Cosmology with minimal length uncertainty relations

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Abstract

We study the effects of the existence of a minimal observable length in the phase space of classical and quantum de Sitter (dS) and Anti de Sitter (AdS) cosmology. Since this length has been suggested in quantum gravity and string theory, its effects in the early universe might be expected. Adopting the existence of such a minimum length results in the Generalized Uncertainty Principle (GUP), which is a deformed Heisenberg algebra between minisuperspace variables and their momenta operators. We extend these deformed commuting relations to the corresponding deformed Poisson algebra in the classical limit. Using the resulting Poisson and Heisenberg relations, we then construct the classical and quantum cosmology of dS and AdS models in a canonical framework. We show that in classical dS cosmology this effect yields an inflationary universe in which the rate of expansion is larger than the usual dS universe. Also, for the AdS model it is shown that GUP might change the oscillatory nature of the corresponding cosmology. We also study the effects of GUP in quantized models through approximate analytical solutions of the Wheeler-DeWitt (WD) equation, in the limit of small scale factor for the universe, and compare the results with the ordinary quantum cosmology in each case.

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1 Introduction

One of the most important predictions of the theories which deal with quantum gravity is that there exists a minimal length below which no other length can be observed [1]-[6]. From perturbative string theory point of view [1, 2], such a minimal observable length, of order of Planck scale, is due to the fact that strings cannot probe distances smaller than the string size. Also, the existence of this minimal observable length has been suggested in quantum gravity [5], quantum geometry [7] and black hole physics [8]. Indeed in the scale of this minimal size $l_p = \sqrt{\frac{G \hbar}{c^3}}$, the quantum effects of gravitation become as important as the electroweak and strong interactions. Clearly, at low energy levels, these quantum gravity effects are not too important, but in high energy physics, that is, the energies of order of Planck mass, $m_p = \hbar/l_p$ such as very early universe or in the strong gravitational fields of a black hole, one cannot neglect these effects. In string theory, as it is indicated in [9], if the energy of a string reaches the Planck mass, excitations of the string cause a non-zero extension and thus it is impossible to measure the position with an uncertainty smaller than $l_p$, [10].

One of the interesting features of the existence of a minimum length described above is the modification it makes to the standard commutation relation between position and momentum in
ordinary quantum mechanics [10, 11], which are called Generalized Uncertainty Principle (GUP). In one dimension the simplest form of such relations can be written as

\[ \Delta p \Delta x \geq \frac{\hbar}{2} \left( 1 + \beta (\Delta p)^2 + \gamma \right), \tag{1} \]

where \( \beta \) and \( \gamma \) are positive and independent of \( \Delta x \) and \( \Delta p \), but may in general depend on the expectation values \( <x> \) and \( <p> \). The usual Heisenberg commutation relation can be recovered in the limit \( \beta = \gamma = 0 \). As is clear from equation (1), this equation implies a minimum position uncertainty of \( (\Delta x)_{\text{min}} = \hbar \sqrt{\beta} \), and hence \( \beta \) must be related to the Planck length. For a more general discussion of such deformed Heisenberg algebras, especially in three dimension, see Ref. [12]. Now, it is possible to realize equation (1) from the following commutation relation between position and momentum operators

\[ [x, p] = i\hbar \left( 1 + \beta p^2 \right), \tag{2} \]

where we take \( \gamma = \beta <p>^2 \). More general cases of such commutation relations are studied in Refs. [10]-[14].

The low energy effects of the modified Heisenberg uncertainty relations have been extensively studied in various phenomena in recent years. For example, the spectrum of Hydrogen atom arisings from modified Schrödinger equation based on GUP is studied in [15] and [16]. In [15], using this approach for the Hydrogen atom results in the splitting of degenerate energy levels. Also, an upper bound for the deformation parameter of Heisenberg algebra is obtained. On the other hand, the generalized Schrödinger equation and its momentum space representation have been studied in [16] and [17]. In this direction, in [18], the authors consider two well-known problems in usual quantum mechanics, a particle restricted to move in a one dimensional box and the free particle, and solved them in the GUP framework. In each case the resulting wave functions from the corresponding modified Schrödinger equation are compared with the ordinary solutions. Applying this formalism to quantum fluctuations in the early universe and the corresponding effects on the inflation is investigated in [19].

