Gravitational Lensing by Traversable Wormholes Supported by Three-Form Fields

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In this paper, the deflection angle of light by traversable wormholes, which are supported by the three-form fields, are studied. The specific forms of the redshift and shape functions that produce results compatible with the energy conditions at throat of the wormholes are used. Having used the well-defined parameter sets of the three-form wormholes, the photon geodesic motion is investigated under the effective potential of the wormhole background. As a result, it is discovered that the radius of the photon sphere can be used to analyze the geometrical structures of a physical wormhole. The analytical form of the effective potential is used to figure out a deflection angle of light caused by wormholes with the three-form fields. In this work, the relativistic images generated from the wormholes are also constructed.

1. Introduction

Over the past few decades, wormholes have become one of the most intensively studied topics in the literature. The wormhole solutions represent a conduit between the points of two parallel universes or even two different points of the same universe. The existence of traversable wormholes is known as Einstein–Rosen bridges proposed by Einstein and Rosen in 1935. However, Wheeler and his colleague later showed that wormholes would not be stable and cannot be traversable. In 1988, Morris, Thorne, and Yurtsever have demonstrated that the traversable wormholes were explicitly constructed. Later, other types of traversable wormholes were studied as plausible solutions to the equations of general relativity, including analyses initiated by Matt Visser and some relevant references on wormhole constructions have been proposed so far, such as, the Casimir wormholes, charged wormholes, B-field non-minimal coupling wormholes, Brans–Dicke wormholes, f(R) gravity wormholes, dilaton Einstein–Gauss–Bonnet wormholes. Searching for a proper form of the exotic matter inside the wormholes is a central topic in the wormhole research. In this work, we will use the three-form fields as the matter source that can support the throat of the traversable wormholes. The three-form fields naturally exist in the string theory. Moreover, it has been used to explain the several problems in cosmology such as inflation in very early universe and structure formation and even the dark energy problem. The traversable wormholes with the three-form fields were constructed by authors of ref. It was shown that the three-form fields are an interesting matter source possibly used to support the wormholes throat. Consequently, the three-form fields are also used to study relativistic stars and black holes. Recently, the three-form wormholes are studied in detail by using the Higgs type three-form potential and the authors in this work have solved the Einstein equation to construct the traversable three-form wormhole. Moreover, they have also studied the location of the photon sphere and the innermost stable circular orbit (ISCO) of the three-form wormholes. This allows to show that the wormholes can mimic the Schwarzschild black hole in certain condition of the parameter.

To prove the existence of the wormholes, there are several methods that can be used to figure the wormholes. Gravitational lensing is one of the interesting phenomena which can be used to probe the astrophysical objects such as black holes. Gravitational lensing in wormholes has been explored in many aspects and compared with the study of black holes. Using the Gauss–Bonnet theorem, gravitational lensing by many types of black holes and wormholes have also been extensively studied, for example, ref. The gravitational lensing tells us about the geometry of space–time around massive object and wormhole is one of them, see also, for example, ref. Therefore, this is a promising idea to search for the existence of the wormholes. In addition, gravitational lensing occurs when the light travel passing through the space–time around the massive objects. Consequently, this leads to the bending of light between source and observers according to the curve space–time around the massive object. One of the striking predictions of the gravitational lensing is the Einstein’s rings commonly observed phenomena of the light bending in astrophysics.
In this work, we use the gravitational lensing as the main tool to search for the existence of the wormholes. Gravitational lensing is one of the classical predictions by GR. This phenomenon tells us about distribution of matter and manifests the curved space–time. This makes the curve geodesic line and leads to the bending of light between the light source and observers due to the massive object in the space–time. [64,65] The trajectory of the light passing through the wormholes will be curved by the space–time geometry of the wormholes. We will calculate and discuss a strong gravitational lensing and an Einstein’s rings which are very useful in the search of the traversable wormholes.

The present work is organized as follows: In Section 2, all relevant equations of the three-form wormholes are briefly constructed and some implications are revisited. Next, in Section 3, we set up the analytical expression of the effective potential and deflection angle of the three-form wormholes. The numerical results are presented and its physical interpretations are given in Section 5. Finally, we conclude our findings in the last Section 6.

2. Background Equations with Three-Form Fields

2.1. Gravitational Three-Form Fields Action and Spherical Symmetric Solution Ansatz

We start with the gravitational action of the Einstein gravity including the three-form fields. In this case, the action is of the form

\[ S = \int \frac{1}{2} \mathcal{K} \mathcal{R} - \frac{1}{48} F_{\mu \nu \rho} F^{\mu \nu \rho} - V(A^3) \]  

(1)

where \( \mathcal{K} \equiv 8 \pi G \), \( A^3 = A_{\mu \nu \rho} \), and \( \mathcal{R} \) is a Ricci scalar. Additionally, \( F_{\mu \nu \rho} \) is a field strength tensor of the three-form fields, \( A_{\mu \nu \rho} \). It is defined as

\[ F_{\mu \nu \rho} = \nabla_{\mu} A_{\nu \rho} - \nabla_{\nu} A_{\mu \rho} + \nabla_{\rho} A_{\mu \nu} - \nabla_{\nu} A_{\rho \mu} \]

(2)

Varying the action in Equation (1), the Einstein field equation (EFE) is given by

\[ G_{\mu \nu} = k^2 \mathcal{T}_{\mu \nu} \]

(3)

where \( G_{\mu \nu} \equiv R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \mathcal{R} \). The energy-momentum tensor of the three-form fields reads

\[ T_{\mu \nu} = \frac{1}{6} F_{\rho \sigma \tau} F^{\rho \sigma \tau} + 6 \frac{\partial V(A^3)}{\partial (A^3)} A_{\rho \sigma} A_{\tau \mu} - \frac{1}{48} F_{\mu \nu \rho} F^{\mu \nu \rho} - V(A^3) \]

(4)

The equation of motion of the three-form is obtained by varying with respect to the \( A_{\mu \nu \rho} \) and we find

\[ \nabla_{\mu} F_{\mu \nu \rho} - 12 \frac{\partial V(A^3)}{\partial (A^3)} A_{\mu \nu \rho} = 0 \]

(5)

The three-form gauge field, \( A_{\mu \nu \rho} \) is represented by the following relation

\[ A_{\mu \nu \rho} = \sqrt{-g} g_{\mu \nu \rho} B^\sigma \]

(6)

where the \( B^\sigma \) is the one-form dual vector of the three-form fields, \( A_{\mu \nu \rho} \). The solution of the \( B^\sigma \) can be expressed by using ansatz as a function of \( \zeta(r) \) via

\[ B^\sigma = \left( 0, \sqrt{1 - \frac{b(r)}{r}} \zeta(r), 0, 0 \right) \]

(7)

where \( \zeta(r) \) is a generic parametrization of the three-form fields and we will solve the EFE in Equation (3) in order to obtain the solution of \( \zeta(r) \).

