REMARKS ON SESHAĐRI CONSTANTS

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Abstract. Given a smooth complex projective variety \( X \) and an ample line bundle \( L \) on \( X \). Fix a point \( x \in X \). We consider the question, are there conditions which guarantee the maxima of the Seshadri constant of \( L \) at \( x \), i.e \( \varepsilon(L, x) = \sqrt[1]{L^n} \)? We give a partial answer for surfaces and find examples where the answer to our question is negative. If \((X, \Theta)\) is a general principal polarized abelian surface, then \( \varepsilon(\Theta, x) = \frac{4}{3} < \sqrt{\frac{2}{3}} = \sqrt{\Theta^2} \) for all \( x \in X \).

Introduction

Let \( X \) be a smooth projective variety and let \( L \) be a line bundle on \( X \). Fix a point \( x \in X \). Demailly [D] introduced a very interesting measure of the local positivity at a point \( x \) of \( L \), namely the real number

\[
\varepsilon(L, x) = \inf_{C \ni x} \frac{L.C}{m_x(C)},
\]

which is called the Seshadri constant of \( L \) at \( x \). Here the infimum is taken over all irreducible curves \( C \) passing through \( x \) and \( m_x(C) \) is the multiplicity of \( C \) at \( x \). For example, if \( L \) is very ample then \( \varepsilon(L, x) \geq 1 \).

There has been recent interest in trying to give lower bounds for this invariant at a general point. Ein and Lazarsfeld [EL] show that if \( X \) is a surface, then \( \varepsilon(L, x) \geq 1 \) for very general \( x \in X \). In higher dimension \((n \geq 3)\) Ein, Küchle and Lazarsfeld [EKL] prove that \( \varepsilon(L, x) \geq \frac{1}{n} \) for a very general point. We say that a point \( x \in X \) is very general if \( x \) is in the complement \( X \setminus Z \) of \( Z \) a countably union of proper subvarieties. Examples of Miranda show that \( \varepsilon(L, x) \) can take arbitrarily small values in codimension two (i.e. codim \( Z = 2 \)), even for an ample line bundle.

One may expect that this general bounds are not optimal. An elementary observation (see Remark 1 below for the proof) shows that \( \varepsilon(L, x) \leq \sqrt[n]{L^n} \). A natural question is, are there conditions which guarantee equality? Even in relative simple cases it turns out to be hard to give an answer.

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Recently, Xu [Xu] improved the surface bound given by Ein and Lazarsfeld. He showed that if \( L^2 \geq \frac{1}{3}(4\alpha^2 - 4\alpha + 5) \) for a given integer \( \alpha > 1 \) and \( LC \geq \alpha \) for every irreducible curve \( C \subset X \), then \( \varepsilon(L, x) \geq \alpha \) for all but finitely many \( x \in X \).

The first result we have given a further improvement and gives a partial answer to the equality question.

**Proposition 1.** Let \( X \) be a surface with \( \rho(X) = \text{rk}(\text{NS}(X)) = 1 \) and let \( L \) be an ample generator of \( \text{NS}(X) \). Let \( \alpha \) be an integer with \( \alpha^2 \leq L^2 \).

If \( x \in X \) is a very general point, then \( \varepsilon(L, x) \geq \alpha \). In particular if \( \sqrt{L^2} \) is an integer, then \( \varepsilon(L, x) = \sqrt{L^2} \).

For example, if \((X, L)\) is a general polarized abelian surface of type \((1, 2d^2)\) for some \( d \geq 1 \), then \( \varepsilon(L, x) = 2d = \sqrt{L^2} \) for all \( x \in X \). Or if \( X \subset \mathbb{P}^3 \) is a general hypersurface of degree \( d^2 \geq 4 \), then \( \varepsilon(\mathcal{O}_X(1), x) = d \) for a very general \( x \in X \).

The proof uses essentially the fact that \( L \) is the generator and in a diophantine way that \( \alpha \) is an integer.

One might be tempted to suppose that the conclusion of Proposition 1 holds allowing \( \alpha \) to be a (possibly non-integral) real number. But the next result shows that the situation is more complicated.

