MINIMAL EXTENSIONS OF TANNAKIAN CATEGORIES IN POSITIVE CHARACTERISTIC

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Abstract. We extend [DGNO1, Theorem 4.5] and [LKW, Theorem 4.22] to positive characteristic (i.e., to the finite, not necessarily fusion, case). Namely, we prove that if $\mathcal{D}$ is a finite non-degenerate braided tensor category over an algebraically closed field $k$ of characteristic $p > 0$, containing a Tannakian Lagrangian subcategory $\text{Rep}(G)$, where $G$ is a finite $k$-group scheme, then $\mathcal{D}$ is braided tensor equivalent to $\text{Rep}(D^\omega(G))$ for some $\omega \in H^3(G, \mathbb{G}_m)$, where $D^\omega(G)$ denotes the twisted double of $G$. We then prove that the group $\mathcal{M}_{\text{ext}}(\text{Rep}(G))$ of minimal extensions of $\text{Rep}(G)$ is isomorphic to the group $H^3(G, \mathbb{G}_m)$. In particular, we use [EG2, FP] to show that $\mathcal{M}_{\text{ext}}(\text{Rep}(\mu_p)) = 1$, $\mathcal{M}_{\text{ext}}(\text{Rep}(\alpha_p))$ is infinite, and if $\mathcal{O}(\Gamma)^* = u(\mathfrak{g})$ for a semisimple restricted $p$-Lie algebra $\mathfrak{g}$, then $\mathcal{M}_{\text{ext}}(\text{Rep}(\Gamma)) = 1$ and $\mathcal{M}_{\text{ext}}(\text{Rep}(\Gamma \times \alpha_p)) \cong \mathfrak{g}^{*(1)}$.

Contents

1. Introduction 2
2. Preliminaries 4
   2.1. Finite tensor categories 4
   2.2. Exact module categories 5
   2.3. Exact sequences of finite tensor categories 5
   2.4. Lagrangian subcategories 6
   2.5. Exact commutative algebras 6
   2.6. Finite group schemes 7
   2.7. De-equivariantization 8
   2.8. The categories $\text{Coh}(G, \omega)$ and $\mathcal{Z}(\text{Coh}(G, \omega))$ 8
   2.9. The group $\mathcal{M}_{\text{ext}}(\text{Rep}(G))$ 9
3. The proof of Theorem 1.1 10
   3.1. The free module functor $F$ 10
   3.2. The central structure on $F$ 11
   3.3. $\mathcal{O}(\mathcal{C}) = G(k)$ 13

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1. Introduction

Let $\mathcal{C}$ be a finite symmetric tensor category over an algebraically closed field $k$ with characteristic 0. A minimal extension of $\mathcal{C}$ is, by definition, a finite non-degenerate braided tensor category $\mathcal{D}$ over $k$, containing $\mathcal{C}$ as a Lagrangian subcategory (see 2.4 below). Given $\mathcal{C}$ as above, it is natural to try to classify its minimal extensions. Indeed, this problem was studied in several papers, e.g., [DN, DGNO1, DGNO2, DMNO, GS, GVR, JFR, LKW, VR, OY].

In particular, in the papers [DGNO1, DGNO2, LKW] the authors classify the minimal extensions of Tannakian categories $\mathcal{C} = \text{Rep}(G)$ over $k$ ($G$ a finite group). Note that in this case, $\mathcal{C}$ and all its minimal extensions are fusion categories. More explicitly, in [DGNO1, Theorem 4.5], [DGNO2, Theorem 4.64], it is proved that $\mathcal{D}$ is a minimal extension of $\text{Rep}(G)$ if and only if $\mathcal{D}$ is braided tensor equivalent to the representation category $\text{Rep}(D^\omega(G))$ of a certain quasi-Hopf algebra $D^\omega(G)$ for some $\omega \in H^3(G, \mathbb{G}_m)$. Then in [LKW, Theorem 4.22], it is shown that minimal extensions of $\text{Rep}(G)$ carry a natural structure of a commutative group, denoted by $\mathcal{M}_{\text{ext}}(\text{Rep}(G))$, and that $\mathcal{M}_{\text{ext}}(\text{Rep}(G)) \cong H^3(G, \mathbb{G}_m)$ as groups.

Our goal in this paper is to extend the above mentioned results of [DGNO1, DGNO2, LKW] to minimal extensions of Tannakian categories in positive characteristic.

Namely, assume from now on that $k$ is an algebraically closed field with characteristic $p > 0$, and let $G$ be a finite $k$-group scheme. Then our goal is to relate minimal extensions of $\text{Rep}(G)$ over $k$, and representation categories of twisted doubles $D^\omega(G)$, $\omega \in H^3(G, \mathbb{G}_m)$ [G].

More precisely, we first prove the following result.

**Theorem 1.1.** Assume $\mathcal{D}$ is a finite non-degenerate braided tensor category over $k$ containing a Tannakian Lagrangian subcategory $\text{Rep}(G)$, where $G$ is a finite group scheme over $k$. Then there exists an element $\omega \in H^3(G, \mathbb{G}_m)$ such that $\mathcal{D} \cong \mathcal{D}(\text{Coh}(G, \omega))$ as braided tensor categories (equivalently, $\mathcal{D}$ is braided tensor equivalent to the representation category $\text{Rep}(D^\omega(G))$ of a twisted double $D^\omega(G)$ of $G$).
Fix a finite non-degenerate braided tensor category $\mathcal{D}$ over $k$, and consider the set of all triples $(G, \omega, F)$, where $G$ is a finite group scheme over $k$, $\omega \in H^3(G, \mathbb{G}_m)$, and $F : \mathcal{D} \xrightarrow{\cong} \mathcal{L}(\text{Coh}(G, \omega))$ is a braided tensor equivalence. We say that $(G_1, \omega_1, F_1)$, $(G_2, \omega_2, F_2)$ are equivalent, if there exists a tensor equivalence $\varphi : \text{Coh}(G_1, \omega_1) \xrightarrow{\cong} \text{Coh}(G_2, \omega_2)$ such that $F_2 \circ F_2 = \varphi \circ F_1 \circ F_1$, where $F_i : \mathcal{L}(\text{Coh}(G_i, \omega_i)) \rightarrow \text{Coh}(G_i, \omega_i)$ are the canonical forgetful functors, $i = 1, 2$. Let $[(G, \omega, F)]$ denote the equivalence class of $(G, \omega, F)$.

Then using Theorem 1.1 we obtain the following corollary.

**Corollary 1.2.** Let $\mathcal{D}$ be a finite non-degenerate braided tensor category over $k$. Then the following sets are in natural one to one correspondence with each other:

1. Equivalence classes $[(G, \omega, F)]$.
2. Tannakian Lagrangian subcategories of $\mathcal{D}$.

Next we fix a finite group scheme $G$ over $k$, and consider the set of pairs $(\mathcal{D}, \iota)$, where $\mathcal{D}$ is a non-degenerate braided tensor category over $k$, and $\iota : \text{Rep}(G) \xrightarrow{1:1} \mathcal{D}$ is an injective braided tensor functor. We say that two pairs $(\mathcal{D}_1, \iota_1)$ and $(\mathcal{D}_2, \iota_2)$ are equivalent, if there exists a braided tensor equivalence $\psi : \mathcal{D}_1 \xrightarrow{\cong} \mathcal{D}_2$ such that $\psi \circ \iota_1 = \iota_2$. Let $\mathcal{M}_{\text{ext}}(\text{Rep}(G))$ denote the set of all equivalence classes of pairs $(\mathcal{D}, \iota)$. Recall that $\mathcal{M}_{\text{ext}}(\text{Rep}(G))$ has a natural structure of a commutative group $\mathbb{LKW}$.

Then using Theorem 1.1 we obtain the following corollary, which extends $\mathbb{LKW}$ Theorem 4.22 to positive characteristic.

**Corollary 1.3.** Let $G$ be a finite group scheme over $k$. There is a group isomorphism

$$\mathcal{M}_{\text{ext}}(\text{Rep}(G)) \cong H^3(G, \mathbb{G}_m).$$

The structure of the paper is as follows. In Section 2 we recall some necessary background on finite tensor categories and their exact module categories (2.1 2.2), exact sequences of finite tensor categories (2.3), non-degenerate braided tensor categories and exact commutative algebras in finite braided tensor categories (2.4 2.5), finite group schemes, de-equivariantization, the finite tensor categories $\text{Coh}(G, \omega)$ and their centers $\mathcal{L}(\text{Coh}(G, \omega))$, and the group $\mathcal{M}_{\text{ext}}(\text{Rep}(G))$ (2.6 - 2.9).

Sections 3, 4 and 5 are devoted to the proofs of Theorem 1.1, Corollary 1.2 and Corollary 1.3, respectively.

