Inside the Schwarzschild-Tangherlini black holes

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Abstract

The first-order semiclassical Einstein field equations are solved in the interior of the Schwarzschild-Tangherlini black holes. The source term is taken to be the stress-energy tensor of the quantized massive scalar field with arbitrary curvature coupling calculated within the framework of the Schwinger-DeWitt approximation. It is shown that for the minimal coupling the quantum effects tend to isotropize the interior of the black hole (which can be interpreted as an anisotropic collapsing universe) for $D = 4$ and 5, whereas for $D = 6$ and 7 the spacetime becomes more anisotropic. Similar behavior is observed for the conformal coupling with the reservation that for $D = 5$ isotropization of the spacetime occurs during (approximately) the first $1/3$ of the lifetime of the interior universe. On the other hand, we find that regardless of the dimension, the quantum perturbations initially strengthen the grow of curvature and its later behavior depends on the dimension and the coupling. It is shown that the Karlhede’s scalar can still be used as a useful device for locating the horizon of the quantum-corrected black hole, as expected.

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I. INTRODUCTION

Although described by the same line element, the classical interior of the Schwarzschild-
Tangherlini black hole has entirely different properties than the region outside the event
horizon and can be better understood as some sort of the anisotropic and nonstatic uni-
verse. This interpretation (not mandatory) is helpful when one is forced to abandon
usual, i.e., referring to the external world, interpretation of the coordinates, metric poten-
tials and so on. In the $D = 4$-dimensional case much work have been done in this direction
and we have a good understanding of the geometry and dynamics of the classical interior
(See e.g., Refs. [5–8] and the references therein). On the other hand, less is known about
quantum processes inside black holes and their influence upon the background geometry.

In the recent paper [9] we have studied influence of the quantized fields on the static
spacetime of the Schwarzschild-Tangherlini black hole using the semi-classical Einstein field
equations. Since the stress-energy tensor constructed in that paper functionally depends
on the metric, one has a rare opportunity to analyze and compare the quantum corrections
to the black hole characteristics (and the geometry itself) calculated for various spacetime
dimensions. The purpose of this paper, which is a natural continuation of Refs. [9, 10], is
to extend the study of the quantized fields to the interiors of the higher-dimensional black
holes. It should be noted however, that now there are problems that do not appear in
the external region. The first one is the problem of the central singularity and its closest
vicinity. It is evident that the semicalssical Einstein field equations cannot be trusted there.
The second difficulty is to some extend related to the previous one and may be stated as
follows. The effective action of the quantized fields for $r_+ \geq 0$ ($r_+$ is the coordinate of
the event horizon) has been constructed for the positive-definite metric signature. Once
the stress-energy tensor is calculated it can be transformed to the physical spacetime by
analytic continuation. In the exterior region it is the familiar Wick rotation, which affects
only the time coordinate. On the other hand, inside the event horizon the problem is more
complicated.

The classical $D-$dimensional solution describing interior of the Schwarzschild-Tangherlini
black hole with the event horizon located at $T = r_+$ is given by the line element

$$
ds^2 = - \left[ \left( \frac{r_+}{T} \right)^{D-3} - 1 \right]^{-1} dT^2 + \left[ \left( \frac{r_+}{T} \right)^{D-3} - 1 \right] dX^2 + T^2 d\Omega_{D-2}^2, \quad (1)$$
where $d\Omega_{D-2}^2$ is a metric on a unit $(D-2)$-dimensional sphere. Since only in $D = 4$ case there is a simple linear relation between the mass and $r_+$ in the present paper we use (almost) exclusively the latter. The radius of the event horizon of the Schwarzschild-Tangherlini black hole characterized by the mass $M$ is given by

$$r_+ = \left( \frac{16\pi G(D) M}{c^2(D-2)\omega_{D-2}} \right)^{1/(D-3)},$$

(2)

where $G(D)$ is $D$-dimensional Newton constant and $\omega_{D-2}$ is the volume of the unit $(D-2)$-dimensional sphere. If the $(D-2)$-dimensional sphere is covered by a standard “angular” coordinates $\theta_1, ..., \theta_{D-2}$ the metric $T^2d\Omega_{D-2}^2$ can be written in the form

$$T^2d\Omega_{D-2}^2 = T^2 \left[d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 (\ldots)\right].$$

(3)

Now, in order to construct the positive-definite metric let us replace $T$ by $iT$, $\theta_1$ by $i\theta_1$ and $r_+$ by $ir_+$. The metric thus becomes:

$$ds^2 = \left[ \left( \frac{r_+}{T} \right)^{D-3} - 1 \right]^{-1} dT^2 + \left[ \left( \frac{r_+}{T} \right)^{D-3} - 1 \right] dX^2 + T^2d\Omega_{D-2}^2,$$

(4)

where

$$T^2d\Omega_{D-2}^2 = T^2 \left[d\theta_1^2 + \sinh^2 \theta_1 d\theta_2^2 (\ldots)\right].$$

(5)

Note that our transformation differs from that of Ref. [11], which results in the negative-definite metric.

Having Euclidean version of the geometry of the black hole interior one can construct the one-loop approximation to the effective action of the quantized massive fields in a large mass limit. Indeed, for a sufficiently massive fields, i.e., when the Compton length, $\lambda_C$, associated with the mass of the field, $m$, is much smaller than the characteristic radius of the curvature of the spacetime $L$, the contribution of the vacuum polarization to the effective action dominates and the contribution of real particles is negligible. One can therefore make use of the Schwinger-DeWitt asymptotic expansion that approximates the effective action $W^{(1)}$. This approach has been successfully applied in a number of interesting cases and the background spacetimes range from black holes [12–10] to cosmology [20, 21] and from wormholes [22] to topologically nontrivial spacetimes [23, 26]. For the purposes of the present paper, the most relevant are the results presented in Refs. [9, 10, 27].

