Self-Duality from New Massive Gravity Holography

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ABSTRACT

The holographic renormalization group (RG) flows in certain self-dual two dimensional QFT’s models are studied. They are constructed as holographic duals to specific New Massive 3d Gravity (NMG) models coupled to scalar matter with “partially self-dual” superpotentials. The standard holographic RG constructions allow us to derive the exact form of their \( \beta \)-functions in terms of the corresponding NMG’s domain walls solutions. By imposing invariance of the free energy, the central function and of the anomalous dimensions under specific matter field’s duality transformation, we have found the conditions on the superpotentials of two different NMG’s models, such that their dual 2d QFT’s are related by a simple strong-weak coupling transformation.

KEYWORDS: New Massive Gravity, Holographic RG Flows, 2d phase transitions, strong-weak coupling duality

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1 Introduction

In the lack of small parameters, the concepts and methods of the strong-weak coupling duality \[1,2\] are known to be the main tool for the description of relevant physical phenomena, and for the derivation of non-perturbative strong coupling results. In all the known examples of self-dual (super-symmetric) QFT\(_d\)'s (with \(d = 2, 3, 4\)), this duality is realized as an inversion transformation or, more generally, as fractional linear transformations of the couplings belonging to certain discrete subgroups of SL(2,C), which leave invariant the corresponding partition functions \[1–4\]. The gauge/gravity duality \[5–8\] on the other hand, together with the holographic Renormalization Group (RG) \[9,10\], establish an equivalence relation between certain limits of the (semi-)classical \(d\)-dimensional gravity models and the strong-coupling regime of \((d − 1)\)-dimensional gauge theories. According to the off-critical holographic RG version of the AdS/CFT correspondence, the QFT\(_{d−1}\)'s dual to certain
asymptotically AdS$_d$ geometries of domain wall (DW’s) type \[10\] may involve — together with the original gauge strong coupling — a few other relevant or/and marginal couplings. These models can be also realized as certain conformal perturbations around a given CFT, thus defining non-conformal theories called pCFT’s \[23, 34\]. We are interested in the specific holographic features of such dual, non-conformal QFT’s, in the case when they belong to the family of the self-dual theories w.r.t. one (or a few) of these couplings. More precisely, we shall address the question of how can one derive the “holographic gravitational” counterparts of certain duality symmetries, such as the above mentioned inversions and fractional linear transformations. We consider the 3-dimensional New Massive Gravity (NMG) model \[17\], coupled to scalar self-interacting matter, and will look for the specific restrictions to be imposed on the form of the matter superpotential in order to ensure the strong-weak coupling self-duality of the corresponding two-dimensional pCFT$_2$, constructed by the methods of the NMG holography \[28–30, 32\].

The recent progress in the understanding of the t’Hooft limits \((N, k \to \infty \text{ but finite } N/L)\) of certain cosets of SU($N$)$_k$ WZW models (as for example the $W_N$ minimal models) as an appropriate higher spin extension of the 3d Einstein gravity \[14–16\] has renewed the interest in the identification of appropriate limits of the most famous family of CFT$_2$’s — the BPZ and the Liouville minimal models \[11\] — as holographic duals of certain extended 3d gravity models \[28–30, 32\]. There exists an indication that the holographic description of these CFT$_2$’s in the case of relatively large, but finite central charges, can be achieved by considering the quantum 3d gravity contributions beyond the (semi-)classical one \[16\], or/and of certain “higher curvature” extensions of the Einstein gravity, including powers of the curvature and of the Ricci tensor at the classical level. The simplest model of such an extended 3d gravity is given by the following action, called New Massive Gravity \[^5\] \[17\]:

$$S_{\text{NMG}}(g_{\mu \nu}, \kappa, \Lambda) = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \left[ \epsilon R + \frac{1}{m^2} \left( R_{\mu \nu} R^{\mu \nu} - \frac{3}{8} R^2 \right) - 2\Lambda \right], \quad (1.1)$$

$$\kappa^2 = 16\pi G, \quad \epsilon = \pm 1.$$  

At the linearised level, it describes a massive graviton with two polarizations. As it was shown by Bergshoeff, Hohm and Townsend (BHT) \[17\], the above model turns out to be unitary consistent (ghost free) for both choices, $\epsilon = \pm 1$, of the “right” and “wrong” signs of the $R$-term, under certain restrictions on the values of the cosmological constant $\Lambda = -m^2 \lambda$, as for example \[18\]:

$$-1 \leq \lambda < 0, \quad \epsilon = -1, \quad m^2 < 0. \quad (1.2)$$

in the case of the negative $\lambda$ BHT-unitary window.

An important feature of the central charges of the CFT$_2$’s dual to these NMG models is the presence of a particular $m^2$-dependent term \[18, 19\]:

$$c_{\text{nmg}} = \frac{3\epsilon L}{2l_{pl}} \left( 1 + \frac{L_{gr}^2}{L^2} \right), \quad L_{gr}^2 = \frac{1}{2\epsilon m^2} \gg l_{pl}^2, \quad \Lambda_{\text{eff}} = -\frac{1}{L^2} = -2m^2(\epsilon \pm \sqrt{1 + \lambda}). \quad (1.3)$$

\[^5\]One may consider the new $R^2$-type terms as one loop counter-terms appearing in the perturbative quantization of 3d Einstein gravity with $\Lambda < 0$. 

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Compared to the standard 3d Einstein gravity case, which one resumes to in the $m^2 \to \infty$ limit, the above central charges yield a remarkable new self-duality property: $c_{\text{ang}}(L) = c_{\text{ang}}(L^2_{gr}/L)$, coinciding with the well known "b to 1/b" duality of the (exact, non-perturbative) central charges $c^\pm(b) = 1 \pm 6(b \pm \frac{1}{b})^2$ of the Liouville ($c^+$) and of the BPZ ($c^-$) minimal models [11,13]. It is then natural to expect that appropriate perturbations of these CFT’s give rise to certain strong-weak coupling self-dual non-conformal pCFT$_2$’s we are interested in. Although the proper identification of the CFT$_2$’s dual to NMG model (1.1), is not yet fully understood, the off-critical AdS/CFT methods based on the DW’s solutions of NMG model coupled to massive self-interacting scalar field with an action [29,31]:

$$S_{\text{NMG}}(g_{\mu\nu}, \kappa, \Lambda) = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \left( \epsilon R + \frac{1}{m^2} \mathcal{K} - \kappa^2 \left( \frac{1}{2} |\nabla \sigma|^2 + V(\sigma) \right) \right); \quad (1.4)$$

as well as the NMG holographic RG results related to them [29,30], provide the necessary tools for the selection of the conditions on the NMG-matter interactions which lead to such self-dual pCFT$_2$’s.

Our main result consists in the explicit construction of the duality transformations between pairs of 3d NMG-matter models (1.4), whose holographic 2d images represent specific strong-weak coupling transformations which keep invariant the free energy, the corresponding C-function and the anomalous dimensions of their pCFT$_2$ duals. We also derive the explicit form of a partially self-dual matter superpotential (with all the vacua within the negative BHT-unitary window (1.2)) giving rise to a holographic, self-dual, pCFT$_2$ model, presenting both strong- and weak-coupling phases and critical points. The practical importance of the concept of partial self-duality, introduced in Sect 3.2., is that it provides an efficient method for the identification of such holographic pCFT$_2$’s of a given exact $\beta$-function, by comparing the results concerning its weak-coupling phases with the standard and well known perturbative CFT$_2$’s calculations around a given (weak-coupling) critical point [23–25,34].

2 NMG holography

The models involved in the “boundary” QFT$_2$’s part of the off-critical AdS$_3$/CFT$_2$ correspondence [22] are usually identified as certain CFT$_2$’s, perturbed by marginal or/and relevant operators that break the conformal symmetry of it’s Poincaré subgroup:

$$S^\text{ren}_{\text{pCFT}_2}(\sigma) = S^\text{UV}_{\text{CFT}_2} + \sigma(L_*) \int d^2x \Phi_\sigma(x^i). \quad (2.1)$$

The scale-radial duality [9,10] allows to further identify the “running” coupling constant $\sigma(L_*)$ of the pCFT$_2$ with the scalar field $\sigma(z)$, and the RG scale $L_*$ with the scale factor $e^{\varphi(z)}$ of the DW’s solutions of the bulk gravity coupled to scalar matter, as follows:

$$ds^2 = dz^2 + e^{\varphi(z)}(dx^2 - dt^2), \quad \sigma(x^i, z) \equiv \sigma(z), \quad L_* = l_p e^{-\varphi/2}. \quad (2.2)$$
The main ingredients of the NMG holography – the NMG’s vacua and DW’s solutions, the values of the central charges of the conjectured dual CFT$^2$’s and the holographic RG flows – were extensively studied by different methods [17, 20, 21, 27, 29, 30]. As is well known from the example of Einstein gravity [35], the properties and the proper existence of the holographic RG flows in its 2d dual QFT$^2$, strongly depend on the form of bulk matter interactions. If they permit DW’s solutions relating two unitary NMG vacua of different $\lambda_A$, then we might have massless RG flows in the dual pCFT$^2$. The explicit construction of all the DWs solutions of the corresponding second order system of equations:

$$\ddot{\sigma} + \dot{\sigma} \dot{\varphi} - V'(\sigma) = 0$$
$$\dot{\varphi} \left(1 - \frac{\dot{\varphi}^2}{8\epsilon m^2}\right) + \frac{1}{2} \dot{\varphi}^2 \left(1 - \frac{\dot{\varphi}^2}{16\epsilon m^2}\right) + \epsilon \kappa^2 \left(\frac{1}{2} \dot{\sigma}^2 + V(\sigma)\right) = 0$$
$$\dot{\varphi}^2 \left(1 - \frac{\dot{\varphi}^2}{16\epsilon m^2}\right) + \epsilon \kappa^2 (-\dot{\varphi}^2 + 2V(\sigma)) = 0 \quad (2.3)$$

is a rather difficult problem, and in general it requires the use of numerical methods. However, one particular class of such solutions which are “stable” and exact can be obtained by introducing an auxiliary function $W(\sigma)$, called superpotential, which allows to reduce the corresponding DW’s gravity-matter equations to an specific BPS-like 1st order system [27, 29]:

$$\kappa^2 V(\sigma) = 2(W')^2 \left(1 - \frac{\kappa^2 W^2}{2\epsilon m^2}\right)^2 - 2\epsilon \kappa^2 W^2 \left(1 - \frac{\kappa^2 W^2}{4\epsilon m^2}\right),$$
$$\dot{\varphi} = -2\epsilon \kappa W, \quad \dot{\sigma} = 2\epsilon \kappa W' \left(1 - \frac{\kappa^2 W^2}{2\epsilon m^2}\right), \quad (2.4)$$

where $W'(\sigma) = dW/d\sigma$, $\dot{\sigma} = d\sigma/dz$ etc. This provides the explicit form of qualitatively new DW’s relating “old” and “new” purely NMG vacua, as well as of the corresponding pCFT$^2$ model’s $\beta$-function [29].

