Research Article

Picture Fuzzy Rough Set and Rough Picture Fuzzy Set on Two Different Universes and Their Applications

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The major concern of this article is to propose the notion of picture fuzzy rough sets (PFRSs) over two different universes which depend on \((δ, ζ, ϑ)\)-cut of picture fuzzy relation \(R\) on different universes (i.e., by combining picture fuzzy sets (PFSs) with rough sets (RSs)). Then, we discuss several interesting properties and related results on the PFRSs. Furthermore, we define some notions related to PFRSs such as (Type-I/Type-II) graded PFRSs, the degree \(α\) and \(β\) with respect to \(R_{[δ, ζ, ϑ]}\) on PFRSs, and (Type-I/Type-II) generalized PFRSs based on the degree \(α\) and \(β\) with respect to \(R_{([δ, ζ, ϑ])}\) and investigate the basic properties of above notions. Finally, an approach based on the rough picture fuzzy approximation operators on two different universes in decision-making problem is introduced, and we give an example to show the validity of this approach.

1. Introduction

In the past few years, Pawlak [1] proposed the notion of RS as a mathematical tool to handle with ambiguity and incomplete information systems. The lower/upper approximations (i.e., rough sets) are firstly described through the equivalence classes. That is to say, many datasets cannot be treated properly by way of classical rough sets. In mild of this, the graded rough sets [2], similarity or tolerance relations [3–5], arbitrary binary relation [6, 7], and variable precision rough sets [8, 9] are a few extensions of the classical rough sets. So, several researchers, for example, Dubois and Prade [10], presented the concept of fuzzy rough set (FRS) (i.e., the fuzzy set (FS) [11] and the RS). Many researchers have worked on fuzzy rough models (see [12–16]).

Wong et al. [17] presented the notion of the RS model over two universes and its application. Several applications and the fundamental properties of the FRS model on two universes are studied [18–32]. Yao and Lin [33, 34] proposed the notion of graded rough sets (GRSs) on one universe. Zhang et al. [35] gave a comparative between the variable precision rough set (VPFRS) and the GRS. In addition to the previous studies, Liu et al. [36] introduced the notion of GRSs on two universes. Yu et al. [37] presented the notion of a variable precision-graded rough set (VPGRSs) over two universes and Yu and Wang [38] presented a novel type of GRS with VP over two inconsistent universes.

In this paper, we propose the notions of PFRSs and RPFRSs over two different universes. The basic properties of PFRSs based on \((δ, ζ, ϑ)\)-cut of picture fuzzy relation \(R\) over two different universes are discussed. Meanwhile, we propose two types of graded picture fuzzy rough sets (GPFRSs) based on \((δ, ζ, ϑ)\)-cut of \(R\) on two different universes: type-I PFRS is according to the graded \(n\) with respect to \(R_{[δ, ζ, ϑ]}\) and type-II PFRS is according to the graded \(n\) with respect to \(R_{([δ, ζ, ϑ])}\). The interesting properties of Type-I/Type-II PFRSs are investigated in detail. Furthermore, we define the notions of PFRS according to the degree \(α\) and \(β\) with respect to \(R_{[δ, ζ, ϑ]}\) and Type-I/Type-II generalized PFRSs according to the degree \(α\) and \(β\) with respect to \(R_{([δ, ζ, ϑ])}\). The main results of the above notions are studied and explored. Finally, an application of rough picture fuzzy model over two different universes is presented to solve the decision-making problem.

Sections of this article are arranged as follows. In Section 2, we gave the concepts of PFSs and picture fuzzy relations. In Section 3, we give the notion of PFRSs based on
(δ, ξ, θ)-cut of picture fuzzy relation \( R \) over two different universes and study some interesting properties on PFSSs. In Section 4, an algorithm is constructed and an application on PFSSs over two different universes in decision-making problem is explored. Lastly, conclusion is discussed in Section 5.

2. Preliminaries

2.1. Picture Fuzzy Sets and Picture Fuzzy Relations. Cuong [39–41] introduced the notion of PFS is an extension of fuzzy

\[
\mathcal{A} = \left\{ \left( p_1 \circ \mathcal{A}(u_1), p_2 \circ \mathcal{A}(u_1), p_3 \circ \mathcal{A}(u_1) \right), \left( p_1 \circ \mathcal{A}(u_2), p_2 \circ \mathcal{A}(u_2), p_3 \circ \mathcal{A}(u_2) \right), \ldots, \left( p_1 \circ \mathcal{A}(u_n), p_2 \circ \mathcal{A}(u_n), p_3 \circ \mathcal{A}(u_n) \right) \right\},
\]

Moreover, \( 1 - p_1 \circ \mathcal{A}(u) - p_2 \circ \mathcal{A}(u) - p_3 \circ \mathcal{A}(u) \) is called the refusal degree of \( u \) (\( u \in U \)). A PFS \( \mathcal{A} \in \mathcal{U}^U \) with refusal degree \( 0 \) at each point \( u \in U \) can be identified with an IFS on \( U \) and \( \mathcal{U} \) with the pointwise order \( \leq \) is the set of all mappings from a set \( U \) (or an universe) to \( \{1, a_1, a_2, a_3\} \in [0,1]^3|a_1 + a_2 + a_3 = 1\}. Then, each element \( \mathcal{A} \) of \( \mathcal{I}^U \) is called an \( l \)-set or a PFS on \( U \), \( p_1 \circ \mathcal{A}(u) \) (i.e., the degree of positive), \( p_2 \circ \mathcal{A}(u) \) (i.e., the degree of neutral), and \( p_3 \circ \mathcal{A}(u) \) (i.e., the degree of negative) of the element \( u \in U \), where \( p_i : [0,1]^3 \rightarrow [0,1] \) (i.e., the \( i \)th projection from \([0,1]^3 \) to \([0,1] \) \( i = 1, 2, 3 \)).

2.2. Picture Fuzzy Relations. Let \( \mathcal{A} \), \( \mathcal{B} \in \mathcal{U}^U \). Then,

\[
\mathcal{A} \circ \mathcal{B} = \left\{ \left( p_1 \circ \mathcal{A}(u_1), p_2 \circ \mathcal{A}(u_2), p_3 \circ \mathcal{A}(u_3) \right), \left( p_1 \circ \mathcal{B}(u_2), p_2 \circ \mathcal{B}(u_2), p_3 \circ \mathcal{B}(u_3) \right), \ldots, \left( p_1 \circ \mathcal{B}(u_n), p_2 \circ \mathcal{B}(u_n), p_3 \circ \mathcal{B}(u_n) \right) \right\}.
\]

The union \( \bigcup_{k \in K} \mathcal{A}_k \) (called also supremum \( \vee_{k \in K} \mathcal{A}_k \)) and the intersection \( \bigcap_{k \in K} \mathcal{A}_k \) (called also infimum \( \wedge_{k \in K} \mathcal{A}_k \)) of a family \( \{ \mathcal{A}_k : k \in K \} \subseteq \mathcal{U}^U \) can be defined by the following formulae:

\[
\bigcup_{k \in K} \mathcal{A}_k (u) = \left( \bigvee_{k \in K} p_1 \circ \mathcal{A}_k (u), \bigvee_{k \in K} p_2 \circ \mathcal{A}_k (u), \bigvee_{k \in K} p_3 \circ \mathcal{A}_k (u) \right),
\]

\[
\bigcap_{k \in K} \mathcal{A}_k (u) = \left( \bigwedge_{k \in K} p_1 \circ \mathcal{A}_k (u), \bigwedge_{k \in K} p_2 \circ \mathcal{A}_k (u), \bigwedge_{k \in K} p_3 \circ \mathcal{A}_k (u) \right).
\]

\( \mathcal{A} \) is a subset of \( \mathcal{B} \) (i.e., \( p_1 \circ \mathcal{A}(u) \leq p_1 \circ \mathcal{B}(u) \), \( p_2 \circ \mathcal{A}(u) \leq p_2 \circ \mathcal{B}(u) \), and \( p_3 \circ \mathcal{A}(u) \leq p_3 \circ \mathcal{B}(u) \), for each \( x \in X \)).