In Ref. [9] it has been shown that the approximate one-loop effective action $W^{(1)}$ of the quantized massive scalar field in a large mass limit can be constructed from the (asymptotic)
Schwinger-DeWitt representation of the Green function and in the lowest order it can be written in the following form:

$$W^{(1)}_{\text{reg}} = \frac{1}{2(4\pi)^{D/2}} \int d^D x \sqrt{g} \frac{[a_k]}{(m^2)^{k-D/2}} \Gamma \left( k - \frac{D}{2} \right),$$

(6)

where $k = \left\lfloor \frac{D}{2} \right\rfloor + 1$ and $[x]$ denote the floor function, i.e., it gives the largest integer less than or equal to $x$. Here $[a_k]$ is the coincidence limit of the $k$-th Hadamard-DeWitt coefficient constructed from the Riemann tensor, its covariant derivatives up to $(2k - 2)$-order and contractions. For the technical details concerning construction of the Hadamard-DeWitt coefficients the reader is referred to Refs. [28–31]. The (regularized) stress-energy tensor can be calculated from the standard definition

$$T^{ab} = \frac{2}{g^{1/2}} \frac{\delta}{\delta g_{ab}} W^{(1)}_{\text{reg}}.$$  

(7)

There is one-to-one correspondence between the order of the WKB approximation and the order of the Schwinger-DeWitt expansion. For example, the sixth-order WKB approximation is equivalent to $m^{-2}$ term in $D = 4$ and to $m^{-1}$ in $D = 5$ whereas for the analogous results in $D = 6$ and $D = 7$ the eight-order WKB approximation is required.

On general grounds one expects that the lowest-order (nonvanishing) term of the Schwinger-DeWitt expansion is the most important. The condition $\lambda_C/L \ll 1$ (with the physical constants reinserted) leads to

$$T \gg \left( \frac{\hbar^2 r_+^{D-3} s^{1/2}}{c^{D+1} m^2} \right)^{1/(D-1)} = \left( \frac{G_{(D)} \hbar^2 M}{c^{D+3} m^2} \right)^{1/(D-1)} \left( \frac{16\pi s^{1/2}}{(D-2) \omega_{D-2}} \right)^{1/(D-1)},$$

(8)

where $s = (D-1)(D-2)^2(D-3)$ and $T$ is given in seconds. For example, taking $D = 4$, $r_+$ equal to the Schwarzschild radius of the Sun and $m = 10^{-30}$ kg one has $T \gg 10^{-16}$ which is many orders of magnitude smaller than the coordinate time of the event horizon. It follows than that in our calculations we can go fairly close to the central singularity. Note that the coordinate time goes form $r_+/c$ to 0. In the rest of the paper we use the geometric units and the adopted conventions are those of Misner, Thorne and Wheeler [32].

The paper is organized as follows. In Sec. II we study some aspects of the classical interior of the $D$-dimensional Schwarzschild-Tangehrlii black holes. In Sec. III we construct and formally solve the $D$-dimensional semiclassical Einstein field equation and analyze the

1 This is a generalization of the condition $T \gg (M/m^2)^{1/3}$ employed in the $D = 4$-dimensional back reaction calculations reported in Ref. [10].
problem of the finite renormalization. In Sec. III A we show how to construct the appropriate measure of anisotropy and investigate the two useful scalars: the Kretschmann scalar and the Karlhede scalar. Finally, taking the stress-energy tensor of the quantized massive scalar field, in Sec. III B we study the semiclassical equations and analyze the influence of quantum perturbations on the black hole interior for $4 \leq D \leq 7$.

II. INTERIOR OF CLASSICAL SCHWARZSCHILD-TANGHERLINI BLACK HOLE

To gain a better understanding of the classical interior of the Schwarzschild-Tangherlini black hole let us introduce the proper time

$$\tau = \int \frac{dT}{\left[\left(\frac{r}{r_0}\right)^{D-3} - 1\right]^{1/2}},$$

and, in the neighborhood of a point $(\theta_{(0)1}, \theta_{(0)2}, ..., \theta_{(0)D-2})$, the locally Euclidean coordinates

$$x_1 = r_+ \left(\theta_1 - \theta_{(0)1}\right)$$
$$x_2 = r_+ \sin \theta_{(0)1} \left(\theta_2 - \theta_{(0)2}\right)$$
$$.........................$$
$$x_{D-2} = r_+ \sin \theta_{(0)1} \cdots \sin \theta_{(0)D-3} \left(\theta_{D-2} - \theta_{(0)D-2}\right).$$

Near the singularity the Schwarzschild-Tangherlini metric can be approximated by the Kasner metric

$$ds^2 = -d\tau^2 + \left(\frac{\tau}{\tau_0}\right)^{-2p_1} dX^2 + \left(\frac{\tau}{\tau_0}\right)^{2p_2} \left(dx_1^2 + \cdots + dx_{D-2}^2\right),$$

where

$$\tau_0 = \frac{2r_+}{D - 1}, \quad p_1 = -\frac{D - 3}{D - 1} \quad \text{and} \quad p_2 = \frac{2}{D - 1}. $$

It can easily be checked that both Kasner conditions are satisfied. Indeed,

$$p_1 + (D - 2)p_2 = 1$$

and

$$p_1^2 + (D - 2)p_2^2 = 1.$$
On the other hand, near the event horizon the Schwarzschild-Tangherlini metric asymptotically approaches
\[
ds^2 = -d\tau^2 + \left(\frac{D-3}{2r_+}\right)^2 \tau^2 dX^2 + dx_1^2 + \ldots + dx_{D-2}^2,
\]
(15)
where
\[
\tau = \frac{2r_+}{D-3} \left(1 - \frac{T}{r_+}\right)^{1/2}.
\]
(16)

Once again it is the Kasner metric with \(p_1 = 1\) and vanishing remaining Kasner exponents.
Finally observe, that the line element (15) can be formally obtained from the Rindler solution
\[
ds^2 = -g_{xx} dt^2 + dx^2 + \ldots
\]
(17)
by using the complex coordinate transformation.

