Semiclassical description of nonlinear electron-positron photoproduction in strong laser fields

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Abstract. The nonlinear Breit-Wheeler process is studied in the presence of strong and short laser pulses. We show that for a relativistically intense plane-wave laser field many aspects of the momentum distribution for the produced electron-positron pair like its extend, region of highest probability and carrier-envelope phase effects can be explained from the classical evolution of the created particles in the background field. To this end we verify that the local constant-crossed field approximation is also appropriate for the calculation of the spectrum if applied on the probability-amplitude level. To compare the exact expressions with the semiclassical approach, we introduce a very fast numerical scheme, which makes it feasible to completely resolve the interference structure of the spectrum over the available multidimensional phase space.

Keywords: nonlinear QED, strong laser fields, Breit-Wheeler pair production, short-pulse effects

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1. Introduction

Due to the continuous improvement of laser technology the experimental observation of the nonlinear Breit-Wheeler process (see Fig. 1) is now within reach, e.g. at upcoming high-power laser facilities like CLF, ELI and XCELS [4–6]. Therefore, this process has recently been considered by many authors [7–23] (see also the reviews [24, 25]).

The decay of a photon into an electron-positron pair is an intrinsic quantum process, which has no classical analogue and must be described in the realm of quantum field theory using the $S$-matrix approach [24, 26]. This implies that we can typically determine only the probability distribution for the asymptotic momenta and all details of the actual production process are hidden. However, it is well known that inside a plane-wave laser field with electric field amplitude $E_0$ and central angular frequency $\omega$ the formation region of the basic QED processes nonlinear Compton scattering and nonlinear Breit-Wheeler pair production in the ultra-relativistic regime $\xi \gg 1$ are $\xi$-smaller than the laser period. Here, $\xi = |e| E_0 / (m \omega c)$ is the classical intensity parameter, with $e < 0$ and $m$ denoting the electron charge and mass, respectively [24, 26]. Hence, the total probability for nonlinear Compton scattering and nonlinear Breit-Wheeler pair production can be calculated by applying the local constant-crossed field approximation, i.e. by averaging the corresponding probability in a constant-crossed field over the laser pulse [3, 24, 26–31]. As pointed out by Ritus [26], this procedure is justified for the calculation of the total probabilities but it may not work for differential probabilities, i.e. for determining the momentum distribution of the final particles (this has also been recently observed numerically in [32] for nonlinear Compton scattering). In this case, in fact, interference effects arising from processes occurring at different space-time points, which are neglected from the beginning when the averaging procedure is applied to the probabilities, can play an important role [26]. Nevertheless, this is the state of the art approach for the implementation of strong-field QED processes in plasma codes [33–35]. Therefore, it is desirable to estimate the error of this procedure and to show how the standard approach could be extended if necessary.

![Figure 1. Leading-order Feynman diagram for electron-positron photoproduction inside a plane-wave background field (nonlinear Breit-Wheeler process). The double lines represent Volkov states (solutions of the interacting Dirac equation, which take the plane-wave background field into account exactly [1, 2]), the wiggly line the incoming photon. As long as the total pair-production probability is much smaller than unity, the spectrum for the final particles is determined to a good accuracy by simply evaluating this diagram and neglecting radiative corrections [3].](image_url)
By applying a stationary-phase analysis to the leading-order $S$-matrix element for electron-positron photoproduction we show here that for $\xi \gg 1$ all significant features of the momentum distribution for the created electron-positron pair are obtainable from the following three-step procedure: at each laser phase the pair-production probability amplitude is calculated using the local constant-crossed field approximation. Then, the asymptotic momenta for the electron and the positron are obtained from the classical equations of motion. Finally, the probability for the production of a pair with given asymptotic momenta is obtained by squaring the probability amplitude, taking the interference between pairs which have the same asymptotic momenta but originating from different formation regions into account. An analogous analysis has been carried out in [36–38] for explaining various features of the emission spectra for nonlinear Compton scattering.

We point out that electron-positron photoproduction has a lot of commonalities with laser-induced ionization processes. In fact, the procedure outlined above is closely related to similar approaches used in atomic physics to describe the time evolution of an electron after tunnel ionization [39, 40].

