Low-complexity Scaling Methods for DCT-II Approximations

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Abstract

This paper introduces a collection of scaling methods for generating $2N$-point DCT-II approximations based on $N$-point low-complexity transformations. Such scaling is based on the Hou recursive matrix factorization of the exact $2N$-point DCT-II matrix. Encompassing the widely employed Jridi-Alfalou-Meher scaling method, the proposed techniques are shown to produce DCT-II approximations that outperform the transforms resulting from the JAM scaling method according to total error energy and mean squared error. Orthogonality conditions are derived and an extensive error analysis based on statistical simulation demonstrates the good performance of the introduced scaling methods. A hardware implementation is also provided demonstrating the competitiveness of the proposed methods when compared to the JAM scaling method.

Keywords

DCT Approximation, Fast algorithms, Image compression

After its inception in 1974 [1], the discrete cosine transform (DCT) has attracted a significant amount of attention leading to a large number of variations and algorithms for its computation [2]. In particular, the DCT of type II (DCT-II) is widely employed for image and video compression [2] codecs such as H.262/MPEG-2 [3], H.263 [4], H.264/AVC [5], H.265/HEVC [6], and the new H.266/VVC standard [7]. The polynomial arithmetic technique—the use of the divide-and-conquer technique to reduce the degree of the polynomial [8–12]—and the matrix factorization technique—direct factorization into the product of sparse matrices [13–21]—are the two main techniques that can be used to factor DCT matrices and produce efficient hardware implementations.

Traditional video and image codecs achieve data decorrelation by means of blockwise image analysis of $8 \times 8$ subimages. As a result, considerable efforts have been put into finding methods for the 8-point DCT-II computation [22–29]. However, the increasing demand for energy-efficient hardware implementations moved the scientific community towards the development of low-complexity integer-based approximate transforms, i.e., transforms whose fast algorithms simply require trivial multiplications (e.g., $\{0, \pm 1, \pm 1/2, \pm 2\}$) and still provide good mathematical properties [22–25]. In particular, deriving good low-complexity approximations has been an actively pursued goal [30–36]. Such approximate approaches have led to robust image and video compression systems capable of significantly reducing the computational cost of the transform stage. Although the 8-point DCT-II is ubiquitously employed, modern codecs, such as the high efficiency video coding (HEVC), require the DCT-II computation for larger blocklengths, including 16- and 32-point transforms [6]. Such demand adds an extra difficulty for deriving good DCT-II approximations.

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In its most general form, designing an \(N\)-point approximate transform requires the solution of a non-linear multivariate integer optimization problem constrained to the mathematical properties of the desired \(N \times N\) matrix \(37\). Because it is often an analytically intractable problem, exhaustive search and brute force calculations are commonly adopted as solution methods \(37,38\). For small blocklengths, such as \(N = 8\), the exhaustive search approach is effective and led to the proposition of several low-complexity approximations for the DCT-II \(39,40\). On the other hand, as \(N\) increases, the number of variables increases quadratically, which leads to large computation times; thus becoming an impractical approach.

To address such issue in designing \(N\)-point DCT-II approximations, matrix scaling has been considered as a feasible approach. In the context of this paper, the term scaling refers to any procedure capable of deriving a larger transformation matrix in terms of smaller transformations of the same kind. In \(41\), Jridi, Alfalou, and Meher (JAM) proposed a scaling method for deriving \(2N\)-point DCT-II approximations based on \(N\)-point DCT-II approximations. Although JAM scaling method is a workable solution, it consists of an ad-hoc method and a clear analytical justification for its operation is still lacking in the literature. Moreover, the JAM scaling is an inexact procedure in the sense that the scaling operation itself introduces errors that are not due to the original \(N\)-point approximation. Nevertheless, due to the fact that (i) the DCT-II matrix is highly symmetrical \(2\) p. 61, 70), (ii) the transform-based compression methods are generally robust to matrix perturbations, and (iii) image compression partially relies on subjective aspects of the human visual system, the approximations derived from the JAM scaling method tended to provide practical transforms for image compression. However, we notice that there is a mathematical gap in the understanding of DCT-II scaling methods, where more comprehensive matrix analyses are required.

The main goal of this paper is two-fold. First we aim at providing a solid mathematical justification to the DCT-II approximation scaling proposed in \(41\). Second, a collection of scaling methods capable of extending current approaches for DCT approximations which render low-complexity hardware implementations is sought. Direct matrix factorizations \(13-21\) and a relation between the DCT-II and the discrete sine (DST) transform of type IV (DST-IV) are employed to derive the sought methods. As contributions, we also provide 16- and 32-point DCT-II approximations with the associated performance analysis compared to the scaled approximations obtained from the JAM method.

The paper is organized as follows. Section 1 reviews the DCT and DST definitions and relationships among them; the JAM scaling method for DCT-II approximations is also revisited. Section 1.4 shows the matrix derivation for obtaining the proposed recursive algorithm to compute the \(2N\)-point DCT-II. Section 2 introduces a family of scaling methods for DCT-II approximations in which the JAM scaling method is identified as a particular case. Sufficient mathematical conditions for the orthogonality of the scaled approximations are also examined. Section 3 presents error and coding performance analyses as well as an arithmetic complexity evaluation of the proposed family of scaling methods. A suit of 16-point DCT-II approximations resulting from the application of the proposed methods to well-known 8-point DCT-II approximations is also introduced and assessed. Section 5 brings final comments and future work directions.
1 Mathematical Background

1.1 Discrete Cosine and Sine Transforms

There are four main variants of DCT and DST, which ranges from types I to IV based on Dirichlet and Neumann boundary conditions [2, p. 29–36]. The entries of the $N$-point DCT-II, DCT-IV, and DST-IV transformation matrices are, respectively, given by [2]

$$
\begin{align*}
[C_{\text{II}}^N]_{k,n} &= \sqrt{\frac{2}{N}} \beta_0 \cos \left( \frac{k(2n+1)\pi}{2N} \right), \\
[C_{\text{IV}}^N]_{k,n} &= \sqrt{\frac{2}{N}} \cos \left( \frac{(2k+1)(2n+1)\pi}{4N} \right), \\
[S_{\text{IV}}^N]_{k,n} &= \sqrt{\frac{2}{N}} \sin \left( \frac{(2k+1)(2n+1)\pi}{4N} \right),
\end{align*}
$$

where $k,n = 0,1,\ldots,N-1$, $\beta_0 = 1/\sqrt{2}$, and $\beta_k = 1$, for $k \neq 0$.

