LOCAL WELL-POSEDNESS OF AXISYMMETRIC FLOWS IN A BOUNDED CYLINDER WITH THE MOVING CONTACT LINES

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Abstract. The aim of this paper is to propose a new formulation of free boundary problems of the Navier–Stokes equations in the three-dimensional Euclidean space with moving contact line in terms of classical balance laws and boundary conditions. The new system does not require any additional boundary conditions on the moving contact line, where a contact angle between a free interface and a rigid surface is oscillating in time. It should be emphasized that the total available energy of the system is conserved along with smooth solutions and the negative total available energy is a strict Lyapunov functional if the velocity field and the free interface are axisymmetric. Furthermore, we show local well-posedness of the formulated system provided that the initial data are axisymmetric. Of crucial importance for the analysis is the property of maximal $L^p - L^q$-regularity for the corresponding linearized problem.

1. Introduction

The study of the well-posedness of free boundary problems of the Navier–Stokes equations with moving contact lines has been one of the challenging mathematical problems for decades. To the best of the author’s knowledge, this problem has been open if a contact angle between a free boundary and a rigid surface is oscillating in time, i.e., the contact angle is not fixed — even for a formulation of the problem has yet not well understood. The present paper aims to formulate this free boundary problem by means of classical balance laws and boundary conditions, in which any additional boundary conditions on the contact line are not needed. Furthermore, we show that the problem is locally well-posed and admits the unique axisymmetric and real analytic solution provided that the initial data are axisymmetric.

1.1. Description of the problem. Let us state our problem precisely. We consider a fixed rigid body with an axisymmetric, cylindrical, and simply connected cavity $V \subset \mathbb{R}^3$, partially filled with an incompressible viscous Newtonian fluid that fills a region $\Omega(t)$ at time $t > 0$. Here, the domain of cavity $V$ is defined by

$$V := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D_R, \ x_3 \in (-2H, 2H)\},$$

where $D_R \subset \mathbb{R}^2$ is a disk centered at the origin with a radius $R > 0$ and $H$ is a given positive constant. A sharp interface $\Gamma(t)$ separates the cavity $V$ into the fluid part $\Omega(t)$ and the vacant part $V \setminus \Omega(t)$. In addition, the boundary of the free interface $\partial \Gamma(t)$ separates the boundary of cavity $\partial V$ into the wetting part $\Sigma(t)$ and the drying part $\Sigma^*(t)$. The contact line $S(t)$ is defined by $S(t) = \partial V \cap \partial \Gamma(t)$. By abuse of notation, we will write $\Omega_t = \Omega(t)$, $\Gamma_t = \Gamma(t)$, $\Sigma_t = \Sigma(t)$, $\Sigma^*_t = \Sigma^*(t)$, and $S_t = S(t)$. If the initial position $\Gamma_0 = \Gamma(0)$ of $\Gamma_t$ can be approximate $D_R$ in the sense that the Hausdorff distance of the second order bundles of $\Gamma_t$ and $D_R$ is small enough, we may assume that the unknown free surface $\Gamma_t$ can be parameterized by means of an unknown height function $\eta$ such that

$$\Gamma_t := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D_R, \ x_3 = \eta(x_1, x_2, t), \ t \geq 0\} \quad (1.1)$$
suppose that \( \Omega \) functions \( \eta \) can be formulated as follows: Given \( \Gamma \) cylinder

Then a free boundary problem of the Navier-Stokes equations with moving contact angles in a right circular cylinder can be formulated as follows: Given \( \Gamma_0 \subset V \) and \( v_0 : \Omega_0 \to \mathbb{R}^3 \), find a family \( \{ \Gamma_t \}_{t \geq 0} \) and a pair of functions \( v(\cdot, t) : \Omega_t \to \mathbb{R}^3 \) and \( p(\cdot, t) : \Omega_t \to \mathbb{R} \) satisfying

\[
\begin{align*}
\partial_t v + (v \cdot \nabla)v &= \text{div} \, T(v, p), & \text{in } \Omega_t, \\
\text{div } v &= 0, & \text{in } \Omega_t, \\
T(v, p)n_{\Gamma_t} &= \sigma \mathcal{H}_\Gamma; n_{\Gamma_t} - p_0 n_{\Gamma_t}, & \text{in } \Gamma_t, \\
V_{\Gamma_t} &= \langle v, n_{\Gamma_t} \rangle, & \text{in } \Gamma_t, \\
P_{\Sigma_t}(2\mu D(v)n_{\Sigma_t}) &= 0 & \text{on } \Sigma_t, \\
\langle v, n_{\Sigma_t} \rangle &= 0 & \text{on } \Sigma_t, \\
P_B(2\mu D(v)n_B) &= 0 & \text{on } B, \\
\langle v, n_B \rangle &= 0 & \text{on } B, \\
v(0) &= v_0 & \text{in } \Omega_0, \\
\Gamma_{t=0} &= \Gamma_0.
\end{align*}
\]  

(1.2)

where we use the notation \( \langle \cdot, \cdot \rangle \) to describe the dot product of vector fields. In this paper, we consider the case where the initial position \( \Gamma_0 \) of the free boundary is the graph of a height function \( \eta_0 \) on \( D_R \), i.e.,

\[
\Gamma_0 := \{ (x_1, x_2, x_3) \mid (x_1, x_2) \in D_R, \ x_3 \in \eta(x_1, x_2), \ t \geq 0 \}.
\]

Here, \( v \) and \( p \) are unknown functions describing the velocity fields and the pressure of incompressible viscous fluid, respectively. In the system (1.2), the symbol \( T(v, p) \) stands for the viscous stress tensor defined by

\[
T(v, p) = 5(v) - pI = 2\mu D(v) - pI,
\]

where \( D(v) = 2^{-1}(\nabla v + (\nabla v)^T) \) is the deformation tensor; \( \mu > 0 \) stands for a constant denoting the viscosity coefficient; \( n_{\Gamma_t} \), \( n_{\Sigma_t} \), and \( n_B \) are the outward unit normal field on \( \Gamma_t \), \( \Sigma_t \), and \( B \), respectively; \( \sigma \) stands for the surface tension coefficient, which is a given positive constant; \( \mathcal{H}_\Gamma \) stands for the double mean curvature of \( \Gamma_t \) given by \( \mathcal{H}_\Gamma = -\text{div } \gamma; n_{\Gamma_t} \), where \( \text{div } \gamma; \) is the surface divergence on \( \Gamma_t \); \( p_0 \) stands for an external pressure, which is a given positive constant; \( V_{\Gamma_t} \) stands for the normal velocity of \( \Gamma_t \); \( P_{\Sigma_t} := I - n_{\Sigma_t} \otimes n_{\Sigma_t} \) and \( P_B := I - n_B \otimes n_B \) stand for the orthogonal projections onto the tangent bundle of \( \Sigma_t \) and \( B \), respectively. The contact angle \( \theta = \theta(x, t) \) is defined by

\[
\cos \theta := -\langle n_{\Gamma_t}, n_{\Sigma_t} \rangle, \quad x \in \partial D_R, \ t \geq 0,
\]

(1.3)

so that we can determine the contact angle if we find \( n_{\Gamma_t} \) because \( n_{\Sigma_t} \) is a priori known. Here, the contact angle should satisfy \( \theta \in [0, \pi] \) due to the geometry of domain. Notice that the total available energy of the system is conserved for smooth solutions and is a strict Lyapunov functional if the velocity field \( v \) of the fluid the free surface \( \Gamma_t \) are axisymmetric. The system (1.2) will be explained in more detail in the next section.

1.2. Historical remarks. If the contact angle \( \theta \) is fixed into the trivial cases, say, \( \theta = \pi \) or \( \theta = \pi/2 \), there are the pioneering contributions by Solonnikov [49] and Wilke [53]. The key observations of their studies were that these contact angles remove the singularities at the contact lines. However, from the classical Young law, the contact angles seem to depend on the time if the initial contact angles are not equal to the contact angles at the equilibria, denoted by \( \theta_\infty \). Hence, this shows that fixing the contact angle is a kind of idealization. Recently, Fricke et al. [12] studied the spreading droplet problem and derive a kinematic evolution law for
Here, the contact line normal vector \( \mathbf{n}_{S_t} \) satisfying
\[
f(0) = \theta_{\infty}, \quad V_{S_t}(f(V_{S_t}) - \theta_{\infty}) \geq 0.
\]

Here, the contact line normal vector \( \mathbf{n}_{S_t} \) is defined via projection \( P_{\partial \Sigma_t} = I - \mathbf{n}_{\partial \Sigma_t} \otimes \mathbf{n}_{\partial \Sigma_t} \) as
\[
\mathbf{n}_{S_t} := \frac{P_{\partial \Sigma_t} \mathbf{n}_{\partial \Sigma_t}}{|P_{\partial \Sigma_t} \mathbf{n}_{\partial \Sigma_t}|},
\]
cf. Fricke [12, Def. 2], where \( \mathbf{n}_{\partial \Sigma_t} \) is a unit outer vector of \( \partial \Sigma_t \) that is perpendicular to the lateral of the cavity \( \mathcal{V} \). The condition \( V_{S_t}(f(V_{S_t}) - \theta_{\infty}) \geq 0 \) ensures the energy dissipation of the system. Although (1.4) will be reasonable if we deal with a spreading droplet problem, the assumption does not seem to be applicable to general contact angle problems because the condition means that the moving direction of the contact line is monotone. We point out that similar condition to (1.4) is also appeared in recent studies of the contact line problem for the two-dimensional Stokes flow [13,54]. To be more precise, as formulated in [13, Sec. 1.3] and [54, Sec. 1], instead of (1.4), they supposed the existence of an increasing function \( F \) such that
\[
F(0) = 0, \quad V_{S_t} = F(\cos \theta - \cos \theta_{\infty}),
\]
which also lead the energy dissipation of the system. Although this form of \( V_{S_t} \) derived with the help of molecular dynamics and thermodynamics by several authors [4,8,34], their models cannot represent oscillating contact angles because the monotonicity of \( F \) and the condition \( F(0) = 0 \) imply that the moving direction of the contact line is monotone. In fact, the assumptions on \( F \) show that slip in the contact line acts to restore the equilibrium angle but the contact line stops slipping when the contact angle reaches \( \theta_{\infty} \); the contact line cannot “go through” the equilibrium contact line. Notice that the latter condition of (1.4) as well as the monotonicity of \( F \) guarantee the relation \( (\cos \theta - \cos \theta_{\infty}) \langle \mathbf{v}, \mathbf{n}_{S_t} \rangle \leq 0 \), which infers
\[
\int_{S_t} (\cos \theta - \cos \theta_{\infty}) \langle \mathbf{v}, \mathbf{n}_{S_t} \rangle \, dl \leq 0,
\]
see also Section 2. To overcome the defect of previous studies, we directly deal with the inequality (1.5), i.e., we show this inequality without considering \( (\cos \theta - \cos \theta_{\infty}) \langle \mathbf{v}, \mathbf{n}_{S_t} \rangle \leq 0 \) on \( S_t \). To this end, we suppose that the velocity field \( \mathbf{v} \) and the free surface \( \Gamma_t \) of the fluid are axisymmetric and that the cavity \( \mathcal{V} \) is axisymmetric, cylindrical, and simply connected. Although this condition is the somewhat restricted situation, the author believes that this assumption is more natural than the previous one since the contact angle may oscillate in time. Furthermore, in our situation, the dynamical contact angle is uniquely determined by the relation (1.3) due to the physical restriction \( \theta \in [0, \pi] \).

Since the position of the free surface \( \Gamma_t \) is a priori unknown, it will be convenient to transform the problem for the velocity and the pressure on a fixed domain. To this end, we apply the direct mapping method via a Hanzawa transformation, where we can obtain precise regularity information for the free surface. Namely, we do not consider Lagrangian coordinates to derive the transformed problem. Here, the position of the fixed surface \( \Gamma_\ast \) should be close to the unknown free surface \( \Gamma_t \) in the sense that the Hausdorff distance of the second-order bundles of \( \Gamma_t \) and \( \Gamma_\ast \) is small enough. In this case, the Hanzawa transformation is a diffeomorphism mapping, so that we may obtain the well-posedness result of the free boundary problem from the fixed boundary problem. This shows that it is crucial to consider the problem in the appropriate fixed domain. For simplicity, in our study, we choose the right circular cylinder as a fixed domain, i.e., the domain \( \Omega_t \) will be given as a perturbation of the right circular cylinder. The advantage of this setting is that we can apply the standard reflection arguments as was applied in [53]. Especially, the fixed contact angle is equal to \( \pi/2 \), and hence there is no singularity on the fixed contact line. Although there is no additional condition on the contact line, no singularity appears on the contact line because the total available energy of the system is still finite and a strict Lyapunov functional, see Section 2.

As we will see in Section 3, the transformed system via the Hanzawa transform becomes a quasilinear parabolic system. Hence, in the present paper, we devote to prove solvability results based on the theory of maximal regularity, in which we can solve the nonlinear problem by the contraction mapping principle.
The main difficulty of maximal $L^p - L^q$-regularity approach is that, if we study the corresponding linearized problem, the boundary data have to be in the intersection space

$$F_{p,q}^s(J; L^q(\partial D)) \cap L^p(J; B^{q/2}_{q,q}(\partial D)) \quad 0 < s < 1,$$

(1.6)

where $F_{p,q}^s$ and $B^{q/2}_{q,q}$ denote the vector-valued inhomogeneous Triebel-Lizorkin space and the scalar-valued inhomogeneous Besov space, respectively, cf., [9, 33]. Hence, it is required to establish estimates for the nonlinear terms in this intersection space, which has not been established so far when $p \neq q$. Notice that if $p = q$, then the vector-valued Triebel-Lizorkin space becomes the vector-valued Sobolev-Slobodeckiï space, where the estimations for the nonlinear terms in this space are well-known, see, e.g., [32, Appendix]. Here, in [18,31,32], they studied the two-phase free boundary problem for the Navier–Stokes equations in the case $p = q$, where the free boundary can be understood in the classical sense. See also, e.g., [24,26,45,47,48] for results of the free boundary problem for the Navier–Stokes equations, which were established in anisotropic Sobolev-Slobodetskiï as well as in Hölder spaces. Compared with their studies, Shibata [37–39] obtained the maximal $L^p - L^q$-regularity results for the linearized problem of the free boundary problem of the one-phase Navier–Stokes equations in the case $p \neq q$, see also [20, 40, 41] for the two-phase case. Unfortunately, in their arguments, the boundary data were not lying in (1.6), and thus their results are not optimal in view of trace theorems. To overcome these fallacious, we use the recent contributions established by Meyries and Veraar [22,23] and Lindemulder [19]. Using their results, we will show the principal linearization has maximal $L^p - L^q$ regularity in the case $p \neq q$, where the boundary data belong to the intersection space (1.6). We also succeed to establish the estimates for the nonlinearities appeared on the boundaries in the intersection space (1.6). It should be emphasized that our approach completely works for refining the previous studies of the free boundary problems for the one-phase or two-phase Navier–Stokes equations in domains surrounded by smooth boundary (e.g., bounded or exterior domains) to obtain optimal $L^p - L^q$-regularity space-time estimates for the corresponding linearized equation with $p \neq q$, which also yields well-posedness results for the free boundary problems with initial data possessing optimal regularity.

To work in the maximal regularity framework, it is crucial to study the model problems in the whole space, a half space, a wedge domain with an angle equals to $\pi/2$, see Section 4. Compared with the arguments in [53], we do not suppose that the Neumann trace of the height function $\eta$ vanishes on the contact line since there is no additional condition on the contact line. The model problems in half spaces are well-studied by many authors, e.g., [32, 39], but we give more sophisticated results in view of the trace theory. For the model problem in a wedge domain with an angle equals to $\pi/2$, there are several studies by [5, 17, 53]. Their arguments relied on the standard refection argument but they seemed to be missing the condition for the existence of “trace of trace.” Hence, in this paper, we give optimal conditions for the existence of trace of trace — this will be related to the compatibility conditions for the Cauchy problem.

1.3. Main result. The purpose of the present paper is to construct a unique axisymmetric solution of (1.2) for axisymmetric initial data $(v_0, \eta_0) \in B^{2(\delta-1/p)}_{q,p}(\Omega_0)^3 \times B^{2+\delta-1/p-1/q}_{q,p}(D_R)$. To this end, we introduce the axial symmetry of vector fields and scalar functions. A vector field $v$ is said to be axisymmetric if

$$v(x, t) = R_\rho^\top v(R_\rho x, t), \quad x \in \mathbb{R}^3$$

for $R_\rho = (e_r(\rho), e_\rho(\rho), e_z)$ and $e_\rho (\rho) = (\cos \rho, \sin \rho, 0)^\top$, $e_r (\rho) = (-\sin \rho, \cos \rho, 0)^\top$, $e_z = (0, 0, 1)^\top$ with $\rho \in [0, 2\pi]$. A scalar function $\eta$ is axisymmetric if $\eta(x, t) = \eta(R_\rho x, t)$ for $x \in \mathbb{R}^3$ and $\rho \in [0, 2\pi]$.

As the main result in the article we establish the local well-posedness of the problem (1.2).

**Theorem 1.1.** Let $p$, $q$, $\delta$ satisfy

$$2 < p < \infty, \quad 3 < q < \infty, \quad \frac{1}{p} + \frac{3}{2q} < \delta - \frac{1}{2} \leq \frac{1}{2}.$$ (1.7)

Then given $T > 0$, there exists $\varepsilon_0 = \varepsilon_0(T) > 0$ such that for any axisymmetric initial data

$$(v_0, \eta_0) \in B^{2(\delta-1/p)}_{q,p}(\Omega_0)^3 \times B^{2+\delta-1/p-1/q}_{q,p}(D_R),$$

...
satisfying the compatibility conditions
\[
\begin{cases}
\text{div } \mathbf{v}_0 = 0, & \text{in } \Omega_0, \\
P_{\Gamma_0}(2\mu \mathbf{D}(\mathbf{v}_0)\mathbf{n}_{\Gamma_0}) = 0, & \text{on } \Gamma_0, \\
P_{\Sigma_0}(2\mu \mathbf{D}(\mathbf{v}_0)\mathbf{n}_{\Sigma_0}) = 0, & \langle \mathbf{v}_0, \mathbf{n}_{\Sigma_0} \rangle = 0, \text{ on } \Sigma_0, \\
P_B(2\mu \mathbf{D}(\mathbf{v}_0)\mathbf{n}_B) = 0, & \langle \mathbf{v}_0, \mathbf{n}_B \rangle = 0, \text{ on } B,
\end{cases}
\]
and the smallness condition
\[
|\mathbf{v}_0|_{B_{q,p}^{2(\delta-1/p)}(\Omega_0)} + |\eta_0|_{B_{q,p}^{2+\delta-1/p-1/q}(\partial \Omega_0)} \leq \varepsilon_0,
\]
the problem (1.2) admits a unique axisymmetric classical solution \((\mathbf{v}, p, \Gamma)\) on \((0, T)\), where \(P_{\Gamma_0} := I - \mathbf{n}_{\Gamma_0} \otimes \mathbf{n}_{\Gamma_0}\) and \(P_{\Sigma_0} := I - \mathbf{n}_{\Sigma_0} \otimes \mathbf{n}_{\Sigma_0}\). Furthermore, the free boundary \(\Gamma_t\) is the graph of a function \(\eta(t)\) on \(D_R\), the set \(M = \bigcup_{t \in (0,T)} (\Gamma_t \times \{t\})\) is a real analytic manifold, and the function \((\mathbf{v}, p)\) : \((x, t) \in \Omega_t \times (0, T)\) \(\rightarrow \mathbb{R}^4\) is real analytic.

**Remark 1.2.** The condition (1.7) induces the embeddings
\[
B_{q,p}^{2(\delta-1/p)}(\Omega_0) \hookrightarrow \text{BUC}^1(\overline{\Omega}_0), \quad B_{q,p}^{2+\delta-1/p-1/q}(D_R) \hookrightarrow \text{BUC}^2(\overline{D}_R).
\]
Furthermore, this restriction also implies
\[
\eta \in \text{BUC}^1([0, T]; \text{BUC}^2(\overline{D}_R)) \cap \text{BUC}^1([0, T]; \text{BUC}^1(\overline{D}_R)),
\]
which means that the condition on the free interface can be understood in the classical sense. For \(\mathbf{v}\) and \(p\), we have
\[
\mathbf{v}(:, t) \in \text{BUC}^1(\overline{\Omega}_t), \quad \nabla \mathbf{v}(:, t) \in \text{BUC}(\overline{\Omega}_t) \quad \text{for } t \in [0, T],
\]
\[
p(:, t) \in \text{UC}(\Omega_t) \quad \text{for } t \in (0, T].
\]
In addition, the solution \((\mathbf{v}, p, \eta)\) depends continuously on the initial data \((\mathbf{v}_0, \eta_0)\).

**Remark 1.3.** The restriction (1.7) implies the existence of “trace of trace.” Hence, we need the compatibility conditions on the initial contact line \(S_0 := \partial \Omega_0 \cap \partial \Sigma_0\) and on the corner of the boundary \(\partial \Sigma_0 \cap \partial B\).

**Remark 1.4.** The smallness condition for the initial velocity \(\mathbf{v}_0\) is due to the nonlinear term \(D(\mathbf{u}', \eta) = -\langle \mathbf{u}', \nabla x \eta \rangle\) appeared in the transformed problem (3.2), where \(\mathbf{u}' = (u_1, u_2)^T\). In fact, this term cannot be small in the norm of \(F_{3,\delta}(J; \Gamma_\ast)\), defined in (4.4), even if \(|\nabla x \eta|_{L^\infty(D_R)}|\) is small. To avoid this harmful nonlinearity, we have to consider the modified term \((\mathbf{b} - \mathbf{u}', \nabla x \eta)\), where \(\mathbf{b}\) is taken such that \(b(0) = \text{Tr}_\Gamma \mathbf{u}_0\). We refer to Priess and Simonett [32] for the details, see also [18,30,39,43,51]. However, to make our arguments simple, we keep the smallness condition for the initial velocity \(\mathbf{v}_0\).

The plan of this paper is constructed as follows: In Section 2, we show that the negative total entropy is a strict Lyapunov functional for the problem. Section 3 introduces the Hanzawa transformation to transform the free boundary problem (1.2) into a domain with a fixed boundary problem. Section 4 gives the results for the model problems in the whole space, a half space, a quarter space. Using a well-known localization technique, we show maximal regularity of the principal linearization in Section 5. Finally, Section 6 proves our main result, Theorem 1.1, for the free boundary problem (1.2) with the help of the result obtained in Section 5. In the appendix, we collect technical results needed to execute the above program.

**Notation.** As usual, \(\mathbb{N}, \mathbb{R}, \mathbb{C}\) denote the set of all natural, real, and complex number, respectively. Moreover, we also denote \(\mathbb{R}_+ = (0, \infty)\).

For \(m \in \mathbb{N}\) and \(D \subset \mathbb{R}^n\), \(n = 1, 2, 3\), we set \(\mathbf{a} \cdot \mathbf{b} = \langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^m a_j b_j\) for \(m\)-vectors \(\mathbf{a} = (a_1, \ldots, a_m)^T\) and \(\mathbf{b} = (b_1, \ldots, b_m)^T\), while we set \(f \mid g_D = \int_D f(x) \cdot g(x) \, dx\) for \(m\)-vector functions \(f(x) = (f_1(x), \ldots, f_m(x))^T\) and \(g(x) = (g_1(x), \ldots, g_m(x))^T\) on \(D\). Besides, \(C_\infty^\ast(D)\) stands for the set of all \(C_\infty\)-functions on \(\mathbb{R}^n\) whose supports are compact and contained in \(D \subset \mathbb{R}^n\).

For \(p, q \in [1, \infty], s > 0, \text{ and } D \subset \mathbb{R}^n\), let \(L^s(D), H^s(D), B_{p,q}^s(D), F_{p,q}^s(D)\) denote the standard \(\mathbb{K}\)-valued Lebesgue, Bessel potential, inhomogeneous Besov spaces, and inhomogeneous Triebel-Lizorkin spaces on \(D,\)
respectively, where $K \in \{ \mathbb{R}, \mathbb{C} \}$. For $I \subset \mathbb{R}$ and $p \in (1, \infty]$, let $L^p(I; X)$ and $H^{1,p}(I; X)$ be the $X$-valued Lebesgue and Sobolev spaces on $I$, respectively. Furthermore, for $p \in (1, \infty)$ and $\delta \in (1/p, 1]$, we set

\[
L^p_s(I; X) := \{ f : I \to X \mid t^{1-\delta} f \in L^p(I; X) \}, \\
H^{1,p}_s(I; X) := \{ f \in L^p_s(I; X) \cap H^{1,1}(I; X) \mid \partial_t f \in L^p_s(I; X) \}.
\]

For $p \in (1, \infty)$, $q \in [1, \infty)$, $s \in \mathbb{R}$, we write the $X$-valued Triebel-Lizorkin spaces with the power weight $|t|^{p(1-\delta)}$ by $F^s_{p,q,\delta}(I; X)$, where $X$ is a Banach space. Furthermore, $\mathcal{O}H^{1,p}_s(I; X)$ and $\mathcal{O}F^s_{p,q,\delta}(I; X)$ denote the subspaces of $H^{1,p}_s(I; X)$ and $F^s_{p,q,\delta}(I; X)$, respectively, consisting of all functions having a vanishing trace at $t = 0$, whenever they exist. For $0 < s < 1$, the $X$-valued Bessel-potential spaces $H^{1,p}_s(I; X)$ as well as $\mathcal{O}H^{1,p}_s(I; X)$ are defined in an analogous way, employing the complex interpolation method. Namely,

\[
\begin{align*}
H^{1,p}_s(I; X) &= [L^p_s(I; X), H^{1,1}_s(I; X)]_s, \\
\mathcal{O}H^{1,p}_s(I; X) &= [L^p_s(I; X), \mathcal{O}H^{1,1}_s(I; X)]_s,
\end{align*}
\]

where $[\cdot, \cdot]_\theta$ is the complex interpolation functor with $0 < \theta < 1$. For the precise definition, see, e.g., [23]. Here, the following characterizations are known (cf. Prüss and Simonett [33, Ch. 3, 6]):

\[
\begin{align*}
f &\in \mathcal{O}H^{1,p}_s(\mathbb{R}^+; X) \quad \iff \quad t^{1-\delta} f \in \mathcal{O}H^{1,p}(\mathbb{R}^+; X), \\
f &\in \mathcal{O}H^{s,p}_{s,p}(\mathbb{R}^+; X) \quad \iff \quad t^{1-\delta} f \in \mathcal{O}H^{s,p}(\mathbb{R}^+; X), \\
f &\in \mathcal{O}F^{s}_{p,q,\delta}(\mathbb{R}^+; X) \quad \iff \quad t^{1-\delta} f \in \mathcal{O}F^{s}_{p,q,1}(\mathbb{R}; X),
\end{align*}
\]

where we have set $t^{1-\delta}_+ = \max\{t^{1-\delta}, 0\}$. The symbol BUC$(I; X)$ stands for the Banach space of all $X$-valued bounded uniformly continuous functions on $I$. In addition, BUC$^m(I; X)$ is the subset of BUC$(I; X)$ that has bounded partial derivatives up to order $m \in \mathbb{N}$. Here, BUC$(D)$ and BUC$^m(D)$ are defined similarly as above. In addition, UC$(I; X)$ denotes the Banach space of all $X$-valued uniformly continuous functions on $I$. For further information on function spaces, we refer to [23,33,35,52] and references therein.

2. Discussion on the model

Let us give a discussion on the model. To investigate a moving contact line problem of incompressible viscos fluid, we suppose that the flow in the bulk phase can be described by the Navier–Stokes equations with a constant density $\rho \equiv 1$ for simplicity. In addition, we assume the Navier slip boundary conditions on the rigid surface. Here, the Navier slip boundary conditions can be read as

\[
\begin{align*}
\alpha v + \beta P_{\Sigma_t}(2\mu D(v)n_{\Sigma_t}) &= 0, & \langle v, n_{\Sigma_t} \rangle &= 0 \quad &\text{on } \Sigma_t, \\
\alpha v + \beta P_{B}(2\mu D(v)n_B) &= 0, & \langle v, n_B \rangle &= 0 \quad &\text{on } B,
\end{align*}
\]

where $0 \leq \alpha, \beta \leq 1$ are constants satisfying $\alpha + \beta = 1$. If $(\alpha, \beta) = (0, 1)$ these boundary conditions can be deduced to (prefect) slip boundary conditions, while if $(\alpha, \beta) = (1, 0)$ we have no-slip boundary conditions. Although, as far as explaining our model, it suffices to consider the case when $(\alpha, \beta) = (0, 1)$, one may observe that the philosophy of our modeling also works for a general choice of $(\alpha, \beta)$. We also remark that our system can be also applied to moving contact point problems, that is, the discussion in this section is also valid for the two-dimensional case. In the following, we will show that smooth solutions to the system

\[
\begin{align*}
\partial_t v + (v \cdot \nabla) v &= \text{div } T(v, p), & \quad &\text{in } \Omega_t, \\
\text{div } v &= 0, & \quad &\text{in } \Omega_t, \\
T(v, p)n_{\Gamma_t} &= \sigma H_{\Gamma_t} n_{\Gamma_t} - p_0 n_{\Gamma_t}, & \quad &\text{in } \Gamma_t, \\
V_{\Gamma_t} &= \langle v, n_{\Gamma_t} \rangle, & \quad &\text{in } \Gamma_t, \\
\alpha v + \beta P_{\Sigma_t}(2\mu D(v)n_{\Sigma_t}) &= 0, & \quad &\text{on } \Sigma_t, \\
\langle v, n_{\Sigma_t} \rangle &= 0, & \quad &\text{on } \Sigma_t, \\
\alpha v + \beta P_{B}(2\mu D(v)n_B) &= 0, & \quad &\text{on } B, \\
\langle v, n_B \rangle &= 0, & \quad &\text{on } B,
\end{align*}
\]
has the energy functional that is a Lyapunov functional. Especially, we will show that the total available energy of (2.2) defined by

$$E = E(t) := \int_{\Omega_t} \frac{1}{2} v^2 \, dx + \sigma |\Gamma_t| + \sigma_0 (|\Sigma_t| + |B|) + \sigma_1 |\Sigma_t^*|,$$

(3.3)
is a Lyapunov functional, where $\sigma_0$ denotes the surface tension coefficient on the wetting surfaces $\Sigma_t$ and $B$ while $\sigma_1$ denotes the surface tension coefficient on the drying surface $\Sigma_t^*$. Notice that, according to the classical Young relation, we have

$$\sigma_0 - \sigma_1 + \sigma \cos \theta_{\infty} = 0,$$

where $\theta_{\infty}$ denotes the contact angle at equilibria. Our aim is to verify the energy dissipation

$$\frac{dE}{dt} \leq 0.$$  \hspace{1cm} (2.4)

To show (2.4) we use the following transport theorems; see the book by Slattery et al. [44] (cf. Bothe and Prüss [6, Sec. 2.3]). For given any bulk fields $\phi$, it hold that

(\text{Volume Transport}) \hspace{1cm} \frac{d}{dt} \int_{\Omega_t} \phi \, dx = \int_{\Omega_t} \partial_t \phi \, dx - \int_{\Gamma_t} \phi \langle v, n_{\Gamma_t} \rangle \, ds

- \int_{\Sigma_t} \phi \langle v, n_{\Sigma_t} \rangle \, ds - \int_B \phi \langle v, n_B \rangle \, ds,$

(Surface Transport) \hspace{1cm} \frac{d}{dt} |\Sigma_t| = - \int_{S_t} \mathcal{H}_{S_t} \langle v, n_{S_t} \rangle \, ds + \int_{\partial S_t} \langle v, n_{\partial S_t} \rangle \, dl,$

where $S_t$ is a surface depending on $t$ and $\mathcal{H}_{S_t} := -\text{div}_{S_t} n_{S_t}$ is the double mean curvature of $S_t$. According to Fricke [12, Def. 2], the contact line normal vector $n_{S_t}$ is defined via projection $P_{\partial S_t} = I - n_{\partial S_t} \otimes n_{\partial S_t}$ as

$$n_{S_t} := \frac{P_{\partial S_t} n_{\partial S_t}}{|P_{\partial S_t} n_{\partial S_t}|},$$

where $n_{\partial S_t}$ stands for a unit outer vector of $\partial S_t$ that is perpendicular to the lateral of the cavity $V$ (cf. Figure 1). Furthermore, $n_{\partial S_t}$ denotes the unit normal vector of $S_t$ by means of the plane tangential to $S_t$.

Using these relations, we have

$$\frac{d}{dt} \int_{\Omega_t} \frac{1}{2} v^2 \, dx = - \int_{\Omega_t} \mu |\nabla v|^2 \, dx + \int_{\Sigma_t} \langle v, \nabla v \rangle n_{\Sigma_t} \, ds

+ \int_B \langle v, \nabla v \rangle n_B \, ds + \int_{\Gamma_t} \langle v, \nabla v \rangle n_{\Gamma_t} \, ds,$$

$$\frac{d}{dt} |\Gamma_t| = - \int_{\Gamma_t} \mathcal{H}_{\Gamma_t} \langle v, n_{\Gamma_t} \rangle \, ds + \int_{S_t} \langle v, n_{S_t} \rangle \, dl,$$

$$\frac{d}{dt} |\Sigma_t| = \int_{S_t} \langle v, n_{S_t} \rangle \, dl,$$

$$\frac{d}{dt} |B| = 0,$$

where $n_{S_t}$ represents the unit tangential vector of $\Gamma_t$ that is perpendicular to $n_{\Gamma_t}$. In fact, by the transport theorem, it holds

$$\frac{d}{dt} \int_{\Omega_t} \frac{1}{2} v^2 \, dx = \int_{\Omega_t} \frac{1}{2} \left( \partial_t v^2 + \text{div} (|v|^2 v) \right) \, dx.$$

Recalling the first relation in (2.2), we see that

$$\frac{1}{2} \left( \partial_t v^2 + \text{div} (|v|^2 v) \right) = \langle v, \partial_t v \rangle + \langle v, (v \cdot \nabla) v \rangle = \langle v, \text{div} \nabla (v, p) \rangle.$$
Hence, we can compute
\[
\frac{d}{dt} \int_{\Omega_t} \frac{1}{2} \mathbf{v}^2 \, dx = \int_{\Omega_t} \langle \mathbf{v}, \text{div} \, \mathbf{T}(\mathbf{v}, p) \rangle \, dx
\]
\[
= -2 \int_{\Omega_t} \mu |\nabla \mathbf{v}|^2 \, dx + \int_{\Sigma_t} \langle \mathbf{v}, \mathbf{T}(\mathbf{v}, p) \mathbf{n}_{\Sigma_t} \rangle \, ds + \int_{\Gamma_t} \langle \mathbf{v}, \mathbf{T}(\mathbf{v}, p) \mathbf{n}_{\Gamma_t} \rangle \, ds.
\]
The rest identities follow from the transport theorem and \( H_{\Gamma_t} = -\text{div} \, \mathbf{n}_{\Sigma_t} \).

According to the boundary conditions of the system (2.2), we find that
\[
\int_{\Sigma_t} \langle \mathbf{v}, \mathbf{T}(\mathbf{v}, p) \mathbf{n}_{\Sigma_t} \rangle \, ds = \beta \int_{\Sigma_t} \langle P_{\Sigma_t}(2\mu \mathbf{D}(\mathbf{v})\mathbf{n}_{\Sigma_t}), \mathbf{v} \rangle \, ds = -\alpha \int_{\Sigma_t} |\mathbf{v}|^2 \, ds.
\]
Since \(|\Omega_t|\) is a constant we have \( \int_{\Gamma_t} \langle \mathbf{v}, \mathbf{n}_{\Gamma_t} \rangle \, ds = \int_{\Gamma_t} V_{\Gamma_t} \, ds = 0 \), which leads us to obtain
\[
\int_{\Gamma_t} \langle \mathbf{v}, \mathbf{T}(\mathbf{v}, p) \mathbf{n}_{\Gamma_t} \rangle \, ds = \sigma \int_{\Sigma_t} H_{\Gamma_t} \langle \mathbf{v}, \mathbf{n}_{\Sigma_t} \rangle \, ds.
\]
Recalling \( \langle \mathbf{v}, \mathbf{n}_{\Sigma_t} \rangle = 0 \) on \( S_t \), we observe
\[
\int_{S_t} \langle \mathbf{v}, \mathbf{n}_{S_t}^* \rangle \, dl = \int_{S_t} (\cos \theta) \langle \mathbf{v}, \mathbf{n}_{S_t} \rangle \, dl.
\]
Therefore, the above observations leads the energy dissipation
\[
\frac{dE}{dt} = -2 \int_{\Omega_t} \mu |\nabla \mathbf{v}|^2 \, dx - \alpha \int_{\Sigma_t} |\mathbf{v}|^2 \, ds + \sigma \int_{S_t} (\cos \theta - \cos \theta_{\infty}) \langle \mathbf{v}, \mathbf{n}_{S_t} \rangle \, dl
\]
\[
\leq 0
\]
whenever the following relation is valid:
\[
\int_{S_t} (\cos \theta - \cos \theta_{\infty}) \langle \mathbf{v}, \mathbf{n}_{S_t} \rangle \, dl \leq 0.
\]
Notice that the second term in (2.5) vanishes if we impose the (homogeneous) no-slip boundary condition on the rigid surface. However, as was pointed out by Huh and Scriven [16] (see also [28, 50]), this philosophy may not be suitable for a moving contact line problem. Notice that Ren and E [34] have also derived the energy dissipation (2.5) by using formal argument. However, they imposed additional condition on the contact line, which seems to be consistent with respect to the latest molecular dynamics. We emphasize that it is possible to derive (2.5) without any additional conditions on the contact line— we just employ the transport theorem and the classical Young law.

