A UNIFIED APPROACH TO WEIGHTED HARDY TYPE INEQUALITIES ON CARNOT GROUPS

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Abstract. We find a simple sufficient criterion on a pair of nonnegative weight functions \(V(x)\) and \(W(x)\) on a Carnot group \(G\), so that the general weighted \(L^p\) Hardy type inequality

\[
\int_G V(x)|\nabla_G \phi(x)|^p \, dx \geq \int_G W(x)|\phi(x)|^p \, dx
\]

is valid for any \(\phi \in C^\infty_0(G)\) and \(p > 1\). It is worth noting here that our unifying method may be readily used both to recover most of the previously known weighted Hardy and Heisenberg-Pauli-Weyl type inequalities as well as to construct other new inequalities with an explicit best constant on \(G\). We also present some new results on two-weight \(L^p\) Hardy type inequalities with remainder terms on a bounded domain \(\Omega\) in \(G\) via a differential inequality.

1. Introduction. Hardy inequalities are of fundamental importance for studying a wide range of problems in various branches of mathematics as well as in other areas of science, and have been comprehensively studied since their discovery, see for example [5], [8], [18], [1], [13], [6], [9], [19], [20], [23], [21] and the references therein.

On the Euclidean space \(\mathbb{R}^n\), the classical Hardy inequality asserts that

\[
\int_{\mathbb{R}^n} |\nabla \phi(x)|^p \, dx \geq \left| \frac{n-p}{p} \right| \int_{\mathbb{R}^n} |\phi(x)|^p \frac{1}{|x|^p} \, dx,
\]

(1)

and holds for every \(\phi \in C^\infty_0(\mathbb{R}^n)\) if \(1 \leq p < n\), and for every \(\phi \in C^\infty_0(\mathbb{R}^n \setminus \{0\})\) if \(n < p < \infty\). Here the subscript zero signifies compact support. It is also known that the positive constant on the right-hand side of (1) is sharp but, for \(p > 1\), that equality is only possible if \(\phi = 0\) a.e. In the critical case \(n = p\) an inequality of

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type (1) fails for every positive constant on the right-hand side, while the following sharp inequality
\[
\int_{B_1(0)} |\nabla \phi(x)|^p \, dx \geq \left( \frac{p-1}{p} \right)^p \int_{B_1(0)} |\phi(x)|^p \left( 1 + \log \frac{1}{|x|} \right)^p \, dx \tag{2}
\]
is valid for all $\phi \in C_0^\infty(B_1(0))$, where $B_1(0)$ is the unit ball in $\mathbb{R}^n$ centered at the origin; see Edmunds and Triebel [12]. The stronger version of (2) was then presented by Adimurthi and Sandeep in [2]. On the other hand in the case of sub-Riemannian spaces, especially on Carnot groups $G$, Hardy type inequalities have been also intensively investigated, see [11], [22], [34], [10], [27], [29], [30]. For instance, D’Ambrosio in [11] and Goldstein and Kombe in [22] established, among the other things, the following $L^p$ Hardy type inequality on polarizable Carnot groups $G$,
\[
\int_G |\nabla_G \phi|^p \, dx \geq \left( \frac{Q-p}{p} \right)^p \int_G \frac{|\nabla_G N|^p}{N^p} |\phi|^p \, dx \tag{3}
\]
for all $\phi \in C_0^\infty(G \setminus \{0\} )$, provided that $Q \geq 3$ and $1 < p < Q$. Here, $Q$ is the homogeneous dimension of $G$, $N : G \to [0, \infty)$ is the homogeneous norm associated with the fundamental solution for the sub-Laplacian $\Delta_G = \nabla_G \cdot \nabla_G = \sum_{j=1}^{m} X_j^2$, where $\nabla_G = (X_1, \ldots, X_m)$ is the horizontal gradient on $G$ and $X_1, \ldots, X_m$ are the generators of $G$ (see Section 2 for definitions and preliminaries).

Later in [27] Kombe discovered the sharp weighted Hardy inequality for the $p = 2$ case on general Carnot groups $G$ having the form
\[
\int_G N^\alpha |\nabla_G \phi|^2 \, dx \geq \left( \frac{Q + \alpha - 2}{2} \right)^2 \int_G N^\alpha |\nabla_G N|^2 \phi^2 \, dx, \tag{4}
\]
where $\phi \in C_0^\infty(G \setminus \{0\} )$ and $\alpha \in \mathbb{R}$, $Q \geq 3$, $2 < Q + \alpha$. Niu and Wang [34] then extended the inequality (4) to the $L^p$ case on polarizable Carnot groups $G$ and showed that for any $\phi \in C_0^\infty(G \setminus \{0\} )$ one has
\[
\int_G N^\alpha |\nabla_G N|^\gamma |\nabla_G \phi|^p \, dx \geq \left( \frac{Q + \alpha - p}{p} \right)^p \int_G N^\alpha |\nabla_G N|^{p+\gamma} \phi^p \, dx, \tag{5}
\]
whenever $\alpha \in \mathbb{R}$, $1 < p < Q + \alpha$, $\gamma > -1$. We note that all the constants appearing in (3), (4) and (5) are sharp but are never achieved.

We also mention that Jin and Shen [26], recently have proved a weighted $L^p$ Hardy inequality on general Carnot groups $G$ by using a special class of weighted $p$-sub-Laplacian and the corresponding fundamental solution. More recently, Lian also has got a similar result on the same groups with a sharp constant, see [29].

