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The research on rumor spreading has a long history, and its wanton flooding has brought huge impact on people's life. In the process of its spreading, the individual's activity plays an important role. However, in the complex and changeable environment, randomness cannot be ignored, not to mention its influence on individual activity. Based on the ISK model of individual activity, this paper explores the stochastic version of the rumor model including fluctuations in the activity. Then, the influence of Stratonovich stochastic noise on the asymptotic behavior of non-linear rumor spreading model is studied. Through the mathematical analysis, we get the critical values to measure whether the deterministic and stochastic models spread or not, as well as the threshold conditions for rumor to spread wantonly. At the same time, the effects of Stratonovich stochastic noise on the asymptotic behavior of rumor-free equilibrium point $E_0$ and endemic equilibrium point $E^*$ are obtained respectively, and the condition that the rumor-free equilibrium is globally asymptotically stable in the presence of noise is given. Finally, the theoretical results are verified by numerical simulation.

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1. Introduction

With the progress of science and technology and the advent of web2.0 era, social software such as Facebook and WeChat have replaced the mode of word of mouth, and which become one of the main tools for interpersonal communication. All kinds of information spread rapidly on social media, which produces a lot of useless junk information. What is more, some malicious criminals spread rumors on the Internet at will, disturbing public order [1], causing serious bullying [2] and social panic [3].

To avoid the harm of rumors, predicting the dynamic behavior of rumors in the virtual digital world has become a hot topic for scholars from all works of life. The earliest history of rumor spreading can be traced back to the 1960s. Delay and Kendall [4] first realized that the social interaction between people was actually based on the Internet, and established a DK network rumor spreading model, which divided people into three categories: ignorant (the people who have not heard of rumors), spreaders (the people who have heard and spread rumors) and stiflers (the people who have heard of rumors but do not spread them). Maki and Thompson [5] improved the defect of DK model in 1973, and extended the reconstructed MK model to the most classical rumor spreading model. That is to say, MK model believes that when two spreaders contact, only one spreader changes to a stifler, while DK model holds that when two spreaders contact, both spreaders will change to stiflers. Since then, most researchers have expanded the spread of rumors on complex networks in different fields. A group of scholars devoted themselves to subdividing the population to enrich the rumor
models [6–8]. In addition, some people are committed to study of non-linear spreading rate of rumors based on individual differences [9,10], while others were engaged in rumor control to overcome the harm of rumors [11–13].

In view of the individual particularity [14] of rumor spreading group, Huo and Liu et al. [15–17] believed that the individual activity greatly affects their behavior choices. Thus, the activity played a crucial role in the enthusiasm of rumor spreading. To this end, they have introduced the concept of individual activity into the rumor model and made outstanding contributions in this field.

However, most of the existing literature is limited to the exploration of individual emotion, thinking mode, education level and other internal characteristics. In fact, our living environment is changing rapidly, the decision-making preferences of rumor spreaders depend strongly on the surrounding environment. Therefore, it is necessary to study the influence of random factors on decision-making. In view of this, Rupp [18] proposed the idea diffusion model containing random effects, and extended the ordinary differential equation model to the stochastic differential equation model. Yao et al. [19] proposed a dynamic I2SR model based on rumor spreading with random noise interference and found that there is a negative correlation between the spreading range of rumors and noise intensity. Coletti et al. [20] explored the sufficient conditions for rumor extinction or survival with a positive probability in an interacting particle system. In order to comprehensively investigate the randomness of Incubator’s impact on the ignorant, Dauhoo et al. [21] extrapolated the deterministic rumor model to a random model by adopting a random coefficient to the term representing the impact within the system. Wang et al. [22] deeply explored the mode of gossip diffusion in social networks when uncertainty exists during users’ decision-making. Zhu et al. [23] proposed a rumor spreading model which considered the change of connectivity as noise interference. It was found that the peak value of infected individuals was positively correlated with noise intensity, while the rumor life cycle was negatively correlated with noise intensity as a whole. Jia et al. [24–26] compared and analyzed the threshold difference between deterministic model and stochastic model with white noise fluctuation, and concluded that the basic reproduction number \( R_0 \) of stochastic model was less than that of deterministic system. Jain et al. [27] introduced the concept of expert interaction into the rumor spreading model to further subdivide the categories of people. At the same time, Gauss white noise was included in the model as a random factor, and it was explored that noise may be a reason for the persistence of rumors.

Individual activity is not only affected by the environment, but also depends on their own interests and personality characteristics to a large extent, so their attitudes towards the same event are quite different. Some people are born lively and willing to spread information actively. Whether it is rumor or truth, they are all actively spreading. That is to say, there are not only spreaders who are willing to spread rumor, but also stiflers who are good at releasing truth, and the ignorant who are more active in exploring new things, i.e. active individuals. On the contrary, other people are naturally indifferent to their surroundings. They are tired of communicating with others, let alone spreading rumors or real information, i.e. inactive individuals. From this point of view, people can be divided into two categories: active state and inactive state. Thus, we introduce a concept of activity as a criterion to distinguish whether it is active or not.

