ON OHTSUKI’S INVARIANTS OF INTEGRAL HOMOLOGY 3-SPHERES, I

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Abstract. An attempt is made to conceptualize the derivation as well as to facilitate the computation of Ohtsuki’s rational invariants $\lambda_n$ of integral homology 3-spheres extracted from Reshetikhin-Turaev $SU(2)$ quantum invariants. Several interesting consequences will follow from our computation of $\lambda_2$. One of them says that $\lambda_2$ is always an integer divisible by 3. It seems interesting to compare this result with the fact shown by Murakami that $\lambda_1$ is 6 times the Casson invariant. Other consequences include some general criteria for distinguishing homology 3-spheres obtained from surgery on knots by using the Jones polynomial.

1. Introduction

In [O1], Ohtsuki extracted a series of rational topological invariants of oriented homology 3-spheres from the $SU(2)$ quantum invariants of Reshetikhin and Turaev [RT]. Physically, they correspond to the coefficients of the asymptotic expansion of Witten’s Chern-Simons path integral at the trivial connection as shown by Rozansky [R1,2]. The original derivation of these invariants, though, seems too complicated to make any practical computation possible. We will present here an attempt to conceptualize as well as to simplify the derivation of Ohtsuki’s invariants. The simplified derivation will lead to formulae for Ohtsuki’s

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invariants, which are very practical and straightforward when it comes to actual computation. Various interesting consequences will follow from our formulae for Ohtsuki’s invariants.

To be more precise, Ohtsuki [O1] defined a formal power series

\[ \tau(M) = 1 + \sum_{n=1}^{\infty} \lambda_n (t - 1)^n \]

with rational coefficients for every oriented integral homology 3-sphere \( M \). It is, in our terminology, the Fermat limit of Reshetikhin-Turaev SU(2) quantum invariants \( \{ \tau_r(M) \} \) of \( M \) (See the definition in Section 3). The notion of Fermat limits has appeared implicitly in many places in the literature on quantum 3-manifold invariants. But it seems that Murakami was the first to use it effectively in identifying \( \lambda_1 \) with 6 times the Casson invariant [Mu].

Ohtsuki’s invariants \( \{ \lambda_n \} \) are closely related with the developing theory of finite type 3-manifold invariants [Ha, G, GLe1, GLe2, GLi, GO, O2]. As we will see in the sequel to this paper, some special properties of the colored Jones polynomial of algebraically split links, i.e., links with all pairwise linking numbers zero, play a key role in this relationship.

One of the consequences of our computation of Ohtsuki’s invariants concerns their integrality. In particular, we have

**Theorem 1.1.** \( \lambda_2(M) \in 3\mathbb{Z} \) for all integral homology 3-spheres \( M \).

This theorem, together with the fact that \( \lambda_1 \in 6\mathbb{Z} \), suggests the following conjecture.

**Conjecture 1.1.** \( n! \cdot \lambda_n(M) \in 6\mathbb{Z} \) for all integral homology 3-spheres \( M \).
See Section 4 for a relevant conjecture about the integrality of coefficients of a certain variation of the Jones polynomial on algebraically split links (Conjecture 4.1). We will say more about this conjecture in the course of our discussion.

The Casson invariant for homology 3-spheres comes originally from an algebraic counting of conjugacy classes of irreducible representations of the fundamental group into $SU(2)$ [AM]. This leads to a general criterion for detecting knots with property $P$ using the Alexander or Jones polynomial. To explain this in some detail, we denote by $J(K, t)$ the 1-variable Jones polynomial of a knot $K$ in $S^3$. We write

$$J(K, e^h) = 1 + \sum_{n=2}^{\infty} (-1)^n \frac{v_n(K)}{n!} h^n.$$ 

We will also denote by $S^3_{K, p/q}$ the 3-manifold obtained from Dehn surgery on the knot $K$ with the surgery coefficient $p/q$. Then, using the Casson invariant, we may conclude that if $v_2(K) \neq 0$, $S^3_{K, p/q}$ will never be a homotopy 3-sphere unless $p/q = 1/0$. We have several similar criteria for distinguishing homology 3-spheres obtained from surgery on knots by using other coefficients of the Jones polynomial in relation with Ohtsuki’s invariant $\lambda_2$. For example, we have the following theorem.

**Theorem 1.2.** Let $K^*$ be the mirror image of $K$. If $v_3(K) \neq 0$, then $S^3_{K, 1/n} \neq S^3_{K^*, 1/n}$ as unoriented manifolds unless $n = 0$.

See Corollary 5.3. Needless to say, we don’t know what kind of geometrical or topological obstruction the invariant $\lambda_2$ represents which prevents these two 3-manifolds $S^3_{K, 1/n}$ and $S^3_{K^*, 1/n}$ from being homeomorphic to each other for $n \neq 0$. 
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2. Fermat functions and their residues

Fermat’s little theorem tells us that for an integer \( a \) and a prime \( r \) such that \( a \not\equiv 0 \pmod{r} \), \( a^{r-1} \equiv 1 \pmod{r} \). Let us view this theorem from a different angle.

Suppose we have a function \( f \) which sends each prime \( r \) to \( f(r) \in \mathbb{Z}_r \), where \( \mathbb{Z}_r \) is the ring of \( r \)-adic integers. Suppose \( a \in \mathbb{Z}_r \) be an \( r \)-adic integer, then \( \bar{a} \) denotes the first digit in the \( r \)-adic expansion of \( a \). We will call \( f(r) \) a Fermat function if there is a rational number \( \lambda \) independent of \( r \) such that

\[
 f(r) \equiv \lambda \quad \pmod{r}
\]

(2.1)

i.e. \( \overline{f(r)} = \overline{\lambda} \) for all sufficiently large primes \( r \). Here we should notice that for the fixed rational number \( \lambda \), we have \( \lambda \in \mathbb{Z}_r \) if \( r \) is sufficiently large. So (2.1) makes sense for sufficiently large \( r \).

If \( f \) is a Fermat function, we will call the rational number \( \lambda \) the residue of \( f \) and it is denoted by \( \text{Res}(f) \).

Lemma 2.1. The residue of a Fermat function is unique.

Proof: If there is another rational number \( \lambda' \) such that \( f(r) \equiv \lambda' \pmod{r} \) for all sufficiently large primes \( r \), then the numerator of the rational number \( \lambda - \lambda' \) will be divisible by all sufficiently large primes \( r \). Therefore, \( \lambda - \lambda' = 0 \), i.e. \( \lambda = \lambda' \). □

With these terminologies, Fermat’s Little Theorem can be rephrased by saying that if \( a \not\equiv 0 \pmod{r} \), then the function \( f(r) = a^{r-1} \) is a Fermat
function with \( \text{Res}(f) = 1 \). There are certainly plenty of other Fermat functions. For example, the function \( f(r) = (r - 1)/2 \) is a Fermat function whose residue is \(-1/2\). Also, the function \( f(r) = (r - 1)! \) is a Fermat function because of Wilson’s theorem, which says that \((r - 1)! \equiv -1 \mod r\) [HW]. On the other hand, an example of a non-Fermat function is given by \( f(r) = (\frac{r-1}{2})! \). See [HW] for a discussion of the residue of \((\frac{r-1}{2})!\), which turns out to be dependent on \( r \).

The following lemma comes directly from the definition.

**Lemma 2.2.** Suppose that \( f \) and \( g \) are Fermat functions. Then

1. for rational numbers \( \alpha \) and \( \beta \), \( \alpha f + \beta g \) is a Fermat function whose residue is \( \alpha \text{Res}(f) + \beta \text{Res}(g) \);
2. \( f \cdot g \) is a Fermat function whose residue is \( \text{Res}(f) \text{Res}(g) \);
3. if \( \text{Res}(g) \neq 0 \), then \( f/g \) is a Fermat function whose residue is \( \text{Res}(f)/\text{Res}(g) \).

Thus, in particular, every polynomial function of \( r \) with rational coefficients is a Fermat function whose residue is its constant term. And every rational function with rational coefficients is a Fermat function if the constant term of the denominator is not zero. Its residue will be the value of the function at \( r = 0 \).

The following examples of Fermat functions and the calculation of their residues are very important to our discussion of Ohtsuki’s invariants. So we single them out first. We will use the notation

\[
\mathbb{Z}_{(m)} = \begin{cases} 
\mathbb{Z}[\frac{1}{m}, \frac{1}{m-1}, \ldots, \frac{1}{m-t}] & \text{for } m \geq 3, \\
\mathbb{Z} & \text{for } m \leq 2
\end{cases}
\]

with \( m \) a positive integer.
Example 2.1. For a fixed integer \( k \), the function
\[
D_k(r) = \frac{\left(\frac{r-1}{2}\right)!}{\left(\frac{r-1}{2} - k\right)!}
\]
is a Fermat function whose residue is given by
\[
\text{Res}(D_k) = \begin{cases} 
(-\frac{1}{2})(-\frac{1}{2} - 1) \cdots (-\frac{1}{2} - (k - 1)) \in \mathbb{Z}_{(3)} & \text{for } k > 0, \\
1 & \text{for } k = 0, \\
\frac{1}{(-\frac{1}{2} + 1) \cdots (-\frac{1}{2} - k)} \in \mathbb{Z}_{(-2k)} & \text{for } k < 0.
\end{cases}
\]

Example 2.2. We denote by \( \sigma_j(x_1, \ldots, x_n) \) \((1 \leq j \leq n)\) the \( j \)-th elementary symmetric polynomial in the indeterminates \( x_1, \ldots, x_n \), and \( \sigma_0(x_1, \ldots, x_n) = 1 \). And we will also need \( s_j(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^j \).

Lemma 2.3. \( s_j(1, 2, \ldots, n - 1) \) is a polynomial in \( n \) with coefficients in \( \mathbb{Z}_{(j+2)} \), and \( s_j(1, 2, \ldots, n - 1) \equiv 0 \) mod \( n \).

