On the time-dependent grade-two model for the
magnetohydrodynamic flow: 2D case

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Abstract

In this paper we discuss the MHD flow of a second grade fluid, in particular we prove the existence and uniqueness of a weak solution of a time-dependent grade two fluid model in a two-dimensional Lipschitz domain. We follow the methodology of [3], i.e., we use a constructive method which can be adapted to the numerical analysis of finite-element schemes for solving this problem numerically.

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1 Introduction

A fluid of grade two is a non-Newtonian fluid of differential type introduced by Rivlin and Ericksen in [8]. An analysis in [1] shows that the equation of a fluid of grade two is given by

\[ \frac{\partial}{\partial t}(u - \alpha \Delta u) - \nu \Delta u + \sum_j (u - \alpha \Delta u)_j \nabla u_j - u \cdot \nabla (u - \alpha \Delta u) = -\nabla p + f \]

\[ \text{div } u = 0 \]

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where $\alpha \geq 0$ is a constant of material, $\nu > 0$ is the viscosity of the fluid, $u$ is the velocity field, and $p$ is pressure. For $\alpha = 0$ the classical Navier-Stokes equation is obtained.

On the other hand, in several situations the motion of incompressible electrical conducting fluid can be modeled by the magnetohydrodynamic equation, which correspond to the Navier-Stokes equations coupled with the Maxwell equations. In presence of a free motion of heavy ions, not directly due to the electrical field (see Schluter [4] and Pikelner [3]), the MHD equation can be reduced to

$$
\frac{\partial u}{\partial t} - \frac{\nu}{\rho_m} \Delta u + u \cdot \nabla u - \frac{\mu}{\rho_m} h \cdot \nabla h = f - \frac{1}{\rho_m} \nabla (p^* + \frac{\mu}{2} h^2)
$$

$$
\frac{\partial h}{\partial t} - \frac{1}{\mu\sigma} \Delta h + u \cdot \nabla h - h \cdot \nabla u = -\text{grad } \omega
$$

(1)

$$
\text{div } u = \text{div } h = 0
$$

with

$$
u \big|_{\partial \Omega} = h \big|_{\partial \Omega} = 0.
$$

Here, $u$ and $h$ are respectively the unknown velocity and magnetic field; $p^*$ is the unknown hydrostatic pressure; $\omega$ is an unknown function related to the heavy ions (in such way that the density of electric current, $j_0$, generated by this motion satisfies the relation $\text{rot } j_0 = -\sigma \nabla \omega$) is the density of mass of the fluid (assumed to be a positive constant); $\mu > 0$ is the constant magnetic permeability of the medium; $\sigma > 0$ is the constant electric conductivity; $\nu > 0$ is the constant viscosity of the fluid; $f$ is a given external force field.

In the case the MHD equation coupled with the equation of an incompressible second grade fluid, the model can be write as

$$
\frac{\partial (u - \alpha \Delta u)}{\partial t} - \nu \Delta u + \text{curl } (u - \alpha \Delta u) \times u - (h \cdot \nabla) h = f - \nabla (p^* + h^2)
$$

$$
\frac{\partial h}{\partial t} - \Delta h + (u \cdot \nabla) h - (h \cdot \nabla) u = -\text{grad } \omega
$$

(3)

$$\text{div } u = \text{div } h = 0$$

with

$$
u \big|_{\partial \Omega} = h \big|_{\partial \Omega} = 0.
$$

(4)

Note that when $\alpha = 0$ we recover the model (1).

One of the first mathematical results for this model type appears in [2], they prove the existence and uniqueness of solutions for a small time and global existence of solutions for small initial data in a conducting domain of $\mathbb{R}^3$, based on the iterative scheme where discretization is performed in the spatial variables. In this paper we discuss the MHD flow of a second grade fluid, in particular we prove the existence and uniqueness of a weak solution of a time-dependent
grade two fluid model in a two-dimensional Lipschitz domain, where we follow the methodology of [3], i.e., we use semi-discretization in time and the work is in a domain of $\mathbb{R}^2$.

\section{Preliminary results}

\subsection{Notation}

Let $(k_1,k_2)$ denote a pair of non-negative integers, set $|k| = k_1 + k_2$ and define the partial derivative $\partial^k$ by

$$\partial^k v = \frac{\partial|k|v}{\partial x_1^{k_1} \partial x_2^{k_2}}.$$ 

Then, for any non-negative integer $m$ and number $r \geq 1$, recall the classical Sobolev space

$$W^{m,r}(\Omega) = \left\{ v \in L^r(\Omega); \partial^k v \in L^r(\Omega) \forall |k| \leq m \right\},$$

equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \left[ \sum_{|k|=m} \int_\Omega |\partial^k v|^r \, dx \right]^{1/r},$$

and norm (for which it is a Banach space)

$$\|v\|_{W^{m,r}(\Omega)} = \left[ \sum_{0 \leq |k| \leq m} \int_\Omega |v|_{W^{k,r}(\Omega)}^r \right]^{1/r},$$

with the usual extension when $r = \infty$. When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $u = (u_1,u_2)$; then we set

$$\|u\|_{L^r(\Omega)} = \left[ \int_\Omega \|u(x)\|^r \right]^{1/r},$$

where $\|\cdot\|$ denotes the Euclidean vector norm.

For functions that vanish on the boundary, we define for any $r \geq 1$,

$$W^{1,r}_0(\Omega) = \left\{ v \in W^{1,r}(\Omega); v|_{\partial\Omega} = 0 \right\},$$

and recall Poincaré’s inequality, there exists a constant $P$ such that

$$\forall v \in H^1_0(\Omega), \quad \|v\|_{L^r(\Omega)} \leq P \|v\|_{H^1(\Omega)}. \tag{5}$$

More generally, recall the inequalities of Sobolev embeddings in two dimension, for each $r \in [2,\infty)$, there exists a constant $S_r$ such that

$$\forall v \in H^1_0(\Omega), \quad \|v\|_{L^r(\Omega)} \leq S_r \|v\|_{H^1(\Omega)}. \tag{6}$$
The case $r = \infty$ is excluded and is replaced by, for any $r > 2$ there exists a constant $M_r$ such that

$$\forall v \in W^{1,r}_0(\Omega), \quad \|v\|_{L^\infty(\Omega)} \leq M_r |v|_{W^{1,r}(\Omega)},$$

(7)

Owing to (5), we use the seminorm $|\cdot|_{H^1(\Omega)}$ as a norm on $H^1_0(\Omega)$ and we use it to define the norm of the dual space $H^{-1}(\Omega)$ of $H^1_0(\Omega)$:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H^1_0(\Omega)} \frac{\langle f, v \rangle}{|v|_{H^1(\Omega)}}.$$

In addition to the $H^1$ norm, it will be convenient to define the following norm with the parameter $\alpha$:

$$\|v\|_\alpha = \left( \|v\|^2_{L^2(\Omega)} + \alpha |v|^2_{H^1(\Omega)} \right)^{1/2}.$$

In the following, we denote by $\| \cdot \|$ the $L^2$ norm.

We shall also use the standard space for incompressible flow:

$$H(\text{div}; \Omega) = \{ v \in L^2(\Omega)^2; \text{div} \, v \in L^2(\Omega) \}$$

$$H(\text{curl}; \Omega) = \{ v \in L^2(\Omega)^2; \text{curl} \, v \in L^2(\Omega) \}$$

$$V = \{ v \in H^1_0(\Omega)^2; \text{div} \, v = 0 \text{ in } \Omega \}$$

$$V^\perp = \{ v \in H^1_0(\Omega)^2; \forall w \in V, \langle \nabla v, \nabla w \rangle = 0 \}$$

$$L^2_0(\Omega) = \{ v \in L^2(\Omega); \int_\Omega q \, dx = 0 \}$$

and the space transport:

$$X_v = \{ f \in L^2(\Omega); v \cdot \nabla f \in L^2(\Omega) \},$$

where $v$ is a given velocity in $H^1(\Omega)^2$.

2.2 Auxiliary theoretical results

To analyze, we shall use the following results. The first theorem concerns the divergence operator in any dimension $d$. Its proof can be found for instance in Girault and Raviart [4].

**Theorem 1** Let $\Omega$ be a bounded Lipschitz-continuous domain of $\mathbb{R}^d$. The divergence operator is an isomorphism from $V^\perp$ onto $L^2_0(\Omega)$ and there exists a constant $\beta > 0$ such that for all $f \in L^2_0(\Omega)$, there exists a unique $v \in V^\perp$ satisfying

$$\text{div} \, v = f \text{ in } \Omega \quad \text{and} \quad \|v\|_{H^1(\Omega)} \leq \frac{1}{\beta} \|f\|.$$

The second result concerns the regularity of the Stokes operator in two dimensions, see [6].
Theorem 2 Let $\Omega$ be a bounded polygon in the plane.

