Rigidity of action of compact quantum groups
III: the general case
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Abstract
If a compact quantum group acts faithfully and smoothly (in the sense of \([2]\)) on a smooth, compact, oriented, connected Riemannian manifold such that the action induces a natural bimodule morphism on the module of sections of the co-tangent bundle, then it is proved that the quantum group is necessarily commutative as a \(C^\ast\) algebra i.e. isomorphic with \(C(G)\) for some compact group \(G\). From this, we deduce that the quantum isometry group of such a manifold \(M\) coincides with \(C(ISO(M))\) where \(ISO(M)\) is the group of (classical) isometries, i.e. there is no genuine quantum isometry of such a manifold.

1 Set-up and statement of the main result
In this note, we complete the programme taken up in \([1]\) to extend the main result of \([1]\) to an arbitrary compact connected oriented Riemannian manifold and thereby proving Conjecture II of that paper which states that there is no genuine quantum isometry of such a manifold. We give only sketches of arguments here, and plan to give full details in an article (or a couple of them) combining \([1]\) with the present one. We will freely use all the notations and terminologies of \([1]\). In particular, we will use the Frechet algebras, (Hilbert) modules over them and topological tensor product (denoted by \(\hat{\otimes}\)) between such algebras or modules, discussed in details in \([1]\) and references therein.

Let \(M\) be a smooth, compact, connected, oriented manifold of dimension \(n\) and assume that a compact quantum group (CQG) \(Q\) has a faithful, smooth (in the sense of \([1]\)) action \(\alpha : C(M) \rightarrow C(M) \hat{\otimes} Q\) which satisfies condition (1) of Theorem 3.4 of \([1]\). Our goal is to prove the following:

**Theorem 1.1** \(Q\) is commutative as \(C^\ast\) algebra, i.e. of the form \(C(G)\) for some compact group \(G\).

From this, we get the following

**Corollary 1.2** The quantum isometry group of any compact connected oriented Riemannian manifold \(M\) coincides with \(C(ISO(M))\) where \(ISO(M)\) denotes the group of Riemannian isometries. In other words, there is no genuine quantum isometry of such manifolds.

Let us explain our strategy. Using the techniques of \([1]\), we can equip \(M\) with a Riemannian structure \((\cdot, \cdot)\) for which \(\alpha\) is isometric. Then, we emulate the

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classical technique of lifting an isometric group action to a smooth action on the total space of the orthonormal frame bundle of $M$, which is always a parallelizable manifold (see, e.g. [3]). We verify that the lifted action satisfies the hypothesis of Theorem 6.6 of [1], hence the conclusion of Theorem 1.1 follows.

2 Sketch of proof

We recall from [1] the equivariant unitary representation $d\alpha$ on the Hilbert module of smooth forms. For $x \in M$ let us denote by $Q_x$ the unital $C^*$-subalgebra of $Q$ generated by elements of the forms $\alpha(f)(x), (\phi \otimes \text{id})(\alpha(g))(x)$, where $f, g \in C^\infty(M)$ and $\phi$ is any smooth vector field. By assumption, $\alpha(f)(x)$ commute among themselves for different $f$’s and also commute with $(\phi \otimes \text{id})(\alpha(g))(x)$’s. We now claim that actually $(\phi \otimes \text{id})(\alpha(g))(x)$’s commute among themselves too, for different choices of $\phi$ and $g$. In other words:

