Properties of hitting times for $G$-martingale

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Abstract

In this article, we consider the properties of hitting times for $G$-martingale and the stopped processes. We prove that the stopped processes for $G$-martingales are still $G$-martingales and that the hitting times for a class of $G$-martingales including $G$-Brownian motion are quasi-continuous. As an application, we improve the $G$-martingale representation theorems in [Song10].

1 Introduction

Recently, [P06], [P08] introduced the notion of sublinear expectation space, which is a generalization of probability space. One of the most important sublinear expectation space is $G$-expectation space. As the counterpart of Wiener space in the linear case, the notions of $G$-Brownian motion, $G$-martingale, and Itô integral w.r.t $G$-Brownian motion were also introduced. These notions have very rich and interesting new structures which nontrivially generalize the classical ones.

As is well known, stopping times play a great role in classical stochastic analysis. However, it is difficult to apply stopping time technique in sublinear expectation space since the stopped process may not belong to the class of processes which are meaningful in the present situation. For example, let $\{M_t\}_{t \in [0,T]}$ be a $G$-martingale and $\tau$ be an $\mathbb{F}$-stopping time, we don’t know whether $M_t^\tau$ has a quasi-continuous version for $t \in [0,T]$.

In this article we consider the properties of hitting times for $G$-martingale and the stopped processes. We prove that the stopped processes for $G$-martingales are still $G$-martingales and that the hitting times for symmetric $G$-martingales with strictly increasing quadratic variation processes are...
quasi-continuous. As an application, we prove that any symmetric random variable can be approximated by bounded random variables that are also symmetric. Besides, we improve the results in [Song10] for $G$-martingale representation by a stopping time technique.

This article is organized as follows: In section 2, we recall some basic notions and results of $G$-expectation and the related space of random variables. In section 3, we give several preliminary lemmas. In section 4, we prove that the stopped processes for $G$-martingales are still $G$-martingales and that the hitting times for a class of $G$-martingales including $G$-Brownian motion are quasi-continuous. In section 5, we give some applications by a stopping time technique.

2 Preliminary

We recall some basic notions and results of $G$-expectation and the related space of random variables. More details of this section can be found in [P07].

2.1 G-expectation

**Definition 2.1** Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$ with $c \in \mathcal{H}$ for all constants $c$. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E}: \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$.

(b) Constant preserving: $\hat{E}(c) = c$.

(c) Sub-additivity: $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$.

(d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X)$, $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

**Definition 2.2** Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, $\forall \varphi \in C_{l,\text{Lip}}(\mathbb{R}^n)$, where $C_{l,\text{Lip}}(\mathbb{R}^n)$ is the space of real continuous functions defined on $\mathbb{R}^n$ such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \forall x, y \in \mathbb{R}^n,$$

where $k$ depends only on $\varphi$. 

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**Definition 2.3** In a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) a random vector \(Y = (Y_1, \cdots, Y_n)\), \(Y_i \in \mathcal{H}\) is said to be independent to another random vector \(X = (X_1, \cdots, X_m)\), \(X_i \in \mathcal{H}\) under \(\hat{E}(\cdot)\) if for each test function \(\varphi \in C_{t,Lip}(\mathbb{R}^m \times \mathbb{R}^n)\) we have \(\hat{E}[^{\varphi}(X,Y)] = \hat{E}[^{\varphi}(x,y)]_{x=\hat{X}}\).

**Definition 2.4** (\(G\)-normal distribution) A d-dimensional random vector \(X = (X_1, \cdots, X_d)\) in a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is called \(G\)-normal distributed if for each \(a, b \in \mathbb{R}\) we have

\[
aX + b\hat{X} \sim \sqrt{a^2 + b^2}X,
\]

where \(\hat{X}\) is an independent copy of \(X\). Here the letter \(G\) denotes the function

\[
G(A) := \frac{1}{2}\hat{E}[(AX,X)] : S_d \to \mathbb{R},
\]

where \(S_d\) denotes the collection of \(d \times d\) symmetric matrices.

The function \(G(\cdot) : S_d \to \mathbb{R}\) is a monotonic, sublinear mapping on \(S_d\) and \(G(A) = \frac{1}{2}\hat{E}[(AX,X)] \leq \frac{1}{2}|A|\hat{E}[[X]^2] =: \frac{1}{2}|A|\sigma^2\) implies that there exists a bounded, convex and closed subset \(\Gamma \subset S_d^+\) such that

\[
G(A) = \frac{1}{2}\sup_{\gamma \in \Gamma} Tr(\gamma A).
\]

If there exists some \(\beta > 0\) such that \(G(A) - G(B) \geq \beta Tr(A - B)\) for any \(A \succeq B\), we call the \(G\)-normal distribution is non-degenerate, which is the case we consider throughout this article.

