Axiom system and completeness expression for quantum mechanics

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(Dated: 05.16.2007)

Abstract

The standard axiomatization of quantum mechanics (QM) is not fully explicit about the role of the time-parameter. Especially, the time reference within the probability algorithm (the Born Rule, BR) is unclear. Using a plausible principle P1, about the role of probability in a physical theory, and a second principle P2 affording a most natural way to make BR precise, a logical conflict with the standard expression for the completeness of QM can be derived. Rejecting P1 is implausible. Rejecting P2 leads to unphysical results and to a conflict with a generalization of P2, a principle P3. It is thus made plausible that the standard expression of QM completeness must be revised. An absolutely explicit form of the axioms is provided, including a precise form of the projection postulate. An appropriate expression for QM completeness, reflecting the restrictions of the Gleason and Kochen-Specker theorems is proposed.
Quantum mechanics (QM), the most elementary of quantum theories, can be shown to be complete in a quite precise sense. It is impossible to assign pre-existing values to suitable physical systems beyond the QM allowances, under two plausible constraints. Take a physical system $S$ such that its QM representation requires a Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) > 2$. (A one-particle spin-1 system is an example.) It is impossible, in this case, to assign pre-existing values to all QM observables under the constraints that (i) algebraic relations among such value assignments mirror the algebraic relations among the (operators representing the) QM observables; (ii) the value assignments are non-contextual, i.e. every submaximal (degenerate) observable gets assigned a unique value, not one relativized to the different maximal (non-degenerate) observables of which it is a function. An indirect proof of this fact is immediately obtained from Gleason’s Theorem [1]. The theorem entails that, when $\{P\}$ is the set of all projection operators on $\mathcal{H}$, every mapping $\mu: \{P\} \rightarrow [0, 1]$ being interpretable as a probability function must be continuous, while a value assignment obeying (i) and (ii) must induce a mapping $\mu': \{P\} \rightarrow [0, 1]$ that is discontinuous [2].
A _direct_, i.e. constructive proof is the Kochen-Specker theorem [3, 4, 5, 6], presenting a finite set of operators for which an assignment obeying (i) and (ii) fails.

But why does the impossibility of assigning pre-existing values under these constraints tell us anything about QM completeness? After all, the theory’s empirical output consists just in probabilities for measurement results and their generalizations: expectation values. In an axiomatic formulation, QM is formally incapable of directly making value assignments, so it cannot generate anything conflicting with any value assignment to S. The natural idea filling this logical gap is the insight that some probability assignments entail value assignments, namely those that predict values with certainty. E.g., a QM prediction to the effect that S, with probability 1, will be found to have a property $a_k$ at time $t$ makes it plausible to conclude that S, at that time, has $a_k$. Working, from now on, in the Schrödinger picture, writing states as density operators, and taking $A$ as a discrete observable on S with values $a_1, a_2, \ldots$, we can express this idea as:

$$\text{if } P_{a_k}(t), \text{ then } a_k(t).$$

(Here, $P_{a_k}(t) = |a(t) > < a(t)|$, ‘$P_{a_k}(t)$’ abbreviates ‘S is in state $P_{a_k}(t)$’, and ‘$a_k(t)$’ abbreviates ‘S has $a_k$ at t’.) Adding such a plausible rule to the QM formalism, we can extract value assignments, but nothing near a set of values big enough to conflict with either the discontinuous assignment used in the corollary of Gleason’s Theorem or the assignment to some Kochen-Specker set of operators. This will be the case only if we _limit_ ourselves to the value assignments following from the QM state as follows:

$$\text{EE } a_k(t) \text{ if and only if } P_{a_k}(t).$$

This condition establishes a logical link between QM and the two theorems and thus makes precise in which sense they prove QM completeness. Indeed, the condition (often called the eigenstate-eigenvalue link, hence the label ‘EE’) embodies the classic definition of QM completeness [7, 8]. EE substantiates the generally accepted and most familiar idea that a QM system in a superposition of $A$ eigenstates does not have a value of $A$. 

