A variance for $k$-free numbers in arithmetic progressions of given modulus

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1 - Introduction

Let

$$\mathcal{S} = \{ n \in \mathbb{N} \mid \text{there is no prime } p \text{ with } p^k | n \},$$

the set of $k$-free numbers. For some suitable main term $\eta(q, a)$ to be defined soon enough we will study in this paper the object

$$\sum_{a=1}^{q} \left( \sum_{\substack{n \in \mathcal{S} \\ \text{mod } \mathcal{S} \equiv a}} 1 - x\eta(q, a) \right)^2,$$

a variance for $k$-free numbers in arithmetic progressions when averaging over a (complete) residue system. One would like to establish for some $q$ that this is

$$\approx q \left( \frac{x}{q} \right)^{1/k},$$

since this would mean that on average

$$\sum_{\substack{n \in \mathcal{S} \\ n \equiv a \text{ mod } (q)}} 1 - x\eta(q, a) \approx \left( \frac{x}{q} \right)^{1/2k}. \quad (1)$$

Since an improvement in the error term in the classical statement

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}}} 1 = \frac{x}{\zeta(k)} + O \left( x^{1/k} \right)$$

is tantamount to a better zero-free region for the zeta function, information as to the true size of the LHS of (1) is relevant.
Averaging just over the reduced classes an asymptotic formula for the variance, in the squarefree case, is already established in [3] with error essentially
\[\ll q \left( \frac{x}{q} \right)^{1/3} + \left( \frac{x}{q} \right)^{23/15}.\] (2)

Before this only upper bound results are recorded (see [1] and the references therein), although these are stronger in the range where the above asymptotic formulas don’t hold and are concerned with more general sequences than the squarefrees. In this paper, we improve the first error term in (2).

**Theorem.** Let \( k \geq 2 \) and denote by \( S \) the set of \( k \)-free numbers. For \( q, a \in \mathbb{N} \) and \( x > 0 \) define
\[
\eta(q,a) = \sum_{d=1 \atop (d,q) = 1}^{\infty} \frac{\mu(d)}{[q,d^k]^{1/k}}, \quad E_x(q,a) = \sum_{n \leq x \atop n \equiv a \pmod{q}} \left[ \frac{1}{n^{1/k}} \right], \quad V_x(q) = \sum_{a=1}^{q} |E_x(q,a)|^2.
\]
and
\[
V_x(q) = \sum_{a=1}^{q} |E_x(q,a)|^2.
\]

Define
\[
C_k = \frac{2k}{(1/k - 1)\zeta(2)} \prod_{p} \frac{1 - 2/(p^k + p^{k-1})}{1 - p^{k-1/k}}
\]
and
\[
f_k(q) = C_k \prod_{p|q} \frac{1 - 2/p^{k} + (q/p^k)^{1/k-1}/p}{1 - 2/p^{k} + 1/p}.
\]

For \( 1 \leq q \leq x \) we have for every \( \epsilon > 0 \)
\[
V_x(q) = q \left( \frac{x}{q} \right)^{1/k} f_k(q) + \mathcal{O}_{k,\epsilon} \left( x^\epsilon \left( \frac{x}{q} \right)^{2/(9-2/k)} + \frac{x^{1+2/(k+1)}}{q} \right).
\]

This is an asymptotic formula for \( k = 2, 3, 4 \). The relevance of our result is the improvement in the first error term, which for \( k = 2 \) seems decently small. This is obtained by a careful analysis of the integrals arising from an application of Perron’s formula. (Our second error term is weaker than in [2] but most likely can be made to be just as small for the squarefrees by arguing, as in that paper, with the square sieve.)

We consider \( k \geq 2 \) and \( q \leq x \) as fixed throughout. Each time \( \epsilon \) appears it is to be understood that it may be taken arbitrarily small at each occurrence. Fix some \( 0 < \delta < 1/2k \). All \( \ll, \Omega \) constants depend on \( \epsilon, k \) and \( \delta \).

## 2 - Lemmas

For \( \Re(s) > 1 \) define
\[
\mathcal{F}(s) = \sum_{d,d' = 1}^{\infty} \frac{\mu(d)\mu(d')}{[d^k,d'^k][q,(d^k,d'^k)]^s}.
\]
and for $\Re(s) \geq -1 + \delta$ define

$$F^*(s) = \prod_{p \mid q} \frac{1 + (q, p^k s/p^{k(1+s)})}{1 + 1/p^{k(1+s)}} \prod_{p} \left(1 - \frac{2}{p^k (1 + (q, p^k s/p^{k(1+s)})\right).$$

The first series converges since the summands are bounded by

$$\frac{1}{[d^k, d^{'k}](d^k, d^{'k})};$$

for $\Re(s) \geq -1 + \delta$

$$\left|\frac{(q, p^k s/p^{k(1+s)})}{1 - s}\right| \leq \begin{cases} 1/p^k & \text{for } \Re(s) \geq 0 \\ 1/p^k \delta & \text{for } \Re(s) < 0 \end{cases}$$

and therefore

$$1 + (q, p^k s/p^{k(1+s)}) \geq 1 - 1/2^k \gg 1$$

so that each Euler factor of the infinite product in $F^*(s)$ is of the form

$$1 + O\left(1/p^k\right)$$

and therefore this product converges and is uniformly bounded for $\Re(s) \geq -1 + \delta$; for $\Re(s) \geq -1 + \delta$ we have

$$\frac{1}{p^{k(1+s)}} \geq \frac{1}{p^k};$$

and therefore

$$1 + 1/p^{k(1+s)} \geq 1 - 1/2^k \gg 1$$

so each factor in the finite product in $F^*(s)$ is from (5) uniformly bounded for $\Re(s) \geq -1 + \delta$, and since we have just said the same is true for the infinite product, we conclude that $F^*(s) \ll q^\epsilon$ for $\Re(s) \geq -1 + \delta$.

**Lemma 2.1.** If $\Re(s) > 1$ then

$$F(s) = \frac{\zeta(k(s + 1)) F^*(s)}{q^\epsilon \zeta(2k(s + 1))}.$$

**Proof.** We have

$$\sum_{d,d'} \frac{\mu(d) \mu(d')(d^k, d'^k)^{1-s}(q, d^k, d'^k)^s}{d^k, d'^k} = \sum_{N=1}^{\infty} \frac{1}{N^k} \sum_{d'd' = N} \mu(d) \mu(d')(d^k, d'^k)^{1-s}(q, d^k, d'^k)^s$$

$$= \sum_{N=1}^{\infty} a_q(N) \frac{N^k}{N^k}. \quad (6)$$

Clearly $a_q(N)$ is multiplicative and simple calculations show

$$a_q(p) = -2,$$

$$a_q(p^2) = p^{k(1-s)(q, p^k)^s}$$
and \(a_q(p^t) = 0\) for \(t \geq 3\). Consequently

\[
\sum_{N=1}^{\infty} \frac{a_q(N)}{N^k} = \prod_p \left( 1 - \frac{2}{p^k} + \frac{(q, p^k)s}{p^k(1+s)} \right)
\]

\[
= \prod_p \left( 1 + \frac{(q, p^k)s}{p^k(1+s)} \right) \prod_p \left( 1 - \frac{2}{p^k (1 + (q, p^k)s/p^k(1+s))} \right)
\]

\[
= \prod_p \left( 1 + \frac{1}{p^k(1+s)} \right) \prod_{p|q} \frac{1 + (q, p^k)s/p^k(1+s)}{1 + 1/p^k(1+s)} \prod_p \left( 1 - \frac{2}{p^k (1 + (q, p^k)s/p^k(1+s))} \right)
\]

\[
= \frac{\zeta((k(1+s))\mathcal{F}^*(s))}{\zeta(2(k(1+s)))}
\]

so that (6) becomes

\[
\sum_{d,d'} \mu(d)\mu(d')\left(\frac{\sum_{k \geq 1} \frac{1}{k^s} (d^k, d'^k)^{1-s}(q, d^k, d'^k)^s}{d^k d'^k}\right) = \frac{\zeta((k(1+s))\mathcal{F}^*(s))}{\zeta(2(k(1+s)))}
\]

and the claim follows. \(\square\)