Given the form of the superpotential $W(\sigma)$ and the 1st order system (2.4) — which describes the radial evolution of the NMG’s scale factor and of the scalar field $\sigma(z)$ —, the scale-radial identifications (2.2) provide the explicit form of the $\beta$-function of the conjectured dual QFT$^2$ [9, 10]:

$$\frac{d\sigma}{dl} = -\beta(\sigma) = \frac{2\epsilon}{\kappa^2 W(\sigma)} W'(\sigma) \left(1 - \frac{W^2(\sigma)}{2\epsilon m^2}\right), \quad l = \ln L_\ast. \quad (2.5)$$

The admissible constant solutions $\sigma_A^*$ of the above RG equation (2.5) are defined by the zeros of the $\beta$-function, and they indeed coincide with the NMG-matter models vacua solutions of AdS$_3$ type. The variety of different non-constant solutions $\sigma_{ij} = \sigma(l; \sigma_{A_i}^*, \sigma_{A_j}^*)$ representing the way the coupling constant $\sigma(l)$ of the dual QFT$^2$ is running (with the RG scale $L_\ast$ increasing) between two consecutive critical points (i.e. for $j = i + 1$) describe the RG flows (and the phase transitions) that occur in the QFT$^2$.

Let us briefly remind how one can extract the information about the critical properties of such QFT$^2$ models from eq.(2.5) and the way the CFT$^2$ data is related to the asymptotic behaviour of the NMG’s domain wall solutions [29], or equivalently to the shape of the matter potential $V(\sigma)$.
2.1 QFT\(_2\) critical behaviour

The zeros \(\sigma_A^*\) of the \(\beta\)-function determine a set of critical points in the coupling space, where the corresponding QFT\(_2\) becomes conformal invariant and the phase transitions of second or infinite order take place. The nature of the observed changes in the behaviour of the thermodynamical (TD) potentials and certain correlation functions at the neighbours of each critical point \(\sigma_A^*\) does depend on the multiplicity \(n_A\) of these zeros. In the case of simple zeros, we have 
\[
y(\sigma_A^*) = -\frac{d\beta}{d\sigma}|_{\sigma=\sigma_A^*} \neq 0
\]
and hence \(\beta(\sigma) \approx -y(\sigma_A^*)(\sigma - \sigma_A^*)\). The corresponding second order phase transitions are characterized by the scaling laws and the critical exponents of their TD potentials as for example
\[
F_A^s(\sigma) \approx (\sigma - \sigma_A^*)^{\frac{2}{\nu_A}}, \quad \xi_A \approx (\sigma - \sigma_A^*)^{-\frac{1}{\nu_A}}, \tag{2.6}
\]
at the neighbourhood of \(\sigma_A^*\). Once the \(\beta\)-function (2.5) is given\(6\), the above “near-critical forms” of \(F_A^s(\sigma)\) and \(\xi_A\) can be easily derived from the following RG equations:
\[
\beta(\sigma) \frac{dF_A^s(\sigma)}{d\sigma} + 2F_A^s(\sigma) = 0, \quad \beta(\sigma) \frac{d\xi(\sigma)}{d\sigma} = \xi(\sigma), \tag{2.7}
\]
which determine the scaling properties of the TD potentials, etc. under infinitesimal RG transformations (see for example \[34\]).

If one divides the coupling space \(\sigma \in R\) into intervals \(p_{k,k+1} = (\sigma_s^k, \sigma_s^{k+1})\) limited by vacua \(\sigma_s\), then each interval will correspond to a different phase. Two such consecutive phases share the same UV critical point \(\sigma_{UV}^k\), where a second order phase transition, driven by a relevant operator \(\Phi_{\sigma}\), may occur. The nature of this phase transition indeed depend on the properties of the neighbours, i.e. if \(\sigma_s^{k+1} = \sigma_{IR}, \sigma_s, \infty\), which also determine the features of the considered the QFT\(_2\) phase: massive, massless, etc. An efficient method for the analytic description of these QFT\(_2\)’s phase transitions is given by the conformal perturbation theory \(p\text{CFT}_2(\sigma_{UV}^k)\), based on the action (2.1) and on the knowledge of the exact correlation functions of \(\Phi_{\sigma}\), once the \(\text{CFT}_2(\sigma_{UV}^k)\) is known and the relevant operator \(\Phi_{\sigma}\) is appropriately chosen \[23\]. In the case of integrable perturbations of \(\Phi_{13}\)-type \[7\] for Virasoro and Liouville (minimal) models (or of \(\Phi_{adj}\)-type for, say, \(W_N\ m.m.s\)) \[23–25\] the calculations involving conformal OPEs:
\[
\Phi(1)\Phi(2) \approx I + C_{\Phi\Phi}\Phi(2) + ...
\]
allow us to derive the \(\beta\)-function at first order in perturbation theory around the critical point:
\[
\beta(\sigma) \approx -y_{13}(\sigma - \sigma_A^*) + C_{\Phi\Phi}\Phi(\sigma - \sigma_A^*)^2 + ...
\]
It is well known that the phase structure of such \(p\text{CFT}_2\) is of massless-to-massive \((\sigma_{IR}, \sigma_{UV}, \infty)\) type \[23\].

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\(6\) Conjectured as in the case of the holographic RG or perturbatively calculated from the explicit form of the \(p\text{CFT}_2\) action (2.1).

\(7\) These, for unitary models, are known to be the only consistent one coupling perturbations.
We begin our short NMG/pCFT correspondence dictionary by remembering one specific “NMG feature” — the existence of two types of distinct critical points: the usual type (a) vacua, given by \( W'(\sigma^*_A) = 0 \), and therefore representing the extrema of \( W(\sigma) \); and the “new” vacua of type (b), given by the real solutions of the equations \( W^2(\sigma^*_b) = 2\epsilon m^2/\kappa^2 \), which exist only in the case when \( \epsilon m^2 > 0 \). Both types of vacuum are extrema of the matter potential, \( V'(\sigma^*_A) = 0 \), for

\[
\kappa^2 V'(\sigma) = 4W'(1 - \frac{W^2\kappa^2}{2\epsilon m^2})\omega(\sigma),
\]

but there are others extrema of \( V(\sigma) \), given by the real constant solutions of the algebraic equation:

\[
\omega(\sigma^*) = W''\left(1 - \frac{\kappa^2W^2}{2\epsilon m^2}\right) - \frac{\kappa^2}{\epsilon m^2}(W')^2W - \epsilon\kappa^2W = 0,
\]

which do not represent (vacuum) solutions of the \( I^{st} \) order eqs. (2.4). We will fix our attention, in what follows, in the vacua of type (a) and (b). As one can see from eqs. (2.3), such vacua are defined by \( \dot{\sigma} = 0 \) and \( \dot{\varphi} = -2\epsilon\kappa W(\sigma^*_A) = \text{const.} \) It is then evident that they both present the geometry of an AdS vacuum \((\sigma^*_A, \Lambda_{\text{eff}}^A)\) of the NMG model:

\[
ds^2 = dz^2 + e^{-2\sqrt{\Lambda_{\text{eff}}^A}|z|}(dx^2 - dt^2), \quad A = a, b.
\]

As usually, the corresponding effective cosmological constants \( \Lambda_{\text{eff}}^A \) are realised as the vacuum values of the 3d scalar curvature \( R(\varphi) \), which, for the considered DW’s and vacua solutions (2.2), is given by

\[
R = -2\ddot{\varphi} - \frac{3}{2}\dot{\varphi}^2 = 8(W')^2\left(1 - \frac{\kappa^2W^2}{2\epsilon m^2}\right) - 6\kappa^2W^2;
\]

hence at a vacuum \( \sigma^*_A \) we have \( R_{\text{vac}} = -6\epsilon^2W^2(\sigma^*_A) = 6\Lambda_{\text{eff}}^A = -6/L_A^2 \). Notice that the NMG vacua of \( W(\sigma^*) = 0 \) have the geometry of flat Euclidean \( E_3 \) or Minkowski \( M_3 \) space.

The critical exponents also play a crucial part on the asymptotic behaviour of the matter field \( \sigma(z) \). In the non-degenerated case we have

\[
\sigma(z) \xrightarrow{z \to \infty} \sigma^*_A - \sigma^*_A e^{-y_A \sqrt{\Lambda_{\text{eff}}^A}|z|}, \quad \Delta_A = 2 - y_A = 1 + \sqrt{1 - \frac{m^2(\Lambda^A_{\text{eff}})}{\Lambda^A_{\text{eff}}}}, \quad m^2 = V''(\sigma^*_A).
\]

Thus \( y_A \neq 0 \) provide the boundary conditions (b.c.’s) for the corresponding DW’s solutions of the NMG model (see ref. [29]), as one can easily verify by considering the near-boundary/horizon approximation of eqs. (2.4): \( \dot{\sigma} = -\epsilon\kappa\beta(\sigma)W(\sigma) \approx y_A(\sigma - \sigma^*_A)e\kappa W(\sigma^*_A) \), and taking into account the identification \( \kappa W_A = -\epsilon/L_A \).

As it is expected, the quantities characterizing the pCFT2 present in the asymptotic limits of these QFT2 can also be described by the geometric properties of the associated NMG-matter model, and written in terms of the superpotential. Let us first consider the critical exponents
\[ y_A = y(\sigma^*_A) = -\beta'(\sigma^*_A) \] in the case when all the critical points have multiplicities \( n_A = 1 \), i.e. both \( \sigma^*_a \) and \( \sigma^*_b \) are first order zeros of \( \beta(\sigma) \), and \( W(\sigma^*_A) \neq 0 \) [29,30]:

\[ y_a = y(\sigma^*_a) = \frac{2\epsilon W''}{\kappa W_a} \left( 1 - \frac{\kappa^2 W^2}{2\epsilon m^2} \right), \quad y_b = y(\sigma^*_b) = -\frac{4\epsilon (W')^2}{\kappa W_b^2}, \quad W_b^2 = \frac{2\epsilon m^2}{\kappa^2}. \quad (2.13) \]

The structure constants \( C^A_{\Phi\Phi\Phi}_{\sigma} = \frac{1}{2} \beta''(\sigma_A) \) can be calculated from eqs. (2.5):

\[ C^a_{\Phi\Phi\Phi} = -\frac{\epsilon W''}{\kappa^2 W_a} \left( 1 - \frac{W_a^2\kappa^2}{2\epsilon m^2} \right), \quad C^b_{\Phi\Phi\Phi} = \frac{2\epsilon}{\kappa^2} \left( 3\frac{W''}{W_b} \frac{W'}{W_b} - \frac{(W'_b)^3}{W_b^3} \right). \quad (2.14) \]

By definition, their \( C_{\Phi_{UV}}(\sigma^*_A) \) counterparts represent the ratio of the constants of 3-point and 2-point functions of the perturbing field \( \Phi_{\sigma} \) [11]. Finally, the Zamolodchikov’s \( C \)-function can be written in terms of the superpotential as follows [23]:

\[ C(\sigma) = \frac{-3}{2G\kappa W(\sigma)} \left( 1 + \frac{\kappa^2 W^2(\sigma)}{2\epsilon m^2} \right). \quad (2.15) \]

This particular form is derived in refs. [21,28,29] by the Brown-Henneaux asymptotic method [33]. The central charge at a vacuum \( c_A = C(\sigma^*_A) \) can therefore be evaluated when the superpotential is given. It is important to note a specific feature of the NMG-induced 2d models, namely that the \( C_{2d} \)'s describing all the type (b) critical points have equal central charges \( c_b = 3L^2_{\text{gr}}/l_{\text{pl}}^2 \) (with \( L^2_{\text{gr}} = 1/2\epsilon m^2 \)) while the type (a) central charges \( c_a = (3\epsilon L_a/2l_{\text{pl}}^2) \left( 1 + L^2_{\text{gr}}/L^2_a \right) \) are parametrized by the corresponding critical values of the superpotential: \( W^2(\sigma^*_a) = 1/\kappa^2 L^2_a \).

### 3 Strong-weak coupling duality

Motivated by the eventual existence of holographic self-dual pCFT’s, we address the problem about the properties of the pairs of the their dual 3d NMG-matter models and about the nature of the ”duality” transformations relating their superpotentials.