**Definition 2.5** i) Let \(\Omega_T = C_0([0,T];\mathbb{R}^d)\) with the supremum norm, \(\mathcal{H}_T^0 := \{\varphi(B_{t_1}, \ldots, B_{t_n})|\forall n \geq 1, t_1, \ldots, t_n \in [0,T], \forall \varphi \in C_{t,Lip}(\mathbb{R}^{n})\}\), \(G\)-expectation is a sublinear expectation defined by

\[
\hat{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]
= \hat{E}[\varphi(\sqrt{t_1 - t_0} \xi_1, \cdots, \sqrt{t_m - t_{m-1}} \xi_m)],
\]

for all \(X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})\), where \(\xi_1, \cdots, \xi_n\) are identically distributed \(d\)-dimensional \(G\)-normal distributed random vectors in a sublinear expectation space \((\hat{\Omega}, \hat{\mathcal{H}}, \hat{E})\) such that \(\xi_{i+1}\) is independent to \((\xi_1, \cdots, \xi_i)\) for each \(i = 1, \cdots, m\). \((\Omega_T, \mathcal{H}_T^0, \hat{E})\) is called a \(G\)-expectation space.

ii) For \(t \in [0,T]\) and \(\xi = \varphi(B_{t_1}, \ldots, B_{t_n}) \in \mathcal{H}_T^0\), the conditional expectation defined by (there is no loss of generality, we assume \(t = t_i\))

\[
\hat{E}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]
\]
\[
\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}),
\]
where
\[
\varphi(x_1, \cdots, x_i) = \hat{E}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}})].
\]

Let \( \|\xi\|_{p,G} = [\hat{E}(|\xi|^p)]^{1/p} \) for \( \xi \in \mathcal{H}_T^0 \) and \( p \geq 1 \), then \( \forall t \in [0,T], \hat{E}_t(\cdot) \) is a continuous mapping on \( \mathcal{H}_T^0 \) with norm \( \|\cdot\|_{1,G} \) and therefore can be extended continuously to the completion \( L_{d,1}^1(\Omega_T) \) of \( \mathcal{H}_T^0 \) under norm \( \|\cdot\|_{1,G} \).

**Theorem 2.6** ([DHP08]) There exists a tight subset \( \mathcal{P} \subset \mathcal{M}_1(\Omega_T) \) such that
\[
\hat{E}(\xi) = \max_{P \in \mathcal{P}} E_P(\xi) \quad \text{for all} \quad \xi \in \mathcal{H}_T^0.
\]

\( \mathcal{P} \) is called a set that represents \( \hat{E} \).

**Remark 2.7** i) Let \( \mathcal{A} \) denotes the sets that represent \( \hat{E} \). \( \mathcal{P}^* = \{P \in \mathcal{M}_1(\Omega_T)|E_P(\xi) \leq \hat{E}(\xi), \forall \xi \in \mathcal{H}_T^0\} \) is obviously the maximal one, which is convex and weak compact. All capacities induced by weak compact sets of probabilities in \( \mathcal{A} \) are the same, i.e. \( c_P := \sup_{P \in \mathcal{P}} P = \sup_{P \in \mathcal{P}'} P =: c_P \) for any weak compact set \( \mathcal{P}, \mathcal{P}' \in \mathcal{A} \).

ii) Let \( (\Omega^0, \{\mathcal{F}_t^0\}, \mathcal{F}, P^0) \) be a filtered probability space, and \( \{W_t\} \) be a \( d \)-dimensional Brownian motion under \( P^0 \). [DHP08] proved that
\[
\mathcal{P}'_{\mathcal{M}} := \{P_0 \circ X^{-1}|X_t = \int_0^t h_s dW_s, h \in L^2_T([0,T];\Gamma^{1/2})\} \in \mathcal{A},
\]
where \( \Gamma^{1/2} := \{\gamma^{1/2}|\gamma \in \Gamma\} \) and \( \Gamma \) is the set in the representation of \( G(\cdot) \).

iii) Let \( \mathcal{P}_M \) be the weak closure of \( \mathcal{P}'_{\mathcal{M}} \). Then under each \( P \in \mathcal{P}_M \), the canonical process \( B_t(\omega) = \omega_t \) for \( \omega \in \Omega_T \) is a martingale.