Before considering (2.6) let us characterize a steady state equilibrium. To this end, we define
\[
\Gamma_{\infty} := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D_R, \ x_3 = \eta_{\infty}(x_1, x_2)\}.\]
Then the set of equilibria is defined by
\[ \mathcal{E} := \{ (\mathbf{v}_\infty, \Gamma_\infty) \mid D(\mathbf{v}_\infty) = 0 \}. \] (2.7)
Notice that if \( \mathbf{v}_\infty \neq 0 \) we observe a rigid rotation of the fluid with a constant angular velocity at a steady state due to the condition \( D(\mathbf{v}_\infty) = 0 \) provided the slip boundary conditions on the rigid surface. For each equilibrium \( (\mathbf{v}_\infty, \Gamma_\infty) \in \mathcal{E} \) we have the relations
\[
\begin{align*}
\sigma \mathcal{H}_{\Gamma_\infty} &= p_0 - p_\infty \quad \text{on } \Gamma_\infty, \\
-\langle \mathbf{n}_{\Gamma_\infty}, \mathbf{n}_{\Sigma_\infty} \rangle &= \cos \theta_\infty \quad \text{on } S_\infty,
\end{align*}
\] (2.8)
where \( p_\infty \) describes the pressure of the fluid at the equilibrium \( (\mathbf{v}_\infty, \Gamma_\infty) \). Here, \( \mathcal{H}_{\Gamma_\infty} \) and \( \mathbf{n}_{\Gamma_\infty} \) are given by
\[ \mathcal{H}_{\Gamma_\infty} = -\text{div}_{\Gamma_\infty} \mathbf{n}_{\Gamma_\infty}, \quad \mathbf{n}_{\Gamma_\infty} = \frac{1}{\sqrt{1 + |\nabla_x \eta_\infty|^2}} \left( -\nabla_x \eta_\infty \right). \]
respectively. In the following, we characterize the quantity of \( \mathcal{H}_{\Gamma_\infty} \).

First, let us consider the case \( \mathbf{v}_\infty \equiv 0 \). In this case, we see that \( \mathcal{H}_{\Gamma_\infty} \) is a constant because there is no gravity effect. If it holds \( p_0 = p_\infty \), i.e., if the pressure of the fluid tends to \( p_0 > 0 \) as \( t \to \infty \), we see that \( \mathcal{H}_{\Gamma_\infty} = 0 \) because \( \sigma > 0 \). We mention that it is possible to observe this physical situation when \( p_0 \) denotes the atmospheric pressure or a saturated vapor pressure. Integrating the first equation of (2.8) over \( \Gamma_\infty \) and using the divergence theorem, we obtain
\[ (p_0 - p_\infty)|\Gamma_\infty| = \sigma \int_{S_\infty} \langle \mathbf{n}_{\Sigma_\infty}, \mathbf{n}_{\Gamma_\infty} \rangle \, dl = -|S_\infty| \cos \theta_\infty, \]
where \( |S_\infty| \) denotes the length of \( S_\infty \) and \( |\Gamma_\infty| \) the area of \( \Gamma_\infty \). Thus, we have the identity
\[ \mathcal{H}_{\Gamma_\infty} = -\frac{|S_\infty|}{|\Gamma_\infty|} \cos \theta_\infty, \]
which deduces that \( \theta_\infty \equiv \pi/2 \) since \( \mathcal{H}_{\Gamma_\infty} = 0 \) and \( |S_\infty| \neq 0 \). Hence, an equilibrium of the free surface can be characterized by a constant \( \eta_\infty \) independent of \( (x_1, x_2) \in D_R \) provided that the pressure of fluid converges to an external pressure and that the gravity potential is absent. In this case, we see that \( \eta_\infty \) is a constant, which minimizes the energy functional
\[ E[\eta_\infty] = \sigma \int_{\Gamma_\infty} \sqrt{1 + |\nabla_x \eta_\infty|^2} \, ds + \sigma (\cos \theta_\infty) \int_{S_\infty} \eta_\infty \, dl, \] (2.9)
see, e.g., Massari and Miranda [21, Sec. 3.8]. Physically, the first term of (2.9) represents the potential energy on the equilibrium of free surface, \( \Gamma_\infty \), while the second term of (2.9) represents the wetting energy due to the adhesion on the rigid surface. It is wildly known that \( \eta_\infty \) is not unique in general if we do not consider the gravitational effects, see Concus and Finn [7] (cf. Finn [11, Thm. 5.1]). In our situation, however, a height function at a steady equilibrium can be determined uniquely because the volume of fluid is conserved following from its incompressibility.

The case \( \mathbf{v}_\infty \neq 0 \) is much more involved. Before characterizing \( \mathcal{H}_{\Gamma_\infty} \), let us recall the set of velocity fields satisfying \( D(\mathbf{v}_\infty) = 0 \). It is known that \( \mathbf{v} \) satisfies \( D(\mathbf{v}) = 0 \) if and only if \( \mathbf{v} = \mathcal{C}x + \mathbf{b} \), where \( \mathcal{C} = \mathcal{C}(t) \) is a \( 3 \times 3 \) anti-symmetric matrix and \( \mathbf{b} = \mathbf{b}(t) \in \mathbb{R}^3 \). Here, \( \mathcal{C} \) and \( \mathbf{b} \) are given functions with respect to \( t \) and independent of \( x \in \mathbb{R}^3 \). In fact, in view of affine transformation, for any \( 3 \times 3 \) orthogonal matrices \( Q = Q(t) \) and any vector fields \( \mathbf{c} = c(t) \in \mathbb{R}^3 \) we can write
\[ x(\xi, t) = Q\xi + c \quad (\xi \in \Omega_0) \]
provided that the fluid is rigid rotating, where \( \xi \in \Omega_0 \) stands for a point of the Lagrangian coordinate. Differentiating this identity with respect to \( t \) implies
\[ \mathbf{v}(x(\xi, t), t) = (\partial_t Q)\xi + \partial_t c. \]
Since \( Q \) is orthogonal, we find that \( \xi = Q^\top x - Q^\top c \), which furnishes that
\[ \mathbf{v}(x(\xi, t), t) = (\partial_t Q)Q^\top x - (\partial_t Q)Q^\top c + \partial_t c \]
\[ \quad := \mathcal{C}x + \mathbf{b}. \]
Besides, differentiating $\mathbb{Q}\mathbb{Q}^\top = I$ with respect to $t$, we obtain $(\partial_t \mathbb{Q})\mathbb{Q}^\top + \mathbb{Q}(\partial_t \mathbb{Q}^\top) = I$, and thus we have $\mathcal{C} = -\mathcal{C}^\top$ for all $t > 0$. Especially, if $\mathcal{C}$ and $\mathbf{b}$ are constant in time, then the motion will be called *uniform* and we see that the pair of functions $(v_\infty, p_\infty)$ defined by

$$v_\infty(x) = \mathcal{C}x + \mathbf{b}, \quad p_\infty(x) = \frac{1}{2}|\mathcal{C}x|^2 + p_0$$

is a solution to the stationary Navier-Stokes equations satisfying the boundary conditions (2.8). Hence, in this case, we observe that $\mathcal{H}_{\Gamma_{\infty}}$ can be written by

$$\mathcal{H}_{\Gamma_{\infty}} = \frac{1}{2}|\mathcal{C}x|^2,$$

(2.10)

that is, $\mathcal{H}_{\Gamma_{\infty}}$ is a function with respect to $x$. We mention that $\mathcal{C} = 0$ implies $\mathbf{b} = 0$ due to the boundary conditions because we suppose the Navier slip boundary conditions on the rigid surface, and the kinematic condition on the free surface at equilibrium can be read as $(v_\infty, \mathbf{n}_{\Gamma_{\infty}}) = 0$. Hence, we have $v_\infty \neq 0$ if and only if $\mathcal{C} \neq 0$. For the corresponding discussions in $n$-dimensional space, we refer to [27, Sec. 2].

It remains to verify the relation (2.6), which causes a moving contact line problem difficult. A *sufficient* condition of ensuring energy dissipation is given by

$$(\theta - \theta_{\infty})\langle \mathbf{v}, \mathbf{n}_{S_t} \rangle \leq 0,$$

(2.11)

which is a physically reasonable condition for a moving contact line problem of a spreading droplet, see Fricke et al. [12, Remark 2]. However, it is not clear whether the condition (2.11) holds true for general contact line which is a physically reasonable condition for a moving contact line problem of a spreading droplet, see Fricke et al. [12, Remark 2].

Let us consider when we can obtain the relation $\theta_{\infty} = \pi/2$. According to the relation (2.10), it is *impossible* to obtain $\theta_{\infty} = \pi/2$ if we are interested in the case $v_\infty \neq 0$. Hence, it is necessary to impose $v_\infty = 0$ to gain $\theta_{\infty} = \pi/2$. Nevertheless, this condition means that the height function $\eta$ is a constant with respect to $t$ due to the Navier slip condition on the rigid surface in general if we have $\theta = \theta_{\infty}$. In fact, the kinematic equation of the contact line can be read as

$$\partial_t \eta = \langle \mathbf{v}, \mathbf{n}_{S_t} \rangle \text{ on } S_t.$$

Using the conditions (2.1) and $\langle \mathbf{n}_{\partial \Gamma_t}, \mathbf{n}_{\Sigma_t} \rangle = -\cos \theta_{\infty} = 0$, we observe

$$-\alpha \langle \mathbf{v}, \mathbf{n}_{S_t} \rangle = \beta \langle 2\mu \mathbf{D}(\mathbf{v})\mathbf{n}_{\Sigma_t}, \mathbf{n}_{S_t} \rangle \text{ on } S_t,$$

which yields

$$-\alpha \langle \partial_t \eta \rangle = \beta \langle 2\mu \mathbf{D}(\mathbf{v})\mathbf{n}_{\Sigma_t}, \mathbf{n}_{S_t} \rangle \text{ on } S_t.$$

Especially, it holds $\langle 2\mu \mathbf{D}(\mathbf{v})\mathbf{n}_{\Sigma_t}, \mathbf{n}_{S_t} \rangle = 0$ because the third component of $\mathbf{D}(\mathbf{v})\mathbf{n}_{\Sigma_t}$ and the first and second component of $\mathbf{n}_{S_t}$ are zero. Namely, we deduce that $\alpha \langle \partial_t \eta \rangle = 0$ on $S_t$. Since the number $\alpha \in [0, 1]$ should be chosen arbitrary, it is necessary to assume $\partial_t \eta = 0$ on $S_t$ when the contact angle is equivalent to $\pi/2$. This concludes that $\eta$ is a constant with respect to $t$. Therefore, if we impose $\theta = \theta_{\infty}$, not only the contact angle but also the contact line are identically equal to equilibrium state. This shows that the condition (2.11) is not suitable for general moving contact line problems.

To guarantee the energy dissipation (2.5), we directly deal with (2.6) provided that the flows are axisymmetric, and that the cavity is axisymmetric, cylindrical, and simply connected. Taking these assumptions into account, we see that $\theta$ and $\theta_{\infty}$ are independent of $x$ and $|S_t|$ is a universal constant independent of $x$ and $t$. In this case, from the transport theorem we arrive at

$$\int_{S_t} (\cos \theta - \cos \theta_{\infty})\langle \mathbf{v}, \mathbf{n}_{S_t} \rangle \, dl = (\cos \theta - \cos \theta_{\infty}) \int_{S_t} \langle \mathbf{v}, \mathbf{n}_{S_t} \rangle \, dl$$

$$= (\cos \theta - \cos \theta_{\infty}) \frac{d}{dt} |S_t|$$

$$= 0.$$
Namely, the condition (2.6) is valid for arbitrary contact angle \( \theta \). Hence, in our case, the relation (2.5) can be deduced to

\[
\frac{dE}{dt} = -2 \int_{\Omega} \mu|\nabla v|^2 \, dx - \alpha \int_{\Sigma^e_t} |v|^2 \, ds \leq 0.
\]

Therefore, the total available energy is a Lyapunov functional for the system due to \( \mu > 0 \) and \( \alpha \geq 0 \).

Summing up, we have shown the following results.

**Theorem 2.1.** Assume that \( \mu > 0 \) and \( \sigma > 0 \) are constants. Let the free interface \( \Gamma_t \) be given by (1.1) and let \( \Omega_t, \Sigma_t \), and \( \Sigma_t^* \) be as above. Define the total available energy \( E \) of the system and the set of equilibrium \( \mathcal{E} \) by (2.3) and (2.7), respectively. If the velocity field \( v \) of the fluid the free surface \( \Gamma_t \) are axisymmetric, the following assertions are valid.

(i) The functional \( E \) is conserved for smooth solutions.

(ii) The functional \( E \) is a strict Lyapunov functional, i.e., \( E \) is strictly decreasing for nonconstant smooth solutions.

(iii) The critical points of \( E \) are the equilibria of the system.

(iv) If \( (\alpha, \beta) \neq (0, 1) \), the equilibria are the constant pressure \( p_0 \), zero velocity, and \( \mathcal{H}_{\Gamma_\infty} = 0 \).

(v) If \( (\alpha, \beta) = (0, 1) \), the equilibria are \( (\mathbf{v}_\infty, p_\infty, \mathcal{H}_\infty) \) characterized by

\[
\mathbf{v}_\infty(x) = Cx + b, \quad p_\infty(x) = \frac{1}{2}|Cx|^2 + p_0, \quad \mathcal{H}_\infty = \frac{1}{2}|Cx|^2,
\]

where \( C \) is a constant \( 3 \times 3 \) anti-symmetric matrix and \( b \) is a constant vector. Especially, it holds \( \mathbf{v}_\infty \equiv 0 \) if and only if \( C = 0 \).

3. Reduction to a fixed reference configuration

In general, if we study the well-posedness of a free boundary problem it is required to transform the free boundary problem into a domain with a fixed boundary problem since the free boundary is a priori unknown.

Suppose that the free interface at time \( t \) is given as a graph over the fixed interface \( \Gamma_\star := \overline{D_R} \times \{0\} \). Namely, we suppose that there exists a height function \( \eta: \Gamma_\star \times J \to (-H, H) \) such that

\[
\Gamma(t) = \Gamma_t := \{ x \in \Gamma_\star \times (-H, H) \mid (x_1, x_2) \in \Gamma_\star, \; x_3 = \eta(x_1, x_2, t) \}
\]

for all \( t \in J \), where \( J = (0, T), \; T > 0 \). Let \( \chi \in C^\infty_c(\mathbb{R}; [0, 1]) \) be a bump function such that

\[
\chi(s) = \begin{cases} 
1 & \text{for } |s| \leq r/2, \\
0 & \text{for } |s| \geq r,
\end{cases}
\]

where \( 0 < r \leq H/3 \). We then define a mapping

\[
\Theta_\eta: \overline{D_R} \times (-H, 0) \times J \to \bigcup_{t \in J} \Omega_t \times \{t\},
\]

\[
\Theta_\eta(x, t) := x + \chi(x_3)\eta(x_1, x_2, t)e_3 =: x + \theta_\eta(x, t).
\]

A simple computation yields

\[
\nabla\Theta_\eta = I + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\partial_{x_3}\eta \chi & (\partial_{x_2}\eta)\chi & \eta\chi'
\end{pmatrix} =: I + \theta_\eta'.
\]

Hence, if the value of \( |\chi'|_{L^\infty(\Gamma_\star \times J)} \) is sufficiently small, we see that \( \nabla\Theta_\eta \) is a regular matrix and thus \( \Theta_\eta \) is invertible. For example, we can achieve this investigation if it holds

\[
|\eta|_{L^\infty(\Gamma_\star \times J)} \leq \frac{1}{2|\chi'|_{L^\infty(\mathbb{R})}}.
\]

We remark that \( |\chi'|_{L^\infty(\mathbb{R})} \) is bounded by a constant depending only on \( r \), and thus \( |\eta|_{L^\infty(\Gamma_\star \times J)} \) is bounded by a constant depending only on \( r \). In the following, we fix \( \chi \), choose \( r_0 \in (0, (2|\chi'|_{L^\infty(\mathbb{R})}^{-1}) \) suitably small, and suppose \( |\eta|_{L^\infty(\Gamma_\star \times J)} \leq r_0 \). Under the conditions stated above, we find that the inverse \( \Theta_\eta^{-1} \) of \( \Theta_\eta \) is well-defined and it transforms the free interface \( \Gamma_t \) to the flat interface \( \Gamma_\star \).
We denote the pull-back of \((\mathbf{u}, p - p_\infty)\) by \((\mathbf{u}, \pi)\), that is,
\[
\mathbf{u}(x, t) = \mathbf{v}(\Theta_\eta(x, t), t), \quad \pi(x, t) = p(\Theta_\eta(x, t), t) - p_\infty
\]
for \(x \in \overline{D_R} \times [-H, H]\) and \(t \in [0, \infty)\). Then, we can compute
\[
[\nabla p] \circ \Theta_\eta = \nabla_x \pi - M_0(\eta)\nabla_x \pi,
\]
\[
[\text{div } \mathbf{v}] \circ \Theta_\eta = \text{div}_x \mathbf{u} - M_0(\eta) \text{div}_x \mathbf{u},
\]
\[
[\Delta \mathbf{v}] \circ \Theta_\eta = \Delta_x \mathbf{u} - M_1(\eta): \nabla^2_x \mathbf{u} - M_2(\eta)\nabla_x \mathbf{u},
\]
\[
[\partial_t \mathbf{v}] \circ \Theta_\eta = \partial_t \mathbf{u} - \chi \partial_t \eta(1 + \chi')^{-1} \partial_{x^3} \mathbf{u},
\]
where we have set
\[
M_0(\eta) := \theta'_\eta (I + \theta'_\eta)^{-1},
\]
\[
M_1(\eta): \nabla^2 \mathbf{u} := \left[ 2 \text{sym}(\theta'_\eta (I + \theta'_\eta)^{-1}) - (I + \theta'_\eta)^{-1} \theta'_\eta (I + \theta'_\eta)^{-1} \right] \nabla^2_x \mathbf{u},
\]
\[
M_2(\eta)\nabla \mathbf{u} := ((\Delta_x \Theta_\eta^{-1}) \circ \Theta_\eta) \text{div } \mathbf{u}.
\]
Here, \text{sym} denotes the symmetric part of a matrix. Notice that the similar calculations can be found in, e.g., Köhne et al. [18, Sec. 2], Prüss and Simonett [33, Ch. 2], and Wilke [53, Sec. 1.1]. Besides, the assumption (3.1) implies \(V_{R_1} = (\partial_t \Theta_\eta) \cdot \mathbf{n}_{R_1} = (\partial_t \eta)/\sqrt{1 + |\nabla_x \eta|^2}\). Hence, we see that \((\mathbf{u}, \pi, \eta)\) satisfies
\[
\begin{aligned}
\partial_t \mathbf{u} - \mu \Delta \mathbf{u} + \nabla \pi &= \mathbf{F}(\mathbf{u}, \pi, \eta), \quad \text{in } \Omega_s \times (0, T), \\
\text{div } \mathbf{u} &= F_{\text{div}}(\mathbf{u}, \eta), \quad \text{in } \Omega_s \times (0, T), \\
\partial_t \eta - u_3 &= D(\mathbf{u}, \eta), \quad \text{in } \Gamma_s \times (0, T), \\
\mu(\partial_3 u_m + \partial_m u_3) &= K_m(\mathbf{u}, \eta), \quad \text{in } \Gamma_s \times (0, T), \\
2\mu \partial_3 u_3 - \sigma \Delta \gamma_{\Sigma} \eta &= K_3(\mathbf{u}, \eta), \quad \text{in } \Gamma_s \times (0, T), \\
P_{\Sigma_\tau}(2\mu D(\mathbf{u}) \mathbf{n}_{\Sigma_\tau}) &= G(\mathbf{u}, \eta) \quad \text{on } \Sigma_\tau \times (0, T), \\
\mathbf{u} \cdot \mathbf{n}_{\Sigma_\tau} &= 0 \quad \text{on } \Sigma_\tau \times (0, T), \\
\mu(\partial_3 u_m + \partial_m u_3) &= H_m(\mathbf{u}, \eta) \quad \text{on } B \times (0, T), \\
u_3 &= 0 \quad \text{on } B \times (0, T), \\
\mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in } \Omega_s, \\
\eta(0) &= \eta_0 \quad \text{on } \Gamma_s
\end{aligned}
\]
with
\[
\mathbf{F}(\mathbf{u}, \pi, \eta) := \chi \partial_t \eta (1 + \chi')^{-1} \partial_{x^3} \mathbf{u} - \mu(M_1(\eta): \nabla^2 \mathbf{u} + M_2(\eta)\nabla \mathbf{u}) + M_0(\eta)\nabla \pi,
\]
\[
F_{\text{div}}(\mathbf{u}, \eta) := M_0(\eta) \text{div } \mathbf{u},
\]
\[
D(\mathbf{u}, \eta) := -\mathbf{u}' \cdot \nabla_x \eta,
\]
\[
K_m(\mathbf{u}, \eta) := 2 \mu D(\mathbf{u}) \partial_3 \eta - |\nabla_x \eta|^2 \mu \partial_{x^3} u_m - \left( (1 + |\nabla_x \eta|^2) \mu \partial_{x^3} u_3 - \nabla_x \eta \cdot (\mu \nabla_x u_3) \right) \nabla_x \eta,
\]
\[
K_3(\mathbf{u}, \eta) := \nabla_x \eta \cdot (\mu \nabla_x u_3) + \sum_{m=1}^{2} \partial_3 \eta (\mu \nabla_x u_m) - |\nabla_x \eta|^2 \mu \partial_{x^3} u_3
\]
\[
+ \sigma \left\{ \text{div}_x \left( \frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}} \right) - \Delta \eta \right\},
\]
\[
G(\mathbf{u}, \eta) := P_{\Sigma_\tau} \left( \mu(M_0(\eta)\nabla \mathbf{u} + (\nabla \mathbf{u})^T (M_0(\eta))^T \mathbf{n}_{\Sigma_\tau} \right),
\]
\[
H_m(\mathbf{u}, \eta) := -K_m(\mathbf{u}, \eta)
\]
for $m = 1, 2$, where $\mathbf{u}' = (u_1, u_2)^T$. Here, $T$ is a positive constant and we have used the assumption $p_0 = p_\infty$ and set
\[
\Omega_* := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D_R, \ x_3 \in (-H, 0)\},
\Gamma_* := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D_R, \ x_3 = 0\},
\Sigma_* := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \partial D_R, \ x_3 \in (-H, 0)\},
\]
\[
B := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D_R, \ x_3 = -H\},
S_* := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \partial D_R, \ x_3 = 0\},
\]
\[(3.4)\]

Notice that the right-hand members in this system are nonlinear and lower order terms. Our aim in this paper is to consider the reformulation of the transformed problem (3.2) in abstract form $Lz = N(z)$ with $z = (\mathbf{u}, \pi, \eta)$, where $L$ is said to be the \textit{principal linearization}. In order to follow the strategy due to Köhne et al. [18] (and see also Prüss and Simonett [33]), we first show that $L$ has maximal regularity.

4. Model problems

The proof of maximal regularity of the principal linearization $L$ is based on a localization technique. In fact, using change of coordinates, we may reduce the problem to the following types of model problems:

(i) The Stokes equations in the whole space (without any boundary conditions);
(ii) The Stokes equations in a half space with slip boundary conditions;
(iii) The Stokes equations in a half space with free boundary conditions;
(iv) The Stokes equations in a quarter space with slip boundary conditions;
(v) The Stokes equations in a quarter space with slip boundary conditions on one part of the boundary and free boundary conditions on the other part.

Here and in the following, a wedge domain with an angle equal to $\pi/2$ is said to be a \textit{quarter space}. Details of a localization procedure will be left to the next section. Compared with the pioneering work by Wilke [53], in the model problem of type (v), the Neumann trace of the height function is not required to be vanished in our discussion below.

This section aims to state the maximal $L^p - L^q$-regularity properties for the linearized problems (i)–(v).

To this end, we define
\[
\dot{H}^{1,q}(D) := \{w \in L^1_{\text{loc}}(D) \mid \nabla w \in L^q(D)\}
\]
for $1 < q < \infty$ and a domain $D \subset \mathbb{R}^3$. Besides, for $S \subset \partial D$, we define
\[
\dot{H}^q_\partial(D) := \{w \in L^1_{\text{loc}}(D) \mid \nabla w \in L^q(D), \ w = 0 \text{ on } S\};
\]
in particular, $\dot{H}^q_\partial(D) := \dot{H}^{1,q}(D)$. Then, the space $\dot{H}^{-1,q}_S(D)$ is defined as
\[
\dot{H}^{-1,q}_S(D) := \left(\dot{H}^{1,q}_0(\partial D, S(D))\right)^*;
\]
with conventions $\dot{H}^{-1,q}(D) = \dot{H}^{-1,q}_0(D)$ and $\partial \dot{H}^{-1,q}(D) = \partial \dot{H}^{-1,q}_0(D)$.

For simplicity of notation, we write $\partial_j$ instead of $\partial_j/\partial x_j$, $j = 1, 2, 3$, if there is no confusion.

4.1. The Stokes equations in the whole space. Consider the problem
\[
\begin{aligned}
\partial_t \mathbf{u} - \mu \Delta \mathbf{u} + \nabla \pi &= \mathbf{f}, \quad \text{in } \mathbb{R}^3 \times J, \\
\text{div } \mathbf{u} &= f_{\text{div}}, \quad \text{in } \mathbb{R}^3 \times J, \\
\mathbf{u}(0) &= \mathbf{u}_0, \quad \text{in } \mathbb{R}^3.
\end{aligned}
\]
\[(4.1)\]

Here and in the following, we use the notation $J = (0, T)$ for $0 < T < \infty$. According to Prüss and Simonett [33, Sec. 7.1], we know the following theorem.

**Theorem 4.1.** Let $1 < p, q < \infty$, $1/p < \delta \leq 1$, and $T > 0$. The problem (4.1) has a unique solution $(\mathbf{u}, \pi)$ satisfying
\[
\mathbf{u} \in H^{1,p}_\delta(J; L^q(\mathbb{R}^3)^3) \cap L^p_\delta(J; H^{2,q}(\mathbb{R}^3)^3),
\]
\[
\pi \in L^p_\delta(J; \dot{H}^{1,q}(\mathbb{R}^3)).
\]
4.2. The Stokes equations in a half space. We next consider the Stokes equations in a half space

\[
\begin{cases}
\partial_t \mathbf{u} - \mu \Delta \mathbf{u} + \nabla \pi = \mathbf{f}, & \text{in } \mathbb{R}_+^3 \times J, \\
\text{div } \mathbf{u} = f_{\text{div}}, & \text{in } \mathbb{R}_+^3 \times J, \\
\mathbf{u} = \mathbf{g}, & \text{on } \partial \mathbb{R}_+^3 \times J, \\
\mathbf{u}(0) = \mathbf{u}_0, & \text{in } \mathbb{R}_+^3.
\end{cases}
\]  

(4.2)

where \( m = 1, 2 \) and we have used the notation

\[
\mathbb{R}_+^3 := \{ x = (x_1, x_2, x_3) | (x_1, x_2) \in \mathbb{R}^2, \quad x_3 > 0 \},
\]

\[
\partial \mathbb{R}_+^3 := \{ x = (x_1, x_2, x_3) | (x_1, x_2) \in \mathbb{R}^2, \quad x_3 = 0 \}.
\]

Besides, in order to describe the compatibility conditions for the problems (4.2) and (4.3), we introduce the space \( \tilde{H}^{-1, q}(\mathbb{R}_+^3) \) as the set of all \( (\varphi_1, \varphi_2) \in L^q(\mathbb{R}_+^3) \times B^{2-1/q}_q(\partial \mathbb{R}_+^3)^3 \) that satisfy the regularity property \( (\varphi_1, \varphi_2 \cdot \mathbf{n}_{\partial \mathbb{R}_+^3}) \in \tilde{H}^{-1, q}(\mathbb{R}_+^3) \). If we adopt the notation

\[
((\varphi_1, \varphi_2 \cdot \mathbf{n}_{\partial \mathbb{R}_+^3}) | \phi)_{\mathbb{R}_+^3} := -(\varphi_1 | \phi)_{\mathbb{R}_+^3} + (\varphi_2 \cdot \mathbf{n}_{\partial \mathbb{R}_+^3} | \phi)_{\mathbb{R}_+^3}
\]

for any \( \phi \in \tilde{H}^{1, q}(\mathbb{R}_+^3) \), then from the divergence equation we have the conditions

\[
((f_{\text{div}}, \mathbf{g} \cdot \mathbf{n}_{\partial \mathbb{R}_+^3}) | \phi)_{\mathbb{R}_+^3} = -(\mathbf{u} | \nabla \phi)_{\mathbb{R}_+^3},
\]

\[
((f_{\text{div}}, h_3) | \phi)_{\mathbb{R}_+^3} = -(\mathbf{u} | \nabla \phi)_{\mathbb{R}_+^3},
\]

for any \( \phi \in \tilde{H}^{1, q}(\mathbb{R}_+^3) \) when we deal with (4.2) and (4.3), respectively. Furthermore, let us introduce function spaces

\[
E_{1, \delta}(J; D) := \mathcal{H}^{1, p}_q(J; L^q(D)^3) \cap L^q_p(J; H^{2, q}(D)^3),
\]

\[
E_{2, \delta}(J; D) := L^q_p(J; H^{1, q}(D)),
\]

\[
F_{0, \delta}(J; D) := L^q_p(J; L^q(D)^3),
\]

\[
F_{1, \delta}(J; D) := H^{1, p}_q(J; \tilde{H}^{-1, q}(D)) \cap L^q_p(J; H^{1, q}(D)),
\]

\[
F_{2, \delta}(J; \partial D) := F^{1/2-1/(2q)}_{p, q, \delta}(J; L^q(\partial D)) \cap L^q_p(J; B^{-1, q}_q(\partial D)),
\]

\[
F_{3, \delta}(J; \partial D) := F^{1/2-1/(2q)}_{p, q, \delta}(J; L^q(\partial D)) \cap L^q_p(J; B^{-1, q}_q(\partial D))
\]

for \( J \subseteq \mathbb{R}_+ \) and \( D \subseteq \mathbb{R}^3 \). Then the following theorems are well-known (cf. Prüss and Simonett [33, Sec. 7.2]).

**Theorem 4.2.** Let \( 1 < p, q < \infty, 1/p < \delta \leq 1, 1/p + 1/(2q) \neq \delta, \quad T > 0, \) and \( J = (0, T) \). Then there exists a unique solution \( (\mathbf{u}, \pi) \) to the equations (4.2) with regularity \( \mathbf{u} \in E_{1, \delta}(J; \mathbb{R}_+^3) \) and \( \pi \in E_{2, \delta}(J; \mathbb{R}_+^3) \) if and only if

(a) \( \mathbf{f} \in F_{0, \delta}(J; \mathbb{R}_+^3) \);

(b) \( f_{\text{div}} \in F_{1, \delta}(J; \mathbb{R}_+^3) \);
(c) \( g \in \mathbb{F}_{3,\delta}^3(J;\partial \mathbb{R}^3_+) \) and \( g(0) = \text{Tr}_{\partial \mathbb{R}^3_+} [u_0] \) if \( 1/p + 1/(2q) < \delta \);
(d) \( (f_{\text{div}}, g, n_{\partial \mathbb{R}^3_+}) \in H_{3}^{1,p}(J;\tilde{H}^{-1,q}((\mathbb{R}^3_+)^3)) \);
(e) \( u_0 \in B_{3,p}^{2,(\delta-1/p)}(\mathbb{R}^3_+)^3 \) and \( \text{div} u_0 = f_{\text{div}}(0) \).

Furthermore, the solution \((u, \pi)\) depends continuously on the data in the corresponding spaces.

**Theorem 4.3.** Suppose \( 1 < p, q < \infty, 1/p < \delta \leq 1, 1/p + 1/(2q) \notin \{\delta - 1/2, \delta\}, T > 0 \), and \( J = (0,T) \). The equations (4.3) admits a unique solution \((u, \pi)\) in the class \( u \in E_{1,\delta}(J;\mathbb{R}^4_+), \pi \in E_{2,\delta}(J;\mathbb{R}^4_+) \) if and only if

(a) \( f \in F_{0,\delta}(J;\mathbb{R}^4_+); \)
(b) \( f_{\text{div}} \in F_{1,\delta}(J;\mathbb{R}^4_+); \)
(c) \( h_m \in F_{2,\delta}(J;\mathbb{R}^4_+) \) and \( h_m(0) = \text{Tr}_{\partial \mathbb{R}^3_+} [\mu(\partial_3 u_{0,m} + \partial_3 u_{0,3})] \) for \( m = 1, 2 \) if \( 1/p + 1/(2q) < \delta - 1/2; \)
(d) \( h_3 \in F_{3,\delta}(J;\mathbb{R}^4_+) \) and \( h_3(0) = \text{Tr}_{\partial \mathbb{R}^3_+} [u_{0,3}] \) for \( m = 1, 2 \) if \( 1/p + 1/(2q) < \delta; \)
(e) \( (f_{\text{div}}, h_3) \in H_{3}^{1,p}(J;\tilde{H}^{-1,q}(\mathbb{R}^4_+)); \)
(f) \( u_0 \in B_{3,p}^{2,(\delta-1/p)}(\mathbb{R}^3_+)^3 \) and \( \text{div} u_0 = f_{\text{div}}(0) \).

In addition, the solution map \((f, f_{\text{div}}, h, u_0) \mapsto (u, \pi)\) is continuous between the corresponding spaces.

**Remark 4.4.** In view of trace theorems, the statements in Prüss and Simonett [33, Thm. 7.2.1] should be corrected as above. See, e.g., [19] for the details.

Consider the Stokes equations with free boundary conditions

\[
\begin{aligned}
\partial_t u - \mu \Delta u + \nabla \pi &= f, \quad \text{in } \mathbb{R}^3_+ \times J, \\
\text{div} u &= f_{\text{div}}, \quad \text{in } \mathbb{R}^3_+ \times J, \\
\partial_t \eta - u_3 &= d, \quad \text{on } \partial \mathbb{R}^3_+ \times J, \\
\mu(\partial_3 u_m + \partial_3 u_{0,3}) &= k_m, \quad \text{on } \partial \mathbb{R}^3_+ \times J, \\
2\mu \partial_3 u_3 - \pi - \sigma \Delta_{\partial \mathbb{R}^3_+} \eta &= k_3, \quad \text{on } \partial \mathbb{R}^3_+ \times J, \\
u(0) &= u_0, \quad \text{in } \mathbb{R}^3_+, \\
\eta(0) &= \eta_0, \quad \text{on } \partial \mathbb{R}^3_+,
\end{aligned}
\]

where \( m = 1, 2 \) and \( \Delta_{\partial \mathbb{R}^3_+} = \sum_{j=1}^2 \partial_j^2 \). The maximal \( L^p - L^q \) regularity theorem for this problem has been studied by Shibata [38,39] (cf. Shibata and Shimizu [42]). However, we will show that more optimal regularity results can be obtained. To this end, we introduce a function space

\[
E_{3,\delta}(J;\partial D) := F_{p,q,\delta}^{1/2-1/(2q)}(J;L^q(\partial D)) \cap L_p^q(J;B_3^{1-1/q}(\partial D)),
\]

\[
E_{4,\delta}(J;\partial D) := F_{p,q,\delta}^{2-1/(2q)}(J;L^q(\partial D)) \cap H_{3}^{1,p}(J;\tilde{H}_0^{2-1/q}(\partial D)) \cap L_p^q(J;B_3^{1-1/q}(\partial D))
\]

for \( J \subset \mathbb{R}^+ \) and \( D \subset \mathbb{R}^3 \).

