The pressureless limits of Riemann solutions to the Euler equations of one-dimensional compressible fluid flow with a source term

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Abstract

In this paper, we study the limits of Riemann solutions to the inhomogeneous Euler equations of one-dimensional compressible fluid flow as the adiabatic exponent $\gamma$ tends to one. Different from the homogeneous equations, the Riemann solutions of the inhomogeneous system are non self-similar. It is rigorously shown that, as $\gamma$ tends to one, any two-shock Riemann solution tends to a delta shock solution of the pressureless Euler system with a source term, and the intermediate density between the two shocks tends to a weighted $\delta$-measure which forms the delta shock; while any two-rarefaction-wave Riemann solution tends to a two-contact-discontinuity solution of the pressureless Euler system with a source term, whose intermediate state between the two contact discontinuities is a vacuum state. Moreover, we also give some numerical results to confirm the theoretical analysis.

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1. Introduction

The Euler equations of one-dimensional compressible fluid flow with the Coulomb-like friction term can be written as

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
u_t + (\frac{1}{2} u^2 + p(\rho))_x &= \beta,
\end{align*}
\]

(1.1)

where $\beta$ is a constant, the nonlinear function $p(\rho) = \frac{\theta}{2} \rho^{\gamma - 1}$, $\theta = \frac{\gamma - 1}{2}$ and $\gamma \in (1, 2)$ is a constant.

Shen [24] considered the pressureless Euler system with the Coulomb-like friction term and obtained the non self-similar Riemann solutions by introducing a new velocity:

\[v(t, x) = u(t, x) - \beta t,\]

(1.2)

which was introduced by Faccanoni and Mangeney [9] to study the Riemann problem of the shallow water equations with the Coulomb-like friction term.

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If $\beta = 0$, then the system (1.1) becomes the homogeneous Euler equations of one-dimensional compressible fluid flow (cf. [8]):

$$\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
u_t + (u^2 + p(\rho))_x &= 0.
\end{align*}$$

System (1.3) was firstly derived by Earnshaw [8] in 1858 for isentropic flow and is also viewed as the Euler equations of one-dimensional compressible fluid flow [14], where $\rho$ denotes the density, $u$ the velocity, and $p(\rho)$ the pressure of the fluid. System (1.3) has other different physical backgrounds. For instance, it is a scaling limit system of Newtonian dynamics with long-range interaction for a continuous distribution of mass in $R$ [20, 21] and also a hydrodynamic limit for the Vlasov equation [1].

The solutions for system (1.3) were widely studied by many scholars (see [4-5, 7-8, 17-18, 22]). In particular, the existence of global weak solutions of the Cauchy problem was first established by DiPerna [7] for the case of $1 < \gamma < 3$ by using the Glimm’s scheme method. Using the result of DiPerna [7], Li [17] obtained a global weak solution to the Cauchy problem for the case $-1 < \gamma < 1$. Using the theory of compensated compactness coupled with some basic ideas of the kinetic formulation, Lu [18] established an existence theorem for global entropy solutions for the case $\gamma > 3$. Cheng [5] also used the same methods as in [18] to obtain the existence of global entropy solutions for the Cauchy problem with a uniform amplitude bound for the case $1 < \gamma < 3$.

When $\gamma \to 1$, the limiting system of (1.1) formally becomes the pressureless Euler system with the Coulomb-like friction term,

$$\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
u_t + (\frac{u^2}{\Delta})_x &= \beta,
\end{align*}$$

which can be also obtained by taking the constant pressure where the force is assumed to be the gravity with $\beta$ being the gravity constant [6].

For the Euler system of power law in Eulerian coordinates,

$$\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p(\rho))_x &= 0,
\end{align*}$$

when the pressure tends to zero or a constant, the Euler system (1.5) formally tends to the zero pressure gas dynamics. In earlier seminal papers, Chen and Liu [2] first showed the formation of $\delta$-shocks and vacuum states of the Riemann solutions to the Euler system (1.5) for polytropic gas by taking limit $\varepsilon \to 0^+$ in the model $p(\rho) = \varepsilon \rho^\gamma / \gamma$ ($\gamma > 1$), which describe the phenomenon of concentration and cavitation rigorously in mathematics. Further, they also obtained the same results for the Euler equations for nonisentropic fluids in [3]. The same problem for the Euler equations (1.5) for isothermal case ($\gamma = 1$) was studied by Li [16]. Recently, Muhammad Ibrahim, Fujun Liu and Song Liu [12] showed the same phenomenon of concentration also exists in the mode $p(\rho) = \rho^\gamma$ ($0 < \gamma < 1$) as $\gamma \to 0$, which is the case that the pressure goes to a constant. Namely, they showed rigorously the formation of delta wave with the limiting behavior of Riemann solutions to the Euler equations (1.5). For some other physical models, there are also many results, the readers are referred to [10, 11, 19, 23, 25-27, 30-32] and the references cited therein.
Motivated by [2-3, 16], in this paper, we focus on the pressureless limits of Riemann solutions to the inhomogeneous Euler system (1.1) of one-dimensional compressible fluid flow. Different from the homogeneous equations, the Riemann solutions are non self-similar, we show the same phenomenon of concentration and cavitation also exists in the case $1 < \gamma < 2$ as $\gamma \to 1$.

The organization of this article is as follows: In section 2 and section 3, we display some results on the Riemann solutions of (1.4), (1.1), respectively. In section 4, we show rigorously the formation of $\delta$-shocks and vacuum states in the pressureless limit of Riemann solutions to (1.1) as $\gamma \to 1$. In Section 5, we present some representative numerical results to demonstrate the validity of the theoretical analysis in Sections 4.

2. Preliminaries

In this section, we give the results on the Riemann problem for system (1.4). For the homogeneous pressureless Euler system corresponding to system (1.4), the results on the Riemann problem can be found in [28, 26, 30, 13].

By a change of variable (1.2), system (1.4) can be rewritten in the conservative form

$$
\begin{align*}
\rho_t + (\rho v)_x &= 0, \\
v_t + (v^2 + \beta t)_x &= 0.
\end{align*}
$$

(2.1)

In this section, we are interested in the Riemann problem for (2.1) with initial data

$$
(\rho, v)(0, x) = \begin{cases} 
(\rho_-, u_-), & x < 0, \\
(\rho_+, u_+), & x > 0,
\end{cases}
$$

(2.2)

where $\rho_\pm > 0$ and $u_\pm$ are given constant states.

