DOUBLE POINTS OF PLANE MODELS IN $\overline{M}_{g,1}$

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Abstract. The aim of this paper is to compute the class of the closure of the effective divisor $D_d^2$ in $M_{g,1}$ given by pointed curves $[C,p]$ with a sextic plane model mapping $p$ to a double point. Such a divisor generates an extremal ray in the pseudoeffective cone of $\overline{M}_{g,1}$ as shown by Jensen. A general result on some families of linear series with adjusted Brill-Noether number 0 or $-1$ is introduced to complete the computation.

The birational geometry of an algebraic variety is encoded in its cone of effective divisors. Nowadays a major problem is to determine the effective cone of moduli spaces of curves.

Let $GP_1^4$ be the Gieseker-Petri divisor in $M_6$ given by curves with a $g^1_4$ violating the Petri condition. The class $[GP_1^4]$ is computed in [EH87] where classes of Brill-Noether divisors and Gieseker-Petri divisors are determined for arbitrary genera in order to prove the general type of $\overline{M}_g$ for $g \geq 24$.

Now let $D_d^2$ be the divisor in $M_{g,1}$ defined as the locus of smooth pointed curves $[C,p]$ with a net $g^2_d$ of Brill-Noether number 0 mapping $p$ to a double point. That is

$$D_d^2 := \{ [C,p] \in M_{g,1} \mid \exists l \in G^2_d(C) \text{ with } l(-p-x) \in G^1_{d-2}(C) \text{ where } x \in C, x \neq p \}$$

for values of $g,d$ such that $g = 3(g - d + 2)$. Recently Jensen has shown that $D_6^2$ and the pull-back of $GP_1^4$ to $\overline{M}_{g,1}$ generate extremal rays of the pseudoeffective cone of $\overline{M}_{g,1}$ (see [Jen10]). Our aim is to prove the following theorem.

**Theorem 1.** The class of the divisor $D_6^2 \subset \overline{M}_{6,1}$ is

$$[D_6^2] = 62\lambda + 4\psi - 8\delta_0 - 30\delta_1 - 52\delta_2 - 60\delta_3 - 54\delta_4 - 34\delta_5 \in \text{Pic}_Q(\overline{M}_{6,1}).$$

A mix of a Porteous-type argument, the method of test curves and a pull-back to rational pointed curves will lead to the result. Following a method described in [Kho07], we realize $D_d^2$ in $M_{g,1}^{\text{int}}$ as the push-forward of a degeneracy locus of a map of vector bundles over $G^2_d(M_{g,1}^{\text{int}})$. This will give us the coefficients of $\lambda$, $\psi$ and $\delta_d$ for the class of $D_d^2$ in general. Intersecting $D_d^2$ with carefully chosen one-dimensional families of curves will produce relations to determine the coefficients of $\delta_1$ and $\delta_{g-1}$. Finally in the case $g = 6$ we will get enough relations to find the other coefficients by pulling-back to the moduli space of stable pointed rational curves in the spirit of [EH87, §3].

To complete our computation we obtain a general result on some families of linear series on pointed curves with adjusted Brill-Noether number $\rho = 0$ that morally excludes further ramifications on such families.
Furthermore let \((C, y)\) be a general pointed curve of genus \(g > 1\). Let \(l\) be a \(g_d^r\) on \(C\) with \(r \geq 2\) and adjusted Brill-Noether number \(\rho(C, y) = 0\). Denote by \((a_0, a_1, \ldots, a_r)\) the vanishing sequence of \(l\) at \(y\). Then \(l(-a_1 y)\) is base-point free for \(i = 0, \ldots, r - 1\).

For instance if \(C\) is a general curve of genus 4 and \(l \in G^3_6(C)\) has vanishing sequence \((0, 1, 3)\) at a general point \(p\) in \(C\), then \(l(-p)\) is base-point free.

Using the irreducibility of the families of linear series with adjusted Brill-Noether number \(-1\) (EH80), we get a similar statement for an arbitrary point on the general curve in such families.

**Theorem 2.** Let \(C\) be a general curve of genus \(g > 2\). Let \(l\) be a \(g_d^r\) on \(C\) with \(r \geq 2\) and adjusted Brill-Noether number \(\rho(C, y) = -1\) at an arbitrary point \(y\). Denote by \((a_0, a_1, \ldots, a_r)\) the vanishing sequence of \(l\) at \(y\). Then \(l(-a_1 y)\) is base-point free.

As a verification of Thm. 1 let us note that the class of \(\Sigma^2_6\) is not a linear combination of the class of the Gieseker-Petri divisor \(\mathcal{GP}_1\) and the class of the divisor \(\mathcal{W}\) of Weierstrass points computed in Cuk89.

\[ [\mathcal{W}] = -\lambda + 21\psi - 15\delta_1 - 10\delta_2 - 6\delta_3 - 3\delta_4 - 5\delta_5 \in \text{Pic}_C(\overline{\mathcal{M}}_{0,1}). \]

After briefly recalling in the next section some basic results about limit linear series and enumerative geometry on the general curve, we prove Thm. 2 and Thm. 3 in section 2. Finally in section 3 we prove a general version of Thm. 1.

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1. **Limit linear series and enumerative geometry**

We use throughout Eisenbud and Harris’s theory of limit linear series (see EH80). Let us recall some basic definitions and results.

1.1. **Linear series on pointed curves.** Let \(C\) be a complex smooth projective curve of genus \(g\) and \(l = (\mathcal{L}, V)\) a linear series of type \(g_d^r\) on \(C\), that is \(\mathcal{L} \in \text{Pic}^d(C)\) and \(V \subset H^0(\mathcal{L})\) is a subspace of vector-space dimension \(r + 1\). The \textit{vanishing sequence} \(a^l(p) : 0 \leq a_0 < \cdots < a_r \leq d\) of \(l\) at a point \(p \in C\) is defined as the sequence of distinct order of vanishing of sections in \(V\) at \(p\), and the \textit{ramification sequence} \(a^l(p) : 0 \leq a_0 \leq \cdots \leq a_r \leq d - r\) as \(\alpha_i := a_i - i\), for \(i = 0, \ldots, r\). The \textit{weight} \(w^l(p)\) will be the sum of the \(\alpha_i\)’s.

