Picard groups of the moduli spaces of semistable sheaves I

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Abstract. We compute the Picard group of the moduli space $U'$ of semistable vector bundles of rank $n$ and degree $d$ on an irreducible nodal curve $Y$ and show that $U'$ is locally factorial. We determine the canonical line bundles of $U'$ and $U'_L$, the subvariety consisting of vector bundles with a fixed determinant. For rank 2, we compute the Picard group of other strata in the compactification of $U'$.

Keywords. Picard groups; semistable sheaves; nodal curve.

1. Introduction

In our previous paper [3] we proved that the Picard group of the moduli space $U'_L(n,d)$ of semistable vector bundles of rank $n$ with fixed determinant $L$ ($L$ being a line bundle of degree $d$) on an irreducible projective nodal curve $Y$ of geometric genus $g \geq 2$ is isomorphic to $\mathbb{Z}$ (except possibly in the case $g = 2, n = 2, d$ even). We used this to show that $U'_L(n,d)$ is locally factorial. Interestingly, the results for irreducible nodal curves are very similar to those for smooth curves. However, the proofs are different and much more difficult. Unlike in the smooth case, the moduli space of vector bundles on a nodal curve is not projective. Moreover its complement in the compactification $U$ (moduli of torsion-free sheaves) has codimension 1. The computation of Picard group needs codimension of the non-semistable and non-stable strata (see [6,11] for smooth case). Since HN-filtrations of vector bundles contain non-locally free sheaves and tensor products of stable bundles are not semistable (on $Y$), in general it is impossible to determine this codimension directly on $Y$. We did it by using parabolic bundles on the normalization $X$ of $Y$ and hence had to assume $g \geq 2$ and exclude the case $g = n = d = 2$.

In this paper, we do a detailed analysis for rank 2 and extend these results to nodal curves of arithmetic genus $g_Y \geq 0$ (rank 2). Combining this with results of [3], we have the following theorem.

Theorem 1. Let $Y$ be an irreducible reduced curve with only ordinary nodes as singularities. Assume that for $n \geq 3$, the geometric genus $g \geq 2$. Then

1. $\text{Pic } U'_L(n,d) \cong \text{Pic } U''_L(n,d) \cong \mathbb{Z},$
2. $U'_L$ is locally factorial.

We also show that the dualising sheaf $\omega_Y$ of $U'_L(n,d)$ is isomorphic to the line bundle $-2\delta L$, where $\delta = \gcd(n,d)$ and $L$ is the ample generator of $\text{Pic } U'_L(n,d)$ (Theorem 4).
We then compute the Picard group of the moduli space $U'(n,d)$ (resp. $U''(n,d)$) of semistable (resp. stable) vector bundles of rank $n$ and degree $d$ on $Y$. Let $J$ denote the generalised Jacobian of degree $d$ on $Y$.

**Theorem (Theorem 3(A)).** Let the assumptions be above.

(a) $\text{Pic } U'' \approx \text{Pic } J \oplus \mathbb{Z}$,
(b) $\text{Pic } U' \approx \text{Pic } J \oplus \mathbb{Z}$,
(c) $U'$ is locally factorial.

This completes the extension of results of [6] to nodal curves.

Let $U = U(n,d)$ denote the moduli space of torsion-free sheaves of rank $n$ and degree $d$ on $Y$. If $Y$ has only a single ordinary node as singularity, then the variety $U(2,d)$ has a stratification, $U = U'' \cup U_1 \cup U_0$, a disjoint union. Points of $U_1$ correspond to torsion-free sheaves $F$ of rank 2 with $F \cong \mathcal{O}_v \oplus m_Y$. Let $L$ be a rank 1 torsion-free sheaf which is not locally free. Let $U_{1,L}(2,d)$ be the subscheme of $U_1$ corresponding to torsion-free sheaves of rank 2 with determinant isomorphic to $L$.

**Theorem (Theorem 2, Theorem 3(B)).** Let $g_Y \geq 2$; if $g_Y = 2$, assume that $d$ is odd for (b), (c), (d). Then

(a) $\text{Pic } U_{1,L}(2,d) \approx \mathbb{Z}$,
(b) $\text{Pic } U_1'(2,d) \approx \text{Pic } J_Y \oplus \mathbb{Z}$,
(c) $\text{Pic } U_1(2,d) \approx \text{Pic } J_Y \oplus \mathbb{Z}$,
(d) $U_1(2,d)$ is locally factorial.

In a subsequent paper, we study the Picard group of a seminormal variety. As an application we compute the Picard groups of the compactified Jacobian and some subvarieties of $U(2,d)$.

**Notation.** Let $Y$ denote an irreducible reduced projective curve with ordinary nodes $y_j, j = 1, \ldots, m$ as only singularities. Let $g$ be the geometric genus and $g_Y$ the arithmetic genus of $Y$. For $y \in Y$, let $(\mathcal{O}_y, m_Y)$ be the local ring at $y$. A torsion-free sheaf $N$ on $Y$ is locally free on the subset $U$ of non-singular points of $Y$. The rank $r(N)$ of $N$ is the rank of the locally free sheaf $N|_U$. The degree $d(N)$ of $N$ is defined by $d(N) = \chi(N) + r(N)(g - 1)$, where $\chi$ denotes the Euler characteristic. Let $N^*$ denote the torsion-free sheaf $\text{Hom}(N, \mathcal{O})$.

Let $J$ and $\overline{J}$ be respectively the generalised Jacobian and the compactified Jacobian of $Y$ (of a fixed degree) and $\mathcal{P}$ the Poincaré bundle. Let $p_J$ denote the projection to $\overline{J}$. Let $U = U(n,d)$ be the moduli space of semistable torsion-free sheaves of rank $n$ and degree $d$ on $Y$. Let $\delta = \gcd(n,d)$. Let $U' \subset U$ be the open subvariety corresponding to vector bundles (i.e. $S$-equivalence classes of $E$ such that $\text{gr}E$ is a vector bundle). Fix a rank $1$ torsion-free sheaf $L$ of degree $d$ on $Y$. Let $U'_L$ (resp. $U_{1,L}$) be the subscheme of $U$ corresponding to vector bundles (resp. torsion-free sheaves) with determinant isomorphic to $L$ and $U_L$ its closure in $U$. Let $U'' \subset U'$, $U''_L \subset U'_L$ etc. be the open subvarieties corresponding to stable torsion-free sheaves. The variety $U$ is seminormal ([13], Theorem 4.2), $U'$ and $U''_L$ are normal being GIT-quotients of non-singular varieties [10]. For $m = 1$, $U$ has a filtration $U \supset W_{m-1} \supset \cdots \supset W_0$, with $W_i$ seminormal closed subvarieties [13]. $W_{i-1}$ is the non-normal locus of $W_i, i = 1, \ldots, n$ and $W_0$ is normal. Let $U'' = U - W_1, U_i = W_i - W_{i-1}(i = 1, \ldots, n-1), U_0 = W_0$. 
2. Torsion-free sheaves of rank 2

In this section we study $U_L(2,d)$ and $U(2,d)$. Throughout the section $E$ will denote a torsion-free sheaf of rank 2 and degree $d$ on $Y$.

**Lemma 2.1.** Let $E$ be a torsion-free sheaf with $\wedge^2 E = L$ torsion-free. Let $N_1$ be a rank 1 subsheaf of $E$ such that the quotient $N_2 = E/N_1$ is torsion-free.

1. If $N_1$ or $L$ is locally free, then $N_2 \cong N_1^* \otimes L$,
2. If $N_2$ is locally free, then $N_1 \otimes N_2 \cong L$.

