A Trajectory-Based Approach to Discrete-Time Flatness

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Abstract: For discrete-time systems, flatness is usually defined by replacing the time-derivatives of the well-known continuous-time definition by forward-shifts. With this definition, the class of flat systems corresponds exactly to the class of systems which can be linearized by a discrete-time endogenous dynamic feedback as it is proposed in the literature. Recently, checkable necessary and sufficient differential-geometric conditions for this property have been derived. In the present contribution, we make an attempt to take into account also backward-shifts. This extended approach is motivated by the one-to-one correspondence of solutions of flat systems to solutions of a trivial system as it is known from the continuous-time case. If we transfer this idea to the discrete-time case, this leads to an approach which also allows backward-shifts. To distinguish the classical definition with forward-shifts and the approach of the present paper, we refer to the former as forward-flatness. We show that flat systems (in the extended sense with backward-shifts) still share many beneficial properties of forward-flat systems. In particular, they still are reachable/controllable, allow a straightforward planning of trajectories and can be linearized by a dynamic feedback.

Keywords: Differential-geometric methods; discrete-time systems; nonlinear control systems; feedback linearization; difference flatness

1. INTRODUCTION

In the 1990s, the concept of flatness has been introduced by Fliess, Lévine, Martin and Rouchon for nonlinear continuous-time systems (see e.g. Fliess et al. (1992), Fliess et al. (1995) and Fliess et al. (1999)). Flat continuous-time systems have the characteristic feature that all system variables can be parameterized by a flat output and its time derivatives. This leads to a one-to-one correspondence of solutions of a flat system to solutions of a trivial system with the same number of inputs. Flat systems form an extension of the class of static feedback linearizable systems and can be linearized by an endogenous dynamic feedback. Their popularity stems from the fact that many physical systems possess the property of flatness and that the knowledge of a flat output allows an elegant solution to motion planning problems and a systematic design of tracking controllers.

For nonlinear discrete-time systems, flatness is usually defined by replacing the time-derivatives of the well-known continuous-time definition by forward-shifts. More precisely, the flat output is a function of the state variables, input variables, and forward-shifts of the input variables. Conversely, the state- and input variables can be expressed as functions of the flat output and its forward-shifts. This point of view has been adopted in Kaldmäe and Kotta (2013), Sira-Ramirez and Agrawal (2004) and Kolar et al. (2016). With this definition, the class of flat systems corresponds exactly to the class of systems which can be linearized by a discrete-time endogenous dynamic feedback as it is proposed e.g. in Aranda-Bricaire and Moog (2008). Recently, checkable necessary and sufficient differential-geometric conditions have been derived in Kolar et al. (2019b) and Kolar et al. (2019a). Furthermore, in Diwold et al. (2020) it has been shown that in the two-input case even a transformation into a certain normal form is always possible.

In this contribution, we focus solely on the one-to-one correspondence of solutions of flat systems to solutions of a trivial system with the same number of inputs, as it is known from the continuous-time case. For discrete-time systems, this would mean that the flat output may depend both on forward- and backward-shifts of the system variables. Conversely, the state- and input variables could be expressed as functions of both forward- and backward-shifts of the flat output. To distinguish the usual definition of Kaldmäe and Kotta (2013), Sira-Ramirez and Agrawal (2004) and Kolar et al. (2016) with forward-shifts from the alternative approach of the present paper, we refer to the former as forward-flatness. A special case of this alternative definition has already been suggested in Guillot and Millérioux (2020), where the flat output may depend also on backward-shifts of the input variables but not on backward-shifts of the state variables. 1 To justify

1 For systems where the Jacobian matrix of the right-hand side of the system equations with respect to the state variables has full rank, both approaches are equivalent.
our alternative approach, we show that flat systems (in the extended sense with backward-shifts) still share many beneficial properties of forward-flat systems. In particular, we show that they still are reachable (and hence controllable) and allow a straightforward planning of trajectories to connect arbitrary points of the state space. Furthermore, we show that they can be linearized by a dynamic feedback. With respect to the classical dynamic feedback linearization problem, the following inclusions hold: (static feedback linearizable systems) $\subset$ (forward-flat systems) $\subset$ (flat systems) $\subset$ (dynamic feedback linearizable systems). We illustrate by examples that the class of forward-flat systems is a strict subset of the class of flat systems in the alternative sense with backward-shifts.

The paper is organized as follows: In Section 2 we introduce the extended concept of flatness with both forward- and backward-shifts, and illustrate it by examples. In Section 3 we first demonstrate the planning of trajectories and prove the reachability of flat systems. Second, we show that every flat system can be linearized by a certain dynamic feedback, and illustrate this property again by an example.