#### 3.1 Pairs of dual NMG models

Given a NMG model coupled to scalar field \( \sigma \) of superpotential \( W(\sigma) \) [1,4], we define its dual as a specific NMG model, whose scalar field \( \tilde{\sigma} \) and superpotential \( \tilde{W}(\tilde{\sigma}) \) are fulfilling the following two conditions:

\[ \varphi(\sigma) = \tilde{\varphi}(\tilde{\sigma}), \quad \tilde{W}(\tilde{\sigma}) = \frac{1}{\kappa^2 L^2_{\text{gr}}} \tilde{W}(\tilde{\sigma}) \quad (3.1) \]

We also impose an additional requirement that all the critical points \( \sigma_k \) and \( \tilde{\sigma}_k \) of the pair of superpotentials, i.e. \( W'(\sigma_k) = 0 = \tilde{W}'(\tilde{\sigma}_k) \) correspond to true AdS\(_3\) vacua : \( W(s_k) \neq 0 \) and

\[ ^8 \text{corresponding to the lower bound } \lambda_0 = -1 \text{ of the negative BHT-unitary window} [1,2] \]

\[ ^9 \text{representing few critical points and having massive and massless phases} \]
\( \hat{W}(\hat{\sigma}_k) \neq 0 \). It is natural to expect that the above pairs of duals \( NMG_3 \) models are mapped by the \( AdS/CFT \) correspondence rules in certain pairs of duals (or self-dual) \( CFT_2 \)'s models. The particular form of the NMG’s matter superpotentials transformations (3.1) is chosen in the way that the coupling space duality transformations between the corresponding pairs of duals \( CFT_2 \)'s, induced by eqs. (3.1), preserve the form of the central charges at the critical points, the form of the central function (2.15) and of the corresponding s.p. of their free energy.

The first requirement, i.e. the invariance of the scale factor of the NMG’s domain walls, ensures the desired invariance of singular part of the reduced free energy \( F^A_s(\sigma) = e^{-\varphi(\sigma)} \) of theirs duals \( pCFT_2 \) models. It is equivalent to the condition \( l(\sigma) = \tilde{l}(\tilde{\sigma}) \) of the invariance of the QFT scales under such transformation. We next recall that the central charge associated with a vacuum \( \sigma_A \) has the form:

\[
\begin{align*}
  c_A &= \frac{3\epsilon L_A}{2l_{pl}} \left[ 1 + \frac{L^2_{gr}}{L^2_A} \right], \\
  \tilde{L}_A &= \frac{L_{gr}}{L_A}. 
\end{align*}
\] (3.2)

Therefore the above NMG’s superpotential transformation ensures the invariance of the \( CFT_2 \)'s central charges\(^{10}\) and it has the same form as the well known central charges duality properties of the Liouville and minimal models, namely \( c(L_A) = c(L^2_{gr}/L_A) \). Notice that this is a direct consequence of the curvature quadratic terms in the action (1.4), which generates the specific form of the central charge (3.2), which \textit{is not present} in EH gravity. The transformation \( \sigma \to \tilde{\sigma} \) maps the AdS\(_3\) spaces of large radii (and small cosmological constants) to certain ”dual” AdS\(_3\) spaces of small radii (and large cosmological constants\(^{11}\)) but the corresponding dual \( CFT_2 \)'s share \textit{equal} central charges.

An important (implicit) element of the above introduced concept of pairs of duals NMG’s (and corresponding pairs of duals pCFT\(_2\)'s) is that the mapping is always between the vacua of the same kind, i.e. \( \sigma_a \to \tilde{\sigma}_a \) and \( \sigma_b \to \tilde{\sigma}_b \). This must be proved however. We first note that according to the condition \( \varphi(\sigma) = \tilde{\varphi}(\tilde{\sigma}) \), the transformation of the beta-functon \( \beta(\sigma) = -d\sigma/dl \), with \( l = -\varphi/2 \) is given by:

\[
\beta(\sigma) = \frac{d\sigma}{d\tilde{\sigma}} \tilde{\beta}(\tilde{\sigma}).
\] (3.4)

The next step is to calculate the derivative \( d\sigma/d\tilde{\sigma} \) in terms of the corresponding superpotentials, by substituting the explicit form (2.5) of the both \( \beta \)-functions into eq. (3.4), and then taking into account eqs. (3.1) to eliminate \( \hat{W} \):

\[
\frac{d\tilde{\sigma}}{d\sigma} = -1/\kappa L_{gr} W(\sigma).
\] (3.5)

\(^{10}\)and of the corresponding pCFT\(_2\)'s C-function (2.15) as well

\(^{11}\)Note that if the types (a) and (b) vacua coincide, then all the scales remain invariant: \( L_a = L_{gr} = \tilde{L}_a \).
Due to the additional requirements $W(\sigma_k) \neq 0$ and $\tilde{W}(\tilde{\sigma}_k) \neq 0$ it is not singular at the critical points, and therefore the zeros of $\beta(\sigma)$ are also zeros of $\tilde{\beta}(\tilde{\sigma})$ and vice-versa. Hence the vacua of one theory are also vacua of its dual, and the transformation \((3.1)\) maps vacua into vacua. As a consequence we find the explicit form of the NMG’s scalar fields duality transformation as follows:

$$\tilde{\sigma}(\sigma) = -\frac{1}{\kappa L_{gr}} \int^{\sigma} dx \frac{W(x)}{W(x)} + \text{constant.} \quad (3.6)$$

The above properties confirm the fact that the type (a) NMG -vacua are mapped into the type (a) vacua of the dual NMG model $\sigma_a \rightarrow \tilde{\sigma}_a$, as one can see from the identity $dW(\tilde{\sigma})/d\tilde{\sigma} = (\kappa L_{gr} W(\sigma))^{-1} dW(\sigma)/d\sigma$. The type (b) vacua remain invariant under the duality transformation, since their defining equation $1 - \kappa^2 L_{gr}^2 W^2(\sigma) = 0$ is mapped by \((3.1)\) into itself.

Taking into account the explicit form of the I-st order eqs.\((2.4)\) for the pairs of NMG dual models, it is not difficult to derive the relation between the dual “radial” coordinates $\tilde{z}(z)$:

$$\tilde{z}(z) = \frac{\kappa^2 L_{gr}^2}{2} \int^{z} dx \frac{W^2(x)}{W'(x) \left[1 - \kappa^2 L_{gr}^2 W^2(x)\right]} + \text{constant,} \quad (3.7)$$

or, in terms of $\sigma$ we get

$$\tilde{z}(\sigma) = \frac{\kappa^2 L_{gr}^2}{2} \int^{\sigma} dx \frac{W^2(x)}{W'(x) \left[1 - \kappa^2 L_{gr}^2 W^2(x)\right]} + \text{constant.} \quad (3.8)$$

It remains to demonstrate one of the most important properties of the duality transformations \((3.1)\): namely, that they keep invariant the critical exponents $y_A, A = a, b$ given by \((2.13)\). Starting by their definitions $y_A = d\beta/d\sigma \mid_{\sigma_A}$ at the corresponding critical points $\sigma_A$, and further by using eqs.\((2.15)\), and the fact that the vacua are the zeros of these $\beta$-functions, we find

$$y_A = -\frac{d}{d\sigma} \beta(\sigma) \mid_{\sigma_A} = - \left\{ \frac{d\tilde{\sigma}}{d\sigma} \frac{d}{d\tilde{\sigma}} \tilde{\beta}(\tilde{\sigma}) \right\}_{\tilde{\sigma}_A} = - \left\{ \frac{d\tilde{\sigma}}{d\sigma} \frac{d}{d\tilde{\sigma}} \frac{d}{d\tilde{\sigma}} \tilde{\beta}(\tilde{\sigma}) \right\}_{\tilde{\sigma}_A} = - \frac{d}{d\tilde{\sigma}} \tilde{\beta}(\tilde{\sigma}) \mid_{\tilde{\sigma}_A}$$

Thus we can conclude that indeed $y_A = \tilde{y}_A$.

Let us summarize the main features of the duality transformations \((3.1)\) between two specific NMG -matter models, whose superpotentials are "inversely proportional": their matter potentials are different, but they do have equal number of vacua such that the pairs of type (a) dual vacua are representing $AdS_3$ spaces of different radii that are inversely proportional to each other and their type (b) vacua are coinciding. The most relevant characteristics of the corresponding pairs of dual $CFT_2$’s models (and of the pairs of p$CFT_2$’s as well) are: (1) they have different holographic $\beta$-functions, whose type (a) critical points have different values (one in the weak-coupling another in the strong coupling regions), but still identical central charges and central functions; (2) the critical exponents $y_A = \tilde{y}_A$ remains invariant under such duality transformations and (3) their s.p. free energies are identical by construction. It remains to answer the important question concerning the explicit construction of relatively simple and physically interesting pairs of such dual NMG models and to describe the nature phase transitions and of the different phases of the corresponding pairs of duals p$CFT_2$’s, whose exact holographic $\beta$-functions are related by the eqs.\((3.4)\).
3.2 Examples of dual and self-dual NMG models

In order to illustrate how the concepts of NMG duality transformations (3.1) introduced above can be realized in practice, we consider few representative simple examples of pairs of NMG dual models. An important problem addressed in this subsection concerns one particular class of duality transformations $\sigma = \sigma(\tilde{\sigma})$, that together with the definitions (3.1) and (3.6) satisfy the new "self-duality" condition: namely, when substituted in the second of the eqs.(3.1) to give rise of a very special self-dual superpotentials:

\begin{itemize}
  \item self-duality : $W(g_k, \sigma) = \tilde{W}(g_k, \tilde{\sigma})$,  \\
  \item partial self-duality : $W(g_k, \sigma) = \tilde{W}(\tilde{g}_k, \tilde{\sigma})$,
\end{itemize}

(3.9)

where the parameters $g_k$ and $\tilde{g}_k$ determine the coupling constants and the masses in the corresponding NMG$_3$ matter potentials $V(g_k, \sigma)$ and $\tilde{V}(\tilde{g}_k, \tilde{\sigma})$. In both cases the shapes of the pairs of duals NMG superpotentials are coinciding, but in the second case the particular “partial self-duality” transformations are mapping the NMG-matter couplings $\tilde{g}_k = \tilde{g}_k(g_k)$ as well. The particular examples analysed in this section are all chosen to provide a kind of "strong-to-weak couplings" duality transformations $\sigma = \sigma(\tilde{\sigma})$ between the corresponding pairs of dual pCFT$_2$’s.

