CONTROLLER SYNTHESIS FOR TIMELINE-BASED GAMES*

RENATO ACAMPORA a, LUCA GEATTI a, NICOLA GIGANTE b, ANGELO MONTANARI a, AND VALENTINO PICOTTI c

a University of Udine, Italy
e-mail address: acampora.renato@spes.uniud.it, luca.geatti@uniud.it, angelo.montanari@uniud.it

b Free University of Bozen-Bolzano, Italy
e-mail address: nicola.gigante@unibz.it

c University of Southern Denmark
e-mail address: picotti@imada.sdu.dk

Abstract. In the timeline-based approach to planning, the evolution over time of a set of state variables (the timelines) is governed by a set of temporal constraints. Traditional timeline-based planning systems excel at the integration of planning with execution by handling temporal uncertainty. In order to handle general nondeterminism as well, the concept of timeline-based games has been recently introduced. It has been proved that finding whether a winning strategy exists for such games is 2EXPTIME-complete. However, a concrete approach to synthesize controllers implementing such strategies is missing. This article fills the gap by providing an effective and computationally optimal approach to controller synthesis for timeline-based games.

1. Introduction

Automated planning is the field of artificial intelligence that studies the development of autonomous agents able of reasoning about how to reach some goals, starting from a high-level description of their operating environment. It is one of the most studied fields of AI, with early work going several decades back [MH69, FN71]. Most of the research by the planning community focuses on the action-based approach, where planning problems are modeled in terms of actions that an agent has to perform to suitably change its state. The task is to devise a sequence of such actions that lead to the goal when executed starting from a given initial state [FN71, FL03].

In this paper, we focus on the alternative paradigm of timeline-based planning, an approach born and developed in the space sector [Mus94]. In timeline-based planning, there is no explicit separation among actions, states, and goals. Planning domains are represented as systems of independent but interacting components, whose behavior over time, the timelines, is governed by a set of temporal constraints, called synchronization rules.

Key words and phrases: Planning, automata, synthesis.

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Over the years, timeline-based planning systems have been developed and successfully exploited by space agencies on both sides of the Atlantic [CCF+06, CCD+07, FJ03, BS07, CRK+00], for short- to long-term mission planning [CRT+15] as well as on-board autonomy [FCO+11]. The main advantage of such a paradigm in these contexts is the ability of these systems of handling both planning and execution in a uniform way: by the use of flexible timelines, timeline-based planners can produce robust plans that, during execution, can be adapted to the current contingency.

However, flexible timelines currently employed in timeline-based systems only handle temporal uncertainty, where the precise timings of events in the plan are unknown, but the causal sequence of the events is determined. In particular, they cannot generate robust plans against an environment empowered with general nondeterminism. To overcome this limitation, the concept of timeline-based games was recently introduced [GMO+20]. In timeline-based games, state variables belong either to the controller or to the environment. The controller aims at satisfying its set of system rules, while the environment can make arbitrary moves, as long as the domain rules that define the game arena are satisfied. A controller’s strategy is winning if it guarantees that the controller wins, regardless of the choices made by the environment. The moves available to the two players can determine both what happens and when it happens, thus handling temporal uncertainty and general nondeterminism in a uniform way.

Determining whether a winning strategy exists for timeline-based games has been proved to be 2EXPTIME-complete [GMO+20]. However, there is currently no effective way to synthesize a controller that implements such strategies. A necessary condition for synthesizing a finite-state strategy and the corresponding controller is the availability of a deterministic arena. Two methods to obtain such an arena have been followed in the literature, but both have limitations and turn out to be inadequate. On the one hand, the complexity result of [GMO+20] relies on the construction of a (doubly exponential) concurrent game structure used to model check some Alternating-time Temporal Logic formulas [AHK02]. Even though such a structure is deterministic and theoretically suitable to solve a reachability game and to synthesize a controller, its construction relies on theoretical nondeterministic procedures that are not realistically implementable. On the other hand, Della Monica et al. [DGMS18] devised an automata-theoretic solution that provides a concrete and effective way to construct an automaton that accepts a word if and only if the original planning problem has a solution plan. Unfortunately, the size of the resulting nondeterministic automaton is already doubly exponential, and its determinization would result in a further blowup and thus in a non-optimal procedure.

The present paper fills the gap by developing an effective and computationally optimal approach to synthesizing controllers for timeline-based games. The proposed method addresses the limitations of previous techniques by directly constructing a deterministic finite-state automaton of an optimal doubly-exponential size, that recognizes solution plans. Such an automaton can be turned into the arena for a reachability game, for which many controller synthesis techniques are available in the literature. The paper is a significantly revised and extended version of [AGG+22]. It provides a detailed account of the general framework, gives some illustrative examples, and fully works out all the proofs.

The rest of the paper is organized as follows. After discussing related work in Section 2, Section 3 introduces timeline-based planning and games. Section 4 presents the main technical contribution of the paper, namely, the construction of the deterministic automaton that recognizes solution plans. Section 5 shows how to turn such an automaton into the
arena of a suitable game from which the controller can be synthesized. Section 6 summarizes the main contributions of the work and suggest future research directions. All the technical proofs are included in the appendix.

2. RELATED WORK

The paradigm of timeline-based planning has been first introduced to plan and schedule scientific operations of the Hubble space telescope [Mus94]. In the following two decades, many timeline-based planning systems have been developed both at NASA and ESA, including EUROPA [BWMB+05], ASPEN [CRK+00], and APSI [DPC+08]. Such systems have been used both for short- to long-term mission planning, e.g., for the renowned Rosetta mission [CRT+15], and for onboard autonomy [FCO+11]. Elements of the timeline-based and the action-based paradigm have been combined into the Action Notation Modeling Language (ANML) [SFC08], extensively used at NASA since then.

Despite the real-world success, the timeline-based planning paradigm lacked a thorough foundational understanding in contrast to the action-based one, which has been extensively studied from a theoretical perspective from the start [MH69, Byl94]. To enable theoretical investigations into timeline-based planning, Cialdea Mayer et al. [COU16] laid down the core features of the paradigm, describing them in a uniform formalism, which has been later studied in several contributions. The formalism was compared to traditional action-based languages like STRIPS, and it was proved that the latter are expressible by timeline-based languages [GMCO16]. The timeline-based plan existence problem was proved to be EXPSPACE-complete [GMCO17] over discrete time in the general case, and PSPACE-complete with qualitative constraints only [DGTM20]. On dense time, the problem goes from being NP-complete to undecidable, depending on the applied syntactic restrictions [BMM+20]. Additionally, logical [DGM+17] and automata-theoretic [DGMS18] counterparts have been investigated to study the expressiveness of timeline-based languages.

The above body of work focuses on deterministic timeline-based planning domains. However, the paradigm also fits to uncertain domains requiring robust plans. Current timeline-based planning systems employ the concept of flexible timelines, described as including uncertainty in the timings of events, representing envelopes of possible executions of the plan. Planners, when possible, produce strongly controllable flexible plans, whose execution is then robust for the given temporal uncertainty. In order to obtain controllers for executing strongly controllable flexible plans, the problem can be simplified by reducing it to timed game automata [OFCF11].

While the current approach works fairly well in handling temporal uncertainty, it does not support scenarios where the environment is fully nondeterministic. Furthermore, as pointed out in [GMO+20], the language of timeline-based planning as formalized in [COU16] allows one to write domains that are not solvable by strongly controllable flexible plans, but that may easily be by strategies coping with general nondeterminism. For this reason, [GMO+20] introduced the concept of timeline-based game, which is the focus of this work. Timeline-based games adopt a game-theoretic point of view, where the controller and the environment play by constructing timelines, with the controller trying to fulfill its synchronization rules independently from the choices of the environment. This setting allows one to handle both temporal uncertainty and general nondeterminism, thus strictly generalizing previous approaches based on flexible timelines. In [GMO+20], the problem of deciding the existence of a winning strategy for a given timeline-based game has been
proved to be 2EXPTIME-complete. The proof is based on the construction of a concurrent game structure where a suitable alternating-time temporal logic (ATL) formula is model checked [AHK02]. However, the construction relies on nondeterministic procedures that are not effectively implementable, and thus it does not solve the problem of synthesizing actual controllers for timeline-based games. This work fills the gap by providing an effective synthesis algorithm.

The devised algorithm builds on classical results in the field of reactive synthesis, which studies how to build correct-by-construction controllers satisfying high-level logical specifications. The original formulation of the problem of reactive synthesis is due to Church [Chu62]. The problem for S1S specifications was later solved by Büchi and Landweber using a non-elementary complexity algorithm [BL90]. As for Linear Temporal Logic (LTL) specifications, the problem is 2EXPTIME-complete [PR89b, Ros92], which, interestingly, is the same complexity as timeline-based games. In both cases, the core of the synthesis algorithm is the construction of a deterministic arena, where the game can be solved with a fix-point computation. This work focuses on constructing such an arena for timeline-based games (Sections 4 and 5).

3. Preliminaries

In this section, we provide an overview of the general framework that underpins our work. We begin by introducing the general features of timeline-based planning, and then we discuss timeline-based games. Next, we introduce the reactive synthesis problem. Finally, we recall the concept of difference bound matrices (DBMs) [Dil89, PH07], which are the data structures that we will use to represent the temporal constraints of a system.

3.1. Timeline-based planning. The first basic notion is that of state variable.

Definition 3.1 (State variable). A state variable is a tuple $x = (V_x, T_x, D_x, \gamma)$, where:

- $V_x$ is the finite domain of $x$;
- $T_x : V_x \rightarrow 2^{V_x}$ is the value transition function of $x$, which maps each value $v \in V_x$ to the set of values that can immediately follow it;
- $D_x : V_x \rightarrow \mathbb{N} \times \mathbb{N}$ is the duration function of $x$, mapping each value $v \in V_x$ to a pair $(d_{x=v}^{\min}, d_{x=v}^{\max})$ specifying respectively the minimum and maximum duration of any interval where $x = v$;
- $\gamma : V_x \rightarrow \{c, u\}$ is the controllability tag, that, for each value $v \in V_x$, specifies whether it is controllable ($\gamma (v) = c$) or uncontrollable ($\gamma (v) = u$).

A state variable $x$ takes its values from a finite domain and represents a finite state machine with a transition function $T_x$. The behavior over time of a state variable $x$ is modeled by a timeline. Intuitively, a timeline for a state variable $x$ is a finite sequence of tokens, that is, contiguous time intervals where $x$ holds a given value.

Following the approach described in [GMO+20], instead of formally defining timelines in terms of tokens, we represent executions of timeline-based systems as single words, called event sequences, where each event describe the start/end of some token in a given time point.

To this end, we first define the notion of action.

Definition 3.2. Let $SV$ be a set of state variables. An action is a term of the form $\text{start}(x, v)$ or $\text{end}(x, v)$, where $x \in SV$ and $v \in V_x$. 

Actions of the form \( \text{start}(x, v) \) are starting actions, and those of the form \( \text{end}(x, v) \) are ending actions. We denote by \( A_{SV} \) the set of all the actions definable over a set of state variables \( SV \).

**Definition 3.3** (Event sequence \([GMO^{+}20]\)). Let \( SV \) be a set of state variables and \( A_{SV} \) be the set of all the actions \( \text{start}(x, v) \) and \( \text{end}(x, v) \), for \( x \in SV \) and \( v \in V_x \). An event sequence over \( SV \) is a sequence \( \overline{p} = (\mu_1, \ldots, \mu_n) \) of pairs \( \mu_i = (A_i, \delta_i) \), called events, where \( A_i \subseteq A_{SV} \) and \( \delta_i \in \mathbb{N}^+ \), such that, for any \( x \in SV \):

1. for all \( 1 \leq i \leq n \), if \( \text{start}(x, v) \in A_i \), for some \( v \in V_x \), then there is no \( \text{start}(x, v') \) in any \( \mu_j \) before the closest event \( \mu_k \), with \( k > i \), such that \( \text{end}(x, v) \in A_k \) (if any);
2. for all \( 1 \leq i \leq n \), if \( \text{end}(x, v) \in A_i \), for some \( v \in V_x \), then there is no \( \text{end}(x, v') \) in any \( \mu_j \) after the closest event \( \mu_k \), with \( k < i \), such that \( \text{start}(x, v) \in A_k \) (if any);
3. for all \( 1 \leq i < n \), if \( \text{end}(x, v) \in A_i \), for some \( v \in V_x \), then \( \text{start}(x, v') \in A_i \), for some \( v' \in V_x \);
4. for all \( 1 < i \leq n \), if \( \text{start}(x, v) \in A_i \), for some \( v \in V_x \), then \( \text{end}(x, v') \in A_i \), for some \( v' \in V_x \).

The first two conditions guarantee correct parenthesis placement by identifying the start and the end of each token in the sequence. Condition 1 prevents a token from starting before the end of the previous one, while condition 2 prevents the occurrence of two consecutive ends not interleaved by a start. Conditions 3 and 4 ensure seamless continuity: each token’s end (resp., start) is consistently followed (preceded) by the start (resp., end) of another, except for the first (resp., last) event in the sequence. These latter conditions prevent gaps in the timeline description of the represented plan.

In event sequences, a token for a variable \( x \) is a maximal interval with at most one occurrence of events \( \mu_i = (A_i, \delta_i) \) and \( \mu_j = (A_j, \delta_j) \), where \( \text{start}(x, v) \in A_i \) and \( \text{end}(x, v) \in A_j \), for some \( v \in V_x \). We say such a token starts at position \( i \) and ends at position \( j \). Note that Definition 3.3 implies that a token that has started is not required to end before the end of the sequence and that it can end without the corresponding starting action ever appearing. If this is the case, we say that an event sequence is open either to the right or to the left. Otherwise, it is said to be closed. An event sequence closed to the left and open to the right is called a partial plan. Notice that the empty event sequence \( \varepsilon \) is closed on both sides for any variable. Furthermore, in closed event sequences, the first event contains only start actions, while the last one contains only end actions, one for each variable \( x \).

