Some Geometry and Analysis on Ricci Solitons

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Abstract

The Bakry-Émery Ricci tensor of a metric-measure space \((M, g, e^{-f} dv_g)\) plays an important role in both geometric measure theory and the study of Hamilton’s Ricci flow. Under a uniform positivity condition on this tensor and with bounded Ricci curvature we show the underlying space has finite \(f\)-volume. As a consequence such manifolds, including shrinking Ricci solitons, have finite fundamental group. The analysis can be extended to classify shrinking solitons under convexity or concavity assumptions on the measure function.

1 Introduction

In this paper we study smooth metric measure spaces \((M, g, f)\), where \(g\) is a smooth complete metric on an \(n\) dimensional manifold \(M\) and \(f\) is a smooth real valued function. We associate to \(M\) the measure \(e^{-f} dv_g\), where \(dv_g\) is the Riemannian volume form on \(M\). The interest of this paper is in studying the Bakry-Émery Ricci tensor \(R_{cf} \equiv Rc + \nabla^2 f\), where \(Rc\) is the usual Ricci tensor and \(\nabla^2 f\) is the hessian of \(f\). We refer the reader to [1] and to Lott’s paper [2] for more information.

Manifolds with constant Bakry-Émery tensor have come to be known as Ricci solitons, and play an important role in the Ricci Flow as they are the result of certain singularity dilations around finite time singularities of the Ricci Flow (see [3] and [4]). With this in mind we will be interested in studying the following

**Definition.** Let \((M, g, f)\) be a smooth metric measure space. We call \(M\) a Ricci soliton if \(R_{cf} = Rc + \nabla^2 f = \lambda g\), where \(\lambda \in \mathbb{R}\). We say the soliton is shrinking, steady, or expanding when \(\lambda > 0, = 0, < 0\), respectively.

Our first result is a form of Myers Theorem for metric measure spaces with uniform positive lower bounds on the \(R_{cf}\) tensor. It is well known that such manifolds need not be compact, and in fact some of the most interesting examples are

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those which are not. Hence bounds on the diameter are not reasonable under such a constraint, however in the following we show that such manifolds do have finite $f$-volume.

**Theorem 1.** Let $(M, g, f)$ be complete with bounded Ricci curvature. Assume $Rc_f \geq \lambda g$ with $\lambda > 0$. Then the measure $e^{-f}dv_g$ is finite, and consequently $M$ has finite fundamental group.

**Remark.** The finiteness of the fundamental group was proved by Lott in [2] under the additional assumption that $M$ is compact.

Next we wish to use the above to understand the structure of shrinking solitons under some simplified conditions. We will prove the following:

**Theorem 2.** Let $(M, g, f)$ be complete with $Rc + \nabla^2 f = \lambda g$, $\lambda > 0$. Assume $Rc \geq 0$ and that $f$ is either convex or concave. Then $(M, g)$ is isometric to a finite quotient of $E \times \mathbb{R}^k$ where $E$ is a compact simply connected Einstein manifold.

**Remark.** The point of the above is that the soliton structure on such an $M$ must be trivial. The noncompactness of $M$ must result purely from an isometric $\mathbb{R}^n$ factor, and on the compact component $f$ behaves trivially. This is not true in the case $\lambda = 0$, and there are nontrivial soliton structures on such manifolds (for instance the cigar and Bryant solitons).

To prove the above we introduce the notion of the $f$-Laplacian of a function $u$. The motivation comes directly from the standard Laplace-Beltrami operator, which is defined as $\Delta = \nabla^* \nabla$ with $\nabla^*$ the adjoint of the covariant derivative with respect to the Riemannian volume form. Similarly we define:

**Definition.** The $f$-Laplacian of a function $u$ is defined by $\triangle_f u \equiv \nabla^* f \nabla u$, where the adjoint is taken with respect to the $f$-measure $e^{-f}dv_g$.

Under a positivity assumption on $Rc_f$ we have the following estimate and Liouville type theorem:

**Theorem 3.** Let $(M, g, f)$ be complete with bounded Ricci curvature. Assume $Rc_f \geq \lambda g$ with $\lambda > 0$. Let $u : M \to \mathbb{R}$. Then

1) If $\triangle_f u = 0$ then $\exists \alpha > 0$ such that if $|u| \leq e^{\alpha d(x,p)^2}$ for some $p \in M$, then $u = \text{constant}$.