It can be seen that the solutions of the Riemann problem to system (1.4) can be obtained from the corresponding ones of (2.1) and (2.2) by using the change of state variables $(\rho, u)(t, x) = (\rho, v + \beta t)(t, x)$ directly.

The system (2.1) has a double eigenvalue $\lambda = v + \beta t$ whose corresponding right eigenvector is $\tau^r = (1, 0)^T$. Since $\nabla \lambda \cdot \tau^r \equiv 0$, so (2.1) is full linear degenerate and elementary waves are contact discontinuities.

For a discontinuity $\sigma(t) = x'(t)$, the Rankine-Hugoniot conditions

$$
\begin{align*}
-\sigma(t)[\rho] + [\rho(v + \beta t)] &= 0, \\
-\sigma(t)[v] + \frac{[v^2 + \beta t]}{2} &= 0,
\end{align*}
$$

(2.3)

hold, where $[\rho] = \rho_+ - \rho_-$, etc. By solving (2.3), we obtain contact discontinuity $J(\rho_-, u_-)$:

$$
\sigma(t) = v + \beta t = u_- + \beta t.
$$

(2.4)

We now can construct the Riemann solutions of (2.1) and (2.2) by contact discontinuities, vacuum or $\delta$-shock wave connecting two constant states $(\rho_\pm, u_\pm)$. 

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For the case \( u_- < u_+ \), the Riemann solution consists of two contact discontinuities with a vacuum between them, which is shown as

\[
\begin{array}{ll}
(\rho, v)(t, x) = \begin{cases}
(\rho_-, u_-), & -\infty < x < u_- t + \frac{1}{2} \beta t^2, \\
\text{Vac}, & u_- t + \frac{1}{2} \beta t^2 \leq x \leq u_+ t + \frac{1}{2} \beta t^2, \\
(\rho_+, u_+), & u_+ t + \frac{1}{2} \beta t^2 < x < +\infty.
\end{cases}
\end{array}
\] (2.5)

The Riemann solution can be expressed by:

\[
(\rho_-, u_-) + J_1 + \text{Vac} + J_2 + (\rho_+, u_+),
\] (2.6)

where \( \text{“+”} \) means “followed by”.

For the case \( u_- = u_+ \), the Riemann solution consists of one contact discontinuity, which is shown as

\[
(\rho, v)(t, x) = \begin{cases}
(\rho_-, u_-), & -\infty < x < u_- t + \frac{1}{2} \beta t^2, \\
(\rho_+, u_+), & u_- t + \frac{1}{2} \beta t^2 < x < +\infty.
\end{cases}
\] (2.7)

The Riemann solution can be expressed by:

\[
(\rho_-, u_-) + J + (\rho_+, u_+).
\] (2.8)

For the case \( u_- > u_+ \), the Riemann solution cannot be constructed by using the classical waves, and the delta shock wave appears. The Riemann solution can be expressed by:

\[
(\rho_-, u_-) + \delta S + (\rho_+, u_+).
\] (2.9)

The delta shock \( \delta S \) satisfies the generalized Rankine-Hugoniot conditions

\[
\begin{align*}
\frac{dx(t)}{dt} &= u_\delta(t), \\
\frac{dw(t)}{dt} &= u_\delta(t)[p] - [\rho(v + \beta t)], \\
u_\delta(t)[v] &= \frac{[\rho + \beta t]^2}{2},
\end{align*}
\] (2.10)

where \([p] = \rho_+ - \rho_-\), \(x(t), w(t)\) and \(u_\delta(t) = v_\delta + \beta t\) respectively denote the location, weight and propagation speed of the delta shock, and \((x, w)(0) = (0, 0)\).

By simple calculation, we have

\[
v_\delta = \frac{1}{2}(u_- + u_+), \quad x(t) = v_\delta t + \frac{1}{2} \beta t^2, \quad w(t) = \frac{1}{2}(\rho_+ + \rho_+)(u_- - u_+) t.
\] (2.11)

We also can justify that the delta shock satisfies the generalized entropy condition

\[
u_+ + \beta t < u_\delta(t) < u_- + \beta t.
\] (2.12)

Thus, we have obtained the Riemann solutions of (2.1) and (2.2).

In summary, we obtain the Riemann solutions to system (1.4) as follows

(1) For \( u_- > u_+ \), the Riemann solution to system (1.4) has the following form:

\[
(\rho, u)(t, x) = \begin{cases}
(\rho_-, u_- + \beta t), & x < x(t), \\
(w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\
(\rho_+, u_+ + \beta t), & x > x(t),
\end{cases}
\] (2.13)
where
\[ x(t) = \frac{1}{2}(u_- + u_+)t + \frac{1}{2} \beta t^2, \quad w(t) = \frac{1}{2}(\rho_- + \rho_+)(u_- - u_+)t, \quad u_s(t) = \frac{1}{2}(u_- + u_+) + \beta t. \] (2.14)

(2) For \( u_- < u_+ \), the Riemann solution can be expressed as
\[ (\rho, u)(t, x) = \begin{cases} 
(\rho_-, u_+ + \beta t), & -\infty < x < u_- t + \frac{1}{2} \beta t^2, \\
Vac, & u_- t + \frac{1}{2} \beta t^2 \leq x < u_+ t + \frac{1}{2} \beta t^2, \\
(\rho_+, u_+ + \beta t), & u_+ t + \frac{1}{2} \beta t^2 < x < +\infty,
\end{cases} \] (2.15)
where the locations and propagation speeds of two contact discontinuities \( J_1 \) and \( J_2 \) are identical with those in the Riemann solution of (2.1) and (2.2).

(3) For \( u_- = u_+ \), the Riemann solution can be expressed as
\[ (\rho, u)(t, x) = \begin{cases} 
(\rho_-, u_+ + \beta t), & -\infty < x < u_- t + \frac{1}{2} \beta t^2, \\
(\rho_+, u_+ + \beta t), & u_- t + \frac{1}{2} \beta t^2 < x < +\infty,
\end{cases} \] (2.16)
where the location and propagation speed of contact discontinuity \( J \) are identical with those in the Riemann solution of (2.1) and (2.2).

3. Riemann problem for Euler equations with a source term (1.1)

In this section, we construct the Riemann solutions of the Euler equations with the Coulomb-like friction term (1.1).