This idea plays a special role when interpreters try to say what goes on when S, being in
superposition of \(A\)-eigenstates, meets an \(A\)-measurement device. It is standardly claimed
that S, during the measurement interaction, takes on one of the \(A\) values, e.g. \(a_k\) [9]. If S is
found to have a value of \(A\), e.g. \(a_k\), at a certain time, then \(EE\) dictates that S’s state is the
pertaining eigenstate, e.g. \(P_{a_k}\), at this time. A slightly less exact form of this consequence
would be: If S is found to have value \(a_k\), of \(A\), then S’s state immediately becomes the
pertaining eigenstate e.g. \(P_{a_k}\). This latter requirement is generally called the projection
postulate [10]. Projection, i.e. S’s adopting an \(A\)-eigenstate during \(A\)-measurement, is
generally thought to be an empirically confirmed fact and with good reason. We can
measure copies of S for \(A\), filter out the non-\(a_k\) results, and then experimentally confirm
the remaining state to be \(P_{a_k}\), e.g. via quantum-state tomography [11, 12]. \(EE\) seems to
have an exact version of the projection postulate as its consequence, thus seems to embody
both the completeness of QM and the necessity of projection upon finding a certain result.
For future reference, I extract from \(EE\) the parts representing, respectively, the complete-
ness condition (COMP) and the simplest candidate for a precise projection postulate (CPP):

\[
\text{COMP} \quad \text{If } S \text{ is not in state } P_{a_k}(t), \text{ then not } a_k(t).
\]

\[
\text{CPP} \quad \text{If } a_k(t), \text{ then } S \text{ is in state } P_{a_k}(t).
\]

(Note that, by contraposition, both COMP and CPP express the same (backward
or ‘only if’-) direction of \(EE\).) The aim of the present paper is to show that COMP
(and, consequently CPP) is in harmony with QM, in its standard axiomatization, and
to provide more appropriate expressions for completeness and projection. More exactly,
I will show COMP to be in conflict with QM as follows. I briefly review the standard
axioms of QM and point out that two of them are not fully unambiguous concerning
the role of the time parameter. I introduce three reasonable principles, P1-P3, where
P3 is a generalized version of P2. The first principle P1 concerns the interpretation of
probabilities in a physical theory, in general, while P2 and P3 refer specifically to the time
parameter in QM and remove the ambiguity in the axioms. (All axioms and principles
are introduced in Sec. II.) However, using P1 and P2 to interpret QM probabilities, we
can produce a contradiction from QM and COMP (Sec. III). It will immediately be clear that P1 is not open to reasonable doubt and that sacrificing P2 leads to an implausible and unphysical consequence. Hence, the standard way to express the completeness of QM, i.e. COMP, must be revised. The latter is not an appropriate expression of the limitations generated by the Gleason and Kochen-Specker theorems. Principle P2 suggests a more precise version of the QM axioms, including a precise version of the projection postulate (Sec. VI). Finally, a version of completeness will be proposed that both represents the limitations due to the two theorems and respects these axioms (Sec. VII).

Some interpretations of QM reject the projection postulate and EE; they are now collected under the title of modal interpretations [13]. This group of interpretations has a weaker expression of completeness at hand:

\[ \text{COMP}^* \quad \text{If } S \text{ is in a pure state } W(t) \neq P_{a_k}(t), \text{ then not } a_k(t). \]

The rationale of COMP* is this: While a measurement may leave S in a mixture (obtained by partial tracing of the state of the S-cum-apparatus-supersystem) such that we can say that S has adopted one of A's values without state projection, we still can express the idea that S, in a pure state W(t) at interaction onset, does not have any value of A. This possibility of implementing a weaker form of completeness into a weaker version of QM must be considered, which is done in a digression comprising Sec.s IV. and V. I show that rejecting P2, despite its plausibility must go along with rejecting the projection postulate, thus the acceptance of a reduced version of QM (the standard axioms without projection postulate), like the one adopted in modal interpretations (Sec. IV). Then I consider the third principle P3, a generalized version of P2, and show that COMP* and the reduced version of QM make P3 implausible (Sec. V). By nature of the foundational and conceptual questions involved, the reasoning will consist of logical, not mathematical, argument throughout.
II. AXIOMS AND PRINCIPLES

Consider the following standard axiomatization of QM, using again the Schrödinger picture and projection operators:

A1 Any QM system S is associated with a unique Hilbert space $\mathcal{H}$ and its state is represented by a unique density operator $W(t)$ on $\mathcal{H}$, a function of time.

A2 Any physical quantity $A$ (called an observable) is represented by a self-adjoint operator $A$ on $\mathcal{H}$ and the possible values of $A$ (possible properties of S) by the numbers in the spectrum of $A$.

A3 S evolves in time according to $W(t) = U(t)W(t_0)U(t)^{-1}$ where $U(t) = \exp[-iHt]$, a unitary operator, is a function of time and $H$ is an operator representing the total energy of S.

A4 If S is in state $W(t)$ and $A$ is an observable on S, then the expectation value $\langle A \rangle$ is: $\langle A \rangle = \text{Tr}(W(t)A)$.

A5 If S is found to have value $a_k$ as a result of an $A$ measurement, then S’s state is $P_{a_k}$ immediately after this measurement.

