APPROXIMATE CONTROLLABILITY OF SOBOLEV TYPE FRACTIONAL EVOLUTION SYSTEMS WITH NONLOCAL CONDITIONS

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Abstract. In this paper, we study the approximate controllability of Sobolev-type fractional evolution systems with non-local conditions in Hilbert spaces. Sufficient conditions of approximate controllability of the desired problem are presented by supposing an approximate controllability of the corresponding linear system. By constructing a control function involving Gramian controllability operator, we transform our problem to a fixed point problem of nonlinear operator. Then the Schauder Fixed Point Theorem is applied to complete the proof. An example is given to illustrate our theoretical results.

1. Introduction. It is well known that controllability problem is one of the most fundamental issues in mathematical control theory. Since the beginning of sixties in the last century, controllability problems for dynamical systems have been reported in many literature and monographs.

Fractional calculus (FC for short) has been introduced since the end of the nineteenth century by famous mathematicians Liouville and Riemann, but the concept of non-integer calculus, as a generalization of the traditional integer order calculus was mentioned already in 1695 by Leibnitz and L'Hospital. The subject of FC has

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become a rapidly growing area and has found applications in diverse fields ranging from physical sciences, engineering to biological sciences and economics.

FC has recently been proved to be a vital tool in the modeling of many physical phenomena, and fractional differential equations (FDE for short) have attracted great attention of several researchers. For more details on this topic, see the literature on fractional calculus and its application [2], [7], [10], [16], [17], [22] and [26], and monographs on FDE [3], [6], [13], [24], [32] and the survey [29]. There are presented fundamental results on FC and on the existence, uniqueness and stability of solutions of FDE with applications [14, 30, 31]. Moreover, discrete fractional equations are also investigated.

Controllability is one of the important fundamental issues in mathematical control theory. Recently, controllability problems of fractional evolution systems in infinite dimensional spaces have been attracted by many researchers. There are some interesting and important controllability results concerning semilinear differential systems involving the Caputo fractional derivative. For example, Debbouche and Baleanu [4], Fečkan et al. [9] and [28] initiated to study complete controllability of two classes of Sobolev-type fractional functional evolution equations by constructing two new characteristic solution operators via the well-known Schauder Fixed Point Theorem. Meanwhile, Sakthivel and Ren [25], Debbouche and Torres [5], Kerboua et al. [11] and [12], Mahmudov [19], Mahmudov and Zorlu [20] and [21] pay attention to studying approximate controllability for different types of fractional evolution systems. The above controllability results are derived with the help of semigroup theory and fixed point technique.

Note that the concept of approximate controllability enables to steer the system to arbitrary small neighborhood of final state. Thus, approximate controllable problems are more prevalent in fields of control engineering. There is no work yet reported on approximate controllability of Sobolev-type fractional evolution equations with non-local conditions. Inspired by the above recent contributions, we propose to discuss approximate controllability of Sobolev-type fractional evolution systems with classical non-local conditions in Hilbert spaces. To handle our task, we first impose an approximate controllability of the corresponding linear system. Then we develop the approach in [18] and rewrite our control problem as a fixed point equation for an appropriate nonlinear operator. Using the Schauder Fixed Point Theorem on this equation, sufficient conditions of approximate controllability of the desired problem are presented.

To end this section, we would like to claim the original ideas of this paper as follows. We establish a uniform framework to investigate approximate controllability for non-local problems for Sobolev-type fractional evolution systems in Hilbert spaces by introducing a suitable Gramian controllability operator and adopting a fixed point theorem, which extend the classical method for Cauchy problems for integer order systems.

2. Preliminaries. Let $X$ be a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. We consider the following Sobolev-type fractional evolution system:

$$\begin{cases}
\frac{C_0^q}{t} D_q^t (E x(t)) + A x(t) = f(t, x(t)) + B u(t), \ t \in J := [0, a], \\
x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0,
\end{cases}$$

(1)

where $\frac{C_0^q}{t} D_q^t$ is the generalized Caputo fractional derivative of order $0 < q < 1$ with the lower limit zero (see Kilbas et al. [13]), $E$ and $A$ are two linear operators with
domains contained in $X$ and ranges still contained in $X$, the pre-fixed points $t_k$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a$ and $a_k$ are real numbers.

In order to guarantee that $-AE^{-1} : X \to X$ generates a semigroup $\{W(t), t \geq 0\}$, we consider that the operators $A$ and $E$ satisfy the following conditions:

$(S_1)$: $A : D(A) \subset X \to X$ and $E : D(E) \subset X \to X$ are linear, $A$ is closed;
$(S_2)$: $D(E) \subset D(A)$ and $E$ is bijective;
$(S_3)$: $E^{-1} : X \to D(E)$ is compact;
$(S_4)$: $E^{-1} : X \to D(E)$ is continuous.

Now we note that $(S_4)$ implies that $E$ is closed; $(S_3)$ implies $(S_4)$. It follows from $(S_1)$, $(S_2)$, $(S_4)$ and the closed graph theorem that $-AE^{-1} : X \to X$ is bounded, which generates a uniformly continuous semigroup $\{W(t), t \geq 0\}$ of bounded linear operators from $X$ to itself.

Denote by $\rho(-AE^{-1})$ the resolvent set of $-AE^{-1}$. If we assume that the resolvent $R(\lambda, -AE^{-1})$ is compact, then $\{W(t), t \geq 0\}$ is a compact semigroup (see Pazy [23, Theorem 3.3, p. 48]).

The state $x(t)$ takes values in $X$ and the control function $u(\cdot)$ is given in $\mathcal{W}$, the Banach space of admissible control functions, where $\mathcal{W} := L^p(J, U)$, for $q \in \left[\frac{1}{p}, 1\right]$ with $1 < p < \infty$ and $U$ is a Hilbert space. Moreover, $B : U \to X$ is a bounded linear operator and $f : J \times X \to X$ will be specified later.

Consider a probability density function $\xi_q$ (see [8]) defined on $[0, \infty[$:

$$\xi_q(\theta) = \frac{1}{q} \theta^{-(1+\frac{1}{q})} \sin(n \pi \theta) \geq 0,$$

where

$$\varpi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-qn-1} \Gamma(nq + 1) \sin(n \pi q).$$

Obviously,

$$\int_0^\infty \xi_q(\theta) d\theta = 1, \quad \text{and} \quad \int_0^\infty \theta \xi_q(\theta) d\theta = \frac{1}{\Gamma(1+q)}.$$  

Then, we define the following two operators:

$$\mathcal{T}_{(A,E)}(t) = \int_0^\infty \xi_q(\theta) W(t^q \theta) d\theta, \quad \mathcal{T}_{(A,E)}(t) = q \int_0^\infty \theta \xi_q(\theta) W(t^q \theta) d\theta.$$  

Similar to the proof in Fečkan et al. [9] and Zhou and Jiao [33], one has the following results.

