UNIQUENESS RESULTS FOR INVERSE SOURCE PROBLEMS OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We study inverse source problems associated to semilinear elliptic equations of the form
\[ \Delta u(x) + a(x, u) = F(x), \]
on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Unlike inverse source problems for linear elliptic equations, we show that for a class of nonlinearities $a(x, u)$, including certain polynomials, both $a(x, u)$ and $F(x)$ can be uniquely determined from the corresponding Dirichlet-to-Neumann map. For general nonlinearities $a(x, z)$, we show that we may recover $a(x, z)$, or a condition for its derivatives $\partial_z^k a(x, z)$, and a source $F(x)$ and up to a gauge symmetry. We also generalize results of [FO20, LLLS20] by not assuming that $u \equiv 0$ is a solution and recovering lower order terms.

Keywords. Inverse problems, inverse source problems, gauge invariance, semilinear elliptic equations, sine-Gordon equation, higher order linearization.

Contents

1. Introduction 1
2. Preliminaries 9
3. Uniqueness for polynomial nonlinearities up to gauge invariances 11
   3.1. Quadratic nonlinearity 11
   3.2. Cubic nonlinearity 13
   3.3. Polynomial and general nonlinearity 16
4. Case studies of Theorem 1.4 20
   4.1. Exponential nonlinearity 20
   4.2. The sine-Gordon equation 21
Appendix A. Proof of Proposition 2.1 22
References 23

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^\infty$-smooth boundary $\partial \Omega$ with $n \geq 2$. In this paper we consider semilinear elliptic equations of the form

\[ \begin{cases} 
\Delta u + a(x, u) = F & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega,
\end{cases} \]

where $a = a(x, z) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is $C^\infty$-smooth in the $z$-variable and

\[ a(x, 0) = 0 \text{ in } \Omega. \]

The above condition is just for presentational purposes and not a restriction of generality. It can be achieved by redefining $F$ in (1.1).
Let us assume for now that (1.1) is well-posed on an open subset \( \mathcal{N} \subset C^{2,\alpha}(\partial \Omega) \). In this case, the Dirichlet-to-Neumann map (DN map) is defined by the usual assignment

\[
\Lambda_{a,F} : \mathcal{N} \rightarrow C^{1,\alpha}(\partial \Omega), \quad f \mapsto \partial_{\nu} u_f|_{\partial \Omega}.
\]

Here \( \nu \) denotes the unit outer normal on \( \partial \Omega \). In Theorem 2.1 we show that if there is

\[
f_0 \in C^{2,\alpha}(\partial \Omega),
\]

such that the equation (1.1) admits a unique solution \( u_0 \in C^{2,\alpha}(\Omega) \) with \( u_0|_{\partial \Omega} = f_0 \), and

\[
0 \text{ is not a Dirichlet eigenvalue of } \Delta + \partial_2 a(x, u_0) \text{ in } \Omega,
\]

then there is an open neighborhood \( \mathcal{N} \subset C^{2,\alpha}(\partial \Omega) \) of \( f_0 \) where (1.1) is well-posed in the following sense: For each \( f \in \mathcal{N} \) there exists a solution \( u_f \) to (1.1) with \( u_f|_{\partial \Omega} = f \) and the solution \( u_f \) is unique in a fixed neighborhood of \( u_0 \in C^{2,\alpha}(\Omega) \). If \( F \) vanishes on \( \Omega \), one can take \( f_0 \equiv 0 \) on \( \partial \Omega \). In this case, Theorem 2.1 reduces to similar well-posedness theorems in the literature, such as the one in [LLLS20].

Consider the equation (1.1) for two sets \( (a_1, F_1) \) and \( (a_2, F_2) \) of coefficients. Let \( \Lambda_1 \) and \( \Lambda_2 \) be the corresponding DN maps defined on \( \mathcal{N}_1 \subset C^{2,\alpha}(\partial \Omega) \) and \( \mathcal{N}_2 \subset C^{2,\alpha}(\partial \Omega) \) respectively. When we write

\[
\Lambda_1(f) = \Lambda_2(f) \text{ for } f \in \mathcal{N}
\]

we have especially assumed that \( \mathcal{N} \subset \mathcal{N}_1 \cap \mathcal{N}_2 \).

- **Inverse source problem:** What can we determine about both \( a \) and \( F \) from the knowledge of the corresponding DN map \( \Lambda_{a,F} \)?

For general nonlinearities \( a(x, z) \) it is impossible to determine both \( a(x, z) \) and \( F(x) \) simultaneously from the corresponding DN map. This is due to an inherit gauge invariance of the problem. For inverse source problems of related linear equations, where the aim is to determine a source function from boundary measurements, the gauge invariance of the problem is well-known:

**Remark 1.1.** Let us consider the inverse source problem for the equation

\[
\begin{align*}
\Delta u + qu &= F \quad \text{in } \Omega, \\
u &= f \quad \text{on } \partial \Omega.
\end{align*}
\]

In this inverse problem one asks if the DN map \( \Lambda_F : C^\infty(\partial \Omega) \rightarrow C^\infty(\partial \Omega) \) associated to the above equation determines \( F \) uniquely. We assume here for simplicity that the potential function \( q \) is known. In general, the answer to the question is negative due to the following observation. Let \( u \) solve (1.4) and let \( \psi \) be an arbitrary \( C^2 \)-function satisfying \( \psi|_{\partial \Omega} = \partial_{\nu}\psi|_{\partial \Omega} = 0 \). Let us also define

\[
\tilde{u} := u + \psi.
\]

Consequently, we have \( (\tilde{u}|_{\partial \Omega}, \partial_{\nu}\tilde{u}|_{\partial \Omega}) = (u|_{\partial \Omega}, \partial_{\nu}u|_{\partial \Omega}) \), and

\[
\begin{align*}
\Delta \tilde{u} + q\tilde{u} &= \Delta(u + \psi) + q(u + \psi) \\
&= F - qu + \Delta \psi + qu + q\psi \\
&= F + \Delta \psi + q\psi.
\end{align*}
\]

Hence \( u \) and \( \tilde{u} \) solve the equations \( \Delta u + qu = F \) and \( \Delta \tilde{u} + q\tilde{u} = F + \Delta \psi + q\psi \) respectively. Since \( u \) and \( \tilde{u} \) also have the same Cauchy data on \( \partial \Omega \), it follows that the corresponding DN maps are the same: \( \Lambda_{F}(f) = \Lambda_{F + \Delta \psi + q\psi}(f) \) on \( \partial \Omega \). It is thus not possible to determine a source function uniquely from the DN map.
In this work, we consider general nonlinearities \( a(x, z) \). For such nonlinearities, we prove (in Theorem 1.4) that the corresponding DN map determines the quantities
\[
\partial_k^s a(x, u_0(x)), \quad x \in \Omega, \quad k \in \mathbb{N}.
\]
Here \( u_0 \) is a solution to (1.1) corresponding to a boundary value \( f_0 \). As already shown in Remark 1.1, it might not be possible to recover \( u_0 \) from the DN map. This means that in general, the condition (1.7) does not determine \( a(x, z) \), or even its derivatives in the variable \( z \).

Due to the above obstruction to determining \( a(x, z) \), and consequently \( F(x) \), in general, we mainly focus on nonlinearities \( a(x, z) \) of the following special types:

- **General polynomial nonlinearity:**
  \[
  a(x, z) = \sum_{k=1}^{N} a^{(k)}(x)z^k, \quad N \in \mathbb{N},
  \]

- **Exponential type nonlinearities:**
  \[
  a(x, z) = q(x)e^z \quad \text{and} \quad a(x, z) = q(x)ze^z,
  \]

- **Sine-Gordon nonlinearity:**
  \[
  a(x, z) = q(x)\sin(z).
  \]

For these nonlinearities, we show that the corresponding inverse source problems are either uniquely solvable or there is a gauge symmetry, which has a specific form. The fact that there are nonlinearities for which the related inverse source problem is uniquely solvable is in contrast to inverse source problem for linear equations, which always have the gauge symmetry presented in Remark 1.1. That is, nonlinearity can make inverse source problems uniquely solvable.

Quadratic nonlinearity
\[
a(x, u) = a^{(1)}(x)u(x) + a^{(2)}(x)u^2(x)
\]
has a specific form gauge symmetry, which we now derive. For this, let us assume that \( u \) solves (1.1), where \( a(x, z) \) is as above. If \( \psi \in C^2(\Omega) \), we denote by \( \tilde{a}^{(1)}, \tilde{a}^{(2)} \) and \( \tilde{F} \) another set of \( C^0(\Omega) \) functions. We allow these functions depend on \( \psi \). Let \( \Lambda \) and \( \tilde{\Lambda} \) be the DN maps corresponding to the coefficients without and with tilde signs respectively. If we define
\[
\tilde{a} := u + \psi,
\]
then we have the chain of equivalences
\[
\begin{align*}
\Delta \tilde{u} + \tilde{a}^{(1)}\tilde{u} + \tilde{a}^{(2)}\tilde{u}^2 &= \tilde{F}, \\
\iff \Delta (u_1 + \psi) + \tilde{a}^{(1)}(u_1 + \psi) + \tilde{a}^{(2)}(u_1 + \psi)^2 &= \tilde{F}, \\
\iff \Delta u + \Delta \psi + \tilde{a}^{(1)}u + \tilde{a}^{(1)}\psi + \tilde{a}^{(2)}u^2 + 2\tilde{a}^{(2)}\psi u + \tilde{a}^{(2)}\psi^2 &= \tilde{F}.
\end{align*}
\]
By using \( \Delta u = -a^{(1)}u - a^{(2)}u^2 + F \) and equating the powers of \( u \) gives the following system
\[
\begin{cases}
F + \Delta \psi + a^{(1)}\psi + a^{(2)}\psi^2 = \tilde{F} & \text{in } \Omega, \\
(a^{(1)} = \tilde{a}^{(1)} + 2\tilde{a}^{(2)}\psi & \text{in } \Omega, \\
(a^{(2)} = \tilde{a}^{(2)} & \text{in } \Omega.
\end{cases}
\]
It the above system is satisfied, then
\[
\Delta u + a(x, u) = F \iff \Delta \tilde{u} + a(x, \tilde{u}) = \tilde{F}.
\]
Consequently, if we additionally require that \( \psi|_\Omega = \partial_u \psi|_{\partial \Omega} = 0 \), then the DN maps \( \Lambda \) and \( \tilde{\Lambda} \) are the same. That is, if we change the coefficients \( (a^{(1)}, a^{(2)}, F) \)
to \((\tilde{a}^{(1)}, \tilde{a}^{(2)}, \tilde{F})\), the DN map is preserved. It is at best possible to determine coefficients and a source from the DN map up to the gauge conditions (1.11).

