Optimization in Structure Population Models through the Escalator Boxcar Train

Rinaldo M. Colombo\textsuperscript{1} \hspace{1cm} Piotr Gwiazda\textsuperscript{2} \hspace{1cm} Magdalena Rosińska\textsuperscript{3}

January 29, 2016

Abstract

The Escalator Boxcar Train (EBT) is a tool widely used in the study of balance laws motivated by structure population dynamics. This paper proves that the approximate solutions defined through the EBT converge to exact solutions. Moreover, this method is rigorously shown to be effective also in computing optimal controls. As preliminary results, the well posedness of classes of PDEs and of ODEs comprising various biological models is also obtained. A specific application to welfare policies illustrates the whole procedure.

Keywords: Escalator Boxcar Train, Structure Population Model.

2010 MSC: 65M75, 35R06, 92D25.

1 Introduction

This paper is devoted to the well posedness, to the numerical approximation and to the optimal control of renewal equations motivated by physiologically structured population models and whose solutions attain values in spaces of measures.

The dynamics of populations which are heterogenous with respect to some individual property can be described through initial – boundary value problems for a class of nonlinear first order partial differential equations (PDE), called renewal equations. Within this class, one of the first PDE models devoted to population biology is the renewal equation introduced by Kermack and McKendrick with reference to epidemiology, see [21, 22]. There, the time since infection, i.e., the age, plays the role of a structure parameter, due to its essential role in the spreading of the epidemic. Equations of the same class are later proposed by von Förster in [30] to describe the process of cell division. The recent monograph [9] provides an extensive theoretical and empirical treatment of the ecology of ontogenetic growth and development of organisms, emphasizing the importance of an individual–based perspective in understanding the dynamics of populations and communities. Classical analytic studies on these equations are settled in $L^1$ and go back, for instance, to the monographs of Webb [31], Iannelli [20] or Thieme [28].

The space of positive Radon measures is introduced in biological applications in [24]. Indeed, whenever the distribution of individuals is concentrated on discrete values of structure

\textsuperscript{1}University of Brescia, rinaldo.colombo@unibs.it
\textsuperscript{2}University of Warsaw, pgwiazda@mimuw.edu.pl
\textsuperscript{3}University of Warsaw, mrosinska@pzh.gov.pl
parameters, for instance at the initial time, the resulting population density may well lack absolute continuity with respect to the Lebesgue measure. One is thus lead to consider the problem

\[
\begin{align*}
\partial_t \mu + \partial_a \left( b(t, \mu) \mu \right) + c(t, \mu) \mu &= 0 \\
(b(t, \mu))(0) D\lambda \mu(0+) &= \int_0^{+\infty} \beta(t, \mu) \, d\mu \\
\mu(0) &= \mu_o
\end{align*}
\] (1.1)

where \( t \in \mathbb{R}_+ \) is time and \( a \in \mathbb{R}_+ \) is a biological parameter, typically age or size. The unknown \( \mu \) is a time dependent, non-negative and finite Radon measure. The growth function \( b \) and the mortality rate \( c \) are strictly positive, while the birth function \( \beta \) is non-negative. By \( D\lambda \mu(0+) \) we denote the Radon–Nikodym derivative of \( \mu \) with respect to the Lebesgue measure \( \lambda \) computed at 0. The initial datum \( \mu_o \) is a non-negative Radon measure.

The analysis of solutions to (1.1) in spaces of positive Radon measures was initiated in [10], where the authors show the weak∗ continuity of solutions with respect to time and initial data. They also point out the key relevance of the dependence of solutions on the various model parameters, which was obtained in [5, 17, 19, 29].

The Lipschitz continuous dependence of solutions in measure spaces from time and initial datum is a preliminary step towards the convergence of the so called particle methods. These are numerical algorithms whose starting idea is the representation of a heterogeneous population as a sum of Dirac masses evolving in time. This representation is consistent with the usual experimental attitude of concentrating real data in discrete cohorts evolving in time. On these grounds, the numerical algorithm usually referred to as the Escalator Boxcar Train (EBT) is introduced back in [8]. Remarkably, in spite of the wide success of this method, a convergence proof of the EBT appears only rather recently in [2]. A key role in this result is played by the bounded Lipschitz distance introduced in [17]. Detailed estimates on the order of convergence are then provided in [16].

Another numerical method effective in the computation of solutions to structured population models is proposed in [6]. Here, a key role is played by the operator splitting method. According to it, the measure valued semigroup generated by renewal equations can be approximated through the iterated application of simpler semigroups. More precisely, a problem involving both transport terms and nonlocal growth terms is approximated through two problems, each involving only one of the two processes. The analytic framework established in [5] allows a detailed control of the convergence rate of the algorithm.

From the measure theoretic point of view, the above mentioned results rely on the use of Wasserstein (or Monge-Kantorovich) type metrics, adapted to the nonconservative character of (1.1). This methodology was proposed in [17] for a flat metric (bounded Lipschitz distance) and in [19] for a Wasserstein metric, suitably modified to deal with nonnegative Radon measures with integrable first moment. A relevant advantage of this approach is in providing a structure of a space appropriate both to compare solutions and to study their stability. Remark that precise estimates on the continuous dependence of solutions on the modeling parameters plays a key role in the numerical approximations and in calibrating the model calibration on the basis of experimental data. We refer to [26] for the definition and properties of a similar metric structure.

Similar techniques based on particle methods are usual tools in simulating kinetic models since more than three decades in physics, see for instance [27] and the references therein. Recent applications include for instance the porous medium equation [28, 52] and the isentropic
Euler equations in fluid mechanics [15, 32]. Other article method are found also in the study of problems related to crowd dynamics and pedestrians flow, see [12, 13, 26], as well as in the description of the collective motion of large groups of agents, see [4]. Differently from the case of structured population models, the original particle methods are mainly designed for problems where the total mass, or number of individuals, is conserved.

Aiming at the optimal control of the solution to (1.1), we introduce therein a control parameter $u$, possibly time and/or state dependent, attaining values in a given set $U$. Therefore, we obtain:

$$
\begin{cases}
\partial_t \mu + \partial_a \left( b(t, \mu; u) \mu \right) + c(t, \mu; u) \mu = 0 \\
(b(t, \mu; u))(0) D\lambda \mu(0+) = \int_0^{+\infty} \beta(t, \mu; u) \, d\mu \\
\mu(0) = \mu_0
\end{cases}
$$

(1.2)

Together with (1.2), we are given a cost functional

$$
\mathcal{J}(u) = \int_0^{+\infty} j(t, u(t), \mu(t)) \, dt
$$

and we provide below a constructive algorithm to find, within a suitable function space, a control function $u_*$ optimal in the sense that

$$
\mathcal{J}(u_*) = \min_{u(t) \in \mathcal{U}} \mathcal{J}(u).
$$

As is well known, solutions to conservation or balance laws typically depend in a Lipschitz continuous way from the initial datum as well as from the functions defining the equation. This does not allow the use of differential tools in the search for the optimal control.

Here, constructive should be understood in the following sense: on the basis of the control problem for (1.2), we define a sequence of control problems for a system of ordinary differential equations and prove that the corresponding sequence of optimal controls converges to an optimal control for the original problem. More precisely, we approximate the solution to (1.2) by means of the EBT algorithm as defined in [16, Section III]. The functional $\mathcal{J}$ computed along approximate solutions is proved to be a smooth, namely $C^1$, function of the control parameter $u$ and this allows to exhibit the existence of an optimal control for each approximate problem. A limiting procedure constructively ensures the existence of the optimal control for the original problem (1.2).

The next section presents results on the well posedness of (1.1) and the results on the escalator boxcar train algorithm that allow to obtain our main result, namely the construction of a sequence of controls that converge to an optimal control for (1.2). Section 3 is devoted to a possible application of the theory here developed. The technical proofs are deferred to Section 4 with a final Appendix that gathers necessary results concerning ordinary differential equations.

## 2 Main Results

Throughout, we denote $\mathbb{R}_+ = [0, +\infty]$. Let $(M, d_M)$ be a metric space and $(V, \| \cdot \|_V)$ be a normed space. Then, $C^0(M; V)$, respectively $C^{0,1}(M; V)$ is the space of continuous, respectively Lipschitz continuous, functions defined on $M$ and attaining values in $V$, equipped with
the norm
\[
\|\varphi\|_{C^0(M,V)} = \sup_{x \in M} \|\varphi(x)\|_V,
\]
respectively
\[
\|\varphi\|_{C^{0,1}(M,V)} = \max \left\{ \sup_{x \in M} \|\varphi(x)\|_V, \sup_{x_1,x_2 \in M} \frac{\|\varphi(x_2) - \varphi(x_1)\|_V}{d_M(x_1,x_2)} \right\}.
\]

Given \(T \in \mathbb{R}\) and a function \(f\colon [0,T] \to V\), we set
\[
TV_V(f) = \sup \left\{ \sum_{i=1}^n \|f(t_i) - f(t_{i-1})\|_V : n \in \mathbb{N} \text{ and } t_i \in [0,T] \text{ for } i = 0, \ldots, n, \quad t_i-1 < t_i \text{ for } i = 1, \ldots, n \right\}.
\]

The space \(\mathcal{M}^+(\mathbb{R}_+)\) of positive Radon measures on \(\mathbb{R}_+\) is equipped with the flat distance
\[
d(\mu', \mu'') = \sup \left\{ \int_{\mathbb{R}_+} \varphi \, d(\mu' - \mu'') : \varphi \in C^1(\mathbb{R}_+; [-1,1]) \text{ with } \text{Lip} (\varphi) \leq 1 \right\},
\]
see [5] Section 2. Below, for positive \(T, \mathcal{L}\) and \(C\), we use the space \(\mathcal{F}\) of functions
\[
f : [0,T] \to C^{0,1}(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R}_+)
\]
with the properties:

\(\mathcal{F}_1\) \(f\) is bounded: \(\|f(t)\|_{C^{0,1}(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \leq \mathcal{L}\) for all \(t \in [0,T]\).

\(\mathcal{F}_2\) \(f\) has bounded total variation in time: \(TV_{C^{0,1}(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})}(f) \leq \mathcal{C}\).