Before imposing the ansatz of the three-form field solution, it is worth introducing the spherical symmetry line element of the traversable wormholes used in the present work. We consider a general spherically symmetric space–time geometry and static traversable wormholes proposed by Morris–Thorne. [4] It reads

\[ ds^2 = -e^{2b(r)/r} dr^2 + \left( 1 - \frac{b(r)}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

(8)

where \( r, \theta, \phi \) are the spherical coordinates, \( \Phi(r) \) and \( b(r) \) are two arbitrary functions of radius, \( \Phi(r) \) is called the redshift function since it determines the gravitational redshift, and \( b(r) \) determines the spatial shape of the wormhole we called the shape function. The traversable wormholes with three-form fields solution \( \Phi(r) \) and \( b(r) \) and its relevant quantities will be solved in the next section. Specifically, we will first use the wormhole line element to specify the ansatz of the three-form field at this moment.

According to Equations (6) and (7), the nonzero components of three-form fields are given by

\[ A_{\mu \rho} = A_{\sigma \rho} = A_{\sigma \phi} = -A_{\phi \rho} = -A_{\phi \sigma} = A_{\phi \phi} = -A_{\phi \phi} = \sqrt{g} \left( \frac{b(r)}{r} - 1 \right) \zeta(r) \]

(9)

and \( A^2 \) is given by

\[ A^2 \equiv A_{\rho \sigma} A_{\rho \sigma} = -6\zeta(r)^2 \]

(10)

Having uses Equations (2) and (6), we can re-write the kinetic term of the three-form fields as

\[ -\frac{1}{48} F^2 = -\frac{1}{48} F_{\sigma \rho \phi} F^{\sigma \rho \phi} = \frac{1}{2} (\nabla_{\mu} B^\mu)^2 = -6Y(r) \]

(11)

where the function \( Y(r) \) with the traversable wormhole line element in Equation (8) reads

\[ Y = 4 \left( 1 - \frac{b}{r} \right) \left[ \zeta (\Phi' + 2 \frac{b}{r} + \zeta') \right]^2 \]

(12)

We have completed a general form of the three-form field solutions in the spherical symmetric background. This will be used to construct the traversable wormholes in the next section.
2.2. Traversable Wormhole with Three-Form Fields and its Energy Conditions

In the present section, we continue to derive the EFE with the three-form field in order to construct the traversable wormholes with the three-form field solution. More importantly, we also include an ordinary matter term, into the gravitational action in Equation (1) and the ordinary matter term is considered as an anisotropic fluid in this work. Taking a standard variation of the action with respect to the metric tensor, $g_{\mu \nu}$, and setting $\kappa^2 \equiv 8\pi G = 1$, one gets

\[ C_{\mu \nu} = T_{\mu \nu}^{\text{eff}} \tag{13} \]

where the effective energy–momentum tensor defined by

\[ T_{\mu \nu}^{\text{eff}} = T_{\mu \nu}^{(A)} + T_{\mu \nu}^{(m)} \tag{14} \]

where an $(A)$ superscript refers to the three-form $A_{\mu \rho \sigma}$ and an $(m)$ denotes an ordinary matter of the energy–momentum tensor, respectively, while the energy–momentum tensor of the anisotropic fluid is given by $T_{\mu \nu}^{(m)} = (-\rho_m, -\tau_m, p_m, p_m)$.

Using the metric tensor in Equation (8), the energy–momentum tensor of the three-form, $T_{\mu \nu}^{(A)}$ in Equation (4) can be calculated for diagonal components as

\[ T_{\mu \nu}^{(A)\rho} = -\rho_A = -\frac{1}{8} Y - V + \zeta \frac{dV}{d\zeta} \tag{15} \]

\[ T_{\mu \nu}^{(A)\tau} = -\tau_A = -\frac{1}{8} Y - V \tag{16} \]

\[ T_{\mu \nu}^{(A)p} = T_{\mu \nu}^{(A)m} = p_A = -\frac{1}{8} Y - V + \zeta \frac{dV}{d\zeta} \tag{17} \]

From a definition of the effective energy–momentum tensor in Equation (14), we can calculate the EFE in Equation (13) for each diagonal component. One finds

\[ b' \frac{1}{r} = \rho_{\text{eff}} = \rho_m + \rho_A \tag{18} \]

\[ 2 \left( 1 - \frac{b}{r} \right) \left\{ \frac{\Phi'}{r} - \frac{b}{r^2} \right\} = \tau_{\text{eff}} = \tau_m + \tau_A \tag{19} \]

\[ \left( 1 - \frac{b}{r} \right) \left( \Phi'' - \frac{b'r - b}{2(r - b)} \Phi' + \Phi' \Phi'' + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r - b)} \right) = p_{\text{eff}} = p_m + p_A \tag{20} \]

According to the conservation of the effective energy–momentum tensor, $\nabla_{\mu} T_{\mu \nu}^{\text{eff}} = 0$, we obtain the equation of motion of the effective energy–momentum tensor as

\[ \tau_{\text{eff}}' + \frac{2}{r} (\tau_{\text{eff}} + p_{\text{eff}}) + \Phi' (\tau_{\text{eff}} - \rho_{\text{eff}}) = 0 \tag{21} \]