**Lemma 2.1** $Q_x$ is commutative.

**Proof:**
The statement of the lemma is clearly equivalent to proving $\tilde{d}\alpha(f)$ and $\tilde{d}\alpha(g)$ commute for $f, g \in C^\infty(M)$, where $\tilde{d} = d \otimes \text{id}$ as in [1]. For $x \in M$, choose smooth one-forms $\{\omega_1, \ldots, \omega_n\}$ such that they form a basis of $T^*M$ at $x$. Let $F_i(x), G_i(x), i = 1, \ldots, n$ be elements of $Q$ (actually in $Q_x$) such that $\tilde{d}\alpha(f)(x) = \sum_i \omega_i F_i(x), \tilde{d}\alpha(g)(x) = \sum_i \omega_i G_i(x)$. Now, in the notation of Lemma 4.5 of [1], $d\alpha(2)$ leaves invariant the submodules of symmetric and antisymmetric tensor product of $\Lambda^1(M)$, thus in particular, $C_{ij} = C_{ji}^s, C_{ij} = -C_{ji}^a$ for all $i, j$, where $C_{ij}^s$ and $C_{ij}^a$ denote the $Q$-valued coefficient of $w_i \otimes w_j$ in the expression of $d\alpha(2)$ leaves invariant the submodules of symmetric and antisymmetric tensor product of $\Lambda^1(M)$, thus in particular, $C_{ij}^s = C_{ji}^s, C_{ij}^a = -C_{ji}^a$ for all $i, j$, where $C_{ij}^s$ and $C_{ij}^a$ denote the $Q$-valued coefficient of $w_i \otimes w_j$ in the expression of $df \otimes dg + dg \otimes df$ and $d\alpha(2)(df \otimes dg - dg \otimes df)$ respectively. By a simple calculation using these relations, we get the commutativity of $F_i(x), G_j(x)$ for all $i, j$. $\square$

Let $O_M$ be the bundle of orthonormal frames of $M$. It is convenient to identify it as a subbundle of $E = \text{Hom}(M \times \mathbb{R}^n, \Lambda^1(M))$ (where $\Lambda^1(M)$ denotes the real vector bundle of smooth one-forms on $M$) with fibres at $x$ isomorphic with linear maps from $\mathbb{R}^n$ to $T^*_x M$. Then, $O_M$ can be viewed as the subset of $E$ consisting of those linear maps $e$ which are orthonormal w.r.t. the canonical Euclidean inner product on $\mathbb{R}^n$ and the Riemannian inner product on $T^*_x M$ where $x = \pi(e), \pi : O_M \to M$ is the projection map from the total space onto the base space $M$. Let $\{b_1, \ldots, b_n\}$ be the canonical basis of $\mathbb{R}^n$. We will also use them as the canonical basis of $\mathbb{C}^n$. Then any element $e \in O_M$ is determined by the orthonormal basis $e(b_1), \ldots, e(b_n)$ of $T^*_\pi(e)M$. Moreover, we can define local coordinates of $O_M$ as follows. Consider any open subset $U$ of $M$ such that $\Lambda^1(U)$ is trivial, diffeomorphic to $U \times \mathbb{R}^n$. Choose $\omega_1, \ldots, \omega_n$ in $\Lambda^1(M)$ such that for each $x \in U$, $\{\omega_1(x), \ldots, \omega_n(x)\}$ is an orthonormal basis w.r.t. the Riemannian inner product $\langle \cdot, \cdot \rangle_x$. Define $t_{ij}(e) := \omega_j(x), e(b_i) > x, i, j = 1, \ldots, n, e \in \pi^{-1}(U) \cong U \times O(n)$. It is clear that $\{(t_{ij}(e))\}$ is in $O(n)$ for all $e \in \pi^{-1}(U)$. Thus, choosing any $n(2n - 1)$'s of $t_{ij}$’s, along with a choice of $n$
local coordinates for \( U \subseteq M \), we get a choice of local coordinates for \( O_M \). Let us call \( t_{ij}(e) \) the ‘matrix coefficients’ of \( e \) corresponding to the basis \( \{\omega_1, \ldots, \omega_n\} \).

