Similarity scaling of the axisymmetric turbulent jet

Chunyue Zhu,1 Yujia Tian,2 Preben Buchhave,3 and Clara M. Velte4

1) College of Water Resources and Environmental Engineering, Zhejiang University of Water Resources and Electric Power, Hangzhou, China.
2) School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore 639798, Singapore.
3) Intarsia Optics, Sonderskovvej 3, 3460 Birkerød, Denmark.
4) Department of Civil and Mechanical Engineering, Technical University of Denmark, Nils Koppels Allé, Building 403, 2800 Kongens Lyngby, Denmark.

(*Electronic mail: cmve@dtu.dk)

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In the current work, we find that a free axisymmetric jet in air displays self-similarity in the fully developed part of the jet. We report accurate measurements of first, second and third order, spatially averaged statistical functions of the axial velocity component performed with a laser Doppler anemometer, including in the outer (high intensity) regions of the jet. The measurements are compared to predictions derived from a simple jet model, described in a separate publication, and we discuss the implications for the further study of self-similarity in a free jet. It appears that all statistical functions included in this study can be scaled with a single geometrical scaling factor – the downstream distance from a common virtual origin.

I. INTRODUCTION

Self-similarity, or self-preservation, is an important concept in the description of turbulent flows. Self-similarity can occur in turbulence when the flow is allowed to develop free of external influences so that the statistical properties are mainly determined by the local scales of the flow. Properties downstream can then, in those instances, be related to upstream properties by simple scaling factors. This significantly facilitates mathematical models and descriptions of such flows. It is also interesting from a more fundamental aspect, not least in terms of understanding the origins and underlying processes of self-similarity. In this paper we shall consider these questions with the turbulent, axisymmetric jet in air as a concrete and important example.

In a companion paper, Buchhave and Velte (2021), we propose a simple model for an axisymmetric jet, free of external influences and issuing into quiescent air, and investigate what can be concluded about its statistical properties based on fundamental symmetry properties of space and time. Based on this analysis, we present a list of expected scaling parameters for some first, second and third order statistical functions, see Table I in section II. We then describe the laboratory jet experiment and the laser Doppler anemometer (LDA) as well as the signal- and data-processing. The LDA is developed to be highly suited for velocity measurement in a flow with high turbulence intensity. We present results of measurements in a high Reynolds number jet in air and throughout the paper, we compare the measured results to the properties expected from the simple model, and finally present some general conclusions.

That turbulence has a tendency in some situations to develop into a state, which to some degree shows self-similarity for all or some of the statistical quantities describing the flow, has been known since the early pioneering theories for turbulence as a stochastic process, c.f. Prandtl (1925); von Kármán (1930); von Kármán and Howarth (1938); Taylor (1939). Among the first to define the concept of self-similarity in general, we can also mention Zel’dovich (1937). The free jet has been studied as a canonical flow, well suited for laboratory experiments, and used as a benchmark for the study of self-similarity. Early studies of the axial jet were performed by Batchelor (1948) and Wygnanski and Fiedler (1969). Theoretical concepts were published by Corrsin (1943) and Antonia and Danaila (2005), who showed by hot-wire measurements in a large, fully developed jet in air, convincing collapse of scaled statistical parameters including second order. Ewing et al. (2007) extended the derivation of similarity variables to two-point statistics, in particular two-point correlations along the jet centerline and showed a similar degree of collapse of scaled two-point correlation functions by measurements with a laser Doppler system upstream and a hot-wire system downstream. Ewing’s results have recently been revisited by Hodžič and Velte (2020).

The theory for self-similarity was, as mentioned, initiated by Zel’dovich (1937), and references to the commonly accepted theory for self-similarity in general and for the theory relating to the free axisymmetric jet in particular, to which we shall limit our description in the following, may be found in textbooks, e.g. Monin and Yaglom (2007); Tennekes and Lumley (1972); Frisch (1995); Pope (2000).
the structure of this theoretical treatment is that the existence of self-similarity is assumed, likely inspired by experiments. The forms of the scaled functions are proposed and inserted into the governing equation, which is taken to be the Navier-Stokes equation in the chosen coordinate system, most often cylindrical coordinates with the z-axis coincident with the jet axis, or some reduced form describing a specific flow, a Reynolds averaged equation or an equation describing only the fluctuating components. The result is then some scaling relations that describe the transformation of the different statistical functions, such as mean velocity profile, second order statistics and higher order statistical functions. With this approach, self-similarity is shown to be possible mathematically, but the existence of self-similarity in real flows must be proven by experiment.

The existence of self-similarity in the fully developed free axisymmetric jet in the inertial subrange with an initial top-hat profile has been amply proven for the first order statistical quantities such as mean velocity profile and second order static moments. Additionally, the linearly expanding conical shape of the jet, defined e.g. by its half velocity contour, and a virtual origin a few jet diameters downstream from the jet orifice, have been verified by experiments. Careful LDA measurements of second order dynamic moments such as velocity power spectrum, correlation function and second order structure function have nicely supported similarity for these quantities in the inertial subrange, also in parts of the developing region, Yaacob, Buchhave, and Velte (2021).

The commonly used turbulent scales such as integral scale, Taylor microscale and Kolmogorov scale also tend to obey a scaling proportional to the distance from the virtual origin, although with some scatter in the value of the origin when extended backwards toward the jet opening. Velocity gradients along the coordinate axes and third order statistical functions that are important for evaluation of the kinetic energy budget are more difficult to measure, and the literature displays considerable scatter in the numerical values. Thus, there are still questions remaining in the literature relating to the number of scaling parameters describing the self-similarity of higher order moments and whether the various statistical functions relate to the same virtual origin, Thiesset, Antonia, and Djenidi (2014).

