Optimal reinsurance and dividends with transaction costs and taxes under thinning structure

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\textbf{ABSTRACT}
In this paper, we investigate the problem of optimal reinsurance and dividends under the Cramér–Lundberg risk model with the thinning-dependence structure which was first introduced by Wang and Yuen (Wang, G. & Yuen, K. C. (2005). On a correlated aggregate claims model with thinning-dependence structure. Insurance: Mathematics and Economics 36(3), 456–468). The optimization criterion is to maximize the expected accumulated discounted dividends paid until ruin. To enhance the practical relevance of the optimal dividend and reinsurance problem, non-cheap reinsurance is considered and transaction costs and taxes are imposed on dividends. These realistic features convert our optimization problem into a mixed classical-impulse control problem. For the sake of mathematical tractability, we replace the Cramér–Lundberg risk model by its diffusion approximation. Using the method of quasi-variational inequalities, we show that the optimal reinsurance follows a two-dimensional excess-of-loss reinsurance strategy, and the optimal dividend strategy turns out to be an impulse dividend strategy with an upper and a lower barrier, i.e. everything above the lower barrier is paid as dividends whenever the surplus goes beyond the upper barrier, and no dividends are paid otherwise. Under the diffusion risk model, closed-form expressions for the value function associated with the optimal dividend and reinsurance strategy are derived. In addition, some numerical examples are presented to illustrate the optimality results.

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1. Introduction
As for listed insurance companies, the distribution of dividends is a main approach to share profits with policy holders, while the purchase of reinsurance is an effective way to reduce risk exposure. Due to the importance of these two features, risk models with reinsurance and dividend payments have received great attention in the actuarial literature in the past few decades.

In the actuarial literature, the optimal dividend problem has been studied for various risk models. For example, Jeanblanc-Picqué & Shiryaev (1995), Asmussen & Takasar (1997), Paulsen (2003), Gerber & Shiu (2004), Løkka & Zervos (2008), Vierkötter & Schmidli (2017), Zhu (2017) considered the optimal dividend problem in the diffusion model; Højgaard (2002), Schmidli (2006), Gerber & Shiu (2006), Albrecher & Thonhauser (2008), Azcue & Muler (2012) derived the optimal dividend strategies under the Cramér–Lundberg model; and some recent research on optimal dividends for the
jump-diffusion model and the Lévy risk model can be found in Avram et al. (2007, 2015), Kyprianou & Palmowski (2007), Loeffen (2008, 2009), Loeffen & Renaud (2010), Czarna & Palmowski (2010), Wang & Hu (2012), Hunting & Paulsen (2013), Hernandez & Junca (2015), Zhao et al. (2017), Pérez et al. (2018), Wang et al. (2018), Wang & Zhou (2018), and Wang & Zhang (2019). For a comprehensive review of optimality results for dividend problems, we refer the reader to the survey paper by Albrecher & Thonhauser (2009) and references therein. Moreover, many existing results indicate that some well-known dividend strategies often turn out to be optimal. For instance, in the diffusion set-up, Højgaard & Taksar (1999) showed that the optimal dividend strategy is a threshold strategy if the rate of dividend payout is bounded by some positive constant, and that the optimal strategy is a barrier strategy when there is no restriction on the rate of dividend payout; and Paulsen (2007, 2008) showed that the optimal dividend strategy is usually an impulse strategy when transaction costs on dividends are included.

Besides dividends, reinsurance is another important control variable that has been studied extensively in the actuarial literature. For the diffusion model with proportional reinsurance control, the optimal dividend problem without transaction costs has been investigated by many authors including Højgaard & Taksar (1999), He & Liang (2008), Chen et al. (2013), Peng et al. (2016), Liang & Palmowski (2018), while the optimal dividend problem with transaction costs has been examined by Wei et al. (2010), Peng et al. (2012), Yao et al. (2014, 2016), Chen & Yuen (2016), and Yao & Fan (2018). For excess-of-loss reinsurance in the diffusion setup, Asmussen et al. (2000), Wu (2013), Liu & Hu (2014) considered the optimal dividend problem without transaction costs, while Bai et al. (2010), Choulli & Taksar (2010), Cheng et al. (2018) studied the one with transaction costs. Mnif & Sulem (2005) investigated the optimal excess-of-loss reinsurance policy and dividend distribution in the compound Poisson model. In the presence of both proportional reinsurance and excess-of-loss reinsurance, Meng & Siu (2011) considered the optimal dividend problem with transaction costs in the diffusion model and Azcue & Muler (2005) obtained the optimal dividend distribution strategy for the Cramér–Lundberg model. A short survey of stochastic models of risk control and dividend optimization techniques for insurance companies can be found in Taksar (2000).

Although most of the research in this direction mainly deals with independent risks, much attention has been paid to the optimization problems in relation to dependent risks in recent years. For the risk model with common shock dependence, Bai et al. (2013) derived the optimal excess-of-loss reinsurance strategies that minimize ruin probability; Yuen et al. (2015), Liang & Yuen (2016) considered the optimal proportional reinsurance strategy under the criterion of maximizing the expected exponential utility; and Zhang & Liang (2017) studied the problem of portfolio optimization for jump-diffusion risky assets with common shock dependence and state-dependent risk aversion. In recent years, this kind of optimality study has been extended to the risk model with the thinning-dependence structure proposed by Wang & Yuen (2005) which embraces the common shock risk model. Such a generalization undoubtedly makes the problem of study more complicated and challenging. For example, under the thinning dependence, Han et al. (2018) used the technique of HJB equation to investigate the optimal proportional reinsurance problem that minimizes the probability of drawdown in the Brownian motion case; and Wei et al. (2018) derived the optimal proportional reinsurance strategy in the compound Poisson case under the criterion of maximizing the adjustment coefficient.

In this paper, the problem of optimal dividend payment and reinsurance under the thinning-dependence structure is studied. We adopt the expected value premium principle and take into account transaction costs and taxes on dividend payments. In order to make our problem mathematically tractable so as to obtain explicit expressions for the optimal dividend and reinsurance strategy and its associated value function, we consider the diffusion approximation of the thinning risk model with two dependent classes of insurance business. In this setup, we show that the optimal reinsurance does not have the form of proportional reinsurance strategy that was studied in Han et al. (2018) and Wei et al. (2018), but follows the excess-of-loss reinsurance strategy. Since fixed transaction costs on dividends are considered, the optimization problem becomes a mixed classical-impulse stochastic
control problem, and hence the methods used in Han et al. (2018) and Wei et al. (2018) cannot be applied. By the method of quasi-variational inequalities (QVI), closed-form expressions for the value function and the corresponding optimal excess-of-loss reinsurance and impulse dividend strategy are derived.

Although there are a lot of optimality studies taking both dividend and reinsurance into account, research with both controls under dependence is still fairly scarce. To the best of our knowledge, Li et al. (2016) is the only paper that studies the problem of optimal dividend and reinsurance with common shock dependence so far. Since the structure of thinning dependence is more general than that of common shock dependence, we extend the work of Li et al. (2016) to a risk model with thinning dependence. Furthermore, Li et al. (2016) studied a classical control problem with no transaction costs and taxes, while the inclusion of transaction costs and taxes in this paper converts the optimal problem to an impulse control problem. For the optimal dividend and reinsurance problem without dependent risks, the reinsurance strategy often turns out to be a one-dimensional excess-of-loss reinsurance strategy. For example, see Bai et al. (2010) and Choulli & Taksar (2010). With dependent classes, the optimal reinsurance strategy in this paper is a two-dimensional excess-of-loss reinsurance strategy, and the two individual reinsurance strategies have a close relationship that complicates the problem very much. It is also worth mentioning that our optimal reinsurance strategy dominates all admissible reinsurance strategies, while Li et al. (2016) characterized the optimal reinsurance strategy only among the sub-class of excess-of-loss reinsurance strategies.