**Theorem 4.5.** Let \( 1 < p, q < \infty, 1/p < \delta \leq 1, 1/p + 1/(2q) \neq \delta, T > 0 \), and \( J = (0,T) \). The problem (4.5) has a unique solution \((u, \pi, \eta)\) with \( u \in E_{1,\delta}(J;\mathbb{R}^4_+), \pi \in E_{2,\delta}(J;\mathbb{R}^4_+), \text{Tr}_{\partial \mathbb{R}^3_+}[\pi] \in E_{3,\delta}(J;\partial \mathbb{R}^3_+), \) and \( \eta \in E_{4,\delta}(J;\partial \mathbb{R}^3_+) \) if and only if

(a) \( f \in F_{0,\delta}(J;\mathbb{R}^4_+); \)
(b) \( f_{\text{div}} \in F_{1,\delta}(J;\mathbb{R}^4_+); \)
(c) \( d \in F_{2,\delta}(J;\partial \mathbb{R}^3_+); \)
(d) \( k_j \in F_{3,\delta}(J;\mathbb{R}^4_+) \) for \( j = 1, 2, 3 \) and \( k_m(0) = \text{Tr}_{\partial \mathbb{R}^3_+} [\mu(\partial_3 u_{0,m} + \partial_3 u_{0,3})] \) for \( m = 1, 2 \) if \( 1/p + 1/(2q) < \delta - 1/2; \)
(e) \( u_0 \in B_{3,p}^{2,(\delta-1/p)}(\mathbb{R}^3_+)^3 \) and \( \text{div} u_0 = f_{\text{div}}(0); \)
(f) \( \eta_0 \in B_{3,p}^{2,(\delta-1/p)}(\partial \mathbb{R}^3_+). \)

Besides, the solution \((u, \pi, h)\) depends continuously on the data in the corresponding spaces.
Proof. We only prove the sufficient part since the necessary part immediately follows from trace theorems. In order to show the sufficient part, we consider the following shifted problem:

\[
\left\{ \begin{array}{ll}
\partial_t u + \omega u - \mu \Delta u + \nabla \pi = f, & \text{in } \mathbb{R}^3_+ \times \mathbb{R}_+ , \\
\text{div } u = f_{\text{div}}, & \text{in } \mathbb{R}^3_+ \times \mathbb{R}_+ , \\
\partial_t \eta + \omega \eta - u_3 = d, & \text{on } \partial \mathbb{R}^3_+ \times \mathbb{R}_+ , \\
\mu (\partial_3 u_m + \partial_m u_3) = k_m, & \text{on } \partial \mathbb{R}^3_+ \times \mathbb{R}_+ , \\
2 \mu \partial_3 u_3 - \pi - \sigma \Delta_{\partial \mathbb{R}^3_+} \eta = k_3, & \text{on } \partial \mathbb{R}^3_+ \times \mathbb{R}_+ , \\
u(0) = u_0, & \text{in } \mathbb{R}^3_+ , \\
\eta(0) = \eta_0, & \text{on } \partial \mathbb{R}^3_+ , \\
\end{array} \right.
\]

(4.6)

where $m = 1, 2$ and $\omega > 0$ denotes a (possibly large) shift parameter. We first show that there exists $\omega_0 > 0$ such that for each $\omega \geq \omega_0$, the system (4.6) admits a unique solution $(u, \pi, \eta)$ in the corresponding regularity classes, and then we will prove that (4.5) has a unique solution in the right regularity classes — this is due to the fact that several trace theorems can be applied not only for $J$ but also for the semi-infinite interval $\mathbb{R}_+$. However, it is known that we can drop off the parameter $\omega$ by restricting the time interval to be finite, and thus it is sufficient to consider (4.6).

Without loss of generality, we may assume that $(f, f_{\text{div}}, d, u_0, \eta_0) = 0$ and $k(0) = 0$ if $1/p + 1/(2q) < \delta$. This can be observed as follows: Let us first consider the problem with $\omega = 0$. Define

\[
\eta_1(t) := \left[ 2 e^{- (I - \Delta_{\partial \mathbb{R}^3_+})^{1/2} t} - e^{- 2 (I - \Delta_{\partial \mathbb{R}^3_+})^{1/2} t} \right] \eta_0
\]

\[
+ \left[ e^{- (I - \Delta_{\partial \mathbb{R}^3_+}) t} - e^{- 2 (I - \Delta_{\partial \mathbb{R}^3_+}) t} \right] \left( I - \Delta_{\partial \mathbb{R}^3_+} \right)^{-1} (u_0|_{\partial \mathbb{R}^3_+} + d|_{t=0})
\]

\[
= \eta_{1.1}(t) + \eta_{1.2}(t)
\]

for any $t \geq 0$. Since the operators $T_{t=0} \circ \Delta_{\partial \mathbb{R}^3_+}$ and $\text{Tr}_{\partial \mathbb{R}^3_+} \circ T_{t=0}$ coincide (cf. Lindemulder [19, pp. 88]), we have $d|_{t=0} \in B^{2(\delta - 1/p - 1/q)}(\partial \mathbb{R}^3_+)$. Since $\eta_0 \in B^{2+\delta - 1/p - 1/q}(\partial \mathbb{R}^3_+)$ and $u_0|_{\partial \mathbb{R}^3_+} \in B^{2(\delta - 1/p - 1/q)}(\partial \mathbb{R}^3_+)$, by the standard semigroup theory, we see that

\[
\eta_1 \in E_{4,\delta}(\mathbb{R}_+; \partial \mathbb{R}^3_+)
\]

with $\eta_1(0) = \eta_0$ and $\partial_t \eta_1(0) = u_0|_{\partial \mathbb{R}^3_+} + d|_{t=0}$. Indeed, we observe that $\eta_{1.1}(0) = \eta_0$, $\partial_t \eta_{1.1}(0) = 0$ and $\eta_{1.2}(0) = 0$, $\partial_t \eta_{1.2}(0) = u_0|_{\partial \mathbb{R}^3_+} + d|_{t=0}$. Hence, by [23, Thm. 4.2], it holds

\[
\eta_{1.1} \in F^{2-\delta/q}_p \left( \mathbb{R}_+; L^q(\partial \mathbb{R}^3_+) \right) \cap F^0_{p,1,\delta} \left( \mathbb{R}_+; B^{2-\delta/q}_q(\partial \mathbb{R}^3_+) \right) \Rightarrow E_{4,\delta}(\mathbb{R}_+; \partial \mathbb{R}^3_+)
\]

due to $\eta_0 \in B^{2+\delta - 1/p - 1/q}(\partial \mathbb{R}^3_+)$. Besides, by $\left( I - \Delta_{\partial \mathbb{R}^3_+} \right)^{-1} u_0|_{\partial \mathbb{R}^3_+} + d|_{t=0} \in B^{2+2(\delta - 1/p - 1/q)}(\partial \mathbb{R}^3_+)$, we also have

\[
\eta_{1.2} \in F^{2-\delta/q}_p \left( \mathbb{R}_+; L^q(\partial \mathbb{R}^3_+) \right) \cap F^0_{p,1,\delta} \left( \mathbb{R}_+; B^{2-\delta/q}_q(\partial \mathbb{R}^3_+) \right) \Rightarrow E_{4,\delta}(\mathbb{R}_+; \partial \mathbb{R}^3_+)
\]

Hence, we have $\eta_1 \in E_{4,\delta}(\mathbb{R}_+; \partial \mathbb{R}^3_+)$. We now extend $u_0$ to all of $\mathbb{R}^3$ in the class $B^{2(\delta - 1/p)}_q(\mathbb{R}^3)$, and extend $f$ by zero, where those are denoted by $c_{\mathbb{R}^3}[u_0]$ and $E^\infty_{x_3}[f]$, respectively. Then for each $\omega > 0$ the problem

\[
\left\{ \begin{array}{ll}
\partial_t u_1 + \omega u_1 - \mu \Delta u_1 = E^\infty_{x_3}[f], & \text{in } \mathbb{R}^3 \times \mathbb{R}_+ , \\
u_1(0) = c_{\mathbb{R}^3}[u_0], & \text{in } \mathbb{R}^3
\end{array} \right.
\]

has a unique solution $u_1 \in H^{1,p}_\omega(\mathbb{R}^3; L^q(\mathbb{R}^3)) \cap L^q_0(\mathbb{R}^3; H^{2,q}(\mathbb{R}^3))$, see, e.g., [33, Thm. 6.1.8, 4.4.4]. Thus $u_2 := u - u_1|_{\mathbb{R}_+^3}$ and $\pi_2 := \pi$ should solve

\[
\left\{ \begin{array}{ll}
\partial_t u_2 + \omega u_2 - \mu \Delta u_2 + \nabla \pi_2 = 0, & \text{in } \mathbb{R}^3 \times \mathbb{R}_+ , \\
\text{div } u_2 = f_{\text{div},2}, & \text{in } \mathbb{R}^3 \times \mathbb{R}_+ , \\
u_2(0) = 0, & \text{in } \mathbb{R}^3
\end{array} \right.
\]
where we have set \( f_{\text{div}, 2} := f_{\text{div}} - \text{div} (u_1|_{\mathbb{R}^3_+}) \in F_{1,4} (\mathbb{R}^+_+; \mathbb{R}^3_+) \). Here, \( f_{\text{div}, 2} \) has vanishing trace at \( t = 0 \).

Extending \( u_2 \) by even reflection with respect to \( x_3 \), we consider

\[
\begin{cases}
\partial_t u_3 + \omega u_3 - \mu \Delta u_3 + \nabla \pi_3 = 0, & \text{in } \mathbb{R}^3 \times \mathbb{R}^+_+,
\text{div } u_3 = E^\infty_{x_3} [f_{\text{div}, 2}], & \text{in } \mathbb{R}^3 \times \mathbb{R}^+_+,
\end{cases}
\]

with \( E^\infty_{x_3} \) denotes even extension with respect to \( x_3 \). According to \([33, \text{Sec. 7.1}]\), we can obtain \((u_3, \pi_3)\) in the right regularity class. Define \( u_4 := u_2 - u_3|_{\mathbb{R}^3_+}, \pi_4 := \pi_2 - \pi_3|_{\mathbb{R}^3_+}, \) and \( \eta_4 := \eta - \eta_1 \). Then we see that the pair \((u_4, \pi_4, \eta_4)\) satisfies the problem \((4.6)\) with \((f, f_{\text{div}}, d, u_0, \eta_0) = 0\) and the boundary data

\[
\begin{align*}
d_4 &= d - \omega \eta_1 + u_{1,3}|_{\mathbb{R}^3_+} + \omega_{3,3}|_{\mathbb{R}^3_+} \\
k_{4,m} &= k_m - \text{Tr}_{\partial \mathbb{R}^3_+} [\mu (\partial_3 (u_{1,3}|_{\mathbb{R}^3_+} + \omega_{3,3}|_{\mathbb{R}^3_+}) + \partial_3 (u_{1,3}|_{\mathbb{R}^3_+} + \omega_{3,3}|_{\mathbb{R}^3_+}))] \\
k_{4,3} &= k_3 - \text{Tr}_{\partial \mathbb{R}^3_+} [(2 \mu \partial_3 (u_{1,3}|_{\mathbb{R}^3_+} + \omega_{3,3}|_{\mathbb{R}^3_+}) - \sigma \Delta_{\partial \mathbb{R}^3_+} \eta_1)].
\end{align*}
\]

Notice that \( \partial_3 \eta_4|_{t=0} = 0 \) due to the construction. It is not difficult to observe that \( k_{4,1}, k_{4,2}, k_{4,3} \) have the right regularity and have vanishing traces at \( t = 0 \) whenever they exist. Summing up, for a fixed parameter \( \omega > 0 \), we see that \((u_4, \pi_4, \eta_4)\) solves

\[
\begin{cases}
\partial_t u_4 + \omega u_4 - \mu \Delta u_4 + \nabla \pi_4 = 0, & \text{in } \mathbb{R}^3_+ \times \mathbb{R}^+_+,
\text{div } u_4 = 0, & \text{in } \mathbb{R}^3_+ \times \mathbb{R}^+_+,
\partial_3 \eta_4 - u_{4,3} = d_4, & \text{on } \partial \mathbb{R}^3_+ \times \mathbb{R}^+_+,
2 \mu \partial_3 u_{4,3} - \pi_4 - \sigma \Delta_{\partial \mathbb{R}^3_+} \eta_4 = k_{4,3}, & \text{on } \partial \mathbb{R}^3_+ \times \mathbb{R}^+_+,
\end{cases}
\]

with \( m = 1, 2 \). This shows that we may suppose \((f, f_{\text{div}}, d, u_0, \eta_0) = 0\) and \( k(0) = 0 \) if \( 1/p + 1/(2q) < \delta \).

In the following, we suppose \( 1/p + 1/(2q) \neq \delta \). We now consider \((4.6)\) with \((f, f_{\text{div}}, d, u_0, \eta_0) = 0\), where \( k \) has vanishing trace at \( t = 0 \) whenever it exists. Let \( E_t \) be an extension operator with respect to \( t \). Notice that we can extend \( k \) trivially to all of \( t \in \mathbb{R} \). We show that the unknowns \( u, \pi, \eta \) vanishes on \( \{t < 0\} \) if \((u, \pi, \eta)\) satisfies

\[
\begin{cases}
\partial_t u + \omega u - \mu \Delta u + \nabla \pi = 0, & \text{in } \mathbb{R}^3_+ \times \mathbb{R}^+_+,
\text{div } u = 0, & \text{in } \mathbb{R}^3_+ \times \mathbb{R}^+_+,
\partial_3 \eta - u_3 = 0, & \text{on } \partial \mathbb{R}^3_+ \times \mathbb{R}^+_+,
2 \mu \partial_3 u_3 - \pi - \sigma \Delta_{\partial \mathbb{R}^3_+} \eta = k_3, & \text{on } \partial \mathbb{R}^3_+ \times \mathbb{R}^+_+,
\end{cases}
\]

with \( \omega > 0 \). This shows that we may suppose \((f, f_{\text{div}}, d, u_0, \eta_0) = 0\) and \( k(0) = 0 \) if \( 1/p + 1/(2q) < \delta \).
with \( k = 0 \) on \( \{ t < 0 \} \). To this end, for \( \tilde{f} \in L^q \left( -\infty, T; L^q(\mathbb{R}_+^3) \right) \) let \((\tilde{u}, \tilde{\pi}, \tilde{\eta})\) be the solution to the dual backward problem

\[
\begin{cases}
-\partial_t \tilde{u} + \omega \tilde{u} - \mu \Delta \tilde{u} + \nabla \tilde{\pi} = \tilde{f}, & \text{in } \mathbb{R}_+^3 \times (-\infty, T), \\
\nabla \cdot \tilde{u} = 0, & \text{in } \mathbb{R}_+^3 \times (-\infty, T), \\
-\partial_t \tilde{\eta} + \omega \tilde{\eta} - \tilde{u}_3 = 0, & \text{on } \partial \mathbb{R}_+^3 \times (-\infty, T), \\
\mu (\partial_3 \tilde{u}_m + \partial_m \tilde{u}_3) = 0, & \text{on } \partial \mathbb{R}_+^3 \times (-\infty, T), \\
2\mu \partial_3 \tilde{u}_3 - \bar{\pi} - \sigma_{\partial \mathbb{R}_+^3} \tilde{\eta} = 0, & \text{on } \partial \mathbb{R}_+^3 \times (-\infty, T), \\
\tilde{u}(T) = 0, & \text{in } \mathbb{R}_+^3, \\
\tilde{\eta}(T) = 0, & \text{on } \partial \mathbb{R}_+^3 \end{cases}
\]

with \( T \in \mathbb{R} \). Integration by parts gives

\[
0 = \int_{-\infty}^{T} \left( \partial_t u - \partial_t \tilde{u} + \omega (u - \tilde{u}) - \mu \Delta u + \nabla \pi \right)_{\mathbb{R}_+^3} dt.
\]

\[
= \int_{-\infty}^{T} (u | -\partial_t \tilde{u} + \omega (u - \tilde{u}) - \mu \Delta u + \nabla \pi)_{\mathbb{R}_+^3} dt + \int_{-\infty}^{T} \left( (u_3 | 2\mu \partial_3 \tilde{u}_3)_{\partial \mathbb{R}_+^3} - (\sigma_{\partial \mathbb{R}_+^3} \eta | \tilde{u}_3)_{\partial \mathbb{R}_+^3} \right) dt
\]

\[
= \int_{-\infty}^{T} (u | -\partial_t \tilde{u} + \omega (u - \tilde{u}) - \mu \Delta u + \nabla \pi)_{\mathbb{R}_+^3} dt + \int_{-\infty}^{T} \left( (u_3 | 2\mu \partial_3 \tilde{u}_3)_{\partial \mathbb{R}_+^3} - (\sigma_{\partial \mathbb{R}_+^3} \eta | -\partial_t \tilde{\eta} + \omega \tilde{\eta})_{\partial \mathbb{R}_+^3} \right) dt,
\]

where we have used the fact \( u(0) = \tilde{u}(T) = 0 \) and \( \eta(0) = \tilde{\eta}(T) = 0 \). We also see that

\[
\int_{-\infty}^{T} (u | \nabla \pi)_{\mathbb{R}_+^3} dt = \int_{-\infty}^{T} (u_3 | \tilde{\pi})_{\partial \mathbb{R}_+^3} dt
\]

\[
= \int_{-\infty}^{T} (u_3 | 2\mu \partial_3 \tilde{u}_3 - \sigma_{\partial \mathbb{R}_+^3} \tilde{\eta})_{\partial \mathbb{R}_+^3} dt
\]

\[
= \int_{-\infty}^{T} \left( (\partial_t \eta + \omega \eta | -\sigma_{\partial \mathbb{R}_+^3} \tilde{\eta})_{\partial \mathbb{R}_+^3} + (u_3 | 2\mu \partial_3 \tilde{u}_3)_{\partial \mathbb{R}_+^3} \right) dt.
\]

Summarizing, we obtain

\[
0 = \int_{-\infty}^{T} (u | -\partial_t \tilde{u} + \omega (u - \tilde{u}) - \mu \Delta u + \nabla \pi)_{\mathbb{R}_+^3} dt = \int_{-\infty}^{T} (u | \tilde{f})_{\mathbb{R}_+^3} dt.
\]

Since \( \tilde{f} \in L^q \left( -\infty, T; L^q(\mathbb{R}_+^3) \right) \) should be chosen arbitrary, we have \( u \equiv 0 \) on \( (-\infty, T) \) for any \( T \in \mathbb{R} \) whenever given data vanishes. Recalling that \( k \) vanishes on \( \{ t < 0 \} \), we can conclude that \( u \equiv 0 \) on \( \{ t < 0 \} \). In addition, we can also show \( \eta = 0 \) and \( \pi = 0 \) on \( \{ t < 0 \} \). Therefore, in the following, let \( E_t \) denotes the zero extension operator with respect to \( t \).

Let \( L \) be the (bilateral) Laplace transform with respect to a time variable \( t \) defined by

\[
L[f(t)](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t) dt = \int_{-\infty}^{\infty} e^{-i\tau t} e^{-\gamma t} f(t) dt = \mathcal{F}_\tau[e^{-\gamma t} f(t)](\tau),
\]

where \( \gamma + i \tau \in \mathbb{C} \) is a parameter with real numbers \( \gamma \) and \( \tau \). Extending the unknowns to \( \{ t < 0 \} \) trivially by \( E_t \), and then applying the Laplace transform to (4.6), we find

\[
\begin{cases}
\\zeta L[E_t[u]] - \mu L[E_t[u]] + \nabla L[E_t[\pi]] = 0, & \text{in } \mathbb{R}_+^3, \\
\nabla L[E_t[u]] = 0, & \text{in } \mathbb{R}_+^3, \\
\zeta L[E_t[\eta]] - L[E_t[u_3]] = 0, & \text{on } \partial \mathbb{R}_+^3, \\
\\mu (\partial_3 L[E_t[u_m]] + \partial_m L[E_t[u_3]]) = L[E_t[k_4,m]], & \text{on } \partial \mathbb{R}_+^3, \\
2\mu \partial_3 L[E_t[u_3]] - L[E_t[\pi]] - \sigma_{\partial \mathbb{R}_+^3} L[E_t[\eta]] = L[E_t[k_4,3]], & \text{on } \partial \mathbb{R}_+^3.
\end{cases}
\]
with \( z := \lambda + \omega \). To simplify the notation, we may denote \( K = E_{\partial R_3^+} [E_t[k_4]] \) and \( K_{\mathcal{L}} = E_{\partial R_3^+} [\mathcal{L}[E_t[k_4]]] \).

From the result due to Shibata, for any \( 0 < \theta < \pi / 2 \) and \( \mathcal{L}[E_t[k_4]] \in B^{1-1/3}(\partial R_3^+) \), there exist a (possibly large) constant \( \omega_0 > 0 \) and operator families

\[
\begin{align*}
U_{R_3^+}(z) &\in \text{Hol}(\omega_0 + \Sigma_{\theta + \pi / 2} \cup \{0\}; B(L^q(R_3^+)^3 \times H^{1-q}(R_3^+)^3, H^{2-q}(R_3^+)^3)), \\
P_{R_3^+}(z) &\in \text{Hol}(\omega_0 + \Sigma_{\theta + \pi / 2} \cup \{0\}; B(L^q(R_3^+)^3 \times H^{1-q}(R_3^+)^3, \dot{H}^{1-q}(R_3^+))), \\
E_{R_3^+}(z) &\in \text{Hol}(\omega_0 + \Sigma_{\theta + \pi / 2} \cup \{0\}; B(L^q(R_3^+)^3 \times H^{1-q}(R_3^+)^3, H^{3-q}(R_3^+))
\end{align*}
\]

such that for each \( \omega > \omega_0 \), the triplet

\[
(\mathcal{L}[E_t[u]], \mathcal{L}[E_t[\pi]], \mathcal{L}[E_t[\eta]]) := (U_{R_3^+}, P_{R_3^+}, \tau \mapsto \mathcal{L}[E_t[Z]](z^{1/2} K_{\mathcal{L}}, K_{\mathcal{L}}))
\]

is a unique solution to (4.9) possessing the estimates

\[
\begin{align*}
\mathcal{R}_{L^q(R_3^+) \times H^{1-q}(R_3^+)} &\to H^{2-j,q}(R_3^+) \left\{ (\tau \partial z^j \mathcal{U}(z)) \mid z \in \omega_0 + \Sigma_{\theta + \pi / 2} \cup \{0\} \right\} \leq C, \\
\mathcal{R}_{L^q(R_3^+) \times H^{1-q}(R_3^+)} &\to L^q(R_3^+) \left\{ (\tau \partial z^j \mathcal{V}(z)) \mid z \in \omega_0 + \Sigma_{\theta + \pi / 2} \cup \{0\} \right\} \leq C, \\
\mathcal{R}_{L^q(R_3^+) \times H^{1-q}(R_3^+)} &\to H^{3-j,q}(R_3^+) \left\{ (\tau \partial z^j \mathcal{H}(z)) \mid z \in \omega_0 + \Sigma_{\theta + \pi / 2} \cup \{0\} \right\} \leq C
\end{align*}
\]

for \( j, j' \neq 0, 1, j = 0, 1, 2, \) and \( \tau \) stands for the imaginary part of \( \lambda \), where a positive constant \( C \) does not depend on \( \lambda \) and \( \omega \). Here, we have set \( \Sigma_0 := \{ z \in \mathbb{C} \setminus \{0\} \mid |\arg(z)| < \theta_0 \} \) with \( \theta_0 \in (0, \pi) \). Furthermore, for Banach spaces \( Z_1 \) and \( Z_2 \), we denote by \( B(Z_1, Z_2) \) the Banach space of all bounded linear operators from \( Z_1 \) to \( Z_2 \), by \( H(U; B(Z_1, Z_2)) \) the set of all \( B(Z_1, Z_2) \)-valued holomorphic functions defined on \( U \subset \mathbb{C} \), and by \( R \to \mathcal{R}_{Z_1 \to Z_2} \{ T \} \) the \( \mathcal{R} \)-bound of a family of operators \( T \subset B(Z_1, Z_2) \) Furthermore, \( E_{\partial R_3^+} \) stands for an extension operator to \( \{ x_3 > 0 \} \), which makes us observe \( K_{\mathcal{L}} \in H^{1-q}(R_3^+) \). Notice that we have taken \( \omega_0 \) so large that the set of \( \lambda \) may include the right half-plane \( \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq 0 \} \). In the following, we may assume that \( \lambda \in \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq 0 \} \). Let \( \mathcal{L}^{-1} \) be the inverse of Laplace transform given by

\[
\mathcal{L}^{-1}[g(\lambda)](t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} g(\lambda) \, d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ir\tau} e^{\gamma t} g(\lambda) \, d\tau = e^{\gamma t} \mathcal{F}^{-1}[g(\lambda)](t)
\]

for \( \gamma = \gamma + it \in \mathbb{C}, t \in \mathbb{R} \), where we choose \( \gamma \) such that \( \gamma \geq 0 \). Setting

\[
\Lambda_{z}^{1/2}[f](t) := \mathcal{L}^{-1}[|z|^{1/2} \mathcal{L}[f](z)](t)
\]

and using the relation \( z^{1/2}[g](z) = \mathcal{L}[\Lambda_{z}^{1/2}[g]](z) \), we define \( (E_t[u], E_t[\pi], E_t[\eta]) \) by

\[
\begin{align*}
E_t[u](t) &= \mathcal{L}^{-1}[U_{R_3^+}(z)(z^{1/2} K_{\mathcal{L}}, K_{\mathcal{L}})] \\
&= e^{\gamma t} \mathcal{F}^{-1}[U_{R_3^+}(z)\mathcal{F}_{\tau}[e^{-\gamma t}(\Lambda_{z}^{1/2}[K], K)](\tau)](t), \\
E_t[\pi](t) &= \mathcal{L}^{-1}[P_{R_3^+}(z)(z^{1/2} K_{\mathcal{L}}, K_{\mathcal{L}})] \\
&= e^{\gamma t} \mathcal{F}^{-1}[P_{R_3^+}(z)\mathcal{F}_{\tau}[e^{-\gamma t}(\Lambda_{z}^{1/2}[K], K)](\tau)](t), \\
E_t[\eta](t) &= \mathcal{L}^{-1}[E_{R_3^+}(z)(z^{1/2} K_{\mathcal{L}}, K_{\mathcal{L}})] \\
&= e^{\gamma t} \mathcal{F}^{-1}[E_{R_3^+}(z)\mathcal{F}_{\tau}[e^{-\gamma t}(\Lambda_{z}^{1/2}[K], K)](\tau)](t)
\end{align*}
\]

for \( t \in \mathbb{R}, \text{Re} \lambda \geq 0 \). Noting \( \lambda \mathcal{L}[f](z) = \mathcal{L}[\partial_z f](z) \), we see that

\[
\begin{align*}
\partial_z E_t[u](t) &= \mathcal{L}^{-1}[(z - \omega)U_{R_3^+}(z)(z^{1/2} K_{\mathcal{L}}, K_{\mathcal{L}})] \\
&= e^{\gamma t} \mathcal{F}^{-1}[(z - \omega)U_{R_3^+}(z)\mathcal{F}_{\tau}[e^{-\gamma t}(\Lambda_{z}^{1/2}[K], K)](\tau)](t), \\
\partial_z E_t[\pi](t) &= \mathcal{L}^{-1}[(z - \omega)P_{R_3^+}(z)(z^{1/2} K_{\mathcal{L}}, K_{\mathcal{L}})] \\
&= e^{\gamma t} \mathcal{F}^{-1}[(z - \omega)P_{R_3^+}(z)\mathcal{F}_{\tau}[e^{-\gamma t}(\Lambda_{z}^{1/2}[K], K)](\tau)](t)
\end{align*}
\]
Since it holds
\[ \Lambda_{\frac{1}{2}}^{-1/2} K = \mathcal{L}^{-1} |\lambda + \omega|^{1/2} \mathcal{L}[K](\lambda)(t) \]
\[ = e^{\gamma t} \mathcal{F}_{\mathcal{X}}^{-1} \left[ ((\gamma + \omega)^2 + \tau^2)^{1/4} \mathcal{F}_{\mathcal{X}}[e^{-\gamma t} K](t) \right] \]
\[ = e^{\gamma t} \mathcal{F}_{\mathcal{X}}^{-1} \left[ \left( 1 + \frac{(\gamma + \omega)^2 - 1}{1 + \tau^2} \right)^{1/4} (1 + \tau^2)^{1/4} \mathcal{F}_{\mathcal{X}}[e^{-\gamma t} K](t) \right], \]
we see that
\[ |e^{-\gamma t} \Lambda_{\frac{1}{2}}^{-1/2} K|_{L^p_t(R;L^q_v(R^3_v))} \]
\[ \leq \sup_{\tau \in \mathbb{R}} \left| 1 + \frac{(\gamma + \omega)^2 - 1}{1 + \tau^2} \right|^{1/4} \left| \mathcal{F}_{\mathcal{X}}^{-1}[(1 + \tau^2)^{1/4} \mathcal{F}_{\mathcal{X}}[e^{-\gamma t} K](t)] \right|_{L^p_t(R;L^q_v(R^3_v))} \]
\[ \leq (\gamma + \omega)^{1/2} |e^{-\gamma t} K|_{H^{1/2,p}_t(R;L^q_v(R^3_v))} \]
provided that \( \omega > \max(\omega_0, 1) \). Hence, employing the operator-valued Fourier multiplier theorem of Prüss [29], for each \( \omega > \max(\omega_0, 1) \), it holds
\[ |e^{-\gamma t} \partial_t E_t[u]|_{L^p_t(R;L^q_v(R^3_v))} \leq C \left( |e^{-\gamma t} \Lambda_{\frac{1}{2}}^{-1/2} [K]|_{L^p_t(R;L^q_v(R^3_v))} + |e^{-\gamma t} K|_{L^p_t(R;H^{1/2,q}(R^3_v))} \right) \]
\[ \leq C (\gamma + \omega)^{1/2} |e^{-\gamma t} K|_{H^{1/2,p}_t(R;L^q_v(R^3_v))} + |e^{-\gamma t} K|_{L^p_t(R;H^{1/2,q}(R^3_v))} \]
with arbitrary \( \gamma \geq 0 \). Without loss of generality, we now assume \( \gamma = 0 \). From the trace theory, there exists a constant \( C \) such that
\[ |K|_{H^{1/2,p}_t(R;L^q_v(R^3_v))} \leq |K|_{H^{1/2,p}_t(R;L^q_v(R^3_v))} \cap L^p_t(R;H^{1/2,q}(R^3_v)) \leq |k|_{\mathcal{L}^p_{2,\delta}(R^3_v;\partial R^3_v)}, \]
\[ |K|_{L^p_t(R;H^{1/2,q}(R^3_v))} \leq |K|_{H^{1/2,p}_t(R;L^q_v(R^3_v))} \cap L^p_t(R;H^{1/2,q}(R^3_v)) \leq |k|_{\mathcal{L}^p_{2,\delta}(R^3_v;\partial R^3_v)} \]
are valid. Thus, for each \( \omega > \max(\omega_0, 1) \) we obtain
\[ |\partial_t u|_{L^p_t(R;L^q_v(R^3_v))} \leq C \omega^{1/2} |k|_{\mathcal{L}^p_{2,\delta}(R^3_v;\partial R^3_v)}, \]
where a constant \( C \) is independent of \( \gamma, \tau, \) and \( \omega \). Similarly, we observe
\[ |u|_{L^p_t(R;H^{1/2,q}(R^3_v))} \leq C \omega^{1/2} |k|_{\mathcal{L}^p_{2,\delta}(R^3_v;\partial R^3_v)}, \]
\[ |\nabla \pi|_{L^p_t(R;L^q_v(R^3_v))} \leq C \omega^{1/2} |k|_{\mathcal{L}^p_{2,\delta}(R^3_v;\partial R^3_v)}, \]
\[ |\partial \eta|_{L^p_t(R;B^{2-1/q}_q(\partial R^3_v))} \leq C \omega^{1/2} |k|_{\mathcal{L}^p_{2,\delta}(R^3_v;\partial R^3_v)}, \]
\[ |\eta|_{L^p_t(R;B^{3-1/q}_q(\partial R^3_v))} \leq C \omega^{1/2} |k|_{\mathcal{L}^p_{2,\delta}(R^3_v;\partial R^3_v)}. \]
Then, we now see that the problem (4.8) admits the unique solution \((u, \pi, \eta)\) with \( u \in E_{1,\delta}(R^3_v;\partial R^3_v), \)
\( \pi \in E_{2,\delta}(R^3_v;\partial R^3_v), \) and
\[ \eta \in H^1_{\delta,p}(R^3_v;B^{2}_{q,q} / (\partial R^3_v)) \cap L^p_{\delta}(R^3_v;B^{3-1/q}_{q,q}(\partial R^3_v)). \]
Especially, from the equation for \( \eta \), we obtain an additional regularity information on \( \eta \), and thus we arrive at \( \eta \in E_{4,\delta}(J;\partial R^3_v) \) due to \( u \in E_{1,\delta}(J;L^p_t(\partial R^3_v)) \cap L^p_t(\partial R^3_v;B^{2-1/q}_q(\partial R^3_v)). \) In addition, since it holds
\[ \pi = k_3 - (2\mu \partial_3 u_3 - \sigma \partial_3^2 \eta) \quad \text{on } \partial R^3_v, \]
we also observe that \( \text{Tr}_{\partial R^3_v} [\pi] \in E_{3,\delta}(R^3_v;\partial R^3_v) \) from the trace theory. Summing up, there exists a unique solution to (4.6). Finally, a uniqueness part follows from the duality argument, which we have used above in this proof. \( \square \)

4.3. The Stokes equations in quarter-spaces.
4.3.1. System with slip-slip boundary conditions. Let us consider the Stokes equations in a quarter-space with slip-slip boundary conditions:

\[
\begin{align*}
\partial_t u - \mu \Delta u + \nabla \pi &= f, \quad \text{in } \mathbb{K}^3 \times J, \\
\text{div } u &= f_{\text{div}}, \quad \text{in } \mathbb{K}^3 \times J, \\
\mu(\partial_2 u_\ell + \partial_3 u_2) &= g_\ell, \quad \text{in } \partial_2 \mathbb{K}^3 \times J, \\
u_2 &= g_2, \quad \text{in } \partial_2 \mathbb{K}^3 \times J, \\
\mu(\partial_3 u_m + \partial_m u_3) &= h_m, \quad \text{in } \partial_3 \mathbb{K}^3 \times J, \\
u_3 &= h_3, \quad \text{in } \partial_3 \mathbb{K}^3 \times J, \\
u(0) &= u_0, \quad \text{in } \mathbb{K}^3 \times J,
\end{align*}
\]

where \(\ell = 1, 3\) and \(m = 1, 2\). Here, to simplify the notation, we have used the following notations:

\[\mathbb{K}^3 := \{(x_1, x_2, x_3) \mid x_1 \in \mathbb{R}, \ x_2 > 0, \ x_3 > 0\},\]

\[\partial_2 \mathbb{K}^3 := \{(x_1, x_2, x_3) \mid x_1 \in \mathbb{R}, \ x_2 = 0, \ x_3 > 0\},\]

\[\partial_3 \mathbb{K}^3 := \{(x_1, x_2, x_3) \mid x_1 \in \mathbb{R}, \ x_2 > 0, \ x_3 = 0\}.
\]

If there is no confusion, we write \(\partial \mathbb{K}^3 := \partial_2 \mathbb{K}^3 \cap \partial_3 \mathbb{K}^3\) for short. Let us introduce the space \(\tilde{H}^{-1,q}(\mathbb{K}^3)\) as the set of all \((\varphi_1, \varphi_2, \varphi_3) \in L^q(\mathbb{K}^3) \times B^{2-1/q}_q(\partial_2 \mathbb{K}^3) \times B^{2-1/q}_q(\partial_3 \mathbb{K}^3)\) with \((\varphi_1, \varphi_2, \varphi_3) \in H^{-1,q}(\mathbb{K}^3)\). Set

\[\langle (\varphi_1, \varphi_2, \varphi_3) | \phi \rangle_{\mathbb{K}^3} := -(\varphi_1 | \varphi)_{\mathbb{K}^3} + (\varphi_2 | \phi)_{\partial_2 \mathbb{K}^3} + (\varphi_3 | \phi)_{\partial_3 \mathbb{K}^3}\]

for any \(\phi \in H^{-1,q}(\mathbb{K}^3)\).

Then, by the divergence equation, we have the condition

\[\langle (f_{\text{div}}, g_2, h_3) | \phi \rangle_{\mathbb{K}^3} = -(u | \nabla \phi)_{\mathbb{K}^3}\]

for any \(\phi \in \tilde{H}^{-1,q}(\mathbb{K}^3)\).

We now prove the following theorem.

**Theorem 4.6.** Let

\[1 < p < \infty, \ 2 < q < \infty, \ \frac{1}{p} < \delta \leq 1, \ \frac{1}{p} + \frac{1}{2q} \not\in \left\{\frac{1}{2}, \frac{1}{2}, \delta, \frac{1}{2}, \delta\right\}, \ \frac{1}{p} + \frac{1}{q} \not\in \left\{\frac{1}{2}, \frac{1}{2}, \delta, \frac{1}{2}, \delta\right\};\]

\[1 + \frac{1}{2q} < \delta - 1/2;\]

\[1/p + 1/(2q) < \delta - 1/2;\]

\[1/p + 1/q < \delta - 1/2;\]

\[1/p + 1/q < \delta - 1/2;\]

\[1/p + 1/q < \delta.\]

The solution \((u, \pi)\) depends continuously on the data in the corresponding spaces.

**Remark 4.7.** The condition \(q \in (2, \infty)\) ensures the existence of “trace of trace.” For instance, using a similar argument as in the proof of \([19, \text{Thm. 4.6}]\), we see that

\[\text{Tr}_{\partial_3 \mathbb{K}^3} \left[ F_{\mu,q,\delta}(J; \partial_2 \mathbb{K}^3) \right] = F_{\mu,q,\delta}^{1/2-1/q}(J; L^q(\partial \mathbb{K}^3)) \cap L^p_{\mu,q,\delta}(J; B^{1/2-1/q}(\partial \mathbb{K}^3))\]
if $2 < q < \infty$. Hence, we have compatibility conditions for $g_\ell$ and $h_m$ on $\partial K^3$ provided $1/p + 1/q < \delta - 1/2$. Similarly, there are compatibility conditions for $g_2$ and $h_3$ on $\partial R^3$ if $1/p + 1/q < \delta$, cf., [19, Rem. 3.5].