1. For each $r \in [1, 4/3[$, the Stokes operator is an isomorphism from
\[
\left[ (W^{2,r}(\Omega))^2 \cap V \right] \times \left[ W^{1,r}(\Omega) \cap L^2_0(\Omega) \right] \text{ onto } L^r(\Omega)^2,
\]
i.e. for each $f \in L^r(\Omega)^2$, there exists a constant $C_r$ and a unique pair
\[(u, p) \in \left[ (W^{2,r}(\Omega))^2 \cap V \right] \times \left[ W^{1,r}(\Omega) \cap L^2_0(\Omega) \right]\]
such that
\[-v\Delta u + \nabla p = f, \quad \text{div } u = 0 \text{ in } \Omega, u = 0 \text{ on } \partial \Omega,
\]
and
\[|u|_{W^{2,r}(\Omega)} + |p|_{W^{1,r}(\Omega)} \leq C_r \|f\|_{L^r(\Omega)}.
\]

2. If in addition, $\Omega$ is a convex polygon, then the Stokes operator is an isomorphism from
\[
\left[ (H^2(\Omega))^2 \cap V \right] \times \left[ H^1(\Omega) \cap L^2_0(\Omega) \right] \text{ onto } L^2_0(\Omega)^2.
\]
Furthermore, there exists a real number $r > 2$, depending on the largest inner angle of $\partial \Omega$ such that for all $t \in [2, r]$, the Stokes operator is an isomorphism from \[
\left[ (W^{2,t}(\Omega))^2 \cap V \right] \times \left[ W^{1,t}(\Omega) \cap L^2_0(\Omega) \right] \text{ onto } L^t(\Omega)^2.
\]

The next result concerns the unique solvability of the steady transport equation in any dimension $d$, see [5].

Theorem 3 Let $\Omega$ be a bounded Lipschitz-continuous domain of $\mathbb{R}^d$ and let $u$ be a given velocity in $V$.

1. For every $f$ in $L^2(\Omega)$ and every constant $\gamma > 0$, the transport equation
\[z + \gamma u \cdot \nabla z = f \quad \text{ in } \Omega,
\]
has a unique solution $z \in X_u$ and
\[\|z\| \leq \|f\|.
\]

2. The following Green's formula holds:
\[\forall z, \theta \in X_u, \quad (u \cdot \nabla z, \theta) = - (u \cdot \nabla \theta, z).
\]

Finally, the last result establishes compact embeddings in space and time. Its proof, due to Simon, see [9].

Theorem 4 (Simon) Let $X, E, Y$ be three Banach spaces with continuous embeddings: $X \subset E \subset Y$, the imbedding of $X$ into $E$ being compact. Then for any number $q \in [1, \infty]$, the space
\[
\left\{ v \in L^q(0, T; X); \frac{\partial v}{\partial t} \in L^1(0, T; Y) \right\}
\]
is compactly imbedded in $L^q(0, T; E)$. 

2.3 Formulation of the problem

Let \([0,T]\) be a time interval for some positive time \(T\), let \(\Omega\) be an domain in two dimensions, with a Lipschitz-continuous boundary \(\partial \Omega\) and let \(\mathbf{n}\) denote the unit normal to \(\partial \Omega\), pointing outside \(\Omega\). Let \(\mathbf{f} \in L^2(0,T; H(\text{curl}; \Omega))\), the initial velocities \(\mathbf{u}_0, \mathbf{h}_0 \in V\) with \(\text{curl} (\mathbf{u}_0 - \alpha \Delta \mathbf{u}_0) \in L^2(\Omega)\), and we expect the velocity \(\mathbf{u} \in L^\infty(0,T; V)\) with \(\partial \mathbf{u}/\partial t \in L^2(0,T; V)\), the magnetic field \(\mathbf{h} \in L^\infty(0,t; V)\) with \(\partial \mathbf{h}/\partial t \in L^2(0,T; V)\), and the pressures \(p, \omega \in L^2(0,T; L_0^2(\Omega))\).

The system (3) can be rewritten by introducing the auxiliary variable \(z = \text{curl} (\mathbf{u} - \alpha \Delta \mathbf{u})\), as

\[
\begin{align*}
\frac{\partial}{\partial t} (\mathbf{u} - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + z \times \mathbf{u} - (\mathbf{h} \cdot \nabla) \mathbf{h} &= \mathbf{f} - \nabla (p^* + \mathbf{h}^2) \\
\frac{\partial \mathbf{h}}{\partial t} - \Delta \mathbf{h} + (\mathbf{u} \cdot \nabla) \mathbf{h} - (\mathbf{h} \cdot \nabla) \mathbf{u} &= -\text{grad} \omega \\
\text{div} \mathbf{u} &= \text{div} \mathbf{h} = 0
\end{align*}
\]

(11)

(12)

where we have used the fact that \(\text{curl} (z \times \mathbf{u}) = \mathbf{u} \cdot \nabla z\), valid in two dimensions. Considering the above equation, we can rewrite the system (11) as follows:

\[
\begin{align*}
\frac{\partial}{\partial t} (\mathbf{u} - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + z \times \mathbf{u} &= \mathbf{f} + (\mathbf{h} \cdot \nabla) \mathbf{h} - \nabla (p^* + \mathbf{h}^2) \\
\frac{\partial \mathbf{h}}{\partial t} - \Delta \mathbf{h} + (\mathbf{u} \cdot \nabla) \mathbf{h} - (\mathbf{h} \cdot \nabla) \mathbf{u} &= -\text{grad} \omega \\
\alpha \frac{\partial z}{\partial t} + \nu z + \alpha (\mathbf{u} \cdot \nabla) z &= \text{curl} (\mathbf{h} \cdot \nabla) \mathbf{h} + \text{curl} \mathbf{f} - \text{curl} \nabla (p^* + \mathbf{h}^2)
\end{align*}
\]

(13)

\[
\text{div} \mathbf{u} = \text{div} \mathbf{h} = 0.
\]

**Semi-discretization in time**

Let \(N > 1\) be an integer, define the time step \(k\) by

\[
k = \frac{T}{N}
\]

and the subdivision points \(t^n = nk\). For each \(n \geq 1\), we approximate \(\mathbf{f}(t^n)\) by the average defined almost everywhere in \(\Omega\) by

\[
f^n(x) = \frac{1}{k} \int_{t^{n-1}}^{t^n} f(x,s)ds.
\]
We set 
\[ u^0 = u_0, \quad h^0 = h_0 \text{ and } z^0 = \text{curl} (u_0 - \alpha \Delta u_0). \]

Then, our semi-discrete problem reads: Find sequences \((u^n)_{n \geq 1}, (h^n)_{n \geq 1}, (z^n)_{n \geq 1}, (p^n)_{n \geq 1}\) and \((\omega^n)_{n \geq 1}\) such that \(u^n, h^n \in V, z^n \in L^2(\Omega), \) and \(p^n, \omega^n \in L^2_0(\Omega),\) solution of:

\[
\begin{align*}
\frac{1}{k}(u^{n+1} - u^n) - \frac{1}{k} \Delta (u^{n+1} - u^n) - \nu \Delta u^{n+1} + z^n \times u^{n+1} &= f^{n+1} + h^{n+1} \cdot \nabla h^{n+1} - \nabla (p^{n+1} + (h^{n+1})^2), \\
\frac{1}{k}(h^{n+1} - h^n) - \Delta h^{n+1} + u^{n+1} \cdot \nabla h^{n+1} - h^{n+1} \cdot \nabla u^{n+1} &= -\text{grad} \omega^{n+1}, \\
\frac{\alpha}{k}(z^{n+1} - z^n) + \nu z^{n+1} + \alpha u^{n+1} \cdot \nabla z^{n+1} &= \nu \text{curl} u^{n+1} + \text{curl} f^{n+1} + \text{curl} (h^{n+1} \cdot \nabla h^{n+1}) - \text{curl} \nabla (p^{n+1} + (h^{n+1})^2),
\end{align*}
\]

(14)

Now, we will make some estimates for \(u^i, h^i, z^i, p^i\) and \(\omega^i\). Multiplying the first Eq. of (14) by \(2k u^{i+1}\), the second Eq. of (14) by \(2kh^{i+1}\) and the third Eq. of (14) by \(2kz^{i+1}\) and observing that \(\text{curl} \nabla F = 0\), for any vector field \(F\), we obtain

\[
\begin{align*}
2 (u^{i+1} - u^i, u^{i+1}) - 2\alpha (\Delta (u^{i+1} - u^i), u^{i+1}) - 2k\nu (\Delta u^{i+1}, u^{i+1}) &= 2k (f^{i+1}, u^{i+1}) + 2k (h^{i+1} \cdot \nabla h^{i+1}, u^{i+1}), \\
2 (h^{i+1} - h^i, h^{i+1}) - 2k (\Delta h^{i+1}, h^{i+1}) &= 2k (h^{i+1} \cdot \nabla u^{i+1}, h^{i+1}), \\
2 (z^{i+1} - z^i, z^{i+1}) + \frac{2\nu k}{\alpha} (z^{i+1}, z^{i+1}) &= 2k (\text{curl} f^{i+1}, z^{i+1}) + \text{curl} (h^{i+1} \cdot \nabla h^{i+1}) \cdot z^{i+1},
\end{align*}
\]

(15)

where we used that fact that \((u^{i+1}, h^{i+1}, h^{i+1}) = 0\).

