Discovering the manifold facets of a square integrable representation: from coherent states to open systems

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Abstract. Group representations play a central role in theoretical physics. In particular, in quantum mechanics unitary — or, in general, projective unitary — representations implement the action of an abstract symmetry group on physical states and observables. More specifically, a major role is played by the so-called square integrable representations. Indeed, the properties of these representations are fundamental in the definition of certain families of generalized coherent states, in the phase-space formulation of quantum mechanics and the associated star product formalism, in the definition of an interesting notion of function of quantum positive type, and in some recent applications to the theory of open quantum systems and to quantum information.

1. Introduction
Symmetries and group representations are fundamental in modern science. E.g., due to Wigner’s theorem on symmetry transformations [1–4], (projective) unitary group representations [5, 6] play a central role in quantum theory. More specifically, several important topics in theoretical physics and applied mathematics — phase-space quantum mechanics, quantization, signal and image processing, various group-theoretical aspects of quantum information science and of the theory of open quantum systems etc. — ultimately rely on a remarkable mathematical tool: the notion of square integrable representation of a locally compact group [7–14]. In the present contribution, we will discuss — without any purpose of completeness, but trying to illustrate how varied the whole subject is — some interesting examples where square integrable representations play a fundamental role. The reader may find further examples in the reference books [15–20].

The paper is organized as follows. In sect. 2, we first recall that the coherent states of the harmonic oscillator are generated by the action on a ‘fiducial vector’ of a square integrable projective representation: the so-called Weyl system. We then argue that the usefulness of square integrable representations is mainly due to certain ‘orthogonality relations’ generalizing Schur’s orthogonality relations for compact topological groups [6]. This generalization is highly nontrivial in the case where the relevant group is not unimodular. Moreover, in the case of a square integrable representation which is genuinely projective, passing to a central extension of the relevant group — so achieving a standard unitary representation of the central extension — one obtains, in general, a representation that is square integrable modulo the center only. We also recall that, by means of a square integrable representation, one can define a linear isometry — sometimes called the (generalized) wavelet transform — mapping the carrier Hilbert space of the representation into the Hilbert space of square integrable functions on the relevant group.
and enjoying nice properties. Standard wavelet analysis involves the (non-unimodular) affine
group of the real line.

By means of a square integrable representation — see sect. 3 — one can also construct a pair
of quantization and dequantization maps. Of course, quantizing is traditionally regarded as an
essential step for switching from the classical picture to the quantum setting [15,17,18,21]. We
stress, however, that the latter map should not be regarded as the ‘poor sister’ of the former:
Dequantizing one obtains a remarkable formulation of quantum mechanics in terms of complex
functions, where the composition of operators is replaced by a ‘non-local’ (i.e., non-pointwise)
star product of functions. Specifically, the harmonic analysis associated with a square integrable
projective representation of the group of translations on phase space — the Weyl system —
allows one to capture, in a very elegant and effective way, some peculiarities of a quantum
system versus a classical one.

This field of research is still in constant progress and there is room for new investigations
that do not fall within the traditional range of applications of abstract harmonic analysis to
theoretical physics; consider, e.g., some new applications to the theory of open quantum systems.
See sect. 4.

Finally, in sect. 5, we argue that square integrable representations are an essential ingredient
in the construction of certain state-preserving products of trace class operators.

2. Generalized coherent states and square integrable representations
Recall that the coherent states \{ |z⟩ \}_{z \in \mathbb{C}} ⊂ \mathcal{L}^2(\mathbb{R}) of the quantum harmonic oscillator [18,20,22]
are generated by a family of unitary operators \{ \mathcal{D}(z) \}_{z \in \mathbb{C}}; the so-called displacement operators:

|z⟩ = \mathcal{D}(z) |0⟩ , \quad z = \frac{1}{\sqrt{2}} (q + ip) ∈ \mathbb{C}; \quad (1)

here, the fiducial vector |0⟩ is the ground state of the harmonic oscillator Hamiltonian. The
vectors \{ |z⟩ \}_{z \in \mathbb{C}} form a tight (continuous) frame [16,18,20,23]; i.e., they give rise to an integral
resolution of the identity of the form