In view of the above discussion, I henceforth denote as QM the theory based on A1-A5, while a reduced version, based on A1-A4 only, will be called QM$^-$. Axiom A2 motivates an identification of physical observables and their mathematical representatives and I will not need to distinguish them. I will also, for simplicity, restrict myself to one discrete observable $A$ throughout. Finally, I will mostly restrict A4 to probabilities, i.e. expectation values of yes-no observables of type $P_{a_i}$. Let $a_k$ always be some fixed value of variable $a_i$. Let ‘$p(a_k)$’ mean the probability that S has $a_k$. Then, since $\langle P_{a_k} \rangle = p(a_k)$, A4 takes on a simpler, very familiar form, called the Born Rule (BR):
If $S$ is in state $W(t)$ and $A$ is an observable on $S$ with eigenvalue $a_k$, then the probability that $S$ has $a_k$ is: $p(a_k) = \text{Tr}(W(t)P_{a_k})$.

It should be emphasized that these axioms, though fairly standard, do not constitute a fully satisfactory axiomatization of QM since A4 and A5, in their present form, leave the role of the time parameter unspecified or vague. The defect in A4 carries over to BR, in whose equation only the right side, but not the left, carries a time-index. Two of the three principles, to be discussed presently, will have the sole purpose of forcing an unambiguous explication of the time-parameter on the left side of ‘$p(a_k) = \text{Tr}(W(t)P_{a_k})$’ and it should be stressed that the interpretations produced from these principles and considered below exhaust all the reasonable options.

Here are three principles, the first concerning the role of probability in a physical theory in general, the second and third its role in QM. The first principle can be motivated by the idea that probability is quantified possibility. More precisely: If a physical theory assigns an event a non-zero probability, then, given the theory’s truth, this event is possible. The weakest form of possibility is logical possibility. Thus, yet more precisely:

$$P1 \quad \text{If, for a proposition } F \text{ (describing an event) a theory } T \text{ yields another proposition } p(F) > 0, \text{ then it is not the case that } T, F \vdash \bot.$$  

(Here ‘$T, F \vdash \bot$’ means that the set of sentences including $F$ and all sentences of $T$ allows to derive a contradiction in first-order logic.) P1 is beyond reasonable doubt, but it also follows from natural assumptions about probability shared by the main interpretations of that notion [14].

The second principle runs:

$$P2 \quad \text{Any expression } ‘a_k’ \text{ such that it names a QM event can be qualified as}$$
‘$a_k(t)$’, where t is a time-parameter.

P2 is motivated by the idea that a fundamental physical theory must explicitly concern spacetime events. A fundamental theory that builds probability spaces over sets of events must be able to explicitly treat these events as spacetime events. Hence, all events that are assigned probabilities in QM must explicitly be spacetime events, here: properties (like $a_k$) possessed at certain sharp times. Fully relativistic versions of QM explicitly treat spacetime events with a finite time-extension $\Delta t$. In the present, non-relativistic, formulation we have $\Delta t = \delta t$: QM events just consist in S having one or more properties at a sharp time $t$. It should be stressed that relativistic generalizations always contain the limiting case $\delta t$ (see, e.g. [15]), hence an argument affecting QM in this respect will affect any relativistic generalization. Note, however, that P2 just says that those events denoted by statements of type ‘$a_k$’ within the QM probabilities are spacetime events such that the expressions can be explicated as ‘$a_k(t)$’. The ‘$a_k$’ may not be appropriate expressions of QM events within BR and P2 may have no application.

The third principle, P3, generalizes P2. It says, in effect, that whatever the QM events are (and whatever expressions denote them) these events can be qualified as spacetime events explicitly. Thus:

**P3**

For any expression ‘F’ such that it QM yields an expression ‘$p(F) = \text{Tr}(W(t)P_{a_k})$’ there is a parameter t in the formalism qualifying ‘F’ as ‘F(t)’.

