BOREL SUMMABILITY OF NAVIER-STOKES EQUATION IN $\mathbb{R}^3$
AND SMALL TIME EXISTENCE

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Abstract. We consider the Navier-Stokes initial value problem,

$$v_t - \Delta v = -\mathcal{P}[v \cdot \nabla v] + f, \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}^3$$

where $\mathcal{P}$ is the Hodge-Protection to divergence free vector fields in the assumption that $\|f\|_{\mu,\beta} < \infty$ and $\|v_0\|_{\mu+2,\beta} < \infty$ for $\beta \geq 0$, $\mu > 3$, where

$$\|f(k)\|_{\mu,\beta} = \sup_{k \in \mathbb{R}^3} e^{\beta|k|(1 + |k|)^{\mu}} |\hat{f}(k)|$$

and $\hat{f}(k) = \mathcal{F}[f(\cdot)](k)$ is the Fourier transform in $x$.

By Borel summation methods we show that there exists a classical solution in the form

$$v(x, t) = v_0 + \int_0^t e^{-p/t} U(x, p) dp$$

$t \in \mathbb{C}$, $\text{Re} \frac{1}{t} > \alpha$, and we estimate $\alpha$ in terms of $\|\hat{v}_0\|_{\mu+2,\beta}$ and $\|f\|_{\mu,\beta}$. We show that $\|\hat{v}(\cdot,t)\|_{\mu+2,\beta} < \infty$. Existence and $t$-analyticity results are analogous to Sobolev spaces ones.

An important feature of the present approach is that continuation of $v$ beyond $t = \alpha^{-1}$ becomes a growth rate question of $U(\cdot, p)$ as $p \to \infty$, $U$ being a known function. For now, our estimate is likely suboptimal.

A second result is that we show Borel summability of $v$ for $v_0$ and $f$ analytic. In particular, we obtain Gevrey-1 asymptotics: $v \sim v_0 + \sum_{m=1}^{\infty} v_m t^m$, where $|v_m| \leq m! A_0 B_0^m$, with $A_0$ and $B_0$ are given in terms of to $v_0$ and $f$ and for small $t$, with $m(t) = [B_0^{-1} t^{-1}]$,

$$|v(x, t) - v_0(x) - \sum_{m=1}^{m(t)} v_m(x) t^m| \leq A_0 m(t)^{1/2} e^{-m(t)}$$

1. Introduction and main results

We consider the Navier-Stokes (NS) initial value problem

(1.1) $v_t - \Delta v = -\mathcal{P}[v \cdot \nabla v] + f(x), \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}^+$

where $v$ is the fluid velocity and $\mathcal{P} = I - \nabla \Delta^{-1}(\nabla \cdot)$ is the Hodge-Protection operator to the space of divergence free vector fields. We rescale $v, x$ and $t$ so that the viscosity is one. The initial condition $v_0$ and the forcing $f(x)$ are chosen to be divergence free. We assume $f$ to be time-independent for simplicity, but a time dependent $f$ could be treated similarly. Moreover, from the analysis presented here, it will be clear that similar results can be obtained for the corresponding periodic problem, i.e. $v(\cdot, t) \in T^3$.

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We first write the equation in the Fourier space. We denote by $\mathcal{F}$ or simply $\hat{\cdot}$ the Fourier transform and $\hat{\cdot}$ is the Fourier convolution. Since $\nabla \cdot v = 0$ we get
\begin{align}
\hat{\dot{v}}_t + |k|^2 \hat{\dot{v}} = -ik_j P_k [\hat{\dot{v}}_j \hat{\dot{v}}] + \hat{f}, \quad \hat{\dot{v}}(k,0) = \hat{v}_0,
\end{align}
where as usual a repeated index $j$ denotes summation over $j (= 1, 2, 3)$. If $P_k = \mathcal{F}(\mathcal{P})$ we get
\begin{align}
P_k \equiv \left(1 - \frac{k(k\cdot)}{|k|^2}\right),
\end{align}

**Definition 1.1.** We introduce the norm $\| \cdot \|_{\mu, \beta}$ by
\begin{align}
\|\hat{v}_0\|_{\mu, \beta} = \sup_{k \in \mathbb{R}^3} (1 + |k|)^\mu e^{\beta|k|} |\hat{v}_0(k)|, \quad \text{where } \hat{v}_0(k) = \mathcal{F}[v_0(t)](k),
\end{align}

We assume $\|\hat{v}_0\|_{2+\mu, \beta} < \infty$, $\|\hat{f}\|_{\mu, \beta} < \infty$ for some $\beta \geq 0$ and $\mu > 3$. Clearly, if $\beta > 0$, then $v_0$ and $f$ are analytic in a strip of width at least $\beta$.

There is considerable mathematical literature for Navier-Stokes equation, starting with Leray’s papers in the 1930s [15], [16], [17]. Global existence and uniqueness are known in 2d (see for instance [4] and reference therein). However, this is not the case in 3d. It is not known whether classical solutions exist globally in time for arbitrary sized smooth or even analytic initial data. While weak solutions in the space of distributions are known since Leray, it is not known if they are unique or not without additional assumptions. Only local existence and uniqueness of classical solutions is known, with a time of existence inversely proportional to $A^\beta$. While weak solutions in the case in 3d. It is not known whether classical solutions exist globally in time for nonlinear evolution PDEs. After Borel transform, NS becomes an integral equation in time for nonlinear evolution PDEs. Borel summability methods have been used by the authors [19] to prove complex sectorial existence of solutions of a rather general class of nonlinear PDEs in $\mathbb{C}^d$.

\begin{equation}
(1) \text{If the equation is first order in time and order } n > 1 \text{ in space, then } p \text{ is dual to } t^{-1/(n-1)}.\end{equation}
for arbitrary $d$. This is in some sense a generalization of the classical Cauchy-Kowalewski theorem to PDEs written as systems that are first order in time and higher order in space.\(^{(2)}\)

The main results in this paper are given by the following two theorems. The results in the first theorem are similar to classical ones, with $\| \cdot \|_{\mu, \beta}$ replacing Sobolev norms.

**Theorem 1.1.** If $\| \hat{\nu}_0 \|_{\mu+2, \beta} < \infty$, $\mu > 3$, $\beta \geq 0$, NS has a unique solution $\nu(., t)$ such that $\| \hat{\nu}(., t) \|_{\mu, \beta} < \infty$ for $\text{Re} \frac{1}{\alpha} > \alpha$. Here $\alpha$ depends on $\hat{\nu}_0$ through (5.39).

Furthermore, $\hat{\nu}(., t)$ is analytic for $\text{Re} \frac{1}{\alpha} > \alpha$ and $\| \hat{\nu}(., t) \|_{\mu+2, \beta} < \infty$ for $t \in [0, \alpha^{-1})$. If $\beta > 0$, this implies that $\nu$ is analytic in $x$ with the same analyticity width as $\nu_0$ and $f$.

**Remark 1.2.** Sobolev space methods give local existence of solutions in $H^m$ for $t \in [0,T)$, where $T$ is proportional to $1/\| \nu_0 \|_{H^m}$. In particular, for $m > \frac{3}{2}$, these solutions are classical solutions (the second derivatives are continuous). The result in Theorem 1.1 is similar, but in a different space. The existence time, $t = \alpha^{-1}$, involves $\| \hat{\nu}_0 \|_{j+\mu}$ for $j = 0, 1, 2$ (see (2.39)). This solution is classical since $\| \hat{\nu}(., t) \|_{\mu+2, \beta} < \infty$ for $\mu > 3$ implies $\nu(., t) \in C^2(\mathbb{R}^3)$.

**Remark 1.3.** If $\nu_0$ has finite suitable Sobolev norms, it was known that $\nu$ is analytic in $t$ in a region in the right half complex $t$ plane. In our setting, $\nu$ is analytic in $\{ t : \text{Re} \frac{1}{\alpha} > \alpha \}$ if $\| \hat{\nu}_0 \|_{\mu+2, \beta} < \infty$.

**Remark 1.4.** Previous results \(^{(12)}\) show that for space-periodic boundary conditions, analytic $f$ and $\nu_0 \in H^1$, the solution $\nu(., t)$ becomes analytic in space, with an analyticity strip improving with time for small time. Moreover, for $f = 0$, a uniform estimate on the analyticity strip width for large time exists under the hypothesis that the local dissipation $\nu \| \nabla \nu(., t) \|_{L^2(\mathbb{T}^3)}$ is bounded \(^{(9)}\). However, we are not aware of similar results in $\mathbb{R}^3$, as is the case in this paper. For $\beta > 0$, our results of Theorem 1.1 show that the analyticity width is preserved for $t \in [0, \frac{1}{\alpha})$.

**Theorem 1.2.** For $\beta > 0$ (analytic initial data) and $\mu > 3$, the solution $\nu$ is Borel summable in $1/t$, i.e. there exists $U(x, p)$, analytic in a neighborhood of $\mathbb{R}^+$, exponentially bounded, and analytic in $x$ for $|\text{Im} x| < \beta$ so that

$$
\nu(x, t) = \nu_0(x) + \int_0^\infty U(x, p) e^{-p/t} dp
$$

Therefore, in particular, as $t \to 0$,

$$
\nu(x, t) \sim \nu_0(x) + \sum_{m=1}^\infty t^m \nu_m(x)
$$

with

$$
|\nu_m(x)| \leq m! A_0 B_0^m,
$$

where $A_0$ and $B_0$ depend on $\nu_0$ and $f$, through (3.58), (3.7) and (3.58).

\(^{(2)}\) Also, Cauchy-Kowalewski theorem usually requires a local expansion in all $d$ independent variables. Our methods accommodate series type expansion in just one variable.
Remark 1.5. Borel summability and classical Gevrey-asymptotic results \[^2\] imply for small \( t \) that

\[
\left| v(x, t) - v_0(x) - \sum_{m=1}^{m(t)} v_m(x)t^m \right| \leq A_0 m(t)^{1/2}e^{-m(t)}
\]

where \( m(t) = [B_0^{-1}t^{-1}] \). Our bounds on \( B_0 \) are likely suboptimal. Formal arguments in the recurrence relation of \( v_{m+1} \) in terms of \( v_m, v_{m-1}, \ldots, v_1 \), indicate that \( B \) only depends on \( \beta \), but not on \( \|\hat{v_0}\|_{\mu, \beta} \).

Remark 1.6. For \( \beta > 0 \) the assumption \( \mu > 3 \) is not restrictive if \( \beta \) is consistent with the analyticity strips of \( v_0 \) and \( f \). This is because \( (1 + |k|)^\mu e^{-|\beta|k} \) is bounded in \( k \) for \( \beta > 0 \).

2. Formulation of Navier Stokes equation: Borel transform

We define \( \hat{w} \) by

\[
\hat{v}(k, t) = \hat{v}_0(k) + t\hat{v}_1(k) + \hat{w}(k, t)
\]

where

\[
\hat{v}_1(k) = (-|k|^2\hat{v}_0 - ik_jP_k [\hat{v}_{0,j}\hat{v}_0]) + \hat{f}(k)
\]

From (1.2) we get for \( \hat{w} \)

\[
\hat{w}_t + |k|^2\hat{w} = -ik_jP_k [\hat{v}_{0,j}\hat{w} + \hat{w}_j\hat{v}_0 + t\hat{v}_{1,j}\hat{w} + t\hat{w}_j\hat{v}_1 + \hat{w}_j\hat{w}]
\]

\[
- t|k|^2\hat{v}_1 - ik_jP_k [\hat{v}_{0,j}\hat{v}_1 + \hat{v}_{1,j}\hat{v}_0 + t\hat{v}_{1,j}\hat{v}_1]
\]

We seek a solution as a Laplace transform

\[
\hat{w}(k, t) = \int_0^\infty \hat{W}(k, p)e^{-p/t}dp
\]

with the property \( \lim_{p \to 0^+} \hat{W}(k, p) = 0 \) and \( \lim_{p \to 0^+} p\hat{W}_p(k, p) = 0 \). The Borel transform of (2.7), which is the same as the formal inverse-Laplace transform in \( 1/t \) gives in the dual variable \( p > 0 \),

\[
p\hat{W}_p + 2\hat{W} + |k|^2\hat{W} + ik_jP_k [\hat{v}_{0,j}\hat{W} + \hat{W}_j\hat{v}_0 + \hat{v}_{1,j}(1*\hat{W}) + (1*\hat{W}_j)\hat{v}_1]
\]

\[
+ ik_jP_k \hat{W}_j \hat{W} + |k|^2\hat{v}_1 + ik_jP_k [\hat{v}_{0,j}\hat{v}_1 + \hat{v}_{1,j}\hat{v}_0 + p\hat{v}_{1,j}\hat{v}_1] = 0,
\]

where \( \hat{W} \) denotes Laplace convolution in \( p \), followed by Fourier convolution in \( k \).

Since the equation \( \mathcal{D}y := [p\partial^2_p + 2\partial_p + |k|^2]y = 0 \) has explicit independent solutions in terms of Bessel functions, \( y = J_1(z)/z \) and \( y = Y_1(z)/z \), where \( z = 2|k|/\sqrt{p} \) which do not vanish at zero, we formally obtain from (2.10) by inverting \( \mathcal{D} \) the Duhamel formulation

\[
\hat{W}(k, p) = \frac{ik_j\pi}{2|k|\sqrt{p}} \int_0^p \mathcal{G}(z, z')\hat{H}[j](k, p')dp',
\]

where

\[
\mathcal{G}(z, z') = z'(-J_1(z)Y_1(z') + Y_1(z)J_1(z')) , \quad z = 2|k|/\sqrt{p} , \quad z' = 2|k|/\sqrt{p'},
\]
where we used the fact that $J(2.18)$.

From (2.12), (2.13) and (2.17) we get

\[ (2.11) \]

and

Writing $y$ of $z$ (2.17) $ik$ $j$

Remark 2.1. \(|G(z, z')|\) is bounded for all real nonnegative \(z' \leq z\). This follows from standard properties of Bessel functions [1]. (The approximate bound is about 0.6.)

