A constrained minimum criterion for variable selection

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Abstract: For variable selection in the sparse linear model setting, we propose a new criterion which chooses the subset with the minimum number of variables under the constraint that the likelihood of the subset is above a certain level. When applied to the best subset selection, this criterion gives users direct control on the probability of choosing the correct subset in an asymptotic sense and a consistent estimator for the model parameter vector. It also has an interpretation as a likelihood-ratio test procedure for detecting the cardinality of the true model. When applied to the lasso, this criterion eliminates the need for selecting the lasso tuning parameter as it is automatically determined by the likelihood level in the criterion. Under mild conditions, the lasso estimator under this criterion is consistent. Simulation results show that this criterion gives the lasso better selection accuracy than the cross-validation criterion in many cases. The criterion also automatically gives the chosen model its level of significance which provides users more information for decision making.

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1 Introduction

Consider the linear regression model

\[ y = X\beta + \varepsilon, \]

where \( y = (y_1, \ldots, y_n)^T, X = [1, x_1, \ldots, x_p], \beta = (\beta_0, \beta_1, \ldots, \beta_p)^T \) and \( \varepsilon \sim N(0, \sigma^2 I) \).

We assume model \( \Pi \) is sparse in the sense that parameters for most variables are

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zero and that $p$ is large and may be larger than $n$. An $x_i$ is said to be active if $\beta_i \neq 0$ and inactive if $\beta_i = 0$. Our objective is to accurately identify or select the active variables.

There are a number of well-known methods of variable selection such as the forward selection, backward selection, stepwise regression, best subset selection, the lasso and the least angle regression. There are also a number of selection criteria that can be used with these methods. These include the partial $F$-test, the adjusted $R^2$, AIC, BIC, Mallows’s $C_p$ statistic and criteria based on expected prediction error. Instead of providing a lengthy review of these methods and criteria, we refer readers to books such as Draper and Smith (1998), Hastie, Tibshirani and Friedman (2009) and references therein for detailed discussions. Here, we only briefly review the lasso which had motivated this work.

The lasso introduced by Tibshirani (1996) solves the following $l_1$-penalized least squares problem

$$
\min_{\beta} \|y - X\beta\|_2^2 \text{ subject to } \sum_{i=1}^{p} |\beta_i| \leq t,
$$

(2)

where $t$ is a non-negative tuning parameter. The lasso solution $\tilde{\beta}$ of (2) often has many elements that are exactly zero, so it is a natural method for variable selection. This and the fact that it can handle $p \gg n$ makes it a popular choice for variable selection in a sparse model situation. The lasso has been generalized in a number of directions [e.g., Zou (2006), Zou and Hastie (2005), Yuan and Lin (2006)]. A powerful R package glmnet by Friedman et al. (2019) is available to handle the computation for very large problems. There are also many theoretical investigations on the lasso [e.g., Knight and Fu (2000), Greenshtein and Ritov (2004), Efron et al. (2004), Wainwright (2009), Raskutti, Wainwright and Yu (2011)]. More recently, Lockhart et al. (2014) developed a test for the significance of a predictor while accounting for the adaptive nature of the lasso fitting. The most important issue in applying the lasso is the selection of the tuning parameter $t$. There are various methods for selecting $t$ but the most commonly used method is the cross-validation which selects the $t$ value that minimizes a $k$-fold cross-validation mean square error [Hastie, Tibshirani and Friedman (2009)]. The availability of the R code cv.glmnet also by Friedman et al. (2019) made the implementation of the cross-validation very
easy. Nevertheless, it is known that by putting the predictive power of a model ahead of its parsimony, the cross-validation based lasso often gives over-fitted models with high false active rates.

To improve the variable selection accuracy of the lasso and to resolve the tuning parameter selection problem, we propose a new variable selection criterion which chooses the subset with the minimum number of variables under the constraint that the likelihood of the subset is above a certain level. For brevity, we call this new criterion the CMC. The CMC is based on two considerations: (i) we should in general choose a subset with a high likelihood level and (ii) due to the sparsity condition, we favor models with a small number of variables and more so than we do in other settings of variable selection without the sparsity condition. These are competing considerations; if we ignore (ii), then we would choose the full model (if $n > p$) which has the highest likelihood; if we ignore (i), we would consider only models with very few variables. The CMC aims to balance these competing considerations through the choice of the likelihood level in the constraint. Although motivated by the objective to reduce the false active rate in the cross-validation based lasso variable selection, the CMC is a general criterion that can be used with other likelihood based methods of variable selection such as the variants of the lasso discussed in Hastie, Tibshirani and Wainwright (2015). The best subset selection provides the simplest setup to illustrate the CMC, so we will first discuss CMC in the context of this method.

