Results of the $O(\alpha_s)$ two-loop virtual corrections to $B \to X_s \ell^+ \ell^-$ in the standard model

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Abstract: We present the results of the $O(\alpha_s)$ two-loop virtual corrections to the differential decay width $d\Gamma(B \to X_s \ell^+ \ell^-)/d\hat{s}$, where $\hat{s}$ is the invariant mass squared of the lepton pair, normalized to $m_b^2$. Those contributions from gluon bremsstrahlung which are needed to cancel infrared and collinear singularities are also included. Our calculation is restricted to the range $0.05 \leq \hat{s} \leq 0.25$ where the effects from resonances are small. The new contributions drastically reduce the renormalization scale dependence of existing results for $d\Gamma(B \to X_s \ell^+ \ell^-)/d\hat{s}$. The renormalization scale uncertainty of the corresponding branching ratio (restricted to $0.05 \leq \hat{s} \leq 0.25$) gets reduced from $\sim \pm 13\%$ to $\sim \pm 6.5\%$.

1. Introduction

After the observation of the penguin-induced decay $B \to X_s \gamma$ [1] and corresponding exclusive channels such as $B \to K^* \gamma$ [2], rare $B$-decays have begun to play an important role in the phenomenology of particle physics. They put strong constraints on various extensions of the standard model. The inclusive decay $B \to X_s \ell^+ \ell^-$ has not been observed so far, but is expected to be detected at the currently running $B$-factories.

The next-to-leading logarithmic (NLL) result for $B \to X_s \ell^+ \ell^-$ suffers from a relatively large ($\pm 16\%$) dependence on the matching scale $\mu_W$ [3, 4]. The NNLL corrections to the Wilson coefficients remove the matching scale dependence to a large extent [5], but leave a $\pm 13\%$-dependence on the renormalization scale $\mu_b$, which is of $O(m_b)$. In order to further improve the result, we have recently calculated the $O(\alpha_s)$ two-loop corrections to the matrix elements of the operators $O_1$ and $O_2$ as well as the $O(\alpha_s)$ one-loop corrections to $O_7, ..., O_{10}$ [6]. Because of large resonant contributions from $\bar{c}c$ intermediate states, we restrict the invariant lepton mass squared $s$ to the region $0.05 \leq \hat{s} \leq 0.25$, where $\hat{s} = s/m_b^2$. In the following we present a summary of the results of these calculations.

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2. Theoretical Framework

The appropriate tool for studies on weak B-mesons decays is the effective Hamiltonian technique. The effective Hamiltonian is derived from the standard model by integrating out the t-quark, the Z0—and the W-boson. For the decay channels $b \rightarrow s \ell^{+}\ell^{-} \ (\ell = \mu, e)$ it reads

$$\mathcal{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{10} C_i O_i,$$

where $O_i$ are dimension six operators and $C_i$ denote the corresponding Wilson coefficients. The operators can be chosen as [3]

$$O_1 = (\bar{s}L\gamma_\mu T^a c_L)(\bar{\ell}L\gamma_\mu T^a b_L) \quad O_2 = (\bar{s}L\gamma_\mu c_L)(\bar{\ell}L\gamma_\mu b_L)$$

$$O_3 = (\bar{s}L\gamma_\mu b_L) \sum_q (\bar{q}q)_\mu q \quad O_4 = (\bar{s}L\gamma_\mu T^a b_L) \sum_q (\bar{q}q)_\mu T^aq$$

$$O_5 = (\bar{s}L\gamma_\mu\gamma_\nu\gamma_\sigma b_L) \sum_q (\bar{q}q)_\mu\gamma_\nu\gamma_\sigma q \quad O_6 = (\bar{s}L\gamma_\mu\gamma_\nu\gamma_\sigma T^a b_L) \sum_q (\bar{q}q)_\mu\gamma_\nu\gamma_\sigma T^aq$$

$$O_7 = \frac{g_s}{g^2} m_b (\bar{s}L\gamma_\mu \gamma_\nu T^a T^b R) F_{\mu\nu} \quad O_8 = \frac{1}{g_s} m_b (\bar{s}L\gamma_\mu T^a b_R) G^{a}_{\mu\nu}$$

$$O_9 = \frac{g_s}{g^2} (\bar{s}L\gamma_\mu b_L) \sum_\ell (\bar{\ell}\gamma_\mu \ell) \quad O_{10} = \frac{g_s}{g^2} (\bar{s}L\gamma_\mu b_L) \sum_\ell (\bar{\ell}\gamma_\mu \gamma_5 \ell).$$

The subscripts $L$ and $R$ refer to left- and right- handed fermion fields. We work in the approximation where the combination $(V_{us}^* V_{ub})$ of Cabibbo-Kobayashi-Maskawa (CKM) matrix elements is neglected. The CKM structure factorizes therefore.