2. DISCRETE-TIME FLATNESS WITH FORWARD- AND BACKWARD-SHIFTS

Throughout this contribution, we consider discrete-time nonlinear systems in state representation of the form

$$x^{i+} = f^i(x,u), \quad i = 1,\ldots,n$$

with $\dim(x) = n$, $\dim(u) = m$ and smooth functions $f^i(x,u)$. We assume that the systems meet the submersivity condition

$$\rank(\partial_{(x,u)}f) = n,$$  \hspace{1cm} (2)

which is necessary for reachability and consequently also for flatness. Furthermore, we assume

$$\rank(\partial_u f) = m,$$

i.e. that the systems do not have redundant inputs. However, we want to emphasize that we do not require $\rank(\partial_x f) = n$. As mentioned in Aranda-Bricaire and Moog (2008), this property is always met by systems which stem from the exact or approximate discretization of continuous-time systems. However, we want to consider discrete-time systems in general, no matter whether they stem from a discretization or not.

2.1 Equivalence of Solutions

To motivate our trajectory-based approach, we want to recall that a continuous-time system $\dot{x} = f(x,u)$ is flat if there exists a one-to-one correspondence between its solutions $(x(t),u(t))$ and solutions $g(t)$ of a trivial system (sufficiently smooth but otherwise arbitrary trajectories) with the same number of inputs (see e.g. Fliess et al. (1999)).

In the following, we attempt to define flatness for discrete-time systems in exactly the same way. Within this paper, we call a discrete-time system (1) flat if there exists a one-to-one correspondence between its solutions $(x(k),u(k))$ and solutions $g(k)$ of a trivial system (arbitrary trajectories that need not satisfy any difference equation) with the same number of inputs.

$$x^+ = f(x,u)$$ \hspace{1cm} trivial system

$$(x(k),u(k)) \overset{\text{one-to-one}}{\iff} (y(k))$$

By one-to-one correspondence, we mean that the values of $x(k)$ and $u(k)$ at some fixed time step $k$ may depend on an arbitrary but finite number of future and past values of $y(k)$, i.e. on the whole trajectory in an arbitrarily large but finite interval \footnote{Note that the time derivatives in the continuous-time case provide via the Taylor-expansion also information about the trajectory both in forward- and backward-direction.}. Conversely, the value of $y(k)$ at some fixed time step $k$ may depend on an arbitrary but finite number of future and past values of $x(k)$ and $u(k)$. Thus, the one-to-one correspondence of the solutions can be expressed by maps of the form

$$(x(k),u(k)) = F(k,y(k−r_1),\ldots,y(k),\ldots,y(k+r_2))$$ \hspace{1cm} (3)

and

$$y(k) = \varphi(k,x(k−q_1),u(k−q_1),\ldots,x(k),u(k),\ldots,x(k+q_2),u(k+q_2)) \hspace{1cm} (4)$$

These maps must satisfy two conditions. First, in order to ensure the one-to-one correspondence, after substituting (4) into (3) or vice versa the resulting equations must be satisfied identically. Second, since the trajectory $y(k)$ of the trivial system is arbitrary, after substituting (3) into the system equations (1) they also must be satisfied identically. Because of the time-invariance of the system (1), within this paper we only consider maps

$$(x(k),u(k)) = F(y(k−r_1),\ldots,y(k),\ldots,y(k+r_2)) \hspace{1cm} (5)$$

and

$$y(k) = \varphi(x(k−q_1),u(k−q_1),\ldots,x(k),u(k),\ldots,x(k+q_2),u(k+q_2)) \hspace{1cm} (6)$$

which do not depend explicitly on the time step $k$.

Remark 1. The number of forward- and backward-shifts in (5) and (6) can of course be different for the individual components of $y$, $x$ and $u$. Thus, where it is necessary we will use appropriate multi-indices.

It is also important to note that the trajectories $x(k)$ and $u(k)$ are of course not independent. Since (1) must hold at every time step $k$, it is obvious that all forward-shifts $x(k+j)$ with $j \geq 1$ of the state variables are determined by $x(k)$ and the forward-shifts $u(k+j−1)$, $j \geq 1$ of the input variables, i.e.

$$x(k+1) = f(x(k),u(k))$$

$$x(k+2) = f(f(x(k),u(k)),u(k+1)). \hspace{1cm} (7)$$

Thus, the forward-shifts of the state variables in (6) are redundant. A similar argument holds for the backward-direction. Since (1) meets the submersivity condition (2), there always exist $m$ functions $g(x,u)$ such that the map

$$x^+ = f(x,u)$$

$$\zeta = g(x,u) \hspace{1cm} (8)$$

is locally a diffeomorphism and hence invertible\footnote{It should be noted that the choice of $g(x,u)$ is not unique.}. If we denote by $(x,u) = \psi(x^+ , \zeta)$ its inverse

$$x = \psi_u(x^+ , \zeta)$$

$$u = \psi_u(x^+ , \zeta), \hspace{1cm} (9)$$
then all backward-shifts $x(k-j)$ and $u(k-j)$ of the state and input variables with $j \geq 1$ are uniquely determined by $x(k)$ and the backward-shifts $(\zeta(j), j \geq 1)$ of the system variables $\zeta$ defined by (8). This can be seen immediately by a repeated evaluation of (9), which yields
\[
\begin{align*}
(x(k-1), u(k-1)) &= \psi(x(k), \zeta(k-1)) \\
(x(k-2), u(k-2)) &= \psi(x(k), \zeta(k-1)), (\zeta(k-2)).
\end{align*}
\] (10)
Consequently, with (7) and (10) the map (6) can be written as
\[
y(k) = \varphi(\zeta(k-q_1), \ldots, \zeta(k-1), x(k), u(k), \ldots, u(k+q_2)).
\] (11)
We conclude that in the trajectory-based approach the flat output (11) is not only a function of $x, u$ and forward-shifts of $u$, but also a function of backward-shifts of $\zeta$. Thus, it extends the usual definition.

