Nonexistence of global positive solutions for p-Laplacian equations with non-linear memory

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Abstract

The Cauchy problem in $\mathbb{R}^d$, $d \geq 1$, for a non-local in time p-Laplacian equations is considered. The nonexistence of nontrivial global weak solutions by using the test function method is obtained.

Keywords: Cauchy problem; p-Laplacian equation; Global nonexistence

1 Introduction

In this paper, we consider the following problem

\[
\begin{aligned}
&u_t - \text{div} (|\nabla u|^{p-2} \nabla u) = \mathcal{I}, \quad t > 0, x \in \mathbb{R}^d, \\
&u \geq 0, \quad t > 0, x \in \mathbb{R}^d, \\
&u(0, x) = u_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]  

(1.1)

where $\mathcal{I} := \int_0^t (t-s)^{-\gamma} u(s)^q \, ds$, $p, q > 1$, $d \geq 1$, $0 < \gamma < 1$, and $u_0 \in L^2_{\text{loc}}(\mathbb{R}^d)$. We are interested in the nonexistence of nontrivial global weak solutions.

The study of nonexistence of global solutions for nonlinear parabolic equations started by Fujita \cite{5}; he studied the Cauchy problem

\[
\begin{aligned}
&u_t - \Delta u = u^p, \quad p > 1, t > 0, x \in \mathbb{R}^d, \\
&u(0, x) = u_0(x) \geq 0, \quad x \in \mathbb{R}^d;
\end{aligned}
\]  

(1.2)

and obtained that

- there is no nontrivial global solution of (1.2) whenever $p < 1 + 2/d$.
- whereas (1.2) admits a global solution whenever $p > 1 + 2/d$ and $u_0(x) \leq \delta e^{-|x|^2}$, $(0 < \delta \ll 1)$.

For the limiting case $p = 1 + 2/d$, it is shown by Hayakawa \cite{6} for $d = 1, 2$ and by Kobayashi, Sirao and Tanaka \cite{8} for $d \geq 1$ that (1.2) has no global solution $u(x,t)$ satisfying $\|u(\cdot, t)\|_\infty < \infty$ for $t \geq 0$.

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Weissler in [13] proved that if (1.2) has no global solution \( u(x,t) \) satisfying \( \|u(\cdot,t)\|_q < \infty \) for \( t > 0 \) and some \( q \in [1, +\infty) \) whenever if \( p = 1 + 2/d \). Therefore, the value

\[
p_F = 1 + 2/d
\]

is the limiting exponent of (1.2); it is called Fujita’s exponent.

Since Fujita’s paper, a sizeable number of extensions in many directions have been published. Recently, Cazenave, Dickstein and Weissler [3] extended the results of Fujita to the non-local in time heat equation

\[
\begin{aligned}
  w_t - \Delta w &= \int_0^t (t-s)^{-\vartheta} |w(s)|^{p-1}w(s) \, ds, \quad t > 0, x \in \mathbb{R}^d, \ p > 1, \\
  w(0, x) &= w_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]

where \( w_0 \in C_0(\mathbb{R}^d) \) and \( \vartheta \in (0, 1) \). In this case, the value of the critical Fujita exponent is

\[
p_c = \max\left\{ \frac{1}{\vartheta}, 1 + \frac{2(2 - \vartheta)}{(d - 2(1 - \vartheta))_+} \right\}.
\]

For the p-Laplacian equation

\[
\begin{aligned}
  w_t - \text{div} \left( |\nabla w|^{p-2} \nabla w \right) &= w^q, \quad t > 0, x \in \mathbb{R}^d, \ p > 2, q > 1, \\
  w(0, x) &= w_0(x) \geq 0, \quad x \in \mathbb{R}^d,
\end{aligned}
\]

Zhao [15] and Mitidieri and Pohozaev [11] obtained the critical exponent

\[
q^* = p - 1 + p/d;
\]

in fact, Zhao [15] proved that if \( p - 1 < q < q^* \), then the Cauchy problem (1.6) has no nontrivial global solution, however if \( q > q^* \) and \( w_0(x) \) is small enough, then (1.6) admits a global solution. Mitidieri and Pohozaev [11] completed the study by proving the nonexistence of nontrivial global solution in the case \( q \leq q^* \) and for all \( p > 2d/(d + 1) \). Andreucci and Tedeve [1, 10] obtained similar results by considering doubly singular parabolic equations.

The test function method was used to prove the nonexistence of global solutions. This method was introduced by Baras and Kersner in [2] and developed by Zhang in [14] and Pohozaev and Mitidieri in [11], it was also used by Kirane et al. in [9].