Given an \(n\)-pointed curve \((C, p_1, \ldots, p_n)\) of genus \(g\) and \(l\) a \(g_d^r\) on \(C\), the \textit{adjusted Brill-Noether number} is

\[ \rho(C, p_1, \ldots, p_n) = \rho(g, r, d, \alpha^l(p_1), \ldots, \alpha^l(p_n)) := g - (r + 1)(g - d + r) - \sum_{i,j} \alpha_i^l(p_j). \]

1.2. **Counting linear series on the general curve.** Let \(C\) be a general curve of genus \(g\) and consider \(r, d\) such that \(\rho(g, r, d) = 0\). Then by Brill-Noether theory, the curve \(C\) admits only a finite number of \(g_d^r\)’s computed by the \textit{Castelnuovo number}

\[ N_{g, r, d} := g! \prod_{i=0}^r \frac{d!}{(g - d + r + i)!}. \]

Furthermore let \((C, p)\) be a general pointed curve of genus \(g\) and let \(\pi = (\alpha_0, \ldots, \alpha_r)\) be a Schubert index of type \(r, d\) (that is \(0 \leq \alpha_0 \leq \cdots \leq \alpha_r \leq d - r\) such that
\[ \rho(g,r,d,\pi) = 0. \] Then by [EH87] Prop. 1.2, the curve \( C \) admits a \( g_d^r \) with ramification sequence \( \pi \) at the point \( p \) if and only if \( \alpha_0 + g - d + r \geq 0 \). When such linear series exist, there is a finite number of them counted by the following formula

\[ N_{g,r,d,\pi} := g^d \prod_{i=0}^{r} (\alpha_i - \alpha_i + j - i). \]

1.3. Limit linear series. For a curve of compact type \( C = Y_1 \cup \cdots \cup Y_s \) of arithmetic genus \( g \) with nodes at the points \( \{p_{ij}\}_{ij} \), let \( \{l_{Y_1}, \ldots, l_{Y_s}\} \) be a limit linear series \( g_d^{r_s} \) on \( C \). Let \( \{q_{ik}\}_k \) be smooth points on \( Y_i, i = 1, \ldots, s \). In [EH86] a moduli space of such limit series is constructed as a disjoint union of schemes on which the vanishing sequences of the aspects \( l_{Y_i}'s \) at the nodes are specified. A key property is the additivity of the adjusted Brill-Noether number, that is

\[ \rho(g,r,d,\{\alpha^r_{ij}(q_{ik})\}_{ik}) \geq \sum_i \rho(Y_i, \{p_{ij}\}_j, \{q_{ik}\}_k). \]

The smoothing result [EH86] Cor. 3.7 assures the smoothability of dimensionally proper limit series. The following facts ease the computations. The adjusted Brill-Noether number for any \( g_d^{r_s} \) on one-pointed elliptic curves or on \( n \)-pointed rational curves is nonnegative. For a general curve \( C \) of arbitrary genus \( g \), one has \( \rho(C,p) \geq 0 \) for \( p \) general in \( C \) and \( \rho(C,y) \geq -1 \) for any \( y \in C \) (see [EH89]).

2. Ramifications on some families of linear series with \( \rho = 0 \) or \( \rho = -1 \)

Here we prove Thm. 2. The result will be repeatedly used in the next section.

Proof of Thm. 2. Clearly it is enough to prove the statement for \( i = r - 1 \). We proceed by contradiction. Suppose that for \( (C,y) \) a general pointed curve of genus \( g \), there exists \( x \in C \) such that \( h^0(l(-a_{r-1}y-x)) \geq 2 \), for some \( l \) a \( g_d^{r_s} \) with \( \rho(C,y) = 0 \). Let us degenerate \( C \) to a transversal union \( C_1 \cup_{\eta} E_1 \), where \( C_1 \) has genus \( g - 1 \) and \( E_1 \) is an elliptic curve. Since \( y \) is a general point, we can assume \( y \in E_1 \) and \( y \in Y_i \) not to be a \( d \)-torison point in \( \text{Pic}^0(E_1) \). Let \( \{l_{C_1}, l_{E_1}\} \) be a limit \( g_d^{r_s} \) on \( C_1 \cup_{\eta} E_1 \) such that \( a^{l_{C_1}}(y) = (a_0, a_1, \ldots, a_r) \). Denote by \( (a_0, \ldots, a_r) \) the corresponding ramification sequence. We have that \( \rho(C_1, y) = \rho(E_1, y, y_1) = 0 \), hence \( w^{l_{C_1}}(y_1) = r + \rho \), where \( \rho = \rho(g,r,d) \). Denote by \( (b_0^1, b_1^1, \ldots, b_r^1) \) the vanishing sequence of \( l_{C_1} \) at \( y_1 \) and by \( (\beta_0^1, \beta_1^1, \ldots, \beta_r^1) \) the corresponding ramification sequence.

Suppose \( x \) specializes to \( E_1 \). Then \( b_1^1 \geq a_r + 1, b_{r-1}^1 \geq a_r + 1 \) and we cannot have both equalities, since \( y - y_1 \) is not in \( \text{Pic}^0(E_1)[d] \) (see for instance [Far06] Prop. 4.1)). Moreover, as usually \( b_k^1 \geq a_k \) for \( 0 \leq k \leq r - 2 \), and again among these inequalities there cannot be more than one equality. We deduce

\[ w^{l_{C_1}}(y_1) \geq w^{l_{C_1}}(y) + 3 + r - 2 > w^{l_{C_1}}(y) + r = r + \rho \]

hence a contradiction. We have supposed that \( h^0(l(-a_{r-1}y-x)) \geq 2 \). Then this pencil degenerates to \( l_{E_1}(-a_{r-1}y) \) and to a compatible sub-pencil \( l_{C_1}' \) of \( l_{C_1}(-x) \). We claim that

\[ h^0(l_{C_1}(-b_{r-1}^1y_1-x)) \geq 2. \]

Suppose this is not the case. Then we have \( a^{l_{C_1}}(-x)(y_1) \leq (b_0^1, \ldots, b_{r-2}^1, b_r^1) \), hence \( b_1^1 \geq a_r, b_{r-2}^1 \geq a_{r-1} \) and \( b_r^1 \geq a_{r-k} \), for \( 0 \leq k \leq r - 3 \). Among these, we cannot have more than one equality, plus \( \beta_1^1 \geq \alpha_{r-1} \) and \( \beta_{r-1}^1 \geq \beta_{r-2}^1 \geq a_{r-2} \geq \alpha_{r-2} \), hence

\[ w^{l_{C_1}}(y_1) \geq w^{l_{C_1}}(y) + 1 + r - 1 + \beta_{r-1}^1 - a_{r-2} > r + \rho \]

a contradiction.

