AUTOMORPHISMS OF $\mathbb{C}^m$ WITH BOUNDED WANDERING DOMAINS

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ABSTRACT. We prove that the Euclidean ball can be realized as a Fatou component of a holomorphic automorphism of $\mathbb{C}^m$, in particular as the escaping and the oscillating wandering domain. Moreover, the same is true for a large class of bounded domains, namely for all bounded simply connected regular open sets $\Omega \subset \mathbb{C}^m$ whose closure is polynomially convex. Our result gives in particular the first example of a bounded Fatou component with a smooth boundary in the category of holomorphic automorphisms.

1. Introduction

With the emergence of new techniques, the study of wandering domains has flourished in recent years. Many strong results have been established, in particular about the existence and the geometry of wandering domains in the category of transcendental holomorphic functions \cite{Ba76, Bi15, EL92, BEGRS19} and in the category of holomorphic endomorphisms of $\mathbb{CP}^2$ \cite{ABDPR16, AsBTP19}. On the other hand there are only few known results in the category of holomorphic automorphisms of $\mathbb{C}^m$ for $m \geq 2$ some of which will be presented later in the introduction. The aim of this paper is to study the geometry of wandering domains. In particular we investigate which bounded domains can be realized as a (wandering) Fatou component of an automorphism. The following are the main results of this paper.

**Theorem 1.** For every $m \geq 2$ there exists an automorphism of $\mathbb{C}^m$ with an escaping wandering domain equal to the Euclidean ball.

**Theorem 2.** For every $m \geq 2$ there exists an automorphism of $\mathbb{C}^m$ with an oscillating wandering domain equal to the Euclidean ball.

As we will argue in the last section, the proofs of these theorems can easily be modified so that the same statements hold for any bounded simply connected regular open sets $\Omega \subset \mathbb{C}^m$ whose closure is polynomially convex, in particular for all bounded convex domains.

Let $F$ be a holomorphic automorphism of $\mathbb{C}^m$, and recall that the Fatou set $\mathcal{F}$ is the largest open subset of $\mathbb{C}^m$ on which the family of iterates $(F^n)_{n \geq 0}$ is locally equicontinuous. Connected component $\Omega$ of the Fatou set is called the Fatou component and we say that such component is wandering if and only if $F^n(\Omega) \cap \Omega = \emptyset$ for all $n \geq 1$. We will call a wandering Fatou component a wandering domain. There are three types of wandering domains:

1. escaping; if all orbits converge to the line at infinity,
2. oscillating; if there exists an unbounded orbit and an orbit with a bounded subsequence,
3. orbitally bounded; if every orbit is bounded.

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The first construction of a holomorphic automorphism of \( \mathbb{C}^2 \) with a wandering domain is due to Fornæss-Sibony [FS98] and their wandering domain is of the oscillating type. More recently Arosio-Benini-Fornæss-Peters [ABTP19] constructed transcendental Hénon maps, i.e. a holomorphic automorphism of \( \mathbb{C}^2 \) of the form \( F(z, w) = (f(z) + aw, az) \) with \( f : \mathbb{C} \to \mathbb{C} \) a transcendental function, that admit wandering domains. In particular they construct examples of wandering domains which are of the escaping and of the oscillating type and they are biholomorphic to \( \mathbb{C}^2 \). The first example of a polynomial automorphism (of \( \mathbb{C}^4 \)) with a wandering domain was given by Hahn-Peters [HP18] and their example was of the orbitally bounded type. In [ABTP19] we have shown that oscillating wandering domains of transcendental Hénon maps can also have different complex structures. In particular we have constructed a wandering domain that supports a non-constant bounded plurisubharmonic function and therefore it can not be biholomorphic to \( \mathbb{C}^2 \). The most recent result is due to Berger-Biebler [BB20] who have solved a long standing problem by proving existence of polynomial Hénon maps which admit wandering domains and note that those can only be of the orbitally bounded type.

Our Theorem 1 and Theorem 2 give new examples of wandering domains in terms of their geometry. In particular, these are also the first examples of bounded Fatou components with a smooth boundary.

Note that the only other known example of an automorphism of \( \mathbb{C}^m \) that has a Fatou component (non-wandering) with a smooth boundary, is a shear automorphism of \( \mathbb{C}^2 \), constructed recently by [BC20], for which the invariant Fatou component is the product of the complex line with the Euclidean unit disc. Also, the only known examples of bounded Fatou components are the Siegel balls for polynomial Hénon maps, and their boundary is very far from being smooth. It is known that such a domain must be biholomorphic to one of the following three the polydisc, the unit ball and a Thullen domain, but which of them can be realised as the Siegel ball for polynomial Hénon maps it is presently unknown.

We believe that the construction behind Theorem 1 will open the way for the study of intrinsic dynamics in wandering domains, as has recently been done for transcendental functions in dimension one [BEGRS19].

Finally let us mention that tools Andersén–Lempert theory, on which our proof rely, apply to the large class of Stein manifolds with the density property, see [For17, Section 4]. Therefore we strongly believe that, in many of those cases, the constructions presented in this paper could easily be modified and used to produce automorphisms with different types of wandering domains.

Our paper is organized as follows:

In Section 2 we introduce the notation and recall the basic ingredients that will be used in the paper.

In Section 3 we modify the constructions from [ABFP18, ABTP19] and reprove the existence of wandering domains biholomorphic to \( \mathbb{C}^m \) and to Short \( \mathbb{C}^m \) using tools of Andersén–Lempert theory. This modification will serve as the basis for the construction of the oscillating wandering ball in Section 5.

In Section 4 we use tools of Andersén–Lempert theory to inductively construct a sequence of automorphisms \( (F_k) \), that converge uniformly on compacts, to an automorphism \( F \) with the following properties: (1) The Euclidean diameter of \( F^k(\mathbb{B}(P_0, 1)) \) is less than 2 for all \( k \geq 0 \), (2) \( F^k(\mathbb{B}(P_0, 1)) \to \infty \) as \( k \to \infty \) and (3) there is a sequence of points \( (T_0^j) \) that
accumulate densely on the \( b\mathbb{B}(P_0, 1) \) and each of them is contained in some attracting basin. This implies that the Fatou component is exactly the ball \( \mathbb{B}(P_0, 1) \) which settles Theorem 1.

In Section 5 we prove Theorem 2 by carefully combining two constructions previously introduced in Section 3 and Section 4.

2. Preliminaries

In this section, we introduce the notation and recall the basic ingredients that will be used in the paper. Throughout this paper we will always assume that \( m \geq 2 \).

The polynomially-convex hull of a compact set \( K \subset \mathbb{C}^m \) is defined as

\[
\hat{K} = \{ z \in \mathbb{C}^m : |p(z)| \leq \sup_{K} |p| \text{ for all holomorphic polynomials } p \}.
\]

We say that \( K \) is polynomially convex if \( \hat{K} = K \).

Given a point \( z_0 \in \mathbb{C}^m \), we denote by \( B(z_0, r) \subset \mathbb{C}^m \) to denote the open \( m \)-dimensional Euclidean ball of radius \( r \) centered at \( z_0 \). We will write \( B = B(0, 1) \).

We shall frequently use the following basic result; see e.g. [St07].

**Lemma 3.** Assume that \( K \subset \mathbb{C}^n \) is a compact polynomially convex set. For any finite set \( p_1, \ldots, p_k \in \mathbb{C}^m \setminus K \) and for all sufficiently small numbers \( r_1 > 0, \ldots, r_k > 0 \), the set \( \bigcup_{j=1}^{k} B(p_j, r_j) \cup K \) is polynomially convex.

We will frequently use the fact that the union of any two disjoint closed Euclidean balls is polynomially convex.

Recall that a domain \( D \subset \mathbb{C}^m \) is called starshaped (in some literature star-like) if there exists a point \( p \in D \) such that the line segment between \( p \) and any other point \( q \in D \) is contained in \( D \). We will say that a domain \( D \subset \mathbb{C}^m \) is starshapelike if there exists \( \Phi \) an automorphism of \( \mathbb{C}^m \) and a starshaped domain \( D' \) so that \( D = \Phi(D') \). For example, any image of the Euclidean ball under an automorphism of \( \mathbb{C}^m \) is a starshapelike domain. It is a well known result that the closure of any bounded starshapelike domain is polynomially convex.