To obtain stronger results with less regularity of $v_0$, it is convenient to introduce $\hat{U}(k, p)$ by:

\[ (2.12) \]

Substituting (2.12) into (2.11), we obtain

\[ (2.13) \]

We can further simplify the integral $\int_0^p G(z, z') H^{(2)}(k, p') dp'$ by noting that the only solution to

\[ (2.14) \]

satisfying $y(k, 0) = 0$, as it is easy to check, is

\[ (2.15) \]

where we used the fact that $J_1(z)/z$ is a solution to the associated homogeneous differential equation and that $\lim_{z \to 0} J_1(z)/z = 1/2$. On the other hand, inversion of $D$ with zero boundary condition at $p = 0$ involves the same kernel $G(z, z')$. Writing $-|k|^2 \hat{v}_1 = ik_j [ik_j \hat{v}_1]$, it follows that

\[ (2.16) \]

Therefore

\[ (2.17) \]

From (2.12), (2.13) and (2.14) we get

\[ (2.18) \]

where $G^{(2)}(k, p)$ is given by (2.13).

We will show that $N'$ is contractive in a suitable space, and hence $\hat{U} = N'[\hat{U}]$ has a unique solution. The solution satisfies $\hat{U}(0, k) = \hat{v}_1(k)$, $\hat{U}$ and $\hat{U}_p$ are bounded for $p \in \mathbb{R}^+$ and exponentially bounded at $\infty$. Then, $\hat{W}(k, p) = \hat{U}(k, p) - \hat{v}_1(k)$ satisfies the integral equation (2.10), and hence the differential equation (2.9) is satisfied, with $\lim_{p \to 0} p \hat{W}_p(k, p) = 0$, $\lim_{p \to 0} \hat{W}(k, p) = 0$, and $\hat{W}$ and $\hat{W}_p$ are exponentially bounded at $\infty$. Thus the Laplace transform $\hat{w}(k, t) = \int_0^\infty e^{-pt} \hat{W}(k, p) dp$ will
Indeed satisfy (2.7) for sufficiently large Re $\frac{1}{\mu}$, and because of the continuity of $\hat{W}$ at $p = 0$ we have $\lim_{t \to 0^+} \hat{w}(k, t) = 0$. Thus,

$$\hat{v}(k, t) = \hat{v}_0 + t\hat{v}_1 + \int_0^\infty e^{-p/t} \hat{W}(k, p) dp = \hat{v}_0 + \int_0^\infty e^{-p/t} \hat{U}(k, p) dp$$

solves the NS equation (1.22) in the Fourier space, with the given initial condition. Furthermore, the sufficiently rapid decay in $k$ of $\hat{U}$ implies that $v(x, t) = \mathcal{F}^{-1}[\hat{v}(\cdot, t)](x)$ is indeed a classical solution to (1.1). It is known (See e.g. [10]) that classical solutions are unique; thus $\hat{v}$ is the only solution to (1.1).

2.1. Existence of a solution to (2.18). First, we prove some preliminary lemmas.

**Lemma 2.2.** If $\|\hat{v}\|_{\mu, \beta}$ and $\|\hat{\omega}\|_{\mu, \beta} < \infty$, then we have

$$\|\hat{v} \hat{\omega}\|_{\mu, \beta} \leq C_0 \|\hat{v}\|_{\mu, \beta} \|\hat{\omega}\|_{\mu, \beta}$$

where $\hat{\omega}$ denotes Fourier convolution,

$$C_0(\mu) = 2^{\mu + 2} \int_{k \in \mathbb{R}^3} \frac{1}{(1 + |k|)^\mu} dk = \frac{32\pi2^\mu}{(\mu - 1)(\mu - 2)(\mu - 3)}$$

**Proof.** From the definition of $\|\cdot\|_{\mu, \beta}$, we get

$$|\hat{v} \hat{\omega}| \leq \|\hat{v}\|_{\mu, \beta} \|\hat{\omega}\|_{\mu, \beta} \int_{k' \in \mathbb{R}^3} \frac{e^{-\beta(|k'| + |k' - k|)} dk'}{(1 + |k'|)^\mu(1 + |k - k'|)^\mu}$$

$$\leq \|\hat{v}\|_{\mu, \beta} \|\hat{\omega}\|_{\mu, \beta} e^{-\beta|k|} \int_{k' \in \mathbb{R}^3} \frac{dk'}{(1 + |k'|)^\mu(1 + |k - k'|)^\mu}$$

For large $|k|$, we break the integral range at $|k'| = |k|/2$. In the inner ball $|k'| < |k|/2$, we have

$$\frac{1}{(1 + |k'|)^\mu(1 + |k - k'|)^\mu} \leq \frac{1}{(1 + |k'|)^\mu(1 + |k|/2)^\mu} \leq \frac{2^\mu}{(1 + |k|)^\mu(1 + |k'|)^\mu}$$

while, in its complement,

$$\frac{1}{(1 + |k'|)^\mu(1 + |k - k'|)^\mu} \leq \frac{1}{(1 + |k|/2)^\mu(1 + |k - k'|)^\mu} \leq \frac{2^\mu}{(1 + |k|)^\mu(1 + |k - k'|)^\mu}$$

Using these estimates, we get for $\mu > 3$,

$$\int_{k' \in \mathbb{R}^3} \frac{dk'}{(1 + |k'|)^\mu(1 + |k - k'|)^\mu} \leq \frac{C_0}{2(1 + |k|)^\mu}$$

**Lemma 2.3.**

$$\|P_k [\hat{\omega} \hat{v}]\|_{\mu, \beta} \leq 2C_0\|\hat{\omega}\|_{\mu, \beta} \|\hat{v}\|_{\mu, \beta}$$

**Proof.** It is easily seen from the representation of $P_k$ in (1.3) that

$$|P_k \hat{g}(k)| \leq 2|\hat{g}(k)|$$

Therefore, using (1.4),

$$\|P_k \hat{g}\|_{\mu, \beta} \leq 2\|g\|_{\mu, \beta}$$

Using Lemma 2.2 with $g = w_j v$, the proof follows.
Lemma 2.4. For \( C_2 = 2\pi C_0 \sup_{z \in \mathbb{R}^+} |G(z, z')| \) \(^{(3)}\), with \( C_0 \) as defined in Lemma 2.2

\[
\| \mathcal{N}[\hat{U}](\cdot, p) \|_{\mu, \beta} \leq \frac{C_2}{\sqrt{p}} \int_0^p \left\{ \| \hat{U}(\cdot, p') \|_{\mu, \beta} + \| \hat{U}(\cdot, p') \|_{\mu, \beta} + \| v_0 \|_{\mu, \beta} \| \hat{U}(\cdot, p') \|_{\mu, \beta} \right\} dp' + v_1 \|_{\mu, \beta}
\]

(2.24) \quad \| \mathcal{N}[\hat{U}^1](\cdot, p) - \mathcal{N}[\hat{U}^2](\cdot, p) \|_{\mu, \beta}

\leq \frac{C_2}{\sqrt{p}} \int_0^p \left\{ \left( \| \hat{U}^1(\cdot, p') \|_{\mu, \beta} + \| \hat{U}^2(\cdot, p') \|_{\mu, \beta} \right) \| \hat{U}^1(\cdot, p') - \hat{U}^2(\cdot, p') \|_{\mu, \beta} + \| v_0 \|_{\mu, \beta} \| \hat{U}^1(\cdot, p') - \hat{U}^2(\cdot, p') \|_{\mu, \beta} \right\} dp'

Proof. From \(| J_1(z)/z | \leq 1/2 \) for \( z \in \mathbb{R}^+ \) and therefore

\[
\| 2\hat{v}_1(k)J_1(z)/z \|_{\mu, \beta} \leq \| \hat{v}_1 \|_{\mu, \beta}
\]

From Lemma 2.3 we have

\[
\mathcal{P}_k \left\{ \hat{U}^1 \ast \hat{U} \right\} (k, p) \leq 2C_0 \| \hat{U}(\cdot, p) \|_{\mu, \beta} \| \hat{U}(\cdot, p) \|_{\mu, \beta} e^{-|k|} \frac{1}{(1 + |k|)^\alpha}
\]

Applying Lemma 2.3 we get

\[
\left| \mathcal{P}_k \left\{ \hat{v}_0 \ast \hat{U} + \hat{U}(\cdot, p) \ast \hat{v}_0 \right\} \right| \leq 4C_0 \| \hat{v}_0 \|_{\mu, \beta} \| \hat{U}(\cdot, p) \|_{\mu, \beta} e^{-|k|} \frac{1}{(1 + |k|)^\alpha}
\]

By Remark 2.4 and the definition of \( \mathcal{N} \) in (2.13), it follows that for \( C_2 \geq 2\pi C_0 |G(z, z')| \) \(^{(2.23)}\) holds.

The second part of the lemma follows by noting that

(2.25) \quad \hat{U}^1 \ast \hat{U}^1 \ast \hat{U}^1 \ast \hat{U}^2 = \hat{U}^1 \ast \left( \hat{U}^1 - \hat{U}^2 \right) + \left( \hat{U}^1 - \hat{U}^2 \right) \ast \hat{U}^2

Applying Lemma 2.3 to (2.25), we obtain

\[
\left\| \mathcal{P}_k \left\{ \hat{U}^1 \ast \hat{U}^1 \ast \hat{U}^1 \ast \hat{U}^2 (\cdot, p) \right\} \right\|_{\mu, \beta} \leq 2C_0 \| \hat{U}^1(\cdot, p) \|_{\mu, \beta} \| \hat{U}^1(\cdot, p) \|_{\mu, \beta} + 2C_0 \| \hat{U}^2(\cdot, p) \|_{\mu, \beta} \| \hat{U}^2(\cdot, p) \|_{\mu, \beta},
\]

from which (2.24) follows easily. \( \blacksquare \)

It is convenient to define a number of different norms for functions of \((k, p)\) on \( \mathbb{R}^3 \times (\mathbb{R}^+ \cup \{0\}) \)

**Definition 2.5.** For \( \alpha \geq 1 \), we define

\[
\| \hat{f} \|^{(\alpha)} = \sup_{p \geq 0} (1 + p^2) e^{-\alpha p} |\hat{f}(\cdot, p)|_{\mu, \beta}
\]

We define \( \mathcal{A}^\alpha \) to be the Banach-space of continuous functions of \((k, p)\) for \( k \in \mathbb{R}^3 \) and \( p \in [0, \infty) \) for which \( \| \cdot \|^{(\alpha)} < \infty \). It is also convenient to consider the Banach

\( ^{(3)}\) Since \( \sup |G| \approx 0.6 \), we get \( C_2 \approx \frac{\frac{32(1,2)2^2}{(\mu-1)(\mu-2)(\mu-3)}}{\frac{2(\mu-2,2)2^2}{(\mu-1)(\mu-2)(\mu-3)}} \).
space $\mathcal{A}_k^p$ of locally integrable ($L^1_{loc}$) functions for $p \in [0, L)$ on $\mathbb{R}^+$, and continuous in $k \in \mathbb{R}^3$ such that

\begin{equation}
\|f\|_1^\alpha = \int_0^L e^{-\alpha p} \|f(\cdot, p)\|_{\mu, \beta} dp < \infty,
\end{equation}

where $L$ is allowed to be finite or $\infty$. It is also convenient to define $\mathcal{A}_k^\infty$ to be the Banach space of continuous functions of $(k, p)$ on $\mathbb{R}^3 \times [0, L]$ such that

\begin{equation}
\|f\|_L = \sup_{p \in [0, L]} \|f(\cdot, p)\|_{\mu, \beta} < \infty
\end{equation}

Lemma 2.6. For $\hat{f}, \hat{g} \in \mathcal{A}_k^\alpha, \mathcal{A}_k^1$ or $\mathcal{A}_k^\infty$, we have the following the following Banach algebra properties:

\[\|\hat{f} \hat{g}\|^{\text{(\alpha)}} \leq M_0 \|\hat{f}\|^{\text{(\alpha)}} \|\hat{g}\|^{\text{(\alpha)}}, \text{ where } M_0 \approx 3.76 \ldots\]

\[\|\hat{f} \hat{g}\|_1^{\text{(\alpha)}} \leq \|\hat{f}\|_1^{\text{(\alpha)}} \|\hat{g}\|_1^{\text{(\alpha)}},\]

\[\|\hat{f} \hat{g}\|_L^{\text{(\alpha)}} \leq L \|\hat{f}\|_L^{\text{\infty}} \|\hat{g}\|_L^{\text{\infty}}\]

Proof. In the following, we take $u(p) = \|\hat{f}(\cdot, p)\|_{\mu, \beta}$ and $v(p) = \|\hat{g}(\cdot, p)\|_{\mu, \beta}$. We observe that

\[\int_0^L u(s)v(p-s)ds \leq e^{\alpha p} \left( \sup_{p \in \mathbb{R}^+} (1 + p^2)e^{-\alpha p} u(p) \right) \left( \sup_{p \in \mathbb{R}^+} (1 + p^2)e^{-\alpha p} v(p) \right) \times \int_0^p \frac{ds}{(1 + s^2)[1 + (p-s)^2]}\]

The first part of the lemma follows since [19]

\[\int_0^p \frac{ds}{(1 + s^2)[1 + (p-s)^2]} \leq \frac{M_0}{1 + p^2}\]

with $M_0 = 3.76 \ldots$. For the second part note that

\begin{equation}
\int_0^L e^{-\alpha p} \int_0^p u(s)v(p-s)ds \\
= \int_0^L \int_0^p e^{-\alpha s}e^{-\alpha (p-s)} u(s)v(p-s)ds \leq \int_0^L e^{-\alpha s} u(s)ds \int_0^L e^{-\alpha \tau} v(\tau)\]

The third part follows from the fact that for $p \in [0, L]$

\[\int_0^p |u(s)||v(p-s)| = \left\{ \sup_{p \in [0, L]} |u(p)| \right\} \left\{ \sup_{p \in [0, L]} |v(p)| \right\} L\]

Lemma 2.7. On $\mathcal{A}_k^\alpha$, the operator $\mathcal{N}$, defined in (2.18), satisfies the following inequalities, with $C_2$ defined in Lemma 2.4.