Throughout, the constants \(T, \mathcal{L}\) and \(C\) are kept fixed and the dependence of \(\mathcal{F}\) on them is omitted. In \(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+\) we use the distance
\[
d_{\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+} ((\mu_1,a_1),(\mu_2,a_2)) = d(\mu_1,\mu_2) + |a_2 - a_1|,
\]
where \(d\) is as in (2.4). Therefore, \(\mathcal{F}_1\) also implies that \(f\) is Lipschitz continuous in \(\mu\) and \(a\) uniformly in \(t\), in the sense that for all \(t \in [0,T]\), \(\mu_1,\mu_2 \in \mathcal{M}^+(\mathbb{R}_+)\) and \(a_1,a_2 \in \mathbb{R}_+\),
\[
\left| (f(t)) (\mu_1,a_1) - (f(t)) (\mu_2,a_2) \right| \leq \mathcal{L} \left( d(\mu_1,\mu_2) + |a_1 - a_2| \right).
\]

### 2.1 PDE – Well Posedness

As a first step, we need to extend the well posedness of (1.1) obtained in [5] Theorem 2.11 to the case of functions \(b\) and \(c\) being only of bounded variation in time. First, recall the definition of solution to (1.1) attaining as values Radon measures.

**Definition 2.1.** [18, Definition 3.1] Fix \(T > 0\) and let \(\mu_0 \in \mathcal{M}^+(\mathbb{R}_+)\). By solution to (1.1) we mean a function \(\mu : [0,T] \to \mathcal{M}^+(\mathbb{R}_+)\) with the following properties:

1. \(\mu\) is Lipschitz continuous with respect to the flat distance (2.4);
2. For all \( \varphi \in (C^1 \cap C^{0,1})([0, T] \times \mathbb{R}_+; \mathbb{R}) \)

\[
\int_{\mathbb{R}_+} \varphi(T, a) \, d\mu_t(a) - \int_{\mathbb{R}_+} \varphi(0, a) \, d\mu_o(a)
= \int_0^T \int_{\mathbb{R}_+} \partial_t \varphi(t, a) \, d\mu_t(a) \, dt
+ \int_0^T \int_{\mathbb{R}_+} (\partial_a \varphi(t, a) \left( b(t, \mu_t) \right)(a) + \varphi(t, a) \left( c(t, \mu_t) \right)(a) ) \, d\mu_t(a) \, dt
+ \int_0^T \int_{\mathbb{R}_+} \varphi(t, 0) \left( \beta(t, \mu_t) \right)(a) \, d\mu_t(a) \, dt.
\]

We now weaken the assumptions on the regularity in time used in [5, Theorem 2.11].

**Theorem 2.2.** Fix \( T > 0 \). Let \( b, c, \beta \in \mathcal{F} \). Then, for any \( \mu_o \in \mathcal{M}^+(\mathbb{R}_+) \), problem (1.1) admits a unique solution in the sense of Definition 2.1. Moreover, there exists a constant \( \mu \) with the properties:

- As above, we remark that (1.2) of functions \( u \), for all \( \varphi \in M \), problem (1.1) with initial data \( \mu_o \) and \( b, c, \beta \) replaced by \( b_i, c_i, \beta_i \), then,

\[
d \left( \mu(t), \mu^2(t) \right) \leq d(\mu_1, \mu_2) e^{C t}
+ C t e^{C t} \left( \sup_{t \in [0, T]} \| b_1(t) - b_2(t) \|_{C^0(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})}
+ \sup_{t \in [0, T]} \| c_1(t) - c_2(t) \|_{C^0(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})}
+ \sup_{t \in [0, T]} \| \beta_1(t) - \beta_2(t) \|_{C^0(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \right).
\]

The proof is deferred to Section 3.

Aiming at the study of (1.2), we extend the definition of \( \mathcal{F} \) as follows. Fix \( T > 0 \) and a compact subset \( \mathcal{U} \) of \( \mathbb{R}^N \), for fixed positive \( T, \mathcal{L}, \mathcal{C} \) and a positive integer \( N \), we introduce the space \( \mathcal{F}^u \) of functions

\[
f : [0, T] \times \mathcal{U} \to C^{0,1}(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})
\]

with the properties:

- \( \mathcal{F}_1^u \) is bounded:

\[
\| f(t, u) \|_{C^0(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \leq \mathcal{L} \text{ for all } t \in [0, T] \text{ and all } u \in \mathcal{U}.
\]

- \( \mathcal{F}_2^u \) has bounded total variation in \( t \) uniformly in \( u \): \( \text{TV}_{C^0(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} f(t, u) \leq \mathcal{L} \text{ for all } u \in \mathcal{U}. \)

- \( \mathcal{F}_3^u \) is Lipschitz continuous in the control uniformly in time: for all \( t \in [0, T] \) and for all \( u_1, u_2 \in \mathcal{U} \),

\[
\| f(t, u_1) - f(t, u_2) \|_{C^0(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \leq \mathcal{L} \| u_1 - u_2 \|.
\]

As above, we remark that \( \mathcal{F}_1^u \) ensures that \( f(t, u) \) is Lipschitz continuous in \( \mu \) and \( a \) uniformly in \( t \) and \( u \): for all \( t \in [0, T] \), \( u \in \mathcal{U} \), \( \mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}_+) \) and \( a_1, a_2 \in \mathbb{R}_+ \),

\[
\left| \left( f(t, u) \right)(\mu_1, a_1) - \left( f(t, u) \right)(\mu_2, a_2) \right| \leq \mathcal{L} \left( d(\mu_1, \mu_2) + |a_1 - a_2| \right).
\]

In \( \mathcal{F}_3^u \), the total variation is computed as in (2.3), keeping \( u \) fixed. Throughout, the constants \( T, \mathcal{L} \) and \( \mathcal{C} \) are kept fixed and the dependence of \( \mathcal{F}^u \) on them is omitted.

The extension of Definition 2.1 from the case of (1.1) to that of (1.2) is immediate.
Corollary 2.3. Fix $T > 0$ and a compact subset $\mathcal{U}$ of $\mathbb{R}^N$. Let $b, c, \beta \in \mathcal{F}^n$. Then, for any $\mu_0 \in \mathcal{M}^+(\mathbb{R}_+)$ and any $u \in \text{BV}(\mathbb{R}_+; \mathcal{U})$, problem (1.2) admits a unique solution. Moreover, there exists a constant a constant $C$ dependent only on $L, C$ and $T$ such that if for $i = 1, 2$, $\mu^i$ is the solution to (1.1) with initial data $\mu_o^i$, and $b, c, \beta, u$ replaced by $b_i, c_i, \beta_i, u_i$, then,

$$d(\mu^1(t), \mu^2(t)) \leq d(\mu^1_o, \mu^2_o) e^{Ct} + Ct e^{Ct} \left( \sup_{t \in [0, T]} \left\| b_1(t) - b_2(t) \right\|_{\mathcal{C}^0(\mathcal{M}^+(\mathbb{R}_+); \mathcal{U})} + \sup_{t \in [0, T]} \left\| c_1(t) - c_2(t) \right\|_{\mathcal{C}^0(\mathcal{M}^+(\mathbb{R}_+); \mathcal{U})} + \sup_{t \in [0, T]} \left\| \beta_1(t) - \beta_2(t) \right\|_{\mathcal{C}^0(\mathcal{M}^+(\mathbb{R}_+); \mathcal{U})} + \sup_{t \in [0, T]} \left\| u_1(t) - u_2(t) \right\| \right).$$

(2.6)

The proof is in Section 4.

2.2 ODE – Well Posedness

We first present the approximation algorithm introduced in [8], see [2, 16] for the present simplified version. Fix a positive time $T$. For any $n \in \mathbb{N} \setminus \{0\}$ and for the time step $\Delta t$, approximate the initial datum $\mu_0$ in (1.2) by means of a linear combination $\mu^0_n$ of Dirac deltas centered at $x_0^i, x_0^1, \ldots, x_0^n$ with masses $m_0^1, \ldots, m_0^n$ and approximate the initial datum with the measure

$$\mu^0_n = \sum_{i=1}^n m_0^i \delta_{x_0^i}.$$ 

On the time interval $[0, \Delta t]$, we approximate the solution to (1.2) with the measure

$$\mu^n(t) = \sum_{i=1}^n m^i(t) \delta_{x^i(t)}$$

where

$$\begin{cases} 
\dot{x}^i = \left( b(t, \mu^n(t); u(t)) \right) (x^i) & i = 0, \ldots, n \\
\dot{m}^0 = -c(t, \mu^n(t); u(t)) (x^0) m^0 + \sum_{i=1}^n \beta(t, \mu^n(t); u(t)) (x^i) m^i \\
\dot{m}^i = -c(t, \mu^n(t); u(t)) (x^i) m^i & i = 1, \ldots, n \\
x^i(0) = x_0^i & i = 1, \ldots, n \\
m^0(0) = 0 & i = 0, \ldots, n \\
m^i(0) = m_0^i & i = 1, \ldots, n 
\end{cases}$$

(2.7)
Define \( x^i_k = \lim_{t \to k\Delta t^-} x^i(t) \) and \( m^i_k = \lim_{t \to k\Delta t^-} m^i(t) \) for \( i = 0, \ldots, n \). Iteratively, for \( k \geq 1 \), we prolong \( \mu^n, x^{-k+1}, \ldots, x^n \) and \( m^{-k+1}, \ldots, m^n \) on the interval \([k \Delta t, (k + 1) \Delta t]\) solving

\[
\begin{align*}
\dot{x}^i &= \left( b(t, \mu^n(t); u(t)) \right) (x^i) & i &= -k, \ldots, n \\
\dot{m}^{-k} &= -c(t, \mu^n(t); u(t)) \left( x^{-k} \right) m^{-k} + \sum_{i=-k+1}^n \beta(t, \mu^n(t); u(t)) (x^i) m^i \\
\dot{m}^i &= -c(t, \mu^n(t); u(t)) (x^i) m^i & i &= -k + 1, \ldots, n \\
x^i(k\Delta t) &= x^i_k & i &= -k, \ldots, n \\
m^i(k\Delta t) &= m^i_k & i &= -k + 1, \ldots, n \\
x^{-k}(k\Delta t) &= 0 \\
m^{-k}(k\Delta t) &= 0
\end{align*}
\]

where \( x^i_k = \lim_{t \to k\Delta t^-} x^i(t) \), \( m^i_k = \lim_{t \to k\Delta t^-} m^i(t) \) for \( i = 0, \ldots, n \) and

\[
\mu^n(t) = \sum_{i=-k+1}^n m^i(t) \delta_{x^i(t)}.
\]

To describe the hypotheses on \( b, c, \beta \) ensuring the well posedness of \((2.7) - (2.8)\) it is of use to introduce, for positive \( T \) and \( L \), the set \( \tilde{F}^a \) of functions

\[
f : [0, T] \times \mathcal{U} \to C^{0,1}(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R}_+)
\]

such that

\[
(f(t; u))(\mu, a) = \tilde{f} \left( t, \int_{\mathbb{R}_+} \tilde{f}(\alpha) d\mu(\alpha), a; u \right)
\]

where

\[
(\tilde{F}^a_1) \quad \text{The map } \tilde{f} \in C^1(\mathbb{R}_+; \mathbb{R}_+) \text{ is bounded.}
\]

\[
(\tilde{F}^a_2) \quad \text{The map } (A, a; u) \to \tilde{f}(t, A, a; u) \text{ is in } C^1(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{U}; \mathbb{R}_+) \text{ for a.e. } t \in [0, T].
\]

\[
(\tilde{F}^a_3) \quad \text{The map } t \to \tilde{f}(t, A, a; u) \text{ is in } L^\infty([0, T]; \mathbb{R}_+) \text{ for all } A \in \mathbb{R}_+, a \in \mathbb{R}_+ \text{ and } u \in \mathcal{U}.
\]

\[
(\tilde{F}^a_4) \quad \text{\( f \) is Lipschitz continuous in } A, a, u \text{ uniformly in } t:
\]

\[
\left| \tilde{f}(t, A_1, a_1; u_1) - \tilde{f}(t, A_2, a_2; u_2) \right| \leq L \left( |A_1 - A_2| + |a_1 - a_2| + \|u_1 - u_2\| \right).
\]

The next result ensures the well posedness of the Cauchy Problem for the system of ordinary differential equations \((2.7) - (2.3)\).