Now, let us consider two points at the same coordinate instant separated by \(\Delta X\).
While the coordinate distance remains constant the physical distance between two points on the
\(X\)– coordinate line is given by
\[
d_{X_1X_2} = \int_{X_1}^{X_2} \left[\left(\frac{r_+}{T}\right)^{D-3} - 1\right]^{1/2} dX
\]
\[
= \left[\left(\frac{r_+}{T}\right)^{D-3} - 1\right]^{1/2} \Delta X
\]
(18)
and grows as the coordinate time decreases. On the other hand, the proper distance between
two points separated by \(d\Omega_{D-2}\) is given by
\[
d_{\Omega} = T d\Omega_{D-2},
\]
(19)
and it decreases as the coordinate time goes from \(r_+\) to 0. This behavior is independent of
the dimension \((D \geq 4)\).

Let us return to the proper time: It should be noted that taking a positive sign of the
root of the equation
\[
d\tau^2 = -g_{TT} dT^2,
\]
(20)
as it has been done in Eq. (9), the proper time monotonically grows with the coordinate time
\(T\). Conversely, taking the negative root, the proper time increases as the coordinate time
goes from \(r_+\) to 0. Since the functional relations between \(\tau\) and \(T\) are not very illuminating
we present them graphically (Fig I), demanding for both types of the universe \(\tau = \pi/2\)
FIG. 1. This graph shows the proper time $\tau$ as a function of the coordinate time $T$ for $4 \leq D \leq 7$. The upper branches are plotted for a negative root of (20) whereas the lower ones for a positive root. The arrows indicate the direction of the flow of the coordinate time. Top to bottom (for the upper branches) the curves are plotted for $D = 4, 5, 6, 7$.

as $T = r_+$. This can always be done by suitable choice of the integration constant. The universe inside the event horizon (in both time scales) has a finite lifetime. From the results collected in Table I one sees that $\tau/r_+$ decreases with the dimension.

| Dimension | Proper time | Coordinate time |
|-----------|-------------|-----------------|
| $D = 4$   | $\tau = \pi r_+/2$ | $T = r_+$       |
| $D = 5$   | $\tau = r_+$    |                 |
| $D = 6$   | $\tau = 0.747 r_+$ |           |
| $D = 7$   | $\tau = 0.599 r_+$ |           |

TABLE I. The lifetime of the interior universe.
III. THE BACK REACTION

The classical Schwarzschild-Tangherlini line element is a solution of the $D$-dimensional vacuum Einstein field equations. In this section we shall analyze the corrections to the characteristics of the classical black hole interior caused by the quantum fields. The semiclassical field equations have the form

$$R_{ab} - \frac{1}{2} R g_{ab} = 8 \pi T_{ab}, \quad (21)$$

where $T_{ab}$ is the properly regularized stress-energy tensor of the quantized field(s) and all remaining symbols have their usual meaning. We have chosen, for simplicity, to work with the minimal generalization of the Einstein equations. Other curvature invariants can be added to the action functional, but the resulting equations can be treated in precisely the same way as the “minimal” theory.

A. General considerations

We shall analyze how far one can go with the semiclassical Einstein field equations without defining explicitly the stress-energy tensor of the quantized fields. The only requirement placed on the stress-energy tensor is its regularity on the event horizon and the absence of the net fluxes. Unfortunately, except for metrically simple manifolds with a high degree of symmetry, the equations (21) cannot be solved exactly. However, assuming the expected quantum corrections to be small, one can try to solve the equations perturbatively and concentrate on the first-order calculations (with the zeroth-order being the classical solution). To achieve this let us consider the general line element

$$ds^2 = -f(T)dt^2 + h(T)dX^2 + T^2 d\Omega_{D-2}^2 \quad (22)$$

with

$$f(T) = f_0(T) \left( 1 + \varepsilon f_1(T) \right) \quad (23)$$

and

$$h(T) = h_0(T) \left( 1 + \varepsilon h_1(T) \right), \quad (24)$$

where $f_1(T)$ and $h_1(T)$ are unknown functions, and $\varepsilon$ is a small dimensionless parameter, which helps to keep track of the order of the terms in complicated expansions. It must not
be confused (in $D = 4$ case) with the small parameter of Ref. [10]. The parameter $\varepsilon$ should be set to 1 at the end of the calculations. The functions $f_0(T)$ and $h_0(T)$ are given by $-g_{TT}$ and $g_{XX}$ of the line element (4), respectively.

The resulting semi-classical Einstein field equations for the line element (22-24) are given by

$$\frac{d}{dT} \left[ \left( r_+^{D-3} - T^{D-3} \right) f_1(T) \right] = \frac{16\pi}{D-2} T^{D-2} T_X^X$$

and

$$\frac{d}{dT} h_1(T) = -\frac{(D-3)T^{D-4}}{r_+^{D-3} - T^{D-3}} f_1(T) - \frac{16\pi}{D-2} \frac{T^{D-2}}{r_+^{D-3} - T^{D-3}} T_T^T.$$  

The first equation can formally be integrated to give

$$f_1(T) = \frac{C_1}{r_+^{D-3} - T^{D-3}} + \frac{16\pi}{(D-2)(r_+^{D-3} - T^{D-3})} \int_{r_+}^{T} T^{D-2} T_X^X dT.$$  