P3 is formulated so wide as to appear vague. But it has only two specifications. The first is to place the time-index ‘inside the probability’, the second ‘outside the probability’. Consider the BR expression ‘$p(a_k) = \text{Tr}(W(t)P_{a_k})$’ made precise as ‘$p(a_k(t)) = \text{Tr}(W(t)P_{a_k})$’. This makes QM fulfill P2. But there is an alternative: Read the BR expression as ‘$p(t)(a_k) = \text{Tr}(W(t)P_{a_k})$’ and interpret the latter in the following way: The probability is a disposition of S at time $t$ to display value $a_k$ (make ‘$a_k$’ true). This idea is discussed widely in the literature and is generally explicated by saying that t is the onset time of a
measurement interaction on \( S \) and \( p(t)(a_k) \)’ quantifies \( S \)’s strength of disposition at \( t \) toward displaying \( a_k \) at some later time [4]. However, while this notion essentially refers to the idea of probabilities as dispositions it does not need to refer to measurement. We should avoid the impression that anything in our principles or axioms makes essential reference to measurement - as this is in fact unnecessary. We can speak more generally of a region of space containing \( S \) and a time \( t \) such that ‘\( E(t) \)’ names the disposition at \( t \) to display \( a_k \) at some later time and call ‘\( E(t) \)’ the triggering event. We can then say that an alternative to the preceding is to disambiguate ‘\( p(a_k) \)’ as ‘\( p(t)(a_k) \)’, which more explicitly reads ‘\( p(a_k) \) given \( E(t) \)’. We thus have an alternative disambiguation of BR that obeys our principle P3. (Note that the argument for only two possibilities is not strict. It would require heavy metalinguistic machinery to show that ‘\( p(t)(a_k) \)’ and ‘\( p(t)(a_k) \)’ are the only ways to specify the time-reference in ‘\( p(a_k) \)’.)

III. QM + COMP CONTRADICT PRINCIPLES P1 AND P2

P1 and P2 now generate the main argument against COMP. Using P2, we can make BR precise in a most natural way. It can now be rendered more exactly:

\[
\text{BR'} \quad \text{If } S \text{ is in state } W(t) \text{ and } A \text{ is an observable on } S \text{ with eigenvalue } a_k, \\
\text{then the probability that } S \text{ has } a_k \text{ at } t \text{ is: } p(a_k(t)) = \text{Tr}(W(t)P_{a_k}).
\]

Now suppose that \( S \) is in a state \( W(t_1) \neq P_{a_k}(t_1) \), for some value \( t_1 \) of \( t \), such that from BR’ it follows that \( 1 > p(a_k(t_1)) > 0 \). (Call this assumption N.) Assuming that a theory contains all its consequences, QM + P2 will contain BR’. Now, let QM + P2, COMP, and N be integrated into one artificial theory, QM’. Then, by simple sentential logic:

\[
\begin{align*}
N & \quad \text{(1)} \quad S \text{ is in state } W(t_1) \quad \text{(N)} \\
N, \text{ BR'} & \quad \text{(2)} \quad p(a_k(t_1)) > 0 \quad \text{(1), (BR')} \\
N & \quad \text{(3)} \quad \neg P_{a_k}(t_1) \quad \text{(N)} \\
N, \text{ COMP} & \quad \text{(4)} \quad \neg a_k(t_1) \quad \text{(3), (COMP)}
\end{align*}
\]
(As usual, the rightmost column indicates the assumptions on which the line in question directly depends and the leftmost column the ones on which the line ultimately depends.) By assumption, BR', COMP, N, are members of QM' which thus entails both line (2), i.e. that a certain proposition is assigned a positive probability, and line (4), i.e. that the negation of that proposition is true. Hence, QM' entails \( p(a_k(t_1)) > 0 \), but also: QM', \( a_k(t_1) \vdash \bot \), in contradiction with P1. Thus given P1, QM' cannot be true [16].

The argument presupposes that QM', the artificial integration of QM, P2 and assumption N is a \textit{theory}. Is the integration of N an innocuous step? Of course, we can add suitable propositions to QM to create a theory that contradicts virtually any other proposition. But N is a trivially admissible state assignment that QM must be consistent with. So, its integration into QM' is innocuous indeed, but the one of P2 is not. BR', COMP, N are in conflict with P1, where BR' is BR, interpreted via P2. Given that P1 is immune to rejection, QM is in conflict with either P2 or COMP.

IV. QM' AS THE SOLE ALTERNATIVE

To reject P2 implies to give up on the most natural disambiguation of BR. The defender of COMP will just say that within the BR equation \( 'p(a_k)' \) cannot be read as \( 'p(a_k(t))' \), the impression of naturalness notwithstanding. But to reject P2 has consequences for the axioms. Recall that A5, like A4 and BR, is vague. Let’s initially apply P2 to A5, yielding:

\begin{equation}
A5'
\end{equation}

If S is found to have value \( a_k(t) \) as a result of an \textbf{A} measurement, then S's state is \( P_{a_k} \) immediately after this measurement.

