Inflationary Braneworld Driven By Bulk Scalar Field

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Abstract

We have developed a formalism to study an inflationary scenario driven by a bulk inflaton in the two-brane system. The 4-dimensional low energy effective action is obtained using the gradient expansion method. It is also found that the dark radiation and the dark scalar source are expressed by the radion. In the single-brane limit, we find these dark components disappear. Therefore, it turns out that the inflation due to the bulk inflaton successfully takes place. Kaluza-Klein corrections are also taken into account in this case.

1. Introduction

Inspired by the recent developments of the superstring theory, Randall and Sundrum have proposed a novel compactification mechanism in the context of the brane world scenario. It is natural to ask if the inflationary universe can be possible or not in this context. Let us recall the well known formula for the cosmological constant in the braneworld:

\[
\Lambda_{\text{eff}} = \frac{k^2 \sigma^2}{12} - \frac{3}{\ell^2},
\]

where \(\sigma\) and \(\ell\) are the tension of the brane and the curvature scale in the bulk which is determined by the bulk vacuum energy, respectively. For \(k^2 \sigma = 6/\ell\), we have Minkowsky spacetime. In order to obtain the inflationary universe, we need the positive effective cosmological constant. In the brane world model, there are two possibilities. One is to increase the brane tension and the other is to increase \(\ell\). The brane tension can be controlled by the scalar field on the brane. The bulk curvature scale \(\ell\) can be controlled by the bulk scalar field. The former case is a natural extension of the 4-dimensional inflationary scenario. The latter possibility is a novel one peculiar to the brane model. Recall that, in the superstring theory, scalar fields are ubiquitous. Indeed, the dilaton and moduli exists in the bulk...
generically, because they arise as the modes associated with the closed string. Moreover, when the supersymmetry is spontaneously broken, they may have the non-trivial potential. Hence, it is natural to consider the inflationary scenario driven by these fields \[2\]. In this paper, we would like to present a formalism to discuss the inflationary scenario in the braneworld context. We also show the bulk inflaton can, in fact, drive the inflation on the brane.

The organization of this paper is as follows. In sec. 2, we describe the model. In sec. 3, we obtain the effective action using the low energy approximation and discuss its implication. In sec. 4, the action with KK effects are presented. In the final section, we summarize our results.

2. Model and Basic Equations

We consider a \(S_1/Z_2\) orbifold spacetime with the two branes as the fixed points. In this first Randall-Sundrum (RS1) model, the two flat 3-branes are embedded in AdS\(_5\) and the brane tensions given by \(\bar{\sigma} = 6/(\kappa^2 \ell)\) and \(\bar{\sigma} = -6/(\kappa^2 \ell)\). Our system is described by the action

\[
S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \mathcal{R} - \int d^5x \sqrt{-g} \left[ \frac{1}{2} g^{AB} \partial_A \varphi \partial_B \varphi + U(\varphi) \right] - \sum_{i=\oplus, \ominus} \int d^4x \sqrt{-g^{\text{brane}}} + \sum_{i=\oplus, \ominus} \int d^4x \sqrt{-g^{\text{brane}}} L_{\text{matter}}^i, \tag{2}
\]

where \(g^{\text{brane}}_{\mu\nu}\) and \(\sigma\) are the induced metric and the brane tension on the \(i\)-brane, respectively. We assume the potential \(U(\varphi)\) for the bulk scalar field takes the form \(U(\varphi) = -\frac{6}{\kappa^2 \ell^2} + V(\varphi)\), where the first term is regarded as a 5-dimensional cosmological constant and the second term is an arbitrary potential function. The brane tension \(\sigma\) is tuned so that the effective cosmological constant on the brane vanishes. The above setup realizes a flat braneworld after inflation ends and the field \(\varphi\) reaches the minimum of its potential.

Inflation in the braneworld can be driven by a scalar field either on the brane or in the bulk. We derive the effective equations of motion which are useful for both models. In this section, we begin with the single-brane system. Since we know the effective 4-dimensional equations hold irrespective of the existence of other branes \[3\], the analysis of the single-brane system is sufficient to derive the effective action for the two-brane system as we see in the next section.

