Abstract. We establish exact sequences in $KK$-theory for graded relative Cuntz–Pimsner algebras associated to nondegenerate right-Hilbert bimodules. We use this to calculate the graded $K$-theory and $K$-homology of relative Cuntz–Krieger algebras of directed graphs for gradings induced by $\{0,1\}$–valued labellings of their edge sets.

Introduction

In the study of $C^*$-algebras, operator $K$-theory, which generalises topological $K$-theory via Gelfand duality, has long been a key invariant. Indeed, it is a knee-jerk reaction for $C^*$-algebraist these days, when presented with a new example, to try to compute its $K$-theory; and the $K$-theory frequently reflects key structural properties. For example, the ordered $K$-theory of an irrational rotation algebra recovers the angle of rotation up to a minus sign [11, 12], while the $K$-theory of a Cuntz–Krieger algebra recovers the Bowen–Franks group of the associated shift space [6, 7]. Cuntz proved that the $K$-theory groups of the $C^*$-algebra of a finite directed graph $E$ with no sources and with $\{0,1\}$-valued adjacency matrix $A_E$ are the cokernel and kernel of the matrix $1-A_E^t$ regarded as an endomorphism of the free abelian group $\mathbb{Z}E^0$ (see [7, Proposition 3.1]). This was generalised to row-finite directed graphs $E$ with no sources in [24], to all row-finite directed graphs $E$ in [28], and to arbitrary graphs (with appropriate adjustments made to the domain and the codomain of $A_E^t$) in [1, 10], see also [13, 36, 32, 33, 27].

Dual to $K$-theory is the $K$-homology theory that emerged in the pioneering work of Brown–Douglas–Fillmore [3, 4]. It is less of an automatic reaction to compute $K$-homology for $C^*$-algebras, but, for example, Cuntz and Krieger computed (in [6, Theorem 5.3]) the Ext-group (that is, the odd $K$-homology group) of the Cuntz–Krieger algebra $O_A$ of $A \in M_n(\mathbb{Z}_+)$ as the cokernel of $1-A$ regarded as an endomorphism of $\mathbb{Z}^n$. The computation was later generalised to graph $C^*$-algebras in [35, 10] (see also [36, 5]).

Both $K$-theory and $K$-homology are unified in Kasparov’s $KK$-theory [18, 19]: the $K$-theory and $K$-homology groups of $A$ are the Kasparov groups $KK_*(\mathbb{C},A)$ and $KK_*(A,\mathbb{C})$ respectively. So Kasparov’s theory provides a unified approach to calculating $K$-theory and $K$-homology. Pimsner exploited this in [26], developing two exact sequences in $KK$-theory for the $C^*$-algebra $O_X$ associated to a right-Hilbert $A$–$A$-bimodule $X$, and using one of them to compute the $K$-theory of $O_X$ in terms of that of $A$. Every graph determines an associated Hilbert module, and while the Pimsner algebra of this module only agrees with the graph $C^*$-algebra when the graph has no sources, Muhly and Tomforde developed...
a modified bimodule [23] whose Pimsner algebra always contains the graph $C^*$-algebra as a full corner. In particular, combining these results provides a new means of computing the $K$-theory and $K$-homology of graph $C^*$-algebras.

Kasparov’s $KK$-theory is most naturally a theory for graded $C^*$-algebras, and the results described above are obtained by endowing the $C^*$-algebras involved with the trivial grading. However, graph $C^*$-algebras admit many natural gradings: by the universal property of $C^*(E)$, every binary labelling $\delta : E^1 \rightarrow \{0, 1\}$ of the edges of $E$ induces a grading automorphism that sends the generator $s_e$ associated to an edge $e$ to $(-1)^{\delta(e)}s_e$. More generally, every grading of a right-Hilbert bimodule induces a grading of the associated Pimsner algebra.

In [20], Kumjian, Pask and Sims investigated graded $K$-theory and $K$-homology, defined in terms of Kasparov theory (other approaches to graded $K$-theory are investigated in, for example, [8, 9, 16]) of graded graph $C^*$-algebras, extending earlier results of [14, 15] for the Cuntz algebras $\mathcal{O}_n$. By extending Pimsner’s arguments to $C^*$-algebras of graded Hilbert bimodules with injective left actions by compacts, they computed the graded $K$-theory of the $C^*$-algebras of row-finite graphs with no sources. They showed (in [20, Colloquary 8.3]) that if $E$ is a row-finite directed graph with no sources, $\alpha_0$ is the grading associated with a given function $\delta : E^1 \rightarrow \{0, 1\}$, and $A_E^0$ is the $E^0 \times E^0$ matrix with entries $A^0_E(v, w) = \sum_{e \in vE^1w} \delta(e)$, then the graded $K$-theory groups are isomorphic to the cokernel and kernel of $1 - (A^0_E)^t$ regarded as an endomorphism of $\mathbb{Z}E^0$.

In this paper we compute both the graded $K$-theory and the graded $K$-homology of relative Cuntz–Krieger algebras of arbitrary graphs: Let $V$ be any subset of regular vertices $E^0_{tg}$ (those which receive a nonzero finite set of edges). The relative Cuntz–Krieger algebra $C^*(E; V)$ is the universal $C^*$-algebra in which the Cuntz–Krieger relation is only imposed at vertices in $V$. In particular $C^*(E) = C^*(E; E^0_{tg})$ for directed graph $E$. Let $A^0_V$ be the $V \times E^0$ matrix with entries given by the same formula as $A^0_E$, regarded as a homomorphism from $\mathbb{Z}E^0$ to $\mathbb{Z}V$. Write $\tilde{A}^0_V$ for the dual homomorphism from $\mathbb{Z}E^0$ to $\mathbb{Z}V$. Let $\iota : \mathbb{Z}V \rightarrow \mathbb{Z}E^0$ be the inclusion map, and let $\pi : \mathbb{Z}E^0 \rightarrow \mathbb{Z}V$ be the projection map. Then the graded $K$-theory groups and $K$-homology groups of the relative Cuntz–Krieger algebra are given by

\begin{align*}
K^0_0(C^*(E; V), \alpha_0) &\cong \ker(\iota - A^0_V)^t, \\
K^0_1(C^*(E; V), \alpha_0) &\cong \ker(\pi - \tilde{A}^0_V), \\
K^0_0(C^*(E; V), \alpha_0) &\cong \ker(\iota - A^0_V)^t, \\
K^1_1(C^*(E; V), \alpha_0) &\cong \ker(\pi - \tilde{A}^0_V).
\end{align*}

To prove this, we use that $C^*(E; V)$ may be realised as a relative Cuntz–Pimsner algebra of a graph module $X(E)$. We verify that the two assumptions (namely injectivity and compactness) imposed on the left actions of $A$ on a Hilbert module $X$ in the arguments of [26, 20] are not needed. As a result we obtain exact sequences in $KK$-theory analogous to those of [20] for relative Cuntz–Pimsner algebras. By calculating the $KK$-groups and the maps between them in the situation where $X$ is the graph module $X(E)$, we obtain the desired calculations of graded $K$-theory and $K$-homology for relative graph $C^*$-algebras, substantially generalising the results in [20].

We then present an alternative calculation using Muhly and Tomforde’s adding-tails construction. In this approach, Pimsner’s exact sequences are needed only for modules where the homomorphism implementing the left action is injective. Given an arbitrary nondegenerate bimodule $X$, we add an infinite direct sum of copies of the Katsura ideal $J_X$ to both the coefficient algebra and the module $X$ to obtain a new module $Y$ over a new
algebra $B$ which acts injectively on the left. We then recover the exact sequences for $\mathcal{O}_Y$ from the ones we already have for $\mathcal{O}_X$ using countable additivity in $KK$-theory. This is automatic in the first variable, so we obtain a complete generalisation of the contravariant exact sequence of [20] for $\mathcal{O}_X$. However, $KK$-theory is not in general countably additive in the second variable. However $KK_1(\mathbb{C}, \cdot)$ is countably additive for graded $C^*$-algebras (we could not find a reference, so we give the details) so we obtain an exact sequence describing the graded $K$-theory $KK_1(\mathbb{C}, \mathcal{O}_X)$.

We begin in Section 1 with some background on $KK$-theory, mostly to establish notation. More-detailed background on $KK$-theory can be found in [20], and of course in Blackadar’s book [2], which is our primary reference. We assume the reader is familiar with Hilbert modules and graph $C^*$-algebras. A convenient summary of the requisite background appears in [20], and more details can be found in [27, 29, 21, 26]. We also provide a little background on the relative Cuntz–Pimsner algebras of Muhly and Solel [22], of which Katsura’s Katsura–Pimsner algebras are a special case. In Section 2 we show how to generalise the results of [20] to relative Cuntz–Pimsner algebras of arbitrary essential graded Hilbert modules. In Section 3 we apply these results to compute the graded $K$-theory and $K$-homology of relative Cuntz–Krieger algebras of arbitrary graphs. In Section 4, we show how Muhly and Tomforde’s adding-tails construction for Hilbert modules can be adapted to graded modules, and reconcile this with our $K$-theory and $K$-homology results for the graded Katsura–Pimsner algebra of a nondegenerate Hilbert module.

1. Background material

In this section we provide some background on $KK$-theory, relative graph $C^*$-algebras and terminology used in the later sections.

1.1. Direct sums and products of groups. Let $S$ be any countable set. We let $Z^S$ denote the direct sum $\bigoplus_{s \in S} \mathbb{Z}$ of copies of $\mathbb{Z}$ (the group of all finitely supported functions from $S$ to $\mathbb{Z}$), and $Z^S$ denotes the direct product $\prod_{s \in S} \mathbb{Z}$ of copies of $\mathbb{Z}$ (the group of all functions from $S$ to $\mathbb{Z}$). More generally we write $\prod_{n=1}^{\infty} G_n$ for the infinite product of groups $G_n$, and $\bigoplus_{n=1}^{\infty} G_n$ for the subgroup generated by the $G_n$.

1.2. Hilbert modules. Given a $C^*$-algebra $B$ and a right Hilbert $B$-module $X$, we write $\mathcal{L}(X)$ for the adjointable operators on $X$, we write $\mathcal{K}(X)$ for the generalised compact operators on $X$, and given $\xi, \eta \in X$ we write $\Theta_{\xi, \eta}$ for the compact operator $\zeta \mapsto \xi \cdot \langle \eta, \zeta \rangle_B$. If $\phi : A \to \mathcal{L}(X)$ is a $C^*$-homomorphism so that $X$ is an $A$–$B$-correspondence, we say that the left action is injective if $\phi$ is injective and that the left action is by compact operators if $\phi(A) \subseteq \mathcal{K}(X)$.

Let $I$ be an ideal of a $C^*$-algebra $A$ and $X$ be a right Hilbert $A$-module $X$. Following [17], we define $XI := \{ x \in X : \langle x, x \rangle \in I \}$. This $XI$ is a right Hilbert $I$-module under the same operations as $Y$, and $XI = X \cdot I := \{ x \cdot i : x \in X, i \in I \}$ justifying the notation.

1.3. Gradings. A grading of a $C^*$-algebra is a self-inverse automorphism $\alpha$ of $A$, and decomposes $A$ into direct summands $A_0 = \{ a : \alpha(a) = a \}$ and $A_1 = \{ a : \alpha(a) = -a \}$. We write $\partial(a) = i$ if $a \in A_i$. If $a, b$ are homogeneous, then their graded commutator is $[a, b]_{\partial} := ab - (-1)^{\partial(a)\partial(b)}ba$, and we extend this formula bilinearly to arbitrary $a, b \in A$. Elements of $A_0 \cup A_1$ are called homogeneous, elements of $A_0$ are even and elements of
A \textit{grading} of a \(C^*\)-correspondence \(X\) over a \(C^*\)-algebra \((A,\alpha_A)\) and \((B,\alpha_B)\) is a map \(\alpha_X : X \to X\) such that \(\alpha_X^2 = \text{id}, \alpha_X(a \cdot x \cdot b) = \alpha_A(a) \cdot \alpha_X(x) \cdot \alpha_B(b),\) and \(\alpha_B((x, y)_B) = \langle \alpha_X(x), \alpha_X(y) \rangle_B\). This induces a grading \(\hat{\alpha}_X\) on \(\mathcal{L}(X)\) given by \(\hat{\alpha}_X(T) = \alpha_X \circ T \circ \alpha_X\). Again, \(X\) decomposes as the direct sum of \(X_0 = \{\xi : \alpha_X(\xi) = \xi\}\) and \(X_1 = \{\xi : \alpha_X(\xi) = -\xi\}\), we call the elements of the \(X_i\) homogeneous, and odd and even as above.

For a graded \(C^*\)-algebra \(B\) set \(IB := C([0, 1]) \hat{\otimes} B\) with the trivial grading on \(C([0, 1])\). For each \(t \in [0, 1]\) the homomorphism \(\epsilon_t : IB \to B\) given by \(\epsilon_t(f \hat{\otimes} b) = f(t)b\) is graded. It follows that \(B\) may be regarded as a graded \(IB-B\)-correspondence \((\epsilon_t, B_B)\).

1.4. \textit{Graded tensor products.} The \textit{graded tensor product} of graded \(C^*\)-algebras \(A\) and \(B\) is the minimal \(C^*\)-completion of the algebraic tensor product \(A \otimes B\) with involution satisfying \((a \hat{\otimes} b)^* = (-1)^{\delta a-\delta b} a^* \hat{\otimes} b^*\) and multiplication satisfying \((a \hat{\otimes} b)(a' \hat{\otimes} b') = (-1)^{\delta a-\delta a'} ab' \hat{\otimes} a'b\) for homogeneous elements \(a, a' \in A\) and \(b, b' \in B\). There is a grading \(\alpha_A \hat{\otimes} \alpha_B\) on \(A \hat{\otimes} B\) such that \((\alpha_A \hat{\otimes} \alpha_B)(a \hat{\otimes} b) = \alpha_A(a) \hat{\otimes} \alpha_B(b)\). The balanced tensor product \(X \hat{\otimes}_B Y\) of graded \(C^*\)-correspondences admits a grading \(\alpha_X \hat{\otimes} \alpha_Y\) on \(X \hat{\otimes}_\psi Y\) such that \((\alpha_X \hat{\otimes} \alpha_Y)(x \hat{\otimes} y) = \alpha_X(x) \hat{\otimes} \alpha_Y(y)\).

1.5. \textit{Relative Cuntz–Pimsner algebras.} Let \(X\) be a graded \(A-A\)-correspondence, and write \(\varphi : A \to \mathcal{L}(X)\) for the homomorphism inducing the left action.