Using (1.2), system (1.1) is rewritten in the conservative form
\[ \begin{cases} 
\rho_t + (\rho v + \beta t) x = 0, \\
v_t + \left( \frac{(\rho + \beta t)^2}{2} + \frac{\gamma - 1}{4} \rho^{\gamma - 1} \right) x = 0.
\end{cases} \] (3.1)

In this section, we are interested in the Riemann problem for (3.1) with initial data
\[ (\rho, v)(0, x) = \begin{cases} 
(\rho_-, v_-), & x < 0, \\
(\rho_+, v_+), & x > 0,
\end{cases} \] (3.2)
where \( \rho_\pm > 0 \) and \( u_\pm \) are given constant states.

The system (3.1) can be reformulated in a quasi-linear form
\[ \begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v + \beta t & \rho \\ \frac{(\rho + \beta t)^2}{2} & v + \beta t \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \] (3.3)

By (3.3), it is easy to see that system (3.1) has two eigenvalues
\[ \lambda_1 = v + \beta t - \frac{\gamma - 1}{2} \rho \frac{\gamma}{\gamma - 1}, \quad \lambda_2 = v + \beta t + \frac{\gamma - 1}{2} \rho \frac{\gamma}{\gamma - 1}, \] (3.4)
with the corresponding right eigenvectors
\[ \varphi_1^\gamma = (1, -\frac{\gamma - 1}{2} \rho \frac{\gamma}{\gamma - 1})^T, \quad \varphi_2^\gamma = (1, \frac{\gamma - 1}{2} \rho \frac{\gamma}{\gamma - 1})^T. \]
monotonic increasing and concave in the $(\rho, v)$ phase plane. Moreover, it can be concluded from (3.6) that the rarefaction wave curves in the phase plane, which are the sets of states that can be connected on the right by a 1-rarefaction or 2-rarefaction wave, are as follows

\[
R_1^\gamma (\rho_-, u_-) : \begin{cases}
\frac{dv}{dt} = \lambda_1^\gamma = v + \beta t - \frac{\gamma - 1}{2} \rho^\frac{\gamma - 1}{\gamma}, \\
v + \rho^\frac{\gamma - 1}{\gamma} = u_- + \rho^\frac{\gamma - 1}{\gamma}, \quad \rho < \rho_-, v > u_-, \\
\lambda_1^\gamma (\rho_-, u_-) < \lambda_1^\gamma (\rho, v),
\end{cases}
\]

and

\[
R_2^\gamma (\rho_-, u_-) : \begin{cases}
\frac{dv}{dt} = \lambda_2^\gamma = v + \beta t + \frac{\gamma - 1}{2} \rho^\frac{\gamma - 1}{\gamma}, \\
v - \rho^\frac{\gamma - 1}{\gamma} = u_- - \rho^\frac{\gamma - 1}{\gamma}, \quad \rho > \rho_-, v > u_-, \\
\lambda_2^\gamma (\rho_-, u_-) < \lambda_2^\gamma (\rho, v).
\end{cases}
\]

Differentiating $v$ with respect to $\rho$ in the second equation of (3.6), we have

\[
\frac{dv}{d\rho} = -\frac{\gamma - 1}{2} \rho^\frac{\gamma - 3}{\gamma} < 0,
\]

\[
\frac{d^2v}{d\rho^2} = -\frac{(\gamma - 1)(\gamma - 3)}{4} \rho^\frac{\gamma - 5}{\gamma} > 0,
\]

which implies that the 1-rarefaction wave curve $R_1^\gamma (\rho_-, u_-)$ is monotonic decreasing and convex in the $(\rho, v)$ phase plane. Similarly, one can also obtain $\frac{dv}{d\rho} > 0$ and $\frac{d^2v}{d\rho^2} < 0$ by differentiating $v$ with respect to $\rho$ in the second equation of (3.7), which implies that the 2-rarefaction wave curve $R_2^\gamma (\rho_-, u_-)$ is monotonic increasing and concave in the $(\rho, v)$ phase plane. Moreover, it can be concluded from (3.6) that $\lim_{\rho \to \rho_-^+} v = u_- + \rho_+^\frac{\gamma - 1}{\gamma}$ for the 1-rarefaction wave curve $R_1^\gamma (\rho_-, u_-)$, which indicates that curve $R_1^\gamma (\rho_-, u_-)$ intersects the $v$-axis at the point $(0, \tilde{v}_1^\gamma)$, where $\tilde{v}_1^\gamma$ is determined by $\tilde{v}_1^\gamma = u_- + \rho_+^\frac{\gamma - 1}{\gamma}$. It can also be seen from (3.7) that $\lim_{\rho \to \rho_-^+} v = +\infty$ for the 2-rarefaction wave curve $R_2^\gamma (\rho_-, u_-)$.

Let $\sigma^\gamma(t) = \frac{dx^\gamma(t)}{dt}$ be the speed of a bounded discontinuity $x = x^\gamma(t)$, then the Rankine-Hugoniot conditions for the conservative system (3.1) are given by

\[
\begin{cases}
-\sigma^\gamma(t) [\rho] + [\rho(v + \beta t)] = 0, \\
-\sigma^\gamma(t) [v] + \left[\frac{v + \beta t^2}{2} + \frac{\gamma - 1}{2} \rho^\gamma - 1\right] = 0,
\end{cases}
\]

where $[\rho] = \rho - \rho_-$, etc. From (3.8) we have

\[
\sigma^\gamma(t) = \frac{[\rho(v + \beta t)]}{6 [\rho]}
\]
Then, substituting (3.11) into the first inequality of (3.10), we have
\[
\begin{align*}
v &= \sqrt{\frac{s_a}{s_b}} - \frac{\gamma - 1}{2} \frac{s_b}{s_a},
\end{align*}
\]
where \((\rho_-, v_-)\) and \((\rho, v)\) are the left state and the right state, respectively.