**Proof.** The canonical alternating form $E \times E \rightarrow L$ induces an alternating form $N_1 \times N_1 \rightarrow L$. We claim that this form is zero. This is clear at $y \in Y$ such that the stalk $(N_1)_y$ is free. If $(N_1)_y \cong \mathcal{O}_{\pi}^n$, then $(N_1)_y = m_{y, \mathcal{O}}$, also $L_y = \mathcal{O}_y$ or $m_y$ (12, Prop. 2, p. 164). Let $u, v$ be the two generators of $(N_1)_y$. Since any $\mathcal{O}_y$-linear map from $m_y$ to $m_y$ (or $\mathcal{O}_y$) is given by the multiplication by $a \in \mathcal{O}_y$ (normalisation of $\mathcal{O}_y$) (12, p. 169), the map $(N_1)_y \rightarrow L_y$ defined by $w \mapsto w \wedge u$ is given by $w \wedge u = wa, a \in \mathcal{O}_y$. In particular, $0 = u \wedge u = uu$. Since $\mathcal{O}_y$ is a domain, this implies $a = 0$. Thus $v \wedge u = 0$ and hence $(N_1)_y \wedge (N_1)_y = 0$. Define an $\mathcal{O}$-bilinear map $b: N_1 \times N_2 \rightarrow L$ by $b(n_1, n_2) = n_1 \wedge n_3$, where $n_3$ is a lift of $n_2$ in $E$. This is well-defined as any two lifts $n_3, n'_3$ differ by an element of $N_1$ and $N_1 \wedge N_1 = 0$ as seen above. The bilinear map $b$ induces an injective sheaf homomorphism $N_2 \rightarrow \text{Hom}(N_1, L)$ which is an isomorphism outside the singular set of $Y$. If $N_1$ or $L$ is locally free, then $d(\text{Hom}(N_1, L)) = d(L) - d(N_1)$ (4, Lemma 2.5(B)) and hence $d(\text{Hom}(N_1, L)) = d(N_2)$. It follows that $N_2 \cong \text{Hom}(N_1, L)$.

If $N_2$ is locally free, the bilinear map $b$ gives an injective homomorphism of torsion-free sheaves $N_1 \otimes N_2 \rightarrow L$. Since $d(N_1 \otimes N_2) = d(N_1) + d(N_2) = d(L)$, this is an isomorphism. This proves the lemma.

We remark that if both $N_1, N_2$ are not locally free then $N_1 \otimes N_2$ has a torsion and $b$ gives a homomorphism $N_1 \otimes N_2 / \text{torsion} \rightarrow L$ which is not an isomorphism.

**Lemma 2.2.** Assume that $Y$ has only one node $y$. Let $\pi: X \rightarrow Y$ be the normalisation map and $\pi^{-1}y = \{x, z\}$. Let $N_1, N_2$ be line bundles of degree $-1$ on $X$.

1. Given a line bundle $L$ on $Y$ with $\pi^*L = N_1 \otimes N_2 (x + z)$, there exists a vector bundle $E$ of rank 2 and determinant $L$ on $Y$ such that $E$ is $S$-equivalent to $\pi, N_1 \oplus \pi, N_2$.
2. There exists a torsion-free sheaf $E$ of rank 2 on $Y$ such that (1) $E_y \cong \mathcal{O}_y \oplus m_y$, (2) determinant of $E$ is isomorphic to $\pi_* (N_1 \otimes N_2 (z))$ and (3) $E$ is $S$-equivalent to $\pi_* N_1 \oplus \pi_* N_2$.

**Proof.**

1. We shall construct a generalised parabolic bundle $(E', F_1(E'))$ on $X$ which gives the required vector bundle $E$ on $Y$. Take $E' = L_1 \oplus L_2, L_1 = N_1 (x + z), L_2 = N_2$. Let $e_1, e_2$ be basis elements of $(L_1)_y, (L_1)_y$, respectively. Let $f_1, f_2$ be basis elements of $(L_2)_y, (L_2)_y$, respectively. Define $F_1(E') = (e_2 - f_1, c e_1 + f_2), c$ being a non-zero scalar. Since the projections $p_1, p_2$ from $F_1(E')$ to $E'_z, E'_c$ are both isomorphisms, $E$ is a vector bundle (11). Choose $c$ such that $L$ corresponds to the generalised parabolic line bundle $(\pi^* L, (c, 1)), (c, 1) \in \mathbb{P}^1(11).$ One has $\det E' = \det (E', (c, 1)) = (\pi^* L, (c, 1)).$
Hence det $E = L$. Since $F_1(L_1) = 0$, $\pi_1(L_1(-x-z))$ is a sub-bundle of $E$. The quotient is $\pi_1(L_2)$ as the projection from $F_1(E')$ to $(L_2)_1 \oplus (L_2)_2$ is onto. Thus $E$ is $S$-equivalent to $\pi_1(N_1 \oplus N_2)$.

(b) Take $E'$ as in the above proof, define $F_1(E') = (e_1 + f_2, f_1)$. Since $p_1$ is an isomorphism and $p_2$ has rank 1, $E_\gamma \cong \mathcal{O}_Y \oplus m_Y$. Since $(e_1 + f_2) \wedge f_1 = 0e_1 \wedge e_2 + f_1 \wedge f_2 + \cdots$, one has $\det(E', F_1(E')) = (L_1 \otimes L_2, (0, 1))$. Hence $\det(E) = \pi_1(L_1 \otimes L_2(-x)) = \pi_1(N_1 \otimes N_2(z))$. The final assertion follows as in the above proof.

PROPOSITION 2.3.

Let $g_Y = 1$. Then one has the following:

1. $U_1(2, 1) = \{ \text{a point} \}$ for $L \in \mathcal{J}$,
   \[
   U(2, 1) \cong \mathcal{Y}, \ U^t(2, 1) \cong \mathcal{J} \cong \mathcal{Y} - \{ \text{node} \}.
   \]

2. $U_1(2, 0) \cong \mathcal{J}/i \cong \mathbb{P}^1$, where $i: \mathcal{J} \to \mathcal{J}$ is defined by $N \mapsto N^*$,
   \[
   U_1(2, 0) \cong \mathbb{P}^1 \text{ and } U^t_1(2, 0) \cong \mathcal{J}, \text{ for } L \in J.
   \]

Proof.

1. For $y \in Y$, let $I_y$ denote the ideal sheaf of $y$. The dual $I_y^*$ is a rank 1 torsion-free sheaf of degree 1 $^{[5]}$. It is well-known that $y \mapsto I_y^*$ gives an isomorphism $Y \to \mathcal{J}$, where $\mathcal{J}$ is the compactified Jacobian of degree 1 torsion-free sheaves.

   Let $E$ be a stable rank 2 torsion-free sheaf of degree 1 on $Y$. Then $h^1(E) = 0$ as $E$ is stable and hence $h^0(E) = 1$. Any non-zero section $s \in H^0(E)$ must be everywhere non-vanishing, otherwise it will generate a rank 1 torsion-free subsheaf of degree $\geq 1$ contradicting the stability of $E$. Hence $s \in H^0(E)$ generates a unique trivial line sub-bundle $\mathcal{O}$ of $E$. The quotient $E/\mathcal{O}$ must be torsion-free, if not then the kernel of $E \to (E/\mathcal{O})/\text{torsion}$ will contradict the stability of $E$. Thus we have a morphism $h: U(2, 1) \to \mathcal{J}$ given by $E \mapsto E/\mathcal{O}$. Conversely, given $L \in \mathcal{J}$, $\Ext(L, \mathcal{O}) = H^1(L^*)$ $^{[4]}$. Proof of Lemma 2.5(B)). Since $h^0(L^*) = 0$, $h^1(L^*) = 1$, any non-zero element in $\Ext(L, \mathcal{O})$ determines a unique (up to isomorphism) torsion-free rank 2 sheaf $E$ of degree 1. It is easy to check that $E$ is stable. This gives the inverse of $h$. Note that $h$ is in fact the determinant map.

2. We first prove that $W_0$ consists of a single point. Any element in $W_0$ has stalk at the node $y$ isomorphic to $m_y \oplus m_y$. By $^{[12]}$, Proposition 10, p. 174, such an element is the direct image of a vector bundle $E_0$ on the desingularisation $\mathbb{P}^1$. Since $\pi_*E_0$ is semistable, so is $E_0$. Hence $E_0 = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. By Lemma 2.2(a), for every line bundle $L$ there exists a vector bundle $E$ with determinant $L$ such that $E$ is $S$-equivalent to $\pi_1(\mathcal{O}(-1)) \oplus \pi_1(\mathcal{O}(-1))$. Thus for any $L \in J$, $U_1(2, 0)$ contains the point $\pi_*E_0$. One has $U_1 \cap W_1 = W_0$ $^{[12]}$. Thus every element of $U_1(2, 0)$ is $S$-equivalent to a vector bundle with determinant $L$. It follows that $U_1 \cong U \mathcal{O}$.

We now prove that $U(2, 0) \cong \mathcal{J}/i \cong \mathbb{P}^1$. Note first that the involution $i$ keeps the unique element $\pi_1(\mathcal{O}(-1))$ of $\mathcal{J} - J$ invariant and under the isomorphism $Y \cong \mathcal{J}$, the map $\mathcal{J} \to \mathcal{J}/i$ is the double cover $Y \to \mathbb{P}^1$ ramified at the image of the node. Let $E$ be a semistable vector bundle of rank 2 with trivial determinant. Let $E_1$ be the vector bundle of degree 2 obtained by tensoring $E$ with a line bundle of degree 1.
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Lemma 2.4.