1.2 JAM Scaling for DCT-II Approximations

Let $\hat{C}_{\text{II}}^N$ be an approximation for the $N$-point DCT-II matrix. The JAM scaling method [41] generates a $2N$-point DCT-II matrix approximation $\hat{C}_{\text{II}}^{2N}$ by using two instantiations of a given $N$-point DCT-II approximation matrix according to

$$
\hat{C}_{\text{II}}^{2N} = P_{2N} \cdot \begin{bmatrix} \hat{C}_{\text{II}}^N & I_N & I_N \\ \hat{C}_{\text{II}}^N & -I_N \end{bmatrix},
$$

(1)

where $I_N$ and $-I_N$ are the identity and counter-identity matrices of size $N$, respectively. The matrix $P_{2N}$ is the permutation whose entries are unitary only at positions $(p_n,n)$, where

$$
p_n = \begin{cases} 
2n, & n = 0,1,\ldots,N-1, \\
2n(\text{mod } 2N) + 1, & n = N,N+1,\ldots,2N-1.
\end{cases}
$$

The permutation $P_{2N}$ is sometimes referred to as the perfect shuffle [42, p. 66], [43] or the transpose of the even-odd permutation matrix [44]. In other word, the JAM scaling method is the mapping described by:

$$
f_{\text{JAM}} : \mathbb{R}^{N^2} \rightarrow \mathbb{R}^{(2N)^2} \\
C_{\text{II}}^N \mapsto \hat{C}_{\text{II}}^{2N}.
$$

The scaling expression (1) stems from the following exact expression based on an odd-even decomposi-
tion [41]:

\[
C_{2N}^{II} = \frac{\sqrt{2}}{2} \cdot P_{2N} \cdot \begin{bmatrix}
C_N^{II} & C_N^{IV} \\
J_N \cdot S_N^{IV} & I_N \quad I_N
\end{bmatrix} \cdot \begin{bmatrix}
I_N & I_N \\
I_N & -I_N
\end{bmatrix},
\]

(2)

where \( J_N = \text{diag} \{((-1)^{n})_{n=0}^{N-1}\} \). In order to introduce (1), the authors in [41], have (i) replaced the lower-right block \( J_N \cdot S_N^{IV} \) matrix in (2) with the \( N \)-point DCT-II matrix; and then (ii) substituted the exact DCT-II matrices with approximate DCT-II matrices.

1.3 Relationships between DCT-II and DCT-IV

The DCT-II computation can be performed by means of the Chen algorithm [45]. Such algorithm is based on a decomposition that expresses \( C_{2N}^{II} \) in terms of \( C_N^{II} \) and \( C_N^{IV} \) [9, 16, 18, 20, 46–48]. The matrix form of the Chen algorithm is given by [2, p. 96]

\[
C_{2N}^{II} = \frac{\sqrt{2}}{2} \cdot R_{2N} \cdot \begin{bmatrix}
R_N \cdot C_N^{II} & I_N \\
R_N \cdot C_N^{IV} & -I_N
\end{bmatrix} \cdot \begin{bmatrix}
I_N & I_N \\
I_N & -I_N
\end{bmatrix}.
\]

where \( R_N \) is the bit-reversal ordering permutation matrix [2, 44]. The above expression can be simplified by noticing that, for any integer \( N \), the perfect shuffle and the bit-reversal permutations satisfy the following expression:

\[
P_{2N} = R_{2N} \cdot \begin{bmatrix}
R_N \\
R_N
\end{bmatrix}.
\]

Therefore, we have the matrix form given below [13]:

\[
C_{2N}^{II} = \frac{\sqrt{2}}{2} \cdot P_{2N} \cdot \begin{bmatrix}
C_N^{II} & C_N^{IV} \\
C_N^{IV} & \overline{I}_N
\end{bmatrix} \cdot \begin{bmatrix}
I_N & I_N \\
I_N & -I_N
\end{bmatrix}.
\]

(3)

In [49] and [50], it has been demonstrated that the matrices \( C_N^{II} \) and \( C_N^{IV} \) are related according to [2, p. 77], [9, 16, 47, 48]

\[
C_N^{IV} = A_N \cdot C_N^{II} \cdot D_N,
\]

(4)

where

\[
D_N = \text{diag} \left\{ 2 \cos \left( \frac{(2n+1)\pi}{4N} \right) \right\}_{n=0}^{N-1}.
\]

\[
A_N = J_N \cdot \text{tril} \left( u_N \cdot \begin{bmatrix}
\frac{\sqrt{2}}{2} \\
u_{N-1}^\top
\end{bmatrix} \right) \cdot J_N,
\]

(5)

\( u_N \) is the \( N \)-point column vector of ones and \( \text{tril}(\cdot) \) returns the lower triangular part of its matrix argument.

\(^1\)Equation 4.50 in [2, p. 96] misses the \( \sqrt{2}/2 \) factor.
setting all other entries to zero \([51]\). Hereafter, let \(U_N = u_N \cdot \left[ \frac{\sqrt{2}}{\omega} \ u_N^\top_{-1} \right] \).

### 1.4 Recursive Computation of the DCT-II

The proposed scaling method relies on finding a suitable relation between the \(2N\)-point DCT-II and the \(N\)-point DCT-II. We seek a scaling expression capable of (i) taking advantage of the DCT-II regular structures and (ii) encompassing the JAM scaling method as a particular case. Thus, we aim at preserving the butterfly stage characterized in the rightmost matrices in (2) and (3). The following proposition due to Hou \([52]\), to which we offer an alternative proof, establishes the sought relationship between the \(2N\)-point DCT-II and the \(N\)-point DCT-II.

