CONSTRUCTIBLE FUNCTIONS AND LAGRANGIAN CYCLES ON ORBIFOLDS

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Abstract. Given a smooth manifold $M$, Kashiwara’s index formula expresses the weighted Euler characteristic of a constructible function in terms of its characteristic cycle. In this note, we generalize this formula to the case when $M$ is a smooth orbifold, answering a question of Behrend.

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1. Introduction

Let $M$ be a smooth compact manifold with cotangent bundle $T^*M$, and let $\zeta_M \subset T^*M$ denote the zero section. It is a classical result that the homological intersection number $\zeta_M \cap \zeta_M$ is equal to the Euler characteristic of $M$. The “index formula” [BDK, K] is a generalization to “weighted” Euler characteristics.

Theorem 1.1 (Kashiwara index formula). Let $M$ be a compact smooth manifold. Let $f$ be a constructible function on $M$, and let $CC(f)$ be its characteristic cycle in $T^*M$. Let $\zeta_M$
denote the zero section in $T^*M$. The intersection number $CC(f) \cap \zeta_M$ and the weighted Euler characteristic $\int f$ coincide:

$$\int f = CC(f) \cap \zeta_M$$

The classical result is the special case when $f$ is identically 1. We will recall the definitions later.

The purpose of this note is to generalize Theorem 1.1 to the case when $M$ is a smooth orbifold. The problem of doing so arose in Behrend’s work [B] on Deligne-Mumford stacks equipped with a symmetric obstruction theory. For a large class of such stacks, he used Theorem 1.1 to express the degree of the virtual fundamental class in terms of the weighted Euler characteristic of a certain constructible function (see Section 5). This “Behrend function” has been central in recent progress on Donaldson-Thomas theory of Calabi-Yau threefolds [Br, JS].

Behrend already observed in loc. cit. that the generalization of Theorem 1.1 to orbifolds would allow one to remove most of the restrictions on $X$. We expect this to have applications in Donaldson-Thomas theory for more general geometries, where nontrivial orbifold structure arises naturally, e.g. when studying sheaves on threefolds relative to divisors.

The problem of generalizing Theorem 1.1 at least has the potential to be somewhat subtle. For instance, when $M$ is an orbifold the homological intersection $\zeta_M \cap \zeta_M$ is usually a rational number, not an integer. In particular it can’t be a finite alternating sum of betti numbers, so one has to be careful about what’s meant by Euler characteristic. Moreover the study of Morse theory and deeper “quantum” questions about Lagrangian intersections are expected to give rise to new phenomena in the orbifold case.

Nevertheless in this paper we show that a suitable orbifold version of the index formula is essentially a formal consequence of the usual index formula. To do this requires us to study the local-to-global behavior of constructible functions and Lagrangian cycles. The fact that both quantities in the formula are ill-defined for non-compactly supported functions is a small obstacle to doing this. To overcome this obstacle we use the fact that constructible functions with compact support, and Lagrangian cycles with compact “horizontal” support, form cosheaves.

1.1. Notation. $\text{Vect}$ denotes the abelian category of rational vector spaces. We write $T^*M$ for the cotangent bundle of $M$. When $N$ is a submanifold of $M$ we write $T^*_N M$ for the conormal bundle to $N$ in $M$. We write $H_i, H_i^{BM}, H^i$ and $H_i^c$ for the functors of respectively homology, Borel-Moore homology, cohomology, and compactly supported cohomology, which we consider with rational coefficients.

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2. Constructible functions and Lagrangian cycles on manifolds

In this section we review the theory of constructible functions and characteristic cycles. We mostly follow [KS1, Chapter 9].
We will consider subanalytic Whitney stratifications of real analytic manifolds $M$, which form a partially ordered set under refinement. An essential result is that this poset is filtered: any two subanalytic Whitney stratifications have a common refinement ([KS1, Theorem 8.3.20]). As a consequence if $G$ is a finite group acting on $M$ then any stratification of $M$ can be refined to a $G$-invariant stratification.

If $S$ is a Whitney stratification, we write $\Lambda_S \subset T^*_M$ for its characteristic variety, thus

$$\Lambda_S = \bigcup_{N \in S} T^*_N M$$

where $N$ runs through the strata of $S$.

2.1. **Constructible functions and Euler integration.** Let $S$ be a subanalytic Whitney stratification of $M$. We will call a compactly supported function $f : M \to \mathbb{Q}$ $S$-constructible if it is constant along the strata of $S$. Write $\text{Fun}_c(M, S)$ for the group of compactly supported $S$-constructible functions, and

$$\text{Fun}_c(M) = \bigcup_S \text{Fun}_c(M, S)$$

We define an operator $\int : \text{Fun}_c(M) \to \mathbb{Q}$ as follows:

**Definition 2.1.** For $f \in \text{Fun}_c(M)$, we set

$$\int_M f = \sum_{t \in \mathbb{Q}} \chi_c(f^{-1}(t)) \cdot t$$

Note that as $f$ is constructible this sum is finite.