Proof: By the Euler-Maclaurin formula for \( s_j \) [HW], we have
\[
s_j(1, 2, \ldots, n - 1) = \sum_{i=0}^{j} \frac{1}{j + 1 - i} \binom{j}{i} n^{j+1-i} \beta_i,
\]
where \( \beta_0 = 1, \beta_1 = -1/2, \) and \( \beta_{2i} = (-1)^{i-1}B_i, \beta_{2i+1} = 0 \) for \( i \geq 1 \) (\( B_i \) is the Bernoulli number). We also have von Staudt’s theorem which determines the fractional part of \( B_i \) [HW]. It follows that \( \beta_i \in \mathbb{Z}_{(i+2)} \).

This implies the lemma. □

For a pair of non-negative integers \( l, m \), we define
\[
H_{l,m}(r) = \frac{\left(\frac{r-1}{2}\right)!}{\left(\frac{r-1}{2} - l + m\right)!} (-1)^{m+1} \sigma_m(1, 2, \ldots, \frac{r-1}{2} - l + m - 1) - 1.
\]

Lemma 2.4. \( H_{l,m} \) is a Fermat function. Moreover, let
\[
g'_{l,m} = \text{Res}(H_{l,m}).
\]
then \( g_{l,m} \in \mathbb{Z}_{(m+2)\left[\frac{1}{2}\right]} \).

Remark: The proof of Lemma 2.4 will also provide us a recursive formula to calculate \( g_{l,m} \). See Formula (2.5).

Proof: We first derive a formula for \( \sigma_j(1,2,\ldots,n-1) \) using a trick of Lagrange. We will abbreviate \( \sigma_j = \sigma_j(1,2,\ldots,n-1) \).

We have

\[
(x - 1)(x - 2) \cdots (x - (n - 1)) = x^{n-1} - \sigma_1 x^{n-2} + \cdots + (-1)^{n-1} \sigma_{n-1}.
\]

If we multiply both sides by \( x \) and change \( x \) into \( x - 1 \), we obtain

\[
(x - 1)^n - \sigma_1 (x - 1)^{n-1} + \cdots + (-1)^{n-1} \sigma_{n-1} (x - 1) = (x - 1)(x - 2) \cdots (x - n) = (x - n)(x^{n-1} - \sigma_1 x^{n-2} + \cdots + (-1)^{n-1} \sigma_{n-1}).
\]

Equating coefficients, we obtain

\[
\sigma_1(1,2,\ldots,n-1) = \binom{n}{2},
\]

\[
2\sigma_2(1,2,\ldots,n-1) = \binom{n}{3} + \binom{n-1}{2}\sigma_1,
\]

\[
3\sigma_3(1,2,\ldots,n-1) = \binom{n}{4} + \binom{n-1}{3}\sigma_1 + \binom{n-2}{2}\sigma_2,
\]

\[
\cdots \cdots \cdots,
\]

\[
j\sigma_j(1,2,\ldots,n-1) = \sum_{i=0}^{j-1} \binom{n-i}{j+1-i}\sigma_i,
\]

\[
\cdots \cdots \cdots,
\]

\[
(n-1)\sigma_{n-1}(1,2,\ldots,n-1) = \sum_{i=0}^{n-2} \sigma_i.
\]

Therefore, we see inductively that \( \sigma_j(1,2,\ldots,n-1) \) is a polynomial of \( n \) with coefficients in \( \mathbb{Z}_{(j+2)} \) and divisible by \( n(n-1) \cdots (n-j) \).
Using these formulae for $\sigma_j$, we have

$$
\frac{j\sigma_j(1, 2, \ldots, n-1)}{n(n-1) \cdots (n-j)} = \frac{1}{(j+1)!} + \sum_{i=1}^{j-1} \left( \frac{n-i}{j+1-i} \right) \frac{\sigma_i(1, 2, \ldots, n-1)}{n(n-1) \cdots (n-j)}
$$

(2.4)

Suggested by (2.4), we define a series of rational numbers $\sigma_i^{l,m} \in \mathbb{Z}_{(m+2)}[\frac{1}{2}]$ for $i \geq 1$ recursively as follows:

$$
\begin{cases}
\sigma_i^{l,m} = \frac{1}{2}, \\
\sigma_j^{l,m} = \frac{1}{(j+1)!} + \sum_{i=1}^{j-1} \frac{(n-i)}{(j+1-i)!} \cdot \frac{\sigma_i^{l,m}}{i} & \text{for } j \geq 2.
\end{cases}
$$

(2.5)

It is clear that we may identify $\sigma_j^{l,m}$ with the residue of

$$
\frac{j\sigma_j(1, 2, \ldots, n-1)}{n(n-1) \cdots (n-j)}
$$

with

$$
n = \frac{r-1}{2} - l + m.
$$

Then we have

$$
\begin{cases}
g'_{0,0} = -1 \\
g'_{l,0} = (-1)^{-\frac{l}{2}} \cdots (-\frac{1}{2} - l + 1) & \text{if } l \geq 1, \\
g'_{l,m} = (-1)^{m+1} (-\frac{1}{2}) \cdots (-\frac{1}{2} - \frac{3}{2} - l) \frac{\sigma_i^{l,m}}{m} & \text{if } m \geq 1.
\end{cases}
$$

(2.5)

Obviously, Formula (2.5) for $g'_{0,0}$ and $g'_{l,0}$ follows from the formula of $\text{Res}(D_k)$ in Example 2.1. To see Formula (2.5) for $m \geq 1$, we only need to notice that

$$
\frac{(-\frac{1}{2}) \cdots (-\frac{1}{2} - l)}{(-\frac{1}{2} - l + m) \cdots (-\frac{1}{2} - \frac{3}{2} - l)} = \text{Res}(D_{l-m}).
$$

This finishes the proof of Lemma 2.4. □
3. Fermat limits and Ohtsuki’s invariants

We fix some notation about the “quantum parameter” first:

- We will use $r$ to denote an odd prime and $q = e^{2\pi \sqrt{-1}/r}$. The quantum integer
  $$[k] = \frac{q^k - q^{-k}}{q^{1/2} - q^{-1/2}}.$$  

  Note that
  $$[2] = q^{1/2} + q^{-1/2}.$$  

- For $R = \mathbb{Z}$ or $\mathbb{Z}(r)$, $O((q - 1)^k; R)$ stands for a complex number of the form $u(q - 1)^k$ for some $u \in R[q]$.

- If $f(x)$ is a $C^{\infty}$ function, we use $\text{Coeff}_n(f; x - a)$ to denote the coefficient of $(x - a)^n$ in the Taylor expansion of $f$ at $x = a$.

- The Gauss sum is $G_0(q) = \sum_{k=0}^{r-1} q^k$. The weighted Gauss sum is $G_2(q) = \sum_{k=0}^{r-1} k^{2l} q^k$.

To define Ohtsuki’s invariants, we need the following definition.

Let $l(r)$ be an integer valued function of $r$ with

$$\lim_{r \to +\infty} l(r) = +\infty \quad \text{and} \quad l(r) \leq r - 2.$$  

Given a sequence of complex numbers $s_r(q) \in \mathbb{Z}(r)[q]$. Fix a non-negative integer $n$; for sufficiently large $r$, we write

$$s_r(q) = a_{r,0} + a_{r,1}(q-1) + \cdots + a_{r,n}(q-1)^n + \cdots + a_{r,l(r)}(q-1)^{l(r)} + O((q-1)^{l(r)+1}, \mathbb{Z}(r))$$

for some $a_{r,n} \in \mathbb{Z}(r)$.

**Definition 3.1.** The sequence of complex numbers $s_r(q)$ has a Fermat limit, denoted by $\text{f-lim} s_r(q)$, if each $a_{r,n}$, thought of as a function of $r$, is a Fermat function. Let $\lambda_n = \text{Res}(a_{r,n})$. We write

$$\text{f-lim} s_r(q) = \sum_{n=0}^{\infty} \lambda_n (t - 1)^n \in \mathbb{Q}[t - 1].$$
Lemma 3.1. The Fermat limit of $s_r(q)$ is well-defined and if it exists, it is unique.

Proof: Notice that although the expansion $s_r(q)$ is not unique as $q$ is not a free variable, $a_{r,n} \in \mathbb{Z}/r\mathbb{Z}$ is well defined for $0 \leq n \leq l(r)$. This is because of $r = O((q - 1)^{r-1}; \mathbb{Z})$, which follows from the only relation in $\mathbb{Z}(r)[q]$: $\sum_{i=0}^{r-1} q^i = 0$. Therefore, each residue $\lambda_n$ is well-defined regardless of the choice of $a_{r,n}$.

The uniqueness of a Fermat limit now comes from the uniqueness of residues of Fermat functions (Lemma 2.1). □

Remark: Notice that if $\lambda_n$ is an integer if and only if the congruence class $a_{r,n} \in \mathbb{Z}/r\mathbb{Z}$, which is well-defined, stabilizes for $r$ sufficient large.

With all the above understood, we summarize the main results of [Mu] and [O1] into the following theorem.

Theorem 3.1. Let $M$ be an oriented homology 3-sphere, and $\tau_r(M)$ be the quantum $SU(2)$ invariant of Reshetikhin and Turaev [RT] at the $r$-th root of unity normalized as in [KM]. Then

1. (H. Murakami) $\tau_r(M) \in \mathbb{Z}[q]$;
2. (T. Ohtsuki) $\text{f-lim} \tau_r(M) = \sum_{n=0}^{\infty} \lambda_n (t - 1)^n$, and $\lambda_n \in \mathbb{Z}_{2n+2}$;
3. (H. Murakami) $\lambda_0 = 1$, and $\lambda_1 = 6\lambda_C$, where $\lambda_C$ is the Casson invariant.