Given an event sequence \( \overline{p} = (\mu_1, \ldots, \mu_n) \) over a set of state variables \( SV \), with \( \mu_i = (A_i, \delta_i) \), we define \( \delta(\overline{p}) \) as \( \sum_{1 \leq i \leq n} \delta_i \), that is, \( \delta(\overline{p}) \) is the time elapsed from the start to the end of the event sequence (its duration). For any subsequence \( (\mu_i, \ldots, \mu_j) \) of \( \overline{p} \), abbreviated \( \overline{p}[i..j] \), we denote by \( \delta_{i,j} \) (or, equivalently, \( \delta(\overline{p}[i..j]) \)) the amount of time spanning that subsequence. Notice that \( \delta_{i,j} \) is defined as \( \sum_{i < k < j} \delta_k \). Finally, given an event sequence \( \overline{p} = (\mu_1, \ldots, \mu_n) \), we define \( \overline{p}_{<i} \) as \( (\mu_1, \ldots, \mu_{i-1}) \), for each \( 1 < i \leq n \).

In timeline-based planning, the objective is to satisfy a set of synchronization rules, that specify the desired behavior of the system (constraints and goal). These rules relate tokens, possibly belonging to different timelines, through temporal relations among their endpoints. Let \( SV \) be a set of state variables and \( N = \{a, b, \ldots \} \) be a set of token names.

**Definition 3.4** (Atom). An atom is a temporal relation between tokens’ endpoints of the form \( \langle \text{term} \rangle \leq_{[l,u]} \langle \text{term} \rangle \), where \( l \in \mathbb{N}, u \in \mathbb{N} \cup \{+\infty \}, l \leq u \), and a term is either \( \text{start}(a) \) or \( \text{end}(a) \), for some \( a \in N \).
As an example, the atom \( \text{start}(a) \leq [3,7] \text{ end}(b) \) constrains token \( a \) to start at least 3 and at most 7 time units before the end of token \( b \), while the atom \( \text{start}(a) \leq [0,\infty) \text{ start}(b) \) simply constrains token \( a \) to start before token \( b \).

**Definition 3.5** (Synchronization rule). A synchronization rule \( R \) has one of the following two forms:

\[
\langle \text{rule} \rangle := a_0[x_0 = v_0] \rightarrow \langle \text{body} \rangle \\
\langle \text{rule} \rangle := \top \rightarrow \langle \text{body} \rangle \\
\langle \text{body} \rangle := E_1 \lor E_2 \lor \cdots \lor E_k \\
E_j := \exists a_1[x_1 = v_1]a_2[x_2 = v_2]\cdots a_n[x_n = v_n] \cdot C_j, \text{ for } 1 \leq j \leq k,
\]

where \( a_i \in \mathbb{N}, x_i \in SV, v_i \in V_{x_i}, \) and \( C_j \) is a conjunction of atoms, for \( 0 \leq i \leq n \).

Terms \( a_i[x_i = v_i] \) are referred to as quantifiers. The term \( a_0[x_0 = v_0] \) is called the trigger. The disjuncts in the body are called existential statements. Quantifiers refer to tokens with the corresponding variable and value. The intuitive semantics of a synchronization rule can be given as follows: for every token satisfying the trigger, at least one of the existential statements must be satisfied as well. Each existential statement \( E_j \) requires the existence of tokens that satisfy the quantifiers in its prefix and the clause \( C_j \). A token that satisfies the trigger of a rule is said to trigger that rule. The trigger of a rule can be empty (\( \top \)). In such a case, the rule is referred to as triggerless and it requires the satisfaction of its body without any precondition.

Let \( a \) and \( b \) be token names. Here are two examples of synchronization rules (relations = and \( \leq \) are syntactic sugar for \( \leq [0,0] \) and \( \leq [0,\infty) \), respectively):

\[
\begin{align*}
& a[x_s = \text{Comm}] \rightarrow \exists b[x_g = \text{Available}] \cdot \text{start}(b) \leq \text{start}(a) \land \text{end}(a) \leq \text{end}(b) \\
& a[x_s = \text{Science}] \rightarrow \exists b[x_s = \text{Slewing}] \cdot c[x_s = \text{Earth}] \cdot d[x_s = \text{Comm}] \\
& \quad \text{end}(a) = \text{start}(b) \land \text{end}(b) = \text{start}(c) \land \text{end}(c) = \text{start}(d)
\end{align*}
\]

where variables \( x_s \) and \( x_g \) represent the state of a spacecraft and the visibility of the communication ground station, respectively. The first synchronization rule requires the satellite and the ground station to coordinate their communications so that when the satellite is transmitting, the ground station is available for reception. The second one instructs the system to send data to Earth after every measurement session, interleaved by the required slewing operation. Triggerless rule can be used to state the goal of the system. As an example, the following rule ensures that the spacecraft performs some scientific measurement:

\[
\top \rightarrow \exists a[x_s = \text{Science}]
\]

Triggerless rules only require the existence of tokens specified by the existential statements, being their universal quantification trivial. In fact, they are syntactic sugar, as it is possible to translate them into triggered rules, as shown in [GMO+20]. From now on, we will not consider them anymore.

We now formalise the above intuitive account of the semantics of synchronization rules.

**Definition 3.6** (Matching functions [Gig19]). Let \( \overline{\mu} = \langle \mu_1, \ldots, \mu_n \rangle \) be a (possibly open) event sequence, \( E \equiv \exists a_1[x_1 = v_1]\cdots a_k[x_k = v_k] \cdot C \) be one of the existential statements of a synchronization rule \( R \equiv a_0[x_0 = v_0] \rightarrow E_1 \lor \cdots \lor E_m \), and \( V \) be a set of terms such that \( \text{start}(a) \in V \) or \( \text{end}(a) \in V \) only if \( a \in \{a_0, \ldots, a_k\} \). A matching function \( \gamma : V \rightarrow [1, \ldots, n] \) maps each term \( T \in V \) to an event \( \mu_{\gamma(T)} \) in \( \overline{\mu} \), such that:
(1) for each $T \in V$, with $T = \text{start}(a)$ (resp., $T = \text{end}(a)$), if $a$ is quantified as $a[x = v]$ in $E$, then the event $\mu_T = (A_T, \delta_T)$ is such that $\text{start}(x, v) \in A_T$ (resp., $\text{end}(x, v) \in A_T$);
(2) if both $T = \text{start}(a)$ and $T' = \text{end}(a)$ belong to $V$ for some token name $a \in \mathbb{N}$, then $\gamma(T)$ and $\gamma(T')$ identify the endpoints of the same token.

As a matter of fact, in [Gig19], matching functions are defined in terms of rule graphs, a data structure that we do not use here. For this reason, we reformulated the original definition in terms of event sequences.

The following definition gives a formal account of the semantics of synchronization rules.

**Definition 3.7** (Semantics of synchronization rules). Let $R \equiv a_0[x_0 = v_0] \rightarrow E_1 \lor \ldots \lor E_m$ and let $\overline{\mu} = \langle \mu_1, \ldots, \mu_n \rangle$ be an event sequence. We say that $R$ is satisfied by $\overline{\mu}$ if, for each event $\mu_i = (A_i, \delta_i)$ such that $\text{start}(x, v_i) \in A_i$, there exist an existential statement $E_j \equiv \exists a_1[x_1 = v_1] \ldots a_k[x_k = v_k]. \ C$ and a matching function $\gamma$ such that if $T \leq [l, u] T'$ appears in $C$, then $l \leq \gamma(T') - \gamma(T) \leq u$, for any pair of terms $T$ and $T'$.

**Definition 3.8** (Timeline-based planning problem). A timeline-based planning problem can be defined as follows.

Timeline-based planning problems can be defined as follows.

Timeline-based games. We are now ready to introduce the notion of timeline-based game, that subsumes that of timeline-based planning with uncertainty given in [COU16].

**Definition 3.9** (Timeline-based game). A timeline-based game is a tuple $G = \langle \text{SV}_C, \text{SV}_E, S, D \rangle$, where $\text{SV}_C$ and $\text{SV}_E$ are the sets of controlled and external state variables, respectively, and $S$ and $D$ are the sets of system and domain synchronization rules, respectively, both involving variables from $\text{SV}_C$ and $\text{SV}_E$.

A partial plan for $G$ is a partial plan over the variables $\text{SV}_C \cup \text{SV}_E$. Let $\Pi_G$ be the set of all possible partial plans for $G$, simply $\Pi$ when there is no ambiguity. Since the empty event sequence $\varepsilon$ is closed and $\delta(\varepsilon) = 0$, the empty partial plan $\varepsilon$ is a good starting point for the game. Players incrementally build onto a partial plan, starting from $\varepsilon$, by playing actions that specify which tokens to start and (or) to end, adding an event that extends the event sequence, or complementing the existing last one.

Formally, we partition the set of all the available actions $A_{\text{SV}}$ into those that are playable by either of the two players.

**Definition 3.10** (Partition of player actions). Let $\text{SV} = \text{SV}_C \cup \text{SV}_E$. The set $A_{\text{SV}}$ of available actions over $\text{SV}$ is partitioned into the sets $A_C$ of Charlie’s actions and $A_E$ of Eve’s actions, which are defined as follows:

\[
A_C = \{ \text{start}(x, v) \mid x \in \text{SV}_C, \ v \in V_x \} \cup \{ \text{end}(x, v) \mid x \in \text{SV}, \ v \in V_x, \ \gamma_x(v) = c \} \quad (1)
\]

\[
A_E = \{ \text{start}(x, v) \mid x \in \text{SV}_E, \ v \in V_x \} \cup \{ \text{end}(x, v) \mid x \in \text{SV}, \ v \in V_x, \ \gamma_x(v) = u \} \quad (2)
\]
Hence, players can start tokens for owned variables and end them for values that they control. Let \( d = \max(L, U) + 1 \), where \( L \) and \( U \) are the maximum lower and (finite) upper bounds appearing in any rule of \( G \). Note that, by Definition 3.10, we may have \( x \in SV_E \) and \( \gamma_x(v) = c \) for some \( v \in V_g \). This means that Charlie may control the duration of a variable that belongs to Eve. This situation is symmetrical to the more common one where Eve controls the duration of a variable that belongs to Charlie, that is, uncontrollable tokens. As an example, Charlie may decide to start a task, without being able to foresee how long it will take. Similarly, the environment may trigger the start of a process, e.g., fixing a plant fault, but Charlie may be able to control, to some extent, how long it will take to end it, e.g., we can decide to fix it today or tomorrow.

Actions combine into moves starting (resp., ending) multiple tokens simultaneously.

**Definition 3.11** (Move). A move \( \mu_C \) for Charlie is a term of the form \( \text{wait}(\delta_C) \) or \( \text{play}(A_C) \), where \( 1 \leq \delta_C \leq d \) and \( \varnothing \neq A_C \subseteq A_C \) is either a set of starting actions or a set of ending actions. A move \( \mu_E \) for Eve is a term of the form \( \text{play}(A_E) \) or \( \text{play}(\delta_E, A_E) \), where \( 1 \leq \delta_E \leq d \) and \( A_E \subseteq A_E \) is either a set of starting actions or a set of ending actions.

By Definition 3.11, moves like \( \text{play}(A_C) \) and \( \text{play}(\delta_E, A_E) \) can play either \( \text{start}(x, v) \) actions only or \( \text{end}(x, v) \) actions only. A move of the former kind is called a starting move, while a move of the latter kind is called an ending move. We consider wait moves as ending moves. Starting and ending moves must alternate during the game.

Let us denote the sets of Charlie’s and Eve’s moves by \( \mathcal{M}_C \) and \( \mathcal{M}_E \), respectively. A round of the game is defined as follows.

**Definition 3.12** (Round). A round \( \rho \) is a pair \((\mu_C, \mu_E) \in \mathcal{M}_C \times \mathcal{M}_E \) of moves such that:

1. \( \mu_C \) and \( \mu_E \) are either both starting or both ending moves;
2. either \( \rho = (\text{play}(A_C), \text{play}(A_E)) \), or \( \rho = (\text{wait}(\delta_C), \text{play}(\delta_E, A_E)) \), with \( \delta_E \leq \delta_C \).

A starting (resp., ending) round is one made of starting (resp., ending) moves. Since Charlie cannot play empty moves and wait moves are ending moves, each round is unambiguously either a starting or an ending round. Moreover, since \( \text{play}(\delta_E, A_E) \) moves are always paired with \( \text{wait}(\delta_C) \) ones, which are ending moves, then \( \text{play}(\delta_E, A_E) \) moves are necessarily ending moves (item 1 of Definition 3.12).

We can now specify how to apply a round to the current partial plan to obtain the new one. The game always starts with a single starting round.

**Definition 3.13** (Outcome of rounds). Let \( \overline{\mu} = \langle \mu_1, \ldots, \mu_n \rangle \) be an event sequence, with \( \mu_n = (A_n, \delta_n) \) (\( \mu_n = (\varnothing, 0) \) if \( \overline{\mu} = \varepsilon \)). Let \( \rho = (\mu_C, \mu_E) \) be a round, \( A_E \) and \( A_C \) be the sets of actions of the two moves (\( A_C \) is empty if \( \mu_C \) is a wait move), and \( \delta_E \) and \( \delta_C \) be the time increments of the moves. We define \( \delta_C = 1 \) (resp., \( \delta_E = 1 \)) for \( \text{play}(A_C) \) (resp., \( \text{play}(A_E) \)).

The outcome of the application of \( \rho \) on \( \overline{\mu} \) is the event sequence \( \rho(\overline{\mu}) \) defined as follows:

1. if \( \rho \) is a starting round, then \( \rho(\overline{\mu}) = \overline{\mu} \circ \mu_n' \), where \( \mu_n' = (A_n \cup A_C \cup A_E, \delta_n) \);
2. if \( \rho \) is an ending round, then \( \rho(\overline{\mu}) = \overline{\mu} \mu_n' \), where \( \mu_n' = (A_C \cup A_E, \delta_E) \).