2.2. **Lagrangian cycles and intersection.** For each stratification $S$ we define the group of conic Lagrangian cycles supported on $\Lambda_S$ to be $H^BM_n(\Lambda_S)$. As $\Lambda_S$ is $n$-dimensional, a member of this group is given by choosing for each component of $\Lambda_S \cap \Lambda_S^\circ$ (the smooth points of $\Lambda_S$) an orientation and multiplicity, satisfying a linear condition. We define the “support” of the cycle to be the closure of the components with nonzero multiplicities.

Write $\mathcal{L}(M)$ for the group of Lagrangian cycles on $T^*M$ that are supported on the conormal variety of some subanalytic Whitney stratification:

$$\mathcal{L}(M) = \bigcup_S H^BM_n(\Lambda_S)$$

We will say that a Lagrangian cycle has compact horizontal support if the projection of its support to $M$ is compact. These form a subgroup $\mathcal{L}_c(M) \subset \mathcal{L}(M)$.

Let $u : M \to \mathbb{R}$ be a smooth function (it is not necessary for it to be subanalytic) and let $\Gamma \subset T^*M$ be the graph of its derivative, i.e.

$$\Gamma = \{(x, \xi) \in T^*M \mid \xi = du_x\}$$

Let us pick an orientation of $M$. If $\Gamma \cap \Lambda_S = \Gamma \cap \Lambda_S^\circ$, these intersection points are isolated, and the intersection is transverse, then we can define a local intersection number $(\eta \cap \Gamma)_x \in \mathbb{Q}$ for any $\eta$ supported on $\Lambda_S$. If furthermore $\eta$ has compact horizontal support then the sum

$$\sum_{x \in \Lambda_S \cap \Gamma} (\eta \cap \Gamma)_x$$
is finite and independent of the $u$ chosen. We denote the map $L_c(M) \rightarrow \mathbb{Q}$ by $\cap \zeta_M$, where $\zeta_M$ denotes the zero section of $T^*M$ endowed with an orientation.

2.3. The characteristic cycle of a constructible function. If $S$ is a stratification of $M$ and $f : M \rightarrow \mathbb{Q}$ is a $S$-constructible function, then we may define a element $CC(f) \in H^B_{BM}(\Lambda_S)$, called the characteristic cycle of $f$. See [KS1, Chapter 9] for details. The support of $CC(f)$ projects onto the support of $f$, so $CC$ carries $Fun_c(M)$ to $L_c(M)$.

**Theorem 2.2** ([KS1, Theorems 9.7.1, 9.7.10]). For each real analytic manifold $M$, the map $CC : Fun_c(M) \rightarrow L_c(M)$ is an isomorphism.

**Theorem 2.3** (Index formula). For each $M$, the triangle

\[
\begin{array}{ccc}
Fun_c(M) & \xrightarrow{CC} & L_c(M) \\
\downarrow f & & \downarrow \cap \\
\mathbb{Q} & \xrightarrow{\cap \zeta} & L_c(M)
\end{array}
\]

commutes

See the remarks at the end of [KS1, Chapter 9] for a history of this result. In the real setting we are working in, the theorem is due to Kashiwara [K].

3. Cosheaves

It is intuitively plausible that constructible functions, conic Lagrangian cycles, and the index formula have a local nature on $M$. A typical way to express this locality is using the theory of sheaves, but for the purposes of this paper we have found that the equivalent but less well-known theory of cosheaves to be more convenient. Actually we will not work with cosheaves on a single manifold but on a Grothendieck site containing all manifolds of a fixed dimension. For simplicity we work with oriented manifolds.

3.1. The subanalytic Grothendieck site. Fix a dimension $n$, and let $C$ be the category whose objects are real analytic manifolds equipped with an orientation, and whose morphisms are real subanalytic maps that are open immersions. We endow $C$ with the following Grothendieck topology: if $\{U_i \rightarrow U\}_{i \in I} \subset C/U$ is a sieve in $C$, we will say it is an covering sieve if there there is a finite list $U_{i_0} \rightarrow U, \ldots, U_{i_m} \rightarrow U$ so that every point of $U$ belongs to one of the $U_i$.

**Remark 3.1.** This is the same topology considered in [KS2]. The finiteness condition makes this different than the usual topology on real analytic manifolds. For instance, the collection of intervals $(\frac{1}{n}, 1)$ do not cover the interval $(0, 1)$ in the topology we are considering.

By a pre-cosheaf on $C$ we will mean a covariant functor $F : C \rightarrow Vect$. A pre-cosheaf $F$ is a cosheaf if the natural map

\[
\lim_{i \in I} F(U_i) \rightarrow F(U)
\]

is an isomorphism whenever $\{U_i\}_{i \in I}$ is a covering sieve of $U$. 
Remark 3.2. There is a dictionary between sheaves and cosheaves: if $F$ is a sheaf on a site $S$, then the assignment $U \mapsto R\Gamma_c(U, F)$ is covariant for open inclusions, and we may interpret $R\Gamma_c(-, F)$ as a complex of cosheaves. When $S$ is a locally compact Hausdorff space, it can be shown that the functor $F \mapsto R\Gamma_c(-, F)$ induces an equivalence of derived categories. We do not know whether this is true for $S = C$, but we will see that some of the cosheaves we consider arise from sheaves in this way.