Proof. A rank 2 vector bundle E which is semistable but not stable contains a torsion-free subsheaf N1 with a torsion-free quotient N2 ≈ Hom (N1, L) = N1 \otimes L, where L is determinant of E (Lemma 2.1(1)). Thus E is S-equivalent to N1 \oplus (N1 \otimes L), hence dim U′_L - U''_L = dim J = g_Y and codim_U\prime L^L - U''_L = 2g_Y - 3 ≥ 3 if g_Y ≥ 3.

Lemma 2.5.

(1) Codim_U\prime L(U_L - U'_L) ≥ 3 for g_Y ≥ 3.
(2) For g_Y = 2, codim_U\prime L(U_L - U'_L) = 3 if d is odd, U_L = U'_L = \mathbb{P}^3 if d is even.

Proof.

(1) The points of U_L - U'_L correspond to torsion-free sheaves which are direct images of semistable vector bundles with fixed determinant on partial normalisations of Y. Hence U_L - U'_L is a finite union of irreducible components each of dimension 3(g_Y - 1) - 3 = 3g_Y - 6 for g_Y ≥ 3. Thus codim_U\prime L(U_L - U'_L) ≥ 3.

(2) For g_Y = 2 the partial normalisations are of arithmetic genus 1. It follows from Proposition 2.3(1) that for d odd, U_L - U'_L consists of one or two points according as g = 1 or g = 0. For d even, U_L = U'_L ≈ \mathbb{P}^3 (\mathbb{Z}, Lemmas 3.3, 3.4, Corollary 3.5). We remark that Proposition 2.3(2) implies that the subset U_{0,L} of non-locally free sheaves in U_L is isomorphic to \mathbb{P}^1 if g = 1 and it consists of two smooth rational curves intersecting in a point if g = 0. The intersection point is the direct image of the unique semistable bundle of degree d - 2 on the desingularisation \mathbb{P}^1. Note also that U_{0,L} = U_L - U'_L in this case.

Lemma 2.6. Codim_U\prime (U'_L - U''_L) ≥ 3 for g_Y ≥ 3 (d even).

Proof. The surjective determinant map U' → J is a fibration with fibres isomorphic to U''_L, L a fixed line bundle of degree d. Hence the lemma follows from Lemma 2.4.

Remark 2.7.

(1) Let g_Y = 1. Then Pic U(2, 1) ≈ G_m \oplus \mathbb{Z}. For L ∈ J, Pic U(L)(2, 0) ≈ \mathbb{Z}, and Pic U'_L(2, 0), Pic U''_L(2, 0), Pic U'(2, 1) are trivial.

(2) If g_Y = 2, then Pic U'_L(2, d) ≈ \mathbb{Z} ≈ Pic U''_L(2, d) for all d.
Proof. Part (1) follows from Proposition 2.3. Part (2) is proved in [3], §2.4.

PROPOSITION 2.8.

For \( \gamma \geq 3 \), one has:

1. \( U_j^g(2, d) \approx \mathbb{Z} \),
2. \( U_j^g(2, d) \approx \mathbb{Z} \).

Proof. Let \( p: U_L \to U_l \) be a (finite) normalisation. Since \( U_L^\prime \) is normal, \( p \) is an isomorphism over \( U_L \) and \( p \) gives a finite map \( \bar{U}_L - p^{-1}U_L^\prime \to U_L - U_L^\prime \). Therefore \( \text{codim} \bar{U}_L - p^{-1}U_L^\prime = \text{codim} U_L - U_L^\prime \geq 3 \) by Lemma 2.5. Since \( \bar{U}_L \) is normal, this implies that \( \text{Pic} U_L \to \text{Pic}(p^{-1}U_L^\prime) \approx \text{Pic} U_L^\prime \). Since \( U_L \) is projective, so \( \bar{U}_L \) and hence \( \text{rank}(\text{Pic} \bar{U}_L) \geq 1 \). It follows that \( \text{rank}(\text{Pic} U_L^\prime) \geq 1 \). Since \( U_L^\prime \) is normal and by Lemma 2.4, \( \text{codim}(U_L^\prime - U_L^\prime) \geq 3 \) we have \( \text{Pic} U_L^\prime \to \text{Pic} U_L^\prime \). Thus \( \text{rank}(\text{Pic} U_L^\prime) \geq 1 \). By [3], Proposition 2.3, one has \( \text{Pic} U_L^\prime \approx \mathbb{Z} \) or \( \mathbb{Z}/m\mathbb{Z} \), \( m \in \mathbb{Z} \). It follows that \( \text{Pic} U_L^\prime \approx \mathbb{Z} \) and hence \( \text{Pic} U_L^\prime \approx \mathbb{Z} \).

Remark 2.9. Putting together the results of [3] and Proposition 2.8, we have Theorem 1.

2.10 Varieties \( U_1 \) and \( U_{1, L} \)

Henceforth we assume that there is only one node \( y \). We first remark that if \( E \) is a rank 2 vector bundle then \( E \) cannot be \( S \)-equivalent to a direct sum of a line bundle and a non-locally free torsion-free rank 1 sheaf. For, then, one has an exact sequence \( 0 \to L_1 \to E \to L_2 \to 0 \) with one of the \( (L_1)_y \) or \( (L_2)_y \) isomorphic to \( \mathcal{O}_y \) and the other isomorphic to \( m_y \). Since \( \text{Ext}^1(m_y, \mathcal{O}_y) = 0 = \text{Ext}^1(\mathcal{O}_y, m_y) \), this means \( E_y \approx \mathcal{O}_y \oplus m_y \), i.e., \( E \) is not locally free. Similarly one sees that if \( E_y \approx \mathcal{O}_y \oplus m_y \), then \( E \) cannot be \( S \)-equivalent to a direct sum of two locally free sheaves. In particular \( E \) with \( E_y \approx \mathcal{O}_y \oplus \mathcal{O}_y \) cannot be \( S \)-equivalent to \( E' \) with \( E'_y \) not free unless \( [E] = [E'] \in W_0 \). Hence taking determinant gives a well-defined morphism \( \det: U' \cup U_1 \to \mathcal{O}_y \) with \( \det(U') = J_y \), \( \det(U_1) = \mathcal{O}_y - J_y \approx J_x \). This morphism induces a morphism of normalisations \( \det: P' \cup P_1 \to \mathcal{O}_y - J_y \approx J_x \). Hence taking determinant gives a well-defined morphism \( \det: U' \cup U_1 \to \mathcal{O}_y \) with \( \det(U') = J_y \), \( \det(U_1) = \mathcal{O}_y - J_y \approx J_x \).

Lemma 2.11. Let \( L \in \mathcal{O}_y \) with degree of \( L \) even.

1. \( \dim(U_{1, L} - U_{1, L}^g) = g_y \), for all \( L \),
2. \( \text{codim} U_{1, L} - U_{1, L}^g \geq 3 \) for \( g \geq 3 \).

Proof.

1. From §2.10, one sees that \( E \in U_{1, L} - U_{1, L}^g \) is \( S \)-equivalent to \( N_1 \oplus N_2 \) with one of \( N_1, N_2 \) locally free and the other torsion-free but not locally free. Also, one of them is a subsheaf and the other is a quotient sheaf. By Lemma 2.1, \( E \sim M \oplus (M' \otimes L), M \in J_y \).

It follows that \( \dim(U_{1, L} - U_{1, L}^g) = g_y \). In fact, one has \( U_{1, L} - U_{1, L}^g \approx J_y \).

2. One has \( \dim U_{1, L} = 3g_y - 3 \). Hence \( \text{codim}(U_{1, L} - U_{1, L}^g) = 2g_y - 3 \geq 3 \) for \( g_y \geq 3 \),
**Lemma 2.12.** For $L \in \mathcal{J}_Y - J_Y$ and $g_Y \geq 2$, one has $\text{codim}_{U_L}(U_L - U_{1, L}) \geq 2$.

**Proof.** The subset $U_L - U_{1, L}$ consists of torsion-free (semistable) rank 2 sheaves $E \approx \pi_* E_0, E_0$ semistable vector bundle of rank 2 on $X$ with $\det E_0 \approx (\pi^* L/\text{torsion})(-x)$ or $(\pi^* L/\text{torsion})(-z)$ [11]. Hence $\dim(U_L - U_{1, L}) = 3g_X - 3$ if $g_X \geq 2$, $\dim(U_L - U_{1, L}) = 0$ if $g_X = 1$ and $d$ is odd, $\dim(U_L - U_{1, L}) = 1$ if $g_X = 1$ and $d$ is even. Therefore, one has for $g_Y \geq 3$, $\dim U_L - U_{1, L} = 3g_Y - 6$ and $\text{codim}_{U_L}(U_L - U_{1, L}) = (3g_Y - 3) - (3g_Y - 6) = 3$. For $g_Y = 2$, $\text{codim}_{U_L}(U_L - U_{1, L}) = 3$ if $d$ is odd and $\text{codim}_{U_L}(U_L - U_{1, L}) = 2$ if $d$ is even.