**Proposition 5** The sequence \((u^n)_{n \geq 1}\) and \((h^n)_{n \geq 1}\) satisfy the following uniform a priori estimates:

\[
\begin{align*}
\sum_{i=0}^{n-1} k \|\nabla u^{i+1}\|^2_{\alpha} &\leq C \frac{2C}{\nu^2} \|f\|^2_{L^2(\Omega, \chi \omega_i)} + \frac{1}{\nu} \|u_0\|^2_{\alpha} + \frac{1}{\nu} \|h_0\|^2, \\
\sum_{i=0}^{n-1} k \|\nabla h^{i+1}\|^2 &\leq C \frac{2C}{2\nu} \|f\|^2_{L^2(\Omega, \chi \omega_i)} + \frac{1}{2} \|u_0\|^2_{\alpha} + \frac{1}{2} \|h_0\|^2, \\
2 \|\nabla h^{i+1}\|^2 &+ \sum_{j=1}^{i} \left( \|\nabla h^{j+1} - \nabla h^j\|^2 + \frac{k}{\mu \sigma} \|A h^{j+1}\|^2 \right) \leq \|\nabla h_0\|^2.
\end{align*}
\]

(16)
Proof: Multiplying the first Eq. of (14) by $2u^{i+1}$ and the second Eq. of (14) by $2h^{i+1}$, we obtain
\[
\frac{2}{k} (u^{i+1} - u^i, u^{i+1}) - \frac{\alpha}{k} \Delta (u^{i+1} - u^i, u^{i+1}) - 2\nu (\Delta u^{i+1}, u^{i+1})
\]
\[
= 2 (f^{i+1}, u^{i+1}) + 2 (h^{i+1} \cdot \nabla h^{i+1}, u^{i+1}),
\]
\[
\frac{2}{k} (h^{i+1} - h^i, h^{i+1}) - 2 (\Delta h^{i+1}, h^{i+1}) - 2 (h^{i+1} \cdot \nabla u^{i+1}, h^{i+1}) = 0
\]
where we should note that
\[
(z^i \times u^{i+1}, u^{i+1}) = 0, \quad \left( \nabla \left( \nu^{i+1} + (h^{i+1})^2 \right), u^{i+1} \right) = 0,
\]
\[
(\nabla \omega^{i+1}, h^{i+1}) = 0, \quad (u^{i+1} \cdot \nabla h^{i+1}, h^{i+1}) = 0.
\]
Using the formula
\[
2(a - b, a) = ||a||^2 - ||b||^2 + ||a - b||^2,
\]
that is true in any Hilbert space, and adding the above equations, adding from $i = 0$ to $n - 1$ and making use the telescopic property, we have
\[
\frac{1}{k} ||u^n||^2 + \frac{1}{k} ||h^n||^2 + \frac{1}{k} \sum_{i=0}^{n-1} \left[ ||u^{i+1} - u^i||^2 \right] + \frac{1}{k} \sum_{i=0}^{n-1} ||h^{i+1} - h^i||^2
\]
\[
+ 2\nu \sum_{i=0}^{n-1} ||\nabla u^{i+1}||^2 + 2 \sum_{i=0}^{n-1} ||\nabla h^{i+1}||^2
\]
\[
\leq 2C \sum_{i=0}^{n-1} ||f^{i+1}|| ||u^{i+1}|| + \frac{1}{k} ||u_0||^2 + \frac{1}{k} ||h_0||^2.
\]
Now taking into account that
\[
2C ||f^{i+1}|| ||u^{i+1}|| \leq
\]
\[
4C^2 \frac{\delta}{2} ||f^{i+1}||^2 + \frac{1}{2\delta} ||u^{i+1}||^2 \leq 4C^2 \frac{\delta}{2} ||f^{i+1}||^2 + \frac{C}{2\delta} ||\nabla u^{i+1}||^2,
\]
than from equation (18) we can write
\[
2\nu \sum_{i=0}^{n-1} ||\nabla u^{i+1}||^2 + 2 \sum_{i=0}^{n-1} ||\nabla h^{i+1}||^2 \leq
\]
\[
\sum_{i=0}^{n-1} \left( 4C^2 \frac{\delta}{2} ||f^{i+1}||^2 + \frac{C}{2\delta} ||\nabla u^{i+1}||^2 \right) + \frac{1}{k} ||u_0||^2 + \frac{1}{k} ||h_0||^2
\]
then, putting $\delta = C/2\nu$, we obtain
\[
\nu \sum_{i=0}^{n-1} ||\nabla u^{i+1}||^2 + 2 \sum_{i=0}^{n-1} ||\nabla h^{i+1}||^2 \leq \frac{C^2C}{\nu k} ||f||^2_{L^2(\Omega, \times \omega^1, \nu^1)} + \frac{1}{k} ||u_0||^2 + \frac{1}{k} ||h_0||^2.
\]
On the other hand, to obtain estimates of the $\|\nabla h^{n+1}\|^2$, we multiply the second equation in (14) by $2Ah^{i+1}$, then we obtain (after applying the projection operator $P$)

$$\frac{2}{k} (\nabla h^{i+1} - \nabla h^i, \nabla h^{i+1}) + \frac{2}{\mu \sigma} \|Ah^{i+1}\|^2 = -2 (u^{i+1} \cdot \nabla h^{i+1}, Ah^{i+1}) + 2 (h^{i+1} \cdot \nabla u^{i+1}, Ah^{i+1}),$$

then bounded each of terms, we have

$$\frac{2}{k} (\nabla h^{i+1} - \nabla h^i, \nabla h^{i+1}) = \frac{1}{k} \|\nabla h^{i+1}\|^2 - \frac{1}{k} \|\nabla h^i\|^2 + \frac{1}{k} \|\nabla h^{i+1} - \nabla h^i\|^2,$$

$$|2 (u^{i+1} \cdot \nabla h^{i+1}, Ah^{i+1})| \leq 2 \|u^{i+1}\|_{L^6} \|\nabla h^{i+1}\|_{L^3} \|Ah^{i+1}\|^2 \leq 2 \|\nabla u^{i+1}\|^2 \|\nabla h^{i+1}\|^{1/2} \|Ah^{i+1}\|^{3/2} \leq 2C_\varepsilon \|\nabla u^{i+1}\|^4 \|\nabla h^{i+1}\|^2 + 2\varepsilon \|Ah^{i+1}\|^2,$$

$$|2 (h^{i+1} \cdot \nabla u^{i+1}, Ah^{i+1})| \leq 2 \|h^{i+1}\|_{L^\infty} \|\nabla u^{i+1}\| \|Ah^{i+1}\| \leq 2C \|\nabla h^{i+1}\|^{1/2} \|Ah^{i+1}\|^{3/2} \|\nabla u^{i+1}\| \leq 2CC_\varepsilon \|\nabla u^{i+1}\|^4 \|\nabla h^{i+1}\|^2 + 2C\varepsilon \|Ah^{i+1}\|^2,$$

where we use the estimate of interpolation $\|h^{i+1}\|_{L^\infty} \leq C \|\nabla h^{i+1}\|^{1/2} \|Ah^{i+1}\|^{1/2}$. From above estimates and taking into account that $\|\nabla u^{i+1}\|$ is bounded, we have

$$\|\nabla h^{i+1}\|^2 + \|\nabla h^{i+1} - \nabla h^i\|^2 + \frac{2k}{\mu \sigma} \|Ah^{i+1}\|^2$$

$$\leq k (2C_\varepsilon + 2CC_\varepsilon) \|\nabla h^{i+1}\|^2 + k (2\varepsilon + 2C\varepsilon) \|Ah^{i+1}\|^2 + \|\nabla h^i\|^2$$

then, there is $k$ and $\varepsilon$ such that (for a sufficiently large $N$) $1 - k (2C_\varepsilon + 2CC_\varepsilon) = 1/2$ and $2k/\mu \sigma - k (2\varepsilon + 2C\varepsilon) = k/\mu \sigma$, thus, from the above inequality we can write

$$2 \|\nabla h^{i+1}\|^2 + \|\nabla h^{i+1} - \nabla h^i\|^2 + \frac{k}{\mu \sigma} \|Ah^{i+1}\|^2 \leq \|\nabla h^i\|^2,$$

from which we get (using the lemma 3.14 pg. 131 in [11] with $\eta = \gamma_i = \xi = 0$)

$$2 \|\nabla h^{i+1}\|^2 + \sum_{j=1}^i \left( \|\nabla h^{i+1} - \nabla h^j\|^2 + \frac{k}{\mu \sigma} \|Ah^{i+1}\|^2 \right) \leq \|\nabla h_0\|^2.$$

\[\square\]