Here, we are concerned with the non-existence of nontrivial global solutions of (1.1); inspired by [11], we choose a suitable test function in the weak formulation of the problem. Our main result is

**Theorem 1.1** Let \( u_0 \in L^2_\text{loc}(\mathbb{R}^d) \), \( 0 < \gamma < 1 \), \( p > 1 \), \( d \geq 1 \), \( q > \max\{1, p - 1\} \). Suppose that

\[
\begin{aligned}
  q &\leq \max\left\{ \frac{p - 1}{\gamma}; q_c \right\}, \quad \text{if} \ p < \frac{d}{1 - \gamma}, \\
  q &< \infty, \quad \text{if} \ p \geq \frac{d}{1 - \gamma},
\end{aligned}
\]

where

\[
q_c := p - 1 + \frac{(d + p)(1 - \gamma)(p - 1) + p - d(1 - \gamma)}{d - (1 - \gamma)p},
\]

then no nontrivial global weak solutions exist for problem (1.1).
Remark 1.2 Note that (1.8) can be seen as follows
\[ q \leq \max \left\{ \frac{p-1}{\gamma} \cdot p - 1 + \frac{(d+p)(1-\gamma)(p-1)+p-d(1-\gamma)}{(d-1-\gamma)p+} \right\}, \]
whence
• when \( p = 2 \) and \( \gamma \to 1 \), then \( q_c \to p_F \) where \( p_F \) is defined in (1.3).
• when \( p = 2 \), then \( q_c = p_c \) where \( p_c \) is defined in (1.5).
• when \( \gamma \to 1 \), and \( p > 2d/(d+1) \), then \( q_c \to q^* \) where \( q^* \) is defined in (1.7).

Remark 1.3 The same result can be obtained for the problem
\[ \begin{cases} u_t - \text{div}A(t,x,u,\nabla u) = \int_0^t (t-s)^{-\gamma}u(s)g \, ds, & t > 0, x \in \mathbb{R}^d, \\ u \geq 0, & t > 0, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \]
where \( q > 1, 0 < \gamma < 1, d \geq 1, u_0 \in L^2_{\text{loc}}(\mathbb{R}^d) \), and \( A : \mathbb{R}^*_+ \times \mathbb{R}^d \times \mathbb{R}^*_+ \times \mathbb{R}^d \to \mathbb{R}^d \) is a Carathéodory function, where one assumes the existence of \( c_1, c_2 > 0 \) and \( p > 1 \) such that
\[ \begin{cases} (A(t,x,u,w),w) \geq c_1 |w|^p, \\ |A(t,x,u,w)| \leq c_2 |w|^{p-1}, \end{cases} \] (1.9)
for all \((t,x,u,w) \in \mathbb{R}^*_+ \times \mathbb{R}^d \times \mathbb{R}^*_+ \times \mathbb{R}^d\). For the study of the non-existence of global solution, the following condition is needed
\[ (A(t,x,u,w),w) \geq c_3 |A(t,x,u,w)|^{\frac{p}{p-1}}, \quad \text{for all} \quad (t,x,u,w) \in \mathbb{R}^*_+ \times \mathbb{R}^d \times \mathbb{R}^*_+ \times \mathbb{R}^d, \] (1.10)
which is a general case of (1.3).

Some words on the structure of the paper: Some definitions and properties on the fractional integrals and derivatives are recalled in Section 2. Section 3 is reserved to the proof of the main result (Theorem 1.1).

2 Preliminaries

Definition 2.1 A function \( \mathcal{A} : [a,b] \to \mathbb{R}, \ -\infty < a < b < \infty, \) is said to be absolutely continuous if and only if there exists \( \psi \in L^1(a,b) \) such that
\[ \mathcal{A}(t) = \mathcal{A}(a) + \int_a^t \psi(s) \, ds. \]
\( AC[a,b] \) denotes the space of these functions. Moreover,
\[ AC^2[a,b] := \{ \varphi : [a,b] \to \mathbb{R} / \varphi' \in AC[a,b] \}. \]
Definition 2.2 Let $f \in L^1(c, d)$, $-\infty < c < d < \infty$. The Riemann-Liouville left-and right-sided fractional integrals are defined by

\[ I_{c_l}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{-\alpha} f(s) \, ds, \quad t > c, \, \alpha \in (0, 1), \]  
\[ I_{c_r}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^d (s-t)^{-\alpha} f(s) \, ds, \quad t < d, \, \alpha \in (0, 1) \tag{2.2} \]

where $\Gamma$ is the Euler gamma function.