**Remark 2.** It is important to emphasize that the flatness of the system (1) does not depend on the choice of the functions $g(x, u)$. Only the representation (11) of the flat output may differ, while the parameterization (5) of $x$ and $u$ is unaffected. If we would restrict ourselves to sampled data systems with rank($\partial_z f$) $= n$, we would always choose $g(x, u) = u$. This approach leads to a definition of flatness as proposed in Guillot and Millérioux (2020), where the flat output is a function of $x, u$, and forward- and backward-shifts of $u$.

Before we give a precise geometric definition of flatness, we also want to mention that considering both forward- and backward-shifts in the parameterizing map (5) is actually not necessary. Indeed, if there exists a parameterizing map (5) and a flat output (11), then one can always define a new flat output as the $r_{1}$-th backward-shift of the original flat output. The corresponding parameterizing map is then of form
\[
(x(k), u(k)) = F(y(k), \ldots, y(k+r))
\] (12)
with $r = r_1 + r_2$. Thus, without loss of generality, in the remainder of the paper we assume that the parameterizing map (5) is of the form (12) and contains only forward-shifts.

### 2.2 Geometric Approach

In order to give a concise definition of flatness including backward-shifts, we use a space with coordinates $(\ldots, \zeta(-1), x, u, u_1[\ldots])$, where the subscript denotes the corresponding shift. Because of (7) and (10), every point of this space corresponds to a unique trajectory $(x(k), u(k))$ of the system (1). In accordance with (8), we have a forward-shift operator $\delta$ defined by the rule
\[
\begin{align*}
\delta(h(\ldots, \zeta(-2), \zeta(-1), x, u, u_1[\ldots])) &= h(\ldots, \zeta(-1), g(x, u), f(x, u), u_{[1]}, u_{[2]}, \ldots) \\
\end{align*}
\] for an arbitrary function $h$. Because of (9), its inverse is given by the backward-shift operator
\[
\delta^{-1}(h(\ldots, \zeta[-1], x, u, u_1[1], u_2[2], \ldots)) = \\
(h(\ldots, \zeta[-2], \psi_+(x, \zeta[-1]), \psi_-(x, \zeta[-1]), u, u_1[\ldots]).
\]
Likewise, every point of a space with coordinates $(\ldots, y[-1], y, y_1[\ldots]$) corresponds to a unique trajectory $y(k)$ of a trivial system. Here the shift operators have the simple form
\[
\delta_y(H(\ldots, y[-1], y, y_1[\ldots])) = H(\ldots, y, y_1[1], y_2[2], \ldots)
\] and
\[
\delta_{y}^{-1}(H(\ldots, y[-1], y, y_1[\ldots])) = H(\ldots, y[-2], y[-1], y, \ldots).
\]
A $\beta$-fold application of $\delta$ and $\delta_y$, or their inverses will be denoted by $\delta^\beta$ and $\delta^\beta_y$, respectively.

With these preliminaries, we can give a geometric characterization for the trajectory-based approach to discrete-time flatness suggested in Section 2.1. In accordance with the literature on static and dynamic feedback linearization for discrete-time systems, we consider a suitable neighborhood of an equilibrium $x_0 = f(x_0, u_0)$, see e.g. Nijmeijer and van der Schaft (1990) or Aranda-Bricaire and Moog (2008). However, we want to emphasize that for many systems the concept may be useful even if the conditions fail to hold at an equilibrium.

**Definition 3.** The system (1) is said to be flat around an equilibrium $(x_0, u_0)$, if the $n + m$ coordinate functions $x$ and $u$ can be expressed locally by an $m$-tuple of functions
\[
y^j = \varphi^j(\zeta[-q_1], \ldots, \zeta[-1], x, u, \ldots, u_{[q_2]})
\] (13) 
$j = 1, \ldots, m$ and their forward-shifts
\[
y_{[1]}^j = \delta(\varphi(\zeta[-q_1], \ldots, \zeta[-1], x, u, \ldots, u_{[q_2]})) \\
y_{[2]}^j = \delta^2(\varphi(\zeta[-q_1], \ldots, \zeta[-1], x, u, \ldots, u_{[q_2]})),
\] (14) up to some finite order. The $m$-tuple (13) is called a flat output.

