A PERIODICITY RESULT FOR TILINGS OF $\mathbb{Z}^3$ BY CLUSTERS OF PRIME-SQUARED CARDINALITY

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Abstract. We show that if $\mathbb{Z}^3$ can be tiled by translated copies of a set $F \subseteq \mathbb{Z}^3$ of cardinality the square of a prime then there is a weakly periodic $F$-tiling of $\mathbb{Z}^3$, that is, there is a tiling $T$ of $\mathbb{Z}^3$ by translates of $F$ such that $T$ can be partitioned into finitely many 1-periodic sets.

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Let $F$ be a finite subset of $\mathbb{Z}^n$, which we will refer to as a cluster. A tiling of $\mathbb{Z}^n$ by translates of $F$, or an $F$-tiling, is, roughly, a covering of $\mathbb{Z}^n$ by non-overlapping translates of $F$. The periodic tiling conjecture (See [2]) states that if there is an $F$-tiling of $\mathbb{Z}^n$ for some cluster $F$, then there is an $F$-tiling which is invariant under translations by a finite index subgroup of $\mathbb{Z}^n$, that is, there is a fully periodic $F$-tiling. It was proved in [1] that if $F$ is a connected cluster (which roughly means that there are no ‘gaps’ in $F$) in $\mathbb{Z}^2$ then every $F$-tiling is periodic in at least one direction. This is sufficient to establish the existence of a fully periodic tiling by a simple pigeon-hole argument. An important progress towards the periodic tiling conjecture was made in [10] in which it was proved that the periodic tiling conjecture holds (in any dimension) for clusters of prime cardinality. Another major progress was made in [2] where it was established that the periodic tiling conjecture holds in $\mathbb{Z}^d$, without any constraint on the cardinality or the geometry of the cluster. The main result of [2] shows that the orbit closure of any $F$-tiling, where $F$ is a cluster in $\mathbb{Z}^2$, has a tiling which satisfies a certain weak notion of periodicity, which was shown to be sufficient to guarantee the existence of a fully periodic tiling. To make this precise, first we define a subset $S$ of $\mathbb{Z}^2$ as 1-periodic if there is a nonzero vector $g$ in $\mathbb{Z}^2$ such that $g + S = S$. We say that a subset $S$ of $\mathbb{Z}^2$ is weakly periodic if $S$ can be partitioned in to finitely many 1-periodic subsets. Bhattacharya [2] shows that the orbit closure of any $F$-tiling has a weakly periodic tiling in it. A stronger version of this result was obtained in [4] which says that every $F$-tiling of a cluster in $\mathbb{Z}^2$ is weakly periodic, thereby removing the need to pass to the orbit closure.

It was recently announced in [6] that the periodic tiling conjecture is false in $\mathbb{Z}^d$ if $d$ is sufficiently large. However, we conjecture that the following weaker statement might still hold.

**Conjecture 1. Weakly Periodic Tiling Conjecture.** Let $F \subseteq \mathbb{Z}^d$ be an exact cluster. Then there is a weakly periodic $F$-tiling. More precisely, there exists an $F$-tiling $T \subseteq \mathbb{Z}^d$ such that there is a finite partition $T = T_1 \cup \cdots \cup T_k$ such that each $T_i$ is 1-periodic.

In this paper we show that if $F \subseteq \mathbb{Z}^3$ is a cluster of cardinality the square of a prime such that there exists an $F$-tiling of $\mathbb{Z}^3$, then there exists a weakly periodic $F$-tiling of $\mathbb{Z}^3$ (See Theorem 4.4). This makes some progress towards the above conjecture beyond the already known result of Szegedy [10]. Unlike in two dimensions, the existence of a weakly periodic tiling alone does not seem (to the author) to imply the existence of a fully periodic tiling. In fact, to upgrade from a 1-periodic tiling of $\mathbb{Z}^3$ to a fully-periodic $F$-tiling also seems hard. The difficulty is genuine, since the problem of this upgradation is a close cousin of the periodic tiling conjecture for the group $\mathbb{Z}^2 \times (\mathbb{Z}/N\mathbb{Z})$ which, as noted in [2], is as yet an unresolved problem.

**1.1. Overview of the Proof.** In [2] the problem of finding a fully periodic $F$-tiling for a cluster $F \subseteq \mathbb{Z}^2$ was first reduced to showing the existence of a weakly periodic $F$-tiling by an elementary combinatorial argument. The existence of a weakly periodic $F$-tiling was then shown to follow from the following dynamical statement: If $\mathbb{Z}^2$ acts ergodically on a probability space $(X, \mu)$ and $A$ is a subset of $X$ such that finitely many $\mathbb{Z}^2$-translates of $A$ partition $X$, then $A$ itself is weakly periodic. To approach this problem, Bhattacharya uses the spectral theorem to transfer the problem to $L^2(T^2, \nu)$, where $\nu$ is the spectral measure associated with the characteristic function of $A$. Then using a dilation lemma and an averaging argument, it was shown that the spectral measure is supported on finitely many 1-dimensional affine subtorii of $T^2$. Then the averages of $f_A$ along the ‘directions’ of these subtorii (when $T^2$ is thought of as $\mathbb{R}^2/\mathbb{Z}^2$, each 1-dimensional subtorus can be thought of as a ‘line’ and hence has a direction) were shown to behave polynomially in $\mathbb{R}/\mathbb{Z}$. Since a polynomial in $\mathbb{R}/\mathbb{Z}$ is either periodic or equidistributes, the space $(X, \mu)$ gets partitioned into ergodic components of a finite index subgroup of $\mathbb{Z}^2$, such
that on each ergodic component, the averages either ‘equidistribute’ or are constant. The rest of the proof goes by carefully dealing with the equidistribution case.

Thus a key step in the argument was to show that the spectral measure is supported on a ‘thin’ subset of $T^2$. We take this as a cue and study the problem for a cluster $F \subseteq \mathbb{Z}^3$. If the corresponding spectral measure again happens to be supported on finitely many lines then Bhattacharya’s proof goes through *mutatis mutandis*. In the adverse case the spectral measure could be supported on ‘planes.’ To deal with these adverse cases we use the hypothesis that the size of the cluster is the square of a prime. This is done by Lemma 3.10 which places a geometric constraint on a cluster of prime-power cardinality in the adverse cases.

1.2. Organization of the Paper. In Section 2 we discuss the basic definitions and the dynamical formulation of the problem. Section 3 collects lemmas needed for the proof of the periodicity result. The results of this section may be read only when needed. In Section 4 we prove the main result. The part of the proof where Bhattacharya’s ideas go through by appropriate modification is collected in Section C.

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2. Preliminaries

2.1. Tilings and Periodicity.

**Definition 2.1.** A finite subset of $\mathbb{Z}^n$ will be referred to as a *cluster*. Given a cluster $F \subseteq \mathbb{Z}^n$ we say that $T \subseteq \mathbb{Z}^n$ is an $F$-tiling if for each $p \in \mathbb{Z}^n$ there exist unique $a \in F$ and $t \in T$ such that $p = a + t$. In other words, $T$ is an $F$-tiling if and only if $\mathbb{Z}^n$ can be partitioned by $T$-translates of $F$. We say that $F$ is exact if there exists an $F$-tiling.

**Definition 2.2.** Let $S$ be a subset of $\mathbb{Z}^n$. We say that $S$ is 1*-periodic* if there exists a rank-1 subgroup $\Lambda$ of $\mathbb{Z}^n$ such that $S$ is invariant under $\Lambda$, that is, $g + S = S$ for all $g \in \Lambda$.

**Definition 2.3.** A subset $S$ of $\mathbb{Z}^n$ is called 1*-weakly periodic*, or simply *weakly periodic* if there exist finitely many 1-periodic subsets $S_1, \ldots, S_k$ such that $S = S_1 \sqcup \cdots \sqcup S_k$.

A subset $S$ of $\mathbb{Z}^n$ is called 2*-weakly periodic* if there exist finitely many 2-periodic subsets $S_1, \ldots, S_k$ such that $S = S_1 \sqcup \cdots \sqcup S_k$.

We will use the following results about tilings of $\mathbb{Z}$ and $\mathbb{Z}^2$.

**Theorem 2.1.** Let $F \subseteq \mathbb{Z}$ be an exact cluster. Then there is a positive integer $n$ such that $n + T = T$ for all $F$-tilings $T$. In other words, every tiling of $\mathbb{Z}$ is 1-periodic.

**Proof.** Note that if $T\mid[a,b]$ is known for some interval $[a,b]$ of length exceeding $\text{diam}(F)$, then $T$ is determined entirely. Now the assertion follows by a pigeonhole argument. ■

**Theorem 2.2.** [10, 8, Example 4] Let $F \subseteq \mathbb{Z}^2$ be an exact cluster of prime cardinality such that the affine span of $F$ has rank 2. Then there is a finite index subgroup $\Lambda$ of $\mathbb{Z}^2$ such that every $F$-tiling is $\Lambda$-invariant.

**Theorem 2.3.** [2, 4] Let $F \subseteq \mathbb{Z}^2$ be an exact cluster, then there is a 2-periodic $F$-tiling.
2.2. Dynamical Formulation. In \[2\] Bhattacharya proved the periodic tiling conjecture (See \[9\]) in two dimensions by developing a dynamical statement which allows to transfer the problem to an ergodic theoretic setting. We discuss the relevant definitions and state the dynamical formulation here which we will use later. The formulation will be given for \(Z^3\).

Let \(X\) be a subshift of \(\{0,1\}^{Z^3}\) and \(\mu\) be a \(Z^3\)-invariant probability measure on \(X\). The action of \(Z^3\) on \(X\) naturally leads to a unitary action of \(Z^3\) on \(L^2(X,\mu)\) as follows. For \(\varphi \in L^2(X,\mu)\) and \(g \in Z^3\) we define \(g \cdot \varphi\) by
\[
(g \cdot \varphi)(x) = \varphi(g^{-1} \cdot x) = \varphi((-g) \cdot x)
\]
The action of \(Z^3\) on \(L^2(X,\mu)\) can be extended to an action of \(Z[u_1^Z, u_2^Z, u_3^Z]\) — the ring of Laurent polynomials in three variables with integer coefficients. To describe this action, first let us set up a convenient notation. For a vector \(v \in Z^3\), we denote the monomial \(u_1^v, u_2^v, u_3^v\) as \(U^v\), and denote the ring \(Z[u_1^Z, u_2^Z, u_3^Z]\) as \(Z[U^Z]\). Now any element of \(Z[U^Z]\) can be written as \(\sum_{v \in Z^3} a_v U^v\), where only finitely many \(a_v\)'s are nonzero. We define, for a Laurent polynomial \(p = \sum_{v \in Z^3} a_v U^v\) and \(\varphi \in L^2(X,\mu)\), the function \(p \cdot \varphi\) in \(L^2(X,\mu)\) as
\[
p \cdot \varphi = \left(\sum_v a_v U^v\right) \cdot \varphi = \sum_{v \in Z^3} a_v (v \cdot \varphi)
\]
We say that \(p\) annihilates \(\varphi\) if \(p \cdot \varphi = 0\).

For \(i \in \{1, 2, 3\}\), we say that \(\varphi \in L^2(X,\mu)\) is \(i\)-periodic if there is a rank-\(i\) subgroup \(\Lambda\) of \(Z^3\) such that \(v \cdot \varphi = \varphi\) for all \(v \in \Lambda\). A measurable subset \(B\) of \(X\) will be called \(i\)-periodic if \(1_B\) is \(i\)-periodic.

Finally, we define a measurable subset \(B\) of \(X\) to be 2-weakly periodic if there exists a partition \(B = B_1 \sqcup \cdots \sqcup B_k\) of \(B\) into finitely many measurable subsets \(B_1, \ldots, B_k\) such that each \(B_i\) is 2-periodic. Similarly, we say that \(B\) is 3-weakly periodic, or simply weakly periodic if \(B\) can be partitioned into finitely many 1-periodic subsets.

The following lemma is the 3-dimensional analog of \([2, \text{Section 2}]\).

**Lemma 2.4.** \([2]\) Section 2 | Write \(A = \{x \in X : x(0) = 1\}\). If \(A\) is \(i\)-periodic then \(\mu\)-almost every point in \(X\) is \(i\)-periodic. If \(A\) is \(i\)-weakly periodic, then \(\mu\)-almost every point in \(X\) is \(i\)-weakly periodic.

2.3. Spectral Theorem. Let \(\mathbb{T}^n\) be the \(n\)-dimensional torus. Let \(\nu\) be a probability measure on \(\mathbb{T}^n\). There is a canonical unitary representation of \(Z^3\) on \(L^2(\mathbb{T}^n, \nu)\) which we now describe. Recall that the characters on \(\mathbb{T}^n\) are in bijection with \(\mathbb{Z}^n\). Let us write \(\chi_g : \mathbb{T}^n \to C\) to denote the character corresponding to \(g \in \mathbb{Z}^n\). Then we define a map \(\sigma_g : L^2(\mathbb{T}^n, \nu) \to L^2(\mathbb{T}^n, \nu)\) by writing \(\sigma_g(\phi) = \chi_g \phi\), where the latter is the pointwise product of \(\chi_g \) and \(\phi\). It can be easily checked that each \(\sigma_g\) is in fact a unitary linear map. Thus we get a map \(\sigma : \mathbb{Z}^n \to \mathcal{U}(L^2(\mathbb{T}^n, \nu))\) which takes \(g\) to \(\sigma_g\).

By the Stone-Weierstrass theorem we have the \(C\)-span of the characters are dense in \(C(\mathbb{T}^n)\), where \(C(\mathbb{T}^n)\) is the set of all the complex valued continuous functions on \(\mathbb{T}^n\) equipped with the sup-norm topology. Also, since \(\mathbb{T}^n\) is a compact metric space, we have \(C(\mathbb{T}^n)\) is dense in \(L^2(\mathbb{T}^n, \nu)\). Therefore the \(C\)-span of the characters are dense in \(L^2(\mathbb{T}^n, \nu)\). From this we see that, if \(1\) denotes the constant map which takes the value 1 everywhere, \(\text{Span}\{\sigma_g : g \in \mathbb{Z}^n\}\) is dense in \(L^2(\mathbb{T}^n, \nu)\). In other words, \(1\) is a cyclic vector for this representation. We now want to state a theorem which dictates that this is a defining property of unitary representations of \(\mathbb{Z}^n\).