**Lemma 2.1.** Assume that $\sup_{t \geq 0} ||W(t)|| \leq M_1$ holds. One has the following properties:

(i): For any fixed $t > 0$, $\mathcal{T}_{(A,E)}(t)$ and $\mathcal{T}_{(A,E)}(t)$ are linear bounded operators on $X$ which satisfy $||\mathcal{T}_{(A,E)}(t)|| \leq M_1$ and $||\mathcal{T}_{(A,E)}(t)|| \leq \frac{M_1}{\Gamma(q)}$.

(ii): If $W(t)$ is compact, then $\mathcal{T}_{(A,E)}(t)$ and $\mathcal{T}_{(A,E)}(t)$ are compact in $X$ for $t > 0$.

(iii): $\mathcal{T}_{(A,E)} : [0, \infty[ \to L(X)$ and $\mathcal{T}_{(A,E)} : [0, \infty[ \to L(X)$ are continuous.

Next, we define

$$\mathcal{T}^*_{(A^*,E^*)}(t) = \int_0^\infty \xi_q(\theta) W^*(t^q \theta) d\theta, \quad \mathcal{T}^*_{(A^*,E^*)}(t) = q \int_0^\infty \theta \xi_q(\theta) W^*(t^q \theta) d\theta,$$

where $\{W^*(t), t \geq 0\}$ is the adjoint semigroup of $\{W(t), t \geq 0\}$. 

Lemma 2.2. The following properties hold:

(i): For any fixed $t \geq 0$, $\mathcal{F}^*(t)$ and $\mathcal{F}^*(t)$ are linear bounded operators on $X$ with $\|\mathcal{F}^*(t)\| \leq M_1$ and $\|\mathcal{F}^*(t)\| \leq M_1 T^*_{\gamma}$. 

(ii): If $W(t)$ is compact, then $\mathcal{F}^*(t)$ and $\mathcal{F}^*(t)$ are compact in $X$ for $t > 0$.

Using Pazy [23, Lemma 10.1, p. 38], we have:

Lemma 2.3. $E^{-1}$ and $B^*$ are bounded operators with $\|E^{-1}\| = \|E^*_1\|$ and $\|B\| = \|B^*\|$.

Assume that there exists a continuous linear operator $\Theta$ on $X$ given by

$$\Theta = \left[ I + \sum_{k=1}^{m} a_k \mathcal{F}(A,E)(t_k) \right]^{-1},$$

where $I$ is the identity operator.

Remark 1. One can give a sufficient condition to guarantee the existence of $\Theta$. For example, set $M_2 \sum_{k=1}^{m} |a_k| < 1$. Indeed, applying Neumann lemma, we get

$$\|\Theta\| \leq \frac{1}{1 - M_2 \sum_{k=1}^{m} |a_k|}.$$

Now we introduce a Green function:

$$G(A,E)(t,s) := E^{-1}G^0_{(A,E)}(t,s)$$

$$= E^{-1} \left( - \sum_{k=1}^{m} \mathcal{F}(A,E)(t) \chi_k(s) \Theta(t_k - s)^{q-1} \mathcal{F}(A,E)(t_k) - \chi_t(t - s)^{q-1} \mathcal{F}(A,E)(t - s) \right), \text{ for } t, s \in J,$$

where

$$\chi_k(s) = \begin{cases} a_k & \text{for } s \in [0, t_k], \\ 0 & \text{for } s \in [t_k, a], \end{cases} \quad \chi_t(s) = \begin{cases} 1 & \text{for } s \in [0, t], \\ 0 & \text{for } s \in [t, a]. \end{cases}$$

Hence, we set $\chi_k(s)(t_k - s)^{q-1} = 0$ for $s \in [t_k, a]$ and $\chi_t(s)(t - s)^{q-1} = 0$ for $s \in [t, a]$.

Inspired by the concept of mild solutions in [27], [28] and [33], we introduce the following definition.

Definition 2.4. For each $u \in \mathcal{Y}$, by a mild solution of the system (1) we mean that there exists a function $x \in C(J, X)$ satisfying

$$x(t) = \int_0^t G(A,E)(t,s)[f(s, x(s)) + Bu(s)]ds, \; t \in J.$$ 

Remark 2. To explain the above formula, like in [9, Lemma 3.1], one can integrate the first equation of the system (1) via Laplace transform to derive

$$Ex(t) = \mathcal{F}(A,E)(t)Ex(0) + \int_0^t (t-s)^{q-1} \mathcal{F}(A,E)(t-s)[f(s, x(s)) + Bu(s)]ds,$$

which implies that

$$x(t) = E^{-1}\mathcal{F}(A,E)(t)Ex(0) + \int_0^t (t-s)^{q-1} E^{-1}\mathcal{F}(A,E)(t-s)[f(s, x(s)) + Bu(s)]ds.$$
Now using the non-local initial condition in the system (1) one can solve
\[ E x(0) = - \sum_{k=1}^{m} a_k \Theta \int_0^{t_k} (t_k - s)^{q-1} \mathcal{J}_{(A,E)}(t_k - s) \left[ f(s, x(s)) + B u(s) \right] ds, \]
which leads to the desired formula of mild solution.

