SEVERI INEQUALITY FOR VARIETIES OF MAXIMAL ALBANESE DIMENSION

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Contents

1. Introduction 1
   1.1. Main result 1
   1.2. Idea of the proof 2
2. Preliminaries 3
   2.1. The reduction process 3
   2.2. Numerical inequalities 4
3. Relative Noether inequality 6
4. Asymptotic behavior of cohomological dimensions 9
5. Proof of Theorem 1.1 11
   References 12

1. Introduction

1.1. Main result. We work over the complex number field \( \mathbb{C} \). Let \( X \) be a projective and irregular variety over \( \mathbb{C} \), i.e., \( h^1(O_X) > 0 \). We say that \( X \) is of maximal Albanese dimension, if the image of \( X \) under its Albanese map has the same dimension as \( X \) itself.

A typical example of a variety of maximal Albanese dimension which is also of general type can be constructed as follows: fix an Abelian variety \( A \) of dimension \( n \) and a very ample line bundle \( H \) on \( A \). Let \( X \) be the double cover of \( A \) branched along a smooth divisor \( L \) such that \( L \sim_{\text{lin}} 2H \). It is easy to see that \( X \) is of maximal Albanese dimension and \( X \) is of general type since \( K_X \) is ample. Moreover, from the double cover formula, we have

\[
K_X^n = 2H^n.
\]

By the Hirzebruch-Riemann-Roch formula,

\[
\chi(K_X) = (-1)^n \chi(O_X) = \chi(H) = \frac{H^n}{n!}.
\]

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Therefore, we have 

\[ K^n_X = 2n!\chi(K_X). \]

In this paper, we prove the following theorem, which shows that 
\( 2n!\chi(K_X) \) turns out to be the optimal lower bound of \( K^n_X \).

**Theorem 1.1.** Let \( X \) be a projective, normal, minimal and Gorenstein \( n \)-dimensional variety of general type. Suppose \( X \) is of maximal Albanese dimension. Then

\[ K^n_X \geq 2n!\chi(K_X). \]

This theorem was known previously for small \( n \). For example, when \( n = 1 \), the result just says that \( \deg K_X = 2g(X) - 2 \) for a smooth curve \( X \) of positive genus. When \( n = 2 \), the theorem, known as Severi inequality for surfaces, is proved by Pardini in [Pa] using the slope inequality in [CH, Xi] and a clever covering (Albanese lifting) method. Pardini’s theorem is reproved in [YZ2] using the relative Noether inequality for fibered surfaces. Moreover, it is proved in [YZ2] that the theorem also holds in positive characteristic.

We point out that Theorem 1.1 is also independently proved by Barja [Ba] during the preparation of this paper. His proof also relies on Pardini’s method. The difference between our approaches is that, he uses a suitable version of Xiao’s method [Xi] on the Harder-Narasimhan filtration and continuous linear series developed by Mendes Lopes, Pardini and Pirola [MPP], while we use the relative Noether inequality which can be viewed as a generalization of [YZ2] in the surface case.

Also, it should be pointed out that using the relative Noether inequality, we can get an upper bound of \( h^0(L) \) for arbitrary nef \( L \), not only the continuous rank of \( L \) (see [Ba]). In particular, using the relative Noether inequality for fibered surfaces, we can get the slope inequality of Cornalba-Harris-Xiao in arbitrary characteristic [YZ2], although it only involves some basic theories on linear series. Furthermore, one should note that the relative Noether formula here still holds in positive characteristic. In an ongoing paper joint with Yuan, we also consider the arithmetic version of this inequality in Arakelov geometry.

1.2. **Idea of the proof.** As pointed before, the idea is similar to [YZ2]. We first prove the relative Noether inequality for fibered \( n \)-folds by induction on the dimension (see Proposition 3.1). As a result, we obtain a Hilbert-Samuel type result as Corollary 3.2. Finally, applying Corollary 3.2 to the \( d \)-th Albanese lifting of (a smooth model of) \( X \), we can get Theorem 1.1 by taking the limit.

