WEYL MODULES FOR $\mathfrak{osp}(1,2)$ AND NONSYMMETRIC MACDONALD POLYNOMIALS

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Abstract. The main goal of our paper is to establish a connection between the Weyl modules of the current Lie superalgebras (twisted and untwisted) attached to $\mathfrak{osp}(1,2)$ and the nonsymmetric Macdonald polynomials of types $A_2^{(2)}$ and $A_2^{(2)*}$. We compute the dimensions and construct bases of the Weyl modules. We also derive explicit formulas for the $t = 0$ and $t = \infty$ specializations of the nonsymmetric Macdonald polynomials. We show that the specializations can be described in terms of the Lie superalgebras action on the Weyl modules.

Introduction

The Weyl modules play important role in the representation theory of infinite-dimensional Lie algebras (see [CP, CL, FoLi1, FeLo1]). These representations pop up in various problems of representation theory and its applications [CFK, FeLo2, FoLi1, FoLi2, SVV]. The most important for us is the established in many cases isomorphism between the Weyl modules and Demazure modules (at least, in types $A, D, E$) for integrable representations of the affine Kac-Moody algebras. The Demazure modules are known to provide finite-dimensional approximation of the infinite-dimensional modules of the affine Kac-Moody Lie algebras and hence so do the Weyl modules. Yet another consequence from the link between the Demazure and Weyl modules is that the characters of both (in the level one case) can be expressed in terms of the nonsymmetric Macdonald polynomials (see [I, S]). The nonsymmetric Macdonald polynomials (see [M1, M2]) are rational functions in parameters $q$ and $t$ and the characters of the Demazure modules are equal to the $t = 0$ specialization. It was conjectured recently (see [CO1, CO2, FM]) that the $t = \infty$ specialization also has representation theoretic realization in terms of the PBW filtration (see [FFL1, FFL2] for the case of simple Lie algebras).

Now let us turn to the case of superalgebras. It is not clear what should be an appropriate definition of a Demazure module for superalgebras. However, the Weyl modules are perfectly well defined (see e.g. [CLS]). So there are two natural questions here. The first one is to compute the characters of the Weyl modules for affine superalgebras and to find a connection with some super analogues of the nonsymmetric Macdonald polynomials. The second question is to figure out if a limit of the Weyl modules (when the highest weight grows) does exist. In this paper we consider the smallest Lie
superalgebra $\mathfrak{osp}(1, 2)$, which plays in the super theory a role similar to that of the Lie algebra $\mathfrak{sl}_2$ in the theory of classical simple Lie algebras. The Weyl modules in this case are parametrized by a non-negative integer $n$; we denote the corresponding $\mathfrak{osp}(1, 2) \otimes \mathbb{C}[t]$ module by $W_{-n}$. We prove the following theorem.

**Theorem 0.1.** $W_{-n}$ as $\mathfrak{osp}(1, 2)$-module is isomorphic to the tensor product of $n$ copies of $3$-dimensional irreducible $\mathfrak{osp}(1, 2)$-module. Moreover, the $\mathfrak{osp}(1, 2) \otimes \mathbb{C}[t]$-module structure is given by the graded tensor product (fusion product) construction.

We show that $W_{-n}$ can be filtered by the Weyl modules for $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$. This allows us to construct bases and compute the characters of $W_{-n}$. Our next goal is to relate the characters of the Weyl modules to the nonsymmetric Macdonald polynomials ([Ch1, Ch2]). Using the Ram-Yip formula for the nonsymmetric Macdonald polynomials (see [RY, OS]), we prove the following theorem.

**Theorem 0.2.** Let $E_{n}^{A_{2}^{(2)\dagger}}(x, q, t)$ be the nonsymmetric Macdonald polynomials of type $A_{2}^{(2)\dagger}$. Then the character of $W_{-n}$ is given by $E_{n}^{A_{2}^{(2)\dagger}}(x, q, 0)$ and the specialization $E_{n}^{A_{2}^{(2)\dagger}}(x, q, \infty)$ coincides with the PBW twisted character of $W_{-n}$.

We close the introduction with several remarks.

First, the $\mathfrak{osp}(1, 2)$ current algebra has the twisted analogue $\mathfrak{osp}(1, 2)[t]^{\sigma}$. We study all the questions described above in the twisted case as well. In particular, we establish a connection with the specializations of the nonsymmetric Macdonald polynomials of type $A_{2}^{(2)}$. We note that both algebras $\mathfrak{osp}(1, 2)[t]$ and $\mathfrak{osp}(1, 2)[t]^{\sigma}$ are Borel’s subalgebras in the affine superalgebra $\hat{\mathfrak{osp}}(1, 2)$.

Second, in both twisted and untwisted cases we define the positive $n$ versions of the Weyl modules $W_n$. We also make a link to the Macdonald polynomials.

Third, we show that there exist embeddings of $\mathfrak{osp}(1, 2) \otimes \mathbb{C}[t]$-modules $W_{-n} \subset W_{-n-1}$ and we compute the character of the (infinite-dimensional) limit. We note that we do not know if there is a structure of the representation of a larger algebra on this limit.

Finally, let us mention that in the Macdonald polynomials part of our paper we follow the ideas and methods of the paper [OS]. In Appendix A we describe the most important for us ingredients of the approach of Orr and Shimozono.

The paper is organized as follows: In Section 1 we study the Weyl modules for $\mathfrak{osp}(1, 2) \otimes \mathbb{C}[t]$ and their twisted version.
In Section 2 we derive explicit formulas for the types $A_2^{(2)}$ and $A_2^{(2)\dagger}$ Macdonald polynomials.
In Section 3 we establish a connection between the two parts of the story.

1. Weyl modules

1.1. The classical case. Let $D_{-n}, n \geq 0$ be the Weyl modules for the current algebra $\mathfrak{sl}_2[t] = \mathfrak{sl}_2 \otimes \mathbb{C}[t]$. They are defined as finite-dimensional cyclic modules with cyclic vector $d_{-n}$, subject to the conditions:

$$\begin{align*}
hd_{-n} &= -nd_{-n}, \quad f \otimes \mathbb{C}[t]d_{-n} = 0, \quad h \otimes t\mathbb{C}[t]d_{-n} = 0,
\end{align*}$$

where $e, h, f$ for the standard basis of $\mathfrak{sl}_2$. These modules are known to be 2$^n$-dimensional with a monomial basis $e_{a_1} \cdots e_{a_k}d_{-n}$, $0 \leq a_1 \leq \cdots \leq a_k \leq n - k$.

For a graded vector space $M = \bigoplus_{s \geq 0} M_s$ with an action of the operator $h$ we define the character $\text{ch} M(x, q)$ as $\sum_{s \geq 0} q^s \text{tr}(x^s|_{M_s})$. The character of $D_{-n}$ is equal to $\sum_{k=0}^{n} x^{−n+2k} \binom{n}{k}/q^k$, where the $q$-binomial coefficients are given by the formula

$$\binom{n}{m}_q = \frac{(1-q)(1-q^2) \cdots (1-q^n)}{(1-q)(1-q^m)(1-q) \cdots (1-q^{n-m})}.$$

The modules $D_{-n}$ are known to be isomorphic to the graded tensor product (the fusion product $[\text{FeLo1}]$) of $n$ copies of standard 2-dimensional $\mathfrak{sl}_2$-modules. Moreover, $D_{-n}$ is isomorphic to a Demazure module in the basic level one representation of the affine Lie algebra $\hat{\mathfrak{sl}}_2$. In particular, there exist embeddings of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-modules

$$D_0 \subset D_{-2} \subset D_{-4} \subset \ldots, \quad D_{-1} \subset D_{-3} \subset D_{-5} \subset \ldots$$

and the inductive limits are isomorphic to the integrable $\hat{\mathfrak{sl}}_2$ modules of level 1. We have the explicit formula for the characters of the limits:

$$\begin{align*}
\text{ch} \lim_{n \to \infty} D_{-2n} &= \sum_{k \in \mathbb{Z}} x^{2k} q^{k^2}/(q)^{k} \\
\text{ch} \lim_{n \to \infty} D_{-2n-1} &= \sum_{k \in \mathbb{Z}} x^{2k+1} q^{(k+1)}/(q)^{k+1}.
\end{align*}$$

