Analytical relations between nuclear symmetry energy and single-nucleon potentials in isospin asymmetric nuclear matter

Chang Xu\textsuperscript{a,b}, Bao-An Li\textsuperscript{a,1}, Lie-Wen Chen\textsuperscript{a,c}, Che Ming Ko\textsuperscript{d}

\textsuperscript{a}Department of Physics and Astronomy, Texas A\&M University-Commerce, Commerce, Texas 75429-3011, USA
\textsuperscript{b}Department of Physics, Nanjing University, Nanjing 210008, China
\textsuperscript{c}Department of Physics, Shanghai Jiao Tong University, Shanghai 200240, China
\textsuperscript{d}Cyclotron Institute and Department of Physics and Astronomy, Texas A\&M University, College Station, TX 77843-3366, USA

Abstract

Using the Hugenholtz-Van Hove theorem, we derive general expressions for the quadratic and quartic symmetry energies in terms of single-nucleon potentials in isospin asymmetric nuclear matter. These analytical relations are useful for gaining deeper insights into the microscopic origins of the uncertainties in our knowledge on nuclear symmetry energies especially at supra-saturation densities. As examples, the formalism is applied to two model single-nucleon potentials that are widely used in transport model simulations of heavy-ion reactions.

Key words: Symmetry energy, Nuclear potential, Heavy-ion collision, Transport model
PACS: 21.30.Fe, 21.65.Ef, 21.65.Cd

1 Introduction

One of the central issues currently under intense investigation in both nuclear physics and astrophysics is the Equation of State (EOS) of neutron-rich nuclear matter \cite{12}. For cold nuclear matter of isospin asymmetry

\footnote{Corresponding author: Bao-An Li@Tamu-Commerce.edu}
\[ \delta = (\rho_n - \rho_p)/(\rho_n + \rho_p) \] at density \( \rho \), the energy per nucleon \( E(\rho, \delta) \) can be expressed as an even series of \( \delta \) that respects the charge symmetry of strong interactions, namely, 

\[ E(\rho, \delta) = E_0(\rho, 0) + \sum_{i=2,4,6...} E_{\text{sym}, i}(\rho) \delta^i \]

where \( E_{\text{sym}, i}(\rho) \) is the so-called symmetry energy of the \( i \)th order and \( E_0(\rho, 0) \) is the EOS of symmetric nuclear matter. The quadratic term \( E_{\text{sym}, 2}(\rho) \) is most important and its value at normal nuclear matter density \( \rho_0 \) is known to be around 30 MeV from analyzing nuclear masses within liquid-drop models. Essentially, all microscopic many-body calculations have indicated that the higher-order terms are usually negligible around \( \rho_0 \), leading to the so-called empirical parabolic law of EOS even for \( \delta \) approaching unity for pure neutron matter. The \( E_{\text{sym}, 2}(\rho) \) is then generally regarded as the symmetry energy. For instance, the value of the quartic term has been estimated to be less than 1 MeV at \( \rho_0 \) [4,5]. However, the presence of higher-order terms at supra-saturation densities can significantly modify the proton fraction in neutron stars at \( \beta \)-equilibrium and thus the cooling mechanism of proto-neutron stars [6,7]. It was also found that a tiny quartic term can cause a big change in the calculated core-crust transition density in neutron stars [8,9]. Therefore, precise evaluations of the quartic symmetry energy in neutron-rich matter are useful. Although much information about the EOS of symmetric nuclear matter \( E_0(\rho, 0) \) has been accumulated over the past four decades, our knowledge about the density dependence of \( E_{\text{sym}, i}(\rho) \) is unfortunately still very poor. It has been generally recognized that the \( E_{\text{sym}, i}(\rho) \), especially the quadratic and quartic terms, is critical for understanding not only the structure of rare isotopes and the reaction mechanism of heavy-ion collisions, but also many interesting issues in astrophysics [9,10,11,12,13,14,15,16,17,18,19,20,21,22]. Therefore, to determine the \( E_{\text{sym}, i}(\rho) \) in neutron-rich matter has recently become a major goal in both nuclear physics and astrophysics. While significant progress has been made recently in constraining the \( E_{\text{sym}, 2}(\rho) \) especially around and below the saturation density, see, e.g., [17,18,19,20], much more work needs to be done to constrain more tightly the \( E_{\text{sym}, i}(\rho) \) at supra-saturation densities where model predictions are rather diverse [23,24,25,26,27,28,29,30,31,32,33,34]. As dedicated experiments using advanced new detectors have now been planned to investigate the high density behavior of \( E_{\text{sym}, 2}(\rho) \) at many radioactive beam facilities around the world, it has become an urgent task to investigate theoretically more deeply the fundamental origin of the extremely uncertain high density behavior of \( E_{\text{sym}, 2}(\rho) \). It is also of great interest to evaluate possible corrections due to the \( E_{\text{sym}, 4}(\rho) \) term to the equation of state of asymmetric nuclear matter.

Among existing proposals for extracting information about \( E_{\text{sym}, i}(\rho) \) using terrestrial laboratory experiments, transport model simulations have shown that many observables in heavy-ion reactions are particularly useful for studying \( E_{\text{sym}, i}(\rho) \) in a broad density range. In these transport model simulations of heavy-ion reactions, the EOS enters the reaction dynamics and affects the final observables through the single-nucleon potential \( U_{n/p}(\rho, \delta, k) \) where \( k \)
is the nucleon momentum. Except in situations where statistical equilibrium is established and thus many observables are directly related to the binding energy $E(\rho, \delta)$ after correcting for finite-size effects, what is being directly probed in heavy-ion reactions is the single-nucleon potential $U_{n/p}(\rho, \delta, k)$. The latter is, however, directly related to the symmetry energy $E_{sym,2}(\rho)$ through the underlying nuclear effective interaction as first pointed out by Brueckner, Dabrowski and Haensel [35,36] using K-matrices within the Brueckner theory. They showed that if one expands $U_{n/p}(\rho, \delta, k)$ to the leading order in $\delta$ as in the well-known Lane potential [37], i.e.,

$$U_{n/p}(\rho, \delta, k) \approx U_0(\rho, k) \pm U_{sym,1}(\rho, k)\delta$$  \hspace{1cm} (1)$$

where $U_0(\rho, k)$ and $U_{sym,1}(\rho, k)$ are, respectively, the nucleon isoscalar and isovector (symmetry) potentials, the quadratic symmetry energy is then [35,36]

$$E_{sym,2}(\rho) = \frac{1}{3} t(k_F) + \frac{1}{6} \left[ \frac{\partial U_0}{\partial k} \right]_{k_F} k_F \cdot k_F + \frac{1}{2} U_{sym,1}(\rho, k_F)$$  \hspace{1cm} (2)$$

where $t(k_F)$ is the nucleon kinetic energy at the Fermi momentum $k_F = (3\pi^2\rho/2)^{1/3}$ in symmetric nuclear matter of density $\rho$. The above equation indicates that the symmetry energy $E_{sym,2}(\rho)$ depends only on the single-particle kinetic and potential energies at the Fermi momentum $k_F$. This is not surprising since the microscopic origin of the symmetry energy is the difference in the Fermi surfaces of neutrons and protons. The first term $E_{sym}^{kin} = \frac{1}{3} t(k_F) = \frac{k_F^2}{6m} (\frac{3\pi^2}{2})^{2/3} \rho^{5/3}$ is the trivial kinetic contribution due to the different Fermi momenta of neutrons and protons; the second term $\frac{1}{6} \left[ \frac{\partial U_0}{\partial k} \right]_{k_F} k_F \cdot k_F$ is due to the momentum dependence of the isoscalar potential and also the fact that neutrons and protons have different Fermi momenta; while the term $\frac{1}{2} U_{sym,1}(\rho, k_F)$ is due to the explicit isospin dependence of the nuclear strong interaction.

For the isoscalar potential $U_0(\rho, k)$, reliable information about its density and momentum dependence has already been obtained from high energy heavy-ion collisions, see, e.g., ref. [13], albeit there are still some rooms for further improvements, particularly at high momenta/densities. On the contrary, the isovector potential $U_{sym,1}(\rho, k)$ is still not very well determined, especially at high densities and momenta, and has been identified as the key quantity responsible for the uncertain high density behavior of the symmetry energy as stressed in ref. [3].