A \textit{representation} of \(X\) in a \(C^*\)-algebra \(B\) is a pair \((\pi, \psi)\) consisting of a homomorphism \(\pi : A \to B\) and a linear map \(\psi : X \to B\) such that \(\pi(a)\psi(\xi)\pi(b) = \psi(a \cdot x \cdot b)\) and \(\pi((\xi, \eta)_A) = \psi(\xi)^* \psi(\eta)\) for all \(a, b \in A\) and \(\xi, \eta \in X\). The \textit{Toeplitz algebra} \(T_X\) of \(X\) is the universal \(C^*\)-algebra generated by a representation of \(X\). Such a representation induces a homomorphism \(\psi^{(1)} : \mathcal{K}(X) \to B\) satisfying \(\psi^{(1)}(\Theta_{\xi, \eta}) = \psi(\xi)^* \psi(\eta)^*\) for all \(\xi, \eta\). The universal property of \(T_X\) gives a grading \(\alpha_T\) of \(T_X\) such that if \((\pi, \psi)\) is the universal representation of \(X\) in \(T_X\), then \(\alpha_T \circ \pi = \pi \circ \alpha_A\), and \(\alpha_T \circ \psi = \psi \circ \alpha_X\).

Let \(C := \varphi^{-1}(\mathcal{K}(X))\); observe that if \((\pi, \psi)\) is a representation of \(X\), then both \(\pi\) and \(\psi^{(1)} \circ \varphi\) are homomorphisms from \(C\) to \(T_X\). Given an ideal \(I \subseteq C\), the relative Cuntz–Pimsner algebra \(\mathcal{O}_{X,I}\) is defined to be the universal \(C^*\)-algebra generated by a representation \((\pi, \psi)\) that is \(I\)-\textit{covariant} in the sense that \(\pi|_I = (\psi^{(1)} \circ \varphi)|_I\). If \(I\) is invariant under the grading \(\alpha_A\) of \(A\), then the universal property of \(\mathcal{O}_{X,I}\) shows that the grading \(\alpha_T\) of \(T_X\) descends to a grading \(\alpha_O\) of \(\mathcal{O}_{X,I}\).

The \textit{Fock space} \(\mathcal{F}_X\) is the internal direct sum \(\mathcal{F}_X := \bigoplus_{n=0}^{\infty} X^{\hat{\otimes} n}\), with the convention that \(X^{\hat{\otimes} 0} = A A\). There is a representation \((\ell_0, \ell_1)\) of \(X\) in \(\mathcal{L}(\mathcal{F}_X)\) such that \(\ell_0(a)\xi = a \cdot \xi\) and such that \(\ell_1(\xi)\eta = \xi \hat{\otimes}_A \eta\). The induced homomorphism \(\pi_0 : T_X \to \mathcal{L}(\mathcal{F}_X)\) is injective.

The ideal \(C = \varphi^{-1}(\mathcal{K}(X))\) of \(A\) induces the submodule \(\mathcal{F}_{X,C} := \mathcal{F}_X C\). The subalgebra \(\mathcal{K}(\mathcal{F}_{X,C}) := \text{span} \{\Theta_{\xi, \eta} : \xi, \eta \in \mathcal{F}_{X,C}\} \subseteq \mathcal{L}(\mathcal{F}_X)\) is contained in \(\pi_0(\mathcal{T}_X)\) (see [22, Lemma 2.17]). Since \(\pi_0 : T_X \to \mathcal{L}(\mathcal{F}_X)\) is injective, it determines an inclusion \(j : \mathcal{K}(\mathcal{F}_{X,C}) \to T_X\). In particular, for any ideal \(I \subseteq C\), \(j\) restricts to a graded inclusion of \(\mathcal{K}(\mathcal{F}_{X,I})\) in \(T_X\). Theorem 2.19 in [22] and a routine application of universal properties show that the quotient map \(T_X \to \mathcal{O}_{X,I}\) induces an isomorphism \(T_X / j(\mathcal{K}(\mathcal{F}_{X,I})) \cong \mathcal{O}_{X,I}\).
1.6. Kasparov modules. If \((A, \alpha_A)\) and \((B, \alpha_B)\) are separable graded \(C^*\)-algebras, then a Kasparov \(A-B\)-module is a quadruple \((X, \phi, F, \alpha_X)\) where \((\phi, X)\) is a countably generated \(A-B\)-correspondence, \(\alpha_X\) is a grading of \(X\), and \(F \in \mathcal{L}(X)\) is an odd element with respect to the grading \(\hat{\alpha}_X\) on \(\mathcal{L}(X)\) such that for all \(a \in A\) the elements \((F-a')\phi(a), (F^2-1)\phi(a), \) and \([F, \phi(a)]^{gr}\) are compact. When these elements are all zero we call \((X, \phi, F, \alpha_X)\) for a degenerate Kasparov module.

Kasparov \(A-B\)-modules \((X_0, \phi_0, F_0, \alpha_{X_0})\) and \((X_1, \phi_1, F_1, \alpha_{X_1})\) are unitarily equivalent if there is a unitary \(U \in \mathcal{L}(X_0, X_1)\) that intertwines \(\phi_0\) and \(\phi_1\), \(F_0\) and \(F_1\), and \(\alpha_0\) and \(\alpha_1\). They are homotopy equivalent if there is Kasparov \(A-\mathbb{I}B\)-module \((X, \phi, F, \alpha_X)\) such that, \((X \hat{\otimes}_{\epsilon_i} B_B, \epsilon_i \circ \phi, \epsilon_i(F), \alpha_X \hat{\otimes} \alpha_B)\) is unitarily equivalent to \((X_i, \phi_i, F_i, \alpha_{X_i})\) for each \(i = 0, 1\). Homotopy equivalence is denoted \(\sim_h\), and is an equivalence relation. The Kasparov group \(KK(A, B)\) is the collection of all homotopy classes of Kasparov \(A-B\)-modules, which is a group under the operation induced by taking direct sums of Kasparov modules. Given a graded homomorphism \(\psi : A \to B\) of \(C^*\)-algebras, and a Kasparov \(B-C\)-module \((X, \phi, F, \alpha_X)\), we obtain a new Kasparov \(A-C\)-module \((X, \phi \circ \psi, F, \alpha_X)\), whose class we denote \(\psi[X]\). For a graded homomorphism \(\psi : B \to C \cong \mathcal{K}(C_C)\) we let \([\psi] := [C_C, \psi, 0, \alpha_C] \in KK(B, C)\). If \(\phi : A \to B\) is a graded homomorphism and \((Y, \psi, G, \alpha_Y)\) is a Kasparov \(C-A\)-module, then \((Y \hat{\otimes}_\phi B_B, \psi \hat{\otimes} 1, G \hat{\otimes} 1, \alpha_Y \hat{\otimes} \alpha_B)\) is a Kasparov \(C-B\)-module whose class we denote by \(\phi_*[Y, \psi, G, \alpha_Y]\).

We will need to use that graded Morita equivalence implies \(KK\)-equivalence. Suppose that \((A, \alpha_A)\) and \((B, \alpha_B)\) are graded \(C^*\)-algebras, and \((X, \alpha_X)\) is a graded imprimitivity \(A-B\)-module. Then in particular the left action of \(A\) on \(X\) is implemented by a homomorphism \(\varphi : A \to \mathcal{K}(X)\), and the right action of \(B\) on the dual module \(X^*\) is implemented by a homomorphism \(\psi : B \to \mathcal{K}(X^*)\). So we obtain \(KK\)-classes \([X] := [X, \varphi, 0, \alpha_X] \in KK(A, B)\) and \([X^*] := [X^*, \psi, 0, \alpha_{X^*}] \in KK(B, A)\). Since the Fredholm operators in both modules of these Kasparov modules are zero, we have \([X] \hat{\otimes}_B [X^*] = [X \hat{\otimes}_B X^*, \varphi \otimes 1, 0, \alpha_X \hat{\otimes} \alpha_{X^*}]\). Since \(X\) is an imprimitivity module, we have \(X \hat{\otimes} X^* \cong A\) via \(\xi \otimes \eta \mapsto A(\xi, \eta)\), and it is routine to check that this isomorphism intertwines the canonical left action \(\hat{\varphi}\) of \(A\) on \(X \otimes X^*\) with the left action \(L_A\) of \(A\) on itself by left multiplication, and intertwines \(\alpha_X \otimes \alpha_{X^*}\) with \(\alpha_A\). So \([X] \hat{\otimes}_B [X^*] = [A, L_A, 0, \alpha_A] = [id_A]\). Similarly \([X^*] \hat{\otimes}_A [X] = [id_B]\), and so \((A, \alpha_A)\) and \((B, \alpha_B)\) are \(KK\)-equivalent.

2. Graded \(K\)-theory of relative Cuntz–Pimsner algebras

In this section we generalise the main results in Section 4 of [20] establishing exact sequences in \(KK\)-theory for graded relative Cuntz–Pimsner algebras associated to an essential graded \(A-A\)-correspondence. We do not assume the action is injective, nor compact nor that \(X\) is full, but we do assume that \(X\) is essential (or nondegenerate) in the sense that \(\overline{\varphi(A)X} = X\).

Set-up. Throughout this section we fix a graded, separable, nuclear \(C^*\)-algebra \(A\) and a graded countably generated essential \(A-A\)-correspondence \(X\) with a left action \(\varphi\) and we fix an ideal \(I \subseteq \varphi^{-1}(\mathcal{K}(X))\).

For readers interested in the Katsura–Pimsner algebra \(O_X\) ([17, Definition 3.5]) we recall that it coincides with the relative Cuntz–Pimsner algebra \(O_{X,I}\) for \(I = \varphi^{-1}(\mathcal{K}(X)) \cap \ker \varphi^\perp\), where \(\ker \varphi^\perp = \{b \in A : b \ker \varphi = \{0\}\}\).

We present terminology of [20] relevant to Lemma 2.1. Let \(F_X\) be the Fock space of \(X\), let \(\alpha_X^\infty\) be the diagonal grading on \(F_X\), and let \(\varphi^\infty : A \to \mathcal{L}(F_X)\) be the diagonal left
action of \( A \) on \( \mathcal{F}_X \). Recall that \( \mathcal{T}_X \) is the Toeplitz algebra associated to \( X \), generated by \( i_A(a) = \varphi^\infty(a) \) and \( i_X(\xi) = T_\xi \), and \( \alpha_T \) is the restriction of \( \tilde{\alpha}_X^\infty \) to \( \mathcal{T}_X \). Let \( \pi_i : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X) \) be the representations determined by

\[
\pi_0(T_\xi)\rho = \begin{cases} 
\xi \otimes \rho, & \rho \in X^{\hat{n}}, n \geq 1, \\
\xi \cdot \rho, & \rho \in A,
\end{cases}
\pi_1(T_\xi)\rho = \begin{cases} 
\xi \otimes \rho, & \rho \in X^{\hat{n}}, n \geq 1, \\
0, & \rho \in A.
\end{cases}
\]

As presented in [20, Section 4] there is a Kasparov \( \mathcal{T}_X-A \)-module given by

\[
M = \left( \mathcal{F}_X \oplus \mathcal{F}_X, \pi_0 \oplus \pi_1 \circ \alpha_T, (\begin{smallmatrix} 0 & \alpha_X^\infty \\
0 & -\alpha_X^\infty \end{smallmatrix}), \left(\begin{smallmatrix} 0 & 0 \\
-\alpha_X^\infty & \alpha_X^\infty \end{smallmatrix}\right) \right).
\]

Recall, for the canonical inclusion \( i_A : A \to \mathcal{T}_X \) and a graded \( C^* \)-algebra \( B \) we have

\[
[i_A] = [(\mathcal{T}_X, i_A, 0, \alpha_T)] \in KK(A, \mathcal{T}_X), \quad \text{and} \quad [\text{id}_B] = [B, \text{id}_B, 0, \alpha_B] \in KK(B, B).
\]

By [20, Theorem 4.2], if \( \varphi \) is injective and by compact operators, then the Kasparov classes \([i_A]\) and \([M]\) are mutually inverse in the sense that \([i_A] \otimesT_X [M] = [i_A]\) and \([M] \otimesA [i_A] = [\text{id}_{\mathcal{T}_X}]\). We prove the result is true without assuming \( \varphi \) is injective or by compact operators.

**Lemma 2.1** (cf. [20, Theorem 4.2]). With notation as above, the pair \([i_A]\) and \([M]\) are mutually inverse. In particular, \((A, \alpha_A)\) and \((\mathcal{T}_X, \alpha_T)\) are \( KK \)-equivalent.

**Proof.** The argument in the proof of [20, Theorem 4.2] showing that \([i_A] \otimesT_X [M] = [i_A]\) does not require injectivity nor compactness of the left action \( \varphi \).

To show \([M] \otimesA [i_A] = [\text{id}_{\mathcal{T}_X}]\), we adjust the proof of [20, Theorem 4.2]. Let \( \pi_0' := \pi_0 \otimes 1_T \) and \( \pi_1' := (\pi_1 \circ \alpha_T) \otimes 1_T \). By [2, Proposition 18.7.2(a)], identifying \( (\mathcal{F}_X \oplus \mathcal{F}_X) \otimesA \mathcal{T}_X \) with \( (\mathcal{F}_X \otimesA \mathcal{T}_X) \oplus (\mathcal{F}_X \otimesA \mathcal{T}_X) \),

\[
[M] \otimesA [i_A] = (i_A)_* [M] = \left( (\mathcal{F}_X \oplus \mathcal{F}_X) \otimesA \mathcal{T}_X, \pi_0' \oplus \pi_1', (\begin{smallmatrix} 0 & 0 \\
-\alpha_X^\infty & \alpha_X^\infty \end{smallmatrix}), \left(\begin{smallmatrix} 0 & 0 \\
-\alpha_X^\infty & \alpha_X^\infty \end{smallmatrix}\right) \right).
\]

Since \( X \) is essential we have \( A \otimesA \mathcal{T}_X \cong \mathcal{T}_X \) as graded \( A-\mathcal{T}_X \)-correspondences, and so \( \alpha_T \) defines a left action of \( \mathcal{T}_X \) on \( A \otimesA \mathcal{T}_X \). Extending by zero, we get an action \( \tau \) of \( \mathcal{T}_X \) on \( \mathcal{F}_X \otimesT \mathcal{T}_X \). The proof of [20, Theorem 4.2] shows that

\[
\left( (\mathcal{F}_X \oplus \mathcal{F}_X) \otimesA \mathcal{T}_X, 0 \oplus \tau, (\begin{smallmatrix} 0 & 0 \\
0 & \alpha_X^\infty \otimesA \alpha_T \end{smallmatrix}), (\begin{smallmatrix} 0 & 0 \\
-\alpha_X^\infty & \alpha_X^\infty \end{smallmatrix}) \right) \]

and hence

(2.1) \( (i_A)_* [M] - [\text{id}_{\mathcal{T}_X}] = \left( (\mathcal{F}_X \oplus \mathcal{F}_X) \otimesA \mathcal{T}_X, \pi_0' \oplus (\pi_1' + \tau), (\begin{smallmatrix} 0 & 0 \\
0 & \alpha_X^\infty \otimesA \alpha_T \end{smallmatrix}), (\begin{smallmatrix} 0 & 0 \\
0 & -\alpha_X^\infty \otimesA \alpha_T \end{smallmatrix}) \right) \).

