Multiparticle quantum walk in the strongly interacting $SU(3)$ Su-Schrieffer-Heeger-Hubbard topological model

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Recent experiments on single-particle quantum walks in photonic structures indicate that the photon dynamics is capable to capture the topological properties of the system’s bulk states. However, the role of interactions in such topological quantum walks has not been explored exhaustively. Here we study multiparticle continuous time quantum walks in interacting Su-Schrieffer-Heeger lattices with multiple flavors, and compute the many-body Berry phase and the single and many-body chiral polarization to assess the effect of interactions on topology. We find that bulk topology survives strong interactions in the presence of inversion symmetry and it is robust to moderate disorder both in equilibrium and during a quantum walk. Our predictions are well within the experimental reach for cold atomic gases and can be used to detect the topological properties of excited states through dynamical probes.

Introduction. Our conventional understanding of the phase transition theory associated with a symmetry breaking and the emergence of a local order parameter [1] needs to be extended with the advent of topological insulators [2, 3]. Such materials are not characterized by a local order parameter and display various topological phases identified by topological invariants [4, 5]. In non-interacting systems, the bulk topological invariants are used to predict robust low-energy physics occurring at the boundaries, by the so-called bulk-boundary correspondence (BBC) [6]. Not only are these used to reveal the topological properties of matter, but also hold the promise to revolutionize quantum computation, quantum technologies, and spintronics [7–9].

Among the large family of topological systems, one dimensional insulators endowed with a chiral symmetry play a special role. Their topological properties are encoded in the Zak phase $\gamma_B = \pi \nu$ (modulo $2\pi$) of the bulk Bloch states [10, 11]. Through BBC, the winding number $\nu$ counts the zero-energy states in the system that are guaranteed to exist at the physical boundaries, and which are protected by the chiral symmetry against various perturbations [12]. Even though by now the basic physics of non-interacting topological insulators is well understood, it is still not clear whether the BBC survives in the presence of strong interactions [13] or even more importantly, how to characterize an interacting topological insulator [14, 15]. The existence of a quantized Berry phase $\gamma_B$ might be the most natural candidate [16–19], as it is generalizes the Zak’s phase of non-interacting models. Other proposals indicate that the degeneracy of the entanglement spectrum [20, 21] or entropy itself [22] are related to the bulk topology, but all these theoretical predictions are difficult to probe experimentally in bulk systems.

An alternative route to probe topology, more appealing from an experimental point of view invokes the quench dynamics of quantum particles in single-particle quantum walk. Recent experiments have revealed the presence of bound states at the interface of systems with different topological phases [23, 24] and allow the detection of topological invariants in cold atoms [25] or in nanophotonic topological lattice through the chiral polarization [26–28]

$$P_1(t) = \sum_x \langle \Psi(t) | \Gamma_n | \Psi(t) \rangle.$$  (1)

Although $P_1(t)$ depends on time, in the long-time limit, it converges to the winding number $\nu$, $P_1(t \to \infty) \approx \nu/2$. By definition, $P_1(t)$ measures the expectation value of the chiral displacement operator, $\Gamma p(x)$, with $\Gamma$ the associated chiral symmetry and $p(x) = x n(x)$ the regular polarization, and $n(x)$ the particle density operator [29]. The average is taken with respect to the time-dependent state of the system $|\Psi(t)\rangle$. So far, the chiral polarization has been measured experimentally only in a non-interacting setup, by performing single-photon quantum walks in topological photonic lattices [26, 28, 30, 31].

By now, experimental quantum walks have been implemented for trapped atoms and ions [32, 33], photons [34–36], or spin impurities [37, 38], and the full control over the dynamics has been achieved. Furthermore, with the
FIG. 1. (a) Multiparticle quantum walk on an SSH lattice. $U$ represents the strength of the on-site interaction. The trion is injected at $t = 0$. At a later times, measuring the single or multiparticle chiral polarization, provides information about the bulk states’ topology. (b) Typical single particle spectrum in the topological regime of the non-interacting spinless SSH model. The red lines indicate the in-gap topological states. (c) Many-body spectrum for the $SU(3)$ model when $U \gg J$. The band structure consists of three bands separated by energy gaps of the order of $U$.

Recent advances in nanophotonics [39, 40], quantum walks of correlated photons, or correlations effects in Bloch oscillations [41, 42] in multiparticle quantum walk have already been measured. In that regard it is interesting to inquire whether the chiral polarization is a suitable quantity to capture the topology in the presence of strong interactions as well [43].

In the present work we address this problem and investigate the effect of interactions on the bulk states’ topology in a multiparticle quantum walk setup. For that, we corroborate the results for the Berry phase, computed using the many-body spectrum in a subspace with a reduced number of particles with the single particle and many-body chiral polarization results.

In general, for the $SU(N)$ SSH model, situations where $N$ is odd or even differ significantly from one another. The Berry phase $\gamma_B$ is marked by a jump of $\pi N$ (modulo $2\pi$) at the topological transition, therefore, when $N = 2n$, $\gamma_B$ is unable to distinguish between the topological phases. This is inherited from the noninteracting version of the model where an even number of flavors dictates the same behavior in $\gamma_B$. When the model supports an odd number of flavors, such as in the $SU(3)$ case, $\gamma_B$ displays a jump of $\pi$ and becomes a good measure for the bulk topology, corroborating the chiral polarization results. Nevertheless, the single and many-body chiral polarization identify correctly the topological phases for any number of flavors [44], are robust to strong interactions and disorder, making them suitable measures to interrogate the bulk states’ topology. In order to clearly substantiate the two approaches, in the present work we explore the simplest non-trivial model corresponding to $N = 3$. The results for the $SU(2)$ SSH model are detailed in the supplementary material [44], where also extensions to the $SU(N > 3)$ are considered.

The prototypical $SU(N = 3)$ SSH model [45] with on-site Hubbard interaction is given by

$$H = -J \sum_{x=\pm L/2}^{L/2} \left\{ \sum_{\alpha=1}^{N} [1 + (-1)^x \delta] (c_{x,\alpha}^\dagger c_{x+1,\alpha}^\dagger + \text{h.c.}) + U \sum_{\alpha \neq \beta} n_{x,\alpha} n_{x,\beta} - \mu \sum_{\alpha=1}^{N} n_{x,\alpha} \right\}.$$  

The first term in Eq. (2) is the dimerized hopping between neighboring sites (see Fig. 1), with the two hoppings given by $J(1 \pm \delta)$, in terms of the dimerization parameter $\delta$ (−1 < $\delta$ < 1). Here, $c_{x,\alpha}^\dagger$ are the annihilation (creation) operator at site $x$ with flavor $\alpha$. The second term is the Hubbard term describing the on-site interaction between fermions with different flavors, and the last term models an external chemical potential.

Another interacting versions of the SSH model have been used so far as a springboard to understand the effects of strong interactions, and have been concerned either with interacting bosonic [46–48], spinless fermionic models [49], or $SU(2)$ fermionic models, i.e. with flavors associated to the $\uparrow$ spin labels [22, 49–51].

In the absence of interactions, $U = 0$, the different flavor channels are decoupled and the model reduces to three copies of the non-interacting spinless SSH model [45]. The non-interacting model with $U = \mu = 0$ has a chiral symmetry $\Gamma$, which transforms the local operators as $\Gamma c_{x,\alpha}^\dagger \Gamma^{-1} \rightarrow (-1)^x c_{x,\alpha}$, $\Gamma i \Gamma^{-1} \rightarrow -i$. In the non-interacting limit, the model displays a topological phase transition at $\delta = 0$, from the trivial $\delta < 0$ to the topologically nontrivial $\delta > 0$ phase. The transition is characterized by a jump of $\pi$ in the Zak phase.