1.2. Weyl modules for superalgebras. Our references here are [P, Mus1, Mus2]. The Lie superalgebra $\mathfrak{osp}(1, 2)$ is isomorphic to the direct sum $\mathfrak{sl}_2 \oplus \pi_1$, where $\mathfrak{sl}_2$ is the even part and the two-dimensional $\mathfrak{sl}_2$ module $\pi_1$ if the odd part. Let $e, h, f$ be the standard basis of $\mathfrak{sl}_2$ and let $g^+, g^-$ be the basis of $\pi_1$. One has the nontrivial commutation relations

$$\begin{align*}
[e, f] &= h, \quad [h, e] = 2e, \quad [h, f] = -2f, \\
[h, g^+] &= g^+, \quad [h, g^-] = -g^-, \quad [g^+, g^-]_+ = h, \\
[g^+, g^+]_+ &= 2e, \quad [g^-, g^-]_+ = -2f, \quad [f, g^+] = g^-, \quad [e, g^-] = -g^+.
\end{align*}$$

We have the Cartan decomposition

$$\mathfrak{osp}(1, 2) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{n}^- = \text{span}(f, g^-), \quad \mathfrak{n}^+ = \text{span}(e, g^+), \quad \mathfrak{h} = \text{span}(h).$$
We consider the current algebra $\mathfrak{osp}(1, 2)[t] = \mathfrak{osp}(1, 2) \otimes \mathbb{C}[t]$ and its Weyl module $W_{-n}$ defined as the cyclic module with a generator $w_{-n}$ subject to the relations

\[(n^- \oplus h) \otimes t \mathbb{C}[t]. w_{-n} = 0, \quad (n^- \otimes 1). w_{-n} = 0, \quad h_0. w_{-n} = -nw_{-n}, \quad (e \otimes 1)^{n+1}. w_{-n} = 0.\]

For $x \in \mathfrak{osp}(1, 2)$ we denote by $x_i \in \mathfrak{osp}(1, 2)[t]$ the element $x \otimes t^i$.

**Lemma 1.1.** One has $\mathfrak{osp}(1, 2) \otimes t^n \mathbb{C}[t]. w_{-n} = 0$ and $W_{-n}$ is spanned by the monomials of the form

\[
ed_1 \cdots e_a g_{b_1}^+ \cdots g_{b_k}^+ . w_{-n},
\]

\[0 \leq b_1 < \cdots < b_k \leq n - 1, \quad 0 \leq a_1 \leq \cdots \leq a_s \leq n - k - s.\]

**Proof.** The condition $\mathfrak{sl}_2 \otimes t^n \mathbb{C}[t]. w_{-n} = 0$ follows from the results on the Weyl modules for $\mathfrak{sl}_2$ (see e.g. [CL]). Now if $e_i w_{-n} = 0$, then $g_0^- e_i w_{-n} = g_i^+ w_{-n} = 0$ and similarly for $g_i^- w_{-n}$.

We note that since $[g_i^+, g_j^+] = 2e_{i+j}$, we have

\[W_{-n} = \sum_{0 \leq b_1 < \cdots < b_k \leq n - 1} U(\mathfrak{sl}_2 \otimes \mathbb{C}[t]) g_{b_1}^+ \cdots g_{b_k}^+ . w_{-n}.\]

We introduce a partial order on the monomials $g_{b_1}^+ \cdots g_{b_k}^+$, $0 \leq b_1 < \cdots < b_k \leq n - 1$, saying that for two different monomials $g_{b_1}^+ \cdots g_{b_k}^+ < g_{c_1}^+ \cdots g_{c_l}^+$ if $k > l$ or ($k = l$ and $b_i \geq c_i$, $i = 1, \ldots, k$). Let us totally order the monomials $g_{b_1}^+ \cdots g_{b_k}^+$, $0 \leq b_1 < \cdots < b_k \leq n - 1$ as $m_1, m_2, \ldots, m_N$ in such a way that if $i < j$ then $m_i < m_j$ (in the sense of the partial order above).

Now let us introduce an increasing filtration $F_i$ on $W_{-n}$ by

\[F_i = \sum_{j=1}^{i} U(\mathfrak{sl}_2 \otimes \mathbb{C}[t]) m_i w_{-n}.\]

We claim that the monomials \((\square)\) span the associated graded space $\text{gr} F_\bullet$. Indeed, let $m_i w_{-n}$ by the image of $m_i w_{-n}$ in the associated graded space. Then for $m_i = g_{b_1}^+ \cdots g_{b_k}^+$ we have

\[(n^- \oplus h) \otimes t \mathbb{C}[t]. m_i w_{-n} = 0, \quad (n^- \otimes 1). m_i w_{-n} = 0, \quad (h \otimes 1). m_i w_{-n} = -(n+k)w_{-n}.\]

Hence all the relations for the $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ Weyl module with lowest weight $-n + k$ are satisfied, which implies the claim of the lemma. \(\square\)

**Lemma 1.2.** The number of elements in the set \((\square)\) is equal to $3^n$ and its character is given by

\[
\sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k} \sum_{s=0}^{n-k} x^{-n+k+2s} \binom{n-k}{s} q^s.
\]

**Proof.** Straightforward. \(\square\)
Now let us give the twisted version of the lemma above. We replace the current algebra \( \mathfrak{osp}(1, 2)[t] \) with its twisted analogue

\[
\mathfrak{osp}(1, 2)[t]^{\sigma} = \bigoplus_{i=0}^\infty \mathfrak{sl}_2 \otimes t^{2i} \oplus \bigoplus_{i=0}^\infty \pi_1 \otimes t^{2i+1}.
\]

This is again a Lie superalgebra. We define its Weyl module

\[
\text{where } a \text{ being the odd part. For a }
\]

and \( h_0, w_{-n}^\sigma = -nw_{-n}^\sigma, e_0^{n+1}w_{-n} = 0. \)

**Lemma 1.3.** One has \( \mathfrak{sl}_2 \otimes t^{2n}\mathbb{C}[t^2], w_{-n} = 0 \) and \( \pi_1 \otimes t^{2n+1}\mathbb{C}[t^2], w_{-n} = 0. \) \( W_{-n}^\sigma \) is spanned by the monomials of the form

\[
(1.2) \quad e_{a_1} \cdots e_{a_s} g_{b_1}^k \cdots g_{b_k} w_{-n},
\]

\[
1 \leq b_1 < \cdots < b_k \leq 2n - 1, \ 0 \leq a_1 \leq \cdots \leq a_s \leq 2(n - k - s),
\]

where \( a_i \) are even and \( b_j \) are odd.

**Lemma 1.4.** The number of elements in the set (1.2) is equal to \( 3^n \) and its character is given by

\[
\sum_{k=0}^{n} q^{k^2} \binom{n}{k} q^{n-k} x^{-n+k+2s} \binom{n-k}{s} q^s.
\]

1.3. **The graded tensor products for superalgebras.** We start with the untwisted case.

Let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra with \( \mathfrak{g}_0 \) being the even part and \( \mathfrak{g}_1 \) being the odd part. For a \( \mathfrak{g} \)-module \( X \) we denote by \( X_0 \) its even part and by \( X_1 \) its odd part. Let \( V \) and \( W \) be cyclic \( \mathfrak{g} \) modules with cyclic vectors \( v \) and \( w \); in what follows we always assume that the cyclic vectors are even. The tensor product of \( V \) and \( W \) is defined by the formula

\[
g(x \otimes y) = gx \otimes y + (-1)^{ah} x \otimes gy, \ g \in \mathfrak{g}, x \in V_b.
\]

Let \( z_1, \ldots, z_n \) be a collection of pairwise distinct complex numbers and let \( V^1, \ldots, V^n \) be cyclic representations of \( \mathfrak{g} \) with cyclic vectors \( v^1, \ldots, v^n \). Let \( V^i(z_i) \) be the evaluation representations of \( \mathfrak{g} \otimes \mathbb{C}[t] \), where \( x \otimes t^k \) acts as \( z_i^k x \).