It follows from Theorem 3.1 (2) and Lemma 3.1 that the $\lambda_n$ are topological invariants of oriented integral homology 3-spheres, and we call $\lambda_n$ the $n$-th Ohtsuki invariant of integral homology 3-spheres.
4. Invariants of Links

Let \( L = K_1 \cup K_2 \cup \cdots \cup K_{\#L} \) be an oriented link in \( S^3 \), where we use \( \#L \) to denote the number of components of \( L \). We will denote the unknot by \( O \), and the empty link by \( \emptyset \).

The Conway polynomial \( \nabla(L; z) \in \mathbb{Z}[z] \) is defined by

\[
\begin{align*}
\nabla(O; z) & = 1 \\
\nabla(\emptyset; z) & = 0 \\
\nabla(L_+; z) - \nabla(L_-; z) & = -z \cdot \nabla(L_0; z).
\end{align*}
\]

Here, as usual, \( L_+ \), \( L_- \) and \( L_0 \) are the links which have plane projections identical to each other except in one small disk where their projections are a positive crossing, a negative crossing and an orientation preserving smoothing of that crossing, respectively. Note the negative sign on the right hand side of the skein relation. Its effect is to change the usual Conway polynomial by a normalization factor.

The Jones polynomial \( V(L; t) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}] \) is defined by

\[
\begin{align*}
V(O; t) & = 1; \\
V(\emptyset, t) & = (t^{\frac{1}{2}} + t^{-\frac{1}{2}})^{-1} \\
tV(L_+; t) - t^{-1}V(L_-; t) & = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0; t).
\end{align*}
\]

Note that our normalization differs from the usual definition of the Jones polynomial in [J]. Actually it is obtained from the usual Jones polynomial by changing \( t \) to \( t^{-1} \) and multiplying by \((-1)^{\#L-1}\). We put

\[
X(L; t) = \frac{V(L; t)}{(t^{\frac{1}{2}} + t^{-\frac{1}{2}})^{\#L-1}}.
\]

Then

\[
X(O; t) = X(\emptyset; t) = 1.
\]
We also put
\[ \Phi(L; t) = \sum_{L' \subset L} (-1)^{\#L - \#L'} X(L'; t) \]
where the sum runs over all sublinks of \( L \) including the empty link, and \( L \) itself. We have \( \Phi(O; t) = 0 \) and define \( \Phi(\emptyset; t) = 0 \).

It will be useful to note that \( X(L; t), \Phi(L; t) \in \mathbb{Z}[[t, t^{-1}, \frac{1}{t+1}]]. \)

The normalized Jones polynomial \( X(L; t) \) and the “averaged” Jones polynomial behave nicely under disjoint unions. Namely, if \( L \) splits geometrically into two links \( L_1 \) and \( L_2 \), then
\[ X(L; t) = X(L_1; t) \cdot X(L_2; t) \]
and
\[ \Phi(L; t) = \Phi(L_1; t) \cdot \Phi(L_2; t). \]

Further, we put
\[ \Phi_i(L) = \frac{d^i \Phi(L; t)}{dt^i} \bigg|_{t=1} \]
so that
\[ \Phi(L; t) = \sum_{i=0}^{\infty} \frac{\Phi_i(L)}{i!} (t - 1)^i. \]

Finally for each \( i \geq 1 \), we set
\[ \phi_i(L) = \frac{(-2)^{\#L}}{(#L + i)!} \cdot \Phi_{#L+i}(L). \]

The invariants \( \phi_i \) will be the basic link invariants we use to express \( \lambda_n \).

Given a link \( L = K_1 \cup K_2 \cup \cdots \cup K_\mu \) with a certain projection. Consider a pair of crossings between two components \( K_i \) and \( K_j, i \neq j \), with different signs. Denote this link by \( L^{++} \). We may modify the link diagram at these two crossings to get various other links, say \( L^{++}, L^{=} \), \( L^{+-}, L^{-+}, L^{-0} \), \( L_{0-} \) and \( L_{0,0} \). We say \( L^{++} \) is obtained from \( L^{=} \) by a double crossing change. A link \( L = K_1 \cup K_2 \cup \cdots \cup K_\mu \) is called an algebraically split link (ASL) if the linking number between every pair of components of \( L \) is 0. In particular, a knot is always an ASL. If there are mutually
disjoint 3-balls \( B_1, B_2, \ldots, B_m \) in \( S^3 \) with \( K_i \subset intB_i \), then \( L \) is called a \textit{geometrically split link} (GSL). Note that every algebraically split link can be changed into a geometrically split link by a finite number of double crossing changes, and the minimum number needed is called the \textit{double unlinking number} of this ASL.

**Lemma 4.1.** (Double crossing change formulae) For any link \( L_{+-} \) with \( \#L_{+-} \geq 2 \), we have

\[
(t^2 + t)[\Phi(L_{+-}; t) - \Phi(L_{-+}; t)] = (t - 1)[\Phi(L_{-0}; t) - \Phi(L_{0-}; t)].
\] (4.1)

and

\[
\phi_i(L_{+-}) - \phi_i(L_{-+}) = -[\phi_i(L_{0-}) - \phi_i(L_{0+})] + \frac{3}{2} [\phi_{i-1}(L_{+-}) - \phi_{i-1}(L_{-+})] - \frac{1}{2} [\phi_{i-2}(L_{+-}) - \phi_{i-2}(L_{-+})].
\] (4.2)

**Proof:** Using the skein relation for \( V(L; t) \), and \( \#L_{+-} = \#L_{-+} = \#L_{0-} + 1 = \#L_{0+} + 1 \), we have

\[
(t^2 + t)X(L_{+-}; t) - (1 + t^{-1})X(L_{-+}; t) = (t - 1)X(L_{0-}; t);
\]

and similarly,

\[
(t^2 + t)X(L_{-+}; t) - (1 + t^{-1})X(L_{+-}; t) = (t - 1)X(L_{0+}; t).
\]

By subtracting the second equation from the first, we obtain

\[
(t^2 + t)[X(L_{+-}; t) - X(L_{-+}; t)] = (t - 1)[X(L_{0-}; t) - X(L_{0+}; t)].
\]

By a direct computation, we get

\[
(t^2 + t)[\Phi(L_{+-}; t) - \Phi(L_{-+}; t)] = (t - 1)[\Phi(L_{-0}; t) - \Phi(L_{0-}; t)].
\]
This is (4.1). Expanding both sides into Taylor series at $t = 1$ and equating coefficients, we will get (4.2) since $\#L_+ = \#L_- = \#L_0 + 1 = \#L_0 + 1$. □

The following lemma partially explains the definition of $\phi_i$. Notice that Lemma 3.3 (2) is a special case of Proposition 3.4 in [O1] (see also Lemma 5.2), proved here using the double crossing change formula.

**Lemma 4.2.** We have

1. $\frac{\Phi_i(L)}{t} \in \mathbb{Z}\left[\frac{1}{2}\right]$ for each $i$;
2. If $L$ is an ASL, then $\Phi_i(L) = 0$ if $i \leq \#L$.

**Proof:** (1) This follows from the fact that $\Phi(L) \in \mathbb{Z}[t, t^{-1}, \frac{1}{t+1}]$.

(2) When $\#L = 1$, (2) follows from the fact that

$$V(L, 1) = 1 \quad \text{and} \quad \left. \frac{dV(L, t)}{dt} \right|_{t=1} = 0.$$ 

The general case can be proved inductively using the double crossing change formula (4.1) together with the fact that

$$\Phi(L; t) = \prod_{i=1}^{\#L} \Phi(K_i; t)$$

if $L = K_1 \cup K_2 \cup \cdots \cup K_{\#L}$ is a GSL. □

Let

$$v_i(L) = \left. \frac{d^iV(L; e^h)}{dh^i} \right|_{h=0}.$$ 

**Lemma 4.3.** Let $K$ be a knot. Then

1. $v_2(K) \in 6\mathbb{Z}$.
2. $v_3(K) \in 9\mathbb{Z}$.
3. $v_4(K) \in 6\mathbb{Z}$.
4. $v_2(K) + v_4(K) \in 18\mathbb{Z}$. 


Proof: Let $K_+$ and $K_-$ be a pair of knots differed by a crossing change. As before, $K_0$ is the two component link obtained by an orientation preserving smoothing of that crossing. We denote by $l$ the linking number of the two components of $K_0$. We may have another knot $K_\infty$ by smoothing the given crossing inconsistent with the orientation. We have
\[
V(K_+; t) - t^{-1}V(K_-; t) = (1 - t^{-1}) t^{-3l} V(K_\infty; t).
\] (4.3)
See Corollary 13.4 in [J].

We use $v_i$ with subscripts $+$, $-$ or $\infty$ to denote the corresponding $v_i$ of $K_+$, $K_-$ and $K_\infty$. By taking derivatives on both side of (4.3), we have:
\[
v_2^+ - v_2^- = -6l
\]
\[
v_3^+ - v_3^- = -3v_2^- + 27t^2 + 9l + 3v_2^\infty
\]
\[
v_4^+ - v_4^- = -4v_3^- + 6v_2^- + 4v_3^\infty - 6v_2^\infty(1 + 6l) - 12l - 54l - 108l^3.
\]
Then the lemma follows inductively. □

Theorem 4.1. Let $L$ be an ASL link. Then
1. $\phi_1(L) \in 6\mathbb{Z}$.
2. $\phi_2(L) \in 3\mathbb{Z}$.