1-shock curve \(S_1^\gamma(\rho_-, u_-)\):

The Lax entropy condition implies that the propagation speed \(\sigma_1^\gamma(t)\) for the 1-shock wave \(S_1^\gamma\) has to be satisfied with
\[
\sigma_1^\gamma(t) < \lambda_1^\gamma(\rho_-, v_-), \quad \lambda_1^\gamma(\rho, v) < \sigma_1^\gamma(t) < \lambda_2^\gamma(\rho, v).
\]
(3.10)

From the first equation of (3.8), we obtain
\[
\sigma_1^\gamma(t) = \frac{\rho(v + \beta t) - \rho_-(v_- + \beta t)}{\rho - \rho_-} = v_- + \beta t + \frac{\rho}{\rho - \rho_-}(v - v_-).
\]
(3.11)

Then, substituting (3.11) into the first inequality of (3.10), we have
\[
\frac{\rho}{\rho - \rho_-}(v - v_-) < \frac{\gamma - 1}{2} \frac{s_b}{s_a} < 0,
\]
which shows that \(v - v_-\) and \(\rho - \rho_-\) have different signs. Thus, from (3.9) we have
\[
v = v_- - \sqrt{\frac{\gamma - 1}{2} \frac{s_a}{s_b}(\rho - \rho_-)}.
\]
If \(v > v_-\), then \(\rho < \rho_-\), and
\[
\sigma_1^\gamma(t) - v_- - \beta t = \frac{\rho}{\rho - \rho_-}(v - v_-) = -\rho \sqrt{\frac{\gamma - 1}{2} \frac{s_a}{s_b}(\rho - \rho_-)} = -\frac{\gamma - 1}{2} \rho \sqrt{\frac{2}{\rho + \rho_-}}.
\]

2-shock curve \(S_2^\gamma(\rho_-, u_-)\):

Similarly, the propagation speed \(\sigma_2^\gamma(t)\) for the 2-shock wave \(S_2^\gamma\) should satisfy
\[
\lambda_1^\gamma(\rho_-, v_-) < \sigma_2^\gamma(t) < \lambda_2^\gamma(\rho_-, v_-), \quad \lambda_2^\gamma(\rho, v) < \sigma_2^\gamma(t).
\]

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Then, given a state \((\rho_-, u_-)\), the 2-shock wave curve \(S^*_2(\rho_-, u_-)\) in the phase plane which is the set of states that can be connected on the right by a 2-shock is as follows

\[
S^*_2(\rho_-, u_-) : \begin{cases} 
\sigma^*_2(t) = u_- + \beta t + \sqrt{ \frac{\gamma - \frac{1}{2}\rho^*_2}{\rho^*_2 - \rho^-} (\rho - \rho^-)}, \\
v = u_- + \sqrt{ \frac{\gamma - \frac{1}{2}\rho^*_2}{\rho^*_2 - \rho^-} (\rho - \rho^-)}.
\end{cases}
\]

(3.13)

Differentiating \(v\) with respect to \(\rho\) in the second equation in (3.12) yields that for \(\rho > \rho_-\),

\[
\frac{dv}{d\rho} = -\frac{1}{2} \sqrt{\frac{\gamma - \frac{1}{2}\rho^*_2}{\rho^*_2 - \rho^-} (\rho - \rho^-)} < 0,
\]

which indicates that the 1-shock wave curve \(S^*_1(\rho_-, u_-)\) is monotonic decreasing in the \((\rho, v)\) phase plane \((\rho > \rho_-)\). Similarly, from (3.13), for \(\rho < \rho_-\) we have \(\frac{dv}{d\rho} > 0\), which indicates that the 2-shock wave curve \(S^*_2(\rho_-, u_-)\) is monotonic increasing in the \((\rho, v)\) phase plane \((\rho < \rho_-)\). It can be seen from (3.13) that \(\lim_{\rho \to 0^+} v = u_- - \sqrt{\frac{\gamma - \frac{1}{2}\rho^*_2}{\rho^*_2 - \rho^-} \rho_-}\) for the 2-shock wave curve \(S^*_2(\rho_-, u_-)\), which implies that \(S^*_2(\rho_-, u_-)\) intersects the \(v\)-axis at the point \((0, \overline{v}^*_*)\), where \(\overline{v}^*_*\) is determined by \(\overline{v}^*_* = u_- - \sqrt{\frac{\gamma - \frac{1}{2}\rho^*_2}{\rho^*_2 - \rho^-} \rho_-}\). It can also be derived from (3.12) that \(\lim_{\rho \to \infty} v = -\infty\) for the 1-shock wave curve \(S^*_1(\rho_-, u_-)\).

In the \((\rho, v)\) phase plane, through a given point \((\rho_-, u_-)\), we draw the elementary wave curves \(R^*_1(\rho_-, u_-)\) and \(S^*_1(\rho_-, u_-)\) \((j=1, 2)\). These elementary wave curves divide the \((\rho, v)\) phase plane into five regions (see Fig. 1). According to the right state \((\rho_+, u_+)\) in the different regions, one can construct the unique global Riemann solution of (3.1) and (3.2) as follows:

1. \((\rho_+, u_+) \in I(\rho_-, u_-) : (\rho_-, u_-) + R^*_1 + (\rho_{\gamma}, v_{\gamma}) + R^*_2 + (\rho_+, u_+);
2. \((\rho_+, u_+) \in II(\rho_-, u_-) : (\rho_-, u_-) + S^*_1 + (\rho_{\gamma}, v_{\gamma}) + R^*_2 + (\rho_+, u_+);
3. \((\rho_+, u_+) \in III(\rho_-, u_-) : (\rho_-, u_-) + R^*_1 + (\rho_{\gamma}, v_{\gamma}) + S^*_2 + (\rho_+, u_+);
4. \((\rho_+, u_+) \in IV(\rho_-, u_-) : (\rho_-, u_-) + S^*_1 + (\rho_{\gamma}, v_{\gamma}) + S^*_2 + (\rho_+, u_+);
5. \((\rho_+, u_+) \in V(\rho_-, u_-) : (\rho_-, u_-) + R^*_1 + \text{Vac} + R^*_2 + (\rho_+, u_+),

where \((\rho_{\gamma}, v_{\gamma})\) is the intermediate state. By using (1.2), we obtain the Riemann solutions of (1.1) as follows:

1. \((\rho_+, u_+) \in I(\rho_-, u_-) : (\rho_-, u_- + \beta t) + R^*_1 + (\rho_{\gamma}, v_{\gamma} + \beta t) + R^*_2 + (\rho_+, u_+ + \beta t);
2. \((\rho_+, u_+) \in II(\rho_-, u_-) : (\rho_-, u_- + \beta t) + S^*_1 + (\rho_{\gamma}, v_{\gamma} + \beta t) + R^*_2 + (\rho_+, u_+ + \beta t);
3. \((\rho_+, u_+) \in III(\rho_-, u_-) : (\rho_-, u_- + \beta t) + R^*_1 + (\rho_{\gamma}, v_{\gamma} + \beta t) + S^*_2 + (\rho_+, u_+ + \beta t);
4. \((\rho_+, u_+) \in IV(\rho_-, u_-) : (\rho_-, u_- + \beta t) + S^*_1 + (\rho_{\gamma}, v_{\gamma} + \beta t) + S^*_2 + (\rho_+, u_+ + \beta t);
5. \((\rho_+, u_+) \in V(\rho_-, u_-) : (\rho_-, u_- + \beta t) + R^*_1 + \text{Vac} + R^*_2 + (\rho_+, u_+ + \beta t).
4. Limits of Riemann solutions to (1.1)

In this section, we study the limiting behavior of the Riemann solutions to system (1.1) as \( \gamma \) tends to one, that is, the formation of delta shock and the vacuum states as \( \gamma \) tends to one, respectively in the case \( u_+ < u_- \) and in the case \( u_+ > u_- \).

4.1. Formation of delta shock wave for system (1.1)

In this subsection, we study the phenomenon of the concentration and the formation of delta shock in the Riemann solutions to (1.1) in the case \( u_+ < u_- \) as \( \gamma \) tends to one.

**Lemma 4.1.** If \( u_+ < u_- \), then there is a sufficiently small \( \gamma_0 > 0 \) such that \( \rho_+, u_+ \in IV(\rho_-, u_-) \) for any \( \gamma \in (1, 1 + \gamma_0) \).

**Proof.** If \( \rho_+ = \rho_- \), then \( (\rho_+, u_+) \in IV(\rho_-, u_-) \) for any \( \gamma \in (1, 2) \). Thus, we only need to consider the case \( \rho_+ \neq \rho_- \).

By (3.12) and (3.13), it is easy to see that all possible states \( (\rho, v) \) that can be connected to the left state \( (\rho_-, u_-) \) on the right by a 1-shock wave \( S_1^+ \) or a 2-shock wave \( S_2^+ \) satisfy

\[
S_1^+: \quad v = u_- - \sqrt{\frac{\gamma - 1}{2} \frac{\rho^{\gamma-1} - \rho_-^{\gamma-1}}{(\rho + \rho_+)(\rho - \rho_-)} (\rho - \rho_-)}, \quad \rho > \rho_-,
\]

\[
S_2^+: \quad v = u_- + \sqrt{\frac{\gamma - 1}{2} \frac{\rho^{\gamma-1} - \rho_-^{\gamma-1}}{(\rho + \rho_+)(\rho - \rho_-)} (\rho - \rho_-)}, \quad \rho < \rho_-.
\]

If \( \rho_+ \neq \rho_- \) and \( (\rho_+, u_+) \in IV(\rho_-, u_-) \), then from Fig. 1, (4.1) and (4.2), we have

\[
u_+ < u_- - \sqrt{\frac{\gamma - 1}{2} \frac{\rho_+^{\gamma-1} - \rho_-^{\gamma-1}}{(\rho_+ + \rho_-)(\rho_+ - \rho_-)} (\rho_+ - \rho_-)}, \quad \rho_+ > \rho_-,
\]

\[
u_+ < u_- + \sqrt{\frac{\gamma - 1}{2} \frac{\rho_+^{\gamma-1} - \rho_-^{\gamma-1}}{(\rho_+ + \rho_-)(\rho_+ - \rho_-)} (\rho_+ - \rho_-)}, \quad \rho_+ < \rho_-.
\]
From (4.3) and (4.4), we derive that

$$\sqrt{\frac{\gamma-1}{\gamma} (\rho_+^{\gamma-1} - \rho_-^{\gamma-1})} = \frac{u_- - u_+}{|\rho_+ - \rho_-|}. \quad (4.5)$$

Since

$$\lim_{\gamma \to 1} \sqrt{\frac{\gamma-1}{\gamma} (\rho_+^{\gamma-1} - \rho_-^{\gamma-1})} = 0, \quad (4.6)$$

it follows that there exists $\gamma_0 > 0$ small enough such that, when $1 < \gamma < 1 + \gamma_0$, we have

$$\sqrt{\frac{\gamma-1}{\gamma} (\rho_+^{\gamma-1} - \rho_-^{\gamma-1})} < \frac{u_- - u_+}{|\rho_+ - \rho_-|}.$$

Then, it is obvious that $(\rho_+, u_+) \in IV(\rho_-, u_-)$ when $1 < \gamma < 1 + \gamma_0$. The proof is completed. \qed

When $1 < \gamma < 1 + \gamma_0$, i.e., $(\rho_+, u_+) \in IV(\rho_-, u_-)$, suppose that $(\rho_{\gamma}, u_{\gamma})$ is the intermediate state connected with $(\rho_-, u_-)$ by a 1-shock wave $S_1^\gamma$ with the speed $\sigma_1^\gamma(t)$, and $(\rho_+, u_+)$ by a 2-shock wave $S_2^\gamma$ with the speed $\sigma_2^\gamma(t)$, then it follows

$$S_1^\gamma: \begin{cases} 
\sigma_1^\gamma(t) = u_- + \beta t - \rho_{\gamma}, \\
v_{\gamma} = u_- - \sqrt{\frac{\gamma-1}{\gamma} (\rho_+^{\gamma-1} - \rho_-^{\gamma-1}) (\rho_{\gamma} - \rho_-)}, \quad \rho_{\gamma} > \rho_-,
\end{cases} \quad (4.7)$$

$$S_2^\gamma: \begin{cases} 
\sigma_2^\gamma(t) = v_{\gamma} + \beta t + \rho_+, \\
u_+ = v_{\gamma} + \sqrt{\frac{\gamma-1}{\gamma} (\rho_-^{\gamma-1} - \rho_+^{\gamma-1}) (\rho_{\gamma} - \rho_+)}, \quad \rho_{\gamma} > \rho_+.
\end{cases} \quad (4.8)$$

From (4.7) and (4.8), we have

$$u_- - u_+ = \sqrt{\frac{\gamma-1}{\gamma} (\rho_+^{\gamma-1} - \rho_-^{\gamma-1}) (\rho_{\gamma} - \rho_-) + \frac{\gamma-1}{\gamma} (\rho_-^{\gamma-1} - \rho_+^{\gamma-1}) (\rho_{\gamma} - \rho_+) + (\rho_{\gamma} - \rho_{\gamma}), \quad \rho_{\gamma} > \rho_. \quad (4.9)$$

Then we have the following lemmas.