Note that A5' still substantially differs from CPP. Both, however, make precise the vague A5. Rejecting P2 would mean that A5 is not so made precise. It would mean, in effect, that expressions like \( 'a_k' \) are not explicated as \( 'a_k(t)' \) throughout QM. In this case, A5 automatically becomes vacuous. In the Schrödinger picture, state evolution cannot start without a precise input state. It is the intention of A5, to generate such an input – for starting post-measurement state evolution, e.g. when a measurement is
a preparation. CPP generates a precise state from the precise ‘$a_k(t)$’ and A5′ at least can be imagined to do so, when the phrase ‘immediately after’ is made precise. If A5 is understood as containing an expression ‘$a_k$’ that must not carry a time-index, it has no such quality. It is a vacuous statement, not only without any empirical content, but also a formally ineffective addition to the rest of QM. So, everyone seeking to escape the argument of Sec. III by rejecting P2 will have to reject the projection postulate in any substantial form.

One might object that applying P2 to BR is one thing and applying it to A5 another. But if we reject P2 for the expression ‘$p(a_k)$’ (refuse to read it as ‘$p(a_k(t))$’) we say that these probabilities do not mean probabilities for ‘$a_k(t)$’, nor that they are tested by observations of type ‘$a_k(t)$’. It is inconsistent then to allow such an observation nevertheless and put it in the antecedent of A5.

All in all, rejecting P2 must go hand in hand with rejecting A5 and the interpretations taking this route are the modal interpretations. Here we must not consider this group of interpretations, in general, but a queer and artificial variant built on negating P2. Note that $\neg$ P2 immediately transforms QM into an unphysical theory. Checking the axioms of QM, we note that the theory (reasonably enough) contains a unique time parameter. (The same goes for QM$^{-}$.) If the theory supplies a time-index for ‘$a_k$’ in ‘$p(a_k)$’, to obey P2, it must be this one. If, vice versa, we claim that ‘$p(a_k)$’ does not inherit the time-index directly from the state, i.e. from the right side of ‘$p(a_k) = \text{Tr}(W(t)P_{a_k})$’, we automatically rule that it does not get any time-reference, at all. QM + $\neg$ P2 does no longer furnish the measurement results, for which it provides probabilities, with exact time-indices. Perhaps we could come up with an additional theory of QM measurement fixing the problem, but any such theory would have to heavily revise the axioms, equipping the formalism with a second time-parameter.

V. QM$^{-}$ + COMP* MAKE P3 IMPLAUSIBLE

Notice that nothing in the previous argument hinges on whether S’s state $W(t) \neq P_{a_k}(t)$ is a pure state or a mixture. So, if we can construct an argument similar to the one of Sec. III, but referring to COMP* instead of COMP, we can disallow the combination QM$^{-}$
and COMP*. Such an argument can indeed be given using P3, but it lacks the rigor of the above one.

Since P2 alone forces the interpretation of BR as BR′, the argument of Sec. III can equally well be applied to COMP*. The defender of COMP* will have to reject P2 and revise BR′. Given the assumption, made plausible above, that there is but one alternative way to specify BR, we will now rewrite it as:

\[\text{BR″} \quad \text{If S is in state } W(t) \text{ and } A \text{ is an observable on S with eigenvalue } a_k, \]
\[\text{then the probability that S has } a_k \text{ given } E(t) \text{ is:} \]
\[p(a_k \text{ given } E(t)) = \text{Tr}(W(t)P_{a_k}).\]

Assuming the triggering event E(t) to be the onset of an A-measurement interaction, we recover the idea, found in classical textbooks [17], that QM probabilities essentially are conditional upon measurement, and the idea that these probabilities are dispositions, possessed by S (or the whole of S and the apparatus) at time t, for S possessing \(a_k\) at some later time. However, as has been pointed out (at the end of Sec. IV), this later time cannot be referred to in QM because the theory, as axiomatized here, does not have the formal resources to refer to two times. (Similarly, again, for QM\textsuperscript{−}.)

So, in the expression ‘\(a_k\) given \(E(t)\)’ the ‘\(a_k\)’, referring to S and the time at which eventually it has \(a_k\), cannot bear a time-index. We have, thus, consciously violated P2, but not necessarily P3, since ‘\(a_k\) given \(E(t)\)’ does contain some time reference, after all. As has been emphasized, however, the discussion at this point takes on an unphysical and academic character.

Probability expressions of the form ‘p(B given A) = z’ (where \(z \in [0, 1]\)) have been thoroughly investigated in the context of QM [18] and three possible analyses have been found: ‘p(B | A) = z’, ‘A \rightarrow p(B) = z’, and ‘p(A \rightarrow B) = z’, where ‘\(\rightarrow\)’ is a conditional connective awaiting further semantic analysis. It should be added that philosophers and logicians have mounted substantial evidence that, in general, p(B | A) \(\neq\) p(A \(\rightarrow\) B), for
standard explications of ‘→’ [19]. So, these two forms of explicating ‘p(B given A) = z’ are indeed logically different and we have (at least) three interpretations for the expression. In the present context, we have the special condition that ‘B’ in ‘p(B given A) = z’ must not bear a time-index, i.e. in the relevant BR” expression ‘p(a_k given E(t)) = z’ ‘a_k’ must not be time-indexed – to escape the contradiction of Sec. III.

