Tensor fundamental theorems of invariant theory

Claudio Procesi

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Abstract

The aim of this paper is to establish a first and second fundamental theorem for \( GL(V) \) equivariant polynomial maps from \( k \)-tuples of matrix variables \( End(V)^k \) to tensor spaces \( End(V)^\otimes n \) in the spirit of H. Weyl’s book \textit{The classical groups} \cite{18} and of symbolic algebra.

1 Introduction

In this paper \( V \) denotes a vector space over a field \( F \) of characteristic 0 of dimension \( d \). In fact since all the formulas developed have rational coefficients it is enough to assume \( F = \mathbb{Q} \).

The aim of this paper is to establish a first and second fundamental theorem (FFT and SFT for short), in the spirit of H. Weyl’s book \textit{The classical groups} \cite{18}, for the algebras of \( GL(V) \) equivariant polynomial maps \( F : End(V)^k \to End(V)^\otimes n \), from \( k \)-tuples of matrix variables \( End(V)^k \) to tensor spaces \( End(V)^\otimes n \).

The FFT, Theorem 2.17 is based on the basic Formula (23), the interpretation Formula from which follows that such maps are the evaluations in matrices of the symbolic twisted group algebra \( T\langle X \rangle \otimes_n \mathbb{Q}[S_n] \), where \( T\langle X \rangle \) is the free algebra with trace in the variables \( X = \{x_1, \ldots, x_k\} \) and \( S_n \) commutes with \( T\langle X \rangle \otimes_n \mathbb{Q}[S_n] \) by exchanging the tensor factors.

The SFT, Theorem 3.16 is in the spirit of \( T \)-ideals of universal algebra.

One defines in Formula (35), for each \( d \), the \( d+2 \) tensor Cayley Hamilton identities \( \mathfrak{C}_{k,d}(x) \), \( k = 0, \ldots, d+1 \), homogeneous of degree \( k \) in the \( x \) and for \( d+1-k \) tensor valued polynomials. One first deduces all identities from these ones, Theorem 3.7. For \( k = 0, d, d+1 \) these have classical interpretation as respectively the antisymmetrizer on \( d+1 \) elements, the Cayley–Hamilton identity and the expression of \( tr(x^{d+1}) \) in term of \( tr(x^i) \), \( i = 1, \ldots, d \).

For the other \( k \) they are new identities. For instance \( \mathfrak{C}_{1,3}(x) \), \( \mathfrak{C}_{2,3}(x) \) are

\[
((1,2,3) + (1,3,2) - (1,2) - (1,3) - (2,3) + 1)(x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x - tr(x))
\]

\[
((1-(1,2))((x^2 \otimes 1 + 1 \otimes x^2 + x \otimes x) - tr(x)[x \otimes 1 + 1 \otimes x] + \det(x)) \cdot x - tr(x)(x \otimes 1 + 1 \otimes x) + \det(x))\cdot x - tr(x).\]

Finally using a further operation \( t : T(X)^\otimes n \times \mathbb{Q}[S_n] \to T(X)^\otimes n-1 \times \mathbb{Q}[S_{n-1}] \), a partial trace, corresponding to the natural trace contraction \( M_d^\otimes n \to M_d^\otimes n-1 \) one shows that

Formula (50) \( t(\mathfrak{C}_{k,d}(x)) = 0 \), \( t(\mathfrak{C}_{k,d}(x) \cdot 1^{n-1} \otimes x) = -n \cdot \mathfrak{C}_{k+1,d}(x) \)
The classical groups \([\text{GL}(V), S_n] \) act on \(V\) and \(V^*\) via the dual action. By convention we will write vectors in \(V\) with Latin letters while covectors in \(V^*\) with Greek letters. We use the bracket notation; for \(\varphi \in V^*\), \(v \in V\) we write often \(\langle \varphi \mid v \rangle := \varphi(v)\), so that the dual action is given by the formula
\[
\langle g\varphi \mid v \rangle := \langle \varphi \mid g^{-1}v \rangle, \forall g \in \text{GL}(V), v \in V, \varphi \in V^*.
\]
In other words \(\langle g\varphi \mid gv \rangle = \langle \varphi \mid v \rangle, \forall g \in \text{GL}(V)\) that is the function \(\langle \varphi \mid v \rangle\) of \(v\) and \(\varphi\) is invariant. For this group the first fundamental theorem, FFT for short states that the ring of invariants of several copies of \(V\) and \(V^*\) the previous functions generate all invariants. That is denoting
\[
(v_1, v_2, \ldots, v_h; \varphi_1, \varphi_2, \ldots, \varphi_k) \in V^\oplus_h \oplus (V^*)^\oplus_k.
\]

**Theorem 1.2.** The polynomial functions on \(V^\oplus_h \oplus (V^*)^\oplus_k\) which are \(\text{GL}(V)\) invariant are generated by the \(h \cdot k\) basic functions \(\langle \varphi_i \mid v_j \rangle\), \(i = 1, \ldots, k; j = 1, \ldots, h\).

One of the remarkable features of the Theory as presented by H. Weyl is the fact that this Theorem is equivalent to a second Theorem, \(h = k\).

**Theorem 1.3.** The algebra of linear operators on \(V^\oplus_h\) which commute with the diagonal action of \(\text{GL}(V)\) is generated by the elements of the symmetric group \(S_h\). The two actions are
\[
g \cdot (v_1 \otimes v_2 \otimes \ldots \otimes v_h) = gv_1 \otimes gv_2 \otimes \ldots \otimes gv_h, \quad g \in \text{GL}(V), \quad (2)
\]
\[
\sigma \cdot (v_1 \otimes v_2 \otimes \ldots \otimes v_h) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \ldots \otimes v_{\sigma^{-1}(h)}; \quad \sigma \in S_h. \quad (3)
\]
1.3.1 The second fundamental theorem

Together with a first fundamental theorem one has a second fundamental theorem, SFT for short. In this theorem one describes the relations among the invariants as generated by basic ones.

For the setting of Theorem 1.2 one should think of

\[ V^\otimes h = \text{hom}(F^h, V), \quad (V^*)^\otimes k = \text{hom}(V, F^k). \]

Then the invariants \( \langle \varphi_i | v_j \rangle \) are the entries of the \( h \times k \) matrix image of the map

\[ \text{hom}(F^h, V) \times \text{hom}(V, F^k) \rightarrow \text{hom}(F^h, F^k), \quad (A, B) \rightarrow B \circ A. \]

The image of this map is the variety of matrices of rank \( \leq d := \dim V \).

The polynomial functions on \( \text{hom}(F^h, F^k) \) are in the variables \( x_{i,j} \) which can be viewed as entries of a generic matrix \( X \) and the SFT is:

**Theorem 1.4.** The ideal of functions on the space of \( h \times k \) matrices vanishing on the matrices of rank \( \leq d \) is generated by the determinants of all the \( d + 1 \times d + 1 \) minors of the matrix \( X \).

This is just the first of a long list of ideas and theorems of geometric nature on special singularities.

Instead for the setting of Theorem 1.3 one has

**Theorem 1.5.** Consider the mapping \( \pi : F[S_h] \rightarrow \text{End}(V^\otimes h) \) from the group algebra to the linear operators, given by Formula (3).

The kernel of \( \pi \) is nonzero only if \( \dim V = d < h \) and then it is the two sided ideal of \( F[S_h] \) generated by an antisymmetrizer \( \sum_{\sigma \in S_{d+1}} \varepsilon_\sigma \).

Here \( S_{n+1} \subset S_h \) and \( \varepsilon_\sigma = \pm 1 \) is the sign of the permutation. This is just the first of a long list of ideas and theorems in representation Theory.

1.6 Matrix invariants

There is still a third way in which the fundamental Theorems appear.

In this case we take as basic representation the direct sum of \( h \) copies of the matrix algebra \( \text{End}(V) \) of linear maps of \( V \), under simultaneous conjugation by \( GL(V) \). Start with a remark.

**Remark 1.7.** Given a vector space \( W \) the symmetric group \( S_n \) acts on \( W^\otimes n \) by the formula (3) thus \( S_n \subset \text{End}(W^\otimes n) = \text{End}(W)^\otimes n \). Thus we have a priori two actions of \( S_n \) on \( \text{End}(W)^\otimes n \), the one given by Formula (3) thinking of \( \text{End}(W)^\otimes n \) as tensors and the conjugation action in \( \text{End}(W)^\otimes n \) as algebra. If \( \sigma \in S_n \), is a permutation of the tensor indices we have for a tensor \( a_1 \otimes \ldots \otimes a_n \in M_n^\otimes \):

\[ \sigma \circ a_1 \otimes \ldots \otimes a_n \circ \sigma^{-1} = a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)} = \sigma \cdot a_1 \otimes \ldots \otimes a_n. \]

Formula (4) states that these two actions coincide, cf. Formula (7).

We need to recall the multilinear invariants of \( m \) matrices, i.e. the invariant elements of the dual of \( \text{End}(V)^\otimes m \). The theorem is, cf. [9]:

**Theorem 1.8.** The space \( T_d(m) \) of multilinear invariants of \( m \) endomorphisms \( (x_1, x_2, \ldots, x_m) \) of a \( d \)-dimensional vector space \( V \) is identified with \( \text{End}_{GL(V)}(V^\otimes m) \). It is linearly spanned by the functions:

\[ t_\sigma(x_1, x_2, \ldots, x_m) := \text{tr}(\sigma^{-1} \cdot x_1 \otimes x_2 \otimes \cdots \otimes x_m), \quad \sigma \in S_m. \]

If \( \sigma = (i_1 i_2 \ldots i_k) \ldots (j_1 j_2 \ldots j_k) (s_1 s_2 \ldots s_l) \) is the cycle decomposition of \( \sigma \) then we have that \( t_\sigma(x_1, x_2, \ldots, x_m) \) equals

\[ = \text{tr}(x_{i_1} x_{i_2} \ldots x_{i_k}) \ldots \text{tr}(x_{j_1} x_{j_2} \ldots x_{j_k}) \text{tr}(x_{s_1} x_{s_2} \ldots x_{s_l}). \]
We need to use the rules Theorem 1.10 way to End(V)⊗m by the pairing formula:

\[ \langle A_1 \otimes A_2 \cdots \otimes A_m | B_1 \otimes B_2 \cdots \otimes B_m \rangle := \text{tr}(A_1 \otimes A_2 \cdots \otimes A_m \circ B_1 \otimes B_2 \cdots \otimes B_m) = \prod \text{tr}(A_i B_i). \]

Under this isomorphism the multilinear invariants of matrices are identified with the GL(V) invariants of End(V)⊗m which in turn are spanned by the elements of the symmetric group, Theorem 1.3, hence by the elements of Formula (5).