In the present work, we first show, using both the differential and integral formulations of the Hugenholtz-Van Hove (HVH) theorem [38], that the relation in Eq. (2) is valid in general. We then derive an expression for the quartic symmetry energy $E_{sym,4}(\rho)$ in terms of the single-nucleon potential by keeping higher-order terms in the expansion of both the EOS and the single-nucleon potential. Applying the HVH formalism to two model single-nucleon potentials, namely, the Bombaci-Gale-Bertsch-Das Gupta (BGBD) potential [16] and a modified Gogny Momentum-Dependent-Interaction (MDI) [23,40],
which are among the most widely used ones in studying isospin physics based on transport model simulations of heavy-ion reactions \[3,14\], we examine the relative contributions from the kinetic and various potential terms to \(E_{\text{sym},2}(\rho)\) and \(E_{\text{sym},4}(\rho)\). We put the emphasis on identifying those terms that dominate the high density behaviors of \(E_{\text{sym},2}(\rho)\). Finally, we evaluate the relative importance of the \(E_{\text{sym},4}(\rho)\) term by studying the \(E_{\text{sym},4}(\rho)/E_{\text{sym},2}(\rho)\) ratio as a function of density.

The paper is organized as follows. In Section 2, based on the HVH theorem we derive general expressions for the higher-order symmetry energy terms \(E_{\text{sym},2}(\rho)\) and \(E_{\text{sym},4}(\rho)\) in terms of the single-nucleon isoscalar and isovector potentials. The derivation is carried out in Section 2.1 using the differential form of the HVH theorem by starting from the neutron and proton chemical potentials and in Section 2.2 using the integral form of the HVH theorem by starting from the total energy density of the system. Numerical results and discussions for both the BGBD and MDI interactions are given in Sections 3.1 and 3.2, respectively. Finally, we give a summary in Section 4.

2 Symmetry energy in terms of the single-nucleon potential

In this section, we present two alternative approaches to derive expressions for the quadratic and quartic symmetry energy terms \(E_{\text{sym},2}(\rho)\) and \(E_{\text{sym},4}(\rho)\). Both are based on the Fermi gas model of interacting nucleons and satisfy the HVH theorem that was first derived in Ref. \[38\]. There are, however, some technical differences between the two approaches in that a Taylor-series expansion is made on the single-nucleon energy in one approach but on the total energy of the system in the other.

2.1 Derivation using the Hugenholtz-Van Hove theorem

The Hugenholtz-Van Hove theorem \[38\] describes a fundamental relation among the Fermi energy \(E_F\), the average energy per particle \(E\) and the pressure of the system \(P\) at the absolute temperature of zero. For a one-component system, in terms of the energy density \(\xi = \rho E\), the general HVH theorem can be written as \[38,39\]

\[
E_F = \frac{d\xi}{d\rho} = \frac{d(\rho E)}{d\rho} = E + \rho \frac{dE}{d\rho} = E + P/\rho.
\] (3)

The above relation has been strictly proven to be valid for any interacting self-bound infinite Fermi system. It does not depend upon the precise nature of the
interaction. In the special case of nuclear matter at saturation density where the pressure $P$ vanishes, the average energy per nucleon becomes equal to the Fermi energy, i.e., $E_F = E$. It is worthwhile to stress that the general HVH theorem of Eq. (3) is valid at any arbitrary density as long as the temperature remains zero [38,39]. In fact, a successful theory for nuclear matter is required not only to describe satisfactorily all saturation properties of nuclear matter but also to fulfill the general HVH theorem at any density. In the following, we use the general HVH theorem to derive the relation between the nuclear symmetry energy and the single-nucleon potential.

According to the HVH theorem, the chemical potentials of neutrons and protons in isospin asymmetric nuclear matter of energy density $\xi(\rho, \delta) = \rho E(\rho, \delta)$ are, respectively [38,39],

$$t(k_F^n) + U_n(\rho, \delta, k_F^n) = \frac{\partial \xi}{\partial \rho_n}, \quad (4)$$
$$t(k_F^p) + U_p(\rho, \delta, k_F^p) = \frac{\partial \xi}{\partial \rho_p}, \quad (5)$$

where $t(k) = \hbar k^2/2m$ is the kinetic energy and $U_{n/p}$ is the neutron/proton single-particle potential. The Fermi momenta of neutrons and protons are $k_F^n = k_F(1 + \delta)^{1/3}$ and $k_F^p = k_F(1 - \delta)^{1/3}$, respectively. Subtracting Eq. (5) from Eq. (4) gives [35,36]

$$[t(k_F^n) - t(k_F^p)] + [U_n(\rho, \delta, k_F^n) - U_p(\rho, \delta, k_F^p)] = \frac{\partial \xi}{\partial \rho_n} - \frac{\partial \xi}{\partial \rho_p}. \quad (6)$$

The nucleon single-particle potentials can be expanded as a power series of $\delta$ while respecting the charge symmetry of nuclear interactions under the exchange of neutrons and protons,

$$U_n(\rho, \delta, k) = U_0(\rho, k) + \sum_{i=1,2,3...} U_{sym,i}(\rho, k)\delta^i$$
$$= U_0(\rho, k) + U_{sym,1}(\rho, k)\delta + U_{sym,2}(\rho, k)\delta^2 + ... \quad (7)$$

$$U_p(\rho, \delta, k) = U_0(\rho, k) + \sum_{i=1,2,3...} U_{sym,i}(\rho, k)(-\delta)^i$$
$$= U_0(\rho, k) - U_{sym,1}(\rho, k)\delta + U_{sym,2}(\rho, k)\delta^2 - ... \quad (8)$$

If one neglects the higher-order terms ($\delta^2, \delta^3,...$), Eq. (7) and Eq. (8) reduce to the Lane potential in Eq. (1). Expanding both the kinetic and potential
energies around the Fermi momentum $k_F$, the left side of Eq.(5) can be further written as

\[
\begin{align*}
[t(k_n^F) - t(k_p^F)] + [U_n(\rho, \delta, k_n^F) - U_p(\rho, \delta, k_p^F)] \\
= \sum_{i=1,2,3,\ldots} \frac{1}{i!} \left( \frac{\partial^i [t(k) + U_0(\rho, k)]}{\partial k^i} \right)_{k_F k_F^i} \\
\times \left[ \left( \sum_{j=1,2,3,\ldots} F(j) \delta^j \right)^i - \left( \sum_{j=1,2,3,\ldots} F(j) (-\delta)^j \right)^i \right] \\
+ \sum_{i=1,2,3,\ldots} U_{sym,i}(\rho, k_F) \delta^i - (-\delta)^i \\
+ \sum_{l=1,2,3,\ldots} \sum_{i=1,2,3,\ldots} \frac{1}{i!} \left( \frac{\partial^i U_{sym,l}(\rho, k)}{\partial k^i} \right)_{k_F k_F^i} \\
\times \left[ \left( \sum_{j=1,2,3,\ldots} F(j) \delta^j \right)^i \delta^l - \left( \sum_{j=1,2,3,\ldots} F(j) (-\delta)^j \right)^i (-\delta)^l \right] \\
= \frac{2}{3} \left( \frac{\partial [t(k) + U_0(\rho, k)]}{\partial k} \right)_{k_F k_F} + 2 U_{sym,1}(\rho, k_F) \delta + \ldots, \quad (9)
\end{align*}
\]

where we have introduced the function $F(j) = \frac{1}{j!} \left( \frac{1}{3} - 1 \right) \ldots \left( \frac{1}{3} - j + 1 \right)$. For the right side of Eq.(5), expanding in powers of $\delta$ gives

\[
\frac{\partial \xi}{\partial \rho_n} - \frac{\partial \xi}{\partial \rho_p} = \frac{2}{\rho} \frac{\partial \xi}{\partial \delta} = \sum_{i=2,4,6,\ldots} 2i E_{sym,i}(\rho) \delta^{i-1} \\
= 4 E_{sym,2}(\rho) \delta + 8 E_{sym,4}(\rho) \delta^3 + 12 E_{sym,6}(\rho) \delta^5 + \ldots \quad (10)
\]

