QUANTUM INSTRUMENTS
AND CONDITIONED OBSERVABLES

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Abstract

Observables and instruments have played significant roles in recent studies on the foundations of quantum mechanics. Sequential products of effects and conditioned observables have also been introduced. After an introduction in Section 1, we review these concepts in Section 2. Moreover, it is shown how these ideas can be unified within the framework of measurement models. In Section 3, we illustrate these concepts and their relationships for the simple example of a qubit Hilbert space. Conditioned observables and their distributions are studied in Section 4. Section 5 considers joint probabilities of observables. We introduce a definition for joint probabilities and discuss why we consider this to be superior to the standard definition.

1 Introduction

This article is a continuation of the author’s work on conditioned observables in quantum mechanics [7]. For the reader’s convenience, we first review the concepts needed in the present paper. We shall only consider quantum systems described by finite-dimensional Hilbert spaces. Although this is a strong restriction, it is general enough to include the important subjects of quantum computation and information theory [8, 12].

In Section 2, we review the definitions of quantum effects, observables and instruments [2, 8, 9, 12]. We consider the sequential product and conditioning of effects and observables [4, 5, 6, 7]. Quantum operations, channels
and instruments are discussed. The idea of different instruments measuring an observable is presented and the special role of the Lüders instrument is emphasized. We also discuss the unifying framework of measurement models

The various concepts presented in Section 2 are illustrated for the simplest case of a qubit Hilbert space in Section 3. In particular, we discuss spin component observables. Complementary observables and their relationship to mutually unbiased bases are presented. We also consider observable probability distributions. Finally, in Section 5 we introduce what we consider to be the natural and correct definition of joint probabilities of observables. Moreover, we discuss why we believe this to be superior to the standard definition.

2 Effects, Observables and Instruments

Let \( \mathcal{L}(H) \) be the set of linear operators on a finite-dimensional complex Hilbert space \( H \). For \( S, T \in \mathcal{L}(H) \) we write \( S \leq T \) if \( \langle \phi, S\phi \rangle = \langle \phi, T\phi \rangle \) for all \( \phi \in H \). We define the set of effects by

\[
\mathcal{E}(H) = \{ a \in \mathcal{L}(H) : 0 \leq a \leq I \}
\]

where 0, \( I \) are the zero and identity operators, respectively. The effects correspond to yes-no experiments and \( a \in \mathcal{E}(H) \) is said to occur when a measurement of \( a \) results in the value yes. We denote the set of projections on \( H \) by \( \mathcal{P}(H) \). It is clear that \( \mathcal{P}(H) \subseteq \mathcal{E}(H) \) and we call the elements of \( \mathcal{P}(H) \) sharp effects. A one-dimensional projection \( P_\phi = |\phi\rangle \langle \phi| \), where \( ||\phi|| = 1 \), is called an atom. If \( \phi \in H \) with \( \phi \neq 0 \), we write \( \hat{\phi} = \phi / ||\phi|| \). We then have

\[
P_\phi = \frac{1}{||\phi||} |\phi\rangle \langle \phi|
\]

We call \( \rho \in \mathcal{E}(H) \) a partial state if \( \text{tr} (\rho) \leq 1 \) and \( \rho \) is a state if \( \text{tr} (\rho) = 1 \). We denote the set of states by \( \mathcal{S}(H) \) and the set of partial states by \( \mathcal{S}_p(H) \). If \( \rho \in \mathcal{S}(H) \), \( a \in \mathcal{E}(H) \) we call \( \mathcal{P}_\rho(a) = \text{tr} (\rho a) \) the probability that \( a \) occurs in the state \( \rho \). Of course, \( 0 \leq \mathcal{P}_\rho(a) \leq 1 \). If \( a, b \in \mathcal{E}(H) \) and \( a + b \leq I \) we write \( a \perp b \). When \( a \perp b \) we have that \( a + b \in \mathcal{E}(H) \) and \( \mathcal{P}_\rho(a+b) = \mathcal{P}_\rho(a) + \mathcal{P}_\rho(b) \). If \( P_\phi \) is an atom, then we call \( P_\phi \) (and \( \phi \)) a pure state. We then write

\[
\mathcal{P}_\phi(a) = \mathcal{P}_{P_\phi}(a) = \text{tr} (P_\phi a) = \langle \phi, a\phi \rangle
\]
If \( \phi \) and \( \psi \) are pure states, we call \( |\langle \phi, \psi \rangle|^2 \) the transition probability from \( \phi \) to \( \psi \).

We denote the unique positive square-root of \( a \in \mathcal{E}(H) \) by \( a^{1/2} \). For \( a, b \in \mathcal{E}(H) \), their sequential product is the effect \( a \circ b = a^{1/2}ba^{1/2} \) where \( a^{1/2}ba^{1/2} \) is the usual operator product [4, 5, 6, 10]. We interpret \( a \circ b \) as the effect that results from first measuring \( a \) and then measuring \( b \). It can be shown that \( a \circ b \leq a \) and that \( a \circ b = b \circ a \) if and only if \( ab = ba \). If \( ab = ba \), we say that \( a \) and \( b \) are compatible and interpret this physically as meaning that \( a \) and \( b \) do not interfere. We also call \( a \circ b \) the effect conditioned on the effect \( a \) and write \( (b | a) = a \circ b \). Notice that if \( b_1, b_2 \in \mathcal{E}(H) \) with \( b_1 \perp b_2 \), then \( (b_1 + b_2 | a) = (b_1 | a) + (b_2 | a) \). Moreover, \( \mathcal{E}(H) \) is convex and if \( \lambda_i \geq 0 \) with \( \sum \lambda_i = 1 \), then
\[
(\sum \lambda_i b_i | a) = \sum \lambda_i (b_i | a)
\]
so \( b \mapsto (b | a) \) is an affine function. Of course, \( a \mapsto (b | a) \) is not an affine function in general.