---

1There is a natural action of \(Z^3\) on \(\{0,1\}^{Z^3}\) by translations. A subshift of \(\{0,1\}^{Z^3}\) is any closed \(Z^3\)-invariant subset.

2We emphasize that this equation is written in \(L^2\) and hence, if \(f\) is an actual function, it only says that \(f\) and \(v \cdot f\) agree almost everywhere and not necessarily everywhere.
Theorem 2.5. **Spectral Theorem.** Let $H$ be a Hilbert space and $\tau : \mathbb{Z}^n \to \mathcal{U}(H)$ be a unitary representation of $\mathbb{Z}^n$. Suppose $v \in H$ is a cyclic vector, that is, $\text{Span}(\tau_v : g \in \mathbb{Z}^n)$ is dense in $H$, and assume that $v$ has unit norm. Then there is a unique probability measure $\nu$ on $\mathbb{T}^n$ and a unitary isomorphism $\theta : H \to L^2(\mathbb{T}^n, \nu)$ with $\theta(v) = 1$ such that the following diagram commutes for all $g \in \mathbb{Z}^n$

$$
\begin{array}{ccc}
H & \xrightarrow{\theta} & L^2(\mathbb{T}^n, \nu) \\
\downarrow{\tau} & & \downarrow{\sigma} \\
H & \xrightarrow{\theta} & L^2(\mathbb{T}^n, \nu)
\end{array}
$$

So the above theorem says that the abstract representation $\tau$ can be thought of as the canonical concrete representation $\sigma$, at the cost of a probability measure $\nu$. Thus understanding the measure $\nu$ is equivalent to understanding $\tau$.

3. Preparatory Lemmas

3.1. Some Combinatorial Lemmas.

**Lemma 3.1.** Let $F$ be a finite subset of $\mathbb{Z}^3$ and $p$ be a prime. Let $g$ be a nonzero vector in $\mathbb{Z}^3$. Assume that whenever $\pi$ is a plane in $\mathbb{R}^3$ parallel to $g$ we have $|\pi \cap F|$ is divisible by $p$. Then whenever $\ell$ is a line parallel to $g$ we have $|\ell \cap F|$ is also divisible by $p$. (The converse is also true.)

**Proof.** We may assume that $g$ is primitive. Using Lemma A.1 after applying a suitable $GL_3(\mathbb{Z})$ transformation, we may assume that $g = (0,0,1)$. Let $S$ be the image of $F$ under the map $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x,y,z) = (x,y)$$

for all $(x,y,z) \in \mathbb{Z}^3$, that is, $f$ is the orthogonal projection on the $x,y$-plane. Let $(a,b)$ be an extreme point of the convex hull of $S$ in $\mathbb{R}^2$ and let $\ell$ be a line in $\mathbb{R}^2$ such that $\ell \cap S = \{(a,b)\}$. Let $\pi$ be a plane parallel to $g$ in $\mathbb{R}^3$ such that the image of $\pi$ under $f$ is $\ell$. Then $\pi \cap F = \{(a,b,z) \in \mathbb{Z}^3 : (a,b,z) \in F\}$. By hypothesis, $|\pi \cap F|$ has size divisible by $p$. Now the set $F \setminus \pi \cap F$ satisfies the same hypothesis as $F$ and now we can finish inductively. ■

**Definition 3.1.** Let $S$ be a subset of $\mathbb{Z}^3$. Let $A$ be a rank $2$-subgroup of $\mathbb{Z}^3$ and $g$ be a nonzero vector such that $Zg + A$ is a rank-$3$ subgroup of $\mathbb{Z}^3$. We say that $S$ is a **prism** with **base** $A$ and **axis** $g$ if there is a vector $h \in \mathbb{Z}^3$ and a finite set $\{0 = n_0, n_1, \ldots, n_k\}$ of integers such that

$$S = \bigcup_{i=0}^k (n_i g + (h + A) \cap S)$$

We refer to $(h + A) \cap S$ as a **foundation** of $S$.

An example of a prism would be any set of the form $A \times B$ where $A \subseteq \mathbb{Z}^2$ and $B \subseteq \mathbb{Z}$.

**Lemma 3.2.** Let $F \subseteq \mathbb{Z}^3$ be a set of size $p^2$, where $p$ is a prime. Suppose that there is a nonzero vector $g$ in $\mathbb{Z}^3$ such that whenever $\pi$ is a plane parallel to $g$ in $\mathbb{R}^3$ we have $|\pi \cap F|$ divisible by $p$. Also, assume that there is a plane $\pi'$ not parallel to $g$ such that $|\pi'' \cap F|$ is divisible by $p$ whenever $\pi''$ is a plane parallel to $\pi'$. Then $F$ is a prism with foundation of size $p$. 

Proof. Using Lemma A.2, after a suitable $GL_3(\mathbb{Z})$ transformation, we may assume that $\pi'$ is the plane $\Lambda = \mathbb{Z}^2 \times \{0\}$. Since $\pi'$ is not parallel to $g$ by hypothesis, we see that $\mathbb{Z}g + \Lambda$ is a subgroup of rank 3 in $\mathbb{Z}^3$. Also, without loss of generality assume that $g$ is a primitive vector.\footnote{See Appendix A for the definition of a primitive vector.} By translating $F$ if necessary, we may assume that $\Lambda$ intersects $F$ but $\mathbb{Z}^2 \times \{k\}$ does not intersect $F$ whenever $k$ is a negative integer. Define $B = \{n \geq 0 : (\mathbb{Z}^2 \times \{n\}) \cap F \neq \emptyset\}$ and say $B = \{0 = b_0, b_1, \ldots, b_k\}$. Since, by hypothesis, each translate of $\Lambda$ intersects $F$ in a set of size divisible by $p$, we see that $(k + 1)p = |B|p \leq |F| = p^2$ and hence $k + 1 \leq p$. By Lemma 3.3, we know that whenever $\ell$ is a line in $\mathbb{R}^3$ parallel to $g$, we have that $\ell \cap F$ has size divisible by $p$. Thus there are at least $p$ translates of $\Lambda$ in the direction of $g$ which intersect $F$. Thus $k + 1 = |B| \geq p$. So we have $k + 1 = p$. This forces that $(\mathbb{Z}^2 \times \{b_i\}) \cap F$ has size $p$ for each $i \in \{0, 1, \ldots, k\}$.

It follows that any translate of $\Lambda$ must intersect each $\mathbb{Z}^2 \times \{b_i\}$ non-trivially. Let $p_0, p_1, \ldots, p_k \in \mathbb{Z}^2$ be such that $(p_i, b_i) - (p_0, b_0)$ are parallel to $g$ for each $i \in \{1, \ldots, k\}$. Since $g$ is a primitive vector, we can find integers $n_1, \ldots, n_k$ such that $n_i g = (p_i, b_i) - (p_0, b_0)$. It follows that

$$F = \bigcup_{i=0}^{k} (n_i g + (\Lambda \cap F))$$

and hence $F$ is prism with base $\Lambda$ and axis $g$, and foundation of size $p$. \hfill \blacksquare

Lemma 3.3. Let $F \subseteq \mathbb{Z}^3$ be a prism with foundation of prime size. Assume that $F$ is an exact cluster. Then there is a 2-periodic $F$-tiling.\footnote{Using this lemma, a simple pigeonhole argument shows that there exists a 3-periodic $F$-tiling.}

Proof. If the affine span of $F$ has dimension smaller than 3 then the result follows by Theorem 2.3.

Thus we may assume that the dimension of the affine span of $F$ is 3.

Let $\Lambda$ and $g$ be the base and axis of $F$ respectively. Let $h$ be a vector in $\mathbb{Z}^3$ and $\{n_0 = 0, \ldots, n_k\}$ be a set of integers such that $F = \bigcup_{i=0}^{k} (n_i g + (h + \Lambda) \cap F)$. By translating $F$ is necessary we may assume that $h = 0$. Also by Lemma A.2 we may without loss of generality assume that $\Lambda = \mathbb{Z}^2 \times \{0\}$.

Thus the affine span of $A := \Lambda \cap F$ has dimension 2. Let $T$ be an $F$-tiling of $\mathbb{Z}^3$. It is easy to see that for any integer $c$ the set $T_c := T \cap (\mathbb{Z}^2 \times \{c\})$ is a tiling of $\mathbb{Z}^2 \times \{c\}$ by translates of $A \times \{c\} \cong A$. By Theorem 2.2 we know that there is a rank-2 subgroup $\Gamma$ of $\mathbb{Z}^3$ such that $T_c$ is $\Gamma$-invariant for each $c$. Thus $T$ is $\Gamma$-invariant and hence 2-periodic. \hfill \blacksquare

Lemma 3.4. Dilation Lemma. \cite{7} Corollary 11] Let $F \subseteq \mathbb{Z}^3$. If $T$ is an $F$-tiling then for all $\alpha$ relatively prime with $|F|$ we have $T$ is also an $\alpha F$-tiling, where $\alpha F = \{\alpha a : a \in F\}$.

It should be noted that various authors (\cite{11, 10, 8, 2, 4}) had discovered the above lemma in one form or another.

3.2. An Analytical Lemma.

Definition 3.2. An element $\gamma \in S^1$ is said to be irrational \footnote{It is irrational if there is no non-trivial character of $S^1$ in whose kernel $\gamma$ lies. More generally, elements $\gamma_1, \ldots, \gamma_m$ in $S^1$ are called rationally independent if there is no non-trivial character $\chi$ of $(S^1)^m$ such that $\chi(\gamma_1, \ldots, \gamma_m) = 1$.} if it is equal to $e^{i\theta}$ for some $\theta$ which is an irrational multiple of $2\pi$. Equivalently, $\gamma \in S^1$ is irrational if there is no non-trivial character of $S^1$ in whose kernel $\gamma$ lies. More generally, elements $\gamma_1, \ldots, \gamma_m$ in $S^1$ are called rationally independent if there is no non-trivial character $\chi$ of $(S^1)^m$ such that $\chi(\gamma_1, \ldots, \gamma_m) = 1$.

Lemma 3.5. Let $\gamma_1, \ldots, \gamma_n$ be irrational elements in $S^1$ and $x_1, \ldots, x_n$ be complex numbers. Assume that

$$\begin{align*}
(\gamma_1^k - 1)x_1 + \cdots + (\gamma_n^k - 1)x_n &\in \mathbb{Z} \\
\end{align*}$$

for all non-negative integers $k$. Then the above expression is 0 for all $k$.\footnote{This is a consequence of the pigeonhole principle.}
Proof. Let \( m \) be the size of a maximal rationally independent subset of \( \{\gamma_1, \ldots, \gamma_n\} \). By renumbering the \( \gamma_i \)'s and \( x_i \)'s if required, we may assume that \( \{\gamma_1, \ldots, \gamma_m\} \) is a maximal rationally independent subset of \( \{\gamma_1, \ldots, \gamma_n\} \). Thus we can find vectors \( v_1, \ldots, v_n \in \mathbb{Z}^m \) such that
\[
\gamma_i = v_i^{(1)} \cdots v_i^{(m)} = \sum_{j=1}^{m} v_{ij}(1) \cdots v_{ij}(m) = \gamma_i
\]
for \( i = 1, \ldots, n \), where \( v(j) \) denotes the \( j \)-th coordinate of any \( v \in \mathbb{Z}^m \). By the algebraic independence of \( \gamma_1, \ldots, \gamma_m \), it follows that for \( i \in \{1, \ldots, m\} \) we have
\[
v_i(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}
\]
Define the Laurent polynomial \( f(U) \) in \( \mathbb{C}[u_1^\pm, \ldots, u_m^\pm] \) as
\[
f(U) = (U_1^{v_1} - 1)x_1 + \cdots + (U_n^{v_n} - 1)x_n
\]
Then we have
\[
f(\gamma_1^{k_1}, \ldots, \gamma_m^{k_m}) = (\gamma_1^{k_1} - 1)x_1 + \cdots + (\gamma_n^{k_n} - 1)x_n
\]
By the rational independence of \( \gamma_1, \ldots, \gamma_m \), we know that the set \( \{(\gamma_1^{k_1}, \ldots, \gamma_m^{k_m}) : k \geq 0\} \) is dense in \( (S^1)^m \). Thus the image of \( f \) on a dense set of \( (S^1)^m \) is contained in \( \mathbb{Z} \). Since \( f \) is continuous, this implies that the image of \( (S^1)^m \) under \( f \) is contained in \( \mathbb{Z} \). The connectedness of \( (S^1)^m \) now yields that the image of \( (S^1)^m \) under \( f \) is a singleton. However, for any \( \varepsilon > 0 \), we can find a \( k \geq 0 \) such that \( |\gamma_i^{k_i} - 1| < \varepsilon \) for each \( 1 \leq i \leq n \), and thus we must have that \( f \) is identically zero on \( (S^1)^m \), finishing the proof. \( \qed \)

3.3. An Elementary Fact about the Kernel of Characters.

**Lemma 3.6.** Let \( g \) and \( h \) be nonzero vectors in \( \mathbb{Z}^3 \) which are linearly independent. Then there is \( v \in \mathbb{Z}^3 \setminus \{0\} \) orthogonal to both \( g \) and \( h \) and a positive integer \( n \) such that
\[
\ker \chi_g \cap \ker \chi_h \subseteq \bigcup_{0 \leq i,j,k < n} \left[ \left( \frac{i}{n}, \frac{j}{n}, \frac{k}{n} \right) + \{tv \in \mathbb{R}^3 / \mathbb{Z}^3 : t \in \mathbb{Z}\} \right]
\]
Proof. Let \( g = (g_1, g_2, g_3) \) and \( h = (h_1, h_2, h_3) \). Consider the \( 2 \times 3 \)-matrix \( M \) over \( \mathbb{Q} \) defined as
\[
M = \begin{bmatrix} g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{bmatrix}
\]
Note that an element \( z \) of \( T^3 = \mathbb{R}^3 / \mathbb{Z}^3 \) is in \( \ker \chi_g \cap \ker \chi_h \) if and only if some (any) representative \( z' \) of \( z \) in \( \mathbb{R}^3 \) satisfies \( Mz' \in \mathbb{Z}^2 \). Since \( g \) and \( h \) are linearly independent, the rank of \( M \) is 2, and its nullity is therefore 1. Thus, since \( M \) is a matrix over \( \mathbb{Q} \), we can find a nonzero vector \( v \) in \( \mathbb{Q}^3 \) such that \( Mv = 0 \). Clearly, any such vector \( v \) also spans the null-space of \( M \) since the null-space of \( M \) is 1-dimensional. By scaling \( v \) if necessary, we may assume that \( v \in \mathbb{Z}^3 \).

