Dynamical topological excitations in parafermion chains

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Topological excitations in many-body systems are one of the paradigmatic cornerstones of modern condensed matter physics. In particular, parafermions are elusive fractional excitations potentially emerging in fractional quantum Hall-superconductor junctions, and represent one of the major milestones in fractional quantum matter. Here, by using a combination of tensor network and kernel polynomial techniques, we demonstrate the emergence of zero modes and finite energy excitations in many-body parafermion chains. We show the appearance of zero energy modes in the many-body spectral function at the edge of a topological parafermion chain, their relation with the topological degeneracy of the system, and we compare their physics with the Majorana bound states of topological superconductors. We demonstrate the robustness of parafermion topological modes with respect to a variety of perturbations, and we show how weakly coupled parafermion chains give rise to in-gap excitations. Our results exemplify the versatility of tensor network methods for studying dynamical excitations of interacting parafermion chains, and highlight the robustness of topological modes in parafermion models.

I. INTRODUCTION

Unconventional excitations in quantum materials are a central research area in modern condensed matter physics. Paradigmatic examples of unconventional excitations are the edge modes of topological insulators, including quantum anomalous Hall insulators and quantum spin Hall insulators. Solely, these systems have attracted a great amount of attention for their potential for dissipationless electronics and spintronics. Topological superconductors represent another instance in which topological excitations have a major role. In particular, the emergence of Majorana zero modes in these systems puts forward the possibility of using superconductors as a noise-resilient platform for topological quantum computing.

Interest in topological superconductors started with the first proposals to artificially realize artificial p-wave superconducting in a variety of platforms, by combining strong-spin orbit coupling effects, superconducting proximity effect and exchange fields. Majorana bound states represent a specific example of a more generic class of topological excitations, known as parafermions. In particular, parafermions realize quantum excitations with generalized commutation relations, providing a powerful platform for topological quantum computing, overcoming a limitation of Majorana bound states.

Although parafermions are substantially more elusive than Majorana bound states, a variety of proposals involving fractional quantum Hall states with superconductivity have been put forward for their artificial engineering. Inspired by the success of proposals for Majorana bound states.

Majorana bound states can be described in an effectively single-particle picture with the Bogoliubov-de-Gennes formalism. In stark contrast, parafermion models represent a much bigger challenge from the theoretical point of view. In particular, models for parafermions become full-fledged many-body problems, requiring a full many-body treatment. As a result, parafermion models are substantially less explored than their Majorana counterparts. In particular, the computation of dynamical
cal excitations in parafermion chains remains a challenging problem due to the genuine many-body nature of the problem, and the lack of exact analytical tools for its generic treatment.

Here, using a combination of tensor network and kernel polynomial techniques, we show the emergence of edge and interface topological excitations in parafermion chains. In particular, here we demonstrate that parafermion chains show strong edge zero modes that are resilient to a variety of parafermion many-body interactions, and that weak coupling between parafermion zero modes give rise to in-gap excitations at a finite energy at the interface (Fig. 1). Furthermore, we compare the phenomenology of these parafermion chains with those of Majorana excitations in topological superconductors. Our manuscript is organized as follows. First, in section II we present a generalized parafermion model, that simultaneously captures conventional fermions and parafermions, together with a quantum many-body procedure used to solve the system. In section III we use this formalism to study the dynamical topological modes in a Majorana chain, including the effects of many-body interactions, decoupling and disorder. In Sec. IV we show that a parafermion chain show emergent zero-modes in the spectral function. In section V we show that perturbations to the parafermion Hamiltonian, including higher order interactions, next neighbor hopping and disorder remain the zero-edge excitation unaffected. In section VI we show how interfacial in-gap excitations at finite energy emerge at the interface between between two parafermion chains. Finally, in section VII we summarize our conclusions.

