A SIMPLE METHOD FOR COMPUTING THE NONLINEAR MASS CORRELATION FUNCTION WITH IMPLICATIONS FOR STABLE CLUSTERING

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ABSTRACT

We propose a simple and accurate method for computing analytically the mass correlation function for cold dark matter and scale-free models that fits N-body simulations over a range that extends from the linear to the strongly nonlinear regime. The method, based on the dynamical evolution of the pair-conservation equation, relies on a universal relation between the pairwise velocity and the smoothed correlation function valid for high- and low-density models, as derived empirically from N-body simulations. An intriguing alternative relation, based on the stable-clustering hypothesis, predicts a power-law behavior of the mass correlation function that disagrees with N-body simulations but conforms well to the observed galaxy correlation function if negligible bias is assumed. The method is a useful tool for rapidly exploring a wide span of models and, at the same time, raises new questions about large-scale structure formation.

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Understanding the origin and evolution of the clustering pattern of galaxies is one of the most important goals of cosmology. Until now, this problem has been investigated using a fourfold path: (1) perturbation theory (for a review of recent advances, see Juszkiewicz & Bouchet 1995 and references therein); (2) a kinetic description, adapted from the BBGKY hierarchy, used in plasma physics (Peebles 1980, § IV); (3) N-body simulations (e.g., Jenkins et al. 1998, hereafter VIRGO); and (4) semianalytic fits to N-body results, based on the so-called universal scaling hypothesis (see Hamilton et al. 1991; Jain, Mo, & White 1995; Peacock & Dodds 1996; Mo, Jing, & Boerner 1997; Ma 1998; Ma et al. 1999). The advantages and limitations of these methods are often complementary. For example, applying perturbation theory often leads to analytic results for a wide class of models, while the N-body simulations allow us to study only one model at a time. On the other hand, perturbation theory works only in the weakly nonlinear regime, while N-body experiments describe the fully nonlinear dynamics, albeit over a limited dynamical range. The subject of the present study is an analytic Ansatz for the evolution of the two-point correlation function of density fluctuations spanning the linear and nonlinear regimes, which builds on all four methods described above.

Our approach is based on the pair-conservation equation, which relates the mean (pair-weighted) relative velocity of a pair of particles to the time evolution of the correlation function in a self-gravitating gas:

\[
\frac{a}{3[1 + \xi(x, a)]} \frac{\partial \tilde{\xi}(x, a)}{\partial a} = -\frac{v_{12}(x, a)}{Hr},
\]

where \(a(t)\) is the expansion factor with \(a = 1\) at present, \(r = ax\) is the proper separation, and \(H(a)\) is the Hubble parameter (see Davis & Peebles 1977 and Peebles 1980). Here \(\xi(x, a)\) represents the two-point correlation function averaged over a ball of comoving radius \(x\):

\[
\tilde{\xi}(x, a) = \frac{3}{x^3} \int_0^x \xi(y, a)y^2 dy.
\]

The approximate solution of equation (1) is known in the large separation limit, where \(\xi \ll 1\) (linear regime); the stable-clustering hypothesis is often invoked to describe the small separation limit, where \(\xi \gg 1\) (nonlinear regime). Hence, equation (1) is “a guide to speculation on the behavior of the correlation function” (Peebles 1980, p. 268) since an assumed \(v_{12}\) implies a function \(\xi\) that should agree with the weak and strong field limits and interpolates between these limits in a reasonable way.

An approximate universal relation between the pairwise velocity and the smoothed correlation function has been conjectured by Hamilton et al. (1991) on the basis of N-body simulation results and has been further explored in Nityananda & Padmanabhan (1994) and Padmanabhan & Engineer (1998). In the past, the relation has been used in attempts to derive a general functional that converts directly from a linear to a nonlinear mass correlation. In this Letter, we present a simple extension of the relation that applies to both high- and low-density models, but we take a different approach to obtaining the nonlinear correlation function. Namely, we use the universal relation to close equation (1); we then evolve the resulting partial differential equation to compute the nonlinear correlation function. This turns out to be a fast and surprisingly accurate method that matches N-body results for a wide variety of cold dark matter (CDM) models. As a stand-alone computer program, the algorithm can be adopted on a programmable calculator, or it can easily be incorporated in more sophisticated programs that predict other cosmological properties, such as CMBFAST (Seljak & Zaldarriaga 1996).