Let \( \Lambda = M(\mathbb{Z}^3) \), that is, \( \Lambda \subseteq \mathbb{Z}^2 \) is the image of \( \mathbb{Z}^3 \) under \( M \). Since \( M \) has rank 2, we see that \( \Lambda \) is a finite index subgroup of \( \mathbb{Z}^2 \). Let \( n \) be the smallest positive integer such that \( (n,0) \) and \( (0,n) \) are both in \( \Lambda \). Say \( u_1 \) and \( u_2 \) in \( \mathbb{Z}^3 \) be such that \( M u_1 = (n,0) \) and \( M u_2 = (0,n) \). We will show that
\[
\ker \chi_g \cap \ker \chi_h \subseteq \bigcup_{0 \leq i,j,k < n} \left[ \left( \frac{i}{n}, \frac{j}{n}, \frac{k}{n} \right) + \{tv \in \mathbb{R}^3 / \mathbb{Z}^3 : t \in \mathbb{Z}\} \right]
\]
Let \( w \in \ker \chi_g \cap \ker \chi_h \) be arbitrary. Let \( w' \) be a representative of \( w \) in \( \mathbb{R}^3 \). Then \( M w' \) is of the form \( (a,b) \) for some \( (a,b) \in \mathbb{Z}^2 \). Therefore
\[
M w' = \frac{a}{n} M u_1 + \frac{b}{n} M u_2
\]
\[5\]See Theorem 4.14 in [3].
giving
\begin{equation}
M \left( w' - \frac{a}{n} u_1 - \frac{b}{n} u_2 \right) = 0
\end{equation}
Thus \( w' - (a/n) u_1 - (b/n) u_2 \) is in the null-space of \( M \), and hence there is \( t \in \mathbb{R} \) such that \( w' - (a/n) u_1 + (b/n) u_2 = tv \). Therefore \( w' = (au_1 + bu_2)/n + tv \). We can find \( 0 \leq i, j, k < n \) such that \((au_1 + bu_2)/n \equiv (i/n, j/n, k/n) \pmod{\mathbb{Z}^3}\). Therefore
\begin{equation}
w' \equiv \left( \frac{i}{n}, \frac{j}{n}, \frac{k}{n} \right) + tv \pmod{\mathbb{Z}^3}
\end{equation}
showing the desired containment. \[\square\]

3.4. Algebraic Lemmas.

**Lemma 3.7.** Let \( p \) be a prime and \( k \) be a positive integer. Let \( \zeta \) be a primitive \( p^k \)-th root of unity. Suppose there are integers \( a_1, \ldots, a_m \) such that
\begin{equation}
\zeta^{a_1} + \cdots + \zeta^{a_m} = 0
\end{equation}
Then \( p \) divides \( m \).

**Proof.** Without loss of generality we may assume that the \( a_i \)'s are all non-negative. Define the polynomial \( Q(z) = z^{a_1} + \cdots + z^{a_m} \). Then since \( \zeta \) is a root of \( Q(z) \), we have \( Q(z) \) is divisible by the \( p^k \)-th cyclotomic polynomial \( \Phi_{p^k}(z) \). Let \( \Psi(z) \) be an integer polynomial such that \( Q(z) = \Phi_{p^k}(z)\Psi(z) \). Now using the fact that \( \Phi_{p^k}(z) = \Phi_p(z^{p^k-1}) \) we have \( Q(z) = \Phi_p(z^{p^k-1})\Psi(z) \). Now substituting \( z = 1 \) gives \( m = p\Psi(1) \) and hence \( p \) divides \( m \). \[\square\]

**Definition 3.3.** For any finite subset \( S \) of \( \mathbb{Z}^3 \), we will write \( Z(\bigcup_{g \in S} \ker \chi_g) \) to mean the set of all the points in \( \mathbb{Z}^3 \) on which \( \sum_{g \in S} \chi_g \) vanishes.

**Lemma 3.8.** (See [2] Lemma 3.2) Let \( F \subseteq \mathbb{Z}^3 \) be a finite set containing the origin. Then there is a finite subset \( \Delta \) of \( \mathbb{Z}^3 \) with pairwise linearly independent elements such that
\begin{equation}
Z := \bigcap_{\alpha \text{ coprime to } |F|} Z \left[ \sum_{g \in F} \chi_{\alpha g} \right] \subseteq \bigcup_{h \in \Delta} \ker \chi_h
\end{equation}

**Proof.** Let \( n \) be the product of all primes which divide \( |F| \). Then for each non-negative integer \( k \) we have \( nk + 1 \) is relatively prime to \( |F| \). Fix \( (a, b, c) \in Z \). Then we have, for all non-negative integers \( k \) that
\begin{equation}
\sum_{g \in F} \chi_{(nk+1)g}(a, b, c) = 0
\end{equation}
giving
\begin{equation}
\sum_{g \in F \setminus \{0\}} \chi_{(nk+1)g}(a, b, c) = \sum_{g \in F \setminus \{0\}} a^{g_1} b^{g_2} c^{g_3} (a^{g_1} b^{g_2} c^{g_3})^k = -1
\end{equation}
Therefore
\begin{equation}
\sum_{g \in F \setminus \{0\}} a^{g_1} b^{g_2} c^{g_3} \left[ \frac{1}{N} \sum_{k=0}^{N-1} (a^{g_1} b^{g_2} c^{g_3})^k \right] = -1
\end{equation}
for all \( N \geq 1 \). Taking the limit \( N \to \infty \) we see that there must exist \( g \in F \setminus \{0\} \) such that \( a^{g_1} b^{g_2} c^{g_3} \) is \( 1 \) and thus \( (a, b, c) \in \ker(\chi_{ng}) \). Therefore \( Z \subseteq \bigcup_{g \in F \setminus \{0\}} \ker \chi_{ng} \).

\footnote{This is because if \( z \in S^1 \) then the average \( (z + z^2 + \cdots + z^N)/N \) does not converge to zero if and only if \( z = 1 \).}
Let \( \Delta \) be a non-empty subset of \( \mathbb{Z}^3 \setminus \{0\} \) of smallest possible size such that \( Z \subseteq \bigcup_{g \in \Delta} \ker \chi_g \). Such a \( \Delta \) exists by the above paragraph. We claim that the elements of \( \Delta \) are pairwise linearly independent. Suppose not. Then there exist distinct \( g, h \in \Delta \) such that \( g \) and \( h \) are linearly dependent. We can thus find a nonzero vector \( v \) such that \( v \in (Zg) \cap (Zh) \). Let \( \Delta' = (\Delta \setminus \{g, h\}) \cup \{v\} \). It is clear that

\[
\bigcup_{u \in \Delta} \ker \chi_u \subseteq \bigcup_{u \in \Delta'} \ker \chi_u
\]

and hence \( Z \subseteq \bigcup_{u \in \Delta'} \ker \chi_u \). But \( \Delta' \) has size strictly smaller than the size of \( \Delta \) which is a contradiction to the choice of \( \Delta \). This finishes the proof.

**Lemma 3.9.** Let \( F \) be a cluster in \( \mathbb{Z}^3 \) containing the origin and \( p \) be a prime. Assume \( |F| \) is a power of \( p \). Let \( h \) be an arbitrary nonzero vector in \( \mathbb{Z}^3 \). Then at least one of the following must happen.

1. Every line in \( \mathbb{R}^3 \) that is parallel to \( h \) intersects \( F \) in a set of size divisible by \( p \).
2. There is a finite set \( \Gamma \subseteq \mathbb{Z}^3 \setminus \{0\} \) such that each element of \( \Gamma \) is linearly independent with \( h \) and

\[
\ker \chi_h \cap \left( \bigcap_{\alpha \text{ coprime to } |F|} \mathbb{Z} \left[ \sum_{g \in F} \chi_{ag} \right] \right) \subseteq \bigcup_{v \in \Gamma} \ker \chi_h \cap \ker \chi_v
\]

**Proof.** Using Lemma 3.8 after applying a suitable \( GL_3(\mathbb{Z}) \) transformation, we may assume that \( h = (0, 0, m) \) for some positive integer \( m \).

Let \( (a, b, c) \) be an arbitrary element in \( \ker \chi_h \cap Z \), where

\[
Z = \bigcap_{\alpha \text{ coprime to } |F|} \mathbb{Z} \left[ \sum_{g \in F} \chi_{ag} \right]
\]

Let \( h_0 = (0, 0, 1) \). Then

\[
\ker \chi_h = \bigcup_{\omega \in S^1, \omega = 1} (1, 1, \omega) \cdot \ker \chi_{h_0}
\]

Thus there is an \( m \)-th root of unity \( \omega \) such that \( (a, b, c) \in ((1, 1, \omega) \cdot \ker \chi_{h_0}) \cap Z \). Then \( c = \omega \) and hence

\[
\sum_{g \in F} \chi_{ag}(a, b, \omega) = 0
\]

for all \( \alpha \) relatively prime with \( |F| \).

Let \( d \) be a positive integer such that \( \omega \) is a primitive \( d \)-th root of unity. Let \( d = p^r \beta \) where \( \beta \) is relatively prime to \( p \). Let \( n \) be the product of all the primes dividing \( |F| \). For all non-negative integers \( k \), we have \( mnk + 1 \) is relatively prime to \( |F| \). Using the fact that \( |F| \) is a power of \( p \), we have \( (mnk + 1) / \beta \) is also relatively prime with \( |F| \). So we have, by substituting \( \alpha = (mnk + 1) \beta / \beta \), that

\[
\sum_{g \in F} \chi_{(mnk+1)\beta g}(a, b, \omega) = \sum_{g \in F} a^{\beta g_1} b^{\beta g_2} \omega^{\beta g_3} \chi_{mnk\beta(g_1, g_2)}(a, b) = 0
\]

for all non-negative integers \( k \). Let \( \zeta = \omega^\beta \), and hence \( \zeta \) is a primitive \( p^r \)-th root of unity. Let \( \sim \) be a relation on \( F \) defined as \( g \sim g' \) for \( g, g' \in F \) if and only if \( (g, g_2) = (g', g_2) \). Then \( \sim \) is

---

7This is because of the following. For any linear map \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) such that \( T \in GL_3(\mathbb{Z}) \), and any \( F \subseteq \mathbb{Z}^3 \) finite, we have \( T \) is \( \mathbb{Z} \) \( \left( \sum_{g \in F} \chi_g \right) \) if and only if \( (T^{-1})^* p \in \mathbb{Z} \left( \sum_{g \in T(F)} \chi_g \right) \).
an equivalence relation, and let $E$ be the set of all the equivalence classes of $\sim$. For each $E \in E$ choose an element $g^E$ of $E$. For any $v \in \mathbb{Z}^3$ let $\hat{v}$ denote the vector in $\mathbb{Z}^2$ obtained by dropping the last coordinate of $v$. Then Equation 3.24 gives

$$\sum_{E \in E} \left( \sum_{g \in E} \zeta^{g^E} \chi_{\beta g^E}(a,b) \chi_{mnk\beta g^E}(a,b) \right) = 0$$

for all $k$ non-negative. Let $\theta_E = \sum_{g \in E} \zeta^{g^E}$ and $\gamma_E = \theta_E \chi_{\beta g^E}(a,b)$ for all $E \in E$. If $\theta_E = 0$ for all $E \in E$, then by Lemma 3.7 we have each $|E|$ is divisible by $p$. This would mean precisely that (1) holds. So we may assume that there is some $E_0 \in E$ such that $\theta_{E_0}$, or equivalently $\gamma_{E_0}$, is nonzero. Then from Equation 3.24 we have

$$\chi_{-mnk\beta g^E}(a,b) \left( \sum_{E \in E} \gamma_E \chi_{mnk\beta g^E}(a,b) \right) = 0$$

which gives

$$\gamma_{E_0} + \sum_{E \in E \setminus \{E_0\}} \gamma_E \chi_{mnk\beta}\hat{g}^{E_0}(a,b) = \gamma_{E_0} + \sum_{E \in E \setminus \{E_0\}} \gamma_E \chi_{mnk\beta}\hat{g}^{E_0}(a,b)^k = 0$$

Averaging we have

$$\gamma_{E_0} + \sum_{E \in E \setminus \{E_0\}} \gamma_E \left[ \frac{1}{N} \sum_{k=0}^{N-1} \chi_{mnk\beta}\hat{g}^{E_0}(a,b)^k \right] = 0$$

Taking limit $N \to \infty$, we must have, since $\gamma_{E_0} \neq 0$, that $(a,b) \in \ker \chi_{mnk\beta}\hat{g}^{E_0}$ for some $E \neq E_0$ in $E$. Define $h^E = mnk\beta(g_1^E - g_1^{E_0}, g_2^E - g_2^{E_0}, 1)$. Then $h^E$ is linearly independent with $h$ since $\hat{g}^{E_0} - \hat{g}^{E_0}$ is nonzero. Also $(a,b,c) \in \ker \chi_{h^E}$. So we have shown that if (1) is assumed to be false then (2) holds and we are done. ■

**Lemma 3.10.** Let $F$ be a cluster in $\mathbb{Z}^3$ containing the origin and $p$ be a prime. Assume $|F|$ is a power of $p$. Let $h$ be an arbitrary nonzero vector in $\mathbb{Z}^3$. Then at least one of the following must happen.