From our assumptions, we have deduced that for \( (C_1,y_1) \) a general pointed curve of genus \( g - 1 \), there exist \( l_{C_1} \) a \( g_d^{r_s} \) and \( x \in C_1 \) such that \( \rho(C_1,y_1) = 0 \) and \( h^0(l_{C_1}(-b_{r-1}^1y_1-x)) \geq 2 \), where \( b_{r-1}^1 \) is as before.
Then we apply the following recursive argument. At the step $i$, we degenerate
the pointed curve $(C_i, y_i)$ of genus $g - i$ to a transversal union $C_{i+1} \cup y_{i+1}$,
where $C_{i+1}$ is a curve of genus $g - i - 1$ and $E_{i+1}$ is an elliptic curve, such
that $y_i \in E_{i+1}$. Let $(l_{C_{i+1}}, E_{i+1})$ be a limit $g^a_d$ on $C_{i+1} \cup y_{i+1}$ $E_{i+1}$ such that
$a^{l_{C_{i+1}}}(y_i) = (b_0, b_1, \ldots, b_t)$. From $\rho(C_{i+1}, y_{i+1}) = \rho(E_{i+1}, y_i, y_{i+1}) = 0$, we compute
that $w^{l_{C_{i+1}}}(y_{i+1}) = (i + 1)r + \rho$. Denote by $(b_0^{l_{C_{i+1}}}, b_1^{l_{C_{i+1}}}, \ldots, b_t^{l_{C_{i+1}}})$ the vanishing sequence of $l_{C_{i+1}}$ at $y_{i+1}$. As before we arrive to a contradiction if $x \in E_{i+1}$, and
we deduce
\[ h^0 \left( l_{C_{i+1}} \left( -b_{r-1}^{l_{C_{i+1}}} y_{i+1} - x \right) \right) \geq 2. \]

At the step $g - 2$, our degeneration produces two elliptic curves $C_{g-1} \cup y_{g-2}, E_{g-1}$, with $y_{g-2} \in E_{g-1}$. Our assumptions yield the existence of $x \in C_{g-1}$ such that
\[ h^0 \left( l_{C_{g-1}} \left( -b_{r-1}^{l_{C_{g-1}}} y_{g-1} - x \right) \right) \geq 2. \]

We compute $w^{l_{C_{g-1}}}(y_{g-1}) = (g - 1)r + \rho$. By the numerical hypothesis, we see that
$(g - 1)r + \rho = (d - r - 1)(r + 1) + 1$, hence the vanishing sequence of $l_{C_{g-1}}$ at $y_{g-1}$
has to be $(d - r - 1, \ldots, d - 3, d - 2, d)$. Whence the contradiction.

The following proves the similar result for some families of linear series with Brill-Noether number $-1$.

Proof of Thm 3. The statement says that for every $y \in C$ such that $\rho(C, y) = -1$
for some $l$ a $g^a_d$, and for every $x \in C$, we have that $h^0(l(-a_1 y - x)) \leq r - 1$. This
is a closed condition and, using the irreducibility of the divisor $D$ of pointed curves
admitting a linear series $g^a_d$ with adjusted Brill-Noether number $-1$, it is enough to prove it for $[C, y]$ general in $D$.

We proceed by contradiction. Suppose for $[C, y]$ general in $D$ there exists $x \in C$
such that $h^0(l(-a_1 y - x)) \geq r$ for some $l$ a $g^a_d$ with $\rho(C, y) = -1$. Let us degenerate
$C$ to a transversal union $C_1 \cup y_1 E_1$ where $C_1$ is a general curve of genus $g - 1$ and $E_1$
is an elliptic curve. Since $y$ is a general point, we can assume $y \in E_1$. Let $(l_{C_1}, E_1)$ be a limit $g^a_d$ on $C_1 \cup y_1 E_1$ such that $a^{l_{C_1}}(y) = (a_0, a_1, \ldots, a_r)$. Then
$\rho(E_1, y_1) \leq -1$ and $\rho(C_1, y_1) = 0$, hence $w^{l_{C_1}}(y_1) = r + \rho$ (see also [Zar70]
Proof of Thm. 4.6). Let $(b_0^{l_{C_1}}, b_1^{l_{C_1}}, \ldots, b_t^{l_{C_1}})$ be the vanishing sequence of $l_{C_1}$ at $y_1$ and
$(\beta_1^{l_{C_1}}, \beta_1^{l_{C_1}}, \ldots, \beta_t^{l_{C_1}})$ the corresponding ramification sequence.

The point $x$ has to specialize to $C_1$. Indeed suppose $x \in E_1$. Then $b_k^{l_{C_1}} \geq a_k + 1$ for
$k \geq 1$. This implies $w^{l_{C_1}}(y_1) \geq w^{l_{E_1}}(y_1) + r > r + \rho$, hence a contradiction. Then $x \in C_1$, and $l(-a_1 y - x)$ degenerates to $l_{E_1}(-a_1 y)$ and to a compatible system
$l'_{C_1} := l_{C_1}(-x)$. We claim that
\[ h^0 \left( l_{C_1} \left( -b_{r-1}^{l_{C_1}} y_1 - x \right) \right) \geq 2. \]

Suppose this is not the case. Then we have $a^{l_{C_1}}(y_1) \leq (b_0, b_1, \ldots, b_{r-2}, b_t^{l_{C_1}})$ and so
$b_k^{l_{C_1}} \geq a_k$ for $0 \leq k \leq r - 2$. Then $\beta_k^{l_{C_1}} \geq \alpha_{k+1} + 1$ for $k \leq r - 2$, and
summing up we receive
\[ w^{l_{C_1}}(y_1) \geq w^{l_{E_1}}(y_1) + r - 1 + \beta_{r-1}^{l_{C_1}} - \alpha_0. \]

Clearly $\beta_{r-2}^{l_{C_1}} \geq \beta_{r-2}^{l_{E_1}} \geq \alpha_{r-2} \geq \alpha_0$. Hence $w^{l_{C_1}}(y_1) > \rho + r$, a contradiction.