With the advent of the network era, the network content is increasingly rich, the development of self-media has accelerated the information update iteration, the network environment becomes more complex and changeable, microblog hot search or instant information may change people’s views on events at any time, the randomness of the network environment shows increasingly important influence. Individual activity is inevitably affected by the randomness of the environment, so the decision-making behavior has greater uncertainty. With the passage of time, the heat of the event may change dramatically, so does the perception of rumors. Affected by this, the individual’s activity also fluctuates. Therefore, the individuals’ activities have great volatility and randomness. Therefore, the activities of users have great uncertainty and randomness. In view of this, it is of great practical significance to consider the noise interference of activity in rumor model. Inspired by this, this paper discusses the spreading process of rumors on the Internet based on the volatility of user activity.

The specific work of this paper is as follows: in Section 2, we introduced a deterministic model without Gaussian noise by mathematical method. In Section 3, the boundary and endemic equilibrium points as well as the basic reproduction number were calculated simply. In Section 4, we discussed the stochastic version of the previous model and deduced the Stratonovich and Itô transformation equations for the model. The existence and uniqueness of positive solutions for stochastic systems were given in Section 5. Next, we deduced the global stability condition for the rumor-free equilibrium of stochastic system. Then, some numerical simulations were carried out to prove the analysis results of the two models. Finally, the results were summarized in the conclusion.

2. Deterministic model

Individual behavior is affected by various subjective factors, such as activity, which is a particularly important factor affecting the spread rate of rumors. In this section, we cite the ISR model of Ref. [15] to consider the impact of active variables on rumor spreading in homogeneous networks. In the ISR model, a node can be in active state with a certain probability \( a \) and in inactive state with probability \( 1 - a \). In terms of whether to understand rumors, the total population can be divided into ignorant (people who do not know rumors), spreaders (people who know and spread rumors), and stiflers (people who know but have no interest in spreading rumors), which are represented by \( I, S, R \), respectively. So each node may have six different states at time \( t \) and their density is as follows: active ignorant \( I^a(t) \), inactive ignorant \( I^d(t) \), active spreaders \( S^a(t) \), inactive spreader \( S^d(t) \), active stiflers \( R^a(t) \), inactive stiflers \( R^d(t) \), in which, the superscript
The basic scheme of population dynamics model for the spread of rumors.

Fig. 1. The basic scheme of population dynamics model for the spread of rumors.

\( a \) denotes active state and the superscript \( d \) represents inactive state. \( N(t) \) denotes the total number of nodes at time \( t \). These fractions of different groups satisfy the following condition:

\[
I^a(t) + I^d(t) + S^a(t) + S^d + R^a(t) + R^d(t) = N(t)
\]

Its spreading rules, process and character meanings (\( b, \mu, \lambda, \sigma, \delta, \theta \)) are consistent with Ref. [15], and the spreading diagram is shown in Fig. 1.

Based on the proposed model, the following differential equations in homogeneous networks can be derived,

\[
\begin{align*}
\frac{dI(t)}{dt} &= b(1 - N(t)) - (2a - a^2)\lambda I(t)S(t) - \mu I(t) \\
\frac{dS(t)}{dt} &= (2a - a^2)\lambda I(t)S(t) - (2a - a^2)\sigma S(t)(S(t) + R(t)) - \mu S(t) - \delta S(t) \\
\frac{dR(t)}{dt} &= (2a - a^2)\sigma S(t)(S(t) + R(t)) + \delta S(t) - \mu R(t)
\end{align*}
\]

where \( k = \sum_{l=1}^{k} lP(l), l = 1, 2, 3, \ldots, k \) indicates the average degree, and \( b(1 - N(t)) \) represents the new born ignorant per unit step, which is proportional to recruitment rate \( b \) and the density of vacant nodes \((1 - N(t))\).

And the initial condition at initial time \( t_0 \) is as follows,

\[
I(t_0) = I_0, \quad S(t_0) = S_0, \quad R(t_0) = R_0
\]

In the following sections, we will explore the dynamical behavior of our improved model.

3. Dynamical behavior of the system

In this section, the dynamic behavior of system (1) is analyzed. Meanwhile, we will discuss the positivity of solution, the existence of equilibrium points, and the stability conditions.

3.1. Positively invariant of the system

First, we study the positivity of social network system (1). As mentioned above, the total number in the system is \( N(t) \) at time \( t \). According to the foregoing hypothesis, we know that \( I^a(t) + I^d(t) + S^a(t) + S^d + R^a(t) + R^d(t) = N(t) \) is satisfied. By summing up the three equations of system (1), we can obtain the following equation,

\[
\frac{dN(t)}{dt} = \frac{d(I(t) + S(t) + R(t))}{dt} = b(1 - N(t)) - \mu N(t) = b - (b + \mu)N(t)
\]

According to Eq. (2), we can get \( N(t) = \frac{b}{b + \mu} + N(0)\exp(-(b + \mu)t) \), where \( N(0) \) represents the initial density of whole population. Hence, \( \lim_{t \to \infty} N(t) = \frac{b}{b + \mu} \), then \( N(t) = I(t) + S(t) + R(t) \leq \frac{b}{b + \mu} \) for all \( t \geq 0 \) and region
\[ \Omega = \left\{ (l(t), S(t), R(t)) \in \mathbb{R}^3_+ : 0 \leq l(t), S(t), R(t) \leq \frac{b}{b+\mu} \right\} \]

can be regarded as a feasible area of the positive invariant of system (1).

What matters most about rumor spreading is when to stop spreading and the conditions of rumor disappearing. Next, in order to find this condition, we will introduce and calculate a critical value for the proposed deterministic system, which is the so-called basic regeneration number \( \varpi_0 \).

