su(1, 1) coherent states and a normal extension of su(1, 1) annihilation operator: squeezed states and the simultaneous measurement of \( Q^{-1}P + PQ^{-1} \) and \( Q^{-2} \)

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Abstract

The over-complete eigenvector system of the operator \( Q^{-1}P \) (\( Q \):position, \( P \):momentum) which consists of the squeezed states \(| 0; \mu, \nu \rangle \) with various \( \mu \) and \( \nu \) are investigated from the viewpoint of the annihilation and creation relations related to the algebra \( su(1,1) \). We derive a positive operator-valued measure(POVM) for the simultaneous measurement between the self-adjoint and anti-self-adjoint parts of \( PQ^{-1} \).

1 Introduction

As is well known, the eigenvectors of the linear combination of the boson annihilation and creation operators are the squeezed states[1], and the eigenvalue of the squeezed state \(| \alpha; \mu, \nu \rangle \) is a function both of the shift parameter \( \alpha \) and of the squeezing parameters \( \mu \) and \( \nu \). In this eigenvalue problem, because the definition of the operator itself depends on the squeezing parameters \( \mu \) and \( \nu \) (i.e. the coefficients of linear combination are these parameters), we can not investigate the estimation problem and the uncertainty relation only with respect to the squeezing parameters \( \mu \) and \( \nu \).

In this paper, for investigating them only with respect to the squeezing parameters, we discuss another type of eigenvector problem related to the squeezed states. The eigenvector system of the non-hermitian operator \( Q^{-1}P \) (\( Q \):position, \( P \):momentum) is the set of the the squeezed states with \( \alpha = 0 \), i.e. \( \{ | 0; \mu, \nu \rangle \mid \mu, \nu: \text{complex}, |\mu|^2 - |\nu|^2 = 1 \} \). The eigenvalue corresponding to the state \(| 0; \mu, \nu \rangle \) is \( i\frac{\mu + \nu}{\mu - \nu} \). Because states \(| 0; \mu, \nu \rangle \) and \(| 0; \mu', \nu' \rangle \) are the same states (with the phase difference neglected) in the case of \( i\frac{\mu + \nu}{\mu - \nu} = i\frac{\mu' + \nu'}{\mu' - \nu'} \), we may regard the eigenvalue \( i\frac{\mu + \nu}{\mu - \nu} \) as the squeezing parameter in the case of \( \alpha = 0 \). Because \( Q^{-1}P = (1/2)(Q^{-1}P + PQ^{-1}) - (i/2)Q^{-2} \), the squeezed states with \( \alpha = 0 \) are the minimum uncertainty states between \( Q^{-1}P + PQ^{-1} \) and \( Q^{-2} \).

These eigenvector relations are closely related to the Lie algebra \( su(1, 1) \), because the three generators of the displacement of the squeezing parameters satisfy the commutation relations of this algebra. The above squeezed states \( \{ | 0; \mu, \nu \rangle \mid \mu, \nu: \text{complex}, |\mu|^2 - |\nu|^2 = 1 \} \) are...
generalized coherent states associated with the Lie group generated by these generators. Its action on the set of squeezed states \( \{|0; \mu, \nu\} \mid \mu, \nu; \text{complex}, |\mu|^2 - |\nu|^2 = 1 \) is covariant with the natural action of \( \mathfrak{su}(1, 1) \) on the left half plane of the eigenvalue \( i\frac{\mu + \nu}{\mu - \nu} \). In addition, we can map this half plane into the unit circle by a Möbius transformation. By this mapping, we will derive the annihilation/creation relations and the number operators related to this algebra.

We will derive these relations from the algebraic structure of the \( \mathfrak{su}(1, 1) \) itself. The unitary representation of this algebra is related to the eigenvector problem of \( Q^{-1}P \), and this type of eigenvector problems can be commonly discussed for general \( \mathfrak{su}(1, 1) \). Moreover, we will show that the Möbius transform is corresponding to the change of the choice of the basis operators of the same algebra where one operator in the triplet is a linear function of the number operator. By using this choice, we will investigate the annihilation and creation relations related to \( \mathfrak{su}(1, 1) \). These relations are different from the boson annihilation and creation relations, because the product between the annihilation and creation operators is not a linear function but a kind of non-linear rational function of the number operator. From these relations, we will derive a non-linear type of reordering relation between the annihilation and creation operators. Moreover we derive the relations these annihilation and creation operators and \( \mathfrak{su}(1, 1) \)-coherent state.

Next we will discuss about the existence of a normal extension. If the overcomplete eigenvector system of \( A \) makes a pseudo-type (not a projection-type) of 'resolution of identity', then we can derive the Positive Operator-Valued Measure (POVM) for the generalized measurement which is the optimal to measure the non-hermitian operator \( A \) (i.e. to simultaneously measure the operators \( (A + A^*)/2 \) and \( (A - A^*)/2 \)). The general formalism proposed above for \( \mathfrak{su}(1, 1) \) has a systematic method for this under a condition on the minimum eigenvalue of a basis operator. This condition is not satisfied in the case of the eigenvalue problem of \( Q^{-1}P \). However, we will show that it is satisfied in the case of the eigenvalue problem of \( PQ^{-1} \).