We say that \( \rho \) is applicable to \( \overline{\mu} \) if:

a) \( \rho(\overline{\mu}) \) complies with Definition 3.3;
b) \( \rho \) is an ending round if and only if \( \overline{\mu} \) is open for all variables that appear in the moves.

A single move by either player is applicable to \( \overline{\mu} \) if there is a move for the other player such that the resulting round is applicable to \( \overline{\mu} \). The game starts from the empty partial plan \( \varepsilon \), and players play in turn, composing a round from the move of each one, which is
applied to the current partial plan to obtain the new one. We can now define the notion of strategy for each player and that of winning strategy for Charlie.

**Definition 3.14** (Strategy). A strategy for Charlie is a function $\sigma_C : \Pi \to M_C$ that maps any given partial plan $\overline{p}$ into a move $\mu_C$ applicable to $\overline{p}$. A strategy for Eve is a function $\sigma_E : \Pi \times M_C \to M_E$ that maps a partial plan $\overline{p}$ and a move $\mu_C \in M_C$ applicable to $\overline{p}$ into a move $\mu_E$ such that the round $\rho = (\mu_C, \mu_E)$ is applicable to $\overline{p}$.

A sequence $\overline{p} = \langle \rho_0, \ldots, \rho_n \rangle$ of rounds is called a play of the game. A play is said to be played according to some strategy $\sigma_C$ for Charlie, if, starting from the initial partial plan $\overline{p}_0 = \varepsilon$, it holds that $\rho_i = (\sigma_C(\Pi_{i-1}), \mu^i_E)$, for some $\mu^i_E$, for all $0 < i \leq n$, and to be played according to some strategy $\sigma_E$ for Eve if $\rho_i = (\mu^i_C, \sigma_E(\Pi_{i-1}, \mu^i_C))$, for all $0 < i \leq n$. It can be easily seen that for any pair of strategies $(\sigma_C, \sigma_E)$ and any $n \geq 0$, there is a unique play $\overline{p}_n(\sigma_C, \sigma_E)$ of length $n$ played according to both $\sigma_C$ and $\sigma_E$.

Then, we say that a partial plan $\overline{p}$ and the play $\overline{p}$ such that $\overline{p} = \overline{p}(\varepsilon)$ are admissible, if the partial plan satisfies the domain rules, and that they are successful if the partial plan satisfies the system rules.

**Definition 3.15** (Admissible strategy for Eve). A strategy $\sigma_E$ for Eve is admissible if for each strategy $\sigma_C$ for Charlie, there is $k \geq 0$ such that the play $\overline{p}_k(\sigma_C, \sigma_E)$ is admissible.

Charlie wins if, assuming that domain rules are respected, he manages to satisfy the system rules no matter how Eve plays.

**Definition 3.16** (Winning strategy for Charlie). Let $\sigma_C$ be a strategy for Charlie. We say that $\sigma_C$ is a winning strategy for Charlie if for any admissible strategy $\sigma_E$ for Eve, there exists $n \geq 0$ such that the play $\overline{p}_n(\sigma_C, \sigma_E)$ is successful.

We say that Charlie wins the game $G$ if he has a winning strategy, while Eve wins the game if a winning strategy for Charlie does not exist.

### 3.3. Synthesis

The synthesis problem is the problem of devising an implementation that satisfies a formal specification of an input-output relation [PR89a]. Such an implementation may be a transducer, a Mealy machine, a Moore machine, a circuit, or the like. In the following, we give a short account of the roles of games and strategies in game-based synthesis.

**Definition 3.17** (Game Graph). A finite game graph $G$ is a triple $(Q, Q_C, E)$, where $Q$ is a finite set of nodes, $Q_C \subseteq Q$ is the subset of Charlie’s nodes, and $E \subseteq Q \times Q$ is a transition relation. The relation $E$ must satisfy the condition: $\forall q \exists q' : (q, q') \in E$ (totality).

A play on a game graph $G$ starting from the initial state $q_0$ is an infinite sequence $p = q_0q_1q_2 \ldots$, where $(q_i, q_{i+1}) \in E$, for all $i \geq 0$. A game is a pair $(G, W)$, where $G$ is a game graph and $W$ is the winning condition of the game. In the general case, $W$ consists of the set of plays won by Charlie.

Here, we focus on reachability winning conditions, which are expressed as $W : = \{ R \subseteq Q \mid R \cap F \neq \emptyset \}$, for a given set $F \subseteq Q$. A play $p$ is said to satisfy $W$ if the set of states visited by $p$, denoted by $\text{occ}(p) = \{ q \in Q \mid \exists i : p(i) = q \}$, intersects $W$, that is, Charlie wins the play $p$ if $p$ visits at least one state in $F$.

**Definition 3.18** (Reachability game). A reachability game is a pair $(G, W)$, where $G = (Q, Q_C, E)$ is a game graph and $W$ is a reachability winning condition.
A strategy for Charlie is a function \( f : Q^* \cdot Q_C \rightarrow Q \). A play \( p \) adheres to strategy \( f \) if, for each \( q_i \in Q_C \), \( q_{i+1} = f(q_0 \ldots q_i) \). Given an initial state \( q \), a strategy for Charlie is a winning strategy if Charlie wins any play from \( q \) that follows the strategy \( f \). The same holds for Eve. Charlie (resp., Eve) wins if a winning strategy exists from \( q \).

Given a game \((G, W)\), with \( G = (Q, Q_C, E) \), the winning region of Charlie is defined as \( W_C := \{ q \in Q \mid \text{Charlie wins from } q \} \). The winning region \( W_E \) for Eve is defined in an analogous way. The two sets are clearly disjoint \((W_C \cap W_E = \emptyset)\). The game is said to be determined if \( W_C \cup W_E = Q \). It is well known that reachability games are determined [Tho08].

The next step is to build a Controller starting from a winning strategy \( f \) such that the specification is met. We use Moore machines as Charlie plays first.

**Definition 3.19** (Moore machine). A Moore machine is a tuple \( M = (Q, \Sigma, \Gamma, q_0, \delta, \tau) \), where \( Q \) is a finite set of states, \( \Sigma \) is a finite input alphabet, \( \Gamma \) is a finite output alphabet, \( q_0 \in Q \) is the initial state, \( \delta : Q \times \Sigma \rightarrow Q \) is the transition function, and \( \tau : Q \rightarrow \Gamma \) is the output function.

By suitably tying \( \delta \) and \( \tau \) to \( f \), one can effectively implement \( f \). We refer the reader to Definition 5.3 for the details on how we do it.

### 3.4. Difference Bound Matrices

**Difference bound matrices** (DBMs) were introduced by Dill [Dil89] as a pragmatic representation of constraints \((x - y \leq c)\). Later on, Péron et al. [PH07] suitably expanded the formalism. The following short account of the formalism is basically borrowed from the latter work.

Let \( Var = \{v_0, v_1, \ldots v_n\} \) be a finite set of variables, \( \mathcal{V} = \mathbb{Z} \cup \{+\infty\} \) be a set of values that variables and constants can take, and \( C \) be a set of constraints of the form \( v_i - v_j \leq c \), where \( v_i, v_j \in Var \) and \( c \in \mathcal{V} \). The DBM that represents \( C \) is an \((n + 1) \times (n + 1)\) matrix defined as follows:

\[
M_{ij} = \inf\{c \mid (v_i - v_j \leq c) \in C\},
\]

where \( \inf(\emptyset) = +\infty \).

\( M_{ij} \) equals the tightest value of \( c \) if there is some constraint \((v_i - v_j \leq c)\) in \( C \); otherwise, it is \( +\infty \). The variable \( v_0 \in Var \) is always valued to 0, and it is used to express bounds on variables, that is, \( v_i \leq c \) is written as \( v_i - v_0 \leq c \). In Section 4, we use DBMs to conveniently represent atoms (see Definition 3.4).

### 4. A Deterministic Automaton for Timeline-based Planning

In this section, we define an encoding of timeline-based planning problems into deterministic finite state automata (DFA). Given a timeline-based planning problem, the corresponding automaton recognizes all and only those event sequences that represent solution plans for the problem. In the next section, we will use such an automaton as the game arena for a timeline-based game.
4.1. Plans as words. Let $P = (SV, S)$ be a timeline-based planning problem and, as already stated in the previous section, let $d = \max(L, U) + 1$, where $L$ and $U$ are the maximum lower and (finite) upper bounds appearing in any rule of $P$. We restrict our attention to event sequences where the distance between two consecutive events is at most $d$. Such a restriction guarantees us the finiteness of the considered alphabet, and it does not cause any loss in generality, as proved by Lemma 4.8 of [Gig19]. Moreover, it agrees with the notion of move of a timeline-based game (see Definition 3.11).

We define the symbols of the alphabet $\Sigma$ as events of the form $\mu = (A, \delta)$, where $A \subseteq A_{SV}$ and $1 \leq \delta \leq d$. Formally, $\Sigma = 2^{A_{SV}} \times [d]$, where $[d] = \{1, \ldots, d\}$. Note that the size of $\Sigma$ is exponential in the size of the problem. Moreover, we define $\text{window}(P)$ as the sum of all the coefficients appearing as upper bounds in the rules of $P$. This value represents the maximum amount of time a rule can “count” far away from the occurrence of the quantified tokens. Consider, for instance, the following rule:

$$a_0[x_0 = v_0] \rightarrow \exists a_1[x_1 = v_1]a_2[x_2 = v_2]a_3[x_3 = v_3].$$ (3)

$$\text{start}(a_1) \leq_{[4,14]} \text{end}(a_0) \land \text{end}(a_0) \leq_{[0,+\infty]} \text{end}(a_2) \land \text{start}(a_2) \leq_{[0,3]} \text{end}(a_3)$$

In this case, assuming the above rule to be the only one in the problem, $\text{window}(P)$ would be $3 + 14 = 17$. Thus, the rule can account for what happens at most 17 time points from the occurrence of its quantified tokens. For instance, if the token $a_1$ appears at a specific distance from $a_0$, it has to be within less than 17 time points, and any modification of the plan that alters this distance can break the rule’s satisfaction. However, what occurs further away from $a_0$ only affects the fulfillment of the rule qualitatively. Suppose that the tokens $a_2$ and $a_3$ are, together, at 100 time points from $a_0$. Changing this distance while maintaining the qualitative order between tokens does not break the rule’s satisfaction. For $\text{window}(P)$’s properties refer to [Gig19].

4.2. Matching structures. A key insight underlying the construction we are going to outline is that every atomic temporal relation $T \leq_{[l,u]} T'$ can be rewritten as the conjunction of two upper bound constraints $T' - T \leq u$ and $T - T' \leq -l$, where we represent a lower bound constraint $T' - T \geq l$ as an upper bound one. This allows us to rewrite the clause $C$ of an existential statement $E$ as a constraint system $\nu(C)$ with constraints of the form $T - T' \leq n$, for $n \in \mathbb{Z} \cup \{+\infty\}$.

The constraint system $\nu(C)$ can be represented by a difference bound matrix $D$ indexed by terms, where the entry $D[T, T']$ gives the upper bound $n$ on $T - T'$. In building $D$, we ensure the right duration of tokens by augmenting the system with constraints of the kind $\text{start}(a_i) - \text{end}(a_i) \leq -d_{\text{min}}^{x_i = v_i}$ and $\text{end}(a_i) - \text{start}(a_i) \leq d_{\text{max}}^{x_i = v_i}$, for any quantified token $a_i[x_i = v_i]$ of $E$. As an example, the constraint system and the DBM for the above rule are the ones in Figs. 1 and 2, respectively.

On top of DBMs, we define the concept of matching structure, a data structure that allows us to monitor and update the fulfillment of atomic temporal relations among terms throughout the execution of the plan. More precisely, it allows us to manipulate and reason about existential statements of which only a portion of the requests has been satisfied by the word read so far, while the rest is potentially satisfiable in the future.

Definition 4.1 (Matching Structure). Let $E \equiv \exists a_1[x_1 = v_1] \ldots a_m[x_m = v_m].C$ be an existential statement of a synchronization rule $R \equiv a_0[x_0 = v_0] \rightarrow E_1 \lor \cdots \lor E_k$ over the set of state variables $SV$. The matching structure for $E$ is a tuple $M_E = (V, D, M, t)$, where:
\[
\begin{align*}
\text{end}(a_0) - \text{start}(a_1) & \leq 14 \\
\text{start}(a_1) - \text{end}(a_0) & \leq -4 \\
\text{end}(a_0) - \text{end}(a_2) & \leq 0 \\
\text{end}(a_3) - \text{start}(a_2) & \leq 3 \\
\text{start}(a_2) - \text{end}(a_3) & \leq 0
\end{align*}
\]

Figure 1: The constraint system of Eq. (3).

| start(a_0) | end(a_0) | start(a_1) | end(a_1) | start(a_2) | end(a_2) | start(a_3) | end(a_3) |
|------------|----------|------------|----------|------------|----------|------------|----------|
| start(a_0) |          |            |          |            |          |            |          |
| end(a_0)   |          | 14         |          |            |          |            |          |
| start(a_1) |          |            | -4       |            |          |            |          |
| end(a_1)   |          |            |          |            |          | 0          |          |
| start(a_2) |          |            |          |            |          |            |          |
| end(a_2)   |          |            |          |            |          |            |          |
| start(a_3) |          |            |          |            |          |            | 3        |
| end(a_3)   |          |            |          |            |          |            |          |

Figure 2: DBM of Eq. (3). Missing entries are intended to be +∞.

- \( V \) is the set of terms \( \text{start}(a) \) and \( \text{end}(a) \), for \( a \in \{a_0, \ldots, a_m\} \);
- \( D \) is a DBM of size \(|V| \times |V|\), indexed by terms of \( V \), whose entries take value over \( \mathbb{Z} \cup \{+\infty\} \), where
  \[
  \begin{cases}
  D[T, T'] = n & \text{if } T - T' \leq n \in \nu(C), \\
  D[T, T'] = 0 & \text{if } T = T', \\
  D[T, T'] = +\infty & \text{otherwise};
  \end{cases}
  \]
- \( M \subseteq V \) and \( 0 \leq t \leq \text{window}(P) \).