It can easily be shown that the integration constant $C_1$ has no independent meaning and can be absorbed into the definition of the renormalized (dressed) radius of the event horizon. Moreover, by the very same procedure, the constant $C_1$ can be absorbed in the second equation. Let us analyze this problem more closely. First, consider the function $f(T)$, which can be written as

$$1/f(T) = -\left[ \left( \frac{r_+}{T} \right)^{D-3} - 1 - \frac{\varepsilon C_1}{T^{D-3}} - \frac{16\pi\varepsilon}{(D-2)T^{D-3}} \int_{r_+}^{T} T^{D-2} T_X^X dT \right] + O(\varepsilon^2)$$

and observe that introducing the renormalized radius of the event horizon, $\bar{r}_+$, defined by the equation

$$r_+ \to \bar{r}_+ = r_+ + \frac{\varepsilon C_1}{(D-3)r_+^{D-4}}$$

the integration constant can be relegated in the first-order calculations. The same transformation can be used to renormalize $r_+$ in the second metric potential

$$h(T) = \left( \frac{r_+}{T} \right)^{D-3} - \frac{\varepsilon C_1}{T^{D-3}} - 1 + \varepsilon C_2 + O(\varepsilon),$$

where $O(\varepsilon)$ terms containing integrals of the stress-energy tensor have not been displayed explicitly. To determine the second integration constant, $C_2$, additional piece of information is needed. Fortunately, considered characteristics of the quantum-corrected interior of the Schwarzschild-Tangherlini black holes are independent of $C_2$. Since $C_1$ and $r_+$ have no independent physical meaning, in what follows, for notational simplicity, we shall replace $\bar{r}_+$ with $r_+$ and treat $r_+$ as the renormalized (dressed) radius of the event horizon.
On general grounds one expects that the components of the stress-energy tensor constructed within the framework of the Schwinger-DeWitt approximation are simple polynomials in \( r_+ / T \), and hence the calculations of the functions \( f_1 \) and \( h_1 \) reduce to two elementary quadratures. Now, in order to better understand the influence of the quantized fields on the black hole interior, we shall study the trace of the rate of the deformations tensor and the ratio of the Hubble parameters. Similarly, to study the influence of the quantized fields on the curvature we calculate the Kretschmann scalar. Additionally we will check if the Karlhede’s scalar is still a useful device for detecting the event horizon.

The interior of the Schwarzschild-Tangherlini black holes is nonstatic and anisotropic. Following Novikov’s paper [2], this can be analyzed using the rate of deformations tensor. Let us introduce the tensor \( p_{ab} \) defined as

\[
p_{ab} = g_{ab} + u_a u_b,
\]

where \( u^a = (-g_{00})^{-1/2} \delta_0^a \). Let the indices from the second half of the Latin alphabet denote spatial coordinates. The deformation rate tensor, which has only spatial components, is given by

\[
D_{rs} = \frac{1}{2\sqrt{-g_{00}}} \frac{\partial}{\partial T} p_{rs},
\]

and its trace is \( D = D_r^r \). Now, let us consider a volume element \( vol = \sqrt{p} \Delta X \Delta \theta_1 \ldots \Delta \theta_{D-2} \), where \( p = det(p_{rs}) \), and construct the quantity

\[
\sqrt{-g_{00}} D = \frac{1}{vol} \frac{\partial}{\partial T} vol
\]

with a natural interpretation as the speed of the relative change of the volume element of the space. For the quantum-corrected Schwarzschild-Tangherlini black hole the trace of the rate of deformation tensor \( D \) is given by

\[
D = D_0 + \varepsilon D_1,
\]

where

\[
D_0 = \frac{1}{T} \left[ \left( \frac{r_+}{T} \right)^{D-3} - 1 \right]^{-1/2} \left[ \left( \frac{D - 1}{2} \right) \left( \frac{r_+}{T} \right)^{D-3} - (D - 2) \right]
\]

and

\[
D_1 = -\frac{1}{2} f_1(T) D_0 + \frac{1}{2} \left[ \left( \frac{r_+}{T} \right)^{D-3} - 1 \right]^{1/2} h_1'(T).
\]

The trace \( D \) is independent of the integration constant \( C_2 \). It should be noted that in the closest vicinity of the event horizon the correction to the trace \( D \) is practically independent.
of the function $h'_1$. On the other hand, the behavior of the correction $D_1$, i.e., the question if the rate of deformations grows or decreases depends on the sign of the $f_1$.

The conclusions that can be drawn from the analysis of the classical part of the tensor $D_{rs}$ and its trace are qualitatively similar to those of the Novikov’s paper.\footnote{It should be noted that English translation of the Novikov’s paper is not always correct.} Regardless of the dimension $D_0$ is always negative in the vicinity of the event horizon, whereas it is always positive for $T < T_{min} \approx 0.75r_+$. $T_{min}$ is the smallest root of $D_0(T) = 0$.

Although the information yielded by the rate of deformation tensor is accurate, it is simultaneously hard to visualize and we need something somewhat simpler. The useful measure of the anisotropy is the ratio of the Hubble parameters

$$\alpha = \frac{H_X}{H_\theta} = \frac{g_{\theta \theta} \frac{d}{dT} g_{XX}}{g_{XX} \frac{d}{dT} g_{\theta \theta}},$$

where $\theta$ is any of the angular coordinates. Making use of (23) and (24), one obtains

$$\alpha = \frac{D - 3}{2} \left( \frac{r_+}{T} \right)^{D-3} \left[ \frac{f_1(T)}{T^4} \right] - 1 \left[ \frac{(D-2)(1-D^2)}{(r_+)^{D-3}} \frac{h'_1(T)}{T^4} \frac{h''_1(T)}{T^4} \right].$$

where the first term in the right hand side of the above equation is the unperturbed part of $\alpha$. Consequently, the second term, which we denote by $\delta \alpha$, depending on the sign can make the black hole interior more isotropic or anisotropic. Further analysis of the role played by $\delta \alpha$ must be postponed until we solve the semicalssical Einstein field equations.