A typical band structure for the spinless SSH model in the topological regime, with the zero-energy modes emphasized is displayed in Fig. 1(b). The interactions in our model generally breaks the chiral symmetry, unless the chemical potential is fine tuned to $\mu = U(N-1)/2$ [44]. Nevertheless, we show that for arbitrary interactions the model still exhibits the two bulk topological phases due to the presence of inversion symmetry, although it has no topological protected edge states. In non-interacting systems, the situation is similar, with inversion symmetry enforcing the Zak phase quantization $\gamma_2 = 0$ or $\pi$ (modulo $2\pi$) [10]. The quantization was recently demonstrated experimentally in a photonic lattice where the chiral symmetry in a non-interacting SSH model is broken by engineering long-range hopping, while preserving
inversion symmetry [52].

Effective model for trions. Before investigating the quench dynamics, let us discuss shortly the band structure of the Hamiltonian (2) in the subspace with three particles. A sketch of the band structure in the limit of strong interactions, $U \gg J$, is displayed in Fig. 1(c). The lowest band is constructed from states where with a higher probability all particles are at different positions in the lattice, and it has a width $W_1 \approx 4J$. The highest in energy band, the trionic band, is constructed mostly from states in which all the three particles are residing at the same site [53]. This band consists of $L$ states (split into two subbands, as we show below), with very large energies $\approx 3U$, and a small bandwidth, of the order $W_3 \approx J^3/U^2$. In between the single particle and the trionic bands there is the doublonic band, well separated in energy from the others. The average bandwidth is of the order $\sim U$ (see Ref. [44] for more details on the band structure). As we shall discuss below, the quantum quench consists in injecting locally a trion at site $x = 0$, in the middle of the chain. Such an object is constructed by the trionic creation operator $\Phi_x^{(3)} = \prod_{\alpha=1}^3 c_{x,\alpha}^\dagger$ so the initial state $|\Psi(0)\rangle = \Phi_x^{(3)}|0\rangle$ has an energy $\sim 3U$, and has most of its weight in the trionic band. By energy conservation, the trion is propagating across the lattice only through quantum fluctuations. In the large $U \gg J$ limit, to a good approximation $\sim \mathcal{O}(J/U^2)$, the dynamics of the trion is dictated by an effective Hamiltonian of the form

$$H_3 = \sum_x \left\{ J_3[1 + \delta_3(-1)^x](\Phi_x^{(3)}\Phi_x^{(3)} + \text{h.c.}) + E_3\Phi_x^{(3)}\Phi_x^{(3)} \right\}$$

(3)

where $E_3$ and $J_3$ are the effective on-site energy and hoppings describing the trionic band alone. The effective Hamiltonian (3) is constructed by projecting out the scattering and the doublonic bands. By construction, $H_3$ takes into account the quantum fluctuations between the bands and renormalizes the effective couplings. It also preserves the bipartite nature of the full Hamiltonian (2) with an effective dimerization $\delta_3 \propto \delta$. The trion is an extremely heavy object, and propagates through the lattice at a low effective velocity $v_3 \approx 2J_3 \ll v_1$, much smaller than the single-particle propagation velocity $v_1 = 2J(1 - |\delta|)$. To propagate through the lattice it must dissociate. With a large probability $p_3 \approx 1 - \mathcal{O}(J/U^2)$, the trionic state with $|\Phi_x^{(3)}(t)\rangle \simeq \sum_a a_x(t)\Phi_x^{(3)}|0\rangle$, and because of the large quantum fluctuations in energy, the trion will break apart only with a small probability amplitude $\sim \mathcal{O}(J/U^2)$, thus reducing considerably the effective velocity $v_3$. In that regard, to order $\mathcal{O}(J/U^2)$, $H_3$ in Eq. (3) precisely describes the dynamics of the trions.

Many-body Berry phase. A numerical analysis performed by computing the winding number using the Green’s functions for the $SU(N = 2)$ version of the model (2), at half filling, indicates that the interactions do not destroy the topology, although the BBC no longer survives [50]. Similar conclusions were drawn from the analysis of two-body physics in spinless bosonic SSH models [46, 47]. In the following, we investigate the bulk properties of the SSH systems using the many-body Berry phase $\gamma_B$.

To assess how strong, local interactions affect the bulk topology we compute the $\gamma_B$ by using an original approach. The main difference when compared to the standard method [17] is that one computes the Berry phase over a subset of excited many-body states instead of simply using the ground state. The justification to use such a subset is because a trion many-body state is built out of highly excited eigenstates, and is not the many-body ground state. To compute $\gamma_B$, we consider the ring geometry and impose twisted boundary conditions on the many-body spectrum. This is done by modifying one of the hopping terms $J \rightarrow Je^{i\theta_n}$ with $\theta_n = 2\pi n/M$, $n \in \{0, 1, \ldots, M - 1\}$, and $M$ controlling the twist angle discretization. For each $\theta_n$, we diagonalize the Hamiltonian (2), $H(\theta_n)\Psi^{(n)}_j = E^{(n)}_j\Psi^{(n)}_j$, with the $N = 3$ subspace and obtain the full many-body spectrum $\{\Psi^{(n)}_j\}$ of the Hamiltonian.

The many-body Berry phase is well-defined over a subset $\{\Psi_i\}$ of the many-body spectrum that is separated by a gap from the rest of the states, for all twist angles. Then $\gamma_B$ follows by generalizing the procedure used to determine it for the ground state [17, 54, 55]:

$$\gamma_B = -\text{Im} \log \prod_{n=0}^{M-1} \det[S^{(n,n+1)}],$$

(4)

where the matrix $S^{(n,n+1)}$ components are

$$S^{(n,n+1)}_{jj'} = \langle \Psi_j^{(n)}|e^{2\pi i X/M L}|\Psi_{j'}^{(n+1)}\rangle,$$

(5)

with $j, j'$ indexing states in the set $\{\Psi_i\}$, and $X$, the many-body position operator along the chain. Under inversion symmetry $\gamma_B \to -\gamma_B$ (modulo $2\pi$). Therefore in the interacting system the inversion symmetry enforces $\gamma_B$ quantization, $\gamma_B = 0$ or $\pi$.

Because of the large energy that is initially injected into the system, the many-body state of the system is mainly constructed from states within the trionic band. Including the scattering and the doublonic states within the set is redundant since that only shifts $\gamma_B$ by a constant irrespective of the sign of $\delta$. As we discussed previously, the trionic band contains $L$ states, divided in two subbands separated by a gap generated to the effective dimerization parameter $\delta_3$. When we include the $L/2$ lowest in energy states from the trionic band into the set $\{\Psi_i\}$, one recovers a jump of $\pi$ in $\gamma_B$ at $\delta = 0$, which indicates a topological phase transition, and that the two regions corresponding to $\delta \lesssim 0$ are topologically distinct. These findings are presented in Fig. 2.
Quantum quench protocol. In what follows we investigate the effect of interactions in the quench dynamics in a multiparticle quantum walk setup. We shall investigate how the strong, local interaction affects the topology. Our findings indicate, that similar to the single-particle quantum walk, in the case of a multiparticle quantum walk the single and multiparticle polarizations are also capable to capture the topological transition in the system.

As the lattice is bipartite, each unit cell contains two sites, labeled $A$ and $B$, and the chiral operator is $\Gamma = \sigma^x$ while the annihilation(creation) operator becomes a spinor $c^{(1)}_x = (c^{(1)}_{x,A}, c^{(1)}_{x,B})^T$ within this basis.

Single and many-body chiral polarizations. The quantity which discriminates between the topological and trivial regimes is the chiral polarization defined Eq. (1). Since the many-body state is mainly constructed from states in the trionic band, it is therefore natural to extend the definition for the non-interacting chiral polarization in Eq. (1) and evaluate the single and the many-body chiral polarizations as

$$\left[ P_(t) \right] \left[ P_3(t) \right] = \frac{1}{\langle n_3(t) \rangle} \sum_x^\prime (\Psi(t)) \frac{c^{(1)}_x \sigma^z c^*_x}{\Phi^{(3)}_x \sigma^z \Phi^{(3)}_x} \langle \Psi(t) \rangle$$

in terms of the spinor creation and annihilation operators for the single particles and for the trions. The primed sum over $x$ indicates that the sum is performed over the lattice unit cells. The results for $\gamma_B$ are confirmed by computing the single-particle and the many-body chiral polarization $P_1(t)$ and $P_3(t)$.

