Integrating the geodesic equations in the Schwarzschild and Kerr space-times using Beltrami’s “geometrical” method

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ABSTRACT - We revisit a little known theorem due to Beltrami, through which the integration of the geodesic equations of a curved manifold is accomplished by a method which, even if inspired by the Hamilton-Jacobi method, is purely geometric. The application of this theorem to the Schwarzschild and Kerr metrics leads straightforwardly to the general solution of their geodesic equations. This way of dealing with the problem is, in our opinion, very much in keeping with the geometric spirit of general relativity. In fact, thanks to this theorem we can integrate the geodesic equations by a geometrical method and then verify that the classical conservation laws follow from these equations.

KEYWORDS - Geodesic, Schwarzschild metric, Kerr metric

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1 Introduction

In the preface to his classic textbook “The Analytical Foundations of Celestial Mechanics” [1], A. Wintner wrote in 1941 “…even the classical literature of the great century of Celestial Mechanics
appears to be saturated with rediscoveries. . .” He then illustrated this remark with some examples of results that had been rediscovered more than once, retracing their history back to the first discoverer. This is also true of the field of differential geometry and especially of its applications to general relativity (GR), where Malcolm MacCallum has used the particularly appropriate term “literature horizon” to describe the loss of familiarity with results that accompanies each new generation of relativists. We propose that an old theorem on geodesics due to Eugenio Beltrami (1835–1900) [2], which as far as we know has not been quoted in texts on differential geometry since the publication of the classic books by L. Bianchi [3] and L.P. Eisenhart [4], deserves to be rediscovered. In his theorem Beltrami showed that the geodesic equations can be integrated through a method substantially analogous to the method of separation of variables used in the integration of the Hamilton-Jacobi equation.

In GR we have some important spacetimes which are exact solutions of the Einstein equations and whose metric tensor components are known explicitly in a given system of coordinates. Starting from these components, one can write the geodesic equations

\[ \frac{d^2x^l}{ds^2} + \Gamma^l_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} = 0, \]

and then try to integrate them to determine the paths of test particles. The Schwarzschild spacetime (whose timelike geodesics can be used to calculate the advance of the perihelion of Mercury) and the Kerr metric (representing the gravitational field outside a rotating body or of a mathematical black hole) are two important examples whose geodesics can yield important physical results.

Two methods are typically used to integrate the geodesic equations. Either one starts with the Lagrangian equations of motion (obtained from a Lagrangian \( L \) given by \( L = g_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} \), where for timelike geodesics \( s \) may be identified with the proper time) or with the corresponding Hamilton-Jacobi equations, in both cases representing a mechanical system governed only by a kinetic energy term, in which the effects of the gravitational field are represented by the curvature of the spacetime associated with the metric which determines this kinetic energy function. Now one is dealing with a mechanical system again instead of pure geometry. In our eyes, this approach seems to be a step backwards with respect to the spirit of GR. The motion of a test particle in a gravitational field is interpreted as the motion of a free particle in a curved spacetime which turns out to follow a geodesic. On the other hand, a completely “geometric” integration of the geodesic equations can be performed without referring to the equivalent point particle mechanical system. Once the geometric problem has been solved, the constants of integration can be interpreted as physical constants that are the first integrals of the motion in the classic approach. For its historical interest we reproduce in Sect. 2 a concise translation of the relevant section of Beltrami’s article, only updating its notation.

2 Beltrami’s integration methods for geodesic equations

We recall Beltrami’s method for obtaining the solutions of geodesic equations [2, Vol. I, p. 366], an extension of the Hamilton-Jacobi integration method used for integrating the equations of motion in dynamics. This method originating with Beltrami in the later 1800s does not appear in recent books [7]; the last exposition of this method is reported in Eisenhart’s classic book [4, p. 59]. Let us consider an \( n \)-dimensional semi-Riemannian manifold \( V_n \) whose metric is represented by

\[ ds^2 = g_{ih} dx^i dx^h \quad (i, h = 1, 2, \ldots n). \]

1For the geodesic equations (1) and the formulas applied below, we refer to the standard texts on general relativity. In particular, in the following we shall refer to two of the most widespread texts [5, 6].
If $U$, $V$ are any real functions of the $x^i$ ($i = 1, 2, ..., n$), the invariants defined by

$$
\Delta_1 U = g^{ih} \frac{\partial U}{\partial x^i} \frac{\partial U}{\partial x^h} \equiv g^{ih} U,_{i} U,_{h},
$$

(3)

$$
\Delta (U, V) = g^{ih} \frac{\partial U}{\partial x^i} \frac{\partial V}{\partial x^h} \equiv g^{ih} U,_{i} V,_{h},
$$

(4)

are called Beltrami's differential parameters of the first order. Since the equations $U = \text{const}$, $V = \text{const}$ represent $(n - 1)$-dimensional hypersurfaces in $V_n$, $\Delta_1 U$ represents the squared length of the gradient of $U$ as well as of a vector orthogonal to the hypersurface $U = \text{const}$; for the same reason, if $\Delta(U, V) = 0$, then the two hypersurfaces $U = \text{const}$ and $V = \text{const}$ are orthogonal.