\( \mathcal{A} \) is an equal of \( \mathcal{B} \) (i.e., \( p_1 \circ \mathcal{A}(u) \leq p_1 \circ \mathcal{B}(u) \), \( p_2 \circ \mathcal{A}(u) \leq p_2 \circ \mathcal{B}(u) \), and \( p_3 \circ \mathcal{A}(u) \leq p_3 \circ \mathcal{B}(u) \), \( p_2 \circ \mathcal{A}(u) \geq p_2 \circ \mathcal{B}(u) \), and \( p_3 \circ \mathcal{A}(u) \geq p_3 \circ \mathcal{B}(u) \), for each \( x \in X \)).

Definition 3 (cf. [39–41, 57]). Let \( \mathcal{R} = (p_1 \circ \mathcal{R}, p_2 \circ \mathcal{R}, p_3 \circ \mathcal{R}) \) be a picture fuzzy relation, denoted by \( \mathcal{U}^{X \times Y} \), where \( p_1 \circ \mathcal{R} \in [0,1]^{U \times U} \), \( p_2 \circ \mathcal{R} \in [0,1]^{U \times U} \), and \( p_3 \circ \mathcal{R} \in [0,1]^{U \times U} \) satisfy \( 0 \leq p_1 \circ \mathcal{R}(u,v) + p_2 \circ \mathcal{R}(u,v) + p_3 \circ \mathcal{R}(u,v) \leq 1 \) for all \( (u,v) \in U \times W \) and \( p_1 : R^3 \rightarrow R, p_2 : R^3 \rightarrow R, \) and \( R^3 \rightarrow R \) are first projection, second projection, and third projection, respectively.

\[
\mathcal{R} \circ \mathcal{R} (u,w) = \bigvee_{v \in Y} \left[ \mathcal{R}(u,v) \wedge \mathcal{R}(v,w) \right], \quad \forall (u,w) \in U \times W.
\]

is said to be a composition of \( \mathcal{R} \) and \( \mathcal{R} \).
3. Picture Fuzzy Rough Sets over Two Different Universes

3.1. Picture Fuzzy Rough Sets Based on $(\delta, \zeta, \emptyset)$-Cut of $R$ on Two Different Universes. We will begin by defining the $(\delta, \zeta, \emptyset)$-cut of $R$ and will subsequently define a picture fuzzy rough set based on $(\delta, \zeta, \emptyset)$-cut.

Definition 5. Let $R \in \mathcal{U}^{X,Y}$ and $(\delta, \zeta, \emptyset) \in \mathcal{I}$. Then,

(i) $\mathcal{R}_{(\delta, \zeta, \emptyset)} = \{(u,v) \in U \times V | R(u,v) \geq (\delta, \zeta, \emptyset)\} = \{(u,v) \in U \times V \mid |p \circ R(u,v) \geq \delta, p_2 \circ R(u,v) \leq \zeta, p_3 \circ R(u,v) \leq 0\}$ is called the $(\delta, \zeta, \emptyset)$-cut of $R$, and

$$\mathcal{R}_{(\delta, \zeta, \emptyset)}(u) = \{v \in V \mid R(u,v) \geq (\delta, \zeta, \emptyset)\}. \quad (5)$$

(ii) Let $A \in 2^{V}$. Then,

$$\mathcal{R}_{(\delta, \zeta, \emptyset)}^-(A) = \{u \in U \mid \mathcal{R}_{(\delta, \zeta, \emptyset)}(u) \subseteq A \wedge \mathcal{R}_{(\delta, \zeta, \emptyset)}(u) \neq \emptyset\}, \quad (6)$$

(i.e., the lower approximation of $A$) and

$$\mathcal{R}_{(\delta, \zeta, \emptyset)}^+(A) = \{u \in U \mid \mathcal{R}_{(\delta, \zeta, \emptyset)}(u) \cap A \neq \emptyset\}, \quad (7)$$

(i.e., the upper approximation of $A$).

The pair $(\mathcal{R}_{(\delta, \zeta, \emptyset)}(A), \mathcal{R}_{(\delta, \zeta, \emptyset)}^+(A))$ is a PFRS approximation of $A$ with respect to $\mathcal{R}_{(\delta, \zeta, \emptyset)}$.

Now, we present some properties based on PFRS as follows.

\[ R = \left\{ (0.2, 0.5, 0.3), (0.6, 0.3, 0), (0.4, 0.3, 0.1), (0.4, 0.2, 0.3), (0.3, 0.3, 0.2), (0.1, 0.6, 0.2), (0.5, 0.2, 0.1), (0.5, 0.5, 0), (0.6, 0.1, 0.1) \right\}. \quad (8) \]

Take $(\delta, \zeta, \emptyset) = (0.5, 0.3, 0.1)$. Then, $\mathcal{R}_{(0.5, 0.3, 0.1)}(x_1) = \{y_2\}$, $\mathcal{R}_{(0.5, 0.3, 0.1)}^+(x_2) = \emptyset$, and $\mathcal{R}_{(0.5, 0.3, 0.1)}(x_3) = \{y_1, y_3\}$. Let $\delta = \{y_1\}$ and $\delta = \{y_3\}$. Then, $\mathcal{R}_{(0.5, 0.3, 0.1)}^-(\delta) = \emptyset$, $\mathcal{R}_{(0.5, 0.3, 0.1)}^+(\delta) = \{x_1, x_2, x_3\}$, and $\mathcal{R}_{(0.5, 0.3, 0.1)}(\delta) = \{x_1, x_2, x_3\} \neq \emptyset$.

Remark 1. In general, $\mathcal{R}_{(\delta, \zeta, \emptyset)}^-(\emptyset) = \emptyset$ and $\mathcal{R}_{(\delta, \zeta, \emptyset)}^+(\emptyset) = U$ do not hold. For example, let $U = \{x_i \mid i = 1, 2\}$ and $V = \{y_i \mid i = 1, 2\}$ be two two-element sets, and $R \in \mathcal{U}^{X,Y}$ is defined by

\[ R = \left\{ (0.2, 0.4, 0.4), (0.4, 0.2, 0.3), (0.5, 0.3, 0.2), (0.7, 0.1, 0.2) \right\}. \quad (9) \]

Take $(\delta, \zeta, \emptyset) = (0.5, 0.1, 0.2)$. Then, $\mathcal{R}_{(0.5, 0.1, 0.2)}(x_1) = \emptyset$ and $\mathcal{R}_{(0.5, 0.1, 0.2)}^+(x_2) = \{y_2\}$. Thus, $\mathcal{R}_{(0.5, 0.1, 0.2)}^-(\emptyset) = \emptyset$ and $\mathcal{R}_{(0.5, 0.1, 0.2)}^+(\emptyset) = \emptyset$.

Remark 2. It should be pointed that some conclusions in [58–60] which are similar to those in Theorem 1 are wrong. Firstly, the equality $\mathcal{R}_{(0.5, 0.1)}^-(\emptyset) = \emptyset$ and $\mathcal{R}_{(0.5, 0.1)}^+(\emptyset) = U$ in Theorem 3.1(2) in [60] are incorrect. Let $U = \{x_i \mid i = 1, 2\}$, $V = \{y_i \mid i = 1, 2\}$, and $R \in \mathcal{U}^{X,Y}$ be defined by

\[ R = \left\{ (0.5, 0.2), (0.7, 0.1), (0.5, 0.3), (0.3, 0.4) \right\}. \quad (10) \]

Take $(\delta, \zeta, \emptyset) = (0.5, 0.1)$. Then, $\mathcal{R}_{(0.5, 0.1)}^-(\emptyset)$ and $\mathcal{R}_{(0.5, 0.1)}^+(\emptyset)$ in the sense of [60] are not $\emptyset$ and $U$, but $\{x_2\}$ and $\{x_1\}$, respectively. Secondly, the inclusion $\mathcal{R}_{(\emptyset)}^-(\delta)$ in Theorem 3.1(1) in [58] is incorrect. Let $U = \{x_1, x_2\}$
$\{x|i = 1, 2, 3\}$ and $V = \{y|i = 1, 2, 3\}$ be two three-element sets, and $\mathcal{R} \in [0, 1]^{U \times V}$, is defined by

$$
\mathcal{R} = \left\{ (x_1, y_1), (x_2, y_2), (x_3, y_3) \mid (x_1, y_1), (x_2, y_2), (x_3, y_3) \right\}.
$$