3.2.1 Self-duality

Consider the following quadratic superpotential:

$$W(\sigma) = B\sigma^2, \quad B > 0.$$  \hfill (3.10)

We assume that there exist at least one (b) vacuum, i.e. $m^2\epsilon > 0$, which is the fixed point of the transformation (3.1). Because of the $Z_2$ symmetry of the superpotential, we can consider the $\sigma > 0$ only. There is no type (a) vacuum for such superpotentials: the vacuum at $\sigma_M = 0$ is of zero cosmological constant, i.e. it represents a Minkowski vacuum. The exact form of the scale factor is easily derived by solving the corresponding I-st order system (2.4) and it determines a particular asymptotically AdS$_3$ (or H$_3$ in the euclidean case) geometry with a naked singularity at $\sigma \to \infty$ [29,30]. The eqs.(3.1) and (3.6) applied for the linear $W$ (3.10) provides the explicit form of the NMG duality transformation:

$$\tilde{\sigma} = \frac{1}{\kappa L_{gr} B \sigma}, \quad \tilde{W}(\tilde{\sigma}) = 1/\kappa^2 L_{gr}^2 B\sigma^2 = B\tilde{\sigma}^2,$$

(3.11)

where the constant of integration has been chosen to be zero. Therefore the dual superpotential has exactly the same shape of the original one and coinciding parameters $B = \tilde{B}$, that determine the coupling constants in the corresponding matter potentials $V(\sigma)$ and $\tilde{V}(\tilde{\sigma})$. The critical points, however, are “interchanged” in the dual model: the original Minkowski vacuum is mapped into the dual naked singularity and the original naked singularity is mapped into the dual Minkowski vacuum.
3.2.2 Partial self-duality

The simplest example of partially self-dual NMGG-models is given by the following hyperbolic superpotential:

\[ W(\sigma) = B \sinh(D\sigma) \quad B > 0. \] (3.12)

It does not lead to physically interesting self-dual pCFT\(_2\), due to the fact that, similarly to the linear superpotential model considered in the beginning of this section, it has only one type (b) vacuum at \( \sigma_b = D^{-1} \sinh^{-1}\{(B\kappa L_{gr})^{-1}\} \), a naked singularities at \( \sigma \to \pm \infty \) and no one type (a) vacua\(^{12}\). The explicit form of the corresponding duality transformation (3.6) can be found by simple integration:

\[ \cosh(D\sigma) = \coth(\kappa L_{gr} BD\tilde{\sigma}), \] (3.13)

By substituting it in the defining equation (3.1), we deduce the following form of the dual superpotential:

\[ \tilde{W}(\tilde{\sigma}) = \tilde{B} \sinh(-\tilde{D}\tilde{\sigma}), \text{ with } \tilde{B} = \frac{1}{\kappa^2 L_{gr}^2 B}, \quad \tilde{D} = \kappa L_{gr} BD. \]

Therefore the original duality transformation (3.1) in the case of the hyperbolic superpotential leaves invariant its shape, but it is changing its parameters. The true self-duality is achieved for a specific "critical" value of \( B \), namely \( B = 1/\kappa L_{gr} \).

We next consider another example of partially self-dual superpotential:

\[ W(\sigma) = [B(\sigma - \sigma_a)^2 + D]^{3/2}, \quad D > 0. \] (3.14)

that give rise to an interesting strong-weak coupling self-dual pCFT\(_2\), representing dual massive and massless phases and also few self-duals CFT\(_2\)'s describing its (a) and (b) type vacua. The type (a) vacuum is placed at the critical value \( \sigma = \sigma_a \) with \( \kappa L_a = D^{-3/2} \) an its type (b) vacua at

\[ \sigma_b^\pm = \sigma_a \pm \sqrt{\frac{1}{B(\kappa L_a)^{2/3}}} \left[ \left( \frac{L_a}{L_{gr}} \right)^{2/3} - 1 \right]. \] (3.15)

Their number depends on how many real values \( \sigma_b^\pm \in \mathbb{R} \) can take. Thus, the existence and the number of the type (b) vacua is determined by the sign of \( B \) and on the values of the ratio \( L_a/L_{gr} \). Notice that, if \( B < 0 \), there are Minkowski vacua at

\[ \sigma_M^\pm = \sigma_a \pm \sqrt{\frac{D}{|B|}}, \] (3.16)

allowing the relation (3.15) to be written as

\[ \frac{(\sigma_b - \sigma_a)^2}{(\sigma_M - \sigma_a)^2} = 1 - \left( \frac{L_a}{L_{gr}} \right)^{3/2}, \] (3.17)

\(^{12}\)Although there is no problem with the geometry, the \( \beta \)-function diverges at \( \sigma = 0 \), so the holographic description is not well defined in this point.
which is valid for $B < 0$ only. Since for $B < 0$ we have $L_a < L_{gr}$, the relation above shows that $0 < (\sigma_b - \sigma_a)/(\sigma_M - \sigma_a) < 1$, i.e. the Minkowski vacua are farther from $\sigma_a$ than the type (b) vacua.

We complete our description of the vacua structure of the NMG model with superpotential (3.14) by listing all the possible different sets of allowed vacua, depending on the signs and the values of the parameters of this superpotential (see fig.1). In all the cases there exists one type (a) vacuum. With regard to the other vacua, we have:

- (I) : $L_a > L_{gr}$
  - I.a . $B > 0$: There are vacua of type (b);
  - I.b . $B < 0$: There are Minkowski vacua;

- (II) : $L_a < L_{gr}$
  - II.a . $B > 0$: There are no Minkowski nor type (b) vacua;
  - II.b . $B < 0$: There are both Minkowski and type (b) vacua;

- (III) (critical case) : $L_a = L_{gr}$
  - III.a . $B > 0$: The only vacuum is $\sigma_a = \sigma_b$;
  - III.b . $B < 0$: There are Minkowski vacua as well as $\sigma_a = \sigma_b$.

The explicit form of the duality transformation (3.6) specific for the considered superpotential (3.14) is given by:

$$
\tilde{\sigma} - \tilde{\sigma}_a = \frac{L_a}{L_{gr}} \frac{\sigma - \sigma_a}{\sqrt{1 + \frac{B}{\tilde{D}}(\sigma - \sigma_a)^2}},
$$

(3.18)

where the (arbitrary) integration constant is denoted by $\tilde{\sigma}_a$. It determines the position of the (a) type vacua dual to the original type (a) one, i.e. we have $\sigma_a \rightarrow \tilde{\sigma}_a$ under the duality transformation (3.18). This arbitrariness can (and will) be used to fix one of the (b) vacua $\sigma_b^\pm$ as a fixed point of the duality transformation.

Substituting eq. (3.18) into (3.1), one derives the form of the dual superpotential

$$
\tilde{W}(\tilde{\sigma}) = \left[ \tilde{B}(\tilde{\sigma} - \tilde{\sigma}_a)^2 + \tilde{D} \right]^{3/2}, \quad \tilde{B} = -(L_{gr}/L_a)^{2/3} B, \quad \tilde{D} = \frac{1}{(\kappa L_{gr})^{1/3} D}.
$$

(3.19)

The last equation for $\tilde{D}$ was to be expected, since it reflects only the fact that $L_a = L_{gr}^2/L_a$ (recall that $L_a = (\kappa D)^{-3/2}$). The difference of sign between the dual superpotentials is not important\textsuperscript{13} and $\tilde{W}(\tilde{\sigma})$ has the same vacua structure as its dual, which is described by cases (Ia), etc. above – but now with the “tilde” quantities $\tilde{B}$, $\tilde{L}_a$, etc. Since the transformation (3.19) changes the sign of $B$, i.e. $B/\tilde{B} < 0$, we establish the duality equivalence between the following models:

(I.a) $\leftrightarrow$ (II.b); \, (I.b) $\leftrightarrow$ (II.a); \, (III.a) $\leftrightarrow$ (III.b)

\textsuperscript{13}The global sign of the superpotential (or its dual) is relevant only in the identification $W(\sigma_A) = \pm 1/L_A$, where the sign must be chosen in order to make $L_A$ positive.
as one can see on fig.2. The most interesting case is the first one, so we will analyse it in more detail. It corresponds to \( L_a > L_{gr} \) and \( B > 0 \), thus \( \tilde{L}_a < L_{gr} \) and \( \tilde{B} < 0 \).

The vacua structure compiled in the cases (I) to (III) above is not complete without the information about the stability (UV versus IR) of the corresponding vacua, according to the sign of the critical exponents \( y_A \) given by (2.13). We have

\[
y_a = -6B \frac{L_{a}^{2/3}}{\kappa^{4/3}} \left[ 1 - \left( \frac{L_{gr}}{L_a} \right)^{2} \right] ; \quad y_b = 4B \frac{L_{gr}^{2/3}}{\kappa^{2}} \left[ 1 - \left( \frac{L_{gr}}{L_a} \right)^{2/3} \right].
\]

and therefore in the cases (I.a) and (II.b) where \( y_a < 0 \) - the type (a) vacuum is an IR critical point. The type (b) vacuum has \( y_b > 0 \) and hence it corresponds to an UV critical point. In the cases (I.b) and (II.a) the sign of \( y_a \) is reversed, i.e. \( y_a > 0 \) and now the type (a) vacua are representing the UV critical points.

The type (b) vacua \( \sigma_b^\pm \) are mapped into the type (b) vacua \( \tilde{\sigma}_B^\pm \) of the dual theory through eq. (3.18):

\[
\tilde{\sigma}_B^\pm - \tilde{\sigma}_a = \left( \frac{L_a}{L_{gr}} \right)^{2/3} (\sigma_b^\pm - \sigma_a).
\]

As said before, the constant of integration \( \tilde{\sigma}_a \) can be chosen in order to set one of the type (b) vacua as a fixed point of the duality transformation, namely \( \sigma_b^- = \tilde{\sigma}_b^- \). Thus we must have

\[
\tilde{\sigma}_a = \left( \frac{L_a}{L_{gr}} \right)^{2/3} \sigma_a - \left[ \left( \frac{L_a}{L_{gr}} \right)^{2/3} - 1 \right] \sigma_b^-. \quad (3.22)
\]

On the other hand, the constant \( \sigma_a \) is also arbitrary, since it can be changed by a translation of \( \sigma \). Hence we can further adjust it in order to put the fixed point \( \sigma_b^- \) at the origin. By taking

\[
\sigma_a = \sqrt{\frac{1}{B(\kappa L_a)^{2/3}}} \left[ \left( \frac{L_a}{L_{gr}} \right)^{2/3} - 1 \right], \quad (3.23)
\]
we get \( \sigma^+_b = 0 \), and also that \( \sigma^+_b = 2\sigma_a \) (cf. eq. (3.15)). An important consequence of this choice is that the values of the corresponding critical couplings \( \tilde{\sigma}_a \) of the dual model

\[
\tilde{\sigma}_a = \left( \frac{L_a}{L_{gr}} \right)^{2/3} \sigma_a,
\]

(3.24)

are greater than \( \sigma_a \), i.e. \( \tilde{\sigma}_a > \sigma_a \) in the cases (I.a) and (I.b), when we have \( L_a > L_{gr} \). Therefore in the asymptotic regime of very large scales \( L_a >> L_{gr} \), we realize that the weak coupling critical point \( \sigma_a - \sigma^+_b = \sigma_a << 1 \), is mapped to the strong coupling regime of the dual model since now we have that \( \tilde{\sigma}_a - \tilde{\sigma}^+_b = \tilde{\sigma}_a >> 1 \).