The rest of this paper is organized as follows. In Section 2, we apply the CMC to the best subset selection resulting in a method of variable selection with direct control on the probability of choosing the correct subset in an asymptotic sense. The method also gives a consistent estimator that can be asymptotically unbiased for the unknown $\beta$ in (1) under certain conditions. We then apply the CMC to the lasso. The CMC eliminates the need to select the lasso parameter as it is automatically determined by the likelihood level in the CMC. The resulting lasso estimator for the unknown $\beta$ is consistent. We also give numerical examples to illustrate the superior selection accuracy of the CMC. We conclude with some remarks in Section 3.
2 The constrained minimum criterion for variable selection

2.1 The constrained minimum criterion for the best subset selection

For this subsection, we assume that \( n > p \) so that the least squares estimator, 
\[
\hat{\beta} = (X^T X)^{-1} X^T y,
\]  
(3)
is available. We denote by \( X_j \) the \( j \)th subset of \( X = \{x_1, \ldots, x_p\} \) and by \( \hat{\beta}_j \) the least squares estimator for the model containing only variables in \( X_j \) for \( j = 1, \ldots, 2^p \). 

For convenience, we view each \( \hat{\beta}_j \) as a \( q = p + 1 \) dimensional vector having zeros in elements or positions of \( \hat{\beta}_j \) corresponding to variables not in \( X_j \). We will use the terms “model” and “subset” interchangeably. We now apply the CMC to identify the best subset among the \( 2^p \) possible subsets for the purpose of classifying the \( p \) variables into active and inactive categories.

The log-likelihood function for \((\beta, \sigma^2)\) is 
\[
l(\beta, \sigma^2) = -\frac{n}{2} \ln(2\pi) - n \ln(\sigma) - \frac{1}{2\sigma^2} \|y - X\beta\|^2_2,
\]  
(4)
which is maximized at \( \beta = \hat{\beta} \) for any fixed \( \sigma^2 \). The likelihood-ratio of \((\beta, \sigma^2)\) is 
\[
\lambda(\beta, \sigma^2) = -2\{l(\beta, \sigma^2) - l(\hat{\beta}, \sigma^2)\} = \frac{1}{\sigma^2} \left\{ \|y - X\beta\|^2_2 - \|y - X\hat{\beta}\|^2_2 \right\}.
\]
The error variance \( \sigma^2 \) is unknown. Since we are only interested in measuring the relative likelihood of models defined by different \( \beta \) and this relative likelihood does not depend on \( \sigma^2 \), we replace it with its unbiased estimator, the residual mean square of the full model \( \hat{\sigma}^2 \). This leads to the following estimated likelihood-ratio 
\[
h(\beta) = \frac{1}{\hat{\sigma}^2} \left\{ \|y - X\beta\|^2_2 - \|y - X\hat{\beta}\|^2_2 \right\}
\]  
(5)
for \( \beta \). A small \( h(\beta) \) value indicates a relatively high likelihood for \( \beta \). For a fixed constant \( \kappa > 0 \), the set \( \{\beta : h(\beta) \leq \kappa\} \) is the collection of \( \beta \) vectors with likelihood above a certain level. Denote by \( z(\beta) \) the number of zeros in elements \( \beta_1, \ldots, \beta_p \) of \( \beta \). The CMC for the best subset selection is
Constrained minimum criterion: for a predetermined constant $\kappa > 0$, we choose the $X_i$ that satisfies (i) $\hat{\beta}_i \in \{\beta : h(\beta) \leq \kappa\}$ and (ii) $z(\hat{\beta}_i) \geq z(\hat{\beta}_j)$ for all $\hat{\beta}_j \in \{\beta : h(\beta) \leq \kappa\}$; that is, among the subsets above a certain likelihood level, we choose the one with the minimum number of variables.

It is possible that there are two or more subsets satisfying the conditions in the above criterion. In this case, we choose the one with the highest likelihood among them. Another formulation of the criterion is to solve

$$\minimize_{\beta} \|\beta\|_0 \quad \text{subject to} \quad h(\beta) \leq \kappa,$$

where $\|\beta\|_0$ is the $l_0$-pseudo-norm that counts the number of non-zero elements in $\beta$. The solution to (6) may not be unique, but since the minimization is over the $2^p$ subsets, we again choose the solution with the highest likelihood. This amounts to minimizing over set $F_B$ in (7) where the solution is unique.