**Theorem 2.4.** Fix \( n, N \in \mathbb{N} \setminus \{0\}, \) \( T, L > 0 \) and a compact subset \( \mathcal{U} \) of \( \mathbb{R}^N \). Let \( b, c, \beta \in \tilde{F}^a \). Then, for any control \( u \in \mathcal{BV}([0, T]; \mathcal{U}) \) and any initial datum \( (x^i_0, \ldots, x^n_0) \in \mathbb{R}_{+1}^n \), \( (m^i_0, \ldots, m^n_0) \in \mathbb{R}^n \), problem \((2.7) - (2.8)\) admits a unique solution \( t \to (x, m)_u(t) \) defined for all \( t \in [0, T] \). Moreover, the map \( u \to (x, m)_u \) is in \( C^1 \left( \mathcal{BV}([0, T]; \mathcal{U}); C^0([0, T]; \mathbb{R}_{+1}^n \times \mathbb{R}^n) \right) \).

The proof directly follows from Lemma \((2.2) \) in § \((2.1)\) and from the usual properties of the Nemitsky operator.
Theorem 2.5. Let $n \in \mathbb{N} \setminus \{0\}$, fix $T > 0$ and a compact subset $\mathcal{U}$ of $\mathbb{R}^N$. Let $b, c, \beta \in \mathcal{F}^u$ and $u \in BV([0,T];\mathcal{U})$. Fix $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$, $(x_0^0, \ldots, x_n^0) \in \mathbb{R}_+^n$, $(m_0^1, \ldots, m_n^1) \in \mathbb{R}_+^n$. Let $\mu$ solve problem (1.2) in the sense of Definition 2.1 and $(m, x)$ solve problem (2.7) – (2.8) with time step $\Delta t$. Then, there exists a positive $C$ independent from $u$, $\Delta t$ and $n$ such that for all $t \in [0,T]$, 
\[
  d \left( \mu(t), \sum_{i=-n}^n m_i^1(t) \delta_{x_i^0(t)} \right) \leq C \cdot \left[ \Delta t + d \left( \mu_o, \sum_{i=0}^n m_0^i \delta_{x_0^0} \right) \right].
\]

In specific numerical implementations of the present method, the quantity $d \left( \mu_o, \sum_{i=0}^n m_0^i \delta_{x_0^0} \right)$ is typically of the same order of the size of the space mesh $\Delta x$.

2.3 Optimal Control

A general cost functional defined on the controls in $BV([0,T];\mathcal{U})$ is
\[
\tilde{J} : BV([0,T];\mathcal{U}) \to \mathbb{R} \\
 u \to \int_0^T j \left( t, u(t), \int_{\mathbb{R}_+} \gamma(\xi) d\mu_u(t)(\xi) \right) dt \tag{2.9}
\]

where $\gamma \in C^{0,1}(\mathbb{R}_+; \mathbb{R}_+)$, $\mu_u$ is the solution to (1.2) corresponding to the control $u$ with $b, \beta, c$ and $\mu_o$ satisfying the assumptions of Theorem 2.2 and $j : [0,T] \times \mathcal{U} \times \mathbb{R}_+ \to \mathbb{R}_+$ being such that:

$(J_1)$ $j \geq 0$

$(J_2)$ the map $t \to j(t,x,u)$ is measurable for all $x \in \mathbb{R}_+$, $u \in \mathcal{U}$ and there exists a $\hat{u} \in BV([0,T];\mathcal{U})$ such that $J(\hat{u}) < +\infty$

$(J_3)$ there exist $L \in L^1([0,T];\mathbb{R}_+)$ and a nondecreasing $\omega \in C^0(\mathbb{R}_+;\mathbb{R}_+)$, with $\omega(0) = 0$, such that
\[
|j(t,x_1,u_1) - j(t,x_2,u_2)| \leq L(t) \omega(|x_1 - x_2| + |u_1 - u_2|)
\]
for a.e. $t \in [0,T]$, for all $x_1, x_2 \in \mathbb{R}_+$ and all $u_1, u_2 \in \mathcal{U}$.

Having to consider also costs related to the adjustments in the values of the control, it is natural to seek the minimization of
\[
J : BV([0,T];\mathcal{U}) \to \mathbb{R} \\
 u \to \tilde{J}(u) + TV_{\mathbb{R}_N}(u). \tag{2.10}
\]

As a first result, we prove the existence of an optimal control.

Theorem 2.6. Fix $T > 0$ and a compact subset $\mathcal{U}$ of $\mathbb{R}^N$. For all $b, c, \beta \in \mathcal{F}^u$, $u \in BV([0,T];\mathcal{U})$ and $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$, let $\mu_u$ be the solution to problem (1.2). With reference to the cost functional (2.9), $\gamma \in C^{0,1}(\mathbb{R}_+;\mathbb{R}_+)$ and $j$ satisfies $(J_1)$, $(J_2)$, $(J_3)$. Then, there exists a control minimizing $J$ as defined in (2.10): 
\[
\exists u^* \in BV([0,T];\mathcal{U}) : J(u^*) = \inf_{u \in BV([0,T];\mathcal{U})} J(u).
\]
We now pass to the discrete counterpart of Theorem 2.6, substituting the evolution described by (1.2) with the approximation provided by the Escalator Boxcar Train (2.7)–(2.8). At the same time, also the functionals (2.7)–(2.8) have to be computed on linear combination of Dirac deltas.

**Theorem 2.7.** Fix $T > 0$ and a compact subset $U$ of $\mathbb{R}^N$. Let $b, c, \beta \in \mathcal{F}_u$ and $u \in \text{BV}([0,T];U)$. For any $n \in \mathbb{N} \setminus \{0\}$ and $\Delta t_n > 0$, fix an initial datum $(x_1^0, \ldots, x_n^0) \in \mathbb{R}_+^{n+1}, (m_1^0, \ldots, m_n^0) \in \mathbb{R}^n$ in (2.7)–(2.8) and call $(x^{-n}, \ldots, x^n), (m^{-n}, \ldots, m^n)$ the corresponding solution. Further, define the cost functionals

\[
\tilde{J}_n : \text{BV}([0,T];U) \to \mathbb{R} \\
u \mapsto \int_0^T j(t, u(t), \int_{\mathbb{R}_+} \gamma(\xi) \, d\mu_n^u(t)(\xi)) \, dt \tag{2.11}
\]

\[
J_n : \text{BV}([0,T];U) \to \mathbb{R} \\
u \mapsto \tilde{J}_n(u) + TV_{\mathbb{R}^N}(u). \tag{2.12}
\]

where $\mu_n^u(t) = \sum_{i=-n}^n m_i^u(t) \delta_{x_i}(t)$, $\gamma \in (C^1 \cap C^{0,1})((\mathbb{R}_+; \mathbb{R}_+))$, $j$ satisfies $(J_1)$, $(J_2)$, $(J_3)$ and there exists a $\hat{u} \in \text{BV}([0,T];U)$ such that $J(\hat{u}) < +\infty$. Then, there exists a control minimizing $J_n$:

\[
\exists u_n^* \in \text{BV}([0,T];U) : \ J_n(u_n^*) = \inf_{u \in \text{BV}([0,T];U)} J_n(u). \tag{2.13}
\]

The above theorems yield the following corollary, which is the main result of the present work. It ensures that the Escalator Boxcar Train algorithm can also be used to solve optimal control problems.

**Corollary 2.8.** With the same assumptions and notation as in Theorem 2.6 and in Theorem 2.7 if

\[
\lim_{n \to +\infty} \Delta t_n = 0 \quad \text{and} \quad \lim_{n \to +\infty} d\left(\mu_{u_n}, \sum_{i=-n}^n m_i^u(t) \delta_{x_i}(t)\right) = 0
\]

then,

\[
\lim_{n \to +\infty} J_n(u_n^*) = \inf_{u \in \text{BV}([0,T];U)} J(u) \tag{2.13}
\]

and, up to a subsequence,

\[
\lim_{n \to +\infty} \|u_n^* - u^*\|_{L^\infty([0,T];\mathbb{R})} = 0 \quad \text{where} \quad J(u^*) = \inf_{u \in \text{BV}([0,T];U)} J(u). \tag{2.14}
\]

**3 The McKendrick – Von Förster Model in Welfare Policies**

The McKendrick – Von Förster model for population growth, equipped with an integral functional to be maximized, provides a first example of a system fitting within (1.2), where the results in the sections 2.1 and 2.2 can be applied.
Consider a population described by the amount \( n = n(t,a) \) of people that at time \( t \) have the age \( a \). Call \(-d\), with \( d = d(a) \), the population mortality rate. We thus obtain:

\[
\begin{cases}
\partial_t n + \partial_a n = -d(a) n \\
n(t,0) = \int_0^{+\infty} \hat{\beta}(a) n(t,a) \, da \\
n(0,x) = n_o(x).
\end{cases}
\]

Here, \( \hat{\beta} \) describes the natality rate of the population of age \( a \) at time \( t \).

Introduce a policy to sustain birth rate. It is then natural to assume that a control parameter, say \( u \), enters the birth functions. The parameter \( u \), possibly vector valued, reflects a government policy to foster natality, helping through ad hoc acts the families with children.

\[
\begin{cases}
\partial_t n_u + \partial_a n_u = -d(a) n_u \\
n_u(t,0) = \int_0^{+\infty} \hat{\beta}(a,u) n_u(t,a) \, da \\
n_u(0,x) = n_o(x).
\end{cases}
\] (3.1)

From the governmental point of view, the income of the state welfare can be described by the functional

\[
\mathcal{J}(u) = \int_0^{+\infty} e^{-\lambda t} \left( \int_0^{+\infty} w(a) n_u(t,a) \, da - u(t) n_u(t,0) \right) \, dt .
\] (3.2)

The weight \( w = w(a) \) is positive all through the active age interval, i.e., all during the period where individuals, paying taxes, sustain the state. On the contrary, \( w \) is negative when individuals receive services from the state, e.g., during childhood and retirement.