The Beltrami's theorem states:

Let us consider the equation

$$
\Delta_1 U = 1,
$$

(5)

the solution of this equation depends on an additive constant and on $N - 1$ essential constants $\alpha_i$ [4]. Now, if we know a complete solution of Eq. (5), we can obtain the equations of the geodesics from the following theorem [3, p. 299], [4, p. 59]: when a complete solution of Eq. (5) is known, the equations of the geodesics are given by

$$
\frac{\partial U}{\partial \alpha_i} = \beta_i,
$$

(6)

where $\beta_i$ are arbitrary constants, and the geodesic arclength is given by the value of $U$.

We demonstrate the theorem following Beltrami's article [2]:

Let us return to differential parameters (3), (4). We can associate contravariant components of the gradient with the covariant ones

$$
U,^h = g^{hi} U,_{i}.
$$

(7)

Since (2) and (3) are reciprocal quadratic forms (since the coefficients in (3) are reciprocal to the ones in (2)), because of their invariance we can write

$$
ds^2 = \Delta_1 U \, dk^2,
$$

(8)

where $dk$ is an infinitesimal scalar quantity introduced to keep the differential homogeneity. From Eq. (8), we have obviously

$$
\frac{ds}{\sqrt{\Delta_1 U}}.
$$

(9)

Moreover, owing to the properties of the reciprocal forms and keeping the differential homogeneity expressed by Eq. (9), we can write

$$
\frac{1}{2} \frac{\partial (ds^2)}{\partial (dx^i)} \equiv g_{ih} dx^h = U,_{i} \, dk,
$$

(10)

which can be rewritten as

$$
dx^h = g^{ih} U,_{i} \, dk \equiv U,^h \, dk.
$$

(11)

Combining (9) with (11), we get

$$
\frac{dx^h}{ds} = \frac{U,^h}{\sqrt{\Delta_1 U}}.
$$

(12)

By multiplying (10) by $dx^i$ and summing, we shall get

$$
ds^2 = dU \, dk,
$$

(13)
where $dU$ is the differential of the function $U$.

In an analogous way, if the increments $\delta x^i$ are the components of a vector indicating another direction from that of components $dx^i$, we shall get by multiplying by $\delta x^i$ and summing

$$g_{ih}\delta x^i dx^h = \delta U \, dk.$$  

(14)

It is evident that, if we take the vector indicated by $\delta x^i$ tangent to a hypersurface $U = \text{const}$, we have $\delta U = 0$ and then the vectors, of components $dx^i$ and $\delta x^i$ respectively, turn out to be orthogonal. Then the “displacement” $dx$ of (10), is orthogonal to the hypersurface $U = \text{const}$.

Finally, if we eliminate $dk$ from Eqs. (8, 13), we find

$$\Delta_1 U = \left(\frac{dU}{ds}\right)^2,$$  

(15)

where $dU$ represents the increment due to a variation $ds$ which, for what we have seen, is orthogonal to the hypersurface $U = \text{const}$.

Let us now consider a curve represented in parametric form as a function of a parameter $t$ and set

$$\dot{s} = ds/dt \quad \text{and} \quad \dot{x}^i = dx^i/dt.$$  

From (2), one has

$$\dot{s}^2 = g_{ik}\dot{x}^i \dot{x}^h.$$  

(16)

For this quadratic form we can repeat the preceding arguments and, in particular, in place of (10) we shall have

$$\frac{1}{2} \frac{\partial \dot{s}^2}{\partial \dot{x}^i} \equiv \frac{\partial \dot{s}}{\partial \dot{x}^i} = g_{ik}\dot{x}^h = U_{,i} k; \text{ for } i = 1, ..., n,$$  

(17)

where now $k$ is a finite constant of proportionality.