**Earlier works.** Before going into our results, we discuss earlier related works.

The standard approach in the study of inverse problems for nonlinear elliptic equations was initiated in [Isa93]. There the author linearized the nonlinear DN map \(C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)\). The linearization reduced the inverse problem of a nonlinear equation to an inverse problem of a linear equation, which the author was able to solve by using methods for linear equations. Later, second order linearizations, where data depends on two independent parameters, were used to solve inverse problems for example in [AZ21, CNV19, KN02, Sun96, SU97].

For the case \(F = 0\) in \(Ω\) in (1.1), inverse problems for semilinear elliptic equations were recently considered in [FO20, LLLS20]. The novelty of these works is that instead of viewing nonlinearity as an additional complication in the inverse problem, the works used nonlinearity as a beneficial tool. The method of these two works originates from the seminal work [KLU18], where inverse problems for nonlinear equations were studied in Lorentzian spacetimes. By using the method where nonlinearity is used as a tool, inverse problems for nonlinear equations have been solved in cases where the corresponding inverse problems for linear equations are still open. The method is by now usually called the higher order linearization method.

After the works [KLU18, FO20, LLLS20], the literature about inverse problems for nonlinear equations, which uses the higher order linearization method, has grown substantially. The works [LLLS20, LLLS21, LLST22, KU20b, KU20a, FLL21] investigated inverse problems for semilinear elliptic equations with general nonlinearities and in the case of partial data. Inverse problems for quasilinear elliptic equations using higher order linearization have been studied in [KKU22, CFK+21, FKU21]. The works [CLLO22, Nur22] studied inverse problems for minimal surface equations on Riemannian surfaces and Euclidean domain. We also mention the works [LL22a, Lin20, LL22b, LL19, LO22, LZ21], where inverse problems for semilinear fractional type equations have been studied.

Inverse source problems for linear equations that regard determination of both unknown sources and coefficients have attracted recent interest. Applications of them include the photo/thermo-acoustic tomography [LU15], magnetic anomaly detection [DL19, DLL20] and quantum mechanics [LLM19, LLM21]. In this paper, we are interested in related nonlinear counterparts of the above works considering linear models.

In our first result we show that a quadratic nonlinearity and a source are determined by the corresponding DN map up to the gauge conditions (1.11).

**Theorem 1.1.** Let \(Ω \subset \mathbb{R}^n\) be a bounded domain with \(C^\infty\)-smooth boundary \(\partial Ω\), \(n \geq 2\). For \(j = 1, 2\), let

\[
a_j(x, z) = a_j^{(1)}(x)z + a_j^{(2)}(x)z^2,
\]

where \(a_j^{(1)}, a_j^{(2)} \in C^\alpha(\overline{Ω})\) for some \(0 < \alpha < 1\). Consider the following semilinear elliptic equation

\[
\begin{align*}
\Delta u_j + a_j(x, u_j) &= F_j \quad \text{in } Ω, \\
u_j &= f \quad \text{on } \partial Ω,
\end{align*}
\]

and let \(Λ_{a_j, F_j}\) be the corresponding DN map of (1.12) for \(j = 1, 2\). Suppose that there is an open set \(N \subset C^{2,\alpha}(\partial Ω)\) such that

\[Λ_{a_1, F_1}(f) = Λ_{a_2, F_2}(f) \text{ for any } f \in N.\]
Then there exists \( \psi \in C^{2,\alpha}(\overline{\Omega}) \) with \( \psi|_{\partial \Omega} = \partial_{\nu} \psi|_{\partial \Omega} = 0 \) in \( \Omega \) such that

\[
\begin{align*}
(a^{(1)}_1 &= a^{(2)}_1 := a^{(3)}_1, \\
(a^{(1)}_1 &= a^{(2)}_1 + 2a^{(3)}_1 \psi, \\
F_1 &= F_2 - \Delta \psi - a^{(1)}_1 \psi - a^{(3)}_1 \psi^2.
\end{align*}
\]

(1.13)

We remark that in the theorem above it is sufficient that the domain \( \mathcal{N} \) of the DN maps is any non-empty open subset of \( C^{2,\alpha}(\partial \Omega) \). Especially \( \mathcal{N} \) can by very small in size. The same holds for other results of this paper.

Interestingly, if one a priori knows the linear term, the gauge symmetry of the inverse source problem vanishes. Our first result in this direction is:

**Corollary 1.2.** Assume as in Theorem 1.1 and adopt its notation. Assume additionally that

\[ a^{(1)}_1 = a^{(2)}_1 \text{ in } \Omega \]

and

\[ a^{(2)}_1(x) \neq 0 \text{ or } a^{(2)}_2(x) \neq 0 \text{ at any } x \in \Omega. \]

Then also

\[ F_1 = F_2 \text{ and } a^{(2)}_1 = a^{(2)}_2 \text{ in } \Omega. \]

In the light of Remark 1.1, the above corollary in particularly says the following.

If we consider inverse source problem for

\[ \Delta u + qu + w^2 = F, \quad q \text{ known}, \]

instead of \( \Delta u + qu = F \) as in Remark 1.1, then the problem of recovering \( F \) becomes uniquely solvable.

For cubic nonlinearities we prove:

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^\infty \)-smooth boundary \( \partial \Omega \), \( n \geq 2 \). For \( j = 1, 2 \), let also

\[ a_j(x, z) = a^{(1)}_j(x)z + a^{(2)}_j(x)z^2 + a^{(3)}_j z^3, \]

where \( a^{(1)}_j, a^{(2)}_j, a^{(3)}_j \in C^\alpha(\overline{\Omega}) \) for some \( 0 < \alpha < 1 \). Let \( \Lambda_{a_j, F_j} \) be the DN map of the equation

\[
\begin{align*}
\Delta u_j + a_j(x, u_j) &= F_j \quad \text{in } \Omega, \\
u_j &= f \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.14)

Suppose that there is an open set \( \mathcal{N} \subset C^{2,\alpha}(\partial \Omega) \) such that

\[ \Lambda_{a_1, F_1}(f) = \Lambda_{a_2, F_2}(f) \text{ for any } f \in \mathcal{N}. \]

Then there exists \( \psi \in C^{2,\alpha}(\overline{\Omega}) \) with \( \psi|_{\partial \Omega} = \partial_{\nu} \psi|_{\partial \Omega} = 0 \) in \( \Omega \) such that

\[
\begin{align*}
(a^{(1)}_1 &= a^{(2)}_1 := a^{(3)}_1, \\
(a^{(1)}_1 &= a^{(2)}_1 + 2a^{(3)}_1 \psi, \\
F_1 &= F_2 - \Delta \psi - a^{(1)}_1 \psi - a^{(2)}_1 \psi^2 - a^{(3)}_1 \psi^3 \text{ in } \Omega.
\end{align*}
\]

(1.15)

We also consider the case of general polynomial nonlinearities. In the following theorem we denote by

\[ \binom{m}{k} = \frac{m!}{(m-k)k!} \]

the usual binomial coefficients. We include a converse statement to the result.
Theorem 1.3. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^\infty \)-smooth boundary \( \partial \Omega \), \( n \geq 2 \). For \( j = 1, 2 \), let \( a_j(x, z) \) be a polynomial of the form

\[
a_j(x, z) = \sum_{k=1}^{N} a_j^{(k)}(x) z^k \quad \text{for} \ (x, z) \in \overline{\Omega} \times \mathbb{R},
\]

for some \( N \in \mathbb{N} \), where \( a_j^{(k)} \in C^\alpha(\overline{\Omega}) \), for \( j = 1, 2 \) and \( k = 1, \ldots, N \). Given \( F_j \in C^\alpha(\overline{\Omega}) \) for some \( 0 < \alpha < 1 \). Let \( \Lambda_{a_j, F_j} \) be the DN map of the equation

\[
\begin{cases}
\Delta u_j + a_j(x, u_j) = F_j & \text{in } \Omega, \\
u_j = f & \text{on } \partial \Omega.
\end{cases}
\]

Suppose that there is an open set \( \mathcal{N} \subset C^{2,\alpha}(\partial \Omega) \) such that

\[
\Lambda_{a_1, F_1}(f) = \Lambda_{a_2, F_2}(f) \quad \text{for any } f \in \mathcal{N}.
\]

Then there exists \( \psi \in C^{2,\alpha}(\overline{\Omega}) \) with \( \psi|_\Omega = \partial_\nu \psi|_\partial \Omega = 0 \) in \( \Omega \) such that

\[
a_1^{(N-k)} = \sum_{m=N-k}^{N} \binom{m}{N-k} a_2^{(m)} \psi^{m-N+k} \quad \text{in } \Omega,
\]

for \( k = 1, 2, \ldots, N \), and

\[
F_1 = F_2 - \Delta \psi - \sum_{k=1}^{N} a_2^{(k)} \psi^k.
\]

Conversely, if (1.18) and (1.19) hold for some \( \psi \in C^{2,\alpha}(\overline{\Omega}) \) with \( \psi|_\Omega = \partial_\nu \psi|_\partial \Omega = 0 \), then \( \Lambda_{a_1, F_1}(f) = \Lambda_{a_2, F_2}(f) \) for all \( f \in C^{2,\alpha}(\partial \Omega) \) for which either side of the equation is defined.

We remark that we could have also let \( N \) to be finite, but otherwise unknown, in the assumptions of the above theorem. That is, \( N \) could be initially assumed to be different for coefficients \( (a_1, F_1) \) and \( (a_2, F_2) \). The determination result, given by (1.18) and (1.19), is the same also in this case.

Also for cubic and general polynomial nonlinearities (1.16) the gauge invariances of the corresponding inverse problems vanish if the linear term is known.

Corollary 1.3. Assume as in Theorem 1.3 and adopt its notation. Suppose additionally that

\[
a_1^{(1)} = a_2^{(1)} \quad \text{in } \Omega
\]

and

\[
a_1^{(N)}(x) \neq 0 \quad \text{or} \quad a_2^{(N)}(x) \neq 0 \quad \text{for all } x \in \Omega.
\]

Then all the coefficients are uniquely determined:

\[
F_1 \equiv F_2 \quad \text{and} \quad a_1^{(k)} \equiv a_2^{(k)} \quad \text{in } \Omega, \quad k = 1, 2, \ldots, N.
\]

In case it is a priori known that \( F_1 = F_2 \), then we have:

Corollary 1.4. Let us adopt the notation and assumptions in Theorem 1.3. If \( F_1 = F_2 \) in \( \Omega \), then we have

\[
a_1^{(k)} = a_2^{(k)} \quad \text{in } \Omega,
\]

for \( k = 1, 2, \ldots, N \).

The above corollary in particularly says the following. If we consider inverse problem for the equation

\[
\Delta u + qu + u^2 = F, \quad F \text{ known},
\]

then we can recover the lower order term \( q \) from the DN map.
We also study an inverse source problem for general semilinear elliptic equations and do not assume that the nonlinearity is necessarily a polynomial. In fact, we will prove Theorem 1.4 below before Theorem 1.3 for convenience.