However, when we take the limit, the intermediate cohomologies are involved. Therefore, we also need to control the asymptotic behavior
of such cohomologies. Fortunately, it can be implied by the generic vanishing theorem [GL] (see Theorem 4.1), with which we can finish the whole proof.

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2. Preliminaries

We first assume $X$ to be a smooth projective variety of dimension $n$. Let $f : X \to Y$ be a fibration from $X$ to a smooth projective curve $Y$ with smooth general fiber $F$. The above notations will be used throughout Section 2 and 3.

2.1. The reduction process. For any nef line bundle $L$ on $X$, we can find an integer $e_L > 0$ such that

- $L - e_L F$ is not nef;
- $L - eF$ is nef for any integer $e < e_L$.

We have the following lemma.

Lemma 2.1. Let $L$ be a nef line bundle on $X$ such that $h^0(L - e_L F) > 0$. Then the base locus of the linear system $|L - e_L F|$ has horizontal part.

Proof. By the definition of $e_L$, $L - e_L F$ is not nef. Then there exists an irreducible and reduced curve $C$ on $X$ such that

$$(L - e_L F) C < 0.$$  

Hence $C$ is contained in the base locus of $|L - e_L F|$. It suffices to prove that $C$ is horizontal. Since $L$ is nef, $LC \geq 0$. It gives $FC > 0$, which implies that $C$ can not be vertical. \qed

Lemma 2.2. Let $L$ be a line bundle on $X$ such that $|L|$ is base point free and that $h^0(L - e_L F) > 0$. Then we have the following decomposition:

$$\pi^*(L - e_L F) = L_1 + Z_1.$$  

Here $\pi : X_1 \to X$ is a composition of blow-ups of $X$ such that the proper transformation of $|L - e_L F|$ is base point free. $L_1$ and $Z_1$ are line bundles on $X_1$. Moreover, we have

1. $|L_1|$ is base point free;
2. $Z_1 \geq 0$;
3. $h^0(L_1) < h^0(L)$. 

Proof. Let $L_1$ be the proper transformation of $|L - e_L F|$ on $X_1$. It is automatically base point free. Now $Z_i$ is the fixed part of $|\pi^*(L - e_L F)|$. So it is effective. Furthermore, since $|L|$ is base point free and $e_L > 0$, it follows that

$$h^0(L) > h^0(L - e_L F) = h^0(L_1).$$

□

We have the following general theorem.

Theorem 2.3. With the above notations. Let $L$ be a nef line bundle on $X$. Then we have the following quadruples

$$\{(X_i, L_i, Z_i, a_i), \; i = 0, 1, \cdots, N\}$$

with the following properties:

1. $(X_0, L_0, Z_0, a_0) = (X, L, 0, e_L)$.
2. For any $i = 0, \cdots, N - 1$, $\pi_i : X_{i+1} \rightarrow X_i$ is a composition of blow-ups of $X_i$ such that the proper transformation of $|L_i - a_i F_i|$ is base point free. Here $F_0 = F$, $F_{i+1} = \pi_i^* F_i$ and $a_i = e_{L_i}$. Moreover, we have the decomposition

$$\pi_i^*(L_i - a_i F_i) = L_{i+1} + Z_{i+1}$$

such that $|L_{i+1}|$ is base point free and $Z_{i+1} \geq 0$.
3. We have $h^0(L_0) > h^0(L_1) > \cdots > h^0(L_N) > h^0(L_N - a_N F_N) = 0$. Here $a_N = e_{L_N}$.