Recall that each presheaf $P$ has a sheafification $P^!$, and there is a natural map $P \to P^!$ that is universal among maps from $P$ to sheaves. Similarly, each pre-cosheaf has a natural cosheafification $P^\dagger \to P$ that is universal among maps from cosheaves to $P$. Our main example is the constant cosheaf, which is the cosheafification of the constant pre-cosheaf given by $P(U) = \mathbb{Q}$. We denote the constant cosheaf by $D$, because in the derived equivalence of 3.2 it corresponds to the Verdier dualizing complex.

3.2. The cosheaf of constructible functions. To each subanalytic inclusion $U \hookrightarrow V$, extension by zero provides an inclusion $\text{Fun}_c(U) \hookrightarrow \text{Fun}_c(V)$. In other words, $\text{Fun}_c$ is a pre-cosheaf on $C$.

Proposition 3.3. $\text{Fun}_c$ is a cosheaf on $C$.

Proof. Let $I = \{U_i \to M\}$ be a covering sieve of $M$. We have to show that the natural map
\[
c : \lim_{\rightarrow \mathcal{I}} \text{Fun}_c(U_i) \to \text{Fun}_c(M)
\]
is an isomorphism. By the definition of covering sieves in $C$, there are finitely many maximal charts $U_{i_1}, \ldots, U_{i_m}$ in the sieve $I$. A standard argument reduces us to the case where $m = 2$, in which case the limit coincides with the pushout of the diagram
\[
\text{Fun}_c(U_{i_1}) \leftarrow \text{Fun}_c(U_{i_1} \cap U_{i_2}) \to \text{Fun}_c(U_{i_2})
\]
To prove the Proposition we therefore only have to show that if $M = U \cup V$ where $U$ and $V$ are subanalytic open charts, then the sequence
\[
\text{Fun}_c(U \cap V) \xrightarrow{f \mapsto (f, f)} \text{Fun}_c(U) \oplus \text{Fun}_c(V) \xrightarrow{(f, g) \mapsto f - g} \text{Fun}_c(M) \to 0
\]
is exact.

If $f \in \text{Fun}_c(U)$ and $g \in \text{Fun}_c(V)$ and $f - g = 0$ in $\text{Fun}_c(M)$, then $f = g$ and the common support must belong to $U \cap V$. This shows that the sequence is exact in the middle. Let us show that it is exact at the right. Let $f : M \to \mathbb{Q}$ be a compactly supported function constructible with respect to a subanalytic Whitney stratification $S$. By refining $S$ if necessary, we may assume that the closure of each stratum of $S$ in the support of $f$ is entirely contained in either $U$ or $V$. The indicator function of each such stratum belongs to either $\text{Fun}_c(U)$ or $\text{Fun}_c(V)$, and since $f$ is a linear combination of such indicator functions it belongs to the image of the map $\text{Fun}_c(U) \oplus \text{Fun}_c(V) \to \text{Fun}_c(M)$.

By definition, the map $\int$ is a morphism of pre-cosheaves from $\text{Fun}_c$ to the constant pre-cosheaf. By the Proposition, $\text{Fun}_c$ is a cosheaf and so this map factors through the cosheafification $D$ of the constant pre-cosheaf. We denote the map by $\int : \text{Fun}_c \to D$. 

□
3.3. The cosheaf of Lagrangian cycles. Each open immersion $U \hookrightarrow V$ induces an open immersion $T^*U \hookrightarrow T^*V$. If the inclusion is subanalytic then we may extend any Whitney stratification of $U$ to one of $V$, and a horizontally compact conic Lagrangian cycle on $T^*U$ extends by zero to one on $T^*V$. This gives us an inclusion $L_c(U) \hookrightarrow L_c(V)$, i.e. $L_c$ is a pre-cosheaf on $\mathcal{C}$.

**Proposition 3.4.** $L_c$ is a cosheaf on $\mathcal{C}$

**Proof.** It is possible to prove this directly, but let us simply note that by Theorem 2.2, there is an isomorphism of pre-cosheaves $L_c \cong \text{Fun}_c$, and since $\text{Fun}_c$ is a cosheaf by Proposition 3.3, $L_c$ is also a cosheaf. □

Fix $\eta \in L_c(U)$. The intersection multiplicity $\eta \cap \zeta$ is defined by choosing a suitably generic smooth function $u : U \to \mathbb{R}$ and adding up local multiplicity of the intersection $\eta \cap \Gamma_{du}$. As $\eta$ has compact support, we may assume that $u$ vanishes outside of a compact set. By extending this function by zero further along an open inclusion $U \hookrightarrow V$, we see that the triangle

$$
\begin{array}{ccc}
L_c(U) & \rightarrow & L_c(V) \\
\cap \zeta & \searrow & \cap \zeta \\
\downarrow \nearrow & & \downarrow \nearrow \\
Q & & Q \\
\end{array}
$$

commutes, i.e. we have a map from $L_c$ to the constant pre-cosheaf. This induces a map $L_c \to D$ that we continue to denote by $\cap \zeta$.