It could be easily shown that the simple differential curvature invariant which is very useful in detecting the location of $r_+$ (sometimes called Karlhede’s invariant)\footnote{It should be noted that English translation of the Novikov’s paper is not always correct.}

$$I = R_{abcd,e} R^{abcd,e}$$

vanishes on the event horizon of the Schwarzschild-Tangherlini black hole and is positive inside and negative outside. Because of their properties such invariants have become popular recently, see e.g., [35–37]. Now, making use of the functions $f$ and $h$ one has

$$I = (D-3)(D-2)(D-1) \left( \frac{r_+}{T} \right)^{D-3} \left[ \frac{f_1(T)}{T^3} \right] - 1 \left[ \frac{(D-2)(1-D^2)}{(r_+)^{D-3}} \frac{h'_1(T)}{T^3} \frac{h''_1(T)}{T^3} \right].$$

where

$$\alpha = \frac{D - 3}{2} \left( \frac{r_+}{T} \right)^{D-3} \left[ \frac{f_1(T)}{T^4} \right] - 1 \left[ \frac{(D-2)(1-D^2)}{(r_+)^{D-3}} \frac{h'_1(T)}{T^4} \frac{h''_1(T)}{T^4} \right].$$

$$\frac{D - 3}{2} \left( \frac{r_+}{T} \right)^{D-3} \left[ \frac{f_1(T)}{T^4} \right] - 1 \left[ \frac{(D-2)(1-D^2)}{(r_+)^{D-3}} \frac{h'_1(T)}{T^4} \frac{h''_1(T)}{T^4} \right].$$
To obtain the classical term it suffices to set to zero the functions \( f_1(T) \) and \( h_1(T) \). Because of the presence of the factor \( h_0(T) \), the invariant \( I \) of the quantum-corrected black hole always vanishes at the event horizon provided the functions \( f_1(T) \) and \( h_1(T) \) are regular in its vicinity. In view of the foregoing discussion it is expected that in the case at hand the condition \( I(r_+) = 0 \) is satisfied.

Finally, let us consider the simplest curvature invariant, namely the Kretschmann scalar, defined as the “square” of the Riemann tensor

\[
K = R_{abcd} R^{abcd},
\]

which, for the quantum-corrected interior of the \( D \)-dimensional Schwarzschild-Tangherlini black hole has the following form:

\[
K = \frac{(D - 1)(D - 2)(D - 3)}{r_+^4} \left( \frac{r_+}{T} \right)^{2D-2} + \frac{2(D - 2)(D - 3)}{r_+^2 T} \left[ \left( \frac{r_+}{T} \right)^{2D-4} - \left( \frac{r_+}{T} \right)^{D-1} \right] h_1''(T) \\
- \frac{2(D - 2)(D - 3)}{r_+^3 T} \left( \frac{r_+}{T} \right)^D \left[ -2 + 2 \left( \frac{r_+}{T} \right)^{D-3} - 3D \left( \frac{r_+}{T} \right)^{D-3} + D^2 \left( \frac{r_+}{T} \right)^{D-3} \right] f_1(T) \\
+ \frac{(D - 2)(D - 3)}{r_+^3} \left( \frac{r_+}{T} \right)^D \left[ -2 - \left( \frac{r_+}{T} \right)^{D-3} + D \left( \frac{r_+}{T} \right)^{D-3} \right] f_1'(T) \\
- \frac{(D - 2)(D - 3)}{r_+^3} \left( \frac{r_+}{T} \right)^D \left[ -2 - 7 \left( \frac{r_+}{T} \right)^{D-3} + 3D \left( \frac{r_+}{T} \right)^{D-3} \right] h_1'(T).
\]

The first term in the right-hand-side of the above equation gives the classical Kretschmann scalar and the remaining ones are the quantum corrections, which we denote by \( \delta K \). Although the semiclassical Einstein field equation are certainly incorrect as \( T \to 0 \), and should be replaced by the (unknown as yet) quantum gravity, it is of some interest to study the tendency exhibited by \( \delta K \) in this very limit. This, however, requires explicit knowledge of the functions \( f_1(T) \) and \( h_1(T) \), which is the subject of the next subsection.

**B. The back reaction of the quantized massive fields**

Now, let us return to the semiclassical Einstein field equations and solve \([25]\) and \([26]\) with the stress-energy tensor of the quantized massive scalar field. The relevant components of \( T_a^b \) are listed in Appendix A. The angular components can easily be calculated form the covariant conservation equation \( \nabla_a T_a^b = 0 \). For any considered dimension, the components of the tensor are simple polynomials (in \( x = r_+/T \)), the difference \( T_T^T - T_X^X \) factorizes as

\[
T_T^T - T_X^X = F(T)g_{XX},
\]

12
and the function $F(T)$ is regular for $T > 0$. Indeed, after some algebra, one has

$$g_{00} = - \left( x^{D-3} - 1 \right)^{-1} (1 + \varepsilon f_1) \quad (44)$$

and

$$g_{11} = \left( x^{D-3} - 1 \right) (1 + \varepsilon h_1), \quad (45)$$

where for $D = 4$

$$f_1 = \alpha_1^{(4)} x(1 + x + x^2 + x^3 + x^4) + \beta_1^{(4)} x^5, \quad (46)$$

$$h_1 = -\alpha_1^{(4)} x(1 + x + x^2 + x^3 + x^4) + \beta_2^{(4)} x^6 + K^{(4)}, \quad (47)$$

and where the coefficients $\alpha_i^{(D)}$ and the integration constants $K^{(D)}$ are listed in Appendix [3]. This result is not new: The stress-energy tensor has been obtained in the early 80’s by Frolov and Zel’nikov [12, 13] and subsequently used in Ref. [10]. To the best of our knowledge the results for higher dimensional geometries are new. Now, making use of the stress-energy tensor constructed in the $D = 5$ Schwarzschild-Tangherlini spacetime, one has

$$f_1 = \alpha_1^{(5)} x^2(1 + x^2 + x^4) + \beta_1^{(5)} x^8, \quad (48)$$

$$h_1 = -\alpha_1^{(5)} x^2(1 + x^2 + x^4) + \beta_2^{(5)} x^8 + K^{(5)}, \quad (49)$$

Both tensors have been calculated from the effective action constructed from the $[a_3]$. Similarly, making use the effective action constructed form the coefficient $[a_4]$, one obtains

$$f_1 = \alpha_1^{(6)} x^3(1 + x^3 + x^6) + \beta_1^{(6)} x^{12} + \gamma_1^{(6)} x^{15}, \quad (50)$$

$$h_1 = -\alpha_1^{(6)} x^3(1 + x^3 + x^6) + \beta_2^{(6)} x^{12} + \gamma_2^{(6)} x^{15} + K^{(6)} \quad (51)$$

and

$$f_1 = \alpha_1^{(7)} \left[ \frac{x^4}{1 + x^2} + x^6(1 + x^4) \right] + \beta_1^{(7)} x^{14} + \gamma_1^{(7)} x^{18}, \quad (52)$$

$$h_1 = -\alpha_1^{(7)} \left[ \frac{x^4}{1 + x^2} + x^6(1 + x^4) \right] + \beta_2^{(7)} x^{14} + \gamma_2^{(7)} x^{18} + K^{(7)}, \quad (53)$$

for $D = 6$ and $D = 7$, respectively. It should be noted that for $D = 7$ the functions loose their polynomial character, but they are still regular except for $T = 0$.