Let \( \mathcal{B} = C(O_M), \mathcal{B}^\infty = C^\infty(O_M), \mathcal{B}_U := C_c(\pi^{-1}(U)) \subset C(O_M), \mathcal{C}_U^\infty = C_c^\infty(\pi^{-1}(U)) \), where \( C_c(V) \) (respectively \( C_c^\infty(V) \)) denotes continuous (respectively smooth) compactly supported functions on a manifold \( V \). Consider the \( * \)-algebra \( \mathcal{P}_U \) consisting of functions of the form below:

\[
\sum_{i=1}^N \hat{f}_i P_i(t_{ij}; i, j = 1, \ldots, n),
\]

where \( N \geq 1, \hat{f}_i \equiv f_i \circ \pi \in \mathcal{B}_U^\infty \) for some \( f_i \in C_c^\infty(U) \), and \( P_i \) denotes a polynomial with complex coefficients of \( n^2 \) variables. It is easy to see that \( \mathcal{P}_U \) is dense in \( \mathcal{B}_U \) and \( \mathcal{B}_U^\infty \) in the norm and Frechet topologies respectively. We shall also use the notation \( \hat{F} \) for \( F \in C(M) \otimes \mathbb{Q} \) to denote the element \( F \circ \pi \) of \( C(O_M) \otimes \mathbb{Q} \).

We want to construct a smooth action \( \eta \) of \( \mathbb{Q} \) on \( C^\infty(O_M) \). For this, we need the concept of symmetric algebra \( \mathcal{S}(\mathcal{E}) \) of a module \( \mathcal{E} \) over a unital commutative algebra say \( \mathcal{M} \). This is by the definition the algebraic direct sum of symmetric tensor products (over \( \mathcal{M} \)) of copies of \( \mathcal{E} \), i.e., \( \mathcal{S}(\mathcal{E}) := \bigoplus_{n \geq 0} \mathcal{E}_{\text{sym}}^{(n)} \), where \( \mathcal{E}_{\text{sym}}^{(0)} := \mathcal{M} \) and for \( n \geq 1 \), \( \mathcal{E}_{\text{sym}}^{(n)} \) is the symmetric tensor product of \( n \) copies (which is possible to define as \( \mathcal{M} \) is commutative). This has a natural algebra structure coming from tensor multiplication. Moreover, it has the following well-known universal property:

**Proposition 2.2** Given any unital commutative algebra \( \mathcal{F} \) with an algebra inclusion \( i \) (unital), which naturally gives \( \mathcal{F} \) an \( \mathcal{M} \equiv i(\mathcal{M}) \)-module structure, and an \( \mathcal{M} \)-module map \( \theta : \mathcal{E} \rightarrow \mathcal{F} \), there is a unique lift \( \hat{\theta} : \mathcal{S}(\mathcal{E}) \rightarrow \mathcal{F} \) which is an algebra homomorphism.

Now, recall the vector bundle \( E = \text{Hom}(M \times \mathbb{R}^n, \Lambda^1_{\mathbb{R}}(M)) \) and consider its complexification \( E^\mathbb{C} = \text{Hom}(M \times \mathbb{C}^n, \Lambda^1(M)) \), where \( \Lambda^1(M) \) stands for the complexification of \( \Lambda^1_{\mathbb{R}}(M) \). Denote by \( \mathcal{E} \) the \( C^\infty(M) \)-module of smooth sections of \( E_C \), viewed as a Hilbert Frechet module as in [11]. There is a natural \( C^\infty(M) \)-valued inner product \( \langle \langle \cdot, \cdot \rangle \rangle^E \) on \( \mathcal{E} \) using matrix traces in fibres: \( \langle \langle \xi, \eta \rangle \rangle^E(x) := \text{Tr}(\xi^*(x)\eta(x)) \). Define \( \theta : \mathcal{E} \rightarrow C^\infty(O_M) \) by

\[
\theta(\xi)(e) := \text{Tr}(e^* \xi(\pi(e))) = \langle \langle \zeta, \xi \rangle \rangle^E(\pi(e)),
\]

for any section \( \zeta \in \mathcal{E} \) such that \( \zeta(\pi(e)) = e \). It is easy to check that it is a module map where we view \( C^\infty(M) \) as a subalgebra of \( C^\infty(O_M) \) in the obvious way, by sending \( f \) to \( \hat{f} \). So, we get algebra homomorphism \( \theta \) from \( \mathcal{S}(\mathcal{E}) \) to \( C^\infty(O_M) \), and it is easy to verify that it is in fact a \( * \)-homomorphism and also its range is dense in \( C^\infty(O_M) \) in the Frechet topology.