In the present work, we describe a high Reynolds number jet experiment in air using a sophisticated laser Doppler anemometer, developed for velocity measurement in a flow with high turbulence intensity, Yaacob et al. (2019). We compare the measured results to the properties expected from the simple model, Buchhave and Velte (2021), and find that the simple jet model indeed concurs with the obtained measurement results.

II. THE JET MODEL

In Buchhave and Velte (2021) we describe a simple model of the jet in a cylindrical coordinate system as depicted in Figure 1. Initially, nothing is assumed about the shape of the jet contour or the form of the velocity distribution, except that the jet is rotationally symmetric about the jet axis and that the average velocity profiles are integrable. z is the axial distance from the jet exit, the radial coordinate is denoted r and the azimuthal angle is given by ϕ.

The jet width, δ, is in the literature usually defined as the radial distance to a certain fraction of the mean streamwise centerline velocity. A common definition is the jet half width, δ1/2(z), corresponding to the radial distance from the centerline to half of the mean streamwise centerline velocity. This assumes, of course, a monotonically decreasing mean velocity with increased radial distance from the jet centerline. In the current work, we will apply these definitions, since the velocity profiles are indeed monotonically decreasing with radial distance. However, in the derivation of our simple model, Buchhave and Velte (2021), we have chosen a different definition to avoid assuming anything about the shape of the velocity profile in order not to implicitly assume self-similarity. Instead, we therein defined the jet width based on a conserved quantity, namely the momentum rate along the z-axis. This only assumes integrability of the velocity profiles.

Based on this simple model and the fundamental symmetry relations in classical space and time, the scaling behavior for some static and dynamic first order, second order and third order statistical functions are derived in Buchhave and Velte (2021). The result of this analysis is listed in Table I, which presents the scaling of the functions along the abscissa and the ordinate relating to plots of the functions in an r-z-coordinate system. In the following, u(z, r, ϕ, t) designates the time varying instantaneous velocity at the position (z, r, ϕ) at time t, and u'(z, r, ϕ, t) is the corresponding fluctuating velocity component. Overbar indicates mean value over a record, whether it be a time record or a spatial record (the choice will be obvious from the context, but in the following, we use exclusively spatial average). Angle brackets indicate ensemble average over a number of records. The combined record and ensemble averaged quantities for the mean, mean square and the variance of the velocity are denoted MEAN{u}, MS{u} and VAR{u}, respectively. For brevity, we have in several cases suppressed non-affected dependent variables (coordinates) in the formulas in Table I.

Of special interest is the slope of the third order structure function S3(l), as it can be used to determine to a good approximation both the average dissipation rate (since the round jet is very close to equilibrium, c.f. Hussein, Capp...
TABLE I. Predicted scaling factors of statistical functions (results from Table 1 in Buchhave and Velte (2021)).

| Statistical Function                      | Definition                                                                 | Dimension | Ordinate Scaling Factor | Abscissa Scaling Factor |
|------------------------------------------|----------------------------------------------------------------------------|-----------|-------------------------|-------------------------|
| Axial Distance                           | \( z \)                                                                  | \( L \)   | +1                      |                         |
| Jet Width                                | \( \delta(z) \)                                                          | \( L \)   | +1                      |                         |
| Top Angle                                | \( \theta_0 = \arctan\left(\frac{\delta(z)}{z}\right) \)               |           |                         |                         |
| Mean Velocity                            | \( MEAN\{u\} = \left\langle \frac{dz}{dt} \right\rangle \)              | \( LT^{-1} \) | −1                      | −1                      |
| Mean Square Velocity                     | \( MS\{u\} = \left\langle u^2 \right\rangle \)                           | \( L^2T^{-2} \) | −2                      | +1                      |
| Velocity Variance                        | \( VAR\{u\} = \left\langle \frac{\partial^2}{\partial t^2} \right\rangle \) | \( L^2T^{-2} \) | −2                      | +1                      |
| Turbulence Intensity                     | \( T = \sqrt{\frac{VAR\{u\}}{MEAN\{u\}}} \)                            |           |                         |                         |
| Wave Number                              | \( k \)                                                                  | \( L^{-1} \) | −1                      |                         |
| Power Spectral Density                   | \( F(k) = \left\langle \frac{1}{L^3} \left| \frac{\partial u}{\partial z} \right|^2 \right\rangle \) | \( L^{-3}T^{-2} \) | −1                      | −1                      |
| Autocovariance Function                  | \( R(l) = \left\langle u'(s)u'(s+l) \right\rangle \)                     | \( L^2T^{-2} \) | −2                      | +1                      |
| Autocorrelation Function                 | \( C(l) = \left\langle \frac{\partial u'(k)}{\partial (k+l)} \right\rangle \) |           |                         | +1                      |
| Integral Length Scale                    | \( I = \int_0^\infty C(l) dl \)                                         | \( L \)   | +1                      |                         |
| Taylor Microscale                        | \( \lambda = \sqrt{\left\langle u''^2 \right\rangle / \left\langle \frac{\partial u}{\partial z} \right\rangle} \) | \( L \)   | +1                      |                         |
| Kolmogorov Microscale                    | \( \eta = \left\langle \frac{\partial u}{\partial z} \right\rangle \right\rangle^{1/4} \) | \( L \)   | +1                      |                         |
| Second Order Structure Function          | \( S_2(l) = \left\langle (u(s+l) - u(s))^2 \right\rangle \)               | \( L^2T^{-2} \) | −2                      | +1                      |
| Third Order Structure Function           | \( S_3(l) = \left\langle (u(s+l) - u(s))^3 \right\rangle \)               | \( L^2T^{-3} \) | −3                      | +1                      |
| \( S_3 \) Slope                          | \( \alpha = \left| \frac{dS_3(l)}{dl} \right| \)                         |           | −3                      | +1                      |
| Mean Dissipation Rate                    | \( \tau = \frac{5}{3} \left| \frac{dS_3(l)}{dl} \right| \)                | \( L^2T^{-3} \) | −3                      | +1                      |