2 Squeezed States and the eigenvalue problem with respect to the squeezing parameters

Let \( a_b := \sqrt{1/2}(Q + iP) \) (\( Q \):position, \( P \):momentum) be the boson annihilation operator, and let

\[
b_{\mu, \nu} := \mu a_b + \nu a_b^* (|\mu|^2 - |\nu|^2 = 1).
\]

Then, as is well known, this operator has complex eigenvalues and the eigenvector of \( b_{\mu, \nu} \) associated with the eigenvalue \( \alpha \) is the squeezed states vector \( |\alpha; \mu, \nu\rangle \). The eigenvalue \( \alpha \) indicates the center of the localization of the wave packet in the phase plane, while the coefficients \( \mu \) and \( \nu \) indicates the squeezing properties[1]. Therefore the set \( \{ |0; \mu, \nu\rangle \mid \mu, \nu; \text{complex} \) is the set of the squeezed states located around the origin of the phase plane with various squeezing parameters.

From

\[
b_{\mu, \nu}|0; \mu, \nu\rangle = 0,
\]

by operation \( Q^{-1} \) from the left, we have

\[
(\mu + \nu)|0; \mu, \nu\rangle - i(\mu - \nu)Q^{-1}P|0; \mu, \nu\rangle = 0,
\]
and hence

\[ Q^{-1}P|0; \mu, \nu\rangle = i \frac{\mu + \nu}{\mu - \nu} |0; \mu, \nu\rangle. \] (1)

This relation is another kind of characteristic equation of the squeezed states. The squeezed states with \( \alpha = 0 \) can be regarded as the eigenvectors of the operator \( Q^{-1}P \) associated with the eigenvalue \( i \frac{\mu + \nu}{\mu - \nu} \). This relation is very convenient for investigating the uncertainty relation and the quantum estimation problem only with respect to the squeezing parameters, because the operator \( Q^{-1}P \) itself does not but the eigenvalue does depend on \( \mu \) and \( \nu \).

The operator \( Q^{-1}P \) is not self-adjoint, and the self-adjoint part and the anti-self-adjoint part of \( Q^{-1}P \) are \( (Q^{-1}P + PQ^{-1})/2 \) and \( Q^{-2}/2 \), respectively. Therefore \( |0; \mu, \nu\rangle \) is a minimum-uncertainty state between \( Q^{-1}P + PQ^{-1} \) and \( Q^{-2} \) in the sense that the equality in the uncertainty relation

\[ (\Delta(Q^{-1}P + PQ^{-1}))^2 \cdot (\Delta(Q^{-2}))^2 = \frac{1}{4} |\langle \psi |[Q^{-1}P + PQ^{-1}, Q^{-2}]|\psi \rangle|^2 \] (2)

holds.

On the other hands, similar relations are shown for the eigenvectors of the operator \( PQ^{-1} \). In this case, the eigenvectors are not the usual squeezed states. However, in Sec.4, we will show that they are the vectors obtained by squeezing the one-boson state. Moreover, because \( PQ^{-1} \) is the adjoint of \( Q^{-1}P \), these vectors are the minimum uncertainty states between the same pair \( Q^{-1}P + PQ^{-1} \) and \( Q^{-2} \).

The eigenvalue problem \( Q^{-1}P \) and \( PQ^{-1} \) has the algebraic structures discussed in the following sections. In the next section, we will start with more general algebraic formalism.

### 3 Annihilation and Creation Relations Related to \( \mathfrak{su}(1, 1) \) and the corresponding \( \mathfrak{su}(1, 1) \) coherent states

The triplet of the anti-self-adjoint operators \( E_0, E_+ \) and \( E_- \) on a Hilbert space \( \mathcal{H} \) which satisfy the commutation relations,

\[ [E_0, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = E_0 \]

is called the unitary representation of the Lie algebra \( \mathfrak{su}(1, 1) \). This algebra is isomorphic to the Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \). From them, define another triplet of the operators \( L_0, L_+ \) and \( L_- \) by

\[ L_0 := i(E_- - E_+), \quad L_\pm := \frac{1}{2}(E_0 \pm i(E_+ + E_-)). \] (3)

Then the same type of commutation relations

\[ [L_0, L_\pm] = \pm 2L_\pm, \quad [L_+, L_-] = L_0 \] (4)

\footnote{The inequality (2) is concerning with the width of wave packet, not but measuring error.}
hold, and these are another basis of the same Lie algebra. However, in this basis system, $L_0$ is self-adjoint while $L_{\pm}$ are neither self-adjoint nor anti-self-adjoint, and $(L_{\pm})^* = -L_{\mp}$. From (3), $E_0$, $E_+$ and $E_-$ are written by $L_0$, $L_+$ and $L_-$, as

$$E_0 = L_+ + L_-, \quad E_{\pm} = \pm \frac{i}{2}(L_0 \mp L_+ \pm L_-).$$  \hspace{1cm} (5)$$

In this paper, we investigate only the cases where the representation is irreducible and is not trivial. Then the corresponding Casimir operator which should be a scalar by the Schur’s lemma,

$$C := L_0^2 + 2(L_+L_- + L_-L_+) = \beta$$ \hspace{1cm} (6)

where the parameter $\beta$ depends on the representation of the Lie algebra. From (4), this relation can be written in other forms

$$L_0^2 + 2L_0 + 4L_-L_+ = \beta, \quad L_0^2 - 2L_0 + 4L_+L_- = \beta.$$  \hspace{1cm} (7)