If \( \rho \in \mathcal{S}(H) \) and \( a \in \mathcal{E}(H) \) with \( \rho \circ a \neq 0 \), since \( \rho \circ a \leq \rho \) we have that
\[
\text{tr} \ [(\rho | a)] = \text{tr} (a \circ \rho) = \text{tr} (\rho \circ a) \leq \text{tr} (\rho) = 1
\]
Hence, \( (\rho | a) \in \mathcal{S}_\rho(H) \) and for \( a \in \mathcal{E}(H) \) we obtain
\[
\mathcal{P}_\rho [(b | a)] = \text{tr} [\rho (b | a)] = \text{tr} (\rho a \circ b) = \text{tr} [(a \circ \rho) b]
\]
\[
= \text{tr} [(\rho | a) b]
\]
If \( \mathcal{P}_\rho(a) = \text{tr} (\rho a) \neq 0 \), we can form the state \( (\rho | a)/\text{tr} (\rho a) \). Then as a function of \( b \)
\[
\hat{\mathcal{P}}_\rho [(b | a)] = \frac{\mathcal{P}_\rho [(b | a)]}{\mathcal{P}_\rho(a)}
\]
becomes a probability measure on \( \mathcal{E}(H) \) and we call \( \hat{\mathcal{P}}_\rho [(b | a)] \) the conditional probability of \( b \) given \( a \).

For a finite set \( \Omega_A \), an observable with value-space \( \Omega_A \) is a subset \( A = \{A_x : x \in \Omega_A \} \) of \( \mathcal{E}(H) \) such that \( \sum_{x \in \Omega_A} A_x = I \). We interpret \( A_x \) as the effect that occurs when \( A \) has the value \( x \). The condition \( \sum A_x = I \) ensures that \( A \) has one of the values \( x \in \Omega_A \) when \( A \) is measured. Defining \( A_X = \sum_{x \in X} A_x \) for all \( X \subseteq \Omega_A \), we see that \( X \mapsto A_X \) is a finite positive operator-valued
measure on $H$. If $A_x \in \mathcal{P}(H)$ for all $x \in \Omega_A$, we call $A$ a sharp observable. The effects $A_x$ for a sharp observable commute and are mutually orthogonal. If the effects $A_x$ are atoms, we say that $A$ is atomic. In this case, $A_x = P_{\phi_x}$ where $\{\phi_x : x \in \Omega_A\}$ is an orthonormal basis for $H$. We denote the set of observables on $H$ by $O(H)$.

For $A,B \in O(H)$ with $A = \{A_x : x \in \Omega_A\}$ and $B = \{B_y : y \in \Omega_B\}$ we define their sequential product $A \circ B$ to be the observables with value-space $\Omega_A \times \Omega_B$ and

$$A \circ B = \{A_x \circ B_y : (x,y) \in \Omega_A \times \Omega_B\}$$

The observable $B$ conditioned by the observable $A$ has value-space $\Omega_B$ and is defined by

$$(B \mid A) = \left\{ \sum_{x \in \Omega_A} A_x \circ B_y : y \in \Omega_B \right\} = \left\{ \sum_{x \in \Omega_A} (B_y \mid A_x) : y \in \Omega_B \right\}$$

We denote the effects in $A \circ B$ and $(B \mid A)$ by $(A \circ B)_x(y) = A_x \circ B_y$ and $(B \mid A)_y = \sum_{x \in \Omega_A} A_x \circ B_y$, respectively. We say that $A$ and $B$ commute if

$$a_x b_y = b_y a_x$$

for all $x \in \Omega_A, y \in \Omega_B$. If $A$ and $B$ commute, then $(B \mid A) = B$. We do not know whether the converse holds. However, we have the following result.

**Lemma 2.1.** If $(B \mid A) = B$ and $A$ is sharp, then $A$ and $B$ commute.

**Proof.** Since $(B \mid A) = B$ and $A$ is sharp, we have that

$$\sum_{x \in \Omega_A} A_x B_y A_x = \sum_{x \in \Omega_A} A_x \circ B_y = B_y$$

for every $y \in \Omega_B$. Since the $A_x$’s are mutually orthogonal we obtain

$$A_x B_y A_x = A_x B_y = B_y A_x$$

For all $x \in \Omega_A, y \in \Omega_B$. Hence, $A$ and $B$ commute. $\square$

If $\rho \in S(H)$ and $A \in O(H)$, we define the state $\rho$ conditioned on $A$ by

$$(\rho \mid A) = \sum_{x \in \Omega_A} (\rho \mid A_x) = \sum_{x \in \Omega_A} A_x^{1/2} \rho A_x^{1/2} \quad (2.2)$$
An operation on $H$ is a completely positive affine map $A: S_p(H) \to S_p(H)$ [8, 12]. Thus, if $\lambda_i \geq 0$, $\sum \lambda_i = 1$ and $\rho_i \in S_p(H)$, $i = 1, 2, \ldots, n$, then

$$A \left( \sum_{i=1}^{n} \lambda_i \rho_i \right) = \sum_{i=1}^{n} \lambda_i A(\rho_i)$$

We call an operation $A$ a channel if $A(\rho) \in S(H)$ for all $\rho \in S(H)$. We denote the set of channels on $H$ by $C(H)$. Notice that if $a \in E(H)$, then the map $\rho \mapsto \rho|a$ is an example of an operation and if $A \in O(H)$ then $\rho \mapsto (\rho | A)$ is a channel. For a finite set $\Omega_I$, an instrument with value-space $\Omega_I$ is a set of operations $I = \{I_x: x \in \Omega_I\}$ such that $\sum_{x \in \Omega_I} I_x \in C(H)$.