As a laser field is oscillatory, each cycle typically contributes two interference paths to the final probability amplitude. Therefore, nonlinear Breit-Wheeler pair production can also be interpreted as an all-optical multi-slit experiment (for the importance of interference effects in Schwinger pair production see e.g. [41]). This intuitive picture provides a clear explanation of many effects, e.g., the strong dependence of the spectrum on the carrier-envelope phase (CEP) for ultra-short laser pulses reported in [15], and it has been exploited in [38] to put forward a scheme for determining the CEP in ultra-short and ultra-intense laser beams via nonlinear single-Compton scattering.

To establish the validity of the outlined semiclassical approach, we compare it with a full numerical calculation of the leading-order $S$-matrix element. In this way we show that already for $\xi \gtrsim 5$ the interference structure obtained from the local constant-crossed field approximation applied on the probability amplitude level is in very good agreement with full numerical predictions.

As the integrals are highly oscillating in the regime $\xi \gg 1$, the numerical calculation is, in principle, a challenging task [14, 15]. We present here a new scheme, which is substantially more efficient than other employed methods. Hence, it becomes feasible to evaluate the three-dimensional differential probability on a grid which is fine enough to completely resolve the interference structure of the spectrum. For nonlinear Compton scattering the final phase space has been studied in [42–45] for moderate values of the parameter $\xi$ ($\xi \lesssim 10$). The numerical computation of total probabilities can also be significantly simplified by employing the method proposed in [46].

From now on we use natural units $\hbar = c = 1$ and Heaviside-Lorentz units for charge ($\alpha = e^2/(4\pi) \approx 1/137$ denotes the fine-structure constant), the notation agrees with [3, 47].
2. Pair production probability

We consider a plane-wave laser field described by the field tensor $F^{\mu\nu}(\phi) = f_1^{\mu\nu} \psi_1'(\phi) + f_2^{\mu\nu} \psi_2'(\phi)$, where $\phi = kx$ is the laser phase, $f_1^{\mu\nu} = k^\mu a_i^\nu - k^\nu a_i^\mu$ ($i = 1, 2$), $k^\mu$ is the typical four-momentum of the laser photons and $a_i^\mu$ and $|\psi_i'(\phi)| \lesssim 1$ characterize the field strength and shape along the two possible polarization directions, respectively [a prime denotes the derivative with respect to the argument, we require $\psi_i'(\pm\infty) = 0$ and introduce the classical intensity parameters $\xi_i = \sqrt{-a_i^2 |e|/m}$].

To leading-order the $S$-matrix element for the decay of a photon with four-momentum $q^\mu$ ($q^2 = 0$) and polarization four-vector $e^\mu$ into an electron and a positron with four-momenta $p_1^\mu$ and $p_2^\mu$ ($p_i^2 = m^2$), respectively, can be written as (see Fig. 1 and e.g. [3])

$$W(q) = \int_0^{+1/2} dR \int_{-1/2}^{+\infty} dt_1 dt_2 \frac{d^3W}{dR dt_1 dt_2},$$

where $iM(p_1, p_2; q) = \epsilon_\mu \bar{u}_{p_1} \mathcal{G}^\mu(p_1, q, -p_2) u_{p_2}$ is the reduced matrix element for the process\footnote{Note that in Eq. (3) and Eq. (A19) of [3] the $i$ is erroneously missing.} and where the sum over “spin” indicates the usual sum with respect to the final spin quantum numbers of the produced particles. Here, $u_{p_1}$ and $v_{p_2}$ denote the Dirac spinors for the electron and the positron, respectively, and $\mathcal{G}^\mu$ the nonsingular part of the laser-dressed vertex (see [3] for more details). For the sake of notational simplicity, the spin quantum numbers are not indicated. In Eq. (1) we excluded the notational trivial case $kq = 0$ and expanded the four-momenta using the following light-cone basis (see [26] and Appendix B)

$$p_1^\mu = r' q^\mu + s' k^\mu + t'_1 m A_1^\mu + t'_2 m A_2^\mu,$$

$$p_2^\mu = -r q^\mu - s k^\mu - t_1 m A_1^\mu - t_2 m A_2^\mu,$$

where $t'_i = t_i$ and $r' = r + 1$ due to momentum conservation, $R = r + 1/2 = (kp_1 - kp_2)/(2kq) = (kp_1 - kp_2)/(2kp_1 + kp_2)$, $w = -1/[r(r + 1)] = 4/(1 - 4R^2)$ and $A_i^\mu = f_i^{\mu\nu} q_\nu/(kq \sqrt{-a_i^2})$. The light-cone basis $q^\mu$, $k^\mu$, $A_1^\mu$, and $A_2^\mu$ fulfills the completeness relation

$$g^{\mu\nu} = \frac{1}{kq} (q^\mu k^\nu + k^\mu q^\nu) - \Lambda_1^\mu \Lambda_1^\nu - \Lambda_2^\mu \Lambda_2^\nu,$$

with $g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ denoting the metric. Furthermore, the quantities

$$s = \frac{1}{2r} \frac{m^2}{kq} (1 + t_1^2 + t_2^2), \quad s' = \frac{1}{2r'} \frac{m^2}{kq} (1 + t_1^2 + t_2^2)$$