We adopt the Gaussian normal coordinate system to describe the geometry of the brane model; \(ds^2 = dy^2 + g_{\mu\nu}(y, x^\mu) dx^\mu dx^\nu\), where the brane is assumed to be located at \(y = 0\). Let us decompose the extrinsic curvature into the traceless part \(\Sigma_{\mu\nu}\) and the trace part \(K\) as \(K_{\mu\nu} = -\frac{1}{2} g_{\mu\nu, y} = \Sigma_{\mu\nu} + \frac{1}{2} g_{\mu\nu} K\). Then, we
can obtain the basic equations off the brane using these variables. First, the Hamiltonian constraint equation leads to

\[
\frac{3}{4} K^2 - \Sigma^\alpha_\beta \Sigma^\beta_\alpha = \left( R - \kappa^2 \nabla^\alpha \varphi \nabla_\alpha \varphi + \kappa^2 (\partial_y \varphi)^2 - 2 \kappa^2 U(\varphi) \right),
\]

where \( R \) is the curvature on the brane and \( \nabla_\mu \) denotes the covariant derivative with respect to the metric \( g_{\mu\nu} \). Momentum constraint equation becomes

\[
\nabla_\lambda \Sigma^\lambda_\mu - \frac{3}{4} \nabla_\mu K = -\kappa^2 \partial_y \varphi \partial_\mu \varphi.
\]

Evolution equation in the direction of \( y \) is given by

\[
\Sigma^\mu_\nu, y - K \Sigma^\mu_\nu = -\left[ R_{\mu\nu} - \kappa^2 \nabla^\mu \varphi \nabla_\nu \varphi \right]_{\text{traceless}}.
\]

Finally, the equation of motion for the scalar field reads

\[
\partial_y^2 \varphi - K \partial_y \varphi + \nabla^\alpha \nabla_\alpha \varphi - U'(\varphi) = 0,
\]

where the prime denotes derivative with respect to the scalar field \( \varphi \).

As we have the singular source at the brane position, we must consider the junction conditions. Assuming a \( Z_2 \) symmetry of spacetime, we obtain the junction conditions for the metric and the scalar field

\[
\left[ \Sigma^\mu_\nu - \frac{3}{4} \delta^\mu_\nu K \right]_{y=0} = -\frac{\kappa^2}{2} \sigma \delta^\mu_\nu + \frac{\kappa^2}{2} T^\mu_\nu,
\]

\[
\left[ \partial_y \varphi \right]_{y=0} = 0,
\]

where \( T^\mu_\nu \) is the energy-momentum tensor for the matter fields on the brane.

3. Low Energy Effective Action

We assume the inflation occurs at low energy in the sense that the additional energy due to the bulk scalar field is small, \( \kappa^2 \ell^2 V(\varphi) \ll 1 \), and the curvature on the brane \( R \) is also small, \( R \ell^2 \ll 1 \). It should be stressed that the low energy does not necessarily implies weak gravity on the brane. Under these circumstances, we can use a gradient expansion scheme to solve the bulk equations of motion.

At zeroth order, we ignore matters on the brane. Then, from the junction condition (7), we have

\[
\left[ \Sigma^\mu_\nu - \frac{3}{4} \delta^\mu_\nu K \right]_{y=0} = -\frac{\kappa^2}{2} \sigma \delta^\mu_\nu.
\]
As the right hand side of (9) contains no traceless part, we get \[ \sum_{\mu\nu}^{(0)} K_{\mu\nu} = 0. \] We also take the potential for the bulk scalar field \( U(\varphi) \) to be \( -\frac{6}{\kappa^2 \ell^2} \). We discard the terms with 4-dimensional derivatives since one can neglect the long wavelength variation in the direction of \( x^\mu \) at low energies. Thus, the equations to be solved are given by

\[ \frac{3}{4} K^{(0)} = \kappa^2 (\partial_y(\varphi)^{(0)})^2 + \frac{12}{\ell^2}, \]
\[ \partial_y^{(0)} \varphi^{(0)} - K \partial_y^{(0)} \varphi = 0. \]

The junction condition (8) at this order \[ \left[ \partial_y^{(0)} \varphi \right]_{y=0} \] tells us that the solution of Eq.(11) must be \( \varphi^{(0)} = \eta(x^\mu) \), where \( \eta(x^\mu) \) is an arbitrary constant of integration. Now, the solution of Eq.(11) yields \( K^{(0)} = 4/\ell \). Other Eqs. (4) and (5) are trivially satisfied at zeroth order. Using the definition \( K_{\mu\nu}^{(0)} = -g^{(0)}_{\mu\nu,y}/2 \), we have the lowest order metric

