DISCONTINUITY POINTS OF A FUNCTION WITH A CLOSED AND CONNECTED GRAPH

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ABSTRACT. The main result of this paper states that for a function $f : \mathbb{R}^2 \to Y$ with a closed, connected and locally connected graph, where $Y$ is a locally compact, second-countable metrisable space, the graph over discontinuity points remains locally connected.

Motivation. It is a classic result that for a function $f : X \to Y$ with a closed graph and $Y$ Hausdorff, a sufficient and necessary condition for being continuous is sub-continuity [Fuller 68, 3.4]. However, it is an interesting question how, for various spaces $X$ and $Y$, additional topological properties of a closed graph are related to the continuity of a function. It is known that, for a function $f : \mathbb{R} \to \mathbb{R}$ with a closed graph, a sufficient and necessary condition for being continuous is the connectedness of the graph [Burgess 90].

In 2001, Michał R. Wójcik and I stated the question whether this result can be extended to $f : \mathbb{R}^2 \to \mathbb{R}$ [Wójcik 2004, 9] – this problem was then propagated by Cz. Ryll-Nardzewski. The answer to this question is negative. The first known discontinuous function $f : \mathbb{R}^2 \to \mathbb{R}$ with a connected and closed graph was shown by J. Jelínek in [Jelínek 2003]. It can be shown that the graph of Jelínek’s function in not locally-connected [Mrwphd 2008, A4]. Therefore a new question was stated, whether connectedness together with the local connectedness of the graph is a sufficient and necessary condition of being continuous for a function $f : \mathbb{R}^2 \to \mathbb{R}$ with a closed graph. This question, as far as I know, remains open.

In this paper, I show some properties of the set of discontinuity of a function $f : \mathbb{R}^2 \to \mathbb{R}$ with a closed, connected and locally connected graph, hoping that they might be useful in the main research.

Result. The main result of this paper states that for a function $f : \mathbb{R}^2 \to Y$ with a closed, connected and locally connected graph, where $Y$ is a locally compact, second-countable metrisable space, the graph over discontinuity points remains locally connected. This result is given as Corollary 16 as a consequence of some deep topological properties of the real plane and the more generic Theorem 13.

1. Notation and terminology

Definition 1. Let $X,Y$ be topological spaces and $f : X \to Y$ be an arbitrary function.

(1) We will denote by $C(f)$ or $C_f$ the set of all points of continuity,
(2) by $D(f)$ or $D_f$ – set of all points of discontinuity,
(3) We will denote by $\pi$ a projection operator $\pi : X \times Y \to X$ and $\pi(x, y) = x$.
(4) For $A \subset X$, by $f|A$ we will denote a restriction of $f$ to the subdomain $A$. 

1
In the context of function \( f : X \to Y \), we will not use a separate symbol to denote the graph of \( f \), for \( f \) itself, in terms of Set Theory, is a graph. So when we use Set Theory operations and relations with respect to \( f \), they should be understood as operations and relations with respect to the graph. Whenever this naming convention might be confusing, we will add the word “graph”, e.g. “\( f \) has a closed graph”.

**Definition 2.** Let \( Y \) be a topological space and \( y_n \in Y \) be an arbitrary net. We will write \( y_n \to \emptyset \) or \( \lim y_n = \emptyset \) iff \( y_n \) has no convergent subnet.

2. Functions with a closed graph

It will be helpful to cite two well-known theorems concerning functions with a closed graph:

**Theorem 3.** If \( X \) is a topological space, \( Y \) is a compact space, \( f : X \to Y \) and the graph of \( f \) is closed, then \( f \) is continuous.

(for proof: e.g. [Wójcik 2004, T2])

**Theorem 4.** If \( X \) is a Bair and Hausdorff space, \( Y \) is a \( \sigma \)-locally compact space, \( f : X \to Y \) and the graph of \( f \) is closed, then \( C(f) \) is an open and dense subset of \( X \).

(for proof: e.g. [Dobos 85, T2])

3. Graph over discontinuity points

I will begin with several well-known facts:

**Fact 5.** Every non-empty metrisable compact space is a continuous image of the Cantor set.

(for proof e.g. [Engelking 89, 4.5.9])

**Fact 6.** If \( X \) is a connected and locally arcwise-connected metrisable space, \( F \) is a closed subset of \( X \), \( C \subset [0,1] \) is the Cantor set and \( f : C \to F \) is continuous and \( f(C) = X \), then \( f \) has a continuous extension \( f^* : [0,1] \to X \).