**Theorem 1.4.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^\infty \)-smooth boundary \( \partial \Omega \), \( n \geq 2 \). For \( j = 1, 2 \), let \( a_j(\cdot, z) \in C^\alpha(\Omega) \) satisfy the condition (1.2) and assume that \( a_j(x, z) \) is \( C^\infty \)-smooth with respect to the \( z \)-variable. Given \( F_j \in C^\alpha(\Omega) \) for some \( 0 < \alpha < 1 \), let \( \Lambda_{a_j, F_j} \) be the DN map of

\[
\begin{aligned}
\Delta u_j + a_j(x, u_j) &= F_j & \text{in } \Omega, \\
u_j &= f & \text{on } \partial \Omega.
\end{aligned}
\]

Suppose that there is an open set \( \mathcal{N} \subset C^{2,\alpha}(\partial \Omega) \) such that

\[
\Lambda_{a_1, F_1}(f) = \Lambda_{a_2, F_2}(f) \text{ for any } f \in \mathcal{N}.
\]

Then, for any \( f_0 \in \mathcal{N} \), we have

\[
\partial_k^k a_1(x, u_1^{(0)}(x)) = \partial_k^k a_2(x, u_2^{(0)}(x)), \quad x \in \Omega,
\]

for any \( k \in \mathbb{N} \). Here \( u_1^{(0)} \) and \( u_2^{(0)} \) are the solutions of (1.21) with boundary condition \( u_j^{(0)}|_{\partial \Omega} = f_0 \).

As a corollary to Theorem 1.4, we do case studies of inverse source problems when the nonlinearity of the model is either of exponential type or \( a(x, z) = q(x) \sin(z) \). These models arise in mathematical modeling of combustion, where the nonlinearity involved is of exponential type (see e.g. [Vol14]). The nonlinearity \( a(x, z) = q(x) \sin(z) \) corresponds to the sine-Gordon equation. The DN map and inverse problems for the sine-Gordon equation have been considered for example in [BK89, FP12]. The models are chosen so to give realistic examples of cases where the inverse source problem is uniquely solvable, or has a specific gauge symmetry.

Let \( q \) and \( F \) belong to \( C^\alpha(\Omega) \), and consider the semilinear elliptic equations

\[
\begin{aligned}
\Delta u + q(x)e^u &= F & \text{in } \Omega, \\
u &= f & \text{on } \partial \Omega.
\end{aligned}
\]

and

\[
\begin{aligned}
\Delta u + q(x)ue^u &= F & \text{in } \Omega, \\
u &= f & \text{on } \partial \Omega.
\end{aligned}
\]

For the corresponding inverse source problems we assume that the above boundary value problems have each a unique solution \( u_0 \) for some boundary value \( f_0 \) such that \( 0 \) is not an eigenvalue of \( \Delta + \partial_k a(x, u_0) \). In this case, it follows from Theorem 2.1 that the DN maps \( \mathcal{N} \to C^{1,\alpha}(\partial \Omega) \) are defined on an open subset \( \mathcal{N} \subset C^{2,\alpha}(\partial M) \) as before by

\[ u \mapsto \partial_{n_j} u_f|_{\partial M}. \]

Here, \( u_f \) is the unique solution on a neighborhood of \( u_0 \) to either (1.23) or (1.24) depending on which of the two models we are considering.

For nonlinearity \( a(x, z) = q(x)e^z \), the inverse source problem is not uniquely solvable due to a gauge symmetry. However, if the nonlinearity is \( q(x)ze^z \), and \( q(x) \neq 0 \) for \( x \in \Omega \), the corresponding inverse source problem has a unique solution.

**Corollary 1.5.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^\infty \)-smooth boundary \( \partial \Omega \), \( n \geq 2 \). Let \( q_j \in C^\alpha(\Omega) \), and suppose additionally that

**Case 1.**

\[ a_j(x, z) = q_j(x)e^z; \]
Case 2.

\[ a_j(x, z) = q_j(x)ze^z, \]

with \( q_j \neq 0 \) in \( \Omega \), for \( j = 1, 2 \).

Suppose that there is an open \( N \subset C^{2,\alpha}(\partial\Omega) \) such that the corresponding DN maps \( \Lambda_{a_j, F_j} \) of the equation

\[
\begin{cases}
\Delta u_j + a_j(x, u_j) = F_j & \text{in } \Omega, \\
u_j = f & \text{on } \partial\Omega
\end{cases}
\]
satisfy

\[ \Lambda_{a_1, F_1}(f) = \Lambda_{a_2, F_2}(f) \]

for any \( f \in N \).

Then we have:

Case 1. Gauge symmetry:

\[ q_1 = q_2 e^\psi \quad \text{and} \quad F_1 = F_2 - \Delta \psi \quad \text{in } \Omega. \]

Conversely, if (1.25) holds for some \( \psi \in C^{2,\alpha}(\Omega) \) with \( \psi|_{\partial\Omega} = \partial_\nu \psi|_{\partial\Omega} = 0 \), then \( \Lambda_{a_1, F_1}(f) = \Lambda_{a_2, F_2}(f) \) for all \( f \in C^{2,\alpha}(\partial\Omega) \) for which either side of the equation is defined.

Case 2. Unique determination:

\[ q_1 = q_2 \quad \text{and} \quad F_1 = F_2 \quad \text{in } \Omega. \]

As the final application of Theorem 1.4, we consider the inverse source problem for the elliptic sine-Gordon equation. Again, let \( q \) and \( F \) belong to \( C^{\alpha}(\Omega) \), and assume that the equation

\[
\begin{cases}
\Delta u + q \sin u = F & \text{in } \Omega, \\
u = f & \text{on } \partial\Omega,
\end{cases}
\]

has a unique solution for some boundary value \( f_0 \in C^{2,\alpha}(\partial\Omega) \) such that 0 is not an eigenvalue of \( \Delta + \partial_z a(x, u_0) \). Then the equation is well-posed on a neighborhood \( N \subset C^{2,\alpha}(\partial\Omega) \) of \( f_0 \) by Theorem 2.1. Hence, the DN map of (1.27) can be again defined by

\[ \Lambda_{q, F} : N \rightarrow C^{1,\alpha}(\partial\Omega), \quad u \mapsto \partial_\nu uf|_{\partial\Omega}, \]

where \( u_f \in C^{2,\alpha}(\Omega) \) is the unique solution to (1.27) on a neighborhood of \( u_{f_0} \).

For the sine-Gordon equation, the inverse source problem is solvable.

Corollary 1.6. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^\infty \)-smooth boundary \( \partial\Omega \), \( n \geq 2 \). Let \( q \in C^\alpha(\Omega) \), and suppose additionally that

\[ a_j(x, z) = q_j(x) \sin z, \]

for \( j = 1, 2 \). Suppose that there is an open set \( N \subset C^{2,\alpha}(\partial\Omega) \) such that the corresponding DN maps \( \Lambda_{q_j, F_j} \) of the equation

\[
\begin{cases}
\Delta u_j + q_j \sin(u_j) = F_j & \text{in } \Omega, \\
u_j = f & \text{on } \partial\Omega
\end{cases}
\]
satisfy

\[ \Lambda_{q_1, F_1}(f) = \Lambda_{q_2, F_2}(f) \]

for any \( f \in N \).

Then

\[ q_1 = q_2 \quad \text{and} \quad F_1 = F_2 \quad \text{in } \Omega. \]
The paper is organized as follows. In Section 2, we prove a well-posedness result for semilinear elliptic equations with sources. Moreover, a local well-posedness result is also given in Section 2, and the proof is left in Appendix A. In Section 3, we prove Theorems 1.1, 1.2, 1.3 and 1.4 by using the higher order linearization method. We prove Corollaries 1.5 and 1.6 in Section 4.

2. Preliminaries

In this section, we prove a well-posedness result for the Dirichlet problem (1.1) on a neighborhood of a given solution. Let \( 0 < \alpha < 1 \) and \( \delta > 0 \) and denote
\[
N_\delta := \{ f \in C^{2,\alpha}(\partial \Omega) : \|f\|_{C^{2,\alpha}(\partial \Omega)} \leq \delta \}.
\]

Note that when the source function \( F \) of the equation \( \Delta u(x) + a(x, u) = F(x) \) does not vanish, zero function is not a solution to the equation (1.1). This is the main reason why our well-posedness result differs from the usual ones, such as the one in [LLLS21, KU20a].

**Theorem 2.1** (Well-posedness). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^\infty \) boundary \( \partial \Omega \) and \( n \geq 2 \). Given \( \alpha \in (0, 1) \), \( F \in C^{2,\alpha}(\overline{\Omega}) \) and \( f_0 \in C^{2,\alpha}(\partial \Omega) \), suppose that there exists a unique solution \( u_0 \in C^{2,\alpha}(\overline{\Omega}) \) to
\[
(2.1)
\Delta u_0 + a(x, u_0) = F \quad \text{in } \Omega,
\]
\[
u = f_0 \quad \text{on } \partial \Omega.
\]

Assume also that
\[
(2.3) \quad 0 \text{ is not a Dirichlet eigenvalue of } \Delta + \partial_2 a(x, u_0) \text{ in } \Omega.
\]

Then there are \( \delta \) and \( C > 0 \) such that for any \( f \in N_\delta \) there exists a unique solution \( u \in C^{2,\alpha}(\overline{\Omega}) \) of
\[
(2.2)
\Delta u + a(x, u) = F \quad \text{in } \Omega,
\]
\[
u = f_0 + f \quad \text{on } \partial \Omega,
\]
within the class \( \{ w \in C^{2,\alpha}(\overline{\Omega}) : \|w - u_0\|_{C^{2,\alpha}(\overline{\Omega})} \leq C \} \). Moreover, there are \( C^\infty \) Fréchet differentiable maps
\[
\mathcal{S} : N_\delta \to C^{2,\alpha}(\overline{\Omega}), \quad f \mapsto u,
\]
\[
\Lambda : N_\delta \to C^{1,\alpha}(\partial \Omega), \quad f \mapsto \partial_\nu u|_{\partial \Omega}.
\]

**Proof of Theorem 2.1.** We use the standard method that uses the implicit function theorem in Banach spaces to prove theorem. A similar proof can be found from the work [LLLS21] where the source \( F \) is assumed to vanish. We refer to that work for additional details of the arguments used. Let
\[
\mathcal{B}_1 = C^{2,\alpha}(\partial \Omega), \quad \mathcal{B}_2 = C^{2,\alpha}(\overline{\Omega}), \quad \mathcal{B}_3 = C^{\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial \Omega)
\]
and assume that \( u_0 \) solves (2.2). Consider the map
\[
\Psi : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_3,
\]
\[
(f, u) \mapsto \left( \Delta u + a(x, u) - F, u|_{\partial \Omega} - (f_0 + f) \right).
\]

Similar to [LLLS21, Section 2], one can show that the map \( u \mapsto a(x, u) \) is a \( C^\infty \) map from \( C^{2,\alpha}(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega}) \).