Finally, in Section 6 we give some examples. In particular, we use the results of $\mathbb{EG2}$ to show that the group $\mathcal{M}_{\text{ext}}(\text{Rep}(\mu_p))$ is trivial (see
Example [6.1] and the group $\mathcal{M}_{\text{ext}}(\text{Rep}(\alpha_p))$ is infinite (see Example 6.2), and use [FP] to conclude that if $\mathcal{O}(\Gamma)^* = u(g)$ for a semisimple restricted $p$-Lie algebra $g$, then $\mathcal{M}_{\text{ext}}(\text{Rep}(\Gamma))$ is the trivial group and $\mathcal{M}_{\text{ext}}(\text{Rep}(\Gamma \times \alpha_p)) \cong g^{(1)}$ (see Examples 6.3 and 6.4).

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2. Preliminaries

Let $k$ be an algebraically closed field with positive characteristic $p > 0$, and let Vec denote the category of finite dimensional $k$-vector spaces.

Let $\mathcal{B}$ be any $k$-linear category. Recall that for every $V \in $ Vec, we have a natural functor

$$V \otimes - : \mathcal{B} \to \mathcal{B}, \ X \mapsto V \otimes X.$$ 

Namely, for $X \in \mathcal{B}$, the object $V \otimes X$ is uniquely defined through the Yoneda lemma by the formula

$$\text{Hom}_{\mathcal{B}}(Y, V \otimes X) \cong V \otimes_k \text{Hom}_{\mathcal{B}}(Y, X)$$

(and the existence of this object is checked by choosing a basis in $V$).

2.1. Finite tensor categories. (See [EGNO Chapter 6].) Let $\mathcal{B}$ be a finite tensor category over $k$. Let $\mathcal{O}(\mathcal{B})$ denote the complete set of isomorphism classes of simple objects of $\mathcal{B}$. Let $\text{Gr}(\mathcal{B})$ denote the Grothendieck ring of $\mathcal{B}$, and let $\text{Pr}(\mathcal{B})$ denote the group of isomorphism classes of projective objects in $\mathcal{B}$. Recall that $\text{Pr}(\mathcal{B})$ is a bimodule over $\text{Gr}(\mathcal{B})$, and we have a natural homomorphism $\tau : \text{Pr}(\mathcal{B}) \to \text{Gr}(\mathcal{B})$.

Recall [EO1 Subsection 2.4] that we have a character

$$\text{FPdim} : \text{Gr}(\mathcal{B}) \to \mathbb{R}, \ X \mapsto \text{FPdim}(X),$$

attaching to $X \in \mathcal{B}$ its Frobenius-Perron dimension. Recall also that

$$\text{FPdim}(\mathcal{B}) := \sum_{X \in \mathcal{O}(\mathcal{B})} \text{FPdim}(X)\text{FPdim}(P(X)),$$

where $P(X)$ denotes the projective cover of $X$ and $\text{FPdim}(P(X))$ is defined to be $\text{FPdim}(\tau(P(X)))$.

Finally, recall that $\mathcal{A}$ is a tensor subcategory of $\mathcal{B}$ if $\mathcal{A}$ is a full subcategory of $\mathcal{B}$, closed under taking subquotients, tensor products, and duality [EGNO Definition 4.11.1].
2.2. **Exact module categories.** Retain the notation from [2.1]. Recall that a left $\mathcal{B}$-module category $\mathcal{M}$ is said to be *indecomposable* if it is not a direct sum of two nonzero module categories, and *exact* if $P \otimes M$ is projective in $\mathcal{M}$, for every $P \in \text{Pr}(\mathcal{B})$ and $M \in \mathcal{M}$.

Let $\mathcal{M}$ be an indecomposable exact $\mathcal{B}$-module category. Let $\text{End}(\mathcal{M})$ be the abelian category of right exact endofunctors of $\mathcal{M}$, and let $\mathcal{B}^* := \text{End}_{\mathcal{B}}(\mathcal{M})$ be the dual category of $\mathcal{B}$ with respect to $\mathcal{M}$, i.e., the category of $\mathcal{B}$-linear (necessarily exact) endofunctors of $\mathcal{M}$. Recall that composition of functors turns $\text{End}(\mathcal{M})$ into a monoidal category, and $\mathcal{B}^*$ into a finite tensor category.

Let $A$ be an algebra object in $\mathcal{B}$, and let $\mathcal{M} := \text{Mod}(A)_\mathcal{B}$ denote the category of right $A$-modules in $\mathcal{B}$. Recall that $\mathcal{M}$ has a natural structure of a left $\mathcal{B}$-module category, and $A$ is called *exact* if $\mathcal{M}$ is indecomposable and exact over $\mathcal{B}$ (see [EGNO, Section 7.5]). Recall [EGNO, Proposition 7.11.6] that in this case, $\mathcal{B}^* \simeq \text{Bimod}_{\mathcal{B}}(A)^{op}$ is a finite tensor category (where $\text{Bimod}_{\mathcal{B}}(A)^{op}$ is $\text{Bimod}_{\mathcal{B}}(A)$ equipped with the opposite tensor product).

The following result will be crucial for us.

**Theorem 2.1.** [EO2, Corollary 12.4] If $\mathcal{C}$ is a tensor subcategory of $\mathcal{B}$, and $A$ is an exact algebra in $\mathcal{C}$, then $A$ is an exact algebra in $\mathcal{B}$. □

Let $\mathcal{M}$ be an exact indecomposable module category over $\mathcal{B}$. Fix a nonzero object $M \in \mathcal{M}$, and let $A := \text{End}_{\mathcal{B}}(M)$ be the internal Hom from $M$ to $M$ in the $\mathcal{B}$-module category $\mathcal{M}$. Recall that $A$ is an exact algebra in $\mathcal{B}$, and there is a canonical equivalence of $\mathcal{B}$-module categories $\mathcal{M} \cong \text{Mod}(A)_\mathcal{B}$ (see, e.g., [EGNO, Theorem 7.10.1]).

2.3. **Exact sequences of finite tensor categories.** Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be finite tensor categories, and let $\iota : \mathcal{A} \xrightarrow{1:1} \mathcal{B}$ be an injective tensor functor. Let $\mathcal{M}$ be an indecomposable exact module category over $\mathcal{A}$, fix a nonzero object $M \in \mathcal{M}$, and consider the algebra $A := \text{End}_{\mathcal{A}}(M)$ (see [2.1, 2.2]). We have $\mathcal{M} \cong \text{Mod}(A)_\mathcal{A}$, and $\mathcal{B} \boxtimes_{\mathcal{A}} \mathcal{M} = \text{Mod}(A)_\mathcal{B}$.

Let $F : \mathcal{B} \to \mathcal{C} \boxtimes \text{End}(\mathcal{M})$ be a surjective \(^1\) monoidal functor, such that $\iota(\mathcal{A}) \subset \mathcal{B}$ is the subcategory consisting of all objects $X \in \mathcal{B}$ such that $F(X) \in \text{End}(\mathcal{M})$. Recall [EG1] that $F$ defines an exact sequence of tensor categories

\begin{equation}
\mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{F} \mathcal{C} \boxtimes \text{End}(\mathcal{M})
\end{equation}

---

\(^1\)Here and below, by “a module category” we will mean a left module category, unless otherwise specified.

\(^2\)I.e., any object of $\mathcal{C} \boxtimes \text{End}(\mathcal{M})$ is a subquotient of $F(X)$ for some $X \in \mathcal{B}$. 

---
with respect to $\mathcal{M}$, if for every object $X \in \mathcal{B}$ there exists a subobject $X_0 \subset X$ such that $F(X_0)$ is the largest subobject of $F(X)$ contained in $\text{End}(\mathcal{M}) \subset \mathcal{C} \boxtimes \text{End}(\mathcal{M})$. Recall [EG1, Theorem 2.9] that the functor $F$ induces on $\mathcal{C} \boxtimes \mathcal{M}$ a structure of an exact $\mathcal{B}$-module category.

Theorem 2.2. [EG1, Theorems 3.4, 3.6] The following are equivalent:

1. $(2.1)$ defines an exact sequence of finite tensor categories.
2. $\text{FPdim}(\mathcal{B}) = \text{FPdim}(\mathcal{A}) \text{FPdim}(\mathcal{C})$.
3. The natural functor $T : \mathcal{B} \boxtimes \mathcal{A} \mathcal{M} \to \mathcal{C} \boxtimes \mathcal{M}$, given by

$$\mathcal{B} \boxtimes \mathcal{A} \mathcal{M} \xrightarrow{F \boxtimes \text{Id}_\mathcal{M}} \mathcal{C} \boxtimes \text{End}(\mathcal{M}) \boxtimes \mathcal{A} \mathcal{M} = \mathcal{C} \boxtimes \mathcal{M} \boxtimes \mathcal{A}^{\text{op}} \mathcal{M} \xrightarrow{\text{Id}_\mathcal{C} \boxtimes \rho} \mathcal{C} \boxtimes \mathcal{M},$$

is an equivalence (where $\rho : \mathcal{M} \boxtimes \mathcal{A}^{\text{op}} \mathcal{M} \to \mathcal{M}$ is the right action of $\mathcal{A}^{\text{op}} \mathcal{M}$ on $\mathcal{M}$).