**Lemma 2.13.**

1. $U_1^s$ is non-singular, $U_1$ is normal.
2. $U_{1, L}$ is normal, $U_{1, L}^s$ is non-singular.
3. $W_0$ is non-singular, $W_0$ is normal.

**Proof.** The moduli space $U$ is the geometric invariant theoretic quotient of $R^a$ by a projective linear group. Let $E$ be the universal quotient sheaf on $R^a \times Y$. Let $R_1 = \{ t \in \mathbb{R}^g \}$, $R_0 = \{ t \in \mathbb{R}^g (\mathcal{O}_Y) \approx m_r \oplus m_r \}$, $R_{1, L} = \{ t \in \mathbb{R}^g (\mathcal{O}_Y) \approx m_r \oplus m_r \}$. At any point $p \in \mathbb{R}^g$, the analytic local model for $R_1 \rightarrow \mathbb{R}^g$ at $p$ is $\text{Spec} A/(u, v) \hookrightarrow \text{Spec} A$ where $A = \mathbb{C}[u, v]/(uv)$ ([9], Theorem 2(2), p. 576). Since the spectrum of a point is a regular scheme, $R_1$ is regular. Since $U_{1, s}$ is a geometric quotient of $R_{1, s}$, it follows that $U_{1, s}$ is a regular scheme. Since $R_1, \mathcal{J}_Y - J_Y$ are regular and $R_{1, L}$ are all isomorphic, $R_{1, L}$ is regular. Hence the assertion (2) follows. We remark here that $R_1, R_{1, L}$ are not saturated for $S$-equivalence; $U_1$ and $U_{1, L}$ are G.I.T. quotients of open subsets of $R_1$ and $R_{1, L}$ consisting of sheaves not $S$-equivalent to elements in $R_0$ and hence are normal. The assertion (3) follows as (2) using ([9], Theorem 2(3)).

**PROPOSITION 2.14.**

Let $Y$ be an irreducible projective curve (with one ordinary node), $g_Y \geq 2$ and $n = 2$. Then

\[ \text{Pic } U_{1, L}^s \approx \mathbb{Z} \text{ or } \mathbb{Z}/m \mathbb{Z}, m \in \mathbb{Z}. \]

**Proof.** The idea of the proof is the same as that of ([6] or [3], Proposition 2.3). Hence we only indicate the necessary modifications. We may assume $d \gg 0$. Then $R^1 p_* \mathcal{P}^s$ is a vector bundle on $\mathcal{J}_Y$. Let $\mathbb{P}^1 = \mathbb{P}(R^1 p_*(\mathcal{P}^s)), \mathbb{P}_L$ be the fibre of $\mathbb{P}$ over $L \in \mathcal{J}_Y$. One has a universal family $E$ of rank 2 torsion-free sheaves $E$ of degree $d$ on $\mathbb{P} \times Y$. Let $\mathbb{P}^1, \mathbb{P}_L^1$ be the subvarieties corresponding to stable sheaves. Since Ext$^1(\mathcal{O}_Y, \mathcal{O}_Y) = 0 \approx$ Ext$^1(m_r, \mathcal{O}_Y)$, one has $E_v \approx \mathcal{O}_Y \oplus \mathcal{O}_Y$ or $\mathcal{O}_Y \oplus m_r$. Hence by the universal property of moduli spaces, one has morphisms $f_e : \mathbb{P}^1 \rightarrow (U - W_0)^s$ and $f_{e, L} : \mathbb{P}_L^1 \rightarrow U_{1, L}^s$ (or $U_{1, L}$) if $L \in J_Y$ (or $L \in \mathcal{J}_Y - J_Y$). By [10], Chapter 7, Lemma 5.2, any semistable torsion-free sheaf $E$ of $d \gg 0$ is generated by global sections. If $E_v \approx \mathcal{O}_Y \oplus \mathcal{O}_Y$ or $\mathcal{O}_Y \oplus m_r$, then by [1], Lemma 2.7, one has an exact sequence $0 \rightarrow \mathcal{O}_Y \rightarrow E \rightarrow G \rightarrow 0$ with $G$ torsion-free. Also $G \approx \det E$ by Lemma 2.1(1). Hence $f_e$ and $f_{e, L}$ are surjective. One shows that the induced map $f_e^*: \mathbb{P} \rightarrow \mathbb{P}(R^1 p_*(\mathcal{P}^s)[J_Y])$ and
PROPOSITION 2.15.

Let the notations be as in Proposition 2.14. Then for $g_Y \geq 3$, $n = 2$ and $g_Y = 2$, $n = 2$, $d$ odd, one has

$$\text{Pic} \, U_{1,L} \approx \text{Pic} \, U_{1,L}^t \approx \mathbb{Z}.$$ 

Proof. For $d$ odd, $U_{1,L} = U_{1,L}^t$. Since $U_{1,L}$ is normal and $\text{codim}(U_{1,L} - U_{1,L}^t) \geq 3$ (Lemma 2.11), $\text{Pic} \, U_{1,L} \hookrightarrow \text{Pic} \, U_{1,L}^t$ for $d$ even, $g_Y \geq 3$ as in the proof of Proposition 2.8. Going to a finite normalisation we see that $\text{rank} \, (\text{Pic} \, U_{1,L}) \geq 1$. We need Lemma 2.12 for this. The result now follows from Proposition 2.14.

2.16

Assume that $g_Y = 2, g_X = 1, n = 2, d = 0$. Let $M$ be the moduli space of $\alpha$-semistable GPBs $(E, F_1(E))$ of rank 2, degree 0 on a smooth elliptic curve $X$, $0 < \alpha < 1$, $\alpha$ being close to 1 [11]. Let $M_\alpha$ be the closed subscheme of $M$ corresponding to $E$ with determinant $L, L \in J_X$. Let $p_1: F_1(E) \to E, p_2: F_1(E) \to E_2$ be the projections. Define $D_L = \{(E, F_1(E)) \in M_\alpha | p_2 \text{ has rank } \leq 1\}$ and $D_{L,1} = \{(E, F_1(E)) \in D_L | \text{rank } p_2 = 1, p_1 \text{ isomorphism}\}$. $D_{L,1}$ is an open subscheme of $D_L$ and $D_{L,1}$ is a closed subscheme of codimension 1 in $D$. There is a surjective birational morphism $f: M \to U$ such that $D_L$ maps onto $U_{1,L}$ inducing an isomorphism $D_{L,1} \approx U_{1,L}$ where $L' = \pi_*(L(-z))$. We shall determine $D_{L,1}$ explicitly and use the explicit description to compute $\text{Pic} \, U_{1,L}$. Note that $D_L \approx D_{\emptyset}$ for all $L$.

PROPOSITION 2.17.

$D_L$ is isomorphic to a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$. Outside $\mathbb{P}^1 - \{4 \text{ points}\}$, this bundle is of the form $\mathbb{P}(\mathcal{O} \oplus \mathcal{E})$, $\mathcal{E}$ being a rank 2 vector bundle.

Proof. It is not difficult to check that $(E, F_1(E))$ of degree 0, rank 2 is $\alpha$-semistable if and only if $E$ is a semistable vector bundle and for any line sub-bundle $L$ of $E$ of degree 0, $F_1(E) \neq L_\alpha \oplus L_z$. Moreover, $(E, F_1(E))$ is $\alpha$-stable if and only if $E$ is semistable and $F_1(E) \cap (L_\alpha \oplus L_z) = 0$ for any sub-bundle of degree 0.
Let \(e_1, e_2\) and \(e_3, e_4\) be the bases of \(E_x\) and \(E_z\) respectively. The subspace \(F_1(E)\) defines a point in the Grassmannian \(Gr\) of two-dimensional subspaces of \(V = E_x \oplus E_z\). Let \(Gr \subset \mathbb{P}(\wedge^2 V)\) be the Plücker embedding, let \((X_1, Y_1, X_2, Y_2, X_3, Y_3)\) be the Plücker coordinates. Any element in \(\wedge^2 V\) is of the form \(X_1 e_1 \wedge e_2 + Y_1 e_3 \wedge e_4 + X_2 e_1 \wedge e_4 + Y_2 e_2 \wedge e_3 + X_3 e_3 \wedge e_1 + Y_3 e_2 \wedge e_4\). The Grassmannian quadric is given by \(X_1 Y_1 + X_2 Y_2 + X_3 Y_3 = 0\). Since \(E\) is semistable, one has either (a) \(E = M \oplus M^*, M \in J_X\) or (b) there is a non-trivial extension \(0 \to M_1 \overset{\phi}{\to} E \overset{h}{\to} M_2 \to 0\) with \(M_1 \approx M_2 \approx M \in J_X, M^2 = \mathcal{O}\). In either case \(E\) is an extension of \(M_2\) by \(M_1; M_1, M_2 \in J_X\). Choose \(e_1, e_2, e_3, e_4\) to be basis elements of \((M_1)_x, (M_2)_x, (M_1)_z, (M_2)_z\) respectively. Let \(D_V \subset Gr\) be defined by \(Y_1 = 0\).