It is a generally accepted practice to introduce GUP either through the coordinates or fields which may be called geometrical or phase space GUP respectively [20]. Applying GUP to ordinary quantum field theories where geometry is considered to obey the GUP relations are interesting to study since they could provide an effective theory bridging the gap between ordinary quantum field theory and string theory, currently considered as the most important choice for quantization of gravity. A different approach to GUP is through its introduction in the phase space constructed by minisuperspace fields and their conjugate momenta [21]. Since cosmology can test physics at energies that are much higher than those on Earth, it seems natural that the effects of quantum gravity could be observed in this context. On the other hand in cosmological systems, since the scale factor, matter fields and their conjugate momenta play the role of dynamical variables of the system, introducing GUP in the corresponding phase space is particularly relevant.

In this paper, we study the well-known classical and quantum de Sitter (dS) and Anti de Sitter (AdS) cosmologies in a canonical framework within the context of GUP in phase space. Here we extend the deformed Heisenberg algebra to the classical limit to get a deformed Poisson algebra, that is, the classical version of GUP. We then resolve the classical dS and AdS cosmologies by using this deformed Poisson brackets. The resulting solutions show some differences between the corresponding cosmologies and the usual ones depending on the deformation parameter \( \beta \). Also, we consider the quantum cosmology of these models and show that applying GUP to the corresponding phase space results in a fourth order Wheeler-DeWitt (WD) equation for the corresponding quantum cosmology. Although, we cannot solve this fourth order differential equation in general, we can provide some approximate analytical solutions of this equation in the limit of small scale factors. Finally, we compare the wave functions of ordinary dS and AdS universes with the wave functions obtained from applying the GUP in this context. It is to be noted that our presentation does not claim to clear the role of GUP in cosmology in a fundamental way because we just study the problem in a special
simple model. However, this may reflect realistic scenarios in similar investigations which deal with this problem in a more fundamental way. In what follows, we work in units where $c = \hbar = 16\pi G = 1$.

2 Classical dS and AdS Cosmologies with Deformed Poisson Algebra

To start, let us make a quick review of the well-known dS and AdS cosmologies and obtain the mini-superspace Lagrangian and Hamiltonian. We assume that the universe is homogeneous and isotropic. Thus, it can be described by the flat Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$  \(3\)

where $a(t)$ is the scale factor of the universe. The scalar curvature corresponding to metric (3) is

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right),$$ \([4]\)$

where a dot represents differentiation with respect to cosmic time $t$. To construct the canonical formalism of the theory, let us start with the Einstein-Hilbert action

$$S = \int (R - 2\Lambda)\sqrt{-g}d^4x,$$ \(5\)

where $g$ is the determinant of the spacetime metric and $\Lambda$ is the cosmological constant representing the vacuum energy. Substituting (3) and (4) into (5) and integrating over the spatial dimension, we are led to the following Lagrangian

$$L = a\dot{a}^2 + \frac{1}{3}\Lambda a^3,$$ \(6\)

which yields the Hamiltonian

$$H = \frac{p_a^2}{4a} - \frac{1}{3}\Lambda a^3,$$ \(7\)

where $p_a$ is the momentum conjugate to $a(t)$. The simplest classical inflationary and oscillatory models, say dS and AdS models respectively, can be obtained in this step using the above Hamiltonian to construct the equations of motion, but a remark about this Hamiltonian is the factor ordering problem when one embarks on construction a quantum mechanical operator equation. This is an indication that in quantizing the system, the ordering problem becomes important. In ordinary canonical quantum cosmology one can deal with a parameter denoting the ambiguity in the ordering of factors to guarantee Hermiticity of the operator corresponding to the Hamiltonian. But as we shall see in the next section, quantization of the model in the GUP framework needs a more complicated representation for momentum operator and thus introducing the factor ordering parameter in this representation is not an easy task. Therefore, to transform Hamiltonian (7) to a more manageable form, consider the change of variables $(a, p_a) \rightarrow (u, p_u)$ as

$$u = a^{3/2} \Rightarrow p_u = \frac{2}{3}a^{-1/2}p_a.$$ \(8\)