The key ingredient in our proofs will be the following result of the Andersén–Lempert theory, which is a combination of Theorem 4.9.2 and Corollary 4.12.4 from [For17].

**Theorem 4.** Let \( A_1, A_2, \ldots, A_n \) be pairwise disjoint compact sets in \( \mathbb{C}^m \) such that all but one are starshapelike. Let \( a_j \in \text{Aut}(\mathbb{C}^m) \) \((j = 1, \ldots, n)\) be such that the images \( B_j = a_j(A_j) \) are pairwise disjoint. If the sets \( K = \bigcup_{j=1}^{n} A_j \) and \( K' = \bigcup_{j=1}^{n} B_j \) are polynomially convex, then for every \( \varepsilon > 0 \) there exists \( g \in \text{Aut}(\mathbb{C}^m) \) such that \( \|g(z) - a_j(z)\| < \varepsilon \) for all \( z \in A_j, j = 1, \ldots, m \). In particular the automorphism of \( g \) can be chosen so that its finite order jets agree with the corresponding jets of \( a_j \) at any given finite set of points in \( A_j \), for \( 1 \leq j \leq m \).

Note that in the above theorem the compact set \( A_j \) can also be a point, since by Lemma 3 we can always find a small closed ball around \( A_j \), such that its union with all the other sets is polynomially convex.

If \( (H_n)_{n \geq 1} \) is a sequence of automorphisms of \( \mathbb{C}^m \), then for all \( 0 \leq n \leq k \) we denote

\[
H_{k,n} := H_k \circ \cdots \circ H_{n+1}.
\]

Notice that with these notations we have for all \( n \geq 0 \),

\[
H_{n+1,n} = H_{n+1} \quad \text{and} \quad H_{n,n} = \text{id}.
\]
If for all \( n \geq 1 \) we have \( H_n(\mathbb{B}) \subset \mathbb{B} \) then we define the basin of the sequence \( (H_n) \) as the domain
\[
\Omega_H := \bigcup_{n \geq 0} H_{n,0}^{-1}(\mathbb{B}).
\]

The following lemma was established in [ABTP19] and will be used in the following section to determine the complex structure of a wandering domain.

**Lemma 5.** To every finite family \((F_1, \ldots, F_n)\) of holomorphic automorphisms of \( \mathbb{C}^m \) satisfying \( F_j(\mathbb{B}) \subset \mathbb{B} \) for all \( 0 \leq j \leq n \) we can associate \( \varepsilon(F_1, \ldots, F_n) > 0 \) such that the following holds:

Given any two sequences \((H_n)_{n \geq 1}\) and \((G_n)_{n \geq 1}\) of holomorphic automorphisms of \( \mathbb{C}^m \) satisfying \( H_n(\mathbb{B}) \subset \mathbb{B} \) and \( G_n(\mathbb{B}) \subset \math{B} \) for all \( n \geq 1 \), and moreover satisfying
\[
\|H_n - G_n\|_{\mathbb{B}} \leq \varepsilon(H_1, \ldots, H_n), \quad \forall n \geq 1,
\]
the basins \( \Omega_G \) and \( \Omega_H \) are biholomorphically equivalent.

### 3. Oscillating wandering domains

The existence holomorphic automorphisms of \( \mathbb{C}^m \) that admit an oscillating wandering domain, has been proven by Fornaess and Sibony [FS98]. Recently constructed examples show that such wandering domains can be biholomorphic to \( \mathbb{C}^m \) [ABFP18] and to a Short \( \mathbb{C}^m \) [ABTP19]. In this section we slightly modify these two constructions and reprove them using tools of Andersén–Lempert theory. This modification will serve as the basis for the construction of the oscillating wandering ball, which will be presented in the last section of this paper.

**Proposition 6.** Let \((H_k)_{k \geq 1}\) be a sequence of holomorphic automorphisms of \( \mathbb{C}^m \) satisfying \( H_k(\mathbb{B}) \subset \mathbb{B} \) for all \( k \geq 1 \) and let \( \varepsilon_k = \varepsilon_k(H_1, \ldots, H_k) \) be as in Lemma 5. There exists a sequence \((F_k)_{k \geq 0}\) of holomorphic automorphisms of \( \mathbb{C}^m \), a sequence of points \((P_n)_{n \geq 0}\), sequences positive real numbers \((\beta_n)_{n \geq 0} \searrow 0\), \((R_k)_{k \geq 0} \nearrow \infty\), \((r_k)_{k \geq 0} \nearrow \infty\), strictly increasing sequences of integers \((n_k)_{k \geq 0}\) and \((N_k)_{k \geq 0}\) satisfying \( n_0 = 0 \) and \( N_k_{k-1} \leq n_k \leq N_k \), and such that the following properties are satisfied:

(a) \( \mathbb{B}(0, \frac{r_{k-1}}{2}) \subset \subset F_k(\mathbb{B}(0, R_k)) \) for all \( k \geq 1 \),
(b) \( \|F_k - F_{k-1}\|_{B(0, R_{k-1})} \leq 2^{-k} \) for all \( k \geq 1 \),
(c) \( F_k(P_n) = P_{n+1} \) for all \( 0 \leq n < N_k \),
(d) \( \|P_{n_k}\| \leq \frac{1}{k} \) for all \( k \geq 1 \),
(e) \( \|P_{N_k}\| > R_k \) for all \( k \geq 1 \),
(f) for all \( k \geq 1 \) we have \( \beta_j < \frac{1}{k+1} \) for \( N_k < j \leq N_{k+1} \),
(g) for all \( 0 \leq s \leq k \),
\[
F_k^j(\mathbb{B}(P_{n_s}, \beta_{n_s})) \subset \subset \mathbb{B}(P_{n_s+j}, \beta_{n_s+j}), \quad \forall 1 \leq j \leq N_k - n_s,
\]
(h) for all \( 1 \leq s \leq k \),
\[
\|\Phi_{n_s}^{-1} \circ F_k^{n_s-n_{s-1}} \circ \Phi_{n_{s-1}} - H_s\|_{\mathbb{B}} \leq \varepsilon_s,
\]
where \( \Phi_n(z) := P_n + \beta_n z \).

In the above proposition, the properties (a) and (b) imply that the sequence \( F_k \) converges uniformly on compacts to an automorphism \( F \). The properties (c)–(g) ensure the existence of an oscillating wandering domain for \( F \). Furthermore, the property (h) determines the intrinsic dynamics and the geometry of such a domain.
Proof. We prove this proposition by induction on \( k \). We start the induction by letting \( F_0(z_1, \ldots, z_m) = (\frac{1}{2}z_1, \ldots, \frac{1}{2}z_k, 2z_{k+1}, \ldots, 2z_m) \) for some \( 1 \leq k < m \). Let \( r_0 = 1 \) and \( R_0 > 0 \) so large that \( \mathcal{B}(0, r_0) \subset F_0(\mathcal{B}(0, R_0)) \) and set \( K_0 = \overline{\mathcal{B}(0, R_0)} \). Moreover we let \( n_0 = N_0 = 0 \), \( \beta_0 = 1 \), and choose any \( P_0 \) with \( \|P_0\| > R_0 + 1 \) such that all conditions are satisfied for \( k = 0 \).

Let us suppose that conditions (a)–(h) hold for certain \( k \), and proceed with the constructions satisfying the conditions for \( k + 1 \).

First let \( R_{k+1} > \|P_N\| + 1 \) such that \( K_k \subset \mathcal{B}(0, R_{k+1}) \), where \( K_k \cup \overline{\mathcal{B}(P_N, \beta_N)} \) is polynomially convex and \( \overline{\mathcal{B}(0, R_k)} \subset K_k \). By the \( \lambda \)-Lemma (see [PdMS2, Lemma 7.1]) there exist a finite \( F_k \) orbit \( (Q_j)_{0 \leq j \leq M} \), i.e. \( F_k^j(Q_0) = Q_j \) for all \( 0 \leq j \leq M \), such that:

\begin{align}
(1) \quad &\|Q_j\| < R_{k+1} \quad \text{for all} \quad 0 \leq j < M, \\
(2) \quad &\|Q_M\| > R_{k+1}, \\
(3) \quad &\|Q_\ell\| < \frac{1}{\ell + 1} \quad \text{for some} \quad 0 < \ell < M.
\end{align}

We can choose small enough \( 0 < \theta < \frac{1}{\ell + 1} \) so that:

(i) the ball \( \mathcal{B}(Q_M, \theta) \) is disjoint from \( \mathcal{B}(0, R_{k+1}) \),

(ii) the balls

\[
\mathcal{B}(P_N, \beta_N), \quad \mathcal{B}(Q_0, \theta), \quad \mathcal{B}(Q_M, \theta)
\]

are pairwise disjoint, and disjoint from the set

\[
L := K_k \cup \bigcup_{0 < i < M} \mathcal{B}(Q_i, \theta),
\]

and their union with \( L \) is a polynomially convex set (see Lemma 6),

(iii) \( \mathcal{B}(P_N, \beta_N) \cup \mathcal{B}(Q_0, \theta) \cup L \subset \subset \mathcal{B}(0, R_{k+1}) \).