Definition 2.3 Let $f \in AC[c, d]$, $-\infty < c < d < \infty$. The Riemann-Liouville left-and right-sided fractional derivatives are defined by

\[ D_{c_l}^\alpha f(t) := \frac{d}{dt} I_{c_l}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_c^t (t-s)^{-\alpha} f(s) \, ds, \quad t > c, \, \alpha \in (0, 1), \]  
\[ D_{c_r}^\alpha f(t) := -\frac{d}{dt} I_{c_r}^{1-\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^d (s-t)^{-\alpha} f(s) \, ds, \quad t < d, \, \alpha \in (0, 1). \tag{2.4} \]

Proposition 2.4 [12, (2.64), p. 46] Let $\alpha \in (0, 1)$ and $-\infty < c < d < \infty$. The fractional integration by parts formula

\[ \int_c^d f(t) D_{c_l}^\alpha g(t) \, dt = \int_c^d g(t) D_{c_r}^\alpha f(t) \, dt, \tag{2.5} \]

is satisfied for every $f \in I_{c_l}^{\alpha}(L^p(c, d))$, $g \in I_{c_l}^{\alpha}(L^q(c, d))$ such that $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, $p, q > 1$, where

\[ I_{c_l}^{\alpha}(L^p(0, T)) := \left\{ f = I_{c_l}^\alpha h, \; h \in L^p(c, d) \right\}, \]

and

\[ I_{c_r}^{\alpha}(L^p(c, d)) := \left\{ f = I_{c_r}^\alpha h, \; h \in L^p(c, d) \right\}. \]

Remark 2.5 A simple sufficient condition for functions $f$ and $g$ to satisfy (2.5) is that $f, g \in C[c, d]$, such that $D_{c_l}^\alpha f(t), D_{c_r}^\alpha g(t)$ exist at every point $t \in [c, d]$ and are continuous.

Proposition 2.6 [12, Chapter 1] For $0 < \alpha < 1$, $-\infty < c < d < \infty$, we have the following identities

\[ D_{c_l}^\alpha I_{c_l}^\alpha \varphi(t) = \varphi(t), \text{ a.e. } t \in (c, d), \text{ for all } \varphi \in L^r(c, d), 1 \leq r \leq \infty, \tag{2.6} \]

and

\[ -\frac{d}{dt} D_{c_r}^\alpha \varphi = D_{c_r}^{1+\alpha} \varphi, \quad \varphi \in AC^2[c, d], \tag{2.7} \]

where $D := \frac{d}{dt}$.

Given $T > 0$, let us define the function $w_1$ by

\[ w_1(t) = (1 - t/T)^\sigma, \quad t \in [0, T], \; \sigma \gg 1 \tag{2.8} \]

The following properties concerning the functions $w_1$ will be used later on.
Lemma 2.7 [IB (2.45), p. 40]
Let \( T > 0, 0 < \alpha < 1 \). For all \( t \in [0, T] \), we have
\[
D_{t_0}^{\alpha} w_1(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + 1 - \alpha)} (1 - t/T)^{\sigma - \alpha}, \quad (2.9)
\]
and
\[
D_{t_0}^{1+\alpha} w_1(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha)} (1 - t/T)^{\sigma - \alpha - 1}. \quad (2.10)
\]
Any constant will be denoted by \( C \).

3 Global nonexistence

In this section, the proof of Theorem 1.1 will be presented.

We set
\[
Q_T = (0, T) \times \mathbb{R}^d, \quad \mathcal{B}_{T,B} = (0, T) \times \Omega(B),
\]
where \( \Omega(B) := \{ x \in \mathbb{R}^n; |x| < 2B \} \), for \( T, B > 0 \).

Definition 3.1 (Weak solution)
Let \( 0 < \gamma < 1 \), and \( T > 0 \). A function
\[
u \in C([0, T], L^2_{\text{loc}}(\mathbb{R}^d)) \cap L^{2\gamma}((0, T), L^2_{\text{loc}}(\mathbb{R}^d)) \cap L^p((0, T), W^{1,p}_{\text{loc}}(\mathbb{R}^d)) \cap W^{1,2}((0, T), L^2_{\text{loc}}(\mathbb{R}^d)),
\]
is said to be a weak solution of (1.1) if \( u \geq 0 \), \( u|_{t=0} = u_0 \) and the following formulation
\[
\Gamma(\delta) \int_{Q_T} I_{0}^{\delta}(u^2) \varphi \, dx \, dt = \int_{Q_T} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \, dt + \int_{Q_T} u_t \varphi \, dx \, dt, \quad (3.1)
\]
holds for all \( \varphi \in L^2(0, \infty; L^2(\mathbb{R}^d)) \cap L^p(0, \infty, W^{1,p}(\mathbb{R}^d)) \) such that \( \text{supp} \varphi \subset [0, T] \times \mathbb{R}^d \) is compact, where \( \delta = 1 - \gamma \). We denote the lifespan for the weak solution by
\[
T_w(u_0) := \{ T \in (0, \infty); \text{there exists a unique weak solution } u \text{ to } (1.1) \}.
\]
Moreover, if \( T > 0 \) can be arbitrary chosen, i.e. \( T_w(u_0) = \infty \), then \( u \) is called a global weak solution of (1.1).