Having constructed the functions $f_1(T)$ and $h_1(T)$ one can analyze the quantum corrections to the Kretschmann scalar and anisotropy of the black hole interior. First, let us consider $\alpha$. Inspection of the unperturbed part of $\alpha$ shows that it is always negative. The
sign of $\alpha$ is positive if the internal geometry expands or contracts in all spatial directions. Of course, for isotropic evolution one has $\alpha = 1$. The negative sign of $\alpha$ means that it is contracting in one dimension and expanding in the other. Depending on the sign, the quantum perturbation can strengthen or weaken the anisotropy. Since $\alpha_0 < 0$ the anisotropy is weaken for $\delta \alpha > 0$ and strengthen in the regions where $\delta \alpha < 0$. This, however, depends on the coupling constant $\xi$ and the coordinate time $T$ in a quite complicated way. The results of the numerical calculations has been plotted in Figs 2 and 3. Specifically, for $D = 4$ and $D = 5$ the $\delta \alpha$ is negative above the $(\delta \alpha = 0)$-curves and positive below. On the other hand, for $D = 6$ and $D = 7$ the perturbation is negative below the curves and positive above.

| Dimension | $\xi = 0$ | $\xi = \frac{D-2}{4D-4}$ |
|-----------|-----------|---------------------------|
| $D = 4$   | more isotropic         | more isotropic            |
| $D = 5$   | more isotropic         | more isotropic for $x > 0.365$ |
| $D = 6$   | more anisotropic       | more anisotropic          |
| $D = 7$   | more anisotropic       | more anisotropic          |

TABLE II. The influence of $\delta \alpha$ on the black hole interior. Depending on the coupling $\xi$ and the dimension the quantum corrections can make the spacetime more isotropic or more anisotropic.

Now, we shall analyze in some details the behavior of the corrections to the Kretschmann scalar and the measure of the anisotropy $\alpha$ for the physical values of the coupling parameter, i.e., for the minimal coupling $\xi = 0$ and the conformal coupling $\xi_c = (D - 2)/(4D - 4)$. (There is no need to perform such analysis for Karlhede’s scalar as its main role is to serve as a useful device for detecting location of the event horizon. Inspection of (10) and (16-53) shows that $I = 0$ at the quantum-corrected event horizon, as expected.)

As have been observed earlier in Ref. [10], the quantum corrections for the minimal and conformal coupling always tend to isotropize the interior of the Schwarzschild black hole. On the other hand however, in higher dimension the pattern is more complicated. Indeed, for $D = 5$ the perturbation $\delta \alpha > 0$ for $\xi = 0$ whereas for $\xi_c = 3/16$ isotropization occurs only for $T > 0.365 r_+$. In turn, for $D = 6$ and $D = 7$, the perturbation $\delta \alpha$ is always negative for the minimal and the conformal coupling, i.e., the quantum effects make the black hole interior more anisotropic. These results are tabulated in Table [11].

Let us analyze how the growth of the curvature are affected by the quantum processes.
| Dimension | $\xi = 0$ | $\xi = \frac{D-2}{4D-4}$ |
|-----------|-----------|------------------|
| $D = 4$   | $\delta K > 0$ for $x > 0.986$ | $\delta K$ always positive |
| $D = 5$   | $\delta K$ always positive | $\delta K$ always positive |
| $D = 6$   | $\delta K$ always positive | $\delta K > 0$ for $x > 0.543$ |
| $D = 7$   | $\delta K$ always positive | $\delta K > 0$ for $x > 0.599$ |

TABLE III. The sign of the quantum corrections to the Kretschmann scalar.

And since the classical part of the Kretschmann scalar is positive, one concludes that the growth of curvature (as $T$ decreases) is weakened if the perturbation is negative. Inspection of the Figs. 4 and 5 as well as the Table III shows that initially, regardless of the dimension, $\delta K$ is always positive for the both types of couplings. This is very important message as it refers to the region where the quality of the approximation is expected to be high.

A few words of comment are in order here. First, it should be emphasized once more that although we have plotted functions $\delta \alpha = 0$ and $\delta K = 0$ for all allowable values of the coordinate time the approximation certainly does not work in the region close to the central singularity. Therefore our results show the tendency in behavior of the quantum corrections as the central singularity is approached (which can be misleading) rather than their actual run. Of course the answer to the question of how close the singularity can be approached depends on many factors, such as dimension, the ‘radius’ of the event horizon and the type of the quantized field. Each case should be analyzed separately. The second observation is less obvious and is, roughly speaking, related to the question of how long the first-order approximation dominates the higher-orders terms inside the event horizon. Once again this problem goes to the very core of the Schwinger-DeWitt asymptotic expansion. And once again there is no better answer than to recall its principal assumptions. Finally, let us observe that although the quantum corrections caused by a solitary field is expected to be small in the domain of applicability of the approximation, they can be made arbitrarily large for a large number of fields.
FIG. 2. The dependence of the curvature coupling parameter, $\xi$, on $T/r_+$ for zero perturbation to the anisotropy ($\delta \alpha = 0$) of the interior of the $D = 4$ (dashed line) and $D = 5$ (solid line) Schwarzschild-Tangherlini black hole. Above the curves, the quantum corrections make the black hole interior more anisotropic, whereas below the curves the spacetime becomes more isotropic.