Comparing the coefficient of each $\delta^i$ term in Eq.(9) with that in Eq.(10) then gives the symmetry energy of any order. For instance, the quadratic term

\[
E_{sym,2}(\rho) = \frac{1}{6} \left( \frac{\partial [t(k) + U_0(\rho, k)]}{\partial k} \right)_{k_F k_F} + \frac{1}{2} U_{sym,1}(\rho, k_F) \\
= \frac{1}{3} t(k_F) + \frac{1}{6} \frac{\partial U_0}{\partial k} |k_F \cdot k_F| + \frac{1}{2} U_{sym,1}(\rho, k_F) \quad (11)
\]

is identical to that in Eq.(2), while the quartic term can be written as
\[ E_{\text{sym},A}(\rho) = \left[ \frac{5}{324} \frac{\partial[t(k) + U_0(\rho, k)]}{\partial k} \right]_{k_F k_F} \]
\[ - \frac{1}{108} \frac{\partial^2 [t(k) + U_0(\rho, k)]}{\partial k^2} \left|_{k_F k_F} \right. + \frac{1}{648} \frac{\partial^3 [t(k) + U_0(\rho, k)]}{\partial k^3} \left|_{k_F k_F} \right. \]
\[ - \frac{36}{36} \frac{\partial U_{\text{sym},1}(\rho, k)}{\partial k} \left|_{k_F k_F} \right. + \frac{1}{72} \frac{\partial^2 U_{\text{sym},1}(\rho, k)}{\partial k^2} \left|_{k_F k_F} \right. \]
\[ + \frac{12}{12} \frac{\partial U_{\text{sym},2}(\rho, k)}{\partial k} \left|_{k_F k_F} \right. + \frac{1}{4} U_{\text{sym},3}(\rho, k_F) \] \hspace{1cm} (12)

2.2 Derivation using the total energy of an interacting Fermi gas

The symmetry energy of any order obtained in the previous subsection can also be derived within the interacting Fermi gas model [41,42]. Although the derivation is more tedious, it is physically interesting and mathematically instructive.

There are different ways to calculate the total energy of a Fermi system at a given density \( \rho \). As explained in detail by Bertsch and Das Gupta [43], one could determine the total energy starting with empty space, adding particles until the desired density is reached. Each added particle would contribute an energy \( k(\rho_x)^2/2m + U(\rho_x, k(\rho_x)) \), where \( k \) is the Fermi momentum corresponding to the density \( \rho_x \) of particles already added to the system. As stressed by Bertsch and Das Gupta, the single-particle potential \( U(\rho_x, k(\rho_x)) \) is not the same as the potential energy per particle as one might at first guess. In this way, the total energy density of the asymmetric nuclear matter written in coordinate space is

\[ \xi = \int_0^{\rho^n} [k(\rho_x)^2/2m + U_n(\rho_x, k(\rho_x))]d\rho_x + \int_0^{\rho^n} [k(\rho_x)^2/2m + U_p(\rho_x, k(\rho_x))]d\rho_x, \] \hspace{1cm} (13)

where \( \rho^n + \rho^p = \rho \). The single particle potential \( U(\rho_x, k(\rho_x)) \) for the neutron or proton in the first or second integral in Eq.(13) can be rewritten as

\[ U(\rho_x, k(\rho_x)) = U(\rho, \delta^*, k(\rho_x)), \] \hspace{1cm} (14)

where \( \rho_x = \rho(1 + \delta^*)/2 \) in terms of the local isospin asymmetry \( \delta^* \) (to be discussed in detail in the following). The single particle potential can be further expanded as a series of \( \delta^* \)

\[ U(\rho, \delta^*, k(\rho_x)) = U_0(\rho, k(\rho_x)) + \sum_{i=1,2,3\ldots} U_{\text{sym},i}(\rho, k(\rho_x))(\delta^*)^i \] \hspace{1cm} (15)

\[ = U_0(\rho, k(\rho_x)) + U_{\text{sym},1}(\rho, k(\rho_x))(\delta^*) + U_{\text{sym},2}(\rho, k(\rho_x))(\delta^*)^2 + \ldots \]
Similar to the interpretation of Eq. (13), one can raise the momenta of all particles in the system from zero to the Fermi momentum corresponding to the density $\rho$. In either the coordinate or momentum space, as one increases the density or momentum for both neutrons and protons to build up the desired Fermi system, the local isospin asymmetry $\delta^*$ changes continuously as a function of density or momentum, i.e.,

$$\delta^* = \left[ \frac{2\rho_x}{\rho} - 1 \right] \rightarrow \delta^* = \left[ \frac{k^3}{k_{F}^3} - 1 \right]$$  \hspace{1cm} (16)$$

where $\rho_x$ is either the neutron or proton density used in the two terms of Eq. (13), and $k$ is the neutron or proton momentum used in the two terms of Eq. (18) below. At the respective Fermi surfaces of neutrons and protons, the local isospin asymmetry $\delta^*$ reduce to the global one

$$\delta^*(k_{F}^n) = \left[ \frac{2\rho^n}{\rho} - 1 \right] = \left[ \frac{(k_{F}^n)^3}{k_{F}^3} - 1 \right] = \delta,$$

$$\delta^*(k_{F}^p) = \left[ \frac{2\rho^p}{\rho} - 1 \right] = \left[ \frac{(k_{F}^p)^3}{k_{F}^3} - 1 \right] = (-\delta).$$  \hspace{1cm} (17)$$

In order to make use of the same technique in expanding both the kinetic and potential terms in $\delta$ series, it is more convenient to express the energy density in momentum space

$$\xi = \frac{1}{\pi^2} \left[ \int_{0}^{k_{F}^n} [t(k) + U_n(\rho, \delta^*, k)]k^2 dk + \int_{0}^{k_{F}^p} [t(k) + U_p(\rho, \delta^*, k)]k^2 dk \right].$$  \hspace{1cm} (18)$$

Before we proceed to derive expressions for $E_{sym,2}(\rho)$ and $E_{sym,4}(\rho)$, it is critical to examine whether the energy density given in Eq. (13) or Eq. (18) satisfies the HVH theorem. Noticing that

$$\frac{\partial \xi}{\partial \rho^n} = \frac{\partial \xi}{\partial k_{F}^n} \frac{\partial k_{F}^n}{\partial \rho^n} = \frac{\partial \xi}{\partial k_{F}^n} / \left( \frac{\partial \rho^n}{\partial k_{F}^n} \right) = \frac{\partial \xi}{\partial k_{F}^n} / \left( \frac{\partial (k_{F}^n)^3}{\partial k_{F}^n} \right) = \frac{\partial \xi}{\partial k_{F}^n} \left[ \frac{\pi^2}{(k_{F}^n)^2} \right],$$  \hspace{1cm} (19)$$

then it is straightforward to show using Eq. (18) that

$$\frac{\partial \xi}{\partial \rho^n} = \left\{ \left[ \frac{1}{\pi^2} \left[ t(k_{F}^n) + U_n(\rho, \delta, k_{F}^n) \right] \right]^2 \right\} \left[ \frac{\pi^2}{(k_{F}^n)^2} \right] = t(k_{F}^n) + U_n(\rho, \delta, k_{F}^n).$$  \hspace{1cm} (20)$$

Similarly, one can show that
\[
\frac{\partial \xi}{\partial \rho} = t(k_F^p) + U_n(\rho, \delta, k_F^p).
\] (21)

Thus, the HVH theorem is indeed satisfied by the energy density expressed in both Eqs. (13) and Eq. (18).