(4)
are determined by the on-shell conditions \( p_1^2 = p_2^2 = m^2 \). Correspondingly, the amount \( n \) of absorbed laser photons is related to the introduced Lorentz invariant momentum parameters as follows

\[
p_1^\mu + p_2^\mu = q^\mu + nk^\mu, \quad n = \frac{1}{2} w \frac{m^2}{kq} (1 + t_1^2 + t_2^2). \tag{5}
\]

For a head-on collision the parameters \( t_i \) are related to the transverse momentum of the created pair, whereas for electrons and positrons created ultra-relativistic the parameter \( R \) is half the asymmetry of the asymptotic energies of the electron and the positron.

To determine the differential pair-production probability from Eq. (1) we note that

\[
\sum_{\text{spin}} |\mathcal{M}(p_1, p_2; q)|^2 = -\epsilon_{\mu} \epsilon_{\nu} \text{tr} \mathcal{G}^\mu(p', q, p)(\not p + m) \mathcal{G}^\nu(p', q, p)(\not p' + m) \tag{6}
\]

[here and in the following \( p_1^\mu = p'^\mu, \ p_2^\mu = -p'^\mu \). Furthermore, the nonsingular part of the dressed vertex can be written as (for on-shell momenta, see [47] for more details)

\[
\mathcal{G}^\nu(p', q, p) = (-ie) \left\{ \gamma_\mu \left[ \mathfrak{G}_0 g^{\mu\nu} + \sum_{j=1,2} (G_1 \mathfrak{G}_{j,1} f_j^{\mu\nu} + G_2 \mathfrak{G}_{j,2} f_j^{2\mu\nu}) \right] + i\gamma_\mu \gamma_5 \sum_{j=1,2} G_3 \mathfrak{G}_{j,1} f_j^{s\mu\nu} \right\}, \tag{7}
\]

where we defined the master integrals \( \mathfrak{G}_0 = \mathfrak{G}_0(w, t_1, t_2) \) and \( \mathfrak{G}_{j,t} = \mathfrak{G}_{j,t}(w, t_1, t_2) \)

\[
\mathfrak{G}_0 = \int_{-\infty}^{+\infty} d\phi \ e^{i\tilde{S}_\Gamma(t_1, t_2; \phi)}, \quad \mathfrak{G}_{j,t} = \int_{-\infty}^{+\infty} d\phi \ [\psi_j(\phi)]^\dagger e^{i\tilde{S}_\Gamma(t_1, t_2; \phi)}. \tag{8}
\]

The nontrivial part of the field-dependent phase \( \tilde{S}_\Gamma(t_1, t_2; \phi) = (w/2)(m^2/kq)\mathfrak{G}_\Gamma(t_1, t_2; \phi) \) is given by

\[
\mathfrak{G}_\Gamma(t_1, t_2; \phi) = (1 + t_1^2 + t_2^2)\phi + \sum_{i=1,2} \int_\phi^{\phi'} d\phi' \left[ \xi_i^2 \psi_i^2(\phi') - 2t_i \xi_i \psi_i(\phi') \right]. \tag{9}
\]

Furthermore, after integrating by parts and neglecting the boundary term, we obtain

\[
\mathfrak{G}_0(w, t_1, t_2) = -\frac{1}{2n} \frac{m^2}{kq} w \sum_{i=1,2} \left[ \xi_i^2 \mathfrak{G}_{i,2}(w, t_1, t_2) - 2t_i \xi_i \mathfrak{G}_{i,1}(w, t_1, t_2) \right]. \tag{10}
\]