All of these works motivate us to investigate a constructive method to derive Hardy type inequalities with different weights on $G$. In this direction, we provide an approach that recovers and improves most of the Hardy type inequalities that have appeared to date. More precisely, we verify that if $V \in C^1(G)$ and $W \in L^1_{loc}(G)$ are nonnegative functions and $\Phi \in C^\infty(G)$ is a positive function such that
\[
-\nabla_G \cdot \left( V(x) |\nabla_G \Phi|^p - \nabla_G \Phi \right) \geq W(x) \Phi^{p-1}
\]
almost everywhere in a general Carnot group $G$, then the inequality
\[
\int_G V(x) |\nabla_G \phi|^p \, dx \geq \int_G W(x) |\phi|^p \, dx + c_p \int_G V(x) \frac{|\nabla_G \phi|}{\Phi} \Phi^p \, dx
\]
is valid for every \( \phi \in C_0^\infty(\mathbb{G}) \), where \( p \geq 2 \) and \( c_p > 0 \). It is worth emphasizing here that one can readily obtain as many weighted Hardy type inequalities as one can construct a weight function \( V \) and a function \( \Phi \) satisfying the above hypotheses (see Applications of Theorem 3.1). We remark that a similar inequality with a different nonnegative remainder term also exists for the case \( 1 < p < 2 \) (see Theorem 3.1).

We also give new results on two-weight \( L^p \) Hardy type inequalities with remainders on a bounded domain \( \Omega \) in polarizable Carnot groups \( \mathbb{G} \). The primary tool which we employ in constructing these type of inequalities is a differential inequality involving a nonnegative general weight function \( V \), the homogeneous norm \( N \) and a positive smooth function \( \delta \) (see Theorem 4.1). We show some concrete examples by specializing the functions \( V \) and \( \delta \) (see Applications of Theorem 4.1).

### 2. Preliminaries and notations

We first give an account of some of the basic definitions, terminology and background results of analysis on Carnot groups \( \mathbb{G} \) that will be used throughout the article. For further details on this topic we refer the interested readers to [3], [4], [7], [15], [17], [32], and the references therein.

A Carnot group is a connected, simply connected, nilpotent Lie group \( \mathbb{G} \equiv (\mathbb{R}^n, \cdot) \) whose Lie algebra \( \mathcal{G} \) admits a stratification. That is, there exist linear subspaces \( V_1, \ldots, V_s \) of \( \mathcal{G} \) such that

\[
\mathcal{G} = V_1 \oplus \cdots \oplus V_s, \quad [V_i, V_j] = V_{i+j}, \quad \text{for} \quad i = 1, 2, \ldots, s-1 \quad \text{and} \quad [V_1, V_s] = 0,
\]

where \([V_i, V_j] \) is the subspace of \( \mathcal{G} \) generated by the elements \([X, Y] \) with \( X \in V_1 \) and \( Y \in V_j \). Here \([X, Y] \) is the commutator \( XY - YX \). This defines an \( s \)-step Carnot group and the integer \( s \geq 1 \) is called the step of \( \mathcal{G} \). Via the exponential map, it is possible to induce on \( \mathcal{G} \) a family of automorphisms of the group, called dilations, \( \delta_{\lambda} : \mathbb{R}^n \rightarrow \mathbb{R}^n (\lambda > 0) \), such that

\[
\delta_{\lambda}(x_1, \ldots, x_n) = (\lambda^{a_1}x_1, \ldots, \lambda^{a_n}x_n),
\]

where \( 1 = a_1 = \cdots = a_m < a_{m+1} \leq \cdots \leq a_n \) are integers and \( m = \dim(V_1) \). The group law can be written in the following form

\[
x \cdot y = x + y + P(x, y), \quad x, y \in \mathbb{R}^n,
\]

where \( P : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) has polynomial components and \( P_1 = \cdots = P_m = 0 \) (see [32], Chapter 12, Section 5). Note that the inverse \( x^{-1} \) of an element \( x \in \mathcal{G} \) has the form \( x^{-1} = -x = (-x_1, \ldots, -x_n) \).

Let \( X_1, \ldots, X_m \) be a family of left invariant vector fields that form an orthonormal basis of \( V_1 \equiv \mathbb{R}^m \) at the origin, that is, \( X_1(0) = \partial_{x_1}, \ldots, X_m(0) = \partial_{x_m} \). The vector fields \( X_j \) have polynomial coefficients and can be assumed to be of the form

\[
X_j(x) = \partial_j + \sum_{i+j=1}^n a_{ij}(x)\partial_i, \quad X_j(0) = \partial_j, \quad j = 1, \ldots, m,
\]

where each polynomial \( a_{ij} \) is homogeneous with respect to the dilations of the group, that is, \( a_{ij}(\delta_{\lambda}(x)) = \lambda^{a_i-a_j}a_{ij}(x) \).

The curve \( \gamma : [a, b] \subset \mathbb{R} \rightarrow \mathbb{G} \) is called horizontal if its tangents lie in \( V_1 \), i.e., \( \gamma'(t) \in \text{span}\{X_1, \ldots, X_m\} \) for all \( t \). Then, the Carnot-Carathéodory distance \( d_{cc}(x, y) \) between two points \( x, y \in \mathbb{G} \) is defined to be the infimum of the lengths \( \int_a^b \sqrt{\gamma'(t)^2} dt \) of all horizontal curves \( \gamma : [a, b] \rightarrow \mathbb{G} \) such that \( \gamma(a) = x \) and \( \gamma(b) = y \). Notice that \( d_{cc} \) is a homogeneous norm and satisfies the invariance property

\[
d_{cc}(z \cdot x, z \cdot y) = d_{cc}(x, y)
\]
for all \(x, y, z \in G\) and is homogeneous of degree one with respect to the dilation \(\delta_x\).