Furthermore, since $\dot{s}^2$ and $\Delta_1 U$ are two scalar quantities in the same metric, we can write

$$\dot{s}^2 = \Delta_1 U \, k^2,$$  

(18)

from which

$$k = \frac{\dot{s}}{\sqrt{\Delta_1 U}}.$$  

(19)

By eliminating $k$ from (17) and (19) we obtain

$$\frac{\partial \dot{s}}{\partial \dot{x}^i} = \frac{U_{,i}}{\sqrt{\Delta_1 U}}.$$  

(20)

The condition for a curve to be a geodesic, as we know, in Lagrangian form (where $\mathcal{L} = \dot{s}$ is the Lagrangian) is given by the $n$ equations

$$\frac{\partial \dot{s}}{\partial x^i} = \frac{d}{dt} \left( \frac{\partial \dot{s}}{\partial \dot{x}^i} \right).$$  

(21)

Let us assume that we know $n$ first integrals of these equations

$$\dot{x}^i = f^i(x^1, ..., x^n)$$  

(22)
and substitute them into \( \dot{s} = \sqrt{g_{hh} \dot{x}^h \dot{x}^h} \) and \( \partial \dot{s} / \partial \dot{x}^i \) of (21) which, in this way, will turn out in the end to be functions only of the variables \( x^i \). If we indicate by \( \left( \frac{d}{dx^i} \right) \) the “total” derivatives with respect to \( x^i \) and make explicit the derivatives of the two sides of (21), we get the equations

\[
\left( \frac{d \dot{s}}{dx^i} \right) = \frac{\partial \dot{s}}{\partial x^i} \frac{\partial x^i}{\partial \dot{x}^r} \frac{\partial \dot{x}^r}{\partial \dot{s}} = 0 \text{, (23)}
\]

where the implicit dependence on \( x^i \), once one have substituted the first integrals into the expression \( s \) of \( \dot{s} \) and \( \partial \dot{s} / \partial \dot{x}^i \), has been exploited.

Now, if we differentiate with respect to \( x^i \) the identity \( \dot{s} = (\partial \dot{s} / \partial \dot{x}^r) \dot{x}^r \), we get

\[
\left( \frac{d \dot{s}}{dx^i} \right) = \dot{x}^r \frac{\partial}{\partial x^i} \frac{\partial \dot{s}}{\partial \dot{x}^r} + \frac{\partial \dot{s}}{\partial \dot{x}^r} \frac{\partial \dot{x}^r}{\partial x^i} \text{. (25)}
\]

From a comparison of (23) and (25) it follows that

\[
\frac{\partial \dot{s}}{\partial x^i} = \frac{\partial}{\partial x^i} \frac{\partial \dot{s}}{\partial \dot{x}^r} \dot{x}^r \text{. (26)}
\]

These equations and (24) allows us to write the system (21) in the pfaffian form

\[
\left( \frac{\partial}{\partial x^i} \frac{\partial \dot{s}}{\partial \dot{x}^r} - \frac{\partial}{\partial x^i} \frac{\partial \dot{s}}{\partial \dot{x}^r} \right) \dot{x}^r = 0 \text{, (i = 1, ..., n) . (27)}
\]

The system (27) is satisfied if it is possible to assign a function \( U \) (potential) such that

\[
\frac{\partial \dot{s}}{\partial \dot{x}^r} = \frac{\partial U}{\partial x^r} \equiv U_{,r} \text{. (28)}
\]

From (20) it follows that this condition is satisfied if \( U \) is a solution of Eq. 5. Then, if we find a solution of the partial differential equation (5), from (12) we obtain the equations of the geodesics as functions of \( s \) by solving a system of first order ordinary differential equations. If (5) is satisfied, we have from (15) that \( U \equiv s \) and then \( U \) represents the line element along the geodesics.

On the basis of what we have summarized above, we can finally enunciate the fundamental result of Beltrami due to which the second integration can be avoided: If we know a complete solution of the partial differential equation (5), we can obtain the geodesic equations by differentiation steps alone.

The demonstration is straightforward. If (5) is satisfied, from (15) it follows that the length of the orthogonal segments between two hypersurfaces \( U = const \) is the same. According to a theorem due to Gauss, the lines which cross orthogonally the fields \( U = const \) are geodesic lines and the hypersurfaces \( U = const \) are said to be geodesically parallel.