In terms of these new variables the Hamiltonian takes the form

$$H = \frac{9}{16}p_u^2 - \frac{1}{3}\Lambda u^2.$$ \(9\)

Now, use of the usual equation of motion $\dot{X} = \{X, H\}$, and that of the canonical variables satisfying the Poisson algebra

$$\{x_i, p_j\} = \delta_{ij}, \quad \{x_i, x_j\} = \{p_i, p_j\} = 0,$$ \(10\)

\footnote{Note that at $a = 0$ we have also $p_u = 0$ and thus $p_u$ is well defined at $a = 0$, i.e. at $u = 0.$}
one obtains the well-known inflationary dS solution in the case of a positive cosmological constant

\[ u(t) = e^{\Omega t}, \quad (11) \]

in which \( \Omega^2 = \frac{3}{4} \Lambda \) and we take \( u(t = 0) = 1 \). Now, we would like to investigate the effects of classical version of GUP, i.e., classical version of commutation relation (2) on the above cosmology. As is well known, in the classical limit the quantum mechanical commutators should be replaced by the classical Poisson brackets as \([P,Q] \to i\hbar \{P,Q\}\). Thus, considering the GUP issue in classical phase space changes the Poisson algebra (10) into their deformed forms as

\[ \{x_i, x_j\} = \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij} \left(1 + \beta p^2\right). \quad (12) \]

Such deformed Poisson algebra is used in [22] to investigate effects of the deformation on the classical orbits of particles in a central force field and on the Kepler third law. Also, the stability of planetary circular orbits in the framework of such deformed Poisson brackets is considered in [23]. Note that here we deal with modifications of a classical cosmology that become important only at the Planck scale, where the classical description is no longer appropriate and a quantum model is required. However, before quantizing the model we shall provide a deformed classical cosmology. In this classical description of the universe in transition from commutation relation (2) to Poisson brackets (12) we keep the parameter \( \beta \) fix as \( \hbar \to 0 \). In string theory this means that the string momentum scale is fixed when its length scale approaches the zero. Therefore, by extending the above Poisson algebra to the phase space constructed by \( u \) and \( p_u \), we should obtain the corresponding cosmology resulting from Hamiltonian (9) where its canonical variables \( u \) and \( p_u \) satisfy the Poisson bracket

\[ \{u, p_u\} = 1 + \beta p^2_u, \quad (13) \]

which yields the following equations of motion

\[ \dot{u} = \{u, \mathcal{H}\} = \frac{9}{8} p_u \left(1 + \beta p^2_u\right), \quad (14) \]

\[ \dot{p}_u = \{p_u, \mathcal{H}\} = \frac{2}{3} \Lambda u \left(1 + \beta p^2_u\right). \quad (15) \]

Differentiation of equation (14) and use of equation (15) yields

\[ \ddot{u} = \frac{3}{4} \Lambda u \left(1 + 4\beta p^2_u\right), \quad (16) \]

where we consider only the terms of first order in \( \beta \). On the other hand from the system of differential equation (14) and (15) we obtain

\[ \frac{\dot{u}}{p_u} = \frac{27}{16\Lambda} \frac{p_u}{u}, \quad (17) \]

which can be immediately integrated with the result

\[ p^2_u = \frac{16}{27} \Lambda u^2, \quad (18) \]

where we take the integration constant to be zero. Now, substituting the result (18) into equation (16), we are led to the following differential equation for \( u(t) \)

\[ \ddot{u} - \frac{3}{4} \Lambda u - \frac{16}{9} \beta \Lambda^2 u^3 = 0. \quad (19) \]