As for Formula (6), since the identity is multilinear, in the variables x, it is enough to prove it on the decomposable tensors of End(V) = V ⊗ V∗ which are the endomorphisms of rank 1, i.e. u ⊗ ϕ : v → ⟨ϕ | v⟩u.

**Lemma 1.9.**

\[ \sigma^{-1} u_1 \otimes ϕ_1 \otimes u_2 \otimes ϕ_2 \otimes \ldots \otimes u_m \otimes ϕ_m = u_{σ(1)} \otimes ϕ_1 \otimes u_{σ(2)} \otimes ϕ_2 \otimes \ldots \otimes u_{σ(m)} \otimes ϕ_m \]

\[ u_1 \otimes ϕ_1 \otimes \ldots \otimes u_m \otimes ϕ_m \circ σ = u_1 \otimes ϕ_{σ(1)} \otimes u_2 \otimes ϕ_{σ(2)} \otimes \ldots \otimes u_m \otimes ϕ_{σ(m)} \] (7)

**Proof.** Given xi := ui ⊗ ϕi and an element σ ∈ Sn in the symmetric group we have

\[ \sigma^{-1} u_1 \otimes ϕ_1 \otimes u_2 \otimes ϕ_2 \otimes \ldots \otimes u_m \otimes ϕ_m = \prod_{i=1}^{m} \langle ϕ_i | v_i \rangle u_{σ(1)} \otimes u_{σ(2)} \otimes \ldots \otimes u_{σ(m)} \]

\[ = u_{σ(1)} \otimes ϕ_1 \otimes u_{σ(2)} \otimes ϕ_2 \otimes \ldots \otimes u_{σ(m)} \otimes ϕ_m \]

similarly for the other formula.

So we need to understand in matrix formulas the invariants

\[ \text{tr}(σ^{-1} u_1 \otimes ϕ_1 \otimes u_2 \otimes ϕ_2 \otimes \ldots \otimes u_m \otimes ϕ_m) = \prod_{i=1}^{m} \langle ϕ_i | u_{σ(i)} \rangle. \] (8)

We need to use the rules

\[ u \otimes ϕ \circ v \otimes ψ = u \otimes ⟨ϕ | v⟩ψ, \quad \text{tr}(u \otimes ϕ) = ⟨ϕ | u⟩ \]

from which the formula easily follows by induction.

**Theorem 1.10 (FFT for matrices).** The ring \( T_n = T_n(X) \) of invariants of matrices under simultaneous conjugation is generated by the elements

\[ \text{tr}(x_{i_1} x_{i_2} \ldots x_{i_k}). \] (9)

This formula means that we take all possible noncommutative monomials in the xi and form their traces.

**Proof.** The ring of invariants of matrices contains the ring generated by the traces of monomials and both rings are stable under polarization and restitution hence, by Aronhold method, see [12], it is enough to prove that they coincide on multilinear elements and this is the content of the previous Lemma.

Finally for the second fundamental Theorem for matrices one should first generalize that Theorem to a statement about the non commutative algebra of equivariant maps \( F : \text{End}(V)^h \rightarrow \text{End}(V) \). This non commutative algebra is generated over the ring of invariants by the coordinates \( X_i \), see [2].

In this case the SFT is formulated in terms of universal algebra and the language of T-ideals, see [10] and [15] or Chapter 12 of [2].
Theorem 1.11. All the relations among the polynomial equivariant maps $F : \text{End}(V)^d \to \text{End}(V)$ are consequences of the Cayley–Hamilton Theorem.

The reader may look at the proof of the second fundamental theorem 1.11 as in [10], [12], [2] or [5] since we have taken this as a model for the more general Theorem 3.7 of this paper.

This is just the first of a long list of ideas and theorems in non commutative algebra, in particular the Theory of Cayley–Hamilton algebras as presented in the two recent preprints [13], [14].

Remark 1.12. The first and second fundamental Theorem for matrix invariants are not as precise as the other cases. In fact from these Theorems for matrices one can only infer estimates, see [15] or §12.2.4 of [2], and not precise description of minimal generators and relations.

For $2 \times 2$ matrices the results are quite precise, due to the fact that in this case the action of $GL(2)$ on $2 \times 2$ matrices with 0 trace is equivalent to that of $SO(3)$ on its fundamental representation and then one can apply the first and second fundamental theorem for this group, see Chapter 9 of [2].

For $3 \times 3$ matrices there are the computations in [1].

2 Equivariant tensor polynomial maps

2.1 Tensor Polynomials

Given a positive integer $d$ and a field $F$ denote by $M_d(F)$ the algebra of $d \times d$ matrices with entries in $F$, or intrinsically $\text{End}(V), \dim_F V = d$. We will in fact work with $F = \mathbb{Q}$ and then denote $M_d := M_d(\mathbb{Q})$. This paper is an introduction to the study of polynomial maps $F : M_d \to M_d$ which are equivariant under the conjugation action of $GL(F,d)$. Or with intrinsic notations $F : \text{End}(V)^d \to \text{End}(V)^{\otimes n} = \text{End}(V^{\otimes n})$. These maps form a non commutative algebra using the algebra structure of $M_d^{\otimes n}$. Among these are the non commutative polynomials in the tensor variables $x_j^{(i)} := 1^i \otimes x_j \otimes 1^{m-i}$, where $x_j$ is a matrix variable.

Definition 2.2. We call the $x_j^{(i)}$ tensor variables and the (non commutative) polynomials they generate tensor polynomials.

The concept of tensor polynomial or trace identities as far as I know was not considered by algebraists. I wish to thank Felix Huber for pointing out this notion which seems to play some role in Quantum Information Theory, see [7] and [16].

Given a free algebra $F(X)$ and an associative algebra $A$ over a field $F$, a map $f : X \to A$ induces a homomorphism $f : F(X) \to A$ and conversely a homomorphism $f : F(X) \to A$ is determined by its values on $X$.

The polynomial identities of $A$ are the elements of $F(X)$ vanishing under all evaluations.

Now though, such a map $f$ defines also, for all integers $n$, a homomorphism of the corresponding $n$-fold tensor products:

$$f^{\otimes n} : F(X)^{\otimes n} \to A^{\otimes n}.$$ 

One can thus define as $n$-fold tensor identity for $A$ an element $G \in F(X)^{\otimes n}$ vanishing for all evaluations in $A$, i.e. $f^{\otimes n}(G) = 0$, $\forall f : X \to A$.

A similar notion holds for $A$ an algebra with trace.

By this we mean an algebra $A$ together with a linear map $\text{tr} : A \to A$ satisfying the following axioms, see [2] Chapter 2.3 or [13].

$$\text{tr}(a \cdot b) = b \cdot \text{tr}(a), \quad \text{tr}(a \cdot b) = \text{tr}(b \cdot a), \quad \text{tr}(\text{tr}(a) \cdot b) = \text{tr}(a) \cdot \text{tr}(b), \quad \forall a, b \in A.$$ 

Then the image $\text{tr}(A)$ of $\text{tr}$ is a central subalgebra, called the trace algebra and $\text{tr}$ is $\text{tr}(A)$ linear.
Trace algebras form a category where maps are trace compatible homomorphisms. Trace algebras admit free algebras. The free algebra with trace $T(X) := F(X)[tr(M)]$, is the polynomial algebra in the elements $tr(M)$. By $tr(M)$ we denote the class of a monomial $M$ up to cyclic equivalence, (cf. [19] for a detailed definition).

**Definition 2.3.** An $n$-fold tensor trace identity of $A$ is an element of $T(X)^{\otimes n}$ vanishing for all evaluations in $A^{\otimes n}$.

In this case by $T(X)^{\otimes n}$ we mean the tensor product over the central subalgebra of traces $F[tr(M)]$. Therefore $T(X)^{\otimes n}$ is a free $F[tr(M)]$ module with basis the tensor monomials $M_1 \otimes M_2 \otimes \ldots \otimes M_n$.

### 2.4 Equivariant maps and permutations

We now restrict to the case $A = M_d(F)$. Together with tensor polynomials we also have the usual invariants of matrices, described in Theorem 1.10, which may be viewed as scalar valued maps to $M_d^{\otimes n}$ i.e. multiples of the identity of $M_d^{\otimes n}$.

Finally one has the constant equivariant maps, that is the $GL(V)$ invariant elements of $M_d^{\otimes n}$. They form the subalgebra $\Sigma_n(V) \subset \text{End}(V)^{\otimes n}$ spanned by the permutations $\sigma \in S_n$ (Formula (3)) described by Theorem 1.5.

Thus we have 3 types of objects to consider:

**Definition 2.5.**

1. The tensor polynomial maps, i.e. the polynomial maps of $A^X \to A^{\otimes n}$ induced by $F(X)^{\otimes n}$.

2. The trace tensor polynomial maps the maps of $A^X \to A^{\otimes n}$ induced by $T(X)^{\otimes n}$.

3. The equivariant tensor polynomial maps, i.e. the polynomial maps of $A^X \to A^{\otimes n}$ equivariant under conjugation by $GL(d, F)$.

Under the multiplication of the algebra $M_d(F)^{\otimes n}$ each one of these spaces forms an algebra.

The way to understand the general form of such equivariant map, item 3., is to associate to such a map an invariant.

Consider an equivariant polynomial map $H(x_1, \ldots, x_k)$ of $k, d \times d$ matrix variables with values in $M_d^{\otimes n}$. To this associate the invariant scalar function of $k + n, d \times d$ matrix variables $x_1, \ldots, x_k, y_1, y_2, \ldots, y_n$

$$T(H)(x_1, \ldots, x_k, y_1, y_2, \ldots, y_n) := tr(H(x_1, \ldots, x_k)y_1 \otimes y_2 \otimes \ldots \otimes y_n).$$

(10)

By Theorem 1.8 we have that Formula (10) is a linear combination of products of elements of type $tr(M)$ with $M$ a monomial in the variables $x_1, \ldots, x_k$ and $y_1, y_2, \ldots, y_n$ and linear in these last variables.