Proof. Define

$$
\begin{align*}
\mathbb{R}^3 &:= \{ x = (x_1, x_2, x_3) \mid x_1 \in \mathbb{R}, \ x_2 > 0, \ x_3 \in \mathbb{R} \}, \\
\partial \mathbb{R}^3 &:= \{ x = (x_1, x_2, x_3) \mid x_1 \in \mathbb{R}, \ x_2 = 0, \ x_3 \in \mathbb{R} \}.
\end{align*}
$$

Let us extend $u_0$ to $\mathbb{R}^{n+}$ in the class $B^{2(\delta - 1/p)}_{q,p}(\mathbb{R}^{n+})$ and denote it by $e_{\mathbb{R}^{n+}}[u_0]$. From Theorem 4.3, there exist functions $u_S = (u_{S,1}, u_{S,2}, u_{S,3})^T \in E_{1,\delta}(J; \mathbb{R}^{n+})$ and $\pi_S \in E_{2,\delta}(J; \mathbb{R}^{n+})$ satisfying

$$
\begin{align*}
\partial_t u_S - \mu \Delta u_S + \nabla \pi_S &= E_x^e[f], & \text{in } \mathbb{R}^{3+} \times J, \\
\text{div } u_S &= E_x^e[f_{\text{div}}], & \text{in } \mathbb{R}^{3+} \times J, \\
\mu(\partial_2 u_{S,\ell} + \partial_\ell u_{S,2}) &= E_x^e[g_\ell], & \text{on } \partial \mathbb{R}^{3+} \times J, \\
u_{S,2} &= u_{x,2}[g_2], & \text{on } \partial \mathbb{R}^{3+} \times J, \\
u_S(0) &= e_{\mathbb{R}^{3+}}[u_0], & \text{in } \mathbb{R}^{3+},
\end{align*}
$$

where $E_x^e$ denotes even extension with respect to $x_3$ and $E_x^o$ is an extension operator defined by

$$
E_{x,3}[f(x_1, x_2, t)] := \begin{cases} E_{\partial \mathbb{K}^3}[f](x_1, x_2, t) & \text{if } x_3 > 0, \\ 2E_{\partial \mathbb{K}^3}[f](x_1, 2x_3, t) - E_{\partial \mathbb{K}^3}[f](x_1, 3x_3, t) & \text{if } x_3 < 0. \end{cases}
$$

for $f(\cdot, t) \in B^{2(\delta - 1/p)}_{q,q}(\partial \mathbb{K}^3)$. We now define $g^* = (g_1^*, g_2^*, g_3^*)$ by

$$
g_m^* := \text{Tr}_{\partial \mathbb{K}^3}[\mu(\partial_3 u_{S,m} + \partial_m u_{S,3})], \quad g_3^* := \text{Tr}_{\partial \mathbb{K}^3}[u_{S,3}]
$$

with $m = 1, 2$. If we write $u = u_S + \overline{u}_S$ and $\pi = \pi_S + \overline{\pi}_S$, we see that a pair of functions $(\overline{u}_S, \overline{\pi}_S)$ satisfies

$$
\begin{align*}
\partial_t \overline{u}_S - \mu \Delta \overline{u}_S + \nabla \overline{\pi}_S &= 0, & \text{in } \mathbb{K}^3 \times J, \\
\text{div } \overline{u}_S &= 0, & \text{in } \mathbb{K}^3 \times J, \\
\mu(\partial_2 \overline{u}_{S,\ell} + \partial_\ell \overline{u}_{S,2}) &= 0, & \text{on } \partial \mathbb{K}^3 \times J, \\
\overline{u}_{S,2} &= 0, & \text{on } \partial \mathbb{K}^3 \times J, \\
\mu(\partial_3 \overline{u}_{S,m} + \partial_m \overline{u}_{S,3}) &= h_m - g_m^*, & \text{on } \partial \mathbb{K}^3 \times J, \\
\overline{u}_{S,3} &= h_3 - g_3^*, & \text{on } \partial \mathbb{K}^3 \times J, \\
\overline{u}_S(0) &= 0, & \text{in } \mathbb{K}^3.
\end{align*}
$$

Notice that we have $\partial_2 \overline{u}_{S,\ell} = 0$ and $\overline{u}_{S,2} = 0$ on the contact line $\partial \mathbb{K}^3$ from the condition (4.11). Hence, we have $\partial_2(h_\ell - g_\ell^*) = 0$, $\ell = 1, 3$, and $h_2 - g_2^* = 0$ on $\partial \mathbb{K}^3 \times \mathbb{R}_+$, which makes us to extend $h_\ell - g_\ell^*$ by even reflection and $h_2 - g_2^*$ by odd reflection to $\{x_2 < 0\}$. We now deal with the half-space problem

$$
\begin{align*}
\partial_t \tilde{u}_S - \mu \Delta \tilde{u}_S + \nabla \tilde{\pi}_S &= 0, & \text{in } \mathbb{R}_+^3 \times J, \\
\text{div } \tilde{u}_S &= 0, & \text{in } \mathbb{R}_+^3 \times J, \\
\mu(\partial_2 \tilde{u}_{S,\ell} + \partial_\ell \tilde{u}_{S,2}) &= E_x^o[h_1 - g_1^*], & \text{on } \partial \mathbb{R}_+^3 \times J, \\
\mu(\partial_3 \tilde{u}_{S,m} + \partial_m \tilde{u}_{S,3}) &= E_x^o[h_2 - g_2^*], & \text{on } \partial \mathbb{R}_+^3 \times J, \\
\tilde{u}_{S,2}(0) &= 0, & \text{in } \mathbb{R}_+^3.
\end{align*}
$$

Here, $E_x^o$ and $E_x^e$ represent odd and even reflection with respect to $x_2$, respectively. According to Theorem 4.3, we can find a unique solution to (4.13). We also see that restricted function $(\tilde{u}_S|_{\mathbb{K}^3}, \tilde{\pi}_S|_{\mathbb{K}^3}) = (\overline{u}_S, \overline{\pi}_S)$
Indeed, it is easy to verify that the function \( (\mu \pi), \pi, u_3, m \) is a solution to (4.14) in the required regularity class. By Theorem 4.2 we know that a solution to (4.14) is unique, so that we observe \( \mathbf{u}_S, \pi_S \) is given by \( \mathbf{u}_S = \mathbf{u}_S + \mathbf{u}_S^* = \mathbf{p}_S, \pi_S = \pi_S + \pi_S^* \), and thus \( \mathbf{u} = \mathbf{u}_S + \mathbf{u}_S^* + \mathbf{u}_S^* \) and \( \pi = \pi_S + \pi_S^* + \pi_S^* \) satisfy (4.10) in the right regularities. The uniqueness of the solution easily follows from its construction. \( \square \)
4.3.2. System with slip-free boundary conditions. Consider the Stokes equations in a quarter-space with slip-free boundary conditions:

$$
\begin{align*}
\partial_t u - \mu \Delta u + \nabla \pi &= f, \quad \text{in } \mathbb{R}^3 \times J, \\
\text{div } u &= f_{\text{div}}, \quad \text{in } \mathbb{R}^3 \times J, \\
\mu(\partial_2 u_\ell + \partial_3 u_2) &= g_\ell, \quad \text{on } \partial_2 \mathbb{R}^3 \times J, \\
u_2 &= g_2, \quad \text{on } \partial_2 \mathbb{R}^3 \times J, \\
\partial_3 \eta - u_3 &= d, \quad \text{on } \partial_3 \mathbb{R}^3 \times J, \\
\mu(\partial_3 u_m + \partial_m u_3) &= k_m, \quad \text{on } \partial_3 \mathbb{R}^3 \times J, \\
2\mu \partial_3 u_3 - \pi - \sigma \Delta g_3 \eta &= k_3, \quad \text{on } \partial_3 \mathbb{R}^3 \times J, \\
u(0) &= u_0, \quad \text{in } \mathbb{R}^3, \\
\eta(0) &= \eta_0, \quad \text{on } \partial_3 \mathbb{R}^3,
\end{align*}
$$

(4.16)

where $\ell = 1, 3$ and $m = 1, 2$. Similarly as we introduced in the previous subsection, we define the space $\tilde{H}^{-1,q}_{\partial_3 \mathbb{R}^3}(\mathbb{R}^3)$ as the set of all $(\varphi_1, \varphi_2) \in L^q(\mathbb{R}^3) \times B^{2-1/q}_{q,2}(\partial_3 \mathbb{R}^3)$ satisfying $(\varphi_1, \varphi_2) \in \tilde{H}^{-1,q}_{\partial_3 \mathbb{R}^3}(\mathbb{R}^3)$. Set

$$
\langle (\varphi_1, \varphi_2) | \phi \rangle_{\mathbb{R}^3} := -\langle \varphi_1 | \phi \rangle_{\mathbb{R}^3} + \langle \varphi_2 | \phi \rangle_{\partial_3 \mathbb{R}^3} \quad \text{for any } \phi \in \tilde{H}^{1,q'}_{\partial_3 \mathbb{R}^3}(\mathbb{R}^3).
$$

Then, by the divergence equation, we obtain the condition

$$
\langle (f_{\text{div}}, g_2) | \phi \rangle_{\mathbb{R}^3} = -\langle u | \nabla \phi \rangle_{\mathbb{R}^3} \quad \text{for any } \phi \in \tilde{H}^{1,q'}_{\partial_3 \mathbb{R}^3}(\mathbb{R}^3).
$$

We will prove the following theorem.

**Theorem 4.8.** Let $T > 0$ and $J = (0, T)$. Suppose that $p$, $q$, and $\delta$ satisfy

$$
1 < p < \infty, \quad 2 < q < \infty, \quad \frac{1}{p} < \delta \leq 1, \quad \frac{1}{p} + \frac{1}{q} - \delta \neq 1, \quad \frac{1}{p} + \frac{1}{q} - \frac{1}{2} < \delta - \frac{1}{2}. \quad (4.17)
$$

Then the problem (4.10) admits a unique solution $(u, \pi, \eta)$ with $u \in E_{1,\delta}(J; \mathbb{R}^3)$, $\pi \in E_{2,\delta}(J; \mathbb{R}^3)$, $\text{Tr}_{\partial_3 \mathbb{R}^3}[\pi] \in F_{2,\delta}(J; \partial_3 \mathbb{R}^3)$, $\eta \in E_{4,\delta}(J; \partial_3 \mathbb{R}^3)$, if and only if

(a) $f \in F_{0,\delta}(J; \mathbb{R}^3)$;
(b) $f_{\text{div}} \in F_{1,\delta}(J; \mathbb{R}^3)$;
(c) $g_\ell \in F_{2,\delta}(J; \partial_3 \mathbb{R}^3)$ for $\ell = 1, 3$;
(d) $g_2 \in F_{3,\delta}(J; \partial_3 \mathbb{R}^3)$;
(e) $d \in F_{4,\delta}(J; \partial_3 \mathbb{R}^3)$;
(f) $k_j \in F_{2,\delta}(J; \partial_3 \mathbb{R}^3)$ for $j = 1, 2, 3$;
(g) $(f_{\text{div}}, g_2, 0) \in H_{1,\delta}^{1,p}(J; \tilde{H}^{-1,q}_{\partial_3 \mathbb{R}^3}(\mathbb{R}^3))$;
(h) $u_0 \in B^{2+\delta-1/p}_{q,2}(\mathbb{R}^3)^3$ and $\text{div } u_0 = f_{\text{div}}(0)$;
(i) $\eta_0 \in B^{2+\delta-1/p-1/q}_{q,2}(\partial_3 \mathbb{R}^3)$;
(j) $g_\ell(0) = \text{Tr}_{\partial_3 \mathbb{R}^3}[\mu(\partial_2 u_\ell + \partial_3 u_{2,0})]$ for $\ell = 1, 2$ and $k_m(0) = \text{Tr}_{\partial_3 \mathbb{R}^3}[\mu(\partial_3 u_{m,0} + \partial_m u_{0,3})]$ for $m = 1, 2$ if $1/p + 1/q < \delta - 1/2$;
(k) $\text{Tr}_{\partial_3 \mathbb{R}^3}[g_\ell(0)] = \text{Tr}_{\partial_3 \mathbb{R}^3}[\mu(\partial_2 u_\ell + \partial_3 u_{2,0})]$ for $\ell = 1, 3$ and $\text{Tr}_{\partial_3 \mathbb{R}^3}[k_m(0)] = \text{Tr}_{\partial_3 \mathbb{R}^3}[\mu(\partial_3 u_{m,0} + \partial_m u_{0,3})]$ for $m = 1, 2$ if $1/p + 1/q < \delta - 1/2$.

Besides, the solution $(u, \pi, h)$ depends continuously on the data in the corresponding spaces.

**Proof.** Necessity follows easily from trace theory. In the following, we will prove sufficiency. Let $e_{\mathbb{R}_+^3}[u_0]$ and $e_{\mathbb{R}_+^3}[\eta_0]$ denote extensions of $u_0$ and $\eta_0$ to $\mathbb{R}_+^3$ in the class $B^{2+\delta-1/p}_{q,2}(\mathbb{R}_+^3)^3$ and $B^{2+\delta-1/p-1/q}_{q,2}(\partial \mathbb{R}_+^3)$,
respectively. Using Theorem 4.5, the problem
\[
\begin{aligned}
\partial_t \mathbf{u}_F - \mu \Delta \mathbf{u}_F + \nabla \pi_F &= f, & \text{in } \mathbb{R}^3_+ \times J, \\
\text{div } \mathbf{u}_F &= f_{\text{div}}, & \text{in } \mathbb{R}^3_+ \times J, \\
\partial_t \eta_F - u_{F,3} &= E_x \eta, & \text{on } \partial \mathbb{R}^3_+ \times J, \\
\mu(\partial_3 u_{F,m} + \partial_m u_{F,3}) &= E_x \eta, & \text{on } \partial \mathbb{R}^3_+ \times J, \\
2 \mu \partial_3 u_{F,3} - \pi_F - \sigma \Delta_{\partial \mathbb{R}^3_+} \eta_F &= E_x \eta, & \text{on } \partial \mathbb{R}^3_+ \times J, \\
\mathbf{u}_F(0) &= e_{\mathbb{R}^3} \eta, & \text{in } \mathbb{R}^3, \\
\eta_F(0) &= e_{\mathbb{R}^3} \eta, & \text{on } \partial \mathbb{R}^3_+.
\end{aligned}
\]

admits a unique solution \((\mathbf{u}_F, \pi_F, \eta_F)\), where \(E_x \eta\) is an extension of \(\eta\) to \(\{x_2 < 0\}\) given by
\[
E_x \eta(x_1, x_3) := \begin{cases} 
\eta(x_1, x_2) & \text{if } x_2 > 0, \\
2\eta(x_1, 2x_3) - \eta(x_1, 3x_3) & \text{if } x_2 < 0.
\end{cases}
\]

Writing \(\mathbf{u} = \mathbf{u}_F + \mathbf{u}_F, \pi = \pi_F + \pi_F, \) and \(\eta = \eta_F + \eta_F, \) we find that \((\mathbf{u}_F, \pi_F, \eta_F)\) solves
\[
\begin{aligned}
\partial_t \mathbf{u}_F - \mu \Delta \mathbf{u}_F + \nabla \pi_F &= 0, & \text{in } \mathbb{K}^3 \times J, \\
\text{div } \mathbf{u}_F &= 0, & \text{in } \mathbb{K}^3 \times J, \\
\mu(\partial_3 \mathbf{u}_F, \partial_3 \pi_F) &= g_F - k_2^*, & \text{on } \partial \mathbb{K}^3 \times J, \\
\mathbf{u}_F(0) &= 0, & \text{in } \mathbb{K}^3, \\
\eta_F(0) &= 0, & \text{on } \partial \mathbb{K}^3.
\end{aligned}
\]

with
\[
k_2^* := \text{Tr}_{\partial \mathbb{K}^3} [\mu(\partial_3 u_{F,3} + \partial_t u_{F,t})], \quad k_2^* := \text{Tr}_{\partial \mathbb{K}^3} [u_{F,3}], \quad d^* := -\text{Tr}_{\partial \mathbb{K}^3} [\partial_2 \eta_F].
\]

Notice that \(d^*\) is well-defined. In fact, by Lemma A.1 and [22, Prop. 3.10], we have
\[
d^* \in C^{3/2-1/(q)}(J; L^q(\mathbb{R}^3)) \cap C^{1-2/q}(J; B^{1-2/q}(\mathbb{R}^3)) \cap L^q(J; B^{2-2/q}(\mathbb{R}^3))
\]
provided the condition (4.17). From Lemma A.2, we see that there exists \(\eta_F \in \mathcal{E}_{4,\eta}(J; \mathbb{K}^3)\) such that \(\partial_2 \eta_F = d^*\) on \(\partial \mathbb{K}^3 \times J\). We next consider the half-space problem
\[
\begin{aligned}
\partial_t \mathbf{u}_F^* - \mu \Delta \mathbf{u}_F^* + \nabla \pi_F^* &= 0, & \text{in } \mathbb{R}^n_+ \times J, \\
\text{div } \mathbf{u}_F^* &= 0, & \text{in } \mathbb{R}^n_+ \times J, \\
\mu(\partial_3 \mathbf{u}_F^*, \partial_3 \pi_F^*) &= E_x \eta, & \text{on } \partial \mathbb{R}^n_+ \times J, \\
\mathbf{u}_F^*(0) &= 0, & \text{in } \mathbb{R}^n_+,
\end{aligned}
\]

where \(E_x \eta\) is the extension operator defined by (4.12). From Theorem 4.3, there exist a pair \((\mathbf{U}_F^*; \pi_F^*)\) that is a solution to (4.19) in the right regularity class. We then see that \(\mathbf{u}_F^* := \mathbf{u}_F - \mathbf{u}_F^*|_{\mathbb{K}^3}, \pi_F^* := \pi_F - \pi_F^*|_{\mathbb{K}^3},\)
and $\eta_F^\ast := \eta_F - \eta_F|_{\partial K^3}$ satisfy the system

\[
\begin{cases}
\partial_t \mathbf{u}_F^\ast - \mu \Delta \mathbf{u}_F^\ast + \nabla \pi_F^\ast = 0, & \text{in } K^3 \times J, \\
\text{div} \mathbf{u}_F^\ast = 0, & \text{in } K^3 \times J, \\
\mu (\partial_3 \mathbf{u}_{F,1}^\ast + \partial_1 \mathbf{u}_{F,2}^\ast) = 0, & \text{on } \partial_2 K^3 \times J, \\
\mathbf{u}_{F,2}^\ast = 0, & \text{on } \partial_2 K^3 \times J, \\
\partial_3 \eta_F^\ast - \pi_F^\ast = d^\ast, & \text{on } \partial_3 K^3 \times J, \\
\mu (\partial_3 \mathbf{u}_{F,m}^\ast + \partial_m \mathbf{u}_{F,3}^\ast) = k_m^\ast, & \text{on } \partial_3 K^3 \times J, \\
2\mu \partial_3 \mathbf{u}_{F,3}^\ast - \pi_F^\ast - \sigma \Delta_{\partial_3 K^3} \eta_F^\ast = k_3^\ast, & \text{on } \partial_3 K^3 \times J, \\
\partial_2 \eta_F^\ast = 0, & \text{on } \partial K^3 \times J, \\
\mathbf{u}_F^\ast (0) = 0, & \text{in } K^3, \\
\eta_F^\ast (0) = 0, & \text{on } \partial K^3,
\end{cases}
\]

(4.20)

where we have set

\[
d^\ast := -\partial_t \eta_F^\ast + \pi_{F,3}, \quad k_m^\ast := \mu (\partial_3 \mathbf{u}_{F,m}^\ast + \partial_m \mathbf{u}_{F,3}^\ast), \quad k_3^\ast := 2\mu \partial_3 \mathbf{u}_{F,3}^\ast - \pi_F^\ast - \sigma \Delta_{\partial_3 K^3} \eta_F^\ast.
\]

To show the existence of solution to (4.20), we consider the reflected half-space problem

\[
\begin{cases}
\partial_t \mathbf{u}_F^\ast - \mu \Delta \mathbf{u}_F^\ast + \nabla \pi_F^\ast = 0, & \text{in } R_+^3 \times J, \\
\text{div} \mathbf{u}_F^\ast = 0, & \text{in } R_+^3 \times J, \\
\partial_t \eta_F^\ast - \mathbf{u}_F^\ast = E_{x2} [d^\ast], & \text{on } \partial R_+^3 \times J, \\
\mu (\partial_3 \mathbf{u}_{F,m}^\ast + \partial_m \mathbf{u}_{F,3}^\ast) = E_{x2} [k_m^\ast], & \text{on } \partial R_+^3 \times J, \\
2\mu \partial_3 \mathbf{u}_{F,3}^\ast - \pi_F^\ast - \sigma \Delta_{\partial_3 K^3} \eta_F^\ast = E_{x2} [k_3^\ast], & \text{on } \partial R_+^3 \times J, \\
\mathbf{u}_F^\ast (0) = 0, & \text{in } R_+^3, \\
\eta_F^\ast (0) = 0, & \text{on } \partial R_+^3.
\end{cases}
\]

(4.21)

According to Theorem 4.5, the problem (4.21) admits a unique solution $(\mathbf{u}_F^\ast, \pi_F^\ast, \eta_F^\ast)$. In addition, we can observe that the function $(\mathbf{v}_F, \tilde{\eta}_F, \tilde{\zeta}_F) = (\mathbf{v}_{F,1}, \mathbf{v}_{F,2}, \mathbf{v}_{F,3}, \tilde{\eta}_F, \tilde{\zeta}_F)$ defined by

\[
\begin{align*}
\mathbf{v}_{F,1}(x,t) &:= \mathbf{u}_{F,1}^\ast (x_1, -x_2, x_3, t), \\
\mathbf{v}_{F,2}(x,t) &:= -\mathbf{u}_{F,2}^\ast (x_1, -x_2, x_3, t), \\
\tilde{\eta}_F(x,t) &:= \eta_F^\ast (x_1, -x_2, t),
\end{align*}
\]

solves the problem (4.21). Since a solution to (4.21) is unique, it follows that

\[
\begin{align*}
\mathbf{u}_{F,1}^\ast (x_1, -x_2, x_3, t) &:= \mathbf{u}_{F,1}^\ast (x_1, x_2, x_3, t), \\
\mathbf{u}_{F,2}^\ast (x_1, -x_2, x_3, t) &:= -\mathbf{u}_{F,2}^\ast (x_1, x_2, x_3, t), \\
\eta_F^\ast (x_1, -x_2, t) &:= \eta_F^\ast (x_1, x_2, t).
\end{align*}
\]

This makes us to obtain

\[
\mu (\partial_2 \mathbf{u}_{F,1}^\ast + \partial_1 \mathbf{u}_{F,2}^\ast) = 0, \quad \mathbf{u}_{F,2}^\ast = 0 \quad \text{on } \partial K^3 \times J
\]

and $\partial_3 \eta_F^\ast = 0$ on $\partial K^3 \times J$. Hence, the restricted function $(\mathbf{u}_F^\ast|_{\partial K^3}, \pi_F^\ast|_{\partial K^3}, \eta_F^\ast|_{\partial K^3})$ is a solution to (4.20). Thus, a solution to (4.18) is given by

\[
\begin{align*}
\mathbf{u}_F &= \mathbf{u}_F^\ast + \mathbf{u}_F^\dagger, \\
\pi &= \pi_F^\ast + \pi_F^\dagger, \\
\eta &= \eta_F^\ast + \eta_F^\dagger.
\end{align*}
\]

The uniqueness of the solution to (4.16) can be obtained by its construction. The proof is complete. \qed
5. Maximal Regularity of the Principal Linearization

The principle part of the linearized system reads as follows

\[
\begin{align*}
\partial_t u - \mu \Delta u + \nabla \pi &= f, & \text{in } \Omega_\ast \times J, \\
\text{div } u &= f_{\text{div}}, & \text{in } \Omega_\ast \times J, \\
\partial_t \eta - u_3 &= d, & \text{in } \Gamma_\ast \times J, \\
\mu(\partial_3 u_m + \partial_m u_3) &= k_m, & \text{in } \Gamma_\ast \times J, \\
2\mu \partial_3 u_3 - \sigma \Delta \eta \eta &= k_3, & \text{in } \Gamma_\ast \times J, \\
P_{\Sigma_\ast}(2\mu D(u)n_{\Sigma_\ast}) &= P_{\Sigma_\ast}g, & \text{on } \Sigma_\ast \times J, \\
u \cdot n_{\Sigma_\ast} &= g \cdot n_{\Sigma_\ast}, & \text{on } \Sigma_\ast \times J, \\
u(0) &= u_0, & \text{in } \Omega_\ast, \\
\eta(0) &= \eta_0, & \text{on } \Gamma_\ast,
\end{align*}
\]

(5.1)

where \(m = 1, 2\) and \(g = (g_1, g_2, g_3)^T\). To identify a hidden regularity coming from the divergence equation, we define the space \(\tilde{H}^{-1,q}(\Omega_\ast)\) as the set of all \((\varphi_1, \varphi_2, \varphi_3) \in L^q(\Omega_\ast) \times B^{2-1/q}_q(\Sigma_\ast) \times B^{2-1/q}_q(B)\) with \((\varphi_1, \varphi_2, \varphi_3) \in \tilde{H}^{1,q}(\Omega_\ast)\). Set

\[
((\varphi_1, \varphi_2, \varphi_3) | \varphi)_\Omega := -(\varphi_1 | \varphi)_{\Omega_\ast} + (\varphi_2 | \varphi)_{\Sigma_\ast} + (\varphi_3 | \varphi)_B \quad \text{for any } \varphi \in \tilde{H}^{1,q}(\Omega_\ast).
\]

By the divergence theorem, the divergence equation yields the condition

\[
((f_{\text{div}}, g_2, h_3) | \varphi)_{\mathbb{R}^3} = -(u | \nabla \varphi)_{\mathbb{R}^3} \quad \text{for any } \varphi \in \tilde{H}^{1,q}(\Omega_\ast).
\]

The aim of this section is to show maximal regularity of the principal linearization.

**Theorem 5.1.** Set \(J = (0, T), T > 0\). Let \(p, q,\) and \(\delta\) satisfy (4.17). There exists a unique solution \((u, \pi, \eta)\) of (5.1) with \(u \in E_{1,\delta}(J; \Omega_\ast), \pi \in E_{2,\delta}(J; \Omega_\ast), \text{Tr}_{\Gamma_{\ast}}[\pi] \in E_{3,\delta}(J; \Gamma_{\ast}), \eta \in E_{4,\delta}(J; \Gamma_{\ast}), \) if and only if

(a) \(f \in F_{0,\delta}(J; \Omega_\ast)\);
(b) \(f_{\text{div}} \in F_{1,\delta}(J; \Omega_\ast)\);
(c) \(F_{\Sigma_\ast}g \in F_{2,\delta}(J; \Sigma_\ast)\) for \(\ell = 1, 3\);
(d) \(g \cdot n_{\Sigma_\ast} \in F_{3,\delta}(J; \Sigma_\ast)\);
(e) \(d \in F_{4,\delta}(J; \Gamma_{\ast})\);
(f) \(k_j \in F_{2,\delta}(J; \Gamma_{\ast})\) for \(j = 1, 2, 3\);
(g) \(h_m \in F_{2,\delta}(J; B)\);
(h) \(h_3 \in F_{3,\delta}(J; B)\);
(i) \((f_{\text{div}}, g \cdot n_{\Sigma_\ast}, h_3) \in H_{\delta}^{1,p}(J; \tilde{H}_{\Sigma_\ast}^{-1,q}(\Omega_\ast))\);
(j) \(u_0 \in B_{2,\delta}^{2-1/p}(\Omega_\ast)^3\) and \(\text{div } u_0 = f_{\text{div}}(0)\);
(k) \(\eta_0 \in B_{2,\delta}^{2-1/p}(\Gamma_{\ast})\);
(l) \(P_{\Sigma_\ast}g(0) = \text{Tr}_{\Sigma_\ast}[P_{\Sigma_\ast}(2\mu D(u_0)n_{\Sigma_\ast})], k_m(0) = \text{Tr}_{\Gamma_{\ast}}[\mu(\partial_3 u_{m_0} + \partial_m u_{3})], \) and \(h_m(0) = \text{Tr}_B[\mu(\partial_3 u_{m_0} + \partial_m u_{3})] \) if \(1/p + 1/(2q) < \delta - 1/2\);
(m) \(g(0) \cdot n_{\Sigma_\ast} = \text{Tr}_{\Sigma_\ast}[u_0 \cdot n_{\Sigma_\ast}] \) and \(h_3(0) = \text{Tr}_B[u_{3}] \) if \(1/p + 1/(2q) < \delta\);
(n) \(\text{Tr}_{\Sigma_\ast}[P_{\Sigma_\ast}g(0)] = \text{Tr}_{\Sigma_\ast}[P_{\Gamma_{\ast}}(2\mu D(u_0)n_{\Sigma_\ast})] \) and \(\text{Tr}_{\Sigma_\ast}[k_m(0)] = \text{Tr}_{\Sigma_\ast}[\mu(\partial_3 u_{m_0} + \partial_m u_{3})] \) if \(1/p + 1/(2q) < \delta - 1/2\);
(o) \(\text{Tr}_{\Sigma_\ast \cap \partial B}[g(0) \cdot n_{\Sigma_\ast}] = \text{Tr}_{\Sigma_\ast \cap \partial B}[u_0 \cdot n_{\Sigma_\ast}] \) and \(\text{Tr}_{\Sigma_\ast \cap \partial B}[h_3(0)] = \text{Tr}_{\Sigma_\ast \cap \partial B}[u_{3}] \) if \(1/p + 1/q < \delta\).

In addition, the solution map is continuous between the corresponding spaces.

5.1. **Bent spaces.** In this subsection, we give results of the Stokes equations in bent spaces.
5.1.1. Bent half spaces. Let $\gamma : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function in $C^3$ class such that $|\nabla x'|_{L^\infty(\mathbb{R}^2)} \leq c$, where $c$ is a small constant. Let $\mathbb{R}^3_\gamma$ be the bent half space defined by
\[
\mathbb{R}^3_\gamma := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in \mathbb{R}^2, \ x_3 > \gamma(x_1, x_2)\}
\tag{5.2}
\]
and $\partial \mathbb{R}^3_\gamma$ be its boundary defined by
\[
\partial \mathbb{R}^3_\gamma := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in \mathbb{R}^2, \ x_3 = \gamma(x_1, x_2)\}.
\]
Introducing the transformation $\Phi_\gamma : \mathbb{R}^3_\gamma \to \mathbb{R}^3_\gamma$ defined by $y = \Phi_\gamma(x) = (x_1, x_2, x_3 - \gamma(x_1, x_2))$, we see that $\Phi_\gamma$ is a bijection with Jacobian equal to 1. We also find that $\mathbb{R}^3_\gamma = \Phi_\gamma(\mathbb{R}^3_\gamma)$ and $\partial \mathbb{R}^3_\gamma = \Phi_\gamma(\partial \mathbb{R}^3_\gamma)$. For the unit normal vector of $\partial \mathbb{R}^3_\gamma$, we have
\[
n_{\partial \mathbb{R}^3_\gamma} = \frac{1}{\sqrt{1 + |\nabla x'|^2}} (\nabla x', -1)^T.
\]
Let $P_{\mathbb{R}^3_\gamma}$ be the tangential projection to $\mathbb{R}^3_\gamma$. Consider the following two systems:
\[
\begin{cases}
\partial_t u - \mu \Delta u + \nabla \pi = f, & \text{in } \mathbb{R}^3_\gamma \times J, \\
\text{div } u = f_{\text{div}}, & \text{in } \mathbb{R}^3_\gamma \times J, \\
\begin{split}
P_{\mathbb{R}^3_\gamma}(2 \mu D(u) n_{\partial \mathbb{R}^3_\gamma}) = h', & \text{on } \partial \mathbb{R}^3_\gamma \times J, \\
u \cdot n_{\partial \mathbb{R}^3_\gamma} = h_3, & \text{on } \partial \mathbb{R}^3_\gamma \times J, \\
u(0) = u_0, & \text{in } \mathbb{R}^3_\gamma.
\end{split}
\end{cases}
\tag{5.3}
\]
Let $\tilde{H}^{-1,q}(\mathbb{R}^3_\gamma)$ be the set of all $(\varphi_1, \varphi_2) \in L^q(\mathbb{R}^3_\gamma) \times B^{2-1/q}_q(\partial \mathbb{R}^3_\gamma)$ that satisfy the regularity property $(\varphi_1, \varphi_2) \in 0\tilde{H}^{-1,q}(\mathbb{R}^3_\gamma)$. With the notation
\[
((\varphi_1, \varphi_2) | \phi)_{\mathbb{R}^3_\gamma} := -(\varphi_1 | \phi)_{\mathbb{R}^3_\gamma} + (\varphi_2 | \phi)_{\partial \mathbb{R}^3_\gamma}
\]
for any $\phi \in \tilde{H}^{1,q}(\mathbb{R}^3_\gamma)$, we have the conditions
\[
((\varphi_1, \varphi_2) | \phi)_{\mathbb{R}^3_\gamma} := -(\varphi_1 | \phi)_{\mathbb{R}^3_\gamma} + (\varphi_2 | \phi)_{\partial \mathbb{R}^3_\gamma}
\]
in which follows from the divergence equation and the divergence theorem. In view of [33, Sec. 7.3.2], there exists a unique solution $(u, \pi)$ to (5.3) in maximal regularity class provided $|\nabla x'|_{L^\infty(\mathbb{R}^2)}$ is bounded above by a small constant $c$.

**Theorem 5.2.** Suppose $1 < p, q < \infty$, $1/p < \delta \leq 1$, $1/p + 1/(2q) \notin \{\delta - 1/2, \delta\}$, $T > 0$, and $J = (0, T)$. Then there exists a constant $c$ such that the problem 5.3 has a unique solution with regularity $u \in E_{p,d}(J; \mathbb{R}^3_\gamma)$, $\pi \in E_{q,d}(J; \mathbb{R}^3_\gamma)$ if and only if
\begin{itemize}
    \item[(a)] $f \in F_{0,d}(J; \mathbb{R}^3_\gamma)$;
    \item[(b)] $f_{\text{div}} \in F_{1,d}(J; \mathbb{R}^3_\gamma)$;
    \item[(c)] $h' \in F_{2,d}(J; \partial \mathbb{R}^3_\gamma)$ and $h'(0) = \text{Tr}_{\partial \mathbb{R}^3_\gamma}[P_{\mathbb{R}^3_\gamma}(2 \mu D(u_0) n_{\partial \mathbb{R}^3_\gamma})]$ if $1/p + 1/(2q) < \delta - 1/2$;
    \item[(d)] $h_3 \in F_{3,d}(J; \partial \mathbb{R}^3_\gamma)$ and $h_3(0) = \text{Tr}_{\partial \mathbb{R}^3_\gamma}[u_0, 3]$ if $1/p + 1/(2q) < \delta$;
    \item[(e)] $(f_{\text{div}}, h_3) \in H^{1,p}(J; \tilde{H}^{-1,q}(\mathbb{R}^3_\gamma))$;
    \item[(f)] $u_0 \in B^{2(\delta - 1/p)}_q(\mathbb{R}^3_\gamma)$ and $\text{div } u_0 = f_{\text{div}}(0)$.
\end{itemize}
Furthermore, the solution map $(f, f_{\text{div}}, h', h_3, u_0) \to (u, \pi)$ is continuous between the corresponding spaces.

5.1.2. Bent quarter spaces. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a function of class $BC^3$ such that $|\partial_1 \gamma|_{L^\infty(\mathbb{R})} \leq c$, where $c$ is a small constant determined later. We define a bent quarter space by
\[
\mathbb{K}^3_\gamma := \{(x_1, x_2, x_3) \mid x_1 \in \mathbb{R}, \ x_2 > \gamma(x_1), \ x_3 > 0\}.
\tag{5.4}
\]
Besides, we denote the boundaries of $\mathbb{K}^3_\gamma$ by
\[
\partial_2 \mathbb{K}^3_\gamma := \{(x_1, x_2, x_3) \mid x_1 \in \mathbb{R}, \ x_2 = \gamma(x_1), \ x_3 > 0\},
\partial_3 \mathbb{K}^3_\gamma := \{(x_1, x_2, x_3) \mid x_1 \in \mathbb{R}, \ x_2 > \gamma(x_1), \ x_3 = 0\}.
We also write $\partial K^3_\gamma := \partial_2 K^3_\gamma \cap \partial_3 K^3_\gamma$ if no confusion occurs. Besides, we set
\[ n_{\partial_2 K^3_\gamma} := b(x)(\partial_1 \gamma, -1, 0)^T, \quad b(x) = \frac{1}{\sqrt{1 + |\partial_1 \gamma|^2}}, \]
which denotes the outward unit normal field on $\partial_2 K^3_\gamma$.