Let us consider a complete solution of (5) which, in addition to an obvious additive constant, will contain \( n - 1 \) other arbitrary constants \( \alpha_l \). By differentiating (5) with respect to these constants, we obtain

\[
\frac{\partial \Delta_l U}{\partial \alpha_l} \equiv 2 \Delta \left( U, \frac{\partial U}{\partial \alpha_l} \right) = 0 \text{, for } l = 1, ..., n - 1 \text{, (29)}
\]

which tells us that the hypersurfaces \( V_l \equiv \partial U / \partial \alpha_l = const \) and \( U = const \) are orthogonal. If we now put

\[
V_l \equiv \frac{\partial U}{\partial \alpha_l} = \beta_l \text{, for } l = 1, ..., n - 1 \text{, (30)}
\]
the curves of intersection of these hypersurfaces, orthogonal to \( U = \text{const} \), are geodesics and have been obtained through differentiation, without being obliged to solve the differential equations (12). The demonstration of Beltrami’s theorem is complete.

Obviously the above theorem is particularly useful for applications when we have a complete solution of (5) at our disposal or this solution is easily obtained.

A general case in which the solution of Eq. (5) is obtained in integral form has been obtained by Bianchi \cite{3, 4}\(^2\):

If the fundamental form (2) can be re-expressed in the generalized Liouville form

\[
\text{d}s^2 = \left[ X_1(x_1) + X_2(x_2) + \ldots + X_N(x_N) \right] \sum_{i=1}^{n} e_i (dx^i)^2, \tag{31}
\]

where \( e_i = \pm 1 \) and \( X_i \) is a function of \( x^i \) alone, a complete integral of Eq. (5) is

\[
U = c + \sum_{i=1}^{n} \int \sqrt{e_i(X_i + \alpha_i)} \, dx^i, \tag{32}
\]

where \( c \) and \( \alpha_i \) are constants, the latter being subject to the condition \( \sum_{i=1}^{n} \alpha_i = 0 \).

In this case the geodesic equations (30) are immediately given by

\[
\frac{\partial U}{\partial \alpha_l} \equiv \frac{1}{2} \int \frac{e_i \, dx_i}{\sqrt{e_i(X_i + \alpha_l)}} = \beta_l. \tag{33}
\]

3 The geodesic equations for the Schwarzschild and Kerr metrics

3.1 The Schwarzschild metric

We start from the standard form of the so-called Schwarzschild metric \cite{8}\(^3\)

\[
\text{d}s^2 = \frac{r - \alpha}{r} \text{d}t^2 - \frac{r}{r - \alpha} \, dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2); \quad \alpha = 2MG \tag{34}
\]

Applied to this metric Eq. (5) gives\(^3\)

\[
\frac{r}{r - \alpha} \left( \frac{\partial \tau}{\partial t} \right)^2 - \frac{r - \alpha}{r} \, \left( \frac{\partial \tau}{\partial r} \right)^2 - \frac{1}{r^2} \left[ \left( \frac{\partial \tau}{\partial \theta} \right)^2 + \sin^{-2} \theta \left( \frac{\partial \tau}{\partial \phi} \right)^2 \right] = 1. \tag{35}
\]

If we set

\[
\tau = A_1 t + A_2 \phi + \tau_1(r) + \tau_2(\theta), \tag{36}
\]

Eq. (35) can be solved by the method of “separation of variables.” We obtain

\[
r^2 \left[ A_1^2 \frac{r}{r - \alpha} - \frac{r - \alpha}{r} \left( \frac{d \tau_1}{dr} \right)^2 - 1 \right] = \left( \frac{d \tau_2}{d \theta} \right)^2 + \frac{A_2^2}{\sin^2 \theta}. \tag{37}
\]

\(^2\)In this paper we do not apply this result and obtain the solution of Eq. (5) directly for metrics that are not in the Liouville form.

\(^3\)We replace the previous symbol \( U \) with \( \tau \), that recalls the physical meaning of proper time along the geodesics.
Since the right hand side is positive, we can set both sides equal to a separation constant \( A_3^2 \) to obtain

\[
r^2 \left[ A_1^2 \frac{r}{r - \alpha} - \frac{r - \alpha}{r} \left( \frac{d \tau_1}{d r} \right)^2 - 1 \right] = A_3^2, \tag{38}
\]

\[
\left( \frac{d \tau_2}{d \theta} \right)^2 + \frac{A_2^2}{\sin^2 \theta} = A_3^2, \tag{39}
\]

which then gives

\[
\tau_1 = \pm \int \frac{\sqrt{A_1^2 r^4 - r(r - \alpha)(r^2 + A_3^2)}}{r(r - \alpha)} \, dr, \tag{40}
\]

\[
\tau_2 = \pm \int \frac{\sqrt{A_3^2 \sin^2 \theta - A_2^2}}{\sin \theta} \, d \theta. \tag{41}
\]