So we say that $H$ is monomial if it is of the following form. There exist monomials $M_i, i = 1, \ldots, n$ and $N_j$ in the variables $x_1, \ldots, x_k$, possibly empty, that is with value 1, such that, setting $z_i = M_iy_i$ we have:

$$T(H) = \prod_j tr(N_j)tr(z_{i_1}z_{i_2}\cdots z_{i_{h_1}})tr(z_{i_{h_1+1}}z_{i_{h_1+2}}\cdots z_{i_{h_1+h_2}})$$

$$\cdots \cdots tr(z_{i_{h_1+\ldots h_{k-1}}\ldots i_n})$$

(11)

If $\sigma \in S_n$ is the permutation of cycles

$$\sigma = (i_1 i_2 \ldots i_{h_1})(i_{h_1+1} i_{h_1+2} \ldots i_{h_1+h_2}) \ldots \ldots (i_{h_1+\ldots h_{k-1}} \ldots i_n)$$

then Formula (2.4) becomes

$$T(H) = \prod_j tr(N_j)tr(\sigma^{-1} \circ M_1y_1 \otimes \cdots \otimes M_ny_n))$$

(12)

$$= tr \left( \prod_j tr(N_j)(\sigma^{-1} \circ M_1 \otimes \cdots \otimes M_n)y_1 \otimes y_2 \otimes \ldots \otimes y_n \right)$$

(13)
\[ H = \prod_j \text{tr}(N_j)\sigma^{-1} \circ M_1 \otimes \cdots \otimes M_n \quad (14) \]

**Theorem 2.6.** Equivariant tensor polynomial maps are linear combinations of maps given by Formula (14).

This Theorem may be viewed as a First Fundamental Theorem for tensor valued equivariant functions on matrices.

**The algebra** \( T\langle X \rangle^{\otimes n} \ltimes \mathbb{Q}[S_n] \) From the point of view of universal algebra the FFT says that the \( n \) tensor valued equivariant maps are the evaluations in matrices of the elements of the *twisted* algebra \( T\langle X \rangle^{\otimes n} \ltimes \mathbb{Q}[S_n] \).

**Definition 2.7.**

1. \( T\langle X \rangle^{\otimes n} \ltimes \mathbb{Q}[S_n] \) is \( T\langle X \rangle^{\otimes n} \otimes \mathbb{Q}[S_n] \) but with the commuting relations

   \[ \sigma \circ M_1 \otimes \cdots \otimes M_n = M_{\sigma^{-1}(1)} \otimes \cdots \otimes M_{\sigma^{-1}(n)} \cdot \sigma, \ \sigma \in S_n. \]

2. The elements of \( \mathbb{Q}[S_n] \) from the point of view of universal algebra are *constants* and are canonically evaluated by the map \( \pi \) of Formula (3) in \( M_d^{\otimes n} \) (cf. Theorem 1.5).

3. The algebra \( T\langle X \rangle^{\otimes n} \ltimes \mathbb{Q}[S_n] \) is an *algebra with trace* (according to the definition of page 5) where trace is defined by Formula (5) and (6), now thought as definitions. Its trace algebra coincides with the central trace algebra \( \mathbb{Q}[\text{tr}(M)] \) of \( T\langle X \rangle \) of Definition 2.3.

**Remark 2.8.** The tensor polynomial maps as algebra are identified to the algebra \( F\langle X \rangle^{\otimes n} \) modulo the vanishing elements, that is the tensor polynomial identities. Similar statement for trace tensor polynomial maps, and \( T\langle X \rangle^{\otimes n} \), and general equivariant maps and the twisted algebra \( T\langle X \rangle^{\otimes n} \ltimes \mathbb{Q}[S_n] \).

**Splitting the cycles** It order to treat the SFT it is useful to understand in a more precise way the multilinear case.

If \( F \) is linear also in the variables \( x_i \) then there is a permutation \( \tau \in S_{n+k} \) such that Formula (2.4) equals

\[
\text{tr}(F(x_1 \otimes y_1 \otimes y_2 \otimes y_3 \otimes x_2 \otimes \cdots \otimes x_k)) := \text{tr}(\tau^{-1} \circ y_1 \otimes y_2 \otimes \cdots \otimes y_k \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_k). \quad (15)
\]

**Definition 2.9.** We denote by \( T^{(n)}_{k,\tau}(x_1, x_2, \ldots, x_k) \) the tensor valued map \( F(x) \) associated to \( \tau \in S_{n+k} \) by Formula (15).

Our next task is to describe the elements \( T^{(n)}_{k,\tau}(x_1, x_2, \ldots, x_k) \) in terms of the permutation \( \tau \). We will use a simple fact on permutations that we call *splitting the cycles*, used in [10] to prove the SFT for matrices, Theorem 1.11.

**Proposition 2.10.** [Splitting the cycles] Decompose \( \{1, 2, \ldots, m\} = A \cup B \) as disjoint union of two subsets \( A, B \).

Every permutation \( \tau \in S_m \) can be uniquely decomposed as the product \( \tau = \tau_1 \tau_2 \tau_3 \)

where:

1. \( \tau_1 \in S_A \) permutes only the indices in \( A \).
2. Each cycle of \( \tau_2 \) contains exactly one element of \( A \).
3. \( \tau_3 \in S_B \) is formed by the cycles of \( \tau \) permuting only the indices in \( B \). It commutes with \( \tau_1 \) and \( \tau_2 \) since its indices are disjoint from the ones appearing in \( \tau_2 \).

**Proof.** First we may split \( \tau = \bar{\tau} \tau_3 \) where \( \tau_3 \) collects all the cycles \( \tau \) permuting only the indices in \( B \). Replacing \( \tau \) with \( \bar{\tau} \) we may assume that \( \tau = \tau_1 \tau_2 \) where \( \tau_1 \) collects all the cycles of \( \tau \) permuting only the indices in \( A \). Thus \( \tau_2 \) collects all the cycles of \( \tau \) involving both the indices in \( A \) and in \( B \).

By construction the 3 permutations \( \bar{\tau} \tau_2 \tau_3 \) commute with each other and the indices appearing in \( \tau_3 \) are disjoint from the ones in \( \tau_2 \) and \( \bar{\tau} \).
Then the construction is based, by induction on the number of cycles, on the following identity. For $a_1, a_2, a_3, \ldots, a_k$ numbers and $C_1, C_2, \ldots, C_k$ strings of numbers, each number appearing only once, consider the permutation cycle:

$$(a_1, C_1, a_2, C_2, a_3, C_3, \ldots, a_k, C_k) = (a_1, a_2, a_3, \ldots, a_k) \circ (a_1, C_1)(a_2, C_2)(a_3, C_3) \ldots (a_k, C_k). \quad (16)$$

This we call splitting the cycles with respect to $A, B$.

The uniqueness is as follows. $\tau_3$ is well defined from $\tau$. So assume we have $\sigma_1 \sigma_2 = \tau_1 \tau_2$ with $\sigma_1, \tau_1$ permuting only the indices in $A$ and each cycle of $\sigma_2, \tau_2$ contains exactly one element of $A$. Multiplying both sides by $\tau_1^{-1}$ we may assume $\tau_1 = 1$. If $\sigma_1 \neq 1$ the nontrivial cycles of $\sigma_1$ cannot be disjoint from those of $\sigma_2$ otherwise they would be cycles of $\tau_1$. But then a cycle $c = (\alpha, i)$ of $\sigma_1$ with $\alpha$ a non trivial string in $A$ times $(\beta, i)$ a cycle of $\sigma_2$ gives rise to the cycle $(\beta, i, \alpha)$ of $\tau_1$. This is a contradiction since the string $i, \alpha$ is in $A$ and has more than one element.

From an algorithmic point of view the two permutations $\tau_1, \tau_2$ are obtained as follows. Write the decomposition of $\tau$ into cycles. Remove the product $\tau_3$ of all the cycles in $B$ and $\tau_4$ the product of all the cycles in $A$, call $\tau_2$ the resulting permutation. Next $\tau_2$ is obtained from $\tau_2$ by breaking each cycle of $\tau_2$ before each occurrence of an element $j \in A$, that is inserting $"("$ unless $j$ is already preceded by $"("$. $\tau_1$ is finally the product of $\tau_1$ by the permutation obtained from $\tau_2$ by simply removing all indices $i \in B$.

The uniqueness is as follows. $\tau_3$ is well defined from $\tau$. So assume we have $\sigma_1 \sigma_2 = \tau_1 \tau_2$ with $\sigma_1, \tau_1$ permuting only the indices in $A$ and each cycle of $\sigma_2, \tau_2$ contains exactly one element of $A$. Multiplying both sides by $\tau_1^{-1}$ we may assume $\tau_1 = 1$. If $\sigma_1 \neq 1$ the nontrivial cycles of $\sigma_1$ cannot be disjoint from those of $\sigma_2$ otherwise they would be cycles of $\tau_1$. But then a cycle $c = (\alpha, i)$ of $\sigma_1$ with $\alpha$ a non trivial string in $A$ times $(\beta, i)$ a cycle of $\sigma_2$ gives rise to the cycle $(\beta, i, \alpha)$ of $\tau_1$. This is a contradiction since the string $i, \alpha$ is in $A$ and has more than one element.

Proposition 2.10. [Splitting the cycles] states that the product map $\pi : S_A \times U_A(B) \rightarrow S_{A,B}$, $(\tau, \sigma) \mapsto \tau \circ \sigma$ is a bijection with inverse $\gamma \mapsto (\gamma_1, \gamma_2 \gamma_3)$.

Remark 2.12. 1) By the uniqueness of the decomposition it follows that, if $\sigma \in S_A$ we have $(\sigma \circ \tau)_1 = \sigma \circ \tau_1$ for all $\tau \in S_{A,B}$.

2) Assume $\tau = \gamma \rho$ with $\gamma$ and $\rho$ permutations on two disjoint sets of indices, $(A_1 \cup B_1), (A_2 \cup B_2)$ and $A = A_1 \cup A_2, B = B_1 \cup B_2$ then, $\tau_i = \gamma_i \rho_i$, $i = \{1, 2, 3\}$ for the respective decompositions.

3) $U_A(B)$ is stable under conjugation by elements of $S_A \times S_B$.

2.12.1 Formulas

We want to apply the previous Proposition 2.10 to Formula (15).

Decompose $\{1, 2, \ldots, n + k\} = A \cup B$ with $A$ the indices of type $y$ and $B$ the ones of type $x$:

$$A = \{1, 2, \ldots, n\}, \quad B = \{n + 1, \ldots, n + k\}.$$
Denote, for simplicity of notations, with $Y_A := y_1 \otimes y_2 \otimes \ldots \otimes y_n$; and $X_B := x_1 \otimes x_2 \otimes \ldots \otimes x_k$.