**Proposition 2.** Let \( X \) be the Jacobian of a hyperelliptic curve of genus \( g \geq 2 \) with \( \text{rk}(\text{NS}(X)) = 1 \). And let \( \Theta \) be the theta divisor on \( X \).

Then \( \varepsilon(\Theta) = \varepsilon(\Theta, x) \leq \frac{2g}{g+1} < \sqrt{g!} = \sqrt{\Theta^g} \).

In particular if \( X \) is an irreducible principal polarized abelian surface, then

\[ \varepsilon(\Theta) = \varepsilon(\Theta, x) = \frac{4}{3} < \sqrt{2} = \sqrt{\Theta^2} \]

Proposition 2 gives an example that the bound \( \varepsilon(L, x) \geq \lfloor \sqrt{L^2} \rfloor \) does not hold for an arbitrarily line bundle \( L \), where \( \lfloor r \rfloor \) denotes the integer part of an real number \( r \). In fact, if \( X \) is the Jacobian of a very general curve of genus two, than \( \text{rk}(\text{NS}(X)) = 1 \). And if \( L = \nu \Theta \) then \( \varepsilon(L, x) = \nu \varepsilon(\Theta, x) \leq \nu \frac{4}{3} < \lfloor \nu \sqrt{2} \rfloor \) for all \( \nu \geq 8 \).

The arguments in the proof of Proposition 2 works more general to show:

**Proposition 3.** Let \( X \) be a general principal polarized abelian variety of dimension \( g \) with theta divisor \( \Theta \). Then

\[ \varepsilon(\Theta) = \varepsilon(\Theta, x) \leq \sqrt[2g-3]{\frac{g!}{(2g-1)}} < \sqrt{g!} = \sqrt{\Theta^g}. \]

In the case of surface, the following result shows that if \( \varepsilon(L, x) \) is non-maximal (i.e. \( \varepsilon(L, x) < \sqrt{L^2} \)) then it is rational. There seems to be no examples known where \( \varepsilon(L, x) \) is irrational.
Proposition 4. In dimension $n$, if the Seshadri constant is non-maximal, then it is a $d$-th root of a rational number for some $1 \leq d \leq n - 1$.

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1. PROOFS AND FURTHER REMARKS

Proof of Proposition 1. Suppose to the contrary that there exists a reduced irreducible curve $C \subset X$ through a general point $x \in X$, such that $C.L > \alpha m_x(C)$. Then by the arguments of [EL] it follows that:

\[(*) \quad C^2 \geq m_x(C)(m_x(C) - 1).\]

To see this we follow [EL]. We may assume that $(C, x)$ moves in a non-trivial continuous family $\{ C_t \ni x_t \}_{t \in \Delta}$ of reduced irreducible curves $C_t \subset X$, plus points $x_t \in C_t$, with

\[m_t = m_{x_t}(C_t) > \alpha C_t.L \quad \text{for all } t \in \Delta.\]

The precise statement we need is:

Proposition [EL]. Let $\{ C_t \ni x_t \}_{t \in \Delta}$ be a 1-parameter family of reduced irreducible curves on a smooth projective surface $X$, such that $m_t = m_{x_t}(C_t) \geq m$ for all $t \in \Delta$.

Then

\[(C_t)^2 \geq m(m - 1).\]

Now back to the proof of Proposition 1. By the condition $\rho(X) = 1$ there exist an integer $d$, such that $C$ is numerically equivalent to $dL$. Since $L^2 \geq \alpha^2$, the assumption that $C.L > \alpha m_x(C)$ gives

\[(**) \quad \alpha d < m_x(C).\]

So it follow from the fact that $\alpha$ is an integer that

\[(***) \quad \alpha d \leq m_x(C) - 1.\]