Let $X_j^*$ be the model with the highest likelihood among the $\binom{p}{j}$ models with exactly $j$ variables. The model chosen by the CMC is from the set

$$F_B = \{X_1^*, \ldots, X_p^*\}. \quad (7)$$

This is desirable as we would not want to select a model when there is another one of the same size but a higher likelihood available. Other criteria such as the AIC and BIC also choose from $F_B$, but there are differences in how the CMC operates. The AIC and BIC consider models of all likelihoods and as such they may be viewed as continuous operations for model selection in term of likelihood of the model. The CMC, on the other hand, considers only models above a certain likelihood level, so it is a truncated version of the continuous operation. Another difference is that AIC and BIC are general purpose model selection criteria not specifically designed to make use of the sparsity information. The CMC is devised to make full use of the sparsity information as it chooses the smallest model above a given level of likelihood without considering any other factors such as prediction accuracy. In this sense, it represents an aggressive way of using the sparsity condition.

We now present asymptotic properties of the model chosen by CMC and discuss the proper selection of the constant $\kappa$ in the process. In the following, for a fixed $\gamma \in (0, 1)$, we use $F_{\gamma,q,n-q}$ to denote the $\gamma$th quantile of the $F$ distribution with $q$
and \((n - q)\) degrees of freedom. We use \(\beta'\) to denote the unknown true parameter vector. We also use the notation \(\beta_1 \overset{z}{=} \beta_2\) (called “z-equal”) to mean that \(\beta_1\) and \(\beta_2\) have zeros at exactly the same positions; for example, \((1, 0, 0)^T \overset{z}{=} (2, 0, 0)^T\) but \((1, 0, 0)^T\) is not z-equal to \((0, 2, 0)^T\) or \((0, 0, 0)^T\). For simplicity, similar to Knight and Fu (2000) we impose the following two conditions on the variables,

\[
D_n = \frac{1}{n} \sum_{i=1}^{n} x_{ri} x_{ri}^T \to D,
\]

where \(x_{ri}\) is the \(i\)th row of \(X\) and \(D\) is a \(q \times q\) positive definite matrix, and

\[
\frac{1}{n} \max_{1 \leq i \leq n} x_{ri}^T x_{ri} \to 0.
\]

Under conditions (8) and (9), the least squares estimator \(\hat{\beta}\) is consistent and asymptotically normal, that is,

\[
\sqrt{n}(\hat{\beta} - \beta') \xrightarrow{d} N(0, \sigma^2 D^{-1}).
\]

The following theorem gives the consistency of the least squares estimator for the model chosen by CMC and an asymptotic lower bound on the probability of all variables being correctly classified.

**Theorem 2.1** Suppose conditions (8) and (9) hold. For a fixed \(\gamma \in (0, 1)\), let

\[
\kappa = q F_{\gamma, q, n-q},
\]

and denote by \(\hat{\beta}_{\gamma,n}\) be least squares estimator of the model chosen by the CMC with \(\kappa\) in (10) and at sample size \(n\). Then

\[
\hat{\beta}_{\gamma,n} \xrightarrow{p} \beta' \quad \text{as } n \to +\infty.
\]

Further, the probability that all variables are correctly classified satisfies

\[
P(\hat{\beta}_{\gamma,n} \overset{z}{=} \beta') \geq \gamma \quad \text{as } n \to +\infty.
\]

**Proof.** For any fixed \(\gamma\) and \(n\), the 100\(\gamma\)% confidence region for \(\beta'\) is

\[
\mathcal{C}_\gamma = \left\{ \beta : (\beta - \hat{\beta})^T X^T X (\beta - \hat{\beta}) \leq q \hat{\sigma}^2 F_{\gamma, q, n-q} \right\}.
\]
It may be verified that the likelihood-ratio from (5) can be expressed as
\[
h(\beta) = \frac{1}{\hat{\sigma}^2} \left\{ \parallel y - X\beta \parallel_2^2 - \parallel y - X\hat{\beta} \parallel_2^2 \right\} = \frac{1}{\hat{\sigma}^2} \left\{ (\beta - \hat{\beta})^T X^T X (\beta - \hat{\beta}) \right\}.
\]
This and (10) imply that \( C_\gamma \) may be expressed in terms of \( h(\beta) \) as
\[
C_\gamma = \{ \beta : h(\beta) \leq \kappa \}.
\]
(14)

It follows that \( \hat{\beta}_{\gamma,n} \in C_\gamma \). Under conditions (8) and (9), \( \parallel \hat{\beta}_{\gamma,n} - \beta \parallel_2 = O_p(n^{-1}) \). This and the consistency of the least squares estimator \( \hat{\beta} \) imply (11).