From (3) and (6),

$$E_0^2 + 2E_0 + 4E_-E_+ = \beta, \quad E_0^2 - 2E_0 + 4E_+E_- = \beta.$$ \hspace{1cm} (8)

From the commutation relation relations (3), we can show that if $v$ is the eigenvector of $L_0$ associated with the eigenvalue value $\kappa$ then $\kappa + 2$ is also its eigenvalue, by

$$L_0(L_+v) = L_+(L_0 + 2)v = (\kappa + 2)(L_+v)$$
$$L_0(L_-v) = L_-(L_0 - 2)v = (\kappa - 2)(L_-v).$$  \hspace{1cm} (9)

From this relation and the self-adjoint property of $L_0$, that the eigenvector system of $L_0$ is a orthogonal system and the eigenvalues of $L_0$ is uniformly spaced real numbers. From the irreducibility and the unitarity, it is easily shown that the dimension of the kernel of $L_{\pm}$ should be not more than one and it is impossible both of dim Ker $L_{\pm}$ are one simultaneously. From now, we are investigating the case where dim Ker $L_+$ = 0 and dim Ker $L_- = 1$. The opposite case can be reduced to this case by the change of the definition of $L_0$ and $L_{\pm}$ without loss of generality(We will not investigate the case where both of dim Ker $L_{\pm}$ are zero). Let $v_0$ be the unit vector in Ker $L_-$, from the relation (3), $v_0$ should be the eigenvector of $L_0$ associated with the minimum eigenvalue $\lambda$ (otherwise the existence of a smaller eigenvalue were contradictory to $L_-v_0 = 0$). Then the characteristic equation

$$L_0((L_+)^nv_0) = (\lambda + 2n)((L_+)^nv_0)$$ \hspace{1cm} (10)

holds. It is well known that this constant $\lambda$ uniquely determine the representation of the algebra $su(1,1)$. In this case, the unitarity means that $\lambda > 0$ [6]. Thus this representation space determined by the constant $\lambda$ is denoted by $H_\lambda$. From the irreducibility and the self-adjointness of $L_0$, we can show easily that $\{(L_+)^nv_0\}_{n=0}^\infty$ is a CONS of $H$. From the relation(4) and $L_-v_0 = 0$ and $L_0v_0 = \lambda v_0$, we have

$$\beta = \lambda(\lambda - 2)$$ \hspace{1cm} (11)
(NB: Similar relations does not hold for \(E_0\) and \(E_+\), because \(E_0\) is anti-self-adjoint.)

Now, we define the \(\mathfrak{su}(1,1)\) number operator \(N\) and \(\mathfrak{su}(1,1)\) number states vector \(|n\rangle_N\) by

\[
N := \frac{1}{2} (L_0 - \lambda) \\
|n\rangle_N := \sqrt{\frac{\Gamma(\lambda)}{n! \Gamma(\lambda + n)}} L_+^n v_0,
\]

then, from (11) we have \(N|n\rangle_N = n|n\rangle_N\), where \(|x\rangle_X\) means the unit eigen vector of an operator \(X\) associated with an eigenvalue \(x\). We can show the right hand side of (13) is an unit vector from the relation

\[
\langle L_+^n v_0, L_+^n v_0 \rangle = -\langle L_+^{n-1} v_0, (L_- L_+) L_+^{n-1} v_0 \rangle = -\frac{1}{4} \langle L_+^{n-1} v_0, (\beta - L_0^2 - 2L_0) L_+^{n-1} v_0 \rangle = n(\lambda + n - 1) \langle L_+^{n-1} v_0, L_+^{n-1} v_0 \rangle
\]

where we utilize (7) with (11).

Next we will define the \(\mathfrak{su}(1,1)\) annihilation operator \(a\). From the relation (10), we can prove that \((L_0 - \lambda)|n\rangle_N\) belongs to the range of \(L_+\) for any \(n\). Because \(\text{dim Ker}(L_+) = 0\), we can define \(\frac{1}{2} L_+^{-1}(L_0 - \lambda)|n\rangle_N\). Therefore, we can define that the \(\mathfrak{su}(1,1)\) annihilation operator

\[
a := \frac{1}{2} L_+^{-1}(L_0 - \lambda) \quad (14)
\]

on the dense subset \(\left\{ \sum_{n=0}^{\infty} x_n |n\rangle_N \right\} \subset \mathcal{H} \left\{ \sum_{m=0}^{\infty} |m\rangle_N |\sum_{n=0}^{\infty} x_n \langle n|_N, \frac{1}{2} L_+^{-1}(L_0 - \lambda)|n\rangle_N \right\} < \infty \}. With the unitary displacement operator

\[
D(\xi) := \exp \left( \xi L_+ - \bar{\xi} L_+^* \right) \quad (15)
\]

define the \(\mathfrak{su}(1,1)\) coherent state

\[
v(\zeta) := D \left( \frac{1}{2} e^{i \text{arg} \zeta} \ln \frac{1 + |\zeta|}{1 - |\zeta|} \right) |0\rangle_N = \exp (\zeta L_+) \exp \left( \frac{1}{2} \ln (1 - |\zeta|^2) L_0 \right) \exp (\bar{\zeta} L_-) |0\rangle_N, \quad (|\zeta| < 1) \quad (16)
\]