From Eq. (8) we see that the master integrals have a very simple dependence on \( w \), implying

\[
\int_{-\infty}^{+\infty} dw \ e^{-i\frac{1}{2} w \frac{m^2}{kq} x^2} \mathfrak{G}_{j,t}(w, t_1, t_2) = 4\pi \frac{kq}{m^2} \frac{[\psi_j(\phi_\hat{x})]^\dagger}{\mathfrak{G}_\Gamma(t_1, t_2; \phi_\hat{x})}, \tag{11}
\]

where the prime denotes the derivative with respect to the laser phase and \( \phi_\hat{x} \) is the (unique) solution of the equation \( \mathfrak{G}_\Gamma(w, t_1, t_2; \phi_\hat{x}) = \hat{x} \). The uniqueness of the solution follows from the fact that

\[
\mathfrak{G}_\Gamma(t_1, t_2; \phi) = 1 + \sum_{i=1,2} \left[ t_i - \xi_i \psi_i(\phi) \right]^2 \tag{12}
\]
Figure 2. Momentum distribution for the created electron-positron pair [see Eq. (1)] for the parameters $\chi = 1$, $\xi = 5$, $N = 5$ and $\phi_0 = \pi/2$ (the longitudinal momentum characterized by $w$ is integrated). The parameters $\xi = 5$ and $\chi = 1$ could be obtained by colliding 17 GeV photons head-on with an optical ($\omega = 1.55$ eV) laser pulses having an intensity of $10^{20}$ W/cm$^2$ (note that few-cycle laser pulses are envisaged e.g. at the PFS in Garching [49]). a) Full numerical calculation of the spectrum [see Eq. (11)]. b) Local constant-crossed field approximation applied on the amplitude level [see Eq. (15)]. The inset shows that the interference pattern is lost if the local constant-crossed field approximation is applied on the probability level. c) Outline for $t_2 = 0$. Solid line: full numerical calculation; dotted (dashed) line: local constant-crossed field approximation applied on the amplitude (probability) level.

is always larger than zero on the real axis. Thus, the calculation of the Fourier-transformed master integrals with respect to the parameter $w$ reduces to a root-finding problem. Correspondingly, the master integrals can be evaluated for different values of $w$ in parallel by using only a single fast Fourier transform (FFT). Therefore, this approach reduces the computation costs substantially, as usually a highly-oscillating integral must be computed for each value of $w$. Since $\mathcal{S}_\Gamma(t_1, t_2; \phi_\bar{\xi}) > 0$ on the real axis [see Eq. (12)], one could equivalently perform the change of variable $\mathcal{S}_\Gamma(w, t_1, t_2; \phi_\bar{\xi}) = \bar{x}$ in Eq. (8) and evaluate the master integrals directly via FFT. This approach has been applied in [48] to the analogous problem of nonlinear Thomson scattering.

3. Semiclassical approximation

From now on we focus on a linearly polarized laser field, i.e. we assume that $\psi_2 = 0$ ($\psi = \psi_1$, $\xi = \xi_1$) and the incoming photon has parallel polarization ($e^\mu = \Lambda_1^\mu$, see Appendix A). Correspondingly, Eq. (1) provides the probability $W_\parallel(q)$, where the trace
in Eq. (6) is given by

\[
\frac{1}{e^2} \Lambda_{\mu \nu} \Lambda_{\lambda \sigma} \text{tr}[\ldots]_{\mu \nu} = 2m^2 (w - 4) \left[ -\xi^2 |\mathcal{G}_{1,1}|^2 + 2\xi t_1 \Re(\mathcal{G}_0^* \mathcal{G}_{1,1}) \right] \\
+ 4m^2 |\mathcal{G}_0|^2 \left[ -(w/2)(1 + t_1^2 + t_2^2) + 2t_1^2 \right].
\]

If \( \xi \gg 1 \), the main contribution to the master integrals [see Eq. (8)] arises from the regions around the phases \( \phi_k \) where \( t_1 \approx \xi \psi(\phi_k) \). Analogously to nonlinear Compton scattering [37], the fact that all three particles are real implies that the true stationary points of the phase \( S_{\Gamma}(t_1, t_2; \phi) \) are complex [see Eq. (12) and [26]]

\[
\psi(\phi_k) = \frac{1}{\xi} \left( t_1 \pm i \sqrt{1 + t_2^2} \right).
\]