So equation (15) is written as

$$tr(F(x)Y_A) := tr(\tau^{-1} \circ Y_A \otimes X_B). \quad (18)$$

If $\sigma \in S_A$, $\tau \in S_B$ we have

$$\sigma \circ \tau \circ Y_A \otimes X_B = (\sigma \circ Y_A) \otimes (\tau \circ X_B)$$

Notice next that:

**Lemma 2.13.** If $\sigma$ permutes the indices $A$ we have

$$tr((\sigma \sigma^{-1})^{-1} \circ Y_A \otimes X_B) = tr(\tau^{-1} \circ Y_A \otimes (\sigma^{-1} \circ X_B \circ \sigma)) \quad (19)$$

and

$$\sigma^{-1} \circ X_B \circ \sigma = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \ldots \otimes x_{\sigma(k)}.$$

so

$$T^{(n)}_{k,\sigma\tau^{-1}}(x_1, x_2, \ldots, x_k) = T^{(n)}_{k,\tau}(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}).$$

If $\gamma$ permutes the indices $A$ we have

$$tr((\gamma \gamma^{-1})^{-1} \circ Y_A \otimes X_B) = tr(\tau^{-1} \circ (\gamma^{-1} \circ Y_A \circ \gamma) \otimes X_B)$$

and

$$T^{(n)}_{k,\gamma\gamma^{-1}}(x_1, x_2, \ldots, x_k) = \gamma^{-1} T^{(n)}_{k,\gamma}(x_1, x_2, \ldots, x_k) \gamma$$

is a permutation of the tensor factors.

Assume that $\tau = \gamma_1 \gamma_2$ with $\gamma_1$ and $\gamma_2$ permutations on two disjoint sets of indices, $(A_1 \cup B_1), (A_2 \cup B_2)$ and $A = A_1 \cup A_2, B = B_1 \cup B_2$. Then, up to conjugating with a permutation of $S_A \times S_B$ we may assume $A_1 = \{1, 2, \ldots, m\}$ the first $m$ indices and similarly for $B_1 = \{n + 1, n + 2, \ldots, n + h\}$ so that

$$Y_A = Y_{A_1} \otimes Y_{A_2}, \quad X_B = X_{B_1} \otimes X_{B_2}.$$ 

$$tr\left((\gamma_1 \gamma_2)^{-1} \circ Y_A \otimes X_B\right) = tr\left(\gamma_1^{-1} \circ Y_{A_1} \otimes X_{B_1}\right) tr\left(\gamma_2^{-1} \circ Y_{A_2} \otimes X_{B_2}\right) \quad (21)$$

We finally deduce

$$T^{(n)}_{k,\gamma_1\gamma_2}(x_1, x_2, \ldots, x_k) = T^{(m)}_{k,\gamma_1}(x_1, x_2, \ldots, x_k) \otimes T^{(n-m)}_{k-h,\gamma_2}(x_{h+1}, x_{h+2}, \ldots, x_k). \quad (22)$$

Here if either $A_1$ or $A_2$ is empty the corresponding element is a scalar and the tensor product is to be understood as multiplication.

We use the following notation.

**Definition 2.14.** If $C = j_1, j_2, \ldots, j_p$ is a string of indices in $B$ we say that the monomial $M := x_{j_1} x_{j_2} \ldots x_{j_p}$ is associated to $C$.

**Proposition 2.15.** If $\tau = \tau_1 \tau_2 \tau_3$ is the splitting relative to the decomposition $(1, 2, \ldots, n) \cup (n + 1, n + 2, \ldots, n + k)$ and $h$ is the number of elements of $B$ appearing in $\tau_2$ we have:

$$T^{(n)}_{k,\tau} = \tau_1 \circ t_{\tau_3}(x) T^{(n)}_{h,\tau_2}(x); \quad t_{\tau_3}(x) = \prod_{\ell} tr(N_{\ell}), \quad T^{(n)}_{h,\tau_2}(x) = M_1 \otimes M_2 \otimes \ldots \otimes M_n. \quad (23)$$

with $M_i$ monomials corresponding to the cycles of $\tau_2$ (as in Formula (6)).

**Proof.** Up to conjugating with a permutation of $S_A \times S_B$, we may apply Remark 2.12, and Formula (22). We thus reduce to $\tau$ is a single cycle $c$. If $c$ consists only of indices in $B$ then it gives a contribution $tr(N)$, otherwise, for $j \geq 1$ and $A = \{1, 2, \ldots, j\}$ we have

$$c^{-1} := (C_1, 1, C_2, 2, C_3, 3, \ldots, C_j, j)$$

$$= (1, C_1)(2, C_2)(3, C_3) \ldots (j, C_j) \circ (1, 2, 3, \ldots, j) \quad (24)$$
If $C_i = j_1, j_2, \ldots, j_{i_p}$ set $M_i := x_{j_1} x_{j_2} \cdots x_{j_{i_p}}$ its associated monomial, so

$$tr(c^{-1} Y_j \otimes X_h) = tr(M_1 y_1 M_2 y_2 M_3 y_3 \cdots M_j y_j)$$

$$= tr((1, 2, 3, \ldots, j)^{-1} M_1 y_1 \otimes M_2 y_2 \otimes M_3 y_3 \cdots \otimes M_j y_j)$$

$$= tr((1, 2, 3, \ldots, j)^{-1} M_1 \otimes M_2 \otimes \cdots \otimes M_j \cdot y_1 \otimes y_2 \otimes y_3 \cdots \otimes y_j)$$

$$\implies T^{(i)}_{k,n}(X) = (1, 2, 3, \ldots, j)^{-1} M_1 \otimes M_2 \otimes \cdots \otimes M_j.$$  

Then remark that, for the decomposition $c = c_1 c_2$ we have:

$$c_1 = (1, 2, 3, \ldots, j)^{-1}, \quad c_2 = [(1, C_1)(2, C_2)(3, C_3)\ldots(j, C_j)]^{-1}.$$

**Corollary 2.16.**

1. Assume $B = B_1 \cup B_2$ and $\rho \in S_{A \cup B_1}$. Take $\gamma = (i, u)$ a cycle, $i \in A$, with $u$ a string of indices of $B_2$. Denote $h = \rho^{-1}(i) \in A$, then if $M$ is the monomial associated to $u$ (Definition 2.14), we have

$$T^{(n)}_{k,\gamma\rho} = T^{(n)}_{k,\rho} \cdot 1^{\otimes h-1} \otimes M \otimes 1^{\otimes n-h} \quad (25)$$

$$T^{(n)}_{k,\rho\gamma} = 1^{\otimes i-1} \otimes M \otimes 1^{\otimes n-i} \cdot T^{(n)}_{k,\rho} \quad (26)$$

2. If $\gamma \in S_A$

$$T^{(n)}_{k,\rho\gamma}(x_1, x_2, \ldots, x_k) = \gamma T^{(n)}_{k,\rho}(x_1, x_2, \ldots, x_k). \quad (27)$$

3. The inclusion $i_n : S_{n-1+k} \subset S_{n+k}$ as permutations fixing $n$ gives for $\tau \in S_{n-1+k}$ that $T^{(n)}_{k,i_n(\tau)}(x_1, x_2, \ldots, x_k) = T^{(n-1)}_{k,\tau}(x_1, x_2, \ldots, x_k) \otimes 1.$

**Proof.**

1) Consider the splitting $\rho = \rho_1 \rho_2 \rho_3$ for $A \cup B_1$. We have

$$\gamma \rho = (u, i) \rho = (u, i) \rho_1 \rho_2 \rho_3 = \rho_1 (u, \rho_1^{-1}(i)) \rho_2 \rho_3 = \rho_1 (u, h) \rho_2 \rho_3.$$  

In order to understand $(u, h) \rho_2$ decompose $\rho_2 = \prod_{j=1}^n (v_j, j)$ into its cycles. Then

$$(u, h) \rho_2 = \prod_{j \neq h} (v_j, j) (u, h) (v_n, h) = \prod_{j \neq h} (v_j, j) (h, v_n, u).$$

Formula (6) gives Formula (25). For Formula (26) we use $\rho(u, i) = \rho_1 \prod_{j \neq i} (v_j, j) (i, u, v_i) \rho_3$. e.g. $A = \{1, 2, 3\}$, $B = \{4, 5, 6\} \cup \{7\}$, $\rho = (1, 2)(2, 5)(3, 4, 6)$, $T^{(3)}_{3,\rho} = (1, 2) \cdot 1 \otimes x_2 \otimes x_1 x_3$

$$\gamma = (1, 7), \gamma \rho = (1, 2)(2, 5, 7)(3, 4, 6), \quad T^{(3)}_{3,\rho} = (1, 2) \cdot 1 \otimes x_2 x_4 \otimes x_1 x_3.$$  

2) follows from Remark 2.12 and 3) is clear.  

**Theorem 2.17.** The space $T^{(n)}_{d,k}(k)$ of multilinear $GL(V)$-equivariant maps of $k$ endomorphism $(x_1, x_2, \ldots, x_k)$ of an $d$-dimensional vector space $V$ to $End(V)^{\otimes n}$ is identified with $End_{GL(V)}(V^{\otimes n+k})$. It is linearly spanned by the elements $T^{(n)}_{k,\tau}, \tau \in S_{n+k}$ of Formula (23).

For instance, $n = 2, k = 1$:

$$tr(x_1 y_1) = tr((1, 2) \circ y_1 \otimes x_1 y_2) \implies T^{(2)}_{1,(2,1,3)}(x_1) = (1, 2) \circ y_1 \implies x_1.$$

$n = k = 3, \quad tr(x_3 x_1 y_1 x_2 y_2) = tr((1, 2, 3) \circ y_1 \otimes x_1 y_2 \otimes x_2 y_3) \implies T^{(3)}_{3,(6,4,2,1,3,5)}(x_1, x_2, x_3) = (1, 2, 3) \circ x_3 x_1 \otimes x_2.$
Definition 2.18. Given \( \sum_{r \in S_{n+k}} a_r \tau \) the equivariant map

\[
T_k^{(n)}( \sum_{r \in S_{n+k}} a_r \tau ) := \sum_{r \in S_{n+k}} a_r T_k^{(n)}(x_1, x_2, \ldots, x_k)
\]

is called the \( n \)-interpretation of \( \sum_{r \in S_{n+k}} a_r \tau \).

In particular the explicit Formula for \( T_k^{(n)} \) in Formula (23), is the \( n \) interpretation of \( \tau \).

By the classical method of polarization and restitution one has that Formula (23) describes a general, not necessarily multilinear \( GL(V) \)-equivariant map.

For \( n = 0 \) this is the classical theorem of generation of invariants of matrices. For \( n = 1 \) the classical theorem of generation of equivariant maps from matrices to matrices. For \( k = 0 \) on the other hand it is also the classical theorem that the span of the symmetric group is the centralizer of the linear group \( G = GL(d, F) \) on \( n^\text{th} \) tensor space, \( V^\otimes n \) i.e.:

\[
(M_d^\otimes n)^G = \text{End}_G(V^\otimes n) = \Sigma_n(V) = \pi(F[S_n]).
\]

3 The Second Fundamental Theorem

3.1 The \( d + 2 \) basic relations

Now, together with the First Fundamental Theorem we have the Second Fundamental Theorem giving the relations among the equivariant tensor valued maps.