**Proposition 1** The \(2N\)-point DCT-II matrix can be factored in the form

\[
C_{2N}^{II} = \frac{\sqrt{2}}{2} \cdot P_{2N} \cdot \begin{bmatrix} I_N & B_N \\ I_N & G_N \end{bmatrix} \cdot \begin{bmatrix} C_N^{II} \\ I_N \end{bmatrix} \cdot \begin{bmatrix} C_N^{II} \\ I_N \end{bmatrix},
\]

where

\[
B_N = -I_N \cdot \text{tril}(U_N) \cdot J_N
\]

and

\[
G_N = \text{diag} \left\{ 2(-1)^n \cos \left( \frac{(2n + 1)\pi}{4N} \right) \right\}_{n=0}^{N-1}.
\]

**Proof:** In \([13]\), Wang has demonstrated the following relationship between the DST-IV and the DCT-IV \([2]\):

\[
S_N^{IV} = I_N \cdot C_N^{IV} \cdot J_N.
\]

The DST-IV and DCT-IV are related according to \([16]\):

\[
C_N^{IV} \cdot I_N = J_N \cdot S_N^{IV}.
\]

Combining the above expression with (7), we obtain:

\[
C_N^{IV} \cdot I_N = J_N \cdot I_N \cdot C_N^{IV} \cdot J_N.
\]

Replacing \(C_N^{IV}\) by (4) yields the expression below:

\[
C_N^{IV} \cdot I_N = J_N \cdot I_N \cdot A_N \cdot C_N^{II} \cdot D_N \cdot J_N.
\]
Setting $B_N = J_N \cdot \tilde{I}_N \cdot A_N$ and $G_N = D_N \cdot J_N$, we get

$$C^{IV}_N \cdot \tilde{I}_N = B_N \cdot C^{II}_N \cdot G_N.$$  

(9)

Here we applied the result $J_N \cdot \tilde{I}_N = -\tilde{I}_N \cdot J_N$ and then we combined the entries of the diagonal matrices $D_N$ and $J_N$ to obtain $G_N$. Also, we used (5) to obtain that $B_N = J_N \cdot \tilde{I}_N \cdot A_N = -\tilde{I}_N \cdot \text{tril}(U_N) \cdot J_N$. Replacing the term $C^{IV}_N \cdot \tilde{I}_N$ in (3) by (9) yields (6).

The structure of the matrix $B_N$ is explicitly given by:

$$B_N = -\tilde{I}_N \cdot \text{tril}(U_N) \cdot J_N = \begin{bmatrix}
-\sqrt{2} & 1 & -1 & \ldots & -1 & 1 \\
-\sqrt{2} & 1 & -1 & \ldots & -1 & 1 \\
-\sqrt{2} & 1 & -1 & \ldots & -1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-\sqrt{2} & 1 & -1 & \ldots & -1 & 1 \\
-\sqrt{2} & 1 & -1 & \ldots & -1 & 1 \\
\end{bmatrix}.$$

2 Approximate Scaling

2.1 The Family of Scaling Methods

In general, a DCT-II approximation $\hat{C}^{II}_N$ can be represented according to the polar decomposition [53, p. 348] consisting of two parts: (i) a low-complexity matrix $T_N$, and (ii) a diagonal matrix $\Sigma_N$ that provides orthogonalization or quasi-orthogonalization [39, 53]. Such matrices are related according to:

$$\hat{C}^{II}_N = \Sigma_N \cdot T_N,$$

where

$$\Sigma_N = \sqrt{\text{diag}\{ (T_N \cdot T_N^\top)^{-1} \}}$$

and $\sqrt{\cdot}$ is the matrix square root [39]. Here the operator diag(·) returns a diagonal matrix with the elements of the diagonal of its matrix argument [39, 53].

Considering the Proposition 1, we introduce the following mapping relating an $N$-point low-complexity matrix $T_N$ to its $2N$-point scaled form $T_{2N}$ as shown below:

$$f(B_N, G_N): \mathbb{R}^{N^2} \longrightarrow \mathbb{R}^{(2N)^2}$$

$$T_N \longrightarrow T_{2N} = P_{2N} \cdot \begin{bmatrix} I_N & \hat{B}_N \\ \hat{C}_N \\ I_N & \tilde{I}_N \\ I_N & -I_N \end{bmatrix} \cdot \begin{bmatrix} T_N \\ \hat{B}_N \end{bmatrix}.$$  

(10)

Matrices $\hat{B}_N$ and $\hat{G}_N$ are parameter matrices. Approximations for $B_N$ and $G_N$ appear as natural candidates for the parameter matrices $\hat{B}_N$ and $\hat{G}_N$, respectively. The actual $2N$-point approximate DCT-II is obtained
Table 1: Scaling Approximations and Its Performance

| Case | $\hat{B}_N$ | $\hat{G}_N$ | $|\hat{C}_{2N}^{II} - C_{2N}^{II}|_F$ | Orth.? |
|------|-------------|-------------|---------------------------------|--------|
| JAM  | $I_N$       | $I_N$       | 3.994, 5.653, 7.997              | Yes    |
| I    | $I_N$       | $I_N$       | 3.826, 5.533, 7.912              | Yes    |
| II   | $-I_N \cdot J_N$ | $I_N$       | 4.001, 5.657, 8.000              | Yes    |
| III  | $-I_N \cdot Z_N \cdot J_N$ | $I_N$       | 4.001, 5.657, 8.000              | Yes    |
| IV   | $I_N$       | $J_N$       | 3.826, 5.533, 7.912              | Yes    |
| V    | $I_N$       | $J_N$       | 4.006, 5.661, 8.003              | Yes    |
| VI   | $-I_N \cdot J_N$ | $J_N$       | 1.954, 3.033, 4.515              | Yes    |
| VII  | $-I_N \cdot Z_N \cdot J_N$ | $J_N$       | 1.954, 3.033, 4.515              | Yes    |

after orthogonalization \([39, 54]\) and is given by:

$$\hat{C}_{2N}^{II} = \Sigma_{2N} \cdot T_{2N},$$  \((11)\)

where $\Sigma_{2N} = \sqrt{\text{diag}\{ (T_{2N} \cdot T_{2N}^T)^{-1} }\}$. Although \((10)\) stems from \([6]\) (Proposition 1), it does not need to inherit the scalar $\sqrt{2}/2$. This is because the orthogonalization in \((11)\) is invariant to the presence of constant scalars \([54]\). Ultimately, depending on the choice of the parameter matrices $\hat{B}_N$ and $\hat{G}_N$, we obtain different methods for scaling DCT-II approximations.