3. Virtual Corrections to the Operators $O_1$, $O_2$, $O_7$, $O_8$, $O_9$ and $O_{10}$

Using the naive dimensional regularization scheme in $d = 4 - 2 \epsilon$ dimensions, ultraviolet and infrared singularities both show up as $1/\epsilon^n$-poles ($n = 1, 2$). The ultraviolet singularities cancel after including the counterterms. Collinear singularities are regularized by retaining a finite strange quark mass $m_s$. They are cancelled together with the infrared singularities at the level of the decay width, when taking the bremsstrahlung process $b \rightarrow s \ell^{+}\ell^{-}g$ into account. Gauge invariance implies that the QCD-corrected matrix elements of the operators $O_i$ can be written as

$$\langle s\ell^{+}\ell^{-}|O_i|b\rangle = \hat{F}_i^{(9)} \langle O_9\rangle_{\text{tree}} + \hat{F}_i^{(7)} \langle O_7\rangle_{\text{tree}},$$

where $\langle O_9\rangle_{\text{tree}}$ and $\langle O_7\rangle_{\text{tree}}$ are the tree-level matrix elements of $O_9$ and $O_7$, respectively.

3.1 Virtual corrections to $O_1$ and $O_2$

For the calculation of the two-loop diagrams associated with $O_1$ and $O_2$ we mainly used a combination of Mellin-Barnes technique [4, 5] and of Taylor series expansion in $s$. For $s < m_b^2$ and $s < 4m_c^2$, most diagrams allow the latter. The unrenormalized form factors $\hat{F}_i^{(7,9)}$ of $O_1$ and $O_2$ are then obtained in the form

$$\hat{F}_i^{(7,9)} = \sum_{i,j,l,m} c_{i,j,l,m}^{(7,9)} s^i \ln^j(s) \left(\tilde{m}_c^2\right)^l \ln^m(\tilde{m}_c),$$
where $\hat{m}_c = \frac{m_c}{m_0}$. The indices $i, j, m$ are non-negative integers and $l = -i, -i+\frac{1}{2}, -i+1, \ldots$.

Besides the counterterms from quark field, quark mass and coupling constant ($g_s$) renormalization, there are counterterms induced by operator mixing. They are of the form

$$C_i \sum_j \delta Z_{ij} \langle O_j \rangle \quad \text{with} \quad \delta Z_{ij} = \frac{\alpha_s}{4\pi} \left[ a_{ij}^0 + \frac{a_{ij}^1}{\epsilon} \right] + \frac{\alpha_s^2}{(4\pi)^2} \left[ a_{ij}^{02} + \frac{a_{ij}^{12}}{\epsilon} + \frac{a_{ij}^{22}}{\epsilon^2} \right] + O(\alpha_s^3) .$$

A complete list of the coefficients $a_{ij}^{lm}$ used for our calculation can be found in [6]. The operator mixing involves also the evanescent operators

$$O_{11} = (\tilde{s}_L \gamma_\mu \gamma_\nu \gamma_\sigma T^a c_L) (\overline{\tilde{c}}_L \gamma_\mu \gamma_\nu \gamma_\sigma T^a b_L) - 16 O_1 \quad \text{and} \quad O_{12} = (\tilde{s}_L \gamma_\mu \gamma_\nu \gamma_\sigma c_L) (\overline{\tilde{c}}_L \gamma_\mu \gamma_\nu \gamma_\sigma b_L) - 16 O_2 .$$

3.2 Virtual corrections to $O_7, O_8, O_9$ and $O_{10}$

The renormalized contributions from the operators $O_7, O_8$ and $O_9$ can all be written in the form

$$\langle s\ell^+\ell^- | C_i O_i | b \rangle = \tilde{C}_i^{(0)} \left[ -\frac{\alpha_s}{4\pi} \right] \left[ F_i^{(0)} (\tilde{O}_9)_{\text{tree}} + F_i^{(7)} (\tilde{O}_7)_{\text{tree}} \right] ,$$

with $\tilde{O}_i = \frac{\alpha_s}{4\pi} O_i$, $\tilde{C}^{(0)}_{7,8} = C^{(1)}_{7,8}$ and $\tilde{C}^{(0)}_9 = \frac{4\pi}{\alpha_s} \left( C^{(0)}_9 + \frac{\alpha_s}{4\pi} C^{(1)}_9 \right)$.