Hence by $(*)$, $(**)$ and $(***)$

\[m_x(C)(m_x(C) - 1) \leq C^2 = C.(dL) < \alpha d m_x(C) \leq m_x(C)(m_x(C) - 1),\]

a contradiction. $\square$
Proof of Proposition 2. Let \( C \) be a hyperelliptic curve of genus \( g \geq 2 \). Then \( C \) has \( 2g + 2 \) Weierstrass points \( p_1, \ldots, p_{2g+2} \), with \( 2p_1 \sim 2p_2 \sim \cdots \sim 2p_{2g+2} \). In terms of the Jacobian \((X, \Theta) = (J(C), \Theta_C)\) of \( C \) this has the following interpretation:

Let \( \mathfrak{W}_C : C \to X \simeq \text{Pic}^1(C) \) be the Abel-Jacobi map and \( 2_X : X \to X \) the multiplication by two, the map determine by the map \( X \simeq \text{Pic}^1(C) \to X \simeq \text{Pic}^2(C) \), \( \eta \to \text{cl}(\eta^{\otimes 2}) \).

Then \( C' = 2_X(\mathfrak{W}_C(C)) \) has a point \( x \) with \( m_x(C') = 2g + 2 \). And \( C \to C' \) is birational.

Assume in contrary that the map \( f = 2_X \mathfrak{W}_C : C \to C' \) is not birational and say has degree \( n \geq 2 \). Let \( \nu : \tilde{C} \to C' \) the normalization of \( C' \). Then \( f \) factors through \( \nu \). By the universal property of the Jacobian, there is a map \( \tilde{f} : X = J(C) \to J(\tilde{C}) \). Since \( rk(\text{NS}(X)) = 1 \) it follows that \( g = \dim X = \dim J(\tilde{C}) = g(\tilde{C}) \). Hence we find by the Riemann-Hurwitz formula
\[
2g - 2 \geq n(2g(\tilde{C}) - 2) > 2g - 2,
\]
a contradiction.

Since \( 2^*_X \Theta \sim 4\Theta \) \cite[II-3 Proposition 3.6]{LB}, we find that
\[
(\Theta, C') = (2^*_X \Theta, \mathfrak{W}_C(C)) = (4\Theta, \mathfrak{W}_C(C)) = 4g.
\]
Hence, we get
\[
\varepsilon(\Theta, x) \leq \frac{2g}{g + 1} < \sqrt[3]{g} = \sqrt[3]{4g}.
\]

Now let \( g = 2 \) and suppose to the contrary that \( \varepsilon(\Theta, x) < \frac{4}{3} \). Then there exists a reduced irreducible curve \( \tilde{C} \) with \( \tilde{C} = aC \) and \( m_x(\tilde{C}) = b \) such that

\[
(*) \quad \frac{a}{b} < \frac{2}{3};
\]

Let \( \varphi : \mathcal{B}_x(X) \to X \) the blow-up of \( X \) at \( x \) with exceptional divisor \( E \). Since \( C' \) and \( \tilde{C} \) have no common components, it follows

\[
(**) \quad 0 \leq (\varphi^*C' - 6E).((\varphi^*C - bE) = 8a - 6b.
\]
Combining (*) and (**) we find \( 9a < 6b \leq 8a \) the desired contradiction. \( \square \)

Sometimes it is useful to use an alternative definition of the Seshadri constant of \( L \) at a point \( x \in X \). If \( \varphi : \mathcal{B}_x(X) \to X \) is the blow-up of \( X \) at \( x \) with exceptional divisor \( E \), then
\[
\varepsilon(L, x) = \sup \{ \delta \in \mathbb{R} \mid \varphi^*L - \delta E \text{ is nef} \}
\]

Remark 1. Let \( Y \subset \mathcal{B}_x(X) \) be a subvariety of dimension \( s = \dim Y \) and \( \delta \leq \varepsilon(L, x) \). Then by Kleiman’s theorem \cite{Kl} we have \( (\varphi^*L - \delta E)^s.Y \geq 0 \). In particular, \( (\varphi^*L - \delta E)^n \geq 0 \). Hence it follows that \( \varepsilon(L, x) \leq \sqrt[n]{L^n} \).