Let $\mathcal{N}$ be an indecomposable exact module category over $\mathcal{C}$. Then $\mathcal{N} \boxtimes \mathcal{M}$ is an exact module category over $\mathcal{C} \boxtimes \text{End}(\mathcal{M})$, and we have $(\mathcal{C} \boxtimes \text{End}(\mathcal{M}))^* \boxtimes \mathcal{M} \cong \mathcal{C}_\mathcal{N}^*$. By [EG1, Theorem 4.1], the dual sequence to $(2.1)$ with respect to $\mathcal{N} \boxtimes \mathcal{M}$ is exact with respect to $\mathcal{N}$.

2.4. Lagrangian subcategories. Let $\mathcal{D}$ be a finite braided tensor category with braiding $c$ (see, e.g., [EGNO, Chapter 8]), and let $\mathcal{E} \subset \mathcal{D}$ be a tensor subcategory of $\mathcal{D}$. Recall that two objects $X, Y \in \mathcal{D}$ centralize each other if

$$(c_{Y, X} \circ c_{X, Y} = \text{Id}_{X \otimes Y},$$

and the (Müger) centralizer of $\mathcal{E}$ is the tensor subcategory $\mathcal{E}' \subset \mathcal{D}$ consisting of all objects which centralize every object of $\mathcal{E}$ (see, e.g., [DGNO1]). The category $\mathcal{D}$ is called non-degenerate if $\mathcal{D}' = \text{Vec}$.

By [S, Theorem 4.9], we have

$$\text{FPdim}(\mathcal{E}) \text{FPdim}(\mathcal{E}') = \text{FPdim}(\mathcal{D}) \text{FPdim}(\mathcal{D}' \cap \mathcal{E})$$

If $\mathcal{D}$ is non-degenerate and $\mathcal{E}' = \mathcal{E}$, then $\mathcal{E}$ is called a Lagrangian subcategory of $\mathcal{D}$. Thus, a Lagrangian subcategory of $\mathcal{D}$ is a maximal symmetric tensor subcategory of $\mathcal{D}$.

2.5. Exact commutative algebras. Retain the notation from 2.4. Let $A$ be an exact algebra in $\mathcal{D}$ (see 2.2), with multiplication and unit morphisms $m_A$ and $u_A$, respectively. Recall that $A$ is commutative if $m_A = m_A \circ c_{A_A, A}$. Assume from now on that $A$ is an exact commutative algebra in $\mathcal{D}$. Consider the category $\mathcal{C} := \text{Mod}(A)_{\mathcal{D}}$ (see 2.2). Let $(X, m_X) \in \mathcal{C},$
where $X \in \mathcal{D}$ and $m_X : X \otimes A \to X$ is the $A$-module structure morphism. Recall that the braiding on $\mathcal{D}$ defines on $X$ two structures of a left $A$-module as follows:

$$A \otimes X \xrightarrow{c_{A,X}} X \otimes A \xrightarrow{m_X} X, \quad A \otimes X \xrightarrow{c^{-1}_{X,A}} X \otimes A \xrightarrow{m_X} X.$$ 

Both structures make $(X, m_X)$ into an $A$-bimodule, denoted by $X_+$ and $X_-$, respectively. By identifying $\mathcal{C}$ with a subcategory of the finite tensor category $\text{Bimod}_{\mathcal{D}}(A)$ (see 2.2) via the full embedding functor $\mathcal{C} \hookrightarrow \text{Bimod}_{\mathcal{D}}(A)$, $(X, m_X) \mapsto X_-$, (see [EGNO, Section 8.8]), we see that $\mathcal{C}$ is a finite tensor category, with the tensor product $\otimes_A$. Namely, the tensor product of $X, Y \in \mathcal{C}$, denoted by $X \otimes_A Y$, is given by $X \otimes_A Y_-$ endowed with the structure map $\text{Id}_X \otimes m_Y : X \otimes_A Y_- \otimes A \to X \otimes_A Y_-.

Now let

$$F : \mathcal{D} \to \mathcal{C}, \quad X \mapsto (X \otimes A, \text{Id}_X \otimes m_A),$$

be the free $A$-module functor, and let

$$I : \mathcal{C} \to \mathcal{D}, \quad (X, m_X) \mapsto X$$

be the forgetful functor. Recall that $F, I$ are adjoint functors [EGNO, Lemma 7.8.12], i.e., we have natural isomorphisms

$$\text{Hom}_\mathcal{C}(F(X), Y) \cong \text{Hom}_\mathcal{D}(X, I(Y)), \quad X \in \mathcal{D}, Y \in \mathcal{C}.$$

Recall that $F$ is a surjective tensor functor [EGNO], with tensor structure $J = \{J_{X,Y} \mid X, Y \in \mathcal{D}\}$, where

$$J_{X,Y} : F(X \otimes Y) \xrightarrow{\cong} F(X) \otimes_A F(Y)$$

is the $\mathcal{C}$-isomorphism given by the composition

$$X \otimes Y \otimes A \xrightarrow{\text{Id}_X \otimes \text{Id}_Y \otimes \text{Id}_A} X \otimes A \otimes_A Y \otimes A.$$

**Lemma 2.3.** We have $\text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{D})/\text{FPdim}(A)$.

**Proof.** Follows from [EGNO, Lemma 6.2.4].

2.6. Finite group schemes. (See, e.g., [W].) Let $G$ be a finite group scheme over $k$, with coordinate algebra $\mathcal{O}(G)$. Then $\mathcal{O}(G)$ is a finite dimensional commutative Hopf algebra, and its representation category $\text{Coh}(G)$ is a finite tensor category over $k$.

We will denote the dimension of $\mathcal{O}(G)$ by $|G|$.

Let $\text{Rep}(G) = \text{Corep}(\mathcal{O}(G))$ denote the representation category of $G$ over $k$. Recall that $\text{Rep}(G)$ is a finite Tannakian category, that
is, a finite symmetric tensor category such that the forgetful functor \( \text{Rep}(G) \to \text{Vec} \) is symmetric.

Set \( \mathcal{E} := \text{Rep}(G) \). Let \( I : \mathcal{E} \to \mathcal{E} \boxtimes \mathcal{E} \) be the adjoint functor of the tensor functor \( \mathcal{E} \boxtimes \mathcal{E} \otimes - \to \mathcal{E} \). Recall that \( I(1) \) is an exact commutative algebra in \( \mathcal{E} \boxtimes \mathcal{E} \), and \( \mathcal{E} \cong \text{Mod}(I(1))_{\mathcal{E} \boxtimes \mathcal{E}} \).

Let \( \Delta : G \overset{1:1}{\to} G \times G \) be the diagonal morphism.

**Lemma 2.4.** We have \( I(1) = O(G \times G / \Delta(G)) \).

**Proof.** Since the tensor functor \( \mathcal{E} \boxtimes \mathcal{E} \otimes - \to \mathcal{E} \) coincides with the functor \( \mathcal{E} \boxtimes \mathcal{E} = \text{Rep}(G \times G) \Delta^* \to \text{Rep}(G) = \mathcal{E} \), the claim follows. \( \square \)

### 2.7. De-equivariantization

Retain the notation of 2.4-2.6. Assume \( \mathcal{D} \) contains \( \mathcal{E} := \text{Rep}(G) \) as a Tannakian subcategory.

Let \( A := O(G) \). The group scheme \( G \) acts on \( A \) via left translations, making \( A \) a commutative algebra in \( \mathcal{E} \). Thus, \( A \) is a commutative algebra in \( \mathcal{D} \), with \( \text{FPdim}(A) = |G| \).

Recall that \( \text{Mod}(A)_\mathcal{E} \) is equivalent to the standard \( \mathcal{E} \)-module category \( \text{Vec} \). Thus, \( \text{Mod}(A)_\mathcal{E} \) is exact, so \( A \) is exact in \( \mathcal{E} \). Thus by Theorem 2.1, \( A \) is exact in \( \mathcal{D} \), so \( \mathcal{C} := \text{Mod}(A)_\mathcal{D} \) and \( \mathcal{D}^* = \text{Bimod}_{\mathcal{D}}(A)^{\text{op}} \) are finite tensor categories (see 2.2).

Let \( \mathcal{C}^0 = \text{Mod}^0(A)_\mathcal{D} \) denote the subcategory of \( \mathcal{C} \) consisting of all objects \( (X, m_X) \) such that \( m_X = m_X \circ c_A,X \circ c_X,A \) (i.e., dyslectic, or local, \( A \)-modules). Recall that \( \mathcal{C}^0 \) is a tensor subcategory of \( \mathcal{C} \), which is moreover braided with the braiding inherited from \( \mathcal{D} \).