**Lemma 2.2.** Suppose \(q\) has \(\omega\) distinct prime factors \(p_1, \ldots, p_\omega\) and let \(\mathcal{F}^*(s)\) be as given at the start of this section. Then for each \(n \in \mathbb{N}\) and each \(l_1, \ldots, l_\omega, l'_1, \ldots, l'_\omega \geq 0\) there are \(\lambda_n, W_n, C_{l', l} \in \mathbb{R}\) with \(W_n, Z_{l', l} > 0\) such that

\[
\mathcal{F}^*(s) = \sum_{l_1, \ldots, l_\omega \geq 0} \sum_{n=1}^{\infty} C_{l_1, l_\omega} Z_{l', l} \lambda_n W_n \mathcal{F}^*(s) \quad \text{for } \Re(s) \geq -1 + \delta.
\]

Moreover for \(-1 + \delta \leq \Re(s) \leq 0\)

\[
\sum_{l_1, \ldots, l_\omega \geq 0} \sum_{n=1}^{\infty} |C_{l_1, l_\omega} Z_{l', l} \lambda_n W_n^{1+s}| \ll \log(q + 1).
\]

**Proof.** From (6) we have \(|(q, p^k)^s/p^k(1+s)| < 1\) and therefore

\[
\prod_p \left( 1 - \frac{2}{p^k (1 + (q, p^k)s/p^k(1+s))} \right) = \prod_p \left( 1 - \frac{2}{p^k} \sum_{t \geq 1} \left(\frac{-(q, p^k)s}{p^k(1+s)}\right)^{t-1} \right)
\]

\[
= \sum_{n=1}^{\infty} f^*_n(n)\]

where \(f^*_n(n)\) is the multiplicative function given on prime powers by

\[
f^*_n(p^t) = -\frac{2}{p^t} \left(\frac{-(q, p^k)s}{p^k(1+s)}\right)^{t-1}.
\]
For any \( n \in \mathbb{N} \) and prime \( p|n \) define \( t = t(p) \) through \( p^t|n \). Then

\[
f^*_s(n) = \prod_{p|n} \left( -\frac{2}{p^k} \right) \left( -\frac{\sigma}{p^k(1+s)} \right)^{t-1} = \left( \prod_{p|n} (-1)^{t-1} \right) \left( \prod_{p|n} \frac{-2}{p^k} \right) \left( \prod_{p|n} \frac{(q, p^k)^{(t-1)(1+s)}}{p^{(t-1)k(1+s)}} \right). \tag{9}
\]

If we now define

\[
\lambda_n = \left( \prod_{p|n} (-1)^{t-1} \right) \left( \prod_{p|n} \frac{-2}{p^k} \right) \left( \prod_{p|n} (q, p^k)^{1-t} \right)
\]

and

\[
W_n = \prod_{p|n} \frac{(q, p^k)^{t-1}}{p^{(t-1)k}}
\]

then (9) becomes

\[
f^*(n) = \lambda_n W_{n^{1+s}}
\]

so (8) becomes

\[
\prod_p \left( 1 - \frac{2}{p^k \left( 1 + (q, p^k)^{s/p^k(1+s)} \right)} \right) = \sum_{n=1}^{\infty} \lambda_n W_{n^{1+s}}. \tag{10}
\]

Just as (8) is true so is

\[
\sum_{n=1}^{\infty} |f^*_s(n)| = \prod_p \left( 1 - \frac{2}{p^k \left( 1 + (q, p^k)^{s/p^k(1+s)} \right)} \right) = \sum_{n=1}^{\infty} \lambda_n W_{n^{1+s}}. \tag{11}
\]

For \(-1 + \delta \leq \Re(s) \leq 0\) the \( t \) sum here is from (9)

\[
\ll \sum_{l \geq 1} \left( \frac{1}{p^k} \right)^{t-1} = \frac{1}{1 - p^k \delta} \ll 1
\]

so the Euler product in (11) is uniformly bounded in this range and therefore

\[
\sum_{n=1}^{\infty} |f^*(n)| \ll 1, \quad \text{for } -1 + \delta \leq \Re(s) \leq 0. \tag{12}
\]

We have for \( \Re(s) \geq -1 + \delta \)

\[
\frac{1}{1 + 1/p^k(1+s)} = \sum_{l \geq 0} \left( \frac{-1}{p^k(1+s)} \right)^l = \sum_{l \geq 0} \frac{C_p(l)}{p^{l(1+s)}}
\]

for some \( C_p(l) \) with

\[
\sum_{l \geq 0} \left| \frac{C_p(l)}{p^{l(1+s)}} \right| \ll \sum_{l \geq 0} \left( \frac{1}{p^k} \right)^l \ll 1. \tag{14}
\]
as well as
\[ 1 + \frac{(q, p^k)^s}{p^{k(1+s)}} = \sum_{l' \geq 0} \frac{C_{p'}(l')(q, p^k)^{sl'}}{p^{k(1+s)l'}} \] (15)
for some \( C_{p'}(l') \) with
\[ \sum_{l' \geq 0} \left| C_{p'}(l')(q, p^k)^{sl'} \right| \leq 1 + 1. \] (16)

From (5). From (13), (14), (15) and (16) there are for each prime \( p \) and \( l, l' \in \mathbb{N} \) some \( C_p(l), C_{p'}(l') \) for which
\[ \frac{1 + (q, p^k)^s}{1 + 1/p^{k(1+s)}} = \frac{C_p(l)C_{p'}(l')(q, p^k)^{sl'}}{p^{k(1+s)l'}} \]
and
\[ \sum_{l', l'' \geq 0} \left| C_{p'}(l')C_{p''}(l'')(q, p^k)^{sl'} \right| \ll 1. \]