Defining $I_X$ for $X \subseteq \Omega_A$ by $I_X = \sum_{x \in X} I_x$ we see that $X \mapsto I_X$ is an operation-valued measure on $H$ [2, 8, 9]. If $A \in O(H)$, we say that an instrument $I$ is $A$-compatible if $\Omega_I = \Omega_A$ and

$$\mathcal{P}_\rho(A_X) = \text{tr} [I_X(\rho)]$$

for all $\rho \in S(H)$, $X \subseteq \Omega_A$. To show that $I$ is $A$-compatible, it is sufficient to show that

$$\mathcal{P}_\rho(A_x) = \text{tr} [I_x(\rho)]$$

for all $\rho \in S(H)$ and $x \in \Omega_A$.

We view an $A$-compatible instrument as an apparatus that can be employed to measure the observable $A$. If $I$ is an instrument, then there is a unique $A^2 \in O(H)$ such that $I$ is $A^2$-compatible [8]. It is clear that $I$ is $A$-compatible if and only if $A^2 = A$. On the other hand, if $A \in O(H)$, then there are many $A$-compatible instruments. For example, if $\eta \in S(H)$ then the trivial instrument $I_X(\rho) = \text{tr} (\rho A_x) \eta$ is $A$-compatible. In this work, an important $A$-compatible instrument is given by the Lüders instrument [8]

$$L^A_x(\rho) = A_x \circ \rho.$$ 

We then have

$$L^A_X(\rho) = \sum_{x \in X} A_x \circ \rho = \sum_{x \in X} A_x^{1/2} \rho A_x^{1/2}$$

Notice that $L^A_{\Omega_A}(\rho) = (\rho \mid A)$ as in (2.2). If $A = \{\phi_x\langle \phi_x \mid: x \in \Omega_A\}$ is atomic, we obtain

$$L^A_X(\rho) = \sum_{x \in X} \langle \phi_x, \rho \phi_x \rangle \phi_x \langle \phi_x \mid = \sum_{x \in X} \mathcal{P}(P_{\phi_x}) P_{\phi_x}$$

5
The duality between observables and instruments is emphasized by the unifying studies of measurement models \[1, 2, 8, 11\]. A measurement model is a 5-tuple \(M = (H, K, \eta, \nu, F)\) where \(H, K\) are Hilbert spaces called the base and probe systems, respectively, \(\eta \in S(K)\) is an initial state, \(\nu: S(H \otimes K) \to S(H \otimes K)\) is a channel describing the measurement interaction between the base and probe systems and \(F \in O(K)\) is the pointer observable. The instrument on \(H\) defined by

\[I_M^X(\rho) = \text{tr}_K[\nu(\rho \otimes \eta)(I \otimes F_X)]\]  \hspace{1cm} (2.3)

is called the model instrument where \(X \subseteq \Omega_F = \Omega_I\) and \(\text{tr}_K\) is the partial trace \[8, 12\]. The unique observable \(B_M \in O(H)\) defined by \(B_M = A^I_M\) is the model observable. We then have the probability reproducing condition

\[\text{tr}(\rho B_M^X) = \text{tr} \left[ I_M^X(\rho) \right] = \text{tr} \left[ \nu(\rho \otimes \eta)(I \otimes F_X) \right]\]  \hspace{1cm} (2.4)

for all \(\rho \in S(H), X \subseteq \Omega_{B_M} = \Omega_F\).

Thus, any measurement model \(\mathcal{M}\) determines a unique instrument \(I^\mathcal{M}\) and a unique observable \(B^\mathcal{M}\). Conversely, for any instrument \(I\) there exist many measurement models \(\mathcal{M}\) such that \(I = I^\mathcal{M}\) and for any observable \(B\) there exist many model measurements \(\mathcal{M}\) such that \(B = B^\mathcal{M}\). These are shown in the next two results.

**Theorem 2.2.** (Ozawa \[8, 12\]) For any instrument \(I\) on \(H\) there exists a measurement model \(\mathcal{M} = (H, K, \eta, \nu, F)\) where \(\eta\) is a pure state, \(\nu\) is a unitary channel \(\nu(\mu) = U\mu U^*\) and \(F\) is a sharp observable such that \(I = I^\mathcal{M}\).

**Corollary 2.3.** For any \(B \in O(H)\) there exists a measurement model \(\mathcal{M}\) as in Theorem 2.2 such that \(B = B^\mathcal{M}\).