FIG. 3. The dependence of the curvature coupling parameter, $\xi$, on $T/r_+$ for zero perturbation to the anisotropy ($\delta \alpha = 0$) of the interior of the $D = 6$ (dashed line) and $D = 7$ (solid line) Schwarzschild-Tangherlini black hole. Above the curves, the quantum corrections make the black hole interior more isotropic, whereas below the curves the spacetime becomes more anisotropic.
FIG. 4. The dependence of the curvature coupling parameter, $\xi$, on $T/r_+$ for zero perturbation to the Kretschmann scalar of the interior of the $D = 4$ (dashed line) and $D = 5$ (solid line) Schwarzschild-Tangherlini black hole. Above the curves, the quantum corrections to the scalar $K$ are positive, whereas below the curves they are negative.

FIG. 5. The dependence of the curvature coupling parameter, $\xi$, on $T/r_+$ for zero perturbation to the Kretschmann scalar of the interior of the $D = 6$ (dashed line) and $D = 7$ (solid line) Schwarzschild-Tangherlini black hole. Above the curves, the quantum corrections to the scalar $K$ are negative, whereas below the curves they are positive.
Appendix A: The stress-energy tensor of the quantized massive scalar field in the interior of the higher-dimensional Schwarzschild-Tangherlini black hole

In this appendix we list the \((T, T)\) and \((X, X)\) components of the stress-energy tensor for \(4 \leq D \leq 7\). The angular components are not displayed as they can easily be calculated from the covariant conservation equation.

\(D = 4\).

\[ T^{(4)T}_{T} = \frac{\xi r^2_+ (3r_+ - 4T)}{80\pi^2 m^2 T^9} + \frac{r^2_+ (63T - 47r_+)}{5760\pi^2 m^2 T^9} \]  
\[ T^{(4)X}_{X} = \frac{r^2_+ (1237r_+ - 1125T)}{40320\pi^2 m^2 T^9} - \frac{\xi r^2_+ (11r_+ - 10T)}{80\pi^2 m^2 T^9} \]

\(D = 5\).

\[ T^{(5)T}_{T} = \frac{\xi r^4_+ (2r^2_+ - 3T^2)}{10\pi^2 m^2 T^{12}} + \frac{r^4_+ (297T^2 - 185r^2_+)}{5040\pi^2 m^2 T^{12}} \]

\[ T^{(5)X}_{X} = \frac{841r^6 - 729r^4 T^2}{5040\pi^2 m^2 T^{12}} - \frac{\xi r^4_+ (7r^2_+ - 6T^2)}{10\pi^2 m^2 T^{12}} \]

\(D = 6\).

\[ T^{(6)T}_{T} = \frac{r^6_+ \left( 53938r^6_+ - 115360r^4_+ T^3 + 48195T^6 \right)}{20160\pi^3 m^2 T^{20}} - \frac{5\xi r^6_+ \left( 6665r^6_+ - 14444r^3_+ T^3 + 6048T^6 \right)}{2688\pi^3 m^2 T^{20}} \]

\[ T^{(6)X}_{X} = \frac{5\xi r^6_+ \left( 11997r^6_+ - 15956r^3_+ T^3 + 4536T^6 \right)}{896\pi^3 m^2 T^{20}} - \frac{r^6_+ \left( 295892r^6_+ - 404570r^3_+ T^3 + 121905T^6 \right)}{20160\pi^3 m^2 T^{20}} \]

\(D = 7\).

\[ T^{(7)T}_{T} = \frac{r^8_+ \left( 30549r^8_+ - 66088r^4_+ T^4 + 26544T^8 \right)}{4480\pi^3 m^2 T^{24}} - \frac{9\xi r^8_+ \left( 198r^8_+ - 435r^4_+ T^4 + 175T^8 \right)}{56\pi^3 m^2 T^{24}} \]

\[ T^{(7)X}_{X} = \frac{9\xi r^8_+ \left( 528r^8_+ - 672r^4_+ T^4 + 175T^8 \right)}{28\pi^3 m^2 T^{24}} - \frac{r^8_+ \left( 4713r^8_+ - 6192r^4_+ T^4 + 1736T^8 \right)}{128\pi^3 m^2 T^{24}} \]

Appendix B: Coefficients of the functions \(f_1(T)\) and \(h_1(T)\)

Here we list the coefficients of the functions \(f_1(T)\) and \(h_1(T)\). (See Eqs. (46 - 53)). The integration constants \(C^{(D)}_2\) are left unspecified and should be determined from the physically
motivated boundary conditions. All quantities considered in the main text, such as $\alpha$, $D$, $K$ and $I$ are independent of the integration constant $C^{(D)}_2$.

\[ \alpha^{(4)}_1 = \frac{113 - 504\xi}{30240\pi m^2 r^4} \quad \text{(B1)} \]

\[ \beta^{(4)}_1 = \frac{-1237 + 5544\xi}{30240\pi m^2 r^4} \quad \text{(B2)} \]

\[ \beta^{(4)}_2 = \frac{-47 - 216\xi}{4320\pi m^2 r^4} \quad \text{(B3)} \]

\[ K^{(4)} = \frac{149 - 672\xi}{5040\pi m^2 r^4} + C^{(4)}_2 \quad \text{(B4)} \]

\[ \alpha^{(5)}_1 = \frac{131 - 504\xi}{7560\pi m^2 r^4} \quad \text{(B5)} \]

\[ \beta^{(5)}_1 = \frac{-841 - 3528\xi}{7560\pi m^2 r^4} \quad \text{(B6)} \]

\[ \beta^{(5)}_2 = \frac{-185 - 1008\xi}{7560\pi m^2 r^4} \quad \text{(B7)} \]

\[ K^{(5)} = \frac{289 - 1260\xi}{3780\pi m^2 r^4} + C^{(5)}_2 \quad \text{(B8)} \]

\[ \alpha^{(6)}_1 = \frac{-13291 - 87300\xi}{151200\pi m^2 r^6} \quad \text{(B9)} \]