Figure 1 clearly illustrates the nearly model-independent relationship between the pairwise velocity and the smoothed correlation function, observed in N-body simulations for a wide
range of perturbation spectra. We define this relation as

\[ V(f \xi) = -\frac{\ln \xi}{Hr}. \]  

(3)

Compared with Hamilton et al. (1991), a novel feature is plotting the relation in terms of \( f(\Omega \xi) \), where \( f \equiv d \ln Dl/d \ln a \) is the standard linear density growth factor rather than \( \xi \) alone—a difference that is essential for extending the relation to low-density models. The evolved, nonlinear clustering of scale-free and OCDM models produces a very similar relation between \(-v_c/2Hr\) and \( \xi \). An excellent fit to the functional relation \( V(f \xi(x, a)) \) in Figure 1, based on the \( n = -1 \) curve, is given by

\[ V(x) = \begin{cases} 
(2/3)x, & x < 0.15, \\
0.7x \exp(-0.31x^{0.61}), & 0.15 \leq x < 20, \\
3.3x^{-0.17}, & x \geq 20,
\end{cases} \]  

(4)

valid for \( x \leq 10^3 \). In this Letter, we use this fitting formula, designed for the \( n = -1 \) curve, as the expression for \( V(x) \) in equation (1) to be applied to all models. We find this to be sufficient to reproduce \( N \)-body results to within 10% accuracy over the range of models and scales shown in Figure 2, which extends deep into the nonlinear regime. To push to lower density models and improve the accuracy further, it would be simple to modify the algorithm for \( V(x) \) to include the \( \Omega \)-dependent rise near \( x \approx 20 \).

Starting with the linear correlation function, and armed only with information about the background cosmological evolution, we propose to obtain dynamically the nonlinear \( \xi \) as a function of separation and time. Here then is our idealized procedure in three steps:

1. Reformulate.—We first rewrite the partial differential equation as

\[ \frac{\partial \ln \xi}{\partial \ln a} = \frac{3}{\xi} \frac{(1 + \xi)}{V(f \xi)}. \]  

(5)

where \( V(x) \) is given in equation (4).

2. Initialize.—The initial conditions are set by the linear correlation function at a redshift \( z = -1 + 1/a \), such that \( \xi(x, a) \) is less than unity for all \( x \) of interest. Our procedure assumes that only the amplitude and not the shape of the linear correlation function has changed over this interval, as occurs for CDM models with a primordial spectrum of adiabatic density perturbations. Hence, this procedure will not apply to cosmological models with a late-time—decaying neutrino but will apply to hot dark matter models wherein the shape of the linear power spectrum is set by redshift \( z \sim 100 \).

3. Evolve.—We numerically solve the partial-differential equation and evolve \( \xi \). At each step in the evolution, we use \( \xi = \xi \times (1 - \gamma/3) \) with \( \gamma \equiv -d \ln \xi /d \ln x \) to determine the
correlation function. The value of $f$ is updated at each step in $a$ as appropriate for the cosmology.

The remarkable results are shown in Figure 2. Here we see that our simple procedure is in excellent agreement with $N$-body simulations. Based on the span of behavior in the cosmic time evolution and the shape of correlation function, we expect that this procedure should be valid for a wide range of cosmological models, including quintessence (Caldwell, Dave, & Steinhardt 1998) and models with tilted spectra.

Figure 2 demonstrates that we have obtained a simple and powerful new tool for rapidly and accurately obtaining the nonlinear power spectrum for a wide range of models. However, the physical origin of the nearly model-independent relation $V(\xi)$ is not understood in detail. In the linear regime, perturbation theory predicts $-v_{12}/Hr = \xi f\xi$. In the nonlinear regime, Padmanabhan et al. (1996) have suggested that insight may be obtained by comparison with the case of the gravitational collapse of a spherical top-hat mass distribution. Using their solution (eqs. [16]–[19] in their paper), we find that $-v_{12}/Hr$ is linearly proportional to $f\xi$ times a slowly decreasing function of $\xi$ for a surprisingly wide range of $\xi \gg 1$, including $\xi = 10$, the turnover point in Figure 1. In particular, for $\xi = 10$, $-v_{12}/Hr = 1.77f(\Omega)$, which is similar to $-v_{12}/Hr \approx 2f(\Omega)$ in Figure 1.

In the strongly nonlinear regime ($\xi \gg 10$), Figure 1 shows a visible difference in the shape of $V(f\xi)$ between the high- and low-density models. This may be due to the suppression of linear growth, which occurs at late times in low-density models and leads to the enhanced clustering on small scales relative to large scales. However, this has a negligible effect on the computed nonlinear correlation function. For example, using the curve for $\Lambda$CDM shown in Figure 1 as the basis of $V$ in our procedure, we find that the amplitude of the nonlinear correlation function differs by only 10% at $r \sim 0.1$ Mpc $h^{-1}$. For the models shown, this accuracy is comparable to what is obtained by Hamilton et al. (1991) and Peacock & Dodds (1996). The advantage here is that our method can be immediately applied to new types of CDM models (e.g., quintessence cosmologies) without having to run new $N$-body simulations to fix fitting parameters.