To show (12), define event \( A = \{ \text{there is a } \beta \in C_r \text{ such that } \beta \text{ has at least one zero in positions where } \beta' \text{ is not zero} \} \). Since \( \hat{\beta} \) is consistent and \( \parallel \beta - \hat{\beta} \parallel_2 = O_p(n^{-1}) \) for all \( \beta \in C_\gamma \), \( P(A) \to 0 \) as \( n \to +\infty \). Thus, \( P(\bar{A}) = P\{ a \beta \in C_\gamma \text{ can only have zeros in the same positions as } \beta' \} \to 1 \). Suppose the true model is \( X_i \). Then, \( \hat{\beta}_i \) is the only vector in \( \{ \hat{\beta}_j \}_{j=1}^p \) that is \( z \)-equal to \( \beta' \). For \( j \neq i \), \( \bar{A} \) implies only \( \hat{\beta}_j \) with fewer zeros than \( \hat{\beta}_i \) can be in \( C_\gamma \), so event \( \{ \hat{\beta}_i \in C_\gamma \} \cap \bar{A} \) implies \( \hat{\beta}_{\gamma,n} = \hat{\beta}_i \). It follows that
\[
P(\hat{\beta}_{\gamma,n} \overset{z}{=} \beta') = P(\hat{\beta}_{\gamma,n} = \hat{\beta}_i) \geq P(\{ \hat{\beta}_i \in C_\gamma \} \cap \bar{A}) \to P(\hat{\beta}_i \in C_\gamma).
\]
Also, \( \hat{\beta}_i \in C_\gamma \) if \( \beta' \in C_\gamma \) since \( \hat{\beta}_i \) has a higher likelihood than \( \beta' \), so
\[
P(\hat{\beta}_i \in C_\gamma) \geq P(\beta' \in C_\gamma) = \gamma.
\]
(16)
Equations (15) and (16) then imply (12).

Noting that the proof for Theorem 2.1 can go through even if we allow \( \gamma \) to increase with \( n \) so long as it increases slowly enough to ensure the size of the confidence region \( C_\gamma \) is \( o_p(1) \), we have the following corollary.

**Corollary 2.1** Suppose conditions (8) and (9) hold. Let \( \gamma_n \) be a sequence in \( (0,1) \) such that \( \gamma_n \to 1 \) and \( F_{\gamma,n,q,n-q} = o(n) \), then
\[
\hat{\beta}_{\gamma_n,n} \xrightarrow{p} \beta' \quad \text{as } n \to +\infty.
\]
(17)
The probability that all variables are correctly classified satisfies
\[
\lim_{n \to +\infty} P(\hat{\beta}_{\gamma_n,n} \overset{z}{=} \beta') = 1,
\]
(18)
and \( \hat{\beta}_{\gamma_n,n} \) is asymptotically unbiased, that is,
\[
\lim_{n \to +\infty} E(\hat{\beta}_{\gamma_n,n}) = \beta'.
\]
(19)
In applications, we use $\hat{\beta}_{\gamma,n} = (\hat{\beta}_{0,\gamma,n}, \hat{\beta}_{1,\gamma,n}, \ldots, \hat{\beta}_{p,\gamma,n})^T$ as a classifier: if $\hat{\beta}_{j,\gamma,n} \neq 0$, we classify $x_j$ as active; if $\hat{\beta}_{j,\gamma,n} = 0$, we classify $x_j$ as inactive, so the $P(\hat{\beta}_{\gamma,n} = \beta')$ in (12) has the interpretation as the probability that all variables are correctly classified. When using the CMC, we first choose the parameter $\gamma$ instead of the $\kappa$ as the latter is easily determined through (11) once $\gamma$ is given. In view of the connection between the CMC and the confidence region for $\beta'$ given by (14), we may set $\gamma$ to a commonly used confidence level such as 0.90. For a chosen $\gamma$, when the sample size $n$ is large the value of $\gamma$ has an interpretation as a lower bound on a confidence level, that is, we are at least $100\gamma\%$ confident that all variables are correctly classified. We may use the bootstrap to check whether this interpretation is valid for the sample size at hand. If it is and $\gamma$ is close to 1, then $\hat{\beta}_{\gamma,n}$ is approximately an unbiased estimator for $\beta'$. Also, the value $\alpha = 1 - \gamma$ has an interpretation as the level of significance of the model chosen by $\hat{\beta}_{\gamma,n}$ because $\hat{\beta}_{\gamma,n}$ is inside but typically very close to the boundary of the $100\gamma\%$ confidence region $C_{\gamma}$. This interpretation is useful; if we want to choose the most sparse model that is still significant at a specific level $\alpha$, we just set $\gamma = 1 - \alpha$ and this provides another way of choosing $\gamma$.