FIG. 2. Disorder-averaged Berry phase $\gamma_B$ as a function of $\delta$, obtained by integrating half of the excited (trion) many-body states in an $L = 8$ lattice with twisted boundary conditions, with $M = 20$ and $U = 3J$, for different disorder strengths $W$’s. For each $W$, the averages are over 100 disorder realizations. The inset represents the Berry phase for the non-interacting SSH model with hopping disorder.

FIG. 3. Chiral polarization in the interacting $SU(3)$ SSH model. (a) Cumulative average polarization $P_3$ for different chiral-symmetry preserving disorder strengths $W$ as a function of dimerization parameter $\delta$. (Inset) Typical evolution of the polarization $P_3$ in the clean system at (right) $\delta = 0.5$ and (left) $\delta = -0.5$. (b) Cumulative disorder-averaged polarization $P_1$ at $t_\infty = 40/J$ for different disorder amplitudes as a function of dimerization parameter $\delta$. In both panels the lattice size is $L = 30$ sites, $U = 3J$ and the average is done over 100 disorder realizations at each $W$. 

The prefactor $\langle n_3(t) \rangle \leq 1$ represents the total number of trions in the lattice at time $t$, which due to quantum fluctuations is not constant in time, and is defined as
\( \langle n_3(t) \rangle = \sum_x \langle \Psi(t) | \Phi^\dagger(x) \Phi_3(x) | \Psi(t) \rangle. \) Since the Hamiltonian Eq. (2) allows single particle tunneling between neighboring sites \( \{ \Phi^\dagger(x) \Phi_3(x), H \} \neq 0, \) and the numbers of trions is not conserved. In line with the discussions following the construction of the effective model, for large \( U \gg J \) in the asymptotic limit, \( t \to \infty, \) \( \langle n_3(t) \rangle \approx 1 - O((J/U)^2). \)

Results for \( \mathcal{P}_3(t) \) are displayed in Fig. 3(a). It indicates that, similar to \( \mathcal{P}_1(t), \) the many-body polarization is a suitable tool to differentiate between the topological distinct regions. The cumulative average polarization, \( \mathcal{P}^\alpha_3(t) \) shows a clear jump at \( \delta = 0, \) and converges to \( 0(0.5) \) in the trivial (topological) regime, and is weakly affected by disorder, making it a robust measure for the bulk topology as well.

**Chiral disorder.** In the absence of any disorder, the effective model (3) resembles the non-interacting version of the SSH model. To test the robustness of \( \gamma_B, \) moderate chiral disorder is introduced to break inversion symmetry. We display in Fig. 2 the results for \( \gamma_B \) for different strengths \( Ws \) of disorder, which indicates that the jump at the transition point remains sharp and that the quantization of \( \gamma_B \) remains intact. Disorder does not affect significantly neither \( \mathcal{P}_1(t_{\infty}), \) nor \( \mathcal{P}_3(t_{\infty}), \) as the chiral polarization remains robust as well against the on-site or hopping disorder. Disorder-averaged polarization are presented in Fig. 3. Further details about the disorder effects are discussed in the supplementary material [44]. However, one notes that very strong interactions reduce the effective gap in the trion bands in (3) as \( 1/U^2, \) and thus disorder destroys more easily the bulk topological phases.

*Extension to the SU(\( N \)) SSH model.* For the sake of clarity, in the present work we address the SU(3) version of the SSH model. Extending the analysis to the SU(\( N \)) case is straightforward. The many-body spectrum has a ladder-like structure with \( N \) bands separated by energy gaps of the order of \( U. \) The highest in energy band, the \( N \)-ion band, has an energy \( E_N \approx N(N - 1)U/2. \) The construction of the effective model that describes the \( N \)-ionic band – similar to Eq. (3) for the trions – is detailed in Ref. [44]. Computing the many-body Berry phase and the polarization is done along the same lines. Introducing the \( N \)-ion creation operator, \( \Phi^\dagger_{3,\alpha} = \prod_{\nu=1}^{N-1} c^\dagger_{\nu,\alpha} \) the many-body polarization is obtained as in Eq. (6) with the substitution \( 3 \to N. \) Details for the Berry phase and polarization in the SU(2) case are presented in Ref. [44].

**Two trions quantum walk.** Our findings carry on to the case when two trions are injected in the lattice. In this case, the effective Hamiltonian (3) needs to be supplemented with an interacting term that describes the effective attraction between the trions. Furthermore, higher-order processes generate a residual interaction between the trions and doublons as well, producing a diffusion of doublons into the lattice [60].

indicate that, irrespective of the strong interaction between the trions, the polarizations \( \mathcal{P}_{1,3}(t_{\infty}) \) remain unaffected in the long-time limit, and correctly capture the topological transition.

**Conclusions.** In the present work we have investigated the quench dynamics in a multiparticle continuous time quantum walk and evaluated the effect of strong interactions on topological properties of bulk states in the SU(3) SSH model. We have shown that strong interactions generically violate the conventional chiral symmetry of SSH models. Nevertheless, a quantized many-body Berry phase survives in the presence of inversion symmetry, and it describes two distinct topological phases separated by a transition, marked by a \( \pi \)-jump. We also found that the topological phases are robust against moderate disorder which breaks inversion symmetry. Although such a jump indicates a topological transition, it is not immediately accessible experimentally. Still, a similar discontinuity shows up in the single-particle and the many-body chiral polarizations during the dynamics of the system. Measuring these quantities in multiparticle quantum walk setups are within experimental reach and could be used to infer the topological properties of the excited states for the underlying lattice. Similar features are expected for the generic SU(\( N \)) case as well.

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[1] L. D. Landau and E. Lifshitz, *Statistical Physics, (Course of Theoretical Physics, Volume 5)* (Butterworth-Heinemann, 1980).

[2] M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).

[3] X.-L. Qi and S.-C. Zhang, *Rev. Mod. Phys.* **83**, 1057 (2011).

[4] A. Kitaev, *AIP Conf. Proc.* **1134**, 22 (2009).

[5] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, *Rev. Mod. Phys.* **88**, 035005 (2016).

[6] A. M. Essin and V. Gurarie, *Phys. Rev. B* **84**, 125132 (2011).
J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, Nat. Phys. 7, 412 (2011).

L. Šmekjral, Y. Mokrousov, B. Yan, and A. H. MacDonald, Nat. Phys. 14, 242 (2018).

Q. L. He, T. L. Hughes, N. P. Armitage, Y. Tokura, and K. L. Wang, Nat. Mater. 21, 15 (2022).

J. Zak, Phys. Rev. Lett. 62, 2747 (1989).

S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New J. Phys. 12, 065010 (2010).

J. K. Asbóth, L. Orozslyány, and A. Pályi, A Short Course on Topological Insulators: Band Structure and Edge States in One and Two Dimensions (Lecture Notes in Physics) (Springer, 2016).

V. Gurarie, Phys. Rev. B 83, 085426 (2011).

A. W. W. Ludwig, Phys. Scripta T168, 014001 (2015).

S. Rachel, Rep. Progr. Phys. 81, 116501 (2018).

M. V. Berry, Proc. Math. Eng. Sci. 392, 45 (1984).

R. Resta, J. Phys.: Condens. Matter 12, R107 (2000).

D. Xiao, M.-C. Chang, and Q. Niu, Rev. Mod. Phys. 82, 12 (1959) (2010).

D. Vanderbilt, Berry Phases in Electronic Structure Theory: Electric Polarization, Orbital Magnetization and Topological Insulators (Cambridge University Press, 2018).

P. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B 81, 064439 (2010).