Consequently
\[ \prod_{p | q} \frac{1 + (q, p^k)^s}{1 + 1/p^{k(1+s)}} = \sum_{i_1, \ldots, i_w \geq 0} \frac{C_{p_1}(l_1)C_{p_2}(l_1') \cdots C_{p_w}(l_w)C_{p_1'}(l_1')(q, p_1^{k_1})^{i_1} \cdots (q, p_w^{k_w})^{i_w}}{p_1^{k_1+l_1+i_1'} \cdots p_w^{k_w+l_w+i_w}} \]
and, for some \( A > 0 \),
\[ \sum_{i_1, \ldots, i_w \geq 0} \left| \frac{C_{p_1}(l_1)C_{p_2}(l_1') \cdots C_{p_w}(l_w)C_{p_1'}(l_1')(q, p_1^{k_1})^{i_1} \cdots (q, p_w^{k_w})^{i_w}}{p_1^{k_1+l_1+i_1'} \cdots p_w^{k_w+l_w+i_w}} \right| \leq A^w \ll \log(q + 1) \]
for \( \Re(s) \geq -1 + \delta \). If we now define
\[ C_{1V}^s = \prod_{i=1}^\omega C_{p_i}(l_i)C_{p_i'}(l_i'), \quad W_{1V} = \left( \prod_{i=1}^\omega p_{i}^{1+i_i'} \right)^k, \quad C_{1V} = \frac{C_{1V}^s}{W_{1V}} \]
\[ D_{1V} = \prod_{i=1}^\omega (q, p_i^{k_i})^{i_i'}, \quad \text{and} \quad Z_{1V} = \frac{D_{1V}}{W_{1V}} \]
then
\[ \prod_{p | q} \frac{1 + (q, p^k)^s}{1 + 1/p^{k(1+s)}} = \sum_{i_1, \ldots, i_w \geq 0} C_{1V}Z_{1V}^s \]
with
\[ \sum_{i_1, \ldots, i_w \geq 0} |C_{1V}Z_{1V}^s| \ll \log(q + 1) \] (17)
for \( \Re(s) \geq -1 + \delta \). The first claim now follows from (10) and the boundedness claim from (12) and (17).
Lemma 2.3. (A) Define
\[
\alpha = \prod_{p|q} \frac{1 + (q, p^k)/p^{2k} - 2/p^k}{1 + 1/p^{2k} - 2/p^k} \prod_{p} \left(1 - \frac{2}{p^k + \frac{1}{p^{2k}}}\right),
\]
\[
\beta = \prod_{p} \left(1 - \frac{1}{p^s}\right)
\]
and
\[
\gamma = \prod_{p|q} \frac{1 + (q, p^k)^{-1+1/k}/p - 2/p^k}{1 + 1/p - 2/p^k} \prod_{p} \frac{1 - 2/(p^k + p^{k-1})}{1 - p^{-1/k}},
\]
and let \(F^*(s)\) be as given at the start of this section. Then
\[
\frac{\zeta(2k)F^*(1)}{\zeta(4k)} = \alpha, \quad \frac{\zeta(k)F^*(0)}{\zeta(2k)} = \beta \quad \text{and} \quad \zeta(-1+1/k)F^*(-1+1/k) = \gamma.
\]

(B) Define \(\eta(q, a)\) as in (3). For any \(q, n \in \mathbb{N}\)
\[
\eta(q, n) = \eta(q, (q, n)) \ll q^{x-1}
\]
and
\[
\sum_{a=1}^{q} \eta(q, a)^2 = \frac{\alpha}{q}.
\]

Proof. (A) So long as there are no problems with zeros of denominators we have
\[
F^*(s) = \prod_{p|q} \frac{1 + (q, p^k)s/p^{k(1+s)} - 2/p^k}{1 + 1/p^{k(1+s)} - 2/p^k} \prod_{p} \left(1 - \frac{2}{p^k(1 + 1/p^{k(1+s)})}\right)
\]
\[
= \prod_{p|q} \frac{1 + (q, p^k)s/p^{2k} - 2/p^k}{1 + 1/p^{2k} - 2/p^k} \prod_{p} \left(1 - \frac{2}{p^{k}(1 + 1/p^{2k})}\right).
\]

For \(s = 1, 0, -1 + 1/k\) there are clearly no problems and therefore from the Euler product expressions for the Riemann zeta function
\[
\frac{\zeta(2k)F^*(1)}{\zeta(4k)} = \prod_{p|q} \frac{1 + (q, p^k)/p^{2k} - 2/p^k}{1 + 1/p^{2k} - 2/p^k} \prod_{p} \left(1 - \frac{2}{p^k(1 + 1/p^{2k})}\right)
\]
\[
= \prod_{p|q} \frac{1 + (q, p^k)/p^{2k} - 2/p^k}{1 + 1/p^{2k} - 2/p^k} \prod_{p} \left(1 + \frac{1}{p^{2k} - 2/p^k}\right),
\]
\[
\frac{\zeta(k)F^*(0)}{\zeta(2k)} = \prod_{p|q} \frac{1 + 1/p^k - 2/p^k}{1 + 1/p^k - 2/p^k} \prod_{p} \left(1 - \frac{2}{p^k(1 + 1/p^k)}\right)
\]
\[
= \prod_{p} \left(1 + \frac{1}{p^k} - \frac{2}{p^k}\right)
\]
\[
\zeta(-1+1/k)F^*(-1+1/k) = \prod_{p \mid q} \frac{1 + (q, p^k)^{-1+1/k}/p - 2/p^k}{1 + 1/p - 2/p^k} \prod_{p} \left(1 - p^{-1-1/k}\right)^{-1} \left(1 - \frac{2}{p^k(1+1/p)}\right).
\]

(B) From (3)
\[
\eta(q, a) = \sum_{D \mid q} \sum_{d \mid a} \mu(d) \left[\frac{q}{d}\right]^{(q, d_k)_{a}} \sum_{d} \mu(d) \left[\frac{q}{d}\right]^{(q, d_k)_{a}} \leq \frac{D}{q l_0^k}
\]

Writing \( l_0 \) for the squarefree part of \( D \) the \( d \) sum must be
\[
\frac{D}{q} \sum_{d \mid a} \frac{\mu(d)}{d^k} = \frac{D}{q} \sum_{d \mid a} \frac{\mu(d l_0)}{(d l_0)^k} \ll q^{-1}
\]

so that
\[
\eta(q, a) \ll \sum_{D \mid q} \frac{D}{q l_0^k} \ll q^{-1}
\]

which is the second claim and the first is trivial. We have
\[
\sum_{a=1}^q \eta(q, a)^2 = \sum_{d, d' \mid a} \frac{\mu(d) \mu(d')}{[q, d_k][q, d'_k]} \sum_{a \mid (q, d_k), (q, d'_k)} 1
\]
\[
= q \sum_{d, d' \mid a} \frac{\mu(d) \mu(d')}{[q, d_k][q, d'_k][(q, d_k), (q, d'_k)]}
\]
\[
= \frac{1}{q} \sum_{d, d' \mid a} \frac{\mu(d) \mu(d')(q, d_k, d'_k)}{d^k d'^k}
\]
\[
= \frac{1}{q} \sum_{d, d' \mid a} \frac{1}{N} \sum_{d'' \mid N} \mu(d) \mu(d')(q, d_k, d'_k)
\]
\[
= \frac{1}{q} \sum_{N=1}^\infty \sum_{d'' \mid N} \mu(d) \mu(d')(q, d_k, d'_k)
\]
\[
= \frac{1}{q} \sum_{N=1}^\infty \frac{b_q(N)}{N^k}.
\]

Clearly \( b_q(N) \) is multiplicative and simple calculations show
\[
b_q(p) = -2,
\]
\[
b_q(p^t) = (q, p^k)
\]

and \( b_q(p^t) = 0 \) for \( t \geq 3 \). Consequently
\[
\sum_{N=1}^\infty \frac{b_q(N)}{N^k} = \prod_{p \mid \alpha} \left(1 - \frac{2}{p^k} + \frac{(q, p^k)}{p^{2k}}\right)
\]
\[
= \prod_{p \mid \alpha} \frac{1 - 2/p^k + (q, p^k)/p^{2k}}{1 - 2/p^k + 1/p^{2k}} \prod_{p} \left(1 - \frac{2}{p^k} + \frac{1}{p^{2k}}\right)
\]

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which with (18) is the third claim.