In this case we want to describe, for given \( d \) and each \( n \), the elements of the twisted algebras \( T(X)^\otimes n \times \mathbb{Q}[S_n] \) vanishing under all evaluations \( T(X)^\otimes n \times \mathbb{Q}[S_n] \rightarrow M_d^\otimes n \) induced by evaluations \( X \rightarrow M_d \) in \( d \times d \) matrices, see Definition 2.7.

One starts from Theorem 1.5 that is \( \sum_{\sigma \in S_{d+1}} \epsilon_{\sigma} = 0 \) as operator on \( V^\otimes d+1 \) or \( \wedge^{d+1} V = \{0\} \). Hence the basic identity, for \( d \times d \) matrices:

\[
\text{tr}\left( \sum_{\sigma \in S_{d+1}} \epsilon_{\sigma} \sigma \circ z_1 \otimes z_2 \otimes z_3 \otimes \ldots \otimes z_{d+1} \right) = 0.
\]

If we use for every \( 0 \leq k \leq d+1 \) for the variables \( z_1, z_2, z_3, \ldots, z_{d+1} \) the \( d \times d \) matrix variables \( x_1, \ldots, x_k, y_1, y_2, \ldots, y_{d+1-k} \) Formula (10) produces from Formula (29), \( d+2 \) relations

\[
F_{k,d}(x_1, \ldots, x_k) := (-1)^k \sum_{\sigma \in S_{d+1}} \epsilon_{\sigma} T_k^{(d+1-k)} = 0; \quad k = 0, 1, \ldots, d+1.
\]

This is a multilinear relation of degree \( k \) in the variables \( x_1, \ldots, x_k \), for \( d+1-k \) tensor valued equivariant maps:

\[
\text{tr}(F_{k,d}(x_1, \ldots, x_k) y_1 \otimes y_2 \otimes \ldots \otimes y_{d+1-k}) = 0.
\]

For \( k = 0, d, d+1 \) these relations have classical interpretations. For \( k = d \) this is the polarized form of the Cayley–Hamilton identity

\[
x^d + \sum_{i=1}^d (-1)^i \sigma_i(x)x^{d-i}
\]

and for \( k = d+1 \) the polarized expression of \( \text{tr}(x^{d+1}) \) in terms of \( \text{tr}(x^i) \), \( i = 1, 2, \ldots, d \), or the expression of the \( d+1 \) Newton symmetric function \( \psi_{d+1}(t_1, \ldots, t_n) = \sum_{i=1}^{d+1} t_i \) in term of the Newton symmetric function \( \psi_i \), \( i = 1, 2, \ldots, d \).

\[
\text{tr}(x^{d+1}) + \sum_{i=1}^d (-1)^i \sigma_i(x)\text{tr}(x^{d-i+1}).
\]
In both cases this is due to the symmetry of formula 29 with respect to permuting the $z_i$. For $k = 0$ it is the relation $\sum_{(\sigma) \in S_{d+1}} \epsilon_{\sigma} = 0$ as operator on $V^\otimes d+1$.

For intermediate $2 \leq k \leq d$, this is still symmetric in the variables $x_1, \ldots, x_k$ so it can still be viewed as the polarized form of a tensor identity, in one variable $x$, for maps to $d + 1 - k$ tensors

\[
\epsilon_{k,d}(x) := \frac{1}{k!} F_{k,d}(x, \ldots, x).
\]

For instance, $d = 2, k = 1$ we have:

\[
\epsilon_{1,2}(x) := (1 - (1, 2)) \circ [x \otimes 1 + 1 \otimes x - tr(x)1 \otimes 1] = 0.
\]

For $d = 3, k = 1$ and $d = 3, k = 2$

\[
\epsilon_{2,3}(x) = (1 - (1, 2)) \left( [x^2 \otimes 1 + 1 \otimes x^2 + x \otimes x] - tr(x)[x \otimes 1 + 1 \otimes x] + \frac{tr(x)^2 - tr(x)}{2} \right)
\]

We see a remarkable factorization theorem through two remarkable factors.

In order to see this in general, Theorem 3.2, let us first make a definition and recall some classical facts.

Given two numbers $i, n \in \mathbb{N}$ consider the set $P(i, n)$ of all partitions $\mathbf{h} \vdash i$ of the form $h_1 \geq h_2 \geq \ldots \geq h_n \geq 0$. For $\mathbf{h} \in P(i, n)$ let $T_{\mathbf{h}}(x)$ be the symmetrization of $x^{h_1} \otimes x^{h_2} \otimes \ldots \otimes x^{h_n}$ as tensor. E.g.

\[
\mathbf{h} = 2, 2, 0; \quad T_{\mathbf{h}} = x^2 \otimes x^2 \otimes 1 + x \otimes 1 \otimes x^2 + 1 \otimes x^2 \otimes x^2, \quad T_{1,1,1} = x \otimes x \otimes x.
\]

\[
\mathbf{h} = 2, 1, 0; \quad T_{\mathbf{h}} = x^2 \otimes x \otimes 1 + x^2 \otimes 1 \otimes x + x \otimes x^2 \otimes 1 + x \otimes 1 \otimes x^2 + 1 \otimes x^2 \otimes x + 1 \otimes x \otimes x^2.
\]

We then define, for $n > 0$:

\[
\mathcal{T}_{i,n}(x) := \sum_{\mathbf{h} \in P(i, n)} T_{\mathbf{h}}(x), \quad \text{e.g. } \mathcal{T}_{2,2}(x) = [x^2 \otimes 1 + 1 \otimes x^2 + x \otimes x].
\]

\[
\mathcal{T}_{i,1}(x) = x^i, \quad \mathcal{T}_{i,3,3}(x) = T_{3,0,0} + T_{2,1,0} + T_{1,1,1}, \quad \mathcal{T}_{i,2}(x) = T_{3,0} + T_{2,1}.
\]

Denote by

\[
\det(t - x) = t^d + \sum_{i=1}^{d} (-1)^i \sigma_i(x) t^{d-i}, \quad \text{the characteristic polynomial of } x.
\]

Recall the formulas

\[
\sigma_i(x) = \sum_{h_1 \geq 0, \ldots, h_i \geq 0, h_1 + 2h_2 + \ldots + rh_r = i} (-1)^{r} \prod_{j=1}^{r} \frac{(-tr(x^j))^{h_j}}{h_j!} = \frac{1}{i!} \sum_{\sigma \in S_r} \epsilon_{\sigma} t_{d}(x).
\]

Now given $d, k$ set $n := d + 1 - k$, decompose $\{1, 2, \ldots, d + 1\} = A \cup B$, $A = \{1, 2, \ldots, n\}$. Recall that, Definition 2.11, $U_A(B)$ denotes the set of permutations in $S_{A \cup B}$ with the property that in each cycle appears at most one element of $A$. Next Proposition 2.10, [Splitting the cycles] states that the product map $S_A \times U_A(B) \rightarrow S_{d+1}$, $(\tau, \sigma) \mapsto \tau \circ \sigma$ is a bijection.

By Remark 2.12 we have, if $\tau \in S_A$ and $\sigma \in S_{d+1}$ that $T_{k,\tau,\sigma} = \tau T_{k,\sigma}$ so that Formula (30) becomes

\[
F_{k,d}(x_1, \ldots, x_k) = (-1)^k \sum_{\gamma \in S_{d+1}} \epsilon_{\gamma} T_{k,\gamma}^{(n)} = (-1)^k \left( \sum_{\tau \in S_A} \epsilon_{\tau} \circ \sum_{\sigma \in U_A(B)} \epsilon_{\sigma} T_{k,\sigma}^{(n)} \right)
\]
Theorem 3.2. The polynomial $F_{k,d}(x_1,\ldots,x_k)$ is the full polarization of the tensor Cayley Hamilton polynomial $(n := d + 1 - k, A = \{1, \ldots, n\})$:

$$C_{k,d}(x) := \left( \sum_{\tau \in S_A} \epsilon_\tau \right) \sigma \left[ \Sigma_{k,n}(x) + \sum_{j=1}^{k} (-1)^j \sigma_j(x) \Sigma_{k-j,n}(x) \right]$$  \hspace{1cm} (35)

Proof. For Formula (35) consider

$$G_{k,d}(x_1,\ldots,x_k) := \sum_{\sigma \in U_A(B)} \epsilon_\sigma T^{(n)}_{k,\sigma}(x_1,\ldots,x_k).$$ \hspace{1cm} (36)

The set $U_A(B)$ is stable under conjugation by the group $S_B$ and such conjugation corresponds to a permutation of the variables $x_i$ in $G_{k,d}(x_1,\ldots,x_k)$. So, since $G_{k,d}(x_1,\ldots,x_k)$ is symmetric, it is the polarization of $\frac{1}{h!}G_{k,d}(x,\ldots,x)$. We thus need to understand $\frac{1}{h!}G_{k,d}(x,\ldots,x)$.

According to Proposition 2.10 each permutation $\sigma$ of $U_A(B)$ is a product $\sigma = \sigma_3 \sigma_2$. These two permutations $\sigma_3$, $\sigma_2$ determine a partition of $B$ in two subsets. With $B_1$ the support of $\sigma_3$, while $A \cup B_2$ is the support of $\sigma_2$. Then $\sigma_3$ a product of cycles in $B_1$ and $\sigma_2$ a product of exactly $n$ cycles $c_j$ of some lengths $h_1 + 1, h_2 + 1, \ldots, h_n + 1$ with $c_j$ containing the index $j \in A$ and the remaining $h_j$ indices in $B_2$.

$$\sigma_2 = c_1 c_2 \ldots c_n, \quad c_a = (i_{a,1}, i_{a,2}, \ldots, i_{a,h_j}, a), \quad a \in A, \quad i_{a,1}, i_{a,2}, \ldots, i_{a,h_j} \in B_2. \hspace{1cm} (37)$$

One has $|B_1| = k - \sum h_j$ and further

$$T^{(n)}_{k,\sigma} = T^{(0)}_{k,\sigma_3} T^{(n)}_{k,\sigma_2}$$

where $T^{(0)}_{k,\sigma_3}$ is an invariant product of traces of monomials, in the $x$ variables $B_1$, while $T^{(n)}_{k,\sigma_2}$ is of the special form $M_1 \otimes \ldots \otimes M_n$.