First, we consider the Stokes equations in $K^3_\gamma$ with slip-slip boundary conditions
\[
\begin{aligned}
\partial_t u - \mu \Delta u + \nabla \pi &= f, \quad \text{in } K^3_\gamma \times J, \\
\text{div } u &= f_{\text{div}}, \quad \text{in } K^3_\gamma \times J, \\
P_{\partial_3 K^3_\gamma}(2\mu D(u)n_{\partial_3 K^3_\gamma}) &= g', \\
\text{in } \partial_2 K^3_\gamma \times J, \\
u \cdot n_{\partial_3 K^3_\gamma} &= g_2, \quad \text{in } \partial_2 K^3_\gamma \times J, \\
\mu(\partial_3 u_m + \partial_m u_3) &= h_m, \quad \text{in } \partial_3 K^3_\gamma \times J, \\
u_3 &= h_3, \quad \text{in } \partial_3 K^3_\gamma \times J, \\
u(0) &= u_0, \quad \text{in } K^3_\gamma, 
\end{aligned}
\]
(5.5)
where $m = 1, 2$ and $g' = (g_1, 0, g_3)$. Since the equations (5.5) is a perturbation of the half-space problem (4.3), we can show the existence of unique solution to (5.5) provided $|\partial_1 \gamma|_{L^\infty(\mathbb{R})}$ is small enough. As we introduced before, let $\tilde{H}^{-1,q}(K^3_\gamma)$ be the set of all $(\varphi_1, \varphi_2, \varphi_3) \in L^q(K^3_\gamma) \times B^{1-1/q}_q(\partial_2 K^3_\gamma) \times B^{2-1/q}_q(\partial_3 K^3_\gamma)$ with $(\varphi_1, \varphi_2, \varphi_3) \in \tilde{H}^{-1,q}(K^3_\gamma)$.

Set
\[ \langle (\varphi_1, \varphi_2, \varphi_3) | \phi \rangle_{K^3_\gamma} := -(\varphi_1 | \phi)_{K^3_\gamma} + (\varphi_2 | \phi)_{\partial_2 K^3_\gamma} + (\varphi_3 | \phi)_{\partial_3 K^3_\gamma}, \quad \text{for any } \phi \in \tilde{H}^{1,q'}(K^3_\gamma). \]

Then, by the divergence equation, we have the condition
\[ \langle (f_{\text{div}}, g_2, h_3) | \phi \rangle_{K^3_\gamma} = -(u | \nabla \phi)_{K^3_\gamma}, \quad \text{for any } \phi \in \tilde{H}^{1,q'}(K^3_\gamma). \]

**Theorem 5.3.** Suppose that $p$, $q$, and $\delta$ satisfy (4.11). Let $T > 0$ and $J = (0, T)$. Then there exists a unique solution $(u, \pi)$ to the problem (5.5) with $u \in E_{1,\delta}(J; K^3_\gamma)$ and $\pi \in E_{2,\delta}(J; K^3_\gamma)$ if and only if

(a) $f \in F_{0,\delta}(J; K^3_\gamma)$;
(b) $f_{\text{div}} \in F_{1,\delta}(J; K^3_\gamma)$;
(c) $g_\ell \in F_{2,\delta}(J; \partial_\ell K^3_\gamma)$, $\ell = 1, 3$;
(d) $g_2 \in F_{3,\delta}(J; \partial_2 K^3_\gamma)$;
(e) $h_m \in F_{2,\delta}(J; \partial_3 K^3_\gamma)$;
(f) $h_3 \in F_{3,\delta}(J; \partial_3 K^3_\gamma)$;
(g) $(f_{\text{div}}, g_2, h_3) \in \tilde{H}^{1,p}(J; \tilde{H}^{-1,q}(K^3_\gamma))$;
(h) $u_0 \in B^{\delta-1/p}_{q,p}(K^3_\gamma)$ and $\text{div } u_0 = f_{\text{div}}(0)$;
(i) $g'(0) = \text{Tr}_{\partial_3 K^3_\gamma}[P_{\partial_2 K^3_\gamma}(2\mu D(u_0)n_{\partial_2 K^3_\gamma})]$ and $h_m(0) = \text{Tr}_{\partial_3 K^3_\gamma}[^{\mu}(\partial_3 u_{0,m} + \partial_m u_{0,3})]$ if $1/p + 1/(2q) < \delta - 1/2$;
(j) $g_2(0) = \text{Tr}_{\partial_2 K^3_\gamma}[u_0 \cdot n_{\partial_3 K^3_\gamma}]$ and $h_3(0) = \text{Tr}_{\partial_3 K^3_\gamma}[u_{0,3}]$ if $1/p + 1/(2q) < \delta$;
(k) $\text{Tr}_{\partial_2 K^3_\gamma}[g'(0)] = \text{Tr}_{\partial_2 K^3_\gamma}[P_{\partial_2 K^3_\gamma}(2\mu D(u_0)n_{\partial_2 K^3_\gamma})]$ and $\text{Tr}_{\partial_3 K^3_\gamma}[h_m(0)] = \text{Tr}_{\partial_3 K^3_\gamma}[^{\mu}(\partial_3 u_{0,m} + \partial_m u_{0,3})]$ if $1/p + 1/q < \delta - 1/2$;
(l) $\text{Tr}_{\partial_3 K^3_\gamma}[g_2(0)] = \text{Tr}_{\partial_3 K^3_\gamma}[u_{0,2}]$ and $\text{Tr}_{\partial_2 K^3_\gamma}[h_3(0)] = \text{Tr}_{\partial_2 K^3_\gamma}[u_{0,3}]$ if $1/p + 1/q < \delta$.

In addition, the solution $(u, \pi)$ depends continuously on the data in the corresponding spaces.

**Proof.** We first reduce the problem (5.5) to the case $u_0 = f = 0$. To this end, let $e_{\mathbb{R}^3}[u_0]$ and $e_{\mathbb{R}^3}[f]$ denote extensions of $u_0$ and $f$ to all of $\mathbb{R}^3$ in the class $B^{2(\delta-1/p)}_{q,p}(\mathbb{R}^3)$ and $F_{0,\delta}(J; \mathbb{R}^3)$, respectively, which solve the heat equation
\[
\begin{cases}
\partial_t \tilde{u} - \mu \Delta \tilde{u} = e_{\mathbb{R}^3}[f], & \text{in } \mathbb{R}^3 \times J, \\
e_{\mathbb{R}^3}[u](0) = e_{\mathbb{R}^3}[u_0], & \text{in } \mathbb{R}^3
\end{cases}
\]
to obtain a unique solution $\tilde{u} \in \mathcal{F}_{0,4}(J; \mathbb{R}^3)$. If $u$ is a solution to (5.5), we see that the restricted function $u^* = u - \tilde{u}$ solves

$$
\begin{cases}
\partial_t u^* - \mu \Delta u^* + \nabla \pi = 0, & \text{in } \mathbb{R}^3 \times J, \\
\text{div } u^* = f^*_\text{div}, & \text{in } \mathbb{R}^3 \times J, \\
P_{\partial_3 \mathbb{K}^3}(2\mu D(u^*)n_{\partial_3 \mathbb{K}^3}) = g^*, & \text{in } \partial_2 \mathbb{K}^3 \times J, \\
u^* \cdot n_{\partial_2 \mathbb{K}^3} = g_2^*, & \text{in } \partial_2 \mathbb{K}^3 \times J, \\
\mu(\partial_3 u^* + \partial_3 u^*_3) = h^*_m, & \text{in } \partial_3 \mathbb{K}^3 \times J, \\
u^*_3 = h^*_3, & \text{in } \partial_3 \mathbb{K}^3 \times J, \\
u^*(0) = 0, & \text{in } \mathbb{K}^3, \\
\end{cases}
$$

(5.6)

with

$$
f^*_\text{div} = f_\text{div} - \text{div } \tilde{u}|_{\mathbb{K}^3}, \quad g'' = g' - P_{\partial_3 \mathbb{K}^3}(2\mu D(\tilde{u}|_{\mathbb{K}^3})n_{\partial_3 \mathbb{K}^3}),
$$

$$
g_2^* = g_2 - \tilde{u}|_{\mathbb{K}^3} \cdot n_{\partial_2 \mathbb{K}^3}, \quad h^*_m = h_m - \mu(\partial_3 \tilde{u}|_{\mathbb{K}^3} + \partial_3 \tilde{u}|_{\mathbb{K}^3}), \quad h^*_3 = h_3 - \tilde{u}|_{\mathbb{K}^3}.
$$

Considering the time trace at $t = 0$, we have $(f^*_\text{div}, g_2^*, h_3^*) \in H^{1,0}(J; \mathbb{H}^{-1,0}(\mathbb{K}^3))$ vanishing at $t = 0$.

Next, we transform (5.5) to the problem in $\mathbb{K}^3$. To this end, we introduce new variables

$$
\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) = (x_1, x_2 - \gamma(x_1), x_3) \quad \text{for } x \in \mathbb{K}^3.
$$

Then, we define the new functions

$$
\bar{u}(\bar{x}) := u^*(\bar{x}_1, \bar{x}_2 + \gamma(\bar{x}_1), \bar{x}_3),
$$

$$
\bar{\pi}(\bar{x}) := \pi(\bar{x}_1, \bar{x}_2 + \gamma(\bar{x}_1), \bar{x}_3),
$$

where $(\bar{u}^*, \bar{\pi})$ is a solution to (5.6). In the same way, we also transform the given data $(f^*_\text{div}, g''^*, g_2^*, h''^*, h_3^*)$ to $(\bar{f^*_\text{div}}, \bar{g''}, \bar{g}_2, \bar{h''}, \bar{h}_3)$, where $\bar{h''} = (h_1, h_2)$ and $\bar{h}'' = (\bar{h}_1, \bar{h}_2)$. Besides, the differential operators are transformed into $\partial_{\bar{x}_2}^* = \partial_{x_2}^*, \partial_{\bar{x}_3}^* = \partial_{x_3}^*$ for $s = 1, 2$ and

$$
\partial_{\bar{x}_1} = \partial_{x_1} - (\partial_{x_1} \gamma) \partial_{\bar{x}_2},
$$

$$
\partial_{\bar{x}_2}^2 = \partial_{x_2}^2 - 2(\partial_{x_1} \gamma) \partial_{x_1} \partial_{x_2} - (\partial_{x_1} \gamma)^2 \partial_{x_2}^2.
$$

Thus, we find that $(\bar{u}, \bar{\pi})$ is a solution to the following problem

$$
\begin{cases}
\partial_t \bar{u} - \mu \Delta \bar{u} + \nabla \bar{\pi} = B_1(\gamma, \bar{u}, \bar{\pi}), & \text{in } \mathbb{K}^3 \times J, \\
\text{div } \bar{u} = \bar{f}^*_\text{div} + B_2(\gamma, \bar{u}), & \text{in } \mathbb{K}^3 \times J, \\
\mu(\partial_{\bar{x}_2} \bar{u}_1 + \partial_{\bar{x}_1} \bar{u}_2) = -\bar{g}_1 + \mu b(x) B_3(\gamma, \bar{u}), & \text{in } \partial_2 \mathbb{K}^3 \times J, \\
\mu(\partial_{\bar{x}_2} \bar{u}_3 + \partial_{\bar{x}_1} \bar{u}_2) = -\bar{g}_3 + \mu b(x) B_3(\gamma, \bar{u}), & \text{in } \partial_2 \mathbb{K}^3 \times J, \\
\bar{u}_2 = -\bar{g}_2 + \mu b(x) B_3(\gamma, \bar{u}), & \text{in } \partial_2 \mathbb{K}^3 \times J, \\
\mu(\partial_{\bar{x}_2} \bar{u}_1 + \partial_{\bar{x}_1} \bar{u}_3) = \bar{h}_1 + B_5(\gamma, \bar{u}), & \text{in } \partial_3 \mathbb{K}^3 \times J, \\
\mu(\partial_{\bar{x}_2} \bar{u}_2 + \partial_{\bar{x}_1} \bar{u}_3) = \bar{h}_2, & \text{in } \partial_3 \mathbb{K}^3 \times J, \\
\bar{u}_3 = \bar{h}_3, & \text{in } \partial_3 \mathbb{K}^3 \times J, \\
\bar{u}(0) = 0, & \text{in } \mathbb{K}^3, \\
\end{cases}
$$

(5.7)
where we have set

\[ B_1(\gamma, \mathbf{u}, \pi) = 2(\partial_{x_1} \gamma)\partial_{x_1} \mathbf{u} + (\partial^{2}_{x_1} \gamma)\partial_{x_2} \mathbf{u} - |\partial_{x_1} \gamma|^2 \partial_{x_2} \mathbf{u} + (\partial_{x_1} \pi)(\partial_{x_1} \gamma, 0, 0)^T, \]
\[ B_2(\gamma, \mathbf{u}) = (\partial_{x_1} \gamma)\partial_{x_1} \mathbf{u}, \]
\[ B_3(\gamma, \mathbf{u}) = 2(\partial_{x_1} \gamma)\partial_{x_2} \mathbf{u}_1 + (\partial_{x_1} \gamma)(1 - |\partial_{x_1} \gamma|^2)\partial_{x_2} \mathbf{u}_2 - |\partial_{x_1} \gamma|^2 \partial_{x_2} \mathbf{u}_1 
- 2(\partial_{x_1} \gamma)(1 - |\partial_{x_1} \gamma|^2)(\partial_{x_1} \gamma)\partial_{x_2} \mathbf{u}_1 - (\partial_{x_1} \gamma)^2 \partial_{x_2} \mathbf{u}_2 
- 2(\partial_{x_1} \gamma)^3 \partial_{x_2} \mathbf{u}_1 + \frac{|\partial_{x_1} \gamma|^2}{1 + \sqrt{1 + |\partial_{x_1} \gamma|^2}} \left( 2\partial_{x_2} \mathbf{u}_1 + \partial_{x_2} \mathbf{u}_2 \right), \]
\[ B_4(\gamma, \mathbf{u}) = (\partial_{x_1} \gamma)(\partial_{x_2} \mathbf{u}_1 + \partial_{x_2} \mathbf{u}_3) - |\partial_{x_1} \gamma|^2 \partial_{x_3} \mathbf{u}_3 + \frac{|\partial_{x_1} \gamma|^2}{1 + \sqrt{1 + |\partial_{x_1} \gamma|^2}} \left( \partial_{x_2} \mathbf{u}_3 + \partial_{x_3} \mathbf{u}_2 \right), \]
\[ B_5(\gamma, \mathbf{u}) = (\partial_{x_1} \gamma)\mathbf{u}_1 + \frac{|\partial_{x_1} \gamma|^2}{1 + \sqrt{1 + |\partial_{x_1} \gamma|^2}} \mathbf{u}_2, \]
\[ B_6(\gamma, \mathbf{u}) = \mu(\partial_{x_1} \gamma)(\partial_{x_2} \mathbf{u}_3), \]

and \( \mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)^T \). Here, to derive \( B_3 \) and \( B_4 \), we have used the following observations: First, by the identity \((P_{\partial_{x_2} \mathbf{K}^3} w) \cdot \mathbf{n}_{\partial_{x_2} \mathbf{K}^3} = 0, w \in \mathbb{R}^3 \), we see that the second component of \( P_{\partial_{x_2} \mathbf{K}^3} w \) is redundant, i.e., the second component of \( P_{\partial_{x_2} \mathbf{K}^3} w \) is given by \((P_{\partial_{x_2} \mathbf{K}^3} w)(\partial_{x_1} \gamma)e_1 \). Next, we read the boundary condition \( P_{\partial_{x_2} \mathbf{K}^3}(2\mu D(\mathbf{u}^*) \mathbf{n}_{\partial_{x_2} \mathbf{K}^3}) = g^* \) componentwise, i.e., this boundary condition is replaced by

\[ [P_{\partial_{x_2} \mathbf{K}^3}(2\mu D(\mathbf{u}^*) \mathbf{n}_{\partial_{x_2} \mathbf{K}^3})] \cdot e_{\ell} = g^* \cdot e_{\ell} \]

for \( \ell = 1, 3 \). The calculation

\[ 2D(\mathbf{u}^*) = 2D(\mathbf{u}) - \begin{pmatrix} 
(\partial_{x_1} \gamma)\partial_{x_2} \mathbf{u}_1 + (\partial_{x_2} \mathbf{u}_1)\partial_{x_1} \gamma & (\partial_{x_2} \mathbf{u}_2)\partial_{x_1} \gamma & (\partial_{x_2} \mathbf{u}_3)\partial_{x_1} \\
0 & 0 & 0
\end{pmatrix}, \]

implies that \( 2D(\mathbf{u}^*) \mathbf{n}_{\partial_{x_2} \mathbf{K}^3} \) is given by

\[ 2D(\mathbf{u}^*) \mathbf{n}_{\partial_{x_2} \mathbf{K}^3} |_{1} = b(x) \left[ - \left( \partial_{x_2} \mathbf{u}_1 + \partial_{x_2} \mathbf{u}_2 \right) + 2(\partial_{x_1} \gamma)\partial_{x_2} \mathbf{u}_1 + (\partial_{x_1} \gamma)\partial_{x_2} \mathbf{u}_2 
- |\partial_{x_1} \gamma|^2 \partial_{x_2} \mathbf{u}_1 - (\partial_{x_1} \gamma)^2 \partial_{x_2} \mathbf{u}_1 \right], \]
\[ 2D(\mathbf{u}^*) \mathbf{n}_{\partial_{x_2} \mathbf{K}^3} |_{2} = b(x) \left[ - 2\partial_{x_2} \mathbf{u}_2 + (\partial_{x_1} \gamma) \left( \partial_{x_2} \mathbf{u}_1 + \partial_{x_2} \mathbf{u}_2 \right) \right], \]
\[ 2D(\mathbf{u}^*) \mathbf{n}_{\partial_{x_2} \mathbf{K}^3} |_{3} = b(x) \left[ - \left( \partial_{x_2} \mathbf{u}_3 + \partial_{x_2} \mathbf{u}_2 \right) + (\partial_{x_1} \gamma) \left( \partial_{x_2} \mathbf{u}_1 + \partial_{x_2} \mathbf{u}_3 \right) - |\partial_{x_1} \gamma|^2 \partial_{x_2} \mathbf{u}_3 \right], \]

and hence we obtain

\[ [P_{\partial_{x_2} \mathbf{K}^3}(2\mu D(\mathbf{u}^*) \mathbf{n}_{\partial_{x_2} \mathbf{K}^3})] \cdot e_1 = b(x) \left[ - \left( \partial_{x_2} \mathbf{u}_1 + \partial_{x_2} \mathbf{u}_2 \right) + 2(\partial_{x_1} \gamma)\partial_{x_2} \mathbf{u}_1 + (\partial_{x_1} \gamma)(1 - |\partial_{x_1} \gamma|^2)\partial_{x_2} \mathbf{u}_2 - |\partial_{x_1} \gamma|^2 \partial_{x_2} \mathbf{u}_1 
- 2(\partial_{x_1} \gamma)(1 - |\partial_{x_1} \gamma|^2)(\partial_{x_1} \gamma)\partial_{x_2} \mathbf{u}_1 - (\partial_{x_1} \gamma)^2 \partial_{x_2} \mathbf{u}_2 - 2(\partial_{x_1} \gamma)^3 \partial_{x_2} \mathbf{u}_1 \right], \]
\[ [P_{\partial_{x_2} \mathbf{K}^3}(2\mu D(\mathbf{u}^*) \mathbf{n}_{\partial_{x_2} \mathbf{K}^3})] \cdot e_3 = b(x) \left[ - \left( \partial_{x_2} \mathbf{u}_3 + \partial_{x_2} \mathbf{u}_2 \right) + (\partial_{x_1} \gamma) \left( \partial_{x_2} \mathbf{u}_1 + \partial_{x_2} \mathbf{u}_3 \right) - |\partial_{x_1} \gamma|^2 \partial_{x_2} \mathbf{u}_3 \right]. \]
Notice that all perturbation operators are linear and analytic with respect to $\mathbf{u}$. The perturbation operators can be estimated as follows:

$$
|B_1(\gamma, \mathbf{u})|_{F_0,\delta(J;K^3)} \leq |\partial_{x_1} \gamma|_{L^\infty(|R|)} \left(2 + |\partial_{x_1} \gamma|_{L^\infty(|R|)}\right) |\nabla^2 \mathbf{u}|_{F_0,\delta(J;K^3)} + |\mathbf{u}|_{E_2,\delta(J;K^3)}
$$

$$
+ |\partial_{x_1} \gamma|_{L^\infty(|R|)} |\nabla \mathbf{u}|_{F_0,\delta(J;K^3)},
$$

$$
|B_2(\gamma, \mathbf{u})|_{L^p_0(J;H^{-1,q}(K^3))} \leq |\partial_{x_1} \gamma|_{L^\infty(|R|)} |\nabla^2 \mathbf{u}|_{F_0,\delta(J;K^3)}
$$

$$
+ \left(|\partial_{x_1} \gamma|_{L^\infty(|R|)} + |\partial_{x_1} \gamma|_{L^\infty(|R|)}\right) |\nabla \mathbf{u}|_{F_0,\delta(J;K^3)},
$$

$$
|\partial_i B_2(\gamma, \mathbf{u})|_{L^p_0(J;H^{-1,q}(K^3))} \leq |\partial_{x_1} \gamma|_{L^\infty(|R|)} |\partial_i \mathbf{u}|_{E_1,\delta(J;K^3)},
$$

where constants $C$ are independent of $T$. Here, we use the trace theorem (cf. Lindemulder [19, pp. 88]) to derive the estimates

$$
|\nabla \mathbf{u}|_{F_{p,q,\delta}^{1/2-1/(2q)}(J;LC^3)} \leq C|\mathbf{u}|_{E_1,\delta(J;K^3)},
$$

$$
|\nabla \mathbf{u}|_{F_{p,q,\delta}^{1/2-1/(2q)}(J;LC^3)} \leq C|\mathbf{u}|_{E_1,\delta(J;K^3)},
$$

$$
|\nabla \mathbf{u}|_{F_{p,q,\delta}^{1/2-1/(2q)}(J;LC^3)} \leq C|\mathbf{u}|_{E_1,\delta(J;K^3)},
$$

Notice that constants appeared in these estimates are independent of $T$ because $\mathbf{u}$ vanishes at $t=0$. To obtain the estimates in $L^p_0(J;B^{1-1/q}_q(S))$ and $L^p_0(J;B^{2-1/q}_q(S))$, $S \in \{\partial K^3, \partial_3 K^3\}$, we use the estimate

$$
|a| \psi|_{B^{q,s}_q(|R|)} \leq |a|_{L^\infty(|R|)} \psi|_{B^{q,s}_q(|R|)} + C|\psi|_{B^{q,s}_q(|R|)} \psi|_{L^s(|R|)}^{1-s},
$$

where $0 < s < 2$, $1 \leq q < \infty$, $0 \leq \theta < 1$, $a \in C^2(|R|)$, and $\psi \in B^{q,s}_q(|R|)$, see [33, Lem. 6.2.8]. Here, the constant $C$ depends linearly on $|a|_{B^{2,\infty}_q(|R|)}$.\footnote{It is known that $C^2(|R|)$ embeds into $B^{2,\infty}_q(|R|)$. Besides, from $0 < s < 2$, it holds $B^{2,\infty}_q(|R|) \hookrightarrow B^{s,\infty}_q(|R|)$ for all $1 \leq q < \infty$, and hence we obtain $C^2(|R|) \hookrightarrow B^{s,\infty}_q(|R|)$ for $1 \leq q < \infty$ and $0 < s < 2$. For the embedding properties, we refer to Triebel [52].} Choosing $\theta$ small such that $\theta \leq |\partial_{x_1} \gamma|_{L^\infty(|R|)}$, we observe

$$
|b(x)B_3,1(\gamma, \mathbf{u})|_{L^p_0(J;B^{1-1/q}_q(\partial_3 K^3))}
$$

$$
\leq \sum_{\tau=0}^{3} |\partial_{x_1} \gamma|_{L^\infty(|R|)} |\nabla \mathbf{u}|_{L^p_0(J;B^{1-1/q}_q(\partial_3 K^3))} + C_{\gamma} |\nabla \mathbf{u}|_{L^p_0(J;L^q(\partial_3 K^3))},
$$

$$
|b(x)B_3,3(\gamma, \mathbf{u})|_{L^p_0(J;B^{1-1/q}_q(\partial_3 K^3))}
$$

$$
\leq \sum_{\tau=0}^{3} |\partial_{x_1} \gamma|_{L^\infty(|R|)} |\nabla \mathbf{u}|_{L^p_0(J;B^{1-1/q}_q(\partial_3 K^3))} + C_{\gamma} |\nabla \mathbf{u}|_{L^p_0(J;L^q(\partial_3 K^3))},
$$

$$
|b(x)B_4(\gamma, \mathbf{u})|_{L^p_0(J;B^{2-1/q}_q(\partial_3 K^3))} \leq C \left(|\partial_{x_1} \gamma|_{L^\infty(|R|)} |\nabla \mathbf{u}|_{L^p_0(J;B^{2-1/q}_q(\partial_3 K^3))} + C_{\gamma} |\nabla \mathbf{u}|_{L^p_0(J;L^q(\partial_3 K^3))}\right),
$$

$$
|b(x)B_5(\gamma, \mathbf{u})|_{L^p_0(J;B^{2-1/q}_q(\partial_3 K^3))} \leq C \left(|\partial_{x_1} \gamma|_{L^\infty(|R|)} |\nabla \mathbf{u}|_{L^p_0(J;B^{2-1/q}_q(\partial_3 K^3))} + C_{\gamma} |\nabla \mathbf{u}|_{L^p_0(J;L^q(\partial_3 K^3))}\right),
$$

where the constants $C$ and $C_{\gamma}$ are independent of $T$. Besides, the constant $C_{\gamma}$ depends on $\gamma$ and $C_{\gamma} \to 0$ as $|\partial_{x_1} \gamma|_{L^\infty(|R|)} \leq c \to 0$.}
It remains to consider \((B_2(\gamma, \overline{u}), b(x)B_4(\gamma, \overline{u}), 0)\) in \(H^{1,p}_\delta(J; \tilde{H}^{-1,q}(\mathbb{K}^3))\). Integrating by parts with respect to the variable \(\tau_2\), it holds
\[
((B_2(\gamma, \overline{u}), b(x)B_4(\gamma, \overline{u}), 0) | \phi)_{\mathbb{K}^3} = -(B_2(\gamma, \overline{u}) | \phi)_{\mathbb{K}^3} + (b(x)B_4(\gamma, \overline{u}) | \phi)_{\partial_2 \mathbb{K}^3}
\]
for any \(\phi \in \tilde{H}^{1,q}(\mathbb{K}^3)\), and thus it is clear that
\[
|(B_2(\gamma, \overline{u}), B_4(\gamma, \overline{u}), 0)|_{H^{1-p}_\delta(J; \tilde{H}^{-1,q}(\mathbb{K}^3))} \leq C|\frac{\partial_x \gamma}{L^\infty}\|u\|_{E_{1,3}(J; \mathbb{K}^3)}.
\]
We now solve (5.7). Let \(Z(J; \mathbb{K}^3) := E_{1,3}(J; \mathbb{K}^3) \times E_{2,3}(J; \mathbb{K}^3)\) be the solution space, while \(Y(J; \mathbb{K}^3)\) be the product space of given data. In addition, let \(\mathcal{D}_Z(J; \mathbb{K}^3)\) and \(\mathcal{D}_Y(J; \mathbb{K}^3)\) denote the solution space \(Z(J; \mathbb{K}^3)\) and the data space \(Y(J; \mathbb{K}^3)\) with vanishing time trace at \(t = 0\). Define
\[
z = (\alpha, \overline{u}) \in \mathcal{D}_Z(J; \mathbb{K}^3),
\]
\[
F := (0, \mathcal{J}_{\nabla}, -\mathcal{J}_{\nabla}, -\mathcal{J}_{\nabla}, -\mathcal{J}_{\nabla}, 0) \in \mathcal{D}_Y(J; \mathbb{K}^3),
\]
\[
Bz = (B_1(\gamma, \overline{u}), B_2(\gamma, \overline{u}), b(x)B_3(\gamma, \overline{u}), b(x)B_4(\gamma, \overline{u}), B_5(\gamma, \overline{u}), 0) : \mathcal{D}_Z(J; \mathbb{K}^3) \to \mathcal{D}_Y(J; \mathbb{K}^3),
\]
where \(B_3(\gamma, \overline{u}) = (B_{3,1}(\gamma, \overline{u}), B_{3,3}(\gamma, \overline{u}))\). Denoting the left-hand side of (5.7) by \(L\), we see that \(L\) is isomorphism from \(\mathcal{D}_Z(J; \mathbb{K}^3)\) to \(\mathcal{D}_Y(J; \mathbb{K}^3)\) and may rewrite (5.7) in the abstract form
\[
Lz = Bz + F.
\]
Recalling the estimates for the perturbation operators, we find
\[
|Bz|_{\mathcal{D}_Z(J; \mathbb{K}^3)} \leq C|\frac{\partial_x \gamma}{L^\infty}||z| + M|u|_{L^2_{1,p}(J; H^{1+1/4,q}(\mathbb{K}^3))}
\]
with some constants \(C, M > 0\) independent of \(T\). By the mixed derivative theorem (cf. [33, Ch. 4]) and the Sobolev embedding (cf. [22, Cor. 1.4]), we have
\[
o E_{1,3}(J; \mathbb{K}^3) \hookrightarrow o H^{1/2-1/4,p}(J; H^{1+1/4,q}(\mathbb{K}^3)) \hookrightarrow L^2_{1,p}(J; H^{1+1/4,q}(\mathbb{K}^3)),
\]
where the embedding constants are independent of \(T\). Hence, the Hölder inequality yields
\[
|\overline{u}|_{L^p_{1,p}(J; H^{1+1/4,q}(\mathbb{K}^3))} \leq T^{1/(2p)}|u|_{L^2_{1,p}(J; H^{1+1/4,q}(\mathbb{K}^3))} \leq CT^{1/(2p)}|\overline{u}|_{E_{1,3}(J; \mathbb{K}^3)},
\]
where \(C > 0\) does not depend on \(T\). Let \(\epsilon\) be a given (small) constant and assume \(|\nabla_x \gamma|_{L^\infty} \leq \epsilon\). Then, combined with (5.9), the Neumann series argument implies that there exists small \(T > 0\) such that (5.8) admits a unique solution \((\overline{u}, \alpha) \in \mathcal{D}_Z(J; \mathbb{K}^3)\). Since \(J = [0, T]\) is compact, we can solve (5.7) for \(J = [0, T]\) by repeating these arguments finitely many times, where \(T > 0\) is now arbitrary. This completes the proof of the theorem. \(\Box\)

We next deal with the Stokes equations in \(\mathbb{K}^3\) with slip-free boundary conditions:

\[
\begin{aligned}
\partial_t u - \mu \Delta u + \nabla \pi &= f, \quad \text{in } \mathbb{K}^3 \times J, \\
\div u &= f_{\div}, \quad \text{in } \mathbb{K}^3 \times J, \\
P_{\partial_2 \mathbb{K}^3}(2\mu D(u)n_{\partial_2 \mathbb{K}^3}) &= g', \quad \text{in } \partial_2 \mathbb{K}^3 \times J, \\
u \cdot n_{\partial_2 \mathbb{K}^3} &= g_2, \quad \text{in } \partial_2 \mathbb{K}^3 \times J, \\
\partial_t \eta - u_3 &= d, \quad \text{in } \partial_3 \mathbb{K}^3 \times J, \\
\mu(\partial_3 u_3 + \partial_m u_3) &= k_m, \quad \text{in } \partial_3 \mathbb{K}^3 \times J, \\
2\mu \partial_3 u_3 - \sigma \partial_3 \Delta_3 \eta &= k_3, \quad \text{in } \partial_3 \mathbb{K}^3 \times J, \\
u(0) &= u_0, \quad \text{in } \mathbb{K}^3, \\
\eta(0) &= \eta_0, \quad \text{on } \partial_3 \mathbb{K}^3,
\end{aligned}
\]
Lemma 5.5. Suppose the assumptions given in Theorem 5.1. Let

Theorem 5.1. However, it is possible to obtain additional time-regularity for

\[ m = 1,2 \text{ and } g' = (g_1,0,g_3)^T. \] Let the space \( \tilde{H}^{-1,q}_{\partial_3 \mathbb{K}^3} \) be the set of all \( (\varphi_1,\varphi_2) \in L^q(\mathbb{K}^3_1) \times B^{2-1/q}_{q,q}(\partial_2 \mathbb{K}^3_3) \) satisfying \( (\varphi_1,\varphi_2) \in \tilde{H}^{-1,q}_{\partial_3 \mathbb{K}^3} \). Setting

\[ \langle (\varphi_1, \varphi_2) \mid \phi \rangle_{\mathbb{K}^3_3} := -\langle \varphi_1 \mid \phi \rangle_{\mathbb{K}^3_3} + \langle \varphi_2 \mid \phi \rangle_{\partial_2 \mathbb{K}^3_3} \text{ for any } \phi \in \tilde{H}^{-1,q}_{\partial_3 \mathbb{K}^3} \),
\]

we observe

\[ \langle (f_{\text{div}},g_2) \mid \phi \rangle_{\mathbb{K}^3_3} = -(u \mid \nabla \phi)_{\mathbb{K}^3_3} \text{ for any } \phi \in \tilde{H}^{-1,q}_{\partial_3 \mathbb{K}^3} \]

as follows from the divergence equation. The next theorem can be proved in the same manner as in the proof of Theorem 5.3, and hence we may omit the details.

**Theorem 5.4.** Let \( T > 0 \) and \( J = (0,T) \). Assume that \( p, q, \) and \( \delta \) satisfy (4.17). The problem (5.10) admits a unique solution \((u,\pi,\eta)\) with \( u \in E_{1,\delta}(J;\mathbb{K}^3_3), \pi \in E_{2,\delta}(J;\mathbb{K}^3_3), \text{Tr}_{\partial_3 \mathbb{K}^3}[\pi] \in E_{3,\delta}(J;\partial_3 \mathbb{K}^3_3), \eta \in E_{4,\delta}(J;\partial_3 \mathbb{K}^3_3), \) if and only if

(a) \( f \in F_{0,\delta}(J;\mathbb{K}^3_3); \)
(b) \( f_{\text{div}} \in F_{1,\delta}(J;\mathbb{K}^3_3); \)
(c) \( g_\ell \in F_{2,\delta}(J;\partial_2 \mathbb{K}^3_3) \) for \( \ell = 1,3; \)
(d) \( g_2 \in F_{3,\delta}(J;\partial_2 \mathbb{K}^3_3); \)
(e) \( d \in F_{3,\delta}(J;\partial_3 \mathbb{K}^3_3); \)
(f) \( k_j \in F_{2,\delta}(J;\partial_2 \mathbb{K}^3_3) \) for \( j = 1,2,3; \)
(g) \( \langle f_{\text{div}},g_2 \rangle \in H^1_{\delta}(J;\tilde{H}^{-1,q}_{\partial_3 \mathbb{K}^3}); \)
(h) \( u_0 \in B^{2\delta-1/p}_{q,q}(\partial_3 \mathbb{K}^3_3)^3 \) and \( \text{div } u_0 = f_{\text{div}}(0); \)
(i) \( \eta_0 \in B^{2\delta-1/p-1/q}_{q,q}(\partial_3 \mathbb{K}^3_3); \)
(j) \( g'(0) = \text{Tr}_{\partial_2 \mathbb{K}^3_3}[P_{\partial_2 \mathbb{K}^3_3}(2\mu D(u_0)n_{\partial_2 \mathbb{K}^3_3})] \) and \( k_m(0) = \text{Tr}_{\partial_3 \mathbb{K}^3_3}[\mu(\partial_3 u_0,m + \partial_m u_0,m) ] \) if \( 1/p + 1/(2q) < \delta - 1/2; \)
(k) \( \text{Tr}_{\partial_2 \mathbb{K}^3_3}[g'(0)] = \text{Tr}_{\partial_2 \mathbb{K}^3_3}[P_{\partial_2 \mathbb{K}^3_3}(2\mu D(u_0)n_{\partial_2 \mathbb{K}^3_3})] \) and \( \text{Tr}_{\partial_3 \mathbb{K}^3_3}[k_m(0)] = \text{Tr}_{\partial_3 \mathbb{K}^3_3}[\mu(\partial_3 u_0,m + \partial_m u_0,m) ] \) if \( 1/p + 1/q < \delta - 1/2. \)

Furthermore, the solution map is continuous between the corresponding spaces.

### 5.2. Regularity of the pressure

In general, the pressure \( \pi \) has no more regularity than one given in Theorem 5.1. However, it is possible to obtain additional time-regularity for \( \pi \) in a special situation.

**Lemma 5.5.** Suppose the assumptions given in Theorem 5.1. Let \((u,\pi,\eta)\) be the solutions to (5.1) with

\[ u_0 = \eta_0 = f_{\text{div}} = 0 \text{ in } \Omega_*, \quad u_3 = 0 \text{ on } \Gamma_*, \]

for a.e. \( t \in J \) and \( f \in H^{\alpha,p}_0(J;L^q(\Omega_*))^3 \) for some \( \alpha \in (0,1/2-1/(2q)) \). Then the following assertions hold.