### 3.2. Basic reproduction number

As an extremely important concept, basic reproduction number \( \varpi_0 \) means the number of new generations per unit of time produced by the spreader in the early stage of rumor spreading, when everyone is ignorant. Usually, \( \varpi_0 = 1 \) can be used as a threshold to determine whether a rumor dies or not.

In order to calculate the basic reproduction number, some necessary substitutions should be done. By substituting the condition \( I(t) = \frac{\varpi}{b+\mu} - S(t) - R(t) \) into system (2), we can simplify the original system (1) into the equivalent 2-dimensional system (3).

\[
\begin{align*}
\frac{dS(t)}{dt} &= (2a - a^2)\lambda S(t) + b + \mu - (a - a^2)\sigma S(t) - S(t) - \delta S(t) \\
\frac{dR(t)}{dt} &= (2a - a^2)\sigma S(t) + R(t) + \delta S(t) - \mu R(t)
\end{align*}
\]

Here, we can calculate the basic reproduction number with fix weight model firstly. System (14) can be written as \( \frac{dx}{dt} = F(x) - x = (S, R)^T \). Applying the next generation matrix method in Ref. [28], system (14) has a rumor-free equilibrium \( E_0 = (0, 0)^T \).

Where the rate of appearance of new infections is

\[ F(x) = \begin{pmatrix} (2a - a^2)\lambda S(t) + b + \mu - (a - a^2)\sigma S(t) - S(t) - \delta S(t) \\ (2a - a^2)\sigma S(t) + R(t) + \delta S(t) - \mu R(t) \end{pmatrix} \]

and transfer rate of individuals out of spreaders subclass is

\[ \nu(x) = \begin{pmatrix} (\mu + \delta)S(t) \\ -\delta S + \mu R(t) \end{pmatrix} \]

The Jacobian matrices of \( F(x) \) and \( \nu(x) \) at the rumor-free equilibrium \( E_0 \) are as showing

\[ F = DF(E_0) = \begin{pmatrix} (2a - a^2)\lambda & b \\ 0 & 0 \end{pmatrix} \]

\[ V = D\nu(E_0) = \begin{pmatrix} (\mu + \delta) \\ -\delta \end{pmatrix} \]

According to the concept of next generation matrix and reproduction number given in Ref. [28], the reproduction number of system (3) equals to \( \varpi_0 = \rho(FV^{-1}) \), where \( \rho(FV^{-1}) \) denotes the spectral radius of the matrix \( FV^{-1} \). Then,

\[ \varpi_0 = \rho(FV^{-1}) = \frac{(2a - a^2)\lambda b k}{(b + \mu)(\delta + \mu)} = \frac{(2a - a^2)\lambda k}{(\delta + \mu)} \]

Next, we will solve all possible equilibrium solutions for our improved ISR model.

### 3.3. Steady states and their stability

The stable and safe operation of social networks is conducive to alleviating rumor spreading, reducing harm of rumor and serving the public. Therefore, we pay more attention to all possible equilibrium states of the above model. The equilibrium state refers to the steady state that all groups of individuals in the system can reach in the limit time in the region \( \Omega \). There are two feasible equilibriums in our system, one rumor-free equilibrium \( E^0 = (I^0, S^0, R^0)^T \) and one rumor endemic equilibrium \( E^* = (I^*, S^*, R^*)^T \). \( E^0 \) and \( E^* \) are also called boundary equilibrium point and internal equilibrium point of the system, respectively.

From the definition of equilibrium state, when the inflow and outflow of each group in the system are equal, that is, when the mobility of each group is 0, the system will reach dynamic equilibrium, which satisfied the following equations,

\[
\begin{align*}
&(b(1 - N(t)) - (2a - a^2)\lambda k I(t)) - S(t) - \mu I(t) = 0 \\
&(2a - a^2)\lambda k I(t)S(t) - (2a - a^2)\sigma k S(t)(S(t) + R(t)) - \mu S(t) - \delta S(t) = 0 \\
&(2a - a^2)\sigma k S(t)(S(t) + R(t)) + \delta S(t) - \mu R(t) = 0
\end{align*}
\]
By solving the above differential equations, we can obtain that there exists a boundary equilibrium solution for system (1) $E^0 = \left( \frac{b}{R + \mu} \ 0 \ 0 \right)^T$, and an endemic equilibrium $E^* = \left( I^* \ S^* \ R^* \right)^T$, which can be summarized as follows,

$$E^* = \left( \begin{array}{c}
\frac{(2a - a^2)b \sigma \bar{k} + \mu (\delta + \mu) (b + \mu)}{(2a - a^2)(\sigma + \lambda) (b + \mu) \bar{k}} \\
\frac{\mu [b \lambda (-2a + a^2) \bar{k} + (\delta + \mu) (b + \mu)]}{\lambda (-2a + a^2) \bar{k} + (\delta + \mu) (b + \mu)} \\
\frac{[\lambda b (a - a^2) \bar{k} - (\delta + \mu) (b + \mu)]}{\lambda (2a - a^2) \bar{k} (\sigma + \lambda) (b + \mu)}
\end{array} \right) \left( \begin{array}{c}
\eta \\
\eta \\
\eta
\end{array} \right)$$

According to Theorem 3 from Ref. [28], we can know that there is a basic reproduction number $\Re_0$, which is the threshold to describe the spreading rate of rumor. It refers to the potential and severity of rumor outbreak. Meanwhile, it represents the number of people who can infect in the average spreading cycle when all people are ignorant at the initial stage of spreading. And if $\Re_0 > 1$, then rumors can spread throughout the whole population, while $\Re_0 < 1$, it tends to die out.