By continuity of \( F_k \) there exists \( 0 < s_\ell < \theta \) small enough such that

\[
F_k^j(\mathcal{B}(Q_\ell, s_\ell)) \subset \subset \mathcal{B}(Q_{\ell+j}, \theta) \quad \text{for all} \quad 0 \leq j \leq M - \ell,
\]

and such that

\[
F_k^{-j}(\mathcal{B}(Q_\ell, s_\ell)) \subset \subset \mathcal{B}(Q_{\ell-j}, \theta) \quad \text{for all} \quad 0 \leq j \leq \ell.
\]

We are now ready to construct the map \( F_{k+1} \). First we define automorphisms

\[
\Phi_\ell := Q_\ell + s_\ell \cdot z \quad \text{and} \quad \varphi := F_k^{-\ell+1} \circ \Phi_\ell \circ H_{k+1} \circ \Phi_n^{-1} \circ F_k^{-N_k-n_k}
\]

and a compact starshapelike domain

\[
W := F_k^{N_k-n_k}(\mathcal{B}(P_N, \beta_N)) \subset \mathcal{B}(P_N, \beta_N).
\]

Observe that the following properties hold:

- \( \varphi(P_N) = Q_1 \),
- \( \varphi(W) \subset F_k(\mathcal{B}(Q_0, \theta)) \),
- \( F_k^j(\varphi(W)) \subset \mathcal{B}(Q_{j+1}, \theta) \) for all \( 0 \leq j < \ell - 1 \),
- \( F_k^{\ell-1}(\varphi(W)) \subset \mathcal{B}(Q_\ell, s_\ell) \).

In the terminology of Theorem 4 we define

\[
A_1 := L \cup \mathcal{B}(Q_M, \theta), \quad A_2 := W.
\]

It follows from above that these two sets are pairwise disjoint and that their union is polynomially convex. Next we define \( q_1(z) := F_k(z) \) on \( A_1 \) and \( q_2(z) := \varphi(z) \) on \( A_2 \). Observe that their images \( B_1 := q_1(A_1) \) and \( B_2 := q_2(A_2) \) are pairwise disjoint and that their union is also polynomially convex. By Theorem 4 there exists an automorphism \( g_{k+1} \) such that
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Choose any point
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\begin{align*}
\text{(iii)} & \text{ for all } 0 \leq j < \ell, \\
\text{(iv)} & \text{ for all } 0 \leq j < M - \ell, \\
\text{(d)} & \text{ for all } 0 \leq s \leq k, \\
\text{(e)} & \| \Phi_{n+1}^{-1} \circ g_{k+1}^j \circ \Phi_{n+1} - H_{s+1} \|_{\mathcal{F}} < \varepsilon_s \text{ for all } 0 \leq s \leq k
\end{align*}

Remark: Note that the map \( g_{k+1} \) already satisfies all the properties (b)—(h) of the Proposition \( \square \). Next we need to post-compose our map \( g_{k+1} \) with an appropriate automorphism to make sure that also property (a) is satisfied. Recall that this property is needed to ensure that the limit map of the sequence \( (F_k) \) is surjective.

Let us continue with induction by choosing \( r_{k+1} > r_k + 1 \) such that
\[ g_{k+1}(B(0, R_{k+1})) \subset B(0, r_{k+1}) \]
Since compact sets \( \overline{B}(Q_M, \theta) \) and \( \overline{B}(0, R_{k+1}) \) are disjoint starshapelike domains whose union is polynomially convex the same holds for
\[ U := g_{k+1}(\overline{B}(Q_M, \theta)), \quad V := g_{k+1}(\overline{B}(0, R_{k+1})). \]
Choose any point \( Q' \in C^n \) such that the ball \( \overline{B}(Q', \theta) \) lies in the complement of \( \overline{B}(0, r_{k+1}) \) and let \( \psi \) be a linear map satisfying \( (\psi \circ g_{k+1})(Q_M) = Q' \) and \( \psi(U) \subset B(Q', \theta) \).

By Theorem \( \square \) there exists an automorphism \( h \) such that
\begin{align*}
\text{(1)} & \| h - \text{id} \|_V \leq \delta'_k, \\
\text{(2)} & \| h - \psi \|_U \leq \delta'_k, \\
\text{(3)} & h(Q_j) = Q_j \text{ for all } 0 < j \leq M, \\
\text{(4)} & (h \circ g_{k+1})(Q_M) = Q', \\
\text{(5)} & h(0) = 0, d_0 h = \text{id}, h(P_j) = P_j \text{ for all } 1 \leq j \leq P_{N_k},
\end{align*}
where we have chosen \( \delta'_k \leq \frac{1}{2^n} \) small enough such that
\begin{align*}
\text{(i)} & (h \circ g_{k+1})^j(W) \subset B(Q_j, \theta), \text{ for all } 0 < j < \ell, \\
\text{(ii)} & (h \circ g_{k+1})^\ell(W) \subset B(Q_{\ell}, \theta), \\
\text{(iii)} & \text{ for all } 0 \leq s \leq k,
\end{align*}
\[ (h \circ g_{k+1})^j(U) \subset B(Q', \theta). \]

We define \( F_{k+1} := h \circ g_{k+1} \), so that the sequences of points
\[ (P_j)_{0 \leq j \leq N_k}, \quad (Q_j)_{0 < j \leq M} \]
together form the start of an $F_{k+1}$-orbit, i.e. $F^j_{k+1}(P_0) = P_j$ for $j \leq N_k + M$ where $P_{N_k + j} = Q_j$ for $0 < j \leq M$.

Set $n_{k+1} := N_k + \ell$ and $N_{k+1} := N_k + M$ and define $\beta_j := \theta$ for $N_k < j < n_{k+1}$ and for $n_{k+1} < j \leq N_{k+1}$ and $\beta_{n_{k+1}} := s_\ell$. 

Finally we define $K_{k+1} := F_{k+1}^{-1}(B(0, r_{k+1}))$ and observe that $B(0, R_{k+1}) \subset K_{k+1}$. Since $F_{k+1}(B(P_{N_{k+1}}, \beta_{N_{k+1}})) \subset \subset B(Q', \theta)$, we make an inductive step. This concludes the inductive step.

It is immediate that conditions (c)–(h) are satisfied for the $(k+1)$-th step. Condition (b) follows from the fact that $B(0, R_k) \subset K_k \subset L \subset B(0, R_{k+1})$ and that $h$ is almost an identity on $g_{k+1}(B(0, R_{k+1}))$, hence $F_{k+1} = h \circ g_{k+1}$ approximates $F_k$ on $B(0, R_k)$ as close as we want. For condition (a) it is enough to prove that $B(0, \frac{r_k}{2}) \subset F_{k+1}(K_k)$ where $K_k = F^{-1}_{k+1}(B(0, r_k)) \subset \subset B(0, R_{k+1})$. Since $g_{k+1}$ is almost $F_k$ on $K_k$ it follows that $B(0, \frac{r_k}{2}) \subset \subset g_{k+1}(F^{-1}_{k+1}(B(0, r_k)))$ and since $h$ is almost an identity on $g_{k+1}(K_n)$ it follows that $B(0, \frac{r_k}{2}) \subset \subset (h \circ g_{k+1})(B(0, R_{k+1}))$. This concludes the inductive step.