From classical considerations we obtain the scalings \( |t_1| \ll \xi \) and \( |t_2| \ll 1 \), which are confirmed by Eq. (14). As \( |\psi(\phi)| \ll 1 \), the stationary-phase condition can be obeyed relatively close to the real axis only if \( |t_1| \ll \xi \). In the regime \( \xi \gg 1 \) the imaginary part in Eq. (14) scales as \( 1/\xi \) and is negligibly small for \( |t_2| \ll 1 \) [26]. Therefore, we define the stationary point \( \phi_k \) by \( t_1 = \xi \psi(\phi_k) \) from now on and obtain the following contribution to the master integrals [see Eq. (8)] [26]

\[
\mathcal{G}_0(w, t_1, t_2) \approx \frac{kq}{m^2} \frac{2}{w} \left[ \frac{w/2}{|\chi(\phi_k)|} \right]^{2/3} 2\pi \text{Ai}(\rho) e^{iS_{\Gamma}(\phi_k)},
\]

\[
\mathcal{G}_{1,1}(w, t_1, t_2) \approx \frac{t_1}{\xi} \mathcal{G}_0(w, t_1, t_2) \\
- i \left( \frac{kq}{m^2} \frac{2}{w} \right)^2 \left[ \frac{w/2}{|\chi(\phi_k)|} \right]^{4/3} 2\pi \text{Ai}'(\rho) \psi'(\phi_k) e^{iS_{\Gamma}(\phi_k)},
\]

where \( \rho = \{w/[2 |\chi(\phi_k)|]\}^{2/3} (1 + t_2^2) \), \( \chi(\phi) = \chi\psi'(\phi) \), with \( \chi = (kq/m^2)\xi \) and \( \text{Ai} \) denotes the Airy function [50]. Since at \( \chi \ll 1 \) pair production is exponentially suppressed [24], we will consider \( \chi = 1 \) in the numerical calculations. Note that the probability is also exponentially suppressed for \( |t_2| \gg 1 \), in agreement with classical considerations and Eq. (14).

Experimentally, the regime \( \chi \gtrsim 1, \xi \gg 1 \) is accessible with presently available technology, i.e. by colliding GeV photons (obtainable e.g. via Compton backscattering [51–58]) with strong optical laser pulses (\( \xi \gtrsim 100 \) [4–6, 59]).

For a single stationary point the well-known result for pair production within a constant-crossed field is obtained [after combining Eq. (13) and Eq. (15), see e.g. [27]]. In general, however, the phase factor in Eq. (15) leads to an interference between contributions of different stationary points. This is demonstrated in Fig. 2, where we compared the full numerical calculation [based on Eq. (11)] with the local constant-crossed field approximation [applied on the level of the master integrals, see Eq. (15)]. If the contribution of each stationary point is taken into account by simply adding the corresponding spectra and neglecting the phase factor (inset), the interference fringes
are lost [for the numerical calculations the pulse shape \(\psi'(\phi) = \sin^2[\phi/(2N)]\) \(\sin(\phi + \phi_0)\) has been used and the numerical values of the parameters are given in the captions of the figures].

From Eq. (8) we conclude that the oscillation frequency of the interference fringes scales as \(\sim \xi^3\) for \(w\), \(\sim \xi^2\) for \(t_1\) and \(\sim \xi\) for \(t_2\). Correspondingly, we used for \(\xi = 10\) a grid with \(\sim 10^5 \times 10^4 \times 10^3 = 10^{12}\) \((w, t_1, t_2)\) data points to resolve it. As a cross-check we ensured that the total probability obtained here by integrating numerically over the phase space agrees with the one calculated, e.g., in [3] using the optical theorem.

To interpret the stationary points we consider the classical equations of motion for the momenta defined in Eq. (2). For an electron with four-momentum \(p_1^\mu(\phi)\) the classical evolution of the component \(t_i(\phi)\) is given by [3, 24, 60]

\[
t_i(\phi') = -\frac{\Lambda_i p_1(\phi')}{m} = t_i(\phi) - \xi_i[\psi_i(\phi') - \psi_i(\phi)]
\]  

[16] [note that \(r'(\phi) = kp_1(\phi)/kq = r'\) is conserved and \(s'(\phi) = qp_1(\phi)/kq = \frac{1}{2}m^2/kp_1\)[1 +