P. Fromholz, G. Magnifico, V. Vitale, T. Mendes-Santos, and M. Dalmon, Phys. Rev. B 101, 085136 (2020).

D. Wang, S. Xu, Y. Wang, and C. Wu, Phys. Rev. B 91, 115118 (2015).

T. Kitagawa, M. A. Broome, A. Fedrizzi, M. S. Rudner, E. Berg, I. Kassal, A. Aspuru-Guzik, E. Demler, and A. G. White, Nat. Commun. 3, 882 (2012).

T. Groh, S. Brakhane, W. Alt, D. Meschede, J. K. Asbóth, and A. Alberti, Phys. Rev. A 94, 013620 (2016).

M. Atala, M. Avidelsburger, J. T. Barreiro, D. Abanin, T. Kitagawa, E. Demler, and I. Bloch, Nat. Phys. 9, 795 (2013).

F. Cardano, A. D’Errico, A. Dauphin, M. Maffei, B. Piccirillo, C. de Lìsio, G. De Filippis, V. Cataudella, E. Santamato, L. Marrucci, M. Lewenstein, and P. Massignan, Nat. Commun. 8, 15116 (2017).

E. J. Meier, F. A. An, A. Dauphin, M. Maffei, P. Massignan, T. L. Hughes, and B. Gadway, Science 362, 929 (2018).

Y. Wang, Y.-H. Lu, F. Mei, J. Gao, Z.-M. Li, H. Tang, S.-L. Zhu, S. Jia, and X.-M. Jin, Phys. Rev. Lett. 122, 193903 (2019).

The chiral polarization defined in Eq. (1) is also dubbed in the literature as mean chiral displacement. The position $x = 0$ is associated with the middle of the chain.

T. Rakovszky, J. K. Asbóth, and A. Alberti, Phys. Rev. B 95, 201407 (2017).

X. Zhan, L. Xiao, Z. Bian, K. Wang, X. Qiu, B. C. Sanders, W. Yi, and P. Xue, Phys. Rev. Lett. 119, 130501 (2017).

H. Schmitz, R. Matjeschk, C. Schneider, J. Glueckert, M. Endelrein, T. Huber, and T. Schaetz, Phys. Rev. Lett. 103, 090504 (2009).

M. Karski, L. Förster, J.-M. Choi, A. Steffen, W. Alt, D. Meschede, and A. Widera, Science 325, 174 (2009).

A. Peruzzo, M. Lobino, J. C. F. Matthews, N. Matsuda, A. Politi, K. Poulios, X.-Q. Zhou, Y. Lahini, N. Ismail, K. Wörhoff, Y. Bromberg, Y. Silberberg, M. G. Thompson, and J. L. O’Brien, Science 329, 1500 (2010).

A. Schreiber, K. N. Cassemiro, V. Potoček, A. Gábris, P. J. Mosley, E. Andersson, I. Jex, and C. Silberhorn, Phys. Rev. Lett. 104, 050502 (2010).

K. Poulios, R. Keil, D. Fry, J. D. A. Meinecke, J. C. F. Matthews, A. Politi, M. Lobino, M. Gräfe, M. Heinrich, S. Nolte, A. Szameit, and J. L. O’Brien, Phys. Rev. Lett. 112, 143604 (2014).

T. Fukuhara, P. Schauß, M. Endres, S. Hild, M. Cheneau, I. Bloch, and C. Gross, Nature 502, 76 (2013).

T. Fukuhara, A. Kintian, M. Endres, M. Cheneau, P. Schauß, S. Hild, D. Bellem, U. Schollwöck, T. Giamarchi, C. Gross, I. Bloch, and S. Kuhr, Nat. Phys. 9, 235 (2013).

L. Lu, J. D. Joannopoulos, and M. Soljačić, Nat. Photonics 8, 821 (2014).

T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, and I. Carusotto, Rev. Mod. Phys. 91, 015006 (2019).

P. M. Preiss, R. Ma, M. E. Tai, A. Lukin, M. Rispoli, P. Zupancic, Y. Lahini, R. Islam, and M. Greiner, Science 347, 1229 (2015).

Z. A. Geiger, K. M. Fujiwara, K. Singh, R. Senaratne, S. V. Rajagopal, M. Lipatov, T. Shimasaki, R. Driben, V. V. Konotop, T. Meier, and D. M. Weld, Phys. Rev. Lett. 120, 213201 (2018).

A. Haller, P. Massignan, and M. Rizzi, Phys. Rev. Research 2, 033200 (2020).

B. Ostahie, D. Sticlet, C. P. Moca, B. Dóra, K. J. Asbóth, and G. Zaránd, Supplementary material.

W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979).

M. Di Liberto, A. Recati, I. Carusotto, and C. Menotti, Phys. Rev. A 94, 062704 (2016).

M. A. Gorlach and A. N. Poddubny, Phys. Rev. A 95, 053866 (2017).

A. A. Stepanenko and M. A. Gorlach, Phys. Rev. A 102, 013510 (2020).

J. Širker, M. Maiti, N. P. Konstantinidis, and N. Sedlmayr, J. Stat. Mech. Theory Exp. 2014, P10032 (2014).

S. R. Mannama, A. M. Essin, R. M. Noack, and V. Gurarie, Phys. Rev. B 86, 205119 (2012).

N. H. Le, A. J. Fisher, N. J. Curson, and E. Ginossar, npj Quantum Inf. 6, 24 (2020).

Z.-Q. Jiao, S. Longhi, X.-W. Wang, J. Gao, W.-H. Zhou, Y. Wang, Y.-X. Fu, L. Wang, R.-J. Ren, L.-F. Qiao, and X.-M. Jin, Phys. Rev. Lett. 127, 147401 (2021).

The trion state itself is not an eigenstate of the Hamiltonian, and it is not to be confused with the eigenstates forming the trionic band, although the overlap between them is large in the $U \gg J$ limit.

R. Resta and S. Sorella, Phys. Rev. Lett. 74, 4738 (1995).

R. Resta and S. Sorella, Phys. Rev. Lett. 82, 370 (1999).

Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A 48, 1687 (1993).

E. Farhi and S. Gutmann, Phys. Rev. A 58, 915 (1998).

A. M. Childs, E. Farhi, and S. Gutmann, Quantum Inf. Process. 1, 35 (2002).

G. Vidal, Phys. Rev. Lett. 98, 070201 (2007).

M. A. Werner, C. P. Moca, M. Kormos, Ő. Legeza, B. Dóra, and G. Zaránd arXiv:2207.00994 (2022).
Supplementary Material for “Multiparticle quantum walk in the strongly interacting $SU(3)$ Su-Schrieffer-Heeger-Hubbard topological model”

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The supplementary material details several of the points in the main text. It derives effective models for $SU(2)$ and $SU(3)$ Hamiltonians in the highest excited states, in the limit of strong interactions. It discusses the role of chiral and inversion symmetry in the Berry phase quantization. It brings further numerical probes for the chiral polarization in $SU(2)$ and $SU(3)$ models, and illustrates the time evolution of a doublon or a trion in $SU(2)$ and $SU(3)$ models, respectively. Additionally, it shows that the chiral polarization is also quantized in the case where two trions are injected in the lattice. Finally, it discusses the many-body Berry phases in $SU(N)$ models, when $N$ is odd or even, and when varying the subset of many-body states over which it is computed.