If in the right hand sides of (40) and (41) we choose the positive sign, the geodesic equations (6) become

\[
\frac{\partial \tau}{\partial A_1} \equiv t + A_1 \int \frac{r^3 \, dr}{(r - \alpha) \sqrt{A_1^2 r^4 - r(r^2 + A_3^2)(r - \alpha)}} = B_1, \tag{42}
\]

\[
\frac{\partial \tau}{\partial A_2} \equiv \phi - A_2 \int \frac{d \theta}{\sin \theta \sqrt{A_3^2 \sin^2 \theta - A_2^2}} \equiv \phi + \sin^{-1} [\sinh \epsilon \cot \theta] = B_2, \tag{43}
\]

\[
\frac{\partial \tau}{\partial A_3} \equiv -A_3 \int \frac{dr}{\sqrt{A_1^2 r^4 - r(r^2 + A_3^2)(r - \alpha)}} + A_3 \int \frac{\sin \theta \, d \theta}{\sqrt{A_3^2 \sin^2 \theta - A_2^2}}
- \sin^{-1} [\cosh \epsilon \cos \theta] = B_3, \tag{44}
\]

where in the explicitly evaluated integrals we have set \( A_3 = A \cosh \epsilon, \ A_2 = A \sinh \epsilon. \)

Eq. (43) is the same as Eq. (4) on p. 645 of Misner et al. [5] which establishes the planar character of the orbit. In fact, with a suitable rotation of the angle \( \phi (\phi_0 = B_2) \), we obtain from (43) the result \( \sin \phi = -\sinh \epsilon \cot \theta \) and by substituting this into the defining equations for polar coordinates, one finds the orbit in \( x, y, z \) space as a function of the parameter \( \theta \):

\[
\begin{align*}
x &= r(\theta) \sin \theta \cos \phi \equiv r(\theta) \sin \theta \sqrt{1 - \sinh^2 \epsilon \cot^2 \theta} ,  \\
y &= r(\theta) \sin \theta \sin \phi \equiv -r(\theta) \sinh \epsilon \cos \theta , \\
z &= r(\theta) \cos \theta .
\end{align*}
\]

It immediately follows that \( y = -\sinh \epsilon z \), i.e., the orbit lies in this plane. We remark that Misner et al. obtain this result by applying the Hamilton-Jacobi method.

We can now calculate the proper time \( \tau \) from the identity

\[
\tau - A_1 \frac{\partial \tau}{\partial A_1} - A_2 \frac{\partial \tau}{\partial A_2} - A_3 \frac{\partial \tau}{\partial A_3} = \tau - A_1 B_1 - A_2 B_2 - A_3 B_3 \tag{45}
\]

obtained by substituting the partial derivatives by the constants \( B_i \). In the left hand side we evaluate \( \tau \) by using Eq. (36) in which \( \tau_1 \) and \( \tau_2 \) are given by Eqs. (40) and (41). We also evaluate the other three terms of the left hand side by using the first identities of Eqs. (42), (43) and (44), respectively.
We group the integrals in \( dr, \ d\theta \) and see that the resulting integral in \( d\theta \) vanishes. Finally we obtain \( \tau \) as a function of \( r \) via an elliptic integral:

\[
\tau = A_1 B_1 + A_2 B_2 + A_3 B_3 - \int \frac{r^2 \, dr}{\sqrt{A_1^2 r^4 - r(r^2 + A_3^2)(r-\alpha)}}. \tag{46}
\]

At this point we must do two things: compare our results with the existing literature and at the same time identify the constants \( A_1, A_2, A_3 \) that we have introduced with the relevant physical constants of the classic approach. To do this, we set \( \theta = \pi/2 \), which is the a priori value everyone assumes. Then from (39), \( A_2^2 = A_3^2 \) and the sum of (43) and (44) we get

\[
\phi = A_3 \int \frac{dr}{\sqrt{A_1^2 r^4 - r(r^2 + A_3^2)(r-\alpha)}} + \text{const}, \tag{47}
\]

which can be re-expressed as

\[
\phi = A_3 \int \frac{dr}{r^2 \sqrt{A_1^2 - (1 + A_3^2/r^2)(1-\alpha/r)}} + \text{const}. \tag{48}
\]

In the same way (42) can be rewritten in the form

\[
t = -A_1 \int \frac{dr}{(1-\alpha/r) \sqrt{A_1^2 - (1 + A_3^2/r^2)(1-\alpha/r)}} + \text{const}. \tag{49}
\]