The set \( M \) contains the set of terms from \( V \) correctly seen in the sequence so far. We say these terms have been matched by the matching structure. We use \( \overline{M} = V \setminus M \) to refer to terms yet to be matched. We say a matching structure \( M \) to be closed if \( M = V \), initial if \( M = \emptyset \), and active if \( \text{start}(a_0) \in M \) and it is not closed. The component \( t \) represents the time elapsed since matching \( \text{start}(a_0) \). As time progresses, we update a matching structure as follows.

In the DBMs of a matching structure, the bounds between any pair of terms \( T \) and \( T' \), with one in \( M \) while the other not, are tightened by the elapsing of time. When \( T \in M \) and \( T' \in \overline{M} \), \( D[T, T'] \) is a lower bound loosened by adding the elapsed time \( \delta \). When \( T \in \overline{M} \) and \( T' \in M \), \( D[T, T'] \) is an upper bound tightened by subtracting \( \delta \). Consider the DBM in Figure 2 and the pair of terms \( \text{start}(a_1) \) and \( \text{end}(a_0) \). We have \( D[\text{start}(a_1), \text{end}(a_0)] = -4 \), implying that \( \text{start}(a_1) - \text{end}(a_0) \leq -4 \) must hold. Suppose that \( \text{start}(a_1) \in M \) (it has been matched), and that \( \text{end}(a_0) \in \overline{M} \) (it needs to be matched). Now, in a time step, the entry in the DBM is incremented and updated to \(-4 + 1 = -3\) reflecting the fact that we now have 3 time steps left to match \( \text{end}(a_0) \). A similar analysis leads us to the conclusion that
the entry $D[\text{end}(a_0), \text{start}(a_1)] = 14$ has to be decremented by 1 and updated to $14 - 1 = 13$. This intuition is formalized as follows.

**Definition 4.2** (Time shifting). Let $\delta > 0$ be a positive amount of time, and let $M = (V, D, M, t)$ be a matching structure. The result of shifting $M$ by $\delta$ time units, written $M + \delta$, is a matching structure $M' = (V, D', M', t')$, where:

- for all $T, T' \in V$:
  $$D'[T, T'] = \begin{cases} 
  D[T, T'] + \delta & \text{if } T \in M \text{ and } T' \in \overline{M} \\
  D[T, T'] - \delta & \text{if } T \in \overline{M} \text{ and } T' \in M \\
  D[T, T'] & \text{otherwise}
  \end{cases}$$

- and
  $$t' = \begin{cases} 
  t + \delta & \text{if } M \text{ is active} \\
  t & \text{otherwise}
  \end{cases}$$

Definition 4.2 specifies how to update the entries of $D$ and how to update $t$ to the trigger occurrence of an active matching structure.

**Definition 4.3** (Matching). Let $M = (V, D, M, t)$ be a matching structure and $I \subseteq \overline{M}$ a set of matched terms. A matching structure $M' = (V, D, M', t)$ is the result of matching the set $I$, written $M \cup I$, with $M' = M \cup I$.

To correctly match an existential statement while reading an event sequence, a matching structure is updated only as long as one witnesses no violation of temporal constraints. As such, we deem an event as *admissible* or not.

**Definition 4.4** (Admissible Event). An event $\mu = (A, \delta)$ is *admissible* for a matching structure $M_E = (V, D, M, t)$ if and only if, for every $T \in M$ and $T' \in \overline{M}$, $\delta \leq D[T', T]$, i.e., the elapsing of $\delta$ time units does not exceed the upper bound of some term $T'$ not yet matched by $M_E$.

Each admissible event $\mu$ that is read can be matched with a subset of terms from the matching structure. However, there can be multiple ways to match events and terms. To make this choice explicit, we introduce the following definition.

**Definition 4.5** (I-match Event). Let $M_E = (V, D, M, t)$ be a matching structure and $I \subseteq \overline{M}$. An \textit{I-match event} is an admissible event $\mu = (A, \delta)$ for $M_E$ such that:

1. for all token names $a \in N$ quantified as $a[x = v]$ in $E$ we have that:
   - (a) if $\text{start}(a) \in I$, then $\text{start}(x, v) \in A$;
   - (b) $\text{end}(a) \in I$ if and only if $\text{start}(a) \in M$ and $\text{end}(x, v) \in A$;
2. and for all $T \in I$ it holds that:
   - (a) for every other term $T' \in V$, if $D[T', T] \leq 0$, then $T' \in M \cup I$;
   - (b) for all $T' \in M$, $\delta \geq -D[T', T]$, i.e., all the lower bounds on $T$ are satisfied;
   - (c) for each other term $T' \in I$, either $D[T', T] = 0$, $D[T, T'] = 0$, or $D[T', T] = D[T, T'] = +\infty$.

We consider an event $\mu$ an I-match event if its actions correspond to the terms in $I$. The definition in Item 1 ensures the correct matching of each term to an action it represents and that the endpoints of a quantified token precisely identify the endpoints of a token in the event sequence. Meanwhile, Item 2 guarantees that matching the terms in $I$ does not
We also write violate any atomic temporal relation. In addition, Item 2a deals with the qualitative aspect of a “happens before” relation, while Items 2b and 2c address the quantitative aspects of the lower bounds of these relations. It is worth noting that an $\emptyset$-event is also considered admissible.

Let $\mathbb{M}_P$ denote the set of all matching structures for a planning problem $P$, and let $\mathbb{I}$ be the set of all possible terms built from token names in $\mathbb{N}$. To describe the evolution of a matching structure, we define a quaternary relation $S \subseteq \mathbb{M}_P \times \Sigma \times \mathbb{I} \times \mathbb{M}_P$ as $(\mathbb{M}, \mu, I, \mathbb{M}') \in S$, for an event $\mu = (A, \delta)$, if and only if $\mu$ is an $I$-match event for $\mathbb{M}$, and $\mathbb{M}' = (\mathbb{M} + \delta) \cup I$. We also write $\mathbb{M} \xrightarrow{\mu, I} \mathbb{M}'$ in place of $(\mathbb{M}, \mu, I, \mathbb{M}') \in S$. Note that, from Definition 4.5, a single event can represent multiple $I$-match events for a matching structure. Therefore, given a matching structure $\mathbb{M}$ and an event $\mu$, automaton states will collect all the matching structures $\mathbb{M}'$ resulting from the relation $S$, for some set of terms $I$. Given a set of matching structures $\Upsilon$, this notion is best described by the function $\text{step}_\mu(\Upsilon) = \{|\mathbb{M}'| (\mathbb{M}, \mu, I, \mathbb{M}') \in S, \text{ for some } \mathbb{M} \in \Upsilon \text{ and } I \in \mathbb{I}\}$. Furthermore, we define $\Upsilon^R_t \subseteq \Upsilon$ as the set of all the active matching structures $\mathbb{M} \in \Upsilon$ with timestamp $t$, associated with any existential statement of $R$. Matching structures in $\Upsilon^R_t$ contribute to fulfilling the same triggering event of $R$, regardless of their existential statement. We also define $\Upsilon^\perp \subseteq \Upsilon$ as the set of non-active matching structures of $\Upsilon$. Lastly, we say that $\Upsilon$ is \textit{closed} if there exists $\mathbb{M} \in \Upsilon$ such that $\mathbb{M}$ is \textit{closed}.

We conclude this section by providing an example of updating a matching structure $\mathbb{M} = (V, D, M, t)$ for the rule discussed at the beginning of the section. Consider the set of timelines in Fig. 3. Before matching any term $\mathbb{M}$ is initial with $M = \emptyset$, $t = 0$, $D$ as the DBM in Fig. 2, and $V$ as the set of term $\text{start}(a)$ and $\text{end}(a)$ for $a \in \{a_0, a_1, a_2, a_3\}$. We begin by matching the terms $\text{start}(a_0)$ and $\text{start}(a_3)$ from the event $\mu = (\{\text{start}(x_0, v_0), \text{start}(x_3, v_3)\}, 0)$ (we do not consider $\text{start}(x_1, v'_1)$ and $\text{start}(x_2, v'_2)$ since they are not in $V$). Such event is an $I$-match event for $I = \{\text{start}(a_0), \text{start}(a_3)\}$: it is an admissible event (Definition 4.4). Item 1a holds, for both $\text{start}(a_0)$ and $\text{start}(a_3)$, there are no terms that should appear before them (Item 2a), there are no related lower bounds (Item 2b), and $D[\text{start}(a_0), \text{start}(a_3)] = D[\text{start}(a_0), \text{start}(a_0)] = +\infty$ (Item 2c). Hence, we update $\mathbb{M} = \mathbb{M} \cup I = \{\text{start}(a_0), \text{start}(a_3)\}$ and $t = t + \delta = 0$; now $\mathbb{M}$ is active. The next term to consider is $\text{start}(a_2)$, which occurs after $\delta = 5$ time steps.

First, we ensure that the event $\mu = (\text{start}(x_2, v_2), 5)$ is admissible. We show that by examining the DBM in Fig. 2, we see that the elapsing of time $\delta$ does not exceed any upper bound related to terms $T \in \mathbb{M}$ and $T' \in \mathbb{M}$. Next, the set $I$ in the current state appears as $I = \{\text{start}(a_2)\}$. Notice that we are in the case of Item 1a, and Item 2 holds because no constraint involves the term $\text{start}(a_2)$ (Item 2a), no lower bounds are

---

**Figure 3:** Example of timelines for variables $x_0$, $x_1$, $x_2$, $x_3$. 

\[
\begin{align*}
\text{x}_0 &= v_0, & \text{x}_0 &= v'_0 \\
\text{x}_1 &= v'_1, & \text{x}_1 &= v_1, & \text{x}_1 &= v''_1 \\
\text{x}_2 &= v'_2, & \text{x}_2 &= v_2, & \text{x}_2 &= v''_2 \\
\text{x}_3 &= v_3, & \text{x}_3 &= v'_3
\end{align*}
\]
related to start($a_2$) (Item 2b), and start($a_2$) is the only term in $I$ (Item 2c). Therefore, from Definitions 4.2 and 4.3, we update $M$ as follows: $M = (M + \delta) \cup I$. Each entry of the DBM will remain unchanged since the third update case of Definition 4.2 applies, $M = M \cup I = \{\text{start}(a_0), \text{start}(a_3), \text{start}(a_2)\}$, and $t = t + \delta = 5$.

Similarly, for the next event is $\mu = (\text{start}(x_1, v_1), 1)$, we check if such an event is admissible, and indeed it is since the upper bound $D[\text{end}(a_0), \text{start}(a_1)] = 6 \geq \delta$. It is also an $I$-match event for $I = \{\text{start}(a_1)\}$, since it respects Item 1a and all the relations in Item 2; thus we update $M$. We decrement $D[\text{end}(a_3), \text{start}(a_2)]$ and increment $D[\text{start}(a_2), \text{end}(a_3)]$ by 1 (see Definition 4.2), update $M$ like follows $M = \{\text{start}(a_0), \text{start}(a_3), \text{start}(a_2)\}$, and $t = t + \delta = 5 + 1 = 6$.

The next event is $\mu = (\text{end}(x_3, v_3), 3)$ after 2 time steps. Note that it is an admissible event and also an $I$-match event for $I = \{\text{end}(a_3)\}$. In this case, we emphasize that Items 1b and 2 are respected. We update the DBM as follows: $D[\text{end}(a_0), \text{start}(a_1)] = 14 - 2 = 12$, $D[\text{start}(a_1), \text{end}(a_0)] = -4 + 2 = -2$, $D[\text{end}(a_3), \text{start}(a_2)] = 2 - 2 = 0$, $D[\text{start}(a_2), \text{end}(a_3)] = 1 + 2 = 3$. Then, we update $M = M \cup I = \{\text{start}(a_0), \text{start}(a_3), \text{start}(a_2)\}$ and $t = t + \delta = 6 + 2 = 8$. Notice that if we did not match end($a_3$) now, at the next time step, the timeline would have violated the rule above because the upper bound $D[\text{end}(a_3), \text{start}(a_2)] = 0$.

The subsequent event is $\mu = (\text{end}(x_1, v_1), \text{start}(x_1 = v_1'), 6)$ for which $I = \text{end}(a_1)$. Since there is no constraint involving end($a_1$), this event is admissible and an $I$-match event. The DBM is shifted by 6 time steps, and $M = \{\text{start}(a_0), \text{start}(a_3), \text{start}(a_2), \text{start}(a_1), \text{end}(a_3)\}$.

The last event $\mu = (\{\text{start}(x_0, v_0'), \text{start}(x_2, v_2'), \text{end}(x_0, v_0), \text{end}(x_2, v_2)\}, 2)$ is admissible and an $I$-match for $I = \{\text{end}(a_0), \text{end}(a_2)\}$; note that there is not an upper bound between end($a_0$) and end($a_2$) and that Items 1b and 2 of the definition of $I$-match event are respected.

4.3. Building the automaton. We can now define the automaton. First, given an existential statement $E$, let $E^E$ be the set of all existential statements in the same rule of $E$. Next, let $F_P$ be the set of functions that map each existential statement of $P$ to a set of existential statements and let $D_P$ be the set of functions that map each existential statement to a set of matching structures $T$. An automaton $T_P$ that checks the transition functions of the variables is easy to define. Then, given a timeline-based planning problem $P = (SV, S)$, we can characterize the corresponding automaton as $A_P = T_P \cap S_P$. Here, $S_P$ checks the fulfillment of the synchronization rules, and we define it as $S_P = (Q, \Sigma, q_0, F, \tau)$ where

1. $Q = 2^{2D_P} \times D_P \times F_P \cup \{\bot\}$ is the finite set of states, i.e., states are tuples of the form $(T, \Delta, \Phi) \in 2^{2D_P} \times D_P \times F_P$, plus a sink state $\bot$;
2. $\Sigma$ is the input alphabet defined above;
3. the initial state $q_0 = (T_0, \Delta_0, \Phi_0)$ is such that $T_0$ is the set of initial matching structures of the existential statements of $P$ and, for all existential statements $E$ of $P$, we have $\Delta_0(E) = \emptyset$ and $\Phi_0(E) = E^E$;
4. $F \subseteq Q$ is the set of final states defined as:
5. $\tau : Q \times \Sigma \to Q$ is the transition function that given a state $q = (T, \Delta, \Phi)$ and a symbol $\mu = (A, \delta)$ computes the new state $\tau(q, \mu)$. Let $\text{step}_E(q) = \{M_E \mid M_E \in \text{step}_E(q)\}$. Moreover, let $\Psi^R_i = \{E \mid M_E \in \text{step}_E(q)\}$. Then, the updated components of the state $\text{update} A(\text{initial state}, (q_0, q_0'))$ we can characterize the corresponding automaton as $A = (Q, \Sigma, q_0, F, \tau)$.
Theorem 4.14 can handle arbitrary event sequences as well, provided that we add suitable 

we introduce the Φ function, mapping existential statements to sets of existential statements, 

This is because there may be other sets, say ∆(E) tracks the satisfaction of the same trigger events as the set ∆(E). This way, when a set ∆(E) is closed, we can discard its matching structures as well as the matching structures in ∆(E').