Proof: (1) Using the double crossing change formula (4.2), we get
\[
\phi_1(L_{+-}) - \phi_1(L_{-+}) = -[\phi_1(L_{0-}) - \phi_1(L_{0+})] = 0.
\]
By induction on the double unlinking number and the number of components, it suffices to prove (1) for GSL’s. If $L$ is a knot, then $\phi_1(L) = -v_2(L) \in 6\mathbb{Z}$. If $L$ is a GSL with $\#L \leq 2$, then $\phi_2(L) = 0$. So (1) holds for GSL’s and, therefore, also for ASL’s.
(2) Using the double crossing formula (4.2) again:

\[
\phi_2(L_{+-}) - \phi_2(L_{-+}) = -[\phi_2(L_{0-}) - \phi_2(L_{-0})] - \frac{3}{2} [\phi_1(L_{+-}) - \phi_1(L_{-+})].
\]

By induction on the double unlinking number and the number of components, it suffices to prove (2) for GSL’s. If \( L \) is a knot,

\[
\phi_2(L) = -\frac{v_3(L) - 3v_2(L)}{3}.
\]

For a two-component GSL \( L = K_1 \cup K_2 \), \( \phi_2(L) = v_2(K_1) \cdot v_2(K_2) \). For a GSL \( L \) with \( \#L \geq 3 \), \( \phi_2(L) = 0 \). Thus (2) holds. \( \square \)

In the light of Theorem 4.1, we was tempted to conjecture originally that \( n! \phi_n(L) \in 6\mathbb{Z} \) for every ASL \( L \). But this was shown by Boden [Bo] to be not true. We therefore strengthen the conjecture to the following form.

**Conjecture 4.1.** For every boundary link \( L \), \( n! \phi_n(L) \in 6\mathbb{Z} \).

5. The second invariant \( \lambda_2 \)

A framed ASL is said to be unit framed if the framing of each component is \( \pm 1 \). Given a link \( L \) and a positive integer \( m \), we use \( L^m \) to denote the 0-framed \( m \)-parallel of \( L \), i.e. each component in \( L \) is replaced by \( m \) parallel copies having linking number zero with each other. So \( L^m \) is an ASL if \( L \) is. Sublinks of \( L^m \) will be assumed to be in one-one correspondence with \( \mu \)-tuples \((i_1, \ldots, i_\mu)\), where \( \mu = \#L \), in such a way that \( L' \) has \( i_\xi \) parallel copies of the \( \xi \)-th component of \( L \), \( 0 \leq i_\xi \leq m \). If \( L \) is a framed link, \( L^m \) and all its sublinks will inherit a framing from \( L \). If the \( \xi \)-th component of \( L \) is framed by \( f_\xi \), \( \xi = 1, \ldots, \mu \), we denote

\[
f_L = \prod_{\xi=1}^\mu f_\xi.
\]
As before, if $L$ is a framed link in $S^3$, then $S^3_L$ denotes the resulting 3-manifold from the Dehn surgery on $L$.

**Theorem 5.1.** Let $L$ be a unit framed ASL and $S^3_L$ be the homology 3-sphere obtained from Dehn surgery on $L$. Let $\lambda_1$ and $\lambda_2$ be the first and second Ohtsuki invariants, respectively. Then

$$\lambda_1(S^3_L) = \sum_{L' \subset L} f_{L'} \phi_1(L')$$  \hspace{1cm} (5.1)

and

$$\lambda_2(S^3_L) = \sum_{L' \subset L} \phi_1(L') f_{L'} \frac{\#L'}{2} + \sum_{L' \subset L^2} \phi_2(L') f_{L'} \frac{1}{2s_2(L')}.$$  \hspace{1cm} (5.2)

Here, if $L'$ corresponds to the $\mu$-tuple $(i_1, \ldots, i_\mu)$, $s_2(L') = \#\{i_\xi; i_\xi = 2\}$.

The formula for $\lambda_1$ is equivalent to Hoste's formula for the Casson invariant [Ho]. This can be proved using Corollary 3.12 [Mu]. Both formulae will be proved in Section 6.

Combining with Theorem 4.1, we obtain:

**Corollary 5.1.** Let $M$ be an oriented homology 3-sphere. Then

1. $\lambda_1(M) \in 6\mathbb{Z}$.
2. $\lambda_2(M) \in 3\mathbb{Z}[\frac{1}{2}]$.

Notice that Corollary 4.1 (2) will be strengthened in Theorem 7.1.

In the rest of this section, we will study the behaviors of the invariant $\lambda_2$ on homology 3-spheres obtained from $1/n$-Dehn surgery on knots. We will use (5.2) to make the relation between $\lambda_2$ with (coefficients of) the Jones polynomial very explicit in this particular case.
Theorem 5.2. Let $n$ be an integer, and $S^3_{K, 1/n}$ be the homology 3-sphere obtained from $1/n$-Dehn surgery on a knot $K$. Then

$$
\lambda_2(S^3_{K, 1/n}) = \frac{n}{2} v_2(K) - \frac{n}{3} v_3(K) - \frac{n^2}{6} [v_2(K) + v_4(K) - 4v_2^2(K)].
$$

(5.3)

To prove this theorem, we need two lemmas.

Lemma 5.1. 1. $\Phi_2(K) = v_2(K)$, and $\Phi_3(K) = v_3(K) - 3v_2(K)$.

2. $\Phi_4(K^2) = -2v_2(K) - 2v_4(K) + 8v_2^2(K)$.

Proof: (1) Exercise.

(2) $\Phi_4(K^2)$ is a Vassiliev invariant of order 4 (see, for example, [MM]). Therefore, for any knot $K$,

$$
\Phi_4(K^2) = a \cdot v_2(K) + b \cdot v_3(K) + c \cdot v_4(K) + d \cdot \phi(K) + e \cdot v_2^2(K),
$$

where $\phi$ is the coefficient of $z^4$ in the Conway polynomial of $K$, and $a, b, c, d, e$ are some constants. If $K$ is a torus knot, then the Conway/Jones polynomials of $K$ and $K^2$ are known (see, for example, [M]). Using these data, we can determine $a, b, c, d, e$. And this gives the above formula. □

Lemma 5.2. Let $L$ be an ASL with $\mu = \#L$. If $L' \subset L^m$ corresponding to $(i_1, \ldots, i_{\mu})$ with a certain $i_{\xi} = m$, then $\Phi_i(L') = 0$ for $i \leq \mu + m - 1$.

Compare it with Lemma 4.2 (2). This is the result of Ohtsuki mentioned above. It is proved in [O1] using the colored Jones polynomial.

Proof of Theorem 5.2: Assume $n \neq 0$. Using Kirby’s calculus on framed links, one can see that $S^3_{K, 1/n}$ is the same as $S^3_{K(n)}$ with the framing of each component equal to $\text{sign}(n)$. This is done explicitly in [G]. By (5.2) and Lemma 5.2, we have

$$
\lambda_2(S^3_{K, 1/n}) = -\frac{n}{2} \cdot \Phi_2(K) - \frac{n}{3} \cdot \Phi_3(K) + \frac{n^2}{12} \cdot \Phi_4(K^2).
$$
Then (5.3) follows from Lemma 5.1. □

Combining (5.3) with Lemma 4.4, we get

**Corollary 5.2.** Let $M$ be the resulting homology 3-sphere of $1/n$-Dehn surgery on a knot $K$. Then $\lambda_2$ is an integer divisible by 3, i.e., $\lambda_2 \in 3\mathbb{Z}$.

Once again, this conclusion will be strengthened in Theorem 7.1.

**Corollary 5.3.** If $v_3(K) \neq 0$ for a knot $K$, and let $K^*$ be the mirror image of $K$. Then $S^3_{K,1/n}$ and $S^3_{K^*,1/n}$ are distinct as unoriented manifolds for each $n \neq 0$.

**Proof:** It suffices to prove that $S^3_{K,1/n}$ is neither homeomorphic to $S^3_{K^*,1/n}$ nor $\overline{S^3_{K^*,1/n}}$.

If $v_2(K) \neq 0$, then $\lambda_1$ will show this. If $v_2(K) = 0$, then $\lambda_2$ takes the same values on $S^3_{K^*,1/n}$ and $\overline{S^3_{K^*,1/n}}$ (see the remark below). As $v_{2n}(K) = v_{2n}(K^*)$, and $v_{2n+1}(K) = -v_{2n+1}(K^*)$, so

$$\lambda_2(S^3_{K,1/n}) - \lambda_2(S^3_{K^*,1/n}) = -\frac{2n}{3} v_3(K).$$

This completes the proof. □

**Corollary 5.4.** Let $n, m$ be two distinct integers, and $K$ be a knot.

1. If $v_2(K) \neq 0$, then $S^3_{K,1/n}$ and $S^3_{K,1/m}$ are distinct as unoriented 3-manifolds.
2. If $v_2(K) = 0$, and $v_3(K) \neq -\frac{n+m}{2} v_4(K)$, then $S^3_{K,1/n}$ and $S^3_{K,1/m}$ are distinct as unoriented 3-manifolds.

**Proof:** (1) follows from comparing $\lambda_1$ of the two manifolds.

(2) follows from comparing $\lambda_2$ of the two manifolds. □

Some examples are as follows.
Example 5.1. (1) If $M$ is the Poincare homology 3-sphere $\Sigma(2, 3, 5)$ (+1-surgery on the right-handed trefoil knot), then $\lambda_2(M) = 39$.

(2) If $M$ is the homology 3-sphere $\Sigma(2, 3, 7)$ (+1-surgery on the left-handed trefoil knot), then $\lambda_2(M) = 63$.

Remark: Suppose $M$ is an oriented homology 3-sphere and $\overline{M}$ is the oriented homology 3-sphere obtained from $M$ by reversing the orientation. Then $\lambda_2(\overline{M}) = \lambda_2(M) + \lambda_1(M)$. If $N$ is the oriented homology 3-sphere obtained from (-1)-surgery on a knot $K$ and $M$ is the oriented homology 3-sphere obtained from (+1)-surgery on the mirror image of $K$, then $N = \overline{M}$. Using these facts, we observe that our computation above agrees completely with that in [L].