Proof. Let \( u \geq 0 \) be a global weak solution of (1.1), then
\[
\Gamma(\delta) \int_{Q_T} I_{0}^{\delta}(u^2) \varphi \, dx \, dt = \int_{Q_T} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \, dt + \int_{Q_T} u_t \varphi \, dx \, dt,
\]
for all \( T > 0 \) and all \( \varphi \in L^2(0, \infty; L^2(\mathbb{R}^d)) \cap L^p(0, \infty, W^{1,p}(\mathbb{R}^d)) \) such that \( \text{supp} \varphi \subset [0, T] \times \mathbb{R}^d \) is compact, where \( \delta = 1 - \gamma \). Let
\[
\varphi(x,t) = D_{t_0T}^{\delta} (\tilde{\varphi}(x,t)) := D_{t_0T}^{\delta} \left( \varphi_1^\delta(x) \varphi_2(t) \right)
\]
with \( \varphi_1(x) := \Phi \left( \frac{|x|}{B} \right), \varphi_2(t) := \left( 1 - \frac{t}{T} \right)^{\eta} \), \( \eta > 0 \).

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where \( \ell, \eta \gg 1 \) and \( \Phi \in C^\infty(\mathbb{R}_+) \) be

\[
\Phi(s) = \begin{cases} 
1, & 0 \leq s \leq 1, \\
\eta, & 1 \leq s \leq 2, \\
0, & s \geq 2.
\end{cases}
\]

Then, using the integration-by-parts formula, we get

\[
\Gamma(\delta) \int_{B_{r,B}} I_{0[B]}^\delta u q D_{I[B]}^\delta \tilde{\varphi} \, dx \, dt + \int_{\Omega(B)} u_0(x) D_{I[T]}^\delta \tilde{\varphi}(0, x) \, dx \\
= \int_{B_{r,B}} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \, dt - \int_{B_{r,B}} u \partial_t \varphi \, dx \, dt.
\]

From (2.5) and Lemma 2.7, we conclude that

\[
\int_{B_{r,B}} D_{I[B]}^\delta \tilde{\varphi} \, dx \, dt + C T^{-\delta} \int_{\Omega(B)} u_0(x) \varphi_1(x) \, dx \\
= C \int_{B_{r,B}} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \, dt - C \int_{B_{r,B}} u \partial_t \varphi \, dx \, dt.
\]

Moreover, using (2.6), \( u_0 \geq 0 \) and \( u \geq 0 \), it follows

\[
\int_{B_{r,B}} u q \tilde{\varphi} \, dx \, dt \leq C \int_{B_{r,B}} |\nabla u|^{p-1} |\nabla \varphi| \, dx \, dt + C \int_{B_{r,B}} u \partial_t \varphi \, dx \, dt =: I_1 + I_2. \tag{3.3}
\]

Next, by introducing \( \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \) in \( I_2 \) and applying the following Young’s inequality

\[
AB \leq \frac{1}{8} A^q + C(q) B^{q'}, \quad A \geq 0, \quad B \geq 0, \quad q + q' = qq',
\]

we get

\[
I_2 \leq \frac{1}{8} \int_{B_{r,B}} u q \tilde{\varphi} \, dx \, dt + C \int_{B_{r,B}} \varphi_1 \varphi_2 \left( D_{I[T]}^\delta \varphi_2 \right)^{q'} \, dx \, dt. \tag{3.4}
\]

In order to obtain a similar estimation on \( I_1 \), let \( \alpha < 0 \) be an auxiliary constant such that \( \alpha > \max\{-1, 1 - p, 1 - \frac{2q}{p+q-2} \} \) and let

\[
u_\epsilon(x, t) = u(x, t) + \epsilon, \quad \epsilon > 0.
\]

By taking \( \varphi_\epsilon(x, t) = \nu_\epsilon(x, t) \varphi(x, t) \) as a test function where \( \varphi \) is given in (3.2), and using the fact that \( u \) is a weak solution, we obtain

\[
\Gamma(\delta) \int_{B_{r,B}} I_{0[B]}^\delta u q \nu_\epsilon \varphi \, dx \, dt = \int_{B_{r,B}} |\nabla u|^{p-2} \nabla u \nabla (\nu_\epsilon \varphi) \, dx \, dt + \int_{B_{r,B}} u_t \nu_\epsilon \varphi \, dx \, dt.
\]