Lemma 2.4. Let $c > 1$, let

$$A(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be absolutely convergent for $\Re(s) > c$, and let

$$A(Q) = \max_{Q/2 \leq n \leq 3Q/2} |a_n|.$$

Then for $T > 1$ and non-integer $Q > 0$

$$\sum_{n \leq Q} a_n (Q-n) = \frac{1}{2\pi i} \int_{c \pm iT} A(s) Q^{s+1} ds + O \left( \frac{Q A(Q)^2}{T} \left( 1 + \frac{Q \log Q}{T} \right) + \left( 1 + \frac{Q^{c+1+c}}{T^2} \right) \sum_{n=1}^{\infty} \frac{|a_n|}{n^c} \right).$$

In particular if $c - 1 \gg 1/\log Q$ then

$$\sum_{n \leq Q} (Q-n) = \frac{1}{2\pi i} \int_{c \pm iT} \zeta(s) Q^{s+1} ds + O \left( Q^c \left( 1 + \frac{Q^2}{T^2} \right) \right).$$

Proof. Take $X > 0$ and define

$$\delta(X) = \begin{cases} 0 & \text{if } 0 < X < 1 \\ X - 1 & \text{if } X > 1 \end{cases}$$

and

$$I_X(T) = \frac{1}{2\pi i} \int_{c \pm iT} X^{s+1} ds.$$

We first prove

$$|I_X(T) - \delta(X)| \ll \frac{X^{c+1}}{T} \min \left\{ 1, \frac{1}{T |\log X|} \right\}. \quad (19)$$

Suppose first $0 < X < 1$ so that for $\sigma > 0$ we have $X^{s+1} \ll 1$. Then for $R > c$

$$\begin{align*}
2\pi i I_x(T) &= - \left( \int_{c+iT}^{R+iT} + \int_{R-iT}^{R+iT} + \int_{c-iT}^{c+iT} \right) X^{s+1} ds \\
&\ll \frac{1}{T^2} \int_c^R X^{\sigma+1} d\sigma + \frac{1}{R^2} \int_{\pm T} dt \\
&\ll \frac{X^{c+1}}{T^2 |\log X|}
\end{align*}$$

with $R \to \infty$. Suppose now that $X > 1$ so that for $\sigma \leq -1$ we have $X^{s+1} \ll 1$. Then for $R < -1$

$$\begin{align*}
2\pi i I_x(T) &= \text{Res}_{s=0} \left( \frac{X^{s+1}}{s(s+1)} \right) + \text{Res}_{s=-1} \left( \frac{X^{s+1}}{s(s+1)} \right) - \left( \int_{c+iT}^{R+iT} + \int_{R-iT}^{R+iT} + \int_{c-iT}^{c+iT} \right) X^{s+1} ds \\
\end{align*}$$

and bounding the integrals as above shows

$$I_X(T) - (X - 1) \ll \frac{X^{c+1}}{T^2 |\log X|}.$$
so that we can conclude that the second bound in (19) is clear; now for the first bound. If
0 < X < 1 and if C is the arc of the circle going from c + iT to c − iT counterclockwise (so a
circle of radius \sqrt{T^2 + c^2} > T, and so that \( X^{c+1} \ll X^{c+1} \) on C) then

\[
2\pi i X(T) = - \int_C \frac{X^{c+1} ds}{s(s + 1)} \leq X^{c+1} \int_C \frac{ds}{|s| + |s + 1|} \leq \frac{X^{c+1}}{T}.
\]

If X > 1 the remaining part of the circle should be taken as the contour so that \( X^{c+1} \ll X^{c+1} \)
holds on the contour, and this gives a similar result. We conclude that the first bound in (19)
also holds and so the proof of (19) is complete. Therefore by absolute convergence

\[
\int_{c \pm iT} \frac{A(s)Q^{s+1} ds}{s(s + 1)} = \sum_{n=1}^{\infty} a_n n \int_{c \pm iT} \frac{1}{s(s + 1)} \left(\frac{Q}{n}\right)^{s+1} ds
= \sum_{n=1}^{\infty} a_n n \delta(Q/n) + O\left(\frac{Q^{c+1}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^\epsilon} \min\left\{1, \frac{1}{T|\log(Q/n)|}\right\}\right).
\]

In general for \( Z > -1 \)

\[
|\log(1 + Z)| \geq \frac{|Z|}{1 + Z}
\]

(take logarithms of a well-known inequality to deduce \( X \geq \log(1 + X) \) for \( X > -1 \) and put in
\( X = -Z/(Z + 1) \), which for \( -1 < Z \leq 0 \) is positive and for \( Z \geq 0 \) satisfies \(|X| \leq 1|\) so that,
since for \( Q/2 \leq n \leq 3Q/2 \) we have \((n - Q)/Q \geq -1|\),

\[
|\log(Q/n)| = \left|\log\left(1 + \frac{n - Q}{Q}\right)\right| \geq \frac{|n - Q|}{n} \geq \frac{|n - Q|}{n}.
\]

Therefore

\[
\sum_{Q/2 \leq n \leq 3Q/2} \frac{|a_n|}{n^\epsilon} \min\left\{1, \frac{1}{T|\log(Q/n)|}\right\} \leq A(Q) \left(\frac{Q}{2}\right)^{-\epsilon} + 2 \left(\frac{Q}{2}\right)^{1-\epsilon} \sum_{h \leq Q/2+1} \frac{1}{h}
\leq Q^{-\epsilon} A(Q) 2^\epsilon \left(1 + \frac{Q^{c+1}}{T}\right)
\]

(assuming that \( Q \geq 1/2|\), as we can since the integral then goes into the last error term) and if
\( n \) is not in this range then \(|\log(Q/n)| \gg 1|\) so we deduce

\[
\frac{Q^{c+1}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^\epsilon} \min\left\{1, \frac{1}{T|\log(Q/n)|}\right\} \leq \frac{QA(Q)^2}{T} \left(1 + \frac{Q \log Q}{T}\right) + \frac{Q^{c+1}}{T^2} \sum_{n=1}^{\infty} \frac{|a_n|}{n^\epsilon}.
\]

Therefore the error term in (20) is of the right order of magnitude and of course the main term is

\[
\sum_{n \leq Q} a_n (Q - n)
\]

and the main claim is proven. For the “in particular claim” the main claim implies an error term

\[
Q^\epsilon \left(1 + \frac{Q}{T} + \left(1 + \frac{Q^{c+1}}{T^2}\right) \zeta(c)\right);
\]

now use \( \zeta(c) \ll 1/(c - 1) \ll \log Q \) and \( Q^\epsilon \ll Q \). \qed
Lemma 2.5. Take $Q > 0$, $L \geq 2$ and $\Delta \in [1/2k, 1/k)$. Let

$$R_1 = -1 + \Delta \quad \text{and} \quad R_2 = \Delta k.$$  

Then

$$\int_1^L \frac{\zeta(R_1 + it) \zeta(R_2 + it) Q^it}{t^2} dt \ll L^{1/4 - 1/2k} \log L.$$

Proof. Take $s = \sigma + it \in \mathbb{C}$ with $t \geq 1$ and take two parameters $N, M \gg 1$ with $NM = t/2\pi$.