On the other hand, identifying the Frechet module \( \mathcal{E} \otimes \mathbb{Q} \) with \( \text{Hom}_{C^\infty(M)} \left( C^\infty(M) \otimes \mathbb{C}^n, \Lambda^1(M) \otimes \mathbb{Q} \right) \),
we can define an equivariant representation \( \beta : E \to E \otimes Q \) by
\[
\beta(\xi)(v) := \alpha(\xi(v)),
\]
for \( \xi \in E, \ v \in C^\infty(M) \otimes \mathbb{C}^n \), say \( v = (v_1, \ldots, v_n) \), \( v_i \in C^\infty(M) \). Now, define
\[
\Gamma(\xi)(e) := (\langle \xi \otimes 1_Q, \beta(\xi) \rangle\rangle_{E \otimes Q}(\pi(e)),
\]
where \( e \in O_M, \xi \in E, \zeta \in E \) such that \( \langle \zeta(\pi(e)) = e \) and \( \langle \cdot, \cdot \rangle_{E \otimes Q} \) denotes the natural \( C^\infty(M) \otimes Q \) valued inner product. In fact, one can easily verify that \( \Gamma(\xi) \in A \subset C^\infty(O_M) \otimes Q \), and consider the inclusion \( i : C^\infty(M) \to A \) given by \( i(f) = \alpha(f) \). We verify that \( \Gamma \) is a module map, hence lifts to a homomorphism \( \hat{\Gamma} : S(E) \to A \).

As in [1], we get a Frechet-dense submodule \( E_0 \) on which the representation \( \beta \) is algebraic, so that in particular \( \beta(E_0) \subseteq E_0 \otimes Q_0 \), where \( Q_0 \) is the canonical Hopf * algebra of \( Q \) and \( \otimes \) (as in [1]) denotes the algebraic tensor product. It is easily seen that \( \hat{\theta}(E_0) \) is dense in \( C^\infty(O_M) \). From the discussion of subsection 2.4 on Hilbert modules over topological algebras, as well as using the commutativity of the algebra \( Q_x \) for all \( x \), we can conclude that the equivariant representation \( \beta \) lifts to an equivariant representation say \( \beta_{\text{sym}}^{(n)} \) of \( Q \) on each \( E_\text{sym}^{(n)} \), hence we get \( \hat{\beta} : S(E_0) \to S(E_0) \otimes Q_0 \), which is a representation and also an algebra homomorphism. We also have on \( S(E_0) \):
\[
(\hat{\theta} \otimes \text{id}) \circ \hat{\beta} = \hat{\Gamma}.
\]

(1)

We claim that there is a well-defined continuous map which sends \( \hat{\theta}(s) \) to \( \hat{\Gamma}(s) \) for \( s \in E(S) \), which will define the desired action of \( Q \). To this end, consider the *-subalgebra \( A \) of \( C(O_M) \otimes Q \equiv C(O_M, Q) \) consisting of those functions \( F \) for which \( F(e) \in Q_{\pi(e)} \) for all \( e \in O_M \). By Lemma 2.1, this is a commutative algebra. Fix the open set \( U \) as before, with \( \omega_1, \ldots, \omega_n \) which give a local section for orthonormal frame bundle on \( U \), and define \( T_{ij} \in C^\infty(O_M) \otimes Q \) by
\[
T_{ij}(e) := \langle \langle \alpha(\omega_j), e(b_i) \otimes 1_Q \rangle \rangle(\pi(e)),
\]
where \( \langle \langle \cdot, \cdot \rangle \rangle \) denotes the \( C^\infty(M) \otimes Q \)-valued inner product of \( \Lambda^1(M) \otimes Q \). If \( \omega_1', \ldots, \omega_n' \) in \( \Lambda^1(M) \) are chosen such that \( \{ \omega_1'(\pi(e)), \ldots, \omega_n'(\pi(e)) \} \) is an orthonormal basis for \( T^*_{\pi(e)} M \), and \( \alpha(\omega_j)(\pi(e)) = \sum_k \omega_k'(\pi(e)) F_{jk}(\pi(e)) \), where \( F_{jk}(\pi(e)) \in Q_{\pi(e)} \), we have
\[
T_{ij}(e) = \sum_k t_{ik}'(e) F_{jk}(\pi(e)),
\]
where \( t_{ik}'(e) \) are the matrix coefficients of \( e \) corresponding to \( \{ \omega_1', \ldots, \omega_n' \} \). It follows that \( T_{ij} \)'s belong to \( A \). Moreover, we have