**Case (a).** Assume that \(E = M_1 \oplus M_2, M_{1}^* = M_2, M_1 \neq M_2\). The group \(H^0 (\text{Aut} E) = H^0 (G_m \times G_m) \approx G_m\) acts on \(D_V \subset \mathbb{P}(\wedge^2 V)\) by \(t (X_1, X_2, Y_2, X_3, Y_3) = (X_1, X_2, Y_2, t X_3, t^{-1} Y_3)\). It is easy to see that \(D_V / G_m \approx \mathbb{P}^2\), the quotient map \(D_V \to \mathbb{P}^2\) being given by \((X_1, X_2, Y_2, X_3, Y_3) \mapsto (X_1, X_2, Y_2)\). Let \(D_1, V = D_V - \{ (X_1 = 0) \cup \{1, 0, 0, 0, 0\}\}\). The image of \(D_1, V\) in \(\mathbb{P}^2\) is given by \(Y_2 (X_1, X_2, Y_2, X_3, Y_3) = 0\).

Let \(\mathcal{P}_X \to J_X \times X\) be the Poincaré bundle, \(\mathcal{P}_x = \mathcal{P}|_{J_x \times x}, \mathcal{P}_z = \mathcal{P}|_{J_z \times z}, J'_X = J_X - J_2\), \(J_2\) being the group of 2-torsion points of \(J_X\). The group \(G_m \times G_m\) acts on the bundles \(\mathcal{V} = (\mathcal{P}_x \oplus \mathcal{P}_z^*) \oplus (\mathcal{P}_x^* \oplus \mathcal{P}_z^*)\) and \(\wedge^2 V\) as above, giving \(G_m\)-action on \(\mathbb{P}(\wedge^2 V)\) and \(D_V / G_m \approx \mathbb{P}^2\)-bundle over \(J'_X\). This \(\mathbb{P}^2\)-bundle is in fact the bundle \(\mathbb{P}(\mathcal{O} \oplus (\mathcal{P}_x \oplus \mathcal{P}_z^*) \oplus (\mathcal{P}_x^* \oplus \mathcal{P}_z^*))\). The involution on \(J_X\) given by \(i (M) = M^*\) lifts to an action on this bundle (switching second and third factors), hence it descends to a bundle on \(J'_X / i = \mathbb{P}^1 - \{4 \text{ points}\}\), of the form \(\mathbb{P}(\mathcal{O} \oplus \mathcal{E}), \mathcal{E}\) a vector bundle of rank 2 on \(J'_X / i\).

**Case (b).** There are, up to isomorphism, exactly four bundles \(E\) given by extension of type (b). Since any automorphism of \(E\) is of the form \(\lambda Id + \mu g \circ h\), one has \(\mathbb{P}(\text{Aut} E) \approx G_m\) under the isomorphism \((\lambda, \mu) \mapsto t = \mu \lambda^{-1} \in G_a\). The action of \(G_a\) on \(V\) is given by \(te_1 = e_1, te_3 = e_3, te_2 = e_2 + te_1, te_4 = e_4 + te_3\) and that on \(D_V\) is given by \(t (X_1, X_2, Y_2, X_3, Y_3) = (X_1, X_2 + t Y_3, X_2 + t Y_3, X_3 - t (X_2 + Y_2) - t^2 Y_3, Y_3)\). It is not difficult to see that the ring of invariants for \(G_a\)-action on \(D_V\) (resp. on the hyperplane \(Y_1 = 0\) of \(\mathbb{P}(\wedge^2 V)\) is generated by \(X_1, X_2 - Y_2, X_3\) (resp. \(X_1, X_2 - Y_2, X_3, X_2 Y_2 + X_3 Y_3\)). The non-semistable points for the \(G_a\)-action are \(\{ X_1 = Y_3 = X_2 - Y_2 = 0 \}\). It follows that \(D_V / G_a \approx \mathbb{P}^2\), the quotient map \(D_V \to \mathbb{P}^2\) being given by \((X_1, X_2, Y_2, X_3, Y_3) \mapsto (X_1, X_2 - Y_2, Y_3)\). Clearly, \(D_1, V / G_a \approx \mathbb{P}^2 - \{ (X_1 = 0) \cup \{1, 0, 0\}\}\). We remark that non-stable GPBs correspond to the line \(Y_3 = 0\) in \(\mathbb{P}^2\). In case \(E = M_1 \oplus M_2, M_1 = M_2\) with \(M_{1}^2 = \mathcal{O}\), one sees that corresponding quotient \(D_V / G_a\) is \(\mathbb{P}^1\) which is identified to the line \(Y_3 = 0\) in the above \(\mathbb{P}^2\). Note that there are no stable GPBs in the last case.

It follows that there is a \(\mathbb{P}^2\)-fibration \(\phi : D_{L} \to \mathbb{P}^1\) which is locally trivial outside the set of four points in \(\mathbb{P}^1\). By Tsen’s theorem (115, p. 108, Case (d)), \(\phi\) is a locally trivial fibration. This completes the proof.

**Corollary 2.18.**

Let \(g_X = 1, g_Y = 2, d\) even, \(n = 2\).

1. \(U_{1,U'}\) is non-singular.
2. \(\text{Pic } U_{1,U'} \approx \mathbb{Z}\).
Proof.

(1) It follows immediately from the proof of Proposition 2.18 that $D_{1, L}$ is a (locally trivial) fibration over $\mathbb{P}^1$ with non-singular fibres isomorphic to $\mathbb{P}^2 - \{(X_1 = 0) \cup (1, 0, 0)\}$. Hence $D_{1, L}$ and $U_{1, L'}$ are non-singular.

(2) $D_{1, L} - D_{1, L} \cong$ (hyperplane $H$) \cup \{a line $\ell$\}, $H \cap \ell = \Phi$, Pic $D_{L} \approx$ Pic $\mathbb{P}^1 \oplus$ Pic $\mathbb{P}^2$. Since $D_{L}$ is non-singular, $0 \to \mathbb{Z}H \to$ Pic $D_{L} \to$ Pic $(D_{L} - H) \to 0$ is exact. It follows that Pic $D_{L} - H \cong$ Pic $\mathbb{P}^1 = \mathbb{Z}$. Since $\ell$ is of codimension 2, Pic $(D_{1, L}) \cong$ Pic $(D_{L} - H) \cong \mathbb{Z}$. Thus Pic $U_{1, L'} \cong$ Pic $D_{1, L} \cong \mathbb{Z}$.

Remark 2.19. Note that $H \to \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle. The fibres of this bundle are given by $X_1 = 0$ in $D_Y$, the restriction of this bundle to $\mathbb{P}^1 - \{4 \text{ points}\}$ is $\mathbb{P}(\mathcal{E})$. Under the map $D_{L} \to U_{L'}$, this $\mathbb{P}^1$-bundle maps onto one component in $U_{L'} - U_{1, L'}$ isomorphic to $J_{Y} / \iota (\approx \mathbb{P}^1)$. This component corresponds to sheaves of the form $\pi, E_0$, det $E_0 \approx L(-x - z)$. The line $\ell$ maps isomorphically onto the other component isomorphic to $\mathbb{P}^1$, it corresponds to $\pi, E_0$, det $E_0 \approx L(-2z)$. Since $g_X = 1$, $E_0$ are semistable but not stable. Thus unlike the case when $L$ is a line bundle ($Y$ smooth or nodal) $U_{L'} - U_{1, L'}$ is not the Kummer variety. It has an open subset isomorphic to $J_Y$ (Proof of Lemma 2.11(1)) whose complement is the union of two disjoint smooth rational curves.