1. If \( \Omega_* \) is bounded domain given by (3.4), then it holds \( \pi \in H^{\alpha,p}_0(J;L^q(\Omega_*)) \) possessing the estimate

\[ |\pi|_{H^{\alpha,p}_0(J;L^q(\Omega_*))} \leq C \left( |u|_{E_{\delta,\delta}(J;\Omega_*)} + |\text{Tr}_\Gamma \pi|_{E_{\delta,\delta}(J;\Gamma_*)} + |f|_{H^{\alpha,p}_0(J;L^q(\Omega_*))} \right) \]

Here, the constant \( C \) does not depend on the length of the interval \( J \).

2. If \( \Omega_* \) is a full space, (a bent) half space, or a (bent) quarter space, then \( (\pi)_{K} \in H^{\alpha,p}_0(J;L^q(K)) \) for each bounded domain \( \Omega^R_K \subset \Omega_* \) with \( \Omega^R = \Omega_* \cap B(0,R) \), \( R > 0 \) large. In addition, it holds the estimate

\[ |P_{0R} \pi|_{H^{\alpha,p}_0(J;L^q(\Omega_*))} \leq C \left( |u|_{E_{\delta,\delta}(J;\Omega_*)} + |\text{Tr}_\Gamma \pi|_{E_{\delta,\delta}(J;\Gamma_*)} + |f|_{H^{\alpha,p}_0(J;L^q(\Omega_*))} \right) \]

with a constant \( C \) independent of the length of the interval \( J \). Here, \( P_{0R} \pi \) denotes the mean zero part of \( \pi \) with respect to \( \Omega^R_* \) in case the pressure \( \pi \) does not appear on the boundary and \( P_{0R} = I \) otherwise.
Proof. (1) Let \( \varphi \in L^q(\Omega) \) be fixed with mean zero. Consider the elliptic problem

\[
\begin{align*}
\Delta \phi &= \varphi, & \text{in } \Omega_s, \\
\phi &= 0, & \text{on } \Gamma_s, \\
\mathbf{n}_{\Sigma_s} \cdot \nabla \phi &= 0, & \text{on } \Sigma_s, \\
\mathbf{n}_B \cdot \nabla \phi &= 0, & \text{on } B.
\end{align*}
\]

(5.11)

By Lemma B.1, we have a unique solution \( \phi \in H^{2,q}(\Omega_s) \). Then, from integration by parts, we observe

\[
(p \mid \varphi)_{\Omega_s} = (p \mid \Delta \phi)_{\Omega_s} = -\langle \nabla p, \nabla \varphi \rangle_{\Omega_s} + \langle p \mathbf{n}_{\partial \Omega_s}, \nabla \varphi \rangle_{\partial \Omega_s}.
\]

Furthermore, we have \( \eta \). In the following, we show that these reductions can be observed. We first extend \( \tilde{\phi} \) to the above identity, and hence, taking the supremum of the left-hand side over all \( \varphi \in L^q(\Omega) \) with \( |\varphi|_{L^q(\Omega)} \leq 1 \), we arrive at

\[
|\partial^\alpha p|_{L^p(\Omega)} \leq C \left( |\partial^\alpha \nabla \varphi|_{L^p(\Omega)} + |\partial^\alpha (\mathbf{n}_{\partial \Omega_s} \cdot \nabla \varphi)|_{L^p(\Omega)} \right)
\]

for each \( \alpha \in \{0, 1/2 - 1/(2q)\} \) because it holds

\[
\hat{p}_{p,q,\delta} \to p_{p,q,\delta} \quad \text{as } \epsilon \to 0,
\]

for arbitrary \( \epsilon > 0 \), see, e.g., [23] for the detail. This shows \( p \in H^{p,q,\delta}(\Omega) \).

(2) The proof of the second statement is essentially the same as in the proof of the first assertion. Let \( \varphi \in L^q(\hat{\Omega}) \) be fixed with mean zero. Extend \( \hat{\varphi} \) by zero to \( \varphi \in L^q(\Omega) \). Then, by Lemma B.1, the elliptic problem (5.11) with \( \tilde{\varphi} \) as an inhomogeneity in the first equation is uniquely solvable. Especially the solution admits the regularity \( \nabla \phi \in H^{2,q}(\Omega_s) \) and the estimate

\[
|\nabla \phi|_{L^q(\Omega_s)} + |\nabla^2 \phi|_{L^q(\Omega_s)} \leq C|\tilde{\varphi}|_{L^q(\Omega_s)} \leq C|\varphi|_{L^q(\Omega)}.
\]

Furthermore, we have \( (P_{\Omega_s} \varphi \mid \varphi)_{\Omega_s} = (P_{\Omega_s} \varphi \mid \tilde{\varphi})_{\Omega_s} \). Thus, employing the same argument employed in the proof of the first assertion yields the desired result.

\[ \square \]

5.3. Reduction of the data. It is convenient to reduce the given data in (5.1) to the case

\[ \mathbf{f} = \mathbf{f}_{\text{div}} = \mathbf{u}_0 = \eta_0 = g = \mathbf{n} \cdot \mathbf{n}_{\Sigma_s} = h_3 = 0. \]

In the following, we show that these reductions can be observed. We first extend \( \eta_0 \in B^{2+\delta-1/p-1/q}(\Gamma_s) \) and \( u_{0,3}|_{\Gamma_s}, d|_{t=0} \in B^{2+\delta-1/p-1/q}(\Gamma) \) to \( \tilde{\eta}_0 \in B^{2+\delta-1/p-1/q}(\partial \mathbb{R}^3_+) \) and \( \tilde{u}_{0,3}|_{\partial \mathbb{R}^3_+}, \tilde{d}|_{t=0} \in B^{2+\delta-1/p-1/q}(\partial \mathbb{R}^3_+) \), respectively. Using these functions, define

\[
\tilde{\eta}_s(t) := 2e^{-\frac{(t-\Delta \tilde{\eta}_s)}{2t}} - e^{-\frac{(t-\Delta \tilde{\eta}_s)}{2t}} \eta_0 + \left[ e^{-\frac{(t-\Delta \tilde{\eta}_s)}{t}} - e^{-\frac{(t-\Delta \tilde{\eta}_s)}{t}} \right] (I - \Delta \tilde{\eta}_s)^{-1}(\tilde{u}_{0,3}|_{\partial \mathbb{R}^3_+} + \tilde{d}|_{t=0}).
\]

for \( t \geq 0 \). Then, as we discussed in the proof of Theorem 4.5, the function \( \tilde{\eta}_s \) has the regularity \( \tilde{\eta}_s \in \mathbb{E}_{4,\delta}(J \cdot \partial \mathbb{R}^3_+) \) satisfying \( \tilde{\eta}_s(0) = \tilde{\eta}_0 \) and \( \partial_t \tilde{\eta}_s(0) = \tilde{u}_{0,3}|_{\partial \mathbb{R}^3_+} + \tilde{d}|_{t=0} \). Setting \( \eta_s = \tilde{\eta}_s|_{\Gamma_s} \), we find that \( \eta_s(0) = \eta_0 \) and \( \partial_t \eta_s(0) = u_{0,3}|_{\Gamma_s} + d|_{t=0} \). Hence, if we set \( \eta_5 := \eta - \eta_s \), we observe \( \eta_5(0) = \partial_t \eta_5(0) = 0 \).
Next, let $q_0 := -2\mu \partial_t u_{0,3} \vert_{\Gamma_3} + \sigma \Delta_{\Gamma_3} \eta_0 + k_3 \vert_{t=0} \in B_{q,p}^{2(\delta - 1/p) - 1 - 1/q}(\Gamma_3)$. Here, it holds

$$k_3 \vert_{t=0} \in B_{q,p}^{2(\delta - 1/p) - 1 - 1/q}(\Gamma_3)$$

because the trace operator $\text{Tr}_{t=0}$ and the boundary operator $\text{Tr}_{\Gamma_3, \partial \Omega}$ may commute (cf. Lindenmuller [19, pp. 88]). We extend $q_0$ to some $\tilde{q}_0 \in B_{q,p}^{2(\delta - 1/p) - 1 - 1/q}(\partial \Omega_3)$ and define $\tilde{q}_s(t) := e^{-(1 - \Delta_{\Gamma_3}) t} \tilde{q}_0$. Then, we obtain

$$\tilde{q}_s \in F_{p,q,\delta}^{1/2 - 1/(2q)}(J; L^q(\partial \Omega_3)) \cap L_p^q(J; B_{q,q}^{1-1/q}(\partial \Omega_3)),$$

see [23, Thm. 4.2]. If we set $q_* := \tilde{q}_s \vert_{\Gamma_*}$, it holds

$$q_* \in F_{p,q,\delta}^{1/2 - 1/(2q)}(J; L^q(\Gamma_*)) \cap L_p^q(J; B_{q,q}^{1-1/q}(\Gamma_*))$$

and $q_*(0) = q_0$. Hence, given $q_*$, there exists a unique solution $\pi_* \in L_p^q(J; \dot{H}^{1,q}(\Omega_*))$ of the weak problem

$$\begin{cases}
(\nabla \pi_* \mid \nabla \varphi) = 0, & \text{for any } \varphi \in H_0^{1,q'}(\Omega_*), \\
\pi_* = q_*, & \text{on } \Gamma_*,
\end{cases}$$

as follows from Lemma B.3, where we suppose that $2 < q < \infty$. Here, we have defined $H_0^{1,q'}(\Omega_*)$ by $H_0^{1,q'}(\Omega_*) := \{ w \in H^{1,q'}(\Omega_*) \mid w = 0 \text{ on } \Gamma_* \}$. Now, we consider the parabolic problem

$$\begin{cases}
\partial_t u_* - \mu \Delta u_* = -\nabla \pi_* + f, & \text{in } \Omega_* \times J, \\
\mu(\partial_3 u_{*,m} + \partial_m u_{*,3}) = k_m, & \text{on } \Gamma_* \times J, \\
2\partial_3 u_{*,3} = k_3 - q_* - \sigma \Delta_{\Gamma_*} \eta_*, & \text{on } \Gamma_* \times J, \\
P_{\Sigma_*}(2\mu D(u_*) n_{\Sigma_*}) = P_{\Sigma_*} g, & \text{on } \Sigma_* \times J, \\
u_* \cdot n_{\Sigma_*} = g \cdot n_{\Sigma_*}, & \text{on } \Sigma_* \times J, \\
\mu(\partial_3 u_{*,m} + \partial_m u_{*,3}) = h_m, & \text{on } B \times J, \\
u_* \cdot n_3 = h_3, & \text{on } B \times J, \\
u_* = u_0, & \text{in } \Omega_*,
\end{cases}$$

(5.12)

where $m = 1, 2$. According to Lemma B.5, this system admits a solution $u_* \in H_0^{1,p}(J; L^q(\Omega_*))^3 \cap L_p^q(J; H^{2,q}(\Omega_*))^3$. Here, all relevant compatibility conditions of the data are valid from the assumption. Setting $u_0 = u - u_*$ and $\pi_0 = \pi - \pi_*$, there is no loss of generality in assuming $u_0 = \eta_0 = f = 0$. To deal with $f_{\text{div}}$, let us consider the elliptic problem

$$\begin{cases}
\Delta \psi = f_{\text{div}} - \text{div } u_*, & \text{in } \Omega_*, \\
\psi = 0, & \text{on } \Gamma_*, \\
n_{\Sigma_*} \cdot \nabla \psi = 0, & \text{on } \Sigma_*, \\
n_B \cdot \nabla \psi = 0, & \text{on } B.
\end{cases}$$

(5.13)

Notice that, by the compatibility conditions for $(f_{\text{div}}, g \cdot n_{\Sigma_*}, h_3)$, we observe that $\int_{\Omega_*} (f_{\text{div}} - \text{div } u_*) \, dx = 0$ and

$$f_{\text{div}} - u_* \in 0 H_0^{1,p}(J; H^{-1,q}(\Omega_*)) \cap L_p^q(J; H^{1,q}(\Omega_*)).$$

Thus, by Lemma B.4, there exists a solution $\psi$ satisfying $\nabla \psi \in 0 E_{1,\delta}(J; \Omega_*)$. Then, setting $u_6 := u_5 - \nabla \psi$ and $\pi_6 = \pi_5 + \partial_t \psi - \mu \Delta \psi$, there is no loss of generality in assuming $f_{\text{div}} = g \cdot n_{\Sigma_*} = h_3 = 0$. Furthermore, all the remaining data have vanishing trace at $t = 0$.

5.4. Localization procedure. To prove Theorem 5.1, we perform a standard localization procedure. However, compared with the discussion due to Wilke [53, Ch. 2], there is no requirement of the condition to the Neumann trace of the height function.
Step 1: Existence of a left inverse. Let \((u, \pi, \eta)\) be a solution to (5.1). Using the result of the last subsection, there exists \(\tilde{(u, \pi, \eta)}\) such that the triplet \(\tilde{(u, \pi, \eta)}\) solves

\[
\begin{aligned}
\partial_t \tilde{u} - \mu \Delta \tilde{u} + \nabla \tilde{\pi} &= 0, & \text{in } \Omega_* \times J, \\
\text{div } \tilde{u} &= 0, & \text{in } \Omega_* \times J, \\
\partial_t \tilde{\eta} - \tilde{u}_3 &= \tilde{d}, & \text{in } \Gamma_* \times J, \\
\mu(\partial_3 \tilde{u}_m + \partial_m \tilde{u}_3) &= \tilde{k}_m, & \text{in } \Gamma_* \times J, \\
2\mu \partial_3 \tilde{u}_3 - \tilde{\pi} - \sigma \Delta_{\Gamma_*} \tilde{\eta} &= \tilde{k}_3, & \text{in } \Gamma_* \times J, \\
P_{\Sigma_*}(2\mu \mathcal{D}(\tilde{u}) n_{\Sigma_*}) &= P_{\Sigma_*} \tilde{g} & \text{on } \Sigma_* \times J, \\
\tilde{u} \cdot n_{\Sigma_*} &= 0 & \text{on } \Sigma_* \times J, \\
\mu(\partial_3 \tilde{u}_m + \partial_m \tilde{u}_3) &= \tilde{h}_m & \text{on } B \times J, \\
\tilde{u}_3 &= 0 & \text{on } B \times J, \\
\tilde{u}(0) &= 0 & \text{in } \Omega_*, \\
\tilde{\eta}(0) &= 0 & \text{on } \Gamma_*
\end{aligned}
\]

with some modified data \(\langle \tilde{d}, \tilde{k}, \tilde{g}, \tilde{h} \rangle\) in the right regularity class, having vanishing traces at \(t = 0\), and satisfying the compatibility conditions stated in Theorem 5.1. Recalling Lemma 5.5, we find that \(\tilde{\pi} \in H^s_{\alpha} (J; L^q(\Omega_*))\) for any \(\alpha \in (0, 1/2 - 1/(2q))\).

Let us consider a small neighborhood \(\mathcal{O}_j\) of \(y_j \in \partial \Omega_*\), \(j = 1, \ldots, N\), defined by \(\mathcal{O}_j := \{ y \in \mathbb{R}^3 \mid |y - y_j| < r \}\). The radii \(r > 0\) is chosen suitably small such that we can apply the result in the previous subsections. From the definition of \(\Omega_*\), we may choose open balls \(\{ O_1, \ldots, O_N \}\) which is a finite open covering of \(\partial \Omega_*\), such that

(i) \(\bigcup_{j=1}^N \mathcal{O}_j \supset \partial \Omega_*\);  
(ii) \(\mathcal{O}_j \cap \Sigma_* \neq \emptyset\), \(\mathcal{O}_j \cap \{ x = (x', x_3) \mid x' = 0, x_3 \in \mathbb{R} \} = \emptyset\), \(\mathcal{O}_j \cap \Gamma_* = \emptyset\), \(\mathcal{O}_j \cap B = \emptyset\), \(j = 1, \ldots, N_1\);  
(iii) \(\mathcal{O}_j \cap B \neq \emptyset\), \(\mathcal{O}_j \cap \Gamma_* = \emptyset\), \(\mathcal{O}_j \cap \Sigma_* = \emptyset\), \(j = N_1 + 1, \ldots, N_2\);  
(iv) \(\mathcal{O}_j \cap \Gamma_* \neq \emptyset\), \(\mathcal{O}_j \cap \Sigma_* = \emptyset\), \(\mathcal{O}_j \cap B = \emptyset\), \(j = N_2 + 1, \ldots, N_3\);  
(v) \(\bigcup_{j=N_2+1}^N \mathcal{O}_j \supset (\partial \Sigma_* \cap \partial \Sigma_*)\), \(\mathcal{O}_j \cap \Gamma_* = \emptyset\), \(j = N_3 + 1, \ldots, N_4\);  
(vi) \(\bigcup_{j=N_4+1}^N \mathcal{O}_j \supset (\partial \Gamma_* \cap \partial \Sigma_*), \mathcal{O}_j \cap B = \emptyset\), \(j = N_4 + 1, \ldots, N\)

for some \(N_1, N_2, N_3, N_4, N \in \mathbb{N}\) and certain \(y_1, y_2, y_3 \in \partial \Omega_*\). Here, the numbers \(N_1, N_2, N_3, N_4, N\) depend on \(r\). Since \(\Omega_*\) is bounded, we may choose an open compact set \(\mathcal{O}_0\) such that \(\bigcup_{j=0}^N \mathcal{O}_j \supset \Omega, \mathcal{O}_0 \subset \Omega_*\), \(\mathcal{O}_0 \cap \partial \Omega_* = \emptyset\) are valid. For these covering \(\{ \mathcal{O}_j \}_{j=0}^N\), it is known that there exists a family \(\{ \varphi_j \}_{j=0}^N\) that is partition of unity in \(\Omega\) subordinate to the covering \(\{ \mathcal{O}_0, \ldots, \mathcal{O}_N \}\), i.e., functions \(\varphi_j, j = 0, \ldots, N\), satisfy

\(0 \leq \varphi_j \leq 1, \varphi_j \in C^\infty(\mathcal{O}_j), \) and \(\sum_{j=0}^N \varphi_j = 1, \) see, e.g., [15, Sec. 2.1] for the proof. In the following, we may use the abbreviations

\[
\begin{aligned}
\Omega_*^j &= \mathcal{O}_j \cap \Omega_* & \text{for } j = 0, \ldots, N, \\
\Gamma_*^j &= \begin{cases} 
\mathcal{O}_j \cap \Gamma_* & \text{for } j = N_2 + 1, \ldots, N_3, N_4 + 1, \ldots, N, \\
\emptyset & \text{otherwise,}
\end{cases} \\
\Sigma_*^j &= \begin{cases} 
\mathcal{O}_j \cap \Sigma_* & \text{for } j = 1, \ldots, N_1, N_2 + 1, \ldots, N_4, \\
\emptyset & \text{otherwise,}
\end{cases} \\
B_*^j &= \begin{cases} 
\mathcal{O}_j \cap \Gamma_* & \text{for } j = N_1 + 1, \ldots, N_2, N_3 + 1, \ldots, N_4, \\
\emptyset & \text{otherwise.}
\end{cases}
\end{aligned}
\]
Multiplying each equation in (5.14) by \( \varphi_j \), we see that \((\bar{u}^j, \pi^j, \tilde{\eta}^j) = (\varphi_j \mathbf{u}, \varphi_j \pi, \varphi_j \eta)\) satisfies the localized equations

\[
\begin{aligned}
\partial_t \bar{u}^j - \mu \Delta \bar{u}^j + \nabla \pi^j &= F^j(\bar{u}, \pi), & \text{in } \Omega^j \times J, \\
\text{div } \bar{u}^j &= F_{\text{div}}^j(\bar{u}), & \text{in } \Omega^j \times J, \\
\partial_t \tilde{\eta}^j - \tilde{\omega}^j &= \tilde{d}^j, & \text{in } \Gamma^j \times J, \\
\mu(\partial_t \bar{u}^j_m + \partial_m \bar{u}^j_\alpha) &= \tilde{k}^j_m + K^j_m(\bar{u}), & \text{in } \Gamma^j \times J, \\
2\mu(\partial_t \tilde{\eta}^j - \pi - \sigma \Delta \pi \tilde{\eta}^j) &= \tilde{k}^j_3 + K^j_3(\bar{u}, \pi), & \text{in } \Gamma^j \times J, \\
P_{\Sigma}(2\mu \mathbf{D}(\bar{u}^j)\mathbf{n}_{\Sigma^j}) &= P_{\Sigma}(\mathbf{g}^j) + G^j(\bar{u}) & \text{on } \Sigma^j \times J, \\
\bar{u}^j \cdot \mathbf{n}_{\Sigma^j} &= 0 & \text{on } \Sigma^j \times J, \\
\mu(\partial_t \tilde{\eta}^j_m + \partial_m \tilde{\omega}^j_\alpha) &= \tilde{h}^j_m + H^j_m(\bar{u}), & \text{on } B^j \times J, \\
\tilde{\eta}^j(0) &= 0 & \text{in } \Omega^j, \\
\tilde{\eta}^j(0) &= 0, & \text{on } \Gamma^j 
\end{aligned}
\]

for \( j = 0, \ldots, N \). Here, the right-hand members denote the remainder terms defined by

\[
\begin{aligned}
F^j(\bar{u}, \pi) &= -\mu[\Delta, \varphi_j] \bar{u} + [\nabla, \varphi_j] \pi, \\
F_{\text{div}}^j(\bar{u}) &= \bar{u} \cdot \nabla \varphi_j, \\
K^j_m(\bar{u}) &= (I - n_{\Gamma^j} \otimes n_{\Gamma^j})(\mu(\nabla \varphi_j \otimes \bar{u} + \bar{u} \otimes \nabla \varphi_j)n_{\Gamma^j} - \sigma[\Delta \pi, \varphi_j] \pi n_{\Gamma^j})n_{\Gamma^j}, \\
K^j_3(\bar{u}, \pi) &= \mu(\nabla \varphi_j \otimes \bar{u} + \bar{u} \otimes \nabla \varphi_j)n_{\Gamma^j} - \sigma[\Delta \pi, \varphi_j] \pi n_{\Gamma^j}, \\
G^j(\bar{u}) &= (I - n_{\Sigma^j} \otimes n_{\Sigma^j})(\mu(\nabla \varphi_j \otimes \bar{u} + \bar{u} \otimes \nabla \varphi_j))n_{\Sigma^j}, \\
H^j_m(\bar{u}) &= (I - n_{B^j} \otimes n_{B^j})(\mu(\nabla \varphi_j \otimes \bar{u} + \bar{u} \otimes \nabla \varphi_j))n_{B^j},
\end{aligned}
\]

respectively, where \([A, B] = AB - BA\). For \( j = 1, \ldots, N \), noting that \([\Delta, \varphi_j]\) are differential operators of order 1, we observe that

\[
[\Delta, \varphi_j] \bar{u} \in H^{1/2,p}_{\text{loc}}(J; L^3(\Omega^j)^3) \cap L^\alpha(J; H^{1,q}(\Omega^j)^3)
\]

since \( \bar{u} \in C_{1,\delta}(J; \Omega^j) \), see, e.g., Lindemulder [19, pp. 88] (cf. Prüss and Simonett [33, Ch. 6]). Since the pressure term \( \pi \) possesses the additional regularity property, we have

\[
[\nabla, \varphi_j] \pi \in H^{\alpha,p}_{\text{loc}}(J; L^3(\Omega^j)),
\]

and thus it holds

\[
F^j(\bar{u}, \pi) \in H^{\alpha,p}_{\text{loc}}(J; L^3(\Omega^j)^3) \cap L^\alpha(J; H^{1,q}(\Omega^j)^3) \quad (\text{5.16})
\]

for some fixed \( \alpha \in (0, 1/2 - 1/(2q)) \).

For \( j = 0 \), we have the standard Stokes equations in \( \mathbb{R}^3 \), where this problem is treated in Theorem 4.1. If \( j = 1, \ldots, N_1 \), we obtain the Stokes equations in bent half-spaces with slip boundary conditions, while if \( j = N_1 + 1, \ldots, N_2 \) and \( j = N_2 + 1, \ldots, N_3 \), we have the Stokes equations in half-spaces with slip boundary conditions and with free boundary conditions, respectively, see Theorem 4.3 and 4.5 for the associated results. Besides, for \( j = N_3 + 1, \ldots, N_4 \) and \( j = N_4 + 1, \ldots, N \), we obtain the Stokes equations in bent quarter-spaces with slip-slip boundary conditions and slip-free boundary conditions, respectively, in which we have already considered in section 5.1. Notice that we have rotated the coordinate system to observe the Stokes equations in the corresponding spaces. Hence, the solution operators for the charts \( \mathcal{O}_j, j = 0, \ldots, N \), are well-defined.
To employ Lemma 5.5, we reduce the problem (5.15) to the case \( F^j_{\text{div}}(\vec{u}) = 0 \). To this end, we consider the following elliptic problem:

\[
\begin{align*}
\Delta \phi^j &= F^j_{\text{div}}(\vec{u}), & \text{in } \Omega^j, \\
\phi^j &= 0, & \text{on } \Gamma^j, \\
\mathbf{n}_{\Sigma^j} \cdot \nabla \phi^j &= 0, & \text{on } \Sigma^j, \\
\mathbf{n}_B \cdot \nabla \phi^j &= 0, & \text{on } B^j.
\end{align*}
\]

Using Lemma B.4, we find that there exists the solution \( \phi^j \) with

\[
\nabla \phi^j \in \partial H^{1,p}_\emptyset(J; H^{1,q}(\Omega^j)^3) \cap L^p_\emptyset(J; H^{3,q}(\Omega^j)^3) =: \partial Z(J; \Omega^j)
\]

and with

\[
|\nabla \phi^j|_{Z(J; \Omega^j)} \leq C|\vec{u}|_{E_{1,1}(J; \Omega^j)}, \tag{5.17}
\]

where \( Z(J; \Omega^j) = H^{1,p}_\emptyset(J; H^{1,q}(\Omega^j)^3) \cap L^p_\emptyset(J; H^{3,q}(\Omega^j)^3) \). Here, the constant \( C \) may depend on \( N \) but independent of the length of \( J \). We now define \( \hat{\vec{u}}^j := \vec{u}^j - \nabla \phi^j, \hat{\pi}^j := \vec{\pi}^j + \partial_t \phi^j - \mu \Delta \phi^j \), and \( \hat{\eta}^j := \vec{\eta}^j \). Then we obtain

\[
\begin{align*}
\partial_t \hat{\vec{u}}^j - \mu \Delta \hat{\vec{u}}^j + \nabla \hat{\pi}^j &= F^j(\hat{\vec{u}}, \hat{\pi}), & \text{in } \Omega^j \times J, \\
\text{div } \hat{\vec{u}}^j &= 0, & \text{in } \Omega^j \times J, \\
\partial_t \hat{\eta}^j - \hat{\pi}^j &= \hat{\delta}^j + \partial_t \phi^j, & \text{in } \Gamma^j \times J, \\
\mu (\partial_t \hat{\vec{u}}^j + \partial_t \hat{\pi}^j) &= \hat{\delta}^j + \hat{\pi}^j, & \text{in } \Omega^j \times J, \\
2\mu \partial_t \hat{\pi}^j - \hat{\pi}^j - \sigma \Delta \hat{\eta}^j &= \hat{\pi}^j + \hat{\pi}^j, & \text{in } \Omega^j \times J, \\
P_{\Sigma^j}(2\mu D(\hat{\vec{u}}^j)\mathbf{n}_{\Sigma^j}) &= P_{\Sigma^j} \hat{\vec{g}}^j + \hat{\mathbf{G}}^j(\hat{\vec{u}}) & \text{on } \Sigma^j \times J, \\
\hat{\vec{u}}^j \cdot \mathbf{n}_{\Sigma^j} &= 0 & \text{on } \Sigma^j \times J, \\
\mu (\partial_t \hat{\vec{u}}^j + \partial_t \hat{\pi}^j) &= \hat{\delta}^j + \hat{\pi}^j, & \text{on } B^j \times J, \\
\hat{\vec{u}}^j(0) &= 0 & \text{in } \Omega^j, \\
\hat{\eta}^j(0) &= 0 & \text{on } \Gamma^j
\end{align*}
\]

where we have set

\[
\begin{align*}
\hat{K}^j_m(\vec{u}) &= K^j_m(\vec{u}) - \mu (I - \mathbf{n}_{T^j} \otimes \mathbf{n}_{T^j}) (2(\nabla \phi^j \otimes \nabla \phi^j) \mathbf{n}_{T^j} - \Delta \phi^j \mathbf{n}_{T^j}), \\
\hat{K}^j_3(\vec{u}, \eta) &= K^j_3(\vec{u}, \eta) - \mu (2(\nabla \phi^j \otimes \nabla \phi^j) \mathbf{n}_{T^j} - \Delta \phi^j \mathbf{n}_{T^j}) \cdot \mathbf{n}_{T^j}, \\
\hat{\mathbf{G}}^j(\vec{u}) &= \mathbf{G}^j(\vec{u}) - 2\mu (I - \mathbf{n}_{\Sigma^j} \otimes \mathbf{n}_{\Sigma^j}) \nabla \phi^j \mathbf{n}_{\Sigma^j}, \\
\hat{H}^j_m(\vec{u}) &= H^j_m(\vec{u}) - 2\mu (I - \mathbf{n}_B \otimes \mathbf{n}_B) \nabla \phi^j \mathbf{n}_B \bigg|_m.
\end{align*}
\]

Setting \( \tilde{z}_j = (\hat{\vec{u}}^j, \hat{\pi}^j, \hat{\eta}^j) \), we write the solution of (5.18) abstractly as

\[
L_j \tilde{z}_j = \tilde{H}_j + \tilde{B}_j \tilde{z}_j,
\]

where \( \tilde{H}_j \) stands for the set of given data and \( \tilde{B}_j \) denotes the remaining part on the right-hand side of (5.18). Since the operator \( L_j : E(T) \to F(T) \) is bounded and linear mapping and that \( L_j : \alpha E(T) \to \alpha F(T) \) is an isomorphism for each \( j = 0, \ldots, N \), there exists a constant \( C_0 \) independent of \( j \) and \( T \) such that the estimate

\[
|\tilde{z}_j|_{E(T)} \leq C_0 \left( |\tilde{H}_j|_{F(T)} + |\tilde{B}_j|_{F(T)} \right).
\]
holds. Here, $\mathcal{E}(T)$ means the space of solutions and $\mathcal{F}(T)$ the space of data. Since all terms in $B_j z_j$ have some extra time regularity, there exists an exponent $\nu > 0$ and a constant $C_1 > 0$ independent of $j$ such that 

$$|B_j z_j|_{\mathcal{F}(T)} \leq C_1 T^\nu |z_j|_{\mathcal{E}(T)}.$$ 

Recalling the definitions of $\tilde{u}^j$ and $\tilde{\pi}^j$, it remains to estimate $|\nabla (\partial_t \phi^j - \mu \Delta \phi^j)|_{\mathcal{F}_o, \delta(J; \Omega_+)}$. To this end, we use Lemma 5.5 and (5.16) to obtain

$$|\partial_t \phi^j - \mu \Delta \phi^j|_{H^{2,p}(J; L^4(\Omega_+))} = |\tilde{\pi}^j - \tilde{\pi}^j|_{H^{2,p}(J; L^4(\Omega_+))} \leq C \left( |\tilde{\pi}|_{\mathcal{E}_1, \delta(J; \Omega_+)} + |\tilde{\pi}^j|_{\mathcal{E}_4, \delta(J; \Gamma_+)} + |\tilde{H}|_{\mathcal{F}(T)} \right),$$

where $H$ stands for the right-hand members of (5.14). However, we have

$$|\Delta \phi^j|_{H^{1,p}(J; L^4(\Omega_+))} \leq C |\tilde{\pi}|_{\mathcal{E}_1, \delta(J; \Omega_+)}$$

and we infer

$$|\Delta \phi^j|_{H^{1,p}(J; L^4(\Omega_+))} \leq C |\Delta \phi^j|_{H^{1,p}(J; L^4(\Omega_+))} \leq C |\tilde{\pi}|_{\mathcal{E}_1, \delta(J; \Omega_+)}.$$ 

Hence, using

$$|\partial_t \phi^j|_{L^p(J; H^{2,\alpha}(\Omega_+))} \leq C |\tilde{\pi}|_{\mathcal{E}_1, \delta(J; \Omega_+)},$$

$$|\Delta \phi^j|_{H^{1,p}(J; L^4(\Omega_+))} \leq C |\tilde{\pi}|_{\mathcal{E}_1, \delta(J; \Omega_+)},$$

we obtain

$$|\nabla (\partial_t \phi^j)|_{L^p(J; L^4(\Omega_+))} \leq CT^{1/(2p)} |\Delta \phi^j|_{H^{1,2,p}(J; H^{1,\alpha}(\Omega_+))} \leq CT^{1/(2p)} |\Delta \phi^j|_{H^{1,p}(J; L^4(\Omega_+))} \leq CT^{1/(2p)} |\tilde{\pi}|_{\mathcal{E}_1, \delta(J; \Omega_+)}$$

and

$$|\nabla (\partial_t \phi^j)|_{L^p(J; L^4(\Omega_+))} \leq CT^{\alpha/p} |\partial_t \phi^j|_{H^{\alpha,p}(J; H^{1,\alpha}(\Omega_+))} \leq CT^{\alpha/p} |\partial_t \phi^j|_{H^{\alpha,p}(J; L^4(\Omega_+))} \leq CT^{\alpha/p} \left( |\tilde{\pi}|_{\mathcal{E}_1, \delta(J; \Omega_+)} + |\tilde{\pi}^j|_{\mathcal{E}_4, \delta(J; \Gamma_+)} + |\tilde{H}|_{\mathcal{F}(T)} \right)$$

for $j = 0, \ldots, N$. Consequently, we have

$$|\nabla (\mu \Delta \phi^j - \partial_t \phi^j)|_{L^p(J; L^4(\Omega_+))} \leq CT^{\alpha/p} \left( |\tilde{\pi}|_{\mathcal{E}_1, \delta(J; \Omega_+)} + |\tilde{\pi}^j|_{\mathcal{E}_4, \delta(J; \Gamma_+)} + |\tilde{H}|_{\mathcal{F}(T)} \right).$$

Especially, if $T \in (0, 1)$, we deduce that

$$|\nabla (\mu \Delta \phi^j - \partial_t \phi^j)|_{L^p(J; L^4(\Omega_+))} \leq C |\tilde{H}|_{\mathcal{F}(T)} + CT^{\alpha/p} \left( |\tilde{\pi}|_{\mathcal{E}_1, \delta(J; \Omega_+)} + |\tilde{\pi}^j|_{\mathcal{E}_4, \delta(J; \Gamma_+)} \right).$$

Summing up, we arrive at the estimate

$$|z_j|_{\mathcal{E}(T)} \leq C_2 |H_j|_{\mathcal{F}(T)} + C_3 T^\nu |z|_{\mathcal{E}(T)},$$

where the constants $C_2$ and $C_3$ are independent of $j$. Thus, taking the sum over all $j$, we can find some constants $C_2$ and $C_4$ such that

$$|z|_{\mathcal{E}(T)} \leq C_4 |H|_{\mathcal{F}(T)} + C_5 T^\nu |z|_{\mathcal{E}(T)}.$$ 

Hence, choosing $T > 0$ sufficiently small, we obtain the a priori estimate $|z|_{\mathcal{E}(T)} \leq C_6 |H|_{\mathcal{F}(T)}$ for the solution to (5.14). Repeating this successive argument for finitely times, the a priori estimate holds on each finite many time interval $J = [0, T]$. We, therefore, see that the operator $L$, defined by the left-hand side of (5.1), maps from $\mathcal{E}(T)$ to $\mathcal{F}(T)$ which is injective and has a closed range. This infers the existence of a left inverse for $L$, i.e., there exists the operator $S$ such that $SLz = z$ for all $z \in \mathcal{E}(T)$. Namely, this shows that the solution of (5.1) is unique.
Step 2: Existence of a right inverse. We next show the existence of a right inverse of $L$, that is, the existence of a solution to (5.1). Let the right-hand members of (5.1), denoted by $F \in F(T)$, and the initial data $(u_0, \eta_0) \in B^{2(\delta-1/p)}(\Omega_\ast)^3 \times B^{\delta+1-1/q}(\Gamma_\ast)$ be given such that the conditions in Theorem 5.1 are valid. Without loss of generality we can assume $u_0 = \eta_0 = 0$ due to the results in the previous section. Notice that this implies that all inhomogeneities have vanishing traces at $t = 0$ whenever they exist.