Considering the influence of various external factors, the perturbation factor is introduced into the original deterministic model, which is transformed into stochastic differential equation model. In this paper, the environmental fluctuation variable is assumed to be Gaussian noise, and the influence of environmental fluctuation variable on system dynamics is analyzed.

4. Stochastic model

When an emergency occurs, people’s perception of the incident is not unchangeable. On the contrary, people’s perception of the event is also fluctuating due to the drastic changes in the surrounding environment. For example, when the COVID-19 pandemic started, people were initially indifferent to relevant rumors, and they always believed that the virus was far away and had nothing to do with them, and so they were not interested in rumors. However, as the number of infected people soared, related rumors spread widely, and the government vigorously promoted, people gradually changed their opinions and became concerned about related events and rumors. It can be seen that the activity of people to the event is a dynamic adjustment process, is a constantly fluctuating variable. That is, the activity has a certain correlation with the evolution and development of the event, and it changes with the environmental conditions and has a certain randomness. In order to make the model more realistic, the stochastic model including the noise term in the individual activity is considered in this paper.

To explore the effects of surrounding environment fluctuations on nodes’ activity during the ignorant, spreaders and stiflers, we allow $a$ to fluctuate around its mean value. Therefore, in this paper, a new way can be used to define the activity rate $a$, as follows,

$$a(t) = a + \eta \Phi(t) \quad (10)$$

where $\Phi(t) \sim N(0, \eta^2)$ expresses noise term with zero mean value and $\eta$ indicates noise intensities which measure amplitude of fluctuation with respect to activity rates. According to system (1), we can know that the effect of activity on the whole model is represented by parameter $2a(t) - a^2(t)$. Combined with Eq. (10), the following results can be obtained,

$$2a(t) - a^2(t) = 2a - a^2 + 2\eta(1 - a)\Phi(t) - \eta^2\Phi^2(t) \quad (11)$$

For the sake of simplification, in view of the value of quadratic term $\eta^2\Phi^2(t)$ is so small that it can be neglected. Therefore, the higher order term $\eta^2\Phi^2(t)$ is omitted and only one-order term is retained, and the following results can be obtained,

$$2a(t) - a^2(t) = 2a - a^2 + 2\eta(1 - a)\Phi(t) \quad (12)$$

Substituting the simple expression of $2a(t) - a^2(t)$ in Eq. (12) into deterministic system (1), we can obtain Stratonovich stochastic system as

$$dI(t) = b ((1 - N(t)) - (2a - a^2)\lambda \bar{k}I(t)S(t) - \mu I(t)) dt - 2\eta(1 - a)\lambda \bar{k}I(t)S(t) \circ dW_1(t)$$

$$dS(t) = ((2a - a^2)\lambda \bar{k}I(t)S(t) - (2a - a^2)\sigma \bar{S}S(t) + R(t)) - \mu S(t) - \delta S(t)) dt + 2\eta(1 - a)\lambda \bar{k}S(t) \circ dW_2(t)$$

$$dR(t) = ((2a - a^2)\sigma \bar{S}S(t) + R(t)) - \mu R(t) + \delta S(t)) dt + 2\eta(1 - a)\sigma \bar{S}S(t) \circ dW_2(t)$$

(13)
With the following initial condition

\[ I(t_0) = I_0, \quad S(t_0) = S_0, \quad R(t_0) = R_0 \]

Here, \( W(t) \) is the standard winner process and symbol ‘\( \circ \)’ represents the Stratonovich integral [29]. For analyzing the behavior of the system (6), first we state following theorem:

**Theorem 1** ([27,29]). The \( N \) dimensional Stratonovich SDE with \( M \) dimensional Wiener process defined as

\[
dX = g(t, X)dt + \sum_{j=1}^{M} b^j(t, X) \circ dW^j(t)
\]

which has same solution as the \( \text{Itô} \) stochastic differential equation, such as the following expression,

\[
dX = a(t, X)dt + \sum_{j=1}^{M} b^j(t, X)dW^j(t)
\]

The above equation has drift coefficient \( a(t, X) \) that is defined in terms of \( g(t, X) \) componentwise as

\[
a^i(t, X) = g^i(t, X)dt + \frac{1}{2} \sum_{k=1}^{N} \sum_{j=1}^{M} b^{k, j}(t, X) \frac{\partial b^{j, k}(t, X)}{\partial x_k}(t, X)
\]