CONTENTS

Effective models 1
Doublonic effective Hamiltonian in a lattice with $SU(2)$ symmetry 2
Trionic effective Hamiltonian in a lattice with $SU(3)$ symmetry 2
Effective model for the $N$-ion in a lattice with $SU(N)$ symmetry 3
Edge states 4
Chiral and inversion symmetry and the Berry phase quantization 4
Chiral polarization 5
Two trions 7
Many-body Berry phase 7
References 8

EFFECTIVE MODELS

We discuss the general situation when the lattice is described by the $SU(N)$ Su-Schrieffer-Heeger-Hubbard (SSHH) Hamiltonian and contains exactly $N$ particles. We develop the effective Hamiltonian for the highest excited band with states constructed from $N$ particles with $N$ flavors, localized on a single site, and moving in a one-dimensional lattice in the presence of Hubbard interactions. Special attention is given to the construction of $SU(N=2)$ doublonic and $SU(N=3)$ trionic effective models. The models constructed in this way are mapped to a non-interacting SSH model with some effective couplings. The general $SU(N)$ SSH Hamiltonian reads

$$H = H_U + H_T, \quad H_U = U \sum_{x=-L/2}^{L/2-1} \sum_{\alpha<\beta} n_{x,\alpha} n_{x,\beta}, \quad H_T = \sum_{x=-L/2}^{L/2-1} J[1 + \delta(-1)^x] \sum_{\alpha} \langle c_{x,\alpha}^\dagger c_{x+1,\alpha} + H.c. \rangle, \quad (1)$$

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where \( c_{x,\alpha}^{\dag} \) are creation (annihilation) operators at site \( x \), and \( n_{x,\alpha} = c_{x,\alpha}^{\dag} c_{x,\alpha} \) represents the number operator. We use Greek indices \((\alpha, \beta \in \{1, \ldots, N\})\) to indicate the flavors. The interaction term \( H_U \) is parameterized by the on-site Hubbard coupling \( U \), while the kinetic term \( H_T \), is characterized by the hopping \( J \), and the lattice dimerization parameter \( \delta \). For numerical calculations we keep the total number of sites \( L \) even and use periodic boundary conditions (PBC).

We are interested in the dynamics of an \( N \)-particle object (hereafter called an \( N \)-ion) injected at \( x = 0 \) in the lattice, in the limit of large Hubbard interactions \( U/J \gg 1 \). The \( N \)-ion resides in the upper band, and has an energy \( E_N \approx N(N - 1)/2 \). In the absence of a dissipative mechanism, the \( N \)-ion has a long lifetime since it propagates through the lattice only through quantum fluctuations.

In order to construct the effective Hamiltonian that describes the dynamics of such an object in the limit of large \( U \), the kinetic term \( H_T \) is treated as a perturbation to \( H_U \). The effective Hamiltonian for the \( N \)-ion dynamics is obtained by projecting out the low-energy states of the rest of the \( N \)-particle states using \( N \)th order perturbation theory in \( H_T \). Below we consider the particular \( SU(2) \) and \( SU(3) \) cases and construct explicitly the effective Hamiltonians that describe the propagation of a doublon and a trion, respectively, through the lattice.

**Doublonic effective Hamiltonian in a lattice with \( SU(2) \) symmetry**

For the \( SU(N = 2) \) case in Eq. (1), the two flavors can be identified with the spin-\( \uparrow \) and spin-\( \downarrow \) projections for a spin-1/2, and the model reduces to the Hubbard model with a dimerized hopping. A doublon is build up of two fermions with opposite spins, located at the same site. The associated creation operator is \( \Phi_x^{(2)} = c_x^{\uparrow \dag} c_x^{\downarrow \dag} \). When a doublon is injected in the lattice, a localized state of the form,

\[
|\Phi^{(2)}_x\rangle = |\Phi^{(2)}_x(\uparrow, \downarrow)\rangle = c_x^{\uparrow \dag} c_x^{\downarrow \dag} |0\rangle = |x\uparrow, x\downarrow\rangle,
\]

is created. Such a state has an energy \( E_2 \approx U \). To construct the effective model, we define the projector \( P \) into the doublonic states (linear superposition of states of the form (2)) and the projector \( S \) into the rest of the biparticle scattering states, respectively,

\[
P = \sum_x |\Phi^{(2)}_x\rangle \langle \Phi^{(2)}_x|, \quad S = \frac{1}{U} \sum_{x<y,\sigma} |x\sigma y\bar{\sigma}\rangle \langle x\sigma y\bar{\sigma}|,
\]

where \( x, y \) run over lattice sites, and \( \sigma, \bar{\sigma} \) are opposite spin projections. The effective Hamiltonian is obtained by projecting out the two-particle scattering states using a second order perturbation theory in \( H_T \),

\[
H_{\text{eff}} = P H_U P + P H_T P + P H_T S H_T P.
\]

After some algebra, the effective Hamiltonian describing the dynamics of the doublon is constructed as

\[
H_{\text{eff}} = \sum_{x=-L/2}^{L/2-1} \left[ U + \frac{4J^2(1 + \delta^2)}{U} \right] \Phi_x^{(2)\dag} \Phi_x^{(2)} + \frac{2J^2}{U} \left[ 1 + \delta(-1)^x \right] |\Phi_x^{(2)}\rangle \langle \Phi_x^{(2)}|_{x+1} + \text{H.c.}.
\]

The model resembles the non-interacting SSH model, with a renormalized hopping and with additional on-site energy. The maximum propagation velocity of a doublon in the lattice follows readily, \( v_2 \approx 4J^2(1 - |\delta|)^2/U \). In Fig. 1(a) we present a comparison for the doublonic band energy spectrum obtained by diagonalizing the full Hamiltonian (1) and the obtained by diagonalizing the effective Hamiltonian Eq. (5) for \( U = 8J \). Apart from a slight deviation, that decreases with increasing \( U \), the effective model provides a good estimate for the band structure.

**Trionic effective Hamiltonian in a lattice with \( SU(3) \) symmetry**

Now we derive the effective model that describes the dynamics of a trion. Following the same line as the previous section, the creation operator for a trion at site \( x \) reads \( \Phi_x^{(3)} = c_x^{\uparrow \dag} c_x^{\downarrow \dag} c_x^{\downarrow \dashv} \) and a localized trionic state \( |\Phi_x^{(3)}\rangle = \Phi_x^{(3)\dag} |0\rangle \) has an energy \( E_3 \approx 3U \). The projector to the trionic subspace is \( P = \sum_x |\Phi_x^{(3)}\rangle \langle \Phi_x^{(3)}| \).

The rest of three-particle states correspond to configurations where each particle sits on different sites, and configurations where two particles form a doublon and the other particle is scattered through the lattice. Therefore the
FIG. 1. Comparison between the energy dispersion for the highest excited states of two interacting SSH models (red dots, $H$) and the dispersion obtained from their respective effective Hamiltonians (blue lines, $H_{\text{eff}}$) as a function of the dimerization parameter for (a) and (b) periodic or (c) and (d) open boundary conditions. Panel (a) presents the $SU(2)$ model’s doublon bands in an $L = 50$ chain, with an inset showing the entire many-body spectrum, while (b) presents the $SU(3)$ model’s trion bands in an $L = 30$ chain, with the inset showing the entire many-body spectrum (for an $L = 14$ chain). (c, d) For the same systems, but with open boundary conditions, both for the full Hamiltonian $H$ and the effective counterpart $H_{\text{eff}}$, there are edge states (green thick line for $H$). In (c) the $SU(2)$ model, the edge states stick to the bulk bands extrema and pass from the upper to lower doublon band at $\delta = 0$, while in (d) the $SU(3)$ model, they form below the trion bands for generic $\delta$. In all cases $U = 8J$.

The flavor index $\gamma$ is always the remaining third flavor, different from summed-over flavors $\alpha$ and $\beta$.

To construct the effective trionic Hamiltonian it requires to perform an expansion in the 3rd order perturbation theory in $H_T$,

\[
H_{\text{eff}} = PH_T P + PH_T SH_T P + PH_T SH_T SH_T P - \frac{1}{2} (PH_T PH_T S^2 H_T P + PH_T S^2 H_T PH_T P).
\]  

The effective trionic Hamiltonian follows after a tedious, but otherwise a straightforward calculation,

\[
H_{\text{eff}} = \sum_{x=-L/2}^{L/2-1} \left( 3U + \frac{3J^2}{U}(1+\delta^2) \right) \phi_x^{(3)} \phi_x^{(3)} + \frac{3J^3}{2U^2} [1 + \delta (-1)^x] (\phi_x^{(3)})^\dagger \phi_{x+1}^{(3)} + \text{H.c.},
\]  

showing again that in this limit the trion Hamiltonian is mapped to a non-interacting SSH-type Hamiltonian, with an additional on-site energy. The parameters of the effective SSH Hamiltonian of Eq. 3 in the main text are: $\delta_3 = \frac{3+\delta^2}{1+3\delta^2} \delta$, $J_3 = 3J^3(1+3\delta^2)/2U^2$, and $E_3 = 3U + 3J^2(1+\delta^2)/U$. The model indicates that the maximum propagation velocity of a trion in the lattice is $v_3 \simeq 3J^3(1-|\delta|)^3/U^2$. Fig. 1(b) shows an almost perfect match between trion bands’ energy dispersion obtained from Eq. (1) and the effective model (8) at $U = 8J$.