3.4. The local index formula. The maps $CC : \text{Fun}_c(M) \to L_c(M)$ clearly assemble to a morphism of cosheaves, we therefore have the “local” or “cosheaf” version of the index formula

**Theorem 3.5.** The triangle

$$
\begin{array}{ccc}
\text{Fun}_c & \rightarrow & L_c \\
\downarrow f & & \downarrow \cap \zeta \\
D & & D \\
\end{array}
$$

of cosheaves on $\mathcal{C}$ commutes.

4. Application to orbifolds

In this section $\mathcal{C}$ continues to denote the subanalytic Grothendieck site of oriented $n$-manifolds. If $\mathcal{C}'$ is any other Grothendieck site equipped with a map to $\mathcal{C}$, the cosheaves $D$, $\text{Fun}_c$, and $L_c$ pull back to $\mathcal{C}'$ and the pullback triangle of Theorem 3.5 still commutes. In this section we will study this triangle when $\mathcal{C}'$ is the Grothendieck topology associated to a real analytic orbifold.

4.1. The subanalytic site of an orbifold. There are many possible foundations for a theory of orbifolds, we will use those set up in [BGNX]. Let $X = [X_1 \Rightarrow X_0]$ be an orbifold in the sense of loc. cit. Section 1.7. We let $\overline{X}$ denote the coarse space of $X$ in the sense of loc. cit. Section 1.1.4. For every topological space $Y$ we have a notion of a map from $Y$ to $X$ (in fact there is a groupoid of such maps), there is a good notion of fiber product of maps over $X$, and it is possible to single out a class of étale maps to $X$. 
By a oriented, real analytic structure on $X$, we will mean a finite collection of étale maps $U_1 \to X, \ldots, U_m \to X$ that cover $\overline{X}$ such that:

- Each $U_i$ is an $n$-dimensional oriented real analytic manifold
- For each $i$ and $j$, the map $U_i \times_X U_j \to U_i$ is real analytic and orientation-preserving.

If $U$ is another oriented real analytic manifold, we say that a map $U \to X$ is real analytic if the induced map $U \times_X U_i \to U_i$ is real analytic and orientation preserving for $i = 1, \ldots, m$.

If $X$ is an oriented real analytic orbifold, define a category $\mathcal{C}/X$ in the following way:

1. The objects are real analytic, orientation-preserving étale maps $U \to X$.
2. The morphisms are orientation preserving open immersions (sic) $V \to U$ over $X$.

Remark 4.1. In (2), we are taking advantage of the fact that the subanalytic topology is fine enough that the “usual” topology and “étale” topologies coincide.

A sieve $\{U_i \to U\}$ in $\mathcal{C}/X$ is a covering sieve if and only if it is a covering sieve in $\mathcal{C}$. There is thus evidently a continuous functor $j_X : \mathcal{C}/X \to \mathcal{C}$. In other words, if $F : \mathcal{C} \to \text{Vect}$ is a cosheaf on $\mathcal{C}$, then $F \circ j_X$ is a cosheaf on $\mathcal{C}/X$.

Remark 4.2. Note that $X$ itself is not always an object of $\mathcal{C}/X$, however we abuse notation and write $F(X)$ for the “global sections” of $F$ over $\mathcal{C}/X$, i.e.

$$F(X) = \lim_{(U \to X) \in \mathcal{C}/X} F(U).$$

Moreover, even if $X$ is not an object of $\mathcal{C}/X$, we still have a notion of a covering sieve on $X$. It is a set $S$ of objects of $\mathcal{C}/X$ with the following properties:

1. (Sieve property) if $(U \to X) \in S$ and $V$ admits a map to $U$, then $V$ is in $S$ as well.
2. (Covering property) there is a finite collection of maps $U_1 \to X, \ldots, U_n \to X$ in $S$ such that the induced maps $U_i \to \overline{X}$ cover $\overline{X}$.

If $S$ is a covering sieve and $F$ is a cosheaf on $\mathcal{C}/X$ then the natural map to $F(X)$ is an isomorphism:

$$\lim_{(U \to X) \in S} F(U) \xrightarrow{\sim} F(X).$$

4.2. The orbifold index formula, abstract version. We make the following definition:

Definition 4.3. Let $X$ be an orbifold, let $\mathcal{C}/X$ be its subanalytic étale site, and let $j_X$ be the natural forgetful functor $\mathcal{C}_X \to \mathcal{C}$.

1. A constructible function on $X$ is a section of the cosheaf $\text{Fun}_c \circ j_X$ on $\mathcal{C}/X$. Write $\text{Fun}_c(X)$ for the group of constructible functions on $X$.
2. A conic Lagrangian cycle on $T^*X$ is a section of the cosheaf $\mathcal{L}_c \circ j_X$ on $\mathcal{C}/X$. Write $\mathcal{L}_c(X)$ for the group of conic Lagrangian cycles on $T^*X$.