In order to determine the optimal two-dimensional reinsurance strategy explicitly, we need to define three auxiliary functions and analyze two zeros associated with these auxiliary functions. The optimal dividend and reinsurance control problem is then solved by considering two opposite scenarios of the relationship between the two zeros. The rest of this paper is organized as follows. In Section 2, the model and mathematical formulation of the problem are introduced. In Section 3, we show that the excess-of-loss reinsurance strategy is the optimal reinsurance form for our optimization problem. In Section 4, the QVI and verification theorem are presented. Section 5 is devoted to the derivation of the solution to the QVI. The value function and the optimal strategy are given in Section 6. Finally, some numerical examples are provided in Section 7.

2. The model

We assume that all stochastic quantities are defined on a large enough complete probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\), where the filtration \(\mathcal{F}_t\) represents the information available at time \(t\), and any decision made is based on this information.

The thinning-dependence structure considered in this paper was first introduced by Wang & Yuen (2005). Suppose that an insurance company has a portfolio of \(n (n \geq 2)\) dependent classes of insurance business, and the stochastic sources that may cause a claim in at least one of the classes are classified into \(m\) groups. It is assumed that each event occurred in the \(k\)th group may cause a claim in the \(l\)th class with probability \(p_{kl}\) for \(k = 1, 2, \ldots, m\) and \(l = 1, 2, \ldots, n\), and that for each \(l\), there exists at least some \(k\) such that \(p_{kl} > 0\). For the \(k\)th group, let \(N^k(t)\) be the number of events occurred up to time \(t\), and \(N_l^k(t)\) be the number of claims of the \(l\)th class up to time \(t\) generated from the events in group \(k\). For the \(l\)th class, let \(X^{(l)}_i\) (\(i = 1, 2, \ldots\)) be the claim size random variables following a common distribution \(F_l\) (corresponding to a random variable \(X_l\)), and denote by \(\mu_l\) and \(\sigma^2_l\) the mean and variance of the distribution \(F_l\), respectively. Then the aggregate claims process of the company is given by

\[
S(t) = \sum_{l=1}^n N_l(t) \sum_{i=1}^{N_l(t)} X^{(l)}_i,
\]

where \(\{X^{(l)}_i; i = 1, 2, \ldots\}\) is a sequence of i.i.d. non-negative random variables for each \(l\), and \(N_l(t) = N^1_l(t) + N^2_l(t) + \cdots + N^m_l(t)\) is the claim-number process of the \(l\)th class. As usual,
we assume that the processes $N^1(t), \ldots, N^m(t)$ are independent Poisson processes with parameters $\lambda_1, \ldots, \lambda_m$, respectively. Furthermore, for $k \neq j$, the two vectors of claim-number processes, $(N^k(t), N^j_1(t), \ldots, N^j_N(t))$ and $(N^j(t), N^j_1(t), \ldots, N^j_N(t))$ are independent; and for each $k$, $N^k_1(t), \ldots, N^k_N(t)$ are conditionally independent given $N^k(t)$. Also, we assume that the $n$ sequences $\{X_i^{(1)}; i = 1, 2, \ldots\}, \ldots, \{X_i^{(n)}; i = 1, 2, \ldots\}$ are mutually independent and are independent of all the claim-number processes.

The reserve process of the insurer without reinsurance is given by

$$U_t = x + ct - S(t),$$

where $x \geq 0$ is the initial reserve and $c > 0$ is the premium rate. In order to manage the underlying insurance risk properly, the insurer would like to buy reinsurance to alleviate the impact of large losses. Suppose that the reinsurance strategy for the $l$th class is $q_l$ (not time-varying) with $0 \leq q_l(x) \leq x$ for $x \geq 0$ and $l = 1, 2, \ldots, n$, and the reinsurance premium rate is denoted by $\delta(q)$ with $q = (q_1, q_2, \ldots, q_n)$. Then the reserve process after reinsurance can be written as

$$U^q_t = x + [c - \delta(q)]t - S^q(t),$$

where

$$S^q(t) = \sum_{l=1}^n \sum_{i=1}^n q_l(X_i^{(l)}).$$

Similar to Wang & Yuen (2005), we know that $S^q(t)$ follows a compound Poisson process with

$$E[S^q(t)] = \sum_{l=1}^n \sum_{i=1}^m \lambda_{ki} q_l(X_i^{(l)}) t,$$

$$Var[S^q(t)] = \sum_{l=1}^n \sum_{i=1}^m \lambda_{ki} q_l(X_i^{(l)})^2 t + \sum_{l=1}^n \sum_{j \neq l}^n \sum_{i=1}^m \sum_{j=1}^m \sum_{j \neq l}^m \lambda_{kj} q_l(X_i^{(l)}) q_j(X_i^{(j)}) \sum_{k=1}^m \lambda_{kj} q_l(X_i^{(l)}) \sum_{k=1}^m \lambda_{kj} q_l(X_i^{(l)}).$$

Then $U^q_t$ can be approximated by a pure diffusion $X^q_t$, which is given by

$$X^q_t = x + [c - \delta(q) - a(q)]t + b(q) W_t,$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion and

$$a(q) = \sum_{l=1}^n \sum_{i=1}^m \lambda_{ki} q_l(X_i^{(l)}) t,$$

$$b^2(q) = \sum_{l=1}^n \sum_{i=1}^m \lambda_{ki} q_l(X_i^{(l)})^2 t + \sum_{l=1}^n \sum_{j \neq l}^n \sum_{i=1}^m \sum_{j=1}^m \sum_{j \neq l}^m \lambda_{kj} q_l(X_i^{(l)}) q_j(X_i^{(j)}) \sum_{k=1}^m \lambda_{kj} q_l(X_i^{(l)}) \sum_{k=1}^m \lambda_{kj} q_l(X_i^{(l)}).$$

From now on, we assume that $q$ changes with time. Besides, the insurer can control the reserves by paying out dividends with both transaction costs and taxes. That is, there will be a fixed transaction cost $K > 0$ and a tax rate $1 - k$ ($0 < k < 1$) when the dividends are paid out. A strategy is described...
by

\[ \alpha = (\mathbf{q}; \tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots), \]

where \( \tau_n \) and \( \xi_n \) denote the times and amounts of dividends. The controlled surplus process with strategy \( \alpha \) is given by

\[ X^\alpha_t = x + \int_0^t [c - \delta(\mathbf{q}_s) - a(\mathbf{q}_s)] \, ds + \int_0^t b(\mathbf{q}_s) \, dW_s - \sum_{n=1}^{\infty} I_{(\tau_n < t)} \xi_n, \]  

and the corresponding ruin time is defined as

\[ \tau^{\alpha} = \inf\{ t \geq 0 : X^\alpha_t < 0 \}. \]

**Definition 2.1:** A strategy \( \alpha \) is said to be admissible if

(i) \( q_l \) (\( l = 1, 2, \ldots, n \)) are \( \mathcal{F}_t \)-adapted processes with \( 0 \leq q_l(x) \leq x \) for all \( x \geq 0 \) and \( t \geq 0 \).

(ii) \( \tau_n \) is a stopping time with respect to \( \{\mathcal{F}_t\}_{t \geq 0} \) and \( 0 \leq \tau_1 < \tau_2 < \cdots < \tau_n < \cdots \) a.s.

(iii) \( \xi_n \) is measurable with respect to \( \mathcal{F}_{\tau_n} \) and \( 0 < \xi_n \leq X^\alpha_{\tau_n}, n = 1, 2, \ldots \).

(iv) \( P(\lim_{n \to \infty} \tau_n \leq T) = 0 \), for all \( T \geq 0 \).