Take $\delta = 0.6$ and $\mathcal{A} = \{y_1, y_3\}$. Then, $\mathcal{R}_{(0.6)}(x_1) = \{y_2\}$, $\mathcal{R}_{(0.6)}(x_2) = \emptyset$, and $\mathcal{R}_{(0.6)}(x_3) = \{y_1, y_3\}$, and thus $\mathcal{R}_{(0.6)}(\mathcal{A})$ (resp., $\mathcal{R}_{(0.6)}(\mathcal{A})$) in the sense of [58] is $\{x_2, x_3\}$ (resp., $\{x_3\}$). Therefore, $\mathcal{R}_{(\delta, 0)}(\mathcal{A}) \subseteq \mathcal{R}_{(\delta, 0)}(\mathcal{A})$. Analogously, assertions $\mathcal{R}_{(\delta, 0)}(\mathcal{A}) \subseteq \mathcal{R}_{(\delta, 0)}(\mathcal{A})$ in Theorem 3.1 [59] and $\mathcal{R}_{(\delta, 0)}^{-}(\mathcal{A}) \subseteq \mathcal{R}_{(\delta, 0)}^{-}(\mathcal{A})$ in Theorem 3.1(1) [60] are incorrect. Let $U = \{x_1|i = 1, 2, 3\}$ and $V = \{y|i = 1, 2, 3\}$ be two three-element sets, and $\mathcal{R} \in (0, 1)^{U \times V}$ and also $\mathcal{R} \in J^{U \times V}$ is defined by

$$
\mathcal{R} = \left\{ (0.2, 0.5), (0.6, 0.3), (0.4, 0.3), (0.4, 0.2), (0.3, 0.3), (0.1, 0.6), (0.5, 0.2), (0.5, 0.5), (0.6, 0.1) \right\}.
$$

To correct some results in [58–60] in above Remark 2, we will give new notations of lower and upper approximations as follow:

**Definition 6.** Let $\mathcal{A} \in 2^V$. Then,

1. For $\mathcal{R} \in [0, 1]^{U \times V}$ and $\delta \in [0, 1]$, we have

   $$
   \mathcal{R}_{(\delta, 0)}^{-}(\mathcal{A}) = \left\{ u \in U \mid \mathcal{R}_{(\delta, 0)}^{-}(\mathcal{A}) \cup \mathcal{R}_{(\delta, 0)}(\mathcal{A}) \cap \mathcal{A} = \emptyset \right\},
   \mathcal{R}_{(\delta, 0)}^{+}(\mathcal{A}) = \left\{ u \in U \mid \mathcal{R}_{(\delta, 0)}(\mathcal{A}) \cap \mathcal{A} \cap \emptyset \cup \mathcal{R}_{(\delta, 0)}(\mathcal{A}) \cap \mathcal{A} = \emptyset \right\}.
   $$

2. For $\mathcal{R} \in (0, 1)^{U \times V}$ and $(\delta, \zeta) \in [0, 1]^2$, we have

   $$
   \mathcal{R}_{(\delta, 0)}^{-}(\mathcal{A}) = \left\{ u \in U \mid \mathcal{R}_{(\delta, 0)}^{-}(\mathcal{A}) \cup \mathcal{R}_{(\delta, 0)}(\mathcal{A}) \cap \mathcal{A} = \emptyset \right\},
   \mathcal{R}_{(\delta, 0)}^{+}(\mathcal{A}) = \left\{ u \in U \mid \mathcal{R}_{(\delta, 0)}(\mathcal{A}) \cap \mathcal{A} \cap \emptyset \cup \mathcal{R}_{(\delta, 0)}(\mathcal{A}) \cap \mathcal{A} = \emptyset \right\}.
   $$

3. For $\mathcal{R} \in J^{U \times V}$ and $(\delta, \zeta) \in \mathcal{J}$, we have (cf. [61])

   $$
   \mathcal{R}_{(\delta, 0)}^{-}(\mathcal{A}) = \left\{ u \in U \mid \mathcal{R}_{(\delta, 0)}^{-}(\mathcal{A}) \cup \mathcal{R}_{(\delta, 0)}(\mathcal{A}) \cap \mathcal{A} = \emptyset \right\},
   \mathcal{R}_{(\delta, 0)}^{+}(\mathcal{A}) = \left\{ u \in U \mid \mathcal{R}_{(\delta, 0)}(\mathcal{A}) \cap \mathcal{A} \cap \emptyset \cup \mathcal{R}_{(\delta, 0)}(\mathcal{A}) \cap \mathcal{A} = \emptyset \right\}.
   $$

**Theorem 2.** Let $\mathcal{R}, \mathcal{A} \in J^{U \times V}$, $(\delta, \zeta, \vartheta) \in \mathcal{I}$, and $\mathcal{A} \in 2^V$. If $\mathcal{R} \subseteq \mathcal{A}$, then $\mathcal{R}_{(\delta, 0)}^{-}(\mathcal{A}) \subseteq \mathcal{R}_{(\delta, 0)}^{-}(\mathcal{A})$ and $\mathcal{R}_{(\delta, 0)}^{+}(\mathcal{A}) \subseteq \mathcal{R}_{(\delta, 0)}^{+}(\mathcal{A})$.

**Proof.** If $\mathcal{R} \subseteq \mathcal{A}$, then $p_1 \circ \mathcal{R}(u, v) \subseteq p_1 \circ \mathcal{A}(u, v) \subseteq p_2 \circ \mathcal{R}(u, v) \subseteq p_2 \circ \mathcal{A}(u, v)$ and $p_3 \circ \mathcal{R}(u, v) \geq p_3 \circ \mathcal{A}(u, v)$ for all $(u, v) \in U \times V$. By Definition 5, we have $\mathcal{R}_{(\delta, 0)}(u) = \{v \in V \mid \mathcal{R}(u, v) \geq (\delta, \zeta, \vartheta)\}$.
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Also, if we take $R_{\delta} \subseteq R_{\delta}^\ast (\mathcal{A})$ (if $R \subseteq \delta$)

Then, $\mathcal{A} = \left\{ \begin{array}{c}
0.6 \\
0.4 \\
1 \\
0.6
\end{array} \right\} (x_1, y_1, x_2, y_2)$. \hspace{1cm} (17)

Table 1: Comparison between some properties by Definition 3.3 in \cite{58} and 6 (1).

\begin{tabular}{|c|c|c|}
\hline
Property & Definition 3.3 in \cite{58} & Definition 6 (1) \\
\hline $R_{\delta} (\emptyset) = \emptyset$ & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta} (\emptyset) = \emptyset$ & $\sqrt{\hspace{1cm}}$ & $\times$ \\
$R_{\delta} (V) = U$ & $\sqrt{\hspace{1cm}}$ & $\times$ \\
$R_{\delta} (V) = U$ & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta} (\mathcal{A}) \subseteq R_{\delta}^* (\mathcal{A})$ (if $R \subseteq \delta$) & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta} (\mathcal{A}) \subseteq R_{\delta}^* (\mathcal{A})$ (if $R \subseteq \delta$) & $\sqrt{\hspace{1cm}}$ & $\times$ \\
$R_{\delta, 1} (\mathcal{A}) \subseteq R_{\delta, 1}^* (\mathcal{A})$ (if $\delta_1 \leq \delta_2$) & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta, 1} (\mathcal{A}) \subseteq R_{\delta, 1}^* (\mathcal{A})$ (if $\delta_1 \leq \delta_2$) & $\sqrt{\hspace{1cm}}$ & $\times$ \\
\hline
\end{tabular}

(\sqrt{\hspace{1cm}}$ indicates that the property is satisfied.

Table 2: Comparison between some properties by Definition 3.2 in \cite{59} and 6 (2).

\begin{tabular}{|c|c|c|}
\hline
Property & Definition 3.2 in \cite{59} & Definition 6 (2) \\
\hline $R_{\delta} (\emptyset) = \emptyset$ & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta} (\emptyset) = \emptyset$ & $\sqrt{\hspace{1cm}}$ & $\times$ \\
$R_{\delta} (V) = U$ & $\sqrt{\hspace{1cm}}$ & $\times$ \\
$R_{\delta} (V) = U$ & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta} (\mathcal{A}) \subseteq R_{\delta}^* (\mathcal{A})$ (if $R \subseteq \delta$) & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta} (\mathcal{A}) \subseteq R_{\delta}^* (\mathcal{A})$ (if $R \subseteq \delta$) & $\sqrt{\hspace{1cm}}$ & $\times$ \\
$R_{\delta, 1} (\mathcal{A}) \subseteq R_{\delta, 1}^* (\mathcal{A})$ (if $\delta_1 \leq \delta_2$) & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta, 1} (\mathcal{A}) \subseteq R_{\delta, 1}^* (\mathcal{A})$ (if $\delta_1 \leq \delta_2$) & $\sqrt{\hspace{1cm}}$ & $\times$ \\
\hline
\end{tabular}

(\sqrt{\hspace{1cm}}$ indicates that the property is satisfied.