(The latter “normal-order” form of the right hand side is obtained from the relation given in pp.73-74 of Ref[2], with the correspondences \(K_0 = L_0/2\), \(K_+ = L_+\) and \(K_- = -L_-\).) Then, from \(K_0|0\rangle_N = 0\) and \(K_0|0\rangle_N = (\lambda/2)|0\rangle_N\),

\[
\exp \left( \frac{1}{2} \ln (1 - |\zeta|^2) L_0 \right) \exp (\bar{\zeta} L_-) |0\rangle_N = (1 - |\zeta|^2)^{\lambda/2} |0\rangle_N.
\]

Because \([a, L_+] = 1/2 L_+^{-1}[L_0, L_+] = 1\),

\[
[a, \exp (\zeta L_+)] = \zeta \exp (\zeta L_+) \quad (18)
\]
These relations show that the vector $v(\zeta)$ is the eigenvector of $a$ associated with the eigenvalue $\zeta$. Thus we can denote $v(\zeta)$ by $|\zeta\rangle$.

Next, we investigate the annihilation and creation relations. Then, from the (3) and (14),

$$[a, N] = \frac{1}{4}[L^+_1, L_0](L_0 - \lambda) = \frac{1}{2}L^{-1}_1(L_0 - \lambda) = a,$$

this implies that the operator $a$ is the annihilation operator of the eigenvector system of $N$. Because we get $aL^+_1|0\rangle_N = nL^+_1|0\rangle_N$ from (10) and (14), by (13), we have the annihilation and creation relations

$$a|n\rangle_N = \sqrt{\frac{n}{n + \lambda - 1}}|n - 1\rangle_N \quad \text{and} \quad a^*|n\rangle_N = \sqrt{\frac{n + 1}{n + \lambda}}|n + 1\rangle_N,$$

where the first equation of (20) means that $a|0\rangle_N = 0$ in the case of $\lambda = 1, n = 0$. Therefore, from the completeness and orthogonality of the eigenvectors of $L_0$, we get the following relations in the case of $\lambda \neq 1$

$$a^*a = (N + \lambda - 1)^{-1}N \quad \text{and} \quad aa^* = (N + \lambda)^{-1}(N + 1).$$

By eliminating $N$ from these, we have the re-ordering relation between the annihilation $a$ and the creator $a^*$,

$$aa^* = -(a^*a + \lambda - 2)^{-1}(\lambda a^*a - 1) \quad \text{and} \quad a^*a = (aa^* - \lambda)^{-1}((2 - \lambda)aa^* - 1).$$

In the case of $\lambda = 1$, we have $aa^* = 1, \quad a^*a = 1 - |0\rangle_N N|0\rangle$. These relations are important for the calculating the quantum characteristic function.

From the relations (3), (4), (7) with (11) and the relation $[L_0, L^{-1}_1] = L^{-1}_1[L_+, L_0]L^{-1}_1 = -2L^{-1}_1$, we can show the following relation

$$2(E_0 - \lambda)(a - 1)L_+ = (L_+ + L_- - \lambda)(L_0 - (\lambda - 2) - 2L_+)$$

$$= \lambda(\lambda - 2) + \lambda(-L_+ - L_- - L_0 + 2L_+)$$

$$+ (L_+L_0 + L_-L_0 - 2L_+L_- - 2L_-L_+ + 2L_+ + 2L_-)$$

$$= - \lambda(L_0 - L_+ + L_-) + L_0L_0 + 2L_+ + 2L_-L_+$$

$$+ (2L_0L_+ - 4L_- - L_+L_0) + L_-L_0 - 2L_+L_- + 2L_+ + 2L_-$$

$$= (L_0 - L_+ + L_-)(-\lambda + L_0 + 2 + 2L_+)$$

$$= -2iE_+ (L_0 - (\lambda - 2) + 2L_+) = -4iE_+(a + 1)L_+$$

$$(E_0 - \lambda)(a - 1)|0\rangle_N = -(E_0 - \lambda)|0\rangle_N = (L_0 - L_+)|0\rangle_N = -2iE_+|0\rangle_N = -2iE_+(a + 1)|0\rangle_N.$$

Hence we get

$$(E_0 - \lambda)(a - 1) = -2iE_+(a + 1).$$

(22)
If the dimension of the kernel of $E_+$ is not 0, then this representation becomes trivial by the relation (8) and the unitarity. Using the relation (20) and (22), we can show that $(E_0 - \lambda)|n\rangle_N$ belongs to the range of $E_+$. Therefore, we can define the operator $A$ like the case of $a$

$$A := \frac{1}{2}E_+^{-1}(E_0 - \lambda).$$

(23)

Hence, we have

$$A = -i(a+1)(a-1)^{-1},$$

$$A|\zeta\rangle_a = -i\frac{\zeta + 1}{\zeta - 1}|\zeta\rangle_a.$$  

(24)  

(25)

