Learning Dictionaries with Bounded Self-Coherence

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Abstract—Sparse coding in learned dictionaries has been established as a successful approach for signal denoising, source separation and solving inverse problems in general. A dictionary learning method adapts an initial dictionary to a particular signal class by iteratively computing an approximate factorization of a training data matrix into a dictionary and a sparse coding matrix. The learned dictionary is characterized by two properties: the coherence of the dictionary to observations of the signal class, and the self-coherence of the dictionary atoms. A high coherence to the signal class enables the sparse coding of signal observations with a small approximation error, while a low self-coherence of the atoms guarantees atom recovery and a more rapid residual error decay rate for the sparse coding algorithm. The two goals of high signal coherence and low self-coherence are typically in conflict, therefore one seeks a trade-off between them, depending on the application. We present a dictionary learning method with an effective control over the self-coherence of the trained dictionary, enabling a trade-off between maximizing the sparsity of codings and approximating an equiangular tight frame.

I. INTRODUCTION

Dictionary learning adapts an initial dictionary to a particular signal class with the help of training observations, such that further observations from that class can be sparsely coded in the trained dictionary with low approximation error. Over-complete dictionaries, consisting of more atoms than dimensions of the feature space, typically support sparser codings by placing more atoms in densely populated regions of the feature space. However, this redundancy increases the self-coherence of the dictionary, i.e. the pairwise similarity of dictionary atoms, as measured by the cosine of the angle between atom pairs. A lower self-coherence permits better support recovery [2] and a more rapid decay of the residual norm when increasing the coding cardinality [12]. Furthermore, bounding the admissible coding cardinality during training can increase the generalization performance of the dictionary, by avoiding over-fitting to the training data and by avoiding atom degeneracy, i.e. two atoms collapsing onto the same vector.

We present a dictionary learning algorithm called IDL($\gamma$), which enables an effective control over the self-coherence of trained dictionaries. Our method is able to span the full spectrum of optimization objectives, from maximizing the sparsity of the resulting codings, to approximating an equiangular tight frame (ETF), which is a dictionary achieving minimal self-coherence for a given number of atoms. We demonstrate the benefits of limiting the self-coherence of the dictionary in terms of better coding support recovery and improved generalization performance (see Sec. III).

A. From Bases to Over-Complete Dictionaries

An orthonormal basis $B \in \mathbb{R}^{D \times D}$ contains $D$ mutually orthogonal unit $\ell_2$ norm atoms spanning the feature space $\mathbb{R}^D$. The unique code $c \in \mathbb{R}^D$ of an observation $x \in \mathbb{R}^D$ is computed by $c = B^T x$ (signal analysis), and the signal is recovered from the code by $x = Bc$ (signal synthesis). The Gram matrix $G = B^T B = I$ of $B$ is the identity matrix.

Although natural signals are approximately sparse in suitably chosen bases, typically a sparser code can be achieved using an over-complete dictionary $D \in \mathbb{R}^{D \times L}$, with $L > D$ unit $\ell_2$ norm atoms, by placing more atoms in densely populated regions of the feature space. However, due to the redundant number of atoms, coding $x$ in $D$ no longer has a unique solution. Therefore, signal analysis in over-complete dictionaries needs to be performed using a sparse coding algorithm, such as orthogonal matching pursuit (OMP) [8].

The non-orthogonality of atoms is measured by the self-coherence of the dictionary, which can be defined as the maximum magnitude over all off-diagonal elements of the Gram matrix $G = D^T D$.

$$\mu(D) = \max_{d,e} |(G - I)_{d,e}| = \max_{d,e} |d_{(\cdot,d)}^T d_{(\cdot,e)}|, \quad d \neq e.$$  

(1)

It therefore holds that $\mu(D) \in [0, 1]$. Note that this definition of the dictionary self-coherence can be misleading when most inner products have small magnitudes [12]. Therefore, the full inner product distribution is considered in Sec. III. $D$ has minimum self-coherence for a given dimension $D$ and dictionary size $L$ if the magnitudes of all the off-diagonal elements of $G$ are equal (see Thm. III below). In
this case, the dictionary is called an *equiangular tight-frame* (ETF) \cite{DFP}. Formally, \( E \in \mathbb{R}^{D \times L} \) is an ETF if there exists an \( \alpha \), \( 0 < \alpha < \pi/2 \), such that
\[
|e_{(c,d)}^T e_{(c,e)}| = \cos(\alpha), \quad d \neq e,
\]
and if
\[
EE^T = \frac{L}{D} I.
\]

Therefore, \( E \) has \( D \) non-zero singular values all equal to \( \sqrt{L/D} \). The following theorem establishes a lower bound on the minimum of the self-coherence.