We claim (2.1) is the class of a degenerate Kasparov module. To show this, for each \( t \in [0, 1] \) define \( \psi_t : X \to \mathcal{L}(\mathcal{F}_X \otimes \mathcal{T}_X) \) by

\[
\psi_t(\xi) = \cos(t\pi/2)(\pi_0'(\alpha_T(T_\xi))) - \pi_1'(T_\xi) + \sin(t\pi/2)\tau(\xi) + \pi_1'(T_\xi).
\]

With \( \varphi^\infty := \varphi^\infty \otimes 1_{\mathcal{T}_X} : A \to \mathcal{L}(\mathcal{F}_X \otimes \mathcal{T}_X) \) we have a Toeplitz representation \( (\varphi^\infty \circ \alpha_A, \psi_t) \) of \( X \). Hence for each \( t \in [0, 1] \) there is a homomorphism \( \pi_t' : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X \otimes \mathcal{T}_X) \) such that \( \pi_t'(T_\xi) = \psi_t(\xi) \) and \( \pi_t'(a) = \varphi^\infty \circ \alpha_A(a) \) for all \( \xi \in X \) and \( a \in A \). We claim that \( K_{t, \xi} := (\pi_t' - \pi_1')(T_\xi) \) is compact for each \( \xi \in X \). To see this, note that \( K_{t, \xi} \) vanishes on \( (\mathcal{F}_X \oplus A) \otimesA \mathcal{T}_X = (\mathcal{F}_X \otimesA \mathcal{T}_X) \oplus (A \otimesA \mathcal{T}_X) \), and has range contained in the subspace \( A \otimesA \mathcal{T}_X \), thus we need only need to show that \( K_{t, \xi} \) is compact on \( A \otimesA \mathcal{T}_X \). To show this recall that for an \( A-B \)-correspondence \( Y \) with an left action \( \psi : A \to \mathcal{L}(Y) \), putting \( J := \psi^{-1}(\mathcal{K}(Y)) \), for each \( k \in \mathcal{K}(A,J) \) the operator \( k \otimes \text{1}_Y \) is compact on \( A \otimes \psi Y \), see [21,
Proposition 4.7. With $Y := T_X$, $\psi := i_A = \varphi^\infty$ and $J := i_A^{-1}(\mathcal{K}(T_X)) = A$ it follows that each $\tilde{\varphi}^\infty(a)|_{A \otimes_A T_X}$ is compact (because $\varphi^\infty(a)|_A$ is compact). Use [29, Proposition 2.31] to express $\xi = y \cdot \langle y, y \rangle$. We compute

\[ K_{\xi, \xi} = \pi'_1(T_{\xi}) - \pi'_1(T_{\xi}) = (\pi'_1(T_y) - \pi'_1(T_y)) \tilde{\varphi}^\infty(\langle y, y \rangle), \]

which is compact since $\tilde{\varphi}^\infty(\langle y, y \rangle)|_{A \otimes_A T_X}$ is compact. As in [20], $\psi_0(\xi) = \pi'_0 \circ \alpha_T(T_{\xi})$ and $\psi_1(\xi) = (\pi'_1 + \tau)(T_{\xi})$, so we can replace $\pi'_1 + \tau$ with $\pi'_0 \circ \alpha_T$ in the expression (2.1) for $(i_A)_* [M] - [\text{id}_{T_X}]$ without changing the class. The latter is the class of a degenerate Kasparov module, proving the claim. Thus $[M] \otimes_A [i_A] = [\text{id}_{T_X}]$. \qed

We introduce the notation used in Lemma 2.1. Let $I \subseteq \varphi^{-1}(\mathcal{K}(X))$ be the ideal of $A$ consisting of elements that act as compact operators on the left of $X$. Let $\mathcal{F}_{X,I}$ denote the right Hilbert $I$-module $\mathcal{F}_{X,I} := \{ \xi \in \mathcal{F}_X : \langle \xi, \xi \rangle_A \in I \}$. Let $\iota_{F_1}: I \hookrightarrow A$ and $\iota_{F_1} : K(\mathcal{F}_{X,I}) \hookrightarrow \mathcal{L}(\mathcal{F}_X)$ be the inclusion maps. As discussed in Section 1, $\iota_{F_1}(K(\mathcal{F}_{X,I}))$ contained in the image of $\pi_0(\mathcal{T}_X) \subseteq \mathcal{L}(\mathcal{F}_X)$ of the Toeplitz algebra under its canonical representation on the Fock space. Thus there is a graded embedding $j : K(\mathcal{F}_{X,I}) \hookrightarrow \mathcal{T}_X$ such that $\pi_0 \circ j = \iota_{F_1}$. We have the induced Kasparov classes

\[ [\iota_I] = [A_A, \iota_I, 0, \alpha_A] \in KK(I, A), \quad [j] = [\mathcal{T}_X, j, 0, \alpha_T] \in KK(K(\mathcal{F}_{X,I}), \mathcal{T}_X). \]

Writing $P : \mathcal{F}_X \to \mathcal{F}_X \ominus A = \bigoplus_{n=1}^{\infty} X^\otimes n$ for the projection onto the orthogonal complement of the $0$th summand, we have

\[ [M] = \left[ F_X \oplus (F_X \ominus A), \pi_0 \oplus \pi_1 \circ \alpha_T, \left( \begin{array}{cc} 0 & 1 \\ \frac{1}{\pi} & 0 \end{array} \right), \left( \begin{array}{cc} \frac{1}{\alpha_X} & 0 \\ 0 & -\frac{1}{\alpha_X} \end{array} \right) \right] \in KK(\mathcal{T}_X, A). \]

We denote by $[j, X]$ the class $[X, \varphi|_I, 0, \alpha_X] \in KK(I, A)$ of the module $X$ with the left action restricted to $I$. We note $\iota_{F_1}$ induces the Kasparov class $[\mathcal{F}_{X,I}, \iota_{F_1}, 0, \alpha_{\infty}^X] \in KK(K(\mathcal{F}_{X,I}), I)$, where $\alpha_{\infty}^X$ and each image of $\iota_{F_1}$ are now considered as operators on $\mathcal{F}_{X,I} \subseteq \mathcal{F}_X$.

Lemma 2.2 (cf. [20, Lemma 4.3]). With notation as above we have

\[ [j] \otimes_{\mathcal{T}_X} [M] = [\mathcal{F}_{X,I}, \iota_{F_1}, 0, \alpha_{\infty}^X] \otimes_I ([\iota_I] - [j, X]). \]

Proof. Using [2, Proposition 18.7.2(b)] we can express $[j] \otimes_{\mathcal{T}_X} [M]$ as

\[ [j] \otimes_{\mathcal{T}_X} [M] = \left[ F_X \oplus (F_X \ominus A), (\pi_0 \oplus (\pi_1 \circ \alpha_T)) \circ j, \left( \begin{array}{cc} 0 & 1 \\ \frac{1}{\pi} & 0 \end{array} \right), \left( \begin{array}{cc} \frac{1}{\alpha_X} & 0 \\ 0 & -\frac{1}{\alpha_X} \end{array} \right) \right]. \]

Let $\alpha_K$ be the restriction of $\alpha_{\infty}^X$ to $K(F_{X,I})$. Since $\pi_0 \circ j = \iota_{F_1}$ and $\pi_1 \circ \alpha_T \circ j = \pi_1 \circ j \circ \alpha_K$ take values in $K(F_{X,I})$, the straight line path from $\left( \begin{array}{cc} 0 & 1 \\ \frac{1}{\pi} & 0 \end{array} \right)$ to 0 determines an operator homotopy of Kasparov modules. Hence we may write

\begin{align*}
[j] \otimes_{\mathcal{T}_X} [M] &= \left[ F_X \oplus (F_X \ominus A), (\pi_0 \oplus (\pi_1 \circ \alpha_T)) \circ j, 0, \left( \begin{array}{cc} \frac{1}{\alpha_X} & 0 \\ 0 & -\frac{1}{\alpha_X} \end{array} \right) \right] \\
&= \left[ F_X, \pi_0 \circ j, 0, \alpha_{\infty}^X \right] + \left[ F_X \ominus A, \pi_1 \circ j \circ \alpha_K, 0, -\alpha_{\infty}^X \right] \\
&\quad + \left[ F_X \ominus A, \pi_1 \circ j, 0, \alpha_X \right] - \left[ F_X \ominus A, \pi_1 \circ j, 0, \alpha_X \right].
\end{align*}

(2.2)

Since $K(F_{X,I}) \mathcal{F}_X = F_{X,I}$ and $K(F_{X,I})(F_X \ominus A) = F_{X,I} \oplus A$ we may (see [2, Section 17.5]) replace each module in (2.2) with its essential submodule without changing the Kasparov classes, to obtain

\[ [j] \otimes_{\mathcal{T}_X} [M] = \left[ F_{X,I}, \iota_{F_1}, 0, \alpha_{\infty}^X \right] - \left[ F_{X,I} \ominus A, \pi_1 \circ j, 0, \alpha_X \right]. \]

(2.3)
The map $(\xi \otimes a) \mapsto \xi \cdot a$ implements a unitary equivalence
\[(\mathcal{F}_{X,I} \hat{\otimes} I A_A, t_{\mathcal{F}I} \hat{\otimes} 1, 0, \alpha_{\mathcal{X},I} \hat{\otimes} \alpha_A) \cong (\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}).\]
So, writing $[\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}]_A$ for the class in $KK(\mathcal{T}_X, A)$ obtained by regarding $\mathcal{F}_{X,I}$ as a right $A$-module, and $[\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}]_I$ for the class obtained by regarding it as a right $I$-module, and recalling that $[i_I] \in KK(I, A)$ is the class of the inclusion of $I$ in $A$, we have
\[[\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}]_I \hat{\otimes} I [i_I] = [\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}]_A.\]
Since $X$ is essential, the map that sends $i \cdot \xi$ to $i \otimes \xi$ for $i \in I$ and $\xi \in X$, and sends $\xi_1 \otimes \cdots \otimes \xi_n$ to $(\xi_1 \otimes \cdots \otimes \xi_{n-1}) \otimes \xi_n$ for $\xi_1, \ldots, \xi_n \in X$ is a unitary equivalence
\[(\mathcal{F}_{X,I} \hat{\otimes} I A, \pi_1 \circ j, 0, \alpha_{\mathcal{X},I}) \cong (\mathcal{F}_{X,I} \hat{\otimes} I X, (\pi_0 \circ j) \hat{\otimes} 1, 0, \alpha_{\mathcal{X},I} \hat{\otimes} \alpha_X).\]
By [2, Proposition 18.10.1], $[\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}] \hat{\otimes} I [i_X] = [\mathcal{F}_{X,I} \hat{\otimes} I X, t_{\mathcal{F}I} \hat{\otimes} 1, 0, \alpha_{\mathcal{X},I} \hat{\otimes} \alpha_X]$ provided that $(t_{\mathcal{F}I} \hat{\otimes} 1)(\mathcal{K}(\mathcal{F}_{X,I})) \subseteq \mathcal{K}(\mathcal{F}_{X,I} \hat{\otimes} I X)$. This containment is a direct consequence of [21, Proposition 4.7] since $I$ consists of elements that act as compact operators on the left of $X$. It follows that
\[[\mathcal{F}_{X,I} \hat{\otimes} A, \pi_1 \circ j, 0, \alpha_{\mathcal{X},I}] = [\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}] \hat{\otimes} I [i_X].\]
Substituting both of these equalities into (2.3) and using distributivity of the Kasparov product gives
\[[j] \hat{\otimes} \mathcal{T}_X [M] = [\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}] - [\mathcal{F}_{X,I} \hat{\otimes} A, \pi_1 \circ j, 0, \alpha_{\mathcal{X},I}] = [\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}] \hat{\otimes} I [i_I] + [\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}] \hat{\otimes} [i_X] = [\mathcal{F}_{X,I}, t_{\mathcal{F}I}, 0, \alpha_{\mathcal{X},I}] \hat{\otimes} I ([i_I] - [i_X]). \quad \Box
\]

Our next result appears in the first author’s honours thesis [25, Theorem 7.0.3] for $I = \varphi^{-1}(\mathcal{K}(X))$, in the context where $\varphi$ is injective. For the definition of the relative Cuntz–Pimsner algebra $\mathcal{O}_{X,I}$ see Section 1.

**Theorem 2.3.** Let $(A, \alpha_A), (B, \alpha_B)$ be graded separable $C^*$-algebras and suppose that $A$ is nuclear. Let $X$ be a countably generated essential $A$–$A$-correspondence with left action $\varphi$, and let $I \subseteq \varphi^{-1}(\mathcal{K}(X))$ be a graded ideal of $A$. Let $i_I : I \to A$ be the inclusion map. Then we have six term exact sequences
\[
\begin{array}{c}
KK_0(B, I) \overset{i_*}{\to} KK_0(B, A) \overset{i_*}{\to} KK_0(B, O_{X,I}) \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
KK_1(B, O_{X,I}) \overset{i_*}{\leftarrow} KK_1(B, A) \overset{i_*}{\leftarrow} KK_1(B, I)
\end{array}
\]
and
\[
\begin{array}{c}
KK_0(I, B) \overset{i_*}{\leftarrow} KK_0(A, B) \overset{i_*}{\leftarrow} KK_0(O_{X,I}, B) \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
KK_1(O_{X,I}, B) \overset{i_*}{\leftarrow} KK_1(A, B) \overset{i_*}{\leftarrow} KK_1(I, B)
\end{array}
\]

**Proof.** As in [20, Theorem 4.4], we shall prove exactness of the first diagram. Exactness of the second follows from a similar argument. Since $A$ is nuclear, so is $\mathcal{T}_X$ by [30, Theorem 6.3]. Hence the quotient map $q : \mathcal{T}_X \to O_{X,I} \cong \mathcal{T}_X/j(\mathcal{K}(\mathcal{F}_{X,I}))$ admits a
completely positive splitting, see [2, Example 19.5.2]. Applying [31, Theorem 1.1] to the graded short exact sequence

$$0 \rightarrow \mathcal{K}(\mathcal{F}_{X,I}) \xrightarrow{j} \mathcal{T}_X \xrightarrow{q} \mathcal{O}_{X,I} \rightarrow 0$$

gives homomorphisms $\delta : KK_*(B, \mathcal{O}_{X,I}) \rightarrow KK_{*+1}(B, \mathcal{K}(\mathcal{F}_{X,I}))$ for which the following six term sequence is exact

$$KK_0(B, \mathcal{K}(\mathcal{F}_{X,I})) \xrightarrow{j_*} KK_0(B, \mathcal{T}_X) \xrightarrow{q_*} KK_0(B, \mathcal{O}_{X,I}) \xrightarrow{\delta}$$

Define $\delta' : KK_*(B, \mathcal{O}_{X,I}) \rightarrow KK_{*+1}(B, A)$ by $\delta' = (\cdot \hat{\otimes} [\mathcal{F}_{X,I}, \iota_{\mathcal{F}_{I}}, 0, \alpha_X^\infty]) \circ \delta$ and let $i : A \rightarrow \mathcal{O}_{X,I}$ be the canonical homomorphism. Consider the following diagram.

$$\begin{array}{ccc}
KK_0(B, I) & \xrightarrow{\hat{\otimes}([I_I]-[I_X])} & KK_0(B, A) & \xrightarrow{i_*} & KK_0(B, \mathcal{O}_{X,I}) \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
KK_0(B, \mathcal{K}(\mathcal{F}_{X,I})) & \xrightarrow{j_*} & KK_0(B, \mathcal{T}_X) & \xrightarrow{q_*} & KK_0(B, \mathcal{O}_{X,I}) \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
KK_1(B, \mathcal{O}_{X,I}) & \xrightarrow{q_*} & KK_1(B, \mathcal{T}_X) & \xrightarrow{j_*} & KK_1(B, \mathcal{K}(\mathcal{F}_{X,I})) \\
\downarrow \id & & \downarrow \id & & \downarrow \id \\
KK_1(B, \mathcal{O}_{X,I}) & \xrightarrow{i_*} & KK_1(B, A) & \xrightarrow{\hat{\otimes}([I_I]-[I_X])} & KK_1(B, I) \\
\end{array}$$

By definition of $\delta'$, the left and right squares commute. Lemma 2.2 shows that the top left and bottom right squares commute. By definition we have $i = q \circ i_A$ as homomorphisms, so $i_* = (q \circ i_A)_* = q_* \circ (i_A)_*$. This shows that the top right and bottom left squares commute.