**Definition 2.8** i) Let \( c \) be the capacity induced by \( \hat{E} \). A map \( X \) on \( \Omega_T \) with values in a topological space is said to be quasi-continuous w.r.t \( c \) if

\( \forall \varepsilon > 0, \text{there exists an open set } O \text{ with } c(O) < \varepsilon \text{ such that } X|_{O^c} \text{ is continuous.} \)

ii) We say that \( X: \Omega_T \to R \) has a quasi-continuous version if there exists a quasi-continuous function \( Y: \Omega_T \to R \) with \( X = Y \), c.q.s.. □

Let \( \|\varphi\|_{p,G} = [\hat{E}(|\varphi|^p)]^{1/p} \) for \( \varphi \in C_b(\Omega_T) \), the completions of \( C_b(\Omega_T) \), \( \mathcal{H}_T^0 \) and \( L_{ip}(\Omega_T) \) under \( \|\cdot\|_{p,G} \) are the same and denoted by \( L_{ip}^p(\Omega_T) \), where
\[
L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n})|\forall n \geq 1, t_1, \ldots, t_n \in [0,T], \forall \varphi \in C_{b,Lip}(R^{d \times n})\}
\]
and \( C_{b,Lip}(R^{d \times n}) \) denotes the set of bounded Lipschitz functions on \( R^{d \times n} \).
Theorem 2.9 [DHP08] For $p \geq 1$ the completion $L^p_G(\Omega_T)$ of $C_b(\Omega_T)$ is

$$L^p_G(\Omega_T) = \{X \in L^0 : X \text{ has a q.c. version}, \lim_{n \to \infty} \hat{E}[|X|^p 1_{\{|X| > n\}}] = 0\},$$

where $L^0$ denotes the space of all R-valued measurable functions on $\Omega_T$.

2.2 Basic notions on stochastic calculus in sublinear expectation space

For convenience of description, we only give the definition of Itô integral with respect to 1-dimensional $G$-Brownian motion. However, all results in the following sections of this article hold for the $d$-dimensional case.

Let $H^0_G(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0, T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N-1$. For each $\eta \in H^0_G(0, T)$, let $\|\eta\|_{H^p_G} = \{\hat{E}(\int_0^T |\eta_s|^2 ds)^{p/2}\}^{1/p}$ and denote $H^p_G(0, T)$ the completion of $H^0_G(0, T)$ under norm $\| \cdot \|_{H^p_G}$.

**Definition 2.10** For each $\eta \in H^0_G(0, T)$ with the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),$$

we define

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).$$

By B-D-G inequality, the mapping $I : H^0_G(0, T) \to L^p_G(\Omega_T)$ is continuous under $\| \cdot \|_{H^p_G}$ and thus can be continuously extended to $H^p_G(0, T)$.

**Definition 2.11** A process $\{M_t\}$ with values in $L^1_G(\Omega_T)$ is called a $G$-martingale if $\hat{E}_s(M_t) = M_s$ for any $s \leq t$. If $\{M_t\}$ and $\{-M_t\}$ are both $G$-martingale, we call $\{M_t\}$ symmetric $G$-martingale.

**Definition 2.12** For two process $\{X_t\}, \{Y_t\}$ with values in $L^1_G(\Omega_T)$, we say $\{X_t\}$ is a version of $\{Y_t\}$ if

$$X_t = Y_t, \text{ q.s. } \forall t \in [0, T].$$
3 Some lemmas

Definition 3.1 We say that a process \( \{M_t\} \) with values in \( L^1_G(\Omega_T) \) is quasi-continuous if

\[
\forall \varepsilon > 0, \text{there exists open set } G \text{ with } c(G) < \varepsilon \text{ such that } M(\cdot) \text{ is continuous on } G^c \times [0, T].
\]

Lemma 3.2 (Song10) Any \( G \)-martingale \( \{M_t\} \) has a quasi-continuous version. \( \square \)

So we shall only consider quasi-continuous \( G \)-martingale in the rest of the article. The following lemma is the counterpart of Doob’s uniform integrability lemma, and the proof is adapted from [Yan98].

Let \( B_i = \sigma\{B_s | s \leq t\} \), \( F_t = \cap_{t \geq \tau} B_r \) and \( F = \{F_t\}_{t \in [0, T]} \). \( \tau : \Omega_T \to [0, T] \)

is called a \( F \) stopping time if \( [\tau \leq t] \in F_t, \forall t \in [0, T] \).

Lemma 3.3 Let \( \{M_t\} \) be a symmetric or negative \( G \)-martingale with \( M_T \in L^p_G(\Omega_T) \) for \( p \geq 1 \), then \( \{[M_{\sigma_i}]^p\}_{i \in I} \) are uniformly integrable under \( \hat{E} \) in the following sense:

\[
\sup_{i \in I} \hat{E}[[M_{\sigma_i}]^p1_{[M_{\sigma_i}] > n}] \to 0,
\]

where \( \{\sigma_i | i \in I\} \) is a family of stopping times w.r.t \( F \).