and George (1994), \( \tau = \frac{5}{3} \left| \frac{dS_3(l)}{dl} \right| \), and the Kolmogorov microscale, \( \eta = \left\langle \frac{\partial u}{\partial z} \right\rangle^{1/4} \). Here \( l \) is the spatial separation in the axial direction between two neighboring measurement points and \( \nu \) is the kinematic viscosity. \( \ell \) is the pseudo integral length scale. All spatial scales are predicted to grow linearly with \( z \), including the Kolmogorov microscale, the spatial Taylor microscale \( \lambda = \sqrt{\left\langle u''^2 \right\rangle / \left\langle \frac{\partial u}{\partial z} \right\rangle} \), and the integral length scale \( I = \int_0^\infty C(l) dl \), where \( C(l) \) is the spatial autocorrelation function.

In the following section, we describe the LDA measurements of the axial velocity component in a high Reynolds number, axisymmetric jet in air and the calculation of the statistical functions. All calculations provide spatial averages based on the convection of the fluid through the measuring control volume. Buchhave and Velte (2017).

III. EXPERIMENTAL METHOD

A. Jet Generation Facility

The jet generator has been designed to produce a top-hat velocity profile at the jet exit using a settling chamber with meshes and baffles transitioning into a nozzle with a smooth fifth order polynomial shape that prevents separation. The jet had an exit diameter of \( D = 10\,\text{mm} \) and was supplied with air from a pressurized air source. The air pressure was kept constant at 1.0 bar, resulting in a jet exit velocity \( u_0 = 43\,\text{m/s} \).
corresponding to a Reynolds number of 32,000 based on the jet exit diameter and velocity.

The experiment was conducted inside an enclosure with dimensions 3 m x 3 m x 2.5 m (length x width x height) to ensure to a good approximation to a free jet flow, Hussein, Capp, and George (1994). A separate pressurized air-line supplied a Laskin-type seeding generator with Glycerine with a pressure of 1.5 bar, which was used to seed both the jet and the ambient air in the enclosure. The resulting seeding had a bell-shaped particle size distribution centered around 2 µm. The seeding generator was operated during at least half an hour prior to a measurement to achieve uniform seed particle distribution throughout the flow.

B. Acquisition

A map of the measurement positions in a cylindrical coordinate system is shown in Figure 2, where the circles indicate the acquired measurement points. Direction 1 and Direction 2 refer to measurements following the natural jet development; Direction 1 parallels the jet centerline (with a top angle \( \theta_{Dir,1} = 0^\circ \)) and Direction 2 follows an off-axis development corresponding to the jet half width, \( \delta_{t/2}(z) \) (top angle \( \theta_{Dir,2} = 5.6^\circ \)).

The axial velocity component was measured with an in-house designed laser Doppler anemometer (Yaacob et al. (2019)), see Figure 3. The transmitting optics was mounted next to the jet aligned to measure the axial velocity component. The detecting optics was mounted in a forward scattering off-axis direction with a scattering angle of 45°. For maximum stability, the LDA was kept at a fixed position while the jet was traversed.

The detector was a Hamamatsu photomultiplier (type H10425) with a load resistor of 270Ω allowing an electronic bandwidth of 10 MHz. The photomultiplier was followed by a low noise preamplifier (FEMTO HVA-200M-40-F) and the signal was digitized with a 12-bit digital oscilloscope (Picoscope 5000). The digitized signal was transferred to computer memory or to an external hard disk.

At each measurement position, 100 – 400 records (number increasing towards the far field, each 1 – 4 seconds long (again increasing towards the far field), were measured. The sample rate of the digital oscilloscope was adjusted to between 2-4 times the Nyquist rate for the frequency shifted Doppler signal. The optical frequency shift is selected for each measurement point to shift the received frequency modulation to a range, which is optimal for applying post detection band-pass filtering in order to obtain optimal signal-to-noise ratio and minimum electronic bias of the received Doppler bursts. The frequency shift was generated by a dual-Bragg cell module (two IntraAction AOM-402AF1 with dual-RF driver DFE-404A4) and the difference frequency shift could be freely adjusted between 0 and 4 MHz.

Several features distinguish this LDA and make it ideal for measurements in a highly turbulent flow, see also Yaacob et al. (2019). The off-axis configuration allows a very small, nearly spherical measuring volume with a diameter of 140 µm. The adjustable frequency shift makes it possible to choose the best frequency range for each measurement position for accurate, bias-free measurements in a highly turbulent flow, including backflow. The low noise amplification with matching bandwidth and 12-bit digitization helps to achieve a signal with a high signal-to-noise ratio.