Putting together Proposition 2.15 and Corollary 2.18, we have proved the following.

Theorem 2. Let $Y$ be an irreducible projective curve of arithmetic genus $\geq 2$ with only a single ordinary node as singularity. Let $L$ be a rank 1 torsion-free sheaf which is not locally free. Then

$$\text{Pic } U_{1, L} \cong \mathbb{Z}.$$ 

3. Pic and local factoriality of $U'(n, d), U_{1, L}(2, d)$

3.1

In this section we prove Theorems 3A and 3B. Throughout the section, we assume that $n \geq 2$ and if $n \geq 3$ then $g \geq 2$. One has a map $U_{L'} \times J \to U'$ given by tensorisation. We first remark that Pic $U'$ cannot be computed easily using this map. The map induces a map of Picard groups Pic $U' \approx$ Pic $U_{L'} \oplus$ Pic $J \to$ Pic $U_{L'} \oplus$ Pic $J$. The induced map Pic $J \to$ Pic $J$ is not identity, it is multiplication by $n$. The right map to consider is the determinant morphism which does induce identity on Pic $J$ as we show below:

Theorem 3A. One has the following:

(a) Pic $U'' \approx$ Pic $J \oplus \mathbb{Z}$,

(b) Pic $U' \approx$ Pic $J \oplus \mathbb{Z}$,

(c) $U'$ is locally factorial.

Proof.

(a) Without loss of generality, we may assume that $d \gg 0$. Then a semistable vector bundle $E$ of degree $d$ is globally generated ([10], Lemma 5.2) and contains a trivial sub-bundle of rank $n - 1$. Let $F = F(\mathcal{O}^{n-1}_{\mathbb{P}^1} \otimes \mathbb{C}^{d-1})$, it is a projective bundle over $J$. 


Let \( \mathbb{P}_L \) denote its fibre over \( L \in J. \mathbb{P}_L \) is a projective space. \( \mathbb{P} \) parametrises a family \( \mathcal{E} \) of vector bundles on \( Y \) of rank \( n \), degree \( d \) and containing a trivial sub-bundle of rank \( n - 1 \). Let \( \mathbb{P}^d = \{ p \in \mathbb{P} | \mathcal{E}_p \text{ stable} \} \). \( \mathbb{P}_L^d = \mathbb{P}^d \cap \mathbb{P}_L \). One has canonical surjective morphisms \( f: \mathbb{P}^d \to U^{(n, d)}(n, d) \), \( f_L: \mathbb{P}_L^d \to U_L^{(n, d)}(n, d) \) such that the induced maps \( f^*: \operatorname{Pic} U^{(n, d)} \to \operatorname{Pic} \mathbb{P}^d \), \( f_L^*: \operatorname{Pic} U_L^{(n, d)} \to \operatorname{Pic} \mathbb{P}_L^d \) are injective (3, Proposition 2.3; 3, Propositions 7.6, 7.8, 7.9). Clearly, \( \operatorname{Pic} \mathbb{P} \approx \operatorname{Pic} \times \operatorname{Pic} \mathbb{P}_L \approx \operatorname{Pic} J \times \mathbb{Z} \). Under the conditions of the theorem we know that (3, Theorem I) \( \operatorname{Pic} U_L^{(n, d)} \approx \mathbb{Z} \) and hence \( \operatorname{Pic} \mathbb{P}_L^d \approx \mathbb{Z} \).

Hence the surjective restriction map \( \operatorname{Pic} \mathbb{P}_L \to \operatorname{Pic} \mathbb{P}_L^p \) is an isomorphism for all \( L \in J \). Hence \( \operatorname{codim}_{\mathbb{P}}(\mathbb{P}_L - \mathbb{P}_L^p) \neq 1 \) and therefore \( \operatorname{codim}_{\mathbb{P}}(\mathbb{P} - \mathbb{P}^p) \geq 2 \). Thus \( \operatorname{Pic} \mathbb{P} \approx \operatorname{Pic} J \oplus \mathbb{Z} \) and hence

\[ \operatorname{Pic} U^{(n, d)} \to \operatorname{Pic} J \oplus \mathbb{Z}. \]

The natural map \( p: \mathbb{P}^p \to J \) factors as \( p = \det \circ f \), where \( \det \) is the determinant map \( E \to \mathbb{P} E \). Since both \( f \) and \( \det \) are surjections, so is \( p \). Note that \( f^* \circ \det^* = p^*: \operatorname{Pic} J \to \operatorname{Pic} \mathbb{P}^p \) is injective. It follows that \( \det^* \) is injective.

One has the following diagram with the last column exact.

\[
\begin{array}{c|c|c|c}
\text{Pic} J & \text{Pic} J & \text{Pic} J \\
\downarrow & \downarrow & \downarrow \\
\text{Pic} U' & \text{Pic} U^{(n, d)} & \text{Pic} J \oplus \mathbb{Z} \\
\downarrow & \downarrow & \downarrow \\
\text{Pic} U_L' & \approx & \text{Pic} U_L^{(n, d)} \approx & \mathbb{Z} \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & \\
\end{array}
\]

Here \( \mathbb{Z} \) denotes the image of \( \text{Pic} U_L^{(n, d)} \) in \( \mathbb{P}_L^d \). The map \( \text{Pic} U_L^{(n, d)} \to \mathbb{P}_L^d \) is the restriction map and is surjective (3, Proposition 3.2 and 3.5). It now follows from the diagram that the injection \( \text{Pic} U^{(n, d)} \to \text{Pic} J \oplus \mathbb{Z} \) is an isomorphism and the second column is exact.

(b) and (c). Since \( \operatorname{codim}_{\mathbb{P}}(U' - U^{(n, d)}) \geq 2 \) under the conditions of the theorem and \( U' \) is normal (3, Proposition 3.4(i)), it follows that the restriction map \( \operatorname{Pic} U' \to \operatorname{Pic} U^{(n, d)} \) is injective. The restriction morphism \( \operatorname{Pic} U' \to \operatorname{Pic} U_L' \) is surjective (3, Propositions 3.2, 3.5). The restriction map \( \operatorname{Pic} U_L' \to \operatorname{Pic} U_L^{(n, d)} \) is an isomorphism (3). It now follows from the commutative diagram that \( \operatorname{Pic} U' \approx \operatorname{Pic} U^{(n, d)} \) under the restriction map. By arguments similar to those in the proof of (3), Proposition 3.6, this implies that \( U' \) is locally factorial.

**Theorem 3B.** Let \( Y \) be an irreducible projective curve of arithmetic genus \( g_Y \geq 2 \) with only a single ordinary node as singularity. If \( g_Y = 2 \), then assume that \( d \) is odd. Let \( L \) be a rank 1 torsion-free sheaf of degree \( d \) which is not locally free. Let \( U_{1, L} \) be the subscheme of \( U \) corresponding to torsion-free sheaves of rank 2 with determinant isomorphic to \( L \).
Proof. The proof is more or less identical with that of Theorem 3A. One has only to replace \( f_1, f_\ell \) by the maps \( f_1, f_\ell, L \) of Proposition 2.14 and use Theorem 2 instead of Theorem 1.

4. The dualising sheaves of \( U' \) and \( U'_L \)

4.1

Let \( K(Y) \) denote the Grothendieck group of vector bundles on \( Y \). Then \( K(Y) \approx \mathbb{Z} \oplus \text{Pic} Y \) under the map \( [E] \mapsto (\text{rank} E, \det E) \), \([E]\) being the class of a vector bundle \( E \) in \( K(Y) \). The inverse map is given by \( n \mapsto [n \cdot \mathcal{O}_Y] \) for \( n \in \mathbb{Z} \) and \( L \mapsto [L] - [\mathcal{O}_X] \) for \( L \in \text{Pic} Y \).

Let \( \chi = d + n(1 - g), P(m) = \chi + rm, \) fix \( m \gg 0 \). Let \( Q = \text{Quot}(\mathbb{C}^p(m) \otimes \mathcal{O}_Y(-m), P) \) be the Hilbert scheme (‘the Quot scheme’) of quotients of \( \mathbb{C}^p(m) \otimes \mathcal{O}_Y(-m) \) with Hilbert polynomial \( P \). Let \( \mathcal{F} \to Q \times Y \) be the universal family. Let \( R_m \subset Q \) be the open subset consisting of \( q \in Q \) such that \( H^1(\mathcal{F}_q(m)) = 0 \), \( H^0(\mathcal{F}_q(m)) \approx H^0(\mathcal{F}(m)) \) under the canonical map, \( \mathcal{F} \) is a semistable torsion-free sheaf is contained in \( R_m \). The subset \( R_{ss} \) of \( R_{st} \) corresponding to semistable vector bundles is a smooth variety, so is the closed subset \( R_{ss}' \subset R_{ss} \) consisting of \( R_{ss} \) with fixed determinant \( L \). (Remark, p. 167).