Proof. The quadruple $(X_{i+1}, L_{i+1}, Z_{i+1}, a_{i+1})$ is obtained by applying Theorem 2.2 to $(X_i, L_i, Z_i, a_i)$. The whole process terminates because $h^0(L_i)$ decreases strictly as $i$ goes larger and they are non-negative. □

2.2. Numerical inequalities. Resume the notations in Theorem 2.3.

Denote

$$L'_i = L_i - a_i F_i$$

for $i = 0, \cdots, N$. Note that we have $L_i|F_i = L'_i|F_i$ following from the construction. Denote

$$r_i = h^0(L_i|F_i), \; d_i = (L_i|F_i)^{n-1}.$$

From our construction, we know that $L'_i + F_i$ is nef for each $i = 0, \cdots, N$. Therefore, we have

Lemma 2.4. With the above notations, for any $i = 0, \cdots, N$,

$$L_i^n + nd_i = (L'_i + F_i)^n \geq 0.$$

Proposition 2.5. For any $j = 0, 1, \cdots, N$, we have the following numerical inequalities:
Proof. We have the following exact sequence:

\[ 0 \rightarrow H^0(L_{i+1} - F_i) \rightarrow H^0(L_{i+1}) \rightarrow H^0(L_{i+1} | F_{i+1}) \]

Then it follows that

\[ h^0(L_{i+1} - F_i) \leq h^0(L_{i+1}) - h^0(L_{i+1} | F_{i+1}) = h^0(L_{i+1}) - r_{i+1}. \]

So by induction, we have

\[ h^0(L_{i+1}) = h^0(L_{i+1} - a_{i+1}F_{i+1}) \leq h^0(L_{i+1} - a_{i+1}r_{i+1}) = h^0(L_i) - a_{i+1}r_{i+1}. \]

Furthermore,

\[ h^0(L_0) \leq h^0(L_0) + a_0r_0. \]

Hence (1) is proved by summing over \( i = 0, \cdots, j - 1. \)

To prove (2), note that both \( L_i + F_i \) and \( L_{i+1} + F_{i+1} \) are nef, and \( Z_i \)

is effective. We get

\[ L_i^n - L_{i+1}^n = (\pi_i^* L_i' - L_{i+1}') \sum_{p=0}^{n-1} (\pi_i^* L_i')^{n-1-p} L_{i+1}^p \]

\[ = (a_{i+1}F_{i+1} + Z_{i+1}) \sum_{p=0}^{n-1} (\pi_i^* L_i')^{n-1-p} L_{i+1}^p \]

\[ = a_{i+1} \sum_{p=0}^{n-1} (\pi_i^* L_i')^{n-1-p} L_{i+1}^p F_{i+1} \]

\[ + \sum_{p=0}^{n-1} (\pi_i^* L_i' + F_{i+1})^{n-1-p} (L_{i+1} + F_{i+1})^p Z_{i+1} \]

\[ - \sum_{p=0}^{n-1} (p(\pi_i^* L_i')^{n-1-p} L_{i+1}^{p-1} + (n - p - 1)(\pi_i^* L_i')^{n-2-p} L_{i+1}^{p-1}) F_{i+1} Z_{i+1} \]

\[ \geq n a_{i+1} d_{i+1} - n \sum_{p=0}^{n-2} (\pi_i^* L_i')^{n-2-p} L_{i+1}^p F_{i+1} Z_{i+1} \]

\[ = n a_{i+1} d_{i+1} - n (\pi_i^* L_i' - L_{i+1}') F_{i+1} \sum_{p=0}^{n-2} (\pi_i^* L_i')^{n-2-p} L_{i+1}^p \]

\[ = n a_{i+1} d_{i+1} - n (d_i - d_{i+1}) \]
Summing over $i = 0, 1, \cdots, j - 1$, we have

$$L_0^n \geq L_j^n + n \sum_{i=1}^{j} a_i d_i - n(d_0 - d_j).$$

Since

$$L_0^n - L_j^n = na_0 d_0,$$

Then (2) follows by applying Lemma 2.4. \qed

We also have the following lemma.

**Lemma 2.6.** With the above notations, we have

$$L_0^n \geq d_0(na_0 + \sum_{i=1}^{N} a_i - n).$$

**Proof.** For $i = 0, \cdots, N - 1$, denote by

$$\tau_i = \pi_i \circ \cdots \circ \pi_{N-1} : X_N \to X_i$$

the composition of blow-ups and denote $\tau_N = \text{id}_{X_N} : X_N \to X_N$.