**Remark:** Note that in the $k$-th step of the induction we chose the radius $\beta_{n_k}$, i.e. $s_\ell$, before we make an $\varepsilon_k$-approximation to $H_k$, i.e. before introducing $\varphi$. This means that we can choose $H_k$ freely, in particular we can choose it so that it satisfies $\|H_k(z)\| \leq \beta_{n_k}^k \|z\|$.

Let us now show that this proposition implies the existence of an oscillating wandering Fatou component. The arguments in the following two paragraphs are similar to those in the proof of [ABTP19, Theorem 1] but we choose to present them here, so that the paper remains self-contained.

Let $(F_k)$ be a sequence of automorphisms of $\mathbb{C}^m$ satisfying conditions (a) – (h) of Proposition [ABTP19]. The sequence $(F_k)$ converges uniformly on compact subsets to an automorphism $F$ of $\mathbb{C}^m$ with an isolated fixed point at the origin and with $d_0 F$ being a diagonal matrix with eigenvalues equal to $\frac{1}{2}$ and $2$. There is an unbounded orbit $(P_n)$, a sequence $\beta_n \to 0$ and a strictly increasing sequences of integers $(n_k)$ such that the following properties are satisfied:

(i) $P_{n_k} \to 0$, 
(ii) for all $k \geq 0$,
\[ F^j(P(P_{n_k}, \beta_{n_k})) \subset B(P_{n_{k+j}}, \beta_{n_{k+j}}), \quad \forall j \geq 0. \] (4)
(iii) if for all $k \geq 1$ we denote
\[ G_k := \Phi_{n_k}^{-1} \circ F^{n_k-n_{k-1}} \circ \Phi_{n_{k-1}} \in \text{Aut}(\mathbb{C}^m), \]
then by combining conditions (g) and (h) it follows that $G_k(B) \subset B$ for all $k$, and
\[ \|G_k - H_k\|_{\text{op}} \leq \varepsilon(H_1, \ldots, H_k), \quad \forall k \geq 1. \] (5)

Next show that
\[ \Omega_F := \bigcup_{k=0}^{\infty} F^{-n_k}(B(P_{n_k}, \beta_{n_k})) \]
is contained in an oscillating Fatou component $F_0$. It suffices to prove that for all $k \geq 0$, the ball $B(n_k, \beta_{n_k})$ is contained in the Fatou set. But this follows from (4) since the Euclidean diameter of $F^j(B(P_{n_k}, \beta_{n_k}))$ is bounded for all $j \geq 0$. For all $j \geq 0$ denote $F_{n_j}$ the
Fatou component containing \( \mathbb{B}(P_{n_j}, \beta_{n_j}) \). Since \( \Omega_F \) is connected, it is contained in the Fatou component \( \mathcal{F}_0 \).

Since \( \beta_n \to 0 \), by (4) it follows that all limit functions on each \( \mathcal{F}_{n_j} \) are constants. We claim that \( \mathcal{F}_{n_i} \neq \mathcal{F}_{n_j} \) iff \( j \neq i \), which implies that they are all oscillating wandering domains. Assume by contradiction that \( \mathcal{F}_{n_j} = \mathcal{F}_{n_i} \), and set \( k := n_i - n_j \). Since the origin a fixed point and \( d_0 F \) is diagonal matrix with eigenvalues equal to \( \frac{1}{2} \) and 2, there exists a neighborhood \( U \) of the origin that contains no periodic points of order less than or equal to \( k \). Since the sequence \( (P_n) \) oscillates, there exists a subsequence \( (P_{m_k}) \) of \( (P_n) \) such that \( P_{m_k} \to z \in U \setminus \{0\} \). But then

\[
F^{m_k - n_j}(P_{n_j}) = F^{m_k - n_j}(P_{m_k}) \to F^k(z) \neq z,
\]

which contradicts \( F^{m_k - n_j}(P_{n_j}) = P_{m_k} \to z \).

This completes the proof of the existence of an oscillating wandering domain.

Next we will see how we can use the sequence \( (H_k) \) to determine the complex structure of the Fatou component \( \mathcal{F}_0 \).

**Theorem 7.** Let \( F \) and \( (\beta_{n_k})_{k \geq 0} \) be as above. If the sequence of automorphisms \( (H_k)_{k \geq 1} \) satisfies \( \|H_k(z)\| \leq \beta_{n_k} \|z\| \) on \( \mathbb{B} \) for all \( k \geq 1 \) then the oscillating wandering Fatou component \( \mathcal{F}_0 \) is biholomorphic to \( \Omega_H \).

**Proof.** Since we have (5), the Lemma (5) implies that the basins \( \Omega_H = \bigcup_{k \geq 0} H_{k,0}^{-1}(\mathbb{B}) \) and \( \Omega_G = \bigcup_{k \geq 0} G_{k,0}^{-1}(\mathbb{B}) \) are biholomorphic. Next observe that \( \Omega_F = \Phi_0(\Omega_G) \), hence \( \Omega_F \) is also biholomorphic to \( \Omega_H \). It remains to prove that \( \Omega_F = \mathcal{F}_0 \).

This will be done using the plurisubharmonic method which was introduced in [ABFP18] and further developed in [ABTP19] Theorem 7.

First observe that by taking smaller \( \varepsilon_k(H_1, \ldots, H_k) \) if necessary we may assume that

\[
\|G_k(z)\| \leq 2\|H_k(z)\|
\]

for all \( z \in \mathbb{B} \) and all \( k \geq 1 \).

Define

\[
\Psi_k(z) := \log \frac{\|F^{n_k}(z) - P_{n_k}\|}{-k \log \beta_{n_k}}.
\]

Let \( \Psi = \limsup_{j \to \infty} \Psi_j \) on \( \mathcal{F}_0 \) and let \( \Psi^* \) be its upper semi-continuous regularization (see [K91]), hence \( \Psi^* \) is a plurisubharmonic function on \( \mathcal{F}_0 \). Since the sequence \( (P_{n_k}) \) is bounded, it follows that for all compact subsets \( K \subset \mathcal{F}_0 \), we have \( \|F^{n_k}(z) - P_{n_k}\| \to 0 \). This implies that \( \Psi \leq 0 \) on \( \mathcal{F}_0 \), and hence \( \Psi^* \leq 0 \) on \( \mathcal{F}_0 \).

Recall that \( \Omega_F = \Phi_0(\Omega_G) \) and observe that

\[
\Psi_k(z) = \frac{\log \|\Phi_{n_k} \circ G_k \circ \ldots \circ G_1 \circ \Phi_0^{-1}(z) - P_{n_k}\|}{-k \log \beta_{n_k}}
= \frac{\log \|G_k \circ \ldots \circ G_1 \circ \Phi_0^{-1}(z)\| + \log \beta_{n_k}}{-k \log \beta_{n_k}}
= \frac{\log \|G_k \circ \ldots \circ G_1 \circ \Phi_0^{-1}(z)\|}{-k \log \beta_{n_k}} + \frac{1}{k}.
\]

Given any \( z \in \mathcal{F}_0 \) we know that the sequence \( G_k \circ \ldots \circ G_1 \circ \Phi_0^{-1}(z) \) is bounded. In particular \( G_k \circ \ldots \circ G_1 \circ \Phi_0^{-1}(z_0) \in \mathbb{B}(0,1) \) for some \( k \geq 1 \) if and only if \( z_0 \in \Omega_F \). This implies that \( \Psi(z) = \Psi^*(z) = 0 \) on \( \mathcal{F}_0 \setminus \Omega_F \).
On the other hand, let $z_0 \in \Omega_F$, and let $k \geq 0$ be large enough such that $z_k := G_k \circ \cdots \circ G_1 \circ \Phi_0^{-1}(z) \in B(0,1)$. Then

$$\Psi_k(z_0) = \frac{\log \|G_k(z_j)\|}{-k \log \beta_{n_k}} - 1 - \frac{\log 2 \|H_k(z_j)\|}{-k \log \beta_{n_k}} \leq \frac{\log 2 + k \log \beta_{n_k}}{-k \log \beta_{n_k}} - 1 \leq -1 + \frac{\log 2}{-k \log \beta_{n_k}}.$$ 

It follows that $\Psi(z) \leq -1$ for all $z \in \Omega_F$, which implies that $\Psi^*(z) \leq -1$ for all $z \in \Omega_F$. Since $\mathcal{F}_0$ is open and connected, it follows from the maximum principle for plurisubharmonic functions that $\mathcal{F}_0 \setminus \Omega_F$ must be empty, which completes the proof.