**Proof.** Given \(B \in O(H)\) there exists a \(B\)-compatible instrument \(I\). By Theorem 2.2 there exists a measurement model satisfying the given conditions such that \(I = I^\mathcal{M}\). Then \(B = A^I = A^I^\mathcal{M}\) so \(B = B^\mathcal{M}\). \(\Box\)

We now continue this study to include sequential products of observables and conditioned observables. If \(A, B \in O(H)\), then \(A \circ B \in O(H)\). By Corollary 2.3 there exists a measurement model \(\mathcal{M} = (H, K, \eta, \nu, F)\)
satisfying the conditions of Theorem 2.2 such that \( A \circ B = B^M \). We then have that
\[
\Omega_F = \Omega_{A \circ B} = \Omega_A \times \Omega_B
\]
and by (2.4)
\[
\text{tr} \left[ \rho (A \circ B)_Z \right] = \text{tr} \left[ U(\rho \otimes P_\phi)U^*(I \otimes F_Z) \right]
\]
for every \( \rho \in S(H) \) and \( Z \subseteq \Omega_A \times \Omega_B \). In particular, for every \((x, y) \in \Omega_A \otimes \Omega_B\), \( \rho \in S(H) \) we obtain
\[
\text{tr} \left( \rho A_x \circ B_y \right) = \text{tr} \left[ \rho (A \circ B)_{(x,y)} \right] = \text{tr} \left[ U(\rho \otimes P_\phi)U^*(I \otimes F_{(x,y)}) \right]
\]
The advantage of (2.5) and (2.6) is that the statistics of \( A \circ B \), which may be unsharp, is described by the sharp observable \( F \) and as we have seen, sharp observables are simpler than general unsharp ones. In particular, the effects \( F_{(x,y)} \) commute and are mutually orthogonal. Applying (2.6), we conclude that
\[
\text{tr} \left( \rho A_x \right) = \text{tr} \left( \rho A_x \circ B_{\Omega_B} \right) = \text{tr} \left[ U(\rho \otimes P_\phi)U^* \left( I \otimes \sum_{y \in \Omega_B} F_{(x,y)} \right) \right]
\]
so \( A \) is described by the sharp observable
\[
\left\{ \sum_{y \in \Omega_B} F_{(x,y)} : x \in \Omega_A \right\} \quad (2.7)
\]
Considering \( (B \mid A) \) we have by (2.6) that
\[
\text{tr} \left[ \rho (B \mid A)_y \right] = \text{tr} \left( \rho \sum_{x \in \Omega_A} A_x \circ B_y \right) = \sum_{x \in \Omega_A} \text{tr} \left( \rho A_x \circ B_y \right)
\]
\[
= \text{tr} \left[ U(\rho \otimes P_\phi)U^* \left( I \otimes \sum_{x \in \Omega_A} F_{(x,y)} \right) \right]
\]
so \( (B \mid a) \) is described by the sharp observable
\[
\left\{ \sum_{x \in \Omega_A} F_{(x,y)} : y \in \Omega_B \right\} \quad (2.8)
\]
where (2.7) and (2.8) commute even though \( A \) and \( (B \mid A) \) need not. But this is taking us away from our primary mission so we leave a further study to later work.
3 Qubit Hilbert Space

This section illustrates the concepts presented in Section 2 for the simplest case of a qubit Hilbert space $H = \mathbb{C}^2$ with the usual inner product. Let $\phi = (1,0)$, $\phi' = (0,1)$ be the standard orthonormal basis for $\mathbb{C}^2$. Relative to this basis, the $Pauli operators$ have the matrix forms

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Letting $\sigma = (\sigma_x, \sigma_y, \sigma_z)$, every $\rho \in S(H)$ has the form

$$\rho = \frac{1}{2} (I + r \cdot \sigma)$$

where $r \in \mathbb{R}^3$ with $||r|| \leq 1$ and $\cdot$ is the usual dot product in $\mathbb{R}^3$. The eigenvalues of $\rho$ are $\lambda_{\pm} = \frac{1}{2} (1 \pm ||r||)$. We have that $\lambda_+ = 1$ and $\lambda_- = 0$ if and only if $||r|| = 1$ so $\rho$ is pure if and only if $||r|| = 1$. Every $a \in E(H)$ has the form

$$a = \frac{1}{2} (\alpha I + n \cdot \sigma)$$

where $||n|| \leq \alpha \leq 2 - ||n||$ and positivity is equivalent to $||n|| \leq \alpha$. For $n \in \mathbb{R}^3$ with $||n|| = 1$, define the atoms $S^n_\pm = \frac{1}{2} (I \pm n \cdot \sigma)$. The atomic observable $S^n = \{S^n_+, S^n_-\}$ is called the spin component observable in direction $n$. Then $S^n_\pm$ is the effect for which the spin component is $+$ and $S^n_-\$ is the effect for which the spin component is $-$ in the direction $n$ and the value-space $\Omega_{S^n} = \{+, -\}$. The effect $S^n_+$ is the 1-dimensional projection

$$S^n_+ = \frac{1}{2} \begin{bmatrix} 1 + n_3 & n_1 - in_2 \\ n_1 + in_2 & 1 - n_3 \end{bmatrix}$$

and $S^n_- = I - S^n_+$. Suppose we measure $S^m$ first and $S^n$ second. Then

$$S^m \circ S^n = \{S^m_+ \circ S^n_+, S^m_+ \circ S^n_-, S^m_- \circ S^n_+, S^m_- \circ S^n_-\} \quad (3.1)$$

and

$$(S^n \mid S^m) = \{S^n_+ \circ S^m_+, S^n_+ \circ S^m_-, S^n_- \circ S^m_+, S^n_- \circ S^m_-\} \quad (3.2)$$