Substituting all components of the effective energy–momentum tensor from Equations (18)–(20) into the equation of motion in Equation (21), we find

\[ \zeta \left[ r \Phi' \left( b' - \frac{b}{r} \right) + 4 + 2b' + 2r^2 \Phi'' \left( \frac{b}{r} - 1 \right) - \frac{b}{r} \right] + 2r^2 \frac{dV}{d\zeta} + \zeta r \left[ 3 \left( \frac{b}{r} - 1 \right) \right] + 2\Phi'' \left( \frac{b}{r} - 1 \right) = 0 \tag{22} \]

In practice, it is very difficult to solve the exact solutions of the wormholes because of the unknown functions such as $\Phi$, $b$, $\rho_m$, $\tau_m$, $p_m$, $\zeta$, and $V$. Therefore, we need to assume a specific form of the redshift function, $\Phi(r)$ and shape function, $b(r)$ which are compatible with all requirements of the traversable wormhole solutions. More importantly, it has been demonstrated in the ref. [37] that the following specific forms of the functions $\Phi(r)$, $b(r)$, and $\zeta(r)$ are consistent to the EFE in Equation (13) with the line element in Equation (8) as well. The specific forms of the solutions in this work are taken from refs. [66–68]. The functions $\Phi(r)$ and $b(r)$ can be parameterized as

\[ \Phi(r) = \Phi_0 \left( \frac{r_0}{r} \right)^\alpha \quad \text{and} \quad b(r) = r_0 \left( \frac{r_0}{r} \right)^\beta \tag{23} \]

where $\alpha$, $\beta$, and $\Phi_0$ are dimensionless and the ranges of the model parameters are assumed to be $\alpha > 0$ and $\beta > -1$. $\Phi(r)$ and $\zeta(r)$ are also dimensionless function, whereas the $b(r)$ and $r_0$ are quantities with dimension of length.

In this work, we use a specific form of the three-form field potential as$^{[71]}

\[ V(r) = \zeta^2 + C \tag{24} \]

where $C$ is an integration constant. The author of ref. [37] has shown that the relevant energy conditions for the three-form wormhole are satisfied by using the form of $\Phi(r)$, $b(r)$, and $V(r)$ in Equations (23) and (24), respectively, with the following sets of the parameters

- Case I

\[ \Phi_0 = -1, \quad \alpha = 1, \quad \beta = 1, \quad C = -0.1 \tag{25} \]

- Case II

\[ \Phi_0 = -2, \quad \alpha = 1, \quad \beta = -1/2, \quad C = 0 \tag{26} \]

Noting that the function $\zeta(r)$ can be solved numerically by substituting $\Phi(r)$ and $b(r)$ in Equation (23) as well as the potential form in Equation (24) to the Equation (22) with parameters in Equations (25) and (26).

Next, let us briefly discuss and show how the two sets of the parameters in Equations (25) and (26) of the three-form fields satisfy the null energy condition (NEC) and weak energy condition (WEC) which are main conditions to support the wormhole’s throat. In general, the NEC reads

\[ T_{\mu \nu} k^\mu k^\nu \geq 0 \tag{27} \]

where $k^\mu$ is null vector with $k_\mu k^\mu = 0$. Together with the flaring out condition of the wormhole, that is, $db(r)/dr < b(r)/r$, this
leads to the violation of the effective energy-momentum tensor in Equation (14) as \( T^{(m)}_{\mu \nu} k^\mu k^\nu < 0 \). However, ref. [37] has imposed that the matter energy–momentum tensor satisfies the NEC, that is, \( T^{(m)}_{\mu \nu} k^\mu k^\nu \geq 0 \) whereas the three-form energy-momentum violates the NEC. According to a generalized NEC proposed by ref. [16], one finds the following constraint

\[
-T^{(m)}_{\mu \nu} k^\mu k^\nu > T^{(m)}_{\mu \nu} k^\mu k^\nu \geq 0
\]  
(28)

This constraint is true in the vicinity of the wormhole throat. Recalling the matter energy–momentum tensor as \( T^{(m)}_{\mu \nu} = \text{diag}(\rho_m, -r_m, p_m, p_m) \) where \( r_m = -p_{r,m} \) (\( p_{r,m} \) is a radial pressure), the NEC is given by

\[
\rho_m - r_m \geq 0
\]  
(29)

As shown in the NEC case previously, the WEC of the matter energy–momentum tensor is read

\[
T^{(m)}_{\mu \nu} \nu^\nu \geq 0
\]  
(30)

where \( \nu^\nu \) is the time-like vector. This implies that

\[
\rho_m - r_m \geq 0, \quad \text{and} \quad \rho_m \geq 0
\]  
(31)

Substituting the parameter sets in Equations (25) and (26) in Equations (29) and (31), it has shown by ref. [37] that these two parameter sets can reproduce the results which are compatible with the NEC and WEC and we will not repeat the numerical results here. Therefore, as a result, these two sets of the parameters are suitable to be used to construct wormhole solutions. Then, we employ above two cases of the parameters to study gravitational lensing of the traversable wormholes in the next section.

Furthermore, interestingly, one might find an analytical solution of the shape function, \( b(r) \) as a special case by using specific forms of the red-shift function in Equation (23) and new ansatz for the scalar function of the three-form field as filed as

\[
\zeta(r) = \zeta_0 \left( \frac{r_0}{r} \right) \gamma
\]  
(32)

Substituting specific forms of the \( \Phi(r) \) and \( \zeta(r) \) mentioned above into the equation of motion of the three-form field in Equation (22) with extended form of the three-form potential as shown in Equation (24) by replacing \( \zeta^2 \rightarrow \zeta^2 + \zeta^4 \), that is

\[
V(r) = \zeta^2 + \zeta^4 + C
\]  
(33)

The analytical solution of the \( b(r) \) function is given by

\[
b(r) = r_0 \frac{p_{11}}{(1 + 10 \rho)^2} \left( \frac{83}{30} p_{12} + p_{10} + 20 p_{39} + 100 p_{18} - \frac{4}{15} p_{12} - \frac{5}{2} p_{36} \right)
\]  
(34)

where \( r \equiv r_0 / r \) and we have set the free parameters of the theory as \( \alpha = 1, \gamma = -8, \gamma = -8, \) and \( C = -10 \). We note that the particular solution of \( b(r) \) above is compatible with all requirements of the well-defined wormhole shape function.