1. Every line in $\mathbb{R}^3$ that is parallel to $h$ intersects $F$ in a set of size divisible by $p$.
2. There is a finite set $R$ of rational points in $\mathbb{R}^3/\mathbb{Z}^3$ and a finite set $V$ of nonzero vectors in $\mathbb{Z}^3$ such that

$$\ker \chi_h \cap \left( \bigcap_{\alpha \text{ coprime to } |F|} Z \left[ \sum_{g \in F} \chi_{\alpha g} \right] \right) \subseteq \bigcup_{\rho \in R} \bigcup_{v \in V} \{ \rho + tv \in \mathbb{R}^3/\mathbb{Z}^3 : t \in \mathbb{T} \}$$

**Proof.** Assume (1) does not hold. Then by Lemma 3.9 we can find a finite set $\Gamma \subseteq \mathbb{Z}^3 \setminus \{0\}$ such that each element of $\Gamma$ is linearly independent with $h$ and

$$\ker \chi_h \cap \left( \bigcap_{\alpha \text{ coprime to } |F|} Z \left[ \sum_{g \in F} \chi_{\alpha g} \right] \right) \subseteq \bigcup_{w \in \Gamma} \ker \chi_h \cap \ker \chi_u$$

Using Lemma 3.6 we know that for each $u \in \Gamma$ there is a finite set of rational points $R_u$ and a nonzero vector $v_u$ such that

$$\ker \chi_h \cap \ker \chi_u \subseteq \bigcup_{\rho \in R_u} \{ \rho + tv_u \in \mathbb{R}^3/\mathbb{Z}^3 : t \in \mathbb{R} \}$$

Note that the equivalence classes are precisely the sets that are obtained by intersection $F$ with a line parallel to $h$. 

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\footnote{Note that the equivalence classes are precisely the sets that are obtained by intersection $F$ with a line parallel to $h$.}
Define $R = \bigcup_{u \in \Gamma} R_u$ and $V = \{ v_u : u \in \Gamma \}$. It follows that
\begin{equation}
(3.32) \quad \bigcup_{u \in \Gamma} \ker \chi_h \cap \ker \chi_u \subseteq \bigcup_{\rho \in R} \bigcup_{v \in V} \{ \rho + tv : t \in \mathbb{R} \}
\end{equation}
whence condition (2) immediately holds.

**Lemma 3.11.** Let $F \subseteq \mathbb{Z}^3$ be a cluster containing the origin and $p$ be a prime. Let $\rho$ be a rational point in $\mathbb{R}^3/\mathbb{Z}^3$ and $v$ be a nonzero vector in $\mathbb{Z}^3$ such that infinitely many points of the set
\begin{equation}
(3.33) \quad \{ \rho + tv : t \in \mathbb{R} \}
\end{equation}
are in
\begin{equation}
(3.34) \quad Z = \bigcap_{\alpha \text{ coprime to } p} \mathbb{Z} \left[ \sum_{g \in F} \chi_{ag} \right]
\end{equation}
Then any plane parallel to $\{ w \in \mathbb{R}^3 : \langle w, v \rangle = 0 \}$ intersects $F$ in a set of size divisible by $p$.

*Proof.* Let $h_0$ be a primitive vector in $\mathbb{Z}^3 \setminus \{0\}$ and $r$ be a rational number such that $\rho = rh_0$ in $\mathbb{R}^3/\mathbb{Z}^3$. Then, by hypothesis, infinitely many points of
\begin{equation}
(3.35) \quad \{ rh_0 + tv : t \in \mathbb{R}^3 : 0 \leq t < 1 \}
\end{equation}
are in $Z$. Let $r = c/d$, where $c$ and $d$ are relatively prime integers. We get, for each $\alpha$ coprime to $p$,
\begin{equation}
(3.36) \quad \sum_{g \in F} e^{2\pi i \alpha(g,h_0)} e^{2\pi i \alpha(g,v)t} = 0
\end{equation}
for infinitely many $t \in [0,1)$. Therefore, the (Laurent) polynomial
\begin{equation}
(3.37) \quad \sum_{g \in F} e^{2\pi i \alpha(g,h_0)} z^{\alpha(g,v)}
\end{equation}
is satisfied by infinitely many $z \in S^1$ for each $\alpha$ coprime to $p$. Let $d = p^m \beta$, where $m$ is a non-negative integer and $\beta$ is coprime to $p$. Thus, by Equation 3.37, the polynomial
\begin{equation}
(3.38) \quad \sum_{g \in F} e^{2\pi i \beta(g,h_0)} z^{\beta(g,v)} = \sum_{g \in F} e^{2\pi i (g,h_0)c/p^m} z^{\beta(g,v)}
\end{equation}
has infinitely many solutions in $S^1$ and is hence identically zero. Define an equivalence relation $\sim_\alpha$ on $F$ by writing $g_1 \sim_\alpha g_2$ for $g_1, g_2 \in F$ if $\langle g_1, v \rangle = \langle g_2, v \rangle$. Let $F_1, \ldots, F_l$ be all the equivalence classes in $F$. The coefficients of the above polynomial are
\begin{equation}
(3.39) \quad \sum_{g \in F_j} e^{2\pi i (g,h_0)c/p^m}, \quad j = 1, \ldots, l.
\end{equation}
and hence each of these terms are 0. If $m = 0$ then this cannot happen since $\sum_{g \in F_j} e^{2\pi i (g,h_0)c} = |F_j|$, and hence we must have $m \geq 1$. This implies that $p$ divides $d$ and hence, since $c$ is relatively prime to $d$, we have $c$ is coprime to $p$. Let $\zeta$ denote the complex number $e^{2\pi i/p^m}$. By Equation 3.39 we have $\zeta$ is a root of each of the (Laurent) polynomials
\begin{equation}
(3.40) \quad \sum_{g \in F_j} z^{(g,h_0)c}, \quad j = 1, \ldots, l
\end{equation}
Therefore by Lemma 3.7, we have that each $|F_j|$ is divisible by $p$. This implies that any plane parallel to $\{ w \in \mathbb{R}^3 : \langle w, v \rangle = 0 \}$ intersects $F$ in a set of cardinality divisible by $p$ and we are done. \qed
4. The Periodicity Result

4.1. Main Theorem.

**Theorem 4.1.** Let \( F \) be an exact cluster in \( \mathbb{Z}^3 \) with cardinality \( p^2 \), where \( p \) is a prime. Then there is a 1-weakly periodic \( F \)-tiling.

4.2. Proof.

4.2.1. Notation. Let \( X \subseteq \{0,1\}^3 \) be the set of all the \( F \)-tilings and \( \mu \) be a \( \mathbb{Z}^3 \)-ergodic probability measure on \( X \), which we may assume to be concentrated on the orbit closure of a given \( F \)-tiling \( T \). We define \( A = \{x \in X : x(0) = 1\} \) and \( f \in L^2(X,\mu) \) as the characteristic function of \( A \). Let \( \nu \) be the spectral measure associated to the unit vector \( f/\|f\|_2 \) in \( L^2(X,\mu) \) and \( \theta \) be the unitary isomorphism between \( H_f \) — the span closure of the orbit of \( f/\|f\|_2 \) — and \( L^2(\mathbb{T}^3,\nu) \) as discussed in Theorem 2.3. We will use \( \mathbb{T} \) to denote the unit circle and not \( \mathbb{R}/\mathbb{Z} \).

By Lemma 3.4 we have \( T \) is an \((\alpha F)\)-tiling whenever \( \alpha \) is relatively prime to \( p \). Thus

\[
(4.1) \quad \sum_{g \in F} 1_{\alpha gA} = \sum_{g \in F} (\alpha g) \cdot 1_A = 1_X
\]

for all \( \alpha \) relatively prime to \( p \). Now \( 1_X \) is \( \mathbb{Z}^3 \)-invariant in \( L^2(X,\mu) \), and hence \( \theta(1_X) \) is \( \mathbb{Z}^3 \)-invariant in \( L^2(\mathbb{T}^3,\nu) \). But the only \( \mathbb{Z}^3 \)-invariant members of \( L^2(\mathbb{T}^3,\nu) \) are the ones in the span of \( \delta_{\{1\}} \)—the Dirac function concentrated at the identity of \( \mathbb{T}^3 \). Thus, applying \( \theta \), we get

\[
(4.2) \quad \sum_{g \in F} \chi_{\alpha g} = c\delta_{\{1\}}, \quad \forall \alpha \text{ coprime to } p
\]

for some constant \( c \). Therefore, the spectral measure \( \nu \) is supported on \( Z \cup \{1\} \), where

\[
(4.3) \quad Z = \bigcap_{\alpha \text{ coprime to } p} Z \left[ \sum_{g \in F} \chi_{\alpha g} \right]
\]

By Lemma 3.8 we know that there is a finite set \( \Delta \subseteq \mathbb{Z}^3 \setminus \{0\} \) whose elements are pairwise linearly independent such that the spectral measure \( \nu \) associated to \( f/\|f\|_2 \) is supported on \( \bigcup_{g \in \Delta} \text{ker}(\chi_g) \).

4.2.2. Case 1: There exist at least two distinct members \( g_0 \) and \( g_1 \) in \( \Delta \) such that every line parallel to either \( g_0 \) or \( g_1 \) intersects \( F \) in a set of cardinality divisible by \( p \). By Lemma 3.2 we deduce that \( F \) is a prism with prime foundation and hence by Lemma 3.3 we see that there is a 2-periodic \( F \)-tiling, which is in particular 1-weakly periodic.

4.2.3. Case 2: There is exactly one \( g_0 \) in \( \Delta \) such that every line parallel to \( g_0 \) intersects \( F \) in a set of cardinality divisible by \( p \). In this case, we see that for each \( g \in \Delta \setminus \{g_0\} \) the condition (1) in Lemma 3.10 is not satisfied, and hence by Lemma 3.10 condition (2) must be satisfied. Thus for each \( g \in \Delta \setminus \{g_0\} \) we can find a finite set \( R_g \) of rational points in \( \mathbb{R}^3/\mathbb{Z}^3 \) and a finite set of vectors \( V_g \) such that

\[
(4.4) \quad Z \cap \text{ker} \chi_g \subseteq \bigcup_{\rho \in R_g} \bigcup_{v \in V_g} \{\rho + tv : t \in \mathbb{R}\}
\]

Let \( R = \bigcup_{g \in \Delta \setminus \{g_0\}} R_g \) and \( V = \bigcup_{g \in \Delta \setminus \{g_0\}} V_g \). Let \( N \) be positive integer such that \( \text{ker} \chi_{N g_0} \) contains \( R \). We further consider two subcases.
Subcase 2.1: Assume that there is \( g_1 \) distinct from \( g_0 \) in \( \Delta \) such that

\[
(Z \cap \ker \chi_{g_1}) \setminus \ker \chi_{N g_0}
\]

is infinite. Then, in particular, \( Z \cap \ker \chi_{g_1} \) is also infinite. Thus, by Equation 4.4 there is \( \rho \in R \) and \( v \in V \) such that infinitely many elements of

\[
\{ \rho + vt \in \mathbb{R}^3/\mathbb{Z}^3 : t \in \mathbb{R} \} \setminus \ker \chi_{N g_0}
\]

are in \( Z \), and none of its elements are in \( \ker \chi_{N g_0} \). Now by Lemma 3.11 we have that whenever \( \pi \) is a plane parallel to \( \pi_1 := \{ w \in \mathbb{R}^3 : \langle v, w \rangle = 0 \} \), we have \( |\pi \cap F| \) is divisible by \( p \). Also, by the property of \( g_0 \) we already know that whenever \( \pi \) is a plane parallel to \( g_0 \) we have \( |\pi \cap F| \) is divisible by \( p \).

If \( v \) is orthogonal to \( g_0 \) then, by the choice of \( N \), the set in Equation 4.6 is empty. Thus \( v \) is not orthogonal to \( g_0 \). By the fact that \( v \) is not orthogonal to \( g_0 \), we have that a plane parallel to \( g_0 \) cannot be parallel to \( \pi_1 \). Then, again, by Lemma 3.2 we deduce that \( F \) is a prism with prime foundation and hence by Lemma 3.3 we see that there is a 2-periodic \( F \)-tiling, which is in particular 1-weakly periodic.

Subcase 2.2: Now assume that \( (Z \cap \ker \chi_{g_1}) \setminus \ker \chi_{N g_0} \) is finite whenever \( h \in \Delta \setminus \{ g_0 \} \). For each \( h \in \Delta \setminus \{ g_0 \} \), let \( S_h = (Z \cap \ker \chi_{g_1}) \setminus \ker \chi_{N g_0} \). Then, by our assumption, each \( S_h \) is finite. Since \( \supp(\nu) \) is contained in \( Z \cap \bigcup_{g \in \Delta} \ker \chi_g \), we see that, in particular,

\[
\supp(\nu) \subseteq \ker \chi_{N g_0} \cup \left( \bigcup_{h \in \Delta \setminus \{ g_0 \}} Z \cap \ker \chi_{g_1} \right) = \ker \chi_{N g_0} \cup \left( \bigcup_{h \in \Delta \setminus \{ g_0 \}} S_h \right)
\]

Therefore

\[
\ker(\chi_{N g_0}) \cup \bigcup_{h \in \Delta \setminus \{ g_0 \}} S_h
\]

is a \( \nu \)-full measure set.