II. MODEL

A. Clock model and parafermions

In the following we will study a one-dimensional model of parafermions exhibiting topological zero modes. Parafermions are generalizations of conventional fermions with $Z_n$ symmetry showing generalized commutation relations. Parafermion models are conveniently written from a so-called clock model, involving operators $\tau$ and $\sigma$. The clock operators $\tau$ and $\sigma$ generalize the Pauli $x$- and $z$-matrices, with the following properties

$$\sigma^n = \tau^n = 1,$$
$$\sigma^\dagger = \sigma^{n-1},$$
$$\tau^\dagger = \tau^{n-1}.\quad (1)$$

The integer $n$ is given by the order of the parafermions considered. In particular, for $n = 2$, the conventional algebra of Pauli matrices is recovered, yielding $\sigma^2 = 1$. In contrast, for $n = 3$, one recovers the same state upon applying the operator three times. The clock operators allow generalizing the notion of fermions, by promoting the typical Jordan-Wigner algebra to $Z_n$ symmetry. The clock operators follow a generalized commutation relation of the form

$$\sigma \tau = z \tau \sigma$$

with $z = e^{2\pi i/n}$. In a parafermion chain, each site is taken to have its own set of parafermion operators $\tau_j$ and $\sigma_j$. With those local clock operators, the operators for parafermions in a parafermion chain are derived from the clock operators as

$$\chi_j = \left(\prod_{k=1}^{j-1} \tau_k\right) \sigma_j \quad (3)$$

$$\psi_j = \left(\prod_{k=1}^{j-1} \tau_k\right) \sigma_j \tau_j \quad (4)$$

where $\psi$ and $\chi$ correspond to the two basic parafermions found at each site of the parafermionic chain. The previous transformation can be a understood as a generalized Jordan-Wigner transformation between conventional spin operators and fermionic operators. The Hamiltonian of the parafermion chain is constructed with $\psi_j$, $\chi_j$, $\psi_j^\dagger$, $\chi_j^\dagger$. The parafermionic commutation relation is derived from the commutation relations of the clock operators and is given by

$$\chi_j \psi_k = z \psi_k \chi_j$$
$$\chi_j \chi_k = z \chi_k \chi_j$$
$$\psi_j \psi_k = z \psi_k \psi_j$$

for $j < k$. These commutation relations are responsible for the exotic quantum statistics of the chain. In particular, taking $n = 2$ recovers the commutation relations for fermions. Given the previous operators, a many-body Hamiltonian for the parafermion chain can be written as

$$\mathcal{H} = i f \sum_n \chi_n^\dagger \psi_n + i \theta \sum_n \psi_n^\dagger \chi_{n+1} + \text{h.c.}$$

where parametrizes $f$ is an on-site coupling between parafermions on the same site, and $\theta$ parametrizes a coupling between parafermions in different sites. The previous Hamiltonian is known to have a rich phase diagram for complex values of $f$ and $\theta$, which in particular hosts a phase with many-body topological order. Here we will focus in this topological phase, which is obtained in particular by taking $\theta = 1$ and $f = 0.5$. In particular, we will be interested in computing the dynamical excitations of the system, that can be obtained by extracting the following dynamical correlator

$$\Xi(\omega) = \langle GS | \chi_N^\dagger \delta(\omega - \mathcal{H} + E_{GS}) \psi_N | GS \rangle$$

(7)
where $|GS\rangle$ is the many-body ground state of the system and $E_{GS}$ the ground state energy. Due to the genuine many-body nature of this model, we will compute this dynamical correlator numerically using the kernel polynomial tensor network as elaborated in the next section.

B. Kernel polynomial tensor network formalism

Due to the many-body nature of the Hamiltonian Eq. 6, a generic analytic solution cannot, in general, be obtained. To tackle this problem, we will here employ the tensor network formalism\textsuperscript{53–58} which is in particular well suited for generic interacting one dimensional problems. In order to compute the dynamical correlators we will use the tensor network kernel polynomial formalism.\textsuperscript{59–63} The kernel polynomial method\textsuperscript{59} (KPM) allows for the computation of spectral functions directly in frequency space by performing expansion in terms of Chebyshev polynomials of Eq. 7. For simplicity, we focus our discussion on the rescaled Hamiltonian $\hat{H} \rightarrow \hat{\bar{H}}$, whose ground state energy is located at $E = 0$ and whose excited states are restricted to the interval $[0,1)$,\textsuperscript{64} which can be generically obtained by shifting and rescaling the original Hamiltonian $\hat{H}$. The dynamical correlator $\Xi$ for the original Hamiltonian $\hat{H}$ can then be recovered by rescaling back the energies in the dynamical correlator $\bar{\Xi}$ of the scaled Hamiltonian $\bar{\hat{H}}$. To compute the dynamical correlator $\bar{\Xi}$, we perform an expansion of the form

$$\bar{\Xi}(\omega) = \frac{1}{\pi \sqrt{1 - \omega^2}} \left( \mu_0 + 2 \sum_{l=1}^{N_L} \mu_l T_l(\omega) \right) \tag{8}$$