The particular solution of \( b(r) \) is compatible with the NEC and WEC in Equations (29) and (31), respectively as shown in

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**Figure 1.** This means that the new analytical solution of the \( b(r) \) with \( \alpha = 1, \Phi_0 = -1, \gamma = -8, \) and \( C = 10 \) is also used to describe the traversable wormhole in three-form field theory. We will employ the particular solution of \( b(r) \) in Equation (34) to study the deflection angle and the relativistic image as well in the next section.

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### 3. Effective Potential and Deflection Angle of the Three-Form Wormhole

In order to study the trajectory of the photon in the three-form wormholes background, the effective potential from the geodesic equation plays the major role leading to analysis of the deflection angle of the photon trajectory. Detailed derivations of the relevant quantities for deflection angle are given in the standard framework and they can be found in general textbooks in General Relativity.[64,65] Then we will not repeat them here. We aim to derive in this section the analytical form of the effective potential and deflection angle of light of wormholes with the three-form fields. We study the numerical calculations in the next chapter.

In the previous section, we have reviewed the wormholes solutions supported by three-form fields. In this section, we use the redshift function \( \Phi(r) = \Phi_0 \left( \frac{r_0}{r} \right)^\beta \) and the shape function \( b(r) = r_0 \left( \frac{r_0}{r} \right)^\gamma \) that can be used to construct wormhole solutions as mentioned previously. We divide our analysis of the effective potential from the geodesic equation and the deflect angle in this section into two cases with two sets of parameters in Equations (25) and (26) as shown in the previous section. In addition, we consider different values of \( \alpha, \beta, \) and \( \Phi_0 \) relevant to the calculation of the effective potential as well as the deflection angle.

Before considering the effective potential and the deflection angle for each case, we shall briefly summarize the derivation of the relevant observables according to the photon geodesic under the traversable wormhole with the three-form field background.
By using the line element in Equation (8), the Lagrangian is used to describe the motion of photon with \( \theta = \pi / 2 \), one finds

\[
2\mathcal{L} = g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = -e^{2\Phi(r)}(1 + \frac{\dot{r}^2}{1 - \frac{2\Phi(r)}{r}}) + r^2 \dot{\phi}^2
\]  

(35)

where a dot represents a derivative with respect to the affine parameter, \( \lambda \). The conjugate momenta in the cyclic coordinates \( t \) and \( \phi \) lead to two constants of the motion and they are given by

\[
p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -e^{2\Phi(r)} \dot{t} = -E, \quad p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2 \dot{\phi} = L
\]  

(36)

where \( E \) and \( L \) are the energy and angular momentum of the photon, respectively.

The null geodesics equation can be written in the following form

\[
E^2 = \frac{e^{2\Phi(r)}}{1 - \frac{2\Phi(r)}{r}} \dot{t}^2 + V_{\text{eff}}
\]  

(37)

where \( V_{\text{eff}} \) is the effective potential and it is given by

\[
V_{\text{eff}} = L^2 \frac{e^{2\Phi(r)}}{r}
\]  

(38)

More importantly, the crucial parameter for evaluating the gravitational lensing is the deflection angle of the light bending due to the traversable wormholes. For an incoming photon from a distance source has a turning point at \( r_p \) and escapes to a distance observer. This leads to a well-known formula of the deflection angle, \( \alpha \) and it is given by \([63,69]\)

\[
\alpha(u) = 2 \int_{r_p}^{\infty} \frac{e^{\Phi(r)}dr}{r^2 \sqrt{1 - \frac{4\Phi(r)}{r} \sqrt{1 - \frac{2\Phi(r)}{r}}}} - \pi
\]  

(39)

where \( u = l/c \), \( r_p \) is the impact parameter, and \( r_p \) is the turning point. With the \( dr/d\phi = 0 \) condition, the relation between \( u \) and \( r_p \) reads

\[
u = r_p e^{-\Phi(r_p)}
\]  

(40)

In addition, it is worth noting that the deflection angle formula in Equation (39) is logarithmic divergence when the turning point approaches to the photon sphere. This leads to an unstable photon orbits. As a result, an infinite number of the relativistic images is formed by the gravitational lensing. However, it has been shown in refs. \([63,70,71]\) that the throat of the symmetric wormhole at the maximum point of the effective potential can be considered as the effective photon sphere (see more detail discussions in Section 5.1). Actually, there is an effective anti-photon sphere (a minimum point of the effective potential) as well but we do not consider in this case. The existence of the photon sphere at and outside the symmetric wormhole must be satisfied to the following condition

\[
\Phi'(r_0) < \frac{1}{r_0}, \quad \frac{e^{2\Phi(r_0)}-r_0^2}{r_{ph}} < \frac{e^{2\Phi(r_0)}}{r_0}
\]  

(41)

Next, we will use the formula in Equation (39) to study observables due to the gravitational lensing effect from the traversable three-form wormholes in the subsequent sections.

### 3.1. Case I: \( \alpha = 1, \beta = 1, \) and \( \Phi_0 = -1 \)

We set \( \Phi_0 = -1, \alpha = 1, \) and \( \beta = 1 \) leading to \( \Phi(r) = -r_0/r \) and \( b(r) = r_1/r \), respectively. From the standard approach in GR as done in refs. \([63,70,71]\), the effective potential, \( V_{\text{eff}} \), is defined by

\[
V_{\text{eff}}(u) = \frac{1}{2} \frac{e^{-2\Phi_0}}{r_0^2}
\]  

where \( l \) is the proper radial coordinate of the traversable wormholes. With the parameters of this case, we find

\[
V_{\text{eff}}(u) = \frac{1}{2} \frac{e^{-2\Phi_0}}{r_0^2}
\]  

(36)

We will display a potential \( V \) with respect to proper radial coordinate, \( l \), in the Section 5.1. The numerical results will be given in the next section.