Using the integration-by-parts formula, we get

\[
\Gamma(\delta) \int_{B_{r,B}} I_{0[B]}^\delta u q \nu_\epsilon \varphi \, dx \, dt = \int_{B_{r,B}} |\nabla u|^{p-2} \nabla u \nabla (\nu_\epsilon \varphi) \, dx \, dt \\
- \int_{B_{r,B}} u \partial_t (\nu_\epsilon \varphi) \, dx \, dt - \int_{\Omega(B)} u_0(x) u_\epsilon^a(x) \varphi(x, 0) \, dx.
\]
Then, as
\[ \nabla (u^\alpha \varphi) = \alpha u^{\alpha-1}_\epsilon \varphi \nabla u + u^\alpha_\epsilon \nabla \varphi \quad \text{and} \quad \partial_t (u^\alpha \varphi) = \alpha u^{\alpha-1}_\epsilon \varphi \partial_t u + u^\alpha_\epsilon \partial_t \varphi, \]
it comes
\[
\Gamma(\delta) \int_{B_{T,B}} J^5_{ul(t)}(u^\alpha_\epsilon) \varphi \, dx \, dt \\
= \alpha \int_{B_{T,B}} |\nabla u|^p u^{\alpha-1}_\epsilon \varphi \, dx \, dt + \int_{B_{T,B}} |\nabla u|^{p-2} \nabla u \nabla \varphi \, u^\alpha_\epsilon \, dx \, dt \\
- \alpha \int_{B_{T,B}} u u^{\alpha-1}_\epsilon \varphi \partial_t u \, dx \, dt - \int_{B_{T,B}} u u^\alpha_\epsilon \partial_t \varphi \, dx \, dt - \int_{\Omega(B)} u_0(x) u^\alpha_\epsilon(x,0) \varphi(x,0) \, dx. \tag{3.5}
\]

Whereupon,
\[
J_1 := \int_{B_{T,B}} uu^{\alpha-1}_\epsilon \varphi \partial_t u \, dx \, dt \\
= \int_{B_{T,B}} (u_\epsilon - \epsilon) u^{\alpha-1}_\epsilon \varphi \partial_t u \, dx \, dt \\
= \int_{B_{T,B}} u^{\alpha}_\epsilon \varphi \partial_t u \, dx \, dt - \epsilon \int_{B_{T,B}} u^{\alpha-1}_\epsilon \varphi \partial_t u \, dx \, dt \\
= \frac{1}{\alpha+1} \int_{B_{T,B}} \partial_t (u^{\alpha+1}_\epsilon) \varphi \, dx \, dt - \frac{\epsilon}{\alpha} \int_{B_{T,B}} \partial_t (u^\alpha_\epsilon) \varphi \, dx \, dt \\
= -\frac{1}{\alpha+1} \int_{B_{T,B}} u^{\alpha+1}_\epsilon \partial_t \varphi \, dx \, dt - \frac{1}{\alpha+1} \int_{\Omega(B)} u^{\alpha+1}(x,0) \varphi(x,0) \, dx \\
+ \frac{\epsilon}{\alpha} \int_{B_{T,B}} u^{\alpha}_\epsilon \partial_t \varphi \, dx \, dt + \frac{\epsilon}{\alpha} \int_{\Omega(B)} u^\alpha_\epsilon(x,0) \varphi(x,0) \, dx.
\]

Similarly,
\[
J_2 := \int_{B_{T,B}} uu^\alpha_\epsilon \partial_t \varphi \, dx \, dt \\
= \int_{B_{T,B}} (u_\epsilon - \epsilon) u^\alpha_\epsilon \partial_t \varphi \, dx \, dt \\
= \int_{B_{T,B}} u^{\alpha+1}_\epsilon \partial_t \varphi \, dx \, dt - \epsilon \int_{B_{T,B}} u^{\alpha}_\epsilon \partial_t \varphi \, dx \, dt,
\]
and
\[
J_3 := \int_{\Omega(B)} u_0(x) u^\alpha_\epsilon(x,0) \varphi(x,0) \, dx \\
= \int_{\Omega(B)} (u_\epsilon(x,0) - \epsilon) u^\alpha_\epsilon(x,0) \varphi(x,0) \, dx \\
= \int_{\Omega(B)} u^{\alpha+1}_\epsilon(x,0) \varphi(x,0) \, dx - \epsilon \int_{\Omega(B)} u^\alpha_\epsilon(x,0) \varphi(x,0) \, dx.
\]
By $J_1$, $J_2$, and $J_3$, it follows from (3.5) that

\[
\Gamma(\delta) \int_{B_{T,B}} I_{0(t)}^{S}(u^\alpha) u^\alpha_\epsilon \varphi \, dx \, dt = \alpha \int_{B_{T,B}} |\nabla u|^p u^\alpha_\epsilon - 1 \varphi \, dx \, dt + \int_{B_{T,B}} |\nabla u|^{p-2} \nabla u \nabla \varphi u^\alpha_\epsilon \, dx \, dt - \frac{1}{\alpha + 1} \int_{B_{T,B}} u^\alpha_\epsilon + 1 \partial_t \varphi \, dx \, dt
\]

\[
- \frac{1}{\alpha + 1} \int_{\Omega(B)} u^\alpha_\epsilon + 1(x,0) \varphi(x,0) \, dx,
\]

then

\[
\Gamma(\delta) \int_{B_{T,B}} I_{0(t)}^{S}(u^\alpha) u^\alpha_\epsilon \varphi \, dx \, dt + |\alpha| \int_{B_{T,B}} |\nabla u|^p u^\alpha_\epsilon - 1 \varphi \, dx \, dt
\]