Lemma 2.1 implies that $(i_A)_*$ and $\hat{\otimes}[M]$ are mutually inverse isomorphisms. Finally, the class of $[\mathcal{F}_{X,I}, \iota_{\mathcal{F}_{I}}, 0, \alpha_X^\infty]$ is induced by the graded Morita equivalence bimodule $\mathcal{F}_{X,I}$ (see [29]), and so induces an isomorphism of $KK$-groups, so exactness of the interior 6-term sequence gives the desired exactness of the exterior one.

\[ \square \]

3. \textsc{Graded \textit{K}-theory and \textit{K}-homology for relative Cuntz–Krieger algebras}

In this section we use our results from the preceding section to calculate the graded $K$-theory and graded $K$-homology of a graded relative graph $C^*$-algebra.

We first recall the key elements of relative graph $C^*$-algebras that we will use here. Given a directed graph $E = (E^0, E^1, r, s)$, we denote by $E^0_{\text{rg}}$ the set

$$E^0_{\text{rg}} := \{ v \in E^0 : vE^1 \text{ is finite and nonempty} \}$$

of regular vertices of $E$. Given a subset $V \subseteq E^0_{\text{rg}}$, the \textit{relative Cuntz–Krieger algebra} $C^*(E; V)$ of $E$ is defined as the universal $C^*$-algebra generated by mutually orthogonal
projections \( \{p_v : v \in E^0\} \) and partial isometries \( \{s_e : e \in E^1\} \) such that \( s_es^*_e = p_{s(e)} \) for all \( e \in E^1 \), and such that \( p_v = \sum_{e \in vE^1} s_es^*_e \) for all \( v \in V \). So \( C^*(E; \emptyset) \) coincides with the usual Toeplitz algebra \( TC(E) \), and \( C^*(E; E^0_{rg}) \) coincides with the graph \( C^* \)-algebra \( C^*(E) \). Given a function \( \delta : E^1 \to \{0,1\} \), the universal property of \( C^*(E; V) \) yields a grading \( \alpha^\delta \) of \( C^*(E; V) \) satisfying \( \alpha^\delta(s_e) = (-1)^{\delta(e)} s_e \) for all \( e \in E^1 \), and \( \alpha^\delta(p_v) = p_v \) for all \( v \in E^0 \).

There is a \( C_0(E^0) \)-valued inner-product on \( C_c(E^1) \) given by

\[
\langle \xi, \eta \rangle_{C_0(E^0)}(w) = \sum_{e \in E^1 w} \xi(e)\eta(e).
\]

The completion \( X(E) \) of \( C_c(E^1) \) in the resulting norm is a \( C_0(E) - C_0(E) \)-correspondence with actions determined by \( (a \cdot \xi \cdot b)(e) = a(r(e))\xi(b(s(e))). \) The ideal \( I \subseteq C_0(E^0) \) of elements that act by compact operators on this module is \( C_0(\{v : |vE^1| < \infty\}) \). A routine argument using universal properties shows that for \( V \subseteq E^0_{rg} \), we have \( \mathcal{O}_{X(E), C_0(V)} \cong C^*(E; V) \); in particular, \( \mathcal{T}_X(E) \cong C^*(E; \emptyset) = TC^*(E) \). Any map \( \delta : E^1 \to \{0,1\} \) determines a grading \( \alpha^\delta \) of \( X(E) \) satisfying \( \alpha^\delta(1_e) = (-1)^{\delta(e)} 1_e \) for all \( e \in E^1 \), and then the induced grading on \( \mathcal{O}_{X(E), C_0(V)} \) matches up with the grading \( \alpha^\delta \) on \( C^*(E; V) \).

Our main result for the section is the following.

**Theorem 3.1.** Let \( E \) be a directed graph. Fix a subset \( V \subseteq E^0_{rg} \) and a function \( \delta : E^1 \to \mathbb{Z}_2 \). Let \( \alpha \in \text{Aut}(C^*(E; V)) \) be the grading such that \( \alpha(s_e) = (-1)^{\delta(e)} s_e \) for all \( e \in E^1 \). Let \( A^\delta_V \) denote the \( V \times E^0 \) matrix such that \( A^\delta_V(v, w) = \sum_{e \in vE^1 w} \xi(e) \) for all \( v \in V \) and \( w \in E^0 \). Let \( \iota : \mathbb{Z}V \to \mathbb{Z}E^0 \) be the inclusion map. Regarding \( (A^\delta_V)^t \) as a homomorphism from \( \mathbb{Z}V \) to \( \mathbb{Z}E^0 \), the graded \( K \)-theory (of \( (C^*(E; V), \alpha) \) is given by

\[
K^0_{gr}(C^*(E; V), \alpha) \cong \ker(\iota - (A^\delta_V)^t) \quad \text{and} \quad K^1_{gr}(C^*(E; V), \alpha) \cong \ker(\iota - (A^\delta_V)^t).
\]

There is a homomorphism \( \tilde{A}^\delta_V : \mathbb{Z}E^0 \to \mathbb{Z} \) given by

\[
\tilde{A}^\delta_V(f)(v) = \sum_{e \in vE^1} A^\delta_V(v, w)f(w) \quad \text{for all } v \in V \text{ and } f \in \mathbb{Z}E^0.
\]

Let \( \pi : \mathbb{Z}E^0 \to \mathbb{Z}E \) be the projection map. Then

\[
K^0_{gr}(C^*(E; V), \alpha) \cong \ker(\pi - \tilde{A}^\delta_V) \quad \text{and} \quad K^1_{gr}(C^*(E; V), \alpha) \cong \ker(\pi - \tilde{A}^\delta_V).
\]

Before proving the theorem, we state an immediate corollary about the graded \( K \)-theory and \( K \)-homology of graph \( C^* \)-algebras.

**Corollary 3.2.** Let \( E \) be a directed graph. Fix a function \( \delta : E^1 \to \mathbb{Z}_2 \), and let \( \alpha \) be the associated grading of \( C^*(E) \). Let \( A^\delta_{rg} \) denote the \( E^0_{rg} \times E^0 \) matrix such that \( A^\delta_{rg}(v, w) = \sum_{e \in vE^1 w} (-1)^{\delta(e)} \) for all \( v \in V \) and \( w \in E^0 \) regarded as a homomorphism from \( \mathbb{Z}E^0_{rg} \) to \( \mathbb{Z}E^0 \) and write \( \tilde{A}^\delta_{rg} \) for the dual homomorphism \( \mathbb{Z}E^0 \to \mathbb{Z}E^0_{rg} \). Then

\[
K^0_{gr}(C^*(E), \alpha) \cong \ker(\iota - (A^\delta_{rg})^t), \quad K^1_{gr}(C^*(E), \alpha) \cong \ker(\iota - (A^\delta_{rg})^t),
\]

\[
K^0_{gr}(C^*(E), \alpha) \cong \ker(\pi - \tilde{A}^\delta_{rg}), \quad \text{and} \quad K^1_{gr}(C^*(E), \alpha) \cong \ker(\pi - \tilde{A}^\delta_{rg}).
\]

**Proof.** Apply Theorem 3.1 to \( V = E^0_{rg} \). \( \square \)

To prove the \( K \)-homology formulas in the theorem, we need some preliminary results; the corresponding results for \( K \)-theory are established in [20].
Proposition 3.3. Let $E$ be a row-finite directed graph. For $n \in \mathbb{Z}^{E_0}$ and $f \in C_0(E^0)$ let $\ell(f) \in \mathcal{L}(\bigoplus_{v \in E^0} \mathbb{C}^{[n_v]})$ be given by
\[
(\ell(f)x)_w = f(w)x_w, \quad \text{for each } w \in E^0.
\]
Then there is an isomorphism $\mu : \mathbb{Z}^{E_0} \to KK_0(C_0(E^0), \mathbb{C})$ such that
\[
\mu(n) = \left[ \bigoplus_{v \in E^0} \mathbb{C}^{[n_v]}, \ell, 0, \bigoplus_{v \in E^0} \text{sign}(n_v) \right].
\]

Proof. Since $E^0$ is discrete we can identify $C_0(E^0)$ with $\bigoplus_{v \in E^0} \mathbb{C}$. For each $w \in E^0$, let $g_w : C \to \bigoplus_{v \in E^0} \mathbb{C}$ be the coordinate inclusion. Then Theorem 19.7.1 of [2] implies that $\theta := \prod_{v \in E^0} g_w^* : KK(\bigoplus_{v \in E^0} \mathbb{C}, \mathbb{C}) \to \prod_{v \in E^0} KK(\mathbb{C}, \mathbb{C}) \simeq \mathbb{Z}^{E^0}$ is an isomorphism of groups, with inverse $\mu$. \(\square\)

Lemma 3.4. Let $E$ be a directed graph and let $\delta : E^1 \to \{0, 1\}$ be a function. Fix a subset $V \subseteq E^0$. Fix $v \in E^0$. For $f \in E^1 v$ and $a \in C_0(E^0)$, define $\psi^v(a) : \ell^2(E^1 v) \to \ell^2(E^1 v)$ on elementary basis vectors $\{e^f \} \subseteq \ell^2(E^1 v)$ by
\[
\psi^v(a)e^f = \begin{cases} a(r(f))e^f & \text{if } r(f) \in V \\ 0 & \text{otherwise.} \end{cases}
\]
Define $\beta : \ell^2(E^1 v) \to \ell^2(E^1 v)$ on basis elements by
\[
\beta(e^f) = (-1)^{\delta(f)} e^f.
\]
Let $\phi_V : C_0(V) \to \mathcal{K}(X(E))$ be the restriction of the left action. Then $(\ell^2(E^1 v), \psi^v, 0, \beta)$ is a Kasparov $C_0(E^0)$-$\mathbb{C}$-module, and for each $n_v \in \mathbb{Z}$ there is a unitary equivalence between the modules
\[
(\ell^2(E^1 v) \otimes \mathbb{C}^{[n_v]}, \psi^v, 0, \beta \otimes \text{sign}(n_v) \text{id})
\]
and
\[
(X(E) \otimes_{e^f} \mathbb{C}^{[n_v]}, \tilde{\phi}_V, 0, \alpha_X \otimes \text{sign}(n_v) \text{id}).
\]

Proof. Throughout the proof, we write $\psi$ for $\psi^v$. For each $w \in V$, the operator $\psi(\delta_w) = \sum_{f \in w E^1} \Theta_{e^f, e^f}$ is compact, and for $w \in E^0 \setminus V$, we have $\psi(\delta_w) = 0$. So each $\psi(a) = \bigoplus_{w \in V} a(w)\psi(\delta_w)$ is compact.

It is immediate that $\beta$ is a grading, and it preserves the grading of the left action since $C_0(E^0)$ carries the trivial grading. So $(\ell^2(E^1 v), \psi^v, 0, \beta)$ is a Kasparov $C_0(V)$-$\mathbb{C}$-module.