According to the Stratonovich \( \text{Itô} \) conversion given by above Theorem 1, combining with the formulas \( R(t) = N(t) - l(t) - S(t), \quad S(t) + R(t) = N(t) - R(t) \), the system (13) will convert into the following equivalent \( \text{Itô} \)'s equivalent

\[
dl(t) = \left( b(1 - N(t)) - (2a - a^2) \lambda \Xi(t)S(t) - \mu l(t) + \frac{1}{2} \left( 2\eta(1-a)\lambda \Xi(t) \right)^2 l(t)S(t)(S(t) - l(t)) \right) dt
\]

\[
dS(t) = \left( (2a - a^2) \lambda \Xi(t)S(t) - (2a - a^2) \sigma \Xi S(t)(S(t) + R(t)) - \mu S(t) - \delta S(t) + \frac{1}{2} \left( 2\eta(1-a)\sigma \Xi(t) \right)^2 S(t)(S(t) + R(t)) R(t) \right) dt
\]

\[
dR(t) = \left( (2a - a^2) \sigma \Xi S(t) ((S(t) + R(t)) - \mu R(t) + \delta S(t)) - \frac{1}{2} \left( 2\eta(1-a)\sigma \Xi(t) \right)^2 S(t)(S(t) + R(t)) R(t) \right) dt + 2\eta(1-a)\sigma \Xi(t) (S(t) + R(t)) dW_2(t)
\]

And it satisfies the initial condition of \( l(t_0) = l_0, \quad S(t_0) = S_0, \quad R(t_0) = R_0 \).

In the next section, we will commitment to the existence and uniqueness of stochastic solutions about system (14)-(16).

5. Existence and uniqueness of positive solution

In order to prove the existence and uniqueness of positive solutions about system (14)-(16), the following definitions are given in this paper.

**Definition 1.** In this paper, we introduce the function \( P(t, u) \in C_{1,2} \), which is a non-negative continuously differentiable function with respect to \( t \) and twice continuously differentiable with respect to \( u \). Let \( D_u \) be a domain defined as \( D_u = |u| < n \), then the differential operator \( L \) for the function \( P(t, u) \in C_{1,2} \) corresponding to the stochastic differential equation with drift and diffusion coefficient \( A(t, u) \) and \( B(t, u) \) respectively is given by,

\[
L_{P}(t, u) = \frac{\partial P(t, u)}{\partial t} + A^T \frac{\partial P(t, u)}{\partial u} + \frac{1}{2} \text{tr} \left[ B^T \frac{\partial^2 P(t, u)}{\partial u^2} B \right].
\]

where

\[
\frac{\partial P(t, u)}{\partial u} = \left( \frac{\partial P(t, u)}{\partial u_1}, \ldots, \frac{\partial P(t, u)}{\partial u_n} \right), \quad \frac{\partial^2 P(t, u)}{\partial u^2} = \left( \frac{\partial^2 P(t, u)}{\partial u_i \partial u_j} \right)_{i,j} , \quad i, j = 1, 2.
\]

In order eliminate the variable \( R \) in system (14)-(16), we replace it with \( R = N - l - S \) to obtain the simplified system in \( (l, S) \).

According to Theorem 3.5 given by Khasminskii et al. [30], we incorporate the following theorem for the simplified system in \( (l, S) \).

**Theorem 2.** Suppose that system (14)-(16) satisfies existence uniqueness theorem in every cylinder \( [a, b] \times U_R \) and a non-negative function \( P(t, u) \in C_{1,2} \) exists defined from \( [0, \Omega] \times R^+_t \rightarrow R^+_t \) such that \( L_{P} \leq cP \) for \( c > 0 \) and
Define function $P: [0, \Omega] \times \mathbb{R}^2_+ \rightarrow \mathbb{R}^+$ as follows,

$$P(I, S) = I - \ln I + S - \ln S$$

Then, according to Itô's formula, differential operator $L(P)$ can be given as follows,

$$L(P) = \left(1 - \frac{1}{I}\right) \left(b(1 - N(t)) - (2a - a^2)\lambda \overline{K}(t)S(t) - \mu I(t) - \frac{1}{2}(2\eta(1 - a)\lambda \overline{K})^2 I(t)S(t)(I(t) - S(t))\right)$$

$$+\left(1 - \frac{1}{S}\right) \left((2a - a^2)\lambda \overline{K}(t)S(t) - (2a - a^2)\sigma \overline{K}S(t)N(t) + I(t) - \mu S(t) - \delta S(t)\right)$$

$$+\left(1 - \frac{1}{S}\right) \left((2\eta(1 - a)\lambda \overline{K})^2 S(t)\{I(t) - S(t)\} \right)$$

Since $x - \ln x \geq 1$ for any $x$, which means $P \geq 2$ and $R = N - I - S \leq 1$. Therefore, $LP \leq cP$ for some positive constant $c$. It is easy to show that $P_n = \inf_{n\geq 0}P(I, S) \rightarrow \infty$, as $n \rightarrow \infty$. This completes the proof.

Applying Corollary 3.1 of Khasminskii [30], the solution remains in the domain $\Omega$.

According to the linearization of the system (14)–(16) in the vicinity of the rumor-free equilibrium $E^0$, we will find out the condition for the system (14)–(16) to evolve into endemic state.