**Lemma 2.3** For any smooth real-valued function \( \chi \) supported in \( U \), we have
\[
\hat{\alpha}(\chi) \sum_k T_{ik} T_{kj} = \hat{\alpha}(\chi) \sum_k T_{ki} T_{kj} = \hat{\alpha}(\chi) \delta_{ij},
\]
for all \( i, j = 1, \ldots, n \), where \( \delta_{ij} \) are the Kronecker’s delta.
Proof:
By a simple calculation using the definition of $F'_{m,n}$s as well as the fact that $d\alpha$ is inner-product preserving, we obtain
\[
\alpha(\chi)  \sum_k T_{kj}T_{kj} = \langle (d\alpha(\psi_1'), d\alpha(\psi_2')) \rangle = \delta_{ij}\alpha(\chi)^2.
\]
Now, as the both sides of the above equality belong to the commutative C*-algebra $\mathcal{A}$, we get the same equality replacing $\alpha(\chi)^2$ by $\alpha(\chi)$, and moreover, the other equality in the statement of the lemma. □

Recall the algebra $\mathcal{P}_U$. Denote by $\mathcal{E}_U$ the smooth sections of $E_U$ supported in $U$ and let $\mathcal{S}_U^0$ be the subalgebra of $\mathcal{S}(\mathcal{E}_U) \subseteq \mathcal{S}(\mathcal{E})$ consisting of elements of the form $\sum_{l=1}^N f_l P_l(\xi_{ij}; i, j = 1, \ldots, n)$, where $P_l, f_l$ as in the definition of $\mathcal{P}_U$, and $\xi_{ij}$ denotes the section $|\omega_i| < b_1$, i.e. $\xi_{ij}(x)$ sends $b \in \mathbb{C}^n$ to $< b_1, b > \omega_j$. Clearly, $\hat{\theta}(\mathcal{S}_U^0) = \mathcal{P}_U$. It is easily seen that
\[
\hat{\theta}(\sum_{l=1}^N f_l P_l(\xi_{ij}; i, j = 1, \ldots, n)) := \sum_{l=1}^N \alpha(\hat{f}_l) P_l(t_{ij}; i, j = 1, \ldots, n),
\]
\[
\hat{\Gamma}(\sum_{l=1}^N f_l P_l(\xi_{ij}; i, j = 1, \ldots, n)) := \sum_{l=1}^N \alpha(\hat{f}_l) P_l(T_{ij}; i, j = 1, \ldots, n).
\]

Lemma 2.4 $\|\hat{\Gamma}(g)\| \leq \|\hat{\theta}(g)\|$ for all $g \in \mathcal{S}_U^0$.

Indeed, as $T_{ij}$’s are self-adjoint elements of the commutative C*-algebra $\mathcal{A}$, note the natural identification of the C*-algebra $\mathcal{B}_U$ with $C(O(n), C_c(U)) \subseteq C(O(n), C(M))$, so that we have $\|\psi\| = \sup_{v \in O(n)} \|\psi^n\|$, for $\psi \in C(O(n) \times M) \equiv C(O(n), C(M))$, where $\psi^n(x) = \psi(v, x)$ for $v \in O(n)$. A similar fact holds if we replace $C(M)$ by $C(M) \circ \mathbb{Q}$. We also denote by $\alpha^2$ the C*-homomorphism sending $\Psi \in C(O(n) \times M)$ to $\alpha^2(\psi)$ in $C(O(n), C(M) \circ \mathbb{Q})$ given by, $\alpha^2(\psi)(v) = \alpha(\psi^n), v \in O(n)$. Clearly, $\|\alpha^2(\psi)\| \leq \|\psi\|$.