Write $b = a_1 + \cdots + a_N$ and $Z = \tau_1^* Z_1 + \cdots + \tau_N^* Z_N$. We have the following numerical equivalence on $X_N$:

$$\tau_0^* L_0' \sim_{\text{num}} L_N' + bF_N + Z.$$

Since $L_0' + F_0$ and $L_N' + F_N$ are both nef, it follows that

$$(L_0' + F_0)^n = (\tau_0^* L_0' + F_N)^{n-1}(L_N' + F_N + bF_N + Z) \geq (\tau_0^* L_0' + F_N)^{n-1}(L_N' + F_N) + b(\tau_0^* L_0' + F_N)^{n-1}F_N \geq bd_0.$$

Combining with

$$L_0^n - (L_0' + F_0)^n = n(a_0 - 1)d_0,$$

the proof is finished. \qed

We end up this section with a remark that when $X$ is a fibered surface, or even $X$ is an arithmetic surface, all the above results have been studied in [YZ1, YZ2].

### 3. Relative Noether Inequality

In this section, we will prove the relative Noether type inequality.

We say a divisor $D$ on a variety $X$ of dimension $n$ is pseudo-effective, if for any nef line bundles $A_1, \cdots, A_{n-1}$ on $X$, we have

$$A_1 \cdots A_{n-1} D \geq 0.$$
Proposition 3.1. Let \( f : X \to Y \) be a fibration from \( X \) to a smooth curve \( Y \) with the general smooth fiber \( F \) of general type. Suppose that \( L \) is a nef line bundle on \( X \). Fix a line bundle \( B \) on \( F \) such that

- \( \delta_i = (L|_F)^{n-i-1}B^i > 0 \) for any \( i \geq 0 \),
- \( |B| \) is base point free on \( F \),
- \( kB - L|_F \) is pseudo-effective on \( F \) for certain integer \( k > 0 \),
- \( L|_F \leq K_F \).

Then one has

\[
h^0(L) - \frac{1}{2n!} L^n \leq c_n \left( \frac{L^n}{\delta_0} + 1 \right) \sum_{j=1}^{n-1} k^{j-1} \delta_j + \frac{\delta_0}{2(n-1)!}.
\]

Here \( c_n \geq 1 \) is a constant depending only on the number \( n \).

Proof. Our proof here is by induction. When \( n = 2 \), since \( \delta_0, \delta_1 > 0 \), the result just says \( h^0(L) \) can be bounded in terms of \( L^2 \). In fact, if \( \delta_0 \geq 2 \), from the relative Noether inequality in [YZ2], we have

\[
h^0(L) \leq \left( \frac{1}{4} + \frac{3}{4\delta_0} \right) L^2 + \frac{\delta_0}{2} + \frac{3}{2}.
\]

If \( \delta_0 = 1 \), then we can use the above inequality to bound \( h^0(2L) \), which is enough to give a bound of \( h^0(L) \).

Now we assume that the result holds for fibered varieties of dimension \( \leq n - 1 \) (\( n \geq 3 \)).

Resume the notations in Theorem 2.3. By Proposition 2.5, one has

\[
h^0(L_0) \leq h^0(L'_N) + \sum_{i=0}^{N} a_i r_i;
\]

\[
L_0^n \geq n(a_0 - 1)d_0 + n \sum_{i=1}^{N} a_i d_i.
\]

Here \( d_0 = \delta_0 \). In the following, we will use induction to compare \( r_i \) and \( d_i \).

For any \( i = 1, \ldots, N \), write

\[
\rho_i = \pi_0 \circ \cdots \circ \pi_{i-1} : X_i \to X_0
\]

and \( \rho_0 = \text{id}_{X_0} : X_0 \to X_0 \). Denote \( B_0 = B \) and \( B_i = \rho^*_i B \).