Notice that \( \Psi(0, u_0) = (0, 0) \), where \( u_0 \in C^{2,\alpha}(\overline{\Omega}) \) is the solution to (2.2). The first linearization of \( \Psi = \Psi(f, u) \) at \( (0, u_0) \) in the variable \( u \) is
\[
D_u \Psi|_{(0, u_0)}(v) = (\Delta v + \partial_\nu a(x, u_0)v, v|_{\partial \Omega}).
\]

This is a homeomorphism \( \mathcal{B}_2 \to \mathcal{B}_3 \) by the condition (2.3), which is guaranteed by well-posedness and Schauder estimates for linear second order elliptic equations.
Using the implicit function theorem in Banach spaces [RR06, Theorem 10.6 and Remark 10.5] yields that there is \( \delta > 0 \) and an open ball \( N_\delta \subset C^{2,\alpha}(\partial \Omega) \) and a \( C^\infty \) map \( \mathcal{S} : N_\delta \rightarrow \mathbb{B}_2 \) such that whenever \( \|f\|_{C^{2,\alpha}(\partial \Omega)} \leq \delta \) we have
\[
\Psi(f, \mathcal{S}(f)) = (0, 0).
\]
Since \( \mathcal{S} \) is smooth and \( \mathcal{S}(0) = u_0 \), the solution \( u = \mathcal{S}(f) \) satisfies
\[
\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|f\|_{C^{2,\alpha}(\partial \Omega)}.
\]
Moreover, by the uniqueness statement of the implicit function theorem, by redefining \( \delta > 0 \) to be smaller if necessary, \( u = \mathcal{S}(f) \) is the only solution to \( \Psi(f, u) = (0, 0) \) whenever \( \|f\|_{C^{2,\alpha}(\partial \Omega)} \leq \delta \) and
\[
\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C.
\]
As in [LLLS21], one can check that the solution operator \( \mathcal{S} : N_\delta \rightarrow C^{2,\alpha}(\overline{\Omega}) \) is a \( C^\infty \) map in the Fréchet sense. Since the normal derivative is a linear map \( C^{2,\alpha}(\overline{\Omega}) \rightarrow C^{1,\alpha}(\partial \Omega) \), then \( \Lambda \) is also a well defined \( C^\infty \) map \( N_\delta \rightarrow C^{1,\alpha}(\partial \Omega) \).

Under the assumptions of the above theorem, the boundary value problem (1.1) is well-posed in the following sense: There is \( f_0 \in C^{\infty}(\partial \Omega) \) and \( \delta > 0 \) such that for each \( f \in f_0 + N_\delta \) there exists a solution \( u_f \) to (1.1) with \( u_f|_{\partial \Omega} = f \). The solution \( u_f \) is unique on a fixed neighborhood of \( u_0 \in C^{2,\alpha}(\Omega) \), where \( u_0 \) solves (1.1) with boundary value \( f_0 \). In this case the corresponding DN map \( f_0 + N_\delta \rightarrow C^{1,\alpha}(\partial \Omega) \) defined by the assignment \( f \mapsto \partial_n u_f|_{\partial \Omega} \) is well-defined and \( C^\infty \) smooth in the Fréchet sense.

Before ending this section, we give a well-posedness result in the case when the Dirichlet data and the source \( F \) are both sufficiently small. In inverse source problems, the source is unknown. Thus the below result still has only limited applicability in the context of this paper. We record it to give an example where the DN map is always defined for small Dirichlet data. Let
\[
\mathcal{A}_\varepsilon := \left\{ F \in C^{2,\alpha}(\overline{\Omega}) : \|F\|_{C^{\alpha}(\overline{\Omega})} \leq \varepsilon \right\},
\]
and we have following result, whose the proof is put in the Appendix A.

**Proposition 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( C^\infty \) boundary \( \partial \Omega \) and \( n \geq 2 \). Assume that \( a(x, 0) = 0 \). There are \( C > 0 \), \( \varepsilon > 0 \) and \( \delta > 0 \) such that for any \( F \in \mathcal{A}_\varepsilon \) and \( f \in N_\delta \), then there is a unique solution \( u \in C^{2,\alpha}(\overline{\Omega}) \) of
\[
\begin{aligned}
\Delta u + a(x, u) &= F & \text{in } \Omega, \\
u &= f & \text{on } \partial \Omega,
\end{aligned}
\]
within the class \( \left\{ w \in C^{2,\alpha}(\overline{\Omega}) : \|w\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\varepsilon + \delta) \right\} \). Moreover, there is a \( C^\infty \) Fréchet differentiable map
\[
\mathcal{S} : \mathcal{A}_\varepsilon \times N_\delta \rightarrow C^{2,\alpha}(\overline{\Omega}), \quad (F, f) \mapsto u.
\]
In particular, for a fixed \( F \in \mathcal{A}_\varepsilon \), the map
\[
\Lambda_F : N_\delta \rightarrow C^{1,\alpha}(\partial \Omega), \quad f \mapsto \partial_n u|_{\partial \Omega}
\]
is also \( C^\infty \) Fréchet differentiable.
3. Uniqueness for polynomial nonlinearities up to gauge invariances

3.1. Quadratic nonlinearity. In the introduction we showed that the inverse source problem for
\[ \Delta u + a(x, u) = F, \]
where \( a(x, u) \) is quadratic,
\[ a(x, u(x)) = a^{(1)}(x)u(x) + a^{(2)}(x)u^2(x), \]
has a gauge invariance given by the gauge conditions (1.11). We show next that these gauge conditions are the only obstruction to uniqueness in the inverse source problem for quadratic nonlinearities. This is Theorem 1.1.

For the quadratic nonlinearity we consider Dirichlet data of the form
\[ f := f(x; \epsilon_1, \epsilon_2) := f_0(x) + \epsilon_1 f_1(x) + \epsilon_2 f_2(x) \quad x \in \partial \Omega, \]
where where \( f_0, f_1, f_2 \in C^{2, \alpha}(\partial \Omega) \), and \( \epsilon_1 \) and \( \epsilon_2 \) are small real parameters.

Proof of Theorem 1.1. By assumption there is \( N \subset C^{2, \alpha}(\partial \Omega) \) such that
\[ \Lambda_{a_1, f_1}(f) = \Lambda_{a_2, f_2}(f), \quad f \in N. \]
Let \( f_0 \in N, f_1, f_2 \in C^{2, \alpha}(\partial \Omega) \) and \( \epsilon_1, \epsilon_2 > 0 \) such that \( f_0 + \epsilon_1 f_1 + \epsilon_2 f_2 \in N. \) We apply the higher order linearization method to the equation
\[ \begin{cases} 
\Delta u_j + a^{(1)}(x)u_j + a^{(2)}(x)u_j^2 = F_j & \text{in } \Omega, \\
u_j = f_0 + \epsilon_1 f_1 + \epsilon_2 f_2 & \text{on } \partial \Omega.
\end{cases} \]
We denote \( \epsilon = (\epsilon_1, \epsilon_2) \), which especially means that \( \epsilon = 0 \) is equivalent to \( \epsilon_1 = \epsilon_2 = 0 \). Below the index \( j \) corresponds to the different sets of coefficients, and an index \( \ell \) to \( \epsilon_\ell \) parameters. Let us denote by \( u_j^{(0)} \) the solution to
\[ \begin{cases} 
\Delta u_j^{(0)} + a^{(1)}(x)u_j^{(0)} + a^{(2)}(x)(u_j^{(0)})^2 = F_j & \text{in } \Omega, \\
u_j^{(0)} = f_0 & \text{on } \partial \Omega.
\end{cases} \]

With the well-posedness holding on a neighborhood \( N \) of \( f_0 \), see Theorem 2.1, we can differentiate (3.2) with respect to \( \epsilon_\ell \), for \( \ell = 1, 2 \). We obtain
\[ \begin{cases} 
(\Delta + a^{(1)} + 2a^{(2)}u_j^{(0)}) v^{(\ell)} = 0 & \text{in } \Omega, \\
v^{(\ell)} = f_\ell & \text{on } \partial \Omega,
\end{cases} \]
where
\[ v^{(\ell)} = \partial_{\epsilon_\ell \epsilon = 0} u_j, \]
for \( j, \ell = 1, 2 \). It also follows from Theorem 2.1 that we know the DN maps of the equation (3.4) for \( j = 1 \) and \( j = 2 \) agree. Thus, by the global uniqueness result for linear inverse boundary value problems (see e.g. [LLLS20, Proposition 2.1] or [SU87] for \( n \geq 3 \) and [Buk08, BTW19] for \( n = 2 \), we have
\[ Q := a^{(1)} + 2a^{(2)}u_1^{(0)} = a^{(1)} + 2a^{(2)}u_2^{(0)} \text{ in } \Omega. \]
It then follows by uniqueness of solutions to the Dirichlet problem (3.3) that
\[ v^{(\ell)} = v_1^{(\ell)} = v_2^{(\ell)} \text{ in } \Omega, \]
for \( \ell = 1, 2 \).

We next derive the equation for the second order linearization of (3.2) at \( u_j^{(0)} \).
For \( j = 1, 2 \), a straightforward computation shows that
\[ \begin{cases} 
(\Delta + a^{(1)} + 2a^{(2)}u_j^{(0)}) w_j + 2a^{(2)}(u^{(1)})u^{(2)} = 0 & \text{in } \Omega, \\
w_j = 0 & \text{on } \partial \Omega,
\end{cases} \]
where
\[ w_j = \partial_{zv}^2 \big|_{z=0} u_j. \]

We show next that \( a_1^{(2)} = a_2^{(2)} \) in \( \Omega \). For that, let us consider \( v^{(\ell)} \) to be the solution of
\[
\begin{cases}
(\Delta + Q)v^{(\ell)} = 0 & \text{in } \Omega, \\
v^{(\ell)} = g_\ell & \text{on } \partial\Omega,
\end{cases}
\]
where \( Q \) is given in (3.5) and \( g_\ell \in H^{1/2}(\partial\Omega) \) will be chosen later for \( \ell = 1, 2 \). We multiply (3.7) by \( v^{(1)} \). Moreover, by using \( \partial_\nu w_1 = \partial_\nu w_2 \) on \( \partial\Omega \), integration by parts yields
\[
0 = \int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2) v^{(1)} dS
= \int_{\Omega} \Delta (w_1 - w_2) v^{(1)} dx + \int_{\Omega} \nabla (w_1 - w_2) \cdot \nabla v^{(1)} dx
\]
\[
= \int_{\Omega} \Delta (w_1 - w_2) v^{(1)} dx + \int_{\partial\Omega} (w_1 - w_2) \partial_\nu v^{(1)} dS
- \int_{\Omega} (w_1 - w_2) \Delta v^{(1)} dx
= \int_{\Omega} \left( a_1^{(2)} - a_2^{(2)} \right) v^{(1)} v^{(2)} v^{(1)} dx.
\]