Example 5.2. If $M$ is the homology 3-sphere $\Sigma(2, 5, 7)$ (this is the boundary of the Mazur manifold), then $\lambda_2(M) = -66$. Notice that the Mazur manifold is contractible and $6|\lambda_2$ in this particular case. This raises the question of whether $\lambda_2/3 \mod 2$ is a homology spin cobordism invariant.

It is not very difficult to prove that there exists two knots $K_1, K_2$ with the same Jones polynomial and such that $1/n$-Dehn surgery on $K_1$ and $K_2$, respectively, for some $n$ give distinct homology 3-spheres. If $M$ is obtained from Dehn surgery on a knot, it follows from Theorem 5.2 that $\lambda_2(M)$ is determined by the Jones polynomial. So $\lambda_2(M)$ can be the same for distinct homology 3-spheres.

6. The Formula for $\lambda_n$

Let $M$ be the resulting oriented homology 3-sphere of Dehn surgery on a unit framed ASL $L$ and $\mu = \#L$. Let $f_\xi = \pm 1$ be the framing on the $\xi$-th component of $L$. Then for each $n \geq 1$, Ohtsuki gave the
following expression for $\lambda_n$:

$$
\lambda_n(M) := \text{Coeff}_n \left\{ q^{\frac{1}{2}} \sum_{l=1}^{n} \sum_{L \subset L'} \frac{\phi_l(L')}{(-2)^{#L'} (l + #L')!} (q - 1)^l \prod_{\xi=1}^{\mu} \left( -f_\xi q^{\frac{3}{2} f_\xi - \frac{1}{2}} \sum_{m_\xi=0}^{n-l} h_{f_\xi, i_\xi, m_\xi} (q - 1)^{m_\xi} \right) ; q - 1 \right\},
$$

where $h_{f_\xi, i_\xi, m_\xi}$'s are some unknown constants whose existence was established by Ohtsuki. Our main result here is a more practical formula for $\lambda_n$. In the following, we will restate several lemmas in [O1] using Definition 3.1. This unifies those lemmas and clarifies the uniqueness of the constants appearing there.

Let the constants $\nu_{f,i,m}$ be defined by the following formal equation, i.e. by equating coefficients on both sides in the expansion of power series in $(t - 1)$:

$$
\sum_{m=0}^{+\infty} \nu_{f,i,m} (t - 1)^m = t^{\frac{3}{4} f - \frac{1}{2}} \sum_{m=0}^{+\infty} h_{f,i,m} (t - 1)^m.
$$

**Theorem 6.1.** Let $M$ be as above. Then

$$
\lambda_n(M) = \sum_{l=1}^{n} \sum_{L \subset L'} \frac{\phi_l(L')}{(-2)^{#L'} (l + #L')!} \left\{ \prod_{\xi=1}^{\mu} (-f_\xi) \left( \sum_{m_1+m_2+\cdots+m_\mu=n-l} \prod_{\xi=1}^{\mu} \nu_{f_\xi, i_\xi, m_\xi} \right) \right\}.
$$

**Proof:** Using Ohtsuki’s expression, we have

$$
\lambda_n(M)
$$

$$
= \text{Coeff}_n \left\{ \sum_{l=1}^{n} \sum_{L \subset L'} \frac{\phi_l(L')}{(-2)^{#L'} (l + #L')!} (q - 1)^l \prod_{\xi=1}^{\mu} \left( -f_\xi q^{\frac{3}{2} f_\xi - \frac{1}{2}} \sum_{m_\xi=0}^{n-l} h_{f_\xi, i_\xi, m_\xi} (q - 1)^{m_\xi} \right) ; q - 1 \right\}
$$

$$
= \text{Coeff}_n \left\{ \sum_{l=1}^{n} \sum_{L \subset L'} \frac{\phi_l(L')}{(-2)^{#L'} (l + #L')!} (q - 1)^l \prod_{\xi=1}^{\mu} \left( -f_\xi q^{\frac{3}{2} f_\xi - \frac{1}{2}} \sum_{m_\xi=0}^{+\infty} h_{f_\xi, i_\xi, m_\xi} (q - 1)^{m_\xi} \right) ; q - 1 \right\}
$$

$$
= \sum_{l=1}^{n} \sum_{L \subset L'} \frac{\phi_l(L')}{(-2)^{#L'}} \text{Coeff}_{n-l} \left\{ \prod_{\xi=1}^{\mu} \left( -f_\xi q^{\frac{3}{2} f_\xi - \frac{1}{2}} \sum_{m_\xi=0}^{+\infty} h_{f_\xi, i_\xi, m_\xi} (q - 1)^{m_\xi} \right) ; q - 1 \right\}
$$

$$
= \sum_{l=1}^{n} \sum_{L \subset L'} \frac{\phi_l(L')}{(-2)^{#L'}} \text{Coeff}_{n-l} \left\{ \prod_{\xi=1}^{\mu} \left( -f_\xi \sum_{m_\xi=0}^{+\infty} \nu_{f_\xi, i_\xi, m_\xi} (q - 1)^{m_\xi} \right) ; q - 1 \right\}.
$$
By picking out the $n$-th coefficient, we get the desired formula. □

**Corollary 6.1.** Let $K$ be a knot. The knot invariant $\theta_n$ induced by $\lambda_n$ is defined to be $\theta_n(K) = \lambda_n(S^3_{K,1})$. Then $\theta_n$ is a Vassiliev invariant of order $2n$.

**Proof:** By Theorem 6.1, $\theta_n$ is a linear combination of $\phi_l(K^j)$ for $1 \leq l \leq n, 1 \leq j \leq l$. As $\phi_l(K^j)$ is a Vassiliev invariant of order $l + j$, $\theta_n$ is an invariant of order $2n$. □

If $\lambda_n$ is indeed a finite invariant of order $3n$ in the sense of [O2] as suggested by [R1,2], then Corollary 6.2 will follow from the result in [Ha].

To get an explicit formula for $\lambda_n$, we have to determine the constants $\nu_{f,i,m}$. Our discussion here is parallel to the discussion of $h_{f,i,m}$ in Section 8 of [O1]. To avoid repetition of many formulae, we will quote some formulae from [O1] and leave the interested reader to consult [O1] for details. We will make some improvement to his main lemma (Lemma 8.3 in [O1]) and the result is more explicit.

To get $\nu_{f,i,m}$, fix a nonnegative integer $i$, and let $l(r) = \frac{r-1}{2} - i - 1$.

Let

$$ s_{r,f,i}(q) = \left( \frac{f}{r} \right) \cdot q^f \cdot \left( q - 1 \right)^{i+1} \cdot \frac{G_0(q)}{q} $$

$$ \times \left\{ \sum_{j=0}^{k-1} q^{\frac{1}{2} f(k^2-1)} \cdot \sum_{k=1}^{l(r)} (-1)^j \binom{k - j - 1}{j} \binom{k - 2j - 1}{i} [2]^{k-2j-1} \right\}, $$

where $G_0(q)$ is the Gauss sum. Note that $s_{r,f,i} \in \mathbb{Z}(r)[q]$ and $\left( \frac{L}{r} \right)$ is the Legendre symbol.

**Theorem 6.2.** With respect to $l(r)$,

$$ \text{f-lim} \ s_{r,f,i}(q) = \sum_{m=0}^{+\infty} \nu_{f,i,m}(t - 1)^m. $$
This is essentially Proposition 3.6 in [O1]. We need to compute the Fermat limit of $s_{r,f,i}(q)$ in order to determine $\nu_{f,i,m}$.

Recall that $g'_{l,m} \in \mathbb{Z}_{(m+2)\left[\frac{1}{2}\right]}$ is the residue of $H_{l,m}$ given by (2.5). Let $G_{2l}(q)$ be the weighted Gauss sum.

**Lemma 6.1.** With respect to $l(r) = \frac{r-3}{2}$, we have

$$\liminf \frac{(r-1)k^{l}G_{2l}(q)}{(q-1)^{\frac{r-1}{2}-l}} = \sum_{m=0}^{+\infty} g'_{l,m}(t-1)^{m}.$$ 

This is the main technical lemma and is an improvement to Lemma 8.3 in [O1]. In [O1], Ohtsuki established the existence of $g'_{l,m}$, but his proof does not give a formula for these numbers. Here, the $g'_{l,m}$ are given explicitly in (2.5). We also have $g'_{l,m} \in \mathbb{Z}_{(m+2)\left[\frac{1}{2}\right]}$, whereas in [O1], it is only known that $g'_{l,m} \in \mathbb{Z}_{(2m+2)\left[\frac{1}{2}\right]}$.