**Lemma 4.2.** \(\lim_{\gamma \to 1} \rho_{\gamma} = +\infty\), and \(\lim_{\gamma \to 1} \gamma^{-1} \rho_{\gamma}^{-1} = a = \frac{(u_- - u_+)^2}{4}.\)

**Proof.** Let \(\lim \inf \rho_{\gamma} = \alpha\), and \(\lim \sup \rho_{\gamma} = \beta\).

If \(\alpha < \beta\), then by the continuity of \(\rho_{\gamma}\), there exists a sequence \(\{\gamma_n\}_{n=1}^{\infty} \subseteq (1, 2)\) such that

$$\lim_{\alpha \to +\infty} \gamma_n = 1, \text{ and } \lim_{\gamma \to 1} \rho_{\gamma} = c,$$

for some \(c \in (\alpha, \beta)\). Then substituting the sequence into the right-hand side of (4.9), and taking the limit \(n \to +\infty\), we have

$$\lim_{n \to +\infty} \sqrt{\frac{\gamma_n^{-1}}{\gamma_n} (\rho_+^{\gamma_n-1} - \rho_-^{\gamma_n-1}) (\rho_{\gamma_n} - \rho_-) - \frac{\gamma_n^{-1}}{\gamma_n} (\rho_-^{\gamma_n-1} - \rho_+^{\gamma_n-1}) (\rho_{\gamma_n} - \rho_+)} = 0. \quad (4.10)$$

Thus, we can obtain from (4.9) that

$$u_- - u_+ = 0,$$

which contradicts with \(u_- > u_+\). Then we must have \(\alpha = \beta\), which implies \(\lim_{\gamma \to 1} \rho_{\gamma} = \alpha\).
If \( \alpha \in (0, +\infty) \), then we can also get a contradiction when taking limit in (4.9). Thus \( \alpha = 0 \) or \( \alpha = +\infty \). By the condition \( \rho_{\gamma} > \max\{\rho_-, \rho_+\} \), it is easy to see that \( \lim_{\gamma \to 1} \rho_{\gamma} = \alpha = +\infty \).

Next taking the limit \( \gamma \to 1 \) at the right-hand side of (4.9), we have

\[
\lim_{\gamma \to 1} \sqrt{\frac{\gamma - 1}{\rho_{\gamma}^2 - \rho_+^2 - \rho_-^2}}(\rho_{\gamma} - \rho_\pm) = \lim_{\gamma \to 1} \sqrt{\frac{(\gamma - 1)(\rho_{\gamma}^2 - \rho_+^2) + \rho_-^2}{\rho_{\gamma}^2 - \rho_+^2 - \rho_-^2}} =: \sqrt{a},
\]
and

\[
u_- - \nu_+ = 2\sqrt{a},
\]
from which we can get \( a = \frac{(\underline{u} - \overline{u})^2}{4} \). The proof is completed. \( \square \)

**Lemma 4.3.** If \( \nu_- > \nu_+ \), then we have

\[
\lim_{\gamma \to 1} \nu_{\gamma} = \lim_{\gamma \to 1} (v_{\gamma} + \beta t) = \lim_{\gamma \to 1} \sigma_1^2(t) = \lim_{\gamma \to 1} \sigma_2^2(t) = \nu(t),
\]
and

\[
\lim_{\gamma \to 1} \int_{u_{\gamma}^1(t)}^{\sigma_1^2(t)} \rho_{\gamma} d\xi = \nu(t)[\rho] - [\rho(v + \beta t)] = \frac{1}{2}(\nu_- + \nu_+)(\nu_- - \nu_+),
\]
where \( \nu(t) = \frac{1}{2}(\nu_- + \nu_+) + \beta t \).

**Proof.** It follows from (1.2), (4.7), (4.8) and Lemma 4.2 that

\[
\lim_{\gamma \to 1} \nu_{\gamma} = \lim_{\gamma \to 1} (v_{\gamma} + \beta t) = \nu_- + \beta t - \frac{1}{2}(\nu_- - \nu_+).
\]

\[
\lim_{\gamma \to 1} \sigma_1^2(t) = \nu_- + \beta t - \sqrt{a} = \nu_- + \beta t - \frac{1}{2}(\nu_- - \nu_+) = \nu(t),
\]
and

\[
\lim_{\gamma \to 1} \sigma_1^2(t) = \lim_{\gamma \to 1} (v_{\gamma} + \beta t + \rho_{\gamma} \sqrt{\frac{\gamma - 1}{\rho_{\gamma}^2 - \rho_+^2 - \rho_-^2}}(\rho_{\gamma} - \rho_-)) = \lim_{\gamma \to 1} (v_{\gamma} + \beta t) = \nu(t),
\]
which immediately lead to \( \lim_{\gamma \to 1} \nu_{\gamma} = \lim_{\gamma \to 1} \sigma_1^2(t) = \lim_{\gamma \to 1} \sigma_2^2(t) = \nu(t) \).

From the first equations of the Rankine-Hugoniot conditions (3.8) for \( S_1^\gamma \) and \( S_2^\gamma \), we have

\[
\sigma_1^2(t)(\rho_\pm - \rho_{\gamma}) = \rho_-(\nu_- + \beta t) - \rho_{\gamma}(v_{\gamma} + \beta t),
\]
and

\[
\sigma_2^2(t)(\rho_\pm - \rho_+)(\rho_{\gamma} - \rho_-) = \rho_+(v_{\gamma} + \beta t) - \rho_+(u_+ + \beta t).
\]

From (4.13), (4.14) and (4.11), we get

\[
\lim_{\gamma \to 1} \rho_{\gamma}(\sigma_2^2(t) - \sigma_1^2(t)) = \lim_{\gamma \to 1} (\rho_-(\nu_- + \beta t) - \sigma_1^2(t)\rho_- + \sigma_2^2(t)\rho_+) - \rho_+(u_+ + \beta t)) = \nu(t)[\rho] - [\rho(v + \beta t)] = \frac{1}{2}(\nu_- + \nu_+)(\nu_- - \nu_+).
\]
Then, from (4.15), we obtain (4.12) immediately. The proof is completed. \( \square \)
Remark 4.1. It can be concluded from Lemmas 4.2-4.3 that, when \( \gamma \to 1 \), the two shock curves \( S_1^\gamma \) and \( S_2^\gamma \) will coincide, the intermediate density \( \rho_{\gamma} \) becomes singular, the limit of \( \rho_{\gamma} \) possesses a singularity which is a weighed Dirac delta function with the speed \( u_3(t) \).