Recall that for an edge $f \in E^1 v$ the element $\delta_f \in X(E)$ denotes the point mass at $f$. Further for $j \leq |n_v|$ let $e^j$ be an orthonormal basis for $\mathbb{C}^{[n_v]}$. We claim there is a unitary equivalence $U : \ell^2(E^1 v) \otimes \mathbb{C}^{[n_v]} \to X(E) \otimes_{e^v} \mathbb{C}^{[n_v]}$ that acts on elementary basis tensors $e^f \otimes e^j$ by the formula
\[
U(e^f \otimes e^j) = \delta_f \otimes e^j.
\]
An elementary calculation shows that this formula preserves inner-products, so extends to a well defined isometry. To see that $U$ is surjective, note that any function $x \in X(E)$ that is zero on $E^1 v$ satisfies $x \otimes w = 0$ for any $w \in \mathbb{C}^{[n_v]}$. Thus it suffices to consider functions contained in the span of $\{\delta_f : f \in E^1 v\} = \overline{C_e(E^1 v)}$, and tensors $\delta^f \otimes e^j$ can be written in the form $U(e^f \otimes e^j)$ by construction.
The definitions of $\phi_V$ and $\psi^v$ show that $U$ intertwines the left actions. To see that it preserves gradings, fix $f \in E^1$ and $j \leq |n_v|$ and compute

$$U((\beta \otimes \text{sign}(n_v) \text{id})(e^f \otimes e^j)) = \text{sign}(n_v)(-1)^{\delta(f)}U(e^f \otimes e^j)$$

$$= \text{sign}(n_v)(-1)^{\delta(f)} \delta^f \otimes e^j$$

$$= (\alpha_X \otimes \text{sign}(n_v) \text{id})(\delta^f \otimes e^j).$$

**Proposition 3.5.** Let $E$ be a directed graph. Fix a subset $V \subseteq E^0_{rg}$ and a function $\delta : E^1 \to \{0, 1\}$. Let $A^0_\delta$ denote the $V \times E^0$ matrix such that $A^0_\delta(v, w) = \sum_{e \in v E^1 w} (-1)^{\delta(e)}$ for all $v \in V$ and $w \in E^0$. Let $[X_{E,V}] = [X(E), \phi_{[C_0(V), 0, \alpha_X]}]$ be the Kasparov class of the graded graph module with left action restricted to $C_0(V)$. Let $\mu$ denote both the isomorphism $\text{KK}_0(C_0(E^0), \mathbb{C}) \to \mathbb{Z}^{E^0}$ and the isomorphism $\text{KK}_0(C_0(V), \mathbb{C}) \to \mathbb{Z}^{V}$ from Proposition 3.3. Let $[i] \in \text{KK}(J_X, A)$ be the class of the inclusion, and let $\pi : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}_{rg}$ be the projection map. Then the following diagrams commute.

\[
\begin{array}{c}
\mathbb{Z}^{V} & \xrightarrow{\mu} & \mathbb{Z}^{E^0} \\
\downarrow{\mu} & \quad & \downarrow{\mu} \\
\text{KK}_0(C_0(V), \mathbb{C}) & \xleftarrow{[X_{E,V}]} & \text{KK}_0(C_0(E^0), \mathbb{C}) \\
\end{array}
\begin{array}{c}
\mathbb{Z}^{V} & \xrightarrow{\pi} & \mathbb{Z}^{E^0} \\
\downarrow{\mu} & \quad & \downarrow{\mu} \\
\text{KK}_0(C_0(V), \mathbb{C}) & \xleftarrow{[i]} & \text{KK}_0(C_0(E^0), \mathbb{C}) \\
\end{array}
\]

**Proof.** By surjectivity of the isomorphism $\mu$ each element of $\text{KK}_0(C_0(E^0), \mathbb{C})$ is of the form

$$\mu(n) = \left[ \bigoplus_{v \in E^0} \mathbb{C}^{[n_v]}, \ell, 0, \bigoplus_{v \in E^0} \text{sign}(n_v) \right]$$

for some $n \in \mathbb{Z}^{E^0}$. In what follows, we write $\phi_V$ for $\phi_{[C_0(V)]}$. We compute using Lemma 3.4

$$[X(E), \phi_V, 0, \alpha_X] \otimes \mu(n) = \left[ \bigoplus_{v \in E^0} (X(E) \otimes \mathbb{C}^{[n_v]}), \bigoplus_{v \in E^0} \psi^v \otimes 1, 0, \bigoplus_{v \in E^0} (\alpha_X \otimes \text{sign}(n_v)) \right]$$

$$= \left[ \bigoplus_{v \in E^0} \ell^2(E^1 v) \otimes \mathbb{C}^{[n_v]}, \bigoplus_{v \in E^0} \psi^v \otimes 1, 0, \bigoplus_{v \in E^0} (\beta \otimes \text{sign}(n_v)) \right].$$

Let $g_v : \mathbb{C} \to C_0(E^0)$ be the coordinate inclusion corresponding to the vertex $v \in E^0$. Theorem 19.7.1 of [2] gives an isomorphism $\theta := \prod_{v \in E^0} g^*_v : \text{KK}(C_0(E^0), \mathbb{C}) \to \prod_{v \in E^0} \text{KK}((C, \mathbb{C}) \cong \mathbb{Z}^{E^0}$. Applying $\theta$ to $[X_{E,V}] \otimes \mu(n)$ gives

$$\theta\left( \left[ \bigoplus_{v \in E^0} \ell^2(E^1 v) \otimes \mathbb{C}^{[n_v]}, \bigoplus_{v \in E^0} \psi^v \otimes 1, 0, \bigoplus_{v \in E^0} (\beta \otimes \text{sign}(n_v)) \right] \right)$$

$$= \left( \left[ \bigoplus_{v \in E^0} \ell^2(E^1 v) \otimes \mathbb{C}^{[n_v]}, \left( \bigoplus_{v \in E^0} \psi^v \otimes 1 \right) \circ g_v, 0, \bigoplus_{v \in E^0} (\beta \otimes \text{sign}(n_v)) \right] \right)_{w \in E^0}. $$
Since the action \((\psi \otimes 1) \circ g_w\) is zero on the submodule \(\ell^2(E^1v \setminus wE^1v) \otimes \mathbb{C}^{[n_v]}\) for each \(v, w \in E^0\), Equation (3.1) becomes
\[
\left( \bigoplus_{v \in E^0} \ell^2(E^1v) \otimes \mathbb{C}^{[n_v]}, \left( \bigoplus_{v \in E^0} \psi^\nu \otimes 1 \right) \circ g_w, 0, \bigoplus_{v \in E^0} (\beta \otimes \text{sign}(n_v)) \right)_{w \in E^0}
\]
\[
= \left( \bigoplus_{v \in E^0} \ell^2(wE^1v) \otimes \mathbb{C}^{[n_v]}, \left( \bigoplus_{v \in E^0} \psi^\nu \otimes 1 \right) \circ g_w, 0, \bigoplus_{v \in E^0} (\beta \otimes \text{sign}(n_v)) \right)_{w \in E^0}
\]
\[
= \left( \bigoplus_{v \in E^0} \mathcal{A}^\delta(w, v) \otimes \mathbb{C}^{[n_v]}, \left( \bigoplus_{v \in E^0} \psi^\nu \otimes 1 \right) \circ g_w, 0, \bigoplus_{v \in E^0} (\text{sign}(A^\delta(w, v)n_v)) \right)_{w \in E^0}.
\]
\[(3.2)\]

Since \(\psi^\nu(\delta_w)\) is zero when \(w \notin V\), we may pass to the essential submodule, so that (3.2) becomes
\[
\left( \bigoplus_{v \in E^0} \mathcal{A}^\delta(w, v) \otimes \mathbb{C}^{[n_v]}, \left( \bigoplus_{v \in E^0} \psi^\nu \otimes 1 \right) \circ g_w, 0, \bigoplus_{v \in E^0} (\text{sign}(A^\delta(w, v)n_v)) \right)_{w \in V}
\]
\[
= \left( \bigoplus_{v \in E^0} \mathcal{A}^\delta(w, v) \otimes \mathbb{C}^{[n_v]}, \left( \bigoplus_{v \in E^0} \psi^\nu \otimes 1 \right) \circ g_w, 0, \text{sign} \left( \sum_{v \in E^0} A^\delta(w, v)n_v \right) \right)_{w \in V}.
\]

This is exactly the representative of \(A^\delta_V n \in \mathbb{Z}E^0\) as a module in \(KK_0(\mathbb{C}, \mathbb{C})^V\). Hence \([X_{E, V}] \hat{\otimes} \mu(n) = \mu(A^\delta_V n)\) as required.

The final statement follows directly from the definition of \(\mu\). \(\square\)

\textbf{Proof of Theorem 3.1.} For the bimodule \(X(E)\), the coefficient algebra \(A\) is \(C_0(E^0) = \bigoplus_{v \in E^0} \mathbb{C}\) and the ideal \(I \subseteq C\) is \(C_0(V) = \bigoplus_{v \in V} \mathbb{C}\). Countable additivity of K-theory (or Proposition 4.1) shows that \(KK_0(\mathbb{C}, C_0(E^0)) \cong \mathbb{Z}E^0\) and \(KK_0(\mathbb{C}, I) \cong \mathbb{Z}V\). The argument of [20, Lemma 8.2] shows that these isomorphisms intertwine the map \(\cdot \hat{\otimes} [\cdot]X(E)\) from \(KK_0(\mathbb{C}, I)\) to \(KK(\mathbb{C}, A)\) with the map \((A^\delta_V)^t : \mathbb{Z}V \rightarrow \mathbb{Z}E^0\). Functoriality of \(KK(\mathbb{C}, \cdot)\) shows that these isomorphisms also intertwine \([\cdot]\) with the inclusion map \(\iota : \mathbb{Z}V \rightarrow \mathbb{Z}E^0\). So the exact sequence of Theorem 4.4 induces the exact sequence
\[
0 \rightarrow K^\text{gr}_1(C^*(E; V), \alpha) \rightarrow \mathbb{Z}V \stackrel{1-(A^\delta_V)^t}{\rightarrow} \mathbb{Z}E^0 \rightarrow K^\text{gr}_0(C^*(E; V), \alpha) \rightarrow 0,
\]
and the first part of the theorem follows.

For the second statement, first note that Proposition 3.3 and Theorem 19.7.1 of [2] give isomorphisms \(KK_0(A, \mathbb{C}) \cong \mathbb{Z}E^0\) and \(KK_0(I, \mathbb{C}) \cong \mathbb{Z}V\), and show that \(KK_1(A, \mathbb{C}) = KK(J_X, \mathbb{C}) = \{0\}\). Proposition 3.5 shows that these isomorphisms intertwine \((1 - [\iota X(E)]) \hat{\otimes} \cdot\) with \((1 - A^\delta_V)^t\). So the exact sequence of Theorem 4.6 gives the exact sequence
\[
0 \rightarrow K^\text{gr}_0(C^*(E; V), \alpha) \rightarrow \mathbb{Z}E^0 \stackrel{1-A^\delta_V}{\rightarrow} \mathbb{Z}V \rightarrow K^\text{gr}_1(C^*(E; V), \alpha) \rightarrow 0.
\]
\(\square\)

4. Adding tails to graded correspondences

Inspired by the technique of ‘adding tails’ to directed graphs which transforms a directed graph into a graph without sources with a Morita equivalent \(C^*-\) algebra, Muhly and Tomforde proved that given an \(A-A\) correspondence \(X\) with non-injective left action, there is a correspondence \(Y\) over a \(C^*-\) algebra \(B\) such that the left action of \(B\) on \(Y\) is implemented by an injective homomorphism, and \(O_X\) is a full corner of \(O_Y\). If \(A\) and \(X\) are graded, these gradings extend naturally to gradings on \(B\) and \(Y\), and the inclusion of \(O_X\) in \(O_Y\) is graded. In particular \((O_X, \alpha_X)\) and \((O_Y, \alpha_Y)\) are \(KK\)-equivalent, and in particular have isomorphic graded \(K\)-theory and \(K\)-homology.
In this section we show how adding tails to a correspondence has applications in $KK$-theory. More specifically we show how to recover Pimsner’s 6-term exact sequences for $KK(\cdot, B)$ and $KK(\mathbb{C}, \cdot)$ for any graded countably generated essential $A$–$A$-correspondence $X$ using only the corresponding sequences for graded correspondences with an injective left action (these were established in the first author’s honours thesis [25]). This yields the calculations of graded $K$-theory and $K$-homology of $C^*$-algebras of arbitrary graphs that first appeared in the first author’s honours thesis [25] and the second author’s honours thesis [34] respectively (see Corollary 3.2). It also provides a useful reality check for our more general results in the preceding sections; we thank Ralf Meyer for pointing us to the direct proof of Pimsner’s sequences for modules with non-injective left actions employed there. To exploit the adding-tails technique, we need to know that graded $K$-theory and $K$-homology are each countably additive in the appropriate sense. We provide details below.

4.1. Countable additivity. In what follows, a direct sum of graded $C^*$-algebras is endowed with the natural direct-sum grading; so given a Kasparov class $[X] = [X, \phi, F, \alpha_X] \in KK(\bigoplus_i A_i, B)$, for each $n$, the inclusion map $\iota_{A_n} : A_n \hookrightarrow \bigoplus_i A_i$ is graded and induces the class $i_{A_n}^*[X] \in KK(A_n, B)$.

Theorem 19.7.1 of [2] shows that for separable $B$, $KK(\cdot, B)$ is countably additive in the following sense. If $A = \bigoplus_{n=1}^\infty A_n$ is a $\sigma$-direct sum of separable $C^*$-algebras $A_n$, then there is an isomorphism $\zeta : KK(A, B) \cong \prod_{n=1}^\infty KK(A_n, B)$ that carries the class of a Kasparov $A$–$B$-module $(X, \phi, F, \alpha_X)$ to the sequence whose $n$th term is the class of $(X, \phi \circ \iota_{A_n}, F, \alpha_X)$; that is, in the notation of Section 1.6, writing $[X]$ for the class of $(X, \phi, F, \alpha_X)$, we have

$$\zeta[X] = \left(\iota_{A_n}^*[X]\right)_{n=1}^\infty.$$  

In particular, identifying $KK(A, \mathbb{C})$ with $K^0_{\mathbb{U}}(A)$, we obtain countable additivity of graded $K$-homology.

On the other hand, as discussed in [2, 19.7.2], the map $KK(B, \cdot)$ is typically not countably additive. There is a natural map $\omega$ from $\bigoplus_{i=1}^\infty KK(A_i, B_i)$ to $KK(A, \bigoplus_{i=1}^\infty B_i)$ such that

$$\omega([X_i, \phi_i, F_i, \alpha_i])_{i=1}^n = [\bigoplus_{i=1}^n X_i, \bigoplus_{i=1}^n \phi_i, \bigoplus_{i=1}^n F_i, \bigoplus_{i=1}^n \alpha_i].$$  

This $\omega$ is always injective (see the proof of Proposition 4.1 below), but is typically not surjective. Since our focus is on graded $K$-theory and graded $K$-homology, we content ourselves with recording the presumably well-known fact that $KK(\mathbb{C}, \cdot)$ is countably additive. For ungraded $C^*$-algebras, this follows from countable additivity of $K$-theory since $KK(\mathbb{C}, A) \cong K_0(A)$.

Proposition 4.1. Let $A_i, \alpha_i$ be a sequence of $\sigma$-unital graded $C^*$-algebras. Then the map $\omega : \bigoplus_{i=1}^\infty KK(C, A_i) \to KK(C, \bigoplus_{i=1}^\infty A_i)$ defined at (4.1) is an isomorphism.

Proof. Write $A_{\infty} := \bigoplus A_i$. Let $(X, \phi, F, \alpha_X)$ be a Kasparov $\mathbb{C}$–$A_{\infty}$-module. Since each $A_i$ is an ideal of $A_{\infty}$, each $X_i := \{\xi : \langle \xi, \xi \rangle \in A_i\}$ is a right-Hilbert $A_i$-module. We identify $X$ and $\bigoplus_i X_i$ as right-Hilbert $A_{\infty}$-modules, so $X = \overline{\text{span}}\{x_i \in X_i : i \geq 1\}$ and $\langle x_i, x_j \rangle_A = 0$ for all $i \neq j$. Since $F$ and $\phi(1)$ are adjointable, they leave each $X_i$ invariant ($FX_i \subseteq X_i$ and $\phi(1)X_i \subseteq X_i$); so $F$ and $\phi$ decompose as $F = \bigoplus F_i$ and $\phi = \bigoplus \phi_i$. Since the inclusion map $\iota_{A_i} : A_i \hookrightarrow A_{\infty}$ is graded and $X_i = X \cdot A_i$, the automorphism $\alpha_X$ also leaves each $X_i$ invariant; so $\alpha_X$ decomposes as a direct sum $\alpha_X = \bigoplus \alpha_i$. Since each compact operator
on $X = \bigoplus X_i$ restricts to a compact operator on $X_i$, each $(X_i, \phi_i, F_i, \alpha_i)$ is a Kasparov $\mathbb{C} - A_i$-module.