Case I. If \( d_i > 0 \), choose two general members \( B_{1,i}, B_{2,i} \in |B_i| \). Let \( \sigma : \widetilde{F}_i \to F_i \) be the blow-up along their intersection. We get a fibration \( \tilde{f}_i : \widetilde{F}_i \to \mathbb{P}^1 \). Denote the general fiber of \( \tilde{f}_i \) by \( \widetilde{B}_i \). Since \( L_i|_{\widetilde{F}_i} \) is big on \( F_i \) and \( L_i \leq \rho^*_i L_0 \), we can easily check the following facts:
For each \( j = 1, \ldots, n - 1 \), we have
\[
\delta_j \geq (L_i|_{F_i})^{n-1-j}B_i^j = (\sigma^*L_i|_{\tilde{B}_i})^{n-1-j}(\sigma^*B_i|_{\tilde{B}_i})^{j-1} > 0;
\]

- \(|\sigma^*B_i|_{\tilde{B}_i}\) is base point free;
- \(\sigma^*(kB_i - L_i)|_{\tilde{B}_i}\) is pseudo-effective;
- Since \(L_i|_{F_i} \leq \rho^*_iL_{F_i} \leq \rho^*K_{F_i} \leq K_{F_i}\), by adjunction formula, one has
\[
\sigma^*L_i|_{\tilde{B}_i} \leq K_{\tilde{B}_i}.
\]

By induction and using the fact that
\[
d_i \left(\frac{(L_i|_{F_i})^{n-1}}{(L_i|_{F_i})^{n-2}}\right)^{n-2}B_i \leq \frac{k(L_i|_{F_i})^{n-2}B_i}{(L_i|_{F_i})^{n-2}B_i} = k,
\]
we have
\[
r_i \leq \frac{1}{2(n-1)!}d_i + 2c_{n-1} \sum_{j=1}^{n-1} k^{j-1}\delta_j.
\]

**Case II.** If \( d_i = 0 \), it implies that \( L_i|_{F_i} \) is not big. Therefore
\[
r_i = h^0((L_i|_{F_i})|_{B_i}) \leq h^0(L_0|_{B}).
\]

Note that \((L_0|_{B})^{n-2} = \delta_1\). So if \( n - 2 \geq 2 \), we can use the blow-up trick as in Case I and get the following inequality by induction:
\[
r_i \leq \frac{1}{2(n-2)!}\delta_1 + 2c_{n-2} \sum_{j=2}^{n-1} k^{j-2}\delta_j.
\]

The only problem is when \( n = 3 \). However, in this case, the above inequality is nothing but Clifford’s inequality.

As a result of the above two case, we are safe to use the following inequality:
\[
r_i \leq \frac{1}{2(n-1)!}d_i + 2(c_{n-1} + c_{n-2}) \sum_{j=1}^{n-1} k^{j-1}\delta_j.
\]

Using the above comparison and Lemma 2.6, it follows that
\[
h^0(L_0) - \frac{1}{2n!}L_0^n \leq 2(c_{n-1} + c_{n-2}) \sum_{j=1}^{n-1} k^{j-1}\delta_j \sum_{i=0}^{N} a_i + \frac{\delta_0}{2(n-1)!}.
\]
\[
\leq 2(c_{n-1} + c_{n-2}) \sum_{j=1}^{n-1} k^{j-1}\delta_j \left(\frac{L_0^n}{\delta_0} + 1\right) + \frac{\delta_0}{2(n-1)!}.
\]

Hence our result is proved by letting \( c_n = 2(c_{n-1} + c_{n-2}) \).

As a corollary, we have the following Hilbert-Samuel type result.
Corollary 3.2. Let $X$ be an $n$-dimensional smooth projective variety of general type and $L$ be a nef line bundle on $X$. Fix a line bundle $B$ on $X$ such that

- $L \leq K_X$;
- $\delta_i = L^{n-i}B_i \geq 0$ for any $1 \leq i \leq n - 1$;
- $|B|$ is base point free on $X$ and $\delta_n = B^n > 0$;
- $kB - L$ is pseudo-effective on $X$ for certain integer $k > 0$.