**Example 1.** For $H_k(z) = \beta_{n_k} z$ it is easy to see that $\Omega_H = \mathbb{C}^m$ and therefore by Theorem 7 the oscillating wandering Fatou component $\mathcal{F}_0$ is biholomorphic to $\mathbb{C}^m$.

**Example 2.** Let $H_k(z_1, \ldots, z_m) = ((\frac{z_j}{k})^{d_k} + 2^{-d_k\cdots-d_1} z_m, 2^{-d_k\cdots-d_1} z_1, \ldots, 2^{-d_k\cdots-d_1} z_{m-1})$ where integers $d_k > 0$ are chosen so that $\|H_k(z)\| \leq \beta_{n_k}^k \|z\|$ on $\mathbb{B}$ for all $k \geq 1$. By [ABTP19, Proposition 3], which is a slightly modified version of [Fo04, Theorem 1.4], we know that $\Omega_H$ is a Short $\mathbb{C}^m$, and therefore by Theorem 7 the oscillating wandering Fatou component $\mathcal{F}_0$ is also a Short $\mathbb{C}^m$. Since such a domain supports a non-constant bounded plurisubharmonic function this implies that $\mathcal{F}_0$ is not biholomorphic to $\mathbb{C}^m$.

## 4. Escaping wandering ball

In this section we prove Theorem 11 using the following proposition.

**Proposition 8.** There exists a sequence $(F_k)_{k \geq 0}$ of holomorphic automorphisms of $\mathbb{C}^m$, disjoint sequences of points $(P_n)_{n \geq 0}$, $(T^j_n)_{n \geq 0}$, $(S_j)_{j \geq 1}$ and sequences positive real numbers $(R_k)_{k \geq 0}$ such that

$a)$ $\mathbb{B}(0, \frac{1}{2k}) \subset F_k(\mathbb{B}(0, R_k))$ for all $k \geq 1$,

$b)$ $\|F_k - F_{k-1}\|_{B(0, R_{k-1})} \leq 2^{-k}$ for all $k \geq 1$,

$c)$ $\|P_k\| > R_k$ for all $k \geq 0$,

$d)$ $F_k(\mathbb{B}(P_0, 1)) \subset \mathbb{B}(P_j, 2)$, for all $1 \leq j \leq k$ and all $k \geq 0$.

$e)$ Points $T^j_n$ accumulate densely on the $b\mathbb{B}(P_0, 1)$.

$f)$ $F_k(T^j_n) = T^j_{n+1}$ for all $1 \leq j \leq k$ and all $0 \leq n \leq j - 1$.

$g)$ $T^j_n \in \mathbb{B}(S_j, 1)$ for all $j \geq 1$.

$h)$ $F_k(S_j) = S_j$ for all $1 \leq j \leq k$.

$i)$ $\|F_k(z) - S_j\| \leq \frac{k}{2(2k+1)} \|z - S_j\|$ on $\mathbb{B}(S_j, 1)$ for all $1 \leq j \leq k$.

Before proving this proposition, let us show that it implies the existence of an escaping wandering Fatou component which is the unit ball.

### 4.1. Proof of Theorem 11

Let $(F_k)_{k \geq 0}$ be a sequence of holomorphic automorphisms of $\mathbb{C}^m$ given by Proposition 8. This sequence converges uniformly on compacts to a holomorphic automorphism $F$ of $\mathbb{C}^m$. Moreover there exist disjoint sequences of points $(P_n)_{n \geq 0}$, $(T^j_n)_{n \geq 0}$, $(S_j)_{j \geq 1}$ and strictly increasing sequence of positive real numbers $(R_k)_{k \geq 0}$ such that the following holds:

$(1)$ $\|P_k\| > R_k$ for all $k \geq 0$, 

$(2)$ $\mathbb{B}(0, \frac{1}{2k}) \subset F_k(\mathbb{B}(0, R_k))$ for all $k \geq 1$, 

$(3)$ $\|F_k - F_{k-1}\|_{B(0, R_{k-1})} \leq 2^{-k}$ for all $k \geq 1$, 

$(4)$ $\|P_k\| > R_k$ for all $k \geq 0$, 

$(5)$ $F_k(\mathbb{B}(P_0, 1)) \subset \mathbb{B}(P_j, 2)$, for all $1 \leq j \leq k$ and all $k \geq 0$. 

$(6)$ Points $T^j_n$ accumulate densely on the $b\mathbb{B}(P_0, 1)$.

$(7)$ $F_k(T^j_n) = T^j_{n+1}$ for all $1 \leq j \leq k$ and all $0 \leq n \leq j - 1$.

$(8)$ $T^j_n \in \mathbb{B}(S_j, 1)$ for all $j \geq 1$.

$(9)$ $F_k(S_j) = S_j$ for all $1 \leq j \leq k$.

$(10)$ $\|F_k(z) - S_j\| \leq \frac{k}{2(2k+1)} \|z - S_j\|$ on $\mathbb{B}(S_j, 1)$ for all $1 \leq j \leq k$. 


(2) $F^k(\mathbb{B}(P_0, 1)) \subset \subset \mathbb{B}(P_k, 2)$ for all $k \geq 0$,

(3) $F(S_j) = S_j$ and $\|F(z) - S_j\| \leq \frac{1}{2}\|z - S_j\|$ on $\mathbb{B}(S_j, 1)$ for all $j \geq 1$

(4) Points $T^j_0$ accumulate densely on the $b\mathbb{B}(P_0, 1)$ and $F^k(T^j_0) \to S_j$ as $k \to \infty$.

Since the Euclidean diameter of $F^j(\mathbb{B}(P_0, 1))$ is bounded for all $j \geq 0$ it follows that $\mathbb{B}(P_0, 1)$ is contained in some Fatou component $F_0$. Assume that $F_0 \neq \mathbb{B}(P_0, 1)$. Since $F_0$ is an open set there exists $j > 0$ so that $T^j_0 \in F_0$. Since $F_0$ contains $\mathbb{B}(P_0, 1)$ whose orbit eventually leaves every compact set, the same must hold for any point in $F_0$. But we know that for $T^j_0$ we have $F^k(T^j_0) \to S_j$ as $k \to \infty$ which brings us to the contradiction. $\square$

4.2. Proof of Proposition $[8]$. We prove this proposition by induction on $k$. We start the induction by letting $F_0 = \text{id}$, $r_0 = 1$ and $R_0 = 2$. We define $K_0 = \overline{\mathbb{B}(0, R_0)}$. Finally we choose a point $P_0$ satisfying $\|P_0\| > R_0 + 3$ and a sequence $(T^j_0)_{j \geq 1} \subset \mathbb{B}(P_0, 2) \backslash \overline{\mathbb{B}(P_0, 1)}$ which accumulates densely on $b\mathbb{B}(P_0, 1)$ and for which the sequence of distances $\|T^j_0 - P_0\| \to 1$ is strictly decreasing. In this setting all conditions (a)–(i) are satisfied for $k = 0$.

