Cosmological imprint of the second law of thermodynamics

Hyeong-Chan Kim\textsuperscript{1}, Jae-Weon Lee\textsuperscript{2} and Jungjai Lee\textsuperscript{3}

\textsuperscript{1} Center for Quantum Spacetime, Sogang University, Seoul 121-742, Korea
\textsuperscript{2} School of Computational Sciences, Korea Institute for Advanced Study, 207-43 Cheongnyangni 2-dong, Dongdaemun-gu, Seoul 130-012, Korea
\textsuperscript{3} Department of Physics, Daejin University, Pocheon 487-711, Korea
E-mail: hyeongchan@sogang.ac.kr, scikid@kias.re.kr and jjlee@daejin.ac.kr

Received 15 June 2008
Accepted 6 August 2008
Published 28 August 2008

Online at stacks.iop.org/JCAP/2008/i=08/a=035
doi:10.1088/1475-7516/2008/08/035

\textbf{Abstract.} We study the evolution of the universe in the presence of inflaton, matter, radiation and holographic dark energy. The time evolution of the scale factor is obtained by solving the Friedmann equation of the universe with a good approximation. We present two independent ways which determine the value of the dark energy constant \(d \sim 1\) from the observational data. The two ways are measuring the deceleration parameter and measuring a universal constant depending only on \(d\). The universal constant is given by a dimensionless combination of three scale factors at the equipartition times of radiation–matter, radiation–dark energy and matter–dark energy. We also discuss that the second law of thermodynamics determines the point of time when the dark-energy-dominated era begins in the universe.

\textbf{Keywords:} dark energy theory, inflation

\textbf{ArXiv ePrint:} 0804.2579
1. Introduction

We show that the decelerating expansion of the universe with holographic dark energy cannot go on forever since it leads to violation of the second law of thermodynamics eventually. Interpreting the area of the cosmological event horizon as the entropy of missing information beyond the horizon, we argue that the second law of thermodynamics restricts the time derivative $\dot{R}_h$ to be non-negative ($d \geq 1$) and determines the point of time when the accelerating expansion begins. To show this, we study the whole history of the universe with holographic dark energy.

Bekenstein [1] formulated the generalized second law which identifies the area of black hole event horizon with its entropy. The law states that the sum of ordinary entropy and one quarter of the horizon area of the black hole cannot decrease with time. This identification of the horizon area as the entropy of the black hole was supported by the presence of thermal Hawking radiation [2] with the black hole temperature $T = (\hbar/\kappa B)(\kappa/2\pi)$, where $\kappa$ is the surface gravity of the black hole. In [3, 4], the authors discussed that the horizon area of a black hole denotes the entropy of information erased by the black hole semiclassically by using Landauer’s principle [5]. It seems natural to relate the area of an event horizon to the quantity of missing information since the horizon, by definition, is the boundary of information. The information always goes beyond the horizon and never returns. Therefore, the corresponding entropy must be a non-decreasing function of time, which coincides with the second law of thermodynamics. In the cosmological case, the de Sitter space has received much attention. Gibbons and Hawking [6] have asserted that the generalized second law extends to de Sitter horizons, and detailed investigation [7] confirms this. The discussion were generalized to quasi-de Sitter spacetime [8] and to more general cosmological models [9].

It is interesting to investigate the role of thermodynamics in the Friedmann–Robertson–Walker spacetime. The spacetime satisfies its first law $dE = T_A dS_A$ by identifying its energy as the Misner-Sharp mass [10] at the apparent horizon in various theories of gravity, including the Einstein, Lovelock, nonlinear and scalar–tensor
The cosmological imprint of the second law of thermodynamics [11]. This result strongly suggests that the relationship between the first law of thermodynamics of the apparent horizon and the Friedmann equation has a profound physical connection, even in the presence of the cosmic microwave background radiation with higher temperature than that of the Hawking temperature of the apparent horizon $T_A$.

The second law of thermodynamics, however, is not guaranteed to be satisfied by the apparent horizon. Rather, it would be natural to relate the second law to the future event horizon of our universe as in the case of a black hole. Following the analogy with the black hole case, we may assume that the cosmological horizon area denotes the quantity of missing information. Since no information returns once it goes out of the event horizon, the quantity of missing information should increase monotonically. This can be written as $\dot{R}_h \geq 0$, and we interpret this as the second law for the cosmological event horizon.

In relation to the black hole event horizon, we present another argument which supports this conclusion: the area of a black hole horizon never decreases with classical processes. We may demand that the property also holds for the cosmological event horizons. In [12], Huang and Li provided several arguments leading to the conclusion $\dot{R}_h \geq 0$. They are the dominant energy condition, the increase of the entropy of the universe and the AdS/CFT correspondence with central charge $d \sim M_p^2/H^2$. In addition, if $d < 1$, the proper size of the future horizon will shrink to zero and the IR cutoff will become shorter than the UV cutoff at a finite time in the future; the very definition of the holographic dark energy breaks down. In addition, if $d < 1$, the size of the Hubble horizon is bigger than the distance to the event horizon ($R_h < H^{-1}$) during the inflationary era. In this case, the spectra of the density perturbation based on the Hubble scale may have trouble with the homogeneity. To avoid these difficulties, in this paper, we assume that the condition $d \geq 1$ holds for our universe.

Therefore, in this case, the entropy bound formulated by Bousso [13] restricting the total entropy inside to its boundary area is inappropriate since the horizon area is not directly related to the total degrees of freedom of the universe. The entropy conjecture was tested in an adiabatically expanding universe in [14].

We briefly introduce the holographic dark energy model. Following the idea that the short distance cutoff is related to the infrared cutoff, the holographic dark energy model was first developed by Li [15] to explain the present accelerating expansion of the universe. The infrared cutoff relevant to the dark energy was shown to be the size of the cosmological event horizon. The stability of the holographic dark energy under small perturbation was also studied [16, 17]. The origin of the holographic dark energy is under investigation. The entanglement energy on the cosmological event horizon related to the Hawking radiation gives the dark energy of the holographic form [18]. It was also shown that the spacetime foam uncertainty relation of the form $\delta l \geq l_p^{d-1}$ leads to the holographic type energy densities [19]. The holographic dark energy model was generalized to have an interaction with matter [20] and was constrained by using the supernova data in [21, 22]. The second law of thermodynamics for several dark energy models was discussed in [23].

The Penrose diagram of the universe with a holographic dark energy with equation of state $-1 \leq w < -1/3$ was given in [24]. The event horizon is a surface such that any light departing from the surface cannot arrive at the origin, however much time passes. If one compares the two Penrose diagrams in figure 1, one may notice that the region we live in is similar to the white hole region of the Kruskal spacetime in the following senses.
First, the singularities are in the past. Second, the scale factor (or space size for fixed \( t \)) increases with time. Third, the asymptotic region (\( r \to \infty \)) lies behind the horizon. If one observe the event horizon from the outside of a black hole, one may see a static metric and the black hole entropy counts the quantity of missing information beyond the horizon [3]. In the cosmological case, we live inside the cosmological horizon and its area must be used to count the quantity of missing information beyond the horizon. Naturally, this entropy (area) is not directly related to the total energy ‘inside’ the horizon. Rather, it enumerates how much information of the universe cannot be determined from the initial condition of the universe due to the information loss beyond the horizon. From our viewpoint, the missing information must be related to the dark energy through Landauer’s principle [18], which presents the holographic form of dark energy. Therefore, considering the second law for the universe, we assume that the cosmological event horizon provides the holographic dark energy. The presence of the holographic dark energy may change the role of the second law of thermodynamics in the universe. In this paper, we are interested in the role of dark energy in the evolution of the universe.

