Transitive $A_6$-invariant $k$-arcs in $PG(2, q)$

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Abstract

For $q = p^r$ with a prime $p \geq 7$ such that $q \equiv 1$ or $19 \pmod{30}$, the desarguesian projective plane $PG(2, q)$ of order $q$ has a unique conjugacy class of projectivity groups isomorphic to the alternating group $A_6$ of degree 6. For a projectivity group $\Gamma \cong A_6$ of $PG(2, q)$, we investigate the geometric properties of the (unique) $\Gamma$-orbit $O$ of size 90 such that the 1-point stabilizer of $\Gamma$ in $O$ is a cyclic group of order 4. Here $O$ lies either in $PG(2, q)$ or in $PG(2, q^2)$ according as $3$ is a square or a non-square element in $GF(q)$. We show that if $q \geq 349$ and $q \neq 421$, then $O$ is a 90-arc, which turns out to be complete for $q = 349, 409, 529, 601, 661$. Interestingly, $O$ is the smallest known complete arc in $PG(2, 601)$ and in $PG(2, 661)$. Computations are carried out by MAGMA.

Keywords: finite desarguesian planes, $k$-arcs, $PSL(2, 9)$.

1 Introduction

Let $GF(q)$ be a finite field of order $q = p^r$, a power of an odd prime $p$. In the projective plane $PG(2, q)$ coordinatized by $GF(q)$, a $k$-arc $K$ is a set of $k$ points no three of which are collinear. If an arc of $PG(2, q)$ is not contained in a larger arc in $PG(2, q)$ then it is called complete. From the theory of linear codes, every $k$-arc of $PG(2, q)$ corresponds to a $[k, 3, k-2]$ maximum distance separable (MDS) code of length $k$, dimension 3 and minimum distance $k-2$. This gives a motivation for the the study of $k$-arcs in $PG(2, q)$; those with many projectivities were investigated in several papers, see [4, 6, 8, 12, 13, 14, 15, 17, 18]

The maximum size of a (complete) arc in $PG(2, q)$ is $q+1$, and the points of an irreducible conic in $PG(2, q)$ form an arc of size $q+1$. Actually, such $(q+1)$-arcs arising from irreducible conics are the unique $(q+1)$-arcs in $PG(2, q)$. This is the famous Segre’s theorem [20]; see also [10] Theorem 8.7. Therefore, the projectivity group which preserves a $(q+1)$-arc $K$ in $PG(2, q)$ is isomorphic to the projective linear group $PGL(2, q)$ and acts on $K$ as $PGL(2, q)$ in its natural 3-transitive permutation representation. In particular, every $(q+1)$-arc $K$ is transitive. Here, the term of a transitive arc of $PG(2, q)$ is adopted to denote a $k$-arc $K$ such that the projectivity group preserving $K$ acts transitively on the points of $K$.
Let $\Gamma$ be a finite group which can act faithfully as a projectivity group in $PG(2, q)$. Actually, this may happen in different characteristics $p$. For instance, $PG(2, q)$ with $p \neq 5$ has a projectivity group isomorphic to the alternating group $A_6$ if and only if $q \equiv 1$ or 19 (mod 30), and in this case such a projectivity group is uniquely determined up to conjugacy in $PGL(3, q)$, see [2]. So the question arises whether or not a $\Gamma$-invariant arc of a fixed size $k$ exists in $PG(2, q)$ for infinitely many values of $p$. From previous work, the answer is affirmative for $\Gamma \cong A_6$ and $k = 72$, see [14], and $\Gamma \cong PSL(2, 7)$ and $k = 42$ see [16]. However the answer is negative for the Hesse-group of order 216 for any $k \geq 9$, see [21].

In this paper we investigate the case of $\Gamma \cong A_6$ and $k = 90$, giving a positive answer to the above question:

**Theorem 1.1.** For a power $q$ of a prime $p \geq 7$ such that $q \equiv 1$ or 19 (mod 30), let $\Gamma \cong A_6$ be a projectivity group of $PG(2, q)$. Let $O$ be the (unique) $\Gamma$-orbit of length 90 in $PG(2, q)$ such that the 1-point stabilizer of $\Gamma$ in $O$ is a cyclic group of order 4. Then $O$ is a 90-arc in $PG(2, q)$ except for a few cases where