All in all from our assumptions we have deduced that for a general pointed curve
$(C_1, y_1)$ of genus $g - 1$, there exist $l_{C_1}$ a $g^a_d$ and $x \in C_1$ such that $\rho(C_1, y_1) = 0$ and
$h^0(l_{C_1}(-b_{r-1}^{l_{C_1}} y_1 - x)) \geq 2$, where $b_k^{l_{C_1}}$ is as before. This contradicts Thm 2.

Therefore we receive the statement.\[ \square \]
3. The divisor $\mathfrak{D}_d^2$

Remember that $\text{Pic}_g (\mathcal{M}_{g,1})$ is generated by the Hodge class $\lambda$, the cotangent class $\psi$ corresponding to the marked point, and the boundary classes $\delta_0, \ldots, \delta_{g-1}$ defined as follows. The class $\delta_i$ is the class of the closure of the locus of pointed irreducible nodal curves, and the class $\delta_i$ as follows. The class $\delta_0$ is the class of the closure of the locus of pointed curves $[C_i \cup C_{g-i}, p]$ where $C_i$ and $C_{g-i}$ are smooth curves respectively of genus $i$ and $g-i$ meeting transversally in one point, and $p$ is a smooth point in $C_i$, for $i = 1, \ldots, g-1$. In this section we prove the following theorem.

**Theorem 4.** Let $g = 3s$ and $d = 2s + 2$ for $s \geq 1$. The class of the divisor $\mathfrak{D}_d^2$ in $\text{Pic}_g (\mathcal{M}_{g,1})$ is

$$[\mathfrak{D}_d^2] = a\lambda + c\psi - \sum_{i=0}^{g-1} b_i \delta_i$$

where

$$a = \frac{48s^4 + 80s^3 - 16s^2 - 64s + 24}{(3s - 1)(3s - 2)(s + 3)} N_{g,2,d}$$

$$c = \frac{2s(s - 1)}{3s - 1} N_{g,2,d}$$

$$b_0 = \frac{24s^4 + 23s^3 - 18s^2 - 11s + 6}{3(3s - 1)(3s - 2)(s + 3)} N_{g,2,d}$$

$$b_1 = \frac{14s^3 + 6s^2 - 8s}{(3s - 2)(s + 3)} N_{g,2,d}$$

$$b_{g-1} = \frac{48s^4 + 12s^3 - 56s^2 + 20s}{(3s - 1)(3s - 2)(s + 3)} N_{g,2,d}.$$

Moreover for $g = 6$ and for $i = 2, 3, 4$, we have that

$$b_i = -7t^2 + 43t - 6.$$

3.1. The coefficient $c$. The coefficient $c$ can be quickly found. Let $C$ be a general curve of genus $g$ and consider the curve $\mathcal{C} = \{ [C, y] : y \in C \}$ in $\mathcal{M}_{g,1}$ obtained varying the point $y$ on $C$. Then the only generator class having non-zero intersection with $\mathcal{C}$ is $\psi$, and $\mathcal{C} \cdot \psi = 2g - 2$. On the other hand, $\mathcal{C} \cdot \mathfrak{D}_d^2$ is equal to the number of triples $(x, y, l) \in C \times C \times G_2^2(C)$ such that $x$ and $y$ are different points and $h^0(l(-x - y)) \geq 2$. The number of such linear series on a general $C$ is computed by the Castelnuovo number (remember that $\rho = 0$), and for each of them the number of couples $(x, y)$ imposing only one condition is twice the number of double points, computed by the Plücker formula. Hence we get the equation

$$\mathfrak{D}_d^2 \cdot \mathcal{C} = 2 \left( \frac{(d - 1)(d - 2)}{2} - g \right) N_{g,2,d} = c(2g - 2)$$

and so

$$c = \frac{2s(s - 1)}{3s - 1} N_{g,2,d}.$$

3.2. The coefficients $a$ and $b_0$. In order to compute $a$ and $b_0$, we use a Porteous-style argument. Let $G_2^2$ be the family parametrizing triples $(C, p, l)$, where $[C, p] \in \mathcal{M}^\text{irr}_{g,1}$ and $l$ is a $g^{\text{st}}_2$ on $C$; denote by $\eta : G_2^2 \rightarrow \mathcal{M}^\text{irr}_{g,1}$ the natural map. There exists $\pi : \mathcal{Y}_d^2 \rightarrow G_2^2$ a universal pointed quasi-stable curve, with $\sigma : G_2^2 \rightarrow \mathcal{Y}_d^2$ the marked section. Let $\mathcal{L} \rightarrow \mathcal{Y}_d^2$ be the universal line bundle of relative degree $d$ together with the trivialization $\sigma^* (\mathcal{L}) \cong G_2^2$, and $\mathcal{V} \subset \pi_*(\mathcal{L})$ be the sub-bundle which over each point $(C, p, l = (L, V))$ in $G_2^2$ restricts to $V$. (See [Kho07], [2] for more details.)
Furthermore let us denote by $\mathcal{Z}_d^2$ the family parametrizing $((C,p), x_1, x_2, l)$, where $[C,p] \in \mathcal{M}_{g,1}^{\text{irr}}$, $x_1, x_2 \in C$ and $l$ is a $g^2_d$ on $C$, and let $\mu, \nu : \mathcal{Z}_d^2 \to \mathcal{Y}_d^2$ be defined as the maps that send $((C,p), x_1, x_2, l)$ respectively to $((C,p), x_1, l)$ and $((C,p), x_2, l)$.

