_1^2 - 3\left| \mu_0 \right|/4\pi^2 \alpha - (3/2)N^2 > 0$. Define $\beta_1$ and $\beta_2$ by
\[
\beta_1 = \frac{1}{2} - 3\alpha \left| \mu_0 \right| > 0,
\]
\[
\beta_2 = \left( \frac{3}{2} \mu_1^2 - \frac{3}{4\pi^2} \alpha - \frac{3}{2} \left| N \right| \right) \int_S u_x^2 dx > 0. \tag{92}
\]
We get
\[
\frac{d}{dt} \int_S u_x^3 dx \geq \beta_1 \int_S u_x^4 dx + \beta_2. \tag{93}
\]
Letting $V(t) = \int_S u_x^3 dx$ with $t \in [0, T)$ derives
\[
\frac{d}{dt} V(t) \geq \beta_1 (V(t))^{4/3}, \quad t \in [t_0, T), \tag{94}
\]
where we used the inequality $(\int_S u_x^3 dx)^{4/3} < \int_S u_x^3 dx$, which leads to
\[
V(t) \geq \left[ \frac{3V^{1/3}(0)}{3 - \beta_1 tV^{1/3}(0)} \right]^{3} \rightarrow +\infty, \tag{95}
\]
as $t \rightarrow \frac{3}{\beta_1 V^{1/3}(0)}$.

On the other hand, we have
\[
\int_S u_x^3 dx \leq \sup_{x \in \bar{S}} u_x(t, x) \int_S u_x^2 dx \leq \mu_1^2 \sup_{x \in \bar{S}} u_x(t, x), \tag{96}
\]
which implies that $0 < T < 3/\beta_1 (V_0)^{1/3}$ satisfying
\[
\limsup_{t \rightarrow T^{-}} \left( \sup_{x \in \bar{S}} u_x(t, x) \right) = +\infty. \tag{97}
\]
(ii) From (90), we get
\[
\frac{d}{dt} \int_S u_x^3 \, dx \geq \left( \frac{1}{2} - 3\alpha |\mu_0| \right) \int_S u_x^2 \, dx - \left( \frac{3}{4\pi^2} \frac{|\mu_0|}{\alpha} + \frac{3}{2} N^2 \right) \alpha^2.
\]  
(98)

Let \( \alpha > 0 \) satisfy
\[
0 < \alpha < \frac{1}{6 |\mu_0|}.
\]  
(99)

it implies that \( 1/2 - 3\alpha |\mu_0| > 0 \). Define \( \alpha_1 \) and \( \alpha_2 \) by
\[
\alpha_1 = \frac{1/2 - 3\alpha |\mu_0|}{\alpha^2} > 0,
\]
\[
\alpha_2 = \left( \frac{3 |\mu_0|}{4\pi^2} \frac{1}{\alpha} + \frac{3}{2} N^2 \right) \alpha^2 > 0.
\]

and let \( \int_S u_x^3 \, dx = V(t) \); we have
\[
\frac{d}{dt} V(t) \geq \alpha_1 V^2(t) - \alpha_2.
\]

(101)

Let \( \sqrt{\alpha_1} / \alpha_1 = K \). If \( V(0) > K \), then \( V(t) > K \). Therefore, we solve (101) to obtain
\[
1 - \frac{V(0) - K}{V(0) + K} e^{2\alpha_1 K t} > \frac{2K}{V(t) + K} > 0.
\]

(102)

It follows that there exists \( T \) satisfying
\[
0 < T < \frac{1}{2\alpha_1 K} \ln \frac{V(0) + K}{V(0) - K},
\]

(103)

such that \( \lim_{t \to -T} V(t) = +\infty \). This contradicts the assumption \( u_x > N \) for all \( (t, x) \in [0, T) \times S \). From Lemma 9 we obtain that the solution \( z \) blows up in finite time.

It completes the proof of Theorem 11.