2) If $\triangle_f u \geq 0$ and $u \leq C'$ for some $C' \in \mathbb{R}$, then $u = \text{constant}$.

The above situation is not typical if we weaken the geometric constraint on $Rc_f$ a little to just $Rc_f \geq 0$. Just for instructional sake we show that not only is the above not true, but that comparison estimates in general, in particular any Harnack type estimate, must depend on $f$ itself and thus $Rc_f \geq 0$ is not a sufficient condition to control many a priori estimates of $\triangle_f$.
Theorem 4. There exists $(M, g, f^k)$ complete with $u^k : M \to \mathbb{R}$, $k \in \mathbb{N}$ such that
1) $Rc + \nabla^2 f^k \geq 0$ for any $k$.
2) $\Delta f^k u^k = 0$ on $M$ with $u^k > 0$.
3) There exists $x, y \in M$ fixed such that $\lim_k u^k(x) = \infty$.

Remark. The above example even has $Rm \geq 0$. So even under a combination of nonnegativity assumptions on $Rm$ and $R_{\gamma}f$ we do not have complete a priori control over solutions of the $f$-Laplacian.

2 Proof of Theorems 1

The key to understanding the metric measure spaces $(M, g, f)$ under geometric assumptions on the $R_{\gamma}f$ is to control and understand the behavior of $f$. On that note we begin by proving the following estimate for $f$.

Lemma 1. Let $(M, g, f)$ be complete with $|Rc| \leq C$ and $Rc + \nabla^2 f \geq \lambda g$ for $\lambda \in \mathbb{R}$. Let $\gamma : [0, L] \to M$ be a weakly minimizing unit speed geodesic, $\gamma(0) = p \in M$. Then $\nabla_{\dot{\gamma}}f(L) \geq \lambda L + a$ and $f(L) \geq \lambda L + aL + b$, where $a = a(\lambda, n, C, f|_{B(p, 2)})$ and $b = b(f|_{B(p, 2)})$ and $B(p, 2)$ is the geodesic ball of radius 2 centered at $p$.

Remark. Note in the above that $a$ and $b$ do not depend on $L$, and hence the estimate on $f$ actually holds for all $t \in [0, L]$

Proof. First assume $L \geq 2$. Let $E^i(p)$ be an orthonormal basis at $p$ with $E^n = \dot{\gamma}$. Define $E^i(t)$ as the parallel transport of $E^i$ over $\gamma(t)$. Let $h : [0, L] \to \mathbb{R}$ be Lipschitz with $h(0) = h(L) = 0$ and $Y^i(t) = h(t)E^i(t)$. Now for some $i$ let $\gamma_s(t) : [0, L] \to M$ be a 1-parameter family of curves with $\gamma_0 = \gamma$ and $\frac{d}{ds}\gamma_s = Y^i$ its variation.

If $l(\gamma_s)$ is defined as the length of $\gamma_s$ we have, because $\gamma$ is a weakly minimizing geodesic, by the usual second variation formula that

$$0 \leq \frac{d^2}{ds^2}l(\gamma_s) = \int_{0}^{L} |\nabla_{\dot{\gamma}}Y^i|^2 - R(Y^i, \dot{\gamma}, Y^i, \dot{\gamma})dt$$

$$= \int_{0}^{L} (h')^2|E^i|^2 + h^2|\nabla_{\dot{\gamma}}E^i|^2 - h^2R(E^i, \dot{\gamma}, E^i, \dot{\gamma})dt$$

$$= \int_{0}^{L} (h')^2 - h^2R(E^i, \dot{\gamma}, E^i, \dot{\gamma})dt.$$ 

Because this holds for each $i$ we can sum and use our assumption that $Rc \geq -\nabla^2 f + \lambda g$ to get

$$\lambda \int_{0}^{L} h^2dt \leq (n - 1) \int_{0}^{L} (h')^2dt + \int_{0}^{L} h^2\nabla^2_{\dot{\gamma}}f dt.$$
Now we define \( h \) by the formula

\[
h(t) = \begin{cases} 
  t & 0 \leq t \leq 1 \\
  1 & 1 \leq t \leq L - 1 \\
  L - t & L - 1 \leq t \leq L
\end{cases}
\]