The formally leading term $\sim g_s^{-2} C^{(0)}_9 (\mu_b)$ to the amplitude for $b \to s\ell^+\ell^-$ is smaller than the NLL term $\sim g_s^{-2} [g_s^2/(16\pi^2)] C^{(1)}_9 (\mu_b)$ [8]. We therefore adapt our systematics to the numerical situation and treat the sum of these two terms as a NLL contribution, as indicated by the expression for $\tilde{C}^{(0)}_9$. The decay amplitude then starts out with a NLL term.

The contribution from $O_8$ is finite, whereas those from $O_7$ and $O_9$ are not, i.e $F_i^{(7)}$ and $F_i^{(9)}$ suffer from the same infrared divergent part $f_{\text{inf}}$.

As the hadronic parts of the operators $O_9$ and $O_{10}$ are identical, the QCD corrected matrix element of $O_{10}$ can easily be obtained from that of $O_9$.

4. Bremsstrahlung Corrections

It is known [2, 3] that the contribution to the inclusive decay width from the interference between the tree-level and the one-loop matrix elements of $O_9$ and from the corresponding bremsstrahlung corrections can be written as

$$\frac{d\Gamma_{99}}{ds} = \left( \frac{\alpha_{em}}{4\pi} \right)^2 \frac{G_F^2 m_b^5}{48 \pi^3} \frac{|V_{ts}^* V_{tb}|^2}{(1-\hat{s})^2 (1+2/\hat{s})} \left[ 2 |\tilde{C}^{(0)}_9| \right]^2 \frac{\alpha_s}{\pi} \omega_{9}(\hat{s}) .$$

Analogous formulas hold true for the contributions from $O_7$ and the interference terms between the matrix elements of $O_7$ and $O_9$:

$$\frac{d\Gamma_{77}}{ds} = \left( \frac{\alpha_{em}}{4\pi} \right)^2 \frac{G_F^2 m_b^5}{48 \pi^3} \frac{|V_{ts}^* V_{tb}|^2}{(1-\hat{s})^2 (1+2/\hat{s})} \left[ 2 |\tilde{C}^{(0)}_7| \right]^2 \frac{\alpha_s}{\pi} \omega_{7}(\hat{s}) ,$$

$$\frac{d\Gamma_{79}}{ds} = \left( \frac{\alpha_{em}}{4\pi} \right)^2 \frac{G_F^2 m_b^5}{48 \pi^3} \frac{|V_{ts}^* V_{tb}|^2}{(1-\hat{s})^2 (1+2/\hat{s})} \left[ 2 |\tilde{C}^{(0)}_9| \right]^2 \frac{\alpha_s}{\pi} \omega_{79}(\hat{s}) \Re \left[ \tilde{C}^{(0)}_7 \tilde{C}^{(0)}_9 \right] .$$
The function $\omega_9(\hat{s}) \equiv \omega(\hat{s})$ can be found eg in [3, 4]. For $\omega_7(\hat{s})$ and $\omega_{79}(\hat{s})$ see [5]. All other bremsstrahlung corrections are finite and will be given in [6].

5. Corrections to the Decay Width for $B \to X_s \ell^+ \ell^-$

Combining the virtual corrections discussed in section 3 with the bremsstrahlung contributions considered in section 4, we find for the decay width

$$\frac{d\Gamma(b \to X_s \ell^+ \ell^-)}{d \hat{s}} = \left( \frac{\alpha_{em}}{4\pi} \right)^2 \frac{G_F^2 m_{b,\text{pole}}^5}{48 \pi^3} |V_{tb}^* V_{tb}|^2 (1 - \hat{s})^2 \times$$