Example 4.4. Let $M$ be a real analytic manifold equipped with a subanalytic action of a finite group $G$, and let $X$ be the quotient stack $M/G$. By definition $(M \to X)$ is an object of $\mathcal{C}/X$ whose automorphism group is $G$. Moreover, every object of $\mathcal{C}/X$ admits a map to $(M \to X)$. It follows that $\text{Fun}_c(X) := \lim_{\mathcal{C}/X} \text{Fun}_c \circ j_X$ is naturally identified with the smaller limit $\lim_{M/G} \text{Fun}_c(M)$, where the limit is over the one-object diagram with automorphism group $G$. In other words, $\text{Fun}_c(X)$ is naturally identified with the coinvariants of the $G$-module $\text{Fun}_c(M)$. An identical argument identifies $\mathcal{L}_c(X)$ with the $G$-coinvariants of $\mathcal{L}_c(M)$. 


Let us also discuss the cosheaf $D$. For simplicity let us assume $X$ is connected. We define
the constant cosheaf $D_X$ on the subanalytic site $\mathcal{C}_X$ of $X$ by $D_X = D \circ \jmath_X$. The global sections
of $D_X$ on $\mathcal{C}_X$ is naturally given by $H_0(X; \mathbb{Q}) = \mathbb{Q}$. The cosheaf-level maps $\int : \text{Fun}_c(X) \to D_X(X)$
and $\cap \zeta : \mathcal{L}_c \to D$ induce maps $\int : \text{Fun}_c(X) \to D_X(X) = \mathbb{Q}$ and $\cap \zeta : \mathcal{L}_c(X) \to D(X) = \mathbb{Q}$.
Similarly the cosheaf map $CC : \text{Fun}_c \to \mathcal{L}_c$ induces a map $\text{Fun}_c(X) \to \mathcal{L}_c(X)$. The following
is an immediate consequence of Theorem 3.5.

**Theorem 4.5.** Let $X$ be a real analytic orbifold. We have a commutative diagram

$$
\begin{array}{ccc}
\text{Fun}_c(X) & \xrightarrow{CC} & \mathcal{L}_c(X) \\
\downarrow f & & \downarrow \cap \zeta \\
\mathbb{Q} & & \\
\end{array}
$$

We will give a more concrete description of the groups $\text{Fun}_c(X)$ and $\mathcal{L}_c(X)$, and of the maps
in the triangle of Theorem 4.5 in Section 4.3.

### 4.3. The role of the coarse space.
Throughout this section, let $X$ denote a real analytic orbifold. We also consider the orbifold cotangent bundle $T^*X \to X$. Let $\overline{X}$ denote the coarse space of $X$ and $\overline{T^*X}$ denote the coarse space of $T^*X$. Let $p : X \to \overline{X}$ denote the natural projection.

**Remark 4.6.** $\overline{X}$ can be realized naturally as a subanalytic variety, i.e. as a closed subanalytic
set in a real analytic manifold, such that the map $p : X \to \overline{X}$ is subanalytic. To see this recall
that there exist subanalytic bump functions of class $C^r$ for any finite $r$ (e.g. [NZ, Section 3.2]). It follows by a partition of unity argument that we can separate points of $\overline{X}$ by finitely
many subanalytic functions $u : X \to \mathbb{R}$ of class $C^r$, i.e. there is a subanalytic map $X \to \mathbb{R}^n$ of class $C^r$ that induces an injection $\overline{X} \to \mathbb{R}^n$. The image of this map is a subanalytic variety.

This remark applies also to $\overline{T^*X}$.

**Remark 4.7.** Locally on $\overline{X}$, $X$ looks like a quotient stack. More precisely, with the subanalytic structure of (1), $\overline{X}$ admits a cover by subanalytic charts $\overline{U}$, so that the fiber product stack $\overline{U} \times_{\overline{X}} X$ is isomorphic to the quotient of a real analytic manifold by a finite group acting
subanalytically.

#### 4.3.1. Constructible functions on the coarse space.
If $\phi : U \to X$ is an object of $\mathcal{C}/X$, and $f \in \text{Fun}_c(U)$, write $(p \circ \phi)_!(f)$ for the function

$$(p \circ \phi)_!(f)(x) = \sum_{u \in (p \circ \phi)^{-1}(x)} f(u)$$

The system of maps $(p \circ \phi)_! : \text{Fun}_c(U) \to \text{Fun}_c(\overline{X})$ assemble to a map from the direct limit $\text{Fun}_c(X) \to \text{Fun}_c(\overline{X})$. Let us denote this map by $p_!$.

**Theorem 4.8.** Let $X$ be a real analytic orbifold, let $\overline{X}$ be its coarse space, and let $p$ denote the quotient map $p : X \to \overline{X}$. The map $p_! : \text{Fun}_c(X) \to \text{Fun}_c(\overline{X})$ defined above is an isomorphism.