Selected choices of parameter matrices $\hat{B}_N$ and $\hat{G}_N$ are shown in Table 1 where $Z_N = \text{diag}[\{1/2, u_{N-1}\}]$. The list is not complete as any choice of $\hat{B}_N$ and/or $\hat{G}_N$ that are regarded to be ‘close enough’ to $B_N$ and $G_N$, respectively, generates a particular scaling method. For instance, the JAM scaling method proposed in \([41]\) is the particular case when $\hat{B}_N = \hat{G}_N = I_N$, furnishing the relationship below:

$$\hat{C}_{2N}^{II} = f_{\text{JAM}}(C_N^{II}) = f(I_N, I_N)(\hat{C}_N^{II}).$$

Notice that the coefficients of the matrices were intentionally chosen to be small magnitude integers which are in the set \([0, \pm 1/2, \pm 1, \pm 2]\). The faithfulness of the $2N$-point DCT approximation will come as a result of how the entries of the parameter matrices $\hat{B}_N$ and $\hat{G}_N$ are chosen. Other choices of parameter matrices could be obtained by bit-expanding the original matrices $B_N$ and $G_N$ or performing multicriteria optimization over the coefficients of $\hat{B}_N$ and $\hat{G}_N$ \([39, 54]\) and taking into account the specifics of the application in hand.

2.2 Orthogonality Condition

The design of approximate transforms often require the transformation matrices to be orthogonal \([2, 39, 54]\). In fact, in contexts such as noise reduction \([55]\), watermarking methods \([56]\), and harmonic detection \([2, 57, 58]\), the invertibility of the DCT is not only desired, but required. This is because the DCT is used to translate the signal to its transform domain, where processing is performed. The resulting signal in the transform domain is then translated back into the time domain, rendering the final desired output. Thus, we show
in Proposition 2, the sufficient conditions but not the necessary conditions for orthogonally of the proposed DCT-II approximation.

It can be shown [54] that if \( T_{2N} \cdot T_{2N}^\top \) is a diagonal matrix, then the DCT-II approximation \( \hat{C}_{2N}^{II} \) obtained according to (11) is orthogonal. Using (10), we can write

\[
T_{2N} \cdot T_{2N}^\top = 2 \cdot P_{2N} \cdot \begin{bmatrix} T_N \cdot T_N^\top & \hat{B}_N \cdot T_N \cdot \hat{G}_N^\top \cdot T_N^\top \cdot \hat{B}_N^\top \\ \end{bmatrix} \tag{12}
\]

Thus, for \( T_{2N} \cdot T_{2N}^\top \) to be a diagonal matrix, we have to ensure that both matrices \( T_N \cdot T_N^\top \) and \( \hat{B}_N \cdot T_N \cdot \hat{G}_N \cdot T_N^\top \cdot \hat{B}_N^\top \) are diagonal matrices. In order to investigate the conditions for orthogonality, we use the following result from [53, p. 151].

**Lemma 1** If \( P \) is a permutation matrix and \( D \) is a diagonal matrix, then \( P \cdot D \cdot P^\top \) is a diagonal matrix.

Therefore, sufficient conditions for orthogonality are furnished by the proposition below.

**Proposition 2** If the following conditions are satisfied:

(i) \( T_N \cdot T_N^\top \) is a diagonal matrix;

(ii) \( \hat{G}_N \cdot \hat{G}_N^\top = a \cdot I_N \), \( a \in \mathbb{R} \);

(iii) \( \hat{B}_N \) is a generalized permutation matrix;

then the scaling method in (10) generates an orthogonal DCT-II approximation.

**Proof:** We need to ensure that the sub-matrices from the block-diagonal matrix in (12) are diagonal matrices. Therefore, the Condition (i) is clearly a necessary condition. Now let us examine the lower-right sub-matrix \( \hat{B}_N \cdot T_N \cdot \hat{G}_N \cdot T_N^\top \cdot \hat{B}_N^\top \). Using (12) and Condition (ii), we obtain:

\[
\hat{B}_N \cdot T_N \cdot \hat{G}_N \cdot T_N^\top \cdot \hat{B}_N^\top = a \cdot \hat{B}_N \cdot T_N \cdot \hat{B}_N^\top.
\]

Condition (iii) ensures that \( \hat{B}_N = D \cdot P \), where \( D \) is a diagonal matrix and \( P \) is a permutation matrix. Thus, we have that: \( \hat{B}_N \cdot T_N \cdot \hat{B}_N^\top = D \cdot P \cdot T_N \cdot \hat{B}_N^\top \cdot P^\top \cdot D \). By applying Lemma 1 it follows that \( P \cdot T_N \cdot \hat{B}_N^\top \cdot P^\top \) is a diagonal matrix. We have then that \( \hat{B}_N \cdot T_N \cdot \hat{B}_N^\top = D \cdot (P \cdot T_N \cdot \hat{B}_N^\top \cdot P^\top) \cdot D \) is also a diagonal matrix. \( \square \)

Notice that the conditions required by Proposition 2 are not too restrictive. In fact, because the exact matrix \( G_N \) is by definition a diagonal matrix, the condition on \( \hat{G}_N \) (Condition (ii)) can be met by approximating the elements of \( G_N \) to a suitable value \( \pm \sqrt{a} \) (e.g., \( a = 1 \)). The methods listed in Table 1 are capable of generating orthogonal approximations, because the selected choices for \( \hat{B}_N \) and \( \hat{G}_N \) are under the conditions prescribed in Proposition 2.
3 Error, Performance, and Complexity Analysis

3.1 Error Analysis and Statistical Modeling

In order to assess the proposed scaling methods, we performed an error analysis based on the Frobenius norm of the difference $\hat{C}_N^{\text{II}} - C_N^{\text{II}}$. To properly isolate the behavior of the scaling method, the required $N$-point matrices were ensured to be identical and equal to the exact DCT matrix, i.e., $\hat{C}_N^{\text{II}} = C_N^{\text{II}}$. Table I shows the computed errors for $N \in \{8, 16, 32\}$. In all cases, Methods I, IV, VI, and VII generated smaller errors, outperforming the JAM scaling method.