Inserting yields

\[
\lambda (L - \frac{4}{3}) \leq (n - 1)(2) + \int_0^L \nabla^2_{\dot{\gamma}} f \, dt - \int_0^L (1 - h^2) \nabla^2_{\dot{\gamma}} f \, dt.
\]

Since \( \gamma \) is a unit speed geodesic and \( \nabla^2_{\dot{\gamma}} f \geq \lambda - C \) we get

\[
\lambda L - 2\left(\frac{2}{3} \lambda + (n - 1) + \frac{2}{3}(C - \lambda)\right) + \nabla_{\dot{\gamma}} f(0) \leq \nabla_{\dot{\gamma}} f(L)
\]

or

\[
\lambda L + a \leq \nabla_{\dot{\gamma}} f(L) \Rightarrow \frac{1}{2} L^2 + aL + b \leq f(L).
\]

Now this is for \( L \geq 2 \). If we replace \( a \) and \( b \) by \( a' = a - 2\lambda - \sup_{B(p,2)}|\nabla f| \) and \( b' = b - 2\lambda - 2a' - \sup_{B(p,2)}|f| \) then our inequality holds for all \( L \).

\[\square\]

The main application of this estimate is the following corollary, which is a direct consequence of the remark following the lemma. It gives us a global lower bound on \( f \) as a quadratic of the distance function.

**Corollary 1.** Let \((M, g, f)\) be a complete manifold with \(|Rc| \leq C\) and \(Rc + \nabla^2 f \geq \lambda g\) for \( \lambda \in \mathbb{R} \), and let \( p \in M \). Then \( \forall x \in M \) we have \( f(x) \geq \lambda d(x, p)^2 + ad(x, p) + b \), where \( d(x, p) \) is the distance function to \( p \) and \( a, b \) depend only on the constants \( \lambda, n, C \) and \( f|_{B(p,2)} \)

The most important special case of the above is when \( \lambda > 0 \). The quadratic growth estimate on \( f \) in this case immediately gives us the following useful facts:

**Corollary 2.** Let \((M, g, f)\) be complete with bounded Ricci curvature and satisfy \( Rc + \nabla^2 f \geq \lambda g \) with \( \lambda > 0 \). Then \( f \) is bounded below and proper.

We may now prove Theorem 1:

**Proof of Theorem 1.** Using exponential coordinates at \( p \) we have, since \( Rc \geq -C \), by the standard comparison that \( \sqrt{\det g} \leq (\sinh(\sqrt{C}r))^{n-1} \lesssim e^{(n-1)\sqrt{C}r} \). Here by definition it is understood that we say \( s \lesssim t \) if \( s \leq At \) and \( A \) is a constant depending only on the dimension and other fixed variables, in this case just the dimension. Now integrating in the tangent space, where it is understood that \( \sqrt{\det g}(x) = 0 \) if \( x \) is outside the segment domain of \( p \), we have

\[
Vol_f(M) = \int_{s_{n-1}} \int_0^\infty e^{-f} \sqrt{\det g} dr ds_{n-1} \lesssim \int_0^\infty e^{-\frac{1}{2}r^2 + (a + (n-1)\sqrt{C})r + b} dr < \infty.
\]

To see that the fundamental group is finite we lift to the universal cover \( \tilde{M} \). Apply the above to see \( \tilde{M} \) must also have finite \( f \)-volume, because it too satisfies the geometric conditions of the theorem. But this is impossible unless the order of \( \pi_1(M) \) is finite. \[\square\]
3 Proofs of Theorems 2 and 3

To prove Theorem 2 we begin by proving some results involving the $f$-Laplace operator.