If we compare our equations (49) and (48) with (101.4) and (101.5) of Landau-Lifshitz [9], which are obtained starting from the Hamilton-Jacobi equation, we find that the equations are the same, apart from the different symbols and units used in them. In the limit \( \alpha/r \ll 1 \) (as is the case for planetary motion), (48) becomes

\[
\phi = A_3 \int \frac{dr}{r^2 \sqrt{(A_1^2 - 1) - A_3^2/r^2}} + \text{const}. \tag{50}
\]

If we compare (50) with the relevant Newtonian equation (see Boccaletti-Pucacco [10], Eq. (2.14), p. 131)

\[
\phi = \int \frac{c \, dr}{r^2 \sqrt{2(h - V(r)) - c^2/r^2}} + \text{const}, \tag{51}
\]

where \( c \) is the angular momentum per unit mass, one immediately identifies \( A_3 \) with the angular momentum \( c \). From (39), one has that in the general case \( \theta \neq \pi/2 \) the constant \( A_2 \) represents the component of the angular momentum along the polar axis.

We can also obtain the physical meaning of the constant \( A_3 \) and the second Kepler law from (46) and (47). Assuming “direct orbits” we have:

\[
\frac{d\phi}{d\tau} = \frac{d\phi}{dr} \frac{dr}{d\tau} = \frac{A_3}{r^2}. \tag{52}
\]

As for the constant \( A_1 \), in our units \((c = 1, m = 1)\), \( A_1^2 - 1 \) must be identified with twice the kinetic energy through the standard relativistic relations \( E^2 - m^2 c^4 = c^2 p^2 \) and \( p^2 = 2mT \), where \( T \) is the classical kinetic energy. Thus in our units \( A_1^2 = E^2 \); a further detailed investigation leads to \( A_1 = -E \).
Regarding the advance of the perihelion of Mercury, it is convenient to start from (48) rewritten in terms of the variable \( u = 1/r \)

\[
\left( \frac{du}{d\phi} \right)^2 = \frac{1}{A_3^2} \left[ A_1^2 - (1 + A_3^2 u^2)(1 - \alpha u) \right].
\]  

(53)

By differentiating this with respect to \( \phi \), we obtain

\[
\frac{d^2 u}{d\phi^2} + u = \frac{1}{2} \frac{\alpha}{A_3^2} + \frac{3}{2} \alpha u^2,
\]  

(54)

which since \( \alpha = 2 G M \) can be rewritten

\[
\frac{d^2 u}{d\phi^2} + u = \frac{G M}{A_3^2} (1 + 3 A_3^2 u^2).
\]  

(55)

A comparison with the classic Binet equation (see [10], Eq. (2.24)) shows that the second term on the right hand side represents the relativistic correction which accounts for the advance of the perihelion. The reader can find the details of the calculations in the well known text by Bergmann [11].

As a final remark, we point out that, returning to the geodesic equations in the form of (42, 43, 44), we can further obtain an expression for \( r \) as a function of \( \phi \) more general than Eq. (47). In fact, without the condition \( \theta = \pi/2 \), we have from Eq. (43), \( \theta \) as a function of \( \phi \) and then, from Eq. (44), \( r \) as a function of \( \phi \). This leads to

\[
\cot \theta = -\frac{\sin(\phi - B_2)}{\sinh \epsilon},
\]  

(56)

and

\[
\sin[f(r) + B_3] = \frac{\cosh \epsilon \sin(\phi - B_2)}{\sqrt{\sinh^2 \epsilon + \sin^2(\phi - B_2)}},
\]  

(57)

where

\[
f(r) = A_3 \int \frac{dr}{\sqrt{A_1^2 r^4 - r(r^2 + A_3^2)(r - \alpha)}}
\]

is the elliptic integral in \( dr \) of Eq. (44). Eqs. (56) and (57) give the geodesics in the space as functions of the parameter \( \phi \).

Note that above we have obtained the result that the orbits are planar without invoking the spherical symmetry of the field and without setting \( \theta = \pi/2 \).