(for proof: e.g. [Kuratowski II 66, 50.I.5])

**Fact 7.** If \( X \) is a topological space and \( A, B \subset X \) are both closed and locally connected, then \( A \cup B \) is locally connected.

(for proof: e.g. [Kuratowski II 66, 49.I.3])

**Fact 8.** If \( X \) is a compact and locally connected space, \( Y \) is a Hausdorff space, \( f : X \to Y \) is continuous and \( f(X) = Y \), then \( Y \) is compact and locally connected.

(for proof: notice that since \( X \) is compact and \( Y \) is Hausdorff, \( f \) is a quotient mapping and local connectedness is invariant under quotient mappings [Whyburn 52, T2])

**Lemma 9.** If \( X \) is a connected metrisable space, and \( F \) is a connected and closed subset, \( X \setminus F = A \cup B \), where \( \text{Clo}(A) \cap B = \text{Clo}(B) \cap A = \emptyset \), then \( F \cup A \) is connected and closed.

*Proof.* [Kuratowski II 66, 46.II.4] □
Lemma 10. If $X$ is a locally connected metrisable space, $F$ is a locally connected and closed subset and $S$ is a sum of some connected components of $X \setminus F$, then $S \cup F$ is locally connected.

Proof. [Kuratowski II, 66, 49.II.11] \hfill \Box

Theorem 11. If $X$ is a connected and locally connected, locally compact, second-countable metrisable space, $E$ is a continuum in $X$ and $U$ is an arbitrary open neighbourhood of $E$, then there exists a locally connected continuum $F$ such that $E \subset F \subset U$ and $X \setminus F$ has finitely many connected components.

Proof. Since $X$ is locally compact, Hausdorff and locally connected, there exists an open neighbourhood $V_x$ of the point $x$ such that $\text{Cl}(V_x)$ is a continuum and $\text{Cl}(V_x) \subset U$ for each $x \in E$. Since $E$ is compact, there exist $x_1, x_2, \ldots, x_n \in E$ such that $E \subset \bigcup_{i=1}^{n} V_{x_i}$. Let $V = \bigcup_{i=1}^{n} V_{x_i}$. Notice that $\text{Cl}(V)$ is compact and $E \subset V \subset \text{Cl}(V) \subset U$. By Fact \ref{fact:continuum} there is a continuous function $f : C \to E$, where $C$ is the Cantor set and $f(C) = E$. Since $X$ is metrisable and locally compact, it is completely metrisable and therefore, by the Mazurkiewicz-Moore theorem, $X$ is locally arcwise-connected, and thus $V$ is locally arcwise-connected. $V$ is connected by construction. Therefore by Fact \ref{fact:extension} there is a function $f^*$ that is a continuous extension of $f$ such that $f^* : [0,1] \to V$. Let $F_0 = f^*([0,1])$. By Fact \ref{fact:continuum} $F_0$ is a locally connected continuum and $E \subset F_0 \subset V$. Let $S_V$ be a family of all the connected components of $X \setminus F_0$ that are subsets of $V$. Let $S_\infty$ be a family of all the other connected components of $X \setminus F_0$. Notice that, due to the connectedness of $S$, (1) $S \cap \partial V \neq \emptyset$ for any $S \in S_\infty$. Since $X$ is locally connected, all the connected components of $X \setminus F_0$ are open in $X$. Since $\partial V \subset \bigcup S_\infty$, by virtue of (1) and the compactness of $\partial V$, the family $S_\infty$ is finite. Let $F = F_0 \cup \bigcup S_V$. By Lemma \ref{lemma:connected} $(F, V)$ is connected and closed. Since $F \subset \text{Cl}(V)$, $F$ is a continuum. By Lemma \ref{lemma:local} $F$ is locally connected. Obviously, $E \subset F \subset V \subset U$. Since $X \setminus F = \bigcup S_\infty$ and $S_\infty$ is finite, the proof is complete. \hfill \Box

Lemma 12. If $X$ is a Hausdorff space, $Y$ is a topological space, $f : X \to Y$ is a function with a closed graph, $D = D(f)$, $E$ is a compact subset of the graph and $U$ is a relatively open subset of the graph such that $U \subset E \subset f$, then $U \cap f[D] = U \cap f[\partial f(E)]$. Moreover, if $(x, f(x)) \in U \cap f[D]$ and $X \setminus \pi(E) \ni x_n \to x$, then $f(x_n) \to \emptyset$.