In what follows, we turn to consider the following linear system:
\[ \begin{cases}
  \frac{D^q_t}{\Gamma(q)} E x(t) = A x(t) + B u(t), & t \in J, \\
  x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0.
\end{cases} \tag{3} \]

Using the mild solution of (3), we get
\[ x(a) = \int_0^a G_{(A,E)}(a, s) B u(s) ds. \]

Define a linear operator \( P : \mathcal{U} \rightarrow X \) by
\[ P u := x(a) = \int_0^a G_{(A,E)}(a, s) B u(s) ds. \]

By using Lemmas 2.1, 2.2 and Hölder inequality, we obtain
\[ \| P u \| \leq \int_0^a \| G_{(A,E)}(a, s) \| B \| u(s) \| ds \]
\[ \leq \frac{M_1 \| E^{-1} \| B}{\Gamma(q)} \int_0^a \left( M_1 \| \Theta \| \sum_{k=1}^{m} \chi_k(s)(t_k - s)^{q-1} \right) ds \]
\[ + \chi_a(s)(a-s)^{q-1} \| u(s) \| ds \]
\[ = \frac{M_1 \| E^{-1} \| B}{\Gamma(q)} \left( M_1 \| \Theta \| \sum_{k=1}^{m} |a_k| \int_0^{t_k} (t_k - s)^{q-1} \| u(s) \| ds \right) \]
\[ + \int_0^a (a-s)^{q-1} \| u(s) \| ds \]
\[ \leq \frac{M_1 \| E^{-1} \| B}{\Gamma(q)} \left( M_1 \| \Theta \| \sum_{k=1}^{m} |a_k| + 1 \right) \left( \frac{p-1}{q(p-1)} a^{\frac{q-p}{p-1}} \right) \| u \|_{\mathcal{U}} \]
\[ := M_P \| u \|_{\mathcal{U}}. \tag{4} \]

Thus, \( P \) is bounded. Furthermore, (3) is approximately controllable if and only if \( \text{cl}(P(\mathcal{U})) = X \). This is equivalent to \( \ker P^* = \{0\} \).

Note that \( U \) is a Hilbert space, so then \( L^p(J, U)^* = L^{p^*}(J, U^*) = L^{p^*}(J, U) \) for \( \frac{1}{p} + \frac{1}{p^*} = 1 \). Next, we compute \( P^* \) as follows. Let \( x^* \in X \), then
\[ \langle x^*, x(a) \rangle = \left( x^* , \int_0^a G_{(A,E)}(a, s) B u(s) ds \right) = \int_0^a \left( B^* G_{(A,E)}^*(a, s) x^*, u(s) \right) ds. \]

Hence, we derive
\[ (P^* x^*)(s) = B^* G_{(A,E)}^*(a, s) x^*, \quad s \in J, \quad x^* \in X, \]
where
\[ G_{(A,E)}(a, s) = \left( - \sum_{k=1}^{m} \chi_k(s)(t_k - s)^{q-1} \mathcal{J}_{(A^*, E^*)}(t_k - s) \Theta^* \mathcal{J}_{(A^*, E^*)}(a) \right) \]
\[ + \chi_a(s)(a-s)^{q-1} \mathcal{J}_{(A^*, E^*)}(a-s) \right) E^{s-1}. \]
Note $P^*: X \to \mathcal{U}^*$. So we need $\mathcal{U}^* = L^p(J,U) \subset L^p(J,U) = \mathcal{U}$, if we want to compose $P$ and $P^*$. This is satisfied, when $p \leq p^* = \frac{p}{p-1}$, so $1 < p \leq 2$. Recall $q \in \left[\frac{1}{2}, 1\right]$ which gives a restriction $\frac{1}{2} < q < 1$.

Define Gramian controllability operator
\[
\Gamma_0^a := PP^* = \int_0^a G_{(A,E)}(a,s)BB^*G_{(A,E)}^*(a,s)ds.
\]

Noting Lemma 2.3, it is straightforward that $\Gamma_0^a$ is a linear bounded operator. In fact, it follows (4) that
\[
\|\Gamma_0^a\| \leq \|P\|\|P^*\| \leq M^2_p.
\]

Now we recall the following result.

**Theorem 2.5.** (See [18, Theorem 2.3]) Assume that $\Gamma : X \to X$ is symmetric. Then the following two conditions are equivalent:

(i): $\Gamma : X \to X$ is positive, that is, $\langle x, \Gamma x \rangle > 0$ for all nonzero $x \in X$.

(ii): For all $\eta \in X$, $x_\epsilon(\eta) = \epsilon(I + \Gamma)^{-1}(\eta)$ strongly converges to zero as $\epsilon \downarrow 0$.

We apply Theorem 2.5 with $\Gamma_0^a$. Then for any $x^* \in X$, we have
\[
\langle x^*, \Gamma_0^a x^* \rangle = \left\langle x^*, \int_0^a G_{(A,E)}(a,s)BB^*G_{(A,E)}^*(a,s)ds x^* \right\rangle
\]
\[
= \int_0^a \left\| BB^*G_{(A,E)}^*(a,s)x^* \right\|^2 ds
\]
\[
= \int_0^a \left\| (P^*x^*)(s) \right\|^2 ds.
\]

Note $P^*: X \to \mathcal{U}^* = L^p(J,U) \subset L^p(J,U) = \mathcal{U} \subset L^2(J,U)$, since $1 < p \leq 2$. So the above last integral is well defined. We also get that $\langle x^*, \Gamma_0^a x^* \rangle > 0$ iff $P^*x^* \neq 0$, i.e., $x^* \notin \ker P^*$. Consequently, $\Gamma_0^a$ is positive iff $\ker P^* = \{0\}$, i.e., $\Gamma_0^a$ is positive if and only if the linear system (3) is approximately controllable on $J$. Setting $R(\epsilon, \Gamma_0^a) := (\epsilon I + \Gamma_0^a)^{-1} : X \to X$, $\epsilon > 0$, by Theorem 2.5, we arrive at the following result (see also [18]).

**Theorem 2.6.** Let $\frac{1}{2} < q < 1$. (3) is approximately controllable on $J$ iff
\[
\epsilon R(\epsilon, \Gamma_0^a) \to 0
\]
in the strong topology as $\epsilon \downarrow 0$.

Finally we note that $R(\epsilon, \Gamma_0^a)$ is continuous with $\|R(\epsilon, \Gamma_0^a)\| \leq \frac{1}{\epsilon}$.

### 3. Main results

In this section, we study the approximate controllability of the system (1) by imposing that the corresponding linear system is approximately controllable and using the Schauder Fixed Point Theorem.

**Definition 3.1.** Let $x(a; x(0), u)$ be the state value of the system (1) at terminal time $a$ corresponding to the control $u \in \mathcal{U}$ and non-local initial condition $x(0)$. The system (1) is said to be approximately controllable on the interval $J$ iff the closure $\text{cl}(\mathcal{R}(a, x(0))) = X$.

Hence, $\mathcal{R}(a, x(0)) = \{x(a; x(0), u) : u \in \mathcal{U}\}$ is called the reachable set of the system (1) at terminal time $a$.