**Effective model for the N-ion in a lattice with $SU(N)$ symmetry**

The effective Hamiltonian for the generic case of an $N$-ion is difficult to determine in closed form since it requires an $N$th order perturbation theory in $H_T$. Instead, we obtain an approximate effective Hamiltonian from Eq. (1) which captures first-order corrections to the kinetic and the on-site energy of $N$-ions. The dominant contribution to the on-site energy is already obtained in the second order perturbation theory for the doublonic band. In contrast, the dominant contribution to the hopping term requires the $N$th order in the perturbation theory, since it requires tunneling of all $N$ particles to an adjacent site.
Therefore the approximate effective Hamiltonian reads
\[ H_{\text{eff}} \approx \sum_{x=-L/2}^{L/2-1} \left[ \frac{UN(N-1)}{2} + \frac{2N}{N-1} J^2 (1 + \delta^2) \right] \phi_x^{(N)} \phi_x^\dagger + \frac{NJ}{N-1} \left[ 1 + \delta (1)^x \right] \phi_x^{(N)} \phi_{x+1}^\dagger + \text{H.c.}, \]

where \( \phi_x^{(N)} = c_{x,1}^\dagger c_{x,2}^\dagger \ldots c_{x,N}^\dagger \) is the \( N \)-ion creation operator. The neglected orders contributing to the on-site term in the perturbation theory behave as \( 1/U^n \) for \( n < N \), and are generally larger than the hopping term correction. Nevertheless, it is the latter which is responsible for the characteristic dimerization that allows to map the highest band in the \( SU(N) \) model the non-interacting SSH Hamiltonian.

**Edge states**

The effective models derived above are all formally equivalent to non-interacting SSH models with additional on-site energy which breaks the chiral symmetry. However, the models present localized states at the edges due to reduced quantum fluctuations at the boundaries. Such non-topological interaction-induced edge states have been seen also in two-body physics in bosonic SSH models \([1, 2]\). Topologic invariants based on the Berry phase do not inform us on the edge physics in such systems, and this indicates a failure of bulk-edge correspondence. Nevertheless, the invariants are measures that probe the bulk topology of the interacting Hamiltonian.

For open boundary conditions, the effective Hamiltonians are complemented by additional edge terms which take into account the renormalization of on-site energy at the edges. Then they reproduce well the exact diagonalization results of the many-body Hamiltonian \( (1) \). The green lines in Fig. 1(c) and (d)] indicate the non-topological localized states at the edges, which for the \( SU(3) \) case decouple from the rest of the trionic band as the dimerization parameter \( \delta \) is increased.

**CHIRAL AND INVERSION SYMMETRY AND THE BERRY PHASE QUANTIZATION**

The chiral symmetry represented by operator \( \Gamma \) acts on the fermion operators as follows \([3]\):
\[ \Gamma c_{x,\alpha} \Gamma^{-1} = (-1)^x c_{x,\alpha}^\dagger, \quad \Gamma i \Gamma^{-1} = -i. \]

Let us consider \( SU(N) \) Hamiltonian \( (1) \), supplemented by a uniform chemical potential
\[ H \rightarrow H - \mu \sum_{x,\alpha} n_{x,\alpha}. \]

Under the chiral symmetry, the new Hamiltonian transforms as
\[ \Gamma H \Gamma^{-1} = \sum_x \left\{ \sum_{\alpha \neq \beta} \frac{U}{2} (1 - n_{x,\alpha})(1 - n_{x,\beta}) + \sum_\alpha \left[ J(1 + \delta (-1)^x)(c_{x,\alpha}^\dagger c_{x+1,\alpha} + \text{H.c.}) - \mu (1 - n_{x,\alpha}) \right] \right\}, \]
\[ = \sum_x \left\{ \frac{U}{2} N(N - 1) - \mu N + \sum_{\alpha \neq \beta} \frac{U}{2} n_{x,\alpha} n_{x,\beta} \right\} \]
\[ + \sum_\alpha J(1 + \delta (-1)^x)(c_{x,\alpha}^\dagger c_{x+1,\alpha} + \text{H.c.}) + (\mu - (N - 1)U) n_{x,\alpha} \right\}. \]

It follows that the interacting Hamiltonian is chiral symmetric \( \Gamma H \Gamma^{-1} = H \) for a chemical potential tuned to
\[ \mu = (N - 1)U/2. \]

The Hamiltonian is also invariant under the inversion symmetry. For open boundary conditions, the inversion symmetry acts as \( I c_{x,\alpha} I^{-1} = c_{-x-1,\alpha} \), with inversion symmetry center taken as the mid position between sites \( x = -1 \) and \( x = 0 \). For a finite \( U \), the inversion symmetry is preserved, while the chiral symmetry is generally broken. Nevertheless, inversion symmetry is sufficient for the many-body Berry phase to be quantized. Under spatial inversion, \( \gamma_B \) transforms as
\[ \gamma_B \rightarrow -\text{Im} \log \prod_n \det[S^{(n+1,n)}(-X)] = -\text{Im} \log \prod_n \det[S^{(n,n+1)}(X)]^* = -\gamma_B \text{ (mod } 2\pi). \]
FIG. 2. Single-particle quantum walk in the non-interacting spinless SSH model. (a) The time evolution of the density profile \( \langle n(x,t) \rangle \) at \( \delta = 0.5 \). The dashed black lines denote the wave front of the propagation with a velocity \( v_1 = 2J(1 - |\delta|) \). (b) Cumulative chiral polarization \( P_1(t) \) as a function of the dimerization parameter \( \delta \). (Insets) Characteristic time evolution of \( P_1(t) \) for \( \delta = -0.5 \) (left) and \( \delta = 0.5 \) (right). In the trivial (topological) regime \( P_1(t) \) converges asymptotically to 0 (0.5). In simulations, we consider 100 sites lattices and the particle is initially injected in the middle of the chain. Time is measured in units of \( 1/J \).

Note the interchange of \( n \) and \( n + 1 \) in the first step, compared to the original formula. This is due to inversion changing \( k_n \rightarrow -k_n \), where \( k_n \) is the momentum associated to \( \theta_n \), \( k_n = 2\pi n/ML \), for \( n \in \{0, \ldots, L - 1\} \). Then a shift by \( 2\pi/L \), brings the momenta back in the first Brillouin zone \([0, 2\pi/L] \), but expectation values are taken in opposite order. In the present case, the system has inversion symmetry, therefore \( \gamma_B = -\gamma_B \pmod{2\pi} \). This is possible only if \( \gamma_B = 0 \) or \( \pi \pmod{2\pi} \). This indicates that the Berry phase is quantized and there are two possible topological phases. We have shown that, similar to the non-interacting model, the interacting model exhibits both phases, and a topological transition exists between them at \( \delta = 0 \).

The robustness of an average \( \gamma_B \) was checked against moderate on-site and hopping disorder which break the inversion symmetry. Hopping disorder or on-site disorder, respectively, are introduced in Eq. (1) as

\[
H \rightarrow H + \sum_{x=-L/2}^{L/2-1} \delta J_{x,x+1}(c_{x,\alpha}^\dagger c_{x+1,\alpha} + \text{H.c.}), \quad \text{or} \quad H \rightarrow H + \sum_{x=-L/2}^{L/2-1} \delta \mu_x c_{x,\alpha}^\dagger c_{x,\alpha},
\]

with \( \delta J_{x,x+1} \in W[-0.5,0.5] \), and \( \delta \mu_x \in W[-0.5,0.5] \), randomly chosen from a uniform distribution with \( W \), the disorder amplitude.