Let us suppose that conditions (a)–(i) hold for certain $k$, and proceed with the constructions satisfying the conditions for $k + 1$. The sets $K_k$ and $\overline{\mathbb{B}(P_k, 2)}$ are disjoint and their union is polynomially convex. Observe that $\overline{\mathbb{B}(0, R_k)} \subset K_k$. Next choose a point $S_{k+1}$ so that the ball $\overline{\mathbb{B}(S_{k+1}, 1)}$ is disjoint from the sets $K_k$, $F_k(K_k)$ and $\overline{\mathbb{B}(P_k, 2)}$ and so that both sets $\overline{\mathbb{B}(S_{k+1}, 1)} \cup K_k \cup \overline{\mathbb{B}(P_k, 2)}$ and $\overline{\mathbb{B}(S_{k+1}, 1)} \cup F_k(K_k)$ are polynomially convex. Choose $R_{k+1} > R_k + 1$ so that

$$\overline{\mathbb{B}(S_{k+1}, 1)} \cup K_k \cup F_k(K_k) \cup \overline{\mathbb{B}(P_k, 2)} \subset \mathbb{B}(0, R_{k+1})$$

and point $P_{k+1}$ such that $\overline{\mathbb{B}(P_{k+1}, 2)}$ and $\overline{\mathbb{B}(P_k, 2)}$ are disjoint. Finally choose $1 < \rho_{k+1} < 2$ such that $\|T^j_0 + 2 - P_0\| < \rho_{k+1} < \|T^j_0 + 1 - P_0\|$ and define a starshapelike compact set

$$W := F^k_k(\overline{\mathbb{B}(P_0, \rho_{k+1})}) \subset \mathbb{B}(P_k, 2).$$

In the terminology of Theorem [4] we define

$$A_1 := K_k, \quad A_2 := \overline{\mathbb{B}(S_{k+1}, 1)}, \quad A_3 := W, \quad A_4 := T^{k+1}_k.$$ (6)

It follows from our construction that all these sets are pairwise disjoint and that their union is polynomially convex. Also note that all these sets are all starshapelike. Next we define $g_1(z) := F_k(z)$ on $A_1$, $g_2(z) := \frac{k+1}{2(k+2)^{k+1}}(z - S_{k+1}) + S_{k+1}$ on $A_2$, $g_3(z) := \frac{k+1}{2(k+2)^{k+1}}z + P_{k+1}$ on $A_3$ and $g_4(z) := z - T^{k+1}_k + Q_0 + \frac{2}{3}$ on $A_4$. Observe that their images $B_j := g_j(A_j)$ where $1 \leq j \leq 4$ are pairwise disjoint and their union is polynomially convex. Moreover we have $B_3 \subset \mathbb{B}(P_{k+1}, 2)$.

Note that by the inductive assumption all balls $\overline{\mathbb{B}(S_j, 1)}$ for $1 \leq j \leq k$ are contained in the compact set $K_k$. By Theorem [4] there exists an automorphism $g_{k+1}$ such that

(1) $\|F_k - g_{k+1}\| \leq \delta_k$ on $K_k$

(2) $\|g_3 - g_{k+1}\| \leq \delta_k$ on $W$.

(3) $g_{k+1}(S_j) = S_j$ for all $1 \leq j \leq k + 1$

(4) $\|g_{k+1}(z) - S_{k+1}\| \leq \frac{k+1}{2(k+2)^{k+1}}\|z - S_{k+1}\|$ on $\overline{\mathbb{B}(S_{k+1}, 1)},$

(5) $g_{k+1}(T^j_n) = F_k(T^j_n)$ for all $1 \leq j \leq k$ and $0 \leq n \leq j - 1$,

(6) $g_{k+1}(T^{k+1}_n) = T^{k+1}_n$ for all $0 \leq n < k$ where $T^{k+1}_n := F^n_k(T^0_n)$

(7) $T^{k+1}_{k+1} := g_{k+1}(T^{k+1}_k) \in \mathbb{B}(S_{k+1}, 1) \backslash \overline{\mathbb{B}(S_{k+1}, \frac{1}{4})}$

where we have chosen $\delta_k \leq \frac{1}{2^k}$ small enough such that:

(a) $g_{k+1}^j(\mathbb{B}(P_0, \rho_{k+1}))) \subset \subset \mathbb{B}(P_j, 2)$, for all $1 \leq j \leq k + 1$
where we have chosen $\delta_k$ verbatim. This concludes the inductive step. $\square$

A proposition which is a hybrid between Proposition 8 and Proposition 6. To construct an oscillating wandering ball. The proof of Theorem 2 is based on the following

used to construct various examples of oscillating wandering domains and also of the escaping

At this point the automorphism $g_{k+1}$ already satisfies properties (b)–(i) and we continue similarly as in the proof of Proposition 8. We choose $r_{k+1} > r_k + 1$ so that

$g_{k+1}(\mathbb{B}(0, R_{k+1})) \subset \mathbb{B}(0, r_{k+1})$.

Since compact sets $\mathbb{B}(P_{k+1}, 2)$ and $\mathbb{B}(0, R_{k+1})$ are disjoint starshapelike domains whose union is polynomially convex the same holds for their images

$U := g_{k+1}(\mathbb{B}(P_{k+1}, 2)), \quad V := g_{k+1}(\mathbb{B}(0, R_{k+1}))$.

Let $Q' \in \mathbb{C}^2$ be a point such that the ball $\mathbb{B}(Q', \theta)$ lies in the complement of $\mathbb{B}(0, r_{k+1})$. Also let $\psi$ be a linear map satisfying $\psi(U) \subset \mathbb{B}(Q', \theta)$.

By Theorem 1 there exists an automorphism $h$ such that

1. $\|h - \text{id}\|_V \leq \delta_k'$,
2. $\|h - \psi\|_U \leq \delta_k'$,
3. $h(S_j) = S_j$ for all $0 < j \leq k + 1$,
4. $h(T^j_\beta) = T^j_\beta$ for all $1 \leq j \leq k + 1$ and $1 \leq n \leq j$

where we have chosen $\delta_k' \leq \frac{1}{2^k}$ small enough such that

1. $(h \circ g_{k+1})^j(\mathbb{B}(P_0, r_{k+1})) \subset \mathbb{B}(P_j, 2)$, for all $1 \leq j \leq k + 1$
2. $\| (h \circ g_{k+1})(z) - S_j \| \leq \frac{k+1}{2(k+2)} \| z - S_j \|$ on $\mathbb{B}(S_j, 1)$ for all $1 \leq j \leq k + 1$,

Finally define $K_{k+1} := F_{k+1}^{-1}(\mathbb{B}(0, r_{k+1}))$ and observe that $\mathbb{B}(0, R_{k+1}) \subset K_{k+1}$. Since $F_{k+1}(\mathbb{B}(P_{k+1}, 2)) \subset \mathbb{B}(Q', 2)$, and since the set $\mathbb{B}(0, r_{k+1}) \cup \mathbb{B}(Q', \theta)$ is polynomially convex it follows that $K_{k+1}$ and $\mathbb{B}(P_{k+1}, 2)$ are disjoint and their union is polynomially convex.

It is immediate that properties (c)–(i) are satisfied for the $(k+1)$-th step. The properties (a) and (b) can be verified by following the last paragraph of the proof of Proposition 8 verbatim. This concludes the inductive step. $\square$

Remark: We believe that by a slight modification of the above proof, in particular by choosing different map $g_3$, one can construct examples of wandering balls with different interior dynamics, as it was recently done for transcendental functions in dimension one [BEGRS19].

5. Oscillating Wandering Ball

In the previous two sections we have seen how the tools of Andersén–Lepert theory can be used to construct various examples of oscillating wandering domains and also of the escaping wandering ball. In this section we will see that by combining these two constructions we can construct an oscillating wandering ball. The proof of Theorem 2 is based on the following proposition which is a hybrid between Proposition 8 and Proposition 6.

**Proposition 9.** There exists a sequence $(F_k)_{k \geq 0}$ of holomorphic automorphisms of $\mathbb{C}^m$, disjoint sequences of points $(P_n)_{n \geq 0}$, $(T^j_n)_{j \geq 1, n \geq 0}$, $(S_j)_{j \geq 1}$ with $(S_j)$ being bounded away from the origin, sequences positive real numbers $(\beta_n)_{n \geq 0} \searrow 0$, $(r_n)_{n \geq 1} \searrow 0$, $(R_k)_{k \geq 0} \nearrow \infty$, \((r_k)_{k \geq 0} \nearrow \infty\), strictly increasing sequences of integers $(n_k)_{k \geq 0}$ and $(N_k)_{k \geq 0}$ satisfying $n_0 = 0$ and $N_{k-1} \leq n_k \leq N_k$, and such that the following properties are satisfied:

(a) $\mathbb{B}(0, \frac{r_n}{2}) \subset F_k(\mathbb{B}(0, R_k))$ for all $k \geq 1$,
5.1. Proof of Theorem Let \((F_k)_{k \geq 0}\) be a sequence of holomorphic automorphisms of \(\mathbb{C}^m\) given by Proposition. This sequence converges uniformly on compacts to a holomorphic automorphism \(F\) of \(\mathbb{C}^m\). Moreover there exists a disjoint sequences of points \((P_n)_{n \geq 0}\), \((T_{n,j})_{j \geq 1, n \geq 0}\), \((S_{j})_{j \geq 1}\), sequences positive real numbers \((\beta_n)_{n \geq 0} \rightarrow 0\), \((\tau_n)_{n \geq 1} \rightarrow 0\), strictly increasing sequences of integers \((n_k)_{k \geq 0}\) and \((N_k)_{k \geq 0}\) such that the following holds:

1. \(F^j(P_0) = P_j\) for all \(j \geq 0\),
2. \(P_{n_k} \rightarrow \infty\) and \(P_{N_k} \rightarrow \infty\) as \(k \rightarrow \infty\),
3. \(F^j(\mathbb{B}(P_0, 1)) \subset \mathbb{B}(P_j, \beta_j)\) for all \(j \geq 0\),
4. \(F(S_j) = S_j\) and \(\|F(z) - S_j\| \leq \frac{1}{j}\) on \(\mathbb{B}(S_j, \tau_j)\) for all \(j \geq 1\)
5. Points \(T_{0,j}^j\) accumulate densely on the \(\partial \mathbb{B}(P_0, 1)\) and \(F^k(T_{0,j}^j) \rightarrow S_j\) as \(k \rightarrow \infty\) for all \(j \geq 1\).