**Proposition 6** The sequence $(u^n)_{n \geq 1}$ and $(h^n)_{n \geq 1}$ satisfy the following uniform a priori estimates, for $1 \leq n \leq N$

$$\|u^n\|_{\alpha}^2 + \sum_{i=0}^{n-1} \|u^{i+1} - u^i\|_{\alpha}^2 \leq \frac{C^2}{2\nu} \|f\|_{L^2(\Omega \times (0, t^n))}^2 + \|u_0\|_{\alpha}^2 + \|h_0\|^2,$$  \hspace{1cm} (19)

$$\|h^n\|_{2}^2 + \sum_{i=0}^{n-1} \|h^{i+1} - h^i\|_{2}^2 \leq \frac{C^2}{2\nu} \|f\|_{L^2(\Omega \times (0, t^n))}^2 + \|u_0\|_{\alpha}^2 + \|h_0\|^2,$$  \hspace{1cm} (20)
\textbf{Proof:} Estimates (19) and (20) are derived by adding the first and second equations of (15) and using the formula (17),
\begin{align*}
&\left\|u^{i+1}\right\|^2 - \left\|u^i\right\|^2 + \left\|u^{i+1} - u^i\right\|^2 + \left\|h^{i+1}\right\|^2 \\
&\quad - \left\|h^i\right\|^2 + \left\|h^{i+1} - h^i\right\|^2 - 2\alpha \left(\Delta (u^{i+1} - u^i), u^{i+1} - u^i\right) \\
&\quad - 2\alpha \left(\Delta (u^{i+1} - u^i), u^i\right) - 2k\nu \left\|\nabla u^{i+1}\right\|^2 + 2k \left\|\nabla h^{i+1}\right\|^2 = 2k \left(f^{i+1}, u^{i+1}\right)
\end{align*}
then adding from $i = 0$ to $i = n - 1$ and making use of the telescopic property, and again using the formulae (17), we have
\begin{align*}
\left\|u^n\right\|^2 - \left\|u_0\right\|^2 + \sum_{i=0}^{n-1} \left\|u^{i+1} - u^i\right\|^2 + \left\|h^n\right\|^2 \\
- \left\|h_0\right\|^2 + \sum_{i=0}^{n-1} \left\|h^{i+1} - h^i\right\|^2 + 2\alpha \sum_{i=0}^{n-1} \left\|\nabla (u^{i+1} - u^i)\right\|^2 \\
+ 2k \sum_{i=0}^{n-1} \left\|\nabla h^{i+1}\right\|^2 &\leq \frac{C^2}{2\nu} \sum_{i=0}^{n-1} \left\|f^{i+1}\right\|^2,
\end{align*}
now, drooping $2\alpha \sum_{i=0}^{n-1} \left\|\nabla (u^{i+1} - u^i)\right\|^2$ and $2k \sum_{i=0}^{n-1} \left\|\nabla h^{i+1}\right\|^2$ we have
\begin{align*}
\left\|u^n\right\|^2 + \sum_{i=0}^{n-1} \left\|u^{i+1} - u^i\right\|^2 + \left\|h^n\right\|^2 + \sum_{i=0}^{n-1} \left\|h^{i+1} - h^i\right\|^2 \\
&\leq \frac{C^2}{2\nu} \left\|f\right\|^2_{L^2(\Omega, t, L^2)} + \left\|u_0\right\|^2 + \left\|h_0\right\|^2,
\end{align*}
where $\sum_{i=0}^{n-1} \left\|f^{i+1}\right\|^2 = \left\|f\right\|^2_{L^2(\Omega, t, L^2)}$. From which we get the result.

\[\square\]

\textbf{Remark 7} From (21) we have
\begin{equation}
2\alpha \sum_{i=0}^{n-1} \left\|\nabla (u^{i+1} - u^i)\right\|^2 + 2k \sum_{i=0}^{n-1} \left\|\nabla h^n\right\|^2 \leq \frac{C^2}{2\nu} \left\|f\right\|^2_{L^2(\Omega, t, L^2)} + \left\|u_0\right\|^2 + \left\|h_0\right\|^2.
\end{equation}

\textbf{Estimate for $z^n$}
Before obtaining an estimate for $z^n$, we will show the following corollary

\textbf{Corollary 8} Give the sequence $(h^n)_{n \geq 1}$ we have the following uniform a priori estimates
\[\|Ah^n\| \leq C \quad \text{and} \quad \|\nabla (h^n \cdot \nabla h^n)\| \leq C \quad \forall n \in \mathbb{N}.\]
\textbf{Proof}: Applying the projector \( P \) to the second equation of (14) we obtain
\[ Ah^{n+1} = -P \left( u^{n+1} \cdot \nabla h^{n+1} \right) + P \left( h^{n+1} \cdot \nabla u^{n+1} \right) - \frac{1}{k} P \left( h^{n+1} - h^{n} \right) \]
then
\[ \| Ah^{n+1} \| \leq \| u^{n+1} \cdot \nabla h^{n+1} \| + \| h^{n+1} \cdot \nabla u^{n+1} \| + \frac{1}{k} \| h^{n+1} - h^{n} \| \] (23)
thus, each term can be estimated as follows:

\[ \text{a) } \]
\[ \| u^{n+1} \cdot \nabla h^{n+1} \| \leq \| u^{n+1} \|_{L^6(\Omega)} \| \nabla h^{n+1} \|_{L^3(\Omega)} \]
\[ \leq C \| \nabla u^{n+1} \| \| \nabla h^{n+1} \|^{1/2} \| Ah^{n+1} \|^{1/2} \]
\[ \leq C_{\varepsilon_1} + \varepsilon_1 \| Ah^{n+1} \| \text{ for } \varepsilon_1 > 0 \text{ small} \]
here was used \( H^1 \hookrightarrow L^6 \) and a result of interpolation.

\[ \text{b) } \]
\[ \| h^{n+1} \cdot \nabla u^{n+1} \| \leq \| h^{n+1} \|_{L^\infty(\Omega)} \| \nabla u^{n+1} \| \]
\[ \leq C \| h^{n+1} \|_{L^\infty(\Omega)} \]
\[ \leq C \| h^{n+1} \|^{1/2} \| Ah^{n+1} \|^{1/2} \]
\[ \leq C \| Ah^{n+1} \|^{1/2} \]
\[ \leq C_{\varepsilon_2} + \varepsilon_2 \| Ah^{n+1} \|. \]
Finally, substituting in (23) we obtain
\[ (1 - \varepsilon_1 - \varepsilon_2) \| Ah^{n+1} \| \leq C_{\varepsilon_1} + C_{\varepsilon_2} + C \]
and considering \((1 - \varepsilon_1 - \varepsilon_2) > 0\) we obtain
\[ \| Ah^{n+1} \| \leq C, \] (24)
where the constant \( C \) is generic.

Now, taking into account the equation (24) and usual estimates, we have
\[ \| \nabla (h^{n+1} \cdot \nabla h^{n+1}) \| \]
\[ \leq C_1 \| \nabla h^{n+1} \cdot \nabla h^{n+1} \| + C_2 \| h^{n+1} \cdot \nabla^2 h^{n+1} \| \]
\[ \leq C_3 \| \nabla h^{n+1} \|_{L^3(\Omega)} \| \nabla h^{n+1} \|_{L^6(\Omega)} + C_4 \| h^{n+1} \|_{L^\infty(\Omega)} \| \nabla^2 h^{n+1} \| \]
\[ \leq C_5 \| Ah^{n+1} \| \| Ah^{n+1} \| + C_6^{1/2} \| Ah^{n+1} \|^{1/2} \| Ah^{n+1} \| \]
\[ \leq C, \]
another consequence of (24) is $\|h^n \cdot \nabla h^{n+1}\| \leq C$. □

**Proposition 9** The sequence $(z^n)_{n \geq 1}$ satisfy the following uniform a priori estimates

$$\|z^n\|^2 + \sum_{i=0}^{n-1} \|z^{i+1} - z^i\|^2 \leq \left( \frac{C^2 C}{\alpha} \|f\|^2_{L^2(\Omega \times [0,T^n])} + \frac{\nu}{2\alpha} \|u_0\|^2_{\alpha} + \frac{1}{\alpha} \|h_0\| \right)$$

$$+ \frac{2\alpha k}{\nu} \|\text{curl } f\|^2_{L^2(\Omega \times (0,T^n))} + \frac{2\alpha}{\nu} CT + \|z_0\|^2.$$  