The observables in (3.1) and (3.2) are not sharp even though $S^m$ and $S^n$ are sharp.
To illustrate, let \( m = (0, 0, 1) \) and \( n = (1, 0, 0) \). These correspond to spin measurements in the \( z \) and \( x \) directions, respectively. Then

\[
S^m_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S^m_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad S^n_+ = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad S^n_- = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},
\]

and we have that

\[
S^m_+ \circ S^m_+ = S^m_+ \circ S^m_- = \frac{1}{2} S^m_+
\]
\[
S^m_- \circ S^m_+ = S^m_- \circ S^m_- = \frac{1}{2} S^m_-
\]

Hence

\[
S^m \circ S^n = \{ \frac{1}{2} S^m_+ S^m_- + \frac{1}{2} S^m_+ S^m_- \}
\]
\[
(S^m \mid S^n) = \{ (S^m \mid S^n)_+ , (S^m \mid S^n)_- \} = \{ \frac{1}{2} I, \frac{1}{2} I \}
\]

and similar formulas hold for \( S^n \circ S^m \) and \( (S^m \mid S^n) \). Notice that \( (S^m \mid S^n) = (S^n \mid S^m) \) but \( S^m \) and \( S^n \) do not commute.

We now find the general form of \( S^m \circ S^n \) and \( (S^n \mid S^m) \). The normalized eigenvector of \( S^n_+ \) with corresponding eigenvalue 1 is

\[
\phi^+_n = \frac{1}{\sqrt{2(1 - n_3)}} \begin{bmatrix} n_1 - in_2 \\ 1 - n_3 \end{bmatrix} \text{ if } n_3 \neq 1, \quad \phi^+_n = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ if } n_3 = 1
\]

and corresponding to eigenvalue 0 we have

\[
\phi^-_n = \frac{1}{\sqrt{2(1 + n_3)}} \begin{bmatrix} in_2 - n_1 \\ 1 + n_3 \end{bmatrix} \text{ if } n_3 \neq -1, \quad \phi^-_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ if } n_3 = -1
\]

Since \( S^n_+ = |\phi^+_n\rangle \langle \phi^+_n| \) and \( S^n_- = |\phi^-_n\rangle \langle \phi^-_n| \), we obtain

\[
S^m \circ S^n = \left\{ |\langle \phi^+_n, \phi^+_n| \rangle^2 S^m_+ , |\langle \phi^+_n, \phi^-_n| \rangle^2 S^m_- , |\langle \phi^-_n, \phi^+_n| \rangle^2 S^m_- , |\langle \phi^-_n, \phi^-_n| \rangle^2 S^m_- \right\}
\]

Moreover, we have that

\[
(S^n \mid S^m)_+ = |\langle \phi^+_n, \phi^+_n| \rangle^2 S^m_+ + |\langle \phi^-_n, \phi^+_n| \rangle^2 S^m_- \\
(S^n \mid S^m)_- = |\langle \phi^+_n, \phi^-_n| \rangle^2 S^m_+ + |\langle \phi^-_n, \phi^-_n| \rangle^2 S^m_- 
\]
In general, \( c \) and corresponding coefficients will be non-zero. The first of these effects becomes

\[
\begin{align*}
(S^m \mid S^m)_+ &= (2a - 1)S^m_+ + (1 - a)I \\
(S^m \mid S^m)_- &= (1 - 2a)S^m_+ + aI
\end{align*}
\]

For another example, the Lüders channel \([10]\) for \( S^m \) becomes

\[
\mathcal{L}^{S^m}(\rho) = S^m_+ \circ \rho + S^m_+ \circ \rho = S^m_+ \rho S^m_+ + S^m_+ \rho S^m_-
\]

\[
= \langle \phi^m_+, \rho \phi^m_+ \rangle P_{\phi^m_+} + \langle \phi^m_-, \rho \phi^m_- \rangle P_{\phi^m_-} = (\rho \mid A)
\]

as in \([2.2]\).

It is also of interest to consider three spin measurements in directions \( m, n \) and \( r \). We then have that

\[
S^m \circ (S^n \circ S^r) = \{ S^m_+ \circ (S^n_+ \circ S^r_+), S^m_+ \circ (S^n_+ \circ S^r_-), S^m_+ \circ (S^n_- \circ S^r_+), S^m_+ \circ (S^n_- \circ S^r_-), S^m_- \circ (S^n_+ \circ S^r_+), S^m_- \circ (S^n_+ \circ S^r_-), S^m_- \circ (S^n_- \circ S^r_+), S^m_- \circ (S^n_- \circ S^r_-) \}
\]

The first of these effects becomes

\[
S^m_+ \circ (S^m_+ \circ S^r_+)^2 = S^m_+ \circ \left( \left| \langle \phi^m_+, \phi^m_+ \rangle \right|^2 S^m_+ \right) = \left| \langle \phi^m_+, \phi^m_+ \rangle \right|^2 S^m_+ \circ S^m_+ = c_{+++} S^m_+
\]

where we have defined \( c_{+++} = \left| \langle \phi^m_+, \phi^m_+ \rangle \right|^2 \). The other effects and corresponding coefficients \( c_{++-}, c_{+-+}, \ldots \), are similar. We conclude that

\[
(S^m \circ S^r \mid S^m) = \{ c_{+++} S^m_+ + c_{++-} S^m_- + c_{+-+} S^m_+ + c_{-+-} S^m_- + c_{++-} S^m_+ + c_{+-+} S^m_- + c_{++-} S^m_+ + c_{++-} S^m_- \}
\]

and

\[
((S^r \mid S^m) \mid S^m) = \{ (c_{+++} + c_{++-}) S^m_+ + (c_{++-} + c_{+-+}) S^m_- + (c_{+++} + c_{+-+}) S^m_+ + (c_{++-} + c_{+-+}) S^m_- \}
\]