Replacing \( N g_0 \) by \( g_0 \) for simplicity of notation, we infer that there is a nonzero vector \( g_0 \) in \( \mathbb{Z}^3 \) and a finite set \( \mathcal{S} \subseteq \mathbb{T}^3 \) disjoint with \( \ker \chi_{g_0} \) such that \( \nu \) is supported on \( \ker(\chi_{g_0}) \cup \mathcal{S} \). We may assume that \( S \) has smallest size with this property and hence each element of \( S \) has positive mass under \( \nu \). The minimality of the size of \( S \) also implies that \( \chi_{g_0}(s) \) is irrational for all \( s \in S \), for otherwise we could replace \( g_0 \) by a scale of itself and reduce the size of \( S \).

We will show that \( S \) is empty. Assume on the contrary that \( S \) is non-empty. Define an equivalence relation \( \sim \) on \( S \) by writing \( p \sim q \) for \( p, q \in S \) if \( \chi_{g_0}(p) = \chi_{g_0}(q) \). Let \( \mathcal{E} \) be the set of all the equivalence classes. Thus

\[
1 = 1_{\ker(\chi_{g_0})} + \sum_{E \in \mathcal{E}} 1_E
\]

in \( L^2(\mathbb{T}^3, \nu) \). Choose a representative \( pE \in E \) for each \( E \in \mathcal{E} \). Now acting both sides of the above equation by \( n g_0 \), where \( n \) is any non-negative integer, we get

\[
\chi_{n g_0} = 1_{\ker(\chi_{g_0})} + \sum_{E \in \mathcal{E}} \chi_{g_0}(pE)^n 1_E
\]

Going back into the \( L^2(\mathbb{T}, \mu) \) world by applying \( \theta^{-1} \), we get that

\[
(n g_0) \cdot f = f^{g_0} + \sum_{E \in \mathcal{E}} \chi_{g_0}(pE)^n \nu_E
\]
where \( \varphi_E = ||f||_2\theta^{-1}(1_E) \), and \( f^{g_0} \) is the orthogonal projection of \( f \) onto the space of \( g_0 \)-invariant functions \( L^2(X, \mu) \).

Therefore, for each \( n \geq 0 \), we have

\[
(n g_0) \cdot f - f = \left( f^{g_0} + \sum_{E \in \mathcal{E}} \chi_{g_0}(pE) n \varphi_E \right) - \left( f^{g_0} - \sum_{E \in \mathcal{E}} \varphi_E \right) = \sum_{E \in \mathcal{E}} (\chi_{g_0}(pE) n - 1) \varphi_E
\]

(4.12)

Let \( Y \subseteq X \) be a \( \mu \)-full measure subset of \( X \) such that

\[
[(n g_0) \cdot f](y) - f(y) = \sum_{E \in \mathcal{E}} (\chi_{g_0}(pE) n - 1) \varphi_E(y)
\]

for all \( y \in Y \) and all \( n \geq 0 \). Therefore

\[
\sum_{E \in \mathcal{E}} (\chi_{g_0}(pE) n - 1) \varphi_E(y) \in \mathbb{Z}
\]

(4.13)

for all \( y \in Y \), and all \( n \geq 0 \). But since each \( \chi_{g_0}(pE) \) is irrational, we may apply Lemma 3.5 to deduce that for any \( y \in Y \) the only value the above expression can take, for any non-negative integer \( n \), and hence in particular for \( n = 1 \), is 0. Therefore \([g_0 \cdot f](y) - f(y)\) is 0 for each \( y \in Y \). But since \( Y \) is a full measure set in \( X \), we infer that \( g_0 \cdot f = f \) in \( L^2(X, \mu) \). Thus \( f \) is 1-periodic, and hence by 2.4 we deduce that the orbit closure of \( T \) has a 1-periodic point in it, which is in particular 1-weakly periodic.

4.2.4. Case 3: There is no \( g \) in \( \Delta \) such that every line parallel to \( g \) intersects \( F \) in a set of size divisible by \( p \). In this case, by Lemma 3.10 we deduce that there exist nonzero vectors \( v_1, \ldots, v_n \in \mathbb{Z}^3 \) and finite subsets \( S_1, \ldots, S_n \subseteq \mathbb{R}^3 / \mathbb{Z}^3 \) such that each member of each \( S_i \) is a rational point and the measure \( \nu \) is supported on

\[
\bigcup_{i=1}^n \left( S_i + \{ tv_i \in \mathbb{R}^3 / \mathbb{Z}^3 : t \in \mathbb{R} \} \right)
\]

(4.15)

We are now done by Theorem C.1.

Appendix A. Algebra

Definition A.1. A nonzero vector \( a = (a_1, \ldots, a_n) \) in \( \mathbb{Z}^n \) is said to be primitive if

\[
gcd(a_1, \ldots, a_n) = 1
\]

Lemma A.1. Let \( n \geq 1 \). Then \( GL_n(\mathbb{Z}) \) acts transitively on the set of all the primitive vectors in \( \mathbb{Z}^n \).

Proof. Let \( a \in \mathbb{Z}^n \) be an arbitrary primitive vector. We will show that there is \( T \in GL_n(\mathbb{Z}) \) such that \( T e_n = a \), where \( e_n = (0, \ldots, 0, 1) \). Let \( \Lambda \) be the subgroup of \( \mathbb{Z}^n \) generated by \( a \). We claim that \( \mathbb{Z}^n / \Lambda \) has no torsion. Assume on the contrary that there is torsion in \( \mathbb{Z}^n / \Lambda \), so that there is a non-trivial element \( g \in \mathbb{Z}^n / \Lambda \) and a positive integer \( k \) such that \( k \cdot g = 0 \). This implies the existence of an element \( v \in \mathbb{Z}^n \setminus \Lambda \) such that \( kv \in \Lambda \). Thus \( kv = ma \) for some nonzero integer \( m \). It follows that \( k \) must divide \( m \), for otherwise \( a \) would not be primitive. But then \( v \in \Lambda \), contrary to the choice of \( v \).

So \( \mathbb{Z}^n / \Lambda \) has no torsion, and thus it is isomorphic to \( \mathbb{Z}^{n-1} \). Let \( v_1, \ldots, v_{n-1} \) in \( \mathbb{Z}^n \) be such that \( v_1 + \Lambda, \ldots, v_{n-1} + \Lambda \) forms a basis of \( \mathbb{Z}^n / \Lambda \). Declare \( v_n = a \) and we get a basis \( v_1, \ldots, v_n \) of \( \mathbb{Z}^n \).

9By the Birkhoff ergodic theorem we see that \( f^{g_0} \) lies in \( H_f \). Also, \( 1_{\ker \chi_{g_0}} \) is the orthogonal projection of 1 onto the space of \( g_0 \)-invariant functions in \( L^2(\mathbb{T}^2, \nu) \). The fact that \( \theta \) is a unitary isomorphism shows that \( \theta^{-1}(1_{\ker \chi_{g_0}}) \) is same as \( f^{g_0} ||f||_2 \).

10This is because for linearly independent vectors \( g \) and \( h \) in \( \mathbb{Z}^3 \) we have \( \ker \chi_g \cap \ker \chi_h \) is equal to \( S + \{ tv \in R^3 / \mathbb{Z}^3 : t \in \mathbb{R} \} \) for some nonzero \( v \in \mathbb{Z}^3 \) orthogonal to both \( g \) and \( h \) and some finite set \( S \) of rational points in \( \mathbb{R}^3 / \mathbb{Z}^3 \).
Now the $\mathbb{Z}$-linear map $T : \mathbb{Z}^n \to \mathbb{Z}^n$ which takes $e_i$ to $v_i$, where $e_i$ is the $i$-th standard basis vector, is an isomorphism and hence an element of $GL_n(\mathbb{Z})$. By definition $T e_n = a$. ■

**Lemma A.2.** Let $n$ be a positive integer and $\Lambda$ be a rank-$k$ subgroup of $\mathbb{Z}^n$. Then there is $T \in GL_n(\mathbb{Z})$ such that $T(\Lambda)$ is contained in $\mathbb{Z}^k \times \{(0, \ldots, 0)\}$. 

Proof. Let $N \geq 1$ be an integer such that there is an isomorphism $\varphi : \mathbb{Z}^n/\Lambda \to \mathbb{Z}^{n-k} \times G$, where $G$ is a finite abelian group. Let $\Gamma = (\varphi \circ \pi)^{-1}(\{0\} \times G)$, where $\pi$ is the natural projection $\mathbb{Z}^n \to \mathbb{Z}^n/\Lambda$. Then $\Gamma$ contains $\Lambda$ and $\mathbb{Z}^n/\Gamma$ is isomorphic to $\mathbb{Z}^{n-k}$. We can now choose vectors $v_1, \ldots, v_{n-k}$ in $\mathbb{Z}^n$ such that $v_1 + \Gamma, \ldots, v_{n-k} + \Gamma$ is a basis of $\mathbb{Z}^n/\Gamma$. If $u_1, \ldots, u_k$ is a basis of $\Gamma$, then $u_1, \ldots, u_k, v_1, \ldots, v_{n-k}$ forms a basis of $\mathbb{Z}^n$. Define a $\mathbb{Z}$-linear map $T : \mathbb{Z}^n \to \mathbb{Z}^n$ by declaring $Te_i = u_i$ for $1 \leq i \leq k$ and $Te_i = v_{i-k}$ for $k + 1 \leq i \leq n$. Then $T$ is a surjective $\mathbb{Z}$-linear map and hence is a member of $GL_n(\mathbb{Z})$. Now $T^{-1}$ is an element of $GL_n(\mathbb{Z})$ which takes $\Gamma$ to $\mathbb{Z}^k \times \{(0, \ldots, 0)\}$, and hence puts $\Lambda$ inside $\mathbb{Z}^k \times \{(0, \ldots, 0)\}$, finishing the proof. ■

**Appendix B.** **Measure Theory**

**Lemma B.1.** Let $X$ and $Z$ be topological spaces. Let $Y$ be a set and $\mathcal{F}$ be a family of functions from $Y$ to $Z$. Equip $Y$ with the initial topology induced by $\mathcal{F}$. Then a map $\varphi : X \to Y$ is measurable if and only if $f \circ \varphi : X \to Z$ is measurable for each $f \in \mathcal{F}$.

Proof. Suppose $\varphi : X \to Z$ is a map such that $f \circ \varphi$ is measurable for each $f \in \mathcal{F}$. Then for each open set $W$ in $X$ and each $f \in \mathcal{F}$ we have $\varphi^{-1}(f^{-1}(W))$ is measurable. But $\{f^{-1}(W) : f \in \mathcal{F}, W \text{ open in } Z\}$ forms a subbasis for the topology on $Y$. Therefore $\varphi$ is measurable. The other direction is clear. ■

**Lemma B.2.** Let $S \subseteq \mathbb{R}^N$ be the set of all the points $x = (x_n : n \in \mathbb{N})$ in $\mathbb{R}^N$ such that $x_n \to 0$. Then $S$ is a measurable set.

Proof. For each $q \geq 1$ and each $k \geq 1$, define

(B.1) $E_{q,k} = \{(x_n : n \in \mathbb{N}) : -1/k < x_n < 1/k\}$

Then each $E_{q,k}$ is measurable and it is easy to check that

$$S = \bigcap_{k \geq 1} \bigcup_{N \geq 1} \bigcup_{q \geq N} E_{q,k}$$

whence $S$ is a measurable set. ■

**Appendix C.** **Weak Periodicity Assuming the Spectral Measure is Supported on the Union of Finitely Many Lines**

The goal of this section is to prove the following.

**Theorem C.1.** Let $F \subseteq \mathbb{Z}^3$ be an exact cluster and $T \subseteq \mathbb{Z}^3$ be an $F$-tiling. Let $X \subseteq \{0,1\}^\mathbb{Z}^3$ be the orbit closure of $T$ and $\mu$ be a $\mathbb{Z}^3$-ergodic probability measure on $X$. Define

$$A = \{x \in X : x(0) = 1\}$$

and $f \in L^2(X,\mu)$ as the characteristic function on $A$. Let $\nu$ be the spectral measure associated to the unit vector $f/\|f\|_2$ in $L^2(X,\mu)$ and $\theta$ be the unitary isomorphism between $H_f$ — the span closure of the orbit of $f/\|f\|_2$ — and $L^2(T^3,\nu)$ as discussed in Theorem 2.3. Assume that there exist nonzero vectors $v_1, \ldots, v_n \in \mathbb{Z}^3$ and finite subsets $S_1, \ldots, S_n \subseteq T^3 = \mathbb{R}^3/\mathbb{Z}^3$ such that each member of each $S_i$ is a rational point and the measure $\nu$ is supported on

(C.1) $\bigcup_{i=1}^n (S_i + \{tv_i \in T^3 : t \in \mathbb{R}\})$
Then $A$ is 1-weakly periodic, and hence there is a 1-weakly periodic F-tiling.

The proof is an adaptation of Bhattacharya’s proof of the periodic tiling conjecture (for $\mathbb{Z}^2$) in [2]. No fundamentally new ideas are needed. However, since Theorem C.1 is not a direct corollary of the work done in [2], we give full details. Wherever possible, we give reference to corresponding results in [2]. Throughout this section we will use the notation in Theorem C.1. We now begin the proof.

We may assume that $n$ is the smallest integer for which one can find vectors $v_1, \ldots, v_n \in \mathbb{Z}^3$ and finite subsets $S_1, \ldots, S_n \subseteq \mathbb{T}^3$ such that the support of $\nu$ is contained in

\[
\bigcup_{i=1}^n (S_i + \{tv_i : t \in \mathbb{R}\})
\]

Then $v_1, \ldots, v_n$ are pairwise linearly independent.

**Lemma C.2.** Let $S \subseteq \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ be a finite set of rational points and $v$ be an arbitrary nonzero vector in $\mathbb{Z}^3$. Then there exist linearly independent vectors $g$ and $h \in \mathbb{Z}^3$ such that both $g$ and $h$ are orthogonal to $v$ and

\[
S + \{tv \in \mathbb{R}^3/\mathbb{Z}^3 : t \in \mathbb{R}\}
\]

is contained in $\ker \chi_g \cap \ker \chi_h$.