### 2.8. The categories \( \text{Coh}(G, \omega) \) and \( \mathcal{Z}(\text{Coh}(G, \omega)) \)

Retain the notation of 2.4-2.6 and let \( \omega \in H^3(G, \mathbb{G}_m) \) be a normalized 3-cocycle. That is, \( \omega \in O(G)^{\otimes 3} \) is an invertible element satisfying the equations

\[
(\text{Id} \otimes \text{Id} \otimes \Delta)(\omega)(\Delta \otimes \text{Id} \otimes \text{Id})(\omega) = (1 \otimes \omega)(\text{Id} \otimes \Delta \otimes \text{Id})(\omega)(\omega \otimes 1),
\]

\[
(\varepsilon \otimes \text{Id} \otimes \text{Id})(\omega) = (\text{Id} \otimes \varepsilon \otimes \text{Id})(\omega) = (\text{Id} \otimes \text{Id} \otimes \varepsilon)(\omega) = 1.
\]

Recall [G Section 5] that the category \( \text{Coh}(G, \omega) \) is just \( \text{Coh}(G) \) as abelian categories, equipped with the same tensor product, but with associativity constraint given by the action of \( \omega \). Then \( \text{Coh}(G, \omega) \) is a finite pointed (i.e., every simple object is invertible) tensor category, and the group of its invertible objects is isomorphic to \( G(k) \).

Let \( \mathcal{Z}(\text{Coh}(G, \omega)) \) denote the center of \( \text{Coh}(G, \omega) \) (see, e.g., [EGNO], [G Section 5]). By [S Theorem 4.2], \( \mathcal{Z}(\text{Coh}(G, \omega)) \) is a finite non-degenerate braided tensor category (see 2.4). Recall moreover that
$\mathcal{Z}(\text{Coh}(G,\omega)) \cong \text{Rep}(D^\omega(G))$ as braided tensor categories, where $D^\omega(G)$ is a quasi-Hopf algebra, called the twisted double of $G$.

Note that for every $\omega_1, \omega_2 \in H^3(G, \mathbb{G}_m)$, we have

\[(2.8) \quad \mathcal{Z}(\text{Coh}(G,\omega_1)) \boxtimes \mathcal{Z}(\text{Coh}(G,\omega_2)) \cong \mathcal{Z}(\text{Coh}(G \times G,\omega_1 \times \omega_2))\]

as braided tensor categories.

Let $F: \mathcal{Z}(\text{Coh}(G,\omega)) \to \text{Coh}(G,\omega)$ be the forgetful functor. The following lemma is well known.

**Lemma 2.5.** The tensor subcategory of $\mathcal{Z}(\text{Coh}(G,\omega))$ consisting of all objects mapped to $\text{Vec}$ under $F$ is tensor equivalent to $\text{Rep}(G)$. Thus, $\mathcal{Z}(\text{Coh}(G,\omega))$ canonically contains $\text{Rep}(G)$ as a Tannakian Lagrangian subcategory.

Now let $\mathcal{I}: \text{Coh}(G,\omega) \to \mathcal{Z}(\text{Coh}(G,\omega))$ be the adjoint functor of $F$.

**Lemma 2.6.** The following hold:

1. $\mathcal{I}(1) = \mathcal{O}(G)$ as objects of $\text{Rep}(G)$.
2. $\mathcal{I}(1) = \mathcal{O}(G)$ is an exact commutative algebra in $\mathcal{Z}(\text{Coh}(G,\omega))$.
3. $\mathcal{I}$ induces a tensor equivalence

   \[\mathcal{I}: \text{Coh}(G,\omega) \xrightarrow{\cong} \text{Mod}(\mathcal{O}(G))_{\mathcal{Z}(\text{Coh}(G,\omega))}.\]

4. $\mathcal{I} \circ F$ coincides with the free $\mathcal{O}(G)$-module functor

   \[F: \mathcal{Z}(\text{Coh}(G,\omega)) \to \text{Mod}(\mathcal{O}(G))_{\mathcal{Z}(\text{Coh}(G,\omega))}, \quad X \mapsto X \otimes \mathcal{O}(G).\]

**Proof.** (1) is clear, and since $\mathcal{O}(G)$ is an exact commutative algebra in $\text{Rep}(G)$ (see 2.7), (2) follows from Theorem 2.1. The proof of (3), (4) is now similar to [EGNO, Lemma 8.12.2].

2.9. The group $\mathcal{M}_{\text{ext}}(\text{Rep}(G))$. Fix a finite group scheme $G$ over $k$. Set $\mathcal{E} := \text{Rep}(G)$, and let $I(1) \in \mathcal{E} \boxtimes \mathcal{E}$ be as in Lemma 2.4.

Consider the set of pairs $(\mathcal{D},\iota)$, where $\mathcal{D}$ is a non-degenerate braided tensor category over $k$, and $\iota: \mathcal{E} \xrightarrow{1_1} \mathcal{E}$ is an injective braided tensor functor. We say that $(\mathcal{D}_1,\iota_1), (\mathcal{D}_2,\iota_2)$ are equivalent, if there exists a braided tensor equivalence $\psi: \mathcal{D}_1 \cong \mathcal{D}_2$ such that $\psi \circ \iota_1 = \iota_2$. Let $\mathcal{M}_{\text{ext}}(\mathcal{E})$ denote the set of all equivalence classes of pairs $(\mathcal{D},\iota)$.

Now for $(\mathcal{D}_1,\iota_1), (\mathcal{D}_2,\iota_2)$ in $\mathcal{M}_{\text{ext}}(\mathcal{E})$, let $B := (\iota_1 \boxtimes \iota_2)(I(1))$. Then $B$ is an exact commutative algebra in $\mathcal{D}_1 \boxtimes \mathcal{D}_2$. Consider the braided
tensor category $\mathcal{D} := \text{Mod}^0(B)_{g_1 \otimes g_2}$ (see [2.7]), and let $\iota : \mathcal{E} \hookrightarrow \mathcal{D}$ be the injective functor

$$\mathcal{E} = \text{Mod}(I(1))_{\mathcal{E} \otimes \mathcal{E}} \xrightarrow{\iota_1 \otimes \iota_2} \text{Mod}^0(B)_{g_1 \otimes g_2} = \mathcal{D}.$$  

Recall [LKW] that the product rule $(\mathcal{D}_1, \iota_1) \circledast (\mathcal{D}_2, \iota_2) = (\mathcal{D}, \iota)$ determines a commutative group structure on $\text{MExt}(\mathcal{E})$, with unit element $(\mathcal{Z}(\text{Coh}(G)), \iota_0)$, where $\iota_0 : \mathcal{E} \to \mathcal{Z}(\text{Coh}(G))$ is the inclusion functor (see Lemma 2.5).

3. The Proof of Theorem 1.1

Let $G$ be a finite $k$-group scheme, and let $\mathcal{E} := \text{Rep}(G)$ (see [2.6]).

Let $\mathcal{D}$ be a finite non-degenerate braided tensor category over $k$, containing $\mathcal{E}$ as a Lagrangian Tannakian subcategory (see [2.4]). By (2.4), we have $\text{FPdim}(\mathcal{D}) = |G|^2$ (see [2.7]).

3.1. The free module functor $F$. Let $A := O(G)$, and consider the $G$-de-equivariantization category $\mathcal{C} := \text{Mod}(A)$ (see [2.7]). Since $A$ is exact in $\mathcal{D}$, $\mathcal{C}$ is a finite tensor category. By Lemma 2.3, we have

$$\text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{D})/\text{FPdim}(A) = |G|^2/|G| = |G|.$$  

Let $F : \mathcal{D} \to \mathcal{C}$ be the free $A$-module functor [2.5], and let $\mathcal{E} \hookrightarrow \mathcal{D}$ be the inclusion functor.

Proposition 3.1. We have an exact sequence of tensor categories

$$\mathcal{E} \hookrightarrow \mathcal{D} \xrightarrow{F} \mathcal{C}$$

with respect to the standard $\mathcal{E}$-module category $\text{Vec}$ (see [2.5]).