Then one has

$$h^0(L) - \frac{1}{2n!}L^n \leq C_n \sum_{j=1}^{n} k^{j-1} \delta_j.$$ 

Here $C_n$ is a constant depending only on the number $n$.

Proof. The idea is very similar to the proof of Proposition 3.1. In fact, in the previous proof, we have shown how to prove this result in lower dimensional case. Here we only need to apply the blow-up trick once more.

Suppose $L^n > 0$. Then $L$ is big. Hence $\delta_i > 0$ for $i \geq 1$. Therefore, using the above blow-up trick on $X$, and the conclusion follows.

If $L^n = 0$, i.e., $L$ is not big. Since $B$ is big, we have $h^0(L) = h^0(L|B)$. By induction, we can find a positive integer $i_0 < n$ such that

$$h^0(L) = h^0(L|B_{i_0}).$$

Otherwise $h^0(L) \leq 1$. But here we can apply the blow-up trick on $B_{i_0}$ to get the conclusion. \qed

At the end, we would like to mention that the same induction method can be applied to prove the relative Noether inequality in positive characteristic, because the relative Noether inequality (resp. Clifford’s inequality) still holds for Gorenstein fibered surfaces (resp. curves) [Li2, YZ2].

4. Asymptotic behavior of cohomological dimensions

Let $X$ be a projective and irregular variety. Let $A$ be the Albanese variety of $X$ of dimension $m = h^1(O_X) > 0$, and $a(X)$ be its Albanese image. Let $\mu_d : A \to A$ be the multiplicative map by $d$. We have the following diagram:

$$
\begin{array}{ccc}
X_d & \xrightarrow{\phi_d} & X \\
\alpha_d \downarrow & & \downarrow \alpha = \text{Alb}_X \\
A & \xrightarrow{\mu_d} & A
\end{array}
$$

Here $X_d = X \times_{\mu_d} A$ is the fiber product. We call $X_d$ the $d$-th Albanese lifting of $X$. 
In this section, we will prove

**Theorem 4.1.** Let $X$ be a projective, smooth and irregular variety and $a(X)$ be its Albanese image. For each $d \in \mathbb{N}$, let $X_d$ be the $d$-th Albanese lifting of $X$. Then for each $i = 0, \ldots, \dim a(X) - 1$, we have

$$\lim_{d \to \infty} \frac{h^i(O_{X_d})}{d^{2m}} = 0$$

Here $m = h^1(O_X)$.

Before the proof, we recall the statement of the generic vanishing theorem first, which is due to Green and Lazarsfeld [GL].

Let $X$ be a projective, smooth and irregular variety over $\mathbb{C}$. Let $S^i(X) = \{ L \in \text{Pic}^0(X) : h^i(L) > 0 \}, \ i = 0, \ldots, \dim X,$

and $a(X)$ be the Albanese image of $X$. By the semi-continuity theorem, we know that $S^i(X)$ is closed for each $i$.

We have the following remarkable theorem:

**Theorem 4.2** (Green-Lazarsfeld). For any $i = 0, \ldots, \dim X$, we have

$$\text{codim}\{S^i(X), \text{Pic}^0(X)\} \geq \dim a(X) - i.$$

With the above theorem, we can prove Theorem 4.1.

**Proof of Theorem 4.1.** Recall our construction of the $d$-th Albanese lifting. The map $\phi_d : X_d \to X$ is a finite abelian cover. Moreover,

$$\text{Gal}(X_d/X) = \text{Gal}(\mu_d) = (\mathbb{Z}/d\mathbb{Z})^{2m}.$$

Thus we have

$$(\phi_d)_* O_{X_d} = \bigoplus_{L \in T_d} L,$$

where $T_d \cong (\mathbb{Z}/d\mathbb{Z})^{2m}$ is the subgroup of $\text{Pic}^0(X)$ consisting of all the $d$-torsion line bundles on $X$.