If (13) is a flat output, then it can be shown that all forward- and backward-shifts of $\varphi$ are functionally independent. The functional independence of $\ldots, \delta^{-1}(\varphi), \varphi, \delta(\varphi), \ldots$ in turn guarantees that the representation of $x$ and $u$ by the flat output is unique and a submersion of the form
\[
x^i = F^i_2(y^i, \ldots, y_{[r-1]}), \quad i = 1, \ldots, n \\
w^j = F^j_2(y^j, \ldots, y_{[R]}), \quad j = 1, \ldots, m.
\] (15)
A detailed proof using the differential-geometric concepts of exterior derivatives and codistributions can be found in the appendix. The multi-index $R = (r_1, \ldots, r_m)$ contains the number of forward-shifts of each component of the flat output which is needed to express $x$ and $u$. The abbreviation $y_{[R]}$ denotes the components $y_{[R]} = (y_{[r_1]}^1, \ldots, y_{[r_m]}^m)$, and the integer $r$ indicates the maximum number of forward-shifts that appear in the parameterization (15), i.e. $r = \max(r_1, \ldots, r_m)$. If we restrict ourselves to forward-shifts in the flat output, then Definition 3 leads to the special case of forward-flatness.

**Definition 4.** The system (1) is said to be forward-flat, if it meets the conditions of Definition 3 with a flat output of the form $y^j = \varphi^j(x, u, \ldots, y_{[q_2]}).$
The class of forward-flat systems has already been analyzed in detail in the literature, see e.g. Kaldmäe and Kotta (2013), Sira-Ramirez and Agrawal (2004) and Kolar et al. (2016). In Kolar et al. (2019b), it has been shown that every forward-flat system can be decomposed into a smaller dimensional forward-flat subsystem and an endogenous dynamic feedback by a suitable state- and input-transformation. Thus, a repeated decomposition allows to check whether a system is forward-flat or not. In Kolar et al. (2019a), this test has been formulated in terms of certain sequences of distributions, similar to the test for static-feedback linearizability in Nijmeijer and van der Schaf (1990). Thus, the property of forward-flatness can be checked in a computationally efficient way. For flat systems that are not forward-flat, the decomposition procedure as stated in Kolar et al. (2019b) necessarily fails in one step, likewise the test as proposed in Kolar et al. (2019a).

We conclude this section with the special cases of single-input systems and linear systems.

**Theorem 5.** For single-input systems (1) with $m=1$, the properties flatness, forward-flatness, and static feedback linearizability are equivalent.

**Proof.** The implication static feedback linearizability $\Rightarrow$ forward-flatness $\Rightarrow$ flatness follows directly from the corresponding definitions. For the other direction, consider a general flat output

$$ y = \varphi(c_{[-q_1]}^1, \ldots, c_{[-1]}^1, x, u^1, \ldots, u^1_{[q_1]}) $$

(16)
of a system with $m=1$ input. Since the forward-shifts of (16) are independent of $c_{[-q_1]}^1$, (16) would be the only function in the parametrization (15) depending on this variable. Thus, $c_{[-q_1]}^1$ could not cancel out and accordingly (16) itself must not be present in the parametrization (15). Repeating this argumentation shows that (15) can only contain forward-shifts of (16) which are already independent of $c_{[-q_1]}^1, \ldots, c_{[-1]}^1$. However, the first such forward-shift of (16) is obviously a flat-output form

$$ y = \varphi(x, u^1, \ldots, u^1_{[q_1]}). $$

(17)

A similar argumentation shows that (17) can actually only depend on $x$, and $u^1$ only appears in the $n$-th forward-shift. Otherwise, the forward-shifts of $u$ could not cancel out and a parameterization (15) would not be possible. Thus, (17) is a linearizing output in the sense of static feedback linearizability.

For linear systems the properties are also equivalent, since flatness implies reachability and every reachable linear system can be transformed into Brunovský normal form.

**2.3 Examples**

In this section we present two examples that meet the conditions of Definition 3. The first example stems from an exact discretization of a continuous-time $(x, u)$-flat system, and is flat but not forward-flat. In fact, it allows not a single decomposition as proposed in Kolar et al. (2019b), and thus does not meet the corresponding necessary condition for forward-flatness. Accordingly, the test for forward-flatness stated in Kolar et al. (2019a) fails in the first step. Hence, the example proves that the class of forward-flat systems is a strict subset of the class of flat systems.