The moduli space \( U' \) (resp. \( U'_L \)) is a geometric invariant theoretic good quotient of the smooth irreducible scheme \( R_{ss}' \) (resp. \( R_{ss}' \)) by the group \( G = \text{P}(\text{Aut} \sum) \approx \text{PGL}(N) \). The restriction of the universal family on \( Q \times Y \) gives a universal family \( \mathcal{F} \to R_{ss}' \times Y \) of vector bundles on \( Y \) of rank \( n \), degree \( d \). Let \( \text{Pic}^G(R_{ss}') \) denote the group of line bundles on \( R_{ss}' \) with \( G \)-action (compatible with the \( G \)-action on \( R_{ss}' \)). For a vector bundle \( E \) on \( Y \), one defines an element \( \lambda_\mathcal{F}(E) \in \text{Pic}^G(R_{ss}') \) by

\[
\lambda_\mathcal{F}(E) := \otimes_i (\text{det} R_{p_1, (\mathcal{F} \otimes p_2^* E)})^{(-1)^{i+1}},
\]

where \( p_1 \) and \( p_2 \) are projections to \( R_{ss}' \) and \( Y \) respectively. \( \lambda_\mathcal{F}(E) \) depends only on the class of \( E \) and \( \lambda_\mathcal{F}: K(Y) \to \text{Pic}^G(R_{ss}') \) is a group homomorphism.

PROPOSITION 4.2.

Let \( E \) be a vector bundle on \( Y \) with \( \text{rank}(E) = n/\delta \), \( \det(E) = \mathcal{O}_Y(-\frac{E}{\delta}) \), \( \chi = d + n(1 - g) \), \( \delta = \gcd(n, d) \). Then \( \lambda_\mathcal{F}(E) \) descends to \( U'_L(n, d) \) as the generator \( L \) of \( \text{Pic} U'_L(n, d) \).

Proof. By \( \lambda_\mathcal{F} \), Propositions 3.2, 3.5, the generator \( L \) is obtained by the descent of the line bundle \( L' \) on \( R_{ss}' \) given by

\[
L' = (\text{det} R_{p_1, \mathcal{F}})^{\frac{1}{\delta}} \otimes (\mathcal{F}|_{R_{ss}' \times Y})^{x/\delta},
\]

\( y_0 \) being a non-singular point of \( Y \). Here \( \text{det} R_{p_1, \mathcal{F}} \) denotes the determinant of cohomology (\( \mathbb{I} \), Ch.VI, pp. 135–136). However, our definition is different from the standard one, it is the inverse of the line bundle defined in \( \mathbb{I} \) as \( \text{det} R_{p_1, \mathcal{F}} \). One has det
Since $\det F(\forall \text{rial Cohen–Macaulay variety is Gorenstein, i.e., its dualising sheaf This proves the claim.

**Proof of the Claim.** For $m \gg 0$ one has the exact sequence

$$0 \to \mathcal{F}(m) \to \mathcal{F}(m+1) \to \mathcal{F}(m)|_{\mathbb{P}L_0} \to 0,$$

$$\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}_V(m), \mathcal{O}_V(1)$$

being a line bundle of degree 1 on $Y$. Since $R^1_{p_1}(\mathcal{F}(m')) = 0$

$\forall m' \geq m, R^1_{p_1}(\mathcal{F}(m)|_{\mathbb{P}L_0}) = 0$, the direct image sequence gives

$$0 \to R^0_{p_1}(\mathcal{F}(m)) \to R^0_{p_1}(\mathcal{F}(m+1)) \to R^0_{p_1}(\mathcal{F}(m)|_{\mathbb{P}L_0}) \to 0.$$

Since $\det p_1(\mathcal{F}(m')) = -\lambda_\mathcal{F}(1+m'h), m' \geq m,$ and

$$\det(p_1, \mathcal{F}(m)|_{\mathbb{P}L_0}) \approx \det(p_1, \mathcal{F}|_{\mathbb{P}L_0}) = n \mathcal{O}|_{\mathbb{P}L_0},$$

one has

$$\det p_1(\mathcal{F}(m+1)) = -\lambda_\mathcal{F}((m+1)h) + \lambda_\mathcal{F}(1+mh)$$

$$= -\lambda_\mathcal{F}(h).$$

This proves the claim.

Thus we have

$$\mathbb{L}' = \frac{n}{\delta} \lambda_\mathcal{F}(1) - \frac{\delta}{\delta} \lambda_\mathcal{F}(h)$$

$$= \lambda_\mathcal{F} \left( \frac{n}{\delta} - \frac{\delta h}{\delta} \right) = \lambda_\mathcal{F}(E).$$

**Remark 4.3.** Note that the line bundle $\mathbb{L}'$ exists on $R^{ss}$ and descends to $U'$ (Theorem 2; Proposition 3.5). Also $\lambda_\mathcal{F}(E)$ makes sense for $\mathcal{F} \to R^{ss} \times Y$, the universal family on $R^{ss} \times Y$. The above relation between $\lambda_\mathcal{F}(E)$ and $\mathbb{L} \in \text{Pic } U'_L(n, d)$ holds for $\lambda_\mathcal{F}(E)$ and $\mathbb{L} \in \text{Pic } U'(n, d) \approx \text{Pic } U'_L \oplus \text{Pic } J$.

### 4.4 Computation of the dualising sheaves

Both $U'$ and $U'_L$ are normal and Cohen–Macaulay as they are quotients of smooth varieties by $\text{PGL}$. They are also locally factorial (Theorem 2; Theorem 1). A locally factorial Cohen–Macaulay variety is Gorenstein, i.e., its dualising sheaf $\omega$ is locally free. The tangent sheaf $T_{U'}$ of $U'$ is locally free on the smooth open subscheme $U'^{ss}$ of codimension $\geq 2$. Hence the determinant of $T_{U'}$ defines a line bundle $\det T_{U'}$ on $U'$. Since it coincides with $\omega^{-1}$ on $U'^{ss}$, it follows that $\omega^{-1} = \det T_{U'}$. Similarly one has a locally free dualising sheaf $\omega_0$ on $U'_L$ with $\omega_0^{-1} = \det T_{U'_L}$.

**Theorem 4.** Let the assumptions be as in Theorem 1. Then one has the following:
(a) $\omega \approx -2\delta L$, $L = \text{generator of } \text{Pic } U'_1(n,d)$.
(b) Let $F_0$ be a vector bundle on $Y$ of rank $2r$ and degree $2(-d + r(g-1))$. Then $\omega \approx \lambda (F_0) \otimes \det \wedge$, where $\wedge$ is a line bundle on $J$ given by

$$\wedge = \det(p_{\text{tr}}[[\mathcal{P}] \otimes \det p_{\text{tr}}[[\mathcal{P}^*]])^{-1} \otimes \det p_{\text{tr}}([\mathcal{P} \otimes F_0]^*)^{-1}.$$

**Proof.** In view of the injective morphism $f'_L: \text{Pic } U'_1 \rightarrow \text{Pic } P'_1$ mapping $L$ to $P'_1(\mathbb{Z}/(r-1))$, it suffices to prove that

$$\det f'_L T_{U'_1} \approx O_{\mathbb{P}^d}(2d(r-1)).$$

One has $f^* T_{U'} \approx R^1_{P'_{U'_1}}(\mathcal{O} \otimes \mathcal{E}^*)$, hence $f'_L T_{U'_1} \approx R^1_{P'_{U'_1}}(\mathcal{O} \otimes \mathcal{E}^*)$. Also, $\det R^1_{P'_{U'_1}}(\mathcal{O} \otimes \mathcal{E}^*) \approx \det R^1_{P'_{U'_1}}(\mathcal{O} \otimes \mathcal{E}^*)$, so that $f'_L T_{U'_1} \approx \det R^1_{P'_{U'_1}}(\mathcal{O} \otimes \mathcal{E}^*)_{[2]}. $

**Computation of $\det R^1_{P'_{U'_1}}(\mathcal{O} \otimes \mathcal{E}^*)$**

There is a universal exact sequence on $\mathbb{P}^d \times Y$.