According to the standard calculations (see refs. \([63,70,71]\) for detailed derivation), the deflection angle with the parameter set given in case I can be obtained as

\[
\alpha(u) = \frac{2}{\sqrt{1 - \frac{2\Phi(r)}{r}} \left(1 + \frac{e^{-2\Phi_0}}{r_0^2} \frac{1}{e^{-2\Phi(r)} - e^{-2\Phi_0}}\right)} - \pi
\]  

(44)

In addition, the impact parameter, \( u \), of the case I is given by

\[
u = r_p e^{-\Phi_0}
\]  

(45)

Moreover, the turning point of the effective potential, \( r_p \) is given by

\[
r_p = \frac{r_0}{W(-\frac{2}{u})}
\]  

(46)

where \( W \) function is the product–log function. The numerical results of the deflection angle will be discussed in the next section.

### 3.2. Case II: \( \alpha = 1, \beta = -\frac{1}{2}, \) and \( \Phi_0 = -2 \)

For the case II, we fix \( \Phi_0 = -2, \alpha = 1, \) and \( \beta = 1 \) to obtain \( \Phi(r) = -2r_0/r \) and \( b(r) = \sqrt{r_0}r \), respectively. Having used the definition of the effective potential as shown previously, one can write the expression of the effective potential, \( V_{\text{eff}} \), as

\[
V_{\text{eff}}(u) = \frac{1}{2} \frac{e^{-2\Phi_0}}{r_0^2}
\]  

(37)

The numerical results of the potential \( V \) with respect to proper radial coordinate \( l \) are displayed in Figure 3 of the next section.
With the same procedure given previously, the deflection angle of case II parameters is given by

\[ \alpha(u) = 2 \int_{r_p}^{\infty} \frac{e^{\frac{2u}{r}}}{r^2 \sqrt{1 - \frac{2u}{\sqrt{1 + e^{\frac{2u}{r}}}}}} \, dr - \pi \]  

(48)

The impact parameter, \( u \), of case II reads

\[ u = r_p e^{\frac{2u}{r}} \]  

(49)

In addition, the turning point, \( r_p \), for case II reads

\[ r_p = -\frac{2r_0}{W(-\frac{2u}{r})} \]  

(50)

The numerical plot of the deflection angle with respect to impact parameter will be depicted in the next section.

### 3.3. Case III: Particular Solution of \( b(r) \) with \( \alpha = 1, \Phi_0 = -1, \gamma = -8, \) and \( C = -10 \)

In this case, we will use the particular solution of \( b(r) \) function in Equation (34) with the parameters with \( \alpha = 1, \Phi_0 = -1, \gamma = -8, \) and \( C = -10 \). We note that the effective potential in this case has the similar form as that of the case I except the size of the throat and it will be shown in the next section, while the deflection angle in this case can be written in the following form

\[ \alpha(u) = 2 \int_{r_p}^{\infty} \frac{\sqrt{30} e^{\frac{2u}{r^2}}}{r^2 \sqrt{\frac{25r_0^2-8r_0^2r_{th}^2+8r_{th}^2}{r_0^2r_{th}^2(r_0^2+2r_{th}^2)^2}} \sqrt{1 + e^{\frac{2u}{r}}}} \, dr \]  

(51)

Noting that the impact parameter, \( u \) and the turning point, \( r_p \), are the same form as that of the case I. We will study the physical consequences of this case numerically in the next section as cases I and II.

In this section, we have computed the effective potential, \( V \), and the deflection angle, \( \alpha \), for the photon trajectories in the space–time background of the three-form wormholes. The numerical results and some physical implication will be presented in the next section.

### 4. Strong Lensing by the Three-Form Wormhole

In this section, we present analytical examination for the strong gravitational lensing generated by the three-form wormholes. The results in this section will be used to study the relativistic images or the Einstein rings created by the wormholes.

For more convenience, we would like to first re-write the line element of the Morris–Thorne wormhole in Equation (8) as

\[ ds^2 = -A(r) \, dt^2 + B(r) \, dr^2 + C(r) \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \]  

(52)

Moreover, the \( A(r) \), \( B(r) \), and \( C(r) \) functions show asymptotic behaviors as

\[ \lim_{r \to \infty} A(r) = 1, \quad \lim_{r \to \infty} B(r) = 1, \quad \lim_{r \to \infty} C(r) = r^2 \]  

(53)

Comparing between line elements (8) and (52), one finds

\[ A(r) = e^{2\Phi(r)}, \quad B(r) = \left(1 - \frac{b(r)}{r}\right)^{-1}, \quad C(r) = r^2 \]  

(54)

In the latter, we will provide the pertinent formulae of the relevant observables and we will not repeat derivations here. For the detailed calculations and discussions, we recommend the readers to refs. [63, 69, 72].

#### 4.1. Strong Lensing at the Wormhole's Throat

We will consider the strong lensing created by the wormholes at its throat. In this case, the wormhole throat behaves as the photon sphere at the maximum position of the effective potential of the photon. As a result, on the one hand, if the observer and the source are on the same side of the throat. The deflection angle of the strong bending of light due to wormhole throat is given by [63,69,72]

\[ \alpha(u) = -\bar{a} \log \left( \frac{u}{u_0} - 1 \right) + \bar{b} \]  

(55)

where

\[ \bar{a} = 2 \sqrt{\frac{A_0}{B_0}} \left( C_0 A_0 - C_0 A_0' \right), \quad \bar{b} = \bar{a} \log \left[ 2r_0 \left( \frac{C_0'}{C_0} - \frac{A_0'}{A_0} \right) \right] \]  

(56)

and \( u_0 = u(r_0), A_0 = A(r_0), B_0 = B(r_0) \) and \( C_0 = C(r_0) \). The \( I_R \) term is defined by

\[ I_R = \int_0^1 f_{0}(z, r_0) \, dz, \quad f_0 = f - f_0, \quad z = 1 - \frac{r_0}{r} \]  