Applying the preceding paragraph with $A_\infty$ replaced by $C([0,1], A_\infty)$ shows that any homotopy of Kasparov $\mathbb{C} - A_\infty$ modules decomposes as a direct sum of homotopies of Kasparov $\mathbb{C} - A_i$-modules, and so $\omega$ is injective.

To show that $\omega$ is surjective, we claim that it suffices to show that if $(X, \phi, F, \alpha)$ is a Kasparov $\mathbb{C} - A_\infty$-module, then there exists $N$ such that $[X_i, \phi_i, F_i, \alpha_i] = 0_{KK(C, A_i)}$ for all $i \geq N$. To see why, suppose that $[X_i, \phi_i, F_i, \alpha_i] = 0_{KK(C, A_i)}$ for all $i \geq N$. Let

$$[X^\infty_N] := \left[0^{N-1} \oplus \bigoplus_{i=N}^{\infty} X_i, 0^{N-1} \oplus \bigoplus_{i=N}^{\infty} \phi_i, 0^{N-1} \oplus \bigoplus_{i=N}^{\infty} F_i, 0^{N-1} \oplus \bigoplus_{i=N}^{\infty} \alpha_i \right] \in K(K(C, A_\infty)).$$

We have

$$[X, \phi, F, \alpha] = \omega(\bigoplus_{i=1}^{N-1} [X_i, \phi_i, F_i, \alpha_i]) + [X^\infty_N].$$

Since, for each $i \geq N$, the module $(X_i, \phi_i, F_i, \alpha_i)$ is a degenerate Kasparov module, so is $(\bigoplus_{i=N}^{\infty} X_i, \bigoplus_{i=N}^{\infty} \phi_i, \bigoplus_{i=N}^{\infty} F_i, \bigoplus_{i=N}^{\infty} \alpha_i)$. So $[X^\infty_N] = 0_{KK(C, A_\infty)}$, and it follows that $[X, \phi, F, \alpha] = \omega(\bigoplus_{i=1}^{N-1} [X_i, \phi_i, F_i, \alpha_i])$ belongs to the range of $\omega$.

To see that there exists $N$ such that $[X_i, \phi_i, F_i, \alpha_i] = 0_{KK(C, A_i)}$ for all $i \geq N$, first note that by [2, Propositions 17.4.2 and 18.3.6], we may assume that $\phi(1) = 1_X$, that $F = F^*$ and that $\|F\| \leq 1$. Using that $K(X) = \text{span} \left\{ \Theta_{x,y} : x, y \in \bigoplus_{n=1}^{\infty} X_n \right\}$ it follows that if $T \in K(X)$, then $\|T|_{X_i}\| \to 0$. Since $\phi_i(1) = 1_X$, and $F^2 - 1 = (F - 1)(F + 1)$, we deduce that $\|F^2 - 1\| \to 0$. So there exists $N$ large enough so that $F^2_i$ is invertible for $i \geq N$. Fix $i \geq N$; we will show that $(X_i, \phi_i, F_i, \alpha_i)$ is degenerate. Since $F^2_i = F_i$ we see that $F_i$ is normal, and so $\sigma(F_i^2) = \sigma(F_i)^2$, and in particular, $F_i$ is invertible. Since $\|F_i\| \leq 1$, we have $\sigma(F_i) \subseteq [-1, 0] \cup [0, 1]$, so for $t \in [0, 1]$ there is a continuous function $f_t \in C(\sigma(F_i))$ given by

$$f_t(x) = \begin{cases} (1-t)x + t & \text{if } x > 0 \\ (1-t)x - t & \text{if } x < 0. \end{cases}$$

Now the path $(F_t)_{t \in [0,1]}$ is a continuous path from $F_i$ to $f_1(F_i)$. Since $\sigma(f_1(F_i))) = f_1(\sigma(F_i)) = \{-1, 1\}$, we have $f_1(F_i)^2 = 1$.

We claim that for each $t$, the tuple $(X_i, \phi_i, f_t(F_i), \alpha_i)$ is a Kasparov module. To see this, first note that each $f_t(F_i)^* = f_t(F_i)$. Since $\phi_i(\mathbb{C}) = \text{span} 1_{X}$ is even-graded and central, we have $[\phi_i(a), f_t(F_i)]^\alpha = a[\phi_i(1), f_t(F_i)] = 0$ for all $t, a$. Since $F_i$ is odd with respect to the grading on $\mathcal{L}(X_i)$, so is $F_i^{2n+1}$ for every $n \geq 0$. So writing $P_{\text{odd}}$ for the space $\{ z \mapsto \sum_{n=0}^{N} a_n z^{2n+1} | N \geq 0 \text{ and } a_n \in \mathbb{C} \}$ of odd polynomials, $f_t(F_i)$ is odd for each $f \in P_{\text{odd}}$. Since the $f_t$ are all odd functions, they can be uniformly approximated by elements of $P_{\text{odd}}$, and we deduce that each $f_t(F_i)$ is odd with respect to $\alpha_i$. So to prove the claim, it remains to show that each $f_t(F_i)^2 - 1 \in K(X_i)$. For this, observe that the functional-calculus isomorphism for $F_i$ carries $F_i^{2} - 1$ to the function $z \mapsto z^2 - 1$. Since this function vanishes only at 1 and −1, the ideal of $C^*(F_i)$ generated by $F_i^{2} - 1$ is $\{ f(F_i) : f(1) = f(-1) = 0, f \in C(\sigma(F_i)) \}$. Since each $f_1^2(1) = 1 = f_2^2(-1)$, we deduce that each $f_t(F_i)^2 - 1$ belongs to the ideal generated by $F_i^{2} - 1$, and since $F_i^{2} - 1 \in \mathcal{K}(X_i)$ it follows that each $f_t(F_i)^2 - 1 \in \mathcal{K}(X_i)$. This proves the claim.

Hence $(X_i, \phi_i, f_t(F_i), \alpha_i)_{t \in [0,1]}$ is an operator homotopy, and it follows that

$$[X_i, \phi_i, F_i, \alpha_i] = [X_i, \phi_i, f_1(F_i), \alpha_i].$$
By construction, $f_1(F_i)^2 = 1$, and we already saw that $f_1(F_i)$ is odd, self-adjoint and commutes with $\phi_i(1)$. So $[X, \phi_i, f_1(F_i), \alpha_i] = 0_{KK(C, A)}$ as required.

**Set-up.** Throughout the remainder of this section we fix a graded, separable, nuclear $C^*$-algebra $A$ and a graded countably generated essential $A$-$A$-correspondence $X$ with left action implemented by $\varphi : A \to \mathcal{L}(X)$.

Recall that if $I$ is an ideal of $A$, then $I^\perp$ is the ideal $\{b \in A : bI = \{0\}\}$. Following Katsura, we define $J_X := \varphi^{-1}(\mathcal{K}(X)) \cap \ker \varphi^\perp \triangleleft A$. Since $ja = aj = 0$ for all $j \in J_X$ and $a \in \ker \varphi$, the ideal $J_X + \ker \varphi \subseteq A$ is the internal direct sum $J_X \oplus \ker \varphi$. We sometimes identify it with the external direct sum via the map $j + a \mapsto (j, a)$.

Since $J_X$ acts compactly on $X$, the quadruple $(X, \varphi|_{J_X}, 0, \alpha_X)$ is a Kasparov module, and we write $[X]$ for the corresponding class in $KK(J_X, A)$.

We define $K^\infty \varphi := \bigoplus_{n=1}^{\infty} \ker \varphi$ as a graded $C^*$-algebra, and $T := (K^\infty \varphi)_{K^\infty \varphi}$ regarded as an $(A \oplus K^\infty \varphi)$-$K^\infty \varphi$-correspondence with left action $(a, f) \cdot g = (ag_1, f_1g_2, f_2g_3, \ldots)$.

We define $Y := X \oplus T$ as a right-Hilbert $A \oplus K^\infty \varphi$-module, so the right action of $A \oplus K^\infty \varphi$ is given by

$$(x, f) \cdot (a, g) = (xa, fg)$$

and the inner product is given by

$$\langle (x, f), (y, g) \rangle = \langle (x, f), (y, g) \rangle^*,$$

for $x, y \in X$, $a \in A$, and $f, g \in T$.

Viewing the left action of $A$ on $X$ as an action of $A \oplus K^\infty \varphi$ in which the second coordinate acts trivially, $Y$ is an $(A \oplus K^\infty \varphi)$-$K^\infty \varphi$-correspondence with left action

$$\varphi^Y(a, f)(x, g) = (\varphi(a)x, ag_1, f_1g_2, \ldots, f_ng_{n+1}, \ldots).$$

The homomorphism $\varphi^Y$ is injective (see [23, Lemma 4.2]).

**Proposition 4.2.** The ideal $(\varphi^Y)^{-1}(\mathcal{K}(Y))$ is equal to $J_X \oplus \ker \varphi \oplus K^\infty \varphi \subseteq A \oplus K^\infty \varphi$.

**Proof.** As discussed just before the statement of the lemma, we have $J_X + \ker \varphi = J_X \oplus \ker \varphi$, so it suffices to show that $(\varphi^Y)^{-1}(\mathcal{K}(Y)) = (J_X + \ker \varphi) \oplus K^\infty \varphi$.

To prove that $(J_X + \ker \varphi) \oplus K^\infty \varphi \subseteq (\varphi^Y)^{-1}(\mathcal{K}(Y))$, fix $j + a \in J_X + \ker \varphi$, and $f \in K^\infty \varphi$. For $x \in X, g \in T$, since $\varphi(a + j) = \varphi(j)$ and since $j \cdot T = 0$, we have

$$\varphi^Y(j + a, f)(x, g) = (\varphi(j)x, ag_1, f_1g_2, f_2g_3, \ldots).$$

Since $j \in J_X$ we have $\varphi(j) \in \mathcal{K}(X)$. Further, letting $F = (a, f_1, f_2, \ldots)$ and letting $L_F \in \mathcal{K}(T)$ denote the left multiplication operator by $F$ we have

$$\varphi^Y(j + a, f)(x, g) = (\varphi(j)x, ag_1, f_1g_2, \ldots) = (\varphi(j)x, 0, 0, \ldots) + (0, L_Fg),$$

so $\varphi^Y(j + a, f) = (\varphi(j), L_F) \in \mathcal{K}(X) \oplus \mathcal{K}(T) \subset \mathcal{K}(X \oplus T)$.

To prove that $(\varphi^Y)^{-1}(\mathcal{K}(Y)) \subseteq J_X \oplus \ker \varphi \oplus K^\infty \varphi$, first note that $\varphi^Y$ decomposes as an (internal) direct sum $\varphi^A \oplus \varphi^T$ of the homomorphisms $\varphi^A$ and $\varphi^T$ given by

$$\varphi^A(a, f)(x, g_1, g_2, \ldots) = (\varphi(a)x, ag_1, 0, 0, \ldots),$$

and $\varphi^T(a, f)(x, g) = (0, 0, f_1g_2, f_2g_3, \ldots)$.

Suppose that $(a, f) \in (\varphi^Y)^{-1}(\mathcal{K}(Y))$; that is, $\varphi^Y(a, f)$ is compact. Since $\varphi^T(a, f)$ is left multiplication by $(0, 0, f_1, f_2, \ldots)$, it is compact. Hence $\varphi^A = \varphi^Y - \varphi^T$ is also compact. In particular $\varphi(a)$ is compact as it is the restriction of $\varphi^A(a, f)$ to the first entry, and so $a \in \varphi^{-1}(\mathcal{K}(X))$. Since $a$ also acts compactly on the first coordinate of $T$, which is the right-Hilbert module $(\ker \varphi)_{\ker \varphi}$, we see that the left-multiplication-by-$a$ operator on $(\ker \varphi)_{\ker \varphi}$
agrees with left-multiplication by some \( a' \in \ker \varphi \). In particular, \( j := a - a' \in (\ker \varphi)_{+} \). Since \( \varphi(j) = \varphi(a) - \varphi(a') = \varphi(a) \), we have \( j \in \varphi^{-1}(\mathcal{K}(X)) \cap (\ker \varphi)_{+} = J_X \). Hence \( a = j + a' \in J_X + \ker \varphi \). \( \Box \)

Having identified \((\varphi^\ast)^{-1}(\mathcal{K}(Y))\) with \( I := J_X \oplus \ker \varphi \oplus K_{\varphi}^\infty \), since \( \varphi^\ast \) is injective, we can apply the graded version of Pimsner’s exact sequence \([25, \text{Theorem 7.0.3}] \) to the \((A \oplus K_{\varphi}^\infty) - (A \oplus K_{\varphi}^\infty)\)-correspondence \( Y \). To avoid overly heavy notation in the resulting sequences \((4.2) \) and \((4.3) \), we employ the following slight abuses of notation: In the following diagrams, although \( i : J_X \oplus \ker \varphi \to A \) is the inclusion map, we write \([u] \) for the class \([I, i \oplus \text{id}_{K_{\varphi}^\infty}, 0, \alpha_I] \in KK(I, A \oplus K_{\varphi}^\infty) \); moreover, in these diagrams we write \([X] \) and \([T] \) for the classes \([X, \varphi|_{J_X + \ker \varphi}, 0, \alpha_X] \) and \([T, \varphi_T|_I, 0, \alpha_T] \) respectively. Both are elements of \( KK(I, A \oplus K_{\varphi}^\infty) \) by letting \( K_{\varphi}^\infty \) act trivially on \( X \) (on the left and right) and letting \( A \) act trivially on \( T \) (on the right but not on the left). While this is slightly at odds with our usual use of, for example, the notation \([X] \) corresponding to a Hilbert module \( X \), the ambient \( KK \)-groups in the diagrams should provide enough context to avoid confusion.

\[
\begin{align*}
KK_0(B, J_X \oplus \ker \varphi \oplus K_{\varphi}^\infty) & \xrightarrow{\hat{\otimes}[I - \varphi]} KK_0(B, A \oplus K_{\varphi}^\infty) \xrightarrow{i_*} KK_0(B, O_X) \\
KK_1(B, O_Y) & \xleftarrow{i^*} KK_1(B, A \oplus K_{\varphi}^\infty) \xleftarrow{\hat{\otimes}[I - \varphi]} KK_1(B, J_X \oplus \ker \varphi \oplus K_{\varphi}^\infty)
\end{align*}
\]
\[(4.2)\]

\[
\begin{align*}
KK_0(J_X \oplus \ker \varphi \oplus K_{\varphi}^\infty, B) & \xrightarrow{[I - \varphi]} KK_0(A \oplus K_{\varphi}^\infty, B) \xleftarrow{i^*} KK_0(O_X, B) \\
KK_1(O_Y, B) & \xleftarrow{i^*} KK_1(A \oplus K_{\varphi}^\infty, B) \xrightarrow{[I - \varphi]} KK_1(J_X \oplus \ker \varphi \oplus K_{\varphi}^\infty, B)
\end{align*}
\]
\[(4.3)\]

### 4.2. The covariant exact sequence

In this section we will use \((4.2) \) to recover our exact sequence describing the graded \( KK \)-theory \( KK(\mathbb{C}, O_X) \).