Let

$$\chi(s) = \frac{2^{s-1} \pi^s \sec(\pi s/2)}{\Gamma(s)}.$$

By formula (4.12.3) of [5] (the definition of $\chi(s)$ comes just before) we have for $-1 \leq \sigma \leq 1$

$$\chi(s) = \left( \frac{t}{2\pi} \right)^{1/2 - \sigma - it} e^{i(t + \pi/4)} \left( 1 + O \left( \frac{1}{t} \right) \right)$$

$$= \left( \frac{t}{2\pi} \right)^{1/2 - \sigma - it} e^{i(t + \pi/4)} + O \left( \frac{1}{t^{1/2 + \sigma}} \right) \tag{20}$$

so that

$$\chi(R_2 + it) \sum_{n \leq M} \frac{1}{n^{1-R_2-it}} = \left( \frac{t}{2\pi} \right)^{1/2-R_2-it} e^{i(t + \pi/4)} \sum_{n \leq M} \frac{1}{n^{1-R_2-it}} + O \left( \frac{RM}{t^{1/2+R_2}} \right)$$

so by the approximate functional equation (formula (4.12.4) of [5])

$$\zeta(R_2 + it) = \sum_{n \leq N} \frac{1}{n^{R_2+it}} + \chi(R_2 + it) \sum_{n \leq M} \frac{1}{n^{1-R_2-it}} + O \left( N^{-R_2 + t^{1/2-R_2} M R_2^{-1}} \right)$$

$$= \sum_{n \leq N} \frac{1}{n^{R_2+it}} + \left( \frac{t}{2\pi} \right)^{1/2-R_2-it} e^{i(t + \pi/4)} \sum_{n \leq M} \frac{1}{n^{1-R_2-it}}$$

$$+ O \left( \left( \frac{M}{t} \right)^{R_2} \left( 1 + \frac{t^{1/2}}{M} \right) \right) \tag{21}.$$  

From the functional equation (this just precedes formula (4.12.1) of [5]) and from (20) we have

$$\zeta(R_1 + it) = \left( \frac{t}{2\pi} \right)^{1/2-R_1-it} e^{i(t + \pi/4)} + O \left( \frac{1}{t^{1/2+R_1}} \right) \zeta(1 - R_1 - it)$$

$$= \left( \frac{t}{2\pi} \right)^{1/2-R_1-it} e^{i(t + \pi/4)} \zeta(1 - R_1 - it) + O \left( \frac{1}{t^{1/2+R_1}} \right)$$

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so that with (21) we get
\[
\zeta(R_1 + it)\zeta(R_2 + it) = \left( \frac{t}{2\pi} \right)^{1/2 - R_1 - it} e^{i(t+\pi/4)}\zeta(1 - R_1 - it) \sum_{n \leq N} \frac{1}{n^{R_1 + it}}
\]
\[
+ \left( \frac{t}{2\pi} \right)^{1 - R_1 - R_2 - 2it} e^{2i(t+\pi/4)}\zeta(1 - R_1 - it) \sum_{n \leq M} \frac{1}{n^{1 - R_2 - it}}
\]
\[
+ O \left( \frac{t^{1/2 - R_1}|\zeta(1 - R_1 - it)|}{M} \right) \left( \frac{R_2}{t} \left( 1 + \frac{t^{1/2}}{M} \right) \right)
\]
\[
= M_1(t) + M_2(t) + O \left( t^{1/2 - R_1} \left( \frac{R_2}{t} \left( 1 + \frac{t^{1/2}}{M} \right) \right) \right).
\]

Write \( N = t^{1/A} \) and \( M = t^{1/B} \) so the above reads
\[
\zeta(R_1 + it)\zeta(R_2 + it) = M_1(t) + M_2(t) + O \left( t^{1/2 - R_1 + R_2 / B - R_2} \left( 1 + t^{1/2 - 1/B} \right) \right). \tag{22}
\]

For some constant \( C \)
\[
M_1(t)Q^{it} = Ct^{1/2 - R_1} \sum_{n \leq N} \sum_{m = 1}^{\infty} \frac{e^{it(-\log t + 1 - \log n + \log m + \log Q)}}{n^{R_2} m^{1 - R_1}}
\]
\[
= Ct^{1/2 - R_1} \sum_{n^{A \leq t}} \sum_{m = 1}^{\infty} \frac{e^{(f_{mQ/n}(t))}}{n^{R_2} m^{1 - R_1}}
\]

where
\[
f_X(t) = \frac{t(-\log t + 1 + \log X)}{2\pi}
\]
and the two summation conditions on \( n \) are equivalent. So for any \( T \geq 1 \)
\[
\int_T^{2T} \frac{M_1(t)Q^{it} dt}{t^2} = C \sum_{m = 1}^{\infty} \frac{1}{m^{1 - R_1}} \sum_{n^{A \leq T}} \frac{1}{n^{R_2}} \int_{\max(2\pi nM,T)}^{2T} e^{(f_{mQ/n}(t))} dt \int_{t^{3/2 + R_1}}^{T^{3/2 + R_1}}. \tag{23}
\]

We now bound this oscillatory integral. We have
\[
2\pi f'_X(t) = -\log t + \log X. \tag{24}
\]

Suppose first that \( T \) is large and \( 0 < X \ll 1 \). For \( \max(2\pi nM,T) < t < 2T \) we have from (24)
\[
f'_X(t) \gg 1
\]
and
\[
t^{3/2 + R_1} \gg T^{3/2 + R_1}
\]
so from Lemma 4.3 of [3]
\[
\int_{\max(2\pi nM,T)}^{2T} e^{(f_X(t))} dt \ll \frac{1}{T^{3/2 + R_1}}, \quad \text{if } 0 < X \ll 1. \tag{25}
\]
Suppose now $X$ is large. Since from (24)

\[
\begin{align*}
    f_X(t) & \gg \left| \log(t/X) \right| \\
    & = \left| \log \left( 1 + \frac{t - X}{X} \right) \right| \\
    & \gg \begin{cases} 
        \frac{t - X}{X} & \text{if } t \in (X/2, 3X/2) \\
        1 & \text{if not} \\
        1/\sqrt{X} & \text{if } t \in (X/2, X - \sqrt{X}) \cup (X + \sqrt{X}, 3X/2) \\
        1 & \text{if } t \notin (X/2, 3X/2)
    \end{cases}
\end{align*}
\]

and since for $t > T$

\[
t^{3/2+R_1} \gg T^{3/2+R_1},
\]

we have from Lemma 4.3 of [5]

\[
\int_{\max(2\pi n M, T)}^{2T} \frac{e(f_X(t)) \, dt}{t^{1/2+R_1}} \ll \frac{1}{T^{1+R_1}}
\]

holds in fact for all $X > 0$, so we deduce from (25)

\[
\int_{\max(2\pi n M, T)}^{2T} \frac{e(f_X(t)) \, dt}{t^{1/2+R_1}} \ll \frac{1}{T^{1+R_1}}
\]

Similarly we have

\[
\int_{T}^{2T} \frac{M_2(t) Q'' dt}{t^2} = C \sum_{m=1}^{\infty} \frac{1}{m^{1-R_1}} \sum_{n \leq T} \frac{1}{n^{1-R_2}} \int_{\max(nN, T)}^{2T} \frac{e(f_{nQ/X}(t)) \, dt}{t^{1+R_1+R_2}}
\]

where the oscillatory integral is

\[
\ll \frac{1}{T^{1/2+R_1+R_2}}
\]

so that

\[
\int_{T}^{2T} \frac{M_2(t) Q'' dt}{t^2} \ll \frac{1}{T^{1/2+R_1+R_2}} \sum_{m=1}^{\infty} \frac{1}{m^{1-R_1}} \sum_{n \leq T} \frac{1}{n^{1-R_2}} \ll \frac{1}{T^{1/2+R_1+R_2}} \left( T^{1/2} \right)^{R_2} \ll \frac{T^{R_2}}{B^{1/2} - R_1 - R_2}.
\]
Note that

\[-1/2 - R_1 - R_2/2 = 1/2 - \Delta - \Delta k/2 \leq 1/2 - \Delta - 1/4\]

so taking \( A = B = 2 \) we see from (27) and (28)

\[\int_T^{2T} \frac{(M_1(t) + M_2(t)) Q^\prime dt}{t^2} \ll T^{-1/2 - R_1 - R_2/2} \ll T^{1/4 - \Delta}.\]