Here we used \( w_1 - w_2 = 0 \) on \( \partial\Omega \) and (3.6) and (3.7). By using that products of pairs of solutions (CGOs) to (3.4) are dense in \( L^1(\Omega) \) for \( n \geq 2 \), we can choose \( v^{(1)} \) and \( v^{(2)} \) so that we obtain
\[
(a_1^{(2)} - a_2^{(2)}) v^{(1)} = 0 \text{ in } \Omega.
\]

Next we take also \( v^{(1)} \) as a CGO solution and multiply the above identity by yet another CGO solution \( v^{(2)} \) with \( v^{(2)}|_{\partial\Omega} = g_2 \), one can integrate the above identity to obtain
\[
\int_{\Omega} \left( a_1^{(2)} - a_2^{(2)} \right) v^{(1)} v^{(2)} dx = 0.
\]

By applying the density of CGOs again shows that
\[
(a_1^{(2)} - a_2^{(2)}) = 0 \text{ in } \Omega.
\]

Let us then define \( \psi \in C^2(\Omega) \) as the difference
\[
\psi := u_2^{(0)} - u_1^{(0)} \text{ in } \Omega.
\]

By plugging (3.9) into (3.5), we obtain
\[
a_1^{(1)} = 2a_2^{(2)} \left( u_2^{(0)} - u_1^{(0)} \right) = 2a_2^{(2)} \psi \text{ in } \Omega.
\]

Moreover, with the relation (3.10) at hand, we calculate
\[
F_2 = \Delta u_2^{(0)} + a_1^{(2)} u_2^{(0)} + a_2^{(2)} \left( u_2^{(0)} \right)^2
= \Delta \left( u_1^{(0)} + \psi \right) + a_1^{(2)} \left( u_1^{(0)} + \psi \right) + a_2^{(2)} \left( u_1^{(0)} + \psi \right)^2
= \left( F_1 + a_1^{(2)} \psi + a_2^{(2)} \psi^2 \right) + \left( a_1^{(2)} - a_1^{(1)} + 2a_2^{(2)} \psi \right) u_1^{(0)}
+ \left( a_2^{(2)} - a_2^{(1)} \right) \left( u_1^{(0)} \right)^2.
\]
Here we also utilized (3.3). By using (3.9) and (3.11), we see that $F_2 = F_1 + a_2^{(2)} \psi + a_2^{(2)} \psi^2$. Finally, the function $\psi$ of the form (3.10) satisfies $\psi|_{\partial \Omega} = (u_2^{(0)} - u_1^{(0)}) = 0$ and $\partial_\nu \psi|_{\partial \Omega} = \partial_\nu (u_2^{(0)} - u_1^{(0)})|_{\partial \Omega} = 0$. We have shown
\[ \begin{cases} 
 a_1^{(2)} = a_2^{(2)} =: a^{(2)} \\
 a_1^{(1)} = a_2^{(1)} + 2a^{(2)} \psi \\
 F_1 = F_2 - \Delta \psi - a^{(2)} \psi - a^{(2)} \psi^2 
\end{cases} \tag{3.13} \]
as desired.

\[\Box\]

**Remark 3.1.** Note that if the coefficients of quadratic terms vanish, $a_1^{(1)} = a_2^{(2)} = 0$ in $\Omega$, then (3.13) describes the gauge symmetry of inverse source problem for linear equation discussed in Remark 1.1.

We also remark that in the above proof we could have alternatively used Runge approximation argument to show that $a_1^{(2)} = a_2^{(2)}$ after (3.8). Indeed, if $x_0 \in \Omega$, there is by Runge approximation (see e.g. [LLS20]) a solution $\psi^{(1)}$ such that $\psi^{(1)}(x_0) \neq 0$. Together with (3.8), and using the above argument for all $x_0 \in \Omega$, shows $a_1^{(2)} = a_2^{(2)}$ in $\Omega$. Runge approximation in similar situations were earlier used in [LLS21].

As discussed in the introduction, if the linear term of a semilinear equation $\Delta u + a(x, u) = F$ is known, then the DN map determines the other coefficients of the equation uniquely. This is Corollary 1.2, which we now prove.

**Proof of Corollary 1.2.** By assumption and Theorem 1.1
\[ a^{(1)} = a^{(1)} + 2a^{(2)} \psi \]
and
\[ F_1 = F_2 - \Delta \psi - a^{(2)} \psi - a^{(2)} \psi^2 \]
hold in $\Omega$ for some gauge function $\psi$. Here $a^{(2)} = a_1^{(2)} = a_2^{(2)}$. Since $a^{(2)} \neq 0$ in $\Omega$ by assumption, the first identity above shows that $\psi = 0$ in $\Omega$. Substituting $\psi = 0$ to latter identity above shows $F_1 = F_2$ in $\Omega$. \[\Box\]

### 3.2. Cubic nonlinearity

We move on to prove our results about cubic nonlinearities. For $j = 1, 2$, we let
\[ a_j(x, z) = a_j^{(1)} z + a_j^{(2)} z^2 + a_j^{(3)} z^3, \]
and let us consider the equation
\[ \Delta u_j + a_j^{(1)} u_j + a_j^{(2)} u_j^2 + a_j^{(3)} u_j^3 = F_j \text{ in } \Omega. \tag{3.14} \]

Theorem 1.2, which we prove in this section shows that the inverse source problems of the above equation has uniqueness property for both coefficients and source up to a gauge.

Before proving Theorem 1.2, let us derive the gauge of the inverse problem. Assume that $u_1$ solves (3.14) with boundary value $u_1|_{\partial \Omega} = f$. If $\psi \in C^2(\overline{\Omega})$, we denote by $a_1^{(j)}, a_2^{(j)}, a_3^{(j)}$ and $F_2$ another set of coefficients and a source, which may depend on $\psi$. If we denote $u_2 = u_1 + \psi$, then we have the chain of equivalences
\[ \Delta u_2 + a_1^{(2)} u_2 + a_2^{(2)} (u_2)^2 + a_3^{(2)} (u_2)^3 = F_2 \]
\[ \iff \Delta (u_1 + \psi) + a_1^{(2)} (u_1 + \psi) + a_2^{(2)} (u_1 + \psi)^2 + a_3^{(2)} (u_1 + \psi)^3 = F_2 \]
\[ \iff \Delta u_1 + \Delta \psi + a_1^{(2)} u_1 + a_2^{(2)} (u_1)^2 + a_3^{(2)} (u_1)^3 + 2a_2^{(2)} \psi u_1 + a_2^{(2)} \psi^2 + a_3^{(2)} u_1^3 + a_2^{(2)} \psi^3 = F_2, \]
\[ + a_3^{(2)} (u_1^3 + 3u_1^2 \psi + 3u_1 \psi^2 + \psi^3) = F_2, \]
which holds in $\Omega$. By using $\Delta u_1 = -a_1^{(1)} u_1 - a_1^{(2)} (u_1)^2 - a_1^{(3)} (u_1)^3 + F_1$ in $\Omega$ and equating the powers of $u$ gives the following system

$$
\begin{align*}
F_1 &= F_2 - \Delta \psi - a_2^{(1)} \psi - a_2^{(2)} \psi^2 - a_2^{(3)} \psi^3 \\
a_1^{(1)} &= a_2^{(1)} + 2a_2^{(2)} \psi + 3a_2^{(3)} \psi^2 \\
a_1^{(2)} &= a_2^{(2)} + 3a_2^{(3)} \psi \\
a_1^{(3)} &= a_2^{(3)}.
\end{align*}
$$  

(3.15)

The above system of equations describes the gauge invariance for the inverse source problem for cubic nonlinearity. If $\psi|_\Omega = \partial_x \psi|_{\partial \Omega} = 0$, the above computation shows that corresponding DN maps $\Lambda_{a_1, f_1}$ and $\Lambda_{a_2, e_2}$ are the same. It is impossible to uniquely determine the coefficients and sources from the DN map at the same time. There is a gauge symmetry given by (3.15).

We next prove Theorem 1.2, which states that the DN map determines the coefficients and source up to the gauge symmetry (3.15).

**Proof of Theorem 1.2.** Let us consider the Dirichlet data

$$
f = f(x; \epsilon) = f_0 + \epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3 \quad \text{on} \quad \partial \Omega,
$$

where the parameters $\epsilon$ are real numbers, $f_0 \in \mathcal{N}$ and $f_\ell \in C^{2,\alpha}(\partial \Omega)$, for $\ell = 1, 2, 3$. By assumption $\Lambda_{a_1, f_1}(f) = \Lambda_{a_2, e_2}(f)$ if the parameters $\epsilon_\ell$ are small enough. We denote $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$.

Let us denote by $u_{j}^{(0)}$ the solution to

$$
\begin{align*}
\Delta u_{j}^{(0)} + a_j(x, u_{j}^{(0)}) &= F_j & \text{in } \Omega, \\
u_{j}^{(0)} &= f_0 & \text{on } \partial \Omega.
\end{align*}
$$

We linearize

$$
\begin{align*}
\Delta u_j + a_j(x, u_j) &= F_j & \text{in } \Omega, \\
u_j &= f_0 + \epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3 & \text{on } \partial \Omega.
\end{align*}
$$

at the solution corresponding to boundary value $f_0$ for $j = 1, 2$. The first linearization at $f_0$ is

$$
\begin{align*}
\left\{ \frac{\Delta + a_j^{(1)} + 2a_j^{(2)} u_j^{(0)} + 3a_j^{(3)} (u_j^{(0)})^2}{v_j^{(0)}} \right\} v_j^{(f)} &= 0 & \text{in } \Omega, \\
v_j^{(0)} &= f_\ell & \text{on } \partial \Omega,
\end{align*}
$$

(3.16)

where $v_j^{(f)} := \partial_\epsilon u_j|_{\epsilon = 0}$ in $\Omega$. By Theorem 2.1, we know that the DN maps of (3.16) for $j = 1$ and $j = 2$ agree. By the global uniqueness result for the Calderón problem for linear equations we have

$$
Q := a_1^{(1)} + 2a_1^{(2)} u_1^{(0)} + 3a_1^{(3)} (u_1^{(0)})^2 = a_2^{(1)} + 2a_2^{(2)} u_2^{(0)} + 3a_2^{(3)} (u_2^{(0)})^2 \quad \text{in } \Omega,
$$

(3.17)

and by the uniqueness of solutions to the Dirichlet problem (3.16) it follows that

$$
v_j^{(f)} = v_1^{(f)} = v_2^{(f)} \quad \text{in } \Omega,
$$

(3.18)

for $\ell = 1, 2, 3$.