In the sequel, we introduce the following assumptions:

**(H_1):** (S_1), (S_2) and (S_3) hold.
(H2): \( f : J \times X \rightarrow X \) is continuous such that \( g_k := \sup_{t \in J, \|x\| \leq k} \|f(t, x)\| < \infty \) with \( \liminf_{k \to \infty} g_k/k = 0 \).

(H3): System (3) is approximately controllable on \( J \).

Recalling the condition (H3) and Theorem 2.6, for any \( x \in C(J, X) \) and \( h \in X \), we define the following control formula:

\[
u_\epsilon(t, x) = B^*G_{(A, E)}^*(a, t)R(\epsilon, \Gamma_0)\Upsilon(x)
\]

with

\[
\Upsilon(x) = h - \int_0^a G_{(A, E)}(a, s)f(s, x(s))ds.
\]

For each \( k > 0 \), define \( B_k = \{ x \in C(J, X) : \|x\|_\infty \leq k \} \).

Of course, \( B_k \) is a bounded, closed, convex subset in \( C(J, X) \). Using the above control \( u \) in (5), we consider an operator \( \mathcal{P} : B_k \rightarrow C(J, X) \) given by

\[
(\mathcal{P}_\epsilon x)(t) = \int_0^a G_{(A, E)}(t, s)[f(s, x(s)) + Bu_\epsilon(s, x)]ds \text{ for } t \in J.
\]

Now we can prove the following important result.

**Theorem 3.2.** Let \( 1/2 < q < 1 \). Assume (H1)–(H3) are satisfied. For any \( \epsilon > 0 \) there is a \( k(\epsilon) > 0 \) such that \( \mathcal{P}_\epsilon \) has a fixed point in \( B_{k(\epsilon)} \).

**Proof.** We divide the proof into several steps.

**Claim 1.** For an arbitrary \( \epsilon > 0 \), there is a \( k = k(\epsilon) > 0 \) such that \( \mathcal{P}_\epsilon \) maps \( B_k \) into \( B_k \).

If it is not true, then for each \( k > 0 \), there would exist \( x \in B_k \) and \( \bar{t}_k \in J \) such that \( \|\mathcal{P}_\epsilon x(\bar{t}_k)\| > k \). Using the following fact

\[
\|G_{(A, E)}^*(a, t)\| = \|G_{(A, E)}(a, t)\|
\]

\[
\leq \frac{M_1 \|E^{-1}\|}{\Gamma(q)} \left( M_1\|\Theta\| \sum_{k=1}^m \chi_k(t)(t_k - t)^{q-1} + \chi_0(t)(a - t)^{q-1} \right),
\]

we derive

\[
\|\Upsilon(x)\| \leq \|h\| + \frac{M_1 \|E^{-1}\|q^q}{\Gamma(q + 1)} \left( M_1\|\Theta\| \sum_{k=1}^m |a_k| + 1 \right)g_k
\]

\[
= \|h\| + M_2g_k := M_{\Upsilon, k},
\]

which implies

\[
\|u_\epsilon(s, x)\| \leq \frac{B\|M_1\|\|E^{-1}\|}{c\Gamma(q)} \left( M_1\|\Theta\| \sum_{k=1}^m \chi_k(s)(t_k - s)^{q-1} + \chi_0(s)(a - s)^{q-1} \right) M_{\Upsilon, k}.
\]

Notice that

\[
\sum_{i=1}^n c_i^2 \leq n \sum_{i=1}^n c_i^2
\]

for \( c_i > 0 \) and then we obtain

\[
k < \|\mathcal{P}_\epsilon x(\bar{t}_k)\| \]
\[
\begin{align*}
\leq & \int_0^a \|G_{(A,E)}(\bar{t}_k,s)\| [g_k + \|B\|\|u_\epsilon(s,x)\|] \, ds \\
\leq & \frac{M_1 \|E^{-1}\|}{\Gamma(q)} \int_0^a \left( M_1 \|\Theta\| \sum_{j=1}^m \chi_j(s)(t_j - s)^{-q-1} + \chi_{\bar{t}_k}(s)(\bar{t}_k - s)^{-q-1} \right) \\
& \times \left[ g_k + \frac{\|B\|^2 M_3 \|E^{-1}\|}{\epsilon \Gamma(q)} \left( M_1 \|\Theta\| \sum_{j=1}^m \chi_j(s)(t_j - s)^{-q-1} \\
& + \chi_a(s)(a-s)^{-q-1} \right) M_T, k \right] \, ds \\
\leq & M_G g_k + \frac{(m+2)\|B\|^2 M_2 \|E^{-1}\|^2 M_T, k}{\epsilon \Gamma(q)^2} \times \int_0^a \left( M_1 \|\Theta\| \sum_{j=1}^m \chi_j(s)^2 \right. \\
& \left. (t_j - s)^{2(q-1)} + \chi_{\bar{t}_k}(s)(\bar{t}_k - s)^{2(q-1)} + \chi_a(s)(a-s)^{2(q-1)} \right) ds \\
\leq & M_G g_k + \frac{(m+2)\|B\|^2 M_2 \|E^{-1}\|^2 (\|h\| + M_G g_k) a^{2q-1}}{\epsilon \Gamma(q)^2(2q-1)} \\
& \times \left( M_1 \|\Theta\|^2 \sum_{j=1}^m a_j^2 + 2 \right) \\
:= & \left( M_G + \frac{M_G'}{\epsilon} \right) g_k + \frac{M_G''}{\epsilon} \|h\|. 
\end{align*}
\]

Dividing both sides by \( k \) and taking the lower limit as \( k \to \infty \), we derive a contradiction \( 1 \leq 0 \).

**Claim 2.** \( \mathcal{P}_\epsilon : C(J, X) \to C(J, X) \) is continuous.

Let \( \{x^m\}_{n \in N} \subseteq C(J, X) \) be a sequence such that \( x^m \to x \) as \( m \to \infty \).