Now given a linear series $l = (L, V)$, the natural map
$$
\varphi : V \to H^0(L|_{p+x})
$$
globalizes to
$$
\tilde{\varphi} : \mathcal{Y} \to \mu_{\ast}(\nu^\ast L \otimes \mathcal{O}/\mathcal{I}_{\Gamma_{\sigma} + \Delta}) =: \mathcal{M}
$$
as a map of vector bundle over $\mathcal{Y}_d^2$, where $\Delta$ and $\Gamma_{\sigma}$ are the loci in $\mathcal{Z}_d^2$ determined respectively by $x_1 = x_2$ and $x_2 = p$. Then $\mathcal{D}_g^2 \cap \mathcal{M}_{g,1}^{\text{irr}}$ is the push-forward of the locus in $\mathcal{Y}_d^2$ where $\tilde{\varphi}$ has rank $\leq 1$. Using Porteous formula, we have
\begin{equation}
\left[\mathcal{D}_d^2|_{\mathcal{M}_{g,1}^{\text{irr}}} \right] = \eta_{\ast}\pi_{\ast}\left[\mathcal{Y} \otimes \mathcal{M}\right] = \eta_{\ast}\pi_{\ast}\left(\pi_{\ast}c_2(\mathcal{Y}) + \pi_{\ast}c_1(\mathcal{Y}) \cdot c_1(\mathcal{M}) + c_1^2(\mathcal{M}) - c_2(\mathcal{M})\right).
\end{equation}

Let us find the Chern classes of $\mathcal{M}$. Tensoring the exact sequence
$$
0 \to \mathcal{I}_{\Delta}/\mathcal{I}_{\Delta + \Gamma_{\sigma}} \to \mathcal{O}/\mathcal{I}_{\Delta + \Gamma_{\sigma}} \to \mathcal{O}_{\Delta} \to 0
$$
by $\nu^\ast L$ and applying $\mu_{\ast}$, we deduce that
$$
ch(\mathcal{M}) = ch(\mu_{\ast}(\mathcal{O}_{\Delta} \otimes \nu^\ast L)) + ch(\mu_{\ast}(\mathcal{O}_{\Delta + \Gamma_{\sigma}} \otimes \nu^\ast L))
$$
$$
= ch(\mu_{\ast}(\mathcal{O}_{\Delta})) + ch(\mu_{\ast}(\mathcal{O}_{\Delta + \Gamma_{\sigma}} \otimes \nu^\ast L))
$$
$$
= e^{-\sigma} + ch(\mathcal{L})
$$

hence
$$
c_1(\mathcal{M}) = c_1(\mathcal{L}) - \sigma
$$
$$
c_2(\mathcal{M}) = -\sigma c_1(\mathcal{L}).
$$

The following classes
$$
\alpha = \pi_{\ast}(c_1(\mathcal{L})^2 \cap [\mathcal{Y}_d^2])
$$
$$
\gamma = c_1(\mathcal{Y}) \cap [\mathcal{D}_d^2]
$$
have been studied in [Kho07 Thm. 2.11]. In particular
$$
\frac{6(g-1)(g-2)}{dN_{g,2,d}} \eta_{\ast}(\alpha)|_{\mathcal{M}_{g,1}^{\text{irr}}} = 6(gd - 2g^2 + 8d - 8g + 4)\lambda + (2g^2 - gd + 3g - 4d - 2)\delta_0 - 6d(g-2)\psi,
$$
$$
\frac{2(g-1)(g-2)}{N_{g,2,d}} \eta_{\ast}(\gamma)|_{\mathcal{M}_{g,1}^{\text{irr}}} = (-g + 3)\xi + 40)\lambda + \frac{1}{6}((g + 1)\xi - 24)\delta_0 - 3d(g-2)\psi,
$$

where
$$
\xi = 3(g-1) + \frac{(g+3)(3g-2d-1)}{g-d+5}.
$$

Plugging into $(1)$ and using the projection formula, we find
\begin{equation}
\left[\mathcal{D}_d^2|_{\mathcal{M}_{g,1}^{\text{irr}}} \right] = \eta_{\ast}\left(-\gamma \cdot \pi_{\ast}c_1(\mathcal{L}) + \gamma \cdot \pi_{\ast}\sigma + \alpha + \pi_{\ast}\sigma^2 - \pi_{\ast}(\sigma c_1(\mathcal{L}))\right)
\end{equation}
$$
= (1 - d)\eta_{\ast}(\gamma) + \eta_{\ast}(\alpha) - N_{g,2,d} \cdot \psi.
$$
Hence

\[
\begin{align*}
a &= \frac{48s^4 + 80s^3 - 16s^2 - 64s + 24}{(3s-1)(3s-2)(s+3)} N_{g,2,d} \\
b_0 &= \frac{24s^4 + 23s^3 - 18s^2 - 11s + 6}{3(3s-1)(3s-2)(s+3)} N_{g,2,d}
\end{align*}
\]

and we recover the previously computed coefficient \(c\).

3.3. The coefficient \(b_1\). Let \(C\) be a general curve of genus \(g - 1\) and \((E,p,q)\) a two-pointed elliptic curve, with \(p - q\) not a torsion point in \(\text{Pic}^0(E)\). Let \(C_1 := \{(C \cup_{y=q} E, p)\}_{y \in C}\) be the family of curves obtained identifying the point \(q \in E\) with a moving point \(y \in C\). Computing the intersection of the divisor \(D_d\) with \(C_1\) is equivalent to answering the following question: how many triples \((x,y,l)\) are there, with \(y \in C\), \(x \in C \cup_{y=q} E \setminus \{p\}\) and \(l = \{l_C, l_E\}\) a limit \(\mathfrak{g}^2_d\) on \(C \cup_{y=q} E\), such that \((p,x,l)\) arises as limit of \((pt,xt,lt)\) on a family of curves \(\{C_t\}_t\) with smooth general element, where \(pt\) and \(xt\) impose only one condition on \(lt\) a \(\mathfrak{g}^2_d\)?

![Diagram](image)

Let \(a^{lu}(q) = (a_0, a_1, a_2)\) be the vanishing sequence of \(l_E \in G^2_d(E)\) at \(q\). Since \(C\) is general, there are no \(\mathfrak{g}^2_d\) on \(C\), hence \(l_C\) is base-point free and \(a_2 = d\). Moreover we know \(a_1 \leq d - 2\). Let us suppose \(x \in E \setminus \{q\}\). We distinguish two cases. If \(\rho(E,q) = \rho(C,y) = 0\), then \(w^{lu}(q) = \rho(1,2,d) = 3d - 8\). Thus \(a^{lu}(q) = (d - 3,d - 2,d)\). Removing the base point we have that \(l_E(-(d-3)q)\) is a \(\mathfrak{g}^2_d\) and \(l_E(-(d-3)q-p-x)\) produces a \(\mathfrak{g}^1_d\) on \(E\), hence a contradiction. The other case is \(\rho(E,q) = 1\) and \(\rho(C,y) \leq -1\). These force \(a^{lu}(q) = (d-4,d-2,d)\) and \(a^{lu}(y) \geq (0,2,4)\). On \(E\) we have that \(l_E(-(d-4)q-p-x)\) is a \(\mathfrak{g}^1_d\).