**Proof.** By Remark 4.7, we may find a covering sieve $\{U_i \to \overline{X}\}$ of $X$ such that each $U_i := \overline{U_i} \times_{\overline{X}} X$ is a quotient stack. Write $p_i : U_i \to \overline{U}_i$. By the cosheaf property $\text{Fun}_c(X) \cong \lim \text{Fun}_c(U_i)$
and \( \text{Fun}_c(\overline{X}) \cong \lim \text{Fun}_c(U_i) \), it suffices to prove that the maps \( p_i : \text{Fun}_c(U_i) \to \text{Fun}_c(U_i) \) are isomorphisms. In other words, we may reduce to the case when \( X \) is a quotient stack.

Thus suppose \( X = M/G \) and \( \overline{X} = M//G \). By Example 4.4, we may identify \( \text{Fun}_c(X) \) with the coinvariants \( \text{Fun}_c(M)_G \). On the other hand, pulling back a constructible function on \( X \) to a constructible function on \( M \) identifies \( \text{Fun}_c(\overline{X}) \) with the \( G \)-invariants of \( \text{Fun}_c(M) \).

Write \( q \) for the map \( M \to X \). For each \( x \in \overline{X} \), the fiber \( (p \circ q)^{-1}(x) \subset M \) is a \( G \)-orbit. Thus, the composite \((p \circ q)_! : \text{Fun}_c(M) \to \text{Fun}_c(M)_G \to \text{Fun}_c(X) \) takes \( h \) to \( \sum_{g \in G} h(g^{-1}) \). In other words, the map \( \text{Fun}_c(M)_G \to \text{Fun}_c(\overline{X}) \cong \text{Fun}_c(M)^G \) is the norm map. The norm map \( V_G \to V^G \) is an isomorphism for every rational vector space \( V \) with a \( G \)-action, so the Theorem follows. \( \square \)

4.3.2. Integration and the coarse space. For simplicity let us assume \( X \) is connected. We define the constant cosheaf \( D \) on the subanalytic site \( C \) of \( X \) by \( D_X = D \circ j_X \). The global sections of \( D_X \) on \( C \) is naturally given by \( H_0(X; \mathbb{Q}) = \mathbb{Q} \). The integration map \( \int : \text{Fun}_c \to D \) between cosheaves on \( C \) induces a map \( \text{Fun}_c(X) \to D_X(X) = \mathbb{Q} \), which we also denote by \( \int \).

Example 4.9. Let \( M, G, \) and \( X \) be as in Example 4.4. If \( h \in \text{Fun}_c(M) \) is of the form \( h_1 - g^* h_1 \), then \( \int_X h = 0 \). It follows that \( \int \) descends to an operator on coinvariants \( \text{Fun}_c(M)_G \to \mathbb{Q} \). If \( f \in \text{Fun}_c(X) \cong \text{Fun}_c(M)_G \), and \( h \) is any lift of \( f \) to \( \text{Fun}_c(M) \), then we have \( \int_X f = \int_M h \).

We wish to describe this operator in terms of the identification \( \text{Fun}_c(X) = \text{Fun}_c(\overline{X}) \) of Theorem 4.8.

Let us first define a constructible function \( \iota : \overline{X} \to \mathbb{Q} \) by

\[
\iota(x) = 1/|G_x| \quad \text{where } G_x \text{ is the inertia group of } X \text{ at } x
\]

Proposition 4.10. Let \( p_i : \text{Fun}_c(X) \to \text{Fun}_c(\overline{X}) \) be the identification of Theorem 4.8. Let \( f \in \text{Fun}_c(X) \). Then \( \int f = \int p_i(f) \cdot \iota \).

Proof. As in the proof of Theorem 4.8 we can reduce to the case when \( X \) is a quotient stack, say \( X = M/G \). For each conjugacy class \( C \) of subgroups of \( G \), call a function \( h \in \text{Fun}_c(M) \) “\( C \)-simple” if it is the indicator function of a locally closed and relatively compact set \( K \) whose isotropy groups are all in \( C \). Call an element of \( \text{Fun}_c(M) \) “simple” if it is \( C \)-simple for some \( C \). And call an element of \( \text{Fun}_c(M)_G \) “simple” if it is represented by a simple function in \( \text{Fun}_c(M) \). We can write any \( f \in \text{Fun}_c(M)_G \) as a linear combination of simple functions, so it suffices to verify the Proposition in case \( f \) itself is simple.

If \( f \in \text{Fun}_c(M)_G \) is represented by the \( C \)-simple indicator function of a set \( K \subset M \), then \( \int_X f = \chi_c(K) \). On the other hand if \( m \) is the order of a group in the conjugacy class \( C \) then \( p_i(f)(x) \) is given by

\[
p_i(f)(x) = \begin{cases} 
  m & \text{if } x \in p(K) \\
  0 & \text{otherwise}
\end{cases}
\]

The Proposition follows. \( \square \)

4.3.3. Lagrangian cycles in \( T^*\overline{X} \). We can analyze \( \mathcal{L}_c(X) \) in terms of the coarse space \( T^*\overline{X} \) of the orbifold \( T^*X \).