Now let us analyze the errors when actual approximations $\hat{C}_N^{\text{II}}$ are considered, i.e. $\hat{C}_N^{\text{II}} \neq C_N^{\text{II}}$. The performance of the proposed scaling methods can be quantified by means of the error of $\hat{C}_N^{\text{II}}$, as a function of the error of $\hat{C}_N^{\text{II}}$, relative to $C_N^{\text{II}}$. Because of the wide popularity and importance of the 8-point DCT-II, we fixed $N = 8$ as the most relevant case for analysis. Thus, the values of $\|\hat{C}_8^{\text{II}} - C_8^{\text{II}}\|_F$ were computed for the 8-point DCT-II approximations discussed below. The recent book [7] by Rao, co-inventor of the DCT, identifies state-of-art transforms such as the series of approximations by Bouguezel-Ahmad-Swamy (BAS) [7, p. 160], the rounded DCT (RDCT) [7, p. 162], and the modified RDCT (MRDCT) [7, p. 162]. Due to its flexibility, we selected the BAS parametric approximation described in [33] for parameter values $a = 0, 1/2, 1$, referred to as BAS$_1$, BAS$_2$, and BAS$_3$. We also included the BAS approximation [34] labeled here as BAS$_4$. In [59], the RDCT and the MRDCT were identified as optimal approximations in terms of output image quality and computing time, respectively, according to a hardware implementation using approximate adder cells for image compression. In addition to the above-mentioned approximations, we included in our comparisons the very recently introduced angle-based approximate DCT in [37] (here termed ABDCT), the traditional signed DCT (SDCT) [22], the Lengwehasatit-Ortega DCT approximation (LODCT) [23], and the low-complexity approximation detailed in [36], which is an improved version of the MRDCT, here denoted as IMRDCT.

The error for the corresponding scaled approximations $\|\hat{C}_{16}^{\text{II}} - C_{16}^{\text{II}}\|_F$ against $\|\hat{C}_8^{\text{II}} - C_8^{\text{II}}\|_F$ for each of the methods in Table I are shown in Figure 1. The errors follow a linear trend that can be quantified according to a linear regression using least-square estimation [60, 61] for the linear model below:

$$g(\|\hat{C}_{16}^{\text{II}} - C_{16}^{\text{II}}\|_F) = m \cdot \|\hat{C}_8^{\text{II}} - C_8^{\text{II}}\|_F + b,$$

where $m$ and $b$ are the slope and intercept to be estimated, respectively. Table 2 shows the estimates of $m$ and $b$, $\hat{m}$ and $\hat{b}$, along with the $\chi^2$ goodness of fit statistic and the residual mean squared error (RMSE) for the model. At the significance level of 0.001, the critical value was approximately 20.1 for all scenarios. The regression models presented $\chi^2$ test statistic values smaller than $\approx 7.9 \cdot 10^{-2}$ and RMSE values smaller than $\approx 1.0 \cdot 10^{-1}$. Such values are much smaller than the critical value for the test at 0.001 significance level; thus we have $p$-values very close to the unit [62], preventing us from rejecting the models.

The quantity $\hat{m}$ determines the average influence of $\|\hat{C}_8^{\text{II}} - C_8^{\text{II}}\|_F$ over $\|\hat{C}_{16}^{\text{II}} - C_{16}^{\text{II}}\|_F$; whereas $\hat{b}$ captures the minimum error due to the scaling method. No matter how good the approximation $\hat{C}_8^{\text{II}}$ is, the resulting approximation $\hat{C}_{16}^{\text{II}}$ has an inherent error due to the approximations $\hat{B}_8$ and $\hat{G}_8$. This floor error is equal to the intercept $\hat{b}$. The JAM method results in the lowest $\hat{m}$ but also in the highest $\hat{b}$; whereas the proposed Method VI and VII presents the highest $\hat{m}$ and lowest $\hat{b}$.

We can compare two fitted models by determining the crossing points of the curves representing the linear
Figure 1: $\|\hat{C}_{16}^{\Pi} - C_{16}^{\Pi}\|_F$ as a function of $\|\hat{C}_8^{\Pi} - C_8^{\Pi}\|_F$ for the methods outlined in Table 1.
Table 2: Linear regression analysis statistics using least-squares estimator

| Method | ̂m | ̂b | χ² | RMSE |
|--------|----|----|----|------|
| JAM    | 0.264 | 3.833 | 7.979 ⋅ 10⁻² | 9.987 ⋅ 10⁻² |
| I      | 0.426 | 3.561 | 3.599 ⋅ 10⁻² | 6.708 ⋅ 10⁻² |
| II     | 0.413 | 3.746 | 3.177 ⋅ 10⁻² | 6.243 ⋅ 10⁻² |
| III    | 0.413 | 3.746 | 3.177 ⋅ 10⁻² | 6.243 ⋅ 10⁻² |
| IV     | 0.431 | 3.555 | 3.636 ⋅ 10⁻² | 6.742 ⋅ 10⁻² |
| V      | 0.562 | 3.636 | 5.220 ⋅ 10⁻² | 8.077 ⋅ 10⁻² |
| VI     | 1.045 | 1.319 | 1.531 ⋅ 10⁻¹ | 1.383 ⋅ 10⁻¹ |
| VII    | 1.045 | 1.319 | 1.531 ⋅ 10⁻¹ | 1.383 ⋅ 10⁻¹ |

Table 3: Approximate ∥̂C^II₈ - C^II₈∥ maximum values for which the proposed methods outperform the JAM method

| Method | I   | II  | III | IV  | V   | VI  | VII |
|--------|-----|-----|-----|-----|-----|-----|-----|
| ∥̂C^II₈ - C^II₈∥ | 1.679 | 0.584 | 1.664 | 1.664 | 0.661 | 3.219 | 3.219 |

models and slope of error regression curve ∥̂C^II₈ - C^II₈∥ as a function of ∥̂C^II₈ - C^II₈∥ for each model. Thus, for instance, the Method VII provides better approximations when compared to the JAM method if ∥̂C^II₈ - C^II₈∥ < 3.219. Table 3 shows the maximum value of ∥̂C^II₈ - C^II₈∥ for which the JAM scaling method is outperformed by each of the proposed methods. The 8-point DCT-II approximations in the literature present Frobenius errors in the range [1.72, 2.68]. Therefore, Methods VI and VII outperform JAM method regardless of the considered 8-point DCT-II approximation. Similar analyses can be applied to larger approximations.