Then \( \|f(\cdot, x^m) - f(\cdot, x)\|_\infty \to 0 \) as \( m \to \infty \). Next, following the above estimations, we first obtain

\[
\|u_\epsilon^m(s, x^m) - u_\epsilon(s, x)\| \leq \frac{M_G}{\epsilon} \|B\| \|G_{(A,E)}(a, s)\| \|f(\cdot, x^m) - f(\cdot, x)\|_\infty,
\]

and then we get

\[
\begin{align*}
\|& (\mathcal{P}_\epsilon x^m)(t) - (\mathcal{P}_\epsilon x)(t)\| \\
\leq & \int_0^a \|G_{(A,E)}(t, s)\| \left( \|f(\cdot, x^m) - f(\cdot, x)\|_\infty + \|B\|\|u_\epsilon^m(s, x^m) - u_\epsilon(s, x)\| \right) \, ds \\
\leq & \int_0^a \|G_{(A,E)}(t, s)\| \left( \|f(\cdot, x^m) - f(\cdot, x)\|_\infty \\
& + \frac{M_G}{\epsilon} \|B\|^2 \|G_{(A,E)}(a, s)\| \|f(\cdot, x^m) - f(\cdot, x)\|_\infty \right) \, ds \\
\leq & \left( M_G + \frac{M_G'}{\epsilon} \right) \|f(\cdot, x^m) - f(\cdot, x)\|_\infty \\
\to & 0 \text{ as } m \to \infty,
\end{align*}
\]
for any $t \in J$. This yields that $P_\epsilon$ is continuous.

**Claim 3.** For every fixed $t \in J$, the set $\Pi(t) = \{(P_\epsilon x)(t) : x \in B_k \}$ is relatively compact in $X$.

Note that

$$(P_\epsilon x)(t) = E^{-1}(P_\epsilon^0 x)(t)$$

and

$$(P_\epsilon^0 x)(t) = \int_0^a G^0_{(A,E)}(t,s)[f(s,x(s)) + Bu_s(x,s)]ds \text{ for } t \in J.$$ 

Next, by Claim 1 for $x \in B_k$, we derive

$$(P_\epsilon^0 x)(t) \leq \|E^{-1}\|^{-1} \left( M_G + \frac{M'_G}{\epsilon} \right) g_k + \frac{M''_G}{\epsilon} \|h\|.$$ 

Thus, $\{(P_\epsilon^0 x)(t) : x \in B_k \}$ is bounded in $X$. By (S$_3$), we know that $\Pi(t) = \{(P_\epsilon x)(t) : x \in B_k \}$ is relatively compact in $X$.

**Claim 4.** $\{(P_\epsilon x) : x \in B_k \}$ is an equicontinuous family of functions on $J$.

Let $x \in B_k$ and $t', t'' \in J$ be such that $t' < t''$. Note that

$$\| (P_\epsilon x)(t'') - (P_\epsilon x)(t') \| \leq \int_0^a \| G_{(A,E)}(t'',s) - G_{(A,E)}(t',s) \| f(s,x(s)) + Bu_s(x,s) \| ds$$

and

$$G_{(A,E)}(t'',s) - G_{(A,E)}(t',s)$$

$$= E^{-1}\left( -\sum_{k=1}^m [\mathcal{J}_{(A,E)}(t'') - \mathcal{J}_{(A,E)}(t')] \chi_k(s) \Theta(t_k - s)^{q-1} \mathcal{J}_{(A,E)}(t_k - s) \right)$$

$$+ E^{-1}\left( \chi_{(A,E)}(t'' - s)^{q-1} - \chi_{(A,E)}(t' - s)^{q-1} \right) \mathcal{J}_{(A,E)}(t' - s).$$

Then,

$$\| (P_\epsilon x)(t'') - (P_\epsilon x)(t') \| \leq \sum_i K_i,$$

where

$$K_i \leq \int_0^a \| E^{-1}\left( \sum_{k=1}^m [\mathcal{J}_{(A,E)}(t'') - \mathcal{J}_{(A,E)}(t')] \chi_k(s) \Theta(t_k - s)^{q-1} \mathcal{J}_{(A,E)}(t_k - s) \right) \| ds$$

$$\leq \frac{\|E^{-1}\|}{\Gamma(q)} \| \mathcal{J}_{(A,E)}(t'') - \mathcal{J}_{(A,E)}(t') \| \int_0^a \sum_{j=1}^m M_1 \| \Theta \| \sum_{j=1}^m \chi_j(s)(t_j - s)^{q-1} \| ds$$

$$\times \left( g_k + \|B\|^2 M_1 \|E^{-1}\| M_1 \| \Theta \| \sum_{j=1}^m \chi_j(s)(t_j - s)^{q-1} \right) ds$$

$$\leq \left( M_G + \frac{M'_G}{\epsilon} \right) g_k + \frac{M''_G}{\epsilon} \|h\|. $$
then
\[
K_2 \leq \int_0^a \left\| E^{-1} \left( \chi_{t'}(s)(t' - s)^{(q-1)} \left( J_{(A,E)}(t'' - s) - J_{(A,E)}(t' - s) \right) \right) \right\| \times \| f(s,x(s)) + B u_t(s,x) \| ds
\]
\[
\leq \max_{s \in [0,t']} \left\| J_{(A,E)}(t'' - s) - J_{(A,E)}(t' - s) \right\| \left\| E^{-1} \right\| \int_0^a \chi_{t'}(s)(t' - s)^{(q-1)}
\]
\[
\times \left( g_k \frac{\| B \|^2 M_1 \| E^{-1} \}}{e \Gamma(q)} \left( M_1 \| \Theta \| \sum_{j=1}^m \chi_j(s)(t_j - s)^{(q-1)}
\right.
\]
\[
+ \chi_a(s)(a - s)^{(q-1)} \right) M_{T,k} \right) ds
\]
\[
\leq \max_{s \in [0,t']} \left\| J_{(A,E)}(t'' - s) - J_{(A,E)}(t' - s) \right\| \left\| E^{-1} \right\| (m + 3)
\]
\[
\times \int_0^a \left( \chi_{t'}(s)(t' - s)^{(2q-1)} + g_k^2 \frac{\| B \|^2 M_1 \| E^{-1} \}}{e^2 \Gamma(q)^2} \left( M_1 \| \Theta \| \right)^2
\]
\[
\sum_{j=1}^m \chi_j(s)(t_j - s)^{(2q-1)} + \chi_a(s)(a - s)^{(2q-1)} \right) M_{T,k}^2 \right) ds
\]
\[
= \max_{s \in [0,t']} \left\| J_{(A,E)}(t'' - s) - J_{(A,E)}(t' - s) \right\| \left\| E^{-1} \right\| (m + 3)\frac{a^{2q-1}}{2q - 1}
\]
\[
\times \left( 1 + g_k^2 \frac{\| B \|^2 M_1 \| E^{-1} \}}{e^2 \Gamma(q)^2} \left( M_1 \| \Theta \| \right)^2 \sum_{j=1}^m a_j^2 + 1 \right) M_{T,k}^2 \right) ds,
\]
and by the H"older inequality, \((c - d)^2 \leq |c - d|(c + d), c, d \geq 0, and
\[
\int_0^a \left( t'' - s \right)^{(2q-1)} - \chi_{t'}(s)(t' - s)^{(2q-1)} \right) ds
\]
\[
= \int_0^a \left( t' - s \right)^{(2q-1)} - \left( t'' - s \right)^{(2q-1)} \right) ds + \int_0^a \left( t'' - s \right)^{(2q-1)} ds
\]
\[
= \frac{t''^{2q-1} - t'^{2q-1}}{2q - 1}
\]
\[
\leq \frac{2(t'' - t')^{2q-1}}{2q - 1},
\]
we have
\[
K_3 \leq \int_0^a \left\| E^{-1} \left( \chi_{t''}(s)(t'' - s)^{(q-1)} - \chi_{t'}(s)(t' - s)^{(q-1)} \right) J_{(A,E)}(t'' - s) \right\|
\]
\[
\times \| f(s,x(s)) + B u_t(s,x) \| ds
\]
Lemma 3.3. Assume \((S_3)\) hold. For any \(r\) such that \(rq > 1\), the operator \(Q : L^r(J, X) \to X\) given by 