We consider the three analyses of ‘p(a_k given E(t)) = z’, in turn. It is easy to see that ‘p(a_k | E(t)) = z’ is not a live option. The standard (Kolmogorov) definition of conditional probability is inapplicable, since this would require ‘p(a_k \land E(t))’ and ‘p(E(t))’ to be well-defined, which they are not. Defining them appropriately would mean to import them into QM from elsewhere – something which is clearly inadmissible in a theory dubbed fundamental and, in addition, breaks the axiomatic closure of the theory. Alternatively, conditional probabilities can be defined as primitive two-place functions from pairs of events into the unit interval [20], but the axioms ruling the interpretation of these functions as probabilities require expressions like ‘p(E(t) | a_k)’ to be well-defined. Again, no version of BR can supply such probabilities and importing them from elsewhere is out of the question.

Consider second ‘E(t) \rightarrow p(a_k) = z’. This variant contracts two problems. ‘z’ is a placeholder for ‘\text{Tr}(W(t)P_{a_k})’, in BR”. Hence, we have the conditional ‘E(t) \rightarrow p(a_k) = \text{Tr}(W(t)P_{a_k})’ containing, as its consequent, an equation ‘p(a_k) = \text{Tr}(W(t)P_{a_k})’. By assumption, this equation is no longer vague, but defined to lack a time index on the left and carry one on the right. For a mathematical function depending on some parameter, this is an inconsistent requirement. Moreover, exporting the time-reference from the set of events that get assigned probabilities via QM violates our principle P3.

Consider third p (E(t) \rightarrow a_k) = z. This possibility respects P3. But the unphysical assumption that its consequent ‘a_k’ must not bear a time-index makes it impossible to distinguish a case where ‘a_k’ is true at some unspecified time directly after E(t) from a case where ‘a_k’ is true at a much later time. This allows constructions of obviously false cases. Suppose that S is a one-particle spin-1/2 system in W(t_1) = P_{am}(t_1), where a_m \neq a_k is another eigenvalue of A. Suppose that E(t_1) is the onset of a measurement interaction
consisting in a series of measurements $A - B - A$ (where $[A, B] \neq 0$). Suppose that, despite the initial state $P_{am}(t_1)$, the second $A$-measurement yields result $'a_k'$. Then $E(t_1) \rightarrow a_k$ is true and yet $p(E(t_1) \rightarrow a_k) = \text{Tr } P_{am}(t_1)P_{a_k} = 0$. Of course, we will understand the physics of the experiment and say that the probability of $'a_k'$ being true directly after $E(t)$ is zero and raises during the course of the whole experiment, but without a time reference we lack the possibility to distinguish different instances of $'a_k'$. The point is not that we cannot come up with an intelligible distinction of instances of $'a_k'$, but rather that we cannot do so within the present (mutilated) version of QM, where BR is interpreted as $BR''$ and, in turn, is read as delivering expressions of type $'p(E(t) \rightarrow a_k) = \text{Tr}(W(t)P_{a_k})'$. 

This argument for a violation of P3 is non-rigorous because it is built on two unproven meta-assumptions: (i) that the two disambiguations sketched in Sec.II and used at the beginning of this section are the only possible ones; (ii) that the three proposed analyses of $'p(a_k \text{ given } E(t)) = z'$ exhaust the possibilities. The argument for QM violating P2 is non-rigorous, too, but is based upon a fairly trivial metalinguistic observation about the axioms: There is a unique time-index in A1–A4.

VI. REVISED AXIOMS FOR QM

As a consequence of the preceding discussion, we cannot express the completeness of QM by COMP or COMP*. Moreover CPP, our simplest candidate for a precise version of the projection postulate A5, cannot be used to make it precise. But amendments to the axioms, guaranteeing harmony with P1-P3, are easily made. A set of axioms for QM respecting P1–P3 will consist of A1–A2 above plus A3*, A4*, and A5*, specified as follows:

$\textbf{A3*}$  
S evolves in time according to $W(t) = U(t)W(t_P)U(t)^{-1}$ where $U(t) = \exp[-iHt]$, a unitary operator, is a function of time and $H$ is an operator representing the total energy of S, where $t_P$, some value of, t is called the preparation time, and $W(t_P)$ the prepared state.

$\textbf{A4*}$  
If $S$ is in state $W(t) \neq W(t_P)$ and $A$ is an observable on $S$, then the expectation value $<A>(t) = \int a(t)p(a(t))d\omega$ is: $<A>(t) = \text{Tr}(W(t)A)$. 