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**Figure 3. Left side:** Numerically calculated differential pair-production probability as a function of the transversal momentum parameters \(t_1\) and \(t_2\) (the longitudinal momentum characterized by \(w\) is integrated). The incoming photon has parallel polarization \((\epsilon^\mu = \Lambda_i^\mu)\), the laser pulse \(N = 2\) cycles (such short pulses are envisaged e.g. at the PFS in Garching [49]) and \(\chi = 1\). We compare two different CEPs and two different intensities: a) \(\phi_0 = 0, \xi = 10\); b) \(\phi_0 = 0, \xi = 5\); c) \(\phi_0 = \pi, \xi = 10\); d) \(\phi_0 = \pi, \xi = 5\). The solid white line confines the phase-space region where the pair can be produced at a phase \(\phi\) with \(|\psi'(\phi)| \geq 0.5\) and the dashed white lines indicate the transverse momenta for which the pair can be produced at a local field peak. After integrating over \(t_1\) and \(t_2\) we obtain for the total pair-production probability \(W_\parallel = 0.09\%\) \(\xi = 10\) and \(W_\parallel = 0.045\%\) \(\xi = 5\), up to this precision it is independent of the CEP. **Right side:** Plot of the laser pulse shape [solid line: \(\psi'(\phi)\), dashed line: \(\psi(\phi)\)].
\[ t_1^2(\phi) + t_2^2(\phi) \] is determined by the on-shell condition. If we require that \( t_1 = \xi \psi(\phi_k) \) (stationary-point condition), we can use the classical equation \( t_1 = t_1(\infty) = t_1(\phi_k) + \xi_1 \psi_1(\phi_k) \) to conclude that \( t_1(\phi_k) = 0 \) is the appropriate initial condition for the classical propagation, i.e. the momentum distribution in \( t_1 \) is obtained solely from the classical acceleration of the particles in the laser field.

From the classical equations we therefore expect that the extend of the spectrum in \( t_1 \) should be proportional to \( \xi \). In Fig. 3 this is verified numerically. Correspondingly, the linear increase of the total pair-production probability as a function of \( \xi \) in the regime \( \xi \gg 1 \) [3] is a pure kinematic effect (size of the available phase space).

Furthermore, it is shown in Fig. 3 that the semiclassical description fully accounts for all qualitative features of the spectrum in \( t_1 \), i.e. its extend and the position of the maxima. In particular, the large CEP effects reported in [15] result from the fact that the classical acceleration of the created particles has a preferred direction (see the plot of \( \psi \), which determines the stationary points).

From a quantum mechanical perspective the initial condition \( t_1(\phi_k) = 0 \) may seem to contradict the Heisenberg uncertainty principle. However, the contradiction is resolved as the application of the local constant-crossed field approximation implies the inclusion of contributions from within the whole formation region around \( \phi_k \), such that the true initial position is blurred on the scale set by the formation region \([the integration in \phi in Eq. (8) exactly expresses the Heisenberg uncertainty relation between the phase formation region and the number of laser photons absorbed in the process (see also [61])].

For monochromatic fields the threshold condition \( \chi \gtrsim 1 \) for the onset of pair production is normally derived using the dressed mass \( (m^* \sim \xi m) \) and the fact that asymptotically the pair can absorb \( n \sim \xi^3/\chi \) photons from the background field [see Eq. (5)] [24, 26]. As already pointed out in [62], the same threshold condition is also obtained by considering the physical mass \( m \) \((4m^2 \leq 2nkq) \) and noting that the pair must become real on the scale set by the formation region \( \delta \phi \sim 1/\xi \) (classically, the electron and the positron may absorb together the four-momentum \( nk^\mu \) with \( n \sim \xi \) during the formation region).

In contrary to \( t_1 \), the two other parameters \( R \) and \( t_2 \) are constants of motion (for a linearly polarized background field with \( F^{\mu\nu} \sim f_1^{\mu\nu} \)) and the associated probability distributions are not changed by the subsequent classical evolution of the produced particles. This is demonstrated in Fig. 4. We point out that after the parameter \( t_1 \) is integrated out, the spectrum corresponds very closely to the one obtained in a constant-crossed field (averaged over the pulse shape). This may not be expected at first sight, as for a fixed value of \( t_1 \) the differential spectrum has a rich interference structure (shown in the inset of Fig. 4).
Figure 4. Differential probability with respect to the parameters $t_2$ and $R$ for $\chi = 1$, $\xi = 10$, $N = 5$ and $\phi_0 = \pi/2$ (full numerical calculation, $t_1$ is integrated). The inset shows $d^3W_{\parallel}/dR dt_1 dt_2$ for $t_1 = 0$ (in arb. units). The pronounced interference pattern vanishes after the integral in $t_1$ is taken. Note that the differential probability does not depend on the sign of $R$ and $t_2$.