\[
Q \ell := \int_0^a G_{(A,E)}(a,s)\ell(s)ds
\]
is compact.

Proof. We can write the linear operator $Q$ as

$$Q = E^{-1}Q_0, \quad Q_0l := \int_0^a G_{(A,E)}^0(a,s)l(s)ds.$$ (7)

Then following computations for (4), we derive

$$\|Q_0\| \leq \frac{M_1}{r(q)} \left( \frac{M_1 \|\Theta\|}{r} \sum_{k=1}^{m} |a_k| + 1 \right) \left( \frac{r - 1}{q^r - 1} a^{1 - \frac{q}{r}} \right)^{r - 1} \|l\|_{L^r}.$$ 

So $Q_0 : L^r(J, X) \to X$ is continuous. Then (S3) and (7) imply that $Q$ is compact. The proof is finished. \qed

Now we are ready to present the main result in this paper.

**Theorem 3.4.** Let all the assumptions of Theorem 3.2 be satisfied. Moreover, there exists a $r$ with $rq > 1$ and $N \in L^r(J, \mathbb{R}^n)$ such that $\|f(t, x)\| \leq N(t)$ for all $(t, x) \in J \times X$. Then the system (1) is approximately controllable on the interval $J$.

Proof. By Theorem 3.2, there is a fixed point $x_\epsilon$ of $P_\epsilon$ in $B_{k_\epsilon}(\cdot)$, which is a mild solution of the system (1) under the control $u_\epsilon(t, x_\epsilon)$ in (5) and satisfies

$$x_\epsilon(a) = \int_0^a G_{(A,E)}(a,s) (f(s, x_\epsilon(s)) + Bu_\epsilon(s, x_\epsilon)) ds $$

$$= \int_0^a G_{(A,E)}(a,s) f(s, x_\epsilon(s)) ds + \int_0^a G_{(A,E)}(a,s) Bu_\epsilon(s, x_\epsilon) ds $$

$$= \int_0^a G_{(A,E)}(a,s) f(s, x_\epsilon(s)) ds + \int_0^a G_{(A,E)}(a,s) BB^* G_{(A,E)}^*(a,s) R(\epsilon, \Gamma_0^a)$$

$$+ \left( h - \int_0^a G_{(A,E)}(a,z) f(z, x_\epsilon(z)) dz \right) ds $$

$$= \int_0^a G_{(A,E)}(a,s) f(s, x_\epsilon(s)) ds + \Gamma_0^a R(\epsilon, \Gamma_0^a)$$

$$+ \left( h - \int_0^a G_{(A,E)}(a,z) f(z, x_\epsilon(z)) dz \right) $$

$$= h - \epsilon R(\epsilon, \Gamma_0^a) \left( h - \int_0^a G_{(A,E)}(a,z) f(z, x_\epsilon(z)) dz \right) $$

$$= h - \epsilon R(\epsilon, \Gamma_0^a) \Upsilon(x_\epsilon) $$

(8)

for

$$\Upsilon(x_\epsilon) := h - \int_0^a G_{(A,E)}(a,z) f(z, x_\epsilon(z)) dz.$$

Furthermore,

$$\int_0^a \|f(s, x_\epsilon(s))\|^r ds \leq \int_0^a |N(s)|^r ds \leq \|N\|_{L^r}^r.$$

From the reflexivity of $L^r(J, X)$, there is a subsequence $\{f(t, x_\epsilon(t))\}_{i=1}^\infty$, $\epsilon_i \to 0$ as $i \to \infty$ that converges weakly to say $f \in L^r(J, X)$. Define

$$w = h - \int_0^a G_{(A,E)}(a,s) f(s) ds.$$
Since
\[ \| \mathcal{Y}(x_e) - w \| = \left\| \int_0^a G(x_e, s)[f(s, x_e(s) - f(s))ds \right\|, \]
(9)
by Lemma 3.3 we find that the right-hand side of (9) tends to zero as \( i \to \infty \). Thus, it follows from Theorem 2.6, (8) and (9) that
\[
\| x_e(a) - h \| \leq \| \epsilon_i R(\epsilon_i, \Gamma_0^a)(w) \| + \| \epsilon_i R(\epsilon_i, \Gamma_0^a)(w) \| \| \mathcal{Y}(x_e) - w \|
\]
\[
\leq \| \epsilon_i R(\epsilon_i, \Gamma_0^a)(w) \| + \| w \| \| \mathcal{Y}(x_e) - w \|
\]
\[
\to 0 \text{ as } i \to \infty.
\]
This proves the approximate controllability of the system (1).

\[ \square \]

**Remark 3.** By applying (ii) of Lemmas 2.1 and 2.2, Theorem 3.4 can be extended to the case when \((S_3)\) is replaced by \((S_4)\) and in addition, compactness of \( \{ W(t) \}_{t>0} \) is assumed.