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**A5***  If S has value $a_k(t_1)$ of $A$, then $t_1 = t_P$ and S’s state is the prepared state $P_{a_k}(t_P)$.

Some remarks on these axioms are in order. In $A4^*$, the integral version of the expectation value serves to clearly specify the meaning of ‘$<A>(t)$’. It explicates that the events, the weights of which go into the QM expectation value, are time-indexed events of type ‘S has $a_k$’. It can indeed be argued (and has been done elsewhere [21]) that requiring the QM expectations to be expected values, as they are usually defined in statistics, forces this form for them. $A4^*$ now allows deriving a final version of BR, $BR^*$, via the familiar identification $p(a_k(t)) = <P_{a_k}>(t)$:

**BR**  If S is in state $W(t) \neq W(t_P)$ and $A$ is an observable on S with eigenvalue $a_k$, then the probability that S has $a_k$ at t is:

$$p(a_k(t)) = \text{Tr}(W(t)P_{a_k}).$$

$BR^*$ is $BR'$, with the restriction that $W(t_P)$ is not an admissible input. More explicitly, $BR^*$ and $A5^*$ in conjunction rule that if ‘$a_k(t_1)$’ is true then no calculation of a number $p(a_k(t_1))$ is allowed. This is not an implausible restriction. It is reasonable indeed to assume that the factual observation of an event at a certain time makes it meaningless to calculate any prediction for that event at that time. Note also that the only axiom making direct reference to the result of a factual observation, i.e. to a property ascription to S, is $A5^*$. Note finally the three crucial virtues of this axiom system: (1) It is absolutely explicit concerning the time parameter; (2) it does not need to use the notion of measurement in any sense; (3) it allows us consistently to describe measurements as preparations because our findings upon measurement can be used, via $A5^*$, as an input for $A3^*$.

**VII. AN EXPRESSION OF COMPLETENESS**

The completeness of QM is embodied in the theorems mentioned above, among others: the corollary from Gleason’s Theorem and versions of the Kochen-Specker Theorem.
We have seen that COMP is not an admissible way to express the impossibility results incorporated in these theorems. Our principle P1 embodies the most reasonable idea that probability is quantified possibility and P2–P3 represent plausible ways to render precise the imprecise A1–A5. Given these principles, COMP cannot be an expression of the impossibility results, hence of the sense in which QM can be proved to be complete. But what is an appropriate expression?

To repeat the first observation of this paper: It is impossible to assign pre-existing values to suitable physical systems beyond the QM allowances, under two plausible constraints. We will now see that QM, made precise in the sense of A1–A5*, does yield probabilities for pre-existing values. Hence, it cannot be the idea of assigning pre-existing values as such, but the one of doing so under conditions (i) and (ii) which we should interpret as disproved by the completeness theorems. One or both of conditions (i) and (ii) for the assignment of pre-existing values must be rejected or modified.

It is easy to see that we have produced a general argument for the existence of pre-existing values. Consider, once more, S being in a state \( W(t_1) \neq P_{a_k}(t_1) \) such that \( p(a_k) \) gets a value other than 1 or 0, where \( t_1 \) is the onset time of an \( A \)-measurement interaction. By BR*, ‘\( p(a_k) \)’ is explicated as ‘\( p(a_k(t_1)) \)’, the probability that S has \( a_k \) at \( t_1 \), the onset time. So \( W(t_1) \), by our new axioms, collects probabilities for values possessed at the time of measurement onset, \( t_1 \). This is nothing but the assumption of pre-existing values. The rationale for BR* can be followed back into our principles. If ‘\( p(a_k) \)’ does not inherit the index \( t_1 \) it cannot bear any time-index, at all – in contradiction with P2 and in obvious contrast with reasonable requirements for a fundamental probabilistic theory of spacetime events. If we sacrifice P2 nevertheless and take the remaining option for explicating a time-reference in ‘\( p(a_k) \)’, i.e. ‘\( p(a_k \text{ given } E(t_1)) \)’, then no established construal of the conditional can both be coherent and respect P3. Respecting both P2 and P3, we end with BR*. Finally, if ‘\( (a_k(t_1)) \)’ receives a positive probability, as it does in our case, it must be logically possible to assume it to be true. This is an instance of P1 and says that it must be logically possible to assume S having a value \( a_k \) of \( A \) at \( t_1 \).