Proof. Since $E$ is compact, by Theorem \ref{lemma:continuity} $f|\pi(E)$ is continuous. If $x \in \text{Int}(E)$, then $f$ is continuous in $x$, so $x \notin D$. Therefore $U \cap f[D] \subset U \cap f[\partial f(E)]$ is obvious. We will show inverse inclusion by contradiction. Assume that $(x, f(x)) \in U \cap f[\partial f(E)]$ and $x$ is a continuity point of $f$. Since $\pi(E)$ is compact and $X$ is Hausdorff, $\pi(E)$ is closed, so $x \in \pi(E)$. Notice that $(x, f(x)) \in U$, so by the continuity of $f$ at point $x$, there is an open neighbourhood $V$ of $x$, such that $f[V] \subset U$. But $U \subset E$, so $x \in V \subset \pi(E)$. This contradicts how $x$ was chosen. Now we will show the “moreover part”. Take any $(x, f(x)) \in U \cap f[D]$ and $X \setminus \pi(E) \ni x_n \to x$. Since $U \subset E$, $(x_n, f(x_n)) \notin U$. So no subnet of $f(x_n)$ is convergent to $f(x)$. But the graph of $f$ is closed, so $f(x_n) \to \emptyset$. \hfill \Box

Theorem 13. If $X$ is a connected and locally connected, locally compact, second-countable metrisable space, $Y$ is a locally compact, second-countable metrisable
space, \( f : X \to Y, \ D = D(f) \) and the graph of \( f \) is closed, connected and locally connected, then for each \( x \in D \) there is an open in the graph topology \( U \) and a locally connected continuum \( E \) such that

1. \((x, f(x)) \in U \subset E \subset f,\)
2. \(U \cap f[D = U \cap f[\partial \pi(E)] \) (so \( x \in \partial \pi(E) \)),
3. if \( X \setminus \pi(E) \ni x_n \to x, \) then \( f(x_n) \to \emptyset, \)
4. \( X \setminus \pi(E) \) has finitely many connected components.

**Proof.** Notice that \( f \) is a connected and locally connected, locally compact, second-countable metrisable subspace of \( X \times Y \). Take an arbitrary \( x \in D \). Choose open in the graph topology set \( U \), such that \((x, f(x)) \in U \) and \( Cl_{f}(U) \) is a continuum. By Theorem 11 there exists a locally connected continuum \( E \subset f \) such that \( f \setminus E \) has finitely many connected components and \( U \subset E \). By Lemma 12, \( f[D \cap U = f[\partial p(E) \cap U \) and for any subsequence \( X \setminus \pi(E) \ni x_n \to x \) we have \( f(x_n) \to \emptyset \). By Fact 8, \( p(E) \) is a locally connected continuum and since \( f \setminus E \) has finitely many connected components and \( \pi \) is continuous, the set \( \pi(f \setminus E) = \pi(f) \setminus \pi(E) = X \setminus \pi(E) \) also has finitely many connected components. \( \square \)

Theorem 13 has an interesting consequence for \( X = \mathbb{R}^2 \), namely \( \partial \pi(E) \) from the above theorem is locally connected, which implies (by Theorem 9) that \( f[D \) has a locally connected graph. To prove this, let me refer to the following theorem:

**Theorem 14.** If \( A \) is a locally connected continuum in \( \mathbb{R}^2 \), and \( S \) is a connected component of \( \mathbb{R}^2 \setminus A \), then \( \partial S \) is a locally connected continuum.

**Proof.** Since \( \mathbb{R}^2 \) is homeomorphic with a unit sphere without one point, it’s enough to apply [Kuratowski II 66, 61.II.4]. \( \square \)

Let’s formulate a simple consequence of the above.

**Theorem 15.** If \( A \) is a locally connected continuum in \( \mathbb{R}^2 \) and \( \mathbb{R}^2 \setminus A \) has finitely many connected components, then \( \partial A \) is locally connected.

**Proof.** Let \( S_1, S_2, \ldots S_n \) be connected components of \( \mathbb{R}^2 \setminus A \). \( S_1, S_2, \ldots S_n \) are open, since \( \mathbb{R}^2 \setminus E \) is an open subset of a locally connected space and thus locally connected. Since we’re dealing only with a finite number of open sets, the below equation holds.

\[
\partial A = \partial(\mathbb{R}^2 \setminus A) = \partial\left(\bigcup_{i=1}^{n} S_i\right) = \bigcup_{i=1}^{n} \partial S_i.
\]

By Theorem 14, \( \partial S_i \) is locally connected for \( i = 1, 2, \ldots, n \). Therefore, by Fact 7, \( \partial A \) is locally connected. \( \square \)

By applying Theorem 13, Theorem 9 and Theorem 8 we immediately get the following corollary.