The second linearization reads

$$
\begin{align*}
\left\{ \Delta + Q \right\} w_j^{(k\ell)} + 2 \left( a_j^{(2)} + 3a_j^{(3)} u_j^{(0)} \right) v^{(k)} v^{(\ell)} &= 0 & \text{in } \Omega, \\
w_j^{(k\ell)} &= 0 & \text{on } \partial \Omega,
\end{align*}
$$

(3.19)
where \( u_j^{(k\ell)} = \partial_{x_i x_j}^2 u_j \big|_{x=0} \) for \( k, \ell \in \{1,2,3\} \) and \( j = 1,2 \). Similar to the proof of Theorem 1.1, multiplying (3.19) by the function \( v \) that solves

\[
\begin{cases}
(\Delta + Q) v = 0 & \text{in } \Omega, \\
v = g & \text{on } \partial \Omega,
\end{cases}
\]

where \( g \in H^{1/2}(\partial \Omega) \) is a function to be chosen later. Multiplying (3.19) by the solution \( v \) and integrating by parts show that

\[
\int_\Omega \left[ \left( a_1^{(2)} + 3a_1^{(3)} u_1^{(0)} \right) - \left( a_2^{(2)} + 3a_2^{(3)} u_2^{(0)} \right) \right] v^{(k\ell)} v dx = 0,
\]

for \( k, \ell = 1,2,3 \). Applying an additional density argument as in the proof of Theorem 1.1 (or the one described in Remark 3.1), one obtains

\[
R := a_1^{(2)} + 3a_1^{(3)} u_1^{(0)} = a_2^{(2)} + 3a_2^{(3)} u_2^{(0)} \text{ in } \Omega.
\]

The uniqueness of solutions to Dirichlet problem of (3.19) and (3.22) imply

\[
w^{(k\ell)} := w_1^{(k\ell)} = w_2^{(k\ell)} \text{ in } \Omega,
\]

for any \( k, \ell \in \{1,2,3\} \).

Now, a computation shows that the third linearized equation is

\[
\begin{cases}
(\Delta + Q) u_j^{(123)} + 2R \left( w_j^{(12)} v^{(3)} + w_j^{(23)} v^{(1)} + w_j^{(13)} v^{(2)} \right) \\
+ 6a_j^{(3)} u_j^{(0)} v^{(3)} = 0 & \text{in } \Omega, \\
u^{(123)} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( R \) is the function given by (3.22). As before, multiplying (3.23) against the solution \( v \) of (3.20) produces the identity

\[
\int_\Omega \left( a_1^{(3)} - a_2^{(3)} \right) v^{(1)} v^{(2)} v^{(3)} dx = 0.
\]

By choosing \( v^{(\ell)} (\ell = 1,2,3) \) and \( v \) to be suitable CGO solutions, we conclude via the above integral identity

\[
a^{(3)} := a_1^{(3)} = a_2^{(3)} \text{ in } \Omega,
\]

which proves the first relation in (1.15).

Let us define \( \psi \in C^2(\overline{\Omega}) \) by

\[
\psi = u_2^{(0)} - u_1^{(0)} \text{ in } \Omega.
\]

Then, the identity (3.22) is equivalent to

\[
a_2^{(2)} = a_2^{(2)} + 3a_2^{(3)} \left( u_2^{(0)} - u_1^{(0)} \right) = a_1^{(2)} + 3a_1^{(3)} \psi,
\]

where we utilized (3.24) and (3.25). This shows the second identity in (1.15). By plugging (3.26) into (3.17), direct computations yield

\[
a_1^{(1)} = a_1^{(1)} + 2a_2^{(2)} u_2^{(0)} + 3a_2^{(3)} \left( u_2^{(0)} \right)^2 - 2a_1^{(2)} u_1^{(0)} - 3a_1^{(3)} \left( u_1^{(0)} \right)^2,
\]

which proves the third identity in (1.15). Finally, by inserting (3.25) into the original nonlinear equation (1.14), and equating the powers of \( u_2^{(0)} \), yield the last identity in (1.15) as desired. This completes the proof. \( \square \)
3.3. Polynomial and general nonlinearity. In order to prove Theorem 1.3, where the nonlinearity is a general polynomial, it is convenient to prove Theorem 1.4 about general nonlinearities first.

Proof of Theorem 1.4. Let $N \in \mathbb{N}$. By using the higher order linearization method, let us take the Dirichlet data to be of the form

$$f(x) = \sum_{\ell=1}^{N} \epsilon_{\ell} f_{\ell}(x) \text{ on } \partial \Omega,$$

where $\epsilon_{\ell}$ are parameters such that $|\epsilon_{\ell}|$ are sufficiently small, and each $f_{\ell} \in C^{2,\alpha}(\partial \Omega)$, for $\ell = 1, \ldots, N$. We first linearize the equation (1.21) around the solution $u_{j}^{(0)}$, so that we can have

$$(3.27) \quad \begin{cases} \left(\Delta + \partial_{x} a_{j}(x, u_{j}^{(0)})\right) v_{j}^{(\ell)} = 0 \text{ in } \Omega, \\ v_{j}^{(\ell)} = f_{\ell} \text{ on } \partial \Omega \end{cases}$$

for $j = 1, 2$, and $\ell = 1, \ldots, N$. The global uniqueness result yields again that

$$\partial_{x} a_{1}(x, u_{1}^{(0)}) = \partial_{x} a_{2}(x, u_{2}^{(0)}) \text{ in } \Omega.$$

Moreover, via the uniqueness of solutions, one can see that $v^{(\ell)} = v_{1}^{(\ell)} = v_{2}^{(\ell)}$ in $\Omega$, for $\ell = 1, 2, \ldots, N$.

To proceed, the second linearized equation can be derived as

$$(3.28) \quad \begin{cases} \left(\Delta + Q\right) w_{j}^{(m(\ell))} + \partial_{x}^{2} a_{j}(x, u_{j}^{(0)}) v^{(m(\ell))} = 0 \text{ in } \Omega, \\ w_{j}^{(m(\ell))} = 0 \text{ on } \partial \Omega, \end{cases}$$

where $Q := \partial_{x} a_{1}(x, u_{1}^{(0)}) = \partial_{x} a_{2}(x, u_{2}^{(0)})$ in $\Omega$, for $\ell, m = 1, 2, \ldots, N$. Similar as before, consider a solution $v$ of

$$\begin{cases} (\Delta + Q) v = 0 \text{ in } \Omega, \\ v = g \text{ on } \partial \Omega, \end{cases}$$

by multiplying (3.28) by the function $v$, then an integration by parts formula yields that

$$\int_{\Omega} \left(\partial_{x}^{2} a_{1}(x, u_{1}^{(0)}) - \partial_{x}^{2} a_{2}(x, u_{2}^{(0)})\right) v^{(m)} v dx = 0,$$

which shows $\partial_{x}^{2} a_{1}(x, u_{1}^{(0)}) = \partial_{x}^{2} a_{2}(x, u_{2}^{(0)})$ in $\Omega$ by utilizing preceding arguments.

Furthermore, by considering higher order linearized equations and using an induction argument, similar to related arguments in proofs of [LLLS20, Proof of Theorem 1.1] and [KU20a, Proof of Theorem 1.3], it is not hard to show that (1.22) holds for any $k \in \mathbb{N}$, where $u_{j}^{(0)}$ are the solutions of (3.27), for $j = 1, 2$. As $N \in \mathbb{N}$ was arbitrary, this completes the proof. \hfill \Box

We now prove Theorem 1.3.

Proof of Theorem 1.3. To prove the theorem, we need to show that there is $\psi \in C^{2,\alpha}(\bar{\Omega})$ with $\psi|_{\partial \Omega} = \partial_{x} \psi|_{\partial \Omega} = 0$ such that

$$(3.29) \quad a_{1}^{(N-k)}(x) = \sum_{m=N-k}^{N} \binom{m}{N-k} a_{2}^{(m)}(x) \psi^{m-N+k} \quad \text{in } \Omega,$$

for $k = 1, \ldots, N$. Since $a_{1}(x, z)$ and $a_{2}(x, z)$ are both polynomials of order $N$, we have by Theorem 1.4

$$a_{1}^{(N)}(x) = \partial_{x}^{N} a_{1}(x, u_{1}^{(0)}) = \partial_{x}^{N} a_{2}(x, u_{2}^{(0)}) = a_{2}^{(N)}(x)$$
for all $x \in \Omega$. Here $u_j^{(0)}$, $j = 1, 2$, is the solution of (1.17) as $u_j^{(0)} \mid_{\partial \Omega} = 0$. Thus the claim holds for $k = 0$. We prove the claim by induction. For this, let us assume that (3.29) holds for all $k = 0, \ldots, L$. It suffices to show that (1.18) holds for $k = L + 1$.

Using Theorem 1.4 again, we have

$$
(3.30) \quad \partial_z^{a_1(N-L-1)} a_1(x, u_1^{(0)}) = \partial_z^{a_2(N-L-1)} a_2(x, u_2^{(0)}) \quad \text{in } \Omega.
$$

Since $a_j(x, z)$ is a polynomial in $a$, this identity is equivalent to

$$
(3.31) \quad \left( \frac{N-L-1}{N-L-1} a_1^{(N-L-1)} + \frac{N-L}{N-L-1} a_1^{(N-L)} u_1^{(0)} \right) \\
+ \frac{N-L+1}{2!} a_1^{(N-L+1)} \left( u_1^{(0)} \right)^2 + \cdots + \frac{N!}{(L+1)!} a_1^{(N)} \left( u_1^{(0)} \right)^{L+1} \\
= \left( \frac{N-L-1}{N-L-1} a_2^{(N-L-1)} + \frac{N-L}{N-L-1} a_2^{(N-L)} u_2^{(0)} \right) \\
+ \frac{N-L+1}{2!} a_2^{(N-L+1)} \left( u_2^{(0)} \right)^2 + \cdots + \frac{N!}{(L+1)!} a_2^{(N)} \left( u_2^{(0)} \right)^{L+1}.
$$

After dividing by $(N-L-1)!$ the above reads

$$
(3.32) \quad a_1^{(N-L-1)} + \sum_{k=0}^{L} \left( \frac{N-L+k}{N-L-1} a_1^{(N-L+k)} \left( u_1^{(0)} \right)^{k+1} \right) \\
= a_2^{(N-L-1)} + \sum_{k=0}^{L} \left( \frac{N-L+k}{N-L-1} a_2^{(N-L+k)} \left( u_2^{(0)} \right)^{k+1} \right).
$$