4. **An example.** Take \( X = U = L^2[0, \pi] \) and \( p = 2 \). Consider the following fractional partial differential equation with control

\[
\begin{cases}
\frac{\partial^2 D_0^\alpha (x(t, y) - x_0(t, y)) = x_0(t, y) + \mu \cos 2\pi t \sin x(t, y) + u(t), & y \in [0, \pi], \ t \in J_1 = [0, 1], \ 0 < \mu < \infty, \ q = \frac{2}{3} \in [\frac{1}{2}, 1], \\
x(t, 0) = x(t, \pi) = 0, & t \geq 0, \\
x(0, y) + \sum_{k=1}^{\infty} a_k x(t_k, y) = 0, & a_k \in \mathbb{R}, \ t_k \in J_1, \ 0 \leq y \leq \pi.
\end{cases}
\]

Define \( A : D(A) \subset X \to X \) by \( Ax = -x_{yy} \) and \( E : D(E) \subset X \to X \) by \( Ex = x - x_0 \) respectively, where \( D(A), D(E) \) is given by \( \{ x \in X : x_y \ \text{is absolutely continuous}, \ x_{yy} \in X, x(0) = x(\pi) = 0 \} \). Then \( Ax = -\sum_{n=1}^{\infty} n^2 \langle x, x_n \rangle x_n \), \( x \in D(A) \) and \( Ex = \sum_{n=1}^{\infty} (1 + n^2) \langle x, x_n \rangle x_n, \ x \in D(E) \), respectively (see [15]), where \( x_n(y) = \sqrt{\frac{2}{\pi}} \sin ny, \ n = 1, 2, \ldots \) is the orthonormal set of eigenfunctions of \( A \). Moreover,
\[
E^{-1} x = \sum_{n=1}^{\infty} (1 + n^2)^{-1} \langle x, x_n \rangle x_n, \ x \in D(E)
\]
is compact since \( \lim_{n \to \infty} \frac{1}{1 + n^2} = 0 \). Thus, \( E^{-1} \) is compact, and bounded. Hence, \((H_d)\) is satisfied.

Next, the bounded operator \( -AE^{-1} \) generates a strongly continuous semigroup \( \{ W(t), t \geq 0 \} \) written as
\[
W(t)x := \sum_{n=1}^{\infty} e^{-\frac{n^2}{1 + n^2} t} \langle x, x_n \rangle x_n,
\]
with \( \| W(t) \| \leq e^{-t} \leq 1 \). Furthermore, \( \mathcal{F}(A, E)(\cdot) \) and \( \mathcal{F}(A, E)(\cdot) \) are now given by
\[
\mathcal{F}(A, E)(t) := \int_0^\infty 0 \xi_2 \left( \frac{2}{3} \right) \sum_{n=1}^{\infty} e^{-\frac{n^2}{1 + n^2} \frac{2}{3} \theta} \langle x, x_n \rangle x_n d\theta
\]
\[
= \sum_{n=1}^{\infty} E_2 \left( -\frac{n^2}{1 + n^2} \frac{2}{3} \theta \right) \langle x, x_n \rangle x_n,
\]
\[
\mathcal{F}(A, E)(t)x = \frac{2}{3} \int_0^\infty \theta \xi_2 \left( \frac{2}{3} \right) \sum_{n=1}^{\infty} e^{-\frac{n^2}{1 + n^2} \frac{2}{3} \theta} \langle x, x_n \rangle x_n d\theta
\]
\[
= \sum_{n=1}^{\infty} E_2^{\frac{2}{3}} \left( -\frac{n^2}{1 + n^2} \frac{2}{3} \theta \right) \langle x, x_n \rangle x_n,
\]
where $E_{\frac{1}{2}}$ and $E_{\frac{1}{2},\frac{3}{2}}$ are the classical and generalized Mittag-Leffler functions \[1\] and \[13\], respectively. Thus, $\|J_{(A,E)}(t)\| \leq 1$ and $\|J_{(A,E)}(t)\| \leq \frac{1}{1^{\frac{1}{2}}}$ for $t \geq 0$.

Supposing $\sum_{k=1}^{m} |a_k| < 1$, by Remark 1, $\Theta = [I + \sum_{k=1}^{m} a_k J_{(A,E)}(t_k)]^{-1}$ exists. Define an operator $f : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ by $f(t,x)(y) = \mu \cos(2\pi t) \sin x(y)$. It is easy to verify (H2). Next, $B : \mathcal{U} \rightarrow \mathcal{X}$ is defined by $B = I$. Now, the system (10) can be abstracted as

$$
\begin{cases}
\mathcal{C}_{\infty} \mathcal{D}_{\infty}^{\frac{1}{2}}(E(t)) = -Ax(t) + f(t,x(t)) + Bu(t), \ t \in J_1, \\
x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0.
\end{cases}
$$

Next, $\Gamma_0 : \mathcal{X} \rightarrow \mathcal{X}$ has the form $\Gamma_0 = \int_{0}^{1} G_{(A,E)}(1,s) G_{(A,E)}^{*}(1,s) ds$, where by (2) we compute

$$
\langle G_{(A,E)}(1,s)x, x_n \rangle = E^{-1} \left( - \sum_{k=1}^{m} J_{(A,E)}(1) \chi_k(s) \Theta(t_k - s)^{-\frac{1}{2}} J_{(A,E)}(t_k - s) \right) \\
+ \chi_1(s)(1-s)^{-\frac{1}{2}} B_{(A,E)}(1-s)
$$

$$
= \langle x, x_n \rangle \left( - \sum_{k=1}^{m} \chi_k(s) \Theta(t_k - s)^{-\frac{1}{2}} E_{\frac{3}{2},\frac{3}{2}} \left( - \frac{n^2 + n^2}{1 + n^2} \right) \right) \\
E_{\frac{3}{2},\frac{3}{2}} \left( - \frac{n^2 (t_k - s)^{\frac{3}{2}}}{1 + n^2} \right) + \chi_1(s)(1-s)^{-\frac{3}{2}} E_{\frac{3}{2},\frac{3}{2}} \left( - \frac{n^2 (1-s)^{\frac{3}{2}}}{1 + n^2} \right)
$$

for any $x \in \mathcal{X}$ and $n \in \mathbb{N}$. To check that $\Gamma_0$ is positive, we consider the equation

$$
\langle \Gamma_0 x, x \rangle = \left\langle \int_{0}^{1} G_{(A,E)}(1,s) G_{(A,E)}^{*}(1,s) ds, x \right\rangle = \int_{0}^{1} \|G_{(A,E)}(1,s)x\|^2 ds = 0,
$$

which implies

$$
G_{(A,E)}^{*}(1,s)x = 0, \ 0 \leq s < 1.
$$

Then (11) gives

$$
0 = \left\langle G_{(A,E)}^{*}(1,s)x, x_n \right\rangle = \frac{\langle x, x_n \rangle}{1 + n^2} (1-s)^{-\frac{3}{2}} E_{\frac{3}{2},\frac{3}{2}} \left( - \frac{n^2 (1-s)^{\frac{3}{2}}}{1 + n^2} \right)
$$