**Lemma 1.5.** The tensor product \( \bigotimes_{i=1}^n V^i(z_i) \) is cyclic \( \mathfrak{g}[t] \)-module with cyclic vector \( \bigotimes_{i=1}^n v^i \).

**Proof.** Let \( x \in \mathfrak{g}_0 \). Then the operator \( x \otimes t^k \) acts on the tensor product \( \bigotimes V^i(z_i) \) by the usual tensor product formula for representations of Lie algebras. Therefore all the operators

\[
x(i) = \underbrace{\text{Id} \otimes \ldots \text{Id} \otimes x \otimes \text{Id} \otimes \ldots \text{Id}}_{i-1}
\]
on $\bigotimes_{i=1}^n V^i(z_i)$ can be written as linear combinations of the operators $x \otimes t^k$
(via the Vandermond determinant).

Now assume that $x \in \mathfrak{g}_k$. Then one has

$$(x \otimes t^k)(u_1 \otimes \cdots \otimes u_n) =
(z_1^k x(1)+z_2^k (−1)^{\deg u_1} x(2)+\cdots+z_n^k (−1)^{\deg u_1+\cdots+\deg u_{n−1}} x(n))(u_1 \otimes \cdots \otimes u_n).$$

Therefore for any $i = 1, \ldots, n$ the operator $(−1)^{\deg u_1+\cdots+\deg u_{i−1}} x(i)$
can be expressed as linear combination of the operators $x \otimes t^j$, $0 \leq j \leq n−1$
(the case $i = 1$ corresponds to just $x(1)$). Therefore the operators $x \otimes t^j$, $0 \leq j \leq n−1$
do generate the whole tensor product $\bigotimes_{i=1}^n V^i(z_i)$ acting on
the tensor product of cyclic vectors. \hfill \Box

The universal enveloping algebra $U(\mathfrak{g}[t])$ has natural grading coming for
the degree of $t$, $U(\mathfrak{g}[t]) = \bigoplus_{s \geq 0} U(\mathfrak{g}[t])_s$ (for example, $U(\mathfrak{g}[t])_0 = U(\mathfrak{g})$). Let
us introduce the increasing filtration $F_s$ on the tensor product $\bigotimes_{i=1}^n V^i(z_i)$
as follows:

$$F_s = U(\mathfrak{g}[t])_s (v^1 \otimes \cdots \otimes v^n).$$

The associated graded space is cyclic $U(\mathfrak{g}[t])$ module. An important feature
is that it is now equipped with the additional grading. We denote the graded
module by $V^1(z_1) \ast \cdots \ast V^n(z_n)$.

Let $V$ be the irreducible 3-dimensional representation of $\mathfrak{osp}(1,2)$.

**Theorem 1.6.** For any pairwise distinct $z_1, \ldots, z_n$ the graded tensor product $V(z_1) \ast \cdots \ast V(z_n)$
is isomorphic to the Weyl module $W_{−n}$ as $\mathfrak{osp}(1,2)[t]$ modules.

**Proof.** It is easy to see that all the defining relations of the Weyl module do
hold in the graded tensor product. Therefore, we have a surjection $W_{−n} \to V(z_1) \ast \cdots \ast V(z_n)$. Since the dimension of the right hand side is $3^n$ and this
is the upper estimate for the dimension of the left hand side (see Lemma 1.2), the surjection is the isomorphism. \hfill \Box

**Corollary 1.7.** The graded tensor product $V(z_1) \ast \cdots \ast V(z_n)$ does not depend
(as $\mathfrak{osp}(1,2)[t]$-module) on the (pairwise distinct) parameters $z_i$.

**Corollary 1.8.** Vectors (6.1) form a basis of the Weyl module $W_{−n}$.

Now let us consider the twisted algebra $\mathfrak{osp}(1,2)[t]^{\sigma}$. For a complex number $z$
one defines the 3-dimensional evaluation $\mathfrak{osp}(1,2)[t]^{\sigma}$-module $V^{\sigma}(z)$
via the same formula as above. We have the following theorem.

**Theorem 1.9.** Assume that the numbers $z_1, \ldots, z_n$ satisfy the conditions $z_i^2 \neq z_j^2$
for $i \neq j$. Then the graded tensor product $V^{\sigma}(z_1) \ast \cdots \ast V^{\sigma}(z_n)$
is well defined and is isomorphic to the Weyl module $W_{−n}^{\sigma}$ as $\mathfrak{osp}(1,2)[t]$ modules.

**Proof.** The only difference with the untwisted case is the condition $z_i^2 \neq z_j^2$,
which guaranties the cyclicity of the tensor product of evaluation modules
in the twisted case. \hfill \Box
Corollary 1.10. The graded tensor product $V^\sigma(z_1) \ast \cdots \ast V^\sigma(z_n)$ does not depend (as $\mathfrak{osp}(1, 2)[t]^\sigma$-module) on the parameters $z_i$, satisfying $z_i^2 \neq z_j^2$ for all $i \neq j$. Vectors (1.2) form a basis of the Weyl module $W_n^\sigma$. 

1.4. The positive $n$ case. In this subsection we define the modules $W_n$ for $n > 0$. We first consider the untwisted case.

We define vector $w_n = e_0^n w_n \subset W_n$. We note that there is a symmetry (the $A_1$ Weyl group action) on $W_{-n}$ interchanging $e$ with $f$ and $g^+$ with $g^-$. This symmetry sends $w_{-n}$ to $w_n$ and vice versa. Therefore, the module $W_{-n}$ enjoys the basis of the form

$$f_{a_1} \cdots f_{a_s} g_{b_1} \cdots g_{b_k} w_{-n}, \quad 0 \leq b_1 < \cdots < b_k \leq n-1, \quad 0 \leq a_1 \leq \cdots \leq a_s \leq n-k-s.$$

Let $W_n = U(n^- \otimes t\mathbb{C}[t]).w_n \subset W_{-n}$, so $W_n$ is a module for the shifted Borel subalgebra, generated by $g_i^-$ and $g_i^+$. In other words, the module $W_n$ is generated from the vector $w_n$ by the action of the operators $f_i$ and $g_i^-$, $i > 0$.

Proposition 1.11. dim $W_n = 3^{n-1}$. The vectors

$$f_{a_1} \cdots f_{a_s} g_{b_1} \cdots g_{b_k} w_{-n}, \quad 1 \leq b_1 < \cdots < b_k \leq n-1, \quad 1 \leq a_1 \leq \cdots \leq a_s \leq n-s-k$$

form a basis of $W_n$.

Proof. The vectors (1.4) belong to the basis of the module $W_{-n}$ and hence are linear independent. Now we know that for the $2^{n-k}$ dimensional Weyl module for $\mathfrak{sl}_2$ the part generated by $f_1, f_2, \ldots$ from the lowest weight vector has basis of the form $f_{a_1} \cdots f_{a_s}$, where $1 \leq a_1 \leq \cdots \leq a_s \leq n-s-k$. Now using filtration from the proof of Lemma 1.1, we obtain that (1.4) is indeed a basis.

Corollary 1.12. The character of $W_n$ is equal to

$$\sum_{k=0}^{n-1} q^k \binom{n-1}{k} q^{n-k-1} \sum_{s=0}^{n-k-2} x^{n-k-2s} q^s \binom{n-k-1}{s} q^s.$$

Now let us work out the twisted case.

We define vector $w_n^\sigma = e_0^n w_n \subset W_n^\sigma$. Let

$$W_n^\sigma = U((\mathfrak{osp}(1, 2) \otimes t\mathbb{C}[t])^\sigma).w_n^\sigma \subset W_{-n},$$

i.e. $W_n^\sigma$ is a cyclic module for the shifted Borel subalgebra, generated by $g_i^-$ and $e_0$. The module $W_n^\sigma$ is generated from the vector $w_n$ by the action of the operators $f_i$, $i = 2, 4, \ldots, 2n-2$ and $g_i^-$, $i = 1, 3, \ldots, 2n-1$.