We define

$$
(3.33) \quad \psi := u_2^{(0)} - u_1^{(0)}.
$$

Then $\psi \in C^2,\alpha(\Omega)$ and $\psi|_{\partial \Omega} = \partial_z \psi|_{\partial \Omega} = 0$. By using the induction assumption, that (3.29) holds for $k = 0, \ldots, L$, we write the identity (3.32) as

$$
(3.34) \quad a_1^{(N-L-1)} \\
= a_2^{(N-L-1)} + \sum_{k=0}^{L} \left( \frac{N-L+k}{N-L-1} \left[ a_2^{(N-L+k)} \left( u_2^{(0)} \right)^{k+1} - a_1^{(N-L+k)} \left( u_1^{(0)} \right)^{k+1} \right] \right) \\
= a_2^{(N-L-1)} + \sum_{k=0}^{L} \left( \frac{N-L+k}{N-L-1} a_2^{(N-L+k)} \left( u_2^{(0)} \right)^{k+1} \right) \\
- \sum_{m=N-L+k}^{N} \left( \frac{m}{N-L+k} a_2^{(m)} \psi^{m-N-L-k} \left( u_1^{(0)} \right)^{k+1} \right).
$$
Here the induction assumption was used in the last equality. By using binomial expansion, the above equality is

\[ a_1^{(N-L-1)} = a_2^{(N-L-1)} + \sum_{k=0}^{L} \left( \frac{N - L + k}{N - L - 1} \right) a_2^{(N-L+k)} \sum_{i=0}^{k+1} \binom{k+1}{i} \psi^i \left( u_1^{(0)} \right)^{k+1-i} \]

Then

\[ = a_2^{(N-L-1)} + S_1 - S_2. \]

Here we have defined

\[ S_1 := \sum_{k=0}^{L} \left( \frac{N - L + k}{N - L - 1} \right) a_2^{(N-L+k)} \sum_{i=0}^{k+1} \binom{k+1}{i} \psi^i \left( u_1^{(0)} \right)^{k+1-i}, \]

\[ S_2 := \sum_{k=0}^{L} \left( \frac{N - L + k}{N - L - 1} \right) \sum_{m=N-L+k}^{N} \binom{m}{N - L + k} a_2^{(m)} \psi^{m-N-L-k} \left( u_1^{(0)} \right)^k. \]

To complete the proof we compare the coefficients of the powers of \( u_1^{(0)} \) of \( S_1 \) and \( S_2 \). We first observe that in the term \( S_1 \), the powers of \( u_1^{(0)} \) range from 0 to \( L + 1 \). In the term \( S_2 \), the powers of \( u_1^{(0)} \) range from 1 to \( L + 1 \). We split the remaining proof into two cases according to powers of \( u_1^{(0)} \).

**Case 1:**

Let us consider the coefficients of the terms \( \left( u_1^{(0)} \right)^J \), \( J = 1, \ldots, L + 1 \), in \( S_1 \) and \( S_2 \). We observe that the coefficient of \( \left( u_1^{(0)} \right)^J \) in \( S_1 \) is

\[ \sum_{k=0}^{L} \left( \frac{N - L + k}{N - L - 1} \right) a_2^{(N-L+k)} \binom{k+1}{k+1-J} \psi^{k+1-J}. \]

Similarly, the coefficient of \( \left( u_1^{(0)} \right)^J \) in \( S_2 \) is

\[ \binom{N - L + J + 1}{N - L - 1} \sum_{m=N-L+J+1}^{N} \binom{m}{N - L + J + 1} a_2^{(m)} \psi^{m-N-L-J+1} \]

\[ = \binom{N - L + J - 1}{N - L - 1} \sum_{k=0}^{L} \binom{N - L + k}{N - L + J - 1} \psi^{k+1-J}. \]

On the other hand, a direct computation shows that

\[ \binom{N - L + k}{N - L - 1} \binom{k+1}{k+1-J} = \binom{N - L + J - 1}{N - L - 1} \binom{N - L + k}{N - L + J - 1}, \]

so that (3.37) and (3.38) are the same.

**Case 2:**

The term \( S_2 \) does not contain the zeroth power of \( u_1^{(0)} \). We express \( S_1 \) as

\[ S_1 := S_0 + \tilde{S}, \]
that which shows (3.38)

\[ S := \sum_{k=0}^{L} \binom{N-L+k}{N-L-1} a_2^{(N-L+k)} \psi^{k+1}. \]

(3.39)

By redefining the summation index of \( S_0 \), we have

\[ S_0 = \sum_{m=N-L}^{N} \binom{m}{N-L-1} a_2^{(m)} \psi^m N L + 1. \]

(3.40)

Therefore, by plugging (3.37)–(3.40) into (3.34), we obtain

\[
a_1^{(N-L+1)} = a_2^{(N-L+1)} + \sum_{m=N-L}^{N} \binom{m}{N-L-1} a_2^{(m)} \psi^m N L + 1
\]

\[ = \sum_{m=N-(L+1)}^{N} \binom{m}{N-L-1} a_2^{(m)} \psi^m N L + 1. \]

This proves the induction step. It remains to prove (1.19).

Recall that the nonlinearity \( a_j(x,z) = \sum_{k=1}^{N} a_j^{(k)} z^k \), for \( j = 1, 2 \), then we can write \( a_1(x, u_1^{(0)}) \) in terms of

\[ a_1(x, u_1^{(0)}) = \sum_{k=0}^{N-1} a_1^{(N-k)} \psi^k (u_1^{(0)})^{N-k} \]

(3.41)

On the other hand, one can also express

\[ a_2(x, u_2^{(0)}) = \sum_{k=1}^{N} a_2^{(k)} (u_2^{(0)})^k = \sum_{k=1}^{N} a_2^{(k)} (u_2^{(0)})^{k} \psi^{k-m}, \]

(3.42)

where we used (3.33) and binomial expansion in the above computation. Similar to the computations of Case 1 in preceding arguments, by comparing the orders of the homogeneous parts \( (u_1^{(0)})^L \), for \( L = 1, 2, \ldots, N \), a direct computation yields that

\[ a_2(x, u_2^{(0)}) - a_1(x, u_1^{(0)}) = \sum_{k=1}^{N} a_2^{(k)} (u_2^{(0)})^k \psi^k. \]

Therefore,

\[ F_1 - F_2 = \Delta \left( u_1^{(0)} - u_2^{(0)} \right) + a_1(x, u_1^{(0)}) - a_2(x, u_2^{(0)}) \]

\[ = - \Delta \psi \sum_{k=1}^{N} a_2^{(k)} (u_2^{(0)})^k \psi^k, \]

which shows (1.19). This proves the assertion. \( \square \)

We next prove that is the sources \( F_1 \) and \( F_2 \) are known in Theorems 1.1–1.3, then it is possible to determine the coefficients uniquely. We have the following corollary, which we formulate in terms of the general polynomial nonlinearity.
Corollary 3.2. Let us adopt the notation and assumptions in Theorem 1.3. If $F_1 = F_2$ in $\Omega$, then we have

$$a^{(k)}_1 = a^{(k)}_2 \text{ in } \Omega,$$

for $k = 1, 2, \ldots, N$.

Proof. By using (1.19), we have

$$\Delta \psi + \sum_{k=1}^{N} a^{(k)}_2 \psi = 0 \text{ in } \Omega,$$

where $\psi \in C^{2, \alpha}(\overline{\Omega})$ is defined via (3.33), which is a bounded function. Since $a^{(k)}_2 \in C^{\alpha}(\Omega)$ for $k = 1, 2, \ldots, N$, (3.43) implies that

$$\left\{ \begin{array}{l}
|\Delta \psi| \leq C|\psi| \text{ in } \Omega, \\
\psi = \partial_\nu \psi = 0 \text{ on } \partial \Omega,
\end{array} \right.$$

for some constant $C > 0$. Applying the unique continuation for differential inequalities (see e.g. [JK85]), one obtains that $\psi = 0$ in $\Omega$. Finally, combining with the relations (1.18), we obtain the uniqueness of coefficients. (To easily see how this final argument goes, see the cubic case and (3.15) first.) \[\square\]

4. Case studies of Theorem 1.4

In the end of this paper, we study special cases Theorem 1.4, which stated that

$$\partial^k z a^{(1)}_1(x,u^{(0)}_1(x)) = \partial^k z a^{(2)}_2(x,u^{(0)}_2(x)), \quad x \in \Omega, \quad k \in \mathbb{N}.$$ (4.1)

In general, given only the conditions (4.1), it is not clear how (or even if it is possible) to find an explicit relation between the coefficients $(a^{(1)}_1(x,z), F^{(1)}_1(x))$ and $(a^{(2)}_2(x,z), F^{(2)}_2(x))$ in terms of $\psi = u^{(0)}_2 - u^{(0)}_1$. This final section of this paper consider examples where the relation is explicit.

4.1. Exponential nonlinearity.

Proof of Theorem Corollary 1.5. We prove cases 1 and 2 separately:

Case 1.

The nonlinearity in this case is $a^{j}_1(x,z) = q^{j}_1(x)e^z$. Let $u^{(0)}_j$ be the solution to

$$\left\{ \begin{array}{l}
\Delta u^{(0)}_j + q^{j}_1(x)e^{u^{(0)}_j} = F^{(j)}_1 \text{ in } \Omega, \\
u^{(0)}_j = f_0 \text{ on } \partial \Omega,
\end{array} \right.$$ (4.2)

for $j = 1, 2$. Here $f_0 \in \mathcal{N}$. Using (4.1) with $k = 1$, we have

$$q^{1}_1 e^{u^{(0)}_1} = \partial_2 a^{1}_1(x, u^{(0)}_1) = \partial_2 a^{2}_2(x, u^{(0)}_2) = q^{2}_2 e^{u^{(0)}_2} \text{ in } \Omega.$$ (4.3)

On the other hand, by taking $u^{(0)}_2 = u^{(0)}_1 + \psi$ in $\Omega$, by (4.3) one has $q^{1}_1 e^{u^{(0)}_1} = q^{2}_2 e^{u^{(0)}_2} + \psi$ which implies $q^{1}_1 = q^{2}_2 e^{\psi}$ in $\Omega$. Then, by using (4.2), we have

$$F^{(2)}_2 - F^{(1)}_1 = \Delta (u^{(0)}_2 - u^{(0)}_1) + q^{2}_2 e^{u^{(0)}_2} - q^{1}_1 e^{u^{(0)}_1} = \Delta \psi \text{ in } \Omega,$$

where we have utilized (4.3). This shows (1.25).

For the converse statement, we note that if

$$q^{1}_1 = q^{2}_2 e^{\psi} \text{ and } F^{(1)}_1 = F^{(2)}_2 - \Delta \psi,$$
and we set \( u_2 = u_1 + \psi \), then
\[
\Delta u_1 + q_1 e^{u_1} = F_1 \iff \Delta u_2 - \Delta \psi + q_2 e^\psi e^{u_2 - \psi} = F_2 - \Delta \psi \\
\iff \Delta u_2 + q_2 e^{u_2} = F_2.
\]
Since \( \psi|_{\partial \Omega} = \partial_n \psi|_{\partial \Omega} = 0 \), we have the converse statement.