(i) $q = 61$ and $O$ is a set of type $(0, 1, 2, 4, 6)$;
(ii) $q = 109$ and $O$ is a set of type $(0, 1, 2, 3)$;
(iii) $q = 181$ and $O$ is a set of type $(0, 1, 2, 3)$;
(iv) $q = 229$ and $O$ is a set of type $(0, 1, 2, 4)$;
(v) $q = 241$ and $O$ is a set of type $(0, 1, 2, 4)$;
(vi) $q = 421$ and $O$ is a set of type $(0, 1, 2, 3)$;
(vii) $q = 7^2$ and $O$ is a set of type $(0, 1, 2, 4)$;
(viii) $q = 11^2$ and $O$ is a set of type $(0, 1, 2, 5)$;
(ix) $q = 13^2$ and $O$ is a set of type $(0, 1, 2, 4)$;
(x) $q = 17^2$ and $O$ is a set of type $(0, 1, 2, 3)$;
(xi) $q = 19^2$ and $O$ is a set of type $(0, 1, 2, 5)$;

An exhaustive computer aided search shows that such a 90-arc may be complete for some particular values of $q$, namely $q = 349, 409, 529, 601, 661$. It is worth mentioning that this gives the smallest known complete arc in $PG(2, 601)$ and in $PG(2, 661)$, see [1, 7].

Notation and terminology are standard, see [10]. Furthermore, $q$ always denotes a power of an odd prime $p \geq 7$ such that $q \equiv 1$ or 19 (mod 30). Then 3 divides $q - 1$ and 5 is a square element in the multiplicative group of $GF(q)$. The latter two requirement are indeed necessary and sufficient for $PGL(3, q)$ to have a subgroup $\Gamma \cong A_6$.

2 Preliminary Results

We give an explicit representation of $\Gamma$ as a subgroup of $PGL(3, q)$ using the well known isomorphism $A_6 \cong PSL(2, 9)$. Following [14], we choose a primitive
element $\eta$ in $GF(9)$ satisfying $\eta^2 = \eta + 1$, and introduce the following matrices over $GF(9)$,

$$U_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & \eta^2 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}. $$

It is easy to show that the above matrices generate $SL(2,9)$. Furthermore, $V^4$ is the identity matrix $I$.

The factor group $SL(2,9)/\langle -I \rangle$ is $PSL(2,9)$.

Let $\Phi : SL(2,9) \to PSL(2,9)$ be the associated natural homomorphism, and set $M = \Phi(M)$ with $M \in SL(2,9)$.

There is a unique conjugacy class of elements of order 4 in $PSL(2,9)$, and the projectivity $W$ with matrix representation $W$ is such an element of order 4 (then $\langle W \rangle$ has order 4 ...si potrebbe aggiungere).

Now, fix a primitive third root $t$ of unity in $GF(q)$ and an element $z$ such that $z^2 = 5$. Let $\Delta = t - t^2$. Define the following matrices over $GF(q)$:

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix}, \quad V = \begin{pmatrix} -2 & 1 + \Delta z & 1 + \Delta z \\ 1 - \Delta z & 4 & -2 \\ 1 - \Delta z & -2 & 4 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 1 & 1 \\ 1 & t & t^2 \\ 1 & t^2 & t \end{pmatrix}. $$

Let $\bar{U}, \bar{\Omega}, \bar{V}$ and $\bar{W}$ be the associated projectivities of $PGL(3,q)$. From [14, Theorem 2.6], the projectivity group generated by $\bar{U}, \bar{\Omega}, \bar{V}$ and $\bar{W}$ is isomorphic to $PSL(2,9)$. More precisely, the map $\varphi$ with

$$\varphi := \begin{cases} U_1 \to \bar{U} \\ U_2 \to \bar{\Omega} \\ V \to \bar{V} \\ W \to \bar{W} \end{cases} $$

extends to an isomorphism from $PSL(2,9)$ into $PGL(3,q)$. Therefore, the group generated by $\bar{U}, \bar{\Omega}, \bar{V}$ and $\bar{W}$ is taken for $\Gamma$; that is,

$$\Gamma = \langle \bar{U}, \bar{\Omega}, \bar{V}, \bar{W} \rangle. $$

3
A representative system of the 90 cosets of \( \langle W \rangle \) in \( PSL(2,9) \) is listed below.

\[ 
\{ 1, V W V U, W V W U, V W U, U W U V W U, V W V O W U, W V O W V U, W O W V U, O W V U, V W O V W U, W O V W U, O V W U, V O W U, O W U, V W U, O W U, V O U, O W U, V \}
\]

Replacing \( U, V, W, \Omega \) with \( \bar{U}, \bar{V}, \bar{W}, \bar{\Omega} \) gives a representative system of \( \langle \bar{W} \rangle \) in \( \Gamma \).