In section 2, we construct the precise cosmological model with a holographic dark energy by dividing the evolution of the universe into three phases, the inflation, the consecutive regime of the radiation-dominated era (RDE) and the first half of the matter-dominated era (MDE), and the consecutive regime of the last half of MDE and the dark-energy-dominated era (DDE). In section 3, we describe the evolution of the physical quantities in detail for each phase. In section 4, we summarize the results and discuss the physical role of the second law of thermodynamics.

2. Cosmological model with the holographic dark energy

In this paper, we consider the flat (\( k = 0 \)) Friedmann universe which is favored by observations [25] and inflationary theory [26]. The model is described by the metric

\[
ds^2 = -dt^2 + a^2(t) d\Omega^2_{(3)},
\]

where \( a(t) \) is the scale factor as usual. We assume that there are four different kinds of energy densities in our universe denoted by the inflaton \( \rho_{\text{inflaton}} \), the radiation \( \rho_r \), the matter \( \rho_m \) and the holographic dark energy \( \rho_h \). Each energy density except for the inflaton
has a specific form of behavior on the scale factor $a(t)$ as
\[
\rho_r(t) = \rho_0 \left( \frac{a_0}{a} \right)^4, \quad \rho_m(t) = \rho_m\left( \frac{a_0}{a} \right)^3, \quad \rho_h(t) = \frac{3M_p^2 a^2}{R_h^2},
\]
where the suffix 0 denotes the value at the present time $t_0$ and $R_h$ represents the distance to the future event horizon
\[
R_h(t) \equiv a(t) \int_t^\infty \frac{da'(t')}{H(t') a(t')} = a(t) \int_t^\infty \frac{dt'}{a(t')}. \tag{2.2}
\]

The Friedmann equation of the universe with inflaton, radiation, matter and holographic dark energy is given by
\[
H^2 = \frac{\dot{a}^2}{a^2} = \frac{\rho_{\text{inflaton}} + \rho_r + \rho_m + \rho_h}{3M_p^2}, \tag{2.3}
\]
For later convenience, we define the portions of the energy densities at a given time $t$ in the universe by
\[
\Omega_h \equiv \frac{\rho_h}{\rho_c} = \frac{\rho_h}{3M_p^2 H^2} = \frac{d^2}{H^2 R_h^2}, \quad \Omega_{\text{inflaton}} \equiv \frac{\rho_{\text{inflaton}}}{\rho_c}, \quad \Omega_r \equiv \frac{\rho_r}{\rho_c}, \quad \Omega_m \equiv \frac{\rho_m}{\rho_c}, \tag{2.4}
\]
where the critical energy density is $\rho_c(t) = 3M_p^2 H^2$. With this definition, the Friedmann equation (2.3) is rewritten in a simple form: $\Omega_h + \Omega_{\text{inflaton}} + \Omega_r + \Omega_m = 1$.

With the condition $\lim_{t \to \infty} a(t) = \infty$, equation (2.2) can be cast into the differential form
\[
\dot{R}_h(t) = HR_h - 1, \tag{2.5}
\]
where the over-dot denotes the derivative with respect to time $t$. Following the assumption $\dot{R}_h(t) \geq 0$, the event horizon is placed outside of the Hubble radius $[R_h(t) \geq H^{-1}(t)]$ always. From equations (2.5) and (2.3), a formula which relates the time derivative of $R_h$ to $\Omega_h$ [12] can be derived:
\[
\dot{R}_h(t) = \frac{d}{\sqrt{\Omega_h}} - 1. \tag{2.6}
\]
This equation implies that the distance to the horizon is a non-decreasing function of time if $d \geq \sqrt{\Omega_h(t)} = \sqrt{1 - \Omega_{\text{inflaton}} - \Omega_m - \Omega_r}$ for all $t$. Comparing equation (2.5) with equation (2.6), we define the number of e-folds $N_h(t)$ of the ratio of the distance to the event horizon and the Hubble radius:
\[
e^{N_h(t)} \equiv \frac{R_h(t)}{H^{-1}(t)} = \frac{d}{\sqrt{\Omega_h}} \geq 1. \tag{2.7}
\]
Note that the second law restricts $d$ to $d \geq 1$ if there is a moment when the holographic dark energy dominates the universe, $\Omega_h = 1$. In this paper, we investigate a possible consequence of the inequality $\dot{R}_h(t) \geq 0$ through the history of our universe.

The Friedmann equation (2.3) is too complex to allow an exact solution. However, we can develop a good approximation of the evolution by dividing the history of the universe
Figure 2. Schematic diagram for each equipartition time and phase. Phase I is the inflationary era located in the region of times $t < t_f$ and is not visualized in this figure. Phases II and III are divided by the equipartition time $t_{md}$ of the radiation and the holographic dark energy.

into three pieces as in figure 2: the inflation, the consecutive era of the RDE and the first half of the MDE, and the consecutive era of the last half of the MDE and the DDE.

The first phase is the inflation of exponential expansion (phase I) with scale factor

$$a(t) = a_i e^{H_i(t - t_i)}; \quad t_i \leq t < t_f - \epsilon,$$

(2.8)

where $a_i$ is the initial scale factor at $t = t_i$, $H_i = M_i^2 / M_P$ is the Hubble parameter with the energy scale $M_i$ of the inflation and $\epsilon$ is a short period of time denoting the reheating process after inflation. The number of e-folds of expansion is $N \equiv H_i(t_f - t_i - \epsilon)$. During the inflationary phase, we ignore the energy densities of the matter and the radiation compared to the densities of the holographic dark energy and the inflaton energy. We also assume that there is the absence of a time-independent cosmological constant. Therefore, the energy density of the inflaton field is relevant only during the inflationary period. At the end of the inflation $(t_f - \epsilon \leq t < t_f)$, there are complex transient phenomena such as preheating and reheating. Rather than dealing with these in detail, we simply assume that the scale factor does not change much during this period.