When the sample size is not large, it is not advisable to set $\gamma$ to a high level as this may substantially increase the chance of false inactive rate especially when there are weak active variables with small parameter values. To see this, when $n$ is not large but $\gamma$ is, $C_{\gamma}$ will be large and thus may contain $\beta$ vectors such that $z(\beta) \geq z(\beta')$. There may be least squares estimates $\hat{\beta}_j$ for wrong models $X_j$ with more zeros than $\beta'$ in $C_{\gamma}$. The CMC will choose the wrong model corresponding to the largest $z(\hat{\beta}_j)$, so some of the weak active variables will be misclassified as inactive. We recommend applying the CMC with a range of $\gamma$ values and compare the resulting selected models to decide on a final model.

The best subset selection is computationally infeasible when $p$ is large. Nevertheless, there is a simple way to reduce the amount of computation. In a truly sparse situation, the true model is likely of a low dimension. We may start the best subset selection process by examining low dimensional models first, beginning with models having 1, 2, 3 variables and so on until we find a $j$-variable model with a likelihood-ratio $h(\beta)$ smaller than the predetermined $\kappa$ and smaller than other models of $j$ variables. This model is the model of choice by the CMC. We stop the selection
process at this point which saves us the trouble of computing models of more than \( j \) variables. If we start the selection process with the full model and gradually reduce the size of the models considered until we find one satisfying the criterion, we will also come to the same choice but this would likely require searching through far more models due to the sparsity condition.

2.2 The constrained minimum criterion for the lasso

The lasso generates a sequence of models with varying cardinalities and likelihoods much like \( F_B \) in (7). The CMC can be used to select a model from this sequence. Denote by \( \mathcal{X}(t') \) the set of variables with non-zero parameter estimates in the lasso solution \( \tilde{\beta} \) when the tuning parameter in (2) takes the value \( t' \). Let \( D = \{ t_i \}_{i=1}^m \) be the set of all discontinuities of \( \mathcal{X}(t) \) (found over a fine grid of tuning parameter values); that is, \( \mathcal{X}(t) \) has either gained or lost at least one variable at each \( t_i \). Suppose \( t_1 < \cdots < t_m \), so \( \mathcal{X}(t_m) \) is the full set of \( p \) variables. We call \( F_L = \{ \mathcal{X}(t_i) \}_{i=1}^m \) the lasso models or lasso subsets. For the purpose of variable selection, it suffices to consider only these lasso subsets since they include all possible subsets on the lasso path found over the grid. Let \( \tilde{\beta}_i \) be the lasso solution to (2) when \( t = t_i \). The likelihood of \( \{ \tilde{\beta}_i \}_{i=1}^m \) is an increasing function of \( t_i \). To see this, for any \( t_i > t_j > 0 \), the corresponding solutions to (2), \( \tilde{\beta}_i \) and \( \tilde{\beta}_j \), satisfy

\[
\| y - X\tilde{\beta}_i \|_2^2 < \| y - X\tilde{\beta}_j \|_2^2
\]

because the constraint region \( \sum_{i=1}^p |\beta_k| \leq t_i \) is larger than \( \sum_{i=1}^p |\beta_k| \leq t_j \), so the objective function \( \| y - X\beta \|_2^2 \) in (2) reaches a smaller minimum in the region defined by \( t_i \). It follows that the likelihood-ratio (5) of the \( \{ \tilde{\beta}_i \}_{i=1}^m \) satisfy

\[
h(\tilde{\beta}_m) < h(\tilde{\beta}_{m-1}) < \cdots < h(\tilde{\beta}_1).
\]