![Figure 2: Symbolic diagram demonstrating the duality between different phases in the case \( L_a > L_b \) and \( B > 0 \).](image)

In order to find the scale factor, we integrate the \( \beta \)-function equation, obtaining

\[
\varphi(\sigma) = -\frac{\kappa^2}{3 \varepsilon B} \int \frac{B(\sigma - \sigma_a)^2 + D}{(\sigma - \sigma_a)} \left\{ 1 - \frac{\kappa^2 L_{gr}^2 [B(\sigma - \sigma_a)^2 + D]^3}{(\sigma - \sigma_a)} \right\} d\sigma + \text{const.}
\]

With the substitution \( g(\sigma) = B(\sigma - \sigma_a)^2 + D \), we get

\[
\varphi(\sigma) = -\frac{1}{6 e B L_{gr}^2} \sum_{i=0}^{3} A_i \log(g - g_i) + \varphi_\infty,
\]

\[
e^{\varphi(\sigma)} = e^{\varphi_\infty} \prod_{i=0}^{3} |g - g_i|^{x_i}, \quad x_i = A_i/6 e B L_{gr}^2,
\]

where

\[
g_0 \equiv g_a = D, \quad A_0 = A_a = -\frac{\kappa^{4/3} L_{gr}^4}{(L_{gr}/L_a)^2},
\]

\[
g_1 = \kappa^{-2/3} L_{gr}^{-2/3}, \quad g_2 = g_3^* = -\left( \kappa L_{gr} \right)^{-2/3} \left( \frac{1+i\sqrt{3}}{2} \right),
\]

\[
A_1 = A^*_1 = \left( \frac{\kappa^{4/3} L_{gr}^{4/3}}{3[1-(L_{gr}/L_a)^{2/3}]}, \quad A_2 = A^*_2 = \frac{\kappa^{4/3} L_{gr}^{4/3}}{3[3+i(1+i\sqrt{3})(L_{gr}/L_a)^{2/3}]}.\]

(3.25)

Notice that although both the exponents \( x_{2,3} \) and the roots \( g_{2,3} \) are complex, the last two terms of the product above are real, and so is the expression for the scale factor. Also notice that \( \sum x_i = 0 \), hence if \( \sigma \to \infty \) we have \( e^\varphi \to e^{\varphi_\infty} \), allowing for the possibility of a naked singularity. This property also allow us to rewrite the scale factor more explicitly as

\[
e^{\varphi(\sigma)} = e^{\varphi_\infty} (\sigma - \sigma_a)^{2x_a} (\sigma - \sigma_b^+)^{x_1} (\sigma - \sigma_b^-)^{x_2} \prod_{i=2}^{3} ((\sigma - \sigma_a)^2 + (D - g_i)/B)^{x_i},
\]

(3.26)
where $\sigma_b^- = 0$ and $\sigma_b^+ = 2\sigma_a$, with the constant $\sigma_a$ being given by (3.23). It is now evident the singular behaviour of the scale factor (and hence of the correlation length) near to the vacua (i.e. the critical points).

It is worthwhile to mention that the particular (degenerate) case $D = 0$ leads to superpotential:

$$W(\sigma) = E(\sigma - \sigma_M)^3, \quad E > 0,$$

which has different vacua structure and in fact it is not any more partially self-dual. The duality transformation in this case has the form:

$$\sigma - \sigma_M = \sqrt{\frac{1}{2L_{gr}E}} \frac{1}{\sqrt{\sigma_M - \sigma}}, \quad \tilde{\sigma}_M > \tilde{\sigma},$$

which allows us to deduce the explicit form of the corresponding dual superpotential:

$$\tilde{W}(\tilde{\sigma}) = \tilde{E}(\tilde{\sigma}_M - \tilde{\sigma})^{3/2}, \quad \tilde{E} = 2^{3/2} \left( \frac{E}{L_{gr}} \right)^{1/3},$$

Evidently we have an example of dual transformation that is not preserving neither the shape of the superpotential nor the values of its parameters.

Let us also mention that the examples we have studied in this subsection, do not exhaust all the possible partially self-dual superpotentials of one or two type (a) vacua. Another physically interesting example of pairs of NMG’s is given by the following periodic superpotential:

$$W(\sigma) = B [D - \cos(\alpha\sigma)], \quad B < 0,$$

whose vacua structure -of two type (a) vacua-, its duality properties and also certain features of the phases of its dual pCFT$_2$ are described in the Appendix below.

The construction of examples of self-duals and partially self-dual NMG’s based on superpotentials having more then two type (a) non-degenerate critical points and the explicit forms (3.18) of the corresponding duality transformations, involves relatively big number of W-parameters (i.e. the matter fields couplings $g_k$ as B,D etc.). It represents a rather complicated open problem and requires better understanding of the group properties of the couplings $\sigma$-transformations and of the group structure behind our definition of the partial self-duality, as well as further investigations of the group-theoretical nature of the W-parameters $\tilde{g}_k = \tilde{g}_k(g_k)$ transformations, see (3.19).

### 3.3 Unitary consistency of duality

An important test for the physical consistency of the pairs of dual vacua, i.e. pair of AdS$_3$’s of dual radii $L_a$ and $\tilde{L}_a = L_{gr}^2 / L_a$, is the verification of whether and under what conditions (if any) they both belong to the same BHT negative unitary window (1.2). Let us first briefly remind the content of the BHT unitarity conditions for the NMG models [17],[18]. Remember that the negative value
of $m_\sigma^2(A)$ for scalar fields (tachyons) in AdS$_3$ backgrounds, which appears in the dimensions of the relevant operators, do not cause problems when the Breitenlohner-Freedman (BF) condition \cite{BF},
\[ \Lambda_{\text{eff}}^A \leq m_\sigma^2(A), \]
is satisfied. The unitarity of the purely gravitational sector of the NMG model \cite{NMG} requires that the following Bergshoeff-Hohm-Townsend (BHT) conditions \cite{BHT}:
\[
m^2 (\Lambda_{\text{eff}}^A - 2\epsilon m^2) > 0, \quad \Lambda_{\text{eff}}^A \leq M_{gr}^2(A) = -\epsilon m^2 + \frac{1}{2}\Lambda_{\text{eff}}^A,
\]
take place. Taking into account that for the each vacua $\sigma_A$ the BHT parameter $\lambda_A$ can be realized as follows:
\[
\lambda_a = \lambda(\sigma_a^*) = \frac{\kappa^2}{2m^2} V(\sigma_a^*) = \frac{L_{gr}^2}{L_a^2} \left( \frac{L_{gr}^2}{L_a^2} - 2 \right),
\]
which for the type (b) vacuum, i.e. $W_{\pm}^2 = \frac{2m^2}{\kappa^2}$, reproduces the lower bound $\lambda = -1$ of BHT-condition \cite{BHT}.

In order to derive the unitarity restrictions on the generic type (a) vacuum we introduce the following notation: $q = \frac{\Lambda_{\text{eff}}^A}{\Lambda_{\text{eff}}^B} = \frac{\kappa^2 W^2}{2m^2}$. Then we have $\lambda_{(a,b)} = q(q - 2) \equiv \lambda(q)$, which makes evident that $\lambda(q) = \lambda(2 - q)$. Therefore the $\lambda_a$ values for which the unitarity condition \cite{BHT} is satisfied are imposing restrictions on the allowed $L_a$ values:
\[
0 \leq \frac{L_{gr}^2}{L_a^2} \leq 2, \quad \epsilon = -1, \quad m^2 < 0
\]
and consequently on the central charges \cite{BHT} of the corresponding CFT’s. The type (b) NMG vacua are known to be always unitary \cite{NMG} of $\lambda = -1$ and whether it represents UV or IR critical points of the dual pCFT$_2$ depends on the sign factor only: UV for $\epsilon = -1$, since we have $y_b > 0$, and IR for $\epsilon = 1$. The properties of the type (a) critical points (UV or IR) do depend on both the sign of $\epsilon$ and the particular form of the matter superpotential, as one can see from eq.\cite{2.13}. The unitarity of the NMG-matter model is still an open problem, and it requires further analysis of the linear fluctuations around the DW’s relating, say, two unitary BHT-vacua from the negative BHT-unitary window: $-1 \leq \lambda < 0, \epsilon = -1, m^2 < 0$. We are however obliged to require that at least all the NMG-matter model’s vacua are BHT-unitary.

In the context of the NMG duality transformations, when applied for the critical points $\tilde{L}_a = \frac{L_{gr}^2}{L_a}$, we impose an additional condition, namely that the ”dual” scales $\tilde{L}_a$ and $L_a$ are both belonging to the same (negative) BHT - unitary window\cite{BHT}. Taking into account the eqs.\cite{3.33} and \cite{3.3} we conclude that the NMG duality \cite{3.1} is compatible with the NMG unitarity only when the following conditions are fulfilled:
\[
\frac{L_{gr}}{\sqrt{2}} \leq \tilde{L}_a \leq L_{gr} \leq L_a \leq L_{gr} \sqrt{2}
\]
Hence when $L_a$ and its dual scale $\tilde{L}_a$ both belong to certain finite interval of values $(L_{gr}/\sqrt{2}, L_{gr} \sqrt{2})$ they describe dual pairs of unitary NMG’s vacua.
3.4 On the group properties of partial self-duality

One of the main features of the strong-weak coupling duality is that in the self-dual 2-, 3- and 4-dimensional (supersymmetric) QFT’s, it is always realized as an inversion transformation and more generally as fractional linear transformations belonging to certain (discrete) subgroups of $SL(2, C)$ \([1–3]\). It is therefore important to verify whether these well known properties of the QFT’s duality (or certain limits of them) take place in the particular examples of $pCFT_2$’s duals to (pairs of) NMG models with appropriately chosen superpotentials \((3.1)\). The question about the gravitational $d = 3$ NMG meaning of the $d = 2$ conditions of strong-weak coupling duality symmetries in the considered two dimensional $pCFT$’s is also addressed.

Let us remind that the requirements on the NMG’s superpotentials \((3.1)\), that select holographic self-dual $pCFT_2$’s, have been introduced by extending the ”critical” duality transformation $\tilde{L}_A = \frac{L^2_{gr}}{L_A}$ (at each critical point $\sigma^*_A$) to its ”off-critical” equivalent \((3.1)\). As a consequence we have deduced the explicit form \((3.35)\) of the corresponding coupling’s $\tilde{\sigma} = \tilde{\sigma}(\sigma)$ transformations that are keeping invariant the central charges, central functions, s.p. of the reduced free energy, but in principal they are changing the form of the exact holographic $\beta$-functions, according to eqs.\((3.31)\). Notice that the $L_A$’s transformation (and the $W$’s as well) represents a particular $G_L \in GL(2, R)$ transformation, i.e.

\[
\tilde{L}_A = \frac{aL_A + b}{cL_A + d}, \quad G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad G_L = \begin{pmatrix} 0 & L^2_{gr} \\ 1 & 0 \end{pmatrix}, \quad G_L^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{1}{L^2_{gr}} & 0 \end{pmatrix} \tag{3.35}
\]

By introducing their dimensionless counterparts, say $l_A = L_A/L_{gr}$ we indeed recover the well known standard large-small radii $Z_2$ inversion transformation:

\[
\tilde{l}_A = 1/l_A, \quad \text{i.e.} \quad G_l = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = G_l^{-1}, \tag{3.36}
\]

such that the large $l_A \gg 1$ (i.e. $L_A \gg L_{gr}$) are mapped to the very small $\tilde{l}_A \ll 1$ ones (in the $L_{gr}$ units of length).

We next consider the problem of the similarities and the differences between the group properties of the particular strong-weak coupling $pCFT_2$’s duality transformations \((3.11)\) and \((3.18)\), present in the specific -one type (a) vacua- examples of self-dual(SD) and partially self-dual(PSD) models, studied in Sect.3.2.

**Self-dual models.** The corresponding SD coupling’s transformation is almost identical of the $L_A$’s ones:

\[
\tilde{\sigma} = \frac{\sigma^2}{\sigma^*_A}, \quad \sigma_+ = 1/\sqrt{\kappa L_{gr} B} \quad \text{with} \quad \sigma_+ = \tilde{\sigma}_+, \tag{3.37}
\]

which takes the standard inversion form $\tilde{u}_{sd} = 1/u_{sd}$ for the rescaled coupling $u_{sd} = \frac{\sigma}{\sigma_+}$. Notice that the strong couplings $\sigma \gg \sigma_+$ are mapped to the weak ones $\tilde{\sigma} \ll \sigma_+$. An important feature of this

\[\text{i.e. of } SL(2, Z) \text{ as for example in the cases of models having discrete spectrum of energies or/and charges- electric and magnetic etc.} \]
self-dual model is that the above SD transformations are leaving invariant the RG equation:

\[
\frac{du_{sd}}{dt} = \frac{4\varepsilon}{\kappa^2 u_{sd}} (1 - u_{sd}^4) = -\beta_{sd}(u_{sd}),
\]

and the form of the corresponding exact \(\beta\)-function, i.e., we have \(\beta_{sd}(u) = \beta_{sd}(\tilde{u})\) as well. It is important to mention that this \(\beta_{sd}\)-invariance property is indeed consistent with the general covariance requirement (3,2). It reflects a very particular form of our SD superpotential and related to it \(\beta_{sd}\).