Phase II $(t_f < t \leq t_{md})$ is composed of the consecutive era of RDE and the initial half of the MDE, where the subscript ‘md’ implies the time of matter dominance. During phase II, the universe is filled with radiation and matter. We ignore the densities of the inflaton and the holographic dark energy. Solving the Friedmann equation (2.3), the scale factor satisfies

$$\left( a - 2 \frac{\rho_{ct}}{\rho_{mf}} \right) \sqrt{a + 2 \frac{\rho_{ct}}{\rho_{mf}}} = b(t) = \frac{3}{2} \sqrt{\frac{\rho_{mf}}{\rho_{ct}}} H_i(t - t_f) - x_f,$$

(2.9)

where we scaled $a(t) = a(t)/a(t_f)$ so that it becomes 1 at the end of inflation, and $H_i$, $\rho_{ct} \equiv 3M_p^2 H_i^2$, $\rho_{mf}$ and $\rho_{ct}$ are the Hubble parameter, the critical energy density, the matter energy density and the radiation energy density at time $t_f$, respectively. The integration
constant $x_t$ is determined by the condition $a(t_t) = 1$:

$$x_t = \left( \frac{2 \rho_{t_f}}{\rho_{\text{mf}}} - 1 \right) \sqrt{\frac{\rho_{t_f}}{\rho_{\text{mf}}} + 1} \simeq 2 \left( \frac{\rho_{t_f}}{\rho_{\text{mf}}} \right)^{3/2}. \tag{2.10}$$

Equation (2.9) allows an explicit exact solution of the scale factor in time:

$$a(t) = \frac{1}{2^{1/3}} \left[ \left( b + \sqrt{b^2 - 4 \left( \frac{\rho_{t_f}}{\rho_{\text{mf}}} \right)^3} \right)^{1/3} + \left( b - \sqrt{b^2 - 4 \left( \frac{\rho_{t_f}}{\rho_{\text{mf}}} \right)^3} \right)^{1/3} \right].$$

At the initial period of the evolution, the scale factor satisfies $1 \leq a(t) \ll \rho_{t_f}/\rho_{\text{mf}}$. Then the scale factor becomes

$$a(t) = y_0(t) + \frac{\rho_{\text{mf}}}{2\rho_{t_f}} \left[ a_0^2(t) - 1 + \frac{1}{3} (a_0(t) - 1)^3 \right] + O \left( \left( \frac{\rho_{t_f}}{\rho_{\text{mf}}} \right)^2 \right),$$

where its zeroth-order solution $a_0(t)$ is the traditional form at RDE:

$$a_0(t) = \left[ 3 \sqrt{\frac{\rho_{t_f}}{\rho_{ct}}} H(t - t_t) + 1 \right]^{1/2}.$$

The transition from the RDE to the MDE happens when the scale factor satisfies $a(t_{eq}) = \rho_{t_f}/\rho_{\text{mf}}$. After the transition, the scale factor in the limit $a \gg \rho_{t_f}/\rho_{\text{mf}} \gg 1$ takes the form of the matter-dominated era:

$$a(t) = \left[ \frac{3}{2} \sqrt{\frac{\rho_{\text{mf}}}{\rho_{ct}}} H(t - t_t) - x_t \right]^{2/3} + \rho_{t_f}/\rho_{\text{mf}} + O(t^{-2/3}). \tag{2.11}$$

Phase II ends when the universe is filled with matter with $\Omega_h(t_{\text{md}}) = \Omega_{\text{r}}(t_{\text{md}}) \ll 1$. Therefore, the time of matter dominance, $t_{\text{md}}$, is the equipartition time of the radiation and the holographic dark energy.

Phase III ($t \geq t_{\text{md}}$) is the consecutive regime of the matter-dominant era and the power-law accelerating era dominated by the dark energy. The radiation energy density is diluted enough so that it can be ignored relative to the dark energy and the matter in this phase. We deal with phase III by using an exact solution as was done by Li in [15]. Li shows that the function $y = 1/\sqrt{\Omega_h}$ satisfies the differential equation

$$y^2 y' = (1 - y^2) \left( \frac{1}{d} + \frac{y}{2} \right), \tag{2.12}$$

where the prime denotes the derivative with respect to $\log a$. He also presented an exact solution to this differential equation for $d = 1$. For arbitrary $d$, the solution of the differential equation (2.12) was given in [21]:

$$\frac{\Omega_h (1 + \sqrt{\Omega_h})^{d/(2-d)}}{(1 - \sqrt{\Omega_h})^{d/(d+2)}(d + \sqrt{\Omega_h})^{8/(4-d^2)}} = x_{\text{md}} a(t). \tag{2.13}$$

The integration constant $x_{\text{md}}$ is determined from the junction condition at $t_{\text{md}}$. 
At the beginning of phase III, the universe is in the MDE and the portion of the holographic dark energy is negligible, \( \Omega_h \ll 1 \). From equation (2.13), it becomes

\[
\Omega_h(t) = d^{3/(4-d^2)} x_{\text{md}} a(t) + O(a^{3/2}).
\]  

(2.14)

We fix the constant \( x_{\text{md}} \) by comparing the values of \( \Omega_h(t_{\text{md}}) \) in phases II and III. Then, we have

\[
x_{\text{md}} \simeq \frac{\Omega_h(t_{\text{md}})}{d^{3/(4-d^2)} a(t_{\text{md}})},
\]  

(2.15)

where \( \Omega_h(t_{\text{md}}) \) and \( a(t_{\text{md}}) \) will be specified by the values in phase II. With equation (2.14), the solution of the Friedmann equation (2.3) becomes

\[
a(t) = \left[ \frac{3}{2} \sqrt{\frac{\rho_m}{\rho_c}} H f(t - \tau_0) \right]^{2/3} + 2d^{3/(4-d^2)} x_{\text{md}} \left[ \frac{3}{2} \sqrt{\frac{\rho_m}{\rho_c}} H f(t - \tau_0) \right]^{4/3} + \cdots,
\]

where the value of \( \tau_0 \) is determined from the condition that \( a(t) \) is continuous at \( t = t_{\text{md}} \).

On the other hand, if the universe is in the DDE, the portion of the holographic dark energy is close to unity. In this case, we have \( x_{\text{md}} a \gg 1 \) and, from equation (2.13), we get the portion of the holographic dark energy density:

\[
\Omega_h = 1 - \frac{2^{4/(2-d)}}{(d+2)^{8/(d(2-d))}} (x_{\text{md}} a)^{(d+2)/d} + O\left(a^{-2(d+2)/d}\right).
\]  

(2.16)

Now, the Friedmann equation (2.3) becomes, to the present accuracy,

\[
\frac{2^{4/(2-d)} H^2}{(d+2)^{8/(d(2-d))} (x_{\text{md}} a)^{(d+2)/d}} = \frac{\rho_m f}{\rho_c f} H^2 f.
\]

The solution of this equation is

\[
x_{\text{md}} a(t) = \frac{(d-1) S}{d} \left( \frac{H_f(t - t_D)}{H_f} \right)^{d/(d-1)}
\]  

(2.17)

where \( t_D \) is a constant of smaller scale than the typical value of \( t \) in the DDE and will be specified from the next-order calculation and

\[
S = \frac{(d + 2)^{4/(d(2-d))}}{2^{4/(2-d)}} \sqrt{\frac{\rho_m f}{\rho_c f}} x_{\text{md}}^{3/2}.
\]

Since \( x_{\text{md}} \) is given in equation (2.15), equation (2.17) determines the asymptotic evolution of \( R(t) \) almost completely. Although it is well known that the scale factor is proportional to \( t^{d/(d-1)} \), the proportionality constant \( S \) is determined here for the first time.