**Theorem 1.** \cite{DFP} The self-coherence of a dictionary \( D \in \mathbb{R}^{D \times L} \) with unit \( \ell_2 \) norm atoms is bounded from below by
\[
\mu(D) \geq \sqrt{\frac{L - D}{D(L - 1)}}. \tag{4}
\]
Equality holds if and only if \( D \) is an ETF and \( L \leq D(D + 1)/2 \).

The self-coherence of a dictionary influences the recovery of the sparse coding support of a signal observation, i.e. the set of atoms that are associated with the non-zero coding coefficients. The *exact recovery condition* (ERC) \cite{D} states that, assuming that the observation in fact has an exact sparse coding \( \hat{e} \) in \( D \), the support of \( \hat{e} \) is recovered if
\[
\|\hat{e}\|_0 < \frac{1}{2} \left( 1 + \frac{1}{\mu(D)} \right). \tag{5}
\]
Furthermore, \( \mu(D) \) also upper bounds the residual error norm decay curve in iterative sparse coding algorithms such as OMP \cite{C}.

**B. Related Work**

Yaghoobi et al. proposed a design algorithm for parametric dictionaries \cite{G}. A *parametric dictionary* \( D_\Gamma \) consists of atoms which have a specific functional form controlled by a small number of parameters. The proposed algorithm accepts a given \( D_\Gamma \) as its input, and optimizes it such that its Gram matrix approximates the optimal properties of an ETF. However, this approach relies on expert knowledge for choosing the appropriate parametric family for a given application, and provides no mechanism to adapt \( D_\Gamma \) if the signal characteristics are not known in advance. Therefore, an analytic dictionary design approach is for instance not suited to source separation of partially coherent sources \cite{D}.

The K-SVD algorithm \cite{D} adapts a non-parametric dictionary to training data. In each iteration of the algorithm, those atoms are replaced which have a too high coherence to another atom in the dictionary. If the coherence to another atom lies above a threshold \( \mu_t \), the atom is replaced by a training observation which does not have a sparse representation in the current dictionary. Therefore, the likelihood that the replacement atom is less coherent to the dictionary is high. However, if multiple atoms are replaced (which is almost always the case in practice), this strategy does not guarantee that the dictionary self-coherence falls below \( \mu_t \). In our experiments, an effective control over the dictionary self-coherence using the proposed atom thresholding step was not possible (see Sec. \[\ref{sec:results}\]).

Very recently, and independently from our own work, Mailhô et al. \cite{G} proposed a more sophisticated atom decorrelation step for the K-SVD algorithm called INK-SVD, where pairs of atoms are decorrelated until the dictionary satisfies the maximum inner product bound \( \mu_t \). After the dictionary update step of the K-SVD algorithm is complete, each pair of atoms which has a coherence above the threshold \( \mu_t \) has its inner angle increased symmetrically until the threshold is satisfied. Because this procedure can inadvertently increase the coherence to other atoms, the pairwise decorrelation step has to be iterated until the self-coherence threshold is satisfied for the complete dictionary. Unfortunately, due to this fact the number of necessary decorrelation steps can grow very large if a small \( \mu_t \) is enforced (see Sec. \[\ref{sec:results}\]).

**C. Our Contribution**

We present a dictionary learning algorithm where a bound on the dictionary self-coherence is enforced directly in the atom update step. Instead of bounding the maximum inner product \( \mu_t \) as in the INK-SVD algorithm, our algorithm enforces an upper bound on the sum of squared inner product values. By varying a Lagrange multiplier \( \gamma \), it is possible to realize any trade-off between maximizing the sparsity of the code and minimizing the self-coherence of the dictionary.