**Example 1.** Consider the sampled data system

$$
\begin{align*}
x^{1+} &= x^1 + Tu^1 \\
x^{2+} &= x^2 + Tu^2 \\
x^{3+} &= x^3 + Tu^1 u^2
\end{align*}
$$

for some sampling time $T > 0$. Since the system meets \( \text{rank}(\partial_x f) = n \), we can choose $c_j^1 = q_j^1(x, u) = u^j$ for $j = 1, 2$ such that the combined map (8) forms a diffeomorphism. We claim that the system has a flat output of the form

$$
y = (x^1 - Tc_{[-1]}^1, x^3 - 2x^2 c_{[-1]}^1).$$

(19)

In order to prove that the system is flat, we need to show that $x$ and $u$ can be expressed by (19) and its forward-shifts. A sequential application of the shift operators to (19) yields the set of equations

$$
\begin{align*}
y^1 &= x^1 - Tc_{[-1]}^1 \\
y^2 &= x^3 - x^2 c_{[-1]}^1 \\
y^3[1] &= x^1 \\
y^3[2] &= x^3 + Tu^1 u^2 \\
y^3[3] &= x^3 + Tu^1 u^2 - y^3[1] \\
y^3[4] &= x^1 + Tu^1 (u^1 + u^1_{[1]}) u^1_1(x^2 + Tu^2),
\end{align*}
$$

which can be solved for $c_{[-1]}^1, x^1, x^2, x^3, u^1, u^2$ and $u^1_1$.

The function $F^2_0(y, y_3)$ is too lengthy to present it here explicitly. We conclude that the system (18) is flat with a flat output (19) and the corresponding parameterization (15) contained in (20).

In our next example, we again consider a system that is flat but not forward-flat. In contrast to the system of Example 1, it does not stem from any discretization and does not meet rank($\partial_x f$) = $n$. Hence, the choice $q(x, u) = u$ as stated in Remark 2 is not possible, since (8) would not result in a diffeomorphism.

**Example 2.** Consider the system

$$
\begin{align*}
x^{1+} &= u^1 \\
x^{2+} &= u^2 \\
x^{3+} &= x^3 + x^1 u^2 + x^2 u^1
\end{align*}
$$

(21)

With the choice $c_j^1 = q_j^1(x, u) = x^j$ for $j = 1, 2$, the combined map (8) forms a diffeomorphism and we claim that the system has a flat output of the form

$$
y = (c_{[-1]}^1, x^3 - x^2 c_{[-1]}^1).$$

(22)

In order to prove that the system is flat, we need to show that $x$ and $u$ can be expressed by (22) and its forward-shifts. A repeated application of the shift operators to (22) yields the set of equations

$$
\begin{align*}
y^1 &= c_{[-1]}^1 \\
y^2 &= x^3 - x^2 c_{[-1]}^1 \\
y^3[1] &= x^1 \\
y^3[2] &= x^3 + x^2 u^1 \\
y^3[3] &= x^3 + x^2 u^1 + u^2 (x^1 + u^1_{[1]}), \\
y^3[4] &= u^1_{[1]}
\end{align*}
$$
which can be solved for \( \zeta_{[-1]}^1, x^1, x^2, x^3, u^1, u^2 \) and \( u_{[1]}^1 \),

\[
\begin{align*}
\zeta_{[-1]}^1 &= y^1 \\
x^1 &= y_{[1]}^1 \\
x^2 &= y_{[2]}^2 - y^2 \\
x^3 &= \frac{y_{[2]}^2 + y_{[2]}^2 y^2}{y^2 + y_{[2]}^2} \\
\end{align*}
\]

(23)

Again we conclude that the system (21) is flat with a flat output and the corresponding parameterization (15) contained in (23).

3. TRAJECTORY PLANNING AND DYNAMIC FEEDBACK LINEARIZATION

In this section, we show that flat systems (in the extended sense with backward-shifts) still allow straightforward trajectory planning and dynamic feedback linearization.

3.1 Trajectory Planning

The popularity of differentially flat systems is mainly due to the fact that the knowledge of a flat output allows an elegant solution to motion planning problems. In this section, we show that also discrete-time flat systems according to Definition 3 allow a straightforward planning of trajectories.

Usually the motion planning problem consists in finding trajectories \((x(k), u(k))\) that satisfy the system equations (1) and some initial and final conditions

\[
\begin{align*}
(x(k_i), u(k_i)) &= (x_i, u_i) \\
(x(k_f), u(k_f)) &= (x_f, u_f),
\end{align*}
\]

with \( k_f > k_i \). For flat systems, this task can be formulated in terms of trajectories \( y(k) \) for the flat output. Every trajectory \( y(k) \) corresponds to a solution of (1), it remains to require that the trajectory \( y(k) \) meets

\[
\begin{align*}
(x_i, u_i) &= F(g(k_i), y(k_i + 1), \ldots, y(k_i + r)) \\
(x_f, u_f) &= F(g(k_f), y(k_f + 1), \ldots, y(k_f + r)).
\end{align*}
\]

(24)

If we assume that \( k_f > k_i + r \) holds, then since the parameterization (15) is a submersion, the set of equations (24) can be solved independently for 2\((n+m)\) values of \( y(k_i), \ldots, y(k_f + r) \).\(^6\) The remaining values of \( y(k_i), \ldots, y(k_f + r) \) can be chosen arbitrarily, and thus the trajectories \( y(k) \) are in general not unique.\(^7\) Once the trajectories \( y(k) \) are determined, the corresponding state- and input-trajectories are also uniquely determined by

\[
(x(k), u(k)) = F(g(k), y(k + 1), \ldots, y(k + r)),
\]

for \( k = k_i, \ldots, k_f \). Since this procedure allows to connect any two points of the state space, we conclude that flatness according to Definition 3 implies the reachability of (1).