The question splits in two: firstly, how many linear series \(l_E \in G^2_d(E)\) and points \(x \in E \setminus \{q\}\) are there such that \(a^{lu}(q) = (0,2,4)\) and \(l_E(-p-x) \in G^2_d(E)\)? The first condition restricts our attention to the linear series \(l_E = (\mathcal{O}(4q), \mathcal{V})\) where \(\mathcal{V}\) is a tridimensional vector space and \(H^0(\mathcal{O}(4q-2q)) \subset \mathcal{V}\), while the second condition tells us \(H^0(\mathcal{O}(4q-p-x)) \subset \mathcal{V}\). If \(x = p\), then we get \(p-q\) is a torsion point in \(\text{Pic}^0(E)\), a contradiction. On the other hand, if \(x \in E \setminus \{p, q\}\), then \(H^0(\mathcal{O}(4q-2q)) \cap H^0(\mathcal{O}(4q-p-x)) \neq \emptyset\) entails \(p+x \equiv 2q\). Hence the point \(x\) and the space \(V = H^0(\mathcal{O}(4q-2q)) + H^0(\mathcal{O}(4q-p-x))\) are uniquely determined.

Secondly, how many couples \((y,l_E)\) \(\in C \times G^2_d(C)\) are there, such that the vanishing sequence of \(l_C\) at \(y\) is greater than or equal to \((0,2,4)\)? This is a particular case of a problem discussed in [Far09]. Proof of Thm. 4.6]. The answer is

\[
(g-1) \left(15N_{g-1,2,d,(0,2,2)} + 3N_{g-1,2,d,(1,1,2)} + 3N_{g-1,2,d,(0,1,3)}\right) = \frac{24(2s^2 + 3s - 4)}{s + 3} N_{g,2,d}
\]

Now let us suppose \(x \in C \setminus \{y\}\). The condition on \(x\) and \(p\) can be reformulated in the following manner. We consider the curve \(C \cup_q E\) as the special fiber \(X_0\) of a family of curves \(\pi : X \rightarrow B\) with sections \(x(t)\) and \(p(t)\) such that \(x(0) = x\), \(p(0) = p\), and with smooth general fiber having \(l = (\mathcal{L}, \mathcal{V})\) a \(\mathfrak{g}^2_d\) such that \(l(-x-p)\) is a \(\mathfrak{g}^1_{d-2}\). Let \(V' \subset V\) be the two dimensional linear subspace formed by those sections \(\sigma \in V\) such that \(\text{div}(\sigma) \geq x + p\). Then \(V'\) specializes on \(X_0\) to \(V'_E \subset V_C\) and \(V'_E \subset V_E\) two-dimensional subspaces, where \(\{l_C = (\mathcal{L}_C, V_C), l_E = (\mathcal{L}_E, V_E)\}\)
is a limit $g_{d-2}$ such that
\[
\begin{align*}
\text{ord}_{y}(\sigma_C) + \text{ord}_{y}(\sigma_E) & \geq d \\
\text{div}(\sigma_C) & \geq x \\
\text{div}(\sigma_E) & \geq p
\end{align*}
\]
for every $\sigma_C \in V_{C}'$ and $\sigma_E \in V_{E}'$. Let $l'_C := (\mathcal{L}_C, V_{C}')$ and $l'_E := (\mathcal{L}_E, V_{E}')$. Note that since $\sigma_E \geq p$, we get $\text{ord}_{y}(\sigma_E) < d$, $\forall \sigma_E \in V_{E}'$. Then $\text{ord}_{y}(\sigma_C) > 0$, hence $\text{ord}_{y}(\sigma_C) \geq 2$, since $y$ is a cuspidal point on $C$. Removing the base point, $l'_C$ is a $g_{d-2}$ such that $l'_C(-x)$ is a $g_{d-3}$. Let us suppose $\rho(E, y) = 1$ and $\rho(C, y) = -1$. Then $a^{i\varepsilon}(y) = (d - 4, d - 2, d)$, $a^{i\varepsilon}(y) = (d - 4, d - 2, d)$ and $a^{i\varepsilon}(y) = (2, 4)$. Now $l_C$ is characterized by the conditions $H^0(l_C(-2y - x)) \geq 3$ and $H^0(l_C(-4y - x)) \geq 1$. By Thm. 3 this possibility does not occur.

Suppose now $\rho(E, y) = \rho(C, y) = 0$. Then $a^{i\varepsilon}(y) = (d - 3, d - 2, d)$, i.e. $l_E(-d - 3) = [3y]$ is uniquely determined. On the $C$ aspect we have that $a^{i\varepsilon}(y) = (0, 2, 3)$ and $H^0(l_C(-2y - x)) \geq 2$. Hence we are interested on $Y$ the locus of triples $(x, y, l_C)$ such that the map
\[
\varphi : H^0(l_C) \to H^0(l_C|_{2y + x})
\]
has rank $\leq 1$. By Thm. 2 there is only a finite number of such triples, and clearly the case $a^{i\varepsilon}(y) > (0, 2, 3)$ cannot occur. Moreover, note that $x$ and $y$ will be necessarily distinct.