In Section 4.4 we state and prove soundness and completeness of the automaton construction. Now, instead, let us address the size of the automaton.

Let us recall that we assumed that the timestamp of each event in an event sequence is bounded. However, it is worth noting that since events may have an empty set of actions, Theorem 4.14 can handle arbitrary event sequences as well, provided that we add suitable empty events. Let us now analyze the size of the automaton.
Theorem 4.6 (Size of the automaton). Let $P = (SV, S)$ be a timeline-based planning problem and let $A_P$ be the associated automaton. Then, the size of $A_P$ is at most doubly-exponential in the size of $P$.

Proof. We define $E$ as the overall number of existential statements in $P$, which is linear in the size of $P$. We can then observe that $|D_P| \in \mathcal{O}((2^{|M_P|})^E) = \mathcal{O}(2^{E|M_P|})$, thus the number of $\Delta$ functions is doubly exponential in the size of $P$. Next, note that $|F_P| \in \mathcal{O}((2^E)^E) = \mathcal{O}(2^{E^2})$. Then, $|S_P| \in \mathcal{O}(|\Sigma| \cdot 2^{|M_P|})$ indicating that the size of $S_P$ is at most exponential in the number of possible matching structures. To bound this number, we define $N$ as the largest finite constant appearing in $P$ in any atom or value duration and $L$ as the length of the longest existential prefix of an existential statement occurring inside a rule of $P$. Note that $N$ is exponential in the size of $P$ since constants are expressed in binary, while $L \in \mathcal{O}(|P|)$. We can then observe that the entries of a DBM for $P$, of which the number is quadratic in $L$, are constrained to take values within the interval $[-N, N]$ (excluding the value $+\infty$), which size is linear in $N$. By Definition 4.1, it follows that $|M_P| \in \mathcal{O}(N^{L^2} \cdot 2^L \cdot \text{window}(P))$ indicating that the number of matching structures is at most exponential in the size of $P$. \qed

Note that our automaton is the same size as the automaton built by Della Monica et al. in [DGMS18]. However, while their automaton is nondeterministic, ours is deterministic: an essential property to achieve the 2EXPTIME optimal asymptotic complexity for the synthesis procedure.

4.4. Soundness and Completeness. In the following, we present auxiliary notation, definitions, and essential lemmas for establishing the soundness and completeness of the automaton construction. For readability, we have included proofs in the appendix.

Definition 4.7 (Run of a matching structure). Let $\mu = \langle \mu_1, \ldots, \mu_n \rangle$ be a (possibly open) event sequence, and let $M_E$ be the initial matching structure of an existential statement $E$. A run of $M_E$ on $\mu$ yielding a matching structure $M_n$ is a sequence $T = \langle I_1, \ldots, I_n \rangle$ of $I$-match events for the matching structures $\langle M_E, M_1, \ldots, M_{n-1} \rangle$, such that for every $i \in [1, \ldots, n]$, $M_{i-1} \mu_i, I_i \rightarrow M_i$. We write $M_E \overset{\mu, T}{\rightarrow} M_n$ when such run exists, or $M_E \overset{\mu}{\rightarrow} M_n$, if $T$ is not relevant.

To link matching structures with the semantics of synchronization rules we establish a connection between matching functions (Definition 3.6) and runs.

Lemma 4.8 (Correspondence between runs and matching functions). Let $\mu = \langle \mu_1, \ldots, \mu_n \rangle$ be a (possibly open) event sequence, and let $M_E$ be the initial matching structure of an existential statement $E = \exists a_1[x_1 = v_1] \ldots a_k[x_k = v_k] \cdot C$, with $C$ augmented with atoms $\text{start}(a_i) \leq d_{\text{min}}^{a_i} \cdot d_{\text{max}}^{a_i}$ end($a_i$), for every $0 \leq i \leq k$. Then, there exists a run $T = \langle I_1, \ldots, I_n \rangle$ of $M_E$ on $\mu$, yielding a matching structure $M_n = (V, D_n, M_n, t_n)$, if and only if there exists a matching function $\bar{\gamma} : M_n \rightarrow [1, \ldots, n]$ such that, for every atom of the form $T \leq [l, u]$ $T'$ in $C$:

- (I) if $T' \in M_n$, then also $T \in M_n$, $\gamma(T) \leq \gamma(T')$, and $l \leq \delta(\mu_{\gamma(T)} \ldots \gamma(T')) \leq u$;
- (II) if $T' \notin M_n$, but $T \in M_n$, then $\delta(\mu_{\gamma(T)} \ldots \gamma(T')) \leq u$.

Furthermore, $\gamma$ and $T$ are such that for every $T \in M_n$, $T \in I_{\gamma(T)}$, i.e., they agree on the matching of the terms of $M_n$. We write $M_E \overset{\mu, \gamma}{\rightarrow} M_n$, if $\gamma$ corresponds to a run of $M_E$, or $\mu, \gamma \models M_n$, if $M_E$ is clear from the context.
Observation 4.9. Note that the existence of the matching function \( \gamma \) stated by Lemma 4.8, if the corresponding matching structure is closed, implies the satisfaction of the given existential statement, and vice versa.

We now state the core technical result of the completeness proof, which ensures no important details are lost when matching structures are discarded.

Lemma 4.10. Let \( \overline{\pi} = \langle \mu_1, \ldots, \mu_n \rangle \) be an event sequence, let \( M_E \) be the initial matching structure of some existential statement \( E \) of a rule \( R \), and let \( M_r \) be an active matching structure resulting from a run \( M_E \xrightarrow{\overline{\pi} ; \gamma} M_r \), such that \( \gamma_r(\text{start}(a_0)) = r \). If there exists a run \( M_E \xrightarrow{\overline{\pi} ; \gamma} M_s \), such that \( \gamma_s(\text{start}(a_0)) < r \), then there exists a run \( M_E \xrightarrow{\overline{\pi} ; \gamma} M \), such that \( \gamma(\text{start}(a_0)) = \gamma_s(\text{start}(a_0)) \) and \( M \) matches at least as many tokens as \( M_r \).

The last needed notion is that of residual matching structure, which is an active matching structure with only infinite bounds.

Definition 4.11 (Residual matching structure). A matching structure \( M = (V, D, M, t) \) is residual if it is active and for every \( T \in M \) and \( T' \in \overline{M} \), \( D[T', T] = +\infty \).

In other words, \( M \) does not impose any finite upper bound on the distance at which terms yet to be matched may appear relative to those already matched. The definition implies that for any residual matching structure, denoted as \( \overline{M} = (V, D, M, t) \), every event \( \mu = (A, \delta) \) is admissible. Additionally, it is never the case that \( \text{start}(a) \in M \) and \( \text{end}(a) \in \overline{M} \) for any quantified token \( a[x = v] \) of \( E \), given that such terms always have a finite upper bound in \( D \) that is at least as strict as the value \( d^{x=v}_{\text{max}} \). As a result, the “if” direction of Item 1b in the Definition 4.5 of I-match never applies to \( \overline{M} \) for any event \( \mu \). Therefore, every event is a valid \( \emptyset \)-match event for \( \overline{M} \).

Observation 4.12. Let \( M_E \xrightarrow{\overline{\pi}_1 ; \overline{T}_1} M \) be a run of the initial matching structure \( M_E \), on an event sequence \( \overline{\pi}_1 \), yielding a residual matching structure \( \overline{M} \). Then, for any event sequence \( \overline{\pi}_2 \), there exists a run \( M_E \xrightarrow{\overline{\pi}_2 ; \overline{T}_2 ; \overline{T}_2'} M' \) such that every I-match event in \( \overline{T}_2 \) is an \( \emptyset \)-match event and \( M' \) differs from \( \overline{M} \) by at most the value of the component \( t \). Consequently, whenever a residual matching structure appears in a run, it has the potential to remain there indefinitely, which is why it is called residual.

Lemma 4.13 (Existence of residual matching structure). Let \( \overline{\pi} = \langle \mu_1, \ldots, \mu_n \rangle \) be an event sequence, and let \( M_n \) be an active matching structure such that \( \overline{\pi}, \gamma \models M_n \) and \( \delta(\overline{\pi};[\gamma(\text{start}(a_0))...n]) > \text{window}(P) \). If we consider the intermediate matching structures \( \langle M_1, \ldots, M_{n-1} \rangle \) of the run \( M_E \xrightarrow{\overline{\pi} ; \gamma} M_n \), then there exists a position \( \gamma(\text{start}(a_0)) \leq k < n \) such that \( M_k \) is a residual matching structure.

We are now ready to prove the final result.

Theorem 4.14 (Soundness and completeness). Let \( P = (SV, S) \) be a timeline-based planning problem and let \( A_P \) be the associated automaton. Then, any event sequence \( \overline{\pi} \) is a solution plan for \( P \) if and only if \( \overline{\pi} \) is accepted by \( A_P \).
5. Controller synthesis

We leverage the deterministic automaton constructed in the previous section to establish a deterministic arena that enables us to solve a reachability game and determine whether a controller exists. If a controller exists, our procedure allows its synthesis.

5.1. From the automaton to the arena. Let $G = (SV_C, SV_E, S, D)$ be a timeline-based game. The automaton construction we used considered a planning problem with a single set of synchronization rules, while in $G$, we have to account for the roles of both $S$ and $D$.

To address this, we define $A_S$ and $A_D$ as the deterministic automata constructed over the timeline-based planning problems $P_S = (SV_C \cup SV_E, S)$ and $P_D = (SV_C \cup SV_E, D)$, respectively. We then construct the automaton $A_G$ by taking the union of $A_S$ with the complement of $A_D$ ($\overline{A_D}$). Note that these are standard automata-theoretic operations over DFAs. An accepting run of $A_G$ represents either a plan that violates the domain rules or a plan that adheres to domain and system rules, according to the definition of winning strategy in Definition 3.16. Furthermore, $A_G$ is deterministic, and its size only polynomially increases when built from $A_D$ and $A_S$.

The $A_G$ automaton is not immediately applicable as a game arena since its transitions' labels only reflect events, not game moves. In $A_G$, a single transition can correspond to various combinations of rounds due to the absence of $\text{wait}(\delta)$ moves in the transition's label. For example, an event $\mu = (A, 5)$ can arise from either a $\text{wait}(5)$ move by $\text{Charlie}$, followed by a play$(5, A)$ move by $\text{Eve}$, or any $\text{wait}(\delta)$ move with $\delta > 5$ followed by a play$(5, A)$ move. To obtain a suitable game arena, we need to modify $A_G$ further.

Let $A_G = (Q, \Sigma, q_0, F, \tau)$ be the automaton constructed as described above. Formally, we define a new automaton $A'_G = (Q, \Sigma, q_0, F, \tau')$ where $\tau'$ is a partial transition function, meaning that the automaton is now incomplete. The function $\tau'$ agrees with $\tau$ on all transitions except those of the form $\tau(q, (A, \delta))$, where $\delta > 1$ and $A$ contains an $\text{end}(x, v)$ action with $x \in SV_C$. In such cases, the transition is undefined in $A'_G$. An example is shown in Figure 4 (left). Note that this removal does not alter the set of plans accepted by the automaton since for each transition $\tau(q, (A, \delta)) = q'$ with $\delta > 1$, there exist two transitions $\tau(q, (\varnothing, \delta - 1)) = q''$ and $\tau(q'', (A, 1)) = q'$ in $A'_G$.

To make the game rounds and moves explicit, we can transform the automaton by splitting each transition into four transitions representing the four moves of the two rounds. Starting from the incomplete automaton $A'_G = (Q, \Sigma, q_0, F, \tau')$, we define a new automaton $A^a_G = (Q^a, \Sigma^a, q_0^a, F^a, \tau^a)$ as the game arena.

1. The set of states $Q^a$ is given by $Q^a = Q \cup \{q_\delta \mid 1 \leq \delta \leq d\} \cup \{q_{\delta, A} \mid 1 \leq \delta \leq d, A \subseteq A\}$.
2. The alphabet $\Sigma^a$ is defined as $\Sigma^a = \mathcal{M}_C \cup \mathcal{M}_E$, which corresponds to the set of moves of the two players.
3. The initial and final states of $A^a_G$ are $q_0^a = q_0$ and $F^a = F$, respectively.
4. The partial transition function $\tau^a$ is defined as follows. Let $w = \tau(q, \mu)$ with $\mu = (A, \delta)$.
   - We distinguish the cases where $\delta = 1$ or $\delta > 1$.
   - (a) if $\delta = 1$, let $A_C \subseteq A$ and $A_E \subseteq A$ be the set of actions in $A$ playable by $\text{Charlie}$ and by $\text{Eve}$, respectively. Then:
     - (i) $\tau(q_1, \text{play}(A'_C)) = q_{1, A'_C}$, where $A'_C$ is the set of $\text{ending}$ actions in $A_C$;
     - (ii) $\tau(q_1, A'_C, \text{play}(A'_E)) = q_{1, A'_C \cup A'_E}$, where $A'_E$ is the set of $\text{ending}$ actions in $A_E$;
     - (iii) $\tau(q_1, A'_C \cup A'_E, \text{play}(A''_C)) = q_{1, A'_C \cup A'_E \cup A''_C}$, where $A''_C$ is the set of $\text{starting}$ actions in $A_C$. 
Figure 4: On the left, the removal of transitions \( \mu = (A, \delta) \) with \( \delta > 1 \) and ending actions of controllable tokens in \( A \). On the right, the transformation of a transition of \( A \) into a sequence of transitions in \( A' \), with \( x \in SV_C \), \( y \in SV_E \), and \( \gamma_x(v_1) = \gamma_y(w_1) = u \).