\(^6\) For certain parameterizations the assumption \( k_f > k_i + r \) may be relaxed. It would be sufficient to require that the integer \( k_f \) is large enough, such that (24) can still be solved for arbitrary 2\((n+m)\) values of the set 0, 1, \ldots, \( y(k_f + r) \).

\(^7\) Like in the continuous-time case, this property can be very beneficial in optimal control problems, e.g. minimizing control effort.

3.2 Dynamic Feedback Linearization

In the continuous-time framework, flatness is closely related to the dynamic feedback linearization problem. To be precise, the class of differentially flat systems is equivalent to the class of systems linearizable via endogenous dynamic feedback. A continuous-time dynamic feedback \( \dot{z} = \alpha(x, z, v) \) with \( u = \beta(x, z, v) \) is said to be endogenous, if there exists a one-to-one correspondence between trajectories of the closed-loop system and trajectories of the original system. As a consequence, \( z, v \) can be expressed as functions of \( x, u \) and time derivatives of \( u \).

According to Aranda-Bricaire and Moog (2008), a discrete-time dynamic feedback is said to be endogenous, if its states \( z \) and inputs \( v \) can be expressed as functions of \( x, u \) and forward-shifts of \( u \). It can be shown that the class of discrete-time systems that is linearizable via endogenous dynamic feedback in the sense of Aranda-Bricaire and Moog (2008) exactly corresponds to the class of forward-flat systems. In this section, we show that also for flat systems according to Definition 3 there always exists a linearizing discrete-time dynamic feedback

\[
\begin{align*}
z^+ &= \alpha(x, z, v) \\
u &= \beta(x, z, v).
\end{align*}
\]

(25)

However, in general the required feedback (25) is not contained within the class of endogenous dynamic feedbacks proposed in Aranda-Bricaire and Moog (2008). In the following, we first illustrate how such a dynamic feedback can be constructed, and show that the resulting closed loop system is transformable into Brunovsky normal form afterwards.

The fact that the parameterizing map (15) is a submersion implies that also the parameterization \( F_z \) is a submersion. Consequently, there exists a map \( z = F_z(y, \ldots, y_{[R-1]}) \), such that the combined map \( (x, z) \) = \( (F_x, F_z) \) forms a diffeomorphism with \( \dim(z) = p \leq m-r \). The choice of \( F_z \) is not unique, and thus also the resulting feedback will not be unique in general. In the next step, we define the map \( \Phi(y, \ldots, y_{[R]}) \) given by

\[
\begin{align*}
x &= F_x(y, \ldots, y_{[R-1]}) \\
\Phi : z &= F_z(y, \ldots, y_{[R-1]}) \\
v &= y_{[R]}
\end{align*}
\]

(26)

and its inverse \( \Phi(x, z, v) \) given by

\[
\begin{align*}
\Phi : (y, \ldots, y_{[R-1]}) &= F_z(x, z) \\
y_{[R]} &= v.
\end{align*}
\]

(27)

Based on (27), we claim that every flat system can be linearized by a dynamic feedback of the form

\[
\begin{align*}
z^+ &= \delta_y(F_z) \circ \Phi(x, z, v) \\
u &= F_u \circ \Phi(x, z, v)
\end{align*}
\]

(28)

In order to prove this statement, we need to show that the closed loop dynamics

\[
\begin{align*}
x^+ &= f(x, F_x \circ \Phi(x, z, v)) \\
z^+ &= \delta_y(F_z) \circ \Phi(x, z, v)
\end{align*}
\]

(29)

is transformable into Brunovsky normal form by a suitable state- and input-transformation. For this purpose, we use the state-transformation \( (x, z) = F_{xz}(y, \ldots, y_{[R-1]}) \) and the input-transformation \( v = y_{[R]} \). Applying the transformations yields
\[(y^+ \ldots y_{[R-1]}^+)^\circ \Phi \circ \delta \circ \Phi = f(x, F_y \circ \Phi(x, z, v))\],

which can be rewritten as

\[(y^+ \ldots y_{[R-1]}^+)^\circ \Phi \circ \delta \circ \Phi = f(F_x, F_u)\]

(30)
The parameterization (15) satisfies the system equations identically, hence by substituting \[F \circ \Phi = f(x, u)\] we get the relation \[\delta \circ \Phi = f(F_x, F_u)\]. Thus, we may rewrite (30) as

\[(y^+ \ldots y_{[R-1]}^+)^\circ \Phi \circ \delta \circ \Phi = f(F_x, F_u)\]

Since \(\hat{F}_{xx} \circ \delta \circ \Phi = \delta \circ \Phi \circ (F_x, F_u)\), and per definition \(\hat{F}_{xx} \circ (F_x, F_u)\) yields identically \((y^+ \ldots y_{[R-1]}^+)\), the Brunovsky normal form follows as

\[(y^+ \ldots y_{[R-1]}^+) = \delta \circ \Phi \circ (y^+ \ldots y_{[R-1]}^+) = (y_{[1]} \ldots y_{[R]}^+)\]