To achieve this goal, we make the following transformation,

$$u = (u_1, u_2, u_3) = (I, S, R) - (I^0, 0, 0)$$

Applying the above transformation, we can get another non-linear form of (14)–(16) as

$$\begin{align*}
    d u_1(t) &= f_1(t, u) - 2\eta(1 - a)\lambda \overline{K}(t)u_1 + I^0u_2dW_1(t) \\
    d u_2(t) &= f_2(t, u) + 2\eta(1 - a)\lambda \overline{K}(t)u_1 + I^0u_2dW_1(t) - 2\eta(1 - a)\sigma \overline{K}u_2(u_2 + u_3)dW_2(t) \\
    d u_3(t) &= f_3(t, u) + 2\eta(1 - a)\sigma \overline{K}u_2(u_2 + u_3)dW_2(t)
\end{align*} \tag{17}$$

where,

$$\begin{align*}
    f_1(t, u) &= b(1 - N(t)) - (2a - a^2)\lambda \overline{K}(t)u_1 + I^0u_2 - \mu (u_1 + I^0) + \frac{1}{2}(2\eta(1 - a)\lambda \overline{K})^2 (u_1 + I^0)u_2(u_2 - u_1 - I^0) \\
    f_2(t, u) &= (2a - a^2)\lambda \overline{K}(t)u_1 + I^0u_2 - (2a - a^2)\sigma \overline{K}u_2(u_2 + u_3) - (\mu + \delta)u_2 \\
    &\quad + \frac{1}{2}(2\eta(1 - a)\lambda \overline{K})^2 (u_1 + I^2) u_2 (u_1 + I^2 - u_2) + \frac{1}{2}(2\eta(1 - a)\sigma \overline{K})^2 u_2u_3 (u_2 + u_3) \\
    f_3(t, u) &= (2a - a^2)\sigma \overline{K}u_2(u_2 + u_3) + \delta u_2 - \mu u_3 - \frac{1}{2}(2\eta(1 - a)\sigma \overline{K})^2 u_2u_3(u_2 + u_3) 
\end{align*}$$

The linearization of system (17) around $u = 0$ is equivalent to linearization of system (14)–(16) around rumor-free equilibrium $E^0$ which can be obtained as,

$$du = (A_1u + A_2)dt + BudW_1(t)u(t_0) = u_0 \tag{18}$$

where

$$A_1 = \begin{bmatrix}
    -\mu & -(2a - a^2)\lambda \overline{K}I^0 - \frac{1}{2}(2\eta(1 - a)\lambda \overline{K})^2 (I^0)^2 & 0 \\
    0 & (2a - a^2)\lambda \overline{K}I^0 - (\mu + \delta) + \frac{1}{2}(2\eta(1 - a)\lambda \overline{K})^2 (I^0)^2 & 0 \\
    0 & 0 & -\mu
\end{bmatrix}$$

$$A_2 = \begin{bmatrix}
    -\mu I^0 \\
    0 \\
    0
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
    0 & -2\eta(1 - a)\lambda \overline{K}I^0 & 0 \\
    0 & 2\eta(1 - a)\lambda \overline{K}I^0 & 0 \\
    0 & 0 & 0
\end{bmatrix}$$

Therefore, the expected value $E[u(t)]$ of Eq. (18) is as follows,

$$E[u(t)] = (E_{u_1}, E_{u_2}, E_{u_3})^T \tag{19}$$
where
\[
E_{a_1} = -\frac{b_1}{a_{11}} + e^{a_{11}x}c[1] + \frac{a_{12} (e^{a_{11}x} - e^{a_{22}x}) c[2]}{a_{11} - a_{22}}
\]
\[
E_{a_2} = e^{a_{22}x}c[2]
\]
\[
E_{a_3} = \frac{a_{32} (e^{a_{22}x} - e^{a_{33}x}) c[2]}{a_{22} - a_{33}} + e^{a_{33}x}c[3]
\]

Here, $c[i]$ are constants for each $i = 1, 2, 3$. The global stability criteria of rumor-free equilibrium for stochastic system (14)–(16) is given in Section 6.

6. Stability analysis of rumor-free equilibrium

Now, we are committed to the global stability [31] of the rumor-free equilibrium $E^0$ for the system (14)–(16). To avoid rumor disaster, inequality $a_{22} < 0$ must be satisfied.

\[
(2a - a^2)\lambda R^0 - (\mu + \delta) + \frac{1}{2} (2\eta(1 - a)\lambda R)^2 (I^0)^2 < 0
\]

After simplification, we can obtain the following inequality,

\[
(2a - a^2)\lambda R^0 + \frac{1}{2} (2\eta(1 - a)\lambda R)^2 \lambda^2 (I^0)^2 < \mu + \delta
\]

For convenience, $\Re_0^*$ is defined as the basic reproduction number as stochastic rumor system, as shown

\[
\Re_0^* = \frac{(2a - a^2)\lambda R^0}{\mu + \delta} + \frac{1}{2} \frac{(2\eta(1 - a)\lambda R)^2}{\mu + \delta} \lambda^2 (I^0)^2
\]

It is noticeable that the basic reproduction numbers of two versions has the following relations,

\[
\Re_0^* = \Re_0 + \frac{1}{2} \frac{(2\eta(1 - a)\lambda R)^2}{\mu + \delta} \lambda^2 (I^0)^2
\]

In view of Eq. (23), it is easy to know that $\Re_0^* \geq \Re_0$. Therefore, in presence of unwanted noise, the basic reproduction number of stochastic rumor model is always greater than that of deterministic rumor model. And then, we concentrate on the following definitions:

**Definition 2.** The solution to linearized system (18), $\dot{u}(t) = 0$, is said to be

i. r-stable ($r > 0$), if $\sup_{0 \leq t \leq h} (E |\dot{u}(t)|^r) \to 0$, as $n \to 0$ here $t \geq 0, h \geq 0$;

ii. asymptotically r-stable, if it is r-stable and $E (|\dot{u}(t)|^r) \to 0$ as $t \to \infty$;

iii. exponentially r-stable, if $E (|\dot{u}(t)|^r) \leq M (|u|^r) e^{-p(t-h)}$ for positive constants $M$ and $p$;

iv. that stability is in the mean if $p = 1$ and in mean square if $p = 2$.