Proof. For every $V = (V, \rho) \in \mathcal{E}$, we have $V \otimes A \cong V \otimes A$ in $\mathcal{E}$. Thus, $F(V) \cong V \otimes A$ lies in $\mathcal{C}$ (as $A$ is the unit object of $\mathcal{C}$), so $F$ maps $\mathcal{E}$ to $\text{Vec}$. Since by (3.1), $\text{FPdim}(\mathcal{D}) = \text{FPdim}(\mathcal{E})\text{FPdim}(\mathcal{C})$, the claim follows from Theorem 2.2. \n
Recall that $\mathcal{C}_e^* \cong \mathcal{C}_e^{\text{op}}$ as tensor categories. Thus, dualizing (3.2) with respect to the exact indecomposable $\mathcal{D}$-module category $\mathcal{C}$ [EG1, Theorem 4.1] (see [2.4]), we obtain an exact sequence

$$\mathcal{C}_e^{\text{op}} \xrightarrow{F^*} \mathcal{D}_e^* \xrightarrow{\iota_*} \mathcal{E}_e^* \boxtimes \text{End}(\mathcal{C})$$

of finite tensor categories with respect to the indecomposable exact $\mathcal{C}_e^{\text{op}}$-module category $\mathcal{C}$. Since $\text{End}(\mathcal{C}) \cong \text{End}(\mathcal{C})^{\text{op}}$ as monoidal categories, and $(\mathcal{E}_e)^{\text{op}}_{\text{vec}} = \mathcal{E}_e^* \cong \text{Coh}(G)$ as tensor categories (see, e.g.,
MINIMAL EXTENSIONS OF TANNAKIAN CATEGORIES

[EGNO, Example 7.9.11], we get an exact sequence

\[ C \xrightarrow{F^*} \mathcal{D}_e^{\text{op}} \xrightarrow{\iota^*} \text{Coh}(G) \boxtimes \text{End}(C) \]

of finite tensor categories with respect to the indecomposable exact $C$-module category $\mathcal{C}$. In particular, $\mathcal{C}$ can be identified with a tensor subcategory of $\mathcal{D}_e^{\text{op}}$ via $F^*$. Also, since $\mathcal{D}_e^{\text{op}} \cong \text{Bimod}_e(A)$ as tensor categories (see 2.2), $\mathcal{D}_e^{\text{op}}$ contains $\text{Coh}(G) = \mathcal{E}_e^* = \text{Bimod}_e(A)$ as a tensor subcategory.

**Lemma 3.2.** We have an equivalence of abelian categories

\[ \Psi : \mathcal{D}_e^{\text{op}} \cong \text{Coh}(G) \boxtimes \mathcal{C} \]

such that

\[ \Psi(X \otimes Y) = X \boxtimes Y, \ X \in \text{Coh}(G), \ Y \in \mathcal{C}. \]

In particular, $P_{\mathcal{D}_e^{\text{op}}}(1) = P_{\text{Coh}(G)}(1) \otimes P_{\mathcal{C}}(1)$.

**Proof.** Applying Theorem 2.2(3) to the exact sequence (3.3) provides the natural equivalence of abelian categories

\[ T : \mathcal{D}_e^{\text{op}} \boxtimes \mathcal{C} \cong \text{Coh}(G) \boxtimes \mathcal{C}, \]

and it is straightforward to verify that composing the natural equivalence $\mathcal{D}_e^{\text{op}} \cong \mathcal{D}_e^{\text{op}} \boxtimes \mathcal{C}$ with $T$ yields an abelian equivalence with the stated property, as claimed.

In particular, it follows from the above that for every simple objects $X \in \text{Coh}(G)$ and $Y \in \mathcal{C}$, the object $X \otimes Y$ is simple in $\mathcal{D}_e^{\text{op}}$, and that every simple object in $\mathcal{D}_e^{\text{op}}$ is of this form. Furthermore, since $P_{\text{Coh}(G)}(X) \boxtimes P_{\mathcal{C}}(Y)$ is the projective cover of $X \boxtimes Y$ in $\text{Coh}(G) \boxtimes \mathcal{C}$, and

\[ \Psi(P_{\text{Coh}(G)}(X) \boxtimes P_{\mathcal{C}}(Y)) = P_{\text{Coh}(G)}(X) \boxtimes P_{\mathcal{C}}(Y), \]

it follows that $P_{\text{Coh}(G)}(X) \boxtimes P_{\mathcal{C}}(Y)$ is the projective cover of $X \otimes Y$ in $\mathcal{D}_e^{\text{op}}$. Thus, we get the last statement for $X = Y = 1$. \(\square\)

### 3.2. The central structure on $F$

For every $X \in \mathcal{D}$, $(Y, m_Y) \in \mathcal{C}$, consider the morphism

\[ \Phi_{X,Y} : F(X) \otimes_A Y \cong Y \otimes_A F(X), \]

given by the composition

\[
\begin{align*}
F(X) \otimes_A Y &= (X \otimes A) \otimes_A Y \\
&\xrightarrow{\text{Id}_X \otimes (m_Y \circ_c Y)} X \otimes Y \\
&\xrightarrow{c_{X,Y} \otimes_A Y} Y \otimes X \\
&\xrightarrow{\text{Id}_Y \otimes_A \text{Id}_X} Y \otimes A \otimes X \\
&\xrightarrow{\text{Id}_Y \otimes_A \circ_A X} Y \otimes (X \otimes A) \\
&\xrightarrow{\Phi_{X,Y}} Y \otimes_A F(X).
\end{align*}
\]
Using that $A$ is commutative, the following lemma can be verified in a straightforward manner.

**Lemma 3.3.** The following hold:

1. $\{\Phi_{X,Y} \mid X \in \mathcal{D}, Y \in \mathcal{C}\}$ is a natural family of $\mathcal{C}$-isomorphisms.
2. For every $X, Z \in \mathcal{D}$ and $Y \in \mathcal{C}$, we have
   \[ \Phi_{X \otimes Z,Y} = (\text{Id}_Y \otimes J_{X,Z}^{-1}) \circ (\Phi_{X,Y} \otimes \text{Id}_{F(Z)}) \circ (\text{Id}_{F(X)} \otimes \Phi_{Z,Y}) \circ (J_{X,Z} \otimes \text{Id}_Y) \]
   (see (2.7)).
3. For every $X \in \mathcal{D}$ and $Y, Z \in \mathcal{C}$, we have
   \[ \Phi_{X,Y \otimes A Z} = (\text{Id}_Y \otimes \Phi_{X,Z}) \circ (\Phi_{X,Y} \otimes \text{Id}_Z). \]

In particular, Lemma 3.3 implies the following corollary.

**Corollary 3.4.** The free $A$-module functor $F$ (2.5) extends to a braided tensor functor

$$ F : \mathcal{D} \to \mathcal{Z}(\mathcal{C}) $$

such that $F$ coincides with the composition of $F$ and the forgetful tensor functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. \qed

Recall that $\mathcal{C}$ is an indecomposable exact module category over $\mathcal{Z}(\mathcal{C})$ via the forgetful functor $\mathcal{Z}(\mathcal{C}) : \mathcal{D} \to \mathcal{C}$, and $\mathcal{Z}(\mathcal{C})^* \cong \mathcal{C} \otimes \mathcal{C}^\text{op}$ as tensor categories (see, e.g., [EGNO, Theorem 7.16.1]).

Recall that since $\mathcal{D}$ is non-degenerate, $\mathcal{D} \otimes \mathcal{D}^\text{op} \cong \mathcal{Z}(\mathcal{D})$ as braided tensor categories. Recall also, that $\mathcal{Z}(\mathcal{D}) \cong \mathcal{Z}(\mathcal{D}^*)$ as braided tensor categories [EGNO, Corollary 7.16.2].

**Proposition 3.5.** [DGNO1, Proposition 4.2] The functor $F$ (3.5) is an equivalence of braided tensor categories.

**Proof.** Consider $\mathcal{C}$ as a module category over $\mathcal{D}$ and $\mathcal{Z}(\mathcal{C})$ via $F$ and the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, respectively. In both cases, $\mathcal{C}$ is indecomposable and exact. Let

$$ F^* : \mathcal{C} \otimes \mathcal{C}^\text{op} \to \mathcal{D}^* $$

be the dual functor to $F$. Recall that $\mathcal{D}^* \cong \text{Bimod}_{\mathcal{D}}(A)^\text{op}$ as tensor categories (see 2.2). Then it is straightforward to verify that we have

$$ F^*(X \otimes Y) = X_+ \otimes_A Y_-, \quad X, Y \in \mathcal{C} $$

(see 2.5).

In particular, we see that the functor

$$ \mathcal{D} \otimes \mathcal{D}^\text{op} \xrightarrow{F \otimes F} \mathcal{C} \otimes \mathcal{C}^\text{op} \xrightarrow{F^*} \mathcal{D}^* = \text{Bimod}_{\mathcal{D}}(A)^\text{op} $$
coincides with the functor
\[ \mathcal{D} \boxtimes \mathcal{D}^{\text{op}} \cong \mathcal{D}(\mathcal{D}) \cong \mathcal{D}(\mathcal{D}^*) \to \mathcal{D}^*. \]
Since the forgetful functor \( \mathcal{D}(\mathcal{D}^*) \to \mathcal{D}^* \) is surjective, we see that the functor \( F^* \) is surjective. Thus, \( F \) is injective [EGNO, Theorem 7.17.4].

Finally, since \( F \) is an injective tensor functor between categories of equal Frobenius-Perron dimension (see (3.1)), it is necessarily an equivalence [EGNO, Proposition 6.3.4], as claimed.