Thus the vector $|\zeta\rangle_a$ can be denoted by $\left| -i\frac{\zeta + 1}{\zeta - 1} \right\rangle_A$. Since the imaginary number $-i$ doesn’t belong to the spectrum of $A$, the relation (24) indicates that

$$a = (A + i)^{-1}(A - i).$$

The operator $A$ has a similar property to (19). Define the ‘normal-ordered affine coherent vector’ as

$$v'_n(s, t) := \exp(sE_+)\exp(tE_0)|0\rangle_n.$$  

(26)

Then, because $[A, E_+] = (1/2)E_+^{-1}[E_0, E_+] = 1$, we have

$$A\exp(sE_+) = \exp(sE_+)A + s\exp(sE_+),$$

$$A\exp(tE_0) = e^{2t}\exp(tE_0)A.$$  

(27)  

(28)

From the relations (23), (26)-(28) and (23) with $\zeta = 0$,

$$Av'_n(s, t) = e^{2t}\exp(sE_+)\exp(tE_0)A|0\rangle_n + s\exp(sE_+)\exp(tE_0)|0\rangle_n = (e^{2t}i + s)v'_n(s, t).$$

We have similar relations under other orderings. Define the ‘anti-normal-ordered affine coherent vector’ as

$$v'_a(s, t) := \exp(tE_0)\exp(sE_+)|0\rangle_n.$$  

(29)

Then, from (23), (27), (28), (29) and (23) with $\zeta = 0$,

$$Av'_a(s, t) = e^{2t}\exp(tE_0)\exp(sE_+)A|0\rangle_n + se^{2t}\exp(tE_0)\exp(sE_+)|0\rangle_n = e^{2t}(i + s)v'_a(s, t).$$  

(30)

These relations show that the vectors $v'_n(s, t)$ and $v'_a(s, t)$ are the eigenvectors of $A$ associated with the eigenvalue $e^{2t}i + s$ and $e^{2t}(i + s)$, respectively. Thus the vectors $v'_n(\text{Re } \eta, 1/2 \ln \text{Im } \eta)$ and $v'_a(\text{Re } \eta)/(\text{Im } \eta), 1/2 \ln \text{Im } \eta)$ are equivalent to $|\eta\rangle_A$ with the phase difference neglected. Note that the operators $\exp(sE_+)$ and $\exp(tE_0)$ are unitary because of $E_0$ and $E_+$ are anti-self-adjoint.
From the viewpoint of the theory of \( \mathfrak{su}(1, 1) \) coherent states\[2\], when \( \lambda > 1 \), these relations implies that the resolutions of unity

\[
\int_U |\zeta_\omega \rangle \langle \zeta_\omega | \mu_\lambda (d\zeta) = I \quad \text{with} \quad \mu_\lambda (d\zeta) := \frac{\lambda - 1}{\pi} \frac{d^2\zeta}{(1 - |\zeta|^2)^2} \tag{31}
\]

\[
\int_H |\eta_\omega \rangle \langle \eta_\omega | \nu_\lambda (d\eta) = I \quad \text{with} \quad \nu_\lambda (d\eta) := \frac{\lambda - 1}{4\pi} \frac{d^2\eta}{(\text{Im} \eta)^2},
\]

where \( U (H) \) denotes the inside of the unit circle (the upper half plane), respectively. However, when \( \lambda < 1 \), these relations are impossible.

4 Squeezed States as the Eigenstates of \( Q^{-1}P \) and ’Odd Squeezed States’ as those of \( PQ^{-1} \)

In this section, we will interpret the eigenfunction problem (1) from the algebraic structure discussed in Sec. 3. For this, we have only to choose the representation

\[
E_0 = i(PQ +QP)/2, \quad E_+ = iQ^2/2, \quad E_- = -iP^2/2.
\]

Then from (3) we get the following relations

\[
L_0 = n_b + \frac{1}{2}, \quad L_+ = -(1/2)a_b^{*2}, \quad L_- = (1/2)a_b^2, \tag{32}
\]

where \( n_b := 1/2(Q^2 + P^2 - 1) = a_b a_b^* \) is the boson number operator and \( a_b \) is the boson annihilation operator defined in Sec. 2. The Casimir operator \( C \) is represented as

\[
\beta = C = -(PQ +QP)^2/4 + (Q^2P^2 + P^2Q^2)/2 = -\frac{3}{4}.
\]

From (11) we have \( \lambda = 1/2 \) or \( 3/2 \). These two solutions are corresponding to the two function spaces of the representation. When we choose the function space with even parity

\[
L_{\text{even}}^2 := \{f(q) \in L^2(R)|f(-q) = f(q)\},
\]

then \( \lambda = 1/2 \). Therefore, there is no POVM defined in (11). From (14), (23), (12) and (32), we have

\[
A = \frac{1}{2}Q^{-2}(PQ +QP +i) = Q^{-2}QP = Q^{-1}P,
\]

\[
a = -a_b^{*2}n_b = a_b^{*2}a_b a_b = a_b^{*1}a_b
\]

\[
N = \frac{1}{4}(Q^2 + P^2 - 1) = \frac{1}{2}n_b.
\]

Then, from (1), (13) and (32),

\[
|0; \mu, \nu \rangle = \frac{\mu + \nu}{\mu - \nu} A, \quad |n\rangle_N = (-1)^n |2n\rangle_{n_b},
\]

\[
|0; \mu, \nu \rangle = \frac{\mu + \nu}{\mu - \nu} A, \quad |n\rangle_N = (-1)^n |2n\rangle_{n_b},
\]