**Proof:** To expand $G_{2l}(q)$ into a power series in $(q - 1)$, we take the Taylor expansion of $\sum_{k=0}^{r-1} k^{2l}x^{k^2}$ at $x = 1$ and put $x = q$. Hence we have

$$\left(\frac{r-1}{2}\right)! G_{2l}(q)$$

$$= \sum_{m=1}^{r-2-l} \frac{(r-1)!}{2^m} \sum_{k=0}^{k^2} \frac{k^{2l}k^2(k^2-1)\cdots(k^2-(m-1))}{m!} (q-1)^m$$

$$+ O((q-1)^{r-1-l}; \mathbb{Z})$$

$$= \sum_{m=1}^{r-2-l} \frac{(r-1)!}{2^m} \sum_{1 \leq m' \leq m} (-1)^{m-m'} \sigma_{m-m'}(1, \ldots, m-1) k^{2l+2m'} (q-1)^m$$

$$+ O((q-1)^{r-1-l}; \mathbb{Z})$$

$$= \sum_{m=1}^{r-2-l} \frac{(r-1)!}{2^m} \sum_{1 \leq m' \leq m} (-1)^{m-m'} \sigma_{m-m'}(1, \ldots, m-1) s_{2l+2m'}(1, \ldots, r-1) (q-1)^m$$

$$+ O((q-1)^{r-1-l}; \mathbb{Z})$$
If \(2l + 2m' \leq r - 2\), i.e., \(m' \leq m \leq \frac{r-3}{2} - l\), then it follows from Lemma 2.3 and \(r = O((q-1)^{r-1}; \mathbb{Z})\) that

\[ s_{2l+2m'}(1, \ldots, r-1) = O((q-1)^{r-1}; \mathbb{Z}(r)). \]

Therefore,

\[
\left(\frac{r-1}{2}\right)! G_2(q) = \sum_{\frac{r-1}{2} - l}^{r-2-l} \frac{\left(\frac{r-1}{2}\right)!}{m!} \sum_{1 \leq m' \leq m} (-1)^{m-m'} \sigma_{m-m'}(1, \ldots, m-1) s_{2l+2m'}(1, \ldots, r-1)(q-1)^m + O((q-1)^{r-1-l}; \mathbb{Z}(r))
\]

We will use Fermat’s Little Theorem \((a^{r-1} \equiv 1 \pmod{r})\) to get rid of \(s_{2l+2m'}(1, \ldots, r-1)\) in the above expression. So if \(2l + 2m' \not\equiv 0 \pmod{r-1}\), then we may assume \(2l + 2m' \leq r - 2\). We have dropped these terms by the preceding argument. So the only non-trivial contribution of \(s_{2l+2m'}(1, \ldots, r-1)\) comes from \(2l + 2m' \equiv 0 \pmod{r-1}\). Since \(\frac{r-1}{2} - l \leq m \leq r - 2 - l\), we have \(2l + 2m' = r - 1\), i.e. \(m' = \frac{r-1}{2} - l\). In this case,

\[ s_{2l+2m'}(1, \ldots, r-1) = \sum_{k=1}^{r-1} k^{r-1} \equiv -1 \pmod{r}. \]
Hence we get
\[
\left(\frac{r - 1}{2}\right)! G_{2l}(q) = \sum_{m=0}^{r-2-l} \frac{(r-1)!}{m!} (-1)^{m-\frac{r-1}{2}-l+1} \sigma_{m-\frac{r-1}{2}-l}(1, \ldots, m-1)(q-1)^m
\]
\[+ O((q - 1)^{r-1-l}; \mathbb{Z}(r))
\]
\[= \sum_{m=0}^{r-3} \frac{(r-1)!}{(r-2 - l + m)!} (-1)^{m+1} \sigma_m(1, \ldots, r-1, -l + m - 1)(q-1)^{\frac{r-1}{2}-l+m}
\]
\[+ O((q - 1)^{r-1-l}; \mathbb{Z}(r))
\]
\[= \sum_{m=0}^{r-3} H_{l,m}(r)(q-1)^{\frac{r-1}{2}-l+m}.
\]
This finishes the proof. □

For a pair of non-negative integers \(l, m\), let \(g_{l,m}\) be a sequence of rational numbers defined by the equation
\[
\sum_{m=0}^{+\infty} g'_{l,m}(t - 1)^m = \sum_{m=0}^{+\infty} g_{l,m}(t - 1)^m + \sum_{m=0}^{+\infty} g'_{0,m}(t - 1)^m.
\]
Comparing the coefficients, we get
\[
\begin{align*}
g_{l,0} &= -g'_{l,0}, \\
g_{l,m} &= -g'_{l,m} + \sum_{k=0}^{m-1} g_{l,k} g'_{0,m-k} \quad \text{if } m > 0.
\end{align*}
\]  
(6.1)
So we can compute \(g_{l,m}\) recursively once the \(g'_{l,m}\) are known. And we also see that
\[g_{l,m} \in \mathbb{Z}(m+2)[\frac{1}{2}].\]

The following lemma is now a direct consequence of the definition.

**Lemma 6.2.** With respect to \(l(r) = \frac{r-3}{2}\), we have
\[
\operatorname{f-lim} (q - 1)^l \frac{G_{2l}(q)}{G_0(q)} = \sum_{m=0}^{+\infty} g_{l,m}(t - 1)^m.
\]
In particular, \(g_{0,m} = \delta_{0,m}\).
Here comes our last technical lemma, whose proof is straightforward.

**Lemma 6.3.** Let $F_{i,l}(x)$ be the polynomial in $x$ defined recursively for $i \geq 0$, $l \geq -1$ and $i \geq l$ by:

\[
F_{0,0}(x) = 1, \quad F_{i,-1}(x) = F_{i,i+1} = 0,
\]

\[
F_{i+1,l}(x) = F_{i,l-1}(x) - (2i + 1 - l)x F_{i,l}(x) + (x^2 - 4)F'_{i,l}(x).
\]

Then

1. $F_{i,i}(x) = 1$;
2. $F_{i+k,i}(x)$ is of the same degree $k$ for any $i \geq 0$; and
3. $F_{i+1,i}(x) = -x \binom{i+2}{2}$.

Now we come to the computation of the Fermat limit of $s_{r,f,i}(q)$, or equivalently, the determination of $\nu_{f,i,m}$. First, note that $s_{r,f,i}(q)$ is the left-hand side of the equation in Proposition 3.6 of [O1] multiplied by \( \left( \frac{L}{f} \right) q^{\frac{i}{2} f - \frac{1}{2}} \). Then the expansion of $s_{r,f,i}(q)$ into a power series in $(q - 1)$ follows from the formulae of Ohtsuki on pages 108-109 of [O1]. There is an omission of $f^l$ on lines 12, 15 and 18 in his formulae on page 109. For $i = 0$, we have $s_{r,f,i} = -f$. So we assume $i \geq 1$. Then we have

\[
s_{r,f,i}(q) = \frac{[2]^i q^{i+\frac{1}{2}+\frac{f}{2}}}{i!} \left\{ \frac{F_{i,0}([2])}{(q - 1)^{i+1}} (q^{-f} - 1) \right. \\
+ q^{-f} \sum_{l=1}^{i} f^l F_{i,l}([2]) (-2)^l q^{-\frac{f}{2}} \sum_{l'=0}^{\frac{f}{2}} \left( \frac{l}{2l'} \right) f^{l'} \sum_{m=0}^{\frac{f}{2}} g_{r,m} (q - 1)^{m-l'-i+1} \\
- \sum_{l=1}^{\frac{f}{2}} F_{i,2l}([2]) 2^{2l} q^{-l} f^l \sum_{m=0}^{\frac{f}{2}} g_{r,m} (q - 1)^{m+l-i-1} \right\} + O((q - 1)^{\frac{r-1}{2} - i}; \mathbb{Z}(r)).
\]
Note that $q^{-f} - 1 = (-f) q^{-f + 1}(q-1)$ and $[2]^i = q^{-\frac{f}{2}}(q+1)^i$. Separating $l' = 0$ from the second term, we get

$$s_{r,f,i}(q) = \frac{(-f)}{i!} q^{\frac{f}{2}} (q + 1)^i F_{i,0}([2])(q - 1)^{-i}$$

$$+ \sum_{l=1}^i f^l \frac{(-2)^l}{l!} q^{\frac{f}{2} + \frac{l}{2} - \frac{i}{2} - \frac{1}{2}} (q + 1)^i F_{i,l}([2])(q - 1)^{-i-1+l}$$

$$+ \sum_{l=1}^i f^l \frac{(-2)^l}{l!} q^{\frac{f}{2} + \frac{l}{2} - \frac{i}{2} - \frac{1}{2}} (q + 1)^i F_{i,l}([2]) \sum_{m=0}^{r-3} \left( \frac{l}{2l'} \right) \sum_{m=0}^{\frac{r-3}{2}} g_{l,m}(q - 1)^{m-l'-i-1} + O((q - 1)^{\frac{r-1}{2} - i}; Z_{(r)}).$$

By changing the order of summations in the third term, and combining the sum with $l = 2l'$ in the new order of summations with the fourth term, we get

$$s_{r,f,i}(q) = \frac{(-f)}{i!} q^{\frac{f}{2}} (q + 1)^i F_{i,0}([2])(q - 1)^{-i}$$

$$+ \sum_{l=1}^i f^l \frac{(-2)^l}{l!} q^{\frac{f}{2} + \frac{l}{2} - \frac{i}{2} - \frac{1}{2}} (q + 1)^i F_{i,l}([2]) \sum_{m=0}^{r-3} \left( \frac{l}{2l'} \right) \sum_{m=0}^{\frac{r-3}{2}} g_{l,m}(q - 1)^{m-l'-i-1}$$

$$+ (q^{-f} - 1) \sum_{l=1}^i f^l \frac{2^{2l}}{l!} q^{\frac{f}{2} + \frac{l}{2} + \frac{1}{2}} (q + 1)^i F_{i,2l}([2]) \sum_{m=0}^{r-3} g_{l,m}(q - 1)^{m+l-i-1}$$

$$+ O((q - 1)^{\frac{r-1}{2} - i}; Z_{(r)}).$$
\[
\begin{align*}
&= \frac{(-f)}{i!} q^i (q + 1)^i F_{i,0}([2])(q - 1)^{i-1} \\
&+ \sum_{l=1}^{i} f^l \frac{(-2)^l}{i!} q^{\frac{l}{2} + \frac{1}{2} - \frac{l}{2} - \frac{i}{2}} (q + 1)^i F_{i,l}([2])(q - 1)^{i-1+l} \\
&+ \sum_{l'=1}^{i} \sum_{l=2l'+1}^{i} f^{l+l'} \frac{(-2)^l}{i!} \left( \frac{l}{2l'} \right) q^{\frac{l}{2} + \frac{1}{2} - \frac{l}{2} - \frac{i}{2}} (q + 1)^i F_{i,l}([2]) \sum_{m=0}^{\frac{r-3}{2}} g_{m,m}(q - 1)^{m-l'-i+1+l} \\
&+ \sum_{l=1}^{i} (-f) f^l \frac{2^{2l}}{i!} q^{\frac{2l}{2} - l} (q + 1)^i F_{i,2l}([2]) \sum_{m=0}^{\frac{r-3}{2}} g_{l,m}(q - 1)^{m+l-i} \\
&\quad + O((q - 1)^{\frac{r-3}{2} - i}; \mathbb{Z}(r)).
\end{align*}
\]

Using the fact that \(g_{0,m} = \delta_{0m}\), we see that the first term is the case \(l = 0\) of the fourth, and the second is \(l' = 0\) of the third. Combining these four terms into two terms, and picking out the coefficient of \((q - 1)^m\) in the expansion above, we obtain

\[
\nu_{f,i,m} = \sum_{l'=0}^{i} \sum_{l=2l'+1}^{i} f^{l+l'} \frac{(-2)^l}{i!} \left( \frac{l}{2l'} \right) \text{Coeff}_m(t^{\frac{l+l'-1}{2}} (t + 1)^i F_{i,l} (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \times \\
\sum_{m'=0}^{+\infty} g_{m,m'} (t - 1)^{m-l'-i-1+l}; t - 1) \\
+ \sum_{l'=0}^{i} (-f) f^l \frac{2^{2l'}}{i!} \text{Coeff}_m(t^{\frac{-l-l'}{2}} (t + 1)^i F_{i,2l'} (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \times \\
\sum_{m'=0}^{+\infty} g_{m,m'} (t - 1)^{m+l'-i}; t - 1).
\]

As \(g_{l,m}\) and \(F_{i,l}(x)\) both can be computed recursively, so we can compute \(\nu_{f,i,m}\).