**Partially self-dual models.** Let us consider the PSD transformation (3.18) for the square of the coupling \(u = (\sigma - \sigma_a)^2\), i.e.,

\[
\tilde{u} = \left(\frac{1}{d^2}\right) \frac{u}{d + bu}, \quad u > 0, \quad d = (\kappa L_{gr})^{2/3}D, \quad b = (\kappa L_{gr})^{2/3}B, \quad d > 0, \quad b > 0.
\]

It is then evident that it represents a two parameters subgroup of the (general) fractional linear transformations \(G_{psd}(d, b) \in GL(2, R)\):

\[
G_{psd} = \begin{pmatrix} 1/d^2 & 0 \\ b & d \end{pmatrix} = \begin{pmatrix} 1/d^2 & 0 \\ b/d & 1 \end{pmatrix} : G_{psd}^{-1} = \begin{pmatrix} d^2 & 0 \\ -bd & 1/d \end{pmatrix} = \begin{pmatrix} 1/d^2 & 0 \\ \tilde{b} & \tilde{d} \end{pmatrix},
\]

composed as a semi-direct product of one specific “dilatation”, of \(\text{Det}G_{dil} = 1/d\), and the special conformal transformation\(^{17}\) of parameter \(b/d\) — with the well known group laws: \(d_3 = d_1d_2\) and \(b_3 = b_2d_1 + b_1/d_2^2\). Notice that, differently from the SD transformation (i.e. the simple inversion), the inverse element \(G_{psd}^{-1}\) in the PSD case is not coinciding with \(G_{psd}\). It is instead providing a group-theoretical meaning of the duality transformations (3.19) for the parameters of the superpotential, that according to our general duality formula (3.6) are parametrizing the group of the duality transformations: \(d\tilde{\sigma}/d\sigma = -\kappa L_{gr} \tilde{W}(\tilde{\sigma})\). Hence the parametric form of the partially self-dual superpotential (3.14) is determined by the PSD duality group elements \(G_{psd}(b, d) \in GL(2, R)\). Thus, our particular choice of the PSD superpotential (3.14) introduces certain group structure on the space of W-parameters, representing the set of couplings in the potential \(V(\sigma, G_{psd})\) of the 3d matter field of the NMG-matter model. The superpotential \(\tilde{W}(\tilde{\sigma}, G_{psd}^{-1})\) of the second member of the dual pair of NMG models is then parametrized by the corresponding inverse elements \(G^{-1}(b, d)\). This is in fact the NMG gravitational counterpart of the \(d = 2\) QFT’s self-duality requirements. It is also in the origin of the important property of the partially self-dual models, namely that the \(\beta\)-functions of such pairs of models have the same form, i.e. the PSD transformations (3.18) and (3.39) are keeping invariant the form of the corresponding RG equation:

\[
\frac{d\sigma}{dl} = \frac{6\varepsilon B(\sigma - \sigma_a)}{\kappa^2 (B(\sigma - \sigma_a)^2 + D)} \left(1 - \kappa^2 L_{gr}^2 (B(\sigma - \sigma_a)^2 + D)^3\right) = -\beta_{psd}(\sigma; \sigma_a, B, D),
\]

\(^{15}\)Together with the free energy, central function and the anomalous dimensions

\(^{16}\)We are simultaneously rescaling the B and D parameters in the way that the equivalent rescaling of the superpotential \(w = \kappa L_{gr} W\) leads to the standard inversion form of the duality condition (3.3): \(\tilde{w}(\tilde{\sigma}) = 1/w(\sigma)\).

\(^{17}\)Remember that one can always realize the special conformal transformation as a product of three consecutive transformations — inversion, translation by \(b/d\) and one more inversion.
but with its W-parameters $\sigma_a, B, D$ replaced by their duals: $\tilde{\sigma}_a, \tilde{B}, \tilde{D}$. Thus, the RG’s equation of the dual $pCFT_2$ has the expected form $d\tilde{\sigma}/dl = -\beta_{psd}(\tilde{\sigma}; \tilde{\sigma}_a, \tilde{B}, \tilde{D})$. The “slight” difference between the invariance conditions of the RG equations and of forms of the $\beta$-functions of the considered SD and PSD models has its origin in the different group properties of their coupling transformations (3.11) and (3.18).

The specific “fractional-linear” form of the PSD transformation (3.39) requires further investigation of the problem of whether strong couplings $u$ are mapped to weak ones $\tilde{u}$ for all the values of the parameters $b$ and $d$. Let us first note one particular feature of our PSD transformation (3.39), namely that $u(0) = 0 = \tilde{u}(0)$ and $u(\infty) = 1/bd^2 = \tilde{u}(1/ bd^2)$ and therefore it is mapping the the positive semi-axis $u \in (0, \infty) = R_+$ to the finite interval $\tilde{u} \in (0, 1/ bd^2)$. It is then clear that in order to transform the large values of $u$ (and of $\sigma$ as well) into the small ones of the $\tilde{u}$ and vice-versa we have to impose the following restriction on the values of the parameters $b$ and $d$:

$$bd^2 \gg 1 \quad \text{or equivalently} \quad BD^2 \gg \frac{1}{\kappa^2 L_{gr}} = 2|m^2|/\kappa^2.$$  \hfill (3.42)

Symmetries of RG equations vs. Duality. As we have shown, the “duality invariance” of the RG equations and of the form of the holographic $\beta$-functions turns out to be one of the main features specific for the class of the SD and PSD models only. It is important however to mention that the SD and PSD duality transformations are not exhausting all the symmetries of the RG equation. In fact one can find more symmetries of the corresponding RG equations, that are not preserving neither the central function nor the free energy. Therefore the invariance of RG equations under a kind of strong-weak coupling transformations can’t be considered as a definition of (partial) self-duality of $pCFT_2$’s under investigation. We shall give an example of such “additional” symmetries of the RG eq. (3.41) for the SD model. Let us first rewrite it in the following equivalent form:

$$\frac{dg}{dl} = g^2 - a^2, \quad g = 8B^2L_{gr}\sigma^2, \quad a = \frac{8BL_{gr}}{\kappa}. \hfill (3.43)$$

Apart of the already discussed duality symmetry $\tilde{g} = \frac{a^2}{g}$, it is also invariant under specific fractional linear transformations

$$g(l) \rightarrow g'(l) = \frac{\cosh(a\gamma)g(l) - a \sinh(a\gamma)}{-\sinh(a\gamma)g(l) + \cosh(a\gamma)}, \hfill (3.44)$$

where $\gamma \in R$ is an arbitrary real parameter. These transformations can be recognized as an $SO(1, 1)$ subgroup of $SO(2, 1)$. In spite of the fact that for certain restrictions on the parameters $a$ and $\gamma$ they are mapping strong to weak couplings, they are not keeping invariant the corresponding central functions, anomalous dimensions and free energy and therefore are not representing duality transformations at all.

18 in the case of generic duality transformations (3.1) and (3.5), the pairs of dual $\beta$-functions are related by the eq. (3.4) and the corresponding RG equations does not remain invariant.

19 notice that the corresponding transformations of the original “coupling variable” $\sigma = \sqrt{\frac{m^2}{2\kappa}}$ are also forming an $SO(1, 1)$ group.
It is worthwhile to mention that the above eq. also appears as RG equation for two rather different models: (a) the RG eq. of the $pCFT_2$ dual to the NMG model of linear superpotential (see ref. [29]); (b) the well known one-loop RG equation with the perturbative $\beta$-function given by (2.9), specific for the perturbations of the so called $\Phi_{13}$ relevant operators of the minimal $CFT_2$'s [33, 34]. In both cases however neither its inversion symmetry $\tilde{g} = a^2 g$ nor the above considered $SO(1, 1)$ symmetry act as proper strong-weak coupling duality.

**Few comments and relevant open questions:**

- The two simple examples of self-dual and partially self-dual superpotentials, that generate very specific (limits of) duality groups and give rise to self-dual $pCFT_2$'s, are indeed not representing all the possible (partial) self-duality transformations. One could considerer, for example, a simple three parameters quadratic superpotential, that turns out to generate (within the NMG context considered in this section) more general $SL(2, R)$ duality transformations.

- The most interesting cases of explicit realizations of the self-duality in the mentioned 2d and 4d QFT's models (see for example [1–3]), that have the $SL(2, Z)$ (sub)group as duality symmetries, are known to be with complex valued coupling constant (or equivalently of two real couplings). In the case of the considered NMG-matter models, it will corresponds to specific two scalar fields matter interactions superpotentials. The problem of the generalizations of the concepts of NMG duality (3.1) to the case of complex fields, based on an appropriate I-st order system of DW’s equations, and of the corresponding constructions of the two $\beta-$functions in terms of these superpotential is under investigation.

4 **Holographic RG flows and self-duality**

The off-critical NMG$_3$/QFT$_2$ conjecture, based on the holographic RG eqs. (2.6), is a natural generalization of the standard ($m^2 \to \infty$) holographic RG [9][10]. Let us remind its content: there exists a family of QFT$_2$ such that their near-critical behaviour and phase structure admit a non-perturbative geometrical description in terms of DW’s solutions of the NMG-matter model (1.4) with an appropriately chosen superpotential $W(\sigma)$. The first part of this statement concerns the identification of the NMG vacua $(\sigma^*_A, L_A, y_A)$ with the critical $CFT_2$-data of the dual QFT$_2$ as we have done in Sect. 2 above. Its second part is about the explicit relation between the set of “consecutive” DW solutions

$$DW_{k,k+1} = (\sigma(z), e^{\sigma(z)}; z \in R \ | \ \sigma^*_k, L_k \to \sigma^*_k, L_{k+1}), \ \sigma \in R,$$

and all the QFT$_2$ phases $p_{k,k+1}^{in} = (\sigma^*_k(\text{IR}), \sigma^*_k(\text{UV}))$ described by the coupling constant $\sigma_{k,k+1}(l)$ and the s.p. of the free energy $F_s(\sigma) \approx e^{-i(\sigma)}$ behaviours. In what follows, our attention is concentrated on the properties of the couples of neighbour DW’s of common boundary $(\sigma^*_\text{UV}, L_\text{UV}, y_\text{UV})$ that have different (IR)-horizons b.c.’s, say for example $(\sigma^*_\text{IR}, \sigma^*_\text{UV})$ and $(\sigma^*_\text{UV}, \infty)$. They represent the main ingredient in the description of the phase transitions and of the nature of the holographic RG flows [29, 30].
4.1 The phases of the self-dual superpotential

Let us recall which of the solutions of the RG eqs. (2.5) and (2.7) – defined within a given interval, say $\sigma \in (\sigma_+, \infty)$ or $\sigma \in (\sigma_-, \sigma_s)$, etc. – can be identified as describing the particular massive RG flows in the related QFT$_2$. The main requirement is that the running coupling $|\sigma(l) - \sigma_+|$ gets its maximal value for a finite RG distance, for example $\sigma(L_{\text{max}}) = \infty$ or $\sigma(L_{\text{max}}) = \sigma_{\text{max}} = |\sigma_s - \sigma_+|$, etc., imposing that the correlation length, say $\xi(\infty) = \xi_{\text{max}} = 1/M_s$ always has a finite maximal value. Then its inverse defines the smallest mass gap in the energy spectrum, and as a consequence of eqs. (2.7), the corresponding 2-point correlation function manifests an exponential decay: $e^{-M_{\text{max}}|x_{12}|}$ – typical for the IR limit of the propagator of a free massive particle. This behaviour has to be compared to the one corresponding to the massless RG flows, where the maximal distance $|\sigma_{\text{IR}} - \sigma_{\text{UV}}|$ from the starting (at $L_s = 0$) UV critical point is reached for $L_{\text{max}} = \infty$, i.e. $\xi(L_{\text{max}}) = \infty$ and therefore no mass gap exists, since $M^2 = 0$. As a result, the correlation functions at an IR critical point have power-like (scale invariant) behaviour.