3.2 Performance Measurements

We assessed the performance of the obtained 16-point DCT approximations according to the following figures of merit: the mean-squared error MSE(·) [2, 63, 64], total error energy c(·) [2, 25], deviation from orthogonality d(·) [2, 26], unified coding gain C₉(·) [2, 63], and transform efficiency η(·) [2, 29]. The total error energy quantifies the error between matrices in a euclidean distance way [2, 63, 64], while the mean square error (MSE) of a given matrix approximation takes into account its proximity to the original transform and its effect on the autocorrelation matrix of the class of signals in consideration [25, 54]. The unified transform coding gain [2, 63] and transform efficiency [2, 29] provide measures to quantify the compression capabilities of a given approximation [54]. Tables 11–13 show the obtained results.

The JAM scaling method is outperformed by Methods VI, and VII in terms of total error energy when considering the BAS₁, BAS₂, BAS₃, BAS₄, SDCT, LO, RDCT, MRDCT, ABDCT, and IMRDCT as shown in Tables 4–13. Methods I, II and III have consistently offered the same coding performance when compared with the JAM scaling method under all considered scenarios.
**Table 4: Metrics for scaling methods using BAS₁ transform**

| Method | $d(\cdot)$ | $\epsilon(\cdot)$ | MSE(·) | $C_g(\cdot)$ | $\eta(\cdot)$ | A(·) | S(·) |
|--------|-------------|--------------------|--------|--------------|--------------|------|------|
| JAM    | 0.00        | 14.62              | 8.16   | 70.98        | 48           | 0    |      |
| I      | 0.00        | 15.04              | 8.16   | 70.98        | 48           | 0    |      |
| II     | 0.00        | 15.79              | 8.16   | 70.98        | 48           | 0    |      |
| III    | 0.00        | 15.79              | 8.16   | 70.98        | 48           | 0    |      |
| IV     | 0.00        | 15.13              | 7.16   | 57.36        | 48           | 0    |      |
| V      | 0.00        | 16.62              | 7.16   | 57.36        | 48           | 0    |      |
| VI     | 0.00        | 13.88              | 7.16   | 57.36        | 48           | 0    |      |
| VII    | 0.00        | 13.88              | 7.16   | 57.36        | 48           | 0    |      |

**Table 5: Metrics for scaling methods using BAS₂ transform**

| Method | $d(\cdot)$ | $\epsilon(\cdot)$ | MSE(·) | $C_g(\cdot)$ | $\eta(\cdot)$ | A(·) | S(·) |
|--------|-------------|--------------------|--------|--------------|--------------|------|------|
| JAM    | 0.00        | 14.58              | 8.37   | 71.83        | 52           | 4    |      |
| I      | 0.00        | 15.19              | 8.37   | 71.83        | 52           | 4    |      |
| II     | 0.00        | 15.61              | 8.37   | 71.83        | 52           | 4    |      |
| III    | 0.00        | 15.61              | 8.37   | 71.83        | 52           | 4    |      |
| IV     | 0.00        | 15.23              | 7.48   | 58.83        | 52           | 4    |      |
| V      | 0.00        | 16.67              | 7.48   | 58.83        | 52           | 4    |      |
| VI     | 0.00        | 13.84              | 7.48   | 58.83        | 52           | 4    |      |
| VII    | 0.00        | 13.84              | 7.48   | 58.83        | 52           | 4    |      |

**Table 6: Metrics for scaling methods using BAS₃ transform**

| Method | $d(\cdot)$ | $\epsilon(\cdot)$ | MSE(·) | $C_g(\cdot)$ | $\eta(\cdot)$ | A(·) | S(·) |
|--------|-------------|--------------------|--------|--------------|--------------|------|------|
| JAM    | 0.00        | 14.67              | 8.16   | 70.80        | 52           | 0    |      |
| I      | 0.00        | 15.36              | 8.16   | 70.80        | 52           | 0    |      |
| II     | 0.00        | 15.57              | 8.16   | 70.80        | 52           | 0    |      |
| III    | 0.00        | 15.57              | 8.16   | 70.80        | 52           | 0    |      |
| IV     | 0.00        | 15.36              | 7.41   | 59.95        | 52           | 0    |      |
| V      | 0.00        | 16.70              | 7.41   | 59.95        | 52           | 0    |      |
| VI     | 0.00        | 13.94              | 7.41   | 59.95        | 52           | 0    |      |
| VII    | 0.00        | 13.94              | 7.41   | 59.95        | 52           | 0    |      |
Table 7: Metrics for scaling methods using BAS₄ transform

| Method | \(d(\cdot)\) | \(\epsilon(\cdot)\) | MSE(\cdot) | \(C_g(\cdot)\) | \(\eta(\cdot)\) | \(A(\cdot)\) | \(S(\cdot)\) |
|--------|--------------|-----------------|------------|----------------|--------------|-------------|-------------|
| JAM    | 0.00         | 13.18           | 0.13       | 8.19           | 70.65        | 64          | 0           |
| I      | 0.00         | 12.65           | 0.34       | 8.19           | 70.65        | 64          | 0           |
| II     | 0.00         | 13.18           | 0.36       | 8.19           | 70.65        | 64          | 0           |
| III    | 0.00         | 13.18           | 0.36       | 8.19           | 70.65        | 64          | 0           |
| IV     | 0.00         | 12.65           | 0.34       | 8.19           | 70.65        | 64          | 0           |
| V      | 0.00         | 13.18           | 0.13       | 8.19           | 70.65        | 64          | 0           |
| VI     | 0.00         | 7.40            | 0.06       | 8.19           | 70.65        | 64          | 0           |
| VII    | 0.00         | 7.40            | 0.06       | 8.19           | 70.65        | 64          | 0           |