As a consequence, it cannot be true that QM is complete in the sense that the QM state
W(t_1) provides all properties S has at t_1. Looking only at the axioms (here BR*), W(t_1) does nothing but collect probabilities for S's values at t_1. It is plausible to supplement the axioms with the rule that predictions with certainty entail value ascriptions (i.e. adopting the forward direction of EE: If \( P_{a_k}(t_1) \), then \( a_k(t_1) \)), but it is implausible to bar all other ascriptions. Let \( A \) and \( B \) be discrete, with values \( a_1, a_2, \ldots, b_1, b_2, \ldots \), and non-degenerate with \([A, B] \neq 0\). Let S be in \( W(t_1) = P_{b_j}(t_1) \). Then, by the rule just adopted, \( 'b_j(t_1)' \) is true and exactly one of \( 'a_1(t_1)', 'a_2(t_1)', \ldots \) is true. Consider now a set of observables \( \{P\}_{AB} \), that contains the projectors \( P_{b_1}, P_{b_2}, \ldots P_{a_1}, P_{a_2}, \ldots \) and forms a Kochen-Specker set (i.e. a set such that a Kochen-Specker contradiction can be derived). What cannot be true, according to the Kochen-Specker Theorem, is that value assignments to all members of \( \{P\}_{AB} \) do both of these two: (i) mirror the algebraic relations of the members of \( \{P\}_{AB} \); (ii) are non-contextual, i.e. are unique for every member of \( \{P\}_{AB} \). There are, then, observables \( A \) and \( B \) such that all of the above assumptions are true, especially \( 'b_j(t_1)' \) is true and exactly one of \( 'a_1(t_1)', 'a_2(t_1)', \ldots \) is true, and yet it cannot be the case that of the \( P_{b_1}, P_{b_2}, \ldots \) exactly one receives value 1, the others 0, and simultaneously, i.e. noncontextually, one of the \( P_{a_1}, P_{a_2}, \ldots \) receives value 1, the others 0. In general, we arrive at the following completeness expression for QM: It is possible to assign pre-existing values to suitable physical systems beyond the QM allowances, but assignments cannot be such that values of submaximal (degenerate) observables mirror the algebraic relations among these observables noncontextually [22, 23].

It is an open question what contextual value assignments would look like. As indicated, the context-dependence of pre-assigned values must be one of pre-existing values rather than one depending on measurement influences on S. The prospects for this type of contextuality (sometimes called ‘ontological contextuality’) have been researched in the past [24, 25], but without much resonance. The present argument clearly shows that this possibility merits renewed attention.

Acknowledgments

I am indebted to audiences at the Spring 2007 conference of the Deutsche Physikalische Gesellschaft at the Universität Heidelberg and at the 15th UK and European Meeting on
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[14] Assume (P1 (a)) that contradictions have probability zero; (P1 (b)) the conditional probability formula: \( p(A \wedge B) = p(A \mid B) p(B) \); (P1 (c)) that the probability space for the probabilities delivered by \( T \) can be expanded so that \( p(T) \) is well-defined; and (P1 (d)) that \( p(T) > 0 \). The non-trivial assumption is P1 (c). However, it can be made plausible for all major con-
ceptions of probability. Consider probability being defined as a subjective degree of belief. (For a meaningful integration of this conception of probability into QM see [11].) Then it is rational to define \( p(T \mid A) \) for a theory in order to be able to express that \( T \)'s prediction \( A \), if it comes out true, raises your degree of belief in it: \( p(T \mid A) > p(T) \). However, if \( p(T \mid A) \) is well-defined, then \( p(A) \) and \( p(T) \) are well-defined on the same space. Consider, alternatively, probability being defined via conditional probabilities understood as ratios of proportions of logically possible worlds. \( p(T) \) then can be defined as \( p(T \mid L) \) where \( L \) is a logical triviality and \( p(A \mid T) \) is defined, on the same space as \( p(T) \), as the ratio of the proportion of logically possible \( T \)-worlds where \( A \) is true to the proportion of logically possible \( T \)-worlds. Consider, finally, probability being defined as the limiting relative frequency of possible outcomes in a hypothetical infinite sequence of trials of an experiment. Let a trial of an ‘experiment concerning \( T \)’ be an explicit statement of \( T \) with possible ‘outcomes’ True (\( T = 1 \)) and Not-true (\( T = 0 \)). Then ‘\( T = 1 \)’ is an outcome as the event reported by \( A \). Thus, \( p(T) \) can be defined as \( p(T = 1) \) on a superspace of the probability space where \( p(A) \) lives. Given P1 (a-d), the argument for P1 is very simple. Assume, by P1 (c) and P1 (d), that \( p(T) > 0 \). Assume also that \( p(A \mid T) > 0 \). Then, by P1 (b), also \( p(A \land T) > 0 \), whence, by P1 (a), \( A \land T \) is not a contradiction.

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