Proposition 1.13. dim $W_n = 2 \cdot 3^{n-1}$. The vectors

$$f_{a_1} \cdots f_{a_s} g_{b_1} \cdots g_{b_k} w_n^\sigma, \quad 1 \leq b_1 < \cdots < b_k \leq 2n-3, \quad 2 \leq a_1 \leq \cdots \leq a_s \leq 2(n-s-k)$$
and

\[(1.6) \quad f_{a_1} \cdots f_{a_s} g_{b_1} \cdots g_{b_{k-1}} g_{2n-1}^{\sigma},\]

\[1 \leq b_1 < \cdots < b_{k-1} \leq 2n - 3, \quad 0 \leq a_1 \leq \cdots \leq a_s \leq 2(n - s - k).\]

form a basis of \(W^\sigma_n\).

**Proof.** We first prove that the elements (1.6) belong to \(W^\sigma_n\) (this is obvious for (1.5), but not for (1.6)). The only problem is the operator \(f_0\) popping up in (1.6). However, it always comes multiplied by \(g_{2n-1}\). Therefore, we only need to check that \(f_0^n g_{2n-1} w_n \in W^\sigma_n\) for all \(m > 0\). First, we note that by the weight reason \(f_0^n g_{2n-1} w_n = 0\). However, up to a nonzero constant, this vector is equal to \(g_{2n-1} w_n\), which does not vanish.

So we know that all the vectors (1.5) and (1.6) belong to \(W^\sigma_n\). We also know that they are linearly independent, since they belong to the basis (1.2) of the whole space \(W_{-n}\). Since the sets (1.5) and (1.6) contain 2 \(3^{n-1}\) elements, we are left to show that the dimension of \(W_{-n}^\sigma/W^\sigma_n \geq 3^{n-1}\). We know that relations in \(W^\sigma_{-n}\) are generated by \(e_0 w_{-n}\). Since the vector \(e_0^n w_{-n}\) is trivial in the quotient, we conclude that

\[\dim W_{-n}^\sigma/W^\sigma_n \geq \dim W_{-n+1}^\sigma = 3^{n-1}.\]

**Corollary 1.14.** The character of \(W^\sigma_n\) is equal to

\[
\sum_{k=0}^{n-1} q^{k^2} \binom{n-1}{k} q^{n-1-k} \sum_{s=0}^{n-k-1} x^{n-k-2s} q^{2s} \binom{n-k-1}{s} q^{2-s} + x^{-1} q^{2n-1} \sum_{k=0}^{n-1} q^{k^2} \binom{n-1}{k} q^{n-k-1} \sum_{s=0}^{n-k-1} x^{n-k-2s} \binom{n-k-1}{s} q^{2-s}. \]

**1.5. The limit procedure.** In this subsection we consider the \(n \to \infty\) limits of the modules \(W_{-n}\) and \(W^\sigma_{-n}\).

**Lemma 1.15.** For \(n > 0\) there exists an embedding of \(\mathfrak{osp}(1, 2)[t]\) modules \(W_{-n} \to W_{-n-1}\), defined by \(w_{-n} \mapsto g_{-n}^+ w_{-n-1}\).

**Proof.** First, we show that the vector \(g_{-n}^+ w_{-n-1}\) satisfies all the annihilation conditions for the cyclic vector of \(W_{-n}\). Clearly, \(h_0 g_{-n}^+ w_{-n-1} = -ng_{-n}^+ w_{-n-1}\). Now, for \(x \in \mathfrak{n}^-\) and \(k \geq 0\) one has \(x_k g_{-n}^+ w_{-n-1} = 0\) (because \(x_k w_{-n-1} = 0\)) and \([x_k, g_{-n}^+] w_{-n-1} = 0\) because

\[\{x_k, g^+_n\} \subset (\mathfrak{n}^- \oplus \mathfrak{h}) \otimes t \mathbb{C}[t] \oplus \mathbb{C} g^+ \otimes t^n \mathbb{C}[t].\]
Finally, for $k > 0 \ h_k g_n^+ w_{n-1} = 0$, because $[h_k, g_n^+] = g_{n+k}^+$. We conclude that there exists a surjective homomorphism of $\mathfrak{osp}(1,2)[t]$-modules $W_n \to U(\mathfrak{osp}(1,2)[t]) g_n^+ w_{n-1}$.

Second, we check that $\dim U(\mathfrak{osp}(1,2)[t]) g_n^+ w_{n-1} \geq 3^n$. Indeed, the vectors

$$e_{a_1} \ldots e_{a_s} g_{b_1}^+ \ldots g_{b_k}^+ g_n^+ w_{n-1}$$

with $0 \leq b_1 < \cdots < b_k \leq n - 1$ and $0 \leq a_1 \leq \cdots \leq a_s \leq n - k - s$ are linearly independent, since they belong to the set of basis vectors (1.1) (with $n + 1$ instead of $n$). The number of these elements is $3^n$.

Let $L = \lim_{n \to \infty} W_n$ be the $\mathfrak{osp}(1,2)[t]$-module, obtained via the embeddings from Lemma 1.15. We define the character of $L$ as follows:

$$\text{ch} L(x, q) = \lim_{n \to \infty} q^{n(n-1)/2} \text{ch} W_n(x, q^{-1}).$$

Our goal is to find a formula for the character of $L$. The following lemma is well known (see e.g. [A]).

**Lemma 1.16.** $\sum_{k \geq 0} \frac{q^{k(k+1)/2}}{(q)_k} x^k = \prod_{i=0}^{\infty} (1 + q^i x)$.

**Theorem 1.17.** $\text{ch} L(x, q) = \prod_{i=0}^{\infty} (1 + q^i x) \prod_{i=0}^{\infty} (1 + q^i x^{-1})$.

**Proof.** Recall the basis (1.1) of the space $W_n$. Let $l_0(n) \in W_n$ be the vector $g_0^+ g_1^+ \ldots g_{n-1}^+ w_{n-1}$. We note that the embedding $W_n \to W_{n-1}$ sends $l_0(n)$ to $l_0(n + 1)$. Therefore, in the limit we obtain the vector $l_0 = \lim_{n \to \infty} l_0(n) \in L$. We note that the summand in the character of $L$ corresponding to $l_0$, is just $1(=x^0 q^0)$. It is convenient to parametrize the images of the elements $g_{b_1}^+ \ldots g_{b_k}^+ w_{n-1}$, $0 \leq b_1 < \cdots < b_k \leq n - 1$ in the limit space $L$ by the set

$$g_{-c_1}^+ \ldots g_{-c_k}^+ l_0, \ 0 \leq c_1 < \cdots < c_k$$

(although we do not have the action of the operators with negative powers of $t$). Then we obtain the basis of $L$ of the form

(1.7) $e_{a_1} \ldots e_{a_s} g_{-c_1}^+ \ldots g_{-c_k}^+ l_0, \ 0 \leq c_1 < \cdots < c_k, \ 0 \leq a_1 \leq \cdots \leq a_s \leq k - s$.

The character of the basis vectors (1.7) with fixed $k$ and $s$ is equal to

$$x^{-k+2s} \frac{q^{k(k-1)/2}}{(q)_k} \frac{k!}{(s)_q^{-1}} = \frac{q^{s(s-1)/2} x^s}{(q)_s} \times \frac{q^{(k-s)(k-s-1)/2} x^{s-k}}{(q)_{k-s}}.$$

Therefore

$$\text{ch} L(x, q) = \sum_{s \leq k} q^{s(s-1)/2} x^s \times q^{(k-s)(k-s-1)/2} x^{s-k} = \prod_{i=0}^{\infty} (1 + q^i x) \prod_{i=0}^{\infty} (1 + q^i x^{-1})$$

(the last equality comes from Lemma 1.16.).
Remark 1.18. We note that the character $\text{ch}L(x, q)$ is equal to the (normalized) theta-function divided by the $\eta$-function $\prod_{i \geq 1}(1 - q^i)$.