**Proof.** The set of all vectors in $\mathbb{Z}^3$ which are orthogonal to $v$ forms a rank-2 subgroup of $\mathbb{Z}^3$. Thus we can find two linearly independent vectors $g_0$ and $h_0$ in $\mathbb{Z}^3$ which are both orthogonal to $v$. Thus

\[
\{tv \in \mathbb{R}^3/\mathbb{Z}^3 : t \in \mathbb{R}\} \subseteq \ker \chi_{g_0} \cap \ker \chi_{h_0}
\]

If $N$ is a positive integer such that $Ns = 0$ in $\mathbb{R}^3/\mathbb{Z}^3$ for each $s \in S$, then we see that

\[
S + \{tv \in \mathbb{R}^3/\mathbb{Z}^3 : t \in \mathbb{R}\} \subseteq \ker \chi_{Ng_0} \cap \ker \chi_{Nh_0}
\]

Thus $g = Ng_0$ and $h = Nh_0$ satisfy the requirement of the lemma. ■

**Lemma C.3.** We can choose vectors $g_1, h_1, \ldots, g_n, h_n \in \mathbb{Z}^3$ such that

a) $S_i + \{tv_i \in \mathbb{R}^3/\mathbb{Z}^3 : t \in \mathbb{R}\} \subseteq \ker \chi_{g_i} \cap \ker \chi_{h_i}$ for each $i$.

b) $\mathbb{Z}g_i + \mathbb{Z}h_i$ is a rank-2 subgroup of $\mathbb{Z}^3$ for each $i$.

c) $g_i$ and $h_i$ are orthogonal to $v_i$ for each $i$.

d) $(\mathbb{Z}g_i + \mathbb{Z}h_i) + (\mathbb{Z}g_j + \mathbb{Z}h_j)$ is a rank-3 subgroup of $\mathbb{Z}^3$ whenever $i \neq j$.

**Proof.** By Lemma C.2 we can find $g_1, h_1, \ldots, g_n, h_n$ such that (a), (b) and (c) are true. We show that (d) also holds. Let $i, j \in \{1, \ldots, n\}$ be distinct. Assume on the contrary that

\[
\Lambda := (\mathbb{Z}g_i + \mathbb{Z}h_i) + (\mathbb{Z}g_j + \mathbb{Z}h_j)
\]

is not a rank-3 subgroup of $\mathbb{Z}^3$. Then $\Lambda$ is a rank-2 subgroup of $\mathbb{Z}^3$. Consequently, both $v_i$ and $v_j$ are orthogonal to each of $g_i, h_i, g_j, h_j$. This forces that $v_i$ and $v_j$ are linearly dependent, giving the required contradiction. ■

Fix $g_1, h_1, \ldots, g_n, h_n$ as in Lemma C.3. Thus, in particular, the measure $\nu$ is supported on the set

\[
\bigcup_{i=1}^n \ker \chi_{g_i} \cap \ker \chi_{h_i}
\]

Let $\Lambda_i$ be the subgroup of $\mathbb{Z}^3$ generated by $g_i$ and $h_i$, that is $\Lambda_i = \mathbb{Z}g_i + \mathbb{Z}h_i$. 
C.1. Very Weak Periodic Decomposition. Let \( H_f = \text{Span}\{v \cdot f : v \in \mathbb{Z}^3\} \). Let \( \nu \) be the spectral measure associated \( f/\|f\|_2 \) and \( \theta : H_f \to L^2(\mathbb{T}^3, \nu) \) be the unitary isomorphism discussed in Section 2.3.

For any subgroup \( \Lambda \) of \( \mathbb{Z}^3 \), let \( f^\Lambda \) denote the projection of \( f \) onto the subspace of all the \( \Lambda \)-invariant functions in \( L^2(X, \mu) \). Writing \( B_N \) to denote the ball of radius \( N \) in \( \mathbb{Z}^3 \), by the ergodic theorem we have

\[
(C.8) \quad f^\Lambda = \lim_{N \to \infty} \frac{1}{|\Lambda \cap B_N|} \sum_{v \in \Lambda \cap B_N} v \cdot f
\]

in \( L^2(X, \mu) \). Therefore \( f^\Lambda \) lies in \( H_f = \text{Span}\{v \cdot f : v \in \mathbb{Z}^3\} \), and it is hence same as the image of \( f \) under the orthogonal projection \( H_f \to H_f \) onto the set of all the \( \Lambda \)-invariant functions in \( H_f \). This implies that \( \theta(f^\Lambda/\|f\|_2) = 1^\Lambda \), where \( 1^\Lambda \) is the orthogonal projection of \( 1 \) onto the space of all the \( \Lambda \)-invariant functions in \( L^2(\mathbb{T}^3, \nu) \).

**Lemma C.4.** Let \( \rho \) be a probability measure on \( \mathbb{T}^3 \). Let \( g \) and \( h \) be two linearly independent vectors in \( \mathbb{Z}^3 \) and \( \Lambda \) be the subgroup of \( \mathbb{Z}^3 \) generated by \( g \) and \( h \). Under the natural unitary action of \( \mathbb{Z}^3 \) on \( L^2(\mathbb{T}^3, \rho) \), we have

\[
(C.9) \quad 1^\Lambda = 1_{\ker \chi_g \cap \ker \chi_h}
\]

where \( 1^\Lambda \) denotes the orthogonal projection of \( 1 \) onto the space of all the \( \Lambda \)-invariant functions in \( L^2(\mathbb{T}^3, \rho) \).

**Proof.** It is clear that \( 1_{\ker \chi_g \cap \ker \chi_h} \) is invariant under \( \Lambda \). Also \( 1 - 1_{\ker \chi_g \cap \ker \chi_h} \) is orthogonal to \( 1_{\ker \chi_g \cap \ker \chi_h} \). Thus

\[
(C.10) \quad 1 = 1_{\ker \chi_g \cap \ker \chi_h} + (1 - 1_{\ker \chi_g \cap \ker \chi_h})
\]

must be the orthogonal decomposition of \( 1 \) into the sum of a vector in the space of \( \Lambda \)-invariant elements in \( L^2(\mathbb{T}^3, \rho) \) and a vector in the orthogonal complement of the same. \( \blacksquare \)

**Lemma C.5.** (See \cite{2} Theorem 3.3) The function

\[
f_0 := f - \sum_{i=1}^n f^\Lambda_i
\]

is 3-periodic, where recall that \( \Lambda_i = \mathbb{Z}g_i + \mathbb{Z}h_i \).

**Proof.** By the remarks made above, we have

\[
(C.11) \quad \frac{1}{\|f\|_2} \theta \left( f - \sum_{i=1}^n f^\Lambda_i \right) = 1 - \sum_{i=1}^n 1^\Lambda_i
\]

By Lemma C.4 we have \( 1^\Lambda_i = 1_{\ker \chi_{g_i} \cap \ker \chi_{h_i}} \) since \( g_i \) and \( h_i \) generate \( \Lambda_i \). So we just need to show that \( 1 - \sum_{i=1}^n 1^\Lambda_i \) is 3-periodic. Note that since \( \nu \) is supported on \( \bigcup_{i=1}^n \ker \chi_{g_i} \cap \ker \chi_{h_i} \), we have

\[
(C.12) \quad 1 = 1_{\bigcup_{i=1}^n \ker \chi_{g_i} \cap \ker \chi_{h_i}}
\]

in \( L^2(\mathbb{T}^3, \nu) \). So we need to show that the function

\[
(C.13) \quad 1_{\bigcup_{i=1}^n \ker \chi_{g_i} \cap \ker \chi_{h_i}} - \sum_{i=1}^n 1_{\ker \chi_{g_i} \cap \ker \chi_{h_i}}
\]

is 3-periodic. But it is easy to check that the finite index subgroup

\[
\Gamma = \bigcap_{i \neq j} (\Lambda_i + \Lambda_j)
\]
of \( Z^3 \) fixes the above function, and we are done. \( \blacksquare \)

**Lemma C.6.** (See [2, Theorem 3.3]) For any \( l \geq 1 \) and \( i \in \{1, \ldots, n\} \) we have \( f^{l \Delta_i} - f^{\Delta_i} \) is 3-periodic.

**Proof.** We have

\[
\theta(f^{l \Delta_i} - f^{\Delta_i}) = \|f\|_2(1_{\ker \chi_{g_i} \cap \ker \chi_{h_i}} - 1_{\ker \chi_{g_i} \cap \ker \chi_{h_i}})
\]

so it suffices to show that

\[
1_{\ker \chi_{g_i} \cap \ker \chi_{h_i}} - 1_{\ker \chi_{g_i} \cap \ker \chi_{h_i}}
\]

is 3-periodic in \( L^2(T^3, \nu) \). This is a simple exercise keeping in mind that \( \nu \) is supported on \( \bigcup_{i=1}^n \ker \chi_{g_i} \cap \ker \chi_{h_i} \). \( \blacksquare \)

### C.2. Polynomial Sequences.

By a bi-infinite sequence in \( R/Z \), we mean a map \( s : Z \to R/Z \). We will write \( s_i \) to mean \( s(i) \). In what follows we will write ‘sequence’ to mean a bi-infinite sequence. For a sequence \( s \), we define another sequence \( \partial s \) by writing \( (\partial s)_i = s_{i+1} - s_i \). The \( k \)-fold composition \( \partial \circ \cdots \circ \partial \) will be denoted by \( \partial^k \). We say that a sequence \( s \) is a **polynomial sequence** if \( \partial^k s = 0 \) for some positive integer \( k \). It is easy to argue by induction that if \( s \) is a polynomial sequence in \( R/Z \), then there exists a non-negative integer \( n \) and \( a_0, a_1, \ldots, a_n \in \mathbb{R}/\mathbb{Z} \) such that \( s_k = a_0 + a_1 k + \cdots + a_n k^n \) for all \( k \), where the terms \( a_i k^i \) have to be interpreted as elements in \( R/Z \) by thinking of \( a_i k^i \) as the \( k \)-fold sum of \( a_i \). This justifies the usage of the phrase ‘polynomial sequence.’

There is a natural action of \( \mathbb{Z}[u, u^{-1}] \), the ring of all the Laurent polynomial over \( \mathbb{Z} \), on the set of all the sequences in \( R/Z \). First note that there is a natural way to add two sequences: For \( s, t : \mathbb{Z} \to \mathbb{R}/\mathbb{Z} \), we have \( s + t : \mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) defined as \( (s + t)_i = s_i + t_i \). Similarly, any sequence can be scaled by an integer. Now for \( n \in \mathbb{Z} \) and a sequence \( s : \mathbb{Z} \to \mathbb{R}/\mathbb{Z} \), we define the sequence \( u^n s : \mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) as \( (u^n s)_i = s_{i+n} \). For an element \( \sum_{n \in S} a_n u^n \) in \( \mathbb{Z}[u, u^{-1}] \), and a sequence \( s : \mathbb{Z} \to \mathbb{R}/\mathbb{Z} \), we define

\[
\sum_{n \in S} a_n u^n \biggl[ \sum_{n \in S} a_n (u^n s) \biggr]
\]

\[
(C.14)
\]

**Lemma C.7.** Let \( s : \mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) be a sequence such that there is \( m \geq 1 \) and \( k \geq 1 \) with the property that \((u^m - 1)^k s = 0\). Then the restriction of \( s \) to any coset of \( m \) in \( \mathbb{Z} \) is a polynomial sequence.

**Proof.** The proposition clearly holds for \( k = 1 \). Fix \( k > 1 \) and inductively assume that the proposition has been proved for all smaller values. Define \( t = (u^m - 1)s \). Now since \((u^m - 1)^{k-1} t = 0\), we deduce by the inductive hypothesis that \( t \) restricted to any coset of \( m \) is a polynomial sequence. Let \( a_0, \ldots, a_n \) be elements of \( \mathbb{R}/\mathbb{Z} \) such that

\[
(C.15)
\]

for all \( j \). Now since \( t = (u^m - 1)s \), we have

\[
(C.16)
\]

for all \( j \).
Using Equation \((\text{C.15})\) we see that the last term above is a polynomial and hence \(s\) restricted to \(m\mathbb{Z}\) is a polynomial sequence. Similarly we can show that the restriction of \(s\) to other cosets of \(m\mathbb{Z}\) are also polynomial sequences and we are done. ■

**Corollary C.8.** Let \(s\) be a sequence and \(m_1, \ldots, m_l\) be positive integers such that
\[
(u^{m_1} - 1) \cdots (u^{m_l} - 1)s = 0
\]
Then there are positive integers \(m\) and \(k\) such that \((u^m - 1)^k\) annihilates \(s\), and thus \(s\) is a polynomial sequence on each coset of \(m\mathbb{Z}\).

**Proof.** The roots of the polynomial \(u^{m_i} - 1\) are roots of unity. Therefore the polynomial \((u^{m_1} - 1) \cdots (u^{m_l} - 1)\) divides \((u^m - 1)^k\) for a suitable choice of positive integers \(m\) and \(k\). Now the desired result follows by the previous lemma. ■

**C.3. Equidistribution.** Let \(\lambda\) denote the Haar measure on \(\mathbb{R}/\mathbb{Z}\). For any sequence \(s : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}\), we define a probability measure \(\lambda_n\) on \(\mathbb{R}/\mathbb{Z}\) as
\[
\lambda_n = \frac{1}{2^n + 1} \sum_{i = -n}^{n} \delta_{s_i}
\]
where \(\delta_{s_i}\) denotes the Dirac mass at \(s_i\). We say that \(s\) is **equidistributed** if \(\lambda_n \to \lambda\) in the weak* sense. Weyl’s equidistribution theorem\(^{[11]}\) states that every polynomial sequence in \(\mathbb{R}/\mathbb{Z}\) is either periodic or equidistributes.