8
where $|n\rangle_{nb}$ denotes the boson number state. From (20) and (21), we have the annihilation and creation relations

$$a|n\rangle_N = \sqrt{\frac{2n}{2n-1}} |n-1\rangle_N, \quad a^*|n\rangle_N = \sqrt{\frac{2n+2}{2n+1}} |n+1\rangle_N$$

and the non-linear re-ordering relations

$$aa^* = -(2a^*a - 3)^{-1} (a^*a - 2) \quad \text{and} \quad a^*a = (2aa^* - 1)^{-1} (3aa^* - 2).$$

On the other hand, when we choose the function space with odd parity

$$L^2_{\text{odd}} := \{ f(q) \in L^2(R) | f(-q) = -f(q) \},$$

then $\lambda = 3/2$. There is the POVM defined as (31). We have

$$A = \frac{1}{2} Q^{-2} (PQ +QP + 3i) = Q^{-2} (QP + i) = PQ^{-1},$$

$$a = -a_b^{-2} (n_b - 1) = a_b^{-2} (a_b^* a_b - 1) = a_b a_b^*$$

and hence, from (13) and (32),

$$|n\rangle_N = (-1)^n |2n + 1\rangle_{nb},$$

In this case the $\text{su}(1,1)$ coherent state is the eigen state of the ‘squeezed number’ operator $D(\xi) n_b D(\xi)^*$ associated with the eigenvalue 1. From the relations (26), (12), (33) and (34) with $n = 0$, we can show that the eigenvector of $PQ^{-1}$ is obtained by squeezing the one-boson state, as

$$|2p + e^{4qi}\rangle_{PQ^{-1}} = \exp(ipQ^2) \exp(iq(PQ + QP)) |1\rangle_{nb}.$$ 

Hence we call it the ‘odd squeezed state’ for convenience in this paper. In this ‘odd’ case, the annihilation and creation relations and the non-linear re-ordering relations are

$$a|n\rangle_N = \sqrt{\frac{2n}{2n+1}} |n-1\rangle_N \quad \text{and} \quad a^*|n\rangle_N = \sqrt{\frac{2n+2}{2n+3}} |n+1\rangle_N$$

$$aa^* = -(2a^*a - 1)^{-1} (3a^*a - 2) \quad \text{and} \quad a^*a = (2aa^* - 3)^{-1} (aa^* - 2).$$

5 Relation to the Cauchy Wavelets

It is known that the system of the wavefunctions of the eigenvectors of the operator $Q-iKP^{-1}$ ($k$: positive real number) is the wavelet system[3] of the Cauchy wavelets i.e. \( \{ \sqrt{\frac{1}{|q|}} h_k(\frac{a-b}{s}) | (a,b) \in (\text{upper} - \text{half plane}) \} \) with $h_k(q) := \frac{\text{(const.)}}{(q+a)^{k+1}}$ [4,5]. This eigenvalue problem also has the same algebraic structure discussed in the previous sections. In this case, we choose

$$E_0 = -i(PQ + QP), \quad E_+ = iP, \quad E_- = -i(QPQ + k^2P^{-1}),$$ (36)
and then

\[ L_0 = (1/2)(QPQ + k^2P^{-1} + P), \]
\[ L_\pm = \pm(i/2)(QPQ \mp (PQ + QP) + k^2P^{-1} - P), \]

and \( \beta = 4k^2 - 1 \). Now we choose the function space \( L^2(\mathbb{R}^+) \) in the momentum representation, which is irreducible. The vector \( H_k \) defined in (37) belongs to the kernel of \( L_- \)

\[ H_k(p) := \frac{1}{\sqrt{2^{2k+1}\Gamma(2k+1)}} \cdot p^k e^{-p}. \] (37)

From a simple calculation, the eigen value of \( L_0 \) for \( H_k \) is \( \lambda = 2k + 1 \). And we get

\[ A = - P^{-1}(PQ + QP - (2k + 1)i) = -2(Q - ikP^{-1}) \]
\[ N = (QPQ + k^2P^{-1} + P - 2k - 1)/2. \]

In the position representation, the corresponding function space is \( \{ f(q) \in L^2 \mid f_H(q) = if(q) \} (f_H : \text{Hilbert trans. of } f) \) or the space of normalizable analytic signals in signal analysis, and the eigenfunction of the vacuum vector is the basic wavelet \( h_k(q) := \frac{(const.)}{(q+i)^{k+1}}. \) For these, we have the similar annihilation and creation relations and the non-linear reordered relations[5]

\[ a|n\rangle_N = \sqrt{\frac{n}{n+2k}} |n-1\rangle_N \quad \text{and} \quad a^*|n\rangle_N = \sqrt{\frac{n+1}{n+2k+1}} |n+1\rangle_N \]
\[ aa^* = -(a^*a + (2k - 1))^{-1}((2k + 1)a^*a - 1) \]
\[ a^*a = -(aa^* + (2k + 1))^{-1}((2k - 1)aa^* - 1). \]

6 Normal extension of the annihilation operator \( a \) and POVM for simultaneous measurement of \( PQ^{-1} + PQ^{-1} \) and \( Q^{-2} \)