Now let us turn to the twisted case. Here we state the twisted analogues of the results from the previous subsection.

Lemma 1.19. For $n > 0$ there exists an embedding of $\mathfrak{osp}(1, 2)[t]^\sigma$ modules $W^\sigma_n \to W^\sigma_{n-1}$, defined by $w_{-n} \mapsto g_{2n-1}^+ w_{-n-1}$.

Let $L^\sigma = \lim_{n \to \infty} W^\sigma_n$ be the $\mathfrak{osp}(1, 2)[t]^\sigma$-module. We define the character of $L$ as follows:

$$\text{ch}L(x, q) = \lim_{n \to \infty} q^n \text{ch}W^\sigma_n(x, q^{-1}).$$

Let $l_0(n) = g_1^+ \ldots g_{2n-1}^+ w_n^\sigma \in W^\sigma_n$ and let $l_0 = \lim_{n \to \infty} l_0(n)$.

Theorem 1.20. $\text{ch}L^\sigma(x, q) = \prod_{i=0}^\infty (1 + q^{2i+1}x) \prod_{i=0}^\infty (1 + q^{2i+1}x^{-1})$.

2. Nonsymmetric Macdonald polynomials of types $A_2^{(2)}$ and $A_2^{(2)\dagger}$

Nonsymmetric Koornwinder polynomials of types $A_2^{(2)}$ and $A_2^{(2)\dagger}$ (see e.g. [OS, II, RY]) are rational functions depending on a parameter $q$ and five independent Hecke parameters $v_1, v_2, v_0, v_z$. Nondegenerate nonsymmetric Macdonald polynomials of types $A_2^{2n}$ ($A_2^{2n\dagger}$ correspondingly) are defined as specializations of Koornwinder polynomials at $v_2 \mapsto 1$ ($v_z \mapsto 1$ correspondingly) and equal Hecke parameters $v_i$'s (we denote them by $v$). Functions thus obtained are rational functions in variables $x, q, t = v^2$, more precisely they belong to $\mathbb{Z}(q, t)[x]$. We study the limits $v \to 0$ and $v \to \infty$ for $n = 1$.

2.1. Ram-Yip formula for type $A_2^{(2)}$ ($A_2^{(2)\dagger}$). We compute specializations of nonsymmetric Macdonald polynomials using methods from the papers [RY, OS]. We use the so called alcove walks, which are certain paths on the set of alcoves. We don’t give the general definition of an alcove walk: one can find it in papers [RY, OS]. However we give an explicit construction in our case.

We consider the real line $\mathbb{R}$ and the set of alcoves $(i, i+1), i \in \mathbb{Z}$ and label the walls of alcoves by simple reflections such that the wall $i$ is labeled by $s_i$ if $i$ is even and by $s_0$ if $i$ is odd.

The Weyl group $W = \langle s_1 \rangle \ast \langle s_0 \rangle$ (the free product of two cyclic groups of order 2) acts on the set of alcoves simply transitively. We identify $W$ with the set of alcoves: $s_1$ acts as a reflection in the wall 0 and $s_0$ acts as a reflection in the wall 1. Hence $2X = s_1s_0$ acts as a shift by 2. We use additive notation for the group generated by $2X$ (i.e. we write elements of this group as $2nX, n \in \mathbb{Z}$). Any element of $W$ can be written in the form $2nXs_i^b, b \in \{0, 1\}$. The following picture illustrates the procedure:

\[
\begin{array}{cccccccc}
-2Xs_1 & -2X & s_1 & 1 & 2Xs_1 & 2X & 4Xs_1 & 6Xs_1 \\
-2 & -1 & 0 & 1 & 2 & 3 & 4 & 6
\end{array}
\]
For any alcove \(a\) denote the even wall of \(a\) by \(2\text{wt}(a)\); if it is the left wall then put \(d(a) = 0\), if it is the right wall, then \(d(a) = 1\). In terms of \(W\), for \(w = 2nXs_1^b\) one has \(\text{wt}(a) = n\), \(d(a) = b\).

An alcove walk is a sequence of simple reflections with some addition information. This sequence is called the type of walk. Take an integer \(J\) to left”. Put folding positive if it is “from left to right” and negative if it is “from right to left”. Put folding on the \(i\)-th step if \(b_i = 0\) and crossing if \(b_i = 1\). We call a folding positive if it is ”from left to right” and negative if it is ”from right to left”. Put \(J = \{i|b_i = 0\}\). Denote by \(p_J\) the final alcove of the walk with such set of foldings. Of course, \(p_J\) also depends on \(w\) but we omit \(w\) to simplify the notation.

Denote by \(B(w)\) the set of alcove walks of type \(w\), i. e. pairs \(p = (w, b)\).

Let \(J_0 = \{i \in J|w_i = s_0\}\) and let \(J_+ (J_-)\) be the subset of positive (correspondingly, negative) foldings of \(J \setminus J_0\).

Put \(\beta_i = s_{k_i} \cdots s_{k_{i+1}} \alpha_{k_i}, 1 \leq i \leq l\), where \(\alpha_{k_i}\) is a simple root. For the walks of types \((s_1, s_0, \ldots, s_1, s_0)\) and \((s_1, s_0, \ldots, s_1, s_0)\) one has:

\[
\begin{align*}
\beta_{l-2i} &= (s_0s_1)^i \alpha_0 = -\alpha_1 + (2i + 1)\delta, \\
\beta_{l-2i-1} &= (s_0s_1)^i s_0 \alpha_1 = -\alpha_1 + (2i + 2)\delta.
\end{align*}
\]

Any \(\beta\) from the affine root lattice of type \(A_2^{(2)}\) (aka \(A_1^{(1)}\)) can be written in the following form:

\[
\beta = \beta' + \text{deg}(\beta)\delta,
\]

where \(\beta'\) is a root of finite dimensional root system. So in our case we have that \(\text{deg}(\beta_{l-i}) = i + 1\).

2.2. Explicit formula in type \(A_2^{(2)}\).

Theorem 2.1. (Ram, Yip, \(A_2^{(2)}\)-case) Put \(v = t^{1/2}\). Let \(w = (s_1, s_0, \ldots, s_1, s_0)\) (with \(-2n\) elements for \(n < 0\)) and \(w = (s_0, \ldots, s_1, s_0)\) (with \(2n-1\) elements
for \( n \geq 0 \). Then:

\[
E^{(2)}_{n} (x, q, t) = \sum_{p \in B(w)} v^{(\text{sign}(n)-1)/2+d(p_j)-|J|} (1-v^2)^{|J|} \prod_{j \in J_0} \frac{\xi_j}{1 - \xi_j^2} \prod_{j \in J_+} \frac{1}{1 - \xi_j} \prod_{j \in J_-} \frac{1}{1 - \xi_j} x^{\text{wt}(p_j)},
\]

where \( \xi_j = q^{\text{deg}(\beta_j)} v^{-(\alpha_1 \cdot \beta_j)} = q^{\text{deg}(\beta_j)} v^2 \).

**Definition 2.2.** Define the elements \( c_r(k_{22}, k_{12}, k_{11}) \in \mathbb{Z}[q] \), \( r = 1, 2 \) by the following recurrent relations:

\[
c_1(k_{22}, k_{12}, k_{11}) = q^{2n} c_2(k_{22} - 1, k_{12}, k_{11}) + q^{2n-1} c_2(k_{22}, k_{12} - 1, k_{11}) + c_1(k_{22}, k_{12}, k_{11} - 1), \\
c_2(k_{22}, k_{12}, k_{11}) = c_2(k_{22} - 1, k_{12}, k_{11}) + q^{2n-1} c_2(k_{22}, k_{12} - 1, k_{11}) + c_1(k_{22}, k_{12}, k_{11} - 1),
\]

where \( n = k_{11} + k_{12} + k_{22} \) in both formulas. The initial values are fixed by \( c_r(k_{22}, k_{12}, k_{11}) = 0 \), if some \( k_{ij} \) is negative, and \( c_r(0, 0, 0) = 1 \).