Before giving the second blow-up result, we firstly claim that, under the condition \( \gamma_1 = 2\gamma_2 \), \( \mu_2 = \int_S (\mu u_x^2 + (1/2) u^2 u_x^2 + (1/2) u^2 \rho^2) \, dx \) is conserved in time. Indeed, like two-component Camassa-Holm equation (see [24–26]), we can easily obtain
\[
\frac{d}{dt} \int_S \left( \mu u_x^2 + \frac{1}{2} u_x^2 + \frac{1}{2} u^2 \rho^2 \right) \, dx
\]
\[
= \int_S \left( 3 \mu u_x + \frac{1}{2} u_x + \frac{1}{2} \rho \right) u_x G_x + u_x u_{xx} \]
\[
+ 3 \mu u_x + \frac{1}{2} u_x + \frac{1}{2} \rho \right) \right) \, dx.
\]

(104)

In view of (23) and (87), we get after some integration by parts
\[
\frac{d}{dt} \int_S \left( \mu u_x^2 + \frac{1}{2} u_x^2 + \frac{1}{2} u^2 \rho^2 \right) \, dx
\]
\[
= \int_S \left( 2 \mu u_x + \frac{1}{2} u_x + \frac{1}{2} \rho \right) u_x G_x + u_x u_{xx} \]
\[
\]
\[
+ \frac{3}{2} \mu \right) \right) \, dx.
\]

(105)

Observe that
\[
\int_S G \, dx - G_{xx} = 2\mu_0 u + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2.
\]

(106)

Therefore, we have
\[
\frac{d}{dt} \int_S \left( \mu u_x^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho \right) \, dx = 0.
\]

(107)

Theorem 12. Let \( z_0 = (\mu_0, \rho_0) \in H^2 \times H^2 \) with \( s \geq 2 \) and \( N \) be a sufficiently large positive real number and let \( T \) be maximal existence time of corresponding solution to system (1) with the initial data \( z_0 \) and \( \int_S \rho_0(x) \, dx = 0 \).

(i) Assume that \( \theta = 12\mu_0 \mu_2 + 6\mu_0^2 \mu_1^2 + \sqrt{3} |\mu_0| \mu_1^2 + (3/2) N^2 \mu_1^2 < 0 \). Then the corresponding solution to system (1) must blow up in finite time \( T \) with
\[
0 \leq T \leq \frac{1}{2\mu_1^2} \ln \frac{\int_S u_{xx}^3 \, dx + K}{\int_S u_{xx}^3 \, dx - K},
\]

(108)

where \( K = \sqrt{2\mu_1^2(12\mu_0 \mu_2 + 6\mu_0^2 \mu_1^2 + \sqrt{3} |\mu_0| \mu_1^2 + (3/2) N^2 \mu_1^2)} \).

Proof. Applying a simple density argument, we only need to consider the case \( s = 3 \). From (88), one has
\[
\frac{d}{dt} \int_S u_x^3 \, dx = \int_S \left( 3 u_x^2 u_{xx} \right) \, dx
\]
\[
= \frac{3}{2} \int_S u_x^2 \, dx + \int_S 3 u_x^2 (u + \gamma_1) u_{xx} \, dx
\]
\[
- 6\mu_0 \int_S u_x^2 u_x \, dx + 6\mu_0 \int_S u_x^2 \, dx
\]
\[
- \frac{3}{2} \mu_0 \int_S \rho^2 u_x^2 \, dx + \frac{3}{2} \mu_0 \int_S u_x^2 \, dx
\]
\[
= \frac{1}{2} \int_S u_x^4 \, dx - 6\mu_0 \int_S u_x^2 \, dx
\]
\[
+ 6\mu_0 \int_S u_x^2 \, dx - \frac{3}{2} \mu_0 \int_S \rho^2 u_x^2 \, dx
\]
\[
+ \frac{3}{2} \mu_0 \int_S u_x^2 \, dx
\]
\[
= \frac{1}{2} \int_S u_x^4 \, dx - 12\mu_0 \mu_2 + 6\mu_0^2 \int_S u_x^2 \, dx
\]
\[
+ 6\mu_0 \int_S u_x^2 \, dx + 6\mu_0 \int_S u_x^2 \, dx
\]
\[
- \frac{3}{2} \mu_0 \int_S \rho^2 u_x^2 \, dx + \frac{3}{2} \mu_0 \int_S u_x^2 \, dx.
\]