Since IDL(\( \gamma \)) maximizes the coherence of a dictionary to a particular signal class, prior expert knowledge to choose the right parametric dictionary family and parameter discretization is not necessary. Furthermore, the IDL(\( \gamma \)) algorithm makes it possible to train an incoherent dictionary even if the number of atoms is large compared to the dimensionality of the signal space. And last but not least, we empirically demonstrate for a speech coding task that training an incoherent dictionary using IDL(\( \gamma \)) improves the sparse coding fidelity of the dictionary on unseen test data.

**II. Method**

A dictionary learning algorithm approximately factorizes a data matrix \( X \in \mathbb{R}^{D \times N} \) into a dictionary matrix \( D \in \mathbb{R}^{D \times L} \) and a coding matrix \( C \in \mathbb{R}^{L \times N} \). The algorithm minimizes the approximation error
\[
\arg \min_{D,C} \| X - D \cdot C \|_F^2, \tag{6}
\]
measured by the squared Frobenius norm, subject to a sparsity constraint on \( C \) and a unit \( \ell_2 \) norm constraint on the atoms (columns) of \( D \). Since \( \| \cdot \|_F^2 \) is not jointly convex in \( D \) and \( C \), many proposed algorithms employ alternating minimization w.r.t. \( C \) and \( D \) until convergence to a local optimum. In the following, we focus our discussion on the dictionary update step.
The K-SVD algorithm minimizes (6) for each atom independently. Given the newly updated dictionary, if there exist atoms \(d_{(c,d)}\) and \(d_{(c,e)}\), such that
\[
|d_{(c,d)}^\top d_{(c,e)}| > \mu_t, \quad d \neq e
\] (7)
d_{(c,e)} is replaced by \(x_{(c,n)}/\|x_{(c,n)}\|_2\), where \(n\) is chosen such that \(\|x_{(c,n)} - DC_{(c,n)}\|_2\) is large. Since observations having a large approximation error are likely incoherent to the current dictionary, the replacement atoms likely have a coherence below \(\mu_t\) to all atoms already in the dictionary. However, if more than one atom is replaced, the coherence between the replacement atoms can potentially be large. This approach therefore does not guarantee that the self-coherence of the updated dictionary falls below \(\mu_t\).

Although updating atoms independently of each other is computationally efficient, it is not well suited to enforcing a self-coherence constraint, which introduces additional dependencies between all atoms. We propose a dictionary update step where the atoms are jointly optimized, and the dictionary self-coherence is minimized along with the approximation error.

Thm. 1 motivates our choice to augment the minimization of the objective (6) w.r.t. \(D\) with a self-coherence penalty,
\[
\text{arg min}_D \|X - DC\|^2_F + \gamma \|D^\top D - I\|^2_F \quad (8)
\]
where the Lagrange multiplier \(\gamma\) controls the trade-off between minimizing the approximation error and minimizing the self-coherence. The second term in (8) penalizes both the average coherence between atoms, as well as a divergence from the unit \(\ell_2\) norm of each atom. However, we still enforce the strict unit \(\ell_2\) norm constraint after the optimization by rescaling each atom.

The gradient of (8) w.r.t. \(D\) is computed by a trace operator expansion, \(\|A\|^2_F = \text{tr}\{A^\top A\}\), of the approximation error term of (8),
\[
\text{tr}\{C^\top D^\top DC\} - 2\text{tr}\{X^\top DC\} + \text{tr}\{X^\top X\}, \quad (9)
\]
and the self-coherence penalty term of (8)
\[
\text{tr}\{D^\top DD^\top D\} - 2\text{tr}\{D^\top D\} + \text{tr}\{I\}. \quad (10)
\]
Taking the partial matrix derivatives of (9) and (10) w.r.t. \(D\) results in the gradient
\[
2\left(DCC^\top - XC^\top\right) + 4\gamma\left(DD^\top D - D\right), \quad (11)
\]
see e.g. [6] how to take partial derivatives of the trace operator.

It is not necessary to find the global minimizer of (8), as long as the objective is sufficiently reduced in each iteration of the dictionary learning algorithm. We therefore run only a few iterations of the limited-memory BFGS algorithm [5], which successively builds an approximation to the Hessian (i.e. the matrix of second order partial derivatives) from evaluating the objective (8) and the gradient (11), without directly computing the Hessian matrix (which is infeasible for large dictionaries).