**Lemma 3.1.** Fix a compact \( \mathcal{U} \) in \( \mathbb{R}^N \) and \( \alpha \in [0,1] \). System (3.1) fits into (2.7)–(2.8) setting

\[
b(t,\mu,u)(a) = 1, \quad c(t,\mu,u)(a) = d(a), \quad \beta(t,\mu,u)(a) = \hat{\beta}(a,u).
\]

Moreover, if

\[
d \in C^{0,1}(\mathbb{R}_+,\mathbb{R}), \quad \hat{\beta} \in C^{0,1}(\mathcal{U} \times \mathbb{R}_+;\mathbb{R})
\]

then, for all \( u \in C^{1,\alpha}_b([0,T];\mathcal{U}) \), Theorem 2.4 applies.

In the present case, equations (2.7)–(2.8) take the form, for \( t \in [0,\Delta t] \)

\[
\begin{cases}
\dot{x}^i = 1 \\
\dot{m}^0 = -d(x^0) m^0 + \sum_{i=0}^n \hat{\beta} \left( x^i, u(t) \right) m^i \\
\dot{m}^i = -d(x^i) m^i \\
x^i(0) = x_o^i \\
\end{cases} \quad i = 0, \ldots, n
\]

while for \( t \in [k\Delta t, (k+1)\Delta t] \) the solution to the above system is extended as follows

\[
\begin{cases}
\dot{x}^i = 1 \\
\dot{m}^{-k} = -d(x^{-k}) m^{-k} + \sum_{i=-k}^n \hat{\beta} \left( x^i, u(t) \right) m^i \\
\dot{m}^i = -d(x^i) m^i \\
x^{-k}(k\Delta t) = 0 \\
\end{cases} \quad i = -k, \ldots, n
\]

\[
\begin{cases}
\dot{x}^i = 1 \\
\dot{m}^{-k} = -d(x^{-k}) m^{-k} + \sum_{i=-k}^n \hat{\beta} \left( x^i, u(t) \right) m^i \\
\dot{m}^i = -d(x^i) m^i \\
x^{-k}(k\Delta t) = 0 \\
\end{cases} \quad i = -k+1, \ldots, n
\]

\[
\begin{cases}
\dot{m}^{-k}(k\Delta t) = 0
\end{cases}
\]
Note that the variables \( x^i \) decouple and it is immediate to obtain

\[
x^i(t) = t - i \Delta t \quad \text{for} \quad t \geq \min\{-i \Delta t, 0\} \quad \text{and} \quad i = -k, \ldots, n.
\]

The discretized version of the cost functional (3.2) is

\[
J^n(u) = \sum_{k=0}^{\infty} \int^{(k+1)\Delta t}_{k\Delta t} e^{-\lambda t} \left( \sum_{i=-k+1}^{n} w(t - i \Delta t) m^i_u(t) - u(t) m^{-i}_u(t) \right) dt
\]

4 Technical Details

4.1 Proofs Related to Section 2.1

Lemma 4.1. Fix \( T > 0 \) and a normed space \( X \). Let \( x \in BV([0, T]; X) \). Then, for any \( \varepsilon > 0 \), there exists \( n \in \mathbb{N} \), \( \{t_1, t_2, \ldots, t_n\} \subset [0, T] \) and \( \{x_1, x_2, \ldots, x_n\} \in X \) such that, setting \( x_\varepsilon(t) = \sum_{i=1}^{n} x_i \chi_{[t_{i-1}, t_i]}(t) \),

\[
\sup_{t \in [0, T]} \|x(t) - x_\varepsilon(t)\|_X \leq \varepsilon, \quad x_\varepsilon([0, T]) \subseteq x([0, T]) \quad \text{and} \quad TV_X(x_\varepsilon) \leq TV_X(x).
\]

Proof. The construction of the function \( x_\varepsilon \) follows, for instance, from [1, Theorem 1.2, Chapter 1]. The inclusion and the bound on the total variation are immediate, since the \( x_i \) are chosen among the values attained by \( x \).

Proof of Theorem 2.2. On the space \( X = C^0(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R}) \), define the norm \( \|\cdot\|_X \) as in (2.1) and apply Lemma 4.1 to the maps \( b, c, \beta : [0, T] \rightarrow X \). For every \( \varepsilon > 0 \), there exists \( n \in \mathbb{N} \), \( \{t_1, t_2, \ldots, t_n\} \subset [0, T] \) and piecewise constant functions \( b_\varepsilon, c_\varepsilon, \beta_\varepsilon : [0, T] \rightarrow X \) such that

\[
\sup_{t \in [0, T]} \|b(t) - b_\varepsilon(t)\|_X \leq \varepsilon, \quad TV_X(b_\varepsilon) \leq TV_X(b),
\]

\[
\sup_{t \in [0, T]} \|c(t) - c_\varepsilon(t)\|_X \leq \varepsilon, \quad TV_X(c_\varepsilon) \leq TV_X(c),
\]

\[
\sup_{t \in [0, T]} \|\beta(t) - \beta_\varepsilon(t)\|_X \leq \varepsilon, \quad TV_X(\beta_\varepsilon) \leq TV_X(\beta).
\]

Moreover, the inclusion proved in Lemma 4.1 ensures that

\[
\sup_{t \in [0, T]} \|b_\varepsilon(t)\|_{C^{0,1}(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \leq \mathcal{L},
\]

\[
\sup_{t \in [0, T]} \|c_\varepsilon(t)\|_{C^{0,1}(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \leq \mathcal{L},
\]

\[
\sup_{t \in [0, T]} \|\beta_\varepsilon(t)\|_{C^{0,1}(\mathcal{M}^+(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \leq \mathcal{L}.
\]

By construction, the sequences \( b_\varepsilon, c_\varepsilon \) and \( \beta_\varepsilon \) converge to \( b, c \) and \( \beta \) uniformly on \([0, T]\). Hence, they are all Cauchy sequences.

Fix \( \varepsilon > 0 \). For all \( i = 1, \ldots, n \), [9, Theorem 2.11], or [18, Theorem 4.6], [19, Theorem 1.3], can be recursively applied on the interval \([t_{i-1}, t_i]\) to the problem

\[
\begin{cases}
\partial_t \mu_i + \partial_a \left( b_\varepsilon(t, \mu_i) \mu_i \right) + c_\varepsilon(t, \mu_i) \mu_i = 0 \\
(b_\varepsilon(t, \mu_i)) (0) D \mu_i(0+) = \int_{0}^{\varepsilon} \beta_\varepsilon(t, \mu_i) \ d \mu_i (c) \\
\mu_i(t_{i-1}) = \mu^0_{i-1}
\end{cases}
\]
where \( \mu_0^i = \mu_0 \) and \( \mu_i^a = \lim_{t \to t_i} \mu_i(t) \) for \( i = 1, \ldots, n - 1 \). Define \( \mu^\varepsilon(t) \) by \( \mu^\varepsilon(t) = \mu_i(t) \) whenever \( t \in [t_i-1, t_i] \).

By [5] (iv) in Theorem 2.8, for any \( \varepsilon, \varepsilon' > 0 \) sufficiently small,

\[
d \left( \mu^\varepsilon(t), \mu^\varepsilon(t) \right) \leq C t e^{C t} \left( \sup_{t \in [0, T]} \| b_{\varepsilon}(t) - b_{\varepsilon'}(t) \|_{C^0(M^+([R_+], \mathbb{R}))} + \sup_{t \in [0, T]} \| c_{\varepsilon}(t) - c_{\varepsilon'}(t) \|_{C^0(M^+([R_+], \mathbb{R}))} + \sup_{t \in [0, T]} \| \beta_{\varepsilon}(t) - \beta_{\varepsilon'}(t) \|_{C^0(M^+([R_+], \mathbb{R}))} \right) .
\]

Therefore, by the completeness of \( C^0([0, T]; M^+([R_+])) \), there exists a measure valued map \( \mu \in C^0([0, T]; M^+([R_+])) \) such that \( \lim_{\varepsilon \to 0} \sup_{t \in [0, T]} d \left( \mu^\varepsilon(t), \mu(t) \right) = 0 \).

To prove that \( \mu \) solves (1.1) in the sense of Definition 2.1 observe that by construction

\[
\int_{\mathbb{R}_+} \varphi(t, a) d\mu(t) - \int_{\mathbb{R}_+} \varphi(0, a) d\mu^\varepsilon(a)
= \int_0^T \int_{\mathbb{R}_+} \partial_t \varphi(t, a) d\mu^\varepsilon(a) dt
+ \int_0^T \int_{\mathbb{R}_+} (\partial_a \varphi(t, a) \left( b_{\varepsilon}(t, \mu^\varepsilon_t) \right) (a) + \varphi(t, a) \left( c_{\varepsilon}(t, \mu^\varepsilon_t) \right) (a)) d\mu^\varepsilon(a) dt
+ \int_0^T \int_{\mathbb{R}_+} \varphi(t, 0) \left( \beta_{\varepsilon}(t, \mu^\varepsilon_t) \right) (a) d\mu^\varepsilon(a) dt .
\]

and the limit \( \varepsilon \to 0 \) can pass inside the integral sign thanks to the uniform convergences \( \mu^\varepsilon \to \mu \), \( b_{\varepsilon} \to b \), \( c_{\varepsilon} \to c \) and \( \beta_{\varepsilon} \to \beta \) on the time interval \([0, T]\).

A further application of [5] (iv) in Theorem 2.1 proves the stability estimate [2.5]. \( \square \)

**Proof of Corollary 2.3.** Note first that if \( b, c, \beta \in F^u \) and \( u \in BV([0, T]; U) \), then the maps \( b^u, c^u, \beta^u \) defined by \( b^u(t) = b(t, u(t)) \), \( c^u(t) = c(t, u(t)) \) and \( \beta^u(t) = \beta(t, u(t)) \) all satisfy \( b^u, c^u, \beta^u \in F^u \). Therefore, Theorem 2.2 applies, ensuring the existence of a solution to (1.2).