(57)

\[ f(z, r_0) = \frac{2r_0}{\sqrt{G(z, r_0)}}, \quad f_0 = \frac{2r_0}{\sqrt{n_0 z^2}}, \quad n_0 = r_0^2 C_0 B_0 \left( \frac{C_0'}{C_0} - \frac{A_0'}{A_0} \right), \]  

\[ \bar{B} = \frac{1}{B}, \quad R = \left( \frac{A_0 C}{A_0 C - 1} - 1 \right), \quad G = \frac{RC}{B} (1 - z)^4 \]  

(58)

where \( \dot{r} \equiv d/dr \). For this case, we consider the photon sphere at the throat in the \( r_0 \to r_0 \lim \). Turning to the observables of the gravitational lensing created by the wormholes, the nth relativistic images of the photon sphere can be represented in terms of the angular positions \( \theta_n \) and the magnifications \( \mu_n \) as [63,73]

\[ \theta_n = \frac{n}{D_\infty} \left( 1 + e_n \right), \quad \mu_n = \frac{1 + e_n}{1 - e_n}, \quad e_n = e^{-\sigma x} \]  

(59)
where $D_{OS}$, $D_{OL}$, and $D_{LS}$ are the distances between the observer and the source, the observer and the lens and the lens and the source, respectively. In addition, the angular separation ($s_n$) between the nth and the $(n-1)$th relativistic images can be written in the following form \([63,73]\)

$$s_n = \theta_n - \theta_{n+1}$$

(61)

On the other hand, for the strong lensing of light coming from the other side of a wormhole throat, the deflection angle of the strong light bending is written by\([63,69,72]\)

$$\alpha(u) = -a \log \left( \frac{u^2}{u_{\text{ph}}^2} - 1 \right) + \tilde{b}$$

(62)

where

$$a = 2 \frac{A_0}{B_0 (C_0 A_0 - C_0 A'_0)} \quad \tilde{b} = \tilde{a} \log \left[ 4 r_0 \left( \frac{C_0}{C_0 - A_0} A'_0 \right) \right]$$

$$+ I_\alpha (r_0) - \pi$$

(63)

The observables in this case are given by\([63,73]\)

$$\theta_{\alpha} = \frac{u_0}{D_{OL}} \left( 1 + \epsilon_{\alpha} \right) = \frac{\theta_{\alpha,0}}{\sqrt{1 + \epsilon_{\alpha}},} \quad \epsilon_{\alpha} = \epsilon_{\alpha,0}$$

(64)

$$\mu_{\alpha} = \frac{u_0^2 D_{OS} \epsilon_{\alpha}}{\tilde{a} \epsilon_{\alpha} D_{OL} \left( 1 + \epsilon_{\alpha} \right)}, \quad D_{OS} = D_{OL} + D_{LS}$$

(65)

$$s_{\alpha} = \left| \theta_{\alpha} - \theta_{\alpha,0} \right|$$

(66)

### 4.2. Strong Lensing at the Photon Sphere

To study the photon sphere outside the wormhole’s throat, the photon trajectories are confined between two turning points outside the throat and form the photon sphere. This leads to a different type of the relativistic images. The deflection angle at the photon sphere outside the throat with the observer and the source are on the same side of the throat is given by

$$\alpha(u) = -\tilde{a} \log \left( \frac{u^2}{u_{\text{ph}}^2} - 1 \right) + \tilde{b}$$

(67)

where

$$\tilde{a} = \sqrt{\frac{2 B_{ph} A_{ph}}{C_{ph} A_{ph}' - C_{ph} A_{ph}'',} \quad \tilde{b} = \tilde{a} \log \left[ r_{ph}^2 \left( \frac{C_{ph}^\prime}{C_{ph} - A_{ph}'} A_{ph}' \right) \right]$$

$$+ I_\alpha (r_{ph}) - \pi$$

(68)

and $u_{\text{ph}} = u(r_{ph}), A_{ph} = A(r_{ph}), B_{ph} = B(r_{ph}),$ and $C_{ph} = C(r_{ph}).$ The radius $r_{ph}$ is known as the photon sphere radius occurring at strong gravity region. The $I_\alpha$ term in this case is defined by

$$I_\alpha = \int_1^{\hat{r}_{\text{ph}}} f_\alpha(z, r_{\text{ph}}) dz, \quad \hat{r}_{\text{ph}} = f - f_\alpha, \quad z = 1 - \frac{r_{\text{ph}}}{r}$$

(69)

$$f (z, r_{\text{ph}}) = \frac{2 r_{\text{ph}}}{\sqrt{G(z, r_{\text{ph}})}}, \quad f_\alpha = \frac{2 r_{\text{ph}}}{\sqrt{\eta_{\text{ph}} z^2}}$$

$$\eta_{\text{ph}} = \frac{r_{\text{ph}}^2 C_{\text{ph}}}{2 B_{\text{ph}} \left( C_{\text{ph}} - A_{\text{ph}}'' \right)} \left( \frac{r_{ph}}{r_0} - 1 \right) = \frac{A_{\text{ph}} C_{\text{ph}}}{AC_{\text{ph}} - 1}$$

$$G = \frac{RC}{B (1 - z)^4}, \quad \text{for the limit } r_p \rightarrow r_{\text{ph}}$$

(70)

The observables of this case are the same in Equations \((59)\)–\((61)\) for $\theta_{\alpha}, \mu_{\alpha},$ and $s_{\alpha},$ respectively.

For the observer and the source located on the other side of the throat, one finds

$$\alpha(u) = -\tilde{a} \log \left( \frac{u_{\text{ph}}^2}{u^2} - 1 \right) + \tilde{b}$$

(71)

where

$$\tilde{a} = \sqrt{\frac{2 B_{ph} A_{ph}}{C_{ph} A_{ph}' - C_{ph} A_{ph}'',} \quad \tilde{b} = \tilde{a} \log \left[ r_{ph}^2 \left( \frac{C_{ph}^\prime}{C_{ph} - A_{ph}'} A_{ph}' \right) \right]$$

$$+ I_\alpha (u_{\text{ph}}) - \pi$$

(72)

In this case, the $\theta_{\alpha}, \mu_{\alpha},$ and $s_{\alpha}$ are given by Equations \((64)\)–\((66)\), respectively.