Table 3: Comparison between some properties by Definition 3.3 in \cite{60} and 6 (3).

\begin{tabular}{|c|c|c|}
\hline
Property & Definition 3.3 in \cite{60} & Definition 6 (2) \\
\hline $R_{\delta, 1} (\emptyset) = \emptyset$ & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta, 1} (\emptyset) = \emptyset$ & $\sqrt{\hspace{1cm}}$ & $\times$ \\
$R_{\delta, 1} (V) = U$ & $\sqrt{\hspace{1cm}}$ & $\times$ \\
$R_{\delta, 1} (V) = U$ & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta, 1} (\mathcal{A}) \subseteq R_{\delta, 1}^* (\mathcal{A})$ (if $R \subseteq \delta$) & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta, 1} (\mathcal{A}) \subseteq R_{\delta, 1}^* (\mathcal{A})$ (if $R \subseteq \delta$) & $\sqrt{\hspace{1cm}}$ & $\times$ \\
$R_{\delta, 1, 1} (\mathcal{A}) \subseteq R_{\delta, 1, 1}^* (\mathcal{A})$ (if $\delta_1 \leq \delta_2, \delta_1 \leq \delta_2$) & $\times$ & $\sqrt{\hspace{1cm}}$ \\
$R_{\delta, 1, 1} (\mathcal{A}) \subseteq R_{\delta, 1, 1}^* (\mathcal{A})$ (if $\delta_1 \leq \delta_2, \delta_1 \leq \delta_2$) & $\sqrt{\hspace{1cm}}$ & $\times$ \\
\hline
\end{tabular}

(\sqrt{\hspace{1cm}}$ indicates that the property is satisfied.

$$R = \left\{ \begin{array}{c}
0.6 \\
0.4 \\
1 \\
0.6
\end{array} \right\} (x_1, y_1, x_2, y_2).$$

3.2. Graded Picture Fuzzy Rough Sets Based on $\mathcal{R}$ on Two Different Universes

Definition 7. Let $\mathcal{R} \in U \times V$, $(\delta, \zeta, \theta) \in l$, $n \in N$, and $\mathcal{A} \in 2^V$. Then,

\begin{equation}
\mathcal{R}^1 \in \{ U \times V, (\delta, \zeta, \theta) \in l, n \in N, and \mathcal{A} \in 2^V \}
\end{equation}

are called the Type-I lower approximation and the Type-I upper approximation of $\mathcal{A}$ according to the graded $n$ with respect to $\mathcal{R} (\delta, \zeta, \theta)$ on $U$ and $V$, respectively, and $(\mathcal{R}^1 (\delta, \zeta, \theta))^n (\mathcal{A})$, $(\mathcal{R}^1 (\delta, \zeta, \theta))^n (\mathcal{A})$ is called the Type-I picture fuzzy rough approximation of $\mathcal{A}$ according to the graded $n$ with respect to $\mathcal{R} (\delta, \zeta, \theta)$ (briefly, a Type-I picture fuzzy rough set according to the graded $n$ with respect to $\mathcal{R} (\delta, \zeta, \theta)$).

Theorem 3. Let $\mathcal{R} \in U \times V$, $(\delta, \zeta, \theta) \in l$, $n \in N$, and $\mathcal{A} \in 2^V$. Then,
(1) If $\mathcal{A} \subseteq \mathcal{B}$, then $\left(\mathcal{R}_{(I,(A,B,c)})^{-1}\right)^n(\mathcal{A}) \subseteq \left(\mathcal{R}_{(I,(A,B,c)})^{-1}\right)^n(\mathcal{B})$ and $\left(\mathcal{R}_{(I,(A,B,c)})^{-1}\right)^n(\mathcal{A}) \subseteq \left(\mathcal{R}_{(I,(A,B,c)})^{-1}\right)^n(\mathcal{B})$.

(2) $\left(\mathcal{R}_{(I,(A,B,c)})^{-1}\right)^n(\mathcal{A} \cap \mathcal{B}) \subseteq \left(\mathcal{R}_{(I,(A,B,c)})^{-1}\right)^n(\mathcal{A}) \cap \left(\mathcal{R}_{(I,(A,B,c)})^{-1}\right)^n(\mathcal{B})$.

(3) $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A} \cup \mathcal{B}) \subseteq \left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) \cup \left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{B})$.

(4) $\left(\mathcal{A}^{\mathcal{R}_{(I,(A,B,c))}^{-1}}\right)^n(\mathcal{A}) = \left(\mathcal{A}^{\mathcal{R}_{(I,(A,B,c))}^{-1}}\right)^n(\mathcal{A})^c$.

(5) If $m \leq n$, then $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^m(\mathcal{A}) \subseteq \left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A})$ and $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) \subseteq \left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{B})$.

Proof. (1) By Definition 7, we have $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \{ u \in U | [\mathcal{R}_{(I,(A,B,c))}(u) - \mathcal{A}] \subseteq \mathcal{A} \}$. Since $\mathcal{A} \subseteq \mathcal{B}$, then $\{ u \in U | [\mathcal{R}_{(I,(A,B,c))}(u) - \mathcal{B}] \subseteq \mathcal{B} \} \subseteq \{ u \in U | [\mathcal{R}_{(I,(A,B,c))}(u) - \mathcal{B}] \subseteq \mathcal{A} \}$.

(2) and (3) Clear.

(4) $\left(\mathcal{A}^{\mathcal{R}_{(I,(A,B,c))}^{-1}}\right)^n(\mathcal{A}) = \{ u \in U | \mathcal{R}_{(I,(A,B,c))}(u) \cap \mathcal{A}^{\mathcal{R}_{(I,(A,B,c))}^{-1}}(\mathcal{A}) \}$.

(5) By Definition 7, we have $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^{-1}(\mathcal{A}) = \{ u \in U | [\mathcal{R}_{(I,(A,B,c))}(u) - \mathcal{A}] \subseteq \mathcal{A} \}$. If $m \leq n$, then we have $\{ u \in U | [\mathcal{R}_{(I,(A,B,c))}(u) - \mathcal{A}] \subseteq \mathcal{A} \} \subseteq \{ u \in U | [\mathcal{R}_{(I,(A,B,c))}(u) - \mathcal{B}] \subseteq \mathcal{B} \}$.

Remark 5. The equality $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A})$ holds. For $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) \subseteq \left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A})$, where $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = U$ does not hold. For example, let $U = \{ x, y | x \neq 1, 2 \}$ and $V = \{ v, y | v \neq 1, 2 \}$ be two two-element sets, and $\mathcal{R} \subseteq U \times V$ is defined by

$$\mathcal{R} = \{ (0.2, 0.6, 0.2), (0.4, 0.2, 0.3), (0.5, 0.2, 0.3), (0.6, 0.1, 0.2) \}.$$  

(19)

Take $(\delta, \zeta, B) = (0.5, 0.1, 0.2)$ and $n = 2$. Then, $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \mathcal{A}$ and $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \mathcal{A}$. Thus, $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \{ x \} \neq \mathcal{A}$.

Remark 6. Let $\mathcal{A} \subseteq \mathcal{B}$, then there does not exist inclusion relation between $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A})$ and $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A})$. Consider $U, V, (\delta, \zeta, B)$ and $\mathcal{R} \subseteq U \times V$ are given in Remark 1. Let $\mathcal{A} = \{ x \}$ and $n = 1$. Then, $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \{ x \}$ and $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \{ x \}$.

We study the notion of Type-II PFRS according to the graded $n$ with respect to $\mathcal{R}_{(I,(A,B,c))}^{-1}(\mathcal{A})$ and $\mathcal{R}_{(I,(A,B,c))}^{-1}(\mathcal{A})$. Let $\mathcal{A} \subseteq \mathcal{B}$.