**Proposition 2.3.** The specializations of nonsymmetric Macdonald polynomials of the type \( A^{(2)}_2 \) can be written in the following way \( (n \geq 0) \):

\[
E^{(2)}_{-n} (x, q, 0) = \sum_{k_{22}+k_{12}+k_{11}=n} c_2(k_{22}, k_{12}, k_{11}) x^{k_{22}-k_{11}}.
\]

\[
E^{(2)}_{n+1} (x, q, 0) = \sum_{k_{22}+k_{12}+k_{11}=n+1} (q^{2n+1} c_2(k_{22} - 1, k_{12}, k_{11}) + c_1(k_{22}, k_{12} - 1, k_{11})) x^{k_{11}-k_{22}+1}.
\]

\[
E^{(2)}_{-n} (x, q^{-1}, \infty) = \sum_{k_{22}+k_{12}+k_{11}=n} c_1(k_{22}, k_{12}, k_{11}) x^{k_{11}-k_{22}}.
\]

\[
E^{(2)}_{n+1} (x, q^{-1}, \infty) = \sum_{k_{22}+k_{12}+k_{11}=n} c_2(k_{22}, k_{12}, k_{11}) x^{k_{11}-k_{22}+1}.
\]

In particular, \( E^{(2)}_{n+1} (x, q^{-1}, \infty) = x E^{(2)}_{-n} (x, q, 0) \).

**Proof.** We first prove \( (2.4), (2.5) \) using Theorem 2.1. Let \( l \) be a length of an element \( w \). We know that \( \beta_{l+1-j} = -\alpha_1 + j \delta \). Therefore \( \xi_{l+1-j} = q^l v^2 = q^l t \). Hence if we study specialization at \( t = 0 \) we can put all denominators to be equal to 1:

\[
E^{(2)}_{n} (x, q, 0) = \lim_{v \to 0} \sum_{p \in B(w)} v^{(\text{sign}(n)-1)/2+d(p_j)-|J|} \prod_{j \in J_0} \xi_j \prod_{j \in J_+} \frac{1}{1 - \xi_j} \prod_{j \in J_-} \frac{1}{1 - \xi_j} x^{\text{wt}(p_j)} =
\]

\[
= \lim_{v \to 0} \sum_{p \in B(w)} v^{(\text{sign}(n)-1)/2+d(p_j)-|J|} \prod_{j \in J_0} \xi_j \prod_{j \in J_+} \frac{1}{1 - \xi_j} \prod_{j \in J_-} \frac{1}{1 - \xi_j} x^{\text{wt}(p_j)}.\]
We claim that the exponent \((\text{sign}(n) - 1)/2 + d(p, j) - |J| + 2|J_0 \cup J_-|\) vanishes iff there are no positive 0-foldings and it is positive if such foldings exist. In fact, let \(J_{0+}\) and \(J_{0-}\) be the sets of positive and negative zero foldings. Then:

\[
(2.8) \quad (\text{sign}(n) - 1)/2 + d(p, j) - |J| + 2|J_0 \cup J_-| - 2|J_{0+}| = \]

\[
(\text{sign}(n) - 1)/2 + d(p, j) - |J_+| - |J_{0+}| + |J_{0-} \cup J_-| = 0.
\]

Therefore, \((\text{sign}(n) - 1)/2 + d(p, j) - |J| + 2|J_0 \cup J_-| = 2|J_{0+}|\). Denote the set of paths with \(|J_{0+}| = 0\) by \(QB\). Thus we have:

\[
E_n^{A_2^{(2)}}(x, q, 0) = \sum_{p \in QB(w)} q^{\sum_{i \in J_0 \cup J_1} i \cdot wt(p, j)}.
\]

Now let us write an alcove path in the following form. We encode it by a sequence \(h = (h_0, \ldots, h_l)\) (see (2.1)) of 1’s and 2’s. We put \(h_0 = 1\) if \(n > 0\) and \(h_0 = 2\) when \(n < 0\). If \(i\)-th step of the path is finished by right arrow (i.e. it is a crossing from left to right or a positive folding) then \(h_i = 1\). If the \(i\)-th step is finished by the left arrow then \(h_i = 2\). Then subsequences \(12\) correspond to negative foldings and subsequences \(21\) correspond to positive.

We consider the sequence \((h_0, \ldots, h_l)\) as a sequence of pairs 11, 12, 21, 22 and possibly the left most element without pair. Then the set of sequences of pairs with no pair 21 inside corresponds to \(QB(w)\). Denote the set of such sequences by \(QS(n)\).

For any sequence \(h\) of length \(l + 1\) denote \(\text{leg}(h) = \sum_{l-j=1, h_{l-j+1}=2} j\). Then:

\[
E_n^{A_2^{(2)}}(x, q, 0) = \sum_{w \in QS(n)} q^{\text{leg}(h)} x^{|\{i: h_i = 1\}| - |\{i: h_i = 2\}|} / 2,
\]

where \([y]\) is the integer part of \(y\).

Let us consider the polynomial \(E_n^{A_2^{(2)}}(x, q, 0)\). Denote by \(QS(i, k_{22}, k_{12}, k_{11}) \subset QS(−n)\) the set of sequences of pairs and the element \(h_0\) such that \(h_0 = i\) and there are \(k_{ij}\) pairs \(ij\). Put

\[
c_i(k_{22}, k_{12}, k_{11}) = \sum_{w \in QS(i, k_{22}, k_{12}, k_{11})} q^{\text{leg}(h)}.
\]

Let us subdivide the sets \(QS(i, k_{22}, k_{12}, k_{11})\) into three subsets of sequences according to the value of the first pair \(h_1, h_2\) ((2, 2), (1, 2) or (1, 1)). Consider the case \(i = 2\). If \((h_1, h_2) = (2, 2)\), then the leg of this sequence will not be changed if we cut \((h_0, h_1)\). It is easy to see that all elements of \(QS(2, k_{22} - 1, k_{12}, k_{11})\) can be obtained by such a procedure. If \((h_1, h_2) = (1, 2)\) then if we cut first two elements then the leg decreases on \(2l - 1\). If \((h_1, h_2) = (1, 1)\) then if we cut first two elements then the leg will not be changed and we obtain \(i = 1\) instead of \(i = 2\). So we obtain a recurrent relation (2.3). Recurrent relations (2.2) can be obtained in the same way. Moreover by definition we have that they satisfy (2.4).
Analogously we obtain equation (2.5).

Now let us consider the polynomials $E_{-n}^{(2)}(x, q^{-1}, \infty)$. At $t \to \infty \frac{c}{1-c} \sim -\frac{1}{c}$, after interchanging $q \to q^{-1}$ we have:

$$E_{-n}^{(2)}(x, q^{-1}, \infty) = \lim_{t \to \infty} \sum_{p \in B(w)} v^{\text{sign}(n)-1/2+d(p_j)+|J|} \prod_{j \in J_0 \cup J_+} (-q^j x^{-2})^{\text{wt}(p_j)} =$$

$$= \sum_{p \in B(w)} v^{-(\text{sign}(n)-1/2+d(p_j)+|J|)\text{sign}(n)-1/2} \sum_{j \in J_0 \cup J_+} j \sum_{x, q} \xi_{\text{sign}(n)-1} / x^{\text{wt}(p_j)}.$$

Similiar to (2.8) we have that $(\text{sign}(n)-1/2+d(p_j)+|J|\text{sign}(n)-1/2) = 0$ iff $J_0 = 0$. So we obtain:

$$E_{-n}^{(2)}(x, q^{-1}, \infty) = \sum_{p \in B(w), |J_0| = 0} q^{\text{sign}(n)-1/2+d(p_j)+|J|\text{sign}(n)-1/2} \sum_{j \in J_0 \cup J_+} j \sum_{x, q} \xi_{\text{sign}(n)-1} / x^{\text{wt}(p_j)}.$$

So in terms of sequences we have that a walk giving nonzero summand correspond to a sequence of pairs 22, 21, 11 with $h_0 = 2$. Denote the set of such sequences by $\mathcal{QS}'(n)$ and put $\text{leg}'(h) = \sum_{h_{l-j}=1, h_{l-j-1}=2} j$. Then:

$$E_{-n}^{(2)}(x, q^{-1}, \infty) = \sum_{p \in B(w), |J_0| = 0} q^{\text{leg}'(h) / x^{\text{wt}(p_j)} \sum_{j \in J_0 \cup J_+} j \sum_{x, q} \xi_{\text{sign}(n)-1} / x^{\text{wt}(p_j)}},$$

Now to obtain (2.6) we interexchange 1 and 2 in all definitions of the previous paragraph.