Let us recall the Nakai-Moishezon criterion for ampleness, which was extended to the case of real divisors by Campana and Peternell. We say that a \( \mathbb{R} \)-divisor is ample if its corresponding real point in the Néron-Severi space \( N^1(X) \) lies in the interior of the ample cone of \( X \).
Nakai-Moishezon criterion for $\mathbb{R}$-Cartier divisors [CP].

Let $D = \sum a_i D_i$ be a $\mathbb{R}$-Cartier divisor on a variety $X$.

Then $D$ is ample if and only if $D^s.Y > 0$ for any $s$-dimensional subvariety $Y \subset X$. In particular if $D$ is numerically effective but not ample, then there exist an irreducible subvariety $Y \subset X$, say of dimension $s$, such that $D^s.Y = 0$.

Proof of Proposition 4. Let $\delta = \varepsilon(L,x) < \sqrt[n]{n}$. Then $\varphi^* L - \delta E$ is numerically effective, but not ample. Hence by the real Nakai-Moishezon criterion, there exist a subvariety $Y \subset \text{Bl}_x(X)$ with $(\varphi^* L - \delta E)^d.Y = 0$, $d = \dim Y$. Since $(\varphi^* L - \delta E)^n > 0$ it follows that $1 \leq d \leq n - 1$. Finally by noting that all mixed terms $\varphi^* L^i.E^{d-i}$ are zero for $1 \leq i \leq d - 1$ we find that $\delta^d$ is rational number. $\square$

Proof of Proposition 3. The general ideology behind the proof is as follows. It might be difficult to bound $\varepsilon(L,x)$ only by using curves, because singular curves are invisible. Nevertheless, any subvariety $Y \subset X$ with high multiplicity $m_x(Y)$ at $x$ forces $\varepsilon(L,x)$ to be small. The precise statement is (c.f.[De, Remark 6.7]): If $Y$ is a $p$-dimensional subvariety of $X$ passing through $x$ then $L^p.Y \geq \varepsilon(L,x)^p m_x(Y)$.

Let $X$ be a principal polarized abelian variety of dimension $g$ and let $2X$ be the multiplication by two. On $X$ there are $2^{g-1}(2^g - 1)$ odd theta characteristics such that $\Theta$ passe through $2^{g-1}(2^g - 1)$ two torsion points ([Mu, Corollary 3.15 in Appendix to II-3]). So there is a divisor $\Theta' = 2X(\Theta)$ with a point $x$ having multiplicity $m = m_x(\Theta') \geq 2^{g-1}(2^g - 1)$ at $x$. And note for late use that $\Theta'$ is numerically equivalent to $4\Theta$, since $2X(\Theta) \sim 4\Theta$ [LB, II-3 Proposition 3.6].

Claim. $2X$ maps $\Theta$ generically 1:1 to its image.

Assume to the contrary that multiplication by two is not generically 1:1 over $\Theta'$. Then for general $x \in \Theta$, there is a $y = y(x) \neq x$ such that $2(x - y) = 0$. Then there is a two torsion point $\eta \in X$ such that $(x - y) = \eta$ for all $x \in X$. But then $\Theta - \eta = \Theta$. But a theta divisor is not invariant under any translations.

Now we are in position to compute an upper bound for $\varepsilon(\Theta,x)$, using the notation before Remark 1. Put $\varepsilon = \varepsilon(\Theta,x)$ and let $\hat{\Theta} = \varphi^* \Theta' - mE$ be the strict transform of $\Theta'$ on $\text{Bl}_x(X)$. Then by the remark at the beginning we find:

$$0 \leq (\varphi^* \Theta - \varepsilon E)^{g-1}.\hat{\Theta} = (\varphi^* \Theta - \varepsilon E)^{g-1}.(4\varphi^* \Theta - mE) = 4(g!) - \varepsilon^{g-1}m.$$  

Hence

$$\varepsilon \leq \sqrt[2^{g-1}(2^g - 1)]{\frac{4(g!)}{2^{g-1}(2^g - 1)}} = \sqrt[2^{g-3}(2^g - 1)]{\frac{g!}{2^{g-3}(2^g - 1)}} \leq \sqrt[2^{g-3}]{{g!}}.$$  

$\square$
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