**Theorem 6.3.** Let \(M\) be an oriented integral homology 3-sphere. Then \(\lambda_n(M) \in \mathbb{Z}(n+1)\).

**Proof:** By Theorem 6.1, we need \(\nu_{f,i,m}\) with \(0 \leq m \leq n - 1\), and \(0 \leq i \leq n - m\) for \(\lambda_n\). Using (6.2), we need \(m' - l' - i - 1 + l \leq m\) and \(m' + l' - i \leq m\). In both cases, the terms with \(l' = 0\) have the right
denominator. So we may assume \( l' \geq 1 \). Then it is easy to check that \( m' \leq n - 1 \). So for the formula of \( \lambda_n \), we need \( g_{l',m'} \) with \( l' \leq n/2 \), and \( m' \leq n - 1 \). As \( g_{l',m'} \in \mathbb{Z}_{(m'+2)} \), our theorem follows. \( \square \)

This theorem tells us that the biggest factor in the denominator of \( \lambda_n \) is indeed \( n \). So it agrees with Conjecture 1.1.

To tie up the loose ends in Section 5, we prove the formulae for \( \lambda_1 \) and \( \lambda_2 \).

**Proof of Theorem 5.1:** First we need the following results for \( \nu_{f,i,m} \) for \( i = 0, 1, 2 \). Use (6.2), for \( i = 0 \)

\[
\nu_{f,0,m} = \text{Coeff}_m(-f; t - 1),
\]

so \( \nu_{f,0,0} = -f \), \( \nu_{f,0,m} = 0 \) if \( m \geq 1 \). For \( i = 1 \),

\[
\nu_{f,1,m} = \text{Coeff}_m(t + 1; t - 1),
\]

so \( \nu_{f,1,0} = 2 \), \( \nu_{f,1,1} = 1 \), and \( \nu_{f,1,m} = 0 \) if \( m \geq 2 \). For \( i = 2 \),

\[
\nu_{f,2,m} = \text{Coeff}_m\left(-\frac{t^2 + 2t + 2}{2}(1 + f + 4 \sum_{m=1}^{\infty} g_{1,m}(t-1)^{m-1}; t-1)\right).
\]

It follows that \( \nu_{f,2,0} = -(2 + 2f + 8g_{1,1}) \). As \( g_{1,1} = -\frac{1}{4} \), so \( \nu_{f,2,0} = -2f \).

For the formula for \( \lambda_1 \), we have \( n = 1 \) and \( l = 1 \). Therefore, \( m_1 = \cdots = m_\mu = 0 \) in Theorem 6.1. Using the values of \( \nu_{f,0,0} \), and \( \nu_{f,1,0} \), we get

\[
\sum_{m_1 + \cdots + m_\mu = 0} \prod_{\xi=1}^\mu \nu_{f_{\xi,1,0},m_\xi} = \prod_{\xi: \xi_1 = 0} (-f_\xi) \prod_{\xi: \xi_1 = 1} 2.
\]

This verifies the formula for \( \lambda_1 \).

For the formula of \( \lambda_2 \), we have two cases: \( n = 2, l = 1 \) and \( n = 2, l = 2 \). If \( n = 2, l = 1 \), then

\[
\sum_{m_1 + \cdots + m_\mu = 1} \prod_{\xi=1}^\mu \nu_{f_{\xi,1,0},m_\xi} = \frac{1}{2} \prod_{\xi: \xi_1 = 0} (-f_\xi) \prod_{\xi: \xi_1 = 1} 2.
\]
If \( n = 2, l = 2 \), then
\[
\sum_{m_1 + \cdots + m_\mu = 0} \prod_{\xi=1}^{\mu} \nu_{f_\xi, i_\xi, m_\xi} = \prod_{\xi: i_\xi = 0} (-f_\xi) \prod_{\xi: i_\xi = 1} 2 \prod_{\xi: i_\xi = 2} (-2f_\xi).
\]
This verifies the formula. \( \square \)

7. Integrality of \( \lambda_n \)

Rozansky predicts that there is an integer constant \( C \) such that 
\( C\lambda_n \in \mathbb{Z} \). The predicted constant \( C \) is very large. Conjecture 1.1 says that \( C = n! \) is sufficient. In this section, we prove the following theorem which is the case \( n = 2 \) of Conjecture 1.1.

**Theorem 7.1.** Let \( M \) be a homology 3-sphere. Then \( \lambda_2(M) \in 3\mathbb{Z} \).

If we write \( \tau_r(M) \) as follows:
\[
\tau_r(M) = a_{r,0} + a_{r,1}(q - 1) + \cdots + a_{r,n}(q - 1)^n + \cdots + a_{r,N}(q - 1)^N
\]
for some \( a_{r,n} \in \mathbb{Z} \). Although \( a_{r,n} \) is not well-defined, \( \overline{a_{r,n}} \in \mathbb{Z}/r\mathbb{Z} \) is well-defined for \( 0 \leq n \leq r - 2 \).

**Corollary 7.1.** For \( n = 1, 2 \), \( \overline{a_{r,n}} \in \mathbb{Z}/r\mathbb{Z} \) stabilizes for large \( r \).

See the remark after Lemma 3.1.

It seems to be an interesting question as of when this magic stabilization starts. We suspect that this may happen from the very first term.

Theorem 7.1 will follow from Theorem 5.1 and the following theorem, where by doubling a component of a link \( L \), we mean to split that component of \( L \) into two parallel copies with zero linking number.

**Theorem 7.2.** Suppose \( L' \) is obtained from an ASL link \( L \) by doubling \( m \) components \((m \geq 1)\) of \( L \). Then \( \phi_2(L') \in 2^m\mathbb{Z} \).
To prove this theorem, we will use the universal invariant (or the colored Jones polynomial) for ASL’s. We will follow the exposition of [O1] (see also [RT, KM, MM]).

As before, let $r$ be an odd prime and $q = e^{2\pi \sqrt{-1}/r}$. We assume

$$q^\frac{1}{2} = -q^{-\frac{r+1}{2}}.$$

Let $U_q$ be the associative algebra over $\mathbb{C}$ with generators $E, F, K^\pm$ subject to the following relations:

$$KK^{-1} = K^{-1}K = 1$$
$$KE = qEK, \quadKF = q^{-1}FK$$
$$EF - FE = \frac{K-K^{-1}}{q^\frac{1}{2} - q^{-\frac{1}{2}}}$$
$$E^r = F^r = 0.$$

Note that we do not impose a relation $K^{2r} = 1$ on $U_q$. The algebra $U_q$ becomes a Hopf algebra with comultiplication $\Delta : U_q \to U_q \otimes U_q$, antipode $S : U_q \to U_q$ and counit $\epsilon : U_q \to \mathbb{C}$ by:

$$\Delta(E) = E \otimes 1 + K \otimes E$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F$$
$$\Delta(K^\pm) = K^\pm \otimes K^\pm$$
$$S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K^\pm) = K^\mp$$
$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K^\pm) = 1.$$

Let $I$ be the vector subspace of $U_q$ generated by $ab - ba$ for any $a, b \in U_q$. Then the universal invariant of a 0-framed ASL $L$, which will be denoted by $\phi(L)$, takes value in $(U_q/I)^{\otimes(\#L)}$. By Proposition 5.6 of [O1], the universal invariant $\phi(L)$ of $L$ has a form obtained by taking sum, product and tensor product in $E, F, K^\pm$ with coefficients in $\mathbb{Z}[q]$. 
Let $\rho : U_q \rightarrow \text{End}(\mathbb{C}^2)$ be the representation given by
\[
\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \rho(K^\pm) = \begin{pmatrix} q^{\pm1/2} & 0 \\ 0 & q^{\mp1/2} \end{pmatrix}.
\]
And let $\chi_V : U_q \rightarrow \mathbb{C}$ be the character of $\rho$. Note that $\chi_V$ descends to a map $U_q/I \rightarrow \mathbb{C}$. Then by Lemma 6.1 of [O1], the Jones polynomial at $q^{-1}$ for $L$ is given by
\[
V(L; q^{-1}) = (-1)^{\#L}\chi_V^{\otimes(\#L)}(\phi(L)).
\]

We collect some facts useful in the proof of Theorem 7.2 into the following lemma.

**Lemma 7.1.** Let $L'$ be as in Theorem 7.2 and assume that the doubled $m$ components of $L$ are the first $m$ components.