### 3.2 The Kerr metric

The method based on Beltrami’s theorem that we have applied so far to study geodesic motion in the Schwarzschild spacetime can clearly be applied to the Kerr spacetime as well. We start from the Kerr metric [6]

\[
d s^2 = \frac{\rho^2 \Delta}{\Sigma^2} d t^2 - \frac{\Sigma^2}{\rho^2} \left( d \phi - \frac{2 a M r}{\Sigma^2} d t \right)^2 \sin^2 \theta - \frac{\rho^2}{\Delta} d r^2 - \rho^2 d \theta^2,
\]  

(58)

where \( \rho^2 = r^2 + a^2 \cos^2 \theta \), \( \Delta = r^2 + a^2 - 2 M r \), \( \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \); \( M \) and \( a \) are constants that in the Newton limit represent the mass and the angular momentum per unit mass.

As is well known (see [6], p. 289) the Kerr metric reduces to the Schwarzschild metric when the constant \( a = 0 \).
The contravariant form of the metric tensor is

\[
g^{ij} = \frac{1}{\rho^2} \begin{pmatrix}
\frac{\Sigma^2}{\Delta} & 0 & 0 & 2aMr/\Delta \\
0 & -\Delta & 0 & 0 \\
0 & 0 & -1 & 0 \\
2aMr/\Delta & 0 & 0 & (a^2\sin^2\theta - \Delta)/\Delta\sin^2\theta
\end{pmatrix},
\]

(59)

where the entries are arranged following the sequence \(dt, dr, d\theta, d\phi\), respectively. Eq. (5) with the contravariant components of the metric tensor given by (59) turns out to be

\[
\frac{\Sigma^2}{\Delta} \left(\frac{\partial \tau}{\partial t}\right)^2 + \frac{2aMr}{\Delta} \left(\frac{\partial \tau}{\partial \phi}\right) - \frac{\Delta - a^2\sin^2\theta}{\Delta\sin^2\theta} \left(\frac{\partial \tau}{\partial \phi}\right)^2 - \Delta \left(\frac{\partial \tau}{\partial r}\right)^2 - \left(\frac{\partial \tau}{\partial \theta}\right)^2 = \rho^2.
\]

(60)

For (58), as for (34), the coefficients of the line element do not depend on \(t\) and \(\phi\), therefore we can seek a solution of (5) in the same form of Schwarzschild equation

\[
\tau = A_1 t + A_2 \phi + \tau_1(r) + \tau_2(\theta).
\]

(61)

By substituting into Eq. (60) we obtain

\[
\frac{1}{\Delta} \left[A_1 (r^2 + a^2) + A_2 a\right]^2 - \frac{1}{\sin^2\theta} \left[A_1 a \sin^2\theta + A_2\right]^2 - \Delta \left(\frac{d \tau_1}{d r}\right)^2 - \left(\frac{d \tau_2}{d \theta}\right)^2 = r^2 + a^2\cos^2\theta,
\]

(62)

and

\[
\frac{1}{\Delta} \left[A_1 (r^2 + a^2) + A_2 a\right]^2 - r^2 - \Delta \left(\frac{d \tau_1}{d r}\right)^2 = \frac{1}{\sin^2\theta} \left[A_1 a \sin^2\theta + A_2\right]^2 + \left(\frac{d \tau_2}{d \theta}\right)^2 + a^2\cos^2\theta.
\]

(63)

The left hand side is a function of \(r\), the right hand side is a positive function of \(\theta\); then introducing a new separation constant \(A_3^2\) we obtain the solutions

\[
\tau_1 = \int \sqrt{\frac{\left[A_1 (r^2 + a^2) + A_2 a\right]^2 - \Delta(r^2 + A_3^2)}{\Delta}} dr,
\]

(64)

\[
\tau_2 = \int \sqrt{\frac{(A_3^2 - a^2\cos^2\theta)\sin^2\theta - (A_1 a \sin^2\theta + A_2)^2}{\sin\theta}} d\theta.
\]

(65)

If we set

\[
R(r) = \left[A_1 (r^2 + a^2) + A_2 a\right]^2 - \Delta(r^2 + A_3^2), \quad \Theta(\theta) = (A_3^2 - a^2\cos^2\theta)\sin^2\theta - (A_1 a \sin^2\theta + A_2)^2,
\]

(66)

Eq. (61) becomes

\[
\tau = A_1 t + A_2 \phi + \int_r^\infty \frac{\sqrt{R(r)}}{\Delta} dr + \int^\theta \frac{\sqrt{\Theta(\theta)}}{\sin\theta} d\theta.
\]

(67)

The equations for the geodesics can now be obtained by the standard procedure of Eq. (6) following from Beltrami’s theorem. We get

\[
\frac{\partial \tau}{\partial A_1} \equiv t + \int \frac{(r^2 + a^2)[A_1 (r^2 + a^2) + A_2 a]}{\Delta \sqrt{R}} dr - \int \frac{(A_1 a \sin^2\theta + A_2)}{\sqrt{\Theta}} a \sin\theta d\theta = B_1,
\]