Denoted by \( \Pi \) the set of all admissible control strategies. For a given admissible strategy \( \alpha \), we define the performance function (or value function) as

\[ V_\alpha(x) = E \left[ \sum_{n=1}^{\infty} e^{-\delta \tau_n} (k_x - K)I_{(\tau_n < \tau^{\alpha})} \mid X_{0-} = x \right] = E_x \left[ \sum_{n=1}^{\infty} e^{-\delta \tau_n} (k_x - K)I_{(\tau_n < \tau^{\alpha})} \right], \]

which represents the expected total discounted dividends received by the shareholders until the ruin time when the initial surplus is \( x \), where \( \delta > 0 \) is a-priori given discount factor. Our aim is to find the optimal performance function given by

\[ V(x) = \sup_{\alpha \in \Pi} V_\alpha(x), \]  

and to find the optimal strategy \( \alpha^* \) such that \( V(x) = V_{\alpha^*}(x) \) for all \( x \geq 0 \).

In this paper, we assume that the premium is calculated according to the expected value premium principle. For the \( l \)th (\( l = 1, 2, \ldots, n \)) class of insurance business, the positive safety loading for the insurer and reinsurer are \( \eta_l \) and \( \theta_l \), respectively. Non-cheap reinsurance is considered, that is, \( \theta_l > \eta_l \).

In order to derive closed-form expressions for the value function \( V(x) \) and the corresponding optimal strategy \( \alpha^* \), we consider the case of \( n = 2 \) only. Let

\[ c_l = \sum_{k=1}^{m} \lambda_k p_{kl}, \quad l = 1, 2, \]  

and \( c_3 = \sum_{k=1}^{m} \lambda_k p_{k1} p_{k2} \).

Then we have

\[ c = \sum_{l=1}^{2} c_l \mu_l (1 + \eta_l), \quad \delta(\mathbf{q}) = \sum_{l=1}^{2} c_l (\mu_l - E[q_l(X^{(l)})])(1 + \theta_l), \]

\[ d(\mathbf{q}) \triangleq c - \delta(\mathbf{q}) - a(\mathbf{q}) = \sum_{l=1}^{2} c_l (\theta_l E[q_l(X^{(l)})] - (\theta_l - \eta_l) \mu_l), \]

\[ b^2(\mathbf{q}) = \sum_{l=1}^{2} c_l E[q_l(X^{(l)})]^2 + 2c_3 E[q_1(X^{(1)})] E[q_2(X^{(2)})]. \]  

(3)
3. The optimal reinsurance form

There exists a variety of reinsurance forms in the literature, such as proportional reinsurance, excess-of-loss reinsurance, stop-loss reinsurance, and so on. In this section, we show that the excess-of-loss reinsurance strategy is the optimal reinsurance form for the problem of study.

Lemma 3.1: For any admissible strategy \( \alpha = (q_1, q_2; \tau_1, \ldots, \tau_m; \xi_1, \ldots, \xi_m, \ldots) \), there exists an admissible strategy \( \alpha^e = (q_1^e, q_2^e; \tau_1, \ldots, \tau_m; \xi_1, \ldots, \xi_m, \ldots) \) such that \( V_\alpha(x) \leq V_{\alpha^e}(x) \), where \( (q_1^e, q_2^e) \) is a two-dimensional excess-of-loss reinsurance strategy.

Proof: Similar to the proof of Proposition 2.1 of Bai et al. (2013), we know that for any one-dimensional reinsurance strategy \( q(\cdot) \) with \( 0 \leq q(x) \leq x \) for \( x \geq 0 \) and a nonnegative random variable \( Z \), there exists an excess-of-loss reinsurance strategy \( q^e(\cdot) = \min[\cdot, m] \) with a retention level \( 0 \leq m \leq \infty \) such that

\[
E[q^e(Z)] = E[q(Z)], \quad E[q^e(Z)]^2 \leq E[q(Z)]^2.
\]

Then, for any \( q = (q_1(\cdot), q_2(\cdot)) \), it follows from (3) that, there exists \( 0 \leq m_i \leq \infty, i = 1, 2 \) and \( q^e = (q_1^e(\cdot), q_2^e(\cdot)) = (\min[\cdot, m_1], \min[\cdot, m_2]) \) such that

\[
d(q^e) = d(q), \quad b^2(q^e) \leq b^2(q).
\]

On the other hand, it is easy to see that both \( d(q^e) \) and \( b^2(q^e) \) are increasing with respect to \( m_1 \) and \( m_2 \), and that \( \lim_{(m_1, m_2) \to (\infty, \infty)} b^2(q^e) \geq b^2(q) \). As a result, there exists \( m'_i \geq m_i, i = 1, 2 \), and \( q^e = (\min[\cdot, m'_1], \min[\cdot, m'_2]) \) such that

\[
d(q^e) \geq d(q), \quad b^2(q^e) = b^2(q).
\]

By (1), we have \( X^a_t \leq X^{a^e}_t \). This implies \( \tau^a \leq \tau^{a^e} \), which in turn yields \( V_\alpha(x) \leq V_{\alpha^e}(x) \).

Due to Lemma 3.1, we only consider the excess-of-loss reinsurance in the rest of this paper. For notational convenience, we define the following functions:

\[
g_l(q) = E(X^{(l)} \wedge q) = \int_0^q \tilde{F}_l(x) \, dx, \quad l = 1, 2,
\]

\[
G_l(q) = E(X^{(l)} \wedge q)^2 = \int_0^q 2x\tilde{F}_l(x) \, dx, \quad l = 1, 2,
\]

where \( q \in [0, \infty] \) and \( \tilde{F}_l(x) = 1 - F_l(x) = P(X^{(l)} > x) \). Then we have

\[
d(q) = \sum_{l=1}^2 c_l[\theta_l g_l(q_l) - (\theta_l - \eta_l) \mu_l],
\]

\[
b^2(q) = \sum_{l=1}^2 c_l G_l(q_l) + 2c_3 g_1(q_1)g_2(q_2), \tag{4}
\]

where \( q = (q_1, q_2) \) with \( 0 \leq q_l \leq \infty \).
4. QVI and verification theorem

Since the optimal control problem (2) is a mixed classical-impulse stochastic control problem, we deal with it by the method of quasi-variational inequalities (QVI).

For a function $\phi : [0, \infty) \mapsto [0, \infty)$, we define the maximum operator $\mathcal{M}$ by

$$\mathcal{M}\phi(x) := \sup\{\phi(x - \eta) + k\eta - K : 0 < \eta \leq x\},$$

and the operator $\mathcal{L}^q$ by

$$\mathcal{L}^q\phi(x) := \frac{1}{2} b^2(q)\phi''(x) + d(q)\phi'(x).$$

Similar to Chen & Yuen (2016), if the value function of (2) is sufficiently smooth, then it satisfies the following QVI:

$$\max \left\{ \max_{0 \leq q_1, q_2 \leq \infty} \mathcal{L}^q V(x) - \delta V(x), \mathcal{M} V(x) - V(x) \right\} = 0, \quad x > 0,$$

with boundary condition $V(0) = 0$. Some economic and intuitive insight of (5) is given in the following remark.

**Remark 4.1:** For $x > 0$, if $\mathcal{M} V(x) = V(x)$, it is optimal for the insurer to pay lump sum dividends rather than buying reinsurance; and if $\mathcal{M} V(x) < V(x)$, it is optimal for the insurer to buy reinsurance to reduce the potential risk rather than paying dividends.

Furthermore, given a solution $v(x)$ to (5), we can construct the following Markov control strategy.