Definition 8. Let $\mathcal{R} \subseteq U \times V$, $(\delta, \zeta, B) \subseteq \mathcal{R}$, $n \in N$ s.t. $n \in [0, n(\mathcal{A} \subseteq \mathcal{B} | (\mathcal{R}_{(I,(A,B,c))}^{-1})(\mathcal{A}) \subseteq \mathcal{A})]$. Then, $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \{ u \in U | \mathcal{R}_{(I,(A,B,c))}(u) - \mathcal{A} \subseteq \mathcal{B} \}$.

Lemma 1. Let $\mathcal{R} \subseteq U \times V$, $(\delta, \zeta, B) \subseteq \mathcal{R}$, $n \in N$, and $\mathcal{A} \subseteq \mathcal{B}$. Then, the following holds:

(1) $n \in [0, n(\mathcal{A} \subseteq \mathcal{B} | (\mathcal{R}_{(I,(A,B,c))}^{-1})(\mathcal{A}) \subseteq \mathcal{A})]$. (2) The limits of the grade $n = \max[0, n(\mathcal{A} \subseteq \mathcal{B} | (\mathcal{R}_{(I,(A,B,c))}^{-1})(\mathcal{A}) \subseteq \mathcal{A})]$. (3) $\mathcal{A} \subseteq \mathcal{B}$.

Theorem 4. Let $\mathcal{A} \subseteq \mathcal{B}$, $(\delta, \zeta, B) \subseteq \mathcal{R}$, $n \in N$ s.t. $n \in [0, n(\mathcal{A} \subseteq \mathcal{B} | (\mathcal{R}_{(I,(A,B,c))}^{-1})(\mathcal{A}) \subseteq \mathcal{A})]$. Then, the following holds:

(1) $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \mathcal{R}_{(I,(A,B,c))}^{-1}(\mathcal{A})$.

(2) $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) \subseteq \left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A})$.

(3) $\mathcal{A} \subseteq \mathcal{B}$.

(4) $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) \subseteq \left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A})$.

(5) $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) \subseteq \left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A})$. 

We are given in Remark 1. Let $\mathcal{A} = \{ x \}$ and $n = 1$. Then, $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \{ x \}$ and $\left(\mathcal{R}_{(I,(A,B,c))}^{-1}\right)^n(\mathcal{A}) = \{ x \}$.
Thus, the element which only belongs to \( R \) is defined by:

\[
\mathcal{R} = \left\{ (0.5, 0.2), (0.7, 0.1), (0.4, 0.3), (0.5, 0.3), (0.3, 0.4), (0.4, 0.5), (0.8, 0.1), (0.1, 0.7), (0.6, 0.2) \right\}.
\]

The main results are as follows.

**Theorem 5.** Let \( \mathcal{R} \in \mathcal{V}_X \), \((\delta, \zeta, \theta) \in \mathbb{N}, \ 0 \leq \beta < \alpha \leq 1, \ \mathcal{R}_{(\delta, \zeta, \theta)}(u) \neq \emptyset, \) and \( \mathcal{R} \subseteq \mathcal{V} \). Then, the following holds:

1. \( \mathcal{R}_{(\delta, \zeta, \theta)}(0) = \emptyset; \)
2. \( \mathcal{R}_{(\delta, \zeta, \theta)}(V) = \emptyset; \)
3. \( \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \subseteq \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \);
4. \( \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \subseteq \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \);
5. \( \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \subseteq \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \);
6. \( \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \subseteq \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \);
7. \( \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \subseteq \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \);

**Proof.** (1) For any \( u \in \mathcal{U}, 0 \leq \beta < \alpha \leq 1 \), we have \( \mathcal{R}_{(\delta, \zeta, \theta)}(u) \cap \mathcal{R}(u) \mathcal{R}_{(\delta, \zeta, \theta)}(u) = \emptyset \) if \( \mathcal{R}_{(\delta, \zeta, \theta)}(u) \neq \emptyset \), and \( u \in \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \). Therefore, \( \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) = \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \).

Remark 7. The equality \( \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) = \emptyset \) and \( \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) = \mathcal{R}_{(\delta, \zeta, \theta)}(\mathcal{R}) \) in Theorem 3.10 in [60] are incorrect. For example, let \( \mathcal{U} = \{ y_1, y_2, y_3 \} \) be two-element sets, and \( \mathcal{R} \subseteq \mathcal{V} \) is defined by:

\[
\mathcal{R} = \left\{ (0.2, 0.6), (0.4, 0.2), (0.5, 0.2), (0.8, 0.1) \right\}.
\]
Similarly, we can obtain 
\((\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \). Thus, the \( \lim_{\mathcal{A} \to \mathcal{R}_{[a,b]}}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) holds.

(2) It is analogous to (1) above.

(3) We know \( (\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = (\mathcal{A} \in U \subseteq \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) if and only if \( \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) holds. In addition, we know that when \( a \) decreased to \( r \) and \( \beta \) increased to \( r \), the boundary region \( \mathcal{B}_{[a,b]}(\mathcal{A}) \) will decrease. Then, we have \( (\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) if and only if \( \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) holds. Conversely, if there exists \( u_0 \in \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) if \( \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) holds. Furthermore, there is \( \mathcal{A} \subseteq \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = (\mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) if and only if \( \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) holds.

Theorem 6. Let \( \mathcal{A} \subseteq \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = (\mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) if and only if \( \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) \) holds.

Proof. Obvious.

Example 3 (continuation of Example 1). Let \( \alpha = 0.4, \beta = 0.3 \), \( \mathcal{A} = \{y_1\} \), and \( \mathcal{A} = \{y_2, y_3\} \). Then,

(1) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(2) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(3) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(4) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(5) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(6) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(7) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(8) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(9) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(10) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(11) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(12) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)

(13) \( \lim_{x \to x_1, y \to y_3} (\mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]}))^n(\mathcal{A}) = \mathcal{A} \cap \mathcal{R}_{[a,b]}(\mathcal{R}_{[a,b]})^n(\mathcal{A}) \)
Lemma 2. Let $\mathcal{B} \in \mathcal{V}_{\delta, \zeta, \theta}^\varepsilon$, $(\delta, \zeta, \theta) \in I$, $0 \leq \beta < \alpha \leq 1$, $\mathcal{B}_{\delta, \zeta, \theta}(u) \neq \emptyset$, and $\mathcal{A} \in 2^V$. Then, the following hold:

1. $(\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{A})$ does not decrease with error $\alpha$ and does not increase with error $\beta$.

2. $(\mu_{\delta, \zeta, \theta})^\alpha : (\mathcal{A})$ does not decrease with error $\alpha$ and does not increase with error $\beta$.

Proof. We only prove (1) and then the proof of (2) can be obtained using similar techniques.

(1) Since $(\mathcal{B}_{\delta, \zeta, \theta}^\alpha)^\alpha : (\mathcal{A}) \subseteq (\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{A}) (0.5 < \alpha < \alpha_{\leq 1})$ by Theorem 5 (7), then there is $((\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{A})) \subseteq ((\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{A}))$ implies that $1 - ((\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{A})) \cap (\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{A})) \geq 1 - ((\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{A})) \subseteq (\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{A}))$. Therefore, $(\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{A}) \subseteq (\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{A})$. Similarly, $(\mu_{\delta, \zeta, \theta})^\alpha : (\mathcal{A}) \subseteq (\mu_{\delta, \zeta, \theta})^\alpha : (\mathcal{A})$.

Theorem 8. Let $\mathcal{B} \in \mathcal{V}_{\delta, \zeta, \theta}^\varepsilon$, $(\delta, \zeta, \theta) \in I$, $0 \leq \beta < \alpha \leq 1$, $\mathcal{B}_{\delta, \zeta, \theta}(u) \neq \emptyset$, $\mathcal{A}, \mathcal{B} \in 2^V$, and $\mathcal{A} \subseteq \mathcal{B}$. Then, the following holds:

1. If $(\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{A}) = (\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{B})$, there is $(\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{A}) \subseteq (\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{B})$ and $(\mu_{\delta, \zeta, \theta})^\alpha : (\mathcal{A}) \subseteq (\mu_{\delta, \zeta, \theta})^\alpha : (\mathcal{B})$.

2. If $(\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{A}) = (\mathcal{B}_{\delta, \zeta, \theta}^{\varepsilon})^\alpha : (\mathcal{B})$, there is $(\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{A}) \subseteq (\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{B})$ and $(\mu_{\delta, \zeta, \theta})^\alpha : (\mathcal{A}) \subseteq (\mu_{\delta, \zeta, \theta})^\alpha : (\mathcal{B})$.