C. Signal and Data Processing

The signal processing is software based, see Yaacob et al. (2019). Upon processing, each measured (burst) signal time record results in a data file containing measured values for particle arrival time, residence time, and velocity. In this section, we introduce the formulas used in the numerical calculations of statistical functions. We use the following definition for the constants and variables:

\[ n : \text{sample index} \]
\[ m : \text{record index} \]
\[ t_n : \text{sampling time} \]
\[ s_n : \text{sampling length (mapped spatial record)} \]
\[ \omega_n : \text{temporal frequency} \]
\[ k_n : \text{spatial wavenumber} \]
\[ u_n : \text{instantaneous velocity sample} \]
\[ \Delta \xi_n : \text{spatial residence length} \]
\[ \Delta t_n : \text{residence time} \]
\[ N : \text{number of samples in a record} \]
\[ M : \text{number of acquired records} \]
\[ \nu : \text{kinematic viscosity} \]

The measured residence time in connection with the known measuring volume diameter allows the conversion of the measured temporal record to a spatial record, the so-called convection record,\[ \text{[Buchhave and Velte (2017)]}\]. With this method, the sweeping effect due to the fluctuating convection velocity is removed, and the record displays the spatial velocity structures as they are convected past the measuring volume. As explained in\[ \text{[Buchhave and Velte (2017)]}\], a time record from a stationary flow phenomenon is by this method converted to a homogeneous spatial record, allowing single point calculations of spatial statistical functions. In the case of an LDA measurement, this only requires that the seed particles are uniformly (albeit randomly) distributed in space.

For LDA systems operating in burst-mode, time averaging must be correctly performed using residence time corrected algorithms (so-called residence time weighting), see e.g.,\[ \text{[Buchhave, George, and Lumley (1979); Velte, George, and Buchhave (2014)]}\]. However, spatial averaging does not require sample rate bias correction and must be carried out using arithmetic averaging.

In the following, unless time average is specifically addressed, all averaging operations and calculations will apply uniformly over all the measured records.

Overbar denotes average over \( N \) samples in a record and angle brackets indicate ensemble average over all \( M \) records.

**Static Moments:**

Mean Velocity:
\[ \bar{u} = \frac{1}{N} \sum_{n=0}^{N-1} u_n, \]
\[ \text{MEAN}\{u\} = \langle \bar{u} \rangle \]

Fluctuating Velocity:
\[ u' = u - \bar{u} \]

Mean Square Velocity:
\[ \overline{u^2} = \frac{1}{N} \sum_{n=0}^{N-1} u_n^2 \]
\[ \text{MS}\{u\} = \langle u'^2 \rangle \]

**Velocity Variance:**
\[ \overline{u^2} = \frac{1}{N} \sum_{n=0}^{N-1} u_n^2 \]
\[ \text{VAR}\{u\} = \langle u'^2 \rangle \]

**Dynamic Moments:**

The data calculations are, as far as possible, performed by array processing. This allows Fourier transforms to be computed from the unaliased, randomly sampled velocity data by means of the discrete Fourier transform (DFT) using, in the current case, an exponentially increasing frequency scale. This results in equally spaced points when the power spectrum is displayed in a logarithmic graph. Hence, far fewer spectral values need to be computed than when a fast Fourier transform (FFT) is used, in which case the Nyquist criterion must be adhered to in order to avoid aliasing.

Spatial Fourier Transform:
\[ \hat{u}'(k) = \sum_{n=0}^{N-1} e^{-ik\xi_n} u_n \Delta \xi_n \]

Power Spectral Density (PSD):
\[ F(k)_m = \frac{1}{L_s} \hat{u}'(k) \hat{u}'(k)^* \]
\[ F(k) = \langle F(k)_m \rangle \]

where \( L_s \) is the spatial record length.

Autocovariance functions (ACF) and autocorrelation functions (normalized ACF) are computed by forming velocity products with lags created from the available randomly sampled data. Note that we use the term lag for both temporal distances, \( \tau \), and spatial separations along the convection record between data points, \( l \). We also use the term ‘frequency’ for both temporal frequency and spatial frequency. Due to the inherent random sampling of the LDA, the velocity products are collected in equidistant slots of width \( \Delta \) and averaged by normalizing each slot by the number of products obtained, \( N_i \) (the so-called slotted autocovariance function, SCAF).

The measured velocities contain a random noise that is apparent in the statistics when a velocity value is correlated with itself (and averages out when velocity samples are correlated with other samples than themselves). Thus, at zero lag, both the measured velocities and the random noise are fully correlated, whilst for non-zero lags the random noise correlations vanish for converged statistics. The zero lag value (or self-products) thus contain high frequency measurement noise not connected to the turbulent velocity, which manifests itself as a ‘spike’ at zero lag. To avoid this noise contribution from the self-products of the velocities, we have formed the zero lag value by second order extrapolation of the first few slots back to zero lag with a horizontal tangent at \( l = 0 \).
Covariance function:
\[ R(l_i) = \frac{1}{N_i} \sum_{n \neq n'} (u'(s_n) \cdot u'(s_{n'})) \, , \, l_i - \frac{\Delta}{2} < \Delta - s_n - s_{n'} < l_i + \frac{\Delta}{2} \]

Correlation function:
\[ C(l_i) = R(l_i)/R(0) \]

where \( N_i \) is the number of product contributions (realizations) in slot \( i \) and the double sum covers all samples (except self-products \( n = n' \)) for the running parameters \( n \) and \( n' \).

Spatial structure functions are likewise formed from velocities displaced in the spatial convection record direction with the actual measured lag values. The velocity differences for each slot are then squared and averaged in case of the second order structure function or elevated to the third power and averaged in case of the third order structure function. Again, we apply the slotting technique and normalize with the number of realizations obtained into each slot, \( N_i \). We assume the turbulence is in equilibrium (which is a reasonable approximation for the axisymmetric jet, see e.g. Hussein, Capp, and George (1994)), so we can apply Kolmogorov’s 4/5—law and compute the average dissipation rate from the initial slope of the third order structure function. We can then estimate the Kolmogorov microscale from the dissipation and the kinematic viscosity of air.