for any $n \in \mathbb{N}$ and $t_m < s < 1$. If $x \neq 0$ then there is an $n_0 \in \mathbb{N}$ so that

$$
E_{\frac{3}{2},\frac{3}{2}} \left( - \frac{n_0^2 (1-s)^{\frac{3}{2}}}{1 + n_0^2} \right) = 0
$$

for any $t_m < s < 1$, which is not possible, since $E_{\frac{3}{2},\frac{3}{2}}(0) = \frac{1}{\Gamma(\frac{3}{2})}$. So we obtain $x = 0$ and $\Gamma_0$ is positive. Finally, we take $r = 2$, so $rq = \frac{4}{3} > 1$, $N(r) = \mu \sqrt{2\pi} \in L^r(J_1, \mathbb{R}^+)$ and obtain

$$
\|f(t,x)\| = \mu \sqrt{\int_{0}^{\pi} \cos^2 2\pi t \sin^2 x(y) dy} \leq \mu \sqrt{2\pi} = N(t).
$$

Summarizing, all the assumptions of Theorem 3.4 are satisfied and thus the system (10) is controllable on $J_1$. 

5. Conclusions. Approximate controllability of Sobolev-type fractional evolution systems, with classical non-local conditions in Hilbert spaces, are investigated. By imposing an approximate controllability of linear system and rewriting the control problem as a fixed point equation for an appropriate nonlinear operator, sufficient conditions of approximate controllability are presented via the Schauder Fixed Point Theorem.

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REFERENCES

[1] C. Atkinson and A. Osseiran, Rational solutions for the time-fractional diffusion equation, *SIAM J. Appl. Math.*, 71 (2011), 92–106.
[2] D. Baleanu, J. A. T. Machado and A. C. Luo, *Fractional Dynamics and Control*, Springer, New York, 2012.
[3] D. Baleanu and O. Mustafa, *Asymptotic Integration and Stability*, (Series on Complexity, Nonlinearity and Chaos) World Scientific, London, 2015.
[4] A. Debbouche and D. Baleanu, Controllability of fractional evolution nonlocal impulsive quasi-linear delay integro-differential systems, *Comput. Math. Appl.*, 62 (2011), 1442–1450.
[5] A. Debbouche and D. F. M. Torres, Approximate controllability of fractional nonlocal delay semilinear systems in Hilbert spaces, *Internat. J. Control*, 86 (2013), 1577–1585.
[6] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics 2004, Springer, New York, 2010.
[7] E. H. Doha, A. H. Bhrawy, D. Baleanu and R. M. Hafez, A new Jacobi rational–Gauss collocation method for numerical solution of generalized pantograph equations, *Appl. Numer. Math.*, 77 (2014), 43–54.
[8] M. M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, *Chaos Solitons Fractals*, 14 (2002), 433–440.
[9] M. Fečkan, J. Wang and Y. Zhou, Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators, *J. Optim. Theory Appl.*, 156 (2013), 79–95.
[10] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.
[11] M. Kerboua, A. Debbouche and D. Baleanu, Approximate controllability of Sobolev type fractional stochastic nonlocal nonlinear differential equations in Hilbert spaces, *Electron. J. Qual. Theory Differ. Equ.*, 58 (2014), 1–16.
[12] M. Kerboua, A. Debbouche and D. Baleanu, Approximate controllability of Sobolev type nonlocal fractional stochastic dynamic systems in Hilbert spaces, *Abstr. Appl. Anal.*, 2013 (2013), Art. ID 262191, 10pp.
[13] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.
[14] M. Li and J. Wang, Finite time stability of fractional delay differential equations, *Appl. Math. Lett.*, 64 (2017), 170–176.
[15] J. H. Lightbourne and S. M. Rankin, A partial functional differential equation of Sobolev type, *J. Math. Anal. Appl.*, 93 (1983), 328–337.
[16] J. T. Machado, V. Kiryakova and F. Mainardi, Recent history of fractional calculus, *Commun. Nonlinear Sci. Numer. Simul.*, 16 (2011), 1140–1153.
[17] R. Magin, X. Feng and D. Baleanu, Solving the fractional order Bloch equation, *Conc. Magn. Reson. Part A*, 34A (2009), 16–23.
[18] N. I. Mahmudov, Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces, *SIAM J. Control Optim.*, 42 (2003), 1604–1622.

[19] N. I. Mahmudov, Approximate controllability of fractional Sobolev-type evolution equations in Banach spaces, *Abstr. Appl. Anal.*, 2013 (2013), Art. ID 502839, 9pp.

[20] N. I. Mahmudov and S. Zorlu, Approximate controllability of fractional integro-differential equations involving nonlocal initial conditions, *Bound. Value Probl.*, 2013 (2013), 16pp.

[21] N. I. Mahmudov and S. Zorlu, On the approximate controllability of fractional evolution equations with compact analytic semigroup, *J. Comput. Appl. Math.*, 259 (2014), 194–204.

[22] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley & Sons, Inc., New York, 1993.

[23] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[24] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[25] R. Sakthivel and Y. Ren, Approximate controllability of fractional differential equations with state-dependent delay, *Results Math.*, 63 (2013), 949–963.

[26] V. E. Tarasov, *Fractional Dynamics*, Springer-HEP, Heidelberg, Beijing, 2010.

[27] J. Wang and Y. Zhou, Analysis of nonlinear fractional control systems in Banach spaces, *Nonlinear Anal.*, 74 (2011), 5929–5942.

[28] J. Wang, M. Fečkan and Y. Zhou, Controllability of Sobolev type fractional evolution systems, *Dyn. Partial. Differ. Equ.*, 11 (2014), 71–87.

[29] J. Wang, M. Fečkan and Y. Zhou, A survey on impulsive fractional differential equations, *Fract. Calc. Appl. Anal.*, 19 (2016), 806–831.

[30] J. Wang, A. G. Ibrahim and M. Fečkan, Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces, *Appl. Math. Comput.*, 257 (2015), 103–118.

[31] J. Wang and Y. Zhang, On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives, *Appl. Math. Lett.*, 39 (2015), 85–90.

[32] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[33] Y. Zhou and F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, *Nonlinear Anal. Real World Appl.*, 11 (2010), 4465–4475.

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