Proof. From Theorem 10, we have $(\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{A}) \subseteq (\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{B})$. Consequently, $(\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{A}) \subseteq (\rho_{\delta, \zeta, \theta})^\alpha : (\mathcal{B})$. We know $|X \cup Y| = |X| + |Y| - |X \cap Y|$. Then,
Hence, (1) is holding. 

(2) It can be easily proved by the relationship 

\[
\mu_{(\mathcal{A}, \mathcal{B})}^\alpha(\mathcal{A}) = 1 - \mu_{(\mathcal{A}, \mathcal{B})}^\beta(\mathcal{A}).
\]

3.3. Generalized Picture Fuzzy Rough Sets Based on 

(δ, ζ, θ)-Cut of R on Two Different Universes

Definition 11. Let \( \mathcal{R} = \{ \mathcal{R}_{i,(\delta, \zeta, \theta)} \mid i \in \mathbb{N} \} \subseteq \mathcal{U}^{\mathcal{U}} \), \( (\delta, \zeta, \theta) \in \mathbb{R} \), \( 0 \leq \delta < \theta \leq 1 \), \( \mathcal{A}, \mathcal{B} \in 2^{\mathcal{U}} \). Then,

\[
(\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) = \{ u \in \mathcal{U} \mid (\mathcal{R}_{i,(\delta, \zeta, \theta)}(u) \cap \mathcal{A}/(\mathcal{R}_{i,(\delta, \zeta, \theta)}(u) \cap \mathcal{A}) \geq \alpha \}) \text{ and } (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) = \{ u \in \mathcal{U} \mid (\mathcal{R}_{i,(\delta, \zeta, \theta)}(u) \cap \mathcal{A}/(\mathcal{R}_{i,(\delta, \zeta, \theta)}(u) \cap \mathcal{A}) \geq \alpha \})
\]

is called the Type-I generalized picture fuzzy rough approximation of \( \mathcal{A} \) according to the degree \( \alpha \) and \( \beta \) with respect to \( \mathcal{R}_{i,(\delta, \zeta, \theta)} \) on \( U \) and \( \mathcal{A} \), respectively.

Remark 9. (1) From Definition 11, we have

(i) \( (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) \subseteq (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) \)

(ii) \( (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) \subseteq (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) \)

(2) If we take \( \alpha = 1 \) and \( \beta = 0 \) in Definition 11, then we can obtain the following definitions:

(i) \( (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) = \{ u \in \mathcal{U} \mid R_{i,(\delta, \zeta, \theta)}(u) \subseteq \mathcal{A} \} \) and \( (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) = \{ u \in \mathcal{U} \mid R_{i,(\delta, \zeta, \theta)}(u) \subseteq \mathcal{A} \} \)

(ii) \( (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) = \{ u \in \mathcal{U} \mid R_{i,(\delta, \zeta, \theta)}(u) \subseteq \mathcal{A} \} \) and \( (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) = \{ u \in \mathcal{U} \mid R_{i,(\delta, \zeta, \theta)}(u) \subseteq \mathcal{A} \} \)

The main results are as follows.

Theorem 10. Let \( \mathcal{R} = \{ \mathcal{R}_{i,(\delta, \zeta, \theta)} \mid i \in \mathbb{N} \} \subseteq \mathcal{U}^{\mathcal{U}} \), \( (\delta, \zeta, \theta) \in \mathbb{R} \), \( 0 \leq \delta < \theta \leq 1 \), \( \mathcal{A}, \mathcal{B} \in 2^{\mathcal{U}} \). Then, the following holds:

1. \( (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) \subseteq (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) \)

2. \( (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) \subseteq (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) \)

3. If \( \mathcal{A} \subseteq \mathcal{B} \), then \( (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) \subseteq (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) \)

4. \( (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) \cap (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) \subseteq (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) \)

5. \( (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) \subseteq (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) \)

6. \( (\mathcal{R}^{-})^\mathcal{A}_{\lambda}(\mathcal{A}) \subseteq (\mathcal{R}^{-})^\mathcal{B}_{\lambda}(\mathcal{A}) \) (if \( 0.5 < \alpha < \alpha_1 \leq 1 \))
Proof. By Definition 11, the result can be similarly proven as Theorem 5.

Remark 10. Let $\mathcal{A} \in 2^V$ and $0 \leq \beta < \alpha \leq 1$. Then,

1. $(\mathcal{R}^-)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) = (\mathcal{R}^-)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$,
2. $(\mathcal{R}^+)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) = (\mathcal{R}^+)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$.

Example 4. Let $U = \{x_i| i = 1, 2\}$ and $V = \{y_i| i = 1, 2\}$ be two two-element set, and $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \in \mathcal{V}^{U \times V}$ is defined by

$\mathcal{R}_1 = \{(0,2,0,4,0.4), (0,4,0,2,0.3), (0,5,0,3,0.2), (0,7,0,1,0.2)\}$,
$\mathcal{R}_2 = \{(0,3,0,2,0.4), (0,1,0,2,0.2), (0,5,0,3,0.1), (0,6,0,1,0.3)\}$,
$\mathcal{R}_3 = \{(0,2,0,3,0.5), (0,6,0,1,0.2), (0,7,0,1,0.1), (0,4,0,3,0.3)\}$.

Take $(\delta, \zeta, \vartheta) = (0.5, 0.3, 0.2)$. Then, $\mathcal{R}_1|_{(0,5,0.3,0.2)}(x_1) = \emptyset$,
$\mathcal{R}_2|_{(0,5,0.3,0.2)}(x_2) = \emptyset$, $\mathcal{R}_1|_{(0,5,0.3,0.2)}(x_1) = \emptyset$,
$\mathcal{R}_3|_{(0,5,0.3,0.2)}(x_1) = \emptyset$, $\mathcal{R}_3|_{(0,5,0.3,0.2)}(x_2) = \{y_1\}$. Let $\mathcal{A} = \{y_1\}$, $\alpha = 0.6$, and $\beta = 0.4$. Thus,

1. $(\mathcal{R}^-)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) = \emptyset 
\neq \mathcal{B}_1 \cup \mathcal{A}$,
2. $(\mathcal{R}^+)^{\mathcal{A}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) = \emptyset 
\neq \mathcal{B}_2 \cup \mathcal{A}$.

From Remark 9 (2), we can conclude the following corollary.

Corollary 2. Let $\mathcal{A} = \{\mathcal{R}_i|_{(\delta, \zeta, \vartheta)}| i \in N\} \subseteq \mathcal{A}^{U \times V}$, $(\delta, \zeta, \vartheta) \in \mathcal{I}$,
$\mathcal{R}_{[\delta,\zeta]}(I)(\mathcal{A}) \neq \emptyset$, and $\mathcal{A}, \mathcal{B} \in 2^V$. Then, the following holds:

1. $(\mathcal{R}^-)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) = (\mathcal{R}^-)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$,
2. $(\mathcal{R}^+)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) = (\mathcal{R}^+)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$.

Theorem 11. Let $\mathcal{A} = \{\mathcal{R}_i|_{(\delta, \zeta, \vartheta)}| i \in N\} \subseteq \mathcal{A}^{U \times V}$, $(\delta, \zeta, \vartheta) \in \mathcal{I}$,
$0 \leq \beta < \alpha \leq 1$, $\mathcal{R}_{[\delta,\zeta]}(I)(\mathcal{A}) \neq \emptyset$, and $\mathcal{A}, \mathcal{B} \in 2^V$. Then, the following holds:

1. $(\mathcal{R}^-)^{\mathcal{A}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) = (\mathcal{R}^-)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$,
2. $(\mathcal{R}^+)^{\mathcal{A}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) \subseteq (\mathcal{R}^+)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$.

Proof. It follows from Definition 11 (2).

Corollary 3. Let $\mathcal{A} = \{\mathcal{R}_i|_{(\delta, \zeta, \vartheta)}| i \in N\} \subseteq \mathcal{A}^{U \times V}$, $(\delta, \zeta, \vartheta) \in \mathcal{I}$,
$\mathcal{R}_{[\delta,\zeta]}(I)(\mathcal{A}) \neq \emptyset$, and $\mathcal{A}, \mathcal{B} \in 2^V$. Then, the following holds:

1. $(\mathcal{R}^-)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) = (\mathcal{R}^-)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$,
2. $(\mathcal{R}^+)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) = (\mathcal{R}^+)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$.
3. If $\mathcal{A} \subseteq \mathcal{B}$, then $(\mathcal{R}^-)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) \subseteq (\mathcal{R}^-)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$,
$(\mathcal{R}^+)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) \subseteq (\mathcal{R}^+)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$.