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^d \times Y} \otimes \mathbb{C}^{-1} \rightarrow \mathcal{E} \rightarrow (1 \times p)^\# \mathcal{P} \otimes p_{\mathbb{P}^d} \mathcal{O}_{\mathbb{P}^d}(-1) \rightarrow 0. \tag{1}$$

For $d \gg 0, H^0(\mathcal{E}^*) = 0$ for $v \in \mathbb{P}^d$, $H^0(\mathcal{E} \otimes \mathcal{E}^*)$ consists of scalars as $\mathcal{E}_v$ is stable. Hence by tensoring (1) with $\mathcal{E}$ and taking direct images, one gets (for $d \gg 0$ and $(1 \times p)^\# = p^\#$)

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^d} \rightarrow \mathcal{O}_{\mathbb{P}^d}(-1) \otimes p_{\mathbb{P}^d}(p^\# \mathcal{P} \otimes \mathcal{E}^*) \rightarrow R^1_{P'_{U'_1}}(\mathcal{E} \otimes \mathcal{E}^*) \rightarrow 0.$$

Hence,

$$\det R^1_{P'_{U'_1}}(\mathcal{E} \otimes \mathcal{E}^*) \approx \det(R^1_{P'_{U'_1}}(\mathcal{E}^*))^{-1} \otimes \det(\mathcal{O}_{\mathbb{P}^d}(-1) \otimes p_{\mathbb{P}^d} p^\# \mathcal{P} \otimes \mathcal{E}^*)^{-1}. \tag{2}$$

$R^1_{P'_{U'_1}}(\mathcal{E}^*)$ is computed by taking dual of (1) and direct images as follows:

$$0 \rightarrow p^\# \mathcal{P} \otimes p_{\mathbb{P}^d} \mathcal{O}_{\mathbb{P}^d}(1) \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^d \times Y} \otimes \mathbb{C}^{-1} \rightarrow 0. \tag{1}\ast$$

Since $p_{\mathbb{P}^d} p^\# \mathcal{P} = p_{\mathbb{P}^d}(\mathcal{S}^*)$ for $d \gg 0$, one has the direct image sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^d} \otimes \mathbb{C}^{-1} \rightarrow \mathcal{O}_{\mathbb{P}^d} \otimes R^1_{P'_{U'_1}}(p^\# \mathcal{P} \otimes \mathcal{E}^*) \rightarrow R^1_{P'_{U'_1}} \mathcal{E}^* \rightarrow 0$$

and hence

$$\det R^1_{P'_{U'_1}} \mathcal{E}^* \approx \det(\mathcal{O}_{\mathbb{P}^d}(1) \otimes R^1_{P'_{U'_1}}(p^\# \mathcal{P} \otimes \mathcal{E}^*).$$

Since $h^1(\mathcal{P}^*) = -\chi(\mathcal{P}^*) = d + g - 1$ for $t \in J$, one gets

$$\det R^1_{P'_{U'_1}} \mathcal{E}^* \approx \mathcal{O}_{\mathbb{P}^d}(d + g - 1) \otimes \det R^1_{P'_{U'_1}}(p^\# \mathcal{P} \otimes \mathcal{E}^*). \tag{3}$$
Tensoring (1) with $p^\# \mathcal{D}$ gives

$$0 \to p^\#_{\mathcal{P}'} \mathcal{O}_{\mathcal{P}'}(1) \to \mathcal{E}^* \otimes p^\# \mathcal{D} \to \mathcal{O}_{\mathcal{P}' \times Y} \otimes \mathcal{C}'^{-1} \otimes p^\# \mathcal{D} \to 0,$$

and hence the direct image sequence

$$0 \to \mathcal{O}_{\mathcal{P}'}(1) \to p_{\mathcal{P}'}^*(\mathcal{E}^* \otimes p^\# \mathcal{D}) \to p_{\mathcal{P}'}^*(\mathcal{C}'^{-1} \otimes p^\# \mathcal{D}) \to 0.$$

By tensoring with $\mathcal{O}_{\mathcal{P}'}(-1)$ and taking det, one has

$$\det(p_{\mathcal{P}'}(\mathcal{E}^* \otimes \mathcal{E}^*) \otimes \mathcal{O}_{\mathcal{P}'}(-1)) \approx \det(p_{\mathcal{P}'}(\mathcal{E}^* \otimes \mathcal{C}'^{-1}) \otimes \mathcal{O}_{\mathcal{P}'}((g-d-1)(r-1))).$$

Since $h^0(\mathcal{F}^t) = d + 1 - g$ for $t \in J$, the latter is isomorphic to $\det p_{\mathcal{P}'}(\mathcal{E}^* \otimes \mathcal{C}'^{-1}) \otimes \mathcal{O}_{\mathcal{P}'}((g-d-1)(r-1)).$ Thus we have

$$\det(p_{\mathcal{P}'}(\mathcal{E}^* \otimes \mathcal{E}^*) \otimes \mathcal{O}_{\mathcal{P}'}(-1)) \approx \det p_{\mathcal{P}'}(\mathcal{E}^* \otimes \mathcal{C}'^{-1}) \otimes \mathcal{O}_{\mathcal{P}'}((r-1)(g-d-1)).$$

(4)

Substituting in (2) from (3) and (4) gives

$$\det R^1 p_{\mathcal{P}'}^*(\mathcal{E}^* \otimes \mathcal{E}^*) \approx \mathcal{O}_{\mathcal{P}'}(2(r-1)d) \otimes \Delta'^{-1},$$

(5)

where $\Delta' = \det(R^1 p_{\mathcal{P}'}^*(\mathcal{E}^* \otimes \mathcal{E}^*)) \otimes \det(p_{\mathcal{P}'} \mathcal{D}).$

Since $\Delta|_{\mathcal{P}'}$ is trivial, from (5) one has

$$\det f^*_L T_{U_L} \approx \mathcal{O}_{\mathcal{P}'}(2(r-1)d),$$

this proves (a).

If $F_0$ is a vector bundle of rank $2r$ and degree $2(-d + r(g - 1))$, then from sequence (1), one sees that

$$\lambda_{\mathcal{E}^*}(F_0) \approx \mathcal{O}_{\mathcal{P}'}(-2d(r-1)) \otimes \det^*(p_{\mathcal{P}'} \mathcal{D} \otimes p^\#_L F_0),$$

so that (5) becomes

$$\det(R^1 p_{\mathcal{P}'}^*(\mathcal{E}^* \otimes \mathcal{E}^*)) \approx \lambda_{\mathcal{E}^*}(F_0)^{-1} \otimes \det^*(p_{\mathcal{P}'} \mathcal{D} \otimes p^\#_L F_0)) \otimes \Delta'^{-1}.$$

Since $p = \det \circ f \circ p^* = f^* \circ \det^*$ and $f^*$ is injective, (b) also follows.

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References

[1] Bhosle Usha N, Generalised parabolic bundles and applications to torsionfree sheaves on nodal curves, *Arkiv for Matematik* 30(2) (1992) 187–215

[2] Bhosle Usha N, Vector bundles of rank 2, degree 0 on a nodal hyperelliptic curve, in: Algebraic geometry (eds) P E Newstead, *Lecture Notes in Pure and Appl. Math.* 200 (1998) 271–281
[3] Bhosle Usha N, Picard groups of the moduli spaces of vector bundles, *Math. Ann.* **314** (1999) 245–263
[4] Bhosle Usha N, Maximal subsheaves of torsionfree sheaves, *TIFR Reprint* (2003)
[5] D’Souza C, Compactification of generalised Jacobians, *Proc. Indian Acad. Sci. (Math. Sci.)* **88** (1979) 419–457
[6] Drézet J M and Narasimhan M S, Groupe de Picard des variétés de modules de fibrés semistable sur les courbes algébriques, *Invent. Math.* **97** (1989) 53–94
[7] Lang S, Introduction to Arakelov theory (Springer-Verlag) (1988)
[8] Milne J S, Etale cohomology (Princeton University Press) (1980)
[9] Narasimhan M S and Ramadas T, Factorisation of generalised theta functions-I, *Invent. Math.* **114** (1993) 565–623
[10] Newstead P E, Introduction to moduli problems and orbit spaces, *TIFR Lecture Notes* **51** (1978)
[11] Ramanan S, The moduli space of vector bundles on an algebraic curve, *Math. Ann.* **200**, (1973) 69–84
[12] Seshadri C S, Fibrés vectoriels sur les courbes algébriques, *Asterisque* **96** (1982) 1–209; Vector bundles on curves, *Contemporary Math.* **153** (1993) 163–200
[13] Sun Xiaotao, Degeneration of moduli spaces and generalized theta functions, *J. Alg. Geom.* **9** (2000) 459–527