In general, \( S^m \circ S^m \circ S^r \neq S^m \circ (S^n \circ S^r) \) and we leave this to the reader.
4 Conditioned Observables and Distributions

Let $A = \{A_x\}$, $B = \{B_y\}$ be observables on $H$ with value-spaces having cardinality $|\Omega_A| = m$, $|\Omega_B| = n$. We say that $A$ and $B$ are complementary if $(B_y \mid A_x) = \frac{1}{n} A_x$ and $(A_x \mid B_y) = \frac{1}{m} B_y$ for all $x \in \Omega_A$, $y \in \Omega_B$. This condition says that when $A$ has a definite value $x$, then $B$ is completely random and vice versa. This is analogous to the complementary position and momentum observables of continuum quantum mechanics. We then have

$$A \circ B = \{A_x \circ B_y : x \in \Omega_A, y \in \Omega_B\} = \{\frac{1}{n} A_x, \ldots, \frac{1}{n} A_x : x \in \Omega_A\} \quad (4.1)$$

and

$$B \circ A = \{B_y \circ A_x : x \in \Omega_A, y \in \Omega_B\} = \{\frac{1}{m} B_y, \ldots, \frac{1}{m} B_y : y \in \Omega_B\} \quad (4.2)$$

where there are $n$ terms $\frac{1}{n} A_x$ in (4.1) and $m$ terms in $\frac{1}{m} B_y$ in (4.2). We also obtain

$$(B \mid A)_y = \sum_{x \in \Omega_A} A_x \circ B_y = \sum_{x \in \Omega_A} \frac{1}{n} A_x = \frac{1}{n} I$$

for all $y \in \Omega_B$ and

$$(A \mid B)_x = \sum_{y \in \Omega_B} B_y \circ A_x = \sum_{y \in \Omega_B} \frac{1}{m} B_y = \frac{1}{m} I$$

for all $x \in \Omega_A$. We conclude that $(B \mid A)$ and $(A \mid B)$ are identity observables. It is also interesting to note that

$$(A_x \circ B_y) \circ A_z = \frac{1}{n} A_x \circ A_z, (A_x \circ B_y) = \frac{1}{n} A_x \circ A_z$$

$$(A_x \circ B_y) \circ B_z = \frac{1}{n} A_x \circ B_z = \frac{1}{nm} A_x$$

$$(A_x \circ B_y) \circ B_z = \frac{1}{n} B_z \circ A_x = \frac{1}{nm} B_z$$

Let $\{\psi_i\}$, $\{\phi_i\}$ be orthonormal bases for $H$ with dim $H = d$, and let $A = \{P_{\psi_i}\}$, $B = \{P_{\phi_j}\}$ be corresponding atomic observables. Since

$$P_{\psi_i} \circ P_{\phi_j} = |\langle \psi_i, \phi_j \rangle|^2 P_{\psi_j}$$

we see that $A$ and $B$ are complementary if and only if $|\langle \phi_i, \psi_j \rangle|^2 = 1/d$ for all $i, j = 1, 2, \ldots, n$. Two orthonormal bases that satisfy this condition are
called mutually unbiased \[3, 8, 13\]. There exist mutually unbiased bases in any finite-dimensional Hilbert space and such bases are important in quantum computation and information studies \[3, 8, 12\].

If \( \rho \in S(H) \) and \( A \in \mathcal{O}(H) \), the distribution of \( A \) in the state \( \rho \) is

\[
\Phi^\rho_A(x) = \text{tr} (\rho A x)
\]

for all \( x \in \Omega_A \). Then \( \Phi^\rho_A \) defines a probability measures on \( \Omega_A \) given by

\[
\Phi^\rho_A(X) = \sum_{x \in X} \Phi^\rho_A(x)
\]

for all \( X \in \Omega_A \). Clearly \( \Phi^\rho_A \) is affine as a function of \( \rho \). Also \( \Phi^\rho_A \) is affine as a function of \( A \) in the following sense. If \( A_i \in \mathcal{O}(H) \) with the same value space \( \Omega \) and \( \lambda_i \geq 0, i = 1, 2, \ldots, n \) with \( \sum \lambda_i = 1 \), then it is easy to verify that \( \sum \lambda_i A_i \in \mathcal{O}(H) \) with value space \( \Omega \) where

\[
\sum \lambda_i A_i = \left\{ \sum \lambda_i A_i x : x \in \Omega \right\}
\]

We then have that

\[
\Phi^\rho_{\sum \lambda_i A_i}(x) = \text{tr} \left( \rho \sum \lambda_i A_i x \right) = \sum \lambda_i \text{tr} (\rho A_i x) = \sum \lambda_i \Phi^\rho_{A_i}(x)
\]

for all \( x \in \Omega \). Hence,

\[
\Phi^\rho_{\sum \lambda_i A_i} = \sum \lambda_i \Phi^\rho_{A_i}
\]

For the sequential product \( A \circ B \) we have that

\[
\Phi^\rho_{A \circ B}(x, y) = \text{tr} (\rho A x \circ B y) = \text{tr} (A^{1/2} \rho A_x^{1/2} B y) = \text{tr} [(\rho | A x) B y]
\]