Like a continuous-time endogenous dynamic feedback, the dynamic feedback (28) has the property that the trajectories of the closed-loop system are in one-to-one correspondence to the trajectories \((x(k), u(k))\) of the original system (1). This follows from the fact that with \((26), z\) and \(v\) can be expressed in terms of the flat output and its forward-shifts \((y^+ \ldots y_{[R]}^+)\), which in turn can be expressed by forward- and backward-shifts \((\ldots, z_{[1]}, x, u_{[1]}, \ldots)\) of variables of the original system. In other words, there exists a one-to-one correspondence between the trajectories of the closed-loop system and the trajectories of the trivial system. By the definition of flatness, the same holds for the trivial system and the original system. It should also be noted that the dynamic feedback (28) preserves reachability and submersivity. This is of course obvious, since the closed-loop system can be transformed into Brunovsky normal form.

Remark 6. In the classical dynamic feedback linearization problem, the one-to-one correspondence between trajectories of the closed loop system and the original system is not required. Thus, the linearizing output of the closed loop system can possibly not be expressed in terms of forward- and backward-shifts of the original system variables.

We conclude this section with an example that illustrates the construction of a linearizing dynamic feedback for a flat system.

Example 3. Consider again the system of Example 1 with the parameterizing map (20). If we choose the map \(z = F_z\) as

\[\begin{align*}
z^1 &= y^1 \\
z^2 &= y^2_1,
\end{align*}\]

then by evaluating (28) we derive the dynamic feedback

\[\begin{align*}
z_{1^+} &= x^1 \\
z_{2^+} &= v^1 \\
u^1 &= \frac{z^2 - z^1}{y} \\
u^2 &= \frac{(x^2 - x^1)z - 2z^2(y^1 - z^2)}{(x^2 - z^2 + v^1 y)}.
\end{align*}\]

By transforming the closed-loop system using the state-transformation \((x, z) = (F_x, F_z)\) and the input-transformation \(v = y_{[R]}\), we obtain the Brunovsky normal form

\[y^1_{[1]} = y^2_{[1]} \quad y^2_{[1]} = y^2_{[2]} \quad y^1_{[2]} = y^3_{[1]} \quad y^2_{[2]} = y^3_{[2]} \quad y^1_{[3]} = y^3_{[3]}\]

4. CONCLUSION

In this contribution, we have investigated the extension of the notion of discrete-time flatness to both forward- and backward-shifts. We have shown that adding backward-shifts fits very nicely with the concept of one-to-one correspondence of solutions of the original system and a trivial system, as it is well-known from the continuous-time case. Even with backward-shifts, reachability and controllability still hold and trajectories can be planned in a straightforward way. Furthermore, such systems still can be linearized by a dynamic feedback. Thus, from an application point of view, the basic properties of forward-flat systems are preserved. Since we expect that the class of flat systems in the extended sense including backward-shifts is significantly larger than the class of forward-flat systems, this opens many new perspectives for practical applications.

Future research will deal with the systematic construction of flat outputs and finding necessary and/or sufficient conditions as they already exist for forward-flat systems. Another open question, which is motivated by the continuous-time case, is whether the class of flat systems is only a subset of or equivalent to the class of systems linearizable by dynamic feedback, see Remark 6.

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Appendix A. PROPERTIES RESULTING FROM DEFINITION 3

In this section, we prove that the functional independence of all forward- and backward-shifts of a flat output \( \ldots, \delta^{-1}(\varphi), \varphi, \delta(\varphi), \ldots \), as well as the property that (15) is a submersion, are direct consequences of Definition 3. Furthermore, we show that the special structure that \( F_x \) is independent of \( y_{[R]} \) follows from the fact that the parameterization (15) needs to satisfy the system equations (1) identically.

By definition, the states \( x \) and inputs \( u \) can be expressed as functions of \( \varphi, \delta(\varphi), \ldots, \delta^{r}(\varphi) \). Thus, there exist some functions \( F(y, \ldots, y_{[R]}) \) such that after substituting (14) the identities
\[
x = F_x(\varphi, \delta(\varphi), \ldots, \delta^{r}(\varphi)) \tag{A.1}
\]
hold. By taking the exterior derivative, relation (A.1) can be written as
\[
\text{span}\{dx, du\} \subset \text{span}\{d\varphi, \delta d(\varphi), \ldots, \delta^r d(\varphi)\}.
\]

A sequential application of the shift operator \( \delta \) to \( u = F_u(\varphi, \delta(\varphi), \ldots, \delta^r(\varphi)) \) of (A.1), yields the identities
\[
u_{[\beta]} = F_u(\delta(\varphi), \ldots, \delta^{r+\beta}(\varphi)), \tag{A.2}
\]
for arbitrary forward-shifts of \( u \) with \( \beta > 0 \). If we consider the codistribution \( \text{span}\{dx, du, \ldots, du_{[\beta]}\} \) for an arbitrary integer \( \beta \), then from (A.2) we get the relation
\[
\text{span}\{dx, du, \ldots, du_{[\beta]}\} \subset \text{span}\{d\varphi, \delta d(\varphi), \ldots, \delta^{r+\beta}(\varphi)\}. \tag{A.3}
\]