Second order structure function:
\[ S_2(l_i)_m = \frac{1}{N_i} \sum_{n \neq n'} (u(s_n) - u'(s_{n'}))^2 \, , \, l_i - \frac{\Delta}{2} < s_n - s_{n'} < l_i + \frac{\Delta}{2} \]

\[ S_2(l_i) = \langle S_2(l_i)_m \rangle \]

Third order structure function:
\[ S_3(l_i)_m = \frac{1}{N_i} \sum_{n \neq n'} (u(s_n) - u'(s_{n'}))^3 \, , \, l_i - \frac{\Delta}{2} < s_n - s_{n'} < l_i + \frac{\Delta}{2} \]

\[ S_3(l_i) = \langle S_3(l_i)_m \rangle \]

Third order structure function slope:
\[ \alpha = \left. \left| \frac{dS_3(l)}{dl} \right| \right|_{l=0} , \]

where the derivative is performed digitally.

Spatial Scales:

Average dissipation rate:
\[ \overline{\epsilon} = \frac{5}{4} |\alpha| = \frac{5}{4} \left| \frac{dS_3(l)}{dl} \right| \bigg|_{l=0} \]

The spatial Kolmogorov microscale is computed from the average dissipation rate by \( \eta = \left( \frac{\epsilon^3}{\overline{\epsilon}} \right)^{1/4} \). Using the slope from the third order structure function we obtain the Kolmogorov microscale.

Kolmogorov microscale:
\[ \eta = \left( \frac{4\nu^3}{5|\alpha|} \right)^{1/4} \]

The spatial integral scale is computed from the integral of the spatial correlation function, \( I = \int_0^\infty C(l) dl \). As the measured records are very long compared to the integral scale, we have, in order to reduce noise, limited the upper limit of the integral to lags with a clear correlation signature, corresponding to two integral length scales, \( l_{3I} \). This corresponds in the discrete case to an upper limit sample number in the sum of \( n_{3I} \).

Integral length scale:
\[ I = \sum_{n=0}^{n_{3I}} C(l_i) \]

The square of the Taylor microscale is defined as the ratio of the mean square fluctuating velocity divided by the mean square of the derivative of the fluctuating velocity,
\[ \lambda^2 = \frac{\left\langle u'(s)^2 \right\rangle}{\left\langle (du'(s)/ds)^2 \right\rangle} \]

However, having available the Fourier components of the velocity field, we have chosen to compute the Taylor scale in k-space:

Taylor microscale:
\[ \lambda^2 = \frac{\left\langle u'(k)^2 \right\rangle}{\left\langle k^2 u'(k) \hat{u}(k)^2 \right\rangle} \]

As the spatial Taylor microscale is very dependent on the wavenumber range, we have, to obtain a well-defined value for the Taylor scale, based the estimates on the data in the wavenumber range with a \(-5/3\) slope, the inertial sub-range. This is simply done by including only Fourier components in this frequency range in the calculation.

Taylor microscale:
\[ \lambda_m = \sqrt{\frac{\sum_{n=-n_{3I/3}} \hat{u}'(k_n)^2 \hat{u}(k_n)^2}{\sum_{n=-n_{3I/3}} k_n^2 \hat{u}'(k_n) \hat{u}(k_n)^2}} , \quad \lambda = \langle \lambda_m \rangle \]

IV. RESULTS

A. Static moments

Time records of the axial velocity component were measured in selected positions (as indicated by circles in Figure 3) in the jet within a matrix covering the axial locations \( z/D = \)
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30:10:100 and transverse locations $r/D = -4.2:2:13.8$.

Figure 4 shows the measured radial profiles of the mean axial velocity component. The measured values are fitted with Gaussian distributions with standard deviation $\sigma$. Figure 5 shows the corresponding mean square profiles of the axial velocity, again fitted with Gaussian distributions.

Figure 5a shows the jet full width at different fractions of the mean axial centerline velocity; $2\delta_{1/3}$, $2\delta_{1/2}$, $2\delta_{2/3}$ and $2\delta_{\sigma}$. The measurements for each definition of the jet width is fitted to a first order polynomial intersecting the $z$-axis at the virtual origin, $z_0$, as expected from the predicted similarity scaling in Buchhave and Velte (2021) (in the following referred to as “Similarity scaling” in the figure legends). A value of $z_0/D = 5$ is seen to represent the virtual origin for the different jet contours quite well.

Figure 5b shows the products of the mean centerline velocity, $u_c$, and the full width at half height (FWHH) of the jet, $2\delta_{1/2}$, as well as the product of $u_c$ and the full width of the jet at the sigma value of the Gaussian fit, $2\delta_{\sigma}$, normalized by the average product along the jet. This product thus represents the product of the large characteristic spatial scale and the large characteristic velocity scale and thereby the constant momentum flux along the axis and thereby indicates a constant large scale Reynolds number and thus self-similarity of the jet.

Figure 6a shows the mean axial centerline velocity, $u_c$, as a function of the downstream distance, $z$. The line is an inverse function of the axial distance from the virtual origin, $z - z_0$ as expected from self-similarity. Figure 6b shows the inverse mean axial centerline velocity normalized by the jet exit velocity, $u_0$, as a function of $z$. The virtual origin $z_0$ again appears to lie around $5D$. The lines in Figures 6a and b are not fitted to the measurement data, but instead constitute first order functions of the distance from the virtual origin, as expected from the simple model in Buchhave and Velte (2021).