Remark 4.11. If \( X_1 \) and \( X_2 \) are two orbifolds with \( \overline{X}_1 \cong \overline{X}_2 \), we may still have \( T^*X_1 \not\cong T^*X_2 \).
Let $U \to X$ be a map that defines an object of $\mathcal{C}_{/X}$. By definition, $U \to X$ is étale, and therefore induces a map $T^*U \to \overline{T^*X}$. Let us say that a subset of $\overline{T^*X}$ is a coarse Whitney Lagrangian if its inverse image under all such maps $T^*U \to \overline{T^*X}$ is of the form $\Lambda_S$ in the sense of Section 2.2. If $\Lambda_1$ and $\Lambda_2$ are coarse Whitney Lagrangians with $\Lambda_1 \subset \Lambda_2$, then we have an inclusion $H_n^{BM}(\Lambda_1) \to H_n^{BM}(\Lambda_2)$, and we will set

$$W(X) = \bigcup H_n^{BM}(\Lambda)$$

where the union runs over coarse Whitney Lagrangians in $\overline{T^*X}$. Let $W_c(X) \subset W(X)$ denote the subgroup of cycles whose support projects onto a compact set in $X$.

**Example 4.12.** Let $X$ be the quotient orbifold $\mathbb{C}/\{\pm 1\}$. Then $\overline{T^*X}$ can be identified with the singular space $\{(u,v,w) \mid uw = w^2\} \subset \mathbb{C}^3$, where $u$ is the square of the base coordinate and $v$ is the square of the fiber coordinate on $T^*\mathbb{C}$. The subset cut out by $u = 0$ belongs to $W_c(X)$.

Let us show that we can identify $\mathcal{L}_c(X)$ with $W_c(X)$. For each object $U \to X$ of $\mathcal{C}_{/X}$, if $S$ is a stratification of $U$ then we may find a Whitney Lagrangian $\Lambda \subset \overline{T^*X}$ such that the map $T^*U \to \overline{T^*X}$ carries $\Lambda_S$ to a subset of $\Lambda$. For instance, we may take $\Lambda = \Lambda_S'$ where $S'$ is a stratification of $X$ whose pullback to $U$ refines $S$. If $\eta \in H_n^{BM}(\Lambda_S)$ has compact horizontal support then its image gives a well-defined element in $H_n^{BM}(\Lambda)$, again with compact horizontal support. This gives us a map $\mathcal{L}_c(U) \to W_c(X)$, and these assemble to a map from the direct limit $w_X : \mathcal{L}_c(X) \to W_c(X)$.

**Theorem 4.13.** For any real analytic orbifold $X$, the map $w_X : \mathcal{L}_c(X) \to W_c(X)$ is an isomorphism.

**Proof.** If $\overline{U}$ is an open subset of the coarse space and $U$ is the corresponding open substack of $X$, we may consider the map $w_U : \mathcal{L}_c(U) \to W_c(U)$. The maps $w_U$ and $w_X$ commute with the extension-by-zero maps $\mathcal{L}_c(U) \to \mathcal{L}_c(X)$ and $W_c(U) \to W_c(X)$. In other words, we may regard $w$ as a map of cosheaves on $\overline{X}$. We may therefore reduce to the case when $X$ is a quotient stack $M/G$.

In that case by Example 4.4 it suffices to show that the map $\mathcal{L}_c(M) \to W_c(X)$ identifies $W_c(X)$ with the $G$-coinvariants of $\mathcal{L}_c(M)$. Let us denote the map $M \to \overline{X}$ by $\pi$. Each $G$-invariant stratification $S$ of $M$ determines a $G$-invariant conic Lagrangian $\Lambda_S \subset T^*M$ and a coarse Whitney Lagrangian $\overline{\Lambda}_S \subset \overline{T^*X}$. Since $G$-invariant stratifications are cofinal among all stratifications of $M$, we are reduced to showing that

$$H_n^{BM}(\Lambda_S) \to H_n^{BM}(\overline{\Lambda}_S)$$

identifies the codomain with the $G$-coinvariants of the domain.

Since $\Lambda_S \to \overline{\Lambda}_S$ is a finite map, we may construct a “transfer” map $H_n^{BM}(\overline{\Lambda}_S) \to H_n^{BM}(\Lambda_S)$, that takes $Z$ to $\sum_{z \in \pi^{-1}(Z)} Z'$. We complete the proof by noting the composite $H_n^{BM}(\Lambda_S) \to H_n^{BM}(\overline{\Lambda}_S) \to H_n^{BM}(\Lambda_S)$ coincides with the norm map.

$\square$
We explain how to use the results here to remove some of the hypotheses in Behrend’s work. We will give a brief review of his definitions, but refer to [B] for a more complete picture. Note that what we have been calling $\int_X f$, Behrend denotes by $\chi(X, f)$.

Let us first make some remarks about algebraically constructible functions. If $U$ is a complex algebraic variety, let $\text{Fun}_{\text{alg}}(U)$ denote the group of $\mathbb{Q}$-valued functions that are constructible with respect to a complex algebraic stratification. If $Y$ is a complex algebraic orbifold let $Y_e$ denote the category of étale maps $U \to Y$ where $U$ is representable, with its natural Grothendieck topology. $\text{Fun}_{\text{alg}}$ forms a sheaf on $Y_e$. Let $\text{Fun}_{\text{alg}}(Y)$ denote its global sections.