The left marginal of \( \Phi^\rho_{A \circ B} \) is defined by

\[
L \Phi^\rho_{A \circ B}(x) = \sum_{y \in \Omega_B} \Phi^\rho_{A \circ B}(x, y) = \text{tr}(\rho A x) = \Phi^\rho_A(x)
\]

for all \( x \in \Omega_A \) so that \( L \Phi^\rho_{A \circ B} = \Phi^\rho_A \). More interestingly, the right marginal of \( \Phi^\rho_{A \circ B} \) becomes

\[
R \Phi^\rho_{A \circ B}(y) = \sum_{x \in \Omega_A} \Phi^\rho_{A \circ B}(x, y) = \text{tr} [\rho (B | A) y] = \Phi^\rho_{(B | A)}(y)
\]
for all $y \in \Omega_B$ so that $R\Phi^\rho_{A|B} = \Phi^\rho_{(B|A)}$. For the conditional observable $(B \mid A)$ we have that

$$\Phi^\rho_{(B|A)}(y) = \text{tr} \left( \rho \sum_{x \in \Omega_A} A_x \circ B_y \right) = \text{tr} \left( \sum_{x \in \Omega_A} A_x^{1/2} \rho A_x^{1/2} B_y \right)$$

$$= \text{tr} \left( (\rho \mid A) B_y \right) = \Phi^\rho_B(y)$$

for every $y \in \Omega_B$. Hence, $R\Phi^\rho_{A|B} = \Phi^\rho_{(B|A)} = \Phi^\rho_B$.

We now consider some special cases. If $A$ and $B$ are complementary, we have that

$$\Phi^\rho_{A|B}(x, y) = \frac{1}{n} \text{tr} (\rho A_x) = \frac{1}{n} \Phi^\rho_A(x)$$

for all $(x, y) \in \Omega_A \times \Omega_B$. Moreover,

$$\Phi^\rho_{(B|A)}(y) = \text{tr} \left( \rho \sum_{x \in \Omega_A} \frac{1}{n} A_x \right) = \frac{1}{n}$$

Thus, $\Phi^\rho_{(B|A)}$ is completely random.

As another example, let $A = \{P_{\phi_x} : x \in \Omega_A\}$ be atomic and let $B \in \mathcal{O}(H)$ be arbitrary. Then $\Phi^\rho_A(x) = \langle \phi_x, \rho \phi_x \rangle$ and

$$\Phi^\rho_{A|B}(x, y) = \text{tr} (P_{\phi_x} \rho P_{\phi_x} B_y) = \langle \phi_x, \rho \phi_x \rangle \text{tr} (|\phi_x\rangle\langle \phi_x| B_y)$$

$$= \langle \phi_x, \rho \phi_x \rangle \langle \phi_x, B_y \phi_x \rangle = \Phi^\rho_A(x) \Phi^\rho_B(\phi_x)(y)$$

We also have that

$$\Phi^\rho_{(B|A)}(y) = \text{tr} \left( \sum_{x \in \Omega_A} P_{\phi_x} \rho P_{\phi_x} B_y \right) = \sum_{x \in \Omega_A} \Phi^\rho_A(x) \Phi^\rho_B(\phi_x)(y)$$

5 Defining Joint Probabilities

“A good definition is worth a hundred theorems.” –Unknown

In classical probability theory, events are represented by sets and if $A$ and $B$ are events and $\mu$ is a probability measure, then $P_{\mu}(A \text{ and } B) = \mu(A \cap B)$ is their joint probability. This definition is not adequate for quantum mechanics. One reason for this is that it does not take account of which event
is observed first. If such a temporal order is considered, then the first measurement may interfere with the second, resulting in quantum interference. Another problem is caused by the joint additivity of $P_{\mu}$ which we shall discuss later.

If $A, B \in \mathcal{O}(H)$, $\rho \in \mathcal{S}(H)$, then the standard definition of the joint probability is

$$P_\rho(A_X \text{ then } B_Y) = \text{tr} \left[ \mathcal{I}_X^A(\rho)B_Y \right] \tag{5.1}$$

where $\mathcal{I}^A$ is an $A$-compatible instrument \[1\] \[2\] \[8\]. We interpret this as meaning that if the system is initially in the state $\rho$ and $A$ is measured first giving a value in $X$ and next $B$ is measured giving a value in $Y$, then their joint probability is the right side of (5.1). We believe that (5.1) is not a satisfactory definition because this joint probability should depend on $A_X$ and not on an instrument measuring $A$. In particular, if $\mathcal{I}^A$ is a trivial $A$-compatible instrument $\mathcal{I}_X^A(\rho) = \text{tr} (\rho A_X) \eta$, then

$$\text{tr} \left[ \mathcal{I}_X^A(\rho)B_Y \right] = \text{tr} \left[ \text{tr} (\rho A_X) \eta B_Y \right] = \text{tr} (\rho A_X) \text{tr} (\eta B_Y) = P_\rho(A_X)P_{\eta}(B_Y)$$

Hence, $P_\rho(A_X \text{ then } B_Y)$ can be any number less than or equal to $\text{tr} (\rho A_X)$ depending on the choice of $\eta$. Also, the conditional output state \[8\] \[11\] becomes

$$\tilde{\rho}_X = \frac{1}{\text{tr} \left[ \mathcal{I}_X^A(\rho) \right]} \mathcal{I}_X^A(\rho) = \eta$$

and this has nothing to do with $A$ which is again unsatisfactory. Moreover, if we include an instrument measuring $A$, why not also include an instrument measuring $B$? We would then define

$$P_\rho(A_X \text{ then } B_Y) = \text{tr} \left[ \mathcal{J}_Y^A \left( \mathcal{I}_X^A(\rho) \right) \right]$$

which gives different results than (5.1).