By dividing the evolution of the universe into three phases we get the evolution of the scale factor in a very accurate form. In particular, the energy densities of the neglected components are extremely small for each phase. Explicitly, for phase III \((t > t_{\text{md}})\), we show in the next section that the maximum value of the portion of the neglected radiation is given by \( \Omega_h(t_{\text{md}}) \sim \rho_{\text{mt}} / \rho_{\text{md}} e^{-N_h(t_{\text{md}})} \) at time \( t = t_{\text{md}} \). Therefore, the relative error \((<10^{-3})\) of the present approximation is very small. This is why we deal the Friedmann equation in this somewhat complex form rather than in taking a simpler approximation.
3. Cosmological history

In the previous section, we have obtained the evolution of the scale factor of the universe with the holographic dark energy by dividing the evolution into three pieces and solving the Friedmann equation. In this section, we calculate the detailed evolution of physical parameters such as the energy densities, the distance to the future event horizon and the Hubble parameter, for each phase.

3.1. The inflationary phase

As usual, we assume that the Hubble parameter in this phase is nearly constant:

\[ H(t) = H_i, \quad t < t_f - \epsilon. \]

At the beginning of inflation, there may be present some portions of matter, radiation, inflaton and holographic dark energy. During the exponential expansion of the scale factor, the densities of the holographic dark energy and the inflaton change slowly. However, the densities of the matter and the radiation decrease exponentially with time, and at some time \( t_i \) they become effectively negligible. We discuss the inflation starting from this time \( t_i \). For \( t \geq t_i \), the portions of the energies satisfy

\[ \Omega_{hi} + \Omega_{\text{inflaton}}(t_i) = 1 = \Omega_h(t_f - \epsilon) + \Omega_{\text{inflaton}}(t_f - \epsilon), \]

where \( \Omega_{hi} \equiv \Omega_h(t_i) \) is the initial portion of the holographic dark energy. The portion of the inflaton energy at the end of inflation is

\[ \Omega_{\text{inflaton}}(t_f - \epsilon) = \frac{\rho_{\text{inflaton}}(t_f - \epsilon)}{3M_p^2 H_i^2} = 1 - \Omega_h(t_f - \epsilon). \quad (3.1) \]

From the scale factor (2.8) and the definition (2.2), the distance to the cosmological event horizon during phase I is

\[ R_h(t) = \frac{1}{H_i} \left( 1 + C e^{H_i(t-t_i)} \right). \quad (3.2) \]

The parameter \( C \) is an integration constant to be determined from the initial portion of the holographic energy at time \( t_i \):

\[ C = H_i R_h(t_i) - 1 = \dot{R}_h(t_i) = \frac{d}{\sqrt{\Omega_{hi}}} - 1 \geq 0, \]

where the second and third equalities come from equations (2.5) and (2.6), respectively. At time \( t_i \), we have no criteria to specify the value of the portion of the holographic dark energy. Since the distance to the horizon grows as time as in equation (3.2), the holographic dark energy will gradually be transferred to the inflaton energy to satisfy the equalities in equation (3.1).

How can we determine the initial distance to the horizon \( R_h(t_i) \)? It may be determined by measuring the initial energy densities of the holographic dark energy and the inflaton field. Since \( \Omega_h(t) \) should not be larger than one, we have the constraint for \( C \geq d - 1 \). This restricts the value of \( d \) to

\[ \sqrt{\Omega_{hi}} \leq d \leq C + 1. \]
If initially the inflaton field is in its vacuum state so that $\Omega_{\text{hi}} = 1$, we may have the identity $C = d - 1$. For $d = 1$ in this case, we may have $C = 0$, which implies a permanent exponential inflation.

In the presence of the inflaton field, the horizon may not be kept at a constant distance but goes away. Therefore, it is natural to assume that the value of $C$ is positive and the distance to the future event horizon grows during the inflation. At $t = t_f - \epsilon$

$$R_h(t_f - \epsilon) = \frac{1}{H_i} \left[ 1 + \left( \frac{d}{\sqrt{\Omega_{\text{hi}}}} - 1 \right) e^N \right] = \frac{e^{N_h(t_f - \epsilon)}}{H_i},$$

(3.3)

where the number of e-folds (2.7) at time $t_f - \epsilon$ is

$$N_h(t_f - \epsilon) = N + \log \left( \frac{d}{\sqrt{\Omega_{\text{hi}}}} - 1 + e^{-N} \right).$$

(3.4)

Note that equation (3.3) restricts the value of $d$ to $d \geq \sqrt{\Omega_{\text{hi}}}$ if we want the distance to the horizon to be positive after sufficient inflation.

We also assume that a sufficient expansion of the horizon happens, $e^{N_h} \gg 1$. In the presence of a fine tuning of $d$ close to $\sqrt{\Omega_{\text{hi}}}$, it would be possible that the distance to the event horizon does not vary much compared to the change of the scale factor. However, we ignore this possibility. Physically, this is correct since the density of the holographic dark energy is negligible just after the inflation. The density of the inflaton at the end of the inflation becomes

$$\rho_{\text{inflaton}}(t_f - \epsilon) = \rho_{\text{ci}}[1 - \Omega_h(t_f - \epsilon)] = \rho_{\text{ci}} - \frac{d^2 \rho_{\text{ci}}}{e^{2N_h(t_f - \epsilon)}},$$

(3.5)

where $\rho_{\text{ci}} = \rho_c(t_i) = 3M_p^2 H_i^2$. In the presence of a sufficient expansion of the scale factor, we ignore the second term on the right-hand side of equation (3.5) and get $\rho_{\text{inflaton}}(t_f - \epsilon) \simeq \rho_{\text{ci}}$.

In reality, one should solve both the inflaton field equation and the Friedmann equation to describe the inflationary period precisely. However, in this paper, we simply assume that there is an exponential inflation. In [27], it was shown that there is an inflationary solution very close to this assumption with a single minimally coupled inflaton field.

At the end of an inflation, many things happen there such as preheating and reheating. We assume that the processes happen during the period $t_f - \epsilon < t < t_f$. We do not deal with these processes in detail and simply present the values of the energy densities after the process. In fact, the quantities of the energy densities after inflation depend on the detailed process of reheating. If the reheating process happens almost instantaneously ($\epsilon \rightarrow 0$; instantaneous reheating approximation), the total energy density does not vary much during the process. Therefore, the physical parameters such as the total energy density, the Hubble parameter and $R_h$ must be continuous at $t_f$.

On the other hand, if the period lasts for somewhat longer (usually the universe during the reheating is assumed to be filled with the matter of inflaton oscillation), the Hubble parameters and the distance to the horizon at times $t_f - \epsilon$ and $t_f$ are different from each other. The changes of these physical parameters should be taken into account...
in this case. Rather than calculating the changes, we write the resulting scale factor to be \( R(t_f) \) and assume that the densities of the holographic dark energy, the matter and the radiation satisfy

\[
\rho_{\text{rf}} \gg \rho_{\text{mf}} \gg \rho_{\text{hf}},
\]

(3.6)

where \( \rho_{\text{hf}} = \rho_{h}(t_f) \) is the density of the holographic dark energy at time \( t_f \). The holographic dark energy at \( t_f - \epsilon \) is exponentially small because of the exponential factor in \( R_h \). Therefore, the number of e-folds \( N_h(t_f) \) in equation (2.7) becomes

\[
N_h(t_f) = \log \left( \frac{H_f R_h(t_f)}{H_i R_h(t_f - \epsilon)} \right).
\]

(3.7)

Since the change of the e-fold of the scale factor during the reheating phase is much smaller than that of the inflationary phase, we may also use the condition for sufficient expansion of \( R_h \) so that \( e^{N_h(t_f)} \gg 1 \). Therefore, the portion of the holographic dark energy at time \( t_f \):

\[
\Omega_h(t_f) = \frac{d^2}{e^{2N_h(t_f)}} \ll 1
\]

(3.8)

is extremely small.