Following Proposition 3.5, we now aim to prove that there exists \( \omega \in H^3(G, \mathbb{G}_m) \) such that \( \mathcal{C} \cong \text{Coh}(G, \omega) \) as tensor categories.

3.3. \( \mathcal{O} (\mathcal{C}) = G(k) \). In this section we show that \( \mathcal{C} \) is pointed, and its group of invertible objects is isomorphic to \( G(k) \).

Recall that \( \text{Bimod}_E(A) = \mathcal{E}_{\text{vec}}^* \cong \text{Coh}(G) \) as tensor categories, and \( \mathcal{O} (\text{Coh}(G)) \cong G(k) \). Let \( A_g, g \in G(k) \), with \( A_e = A = 1 \), denote the invertible objects of \( \text{Coh}(G) = \text{Bimod}_E(A) \subset \text{Bimod}_F(A) \). Namely, \( A_g = A \) as right \( A \)-modules, and the left \( A \)-action is determined by \( g \in G(k) \).

Let \( G(\mathcal{C}) \) denote the group of invertible objects of \( \mathcal{C} \).

Proposition 3.6. The following hold:

1. The category \( \mathcal{C} \) is pointed (i.e., \( \mathcal{O} (\mathcal{C}) = G(\mathcal{C}) \)).
2. For every \( X \in G(\mathcal{C}) \), we have
\[ \text{FPdim}_\mathcal{C}(P_\mathcal{C}(X)) = \text{FPdim}_{\text{Coh}(G)}(P_{\text{Coh}(G)}(1)). \]
(See 2.1)
3. The tensor equivalence \( F^* \) \( (3.4) \) induces a group isomorphism
\[ G(k) \cong G(\mathcal{C}), \quad g \mapsto X_g. \]
4. The universal faithful group grading of \( \mathcal{C} \) is given by
\[ \mathcal{C} = \bigoplus_{g \in G(k)} \mathcal{C}_g, \]
where \( \mathcal{C}_g \subset \mathcal{C} \) denotes the smallest Serre subcategory of \( \mathcal{C} \) containing the invertible object \( X_g \) (see [EGNO Section 4.14]).

Proof. (1) Let \( X \) be simple in \( \mathcal{C} \). By [EGNO Proposition 6.3.1], \( X_+ \) is simple in \( \mathcal{D}^*_\mathcal{E} \) (see 2.5). Since \( X^* \) is simple in \( \mathcal{C} \), and \( F^* \) \( (3.6) \) is an equivalence, it follows from (3.7) that \( F^*(X \boxtimes X^*) = X_+ \otimes_A (X^*)_+ \) is simple in \( \mathcal{D}^*_\mathcal{E} \). Since
\[ \text{Hom}_{\mathcal{D}^*_\mathcal{E}}(X_+ \otimes_A (X^*)_+, A) = \text{Hom}_{\mathcal{D}^*_\mathcal{E}}(X_+, X_+) = k, \]
it follows that \( X_+ \otimes_A (X^*)_+ = A \) in \( \mathcal{D}^*_\mathcal{E} \), hence \( X \otimes_A X^* = A \) in \( \mathcal{C} \).
Thus, \( X \) is invertible in \( \mathcal{C} \), as claimed.
(2) Since the equivalence $F^\ast$ maps $P_{\mathscr{C}}(1)$ to $P_{\mathscr{E}}(1) \boxtimes P_{\mathscr{E}}(1)$, it follows that
\[
\text{FPdim}_{\mathscr{C}}(P_{\mathscr{C}}(1)) = \text{FPdim}_{\mathscr{E}}(P_{\mathscr{E}}(1))^2.
\]
On the other hand, by Lemma 3.2, we have
\[
\text{FPdim}_{\mathscr{C}}(P_{\mathscr{C}}(1)) = \text{FPdim}_{\text{Coh}(G)}(P_{\text{Coh}(G)}(1)) \text{FPdim}_{\mathscr{C}}(P_{\mathscr{C}}(1)).
\]
Therefore, we get
\[
\text{FPdim}_{\mathscr{E}}(P_{\mathscr{E}}(1)) = \text{FPdim}_{\text{Coh}(G)}(P_{\text{Coh}(G)}(1)) \text{FPdim}_{\mathscr{E}}(P_{\mathscr{E}}(1)).
\]
Finally, since for every invertible object $X \in G(\mathscr{C})$,
\[
\text{FPdim}_{\mathscr{C}}(P_{\mathscr{C}}(X)) = \text{FPdim}_{\text{Coh}(G)}(P_{\text{Coh}(G)}(X)) \text{FPdim}_{\mathscr{C}}(P_{\mathscr{C}}(1)),
\]
the claim follows.

(3) Since $F^\ast$ is an equivalence, there exist unique simple objects $X_g, Y_g \in \mathscr{C}$ such that
\[
(X_g)_+ \otimes_A (Y_g)_- = F^\ast(X_g \boxtimes Y_g) = A_g.
\]
Since $X_g, Y_g \in G(\mathscr{C})$ by (1), it follows that $(X_g)_+ = A_g \otimes_A (Y_g)_+$, so forgetting the left $A$-module structure yields $X_g = Y_g^\ast$. Thus, the functor $F^\ast$ induces an injective group homomorphism
\[
(3.8) \quad G(k) \xrightarrow{1:1} G(\mathscr{C}), \quad g \mapsto X_g.
\]
Now it follows from (1), (2) and 3.1 that
\[
|G| = \text{FPdim}(\mathscr{C})
= \sum_{X \in G(\mathscr{C})} \text{FPdim}(P_{\mathscr{C}}(X)) \geq \sum_{X_g \in G(\mathscr{C})} \text{FPdim}(P_{\mathscr{C}}(X_g))
= \sum_{g \in G(k)} \text{FPdim}(P_{\text{Coh}(G)}(1)) = \text{FPdim}(\text{Coh}(G)) = |G|.
\]
Therefore, we have an equality
\[
\text{FPdim}(\mathscr{C}) = \sum_{X_g \in G(\mathscr{C})} \text{FPdim}(P_{\mathscr{C}}(X_g)),
\]
which implies that the map (3.8) is also surjective, as claimed.

(4) By (1), $\mathscr{C}_{\text{ad}}$ is the smallest Serre tensor subcategory of $\mathscr{C}$ containing $1$ (see [EGNO] Section 4.14). Thus, (3) implies that $G(k)$ is the universal group of $\mathscr{C}$, as claimed. □

**Remark 3.7.** Note that if $G$ is an abstract group such that $p$ does not divide $|G|$, then Theorem [1.1] and hence Corollaries [1.2] and [1.3] follow already from Proposition 3.6 (see [DGNO1] Theorem 4.5) and [LKW, Theorem 4.22]).
3.4. $\mathcal{C} \cong \text{Coh}(G, \omega)$. In this section we show that $\mathcal{C} \cong \text{Coh}(G, \omega)$ as tensor categories.

For every $(X, m_X) \in \mathcal{C}$, $V = (V, \rho) \in \text{Rep}(G) \subset \mathcal{D}$, consider the $\mathcal{C}$-isomorphism

$$\Phi_{V,X} : F(V) \otimes_A X \xrightarrow{\cong} X \otimes_A F(V),$$

given by the central structure on $F$ (3.4):

$$\Phi_{V,X} = (\text{Id}_X \otimes c^{-1}_A, V) \circ (\text{Id}_X \otimes u_A \otimes \text{Id}_V) \circ c_{V,X} \circ (\text{Id}_V \otimes (m_X \circ c^{-1}_{X,A})) .$$

Since by Proposition 3.1, $F(V) = V$, we obtain a family of $\mathcal{C}$-isomorphisms

$$\Phi_{V,X} : V \otimes X \xrightarrow{\cong} V \otimes X, \quad V \in \text{Vec},$$

given by

$$V \otimes X = V \otimes A \otimes_A X \xrightarrow{\Phi_{V,X}} X \otimes_A V \otimes A = X \otimes V = V \otimes X .$$

Thus by (3.10), we can view $\Phi_{V,X}$ as an element of

$$\text{Hom}_\mathcal{C}(X, (V^* \otimes_k V) \otimes X) \cong \text{Hom}_\mathcal{C}(X, \text{End}_k(V) \otimes X),$$

so by Lemma 3.3 and the fact that $\text{End}(F) = kG$ as Hopf algebras, we obtain a natural family $\Phi := \{\Phi_X | X \in \mathcal{C}\}$ of $\mathcal{C}$-morphisms

$$\Phi_X : (X, m_X) \rightarrow (kG \otimes X, \text{Id}_{kG} \otimes m_X),$$

or equivalently, $k$-algebra maps

$$\Phi_X : \mathcal{O}(G) \rightarrow \text{End}_\mathcal{C}(X).$$

**Lemma 3.8.** For every $X \in \mathcal{C}$, the morphism $\Phi_X$ (3.11) equips $X$ with a structure of a $kG$-comodule in $\mathcal{C}$ (equivalently, an $\mathcal{O}(G)$-module in $\mathcal{C}$).