Let $u_\ast, \nabla \psi \in \mathbb{V}_{1,3}(J; \Omega_\ast)$ stand for the unique solution of (5.12) and (5.13), respectively, provided that $\tau_\ast = q_\ast = \eta_\ast = 0$. Set \( \bar{u} := u_\ast - \nabla \psi, \bar{\pi} := \mu \Delta \psi - \partial_t \psi, \) and \( \bar{\eta} = 0 \). Defining the operator \( \mathcal{S} \) by \( \mathcal{S}F := (\bar{u}, \bar{\pi}, \bar{\eta}) \), we see that

\[
L \mathcal{S}F = L(\bar{u}, \bar{\pi}, \bar{\eta}) = \begin{pmatrix}
  f \\
  f_{\text{div}} \\
  d + \bar{d}(u_\ast, \psi) \\
  k' + \mathcal{E}(\psi) \\
  k_3 + \mathcal{E}_3(\psi) \\
  P_{\Sigma_3} g + \mathcal{G}(\psi) \\
  g \cdot n_{\Sigma_3} \\
  h' + \mathcal{H}(\psi) \\
  h_3
\end{pmatrix},
\]

where we have set

\[
\begin{align*}
  k' &= (k_1, k_2)^T, \\
  \bar{d}(u_\ast, \psi) &= -(u_\ast - \nabla \psi) \cdot n_{\Gamma'_j}, \\
  \mathcal{E}_3(\psi) &= -2\mu(\nabla^2 \psi n_{\Gamma'_j})n_{\Gamma'_j} - \mu \Delta \psi, \\
  \mathcal{G}(\psi) &= -2\mu(I - n_{\Sigma'_j} \otimes n_{\Sigma'_j})(\nabla^2 \psi n_{\Sigma'_j}), \\
  \mathcal{H}(\psi) &= -\nabla \psi.
\end{align*}
\]

We next consider the following problem:

\[
\begin{align*}
  \partial_t \bar{u}_j - \mu \Delta \bar{u}_j + \nabla \bar{\pi}_j &= 0, & \text{in } \Omega'_j \times J, \\
  \nabla \bar{u}_j &= 0, & \text{in } \Omega'_j \times J, \\
  \partial_t \bar{\pi}_j - \bar{\pi}_j &= \bar{d}(u_\ast, \psi), & \text{in } \Gamma'_j \times J, \\
  \mu(\partial_m \bar{u}_m + \partial_m \bar{u}_j) &= \mathcal{E}_3(\psi), & \text{in } \Gamma'_j \times J, \\
  2\mu \partial_m \bar{u}_m - \bar{\pi}_j - \sigma \Delta \bar{\pi}_j &= \mathcal{E}_3(\psi), & \text{in } \Gamma'_j \times J, \\
  P_{\Sigma_3}(2\mu D(\bar{u}_j)n_{\Sigma'_j}) &= \mathcal{G}(\psi), & \text{on } \Sigma'_j \times J, \\
  \bar{u}_j \cdot n_{\Sigma'_j} &= 0 & \text{on } \Sigma'_j \times J, \\
  \mu(\partial_m \bar{u}_m + \partial_m \bar{u}_j) &= \mathcal{E}_3(\psi), & \text{on } B^j \times J, \\
  \bar{u}_j &= 0 & \text{on } B^j \times J, \\
  \bar{u}_j(0) &= 0 & \text{in } \Omega'_j, \\
  \bar{\pi}_j(0) &= 0, & \text{on } \Gamma'_j,
\end{align*}
\]

where \( \bar{d}(u_\ast, \psi) = \bar{d}(u_\ast, \psi) \varphi_j, \mathcal{E}_m(\psi) = \mathcal{E}_m(\psi) \varphi_j, \mathcal{E}_3(\psi) = \mathcal{E}_3(\psi) \varphi_j, \mathcal{G}(\psi) = \mathcal{G}(\psi) \varphi_j, \mathcal{H}(\psi) = \mathcal{H}(\psi) \varphi_j, \mathcal{H}_m(\psi) = \mathcal{H}_m(\psi) \varphi_j. \)

Here, the indices $j$ run from 1 through $N$ and let $\Omega'_j, \Gamma'_j, \Sigma'_j, B^j$, and $\varphi_j$ be defined as before. The right-hand members of (5.19) belong to the right regularity class with vanishing trace at $t = 0$ whenever they exist, and hence for each $j = 0, \ldots, N$, there exists a unique solution \( (\bar{u}_j, \bar{\pi}_j, \bar{\eta}_j) \) of (5.19) due to the results given before. Now, for $j = 0, \ldots, N$ let $\theta_j$ denote cut-off functions with support in $O_j$ such that $\theta_j = 1$ on the
support of $\varphi_j$. To keep the divergence-free condition, we consider the elliptic problem

$$
\begin{cases}
\Delta \psi^j = (\nabla \theta_j \cdot \nabla) \psi^j|_{\Omega_\ast}, & \text{in } \Omega_\ast, \\
\psi^j = 0, & \text{on } \Gamma_\ast, \\
n_{\Sigma_\ast} \cdot \nabla \psi^j = 0, & \text{on } \Sigma_\ast, \\
n_{B_\ast} \cdot \nabla \psi^j = 0, & \text{on } B.
\end{cases}
$$

Notice that $(\nabla \theta_j \cdot \nabla) \psi^j|_{\Omega_\ast}$ is mean value free due to the conditions for $\tilde{u}^j$. According to Lemma B.4, this elliptic problem admits unique solutions $\psi^j$ with the regularity

$$
\nabla \psi^j \in H^{1, p}(J; H^{1, q}(\Omega_\ast)^3) \cap L^p(J; H^{3, q}(\Omega_\ast)^3).
$$

Then, defining the operator $\tilde{S}$ by

$$
\tilde{S}F := \sum_{j=0}^{N} (\theta_j \tilde{u}^j - \nabla \psi^j, \theta_j \tilde{u}^j + \partial_\psi \psi^j - \mu \Delta \psi^j, \theta_j \tilde{u}^j)
$$

yields the identity

$$
L\tilde{S}F = \sum_{j=0}^{N} \left( \begin{array}{c}
-\mu [\Delta, \theta_j] \tilde{u}^j + [\nabla, \theta_j] \tilde{u}^j \\
\theta_j \tilde{d}^j(u_\ast, \psi) + \nabla \psi^j \cdot n_{\Gamma_\ast} \\
\theta_j K^j(\psi) + (I - n_{\Gamma_\ast} \otimes n_{\Gamma_\ast}) \tilde{K}(\tilde{u}^j, \tilde{\eta}^j) + \theta_j \tilde{K}^j(\psi) + \tilde{K}^j(\eta^j) \\
\theta_j \tilde{g}^j(\psi) + P_{\Sigma_\ast} \left( \mu \nabla \theta_j \otimes \tilde{u}^j + \tilde{u}^j \otimes \nabla \psi^j n_{\Sigma_\ast} \right) + \tilde{g}(\psi^j) \\
0 \\
\theta_j \tilde{K}^j(\psi) + (I - n_{B_\ast} \otimes n_{B_\ast}) \left( \mu \nabla \theta_j \otimes \tilde{u}^j + \tilde{u}^j \otimes \nabla \psi^j n_{B_\ast} \right) + \tilde{K}(\psi^j)
\end{array} \right)
$$

where we have set

$$
\tilde{K}(\tilde{u}^j, \tilde{\eta}^j) = \mu (\nabla \psi^j \otimes \tilde{u}^j + \tilde{u}^j \otimes \nabla \psi^j) n_{\Gamma_\ast} + \sigma [\Delta \psi^j, \theta_j] \tilde{u}^j n_{\Gamma_\ast}.
$$

Recalling $\theta_j = 1$ on the support of $\varphi_j$, we see that $\theta_j \tilde{d}^j(u_\ast, \psi) = \tilde{d}^j(u_\ast, \psi)$, $\theta_j \tilde{K}^j(\psi) = \tilde{K}^j(\psi)$, $\theta_j \tilde{K}^j(\psi) = \tilde{K}^j(\psi)$, $\theta_j \tilde{g}^j(\psi) = \tilde{g}^j(\psi)$, $\theta_j \tilde{K}^j(\psi) = \tilde{K}^j(\psi)$. Hence, by summing $\sum_{j=0}^{N} \theta_j \tilde{d}^j(u_\ast, \psi) = \tilde{d}(u_\ast, \psi)$, $\sum_{j=0}^{N} \theta_j \tilde{K}^j(\psi) = \tilde{K}(\psi)$, $\sum_{j=0}^{N} \theta_j \tilde{g}^j(\psi) = \tilde{g}(\psi)$, $\sum_{j=0}^{N} \theta_j \tilde{K}^j(\psi) = \tilde{K}(\psi)$. Finally, we set $\tilde{S}F := \tilde{S}F - \tilde{S}F = (I - R)F$, where we have set

$$
RF := \sum_{j=0}^{N} \left( \begin{array}{c}
-\mu [\Delta, \theta_j] \tilde{u}^j + [\nabla, \theta_j] \tilde{u}^j \\
\nabla \psi^j \cdot n_{\Gamma_\ast} \\
(I - n_{\Gamma_\ast} \otimes n_{\Gamma_\ast}) \tilde{K}(\tilde{u}^j, \tilde{\eta}^j) + \theta_j \tilde{K}^j(\psi) \\
\tilde{K}(\tilde{u}^j, \tilde{\eta}^j) \cdot n_{\Gamma_\ast} + \tilde{K}(\psi^j) \\
P_{\Sigma_\ast} \left( \mu \nabla \theta_j \otimes \tilde{u}^j + \tilde{u}^j \otimes \nabla \psi^j n_{\Sigma_\ast} \right) + \tilde{g}(\psi^j) \\
(I - n_{B_\ast} \otimes n_{B_\ast}) \left( \mu \nabla \theta_j \otimes \tilde{u}^j + \tilde{u}^j \otimes \nabla \psi^j n_{B_\ast} \right) + \tilde{K}(\psi^j)
\end{array} \right)
$$

All terms involving $\tilde{u}^j$ and $\tilde{\eta}^j$ are lower-order, and hence these terms possess additional time-regularity. In addition, the terms including $\psi^j$ also carry additional time-regularity because $\nabla \psi^j$ is regular enough. As for
Then it holds
\[ T > \pi \]
\[ \sum \text{estimating} \]
\[ \bar{\pi}^j \]
\[ \text{Hence, we can find some constants } C, \nu > 0 \text{ being independent of } T \text{ such that the estimate} \]
\[ |RF|_{F(T)} \leq CT^\nu |F|_{F(T)} \]
holds. Choosing \( T > 0 \) suitably small, we see that the operator \((I - R)\) has its inverse. Thus, the right inverse \( S \) for \( L \) is given by \( S := \hat{S}(I - R)^{-1} \). This completes the proof of Theorem 5.1. \( \square \)

6. Local well-posedness

6.1. Nonlinearity. As an application of the contraction mapping principle, we construct a unique strong solution to (3.2). To this end, we rewrite the system as
\[ Lz = N(z), \quad (u(0), \eta(0)) = (u_0, \eta_0), \quad (6.1) \]
where we have set \( z = (u, \pi, \eta) \) and \( L \) stands for the linear operator representing the left-hand side of (3.2).

The nonlinear mapping \( N = N(z) := (N_1, N_2, N_3, N_4, N_5, N_6) \) is given by
\[ N_1 := F(u, \pi, \eta), \]
\[ N_2 := F_{\text{div}}(u, \eta), \]
\[ N_3 := D(u, \eta), \]
\[ N_4 := (K_1(u, \eta), K_2(u, \eta), K_3(u, \eta)), \]
\[ N_5 := G(u, \eta), \]
\[ N_6 := (H_1(u, \eta), H_2(u, \eta)). \]

For shake of simplicity, we define
\[ E(T) := \{ (u, \pi, \eta) \in E_{1, \delta}(J; \Omega_*) \times E_{2, \delta}(J; \Omega_*) \times E_{5, \delta}(J; \Gamma_*) \mid \text{Tr}_{\Gamma_*}[\pi] \in E_{4, \delta}(J; \Gamma_*), \} \]
\[ E_{5, \delta}(J; \Gamma_*) := F_{p, q, \delta}^2(J; L^q(\Gamma_*)) \cap H_{\delta}^1(J; B_{p, q}^{2-1/q}(\Gamma_*)) \]
\[ \cap F_{p, q, \delta}^{1/2-1/(2q)}(J; H^{2, q}(\Gamma_*)) \cap L_0^2(J; B_{p, q}^{2-1/q}(\Gamma_*)), \]
\[ F(T) := \mathbb{F}_0(J; \Omega_*) \times F_1(J; \Omega_*), \]
\[ \mathbb{F}(T) := \mathbb{F}(J; \Omega_*) \times \mathbb{F}_2(J; \Gamma_*), \]
\[ \mathbb{F}(T) \cap \mathbb{F}_2(J; \Gamma_*), \]
\[ \text{for } T > 0, \text{ where } J = (0, T). \]
Here, the generic elements of \( \mathbb{F}(T) \) are the function \( N(z) \). With sufficiently small \( \varepsilon > 0 \), we set \( U_T := \{ z = (u, \pi, \eta) \in \mathbb{E}(T) \mid \| \eta \|_{L^\infty(J; L^\infty(\Gamma_*))} < \varepsilon \} \).

We first derive suitable estimates for the nonlinearity \( N(z) \).

**Proposition 6.1.** Let \( p, q, \delta \) satisfy
\[ 2 < p < \infty, \quad 3 < q < \infty, \quad \frac{1}{p} + \frac{3}{2q} < \delta - \frac{1}{2} \leq 1, \quad (6.2) \]
Then it holds
\[ (1) \text{ } N \text{ is a real analytic mapping from } U_T \text{ to } \mathbb{F}(T) \text{ and } N(0) = DN(0). \]
\[ (2) \text{ } DN(z) \in B(U_T, \mathbb{F}(T)) \text{ for any } z \in E(T). \]

Here, \( DN \) denotes the Fréchet derivative of \( N \).

**Remark 6.2.** (1) The last condition of (6.2) guarantees the embedding
\[ B_{q,p}^{2(\delta - 1/p)}(\Omega_*) \hookrightarrow \text{BUC}(J; \text{BUC}^1(\Omega_*)). \]
Furthermore, this condition also induces the conditions
\[ \frac{1}{p} + \frac{1}{2q} < \delta - \frac{1}{2}, \quad \frac{1}{p} + \frac{1}{q} < \delta - \frac{1}{2}, \]
and thus we need all compatibility conditions on the boundaries and the contact lines given before whenever \( p, q, \delta \) satisfy (6.2).

To prove Proposition 6.1, we introduce the following useful lemma.

**Lemma 6.3.** Assume that \( p, q, \delta \) satisfy (6.2) and let \( J = [0, T] \). Then the following assertions are valid.
(1) \( E_{1,\delta}(J; \Omega_\ast) \hookrightarrow \text{BUC}^1(J; \text{BUC}(\Gamma_\ast)) \).
(2) \( E_{3,\delta}(J; \Gamma_\ast) \hookrightarrow \text{BUC}(J; \text{BUC}(\Gamma_\ast)) \).
(3) \( E_{4,\delta}(J; \Gamma_\ast) \hookrightarrow E_{5,\delta}(J; \Gamma_\ast) \hookrightarrow \text{BUC}^1(J; \text{BUC}^1(\Gamma_\ast)) \cap \text{BUC}(J; \text{BUC}^2(\Gamma_\ast)) \).
(4) \( E_{3,\delta}(J; \Gamma_\ast) \) and \( E_{5,\delta}(J; \Gamma_\ast) \) are multiplication algebras.

Here, in the assertions (1)–(3), the embedding constants are independent of \( T > 0 \) if the time scales vanish at \( t = 0 \).

**Remark 6.4.** The assertions of Lemma 6.3 is also valid for the case when the domain \( \Omega_\ast \subset \mathbb{R}^n \), \( n \geq 2 \), is surrounded by smooth boundary \( \Gamma_\ast \) (at least of class \( C^{2-} \)) if we replace (6.2) by

\[
2 < p < \infty, \quad n < q < \infty, \quad \frac{1}{p} + \frac{n}{2q} < \delta - \frac{1}{2} \leq \frac{1}{2}.
\]

**Proof.** Using extensions and restrictions, and employing the standard localization procedure, it suffices to consider the cases \( \Gamma_\ast \times J = \mathbb{R}^2 \times \mathbb{R}, \Omega_\ast \times J = \mathbb{R}^3 \times \mathbb{R} \), see also [19].

(1) This is a direct consequence of the trace method of real interpolation (cf. Amann [2, Thm. III.4.10.2]). Here, the last condition of (6.2) ensures \( B_{q,p}^{2(\delta-1)/p}(\mathbb{R}^2) \hookrightarrow \text{BUC}^1(\mathbb{R}^3) \), see Triebel [52, Thm. 2.5.7] (cf. Sawano [35, Prop. 2.4]).

(2) From [23, Thm. 3.1], we have

\[
E_{3,\delta}(\mathbb{R}; \mathbb{R}^2) \hookrightarrow F_{p,q,\delta}^{(1-\theta_1)(1/2-1/(2q))}(\mathbb{R}; B_{q,q}^{\theta_1(1-1/q)}(\mathbb{R}^2))
\]

for any \( 0 < \theta_1 < 1 \), where we have used \( L^q(\mathbb{R}^2) = F_{q,2}^0(\mathbb{R}^2) \) and the real interpolation

\[
(F_{p,q,\delta}^0(\mathbb{R}^2), F_{q,q}^{1-1/q}(\mathbb{R}^2))_{\theta_1,q} = B_{q,q}^{\theta_1(1-1/q)}(\mathbb{R}^2),
\]

cf., [52, Thm. 2.4.1]. Then, by [22, Prop. 7.4] and [52, Thm. 2.5.7] (cf. [35, Prop. 2.4]), we see that

\[
F_{p,q,\delta}^{(1-\theta_1)(1/2-1/(2q))}(\mathbb{R}; B_{q,q}^{\theta_1(1-1/q)}(\mathbb{R}^2)) \hookrightarrow \text{BUC}(\mathbb{R}; \text{BUC}(\mathbb{R}^2))
\]

if \( \theta_1 \) satisfies

\[
(1 - \theta_1) \left( \frac{1}{2} - \frac{1}{2q} \right) - \left( 1 - \delta + \frac{1}{p} \right) > 0, \quad \theta_1 \left( 1 - \frac{1}{q} \right) - \frac{2}{q} > 0.
\]

Noting the last condition of (6.2), these both inequalities are equivalent to finding the constant \( 0 < \theta_1 < 1 \) such that

\[
\frac{2q-1}{1-q-1} < \theta_1 < \frac{2(\delta - p) - (2q)^{-1} - 2^{-1}}{1-q^{-1}}.
\]

Notice that this set is nonempty due to the last condition of (6.2). Hence, for any \( \theta_1 \) satisfying this condition, we obtain the embedding \( E_{3,\delta}(\mathbb{R}; \mathbb{R}^2) \hookrightarrow \text{BUC}(\mathbb{R}; \text{BUC}(\mathbb{R}^2)) \).

(3) By [23, Thm. 3.1], we find that

\[
H_{1,p}^{1,\rho}(\mathbb{R}; B_{q,q}^{2-1/q}(\mathbb{R}^2)) \cap L_p^\rho(\mathbb{R}; B_{q,q}^{3-1/q}(\mathbb{R}^2))
\]

\[
\hookrightarrow F_{p,q,\delta}^{1/2}(\mathbb{R}; B_{q,q}^{2-1/q}(\mathbb{R}^2)) \cap F_{p,\infty,\delta}^0(\mathbb{R}; B_{q,q}^{3-1/q}(\mathbb{R}^2))
\]

\[
\hookrightarrow F_{p,q,\delta}^{1/2-1/(2q)}(\mathbb{R}; B_{q,q}^2(\mathbb{R}^2))
\]

\[
\hookrightarrow F_{p,q,\delta}^{1/2-1/(2q)}(\mathbb{R}; H^{2,q}(\mathbb{R}^2))
\]

where we have used the real interpolation \( (B_{q,q}^{2-1/q}(\mathbb{R}^2), B_{q,q}^{3-1/q}(\mathbb{R}^2))_{1,1} = B_{q,q}^2(\mathbb{R}^2) \) (cf. Triebel [52, Thm. 2.4.1]). This yields \( E_{4,\delta}(J; \Gamma_\ast) \hookrightarrow E_{5,\delta}(J; \Gamma_\ast) \). Similarly, from [23, Thm. 3.1], we see that

\[
E_{5,\delta}(\mathbb{R}; \mathbb{R}^2) \hookrightarrow F_{p,q,\delta}^{2-1/(2q)}(\mathbb{R}; F_{q,q}^0(\mathbb{R}^2)) \cap F_{p,\infty,\delta}^1(\mathbb{R}; B_{q,q}^{2-1/q}(\mathbb{R}^2))
\]

\[
\hookrightarrow \left( F_{p,q,\delta}^{2-1/(2q)}(\mathbb{R}; F_{q,q}^0(\mathbb{R}^2)) \cap F_{p,q,\delta}^1(\mathbb{R}; B_{q,q}^{2-1/q}(\mathbb{R}^2)) \right)
\]

\[
\hookrightarrow F_{p,q,\delta}^{2-1/(2q)-\theta_2(1-1/(2q))}(\mathbb{R}; B_{q,q}^{\theta_2(2-1/q)}(\mathbb{R}^2)) \cap F_{p,\infty,\delta}^{1-\theta_3}(\mathbb{R}; B_{q,q}^{\theta_3+2-1/q}(\mathbb{R}^2))
\]
with $0 < \theta_2, \theta_3 < 1$ because $L^q(\mathbb{R}^2) = F_{q,2}^0(\mathbb{R}^2)$ and $B_{q,q}^{2-1/q}(\mathbb{R}^2) = F_{q,2}^{2-1/q}(\mathbb{R}^2)$. Here, we have used the interpolation properties
\[
(F_{q,2}^0(\mathbb{R}^2), F_{q,q}^{2-1/q}(\mathbb{R}^2))_{\theta_2,q} = B_{q,q}^{\theta_2(2-1/q)}(\mathbb{R}^2),
\]
\[
(B_{q,q}^{2-1/q}(\mathbb{R}^2), B_{q,q}^{3-1/q}(\mathbb{R}^2))_{\theta_3,q} = B_{q,q}^{\theta_3+2-1/q}(\mathbb{R}^2),
\]
see, e.g., [52, Thm. 2.4.1]. Using [22, Prop. 7.4] and [52, Thm. 2.5.7] (cf. [35, Prop. 2.4]), we obtain
\[
F_{p,q,\delta}^{-1/(2q)} \to (1-1/(2q)) \rightarrow \text{BUC}^1(\mathbb{R}; \text{BUC}^1(\mathbb{R}^2)),
\]
\[
F_{p,q,\delta}^{-1/(2q)} \to (1-1/(2q)) \rightarrow \text{BUC}(\mathbb{R}; \text{BUC}^2(\mathbb{R}^2))
\]
whenever
\[
2 - \frac{1}{2q} - \theta_2 \left(1 - \frac{1}{2q}\right) - \left(1 - \delta + \frac{1}{p}\right) > 1, \quad \theta_2 \left(2 - \frac{1}{q}\right) - \frac{2}{q} > 1,
\]
\[
1 - \theta_3 - \left(1 - \delta + \frac{1}{p}\right) > 0, \quad \theta_3 + 2 - \frac{1}{q} - \frac{2}{q} > 2.
\]
Using the last condition of (6.2), we deduce that the constants $\theta_2$ and $\theta_3$ enjoy the conditions
\[
\frac{2^{-1} + 3(2q)^{-1} - (2q)^{-1}}{1 - (2q)^{-1}} < \theta_2 < \frac{\delta - p^{-1} - (2q)^{-1}}{1 - (2q)^{-1}},
\]
\[
\frac{3}{q} < \theta_3 < \delta - \frac{1}{p},
\]
respectively. Hence, for $\theta_2$ and $\theta_3$ both satisfying these inequalities, we arrive at the desired assertion.

(4) Let $f, g \in \mathfrak{D}_{3,\delta}(J; \Gamma_\ast)$. From well-known paraproduct estimates (cf. Bahouri et. al [3, Cor. 2.86.]), it holds
\[
|fg|_{L^p_\delta(\mathbb{R}; B_{q,q}^{1-1/q}(\mathbb{R}^2))} \leq C \left( \|f\|_{L^\infty(\mathbb{R}^2)} |g|_{B_{q,q}^{1-1/q}(\mathbb{R}^2)} + |f|_{B_{q,q}^{1-1/q}(\mathbb{R}^2)} |g|_{L^\infty(\mathbb{R}^2)} \right)_{L^p_\delta(\mathbb{R})},
\]
\[
\leq C \left( \|f\|_{L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^2))} |g|_{L^p_\delta(\mathbb{R}; B_{q,q}^{1-1/q}(\mathbb{R}^2))} + |f|_{L^p_\delta(\mathbb{R}; B_{q,q}^{1-1/q}(\mathbb{R}^2))} |g|_{L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^2))} \right).
\]
For $f \in L^p_\delta(\mathbb{R}; L^2(\mathbb{R}^2))$ we define
\[
[f]_{F_{p,q,\delta}^1(\mathbb{R}; L^2(\mathbb{R}^2))}^{(1)} = \left( \int_0^\infty y^{-sq} \left( \int_{|h| \leq y} |\tau_h f - f|_{L^q(\mathbb{R}^2)} dh \right)^q \frac{dy}{y} \right)^{1/q},
\]
with $s = 1/2 - 1/(2q) < 1$ and set
\[
\|f\|_{F_{p,q,\delta}^1(\mathbb{R}; L^2(\mathbb{R}^2))} = |f|_{L^p_\delta(\mathbb{R}; L^2(\mathbb{R}^2))} + [f]_{F_{p,q,\delta}^1(\mathbb{R}; L^2(\mathbb{R}^2))}^{(1)}.
\]
Here \(\{\tau_h\}_{h \in \mathbb{R}}\) denotes the group of translations defined by
\[
(\tau_h f)(x) = f(x + h) \quad (x, h \in \mathbb{R}).
\]
Then, according to [23, Prop. 2.1], we know that $F_{p,q,\delta}^1(\mathbb{R}; L^2(\mathbb{R}^2))$ can be characterized as
\[
C^{-1} \|f|_{F_{p,q,\delta}^1(\mathbb{R}; L^2(\mathbb{R}^2))} \leq \|f\|_{F_{p,q,\delta}^1(\mathbb{R}; L^2(\mathbb{R}^2))} \leq C \|f|_{F_{p,q,\delta}^1(\mathbb{R}; L^2(\mathbb{R}^2))}
\]
with some constant $C > 0$. Writing
\[
\tau_h(fg) - fg = \tau_h(fg - g) + (\tau_h f - f)g
\]
and using the Hölder inequality, we have
\[
[f g]_{F_{p,q,\delta}^1(\mathbb{R}; L^2(\mathbb{R}^2))} \leq \|f\|_{L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^2))} |g|_{F_{p,q,\delta}^1(\mathbb{R}; L^2(\mathbb{R}^2))} + [f]_{F_{p,q,\delta}^1(\mathbb{R}; L^2(\mathbb{R}^2))} |g|_{L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^2))}.
\]
Noting
\[
|fg|_{L^p_\delta(\mathbb{R}; L^2(\mathbb{R}^2))} \leq |f|_{L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^2))} |g|_{L^p_\delta(\mathbb{R}; L^2(\mathbb{R}^2))},
\]
we achieve at the inequality
\[
|fg|_{F_{p,q,d}(\mathbb{R};L^s(\mathbb{R}^2))} \leq C \|fg\|_{L^1(\mathbb{R};L^s(\mathbb{R}^2))}^{(1)} \leq C \left( \|f\|_{L^\infty(\mathbb{R};L^\infty(\mathbb{R}^2))} \|g\|_{F_{p,q,d}(\mathbb{R};L^s(\mathbb{R}^2))} \right) + \|f\|_{F_{p,q,d}(\mathbb{R};L^s(\mathbb{R}^2))} |g|_{L^\infty(\mathbb{R};L^\infty(\mathbb{R}^2))}^{(1)}
\]
(6.4)
Consequently, by (6.3) and (6.4), it holds
\[
|fg|_{E_3,\delta(\mathbb{R};\mathbb{R}^2)} \leq C \left( |f|_{L^\infty(\mathbb{R};L^\infty(\mathbb{R}^2))} |g|_{E_3,\delta(\mathbb{R};\mathbb{R}^2)} + |f|_{E_3,\delta(\mathbb{R};\mathbb{R}^2)} |g|_{L^\infty(\mathbb{R};L^\infty(\mathbb{R}^2))} \right).
\]
Since $E_{3,\delta}(\mathbb{R};\mathbb{R}^2)$ embeds continuously into $\text{BUC}(\mathbb{R};\text{BUC}(\mathbb{R}^2))$, we see that $E_{3,\delta}(\mathbb{R};\mathbb{R}^2)$ is a multiplication algebra.

It remains to prove that $F_{3,\delta}(\mathbb{R};\mathbb{R}^2)$ is a multiplication algebra. Employing the same argument as above, we only need to prove that $F_{3,\delta}(\mathbb{R};\mathbb{R}^2)$ embeds continuously into $\text{BUC}(\mathbb{R};\text{BUC}(\mathbb{R}^2))$. For arbitrary $0 < \theta_4 < \infty$, it holds
\[
F_{3,\delta}(\mathbb{R};\mathbb{R}^2) \hookrightarrow F_{p,q,d}^{(1-\theta_4)(1-1/(2q))}(\mathbb{R}; B_{q,q}^{\theta_4(2-1/q)}(\mathbb{R}^2)).
\]
If $\theta_4$ satisfies
\[
(1 - \theta_4) \left( 1 - \frac{1}{2q} \right) - \left( 1 - \delta + \frac{1}{p} \right) > 0, \quad \theta_4 \left( 2 - \frac{1}{q} \right) - \frac{2}{q} > 0,
\]
we obtain
\[
F_{p,q,d}^{(1-\theta_4)(1-1/(2q))}(\mathbb{R}; B_{q,q}^{\theta_4(2-1/q)}(\mathbb{R}^2)) \hookrightarrow \text{BUC}(\mathbb{R};\text{BUC}(\mathbb{R}^2)).
\]
The conditions on $\theta_4$ are rewritten as
\[
\frac{q^{-1}}{1 - (2q)^{-1}} < \theta_4 < \frac{\delta - p^{-1} - (2q)^{-1}}{1 - (2q)^{-1}},
\]
and thus for any $\theta_4$ satisfying this inequality, we obtain the required property. This completes the proof.

**Proof of Proposition 6.1.** Since the mapping $z \mapsto N(z)$ is polynomial, it suffices to show that $N: U_T \to \mathbb{R}(T)$ is well-defined and continuous. Noting the mapping properties of the differential operators, cf., [22, Prop. 3.10] and [19, pp. 88], the assertions can be proved as in [31, Prop. 6.2].

**6.2. Nonlinear well-posedness.** We are now ready to prove the existence result for the transformed problem (3.2).

**Theorem 6.5.** Let $T > 0$ be a given constant. Suppose that (6.2) holds. Then there exists a constant $\varepsilon = \varepsilon(T) > 0$ such that for all initial data $(u_0, \eta_0) \in B_{q,p}^{2(\delta-1/p)}(\Omega_*)^3 \times B_{q,p}^{2+\delta-1/p-1/q}(\Gamma_*)$ satisfying the compatibility conditions
\[
\begin{cases}
\text{div } u_0 = \text{div}(u_0, \eta_0), & \text{in } \Omega_*, \\
\mu(\partial_3 u_{0,m} + \partial_m u_{0,3}) = K_m(u_0, \eta_0), & \text{on } \Gamma_*, \\
u_0 \cdot n_{\Sigma_*} = 0, F_{3m}(2\mu D(u_0)n_{\Sigma_*}) = G(u_0, \eta_0), & \text{on } \Sigma_*, \\
u_{0,3} = 0, \mu(\partial_3 u_{0,m} + \partial_m u_{3}) = H_m(u_0, \eta_0), & \text{on } B,
\end{cases}
\]
(6.5)
where $m = 1, 2$, and the smallness condition
\[
|u_0|_{B_{q,p}^{2(\delta-1/p)}(\Omega_*)} + |\eta_0|_{B_{q,p}^{2+\delta-1/p-1/q}(\Gamma_*)} \leq \varepsilon
\]
the transformed problem (3.2) has a unique solution $(u, \pi, \eta) \in \mathbb{R}(T)$. Furthermore, the solution $(u, \pi)$ is real analytic in $\Omega_* \times J$. Especially, $M = \bigcup_{t \in (0,T)} (\Gamma_t \times \{ t \})$ is a real analytic manifold.
Proof. We introduce an auxiliary function \( z_* \in \mathcal{E}(T) \) that resolves the compatibility conditions and the initial conditions. This makes us to reduce the problem (6.1) into

\[
Lz = N(z + z_*) - Lz_* =: K_0(z), \quad z \in \mathcal{E}(T).
\]

Since the mapping \( L : \mathcal{E}(T) \to \mathcal{E}(T) \) is an isomorphism, we can solve this reduced problem by means of the contraction mapping principle.

**Step 1.** Set \( I_0 := B^2_{q,p}(\delta - 1/p)(\Omega_\ast)^3 \times B^2_{q,p}(\delta - 1/p - 1/q)(\Gamma_\ast) \). Let \((u_0, \eta_0) \in I_0 \) satisfy the compatibility conditions (6.5). Set

\[
q_1 := -2\mu \partial_3 + \sigma \Delta \eta_0 + K_3(u_0, \eta_0).
\]

Then, from Proposition 6.1, we see that

\[
K_3^0 := K_3(u_0, \eta_0) - q_1 + \sigma \Delta \eta_0 \in B^2_{q,p}(\delta - 1/p - 1/q)(\Gamma_\ast),
\]

cf., Section 5.3 for the similar argument. Extend \( K_3^0 \) to an appropriate function \( \tilde{K}_3^0 \in B^2_{q,p}(\delta - 1/p - 1/q)(\partial_3 \mathbb{R}^3) \) and define \( \tilde{K}_3^0(t) := e^{-(t - \Delta_3)^{\frac{3}{2}}} \tilde{K}_3^0 \). Then, by [23, Thm. 4.2], we have

\[
\tilde{K}_3^* \in F^1_{p,q,\delta,j}(J; \mathcal{L}(\partial_3 \mathbb{R}^3)) \cap L^2_{p,j}(J; B^2_{q,q}(\partial_3 \mathbb{R}^3)).
\]

Hence, if we set \( K_3 := \tilde{K}_3^0 \), it holds

\[
K_3^0 \in F^1_{p,q,\delta,j}(J; \mathcal{L}(\Gamma_\ast)) \cap L^2_{p,j}(J; B^2_{q,q}(\Gamma_\ast))
\]

with \( K_3(0) = K_3^0 \). Similarly, we can construct the functions \( K_1^*, K_2^*, H_1^*, \) and \( H_2^* \) satisfying

\[
K_1^*, K_2^* \in F^1_{p,q,\delta,j}(J; \mathcal{L}(\Gamma_\ast)) \cap L^2_{p,j}(J; B^2_{q,q}(\Gamma_\ast)),
\]

\[
H_1^*, H_2^* \in F^1_{p,q,\delta,j}(J; \mathcal{L}(B)) \cap L^2_{p,j}(J; B^2_{q,q}(B)),
\]

where \( K_1^*(0) = K_1(u_0, \eta_0), K_2^*(0) = K_2(u_0, \eta_0), H_1^*(0) = H_1(u_0, \eta_0), \) and \( H_2^*(0) = H_2(u_0, \eta_0) \).

We next deal with the compatibility condition on \( \Sigma_\ast \). Since \( \Sigma_\ast \) has a boundary, we consider a compact hypersurface \( \Sigma_\ast \) of class \( C^3 \) without boundary but containing \( \Sigma_\ast \). Let \( \Delta_{\Sigma_\ast} \) be the Laplace-Beltrami operator on \( \Sigma_\ast \). It is well-known that the negative of the operator \( I - \Delta_{\Sigma_\ast} \) generates an exponentially stable analytic semigroup \( \{ e^{-(t - \Delta_{\Sigma_\ast})t} \}_{t \geq 0} \) on \( L^q(\Sigma_\ast) \), see [33, Thm. 6.4.3]. As we have seen in Section 5.3, we see that \( G_j(u_0, \eta_0) \in B^2_{q,p}(\delta - 1/p - 1/q)(\Sigma_\ast) \), which can be derived from Proposition 6.1 and the fact that \( T_{t=0} \) and the boundary operator commute (cf. Lindemulder [19, pp. 88]). Now, we extend \( G(u_0, \eta_0) \) to \( \tilde{G}(u_0, \eta_0) \) satisfies the compatibility conditions and the initial conditions, denoted by \( G_\ast \), it holds

\[
G_\ast \in F^1_{p,q,\delta,j}(J; \mathcal{L}(\Sigma_\ast)) \cap L^2_{p,j}(J; B^2_{q,q}(\Sigma_\ast))
\]

for \( j = 1, 2, 3 \), where \( \tilde{G} = (\tilde{G}_1, \tilde{G}_2, \tilde{G}_3) \). Restricting \( e^{-(t - \Delta_{\Sigma_\ast})t} \tilde{G}(u_0, \eta_0) \) to \( \Sigma_\ast \), denoted by \( G_\ast \), it holds

\[
G_\ast \in F^1_{p,q,\delta,j}(J; \mathcal{L}(\Sigma_\ast)) \cap L^2_{p,j}(J; B^2_{q,q}(\Sigma_\ast))
\]

with \( G_\ast(0) = G(u_0, \eta_0) \).
Consider the parabolic problem

\[
\begin{aligned}
\partial_t u^{**} - \mu \Delta u^{**} &= 0, & \text{in } \Omega \times J,
\mu(\partial_3 u_1^{**} + \partial_3 u_3^{**}) &= K_1^*, & \text{on } \Gamma \times J,
\mu(\partial_3 u_2^{**} + \partial_2 u_3^{**}) &= K_2^*, & \text{on } \Gamma \times J,
2\mu \partial_3 u_3^{**} &= K_3^*, & \text{on } \Gamma \times J,
u, u^{**} \cdot n_{\Sigma^*} &= 0, & \text{on } \Sigma \times J,
P_{\Sigma^*}(2\mu D(u^{**})n_{\Sigma^*}) &= G^*, & \text{on } \Sigma \times J,
\end{aligned}
\]

Then, by Lemma B.5, there exists a unique solution \( u^{**} \in E_{1,\delta}(J; \Omega) \) due to \( u_0 \in B_{q,p}^{2(\delta - 1/p)}(\Omega_3^3) \). Using this solution, we define \( F_{\text{div}}^* := \text{div } u^{**} \) with \( F_{\text{div}}^*(0) = \text{div } u_0 \), where \( F_{\text{div}}^* \) belongs to the same regularity class as \( f_{\text{div}} \) described in Theorem 5.1. Notice that, from the compatibility condition of \( F_{\text{div}} \), it holds \( F_{\text{div}}^*(0) = F_{\text{div}}(u_0, \eta_0) \). Besides, in the following, we set \( K^* := (K_1^*, K_2^*, K_3^*)^T \) and \( H^* := (H_1^*, H_2^*, H_3^*)^T \). Employing the similar argument in Section 5.3, we know \( D(u_0, \eta_0) \in B_{q,p}^{2(\delta - 1/p) - 1/q}(\Gamma_*) \) due to Proposition 6.1. Extend \( D(u_0, \eta_0) \) to \( \tilde{D}(u_0, \eta_0) \in B_{q,p}^{2(\delta - 1/p) - 1/q}(\partial \Omega^3) \). From [23, Thm. 4.2], we deduce that

\[
\tilde{D} := e^{-(I - \Delta_{\mu_0})h} \tilde{D}(u_0, \eta_0) \in \mathcal{F}_{3,\delta}(J; \partial \Omega^3).
\]

Setting \( D^* := \tilde{D}^*|_{\Gamma_*} \), it holds \( D^* \in F_{3,\delta}(J; \Gamma_*) \) and \( D^*(0) = D(u_0, \eta_0) \).