In this section, we consider a normal extension of \( \mathfrak{su}(1, 1) \) annihilation operator \( a \). In this case, it is sufficient to construct a normal extension of \( A \). Let \( Z \) be a operator on \( \mathcal{H} \) satisfying

\[ [Z, Z^*] \geq 0. \] (38)

If a triplet of a Hilbert space \( \mathcal{H}' \), a state \( \phi \in \mathcal{H}' \) and a normal operator \( \tilde{Z} \) on \( \mathcal{H} \otimes \mathcal{H}' \) satisfies the following relations (39) and (40) for any density operator \( \rho \) on \( \mathcal{H} \), the triplet \( (\mathcal{H}', \phi, \tilde{Z}) \) is called a normal extension of the operator \( Z \)

\[ \text{tr} \tilde{Z} \tilde{Z}^* \rho \otimes |\phi\rangle\langle \phi | = \text{tr} ZZ^* \rho \]
\[ \text{tr} \tilde{Z} \rho \otimes |\phi\rangle\langle \phi | = \text{tr} Z \rho. \] (39) (40)

Not satisfying the condition (38), there is no normal extension of \( Z \). But, the converse is not true. Now, we consider the relation between a normal extension of \( Z \) and a simultaneous measurement of \( X \) and \( Y \), where \( X \) and \( Y \) are self-adjoint operators such that \( Z = X + iY \).
If a POVM \( M \) satisfies the condition \((11)\), the POVM \( M \) is called a simultaneous measurement between \( X \) and \( Y \) \[^8\]

\[
Z = \int_\mathbb{C} zM(dz). \tag{41}
\]

If a POVM \( M \) satisfies the condition \((11)\), then we have the lower bound of the second moment for any density operator \( \rho \) as

\[
\int_\mathbb{C} |z|^2 \text{tr} \rho M(dz) \geq \text{tr} \rho ZZ^*. \tag{42}
\]

The inequality \((12)\) is derived by the following. Using the relation \((11)\), we have

\[
0 \leq \int_\mathbb{C} (Z - z)M(dz)(Z - z)^* = \int_\mathbb{C} |z|^2 M(dz) - ZZ^*. \tag{43}
\]

The inequality \((12)\) can be derived from the inequality \((13)\). Let \( \tilde{Z} \) be the spectral measure of \( Z \). A POVM (Positive Operator-Valued Measure) \( \tilde{M}_Z \) on \( \mathcal{H} \) is defined as

\[
\tilde{M}_Z(dz) := \text{tr}_{\mathcal{H}} \ I \otimes |\phi\rangle \langle \phi| \tilde{E}_z(dz), \tag{44}
\]

where \( \text{tr}_{\mathcal{H}} \) denotes the partial trace with respect to \( \mathcal{H}' \). From \((39)\) and \((40)\), the POVM \( \tilde{M}_Z(dz) \) satisfies the condition \((11)\), and attains the lower bound of \((12)\). Therefore it is the optimal simultaneous measurement between \( X \) and \( Y \) in the sense of the second moment. Conversely, It can be shown that if there is a POVM attaining the lower bound of \((12)\), there is a normal extension.

Next, the commutator of \( A \) and \( A^* \) is calculated as

\[
[A, A^*] = \frac{1}{4} \left[ E_+^{-1}(E_0 - \lambda), (E_0 + \lambda)E_+^{-1} \right]
\]

\[
= \frac{1}{4} E_+^{-1}(E_0 + \lambda) \left[ E_0, E_+^{-1} \right] + \frac{1}{4} \left[ E_+^{-1}, E_0 \right] E_+^{-1}(E_0 - \lambda)
\]

\[
= \frac{1}{2} E_+^{-1} \left[ E_0^{-1}, E_0 \right] - \lambda E_+^{-2} = -(\lambda - 1)E_+^{-2} = (\lambda - 1)E_+^{-1}(E_+^{-1})^*. \]

Since \( E_+^{-1}(E_+^{-1})^* \geq 0 \), there may be a normal extension of \( A \) in the case of \( \lambda > 1 \), and there may be one of \( A^* \) in the case of \( 0 < \lambda < 1 \). Remark that \( E_+^{-1}(E_0 - 1) \) is a symmetric operator. Now, we construct a normal extension \((\mathcal{H}', \phi, \tilde{A})\) of \( A \) in the case of \( \lambda > 1 \) as the following method. Let \( \tilde{A} \) defined as \( \tilde{A} := A \otimes I' + E_+^{-1} \otimes F' \). Since \( [A, E_+^{-1}] = (1/2)E_+^{-1}[E_0, E_+^{-1}] = -E_+^{-2} \), we have

\[
\left[ \tilde{A}, \tilde{A}^* \right] = [A, A^*] \otimes I' - [A, E_+^{-1}] \otimes F'' + [E_+^{-1}, A^*] \otimes F' - E_+^{-2} \otimes [F', F'']
\]

\[
= E_+^{-2} \otimes (-(\lambda - 1) + F' + F'' - [F', F'']). \tag{45}
\]