Remark 4.2. It can be concluded from Lemma 4.3 that, when \( \gamma \to 1 \), the velocities of two shocks \( S_1^\gamma \) and \( S_2^\gamma \) and the intermediate \( u_{\gamma} \) of (1.1) approach to \( u_3(t) \), which determines the delta shock solution of the pressureless Euler system with the Coulomb-like friction term, and the intermediate density \( \rho_{\gamma} \) between the two shocks tends to a weighted \( \delta \)-measure which forms the delta shock.

From above analysis, we have the following result.

Theorem 4.4. For \( u_+ < u_- \), as \( \gamma \to 1 \), the Riemann solution containing two shocks of (1.1) with the Riemann initial data \((\rho_\pm, u_\pm)\) constructed in Section 3 converges to a delta shock solution of system (1.4) with the same Riemann initial data \((\rho_\pm, u_\pm)\).

4.2. Formation of vacuum state for system (1.1)

In this subsection, we study the formation of vacuum state for the Riemann solutions containing two rarefaction waves of (1.1) with the Riemann initial data \((\rho_\pm, u_\pm)\) as \( \gamma \to 1 \).

Lemma 4.5. If \( u_- < u_+ < u_- + 2 \), then there exists \( \gamma_1 > 0 \) such that \( (\rho_+, u_+) \in I(\rho_-, u_-) \) when \( 1 < \gamma < 1 + \gamma_1 \).

Proof. It can be derived from (3.6) and (3.7) that all possible states \((\rho, v)\) that can be connected to the left state \((\rho_-, u_-)\) on the right by a 1-rarefaction wave \( R_1^\gamma \) or a 2-rarefaction wave \( R_2^\gamma \) should satisfy

\[
R_1^\gamma(\rho_-, u_-) : v + \rho \frac{\gamma+1}{2} = u_- + \rho \frac{\gamma+1}{2}, \quad v > u_-, \quad \rho < \rho_-, \quad (4.16)
\]

\[
R_2^\gamma(\rho_-, u_-) : v - \rho \frac{\gamma+1}{2} = u_- - \rho \frac{\gamma+1}{2}, \quad v > u_-, \quad \rho > \rho_- \quad (4.17)
\]

Similarly, it can be derived from (3.7) that all possible states \((\rho, v)\) that can be connected to the left state \((0, \bar{v})\) on the right by a 2-rarefaction wave \( R_2^\gamma \) should satisfy

\[
R_2^\gamma(0, \bar{v}) : v - \rho \frac{\gamma+1}{2} = u_- + \rho \frac{\gamma+1}{2}, \quad v > u_- + \rho \frac{\gamma+1}{2}, \quad \rho > 0 \quad (4.18)
\]

If \( u_- < u_+ < u_- + 2 \), \( \rho_+ \neq \rho_- \) and \( (\rho_+, u_+) \in I(\rho_-, u_-) \), then we can see intuitively from Figure 1 together with (4.16)-(4.18) that

\[
u_+ > u_- + \rho \frac{\gamma+1}{2} - \rho_+, \quad \rho_+ < \rho_- \quad (4.19)
\]

\[
u_+ > u_- - \rho \frac{\gamma+1}{2} + \rho_+, \quad \rho_+ > \rho_- \quad (4.20)
\]

and

\[
u_+ < u_- + \rho_+_+ \frac{\gamma+1}{2} + \rho_+, \quad \rho_+ > 0 \quad (4.21)
\]

According to (4.19)-(21), we obtain that

\[
|\rho_\pm \gamma+1 \rho_+ | < u_+ - u_- < \rho_\pm \gamma+1 + \rho_\pm \gamma+1, \quad \rho_+ > 0, \quad \rho_- > 0
\]
5. Numerical results for (1.1)

From \( \lim_{\gamma \to 1} (\rho_{\gamma}^{-1} - \rho_{\gamma}^{+1}) = 0 \) and \( \lim_{\gamma \to 1} (\rho_{\gamma}^{-1} + \rho_{\gamma}^{+1}) = 2 > u_+ - u_- \), it follows that there exists \( \gamma_1 > 0 \) small enough such that, when \( 1 < \gamma < 1 + \gamma_1 \), we have

\[
|\frac{\rho_{-\gamma}^{\frac{1}{\gamma}} - \rho_{-\gamma}^{+\frac{1}{\gamma}}}{\rho_{-\gamma}^{\frac{1}{\gamma}} - \rho_{-\gamma}^{+\frac{1}{\gamma}}}| < u_- - u_+ < \rho_{-\gamma}^{\frac{1}{\gamma}} + \rho_{+\gamma}^{\frac{1}{\gamma}}, \quad \rho_+ > 0, \rho_- > 0.
\]

(4.22)

Then, it is obvious that \( (\rho_+, u_+) \in I(\rho_-, u_-) \) when \( 1 < \gamma < 1 + \gamma_1 \). The proof is completed. \( \square \)

When \( u_- < u_+ < u_- + 2 \), by Lemma 4.5, for any given \( \gamma \in (1, 1 + \gamma_1) \), the Riemann solution of (1.1) with the Riemann initial data \( (\rho_\pm, u_\pm) \) is as follows

\[
(\rho_-, u_- + \beta t) + R_1^\gamma + (\rho_\gamma, v_\gamma + \beta t) + R_2^\gamma + (\rho_+, u_+ + \beta t),
\]

(4.23)

where

\[
R_1^\gamma : \begin{cases} 
\frac{dx}{dt} = \lambda_1^\gamma = v + \beta t - 2\frac{1}{\gamma - 1}\rho^{\frac{1}{\gamma}} \rho_\gamma, \\
v + \rho^{\frac{1}{\gamma}} = u_- + \rho_-, \quad \rho_\gamma \leq \rho \leq \rho_-.
\end{cases}
\]

(4.24)

and

\[
R_2^\gamma : \begin{cases} 
\frac{dx}{dt} = \lambda_2^\gamma = v + \beta t + 2\frac{1}{\gamma - 1}\rho^{\frac{1}{\gamma}} \rho_\gamma, \\
v - \rho^{\frac{1}{\gamma}} = u_+ + \rho_+, \quad \rho_\gamma \leq \rho \leq \rho_+.
\end{cases}
\]

(4.25)

Thus, from (4.24) and (4.25), we can derive that

\[
u_+ - u_- = \frac{\gamma - 1}{\gamma} \rho^{\frac{1}{\gamma}} + \frac{\gamma - 1}{\gamma} \rho_\gamma - 2\rho_\gamma \rho^{\frac{1}{\gamma}}, \quad \rho_\gamma \leq \rho \leq \rho_+.
\]

(4.26)

which implies the phenomenon of vacuum occurs as \( \gamma \to 1 \).