\begin{equation}
\|\mathcal{N}[\hat{U}]\|_1^{\text{(\alpha)}} \leq C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ \left( \|\hat{U}\|_1^{\text{(\alpha)}} \right)^2 + \|\hat{\varepsilon}_0\|_{\mu, \beta} \|\hat{U}\|_1^{\text{(\alpha)}} \right\} + \alpha^{-1} \|\hat{\varepsilon}_1\|_{\mu, \beta}
\end{equation}

\begin{equation}
\|\mathcal{N}[\hat{U}]^1 - \mathcal{N}[\hat{U}]^2\|_1^{\text{(\alpha)}} \leq C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ \left( \|\hat{U}^1\|_1^{\text{(\alpha)}} + \|\hat{U}^2\|_1^{\text{(\alpha)}} \right) \|\hat{U}^1 - \hat{U}^2\|_1^{\text{(\alpha)}} + \|\hat{\varepsilon}_0\|_{\mu, \beta} \|\hat{U}^1 - \hat{U}^2\|_1^{\text{(\alpha)}} \right\}
\end{equation}
while in $\mathcal{A}^\infty_v$, we have

$$
\|\mathcal{N}[\hat{U}]\|_{L}^{(\infty)} \leq C_2 L^{1/2} \left\{ L \left( \|\hat{U}\|_{L}^{(\infty)} \right)^2 + \|\hat{v}_0\|_{\mu,\beta} \|\hat{U}\|_{L}^{(\infty)} \right\} + \|\hat{v}_1\|_{\mu,\beta}
$$

(2.32)

$$
\|\mathcal{N}[\hat{U}^{[1]}] - \mathcal{N}[\hat{U}^{[2]}]\|_{L}^{(\infty)} \leq C_2 L^{1/2} \left\{ L \left( \|\hat{U}^{[1]}\|_{L}^{(\infty)} + \|\hat{U}^{[2]}\|_{L}^{(\infty)} \right) \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_{L}^{(\infty)} + \|\hat{v}_0\|_{\mu,\beta} \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_{L}^{(\infty)} \right\}
$$

(2.33)

Proof. For the space $\mathcal{A}^p_\alpha$, for any $L > 0$, including $L = \infty$, we note that

$$
\int_0^L e^{-\alpha p} \|\hat{v}_1\|_{\mu,\beta} dp \leq \alpha^{-1} \|\hat{v}_1\|_{\mu,\beta},
$$

while

$$
\int_0^L p^{-1/2} e^{-\alpha p} dp \leq \Gamma \left( \frac{1}{2} \right) \alpha^{-1/2} = \sqrt{\pi} \alpha^{-1/2}
$$

Furthermore, we note that for $u(p) \geq 0$ we have

$$
\int_0^L e^{-\alpha p} p^{-1/2} \left( \int_0^p u(p') dp' \right) = \int_0^L u(p') e^{-\alpha p'} \left( \int_{p'}^L p^{-1/2} e^{-\alpha(p-p')} dp' \right) dp' \leq \int_0^L e^{-\alpha p'} u(p') \int_0^L s^{-1/2} e^{-\alpha s} ds dp'
$$

Therefore, it follows from (2.24) that

$$
\int_0^L e^{-\alpha p} \|\mathcal{N}[\hat{U}](\cdot, p)\|_{\mu,\beta} dp \leq C_2 \sqrt{\pi} \alpha^{-1/2} \left( \left( \|\hat{U}\|_{L}^{(\alpha)} \right)^2 + \|\hat{v}_0\|_{\mu,\beta} \|\hat{U}\|_{L}^{(\alpha)} \right) + \alpha^{-1} \|\hat{v}_1\|_{\mu,\beta}
$$

(2.35)

Furthermore, from (2.23), it follows that

$$
\int_0^L \|\mathcal{N}[\hat{U}^{[1]}] - \mathcal{N}[\hat{U}^{[2]}]\|_{\mu,\beta} e^{-\alpha p} dp \leq C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ \left( \|\hat{U}^{[1]}\|_{L}^{(\alpha)} + \|\hat{U}^{[2]}\|_{L}^{(\alpha)} \right) \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_{L}^{(\alpha)} + \|\hat{v}_0\|_{\mu,\beta} \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_{L}^{(\alpha)} \right\}
$$

Hence the first part of the lemma follows.

For the second part, we first note that for any $p \in [0, L]$ we have

$$
\left| p^{-1/2} \int_0^p u(p') dp' \right| \leq \sup_{p \in [0, L]} |u(p)| \sqrt{L}
$$

(2.36)

We note that

$$
\left| \int_0^p y_1(s)y_2(p-s) ds \right| \leq L \left( \sup_{p \in [0, L]} |y_1(p)| \right) \left( \sup_{p \in [0, L]} |y_2(p)| \right)
$$

(2.37)

Taking

$$
u(p) = \|\hat{U}(\cdot, p)\|_{\mu,\beta} \ast \|\hat{U}(\cdot, p)\|_{\mu,\beta} + \|\hat{v}_0\|_{\mu,\beta} \|\hat{U}(\cdot, p)\|_{\mu,\beta}
$$

$$
y_1(p) = y_2(p) = \|\hat{U}(\cdot, p)\|_{\mu,\beta}
$$
(2.32) follows from (2.24). To bound \(\mathcal{N}[\bar{U}^1] - \mathcal{N}[\bar{U}^2]\) in \(\mathcal{A}_L^\infty\), we take
\[
(2.38) \quad u(p) = \left(\|\bar{U}^1(\cdot, p)\|_{\mu, \beta} + \|\bar{U}^2(\cdot, p)\|_{\mu, \beta}\right) * \|\bar{U}^1(\cdot, p) - \bar{U}^2(\cdot, p)\|_{\mu, \beta} \\
+ \|v_0\|_{\mu, \beta} \|\bar{U}^1(\cdot, p) - \bar{U}^2(\cdot, p)\|_{\mu, \beta}
\]
\[
y_1(p) = \left(\|\bar{U}^1(\cdot, p)\|_{\mu, \beta} + \|\bar{U}^2(\cdot, p)\|_{\mu, \beta}\right) ; \quad y_2(p) = \|\bar{U}^1(\cdot, p) - \bar{U}^2(\cdot, p)\|_{\mu, \beta}
\]
in (2.36) and (2.37). The proof now follows from (2.24). □

**Lemma 2.8.** Equation (2.18) has a unique solution in \(\mathcal{A}_L^0\) for any \(L > 0\) (including \(L = \infty\)) in a ball of size \(2\alpha^{-1}\|v_1\|_{\mu, \beta}\), for \(\alpha\) large enough to ensure
\[
(2.39) \quad 2C_2\sqrt{\pi\alpha}^{-1/2} \left(\|v_0\|_{\mu, \beta} + 2\alpha^{-1}\|v_1\|_{\mu, \beta}\right) < 1,
\]
where \(C_2 \approx \frac{32(1.2)^{-2\mu}}{(\alpha^{-1})^{(\mu-\beta)(\mu-2)}}\) is the same as in Lemma 2.4. Furthermore, this solution belongs to \(\mathcal{A}_L^\infty\) for \(L\) small enough so that
\[
(2.40) \quad 2C_2L^{1/2} \left(\|v_0\|_{\mu, \beta} + 2L\|v_1\|_{\mu, \beta}\right) < 1,
\]
In particular, \(\lim_{p \to 0} \bar{U}(k, p) = \hat{v}_1(k)\). Also, \(W(k, p) = \bar{U}(k, p) - \hat{v}_1(k)\) is the unique solution to (2.5) which is zero at \(p = 0\).

**Proof.** The estimates of Lemma 2.7 imply that \(\mathcal{N}\) maps a ball of size \(2\alpha^{-1}\|v_1\|_{\mu, \beta}\) in \(\mathcal{A}_L^0\) back to itself and that \(\mathcal{N}\) is contractive in that ball when \(\alpha\) satisfies (2.39). From Lemma 2.7 in space \(\mathcal{A}_L^\infty\), it follows that \(\mathcal{N}\) maps a ball of size \(2\|v_1\|_{\mu, \beta}\) to itself and that \(\mathcal{N}\) is also contractive in this ball if \(L\) is small enough to ensure (2.40).

Thus, there is a unique solution in this ball. Since \(\mathcal{A}_L^\infty \subset \mathcal{A}_L^0\), it follows that the solutions are in fact the same.

Using Lemma 2.7 with \(\bar{U}^1 = \bar{U}\) and \(\bar{U}^2 = 0\), we obtain from (2.18),
\[
\left\|\bar{U}(k, p) - \hat{v}_1(k)\frac{2J_1(z)}{z}\right\|_{L}^{(\infty)} \leq C_2L^{1/2} \left(L \left[\|\bar{U}\|_{L}^{(\infty)}\right]^2 + \|v_0\|_{\mu, \beta}\|\bar{U}\|_{L}^{(\infty)}\right)
\]
Since \(\|\bar{U}\|_{L}^{(\infty)} < 2\|\hat{v}_1\|_{\mu, \beta}\), it follows that as \(L \to 0\),
\[
\|\bar{U}(k, p) - 2\hat{v}_1(k)\frac{2J_1(z)}{z}\|_{L}^{(\infty)} \to 0
\]
Since \(\lim_{z \to 0} 2J_1(z)/z = 1\), it follows that for fixed \(k\), \(\lim_{p \to 0} \bar{U}(k, p) = \hat{v}_1(k)\). By construction, \(\bar{U}\) satisfies (2.18) iff \(\bar{U} - \hat{v}_1\) satisfies (2.10). From the properties of \(G\) and \(H^{[j]}\), it follows that \(\bar{U}\) will indeed satisfy (2.9) and that it is the only solution which is zero at \(p = 0\). □

**Proposition 2.9.** If \(\alpha\) is large enough so that (2.39) holds, then for an absolute constant \(C_3 > 0\), the solution \(\bar{U}(k, p)\) in Lemma 2.8 and its \(p\)-derivative satisfy
\[
|\bar{U}(k, p)| \leq \frac{2e^{-\beta|k|+\alpha p}\|\hat{v}_1\|_{\mu, \beta}}{(1 + |k|)^\alpha}
\]
\[
|\bar{U}_p(k, p)| \leq \frac{C_3e^{-\beta|k|}\|\hat{v}_1\|_{\mu, \beta}}{(1 + |k|)^\alpha} \left\{ \frac{\sqrt{\alpha}}{C_2} |k|e^{\alpha p} + |k|^2 \right\}
\]
In particular, \( \hat{U} \in A^{\alpha'} \) for any \( \alpha' > \alpha \), and

\[
|\hat{U}(k, p)| \leq \left( \sup_{p \in \mathbb{R}^+} (1 + p^2) e^{-(\alpha'-\alpha)p} \right) \frac{2e^{-\beta|k|+\alpha'p}}{(1 + p^2)(1 + |k|)^\mu} \|\hat{v}_1\|_{\mu, \beta}
\]

**Proof.** With \( L = L_0 = \alpha^{-1} \), then (2.41) holds, and therefore \( \hat{U} \in A^{\infty}_{L_0} \). For \( p \in [0, L_0] \), we obtain

\[
e^{-\alpha p} \|\hat{U}(\cdot, p)\|_{\mu, \beta} < 2e^{-\alpha p} \|\hat{v}_1\|_{\mu, \beta}
\]

We now consider \( p \in [L_0, \infty) \). We define

\[y(p) = \|\hat{U}(\cdot, p)\|_{\mu, \beta} \ast \|\hat{U}(\cdot, p)\|_{\mu, \beta} + \|\hat{v}_0\|_{\mu, \beta} \|\hat{U}(\cdot, p)\|_{\mu, \beta}\]

We note that

\[
\left| \frac{1}{\sqrt{p}} e^{-\beta|z|} \int_0^p y(p')dp' \right| \leq L_0^{-1/2} \left| \int_0^p e^{-\alpha p} y(p')dp' \right| \leq \alpha^{1/2} \|y\|_1^{(\alpha)}
\]

From (2.18) and (2.36), it follows that for \( p \in [L_0, \infty) \)

\[
|\hat{U}(k, p)| \leq \frac{e^{-\beta|k|+\alpha p}}{(1 + |k|)^\mu} \left( C_2 a^{1/2} \left( \|\hat{U}\|_{1}^{(\alpha)} \right)^2 + C_2 a^{1/2} \|\hat{v}_0\|_{\mu, \beta} \|\hat{U}\|_{1}^{(\alpha)} + e^{-\alpha p} \|\hat{v}_1\|_{\mu, \beta} \right)
\]

\[
\leq \frac{2e^{-\beta|k|+\alpha p}}{(1 + |k|)^\mu} \|\hat{v}_1\|_{\mu, \beta}
\]

By (2.11), (2.36) holds for \( p \in [0, L_0] \) as well; hence the bound for \( |\hat{U}| \) follows. For \( \alpha' > \alpha \), \( \|\hat{U}\|^{(\alpha')} < \infty \) because \( e^{-(\alpha'-\alpha)p}(1 + p^2) \) is bounded if \( \alpha' > \alpha \).

Since \( \hat{U} \) is a solution to (2.18), differentiation with respect to \( p \) implies that

\[
\hat{U}_p(k, p) = \hat{v}_1(k) \left( \frac{J_1(z)}{z} \right)' \frac{4|k|^2}{z} + \frac{i k \pi}{p} \int_0^p \{ G_{\cdot, \cdot}(z, z') - \frac{G(z, z')}{z} \} \hat{G}^{(1)}(k, p')dp'
\]

Since the functions \( G_{\cdot, \cdot}(z, z'), G(z, z')/z \) and \( z^{-1} (J_1(z)/z)' \) are easily checked to be bounded for \( z \geq z' \in \mathbb{R}^+ \), there exists \( C_3 > 0 \), independent of any parameter, so that

\[
|\hat{U}_p(k, p)| \leq \frac{C_3|k|}{p} \left| \int_0^p |G^{(1)}(k, p')|dp' + C_3 |k|^2 |\hat{v}_1(k)| \right| \leq \frac{C_3 |e^{-\beta|k|}|}{(1 + |k|)^\mu}
\]

\[
\times \left[ \frac{1}{p} \int_0^p (\|U(\cdot, p')\|_{\mu, \beta} \ast \|U(\cdot, p')\|_{\mu, \beta} + \|\hat{v}_1\|_{\mu, \beta} \|U(\cdot, p')\|_{\mu, \beta} dp' + |k||\hat{v}_1\|_{\mu, \beta} \right]
\]

For \( p \in [0, L_0] \), with \( L = L_0 = \frac{1}{\alpha} \) satisfying (2.40), we have

\[
|\hat{U}_p(k, p)| \leq \frac{C_3 |e^{-\beta|k|}|}{(1 + |k|)^\mu} \left\{ L_0 \left( \|U\|_{L_0}^{(\infty)} \right)^2 + \|\hat{v}_0\|_{\mu, \beta} \|U\|_{L_0}^{(\infty)} + |k||\hat{v}_1\|_{\mu, \beta} \right\}
\]

\[
\leq \frac{C_3 e^{-\beta|k|}}{(1 + |k|)^\mu} \left\{ \frac{\sqrt{\alpha}}{C_2} |k| + |k|^2 \right\} \|\hat{v}_1\|_{\mu, \beta}
\]
For $p \in [L_0, \infty)$ and $\alpha$ satisfying (2.39), we have

$$|\hat{U}_p(k, p)| \leq \frac{C_3 |k| e^{-\beta |k| + \alpha p}}{L_0 (1 + |k|)^\mu} \left\{ \left( (\|U\|_1^{(0)})^2 + \|\hat{v}_0\|_{\mu, \beta} \|U\|_1^{(0)} + |k| L_0 e^{-\alpha p} \|\hat{v}_1\|_{\mu, \beta} \right)^{1/2} \right.$$  

$$\leq \frac{C_3 e^{-\beta |k|} \|\hat{v}_1\|_{\mu, \beta}}{(1 + |k|)^\mu} \left\{ \sqrt{\frac{\alpha}{C_2}} |k| e^{\alpha p} + |k|^2 \right\}$$

Continuity of $\hat{U}$ in $p$ follows from the boundedness of $\hat{U}_p$ for $p \in \mathbb{R}^+$ for fixed $k$.