3 The fixed points of \( \bar{W} \)

The characteristic polynomial of \( W \) is \((\lambda^2 - 3)(\lambda - (1 + 2t))\) which has three pairwise distinct roots, as \( p \neq 3 \). Let \( s \) be an element in \( GF(q) \) or in a quadratic extension \( GF(q^2) \) such that \( s^2 = 3 \). Then

\[ v_1 = (1, \frac{1}{2}(s - 1), \frac{1}{2}(s - 1)), \quad v_2 = (1, -\frac{1}{2}(s + 1), -\frac{1}{2}(s + 1)), \quad v_3 = (0, 1, -1) \]

are three independent eigenvectors of \( W \). For \( i = 1, 2, 3 \), let \( P_i \) be the point represented by \( v_i \). Then \( P_i \) are the fixed points of \( \bar{W} \) in \( PG(2, q) \) (or in \( PG(2, q^2) \) when \( s \in GF(q^2) \setminus GF(q) \)). The subgroup \( S_2 \) of \( \Gamma \) generated by \( V \) and \( W \) is a dihedral group of order 8. Since \( \bar{V} \) fixes \( P_3 \), this shows that \( S_2 \) is contained in the stabilizer of \( P_3 \) in the action of \( \Gamma \). But this is not consistent with the hypothesis on the 1-point stabilizer in Theorem 1. Further, \( \bar{V} \) interchanges the points \( P_1 \) and \( P_2 \). Therefore, the \( \Gamma \)-orbit of \( P_1 \) contains \( P_2 \). From the classification of subgroups of \( A_6 \), every proper subgroup of \( \Gamma \) containing \( W \) also contains \( V \). From this, the stabilizer of \( P_1 \) under the action of \( \Gamma \) is the group of order 4 generated by \( W \). So, from now on we may limit ourselves to consider the \( \Gamma \)-orbit \( \mathcal{O} \) of \( P_1 \). We stress that \( \mathcal{O} \) is in \( PG(2, q) \) (or in \( PG(2, q^2) \) when \( s \not\in GF(q) \)). The 90 points in \( \mathcal{O} \) can be computed as the images of the \( P_1 = (1, \frac{1}{2}(s - 1), \frac{1}{2}(s - 1)) \) by the projectivities in the list in (3) after replacing \( U, V, W, \Omega \) with \( \bar{U}, \bar{V}, \bar{W}, \bar{\Omega} \). These points are listed below.