**Theorem 4.6.** Let \( u_- < u_+ < u_- + 2 \). For any fixed \( \gamma \in (1, 2) \), assume that \((\rho_\gamma, u_\gamma)(t, x)\) is a Riemann solution containing two rarefaction waves of (1.1) with the Riemann initial data \((\rho_\pm, u_\pm)\) constructed in Section 3. Then, as \( \gamma \to 1 \), the vacuum state occurs, and two rarefaction waves become two contact discontinuities connecting the states \((\rho_\pm, u_\pm + \beta t)\) and the vacuum \((\rho = 0)\), which form a vacuum solution of system (1.4) with the same initial data \((\rho_\pm, u_\pm)\).

**Proof.** If \( \lim_{\gamma \to 1} \rho_\gamma = K \in (0, \min(\rho_-, \rho_+)) \), then taking the limit \( \gamma \to 1 \) in (4.27), we have \( u_+ = u_- \), which contradicts with \( u_- < u_+ \). Thus \( \lim_{\gamma \to 1} \rho_\gamma = 0 \), which means the vacuum occurs as \( \gamma \to 1 \).

Moreover, as \( \gamma \to 1 \), one can directly derive from (4.24) and (4.25) that

\[
\lim_{\gamma \to 1} v = u_- \text{ on } R_1^\gamma, \quad \lim_{\gamma \to 1} v = u_+ \text{ on } R_2^\gamma,
\]

(4.27)

and

\[
\begin{cases} 
\lambda_1^\gamma = \frac{\gamma - 1}{\gamma} v - \frac{\gamma - 1}{\gamma} u_- - \frac{\gamma - 1}{\gamma} \rho^{\frac{1}{\gamma}} \rho_\gamma + \beta t, \\
\lambda_2^\gamma = \frac{\gamma - 1}{\gamma} v - \frac{\gamma - 1}{\gamma} u_+ + \frac{\gamma - 1}{\gamma} \rho^{\frac{1}{\gamma}} \rho_\gamma + \beta t.
\end{cases}
\]

(4.28)

(4.27) and (4.28) imply that

\[
\lim_{\gamma \to 1} \lambda_1^\gamma = u_- + \beta t, \quad \lim_{\gamma \to 1} \lambda_2^\gamma = u_+ + \beta t.
\]

(4.29)

The proof is completed. \( \square \)

5. Numerical results for (1.1)
In this section, in order to verify the validity of the formation of $\delta$-shocks and vacuum states for system (1.1) mentioned in section 4, we present two selected groups of representative numerical simulations. A number of iterative numerical trials are executed to guarantee what we demonstrate are not numerical objects. To discretize the system, we use the fifth-order weighted essentially non-oscillatory scheme and third-order Runge-Kutta method \cite{15, 29} with the mesh 200 cells. The numerical simulations are consistent with the theoretical analysis.

5.1. Formation of delta shock wave

When $u_+ < u_-$, we compute the solution of the Riemann problem of (1.1) with $\beta = 2$ and take the initial data as follows:

\[
(\rho, u)(0, x) = \begin{cases}
(1.5, 2), & x < 0, \\
(2, -1), & x > 0.
\end{cases}
\]  

The numerical simulations for different choices of $\gamma$ ($\gamma = 1.7, 1.05, 1.001$, and the time $t = 0.2$) are presented in Figs. 2-4 which show the process of concentration and formation of the delta shock wave in the pressureless limit of solutions containing two shocks.

Fig. 2. Density (left) and velocity (right) for $\gamma = 1.7$.

Fig. 3. Density (left) and velocity (right) for $\gamma = 1.05$. 

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We can clearly see from these numerical results that, as $\gamma$ decreases, the locations of the two shocks become closer and closer, and the density of the intermediate state increases dramatically, while the velocity becomes a piecewise constant function. Finally, as $\gamma$ tends to one, along with the intermediate state, the two shocks coincide to form the delta shock wave of the pressureless Euler system with the Coulomb-like friction term (1.4), while the velocity keeps a step function. The numerical simulations are in complete agreement with the theoretical analysis in section 4.1.

5.2. Formation of the vacuum state

When $u_- < u_+$, we compute the solution of the Riemann problem of (1.1) with $\beta = 2$ and take the initial data as follows:

$$
\begin{align*}
(\rho, u)(0, x) = \begin{cases} 
(1, -0.1), & x < 0, \\
(4, 1), & x > 0.
\end{cases}
\end{align*}
$$

The numerical simulations for different choices of $\gamma$ ($\gamma = 1.8$, 1.18, 1.01 and the time $t = 0.2$), are presented in Figs. 5-7 which show the process of cavitation and formation of the vacuum state in the pressureless limit of solutions containing two rarefaction waves.

From these numerical results, we can clearly observe that, when $\gamma$ decreases, the boundaries of two rarefaction waves become closer and closer, along with the intermediate state, the density tends to zero, while the velocity becomes a linear function. In the end, as $\gamma$ tends to one, a two-rarefaction-wave solution tends to a two-contact-discontinuity solution with a vacuum state of the pressureless Euler system with the Coulomb-like friction term (1.4). The numerical simulations are in complete agreement with the theoretical analysis in section 4.2.
Fig. 5. Density (left) and velocity (right) for $\gamma = 1.8$.

Fig. 6. Density (left) and velocity (right) for $\gamma = 1.18$.

Fig. 7. Density (left) and velocity (right) for $\gamma = 1.01$. 
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