This is the deformed version of equation of motion and the last term is the GUP correction, had it not been for this term, we would have got a pure dS solution as (11). The nonlinear term in the above equation, because of the presence of the deformation parameter \( \beta \), is very small and thus on
this assumption we may solve the equation approximately by a perturbation method. The classical solution (11) with a good approximation is equal to the solution of the above equation, hence, to first order of approximation we can substitute solution (11) into the $\beta$-term to obtain the following equation

$$\ddot{u} - \Omega^2 u = \left(\frac{16}{9}\right)^2 \beta \Omega^4 e^{3\Omega t},$$

where as before $\Omega^2 = \frac{\Lambda}{3\Lambda}$, $\Lambda > 0$. Thus, we see that in our linearization mechanism, the effect of the deformed Poisson algebra is to change the equation of motion to a nonhomogeneous differential equation. The complete solution of equation (20) can be easily obtained by adding its particular solution to the homogeneous solution (11), with the result

$$u(t) = e^{\Omega t} + \frac{32}{81} \beta \Omega^4 e^{3\Omega t}. \quad (21)$$

Figure 1 shows the effect of $\beta$ on the expansion rate of the universe. As it is clear from this figure, increasing the value of $\beta$ results in larger expansion rate and thus larger size for the universe compared to the case when $\beta = 0$, at equal times. This phenomenon may be interesting in the inflationary cosmological scenarios. Now, let us deal with the case of a negative cosmological constant, i.e., the AdS model. In this case the solutions can be obtained with the replacement $\Omega^2 \rightarrow -\omega^2 = -\frac{3}{4}\Lambda$. Thus, the ordinary cosmology is an oscillatory universe with

$$u(t) = \sin \omega t, \quad (22)$$

where we have taken the initial condition $u(0) = 0$. Substituting this solution into the negative cosmological constant version of (20), that is,

$$\ddot{u} + \omega^2 u = \left(\frac{16}{9}\right)^2 \beta \omega^4 \sin^3 \omega t, \quad (23)$$

we are led to the following solution

$$u(t) = \sin \omega t - \frac{32}{27} \beta \omega^3 t \cos \omega t + \frac{8}{81} \beta \omega^2 \sin 3\omega t. \quad (24)$$

We show the corresponding scale factors for some values of $\beta$ in figure 2. As this figure shows, increasing the value of $\beta$ may disturb the oscillatory nature of the universe. Also, in the presence of $\beta$ terms, the period of oscillations become larger and thus the Big-Crunch in the corresponding cosmological model occurs later in comparison with the ordinary AdS model in which $\beta = 0$. 

Figure 1: The effect of $\beta$ on the expansion rate of the universe for a positive cosmological constant. We take the numerical values $\Omega = 1$ and $\frac{32}{81} \beta = 0, 0.005, 0.01, 0.1$ from bottom to top respectively.
3 Quantum dS and AdS Cosmologies with GUP

In general GUP in its original form (see [10, 11]) implies a noncommutative underlying geometry for space time. But formulation of gravity in a noncommutative space time is highly nonlinear and setting up cosmological models is not an easy task. Here our aim is to study some aspects regarding the application of the GUP framework in quantum cosmology, i.e., in the context of a minisuperspace reduction of the dynamics. As is well-known in the minisuperspace approach of quantum cosmology, which is based on the canonical quantization procedure, one first freezes a large number of degrees of freedom by imposition of symmetries on the spacial part of the metric and then quantizes the remaining ones. Therefore, in the absence of a full theory of quantum gravity, quantum cosmology is a quantum mechanical toy model with a finite degrees of freedom which is a simple arena to test ideas and constructions which can be introduced in quantum general relativity. In this respect, the GUP approach to quantum cosmology appears to have physical grounds. In fact, one notes that a deformation of the canonical Heisenberg algebra immediately leads to a generalized uncertainty principle. In other words, the GUP scheme relies on a modification of the canonical quantization prescriptions and, in this respect, it can be reliably applied to any dynamical system (see [24] for a more clear explanation on the GUP in the minisuperspace dynamics). In this sense, as we have mentioned in introduction there are various realizations of GUP and one can introduce deformation between different dynamical variables of the corresponding minisuperspace and of course get different results. But we use the most common approaches which are used in the literature.