Figure 7a and b show along Directions 1 and 2, respec-
FIG. 6. Downstream development of a) the mean axial centerline velocity, $u_c$, and b) the inverse of the mean axial centerline velocity normalized by the jet exit velocity, $u_0 = 43 \text{ m/s}$. The blue lines express the expected respective similarity scaling ($\propto x^{-1}$ and $\propto x$), from Buchhave and Velte (2021).

tively, the mean square (MS) and the root-mean-square (RMS) of the axial velocity component. These quantities have been scaled by multiplication by the distance from the virtual origin $(z-z_0)$ (for the RMS) and, by the square of this distance, $(z-z_0)^2$ (for the MS), in accordance with the expected similarity scaling in Buchhave and Velte (2021). The constant values of the products thus indicate self-similarity along both Directions 1 and 2. These products are, in turn, normalized by their average value along the axis to scale the values so they are distributed around a value of unity.

Figure 8 illustrates the turbulence intensity of the axial velocity component along Directions 1 and 2. Direction 2 is of particular interest, since it indicates the usefulness of a spherical coordinate system as originally proposed by Hodžić (2018). The turbulence intensity is, as highlighted by the fitted constant values, to within the experimental accuracy constant along radial lines and thus adhere to the similarity scaling as expected from the simple model.

B. Dynamic moments

In this section we consider the measured second and third order dynamic moments. We shall investigate how these statistical functions scale along the jet axis and along the direction transverse to the jet axis and compare to the scaling expected from the simple model in Buchhave and Velte (2021). Again, these self-similarity predictions from the simple model are denoted “Similarity scaling”.

We shall compute the dynamic spatial moments based on the convection record Buchhave and Velte (2017). As the convection record for a stationary flow can be considered a spatial, homogeneous record measured at one point in the jet, we can directly compute the local, single point spatial power
FIG. 8. Turbulence intensity of the axial velocity component along Direction 1 (25%) and Direction 2 (41%), respectively. The measurements have been compared to constant values, in accordance with the expected similarity scaling.

spectrum.

1. Power spectral density

Because of the finite record length, the computed power spectral density, $F(k)$, is the convolution of the true spectrum, $F(k)_{\text{true}}$, and the box car spectral window (a sinc-squared function):

$$F(k) = F(k)_{\text{true}} \otimes L_m \text{sinc}^2 \left( \frac{1}{2} k L_m \right)$$

where $L_m$ is the length of the $m$'th spatial record. In order to compute the correct power spectral density, we must correct the computed power spectrum with the weight of the spectral window, also called the equivalent noise bandwidth (ENBW). However, the lengths of the spatial records vary because of the fluctuating velocity, and we must compute the ENBW for each record and correct all the records individually. Random sampling causes a low, flat noise level in each PSD, which we remove in the final plot. Figure 9a shows the measured PSD along the jet centerline where the downstream development follows the shift in spectra from right to left, $z/D = 30 \rightarrow 100$.

To test these ideas, we have extracted the values of the factors needed to scale all the measured PSDs to collapse with the PSD at 30D, as shown in Figure 9. Figure 9a shows the raw measured spectra. Figure 9b shows the result of multiplying the ordinate of the downstream measured PSDs by a factor $F_{\text{ord}}$, to the same ordinate value as the PSD at 30D. Figure 9c shows similarly the result of also scaling the downstream PSDs along the abscissa by a factor, $F_{\text{abs}}$, to collapse with the PSD at 30D along the $k$-axis.

Figure 10a shows the measured values of $F_{\text{ord}}$ as well as its inverse $iF_{\text{ord}}$. Also shown is the ordinate scaling factor expected from the simple model, namely an inverse first order dependence on the distance to the virtual origin, $(z - z_0)$:

$$iF_{\text{ord}, \text{sim}} = 320 \cdot (z - z_0)^{-1} - 3.0 \text{ with } z_0/D = 5.$$ 

Figure 10b shows correspondingly the measured values of $F_{\text{abs}}$ as well as its inverse $iF_{\text{abs}}$ needed to collapse the PSDs along the abscissa. Also shown is the fitting function following the expected form from the simple model: $iF_{\text{abs}, \text{sim}} =...$
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30·(z − z₀)−1 with z₀/D = 5.

These results support that the power spectral density scales along the jet center axis according to a single, geometric scaling factor that equals, to within experimental uncertainty, the distance from a single virtual origin approximately located five jet exit diameters downstream of the jet orifice.

As illustrated in Figures 2 and 10, these first order scaling functions convincingly collapse all the downstream spectra to a single PSD curve, in this case the PSD at 30D. In other words, this scaling of the PSD appears to confirm that the scaling of the free, round jet, in agreement with the predictions of Buchhave and Velte (2021), can be traced to basic physical conservation laws and does not need to rely on ad hoc assumptions about the existence of similarity transformations or on previous experimental data. We also note that we see a scaling from the same value of the virtual origin as we saw in the first order statistical functions such as mean velocity and the jet width.

To further highlight the similarity of the PSDs along the natural jet development directions, we present PSD plots along Direction 1 (Figure 11a) and Direction 2 (Figure 11b) where the maximum values of each PSD have been normalized to unity.

The PSD variations at a fixed distance z from the origin is of great interest, as it describes the distribution of the kinetic energy fluctuations across the jet. In Figure 12a, we plot the raw PSDs at z/D = 60 for the measured radial positions. In Figure 12b, we have normalized the ordinate of the measured PSDs by adjusting the total power to be equal to that at the jet centerline at z/D = 60 in order to better be able to compare the forms of the spectra. Obviously, the spatial fluctuations are similar across the jet at a fixed distance from the origin, indicating an efficient mixing of spatial scales.