where $T_l$ are Chebyshev polynomials. The coefficients of the expansion $\mu_l$ can be then computed as $\mu_l = \langle GS|\chi_N T_l(\bar{\hat{H}})|\psi_N|GS\rangle$. Taking into account the recursion relation of the Chebyshev polynomials $T_l(\omega) = 2\omega T_{l-1}(\omega) - T_{l-2}(\omega)$, with $T_1(\omega) = \omega$ and $T_0(\omega) = 1$, the different coefficients $\mu_l$ can be computed by iteratively defining the vectors

$$|w_0\rangle = \psi_N|GS\rangle \tag{9}$$
$$|w_1\rangle = \bar{\hat{H}}|w_0\rangle \tag{10}$$
$$|w_{l+1}\rangle = 2\bar{\hat{H}}|w_l\rangle - |w_{l-1}\rangle \tag{11}$$

so that $|w_l\rangle = T_l(\bar{\hat{H}})|\psi_N|GS\rangle$.

In this way, the coefficients $\mu_l$ are computed as $\mu_l = \langle GS|\chi_N|w_l\rangle$. To improve the convergence rate of the expansion, we perform an autoregressive extrapolation\textsuperscript{65} and we quench the Gibbs oscillations with the Jackson kernel.\textsuperscript{66}

III. DYNAMICAL EXCITATIONS IN AN INTERACTING TOPOLOGICAL SUPERCONDUCTOR

A. Zero modes in interacting topological superconductors

In this section, we first show how the previous formalism allows to capture the robustness of Majorana zero modes,\textsuperscript{67} a well-studied topological state that emerges taking $n = 2$ in the generalized parafermion model. In order to go beyond the single-particle Majorana limit, we will benchmark our tensor network formalism with an interacting topological superconductor. In particular, it is well known that the ground state degeneracy of a finite island is not lifted by the introduction of many-body interactions,\textsuperscript{58,69} and that the zero-bias peak structure
survives.\textsuperscript{70} We take the following many-body Hamiltonian for an interacting for a one-dimensional topological superconductor

$$H = \mu \sum_n c_n^\dagger c_n + t \sum_n c_n^\dagger c_{n+1} + \Delta \sum_n c_n c_{n+1} + V \sum_n (c_n^\dagger c_n - \frac{1}{2}) (c_{n+1}^\dagger c_{n+1} - \frac{1}{2}) + \text{h.c.}$$

(12)

where $c_n^\dagger$, $c_n$ and the creation and annihilation fermionic operators, $\mu$ is the chemical potential, $t$ the hopping, $\Delta$ the p-wave superconducting order and $V$ the electron-electron interaction. In the case of $V = 0$, the previous Hamiltonian corresponds to a non-interacting one-dimensional topological superconductor, whose eigenstates can be solved with a conventional Bogoliubov-de Gennes transformation.\textsuperscript{67} This limit of $V = 0$ corresponds to the Hamiltonian of Eq. 6 when taking $Z_2$ operators. In this limit $V = 0$, the previous Hamiltonian of Eq. 12 is known to show edge zero-modes. In particular, those zero modes are associated with Majorana excitation, one in each edge of the chain, that together encode a net two-fold degeneracy of the ground state. In the non-interacting regime of $V = 0$, these zero-modes can be understood as arising from a non-trivial topological invariant of the associated Bogoliubov-de-Gennes Hamiltonian.\textsuperscript{7,67,71}

In the presence of interactions $V \neq 0$, the conventional single-particle classification no longer holds, and the Hamiltonian becomes purely many-body. However, it is known that interactions do not lift the two-fold degeneracy of an open Majorana chain.\textsuperscript{68,69} The existence of two-fold degeneracy is associated with the emergence of a zero-energy peak at the edge coexisting with a gaped bulk spectra in the spectral function

$$A(\omega) = \langle GS | c_n \delta(\omega - H + E_{GS}) c_n^\dagger | GS \rangle$$

(13)

where $E_{GS}$ is the ground state energy. This can be observed by computing the dynamical correlator of Eq. 13 at the edge and the bulk of the sample as the interaction $V$ is turned on Fig. 2a. In particular, for $V = 0$ the gaped bulk and zero energy peak can be understood from the single-particle picture as mentioned above. As the interaction $V$ is increased, a finite gap remains in the bulk (Fig. 2a), and the zero-energy peak remains (Fig. 2b). At large enough interaction strengths, the bulk gap would close, and the zero-energy peak would get mixed the bulk states. The previous phenomenology shows that, as long as interactions are not strong enough to close the bulk gap, the Majorana zero-energy edge mode is robust. This can be also observed by computing the spectral function of Eq. 13 in the different sites of the chain at an intermediate interaction $V$ as shown in Fig. 2c. In particular, it is clearly observed that the zero-energy modes are strongly located at the edge and that they rapidly decay inside the chain, leading to a gaped bulk spectra (Fig. 2c).