Let $\mu = \pi_{1, 2, 4} : C \times C \times C \times W^2_3(C) \to C \times C \times W^2_3(C)$ and $\nu = \pi_{3, 4} : C \times C \times C \times W^2_3(C) \to C \times C \times W^2_3(C)$ be the natural projections respectively on the first, second and forth components, and on the third and forth components. Let $\pi : C \times C \times W^2_3(C) \to W^2_3(C)$ be the natural projection on the third component. Now $\varphi$ globalization to
\[
\tilde{\varphi} : \pi^* \mathcal{E} \to \mu_* (\nu^* \mathcal{L} \otimes \mathcal{O} / \mathcal{I}_D) =: \mathcal{M}
\]
as a map of rank 3 bundles over $C \times C \times W^2_3(C)$, where $D$ is the pullback to $C \times C \times C \times W^2_3(C)$ of the divisor on $C \times C \times C$ that on $(x, y, C)$ restricts to $x + 2y + 2$ is a Poincaré bundle on $C \times W^2_3$ and $\mathcal{E}$ is the push-forward of $\mathcal{L}$ to $W^2_3(C)$. Then $Y$ is the degeneracy locus where $\tilde{\varphi}$ has rank $\leq 1$. Let $c_i := c_i(\mathcal{E})$ be the Chern classes of $\mathcal{E}$. By Porteous formula, we have
\[
[Y] = \begin{bmatrix} e_2 & e_3 \\ e_1 & e_2 \end{bmatrix}
\]
where the $e_i$’s are the Chern classes of $\pi^* \mathcal{E} \cdot \mathcal{M}^\vee$, i.e.
\[
\begin{align*}
e_1 &= c_1 + c_1(\mathcal{M}) \\
e_2 &= c_2 + c_1c_1(\mathcal{M}) + c_1^2(\mathcal{M}) - c_2(\mathcal{M}) \\
e_3 &= c_3 + c_2c_1(\mathcal{M}) + c_1(\mathcal{M})^2 - 2c_1(\mathcal{M})c_2(\mathcal{M})
\end{align*}
\]

Let us find the Chern classes of $\mathcal{M}$. First we develop some notations (see also [ACGH85, VIII.2]). Let $\pi_i : C \times C \times C \times W^2_3(C) \to C$ for $i = 1, 2, 3$ and $\pi_4 : C \times C \times C \times W^2_3(C) \to W^2_3(C)$ be the natural projections. Denote by $\theta$ the pull-back to $C \times C \times W^2_3(C)$ of the class $\theta \in H^2(W^2_3(C))$ via $\pi_3$, and denote by $\eta_i$ the cohomology class $\pi_i^*[\text{point}] \in H^2(C \times C \times C \times W^2_3(C))$, for $i = 1, 2, 3$. Note that $\delta_2 = 0$. Furthermore, given a symplectic basis $\delta_1, \ldots, \delta_{2g - 1}$ for $H^1(C, \mathbb{Z}) \cong H^1(W^2_3(C), \mathbb{Z})$, denote by $\delta_0$ the pull-back to $C \times C \times C \times W^2_3(C)$ of $\delta_0$ via $\pi_4$, for $i = 1, 2, 3, 4$. Let us define
\[
\gamma_{ij} := - \sum_{\alpha=1}^{g-1} \left( \delta_0 \delta_{y-1+\alpha} - \delta_{y-1+\alpha} \delta_0 \right).
\]
Note that
\[
\begin{align*}
\gamma_{ij}^2 &= -2(g-1)\eta_i\eta_j \quad \text{and} \quad \eta_i\gamma_{ij} = \gamma_{ij}^3 = 0 \quad \text{for} \quad 1 \leq i < j \leq 3, \\
\gamma_{k4}^2 &= -2\eta_k\theta \quad \text{and} \quad \eta_k\gamma_{k4} = \gamma_{k4}^3 = 0 \quad \text{for} \quad k = 1, 2, 3.
\end{align*}
\]
Moreover
\[
\gamma_{ij}\gamma_{jk} = \eta_j\gamma_{ik},
\]
for \(1 \leq i < j < k \leq 4\). With these notations, we have
\[
ch(\nu^*\ell \otimes \mathcal{O}/\mathcal{I}_D) = (1 + d\eta_3 + \gamma_{34} - \eta_3\theta) \left(1 - e^{-((\eta_1 + \gamma_{13} + \gamma_{32} + 2\gamma_{23} + 2\gamma_3)}\right),
\]
hence by Grothendieck-Riemann-Roch
\[
ch(\mathcal{A}) = \mu_{\nu}((1 + (2 - g)\eta_3)ch(\nu^*\ell \otimes \mathcal{O}/\mathcal{I}_D))
\]
\[
= 3 + (d - 2)\eta_1 + (2g + 2d - 6)\eta_2 - 2\gamma_{12} + \gamma_{14} + 2\gamma_{24} + (2d - 4)\eta_2\gamma_{14} + 2\gamma_{14}\gamma_{24} - 2\eta_2\theta,
\]
Using Newton’s identities, we recover the Chern classes of \(\mathcal{A}\):
\[
\begin{align*}
\chi_1(\mathcal{A}) &= (d - 2)\eta_1 + (2g + 2d - 6)\eta_2 - 2\gamma_{12} + \gamma_{14} + 2\gamma_{24}, \\
\chi_2(\mathcal{A}) &= (2d^2 - 8d + 2gd + 4 - 4g)(s - 1) + (2d - 4)\eta_2\gamma_{14} + 2\gamma_{14}\gamma_{24} - 2\eta_2\theta, \\
\chi_3(\mathcal{A}) &= (4 - 2d)\eta_1\eta_2\theta - 2\eta_2\gamma_{14}\theta.
\end{align*}
\]
We finally find
\[
\begin{align*}
[Y] &= \eta_1\eta_2(c_1^2(2d^2 - 8d + 2gd + 4 - 4g(s - 1)) \\
&\quad + c_1\theta(-12d - 4g + 40) + c_2(-4d + 16 - 8g) + 12\theta^2) \\
&= \frac{(28s + 48)(s - 2)(s - 1)}{(s + 3)}N_{g,2,d} \cdot \eta_1\eta_2\theta^{g - 1},
\end{align*}
\]
where we have used the following identities proved in [Far09, Lemma 2.6]
\[
\begin{align*}
c_1^2 &= \left(1 + \frac{2s + 2}{s + 3}\right)c_2, \\
c_1\theta &= (s + 1)c_2, \\
\theta^2 &= \frac{(s + 1)(s + 2)}{3}c_2, \\
c_2 &= N_{g,2,d} \cdot \theta^{g - 1}.
\end{align*}
\]
We are going to show that we have already considered all non zero contributions. Indeed let us suppose \(x = y\). Blowing up the point \(x\), we obtain \(\mathcal{C} \cup_{q} \mathbb{P}^1 \cup_{q} \mathcal{E}\) with \(x \in \mathbb{P}^1 \setminus \{y, q\}\) and \(p \in E \setminus \{q\}\). We reformulate the condition on \(x\) and \(p\) viewing our curve as the special fiber of a family of curves \(\pi : X \to B\) as before. Let \(\{l_C, l_{E,1}, l_E\}\) be a limit \(g^2_{g,2}\). Now \(V'\) specializes to \(V'_C, V'_{E,1}\) and \(V'_{E}\). There are three possibilities: either \(\rho(C, y) = \rho(\mathbb{P}^1, x, y, q) = \rho(E, p, q) = 0\), or \(\rho(C, y) = -1, \rho(\mathbb{P}^1, x, y, q) = 0, \rho(E, p, q) = 1\), or \(\rho(C, y) = -1, \rho(\mathbb{P}^1, x, y, q) = 1, \rho(E, p, q) = 0\). In all these cases \(a^c_{E}(y) = (0, 2, a_{E}^c(y))(\text{remember that } l_C \text{ is base field})\) and \(a^c_{E}(q) = (a_{E}^c(q), d - 2, d)\). Hence \(a_{E}^c(x) = (a_{E}^c(y), d - 2, d)\) and \(a_{E}^c(q) = (0, 2, a_{E}^c(q))\). Let us restrict now to the sections in \(V'_C, V'_{E,1}\) and \(V'_{E}\). For all sections \(\sigma_{C} \in V'_C\) since \(\text{div}(\sigma_{C}) \geq x\), we have that \(\text{ord}_p(\sigma_{C}) < d\) and hence \(\text{ord}_q(\sigma_{C}) \leq d - 2\). On the other side, since for all \(\sigma_E \in V'_E, \text{div}(\sigma_E) \geq p\), we have that \(\text{ord}_q(\sigma_{E}) < d\) and hence \(\text{ord}_q(\sigma_{E}) \geq 2\). Let us take one section \(\tau \in V'_{E,1}\) such that \(\text{ord}_p(\tau) = d - 2\). Since \(\text{div}(\tau) \geq (d - 2)y + x\), we get \(\text{ord}_q(\tau) \leq 1\), hence a contradiction.
Thus we have that
$$\mathfrak{N}_d^2 \cdot C_1 = \frac{24(2s^2 + 3s - 4)}{s + 3} N_{g,2,d} + \frac{(28s + 48)(s - 2)(s - 1)}{(s + 3)} N_{g,2,d}.$$  
while considering the intersection of the test curve $C_1$ with the generating classes we have
$$\mathfrak{N}_d^2 \cdot C_1 = b_1(2g - 4),$$  
whence
$$b_1 = \frac{14s^3 + 6s^2 - 8s}{(3s - 2)(s + 3)} N_{g,2,d}.$$