\frac{1}{\pi} \int d^2z \ |z⟩⟨z| = I . \quad (2)

We stress that the displacement operators form a projective representation [5] — \( z \mapsto \mathcal{D}(z) \) —
which is often called the Weyl system [23–25]. Adopting phase-space coordinates \( q,p \) — see (1) —
and considering the general case of \( 2n \) degrees of freedom, the Weyl system is given by

\[ G = \mathbb{R}^n × \mathbb{R}^n \ni (q,p) \mapsto U(q,p) := \exp(i(p \cdot \hat{q} - q \cdot \hat{p})) . \quad (3) \]

Here, \( G = \mathbb{R}^n × \mathbb{R}^n \) is the (additive) group of phase-space translations, and \( \hat{q}, \hat{p} \) are the position
and momentum operators in \( \mathcal{L}^2(\mathbb{R}^n) \) (we will always set \( \hbar = 1 \), respectively; moreover:

\[ U(q + \hat{q}, p + \hat{p}) = e^{\frac{i}{\hbar}(q \cdot \hat{p} - p \cdot \hat{q})} U(q,p) U(\hat{q}, \hat{p}). \quad (4) \]

This formula is intimately related to the canonical commutation relations, as it is clear from the
rigorous expression à la Weyl of these relations [6,25]. Note that the multiplier [5] of \( U \) — i.e., the
function \( (q,p; \hat{q}, \hat{p}) \mapsto e^{\frac{i}{\hbar}(q \cdot \hat{p} - p \cdot \hat{q})} \) on the rhs of (4) —
entails the standard symplectic form in \( \mathbb{R}^{2n} \).
Hence, it is not exact, so that the representation \( U \) is genuinely projective; namely, it cannot be
‘converted’ into a standard unitary representation of the same group [5,25]. Accordingly, another
important property of the Weyl system — it is an irreducible representation of an abelian group
in an infinite-dimensional Hilbert space — is only compatible with the fact that \( U \) is genuinely
projective (all irreducible unitary representations of abelian groups are characters [6]).
On the other hand, by a standard procedure [5], one can replace the projective representation \( U \) with a unitary representation \( \mathcal{S} \) of a non-abelian group; i.e., the central extension \( \mathbb{H}_n \) of \( G \), the so-called Heisenberg-Weyl group \([6, 15]\). This is the group \( \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \), with composition law
\[
(\tau, q, p) (\tilde{\tau}, \tilde{q}, \tilde{p}) = (\tau + \tilde{\tau} + (q \cdot \tilde{p} - p \cdot \tilde{q})/2, q + \tilde{q}, p + \tilde{p}), \quad \tau, \tilde{\tau} \in \mathbb{R}, \, q, \tilde{q}, p, \tilde{p} \in \mathbb{R}^n.
\]
Precisely, one stipulates that \( U(q, p) = \mathcal{S}(0, q, p) \), where \( \mathcal{S} \) is an irreducible unitary representation of the extended group \( \mathbb{H}_n \). It turns out that
\[
(\mathcal{S}(\tau, q, p) f) (x) := e^{-i\tau(\frac{1}{2}(q+\tilde{q}) \cdot x)} e^{ip \cdot x} f(x - q), \quad f \in L^2(\mathbb{R}^n).
\]
As already noted, the coherent states generate a resolution of the identity, i.e., \( U \) is such that
\[
\frac{1}{(2\pi)^n} \int dq \, dp \, U(q, p) |0\rangle \langle 0|^* U(q, p) = I;
\]
here, the fiducial vector \( |0\rangle \) may actually be replaced with any other (normalized) nonzero vector.

The integral decomposition (7) holds true because the projective representation \( U \) is square integrable or, equivalently, the related unitary representation \( \mathcal{S} \) is square integrable modulo the center \( \mathcal{Z}(\mathbb{H}_n) = \{(\tau, 0, 0) \in \mathbb{H}_n; \, \tau \in \mathbb{R}\} \). Moreover, the possibility of replacing the fiducial vector \( |0\rangle \) with any other normalized vector is a consequence of the fact that the group \( \mathbb{R}^n \times \mathbb{R}^n \) is abelian (hence, unimodular).