In the discussion above we have considered an interacting Hamiltonian whose terms are uniform in space. It is however worth to note that this topological zero energy modes remain robust in the presence of disorder in the Hamiltonian, both in the non-interacting and in the interacting regime. This can be explicitly shown by adding a disorder term to the Hamiltonian of Eq. 12 of the form $H_d = \sum_n \epsilon_n c_n^\dagger c_n$ where $\epsilon_n$ is a different random number for each site in the interval $(-\epsilon, \epsilon)$. As shown in Fig. 2d, it is observed that the edge zero modes survive in the presence of this random disorder and interactions, whereas the gaped bulk states are heavily affected by it. This resilience of the zero modes is associated to their topological nature, signaling that for a moderate disorder strengths the topological degeneracy of the ground state remains invariant.
B. Interface excitations in coupled topological superconductors

Previously we showed that the topological zero-modes appear at the edge of the one-dimensional chain, both in the presence of electronic interactions and disorder. We will now address how these topological zero-modes would emerge a single chain is decoupled into two, which will lead to edge modes at each end of each subsystem. For this goal, we now define a parametric Hamiltonian, in which the coupling between the left and right parts is controlled by $\lambda$.

\[
\mathcal{H}_\lambda = \mu \sum_n c_n^\dagger c_n + t \sum_{n \neq L/2} c_n^\dagger c_{n+1} + \Delta \sum_{n \neq L/2} c_n c_{n+1} + V \sum_{n \neq L/2} \left( c_n^\dagger c_n - \frac{1}{2} \right) \left( c_{n+1}^\dagger c_{n+1} - \frac{1}{2} \right) + \lambda \left[ tc_{L/2}^\dagger c_{L/2+1} + \Delta c_{L/2} c_{L/2+1} \right] + \lambda \left[ V \left( c_{L/2}^\dagger c_{L/2} - \frac{1}{2} \right) \left( c_{L/2+1}^\dagger c_{L/2+1} - \frac{1}{2} \right) \right] + \text{h.c.}
\]  

By definition, $\lambda = 1$ corresponds to the pristine limit of Eq. 12, whereas $\lambda = 0$ corresponds to the fully decoupled limit in which the system consists of two independent chains. In this limit, the Hamiltonian consists on two fully-decoupled chains, and therefore each chain develops its own pair of Majorana edge modes. The evolution from the fully coupled to the fully decoupled limit can be systematically explored by computing the spectral function in the chain for different strengths of the coupling $\lambda$ as shown in Fig. 3abcd. In particular, it is observed that as the chains are decoupled, an interface state emerges and drifts to lower energies (Figs. 3e). This can be systematically studied by looking at the evolution of the spectral function at the interface as a function of the coupling $\lambda$, as shown in Fig. 3c. In particular, it is observed that a zero mode in the fully decoupled regime becomes a finite energy excitation as the coupling between the two chains is increased. Similar phenomenology is known in non-interacting Majorana chains, highlighting that the emergence of finite energy excitations from coupled topological zero-modes also holds in the purely many-body regime. In the following we will show that an analogous phenomenology happens in interacting parafermion chains.

IV. ZERO-MODE EXCITATIONS IN PARAFERMIon CHAINS

We now move on to consider chains of $Z_3$ parafermions, in particular building on top of the previous results for an interacting topological superconductor. The first interesting issue to consider is the many-body degeneracy of the parafermion chain, in comparison with the one of the topological superconductor. This can be observed by analyzing the excitation energies as a function of the system size, as shown in Fig. 4a. It is observed that as the system size becomes bigger, the energies of the first two excited states become arbitrarily close to the ground state energy, with the next excited state presenting a finite gap. This very same phenomenology takes place for the Majorana model, in which the finite splitting of the states for small chains is rationalized in terms of the hybridization between the edge modes. It is important to note that, in contrast with the Majorana model, the ground state of the $Z_3$ parafermion chain becomes three-fold degenerate, in comparison with the two-fold degeneracy of the Majorana chain.