\[
= \int_{B_{T,B}} |\nabla u|^{p-2} \nabla u \nabla \varphi u^\alpha_\epsilon \, dx \, dt - \frac{1}{\alpha + 1} \int_{B_{T,B}} u^\alpha_\epsilon + 1 \partial_t \varphi \, dx \, dt
\]

\[
- \frac{1}{\alpha + 1} \int_{\Omega(B)} u^\alpha_\epsilon + 1(x,0) \varphi(x,0) \, dx
\]

\[
\leq \int_{B_{T,B}} |\nabla u|^{p-1} |\nabla \varphi| u^\alpha_\epsilon \, dx \, dt + \frac{1}{\alpha + 1} \int_{B_{T,B}} u^\alpha_\epsilon + 1 |\partial_t \varphi| \, dx \, dt
\]

\[
= \int_{B_{T,B}} (|\nabla u|^{p-1} u^\alpha_\epsilon) \left( |\nabla \varphi| \frac{(\alpha - 1)(p-1)}{p} \varphi - \frac{p-1}{p} \right) \, dx \, dt
\]

\[
+ \frac{1}{\alpha + 1} \int_{B_{T,B}} u^\alpha_\epsilon + 1 |\partial_t \varphi| \, dx \, dt
\]

\[
\leq \frac{|\alpha|}{2} \int_{B_{T,B}} |\nabla u|^p u^\alpha_\epsilon - 1 \varphi \, dx \, dt + C(\alpha) \int_{B_{T,B}} u^p_\epsilon - 1 + \alpha |\nabla \varphi|^p \varphi^{1-p} \, dx \, dt
\]

\[
+ \frac{1}{\alpha + 1} \int_{B_{T,B}} u^\alpha_\epsilon + 1 |\partial_t \varphi| \, dx \, dt,
\]

where Young’s inequality

\[
AB \leq \frac{|\alpha|}{2} A^{\frac{p}{p-1}} + C(\alpha) B^p, \quad A \geq 0, \ B \geq 0
\]

has been used. Consequently, as $u \geq 0$, we conclude that

\[
\int_{B_{T,B}} |\nabla u|^p u^\alpha_\epsilon - 1 \varphi \, dx \, dt \leq C \int_{B_{T,B}} u^p_\epsilon - 1 + \alpha |\nabla \varphi|^p \varphi^{1-p} \, dx \, dt + C \int_{B_{T,B}} u^\alpha_\epsilon + 1 |\partial_t \varphi| \, dx \, dt.
\]
Young’s inequality and the last inequality, allow to get
\[
\int_{B_{T,B}} |\nabla u|^{p-1} |\nabla \varphi| \, dx \, dt
\]
\[
= \int_{B_{T,B}} \left( |\nabla u|^{p-1} u_\epsilon^{(\alpha-1)(p-1)} \frac{1}{p} \right) \left( u_\epsilon^{(1-\alpha)(p-1)} \frac{1}{p} |\nabla \varphi| \right) \, dx \, dt
\]
\[
\leq \int_{B_{T,B}} |\nabla u|^{p} u_\epsilon^{(\alpha-1)} \varphi \, dx \, dt + C \int_{B_{T,B}} u_\epsilon^{(1-\alpha)(p-1)} \varphi^{1-p} |\nabla \varphi|^{p} \, dx \, dt
\]
\[
\leq C \int_{B_{T,B}} u_\epsilon^{p-1+\alpha} |\nabla \varphi|^{p-1} \varphi \, dx \, dt + C \int_{B_{T,B}} u_\epsilon^{\alpha+1} |\partial_t \varphi| \, dx \, dt
\]
\[+ C \int_{B_{T,B}} u_\epsilon^{(1-\alpha)(p-1)} \varphi^{1-p} |\nabla \varphi|^{p} \, dx \, dt.\]

Applying Fatou’s and Lebesgue’s theorems, as \( \epsilon \to 0 \), we get
\[
I_1 \leq C \int_{B_{T,B}} u^{p-1+\alpha} |\nabla \varphi|^{p-\alpha} \varphi \, dx \, dt + C \int_{B_{T,B}} u^{\alpha+1} |\partial_t \varphi| \, dx \, dt
\]
\[+ C \int_{B_{T,B}} u^{(1-\alpha)(p-1)} \varphi^{1-p} |\nabla \varphi|^{p} \, dx \, dt =: K_1 + K_2 + K_3.\]