(iv) \( \tau(q_1, A_C^c \cup A_E^c \cup A_s^c, \text{play}(A_E^c)) = w \), where \( A_E^c \) is the set of starting actions in \( A_E \); here, the states mentioned are added to \( Q^a \) as needed.

(b) if \( \delta > 1 \), let \( A_C \subseteq A \) and \( A_E \subseteq A \) be the set of actions in \( A \) playable by Charlie and Eve, respectively. Note that by construction, \( A_C \) only contains starting actions. Then:

(i) \( \tau(q, \text{wait}(\delta)) = q_\delta \) for all \( \delta \leq \delta_C \leq d \);
(ii) \( \tau(q_\delta, A_E^e) = q_\delta A_E^c \), where \( A_E^c \) is the set of ending actions in \( A_E \);
(iii) \( \tau(q_\delta A_E^c, \text{play}(A_C)) = q_\delta A_E^c \cup A_C \);
(iv) \( \tau(q_\delta A_E^c \cup A_C, \text{play}(A_E^c)) = w \) where \( A_E^c \) is the set of starting actions in \( A_E \); where the mentioned states are added to \( Q^a \) as needed.

All the transitions not explicitly defined above are undefined.

We present a graphical illustration of the above construction in Fig. 4. It is worth noting that the automaton preserves the structure of the original automaton \( A_G \). For any state, \( q \in Q \) and event \( \mu = (A, \delta) \), any sequence of moves that would result in appending \( \mu \) to the partial plan (see Definition 3.13) reaches the same state \( w \) in \( A_G^a \) as it does in \( A_G \) by reading \( \mu \). Therefore, we can consider \( A_G^a \) as being able to read event sequences, even though its alphabet is different. We use the notation \( [\mu] \) to represent the state \( q \in Q^a \) reached by reading \( \mu \) in \( A_G^a \). Furthermore, note that, with a slight abuse of notation, any play \( \rho \) in the game \( G \) is a readable word by the automaton \( A_G^a \). Thus, we can establish the following result.

**Theorem 5.1.** If \( G \) is a timeline-based game, for any play \( \rho \) for \( G \), \( \rho \) is successful if and only if it is accepted by \( A_G^a \).
5.2. Computing the Winning Strategy and Building the Controller. Let us define $Q^a_C \subseteq Q^a$ as the set of states in which Charlie can make a move, and $Q^a_E = Q^a \setminus Q^a_C$ as the set of states where Eve can make a move. Additionally, we define $E = \{(q, q') \in Q^a \times Q^a | \exists \mu . \tau^a(q, \mu) = q'\}$ as the set of edges in $A^a_G$. By solving the reachability game $(G^a_R, W)$, where $G^a_R = (Q^a, Q^a_C, E^a)$ and $W = \{R \subseteq Q^a | R \cap F^a \neq \emptyset\}$, we aim to determine the winning region $W^a_C$ and the winning strategy $s_C$ for Charlie, provided they exist. In the following discussion, we will show that the winning strategy $\sigma_C$ for the timeline-based game $G$ is derivable from strategy $s_C$ when $q^a_0 \in W^a_C$.

To determine the winning region $W^a_C$, we use the well-known attractor construction. We are interested to the attractor set of $F^a$ for Charlie, written $Attr^i_C(F^a)$, thus given $i \geq 0$ we compute the set of states from which Charlie can reach a state $q \in F^a$ within $i$ moves, defined as $Attr^i_C(F^a)$:

$$Attr^0_C(F^a) = F^a$$

$$Attr^{i+1}_C(F^a) = Attr^i_C(F^a)$$

$$\cup \{q^a \in Q^a_C | \exists r((q^a, r) \in E \land r \in Attr^i_C(F^a))\}$$

$$\cup \{q^a \in Q^a_E | \forall r((q^a, r) \in E \rightarrow r \in Attr^i_C(F^a))\}.$$  

The sequence $Attr^0_C(F^a) \subseteq Attr^1_C(F^a) \subseteq Attr^2_C(F^a) \subseteq \ldots$ eventually becomes stationary for some index $k \leq |Q^a|$, hence we can define $Attr^*_C(F^a) = \bigcup_{i=0}^{Q^a} Attr^i_C(F^a)$ as the attractor set. Note that $W^a_C = Attr^*_C(F^a)$ is a known fact for which proof is available in [Tho08]. Next, we want that $q^a_0 \in W^a_C$ since we are interested in a winning strategy $\sigma_C$ for the timeline-based game $G$. If it is the case, by defining $s_C(q) = \mu$ for any $\mu$ such that $\tau^a(q, \mu) = q'$ with $q, q' \in W^a_C$, which is guaranteed to exist by the attractor construction, we can define $\sigma_C$ for Charlie in $G$ as $\sigma_C(\overline{p}) = s_C(\overline{p})$ for any event sequence $\overline{p}$. We prove this claim in the following:

**Theorem 5.2.** Given $A^a_G = (Q^a, \Sigma^a, q^a_0, F^a, \tau^a)$, $q^a_0 \in W^a_C$ if and only if $\sigma_C$ is a winning strategy for Charlie in $G$.

**Proof.** ($\rightarrow$). From the definition of a winning strategy for Charlie in $G$ (Definition 3.16), we know that for every admissible strategy $\sigma_E$ for Eve, there exists $n \geq 0$ such that the plays $\overline{p}_n(\sigma_C, \sigma_E)$ is successful. By the soundness of the arena construction (Theorem 5.1), we know that the event sequence $\overline{p}_n$ representing $\overline{p}_n(\sigma_C, \sigma_E)$, when seen as a word over $\Sigma^a$, is accepted by $A^a_G$. Therefore, $\overline{p}_n$ reaches a state in the set $F^a$ starting from $q^a_0$. By the definition of the reachability game, this means that $q^a_0 \in W^a_C$. Thus, we have proved that if $\sigma_C$ is a winning strategy Charlie in $G$, then $q^a_0 \in W^a_C$.

($\leftarrow$). If $q^a_0 \in W^a_C$, then by definition, $s_C$ is a winning strategy for Charlie in the reachability game over the arena $A^a_G$. Hence, any word over $\Sigma^a$ obtained by playing with $s_C$ is accepted by $A^a_G$, and therefore, by the soundness of the arena construction (Theorem 5.1), any corresponding play $\overline{p}$ is successful in $G$. Now, recall that $\sigma_C(\overline{p}) = s_C(\overline{p})$ for any event sequence $\overline{p}$. Hence, $\overline{p} = p(\sigma_C, \sigma_E)$ for some strategy $\sigma_E$ of Eve. As a result, we can conclude that $\sigma_C$ is a winning strategy for Charlie in $G$. □

Finally, we build a Controller that implements the winning strategy $\sigma_C$, provided it exists. First, by Theorem 5.2, the existence of $\sigma_C$ implies that $q^a_0 \in W^a_C$. Next, we define the following Moore machine (Definition 3.19) based on $s_C$:  

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Definition 5.3 (Controller). Given $A^a_G = (Q^a, \Sigma^a, q^a_0, F^a, \tau^a)$, we define a Controller as $M = (Q, \Sigma, \Gamma, q_0, \delta, \tau)$, where $Q = Q^a_C \cap \mathcal{W}_C$ represents the set of states, $q_0 = q^a_0$ is the initial state, $\Sigma = \mathcal{M}_E$ is the input alphabet, $\Gamma = \mathcal{M}_C$ is the output alphabet, $\delta: Q \times \Sigma \rightarrow Q$ is the transition function, and $\tau: Q \rightarrow \Gamma$ is the output function. The transition function $\delta$ and the output function $\tau$ are defined as follows:

$$\delta(q_C, \mu_E) = \tau^a(s_C(q_C), \mu_E)$$
$$\tau(q_C) = s_C(q_C).$$

Note that by construction the states of $M$ belong to the winning region $W_C$ of $A^a_G$, and $\delta$ follows the transition function $\tau^a$ of $A^a_G$. Hence, the output of $M$ after reading a word $\bar{\mu}$ is exactly $\sigma_C(\bar{\mu}) = s_C(\mu_E)$ and $M$ implements $\sigma_C$, which is a winning strategy by Theorem 5.2.

6. Conclusions and Future Work

Our article presents an effective procedure for synthesizing controllers for timeline-based games, whereas previously, only a proof of the $2\text{EXPTIME}$-completeness of the problem of determining the existence of a strategy was available in the literature. We use a novel construction of a deterministic automaton of doubly-exponential (thus optimal) size, which is then adapted to serve as the arena for the game. Then, with standard methods, we solve a reachability game on the arena to effectively compute the winning strategy for the game, if it exists.

This work paves the way for future developments. First, the procedure provided in this article can be realistically implemented and tested. It is conceivable, though, that to avoid the state explosion problem due to the doubly-exponential size of the automaton, it will be necessary to apply symbolic techniques. Moreover, an implementation would also need a concrete syntax to specify timeline-based games. Existing languages supported by timeline-based systems (e.g., NDDL [CO96] or ANML [SFC08]) might be inadequate for this purpose. Next, as in the case of LTL, the high complexity makes one wonder whether simpler but still expressive fragments can be found. One possibility might be restricting the synchronization rules to only talk about the past concerning the rule’s trigger. For co-safety properties (i.e., properties expressing the fact that something good will eventually happen) expressed in pure-past LTL, the realizability problem goes down to being EXPTIME-complete, and by analogy, this might happen to pure-past timeline-based games as well.

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Let $\overline{\pi} = \langle \mu_1, \ldots, \mu_n \rangle$ be a (possibly open) event sequence, and let $M_E$ be the initial matching structure of an existential statement $E \equiv \exists x_1, \ldots, x_k \ [x_1 = v_1] \ldots [x_k = v_k].$ If $\overline{\pi}$ was augmented with atoms $\text{start}(a_i) \leq [d^0_{\mu} = v_i, d^\infty_{\mu} = v_i] \text{end}(a_i)$ for every $0 \leq i \leq k.$ Then, there exists a run $T = \langle I_1, \ldots, I_n \rangle$ of $M_E$ on $\overline{\pi}$, yielding a matching structure $M_n = (V, D_n, M_{\overline{\pi}}, t_n)$, if and only if there exists a matching function $\gamma : M_n \rightarrow [1, \ldots, n]$ such that, for every atom of the form $T \leq [\mu, u, \gamma]$, $T'$ in $C$:

(I) if $T' \in M_n$, then also $T \in M_n$, $\gamma(T) \leq \gamma(T')$, and $l \leq \delta(\overline{\pi}[\gamma(T) \ldots (T')]) \leq u$;

(II) if $T' \notin M_n$, but $T \in M_n$, then $\delta(\overline{\pi}[\gamma(T) \ldots n]) \leq u.$

Furthermore, $\gamma$ and $T$ are such that for every $T \in M_n$, $T \in I_{\gamma(T)}$, i.e., they agree on the matching of the terms of $M_n$. We write $M_E \overline{\pi} \gamma \rightarrow M_n$, if $\gamma$ corresponds to a run of $M_E$, or $\overline{\pi}, \gamma \models M_n$, if $M_E$ is clear from the context.

Proof. ($\Leftarrow$). We proceed by induction on the length of the event sequence $\overline{\pi} = \langle \mu_1, \ldots, \mu_n \rangle$.

Base case. If $n = 0$, the only well defined function on an empty codomain is the function $\gamma_0 : \emptyset \rightarrow \emptyset$ with an empty domain, which vacuously satisfies the definition of matching function and Items (I) and (II). Then, the only run of $M_E = (V, D, \emptyset, 0)$ on an empty event sequence $\overline{\pi}$ is the empty run $\overline{T}$ yielding $M_E$ itself, which vacuously satisfies the definition of run.

Inductive step. Let $\gamma : M_n \rightarrow [1, \ldots, n]$ be a matching function satisfying Items (I) and (II), and let $\gamma|_{<n} : M_{n-1} \rightarrow [1, \ldots, n-1]$ be the restriction of $\gamma$ on the domain $M_{n-1}$ defined as the inverse image of $[1, \ldots, n-1]$ under $\gamma$, i.e., $M_{n-1} = \gamma^{-1}([1, \ldots, n-1])$. $\gamma|_{<n}$ is a matching function for the event sequence $\overline{\pi}[1 \ldots n-1]$ and satisfies Items (I) and (II). By the inductive hypothesis, there exists a run $\langle I_1, \ldots, I_{n-1} \rangle$ of $M_E$ on $\overline{\pi}[1 \ldots n-1]$, yielding a matching structure $M_{n-1} = (V, D_{n-1}, M_{n-1}, t_{n-1})$. Let $I_n = \gamma^{-1}(n)$, and note that $I_n \subseteq M_{n-1}$. We show that $\mu_n = (A_n, \delta_n)$ is an $I_n$-match event for $M_{n-1}$ by breaking the proof in steps.