We assumed that \( T \) is large but the bound is trivial for \( T \) not large so we conclude

\[\int_1^L \frac{(M_1(t) + M_2(t)) Q^\prime dt}{t^2} \ll L^{1/4 - \Delta} \log L\]

and so from (22) and (29)

\[\int_1^L \frac{\zeta(R_1 + it) \zeta(R_2 + it) Q^\prime dt}{t^2} \ll L^{1/4 - \Delta} \log L + \int_1^L \frac{t^{-3/2 - R_1 - R_2/2} dt}{t^2} \ll L^{1/4 - \Delta} \log L.\]

\[\square\]

**Lemma 2.6.** Let \( \alpha, \beta, \) and \( \gamma \) be as in Lemma 2.3 and let \( F^+(s) \) be as given at the start of this section. For \( X > 0 \) and \( T, c > 1 \)

\[\int_{c \pm iT} \frac{\zeta(s) \zeta(k(s + 1)) F^+(s) X^{s+1} ds}{s(s + 1) \zeta(2k(s + 1))} = \frac{\alpha X^2}{2} + \frac{\beta X}{2} + \frac{k \gamma X^{1/k}}{-1 + 1/k} \zeta(2) + \mathcal{O}\left(T^c \left(q^c T^{1/4} \left(X^{1/2k} + X^{c+1} \frac{T}{T^2} + 1 \right)\right)\right).\]

**Proof.** For \( s \in \mathbb{C} \) write always \( s = \sigma + it \) for \( \sigma, t \in \mathbb{R} \) and let

\[\mathcal{I}(s) = \frac{\zeta(s) \zeta(k(s + 1)) F^+(s)}{\zeta(2k(s + 1))}.\]

Let \( R_1 = -1 + 1/2k + \tau \) for some \( 0 < \tau < 1/k \). We have already established (just before Lemma 2.1) that \( F^+(s) \ll q^c \) for \( \sigma \geq -1 + \delta \), therefore

\[\mathcal{I}(s) \ll q^c \left|\frac{\zeta(s) \zeta(k(s + 1))}{\zeta(2k(s + 1))}\right|, \quad \text{for } \sigma \geq R_1.\]

On \( \Re(s) \geq -1 + \delta \) we know by Lemma 2.1 that \( \mathcal{I}(s) \) is holomorphic except for simple poles at \( s = 1 \) and \( s = -1 + 1/k \) so by the Residue Theorem

\[2\pi i \int_{c \pm iT} \frac{\mathcal{I}(s) X^{s+1} ds}{s(s + 1)} = \frac{X^2 \text{Res}_{s=1} \mathcal{I}(s)}{2} + \mathcal{I}(0) X + \frac{k X^{1/k} \text{Res}_{s=-1+1/k} \mathcal{I}(s)}{-1 + 1/k} - 2\pi i \left(\int_{c+iT}^{R_1+iT} + \int_{R_1-iT}^{c-iT} + \int_{c-iT}^{R_1+iT}\right) \frac{\mathcal{I}(s) X^{s+1} ds}{s(s + 1)}.\]
It is standard that for \( t \geq 1 \)

\[
\zeta(s) \ll t^\epsilon \left\{
\begin{array}{ll}
t^{1/2-\sigma} & \text{for } \sigma \leq 0 \\
\max\{1, t^{1/2-\sigma/2}\} & \text{for } \sigma \geq 0 \\
t^{1/4} & \text{for } \sigma \geq 1/2
\end{array}
\right.
\]

and

\[
\zeta(\sigma) \ll \begin{cases} 
1 & \text{for } \sigma \geq 2k \\
1/|\sigma - 1| & \text{for } 1 \leq \sigma \leq 2;
\end{cases}
\]

we will now use these bounds freely without comment. If \( 0 \leq \sigma \leq 2 \) and \( t \geq 1 \) we have

\[
\zeta(s) \ll t^\epsilon \max\{1, t^{1/2-\sigma/2}\},
\]

and

\[
\frac{1}{\zeta(2k(s+1))} \ll \zeta(2k(\sigma + 1)) \ll 1,
\]

so from (33)

\[
I(s) \ll t^\epsilon \max\{1, t^{1/2-\sigma/2}\}
\]

and therefore

\[
\int_{c-iT}^{c+IT} \frac{I(s)X^{s+1}ds}{s(s+1)} \ll T^\epsilon \left( \frac{X}{T^{3/2} + X^{c+1}} \right). \quad (33)
\]

If \( R_1 \leq \sigma \leq 0 \) then for \( t \geq 1 \)

\[
\zeta(s) \ll t^{1/2-\sigma},
\]

\[
\zeta(k(s+1)) \ll t^{1/2}
\]

and

\[
\frac{1}{\zeta(2k(s+1))} \ll \zeta(2k(\sigma + 1)) \ll \frac{1}{|2k(\sigma + 1) - 1|} \ll \frac{1}{\tau},
\]

so from (33)

\[
I(s) \ll \frac{t^{1-\sigma}}{\tau} \quad (34)
\]

and therefore

\[
\int_{R_1 + iT}^{iT} \frac{I(s)X^{s+1}ds}{s(s+1)} \ll \frac{1}{\tau} \left( \frac{XR_{1+1}}{T^{1+R_{1+1}}} + \frac{X}{T} \right) \ll \frac{1}{\tau} \left( 1 + \frac{X}{T} \right). \quad (35)
\]

From (33) and (35) we have

\[
\left( \int_{c-iT}^{c+IT} + \int_{R_1-iT}^{R_1+iT} \right) \frac{I(s)X^{s+1}ds}{s(s+1)} \ll \frac{1}{\tau} \left( 1 + \frac{X}{T} + \frac{X^{c+1}}{T^2} \right) \ll \frac{T^\epsilon}{\tau} \left( 1 + \frac{X^{c+1}}{T^2} \right). \quad (36)
\]

a similar argument for the second integral obviously valid. We now turn to the vertical contribution in (32). Denote by \( \omega \) the number of prime factors of \( q \). For given integers \( n, l_1, \ldots, l_\omega, l'_1, \ldots, l'_{\omega} \geq 0 \) write \( n = (n, l_1, \ldots, l_\omega, l'_1, \ldots, l'_{\omega}) \). Let \( W_n, Z_{1\nu} \) be as in Lemma 2.2. Then that lemma says that for given \( n \) there are \( a_n = a_n(\sigma) \in \mathbb{R} \) such that for \( -1 + \delta \leq \sigma \leq 0 \)

\[
F^*(s) = \sum_n a_n (W_n Z_{1\nu})^\sigma
\]

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and
\[ \sum_n |a_n| \ll 1. \] (37)

Therefore
\[ \frac{F^*(R_1 + it)X^it}{\zeta(2k(R_1 + it + 1))} = \sum_{m,n} \frac{\mu(m)a_n}{m^{2k(R_1+1)}} \left( \frac{XW_nZ_{1.1}}{m^{2k}} \right)^{it}, \]

so from (30), Lemma 2.5 and (37)
\begin{align*}
\int_1^T \frac{I(R_1 + it)X^itdt}{t^2} & = \sum_{m,n} \frac{\mu(m)a_n}{m^{2k(R_1+1)}} \int_1^T \frac{\zeta(R_1 + it)\zeta(k(R_1 + it + 1))}{t^2} \left( \frac{XW_nZ_{1.1}}{m^{2k}} \right)^{it} dt \\
& \ll T^{1/4-1/2k} \log T \sum_{m,n} \left| \frac{\mu(m)a_n}{m^{2k(R_1+1)}} \right| \\
& \ll T^{1/4-1/2k} (\log T) \zeta(1 + 2k\tau) \ll \frac{T^{1/4-1/2k} \log T}{\tau}. \quad (38)
\end{align*}