We now focus attention on the study of the quantum cosmology of the models described above. For this purpose we quantize the dynamical variables of the model with the use of the WD equation, that is, $\mathcal{H}\Psi = 0$, where $\mathcal{H}$ is the operator form of the Hamiltonian given by equation (9), and $\Psi$ is the wave function of the universe, a function of the scale factors and the matter fields, if they exist. In ordinary canonical quantum cosmology, use of usual commutation relation $[u, p_u] = i$, results in the well-known representation $p_u = -i\partial/\partial u$, to construct the WD equation. But in the GUP framework, as we have mentioned in the introduction, the existence of a minimum observable length requires the following commutation relation between $u$ and $p_u$,

$$[u, p_u] = i \left(1 + \beta p_u^2\right). \tag{25}$$

Of course, this new algebra between canonical variables changes the representations of the corresponding operators. Indeed, in the GUP formalism the following representation of the momentum operator in $u$-space fulfills the relation (25) up to first order in $\beta$, [17]

$$p_u = -i \left(1 - \frac{\beta}{3} \frac{\partial^2}{\partial u^2}\right) \frac{\partial}{\partial u}. \tag{26}$$
Now, using this representation for the momentum operator, the WD equation can be written up to first order in $\beta$ as

$$\left[ \frac{2}{3} \beta \frac{d^4}{du^4} - \frac{d^2}{du^2} - \frac{16}{27} \Lambda u^2 \right] \Psi(u) = 0.$$  \hfill (27)

The appearance of a differential equation of fourth order to describe a physical phenomenon is interesting, since it requires to investigate the corresponding new boundary conditions, which is not the goal of our study in this paper. Instead, since we cannot solve the above equation analytically, we provide an approximate method which in its validity domain we need to solve only a second order differential equation. Before trying this, we should give some comments about the quantum mechanical wave functions in GUP formalism. As it is clearly discussed in [10], in the GUP framework, because of the existence of a minimal observable length, we cannot have localized quantum states. In this formalism one resolves this problem by considering maximal localization which are proper quasi-position wave functions $\Psi(u)$. Indeed, these states can be used to define a quasi-position representation, which has a direct interpretation in terms of position measurements. For a more detailed description of this important issue see [10, 15, 16, 17].

Taking $\beta = 0$ in equation (27) yields the ordinary WD equation where its solutions, in the case of a positive cosmological constant, can be written in terms of Bessel functions as

$$\Psi(u) = u^{1/2} \left[ c_1 J_{1/4} \left( \frac{\omega}{2} u^2 \right) + c_2 Y_{1/4} \left( \frac{\omega}{2} u^2 \right) \right],$$  \hfill (28)

where $\omega^2 = \frac{16}{27} \Lambda$, $\Lambda > 0$. To avoid divergent behavior of the wave function in the limit $u \to 0$, we take $c_2 = 0$. \hfill (29)