From Theorem 5.1, we can find the unique solution \( z^* \in E(T) \) of the linear problem

\[
Lz^* = (0, F_{\text{div}}^*, D^*, K^*, G^*, H^*, u_0, \eta_0), \quad z^*(0) = (u_0, \eta_0) \in \mathbb{I}_0,
\]

where \( z^* \) resolves the compatibility conditions (6.5). Notice that \( z^* \) enjoys the estimate

\[
|z^*|_{E(T)} \leq C_0(|u_0, \eta_0|)_{\mathbb{I}_0}
\]

with some constant \( C_0 \) that does not depend on \( (u_0, \eta_0) \).

**Step 2.** By Theorem 5.1, the operator \( L: \mathcal{F}(T) \to \mathcal{F}(T) \) is an isomorphism, and hence the solution of (3.2) is given by \( z = L^{-1}K_0(z) \). From the constructions of \( 0, F_{\text{div}}^*, D^*, K^*, G^*, H^* \), we see that \( K_0(z) \in \mathcal{F}(T) \) for any \( z \in \mathcal{F}(T) \). Hence, by Proposition 6.1, the mapping \( K_0 \) is real analytic from \( \mathcal{F}(T) \) to \( \mathcal{F}(T) \) yielding that \( L_0^{-1}K_0: \mathcal{F}(T) \to \mathcal{F}(T) \) is well-defined and smooth.

In the following, for a given Banach space \( W \), we set

\[
e_{B_{W}} := \{ w \in W \mid |w|_W \leq c \},
\]

where \( c \) is a positive number. Let \( M_0 := |L^{-1}|_{B(\mathcal{F}(T), \mathcal{F}(T))} \) and \( M_1 := |L|_{B(\mathcal{F}(T), \mathcal{F}(T))} \). Based on Proposition 6.1, there exists \( \delta_1 > 0 \) such that

\[
|DN(z + z^*)|_{B(\mathcal{F}(T), \mathcal{F}(T))} \leq \frac{1}{2M_0}
\]

for \( (z + z^*) \in \delta_1 B_{\mathcal{F}(T)} \). We further suppose that \( z \in (\delta_1/2)B_{\mathcal{F}(T)} \). Then, for \( (u_0, \eta_0) \in \varepsilon B_{\mathcal{I}_0} \), we obtain

\[
|z + z^*|_{\mathcal{F}(T)} \leq |z|_{\mathcal{F}(T)} + |z^*|_{\mathcal{F}(T)} \leq \frac{\delta_1}{2} + C_0 \varepsilon < \delta_1.
\]

provided that \( \varepsilon \) is sufficiently small such that

\[
\varepsilon \leq \frac{\delta_1}{2C_0(1 + 2M_0M_1)}.
\]
Therefore, by the mean value theorem, we have
\[
|L^{-1}K_0(z)|_{\mathcal{E}(T)} \leq M_0|K_0(z)|_{\mathcal{E}(T)}
\leq M_0 \left( |N(z + z^*)|_{\mathcal{E}(T)} + M_1|z^*|_{\mathcal{E}(T)} \right)
\leq M_0 \left( \frac{1}{2M_0}|z + z^*|_{\mathcal{E}(T)} + M_1C_0(\|u_0, \eta_0\|_{L_0}) \right)
\leq \frac{\delta_1}{4} + \left( \frac{1}{2} + M_0M_1 \right)C_0\varepsilon
\leq \frac{\delta_1}{2}.
\]

Hence, the mapping \( L^{-1}K_0 : (\delta_1/2)\mathbb{B}_{\varepsilon}\mathcal{E}(T) \to (\delta_1/2)\mathbb{B}_{\varepsilon}\mathcal{E}(T) \) is a self-mapping. In addition, using the mean value theorem again, it holds
\[
|L^{-1}K_0(z_1) - L^{-1}K_0(z_2)|_{\mathcal{E}(T)} \leq \frac{1}{2}|z_1 - z_2|_{\mathcal{E}(T)}
\]
for all \( z_1, z_2 \in (\delta_1/2)\mathbb{B}_{\varepsilon}\mathcal{E}(T) \). This shows that \( L^{-1}K_0 : (\delta_1/2)\mathbb{B}_{\varepsilon}\mathcal{E}(T) \to (\delta_1/2)\mathbb{B}_{\varepsilon}\mathcal{E}(T) \) is contractive. Hence, the contraction mapping principle implies a unique fixed point \( z_\ast \in (\delta_1/2)\mathbb{B}_{\varepsilon}\mathcal{E}(T) \) of \( L^{-1}K_0 \), i.e., \( z_\ast = L^{-1}K_0(z_\ast) \). Accordingly, we find that \( \tilde{z} := z^* + z_\ast \) solves \( L(\tilde{z}) = N(\tilde{z}) \).

**Step 3.** From Lemma 6.1, the right-hand side of (3.2) is real analytic. Therefore, by the parameter trick as employed in [31, Thm. 6.3] (cf. [10, Sec. 8] and [33, Ch. 9]), we can show that the solution \((u, \pi, \eta)\) is real analytic as well. This completes the proof. \( \square \)

### 6.3. Axial symmetry

Finally, we provide the proof of our main result, Theorem 1.1.

**Proof of Theorem 1.1.** First, we see that the compatibility conditions of Theorem 6.5 are satisfied if and only if (6.5) is satisfied. Define the mapping \( \Theta_{\eta_0} \) by \( \Theta_{\eta_0}(x) = x + \theta_{\eta_0}(x') = x + \chi(x_3)\eta_0(x_1, x_2) \). This defines a \( C^2 \)-diffeomorphism from \( \Omega_\ast \) onto \( \Omega_0 \) with inverse \( \Theta_{\eta_0}^{-1}(x) := (x - \theta_{\eta_0}(x')) \), which follows from the Sobolev embedding theorem. Therefore, there exists some constant \( C_{\eta_0} \) depending on \( \eta_0 \) such that
\[
C_{\eta_0}^{-1}|u_0|_{B^{2(4-1/p)}_{2,p}(\Omega_0)} \leq |u_0|_{B^{2(4-1/p)}_{2,p}(\Omega_0)} \leq C_{\eta_0}|u_0|_{B^{2(4-1/p)}_{2,p}(\Omega_0)}.
\]

Hence, there exists \( \varepsilon_0 > 0 \) such that the smallness condition of Theorem 1.1 implies the smallness condition in Theorem 6.5. Theorem 6.5 yields the existence of a unique solution \((u, \pi, \eta) \in \mathcal{E}(T) \) of (3.2) that satisfies the additionally regularity properties denoted in Theorem 6.5. Furthermore, setting
\[
(v, p)(x, t) = (u, \pi)(x - \theta_{\eta}(x, t), t), \quad (x, t) \in \mathcal{O} := \{(x, t) \in \Omega_t \times (0, T)\},
\]
we find that \((v, p, \eta) \) is a unique solution to (1.2), where \((v, p) \) is real analytic in \( \mathcal{O} \) and \( \mathcal{M} = \bigcup_{t \in (0, T)} (\Gamma_t \times \{t\}) \) is a real analytic manifold. The regularity properties for the pressure are derived from the embeddings
\[
\hat{H}^{1,q}(\Omega_t) \hookrightarrow \hat{B}^{1,1-q/2}_{2,q}(\Omega_t) \hookrightarrow \hat{B}^{1-n/q}_{\infty,\infty}(\Omega_t) \hookrightarrow \hat{B}^{1-n/q}_{\infty,\infty}(\Omega_t) \hookrightarrow C^{0,1-n/q}(\Omega_t) \hookrightarrow UC(\Omega_t)
\]
for every \( t \in (0, T) \) provided \( n < q < \infty \), cf. [36, Thm. 3.14]. Here, the *homogeneous* Hölder space \( \mathcal{C}^{0,s}(\mathbb{R}^3) \) of order \( s \in (0, 1) \) is the set of all continuous functions \( f \) on \( \mathbb{R}^3 \) endowed with the seminorm
\[
|f|_{\mathcal{C}^{0,s}(\mathbb{R}^3)} := \sup_{x, y \in \mathbb{R}^3, x \neq y} \frac{|f(y) - f(x)|_{L_\infty(\mathbb{R}^3)}}{|y - x|^s} < \infty,
\]
and we define
\[
|g|_{\mathcal{C}^{0,s}(D)} := \inf\{|f|_{\mathcal{C}^{0,s}(\mathbb{R}^3)} : f \in \mathcal{C}^{0,s}(\mathbb{R}^3), \ f|_D = g \text{ in } \mathcal{D}'(D)\},
\]
where \( D \) is a domain in \( \mathbb{R}^3 \) and \( \mathcal{D}'(D) \) denotes the collection of all complex-valued distributions on \( D \). We remark that \( |f|_{\mathcal{C}^{0,s}(D)} \) can be rewritten as
\[
|f|_{\mathcal{C}^{0,s}(D)} = \sup_{x, y \in D, x \neq y} \frac{|f(y) - f(x)|_{L_\infty(D)}}{|y - x|^s} \quad \text{for } f \in \mathcal{C}^{0,s}(D),
\]
see [25, Sec. 2].
Finally, we prove the axial symmetry of the solution \((v, p, \Gamma)\). Recall that \(R_\rho\) is the linear operator defined by \(R_\rho = (e_r(\rho), e_\vartheta(\rho), e_z)\), where \(e_r(\rho) = (\cos \rho \sin \vartheta, 0, 0)^T\), \(e_\vartheta(\rho) = (-\sin \rho \cos \vartheta, 0, 0)^T\), and \(e_z = (0, 0, 1)^T\) with \(\rho \in [0, 2\pi]\). Consider the new variables \(y\) defined by \(y = R_\rho x\). Besides, define the rotation operator \(U_\rho\) by \(f(x) \mapsto R_\rho f(R_\rho x)\). By the direct computations, the momentum and mass balance can be read as

\[
\begin{align*}
\partial_t (U_\rho v) + (U_\rho v \cdot \nabla)(U_\rho v) - \mu \Delta (U_\rho v) + \nabla p &= 0, & \text{in } R_\rho \Omega_t, \\
\text{div} (U_\rho v) &= 0, & \text{in } R_\rho \Omega_t,
\end{align*}
\]

respectively, where \(R_\rho \Omega_t\) is defined by

\[
R_\rho \Omega_t := \{ y \in \mathbb{R}^3 \mid x \in \Omega_t, \ R_\rho x = y, \ t \geq 0 \}.
\]

In fact, we have \(\nabla_y = R_\rho^T \nabla_x \) and \(\Delta_y = \Delta_x\). Besides, we also obtain \((v \cdot \nabla_x) v = R_\rho (U_\rho v \cdot \nabla_y)(U_\rho v))\). Hence, changing of variables \(y = R_\rho x\) and applying \(R_\rho^T\) yield \((6.6)_1\). Similarly, by the change of variables \(y = R_\rho x\), we obtain \((6.6)_2\).

Next, we consider the boundary conditions on \(\Gamma_t\). To this end, define

\[
R_\rho \Gamma_t := \{ y \in \mathbb{R}^3 \mid x \in \Gamma_t, \ R_\rho x = y, \ t \geq 0 \}.
\]

The stress boundary condition can be decomposed into the tangential part

\[
P_{\Gamma_t}(2\mu D(v) n_{\Gamma_t}) = 0
\]

and the normal part

\[
(2\mu D(v) n_{\Gamma_t}, n_{\Gamma_t}) - p = \sigma \mathcal{H}_{\Gamma_t} n_{\Gamma_t} - p_0.
\]

By the direct computation, we observe that \(R_\rho^T \nabla_y v = \nabla_y (R_\rho^T v)\) yielding \(D(v) = D(U_\rho v)\). Besides, the double mean curvature is given by

\[
\mathcal{H}_{\Gamma_t} = \text{div}_x \left( \frac{\nabla x' \eta}{1 + |\nabla x' \eta|^2} \right),
\]

and hence, by the change of variables \(y = R_\rho x\), we find that \(\mathcal{H}_{\Gamma_t} = \mathcal{H}_{R_\rho \Gamma_t}\), because \(\nabla x' \eta = \Delta_y \eta\) and \(\langle \nabla x' \eta, \nabla x'(1 + |\nabla x' \eta|^2)^{-1} \rangle = \langle \nabla y' \eta, \nabla y'(1 + |\nabla x' \eta|^2)^{-1} \rangle\). Summarizing, it holds

\[
\mathbb{T}(U_\rho v, p) n_{R_\rho \Gamma_t} = \sigma \mathcal{H}_{R_\rho \Gamma_t} n_{R_\rho \Gamma_t} - p_0 n_{R_\rho \Gamma_t}, \quad \text{on } R_\rho \Gamma_t.
\]

Noting that the transform \(R_\rho\) has no contribution to the \(z\)-direction (i.e., \(x_3\)-direction), the change of coordinates \(y = R_\rho x\) infers

\[
V_{R_\rho \Gamma_t} = \langle U_\rho v, n_{R_\rho \Gamma_t} \rangle, \quad \text{on } R_\rho \Gamma_t
\]

To verify the slip boundary conditions on \(\Sigma_t\), it will be convenient to use the fact that

\[
P_{\Sigma_t}(D(v) n_{\Sigma_t}) = 0, \quad \langle v, n_{\Sigma_t} \rangle = 0 \quad \text{on } \Sigma_t \quad \Leftrightarrow \quad \nabla \times v \times n_{\Sigma_t} = 0, \quad \langle v, n_{\Sigma_t} \rangle = 0 \quad \text{on } \Sigma_t.
\]

Since the vector field \(v\) can be written as \(v = v^r(r, \vartheta, z) e_r(\vartheta) + v^\vartheta(r, \vartheta, z) e_\vartheta(\vartheta) + v^z(r, \vartheta, z) e_z\), where we have set \(e_\vartheta := (-\sin \vartheta, \cos \vartheta, 0)^T\) with \(\vartheta \in [0, 2\pi]\). By fundamental calculations using the cylindrical coordinate, we see that \(v\) satisfies \(\nabla \times v \times n_{\Sigma_t} = 0\) and \(\langle v, n_{\Sigma_t} \rangle = 0\) on \(\Sigma_t\) if and only if \((v^r, v^\vartheta, v^z)\) satisfies

\[
v^r = 0, \quad \partial_r v^\vartheta - v^\vartheta = 0, \quad \partial_r v^z = 0 \quad \text{on } \Sigma_t,
\]

see, e.g., [1, Prop. 2.5]. Since it follows that

\[
U_\rho v = R_\rho^T v(R_\rho x, t) = v^r(r, \vartheta + \rho, z) e_r(\vartheta) + v^\vartheta(r, \vartheta + \rho, z) e_\vartheta(\vartheta) + v^z(r, \vartheta + \rho, z) e_z,
\]

we observe that \(U_\rho v\) satisfies the slip boundary conditions on \(\Sigma_t\).

Finally, we deal with the boundary conditions on \(B\). Recalling \(D(v) = D(U_\rho v)\), we have \(P_B(2\mu D(U_\rho v) n_B) = 0\) on \(B\). Since \(n_B = (0, 0, -1)^T\), we easily obtain \(\langle U_\rho v, n_B \rangle = 0\) on \(B\).
Based on the calculations described above, we observe that \((\tilde{v}, \tilde{p}, \tilde{\Gamma})(y, t) := ((U_\rho, v)(x, t), p(R_\rho x, t), R_\rho \Gamma(t))\) solves the Cauchy problem

\[
\begin{align*}
\partial_t \tilde{v} + (\tilde{v} \cdot \nabla)\tilde{v} - \mu \Delta \tilde{v} + \nabla \tilde{p} &= 0, & \text{in } \tilde{\Omega}_t, \\
\text{div } \tilde{v} &= 0, & \text{in } \tilde{\Omega}_t, \\
T(\tilde{v}, \tilde{p})_\Gamma = &\sigma \mathcal{H}_\Gamma n_\Gamma - p_0 n_\Gamma, & \text{in } \Gamma_t, \\
\mathcal{V}_\Gamma &= \langle \tilde{v}, n_\Gamma \rangle, & \text{in } \tilde{\Gamma}_t, \\
P_{\Sigma_t}(2\mu D(\tilde{v})n_{\Sigma_t}) &= 0 & \text{on } \Sigma_t, \\
\langle \tilde{v}, n_{\Sigma_t} \rangle &= 0 & \text{on } \Sigma_t, \\
P_B(2\mu D(\tilde{v})n_B) &= 0 & \text{on } B, \\
\tilde{v}(0) &= \tilde{R}_\rho^{-1}v_0 & \text{in } \tilde{\Omega}_0, \\
\tilde{\Gamma}_{t=0} &= R_\rho \Gamma_0, & \text{in } \mathbb{R}^2.
\end{align*}
\]

If the initial data are axisymmetric, it holds \((U_\rho v_0, R_\rho \Gamma_0) = (v_0, \Gamma_0)\). We also observe \(\tilde{\Omega}_0 = \Omega_0\). Therefore, from the uniqueness of the solution, we have \((\tilde{v}, \tilde{p}, \tilde{\Gamma}) = (v, p, \Gamma)\), i.e., the solution is axisymmetric. This completes the proof. 

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\appendix

\section*{Appendix A. Extension operators}

We collect some technical results that are needed for the execution of the Stokes equations with slip-free boundary conditions (cf. Theorem 4.8). The following lemmas are generalization of Propositions A.1 and A.2 in [53].

\begin{lemma}
Let \(1 < p < \infty, 2 < q < \infty, 1/p < \delta \leq 1\), and \(J = (0, T)\). Then there exists a bounded linear extension operator \(\text{ext} \) from

\begin{align*}
0^{F^{1/2 - 1/q}_p, 0} (J; L^q(\mathbb{R})) &\cap L^p_\delta (J; B^{1/2 - 1/q}_{q, q}(\mathbb{R})) := X_1 \\
g \in 0^{F^{1/2 - 1/(2q)}_p, 0} (J; L^q(\mathbb{R} \times \mathbb{R}^+)) &\cap L^p_\delta (J; B^{1/2 - 1/q}_{q, q}(\mathbb{R} \times \mathbb{R}^+)) := Y_1
\end{align*}

such that \(\text{ext}[v]_{\mathbb{R} \times \{0\}} = v\) for all \(v \in Y_1\). Furthermore, if \(v = v(x_1, y, t) \in Y_1\), then \(\text{Tr}_{y=0}[v] \in X_1\) such that the trace map is bounded from \(Y_1\) to \(X_1\).

\textbf{Proof.} This is a direct consequence of [19, Thm. 4.6].
\end{lemma}

\begin{lemma}
Suppose \(1 < p < \infty, 2 < q < \infty, 1/p < \delta \leq 1\) and set \(J = (0, T)\) with \(T > 0\). If

\begin{align*}
f \in 0^{F^{3/2 - 1/q}_p, 0} (J; L^q(\mathbb{R})) &\cap \partial_\delta H^{1/p}_\delta (J; B^{1/2 - 1/q}_{q, q}(\mathbb{R})) \cap L^p_\delta (J; B^{2/2 - 1/q}_{q, q}(\mathbb{R})) := X_2, \\
g \in 0^{F^{2 - 1/(2q)}_p, 0} (J; L^q(\mathbb{R} \times \mathbb{R}^+)) &\cap \partial_\delta H^{1/p}_\delta (J; B^{2/2 - 1/q}_{q, q}(\mathbb{R} \times \mathbb{R}^+)) \cap L^p_\delta (J; B^{1/2 - 1/(2q)}_{q, q}(\mathbb{R} \times \mathbb{R}^+)) := Y_2
\end{align*}

then there exists \(g = g(t, x_1, y)\) with the regularity

\begin{align*}
g \in 0^{F^{3/2 - 1/q}_p, 0} (J; L^q(\mathbb{R})) &\cap \partial_\delta H^{1/p}_\delta (J; B^{2/2 - 1/q}_{q, q}(\mathbb{R})) \cap L^p_\delta (J; B^{1/2 - 1/q}_{q, q}(\mathbb{R})) := X_2
\end{align*}

such that \(\partial_y g = f\) at \(y = 0\).

\textbf{Proof.} Consider the operator \(L_0 := \partial_t - \partial_{x_1}^2\) in the space \(X_0 := L^q_\delta (J; L^q(\mathbb{R}))\) with domain

\begin{align*}
D(L_0) &= \partial_\delta H^{1/p}_\delta (J; L^q(\mathbb{R})) \cap L^p_\delta (J; H^{2/q}(\mathbb{R})),
\end{align*}

in which \(L_0\) is invertible on \(X_0\) and admits bounded imaginary powers with power angle not exceeding \(\pi/2\).

It is not difficult to see that \(L_0\) is the negative generator of an analytic semigroup \(\{e^{-yL_0^1/2}\}_{y \geq 0}\) in \(X_0\) with domain \(D(L_0^{1/2}) = [X_0, D(L_0)]_{1/2}\), where \([\cdot, \cdot]_\theta\) is the complex interpolation functor with \(\theta \in (0, 1)\), see,
e.g., [33, pp. 255]. Here, \( L_0^{1/2} \) has bounded imaginary powers with angle not larger than \( \pi/4 \). Now, set \( g(y) = -e^{it_0}L_0^{-1/2}f \) for \( f \in X_2, y > 0 \). Recalling \( f \in X_2 \), we easily see that
\[
f, \partial_y f, L_0^{-1/2}f, L_0^{-1/2}\partial_y f \in \mathcal{O}_{p,p,\delta}^{1/2-1/(2q)}(J; L^q(\mathbb{R})) \cap L_0^p(J; B_{q,q}^{1-2/q}(\mathbb{R})). \tag{A.1}
\]
In fact, the operator \( L_0^{-1/2} \) induces an isomorphism from \( D_{L_0}(1/2 - 1/(2q), p) \) onto \( D_{L_0}(1 - 1/(2q), p) \), where we have set
\[
D_L(1/2 - 1/(2q), p) = a^1_{p,p,\delta}^{1/2-1/(2q)}(J; L^q(\mathbb{R})) \cap L_0^p(J; B_{q,q}^{1/2-1/(2q)}(\mathbb{R})),
\]
and used [52, Thm. 1.15.2 (e)] and [33, Thm. 3.4.7]. Here, we have also set \( D_{L_0}(\alpha, r) := (X, D(L_0))_{\alpha, r} \) for \( \alpha \in (0, 1) \) and \( r \in [1, \infty) \), where \((\cdot, \cdot)_{\alpha, r}\) is the real interpolation functor. Then, recalling \( q > 2 \), we have the embedding properties
\[
\begin{align*}
a^1_{p,p,\delta}^{1/2-1/(2q)}(J; L^q(\mathbb{R})) & \hookrightarrow a^1_{p,p,\delta}^{1/2-1/(2q)}(J; L^q(\mathbb{R})) \cap L_0^p(J; B_{q,q}^{1-2/q}(\mathbb{R})) , \\
a^1_{p,p,\delta}^{1/2-1/(2q)}(J; L^q(\mathbb{R})) & \hookrightarrow a^1_{p,p,\delta}^{1/2-1/(2q)}(J; L^q(\mathbb{R})) \cap L_0^p(J; B_{q,q}^{1-2/q}(\mathbb{R})) , \\
\end{align*}
\]
yielding that \( L_0^{-1/2} \) maps continuously from \( a^1_{p,p,\delta}^{1/2-1/(2q)}(J; L^q(\mathbb{R})) \cap L_0^p(J; B_{q,q}^{1-2/q}(\mathbb{R})) \) to itself. This infers (A.1). Using Lemma A.1, from \( \partial_y g = f \) at \( y = 0 \), we deduce that
\[
g, \partial_y g \in a^1_{p,p,\delta}^{1/2-1/(2q)}(J; L^q(\mathbb{R} \times \mathbb{R}_+)) \cap L_0^p(J; B_{q,q}^{2-1/q}(\mathbb{R} \times \mathbb{R}_+)),
\]
see also [19, pp. 88]. Especially, from [22, Prop. 3.10], we arrive at
\[
g \in a^{2-1/(2q)}_{p,p,\delta}(J; L^q(\mathbb{R} \times \mathbb{R}_+)) \cap H_0^{1-p}(J; B_{q,q}^{2-1/q}(\mathbb{R} \times \mathbb{R}_+)).
\]
Finally, by \( \partial_y g = f \) at \( y = 0 \) and \( f \in L_0^p(J; B_{q,q}^{2-2/q}(\mathbb{R})) \), we have \( g \in L_0^p(J; B_{q,q}^{3-1/q}(\mathbb{R} \times \mathbb{R}_+)) \) in view of the trace theory. Summing up, we observe \( g \in Y_2 \). This completes the proof.

\section*{Appendix B. Auxiliary elliptic and parabolic problems}

\subsection*{B.1. Elliptic problems}

We start with considering the auxiliary problem
\[
\begin{aligned}
\Delta \phi &= f, & \text{in } \Omega_*, \\
\phi &= g, & \text{on } \Gamma_*, \\
n_{\Sigma_*} \cdot \nabla \phi &= h_1, & \text{on } \Sigma_*, \\
n_B \cdot \nabla \phi &= h_2, & \text{on } B.
\end{aligned} \tag{B.1}
\]
The domain \( \Omega_* \) stands for either a whole space, a (bent) half space, a (bent) quarter space, or the cylindrical domain given in (3.4). We look for solutions \( \phi \) satisfying \( \nabla \phi \in H^{1,q}(\Omega_*) \) since we cannot expect to seek \( \phi \in L^q(\Omega_*) \) when \( \Omega_* \) is unbounded. Notice that if \( \phi \) is a solution to (B.1) with \( g = 0 \) and \( \nabla \phi \in H^{1,q}(\Omega_*) \), then we have \( f \in L^q(\Omega_*), h_1 \in B_{q,q}^{1-1/q}(\Sigma_*), \) and \( h_2 \in B_{q,q}^{1-1/q}(B) \). As we introduced before, let \( H_{\Sigma_*,\Omega_0}^{-1,q}(\Omega_*) \) be the set of all \( (f, h_1, h_2) \in L^q(\Omega_*) \times B_{q,q}^{1-1/q}(\Sigma_*) \times B_{q,q}^{1-1/q}(B) \) such that \( (f, h_1, h_2) \) belongs to \( H_{\Sigma_*,\Omega_0}^{-1,q}(\Omega_*) \). The following lemma can be shown along the same lines of Lemmas A.6 in [53], and hence we do not repeat the argument.

\begin{lemma}
Let \( 2 < q < \infty \). Assume the compatibility conditions
\[
\begin{align*}
\text{Tr}_{\Sigma_*} [g] &= \text{Tr}_{\Sigma_*} [h_1] & \text{on } S_*, \\
\text{Tr}_{\Sigma_* \cap \partial B} [h_1] &= \text{Tr}_{\Sigma_* \cap \partial B} [h_2] & \text{on } \partial \Sigma_* \cap \partial B
\end{align*}
\tag{B.2}
\end{lemma}
when \( \Omega_* \) is a (bent) quarter space or the cylindrical domain defined in (3.4). Then the following assertions hold.
Remark B.2. Although the result in Lemma A.6 in [53] includes the case $q > 2$, we emphasize that it is required and crucial to assume $q > 2$. Indeed, we need this assumption to employ the reflection argument because there exists the trace onto the contact lines whenever $q > 2$.

As a corollary of Lemma B.1, we can prove the existence of weak solution to (B.1) provided that $h_1 = h_2 = 0$ and (B.2).

Lemma B.3. Let $2 < q < \infty$ and $\Omega_*$ be the cylindrical domain defined in (3.4). Suppose the compatibility conditions (B.2) with $h_1 = h_2 = 0$. Furthermore, let $f \in \dot{H}^{-1-q}_{\Sigma_\Gamma B}(\Omega_*)$ and $g \in B^{1-1/q}_q(\Gamma_*)$ be given. Then there exists a unique solution $\phi \in \dot{H}^{1,q}(\Omega_*)$ to the weak version of (B.1)

$$
\left\{ (\nabla \phi \mid \nabla \varphi)_{\Omega_*} = (f \mid \varphi)_{\Omega_*}, \quad \phi = g, \quad \text{on } \Gamma_*
\right\}
$$

for any $\varphi \in \dot{H}^{1,q}_{\Gamma_*}(\Omega_*)$, where we have set $H^{1,q}_{\Gamma_*}(\Omega_*) := \{ w \in H^{1,q}(\Omega_*) \mid w = 0 \text{ on } \Gamma_* \}$.

Proof. We follow the proof of [53, Lem. A.7]. However, the argument in [53, Lem. A.7] requires the result of [18, Sec. 8], so that we instead use the result due to [33, Sec. 7.4]. Besides, the space of test functions in [53, Lem. A.7] was $W^{1,p}(\Omega)$, but we point out that the test functions have to vanish on the “transformed” boundary where the pressure term appears, which will play an important role in integration by parts; related to the divergence equation. In fact, owing to this investigation, by (B.1) and integration by parts, we find that $(\nabla \phi \mid \nabla \varphi)_{\Omega_*} = (f \mid \varphi)_{\Omega_*}$ holds for any $\varphi \in \dot{H}^{1,q}_{\Gamma_*}(\Omega_*) \hookrightarrow H^{1,q}_{\Gamma_*}(\Omega_*)$ provided $h_1 = h_2 = 0$. Then, the required property can be shown in a same way as in the proof of [53, Lem. A.7].

As a consequence of Lemma B.3, we can show the higher regularity result for the solution $\phi$.

Lemma B.4. Let $2 < q < \infty$ and $J = (0,T)$. Suppose that $g = h_1 = h_2 = 0$. Then the following statements are valid.

(1) If $\Omega_*$ is a whole space, a (bent) half space, or a (bent) quarter space, then the problem (B.1) admits a unique solution $\phi$ with

$$
\nabla \phi \in H^{1,p}_\delta(J; H^{1,q}(\Omega_*)) \cap L^p_\delta(J; H^{3,q}(\Omega_*)),
$$

if and only if

$$
f \in H^{1,p}_\delta(J; \dot{H}^{-1,q}_{\Sigma_\Gamma B}(\Omega_*)) \cap L^p_\delta(J; H^{1,q}(\Omega_*)).
$$

(2) If $\Omega_*$ is the cylindrical domain defined in (3.4), then there (B.1) has a unique solution

$$
\phi \in H^{1,p}_\delta(J; H^{1,q}(\Omega_*)) \cap L^p_\delta(J; H^{3,q}(\Omega_*)),
$$

if and only if

$$
f \in H^{1,p}_\delta(J; \dot{H}^{-1,q}_{\Sigma_\Gamma B}(\Omega_*)) \cap L^p_\delta(J; H^{1,q}(\Omega_*)).
$$

Proof. From Lemma B.1 and Lemma B.3, we immediately obtain the regularity $\nabla \phi \in H^{1,p}(J; H^{1,q}(\Omega_*))$ in the first assertion and $\phi \in H^{1,p}(J; H^{1,q}(\Omega_*))$ in the second assertion, respectively. As for the additional spacial regularity of $\phi$, we use Lemmas B.1 and B.3. By means of local coordinates, we may reduce each localized problem to one of the model problems in a whole space, a (bent) half space, and a (bent) quarter space. Employing the perturbation and reflection arguments as in the proof of [53, Lem. A.3] (cf. [18, Thm. 8.6]), we readily obtain the desired result. Hence, we will not repeat the arguments. \qed
B.2. Parabolic problems. Finally, we provide the existence and uniqueness of solutions to the parabolic problem

\[
\begin{align*}
\partial_t u - \mu \Delta u &= f, & \text{in } \Omega \times J, \\
\mu(\partial_{3} u_m + \partial_m u_3) &= k_m, & \text{on } \Gamma_s \times J, \\
2 \mu \partial_3 u_3 &= k_3, & \text{on } \Gamma_s \times J, \\
u \cdot n_{\Sigma_2} &= g \cdot n_{\Sigma_1}, & \text{on } \Sigma_s \times J, \\
P_{\Sigma_2}(2 \mu D(u) n_{\Sigma_1}) &= P_{\Sigma_2} g, & \text{on } \Sigma_s \times J, \\
u_3 &= h_3, & \text{on } B \times J, \\
\mu(\partial_{3} u_m + \partial_m u_3) &= h_m, & \text{on } B \times J, \\
u(0) &= u_0, & \text{in } \Omega_s,
\end{align*}
\]

(B.3)

where \( m = 1, 2 \). The following lemma can be established in the same way as [53, Lem. A.10] with the help of the argument in Theorem 4.6.

Lemma B.5. Let \( T > 0 \) and \( J = (0, T) \). Assume that \( p, q, \delta \) satisfy (4.11). Then, there exists a unique solution

\[ u \in H^{1,p}_0(J; L^q(\Omega_s)^3) \cap L^p_0(J; H^{2,q}(\Omega_s)^3) \]

of (B.3) if and only if

(a) \( f \in F_{0,\delta}(J; \Omega_s) \);
(b) \( P_{\Sigma_2} g \in F_{2,\delta}(J; \Sigma_s) \) for \( \ell = 1, 3 \);
(c) \( g \cdot n_{\Sigma_2} \in F_{3,\delta}(J; \Sigma_s) \);
(d) \( k_j \in F_{2,\delta}(J; \Gamma_s) \) for \( j = 1, 3 \);
(e) \( h_m \in F_{2,\delta}(J; B) \) for \( m = 1, 2 \);
(f) \( h_3 \in F_{3,\delta}(J; B) \);
(g) \( u_0 \in F^{(\delta-1/p)}_{2,\delta}(\Omega_s)^3 \);
(h) \( P_{\Sigma_2} g(0) = Tr_{\Sigma_s} [P_{\Sigma_2}(2 \mu D(u_0) n_{\Sigma_1})] \), \( k_m(0) = Tr_{\Sigma_s} [\mu(\partial_{3} u_0 + \partial_m u_0)], \) \( m = 1, 2 \), \( k_3(0) = Tr_{\Sigma_s} [2 \mu \partial_3 u_0], \)
and \( h_m(0) = Tr_{B} [\mu(\partial_{3} u_0 + \partial_m u_0)], \) \( m = 1, 2 \), if \( p + 1/(2q) < \delta + 1/2 \);
(i) \( g(0) \cdot n_{\Sigma_1} = Tr_{\Sigma_s} [u_0 \cdot n_{\Sigma_1}], \) and \( h_3(0) = Tr_{B} [u_0], \) if \( p + 1/(2q) < \delta \);
(j) \( Tr_{\Sigma_s} [P_{\Sigma_2} g(0)] = Tr_{\Sigma_s} [P_{\Sigma_2}(2 \mu D(u_0) n_{\Sigma_1})], \) \( Tr_{\Sigma_s} [k_m(0)] = Tr_{\Sigma_s} [\mu(\partial_{3} u_0 + \partial_m u_0)] \) for \( m = 1, 2 \), and
\( Tr_{\Sigma_s} [k_3(0)] = Tr_{\Sigma_s} [2 \mu \partial_3 u_0], \) if \( 1/p + 1/q < \delta - 1/2 \);
(k) \( Tr_{\Sigma_s \cap \partial B} [g(0) \cdot n_{\Sigma_2}] = Tr_{\Sigma_s \cap \partial B} [u_0 \cdot n_{\Sigma_2}], \) and \( Tr_{\Sigma_s \cap \partial B} [h_3(0)] = Tr_{\Sigma_s \cap \partial B} [u_0], \) if \( 1/p + 1/q < \delta \).

Especially, the solution map is continuous in the corresponding spaces.

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