Examples of such massless phases are found in the self-dual superpotential $W = B\sigma^2$. Taking $B > 0$, we have two massive phases $p_{\text{flat}}^{\text{ms}} = (0, \sigma_+)$ and $p_{\text{n.s.}}^{\text{ms}} = (\sigma_+, \infty)$, described holographically by two DW’s, one of $E_3/\text{AdS}_3$ type and the other of $\text{AdS}_3/\text{n.s.}$ type, with a common boundary at the type (b) vacuum $\sigma_+ = 1/\sqrt{\kappa L_{\text{gr}}B}$. We consider here only positive values of $\sigma$ because of the $Z_2$ symmetry of the superpotential. This massive nature of the phases can be apprehended by the correlation length $\xi(\sigma)$, which can be found through the corresponding RG equation:

$$\frac{d\sigma}{dl} = -\beta_\sigma(l, \sigma) = \frac{4\epsilon}{\kappa^2 l^2}(1 - \kappa^2 L_{\text{gr}}^2 B^2 \sigma^4).$$

(4.1)

It has as solution $\sigma^2(l) = \sigma_+^2 \coth(l_0 - \frac{B\sigma}{l})$, leading to

$$e^{-l} \approx \xi(\sigma) = \left[\frac{(\sigma_+^2/\sigma^4) + 1}{(\sigma_+^2/\sigma^4) - 1}\right]^{\frac{1}{2}} \left[\frac{(\sigma_0^2/\sigma^4) - 1}{(\sigma_0^2/\sigma^4) + 1}\right]^{\frac{1}{2}}, \quad y_+ = -\frac{16\epsilon BL_{\text{gr}}}{\kappa}.$$  

(4.2)

This expression is singular at $\sigma_+$, and the divergence (for $\epsilon = -1$) of the scale factor shows that it is an UV vacuum. On the other hand, the correlation length takes finite values at both the singular point $\sigma_0 = 0$, which is a flat vacuum in the weak-coupling “massive-flat” phase, and at $\sigma \to \infty$, in the standard strong-coupling massive phase. It is easy to calculate the corresponding mass gaps, say

$$M_{\text{n.s.}}(\sigma_0) = 1/\xi(\infty) = (\kappa L_{\text{gr}} B\sigma_0^2 - 1)^{\frac{1}{2\sigma_+}} (\kappa L_{\text{gr}} B\sigma_0^2 + 1)^{-\frac{1}{2\sigma_+}},$$

thus confirming the massive nature of $p_{\text{flat}}^{\text{ns}} = (0, \sigma_+)$ and $p_{\text{n.s.}}^{\text{ms}} = (\sigma_+, \infty)$. The duality transformation here is known from Sect.3.2.1 to be $\tilde{\sigma} = \sigma_+^2/\sigma$, leaving the superpotential invariant: $\tilde{W}(\tilde{\sigma}) = B\tilde{\sigma}^2$, as well as the vacuum $\sigma_+ = \tilde{\sigma}_+$. But the singular points are “exchanged” through $\tilde{\sigma}(\sigma_s = 0) = \infty$ and $\tilde{\sigma}_s(\sigma \to \infty) = 0$, and so there is a correspondence between the two massive phases with strong and weak coupling: $M_{\text{n.s.}}(\sigma_0) = M_{\text{flat}}(\tilde{\sigma}_0)$.  

21
4.2 Phase transitions and partial self-duality

The phase structure of the partially self-dual superpotential \( W(\sigma) = |B(\sigma - \sigma_a)^2 + D|^{3/2} \), studied in Sect.3.2., depend on the range of the values of the parameters \( B \) and \( L_a^{-1} = \kappa D^{3/2} \), as shown on fig.1. In the case (I.a), corresponding to \( L_a > L_{gr} \) and \( B > 0 \), we have an IR critical point at \( \sigma_a \) and two UV critical points at \( \sigma^\pm_b \). There are four DW’s, which describe the four different phases of the corresponding dual QFT2 in this region of the parameter space:

\[
p_{\text{n.s.}}^{\text{ms}} = (-\infty, \sigma_b^-); \quad p_{\text{UV/IR}}^{\text{ml}} = (\sigma_b^-, \sigma_a); \quad p_{\text{IR/UV}}^{\text{ml}} = (\sigma_a, \sigma_b^+); \quad p_{\text{n.s.}}^{\text{ms}} = (\sigma_b^+, \infty). \tag{4.3}
\]

The nature – massive (ms) or massless (ml) – of the phases can be easily read from the scale factor’s (3.20) analytic properties, which determines the correlation length of the dual pCFT2:

\[
\xi(\sigma) \approx \left( \frac{\sigma_0 - \sigma_a}{\sigma - \sigma_a} \right)^{\frac{1}{y_a}} \left( \frac{\sigma_0 - \sigma_b^+}{\sigma - \sigma_b^+} \right)^{\frac{1}{y_+}} \left( \frac{\sigma_0 - \sigma_b^-}{\sigma - \sigma_b^-} \right)^{\frac{1}{y_-}} \prod_{i=2}^{3} \left[ \frac{(\sigma_0 - \sigma_a)^2 + (D - g_i)/B}{(\sigma - \sigma_a)^2 + (D - g_i)/B} \right]^{-x_i}. \tag{4.4}
\]

The critical exponents are given by eqs. (3.23). They also satisfy the remarkable NMG ”resonance” condition

\[
\frac{1}{y_a} + \frac{1}{y_+} + \frac{1}{y_-} = \sum_{i=2}^{3} x_i,
\]

that turns out to hold for all the QFT’s models, obtained by NMG3 holography [29]. The “initial condition” \( \sigma_0 \equiv \sigma|_{\epsilon=0} \) of RG rescaling can be further fixed by requiring that \( L_{(e)}^0 \approx 1 \). As we have shown in Sect.3.2. for \( \epsilon = -1 \) we have \( \sigma_a < 0 \) and consequently \( \xi(\sigma_a) \rightarrow 0 \); therefore \( \sigma_a \) is an IR critical point, while for the (b) type critical points: \( y_\pm > 0 \) hence \( \xi(\sigma_\pm_b) \rightarrow \infty \) and \( \sigma^\pm_b \) are UV critical points. Notice that the finite values of \( \xi(\sigma) \) when \( \sigma \rightarrow \pm \infty \) and, as a consequence, the existence and properties of the massive phase are due to the above mentioned NMG resonance condition, i.e. the fact that the sum of the critical exponents \( \nu_k \) (of all the critical points) vanishes. The corresponding values of the mass gaps for the massive phases can be evaluated at these limits \( \sigma \rightarrow \pm \infty \), which correspond to naked singularities in the NMG-geometry. For example, the strong-coupling massive phase \( p_{\text{n.s.}}^{\text{ms}} = (\sigma_b^+, \infty) \), is characterized by the asymptotic value of the correlation length (4.4), which determines the smallest mass in the dual model:

\[
M_{(ms)} \approx \xi^{-1}|_{\sigma \rightarrow \infty} = (\sigma_0 - \sigma_a)^{\frac{1}{y_a}} (\sigma_0 - \sigma_b^+)^{\frac{1}{y_+}} (\sigma_0 - \sigma_b^-)^{\frac{1}{y_-}} \prod_{i=2}^{3} \left[ (\sigma_0 - \sigma_a)^2 + (D - g_i)/B \right]^{-x_i}. \tag{4.5}
\]

We next describe the duality between the strong- and weak-coupling phases of the considered partially self-dual pCFT2 model, i.e. how the duality transformation \([3.18-3.19]\) is effectively mapping the phases of this model. As we have demonstrated in Sect.3.2., the phases duals of the above considered (I.a) case are those of the (II.b)- model (see fig.2.):

\[
p_{\text{flat}}^{\text{ms}} = (\tilde{\sigma}_M, \tilde{\sigma}_b^-); \quad p_{\text{UV/IR}}^{\text{ml}} = (\tilde{\sigma}_b^-, \tilde{\sigma}_a); \quad p_{\text{IR/UV}}^{\text{ml}} = (\tilde{\sigma}_a, \tilde{\sigma}_b^+); \quad p_{\text{flat}}^{\text{ms}} = (\tilde{\sigma}_b^+, \tilde{\sigma}_M). \tag{4.6}
\]

i.e. of our original partially self-dual model, but now with different range of the values of the parameters: \( \tilde{B} < 0 \) and \( \tilde{L}_a < L_{gr} \). The correlation length \( \tilde{\xi}(\tilde{\sigma}) \) has the same form (4.4) as above,
but with the parameters exchanged by the duality according to eq.\((3.19)\). Notice that although \(B\) changes its sign, the critical exponents do not, since the ratio \(L_{gr}/L_a\) is now greater than unity. We have to remind that the corresponding "dual massive" phases correspond to non-singular, \(E_3/AdS_3\) DW's solutions, with a mass gap given by

\[
\tilde{M}_{ms} \approx \tilde{\xi} |_{\tilde{\sigma} \rightarrow \tilde{\sigma}_0} = \left( \frac{\tilde{\sigma}_0 - \tilde{\sigma}_a}{\tilde{\sigma}_0 - \tilde{\sigma}_b} \right)^{-\frac{1}{w_m}} \left( \frac{\tilde{\sigma}_0 - \tilde{\sigma}_a}{\tilde{\sigma}_0 - \tilde{\sigma}_b} \right)^{-\frac{1}{w_m}} \times \\
\times \left( \frac{\tilde{\sigma}_0 - \tilde{\sigma}_a}{\tilde{\sigma}_0 - \tilde{\sigma}_b} \right)^{-\frac{1}{w_m}} \prod_{i=2}^{3} \left[ \frac{(\tilde{\sigma}_a - \tilde{\sigma}_0)^2 + (D-g_i)/\tilde{B}}{(\tilde{\sigma}_0 - \tilde{\sigma}_a)^2 + (D-g_i)/\tilde{B}} \right]^{\tilde{x}_i}.
\]

while in the (Ia) case they are related to the singular \(AdS_3/n.s.\) DW's, interpolating between one \(AdS_3\) vacua and a naked singularity. The large values of the formerly unbounded coupling \(\sigma\) is now mapped at the (small) finite values of \(\tilde{\sigma}\) in the neighbours of the Minkowski vacua.\(^{20}\) We can conclude that in the dual theory the "infinitely strong" couplings are mapped into a finite values, both however corresponding to massive phases: hence the strong coupling massive phase is mapped to certain "dual" weak coupling massive phase. The dual massless phases, on the other hand, are "stretched" by the duality transformation, as one can see from eq.\((3.21)\): the interval \((\tilde{\sigma}_a, \tilde{\sigma}_+\)\) is "longer" than its dual, for \(L_a/L_{gr} > 1\).