In the case of a topological superconductor, the degeneracy of the ground state is associated with the emergence of Majorana zero modes at the edges. The degeneracy of the ground state with open boundary conditions for the $Z_3$ parafermion chain is again rationalized in terms of emergent topological edge modes, but now encoding a three-fold degeneracy. This can be observed in the dynamical correlator computed at the edge of a parafermion chain as a function of the chain length, as showed in Fig. 4b. In this fashion, the finite splitting between the lowest three energy levels for small chains can be rationalized in terms of a finite hybridization between
the topological zero modes located at opposite edges. Due to the existence of a finite gap in the bulk of the chain, the zero modes are exponentially localized, leading to an exponential dependence of the hybridization between the states. This can be verified by looking at the spectral function for every site in the parafermion chain, as shown in Fig. 4c. In particular, it is observed that topological zero modes are strongly localized at edges of the chain, whereas the spectral function remains gaped in the bulk of the chain (Fig. 4c). In the next section we will address the robustness the edge zero-mode excitations, showing that the previous phenomenology is robust towards perturbations.

V. PERTURBATIONS AND DISORDER IN PARAFERMION CHAINS

Previously we focused on the pristine parafermionic chain showing the emergence of topological excitations at zero energy at the edge. In the following we will assess the robustness of previous zero modes with respect to perturbations. In particular, we will focus on two different interaction terms, one realizing a biquadratic interaction between parafermions and another one realizing a next to nearest neighbor hopping in the parafermion chain. We will examine the impact of this perturbations by computing the edge and bulk spectral function as the interaction term is increased, similarly as it was shown in the interacting topological superconductor above.

Let us first address the case of biquadratic interactions. In particular, we now include a term in the Hamiltonian that involves four parafermionic operators, leading to a Hamiltonian of the form

\[
H_W = i \sum_n f \chi_n^\dagger \psi_n + i \theta \sum_n \psi_n^\dagger \chi_{n+1} + \text{W} \sum_n \psi_n^\dagger \chi_n \psi_{n+1}^\dagger \chi_{n+1} + \text{h.c.} \tag{15}
\]

where \(W\) controls the strength of the biquadratic interaction. We compute the spectral function in the bulk and at the edge as a function of the coupling parameter \(W\), as shown in Fig. 5ab. In particular, we observe that as the interaction term is ramped up, the bulk spectral gap decreases. However, as long as the bulk gap remains open, the topological edge excitation remained pinned at zero energy. This phenomenology emphasizes that the biquadratic interaction parametrized by \(W\) competes with the topological gap. However, as long as such perturbation is not strong enough to close the bulk gap, the topological edge excitations will remain pinned at zero energy. From the point of view of the degeneracy of the ground state of the parafermion chain, this means that a three-fold degeneracy is robust against the biquadratic perturbation. It is interesting to note that this is an analogous phenomenology as the one shown above for the Majorana chain.

After showing that first neighbor interactions compete with the topological phase, we now turn to a different perturbation whose effect is dramatically different. We now consider a bilinear term in the parafermion Hamiltonian, giving rise to a second neighbor hopping. The full Hamiltonian now becomes

\[
H_\gamma = i \sum_n f \chi_n^\dagger \psi_n + i \theta \sum_n \psi_n^\dagger \chi_{n+1} + \gamma \sum_n \psi_n^\dagger \chi_{n+2} + \text{h.c.} \tag{16}
\]

where \(\gamma\) parametrizes the strength of a second-neighbor hopping between the parafermion operators. We show the spectral function in the bulk as a function of the coupling parameter \(\lambda\) in Fig. 6. In particular, we see that the spectral function in the bulk increases its gap as \(\gamma\) is ramped up. At the same time, the edge spectral function keeps showing a zero energy resonance corresponding to
FIG. 6. (a-d) Spectral function in the different sites of two coupled parafermion chains, for different values of the interface coupling $\lambda$ between the left and right parts. We took $\lambda = 0.7$ for (a), $\lambda = 0.5$ for (b), $\lambda = 0.3$ for (c), and $\lambda = 0.1$ for (d). As the coupling $\lambda$ is decreased, interface modes move towards lower energies, eventually giving rise to topological modes in the two decoupled chains. Panel (e) shows the spectral function at the interface as a function of the coupling, highlighting the emergence of the interfacial zero mode at $\lambda = 0$.

the topological edge state. This phenomenology highlights that perturbations to the parafermionic Hamiltonian can also enhance the topological gap, and more importantly keeping the excitations pinned at zero energy at the edge.