Proof. Fix \( P \in \mathcal{P}_M \) and \( i \in I \).

\[
E_P[[M_{\sigma_i}]^p1_{[M_{\sigma_i}] > n}] 
\leq E_P[[M_T]^p1_{[M_T] > n}] 
\leq \delta^p P([M_{\sigma_i}] > n) + E_P[[M_T]^p1_{[M_T] > \delta}] 
\leq \delta^p n^{-p} E_P([M_{\sigma_i}]^p) + E_P[[M_T]^p1_{[M_T] > \delta}] 
\leq \delta^p n^{-p} E_P([M_T]^p) + E_P[[M_T]^p1_{[M_T] > \delta}].
\]

So \( \sup_{i \in I} \hat{E}[[M_{\sigma_i}]^p1_{[M_{\sigma_i}] > n}] \leq \delta^p n^{-p} \hat{E}([M_T]^p) + \hat{E}[[M_T]^p1_{[M_T] > \delta}] \). First let \( n \to \infty \), then let \( \delta \) go to infinity, we get the result. \( \square \)

Lemma 3.4 Let \( E \) be a metric space and a mapping \( E \times [0, T] \ni (\omega, t) \to M_t(\omega) \in R \) be continuous on \( E \times [0, T] \).

Define \( \underline{\tau}_a = \inf \{t \geq 0 | M_t \geq a\} \land T \) and \( \overline{\tau}_a = \inf \{t \geq 0 | M_t > a\} \land T \).

Then

i) \( M_{t \land \underline{\tau}_a} \) is continuous at any \( \omega \in E \) with \( M_{t \land \underline{\tau}_a}(\omega) < a \) and \( M_{t \land \overline{\tau}_a} \) is continuous at any \( \omega \in E \) with \( M_{t \land \overline{\tau}_a}(\omega) = a \). Moreover, \( -M_{t \land \underline{\tau}_a}, M_{t \land \overline{\tau}_a} \) are both lower semi-continuous.

ii) \( -\overline{\tau}_a \) and \( \underline{\tau}_a \) are both lower semi-continuous.
Proof. i) For $\omega$ with $M_{t\wedge\tau_a}(\omega) = a$, $M_{t\wedge\tau_a}(\cdot)$ is obviously continuous at $\omega$. Also, we claim that for $\omega$ with $M_{t\wedge\tau_a}(\omega) < a$, $M_{t\wedge\tau_a}(\cdot)$ is continuous at $\omega$. Otherwise, there exists a sequence $\{\omega_n\} \subset \Omega_T$ and a sequence $\{t_n\} \subset [0, t]$ such that $\omega_n \to \omega$ and $M_{t_n}(\omega_n) \geq a$. Assume $t_n \to t' \in [0, t]$, then

$$|M_{t_n}(\omega_n) - M_{t'}(\omega)| \to 0.$$ 

So $M_{t'}(\omega) \geq a$ and $M_{t\wedge\tau_a}(\omega) \geq a$, which contradicts the assumption.

For any $b \in R$, we claim that $[M_{t\wedge\tau_a} < b]$ and $[M_{t\wedge\tau_a} > b]$ are both open. If $b > a$, $[M_{t\wedge\tau_a} < b]$ is obvious open. Assume $b \leq a$. For any $\omega \in [M_{t\wedge\tau_a} < b]$, there exists an open set $O$ such that $\omega \in O \subset [M_{t\wedge\tau_a} < b]$ since $M_{t\wedge\tau_a}$ is continuous at $\omega$. So $[M_{t\wedge\tau_a} < b]$ is open. Also, $[M_{t\wedge\tau_a} > b]$ is obvious open for $b \geq a$. Assume $b < a$. If $M_{t\wedge\tau_a}(\omega) = a$, there exists an open set $O$ such that $\omega \in O \subset [M_{t\wedge\tau_a} > b]$ since $M_{t\wedge\tau_a}$ is continuous at $\omega$. For $b < M_{t\wedge\tau_a}(\omega) < a$, we have $b < M_t(\omega)$. Then there exists an open set $O$ such that $\omega \in O \subset [M_t > b] \subset [M_{t\wedge\tau_a} > b]$ since $M_t$ is continuous at $\omega$. So $[M_{t\wedge\tau_a} > b]$ is open.

ii) For any $t \in [0, T]$, $[\tau_a < t]$ is obviously open. For any $t \in [0, T)$, $[\tau_a > t] = [M_{t\wedge\tau_a} < a]$ is open by i). □

Lemma 3.5 For any closed set $F$, we have

$$c(F) = \inf\{c(O) | F \subset O\},$$

where $c$ is the capacity induced by $\hat{E}$.

Proof. It suffices to prove that for any closed set $F \subset \Omega_T$, $c(F) \geq \inf\{c(O) | F \subset O\}$. In fact, for any closed set $F \subset \Omega_T$, there exists $\{\varphi_n\} \subset C_b(\Omega_T)$ such that $1 \geq \varphi_n \downarrow 1_F$. By Theorem 28 in [DHP08], we have $c(F) = \lim_{n \to \infty} \hat{E}(\varphi_n)$. Let $O_n = [\varphi_n > 1 - 1/n]$. Then $O_n \supset F$ and $c(O_n) \leq \frac{1}{n-1} \hat{E}(\varphi_n) \to c(F)$. So $c(F) \geq \inf_n c(O_n) \geq \inf\{c(O) | F \subset O\}$. □

4 Hitting times for $G$-martingale

4.1 Hitting times for symmetric $G$-martingale

In this section, we try to define stopped processes for symmetric $G$-martingale.