4. Summary and conclusions

In the present paper the momentum distribution for electron-positron pairs produced via the nonlinear Breit-Wheeler process in short laser pulses has been investigated. Using a newly developed numerical scheme we have calculated for the first time the spectrum on a three-dimensional lattice that is fine enough to fully resolve the interference structure even in the ultra-relativistic regime $\xi \gg 1$. Furthermore, we have investigated the local constant-crossed field approximation and showed that it reproduces the spectrum (including the interference fringes) for $\xi \gg 1$ if it is applied on the probability-amplitude level. Correspondingly, three effects determine the final momentum distribution in the regime $\xi \gg 1$: The production of an electron and a positron with physical mass $m$ inside a constant-crossed field, their subsequent classical acceleration by the laser field and the interference between all production channels (corresponding to the various stationary points in the master integrals), which lead to the same asymptotic quantum numbers. Accordingly, we verified that the produced electron and positron behave like classical particles after they have left the formation region and that the substructure of the spectrum can be attributed to interferences similar to those in a multi-slit experiment.

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Appendix A. Photon polarization density matrix

The (complex) polarization four-vector $\epsilon^\mu$ of a photon with four-momentum $q^\mu \ (q^2 = 0)$ must obey $\epsilon^\mu \epsilon_\mu = -1$ and $q \epsilon = 0$. In the light-cone basis $k^\mu$, $q^\mu$, $\Lambda_i^\mu$ we obtain for the metric [see Eq. (3)]

$$g^{\mu\nu} = \frac{1}{kq} \left( k^\mu q^\nu + q^\mu k^\nu \right) - \Lambda_1^\mu \Lambda_1^\nu - \Lambda_2^\mu \Lambda_2^\nu \quad (A.1)$$

and thus the polarization four-vector is given by

$$\epsilon^\mu = c_1 \Lambda_1^\mu + c_2 \Lambda_2^\mu + c_3 q^\mu, \quad c_1 = -(\epsilon \Lambda_1), \quad c_2 = -(\epsilon \Lambda_2), \quad c_3 = \frac{k \epsilon}{kq} \quad (A.2)$$

with the normalization condition $|c_1|^2 + |c_2|^2 = 1$. As the contraction of the matrix element with the four-momentum $q^\mu$ must vanish due to gauge symmetry, we can restrict us to the vectors $\Lambda_i^\mu$ and replace the density matrix by

$$\rho^{\mu\nu} = \epsilon^\mu \epsilon^{*\nu} \rightarrow \sum_{i,j=1,2} \rho_{ij} \Lambda_i^\mu \Lambda_j^\nu, \quad (A.3a)$$

$$\rho_{11} = |c_1|^2, \quad \rho_{22} = |c_2|^2, \quad \rho_{12} = c_1^* c_2, \quad \rho_{21} = c_2^* c_1. \quad (A.3b)$$

The $2 \times 2$ density matrix $\rho_{ij}$ is Hermitian and has unit trace

$$\rho_{ij} = \Lambda_i^\mu \Lambda_j^\nu \rho^{\mu\nu}, \quad \rho_{ji} = \rho_{ij}^*, \quad \text{tr} \rho = \sum_{i=1,2} \rho_{ii} = 1. \quad (A.4)$$

Any Hermitian $2 \times 2$ matrix can be expanded using the Pauli matrices $\sigma^i$ and the identity $1$ (with real parameters). Since $\text{tr} \sigma^i = 0$, we obtain (see [1], Eq. 8.9)

$$\rho = \frac{1}{2} \left( 1 + s_i \sigma^i \right) = \frac{1}{2} \left( \begin{array}{cc} 1 + s_3 & s_1 - is_2 \\ s_1 + is_2 & 1 - s_3 \end{array} \right) = \left( \begin{array}{cc} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{array} \right), \quad (A.5)$$

where $s_i$ are called Stokes parameters. The Stokes vector $s = (s_1, s_2, s_3)$ is a unit vector, which can be seen from

$$\det \rho = 0 = \frac{1}{4} \left( 1 - s^2 \right). \quad (A.6)$$

Correspondingly, it can be described by two Stokes angles

$$s_1 = \cos(\varphi) \sin(\theta), \quad s_2 = \sin(\varphi) \sin(\theta), \quad s_3 = \cos(\theta). \quad (A.7)$$