**Proof.** By Proposition 3.1 and Lemma 3.3(2), for every $V, U \in \text{Vec}$ and $X \in \mathcal{C}$, we have

$$\Phi_{V \otimes U, X} = (\Phi_{V,X} \otimes \text{Id}_U) \circ (\text{Id}_V \otimes \Phi_{U,X}).$$

Now it is straightforward to verify that this translates to the claim. \qed

Now by Proposition 3.6(1) and [EO1 Proposition 2.6], $\mathcal{C}$ admits a quasi-tensor functor to $\text{Vec}$. Let $Q : \mathcal{C} \rightarrow \text{Vec}$ be such a functor. Then by Lemma 3.8, for every $X \in \mathcal{C}$, we get a $k$-algebra map

$$\tilde{\Phi}_X : \mathcal{O}(G) \xrightarrow{\Phi_X} \text{End}_\mathcal{C}(X) \xrightarrow{Q} \text{End}_k(Q(X)),$$

so we see that we have defined a functor

$$\tilde{\Phi} : \mathcal{C} \rightarrow \text{Coh}(G), \quad X \mapsto (Q(X), \tilde{\Phi}_X).$$
Lemma 3.9. For every $g \in G(k)$, $\tilde{\Phi}_{X_g} = g$ (where $g$ is viewed as an $\mathcal{O}(G)$-module). Thus, $\tilde{\Phi}$ induces an injective group homomorphism $G(\mathcal{C}) \hookrightarrow G(k)$.

Proof. The first claim follows from Proposition 3.6 and Lemma 3.8. Also, it is straightforward to verify that for every $g \in G(k)$, the map $\Phi_{V,X}g \in \text{Aut}_\mathcal{C}(V \otimes X_g)$ is given by $\rho(g) \otimes \text{Id}_{X_g}$ for every $V \in \text{Vec}$, which proves the second claim.

Corollary 3.10. The functor $\tilde{\Phi}$ is a quasi-tensor equivalence.

Proof. By Lemma 3.3(3), we have

$$\Phi_{V,X \otimes Y} = (\text{Id}_V \otimes \Phi_{V,Y}) \circ (\Phi_{V,X} \otimes \text{Id}_Y)$$

for every $V \in \text{Vec}$ and $X,Y \in \mathcal{C}$. Now it is straightforward to verify that this implies that $\tilde{\Phi}(A) = (k, \tilde{\Phi}_A)$ is the unit object (since $\text{End}_\mathcal{C}(A) = k$, so $\tilde{\Phi}_A : \mathcal{O}(G) \to k$ is the trivial homomorphism), and we have an isomorphism

$$\Phi(X \otimes Y) = (X \otimes Y, \Phi_{X \otimes Y}) \cong (X, \Phi_X) \otimes (Y, \Phi_Y) = \Phi(X) \otimes \Phi(Y)$$

for every $X,Y \in \mathcal{C}$. Thus, $\Phi$ is a quasi-tensor functor, and hence so is $\tilde{\Phi}$.

Moreover, by Proposition 3.6(3) and Lemma 3.8, $\tilde{\Phi}$ induces a group isomorphism $G(\mathcal{C}) \cong G(k)$, which implies it is an equivalence.

Finally, by Corollary 3.10, the associativity structure on $\mathcal{C}$ determines an associativity structure on $\text{Coh}(G)$, i.e., a class $\omega \in H^3(G, \mathbb{G}_m)$, such that $\tilde{\Phi} : \mathcal{C} \to \text{Coh}(G, \omega)$ is a tensor equivalence, as desired.

This completes the proof of Theorem 1.1.

4. The proof of Corollary 1.2

Consider a braided tensor equivalence $F : \mathcal{D} \cong \mathcal{Z}(\text{Coh}(G, \omega))$, and let $\mathcal{F} : \mathcal{Z}(\text{Coh}(G, \omega)) \to \text{Coh}(G, \omega)$ be the forgetful functor. Let $f(G, \omega, F)$ denote the subcategory of $\mathcal{D}$ consisting of all objects sent to $\text{Vec}$ under $F \circ F$. In other words, $f(G, \omega, F)$ is the preimage of $\text{Rep}(G)$ under $F$ (see Lemma 2.5). Then $f(G, \omega, F)$ is a Tannakian Lagrangian subcategory of $\mathcal{D}$, which is clearly independent of the equivalence class of $(G, \omega, F)$. Thus, every equivalence class $[(G, \omega, F)]$ gives rise to a Tannakian Lagrangian subcategory $f([G, \omega, F]) := f(G, \omega, F)$ of $\mathcal{D}$.

Conversely, assume $\text{Rep}(G) \subset \mathcal{D}$ is a Tannakian Lagrangian subcategory of $\mathcal{D}$ for some finite group scheme $G$ over $k$. Then by Theorem 1.1 we have a braided tensor equivalence $F : \mathcal{D} \cong \mathcal{Z}(\text{Coh}(G, \omega))$ for
some \( \omega \in H^3(G, \mathbb{G}_m) \). Thus, every Tannakian Lagrangian subcategory \( \mathcal{O}(G) \) of \( \mathcal{D} \) gives rise to an equivalence class \([G, \omega, F]\). Set \( h(\mathcal{O}(G)) := [(G, \omega, F)] \).

We claim that the assignments \( f, h \) constructed above are inverse to each other.

Given a Tannakian Lagrangian subcategory \( \mathcal{E} = \mathcal{O}(G) \) of \( \mathcal{D} \), let \( A := \mathcal{O}(G) \) and \( \mathcal{C} := \text{Mod}(A)_\mathcal{D} \) (see [2.3]). Since by Corollary 3.31 the functor \( \mathcal{D} \to \mathcal{O}(\text{Coh}(G_1, \omega_2)) \) coincides with the free \( A \)-module functor, it follows that the category \( f(h(\mathcal{E})) = f(G, \omega, F) \) consists of all objects \( X \) in \( \mathcal{D} \) such that \( X \otimes A \) is a multiple of \( A \). Since \( A \) is the regular object of \( \mathcal{E} \), it follows that \( \mathcal{E} \subset f(h(\mathcal{E})) \), so \( \mathcal{E} = f(h(\mathcal{E})) \) (as both categories have \( FP \) dimension \(|G|\)). Thus, \( f \circ h = \text{Id} \), as desired.

Given an equivalence class \([G_1, \omega_1, F_1]\), we have to show now that \((h \circ f)([G_1, \omega_1, F_1]) = [(G_2, \omega_2, F_2)]\), where \( F_2 : \mathcal{D} \cong \mathcal{O}(\text{Coh}(G_2, \omega_2)) \) is a braided tensor equivalence for some \( \omega_2 \in H^3(G_2, \mathbb{G}_m) \). Since \( F_1(\mathcal{O}(G_2)) = \mathcal{O}(G_1) \), it follows that \( F_1 \) induces a tensor equivalence

\[
F_1 : \text{Mod}(\mathcal{O}(G_2))_\mathcal{D} \xrightarrow{\cong} \text{Mod}(\mathcal{O}(G_1))_\mathcal{D}.
\]

Also, \( F_2 \) induces a tensor equivalence

\[
F_2 : \text{Mod}(\mathcal{O}(G_2))_\mathcal{D} \xrightarrow{\cong} \text{Mod}(\mathcal{O}(G_2))_\mathcal{D}.
\]

Now, let

\[
\mathcal{I}_i : \text{Coh}(G_i, \omega_i) \xrightarrow{\cong} \text{Mod}(\mathcal{O}(G_i))_\mathcal{D} \quad i = 1, 2
\]

be the tensor equivalences given by Lemma 2.63, and consider the tensor equivalence

\[
\varphi := \mathcal{I}_2^{-1} \circ F_2 \circ F_1^{-1} \circ \mathcal{I}_1 : \text{Coh}(G_1, \omega_1) \xrightarrow{\cong} \text{Coh}(G_2, \omega_2).
\]

Then we have \( \mathcal{F}_1 \circ F_2 = \varphi \circ \mathcal{F}_1 \circ F_1 \), where

\[
\mathcal{F}_i : \mathcal{D}(\text{Coh}(G_i, \omega_i)) \to \text{Coh}(G_i, \omega_i) \quad i = 1, 2
\]

are the forgetful functors. Thus, \([G_1, \omega_1, F_1] = [(G_2, \omega_2, F_2)]\), so we have \( h \circ f = \text{Id} \), as desired.

The proof of Corollary 3.2 is complete. \( \square \)
5. The proof of Corollary 1.3

Fix a finite group scheme $G$ over $k$. Set $\mathcal{E} := \text{Rep}(G)$, and retain the notation of 2.9.