Let

$$\mathcal{Q}_T = \{(r, s) | T \geq r > s \geq 0, \ r, s \text{ are rational}\}$$

and

$$\mathcal{S}_a(M) = \{\omega \in \Omega_T | \exists (r, s) \in \mathcal{Q}_T \text{ such that } M_t(\omega) = a \ \forall t \in [s, r]\}.$$
Theorem 4.1 Let \( \{ M_t \}_{t \in [0, T]} \) be a symmetric \( G \)-martingale. Then for all \( a > M_0 \) and \( \tau_a, \zeta_a \) defined above,

i) \( \forall t \in [0, T] \), \( M_{t \wedge \tau_a} \) and \( M_{t \wedge \zeta_a} \) are both quasi-continuous. Consequently, \( \{ M_{t \wedge \tau_a} \} \) and \( \{ M_{t \wedge \zeta_a} \} \) are both symmetric \( G \)-martingale.

ii) If in addition \( c(S_a(M)) = 0 \), then \( \tau_a, \zeta_a \) are both quasi-continuous.

Proof. i) Since \( \{ M_t \}_{t \in [0, T]} \) is a symmetric \( G \)-martingale, it is a martingale under each \( P \in \mathcal{P}_M \). Therefore, \( E_P(M_{t \wedge \tau_a}) = M_0 = E_P(M_{t \wedge \zeta_a}) \) for each \( P \in \mathcal{P}_M \). Consequently \( \hat{E}(M_{t \wedge \tau_a} - M_{t \wedge \zeta_a}) = 0 \). Noting that \( M_{t \wedge \zeta_a} \), \( M_{t \wedge \tau_a} \) both continuous on \( G^c \times [0, T] \). Let \( Q = \{(r, s) \mid r > s, \ r, s \ \text{are rational}\} \). Noting that

\[
[M_{t \wedge \zeta_a} > M_{t \wedge \tau_a}] = \bigcup_{(r, s) \in Q}[M_{t \wedge \zeta_a} \geq r, s \geq M_{t \wedge \tau_a}],
\]

we have

\[
[M_{t \wedge \zeta_a} > M_{t \wedge \tau_a}] \subset G \bigcup \bigcup_{(r, s) \in Q}([M_{t \wedge \zeta_a} \geq r, s \geq M_{t \wedge \tau_a}] \cap G^c).
\]

By Lemma 3.4, \( [M_{t \wedge \zeta_a} \geq r, s \geq M_{t \wedge \tau_a}] \cap G^c \) is closed for any \( (r, s) \in Q \). Since \( c([M_{t \wedge \zeta_a} \geq r, s \geq M_{t \wedge \tau_a}] \cap G^c) = 0 \), by Lemma 3.5 there exists open set \( O \) with \( c(O) < \varepsilon/2 \) such that

\[
\bigcup_{(r, s) \in Q}([M_{t \wedge \zeta_a} \geq r, s \geq M_{t \wedge \tau_a}] \cap G^c) \subset O.
\]

By Lemma 3.3, \( M_{t \wedge \tau_a} \) and \( M_{t \wedge \zeta_a} \) are both continuous on \( O^c \cap G^c \).

ii) By the quasi-continuity of \( \{ M_t \} \), for any \( \varepsilon > 0 \) there exists open set \( G \) such that \( c(G) < \varepsilon/2 \) and \( M_t(\omega) \) is continuous on \( G^c \times [0, T] \). So

\[
G^c \cap [\tau_a > \zeta_a] = S_a(M) \bigcup_{r \in Q \cap [0, T]}[M_{r \wedge \tau_a} < M_{r \wedge \zeta_a}],
\]

where \( Q \) denotes the totality of rational numbers. Then \( c(G^c \cap [\tau_a > \zeta_a]) = 0 \).

Since

\[
G^c \cap [\tau_a > \zeta_a] = \bigcup_{(r, s) \in Q}([\tau_a \geq r, s \geq \zeta_a] \cap G^c)
\]

and \( [\tau_a \geq r, s \geq \zeta_a] \cap G^c \) is closed by Lemma 3.4, there exists open set \( O \) such that \( c(O) < \varepsilon/2 \) and \( G^c \cap [\tau_a > \zeta_a] \subset O \). So on \( O^c \cap G^c \), \( \tau_a = \zeta_a \) are both continuous. \( \square \)