**Corollary 16.** If \( Y \) is a locally compact, second-countable metrisable space, \( f : \mathbb{R}^2 \to Y \) has a closed, connected and locally connected graph, then \( f[\mathbb{R}^2] \) has a locally connected graph.

One might propose that as the local connectedness of \( f[\mathbb{R}^2] \) is a local property, it might be enough to assume only local connectedness of \( f \). Unfortunately, there is a simple example that shows that the connectedness of \( f \) is necessary in Corollary 16 and Theorem 13.
Example 17. Let \( r_n = \frac{1}{4n(n+1)} \), \( B_n = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + (y - \frac{1}{n})^2} < r_n\} \).

\[
f(x, y) = \begin{cases} 
0 & \text{for } y \geq 0 \text{ and } (x, y) \notin \bigcup_{n=1}^{\infty} B_n, \\
\frac{1}{y} & \text{for } y < 0, \\
 n + \tan\left(\frac{x}{\sqrt{2}r_n}\right) \sqrt{x^2 + (y - \frac{1}{n})^2} & \text{for } (x, y) \in B_n \text{ for } n = 1, \ldots
\end{cases}
\]

Note that in the above example \( B_n \) is a sequence of pairwise disjoint open discs convergent to the point \((0, 0)\). \( f = 0 \) on the whole half plane \( \mathbb{R} \times [0, \infty) \) except discs \( B_n \). \( f \geq n \) on \( B_n \) and converges to infinity on \( \partial B_n \). Therefore, it is easy to notice that the graph of \( f \) is closed and not connected. The local connectedness of the graph is obvious everywhere except the point \((0, 0, 0)\). But as \( f \geq n \) on \( B_n \) and \( f = 0 \) on \( \mathbb{R} \times [0, \infty) \) \( \setminus \bigcup_{n=1}^{\infty} B_n \), it’s enough to see that \( \mathbb{R} \times [0, \infty) \setminus \bigcup_{n=1}^{\infty} B_n \) is locally connected at the point \((0, 0)\). Thus the graph of \( f \) is locally connected. However, \( f|D_f = (\mathbb{R} \times \{0\} \cup \bigcup_{n=1}^{\infty} \partial B_n) \times \{0\} \) and is not locally connected in \((0, 0, 0)\). It’s also easy to notice that for any open in the graph topology set \( U \) such that \((0, 0, 0)\) \( \in U \) and for any locally connected continuum \( E \) such that \( U \subset E \subset f, \mathbb{R}^2 \setminus \pi(E) \) has infinitely many connected components, since \( B_n \cap \pi(E) = \emptyset \) and \( \partial B_n \subset E \) for almost all \( n \).

References

[Burgess 90] C. E. Burgess, Continuous Functions and Connected Graphs, The American Mathematical Monthly, Vol. 97, No. 4, 337–339, 1990.

[Dobos 85] J. Dobos, On the set of points of discontinuity for functions with closed graphs, Časopis pro pěstování matematiky, Vol. 110, No. 1, 60–68, 1985.

[Engelking 89] R. Engelking, General Topology, Berlin: Heldermann, 1989.

[Fuller 68] R. V. Fuller, Relations among continuous and various non-continuous functions, Pacific Journal of Mathematics, Vol. 25, No. 3, 495–509, 1968.

[Jelinek 2003] J. Jelínek, A discontinuous function with a connected closed graph, Acta Universitatis Carolinae, 44, No. 2, 73–77, 2003.

[Kuratowski I 66] K. Kuratowski, Topology Volume I, New York 1966.

[Kuratowski II 66] K. Kuratowski, Topology Volume II, New York 1966.

[Mrwphd 2008] M. R. Wójcik, Closed and connected graphs of functions; examples of connected punctiform spaces, PhD Thesis in Institute of Mathematics University of Silesia, 2008

[Whyburn 52] G. T. Whyburn, On quasi-compact mappings, Duke Math J.19, 445–446, 1952

[Wójcik 2004] M. R. Wójcik, M. S. Wójcik, Separately continuous functions with closed graphs, Real Analysis Exchange, Vol. 30, No. 1, 23–28, 2004/2005.

[Wójcik 2007] M. R. Wójcik, M. S. Wójcik, Characterization of continuity for real-valued functions in terms of connectedness, Houston Journal of Mathematics, Vol. 33, No. 4, 1027–1031, 2007.