**Definition 4.1:** The strategy $\alpha^v = (q^v_1, q^v_2; \tau^v_1, \tau^v_2, \ldots; \tau^v_n, \ldots; \xi^v_1, \xi^v_2, \ldots, \xi^v_n, \ldots)$ is called the QVI strategy associated with $v$ if the associated process $X^v$ given by (1) with $x \geq 0$ satisfies

$$(q^v_1, q^v_2) = \arg \max_{0 \leq q_1, q_2 \leq \infty} \mathcal{L}^q v(X^v_t) \text{on}\{v(X^v_t) > \mathcal{M} v(X^v_t)\},$$

$$\tau^v_1 = \inf\{t \geq 0 : v(X^v_t) = \mathcal{M} v(X^v_t)\},$$

$$\xi^v_1 = \arg \sup_{0 < \eta \leq X^v_{\tau^v_1}} \{v(X^v_{\tau^v_1} - \eta) + k\eta - K\},$$

and for every $n \geq 2$,

$$\tau^v_n = \inf\{t > \tau^v_{n-1} : v(X^v_t) = \mathcal{M} v(X^v_t)\},$$

$$\xi^v_n = \arg \sup_{0 < \eta \leq X^v_{\tau^v_n}} \{v(X^v_{\tau^v_n} - \eta) + k\eta - K\}.$$

Mimicking the proof of Theorem 3.2 in Chen & Yuen (2016), one can prove the following verification theorem.

**Theorem 4.2 (Verification Theorem):** Let $v(x) \in C^1([0, \infty))$ be a solution to (5) at all the points with the possible exception of some point where the second derivative may not exist. Suppose there exists $U > 0$ such that $v(x)$ is twice continuously differentiable on $(0, U)$ and $v(x)$ is linear on $[U, \infty)$. Then $V(x) \leq v(x), \ x \geq 0$. Furthermore, if the QVI strategy $\alpha^v$ associated with $v(x)$ is admissible, then $v(x)$ coincides with the value function $V(x)$ and $\alpha^v$ is the optimal strategy, i.e., $V(x) = v(x) = V_{\alpha^v}(x), \ x \geq 0.$
5. Solution to QVI

Inspired by Theorem 4.2, we first assume that there exists a strictly increasing solution $W(x)$ to (5) which is continuously differentiable on $(0, \infty)$ and twice continuously differentiable on $(0, x_1)$, where $x_1 = \inf\{x \geq 0 : MV(x) = V(x)\}$ (all of these will be proved later). Then (5) with $V$ replaced by $W$ for $0 \leq x < x_1$ can be rewritten as

$$\max_{0 \leq q_1, q_2 \leq \infty} \left\{ \frac{1}{2} b^2(q) W''(x) + d(q) W'(x) - \delta W(x) \right\} = 0. \tag{6}$$

Let $q_1(x)$ and $q_2(x)$ be the maximizer of the left-hand side of (6). Assume that $q_1(x)$ and $q_2(x)$ fall in the interval $(0, \infty)$. Differentiating (6) with respect to $q_1$ and $q_2$ respectively, we obtain

$$- \frac{W''(x)}{W'(x)} = \frac{c_1 \theta_1}{c_1 q_1(x) + c_3 g_2[q_2(x)]}, \tag{7}$$

$$- \frac{W''(x)}{W'(x)} = \frac{c_2 \theta_2}{c_2 q_2(x) + c_3 g_1[q_1(x)]}. \tag{8}$$

It follows that

$$\theta_2 q_1(x) - \frac{c_3}{c_2} \theta_1 g_1[q_1(x)] = \theta_1 q_2(x) - \frac{c_3}{c_1} \theta_2 g_2[q_2(x)]. \tag{9}$$

Let

$$l_1(q) = \theta_2 q - \frac{c_3}{c_2} \theta_1 g_1(q), \quad l_2(q) = \theta_1 q - \frac{c_3}{c_1} \theta_2 g_2(q), \quad q \geq 0.$$

Without loss of generality, we assume that $\theta_1 \geq \theta_2$. We further assume that $0 = F_l(0) < F_l(x) < 1$ for $x > 0$ and $l = 1, 2$. Then it is easy to see that $l_2(q)$ is strictly increasing on $[0, \infty)$, so the inverse function $l_2^{-1}(q)$ exists. By (9), we have $q_2(x) = l_2^{-1}[l_1(q_1(x))] \geq 0$ if $l_1(q_1(x)) \geq 0$. Let

$$z_l = \sup\{x \geq 0 : l_1(x) = 0\}.$$

It is easy to see that $0 \leq z_l < \infty$ and $l_1(q) \leq 0$ for $q \leq z_l$ since $l_2'(q) \geq 0$. Naturally, we need to find some $q_1(x) \geq z_l$ to guarantee that $q_2(x) \geq 0$ for $x \geq 0$.

Substituting (7) into (6) and replacing $q_2(x)$ with $l_2^{-1}[l_1(q_1(x))]$, we obtain

$$H(q_1(x)) W'(x) - \delta W(x) = 0, \tag{10}$$

where

$$H(q) = \sum_{l=1}^{2} c_l (\eta_l - \theta_l) \mu_l + c_1 \theta_1 g_1(q) + c_2 \theta_2 g_2[l_2^{-1} l_1(q)]$$

$$- \frac{\theta_1 c_1 G_1(q) + c_2 G_2[l_2^{-1} l_1(q)] + 2 c_3 g_1(q) g_2[l_2^{-1} l_1(q)]}{q + \frac{c_3}{c_2} g_2[l_2^{-1} l_1(q)]}. \tag{11}$$

In view of $W(0) = 0$ and (10), we see that $H(q_1(0)) = 0$. So we should discuss the existence of the solution to $H(q) = 0$. Now we define an auxiliary following function:

$$k(x) = c_1 \theta_1 \left[ g_1(x) - \frac{G_1(x)}{2x} \right] + k_0,$$

where $k_0 = \sum_{l=1}^{2} c_l (\eta_l - \theta_l) \mu_l < 0$. Since $k'(x) = c_1 \theta_1 \frac{G_1(x)}{2x^2} > 0$ for all $x > 0$, the inverse function $k^{-1}(x)$ exists. Note that $k(0+) = k_0 < 0$ and $k(\infty) = c_1 \eta_1 \mu_1 + c_2 (\eta_2 - \theta_2) \mu_2$. Define the zero of
k(x) as
\[ z_k = \begin{cases} 
    k^{-1}(0), & \theta_2 \leq \eta_2 + \frac{c_1 \mu_1 \eta_1}{c_2 \mu_2}, \\
    \infty, & \text{otherwise}. 
\end{cases} \]

**Lemma 5.1:** There exists a unique solution \( q_0 \) to \( H(q) = 0 \) on \([z_l, \infty)\) if and only if \( z_l \leq z_k \). Furthermore, we have \( q_0 > 0 \) if it exists.

**Proof:** By some direct calculation, one can show that for \( q \geq z_l \),
\[ H'(q) = \frac{\theta_1}{2} [c_1 g_1(q) + c_2 g_2[l_2^{-1} l_1(q)] + 2 c_3 g_1(q) g_2[l_2^{-1} l_1(q)]] \times \frac{c_1 + c_3 F_2[l_2^{-1} l_1(q)](l_2^{-1} l_1)'(q)}{(c_1 q + c_3 g_2[l_2^{-1} l_1(q)])^2}. \]

On the other hand, for \( q > z_l \), we have
\[ 0 < l_1(q) = \theta_2 q - \frac{c_2}{c_2} \theta_1 g_1(q) \leq |\theta_2 - \frac{c_2}{c_2} \theta_1 F_1(q)| q = l_1'(q) q, \]
which implies that \( l_1'(q) > 0 \). As a result, we get \( H'(q) > 0 \) for \( q > z_l \), which in turn implies that \( H(q) \) is strictly increasing on \([z_l, \infty)\). Since \( l_1(z_l) = 0 \), we have
\[ H(z_l) = k_0 + c_1 \left[ \theta_1 g_1(z_l) - \frac{c_2 \theta_2 G_1(z_l)}{2 c_3 g_1(z_l)} \right] = k(z_l). \]

Besides, we note that \( H(\infty) = \sum_{i=1}^{2} c_i \eta_i \mu_i > 0 \) and \( k(x) \) is strictly increasing. It is easy to see that there exists a unique solution \( q_0 \) to \( H(q) = 0 \) on \([z_l, \infty)\) if and only if \( z_l \leq z_k \).