**Lemma 2.10.** Let $\|\hat{v}_0\|_{\mu+2, \beta} < \infty$ and $\|\hat{f}\|_{\mu, \beta} < \infty$, with $\mu > 3$, $\beta \geq 0$. Then NS has a unique solution with $\|\hat{\varphi}(\cdot, t)\|_{\mu, \beta} < \infty$ and $\hat{\varphi}(\cdot, t)$ analytic in $t$ for $\text{Re} \frac{1}{t} > \alpha$, where $\alpha$ depends on the initial data (see (2.39)). For $\beta > 0$, this implies $\varphi$ is analytic in $x$ in the same analytic strip as $\varphi_0, f$.

**Proof.** From (2.6) we see that $\|\hat{v}_1\|_{\mu, \beta} < \infty$, since

$$\|v_1\|_{\mu, \beta} \leq \|\hat{v}_0\|_{\mu+2, \beta} + 2C_0 \|\hat{v}_0\|_{\mu, \beta} \|\hat{v}_0\|_{\mu+1, \beta} + \|\hat{f}\|_{\mu, \beta}$$

Therefore, when $\alpha$ is large enough to ensure (2.39), it follows that $\hat{U}(k, \cdot)$ and $\hat{W}(k, \cdot) \equiv \hat{U}(k, \cdot) - \hat{v}_1(k)$ are in $L^1(e^{-\alpha p} dp)$. From Lemma 2.8 it follows that $\lim_{p \to 0} \hat{W}(k, p) = 0$ and Proposition 2.7 implies $\hat{W}_p(k, p)$ (same as $\hat{U}_p(k, p)$) is bounded for $p \in \mathbb{R}^+$ and hence $\lim_{p \to 0} p \hat{W}_p = 0$. Since $\hat{U}$ satisfies (2.18), it follows that $\hat{W}$ will satisfy (2.10) and hence (2.9). For $\text{Re} t^{-1} > \alpha$, we take the Laplace transform of (2.9) in $p$, using the fact $\partial_p[p\hat{W}]$ and $p\hat{W}$ vanish at $p = 0$. There is no contribution at $\infty$ because of boundedness of $e^{-\alpha p} (|\hat{W}| + \hat{W}_p)$ which follows from Proposition 2.7. It can be checked that $\hat{w}(k, t) = \int_0^\infty \hat{W}(k, p) e^{-p^\alpha t} dp$ satisfies (2.7). Therefore,

$$\hat{v}(k, t) = \hat{v}_0 + \hat{t} \hat{v}_1 + \int_0^\infty \hat{W}(k, p) e^{-p^\alpha t} dp = \hat{v}_0 + \int_0^\infty \hat{U}(k, p) e^{-p^\alpha t} dp$$

satisfies NS in Fourier space. Since $|\hat{U}(\cdot, p)|_{\mu, \beta} < \infty$, it follows that $|\hat{\varphi}(\cdot, t)|_{\mu, \beta} < \infty$ if $\text{Re} \frac{1}{t} > \alpha$.

**Proposition 2.11 (Bounds on $|\hat{\varphi}(\cdot, t)|_{\mu+2, \beta}$).** For the solution $\hat{v}(k, t)$ given in Lemma 2.10 for $t \in [0, \alpha^{-1}]$, we have

$$\sup_{t \leq T} \|\hat{v}(\cdot, t)\|_{\mu+2, \beta} < C \left( \|\hat{v}_0\|_{\mu+2, \beta}, T \right) < \infty$$

**Proof.** We note from (1.2) that if we define $\hat{V} = \hat{\nabla} v$, then $\hat{\nabla} = \mathcal{F}[V] = ik \hat{v}$ satisfies

$$\hat{V} + |k|^2 \hat{V} = -ik P \left[ \hat{v}_j \hat{V}^{[j]} \right] + ik \hat{f}, \quad \hat{V}_0(k) = \mathcal{F}[\nabla v_0]$$

where $\hat{V}^{[j]} = ik_j \hat{v}_j$. Therefore,

$$\hat{V}(k, t) = e^{-|k|^2 t} \hat{V}_0(k) - ik \int_0^t e^{-|k|^2 (t-\tau)} \left\{ P \left[ \hat{v}_j \hat{V}^{[j]} \right](k, \tau) \right\} - \hat{f}(k)$$
Therefore,

\begin{equation}
(2.47) \quad |\dot{V}(\cdot, t)| \leq \frac{e^{-\beta|k|}}{(1 + |k|)^\mu} \left\{ \|V_0\|_{\mu, \beta} + |k| \int_0^t e^{-|k|^2(t-\tau)} \left( \|f\|_{\mu, \beta} + 2C_0\|\dot{v}(\cdot, \tau)\|_{\mu, \beta} \|\dot{V}(\cdot, \tau)\|_{\mu, \beta} \right) d\tau \right\}
\end{equation}

Let \(V_{T_1}\) be the Banach space of continuous functions \(g\) of \(k \in \mathbb{R}^3\) and \(t \in [0, T_1]\) for which

\[\|g\|_{T_1} = \sup_{t \in [0, T_1]} \|g(\cdot, t)\|_{\mu, \beta} < \infty\]

Then, the estimates in (2.47), together with the fact that for any \(t \in [0, T]\),
\[2C_0\|\dot{v}(\cdot, t)\|_{\mu, \beta} \leq C(T, \|\dot{v}_0\|_{\mu + 2, \beta})\]

imply there exists \(C_1(T, \|\dot{v}_0\|_{\mu + 2, \beta}) > 0\) so that

\begin{equation}
(2.48) \quad \|\dot{V}\|_{T_1} \leq C_1 \left\{ \sqrt{T_1}\|\dot{V}_1\|_{T_1} + \|\dot{V}_0\|_{\mu, \beta} + \sqrt{T_1}\|\dot{f}\|_{\mu, \beta} \right\},
\end{equation}

where we have used the fact that

\[|k| \int_0^t e^{-|k|^2(t-\tau)} d\tau = \frac{1 - e^{-|k|^2t}}{|k|} \leq \sqrt{T_1} \sup_{\gamma \in \mathbb{R}^+} \frac{1 - e^{-\gamma}}{\gamma^{1/2}} \leq C_\ast \sqrt{T_1},\]

for some \(C_\ast > 0\). Thus, thinking of \(\dot{v}\) as given in (2.45), the estimates in (2.48) and similar estimates on \(V^{[1]} - \dot{V}^{[2]}\) show that for \(C_1\sqrt{T_1} < 1\) the right hand side of (2.48) is contractive in \(V_{T_1}\). We choose \(T_1 \leq T\). Therefore, \(\sup_{t \in [0, T_1]} \|\dot{V}(\cdot, t)\|_{\mu, \beta} < \infty\).

Since the choice of \(T_1\) depends on \(C_1\), which is independent of \(\|\dot{V}_0\|_{\mu, \beta}\), we can repeat the same argument in another interval \([T_1, 2T_1]\) and so on until we span the whole interval \([0, T]\) over which \(\|\dot{v}(\cdot, t)\|_{\mu, \beta}\) is uniformly bounded.

We can take additional derivative and repeat the same type argument for \(F[D^2\dot{v}] = -kk\dot{v}\) to show that in \(\|k|^2\dot{v}(\cdot, t)\|_{\mu, \beta}\) is also bounded uniformly for \(t \in [0, T]\). In this part of the argument, we use the prior knowledge that both \(\|\dot{v}(\cdot, t)\|_{\mu, \beta}\) and \(\|k\dot{v}(\cdot, t)\|_{\mu, \beta}\) are uniformly bounded in \([0, T]\) and that

\[|k|^2 \int_0^t e^{-|k|^2(t-\tau)} \|\dot{f}\|_{\mu, \beta} d\tau \leq \|\dot{f}\|_{\mu, \beta} \sup_{\gamma \in \mathbb{R}^+} [1 - e^{-\gamma}] \leq C\|\dot{f}\|_{\mu, \beta}\]

Combining all the results, it follows that \(\|\dot{v}(\cdot, t)\|_{\mu + 2, \beta}\) is bounded for \(t \in [0, T]\)

**Proof of Theorem 1.1**  
This follows from Lemma 2.10 and Proposition 2.11 noting that \(\|\dot{v}(k, t)\|_{\mu + 2, \beta} < \infty\) implies \(v(x, t) = F^{-1}[\dot{v}(\cdot, t)](x) \in C^2(\mathbb{R}^3)\) and \(v\) is a classical solution to (1.1) for \(Re \frac{1}{\alpha} > \alpha\), which is known to be unique. From the definition of \(\|\cdot\|_{\mu, \beta}\) it follows that \(\|\dot{v}_0\|_{\mu + 2, \beta} < \infty\) and \(\|\dot{f}\|_{\mu, \beta} < \infty\) for \(\beta > 0\) imply \(\|\dot{v}(\cdot, t)\|_{\mu + 2, \beta} < \infty\). Thus \(v\) preserves the analyticity strip width for \(t \in [0, \frac{1}{\alpha})\).

3. **Analyticity of \(\dot{U}(k, p)\) at \(p = 0\)**

We now consider the case \(\beta > 0\). We note that by Remark 1.6 we can choose \(\mu > 3\). The starting point of this section is (2.9), which is satisfied by \(\dot{W}(k, p) =\)
\( \hat{U}(k,p) - \hat{v}_1(k) \). From Lemma 2.8 this is the only solution to (2.9) satisfying \( \tilde{W}(k,0) = 0 \). We seek an potentially alternate solution to (2.9) as a power series,

\[
(3.49) \quad \tilde{W}(k,p) = \sum_{l=1}^{\infty} \tilde{W}^{[l]}(k)p^l
\]

Substituting (3.49) into (2.9) and identifying the coefficients of \( p^l, l = 0, 1 \) we get

\[
(3.50) \quad 2\tilde{W}^{[1]} = -|k|^2 \hat{v}_1 - i k_j P_k [ \hat{v}_{0,j} \hat{\hat{v}}_1 + \hat{v}_{1,j} \hat{\hat{v}}_0 ] ,
\]

\[
(3.51) \quad 6\tilde{W}^{[2]} = -k^2 \tilde{W}^{[1]} - i k_j P_k \left[ \hat{v}_{0,j} \hat{\hat{W}}^{[1]} + \hat{W}^{[1]}_j \hat{\hat{v}}_0 + \hat{v}_{1,j} \hat{\hat{v}}_1 \right]
\]

It follows from (3.50) and Lemma (2.3) that

\[
(3.52) \quad |\tilde{W}^{[1]}(k,p)| \leq \frac{e^{-\beta |k|}}{2(1+|k|)^\mu} \left( |k|^2 \|v_1\|_{\mu,\beta} + 4C_0 \|v_0\|_{\mu,\beta} \|v_1\|_{\mu,\beta} \right)
\]

The coefficient of \( p^l \) for \( l \geq 2 \) in (2.9) can be computed as well, using \( p^{l_1} \ast p^{l_2} = p^{l_1 + l_2 + 1} \hat{v}_l/(l_1 + l_2 + 1)! \). Interpreting \( \tilde{W}^{[0]} = 0 \), we get

\[
(3.53) \quad (l + 1)(l + 2)\tilde{W}^{[l+1]} = -k^2 \tilde{W}^{[l]} - i k_j P_k \left[ \sum_{l_1=1}^{l-2} l_1!(l - l_1)! \tilde{W}^{[l_1]}_j \hat{\hat{W}}^{[l-l_1]} + \right]
\]

\[
- i k_j P_k \left[ \hat{v}_{0,j} \hat{\hat{W}}^{[l]} + \hat{W}^{[1]}_j \hat{\hat{v}}_0 + \hat{v}_{1,j} \hat{\hat{v}}_1 \right]
\]

**Definition 3.1.** It is convenient to define the \( n \)-th order polynomial \( Q_n \):  

\[
Q_n(y) = \sum_{j=0}^{n} 2^{n-j} y^j / j!
\]

**Lemma 3.2.** If \( \|v_0\|_{\mu+2,\beta} < \infty \), for \( \mu > 3, \beta > 0 \), then there exist positive constants \( A_0, B_0 > 0 \) independent of \( l \) and \( k \) so that for any \( l \geq 1 \) we have

\[
(3.54) \quad |\tilde{W}^{[l]}(k)| \leq e^{-\beta |k|} A_0 B_0 (1 + |k|)^{-\mu} Q_{2l+2}(\beta |k|) / (2l + 1)^2
\]

and

\[
|W^{[l]}(x)| \leq \frac{8\pi A_0 (4B_0)^l}{(2l + 1)^2}, \quad |D W^{[l]}(x)| \leq \frac{8\pi A_0 (4B_0)^l}{\beta (2l + 1)^2}, \quad |D^2 W^{[l]}(x)| \leq \frac{16\pi A_0 (4B_0)^l}{\beta^2 (2l + 1)^2}
\]

Furthermore, the solution in Lemma 2.8 §2 has a convergent series representation in \( p \): \( \hat{U}(k,p) = \hat{v}_1(k) + \sum_{l=1}^{\infty} \tilde{W}^{[l]}(k)p^l \) for \( |p| < (4B_0)^{-1} \).