\begin{align*}
(2, -s - 1, -s - 1); & \quad ((-12 * s - 12) * z + t + (-6 * s - 6) * z - 6 * s - 18, (6 * z + 6 * s) * t - 6 * z + 6 * s, (6 * z - 6 * s) * t + 12 * z); \\
((6 * s + 18) * z + 18 * s + 54) * t + (12 * s + 36) * z - 36 * s + 36, ((6 * s + 18) * z + 18 * s - 18) * t + (-6 * s - 18) * z + 18 * s + 54, (6 * s - 18) * z + 18 * s + 18) * t + (-24 * s - 36) * z - 36); \\
((2 * s + 6) * z + t + (s + 3) * z + 9 * s + 3, (2 * s + 6) * z + t + (s + 3) * z - 9 * s - 15, (2 * s + 6) * z + t + (s + 3) * z + 3 * s + 3); \\
((s + 3) * z + 9 * s + 3) * t + (2 * s + 6) * z + 6 * s - 6, ((-5 * s - 3) * z + 3 * s + 9) * t + (4 * s - 6) * z + 6, ((-2 * s - 6) * z - 12) * t + (-s - 3) * z - 3 * s - 9); \\
((2 * s + 6) * z - 6) * t + 2 * s * z + 6 * s + 12, (-2 * s - 6) * z * t + (s - 3) * z + 3 * s - 3); \\
\end{align*}
((−36 * s − 36) * z * t + (−18 * s − 18) * z + 90 * s + 162, (−36 * s − 36) * z * t + (−18 * s − 18) * z + 90 * s + 162, (36 * s + 36) * z * t + (18 * s − 18) * z + 18 * s − 18);
((−18 * s − 54) * z * t + (18 * s − 90) * t + (−18 * s − 54) * z + 90 * s − 18, (18 * s + 54) * z + (18 * s − 90) * t + 36 * s + 36, ((−54 * s − 54) * z − 18 * s − 90) * t + 36 * z − 36 * z − 72);
(−12 * s + 6 * s + 6 * s, ((−12 * s − 12) * z + t + (−6 * s + 6) * z − 6 * s − 18, −12 * s + t + 6 * z + 6 * s);
((36 * s + 36) * z * t + (18 * s + 18) * z + 18 * s − 54, (36 * s + 36) * t − 36 * z − 72 * s − 108, (36 * z − 36) * t + 72 * z − 108 * s + 108);
((6 * s + 6) * t + (6 * s − 6) * s + t + 12 * z, (−12 * s − 12) * z + t + (−6 * s − 6) * s + z − 6 * s − 18);
((−36 * s − 108) * t + (−36 * s − 108) * t, −72 * s) ;
(((−s − 3) * z + 3 * s − 3) * t + (−2 * s − 6) * z + 12, (−(s + 3) * z + (3 * s + 3)) * t + (−5 * s + 3) * z + 3 * s + 9, (−(s + 3) * z + (3 * s + 9)) * t + (s + 3) * z + 9 * s + 3);
((−4 * z − 6 * s − 6) * t + 2 * s + 2) * z * t + (s + 1) * z + s − 3, (−4 * z + (6 * s + 6)) * t + 2 * z + 4 * s + 6);
((−36 * s − 108) * z + (324 * s + 108)) * t + (−72 * s − 216) * z, ((−36 * s − 108) * z + (324 * s + 540)) * t + (36 * s + 108) * z + 324 * s + 540, (72 * s − 216) * z * t + (36 * s − 108) * z + 108 * s + 108);
((6 * s − 18) * z * t + (3 * s − 9) * z + 9 * s + 9, (−(3 * s − 9) * z + (27 * s + 45)) * t + (3 * s + 9) * z + 27 * s + 45, ((−3 * s − 9) * z + (27 * s + 9)) * t + (−6 * s + 18) * z);
((−72 * s + 72) * z * t + ((−36 * s + 36) * z + 36 * s + 108, (−(36 * s − 36) * z − 36 * s − 324) * t + (36 * s + 36) * z + 36 * s + 216, (−36 * s − 36) * z + 180 * s − 324) * t + (−72 * s − 72) * z);
((18 * s + 18) * z + 180 * s − 180, (18 * s + 18) * z + 36 * s + 54, (18 * s + 18) * z + 36 * s + 54) * t + ((−18 * s − 18) * z − 54 * s − 54) * t + (−18 * s − 18) * z − 54 * s − 54);
((−12 * s − 36) * z * t + ((−6 * s − 18) * z − 54 * s − 18, (16 * s − 18) * z − 18 * s − 18) * t + (12 * s + 36) * z, (6 * s + 18) * z + 54 * s + 90) * t + (−6 * s − 18) * z − 54 * s − 90);
((−36 * s + 108) * z + (108 * s + 108)) * t + (−180 * s − 108) * z + 108 * s + 324, (72 * s + 216) * z + 216 * s + 216) * t + (36 * s + 108) * z + 108 * s − 324, (72 * s + 216) * z + 3 + 324);
(((s + 3) * z + 3 * s − 9) * t + (2 * s + 6) * z − 12, ((s + 3) * z + (3 * s + 9)) * t + (2 * s + 6) * z − 6 * s + 6, (−5 * s + 3) * z + 3 * s − 9) * t + (−s + 3) * z − 3 * s − 3);
(((108 * s + 108) * z + 180 * s + 108) * t + 72 * s + 216, (144 * s + 432) * t + (108 * s + 108) * z + 180 * s − 108, (216 * s + (144 * s + 216)) * t + (−108 * s − 108) * z − 36 * s + 108);
((−4 * s + z + (6 * s + 18)) * t + 2 * s + z + 6 * s + 12, (−4 * s + z + 6 * s + 18) * t − 2 * s + z − 6, (2 * s + 6) * z + t + (s + 3) * z − 3 * s + 3);
((−6 * s + 6) * z * t + (−3 * s + 3) * z + 3 * s + 9, (−3 * s − 3) * z − 15 * s − 27) * t + (−6 * s − 6) * z, (−3 * s − 3) * z − 3 * s − 27) * t + (3 * s + 3) * z − 3 * s − 27);
((−36 * s − 108) * s + t, (36 * s + 108) * s + 108 * s + 108, 72 * s);
(((6 * s + 6) * t + (6 * s + 18) * t + (12 * s + 24) * z + 12 * s, ((−6 * s − 6) * z + 18 * s − 18) * t + (−12 * s − 12) * z + 12 * s + 36, ((−6 * s + 6) * z + (6 * s − 18)) * t + (6 * s + 6) * z − 18 * s − 18) * t + (18 * s + 15) * z + 9 * s + 9) * t + ((−3 * s + 3) * z + 3 * s + 9, (−3 * s − 3) * z + 9 * s + 9) * t + (−6 * s + 6) * z + 12 * s + 18, (72 * s + s − 72 * s − 288) * t + 72 * s + z + 144 * s − 360, (72 * s + 72 * s + t + (36 * s + 108) * z + 36 * s − 36 − 72 * s + 288) * t + 72 * s + z + 288;
(((3 * s + 15) * z + 9 * s + 9) * t + (6 * s + 12) * z + 6 * s, ((−3 * s − 3) * z − 9 * s − 9) * t + (3 * s + 3) * z + 3 * s + 9, (−3 * s + 3) * z + 9 * s + 9) * t + (3 * s + 3) * z + 3 * s + 9, (−3 * s − 3) * z − 9 * s − 9) * t + (−6 * s + 6) * z − 6 * s − 18, (−3 * s + 3) * z + (9 * s + 9) * t + (−6 * s + 6) * z + 12 * s);