The above formulae will be used to numerically study the strong gravitational lensing created at the wormhole’s throat in the next section.

### 5. Numerical Results

In this section, we will present the numerical results for the effective potential and the deflection angle by using the parameters given in cases I and case II, where they are satisfied the NEC and WEC for the three-form wormholes as demonstrated in ref. [37].

#### 5.1. Effective Potential

We start with a numerical study of the effective potential in Equations \((43)\) and \((47)\) for parameters of cases I and II, respectively. Before discussing the numerical of the effective potential of the photon trajectory in the three-form wormholes background, it is worth considering the photon moving in the potential until reaching the “turning point” where $dV/dr = 0.$ Here $V_{eff} \equiv L^2 V_{eff}$ with $L$ is the angular momentum of the photon circular motion. The constant radius $r_{ph}$ is the constant radius of...
Figure 2. Plot for effective potential $V$ as a function of radial coordinate $l$ with different values of $r_0$. In this figure, we have used the parameters of case I, $\Phi_0 = -1$, $\alpha = 1$, and $\beta = 1$, with the ansatz of $\Phi(r)$ and $b(r)$ in Equation (23).

the photon motion in the circular orbit. The circular orbit of the photon occurs when

$$2e^{2\Phi_0}b(r_{ph})r_{ph}^2 - 2e^{2\Phi_0}r_{ph}^3 = 0$$

(73)

The radius of the photon circular orbits is given by

$$r_{ph} = \frac{1}{\Phi'(r_{ph})}$$

(74)

In general, the conditions for examining the photon sphere radius are written as

$$\hat{V}_{eff}(r_{ph}) = E^2, \quad \frac{d\hat{V}_{eff}}{dr}_{|_{r_{ph}} = 0}, \quad \frac{d^2\hat{V}_{eff}}{dr^2}_{|_{r_{ph}}} < 0$$

(75)

where $r_{ph}$ and $E$ are the photon sphere radius corresponding to the maximum of the effective potential and the energy of the photon, respectively.

We have plotted effective potential for different values of $r_0$. As a result, we found that the numerical results in Figure 2 are compatible with the conditions (75) for all values of $r_0$ in the plots.

The light deflection occurs when the turning point of the photon approaches the throat $r_{tp} = r_0$ of the wormholes. This implies the occurrence of the bending of light due to the photon sphere at the throat of wormholes. In case II, as a result, Figure 3 shows the maximum value of $\hat{V}$ corresponding to the photon sphere radius, $r_{ph}$. The light with turning point $r_{tp} > r_0$ always take a turn outside the photon sphere. This implies that light diverges at the photon sphere outside the throat as the interpretation from ref. [63]. In the case III, finally, the effective potential shape is similar to the case I, that is, the photon sphere occurs at the throat of the wormholes. In order to obtain the physical values, however, this case requires that the throat of case III is larger than that of the case I as shown in Figure 4.

5.2. Deflection Angle

We have plotted numerical integration of the deflection angle from Equations (44) and (48) with respect to the impact parameter. Figures 5 and 6 show the divergence of $\alpha$ as the $u$ increase. The deflection of light occurs when the impact parameter approaches the critical value $u \to u_0$. It is diverging to infinity corresponding to the throat radius $r_0$. When the throat of wormhole $r_0$ is large $\alpha$ is diverging slowly. We close this section by discussing the implications of the gravitational lensing created by the three-form wormholes. In case I, on the one hand, the numerical results from Figures 2 and 5 show that the potential, $\hat{V}$, has the maximum value at the throat and deflection angle of the photon getting diverge at the throat. This implies that the photon sphere can be used to figure out the throat of the three-form wormholes. On the other hand, the numerical results of case II shown in Figures 3 and 6 show that the potential $\hat{V}$ has the maximum value at the photon sphere radius. This means that the light bending...
occurs at the photon sphere outside the throat. In addition, the results of case II tell us that the observers will see two sets of Einstein’s relativistic rings at and outside the throat. Comparing the numerical results of the deflection angle from two case studies, the solution from case I with \( \Phi_0 = -1 \) has the deflection angle less than that of case II with \( \Phi_0 = -2 \), meaning that the coefficient of the redshift function, \( \Phi_0 \), plays an important role in of the deflection angle analysis. Moreover, the deflection angles of both sets of the parameters from Figures 5 and 6 imply that the larger wormholes produce the bigger deflection angles. In the case III, we obtain the \( \Phi_0 = -1 \) as the case 1. But the physical results require that the throat \( r_0 \geq 1.9 \). This shows that the deflection angle in the particular solution of \( b(r) \) function in case III is larger than the case I in order one of magnitude. The numerical plots of the deflection angle in the case III are given by Figure 7. In addition, we close this subsection by comparing the results of the deflection angle in this work with the charged wormholes in ref. [71]. As a result, we found that the effective potentials in the charge wormholes have two possibilities depending of the red-shift functions as the three-form wormholes, that is, one and two maximum points of the effective potential while the deflection angles for charge and three-form wormholes have a similar tendencies. The magnitudes of the deflection angles increasing proportional to the sizes of the wormhole’s throat. The deflection angle results in this work are also similar to the black-bounce wormholes[74] when increasing the ratio of black-bounce length scale and the wormhole mass.

### 5.3. Einstein Rings

To complete our study on the strong lensings from the three-form wormholes, we will show the numerical results of the relativistic images or the Einstein rings for two cases of the wormhole parameter sets. First of all, the three-form wormhole’s parameters of the case I, that is, \( \Phi_0 = -1, \alpha = 1, \) and \( \beta = 1 \) revealed the effective potential as shown in Figure 2 that the turning point of the photon trajectories existed at the wormhole’s throat. This leads to the relativistic images at the throat and they are created at the same position for all angular positions (\( \theta_{\text{max}} \)) on the \( x-y \) plane. The numerical plots of Equation (59) with Equation (56) and Equation (64) with Equation (63) for the images at the throat are shown in Figure 8.