$$\left(1 + 2 \hat{s} \left| \tilde{C}_9^{\text{eff}} \right|^2 + \left| \tilde{C}_{10}^{\text{eff}} \right|^2 \right) + 4 (1 + 2/\hat{s}) \left| \tilde{C}_7^{\text{eff}} \right|^2 + 12 \text{Re} \left[ \tilde{C}_7^{\text{eff}} \tilde{C}_{9,10}^{\text{eff}} \right] , \quad (5.1)$$

where the effective Wilson coefficients $\tilde{C}_7^{\text{eff}}, \tilde{C}_9^{\text{eff}}$ and $\tilde{C}_{10}^{\text{eff}}$ can be written as

$$\tilde{C}_9^{\text{eff}} = \left[ 1 + \frac{\alpha_s(\mu)}{\pi} \omega_9(\hat{s}) \right] \left( A_9 + T_9 h(m_c^2, \hat{s}) + U_9 h(1, \hat{s}) + W_9 h(0, \hat{s}) \right)$$

$$- \frac{\alpha_s(\mu)}{4\pi} \left( C_1^{(0)} F_1^{(9)} + C_2^{(0)} F_2^{(9)} + A_8^{(0)} F_8^{(9)} \right) ,$$

$$\tilde{C}_7^{\text{eff}} = \left[ 1 + \frac{\alpha_s(\mu)}{\pi} \omega_7(\hat{s}) \right] A_7 - \frac{\alpha_s(\mu)}{4\pi} \left( C_1^{(0)} F_1^{(7)} + C_2^{(0)} F_2^{(7)} + A_8^{(0)} F_8^{(7)} \right) ,$$

$$\tilde{C}_{10}^{\text{eff}} = \left[ 1 + \frac{\alpha_s(\mu)}{\pi} \omega_9(\hat{s}) \right] A_{10} .$$

The function $h(m_c^2, \hat{s})$ is defined in [6], where also the values of $A_7, A_9, A_{10}, T_9, U_9$ and $W_9$ can be found. $C_1^{(0)}, C_2^{(0)}$ and $A_8^{(0)} = \tilde{C}_8^{(0,\text{eff})}$ are taken from [7].

6. Numerical Results

The decay width in eq (5.1) has a large uncertainty due to the factor $m_{b,\text{pole}}^5$. Following common practice, we consider the ratio

$$R_{\text{quark}}(\hat{s}) = \frac{1}{\Gamma(b \to X_c e \bar{\nu}_e)} \frac{d\Gamma(b \to X_s \ell^+ \ell^-)}{d \hat{s}} ,$$

in which the factor $m_{b,\text{pole}}^5$ drops out. $\Gamma(b \to X_c e \bar{\nu}_e)$ can be found eg in [3].

In Fig. 1 we investigate the dependence of $R_{\text{quark}}(\hat{s})$ on the renormalization scale $\mu_b$ for $0.05 \leq \hat{s} \leq 0.25$. The solid lines take the new NNLL contributions into account, whereas the dashed lines include the NLL results combined with the NNLL corrections to the matching conditions [6], only. The lower, middle and upper line each correspond to $\mu_b = 2.5, 5$ and $10$ GeV, respectively, and $\hat{m}_c = 0.29$. From this figure we conclude that the renormalization scale dependence gets reduced by more than a factor of 2. For the integrated quantity we get

$$R_{\text{quark}} = \int_{0.05}^{0.25} d\hat{s} R_{\text{quark}}(\hat{s}) = (1.25 \pm 0.08) \times 10^{-5} ,$$

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where the error is obtained by varying $\mu_b$ between 2.5 GeV and 10 GeV. Not including our corrections, one finds $R_{\text{quark}} = (1.36 \pm 0.18) \times 10^{-5}$. In other words, the renormalization scale dependence got reduced from $\sim \pm 13\%$ to $\sim \pm 6.5\%$. The largest uncertainty due to the input parameters is induced by $\hat{m}_c$. Fig. 2 illustrates the dependence of $R_{\text{quark}}(\hat{s})$ on $\hat{m}_c$. The dashed, solid and dash-dotted lines correspond to $\hat{m}_c = 0.27$, $\hat{m}_c = 0.29$ and $\hat{m}_c = 0.31$, respectively, and $\mu_b = 5$ GeV. We find an uncertainty of $\pm 7.6\%$ due to $\hat{m}_c$.

We conclude with the remark that the results presented in this exposition have recently been included in a systematic description of the corresponding exclusive decay mode $B \to K^* \ell^+ \ell^-$ [10, 11].

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