\[ g^{(0)}_{\mu\nu}(y, x^\mu) = b^2(y) h_{\mu\nu}(x^\mu), \quad b(y) \equiv e^{-y/\ell}, \]

where the induced metric on the brane, \( h_{\mu\nu} \equiv g_{\mu\nu}(y = 0, x^\mu) \), arises as a constant of integration. The junction condition for the induced metric (9) merely implies well known relation \( \kappa^2 \sigma = 6/\ell \) and that for the scalar field (8) is trivially satisfied. At this leading order analysis, we can not determine the constants of integration \( h_{\mu\nu}(x^\mu) \) and \( \eta(x^\mu) \) which are constant as far as the short length scale \( \ell \) variations are concerned, but are allowed to vary over the long wavelength scale. These constants should be constrained by the next order analysis.

Now, we take into account the effect of both the bulk scalar field and the matter on the brane perturbatively. Our iteration scheme is to write the metric \( g_{\mu\nu} \) and the scalar field \( \varphi \) as a sum of local tensors built out of the induced metric and the induced scalar field on the brane, in the order of expansion parameters, that is, \( O((R\ell^2)^n) \) and \( O(\kappa^2 \ell^2 V(\varphi))^n, n = 0, 1, 2, \cdots \) \[ 3 \]. Then, we expand the metric and the scalar field as

\[ g_{\mu\nu}(y, x^\mu) = b^2(y) \left[ h_{\mu\nu}(x^\mu) + (1)^{g_{\mu\nu}(y, x^\mu)} + (2)^{g_{\mu\nu}(y, x^\mu)} + \cdots \right], \]
\[ \varphi(y, x^\mu) = \eta(x^\mu) + (1)^{\varphi(y, x^\mu)} + (2)^{\varphi(y, x^\mu)} + \cdots. \]

Here, we put the boundary conditions \( g^{(i)}_{\mu\nu}(y = 0, x^\mu) = 0 \) \( \varphi(y = 0, x^\mu) = 0 \), \( i = 1, 2, 3, \cdots \) so that we can interpret \( h_{\mu\nu} \) and \( \eta \) as induced quantities.
Extrinsic curvatures can be also expanded as
\[ K = \frac{4}{\ell} + K^{(1)} + K^{(2)} + \cdots, \quad \Sigma^\mu_\nu = \Sigma^{(1)}_\mu + \Sigma^{(2)}_\mu + \cdots. \] (14)

Using the formula such as \( (4) \bigl( g^{\mu\nu} \bigr) = R(h_{\mu\nu})/b^2 \), we obtain the solution
\[ K^{(1)} = \frac{\ell}{6b^2} \left( R(h) - \kappa^2 \eta^\alpha \eta_\alpha - \frac{\ell}{3} \kappa^2 V(\eta) \right), \] (15)
where \( R(h) \) is the scalar curvature of \( h_{\mu\nu} \) and \( | \) denotes the covariant derivative with respect to \( h_{\mu\nu} \). Substituting the results at zeroth order solutions into Eq. (5), we obtain
\[ \Sigma^{(1)}_\mu_\nu = \frac{\ell}{2b^2} \left[ R^{(1)}_{\nu}(h) - \kappa^2 \eta^{\mu}_\eta_\nu \right]_{\text{traceless}} + \frac{\chi^{\mu}_\nu}{b^4}, \] (16)
where \( R^{(1)}_{\nu}(h) \) denotes the Ricci tensor of \( h_{\mu\nu} \) and \( \chi^{\mu}_\nu \) is a constant of integration which satisfies the constraint \( \chi^{\mu}_\mu = 0 \). Hereafter, we omit the argument of the curvature for simplicity. Integrating the scalar field equation (4) at first order, we have
\[ \partial_y^{(1)} \varphi = \frac{\ell}{2b^2} \Box \eta - \frac{\ell}{4} V'(\eta) + \frac{C}{b^4}, \] (17)
where \( C \) is also a constant of integration. At first order in this iteration scheme, we get two kinds of constants of integration, \( \chi^{\mu}_\nu \) and \( C \).

Given the matter fields \( T_{\mu\nu} \) on the brane, the junction condition (8) becomes
\[ \left[ \Sigma^{(1)}_\mu_\nu - \frac{3}{4} \delta^{(1)}_\mu K \right] \big|_{y=0} = \frac{\kappa^2}{2} T^{\mu}_\nu. \] (18)

At this order, the junction condition (9) yields
\[ \left[ \partial_y^{(1)} \varphi \right] \big|_{y=0} = 0. \] (19)
These junction conditions give the effective equations of motion on the brane.