**Remark 3.3.** Lemma 3.2 is proved by induction on \( l \). For \( l = 1 \), by (3.52) we just choose

\[
(3.55) \quad A_0 B_0 \geq \frac{18}{\beta^2} \|v_1\|_{\mu,\beta}(1 + \beta C_0 \|v_0\|_{\mu,\beta})
\]

Let now \( l \geq 2 \). For the induction step, we will estimate each term on the right of (3.55).

**Lemma 3.4.** If for \( l \geq 1 \), \( W^{[l]} \) satisfies (3.54), then

\[
\frac{|k|^2 |W^{[l]}|}{(l + 1)(l + 2)} \leq \frac{6A_0 B_0 e^{-\beta |k|}}{\beta^2 (1 + |k|)^{\mu}} Q_{2l+2}(\beta |k|) / (2l + 3)^2
\]
Proof. The proof simply follows from the (3.54) and noting that for $y \geq 0$

\[
\frac{y^2}{(2l+2)(2l+1)} Q_{2l}(y) \leq Q_{2l+2}(y), \quad \frac{(2l+3)^2}{(2l+1)(l+2)} \leq 3
\]

\[\Box\]

Lemma 3.5. If $W[l]$ satisfies (3.54), then for $l \geq 1$,

\[
\frac{1}{(l+1)(l+2)} |k_j P_k \hat{u}_{0,j} \hat{W}[l]| \leq 2^\mu \|v_0\|_{\mu, \beta} A_0 B_0 e^{-\beta |k|} \frac{9\pi A_0 B_0^l e^{-\beta |k|}}{\beta^3 (2l+3)^2 (1 + |k|)^\mu} Q_{2l+2}(\beta |k|)
\]

\[
\frac{1}{(l+1)(l+2)} |k_j P_k \hat{W}[l] \hat{u}_{0,j}| \leq 2^\mu \|v_0\|_{\mu, \beta} A_0 B_0 e^{-\beta |k|} \frac{9\pi A_0 B_0 e^{-\beta |k|}}{\beta^3 (2l+3)^2 (1 + |k|)^\mu} Q_{2l+2}(\beta |k|)
\]

Proof. We use the estimate (3.54) on $\hat{W}[l]$. From Lemma 5.7 for $n = 0$, we obtain

\[
|k_j \hat{W}[l] \hat{u}_{0,j}| \leq \|v_0\|_{\mu, \beta} A_0 B_0 \left(\frac{2l+2}{(2l+1)^2}\right)^{1/2} \left|k\right| \int_{l+2}^\infty e^{-\beta (|k'|+|k-k'|)} (1+|k'|)^\mu (1+|k-k'|)^\mu Q_{2l+2}(\beta |k'|) dk'
\]

\[
\leq \frac{2^\mu \|v_0\|_{\mu, \beta} A_0 B_0 (2l+1)^2}{(2l+1)^2 \beta^3 (1 + |k|)^\mu} \sum_{m=0}^{2l-2} \frac{2l-2m}{m!} |k| \int_{l+2}^\infty e^{-\beta (|k'|+|k-k'|)} (1+|k'|)^\mu (1+|k-k'|)^\mu Q_{2l+2}(\beta |k'|) dk'
\]

\[
\leq \frac{2^\mu + 3}{(2l+1)^2 \beta^3 (1 + |k|)^\mu} \|v_0\|_{\mu, \beta} A_0 B_0 e^{-\beta |k|} (l + 2) Q_{2l+2}(\beta |k|)
\]

The first part of the lemma follows by using (1.4) and checking that $\frac{2(2l+3)^2}{(2l+1)(l+1)} \leq 9$ for $l \geq 1$. The proof of the second part is essentially the same since $|\hat{W}[l]| \leq |\hat{W}[l]|$.

\[\Box\]

Lemma 3.6. If $W[l-1]$ satisfies (3.54) for any $l \geq 2$, then

\[
\frac{1}{l(l+1)(l+2)} |k_j P_k \hat{u}_{1,j} \hat{W}[l-1]| \leq 2^\mu \|v_1\|_{\mu, \beta} A_0 B_0^l (1+|k|)^{-\mu} e^{-\beta |k|} \frac{Q_{2l}(\beta |k|)}{\beta^3 (l+2)(2l+1)^2}
\]

\[
\frac{1}{l(l+1)(l+2)} |k_j P_k \hat{W}[l-1] \hat{u}_{1,j}| \leq 2^\mu \|v_1\|_{\mu, \beta} A_0 B_0^{l-1} (1+|k|)^{-\mu} e^{-\beta |k|} \frac{Q_{2l}(\beta |k|)}{\beta^3 (l+2)(2l+1)^2}
\]

Proof. The proof is identical to that of Lemma 3.5 with $l$ replaced by $l - 1$ and $v_0$ by $v_1$.

\[\Box\]

Lemma 3.7. If for $l \geq 3$, $\hat{W}[l]$ and $\hat{W}[l-1]$ for $l_1 = 1, \ldots, (l - 2)$ satisfy (3.54), then

\[
\left| \frac{k_j}{(l+1)(l+2)} P_k \left[ \sum_{i=1}^{l-2} \frac{l_1!(l-1-l_1)!}{l} \hat{W}[l_1] \hat{W}[l-1-l_1] \right] \right| \leq 2^\mu A_0^2 B_0^{l-1} (1+|k|)^{-\mu} e^{-\beta |k|} \frac{Q_{2l}(\beta |k|)}{\beta^3 (2l+3)^2}
\]
Proof. First note that if we define \( l_2 = l - 1 - l_1 \), then for \( l \geq 3 \), Lemma 6.4 implies

\[
\frac{l_1!l_2!}{l!} |k_j \hat{W}_j^{[l_1]} \hat{W}^{[l_2]}| \leq A_0^2 B_0^{-1} \frac{(l_1)!}{l!(2l_1 + 1)^2(2l_2 + 1)^2} e^{-\beta(|k' + |k - k'|)} (1 + |k'|)^{-\mu} (1 + |k - k'|)^{-\mu} Q_{2l_1} (\beta|k'|) Q_{2l_2} (\beta|k - k'|) \int_{k' \in \mathbb{R}^d} \exp(-\beta |k'|)
\]

and therefore

\[
\sum_{l_1=1}^{l-2} \frac{l_1!l_2!}{l!(l + 1)(l + 2)} |k_j \hat{W}_j^{[l_1]} \hat{W}^{[l_2]}| \leq \frac{2^{\mu+1} \pi A_0^2 B_0^{-1} e^{-\beta|k|}}{3(1 + |k|)^{\mu}} \frac{(2l - 1)(2l + 1)!l_1!l_2!}{l!(2l_1 + 1)^2(2l_2 + 2)^2} Q_{2l} (\beta|k|)
\]

and the proof follows noting that \( \frac{l_1!l_2!}{l!(l + 1)(l + 2)} \leq 1 \) and checking \( \frac{4(2l - 1)(2l + 1)}{(l - 1/2)(l + 1)} \leq 16 \); by breaking up the sum in the ranges: \( l_1 \leq (l - 1)/2 \) and \( l_1 > (l - 1)/2 \) (in which \( l_2 \leq (l - 1)/2 \)) it is easily seen that for some \( C_* > 0 \) and any \( l \geq 3 \) we have

\[
\sum_{l_1=1}^{l-2} \frac{1}{(2l_1 + 1)^2(2l_2 + 2)^2} \leq \frac{C_*}{(2l + 3)^2},
\]

where \( C_* = 1.07555 \cdots \) (the upper-bound being achieved at \( l = 4 \)).

Lemma 3.8.

\[(3.56) \quad |\hat{W}^{[2]}| \leq \frac{e^{-\beta|k|}}{(1 + |k|)^{\mu}} \frac{Q_4 (|\beta| |k|)}{7^2} \left( \frac{A_0 B_0}{\beta^2} + A_0 B_0 \|v_0\|_{\mu, \beta} \frac{2^\mu 36 \pi}{\beta^2} + \|v_1\|_{\mu, \beta}^2 \right)
\]

and therefore \( \hat{W}^{[2]} \) satisfies (3.54) if

\[(3.57) \quad A_0 B_0^2 \geq \frac{3A_0 B_0}{\beta^2} + A_0 B_0 \|v_0\|_{\mu, \beta} \frac{2^\mu 36 \pi}{\beta^2} + C_0 \|v_1\|_{\mu, \beta}^2
\]

Proof. We use Lemmas 3.4, 3.5 and 3.7 to estimate different terms on the right hand side of (3.56) for \( l = 1 \).

Proof of Lemma 3.2

We use Lemmas 3.4, 3.5, 3.6 and 3.7 to estimate the terms on the right hand side of (3.55) and note that \( Q_{2l} (y) \leq \frac{1}{2} Q_{2l+2} (y) \). Hence, combining all the estimates, we obtain for \( l \geq 2 \),

\[
|\hat{W}^{[l+1]}| \leq A_0 B_0^l \frac{Q_{2l+2} (|\beta| |k|) e^{-\beta|k|}}{(2l + 3)^2(1 + |k|)^{\mu}} \left\{ \frac{6}{\beta^2} B_0 + 2^\mu 18 \pi \beta^2 B_0 \|v_0\|_{\mu, \beta} + \frac{2^\mu 18 \pi (2l + 3)^2}{(l + 2)(2l + 1)^2 \beta^2} \|v_1\|_{\mu, \beta} + \frac{9 A_0^2}{\beta^4} \right\}
\]

\[
\leq \frac{A_0 B_0^{l+1} e^{-\beta|k|}}{(1 + |k|)^{\mu} (2l + 3)^2} Q_{2l+2} (|\beta| |k|)
\]
It is convenient to define

\[ \hat{W}^{(l)}(k, p) = \hat{W}(x, p), \]

It is also convenient to define \( \hat{W}^{(0)}(k, p) = \hat{W}(k, p) - \hat{v}_1(k). \)

The proof therefore reduces to finding appropriate bounds on \( \hat{W}^{(l)}(k, p) \). The main result proved in this section is the Lemma 4.12 which, using Lemma 4.13 leads directly to the proof of Theorem 1.2.

Proposition 2.10 implies that \( \hat{W} \in A^\alpha \) for \( \alpha' > \alpha \), with \( \alpha \) chosen large enough to satisfy (2.39). In particular, if we choose \( \alpha' = \alpha + 1 \), it follows that \( \hat{W}^{(0)}(k, p) = \hat{U}(k, p) - \hat{v}_1(k) \) satisfies

\[ |\hat{W}^{(0)}(k, p)| \leq \frac{3e^{-\beta |k| + \alpha' p} \|v_1\|_{\mu, \beta}}{(1 + p^2)(1 + |k|)^{\alpha}} \]

In the rest of this section, with some abuse of notation, we will replace \( \alpha' \) by \( \alpha \).

**Lemma 4.2.** If \( \|v_0\|_{\mu + 2, \beta} < \infty, \mu > 3 \), there exists positive constants \( A, B \) independent of \( l, k \) and \( p \) so that for any \( l \geq 0 \)

\[ |\hat{W}^{(l)}(k, p)| \leq \frac{e^{\alpha p} e^{-\beta |k|}}{(1 + p^2)(1 + |k|)^{\alpha}} AB^{l} Q_2(|\beta k|) \]

The series (4.72) converges uniformly for any \( p_0 \geq 0 \) for \( |p - p_0| < \frac{1}{4B} \).
Remark 4.3. The proof requires some further lemmas. We will use induction on \( l \). Clearly, from (4.59), the conclusion is valid for \( l = 0 \), when
\[
A = 3\|\tilde{v}_1\|_{\mu, \beta}
\]
We assume (4.60) for \( l \geq 0 \) and then establish it for \( l + 1 \). We obtain a recurrence relation for \( W^{[l+1]}(k, .) \) for any \( k \in \mathbb{R}^3 \) in terms of \( \tilde{W}^{[j]}(k, .) \) for \( j \leq l \).

Taking \( \partial_p^l \) in (4.60) and dividing by \( l! \), we obtain
\[
\begin{align*}
&\frac{\partial^2 \tilde{W}^{[l]} + (l + 2)\partial_p \tilde{W}^{[l]} + |k|^2 \tilde{W}^{[l]} = }{
-ik_j P_k \int_0^p \left\{ \tilde{W}^{[l]}(\cdot, p - s) \tilde{W}^{[0]}(\cdot, s) \right\} ds + \sum_{i=1}^{l-1} \frac{l!}{i!} \tilde{W}^{[l]}(\cdot, 0) \tilde{W}^{[l-1-i]}(\cdot, p) \right. \\
&\left. - ik_j P_k \left[ \tilde{v}_{0j} \hat{W}^{[l]} + \tilde{v}_0 \hat{W}^{[l]} \hat{v}_0 + 1 \hat{v}_{1j} \tilde{W}^{[l+1]} + \frac{1}{l} \tilde{v}_1 \hat{v} \hat{W}^{[l]} + \tilde{v}_{1j} \hat{v}_1 \delta_{l,1} \right] \right.
\end{align*}
\]

Lemma 4.4. For any \( l \geq 0 \), for some absolute constant \( C_6 > 0 \), if \( \tilde{W}^{[l]}(k, p) \) satisfies (4.62) and is bounded at \( p = 0 \), then \( W^{[l+1]}(k, p) \) is bounded in terms of \( \tilde{R}^{(l)}(k, p) \), defined in (4.62):
\[
|\tilde{W}^{[l+1]}(k, p)| \leq \frac{C_6}{(l + 1)^{5/3}} \sup_{p' \in [0, p]} |\tilde{R}^{(l)}(k, p')| + \frac{|k|^2 |\tilde{W}^{[0]}(k, 0)|}{(l + 1)(l + 2)}
\]