**Proposition 4.3.** For any graded \( C^* \)-algebra \( B \), define the
\[
U : KK(B, J_X) \oplus KK(B, \ker \varphi) \oplus \bigoplus_{n=1}^{\infty} KK(B, \ker \varphi)
\arrow{\otimes T} KK(B, A \oplus K_{\varphi}^\infty)
\]
by \( U(j, a, (f_1, f_2, \ldots)) = (0, (a, f_1, f_2, \ldots)) \). Let \( \omega : \bigoplus_{n=1}^{\infty} KK(B, \ker \varphi) \to KK(B, K_{\varphi}^\infty) \) be the canonical homomorphism of \((4.1) \), and let \([T] = [T, \varphi^\ast, 0, \alpha_T] \) be the Kasparov class of \( T \). Then the following diagram commutes
\[
\begin{array}{ccc}
KK(B, J_X \oplus \ker \varphi \oplus K_{\varphi}^\infty) & \xrightarrow{\otimes[T]} & KK(B, A \oplus K_{\varphi}^\infty) \\
\downarrow{\omega} & & \uparrow{\omega} \\
KK(B, J_X) \oplus KK(B, \ker \varphi) \oplus \bigoplus_{n=1}^{\infty} KK(B, \ker \varphi) & \xrightarrow{U} & KK(B, A) \oplus \bigoplus_{n=1}^{\infty} KK(B, \ker \varphi)
\end{array}
\]

**Proof.** Denote the \( n \)th copy of \( \ker \varphi \) in \( K_{\varphi}^\infty \) by \( K_n \), and let \( K_0 \) be the copy of \( \ker \varphi \) in \( A \). Since classes of Kasparov \( B - J_X \)-modules and Kasparov \( B - K_n \)-modules generate \( KK(B, J_X \oplus K_0 \oplus K_{\varphi}^\infty) \) it suffices to show that the diagram commutes on such elements.
Let \([J, ψ_J, F_j, α_J]\) be a \(B-J_X\)-module. Since the Fredholm operator defining the Kasparov class \([T]\) is zero, there is a 0-connection \(F_j \otimes 1\) for \(J\) such that

\[
[J, ψ_J, F_j, α_J] \otimes [T, ϕ^Y, 0, α_T] = [J \otimes T, ψ_J \otimes 1, F_j \otimes 1, α_J \otimes α_T].
\]

We claim this is the zero module. To see this, fix \(j \in J\) and \(f \in T\), and use [29, Proposition 2.31] to write \(j = k \cdot ⟨k, k⟩\). Using that \(⟨k, k⟩ \in \ker ϕ^⊥\) at the last equality, we compute

\[
j \otimes f = k \cdot ⟨k, k⟩ \otimes f = k \otimes ϕ^Y(⟨k, k⟩)f = k \otimes (0, ⟨k, k⟩f_1, 0) = 0.
\]

Since simple tensors span \(J \otimes T\), it follows the tensor product is zero. Hence \(ω([J]) \otimes [T] = ω \circ U([J]) = 0\).

Now consider a Kasparov \(B-K_n\)-module \((Z, ψ_n, F_n, α_n)\). Since \(Z\) is a \(B-K_n\)-correspondence there exists \(a_z \in \ker ϕ\) such that \(⟨z, z⟩ = a_zδ_{i,n}\) for all \(z \in Z\). Hence for \(z \in Z\) and \(f \in T\) we have

\[
⟨z \otimes f, z \otimes f⟩ = ⟨f, ϕ^Y(⟨z, z⟩)f⟩ = f^∗(z, z)_{i−1}f_i = f^∗a_zδ_{i−1,n}f_i,
\]

which is non-zero only if \(i = n + 1\). Hence \(⟨z \otimes f, z \otimes f⟩ \in K_{n+1}\). Thus \(⟨y, y⟩ \in K_{n+1}\) for all \(y \in Z \otimes T\). With \(j_n : KK(B, K_n) → KK(B, J_X ∪ K_0 ∪ K_∞)\) denoting the canonical inclusion,

\[
ω \circ j_n([Z_n]) \otimes [T] = ω((0, \ldots, 0, [Z_n \otimes T], 0, \ldots)) = ω \circ j_{n+1}([Z_n \otimes T])
\]

There is an isomorphism \(Z_n \otimes T \cong Z_n\) that carries an elementary tensor \(z \otimes f\) to \(z \cdot f\). Thus, for \(f_n = j_n[Z_n]\),

\[
ω(U(f_n)) = ω(j_{n+1}[Z_n]) = ω(j_n([Z_n]) \otimes [T]) = ω(f_n) \otimes [T]. \quad \Box
\]

**Theorem 4.4.** Let \((A, α_A)\) be a graded separable nuclear \(C^∗\)-algebra. Let \(X\) be an essential graded \(A-A\)-correspondence with left action \(ϕ\). Let \(J_X = ϕ^{-1}(K(X)) \cap \ker ϕ^⊥\), and let \(ι_{J_X} : J_X → A\) be the inclusion map. Then there is an exact sequence

\[
KK_0(\mathbb{C}, J_X) \xrightarrow{⊗_A[ι_{J_X}]-[X]} KK_0(\mathbb{C}, A) \xrightarrow{i∗} KK_0(\mathbb{C}, O_X) \quad KK_1(\mathbb{C}, O_X) \xleftarrow{⊗_A[ι_{J_X}]-[X]} KK_1(\mathbb{C}, A) \xleftarrow{i∗} KK_1(\mathbb{C}, J_X)
\]

**Proof.** By [23, Lemma 4.2] the left action \(ϕ^Y\) on \(Y = X ⊕ T\) is injective. Let \(I := (ϕ^Y)^{-1}(K(Y))\) and let \(ι_I : I → A ⊕ K_∞\) be the inclusion map. Consider the resulting exact sequence (4.2). Let \(P : KK(C, (J_X + \ker ϕ) ⊕ K_∞) → KK(C, J_X)\) be the projection map given by \(P[Z, ψ, F, α_Z] = [Z \cdot J_X, ψ, F, α_Z]\), and let \(ℓ : KK(C, A) → KK(C, A ⊕ K_∞)\) be the inclusion map. Consider the following ten-term diagram with (4.2) as its central
6-term rectangle.

\[
\begin{array}{c}
\xymatrix{ KK_0(\mathbb{C}, J_X) & KK_0(\mathbb{C}, A) \\
KK_0(\mathbb{C}, J_X \oplus \ker \varphi \oplus K^\infty_\varphi) \ar[u]_{\partial} \ar[r]^{\widehat{\otimes}[|X| - |T|]} & KK_0(\mathbb{C}, A \oplus K^\infty_\varphi) \ar[d]^{\ell} \ar[r]^{i_*} & KK_0(\mathbb{C}, \mathcal{O}_Y) \\
KK_1(\mathbb{C}, \mathcal{O}_Y) \ar[u]_{\partial} & KK_1(\mathbb{C}, A \oplus K^\infty_\varphi) \ar[d]^{\ell} \ar[r]^{\widehat{\otimes}[|X| - |T|]} & KK_1(\mathbb{C}, J_X \oplus \ker \varphi \oplus K^\infty_\varphi) \\
KK_1(\mathbb{C}, A) \ar[u]_{\partial} & \widehat{\otimes}[|X|] \ar[u]_{\widehat{\otimes}[|X| - |T|]} \ar[r]^{i_* \circ \ell} & KK_1(\mathbb{C}, J_X) \ar[u]_{\partial}
}
\end{array}
\]

We show that the sequence

\[
(4.4) \quad \begin{array}{c}
KK_0(\mathbb{C}, J_X) \ar[r]^{\widehat{\otimes}[|X|]} & KK_0(\mathbb{C}, A) \ar[r]^{i_* \circ \ell} & KK_0(\mathbb{C}, \mathcal{O}_Y) \\
KK_1(\mathbb{C}, \mathcal{O}_Y) \ar[u]_{P \circ \partial} & KK_1(\mathbb{C}, A) \ar[r]^{\widehat{\otimes}[|X|]} & KK_1(\mathbb{C}, J_X) \ar[u]_{P \circ \partial}
\end{array}
\]

obtained by traversing the outside of the ten-term diagram is exact. First we show that (4.4) is exact at \(KK_* (\mathbb{C}, \mathcal{O}_Y)\). We claim that \(\ker(P \circ \partial) = \ker \partial\). To see this, observe that \(\operatorname{Img}(\partial) = \ker(\cdot \widehat{\otimes}([|X| - |T|]))\) by exactness of (4.2). Identifying direct sums as in Proposition 4.1, suppose that \((j, a, f) \widehat{\otimes}([|I|] - [|X| - |T|]) = 0\) for some \(j \in KK(\mathbb{C}, J_X), a \in KK(\mathbb{C}, \ker \varphi), \) and \(f = (f_1, f_2, \ldots) \in \bigoplus_{n=1}^\infty KK(\mathbb{C}, \ker \varphi)\). Since \(|I|\) is just the inclusion of \(I\) into \(A \oplus K^\infty_\varphi\), the map \((|I|)_* := \cdot \widehat{\otimes} [I] : KK(\mathbb{C}, I) \to KK(\mathbb{C}, A)\) is just the natural inclusion. Hence under the identification of Proposition 4.2, \(j \widehat{\otimes} [I], a \widehat{\otimes} [I], \) and \(f \widehat{\otimes} [I]\) are the natural images of \(j, a\) and \(f\) in \(KK(\mathbb{C}, A \oplus K^\infty_\varphi)\). Hence \((j, a, f) \widehat{\otimes} [I] = (j + a, f)\).

Since \(\ker \varphi\) acts trivially on \(X\), we have \(a \widehat{\otimes} [X] = f \widehat{\otimes} [X] = 0\). Hence \((j, a, f) \widehat{\otimes} [X] = (j \widehat{\otimes} [X], 0, 0)\). Using Proposition 4.3, \((j, a, (f_1, f_2, \ldots)) \widehat{\otimes} [T] = (0, (a, f_1, f_2, \ldots))\), so

\[
(4.5) \quad 0 = (j, a, f) \widehat{\otimes} ([|I|] - [|X| - |T|]) = (j + a - j \widehat{\otimes} [X], f_1 - a, f_2 - f_1, \ldots, f_n - f_{n-1}, \ldots).
\]

Proposition 4.1 shows that \(f = (f_1, f_2, \ldots) \in \bigoplus_{n=1}^\infty KK(\mathbb{C}, \ker \varphi), \) so there exists \(N \in \mathbb{N}\) such that \(f_n = 0\) for all \(n > N\). Hence (4.5) becomes

\[
0 = (j + a - j \widehat{\otimes} [X], f_1 - a, f_2 - f_1, \ldots, f_N - f_{N-1}, -f_N, 0, 0, \ldots)
\]

forcing \(f_N = 0\). Continuing recursively down the sequence we have \(f_k = 0\) for each \(k \leq N\), and \(a = 0\). Hence \((j, a, f) = (j, 0, 0)\). We conclude

\[
(4.6) \quad \operatorname{Img}(\partial) \subseteq KK(\mathbb{C}, J_X) \oplus 0 \oplus 0,
\]

so \(P|_{\ker \partial}\) is injective, and therefore \(\ker(P \circ \partial) = \ker(\partial)\).

To establish exactness of (4.4) at \(KK_*(\mathbb{C}, \mathcal{O}_Y)\), it now suffices to show that \(\operatorname{Img}(\iota_X \circ \ell) = \operatorname{Img}(\iota_X)\). For this, we first claim that

\[
(4.7) \quad \operatorname{Img}(\widehat{\otimes}[|X| - |T|])|_{KK(\mathbb{C}, \ker \varphi \oplus T)} = \left\{ - \sum f_j, f : f \in KK(\mathbb{C}, T) \right\}.
\]
For all \( a \in KK(\mathbb{C}, \ker \varphi) \) and \( f \in KK(\mathbb{C}, T) \), the product 
\[(0, a, f) \otimes ([t] - [X] - [T])\] 
is of the form 
\[- \sum g_j, g\], where \( g_j = fj - fj_{j-1} \) for \( j > 1 \), and \( g_1 = f_1 - a \). Conversely, we have 
\[\sum f_j, f = (0, a, g) \otimes ([t] - [X] - [T])\] 
for \( g_j = - \sum_{k=0}^{n-j-1} f_{n-k} \) and \( a = - \sum_{k=1}^{n} f_k \), where \( n \) is the index of the last non-zero entry of \( f \). This proves (4.7). By exactness of the inner rectangle (4.2) of the ten-term diagram, we have \( \text{Img}(\otimes([t] - [X] - [T]))_{KK(\mathbb{C}, \ker \varphi \oplus T)} \subseteq \text{Img}(\otimes([t] - [X] - [T])) = \ker \iota \), so we deduce that \( \iota \left( - \sum f_j, f \right) = 0 \) for all \( f \in KK(\mathbb{C}, T) \). 