**Proof:** Multiplying the third equation of (14) by $\frac{2k}{\alpha} z^{i+1}$, we get

$$2(z^{i+1} - z^i, z^{i+1}) + \frac{2k\nu}{\alpha} (z^{i+1}, z^{i+1}) = \frac{2k\nu}{\alpha} (\text{curl } u^{i+1}, z^{i+1})$$

$$+ 2k (\text{curl } f^{i+1}, z^{i+1}) + 2k (h^{i+1} \cdot \nabla h^{i+1}, z^{i+1})$$

then, using the formula (17), we obtain

$$\|z^{i+1}\|^2 - \|z^i\|^2 + \|z^{i+1} - z^i\|^2 \leq \frac{2k\nu}{\alpha} \|\nabla u^{i+1}\| \|z^{i+1}\| + 2k \|\text{curl } f^{i+1}\| \|z^{i+1}\|$$

$$+ 2k \|\nabla (h^{i+1} \cdot \nabla h^{i+1})\| \|z^{i+1}\|$$

$$\leq \frac{2k\nu}{\alpha} \left( C_{\varepsilon_1} \|\nabla u^{i+1}\|^2 + \varepsilon_1^2 \|z^{i+1}\|^2 \right) + 2k C_{\varepsilon_2} \|\text{curl } f^{i+1}\|^2$$

$$+ 2k \varepsilon_2 \|z^{i+1}\|^2 + 2kC \|z^{i+1}\|$$

$$\leq \frac{2k\nu}{2\alpha \varepsilon_1} \|\nabla u^{i+1}\|^2 + \frac{k\nu \varepsilon_1}{\alpha} \|z^{i+1}\|^2 + 2 \frac{2k}{2\varepsilon_2} \|\text{curl } f^{i+1}\|^2$$

$$+ k \varepsilon_2 \|z^{i+1}\|^2 + \frac{k}{\varepsilon_3} + k \varepsilon_3 \|z^{i+1}\|^2.$$  

Now adding to $i = 0$ to $n - 1$ we obtain

$$\sum_{i=0}^{n-1} \left( \|z^{i+1}\|^2 - \|z^i\|^2 \right) \leq \sum_{i=0}^{n-1} \|z^{i+1} - z^i\|^2$$

$$+ \left( \frac{2k\nu}{\alpha} - \frac{k\nu \varepsilon_1}{\alpha} - k \varepsilon_2 - kC \varepsilon_3 \right) \sum_{i=0}^{n-1} \|z^{i+1}\|^2$$

$$\leq \frac{k\nu}{\alpha \varepsilon_1} \sum_{i=0}^{n-1} \|\nabla u^{i+1}\|^2 + \frac{k}{\varepsilon_2} \sum_{i=0}^{n-1} \|\text{curl } f^{i+1}\|^2 + C \frac{T}{\varepsilon_3}.$$
Where the term \( \frac{k}{\varepsilon_3} \sum_{i=0}^{n-1} C \leq \frac{k}{\varepsilon_3} C n \leq \frac{k}{\varepsilon_3} C \frac{T}{k} = C \frac{T}{\varepsilon_3} \). Now considering \( \varepsilon_1 = 1 \) and \( \varepsilon_2 = \varepsilon_3 = \nu/2\alpha \) the above equation can be written as

\[
\sum_{i=0}^{n-1} \left( \|z^{i+1}\|^{2} - \|z^{i}\|^{2} \right) + \sum_{i=0}^{n-1} \|z^{i+1} - z^{i}\|^{2} \leq \frac{k\nu}{\alpha} \sum_{i=0}^{n-1} \|\nabla u^{i+1}\|^{2} + \frac{2\alpha k}{\nu} \sum_{i=0}^{n-1} \|\text{curl} f^{i+1}\|^{2} + \frac{2\alpha C T}{\nu},
\]

consequently,

\[
\|z^{n}\|^{2} + \sum_{i=0}^{n-1} \|z^{i+1} - z^{i}\|^{2} \leq \left( \frac{C^2 C}{\alpha} \|\mathbf{f}\|_{L^2(\Omega, \times [0, t^n])}^{2} \frac{\nu}{\alpha^{2}} \|u_0\|^{2} + \frac{\nu}{2\alpha} \|h_0\|^{2} \right) + \frac{2\alpha k}{\nu} \|\text{curl} f\|_{L^2(\Omega \times [0, t^n])}^{2} + \frac{2\alpha C T}{\nu} + \|z_0\|^{2}.
\]

\[\square\]

**Proposition 10** Let

\[ C_2 = \sup_{0 \leq n \leq N-1} \|z^{n}\|^{2} \]

The sequences \((u^{n+1} - u^{n})/k\)_{n \geq 1} and \((h^{n+1} - h^{n})/k\)_{n \geq 1}, satisfy the following uniform a priori estimates:

\[
\sum_{i=0}^{n-1} \frac{1}{2k} \|u^{i+1} - u^{i}\|_{\alpha}^{2} \leq \frac{(C_1 \nu + C_2 S^2_{\alpha})^{2}}{2\alpha} \left( \frac{C^2 C}{\nu} \|\mathbf{f}\|_{L^2(\Omega, \times [0, t^n])}^{2} \right) + \frac{1}{\nu} \|u_0\|_{\alpha}^{2} + \frac{1}{\nu} \|h_0\|^{2} + C_2 \|\mathbf{f}\|_{L^2(\Omega, \times [0, t^n])}^{2} + DT
\]

\[
\sum_{i=0}^{n-1} \frac{1}{2k} \|h^{i+1} - h^{i}\|_{\alpha}^{2} \leq \frac{(C_1 \nu + C_2 S^2_{\alpha})^{2}}{2\alpha} \left( \frac{C^2 C}{\nu} \|\mathbf{f}\|_{L^2(\Omega, \times [0, t^n])}^{2} \right) + \frac{1}{\nu} \|u_0\|_{\alpha}^{2} + \frac{1}{\nu} \|h_0\|^{2} + C_2 \|\mathbf{f}\|_{L^2(\Omega, \times [0, t^n])}^{2} + DT.
\]

**Proof:** Multiplying the first Eq. (14) by \((u^{i+1} - u^{i})\) and the second Eq. of (14) by \((h^{i+1} - h^{i})\), we obtain

\[
\frac{1}{k} \left( u^{i+1} - u^{i}, u^{i+1} - u^{i} \right) - \frac{\alpha}{k} \Delta \left( u^{i+1} - u^{i}, u^{i+1} - u^{i} \right) - \nu \left( \Delta u^{i+1}, u^{i+1} - u^{i} \right)
\]

\[
+ \left( z^{i} \times u^{i+1}, u^{i+1} - u^{i} \right) = (f^{i+1}, u^{i+1} - u^{i}) + (h^{i+1} \cdot \nabla h^{i+1}, u^{i+1} - u^{i}) + (u^{i+1} \cdot \nabla h^{i+1}, h^{i+1} - h^{i})
\]

\[
- (h^{i+1} \cdot \nabla u^{i+1}, h^{i+1} - h^{i}) = 0,
\]

(26)
then adding the above equations and using that
\[ |(e^i \times u^{i+1}, u^{i+1} - u^i)| \leq C_z S_4^2 \|u^{i+1}\|_{H^1(\Omega)} \|u^{i+1} - u^i\|_{H^1(\Omega)}, \]
we have
\[
\frac{1}{k} \|u^{i+1} - u^i\|^2 + \frac{1}{k} \|h^{i+1} - h^i\|^2 \\
\leq (C_1 \nu + C_z S_4^2)^2 \frac{\varepsilon_1}{2\alpha} \|u^{i+1}\|^2_{H^1(\Omega)} + \frac{\alpha}{2C_1} \|u^{i+1} - u^i\|^2_{H^1(\Omega)} \\
+ \frac{C_2^2 \varepsilon_2}{2} \|f^{i+1}\|^2 + \frac{1}{2\varepsilon_2} \|u^{i+1} - u^i\|^2 + \frac{C_2^2 k \varepsilon_3}{2} \|h^{i+1} \cdot \nabla h^{i+1}\|^2 \\
+ \frac{1}{2k \varepsilon_3} \|u^{i+1} - u^i\|^2 + \frac{C_4^2 k \varepsilon_4}{2} \|Ah^{i+1}\|^2 \\
+ \frac{1}{2k \varepsilon_4} \|h^{i+1} - h^i\|^2 + \frac{C_5^2 \varepsilon_5}{2} \|u^{i+1} \cdot \nabla h^{i+1}\|^2 \\
+ \frac{1}{2k \varepsilon_5} \|h^{i+1} - h^i\|^2 + \frac{C_6^2 k \varepsilon_6}{2} \|h^{i+1} \cdot \nabla u^{i+1}\|^2 + \frac{1}{2k \varepsilon_6} \|h^{i+1} - h^i\|^2,
\]
then put \( \varepsilon_1 = 1, \varepsilon_2 = 2, \varepsilon_3 = 2, \varepsilon_4 = 2, \varepsilon_5 = 4 \) and \( \varepsilon_7 = 4 \), of the above equation can be written
\[
\frac{1}{2k} \|u^{i+1} - u^i\|^2 + \frac{1}{2k} \|h^{i+1} - h^i\| \\
\leq (C_1 \nu + C_z S_4^2)^2 \frac{k}{2\alpha} \|u^{i+1}\|^2_{H^1(\Omega)} + C_2^2 k \|f^{i+1}\|^2 + C_3^2 k \|h^{i+1} \cdot \nabla h^{i+1}\|^2 \\
+ C_4^2 k \|Ah^{i+1}\|^2 + 2C_5^2 k \|u^{i+1} \cdot \nabla h^{i+1}\|^2 \leq 2C_6^2 k \|h^{i+1} \cdot \nabla u^{i+1}\|^2.
\]
On the other hand, recalling the Corollary 7, we have
\[
\|h^{i+1} \cdot \nabla h^{i+1}\|^2 \leq d_1, \quad \|Ah^{i+1}\|^2 \leq d_2 \\
\|u^{i+1} \cdot \nabla h^{i+1}\|^2 \leq d_3, \quad \|h^{i+1} \cdot \nabla u^{i+1}\|^2 \leq d_4
\]
and summing from \( i = 0 \) to \( n-1 \) in (27), we obtain
\[
\sum_{i=0}^{n-1} \frac{1}{2k} \|u^{i+1} - u^i\|^2 + \sum_{i=0}^{n-1} \frac{1}{2k} \|h^{i+1} - h^i\| \\
\leq \sum_{i=0}^{n-1} (C_1 \nu + C_z S_4^2)^2 \frac{k}{2\alpha} \|u^{i+1}\|^2_{H^1(\Omega)} + C_2^2 \sum_{i=0}^{n-1} k \|f^{i+1}\|^2 \\
+ \sum_{i=0}^{n-1} Dk,
\]
where $D = C_3^2 d_1 + C_4^2 d_2 + 2C_5^2 d_3 + 2C_6^2 d_4$, then observed $n \leq N = T/k$ we can write $\sum_{i=0}^{n-1} Dk = Dnk \leq DT$, then from the above inequality we obtain