The Carnot-Caratheodory balls are defined by \(B(x, R) = \{y \in G|d_{cc}(x, y) < R\}\).

The \(n\)-dimensional Lebesgue measure, \(\mathcal{L}^n\), is the Haar measure of group \(G\). This means that if \(E \subset \mathbb{R}^n\) is measurable, then \(\mathcal{L}^n(x \cdot E) = \mathcal{L}^n(E)\) for all \(x \in G\). Moreover, if \(\lambda > 0\) then \(\mathcal{L}^n(\delta_x(E)) = \lambda^Q \mathcal{L}^n(E)\). Clearly

\[
\mathcal{L}^n(B(x, R)) = R^Q \mathcal{L}^n(B(x, 1)) = R^Q \mathcal{L}^n(B(0, 1)),
\]

where

\[
Q = \sum_{j=1}^{s} j(\dim V_j)
\]

is the homogeneous dimension of \(G\).

The nonlinear operator

\[
\Delta_{G, p} = \nabla_G \cdot (|\nabla_G|^p \nabla_G)
\]

is the \(p\)-sub-Laplacian on Carnot group \(G\). If \(p = 2\) then we have the linear sub-Laplacian \(\Delta_G = \nabla_G \cdot \nabla_G = \sum_{j=1}^{m} X_j^2\), where \(\nabla_G = (X_1, \ldots, X_m)\) is the horizontal gradient on \(G\) and \(X_1, \ldots, X_m\) are the generators of \(G\). The fundamental solution \(u\) for \(\Delta_G\) is defined to be a weak solution to the equation \(-\Delta_G u = \delta_0\), where \(\delta_0\) denotes the Dirac distribution with singularity at the neutral element 0 of \(G\). In [F14], Folland proved that in any Carnot group \(G\), there exists a homogeneous norm \(N\) such that \(u = N^{2-Q}\) is harmonic in \(G \setminus \{0\}\) and is a positive multiple of the fundamental solution for \(\Delta_G\). We now start with \(u\) and set

\[
N(x) := \begin{cases} 
\frac{1}{|x|^{-Q/p}} & \text{if } x \neq 0, \\
0 & \text{if } x = 0,
\end{cases}
\]

and recall that a homogeneous norm on \(G\) is a continuous function \(N : G \rightarrow [0, \infty)\), smooth away from the origin, which satisfies the conditions \(N(\delta_x(x)) = \lambda N(x)\), \(N(x^{-1}) = N(x)\) and \(N(x) = 0\) iff \(x = 0\). Using the homogeneous norm \(N\), we define the \(N\)-ball \(B_N \subset G\) with center zero and radius \(R\) by

\[
B_N := \{x \in G : N(x) < R\}.
\]

A Carnot group \(G\) is called polarizable if the homogeneous norm \(N = u^{1/(2-Q)}\), associated to Folland’s solution \(u\) for the sub-Laplacian \(\Delta_G\), satisfies the following \(\alpha\)-sub-Laplace equation

\[
\Delta_{G, \infty} N := \frac{1}{2} \langle \nabla_G(|\nabla_G N|^2), \nabla_G N \rangle = 0, \text{ in } G \setminus \{0\}.
\]

This class of groups was introduced by Balogh and Tyson [B3] and admits the analogue of polar coordinates. It is known that the Euclidean space, the Heisenberg group \(\mathbb{H}^n\) and Kaplan’s \(H\)-type group are polarizable Carnot groups. In [B3], the same authors also proved that for every \(1 < p < \infty\) the function

\[
u_p = \begin{cases} \frac{N}{|x|^{Q/p} - \log N}, & \text{if } p \neq Q \\
-\log N, & \text{if } p = Q
\end{cases}
\]

is \(p\)-harmonic in \(G \setminus \{0\}\), i.e.

\[
\Delta_{G, p} \nu_p = \nabla_G \cdot (|\nabla_G \nu_p|^p \nabla_G \nu_p) = 0 \text{ in } G \setminus \{0\}.
\]

Moreover, for each \(1 < p < \infty\) there exists a constant \(l_p > 0\) such that \(-\Delta_{G, p} \nu_p = l_p \delta_0\) in the sense of distributions.
3. **Weighted Hardy type inequalities.** Here is the main result of this section.

**Theorem 3.1.** Let \( V \in C^1(\mathbb{G}) \) and \( W \in L^1_{\text{loc}}(\mathbb{G}) \) be nonnegative functions and \( \Phi \in C^\infty(\mathbb{G}) \) be a positive function such that

\[
- \nabla_G \cdot \left( V(x) |\nabla_G \Phi|^p \nabla_G \Phi \right) \geq W(x) \Phi^{p-1}
\]  

almost everywhere in a general Carnot group \( \mathbb{G} \). There exists a positive constant \( c_p \) depending only on \( p \) such that, if \( p \geq 2 \), then

\[
\int_G V(x) |\nabla_G \phi|^p dx \geq \int_G W(x) |\phi|^p dx + c_p \int_G V(x) |\nabla_G \phi|^p \Phi dx,
\]  

and if \( 1 < p < 2 \), then

\[
\int_G V(x) |\nabla_G \phi|^p dx \geq \int_G W(x) |\phi|^p dx + c_p \int_G V(x) |\nabla_G \phi|^p \Phi dx \geq \int_G W(x) |\phi|^p dx + c_p \int_G V(x) |\nabla_G \phi|^p \Phi dx,
\]  

for all \( \phi \in C^\infty(\mathbb{G}) \).