### CHIRAL POLARIZATION

Here we discuss in more detail the single particle and the many-body polarization. First, we present results for the non-interacting SSH model, and then discuss the doublonic and trionic many-body polarizations.

When a single particle is injected in the lattice, the interactions play no role and the underlying lattice Hamiltonian reduces simply to the non-interacting spinless SSH model [4]. The lattice is bipartite, each unit cell contains two sites, labeled \( A \) and \( B \), and the chiral operator is \( \Gamma = \sigma^z \) within the site basis.

Following the quench, the system state at \( t = 0 \) becomes \( |\Psi(t=0)\rangle = c_{0,\beta}^\dagger |0\rangle \), with \( \beta \) a sublattice index. The wave function follows a unitary evolution in time according to the Schrödinger equation \( |\Psi(t)\rangle = \exp(-iHt)|\Psi(t=0)\rangle \). The density profile \( \langle n(x,t) \rangle \) develops a light-cone propagation, with a maximum Lieb-Robinson velocity \( v_1 = 2J(1 - |\delta|) \). A typical behavior for \( \langle n(x,t) \rangle \) in the topological regime \( \langle \delta > 0 \rangle \) is displayed in Fig. 2 (a). This behavior remains qualitatively the same in the trivial regime, as \( \langle n(x,t) \rangle \) does not differentiate between the two phases. The quantity that discriminates between the two regimes is the chiral polarization Eq. (1) (main text). In the present setup, we evaluate it as

\[
P_1(t) = \sum_x \sum_{\alpha,\beta,\gamma} \langle 0 | c_{0,\alpha}^\dagger e^{iHt} c_{x,\beta}^\dagger \sigma^z_{\beta\gamma} c_{x,\gamma} e^{-iHt} c_{0,\alpha}^\dagger |0\rangle,
\]

where the primed sum over \( x \) indicates that the sum is performed over the lattice unit cells, and the second sum is over the sublattice labels.
FIG. 3. Time evolution of the density profile in a multiparticle quantum walk in (a) the SU(2), and (b) SU(3) models. Dotted lines denote propagation front corresponding to a free particle maximum velocity \(v_1\), in the lattice, dashed lines, the doublon maximum velocity \(v_2\), and dash-dot lines, the trion maximum velocity \(v_3\), estimated in the effective models. In time, we see in (a) and (b) that a fraction the N-ion density breaks away and propagates with faster velocity \(v_n\), \(n < N\), corresponding to the N-ion’s possible subsystems. The simulation parameters are (a) \(L = 100, U = 8J, \delta = 0.1\), and (b) \(L = 30, \delta = 0.1 U = 3J\), and the plots show a maximum density of \(\simeq 0.3\) such that the breakaway probability amplitudes with \(v_n\) are visible. Time is in units \(1/J\). The cumulative average polarization for several initial conditions (A, B, C, D, E) for (c) SU(2), and (d) SU(3) models, for \(L = 100\) and \(L = 30\), respectively, at \(U = 8J\). (c) In the SU(2) model, setup A consists in injecting a doublon at \(x = 0\), and B, a pair of spin up and down at different sites near \(x = 0\). The panel shows \(P_1(t_\infty)\) and \(P_2(t_\infty)\) for A and B. (d) In the SU(3) model, setup C consists in injecting a trion at \(x = 0\), D, three particles at three different adjacent sites around \(x = 0\), and E, a doublon and single particle at different sites in the middle of the lattice. The panel shows \(P_1(t_\infty)\) and \(P_3(t_\infty)\) for C, D, and E. While \(P_1^c\) tends close to 0 or 0.5 in all setups, the specific \(P_2\) and \(P_3\) exist only when a doublon and a trion, respectively, are injected in the lattice.

The numerical results are confirmed by evaluating the chiral polarization in Eq. (16) analytically. As the model is non-interacting, we compute the average in (16), by using the Wick’s theorem, in terms of the non-interacting Green’s functions. Introducing the greater Green’s functions \(iG_{\alpha\beta}^{>}(0) = \langle c_{x,\alpha}(t)c_{x,\beta}^\dagger(0)\rangle\), the polarization reduces to \(P_1(t) = \sum_x x\text{Tr}\{G^{>}(0)(-x,-t)\sigma^zG^{>}(0)(x,t)\}\). By performing the Fourier transform we get

\[
P_1(t) = i \int \frac{dk}{2\pi} \text{Tr}\{G^{>}(0)(k,-t)\sigma^z\partial_kG^{>}(0)(k,t)\}.
\]

A straightforward calculation shows that the polarization is

\[
P_1(t) \simeq \frac{\nu}{2} - \int \frac{dk}{4\pi} \cos(2E_k t)(\mathbf{n}_k \times \partial_k \mathbf{n}_k)_z, \tag{17}
\]

with the winding number \(\nu = \int \frac{dk}{2\pi} (\mathbf{n} \times \partial_k \mathbf{n})_z\). In the long time limit \(P_1(t) \to \nu/2\), as the highly oscillating correction averages to zero for \(t_\infty \to \infty\), thus confirming our numerical results. The Zak phase \(\gamma_Z = \pi \nu \text{ (modulo } 2\pi\text{)}\), so it will not be able to distinguish nontrivial phases characterized by an even \(\nu\).

For a single-particle quantum walk the theoretical findings corroborate the experimental measurements for \(P_1(t)\) performed in photonic lattices [5, 6] thus confirming that \(P_1(t)\) is able to capture the bulk states’ topology (see Fig. 2). Evaluating analytically \(P_1(t)\) when a doublon or a trion are injected into the lattice is not an easy task, since it requires evaluation of various combinations of full Green’s functions. The presence of strong on-site interaction requires a careful analysis, and there is no guarantee that perturbation theory works for large \(U\).

In SU(\(N\)) models, a fraction of an \(N\)-ion amplitude can break away due to quantum fluctuations. The breakaway parts correspond to possible subsystems of an \(N\)-ion. These excitations will propagate at a faster maximum velocity due to a smaller mass. Figure 3 shows the time evolution for the density profile for (a) SU(2) and (b) SU(3) models. We identify for the SU(2) case, excitations propagating with the single-particle velocity \(v_1\), while for an SU(3) model, there are offshoots with velocities corresponding to single particles and to doublons with \(v_2\). The figures eliminate all amplitudes above a certain threshold, such that the small losses propagating with \(v_1\) or \(v_2\) are visible.

In the SU(2) model we record \(P_1(t)\) and

\[
P_2(t) = \langle n_2(t) \rangle^{-1} \sum_{x} \langle \Psi(t) | \phi^{(2)}_{x} \rangle \sigma^z \phi^{(2)}_{x} | \Psi(t) \rangle, \tag{19}
\]
FIG. 4. (a) Time-evolution of the density profile for two trions at \( \delta = 0.2 \). (b) Cumulative average polarization \( P_2(t) \) for two trions. Left and right insets represent the full time evolution of \( P_3(t) \) at \( \delta = -0.2 \) and \( \delta = 0.2 \), respectively. The two trions are injected at site 14 and 16 in an \( L = 30 \) lattice. \([ U = 3J \) and time \( t \) is in units of \( 1/J \).\]

while in the \( SU(3) \) model, \( P_1(t) \) and \( P_3(t) \), in Fig. 3(c) and (d), respectively. In simulations we also inject particles in the lower energy bands. This is done in \( SU(2) \) model by injecting a pair of particles at different sites. Then, modulo Hubbard local repulsion between them, the two particles propagate freely, such that \( P_1(t) \), properly normalized, has a response similar to the noninteracting SSH model. Similarly in the \( SU(3) \) model, we inject the particles at three different sites, or as a doublon plus a single particle. In all cases we see that \( P_1(t) \) approximates the quantized response in the noninteracting SSH model. In theses cases, due to repulsive Hubbard interactions, formations of a doublon in \( SU(2) \) model or a trion in \( SU(3) \) is suppressed such that (unnormalized) \( P_2(t) \) and \( P_3(t) \), respectively, are zero [datasets B, D, E in Fig. 3(c) and (d)]. Therefore \( P_2(t) \) and \( P_3(t) \) are specific measures when injecting a doublon in the \( SU(2) \) model, or a trion in \( SU(3) \) model.