To obtain expressions for \(E_{\text{sym},2}(\rho)\) and \(E_{\text{sym},4}(\rho)\), we make a Taylor-series expansion of the energy density in Eq. (18). To proceed, it is useful to first recall that for any continuous function \(Y(k)\) the Taylor-series expansion of the integral \(f(k_F) = \int_0^{k_F} Y(k)k^2dk\) around \(k_F\) leads to \(41\)

\[
f(k_F) = f(k_F) + Y(k_F) \cdot k_F^2 \cdot (k_F^i - k_F)\]
\[+
\frac{1}{2} \left[ \frac{\partial Y}{\partial k} \right]_{k_F} \cdot k_F^2 + 2k_F \cdot Y(k_F) \cdot (k_F^i - k_F)^2 + \ldots, \] (22)

Moreover, it is easy to show that

\[
k_F^n - k_F = \left[ \frac{1}{3} \delta - \frac{1}{9} \delta^2 + \frac{5}{81} \delta^3 - \frac{10}{243} \delta^4 + \ldots \right] k_F
\]
\[k_F^p - k_F = \left[ -\frac{1}{3} \delta - \frac{1}{9} \delta^2 - \frac{5}{81} \delta^3 - \frac{10}{243} \delta^4 + \ldots \right] k_F. \] (23)

The kinetic energy per nucleon can then be expanded as

\[
T = \alpha \int_0^{k_F^p} t(k)k^2dk + \alpha \int_0^{k_F^p} t(k)k^2dk
= 2\alpha \int_0^{k_F} t(k)k^2dk + \alpha t(k_F)k_F^2 [(k_F^n - k_F) + (k_F^p - k_F)]
+ \frac{\alpha}{2!} \frac{\partial^2 t(k)k^2}{\partial k^2} \frac{1}{k_F} [(k_F^n - k_F)^2 + (k_F^p - k_F)^2]
+ \frac{\alpha}{3!} \frac{\partial^3 t(k)k^2}{\partial k^3} k_F [(k_F^n - k_F)^3 + (k_F^p - k_F)^3]
+ \frac{\alpha}{4!} \frac{\partial^4 t(k)k^2}{\partial k^4} k_F [(k_F^n - k_F)^4 + (k_F^p - k_F)^4]
= 2\alpha \int_0^{k_F} t(k)k^2dk + \frac{1}{2!} \frac{\partial t(k)}{\partial k} \frac{1}{k_F} k_F^2 \delta^2
+ \left[ \frac{5}{324} \frac{\partial t(k)}{\partial k} \right]_{k_F} k_F^2 \delta^2 + \frac{1}{108} \frac{\partial^2 t(k)}{\partial k^2} \frac{1}{k_F^2} k_F^2 \delta^2
+ \frac{1}{648} \frac{\partial^3 t(k)}{\partial k^3} \frac{1}{k_F} k_F^2 \delta^2 + \ldots, \] (24)

where \(\alpha = 3/(2k_F^3)\). Thus, the kinetic contribution to the \(E_{\text{sym},2}(\rho)\) is
\[ E_{\text{sym},2}(\rho) = \frac{1}{6} \frac{\partial t(k)}{\partial k} \bigg|_{k_F} k_F = \frac{1}{3} t(k_F) = \frac{\hbar^2}{6m} \left( \frac{3\pi^2}{2} \right)^{2/3} \rho^{2/3}, \]  

and its contribution to \( E_{\text{sym},4}(\rho) \) is

\[ E_{\text{sym},4}(\rho) = \frac{5}{324} \frac{\partial t(k)}{\partial k} \bigg|_{k_F} k_F - \frac{1}{108} \frac{\partial^2 t(k)}{\partial k^2} \bigg|_{k_F} k_F^2 + \frac{1}{648} \frac{\partial^3 t(k)}{\partial k^3} \bigg|_{k_F} k_F^3 \]

\[ = \frac{1}{81} t(k_F) = \frac{\hbar^2}{162m} \left( \frac{3\pi^2}{2} \right)^{2/3} \rho^{2/3}. \]

For the potential energy per nucleon written in momentum space, we first expand the single-nucleon potential \( U_{n/p} \) in local isospin asymmetry \( \delta^*(k) \)

\[ \mathcal{U} = \alpha \int_0^{k_F^p} U_n(\rho, \delta^*, k) k^2 dk + \alpha \int_0^{k_F^p} U_p(\rho, \delta^*, k) k^2 dk \]

\[ = [\alpha \int_0^{k_F^p} U_0(\rho, k) k^2 dk + \alpha \int_0^{k_F^p} U_0(\rho, k) k^2 dk] + \sum_{i=1,2,3...} \left[ \alpha \int_0^{k_F^p} U_{\text{sym},i}(\rho, k)(\delta^*)^i k^2 dk + \alpha \int_0^{k_F^p} U_{\text{sym},i}(\rho, k)(\delta^*)^i k^2 dk \right]. \]

We then expand each integral in the above equation around the Fermi momentum \( k_F \) using Eq. (22). The \( U_0 \) contribution to \( \mathcal{U} \) is

\[ E_{\text{pot}}^0(\rho) = \alpha \int_0^{k_F^p} U_0(\rho, k) k^2 dk + \alpha \int_0^{k_F^p} U_0(\rho, k) k^2 dk \]

\[ = 2\alpha \int_0^{k_F^p} U_0(\rho, k) k^2 dk + \frac{1}{6} \frac{\partial U_0(\rho, k)}{\partial k} \bigg|_{k_F} k_F^2 \delta^2 \]

\[ + \left[ \frac{5}{324} \frac{\partial U_0(\rho, k)}{\partial k} \bigg|_{k_F} k_F^2 - \frac{1}{108} \frac{\partial^2 U_0(\rho, k)}{\partial k^2} \bigg|_{k_F} k_F^2 + \frac{1}{648} \frac{\partial^3 U_0(\rho, k)}{\partial k^3} \bigg|_{k_F} k_F^3 \right] \delta^4 + ... \]

(28)

Contributions from the higher-order potential terms in \( \delta \) can be obtained similarly. For the second order-symmetry energy \( E_{\text{sym},2}(\rho) \), only the \( U_{\text{sym},1} \) term contributes.
\[ E_{1}^{\text{pot}}(\rho) = \alpha \int_{0}^{k_F} U_{\text{sym},1}(\rho, k) \delta^* k^2 dk + \alpha \int_{0}^{k_F} U_{\text{sym},1}(\rho, k) \delta k^2 dk \]

\[ = \alpha \int_{0}^{k_F} U_{\text{sym},1}(\rho, k) \left( \frac{k}{k_F} \right)^3 - 1 \delta^* k^2 dk + \alpha \int_{0}^{k_F} U_{\text{sym},1}(\rho, k) \left( \frac{k}{k_F} \right)^3 - 1 \delta k^2 dk \]

\[ = 2\alpha \int_{0}^{k_F} U_{\text{sym},1}(\rho, k) \left( \frac{k}{k_F} \right)^3 - 1 \delta k^2 dk \]

\[ + \frac{1}{2} U_{\text{sym},1}(\rho, k_F) \delta^2 + \left[ -\frac{1}{36} \frac{\partial U_{\text{sym},1}(\rho, k)}{\partial k} |_{k_F} k_F^2 + \frac{1}{72} \frac{\partial^2 U_{\text{sym},1}(\rho, k)}{\partial k^2} |_{k_F} k_F^2 \right] \delta^4 + \ldots \]

(29)

To obtain the fourth-order symmetry energy \( E_{\text{sym},4}(\rho) \), all \( U_{\text{sym},1}, U_{\text{sym},2}, \) and \( U_{\text{sym},3} \) terms are needed, i.e.

\[ E_{2}^{\text{pot}}(\rho) = \alpha \int_{0}^{k_F} U_{\text{sym},2}(\rho, k) \delta^* k^2 dk + \alpha \int_{0}^{k_F} U_{\text{sym},2}(\rho, k) \delta k^2 dk \]

\[ = \alpha \int_{0}^{k_F} U_{\text{sym},2}(\rho, k) \left( \frac{k}{k_F} \right)^3 - 1 k^2 dk + \alpha \int_{0}^{k_F} U_{\text{sym},2}(\rho, k) \left( \frac{k}{k_F} \right)^3 - 1 \delta k^2 dk \]

\[ = 2\alpha \int_{0}^{k_F} U_{\text{sym},2}(\rho, k) \left( \frac{k}{k_F} \right)^3 - 1 \delta k^2 dk + \frac{1}{12} \frac{\partial U_{\text{sym},2}(\rho, k)}{\partial k} |_{k_F} k_F^2 \delta^4 + \ldots \]