Denote by $U_{A,B_2}$ the set of permutations of $A \cup B_2$ which decompose in exactly $n$ cycles each containing one index $i = 1, \ldots, n$ or $i \in A$. So we have a decomposition

$$U_A(B) = \bigcup_{B=B_1 \cup B_2} S_{B_1} \times U_{A,B_2}$$

and the following expansion of Formula (36) into the various decompositions $B = B_1 \cup B_2$:

$$G_{k,d}(x_1,\ldots,x_k) = \sum_{B=B_1 \cup B_2} \sum_{\sigma \in S_{B_1}} \epsilon_\sigma T^{(0)}_{j,\sigma} \sum_{\tau \in U_{A,B_2}} \epsilon_\tau T^{(n)}_{k-j,\tau}$$ \hspace{1cm} (38)

When we evaluate all variables $x_i$ in a single variable $x$ all the contributions relative to the subsets $B_1$ with the same cardinality $j$ become equal so that

If $|B_1| = j$, \hspace{1cm} $\sum_{\sigma \in S_{B_1}} \epsilon_\sigma T^{(0)}_{j,\sigma}(x) \overset{(33)}{=} j! \sigma_j(x)$. \hspace{1cm} (39)

Next compute $\sum_{\sigma \in U_{A,B_2}} \epsilon_\sigma T^{(n)}_{k,d}(x)$. An element $\sigma \in U_{A,B_2}$ is uniquely of the form

$$c_1 c_2 \ldots c_n, \quad c_a = (i_{a,1}, i_{a,2}, \ldots, i_{a,h_j}, a), \quad a = 1, \ldots, n$$

with the elements $i \in B_2$ and $B_2$ has cardinality $k - j$.

Given $n$ integers $h_1, \ldots, h_n$ summing to $k - j$ we have exactly

$$\prod h_1 \left( \begin{array}{c} k - j \\ h_1, \ldots, h_n \end{array} \right) = (k - j)!$$

such permutations which have sign $(-1)^{k-j}$. When evaluated all variables $x_i$ in a single variable $x$ all the contributions become equal giving $(h - j)!$ times the summand
\((-1)^{k-j}T_{k-j,o}(x) = x^{h_1} \otimes \cdots \otimes x^{h_n}\). The sequence \(h_1, \ldots, h_n\) is obtained by reordering a partition \(h \in P(k-j,n)\) so

If \(|B_2| = k-j, \sum_{\tau \in Q_{A,B_2}} \epsilon_\tau T_{k-j,\tau}^{(n)}(x) = (-1)^{k-j} (k-j)! \Sigma_{k-j,n}(x)\).

Formula (38) for \(G_{k,d}(x, \ldots, x)\) becomes

\[
\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j! \sigma_j(x)(k-j)! \Sigma_{k-j,n}(x) = k! (-1)^{k} \sum_{j=1}^{k} (-1)^j \sigma_j(x) \Sigma_{k-j,n}(x).
\]

Substituting in Formula (34) we finally have

\[
F_{k,d}(x) = k! \left( \sum_{\tau \in S_n} \epsilon_\tau \tau \right) \circ \left[ \Sigma_{k,n}(x) + \sum_{j=1}^{k} (-1)^j \sigma_j(x) \Sigma_{k-j,n}(x) \right]
\]

is the desired formula. \(\square\)

### 3.3 The second fundamental theorem

#### 3.3.1 \(T\)–ideals

We have already remarked the relationship between the antisymmetrizer and the \(d\)–Cayley–Hamilton identity. A well known result of Razmyslov and Procesi states that, the \(T\)–ideal in the free algebra with trace, of relations for \(d \times d\) matrices is generated by the \(d\)–Cayley–Hamilton identity \((\text{and } tr(1) = d)\) (see [5]).

We start from Remark 2.8 stating that the equivariant maps are the evaluations in matrices of the elements of the twisted algebra \(T(X)^{\otimes n} \ltimes Q[S_n]\). Here by \(X = \{x_1, x_2, \ldots, x_i, \ldots\}\) we indicate variables indexed by \(N\). Thus

**Definition 3.4.** A tensor identity or relation for \(d \times d\) matrices is an element of the algebra \(T(X)^{\otimes n} \ltimes Q[S_n]\) vanishing under all evaluations of \(X\) in \(d \times d\) matrices.

We denote by \(I_d(n) \subset T(X)^{\otimes n} \ltimes Q[S_n]\) this set of tensor identities.

Clearly \(I_d(n)\) is a two sided ideal of \(T(X)^{\otimes n} \ltimes Q[S_n]\).

Now there are certain operations under which tensor identities map to tensor identities.

First consider the endomorphisms, as trace algebra, of \(T(X)\), which are given by substitution maps \(X \rightarrow T(X)\). Such a map \(g\) induces the map \(g^{\otimes n} : T(X)^{\otimes n} \rightarrow T(X)^{\otimes n}\) which commutes with \(S_n\) and hence finally induces a map, identity on \(S_n\)

\[
g^{\otimes n} \times 1 : T(X)^{\otimes n} \times Q[S_n] \rightarrow T(X)^{\otimes n} \times Q[S_n].
\]

Next the natural inclusion \(S_m \times S_n \subset S_{m+n}\) induces a homomorphism of algebras, in fact an inclusion:

\[
T(X)^{\otimes m} \otimes T(X)^{\otimes n} \otimes Q[S_n] \rightarrow T(X)^{\otimes m+n} \otimes Q[S_{m+n}]
\]

\[A \in T(X)^{\otimes m}, B \in T(X)^{\otimes n}, \sigma \in S_m, \tau \in S_n; \quad A\sigma \otimes B\tau \mapsto A \otimes B\sigma \tau
\]

and we have

\[
I_d(m) \otimes T(X)^{\otimes n} \times Q[S_n] + T(X)^{\otimes m} \times Q[S_m] \otimes I_d(n) \subset I_d(m+n).
\]

Denote for simplicity \(T(X,m) := T(X)^{\otimes m} \times Q[S_m]\).

**Definition 3.5.** A sequence \(\{J(n)\}\) of ideals \(J(n) \subset T(X,n)\) will be called a \(T\)–ideal if

\[
g^{\otimes n} \times 1(J(n)) \subset J(n), \forall g : T(X) \rightarrow T(X)
\]

\[
J(n) \otimes T(X,n) + T(X,m) \otimes J(n) \subset J(m+n), \forall m, n.
\]

Clearly the intersection of \(T\)–ideals is still a \(T\)–ideal, so we define.

A \(T\)–ideal \(\{J(n)\}\) is generated by a subset \(S \subset \bigcup_I T(X,m)\) if it is the minimal \(T\)–ideal containing \(S\).
3.5.1 The $T$–ideal of tensor identities

Clearly the relations for $d \times d$ matrices $\{I_d(n)\}$ form a $T$–ideal. By the classical method of polarization and restitution one can, studying relations or $T$–ideals, restrict to multilinear elements. That is:

**Proposition 3.6.** If $\{J_1(n)\}$ and $\{J_2(n)\}$ are two $T$–ideals having the same multilinear elements they coincide.

Here by the space $T_{\text{mult}}(k,n)$ of multilinear elements of degree $k$ in $T(X,n)$ we may take as definition the span of the elements depending linearly only upon the first $k$ variables $x_1, \ldots, x_k$. We should remark that this subspace can be identified to $\mathbb{Q}[S_{k+n}]$ by the Formula (23), mapping $\psi_k : \tau \mapsto T_{k,n}^{(n)}$.

As for the $T$–ideal $\{I_d(n)\}$ of tensor identities for $d \times d$ matrices, we start from the $d + 2$ interpretations $F_{k,d}(x_1, \ldots, x_k)$ of the antisymmetrizer as tensor identities for $d \times d$ matrices given by Formula (30). Equivalently one could start with the 1–variable relations given by Formula (35). We claim

**Theorem 3.7.** $\{I_d(n)\}$ is generated, as $T$–ideal, from the $d + 2$ interpretations of the antisymmetrizer.

In other words we may say that, every relation for equivariant tensor valued polynomials maps from $d \times d$ matrices to tensor products of $d \times d$ matrices can be deduced from the $d + 2$ identities of Formula (35).

**Proof.** From Proposition 3.6 it is enough to restrict to multilinear relations.

By Theorem (3.7) one sees that the space $T_{\text{mult}}(k,n) \cap I_d(n)$ of multilinear relations is 0 unless $m := k + n \geq d + 1$.

In this case it is the image, under the mapping $\psi_k : \tau \mapsto T_{k,n}^{(n)}$, $\tau \in S_{k+n}$ of the two sided ideal of $\mathbb{Q}[S_{k+n}]$ generated by the antisymmetrizer $A_{d+1}$. Thus it is formed by linear combinations of the $k$–interpretation (Definition 2.18) in terms of tensor valued maps of the basic relations, which we write as $\sigma \circ \tau \circ A_{d+1} \circ \tau^{-1}$. For $m = k + n = d + 1$ we just have the $d + 2$ basic relations, each homogeneous of degree $k$ in $I_d(n) = I_d(d + 1 - k)$.

For fixed $m > d + 1$, $k$ we then decompose $\{1, 2, \ldots, m\} = A \cup B$ with $B$ the last $k$ indices (the $x$ indices) and $A$ the first $n = m - k$ indices (the $y$ indices).

Finally we see that, the conjugation action by elements of $S_A \times S_B$, by Lemma 2.13, commutes with the interpretation, where $S_A$ permutes the tensor factors while $S_B$ permutes the $x$ variables.

We need thus to understand, for $m > d + 1$ and $\sigma \circ \tau \circ A_{d+1} \circ \tau^{-1} \in \mathbb{Q}[S_m]$, the equivariant maps $T_k(\sigma \circ \tau \circ A_{d+1} \circ \tau^{-1})$, $k = 0, 1, \ldots, m$, Formula (28), interpretations of $\sigma \circ \tau \circ A_{d+1} \circ \tau^{-1}$, and prove that they are deduced from the basic relations.

Given any set $I$ of $d + 1$ indices out of the set $\{1, 2, \ldots, m\}$ we denote by $A_{d+1}(I) = \sum_{\sigma \in S_d, \tau \in S_m} \epsilon_\sigma \epsilon_\tau \in \mathbb{Q}[S_m]$ the antisymmetrizer in those indices.

The element $\tau \circ A_{d+1} \circ \tau^{-1}$ is, up to sign, the antisymmetrizer on the $d + 1$ elements of $C := \tau(1, 2, \ldots, d + 1)$ so it is $A_{d+1}(C)$.

We need to understand $T_k(\sigma A_{d+1})$. Decompose

$\{1, 2, \ldots, m\} = C \cup D$.

Apply Proposition 2.10 to $\gamma := \sigma^{-1}$, using the splitting $\gamma = \gamma_1 \gamma_2 \gamma_3$ we have $\sigma = \sigma_1 \sigma_2 \sigma_3$, $\sigma_1 = \gamma_1^{-1}$. Since $\sigma_1$ is a permutation of the indices $C$ we have $\sigma_1 A_{d+1} = \pm A_{d+1}$ so we need only analyze $\sigma_2 \sigma_3 A_{d+1}$. Now since the indices of $\sigma_3$ are disjoint from those of $\sigma_2 A_{d+1}$ the interpretation of $\sigma_2 \sigma_3 A_{d+1}$ is up to permuting the tensor factors, the tensor product of the two interpretations, Formula (22).