TWO TRIONS

We have also addressed the case when two trions are injected in the lattice. Numerical results for the average density \( \langle n(x,t) \rangle \) and for the chiral and cumulative polarizations \( P_{1,3}(t) \), using TEBD, are presented in Fig. 4 for a disorder-free lattice. The results show that the cumulative polarization remains a good indicator of topological properties. Still, because the lattice is populated by more than a single trion, an effective interaction of the form

\[
H_{\text{eff}}^{\text{int}} \approx -|V_3| \sum_{\langle x,x' \rangle} \Phi_{x}^{(3)\dagger} \Phi_{x'}^{(3)} \Phi_{x'}^{(3)*} \Phi_{x}^{(3)\dagger} = -|V_3| \sum_{\langle x,x' \rangle} n_x^{(3)} n_{x'}^{(3)},
\]

with \( n_x^{(3)} = \Phi_{x}^{(3)\dagger} \Phi_{x}^{(3)} \) the trion number operator, and \( V_3/J_3 \sim U/J \). Therefore, in the strong interaction limit, the trions introduced at different positions in the lattice become heavy and attract each other. Furthermore, higher order processes generate a residual interaction between the trions and doublons as well, producing a diffusion of doublons into the lattice [7].

MANY-BODY BERRY PHASE

The many-body Berry phase was defined in Eqs. (3) and (4) in the main text. It allows to investigate numerically the topological properties of the interacting models in equilibrium. The Berry phase shows however limitations in describing the topological properties for \( SU(N) \) Hamiltonians with an even number of flavors \( N = 2n \). Such a case is exemplified in Fig. 5 for \( SU(2) \), with two interacting fermions of opposite spin projection. The problem is similar to the noninteracting case, where the spinful model is just two decoupled chains for each spin projection. Since there is a \( \pi \)-jump at \( \delta = 0 \) for each chain, there is a \( 2\pi \)-jump in the total model. Since the Berry phases are defined modulo 2\( \pi \), it is impossible distinguish in the Berry phase between \( \delta < 0 \) and \( \delta > 0 \) regimes. This is generic for \( SU(2n) \) noninteracting systems, since the chiral polarization is proportional to the winding number \( \nu \), while the Berry phases are defined \( \pi \nu \) (modulo 2\( \pi \)), and cannot discriminate phases with even \( \nu \) from the trivial state. This remains true in the interacting \( SU(2) \) models (see Fig. 5(a), dataset B), where the Berry phase is zero at all \( \delta \), while \( P_2(t) \) correctly distinguishes in Fig. 3(c) the topological phases.
FIG. 5. Many-body Berry phases in the (a) interacting SU(2) and (b) SU(3) models when summing Berry phases for various ranges of many-body states starting from the ground state for clean $L = 8$ chains and $U = 8J$. In panel (a), B (green) represents integration over all low-energy states, including the lower doublon band, showing that a $\pi$-jump is not detectable in SU(2$N$) models. A (red) and C (violet) are the results when subtracting the highest state or adding a new highest state in energy to the integration in B. A $\pi$-jump is obtained in D (blue), when numerically imposing twisted boundary conditions only for spin up. In panel (b), we show the constant $\gamma_B$ obtained when summing over all non-trionic states, A (red), or over all three-particle states, E (teal). Summing all states including the lower trion band, C (blue), shows the expected transition at $\delta = 0$. Finally, subtracting or adding a single state to C, in B and D, respectively, gives a unquantized $\gamma_B$.

Nevertheless, it is still possible to reveal the Berry phase $\pi$-jump at the transition for interacting models. Adding twisted boundary conditions for only one spin channel, allows one to observe the $\pi$-jump in the model even if interactions mix the two flavors (see Fig. 5, dataset D). We conclude that without such numerical probes, it will not be possible to record the topological transition in the Berry phase for SU(2$N$) models.

The case of SU(2$n + 1$) is expected to show an unambiguous $\pi$-jump in $\gamma_B$ at $\delta = 0$. Here we add further proof to the results in the main text for the SU(3) model (see Fig. 5(b)). We show that the integration of the non-trion, low-energy states, gives a constant Berry phase 0, Fig. 5(b), dataset A. Therefore carrying integration for all states from ground state to the dimerization gap (including the lower trion band) shows the expected $\pi$-jump, seen also in the main text from integration only over the lower trion band. We also show in Fig 5(b) that adding or subtracting even a single state to the lower trion band results in an unquantized Berry phase with no trace of the jump. This is a more general result, which holds for most subsets of states, and shows that the many-body Berry phase has the expected quantization characteristic for an SSH model, only when integrating over the states visited by the trion.

In the limit of large $M$ the determinant formula from Eq. (3) in the main text converges to a sum of Berry phases over the many-body states,

$$\gamma_B = -\text{Im} \ln \prod_{n=0}^{M-1} \text{det}[S^{(n,n+1)}] = -\text{Im} \sum_{n=0}^{M-1} \text{tr} \ln[S^{(n,n+1)}] \approx -\text{Im} \sum_{n=0}^{M-1} \sum_j \ln(\psi_j^{(n)}) e^{2\pi i X / ML} \langle \psi_j^{(n)} | \psi_j^{(n+1)} \rangle,$$  \hspace{1cm} (21)

where $j$ indexes states in the chosen subset of many-body states. The last approximate expression is obtained in the limit of large $M$, in which the exponential operator is close unity, and states $\psi_j^{(n)}$ and $\psi_j^{(n+1)}$ are nearly orthogonal for $j \neq j'$, such that the overlap matrix $S$ is close to a diagonal matrix. Therefore, in the limit of large $M$, it follows that

$$\gamma_B \approx -\sum_j \text{Im} \ln \prod_{n=0}^{M-1} \langle \psi_j^{(n)} | e^{2\pi i X / ML} | \psi_j^{(n+1)} \rangle.$$  \hspace{1cm} (22)

Equality is used everywhere modulo $2\pi$. The latter $\gamma_B$ corresponds thus to a sum of Berry phases for the excited states of $M$ trions in an $ML$ lattice, where the trions in different lattices of size $L$ do not interact with each other [8, 9].

We find numerically that the determinant formula is more robust, and for disorder-free systems the single-point Berry phase $M = 1$ is sufficient to identify the topological phases, and shows good quantization. In the interacting disordered models, exact diagonalization is restricted to small sizes $L$. In such cases, increasing $M$ is useful to suppress finite size effects, as the effective length of the system is $ML$.

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[1] M. Di Liberto, A. Recati, I. Carusotto, and C. Menotti, Phys. Rev. A 94, 062704 (2016).
[2] M. A. Gorlach and A. N. Poddubny, Phys. Rev. A 95, 053866 (2017).
[3] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. 88, 035005 (2016).
[4] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979).
[5] F. Cardano, A. D’Errico, A. Dauphin, M. Maffei, B. Piccirillo, C. de Lisio, G. De Filippis, V. Cataudella, E. Santamato, L. Marrucci, M. Lewenstein, and P. Massignan, Nat. Commun. 8, 15516 (2017).
[6] Y. Wang, Y.-H. Lu, F. Mei, J. Gao, Z.-M. Li, H. Tang, S.-L. Zhu, S. Jia, and X.-M. Jin, Phys. Rev. Lett. 122, 193903 (2019).
[7] M. A. Werner, C. P. Moca, M. Kormos, Ö. Legeza, B. Dóra, and G. Zaránd arXiv:2207.00994 (2022).
[8] R. Resta and S. Sorella, Phys. Rev. Lett. 82, 370 (1999).
[9] R. Resta, J. Phys.: Condens. Matter 12, R107 (2000).