Remark 4.2 If the quadratic variation process of \( \{ M_t \} \) is strictly increasing except on a polar set, then \( c(S_a(M)) = 0 \) for any \( a \in \mathbb{R} \).
Example 4.3 Let \( \{B_t\}_{t \in [0,T]} \) be a 1-dimensional \( G \)-Brownian motion. For \( a > 0 \), let \( \tau_a = \inf\{t \geq 0 \mid B_t \geq a\} \wedge T \) and \( \overline{\tau}_a = \inf\{t \geq 0 \mid B_t > a\} \wedge T \). Then we have

i) For any \( t \in [0,T] \), \( -B_t \wedge \tau_a, B_t \wedge \tau_a, -\tau_a \) and \( \overline{\tau}_a \) are all lower semi-continuous.

ii) For any \( t \in [0,T] \), \( B_t \wedge \tau_a, B_t \wedge \overline{\tau}_a, \tau_a \) and \( \overline{\tau}_a \) are all quasi-continuous.

iii) \( \{B_t \wedge \tau_a\} \) and \( \{B_t \wedge \overline{\tau}_a\} \) are both symmetric \( G \)-martingale.

4.2 Hitting times for \( G \)-martingale (non-symmetric)

For each \( P \in \mathcal{P}_M \) and \( t \in [0,T] \), let \( A_{t,P} := \{Q \in \mathcal{P}_M \mid Q = P \mid \mathcal{F}_t\} \). Theorem 2.3 in [STZ09] implies the following result: For \( t \in [0,T] \) and \( \xi \in L^1_G(\Omega_T) \), \( \eta \in L^1_G(\Omega_t) \), \( \eta = \hat{E}_t(\xi) \) if and only if for each \( P \in \mathcal{P}_M \)

\[
\eta = \text{ess sup}^P_{Q \in A_{t,P}} E_Q(\xi | \mathcal{F}_t), \quad P - a.s.
\]

Theorem 4.4 Let \( \{M_t\}_{t \in [0,T]} \) be a quasi-continuous \( G \)-martingale. For all \( a > |M_0| \), \( M_t \wedge \sigma_a \) and \( M_t \wedge \overline{\sigma}_a \) are both \( G \)-martingale, where \( \sigma_a = \inf\{t \geq 0 \mid M_t \leq -a\} \wedge T \) and \( \overline{\sigma}_a = \inf\{t \geq 0 \mid M_t < -a\} \wedge T \).

Proof. For each \( P \in \mathcal{P}_M \), \( \{M_t\}_{t \in [0,T]} \) is a supermartingale, so for each \( t \in [0,T] \)

\[
E_P(M_t \wedge \sigma_a | \mathcal{F}_{t \wedge \sigma_a}) \leq M_{t \wedge \sigma_a}
\]

by Doob optimal stopping theorem and noting that \( \overline{\sigma}_a \leq \sigma_a \). This implies \( E_P(M_{t \wedge \sigma_a} - M_{t \wedge \sigma_a}) \geq 0 \). On the other hand, it’s obvious to see that \( M_{t \wedge \sigma_a} \leq M_{t \wedge \overline{\sigma}_a} \). So \( M_{t \wedge \sigma_a} = M_{t \wedge \overline{\sigma}_a} \) q.s. By the same arguments as in Theorem 4.1, for any \( \varepsilon > 0 \), there exists an open set \( O \) such that \( c(O) < \varepsilon \) and \( M_{t \wedge \sigma_a} = M_{t \wedge \overline{\sigma}_a} \) are continuous on \( O^c \). So \( M_{t \wedge \sigma_a} \) and \( M_{t \wedge \overline{\sigma}_a} \) are both quasi continuous.

Let \( \sigma = \overline{\sigma}_a \) or \( \sigma_a \).

For \( 0 \leq s < t \leq T \) and \( P \in \mathcal{P}_M \),

\[
\hat{E}_s(M_{t \wedge \sigma}) = \text{ess sup}^P_{Q \in A_{s,P}} E_Q(M_{t \wedge \sigma} | \mathcal{F}_s) \leq M_{s \wedge \sigma}, \quad P - a.s.
\]
On the other hand,
\[
E_Q(M_{t∧σ}|F_s) = -a1_{[σ≤s]} + E_Q(M_{t∧σ}|F_s)1_{[σ>s]}
\]
\[
≥ -a1_{[σ≤s]} + E_Q(E_Q(M_t|F_{t∧σ})|F_s)1_{[σ>s]}
\]
\[
= -a1_{[σ≤s]} + E_Q(M_t|F_s∧σ)1_{[σ>s]}
\]
\[
= -a1_{[σ≤s]} + E_Q(M_t|F_s)1_{[σ>s]}
\]

So
\[
\hat{E}_s(M_{t∧σ}) = \text{ess sup}_{P\in A_σ} E_Q(M_{t∧σ}|F_s)
\]
\[
≥ -a1_{[σ≤s]} + \text{ess sup}_{P\in A_σ} E_Q(M_t|F_s)1_{[σ>s]}
\]
\[
= -a1_{[σ≤s]} + M_s1_{[σ>s]}
\]
\[
= M_{σ∧s} P - a.s.
\]
\[
\hat{E}_s(M_{tσ}) = M_s σ.s. \Box
\]