Concerning the stability estimates, with obvious notations, by (2.5) we have:

\[
d \left( \mu^1(t), \mu^2(t) \right) \leq d(\mu^1_0, \mu^2_0) e^{C t} + C t e^{C t} \left( \sup_{t \in [0, T]} \| b^1(t) - b^2(t) \|_{C^0(M^+([R_+], \mathbb{R}))} + \sup_{t \in [0, T]} \| c^1(t) - c^2(t) \|_{C^0(M^+([R_+], \mathbb{R}))} + \sup_{t \in [0, T]} \| \beta^1(t) - \beta^2(t) \|_{C^0(M^+([R_+], \mathbb{R}))} \right) .
\]

Observe now that

\[
\| b^1(t) - b^2(t) \|_{C^0(M^+([R_+], \mathbb{R}))} \leq \| b^1(t) - b^2(t) \|_{C^0(M^+([R_+], \mathbb{R}))} + \| b^2(t) - b^2(t) \|_{C^0(M^+([R_+], \mathbb{R}))} \\
\leq L_u \| u_1(t) - u_2(t) \| + \| b^1(t) - b^2(t) \|_{C^0(U \times M^+([R_+], \mathbb{R}))} .
\]

Entirely analogous estimates hold for the term \( \| c^1(t) - c^2(t) \|_{C^0(M^+([R_+], \mathbb{R}))} \) as well as for \( \| \beta^1(t) - \beta^2(t) \|_{C^0(M^+([R_+], \mathbb{R}))} \) allowing to obtain (2.6). \( \square \)
4.2 Proofs Related to Section 2.2

Aiming at the well posedness of (2.7)-(2.8) we rewrite it as

\[
\begin{aligned}
\dot{x} &= f(t, x, m, u) \\
\dot{m} &= g(t, x, m, u) \\
x(0) &= x_o \\
m(0) &= m_o
\end{aligned}
\]  

(4.1)

where

\[x = (x^{-n}, \ldots, x^n)\] \quad \[m = (m^{-n}, \ldots, m^n)\]

\[x_o = \begin{cases}
  i \Delta t & i = 0, \ldots, n \\
  0 & i = -n, \ldots, -1
\end{cases} \quad m_o = \begin{cases}
  \mu_o \left( \left[ i \Delta t, (i+1)\Delta t \right) \right) & i = 0, \ldots, n \\
  0 & i = -n, \ldots, -1
\end{cases}
\]

\[f : [0, T] \times \mathbb{R}^{2n+1}_+ \times \mathbb{R}^{2n+1}_+ \times \mathcal{U} \to \mathbb{R}^{2n+1} \quad g : [0, T] \times \mathbb{R}^{2n+1}_+ \times \mathbb{R}^{2n+1}_+ \times \mathcal{U} \to \mathbb{R}^{2n+1}\]

the functions \(f_i, g_i\) being defined, for \(i = -n, \ldots, n\), by

\[
f_i(t, x, m; u) = \begin{cases}
  b \left( t, \sum_{j=-n}^{n} m^j \delta_{x^j}; u \right) (x^i) & t \geq \max\{-i \Delta t, 0\} \\
  0 & t < \max\{-i \Delta t, 0\}
\end{cases}
\]  

(4.2)

and

\[
g_i(t, x, m; u) = \begin{cases}
  -c \left( t, \sum_{j=-n}^{n} m^j \delta_{x^j}; u \right) (x^i) m^i & t > \max\{(1-i) \Delta t, 0\} \\
  -c \left( t, \sum_{j=-n}^{n} m^j \delta_{x^j}; u \right) (x^i) m^i & t \in [\max\{-i \Delta t, 0\}, \max\{(1-i) \Delta t, 0\}] \\
  + \sum_{\ell=-n}^{n} \beta \left( t, \sum_{j=-n}^{n} m^j \delta_{x^j}; u \right) (x^\ell) m^\ell & t < \max\{-i \Delta t, 0\}
\end{cases}
\]  

(4.3)

Lemma 4.2. Fix positive \(T, L\) and let \(b, c, \beta \in \mathcal{F}^u\). Then, the map \(f\) and \(g\) defined in 4.2 and 4.3 satisfy the following conditions:

(f1) \(t \to (f, g)(t, x, m; u)\) is measurable for all \(x \in \mathbb{R}_+, m \in \mathbb{R}_+\) and \(u \in \mathcal{U}\);

(f2) \((x, m; u) \to (f, g)(t, x, m; u)\) is in \(C^1\) for a.e. \(t \in [0, T]\);

(f3) \((x, m; u) \to (f, g)(t, x, m; u)\) is sublinear, uniformly in \(t\).

Proof. We detail the proof that \(f\) satisfies the above properties, the case of \(g\) being entirely similar.
The measurability of \( t \to f(t, x, y; u) \) is immediate. To verify the differentiability, introduce the standard base \( (e_{-n}, e_{-n+1}, \ldots, e_{-n+1}, e_n) \) of \( \mathbb{R}^{2n+1} \) and compute for \( i = -n, \ldots, n \), for \( t > \max\{-i\Delta t; 0\} \) and for a (small) \( h \in \mathbb{R} \)

\[
f_i(t, x + he_i, m; u) - f_i(t, x, m; u) = \left( b \left( t, \sum_{j=-n}^{n} m^j \delta_{x^j + \delta_{x^j} he_i}; u \right) \right) (x^i) = \left( b \left( t, \sum_{j=-n}^{n} m^j \delta_{x^j}; u \right) \right) (x^i)
\]

as \( h \to 0 \), while for \( \ell \neq i \) and for \( t > \max\{-i\Delta t; -\ell\Delta t, 0\} \)

\[
f_i(t, x + he_\ell, m; u) - f_i(t, x, m; u) = \left( b \left( t, \sum_{j=-n}^{n} m^j \delta_{x^j + \delta_{x^j} he_\ell}; u \right) \right) (x^i) = \left( b \left( t, \sum_{j=-n}^{n} m^j \delta_{x^j}; u \right) \right) (x^i)
\]

proving the differentiability of \( f_i \) with respect to \( x \). Let now \( i, \ell = -n, \ldots, n \):

\[
f_i(t, x, m + he_\ell; u) - f_i(t, x, m; u) = \left( \bar{b} \left( t, \sum_{j=-n}^{n} m^j \delta_{x^j + \delta_{x^j} he_\ell}; u \right) \right) (x^i) = \left( \bar{b} \left( t, \sum_{j=-n}^{n} m^j \delta_{x^j}; u \right) \right) (x^i)
\]

so that \( f_i \) is differentiable also with respect to \( m \). The differentiability with respect to \( u \) is immediate.

Finally, we prove that \( (x, m; u) \to (f, g)(t, x, m; u) \) is sublinear:

\[
|f_i(t, x, m; u)|
\]
\[ \begin{align*}
&\leq |f_1(t, 0; u)| + |f_1(t, x, m; u) - f_1(t, 0; u)| \\
&= \left| \left(b(t, 0; u)(0)\right) + \left(b\left(t, \sum_{j=-n}^{n} m^j \delta_{x_j}; u\right)(x^j) \right) \right| \tag{b(t, 0; u)(0)} \\
&\leq \left| b(t, 0; u) + \left(b\left(t, \sum_{j=-n}^{n} m^j \delta_{x_j}; u\right) - b(t, 0; u)\right) \right| \\
&\leq |b(t, 0; u)| + L \left( \left| \sum_{j=-n}^{n} m^j \delta_{x_j}(x^j) \right| + x^j \right) \\
&\leq |b(t, 0; u)| + L \sqrt{n} (|m| + |x|) \\
\end{align*} \]

and the first summand above is bounded by \((\tilde{F}_1^u)\), completing the proof. \(\square\)

Below, we call semiflow (or process) on the set \(M\) a map \(S: M \times [0, \delta] \times [0, T] \to M\) such that \(S(0, t) = 1d\) for all \(t \in [0, T]\), and \(S(t_3, t_1 + t_2) \circ S(t_2, t_1) = S(t_2 + t_1, t_1)\) for all \(t_1, t_2, t_3\) such that \(t_1, t_1 + t_2 \in [0, T]\), \(t_2, t_3, t_2 + t_3 \in [0, \delta]\). We say that the semiflow is Lipschitz continuous if the map \(\mu \to S(t, t_0)\mu\) is Lipschitz continuous, uniformly in \(t_0 \in [0, T]\) and in \(t \in [0, \delta]\).

**Lemma 4.3.** Let \((M, d_M)\) be a metric space and \(S: M \times [0, \delta] \times [0, T] \to M\) a Lipschitz semiflow with Lipschitz constant \(L\). For every Lipschitz continuous map \(\mu: [0, T] \to M\), the following estimate holds:

\[ d_M\left(\mu_t, S(t, 0)\mu_0\right) \leq L \int_0^t \liminf_{h \to 0^+} \frac{d_M \left(S(h, \tau) \mu_t\right)}{h} \, d\tau \tag{4.4} \]

For a proof, see [3, Theorem 2.9] or, in the present non-autonomous case, [16, Proposition 4.1] or [7, Proof of Theorem 3.15].

**Lemma 4.4.** [16, Lemma 7.3] Let \(n \in \mathbb{N}\), \(m, m' \in \mathbb{R}^n\) and \(x, x' \in \mathbb{R}^n\). Then, with reference to the distance \(d\) defined in (2.4),

\[ d \left( \sum_{i=1}^{n} m_i \delta_{x_i}, \sum_{i=1}^{n} m'_i \delta_{x'_i} \right) \leq \max \left\{ 1, \sum_{i=1}^{n} |m_i| \right\} \sum_{i=1}^{n} \left( |m_i - m'_i| + |x_i - x'_i| \right) \]

**Proof of Theorem 2.5.** The proof relies on Lemma 4.4. First, we prove that the map

\[ \mu^\pi: [0, T] \to \mathcal{M}^+(\mathbb{R}^+) \]

\[ t \mapsto \sum_{i=-n}^{n} m^i(t) \delta_{x^i(t)} \]

where \(t \to (x^i, m^i)(t)\) solves (4.1)–(4.2)–(4.3), is Lipschitz continuous with respect to the metric \(d\) defined in (2.4). Indeed, by Lemma 4.4

\[ d \left( \mu^\pi(t), \mu^\pi(s) \right) \leq \max \left\{ 1, \sum_{i=-n}^{n} |m^i(t)| \right\} \sum_{i=-n}^{n} \left( |m^i(t) - m^i(s)| + |x^i(t) - x^i(s)| \right) \]

15
Moreover,

\[ \text{Lip}(x^i) \leq \| f_i \|_{C^0([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{U}; \mathbb{R})} \]

\[ \leq \sup_{(t,u) \in [0,T] \times \mathbb{U}} \| b(t,u) \|_{C^{0,1}((\mathcal{M}^+)(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \]

\[ \leq \mathcal{L} \] [by (4.1)]

\[ |m^i(t)| \leq |m_a| \exp \left[ T \sup_{(t,u) \in [0,T] \times \mathbb{U}} \| c(t,u) \|_{C^{0,1}((\mathcal{M}^+)(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \right] \]

\[ \times \exp \left( (2n+1)T \sup_{(t,u) \in [0,T] \times \mathbb{U}} \| \beta(t,u) \|_{C^{0,1}((\mathcal{M}^+)(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \right) \] [by (4.3)]

\[ \leq |m_a|e^{(2(n+1)\mathcal{L})T} \] [by (4.1)]

\[ \text{Lip}(m^i) \leq \| g \|_{C^0([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{U}; \mathbb{R})} \]

\[ \leq \sup_{(t,u) \in [0,T] \times \mathbb{U}} \| c(t,u) \|_{C^{0,1}((\mathcal{M}^+)(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \sup_{t \in [0,T]} \left| m^i(t) \right| \]

\[ + (2n+1)\| \beta(t,u) \|_{C^{0,1}((\mathcal{M}^+)(\mathbb{R}_+) \times \mathbb{R}_+; \mathbb{R})} \sup_{t \in [0,T]} \left| m^i(t) \right| \] [by (4.3)]

\[ \leq 2(n+1)\mathcal{L}|m_a|e^{(2(n+1)\mathcal{L})T} \] [by (4.1)]

These estimates, inserted in (4.6), complete the proof of the Lipschitz continuity of \( \mu^n \) with respect to the metric \( d \) defined in (2.3).