An important issue raised by our Ansatz is the validity of the stable-clustering hypothesis. The stable-clustering regime corresponds to the limit where particle pairs detach from the Hubble flow and $-v_{12}/Hr \to 1$. Figure 1 shows that $-v_{12}/Hr$ first overshoots unity by a factor of 2 and then rebounds toward unity. However, it is not clear whether it converges to unity at $\xi \approx 1000$ or whether it possibly oscillates if the simulations are extended to higher values of $\xi$.

It is interesting to compare the predictions of our Ansatz if the relation between $-v_{12}/Hr$ and $\xi$ is modified to enforce more rapid convergence to stable clustering. A ready example is an alternative Ansatz based on the pair-conservation equation recently proposed by Juszkiewicz, Springel, & Durrer (1999, hereafter JSD). Their Ansatz for $v_{12}$, based on an interpolation between the behavior predicted by perturbation theory in the weakly nonlinear regime and stable clustering in the strongly nonlinear regime, is given by

$$v_{12}(x, a) = -\frac{2}{3}Hrf\bar{\xi}(x, a)[1 + \alpha \bar{\xi}(x, a)],$$

where $\bar{\xi} = \xi/(1 + \xi)$ and $\alpha$ is a function that controls the strength of the nonlinear feedback. Here we use $\alpha = 1.8 - 1.1\gamma$, based on perturbation theory (see Scoccimarro & Frieman 1998), where $\gamma$ is the slope of the correlation function at $\xi = 1$. The key point, as shown in Figure 3, is that the pairwise velocity rapidly approaches the stable-clustering limit by $\xi \sim 10$ and remains there to within $\sim 20\%$ on smaller scales in the more strongly nonlinear regime. This means particle pairs separated by $\lesssim 1$ Mpc $h^{-1}$ have the rough behavior of virialized objects, such as clusters and galaxies.

The correlation function $\xi$ obtained by closing the pair-conservation equation with equation (6), as shown in Figure 3, displays a power-law behavior in the nonlinear regime with index $-1.7$, in disagreement with $N$-body simulations of CDM but curiously similar to the galaxy correlation function observed in the APM survey (Maddox, Efstathiou, & Sutherland 1996). This result is not unique to the JSD Ansatz; substituting any shape similar to that shown in the left panel of Figure 3 for $V(\xi\xi)$ into our Ansatz would produce a similar effect on the mass correlation function. This leads us to speculate on the possibility that we can directly apply the pair-conservation equation to galaxies. The Ansatz illustrated in Figure 3, which enforces stable clustering by fiat, may be implicitly describing a model of galaxy clustering that incorporates the necessary features leading to the observed power-law behavior. That is, the relation between $-v_{12}/Hr$ and $f\xi$ that enforces stable clustering may include the requisite source and sink terms (some important physical feature; e.g., merging or the mechanics of galaxy formation, or some property of the dark matter such as collisional or dissipative interactions) to permit the pair-conservation equation to describe galaxy clustering. Indeed, recent attempts at semianalytic modeling of the galaxy-galaxy correlation function (e.g., Benson et al. 2000) find that the processes that lead to the power-law behavior are coincident with reduced pairwise velocities for galaxies relative to the dark matter on small scales. The difference between $N$-body simulations and observations, or the difference between dark matter and galaxies, whether due to bias in the conventional picture or to something more radical, such as a modification to the collisionless CDM scenario, can perhaps be determined empirically by studying redshift distortion on the scales in which the $N$-body simulation suggests an overshoot in $-v_{12}/Hr$ and the Ansatz of Figure 3 does not.

In sum, our studies have produced a simple recipe for computing the nonlinear power spectrum for a wide range of
A key feature of the method is the universal function relating the pair velocity to the mass correlation function that does not converge rapidly to the stable-clustering limit but rather overshoots it by a factor of 2. This feature is responsible for the mass correlation function not approaching a power law. Our studies have also raised several interesting issues in structure formation. For example, why is $-v_{12}/Hr$ versus $f_k^2$ so similar for a wide range of models? Can the universal relation be derived from theory? For what range of models is the relation model-independent? How does the universal relation ultimately approach the stable-clustering limit at small scales (if it does at all)? Does the success of a universal relation based on the stable-clustering hypothesis, as in Figure 3, suggest a viable alternative explanation for the power-law behavior of the galaxy correlation function?

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