Now, consider $g \in \mathcal{P}_U$ of the form $\sum_{l=1}^N f_l P_l(t_{ij}; i, j = 1, \ldots, n)$, and assume that $K$ is a compact subset of $U$ in which supports of all $f_l$’s are contained. Let $\chi \in C^\infty(U)$ be such that $0 \leq \chi \leq 1$ and $\chi|_K \equiv 1$. Fix $e \in O_M$ and let $\theta$ be a character (multiplicative functional) of the commutative C*-algebra $\mathcal{Q}_{\pi(e)}$. By Lemma 2.3 we see that if $\theta(\alpha(\chi)(e))$ is nonzero, then $(\tau_{ij} := \theta(T_{ij}(e)))$ is an orthogonal matrix, i.e. in $O(n)$. On the other hand, we have (using the fact that $f_l = \chi f_l$ for each $l$):
\[
\theta(\eta_U(g)(e)) = \theta(\alpha(\chi)(e))\theta(\eta_U(g)(e))
\]
\[
\sum_{l=1}^{N} \theta(\alpha(f_l)(\pi(e)))P_l(\tau_{ij}; i, j = 1, \ldots n) = \theta \left( \sum_{l=1}^{N} \alpha(f_l)(\pi(e))P_l(\tau_{ij}; i, j = 1, \ldots n) \right). 
\]

Thus, if \( \theta(\alpha(\chi)(e)) \) is zero, \( \theta(\eta_U(g)(e)) = 0 \), and when it is nonzero, we obtain
\[
|\theta(\eta_U(g)(e))| \leq \sup_{v=(v_{ij}) \in O(n)} \left\| \left( \sum_{l=1}^{N} \alpha(f_l)(\pi(e))P_l(v_{ij}; i, j = 1, \ldots n) \right) \right\|, 
\]
which is equal to \( \|\alpha^2(g)\| \leq \|g\| \).}

This allows us to get a \( C^* \)-homomorphism \( \eta_U \) on \( B_U \) satisfying \( \eta_U(\hat{\theta}(g)) = \hat{\Gamma}(g) \). We can now ‘patch’ \( \eta_U \)’s by \( C^\infty \)-partition of unity to get \( C^* \) homomorphism \( \eta \) from \( C(O_M) \) to \( C(O_M) \hat{\otimes} \mathbb{Q} \) satisfying \( \eta(\hat{\theta}(g)) = \hat{\Gamma}(g) \) for all \( g \in S(E) \). Indeed, as \( T_{ij} \) are in \( C^\infty(O_M) \hat{\otimes} \mathbb{Q} \), we do get \( \eta(C^\infty(O_M)) \subseteq C^\infty(O_M) \hat{\otimes} \mathbb{Q} \), and also that \( \eta \) is Frechet continuous. Moreover, we have
\[
(\hat{\theta} \otimes \text{id}) \circ \hat{\beta} = \hat{\Gamma} = \eta \circ \hat{\theta}
\]
on \( S(\mathcal{E}_0) \). From this, and the Frechet density of \( \hat{\Gamma}(S(\mathcal{E}_0) \otimes \mathbb{Q}_0) \) in \( C^\infty(O_M) \hat{\otimes} \mathbb{Q} \), we get the Frechet density of \( \eta(S(\mathcal{E}_0))(1 \otimes \mathbb{Q}_0) \) in \( C^\infty(O_M) \hat{\otimes} \mathbb{Q} \). It follows that \( \eta \) is a smooth faithful action of \( \mathbb{Q} \) in the sense of [1].

Finally, the condition (1) of Theorem 3.4 of [1] follows by verifying it on \( \mathcal{P}_U \) for every \( U \), with the help of the commutativity of \( \mathcal{A} \) and the expression of \( \eta_U \) in terms of \( F_{ij} \)’s and the coordinate functions \( t'_{ij} \)’s. This completes the proof.

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