Table 8: Metrics for scaling methods using RDCT

| Method | \(d(\cdot)\) | \(\epsilon(\cdot)\) | MSE(\cdot) | \(C_g(\cdot)\) | \(\eta(\cdot)\) | \(A(\cdot)\) | \(S(\cdot)\) |
|--------|--------------|-----------------|------------|----------------|--------------|-------------|-------------|
| JAM    | 0.00         | 12.93           | 0.12       | 8.43           | 72.23        | 60          | 0           |
| I      | 0.00         | 12.25           | 0.31       | 8.43           | 72.23        | 60          | 0           |
| II     | 0.00         | 12.82           | 0.30       | 8.43           | 72.23        | 60          | 0           |
| III    | 0.00         | 12.82           | 0.30       | 8.43           | 72.23        | 60          | 0           |
| IV     | 0.00         | 12.25           | 0.34       | 7.50           | 59.87        | 60          | 0           |
| V      | 0.00         | 12.65           | 0.14       | 7.50           | 59.87        | 60          | 0           |
| VI     | 0.00         | 6.80            | 0.07       | 7.50           | 59.87        | 60          | 0           |
| VII    | 0.00         | 6.80            | 0.07       | 7.50           | 59.87        | 60          | 0           |

Table 9: Metrics for scaling methods using MRDCT

| Method | \(d(\cdot)\) | \(\epsilon(\cdot)\) | MSE(\cdot) | \(C_g(\cdot)\) | \(\eta(\cdot)\) | \(A(\cdot)\) | \(S(\cdot)\) |
|--------|--------------|-----------------|------------|----------------|--------------|-------------|-------------|
| JAM    | 0.00         | 12.77           | 0.13       | 7.58           | 66.07        | 44          | 0           |
| I      | 0.00         | 13.19           | 0.34       | 7.58           | 66.07        | 44          | 0           |
| II     | 0.00         | 13.72           | 0.34       | 7.58           | 66.07        | 44          | 0           |
| III    | 0.00         | 13.72           | 0.34       | 7.58           | 66.07        | 44          | 0           |
| IV     | 0.00         | 13.19           | 0.36       | 6.48           | 52.20        | 44          | 0           |
| V      | 0.00         | 14.39           | 0.25       | 6.48           | 52.20        | 44          | 0           |
| VI     | 0.00         | 9.67            | 0.18       | 6.48           | 52.20        | 44          | 0           |
| VII    | 0.00         | 9.67            | 0.18       | 6.48           | 52.20        | 44          | 0           |
| Method | $d(\cdot)$ | $\epsilon(\cdot)$ | MSE(\cdot) | $C_g(\cdot)$ | $\eta(\cdot)$ | $A(\cdot)$ | $S(\cdot)$ |
|--------|-----------|----------------|------------|------------|--------------|----------|---------|
| JAM    | 0.00      | 12.63         | 0.12       | 8.88       | 76.81        | 64       | 12      |
| I      | 0.00      | 12.21         | 0.31       | 8.88       | 76.81        | 64       | 12      |
| II     | 0.00      | 12.75         | 0.32       | 8.88       | 76.81        | 64       | 12      |
| III    | 0.00      | 12.75         | 0.32       | 8.88       | 76.81        | 64       | 12      |
| IV     | 0.00      | 12.21         | 0.34       | 8.18       | 63.79        | 64       | 12      |
| V      | 0.00      | 12.81         | 0.14       | 8.18       | 63.79        | 64       | 12      |
| VI     | 0.00      | 6.56          | 0.07       | 8.18       | 63.79        | 64       | 12      |
| VII    | 0.00      | 6.56          | 0.07       | 8.18       | 63.79        | 64       | 12      |

| Method | $d(\cdot)$ | $\epsilon(\cdot)$ | MSE(\cdot) | $C_g(\cdot)$ | $\eta(\cdot)$ | $A(\cdot)$ | $S(\cdot)$ |
|--------|-----------|----------------|------------|------------|--------------|----------|---------|
| JAM    | 0.20      | 12.83         | 0.13       | 6.27       | 68.82        | 64       | 0       |
| I      | 0.20      | 12.42         | 0.34       | 6.27       | 68.82        | 64       | 0       |
| II     | 0.20      | 12.96         | 0.36       | 6.27       | 68.82        | 64       | 0       |
| III    | 0.20      | 12.96         | 0.36       | 6.27       | 68.82        | 64       | 0       |
| IV     | 0.20      | 12.42         | 0.38       | 5.57       | 58.11        | 64       | 0       |
| V      | 0.20      | 13.12         | 0.16       | 5.57       | 58.11        | 64       | 0       |
| VI     | 0.20      | 7.29          | 0.09       | 5.57       | 58.11        | 64       | 0       |
| VII    | 0.20      | 7.29          | 0.09       | 5.57       | 58.11        | 64       | 0       |

| Method | $d(\cdot)$ | $\epsilon(\cdot)$ | MSE(\cdot) | $C_g(\cdot)$ | $\eta(\cdot)$ | $A(\cdot)$ | $S(\cdot)$ |
|--------|-----------|----------------|------------|------------|--------------|----------|---------|
| JAM    | 0.00      | 12.67         | 0.12       | 8.64       | 73.11        | 64       | 4       |
| I      | 0.00      | 12.15         | 0.30       | 8.64       | 73.11        | 64       | 4       |
| II     | 0.00      | 12.69         | 0.31       | 8.64       | 73.11        | 64       | 4       |
| III    | 0.00      | 12.69         | 0.31       | 8.64       | 73.11        | 64       | 4       |
| IV     | 0.00      | 12.15         | 0.34       | 7.83       | 61.49        | 64       | 4       |
| V      | 0.00      | 12.68         | 0.14       | 7.83       | 61.49        | 64       | 4       |
| VI     | 0.00      | 6.30          | 0.07       | 7.83       | 61.49        | 64       | 4       |
| VII    | 0.00      | 6.30          | 0.07       | 7.83       | 61.49        | 64       | 4       |
Table 13: Metrics for scaling methods using the IMRDC

| Method | d(·) | c(·) | MSE(·) | Cg(·) | η(·) | A(·) | S(·) |
|--------|------|------|--------|-------|------|------|------|
| JAM    | 0.00 | 13.21| 0.15   | 7.58  | 66.07| 44   | 0    |
| I      | 0.00 | 13.51| 0.39   | 7.58  | 66.07| 44   | 0    |
| II     | 0.00 | 14.03| 0.39   | 7.58  | 66.07| 44   | 0    |
| III    | 0.00 | 14.03| 0.39   | 7.58  | 66.07| 44   | 0    |
| IV     | 0.00 | 13.51| 0.37   | 6.48  | 52.20| 44   | 0    |
| V      | 0.00 | 14.58| 0.26   | 6.48  | 52.20| 44   | 0    |
| VI     | 0.00 | 9.94 | 0.20   | 6.48  | 52.20| 44   | 0    |
| VII    | 0.00 | 9.94 | 0.20   | 6.48  | 52.20| 44   | 0    |