Figure 13a shows the mean streamwise velocity profile in the radial direction at z/D = 60 along with a Gaussian fit to the mean velocity. In Figure 13b, corresponding radial profiles of the total power (integral) of the PSDs and the respec-
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FIG. 12. Raw PSD spectra measured across the jet at $z/D = 60$ (increasing radial distance from right to left). a) PSDs across the jet at $z/D = 60$, normalized by the variance at $z/D = 60$ at the centerline. The difference can be attributed to the self-products in the computation of the variance. The noise from the self-products results from high frequency noise unrelated to the measured turbulence. This noise has been subtracted from the PSDs and is thus not present in the profile of the total power of the PSDs. Figure 12 also shows the local turbulence intensity (multiplied by 5) in the transverse direction, for reference. From this figure, we see that the spatial velocity scales are efficiently mixed and has the same shape across the jet, and that the fluctuations relative to the mean velocity are greater at the edge of the jet.

Figures 14a–c show the normalized PSDs collapsing in cross sections at $z/D = 30, 60$ and 90, respectively.

2. Covariance function

We compute the spatial autocovariance function (ACF) using the slotting method (c.f. Buchhave, George, and Lumley [1979]) by sorting products of velocities at the actually occurring random arrival convection record points into equidistant displacement slots (the so-called slotted autocovariance function, SACF).

Again, we emphasize that the mapping from time lags to spatial displacements that we obtain by using the residence times means that we can compute the single point covariance of a homogeneous record as long as the flow is stationary, [Buchhave and Velte (2017)]. This means that we fulfill the prerequisite for identical joint velocity probability distributions (homogeneity) of the two velocities (points) in the covariance product required for a meaningful statistical function, Monin and Yaglom [2007] or Frisch [1995]. This is in contrast to many two-point covariance measurements in jets, where the joint probability distributions cannot be considered identical as the jet is not a homogeneous flow along the axial direction.
FIG. 14. Transverse development of the normalized PSDs at the respective measured cross sectional positions at axial distances: a) $z/D = 30$, b) $z/D = 60$ and c) $z/D = 90$.

FIG. 15. Transverse measurements of the ACF at the respective measured cross sectional positions at axial distances: a) $z/D = 30$, b) $z/D = 60$ and c) $z/D = 90$ as a function of spatial lag, $l$.

Figures 15a–c show the autocorrelation function, $C(l)$, computed at cross sectional scans at $z/D = 30$, 60 and 90, respectively. As before, we observe that the structures are the same across the jet at different downstream positions.

In all the computed ACFs, a spike, which is approximately 30% higher than that at the following lag, occurs at zero lag. We attribute the added contribution to $C(l = 0)$ to self-products from high frequency noise not related to the turbulence. This is the same self-product effect that causes the variance to be higher than the total power computed from the PSD. In Figure 15, we have extrapolated the ACF from the first couple of points back to zero lag with a second order polynomial with horizontal tangent.
In order to make $S_2(l)$ plots collapse to the same ordinate value as that of $S_2(l)$ at $z/D = 30$, we have multiplied the ordinate of $S_2(l)$ at other positions along the jet axis with an ordinate factor, $S_{2,ord}$. Figure 17a shows this scaling factor for all measured second order structure functions (red squares). The blue squares show the inverted scaling factor, $iS_{2,ord}$ (multiplied by 10). Also shown is the $S_2(l)$ scaling expected from the simple model, $iS_{2,ord} = 6200 \cdot (z - z_0)^{-2}$ with $z_0/D = 5$ (blue curve in Figure 17a).

Correspondingly, the factors needed to make $S_2(l)$ collapse along the abscissa with $S_2(l)$ at $z/D = 30$, $S_{2,abs}$, is shown in Figure 17b (red squares) along with the inverted scaling factor, $iS_{2,abs}$ (blue squares). The blue line in Figure 17b shows the similarity scaling expected from the simple model, $iS_{2,abs, sim} = 0.0295 \cdot (z - z_0)$, with $z_0/D = 5$ (blue line in Figure 17b). The blue curve shows the growth of the spatial structures. The deviation at $z/D = 30$ and 40 can be attributed to the difficulty in visually distinguishing the very small shifts of the $S_2(l)$ plots at low displacement values.

In conclusion, the scaling of $S_2(l)$ along both the abscissa direction and the ordinate direction shows excellent agreement with the simple geometrical one-factor scaling expected from the predictions of the simple jet model presented in Buchhave and Velte (2021).

Figure 18 shows the third order structure function, $S_3(l)$, along the centerline of the jet as a function of the displacement, $l$. Figure 18 shows the raw measured data. According to the K41 theory, for an equilibrium turbulent flow as is well approximated by the round jet, the third order structure function is expected to show a linear increase for small displacements with a slope proportional to the average rate of dissipation of turbulent kinetic energy, $\overline{\varepsilon}$: $S_3(l) = -\frac{\varepsilon}{\overline{\varepsilon}} l$. Figure 18 shows a closeup of $S_3(l)$ for small lags rescaled to fit an initial slope of 4/5 (yellow line): $\alpha S_3(l)/l = \frac{4}{5}$. Thus, the average dissipation rate is estimated as $\overline{\varepsilon} = 1/\alpha$.

According to the dimension of $S_3(l)$, $L^3T^{-3}$, we can expect the $S_3(l)$-ordinate to scale as the inverse third power of the axial distance and for the spatial scales (the abscissa) to grow linearly with this distance.