Proof. We invert the operator on the left hand side of (4.62). With the requirement that \( \tilde{W}^{[l]} \) is bounded at \( p = 0 \), we obtain
\[
\tilde{W}^{[l]}(k, p) = \int_0^p Q \left( z(p), 2|k| \sqrt{p'} \right) \tilde{R}^{(l)}(k, p') dp'
\]
and
\[
Q(z, z') = \pi z^{-(l+1)} \left[ -J_{l+1}(z) z'^{(l+1)} Y_{l+1}(z') + z'^{(l+1)} J_{l+1}(z') Y_{l+1}(z) \right]
\]
On taking the first derivative with respect to \( p \), we obtain
\[
(l + 1) \tilde{W}^{[l+1]}(k, p) = \frac{|k|}{\sqrt{p}} \int_0^p Q \left( 2|k| \sqrt{p}, 2|k| \sqrt{p'} \right) \tilde{R}^{(l)}(k, p') dp' - 2^{l+2} (l + 1)! |k|^2 J_{l+2}(z) \tilde{W}^{[0]}(k, 0)
\]
Using again the properties of Bessel functions, we get
\[
\frac{1}{z} Q(z, z') = \frac{\pi}{2} \left[ - \left( \frac{J_{l+1}(z)}{z^{l+1}} \right)^r z'^{(l+1)} Y_{l+1}(z') + z'^{(l+1)} J_{l+1}(z') \left( \frac{Y_{l+1}(z)}{z^{l+1}} \right)^r \right]
\]
}\[
= \pi \left[ \frac{J_{l+2}(z)}{z^{l+2}} z'^{(l+1)} Y_{l+1}(z') - z'^{(l+1)} J_{l+1}(z') \frac{Y_{l+2}(z)}{z^{l+2}} \right]
\]
It is also known \[1\] that
\[
2^{l+2}(l+1)! \left| \frac{J_{l+2}(z)}{z^{l+2}} \right| \leq \frac{1}{(l+2)}
\]

Using (4.66) and the known uniform asymptotics of Bessel functions for large \(l\) \[1\], it is easily seen that \(C_0\) independent of \(l\) so that
\[
\int_0^{\bar{z}} \frac{z'}{z} |Q_z(z,z')|dz' \leq \frac{C_0}{(l+1)^{2/3}}
\]

It follows that
\[
(l+1)|\hat{W}[l+1](k,p)| \leq \sup_{p' \in [0,p]} |\hat{R}(l)(k,p')| \int_0^{\bar{z}} \frac{z'}{z} |Q_z(z,z')|dz' + \frac{|k|^2}{(l+2)} |\hat{W}[l](k,0)|
\]

Therefore, it follows that
\[
|\hat{W}[l+1](k,p)| \leq \frac{C_0}{(l+1)^{2/3}} \sup_{p' \in [0,p]} |\hat{R}(l)(k,p')| + \frac{|k|^2 |\hat{W}[l](k,0)|}{(l+1)(l+2)}
\]

Remark 4.5. We now find bounds on the different terms in \(\hat{R}(l)(k,p)\).

Lemma 4.6. If \(W[l]\) satisfies (4.60), for \(l \geq 0\) then
\[
|k_j P_k \left( \hat{v}_{0,j} \hat{W}[l] \right) | \leq C_1 \|\hat{v}_0\|_{1,\beta} \frac{(l+1)^{2/3} AB^l e^{-\beta |k|+\alpha p Q_2 l + 2(\beta |k|)}}{(2l+1)(1+|k|)^\mu (1+p^2)}
\]

Proof. We use (4.60). From Lemma 6.10 we obtain
\[
(1+p^2) e^{-\alpha |k|} |k_j \hat{W}_j^l \hat{v}_0| \leq \|\hat{v}_0\|_{1,\beta} \frac{AB^l}{(2l+1)} |k| \int_{k' \in \mathbb{R}^3} e^{-\beta (|k'|+|k-k'|)} (1+|k'|)^\mu (1+|k-k'|)^\mu Q_2 l (\beta |k'|) dk'
\]
\[
\leq C_1 (l+1)^{2/3} \|\hat{v}_0\|_{1,\beta} \frac{AB^l}{(2l+1)} e^{-\beta |k|} Q_2 l (\beta |k|)
\]

The first part of the Lemma follows. The proof of the second part is essentially the same since \(|\hat{W}_j^l| \leq |\hat{W}[l]|\).

Lemma 4.7. If \(W[l-1]\) satisfies (4.60) for \(l \geq 1\), then
\[
\left| \frac{k_j}{l} P_k \left( \hat{v}_{1,j} \hat{W}[l-1] \right) \right| \leq C_1 \|v_1\|_{1,\beta} AB^{l-1} \frac{e^{-\beta |k|+\alpha p}}{(1+p^2)(1+|k|)^\mu} \frac{l^{2/3} Q_2 l (\beta |k|)}{l(2l-1)!(l-1)}
\]

Proof. The proof is identical to Lemma 4.6 replacing \(l\) by \(l-1\) and \(\hat{v}_0\) by \(\hat{v}_1\).

Lemma 4.8. If \(W[l]\) satisfies (4.60), then for \(l \geq 1\),
\[
\left| \frac{k_j}{l} P_k \left( \hat{W}[l-1](\cdot,0) \hat{W}[0](\cdot,p) \right) \right| \leq C_1 \frac{(l+1)^{2/3} AB^{l-1} e^{-\beta |k|+\alpha p Q_2 l (\beta |k|)}}{l(2l-1)(1+|k|)^\mu (1+p^2)}
\]
Proof. Noting that
\[ |\hat{W}^{[\beta]}(k, p)| \leq A \frac{e^{-\beta|k|+\alpha p}}{(1 + |k|)^\mu(1 + p^2)} \]
and
\[ |\hat{W}^{[l-1]}(k, 0)| \leq \frac{e^{-\beta|k|}}{(2l - 1)^2(1 + |k|)^\mu} AB^{l-1}Q_{2l-2}(\beta|k|) \]
the rest of the proof is very similar to the proof of Lemma 4.6.

Lemma 4.9. If \( \hat{W}^{[l_1]} \) and \( \hat{W}^{[l_1-1]} \) for \( l_1 = 1, \ldots, (l - 2) \) for \( l \geq 2 \) satisfy (4.60), then
\[
\left| k_j P_k \left[ \sum_{l_1=1}^{l-2} \frac{l_1!(l-1-l_1)!}{l!} \hat{W}^{[l_1]}(\cdot, 0) \hat{W}^{[l_1-1]}(\cdot, p) \right] \right|
\leq C_8 2^{\mu+1} \pi A^2 B^{l-1} \frac{e^{-\beta|k|+\alpha p}}{3\beta^4(1 + p^2)(1 + |k|)^\mu} \frac{lQ_{2l}(\beta|k|)}{(2l + 3)^2} ; \text{ where } C_8 = 82
\]

Proof. First note that if we define \( l_2 = l - 1 - l_1 \), then for \( l \geq 2 \), using Lemma 6.9 we get
\[
\frac{l_1!l_2!}{l!} |k_j W_j^{[l_1]}(\cdot, 0) \hat{W}^{[l_2]}(\cdot, p)| \leq \frac{e^{\alpha p}}{(1 + p^2)} A^2 B^{l-1} \frac{l_1!l_2!}{l!(2l_1 + 1)^2(2l_2 + 1)^2} \times |k| \int_{k' \in \mathbb{R}^3} e^{-\beta|k+k'|}(1 + |k'|)^{-\mu}(1 + |k-k'|)^{-\mu} Q_{2l}(\beta|k|)Q_{2l_1}(\beta|k'|)Q_{2l_2}(\beta|k-k'|)dk'
\leq \frac{A^2 B^{l-1} 2^{\mu+1} \pi e^{-\beta|k|+\alpha p} l_1!l_2!(2l)(2l-1)(2l+1)}{3\beta^4(1 + p^2)(1 + |k|)^\mu} \frac{lQ_{2l}(\beta|k|)}{(2l + 3)^2}
\]
Therefore,
\[
\sum_{l_1=1}^{l-2} \frac{l_1!l_2!}{l!} |k_j W_j^{[l_1]}(\cdot, 0) \hat{W}^{[l_2]}(\cdot, p)|
\leq \frac{2^\mu A^2 B^{l-1} 2^{\mu+1} \pi Q_{2l}(\beta|k|) \sum_{l_1=1}^{l-2} l_1!l_2!(2l)(2l-1)(2l+1)}{\beta^4(1 + p^2)(1 + |k|)^\mu} \frac{1}{l!(2l_1 + 1)^2(2l_2 + 1)^2}
\]
We claim that for \( l \geq 2 \), with \( l_1 \geq 1, l_2 = l - l_1 - 1 \geq 1 \),
\[
\sum_{l_1=1}^{l-1} \frac{l_1!l_2!(2l)(2l-1)(2l+1)}{l!(l-1)(2l_1 + 1)^2(2l_2 + 1)^2} \leq \frac{C_8l}{(2l + 3)^2}
\]
for some \( C_8 \) independent of \( l \); \( C_8 \) is bounded by 82.

Proving the above bound only requires consideration for sufficiently large \( l \). We will therefore assume \( l \geq 5 \). Further, consider summation terms other than \( l_1 = 1 \) and \( l_2 = 1 \). So, we may assume \( l_1, l_2 \geq 2 \). Then, we claim that
\[
(4.69) \quad \frac{l_1!l_2!(2l)(2l-1)(2l+1)}{l!(2l+1)^2(2l_2 + 1)^2} = \left( \frac{l_1!l_2!(l_2-2)!}{(l-5)!} \right) \left( \frac{l_1(l_1-1)l_2(l_2-1)}{(2l_1 + 1)^2(2l_2 + 2)^2} \right) \times \left( \frac{2l(2l+1)}{(l-1)(l-2)(l-3)(l-4)} \right) \leq \frac{12}{(2l+3)^2}
\]
This follows since the first two parenthesis term on the right of (4.69) is clearly bounded, while the last term is a cubic in \( l \) divided by fifth order polynomial, and simple estimates give the upperbound of 12. Therefore, for \( l \geq 5 \),

\[
\sum_{l_1=2}^{l-3} \frac{l_1!l_2!2(l+1)(2l-1)}{l!(2l+1)^2(2l+1)^2} \leq 12 \frac{(l-4)}{(2l+3)^2}
\]

For \( l_1 = 1 \) or \( l_2 = 1 \), clearly

\[
\frac{l_1!l_2!2(l+1)(2l-1)}{l!(2l+1)^2(2l+1)^2} = \frac{(l-2)!2l(2l+1)(2l-1)}{9l!(2l-3)^2} = \frac{2l(2l+1)(2l-1)}{9l(l-1)(2l-3)^2} \leq 82 \frac{l}{(2l+3)^2}
\]

**Lemma 4.10.** If \( \tilde{W}^{[l]} \) satisfies (4.60), then for \( l \geq 0 \),

\[
\left| k_j \left( 1 - \frac{k(k\cdot)}{|k|^2} \right) \int_0^p \tilde{W}^{[l]}(\cdot; p-s)\tilde{W}^{[0]}(\cdot; s) ds \right| 

\leq C_1 (l+1)^{2/3} A^2 B^L \frac{e^{-\beta |k|+\alpha p} Q_{2l+2}(\beta |k|)}{(1+p^2)(1+|k|)^\mu (2l+1)}
\]

**Proof.** We note that Lemma 6.10 implies

\[
\left| k_j \int_{k'\in\mathbb{R}^3} \int_0^p \tilde{W}^{[l]}(k', p-s)\tilde{W}^{[0]}(k-k', s) ds dk' \right| \leq \frac{A^2 B^L e^{\alpha p}}{(1+p^2)(2l+1)^2}
\]

\[
\times |k| \int_{k'\in\mathbb{R}^3} e^{-\beta |k'|-\beta |k-k'|}(1+|k'|)^\mu (1+|k-k'|)^\mu Q_{2l}(\beta k') dk'
\]

\[
\leq \frac{C_1 (l+1)^{2/3} A^2 B^L e^{\alpha p-\beta |k|}}{(2l+1)(1+p^2)(1+|k|)^\mu} Q_{2l+2}(\beta |k|)
\]

**Lemma 4.11.**

\[
\left| k_j \left( 1 - \frac{k(k\cdot)}{|k|^2} \right) \hat{\psi}_{0,j} \hat{\psi}_0 + \hat{\psi}_{1,j} \hat{\psi}_1 \right| \leq 4C_0|k| e^{-\beta |k|} \frac{1}{(1+|k|)^\mu} \|\hat{\psi}_0\|_{\mu,\beta} \|\hat{\psi}_1\|_{\mu,\beta}
\]

\[
\left| k_j \left( 1 - \frac{k(k\cdot)}{|k|^2} \right) \hat{\psi}_{1,j} \hat{\psi}_1 \right| \leq 2C_0|k| e^{-\beta |k|} \frac{1}{(1+|k|)^\mu} \|\hat{\psi}_1\|_{\mu,\beta}^2
\]

**Proof.** This follows simply from the observation that

\[
\left| k_j \left( 1 - \frac{k(k\cdot)}{|k|^2} \right) \hat{\psi}_{0,j} \hat{\psi}_0 \right| \leq 2|k| \|\hat{\psi}_0\|_{\mu,\beta} \|\hat{\psi}_0\|_{\mu,\beta} \int_{k'\in\mathbb{R}^3} e^{-\beta |k'|-\beta |k-k'|}(1+|k'|)^\mu (1+|k-k'|)^\mu dk'
\]

and using (2.21) to bound the convolution. Other parts of the Lemma follow similarly.