In the case when $\beta \neq 0$, as we mentioned before, equation (27) cannot be solved exactly, but if we use the solution (29) in the $\beta$-term of (27), we may obtain some approximate analytical solutions in the region $u \to 0$. To this end, note that the effects of $\beta$ are important at Planck scales, \textit{i.e.} in cosmology language in the very early universe, that is, when the scale factor is small $u \sim 0$. The limiting behavior of solution (29) in the region $u \sim 0$ is \hfill (30)

$$\Psi(u) = u^{1/2} J_{1/4} \left( \frac{\omega}{2} u^2 \right) \to$$

$$u^{1/2} \left\{ \frac{1}{\Gamma(5/4)} \left( \frac{\omega}{4} u^2 \right)^{1/4} - \frac{1}{\Gamma(9/4)} \left( \frac{\omega}{4} u^2 \right)^{1/4+2} + ... \right\} =$$

$$u^{1/2} \left\{ \left( \frac{\omega}{4} \right)^{1/4} \frac{1}{\Gamma(5/4)} u^{1/2} - \left( \frac{\omega}{4} \right)^{9/4} \frac{1}{\Gamma(9/4)} u^{9/2} \right\}.$$  \hfill (31)

Thus, in the limit $u \to 0$, we have a wave function of the form

$$\Psi(u) = C_1 u - C_2 u^5,$$  \hfill (32)

and therefore its fourth derivative is equal to $d^4\Psi/du^4 = -120C_2 u$. Now, if we consider only the first order term with respect to $u$ in (31), we get $d^4\Psi/du^4 = -6\omega^2 \Psi$. Substituting this result into equation (27) leads us to the following WD equation for the corresponding dS quantum cosmology in the GUP framework

$$\frac{d^2\Psi}{du^2} + \omega^2 u^2 \Psi + 4\beta \omega^2 \Psi = 0.$$  \hfill (33)

This equation, after a change of variable $v = i\omega u^2$ and transformation $\Psi = v^{-1/4} \phi$, takes the form

$$\frac{d^2\phi}{dv^2} + \left( -\frac{1}{4} + \frac{\kappa}{v} + \frac{1/4 - \mu^2}{v^2} \right) \phi = 0,$$  \hfill (34)
Figure 3: The square of wave function of the dS quantum universe. We take the numerical values $\omega = 2$ and $\beta = 0, 1$ for solid and dashed curves respectively.

where $\kappa = -i\beta\omega$ and $\mu = 1/4$. The above equation is the well-known Whittaker differential equation and its solutions can be written in terms of Confluent Hypergeometric functions $M(a, b; x)$ and $U(a, b; x)$ as

$$\phi(v) = e^{-v^2/2}v^{\mu+1/2} \left\{ c_1 M \left( \mu - \kappa + \frac{1}{2}, 2\mu + 1; v \right) + c_2 U \left( \mu - \kappa + \frac{1}{2}, 2\mu + 1; v \right) \right\}, \quad (34)$$

Therefore the solutions of equation (32) read

$$\Psi(u) = e^{-i\omega u^2/2} \left( i\omega u^2 \right)^{1/2} \left\{ c_1 M \left( \frac{3}{4} + i\beta\omega, \frac{3}{2}; i\omega u^2 \right) + c_2 U \left( \frac{3}{4} + i\beta\omega, \frac{3}{2}; i\omega u^2 \right) \right\}. \quad (35)$$

Bearing in the mind that if $\beta = 0$, the solution (29) should be recovered, we take $c_2 = 0$ and retain only the function $M(a, b; x)$ [25]. Therefore, we are led to the final form of the GUP wave function of the early universe as follows

$$\Psi(u) = e^{-i\omega u^2/2} \left( i\omega u^2 \right)^{1/2} M \left( \frac{3}{4} + i\beta\omega, \frac{3}{2}; i\omega u^2 \right). \quad (36)$$

In figure 3 we have plotted the square of wave functions (29) and (36), which are related to the cases $\beta = 0$ and $\beta \neq 0$, respectively. As is clear from this figure, in both cases the probability (which is proportional to $|\Psi(u)|^2$), of the emerging universe from $u = 0$ is zero, which is in agreement with classical solutions. Indeed the peaks in $|\Psi(u)|^2$ occur after $u = 0$ which shows an expanding universe as the classical solutions also predict. But as the figure shows, the square of the wave function in the case of $\beta \neq 0$ has a more rapid slope in comparison with the case $\beta = 0$. We recall the same behavior for the classical solutions for which the $\beta \neq 0$ solutions have a larger rate of expansion.