By the above computations and Corollary 2.3, we can thus use Lemma 4.4, where \( S \) is the semiflow generated by (1.2) and \( \mu \) is replaced by \( \mu^n \) as defined in (4.5), obtaining

\[ d \left( \sum_{i=-n}^{n} m^i(t) \delta_{x^i(t)}, \mu_a(t) \right) = d \left( \mu^n(t), S(t,0)\mu_o \right) \]

\[ \leq d \left( \mu^n(t), S(t,0)\mu^n(0) \right) + d \left( S(t,0)\mu^n(0), S(t,0)\mu_o \right) \]

\[ \leq e^{\mathcal{C}\mathcal{L}} \left[ \int_0^t \liminf_{h \to 0^+} \frac{d \left( \mu^n(\tau+h), S(h,\tau)\mu_o \right)}{h} d\tau + d \left( \mu^n(0), \mu_o \right) \right]. \] (4.7)

The rest of the proof is devoted to estimate the integrand in the latter term above.

Without loss of generality, we may assume that \( \tau \in [0,\Delta t] \) and that \( h \) is so small that \([\tau, \tau + h] \subset [0, \Delta t]\). Define

\[ \mu_\tau(t) := S(t,\tau) \mu^n(\tau). \]

Then, for \( t \in [\tau, \Delta t] \), the map \( t \to \mu_\tau(t) \) solves problem (1.2) with initial datum \( \mu^n(\tau) = \sum_{i=-n}^{n} m^i(\tau) \delta_{x^i(\tau)} = \sum_{i=0}^{n} m^i(\tau) \delta_{x^i(\tau)} \) assigned at time \( \tau \).

As in [10] Proof of Theorem 4.3, \( \mu_\tau(t) \) can be represented as

\[ \mu_\tau(t) = \sum_{i=0}^{n} M^i(\tau + h) \delta_y(y^i(\tau+h) + \pi(t,\cdot)) \ dx \]

for suitable maps \( M^0, \ldots, M^n, y^0, \ldots, y^n \), the density \( \pi(t,\cdot) \) arising from the boundary and supported inside \([x^0_\tau, y^0(\tau)]\). Denote the total mass of \( \pi(t,\cdot) \) by

\[ M^n(t) = \int_{x^0_\tau}^{y^0(t)} \pi(t,x) \ dx. \]
Using suitable test functions in Definition 2.1 we obtain:

\[ y^i(\tau + h) = x^i(\tau) + \int_{\tau}^{\tau+h} b(t, \mu_\tau(t - \tau), u)(y^i(t)) \, dt \]

\[ M^i(\tau + h) = m^i(\tau) + \int_{\tau}^{\tau+h} c(t, \mu_\tau(t - \tau), u)(y^i(t)) M^i(t) \, dt \]

\[ M^\pi(\tau + h) = M^\pi(\tau) + \int_{\tau}^{\tau+h} \left( \int_{x_0^i}^{y^0(t)} -c(t, \mu_\tau(t - \tau), u)(x) \, d\mu_\tau(t - \tau)(x) \right. \]

\[ \left. + \int_{x_0^i}^{+\infty} \beta(t, \mu_\tau(t - \tau), u)(x) \, d\mu_\tau(t - \tau)(x) \right) \, dt \]

\[ = M^\pi(\tau) + \int_{\tau}^{\tau+h} \int_{x_0^i}^{y^0(t)} \left[ \left( -c(t, \mu_\tau(t - \tau), u)(x) + \beta(t, \mu_\tau(t - \tau), u)(x) \right) \right. \]

\[ \left. \cdot \left. d\mu_\tau(t - \tau)(x) \right] \, dt \]

\[ + \sum_{i=0}^{n} \int_{\tau}^{\tau+h} \beta(t, \mu_\tau(t - \tau), u)(y^i(t)) \, M^i(t) \, dt \]

\[ \leq M^\pi(\tau) + 2C\int_{\tau}^{\tau+h} \int_{x_0^i}^{y^0(t)} \pi(t, x) \, dx \, dt + \sum_{i=0}^{n} \int_{\tau}^{\tau+h} \beta(t, \mu_\tau(t - \tau), u)(y^i(t)) M^i(t) \, dt \]

\[ = M^\pi(\tau) + O(h^2) + \sum_{i=0}^{n} \int_{\tau}^{\tau+h} \beta(t, \mu_\tau(t - \tau), u)(y^i(t)) M^i(t) \, dt , \]

where with \( O(h^k) \) we denote a quantity that can be bounded the product of \( h^k \) with a constant dependent only on \( T, \mathcal{L} \) and \( C \).

Above, \( (\mathcal{F}_t^\pi) \) ensures a bound on \( c \) and \( \beta \). We also used the uniform boundedness of \( \pi(t, \cdot) \) on \([0, T]\) and the estimate

\[ \left| y^0(t) - x_0^i \right| \leq \sup_{t \in [0, T]} \sup_{u \in \mathcal{U}} \left\| b(t, u) \right\|_{C^0_0(\mathbb{R}^+ \times \mathbb{R}^+_+ : \mathbb{R}^n)} \leq \mathcal{L} h \]

for \( t \in [\tau, \tau+h) \). For \( t \in [0, \Delta t] \) define the time dependent measure

\[ \xi(t) = \sum_{i=0}^{n} p^i(t) \delta y^i(t) \]

where \( \begin{cases} p^0(t) = M^0(\tau) + M^\pi(\tau), \\ p^i(t) = M^i(\tau), \quad & \text{for } i = 1, \ldots, n \end{cases} \) (4.8)

in other words, in the measure \( \xi(t) \) the mass created due to the boundary condition, described by the density \( \pi(t, \cdot) \), is shifted to the closest Dirac delta. We note that:

\[ d \left( \mu_\tau(t), \mu^n(t + \tau) \right) \leq d \left( \mu_\tau(t), \xi(t + \tau) \right) + d \left( \xi(t + \tau), \mu^n(t + \tau) \right) . \]

Recalling that \( t \to y^i(t) \) is Lipschitz continuous with Lipschitz constant

\[ \text{Lip}(y^i) \leq \sup_{t \in [0, T]} \sup_{u \in \mathcal{U}} \left\| b(t, u) \right\|_{C^0_0(\mathbb{R}^+ \times \mathbb{R}^+_+ : \mathbb{R}^n)} \leq \mathcal{L} \]

and that the total mass is uniformly bounded on \([0, T]\), the first term in the right hand side of (4.9) is estimated as follows:

\[ d(\mu_\tau(h), \xi(\tau + h)) = d \left( \pi(\tau + h, \cdot), M^\pi(\tau + h) \delta y^0(\tau + h) \right) \]
\[
\begin{align*}
&\leq y^0(\tau + h) |M^\tau(\tau + h)| \\
&\leq \mathcal{L} \Delta t \left[ \sup_{t \in [0,T], u \in U} \| \beta(t, u) \|_{C^0([\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R})}} \int_\tau^{\tau + h} \sum_{i=0}^{n} M^i(t) \, dt + \mathcal{O}(h^2) \right] \\
&\leq \mathcal{L} \Delta t \left( \mathcal{L} C(T) h + \mathcal{O}(h^2) \right) \\
&= \Delta t \left( \mathcal{O}(h) + \mathcal{O}(h^2) \right). 
\end{align*}
\]

To bound the second term in (4.9), we want to use Lemma 4.4. Hence, we preliminarily obtain the following estimates on \( |x^i(\tau + h) - y^i(\tau + h)| \) and \( |m^i(\tau + h) - p^i(\tau + h)| \):

\[
\begin{align*}
|x^i(\tau + h) - y^i(\tau + h)| &\leq \int_\tau^{\tau + h} \left| b(t, \mu^n(t), u) \left( x^i(t) \right) - b(t, \mu_\tau(t - \tau), u) \left( y^i(t) \right) \right| \, dt \\
&\leq \int_\tau^{\tau + h} \left| b(t, \mu^n(t), u) \left( x^i(t) \right) - b(t, \mu_\tau(t - \tau), u) \left( x^i(t) \right) \right| \, dt \\
&\quad + \int_\tau^{\tau + h} \left| b(t, \mu_\tau(t - \tau), u) \left( x^i(t) \right) - b(t, \mu_\tau(t - \tau), u) \left( y^i(t) \right) \right| \, dt \\
&\leq \mathcal{L} \int_\tau^{\tau + h} d(\mu^n(t), \mu_\tau(t - \tau)) \, dt + \mathcal{L} \int_\tau^{\tau + h} \left| x^i(t) - y^i(t) \right| \, dt \\
&\leq \mathcal{L} \int_\tau^{\tau + h} \left( \text{Lip}(\mu^n) h + d(\mu^n(t), \mu_\tau(0)) + \text{Lip}(\mu_\tau(t)) h \right) \, dt \\
&\quad + \int_\tau^{\tau + h} \left( \text{Lip}(x^i) h + \left| x^i(\tau) - y^i(\tau) \right| + \text{Lip}(y^i) h \right) \, dt \\
&\leq \mathcal{O}(h^2),
\end{align*}
\]

since \( \mu^n(\tau) = \mu_\tau(0) \) and \( x^i(\tau) = y^i(\tau) \). Analogous estimates can be used to bound the term \( \sum_{i=0}^{n} |m^i(\tau + h) - p^i(\tau + h)| \), taking into account (4.8) and the estimate for \( M^\tau(t) \):

\[
\begin{align*}
\sum_{i=0}^{n} |m^i(\tau + h) - p^i(\tau + h)| \\
&\leq \sum_{i=0}^{n} \int_\tau^{\tau + h} \left| c(t, \mu^n(t), u) \left( x^i(t) \right) - c(t, \mu_\tau(t - \tau), u) \left( y^i(t) \right) \right| \, dt \\
&\quad + \sum_{i=0}^{n} \int_\tau^{\tau + h} \left| \beta(t, \mu^n(t), u) \left( x^i(t) \right) - \beta(t, \mu_\tau(t - \tau), u) \left( y^i(t) \right) \right| \, dt + \mathcal{O}(h^2) \\
&\leq \sum_{i=0}^{n} \int_\tau^{\tau + h} \left| c(t, \mu^n(t), u) \left( x^i(t) \right) \right| \, dt \\
&\quad + \sum_{i=0}^{n} \int_\tau^{\tau + h} \left| \beta(t, \mu^n(t), u) \left( x^i(t) \right) \right| \, dt \\
&\quad + \sum_{i=0}^{n} \int_\tau^{\tau + h} \left| c(t, \mu^n(t), u) \left( y^i(t) \right) - c(t, \mu_\tau(t - \tau), u) \left( y^i(t) \right) \right| \, dt \\
&\quad + \sum_{i=0}^{n} \int_\tau^{\tau + h} \left| \beta(t, \mu^n(t), u) \left( y^i(t) \right) \right| \, dt.
\end{align*}
\]