**Proof.** We now recall the following inequalities that will be used in this article (see, for example, [30]). For any \( 1 < p < \infty \) there exists a positive constant \( c_p \) depending only on \( p \) such that for all \( a, b \in \mathbb{R}^n \) we have

\[
|a + b|^2 \geq |a|^p + p|a|^{p-2}a \cdot b + c_p |b|^p, \quad \text{for } p \geq 2
\]  

and

\[
|a + b|^2 \geq |a|^p + p|a|^{p-2}a \cdot b + c_p \frac{|b|^2}{(|a| + |b|)^{2-p}}, \quad \text{for } 1 < p < 2.
\]  

Let \( \varphi \) be a new variable \( \varphi := \frac{\varphi}{\Phi} \), where \( 0 < \Phi \in C^\infty(\mathbb{G}) \) and \( \varphi \in C^\infty(\mathbb{G}) \). Applying the inequality \( |a + b|^2 \) with \( a = \varphi \nabla_G \Phi \) and \( b = \Phi \nabla_G \varphi \), we get

\[
|\nabla_G \phi|^p = |\varphi \nabla_G \Phi + \Phi \nabla_G \varphi|^p \geq |\nabla_G \Phi|^p \varphi^p + \Phi |\nabla_G \Phi|^{p-2} \nabla_G \Phi \cdot \nabla_G (|\varphi|^p) + c_p |\nabla_G \varphi|^p \Phi^p.
\]  

Multiplying the inequality \( |\nabla_G \phi|^p \) by \( V(x) \) on both sides and integrating by parts over \( \mathbb{G} \) yield

\[
\int_G V(x) |\nabla_G \phi|^p dx \geq \int_G V(x) |\nabla_G \Phi|^p |\varphi|^p dx + c_p \int_G V(x) |\nabla_G \phi|^p \Phi^p dx
\]  

\[
= \int_G V(x) \Phi \cdot \left( V(x) |\nabla_G \Phi|^p - 2 \nabla_G \Phi \right) |\varphi|^p dx
\]  

\[
+ c_p \int_G V(x) |\nabla_G \phi|^p \Phi^p dx.
\]  

As a next step, by using the weighted \( p \)-Laplace inequality \( |a + b|^2 \), we conclude that

\[
\int_G V(x) |\nabla_G \phi|^p dx \geq \int_G W(x) |\varphi|^p \Phi^p dx + c_p \int_G V(x) |\nabla_G \varphi|^p \Phi^p dx.
\]  

Making the change of variable \( \varphi = \frac{\Phi}{\Phi} \) in the above integrals, we obtain the desired result \( |\nabla_G \phi|^p \). Note that the Theorem 3.1 holds also for \( 1 < p < 2 \) and in this case we
use the inequality (10) with the same choices of $a$ and $b$ as in the above derivation. This finishes the proof of Theorem 3.1 \( \square \)

**Remark 1.** Observe that if $p = 2$, then (9) is an equality with $c_2 = 1$.

**Applications of Theorem 3.1.** Let $\epsilon > 0$ be given. To make following arguments rigorous we should replace the function $N$ with its regularization $N_\epsilon := (u + \epsilon)^{\frac{1}{2}}$ and after the compution take the limit as $\epsilon \to 0$. However, for the sake of simplicity we will proceed formally.

As we have already mentioned most of the known Hardy type inequalities on polarizable Carnot groups $G$ such as (3), (4) and (5), and as well as other new results can be obtained, via the above approach, by making suitable choices for $V$ and $\Phi$. As a first example, note that the choice

$$V = N^\alpha |\nabla_G N|^\gamma$$

and

$$\Phi = N^{-\left(\frac{p + \alpha - p}{p}\right)}$$

easily yields the following important result due to J. Wang and P. Niu [34].

**Corollary 1.** Let $G$ be a polarizable Carnot group with homogeneous norm $N = u^{\frac{1}{p}}$ and let $\alpha \in \mathbb{R}$, $1 < p < Q + \alpha, \gamma > -1$. Then the inequality

$$\int_G N^\alpha |\nabla_G N|^\gamma |\nabla_G \phi|^p \, dx \geq \left(\frac{Q + \alpha - p}{p}\right)^p \int_G N^\alpha |\nabla_G N|^{p+\gamma} |\phi|^p \, dx$$

is valid for all $\phi \in C^\infty_0(G \setminus \{0\})$.

It is worth stressing here that, by setting the pair as

$$V = \left(\log \frac{R}{N}\right)^{\alpha+p} \quad \text{and} \quad \Phi = \left(\log \frac{R}{N}\right)^{-\left(\frac{\alpha + 1}{p}\right)}$$

on $B_N$, we recover the weighted $L^p$ Hardy type inequality (3.40) presented in [11].

**Corollary 2.** Let $G$ be a polarizable Carnot group with homogeneous norm $N = u^{\frac{1}{p}}$ and let $Q = p > 1, \alpha < -1$. Then the inequality

$$\int_{B_N} \left(\log \frac{R}{N}\right)^{\alpha+p} |\nabla \phi|^p \, dx \geq \left(\frac{\alpha + 1}{p}\right)^p \int_{B_N} \left(\log \frac{R}{N}\right)^\alpha \frac{|\nabla_G N|^{p} \, dx}{N^p} |\phi|^p$$

is valid for all $\phi \in C^\infty_0(B_N)$.