We clearly have for \( \sigma \geq -1 + \delta \)
\[ I(s) \ll q^\epsilon \left\{ \begin{array}{ll}
\frac{1}{t^{\sigma/4}} & \text{for } 0 \leq t \leq 1 \\
& \text{for } t \geq 1
\end{array} \right. \]

and for \( t \geq 1 \) we have
\[ \frac{1}{s(s+1)} = \frac{1}{t^2} + O\left( \frac{1}{t^3} \right), \]

therefore from (38)
\begin{align*}
\int_{R_1}^{R_1+iT} \frac{I(s)X^{s+1}ds}{s(s+1)} & = X^{R_1+1} \int_1^T \frac{I(R_1 + it)X^it ds}{t^2} \\
& \quad + O \left( X^{R_1+1} \int_{R_1}^{R_1+i\infty} \frac{|I(s)| ds}{t^3} + X^{R_1+1} \int_{R_1}^{R_1+i\infty} \frac{|I(s)| ds}{|s(s+1)|} \right) \\
& \ll \frac{X^{R_1+1}T^{1/4-1/2k} \log T}{\tau} + q^\epsilon X^{R_1+1} \\
& = \frac{X^{R_1+1}T^{1/4+\epsilon}}{\tau} \left( \frac{X}{T} \right)^{1/2k} q^\epsilon.
\end{align*}

A similar bound obviously holding for \( t \) negative we conclude
\[ \int_{R_1-iT}^{R_1+iT} \frac{I(s)X^{s+1}ds}{s(s+1)} \ll X^{\tau T^{1/4+\epsilon}} \frac{X}{T} \left( \frac{X}{T} \right)^{1/2k} q^\epsilon. \] (39)

From Lemma 2.3 (A) we have
\[ Res_{s=1} I(s) = \frac{\zeta(2k)f^*(1)}{\zeta(4k)} = \alpha, \]
\[ I(0) = \frac{\zeta(0)\zeta(k)f^*(0)}{\zeta(2k)} = -\frac{\beta}{2}. \]
and

$$R_{es}=(-1+1/k)^J(-1+1/k)\zeta(2)$$

so the main terms in (32) are

$$\frac{\alpha X^2}{2} - \frac{\beta X}{2} + \frac{k\gamma X^{1/k}}{(-1+1/k)\zeta(2)} =: M(X).$$

This with (36) and (39) means (32) becomes

$$\int_{c+iT}^{c+iT} \frac{I(s)}{s(s+1)} ds = M(X) + O\left(\frac{T^{c}}{T} \left( q^c X^{T^{1/4}} \left( \frac{X}{T} \right)^{1/2k} + 1 + \frac{X^{c+1}}{T^2} \right) \right)$$

on taking \( \tau = 1/\log X \), so long as \( X \) is large. If \( X \) is not large then the claim is trivial, the integrand being trivially \( \ll t^{-2} \) for \( \sigma = c \).

Lemma 2.7. For any \( x, y > 0 \)

\[
\sum_{|d,d'| \leq y} 1 \ll y^{1+\epsilon},
\]

\[
\sum_{|d,d'| > y} \frac{1}{|d^k, d'^k|} \ll y^{1-k+\epsilon}
\]

and, for \( N \leq x \),

\[
\sum_{d,d' \geq Z} \sum_{n \leq y \land n \equiv -N(d^k)} 1 \ll xy^{-k+\epsilon} + x^{2/(k+1)+\epsilon}.
\]

Proof. Since

\[
\sum_{|d,d'| = n} 1 \ll n^\epsilon
\]

we have

\[
\sum_{|d,d'| \leq y} 1 \ll y^{1+\epsilon}
\]

and

\[
\sum_{|d,d'| > y} \frac{1}{|d^k, d'^k|} = \sum_{n \geq y} \frac{1}{n^k} \sum_{|d,d'| = n} 1 \ll y^{1-k+\epsilon}
\]

which are the first two claims. Let \( Z \) be a parameter. We have with a divisor estimate

\[
\sum_{d,d' \geq Z} \sum_{n \leq y \land n \equiv -N(d^k)} 1 \ll x^\epsilon \sum_{d \leq y} \sum_{d^k \leq z} \frac{1}{d^k} \ll x^{1+\epsilon} \sum_{d > Z} \frac{1}{d^k} \ll x^{1+\epsilon} Z^{1-k}
\]

and similarly for the terms with \( d' > Z \). On the other hand the second claim implies

\[
\sum_{d,d' \geq Z} \sum_{n \leq y \land n \equiv -N(d^k)} 1 \ll \sum_{d,d' \geq Z} \left( \frac{x}{|d^k, d'^k|} + 1 \right) \ll xy^{1-k+\epsilon} + Z^2
\]
and therefore
\[
\sum_{[d, d'] > y} \sum_{\substack{n \leq x \atop n \equiv d \pmod{d'}}} 1 \ll xy^{1-k+\epsilon} + Z^2 + x^{1+\epsilon}Z^{1-k}
\]
which gives the claim on choosing \(Z = x^{1/(k+1)}\).

3.2 - Proof of theorem

Let \(1 \leq q \leq x\) and define \(\eta(q, a)\) and \(V_x(q)\) as in (3) and (4). Opening the square we have

\[
V_x(q) = \sum_{a=1}^{q} \sum_{\substack{n,n' \leq x \atop n \equiv a \pmod{n} \land n' \equiv a \pmod{n}}} 1 - 2x \sum_{a=1}^{q} \eta(q, a) \sum_{\substack{n \leq x \atop n \equiv a \pmod{n}}} 1 + x^2 \sum_{a=1}^{q} \eta(q, a)^2
\]

\[
= \sum_{\substack{n,n' \leq x \atop n \equiv a \pmod{n} \land n' \equiv a \pmod{n}}} 1 - 2x \sum_{n \leq x} \eta(q, n) + x^2 \sum_{a=1}^{q} \eta(q, a)^2
\]

\[
=: \quad A_x(q) - 2xB_x(q) + x^2 \sum_{a=1}^{q} \eta(q, a)^2.
\]

From Lemma 2.3 (B) we have \(\eta(q, n) = \eta(q, (q, n))\) and \(\eta(q, d) \ll 1\). Therefore from Lemma 2.2 (ii) of [6] we have for some constants \(c_{dh}, c_q\) and a new parameter \(X \geq 1\)

\[
B_X(q) = \sum_{d|q} \eta(q, d) \sum_{\substack{n \leq X \atop n \equiv \frac{q}{d} \pmod{n}}} 1
\]

\[
= \sum_{d|q} \eta(q, d) \sum_{h|\frac{q}{d}} \mu(h) \sum_{\substack{n \leq X \atop n \equiv \frac{h}{d} \pmod{n}}} 1
\]

\[
= X \sum_{d|q} \eta(q, d) \sum_{h|\frac{q}{d}} \mu(h)c_{dh} + O \left( X^{1/k+\epsilon} \sum_{d|q} |\eta(q, d)| \sum_{h|\frac{q}{d}} |\mu(h)| \right)
\]

\[
= Xc_q + O \left( X^{1/k+\epsilon} \right)
\]

\[
\sim \quad Xc_q
\]

(41)

with \(X \to \infty\). But it is easy to establish

\[
\sum_{\substack{n \leq X \atop n \equiv a \pmod{n}}} 1 \sim X\eta(q, a)
\]

so that evidently

\[
B_X(q) = \sum_{a=1}^{q} \eta(q, a) \sum_{\substack{n \leq X \atop n \equiv a \pmod{n}}} 1 \sim X \sum_{a=1}^{q} \eta(q, a)^2
\]
so (41) implies

\[ c_q = \sum_{a=1}^{q} \eta(q,a)^2 \]

and therefore the last but one line of (41) says

\[ B_x(q) = x \sum_{a=1}^{q} \eta(q,a)^2 + O(x^{1/k+\epsilon}). \]  