Now, we use Young’s inequality, \( \nabla (\varphi^{q}) = \ell \varphi^{q-1} \nabla \varphi_1 \), and the conditions that \( q > \max\{1, p-1\} \), \( \alpha < 0 \) to get estimations of \( K_1, K_2 \) and \( K_3 \); we have
\[
K_1 = \int_{B_{T,B}} (u^{p-1+\alpha} \varphi^{(p-1+\alpha)/q}(C \varphi^{-(p-\alpha-1)/q} \nabla \varphi_1 \varphi^{1-p}) \, dx \, dt
\]
\[
\leq \frac{1}{8} \int_{B_{T,B}} u^{q} \varphi \, dx \, dt
\]
\[+ C \int_{B_{T,B}} \varphi_1^{(q-1)^{-1+\alpha}} \nabla \varphi^{(q-1)^{-\alpha}} \varphi_2^{(q-1)^{-\alpha}} \left( D_{\mathcal{I}_{T}} \varphi_2 \right) \, dx \, dt,
\]
\[
K_2 = \int_{B_{T,B}} (u^{\alpha+1} \varphi^{(\alpha+1)/q}(C \varphi^{-(\alpha+1)/q} \partial_t \varphi_2)) \, dx \, dt
\]
\[
\leq \frac{1}{8} \int_{B_{T,B}} u^{q} \varphi \, dx \, dt + C \int_{B_{T,B}} \varphi_1^{(q-1)^{-1+\alpha}} \left( D_{\mathcal{I}_{T}} \varphi_2 \right) \, dx \, dt,
\]
and
\[
K_3 = \int_{B_{T,B}} (u^{(1-\alpha)(p-1)} \varphi^{(1-\alpha)(p-1)/q}(C \varphi^{-(1-\alpha)(p-1)/q} \nabla \varphi^{1-\alpha} |\nabla \varphi|^{p}) \, dx \, dt
\]
\[
\leq \frac{1}{8} \int_{B_{T,B}} u^{q} \varphi \, dx \, dt
\]
\[+ C \int_{B_{T,B}} \varphi_1^{(q-1)^{-1}(1-\alpha)(p-1)} \nabla \varphi_1^{(q-1)^{-1}(1-\alpha)(p-1)} \varphi_2^{(1-\alpha)(p-1)} \left( D_{\mathcal{I}_{T}} \varphi_2 \right) \, dx \, dt.
\]
Therefore, we conclude that
\[
I_1 \leq \frac{3}{8} \int_{B,T} u^q \tilde{\varphi} \, dx \, dt \\
+ C \int_{B,T} \varphi_1 x \phi_2^{q-1} \frac{p-1}{q-1-\alpha} \, \varphi_2 dx \, dt \\
+ C \int_{B,T} \varphi_1 x \phi_2^{q-1} \frac{p+1}{q-1-\alpha} \, \varphi_2 dx \, dt \\
+ C \int_{B,T} \varphi_1 x \phi_2^{q-1} \frac{p+1}{q-1-\alpha} \, \varphi_2 dx \, dt.
\]

Using the estimates of $I_1$ and $I_2$ into (3.3), we obtain
\[
\int_{B,T} u^q \tilde{\varphi} \, dx \, dt \\
\leq C \int_{B,T} \varphi_1 x \phi_2^{q-1} \frac{p+1}{q-1-\alpha} \, \varphi_2 dx \, dt \\
+ C \int_{B,T} \varphi_1 x \phi_2^{q-1} \frac{p+1}{q-1-\alpha} \, \varphi_2 dx \, dt \\
+ C \int_{B,T} \varphi_1 x \phi_2^{q-1} \frac{p+1}{q-1-\alpha} \, \varphi_2 dx \, dt.
\]

At this stage, we choose $B = T^\theta$, $\theta > 0$. Taking $\alpha$ small enough and passing to $s = T^{-1}t$, $y = T^{-\theta}x$, we get from (3.7) that
\[
\int_0^T \int_{\Omega(T^s)} |u|^q \tilde{\varphi} \, dx \, dt \leq CT^{-\delta t \alpha q - \frac{1}{q-1} + \theta} + CT^{-\delta t \alpha q - \frac{1}{q-1} + \theta} + C T^{-\delta t \alpha q - \frac{1}{q-1} + \theta}.
\]

If all exponents of $T$ are negative, by taking $T \to +\infty$ and using the dominated convergence theorem, we conclude that $u = 0$. In order to ensure the negativity of the exponents of $T$, it is sufficient to require
\[
q < \frac{1 + d\theta}{(d\theta - \delta)_+} =: q_1(\theta) \quad \text{and} \quad q < \frac{(p - 1)(1 + d\theta)}{(d\theta - \theta p + 1 - \delta)_+} =: q_2(\theta).
\]

which is equivalent to
\[
q < \max_{\theta > 0} \min \{q_1(\theta), q_2(\theta)\}.
\]

To take into consideration $q_1(\theta)$ and $q_2(\theta)$, we first look at $(d\theta - \delta)_+$ and $(d\theta - \theta p + 1 - \delta)_+$ and try to compare them in terms of $\theta$, i.e., to compare between $\delta/d$ and $(1 - \delta)/(p - d)_+$; this requires to study two cases: $p \geq d/\delta$ and $p < d/\delta$.