One can however apply the Theorem 3.1 to obtain other new inequalities on $G$. For instance, let us take

$$V = N^\alpha \quad \text{and} \quad \Phi = \left(1 + N^\frac{p}{p+\alpha}\right)^{\left(-\frac{p + \alpha - p}{p}\right)}$$

then we readily get the following result.

**Corollary 3.** Let $G$ be a polarizable Carnot group with homogeneous norm $N = u^{\frac{1}{p}}$ and let $\alpha \in \mathbb{R}$, $Q + \alpha > p > 1$. Then the inequality

$$\int_G N^\alpha |\nabla_G \phi|^p \, dx \geq \left(\frac{Q + \alpha - p}{p - 1}\right)^{p-1} (Q + \alpha) \int_G N^\alpha \frac{|\nabla_G N|^p}{\left(1 + N^\frac{p}{p+\alpha}\right)^p} |\phi|^p \, dx$$

holds for every $\phi \in C^\infty_0(G)$. 
On the other hand, by considering the functions
\[ V = \left( 1 + N \frac{p}{p-1} \right)^{\alpha(p-1)} \quad \text{and} \quad \Phi = \left( 1 + N \frac{p}{p-1} \right)^{1-\alpha}, \]
we obtain Carnot version of the inequality (5.1) proved in [31] for the Euclidean context.

**Corollary 4.** Let \( \mathbb{G} \) be a polarizable Carnot group with homogeneous norm \( N = u^{\frac{1}{p-1}} \) and let \( 1 < p < Q, \alpha > 1 \). Then for every \( \phi \in C_0^\infty(\mathbb{G}) \), one has
\[ \int_{\mathbb{G}} \left( 1 + N \frac{p}{p-1} \right)^{\alpha(p-1)} |\nabla G \phi|^p \, dx \geq C(Q,p,\alpha) \int_{\mathbb{G}} \frac{|\nabla G N|^p}{\left( 1 + N \frac{p}{p-1} \right)^{(1-\alpha)(p-1)}} |\phi|^p \, dx, \]
where \( C(Q,p,\alpha) := Q \left( \frac{p(\alpha-1)}{p-1} \right)^{p-1} \).

Another application of Theorem 3.1 with the special functions
\[ V = \frac{(a+bN^\alpha)^\beta}{N^{2m}} \quad \text{and} \quad \Phi = N^{-\left(\frac{Q-2m-2}{2}\right)} \]
leads us to the subsequent improved Carnot analogue of the inequality (42) established in [19] for the Euclidean setting.

**Corollary 5.** Let \( \mathbb{G} \) be a polarizable Carnot group with homogeneous norm \( N = u^{\frac{1}{p-1}} \). Let \( a, b > 0 \) and \( \alpha, \beta, m \) be real numbers. If \( \alpha \beta > 0 \) and \( m \leq \frac{Q-2}{2} \), then for all \( \phi \in C_0^\infty(\mathbb{G}) \) one has
\[ \int_{\mathbb{G}} \frac{(a+bN^\alpha)^\beta}{N^{2m}} |\nabla_G \phi|^2 \, dx \geq C(Q,m)^2 \int_{\mathbb{G}} \frac{(a+bN^\alpha)^\beta}{N^{2m+2}} |\nabla_G N|^2 \phi^2 \, dx \]
\[ + C(Q,m) \alpha \beta b \int_{\mathbb{G}} \frac{(a+bN^\alpha)^{\beta-1}}{N^{2m-\alpha+2}} |\nabla_G N|^2 \phi^2 \, dx, \]
where \( C(Q,m) := \left( \frac{Q-2m-2}{2} \right) \).

We now take the units
\[ V \equiv 1 \quad \text{and} \quad \Phi = (R-N)^{\frac{p-1}{p}} \]
on \( B_N \), and we have the following Hardy type inequality [12] that was first proved in [24] for the Heisenberg group \( \mathbb{H}^n \) and then in [11] for polarizable Carnot groups \( \mathbb{G} \) by slightly different methods.

**Corollary 6.** Let \( \mathbb{G} \) be a polarizable Carnot group with homogeneous norm \( N = u^{\frac{1}{p-1}} \) and let \( Q = p > 1 \). Then for every \( \phi \in C_0^\infty(B_N) \), one has
\[ \int_{B_N} |\nabla_G \phi|^p \, dx \geq \left( \frac{p-1}{p} \right)^p \int_{B_N} \frac{|\nabla_G N|^p}{(R-N)^p} |\phi|^p \, dx. \quad (12) \]

**Uncertainty Principle Inequalities.** The first and most famous uncertainty principle goes back to Heisenberg’s seminal work, which was developed in the context of quantum mechanics [25]. The mathematical details of this principle were provided by Pauli and Weyl [35] and hence it is sometimes referred to as the Heisenberg-Pauli-Weyl inequality. In the Euclidean setting, the uncertainty principle inequality with sharp constant can be stated as
\[ \left( \int_{\mathbb{R}^n} |\nabla \phi|^2 \, dx \right) \left( \int_{\mathbb{R}^n} |x|^2 \phi^2 \, dx \right) \geq \frac{n^2}{4} \left( \int_{\mathbb{R}^n} \phi^2 \, dx \right)^2, \quad (13) \]
where $\phi \in C_0^\infty(\mathbb{R}^n)$. There exists much literature devoted to deriving various uncertainty principle type inequalities in the Euclidean and other settings (see [16], [27]). For instance, in polarizable Carnot groups $G$, Kombe [27] showed that for any $\phi \in C_0^\infty(G)$, the following inequality is valid

$$\left( \int_G \frac{|\nabla_G \phi|^2}{|\nabla_G N|^2} dx \right) \left( \int_G N^2 \phi^2 dx \right) \geq \frac{Q^2}{4} \left( \int_G \phi^2 dx \right)^2. \tag{14}$$