Similarly, we say that a \(k\)-dimensional sequence \(t : \mathbb{Z}^k \to \mathbb{R}/\mathbb{Z}\) is **equidistributed** if the probability measures
\[
\lambda_n = \frac{1}{(2n + 1)^k} \sum_{p \in [-n,n]^k} \delta_{t_p}
\]
converges to \(\lambda\) in the weak* sense. For later use we need the following lemma.

**Lemma C.9.** Let \((Y, \mathcal{Y}, \nu)\) be a probability space equipped with a measure preserving \(\mathbb{Z}^3\)-action. Let \(\Gamma\) be a finite index subgroup of \(\mathbb{Z}^3\) and \(\varphi : Y \to \mathbb{R}/\mathbb{Z}\) be a measurable function. For each \(y \in Y\) we have the function \(\varphi_y : \Gamma \to \mathbb{R}/\mathbb{Z}\) which takes \(g \in \Gamma\) to \(\varphi(g \cdot y)\). Then the sets
\[
E = \{y \in Y : \varphi_y : \Gamma \to \mathbb{R}/\mathbb{Z} \text{ equidistributes}\}, \quad K = \{y \in Y : \varphi_y : \Gamma \to \mathbb{R}/\mathbb{Z} \text{ is constant}\}
\]
are measurable.

**Proof.**\(^{[12]}\) We give the proof for \(\Gamma = \mathbb{Z}^3\) and leave the general case as an exercise. First we show that \(E\) is measurable. We know that \(\mathcal{P}(\mathbb{R}/\mathbb{Z})\), the set of all the probability measures in \(\mathbb{R}/\mathbb{Z}\), is metrizable (in the weak* topology). Choose a metric \(d\) on \(\mathcal{P}(\mathbb{R}/\mathbb{Z})\). For each \(n \geq 1\) and each \(y \in Y\), define probability measure \(\lambda_{n,y}\) by writing
\[
\lambda_{n,y} = \frac{1}{(2n + 1)^3} \sum_{p \in [-n,n]^3} \delta_{\varphi_y(p)}
\]
By Lemma \([B.1]\) the map \(Y \to \mathcal{P}(\mathbb{R}/\mathbb{Z})\) taking \(y\) to \(\lambda_{n,y}\) is measurable for any \(n \geq 0\). Therefore, the composite

\(^{[11]}\)See Theorem 1.4 in \([3]\)

\(^{[12]}\)Thanks to Ronnie Pavlov and Nishant Chandgotia for helping with this proof.
is measurable. If $S$ is the set of all the elements $(x_n : n \geq 1)$ in $\mathbb{R}^n$ such that $x_n \to 0$, then by Lemma [B.2] we have $S$ is measurable. It is clear that $E$ is nothing but the preimage of $S$ under the composition above, and hence $E$ is measurable.

Now we show that $K$ is measurable. For each $n,k \geq 1$, define

(C.22) $K_{n,k} = \{ y \in Y : |\varphi(y) - \varphi(p \cdot y)| < 1/k \text{ for } p \in [-n,n]^3 \}$

which is measurable. Now $K = \bigcap_{n,k\geq 1} K_{n,k}$ and hence $K$ is measurable. \hfill \blacksquare

C.4. Pure and Mixed Statistics. Now suppose $(Y,\mathcal{F},\nu)$ is a probability measure space equipped with an ergodic $\mathbb{Z}^3$-action. Let $\varphi : Y \to \mathbb{R}/\mathbb{Z}$ be a measurable function. For each $y \in Y$ we get a map $\varphi_y : \mathbb{Z}^3 \to \mathbb{R}/\mathbb{Z}$ defined as $\varphi_y(v) = \varphi(v \cdot y)$.

Lemma C.10. Pure Statistics Lemma. Suppose the $3$-dimensional sequence $\varphi_y : \mathbb{Z}^3 \to \mathbb{R}/\mathbb{Z}$ is either a constant or equidistributed for $\nu$-a.e. $y \in Y$. Then $\varphi_*\nu$ is either a Dirac measure or the Haar measure on $\mathbb{R}/\mathbb{Z}$.

Proof. The set $E$ of points $y$ in $Y$ such that $\varphi_y$ equidistributes form a measurable subset of $Y$. It is easy to check that this set is $\mathbb{Z}^3$-invariant. Thus this set either has full measure or has zero measure. Similarly, the set $K$ of points $y$ in $Y$ for which $\varphi_y$ is constant either has full measure or zero measure. Now we have a full measure subset $S$ of $Y$ such that for each $y \in S$ we have

(C.23) $\lim_{n \to \infty} \int_{\mathbb{R}/\mathbb{Z}} F \, d\left( \frac{1}{(2n+1)^3} \sum_{p \in [-n,n]^3} \delta_{\varphi_y(p)} \right) = \lim_{n \to \infty} \frac{1}{(2n+1)^3} \sum_{p \in [-n,n]^3} F \circ \varphi(p \cdot y)$

which by the Birkhoff ergodic theorem is equal to

(C.24) $\int_Y F \circ \varphi \, d\nu = \int_{\mathbb{R}/\mathbb{Z}} F \, d(\varphi_*\nu)$

for all $F \in C(\mathbb{R}/\mathbb{Z})$. If $E$ has full measure, then, by the Riesz representation theorem we must have $\varphi_*\nu$ is the Lebesgue measure and if $K$ has full measure then we must have $\varphi_*\nu$ is a Dirac measure. Since one of these two possibilities must occur, we are done. \hfill \blacksquare

Lemma C.11. Mixed Statistics Lemma. Suppose for $\nu$-a.e. $y$ in $Y$ there is a finite index subgroup $\Gamma_y$ of $\mathbb{Z}^3$ such that the $3$-dimensional sequence obtained by restricting $\varphi_y$ to any coset of $\Gamma_y$ is either a constant or equidistributed. Then there is a finite index subgroup $\Gamma$ of $\mathbb{Z}^3$ such that for any $\Gamma$-ergodic component $(E,\nu_E)$ of $Y$ we have $\varphi_*\nu_E$ is either a Dirac measure or the Haar measure on $\mathbb{R}/\mathbb{Z}$.

\textsuperscript{13}First one may establish this for a countable dense subset of $C(\mathbb{R}/\mathbb{Z})$ and then upgrade to all of $C(\mathbb{R}/\mathbb{Z})$.  

Proof. Let $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ be an enumeration of all the finite index subgroups of $\mathbb{Z}^3$. For each $i \geq 1$ let

(C.25) $Y_i = \bigcap_{g \in \mathbb{Z}^3} \{ \{ y \in Y \mid \varphi_{y,g} : \Gamma_i \to \mathbb{R}/\mathbb{Z} \text{ equidistributes} \} \cup \{ y \in Y \mid \varphi_{y,g} : \Gamma_i \to \mathbb{R}/\mathbb{Z} \text{ is constant} \} \}$

By Lemma C.9 we know that each $Y_i$ is a measurable subset of $Y$. Note that each $Y_i$ is $\mathbb{Z}^3$-invariant. Define a map $\Psi : Y \to S(\mathbb{Z}^3)$, where $S(\mathbb{Z}^3)$ is the set of all the finite index subgroups of $\mathbb{Z}^3$, by writing

(C.26) $\Psi(y) = \Gamma_n$

where $n$ is the smallest positive integer $i$ such that on every coset of $\Gamma_i$, the sequence $\varphi_{y,\nu}$ is either a constant or is equidistributed. Then note that

(C.27) $\Psi^{-1}(\Gamma_n) = Y_n \cap \bigcap_{i=1}^{n-1} Y_i^c$

Therefore each fiber of $\Psi$ is measurable. Since there are only countably many fibers of $\Psi$, there must exist a fiber with positive measure, whence by the ergodicity of the $\mathbb{Z}^3$ action we deduce that some fiber of $\Psi$ has full measure. Therefore, there is a finite index subgroup $\Gamma$ of $\mathbb{Z}^3$ such that for $\nu$-a.e. $y$ in $Y$ we have $\varphi_{y,\nu}$ restricted to any coset of $\Gamma$ is either constant or equidistributed. Since $\Gamma$ acts ergodically on each $\Gamma$-ergodic component, the desired result follows from the Pure Statistics Lemma.

C.5. Behaviour of Averages on Suitable Ergodic Components. For a map $\varphi : X \to \mathbb{R}$ we define a map $\bar{\varphi} : X \to \mathbb{R}/\mathbb{Z}$ obtained by composing $\varphi$ with the natural map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. There is a natural action of $\mathbb{Z}[u_1^+, u_2^+, u_3^+]$ on the set of maps taking $X$ into $\mathbb{R}/\mathbb{Z}$. We have

Lemma C.12. (See [2, Lemma 4.2]) Let $j \in \{1, \ldots, n\}$ be arbitrary. Then there is $h$ in $\mathbb{Z}^3$ not in any of the $\Lambda_i$’s such that

(C.28) $$(U^h - 1)^k \mathcal{F}^\Lambda_j = 0$$

for some $k \geq 1$.

Proof. Let $h$ be a vector in $\mathbb{Z}^3$ that is not in any of the $\Lambda_i$’s. Let $\Gamma_0$ be a finite index subgroup of $\mathbb{Z}^3$ such that $f_0$ is $\Gamma_0$-invariant (recall the definition of $f_0$ from Lemma C.5). Let $l$ be a positive integer such that $l\mathbb{Z}^3$ is contained in $(Zh + \Lambda_j) \cap \Gamma_0$. For each $i \in \{1, \ldots, n\}$, $i \neq j$, let $w_i \in \Lambda_i$ be a nonzero vector such that $w_i \notin \Lambda_j$. Consider the Laurent polynomial

(C.29) $q(U) = \prod_{i: i \neq j} (U^{lw_i} - 1)$

It is clear that $q$ annihilates $f_0$ as well as each $\mathcal{F}^\Lambda_i$ for $i \neq j$. For each $i \neq j$, we can find a nonzero integer $a_i$ and a vector $v_i$ in $\Lambda_j$ such that $lw_i = a_i h + w_{ij}$ for some $w_{ij} \in \Lambda_j$. Therefore

(C.30) $U^{lw_i} \mathcal{F}^\Lambda_i = U^{a_i h} \mathcal{F}^\Lambda_j$

for each $i \neq j$, and thus we have

(C.31) $q(U) \mathcal{F}^{\Lambda_j} = \left[ \prod_{i: i \neq j} (U^{a_i h} - 1) \right] \mathcal{F}^\Lambda_j = 0$

\footnote{Thanks to Nishant Chandgotia for the argument showing the measurability of $\Psi$.}

\footnote{Recall that we write a typical element of $\mathbb{Z}[u_1^+, u_2^+, u_3^+]$ as $\sum_{\nu \in \mathbb{Z}^3} a_{\nu} U^{\nu}$.}
Therefore \((U^{rh} - 1)^k\) annihilates \(fN\) for a suitable choice of integers \(n\) and \(k\) and are done. ■

**Lemma C.13.** (See [2] Lemma 4.3) Let \(i \in \{1, \ldots, n\}\) be given. Then there is a finite index subgroup \(\Gamma_i\) of \(\mathbb{Z}^3\) with the property that if \((E, \mu_E)\) is a \(\Gamma_i\)-ergodic component of \(X\) then we have \(fN\mu_E\) is either a Dirac measure or the Haar measure on \(R/Z\).

**Proof.** Write \(\varphi = fN\). By Lemma C.12 we can find \(h \in \mathbb{Z}^3\) not in \(\Gamma_i\) and a positive integer \(k\) such that \((U^{rh} - 1)^k\varphi = 0\). Also, \((U^{rh} - 1)\varphi = 0\) for all \(g \in \Lambda_i\). Therefore, for \(\mu\)-a.e. \(x \in X\) we have \((U^{rh} - 1)^k\varphi_x = 0\) and \((U^{rh} - 1)\varphi_x = 0\) for all \(g \in \Lambda_i\). By Lemma C.7 we deduce that for \(\mu\)-a.e. \(x \in X\) we can find a finite index subgroup \(\Gamma_x\) of \(\mathbb{Z}^3\) such that \(\varphi_x\) is either equidistributed or is constant on any given coset of \(\Gamma_x\). Now by **Mixed Statistics Lemma** we can find a finite index subgroup \(\Gamma_i\) of \(\mathbb{Z}^3\) such that \(\varphi_x\mu_E\) is either a Dirac measure or the Haar measure for any \(\Gamma_i\)-ergodic component \((E, \mu_E)\) of \(X\).

■

**Remark C.14.** It can be easily argued that the conclusion of the above lemma does not change if we pass to any finite index subgroup of \(\Gamma_i\). More precisely, the above lemma can be made more nuanced by saying that there is a finite index subgroup \(\Gamma_i\) such that whenever \(\Gamma\) is any finite index subgroup of \(\Gamma_i\) we have \(fN\mu_E\) is either the Haar measure on \(R/Z\) or a Dirac measure on \(R/Z\) for any \(\Gamma\)-ergodic component \(E\) of \(X\). We will make use of this observation in what follows.

The strategy to show that \(A\) is \(1\)-weakly periodic is to show that there is a finite index subgroup \(\Gamma\) of \(\mathbb{Z}^3\) such that the intersection of \(A\) with any \(\Gamma\)-ergodic component of \(X\) is \(1\)-weakly periodic. Our next lemma almost achieves this. Since the conclusion of the above lemma does not change when \(\Gamma_i\) is replaced by any finite index subgroup of \(\Gamma_i\), this allows us to prove the following.

**Lemma C.15.** (See [2] Theorem 4.4) There is a finite index subgroup \(\Gamma\) of \(\mathbb{Z}^3\) such that for each \(\Gamma\)-ergodic component \((E, \mu_E)\) of \(X\), we have either \(\mu_E(A \cap E) = 1/2\) or \(A \cap E\) is \(2\)-periodic (or both).