For each $i = 0, \ldots, \dim a(X) - 1$, it follows from the generic vanishing theorem that $h^i(L) = 0$ for each $L \in \text{Pic}^0(X) \setminus S^i(X)$. Also from the semi-continuity theorem, we can find an $M_i > 0$ such that

$$M_i = \max_{L \in \text{Pic}^0(X)} h^i(L).$$

Then we have

$$h^i(O_{X_d}) = h^i((\phi_d)_* O_{X_d}) = \sum_{L \in T_d} h^i(L) \leq l_d M_i,$$
where \( l_d = \#(T_d \cap S^i(X)) \). Because \( S^i(X) \) has positive codimension, one has

\[
\lim_{d \to \infty} \frac{\#(T_d \cap S^i(X))}{#T_d} = 0.
\]

Hence

\[
\lim_{d \to \infty} \frac{h^i(\mathcal{O}_{X_d})}{d^{2m}} = 0.
\]

\[\square\]

5. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1 using Corollary 3.2 and Theorem 4.1 by constructing the Albanese lifting of \( X \). This construction has been used by Pardini [Pa] in the proof of the surface case.

Let \( X \) be as in Theorem 1.1. Let \( \varepsilon : X' \to X \) be the resolution of singularities of \( X \). Since \( X \) is minimal, \( X \) has only terminal singularities [KMM]. It is known that terminal singularities are rational, so we have

\[
\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X'), \quad h^0(K_X) = h^0(K_{X'}).$

Furthermore, since \( X \) is of maximal Albanese dimension, so is \( X' \).

Denote \( \text{Alb}_{X'} : X' \to A \) to be the Albanese map of \( X' \), where \( A = \text{Alb}(X') \) is the Albanese variety of \( X' \) whose dimension is \( m = h^1(\mathcal{O}_X) \).

Let \( H \) be a very ample line bundle on \( A \), and \( L \) be the pull-back of a very general member of \( |H| \) on \( X' \) by \( \text{Alb}_{X'} \). Set

\[
\delta_i = (\varepsilon^*K_X)^{n-i}L^i
\]

for \( i = 0, \ldots, n \). Since \( L \) is big, we are able to find a positive integer \( k \) such that \( h^0(kL - K_{X'}) > 0 \).

Let \( X_d \) (resp. \( X'_d \)) be the \( d \)-th Albanese lifting of \( X \) (resp. \( X' \)). It follows that \( X_d \) is still normal, minimal and Gorenstein. So we have

\[
\chi(\mathcal{O}_{X_d}) = \chi(\mathcal{O}_{X'_d}) = d^{2m} \chi(\mathcal{O}_X), \quad h^0(K_{X_d}) = h^0(K_{X'_d}).
\]

Write \( \varepsilon_d : X'_d \to X_d \) to be the resolution of singularities of \( X_d \). Recall the diagram

\[
\begin{array}{ccc}
X'_d & \xrightarrow{\phi_d} & X' \\
\downarrow{\alpha_d} & & \downarrow{\alpha = \text{Alb}_{X'}} \\
A & \xrightarrow{\mu_d} & A
\end{array}
\]

We have the following numerically equivalence on \( A \) [BL]:

\[
\mu_d^*H \sim \text{num} d^2 H,
\]
which yields
\[ \phi^*_d L \sim_{\text{num}} d^2 L_d. \]

Here \( L_d = \alpha^*_d H \). It follows that for any \( i = 0, \ldots, n \),
\[ (\varepsilon^*_d K_{X_d})^{n-i} L_d^i = (\phi^*_d (\varepsilon^* K_X))^{n-i} L_d^i = d^{2m-2i} \delta_i. \]

Furthermore, from the above numerical equivalence, we know that
\[ d^2 k L_d - K_{X_d} \sim_{\text{num}} \phi^*_d (kL - K_{X_d}) \]
is pseudo-effective. Since \( X \) is minimal and \( \varepsilon^*_d K_{X_d} \leq K_{X_d} \), it implies that \( d^2 k L_d - \varepsilon^*_d K_{X_d} \) is pseudo-effective.