We should mention that Theorem 3.1 does not only give us weighted Hardy inequalities but also gives the Heisenberg-Pauli-Weyl type inequalities with the best constant. For instance, we now consider the pair $V \equiv \frac{1}{|\nabla_G N|^2}$ and $\Phi = e^{-\alpha N^2}$, where $\alpha > 0$, and we immediately obtain

$$\int_G \frac{|\nabla_G \phi|^2}{|\nabla_G N|^2} dx \geq 2\alpha Q \int_G \phi^2 dx - 4\alpha^2 \int_G N^2 \phi^2 dx.$$

Let $A = -4 \int_G N^2 \phi^2 dx$, $B = 2Q \int_G \phi^2 dx$ and $C = -\int_G \frac{|\nabla_G \phi|^2}{|\nabla_G N|^2} dx$. Then the above inequality takes the form $A\alpha^2 + B\alpha + C \leq 0$ for every $\alpha \in \mathbb{R}$ which implies that $B^2 - 4AC \leq 0$. In other words, we have the inequality (14).

Now we make the following special choices of functions $V$ and $\Phi$ in Theorem 3.1

$$V \equiv 1 \text{ and } \Phi = e^{-\alpha N^2},$$

where $\alpha > 0$. We get

$$\int_G \frac{|\nabla_G \phi|^2}{|\nabla_G N|^2} dx \geq 2\alpha Q \int_G \phi^2 dx - 4\alpha^2 \int_G N^2 \phi^2 dx.$$

Arguing as above, we have the following version of the Heisenberg uncertainty principle inequality.

**Corollary 7.** Let $G$ be a polarizable Carnot group with homogeneous norm $N = u^{\frac{1}{\dim G}}$. Then for every $\phi \in C_0^\infty(G)$, one has

$$\left( \int_G |\nabla_G \phi|^2 dx \right) \left( \int_G N^2 |\nabla_G N|^2 \phi^2 dx \right) \geq \frac{Q^2}{4} \left( \int_G |\nabla_G N|^2 \phi^2 dx \right)^2.$$

Finally, let us consider the pair

$$V \equiv 1 \text{ and } \Phi = e^{-\alpha N}, \quad \alpha > 0$$

then we get following inequality.

**Corollary 8.** Let $G$ be a polarizable Carnot group with homogeneous norm $N = u^{\frac{1}{\dim G}}$. Then for every $\phi \in C_0^\infty(G)$, one has

$$\left( \int_G |\nabla_G \phi|^2 dx \right) \left( \int_G |\nabla_G N|^2 \phi^2 dx \right) \geq \left( \frac{Q - 1}{4} \right)^2 \left( \int_G \frac{|\nabla_G N|^2 \phi^2 dx}{N^2} \right)^2.$$
4. Two-Weight Hardy type inequalities with remainders. We now prove an improved two-weight $L^p$ Hardy type inequality via a differential inequality involving a general nonnegative weight function $V$, the homogeneous norm $N$ and a positive smooth function $\delta$.

**Theorem 4.1.** Let $G$ be a polarizable Carnot group with homogeneous norm $N = u^{-\frac{1}{\alpha}}$ and let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $G$. Assume $V$ is a nonnegative $C^1$—function and $\delta$ is a positive $C^\infty$—function such that

$$-\nabla_G \cdot (V (x) N^{p-Q} \frac{\left|\nabla_G \delta\right|^{p-2} \nabla_G \delta}{\delta^{p-2}}) \geq 0$$

almost everywhere in $\Omega$. Then for any $\phi \in C^\infty_0 (\Omega)$, one has

$$\int_\Omega V (x) N^\alpha \left|\nabla_G \phi\right|^p dx \geq \left(\frac{Q + \alpha - p}{p}\right)^p \int_\Omega V (x) N^\alpha \frac{\left|\nabla_G N\right|^p}{N^p} \left|\phi\right|^p dx + \frac{c_p}{p^p} \int_\Omega V (x) N^\alpha \frac{\left|\nabla_G \delta\right|^p}{\delta^p} \left|\phi\right|^p dx,$$

where $Q + \alpha > p \geq 2$, $\alpha \in \mathbb{R}$ and $c_p = c (p) > 0$.

**Proof.** For any $\phi \in C^\infty_0 (\Omega)$ we set $\varphi := N^{-\gamma} \phi$ with $\gamma < 0$, a constant that will be chosen later. By direct computation we have

$$\nabla_G (N^\gamma \varphi) = \gamma N^{\gamma-1} \varphi \nabla_G N + N^\gamma \nabla_G \varphi.$$ 

Applying the inequality (9) with $a = \gamma N^{\gamma-1} \varphi \nabla_G N$ and $b = N^\gamma \nabla_G \varphi$ yields

$$\left|\nabla_G \varphi\right|^p \geq \left|\gamma\right|^p N^{p(\gamma-1)} \left|\nabla_G N\right|^p \left|\varphi\right|^p + \gamma \left|\gamma\right|^{p-2} N^{p(\gamma-1)+1} \left|\nabla_G N\right|^{p-2} \nabla_G N \cdot \nabla_G \left(\left|\varphi\right|^p\right) + c_p N^{p\gamma} \left|\nabla_G \varphi\right|^p.$$