By Theorem 1.1, we have a surjective map of sets

$$\alpha : \mathcal{M}_{\text{ext}}(\mathcal{E}) \cong H^3(G, \mathbb{G}_m), (\mathcal{D}, \iota) \mapsto \omega,$$

where $\omega \in H^3(G, \mathbb{G}_m)$ is such that $\mathcal{D} \cong \mathcal{Z}(\text{Coh}(G, \omega))$ as braided tensor categories. Moreover, it is clear that $\alpha(\mathcal{D}, \iota) = 1$ if and only if $\mathcal{D} \cong \mathcal{Z}(\text{Coh}(G))$ as braided tensor categories. Thus, it remains to show that $\alpha$ is a group homomorphism.

Set $A := O(G \times G)$ and $B := O(G \times G/\Delta(G))$ (see Lemma 2.4). Then $B \subset A$ are exact commutative algebras in the Tannakian category $\text{Rep}(G \times G) = \text{Rep}(G) \boxtimes \text{Rep}(G)$, and we have to prove that

$$\text{Mod}^0(B) \mathcal{Z}(\text{Coh}(G, \omega_1 \omega_2)) \cong \mathcal{Z}(\text{Coh}(G, \omega_1 \omega_2))$$

as braided tensor categories. By (2.8), it suffices to prove that

$$\text{Mod}^0(B) \mathcal{Z}(\text{Coh}(G \times G, \omega_1 \omega_2)) \cong \mathcal{Z}(\text{Coh}(G, \omega_1 \omega_2))$$

as braided tensor categories.

To this end, note that similarly to [DMNO, Example 4.11(ii)], one shows that exact subalgebras $B$ of $A$ are in one to one correspondence with tensor subcategories of $\mathcal{Z}(\text{Coh}(G \times G, \omega_1 \omega_2))$, such that $B$ corresponds to the image of the functor

$$\text{Mod}^0(B) \mathcal{Z}(\text{Coh}(G \times G, \omega_1 \omega_2)) \leftarrow \mathcal{Z}(\text{Coh}(G, \omega_1 \omega_2)).$$

Since the image of the functor (5.1) is

$$\text{Coh}(\Delta(G), (\omega_1 \omega_2)_{|\Delta(G)}) \cong \text{Coh}(G, \omega_1 \omega_2),$$

we are done.

6. Examples

Recall that a finite group scheme $G$ over $k$ is commutative if and only if $\mathcal{O}(G)$ is a finite dimensional commutative and cocommutative Hopf algebra. Thus, if $G$ is a finite commutative group scheme over $k$ then its group algebra $kG := \mathcal{O}(G)^*$ is also a finite dimensional commutative and cocommutative Hopf algebra, so it represents a finite commutative group scheme $G^D$ over $k$, which is called the Cartier dual of $G$.

Example 6.1. Let $G$ be a finite abelian $p$-group. Then by [EG2, Corollary 5.8], $H^3(G^D, \mathbb{G}_m) = 1$, thus by Corollary 1.3, $\mathcal{M}_{\text{ext}}(\text{Rep}(G^D)) = 1$ is the trivial group. For example, if $G = \mathbb{Z}/p\mathbb{Z}$ then $G^D = \mu_p$ is the Frobenius kernel of the multiplicative group $\mathbb{G}_m$ (see, e.g., [G, Section 2.2]), so $\mathcal{M}_{\text{ext}}(\text{Rep}(\mu_p)) = 1$. 
Example 6.2. Let $\alpha_{p,r}$ denote the $r$-th Frobenius kernel of the additive group $\mathbb{G}_a$ (see, e.g., [EG2, Section 2.6]). Let $G := \prod_{i=1}^n \alpha_{p,r_i}$. Then by [EG2, Corollary 5.10], we have $H^3(G^D, \mathbb{G}_m) = H^3(G^D, \mathbb{G}_a)$. Thus by Corollary 1.3, we have a group isomorphism

$$\mathcal{M}_{\text{ext}}(\text{Rep}(G^D)) = H^3(G^D, \mathbb{G}_a).$$

For example, if $G = \alpha_p^n$ then $(\alpha_p^n)^D = \alpha_p^n$, so we have

$$\mathcal{M}_{\text{ext}}(\text{Rep}(\alpha_p^n)) = H^3(\alpha_p^n, \mathbb{G}_a).$$

Thus by [EG2, Proposition 2.2], we have

$$\mathcal{M}_{\text{ext}}(\text{Rep}(\alpha_p^n)) \cong \wedge^3 \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}^{(1)}), \ p > 2,$$

where $\mathfrak{g} := \text{Lie}(\alpha_p^n)$ and $\mathfrak{g}^{(1)}$ is the Frobenius twist of $\mathfrak{g}$.

Example 6.3. Let $\mathfrak{g}$ be a finite dimensional restricted $p$-Lie algebra over $k$, and let $u(\mathfrak{g})$ be its restricted universal enveloping algebra (see, e.g., [C, Section 2.2]). Since $u(\mathfrak{g})^*$ is a finite dimensional commutative Hopf algebra over $k$, $u(\mathfrak{g})^* = \mathcal{O}(\Gamma)$ is the coordinate algebra of a finite group scheme $\Gamma$ over $k$. Recall that $\mathfrak{g}$ is the Lie algebra of $\Gamma$, and $\text{Rep}(\Gamma) \cong \text{Rep}(u(\mathfrak{g}))$ as symmetric tensor categories. Now by [EG2, Theorem 5.4], we have $H^3(\Gamma, \mathbb{G}_m) = H^3(\Gamma, \mathbb{G}_a)$. Thus by Corollary 1.3, we have a group isomorphism

$$\mathcal{M}_{\text{ext}}(\text{Rep}(\Gamma)) = H^3(\Gamma, \mathbb{G}_a).$$

For example, if $\mathfrak{g}$ is semisimple (i.e., $\mathfrak{g}$ is the Lie algebra of a simple, simply connected algebraic group defined and split over $\mathbb{F}_p$) then by [FP, Corollary 1.6], $H^\bullet(\Gamma, \mathbb{G}_a)$ is zero in odd degrees. Thus, by (6.1), we have $\mathcal{M}_{\text{ext}}(\text{Rep}(\Gamma)) = H^3(\Gamma, \mathbb{G}_a) = 1$ is trivial in this case.

Example 6.4. Let $\mathfrak{g}$ be a semisimple restricted $p$-Lie algebra over $k$ (see Example 6.3). Let $\mathfrak{t}$ be the 1-dimensional abelian restricted $p$-Lie algebra over $k$ with $u(\mathfrak{t})^* = \mathcal{O}(\alpha_p) = k[x]/x^p$, where $x$ is a primitive element. Recall that $H^1(\alpha_p, \mathbb{G}_a) = \text{sp}_k \{ x \}$.

Consider now the restricted $p$-Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{t}$, and let $\tilde{\Gamma}$ be the finite group scheme over $k$ such that $u(\tilde{\mathfrak{g}})^* = \mathcal{O}(\tilde{\Gamma})$. Then $\tilde{\Gamma} = \Gamma \times \alpha_p$, and the cup product induces an isomorphism

$$H^2(\Gamma, \mathbb{G}_a) \otimes H^1(\alpha_p, \mathbb{G}_a) \xrightarrow{\cong} H^3(\tilde{\Gamma}, \mathbb{G}_a).$$

3Namely, $\mathfrak{g}^{(1)}$ is an abelian group, and $a \in k$ acts on $\mathfrak{g}^{(1)}$ as $a^{p^{-1}}$ does on $\mathfrak{g}$.

4I.e., $\Delta(x) = x \otimes 1 + 1 \otimes x$. 


Equivalently, since $x$ is a basis of $H^1(\alpha_p, G_a)$, we have an isomorphism

$$H^2(\Gamma, G_a) \xrightarrow{\cong} H^2(\tilde{\Gamma}, G_a), \quad \xi \mapsto \xi \otimes x.$$ 

Now recall (see, e.g., [FP]) that for $p \neq 2, 3$, we have an isomorphism

$$\tau : g^{*(1)} \xrightarrow{\cong} H^2(\Gamma, G_a), \quad f \mapsto g_f,$$

where $g^{*(1)}$ is the Frobenius twist of $g^*\tilde{\Gamma}$, and $g_f = g \oplus k$ (as an abstract Lie algebra) with $[p]$-operator defined by $(v, a)^{[p]} := (v^{[p]}, f(v))$. Thus, we obtain an isomorphism

$$\Omega : g^{*(1)} \xrightarrow{\cong} H^3(\tilde{\Gamma}, G_a), \quad f \mapsto \tau(f) \otimes x.$$ 

In particular, by (6.1), we have

$$\mathcal{M}_{\text{ext}}(\text{Rep}(\tilde{\Gamma})) \cong g^{*(1)}.$$ 

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5Namely, $g^{*(1)}$ is the space of semi-linear maps $g \to k$. 
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