18
Proposition 4.5. (Helly). Fix \( N \in \mathbb{N} \) with \( N > 0 \). Let \( \mathcal{U} \) be a compact subset of \( \mathbb{R}^N \) and \( T \) be a positive scalar. Consider a sequence of functions \( u_n \in \text{BV}([0,T], \mathcal{U}) \) such that \( \sup_{n \in \mathbb{N}} TV_{\mathbb{R}^N}(u_n) < +\infty \). Then, there exists a subsequence \( u_{n_k} \) and a function \( u \in \text{BV}([0,T]; \mathcal{U}) \) such that

\[
\lim_{k \to +\infty} \|u_{n_k} - u\| = 0 \quad \text{and} \quad TV_{\mathbb{R}^N}(u) \leq \sup_{n \in \mathbb{N}} TV_{\mathbb{R}^N}(u_n).
\]

For a proof, see for instance [3, Chapter 2, Theorem 2.3].

Lemma 4.6. Fix \( T > 0 \) and for all \( u \in \text{BV}([0,T]; \mathcal{U}) \) call \( \mu_u \) the corresponding solution to (1.2). Assume there exists a constant \( L \) such that for all \( u_1, u_2 \in \text{BV}([0,T]; \mathcal{U}) \)

\[
d(\mu_{u_1}, \mu_{u_2}) \leq L \|u_1 - u_2\|_{L^\infty([0,T]; \mathbb{R})}.
\]

Let \( \gamma \in C^{0,1}(\mathbb{R}_+; \mathbb{R}_+) \) and \( j \) satisfy (J1)–(J3). Then, the functional \( J \) defined in (2.9)–(2.10) is lower semi-continuous with respect to the \( L^\infty \)-norm.

19
Proof. Let \( u_n \in BV([0, T]; \mathcal{U}) \) be a sequence converging to \( u \in BV([0, T]; \mathcal{U}) \) in the \( L^\infty \)-norm. First we recall that by [11, Theorem 1, § 5.2.1] \( TV_{R^N}(u_n) \leq \liminf_{n \to \infty} TV_{R^N}(u_n) \).

Next we show the sequential continuity of the map \( \mathcal{J} \) defined in (2.4), using (J3) and the fact that \( \omega \) is a nondecreasing function by (J3).

\[
|\mathcal{J}(u_n) - \mathcal{J}(u)| \leq \int_0^T \left| j \left( t, u_n(t), \int_{R^+} \gamma(\xi) d\mu_n(t) (\xi) \right) - j \left( t, u(t), \int_{R^+} \gamma(\xi) d\mu_u(t) (\xi) \right) \right| dt
\]
\[
\leq \int_0^T L(t) \omega \left( \int_{R^+} \gamma(\xi) d\mu_n(t) (\xi) - \int_{R^+} \gamma(\xi) d\mu_u(t) (\xi) \right) \ dt
\]
\[
\leq \omega \left( \|\gamma\|_{L^\infty(R^+; R^+)} \sup_{t \in [0, T]} d(\mu_n(t), \mu_u(t)) + \|u_n - u\|_{L^\infty([0, T]; R^N)} \right) \int_0^T L(t) \ dt
\]
\[
\leq \omega \left( 1 + L \|\gamma\|_{L^\infty(R^+; R^+)} \|u_n - u\|_{L^\infty([0, T]; R^N)} \right) \int_0^T L(t) \ dt
\]
\[
\to 0 \quad \text{in} \quad L^\infty([0, T]; R^N) \quad \text{as} \quad n \to +\infty ,
\]
completing the proof. \( \square \)

**Proof of Theorem 2.6.** Let \( \varepsilon_n \) be a strictly decreasing sequence converging to 0. Correspondingly, there exists a sequence \( u_{\varepsilon_n} \in BV([0, T]; \mathcal{U}) \) such that

\[
\mathcal{J}(u_{\varepsilon_n}) \leq \inf_u \mathcal{J}(u) + \varepsilon_n
\]

and, without loss of generality, we may also assume that \( \mathcal{J}(u_{\varepsilon_n}) \leq \mathcal{J}(\hat{u}) + 1 \) for all \( n \). Moreover, by (J1) and (2.10)

\[
TV(u_{\varepsilon_n}) \leq \mathcal{J}(u_{\varepsilon_n}) \leq \mathcal{J}(\hat{u}) + 1
\]

So that Proposition 4.5 can be applied, showing that, up to a subsequence, \( u_{\varepsilon_n} \) converges pointwise a.e. to a function \( u^* \in BV([0, T]; \mathcal{U}) \). Therefore,

\[
\mathcal{J}(u^*) = \mathcal{J} \left( \lim_{n \to +\infty} u_{\varepsilon_n} \right) \quad \text{[by the definition of} \ u^* \]
\[
\leq \liminf_{n \to +\infty} \mathcal{J}(u_{\varepsilon_n}) \quad \text{[by Lemma 4.6]}
\]
\[
= \inf_{u \in BV([0, T]; \mathcal{U})} \mathcal{J}(u) \quad \text{[by the definition of} \ u_{\varepsilon_n} \]
\]
completing the proof. \( \square \)

**Proof of Theorem 2.7.** The proof follows the same lines as that of Theorem 2.6. \( \square \)

**Proof of Corollary 2.8.** We first prove the uniform convergence \( \mathcal{J}_n \to \mathcal{J} \) of the costs on \( BV([0, T]; \mathcal{U}) \), using (J3), (2.9), (2.10), (2.11), (2.12) and Theorem 2.5 for all \( n \), we have:

\[
\sup_{u \in BV([0, T]; \mathcal{U})} |\mathcal{J}_n(u) - \mathcal{J}(u)| = \sup_{u \in BV([0, T]; \mathcal{U})} |\mathcal{J}_n(u) - \mathcal{J}(u)|
\]

20
\[
\begin{align*}
&\leq \sup_{u \in BV([0,T])} \int_0^T \left| j \left(t, u(t), \int_{R^+} \gamma(\xi) \, d\mu_n^u(t)(\xi) \right) - j \left(t, u(t), \int_{R^+} \gamma(\xi) \, d\mu_u(t)(\xi) \right) \right| dt \\
&\leq \sup_{u \in BV([0,T])} \int_0^T L(t) \omega \left( \left| \int_{R^+} \gamma(\xi) \, d\mu_n^u(t)(\xi) - \int_{R^+} \gamma(\xi) \, d\mu_u(t)(\xi) \right| \right) dt \\
&\leq \int_0^T L(t) \, dt \sup_{u \in BV([0,T])} \omega \left( \|\gamma\|_{L^\infty} \sup_{t \in [0,T]} d \left( \mu_n^u(t), \mu_u(t) \right) \right) \\
&\to 0 \text{ as } n \to +\infty,
\end{align*}
\]
which immediately implies (2.13).

Using (J_2), the same procedure used in the proof of Theorem 2.6 ensures that TV(u_n) is bounded uniformly in n. By Proposition 4.5, up to a subsequence, u_n \to \bar{u} a.e. on [0, T], proving (2.14). Using Lemma 4.6 and the uniform convergence of J_n to J proved above, we have:

\[
\begin{align*}
J(\bar{u}) &\leq \liminf_{n \to \infty} J(u_n) \\
&= \liminf_{n \to \infty} \left( J(u_n) + J_n(u_n) - J_n(u_n) \right) \\
&\leq \liminf_{n \to \infty} J_n(u_n) + \lim_{n \to \infty} \left( \sup_{u \in BV([0,T])} \left| J_n(u) - J(u) \right| \right) \\
&\leq \lim_{n \to \infty} J_n(u_n) \\
&= \inf_{u \in BV([0,T])} J(u),
\end{align*}
\]
where (2.13) was used to obtain the last equality. \qed

A Appendix: ODE Results

For completeness, we collect here a few basic ODE results using exactly the spaces and norms of use above.

Lemma A.1. Fix T > 0 and a compact \( U \subset \mathbb{R}^M \). Let \( f : [0, T] \times \mathbb{R}^N \times U \to \mathbb{R}^N \) be such that

\((f_1)\) \( t \to f(t, x; u) \) is measurable for all \( x \in \mathbb{R}^N \) and \( u \in U \);

\((f_2)\) \( (x, u) \to f(t, x; u) \) is in \( C^1 \) for a.e. \( t \in [0, T] \) and there exists a constant \( L > 0 \) such that for a.e. \( t \in [0, T] \), for all \( x_1, x_2 \in \mathbb{R}^N \) and for all \( u_1, u_2 \in U \),

\[
\| f(t, x_1; u_1) - f(t, x_2; u_2) \| \leq L \left( \| x_1 - x_2 \| + \| u_1 - u_2 \| \right).
\]
Then, for all \( x_o \in \mathbb{R}^N \) and all \( u \in L^\infty([0,T];\mathcal{U}) \), the problem

\[
\begin{align*}
\dot{x} &= f(t,x;u) \\
x(0) &= x_o
\end{align*}
\]  

(A.1)

admits a unique solution \( X(u) : [0,T] \rightarrow \mathbb{R}^N \). The map \( X : L^\infty([0,T];\mathcal{U}) \rightarrow C^1([0,T];\mathbb{R}^N) \) is Gateaux differentiable in any direction \( v \in L^\infty([0,T];\mathcal{U}) \) and the directional derivative \( D_v X(u) \) solves the Cauchy problem

\[
\begin{align*}
\frac{d}{dt} D_v X(u) &= \partial_x f(t,X(u);u) D_v X(u) + \partial_v f(t,X(u);u) v \\
(D_v X(u))(0) &= 0.
\end{align*}
\]  

(A.2)

Proof. The map \( X \) is well defined by the standard theory of Caratheodory ODEs, see for instance [14]. Moreover, there exists a compact \( \Omega \subset \mathbb{R}^N \) such that for all \( u \in L^\infty([0,T];\mathcal{U}) \), \( (X(u))(0,T] \subset \Omega \).