Finally:

$$E_{n}^{(2)}(x, q^{-1}, \infty) = \lim_{t \to \infty} \sum_{p \in B(w)} v^{d(p_j)+|J|} \prod_{j \in J_0 \cup J_+} (-q^j x^{-2})^{\text{wt}(p_j)} =$$

$$= \sum_{p \in B(w)} v^{d(p_j)+|J|-2|J_0|-2|J_+|} \sum_{j \in J_0 \cup J_+} j \sum_{x, q} \xi_{\text{sign}(n)-1} / x^{\text{wt}(p_j)},$$

and we obtain (2.7). \hfill \Box

**Lemma 2.4.** The unique solution for the quantities $c_r(k_{22}, k_{12}, k_{11})$ from Definition 2.2 is given by the formulas:

\begin{align*}
(2.9) & \quad c_1(k_{22}, k_{12}, k_{11}) = q^{k_{12}^2 + 2k_{22}} \begin{pmatrix} k_{22} + k_{12} + k_{11} \\ k_{22}, k_{12}, k_{11} \end{pmatrix} q^2, \\
(2.10) & \quad c_2(k_{22}, k_{12}, k_{11}) = q^{k_{12}^2} \begin{pmatrix} k_{22} + k_{12} + k_{11} \\ k_{22}, k_{12}, k_{11} \end{pmatrix} q^2.
\end{align*}

**Proof.** Direct computation. \hfill \Box
2.3. Dual Macdonald polynomials. In this section we work with Macdonald polynomials of type $A_2^{(2)\dagger}$. We keep the notation from subsection 2.1 and 2.2.

**Theorem 2.5.** (Ram, Yip, $A_2^{(2)\dagger}$-case) Put $v = t^{1/2}$. Then:

$$E_{n}^{A_2^{(2)\dagger}}(x, q, t) = \sum_{p \in \mathcal{B}(w)} v^{(\text{sign}(n) - 1)/2 + d(p)} - |J| (1 - q^2)^{|J|} \times$$

$$\prod_{j \in J_0} \frac{1}{1 - \xi_j} \prod_{j \in J_{-}} \frac{\xi_j^2}{1 - \xi_j} \prod_{j \in J_{+}} \frac{1}{1 - \xi_j} \prod_{j \in J_{-}} \frac{\xi_j}{1 - \xi_j} x^{v^t(p_j)}.$$

**Definition 2.6.** We define elements $c_r^\dagger(k_{22}, k_{21}, k_{11}) \in \mathbb{Z}[q]$, $r = 1, 2$ by the following recurrent relations:

(2.11) \[ c_r^\dagger(k_{22}, k_{21}, k_{11}) = q^{2n} c_2^\dagger(k_{22} - 1, k_{21}, k_{11}) + q^{2n} c_1^\dagger(k_{22}, k_{21} - 1, k_{11}) + c_1^\dagger(k_{22}, k_{21}, k_{11} - 1), \]

(2.12) \[ c_2^\dagger(k_{22}, k_{21}, k_{11}) = c_1^\dagger(k_{22} - 1, k_{21}, k_{11}) + c_1^\dagger(k_{22}, k_{21} - 1, k_{11}) + c_1^\dagger(k_{22}, k_{21}, k_{11} - 1), \]

where $n = k_{11} + k_{21} + k_{22}$. The initial conditions are $c_r^\dagger(k_{22}, k_{21}, k_{11}) = 0$ if any $k_{ij} < 0$ and $c_0^\dagger(0, 0, 0) = 1$.

**Proposition 2.7.** We have the following equations ($n \geq 0$):

(2.13) \[ E_{n}^{A_2^{(2)\dagger}}(x, q, 0) = \sum_{k_{22} + k_{21} + k_{11} = n} c_1^\dagger(k_{22}, k_{21}, k_{11}) x^{k_{11} - k_{22}}. \]

(2.14) \[ E_{n+1}^{A_2^{(2)\dagger}}(x, q, 0) = \sum_{k_{22} + k_{21} + k_{11} = n} c_1^\dagger(k_{22}, k_{21}, k_{11}) x^{k_{11} - k_{22} + 1}. \]

(2.15) \[ E_{n}^{A_2^{(2)\dagger}}(x, q^{-1}, \infty) = \sum_{k_{22} + k_{21} + k_{11} = n} c_1^\dagger(k_{22}, k_{21}, k_{11}) x^{k_{11} - k_{22}}. \]

(2.16) \[ E_{n+1}^{A_2^{(2)\dagger}}(x, q^{-1}, \infty) = \sum_{k_{22} + k_{21} + k_{11} = n} \left( c_2^\dagger(k_{22}, k_{21}, k_{11}) + c_1^\dagger(k_{22}, k_{21}, k_{11}) \right) x^{k_{11} - k_{22} + 1}. \]

In particular $E_{n+1}^{A_2^{(2)\dagger}}(x, q, 0) = x E_{n}^{A_2^{(2)\dagger}}(x, q^{-1}, \infty)$.

**Proof.** The proof is completely analogous to the proof of Proposition 2.3. We have the same elements $\xi_j = q^2 v^t = q^2 t$. Hence if we study the specialization at $t = 0$ we can put all denominators to be equal to 1:

$$E_{n}^{A_2^{(2)\dagger}}(x, q, 0) = \lim_{v \to 0} \sum_{p \in \mathcal{B}(w)} x^{v^t(p_j)} v^{(\text{sign}(n) - 1)/2 + d(p)} - |J| \prod_{j \in J_0} \xi_j \prod_{j \in J_{-}} \xi_j =$$
\begin{align*}
&= \lim_{v \to 0} \sum_{p \in B(w)} x^{wt(p_J)} \psi(v(\text{sign}(n) - 1)/2 + d(p_J) - |J|+2|J_+|+4|J_0+| q^{\sum_{i \in J_0\cup J_-} i}. \\
\text{The exponent } (\text{sign}(n) - 1)/2 + d(p_J) - |J|+2|J_+|+4|J_0+| \text{ vanishes iff there are no negative 0-foldings. Encode alcove paths by binary words in the same way as in the proof of Proposition 2.3. So the nonzero summands correspond to words with no pairs 12. This is the only difference with the previous proof.} \quad \square
\end{align*}

\textbf{Lemma 2.8.} The unique solution for the quantities $c_i^\dagger(k_{22}, k_{12}, k_{11})$ from Definition 2.6 is given by the formulas:

\begin{align*}
(2.17) \quad c_1^\dagger(k_{22}, k_{21}, k_{11}) &= q^{k_{21}(k_{21}-1)+2k_{22}+2k_{21}} \binom{k_{22} + k_{21} + k_{11}}{k_{22}, k_{21}, k_{11}} q^2, \\
(2.18) \quad c_2^\dagger(k_{22}, k_{21}, k_{11}) &= q^{k_{21}(k_{21}-1)} \binom{k_{22} + k_{21} + k_{11}}{k_{22}, k_{21}, k_{11}} q^2.
\end{align*}

\textit{Proof.} Direct computation. \quad \square

\section{3. Comparison}

In this section we establish a link between the characters of the Weyl modules and the specialized Macdonald polynomials.