5 Applications

**Theorem 5.1** Let \( ξ ∈ L^β(Ω_T) \) for some \( β ≥ 1 \) be symmetric, then there exist a sequence \( \{ξ^n\} \subset L^1(Ω_T) \) which are bounded and symmetric such that \( \hat{E}[|ξ - ξ^n|^β] → 0. \)

**Proof.** Let \( M_t = \hat{E}_t(ξ) \) for \( t ∈ [0,T] \) be the quasi-continuous version. For each \( n ∈ N, \) let \( σ_n = \inf\{t ≥ 0| |M_t| > n\} ∧ T, \) and \( τ_n = \inf\{t ≥ 0| M_t > n\} ∧ T. \) By Theorem 4.1, \( \{M_{tσ}^n\}_{t\in[0,T]} \) is a symmetric G-martingale. Let \( \{N_t\} \) be the quasi-continuous version of \( M_t^n \) and \( ς_n = \inf\{t ≥ 0| -N_t > n\} ∧ T. \)

By the same arguments, \( \{N_{tσ}^n\} \) is a bounded symmetric G-martingale. Since the paths of \( \{M_{tσ}^n\} \) and \( \{N_{tσ}^n\} \) are continuous except on a polar set,

\[
\{ω ∈ Ω_T|∃ \text{ some } t ∈ [0,T], s.t. N_t(ω) ≠ M_{tσ}^n(ω)\}
\]

is a polar set. So \( N_{tσ}^n = M_{tσ}^n∧σ_n = M_{tσ}^n, σ.s. \)

\[
|M_{σ_n} - M_T|^β \leq 2^{β-1}|(M_{σ_n} - (M_T ∧ n) ∨ (-n))|^β + (|M_T ∧ n) ∨ (-n) - M_T|^β
\]
\[
≤ 2^{2β-1}|M_{σ_n}|^β 1_{||M_{σ_n}|≥n} + 2^{β-1}|M_T|^β 1_{||M_T|>n}.
\]

Hence, by Lemma 3.3,
\[
\hat{E}(|M_{σ_n} - M_T|^β) ≤ 2^{2β-1} \sup_i \hat{E}[(|M_{σ_i}|^β 1_{||M_{σ_i}|≥n}] + 2^{β-1} \hat{E}[(|M_T|^β 1_{||M_T|>n}] → 0.
\]
The following Corollary improved Theorem 4.6 in [Song10].

**Corollary 5.2** Let $\xi \in L^\beta_G(\Omega_T)$ for some $\beta > 1$ with $\hat{E}(\xi) + \hat{E}(-\xi) = 0$, then there exists $\{Z_t\}_{t \in [0,T]} \in H^\beta_G(0,T)$ such that
\[
\xi = \hat{E}(\xi) + \int_0^T Z_s dB_s.
\]

**Proof.** By Theorem 5.1, there exist a sequence $\{\xi^n\} \subset L^1(\Omega_T)$ which are bounded and symmetric such that $\hat{E}[|\xi - \xi^n|^{\beta}] \to 0$. By Theorem 4.6 in [Song10], there exists $\{Z^n_t\}_{t \in [0,T]} \in H^\beta_G(0,T)$ such that $\xi^n = \hat{E}(\xi^n) + \int_0^T Z^n_s dB_s$.

By B-D-G and Doob’s maximal inequality, $\{Z^n_t\}_{t \in [0,T]}$ is a Cauchy sequence in $H^\beta_G(0,T)$. So there exists $\{Z_t\} \in H^\beta_G(0,T)$ such that $\|\xi^n - Z\|_{H^\beta_G} \to 0$. Then
\[
\xi = \lim_{L^\beta_G,n \to \infty} \xi^n = \lim_{L^\beta_G,n \to \infty} [\hat{E}(\xi^n) + \int_0^T Z^n_s dB_s] = \hat{E}(\xi) + \int_0^T Z_s dB_s.
\]

**Theorem 5.3** Let $\xi \in L^2_G(\Omega_T)$ be bounded above, then $M_t = \hat{E}_t(\xi), t \in [0,T]$ has the following representation:
\[
M_t = M_0 + \int_0^t Z_s dB_s - K_t, \quad (5.0.1)
\]
where $\{Z_t\} \in M^2_G(\Omega_T)$, $\{K_t\}$ is a quasi-continuous increasing process with $K_0 = 0$ and $\{-K_t\}_{t \in [0,T]}$ a $G$-martingale.