Let \( \mathcal{O} = \{P_1, Q_1, \ldots, Q_{3916}\} \). The points \( P_1, Q_1 \) and \( Q_2 \) are collinear if and only if the determinant \( D_{i,j} \) of the coordinates of these points vanishes. There are 3916 triples \( \{P_1, Q_i, Q_j\} \) with \( 1 \leq i < j \leq 89 \). Observe that \( D_{i,j} \) can be viewed as a polynomial in \( t, s \) and \( z \), say \( D_{i,j}(t, s, z) \), with coefficients in \( \mathbb{Z} \). Therefore a necessary and sufficient condition for the points \( Q_i, Q_j \in \mathcal{O} \) to produce together with \( P_1 \) a collinear triple is that \( (t, s, z) \) be a solution of the
system of equations
\[
\begin{align*}
  t^2 + t + 1 &= 0; \\
  s^2 &= 3; \\
  z^2 &= 5; \\
  D_{i,j}(t, s, z) &= 0.
\end{align*}
\]

We look at the above system over \( \mathbb{Z} \) with unknowns \( t, s, z \) and use Sylvester's resultant to discuss solvability. Eliminating \( t \) from the first and the forth equations produces an equation in \( s, z \) over \( \mathbb{Z} \); then eliminating \( s \) from this and the second equation provides an equation in \( z \) over \( \mathbb{Z} \); finally eliminating \( z \) from this and the third equation gives an integer, the resultant of the system. A sufficient condition for a triple of points not to be collinear is that this resultant does not vanish in \( \mathbb{Z}_p \).

A computer aided search shows that such a resultant is a non zero integer for any \( \delta \) of the above 3916 cases. Now, let \( \delta \) vanish in \( \mathbb{Z} \) condition for a triple of points not to be collinear is that this resultant does not and the third equation gives an integer, the resultant of the system. A sufficient condition for a triple of points not to be collinear is that this resultant does not vanish in \( \mathbb{Z}_p \).

An exhaustive computer-aided computation shows that \( \delta \) has size 14, namely \( \delta = \{2, 3, 5, 7, 11, 13, 17, 19, 61, 109, 181, 229, 241, 421\} \). Therefore, the following result holds.

**Proposition 3.1.** The \( \Gamma \)-orbit \( \mathcal{O} \) of the point \( P_1 \) has length 90 and the stabilizer of \( P_1 \) in \( \Gamma \) is a cyclic group of order 4. Furthermore, \( \mathcal{O} \) is a 90-arc on \( \text{PG}(2, q) \) with \( q = p^h \) and \( p \geq 7 \) apart from finitely many values of \( p \) which are
\[
7, 11, 13, 17, 19, 61, 109, 181, 229, 241, 421.
\]

Now, we discuss the exceptional cases.