For the case of \( n > p \), the CMC for the lasso is

\begin{center}
Constrained minimum criterion for the lasso: for a given \( \kappa > 0 \), suppose \( h(\tilde{\beta}_j) \leq \kappa < h(\tilde{\beta}_{j-1}) \). We choose the \( \mathcal{X}(t_k) \) from \( \{ \mathcal{X}(t_i) \}_{i=1}^m \) that satisfies \( z(\tilde{\beta}_k) \geq z(\tilde{\beta}_i) \) for \( i = j, \ldots, m \); that is, among the lasso subsets that are above a certain likelihood level, we choose the smallest one.
\end{center}
The lasso subsets with higher likelihood (or lower likelihood-ratio) values tend to have more variables, so model $X(t_j)$ is usually the choice of the CMC. When this is the case, variables in $X(t_j)$ are classified as active and those not in this set are classified as inactive, $\hat{\beta}_j$ is the lasso estimate for $\beta^t$ and $t_j$ is the tuning parameter determined by $\kappa$.

For the case of $n \leq p$, $\hat{\beta}$ in (3) is not available, so $h(\beta)$ in (5) is undefined. In this case, we make the following “$k$-efficient” assumption about the lasso: for some $k < n$, there exists a large $t^*$ such that $X(t^*)$ is a subset of $k$ variables that contains all active variables. We then apply the lasso and CMC to this subset of $k$ variables. To find such a $k$ and $t^*$, we may plot $SS^i_{Res} = \|y - X\hat{\beta}_i\|^2_2$ against $t_i$ and find the value $t^*_i$ after which $SS^i_{Res}$ drops slowly. Then, use $X(t^*_i)$ as the $X(t^*)$ and the number of variables in $X(t^*_i)$ as $k$. This $k$-efficient assumption is not an excessively strong assumption in the context of the lasso based variable selection because the lasso selects at most $n$ variables. If this assumption is not true for an application, then the lasso may not be an effective tool for variable selection for this application. In this sense, the $k$-efficient assumption is implicitly made whenever the lasso is applied to perform variable selection, whether or not it is explicitly used.

We now present asymptotic properties of the CMC based lasso estimator under the assumption of either $n > p$ or $n \leq p$ but the $k$-efficient assumption holds. In the later case, the lasso and CMC are applied to the $k$ variables in $X(t^*)$. We write $\beta^1 \geq \beta^2$ if $\beta^1$ has zeros in all positions where $\beta^2$ does and possibly also zeros in positions where $\beta^2$ does not; for example, $(1, 0, 0)^T \geq (2, 1, 0)^T$. We have

**Corollary 2.2** Suppose conditions (8) and (9) hold. For a fixed $\gamma \in (0, 1)$ let

$$\kappa = qF_{\gamma,q,n-q}$$

and denote by $\tilde{\beta}_{\gamma,n}$ be the lasso estimator of the model $X_{\gamma,n}$ chosen by the CMC with $\kappa$ in (21) and at sample size $n$. Then

$$\tilde{\beta}_{\gamma,n} \xrightarrow{p} \beta^t \quad \text{as } n \to +\infty. \quad (22)$$

Further, the probability of no false active classification satisfies

$$\lim_{n \to +\infty} P(\beta^t \geq \tilde{\beta}_{\gamma,n}) = 1 \quad \text{as } n \to +\infty. \quad (23)$$
Table 1: Variable selection accuracy comparison: simulated (false inactive rate, false active rate) of the CMC and cross-validation (CV) methods for various scenarios

| (p∗, p, n)     | Model 1          |            | Model 2          |            |
|----------------|------------------|------------|------------------|------------|
|                | CMC              | CV         | CMC              | CV         |
| (10, 100, 110) | (0.01, 0.03)     | (0.00, 0.27)| (0.16, 0.03)     | (0.00, 0.28)|
| (10, 500, 550) | (0.00, 0.01)     | (0.00, 0.07)| (0.00, 0.03)     | (0.00, 0.09)|
| (10, 1000, 1100)| (0.00, 0.01)    | (0.00, 0.05)| (0.00, 0.02)     | (0.00, 0.05)|
|                | (50, 100, 110)   | (0.03, 0.35)| (0.00, 0.65)     | (0.18, 0.33)| (0.04, 0.66)|
|                | (50, 500, 550)   | (0.00, 0.04)| (0.00, 0.26)     | (0.00, 0.10)| (0.00, 0.26)|
|                | (50, 1000, 1100) | (0.00, 0.03)| (0.00, 0.15)     | (0.00, 0.08)| (0.00, 0.16)|