We are left to understand the interpretation of $\sigma_2 A_{d+1}$ where $\sigma_2 = \prod_i c^i$ is a product of cycles $c^i$ each containing exactly one element of $C$. For simplicity now we write $\sigma_2 = \varphi$. 

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Let us denote by $E$ the set of indices appearing in $\varphi$ and decompose $E = E_1 \cup E_2$ with $E_1 = E \cap A$ the set of indices in $E$ of type $y$ and $E_2 = E \cap B$ is formed by indices of type $x$. Next split $\varphi = \varphi_1 \varphi_2 \varphi_3$ as in Proposition 2.10 with respect to the decomposition of $E$. Since $\varphi_1$ is a permutation of indices of type $y$ by Corollary 2.16 we have

$$T_{k} (\varphi_1 \varphi_2 \varphi_3 A_{d+1}) = \varphi_1 T_{k} (\varphi_2 \varphi_3 A_{d+1}) = \varphi_1 T_{k} (\varphi_1 \varphi_2 A_{d+1}).$$

The permutation $\varphi_3$ is a product of cycles all of type $x$, i.e. with indices in $B$ and disjoint from the indices appearing in $\varphi_2$ and commutes with $\varphi_2$. Moreover for each $i = 1, 2, 3$ we have that $\varphi_i = \prod c_i^j$ (cf. Formula (17)).

Write $\varphi_2 = \alpha \circ \beta$ with $\alpha$, permuting $h$ elements, the product of those cycles which are supported in $D \cap B$ and $\beta = \prod_{i \in C \cap B} (i, u_i)$ which contain exactly one index in $C \cap B$ and $u_i$ is a string of indices in $D \cap B$.

We have $T_k (\alpha \beta \varphi_3 A_{d+1}) = T_k (\alpha) T_{k-h} (\beta \varphi_3 A_{d+1})$ with $T_k (\alpha)$ an invariant. As for $T_{k-h} (\beta \varphi_3 A_{d+1})$ we have that $T_{k-h} (\beta \varphi_3 A_{d+1})$ is obtained from $T_{k-h} (\varphi_3 A_{d+1})$ by replacing $x_i$ with the monomial in the $x$'s associated to the string $(i, u_i)$.

We pass to analyze $\varphi_2 A_{d+1}$, where by the algorithm of splitting the cycles we have that each cycle $c$ of $\varphi_2$ contains a unique index $i$ of type $y$.

Decompose $\varphi_2 = \gamma \rho$ with $\gamma$ the product of all the cycles of $\varphi_2$ which do not contain indices in $C$. Thus, by Corollary 2.16 for some $a, b$ with $a + b = k - h$ up to conjugating with some permutation in $S_4$:

$$T_{k-h} (\gamma \rho A_{d+1}) = T_{a}(\gamma) \otimes T_{b}(\rho A_{d+1}).$$

We are left to analyze $\rho A_{d+1}$ where $\rho$ is a product of cycles each containing exactly one index $j \in A$ and also, since $\varphi$ is a product of cycles each containing exactly one element of $C$, one index $h \in C$.

I) If $j = h \in A \cap C$ then $c = (j, u_j)$ with $u_j$ a string in $D_x$.

II) If $j \neq h$ then $c = (j, a_j, h, b_h)$ with $h \in C \cap D_y$, $j \in D_x$ and $a_j, b_h$ are strings in $D_x$.

For the first type of cycles we may apply Formula 25 of Corollary 2.16 so we may assume that all the cycles of $\rho$ are of type $\Pi$.

II) If $c = (j, a_j, h, b_h)$ split

$$(j, a_j, h, b_h) = (j, b_h)(h, a_j)(j, h).$$

Let $\sigma = \prod_{j} (j, h) = \sigma^{-1}$ over all cycles so that

$$\rho A_{d+1} = \rho \sigma (\sigma A_{d+1} \sigma^{-1}) \sigma.$$ 

Then $\sigma A_{d+1} \sigma^{-1} = A_{d+1}(\sigma \tau(\{1, 2, \ldots, d + 1\})) = A_{d+1}(\sigma (C))$ is also an antisymmetrizer on $d + 1$ indices. Only now the $x$ indices $h$ corresponding to the $j$ have been replaced by the $y$ indices $p$ and $\rho A_{d+1}$ has been replaced by $\rho \sigma A_{d+1}(\sigma (C)) \sigma$. The permutation $\rho \sigma = \prod_{j} (j, b_h)(h, a_j) = \prod_{j} (j, b_h) \prod_{j} (h, a_j)$ is such that the indices of $\prod_{j} (h, a_j)$ are all $x$ indices disjoint from the indices in $\prod_{j} (j, b_h) A_{d+1} \sigma$ therefore the factor $T(\prod_{j} (h, a_j))$ can be taken out of the interpretation. We are left with $\prod_{j} (j, b_h) A_{d+1} \prod_{j} (j, h)$, where $\prod_{j} (j, b_h)$ and $\prod_{j} (j, h)$ are formed by a product of cycles of type $I$.

We can now apply the two Formulas (25) and (26) and conclude that $T(\rho \sigma A_{d+1}) \sigma (C))$ is obtained from $T(\sigma A_{d+1}(\sigma (C)))$, by multiplying from the right and from the left by tensor products of monomials.

Finally denote the cardinality of $\sigma (C) \cap A$ by $p \leq d + 1$ we have $T(A_{d+1}(\sigma (C))) \in I_{d}(p)$ is obtained from $T_{p}(A_{d+1}) \in I_{d}(p)$ by tensor multiplying by $1^{n-(d+1)}$ and then conjugating by a permutation in $S_A$ to rearrange the tensor factors.

\[\square\]

3.8 The final theorem

3.8.1 Operations on equivariant maps

Some operations on equivariant maps from matrices to tensors can be interpreted as operations on permutations.
Consider the following basic operations on elements of $M_d^{\otimes n}$.

\[
\sigma \in S_n, \quad \sigma \circ X_1 \otimes \cdots \otimes X_n = X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(n)} \in M_d^{\otimes n},
\]

(43)

\[
m : X_1 \otimes X_2 \otimes \cdots \otimes X_{n-1} \otimes X_n \mapsto X_1 \otimes X_2 \otimes \cdots \otimes X_n X_{n-1} \in M_d^{\otimes n-1}
\]

(44)

\[
t : X_1 \otimes X_2 \otimes \cdots \otimes X_{n-1} \otimes X_n \mapsto tr(X_n)X_1 \otimes \cdots \otimes X_{n-1} \in M_d^{\otimes n-1}.
\]

(45)

One obtains many similar operations by combining these basic ones.

**Lemma 3.9.**

\[
t((n, i) \circ X_1 \otimes X_2 \otimes \cdots \otimes X_n) = X_1 \otimes \cdots \otimes X_n X_i \otimes \cdots \otimes X_{n-1}
\]

(46)

**Proof.** We may assume that $X_i := u_i \otimes \varphi_i$ be $n$ decomposable endomorphisms,

\[
t((n, i) \circ X_1 \otimes X_2 \otimes \cdots \otimes X_n) = \, \, t(u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \cdots \otimes u_n \otimes \varphi_n \otimes \cdots \otimes u_i \otimes \varphi_i)
\]

\[
= u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n \mid u_i \otimes \varphi_i \otimes \cdots \otimes u_{n-1} \otimes \varphi_{n-1}
\]

\[
= X_1 \otimes \cdots \otimes X_n X_i \otimes \cdots \otimes X_{n-1}
\]

\[
\square
\]

In particular $m = t \circ (n, n-1)$. Then remark that if $\sigma \in S_n$ fixes $n$ then $\sigma \circ t = t \circ \sigma$.

So consider $S_{n-1} \subset S_n$ the permutations fixing $n$. We have the coset decomposition

\[
S_n = S_{n-1} \cup \bigcup_{i=1}^{n-1} S_{n-1}(n, i).
\]

From the previous lemma we deduce, for $n \geq 2$:

**Proposition 3.10.** If $\sigma \in S_{n-1}$ then $t \circ \sigma = \sigma \circ t$ and

\[
t(\sigma \circ X_1 \otimes X_2 \otimes \cdots \otimes X_n) = \sigma \circ tr(X_n)X_1 \otimes \cdots \otimes X_{n-1},
\]

(47)

in particular $t(\sigma) = \sigma \cdot tr(1)$.

If $\sigma = \tau(n, i), \tau \in S_{n-1}$ then

\[
t(\sigma \circ X_1 \otimes X_2 \otimes \cdots \otimes X_n) = \tau \circ X_1 \otimes \cdots \otimes X_i X_n \otimes \cdots X_{n-1}.
\]

(48)

In particular $t(\sigma) = \tau$.

**Proposition 3.11.** Using Formulas (47) and (48) one can define $t$ as a formal operation

\[
t : T(X)^{\otimes n} \times Q[S_n] \to T(X)^{\otimes n-1} \times Q[S_{n-1}],
\]

a partial trace which is linear with respect to multiplication by invariants.

From formulas (47) and (48) we have that $t(I_d(n)) \subset I_d(n-1)$.

Recall that the element, with $A = \{1, 2, \ldots, n\}$

\[
\mathcal{C}_{k,d}(x) := \left( \sum_{\tau \in S_A} \epsilon_{\tau, \tau} \sigma_{\tau} \right) \circ \left[ \mathcal{T}_{k,n}(x) + \sum_{j=1}^{k} (-1)^j \sigma_j(x) \mathcal{T}_{k-j,n}(x) \right]
\]

(49)

of Formula (35) is an $n$–tensor identity of degree $k$ when evaluated in $d \times d$ matrices, $d = n + k - 1$.

**Remark 3.12.** From Theorem 3.7 follows in particular that there are no identities in degree $k$ on $s < d + 1 - k$ tensors and furthermore, up to a scalar constant, $\mathcal{C}_{k,d}(x)$ is the unique identity in degree $k$ on $n = d + 1 - k$ tensors.