By using **Definition 2** for the system Eq. (18), we can expand the following theorem:

**Theorem 3.** The solution $u(t) = 0$ to the linearized system (18) is globally asymptotically stable if $\Re_0^* < 1$ with Eq. (20) and unstable if $\Re_0^* > 1$.

**Proof.** If $\Re_0^* < 1$ with Eq. (20), then $\Re_0 < 1$ and on account of the expected value of $u(t)$ shown in Eq. (19), we can get $\lim_{t \to \infty} E (u(t)) \to 0$, which shows the global asymptotical stability of the system.

If $\Re_0 > 1$, then $\Re_0^* > 1$, which means $\lim_{t \to \infty} E (u(t)) \to \infty$. so, the system is unstable if $\Re_0^* > 1$. $\square$

In the case of $\Re_0^* = 1$, i.e.,

\[
1 - \frac{1}{2} \frac{(2\eta(1 - a)\lambda R)^2}{\mu + \delta} \lambda^2 (I^0)^2 \leq \Re_0 \leq 1
\]

Rumors will grow in the initial stage, which may be the reason for temporary spread. This also due to the influence of the noise intensity $\eta$ in the activity rate $a$.

In this case, $a_{22} > 0$ always holds, which implies $\lim_{t \to \infty} E (|u(t)|) \to \infty$. The situation is controllable when equation $c[1] = E (u_2(t_0)) = E (S(t_0)) = 0$ is satisfied. Therefore, we can establish the following theorem.

**Theorem 4.** If $\Re_0^* = 1$, the system (18) is globally asymptotically stable if and only if $c[1] = E (u_2(t_0)) = E (S(t_0)) = 0$. 
Next, we focus on proving the global stability of the rumor-free equilibrium $E^0$ and solution $u(t) = 0$ to the non-linearized systems (14)–(16) and (17), respectively. Thus, we propose the following theorem:

**Theorem 5.** The solution $u(t) = 0$ to the non-linearized system (17) with drift and diffusion $f(t, u)$ and $g(t, u)$ respectively, is globally asymptotically stable if the linearized system (18) with drift and diffusion coefficients $F(t, u)$ and $G(t, u)$ respectively, is globally asymptotically stable and

$$|f(t, u) - F(t, u)| + |g(t, u) - G(t, u)| < \varepsilon |u|$$

in a sufficiently small neighborhood of $u = 0$ and with a sufficiently small constant $\varepsilon$.

**Proof.** After selecting a small neighborhood around $u = 0$, and a sufficiently small $\zeta$ satisfied the condition $|u| < \zeta$, we can easily obtain that,

$$|f(t, u) - F(t, u)| + |g(t, u) - G(t, u)| = \sqrt{f_4^2 + f_5^2 + f_6^2 + \sqrt{g_4^2 + g_5^2 + g_6^2}} < \zeta |u|$$

for some positive constant $\zeta$, where

$$\begin{align*}
f_4 &= b (1 - N(t)) - (2a - a^2) \lambda k u_1 u_2 - \mu t^2 + \frac{1}{2} (2\eta(1 - a)\lambda \bar{k})^2 (u_1 u_2^2 - u_1^2 u_2 - 2u_1 u_2^0 + u_2^0) \\
f_5 &= (2a - a^2) \lambda k u_1 u_2 - (2a - a^2) \sigma k u_2 (u_2 + u_3) \\
&\quad + \frac{1}{2} (2\eta(1 - a)\lambda \bar{k})^2 (u_1^2 u_2 + 2u_1 u_2^0 - 2u_1 u_2 - u_2^0)^2 - \frac{1}{2} (2\eta(1 - a)\sigma \bar{k})^2 (u_2^2 u_3 + u_2 u_3^2) \\
f_6 &= (2a - a^2) \sigma k u_2 (u_2 + u_3) + \frac{1}{2} (2\eta(1 - a)\sigma \bar{k})^2 (u_2^2 u_3 + u_2 u_3^2) \\
g_4 &= -2\eta(1 - a)\lambda k u_1 u_2 \\
g_5 &= 2\eta(1 - a)\lambda k u_1 u_2 - 2\eta(1 - a)\sigma \bar{k} (u_2^2 + u_2 u_3) \\
g_6 &= 2\eta(1 - a)\sigma \bar{k} (u_2^2 + u_2 u_3)
\end{align*}$$

In the part of numerical simulation, we will illustrate the correctness of analytical results obtained in mathematical analysis and also compare the stochastic and the deterministic trajectories.
Fig. 3. Densities variation trajectories of ignorant, spreaders and stiflers for deterministic and stochastic model. (Fig. (a), (b), (c) for $\Re_0 < 1$, $\Re_0^* < 1$ and Fig. (d), (e), (f) for $\Re_0 > 1$, $\Re_0^* > 1$.)
Fig. 3. (continued).
Fig. 4. Densities variation trajectories of spreaders and stiflers for deterministic and stochastic model. (Fig. (a), (c) for $R_0 < 1, R_0^* < 1$ and Fig. (b), (d) for $R_0 > 1, R_0^* > 1$.)
7. Simulations

In this chapter, we will conduct numerical simulations of deterministic and stochastic models and demonstrate the validity of the analysis discussed in the previous section. In order to distinguish the effect of Gaussian white noise from noise-free on the asymptotic behavior of the model, the sample trajectories of the deterministic model and stochastic model are simulated. And then, the conclusions drawn in this paper are verified by setting appropriate parameters.