3.2. From RDE to MDE

Now consider phase II (\( t_f \leq t \leq t_{md} \)) which is composed of the whole RDE and the first half of the MDE. At the end of the inflationary phase, most of the inflaton energy has changed to radiation. During this phase, we ignore the holographic dark energy when we calculate the evolution of \( a(t) \). We calculate the evolution of the distance to the horizon simply as if we are in a Robertson–Walker universe with radiation and matter. The portion of the holographic dark energy density gradually increases and will be maximized at the end of this phase, \( t = t_{md} \). However, the maximal value is of the order of \( \Omega_{\text{mf}} e^{-N_h(t_f)} \), justifying the present approximation to the accuracy.

The Hubble parameter in this phase is

\[
H(t) = \frac{\dot{a}}{a} = H_f \sqrt{\frac{\rho_{\text{mf}}}{\rho_{\text{cf}}} \sqrt{\frac{a + \rho_{\text{rf}}/\rho_{\text{mf}}}{a^2}}}, \quad t_f < t \leq t_{md}.
\]

(3.9)

At the beginning of this phase \( t = t_f \), the Hubble parameter becomes

\[
H(t_f) = H_f,
\]

(3.10)

where we use \( \rho_{\text{rf}} + \rho_{\text{mf}} = \rho_{\text{cf}} \). The holographic dark energy density is ignored in this equation.

In the instantaneous reheating approximation, both the Hubble parameter and \( \dot{R_h} \) must be continuous at \( t_f \) and we obtain the total energy density at the beginning of RDE from the initial conditions:

\[
\rho_{\text{rf}} + \rho_{\text{mf}} = \rho_{\text{inflaton}}(t_f - \epsilon) \simeq \rho_{ci} - \frac{d^2 \rho_{ci}}{e^{2N_h(t_f)}}.
\]

(3.11)
For large $N_h(t_f)$, we have $\rho_{t} + \rho_{\text{mf}} \simeq \rho_{\text{cd}}$. On the other hand, if the period lasts for a non-negligible period of time, we use the assumption in equation (3.6), which leads to the inequality

$$e^{2N_h(t_f)} \gg \frac{1}{\Omega_{\text{mf}}} \gg 1,$$

(3.12)

where $\Omega_{\text{mf}} \equiv \rho_{\text{mf}}/\left(\rho_{\text{mf}} + \rho_{\text{cd}}\right) \simeq \rho_{\text{mf}}/\rho_{\text{cd}}$ denotes the portion of the matter energy at time $t_f$ and $\rho_{\text{cd}} = 3M_p^2 H^2(t_f)$. From this point on in this paper, we assume that $\rho_{\text{cd}} \simeq \rho_{\text{cd}}$ for simplicity and the stronger constraint

$$e^{N_h(t_f)} \gg \frac{d}{\Omega_{\text{mf}}},$$

(3.13)

is satisfied with the parameters $N_h(t_f)$, $d$ and $\Omega_{\text{mf}}$. In fact, in the next subsection, it turns out that equation (3.13) guarantees the presence of the MDE between the RDE and DDE. If we have explicit model of inflation and reheating, we may determine $H_f$ and $t_f$ from the initial conditions.

The densities of the radiation and the matter decrease as $1/a^4$ and $1/a^3$, respectively. Therefore, the densities at time $t$ become

$$\rho_{t}(t) = \frac{\rho_{t}}{a^2(t)}, \quad \rho_{m}(t) = \frac{\rho_{\text{mf}}}{a^3(t)}.$$  (3.14)

The transition to the MDE happens at time $t_{\text{eq}}$ when $\rho_{t}(t_{\text{eq}}) = \rho_{\text{mf}}(t_{\text{eq}})$. The scale factor at this time is

$$a(t_{\text{eq}}) = \frac{1}{\Omega_{\text{mf}}}.  \quad (3.15)$$

Using $a(t)$ in equations (2.2) and (3.9), we obtain the distance to the horizon from equation (2.2):

$$R_{h}(t) = a \left( \int_{1}^{\infty} \frac{da}{H a^2} - \int_{1}^{a} \frac{da}{H a^2} \right)$$

$$= \frac{1}{H_f \Omega_{\text{mf}}} \left( g_N - 2\sqrt{\Omega_{\text{mf}} a + 1} \right) a,  \quad (3.16)$$

where we use $\int_{1}^{\infty} \frac{da}{H a^2} = R_{h}(t_f)$ and $g_N$ is given by

$$g_N = \Omega_{\text{mf}} e^{N_h(t_f)} + 2\sqrt{1 + \Omega_{\text{mf}}} \simeq \Omega_{\text{mf}} e^{N_h(t_f)}.  \quad (3.17)$$

In the second equality, we use equation (3.13).

In phase II, we cannot use the formula (2.6) because we have ignored the holographic dark energy to get the solution of the Friedmann equation. Instead, the density of the holographic dark energy is given by scaling $R_{h}(t)$ and it becomes

$$\rho_{h}(t) = \rho_{h}(t_f) \frac{R^2_{h}(t_f)}{R^2_{h}(t)} = \frac{\rho_{\text{cd}} d^2 \Omega_{\text{mf}}}{a^2 \left( g_N - 2\sqrt{\Omega_{\text{mf}} a + 1} \right)^2}.  \quad (3.18)$$

Phase II ends at time $t_{md}$ when $\rho_{h}(t_{md}) = \rho_{t}(t_{md})$. The scale factor at this time is

$$a(t_{md}) = \frac{g_N + 2/d - \sqrt{d^{-1} g_N + 1 + d^{-2}}}{d \Omega_{\text{mf}}} \simeq \frac{e^{N_h(t_f)}}{d}.  \quad (3.19)$$
Interestingly, the scale factor $a(t_{md})$ is dependent on $N_h(t_f)$ rather than $N$. In addition, it is almost independent of the other physical parameters such as $\Omega_{mf}$. Using the approximate formula for $a(t)$ in equations (2.11) and (3.19) we get the time of full matter dominance:

$$t_{md} \simeq \frac{2e^{3N_h(t_f)/2}}{3H(t)d^{3/2}\Omega_{mf}}.$$  \hfill (3.20)

Note that this time is dependent on $N_h(t_f)$ rather than $N$ itself.