Furthermore, we have \( q_0 = z_l \) if \( z_l = z_k \) and \( q_0 > z_l \) if \( z_l < z_k \). Note that \( z_l \geq 0 \) and \( z_k > 0 \). Then we obtain \( q_0 > 0 \).

According to Lemma 5.1, we consider the problem in two cases: (1) \( z_l \leq z_k \) and (2) \( z_l > z_k \).

### 5.1. Case 1: \( z_l \leq z_k \)

In this case, it follows from Lemma 5.1 and (10) that \( q_1(0) = q_0 \). Furthermore, differentiating (10) with respect to \( x \), we have
\[ [H'(q_1(x)) q_1'(x) - \delta] W'(x) + H(q_1(x)) W''(x) = 0. \]

Using (7) and \( q_2(x) = l_2^{-1}[l_1(q_1(x))] \) once again, we obtain
\[ W'(x) \left\{ H'(q_1(x)) q_1'(x) - \delta - H(q_1(x)) \frac{c_1 \theta_1}{c_1 q_1(x) + c_3 g_2[l_2^{-1} l_1(q_1(x))]} \right\} = 0. \]

Since \( W'(x) > 0 \), (14) gives
\[ q_1'(x) = \frac{\delta + H(q_1(x))}{H'(q_1(x))} \frac{c_1 \theta_1}{c_1 q_1(x) + c_3 g_2[l_2^{-1} l_1(q_1(x))]} . \]

(15)
Let
\[ G(q) = \int_{q_0}^{q} \frac{H'(y)}{\delta + H(y) \frac{c_1 \theta_1}{c_1 y + c_3 \theta_2 [l_2^{-1} l_1'(y)]}} \, dy, \quad q \geq q_0. \tag{16} \]

Since the integrand on the right-hand side of (16) is positive on \([q_0, \infty)\), we see that \(G(q)\) is increasing on \([q_0, \infty)\), and hence the inverse of \(G(q)\) exists on \([q_0, \infty)\). As a result, we have
\[ q_1(x) = G^{-1}(x), \quad q_2(x) = l_2^{-1} [l_1(G^{-1}(x))]. \]

**Lemma 5.2:** Let \(G(q)\) be given by (16). Then we have \(G(\infty) < \infty\), which implies that there exists \(x_0 = G(\infty) < \infty\) such that \(q_1(x_0) = \infty\).

**Proof:** Note that
\[ (l_2^{-1} l_1)'(q) = \frac{1}{\theta_1 - c_3 \theta_2 [l_2^{-1} l_1'(q)]} \times \left( \theta_2 - \frac{c_3 \theta_1}{c_2} \bar{F}_1(q) \right) \rightarrow \frac{\theta_2}{\theta_1}, \]
as \(q \rightarrow \infty\). Then it follows from (12) that \(H'(y)\) tends to 0 at the rate \(y^{-2}\) as \(y \rightarrow \infty\). On the other hand, the denominator of the integrand of (16) tends to \(\delta\) as \(y \rightarrow \infty\). Then it is easy to see that \(G(\infty) < \infty\), which in turn implies that there exists a \(x_0 < \infty\) such that \(q_1(x_0) = \infty\). \(\blacksquare\)

**Remark 5.1:** Lemma 5.2 suggests that the insurer will not buy reinsurance if the reserve is no less than \(x_0\).

Assume that \(x_0 < x_1\) (this will be proved later). Then for \(0 < x < x_0\), it follows from (7) that
\[ W(x) = c_4 \int_0^{x} \exp \left( - \int_0^{z} \frac{c_1 \theta_1}{c_1 G^{-1}(y) + c_3 \theta_2 [l_2^{-1} l_1'(G^{-1}(y))]} \, dy \right) \, dz, \tag{17} \]
where \(c_4 > 0\) is a constant.

For \(x_0 \leq x \leq x_1\), we guess that \(q_1(x) = q_2(x) = \infty\). Let
\[ K_1 = \frac{1}{2} \sum_{l=1}^{2} c_l (\mu_l^2 + \sigma_l^2) + c_3 \mu_1 \mu_2, \quad K_2 = \sum_{l=1}^{2} c_l \eta_l \mu_l. \]

Then (6) becomes
\[ K_1 W''(x) + K_2 W''(x) - \delta W(x) = 0, \]
which has the following general solution:
\[ W(x) = c_5 e^{r_+ (x-x_0)} + c_6 e^{r_- (x-x_0)}, \tag{18} \]
where \(c_5\) and \(c_6\) are constants, and
\[
 r_+ = \frac{-K_2 + \sqrt{K_2^2 + 4 \delta K_1}}{2 K_1}, \quad r_- = \frac{-K_2 - \sqrt{K_2^2 + 4 \delta K_1}}{2 K_1}.
\]

For \(x > x_1\), by the definition of \(x_1\), we guess that
\[ W(x) = W(\bar{x}) + k(x - \bar{x}) - K, \tag{19} \]
where \(\bar{x} < x_1\) is a constant.
By the continuity of $W'$ and $W''$ at $x_0$, it is easy to see that
\[ c_5 r_+ + c_6 r_- = c_4, \quad c_5 r_+^2 + c_6 r_-^2 = 0, \]
which results in $c_5 = c_4 b_1$ and $c_6 = c_4 b_2$, where
\begin{align*}
  b_1 &= \frac{r_-}{r_+(r_- - r_+)} > 0, & b_2 &= \frac{r_+}{r_-(r_+ - r_-)} < 0.
\end{align*}

The unknown constants $c_4, \bar{x}$ and $x_1$ can be determined in the same way as that in Chen & Yuen (2016). For details, see Chen & Yuen (2016). The following steps briefly describe how these constants can be determined:

(i) Define an auxiliary function $U(x)$ as

\[
U(x) = \begin{cases} 
  \exp \left( - \int_{x_0}^{x} \frac{c_1 \theta_1}{c_1 G^{-1}(y) + c_3 g_2[I_{l_2}^{-1} l_1(G^{-1}(y))] \, dy} \right), & 0 \leq x \leq x_0, \\
  b_1 r_+ e^{r_+(x-x_0)} + b_2 r_- e^{r_-(x-x_0)}, & x > x_0,
\end{cases}
\]

which is convex on $(0, \infty)$, and attains its minimum at $x = x_0$ with $U(x_0) = 1$.

(ii) For any fixed $c \in (0, k]$, there exists a unique $\hat{x}_c \geq x_0$ such that $cU(\hat{x}_c) = k$. Let $\hat{c} = k/U(0) < k$. If $c \in [\bar{c}, k]$, then there exists a unique $\hat{x}_c \in [0, x_0]$ such that $cU(\hat{x}_c) = k$.

(iii) Let
\[
I_1(c) = \int_{\hat{x}_c}^{\hat{c}} (k - cU(y)) \, dy, \quad c \in [\bar{c}, k],
\]
\[
I_2(c) = \int_{0}^{\hat{x}_c} (k - cU(y)) \, dy, \quad c \in [0, k].
\]

If $I_1(\hat{c}) > K$, then there exists a unique $c^* \in (\bar{c}, k)$ such that $I_1(c^*) = K$. If $I_1(\hat{c}) \leq K$, then there exists a unique $c^* \in (0, k)$ such that $I_2(c^*) = K$.

(iv) Let $c_4 = c^*$, $x_1 = \hat{x}_c^+ > x_0$, and $\bar{x} = \bar{x}_c^*$, where $\bar{x}_c^+ = 0$ if $I_1(\hat{c}) \leq K$.