### 3.1 \( p = 7, 11, 13, 17 \)

In this case \( p^2 \equiv 1 \) or 19 (mod 30). Therefore, \( \mathcal{O} \) lies in \( \text{PG}(2, p^2) \setminus \text{PG}(2, p) \). By a MAGMA computation, some \( \delta_{i,j} \) is divisible by \( p \). Hence \( \mathcal{O} \) is not an arc. Some more effort allows to compute the intersection numbers of \( \mathcal{O} \) with lines. The results are reported below.

- For \( q = 7^2 \) a square root of 5 is \( w^{20} \), where \( w \) is a primitive element of \( GF(7^2) \) such that \( w^2 + 6w + 3 = 0 \). In this case \( \mathcal{O} \) is a complete \( (90, 4) \)-arc with 336 external lines, 810 tangents, 765 bi-secants, 540 four-secants.

- For \( q = 11^2 \) a primitive cubic root of unity is \( w^{40} \), where \( w \) is a primitive element of \( GF(11^2) \). In this case \( \mathcal{O} \) is a non-complete \( (90, 5) \)-arc with 7248 external lines, 4320 tangents, 3105 bi-secants, 90 five-secants.

- For \( q = 13^2 \) a square root of 5 is \( w^{63} \), where \( w \) is a primitive element of \( GF(13^2) \) such that \( w^2 + 12w + 2 = 0 \). In this case \( \mathcal{O} \) is a non-complete \( (90, 4) \)-arc with 16896 external lines, 8730 tangents, 2925 bi-secants, 180 four-secants.

- For \( q = 17^2 \) a primitive cubic root of unity is \( w^{88} \) and a square root of 5 is \( w^{45} \), where \( w \) is a primitive element of \( GF(17^2) \) such that \( w^2 + 16w + 3 = 0 \). In this case \( \mathcal{O} \) is a non-complete \( (90, 3) \)-arc with 61356 external lines, 19170 tangents, 2925 bi-secants, 360 three-secants.
3.2 \( p = 19 \)

\( \Gamma \) is a projectivity group of \( PG(2, 19) \). However, \( s \in GF(19^2) \setminus GF(19) \), whence \( O \) lies in \( PG(2, 19^2) \setminus PG(2, 19) \). Furthermore, \( O \) is a non-complete \((90, 5)\)—arc with 101676 external lines, 25650 tangents, 3285 bi-secants, 72 five-secants. Here, \( s = w^{130} \) where \( w^2 + 18w + 2 = 0 \).

3.3 \( p = 61, 109, 181, 229, 241, 421 \)

In this case, \( \Gamma \) is a projective group of \( PG(2, p) \) and \( s \in GF(p) \). Therefore \( O \) lies in \( PG(2, p) \). Again, some \( \delta_{i,j} \) is divisible by \( p \), and \( O \) is not an arc. By a MAGMA computation, the intersection numbers of \( O \) lines can be computed, and the results are reported below.

- For \( p = 61 \) \( O \) is a non-complete \((90, 6)\)—arc with 1068 external lines, 450 tangents, 2025 bi-secants, 180 four-secants, 60 six-secants.
- For \( p = 109 \) \( O \) is a non-complete \((90, 3)\)—arc with 5736 external lines, 2970 tangents, 2925 bi-secants, 360 three-secants.
- For \( p = 181 \) \( O \) is a non-complete \((90, 3)\)—arc with 20208 external lines, 9450 tangents, 2925 bi-secants, 360 three-secants.
- For \( p = 229 \) \( O \) is a non-complete \((90, 4)\)—arc with 35436 external lines, 14130 tangents, 2925 bi-secants, 180 four-secants.
- For \( p = 241 \) \( O \) is a non-complete \((90, 4)\)—arc with 40008 external lines, 15210 tangents, 2925 bi-secants, 180 four-secants.
- For \( p = 421 \) \( O \) is a non-complete \((90, 3)\)—arc with 143328 external lines, 31050 tangents, 2925 bi-secants, 360 three-secants.

The results of the present section provide a proof of Theorem 1.1.

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