(Step: $\mu_n$ is an admissible event for $M_{n-1}$). Let $T \in M_{n-1}$ and $T' \notin M_{n-1}$. If $D_{n-1}[T', T] = +\infty$, $\delta_n \leq D_{n-1}[T', T]$ trivially holds. Otherwise, there exists an atom $T \leq [\mu, u]$ in $C$ and $D_{n-1}[T', T] = u - \delta(\overline{\pi}[\gamma|_{<n}(T) \ldots n-1])$. We consider two cases based on whether $T'$ belongs to the domain of $\gamma$, or not. In the first case, $\gamma(T') = n$ and $\delta(\overline{\pi}[\gamma(T) \ldots n-1]) + \delta_n = \delta(\overline{\pi}[\gamma(T) \ldots n]) \leq u$, by Item (I). In the second case, $\delta(\overline{\pi}[\gamma(T) \ldots n-1]) + \delta_n = \delta(\overline{\pi}[\gamma(T) \ldots n]) \leq u$, by Item (II). In either case, $\delta_n \leq u - \delta(\overline{\pi}[\gamma(T) \ldots n-1]) = D_{n-1}[T', T]$.
(Step: Item 1a of Definition 4.5). Let $a[x = v]$ be a quantified token of $E$. If $\text{start}(a) \in I_n$, then $\gamma(\text{start}(a)) = n$ and by definition of matching function $\text{start}(x, v) \in A_n$. 

(Step: Item 1b of Definition 4.5). $(\leftarrow)$. Let $\text{end}(a) \notin M_{n-1}$ be a possible candidate for inclusion in $I_n$. If $\text{start}(a) \in M_{n-1}$ and $\text{end}(x, v) \in A_n$, then $\text{end}(x, v)$ ends the token started at $\mu_{\text{start}(a)}$; otherwise, there would exist $\mu_i = (A_i, \delta_i)$ prior to $\mu$, such that $\text{end}(x, v) \in A_i$, contradicting that $\gamma(\mu) \neq n$ is undefined on $\text{end}(a)$. By definition of matching function, since $\text{end}(x, v) \in A_n$, ends the token started at $\mu_{\text{start}(a)}$, we have $\gamma(\text{end}(a)) = n$ and $\text{end}(a) \in I_n$. 

(Step: Item 1b of Definition 4.5). $(\rightarrow)$. If $\text{end}(a) \in I_n$, then by definition of matching function $\text{end}(x, v) \in A_n$. Furthermore, since $\text{end}(a) \in M_n$, Item (I) gives $\gamma(\text{start}(a)) \leq \gamma(\text{end}(a))$ for the atom start($a$, $l$, $u$) $\leq [n, u]$ end($a$) in $C$. By definition of event sequence, start($x$, $v$) and $\text{end}(x, v)$ cannot appear in the same event; hence, $\gamma(\text{start}(a)) < \gamma(\text{end}(a)) = n$ and $\text{start}(a) \in M_{n-1}$.

(Step: Item 2a of Definition 4.5). Let $T$ be a term in $I_n$, and let $\gamma(T') \in V$ be any other term such that $D_{n-1}[T', T] \leq 0$. Then, $D_{n-1}[T', T]$ can either be the lower bound of an atom $T' \leq [l, u]$ $T$, or the upper bound of an atom $T \leq [l, u]$ $T'$ in $C$. In the first case, we can directly conclude that $T' \in M_{n-1} \cup I_n$, because $T' \in M_n$ by Item (I) of $\gamma$ and $M_n = M_{n-1} \cup I_n$ by definition of $M_{n-1}$ and $I_n$. In the second case, note that $D_{n-1}[T', T] = u$, i.e., it has never been decremented because $T \notin M_{n-1}$, and that upper bounds $u$ can never be negative. Thus, $u$ is equal to 0 and $\gamma$ satisfies $0 \leq \delta(\gamma(T')) \leq 0$ (Item (I)), meaning that $\gamma(T') = \gamma(T)$ and $T' \in I_n$.

(Step: Item 2b of Definition 4.5). Let $T$ be a term in $I_n$ and $\gamma(T') \in V$ be any other term such that $D_{n-1}[T', T] \leq 0$. Then, $D_{n-1}[T', T]$ cannot be the upper bound of an atom $T' \leq [l, u]$ $T'$; otherwise, Item (I) would imply $T \in M_{n-1}$, contradicting that $\text{start}(a) \in M_{n-1}$. Thus, $D_{n-1}[T', T]$ must either represent the lower bound of an atom $T' \leq [l, u]$ $T$ in $C$, or be equal to $+\infty$. In the latter case, $\delta_n \geq -D_{n-1}[T', T]$ trivially holds. In the former case, $D_{n-1}[T', T] = -l + \delta(\gamma(T')) \geq l$. Hence, $\delta_n \geq l - \delta(\gamma(T')) \geq -D_{n-1}[T', T]$.

(Step: Item 2c of Definition 4.5). Let $T, T' \in I_n$ be two distinct terms. Then, $\gamma(T') = \gamma(T)$ and $\delta(\gamma(T')) = 0$. If $T \leq [l, u]$ $T'$ (resp., $T' \leq [l, u]$ $T$) belongs to $C$, then $D_{n-1}[T', T]$ (resp., $D_{n-1}[T', T]$) is the lower bound $l$ and equals 0 by Item (I). Otherwise, $D_{n-1}[T', T] = D_{n-1}[T', T] = +\infty$.

Hence, $M_{n-1} \mu_n \rightarrow M_n$ is well defined and $\langle I_1, \ldots, I_n \rangle$ is a run of $M_E$ on $\overline{\mu}$ yielding $M_n$.

$(\rightarrow)$. We proceed by induction on the length of the event sequence $\overline{\mu} = \langle \mu_1, \ldots, \mu_n \rangle$.

**Base case.** An empty run $\overline{T}$ yields $M_E = (V, D, \varnothing, 0)$ itself. Then the function $\gamma_0 : \varnothing \rightarrow \varnothing$ vacuously satisfies the definition of matching function and Items (I) and (II).

**Inductive step.** Let $\overline{T} = \langle I_1, \ldots, I_n \rangle$ be a run of $M_E$ on $\overline{\mu}$ yielding a matching structure $M_n = (V, D, M_n, t_n)$. Note that $\overline{T}_{[1, \ldots, n-1]}$ is a run of $M_E$ on $\overline{\mu}_{[1, \ldots, n-1]}$ yielding a matching structure $M_{n-1} = (V, D_{n-1}, M_{n-1}, t_{n-1})$. By the inductive hypothesis, there exists a matching function $\gamma_{<n} : M_{n-1} \rightarrow [1, \ldots, n-1]$ satisfying Items (I) and (II). Let $\gamma : M_n \rightarrow [1, \ldots, n]$ be the extension of $\gamma_{<n}$ to $M_n$, such that $\gamma(T) = n$, for all $T \in I_n$.

(Step: $\gamma$ is a matching function). Items 1 and 2 hold for all the terms already present in the domain of $\gamma_{<n}$. For every term in $I_n$, Item 1 for $\gamma$ follows from Item 1 of $I_n$-match event. Let $\text{start}(a), \text{end}(a) \in M_n$ be two terms not both already present in $M_{n-1}$, meaning that $\text{start}(a) \in M_{n-1}$ and $\text{end}(a) \in I_n$, for some quantified token $a[x = v]$ in $E$. By definition of $I_n$-match event, $\mu_n = (A_n, \delta_n)$ is such that $\text{end}(x, v) \in A_n$ and no other event in
\(\mu[\gamma_{\leq}(T)_{-n}^{-1}]\) contains an action end\((x,v)\), otherwise end\((a)\) would already belong to \(M_{n-1}\) (by Item 1b of I-match event). Thus, end\((x,v)\in A_n\) ends the token started at \(\mu_{\gamma}(\text{start}(a))\), and \(\gamma(\text{start}(a))\) and \(\gamma(\text{end}(a))\) correctly identify the endpoints of such token.

(Step: Item (I) of Lemma 4.8). Let \(T \leq [l,u]\) \(T'\) be an atom in \(C\), and note that \(\gamma\) already satisfies Item (I) for every \(T' \in M_{n-1}\). If \(T' \in \mathcal{I}_n\) instead, consider the entry \(D_{n-1}[T,T']\) representing the lower bound \(l\) of the aforementioned atom. If \(D_{n-1}[T,T'] \leq 0\), Item 2a of I-match event gives \(T \in M_{n-1} \cup \mathcal{I}_n = M_n\). If \(D_{n-1}[T,T'] > 0\), \(D_{n-1}[T,T']\) no longer stores its initial value \(-l \leq 0\), meaning that \(T\) must have been previously matched and \(T \in M_{n-1} \subseteq M_n\). In either case, \(T \in \mathcal{I}_n\) and \(\gamma(T) \leq \gamma(T')\), because \(\gamma(T) \leq n\). If \(T \in \mathcal{I}_n\), then \(\delta(\mathcal{P}_{[\gamma(T)_{-n}]}) = 0 \leq u\), is trivially satisfied by any upper bound \(u\). Furthermore, by Item 2c of I-match event, either the lower bound \(D_{n-1}[T,T'] = 0\) or the upper bound \(D_{n-1}[T',T] = 0\), and they both equal their initial values \(l\) and \(u\). Note that the former case is also implied by the latter, so that \(l = 0\) and \(l \leq \delta(\mathcal{P}_{[\gamma(T)_{-n}]})\). If \(T \in \mathcal{I}_n\), by Item 2b of I-match event, \(\delta_n \geq -D[T,T'] = l - \delta(\mathcal{P}_{[\gamma(T)_{-n}]})\). Hence, \(l \leq \delta(\mathcal{P}_{[\gamma(T)_{-n}]}) + \delta_n = \delta(\mathcal{P}_{[\gamma(T)_{-n}]})\). While \(\delta_n \leq D_{n-1}[T',T] = u - \delta(\mathcal{P}_{[\gamma(T)_{-n}]})\), since \(\mu_n\) is an admissible event for \(M_{n-1}\). Hence, \(\delta(\mathcal{P}_{[\gamma(T)_{-n}]}) + \delta_n = \delta(\mathcal{P}_{[\gamma(T)_{-n}]}) \leq u\).

(Step: Item (II) of Lemma 4.8). Let \(T \leq [l,u]\) \(T'\) be an atom in \(C\) such that \(T \in \mathcal{I}_n\) and \(T' \notin M_n\). Since \(\mu_n\) is an admissible event for \(M_{n-1}\), \(\delta_n \leq D_{n-1}[T',T] = u - \delta(\mathcal{P}_{[\gamma(T)_{-n}]})\).

Lemma 4.10. Let \(\mathcal{P} = \langle \mu_1, \ldots, \mu_n \rangle\) be an event sequence, let \(M_E\) be the initial matching structure of some existential statement \(E\) of a rule \(R\), and let \(M_r\) be an active matching structure resulting from a run \(M_E \xrightarrow{\pi_r} M_r\), such that \(\gamma_r(\text{start}(a_0)) = r\). If there exists a run \(M_E \xrightarrow{\pi_s} M_s\), such that \(\gamma_s(\text{start}(a_0)) < r\), then there exists a run \(M_E \xrightarrow{\pi_s} M\), such that \(\gamma(\text{start}(a_0)) = \gamma_s(\text{start}(a_0))\) and \(M\) matches at least as many tokens as \(M_r\).

Proof. Let \(M_E \xrightarrow{\pi_r} M_r = (V, D_r, M_r, t_r)\) and \(M_E \xrightarrow{\pi_s} M_s = (V, D_s, M_s, T_s)\), with \(\gamma_s(\text{start}(a_0)) \leq \gamma_r(\text{start}(a_0))\). Let \(M = M_r \cup M_s\) and \(\gamma : M \rightarrow [1, \ldots, n]\) be a function defined as:

\[
\gamma(T) = \begin{cases}
\gamma_s(T) & \text{if } T \in M_s \cap M_r \text{ and } \gamma_s(T) \leq \gamma_r(T) \\
\gamma_r(T) & \text{if } T \in M_s \cap M_r \text{ and } \gamma_s(T) > \gamma_r(T) \\
\gamma_s(T) & \text{if } T \in M_s \setminus M_r \\
\gamma_r(T) & \text{if } T \in M_r \setminus M_s
\end{cases}
\]

(Step: \(\gamma\) is a matching function). Item 1 of Definition 3.6 for \(\gamma\) follows from our hypothesis on \(\gamma_s\) and \(\gamma_r\). Regarding Item 2, let \(\text{start}(a), \text{end}(a) \in M\) for some quantified token \(a[x=v]\) in \(E\). If \(\gamma_s\) and \(\gamma_r\) map the endpoints of \(a\) to the same token in \(\mathcal{P}\), then \(\gamma(\text{start}(a))\) and \(\gamma(\text{end}(a))\) correctly identify the endpoints of that token. If instead \(\gamma_s\) and \(\gamma_r\) map \(a\) to two distinct tokens in \(\mathcal{P}\), then \(\gamma\) would match \(a\) according to the function whose token comes first, correctly identifying the endpoints of such token.

(Step: \(\gamma\) satisfies Items (I) and (II) of Lemma 4.8). Let \(T \leq [l,u]\) \(T'\) be an atom in \(C\). If \(T' \in M\), then either \(T' \in M_s\), and \(T \in M_s \subseteq M\), or \(T' \in M_r\), and \(T \in M_r \subseteq M\). If \(\gamma\) maps both terms with either \(\gamma_s\) or \(\gamma_r\), then \(\gamma(T) \leq \gamma(T')\) and \(l \leq \delta(\mathcal{P}_{[\gamma(T)_{-n}]}) \leq u\) immediately follows. If instead \(\gamma(T) = \gamma_s(T)\) and \(\gamma(T') = \gamma_r(T')\), then \(T' \in M_r\) and \(T \in M_s \cap M_r\). By definition of \(\gamma\), \(\gamma_s(T) \leq \gamma_r(T)\), and, by Item (I) for \(\gamma_r\), \(\gamma_r(T) \leq \gamma_r(T')\). Hence, \(\gamma(T) \leq \gamma(T')\).
If \( T' \in M_s \), then \( \gamma_s(T') > \gamma_r(T') \), and:

\[
\begin{aligned}
l &\leq \delta_{\gamma_r(T), \gamma_r(T')} & \text{Item (I) for } \gamma_r \\
\leq \delta_{\gamma_s(T), \gamma_r(T')} & \gamma_s(T) \leq \gamma_r(T) \\
< \delta_{\gamma_s(T), \gamma_s(T')} & \gamma_s(T') > \gamma_r(T') \\
\leq u & \text{Item (I) for } \gamma_s
\end{aligned}
\]

otherwise:

\[
\begin{aligned}
l &\leq \delta_{\gamma_r(T), \gamma_r(T')} & \text{Item (I) for } \gamma_r \\
< \delta_{\gamma_s(T), \gamma_r(T')} & \gamma_s(T) \leq \gamma_r(T) \\
\leq \delta_{\gamma_r(T), n} & \gamma_r(T') \leq n \\
\leq u & \text{Item (II) for } \gamma_s
\end{aligned}
\]

The case for \( \gamma(T) = \gamma_r(T) \) and \( \gamma(T') = \gamma_s(T') \) is completely symmetrical.