Using the trigonometric identities

$$\frac{1}{2} (1 + \cos \theta) = \cos^2(\theta/2), \quad \frac{1}{2} (1 - \cos \theta) = \sin^2(\theta/2), \quad (A.8a)$$

$$2 \cos(\theta/2) \sin(\theta/2) = \sin(\theta) \quad (A.8b)$$
we conclude that the complex coefficients $c_1$ and $c_2$ can be expressed in terms of the stokes angles as

$$c_1 = \cos(\theta/2) e^{-i\varphi/2}, \quad c_2 = \sin(\theta/2) e^{+i\varphi/2},$$

(A.9)

implying the representations

$$|c_1|^2 = \cos^2(\theta/2), \quad |c_2|^2 = \sin^2(\theta/2),$$

(A.10a)

$$c_1 c_2^* = \frac{1}{2} \sin(\theta) [\cos(\varphi) - i \sin(\varphi)]$$

(A.10b)

[note that we can always multiply by a total phase in Eq. (A.9)].

**Appendix B. Equivalence of different light-cone bases**

We call the four four-vectors $k^\mu, \bar{k}^\mu, e_i^\mu \ (i \in 1, 2)$ a light-cone basis if they obey [47]

$$k^2 = \bar{k}^2 = 0, \quad k e_i = \bar{k} e_i = 0, \quad k \bar{k} = 1, \quad e_i e_j = -\delta_{ij}. \quad (B.1)$$

Using the above properties and the determinant identity for $\epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta}$ one can show that any such light-cone basis obeys $\Omega^2 = 1$, where

$$\Omega = \epsilon^{\mu\nu\rho\sigma} k^\mu \bar{k}^\nu e_1^\rho e_2^\sigma \quad (B.2)$$

is called the orientation of the basis. In light-cone coordinates the metric is given by

$$g^{\mu\nu} = k^\mu \bar{k}^\nu + \bar{k}^\mu k^\nu - e_1^\mu e_1^\nu - e_2^\mu e_2^\nu. \quad (B.3)$$

As any set of four four-vectors $e_i^\mu \ (i \in 1, 2), \ \bar{k}^\mu$ and $k^\mu$ which obeys the relations given in Eq. (B.1) represents a light-cone basis, it is natural to ask which expressions are invariant under a change of the underlying light-cone basis. To this end we consider two different bases $\bar{k}^\mu, e_i^\mu$ and $\bar{k}^\mu, e'_i^\mu$ and denote the corresponding components of a four-vector $v^\mu$ by

$$v^+ = \bar{k}^\mu v_\mu, \quad v^I = e_i^\mu v_\mu, \quad v^\parallel = e_2^\mu v_\mu,$$

$$v'^+ = \bar{k}^\mu v'_\mu, \quad v'^I = e'_i^\mu v'_\mu, \quad v'^\parallel = e'_2^\mu v'_\mu, \quad (B.4)$$

($v^- = v'^- = k v$). The three coordinates $(-, \perp = 1, \parallel)$ define a closed subspace and we obtain the relation

$$\begin{pmatrix} v'^- \\ v'^I \\ v'^\parallel \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ e'_1 \bar{k} & -e'_1 e_1 - e'_2 e_2 \\ e'_2 \bar{k} & -e'_2 e_1 - e'_2 e_2 \end{pmatrix} \cdot \begin{pmatrix} v^- \\ v^I \\ v^\parallel \end{pmatrix}. \quad (B.5)$$

In order to show that its determinant has magnitude one, we write

$$e'_1^\mu = ae_1^\mu + be_2^\mu + \lambda k^\mu, \quad e'_2^\mu = ce_1^\mu + de_2^\mu + \mu k^\mu. \quad (B.6)$$
As $e_1'^2 = e_2'^2 = -1$ and $e_1'e_2' = 0$, we obtain $a^2 + b^2 = c^2 + d^2 = 1$ and $ac + bd = 0$. Without restricting generality, we set $a = \cos \varphi$, $b = \sin \varphi$ and $d = \cos \theta$, $c = \sin \theta$. Finally, we obtain the two solutions $\theta = -\varphi$ and $\theta = -\varphi + \pi$, which correspond to $ad - bc = \pm 1$. Therefore, the measure $dv^-dv^\perp = dv'^-dv'^\perp$ and the delta function

$$\delta^{(-,\perp)}(v) = \delta(v^-)\delta(v'^i) = \delta(v'^-)\delta(v'^\text{II})\delta(v'^\text{II}) = \delta^{(-,\perp)}(v')$$

(B.7)

are invariant under a change of the light-cone basis. If $v^- = v^\perp = 0$ also the component $v^+ = \bar{k}^\mu v_\mu$ is invariant.

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