(30)

and

\[ E_{3}^{\text{pot}}(\rho) = \alpha \int_{0}^{k_F} U_{\text{sym},3}(\rho, k) \delta^* k^2 dk + \alpha \int_{0}^{k_F} U_{\text{sym},3}(\rho, k) \delta k^2 dk \]

\[ = \alpha \int_{0}^{k_F} U_{\text{sym},3}(\rho, k) \left( \frac{k}{k_F} \right)^3 - 1 k^2 dk + \alpha \int_{0}^{k_F} U_{\text{sym},3}(\rho, k) \left( \frac{k}{k_F} \right)^3 - 1 \delta k^2 dk \]

\[ = 2\alpha \int_{0}^{k_F} U_{\text{sym},3}(\rho, k) \left( \frac{k}{k_F} \right)^3 - 1 \delta k^2 dk + \frac{1}{4} U_{\text{sym},3}(\rho, k_F) \delta^4 + \ldots \]

(31)

Combining all coefficients of the \( \delta^2 \) and \( \delta^4 \) terms in both the kinetic and potential parts given above, one sees that the \( E_{\text{sym},2}(\rho) \) and \( E_{\text{sym},4}(\rho) \) obtained here are exactly the same as those given in using directly the HVH theorem. Moreover, they agree with earlier results obtained by Brueckner et al. [35,36].
3 Applications and discussions

As shown in the previous section, the symmetry energy can be explicitly separated into the kinetic energy term $T$ and the potential terms $U_0$ and $U_{\text{sym},i}$ at the Fermi momentum $k_F$. To evaluate their relative contributions to the symmetry energies, especially for the second-order and fourth-order terms $E_{\text{sym},2}(\rho)$ and $E_{\text{sym},4}(\rho)$, we consider in this section two typical single-nucleon potentials that have been widely used in tansport model simulations of heavy-ion reactions.

3.1 The Bombaci-Gale-Bertsch-Das Gupta potential

As a first example, we use the phenomenological potential of Bombaci-Gale-Bertsch-Das Gupta \cite{16}

$$U_{\tau}(u, \delta, k) = Au + Bu^\sigma - \frac{2}{3}(\sigma - 1) \frac{B}{\sigma + 1} \left( \frac{1}{2} + x_3 \right) u^\sigma \delta^2$$

$$ \pm \left[ - \frac{2}{3} A \left( \frac{1}{2} + x_0 \right) u - \frac{4}{3} \frac{B}{\sigma + 1} \left( \frac{1}{2} + x_3 \right) u^\sigma \right] \delta$$

$$ + \frac{4}{5\rho_0} \left[ \frac{1}{2} (3C - 4z_1) I_\tau + (C + 2z_1) I_{\tau'} \right] + \left( C \pm \frac{C - 8z_1}{5} \right) u \cdot g(k), \quad (32)$$

where $u = \rho/\rho_0$ is the reduced density and $\pm$ is for neutrons/protons. In the above, we have $I_{\tau} = \left[ 2/(2\pi)^3 \right] \int d^3k f_{\tau}(k) g(k)$ with $g(k) = 1/[1 + (k/\Lambda)^2]$ being a momentum regulator and $f_{\tau}(k)$ being the phase space distribution function. The parameter $\Lambda$ has the value $\Lambda = 1.5k_F^0$, where $k_F^0$ is the nucleon Fermi wave number in symmetric nuclear matter at $\rho_0$. With $A=-144$ MeV, $B=203.3$ MeV, $C=-75$ MeV and $\sigma = 7/6$, the BGBD potential reproduces all ground state properties including an incompressibility $K_0=210$ MeV for symmetric nuclear matter \cite{16}. The three parameters $x_0, x_3$ and $z_1$ can be adjusted to give different symmetry energy $E_{\text{sym},2}(\rho)$ and the neutron-proton effective mass splitting $m_n^* - m_p^*$ \cite{16,44,45}. For example, the parameter set $z_1 = -36.75$ MeV, $x_0 = -1.477$ and $x_3 = -1.01$ leads to $m_n^* > m_p^*$ while the one with $z_1 = 50$ MeV, $x_0 = 1.589$ and $x_3 = -0.195$ leads to $m_n^* < m_p^*$ at all non-zero densities and isospin asymmetries.

On expanding the BGBD potential in $\delta$, the coefficients of the first few terms are

$$12$$
\[ U_0(\rho, k) = U_{n/p} \big|_{\delta=0} =Au + Bu^\sigma + \frac{2C}{\rho_0} \frac{A^2}{\pi^2} \left[ k_F - \Lambda \tan^{-1}\left( \frac{k_F}{\lambda} \right) \right] + Cu \cdot g(k), \quad (33) \]

\[ U_{\text{sym,1}}(\rho, k) = \pm \frac{1}{1!} \frac{\partial U_{n/p}}{\partial \delta} \big|_{\delta=0} = \left[ \frac{2}{3} A \left( \frac{1}{2} + x_0 \right) u - \frac{4}{3} B \frac{1}{\sigma + 1} \left( \frac{1}{2} + x_3 \right) u^\sigma \right] \\
+ C - 8z_1 \frac{5}{u} \cdot g(k_F) + C - 8z_1 \frac{5}{u} \cdot g(k), \quad (34) \]

\[ U_{\text{sym,2}}(\rho, k) = \frac{1}{2!} \frac{\partial^2 U_{n/p}}{\partial \delta^2} \big|_{\delta=0} = -\frac{2}{3} (\sigma - 1) \frac{B}{\sigma + 1} \left( \frac{1}{2} + x_3 \right) u^\sigma - \frac{C}{3} \frac{k_F^2}{\Lambda^2} g(k_F)^2, \quad (35) \]

\[ U_{\text{sym,3}}(\rho, k) = \pm \frac{1}{3!} \frac{\partial^3 U_{n/p}}{\partial \delta^3} \big|_{\delta=0} = \frac{C - 8z_1}{135} \frac{k_F^2}{\Lambda^2} \left( 5 \frac{k_F^2}{\Lambda^2} + 1 \right) g(k_F)^3. \quad (36) \]

Thus, the second-order symmetry energy \( E_{\text{sym,2}}(\rho) \) is given by

\[ E_{\text{sym,2}}(\rho) = \frac{1}{3} t(k_F) + \frac{1}{6} \frac{\partial U_0}{\partial k} \big|_{k_F} \cdot k_F + \frac{1}{2} U_{\text{sym,1}}(\rho, k_F) \]
\[ = \frac{\hbar^2}{6m} \left( \frac{3\pi^2}{2} \right)^{2/3} \rho^{2/3} - \frac{C}{3} \frac{k_F^2}{\Lambda^2} g(k_F)^2 \]
\[ + \left[ -\frac{1}{3} A \left( \frac{1}{2} + x_0 \right) u - \frac{2}{3} B \frac{1}{\sigma + 1} \left( \frac{1}{2} + x_3 \right) u^\sigma \right] \\
+ C - 8z_1 \frac{5}{u} \cdot g(k_F), \quad (37) \]

and the fourth-order symmetry energy \( E_{\text{sym,4}}(\rho) \) is
\[ E_{\text{sym},4}(\rho) = \frac{\hbar^2}{162m} \left( \frac{3\pi^2}{2} \right)^{2/3} \rho^{2/3} \]

\[ + \left[ \frac{5}{324} \frac{\partial U_0(\rho, k)}{\partial k} \right]_{k_F} \left[ k_F \right] - \frac{1}{108} \frac{\partial^2 U_0(\rho, k)}{\partial k^2} \left[ k_F \right] + \frac{1}{648} \frac{\partial^3 U_0(\rho, k)}{\partial k^3} \left[ k_F \right] \]

\[ - \frac{1}{36} \frac{\partial U_{\text{sym},1}(\rho, k)}{\partial k} \left[ k_F \right] + \frac{1}{72} \frac{\partial^2 U_{\text{sym},1}(\rho, k)}{\partial k^2} \left[ k_F \right] + \frac{1}{12} \frac{\partial U_{\text{sym},2}(\rho, k)}{\partial k} \left[ k_F \right] + \frac{1}{4} U_{\text{sym},3}(\rho, k, F) \]

\[ = \frac{\hbar^2}{162m} \left( \frac{3\pi^2}{2} \rho \right)^{2/3} + \frac{C}{81} \frac{k_F}{\Lambda^2} \left( \frac{10k_F^4}{\Lambda^4} + 5k_F^2 \right) + 1 \right) \frac{1}{g(k_F)} + \frac{C}{135} \frac{k_F^2}{\Lambda^2} \left( 5k_F^2 + 1 \right) \frac{1}{g(k_F)} \frac{1}{3}. \]

(38)

Fig. 1. The kinetic energy part (T), the isoscalar potential part (U₀) and the isovector potential part (U_{sym,1}) of the symmetry energy E_{sym,2} from the BGBD potential with m_n^* > m_p^* (left) and for m_n^* < m_p^* (right).