**Proof of Theorem 3 (1).** Let $x \in M$ be arbitrary. Note by multiplying by $e^{-f}$ we get

$$\nabla^i(e^{-f}\nabla_i u) = 0.$$ (1)

Let $\phi : M \to \mathbb{R}$ be a cutoff function with

$$\phi = \begin{cases} 1 \text{ on } B(x, 1) \\ 0 \leq \phi \leq 1 \text{ on } B(x, 1 + r) - B(x, 1) \\ 0 \text{ on } M - B(x, 1 + r) \end{cases}$$

where $r > 0$ and $|\nabla \phi| \leq \frac{C}{r}$ for some $C$. Multiplying the above by $\phi^2 u$ and integrating we get

$$-\int (2\phi u \nabla^i \phi \nabla_i u + \phi^2 |\nabla u|^2)e^{-f}dv_g = 0$$

$$\int \phi^2 |\nabla u|^2 e^{-f}dv_g = -2 \int (\phi u \nabla^i \phi \nabla_i u)e^{-f}dv_g$$

$$\leq \int \left(\frac{1}{2}\phi^2 |\nabla u|^2 + 2u^2 |\nabla \phi|^2\right)e^{-f}dv_g$$

so that

$$\int_{B(x,1)} |\nabla u|^2 e^{-f}dv_g \leq 4 \int M u^2 |\nabla \phi|^2 e^{-f}dv_g$$

$$\leq \frac{4C^2}{r^2} \int_{B(x,1)} u^2 e^{-f}dv_g \leq \frac{4C^2}{r^2} \int M u^2 e^{-f}dv_g.$$  

But let $\alpha < \frac{A}{4}$, and thus $u^2(x) \lesssim e^{2\alpha d(x,p)}$. So in exponential coordinates we compute

$$\int M u^2 e^{-f}dv_g \leq \int_{S^{n-1}} \int_0^\infty e^{-(\frac{A}{2}-2\alpha)r^2+ar+b}drds_{n-1} < \infty$$

for some constants $a$ and $b$. Thus we can tend $r \to \infty$ to get

$$\int_{B(x,1)} |\nabla u|^2 e^{-f}dv_g = 0$$

Since $x$ was arbitrary, $|\nabla u| = 0$ and thus $u$=constant. \hfill \Box

**Proof of Theorem 3 (2).** This is much the same. Since $u$ is bounded above we can assume, by adding a constant, that $\sup u = 1$. Let $u^+(x) = \max(u(x),0)$. Let $x \in M$ such that $u(x) > 0$ and $\phi$ as in the last part with center $x$. Then our equation $\nabla^i(e^{-f}\nabla_i u) \geq 0$ gives

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\[-\int (2\phi u^+ \nabla^i \phi \nabla_i u + \phi^2 \nabla^i u^+ \nabla_i u)e^{-f}dv_g \geq 0\]

so that
\[\int_M \phi^2 |\nabla u^+|^2 e^{-f}dv_g \leq \frac{4C^2}{r^2} \int_M (u^+)^2 e^{-f}dv_g\]

But \(u^+\) is bounded and \(\int_M e^{-f}dv_g\) is finite. So we may limit out, using monotone convergence, to get \(\int_M |\nabla u^+|^2 e^{-f}dv_g = 0\). So \(u^+\) is constant. Since \(u(x) > 0\), \(u\) is constant.

Proof of Theorem 4. Let \(M = \mathbb{R} \times S^n\) with the standard product metric. Let \(f^k(t, s) = kt\) for \(k\) a constant, \(t \in \mathbb{R}\) and \(s \in S^n\). Clearly \(Rc + \nabla^2 f \geq 0\). Looking for a solution of \(\Delta f^k u^k = 0\) which is a function of only \(t\) as well we find \(u^k(t)\) must satisfy the ode \(u_{tt} - ku_t = 0\). So \(u^k(t, s) = e^{kt}\) is a solution of our equation. Notice \(u^k > 0\). Let \(x\) be on the \(t = 1\) slice and \(y\) on the \(t = 0\) slice, then we see that \(\frac{u^k(x)}{u^k(y)} = e^k\). Let \(k\) tend to \(\infty\).

Remark. The key point in the above is that \(f\) is only well defined up to a linear function on \(M\), thus if \(f\) itself does not growth faster than linearly then we can add large linear terms to \(f\) which will have a significant impact on the solutions of the \(f\)-Laplacian.

Now we apply the above to prove Theorem 3:

Proof of Theorem 3. First we assume \(f\) is convex. Then \(Rc \leq Rc + \nabla^2 f = \lambda g\).