Similar statements are valid for all the other pairs of dual models described in Sect.3.2.:

\((I.a) \leftrightarrow (II.b); \ (I.b) \leftrightarrow (I.a); \ (III.a) \leftrightarrow (III.b)\)

Let us mention that the behaviour of the correlation length and the properties of the marginally degenerate cases \((III.a)\) and \((III.b)\), that in fact describe a pair of dual models with an infinite order phase transition at the critical point \(\sigma_a = \sigma_b\) and having two massive phases, are quite similar to the ones of the NMG model of quadratic superpotential, studied in ref.\(^{30}\).

Few comments are now in order:

(a) the phase structure, the corresponding RG flows and the duality relations between different phases of our second example of partially self-dual "periodic" superpotential \((6.1)\) (we have introduced in App.A.), are rather similar to the one we have described in this subsection;

(b) the holographic RG flows in the pCFT\(_2\) model dual to the NMG of quadratic superpotential

\[
W(\sigma) = B(\sigma - \sigma_a)^2 + D, \quad D \neq 0
\]

\((4.7)\)
can be easily found by applying the methods developed in Sect.3.1. and by using the results of refs.\(^{29}, 30\). Although (for \(D \neq 0\)) it is neither self-dual (as in the \(D = 0\) case) nor partially self-dual, it possess a rich and interesting phase structure\(^{29}\).\(^{30}\). It is worthwhile to also mention the well known fact that it represents the near-critical behaviour of an arbitrary (even) superpotential.

\(^{20}\) Notice that such vacua of \(W(\sigma_M) = 0\) are not representing "conformal critical" points, but instead are defining a particular massive phase\(^{29}, 30\).
5 Discussion

The holographic RG methods, when applied to the NMG-matter models with appropriate superpotentials, provide important critical (about certain CFT$_2$’s) and off-critical (of the corresponding pCFT$_2$’s) data, which can be used for their identification with the — already known — perturbative and exact CFT$_2$ and pCFT$_2$’s results [11, 23, 25]. It is worthwhile to remind once more that all the information about the holographic RG flows and phase transitions in the QFT$_2$’s dual to the NMG model (1.4) are not sufficient for the complete identification of the pCFT$_2$ dual to a given NMG-matter model. One has to further consider the difficult problem of the construction of the off-critical correlation functions of 2d fields dual to the 3d matter scalar, by studying the linear fluctuations of the metrics and of the scalar field around the DW’s solutions [10, 21, 22, 37]. The real problem with the verification of the validity of the off-critical (a)AdS$_3$/pCFT$_2$ conjecture consists however in the comparison of the strong-coupling holographic results, based on the exact $\beta$-functions, with the known perturbative, near-critical calculations of the corresponding 2d models [13, 23, 24, 34].

The construction of a particular class of strong-weak coupling self-dual pCFT$_2$’s models, i.e. the holographic duals of selected pairs of NMG-matter models with partially self-dual superpotentials, described in Sects.3 and 4 represents an important exception. In this case it becomes possible to compare the holographic non-perturbative results with the ones obtained by the conformal perturbation theory [23, 34].

Another important problem concerning the (a)AdS$_3$/pCFT$_2$ correspondence, in the particular case of the NMG model (1.4), is related to the negative values of the central charges (1.3) for $\epsilon = -1$ and $m^2 < 0$. These are usually interpreted as non-unitary CFT$_2$’s. Let us assume that all these CFT$_2$’s, without any extra symmetries present, are described by the representations of two commuting Virasoro algebras, characterized by their central charges $c_L = c_R = c$, and the set of scaling dimensions and spins [11]. In all the cases when $c < 0$, the corresponding CFT$_2$’s contain primary fields (states) of negative dimensions (and negative norms), and hence they represent non-unitary QFT$_2$’s [21]. As it is well known, in the interval $0 < c < 1$ there exists an infinite series of “minimal” unitary quantum models corresponding to $c_{\text{quant}}^- (p) = 1 - 6Q^2_p$, with $Q_p = \sqrt{\frac{p+1}{p}} - \sqrt{\frac{p}{p+1}}$ and $p = 3, 4, 5, \ldots$, while the models with $c > 25$ give rise to unitary representations used in the quantization of the Liouville model [13]: $c_+ (b) = 1 + 6(b + \frac{1}{b})^2$, where the parameter $b$ is related to the Liouville coupling constant. On the other hand, the derivation of the Brown-Henneaux [33] central charge formula $c = \frac{3L}{2G}$, as well as its NMG generalizations (1.3), are based on the “Dirac quantization” of the classical Poisson brackets of the Virasoro algebra, and by further identifying the classical central charge $c_{\text{class}}$ for $L \gg l_{\text{pl}}$ with the “quantum” central charge $c_{\text{quant}}$ of the “dual” boundary CFT$_2$. The well known fact, coming from the standard procedure of the Liouville models [13] and of the “minimal” models quantizations [24], is that this classical central charge is receiving quantum corrections, i.e. starting from the $c_{\text{class}}^\pm = \pm 6b^2$ we are getting their “corrected”, exact

\footnote{Some of them turn out to describe interesting 2d statistical models, as for example the one of central charge $c = -22/5$, known as Lee-Yang edge singularity [12].}
values $c_{\text{quant}}^{\pm} = 1 \pm 6(b \pm \frac{1}{6})^2$. In the classical limit $\hbar \to 0$ one obtains $c_{\text{quant}}^- \to c_{\text{class}}^- \approx -\infty$, i.e. the corresponding classical (and semiclassical) central charges are very big, negative numbers $[23]$. Similarly, for the limits of the central charges of the Liouville’s model $[13]$, we have $c_{\text{class}}^+ \approx \infty$.

Hence the classical (and semi-classical) large negative central charges are a common feature of all the $c_{\text{quant}}^- < 1$ models and of their supersymmetric $N = 1$ extensions. It is therefore important to bear in mind that given the values of the (semi-)classical limits of the central charges of certain class of CFT$_2$’s, further investigations of the limiting properties of the anomalous dimensions of the primary fields are also needed, in order to conclude whether such 2d CFT’s belong to the non-unitary ($c_{\text{quant}}^- < 0$) case, or else to the interval $0 < c_{\text{quant}}^- < 1$, where unitary models are known to exist.

Our final comment concerns the eventual higher dimensional $d > 3$ generalizations of the duality concepts and of the specific examples we have considered in the present paper. It should be stressed that the presence of the $R^2$ terms (specific for the NMG gravity) and the knowledge of the corresponding 1-st order system of eqs. (2.4) were essential in the derivation of our 3d NMG duality conditions $[31]$. Due to the specific form of the NMG central function, it is clear that the pure EH action coupled to scalar matter, and the corresponding dual pCFT$_{d-1}$, do not provide examples of dual and self-dual models (even in the 3-dimensional case); at least not in the context proposed in Sect.3 above. Therefore one has to look for appropriate higher dimensional “higher curvature” gravitational actions of Lovelock type, as for example the ones containing the Gauss-Bonnet term and/or specific combinations of cubic or quartic powers of the curvature tensors similar to the actions of Quasi-Topological gravities $[38–40]$. As in the case of 3d NMG models studied in the present paper, the main ingredients of such holographic duality constructions are again the explicit forms of the corresponding $a$- and $c$-central functions, of the exact $\beta$-functions and of the holographic free energy. There exist many indications of how one can formulate an appropriate generalization of the considered NMG duality conditions in certain higher dimensional models, for which the holographic RG methods, based on the DW’s solutions and on the first order order system of equations $[41, 42]$ are well established. Our preliminary results $[43]$ provide convincing arguments that the NMG-like duality conditions $[31]$ can be realised only in a very particular class of higher dimensional gravity models: For $d = 4$, i.e. for the construction of self-dual pCFT$_3$’s, the appropriate model allowing such partial self-dualities is the $d = 4$ cubic Quasi-Topological gravity $[38–41, 42]$; while for the $d = 5$ case it turns out to be the recently constructed quartic Quasi-Topological Gravity, with the linear and the quartic terms only $[40]$.

6 Appendix. Partially self-dual NMG’s with periodic superpotential

The vacua structure of the following superpotential

$$W(\sigma) = B \left[D - \cos(\alpha \sigma)\right], B < 0 \quad (6.1)$$
consists in two type (a) vacua at \( \sigma^{(0)}_a = 0 \) and \( \sigma^{(a)}_a = \pi/\alpha \) and few type (b) ones (within the interval \( \sigma \in (0, \pi/\alpha) \)). We can further rewrite the parameters \( B \) and \( D \) in an equivalent form as

\[
B = -\frac{L_0 - L_\alpha}{2\kappa L_0 L_\alpha}, \quad D = \frac{L_0 + L_\alpha}{L_0 - L_\alpha},
\]

by introducing an obvious notation for the vacua scales \( L_{0,\alpha} \). The condition \( B < 0 \), i.e. \( L_0 > L_\alpha \), implies that \( D > 1 \), hence Minkowski vacua or Janus-type geometries are excluded.

Using (6.3), we have

\[
\tan \left[ \frac{B \alpha \kappa L_\text{gr} \sqrt{D^2 - 1} \hat{\sigma}}{2} \right] = \sqrt{\frac{D + 1}{D - 1}} \tan \left[ \frac{\alpha \sigma}{2} \right].
\]

This gives:

\[
\tilde{W}(\tilde{\sigma}) = \tilde{B} \left[ \tilde{D} - \cos(\tilde{\alpha} \tilde{\sigma}) \right],
\]

where

\[
\tilde{B} = -\frac{1}{\kappa^2 L_\text{gr}^2 B(D^2 - 1)}, \quad \tilde{D} = -D, \quad \tilde{\alpha} = B \kappa L_\text{gr} \sqrt{D^2 - 1} \alpha.
\]

Thus, we see that the case considered: \( B < 0, D > 1 \), is dual to other case: \( \tilde{B} > 0, \tilde{D} < 1 \). We can integrate the scale factor, to find

\[
e^{\varphi(\sigma)} = e^{\varphi_0} (1 + \cos \alpha \sigma)^{x_1} (1 - \cos \alpha \sigma)^{x_2} |\delta_+ - \cos \alpha \sigma|^{x_3} |\delta_- - \cos \alpha \sigma|^{x_4},
\]

where

\[
x_1 = -\frac{L_0 L_\alpha^2}{2\alpha^2 [L_0 - L_\alpha] [L_0^2 - L_\text{gr}^2]}, \quad x_2 = \frac{L_\alpha L_0^2}{2\alpha^2 [L_0 - L_\alpha] [L_0^2 - L_\text{gr}^2]},
\]

\[
x_3 = \frac{L_0 L_\alpha}{4\alpha^2 [L_\text{gr} + L_\alpha] [L_0 + L_\text{gr}]}, \quad x_4 = \frac{L_0 L_\alpha}{4\alpha^2 [L_0 - L_\text{gr}] [L_\alpha - L_\text{gr}]}.
\]

\[
\delta_\pm = \frac{1}{(L_0 - L_\alpha)} \left[ L_0 + L_\alpha \pm 2\frac{L_0 L_\alpha}{L_\text{gr}} \right].
\]

The condition for the existence of a DW solution connecting two type (a) vacua, i.e. the condition for the absence of singularities of the scale factor for \( \sigma \in (\sigma^{(0)}_a, \sigma^{(a)}_a) \), is that \( \delta_+ > 1 \) and \( \delta_- < -1 \), implying \( L_{0,\alpha} > L_\text{gr} \), thus

\[
0 < L_\text{gr} < L_\alpha < L_0.
\]

In this case, we have a DW connecting a boundary at \( \sigma = 0 \) and a horizon at \( \sigma = \pi/\alpha \).

The description of its phase structure, the nature of the phase transitions as well as the duality relations between the different phases (for different "dual" values of the superpotential parameters), following the methods developed in Sect.3.2.2. and Sect.4.2, is straightforward.
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