**Lemma 4.12.**

\[
|\tilde{W}^{[1]}(\cdot; p)| \leq \frac{e^{-\beta |k|+\alpha p}}{(1+|k|)^\mu (1+p^2)} ABQ_2(\beta |k|)
\]

with

\[
AB \geq \left( 2C_1 \|v_0\|_{\mu,\beta} A + C_1 A^2 + \frac{2C_0}{\beta} \|\hat{\psi}_0\|_{\mu,\beta} \|v_1\|_{\mu,\beta} \right)
\]

\[
(4.70)
\]
Proof. Combining Lemmas 4.6, 4.10 and 4.11 with (4.68) for \( l = 0 \), we obtain
\[
|\tilde{W}^{[l]}(\cdot, p)| \leq \frac{e^{-\beta|\kappa|+\alpha p}}{(1 + |\kappa|)^\mu(1 + p^2)} \beta_2(\beta|\kappa|) \left( 2C_1\|v_0\|_{\mu,\beta}A + C_1A^2 + \frac{2C_0}{\beta} \|\hat{v}_0\|_{\mu,\beta}\|\hat{v}_1\|_{\mu,\beta} \right) \\
\leq \frac{e^{-\beta|\kappa|+\alpha p}}{(1 + |\kappa|)^\mu(1 + p^2)} AB\beta_2(\beta|\kappa|)
\]

Proof of Lemma 4.2

From Lemmas 4.6, 4.10 and 4.11 (the latter is only needed for \( l = 1 \)), it follows that \( \tilde{R}^{(l)} \) (cf. (4.60)) satisfies
\[
|\tilde{R}^{(l)}| \leq AB^{l-1} \frac{e^{-\beta|\kappa|+\alpha p}}{(2l + 3)^2(1 + p^2)(1 + |\kappa|)^\mu} Q_{2l+2}(\beta|\kappa|) \\
\times \left[ ABC_1 \frac{(l + 1)^2(2 + 3)^2}{(2l + 1)} + \frac{AC_1(l + 1)^2(2 + 3)^3}{4l(l + 1)(2l - 1)} + \frac{82 2^\mu(1 + p^2)\pi A}{12 \beta^3} + \frac{2C_1\|\hat{v}_1\|_{\mu,\beta}(2l + 3)^3}{4l^{1/2}(l + 1)(2l + 1)} \\
+ \frac{2C_0(2 + 3)^2\|\hat{v}_0\|_{\mu,\beta}}{(l + 1)(2l + 1)} + \frac{25C_0}{\beta(1 + p^2)} e^{-\alpha p}\|\hat{v}_1\|_{\mu,\beta}^2 \right]
\]
Noting that \( e^{-\alpha p}(1 + p^2) \leq 1 \) and
\[
\sup_{p' \in [0, p]} \frac{e^{\alpha p'}}{1 + p^2} = \frac{e^{\alpha p}}{1 + p^2}
\]
for \( \alpha \geq 1 \), it follows from Lemma 4.4 and the above bounds that (4.60) holds when \( l \) is replaced by \( l + 1 \), provided \( B \) is chosen large enough to satisfy (4.70) and
\[
C_6 \left[ \frac{AB(2 + 3)^3}{(l + 1)(2l + 1)} + \frac{AC_1(2 + 3)^3}{4(l + 1)(2l - 1)} + \frac{(82)(1 + p^2)\pi A}{12 \beta^3} + \frac{2C_1\|\hat{v}_1\|_{\mu,\beta}(2l + 3)^3}{4l^{1/2}(l + 1)(2l + 1)} \\
+ \frac{2C_0(2 + 3)^2\|\hat{v}_0\|_{\mu,\beta}}{(l + 1)(2l + 1)} + \frac{25C_0}{\beta(1 + p^2)} e^{-\alpha p}\|\hat{v}_1\|_{\mu,\beta}^2 \right] + \frac{100B}{9 \beta^2} \leq B^2,
\]
for any \( l \geq 1 \), with \( A \) given by (4.61). From the asymptotic behavior of the left hand side of (4.71) as \( l \to \infty \) and recalling that constants \( C_0, C_1 \) and \( C_6 \) are independent of \( l \), it follows that \( B \) can be chosen independent of \( l \). Therefore, by induction, (4.60) follows for all \( l \). The proof of Lemma 4.2 is complete.

From (4.60), after noting that that \( Q_{2l}(\|q\|) \leq 4e^{-\|q\|/2} \), it follows that
\[
\tilde{W}(k, p; p_0) = \sum_{l=0}^{\infty} \tilde{W}^{[l]}(k, p_0)(p - p_0)^l := \tilde{W}_1(k, p)
\]
is convergent for \( |p - p_0| < \frac{1}{2B} \) for \( B \) independent of \( p_0 \in \mathbb{R}^+ \). The following Lemma shows that \( \tilde{W}(k, p; p_0) \) is indeed the local representation of the solution \( \tilde{W}(k, p) \) to (2.40).

Lemma 4.13. The unique solution to (2.40) satisfying \( \tilde{W}(k, 0) = 0 \), given by \( \tilde{W}(k, p) = \tilde{U}(k, p) - \hat{v}_1(k) \), where \( \tilde{U}(k, p) \) is determined in \( \S 2 \) in Lemma 2.8, has the
local representation $\hat{W}(k, p; p_0)$ in a neighborhood of $p_0 \in \mathbb{R}^+$. Therefore, $\hat{W}(k, \cdot)$ (and therefore $\hat{U}(k, \cdot)$) is analytic in $\mathbb{R}^+ \cup \{0\}$.

**Proof.** First, by permanence of relations (for analyticity of convolutions, see e.g., [5]), it follows that if $\hat{V}$ is an analytic solution of an equation of the form (2.9) on an interval $[0, L]$ and $\hat{V}$ has analytic continuation on $[0, L']$ with $L' > L$, then the equation is automatically satisfied in the larger interval. Therefore, if we analytically continue $\hat{W}$ to $\mathbb{R}^+$, the analytic continuation will automatically satisfy (2.9) and will therefore be the same as $\hat{W}(k, p)$.

From §3, Lemma 3.7, we know that the actual solution to (2.9) satisfying $\hat{W}(k, 0) = 0$, is unique, and given by

$$\hat{W}(k, p) = \hat{W}(k, p; 0)$$

for $|p| < (4B)^{-1}$.

We now choose a sequence of $\{p_{0,j}\}^\infty_{j=0}$, with $p_{0,j} = j/(8B)$ and define the intervals $I_j = (p_{0,j} - 1/(4B), p_{0,j} + 1/(4B))$. Consider the sequence of analytic functions $\{\hat{W}(k, p; p_{0,j})\}^\infty_{j=0}$. Since $p_{0,1} \in I_0 \cap I_1$, it follows from (4.72) that $\hat{W}(k, p)$ has analytic continuation to $I_1$, namely $\hat{W}(k, p; p_{0,1})$. Again $p_{0,2} \in I_1 \cap I_2$. Hence $\hat{W}(k, p; p_{0,2})$ provides analytic continuation of $\hat{W}(k, p)$ to the interval $I_2$. We can continue this process to obtain analytic continuation of $\hat{W}$ to any interval $I_j$. Since the union of $\{I_j\}^\infty_{j=0}$ contains $\mathbb{R}^+ \cup \{0\}$, it follows that $\hat{W}(k, \cdot)$ is analytic in $\mathbb{R}^+$. In particular, (4.72) provides the local Taylor series representation of $\hat{W}(k, p)$ near $p = p_0$.

**Proof of Theorem 1.2**

Using Lemma 4.2 it follows from the inequality $\|W^{[l]}(\cdot, p_0)\|_\infty \leq \|\hat{W}^{[l]}(\cdot, p_0)\|_{L^1}$ by integration in $k$ that

$$|W^{[l]}(x, p_0)| \leq \frac{8\pi A(4B)^l e^{\alpha p_0}}{\beta(2l + 1)^2(1 + p_0^2)}$$

$$|DW^{[l]}(x, p_0)| \leq \frac{8\pi A(4B)^l e^{\alpha p_0}}{\beta^2(2l + 1)^2(1 + p_0^2)}$$

$$|D^2W^{[l]}(x, p_0)| \leq \frac{16\pi A(4B)^l e^{\alpha p_0}}{\beta^2(2l + 1)^2(1 + p_0^2)}$$

and therefore, the series (4.72) converges for $|p - p_0| < B^{-1}/4$ and, from Lemma 4.13 it is the local representation of the solution $W(k, p)$ to (2.9) satisfying $W(k, 0) = 0$ for any $p_0 \geq 0$. These estimates on $W$ in terms of $\hat{W}$, and the fact that $W(x, p)$ is analytic in a neighborhood of for $p \in \{0\} \cup \mathbb{R}^+$ and is exponentially bounded in $p$ for large $p$ (recall $\hat{W} \in A^\infty$) implies Borel summability of $v$ in $1/t$. Watson’s Lemma [21] implies $\omega(x, t) = \int_0^\infty e^{-p/t}W(x, p)dp \sim \sum_{m=2}^\infty v_m(x)t^m$, implying

$$v(x, t) = v_0(x) + tv_1(x) + \sum_{m=2}^\infty v_m(x)t^m,$$
where \( v_m(x) = m!W^{[m-1]}(x; 0) = m!U^{[m-1]}(x; 0) \) for \( m \geq 2 \). It follows from the bounds on \( \hat{W}^{[m-1]}(k) \) in §3, that for \( m \geq 2 \), \( |W^{[m-1]}(x; 0)| \leq A_0B_0^m \), where \( A_0 \) and \( B_0 \) are chosen to ensure (3.55), (3.57) and (3.58).

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6. Appendix

6.1. Some Fourier convolution inequalities. The following lemmas are relatively straightforward.

**Definition 6.1.** Consider the polynomial

\[
P_n(z) = \sum_{j=0}^{n} \frac{n!}{j!} z^j
\]

**Remark 6.2.** Integration by parts yields

\[
\int_{0}^{z} e^{-\tau} \tau^m d\tau = -e^{-z}P_n(z) + n!
\]

**Lemma 6.3.** For all \( y \geq 0 \) and nonnegative integers \( m, n \geq 0 \) we have

\[
y^{m+1} \int_{0}^{1} \rho^m P_n(y(1-\rho))d\rho = n! \sum_{j=0}^{n} \frac{y^{m+j+1}}{(m+j+1)!}
\]

**Proof.** This follows from a simple computation:

\[
y^{m+1} \int_{0}^{1} \rho^m P_n(y(1-\rho))d\rho = \sum_{j=0}^{n} \frac{n!}{j!} y^{j+m+1} \int_{0}^{1} (1-\rho)^j \rho^m d\rho = n! \sum_{j=0}^{n} \frac{y^{j+m+1}}{(m+j+1)!}
\]

**Lemma 6.4.** For all \( y \geq 0 \) and nonnegative integers \( n \geq m \geq 0 \) we have

\[
y^{m+1} \int_{1}^{\infty} e^{-2y(\rho-1)} \rho^m P_n(y(\rho - 1))d\rho \leq 2^{-m}(m+n)! \sum_{j=0}^{m} \frac{y^{j}}{j!}
\]

**Proof.** First we note that

\[
y^{m+1+l} \int_{1}^{\infty} e^{-2y(\rho-1)} \rho^m (\rho - 1)^l d\rho = y^{m+1+l} \int_{0}^{\infty} e^{-2y\rho} (1+\rho)^m \rho^l d\rho
\]

\[
= y^{m+1+l} \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} \int_{0}^{\infty} e^{-2y\rho} \rho^l d\rho = 2^{l-1} \sum_{j=0}^{m} \frac{y^{m-j}m!(l+j)!}{j!(m-j)!2^j}
\]

\[
= 2^{l-1} \sum_{j=0}^{m} \frac{y^j m!(l+m-j)!}{(m-j)!j!2^{m-j}}
\]

(4) We may express it in terms of \( A \) and \( B \) as well, however, the estimates \( A_0 \) and \( B_0 \) found in §3, are better.
Therefore, from the definition of $P_n$, it follows that
\[
y^{m+1} \int_1^\infty \frac{y^j}{j!(m-j)!2^{m-j}} \left( \sum_{l=0}^n \frac{(l+m-j)!}{2^{l+1}l!} \right) d\rho = m!n! \sum_{j=0}^m \frac{y^j}{j!(m-j)!2^{m-j}} \left( \sum_{l=0}^n \frac{(l+m-j)!}{2^{l+1}l!} \right)
\]
Taking the ratio of two consecutive terms we see that \((l + m - j)! / l!\) is nondecreasing with \(l\) since \(m - j \geq 0\). Therefore the \(l = n\) term is the largest term in the summation over \(l\). Further, \(\sum_{l=0}^n 2^{l-1} l! \leq 1\). Therefore, \(\sum_{l=0}^n 2^{l-1} (l + m - j)! / l! \leq (m - j + n)! / n!\), and hence
\[
y^{m+1} \int_1^\infty e^{-2y(\rho-1)} \rho^m P_n(y(\rho-1)) d\rho \leq 2^{-m} m!n! \sum_{j=0}^m \frac{y^j 2^j (m-j+n)!}{n!(m-j)!}
\]
The ratio of two consecutive (in \(j\)) terms in \(2^j (m-j+n)! / (m-j)!\) is \(\leq 1\) for \(m \leq n\), hence the largest value is attained at \(j = 0\) and thus
\[
y^{m+1} \int_1^\infty e^{-2y(\rho-1)} \rho^m P_n(y(\rho-1)) d\rho \leq 2^{-m} (m+n)! \sum_{j=0}^m \frac{y^j}{j!}
\]

**Lemma 6.5.** For all \(y \geq 0\) and nonnegative integers \(n \geq m \geq 0\) we have
\[
y^{m+1} \int_0^1 e^{-y(\rho-1)[1+\text{sgn}(\rho-1)]} \rho^m P_n(y(1-\rho)) d\rho \leq m!n!Q_{m+n+1}(y)
\]

**Proof.** By breaking up the integral range into \([0, 1]\) and \([1, \infty)\) and using the two previous Lemmas, we obtain
\[
y^{m+1} \int_0^1 e^{-y(\rho-1)[1+\text{sgn}(\rho-1)]} \rho^m P_n(y(1-\rho)) d\rho \leq m!n! \left( \sum_{j=m+1}^{m+n+1} \frac{y^j}{j!} \right)
\]
\[
+ 2^{m-n} \left( \frac{(m+n)!}{m!n!} \right) \sum_{j=0}^m 2^n \frac{y^j}{j!} \leq m!n! \sum_{j=0}^{m+n+1} 2^{m+n+1-j} \frac{y^j}{j!} = m!n!Q_{m+n+1}(y)
\]
where we used \(2^{m-n} \frac{(m+n)!}{m!n!} \leq 1\).

**Lemma 6.6.** If \(m\) and \(n\) are integers no less that \(-1\) we obtain
\[
|q| \int_{q' \in \mathbb{R}^3} e^{\left| q - q' \right| - |q' Q'|} |q' |^m |q - q'|^n d q' \leq 2\pi (m+1)!(n+1)! Q_{m+n+3}(|q|)
\]

**Proof.** We note that we may assume \(m \leq n\) without loss of generality since changing variable \(q' \mapsto q - q'\) switches the roles of \(m\) and \(n\).