(3) If $\mathcal{A} \subseteq \mathcal{B}$, then $(\mathcal{R}^-)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) \subseteq (\mathcal{R}^-)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$,
$(\mathcal{R}^+)^{\mathcal{A}\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A}) \subseteq (\mathcal{R}^+)^{\mathcal{B}}_{\mathcal{R}_{[\delta,\zeta]}(I)}(\mathcal{A})$.

If $0 \leq \beta < \beta_1 < 0.5$, the result does not hold by the following example.
(4) \( (\mathcal{R}^+)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \subseteq (\mathcal{R}^+)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \cap (\mathcal{R}^-)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \cap (\mathcal{A}) \cap \mathcal{B} \).

(5) \( (\mathcal{R}^+)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \subseteq (\mathcal{R}^+)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \cup (\mathcal{R}^-)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \cup (\mathcal{A}) \cap \mathcal{B} \cap (\mathcal{R}^+)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \cap (\mathcal{A}) \cap \mathcal{B} \).

\[ \mathcal{R}_{\{u \in \mathcal{R} \}} \cap \mathcal{A} \cup (\mathcal{A}) \cap \mathcal{B} \cup (\mathcal{A}) \cap \mathcal{B} \cap (\mathcal{R}^+)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \cap (\mathcal{A}) \cap \mathcal{B} \]

\[ \mathcal{R}^- \cap (\mathcal{A}^+)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \] is the lower approximation of \( \mathcal{A} \).

\[ \mathcal{R}^+ \cup (\mathcal{A}) \cap \mathcal{B} \cap (\mathcal{R}^+)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \cap (\mathcal{A}) \cap \mathcal{B} \]

\[ \mathcal{R}^+ \cup (\mathcal{A}) \cap \mathcal{B} \cap (\mathcal{R}^+)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \cap (\mathcal{A}) \cap \mathcal{B} \]

\[ \mathcal{R}^- \cap (\mathcal{A}^+)^{\setminus}_{\mathcal{V} \setminus_{\mathcal{R}(\mathcal{A})}} \] is the upper approximation of \( \mathcal{A} \).

A pair \((\mathcal{R}^-, \mathcal{A})\) is said to be the rough picture fuzzy approximation of \( \mathcal{A} \) with respect to \( \mathcal{R} \).

Example 5. Let \( U = \{x_i | i = 1, 2, 3\} \) and \( V = \{y_i | i = 1, 2, 3\} \) be three-element sets, \( \mathcal{R} \in \mathcal{U}^{\times \mathcal{V}} \) defined by

\[
\mathcal{R} = \begin{cases}
(0.2, 0.5, 0.2) & (0.6, 0.3, 0.1) & (0.4, 0.2, 0.3) & (0.4, 0.2, 0.3) & (0.3, 0.3, 0.4) & (0.1, 0.6, 0.2) & (0.5, 0.2, 0.3) & (0.5, 0.4, 0.1) & (0.4, 0.1, 0.2)
\end{cases}
\]

and \( \mathcal{A} \in \mathcal{V} \) defined by

\[
\mathcal{A} = \begin{cases}
(0.4, 0.3, 0.2) & (0.2, 0.1, 0.5) & (0.3, 0.5, 0.2)
\end{cases}
\]
By Definition 12, we obtain

$$R^- (\mathcal{A}) (x_1) = \left( \bigwedge_{v \in V} [p_3 \circ R (x_1, v) \cap p_1 \circ \mathcal{A} (v)], \bigvee_{v \in V} [p_2 \circ R (x_1, v) \cap p_2 \circ \mathcal{A} (v)] \right)$$

$$= ([(0.2 \vee 0.4) \cap (0.1 \vee 0.2) \cap (0.2 \vee 0.3)], [(0.5 \vee 0.3) \cap (0.3 \vee 0.1) \cap (0.3 \vee 0.5)], [(0.2 \vee 0.2) \cap (0.6 \vee 0.5) \cap (0.4 \vee 0.2)])$$

$$= (0.2, 0.3, 0.5).$$

Similarly, $R^- (\mathcal{A}) (x_2) = (0.3, 0.3, 0.3), R^- (\mathcal{A}) (x_3) = (0.2, 0.3, 0.5), R^+ (\mathcal{A}) (x_1) = (0.3, 0.3, 0.2), R^+ (\mathcal{A}) (x_2) = (0.4, 0.3, 0.2),$ and $R^+ (\mathcal{A}) (x_3) = (0.4, 0.3, 0.2).

Therefore,

$$R^- (\mathcal{A}) = \left( \frac{(0.2, 0.3, 0.5)}{x_1}, \frac{(0.3, 0.3, 0.3)}{x_2}, \frac{(0.2, 0.3, 0.5)}{x_3} \right),$$

$$R^+ (\mathcal{A}) = \left( \frac{(0.3, 0.3, 0.2)}{x_1}, \frac{(0.4, 0.3, 0.2)}{x_2}, \frac{(0.4, 0.3, 0.2)}{x_3} \right).$$

(33)

Next, we present the RPFS decision-making medical diagnosis problem over two different universes as indicated below.

Suppose that $U = \{u_1, u_2, \ldots, u_n\}$ (a $p$-element set) be the set of a disease, $V = \{v_1, v_2, \ldots, v_q\}$ the set of symptoms (where $p, q \in \mathbb{N}$), and $\mathcal{R} \in U \cup V$ be picture fuzzy relation

$$\mathcal{A} = \left\{ \left( p_1 \circ \mathcal{A} (v_1), p_2 \circ \mathcal{A} (v_1), p_3 \circ \mathcal{A} (v_1), \ldots, p_1 \circ \mathcal{A} (v_q), p_2 \circ \mathcal{A} (v_q), p_3 \circ \mathcal{A} (v_q) \right) \right\},$$

where $p_1 \circ \mathcal{A} (v_q) \in [0, 1]$ (i.e., the degree of positive membership) to the symptom $v_q \in V$ of $\mathcal{A}, p_2 \circ \mathcal{A} (v_q) \in [0, 1]$ (i.e., the degree of neutral membership) to the symptom $v_q \in V$ of $\mathcal{A},$ and $p_3 \circ \mathcal{A} (v_q)$ (i.e., the degree of negative membership) to the symptom $v_q \in V$ of $\mathcal{A}.$

From Definition 14, we calculate the cosine similarity measure between the $(\mathcal{R}^- (\mathcal{A}) \circ \mathcal{R}^+ (\mathcal{A})) u_i$ corresponding to $u_i$ and the ideal $(\mathcal{R}^- (\mathcal{A}) \circ \mathcal{R}^+ (\mathcal{A})) u_i.$ Finally, we determine $\tilde{R} = \arg \max \left\{ C ((\mathcal{R}^- (\mathcal{A}) \circ \mathcal{R}^+ (\mathcal{A})) u_i), (\mathcal{R}^- (\mathcal{A}) \circ \mathcal{R}^+ (\mathcal{A})) u_i \right\}.$

(36)

From Definition 14, we calculate the cosine similarity measure between the $(\mathcal{R}^- (\mathcal{A}) \circ \mathcal{R}^+ (\mathcal{A})) u_i \cap u_i \in U.$

$$R^- (\mathcal{A}) \circ \mathcal{R}^+ (\mathcal{A}) = \left\{ R^- (\mathcal{A}) (u) \cap \mathcal{R}^+ (\mathcal{A}) (u) \right\}.$$

(37)
Step 1. Input the picture fuzzy relation $R \in \mathbb{R}^{U \times V}$, where $R(u_p, v_q)$ represents the picture fuzzy relation between the disease $u_p$ ($u_p \in U$) and the symptom $v_q$ ($v_q \in V$), which is evaluated by a doctor in advance.