### 3.3 Arithmetic Complexity

The only matrix structures in (10) that contribute to the arithmetic complexity are: (i) the two instantiations of \( T_N \); (ii) the butterfly matrix of size \( 2N \); (iii) the diagonal matrix \( \hat{G}_N \); and (iv) the matrix \( \hat{B}_N \). The matrices \( \hat{G}_N \) and \( \hat{B}_N \) requires no multiplication. Thus, if \( T_N \) is selected to be a multiplierless approximation, then the proposed scaling methods are ensured to have null multiplicative complexity. Therefore the arithmetic complexity is fully characterized by the number of additions and bit-shifting operations, which are given by:

\[
A(T_{2N}) = 2A(T_N) + A(\hat{B}_N) + A(\hat{G}_N) + 2N
\]

and

\[
S(T_{2N}) = 2S(T_N) + S(\hat{B}_N) + S(\hat{G}_N),
\]

where functions \( A(\cdot) \) and \( S(\cdot) \) return the number of additions and bit-shifting operations required by its arguments, respectively [44]. Tables 11–13 shows the arithmetic complexity for the considered methods. The proposed scaling methods does not incur in higher arithmetic complexity when compared to the JAM scaling method.

### 4 Hardware Implementation

The JAM method and the proposed methods listed in Table 1 were implemented on a field programmable gate array (FPGA). The device used for the hardware implementation was the Xilinx Artix-7 XC7A35T-1CPG236C. Because of its high coding performance (see Table 10), we selected the ABDCT [37] to be submitted to the discussed methods. Each scaled transform \( T_{2N} \) employed two pipelined instances of the ABDCT core, as described in Figure 2 showing the internal architecture for the resulting transformation matrix \( T_{2N} \). For each of the methods outlined in Table 1 the respective matrices \( \hat{B}_N \) and \( \hat{G}_N \) are generalized permutations [53, p. 151]. Therefore, the implementation of such matrices solely requires combinational logic leading to a reduced overall design latency. Each sub-block implementing the ABDCT was implemented according to the fast algorithm.
Figure 2: Block diagram for the proposed scaling methods for DCT approximation. For the hardware implementation using the ABDCT, the $T_N$ sub-blocks implement the ABDCT.

Figure 3: Testbed architecture for testing the implemented designs.

outlined in [37], where arithmetic operations were pipelined for achieving higher maximum operating frequency of each block. The architecture was implemented using input wordlength of 8 bits.

The designs were tested employing the scheme depicted in Figure 3 together with a controller state-machine and connected to a universal asynchronous receiver-transmitter (UART) block. The UART core interfaces with the controller state machine using the ARM Advanced Microcontroller Bus Architecture Advanced eXtensible Interface 4 (AMBA AXI-4) protocol. A personal computer (PC) communicates with the controller through the UART by sending a set of 16 coefficients corresponding to the input submitted to the transform under evaluation. The 16 coefficients are passed to the design and processed, then the controller state machine sends the resulting 16 coefficients back to the personal computer. This operation is performed several times and then to the output of processing the original coefficients with a software model implemented in Python.

The metrics examined to evaluate the hardware implementations were: number of occupied slices, number of look-up tables (LUT), flip-flop (FF) count, critical path delay ($T_{cpd}$), maximum operating frequency $F_{max} = T_{cpd}^{-1}$, and dynamic power ($D_p$) normalized by $F_{max}$.

Table 14 shows the hardware metrics for the implemented designs. In terms of resources consumption, the

| Metric        | JAM | I   | II  | III | IV  | V   | VI  | VII |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|
| # Slices      | 224 | 230 | 246 | 265 | 241 | 249 | 251 | 270 |
| # LUT         | 673 | 672 | 724 | 723 | 684 | 684 | 738 | 736 |
| # FF          | 1061| 1061| 1061| 1058| 1061| 1061| 1061| 1058|
| $T_{cpd}$ (ns)| 4.558| 4.202| 4.407| 4.485| 4.606| 4.694| 4.897| 4.620|
| $F_{max}$ (MHz)| 219.394| 237.982| 226.917| 222.965| 217.108| 213.038| 204.207| 216.450|
| $D_p$ (µW/MHz)| 86.389| 79.838| 88.138| 89.700| 89.186| 93.043| 87.780|     |
JAM method requires the least amount of slices and LUTs. Method VII, however, has the least amount of FFs, while using the largest amount of slices a high number of LUTs.

Method I is the one achieving the lowest critical path delay, followed by Method II, III, and then JAM, respectively. Because of that, Method I, II, III, and JAM are the ones achieving the highest maximum operating frequency, respectively. Method I is also the one achieving the most efficient implementation in terms of dynamic power, representing a reduction of 7.58% when compared to the JAM implementation, which is the second most efficient implementation.

5 Conclusions

We provided an alternative derivation to the recursive algorithm proposed by Hou [52].

By judiciously approximating specific matrix factors of Hou recursive DCT-II factorization, we introduced a framework for approximate scaling that is capable of deriving $2N$-point DCT approximations based on $N$-point DCT approximations. The proposed collection of scaling DCT approximations is flexible and generates several methods, encompassing the JAM scaling method as a particular case. Conditions for orthogonality—a common property in the context of image/data compression—were identified. An error analysis and statistical modeling of the scaling methods were derived. The proposed scaling methods are inherently multiplierless, i.e., they do not contribute to any multiplication. An arithmetic complexity analysis was derived and expressions for the additive and bit-shifting costs were furnished. The new proposed scaling methods were able to outperform the competing JAM method in terms of Frobenius errors and coding gain, paving the way for promising hardware implementations.

As a topic for future research, the authors are aware that the work in [65] proposed a relationship between the DCT-II and DCT-IV with matrix factors that possess the highest sparsity in the literature. This relation can be used in connection with (7) in place of (8), leading to alternative derivations and a different family of scaling methods. Possible research fronts are also the investigation of how the resulting DCT-II approximations behave after several uses of the recursion in (10) and the use of matrix parameterizations based on the generalization in [66], which is derived from a generalization of different five algorithms in the literature [50, 52, 67–69].
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