These together with (17)–(19) yield
\begin{equation}
W(x) = \begin{cases} 
  c^* \int_{0}^{x} \exp \left( - \int_{x_0}^{z} \frac{c_1 \theta_1}{c_1 G^{-1}(x) + c_3 g_2[I_{l_2}^{-1} l_1(G^{-1}(x))] \, dy} \right) \, dz, & 0 \leq x < x_0, \\
  c^*[b_1 e^{r_+(x-x_0)} + b_2 e^{r_-(x-x_0)}], & x_0 \leq x \leq \hat{x}_c^+, \\
  W(\hat{x}_c^+) + k(x - \hat{x}_c^+) - K, & x \geq \hat{x}_c^*,
\end{cases}
\end{equation}

where $b_1$ and $b_2$ are given in (20).

**Theorem 5.3:** If $z_l \leq z_k$, then the function $W(x)$ of (21) is continuously differentiable on $(0, \infty)$ and twice continuously differentiable on $(0, \hat{x}_c^+) \cup (\hat{x}_c^*, \infty)$. Furthermore, $W(x)$ is a solution to the QVI of (5).

**Proof:** One can prove the theorem by replacing $G(1)$ and $\max_{0 \leq t \leq 1, 0 \leq u \leq 1} \mathcal{L}^{b,u} W(x)$ by $x_0$ and $\max_{0 \leq t \leq q_1, q_2 \leq \infty} \mathcal{L}^q W(x)$, respectively, and then mimicking the steps in the proof of Theorem 4.1 of Chen & Yuen (2016).
5.2. Case 2: $z_l > z_k$

In this case, it follows from Lemma 5.1 that the equation $H(q) = 0$ on $[z_l, \infty)$ has no solution. Then we guess that $q_2(x) = 0$. Then (6) becomes

$$\max_{0 \leq q_1 \leq \infty} \left\{ \frac{1}{2} c_1 G_1(q_1) W''(x) + [c_1 \theta_1 g_1(q_1) + k_0] W'(x) - \delta W(x) \right\} = 0. \quad (22)$$

Differentiating (22) with respect to $q_1$, we obtain

$$c_1 F_1(q_1) [q_1 W''(x) + \theta_1 W'(x)] = 0,$$

which yields

$$\frac{W''(x)}{W'(x)} = -\frac{\theta_1}{q_1(x)}. \quad (23)$$

Substituting (23) into (22), we obtain

$$k(q_1(x)) W'(x) - \delta W(x) = 0. \quad (24)$$

Differentiating (24) with respect to $x$ and using (23) once again, we have

$$W'(x) \left\{ k'(q_1(x)) q_1'(x) - \delta - \frac{\theta_1 k(q_1(x))}{q_1(x)} \right\} = 0. \quad (25)$$

Since $W'(x) > 0$, (25) gives

$$q_1'(x) = \frac{\delta + \frac{\theta_1 k(q_1(x))}{q_1(x)}}{k'(q_1(x))}.$$

In view of $W(0) = 0$ and (24), we see that $k(q_1(0)) = 0$, which implies that $q_1(0) = z_k > 0$. Let

$$R_1(q) = \int_{z_k}^q \frac{k'(y)}{\delta + \frac{\theta_1 k(y)}{y}} \, dy, \quad q \geq z_k. \quad (26)$$

Since the integrand on the right-hand side of (26) is positive on $[z_k, \infty)$, $R_1(q)$ is increasing on $[z_k, \infty)$, which implies that the inverse of $R_1(q)$ exists on $[z_k, \infty)$. Let $\tilde{x}_0 = R_1(z_l)$. Then for $0 < x \leq \tilde{x}_0$, we have $q_1(x) = R_1^{-1}(x)$, $q_2(x) = 0$, and it follows from (23) that

$$W(x) = C_1 \int_0^x \exp \left( -\int_{\tilde{x}_0}^z \frac{\theta_1}{R_1^{-1}(y)} \, dy \right) \, dz, \quad (27)$$

where the constant $C_1 > 0$ will be determined later.

For $x > \tilde{x}_0$, similar to the case of $z_l \leq z_k$, it can be shown that $q_1(x)$ satisfies (15). Note that $q_1(\tilde{x}_0) = z_l$. Define

$$R_2(q) = \int_{z_l}^q \frac{H'(y)}{\delta + H(y) \left[ c_1 y + c_3 g_2[l_2^{-1}(y)] \right]} \, dy, \quad q \geq z_l. \quad (28)$$

Let

$$q_1(x) = R_2^{-1}(x - \tilde{x}_0), \quad x > \tilde{x}_0.$$ 

Similar to Lemma 5.2, there exists a $x_0 \in (\tilde{x}_0, \infty)$ such that $q_1(x_0) = \infty$. Then for $\tilde{x}_0 < x < x_0$, we have

$$q_1(x) = R_2^{-1}(x - \tilde{x}_0), \quad q_2(x) = l_2^{-1}[l_1(R_2^{-1}(x - \tilde{x}_0))].$$
\[ W(x) = C_2 \int_{\tilde{x}_0}^{x} \exp \left( - \int_{x_0}^{z} \frac{c_1 \theta_1}{c_1 R_2^{-1}(y - \tilde{x}_0) + c_3 g_2(l_2^{-1}l_1(R_2^{-1}(y - \tilde{x}_0)))} \, dy \right) \, dz + C_3, \]  

(28)

where the constants \( C_2 \) and \( C_3 > 0 \) will be determined later.

For \( x \geq x_0 \), we guess that \( q_1(x) = q_2(x) = \infty \), and \( W(x) \) is the same as (18) and (19) for \( x_0 \leq x \leq x_1 \) and \( x > x_1 \), respectively.

As a result, we have

\[
W(x) = \begin{cases} 
C_1 \int_0^x \exp \left( - \int_{\tilde{x}_0}^{z} \frac{\theta_1}{R_1^{-1}(y)} \, dy \right) \, dz, & 0 \leq x < \tilde{x}_0, \\
C_2 \int_{\tilde{x}_0}^{x} \exp \left( - \int_{x_0}^{z} \frac{c_1 \theta_1}{c_1 R_2^{-1}(y - \tilde{x}_0) + c_3 g_2(l_2^{-1}l_1(R_2^{-1}(y - \tilde{x}_0)))} \, dy \right) \, dz + C_3, & \tilde{x}_0 \leq x < x_0, \\
W(\tilde{x}) + k(x - \tilde{x}) - K, & x \geq x_1.
\end{cases}
\]

(29)

We now need to determine the unknown constants mentioned above. By the continuity of \( W' \) at \( \tilde{x}_0 \), we have

\[ C_1 = C_2 \exp \left( \int_{\tilde{x}_0}^{x_0} \frac{c_1 \theta_1}{c_1 R_2^{-1}(y - \tilde{x}_0) + c_3 g_2(l_2^{-1}l_1(R_2^{-1}(y - \tilde{x}_0)))} \, dy \right). \]

Besides, (28) and (24) imply that \( C_3 = W(\tilde{x}_0) = \frac{k(\tilde{x}_0)}{\delta} C_1 \). Moreover, the continuity of \( W' \) and \( W'' \) at \( x_0 \) implies that \( c_5 = C_2 b_1 \) and \( c_6 = C_2 b_2 \), where \( b_1 \) and \( b_2 \) are given in (20). Then it is enough to determine the constants \( C_2, \tilde{x} \) and \( x_1 \), which can be obtained by using steps similar to those presented in Section 5.1. Analogous to Theorem 5.3, we have the following result.

**Theorem 5.4:** If \( z_l > z_k \), then the function \( W(x) \) of (29) is continuously differentiable on \((0, \infty)\) and twice continuously differentiable on \((0, \tilde{x}_c) \cup (\tilde{x}_c, \infty)\). Furthermore, \( W(x) \) is a solution to the QVI of (5).