The following computation is useful:
\[
\nabla^i R_{ij} + \nabla^i \nabla_j f = 0
\]
\[
\frac{1}{2} \nabla_j R + \nabla_j (-R - n\lambda) + Rc_{jk} \nabla^k f = 0
\]
\[
\nabla_i R = 2Rc_{ij} \nabla^j f
\]

Now if we take the divergence of this we get
\[
\Delta_f R = 2(\lambda R - |Rc|^2)
\]

(3)

A similar computation gives us
\[
\Delta_f |Rc|^2 = 2|\nabla Rc|^2 + 4(\lambda |Rc|^2 - R_{ijkl} R^{ik} R^{jl})
\]

(4)

Now if \(\partial_i\) is an eigenbasis for \(Rc\) we write the rhs of (3) as \((\lambda R - |Rc|^2) = \Sigma R_{ii}(\lambda - R_{ii}) \geq 0\) under our assumptions. In particular the scalar curvature is a bounded subsolution to \(\Delta_f\), and thus must be constant. Plugging this in we see that \(\Sigma R_{ii}(\lambda - R_{ii}) = 0\), which under our assumptions implies that each term is zero and thus every eigenvalue of \(Rc\) is either 0 or \(\lambda\). By continuity the number of 0 eigenvalues
must be constant, and thus \(|Rc| = \text{const.} \) So writing
\(\langle |Rc|^2 - R_{ijkl} R^{ijkl} \rangle = \sum p,q \text{sec}(\partial_p, \partial_q) R_{pp}(\lambda - Rc_{qq}) = 0\) we then see that we must have \(|\nabla Rc| = 0\) from (4).

By using deRham’s Theorem we then have an isometric splitting of the universal cover (which is a finite cover by the previous theorem) into \(E \times N\), where \(E\) has Einstein constant \(\lambda\) and \(N\) is simply connected and Ricci flat. By restricting \(f\) to \(N\) we see \(N\) has a soliton structure. We finally show that \(N\) is \(R^k\).

We know by Ricci flatness that on \(N\), \(\nabla^2 f = \lambda g\). Now by Lemma 1 we know \(f\) always has a global minimum point, say \(p \in N\). If \(x \in N\) and \(\gamma\) a geodesic connecting \(x\) to \(p\) we see by integration that \(\nabla f(x) = \lambda d(x, p)\) and \(f(x) = \frac{1}{2} d(x, p)^2 + f(p)\). Hence \(f\) has a unique nondegenerate minimum point at \(p\). Now we compute
\[
R_{ijkl} \nabla^4 f = (\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f) = \nabla_j R_{ik} - \nabla_i R_{jk} = 0. \tag{5}
\]

In particular, because \(p\) is a nondegenerate critical point, for any unit vector \(X \in T_p M\) we can find \(x \to p\) such that \(\frac{\nabla f(x)}{|\nabla f|} \to X\) (just use Taylor’s theorem in normal coordinates to see this). Dividing both sides of (5) by \(|\nabla f|\), taking \(X = \partial_i\) and limiting out we get that \(Rm(p) = 0\). A final computation now gives us that
\[
\nabla^2 f|RM|^2 = \nabla^p f \nabla_p|RM|^2 = 2 \nabla^p f R^{ijkl} \nabla_p R_{ijkl}
\]
\[
= -2 \nabla^p f R^{ijkl} (\nabla_i R_{jk} + \nabla_j R_{pi})
\]
\[
= -2 R^{ijkl} (\nabla_i (\nabla^p f R_{jk}) + \nabla_j (\nabla^p f R_{pi}) - R_{jk} \nabla_i \nabla^p f - R_{pi} \nabla_j \nabla^p f)
\]
\[
= 4 \lambda R^{ijkl} (R_{ijkl}) = -4 \lambda |RM|^2 \leq 0
\]

Our explicit formula for \(f\) tells us that the negative gradient flow from any \(x \in N\) converges to \(p\), and hence from the above \(|RM|\) takes a maximum at \(p\). But we showed \(Rm(p) = 0\). Hence \(Rm = 0\). Since \(N\) is simply connected, \(N\) is isometric to \(R^k\).

If we now instead assume that \(f\) is concave the situation is more simple. We get \(Rc \geq Rc + \nabla^2 f = \lambda g\). In particular \(M\) is compact and so \(f\) must be a constant and so we are done. \(\square\)

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