The consistency result in (22) is again the consequence of \( \hat{\beta}_{\gamma,n} \) being in the confidence region \( C_\gamma \) in (13). There are stronger results concerning \( P(\beta^t \overset{\leq}{\sim} \hat{\beta}_{\gamma,n}) \) in Wainwright (2009) and (23) holds trivially for all consistent estimators of \( \beta^t \). We include it here for comparing (12) with (23). The latter is a weaker statement, and the reason we are unable to put a lower bound on \( P(\beta^t \overset{\leq}{\sim} \beta) \) in the present case is that the true model may not be in the set \( F_L \) which contains all the models considered for variable selection by the lasso. In contrast, the best subset selection considers all \( 2^p \) models, so it always includes the (unknown) true model in the model selection process. This point is used in (15) in the proof of (12). Nevertheless, (23) is still a useful statement as it lends support to the use of the lasso-CMC estimator \( \hat{\beta}_{\gamma,n} \) for variable selection; when the sample size is large we can drop variables not in the model \( X_{\gamma,n} \) with confidence because of (23). For lasso estimators defined by other choices of tuning parameters, we would have to examine the consistency issue of such estimators before we can be confident with using them as classifiers.

Since (23) does not depend on \( \gamma \), it may looks as if we can set \( \gamma \) and hence \( \kappa \) to large values. This again is not advisable as all asymptotic results depend on the confidence region (13) being sufficiently small and \( \hat{\beta}_{\gamma,n} \) being close to \( \hat{\beta} \). We still
need the condition that $F_{\gamma,q,n-q} = o(n)$ to guarantee these, so $\gamma$ must not be too large relative to $n$.

We now present some numerical results to demonstrate the accuracy of the CMC. We consider two models (1) both with $p$ variables in all but only $p^*$ of them are active:

Model 1: the $p^*$ active variables all have parameters equal to 1.
Model 2: the $p^*$ active variables all have parameters equal to 0.5.

The variance for both models is set to 1, so Model 1 represents a model having active variables of moderate effects (moderate signal-to-noise ratio) and Model 2 represents a model with weak effects (low signal-to-noise ratio). The elements of the design matrix $X$ are randomly generated from a standard normal distribution. The intercepts of both models are set to 1. Table 1 contains the simulated (false inactive rate, false active rate) pairs of the CMC method and the cross-validation method (CV) for various combinations of $(p^*, p, n)$ computed using glmnet. The false inactive rate is the proportion of active variables misclassified as inactive, and false active rate is that of inactive variables misclassified as active. For the cross-validation method, the $\lambda$ value used is the optimal value found in a 10-fold cross-validation by cv.glmnet. We apply the CMC under the $k$-efficient assumption which, in this case, is represented by the sample size $n$ being not much bigger than $p$; we set $n$ to be 10% higher than $p$ in all cases. For Model 1, we set $\gamma = 0.95$ to capture the $p^*$ variables with moderate effects. For Model 2, we set it to $\gamma = 0.8$ as the effects are weaker. In real applications, the strength of the effects of active variables is unknown and there are various techniques for choosing a $\gamma$ value. For example, we may choose the $\gamma$ value such that effects of a certain size will be detectable at a high probability when the variables are uncorrelated. We may also use the cross-validation based tuning parameter as a reference (see point-3 below).

Each simulated rate-pair in Table 1 is based on 100 simulation runs. We make the following observations about the CMC and the CV criterion based on Table 1.

1. When effects (absolute values of parameters) of the active variables are all moderately high (Model 1), the CMC with a $\gamma = 0.95$ worked very well. Its false inactive rate is zero or nearly zero in all cases, and its false active rate is substantially lower than that of the CV. Thus to capture effects of moderate or large sizes and to avoid a high false active rate at the same time, the CMC
with a high $\gamma$ value is preferred.

2. When the effects of the active variables are weak (Model 2), setting $\gamma$ to a high value is too aggressive and risks a high false inactive rate. At $\gamma = 0.8$, the CMC worked well in large sample situations ($n = 550, 1100$) with zero false inactive rate and low false active rate. But at the smaller sample size of $n = 110$, the CMC has a substantially higher false inactive rate than the CV. This is undesirable. The CV has nearly zero false inactive rates in this case but a high false active rate. Since it is usually more important to have a low false inactive rate, the CV is preferred.