(42)

It is well known that

\[ \sum_{\substack{n \leq x \\n \in S}} 1 = \frac{x}{\zeta(k)} + O(x^{1/k}) \]

therefore

\[ A_x(q) = 2 \sum_{\substack{n \leq x \\n \in S}} 1 + \sum_{n \leq x \\n \in S} 1 \]

\[ = 2 \sum_{l \leq x/q} \sum_{\substack{n \leq x \\n \in S \\n n \equiv n'} 1 + \frac{x}{\zeta(k)} + O(x^{1/k}) \]

\[ =: 2C_x(q) + \frac{x}{\zeta(k)} + O(x^{1/k}) \]  

(43)

so we deduce from (40) and (42)

\[ V_x(q) = 2C_x(q) + \frac{x}{\zeta(k)} - x^2 \sum_{a=1}^{q} \eta(q,a)^2 + O(x^{1/k+\epsilon}). \]  

(44)

Take a parameter \( y \leq x^{1/k} \) so that \([d,d'] \leq y\) is a stronger condition than \(d^k, d'^k \leq x\). Using

\[ \sum_{d^k | n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is } k\text{-free} \\ 0 & \text{if not} \end{cases} \]
we see that
\[
\sum_{n, n' \leq x \atop n, n' \in \mathbb{S}} 1 = \sum_{d, d' \leq x} \mu(d) \mu(d') \sum_{n, n' \leq x \atop \gcd(n, d) = 1, \gcd(n', d') = 1} 1
\]
\[
= \sum_{d, d' \leq x} \mu(d) \mu(d') \sum_{n, n' \leq x \atop \gcd(n, d) = 1, \gcd(n', d') = 1} 1
\]
\[
= \sum_{(d, d') \leq y \atop \gcd(d, d') = 1} \mu(d) \mu(d') \left(\frac{x - ql}{d^k, d'^k} + O(1)\right) + O\left(\sum_{(d, d') > y \atop \gcd(d, d') = 1} 1\right)
\]
\[
= (x - ql) \sum_{d, d' \leq x \atop \gcd(d, d') = 1} \mu(d) \mu(d') \left(\frac{1}{d^k, d'^k}\right) + O\left(\sum_{(d, d') > y \atop \gcd(d, d') = 1} 1\right) + O\left(\sum_{(d, d') > y \atop \gcd(d, d') = 1} 1\right).
\]
From Lemma 2.27 the error terms here are for \( ql \leq x \)
\[y^{1+\varepsilon} + xy^{1-k+\varepsilon} + x^{2/(k+1)+\varepsilon} \ll x^{2/(k+1)+\varepsilon}\]

after setting \( y = x^{1/k} \), so that
\[
\sum_{n, n' \leq x \atop n, n' \in \mathbb{S}} 1 = (x - ql) \sum_{d, d' \leq x \atop \gcd(d, d') = 1} \mu(d) \mu(d') \left(\frac{1}{d^k, d'^k}\right) + O\left(x^{2/(k+1)+\varepsilon}\right)
\]
so from (43)
\[
C_x(q) = \sum_{d, d' \leq x \atop \gcd(d, d') = 1} \mu(d) \mu(d') \left(\frac{x - ql}{d^k, d'^k}\right) + O\left(x^{2/(k+1)+\varepsilon}\right)
\]
\[
= \sum_{d, d' \leq x \atop \gcd(d, d') = 1} \mu(d) \mu(d') \left[\frac{1}{d^k, d'^k}\right] \sum_{l \leq x/q \atop \gcd(l, d, d') = 1} \left(\frac{x}{d^k, d'^k} - l\right) + O\left(x^{1+2/(k+1)+\varepsilon}\right)
\]
\[
=: \mathcal{J}(x) + O\left(x^{1+2/(k+1)+\varepsilon}\right).
\]
From now on all \( \ll \) symbols will denote bounds up to \( x^\varepsilon \) bounds so that (44) and (45) read
\[
V_q(x) = 2\mathcal{J}(x) + \frac{x}{\zeta(k)} - x^2 \sum_{a=1}^q \eta(q, a)^2 + O\left(x^{1+2/(k+1)}\right).
\]
Assuming as we can that $x$ is not an integer, write $Q = x/[q, (d^k, d^k)]$ and let $c = 1 + 1/\log Q$. From Lemma 2.4 the inner sum in $J$ coefficient is

$$
\int_{c\times T} \frac{\zeta(s)}{s(s+1)} \left( \frac{x}{[q, (d^k, d^k)]} \right)^{s+1} ds + O \left( 1 + \left( \frac{x}{T[q, (d^k, d^k)]} \right)^2 \right)
$$

so from Lemma 2.1 and Lemma 2.6

$$
\mathcal{J}(x) = \int_{c\times T} \frac{\zeta(s)x^{s+1}}{s(s+1)} \left( \sum_{d,d'} \frac{\mu(d)\mu(d')}{[d^k, d^k]} \right) ds + O \left( \sum_{d,d'} \left| \frac{\mu(d)\mu(d')}{[d^k, d^k]} \right| (1 + \frac{x^2}{T^2[q, (d^k, d^k)]}) \right)
$$

$$
= q \int_{c\times T} \frac{\zeta(s)\zeta(k(s+1))\mathcal{F}^*(s)}{s(s+1)\zeta(2k(s+1))} \left( \frac{x}{q} \right)^{s+1} ds + O \left( 1 + \frac{x^2}{T^2} \sum_{d,d'=1}^\infty \frac{(q, d^k, d^k)}{d^k d^k} \right)
$$

$$
= \frac{\alpha x^2}{2q} - \frac{\beta x}{2} + k\gamma q^{1-1/k} x^{1/k} (-1 + 1/k)\zeta(2) + O \left( q \left( T^{1/4} \left( \frac{x}{qT} \right)^{1/2k} + \left( \frac{x}{qT} \right)^2 + 1 \right) \right)
$$

where $\alpha, \beta, \gamma$ are as in Lemma 2.3 and assuming $T \leq x^2$. Setting

$$
T = \left( \frac{x}{q} \right)^V
$$

where

$$
V = \frac{2 - 1/2k}{9/4 - 1/2k}
$$

the error term becomes

$$
\ll q \left( \frac{x}{q} \right)^{2(9-2/k)}
$$

and so from (16)

$$
V_2(q) = \left( \frac{\alpha}{q} - \sum_{a=1}^q \eta(q, a)^2 \right) x^2 + \left( \frac{1}{\zeta(k)} - \beta \right) x + \frac{2k\gamma q^{1-1/k} x^{1/k}}{(-1 + 1/k)\zeta(2)} + O \left( q \left( \frac{x}{q} \right)^{2(9-2/k)} \right) + O \left( \frac{x^{1/2(k+1)}}{q} \right).
$$

From Lemma 2.3 (B) the $x^2$ coefficient vanishes. Directly from the definitions (Lemma 2.3) we see that $\beta = \zeta(k)^{-1}$ so the $x$ coefficient also vanishes. Again from the definitions the $x^{1/k}$ coefficient is

$$
\frac{2kq^{1-1/k}}{(-1 + 1/k)\zeta(2)} \prod_{p} \left( 1 - 2/(p^k + p^{k-1}) \right) \prod_{p/q} \left( \frac{1 + (q,p^k)^{-1+1/k}/p - 2/p^k}{1 + 1/p - 2/p^k} \right)
$$

and we have our theorem.

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