**Remark 5.** The previous class $[Y]$ being nonzero, it implies together with Thm. 2 that the scheme $\mathcal{G}_2^2((0,2,3))$ over $\mathcal{M}_{g-1,1}$ is not irreducible.

3.4. The coefficient $b_{g-1}$. We analyze now the following test curve $\bar{E}$. Let $(C,p)$ be a general pointed curve of genus $g - 1$ and $(E,q)$ be a pointed elliptic curve. Let us identify the points $p$ and $q$ and let $y$ be a movable point in $E$. We have
$$0 = \mathfrak{N}_d^2 \cdot \bar{E} = c + b_1 - b_{g-1},$$  
whence
$$b_{g-1} = \frac{48s^4 + 12s^3 - 56s^2 + 20s}{(3s - 1)(3s - 2)(s + 3)} N_{g,2,d}.$$

3.5. A test. Furthermore, as a test we consider the family of curves $R$. Let $(C,p,q)$ be a general two-pointed curve of genus $g - 1$ and let us identify the point $q$ with the base point of a general pencil of plane cubic curves. We have
$$0 = \mathfrak{N}_d^2 \cdot R = a - 12b_0 + b_{g-1}.$$  

3.6. The remaining coefficients in case $g = 6$. Denote by $P_g$ the moduli space of stable $g$-pointed rational curves. Let $(E,p,q)$ be a general 2-pointed elliptic curve and let $j : P_g \to \overline{\mathcal{M}}_{g,1}$ be the map obtained identifying the first marked point on a rational curve with the point $q \in E$ and attaching a fixed elliptic tail at the other marked points. We claim that $j^*(\mathfrak{N}_d^2) = 0$.

Indeed consider a flag curve of genus 6 in the image of $j$. Clearly the only possibility for the adjusted Brill-Noether numbers is to be zero on each aspect. In particular the collection of the aspects on all components but $E$ smooths to a $g^2$ on a general 1-pointed curve of genus 5. As discussed in section 3.3 the point $x$ is in the rest of the curve. Then smoothing we get a $g^2$ on a general pointed curve of genus 5 such that $l(l(-2q - x))$ is a $g^2$, a contradiction.

Now let us study the pull-back of the generating classes. As in [EH87, §3] we have that $j^*(\lambda) = j^*(\delta_0) = 0$. Furthermore $j^*(\psi) = 0$.

For $i = 1, \ldots , g - 3$ denote by $e_i^{(1)}$ the class of the divisor which is the closure in $P_g$ of the locus of 2-component curves having exactly the first marked point and other $i$ marked points on one of the two components. Then clearly $j^*(\delta_i) = e_i^{(1)}$ for $i = 2, \ldots , g - 2$. Moreover adapting the argument in [EH89, pg. 49], we have that
$$j^*(\delta_{g-1}) = - \sum_{i=1}^{g-3} \frac{i(g - i - 1)}{g - 2} e_i^{(1)}$$  
while
$$j^*(\delta_1) = - \sum_{i=1}^{g-3} \frac{(g - i - 1)(g - i - 2)}{(g - 1)(g - 2)} e_i^{(1)}.$$
Finally since \( j^*(\Sigma^2) = 0 \), checking the coefficient of \( e_i^{(1)} \) we obtain

\[
b_{i+1} = \frac{(g-i-1)(g-i-2)}{(g-1)(g-2)}b_i + \frac{i(g-i-1)}{g-2}b_{g-1}
\]

for \( i = 1, 2, 3 \).

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