Instead of an arbitrary $A$-compatible instrument, we suggest employing the unique $A$-compatible Läders instrument $\mathcal{L}^A$. This overcomes the previously discussed problems. Moreover, $\mathcal{L}^A$ is the canonical $A$-compatible instrument because any $A$-compatible instrument has the form $\mathcal{I}_X = \mathcal{E}_x \circ \mathcal{L}_x^A$ where $\{\mathcal{E}_x : x \in \Omega_A\}$ is a set of channels \[8\] \[10\]. With this assumption (5.1) becomes

$$P_\rho(A_X \text{ then } B_Y) = \text{tr} \left[ \mathcal{L}_X^A(\rho)B_Y \right] = \text{tr} \left( \sum_{x \in X} A_x^{1/2} \rho A_x^{1/2} B_Y \right)$$

14
\[
\begin{align*}
= \sum_{x \in X} \text{tr} (\rho A^{1/2} B Y A^{1/2}) = \sum_{x \in X} \text{tr} (\rho A_x \circ B Y) \quad (5.2)
= \sum_{x \in X} \mathcal{P}_\rho(A_x \circ B Y)
\end{align*}
\]

Moreover, the conditional output state becomes
\[
\tilde{\rho}_X = \frac{1}{\text{tr} [\mathcal{L}_X^A(\rho)]} \mathcal{L}_X^A(\rho) = \frac{1}{\text{tr} (\rho A_X)} \sum_{x \in X} (A_x \circ \rho)
\]

Even this last definition of \(\mathcal{P}_\rho(A_X \text{ then } B_Y)\) is not satisfactory. This is because (5.1) and (5.2) are additive in the first variable. That is
\[
\mathcal{P}_\rho(A \cup X, \text{ then } B_Y) = \sum_i \mathcal{P}_\rho(A_{X_i}, \text{ then } B_Y) \quad (5.3)
\]
Whenever \(X_i \cap X_j = \emptyset\) for \(i \neq j\). Now a measurement of \(A\) can interfere with a later measurement of \(B\) so one should not expect (5.3) to hold. In fact, (5.3) is the defining property of classical probability theory in which events do not interfere. Of course, (5.1) and (5.2) are also additive in the second variable, but this is not a problem because the measurements of \(B\) is after the measurement of \(A\) so there is no interference.

We believe that the natural and correct definition of the joint probability is:
\[
\mathcal{P}_\rho(A_X \text{ then } B_Y) = \mathcal{P}_\rho(B_Y \mid A_X) = \text{tr} (\rho A_X \circ B_Y) \quad (5.4)
\]
This is just the definition that we have used in the previous sections of this article. Notice that the difference between (5.1) and the last expression in (5.2) is the lack of additivity in (5.4).

In order to investigate additivity more closely, we make the following definition. For \(a, b, c \in \mathcal{E}(H)\) with \(a \perp b\), we say that \(a\) and \(b\) are additive relative to \(c\) if
\[
(a + b) \circ c = a \circ c + b \circ c \quad (5.5)
\]
and when (5.5) holds we write \((a, b; c)\). We can rewrite (5.5) as
\[
(a + b)^{1/2} c (a + b)^{1/2} = a^{1/2} c a^{1/2} + b^{1/2} c b^{1/2} \quad (5.6)
\]
We show in the next lemma that \((a, b; c)\) is a weakening of the compatibility of \(a, c\) and \(b, c\). This makes sense because compatibility corresponds physically to noninterference which, as mentioned previously, is related to additivity.
Lemma 5.1. If \( a, c \) an \( b, c \) are compatible, the \((a, b; c)\).

Proof. If \( ac = ca \) and \( bc = cb \), then \( a, c \) and \( b, c \) can be simultaneously diagonalized. Hence \( a^{1/2}c = ca^{1/2} \) and \( b^{1/2}c = cb^{1/2} \). Also, \( c \) and \( a + b \) can be simultaneously diagonalized so \( (a + b)^{1/2}c = c(a + b)^{1/2} \). Therefore,

\[
(a + b)^{1/2}c(a + b)^{1/2} = c(a + b) = ca + cb = a^{1/2}ca^{1/2} + b^{1/2}cb^{1/2}
\]
so \((a, b; c)\). \(\square\)

The next example shows that the converse of Lemma 5.1 does not hold. Thus, there are noncompatible pairs that are still additive.

Example. Let \( a, b, c \in \mathcal{E} (\mathbb{C}^3) \) be the following effects

\[
a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad c = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Then \( cb = 0 \) and \( ac \neq ca \) so \( a \) and \( c \) are not compatible. However,

\[
(a + b) \circ c = a \circ c + b \circ c = \frac{1}{2} a \quad \square
\]

The next lemma characterizes additivity for sharp \( a \) and \( b \).

Lemma 5.2. If \( a, b \in \mathcal{E}(H) \) are sharp with \( a \perp b \), then \((a, b; c)\) if and only if \( acb = 0 \).

Proof. Since \( a \) and \( b \) are sharp and \( a \perp b \), we have that \( ab = 0 \) \(\mathbb{S}\) so \( a + b \) is sharp. Hence,

\[
(a + b) \circ c = (a + b)c(a + b) = ac + bc + acb + bca
\]

\[
= a \circ c + b \circ c + acb + bca
\]

Therefore, \((a, b; c)\) if and only if \( acb + bca = 0 \) which is equivalent to \( acb = 0 \). \(\square\)

We do not know a generalization of this lemma for unsharp \( a, b \in \mathcal{E}(H) \).
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