**Proof.** There is no loss of generality, we only consider the $\xi \leq 0$ case. Let $\{M_t\}$ be the quasi-continuous version. For $n \in N$, let $\sigma_n = \sigma_n$ defined in Theorem 4.4. Then $M_{\sigma_n}$ is bounded. By Theorem 4.4 and Theorem 4.5 in [Song10], we have the following representation
\[
M^\sigma_{t,n} = \hat{E}(M_T) + \int_0^t Z^n_s dB_s - K^n_t =: N^n_t - K^n_t, \quad q.s.
\]
where $\{Z^n_t\}_{t \in [0,T]} \in H^2_G(0,T)$ and $\{K^n_t\}_{t \in [0,T]}$ is a continuous increasing process with $K^n_0 = 0$ and $\{-K^n_t\}_{t \in [0,T]}$ a $G$-martingale. For $m > n$, by uniqueness of decomposition of semimartingale, $N^n_t = (N^m)^\sigma_t$ and $K^n_t = (K^m)^\sigma_t$.

So
\[
\tilde{M}_t := M^\sigma_{t,n} - M^\sigma_{t,n} = (N^n_t - N^n_t) - (K^n_t - K^n_t) =: \tilde{N}_t - \tilde{K}_t
\]
with \( \{ \hat{K}_t \} \) a continuous increasing process.

By Itô formula, we have
\[
\hat{E}(\hat{N}^2_T) \leq \hat{E}(\hat{M}^2_T) + 2\hat{E}\left( \int_0^T \hat{M}^+_s \, d\hat{K}_s \right).
\]
Noting that \( \hat{M}^+_s \leq n \), we have
\[
\hat{E}(\hat{N}^2_T) \leq \hat{E}(\hat{M}^2_T) + 2n\hat{E}(\hat{K}_T).
\]
\[
\hat{E}(\hat{M}^2_T) \leq 2\{ \hat{E}(M_T - M_{\sigma_n})^2 \} + \hat{E}(M_T - M_{\sigma_n})^2 \}
\]
\[
2n\hat{E}(\hat{K}_T)
\]
\[
= 2n[\hat{E}(M_{\sigma_n} - M_{\sigma_m}) + \hat{E}(M_{\sigma_m} - M_{\sigma_n})]
\]
\[
\leq 4n[\hat{E}(|M_{\sigma_n} - M_T|) + \hat{E}(|M_{\sigma_m} - M_T|)].
\]
By the same arguments as in Theorem 5.1, \( \hat{E}(\hat{M}^2_T) \to 0 \) as \( m, n \to \infty \).

\[
|M_{\sigma_n} - M_T| \\
\leq |M_{\sigma_n} - M_T \lor (-n)| + |M_T \lor (-n) - M_T| \\
\leq 2M_{\sigma_n} 1[|M_{\sigma_n}| \geq n] + M_T 1[|M_T| > n].
\]
So
\[
2n\hat{E}(\hat{K}_T)
\]
\[
\leq 8n\hat{E}(M_{\sigma_n} 1[|M_{\sigma_n}| \geq n]) + 4n\hat{E}(M_T 1[|M_T| > n]) + \\
8m\hat{E}(M_{\sigma_m} 1[|M_{\sigma_m}| \geq m]) + 4m\hat{E}(M_T 1[|M_T| > m])
\]
\[
\leq 8\hat{E}(M_{\sigma_n}^2 1[|M_{\sigma_n}| \geq n]) + 4\hat{E}(M_T^2 1[|M_T| > n]) + \\
8\hat{E}(M_{\sigma_m}^2 1[|M_{\sigma_m}| \geq m]) + 4\hat{E}(M_T^2 1[|M_T| > m]).
\]
So \( 2n\hat{E}(\hat{K}_T) \to 0 \) as \( m, n \to \infty \) by Lemma 3.3. Consequently, we conclude that \( \hat{E}(\hat{N}^2_T) \to 0 \) and \( \hat{E}(\hat{K}_T^2) \to 0 \) as \( m, n \to \infty \).

So \( \{ Z^{t^n}_t \} \) and \( \{ K^n_t \} \) be Cauchy sequences in \( H^2_G(0, T) \) and \( L^2_G(\Omega_T) \) respectively. Let \( \{ Z_t \}_{t \in [0, T]} \), \( \{ K_t \} \) be the corresponding limits of \( \{ Z^{t^n}_t \}_{t \in [0, T]} \), \( \{ K^{t^n}_t \} \). Then
\[
M_t = \lim_{L^2_G, n \to \infty} M^{t^n}_t = \lim_{L^2_G, n \to \infty} \int_0^t Z^n_s \, dB_s - \lim_{L^2_G, n \to \infty} K^n_t = \int_0^t Z_s \, dB_s - K_t.
\]
\( \square \)

In this theorem, for \( \xi \in L^2_G(\Omega_T) \) and bounded above, we have \( K_T \in L^2_G(\Omega_T) \). In this sense, this result improved Theorem 4.5 in [Song10].
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