In Fig. 1 we compare E_{sym,2}(\rho) and its three components in the two cases of m_n^* > m_p^* and m_n^* < m_p^*. It is seen that the kinetic and isoscalar contributions are the same in both cases. However, they have significantly different isovector potentials U_{sym,1}, leading thus to different E_{sym,2}(\rho) especially at supra-saturation densities.

Various contributions to the fourth-order symmetry energies E_{sym,4}(\rho) in the two cases are compared in Fig. 2. Similar to E_{sym,2}(\rho), the contributions of the T and U₀ terms to E_{sym,4}(\rho) are positive and they are the same in both cases. Interestingly, the U_{sym,1} term also plays the most important role in determining the high-density behavior of E_{sym,4}(\rho). It is positive in the case of m_n^* > m_p^* but negative in the case of m_n^* < m_p^*, resulting in very different behaviors of E_{sym,4}(\rho) at supra-saturation densities. Moreover, it is interesting
to note that $E_{\text{sym},4}(\rho)$ receives no contribution from the $U_{\text{sym},2}$ term. This is not surprising because the $U_{\text{sym},2}$ term in the BGBD interaction is momentum independent and its contribution to $E_{\text{sym},4}(\rho)$ is actually $\frac{1}{12} \frac{\partial U_{\text{sym},2}(k)}{\partial k}|_{k_F=0}$. On the contrary, the $U_{\text{sym},3}$ term still contributes to $E_{\text{sym},4}(\rho)$ via $\frac{1}{4} U_{\text{sym},3}(k_F)$ although it is momentum independent too. In the two cases considered here, the contributions from the $U_{\text{sym},3}$ term also have opposite sign.

To compare the fourth-order term $E_{\text{sym},4}(\rho)$ with the second-order term $E_{\text{sym},2}(\rho)$ more clearly, we show in Fig. 3 their ratio $E_{\text{sym},4}(\rho)/E_{\text{sym},2}(\rho)$ as a function of the reduced density $\rho/\rho_0$. Obviously, the relative value of $E_{\text{sym},4}(\rho)$ is generally small. However, it can reach up to about $\pm10\%$ at high densities for both cases of $m_n^* > m_p^*$ and $m_n^* < m_p^*$. It may thus lead to an appreciable modification in the proton fraction and therefore the properties of neutron stars at $\beta$-equilibrium.

### 3.2 A modified Gogny Momentum-Dependent-Interaction

In this subsection, we discuss the symmetry energy obtained from the MDI interaction [23], which is derived from the Hartree-Fock approximation using a modified Gogny effective interaction [40].
Fig. 3. The ratio of $E_{\text{sym},4}$ over $E_{\text{sym},2}$ with the BGBD potential for $m^*_n > m^*_p$ (left) and for $m^*_n < m^*_p$ (right).

\[ U(\rho, \delta, \vec{p}, \tau) = A_u(x) \frac{\rho_x}{\rho_0} + A_l(x) \frac{\rho_x}{\rho_0} \]
\[ + B \left( \frac{\rho}{\rho_0} \right)^\sigma (1 - x\delta^2) - 8\rho_x B \frac{\rho^{\sigma-1}}{\rho_0^\sigma} \delta \rho_x \]
\[ + \frac{2C_{\tau,\tau'}}{\rho_0} \int d^3 p' \frac{f_\tau(\vec{r},\vec{p})}{1 + (\vec{p} - \vec{p'})^2/\Lambda^2} + \frac{2C_{\tau',\tau'}}{\rho_0} \int d^3 p' \frac{f_{\tau'}(\vec{r},\vec{p})}{1 + (\vec{p} - \vec{p'})^2/\Lambda^2}. \] (39)

In the above, $\tau = 1/2$ ($-1/2$) for neutrons (protons) and $\tau \neq \tau'$; $\sigma = 4/3$ is the density-dependence parameter; $f_\tau(\vec{r},\vec{p})$ is the phase space distribution function at coordinate $\vec{r}$ and momentum $\vec{p}$. The parameters $B, C_{\tau,\tau}, C_{\tau,\tau'}$ and $\Lambda$ are obtained by fitting the nuclear matter saturation properties \[23\]. The momentum dependence of the symmetry potential stems from the different interaction strength parameters $C_{\tau,\tau'}$ and $C_{\tau,\tau'}$ for a nucleon of isospin $\tau$ interacting, respectively, with unlike and like nucleons in the background fields. More specifically, $C_{\text{unlike}} = -103.4$ MeV while $C_{\text{like}} = -11.7$ MeV. The quantities $A_u(x) = -95.98 - x\frac{2B}{\sigma+1}$ and $A_l(x) = -120.57 + x\frac{2B}{\sigma+1}$ are parameters. The parameters $B$ and $\sigma$ in the MDI single-particle potential are related to the $t_0$ and $\alpha$ in the Gogny effective interaction via $t_0 = \frac{8B}{3(\sigma+1)\rho_0}$ and $\sigma = \alpha + 1$ \[40\]. The parameter $x$ is related to the spin(isospin)-dependence parameter $x_0$ via $x = (1 + 2x_0)/3$ \[46\]. On expanding the single-nucleon potential in $\delta$, the first four terms are
\[ U_0(\rho, k) = U_{n/p} \big|_{\delta=0} \]
\[ = \frac{(A_l + A_u)}{2} \frac{\rho}{\rho_0} + B \left( \frac{\rho}{\rho_0} \right)^{\sigma} + \frac{2(C_{\tau,\tau} + C_{\tau,\tau'})}{2} \frac{\rho}{\rho_0} \frac{2p_F^2 + \Lambda^2 - \rho^2}{2p\Lambda} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) + \frac{2p_F\pi\Lambda^2}{3h^3p} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \]
\[ \times \left[ \frac{p_F^2 + \Lambda^2 - p^2}{2p\Lambda} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) + \frac{2p_F\pi\Lambda^2}{3h^3p} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \right] , \tag{40} \]

\[ U_{n/m,1}(\rho, k) = \pm \frac{1}{1!} \frac{\partial U_{n/p}}{\partial \delta} \big|_{\delta=0} \]
\[ = \frac{(A_l - A_u)}{2} \frac{\rho}{\rho_0} - 2x \frac{B}{\sigma + 1} \frac{\rho}{\rho_0}^{\sigma} + \frac{2(C_{\tau,\tau} - C_{\tau,\tau'})}{2} \frac{2p_F^2\pi\Lambda^2}{3h^3p} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \]
\[ - \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \]
\[ \times \left[ \frac{p_F^2 + \Lambda^2 - \rho^2}{2p\Lambda} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) + \frac{2p_F\pi\Lambda^2}{3h^3p} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \right] , \tag{41} \]