**Proposition 5.1.** The pullback map $\text{Fun}_{\text{alg}}(Y) \to \text{Fun}_{\text{alg}}(Y)$ is an isomorphism.

**Proof.** As in the proof of Theorem 4.8, we may reduce to the case where $Y$ is a quotient stack $M/G$, in which case $\text{Fun}_{\text{alg}}(Y)$ can be naturally identified with the group of $G$-invariant algebraically constructible functions on $Y$. □

If $Y$ is proper, we may therefore use Theorem 4.8 to identify $\text{Fun}_{\text{alg}}(Y)$ with a subgroup of $\text{Fun}_c(Y)$. In particular, we can construct elements of $\text{Fun}_c(Y)$ by constructing elements of $\text{Fun}_{\text{alg}}(Y)$ étale locally.

Similar remarks apply to $L_{\text{c}}(Y)$. Write $L_{\text{alg}}(X) \subset L(X)$ for the group of conic Lagrangian cycles that are supported on the conormal variety of some algebraic stratification of $U$. The assignment $U \mapsto L_{\text{alg}}(U)$ is contravariant for étale maps and forms a sheaf on $X_e$. Let $L_{\text{alg}}(X)$ denote the global sections of this sheaf. Let $W_{\text{alg}}(X)$ denote the subgroup of $W(X)$ consisting of cycles whose inverse image along every map $U \to X$ of $X_e$ is algebraic, or equivalently whose support (as a subset of the algebraic space $T^*X$) is algebraic. The proof of Proposition 5.1 may be repeated to establish

**Proposition 5.2.** The pullback map $W_{\text{alg}}(X) \to L_{\text{alg}}(X)$ is an isomorphism.

Because of this and Theorem 4.13 when $X$ is proper we may identify $L_{\text{alg}}(X)$ with a subspace of $L_c(X)$.

We now explain the application to symmetric obstruction theories. Let $X$ be a quasiprojective Deligne-Mumford stack over $\mathbb{C}$. A perfect obstruction theory on $X$ is a perfect complex $E \in D^{b}_{\text{coh}}(X)$ in the derived category of coherent sheaves, equipped with a map to the cotangent complex

$$\phi : E \to L_X$$

such that $E$ has cohomology only in degrees $[-1, 0]$ and such that $\phi$ induces a surjection in degree $-1$ and an isomorphism in degree $0$. The obstruction theory is symmetric if, in addition, there exists an isomorphism

$$\theta : E \to E^\vee[1]$$

such that $\theta^\vee[1] = \theta$. Symmetric obstruction theories arise naturally when studying moduli spaces of stable sheaves on Calabi-Yau threefolds. The symmetric structure is induced from Serre duality.

If $X$ carries a symmetric perfect obstruction theory, it also has a virtual fundamental class in degree $0$

$$[X]^{\text{vir}} \in A_0(X),$$
constructed using the data of $(E, \phi)$ above. Here, $A_0(X)$ denotes the Chow group of zero-cycles on $X$. In the geometric setting of sheaves on Calabi-Yau threefolds, the degrees of these virtual classes for varying choice of numerical invariants define the Donaldson-Thomas theory of the threefold.

In [B], for an arbitrary Deligne-Mumford stack $X$, Behrend also defines a constructible function

$$\nu_X \in \text{Fun}^{\text{alg}}(X)$$

using the intrinsic normal cone of $X$. Behrend uses the index formula for manifolds to prove the following theorem, under the additional conditions that $X$ is either smooth, or else a global finite quotient, or else a gerbe over a scheme.

**Theorem 5.3.** If $X$ is proper, then

$$\deg[X]^\text{vir} = \chi(X, \nu_X).$$

**Proof.** Choose a closed embedding

$$X \hookrightarrow Y$$

into a smooth and proper Deligne-Mumford stack $Y$. We may regard $\nu_X$ as an element of $\text{Fun}^{\text{alg}}(Y)$ via extension-by-zero. Since $Y$ is proper we may by Proposition 5.1 identify $\nu_X$ with an element of $\text{Fun}_s(Y)$, with associated characteristic cycle $CC(\nu_Y) \in L_c(Y)$. The restriction of $CC(\nu_Y)$ to a representable étale open set $U \to Y$ belongs to $L^\text{alg}(Y)$, so by Proposition 5.2 and the properness of $Y$ we have $CC(\nu_Y) \in L^\text{alg}(Y) \subset L_c(Y)$. It follows from [B, Proposition 4.16] that

$$[X]^\text{vir} = 0^! CC(\nu_X) \in A_0(X)$$

Here $0^!$ denotes refined (to the Chow group) intersection with the zero section of $T^*M$. Using Theorem 4.15, we have

$$\deg[X]^\text{vir} = \deg 0^! CC(\nu_X) = \zeta_Y \cap CC(\nu_X) = \chi(X, \nu_X).$$

□

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