The point is the fact that the equations of motion on each brane take the same form if we use the induced metric on each brane [5]. The effective Einstein equations on each positive (⊕) and negative (⊖) tension brane at low-energies yield
\[ G^{\mu}_\nu(h) = \kappa^2 \left( \eta^{\mu}_\eta_\nu - \frac{1}{2} \delta^\mu_\eta \eta^{\alpha}_\eta_\alpha - \frac{1}{2} \delta^\nu_\eta \right) - \frac{2}{\ell} \chi^{\mu}_\nu + \frac{\kappa^2}{\ell} T^{\mu}_\nu, \] (20)
\[ G^{\mu}_\nu(f) = \kappa^2 \left( \eta^{\mu}_\eta_\nu - \frac{1}{2} \delta^\mu_\eta \eta^{\alpha}_\eta_\alpha - \frac{1}{2} \delta^\nu_\eta \right) - \frac{2}{\ell} \chi^{\mu}_\nu - \frac{\kappa^2}{\ell} T^{\mu}_\nu. \] (21)
where \( f_{\mu \nu} \) is the induced metric on the negative tension brane and \( ; \) denotes the covariant derivative with respect to \( f_{\mu \nu} \). When we set the position of the positive tension brane at \( y = 0 \), that of the negative tension brane \( \bar{y} = \bar{y}(x^\mu) \). Hence, the warp factor at the negative tension brane \( \Omega(x^\mu) \equiv b(\bar{y}(x)) \) also depends on \( x^\mu \). Because the metric always comes into equations with derivatives, the zeroth order relation is enough in this first order discussion. Hence, the metric on the positive tension brane is related to the metric on the negative tension brane as \( f_{\mu \nu} = \Omega^2 h_{\mu \nu} \).

Although Eqs. (20) and (21) are non-local individually, with undetermined \( \chi^{\mu \nu} \), one can combine both equations to reduce them to local equations for each brane. We can therefore easily eliminate \( \chi^{\mu \nu} \) from Eqs. (20) and (21), since \( \chi^{\mu \nu} \) appears only algebraically. Eliminating \( \chi^{\mu \nu} \) from both Eqs. (20) and (21), we obtain

\[
G^\nu_\mu = \frac{\kappa^2}{\ell} T^\mu_\nu + \frac{\kappa^2(1 - \Psi)^2}{\ell} T^\mu_\nu + \frac{1}{\Psi}\left[\Psi^{\mu \nu}_\mid - \delta^{\mu}_\nu \Psi^{\alpha \mid}_\mid_\alpha + \frac{3}{2(1 - \Psi)} \left(\Psi^{\mu \nu} - \frac{1}{2} \delta^{\mu}_\nu \Psi^{\alpha \mid}_\mid_\alpha\right)\right] + \frac{\kappa^2}{\ell} \left(\eta^{\mu \nu}_\mid - \frac{1}{2} \delta^{\mu}_\nu \eta^{\alpha \mid}_\mid_\alpha - \delta^{\mu}_\nu V_{\text{eff}}\right), \quad V_{\text{eff}} = \frac{2 - \Psi}{2} V, \tag{22}
\]

where we defined a new field \( \Psi = 1 - \Omega^2 \) which we refer to by the name “radion”. The bulk scalar field induces the energy-momentum tensor of the conventional 4-dimensional scalar field with the effective potential which depends on the radion.

We can also determine the dark radiation \( \chi^{\mu \nu} \) by eliminating \( G^\mu_\nu(h) \) from Eqs. (20) and (21),

\[
\frac{2}{\ell} \chi^{\mu \nu} = -\frac{1}{\Psi}\left[\Psi^{\mu \nu}_\mid - \delta^{\mu}_\nu \Psi^{\alpha \mid}_\mid_\alpha + \frac{3}{2(1 - \Psi)} \left(\Psi^{\mu \nu} - \frac{1}{2} \delta^{\mu}_\nu \Psi^{\alpha \mid}_\mid_\alpha\right)\right] + \frac{\kappa^2}{2} (1 - \Psi) \delta^{\mu}_\nu V - \frac{\kappa^2}{\ell} \left(\bar{T}^\mu_\nu + (1 - \Psi) \bar{T}^\mu_\nu\right) \tag{23}
\]

Due to the property \( \chi^{\mu \nu}_\mid = 0 \), we have

\[
\Box \Psi = \frac{\kappa^2}{3\ell} (1 - \Psi) \left[\bar{T}^\mu + (1 - \Psi) \bar{\Box} T\right] - \frac{1}{2(1 - \Psi)} \Psi^{\alpha \mid}_\mid_\alpha - \frac{2\kappa^2}{3} \Psi (1 - \Psi) V. \tag{24}
\]

Note that Eqs. (22) and (24) are derived from a scalar-tensor type theory coupled to the additional scalar field.