$$\sum_{i=0}^{n-1} \frac{1}{2k} \|u^{i+1} - u^i\|^2_\alpha + \sum_{i=0}^{n-1} \frac{1}{2k} \|h^{i+1} - h^i\|^2$$

$$\leq \sum_{i=0}^{n-1} \left( C_1 \nu + C_2 S_4^2 \right)^2 \frac{k}{2\alpha} \|u^{i+1}\|^2_{H^1(\Omega)} + C_2^2 \sum_{i=0}^{n-1} k \|f^{i+1}\|^2 + DT.$$ 

Indeed, from (16) we obtain

$$\sum_{i=0}^{n-1} \frac{1}{2k} \|u^{i+1} - u^i\|^2_\alpha + \sum_{i=0}^{n-1} \frac{1}{2k} \|h^{i+1} - h^i\|^2$$

$$\leq \frac{(C_1 \nu + C_2 S_4^2)^2}{\alpha} \left( \frac{C_2^2 C}{\nu^2} \|f\|^2_{L^2(\Omega, \times [0, t^m])} + \frac{1}{\nu} \|u_0\|^2 + \frac{1}{\nu} \|h_0\|^2 \right)$$

$$+ 2C_2^2 \|f\|^2_{L^2(\Omega, \times [0, t^m])} + 2DT.$$ 

From which we get the result. □

**Proposition 11** The sequence $(p^n)_{n \geq 1}$ and $(\omega^n)_{n \geq 1}$ satisfy the following uniform a priori estimates:

$$\sum_{i=0}^{n-1} \left[ k \|p^{i+1}\|^2 + k \|\omega^{i+1}\|^2 \right] \leq$$

$$\left[ \frac{(C_1 \nu + C_2 S_4^2)^2 L}{\alpha} + 4C_2^2 S_4^4 \right] \left( \frac{C_2^2 C}{\nu^2} \|f\|^2_{L^2(\Omega, \times [0, t^m])} + \frac{1}{\nu} \|u_0\|^2 + \frac{1}{\nu} \|h_0\|^2 \right)$$

$$+ \left[ 4C_2^2 C + 2C_2^2 L \right] \|f\|^2 + (\mathcal{D} + 2LD)T, \quad 1 \leq n \leq N.$$

**Proof:** Let $v_1, v_2 \in V^\perp = \{ v \in H_0^1(\Omega) ; \forall w \in V, (\nabla v, \nabla w) = 0 \}$ such that div $v_1 = p^{i+1}$ and grad $v_2 = \omega^{i+1}$. Then by multiplying the first and second eq.(14) by $v_1$ and $v_2$ respectively, and adding the results, we have

$$\frac{1}{k} (u^{i+1} - u^i, v_1) + (z^i \times u^{i+1}, v_1) + (p^{i+1}, p^{i+1})$$

$$- (h^{i+1} \cdot \nabla h^{i+1}, v_1) + \frac{1}{k} (h^{i+1} - h^i, v_2) + (u^{i+1} \cdot \nabla h^{i+1}, v_2)$$

$$- (h^{i+1} \cdot \nabla u^{i+1}, v_2) + (\omega^{i+1}, \omega^{i+1}) = (f^{i+1}, v_1).$$
where, we consider that $v_1, v_2 \in V^\perp$. Thus, we can write

$$
k \|p_{i+1}\|^2 + k \|\omega_{i+1}\|^2 \leq C_1 \frac{\delta_1}{2} \|u_{i+1} - u^i\|^2 + \frac{1}{2\delta_1} \|v_1\|^2
$$

$$+ kC_z^2 \delta_4 \frac{\delta_2}{2} \|u_{i+1}\|_{H^1}^2 + \frac{k}{2\delta_2} \|v_1\|_{H^1}^2 + kC_3^2 \delta_3 \frac{\delta_2}{2} \|h_{i+1} \cdot \nabla h_{i+1}\|^2
$$

$$+ \frac{k}{2\delta_3} \|v_1\|^2 + C_3 \frac{\delta_4}{2} \|h_{i+1} - h^i\|^2 + \frac{1}{2\delta_4} \|v_2\|^2
$$

$$+ kC_4^2 \delta_7 \frac{\delta_2}{2} \|u_{i+1} \cdot \nabla h_{i+1}\|^2 + \frac{k}{2\delta_5} \|v_2\|^2 + kC_6 \frac{\delta_7}{2} \|h_{i+1} \cdot \nabla u_{i+1}\|^2
$$

$$+ \frac{k}{2\delta_6} \|v_2\|^2 + kC_6^2 \frac{\delta_7}{2} \|f_{i+1}\|^2 + \frac{k}{2\delta_7} \|v_1\|^2.
$$

Now, considering that $H^1 \hookrightarrow L^2$, i.e.,

$$
\|v_1\| \leq C \|\nabla v_1\| = C \|p_{i+1}\|
$$

and

$$
\|v_2\| \leq C \|\nabla v_2\| = C \|w_{i+1}\|.
$$

From which, we obtain

$$
k \|p_{i+1}\|^2 + k \|\omega_{i+1}\|^2 \leq C_1 \frac{\delta_1}{2} \|u_{i+1} - u^i\|^2 + \frac{C}{2\delta_1} \|p_{i+1}\|^2
$$

$$+ kC_z^2 \delta_4 \frac{\delta_2}{2} \|u_{i+1}\|_{H^1}^2 + \frac{k}{2\delta_2} \|p_{i+1}\|_{H^1}^2 + kC_3^2 \delta_3 \frac{\delta_2}{2} \|h_{i+1} \cdot \nabla h_{i+1}\|^2
$$

$$+ \frac{Ck}{2\delta_3} \|p_{i+1}\|^2 + C_3 \frac{\delta_4}{2} \|h_{i+1} - h^i\|^2 + \frac{C}{2\delta_4} \|\omega_{i+1}\|^2
$$

$$+ kC_4^2 \delta_7 \frac{\delta_2}{2} \|u_{i+1} \cdot \nabla h_{i+1}\|^2 + \frac{Ck}{2\delta_5} \|\omega_{i+1}\|^2 + kC_6^2 \frac{\delta_7}{2} \|h_{i+1} \cdot \nabla u_{i+1}\|^2
$$

$$+ \frac{Ck}{2\delta_6} \|\omega_{i+1}\|^2 + kC_6^2 \frac{\delta_7}{2} \|f_{i+1}\|^2 + \frac{Ck}{2\delta_7} \|p_{i+1}\|^2,
$$

then, taking $\delta_1 = 4C/k$, $\delta_2 = 4$, $\delta_3 = \delta_7 = 4C$ and $\delta_4 = 2C/k$, $\delta_5 = \delta_6 = 4C$, adding from $i = 0$ to $n - 1$ and multiplying by $2$, we have

$$
\sum_{i=0}^{n-1} \left[ k \|p_{i+1}\|^2 + k \|\omega_{i+1}\|^2 \right] \leq 4C_1^2 C \sum_{i=0}^{n-1} \frac{1}{k} \|u_{i+1} - u^i\|^2
$$

$$+ 4C_z^2 \delta_4 \sum_{i=0}^{n-1} k \|u_{i+1}\|_{H^1}^2 + 2C_3^2 C \sum_{i=0}^{n-1} \frac{1}{k} \|h_{i+1} - h^i\|^2 + 4C_5^2 C \|f\|^2_{L^2(\Omega, T)}
$$