Figure 19a shows the factor, $S_{3,ord}$, needed to collapse the $S_3(l)$ ordinate to the $S_3(l)$-values at $z/D = 30$, (red squares) and the inverse scaling factor, $iS_{3,ord}$, multiplied by 40, (blue squares) and a similarity scaling function in agreement with the geometrical scaling expected from the simple model, $iS_{3,ord, sim} = 6.2 \cdot 10^{-5} \cdot (z - z_0)^{-3}$ with $z_0/D = 5$ (blue curve).

Figure 19b shows the factor, $S_{3,abs}$, needed to collapse $S_3(l)$ along the abscissa to the $S_3(l)$-values at $z/D = 30$ (red squares), the inverse of this function, $iS_{3,abs}$ (blue squares), and a similarity scaling function, $iS_{3,abs, sim} = 0.034 \cdot (z - z_0)$ (blue curve), again with $z_0/D = 5$. The results of the $S_3(l)$-scaling show excellent agreement with a single geometrical scaling factor, i.e., the axial distance from a single virtual origin around five jet orifice diameters downstream.

Figure 20 shows the average dissipation rate, $\overline{\varepsilon}$, computed from the $S_3$-slope, $\alpha$. The data point at $z/D = 100$ was excluded due to the difficulty to fit the $S_3(l)$-slope accurately.

3. Structure functions

As described in section IIIC, the second and third order structure functions are computed from the measured axial velocity records at the random spatial separations or displacements, $s_l = s_{ord}$, in the convection record, raised to the second power and third power, respectively, and averaged over the record. The displacements are sorted into equal size bins or slots. The computations are repeated for each record and then averaged over all records.

Figure 16a shows the development of the second order spatial structure function $S_2(l)$ along the centerline of the jet from $z/D = 30 : 10 : 100$. The $S_2(l)$-functions are observed to begin to level off at large separations, but they follow the expected 2/3-slope predicted by the K41 theory for small separations $l$. Figure 16b shows $S_2(l)$ scaled along the ordinate and the abscissa to convincingly collapse with $S_2(l)$ at $z/D = 30$.

According to the simple model and the expected scaling of $S_2(l)$ from its dimension, $L^2T^{-2}$, we should expect the ordinate, $S_2$, to scale as the inverse second power of the local velocity scale, which we have taken to be the local mean velocity. The structures themselves, the abscissa $l$, we would expect to scale with the local spatial scale, which we have taken to be the local half width of the jet.
4. Spatial scales

The measured time series in a stationary turbulent flow along with the convection record technique allow the computation of the spatial velocity scales as a function of the downstream distance. In all instances, the scales are computed for each convection record and afterwards averaged over all records. The downstream development along the centerline of the three length scales are displayed together in Figure 21; the integral scale, $I(z)$, the Taylor microscale, $\lambda(z)$ and the Kolmogorov microscale, $\eta(z)$.

The integral length scale is defined as the integral of the spatial autocorrelation function, $I \equiv \int_0^\infty C(l) \, dl$. We have used the ACFs, e.g., as shown in Figure 15 as basis for the calculation. The square of the Taylor microscale is computed in $k$-space, $\lambda^2 \equiv \frac{\langle \hat{u}(k) \hat{u}(k)^* \rangle}{\langle \hat{u}^2(k) \rangle}$. The Kolmogorov microscale is computed from the third order structure function, $\eta = \left( \frac{4\nu^3}{5\alpha} \right)^{1/4}$, where $\alpha$ is the initial slope of $S_3(l)$. Figure 21 shows that all three scales have a linear evolution along the jet centerline with approximately the same virtual origin, $z_0/D = 5$, for all three scales, as expected from the model.

V. CONCLUSIONS

The purpose of the current work is to evaluate to what extent the statistical properties of a “real” jet in air correspond to predictions from a simple model governed by some basic symmetry properties and corresponding conservation laws (Buchhave and Velte (2021)). The predictions can be based on the dimension of the quantity as expressed by the order of velocity and length scales entering the quantity. Or stated differently, the abscissa and ordinate of the plots of the statistical
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FIG. 19. a) Factors collapsing $S_3(\ell)$ along the ordinate, $S_{3,ord}$ (red squares), the inverted $S_3$ scaling factor, $iS_{3,ord}$ (blue squares), and the expected ordinate similarity scaling from the simple model, $iS_{3,ord,sim}$ (blue line). b) The factors collapsing the abscissa, $S_3(abs)$ (red squares), the inverted scaling factor, $iS_{3,abs}$ (blue squares), and the expected $S_3$ abscissa similarity scaling, $iS_{3,abs,sim}$ (blue line). Virtual origin $z_0/D = 5$ in both cases.

functions shown in Table I. The overall conclusion is that the scaling of all the quantities investigated in this work can be expressed by velocity, $u(z)$, and distance, $z$. Since the velocity is itself a function of $z$, the scaling of all the investigated statistical quantities can be said to depend on only one scaling parameter, namely the distance from the virtual origin, $z - z_0$.

Careful laser Doppler anemometer measurements in a free, round jet in air have allowed collection of high-quality data, even in very high turbulence intensity and high shear regions of a high Reynolds number, turbulent jet. Conversion of the randomly sampled time records to spatial records, using the so-called convection record method, has allowed computation of single point static and dynamic first, second and third order spatially averaged statistical functions. Comparison of the measured statistical functions and the way they scale from one point in the jet to other points in the jet has convincingly shown that the jet, when isolated from external influence, appears to develop according to basic underlying symmetry properties governing Newtonian space and time and scale with a single geometrical scaling factor, the distance from a common virtual origin, $z_0$, which is to be determined from experiments.

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