3. It is helpful to view the CMC and CV not as competitors but as complements to each other. The value of the tuning parameter $t$ corresponding to the CMC with $\gamma = 0.95$ may be considered the lower limit and that given by the optimal 10-fold cross-validation may be considered the upper limit. One may choose a $t$ value between these two limits depending on the priority regarding the two rates.

We did not include numerical results about the lasso-CMC estimator $\tilde{\beta}_{\gamma,n}$ for $\beta^t$. It is as expected more biased than the lasso estimator based on the cross-validation criterion, especially for smaller sample sizes, as the CMC corresponds to a smaller tuning parameter than the one chosen by the cross-validation criterion. In general, $\tilde{\beta}_{\gamma,n}$ should only be used as a classifier, not as an estimator, unless $n$ is very large. We recommend users of the CMC to perform the least squares regression on the active variables identified by the CMC. Such a follow-up least squares regression has been suggested by a number of authors [e.g., Efron et al. (2004), Hastie, Tibshirani and Wainwright (2015)]. In situations where the number of active variables is expected to be small, say less than 30, one may apply the lasso-CMC variable selection repeatedly as a variable screening method until the total number of variables remaining becomes manageable by the best subset selection. Then, apply the CMC with the best subset selection on these variables to find the final model. The CMC estimator from the best subset selection can be used as an estimator for $\beta^t$. 
3 Concluding remarks

Apart from its motive of minimizing the size of the subset under a constraint on the likelihood, the CMC has two other simple interpretations. [1] The proof of Theorem 2.1 shows it amounts to choosing among the least squares estimators for the $2^p$ models that are in the confidence region (13) the one that has the most zeros. [2] It is also equivalent to a likelihood-ratio test procedure for determining the cardinality of $\beta^*_t$ at significance level $\alpha \leq 1 - \gamma$. To see this, consider testing the sequence of hypotheses $H_0^j : |\beta^*_t| = j$ for $j = 1, \ldots, p$ sequentially with the likelihood-ratio test and stopping at the first $j$ value for which the null hypothesis is accepted (denote this value by $j^*$). Here, “accepted” means the maximum of $h(\beta)$ in (5) under $H_0^{j^*}$ is less than $\kappa$ or equivalently there exists a $j^*$-variable model for which the least squares estimator is in the confidence region $C_\gamma$. The CMC amounts to choosing the $j^*$-variable model that maximizes the likelihood among all $j^*$-variable models. This connection to the likelihood-ratio test shows the CMC is accurate when used with the best subset selection because the likelihood-ratio test is in general powerful, as such the accepted $H_0^{j^*}$ has a good chance of being true.

Although the likelihood-ratio test interpretation is not strictly valid when the CMC is applied to the lasso because the lasso does not optimize over all subsets under each $H_0^j$, numerical evidences such as the examples in the last section show that the CMC is also accurate, provided the variables are not strongly correlated. This is because the lasso and the best subset selection are using similar subsets for variable selection under the condition, so the likelihood-ratio test interpretation is approximately valid. To illustrate this point, consider the extreme case where the $p$ variables are uncorrelated and variable selection performed in the standardized variable space where “uncorrelated” is equivalent to “orthogonal”. In this case, $\mathcal{F}_L$ coincides with $\mathcal{F}_B$; the lasso and the best subset selection are choosing from the same sets of models, so the likelihood-ratio test interpretation of the CMC is strictly valid. In practice, perfectly uncorrelated variables will not appear in sparse models and there may be strong correlations among variables, possibly spurious strong correlations, when there are a large number variables. We observed that so long as the underlying active variables are not strongly correlated, the CMC is accurate.
Finally, in situations where \( p \) is not large so that the best subset selection is feasible, it makes a powerful combination with the CMC for variable selection. Unlike the computational methods such as the forward selection and backward selection which test the significance of variables one at a time, the best subset selection and the CMC combination implicitly tests the significance of all combinations of the variables at the same time which gives this combination a better chance of finding the true model. A larger numerical study on the performance of the CMC, aimed at producing a comprehensive comparison involving several commonly used variable selection criteria and variable selection methods, is in progress. Final results of this study will be reported elsewhere but preliminary results indicate that the CMC is very competitive, especially in the sparse model setting.

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