Since the Euclidean diameter of \(F^j(\mathbb{B}(P_0, 1))\) is bounded for all \(j \geq 0\) it follows that \(\mathbb{B}(P_0, 1)\) is contained in some Fatou component \(\mathcal{F}_0\). Assume that \(\mathcal{F}_0 \neq \mathbb{B}(P_0, 1)\). Since \(\mathcal{F}_0\) is an open set there exists \(j > 0\) so that \(T_{0,j}^j \in \mathcal{F}_0\). Since \(\mathcal{F}_0\) contains \(\mathbb{B}(P_0, 1)\) we have \(F^{n_k}(z) \rightarrow 0\) on \(\mathcal{F}_0\) as \(k \rightarrow \infty\) but on the other hand we have \(F^{n_k}(T_{0,j}^j) \rightarrow S_j \neq 0\) as \(k \rightarrow \infty\) which brings us to a contradiction.

5.2. Proof of Proposition. We prove this proposition by induction on \(k\). We start the induction by letting \(F_0(z_1, \ldots, z_m) = (\frac{1}{2}z_1, \ldots, \frac{1}{2}z_k, 2z_{k+1}, \ldots, 2z_m)\) for some \(1 \leq k < m\). Let \(r_0 = 1\) and let \(R_0 > 0\) be so large that \(\mathbb{B}(0, r_0) \subset \subset F_0(\mathbb{B}(0, R_0))\). Define \(K_0 = \mathbb{B}(0, R_0)\), \(n_0 = N_0 = 0\), \(\beta_0 = 2\), and choose any \(P_0\) with \(\|P_0\| > R_0 + 3\). Finally choose a sequence \((T_{0,j}^j)_{j \geq 1} \subset \mathbb{B}(P_0, 2) \setminus \mathbb{B}(P_0, 1)\) which accumulates densely on \(\partial \mathbb{B}(P_0, 1)\) and for which the sequence of distances \(\|T_{0,j}^j - P_0\| \rightarrow 1\) is strictly decreasing. Observe that in this setting all conditions (a)—(l) are satisfied for \(k = 0\).

Let us suppose that conditions (a)—(l) hold for certain \(k\), and proceed with the constructions satisfying the conditions for \(k + 1\).
First let $R_{k+1} > \|P_{N_k}\| + 1$ such that $K_k \subset B(0, R_{k+1})$, where $K_k \cup \overline{B}(P_{N_k}, \beta_{N_k})$ is polynomially convex and $\overline{B}(0, R_k) \subset K_k$. By the $\lambda$-Lemma (see [PdM2, Lemma 7.1]) there exist a finite $F_k$ orbit $(Q_j)_{-1 \leq j \leq M}$, i.e. $F_k(Q_{j-1}) = Q_j$ for $0 \leq j \leq M$, such that:

1. $\|Q_j\| < R_{k+1}$ for all $-1 \leq j < M$,
2. $\|Q_M\| > R_{k+1}$,
3. $\|Q_\ell\| < \frac{1}{k+1}$ for some $0 < \ell < M$.

By increasing $R_{k+1}$ if necessary, we can choose $0 < \theta \leq \eta < \frac{1}{k+1}$ so that:

(i) the ball $\overline{B}(Q_M, \theta)$ is disjoint from $\overline{B}(0, R_{k+1})$,
(ii) $\overline{B}(Q_0, \theta) \subset F_k(\overline{B}(Q_{-1}, \eta))$
(iii) the balls
\[
\overline{B}(P_{N_k}, \beta_{N_k}), \quad \overline{B}(Q_0, \theta), \quad \overline{B}(Q_M, \theta), \quad \overline{B}(Q_{-1}, \eta)
\]
are pairwise disjoint, and disjoint from the set
\[
L := K_k \cup \bigcup_{0 < i < M} \overline{B}(Q_i, \theta),
\]
and their union with $L$ is a polynomially convex set (see Lemma 3),
(iv) $\overline{B}(P_{N_k}, \beta_{N_k}) \cup \overline{B}(Q_0, \theta) \cup \overline{B}(Q_{-1}, \eta) \cup L \subset \subset B(0, R_{k+1})$.

By continuity of $F_k$ there exists $0 < s_\ell < \theta$ small enough such that for all $0 \leq j \leq M - \ell$,
\[
F_k^j(\overline{B}(Q_\ell, s_\ell)) \subset \subset \overline{B}(Q_{\ell-j}, \theta)
\]
and such that for all $0 \leq j \leq \ell$,
\[
F_k^{-j}(\overline{B}(Q_\ell, s_\ell)) \subset \subset \overline{B}(Q_{\ell+j}, \theta).
\]

Choose $1 < \rho_{k+1} < 2$ such that $\|T_0^{k+2} - P_0\| < \rho_{k+1} < \|T_0^{k+1} - P_0\|$ and define a starshapelike compact set
\[
W := F_k^{N_k}(\overline{B}(P_0, \rho_{k+1})) \subset \overline{B}(P_{N_k}, \beta_{N_k}).
\]

Next define linear automorphisms
\[
\Phi_1(z) = \frac{z - P_0}{2 \rho_{k+1}}, \quad \Phi_2(z) = s_\ell \cdot z + Q_\ell
\]
and an automorphism
\[
\varphi = F_k^{-\ell+1} \circ \Phi_2 \circ \Phi_1 \circ F_k^{-N_k}.
\]

Observe that the following holds:

- $\varphi(P_{N_k}) = Q_1$,
- $\varphi(W) \subset \subset F_k(\overline{B}(Q_0, \theta))$,
- $F_k^j(\varphi(W)) \subset \subset \overline{B}(Q_{j+1}, \theta)$ for all $0 \leq j < \ell - 1$,
- $F_k^{\ell-1}(\varphi(W)) \subset \subset \overline{B}(Q_\ell, s_\ell)$.

Let us write $S_{k+1} := Q_0$ and $\tau_{k+1} := \theta$. In the terminology of Theorem 4 we define
\[
A_1 := L \cup \overline{B}(Q_M, \theta), \quad A_2 := \overline{B}(S_{k+1}, \tau_{k+1}), \quad A_3 := W, \quad A_4 := T_0^{k+1}.
\]

It follows from our construction that these sets are pairwise disjoint and that their union is polynomially convex. Observe the sets $A_2$, $A_3$ and $A_4$ are all starshapelike. Next we define $q_1(z) := F_k(z)$ on $A_1$, $q_2(z) := \frac{(k+1)}{2(k+2)+1}(z - S_{k+1}) + S_{k+1}$ on $A_2$, $q_3(z) = \varphi(z)$ on $A_3$ and $q_4(z) = z - T_0^{k+1} + Q_0 + \frac{2\rho_{k+1}}{3}$ on $A_4$. Observe that their images $B_j := q_j(A_j)$ where $1 \leq j \leq 4$ are also pairwise disjoint and their union is polynomially convex.
Note that by the inductive assumption all balls $\overline{B}(S_j, \tau_j)$ for $j \leq k$ are contained in the compact set $K_k$ and therefore in $A_1$. By Theorem 4 there exists an automorphism $g_{k+1}$ such that