The Hubble parameter (3.9) at this time is

$$H(t_{md}) = H_t \sqrt{\frac{\rho_{mf}}{\rho_{cf}} \frac{a(t_{md}) + \Omega_{mf}^{-1}}{a^2(t_{md})}} \simeq H_t \frac{d^{3/2} \sqrt{\Omega_{mf}}}{e^{3N_h(t_f)/2}}.$$ \hfill (3.21)

The energy densities at $t_{md}$ become

$$\rho_r(t_{md}) = \rho_h(t_{md}) \simeq \rho_{cf} d^4 e^{-4N_h(t_f)};$$

$$\rho_m(t_{md}) \simeq \rho_{cf} d^3 \Omega_{mf} e^{-3N_h(t_f)}.$$ \hfill (3.22)

At the time of matter dominance $t_{md}$, we should have $\rho_r(t_{md}) \ll \rho_m(t_{md})$. This provides the condition (3.13). The portions of the holographic dark energy, the radiation and the matter at time $t_{md}$ are

$$\Omega_h(t_{md}) = \Omega_r(t_{md}) = \frac{\rho_r(t_{md})}{3M_p^2 H^2(t_{md})} \simeq \frac{d}{\Omega_{mf} e^{N_h(t_f)}} \ll 1,$$

$$\Omega_m(t_{md}) \simeq 1 - \frac{d}{\Omega_{mf} e^{N_h(t_f)}}.$$ \hfill (3.23)

The time derivative of $R_h$ is

$$\dot{R}_h(t) = \frac{g_N \sqrt{\Omega_{mf} a + 1}}{\Omega_{mf} a} - 3 - \frac{2}{\Omega_{mf} a}.$$ \hfill (3.24)

The second law of thermodynamics says that the value of $\dot{R}_h(t)$ should be non-negative. However, equation (3.24) becomes negative since $a$ indefinitely increases with time. The time derivative $\dot{R}_h(t)$ vanishes at time $t_{max}$ where the scale factor becomes

$$a(t_{max}) = \frac{g_N^2 + 6 + g_N \sqrt{g_N^2 + 12}}{18 \Omega_{mf}} \simeq \frac{1}{9} \Omega_{mf} e^{2N_h(t_f)}.$$ \hfill (3.25)

If the universe is still in the MDE after a time $t_{max}$, the distance to the horizon decreases for $t > t_{max}$. This faulty behavior is due to the failure of the present approximation scheme ignoring the holographic dark energy in phase II. Therefore, the holographic dark energy should be included before the time $t_{max}$ to have a accurate solution of the Friedmann equation. Naturally, phase II should not include this time region and we have the restriction $a(t_{md}) \ll a(t_{max})$, which is respected by the condition (3.13). As seen in the present calculation, the decelerating expansion makes $\dot{R}_h$ decrease and the accelerating expansion increase. This conclusion is in contrast with the apparent facts: in the permanent exponential expansion of de Sitter space, the distance to the cosmological event horizon $\dot{R}_h$ is a constant of time. In decelerating power law expansion, the event horizon is present at infinity.
3.3. From matter-dominant era to dark-energy-dominant era

At the beginning of phase III, the portion of the radiation energy is already negligible \((\Omega_r(t_{md}) \sim d\Omega_{rmd}^{-1} e^{-N_r(t_{md})})\) and keeps decreasing throughout the whole evolution. The matter energy density dominates the first stage evolution. The portion of the holographic dark energy is negligible at the beginning: however, it keeps increasing throughout the whole evolution of phase III. It is the same as that of the portion of the radiation at \(t_{md}\) and becomes the same as that of the matter at \(t_h(> t_{md})\). In this sense, we ignore the radiation in phase III compared to the dark energy and the matter.

The portion of the holographic dark energy satisfies \(\Omega_h(t_{md}) \ll 1\) since we are in the MDE at this time. Since \(\Omega_h(t)\) is continuous at \(t_{md}\), from equations (2.13) and (3.26), the relative scale factor \(a\) can be rewritten in terms of \(\Omega_h\) as

\[
a(t_{md}) \approx \frac{e^{N_h(t_{md})}}{d}, \quad x_{md} = \frac{1}{\Omega_{md} d^{2-2d/(4-d^2)}}. \tag{3.26}
\]

In this phase, it would be better to use \(\Omega_h\) instead of \(a\) as a parameter characterizing a given moment of time. From equations (2.13) and (3.26), the relative scale factor \(a\) can be rewritten in terms of \(\Omega_h\) as

\[
a(\Omega_h) = \frac{\Omega_{md} e^{2N_h(t_{md})}}{d^2 (1 - \sqrt{\Omega_h}) d/(d+2) (1 + 2\sqrt{\Omega_h}/d)^{8/(4-d^2)}} \tag{3.27}
\]

From the Friedmann equation, the Hubble parameter becomes

\[
H = \frac{H_f}{\Omega_{md} e^{3N_h(t_{md})}} \left[ d^2 (1 - \sqrt{\Omega_h}) d/(d+2) (1 + 2\sqrt{\Omega_h}/d)^{8/(4-d^2)} \right]^{3/2} \frac{\Omega_h (1 + \sqrt{\Omega_h}) d^{2/(2-d)}}{(1 - \Omega_h)^{1/3} (1 + \sqrt{\Omega_h}) d^{2/(2-d)}}. \tag{3.28}
\]

The explicit value of \(d\) should be taken to be larger than 1 since there is no inflaton field.

In the final stage of the evolution, with \(a \rightarrow \infty\) and \(d \neq 1\), from equation (2.17), we have

\[
H(t) = \frac{d}{(d-1)(t-t_D)}, \quad R(t) = (d-1)(t-t_D). \tag{3.29}
\]

Therefore, the value of \(d\) governs the final evolution.

Now let us consider the physics at the equipartition time \(t_h\) of the matter and the holographic dark energy. The holographic dark energy is treated exactly in phase III. Therefore, the time derivative \(\dot{R}_h\) is related to the holographic dark energy density through equation (2.6). Since we ignore the radiation energy density, we have \(\Omega_h + \Omega_m = 1\). At the time \(t_h\), the portions of the holographic dark energy and the matter are the same: \(\Omega_h(t_h) = 1/2 = \Omega_m(t_h)\). After this time \((t > t_h)\), the holographic dark energy starts to dominate the universe. The relative scale factor at this time is given by

\[
a(t_h) = \Omega_{md} e^{2N_h(t_{md})} \frac{c(d)}{d^2}, \tag{3.30}
\]

where \(c(d)\) is a non-decreasing function of \(d \geq 0\) only:

\[
c(d) = \left( \frac{1 + \sqrt{2}}{2^{d^2/4}(1 + \sqrt{2}/d)^2} \right)^{4/(4-d^2)}.
\]
Cosmological imprint of the second law of thermodynamics

which varies from 0 to 2 as \( d \) changes from 0 to \( \infty \). The Hubble parameter and the
distance to the horizon at \( t_h \) are given by

\[
H(t_h) = \frac{\sqrt{2}H_t}{\Omega_{mf} \ e^{3/2(d) \ e^{3N_h(t_f)}}}, \quad R_h(t_h) = \frac{\Omega_{mf} \ e^{3/2(d) \ e^{3N_h(t_f)}}}{H_t}.
\]  

(3.31)

The ratio of the distance to the event horizon to the Hubble radius is \( H(t_h)R_h(t_h) = \sqrt{2}d \).