Similarly, the equations for the scalar field on branes become

\[
\Box h \eta - \frac{V'}{2} \ell C = 0, \tag{25}
\]

\[
\Box f \eta - \frac{V'}{2} \ell \Omega^4 = 0, \tag{26}
\]
where the subscripts refer to the induced metric on each brane. Notice that the scalar field takes the same value for both branes at this order. Eliminating the dark source $C$ from these Eqs. (25) and (26), we find the equation for the scalar field takes the form

$$\Box h \eta - V'_{\text{eff}} = -\frac{\Psi|\mu}{\Psi} \eta_{\mu}.$$  \hspace{1cm} (27)

Notice that the radion acts as a source for $\eta$. And we can also get the dark source as

$$\frac{2}{\ell} C = -\frac{V'}{2} (1 - \Psi) + \frac{\Psi|\mu}{\Psi} \eta_{\mu}.$$  \hspace{1cm} (28)

Now the effective action for the positive tension brane which gives Eqs. (22), (24) and (27) can be read off as

$$S = \frac{\ell}{2\kappa^2} \int d^4x \sqrt{-h} \left[ \Psi R - \frac{3}{2(1 - \Psi)} \Psi|\alpha \Psi|\alpha - \kappa^2 \Psi \left( \eta|\alpha \eta_{\alpha} + 2V_{\text{eff}} \right) \right]$$

$$+ \int d^4x \sqrt{-h} \mathcal{L} + \int d^4x \sqrt{-h} (1 - \Psi)^2 \mathcal{L},$$  \hspace{1cm} (29)

where the last two terms represent actions for the matter on each brane. Thus, we found the radion field couples with the induced metric and the induced scalar field on the brane non-trivially. Surprisingly, at this order, the nonlocality of $\chi_{\mu\nu}$ and $C$ are eliminated by the radion.

As this is a closed system, we can analyze a primordial spectrum to predict the cosmic background fluctuation spectrum. Interestingly, $\chi_{\mu}^\mu$ and $C$ vanishes in the single brane limit, $\Psi \to 1$, as can be seen from (23) and (28). The dynamics is simply governed by Einstein theory with the single scalar field. Therefore, we can conclude that the bulk inflaton can drive inflation when the slow role conditions are satisfied.

4. KK corrections

It would be important to take into account the KK effects as corrections to the leading order result. Using our approach, in the single brane limit, we can deduce the effective action with KK corrections as (see also [6] for recent developments)

$$S = \frac{\ell}{2\kappa^2} \int d^4x \sqrt{-h} \left[ \left( 1 + \frac{\ell^2}{12\kappa^2} V \right) R - \kappa^2 \left( 1 + \frac{\ell^2}{12\kappa^2} V - \frac{\ell^2}{4} V'' \right) \eta|\alpha \eta_{\alpha} - 2\kappa^2 V_{\text{eff}} \right.$$

$$- \frac{\ell^2}{4} \left( R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{3} R^2 \right) + \int d^4x \sqrt{-h} \mathcal{L}_{\text{matter}} + S_{\text{CFT}},$$  \hspace{1cm} (30)
where the effective potential at this order is defined by

$$V_{\text{eff}} = \frac{1}{2} V + \frac{\ell^2 \kappa^2}{48} V^2 - \frac{\ell^2}{64} V'^2.$$

(31)

and the last term comes from the energy-momentum tensor of CFT matter $\tau_{\mu \nu}$.

5. Conclusion

We derived the non-linear low energy effective action for the dilatonic braneworld. We considered the bulk scalar field with a nontrivial potential. Then, the constant of integration is determined completely. As a result, the effective theory reduces to the scalar-tensor theory with the non-trivial coupling between the radion and the bulk scalar field. It turns out that $\chi_{\mu \nu}$ and $C$ becomes zero when two branes get separated infinitely. This implies that the bulk inflaton can drive the inflation on the brane as far as the slow role conditions are satisfied. We also obtained KK corrections in the single brane limit.

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