Since the forward-shifts \( u_{[\beta]} \), with \( \text{dim}(u_{[\beta]}) = \text{dim}(u) = m \), are independent coordinates, the dimension of \( \text{span}\{dx, du, \ldots, du_{[\beta]}\} \) grows by \( m \) for each increment of \( \beta \). Consequently, the dimension of \( \text{span}\{d\varphi, \delta d(\varphi), \ldots, \delta^{r+\beta}(\varphi)\} \) must grow by \( m \) for each increment of \( \beta \). If in some step the dimension of \( \text{span}\{d\varphi, \ldots, \delta^{r+\beta}(\varphi)\} \) would only grow by \( m < m \), then also in all following steps the dimension could only grow at most by \( m \). However, this would necessarily result in a contradiction to (A.3). Therefore, the condition
\[
\text{dim}(\text{span}(d\varphi, \delta d(\varphi), \ldots, \delta^{r}(\varphi))) = (\beta + 1)m
\]
must hold for arbitrary \( \beta \) which means that the functions \( \varphi, \delta(\varphi), \ldots, \delta^{r}(\varphi) \) are functionally independent. A similar argument holds for the backward-shifts of \( \varphi \). With \( F_{\zeta} := g \circ F(y, \ldots, y_{[R]}) \) we have the identities
\[
\zeta = F_{\zeta}(\varphi, \ldots, \delta^{r}(\varphi)).
\]

By a sequential application of the backward-shift operator \( \delta^{-1} \), we derive the identities
\[
\zeta_{[-\alpha]} = F_{\zeta}(\delta^{-\alpha}(\varphi), \ldots, \delta^{-r-\alpha}(\varphi)). \tag{A.4}
\]

for arbitrary backward-shifts of \( \zeta \) with \( \alpha > 0 \). If we consider the codistribution \( \text{span}\{d\zeta_{[-\alpha]}, \ldots, d\zeta_{[-1]}, dx, du, \ldots, du_{[\beta]}\} \), then we get the relation
\[
\text{span}\{d\zeta_{[-\alpha]}, \ldots, d\zeta_{[-1]}, dx, du, \ldots, du_{[\beta]}\} \subset \text{span}\{\delta^{-\alpha}(\varphi), \ldots, \delta^{r+\beta}(\varphi)\}.
\]

Since the backward-shifts \( \zeta_{[-\alpha]} \), with \( \text{dim}(\zeta_{[-\alpha]}) = \text{dim}(\varphi) = m \), are independent coordinates, the dimension of \( \text{span}\{d\zeta_{[-\alpha]}, \ldots, d\zeta_{[-1]}, dx, du, \ldots, du_{[\beta]}\} \) grows by \( m \) for each increment of \( \alpha \). Consequently also the dimension of \( \text{span}\{\delta^{-\alpha}(\varphi), \ldots, \delta^{r+\beta}(\varphi)\} \) must grow by each increment of \( \alpha \). From
\[
\text{dim}(\text{span}(\delta^{-\alpha}(\varphi), \ldots, \delta^{r}(\varphi))) = (\alpha + \beta + 1)m,
\]
we conclude that the functions \( \delta^{-\alpha}(\varphi), \ldots, \delta^{r+\beta}(\varphi) \) for arbitrary \( \alpha, \beta > 0 \), are functionally independent, which entails the uniqueness of (15).

Next, we prove that the parameterizing map (15) is indeed a submersion. By taking the exterior derivative of identity (15) we get the relation
\[
\text{span}(dx, du) = \text{span}\{\delta \circ F^i \circ \varphi, \delta d(\varphi), \ldots, \delta^{r+\beta}(\varphi)\}, \tag{A.5}
\]
for \( i = 1, \ldots, n + m \). Since \( \text{dim}(\text{span}(dx, du)) = n + m \), the identity (A.5) can only be satisfied if the rows of the Jacobian-matrix \( \delta \circ F^i \), \( i = 1, \ldots, n + m \), are linearly independent. Hence, the map (15) must be a submersion.

Finally, we show that the special structure that the parameterization of the states is independent of \( y_{[R]} \) is a consequence of the identity
\[
\delta(x^i) = F^i(x, u), \quad i = 1, \ldots, n. \tag{A.6}
\]
If we substitute the parameterization into (A.6) we get
\[
\delta_y(F^i(y, \ldots, y_{[R-1]})) = F^i \circ F(y, \ldots, y_{[R]}),
\]
which can only be satisfied if \( F_x \) is independent of \( y_{[R]} \). Otherwise, \( \delta_y(F_x) \) would depend on \( (R + 1) \)-th forward-shifts of \( y \).