**Proof.** Let \(\Gamma_0\) be a finite index subgroup of \(\mathbb{Z}^3\) such that \(f_0\) is \(\Gamma_0\)-invariant. By Lemma C.13 we know that for each \(i \in \{1, \ldots, n\}\) we have a finite index subgroup \(\Gamma_i\) of \(\mathbb{Z}^3\) such that for each \(\Gamma_i\)-ergodic component \(E\) of \(X\) we have \(fN\mu_E\) is either the Haar measure on \(R/Z\) or a Dirac measure on \(R/Z\) for any \(\Gamma_i\)-ergodic component \(E\) of \(X\). There are two cases.

**Case 1:** There is \(i \in \{1, \ldots, n\}\) such that \(fN\mu_E\) is the Haar measure \(\lambda\) on \(R/Z\). We will show that \(\mu_E(A \cap E) = 1/2\). Note that since \(f\), and hence \(f^\lambda\), is valued in the unit interval \(I\), the fact that \(fN\mu_E\) is the Haar measure on \(R/Z\) implies that \(f^\lambda\mu_E\) is the Haar measure \(\lambda^1\) on the unit interval \(I\). Let \(l \geq 1\) be such that \(l\mathbb{Z}^3 \subseteq \Gamma\). By Lemma C.6 we have \(f^\lambda - f^\lambda\) is \((l+1)\)-periodic. So there is a finite index subgroup \(\Gamma'\) of \(\mathbb{Z}^3\) such that \(f^\lambda - f^\lambda\) is \(l\)-invariant under \(\Gamma'\). Let \(\Theta = \Gamma \cap \Gamma'\) and \((E_1, \mu_1), \ldots, (E_k, \mu_k)\) be all the \(\Theta\)-ergodic components of \(X\) which are contained in \(E\). For each \(j \in \{1, \ldots, k\}\) let \(f^\lambda\mu_j\) be \(\Theta\)-invariant, and hence \(f^\lambda\mu_j\) is the Haar measure on the unit interval. Therefore \(f^\lambda\mu_j\) is the Haar measure on the interval \([c_i, c_i+1]\) and \(f^\lambda\mu_j\) is valued in \(I\), and hence \(c_i\) must be 0. This implies that \(f^\lambda\mu_E\) is also the Haar measure on the unit interval. Now

\[
(\text{C.32}) \quad \mu_E(A \cap E) = \int_E 1_A \ d\mu_E = \int_E f \ d\mu_E = \int_E f^\lambda \ d\mu_E
\]

where the last equality is because of the ergodic theorem coupled with the fact that \(E\) is \(\Gamma\)-invariant, and hence \(l\)-invariant. But since \(f^\lambda\mu_E\) is the Haar measure on \(I\), we see that \(\mu_E(A \cap E) = 1/2\).
Case 2: There is no \( i \in \{1, \ldots, n\} \) such that \( f^{\Lambda_i}_E \) is the Haar measure on \( \mathbb{R}/\mathbb{Z} \). Thus \( f^{\Lambda_i}_E \) is a Dirac measure for each \( i \). We divide this into two subcases.

Subcase 2.1: There is \( i \in \{1, \ldots, n\} \) such that \( f^{\Lambda_i}_E \) is the Dirac measure at 0. We will show that \( A \cap E \) is 2-periodic. It follows that \( f^{\Lambda_i}_E \) takes values in the two-element set \( \{0, 1\} \).

Choose \( l \geq 1 \) such that \( l\Lambda_i \subseteq \Gamma \). By the Birkhoff ergodic theorem we see that \( f^{l\Lambda_i} = 1 \) whenever \( f^{\Lambda_i} = 1 \) and \( f^{l\Lambda_i} = 0 \) whenever \( f^{\Lambda_i} = 0 \) on \( E \). Thus \( f^{l\Lambda_i}_E \) coincides with \( f^{\Lambda_i}_E \) on \( E \). Now let

\[
(C.33) \quad S = \{ x \in E : f^{\Lambda_i}(x) = 1 \} = \{ x \in E : f^{l\Lambda_i}(x) = 1 \}
\]

Then

\[
(C.34) \quad \mu_E(S) = \int_E 1_S \, d\mu_E = \int_E f^{l\Lambda_i}_E \, d\mu_E = \int_E f \, d\mu_E = \mu_E(A)
\]

where \((*)\) is because \( E \) is \( \Gamma \)-invariant and hence is \( l\Lambda_i \)-invariant. We also have

\[
(C.35) \quad \int_E f^{l\Lambda_i}_E \, d\mu_E = \int_S f^{l\Lambda_i}_E \, d\mu_E = \int_S f \, d\mu_E = \mu_E(A \cap S)
\]

because \( f^{l\Lambda_i}_E \) is the same as \( 1_S \) and \( S \) is \( l\Lambda_i \)-invariant. We deduce that \( \mu_E(A) = \mu_E(S) = \mu_E(A \cap S) \) and therefore \( A \cap E = S \). Since \( S \) being \( l\Lambda_i \)-invariant is 2-periodic, we have \( A \cap E \) is 2-periodic.

Subcase 2.2: There is no \( i \in \{1, \ldots, n\} \) such that \( f^{\Lambda_i}_E \) is the Dirac measure at 0. In this case we have \( f^{\Lambda_i}_E \) is the Dirac measure at a point other than 0 in \( \mathbb{R}/\mathbb{Z} \). This implies that each \( f^{\Lambda_i}_E \) is constant on \( E \). Since \( f = f_0 + \sum_{i=1}^n f^{\Lambda_i} \), with \( f_0 \) being \( \Gamma \)-invariant, we clearly have \( f |_E \) is 3-periodic, and thus \( A \cap E \) is 3-periodic (which in particular implies that \( A \cap E \) is 2-periodic).

This completes the proof.

C.6. Weakly Periodic Decomposition.

Lemma C.16. (See [2] Lemma 5.1) Let \((X, \mathcal{Y}, \nu)\) be a probability space equipped with an ergodic \( \mathbb{Z}^2 \)-action. If \( E \subseteq Y \) is 3-periodic and \( B \subseteq E \) is 2-weekly periodic, then \( E \setminus B \) is also 2-weekly periodic.

Proof. Let \( B = B_1 \sqcup \cdots \sqcup B_k \) be a 2-weekly periodic decomposition of \( B \) and let \( \Theta_1, \ldots, \Theta_k \) be rank-2 subgroups of \( \mathbb{Z}^3 \) such that each \( B_i \) is \( \Theta_i \)-invariant. We may assume that \( k \) is the smallest positive integer satisfying the above. Thus \( \Theta_i + \Theta_j \) is a finite index subgroup of \( \mathbb{Z}^3 \) whenever \( i \neq j \). Let \( \Gamma_0 \) be a finite index subgroup of \( \mathbb{Z}^3 \) such that \( E \) is \( \Gamma_0 \)-invariant. Define \( \Gamma_{ij} = \Gamma_0 \cap (\Theta_i + \Theta_j) \) whenever \( i \neq j \) and note that each \( \Gamma_{ij} \) is a finite index subgroup of \( \mathbb{Z}^3 \).

Define \( \Gamma = \bigcap_{i \neq j} \Gamma_{ij} \). Note that \( E \) is partitioned by the \( \Gamma \)-ergodic components contained in \( E \) since \( E \) is \( \Gamma \)-invariant.

We will show that for every \( \Gamma \)-ergodic component \( Q \) of \((X, \mu)\), the set \( Q \cap (E \setminus B) \) is 2-periodic. Indeed, fix a \( \Gamma \)-ergodic component \( Q \) such that \( Q \cap B \) has positive measure. Then \( Q \) must be contained in \( E \). Suppose there exists \( i, j \in \{1, \ldots, k\}, i \neq j \), such that \( Q \cap B_i \) and \( Q \cap B_j \) both have positive measure. Let \( S = \bigcup_{j \in \Gamma_{ij}} g(Q \cap B_j) \). Then \( S \) is \( \Gamma_{ij} \)-invariant and hence \( \Gamma \)-invariant.

Thus \( S \) must contain \( Q \). But note that if \( g \in \Gamma_{ij} \), then \( g \) can be written as \( g = g_i + g_j \) for some \( g_i \in \Gamma_0 \cap \Theta_i \) and \( g_j \in \Gamma_0 \cap \Theta_j \). Therefore

\[
(C.36) \quad g(Q \cap B_j) = (g_i + g_j)(Q \cap B_j) = g_i(g_j(Q \cap B_j)) \subseteq g_i(g_j B_j) \subseteq g_i B_j
\]

Now since \( B_i \) and \( E \) are both \( g_i \)-invariant, and \( B_j \subseteq E \) is disjoint with \( B_i \), we must have \( g_i B_j \subseteq E \setminus B_i \). This means that the set \( S \) is disjoint with \( B_i \), contradicting the fact that \( S \) contains \( Q \).

So we see that each \( \Gamma \)-ergodic component can intersect at most one of the \( B_i \)'s in a positive measure set. Finally, if \( Q \) is a \( \Gamma \)-ergodic component intersecting \( B_i \) in a positive measure set, and \( l \) is a positive integer such that \( l\Theta_i \subseteq \Gamma \), then \( Q \) and \( B_i \) are both \( l\Theta_i \)-invariant. Thus \( Q \setminus B = Q \setminus B_i \) is also \( l\Theta_i \)-invariant, and is hence 2-periodic.
By translating $F$ if necessary, we can manage that $0 \in F$ along with the property that the sum of any set of nonzero vectors in $F$ is nonzero. With this in mind we now finish the proof of the 1-weak periodicity of $A$, thereby establishing the existence of a 1-weakly periodic $F$-tiling.

**Lemma C.17.** Let $\Gamma$ be as in Lemma C.15. If $A \cap E$ is not 2-weakly periodic for some $\Gamma$-ergodic component $(E, \mu_E)$ of $\mu$, then $A \cap E$ is 1-periodic.

**Proof.** (See [2, Proof of Lemma 1.3, pg 13]) For any vector $v \in \mathbb{Z}^3$, we will write $\bar{v}$ to denote $-v$. Let $E$ be a $\Gamma$-ergodic component of $X$ such that $A \cap E$ is not 2-weakly periodic. Then $\mu_E(A \cap E) = 1/2$. Now we have

\[
E \setminus (A \cap E) = \bigcup_{g \neq 0, g \in F} (gA) \cap E
\]

since $\{gA : g \in F\}$ is a partition of $X$. If each $(gA) \cap E$ is 2-weakly periodic then so is $E \setminus (A \cap E)$, and hence, by Lemma C.16, $A \cap E$ is also 2-weakly periodic, contrary to our assumption. Thus there is $b_1 \in F \setminus \{0\}$ such that $(b_1A) \cap E$ is not 2-weakly periodic. This implies that $A \cap b_1E$ is also not 2-weakly periodic, and hence

$$1/2 = \mu_{b_1E}(A \cap b_1E) = \mu_E((b_1A) \cap E)$$

So we have $E = (A \cap E) \cup ((b_1A) \cap E)$, which implies that

$$b_1E = (A \cap b_1E) \cup (b_1(A \cap E))$$

As noted earlier, we have $A \cap b_1E$ is not 2-weakly periodic. By the same argument as above applied to $A \cap b_1E$, we deduce that there is $b_2 \in F \setminus \{0\}$ such that $b_1E = (A \cap b_1E) \cup ((b_2A) \cap (b_1E))$, which shows that

\[
(b_2A) \cap b_1E = b_1(A \cap E)
\]

and hence $A \cap (b_1 + b_2)E = (b_1 + b_2)(A \cap E)$. Thus, continuing this way, for each $n \geq 1$ we can find $b_1, \ldots, b_{2n} \in F \setminus \{0\}$ such that

\[
A \cap (b_1 + \cdots + b_{2n})E = (b_1 + \cdots + b_{2n})(A \cap E)
\]

Since there are only finitely many $\Gamma$-ergodic components, we see that there must exist natural numbers $m$ and $n$ with $m < n$ such that

\[
(b_1 + \cdots + b_{2m})E = (b_1 + \cdots + b_{2n})E
\]

and hence

\[
(b_1 + \cdots + b_{2m})(A \cap E) = (b_1 + \cdots + b_{2n})(A \cap E)
\]

giving

\[
(b_{2m+1} + \cdots + b_{2n})(A \cap E) = A \cap E
\]

and hence, since $b_{2m+1} + \cdots + b_{2n}$ is nonzero, $A \cap E$ is 1-periodic. This proves the lemma. □

**Corollary C.18.** Then $A$ is 1-weakly periodic.

**Proof.** Let $\Gamma$ be as in Lemma C.15 and apply Lemma C.16 □

**Data Availability Statement**

This manuscript has no associated data.

**Conflict of Interest**

On behalf of all authors, the corresponding author states that there is no conflict of interest.
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References

[1] D. Beauquier and M. Nivat. On translating one polyomino to tile the plane. *Discrete Comput. Geom.*, 6(6):575–592, 1991.
[2] S. Bhattacharya. Periodicity and decidability of tilings. *arXiv 1602.05738v1*, 2016.
[3] M. Einsiedler and T. Ward. *Ergodic theory with a view towards number theory*, volume 259 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.
[4] R. Greenfeld and T. Tao. The structure of translational tilings in $\mathbb{Z}^d$. *arXiv 2010.03254*, 2020.
[5] R. Greenfeld and T. Tao. Undecidable translational tilings with only two tiles, or one nonabelian tile. 2021.
[6] R. Greenfeld and T. Tao. A counterexample to the periodic tiling conjecture (announcement). 2022.
[7] P. Horak and D. Kim. Algebraic method in tilings. *arXiv 1603.00051v1*, 2016.
[8] J. Kari and M. Szabados. An algebraic geometric approach to Nivat’s conjecture. *Inform. and Comput.*, 271:104481, 25, 2020.
[9] J. C. Lagarias and Y. Wang. Tiling the line with translates of one tile. *Invent. Math.*, 124(1-3):341–365, 1996.
[10] M. Szegedy. Algorithms to tile the infinite grid with finite clusters. *Foundations of Computer Science*, 1998.
[11] R. Tijdeman. Decomposition of the integers as a direct sum of two subsets. 215:261–276, 1995.

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