Now suppose \( \iota \left( a, f \right) \in \text{Img}(\iota \ast) \). Then
\[\iota \ast (a, f) = \iota \ast (a + \sum f_j, 0) + \iota \ast (- \sum f_j, f) = \iota \ast (a + \sum f_j, 0) = \iota \ast \ell (a + \sum f_j).\]

Hence \( \text{Img}(\iota \ast \ell) = \text{Img}(\iota) = \ker(\partial) \) by exactness of (4.2). This completes the proof of exactness of (4.4) at \( KK_{s}(\mathbb{C}, O_{Y}). \)

We now establish exactness of (4.4) at \( KK_{s}(\mathbb{C}, J_{X}). \) We have already demonstrated in (4.6) that \( \ker(\otimes([t] - [X] - [T])) = \ker(\otimes([t] - [X] - [T]))_{KK(\mathbb{C}, J_{X})} \), and further by Proposition 4.3 we have that \( \cdot \otimes ([t] - [X] - [T])_{KK(B, J_{X})} = \cdot \otimes ([t]_{J_{X}} - [X]). \) Hence
\[\text{Img}(\iota \ast \ell) = \text{Img}(\iota) = \ker(\otimes([t] - [X] - [T])) = \ker(\otimes([t]_{J_{X}} - [X])),\]
giving exactness at \( KK_{s}(\mathbb{C}, J_{X}). \)

Next, we establish exactness of (4.4) at \( KK_{s}(\mathbb{C}, A). \) By definition we have
\[\ker(\iota \ast \ell) = \{ a \in KK(\mathbb{C}, A), \iota \ast (a, 0) = 0 \} = \ker(\iota) \cap (KK(\mathbb{C}, A) \oplus \{0\}).\]

Suppose that for \( j \in KK(\mathbb{C}, J_{X}), a \in KK(\mathbb{C}, \ker \varphi) \) and \( f \in \bigoplus_{n=1}^{\infty} KK(\mathbb{C}, \ker \varphi) \) we have 
\[(j, a, f) \otimes ([t] - [X] - [T]) = (b, 0) \] 
for some \( b \in KK(\mathbb{C}, A). \) Then
\[(j, a, f) \otimes ([t] - [X] - [T]) = (j + a - j \otimes [X], f_{1} - a, f_{2} - f_{1}, \ldots, -f_{N}, 0, \ldots)\]
where \( N \in \mathbb{N} \) is the index of the last non-zero component of \( f \). We have \( f_{N} = 0 \), and so recursively, \( f_{j} = 0 \) for each \( j > 0 \), and \( a = 0 \). Hence \( (j, a, f) \otimes ([t] - [X] - [T]) = (j, 0) \otimes ([t]_{J_{X}} - [X]) \in \text{Img}(\otimes([t]_{J_{X}} - [X])). \) Thus \( \text{Img}(\otimes([t] - [X] - [T]))_{KK(\mathbb{C}, A) \oplus \{0\}} = \text{Img}([t]_{J_{X}} - [X]), \) and so by exactness of (4.2), we deduce that \( \text{Img}(\otimes([t]_{J_{X}} - [X])) = \ker(\iota \ast \ell). \) This proves exactness of (4.4).

The inclusion \( i : A \oplus K_{\mathbb{P}}^{\infty} \to O_{Y} \) is nondegenerate and so extends to a homomorphism \( \tilde{i} : \mathcal{M}(A \oplus K_{\mathbb{P}}^{\infty}) \to \mathcal{M}(O_{Y}). \) Theorem 4.3 of [23] shows that \( Q := \tilde{i}(1_{\mathcal{M}(A)}) \in \mathcal{M}(O_{Y}) \) is a full projection and that \( QO_{Y}Q \cong O_{X}. \) Since \( Q \) is trivially graded with respect to the grading on \( A \oplus K_{\mathbb{P}}^{\infty}, \) the space \( O_{Y}Q \) is a graded imprimitivity \( O_{Y} - O_{X} \)-module. So \( (O_{Y}, \alpha_{O_{Y}}) \) and \( (O_{X}, \alpha_{O_{X}}) \) are \( KK \)-equivalent as discussed in Section 1.6. In particular, \( KK_{s}(\mathbb{C}, O_{X}) \cong KK_{s}(\mathbb{C}, O_{Y}) \). Now \( \tilde{i} \ast \ell : KK(\mathbb{C}, A) \to KK(\mathbb{C}, O_{Y}) \) restricts to \( \iota \ast \ell : KK(\mathbb{C}, A) \to KK(\mathbb{C}, O_{X}), \) giving the desired sequence. 

4.3. The contravariant exact sequence. We now use similar techniques to those used in the preceding subsection to obtain an exact sequence describing \( KK(O_{X}, B) \) using the contravariant exact sequence (4.3).
Proposition 4.5. Define

\[ \overline{U} : KK(A, B) \oplus \left( \prod_{n=1}^{\infty} KK(\ker \varphi, B) \right) \to KK(J_X, B) \oplus KK(\ker \varphi, B) \oplus \left( \prod_{n=1}^{\infty} KK(\ker \varphi, B) \right) \]

by \( \overline{U}(a, f_1, f_2, \ldots) = (0, f_1, f_2, \ldots) \). Let \( \overline{\zeta} : KK(K^\infty_{\varphi}, B) \to \prod_{n=1}^{\infty} KK(\ker \varphi, B) \) be the isomorphism discussed at the beginning of Section 4.1, and let \( [T] = [T, \varphi^Y, 0, \alpha_T] \) be the class of the module associated to \( T \). Then the following diagram commutes.

\[ \begin{array}{ccc}
KK((J_X \oplus \ker \varphi) \oplus K^\infty_{\varphi}, B) & \xrightarrow{\overline{U}} & KK(A \oplus K^\infty_{\varphi}, B) \\
\zeta \downarrow & & \downarrow \overline{\zeta} \\
KK(J_X, B) \oplus KK(\ker \varphi, B) \oplus \prod_{n=1}^{\infty} KK(B, \ker \varphi) & \xrightarrow{\overline{U}} & KK(A, B) \oplus \prod_{n=1}^{\infty} KK(\ker \varphi, B)
\end{array} \]

Proof. As discussed at the beginning of Section 4.1, if \( \iota_i : \ker \varphi \to K^\infty_{\varphi} \) is the inclusion into the \( i \)th coordinate, then the right-hand map \( \zeta \) carries the class \([Z]\) of a Kasparov \((A \oplus K^\infty_{\varphi})\)-\(B\)-module to \(([A \cdot Z], (\iota_i(\ker \varphi) \cdot Z)_{i=1}^{\infty})\). Likewise, the right-hand map takes \([W]\) to \(([J_X \cdot W], [\ker \varphi \cdot W], (\iota_i(\ker \varphi) \cdot W)_{i=1}^{\infty})\).

Regard \( T \) as a right \((J_X \oplus K^\infty_{\varphi})\)-module, and take \( W = T \otimes Z \). The left action of \( K^\infty_{\varphi} \) on \( Z \) is given by the inclusion \( K^\infty_{\varphi} \hookrightarrow A \oplus K^\infty_{\varphi} \), and so \( T \otimes (A \oplus 0) \cdot Z = 0 \). For each \( i \geq 1 \), we have \( T \otimes (\iota_i(\ker \varphi)) \cdot Z \cong \ker \varphi \otimes_{\ker \varphi} (\iota_i(\ker \varphi)) \cdot Z \cong \iota_i(\ker \varphi) \cdot Z \) as a right module. The left action of \( J_X \oplus \ker \varphi \) on \( T \otimes (0 \oplus \iota_i(\ker \varphi)) \cdot Z \) restricts to the zero action of \( J_X \) because \( J_X \subseteq \ker \varphi \), and restricts to the standard action of \( \ker \varphi \). So we see that \( W \) is isomorphic to \( 0 \oplus \bigoplus_{i=1}^{\infty} \iota_i(\ker \varphi) \cdot Z \) as a right-Hilbert \( J_X \oplus \ker \varphi \oplus K^\infty_{\varphi} \)-module. This isomorphism preserves gradings because \( \iota \) is a graded homomorphism.

In particular, \( \iota_{J_X} \cdot W = 0 \), each \( \iota_{K^\infty_{\varphi}}(\iota_i(\ker \varphi)) \cdot W \cong \iota_{i+1}(\ker \varphi) \cdot W \), and \( \iota_{\ker \varphi} \cdot W \cong \iota_1(\ker \varphi) \cdot Z \). Thus \( \zeta([W]) = (0, [\iota_1(\ker \varphi) \cdot Z], [\iota_2(\ker \varphi) \cdot Z], \ldots) \). Since \( \zeta([Z]) = ([A \cdot Z], [\iota_1(\ker \varphi) \cdot Z], [\iota_2(\ker \varphi) \cdot Z], \ldots) \), the result follows.

\[ \square \]

Theorem 4.6. Let \((A, \alpha_A), (B, \alpha_B)\) be a graded separable \( C^* \)-algebras, and suppose \( A \) is nuclear. Let \( X \) be an essential graded \( A \)-\( A \)-correspondence with left action \( \varphi \). Let \( J_X = \varphi^{-1}(K(X)) \cap \ker \varphi \), and let \( \iota_{J_X} : J_X \to A \) be the inclusion map. Then there is an exact sequence

\[ KK_0(J_X, B) \leftarrow \frac{([\iota_{J_X} \cdot -[X]])_{\otimes A}}{i^*} \xrightarrow{i^*} KK_0(A, B) \xrightarrow{i^*} KK_0(O_X, B) \]

\[ KK_1(O_X, B) \xrightarrow{i^*} KK_1(A, B) \xrightarrow{([\iota_{J_X} \cdot -[X]])_{\otimes A}} KK_1(J_X, B) \]

Proof. The argument is similar to that of Theorem 4.4. Since the left action \( \varphi^{A \oplus K^\infty_{\varphi}} \) on \( Y \) is injective, if we write \( I := (\varphi^{A \oplus K^\infty_{\varphi}})^{-1}(K(Y)) \) and \( \iota_I : I \to A \oplus K^\infty_{\varphi} \) for the inclusion, then the exact sequence (4.3) fits as the central rectangle in the following diagram whose
top and bottom rectangles commute:

\[
\begin{array}{ccc}
KK_0(J_X, B) & \xrightarrow{(\iota_{J_X} - [X]) \otimes A} & KK_0(A, B) \\
\downarrow \iota & & \downarrow P \\
KK_0(J_X \oplus \ker \phi \oplus K^\infty, B) & \xleftarrow{(\iota_{J_X} - [X] - [T]) \otimes A} & KK_0(A \oplus K^\infty, B) & \leftarrow i^* & KK_0(O_Y, B) \\
\downarrow \partial & & \downarrow \partial \\
KK_1(O_Y, B) & \xrightarrow{i^*} & KK_1(A \oplus K^\infty, B) & \xrightarrow{(\iota_{J_X} - [X]) \otimes A} & KK_1(J_X \oplus \ker \phi \oplus K^\infty, B) \\
\downarrow P & & \downarrow \iota \\
KK_1(A, B) & \xrightarrow{(\iota_{J_X} - [X]) \otimes A} & KK_1(J_X, B)
\end{array}
\]

To prove the result, we show that the six-term sequence consisting of the six extreme points of this diagram is exact; the result will again follow from the graded Morita equivalence of \( O_X \) and \( O_Y \).

Throughout this proof, without further comment, we identify \( KK_*(J_X \oplus \ker \phi \oplus K^\infty, B) \) with \( KK_*(J_X, B) \oplus KK_*(\ker \phi, B) \oplus \prod_{i=1}^\infty KK_i(\ker \phi, B) \) and we identify \( KK_*(A \oplus K^\infty, B) \) with \( KK_*(A, B) \oplus \prod_{i=1}^\infty KK_i(\ker \phi, B) \) as discussed at the beginning of Section 4.1.

For exactness at \( KK_*(A, B) \), observe that \( \text{Img}(P \circ i^*) = P(\ker((\iota_{J_X} - [X] - [T]) \otimes \cdot)) \).

By Proposition 4.5, and using that \( \ker \phi \) annihilates \( X \), we see that for any \((a, j_1, j_2, \ldots) \in KK_*(A \oplus K^\infty, B)\), we have

\[
(\iota_{J_X} - [X] - [T]) \otimes A (a, j_1, j_2, \ldots) = ((\iota_{J_X} - [X]) \otimes a, [\ker \phi] \otimes a, j_1, j_2, \ldots).
\]

So \( \ker((\iota_{J_X} - [X] - [T]) \otimes \cdot) \) is the set of sequences \((a, j, j, j, \ldots)\) such that \((\iota_{J_X} - [X]) \otimes a = 0\) and \( j = [\ker \phi] \otimes a \). In particular, \( P(\ker((\iota_{J_X} - [X] - [T]) \otimes \cdot)) = \ker(\iota_{J_X} - [X]) \) as required.

For exactness at \( KK_*(J_X, B) \) first observe that \( \iota \) is given by \( \iota([j]) = ([j], 0, 0, \ldots) \). So \( \ker(\partial \circ \iota) = \{[j] : ([j], 0, 0, \ldots) \in \ker(\partial)\} = \{([j], 0, 0, \ldots) : [j] \in \text{Img}((\iota_{J_X} - [X] - [T]) \otimes_A \cdot)\} \).

By the description of the map \((\iota_{J_X} - [X] - [T]) \otimes \cdot\) in the preceding paragraph, we see that if \(([j], 0, 0, \ldots) = (\iota_{J_X} - [X] - [T]) \otimes_A (a, j_1, j_2, \ldots)\), then \([j] = (\iota_{J_X} - [X]) \otimes_A a\). Conversely, given \( a \in A \), since the rectangles involving \( P \) and \( \ell \) commute and since the maps \( P \) are surjective, \( \ell((\iota_{J_X} - [X]) \otimes_A a) \in \text{Img}((\iota_{J_X} - [X] - [T]) \otimes_A \cdot) = \ker(\partial) \), and so the image of \((\iota_{J_X} - [X]) \otimes_A \cdot\) is contained in the kernel of \( \partial \circ \ell\).

It remains to establish exactness at \( KK_*(O_Y, B) \). By exactness of (4.3), \( \text{Img}(i^*) = \ker((\iota_{J_X} - [X] - [T]) \otimes \cdot)\). As we saw earlier, this is the collection of sequences \((a, j, j, \ldots)\) such that \((\iota_{J_X} - [X]) \otimes A a = 0\) and \( j = [\ker \phi] \otimes a \). In particular, if \((P \circ i^*)(x) = 0\), then \( i^*(x) = (0, j, j, \ldots) \) with \( j = [\ker \phi] \otimes 0 = 0\), and hence \( i^*(x) = 0\). That is, \( \ker(P \circ i^*) = \ker(i^*)\). Since (4.3) is exact, it now suffices to show that \( \text{Img}(\partial \circ \ell) = \text{Img} \partial \).

We clearly have \( \text{Img}(\partial \circ \ell) = \text{Img} \partial\), so we must show the reverse containment. For this, fix \( \theta = (j_X, j_0, j_1, j_2, \ldots) \in KK_*(J_X \oplus \ker \phi \oplus K^\infty, B)\), so that \( \partial \theta \) is a typical element of \( \text{Img}(\partial) \).

Consider the element \( \eta := (0, -j_0, -j_0 - j_1, -j_0 - j_1 - j_2, \ldots) \in KK_*(A \oplus K^\infty, B)\). We have \((\iota_{J_X} - [X] - [T]) \otimes \eta = (0, j_0, j_1, j_2, \ldots)\). In particular, \( \theta = (\iota_{J_X} - [X] - [T]) \otimes \eta + (j_X, 0, 0, 0, \ldots) = (\iota_{J_X} - [X] - [T]) \otimes \eta + \ell(j_X)\). Since (4.3) is
exact, $\partial(\theta) = \partial((|I_1| - [X] - [T]) \hat{\otimes} \eta) + \partial(\ell(j_X)) = \partial \circ \ell(j_X)$. So $\partial(\theta) \in \text{Img}(\partial \circ \ell)$ as required.

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