Lastly, if \( T' \notin M \), but \( T \in M \), then either \( \gamma(T) = \gamma_s(T) \) or \( \gamma(T) = \gamma_r(T) \), and Item (II) for \( \gamma \) follows from Item (II) for \( \gamma_s \) and \( \gamma_r \).

\[
\text{Lemma 4.13 (Existence of residual matching structure). Let } \vec{\mu} = \langle \mu_1, \ldots, \mu_n \rangle \text{ be an event sequence, and let } M_n \text{ be an active matching structure such that } \vec{\mu}, \gamma \models M_n \text{ and } \delta(\vec{\mu}[(\gamma_{(T)}(0))...]) > \text{window}(P). \text{ If we consider the intermediate matching structures } \langle M_1, \ldots, M_{n-1} \rangle \text{ of the run } M_E \xrightarrow{\vec{\mu},\gamma} M_n, \text{ then there exists a position } \gamma(\text{start}(a_0)) \leq k < n \text{ such that } M_k \text{ is a residual matching structure.}
\]

\[
\text{Proof. Let } \gamma(\text{start}(a_0)) = s, \text{ assuming there is no residual matching structure } M_k \text{ in the sequence } \langle M_s, \ldots, M_{n-1} \rangle, \text{ then for every matching structure } M_i = (V, D_i, M_i, t_i), \text{ where } s \leq i < n, \text{ there exists a pair of terms } (T, T') \text{ such that } T \in M_i \text{ and } T' \notin M_i, \text{ and their distance } D_i[T', T] \text{ has a finite upper bound. Let } E \subseteq V \times V \text{ be the set that collects all pairs } (T, T') \text{ for the matching structures } M_i. \text{ We define } \delta_{T, T'} \text{ as } \delta(\vec{\mu}[(\gamma_{(T)}(0))...]) \text{ if } \gamma(T') \text{ is defined, or as } \delta(\vec{\mu}[(\gamma_{(T)}(0))...]) \text{ otherwise. Let } \delta_E = \sum_{(T, T') \in E} \delta_{T, T'} \text{ and note that } \delta_E \geq \delta(\vec{\mu}[(\gamma_{(T)}(0))...]), \text{ because every position in } \vec{\mu}[(\gamma_{(T)}(0))...] \text{ is covered by some distance } \delta_{T, T'}. \text{ Moreover, each pair } (T, T') \in E \text{ corresponds to an atom of the form } T \leq l[T', T] \text{ in } C. \text{ According to Lemma 4.8, we have } \delta_{T, T'} \leq u, \text{ and therefore, } \delta_E \leq \text{window}(P). \text{ Hence, we have } \delta(\vec{\mu}[(\gamma_{(T)}(0))...]) \leq \delta_E \leq \text{window}(P): \text{ a contradiction hence proving the existence of a residual matching structure } M_k. \quad \square
\]

\[
\text{Theorem 4.14 (Soundness and completeness). Let } P = (SV, S) \text{ be a timeline-based planning problem and let } A_P \text{ be the associated automaton. Then, any event sequence } \vec{\mu} \text{ is a solution plan for } P \text{ if and only if } \vec{\mu} \text{ is accepted by } A_P.
\]

\[
\text{Proof. } (\rightarrow). \text{ Let } \vec{\mu} = \langle \mu_1, \ldots, \mu_n \rangle \text{ be a solution plan for } P, \text{ and let } \vec{q} = \langle q_0, \ldots, q_n \rangle \text{ be the run of } A_P \text{ on } \vec{\mu}. \text{ We first show that the sink state is never reached, and then that } q_n \text{ is a final state.}
\]

\[
\text{Let } \mu_s = (A_s, \delta_s) \text{ be the trigger event of a rule } R \equiv a_0[x_0 = v_0] \rightarrow E_1 \lor \cdots \lor E_m, \text{ i.e., start}[x_0, v_0] \in A_s. \text{ Since } \vec{\mu} \text{ is a solution plan, there exist tokens satisfying an existential statement } E \text{ of } R \text{ for the trigger } \mu_s. \text{ Hence, by Lemma 4.8 and Observation 4.9 there exists a run } M_E \xrightarrow{\vec{\mu},\gamma} M_n, \text{ yielding a closed matching structure } M_n, \text{ such that } \gamma(\text{start}(a_0)) = s.
\]

\[
\text{Let } \overline{M} = \langle M_E, M_1, \ldots, M_n \rangle \text{ be the sequence of all the matching structures involved in such run. Note that, by construction (Section 4.3), the states of } \overline{q} \text{ induce all the possible}
\]
runs for the initial matching structures of \( P \) that can be defined on \( \overline{\mu} \). In particular, the run \( \gamma \) must be one of them. However, only a subsequence of the matching structures \( \overline{M} \) will appear in the states of the run \( \overline{\gamma} \). Indeed, we can identify three key points for the sequence \( \overline{M} \): the least position \( s \) such that \( M_s \) is active (corresponding to \( \gamma(\text{start}(a_0)) \)), the least position \( h \) following \( s \) such that \( M_h \) no longer belongs to the component \( \Upsilon \) of the states in \( \overline{\gamma}[s...h] \), either because \( M_h \) is closed or because \( \delta(\overline{\gamma}[s...h]) > \overline{\text{window}}(P) \), and the least position \( k \) following \( h \) such that \( M_k \) no longer belongs to the component \( \Delta \) of the states in \( \overline{\gamma}[k...n] \), either because \( M_k \) is closed or because it gets discarded in favour of the matching structures of a later trigger event.

Every matching structure in \( \overline{M}[1...h-1] \) belongs to the component \( \Upsilon \) of a corresponding state in \( \overline{\gamma}[1...h-1] \), so the set \( \Upsilon \) of the state \( q_{s-1} \) is such that \( M_s \in \text{step}_{\mu_s}(\Upsilon_{⊥}) \), satisfying condition 5b of Section 4.3 for the trigger event \( \mu_s \). Matching structures \( \overline{M}[s+1...h] \) instead belong to the set \( \text{step}_{\mu_s}(\Upsilon_{R}) \), for the partition \( \Upsilon_{R}^R \) tracking the satisfaction of the trigger event \( \mu_s \) of every state \( \overline{\gamma}[s...h] \). Hence, all such states satisfy condition 5a of Section 4.3.

We now show that no active matching structures for the trigger event \( \mu_s \) exists after some state \( q_h \), following \( q_s \) in \( \overline{\gamma} \). Note that the run \( \gamma \) yields a closed matching structure, and if it does so within window(\( P \)) time units from the event \( \mu_{\text{start}(a_0)} \), we identified such position as the closed matching structure \( M_h \). So that the state \( q_{h-1} \) is such that \( M_h \in \text{step}_{\mu_h}(\Upsilon^R_t) \), for the partition \( \Upsilon^R_t \) tracking the trigger event \( \mu_s \), and step_{\mu_h}(\Upsilon^R_t) is discarded from \( q_h \).

If instead \( \gamma \) yields a closed matching structure after window(\( P \)) time units from the event \( \mu_{\text{start}(a_0)} \), lets identify such position as \( M_j \), with \( j \leq n \). If \( M_{j-1} \) belongs to the set \( \Delta(\Upsilon_t) \) of the state \( q_{j-1} \), then \( M_j \in \text{step}_{\mu_j}(\Delta(\Upsilon_t)) \), so that step_{\mu_j}(\Delta(\Upsilon_t)) is closed and discarded from \( q_j \), alongside all the other matching structures in \( \Delta(\Upsilon_t') \), for every \( \Upsilon_t' \in \Phi(\Upsilon_t) \), i.e., for every other existential statement \( \Upsilon_t' \) of \( R \) still tracking the trigger \( \mu_s \). If instead \( M_{j-1} \) for the trigger event \( \mu_s \) does not belong to the set \( \Delta(\Upsilon_t) \) of the state \( q_{j-1} \), by construction (Section 4.3), there exist a state \( q_h \) in which the matching structures tracking \( \mu_s \) have been replaced by those of a later event, and they no longer appear in \( \Delta(\Upsilon_t) \) from \( q_h \) onwards.

Since \( \overline{\mu} \) is a solution plan, the previous argument holds for all the trigger events in \( \overline{\mu} \) of any rules in \( S \). Hence, conditions 5a and 5b are always met, i.e., the sink state is never reached, and no active matching structures belong to \( q_n \), making it a final state.

\((\leftarrow\rightarrow)\). Let \( \overline{\mu} = (\mu_1, \ldots, \mu_n) \) be an event sequence accepted by \( A_P \) and let \( \rho = \langle q_0, \ldots, q_n \rangle \) be its accepting run. We have to show that the plan corresponding to \( \overline{\mu} \) is a solution plan for \( P \), i.e., for every event triggering a rule \( R \) in \( S \), at least one of the existential statements of \( R \) is satisfied by \( \overline{\mu} \).

Let \( \mu_s = (A_s, \delta_s) \) be an event in \( \overline{\mu} \) triggering a rule \( R \equiv a_0[x_0 = v_0] \rightarrow E_1 \lor \cdots \lor E_m \), i.e., \( \text{start}(x_0, v_0) \in A_s \). Since the sink state is never visited in an accepting run, the state \( q_s \), reached upon reading the event \( \mu_s \), is such that the partition \( \Upsilon^R_{t_0} \), tracking the satisfaction of the trigger event \( \mu_s \), is not empty. For the same reason, the partition \( \Upsilon^R_{t_k} \) tracking \( \mu_s \) in every state following \( q_s \) can never be empty as a result of the function \( \text{step}_{\mu} \). However, since the final state \( q_n \) does not contain any active matching structure, there must exists a state \( q_h \) in \( \overline{\gamma} \) whose partition \( \text{step}_{\mu_{h+1}}(\Upsilon^R_{t}) \) gets discarded from \( q_{h+1} \). This can happen either because \( \text{step}_{\mu_{h+1}}(\Upsilon^R_{t}) \) is a closed set, or because the matchings structures in \( \text{step}_{\mu_{h+1}}(\Upsilon^R_{t}) \) get promoted to the component \( \Delta \). In the first case, we can conclude that there exists a run \( M_E \xrightarrow{\overline{\mu} \gamma} M_n \) for the initial matching structure \( M_E \) of an existential statement \( E \) of \( R \).
such that $M_n$ is closed and $\gamma(\text{start}(a_0)) = s$, hence, by Observation 4.9 and Lemma 4.8, the trigger event $\mu_s$ satisfies $R$.

In the second case, let $\Psi$ be the set of existential statements having an active matching structure in step $\mu_{k+1}(\Upsilon_R)$, so that we can identify them as the sets $\Delta(E)$, for $E \in \Psi$, in the states from $q_{h+1}$ onwards. By Lemma 4.13, every such set contains a residual matching structure. Hence, by Observation 4.12, they can become empty only if, at some state $q_k$ following $q_h$, step $\mu_{k+1}(\Delta(E))$ contains a closed matching structure for some existential statement $E \in \Psi$. Note that the run $\bar{q}$ is an accepting run, so every non-empty set $\Delta(E)$ must become empty before the end of the run. Hence, $q_k$ is guaranteed to exist.

However, it may be the case that, by the time step $\mu_{k+1}(\Delta(E))$ is closed, $\Delta(E)$ no longer contains the matching structures for the trigger event $\mu_s$, but those for a later trigger event $\mu_r$ of $R$. Since the sets $\Delta(E)$ store only the matching structures tracking the most recent trigger event older than window($P$). Thus, if step $\mu_{k+1}(\Delta(E))$ contains a closed matching structure for $\mu_r$, we can directly assert the existence of a run for $M_E$ implying the satisfaction of $R$ for the trigger event $\mu_r$. If instead step $\mu_{k+1}(\Delta(E))$ contains a closed matching structure $M_r$ for a later event $\mu_r$, there exists a run $M_E \xrightarrow{\bar{p} \gamma_r} M_r$, such that $\gamma_r(\text{start}(a_0)) = r$. Furthermore, by a previous consideration on $q_{h+1}$, there exists a run $M_E \xrightarrow{\bar{p}_{[h+1]} \gamma_s} M_{h+1}$, yielding a residual matching structure $M_{h+1}$, and, by Observation 4.12, such run can be extended on the entire event sequence $M_E \xrightarrow{\bar{p} \gamma_s} M_n$, to yield a residual matching structure $M_n$. Given $M_E \xrightarrow{\bar{p} \gamma_s} M_r$ and $M_E \xrightarrow{\bar{p} \gamma_s} M_n$, with $\gamma_s(\text{start}(a_0)) \leq \gamma_r(\text{start}(a_0))$, by Lemma 4.10, there exists a run $M_E \xrightarrow{\bar{p} \gamma_s} M_n$ yielding a matching structure $M_n$ matching as many terms as $M_r$ and such that $\gamma_s(\text{start}(a_0)) = s$. Hence, $M$ is a closed matching structure for the existential statement $E$, and, by Observation 4.9 and Lemma 4.8, $R$ satisfies the trigger event $\mu_s$.

Furthermore, all the value duration functions are satisfied by the tokens in $\bar{p}$, being encoded as synchronisation rules by the automaton $S_P$. Meanwhile, the automaton $T_P$ guarantees the fulfilment of the value transition functions. Hence, we can conclude that $\bar{p}$ is a solution plan for $P$, because every rule in $S$ is satisfied, as well as the value duration and value transition functions of every state variable. 

\[\square\]