\textbf{a) If $p \geq d/\delta$:} In this case we have $(1 - \delta)/(p - d) \leq \delta/d$. As $q_1(\theta)$ is non-increasing and $q_2(\theta)$ is non-decreasing as a function of $\theta$, (3.9) can be read as
\[
q < \max \left\{ \max_{\theta \in (d/\delta, +\infty]} q_1(\theta), \max_{\theta \in [0, (1 - \delta)/(p - d)_+]} q_2(\theta) \right\} = +\infty.
\]
b) If $p < d/\delta$: Technical calculations lead us to to distinguish 3 cases.

i) **Case** $p > d$. In this case we have $\delta/d < (1 - \delta)/(p - d)$. So, (3.9) can be read as

$$q < \max \left\{ \frac{p - 1}{1 - \delta}, q_c \right\},$$

with

$$\theta_0 := \frac{\delta p + 1 - 2\delta}{dp + p - 2d} > 0,$$

using that $q_1(\theta)$ is non-increasing and $q_2(\theta)$ is non-decreasing.

ii) **Case** $2d/(d + 1) < p \leq d$. In this case, (3.9) can be read as

$$q < \max \left\{ \frac{p - 1}{1 - \delta}, q_c \right\}.$$ 

As $p_2(\theta)$ is non-decreasing when $p \geq d\delta$, and non-increasing when $p \leq d\delta$, we can see that (3.10) is equivalent to

$$q < \max \left\{ \frac{p - 1}{1 - \delta}, q_c \right\}.$$ 

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\[ K_2 \leq \frac{1}{3} \int_{B_{T,B}} u^q \varphi \, dx \, dt + C \int_{B_{T,B}} \varphi_1 \varphi_2^{1+\frac{\alpha}{q-1}} \left( D_{1[T]} \varphi_2 \right)^{\frac{q}{q-1} \alpha} \, dx \, dt, \]

and

\[ K_3 \leq C \left( \int_0^T \int_{\Delta(B)} u^q \varphi \, dx \, dt \right)^{\frac{(1-\alpha)(p-1)}{q}} \left( \int_{B_{T,B}} \varphi_1 \varphi_2^{1/(q-1)} \left( D_{1[T]} \varphi_2 \right)^{\frac{q}{q-1} \alpha} \, dx \, dt \right) \]

\[ \left( \int_{B_{T,B}} \varphi_1 \left| \nabla \varphi_1 \right|^{q^{1-p}+1} \varphi_2^{\frac{p-1}{q^{1-p}+1}} \left( D_{1[T]} \varphi_2 \right)^{\frac{q}{q-1} \alpha} \, dx \, dt \right)^{\frac{q}{q-1} \alpha} \left( D_{1[T]} \varphi_2 \right)^{\frac{q}{q-1} \alpha} \, dx \, dt \right)^{\frac{q-1}{q} \alpha}. \]

Whereupon

\[ \int_{B_{T,B}} u^q \varphi \, dx \, dt \]

\[ \leq C \int_{B_{T,B}} \varphi_1 \varphi_2^{1/(q-1)} \left( D_{1[T]} \varphi_2 \right)^{\frac{q}{q-1} \alpha} \, dx \, dt \]

\[ + C \left( \int_0^T \int_{\Delta(B)} u^q \varphi \, dx \, dt \right)^{\frac{p-1}{q}} \left( \int_{B_{T,B}} \varphi_1 \left| \nabla \varphi_1 \right|^{q^{1-p}+1} \varphi_2^{\frac{p-1}{q^{1-p}+1}} \left( D_{1[T]} \varphi_2 \right)^{\frac{q}{q-1} \alpha} \, dx \, dt \right)^{\frac{q}{q-1} \alpha} \left( D_{1[T]} \varphi_2 \right)^{\frac{q}{q-1} \alpha} \, dx \, dt \right)^{\frac{q}{q-1} \alpha}. \]

Taking into account that \( q = \max \{ \frac{p-1}{1-\delta}, q_c \} \) and \( s = T^{-1}t, y = R^\theta T^{-\theta}x \), we get

\[ \int_{B_{T,B}} |u|^q \varphi \, dx \, dt \leq C R^{-d\theta} + C R^{-d\theta + \frac{pd\theta}{q-p+1}} \left( \int_0^T \int_{\Delta(B)} u^q \varphi \, dx \, dt \right)^{\frac{p-1}{q}}. \]

Taking the limit when \( T \to \infty \), and using (3.11), we obtain

\[ \int_0^\infty \int_{\mathbb{R}^d} |u|^q \, dx \, dt \leq C R^{-d\theta}. \]

Finally, letting \( R \to \infty \), it comes that \( u = 0 \). The proof of Theorem 1 is complete.
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