\[ U_{n/m,2}(\rho, k) = \frac{1}{2!} \frac{\partial^2 U_{n/p}}{\partial \delta^2} \big|_{\delta=0} \]
\[ = -B \frac{\rho}{\rho_0}^{\sigma} \left[ \frac{\rho}{\rho_0} \frac{2p_F^2 \pi \Lambda^2}{3\rho_0} \right] \frac{4p_F^2(2p^2 - p_F^2 + \Lambda^2)}{9h^3p} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \]
\[ - \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \]
\[ \times \left[ \frac{p_F^2 + \Lambda^2 - \rho^2}{2p\Lambda} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) + \frac{2p_F\pi\Lambda^2}{3h^3p} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \right] . \tag{42} \]

\[ U_{n/m,3}(\rho, k) = \pm \frac{1}{3!} \frac{\partial^3 U_{n/p}}{\partial \delta^3} \big|_{\delta=0} \]
\[ = -\frac{(C_{\tau,\tau} - C_{\tau,\tau'})}{3\rho_0} \frac{4p_F^2 \pi \Lambda^2}{81h^3p} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \]
\[ \times \left[ \frac{2pp_F(2p^6 - 3p_F^6 + 5p_F^2\Lambda^4 + 2\Lambda^6 + p^4(-7p_F^2 + 6\Lambda^2) + p^2(8p_F^4 - 2p_F^2\Lambda^2 + 6\Lambda^4))}{(p + p_F)^2 + \Lambda^2} \right] \]
\[ - \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \]
\[ - \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \]
\[ \times \left[ \frac{p_F^2 + \Lambda^2 - \rho^2}{2p\Lambda} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) + \frac{2p_F\pi\Lambda^2}{3h^3p} \ln \left( \frac{(p + p_F)^2 + \Lambda^2}{(p - p_F)^2 + \Lambda^2} \right) \right] . \tag{43} \]

According to Eq. (43), the second-order symmetry energy \( E_{sym,2}(\rho) \) is
\[ E_{\text{sym},2}(\rho) = \frac{1}{3} t(k_F) + \frac{1}{6} \frac{\partial U_0}{\partial k} |_{k_F} k_F + \frac{1}{2} U_{\text{sym},1}(\rho, k_F) \]

\[ = \frac{\hbar^2}{6m} \left( \frac{3\pi^2}{2} \right)^{2/3} \rho^{2/3} \]

\[ + \frac{(C_{\tau,\tau} + C_{\tau,\tau'}) \pi \Lambda^2}{3\rho_0} \left[ \frac{4p_F}{h^3} - \left( 2p_F + \frac{\Lambda^2}{p_F} \right) \ln \left( \frac{4p_F^2 + \Lambda^2}{\Lambda^2} \right) \right] \]

\[ + \frac{(A_l - A_u) \rho}{4} \frac{B}{\rho^\sigma + 1} \rho^\sigma \]

\[ + \frac{(C_{\tau,\tau} - C_{\tau,\tau'}) \pi \Lambda^2}{3\rho_0} \frac{2p_F \ln \left( \frac{4p_F^2 + \Lambda^2}{\Lambda^2} \right)}{h^3} \]

and according to Eq. (42) the fourth-order symmetry energy \( E_{\text{sym},4}(\rho) \) is

\[ E_{\text{sym},4}(\rho) = \frac{\hbar^2}{162m} \left( \frac{3\pi^2}{2} \right)^{2/3} \rho^{2/3} \]

\[ + \left[ \frac{5}{324} \frac{\partial U_0(\rho, k)}{\partial k} |_{k_F} k_F - \frac{1}{108} \frac{\partial^2 U_0(\rho, k)}{\partial k^2} |_{k_F} k_F^2 + \frac{1}{648} \frac{\partial^3 U_0(\rho, k)}{\partial k^3} |_{k_F} k_F^3 \right] \]

\[ - \frac{1}{36} \frac{\partial U_{\text{sym},1}(\rho, k)}{\partial k} |_{k_F} k_F + \frac{1}{72} \frac{\partial^2 U_{\text{sym},1}(\rho, k)}{\partial k^2} |_{k_F} k_F^2 + \frac{1}{12} \frac{\partial U_{\text{sym},2}(\rho, k)}{\partial k} |_{k_F} k_F + \frac{1}{4} U_{\text{sym},3}(\rho, k_F) \]

\[ = \frac{\hbar^2}{162m} \left( \frac{3\pi^2 \rho}{2} \right)^{2/3} - \frac{C_{\tau,\tau'}}{3^5 \rho_0} \left( \frac{4\pi}{h^3} \right)^2 \Lambda^2 \left[ 7\Lambda^2 p_f^2 \ln \frac{4p_f^2 + \Lambda^2}{\Lambda^2} - \frac{4(7\Lambda^4 p_f^4 + 42\Lambda^2 p_f^6 + 40p_f^8)}{(4p_f^2 + \Lambda^2)^2} \right] \]

\[ - \frac{C_{\tau,\tau'}}{3^5 \rho_0} \left( \frac{4\pi}{h^3} \right)^2 \Lambda^2 \left[ (7\Lambda^2 p_f^2 + 16p_f^4) \ln \frac{4p_f^2 + \Lambda^2}{\Lambda^2} - 28p_f^4 - \frac{8p_f^6}{\Lambda^2} \right] \]

As one expects, the above expressions are identical to those derived directly from the exact MDI EOS using [47]

\[ E_{\text{sym},2}(\rho) = \frac{1}{2!} \frac{\partial^2 E(\rho, \delta)}{\partial \delta^2} |_{\delta=0} \]

\[ E_{\text{sym},4}(\rho) = \frac{1}{4!} \frac{\partial^4 E(\rho, \delta)}{\partial \delta^4} |_{\delta=0} \] (46)

In Fig. 4 we show the kinetic (T), isoscalar (\( U_0 \)) and isovector (\( U_{\text{sym},1} \)) potential contributions to \( E_{\text{sym},2} \) for the three different spin (isospin)-dependence parameter \( x = 1, 0, \) and -1. We notice that the kinetic (T) and the isoscalar potential (\( U_0 \)) contributions are the same for the three different \( x \) values. As pointed out in Ref. [23], it is the isovector potential \( U_{\text{sym},1} \) that is causing the different density dependence of \( E_{\text{sym},2} \). For instance, with \( x = 1 \) the \( U_{\text{sym},1} \) term decreases very quickly with increasing density and thus results in
Fig. 4. The kinetic energy part (T), the isoscalar potential part ($U_0$) and the isovector potential part ($U_{sym,1}$) of the symmetry energy $E_{sym,2}$ from the MDI interaction with $x = 1$, 0 and -1.

a super-soft symmetry energy at supra-saturation densities. On the contrary, the symmetry energy $E_{sym,2}$ at supra-saturation densities is very stiff for both $x = 0$ and $x = -1$ as the contribution of the $U_{sym,1}$ term becomes very positive with smaller values of $x$.

Fig. 5. The kinetic energy and potential contributions to the fourth-order symmetry energy $E_{sym,4}$ from the MDI interaction.
Unlike the second-order term $E_{\text{sym},2}$, the fourth-order symmetry energy $E_{\text{sym},4}$ is independent of the spin (isospin)-dependence parameter $x$. Shown in Fig. 5 are the various contributions to the fourth-order symmetry energy $E_{\text{sym},4}$. Comparing these with the results obtained using the BGBD in Fig. 2, we find that the $T$ and $U_0$ terms from these two interactions are almost identical. However, there exists some differences for other terms. For the MDI interaction, the $U_{\text{sym},2}$ term is negative and becomes very important for determining $E_{\text{sym},4}$. One the contrary, the contributions from the $U_{\text{sym},1}$ and $U_{\text{sym},3}$ terms are positive and they are relatively small as compared to $U_{\text{sym},2}$. Generally, the behavior of $E_{\text{sym},4}$ from the MDI interaction is very similar to that from the BGBD interaction for the case of $m_n^* > m_p^*$.