$$+ \sum_{i=0}^{n-1} \left( 4C_2 C d_1 + 4C_2^2 C d_3 + 4C_5^2 C d_4 \right) k,
$$
thus, put $D = 4C_2^2 C_1 + 4C_2^2 C_2 + 4C_5^2 C_4$, noting that $n \leq T/k$ and using the inequalities (28) and (16), we have

$$
\sum_{i=0}^{n-1} \left[ k \|p_i^{i+1}\|^2 + k \|\omega^{i+1}\|^2 \right] \leq
$$

$$
\frac{(C_1 \nu + C_5 S_1^2)^2 L}{\alpha} + 4C_2^2 S_4^4 \left( \frac{C_2 C}{\nu^2} |f|^2_{L^2(\Omega \times [0, t^n])} + \frac{1}{\nu} \|u_0\|^2 + \frac{1}{\nu} \|h_0\|^2 \right)
$$

$$+ \left[ 4C_2^2 + 2C_2^2 L \right]|f|^2_{L^2(\Omega \times [0, T])} + |D + 2LD| T.$$

where $L = \max \{4C_2^2 C, 4C_5^2 C\}$

$\square$

### 2.4 Existence of solutions

Here, it is convenient to transform the sequence $(u^n), (h^n), (p^n), (\omega^n)$ and $(z^n)$ into functions. Since $(u^n), (h^n)$ and $(z^n)$ need to be differentiated, we define the piecewise linear functions in time:

$$\forall t \in [t^n, t^{n+1}], \quad u_k(t) = u^n + \frac{t - t^n}{k} (u^{n+1} - u^n), \quad 0 \leq n \leq N - 1$$

$$\forall t \in [t^n, t^{n+1}], \quad h_k(t) = h^n + \frac{t - t^n}{k} (h^{n+1} - h^n), \quad 0 \leq n \leq N - 1$$

$$\forall t \in [t^n, t^{n+1}], \quad z_k(t) = z^n + \frac{t - t^n}{k} (z^{n+1} - z^n), \quad 0 \leq n \leq N - 1.$$

Next, in view of the other terms in (14), we define the step functions:

$$\forall t \in [t^n, t^{n+1}] \quad \text{and} \quad 0 \leq n \leq N - 1;$$

$$f_k(t) = f^{n+1}, \quad w_k(t) = u^{n+1}, \quad g_k(t) = h^{n+1}, \quad p_k(t) = p^{n+1},$$

$$\omega_k(t) = \omega^{n+1}, \quad \zeta_k(t) = z^{n+1}, \quad \lambda_k(t) = z^n.$$

Then we have the following convergences.

**Proposition 12** The exist functions $u, h \in L^\infty(0, T; V)$ with $\partial u/\partial t, \partial h/\partial t \in L^2(0, T; V), p, \omega \in L^2(0, T; \text{L}^2_0(\Omega))$ and $z \in L^\infty(0, T; L^2(\Omega))$ such that a subsequence of $k$, still denoted by $k$, satis-
\[
\begin{align*}
\lim_{k \to 0} u_k &= \lim_{k \to 0} w_k = u \quad \text{weakly \ast in } L^\infty(0, T; V), \\
\lim_{k \to 0} h_k &= \lim_{k \to 0} g_k = h \quad \text{weakly \ast in } L^\infty(0, T; V), \\
\lim_{k \to 0} z_k &= \lim_{k \to 0} \zeta_k = \lim_{k \to 0} \lambda_k = z \quad \text{weakly \ast in } L^\infty(0, T; L^2(\Omega)), \\
\lim_{k \to 0} p_k &= p \quad \text{weakly in } L^2(0, T; L^2_0(\Omega)), \\
\lim_{k \to 0} \omega_k &= \omega \quad \text{weakly in } L^2(0, T; L^2_0(\Omega)), \\
\lim_{k \to 0} \frac{\partial}{\partial t} u_k &= \frac{\partial}{\partial t} u \quad \text{weakly in } L^2(0, T; V), \\
\lim_{k \to 0} \frac{\partial}{\partial t} h_k &= \frac{\partial}{\partial t} h \quad \text{weakly in } L^2(0, T; V). \\
\end{align*}
\]

Furthermore,
\[
\begin{align*}
\lim_{k \to 0} u_k &= \lim_{k \to 0} w_k = u \quad \text{strongly in } L^\infty(0, T; L^4(\Omega)^2), \\
\lim_{k \to 0} h_k &= \lim_{k \to 0} g_k = h \quad \text{strongly in } L^\infty(0, T; L^4(\Omega)^2)
\end{align*}
\]

**Proof:** Due to the uniform estimates given in Propositions 5 -10, we can extract a subsequence (still denoted by \(k\)) such that:
\[
\begin{align*}
\lim_{k \to 0} u_k &= u; \quad \lim_{k \to 0} h_k = h \quad \text{weakly \ast in } L^\infty(0, T; V), \\
\lim_{k \to 0} z_k &= z \quad \text{weakly \ast in } L^\infty(0, T; L^2(\Omega)), \\
\lim_{k \to 0} p_k &= p; \quad \lim_{k \to 0} w_k = w \quad \text{weakly in } L^2(0, T; L^2_0(\Omega)), \\
\lim_{k \to 0} \frac{\partial}{\partial t} u_k &= \frac{\partial}{\partial t} u; \quad \lim_{k \to 0} \frac{\partial}{\partial t} h_k = \frac{\partial}{\partial t} h \quad \text{weakly in } L^2(0, T; V), \\
\lim_{k \to 0} w_k &= w; \quad \lim_{k \to 0} g_k = g \quad \text{weakly \ast in } L^\infty(0, T; V), \\
\lim_{k \to 0} \zeta_k &= \zeta; \quad \lim_{k \to 0} \lambda_k = \lambda \quad \text{weakly in } L^2(0, T; L^2(\Omega)).
\end{align*}
\]
As far as the function \( w, g, \zeta \) and \( \lambda \) are concerned, observe that

\[
\forall t \in [t^n, t^{n+1}], \quad w_k(t) - u_k(t) = \frac{t^{n+1} - t}{k} (u^{n+1} - u^n), \quad 0 \leq n \leq N - 1
\]

\[
\forall t \in [t^n, t^{n+1}], \quad g_k(t) = h_k(t) = \frac{t^{n+1} - t}{k} (h^{n+1} - h^n), \quad 0 \leq n \leq N - 1
\]

\[
\forall t \in [t^n, t^{n+1}], \quad \zeta_k(t) - z_k(t) = \frac{t^{n+1} - t}{k} (z^{n+1} - z^n), \quad 0 \leq n \leq N - 1
\]

\[
\forall t \in [t^n, t^{n+1}], \quad \lambda_k(t) - z_k(t) = \frac{t^{n+1} - t}{k} (z^{n+1} - z^n), \quad 0 \leq n \leq N - 1.
\]

Therefore

\[
\| w_k - u_k \|^2_{L^2(0,T;V)} = \frac{k}{3} \sum_{n=0}^{N-1} \| u^{n+1} - u^n \|^2_{H^1(\Omega)},
\]

\[
\| g_k - h_k \|^2 = \frac{k}{3} \sum_{n=0}^{N-1} \| h^{n+1} - h^n \|^2_{H^1(\Omega)}, \tag{30}
\]

\[
\| \zeta_k - z_k \|^2_{L^2(\Omega \times [0,T])} = \| \lambda_k - z_k \|^2_{L^2([0,T])} = \frac{k}{3} \sum_{n=0}^{N-1} \| z^{n+1} - z^n \|^2.
\]

Then, using the estimates (19),(20),(25) and the uniqueness of the limit, we have \( w = u, g = h \) and \( \zeta = \lambda = z \). It remains to prove the strong convergence (29). In view of (30), it suffices to prove the strong convergence of \( u_k \) and \( h_k \). Note that, \( (u_k) \) and \( (h_k) \) are bounded uniformly in the space

\[
\left\{ v \in L^2(0,T;H_0^1(\Omega)^2); \frac{\partial v}{\partial t} \in L^2(0,T;L^4(\Omega)^2) \right\},
\]

and as the imbedding of \( H^1(\Omega) \) into \( L^4(\Omega) \) is compact, the Simon’s theorem implies that \( u_k \) and \( h_k \) converges strongly to \( u \) and \( h \) respectively in \( L^2(0,T;L^4(\Omega)^2) \).

\[ \square \]