\[ \beta^{(6)}_1 = \frac{-419641 - 1788300\xi}{151200\pi m^2 r^6} \quad \text{(B10)} \]

\[ \gamma^{(6)}_1 = \frac{591784 - 2699325\xi}{151200\pi m^2 r^6} \quad \text{(B11)} \]

\[ \beta^{(6)}_2 = \frac{-5609 - 54450\xi}{151200\pi m^2 r^6} \quad \text{(B12)} \]

\[ \gamma^{(6)}_2 = \frac{107876 - 499875\xi}{151200\pi m^2 r^6} \quad \text{(B13)} \]

\[ K^{(6)} = \frac{-9476 - 47155\xi}{10080\pi m^2 r^6} + C^{(6)}_2 \quad \text{(B14)} \]

\[ \alpha^{(7)}_1 = \frac{-1439 - 10080\xi}{8400\pi m^2 r^6} \quad \text{(B15)} \]

\[ \beta^{(7)}_1 = \frac{7579 - 32256\xi}{1680\pi m^2 r^6} \quad \text{(B16)} \]

\[ \gamma^{(7)}_1 = \frac{10997 - 50688\xi}{1680\pi m^2 r^6} \quad \text{(B17)} \]

\[ \beta^{(7)}_2 = \frac{479 + 720\xi}{8400\pi m^2 r^6} \quad \text{(B18)} \]
\begin{align}
\gamma_2^{(7)} &= \frac{10183 - 47520\xi}{8400\pi^2 r_+^6} \\
K^{(7)} &= -\frac{28519 - 144000\xi}{16800\pi^2 r_+^6} + C_2^{(7)}
\end{align}

\[\text{(B19)}\]

\[\text{(B20)}\]

[1] F. R. Tangherlini, Il Nuovo Cimento 27, 636 (1963).
[2] I. Novikov, Soviet Astronomy 5, 423 (1961).
[3] R. W. Brehme, Am. J. Phys 45, 423 (1977).
[4] R. Doran, F. S. Lobo, and P. Crawford, Foundations of Physics 38, 160 (2008), gr-qc/0609042.
[5] B. S. DiNunno and R. A. Matzner, Gen. Rel. Grav. 42, 63 (2010), ArXiv:0801.1734.
[6] V. P. Frolov and A. A. Shoom, Phys. Rev. D76, 064037 (2007), ArXiv:0705.1570.
[7] M. Christodoulou and C. Rovelli, Phys. Rev. D91, 064046 (2015), ArXiv:1411.2854.
[8] I. Bengtsson and E. Jakobsson, Mod. Phys. Lett. A30, 1550103 (2015), ArXiv:1502.01907.
[9] J. Matyjasek and P. Sadurski, Phys. Rev. D91, 044027 (2015), ArXiv:1412.2389.
[10] W. A. Hiscock, S. L. Larson, and P. R. Anderson, Phys. Rev. D56, 3571 (1997), gr-qc/9701004.
[11] P. Candelas and B. Jensen, Phys. Rev. D33, 1596 (1986).
[12] V. P. Frolov and A. I. Zel’nikov, Phys. Lett. B 115, 372 (1982).
[13] V. P. Frolov and A. I. Zel’nikov, Phys. Lett. B 123, 197 (1983).
[14] V. P. Frolov and A. I. Zelnikov, Phys. Rev. D29, 1057 (1984).
[15] J. Matyjasek, Phys. Rev. D61, 124019 (2000), gr-qc/9912020.
[16] J. Matyjasek, Phys. Rev. D63, 084004 (2001), gr-qc/0010097.
[17] J. Matyjasek, Phys. Rev. D74, 104030 (2006), gr-qc/0610020.
[18] R. Thompson and J. P. Lemos, Phys. Rev. D80, 064017 (2009), ArXiv:0811.3962.
[19] A. Belokogne and A. Folacci, Phys. Rev. D 90, 044045 (2014), ArXiv:1404.7422.
[20] J. Matyjasek and P. Sadurski, Phys. Rev. D88, 104015 (2013), ArXiv:1309.0552.
[21] J. Matyjasek, P. Sadurski, and M. Telecka, Phys. Rev. D89, 084055 (2014).
[22] B. E. Taylor, W. A. Hiscock, and P. R. Anderson, Phys. Rev. D55, 6116 (1997), gr-qc/9608036.
[23] L. A. Kofman and V. Sahni, Phys. Lett. B127, 197 (1983).
[24] V. Sahni and L. A. Kofman, Phys. Lett. A117, 275 (1986).
[25] L. Kofman, V. Sahni, and A. A. Starobinsky, Sov. Phys. JETP 58, 1090 (1983).
[26] J. Matyjasek and D. Tryniecki, Mod. Phys. Lett. A24, 2517 (2009), ArXiv:0908.2648.
[27] J. Matyjasek, P. Sadurski, and D. Tryniecki, Phys. Rev. D87, 124025 (2013), ArXiv:1304.6347.
[28] T. Sakai, Tôhoku Math. J. 23, 589 (1971).
[29] P. B. Gilkey, J. Differential Geometry 10, 601 (1975).
[30] P. Amsterdamski, A. Berkin, and D. O’Connor, Class. Quant. Grav. 6, 1981 (1989).
[31] I. G. Avramidi, Heat kernel and quantum gravity, (Springer-Verlag, Berlin, 2000).
[32] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (W. H. Freeman., San Francisco, 1973).
[33] V. P. Frolov and I. D. Novikov, Black hole physics (Kluwer Academic Publishers, Dordrecht, 1998).
[34] A. Karlhede, U. Lindstrom, and J. E. Aman, Gen. Rel. Grav. 14, 569 (1982).
[35] D. N. Page and A. A. Shoom, Phys. Rev. Lett. 114, 141102 (2015), ArXiv:1501.03510.
[36] M. Abdelqader and K. Lake, Phys. Rev. D91, 084017 (2015), ArXiv:1412.8757.
[37] J. Moffat and V. Toth (2014), ArXiv:1404.1845.