Multiplying both sides of (16) by $V (x) N^\alpha$ and then using integration by parts gives

$$\int_\Omega V (x) N^\alpha \left|\nabla_G \phi\right|^p dx \geq \left|\gamma\right|^p \int_\Omega V (x) N^{\alpha+p(\gamma-1)} \left|\nabla_G N\right|^p \left|\varphi\right|^p dx - \gamma \left|\gamma\right|^{p-2} \int_\Omega \nabla_G \cdot \left( V (x) N^{\alpha+p(\gamma-1)+1} \left|\nabla_G N\right|^{p-2} \nabla_G N\right) \left|\varphi\right|^p dx + c_p \int_\Omega V (x) N^{\alpha+p\gamma} \left|\nabla_G \varphi\right|^p dx.$$

Taking into account that $\Delta_G N = (Q - 1) \frac{\left|\nabla_G N\right|^2}{N}$ and $\Delta_{G, \infty} N = 0$ we obtain

$$\nabla_G \cdot \left( V (x) N^{\alpha+p(\gamma-1)+1} \left|\nabla_G N\right|^{p-2} \nabla_G N\right) = N^{\alpha+p(\gamma-1)+1} \left|\nabla_G N\right|^{p-2} \nabla_G N \cdot \nabla_G V + [Q + \alpha + p (\gamma - 1)] V (x) N^{\alpha+p(\gamma-1)} \left|\nabla_G N\right|^p.$$
Together with the equality \( \text{(18)} \) the above expression \( \text{(17)} \) can be written as

\[
\begin{align*}
\int_\Omega V(x) N^\alpha |\nabla G\phi|^p \, dx &\geq \zeta(Q, \alpha, p, \gamma) \int_\Omega V(x) N^{\alpha + p(\gamma - 1)} |\nabla G N|^p |\varphi|^p \, dx \\
&\quad - \gamma |\varphi|^p - 2 \int_\Omega N^{\alpha + p(\gamma - 1) + 1} |\nabla G N|^{p - 2} \nabla G N \cdot \nabla G V |\varphi|^p \, dx \\
&\quad + c_p \int_\Omega V(x) N^{\alpha + p\gamma} |\nabla G \varphi|^p \, dx,
\end{align*}
\]

where \( \zeta(Q, \alpha, p, \gamma) = |\varphi|^p - \gamma |\varphi|^{p - 2} (Q + \alpha + \gamma p - p) \). Note that we can make the choice \( \gamma = (p - \alpha - Q) / p \), since \( \gamma < 0 \) holds. Therefore, we have

\[
\begin{align*}
\int_\Omega V(x) N^\alpha |\nabla G\phi|^p \, dx &\geq \left( \frac{Q + \alpha - p}{p} \right)^p \int_\Omega V(x) \frac{|\nabla G N|^p}{N^Q} |\varphi|^p \, dx \\
&\quad + \left( \frac{Q + \alpha - p}{p} \right)^{p - 1} \int_\Omega \frac{|\nabla G N|^{p - 2}}{N^{Q - 1}} |\nabla G N|^{p - 2} \nabla G \cdot \nabla G V |\varphi|^p \, dx \\
&\quad + c_p \int_\Omega V(x) N^{p - Q} |\nabla G \varphi|^p \, dx.
\end{align*}
\]

We now focus on the integral expression \( c_p \int_\Omega V(x) N^{p - Q} |\nabla G \varphi|^p \, dx \) on the right hand side of \( \text{(20)} \). Let \( \vartheta \) be the new function \( \vartheta := \delta^{-1/p} \varphi \), where \( 0 < \delta \in C^\infty(\Omega) \) and \( \varphi \in C_0^\infty(\Omega) \). It follows from the inequality \( \text{(9)} \) that

\[
|\nabla G \varphi|^p = \frac{1}{p} \delta^{1 - \frac{p}{2}} |\nabla G \vartheta|^p + \frac{1}{p^{p - 1}} |\nabla G \vartheta|^{p - 2} \nabla G \cdot \nabla G |\vartheta|^p + c_p \delta^p |\nabla G \vartheta|^p.
\]

By integration by parts, since \( c_p \delta^p |\nabla G \vartheta|^p \geq 0 \), we can infer from \( \text{(21)} \) that

\[
c_p \int_\Omega V(x) N^{p - Q} |\nabla G \varphi|^p \, dx \geq \frac{c_p}{p^p} \int_\Omega V(x) N^{p - Q} |\nabla G \vartheta|^p \delta^{-1/p} |\vartheta|^p \, dx \\
- \frac{c_p}{p^{p - 1}} \int_\Omega \nabla G \cdot (V(x) N^{p - Q} |\nabla G \vartheta|^{p - 2} \nabla G \vartheta) |\vartheta|^p \, dx.
\]

Using the inequality \(- \nabla G \cdot (V(x) N^{p - Q} |\nabla G \vartheta|^{p - 2} \nabla G \vartheta) \geq 0 \) and the substitution \( \vartheta := \delta^{-1/p} N^{\frac{Q + p - 2}{p}} \varphi \), we conclude that

\[
c_p \int_\Omega V(x) N^{p - Q} |\nabla G \varphi|^p \, dx \geq \frac{c_p}{p^p} \int_\Omega V(x) N^\alpha |\nabla G \vartheta|^p \delta^p |\varphi|^p \, dx.
\]

Combining \( \text{(20)} \) and \( \text{(22)} \), and then taking into account that \( \varphi = N^{\frac{Q + p - 2}{p}} \varphi \), we deduce the claimed inequality \( \text{(15)} \).

\[\square\]

**Remark 2.** We note that the result stated in Theorem \( \text{(1.1)} \) holds also for \( 1 < p < 2 \) with a different reminder term and in this case we use the convexity inequality \( \text{(10)} \).