Now, apply Corollary 3.2 to \( \varepsilon^*_d K_{X_d} \) and it follows that
\[
  h^0(K'_{X_d}) \leq \frac{1}{2n!} K^n_{X_d} + C_n \sum_{j=1}^{n} (d^2 k)^{j-1} (\varepsilon^*_d K_{X_d})^{n-j} L_d^j \\
  \leq \frac{d^{2m}}{2n!} K^n_{X} + C_n d^{2m-2} \sum_{j=1}^{n} k^{j-1} \delta_j.
\]

On the other hand,
\[
  h^0(K'_{X_d}) = \chi(K'_{X_d}) - \sum_{j=0}^{n-1} (-1)^{n-j} h^j(O_{X_d}) \\
  = d^{2m} \chi(K_X) - \sum_{j=0}^{n-1} (-1)^{n-j} h^j(O_{X_d}).
\]

So the proof of Theorem 1.1 is completed by letting \( d \to \infty \) and applying Theorem 4.1.

References

[Ba] M. A. Barja, Generalized Clifford-Severi inequality and the volume of irregular varieties, arXiv: 1303.3045

[BL] Ch. Birkenhake, H. Lange, Complex abelian varieties, Grundlehren der Mathematischen Wissenschaften, vol. 302. Berlin, Heidelberg: Springer 1992.

[Ch] X. Chen, Private communication.

[CH] M. Cornalba, J. Harris, Divisor classes associated to families of stable varieties, with applications to the moduli space of curves, Ann. Sci. École Norm. Sup. (4) 21 (1988), no. 3, 455–475.

[GL] M. Green, R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, Invent. Math. 90 (1987), no. 2, 389–407.

[Ha] R. Hartshorne, Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.
SEVERI INEQUALITY FOR VARIETIES OF MAXIMAL ALBANESE DIMENSION

[HK] C. Hacon, S. Kovács, Generic vanishing fails for singular varieties and in characteristic $p > 0$, arXiv: 1212.5105.

[KMM] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model program, Algebraic Geometry, Sendai (T. Oda, ed.), Kinokuniya-NorthHolland, 1987, Adv. Stud. Pure Math, vol 10, pp. 283–360.

[Li1] C. Liedtke, Algebraic surfaces in positive characteristic, to appear in F. Bogomolov, B. Hassett, Y. Tschinkel (editors), Birational Geometry, Rational Curves, and Arithmetic, Springer (2013).

[Li2] C. Liedtke, Algebraic surfaces of general type with small $c_1^2$ in positive characteristic, Nagoya Math. J. 191 (2008), 111–134.

[MP] M. Mendes Lopes, R. Pardini, The geography of irregular surfaces, Current developments in algebraic geometry, Math. Sci. Res. Inst. Publ. 59, Cambridge Univ. Press (2012), 349–378.

[MPP] M. Mendes Lopes, R. Pardini, G. Pirola, Continuous families of divisors, paracanonical systems and a new inequality for varieties of maximal Albanese dimension, arXiv: 1207.4516

[Pa] R. Pardini, The Severi inequality $K^2 \geq 4\chi$ for surfaces of maximal Albanese dimension, Invent. Math. 159 (2005), no. 3, 669–672.

[Se] F. Severi, La serie canonica e la teoria delle serie principali de gruppi de punti sopra una superficie algebrica, Comment. Math. Helv. 4 (1932), 268–326.

[Si] C. Simpson, Subspaces of moduli spaces of rank one local systems, Ann. Sci. École Norm. Sup. (4) 26 (1993), 361–401.

[Xi] G. Xiao, Fibered algebraic surfaces with low slope, Math. Ann. 276 (1987), no. 3, 449–466.

[YZ1] X. Yuan, T. Zhang, Effective bound of linear series on arithmetic surfaces, Duke Math. J., to appear.

[YZ2] X. Yuan, T. Zhang, Relative Noether inequality on fibered surfaces, submitted.

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