**Proof:** For \( x \geq \tilde{x}_0 \), the proof is similar to that of Theorem 5.3. So, we only prove that \( W(x) \) is a solution to (5) for \( 0 \leq x < \tilde{x}_0 \). Since one can show that \( W(x) \) of (29) satisfies \( \mathcal{L}^q W(x) - \delta W(x) = 0 \) with \( q^* = (q_1^*, q_2^*) = (R_1^{-1}(x), 0) \). As a consequence, we need to show that \( \mathcal{L}^q W(x) - \delta W(x) \leq 0 \) for any \( q_1, q_2 \in [0, \infty] \), which is equivalent to verify that \( \mathcal{L}^q W(x) - \mathcal{L}^q W(x) \leq 0 \) for any \( q_1, q_2 \in [0, \infty] \). By (23), the latter is then equivalent to

\[ [d(q) - d(q^*)] - \frac{\theta_1}{2q_1^*} [b_1^2(q) - b_2^2(q^*)] \leq 0. \]

Let

\[ \varphi(q) = d(q) - \frac{\theta_1}{2q_1^*} b_2^2(q), \quad q \in [0, \infty] \times [0, \infty]. \]

Then it is enough to show that \( \varphi(q) \) attains its maximum at \( q = q^* = (q_1^*, 0) \). Note that

\[ \frac{\partial \varphi(q)}{\partial q_1} = \frac{c_1 \theta_1 \bar{F}_1(q_1)}{q_1^*} \left[ q_1^* - q_1 - \frac{c_3}{c_1} g_2(q_2) \right], \]

where

\[ \frac{\partial \varphi(q)}{\partial q_2} = \frac{c_1 \theta_1 \bar{F}_1(q_1)}{q_1^*} \left[ q_1^* - q_1 - \frac{c_3}{c_1} g_2(q_2) \right], \]
\[
\frac{\partial \varphi(q)}{\partial q_2} = \frac{\tilde{F}_2(q_2)}{q_1^1} \left[ c_2 \theta_2 q_1^* - c_2 \theta_1 q_2 - c_3 \theta_1 g_1(q_1) \right].
\]

Since \( \frac{\partial \varphi(q)}{\partial q_1} < 0 \) for \( q_1 > q_1^* \), we only consider the case of \( q_1 \leq q_1^* \). If \( \frac{\partial \varphi(q)}{\partial q_1} = 0 \), then \( q_2 = g_2^{-1} \left( \frac{c_1}{c_3} (q_1^* - q_1) \right) \), and (30) yields \( \frac{\partial \varphi(q)}{\partial q_2} = \frac{\tilde{F}_2(q_2)}{q_1} L(q_1) \), where

\[
L(q_1) = c_2 \theta_2 q_1^* - c_3 \theta_1 g_1(q_1) - c_2 \theta_1 g_2^{-1} \left[ \frac{c_1}{c_3} (q_1^* - q_1) \right], \quad 0 \leq q_1 \leq q_1^*.
\]

It is easy to see that

\[
L'(q_1) = -c_3 \theta_1 \tilde{F}_1(q_1) + c_2 \theta_1 \frac{1}{\tilde{F}_2} \left( \frac{c_1}{c_3} (q_1^* - q_1) \right) c_3
\]

\[
= \frac{\theta_1}{c_3 \tilde{F}_2} \left( \frac{c_1}{c_3} (q_1^* - q_1) \right) \left[ \frac{c_1}{c_3} - \frac{2}{c_3} \tilde{F}_1(q_1) \tilde{F}_2 \left( \frac{c_1}{c_3} (q_1^* - q_1) \right) \right] > 0.
\]

On the other hand, for \( 0 \leq x < \tilde{x}_0 \), we have \( z_k \leq q_1^* < z_l \), and

\[
L(q_1^*) = c_2 l_1(q_1^*) \leq c_2 l_1(z_l) = 0.
\]

Therefore, we obtain

\[
\frac{\partial \varphi(q)}{\partial q_2} \bigg|_{(q_1, q_2) \in \{ (q_1, q_2) : \frac{\partial \varphi(q)}{\partial q_1} = 0 \}} \leq 0.
\]

As a result, we see that \( \varphi(q) \) attains its maximum at \( (q_1^*, 0) \).

**6. The value function and the optimal policy**

Let

\[
(q_1^*, q_2^*) = \begin{cases} (G^{-1}(X_1^*), l_2^{-1}[l_1(G^{-1}(X_1^*))]), & 0 \leq X_1^* \leq x_0, \quad X_1^* > x_0, \\ (\infty, \infty), & \end{cases}
\]

when \( z_l \leq z_k \); and

\[
(q_1^*, q_2^*) = \begin{cases} (R_1^{-1}(X_1^*), 0), & 0 \leq X_1^* < \tilde{x}_0, \\ (R_2^{-1}(X_1^* - \tilde{x}_0), l_2^{-1}[l_1(R_2^{-1}(X_1^* - \tilde{x}_0))]), & \tilde{x}_0 \leq X_1^* < x_0, \\ (\infty, \infty), & X_1^* \geq x_0, \\ \end{cases}
\]

when \( z_l > z_k \). Recall \( X_1^* \) of (1) with \( \alpha = \alpha^* = (q_1^*, q_2^*, r_1^*, \ldots; \xi_1^*, \xi_2^*, \ldots) \). Define \( \{ r_n^*, \xi_n^*, n \geq 1 \} \) as follows:

(i) If \( I_1(\tilde{c}) > K \), then we define

\[
\tau_1^* = \inf \{ t > 0 : X_t^* = \hat{x}^* \}, \quad \xi_1^* = \hat{x}^* - \xi_c^*,
\]

when the initial surplus \( 0 < x < \hat{x}^* \),

\[
\tau_1^* = 0, \quad \xi_1^* = x - \hat{x}^*,
\]
Figure 1. Impact of $\lambda_3$ on the optimal reinsurance policy $q_1(x)$.

when the initial surplus $x \geq \hat{x}_c^*$, and

$$
\tau_n^* = \inf\{t > \tau_{n-1}^* : X_t^* = \hat{x}_c^*\}, \quad \xi_n^* = \hat{x}_c^* - \tilde{x}_c^*,
$$

for every $n \geq 2$.

(ii) If $I_1(\bar{c}) \leq K$, then we define

$$
\tau_1^* = \inf\{t > 0 : X_t^* = \hat{x}_c^*\}, \quad \xi_1^* = \hat{x}_c^*,
$$

when the initial surplus $0 < x < \hat{x}_c^*$,

$$
\tau_1^* = 0, \quad \xi_1^* = x,
$$

when the initial surplus $x \geq \hat{x}_c^*$, and

$$
\tau_n^* = \infty, \quad \xi_n^* = 0,
$$

for every $n \geq 2$.

Theorem 6.1: The value function $V(x)$ is given by (21) when $z_l \leq z_k$, and by (29) when $z_l > z_k$; and the strategy $\alpha^*$ is the corresponding optimal policy.

Proof: The proof is similar to that of Theorem 4.2 of Chen & Yuen (2016).

7. Numerical examples

In this section, we give some numerical examples to assess the impact of some parameters on the optimal reinsurance policy. We assume that the claim sizes $X^{(1)}$ and $X^{(2)}$ are exponentially distributed with parameters $\beta_1$ and $\beta_2$, respectively. Then, for $l = 1, 2$, we have $\mu_l = \frac{1}{\beta_l}, \sigma_l^2 = \frac{1}{\beta_l^2}, g_l(q) = \frac{1}{\beta_l} (1 -$
Figure 2. Impact of $\lambda_3$ on the optimal reinsurance policy $q_2(x)$.