First, we will show that
\[
(6.74) \quad \frac{|q|}{2\pi} \int_{q' \in \mathbb{R}^3} e^{\left| q - q' \right| - |q' Q'|} |q' |^m |q - q'|^n d q' \leq |q|^{m+2} \int_0^\infty e^{-\left| q (\rho-1) [1+\text{sgn}(\rho-1)] \right|} \rho^{m+1} P_{n+1}(|q|(|\rho - 1|) d \rho
\]
We scale \( q' \) with \(|q| \) and use a polar representation \((\rho, \theta, \phi)\) for \( q'/|q| \), where \( \theta \) is the angle between \( q \) and \( q' \). As a variable of integration however, we prefer to use 
\[
z = \sqrt{1 + \rho^2 - 2\rho \cos \theta} \to \theta.
\]
Then, it is clear that
\[
|q - q'| = |q|\sqrt{1 + \rho^2 - 2\rho \cos \theta} = |q|z, \text{ and } dz = \frac{\rho \sin \theta \, d\theta}{\sqrt{1 + \rho^2 - 2\rho \cos \theta}}.
\]
Therefore,
\[
|q| \int_{q' \in \mathbb{R}^3} e^{\left|q'\right| - \left|q - q'\right|} |q'|^m |q - q'|^n \, dq' 
\]
\[
= 2\pi |q|^{m+n+4} \int_0^\infty d\rho \rho^{m+1} e^{-|q|(\rho-1)} \left\{ \int_{|\rho-1|}^{1+\rho} dze^{-|q|z} z^{n+1} \right\} 
\]
\[
= 2\pi |q|^{m+2} \int_0^\infty d\rho \rho^{m+1} e^{-|q|(\rho-1)} \left[ e^{-|q|\rho-1} P_{n+1}(|q|\rho-1) - e^{-|q|(1+\rho)} P_{n+1}(|q|(1+\rho)) \right]
\]
Inequality (6.74) follows since \( e^{-|q|(1+\rho)} P_{n+1}(|q|(1+\rho)) \geq 0 \). The rest of the Lemma follows from Lemma 6.5 with \( y = |q| \), and \( m \) replaced by \( m+1 \), \( n \) by \( n+1 \) respectively.

**Lemma 6.7.** For any \( \mu \geq 1 \), and nonnegative integers \( m, n \) we have
\[
|k| \int_{k' \in \mathbb{R}^3} \frac{e^{-\beta(|k'|+|k-k'|)}}{(1+|k'|)^\mu (1+|k-k'|)^\mu |\beta k'|^m |\beta(k-k')|^n} \, dk' 
\]
\[
\leq \frac{\pi 2^{\mu+1} e^{-\beta|k|} m! n!}{\beta^3 (1+|k|)^\mu} (m+n+2) Q_{m+n+2}(\beta|k|)
\]
**Proof.** We break up the integral into two ranges:
\[
(6.75) \quad \int_{|k'| \leq |k|/2} + \int_{|k'| > |k|/2}
\]
In the first integral we have
\[
\frac{1}{(1+|k-k'|)^\mu (1+|k'|)^\mu} \leq \frac{1}{(1+|k|/2)^\mu (1+|k'|)^\mu} \leq \frac{\beta}{(1+|k|/2)^\mu |\beta k'|}
\]
While in the second integral we have
\[
\frac{1}{(1+|k-k'|)^\mu (1+|k'|)^\mu} \leq \frac{1}{(1+|k|/2)^\mu (1+|k-k'|)^\mu} \leq \frac{\beta}{(1+|k|/2)^\mu |\beta(k-k')|}
\]
Introducing in the first integral \( q = \beta k \) and \( q' = \beta k' \), we obtain
\[
|k| \int_{|k'| \leq |k|/2} \frac{e^{-\beta(|k'|+|k-k'|)}}{(1+|k'|)^\mu (1+|k-k'|)^\mu |\beta k'|^m |\beta(k-k')|^n} \, dk' 
\]
\[
\leq \frac{2^{\mu} e^{-\beta|k|}}{\beta^3 (1+|k|)^\mu} |q| \int_{q' \in \mathbb{R}^3} e^{-|q'|-|q-q'|+|q|} |q'|^{m-1} |q - q'|^n \, dq'
\]
while in the second integral, with \( q = \beta k \) and \( q - q' = \beta k' \), we obtain

\[
|k| \int_{|k'|>|k|/2} e^{-\beta|k'|+|k-k'|} \left( 1 + |k'| \right) \mu \left( 1 + |k - k'| \right)^\mu |\beta(k - k')|^n dk'
\]

\[
\leq \frac{2^\mu e^{-\beta|k|}}{\beta^2 \left( 1 + |k| \right)^\mu} \left| q \right| \int_{q' \in \mathbb{R}^3} e^{-|q'| - |q - q'| + |q|} |q - q'|^{-\mu} dq'
\]

We now use Lemma 6.6 to bound the first integral, with \( m \) replaced by \( m - 1 \). We also use Lemma 6.6 to bound the second integral, with \( n - 1 \) replacing \( n \). The proof is completed by adding the two bounds.

**Lemma 6.8.** For any \( \mu \geq 2 \), and \( n \in \mathbb{N} \setminus \{0\} \) we have

\[
|k| \int_{|k'|<|k|/2} e^{-\beta|k'|+|k-k'|} \left( 1 + |k'| \right) \mu \left( 1 + |k - k'| \right)^\mu |\beta(k - k')|^n dk'
\]

\[
\leq \frac{2^\mu e^{-\beta|k|}}{\beta^2 \left( 1 + |k| \right)^\mu} \left| q \right| \int_{q' \in \mathbb{R}^3} e^{-|q'| - |q - q'| + |q|} |q - q'|^{-\mu} dq' \leq \frac{2^\mu e^{-\beta|k|}}{\beta^2 \left( 1 + |k| \right)^\mu} \left( n - 1 \right) ! Q_{n+1} (|q|)
\]

**Proof:** We break up the integral into \( \int_{|k'|<|k|/2} + \int_{|k'|\geq|k|/2} \). In the first integration range we have \( 1 + |k - k'| - \mu \leq 2^\mu \left( 1 + |k| \right)^{-\mu} \), whereas in the second range \( 1 + |k'| - \mu \leq 2^\mu \left( 1 + |k| \right)^{-\mu} \). Therefore, using Lemma 6.6

\[
|k| \int_{|k'|\geq|k|/2} e^{-\beta|k'|+|k-k'|} \left( 1 + |k'| \right) \mu \left( 1 + |k - k'| \right)^\mu |\beta(k - k')|^n dk'
\]

\[
\leq \frac{2^\mu e^{-\beta|k|}}{\beta^2 \left( 1 + |k| \right)^\mu} \left| q \right| \int_{q' \in \mathbb{R}^3} e^{-|q'| - |q - q'| + |q|} |q - q'|^{-\mu} dq' \leq \frac{2^\mu e^{-\beta|k|}}{\beta^2 \left( 1 + |k| \right)^\mu} \left( n - 1 \right) ! Q_{n+1} (|q|)
\]

On the other hand, using \( (1 + |k'|)^{-\mu} \leq (1 + |k'|)^{-2+2/3} \leq |k'|^{-2+2/3} \leq |k'|^{-2+2/3} \) we get

\[
|k| \int_{|k'|<|k|/2} e^{-\beta|k'|+|k-k'|} \left( 1 + |k'| \right) \mu \left( 1 + |k - k'| \right)^\mu |\beta(k - k')|^n dk'
\]

\[
\leq \frac{2^\mu e^{-\beta|k|}}{\beta^2 \left( 1 + |k| \right)^\mu} \left| q \right| \int_{|q'\leq|q|/2} |q'|^{-2+2/3} e^{-|q'| - |q - q'| + |q|} |q - q'|^{-\mu} dq'
\]

We note that

\[
\frac{|q|}{2\pi} \int_{|q'|<|q|/2} |q'|^{-2+2/3} e^{-|q'| - |q - q'| + |q|} |q - q'|^{-\mu} dq'
\]

\[
= |q|^{n+2+2/3} \int_{0}^{1/2} \rho^{-1+2/3} e^{-|q|/\rho} \left( \int_{1-\rho}^{1+\rho} dz e^{-|q|z^{n+1}} \right) d\rho
\]

\[
\leq |q|^{2/3} \int_{0}^{1/2} \rho^{-1+2/3} P_{n+1} (|q|(1-\rho)) d\rho
\]

\[
\leq |q|^{2/3} \sum_{j=0}^{n+1} \frac{(n+1)!}{j!} |q|^j \int_{0}^{1} \rho^{-1/3} (1-\rho)^j d\rho \leq \frac{3}{2} |q|^{2/3 (n+1)} \sum_{j=0}^{n+1} \frac{|q|^j}{j!}
\]
Lemma 6.9. For any $\mu \geq 1$ and nonnegative integers $l_1, l_2 \geq 0$ we have

$$(6.76) \quad |k| \int_{k' \in \mathbb{R}^3} \frac{e^{-\beta|k'|+|k-k'|}}{(1 + |k'|)^\mu(1 + |k - k'|)^\mu} Q_{2l_1}(|\beta k'|)Q_{2l_2}(|\beta(k - k')|) dk' \leq \frac{2^{\mu+1}\pi e^{-\beta|k|}}{3\beta^3(1 + |k|)^\mu} (2l_1 + 2l_2 + 1)(2l_1 + 2l_2 + 2)(2l_1 + 2l_2 + 3) Q_{2l_1+2l_2+2}(\beta|k|)$$

Proof. As before, we define $q = \beta k$. Also, for notational convenience, we define

$$k^m \otimes k^n = |k| \int_{k' \in \mathbb{R}^3} \frac{e^{-\beta|k'|+|k-k'|}}{(1 + |k'|)^\mu(1 + |k - k'|)^\mu} |\beta k'|^m |\beta(k - k')| \mu dk'$$

$$K = \frac{2^{\mu+1}\pi e^{-\beta|k|}}{\beta^3(1 + |k|)^\mu}$$

Lemma 6.7 and $2^{2l_1+2l_2+2} Q_{j+2}(|q|) \leq Q_{2l_1+2l_2+2}$ for $j \leq 2l_1 + 2l_2$ imply that the left side of (6.76) is given by

$$\sum_{m=0}^{2l_1} \sum_{n=0}^{2l_2} \frac{2^{2l_1+2l_2-m-n} m! n!}{m! n!} k^m \otimes k^n \leq K \sum_{m=0}^{2l_1} \sum_{n=0}^{2l_2} 2^{2l_1+2l_2-m-n} (m+n+2) Q_{m+n+2}(\beta|k|)$$

$$\leq K \sum_{j=0}^{2l_1+2l_2} 2^{2l_1+2l_2+2-j} \binom{j}{j} (j + 1) Q_{j+2}(|q|)$$

$$\leq K \frac{3}{2}(2l_1 + 2l_2 + 1)(2l_1 + 2l_2 + 2)(2l_1 + 2l_2 + 3) Q_{2l_1+2l_2+2}(|q|),$$

which imply the result. \[\square\]

Lemma 6.10. If $\mu \geq 2$ and $l \geq 0$, then

$$\frac{|k|}{(l + 1)^{2/3}} \int_{k' \in \mathbb{R}^3} \frac{e^{-\beta|k'|+|k-k'|}}{(1 + |k'|)^\mu(1 + |k - k'|)^\mu} Q_{2l}(|\beta(k - k')|) dk' \leq C_1 \frac{e^{-\beta|k|}}{(1 + |k|)^\mu}(2l + 1) Q_{2l+2}(|q|),$$

where

$$C_1 = 12\pi^2 2^\mu \beta^{-8/3} + 2\pi 2^\mu \beta^{-2} + \frac{1}{2} C_0(\mu) \beta^{-1}$$

Proof. The case $l = 0$ follows easily by using (2.21) and the fact that

$$|k| = \beta^{-1} |q| \leq \frac{1}{2} \beta^{-1} Q_2(|q|)$$

For $l \geq 1$, it is convenient to separate out the constant term $2^{2l}$ in $Q_{2l}$ and note that from (2.21) and the definition of $Q_n(z)$ we have

$$|k| \int_{k' \in \mathbb{R}^3} \frac{e^{-\beta|k'|+|k-k'|}}{(1 + |k'|)^\mu(1 + |k - k'|)^\mu} 2^{2l} dk' \leq C_0 \frac{|k| e^{-\beta|k|}}{(1 + |k|)^{\mu}} 2^{2l} \leq C_0 \frac{e^{-\beta|k|}}{2\beta(1 + |k|)^\mu} Q_{2l}(|q|)$$
As in previous Lemma, for notational convenience, we define
\[ k^m \otimes k^n = |k| \int_{k' \in \mathbb{R}^3} \frac{e^{-\beta|k'|+|k-k'|}}{(1 + |k'|)^\mu (1 + |k-k'|)^\mu} |\beta k'|^m |\beta (k - k')|^n \, dk' \]
Then, it is clear from Lemma 6.8 that
\[
\{Q_{2l}(\beta|k|) - 2^{2l}\} \otimes k^0 = \sum_{n=1}^{2l} \frac{2^{2l-n}}{n!} k^n \hat{e}_k k^0
\]
\[
\leq \frac{2^{n+1} \pi e^{-\beta |k|}}{\beta^2 (1 + |k|)^\mu} \sum_{n=1}^{2l} \frac{2^{2l-n}}{n!} \left\{ (n - 1)! \sum_{j=0}^{n+1} \frac{2^{n+1-j} (\beta |k|)^j}{j!} + \frac{3(n+1)! 2^{3/2}}{2^{2/3} \beta^2 j!} \sum_{j=0}^{2^{l+1}-1} \frac{(n'-2)!}{(n'-1)!} \right\}
\]
\[
\leq \frac{2^{n+1} \pi e^{-\beta |k|}}{\beta^2 (1 + |k|)^\mu} \left\{ \beta Q_{2l+1}(\beta |k|) \log(2l + 2) + 3(2l + 1) \beta^{1/3} |\beta k|^{2/3} Q_{2l+1}(\beta |k|) \right\}
\]
The lemma follows since \( \log(2l + 2)/(2l + 1) \leq 1 \), while if \( |\beta | \leq (l + 1) \),
\[
\left( \frac{|\beta |}{(l + 1)} \right)^{2/3} Q_{2l+1}(\beta |k|) \leq Q_{2l+1}(\beta |k|) \leq \frac{1}{2} Q_{2l+2}(\beta |k|)
\]
wheras for \( |\beta | \geq (l + 1) \) we have
\[
\left( \frac{|\beta |}{(l + 1)} \right)^{2/3} Q_{2l+1}(\beta |k|) \leq \frac{2|\beta |}{2l + 2} Q_{2l+1}(\beta |k|) \leq 2Q_{2l+2}(\beta |k|)
\]

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