Fig. 2 show that the system presents bifurcations at $R_0 = 1$ and $R_0^* = 1$, which is consistent with the analysis conclusion. When $R_0 = 1$, the spreaders in stable state increase with the increase of $R_0$, that is, the density of spreaders at the equilibrium point $E^*$ increases gradually. By comparing Figs. 2(a) and 2(b), it can be seen that with the increase of $R_0$ and $R_0^*$, the density of spreaders in the deterministic model increases steadily, while the stochastic model oscillates upward, which fully reflects its randomness.

Fig. 3 shows the time evolution diagram of the density for different categories, Fig. 3(a, b, c) correspond to the case $R_0 < 1$ and $R_0^* < 1$, while Fig. 3(d, e, f) correspond to the case $R_0 > 1$ and $R_0^* > 1$. It can be seen from Fig. 3 that, after the introduction of random intervention, the trajectories of various groups for stochastic versions fluctuate around the trajectories of deterministic versions. Although the stochastic trajectories fluctuate wildly, they do not completely deviate completely from the deterministic trajectories. For the deterministic model, it can be seen from the solid red line in Fig. 3(a, d) that the ignorant gradually approach a stable constant after experiencing a series of peaks and valleys. And the stiflers oscillate upward from zero and then stabilize to a constant (see in Fig. 3(c, f)). At the same time, it can be found that no matter whether there is noise interference or not, as long as the basic reproduction number of the system is less than 1 (i.e. $R_0 < 1$ and $R_0^* < 1$, shown in Fig. 3(b)), the spreaders will gradually tend to be stable. It is bound to pass through several peaks and valleys and then stabilize at zero, which means that rumors eventually die out. On the contrary, if $R_0 > 1$ and $R_0^* > 1$, the spreaders tended to be a constant, and rumors will always exist and spread in the system (shown in Fig. 3(e)). Under the interference of Gaussian white noise, the individual’s gradual behavior will produce a large fluctuation. As Gaussian noise is a right continuous Brownian Process (or Wiener Process), the trajectories deviates from its corresponding deterministic model to some extent, which is consistent well with the theoretical results.

Fig. 4 depicts the time evolution trajectories of some groups in the system under different noise intensities, such as $\eta = 0, 0.1, 0.2, 0.3$. And Fig. 4(a, c) represents $R_0 < 1$ and $R_0^* < 1$, while Fig. 4(b, d) indicates $R_0 > 1$ and $R_0^* > 1$.

A comparison between Figs. 3 and 4 yields the following commonalities: no matter what the parameters are set, as long as $R_0 < 1$ and $R_0^* < 1$, rumor will disappear, however, when $R_0 > 1$ and $R_0^* > 1$, the rumor will be permanent. As can be seen from Fig. 4, the greater the noise intensity, the greater the amplitude of the shock, the greater the deviation degree from the trajectories of deterministic model, and the more difficult the rumor is to control.

8. Conclusions

This paper analyzed the deterministic and stochastic version of the dynamic mathematical model for rumors in homogeneous mixed groups of social networks considering the effect of activity. We deduced the basic reproduction numbers $R_0$ and $R_0^*$ defined by the Eqs. (8) and (22) respectively. Fig. 2 shows that the basic reproduction numbers $R_0 = 1$ and $R_0^* = 1$ are the critical condition to measure whether the rumor broke out. As shown in Fig. 3, if $R_0 < 1$ and $R_0^* < 1$, the rumor-free equilibrium is locally asymptotically stable, and no spreaders exist in the system. However, if $R_0 > 1$ and $R_0^* > 1$, rumors will erupt in social networks and exist in the system forever. At the same time, the global stability criterion for rumor-free equilibrium of stochastic versions model was obtained by mathematical calculation. In the presence of noise ($\eta$), if $R_0 < 1$ and condition (20) held, the rumor-free equilibrium is globally asymptotically stable.

The simulation results show that no matter the deterministic model or the stochastic model, as long as the basic reproduction number of the system is less than 1, the rumor spreaders will gradually stabilize at zero after a period of shock, and the rumor will die out in the system. When the basic reproduction number is greater than 1, the rumor spreaders will eventually tend to a stable constant, and the rumor will exist in the system forever. At the same time, we find that the evolution trajectory of the stochastic model fluctuates around the deterministic model at some extent. According to the simulations, we can conclude that the bigger the noise intensity is, the bigger the deviation of trajectory is, and the more difficult it is to control rumor.

CRediT authorship contribution statement

Yingying Cheng: Data curation, Writing - original draft, Visualization, Writing - review & editing, Investigation, Software, Validation. Liang’an Huo: Conceptualization, Methodology, Supervision, Writing - review & editing, Formal analysis, Funding acquisition, Investigation, Project administration. Laijun Zhao: Supervision, Writing - review & editing, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
Acknowledgments

This work was partially supported by the National Natural Science Foundation of China (71774111, 61702331), the project for Shanghai Municipal Government Development Research Center, China (2019-YJ-L04-B). The authors are very grateful to the anonymous referees for their valuable comments and suggestions, helping them to improve the quality of this paper.

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