Step 2. Define $\mathcal{A}$ (patient set) is PFS on symptom $V$, that is, $\mathcal{A} = \{(p_1 \circ \mathcal{A}(v_1), p_2 \circ \mathcal{A}(v_1), p_3 \circ \mathcal{A}(v_1)) \mid v_1 \mapsto \mathcal{A}(v_1), (((p_1 \circ \mathcal{A}(v_2), p_2 \circ \mathcal{A}(v_2), p_3 \circ \mathcal{A}(v_2))) \mapsto \mathcal{A}(v_2), \ldots, (((p_1 \circ \mathcal{A}(v_9), p_2 \circ \mathcal{A}(v_9), p_3 \circ \mathcal{A}(v_9))) \mapsto \mathcal{A}(v_9))\}$.

Step 3. Compute and write $R^-(\mathcal{A})$ and $R^+(\mathcal{A})$, respectively, of $\mathcal{A}$.

Step 4. By Definition 13, we calculate $R^-((\mathcal{A}) \oplus R^+(\mathcal{A})))$ as $R^-((\mathcal{A}) \oplus R^+(\mathcal{A}))) = R^-((\mathcal{A}) (u) + R^+(\mathcal{A})(u)) \cup u \in U$.

Step 5. By Definition 14, we compute the cosine similarity measure between the $(R^-((\mathcal{A}) \oplus R^+(\mathcal{A})))_{u_p}$ corresponding to $u_p$ and the ideal $(R^-((\mathcal{A}) \oplus R^+(\mathcal{A})))^*$.

Step 6. Get the best choice to select $u_{p}$, that is, we can conclude that patient $\mathcal{A}$ is suffering from the diseases.

**Algorithm 1:** Determine the best choice of rough picture fuzzy soft sets over two different universes.

Example 6. Assume that five diseases $(U = \{x_i \mid i = 1, 2, 3, 4, 5\})$, where $x_1$ stands for the "malaria," $x_2$ stands for the "typhoid," $x_3$ stands for the "stomach problem," and $x_4$ stands for the "chest problem." The five symptoms are in clinic (let $V = \{y_i \mid i = 1, 2, 3, 4, 5\}$), where $y_1$ stands for the "temperature," $y_2$ stands for the "headache," $y_3$ stands for the "stomach Pain," $y_4$ stands for the "cough," and $y_5$ stands for the "Chest-Pain." Let $R \in \mathbb{R}^{U \times V}$ be a picture fuzzy relation from $U$ to $V$, where $R$ is a medical knowledge statistic data of the relationship of the disease $x_p (x_p \in U)$ and the symptom $y_q (y_q \in V)$ (where $p, q = 1, 2, 3, 4, 5$). The statistic data are given in Table 4.

By the Step 2 of Algorithm 1, suppose the symptoms of a patient $\mathcal{A}$ are defined by a PFS on $V$, and

$\mathcal{A} = \left\{ \begin{array}{ll}
(0.4, 0.3, 0.2) & \text{y}_1 \\
(0.2, 0.1, 0.5) & \text{y}_2 \\
(0.3, 0.5, 0.2) & \text{y}_3 \\
(0.6, 0.2, 0.1) & \text{y}_4 \\
(0.4, 0.4, 0.2) & \text{y}_5
\end{array} \right\}.$ \hspace{1cm} (38)

Then, by Definition 12, we obtain on the $R^-((\mathcal{A})$ and $R^+(\mathcal{A})$ of patient $\mathcal{A}$ in Step 3 of Algorithm 1, respectively, as follows:

\[
R^-((\mathcal{A}) = \left\{ \begin{array}{ll}
(0.4, 0.3, 0.3) & \text{x}_1 \\
(0.3, 0.2, 0.5) & \text{x}_2 \\
(0.3, 0.1, 0.5) & \text{x}_3 \\
(0.2, 0.3, 0.5) & \text{x}_4 \\
(0.2, 0.1, 0.4) & \text{x}_5
\end{array} \right\},
\hspace{1cm} (39)
\]

$R^+(\mathcal{A}) = \left\{ \begin{array}{ll}
(0.5, 0.3, 0.2) & \text{x}_1 \\
(0.4, 0.2, 0.3) & \text{x}_2 \\
(0.5, 0.1, 0.2) & \text{x}_3 \\
(0.4, 0.3, 0.1) & \text{x}_4 \\
(0.4, 0.1, 0.1) & \text{x}_5
\end{array} \right\}.$

By Step 4 of Algorithm 1, we obtain

\[
R^-((\mathcal{A}) \oplus R^+(\mathcal{A})) = \left\{ \begin{array}{ll}
(0.7, 0.09, 0.06) & \text{x}_1 \\
(0.58, 0.04, 0.15) & \text{x}_2 \\
(0.65, 0.01, 0.1) & \text{x}_3 \\
(0.52, 0.09, 0.05) & \text{x}_4 \\
(0.52, 0.01, 0.04) & \text{x}_5
\end{array} \right\}. \hspace{1cm} (40)
\]
Then, by Step 4 of Algorithm 1, we can get the cosine similarity measure between the \((\mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A})) u_\alpha\) corresponding to \(u_\alpha\) and the ideal \((\mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}))^*\) as follows:

\[
C \left( \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right) u_\alpha, \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right)^* \right) = 0.141,
\]

\[
C \left( \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right) u_\alpha, \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right)^* \right) = 0.181,
\]

\[
C \left( \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right) u_\alpha, \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right)^* \right) = 0.198,
\]

\[
C \left( \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right) u_\alpha, \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right)^* \right) = 0.196,
\]

\[
C \left( \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right) u_\alpha, \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right)^* \right) = 0.199.
\]

Thus, according to Step 6, we conclude the maximum value is

\[
C \left( \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right) u_\alpha, \left( \mathcal{R}^- (\mathcal{A}) \mathcal{B} \mathcal{R}^+ (\mathcal{A}) \right)^* \right) = 0.199.
\]

Thus, the patient is suffering from the disease chest problem \((x_5)\).

### 5. Conclusions

In this paper, we suggest novel notion of picture fuzzy rough sets (PFRSs) over two different universes which depend on \((\delta, \zeta, \theta)\)-cut. Also, we discussed some interesting properties and related results on the PFRSs. Furthermore, we presented several notions related to PFRSs such as Type-I-/Type-II-graded PFRSs, the degree \(\alpha\) and \(\beta\) with respect to \(\mathcal{R} (\delta, \zeta, \theta)\) on PFRSs, and Type-I-/Type-II-generalized PFRSs based on the degree \(\alpha\) and \(\beta\) with respect to \(\mathcal{R} (\delta, \zeta, \theta)\) and investigate the basic properties of above notions. Lastly, we gave an approach based on the rough picture fuzzy approximation RPFA operators on two different universes in decision-making problem is introduced, and we present an example to show the validity of this approach.

### Data Availability

All data required for this paper are included within the article.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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### Table 4: Picture fuzzy relation \(\mathcal{R} \in \mathcal{B}^{|V|}\) between the diseases and symptoms.

| Disease          | Temperature \((y_1)\) | Headache \((y_2)\) | Stomach pain \((y_3)\) | Cough \((y_4)\) | Chest-pain \((y_5)\) |
|------------------|-----------------------|-------------------|-------------------------|---------------|---------------------|
| Viral fever \((x_1)\) | \((0.1, 0.5, 0.4)\)    | \((0.3, 0.3, 0.4)\) | \((0.3, 0.2, 0.4)\)     | \((0.5, 0.3, 0.2)\) | \((0.1, 0.4, 0.4)\) |
| Malaria \((x_2)\)    | \((0.2, 0.4, 0.3)\)    | \((0.5, 0.2, 0.3)\) | \((0.5, 0.1, 0.3)\)     | \((0.2, 0.3, 0.5)\) | \((0.4, 0.2, 0.4)\) |
| Typhoid \((x_3)\)    | \((0.7, 0.2, 0.1)\)    | \((0.6, 0.1, 0.3)\) | \((0.3, 0.6, 0.1)\)     | \((0.5, 0.1, 0.4)\) | \((0.7, 0.2, 0.1)\) |
| Stomach problem \((x_4)\) | \((0.4, 0.4, 0.2)\)    | \((0.7, 0.3, 0)\)  | \((0.8, 0.1, 0.1)\)    | \((0.2, 0.3, 0.1)\) | \((0.6, 0.2, 0.2)\) |
| Chest problem \((x_5)\) | \((0.1, 0.8, 0.1)\)    | \((0.4, 0.1, 0.2)\) | \((0.4, 0.2, 0.3)\)    | \((0.1, 0.8, 0.1)\) | \((0.9, 0, 0.1)\) |

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