**Theorem 3.13.** Upon specializing $tr(1) = d$ we have, for $n \geq 1$ $\iff k \leq d$:

\[
t(\mathcal{C}_{k,d}(x)) = 0, \quad t(\mathcal{C}_{k,d}(x) \cdot 1^{n-1} \otimes x) = -n \cdot \mathcal{C}_{k+1,d}(x)
\]

(50)
Proof. For $n = 1$, $k = d$ we have $\mathfrak{C}_{d,d}(x) = x^d + \sum_{i=1}^{d} (-1)^i \sigma_i(x)x^{d-i}$ and
\[
\mathfrak{t}(x^d + \sum_{i=1}^{d} (-1)^i \sigma_i(x)x^{d-i}) = tr(x^d) + \sum_{i=1}^{d} (-1)^i \sigma_i(x)tr(x^{d-i}) = 0,
\]
\[
\mathfrak{t}(x^{d+1} + \sum_{i=1}^{d} (-1)^i \sigma_i(x)x^{d+i}) = tr(x^{d+1}) + \sum_{i=1}^{d} (-1)^i \sigma_i(x)tr(x^{d+i+1})
\]
are the two desired Formulas, so assume $n \geq 2$. Both $\mathfrak{t}(\mathfrak{C}_{k,d}(x))$ and $\mathfrak{t}(\mathfrak{C}_{k,d}(x) \cdot 1^{n-1} \otimes x)$ are tensor identities on $n-1 = d - k$ tensors, respectively of degree $k$ and $k+1$. Thus by the previous remark we have $\mathfrak{t}(\mathfrak{C}_{k,d}(x)) = 0$ and $\mathfrak{t}(\mathfrak{C}_{k,d}(x) \cdot 1^{n-1} \otimes x) = \alpha \cdot \mathfrak{C}_{k,d}(x)$ for some scalar $\alpha$.

In order to compute $\alpha$ it is enough to compare some leading terms.

For $\mathfrak{C}_{k+1,d}(x)$ we take $(\sum_{\tau \in S_{1,2,\ldots,n-1}} \epsilon_{\tau}) \circ \mathfrak{I}_{k+1,n-1}(x)$ while for $\mathfrak{t}(\mathfrak{C}_{k,d}(x) \cdot 1^{n-1} \otimes x)$ we look at $\mathfrak{t}(\sum_{\tau \in S_{1,2,\ldots,n}} \epsilon_{\tau}) \circ \mathfrak{I}_{k,n}(x) \cdot 1^{n-1} \otimes x)$.

Observe that
\[
\sum_{\tau \in \Delta_{1,2,\ldots,n}} \epsilon_{\tau} = (\sum_{\tau \in \Delta_{1,2,\ldots,n-1}} \epsilon_{\tau})(1 - \sum_{i=1}^{n-1} (i, n))
\]
\[
\implies \mathfrak{t}((\sum_{\tau \in \Delta_{1,2,\ldots,n}} \epsilon_{\tau}) \circ \mathfrak{I}_{k,n}(x) \cdot 1^{n-1} \otimes x)
\]
\[
= \sum_{\tau \in \Delta_{1,2,\ldots,n}} \epsilon_{\tau} \mathfrak{t}((1 - \sum_{i=1}^{n-1} (i, n)) \epsilon_{\tau}) \circ \mathfrak{I}_{k,n,n-1}(x) \cdot 1^{n-1} \otimes x).
\]

We need to compute the coefficient of the term $1^{\otimes n-2} \otimes x^{k+1}$ of $\mathfrak{C}_{k+1,n-1}(x)$ in the sum $\mathfrak{t}((1 - \sum_{i=1}^{n-1} (i, n)) \epsilon_{\tau}) \circ \mathfrak{I}_{k,n}(x) \cdot 1^{n-1} \otimes x)$.

Now $\mathfrak{I}_{k,n}(x)$ is the sum of the $n$ terms $1^{\otimes i-1} \otimes x \otimes 1^{\otimes n-1}$ so we have a contribution $-1^{\otimes n-2} \otimes x^{k+1}$ from each $\mathfrak{t}((i, n) \cdot 1^{\otimes i-1} \otimes x \otimes 1^{\otimes n-1})$ for $i = 1, \ldots, n-2$.

For $i = n - 1$ we have two contributions, the first from
\[
\mathfrak{t}((n-1, n) \cdot 1^{\otimes n-2} \otimes x \otimes 1^{\otimes n-1} \otimes x) = t((n-1, n) \cdot 1^{\otimes n-2} \otimes x \otimes x)
\]
and then from
\[
\mathfrak{t}((n-1, n) \cdot 1^{\otimes n-2} \otimes x \otimes 1^{\otimes n-1} \otimes x) = t((n-1, n) \cdot 1^{\otimes n-2} \otimes x^{k+1}).
\]

Hence $\alpha = -n$. \qed

The specialization $tr(1) = d$ is necessary since for instance formally
\[
\mathfrak{t}(\mathfrak{C}_{1,2}(x)) = \mathfrak{t}(1 - (1,2)) \circ [x \otimes 1 + 1 \otimes x - tr(x))
\]
\[
= tr(1)x + tr(x) - tr(x)tr(1) - 2x + tr(x) = (tr(1) - 2)(x - tr(x)).
\]

Exercise 3.14. $\mathfrak{t}(\mathfrak{C}_{k,d}(x)) = (tr(1) - d)\mathfrak{C}_{k,d-1}(x)$, $\forall k \leq d$.

Remark 3.15. For the multilinear identities of Formula (30) we have
\[
F_{k+1,d}(x_1, \ldots, x_{k+1}) = t(F_{k,d}(x_1, \ldots, x_k) \cdot 1^{\otimes d-k} \otimes x_{k+1}).
\]

At this point one should introduce the operation $t$ in the operations used to deduce an identity from another. So we change the definition of $T$ ideal asking that it should also be stable under $t$.

Under this new definition we finally have the result.

Theorem 3.16. [SFT for equivariant maps] $\{I_d(n)\}$ is generated, as $T$–ideal, by the antisymmetrizer $\sum_{\sigma \in S_{d+1}} \epsilon_{\sigma} \sigma$ and $tr(1) = d$.  

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Theorem 3.17. \[ T(X)^{\otimes n} \cong \mathbb{Q}[S_n]/\bar{I}_d(n) \] modulo the \( T \)-ideal \( \bar{I}_d \) generated by the antisymmetrizer \( A_{d+1} \) and no condition on \( tr(1) \). From Exercise 2.6 we have \[ t(A_{d+1}) = (tr(1) - d)A_d \implies t^d(A_{d+1}) = \prod_{i=1}^d (tr(1) - i) \in \bar{I}_d. \] The algebra \( \mathbb{Q}[\lambda]/\prod_{i=1}^d (\lambda - i) = \mathbb{Z}_d \mathbb{Q} \) and so \( T(X)^{\otimes n} \cong \mathbb{Q}[S_n]/\bar{I}_d(n) \) decomposes as a direct sum of \( d \) summands, in the \( i^{th} \) summand we have \( tr(1) = i \). But now by the same formula \( t(A_{d+1}) = (tr(1) - d)A_d \) we deduce from \( A_{d+1} \) and \( tr(1) = i \) that in the \( i^{th} \) summand we have also \( A_{i+1} = 0 \). Therefore we deduce the decomposition as direct sum of the \( d \) algebras of equivariant maps for \( i \times i \) matrices, \( i = 1, \ldots , d \).

Theorem 3.17.
\[ T(X)^{\otimes n} \cong \mathbb{Q}[S_n]/\bar{I}_d(n) = \oplus_{i=1}^d T(X)^{\otimes n} \cong \mathbb{Q}[S_n]/I_s(n). \]

A comment 1) Most of the results of this paper hold in a characteristic free way. In particular all identities with integer coefficients continue to hold. Theorem 2.6 still holds, from the Theory of Donkin [6], provided in Formula (14) one replaces the factors \( tr(N_i) \) by \( \sigma_j(N_i) \).

The only result which should require a particular care is Theorem 3.7.

In fact in order to carry out the proof in positive characteristic one would need to follow closely the rather difficult and non trivial calculations of Zubkov, see [19] or [5]. I have not tried to do this since it would have made the treatment very hard to follow but I believe that the argument can be generalized to this setting.

2) The algebra \( T(X)^{\otimes n} \cong \mathbb{Q}[S_n] \) contains the two subalgebras \( T(X)^{\otimes n} \) and \( \mathbb{Q}(X)^{\otimes n} \). The identities belonging to the first subalgebra are the tensor trace identities, the ones belonging to the second subalgebra are the tensor polynomial identities. Although it is true that these can be deduced from the antisymmetrizer their structure is far from being understood.

A start in the study of tensor polynomial identities appears in the preprint with F. Huber [8].

As for tensor trace identities we know that for \( n = 1 \) they are generated by the \( d \)-Cayley Hamilton identity.

For higher \( n \) the situation is more complex since the algebra of equivariant maps is not flat over its trace algebra. Let us explain what happens. The algebra \( T(X) \) modulo the \( d \)-Cayley Hamilton identity is a domain \( T_d(\Xi) \), generated by \( k \) generic matrices \( \Xi = \{ \xi_1 , \ldots , \xi_k \} \) and the traces of their monomials. Its center is the algebra of invariants \( T_d(\Xi) \). If \( Q_d(\Xi) \) is the field of fractions of \( T_d(\Xi) \) then \( Q_d(\Xi) := T_d(\Xi) \otimes_{T_d(\Xi)} Q_d(\Xi) \) is a division algebra of dimension \( d^2 \) over its center the equivariant rational functions. If we take the tensor product \( Q_d(\Xi)^{\otimes n} \) of \( n \) copies of \( Q_d(\Xi) \) over its center \( Q_d(\Xi) \) we have the space of tensor valued equivariant rational functions. We also have a map \( j : T(X)^{\otimes n} \to Q_d(\Xi)^{\otimes n} \). A simple argument shows that its Kernel is formed by the tensor trace identities since we may view this as a specialization to generic matrices. Now when we use the \( d \)-Cayley Hamilton identity this map factors through a map \( j_d : T_d(X)^{\otimes n} \to Q_d(\Xi)^{\otimes n} \). By simple localization arguments we then see that if \( a \in T_d(X)^{\otimes n} \) is in the Kernel of \( j_d \) then there is an invariant \( b \) so that \( ba = 0 \). In fact if \( b \) is any central polynomial we have \( b^a = 0 \) for some \( k \).

We can also take any nonzero discriminant \( \delta := \det(tr(b_i b_j)) \) of \( d^2 \) elements \( b_i \in T_d(X) \). Then after localizing \( T_d(X)[\delta^{-1}] \) is a free module with basis the \( b_i \) over the localized trace algebra \( T_d(\Xi)[\delta^{-1}] \), so its \( n^{th} \) tensor power is also a free module and embeds in \( Q_d(\Xi)^{\otimes n} \).
So we may say that up to multiplication by some power of this discriminant a tensor trace identity can be deduced from the $d$–Cayley Hamilton identity.

On the other hand I think that the situation is similar to that of functional identities, so, as in the paper [3], one should have tensor trace identities not deduced from the $d$–Cayley Hamilton identity.

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