![Fig. 6. The ratio of $E_{\text{sym},4}$ over $E_{\text{sym},2}$ obtained from the MDI interaction as a function of reduced density $\rho/\rho_0$ for $x = 1, 0, \text{ and } -1$.](image)

To compare more directly $E_{\text{sym},4}$ with $E_{\text{sym},2}$, their ratio $E_{\text{sym},4}/E_{\text{sym},2}$ is plotted in Fig. 6 as a function of reduced density for $x = 1, 0, \text{ and } -1$. It is seen that with $x = 1$ there is a sharp break in the curve around $3\rho_0$. This is because the second-order symmetry energy $E_{\text{sym},2}$ changes from positive to negative around $3\rho_0$ in this case. However, this is not the case for both $x = -1$ and $x = 0$ where $E_{\text{sym},2}$ remains positive at all densities. In all cases, $E_{\text{sym},4}$ is very small compared to $E_{\text{sym},2}$. For both BGBD and MDI interactions, the small values of $E_{\text{sym},4}$ up to several times the normal density clearly shows that the parabolic approximation of the EOS is well justified for most purposes. However, cares have to be taken in evaluating the core-crust transition density where the energy curvatures are involved.
4 Summary

In summary, using the Hugenholtz-Van Hove theorem we have derived general expressions for the quadratic and quartic symmetry energies in terms of single-particle potentials in isospin asymmetric nuclear matter. Identical results are obtained by using two approaches, i.e., one based on the single-particle potential and the other based on the total energy, that are physically identical although mathematically different. By using the derived analytical formulas, the symmetry energies are explicitly separated into the kinetic and several potential parts. The formalism is applied to two typical single-nucleon potentials, namely the Bombaci-Gale-Bertsch-Das Gupta (BGBD) potential and the modified Gogny Momentum-Dependent-Interaction (MDI), that are widely used in transport model simulations of heavy-ion reactions. We find that for both interactions the isovector potential is responsible for the uncertain high density behavior of the quadratic symmetry energy. Also, the magnitude of the quartic symmetry energy in both cases is found to be significantly smaller than that of the quadratic symmetry energy. We expect that the analytical formulas for the nuclear symmetry energies derived in the present study will be useful in extracting reliable information about the EOS of neutron-rich nuclear matter from heavy-ion reactions.

Acknowledgements

This work is supported in part by the US National Science Foundation grants PHY-0757839 and PHY-0758115, the Research Corporation under grant No.7123, the Welch Foundation under grant No. A-1358, the Texas Coordinating Board of Higher Education grant No.003565-0004-2007, the National Natural Science Foundation of China grants 10735010, 10775068, 10805026, and 10975097, Shanghai Rising-Star Program under grant No. 06QA14024, the National Basic Research Program of China (973 Program) under Contract No. 2007CB815004 and 2010CB833000.

References

[1] J. M. Lattimer, M. Prakash, Science 304 (2004) 536.

[2] A. W. Steiner et al., Phys. Rep. 411 (2005) 325.

[3] B. A. Li, L. W. Chen and C. M. Ko, Phys. Rep. 464 (2008) 113.

[4] P. J. Siemens, Nucl. Phys. A 141 (1970) 225.

[5] C. -H. Lee, T. T. Kuo, G. Q. Li, and G. E. Brown, Phys. Rev. C 57 (1998) 3488.
[6] A. W. Steiner, Phys. Rev. C 74 (2006) 045808.
[7] F. S. Zhang and L. W. Chen, Chin. Phys. Lett. 18 (2001) 142.
[8] O. Sjöberg, Nucl. Phys. A 222 (1974) 161.
[9] J. Xu, L.W. Chen, B.A. Li, and H.R. Ma, Phys. Rev. C 79 (2009) 035802; Astrophys. J. 697 (2009) 1549.
[10] B. A. Li, C. M. Ko and W. Bauer, Int. Jour. Mod. Phys. E 7 (1998) 147.
[11] B. A. Brown, Phys. Rev. Lett. 85 (2000) 5296.
[12] Isospin Physics in Heavy-Ion Collisions at Intermediate Energies, Eds. Bao-An Li and W. Udo Schröer (Nova Science Publishers, Inc, New York, 2001).
[13] P. Danielewicz, R. Lacey and W.G. Lynch, Science 298 (2000) 1592.
[14] V. Baran et al., Phys. Rep. 410 (2005) 335.
[15] K. Sumiyoshi and H. Toki, Astrophys. J. 422 (1994) 700.
[16] I. Bombaci, Chapter 2 in Ref.[6].
[17] L. W. Chen, C. M. Ko and B. A. Li, Phys. Rev. Lett. 94 (2005) 032701; B. A. Li and L. W. Chen, Phys. Rev. C 72 (2005) 064611.
[18] M. B. Tsang, Yingxun Zhang, P. Danielewicz, M. Famiano, Zhuxia Li, W. G. Lynch, and A. W. Steiner, Phys. Rev. Lett. 102 (2009) 122701.
[19] M. Centelles, X. Roca-Maza, X. Vinas and M. Warda, Phys. Rev. Lett. 102 (2009) 122502.
[20] J. B. Natowitz et al., Phys. Rev. Lett. (2010) in press.
[21] Z. G. Xiao, B. A. Li, L. W. Chen, G. C. Yong and M. Zhang, Phys. Rev. Lett. 102 (2009) 062502.
[22] D. H. Wen, B. A. Li and L. W. Chen, Phys. Rev. Lett. 103 (2009) 211102.
[23] C. B. Das, S. Das Gupta, C. Gale, B. A. Li, Phys. Rev. C 67 (2003) 034611.
[24] S. Ulrych and H. Münther, Phys. Rev. C 56 (1997) 1788.
[25] E. N. E. van Dalen, C. Fuchs, and A. Faessler, Nucl. Phys. A 744 (2004) 227.
[26] W. Zuo, L. G. Cao, B. A. Li, U. Lombardo, and C. W. Shen, Phys. Rev. C 72 (2005) 014005.
[27] S. Fritsch, N. Kaiser, W. Weise, Nucl. Phys. A 750 (2005) 259.
[28] J. A. McNeil, J. R. Shepard, S. J. Wallace, Phys. Rev. Lett. 50 (1983) 1439.
[29] L. W. Chen, C. M. Ko, B. A. Li, Phys. Rev. C 72 (2005) 064606.
[30] Z. H. Li, L. W. Chen, C. M. Ko, B. A. Li, and H. R. Ma, Phys. Rev. C 74 (2006) 044613.
[31] J.R. Stone, J.C. Miller, R. Koncewicz, P.D. Stevenson, M.R. Strayer, Phys. Rev. C 68 (2003) 034324.

[32] V. R. Pandharipande, V.K. Garde, Phys. Lett. B 39 (1972) 608.

[33] R. B. Wiringa et al., Phys. Rev. C 38 (1988) 1010.

[34] M. Kutschera, Phys. Lett. B 340 (1994) 1.

[35] K. A. Brueckner and J. Dabrowski, Phys. Rev. 134 (1964) B722.

[36] J. Dabrowski and P. Haensel, Phys. Lett. B 42 (1972) 163; Phys. Rev. C 7 (1973) 916; Can. J. Phys. 52 (1974) 1768.

[37] A. M. Lane, Nucl. Phys. 35 (1962) 676.

[38] N. M. Hugenholtz and L. Van Hove, Physica 24, 363 (1958).

[39] L. Satpathy, U.S. Uma maheswari and R.C. Nayak, Phys. Rep. 319 (1999) 85.

[40] J. Decharge and D. Gogny, Phys. Rev. C 21 (1980) 1568.

[41] M. A. Preston and R. K. Bhaduri, Structure of the Nucleus (Addison-Wesley, Reading, MA, 1975), p. 191-202.

[42] C. Xu and B. A. Li, arXiv:0910.4803.

[43] G. F. Bertsch and S. Das Gupta, Phys. Rep. 160 (1988) 189.

[44] J. Rizzo, M. Colonna, M. DiToro, and V. Greco, Nucl. Phys. A 732 (2004) 202.

[45] B. A. Li, Phys. Rev. C 69 (2004) 064602.

[46] C. Xu and B. A. Li, Phys. Rev. C 81 (2010) 044603.

[47] L. W. Chen, B. J. Cai, C. M. Ko, B. A. Li, C. Shen and J. Xu, Phys. Rev. C 80 (2009) 014322.