To prove the directional differentiability, call \( g \) the solution to the linear problem (A.2) and use the integral form of the Cauchy problem (A.1) to obtain

\[
\begin{align*}
\frac{1}{h} (X(u+vh) - X(u))(t) - g(t) &= \int_0^t f(t, X(u+vh)(\tau), (u+vh)(\tau)) - f(t, X(u)(\tau), u(\tau)) d\tau - g(t) \\
&= \int_0^t \int_0^1 \partial_x f \left( \tau, X(u+\theta vh)(\tau), (u+\theta vh)(\tau) \right) d\theta \frac{(X(u+vh) - X(u))(\tau)}{h} d\tau \\
&\quad + \int_0^t \int_0^1 \partial_v f \left( \tau, X(u+\theta vh)(\tau), (u+\theta vh)(\tau) \right) d\theta v(\tau) d\tau - g(t) \\
&= \int_0^t \int_0^1 \left[ \partial_x f \left( \tau, X(u+\theta vh)(\tau), (u+\theta vh)(\tau) \right) - \partial_x f \left( \tau, X(u); u \right) \right] d\theta \\
&\quad \times \frac{(X(u+vh) - X(u))(\tau)}{h} d\tau \\
&\quad + \int_0^t \partial_x f \left( \tau, X(u); u \right) \left( \frac{(X(u+vh) - X(u))(\tau)}{h} - g(\tau) \right) d\tau \\
&\quad + \int_0^t \int_0^1 \left[ \partial_v f \left( \tau, X(u+\theta vh)(\tau), (u+\theta vh)(\tau) \right) - \partial_v f \left( \tau, X(u); u \right) \right] d\theta v(\tau) d\tau
\end{align*}
\]

Lusin’s Theorem [23, Theorem 1], applied to \( (\partial_x f, \partial_v f) \in L^\infty([0,T]; C^{0,1}(\mathbb{R}^N \times \mathcal{U}; \mathbb{R}^{2N^2}) \),

ensures that for any \( \varepsilon > 0 \), there exists a compact set \( K \subset [0,T] \) such that the Lebesgue measure of \( [0,T] \setminus K \) is smaller than \( \varepsilon \) and both \( \partial_x f \) and \( \partial_v f \) are continuous on \( K \times \Omega \times \mathcal{U} \), hence also uniformly continuous. Therefore, if \( h \) is sufficiently small,

\[
\begin{align*}
\sup_{K \times \Omega \times \mathcal{U}} \sup_{\theta \in [0,1]} \| \partial_x f \left( \tau, X(u+\theta vh)(\tau), (u+\theta vh)(\tau) \right) - \partial_x f \left( \tau, X(u); u \right) \| &\leq \varepsilon, \\
\sup_{K \times \Omega \times \mathcal{U}} \sup_{\theta \in [0,1]} \| \partial_v f \left( \tau, X(u+\theta vh)(\tau), (u+\theta vh)(\tau) \right) - \partial_v f \left( \tau, X(u); u \right) \| &\leq \varepsilon.
\end{align*}
\]

22
Introduce now the quantity
\[ \delta_h(t) = \left\| \frac{1}{h} \left( X(u + hv) - X(u) \right)(t) - g(t) \right\| . \]

Since \( \| \partial_x f \| \leq L, \| \partial_v f \| \leq L, \| f \|_{L^\infty([0,T] \times \Omega \times \mathbb{R}^N)} < +\infty \), the above estimates lead to
\[ \delta_h(t) \leq (2L \varepsilon + \varepsilon t)\| f \|_{L^\infty([0,T] \times \Omega \times \mathbb{R}^N)} + \int_0^t L \delta_h(\tau) \, d\tau + (2L \varepsilon + \varepsilon t)\| v \|_{L^\infty([0,T],\mathbb{R}^M)} \]
\[ = (2L + t) \left( \| f \|_{L^\infty([0,T] \times \Omega \times \mathbb{R}^N)} + \| v \|_{L^\infty([0,T],\mathbb{R}^M)} \right) \varepsilon + \int_0^t L \delta_h(\tau) \, d\tau . \]

An application of Gronwall Lemma yields that for all \( \varepsilon > 0 \), if \( h \) is sufficiently small
\[ \delta_h(t) \leq (2L + t) \left( \| f \|_{L^\infty([0,T] \times \Omega \times \mathbb{R}^N)} + \| v \|_{L^\infty([0,T],\mathbb{R}^M)} \right) \varepsilon e^{Lt} \]
completing the proof.

**Acknowledgment:** R.M.C. was partially supported by the 2015 GNAMPA project *Balance Laws in the Modeling of Physical, Biological and Industrial Processes* and by the CaRiPLO project 2013-0893. The research of M.R. received funding from the National Science Centre, DEC-2012/05/E/ST1/02218. The research of P.G. received funding from the National Science Centre, Poland, 2014/13/B/ST1/03094.

**References**

[1] H. Amann and J. Escher. *Analysis. II*. Birkhäuser Verlag, Basel, 2008. Translated from the 1998 German original by Silvio Levy and Matthew Cargo.

[2] Å. Brännström, L. Carlsson, and D. Simpson. On the convergence of the escalator boxcar train. *SIAM J. Numer. Anal.*, 51(6):3213–3231, 2013.

[3] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[4] J. A. Cañizo, J. A. Carrillo, and J. Rosado. A well-posedness theory in measures for some kinetic models of collective motion. *Math. Models Methods Appl. Sci.*, 21(3):515–539, 2011.

[5] J. A. Carrillo, R. M. Colombo, P. Gwiazda, and A. Ulikowska. Structured populations, cell growth and measure valued balance laws. *J. Differential Equations*, 252(4):3245–3277, 2012.

[6] J. A. Carrillo, P. Gwiazda, and A. Ulikowska. Splitting-particle methods for structured population models: convergence and applications. *Math. Models Methods Appl. Sci.*, 24(11):2171–2197, 2014.

[7] R. M. Colombo and G. Guerra. Hyperbolic balance laws with a non local source. *Comm. Partial Differential Equations*, 32(10-12):1917–1939, 2007.

[8] A. M. de Roos. Numerical methods for structured population models: the escalator boxcar train. *Numer. Methods Partial Differential Equations*, 4(3):173–195, 1988.

[9] A. M. de Roos and L. Persson. *Population and community ecology of ontogenetic development*. Monographs in population biology. Princeton University Press, Princeton, 2013.
[10] O. Diekmann and P. Getto. Boundedness, global existence and continuous dependence for nonlinear dynamical systems describing physiologically structured populations. *J. Differential Equations*, 215(2):268–319, 2005.

[11] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[12] J. H. M. Evers, S. C. Hille, and A. Muntean. Well-posedness and approximation of a measure-valued mass evolution problem with flux boundary conditions. *C. R. Math. Acad. Sci. Paris*, 352(1):51–54, 2014.

[13] J. H. M. Evers, S. C. Hille, and A. Muntean. Mild solutions to a measure-valued mass evolution problem with flux boundary conditions. *J. Differential Equations*, 259(3):1068–1097, 2015.

[14] A. F. Filippov. *Differential equations with discontinuous righthand sides*. Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian.

[15] W. Gangbo and M. Westdickenberg. Optimal transport for the system of isentropic Euler equations. *Comm. Partial Differential Equations*, 34(7-9):1041–1073, 2009.

[16] P. Gwiazda, J. Jabłoński, A. Marciniak-Czochra, and A. Ulikowska. Analysis of particle methods for structured population models with nonlocal boundary term in the framework of bounded Lipschitz distance. *Numer. Methods Partial Differential Equations*, 30(6):1797–1820, 2014.

[17] P. Gwiazda, T. Lorenz, and A. Marciniak-Czochra. A nonlinear structured population model: Lipschitz continuity of measure-valued solutions with respect to model ingredients. *J. Differential Equations*, 248(11):2703–2735, 2010.

[18] P. Gwiazda, T. Lorenz, and A. Marciniak-Czochra. A nonlinear structured population model: Lipschitz continuity of measure-valued solutions with respect to model ingredients. *J. Differential Equations*, 248(11):2703–2735, 2010.

[19] P. Gwiazda and A. Marciniak-Czochra. Structured population equations in metric spaces. *J. Hyperbolic Differ. Equ.*, 7(4):733–773, 2010.

[20] M. Iannelli. *Mathematical theory of age-structured population dynamics*, volume 7 of *Applied mathematics monographs*. Giardini editori e stampatori in Pisa, 1995.

[21] W. Kermack and A. McKendrick. Contributions to the mathematical theory of epidemics–I. *Bull. Math. Biol.*, 53:33–55, 1991.

[22] W. Kermack and A. McKendrick. Contributions to the mathematical theory of epidemics–III. *Bull. Math. Biol.*, 53:57–87, 1991.

[23] P. A. Loeb and E. Talvila. Lusin’s theorem and Bochner integration. *Sci. Math. Jpn.*, 60(1):113–120, 2004.

[24] J. A. J. Metz and O. Diekmann, editors. *The dynamics of physiologically structured populations*, volume 68 of *Lecture Notes in Biomathematics*. Springer-Verlag, Berlin, 1986. Papers from the colloquium held in Amsterdam, 1983.

[25] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.

[26] B. Piccoli and F. Rossi. Generalized Wasserstein distance and its application to transport equations with source. *Arch. Ration. Mech. Anal.*, 211(1):335–358, 2014.

[27] P.-A. Raviart. *An analysis of particle methods*, volume 1127 of *Lecture Notes in Math*. Springer, Berlin, 1985.

[28] H. R. Thieme. *Mathematics in population biology*. Princeton Series in Theoretical and Computational Biology. Princeton University Press, Princeton, NJ, 2003.
[29] A. Ulikowska. An age-structured two-sex model in the space of Radon measures: well posedness. *Kinet. Relat. Models*, 5(4):873–900, 2012.

[30] H. Von Förster. *Some remarks on changing populations. The Kinetics of Cellular Proliferation*. Shalton Press, New York, 1959.

[31] G. F. Webb. *Theory of nonlinear age-dependent population dynamics*, volume 89 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1985.

[32] M. Westdickenberg and J. Wilkening. Variational particle schemes for the porous medium equation and for the system of isentropic Euler equations. *M2AN Math. Model. Numer. Anal.*, 44(1):133–166, 2010.