### III. Experiments

We compare the proposed dictionary learning algorithm, denoted \(\text{IDL}(\gamma)\), to the K-SVD algorithm with atom replacement and the INK-SVD algorithm. The difference of our algorithm lies in the dictionary update: it jointly minimizes both the data approximation error and the coherence of all pairs of atoms. In contrast, the K-SVD and the INK-SVD algorithm first perform a dictionary update step to minimize the data approximation error, and then sequentially minimize the coherence of pairs of atoms.

The effectiveness of all algorithms to upper bound the dictionary self-coherence was evaluated for a speech coding task, as follows. The audio recordings of the first male speaker of the GRII\[1\] corpus were randomly sub-sampled to obtain \(N = 30000\) training signals, each \(D = 160\) samples long. A dictionary with \(L = 1000\) atoms was initialized using random sampling of training observations. The LARC algorithm [9] (an extension of the LARS algorithm [4]) was used for the sparse coding step of all dictionary learning algorithms, with the LARC residual coherence threshold set to \(\mu_d = 0.2\) (not to be confused with the self coherence threshold \(\mu_t\)). The number of dictionary learning iterations was set to 25, which resulted in approximate convergence to a local optimum in all experiments.

Figure 1 plots the singular value spectra of the trained dictionaries. As a reference, the constant line at \(\sqrt{L/D} = 2.5\) indicates the flat spectrum of a corresponding ETF. For the K-SVD algorithm (left figure), setting \(\mu_t = 1\) implies that the upper bound on the self-coherence is inactive. Note that decreasing \(\mu_t\) below unity proved to be counterproductive, i.e. the singular value spectrum decreases even more rapidly. As desired, lowering \(\mu_t\) for the INK-SVD algorithm resulted in a flatter spectrum (middle figure), but the computational cost is increasingly dominated by the growing number of decorrelation steps. Thus we were unable to train a dictionary with \(\mu_t = 0.1\) (or smaller) in the available time frame (24 hours on an Intel Core 2 Duo CPU). The results for \(\text{IDL}(\gamma)\) (right figure) show that by increasing the influence of the self-coherence penalty in (8), it is possible to approximate the flat spectrum of an ETF. Setting \(\gamma > 50\) resulted in even flatter spectra (not shown). Atom coherence histograms and atom recovery percentages are available from the paper companion webpage\[2].

Figure 2 plots the generalization performance of the trained dictionaries, in terms of the trade-off between the residual norm and the cardinality of the coding. Twenty test utterances were coded using OMP with a cardinality stopping criterion, and the median residual norm is reported. For the K-SVD algorithm, decreasing \(\mu_t < 1\) resulted in a deteriorating generalization performance. For the INK-SVD algorithm, decreasing the residual norm is possible for \(0.7 < \mu_t < 0.2\) at cardinalities beyond 80, but only at the cost of increasing the residual norm at smaller cardinalities. While the curves are nearly identical for all algorithms if no

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1. [http://www.dcs.shef.ac.uk/spandh/gridcorpus/](http://www.dcs.shef.ac.uk/spandh/gridcorpus/)
2. [http://sigg-iten.ch/research/spl2012/]
coherence bound is enforced, the generalization performance improves consistently only in the case of IDL(γ). We conjecture that the difference is due to joint minimization of the residual norm and the dictionary self-coherence in IDL(γ), whereas the atom decorrelation of K-SVD and INK-SVD is independent of the dictionary update.

IV. CONCLUSIONS AND DISCUSSION

We present a dictionary learning algorithm which enables an effective control over the self-coherence of the trained dictionary, enabling a trade-off between maximizing the sparsity of the code and approximating an equiangular tight frame. Neither a simple replacement of too similar atoms or a pairwise decorrelation of atoms can both effectively and efficiently control the dictionary self-coherence. We propose a joint atom update step instead, simultaneously minimizing the approximation error and the coherence of all pairs of atoms.

We show for a speech coding task that our method is able to achieve the full range of optimization objectives, from maximizing the coding sparsity to approximating the properties of an ETF. Furthermore, we demonstrate the benefits of bounding the dictionary self-coherence on the generalization performance of the dictionary.

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