\textbf{Theorem 3.1.} For any $n \in \mathbb{Z}$ one has

$$
\text{ch} W_n(x, q^2) = E_n^{A_2(2)}(x, q, 0), \quad \text{ch} W_n^\sigma(x, q^2) = E_n^{A_2(2)}(x, q, 0).
$$

\textit{Proof.} Lemma 1.2 and Lemma 1.4 give for $n \geq 0$

\begin{align*}
(3.1) \quad &\text{ch} W_{-n}(x, q^2) = \sum_{k+s \leq n} q^{k(k-1)} x^{n+k+2s} \binom{n}{k, s, n-k-s} q^2, \\
(3.2) \quad &\text{ch} W_{-n}^\sigma(x, q) = \sum_{k+s \leq n} q^{k^2} x^{n+k+2s} \binom{n}{k, s, n-k-s} q^2.
\end{align*}

Formula (3.1) agrees with formulas (2.13), (2.18). Formula (3.2) agrees with formulas (2.4), (2.10).

Using Corollary 1.12 and Corollary 1.14 we obtain the following formulas, $n \geq 0$

\begin{align*}
(3.3) \quad &\text{ch} W_{n+1}(x, q^2) = \sum_{k+s \leq n} q^{k(k+1)+2s} x^{n+1-k-2s} \binom{n}{k, s, n-k-s} q^2, \\
(3.4) \quad &\text{ch} W_{n+1}^\sigma(x, q) = \sum_{k+s \leq n} x^{n+1-k-2s} \left( q^{k^2+2s} + x^{-1} q^{k^2+2n+1} \right) \binom{n}{k, s, n-k-s} q^2.
\end{align*}

Formula (3.3) agrees with formulas (2.14), (2.17).
The nontrivial part is formula (3.4), which we want to compare with formulas (2.5), (2.9) and (2.10). We have conditions (1.5) and (1.6) on elements of basis of $W_{n+1}$. Elements that satisfy condition (1.5) are parametrized by the data very similar to the parametrization data of (1.2). More precisely we obtain (1.5) if we increase the $t$-degree of the elements $e_i$ in (1.2) by 2. Thus, under the identification $s = k_{11}$, $k = k_{12}$, $k_{22} + k_{12} + k_{11} = n + 1$, the character of the elements (1.5) is equal to

$$\sum_{k_{22} + k_{12} + k_{11} = n + 1} c_2(k_{22} - 1, k_{12}, k_{11}) q^{2k_{22}} x^{k_{11} - k_{22} + 1} =$$

$$\sum_{k_{22} + k_{12} + k_{11} = n + 1} c_1(k_{22} - 1, k_{12}, k_{11}) x^{k_{11} - k_{22} + 1}.$$

Analogously the elements satisfying condition (1.6) are elements that satisfy formulas (2.6), (2.9) and (2.10). We have conditions (1.5) and (1.6) on elements (1.2). More precisely we obtain (1.5) if we increase the $t$-degree of the elements $e_i$ in (1.2) by 2. Thus, under the identification $s = k_{11}$, $k = k_{12}$, $k_{22} + k_{12} + k_{11} = n + 1$, the character of the elements (1.5) is equal to

$$\sum_{k_{22} + k_{12} + k_{11} = n + 1} c_2(k_{22}, k_{12} - 1, k_{11}) q^{2n+1} x^{k_{11} - k_{22}}.$$

□

Let us introduce the PBW filtration on the Weyl module $W_n$. Namely, we define $F_0 = \mathbb{C}w_{-n}$, $F_{s+1} = F_s + n^{-}[t].F_s$. The associated graded space is a cyclic module for the algebra $\mathbb{C}[e_0, e_1, \ldots] \otimes \Lambda(g_0^+, g_1^+, \ldots)$. Let us attach degree 1 to the variables $e_i$ and $g_i^+$. We thus obtain an additional grading on the module $grW_n$. We denote the character by $\text{ch}(grW_n)(x, q, t)$.

For the twisted Weyl modules we make a similar procedure. First we pass to the graded $\mathbb{C}[e_0, e_1, \ldots] \otimes \Lambda(g_0^+, g_1^+, \ldots)$ module $grW_\sigma_n$. Then we attach degree 1 to the variables $e_i$ and degree 0 to the variables $g_i^+$. Using this new grading we define the new character depending on $x, q, t$ and denote it by $\text{ch}(grW_\sigma_n)(x, q, t)$.

**Theorem 3.2.** Let $n \geq 0$. Then

$$\text{ch}(grW_n)(x, q^2, q^2) = E_n^{22}(x, q^{-1}, \infty), \quad \text{ch}(grW_\sigma_n)(x, q, q) = E_n^{\sigma(2)}(x, q^{-1}, \infty).$$

**Proof.** It is easy to see that the set of vectors (1.1) forms basis of $grW_n$ and the set of vectors (1.2) forms basis of $grW_\sigma_n$. Hence, we derive the following formulas for the graded characters:

(3.5) $$\text{ch}(grW_n)(x, q^2, q^2) = \sum_{k+s \leq n} q^{k(k-1)+2k+2s} x^{-n+k+2s} \binom{n}{k, s, n-k-s} q^2,$$

(3.6) $$\text{ch}(grW_\sigma_n)(x, q, q) = \sum_{k+s \leq n} q^{k^2+s} x^{-n+k+2s} \binom{n}{k, s, n-k-s} q^2.$$

Now formula (3.5) agrees with formulas (2.15), (2.17). Formula (3.6) agrees with formulas (2.6), (2.9). □
Appendix A. Quantum Bruhat graph

Here we briefly describe the methods of a paper [OS]. Although we don’t use their techniques, the ideas of [OS] are very important for the content of our paper.

Consider the Weyl group \( W = \langle s_0 \rangle \ast \langle s_1 \rangle \) of a root system \( A_2^{(2)} \).

**Definition A.1.** Let \( W(Y) \) be the Coxeter group of the root system \( Y \), \( s_\alpha \) be a reflection in the root \( \alpha \), \( l \) be the length function on \( W(Y) \). Then the quantum Bruhat graph is the following ordered labelled graph:

- the set of vertices is \( W(Y) \);
- we have a Bruhat arrow from \( g \) to \( gs_\alpha \), if \( l(gs_\alpha) = l(g) + 1 \);
- we have a quantum arrow from \( g \) to \( gs_\alpha \), if \( l(gs_\alpha) = l(g) - \langle 2\rho, \alpha \rangle + 1 \).

Consider the quantum Bruhat graph of type \( \widehat{A}_1 \). We want to make a difference between \( s_{\alpha_1} = s_1 \) and \( s_{\alpha_0} = s_0 \). So we have the following labeled graph on two vertices:

where arrows from id to \( s \) are Bruhat and arrows from \( s \) to id are quantum.

Put \( \beta_i = s_{k_{i-1}} \cdots s_{k_1 + 1} \alpha_{k_i} \), where \( \alpha_{k_i} \) is a simple root. We write \( \beta \) in the following form \( \beta = \beta' + \deg(\beta) \delta \), where \( \beta' \in \mathbb{Z}_\alpha \).

For any alcove walk \( \langle w, b \rangle \) let \( J = \{ i | b_i = 0 \} \), i.e. the set of foldings of a walk. Then we consider the following path on the quantum Bruhat graph started at element id:

\[
\text{dir}(\beta_{i_1}) \cdots \text{dir}(\beta_{i_r})
\]

It is easy to see that any odd arrow of this path is quantum and any even is Bruhat. The Bruhat arrows correspond to negative foldings and quantum arrows correspond to positive ones. Define quantum Bruhat paths as paths such that they have no Bruhat (quantum for \( A_2^{(2)} \)) edges labeled by \( s_0 \). It is proved in [OS] that \( E_n^{A_2^{(2)}}(x, q, 0) \) is obtained as a sum of some summands which are in one-to-one correspondence with the quantum Bruhat paths on the quantum Bruhat graph.

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