1. $g_{k+1}(0) = 0$ and $d_0g_{k+1} = d_0F_k$,
2. $\|F_k - g_{k+1}\| \leq \delta_k$ on $L \cup \overline{B}(Q_M, \theta)$,
3. $g_{k+1}(P_j) = F_k(P_j)$ for all $0 \leq j < N_k$,
4. $g_{k+1}(Q_j) = F_k(Q_j)$ for all $1 \leq j < M$,
5. $g_{k+1}(P_Nk) = Q_1$
6. $\|\varphi - g_{k+1}\| \leq \delta_k$ on $W$.
7. $g_{k+1}(S_j) = S_j$ for all $1 \leq j \leq k + 1$
8. $\|g_{k+1}(z) - S_{k+1}\| \leq \frac{k+1}{2(k+2)} \|z - S_{k+1}\|$ on $\overline{B}(S_{k+1}, \tau_{k+1})$,
9. $g_{k+1}(T_n^k) = F_k(T_n^k)$ for all $1 \leq j \leq k$ and $0 \leq n \leq N_{j-1}$,
10. for all $0 \leq n < N_k$ we have $g_{k+1}(T_{n+1}^k) = T_{n+1}^{k+1}$, where $T_{n+1}^{k+1} = F_n(T_0^{k+1})$
11. $T_{N_k+1}^{k+1} := g_{k+1}(T_{N_k+1}^k) \in \overline{B}(S_{k+1}, \tau_{k+1}) \setminus \overline{B}(S_{k+1}, \frac{\tau_{k+1}}{2})$

where we have chosen $\delta_k \leq \frac{1}{2^k}$ small enough such that:

(a) $g_{k+1}^{-1}(B(P_0, \rho_{k+1})) \subset B(P_j, \beta_j)$, for all $1 \leq j \leq N_k$
(b) $g_{k+1}^{-1}(B(P_0, \rho_{k+1})) \subset B(Q_j, \theta)$, for all $0 < j \leq M$
(c) $g_{k+1}^{-1}(B(P_0, \rho_{k+1})) \subset B(Q_\ell, s_\ell)$,
(d) $\|g_{k+1}(z) - S_j\| \leq \frac{k+1}{2(k+2)} \|z - S_j\|$ on $\overline{B}(S_j, \tau_j)$ for all $1 \leq j \leq k$,

To make sure that the newly constructed automorphism satisfies property (a) we need to correct $g_{k+1}$ by pre-composing it with an appropriate automorphism.

Let $r_{k+1} > r_k + 1$ such that

$$g_{k+1}(B(0, R_{k+1})) \subset B(0, r_{k+1}).$$

Since the compact sets $\overline{B}(Q_M, \theta)$ and $\overline{B}(0, R_{k+1})$ are disjoint starshapelike domains whose union is polynomially convex the same holds for their images

$$U := g_{k+1}(\overline{B}(Q_M, \theta)), \quad V := g_{k+1}(\overline{B}(0, R_{k+1})).$$

Let $Q' \in \mathbb{C}^2$ be a point such that the ball $\overline{B}(Q', \theta)$ lies in the complement of $\overline{B}(0, r_{k+1})$. Moreover let $\psi$ be a linear map satisfying $(\psi \circ g_{k+1})(Q_M) = Q'$ and

$$\psi(U) \subset B(Q', \theta).$$

By Theorem 4 there exists an automorphism $h$ such that

1. $\|h - id\|_V \leq \delta_k'$,
2. $\|h - \psi\|_U \leq \delta_k'$,
3. $h(Q_j) = Q_j$ for all $0 < j \leq M$,
4. $(h \circ g_{k+1})(Q_M) = Q'$
5. $h(S_j) = S_j$ for all $0 < j \leq k + 1$,
6. $h(0) = 0, d_0h = id, h(P_j) = P_j$ for all $1 \leq j \leq P_Nk$,
7. $h(T_n^k) = T_n^k$ for all $1 \leq j \leq k + 1$ and $1 \leq n \leq N_{j-1} + 1$

where we have chosen $\delta_k' \leq \frac{1}{2^k}$ small enough such that

(i) $(h \circ g_{k+1})^{-1}(B(P_0, \rho_{k+1})) \subset B(P_j, \beta_j)$, for all $1 \leq j \leq N_k$
(ii) $(h \circ g_{k+1})^{-1}(B(P_0, \rho_{k+1})) \subset B(Q_j, \theta)$, for all $0 < j \leq M$
(iii) $(h \circ g_{k+1})^{-1}(B(P_0, \rho_{k+1})) \subset B(Q_\ell, s_\ell)$,
(iv) \( \| (h \circ g_{k+1})(z) - S_j \| \leq \frac{k+1}{2(k+2)} \| z - S_j \| \) on \( \overline{B}(S_j, \tau_j) \) for all \( 1 \leq j \leq k+1 \).

Similarly as in the proof of Proposition \( \text{Proposition 8} \) we define \( F_{k+1} := h \circ g_{k+1} \), so that the sequences of points \( (P_j)_{0 \leq j \leq N_k} \), \( (Q_j)_{0 < j \leq M} \) together form the start of an \( F_{k+1} \)-orbit, t.i. \( F_j^{k+1}(P_0) = P_j \) for \( j \leq N_k + M \) where \( P_{N_k + j} = Q_j \) for \( 0 < j \leq M \).

Set \( n_{k+1} := N_k + \ell \) and \( N_{k+1} := N_k + M \). Define \( \beta_j := \theta \) for \( N_k < j < n_{k+1} \) and for \( n_{k+1} < j \leq N_{k+1} \) and \( \beta_{n_{k+1}} := s \ell \).

Finally we define \( K_{k+1} := F_{k+1}^{-1}((0, r_{k+1})) \) and observe that \( \overline{B}(0, R_{k+1}) \subset K_{k+1} \). Since \( F_{k+1} \) is a union of \( \overline{B}(0, r_{k+1}) \cup \overline{B}(Q', \theta) \) is polynomially convex it follows that \( K_{k+1} \) and \( \overline{B}(P_{N_{k+1}}, \beta_{N_{k+1}}) \) are disjoint and their union is polynomially convex.

It is immediate that properties (c)–(l) are satisfied for the \( (k+1) \)-th step. The properties (a) and (b) can be verified by following the last paragraph of the proof of Proposition \( \text{Proposition 6} \) verbatim. This concludes the inductive step. \( \Box \)

5.3. Concluding remarks. In Theorem \( \text{Theorem 1} \) and Theorem \( \text{Theorem 2} \) the term unit ball can be replaced by any bounded simply connected regular open set \( \Omega \subset \mathbb{C}^m \) whose closure is polynomially convex.

Here we explain how can one adapt the proof of Proposition \( \text{Proposition 9} \) to include also these domains and note that similarly can be done for the proof of Proposition \( \text{Proposition 5} \).

Since each compact polynomially convex set admits a basis of Stein neighborhoods that are Runge in \( \mathbb{C}^m \), there exists a decreasing sequence of compact polynomially convex neighborhoods \( (U_k) \) of \( \overline{\Omega} \).

In the above proof we simply replace the role of \( \overline{B}(P_0, 1) \) with \( \overline{\Omega} \) and \( \overline{B}(P_0, \rho_{k+1}) \) with \( U_{k+1} \) and choose a point \( \tilde{P}_0 \in \Omega \). Recall that the automorphism \( \Phi_1 \) defined in \( \text{Proposition 9} \) and used in \( \text{Proposition 10} \) maps \( \overline{B}(P_0, \rho_{k+1}) \) into \( \overline{B}(0, 1) \) with \( \Phi_1(P_0) = 0 \). We replace this with a automorphism \( \tilde{\Phi}_1 \) which maps \( U_{k+1} \) into \( \overline{B}(0, 1) \) and satisfies \( \tilde{\Phi}_1(\tilde{P}_0) = 0 \).

By choosing \( \theta \) sufficiently small we may assume that for every \( 0 < j < M \) the ball \( \overline{B}(Q_j, \theta) \) is either contained in \( K_j \) or else they are disjoint, hence set \( L \) defined in \( \text{Proposition 5} \) is a union of starshapelike domains. Finally in \( \text{Proposition 11} \) we get finitely many sets \( A_j \), so that all but one are starshapelike. The only one that might not be starshapelike is the set \( W = F^{N_{k}}(U_{k+1}) \). The rest of the proof follows verbatim.

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