Figure 3. The difference of $q_1(x)$ and $q_2(x)$ for $\lambda_3 = 1.5$.

e^{-\beta lq}, \text{ and } G_l(q) = \frac{2}{\beta l} [1 - (1 + \beta l q) e^{-\beta l q}]$. We take $m = 3, n = 2, p_{11} = p_{22} = 1, p_{12} = p_{21} = 0$ and $p_{31} = p_{32} = 1$ so that the resulting model reduces to the common shock model. Besides, we set $\beta_1 = 1, \beta_2 = 2, \eta_1 = 1, \eta_2 = 0.8, \delta = 0.5, \lambda_1 = 3, \lambda_2 = 4, \theta_2 = 1$. For $\theta_1 = 1.2$, the effect of $\lambda_3$ on the optimal reinsurance policy is studied in Example 7.1. Example 7.2 shows the effect of $\theta_1$ on the optimal reinsurance policy for $\lambda_3 = 2$. 
**Example 7.1:** In this example, we set $\theta_1 = 1.2$, and take $\lambda_3 = 1, 1.5, 2$, respectively. The effect of $\lambda_3$ on the optimal reinsurance strategies $q_1(x)$ and $q_2(x)$ are shown in Figures 1–3.

Table 1 shows that the critical point $x_0$ increases as $\lambda_3$ increases. We see from Figures 1 and 2 that both $q_1(x)$ and $q_2(x)$ are strictly increasing functions, and they decrease as $\lambda_3$ increases. This means that the optimal retention level is higher for larger reserve and is lower when the insurers face higher risk. The result coincides with our intuition. We also observe from Figure 3 that the difference of two reinsurance strategies is quite small, and both of the reinsurance strategies change slowly for small reserve, while they are quite sensitive to the change of surplus when the surplus near the critical point $x_0$.

**Example 7.2:** In this example, we set $\lambda_3 = 2$, and take $\theta_1 = 1.2, 1.5, 2.1$, respectively.

We see from Table 2 that the critical point $x_0$ also increases as $\theta_1$ increases, which means that the insurer should hold a larger reserve when the reinsurance premium becomes more expensive. Figures 4 and 5 indicate that both $q_1(x)$ and $q_2(x)$ are not strictly decreasing with respect to $\theta_1$. We can also see that the change of $\theta_1$ has larger effect on $q_1(x)$ than that on $q_2(x)$. When the reinsurance premium is more expensive, the insurer with small reserve tends to buy less reinsurance, and vice versa.
Figure 5. Impact of $\theta_1$ on the optimal reinsurance policy $q_2(x)$.

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References

Albrecher, H. & Thonhauser, S. (2008). Optimal dividend strategies for a risk process under force of interest. *Insurance: Mathematics and Economics* 43(1), 134–149.
Albrecher, H. & Thonhauser, S. (2009). Optimality results for dividend problems in insurance. *Revista De La Real Academia De Ciencias Exactas, Fisicas Y Naturales-Serie A: Matematicas* 103(2), 295–320.
Asmussen, S., Højgaard, B. & Taksar, M. (2000). Optimal risk control and dividend distribution policies. Example of excess-of-loss reinsurance. *Finance and Stochastics* 4(3), 299–324.
Asmussen, S. & Taksar, M. (1997). Controlled diffusion models for optimal dividend pay-out. *Insurance: Mathematics and Economics* 20(1), 1–15.
Avram, F., Palmowski, Z. & Pistorius, M. (2007). On the optimal dividend problem for a spectrally negative Lévy process. *The Annals of Applied Probability* 17(1), 156–180.
Avram, F., Palmowski, Z. & Pistorius, M. (2015). On gerber-Shiu functions and optimal dividend distribution for a Lévy risk process in the presence of a penalty function. *The Annals of Applied Probability* 25(4), 1868–1935.
Azcue, P. & Muler, N. (2005). Optimal reinsurance and dividend distribution policies in the cramér-Lundberg model. *Mathematical Finance* 15(2), 261–308.
Paulsen, J. (2008). Optimal dividend payments and reinvestments of diffusion processes with both fixed and proportional costs. *SIAM Journal on Control & Optimization* **47**(5), 2201–2226.

Peng, X., Bai, L. & Guo, J. (2016). Optimal control with restrictions for a diffusion risk model under constant interest force. *Applied Mathematics & Optimization* **73**(1), 115–136.

Peng, X., Chen, M. & Guo, J. (2012). Optimal dividend and equity issuance problem with proportional and fixed transaction costs. *Insurance: Mathematics and Economics* **51**(3), 576–585.

Pérez, J., Yamazaki, K. & Yu, X. (2018). On the bail-out optimal dividend problem. *Journal of Optimization Theory and Applications* **179**(2), 553–568.

Schmidli, H. (2006). Optimisation in non-life insurance. *Stochastic Models* **22**(4), 689–722.

Taksar, M. (2000). Optimal risk and dividend distribution control models for an insurance company. *Mathematical Methods of Operations Research* **51**(1), 1–42.

Vierkötter, M. & Schmidli, H. (2017). On optimal dividends with exponential and linear penalty payments. *Insurance: Mathematics and Economics* **72**(1), 265–270.

Wang, G. & Yuen, K. C. (2005). On a correlated aggregate claims model with thinning-dependence structure. *Insurance: Mathematics and Economics* **36**(3), 456–468.

Wang, W. & Hu, Y. (2012). Optimal loss-carry-forward taxation for the Lévy risk model. *Insurance: Mathematics and Economics* **50**(1), 121–130.

Wang, W., Wang, Y. & Wu, X. (2018). Dividend and capital injection optimization with transaction cost for spectrally negative Lévy risk processes. arXiv:1807.11171.

Wang, W. & Zhang, Z. (2019). Optimal loss-carry-forward taxation for Lévy risk processes stopped at general draw-down time. *Advances in Applied Probability* **51**(3), 865–897.

Wang, W. & Zhou, X. (2018). General draw-down based de Finetti optimization for spectrally negative Lévy risk processes. *Journal of Applied Probability* **55**(2), 513–542.

Wei, J., Yang, H. & Wang, R. (2010). Classical and impulse control for the optimization of dividend and proportional reinsurance policies with regime switching. *Journal of Optimization Theory and Applications* **147**(2), 358–377.

Wei, W., Liang, Z. & Yuen, K. C. (2018). Optimal reinsurance in a compound Poisson risk model with dependence. *Journal of Applied Mathematics and Computing* **58**(2), 389–412.

Wu, Y. (2013). Optimal reinsurance and dividend strategies with capital injections in Cramer–Lundberg approximation model. *Bulletin of the Malaysian Mathematical Sciences Society* **36**(1), 193–210.

Yao, D. & Fan, K. (2018). Optimal risk control and dividend strategies in the presence of two reinsurers: variance premium principle. *Journal of Industrial & Management Optimization* **14**(3), 1055–1084.

Yao, D., Yang, H. & Wang, R. (2014). Optimal risk and dividend control problem with fixed costs and salvage value: variance premium principle. *Economic Modelling* **37**(1), 53–64.

Yao, D., Yang, H. & Wang, R. (2016). Optimal dividend and reinsurance strategies with financing and liquidation value. *ASTIN Bulletin* **46**(2), 365–399.

Yuen, K. C., Liang, Z. & Zhou, M. (2015). Optimal proportional reinsurance with common shock dependence. *Insurance: Mathematics and Economics* **64**(1), 1–13.

Zhang, C. & Liang, Z. (2017). Portfolio optimization for jump-diffusion risky assets with common shock dependence and state dependent risk aversion. *Optimal Control Applications and Methods* **38**(2), 229–246.

Zhao, Y., Chen, P. & Yang, H. (2017). Optimal periodic dividend and capital injection problem for spectrally positive Lévy processes. *Insurance: Mathematics and Economics* **74**(1), 135–146.

Zhu, J. (2017). Optimal financing and dividend distribution with transaction costs in the case of restricted dividend rates. *ASTIN Bulletin* **47**(1), 239–268.