From \((15)\), the normality condition of \( \tilde{A} \) means

\[
\left[ F' - F'', \frac{-i}{2} (F'' + F' - (\lambda - 1)) \right] = 2 \left( \frac{-i}{2} (F'' + F' - (\lambda - 1)) \right).
\]
Remark that the operators $F'^* - F'$ and $\frac{-i}{2} (F'^* + F' - (\lambda - 1))$ are anti-self-adjoint. This relation is the commutation relation of $E_0$ and $E_+$ of the Lie algebra $\mathfrak{su}(1,1)$. Let a triplet $E_0', E_+'$ and $E_-'$ be a unitary and irreducible representation of $\mathfrak{su}(1,1)$ on $\mathcal{H}'$. Choosing

$$F' - F'^* = E_0' \quad \text{and} \quad \frac{-i}{2} (F'^* + F' - (\lambda - 1)) = E_+',$$

$\tilde{A}$ satisfies the normality condition. Let

$$L'_0 := i(E_0' - E_+'), \quad L'_+ := \frac{1}{2}(E_0' \pm i(E_+ + E_-')),$$

then the operators $F'$ and $F'^*$ are calculated as

$$F' = -\frac{1}{2} (L'_0 - (\lambda - 1)) + L'_+, \quad F'^* = -\frac{1}{2} (L'_0 - (\lambda - 1)) - L'_-.$$

Next we consider the conditions (39) and (40). The conditions (39) and (40) mean that

$$\langle \phi, F'^* \phi \rangle \text{tr} A^* E_{+1}' \rho - \langle \phi, F' \phi \rangle \text{tr} E_{+1}' A \rho - \langle \phi, F' F'^* \phi \rangle \text{tr} E_{+2}' \rho = 0, \quad (46)$$

$$\langle \phi, F' \phi \rangle \text{tr} E_{+1}' \rho = 0. \quad (47)$$

If $F'^* \phi = 0$, then the conditions (46) and (47) are satisfied. If the lowest eigenvalue of $L'_0$ is $\lambda - 1$, then the $\mathfrak{su}(1,1)$ vacuum state $|0\rangle_{N'}$ belongs to the kernel of $F$. Therefore we proved that the triplet $(\mathcal{H}_{\lambda-1}, |0\rangle_{N'}, A \otimes I + E_{+1}' \otimes (-\frac{1}{2} (L'_0 - (\lambda - 1)) + L'_+))$ is a normal extension of $A$. In addition, the POVM $|\eta\rangle_A \langle \eta| \nu_{\lambda}(d\eta)$ is constructed by this normal extension.

Moreover a normal extension of the $\mathfrak{su}(1,1)$ annihilation operator is given by the triplet $(\mathcal{H}_{\lambda-1}, |0\rangle_{N'}, \tilde{a})$, where $\tilde{a}$ is defined as

$$\tilde{a} := \left( (A - i) \otimes I + E_{+1}' \otimes \left(-\frac{1}{2} (L'_0 - (\lambda - 1)) + L'_+ \right) \right)$$

$$\cdot \left( (A + i) \otimes I + E_{+1}' \otimes \left(-\frac{1}{2} (L'_0 - (\lambda - 1)) + L'_+ \right) \right)^{-1}.$$

Next, we consider the case of $0 < \lambda < 1$. In this case, we can similarly prove that the triplet $(\mathcal{H}_{\lambda-1}, |0\rangle_{N'}, A^* \otimes I' + E_{+1}' \otimes (-\frac{1}{2} (L'_0 - (1 - \lambda)) + L'_+))$ is a normal extension of $A^*$.

Now we apply these normal extensions to a simultaneous measurement of $(Q^{-1}P + PQ^{-1})/2$ and $Q^{-2}$ in the boson fock space $L^2(\mathbb{R})$. Since $PQ^{-1} = (Q^{-1}P + PQ^{-1})/2 + iQ^{-2}$, we construct normal extension of $PQ^{-1}$. In the even functions space $L^2_{\text{even}}$ we have $A^* = PQ^{-1}, \quad \lambda = 1/2$, and in the odd functions space $L^2_{\text{odd}}$ we have $A = PQ^{-1}, \quad \lambda = 3/2$. Therefore, we can show that the triplet $(\mathcal{H}_{1/2}, |0\rangle_{N'}, PQ^{-1} \otimes I' + E_{+1}' \otimes (-\frac{1}{2} (L'_0 - 1/2)) + L'_+))$ is a normal extension of $PQ^{-1}$ in $L^2(\mathbb{R})$. Let $\mathcal{H}_{1/2}$ be the even functions space in $L^2(\mathbb{R})$, then the normal extension of $PQ^{-1}$ is rewritten by $(L^2, |0\rangle_{n_b}, PQ^{-1} \otimes I' - \frac{i}{2} Q^{-2} \otimes (n'_b + (a^*_b)^2))$.

### 7 Conclusions

We have investigated the squeezed states with various squeezing parameters from the viewpoint of the eigenvector system of a kind of annihilation operator related to the algebra $\mathfrak{su}(1,1)$. It
turned out that the annihilation and creation relations for this have non-linear properties, which
is different from those of the boson case. By utilizing these algebraic structures, we derive an
optimal POVM for the simultaneous measurement for $Q^{-1}P + PQ^{-1}$ and $Q^{-2}$. We will continue
more investigations of wider class of similar eigenvector systems based on this type of algebra.

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