For the wormhole with the parameters of case II, that is, \( \Phi_0 = -2, \alpha = 1 \) and \( \beta = -1/2 \), the effective potential of this case presented the two maximum points of the potential in Figure 3. This means that the photon trajectories are bounded between two turning points and created the photon sphere outside the throat of the wormhole. We have depicted the numerical results for the photon sphere via the relativistic images outside the wormhole’s throat in Figure 9 with the observer using Equations (59) and (68), and the source are the same and other side of the wormhole using Equations (64) and (72), respectively. In addition, the higher angular positions of the relativistic images for this case are almost identical to \( \theta_{\text{max}} \). For the case III, the Einstein’s ring profiles are similar to the case I. However, the magnitudes of the relativistic images of the wormholes in this case are one order of magnitude.
Figure 8. The numerical plots represent the relativistic images creating from the wormhole’s throat for three different values of $r_0 = 0.10\, \text{GM}/c^2$, $0.15\, \text{GM}/c^2$, $0.20\, \text{GM}/c^2$ (from left to right) by using the ansatz of $\Phi(r)$ and $b(r)$ and the parameter set in case I in Equations (23) and (25), respectively. The upper panels are the relativistic images with the observer and the source are on the same side of the wormhole’s throat whereas the lower panels are the relativistic images with the observer and the source are on the other side of the wormhole’s throat. Here the distance from the observer to the lens, $D_{O L} = 7.86\, \text{kpc}$ and the supermassive black hole $\text{Sgr A}^*$, $M = 4.31 \times 10^6\, \text{M}_\odot$ are used in this work.

Figure 9. The numerical plots represent the relativistic images creating from the photon sphere for three different values of $r_0 = 0.10\, \text{GM}/c^2$, $0.15\, \text{GM}/c^2$, $0.20\, \text{GM}/c^2$ (from left to right) by using the ansatz of $\Phi(r)$ and $b(r)$ and the parameter set in case II in Equations (23) and (26), respectively. The upper panels are the relativistic images with the observer and the source are on the same side of the wormhole’s throat whereas the lower panels are the relativistic images with the observer and the source are on the other side of the wormhole’s throat.
larger than that of the case I. The numerical plots of the relativistic images are depicted in Figure 10. More importantly, it is worth discussing about the black hole mimicker of the wormholes. In the case III, we found that the three-form wormholes with the new potential ansatz, \( V(\tau) \) and the particular solution, \( b(\tau) \) for the parameters \( \alpha = 1, \Phi_0 = -1, \gamma = -8, \) and \( C = -10 \) cannot mimic the Schwarzschild black hole. One can easily determine via the following black hole shadow radius, \( r_{sh} \) formula\(^{[5]}\) with the photon sphere of the black holes, \( r_{ph} = 3M \). We find

\[
r_{sh} = \frac{r}{\sqrt{-\frac{b(\tau)}{c^2}}} |_{\tau=\tau_{ph}, \gamma=3M} = \frac{3M}{\epsilon} \tag{76}
\]

This means that the shadow radius of the wormholes in this case cannot reproduce the shadow radius of the black holes, that is, \( r_{sh}^{BH} = 3\sqrt{3}M \).

We close this section by summarizing all observables of the strong lensing from the wormhole, that is, \( \theta_{sh}, \mu_{sh}, \) and \( s_{sh} \) in Table 1. Before closing this section, we note that the \( \theta_{sh} \) from the black-bounce wormholes\(^{[74]}\) is bigger than the \( \theta_{sh} \) in our three-form wormholes case with the same order of the distance from the observer to the lens, \( D_{OL} \) and the supermassive black hole Sgr A*, \( M = 4.31 \times 10^6 M_\odot \) are used in this work.

6. Conclusions

In this work, we focus on the study of gravitational lensing from the traversable wormholes with the three-form fields. We used specific solutions to construct the wormhole solutions. We employ two sets of parameters and specific form of the three-form potential that are given by ref. [37] which are most suitable to construct the energy densities and NEC profiles of the three-form fields and the matter source which does not violate the energy conditions. The null geodesics condition is used to find the geodesics of the wormhole metric to obtain effective potential. We have used the ansatz of the redshift and the shape functions in Equation (23) with the parameters of case I and case II to explain the existence of the photon sphere occurring in the wormhole background. As the results present in the present work, the geometrical shapes of the wormholes can be represented by using the observed photon sphere. Moreover, the Einstein’s ring which is an useful physical observable in astronomy and cosmology can be used to probe the existence of the wormholes, see, for example,

![Figure 10](image-url) The numerical plots represent the relativistic images creating from the wormhole’s throat for two different values of \( \gamma = 1.9 \text{ GM}/\text{c}^2, \) \( 3.0 \text{ GM}/\text{c}^2 \) (from left to right) by using the ansatz of \( \Phi(\tau) \) and \( b(\tau) \) in Equations (34) and the parameters \( \alpha = 1, \Phi_0 = -1, \gamma = -8, \) and \( C = -10 \). The upper panels are the relativistic images with the observer and the source are on the same side of the wormhole’s throat whereas the lower panels are the relativistic images with the observer and the source are on the other side of the wormhole’s throat. Here the distance from the observer to the lens, \( D_{OL} = 7.86 \text{ kpc} \) and the supermassive black hole Sgr A*, \( M = 4.31 \times 10^6 M_\odot \) are used in this work.

![Table 1](image-url) The table summarize all observables from the strong lensing created by the three-form wormhole. The angular positions are in \( \mu\)-arcsec, the distance from the observer to the lens, \( D_{OL} = 7.86 \text{ kpc} \) and the supermassive black hole Sgr A* mass, \( M = 4.31 \times 10^6 M_\odot \) are used in this work.
In this work, we have provided a useful information that can be developed to predict the Einstein ring in the three-form wormholes. In addition, the gravitational signal of the wormholes has been drawn a lot of attention due to the active research fields in the gravitational wave observations, for instances, a study of the quasi normal modes of the wormholes in various models see refs. [77–81]. We will leave this interesting topic for the future work.

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**Conflict of Interest**

The authors declare no conflict of interest.

**Data Availability Statement**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Keywords**

gravitational lensing, three-form fields, traversable wormholes

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