The solutions of the WD equation in the case of $\Lambda < 0$, i. e. the AdS quantum cosmology wave functions can be obtained by replacing $\omega \to i\omega$. Thus, when $\beta = 0$ the wave function of the WD equation can be written in terms of modified Bessel functions as

$$\Psi(u) = u^{1/2} \left\{ c_1 K_{1/4} \left( \frac{\omega}{2} u^2 \right) + c_2 I_{1/4} \left( \frac{\omega}{2} u^2 \right) \right\}. \quad (37)$$

The functions $I_\nu(x)$ are usually omitted because of their divergent behavior in the limit $u \to \infty$. Therefore, the wave function in this case reads

$$\Psi(u) = u^{1/2} K_{1/4} \left( \frac{\omega}{2} u^2 \right). \quad (38)$$

Now, with the GUP consideration in this quantum cosmology model, i. e., taking $\beta \neq 0$ and following the same procedure as in the positive cosmological constant case, we obtain the corresponding WD equation in the limit $u \sim 0$, as

$$\frac{d^2 \Psi}{du^2} - \omega^2 u^2 \Psi + 4\beta\omega^2 \Psi = 0, \quad (39)$$
where its solutions can be again written in terms of the Confluent Hypergeometric functions as

$$
\Psi(u) = e^{-\omega u^2/2} \left(\omega u^2\right)^{1/2} U\left(\frac{3}{4} - \beta\omega, \frac{3}{2}; \omega u^2\right).
$$

(40)

However this time to recover the solution (38) in the limit $\beta = 0$, we retain only $U(a, b; x)$ function [25]. Figure 4 shows $|\Psi(u)|^2$ for typical values of parameters in both $\beta = 0$ and $\beta \neq 0$ cases. We see that in ordinary AdS quantum cosmology, the universe with highest probability emerges at $u = 0$. This behavior also occurs in the model with GUP, but in spite of taking $\beta = 0$, in this case the square of the wave function has several peaks. The emergence of new peaks in the GUP model wave function may be interpreted as a representation of different quantum states that may communicate with each other through tunnelling. This means that there are different possible states from which our present universe could have evolved and tunneled in the past. Similar behavior also occurs in noncommutative cosmological scenarios [26], which may mean that there is a close relationship between noncommutativity and GUP [27].

4 Conclusions

In this paper we have studied the effects of generalized uncertainty relation on phase space, in classical and quantum dS and AdS cosmologies. This yields a deformed Heisenberg commutation relation between the phase space operators called GUP. We extend this new algebra to the classical limit to get the corresponding Poisson brackets. With this classical version of GUP, we obtain the corrections to dS and AdS cosmologies. We have shown that if the deformation parameter $\beta$ is not equal to zero, the resulting dS cosmology exhibits an inflationary universe in which the rate of expansion of the universe is larger than the ordinary dS model. Also, in the classical AdS model there is some differences between ordinary and deformed models. In the early universe limit ($t \to 0$), both models have oscillatory behavior but the model based on the deformed Poisson algebra has a larger period, which means that in such models the corresponding Big-Crunch occur later in comparison to the ordinary model. The oscillatory behavior of the classical AdS model can be disturbed in the presence of a $\beta \neq 0$, when time increases. We have also studied the corresponding quantum cosmologies through finding the exact solutions of the WD equation in ordinary dS and AdS models and its approximate analytical solutions in the case when GUP is considered. We have seen that in the quantum dS model the square of wave function has a more rapid slope if the model has a nonzero deformation parameter which may be interpreted as a universe with larger rate of expansion, in agreement with the classical model. Finally, in the quantum AdS model we have shown that in the presence of GUP, the square of wave function of the universe has several peaks which may be related to different states in the early universe from which our present universe could have evolved.
The ordinary AdS models do not exhibit such behavior.

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