On the other hand, one may calculate when the acceleration of the scale factor \( a(t) \)
becomes positive by calculating the deceleration parameter. Instead of direct calculation,
to get \( \ddot{a} = a(H + H^2) \) we use

\[
-2 \left( \frac{\dot{H}}{H^2} + 1 - \frac{\sqrt{\Omega_h}}{d} \right) = \frac{\dot{\Omega}_h}{H\Omega_h} = \frac{1}{g(d, \Omega_h)};
\]

\[
g(d, \Omega_h) = 1 + \frac{d}{2(2-d)} \frac{\sqrt{\Omega_h}}{1 + \sqrt{\Omega_h}} + \frac{d}{2(d+2)} \frac{\sqrt{\Omega_h}}{1 - \sqrt{\Omega_h}} - \frac{8}{4 - d^2} \frac{\sqrt{\Omega_h}}{2\sqrt{\Omega_h}}.
\]  

(3.32)

where the first equality of the first line of equation (3.32) comes from the definition of
holographic dark energy density (2.4) and the second equality from equation (2.13). The
deceleration parameter now becomes

\[
q(t) \equiv -\frac{\ddot{a}}{aH^2} = -\frac{\sqrt{\Omega_h}}{d} + \frac{1}{2g(d, \Omega_h)}.
\]  

(3.33)

Even though the equation \( \ddot{a}(t) = 0 \) allows a closed form of solution, we write an
approximate solution \( \Omega_t \equiv \Omega_h \simeq 0.432 + 0.145(d - 1) \) around \( d \sim 1 \). This result implies
that the value of \( a(t_h) \) will be of the same order as \( a(\Omega_t) \).

The radiation energy density in phase III is

\[
\rho_r(t) = \frac{\rho_{ct}}{\Omega_{mf}^2 e^{6N_h(t_f)}} \left[ d^2 \left( 1 - \sqrt{\Omega_h} \right)^{d/(d+2)} (1 + 2\sqrt{\Omega_h}/d)^{8/(4-d^2)} \right].
\]  

(3.34)

The matter energy density is

\[
\rho_m(t) = \frac{\rho_{ct}}{\Omega_{mf}^2 e^{6N_h(t_f)}} \left[ d^2 \left( 1 - \sqrt{\Omega_h} \right)^{d/(d+2)} (1 + 2\sqrt{\Omega_h}/d)^{8/(4-d^2)} \right].
\]  

(3.35)

The density of the holographic dark energy is

\[
\rho_h(t) = \frac{\rho_{ct}}{\Omega_{mf}^2 e^{6N_h(t_f)}} \left[ d^2 \left( 1 - \sqrt{\Omega_h} \right)^{d/(d+2)} (1 + 2\sqrt{\Omega_h}/d)^{8/(4-d^2)} \right].
\]  

(3.36)

The ratio of the holographic dark energy and the matter energy density \( \rho_h(t)/\rho_m(t) = \Omega_h(t)/(1 - \Omega_h(t)) \) is independent of \( d \) and well met with the criteria \( \Omega_h + \Omega_m = 1 \).

We may determine the parameters \( \Omega_{mf} \), \( N_h(t_f) \) and \( \rho_{ct} \) from the present data of
the universe. Let us set the present values of the holographic dark energy, the ratio of the
densities of the radiation and matter, the Hubble parameter, the deceleration parameter
and the relative scale factor to be \( \Omega_h(t_0) \equiv \Omega_{h0} \), \( \Omega_r(t_0)/\Omega_m(t_0) = r_0 \), \( H(t_0) = H_0 \),
\( q(t_0) = q_0 \) and \( a(t_0) = a_0 \). Note that the deceleration parameter (3.37) is dependent
on \( d \) and \( \Omega_h \) and independent of the other energy densities. Therefore, once we measure
the deceleration parameter and the portion of the holographic dark energy at the present time, we may get the explicit value of $d$ from

$$ q(t_0) = -\sqrt{\Omega_{h0}} \frac{1}{d} + \frac{1}{2g(d, \Omega_{h0})}. \quad (3.37) $$

If we use the present data $q(t_0) \sim -1$ and $\Omega_{h0} \sim 0.72$, we get $d \sim 0.6$. On the other hand, if we want to have $d \sim 1$, we need $q(t_0) \sim -0.5$. This value is within 1σ error of the present experimental data $(-0.96 \pm 0.43)$ \cite{28}.

From the ratio $r_0 = \rho_r(t_0)/\rho_m(t_0)$, we have

$$ \Omega_{\text{mfe}} e^{N_h(t_f)} = \frac{1}{\sqrt{r_0}} \left[ \frac{d^2 \left( \frac{1 - \sqrt{\Omega_{h0}}}{{\Omega_{h0}}^{d/(d+2)}(1 + 2\sqrt{\Omega_{h0}}/d)^{8/(4-d^2)}} \right)^{1/2}} {{\Omega_{h0}}^{1 + \sqrt{\Omega_{h0}} d/(2-d)}} \right]. \quad (3.38) $$

Using equations (3.27) and (3.38), we may identify the e-fold of the ratio between the distance to the horizon and the Hubble radius:

$$ e^{N_h(t_i)} = \sqrt{r_0} a_0 \left[ \frac{d^2 \left( \frac{1 - \sqrt{\Omega_{h0}}}{{\Omega_{h0}}^{d/(d+2)}(1 + 2\sqrt{\Omega_{h0}}/d)^{8/(4-d^2)}} \right)^{3/2}} {{\Omega_{h0}}^{1 + \sqrt{\Omega_{h0}} d/(2-d)}} \right]. \quad (3.39) $$

From equations (3.28) and (3.38), we get

$$ H_f = H_0 \sqrt{r_0} a_0 \left[ \frac{d^2 \left( \frac{1 - \sqrt{\Omega_{h0}}}{{\Omega_{h0}}^{d/(d+2)}(1 + 2\sqrt{\Omega_{h0}}/d)^{8/(4-d^2)}} \right)^{2}} {{\Omega_{h0}}^{1 + \sqrt{\Omega_{h0}} d/(2-d)}} \right]. \quad (3.40) $$

In this way, we may identify all initial parameters at time $t_f$ from the data today.

Using the present data, $\Omega_{h0} \simeq 0.72$ and $r_0 \simeq 10^{-4}$, we may get more explicit value. For example, equation (3.38) becomes

$$ \Omega_{\text{mfe}} e^{N_h(t_i)} \simeq 138.9 \cdot d(0.389)^{d/(d+2)} \left( \frac{1 + 1.697/d}{(1.8485)^{d/(d+2)/8}} \right)^{4/(4-d^2)}. $$

The function $\Omega_{\text{mfe}} e^{N_h(t_i)}$ is approximated by $280.0 + 61.2(d-1)$ around $d = 1$. This shows that the stronger constraint (3.13) is valid for our universe.