**Theorem 13** Let \( \Omega \) be a bounded Lipschitz-continuous domain in two dimensions. Then for any \( \alpha > 0, \nu > 0, f \in L^2(0,T;H(\text{curl};\Omega)) \) and \( u_0, h_0 \in V \) with \( \text{curl} (u_0 - \alpha \Delta u_0) \in L^2(\Omega) \), problem (11) has at least one solution \( u, h \in L^\infty (0,T;V) \) with \( \frac{\partial u}{\partial t}, \frac{\partial h}{\partial t} \in L^2(0,T;V) \) and \( p, \omega \in L^2(0,T;L^2(\Omega)) \).
Proof: Let \( k \) be a subsequence satisfying the convergences of above Proposition. It is easy to check that the functions \( u_k, h_k, z_k, p_k, \omega_k, w_k, g_k, \zeta_k \) and \( \lambda_k \) satisfy the following formulations:

\[
\forall v \in H_0^1(\Omega), \forall \varphi \in C^0([0, T]), \quad \int_0^T \left[ \left( \frac{\partial}{\partial t} u_k(t), v \right) + \alpha \left( \nabla u_k(t), \nabla v \right) \right] + v \left( \nabla w_k(t), \nabla v \right) + (\lambda_k(t) \times w_k(t), \varphi) - (p_k(t), \text{div} v) \right] \varphi(t) dt
\]

\[
+ \int_0^T (g_k(t) \cdot \nabla v, g_k(t)) \varphi(t) dt = \int_0^T (f_k(t), v) \varphi(t) dt,
\]

\[
\forall g \in H_0^1(\Omega), \forall \phi \in C^0([0, T]), \quad \int_0^T \left[ \left( \frac{\partial}{\partial t} h_k(t), g \right) + (\nabla g_k(t), \nabla g) \right] - (w_k(t) \cdot \nabla g, g_k(t)) + (g_k(t) \cdot \nabla g, w_k(t)) - (w_k, \text{div} g) \phi(t) dt = 0,
\]

\[
\forall \theta \in W^{1,4}(\Omega), \forall \psi \in C^1([0, T]) \text{ with } \psi(T) = 0,
\]

\[
- \alpha \int_0^T (z_k(t), \theta) \frac{\partial}{\partial t} \psi(t) dt + \int_0^T \left[ \left( \frac{\partial}{\partial t} \zeta_k(t), \theta \right) - \alpha (w_k(t) \cdot \nabla \theta, \zeta_k(t)) \right] \psi(t) dt
\]

\[
- (z_0, \theta) \psi(0) = \int_0^T \left[ \left( \text{curl} w_k(t), \theta \right) + \alpha (\text{curl} f_k(t), \theta) \right] \psi(t) dt
\]

\[
+ \int_0^T \alpha (\text{curl} (g_k(t) \cdot \nabla g_k(t)), \theta) \psi(t) dt,
\]

where we note that

\[
\frac{\partial}{\partial t} u_k = \frac{\partial}{\partial t} \left[ u^n + \frac{t - t^n}{k} (u^{n+1} - u^n) \right] = \frac{1}{k} (u^{n+1} - u^n)
\]

Note that, the weak convergences of the proposition above, imply the convergences of all the linear terms in (31),(32) and (33) and the terms involving \( f \) also converge, from standard integration results. Thus, it suffices to check the convergence of the non-linear terms. Then, for all indices \( i \) and \( j, 1 \leq i, j \leq 2, \)

\[
\lim_{k \to 0} (w_k)_i v_j \varphi = u_i v_j \varphi; \quad \lim_{k \to 0} (g_k)_i g_j \phi = h_i g_j \phi \quad \text{strongly in} \ L^2(\Omega \times [0, T])
\]

and

\[
\lim_{k \to 0} \lambda_k = z \quad \text{weakly in} \ L^2(\Omega \times [0, T]),
\]
then, we have
\[
\lim_{k \to 0} \int_0^T (\lambda_k(t) \times w_k(t), \varphi(t)) dt = \int_0^T (\lambda(t) \times u(t), \varphi(t)) dt,
\]
\[
\lim_{k \to 0} \int_0^T (g_k(t) \cdot \nabla v, g_k(t)) \varphi(t) dt = \int_0^T (h(t) \cdot \nabla v, h(t)) \varphi(t) dt,
\]
\[
\lim_{k \to 0} \int_0^T (w_k(t) \cdot \nabla g, g_k(t)) \phi(t) dt = \int_0^T (u(t) \cdot \nabla g, h(t)) \phi(t) dt,
\]
\[
\lim_{k \to 0} \int_0^T (g_k(t) \cdot \nabla g, w_k(t)) \phi(t) dt = \int_0^T (h(t) \cdot \nabla g, u(t)) \phi(t) dt.
\]

Similarly
\[
\lim_{k \to 0} (w_k \cdot \nabla \psi) = (u \cdot \nabla \psi) \quad \text{strongly in } L^2(\Omega \times ]0,T[)
\]
\[
\lim_{k \to 0} (g_k \cdot \nabla \text{curl} \theta) = (h \cdot \nabla \text{curl} \theta) \quad \text{strongly in } L^2(\Omega \times ]0,T[)
\]

Therefore
\[
\lim_{k \to 0} \int_0^T (w_k \cdot \nabla \psi, \zeta_k) \psi(t) dt = \int_0^T (u \cdot \nabla \psi, z(t)) \psi(t) dt,
\]
\[
\lim_{k \to 0} \int_0^T (g_k \cdot \nabla \text{curl} \theta, g_k) \psi(t) dt = \int_0^T (h \cdot \nabla \text{curl} \theta, h) \psi(t) dt.
\]

Hence we can pass to the limit in (31), (32) and (33) and we obtain
\[
\forall v \in H^1_0(\Omega), \forall \varphi \in C^0([0,T]), \quad \int_0^T \left[ \left( \frac{\partial}{\partial t} u(t), v \right) + \alpha \left( \frac{\partial}{\partial t} \nabla u(t), \nabla v \right) 
\right.
\]
\[
+ \nu (\nabla u(t), \nabla v) + (z(t) \times u(t), v) - (p(t), \text{div} v) \right] \varphi(t) dt
\]
\[
+ \int_0^T (h(t) \cdot \nabla v, h(t)) \varphi(t) dt = \int_0^T (f(t), v) \varphi(t) dt,
\]
\[
\forall g \in H^1_0(\Omega), \forall \phi \in C^0([0,T]), \quad \int_0^T \left[ \left( \frac{\partial}{\partial t} h(t), g \right) + (\nabla h(t), \nabla g) 
\right.
\]
\[
- (u(t) \cdot \nabla g, h(t)) + (h(t) \cdot \nabla g, u(t)) - (\omega, \text{div} g) \right] \phi(t) dt = 0,
\]
\[
\forall \theta \in W^{1,4}(\Omega), \forall \psi \in C^1([0,T]) \quad \text{with } \psi(T) = 0,
\]
\[
- \alpha \int_0^T (z(t), \theta) \frac{\partial}{\partial t} \psi(t) dt + \int_0^T \left[ \nu (z(t), \theta) - \alpha (u(t) \cdot \nabla \theta, z(t)) \right] \psi(t) dt
\]
\[
- (z^0, \theta) \psi(0) = \int_0^T \left[ \nu (\text{curl} u(t), \theta) + \alpha (\text{curl} f(t), \theta) \right] \psi(t) dt
\]
\[
+ \int_0^T \alpha (\text{curl} (h(t) \cdot \nabla h(t)), \theta) \psi(t) dt.
\]
By choosing \( v, g \in D(\Omega)^2 \), \( \phi, \psi \) and \( \theta \in D(\Omega) \), we easily recover (13). It remains to recover the initial data, for this note for any \( g \in L^2(\Omega)^2 \) and any \( \phi \in H^1(0, T) \) satisfying \( \phi(T) = 0 \), and used the formula \( h_k(t) = h^n + \frac{t-n}{n} (h^{n+1} - h^n) \) we have

\[
\int_0^T \left( \frac{\partial}{\partial t} h_k(t), g \right) \phi(t) dt = -\int_0^T (h_k(t), g) \frac{\partial}{\partial t} \phi(t) dt - (h^0, g) \varphi(0)
\]

Passing to the limit in the equality above, we have

\[
\int_0^T \left( \frac{\partial}{\partial t} h(t), g \right) \phi(t) dt = -\int_0^T (h(t), g) \frac{\partial}{\partial t} \phi(t) dt - (h^0, g) \varphi(0).
\]

On the other hand, we have

\[
\int_0^T \left( \frac{\partial}{\partial t} h(t), g \right) \phi(t) dt = -\int_0^T (h(t), g) \frac{\partial}{\partial t} \phi(t) dt - (h(0), g) \varphi(0)
\]

where we conclude that \( h^0 = h(0) \), similarly we obtain \( u^0 = u(0) \) and \( z^0 = z(0) \). \( \square \)

With respect to the uniqueness it is possible to show an analogous to the [3]. In fact, we have

\textbf{Theorem 14} Assume that \( \Omega \) is a convex polygon. Then for any \( \alpha > 0, \nu > 0, f \) in \( L^2(0, T; H(\text{curl}; \Omega)) \) and \( u_0 \in V, h \in V \) with \( \text{curl}(u_0 - \alpha \Delta u_0) \in L^2(\Omega) \), problem (3)-(4) has exactly one solution \( (u, h, p, \omega) \in W \).

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