**Case 2.**

In this case \( a_j(x, z) = q_j(x) z e^z \). Let \( u_j^{(0)} \) be the solution of
\[
(4.4) \quad \begin{cases}
\Delta u_j^{(0)} + q_j u_j^{(0)} e^{u_j^{(0)}} = F_j & \text{in } \Omega, \\
u_j^{(0)} = f_0 & \text{on } \partial \Omega,
\end{cases}
\]
for \( j = 1, 2 \). The condition (4.1) for \( k = 1 \) yields
\[
(4.5) \quad Q := q_1 \left( u_1^{(0)} + 1 \right) e^{u_1^{(0)}} = q_2 \left( u_2^{(0)} + 1 \right) e^{u_2^{(0)}} \text{ in } \Omega,
\]
and for \( k = 2 \) it yields
\[
(4.6) \quad q_1 \left( u_1^{(0)} + 2 \right) e^{u_1^{(0)}} = q_2 \left( u_2^{(0)} + 2 \right) e^{u_2^{(0)}} \text{ in } \Omega.
\]
Combining (4.5) and (4.6), we obtain
\[
(4.7) \quad q_1 e^{u_1^{(0)}} = q_2 e^{u_2^{(0)}} \quad \text{and} \quad q_1 u_1^{(0)} e^{u_1^{(0)}} = q_2 u_2^{(0)} e^{u_2^{(0)}} \text{ in } \Omega.
\]
By the first identity of (4.7), we have \( q_1 = q_2 e^{u_2^{(0)} - u_1^{(0)}} \) in \( \Omega \). The second identity of (4.7) shows that \( q_2 u_1^{(0)} e^{u_1^{(0)}} = q_2 u_2^{(0)} e^{u_2^{(0)}} \) in \( \Omega \). Since \( q_2 \neq 0 \) in \( \Omega \), we must have \( u_1^{(0)} = u_2^{(0)} \) in \( \Omega \), which implies that \( F_1 = F_2 \) in \( \Omega \), where we utilized the equation (4.4). Moreover, by the first identity of (4.7) and \( u_1^{(0)} = u_2^{(0)} \) in \( \Omega \), we can derive \( q_1 = q_2 \) in \( \Omega \). This proves the assertion.

If \( F_1 = F_2 \) in \( \Omega \) in Corollary 1.5, we have the following uniqueness result regarding the Case 1 in the above corollary.

**Corollary 4.1.** Let us assume as in the Case 1 of Corollary 1.5 and adopt its notation. If additionally \( F_1 = F_2 \), then
\[
q_1 = q_2 \text{ in } \Omega.
\]

**Proof.** Since the source terms in Corollary 1.5 satisfy \( F_1 = F_2 \) in \( \Omega \), it follows from (1.25) that \( \Delta \psi = 0 \) in \( \Omega \) with \( \psi|_{\partial \Omega} = \partial_n \psi|_{\partial \Omega} = 0 \). By using the unique continuation principle, we conclude that \( \psi \equiv 0 \) in \( \Omega \). Therefore, combining with (4.3), we must have \( q_1 = q_2 \) in \( \Omega \) as desired.

4.2. **The sine-Gordon equation.** It remains to prove Corollary 1.6.

**Proof of Corollary 1.6.** We divide the proof into two steps:

**Step 1. Gauge invariance.**

Let \( u_j^{(0)} \) be the solution of
\[
(4.8) \quad \begin{cases}
\Delta u_j^{(0)} + a_j(x, u_j^{(0)}) = F_j & \text{in } \Omega, \\
u_j^{(0)} = f_0 & \text{on } \partial \Omega,
\end{cases}
\]
for \( j = 1, 2 \) and where \( f_0 \in \mathcal{V} \). By Theorem 1.4, we have \( \partial_k^2 a_1(x, u_1^{(0)}) = \partial_k^2 a_2(x, u_2^{(0)}) \), for \( k = 1, 2 \), which implies that
\[
(4.9) \quad q_1 \cos u_1^{(0)} = q_2 \cos u_2^{(0)} \quad \text{and} \quad q_1 \sin u_1^{(0)} = q_2 \sin u_2^{(0)} \text{ in } \Omega.
\]
By the Euler identity, we have $e^{iy} = \cos y + i \sin y$, where $i = \sqrt{-1}$. Then (4.9) is equivalent to
\begin{equation}
q_1 e^{iu_1^{(0)}} = q_2 e^{iu_2^{(0)}} \text{ in } \Omega.
\end{equation}
By defining $\psi = u_2^{(0)} - u_1^{(0)}$, we have that $\psi \in C^{2,\alpha}(\bar{\Omega})$ and $\psi = \partial_y \psi = 0$ on $\partial \Omega$. Via the second identity of (4.9) and (4.8), one has
\[ \Delta \psi = \Delta (u_2^{(0)} - u_1^{(0)}) = F_2 - F_1 \text{ in } \Omega, \]
and by (4.10),
\[ q_1 e^{iu_1^{(0)}} = q_2 e^{iu_2^{(0)} + \psi} \text{ in } \bar{\Omega}, \]
which implies $q_1 = q_2 e^{i\psi}$ in $\bar{\Omega}$. Furthermore, since $q_1$ and $q_2$ are real-valued functions and $\psi$ is continuous, we must have either $e^{i\psi} = -1$ or $e^{i\psi} = 1$ in $\Omega$. Thus
\begin{equation}
q_1 = \pm q_2 \text{ in } \bar{\Omega}.
\end{equation}
It remains to show that
\begin{equation}
e^{i\psi} = 1 \text{ in } \bar{\Omega}.
\end{equation}

**Step 2. Boundary determination.**

We show by using boundary determination that $\psi \equiv 1$ in $\bar{\Omega}$. Let $\epsilon$ be a small real parameter, $g \in C^{2,\alpha}(\partial \Omega)$ and $f = f_0 + \epsilon g$. By linearizing (1.29) around the solution $u_j^{(0)}$ of (4.8), one has
\begin{equation}
\begin{cases}
\left( \Delta + q_j \cos u_j^{(0)} \right) v_j = 0 & \text{in } \Omega, \\
v_j = g & \text{on } \partial \Omega,
\end{cases}
\end{equation}
for $j = 1, 2$. Now, by applying standard boundary determination for the linear Schrödinger equation (4.13), one can determine that
\[ q_1 \cos(f_0 + \epsilon g) = q_1 \cos u_1^{(0)} = q_2 \cos u_2^{(0)} = q_2 \cos(f_0 + \epsilon g) \text{ on } \partial \Omega. \]
In particular, for $\epsilon = 0$, the above identity shows that
\[ q_1 \cos(f_0) = q_2 \cos(f_0) \text{ on } \partial \Omega. \]
If $\cos(f_0) \equiv 0$ on $\partial M$, we can slightly perturb $f_0$ so that there is $x_0 \in \partial \Omega$ with $\cos(f_0(x_0)) \neq 0$ and repeat the above argument again. We deduce that $e^{i\psi(x_0)} = 1$, and since $\psi$ is constant, we conclude that
\[ q_1 = q_2 \text{ in } \Omega. \]
This proves the claim. \qed

**Appendix A. Proof of Proposition 2.1**

In the end of this work, let us prove Proposition 2.1. The proof is almost identical to the proof of Theorem 2.1, but Proposition 2.1 does not exactly follow from Theorem 2.1. A very similar proof can be found from the work [LLLS21, Section 2].

**Proof of Proposition 2.1.** Let
\[ B_1 = C^{2,\alpha}(\partial \Omega), \quad B_2 = C^{2}(\bar{\Omega}), \quad B_3 = C^{2,\alpha}(\bar{\Omega}), \quad B_4 = C^{2,\alpha}(\bar{\Omega}) \times C^{2,\alpha}(\partial \Omega) \]
and consider the map
\[ \Psi : B_1 \times B_2 \times B_3 \to B_4, \]
\[ (f, F, u) \mapsto (\Delta u + a(x, u) - F, u|_{\partial \Omega} - f). \]
Similar to [LLLS21, Section 2], one can show that the map $u \mapsto a(x, u)$ is a $C^\infty$ map from $C^{2,\alpha}(\bar{\Omega}) \to C^{2,\alpha}(\bar{\Omega})$. 

Notice that $\Psi(0,0,0) = (0,0)$, where we have used condition (1.2). The first linearization of $\Psi = \Psi(f,F,u)$ at $(0,0,0)$ with respect to the variable $u$ is

$$D_u\Psi|_{(0,0,0)}(v) = (\Delta v + \partial_z a(x,0)v, v|_{\partial\Omega}),$$

which is a homeomorphism $B_3 \to B_4$ by the condition

$$0 \text{ is not a Dirichlet eigenvalue of } \Delta + \partial_z a(x,0) \text{ in } \Omega.$$

This is guaranteed by well-posedness and Schauder estimates for the linear second order elliptic equation.

Now, the implicit function theorem in Banach spaces [RR06, Theorem 10.6 and Remark 10.5] yields that there are $\varepsilon, \delta > 0$ and a neighborhood $N_\delta \times A_\varepsilon \subset C^{2,\alpha}(\partial\Omega) \times C^\alpha(\overline{\Omega})$ and a $C^\infty$ map $S : N_\delta \times A_\varepsilon \to B_3$ such that

$$\Psi(f,F,S(f,F)) = (0,0),$$

whenever $\|f\|_{C^{2,\alpha}(\partial\Omega)} \leq \delta$ and $\|F\|_{C^\alpha(\overline{\Omega})} \leq \varepsilon$. Since $S$ is smooth and $S(0,0) = 0$, the solution $u = S(f,F)$ satisfies

$$\|u\|_{C^{2,\alpha}(\partial\Omega)} \leq C \left( \|f\|_{C^{2,\alpha}(\partial\Omega)} + \|F\|_{C^\alpha(\overline{\Omega})} \right).$$

Furthermore, by the uniqueness statement of the implicit function theorem, $u = S(f,F)$ is the only solution to $\Psi(f,F,u) = (0,0)$ whenever $\|f\|_{C^{2,\alpha}(\partial\Omega)} + \|F\|_{C^\alpha(\overline{\Omega})} \leq \delta + \varepsilon$, and

$$\|u\|_{C^{2,\alpha}(\partial\Omega)} \leq C (\varepsilon + \delta).$$

This can be achieved by redefining $\varepsilon, \delta > 0$ to be smaller if necessary. As in [LLLS21], one can check that the solution operator $S : N_\delta \times A_\varepsilon \to C^{2,\alpha}(\overline{\Omega})$ is a $C^\infty$ map in the Fréchet sense. The normal derivative is a linear map $C^{2,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$. Thus for a fixed $F \in A_\varepsilon$, $A_F : f \mapsto \partial_n u_{f,F}$, where $u_{f,F}$ solves $\Delta u_{f,F} + a(x, u_{f,F}) = f$ with $u_{f,F}|_{\partial\Omega} = f$, is also a well defined $C^\infty$ map $N_\delta \to C^{1,\alpha}(\partial\Omega)$. \(\square\)

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