(68)

\[
\frac{\partial \tau}{\partial A_2} \equiv \phi + \int a \frac{A_1 (r^2 + a^2) + A_2 a}{\Delta \sqrt{R}} dr - \int \frac{(A_1 a \sin^2\theta + A_2)}{\sin\theta \sqrt{\Theta}} d\theta = B_2,
\]

(69)

\[
\frac{\partial \tau}{\partial A_3} \equiv A_3 \left[\int \frac{dr}{\sqrt{R}} + \int \frac{\sin\theta}{\sqrt{\Theta}} d\theta\right] = B_3.
\]

(70)
Moreover, we can calculate the proper time with the same procedure we have used for the Schwarzschild metric. We start from the expression (45) and obtain

\[ \tau = \text{const} - \int \frac{r^2}{\sqrt{R}} \, dr - \int \frac{a^2 \cos^2 \theta \sin \theta}{\sqrt{\Theta}} \, d\theta. \]  

(71)

Now the integral in $d\theta$ does not vanish, while from (70) we can find $\theta$ as a function of $r$ and then find $\tau(r)$. In any case from Eqs. (69, 70, and 71) we can obtain the relations analogous to those obtained for the Schwarzschild metric.

4 Conclusions

It turns out, as Beltrami himself was the first to point out, that Beltrami’s method is formally analogous to the method of integration of the Hamilton-Jacobi equation. On the other hand this is even more evident if we consider that the differential parameter $\Delta_1 \tau$ is formally analogous to the expression

\[ H = g^{ij} p_i p_j \]

which represents the Hamiltonian and then $\Delta_1 \tau = 1$ is equivalent to the Hamilton-Jacobi equation.

However, in spite of this formal analogy, by using Beltrami’s theorem we remain in a geometric context to obtain the geodesics, i.e., the orbits of a test particle in the gravitational field, without being obliged to resort to concepts copied from classical mechanics. In fact it must be remarked that very often one uses too freely in general relativity procedures which obtain a true legitimacy only for $r \to \infty$ or weak gravitational fields. The fact that, a posteriori, things turn out to be correct in the nonrelativistic limit does not always remove the ambiguity from certain formulations.

Finally we note that the “geometrical integration” here described allows us to recover the classical conservation laws instead of introducing them a priori.

Acknowledgments

We are grateful to Robert Jantzen for fruitful discussions.

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[1] A. Wintner, *The Analytical Foundations of Celestial Mechanics* (Princeton University Press, Princeton, 1941)

[2] The paper *Sulla Teorica Generale dei Parametri Differenziali* by Eugenio Beltrami was published in *Memorie dell’Accademia delle Scienze dell’Istituto di Bologna*, serie II, tomo VIII, pp. 551–590 (1868), and reprinted in *Opere Matematiche*, Vol. II, pp. 74–118 (Hoepli, Milano, 1904). For the theorem considered here, which generalizes to $n$-dimensions previous results relative to ordinary surfaces, see pp. 366–373 op. cit. Vol. I, and also A.R. Forsyth, *Lectures on the Differential Geometry of Curves and Surfaces*, pp. 163–165, (Cambridge University Press, 1912). We summarize the Beltrami contribution: in the first paper Beltrami follows Jacobi up to the remark that the results obtained can be expressed using a partial differential equation via the differential parameter ($\Delta_1 U = 1$) that he had introduced in previous work studying surfaces. This correspondence and a theorem of Gauss allows one to obtain the geodesics by differentiation alone (as we have summarized in Sect. 2). In the second article Beltrami extends the introduction
of the differential parameters to $n$-dimensional Riemann spaces. After this he tries and succeeds to extend the theorem for the integration of geodesics to an $n$-dimensional space. This method is included in a long article as an application of the differential parameter. This may be the reason it is not as well known as it deserves to be, like other contributions of Beltrami.

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The theorem is discussed in pp. 336–338 of Vol. I. In the 3rd edition of Bianchi’s treatise (Zanichelli, Bologna, 1924) the theorem is given for two dimensions in Vol. I, p. 299, and for $n$ dimensions in pp. 423–426 of Vol. II, part 2. For the theorem in two dimensions Bianchi credits Jacobi. As far as $n$ dimensions are concerned he credits Beltrami for the introduction of the differential parameter and the theorem is considered as an important consequence of this introduction, but without reference to any author for this theorem.

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