**Applications of Theorem \( \text{(4.1)} \).** Different choices of the functions \( V \) and \( \delta \) satisfying the assumptions of the above theorem, produce new weighted improved \( L^p \) Hardy type inequalities. For instance, consider the two functions

\[
V \equiv 1 \quad \text{and} \quad \delta = \log\left( \frac{R}{\Omega} \right)
\]
on a bounded domain $\Omega$ with smooth boundary in $\mathbb{G}$, where $R > \sup_{x \in \Omega} N(x)$. It is obvious that they fulfill all hypotheses in the Theorem 4.1, hence we have the weighted $L^p$ Hardy type inequality containing a logarithmic remainder.

**Corollary 9.** Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous norm $N = u^{-\frac{1}{\alpha}}$ and let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{G}$. Then for all $\phi \in C_0^\infty(\Omega)$, we have

\[
\int_\Omega N^\alpha |\nabla_G \phi|^p dx \geq \left( \frac{Q + \alpha - p}{p} \right)^p \int_\Omega N^\alpha \frac{|\nabla_G N|^p}{Np} |\phi|^p dx + c_p \int_\Omega N^\alpha \frac{|\nabla_G N|^p}{(N \log \frac{\delta}{N})^p} |\phi|^p dx,
\]

where $Q + \alpha > p \geq 2$, $\alpha \in \mathbb{R}$, $c_p > 0$ and $R > \sup_{x \in \Omega} N(x)$.

**Remark 3.** In the Abelian case, when $\mathbb{G} = \mathbb{R}^n$, with the ordinary dilations, one has $\mathbb{G} = V_1 = \mathbb{R}^n$ so that $Q = n$. Now it is clear that the above inequality with the homogeneous norm $N(x) = |x|$ and $\alpha = 0$ recovers the inequality (1.4) proved by Adimurthi et al. in [1].

We now apply Theorem 4.1 with the pair

\[
V \equiv 1 \quad \text{and} \quad \delta = \log(\log \frac{R}{N}), \quad R > e \sup_{x \in \Omega} N(x),
\]

and we obtain the following result including a different logarithmic remainder.

**Corollary 10.** Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous norm $N = u^{-\frac{1}{\alpha}}$ and let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{G}$. Then for all $\phi \in C_0^\infty(\Omega)$, we have

\[
\int_\Omega N^\alpha |\nabla_G \phi|^p dx \geq \left( \frac{Q + \alpha - p}{p} \right)^p \int_\Omega N^\alpha \frac{|\nabla_G N|^p}{Np} |\phi|^p dx + c_p \int_\Omega N^\alpha \frac{|\nabla_G N|^p}{Np} \frac{1}{(\log \frac{\delta}{N})^p} |\phi|^p dx,
\]

where $Q + \alpha > p \geq 2$, $\alpha \in \mathbb{R}$, $c_p > 0$ and $R > e \sup_{x \in \Omega} N(x)$.

On the other hand, by making the choices

\[
V = e^N \quad \text{and} \quad \delta = e^{-N},
\]

we derive the subsequent two-weight $L^p$ Hardy type inequality involving two non-negative remainders.

**Corollary 11.** Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous norm $N = u^{-\frac{1}{\alpha}}$ and let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{G}$. Then for all $\phi \in C_0^\infty(\Omega)$, we have

\[
\int_\Omega e^N N^\alpha |\nabla_G \phi|^p dx \geq \left( \frac{Q + \alpha - p}{p} \right)^p \int_\Omega e^N N^\alpha \frac{|\nabla_G N|^p}{Np} |\phi|^p dx + \left( \frac{Q + \alpha - p}{p} \right)^{p-1} \int_\Omega e^N N^\alpha \frac{|\nabla_G N|^p}{Np-1} |\phi|^p dx + c_p \int_\Omega e^N N^\alpha |\nabla_G N|^p |\phi|^p dx,
\]

where $Q + \alpha > p \geq 2$, $\alpha \in \mathbb{R}$ and $c_p > 0$. 

Another consequence of the Theorem 4.1 with the special functions
\[ V \equiv 1 \quad \text{and} \quad \delta = R - N \]
on the \( N \)-ball \( B_N \) in \( G \) is the following inequality.

**Corollary 12.** Let \( G \) be a polarizable Carnot group with homogeneous norm \( N = u^{\frac{Q}{Q + \alpha}} \). Then for every \( \phi \in C_0^\infty(B_N) \), we have
\[
\int_{B_N} N^\alpha |\nabla G \phi|^p dx \geq \left( \frac{Q + \alpha - p}{p} \right)^p \int_{B_N} N^\alpha \frac{\nabla G N^p}{N^p} |\phi|^p dx
\]
\[ + c_p \int_{B_N} N^\alpha \frac{\nabla G N^p}{(R - N)^p} |\phi|^p dx, \]
where \( Q + \alpha > p \geq 2 \), \( \alpha \in \mathbb{R} \) and \( c_p > 0 \).

**Remark 4.** The lack of regularity on the above choices can be readily handled by replacing the function \( N \) with a suitable \( N_\epsilon \) and then passing to the limit as \( \epsilon \to 0 \).
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