1. $(-q^{1/2} - q^{-1/2})^{\#L'}\Phi(L'; q^{-1}) = (\chi_V + (q^{1/2} + q^{-1/2})\epsilon)^{\otimes(\#L')}(\phi(L'))$;
2. $\phi(L') = (\Delta \otimes \cdots \otimes \Delta \otimes \text{id} \otimes \cdots \otimes \text{id})(\phi(L))$ (there are $m$ $\Delta$’s here);
3. $\phi(L)$ is a $\mathbb{Z}[q]$-linear combination of

\[
(q - 1)^{2j_1 + \sum_{\nu=1}^{m} j_{\nu}} K^{i_1} E^{j_1} F^{j_1} \otimes K^{i_2} E^{j_2} F^{j_2} \otimes \cdots \otimes K^{i_\nu} E^{j_\nu} F^{j_\nu}
\]

such that (a) $i_\nu + j_\nu$ is even; and (b) if $j_1 = 0$, then

\[
K^{i_1} \otimes K^{i_2} E^{j_2} F^{j_2} \otimes \cdots \otimes K^{i_\nu} E^{j_\nu} F^{j_\nu}
\]

and

\[
K^{-i_1} \otimes K^{i_2} E^{j_2} F^{j_2} \otimes \cdots \otimes K^{i_\nu} E^{j_\nu} F^{j_\nu}
\]

appear in pairs with the same coefficient.

The facts in (1), (2) and part (a) of (3) follow from Lemmas 6.2, 6.3 and 6.5 of [O1]. Part (b) of (3) follows from the proof of Lemma 6.3 of [O1].
Proof of Theorem 7.2: Theorem 7.2 follows from a careful study of the proof of Proposition 3.4 of [O1] (see Lemma 5.2). We will examine how the universal invariant changes when we double one component of $L$. Note that $\varphi_2(L')$ is the first possibly non-vanishing coefficient in the Taylor expansion of $\Phi(L'; t)$ at $t = 1$ by Lemma 5.2. Since

$$(-q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{\#L'} \Phi(L'; q^{-1}) = (\chi_V + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \epsilon)^{\otimes (\#L')} (\phi(L')),$$

we need to show that the first possibly non-vanishing coefficient in the expansion of $(\chi_V + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \epsilon)^{\otimes (\#L')} (\phi(L'))$ is always a multiple of 2 when some component of $L$ is doubled. As observed in [O1], we can treat $q$ as an indeterminate up to a certain order of the expansion. Therefore, according to the argument which proves Proposition 3.4 of [O1], it is sufficient to show that after factoring out $(q - 1)^2$ when $\nu \neq 1$ or $(q - 1)^4$ when $\nu = 1$ from each term

$$
\begin{cases}
(q - 1)^{2\nu} (\chi_V + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \epsilon)^{\otimes 2} \Delta(K^{i\nu} E^{j\nu} F^{j\nu}) & \text{for } \nu \neq 1 \\
(q - 1)^{2j_1} (\chi_V + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \epsilon)^{\otimes 2} \Delta(K^{i_1} E^{j_1} F^{j_1}) & \text{for } \nu = 1,
\end{cases}
$$

respectively, we always have a factor 2 in the constant coefficient.

First we consider $\nu \neq 1$. There are three cases: $j_\nu = 0, 1, 2$.

If $j_\nu = 0$, then

$$(\chi_V + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \epsilon)^{\otimes 2} \Delta(K^{i\nu}) = O(((q - 1)^4; \mathbb{Z}).$$

Since this is a higher order term, its contribution to $\varphi_2(L')$ is 0.

If $j_\nu = 1$, then

$$(q - 1)(\chi_V + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \epsilon)^{\otimes 2} \Delta(K^{i\nu} EF) = O(((q - 1)^3; \mathbb{Z}).$$

Again this is a higher order term.

If $j_\nu = 2$, then

$$(q - 1)^2(\chi_V + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \epsilon)^{\otimes 2} \Delta(K^{i\nu} E^2 F^2) = (q - 1)^2(2 + q + q^{-1})q^{j_\nu}.$$
So we have the desired factor 2.

Now consider $\nu = 1$. There are also three cases: $j_\nu = 0, 1, 2$.

If $j_1 = 0$, then
\[
(\chi_V + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\epsilon)^{\otimes 2} \Delta(K^{i_1}) = \left[\left(\frac{i_1}{2}\right)^2 - \left(\frac{r+1}{2}\right)^2\right](q-1)^4 + O((q-1)^5; \mathbb{Z}).
\]

As $K^{i_1}$ and $K^{-i_1}$ appear in pairs with the same coefficient, so we have a factor 2.

If $j_1 = 1$, then
\[
(q-1)^2(\chi_V + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\epsilon)^{\otimes 2} \Delta(K^{i_1} E F) = 2(q-1)^2q^{\frac{i_1}{2}}(q^{\frac{i_1-1}{2}} + q^{-\frac{i_1-1}{2}} - q^{-\frac{r+1}{2}} - q^{\frac{r+1}{2}}).
\]

If $j_1 = 2$, then
\[
(q-1)^4(\chi_V + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})\epsilon)^{\otimes 2} \Delta(K^{i_1} E^2 F^2) = (q-1)^4(2 + q + q^{-1})q^{i_1}.
\]

In both cases, we have the desired factor 2. This completes the proof of Theorem 7.2. □

**Proof of Theorem 7.1:** Combine (5.2), Corollary 5.1 (2) and Theorem 7.2, we get Theorem 7.1. □

8. Some problems

Here are several problems raised during the course of this work.

**Problem 1:** The motivation of [O2] is to give a definition of “finite type invariants of homology 3-spheres” which will include $\lambda_n$. To the best of our knowledge, it is still unknown whether $\lambda_n$’s are of finite type. See [R1,2] for the physical evidence indicating that $\lambda_n$’s are likely to be of “finite type”. We will address this problem in a sequel to this paper.
Problem 2: As $\lambda_1$ is essentially the Casson invariant and $\lambda_2$ is an integer, it is natural to ask if there is an analogous interpretation for $\lambda_2$ as the Casson invariant; or whether $\lambda_2/3$ counts algebraically any geometrical or topological objects related with the manifolds in question?

Problem 3: The series $\sum \lambda_n(t - 1)^n$ is only a formal series by definition. It seems interesting to know if this series actually converges to an analytic function for a fixed homology 3-sphere. See [L] for some examples where we do have the convergence. This is related to Problem 1 as follows. As conjectured in [LW] and proved in [B], for any Vassiliev invariant $v_k$ of order $k$, there is a constant $C$ such that for any knot of $n$ crossings, $|v_k(K)| \leq Cn^k$. If $\lambda_n$’s are indeed finite type invariants, it seems possible that there are analogous inequalities for $n! \cdot \lambda_n$. With some control over the constants, this will imply that $\sum \lambda_n(t - 1)^n$ converges to an analytic function for a fixed homology 3-sphere.

Problem 4: The Reshetikhin-Turaev invariant $\tau_r(M)$ for an arbitrary 3-manifold $M$ normalized as in [KM] has the property $\tau_r(M) \in \mathbb{Z}_r[q]$. Does their Fermat limit exist?

References

[AM] S. Akbulut and J. McCarthy, Casson’s invariant for oriented homology 3-spheres: An exposition, Math. Notes, vol. 36, Princeton University Press, 1990.

[B] D. Bar-Natan, Polynomial invariants are polynomial, Math. Res. Letter, 2(1995).

[Bo] H. Boden, Note on a conjecture of Lin and Wang, preprint, April 1996.

[J] V. Jones, Hecke algebra representations of braid groups and link polynomials, Ann. of Math., 186(1987).

[Ha] N. Habegger, Finite type 3-manifold invariants: a proof of a conjecture of Garoufalidis, preprint, July 1995.
G. Hardy and E. Wright, An introduction to the theory of numbers, Oxford University press, Fifth Edition, 1979.

J. Hoste, A formula for Casson’s invariant, Tran. AMS, 297(1986).

R. Kirby and P. Melvin, The 3-manifold invariants of Witten and Reshetikhin-Turaev for sl(2,C), Invent. Math., 105(1991).

S. Garoufalidis, On finite type 3-manifold invariants I, preprint, February 1995.

S. Garoufalidis and J. Levine, On finite type 3-manifold invariants II, preprint, June 1995.

S. Garoufalidis and J. Levine, On finite type 3-manifold invariants IV: comparison of definitions, preprint, September 1995.

M. Greenwood and X.-S. Lin, On Vassiliev knot invariants induced from finite type 3-manifold invariants, preprint, May 1995.

S. Garoufalidis and T. Ohtsuki, On finite type 3-manifold invariants III: manifold weight systems, preprint, August 1995.

R. Lawrence, Asymptotic expansion of Witten-Reshetikhin-Turaev invariants for some simple 3-manifolds, to appear in Jour. Math. Phys..

X.-S. Lin and Z. Wang, Integral geometry of plane curves and knot invariants, to appear in Jour. of Diff. Geom..

H. Morton, The coloured Jones function and Alexander polynomial for torus knots, Math. Proc. Camb. Phil. Soc., 117(1995).

P. Melvin and H. Morton, The coloured Jones function, Comm. Math. Phys., 169(1995).

H. Murakami, Quantum SU(2)-invariants dominate Casson’s SU(2)-invariant, Math. Proc. Camb. Phil. Soc., 115(1994).

T. Ohtsuki, A polynomial invariant of integral homology 3-spheres, Math. Proc. Camb. Phil. Soc., 117(1995).

T. Ohtsuki, Finite type invariants of integral homology 3-spheres, preprint, 1994.

L. Rozansky, The trivial connection contributions to Witten’s invariants of rational homology spheres, preprint, 1995.

L. Rozansky, Witten’s invariants of rational homology spheres at prime values of K and trivial connection contribution, preprint, 1995.
[RT] N.Y. Reshetikhin and V.G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math., **103**(1991).

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