(110)
Applying the assumption of the theorem and Hölder inequality give rise to

$$
\frac{d}{dt} \int_S u_x^3 \, dx \geq \frac{1}{2} \left( \int_S u_x^2 \, dx \right)^{4/3} - 12 \mu_0 \mu_2 - 6 \mu_0^2 \mu_1^2 - \sqrt{3} |\mu_0| \mu_1^3 - \frac{3}{2} N^2 \mu_1^2.
$$

Define $V(t) = \int_S u_x^3 \, dx$, $t \in [0, T)$. It is clear that

$$
\frac{d}{dt} V(t) \geq \frac{1}{2} (V(t))^{4/3} + \theta,
$$

(112)

where $\theta = - (12 \mu_0 \mu_2 + 6 \mu_0^2 \mu_1^2 + \sqrt{3} |\mu_0| \mu_1^3 + (3/2) N^2 \mu_1^2) > 0$. This implies that $V(t)$ increases strictly in $[0, T)$. Let $t_0 = (1 + |V(0)|)/\theta$; we assume $t_0 < T$. Otherwise, $T \leq t_0 < \infty$ and the theorem is proved.

From (112), we have

$$
\frac{d}{dt} V(t) \geq \theta.
$$

(113)

Now, integrating (113) over $[0, t_0)$ yields

$$
V(t_0) \geq V_0 + \theta t_0 \geq -|V(0)| + \theta t_0 = 1.
$$

(114)

Using again (112), one has

$$
\frac{d}{dt} V(t) \geq \frac{1}{2} (V(t))^{4/3}, \quad t \in [t_0, T),
$$

(115)

which leads to

$$
V(t) \geq \left[ \frac{6}{6 - (t - t_0)} \right]^3 \to +\infty, \quad \text{as } t \to 6 + t_0.
$$

(116)

On the other hand, we have

$$
\int_S u_x^3 \, dx \leq \sup_{x \in S} u_x(t, x) \int_S u_x^2 \, dx \leq \mu_1 \sup_{x \in S} u_x(t, x),
$$

(117)

which implies that $0 < T < 6 + t_0$ satisfying

$$
\limsup_{t \to T^-} \left( \sup_{x \in S} u_x(t, x) \right) = +\infty.
$$

(118)

(ii) Applying the assumption of the theorem, (110) and Hölder inequality give rise to

$$
\frac{d}{dt} \int_S u_x^3 \, dx \geq \frac{1}{2 \mu_1} \left( \int_S u_x^2 \, dx \right)^2 - 12 \mu_0 \mu_2 - 6 \mu_0^2 \mu_1^2 - \sqrt{3} |\mu_0| \mu_1^3 - \frac{3}{2} N^2 \mu_1^2.
$$

(119)

Define $V(t) = \int_S u_x^3 \, dx$, $t \in [0, T)$. It is clear that

$$
\frac{d}{dt} V(t) \geq \frac{1}{2 \mu_1} (V(t))^2 - \phi,
$$

(120)

where $\phi = 12 \mu_0 \mu_2 + 6 \mu_0^2 \mu_1^2 + \sqrt{3} |\mu_0| \mu_1^3 + (3/2) N^2 \mu_1^2 > 0$. Let $\sqrt{2 \mu_1^2 \phi} = K$. If $V(0) > K$, then $V(t) > K$. Therefore, we solve (101) to obtain

$$
1 - \frac{V(0) - K}{V(0) + K} \frac{K \gamma \mu_1^2}{V(t) + K} > 0.
$$

(121)

It follows that there exists $T$ satisfying

$$
0 < T < \frac{\mu_1^2}{K} \ln \frac{V(0) + K}{V(0) - K},
$$

(122)

such that $\lim_{t \to T} V(t) = +\infty$. This contradicts the assumption $u_x > N$ for all $(t, x) \in [0, T) \times S$. From Lemma 9 we obtain that the solution $z$ blows up in finite time. It finishes the proof of the Theorem 12.

Remark 13. If we let $\rho_0(x) = 0$, then we can obtain $\rho(t, x) = 0$ easily. Then system (1) is degenerated into $\mu$-version Camassa-Holm equation under some transform. The blow-up results related to $\mu$-version Camassa-Holm equation; the reader is referred to [5].

Remark 14. It is worthwhile to mention that, comparing with the results in [18], our blow-up results are new and quite different from the ones in [18].

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

There are no conflicts of interest.

**Authors’ Contributions**

The two authors contributed to the work equally.

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