It is finally interesting to show that the emergence of zero-modes is not associated to the translational symmetry of the lattice. In particular, we now consider a Hamiltonian parafermion lattice where the couplings are disordered

$$\mathcal{H}_{\text{dis}} = i \sum_n f_n \chi_n \psi_n + i \theta \sum_{n \neq L/2} \psi_{n+1} \chi_{L/2} + \text{h.c.}$$

(17)

where $f_n$ takes random values between $(0.37, 0.62)$. We compute the spatially resolved spectral function as shown in Fig. 5e. It is clearly seen that the edge zero-modes survive in this disordered chain, despite the strong effect on the bulk states. This phenomenology demonstrates the robustness of the zero-energy modes of the parafermion chains, and in particular that their existence is not associated to a lattice symmetries. Having demonstrate the robustness of the topological edge modes in parafermion chains, in the next section we address how coupling different topological modes between different chains allows to lift those excitations from zero energy.

VI. INTERFACE EXCITATIONS IN COUPLED PARAFERMION CHAINS

Previously, we showed that weak perturbations to the parafermion Hamiltonian to not lift the edge excitations from zero energy. We now explore how topological excitations at zero energy can be created by weakly coupling two parafermion chains. For this sake, we define a parametric Hamiltonian of the form

$$\mathcal{H}_{\text{dis}} = i f \sum_n \chi_n^\dagger \psi_n + i \theta \sum_{n \neq L/2} \psi_{n+1} \chi_{L/2} + \text{h.c.}$$

(18)

where $\lambda$ controls the decoupling between two halves of the chain. In particular, for $\lambda = 1$ the system corresponds to a uniform chain, whereas for $\lambda = 0$ the system is formed for two decoupled chains.

Let us now look at the spectral function at every site as a function of the coupling strength between the two chains $\lambda$, as shown in Fig. 6abcd. In the pristine case $\lambda = 0.7$ zero edge excitations emerge at the two-edges, co-existing with a fully gaped bulk. Starting with a finite but not perfect coupling $\lambda = 0.7$ (Fig. 6a), we observe that a finite energy excitation starts to appear at the interface between the two chains. As the coupling between the two halves in weakened, an in-gap state drifts towards lower energies (Fig. 6bcd), ultimately creating zero modes at the edges of the now two decoupled chains. This can be also systematically explored by computing the spectral function at the interface between the two chains as a function of $\lambda$, as shown in Fig. 6e. It is clearly observed that the two topological edge modes, originally located at zero energy, become finite energy excitations as the coupling between the two chains is increased. This shares the same phenomenology as conventional Majorana chains, highlighting that the hybridization between topological zero modes generically give rise to finite energy excitations. It is finally interesting to note that for $\lambda \neq 0$ the collective ground state of the two chains will be three-fold degenerate in the thermodynamic limit, whereas for $\lambda = 0$ the ground state becomes nine-fold degenerate. For $\lambda \neq 0$, the first excited state will then correspond to the interface excitation that arises from the coupled edge modes at the junction, whose energy
can be inferred from the spectral function of Fig. 6e. Finally, these results highlight that whereas perturbations to the many body Hamiltonian do not lift the zero edge modes due to their topological protection, coupling topological excitations is an effective way of creating topological modes at finite energy.

VII. CONCLUSIONS

To summarize, we have shown the emergence of zero modes and excitations at finite energies in a parafermion chain. To study this interacting model, we employed a combination of tensor network and kernel polynomial techniques that allow addressing the full excitation spectra of the interacting Hamiltonian. We have shown that topological parafermion chains feature robust zero-energy excitations, that encode the three-fold degeneracy of the ground state in the thermodynamic limit. We demonstrated that these excitations are robust against perturbations, including biquadratic interactions, second neighbor hopping and disorder. We then showed how interfacial modes at finite energies can be created by weakly coupling different parafermion chains, with an excitation energy controllable by the coupling between the chains. Our results demonstrate the robustness of these topological excitations in parafermion chains, and put forward kernel polynomial tensor networks as a versatile technique to study finite-energy excitations in highly interacting models.

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