4. Summary and discussions

The precise history of the universe is presented in the presence of the inflaton, the matter, the radiation and the holographic dark energy by dividing the whole evolution into three pieces, the inflation, the consecutive period of the radiation-dominated era and the first half of the matter-dominated era, and the consecutive period of the last half of the matter-dominated era and the dark-energy-dominated era. Identifying the area of the event horizon with the logarithm of the content of missing information, we discuss that the second law of thermodynamics restricts the value of the constant $d$ to be larger than the square root of the portion of the holographic dark energy $\sqrt{\Omega_h}$. The scale factors at the three equipartition times $t_{eq}$, $t_{md}$ and $t_h$ of the matter–radiation, the holographic dark energy–radiation and the matter–holographic dark energy, respectively, are given by

$$ a(t_{eq}) = \frac{1}{\Omega_{\text{mfe}}}, \quad a(t_{md}) \simeq e^{N_h(t_f)} \frac{c(t)}{d}, \quad a(t_h) = \Omega_{\text{mfe}} e^{2N_h(t_f)} \frac{c(d)}{d^2}. $$
Since $c(d)$ is an $O(1)$ number, the ratio of two scale factors of neighboring equipartition times are roughly $\Omega_{\text{me}} N_{t_h} / d$. In addition, these three scales provide a very interesting dimensionless constant:

$$\frac{a(t_{eq}) a(t_{h})}{a^2(t_{md})} = c(d) = \left( \frac{2^{d^2/4} (1 + \sqrt{2/d})^2}{2^{d^2/4} (1 + \sqrt{2/4})^2} \right)^{4/(1-d^2)}.$$ (4.1)

Note that $c(d)$ is a non-decreasing function of $d$ of the order of unity and is independent of all other physical parameters. Since the three scales will be measurable from experiments, the value of $d$ can be determined from equation (4.1). Interestingly, the scale factor at the time of full matter dominance is roughly given by the geometric average of the scale factors at the two other equipartition times since $c(d) \sim 1$. Independently, we have presented another way which determines the value of $d$ by measuring the deceleration parameter $q$ in equation (3.37).

In the presence of a holographic dark energy, the universe must go into the dark-energy-dominant era eventually. It is natural to ask why the transition to the dark-energy-dominant era should happen. From the point of view of the energy, it is because that the rate of changes of the energy densities are different for each component of the energies, the matter, the radiation and the holographic dark energy as in equation (2.1). If the holographic dark energy decreases slower than other densities, it will determine the final fate of the universe. We have restricted $d \geq 1$ which gives the condition for the velocity of the distance to the horizon, $\dot{R}_h \geq 0$. This condition, in fact, determines when DDE begins. A convincing evidence for this is given by comparing equation (3.25) and equation (3.30). The maximum value of the scale factor $a(t_{max})$ determined from the condition $\dot{R}_h \geq 0$ in phase II is almost the same as the scale factor $a(t_h)$ at which the transition to DDE really happens.

In [7], the generalized second law of thermodynamics was studied with the quasi-de Sitter space filled with a viscous fluid in Einstein gravity with a cosmological constant. Interestingly, they showed that there is a process in which the decrease of the horizon area is supplemented by the increase of the matter entropy to satisfy the generalized second law. It is an interesting question to ask whether this process is possible or not in the presence of a holographic dark energy. In the absence of such a process, the cosmological arrow of time becomes the same as the thermodynamical one because of the entropy interpretation of the horizon area.

Acknowledgments

This work was supported by the SRC Program of the KOSEF through the CQUEST grant R11-2005-021 (H-CK) and the Korea Research Foundation Grant funded by Korea Government (MOEHRD, Basic Research Promotion Fund) (KRF-2006-312-C00095;JJL).

References

[1] Bekenstein J D, 1973 Phys. Rev. D 7 2333 [SPIRES]
[2] Bekenstein J D, 1974 Phys. Rev. D 9 3292 [SPIRES]
[3] Hawking S W, 1975 Commun. Math. Phys. 43 199 [SPIRES]
[4] Kim H-C, Lee J-W and Lee J, Black hole as an information Eraser, 2007 Preprint 0709.3573
[5] Song D D and Winstanley E, Information erasure and the generalized second law of black hole thermodynamics, 2000 Preprint gr-qc/0009083
Cosmological imprint of the second law of thermodynamics

[5] Landauer R, 1961 IBM J. Res. Dev. 5 183
[6] Gibbons G W and Hawking S W, 1977 Phys. Rev. D 15 2738 [SPIRES]
[7] Davies P C W, 1987 Class. Quantum Grav. 4 L225 [SPIRES]
Davies P C W, 1988 Class. Quantum Grav. 5 1349 [SPIRES]
Davies P C W, 1988 Ann. Inst. Henri Poincaré 49 297
[8] Pollock M D and Singh T P, 1989 Class. Quantum Grav. 6 901 [SPIRES]
[9] Brustein R, 2000 Phys. Rev. Lett. 84 2072 [SPIRES]
Davis T M, Davies P C W and Lineweaver C H, 2003 Class. Quantum Grav. 20 2753 [SPIRES]
Izquierdo G and Pavón D, 2006 Phys. Lett. B 633 420 [SPIRES]
[10] Misner C M and Sharp D H, 1964 Phys. Rev. 136 B571 [SPIRES]
[11] Huang Q-G and Li M, 2004 J. Cosmol. Astropart. Phys. JCAP08(2004)013 [SPIRES]
[12] He S and Zhang H, 2007 J. High Energy Phys. JHEP12(2007)052 [SPIRES] [0712.1313]
He S and Zhang H, 2007 J. High Energy Phys. JHEP10(2007)077 [SPIRES] [0708.3670]
[13] Lee J-W, Lee J and Kim H-C, 2007 J. Cosmol. Astropart. Phys. JCAP08(2007)005 [SPIRES]
[hep-th/0701199]
Lee J-W, Lee J and Kim H-C, Quantum informational dark energy: dark energy from forgetting, 2007 Preprint 0709.0047
Lee J-W, Lee J and Kim H-C, Is dark energy from cosmic Hawking radiation?, 2008 Preprint 0803.1987
[14] Myung Y S, Instability of holographic dark energy models, 2007 Preprint 0706.3757
[15] Myung Y S and Seo M-G, Origin of holographic dark energy models, 2008 Preprint 0803.2913
[16] Wang B, Gong Y G and Abdalla E, 2005 Phys. Lett. B 624 141 [SPIRES]
Wang B, Lin C-Y and Abdalla E, 2006 Phys. Lett. B 637 357 [SPIRES]
[17] Zhang X and Wu F-Q, 2005 Phys. Rev. D 72 043524 [SPIRES] [astro-ph/0506310]
Zhang X and Wu F-Q, 2007 Phys. Rev. D 76 023502 [SPIRES] [astro-ph/0701405]
[18] Setare M R, 2007 J. Cosmol. Astropart. Phys. JCAP01(2007)023 [SPIRES] [hep-th/0701242]
Setare M R, 2006 Phys. Lett. B 641 130 [SPIRES] [hep-th/0611165]
Setare M R and Vagenas E C, 2008 Preprint 0801.4478
[19] Chiba T, Takahashi R and Sugiyama N, 2005 Class. Quantum Grav. 22 3745 [SPIRES] [astro-ph/0501661]
[20] Spergel D N et al, 2003 Astron. J. 148 175 [SPiRES]
[21] Bassett B A, Tsujikawa S and Wands D, 2006 Rev. Mod. Phys. 78 537 [SPIRES]
[22] Chen B, Li M and Wang Y, 2007 Nucl. Phys. B 774 256 [SPIRES]
[23] Gong Y and Wang A, 2007 Phys. Rev. D 75 04352 [SPIRES]