Let us now denote by \( U \) a generic irreducible — in general, projective — representation of a locally compact group \( G \), with multiplier \( \gamma: G \times G \to \mathbb{T} \), acting in a separable complex Hilbert space \( \mathcal{H} \). We assume the scalar product \( \langle \cdot, \cdot \rangle \) in \( \mathcal{H} \) to be linear in its second argument. We denote by \( U(\mathcal{H}) \) the unitary group of \( \mathcal{H} \), by \( \nu_G \) (a normalization of) the left Haar measure on \( G \), and by \( \Delta_G \) the modular function on \( G \) \([5, 6]\). For \( \psi, \phi \in \mathcal{H} \), consider the bounded and continuous ‘coefficient’ function
\[
c_{\psi\phi}: G \ni g \mapsto \langle U(g) \psi, \phi \rangle \in \mathbb{C}.
\]
Functions of this form allow us to define the set
\[
\mathcal{A}(U) := \{ \psi \in \mathcal{H} \mid \exists \phi \in \mathcal{H}; \, \phi \neq 0, \, c_{\psi\phi} \in L^2(G, \nu_G; \mathbb{C}) \}
\]
of all admissible vectors for \( U \). The representation \( U \) is called square integrable if \( \mathcal{A}(U) \neq \{0\} \).

Square integrable representations are ruled by the following fundamental result \([7–13]\):

**Theorem 1.** Let \( U: G \to U(\mathcal{H}) \) be a square integrable projective representation. The set \( \mathcal{A}(U) \) of all admissible vectors is a dense linear subspace of \( \mathcal{H} \), stable under the action of \( U \). For every pair of vectors \( \phi, \psi \in \mathcal{H} \), the coefficient \( c_{\psi\phi}: G \to \mathbb{C} \) is square integrable wrt the Haar measure \( \nu_G \). Moreover, there exists a unique positive selfadjoint, injective linear operator \( D_U \) in \( \mathcal{H} \) — the so-called Duflo-Moore operator — such that \( \mathcal{A}(U) = \text{Dom}(D_U) \), and satisfying the orthogonality relations
\[
\int_G c_{\psi\phi_1}(g) \, c_{\psi\phi_2}(g) \, d\nu_G(g) = \langle \phi_1, \phi_2 \rangle \langle D_U \psi_2, D_U \psi_1 \rangle,
\]
for all \( \phi_1, \phi_2 \in \mathcal{H} \) and all \( \psi_1, \psi_2 \in \mathcal{A}(U) \). \( D_U \) is bounded if and only if \( G \) is unimodular — i.e., \( \Delta_G \equiv 1 \) — and, in such case, it is a multiple of the identity: \( D_U = d_U I \), \( d_U > 0 \).

A few comments are in order:

(i) The square-integrability of a representation extends to its unitary equivalence class.
(ii) Every irreducible unitary representation of a compact group is square integrable, as the Haar measure of such a group is finite. If this measure is normalized as a probability measure, then the Duflo-Moore operator is of the form $\dim(H)^{-1/2} I$ (Peter-Weyl theorem [6]).

(iii) A locally compact group having a non-compact center does not admit square integrable unitary representations [13]. It may admit square integrable projective representations, or unitary representations that are square integrable modulo the center; see the example of the group of translations on phase space with the Weyl system, or of the Heisenberg-Weyl group with the Schrödinger representation. The fact that various groups of interest in physics, like the Poincaré group, do not admit square integrable representations has motivated the investigation of useful alternative approaches [13, 17, 18, 26, 27].

(iv) If $U: G \to \mathcal{U}(H)$ is a square integrable representation, then, for every $\psi \in \mathcal{H}$ such that $0 \neq \psi \in \mathscr{A}(U)$, the following resolution of the identity holds (compare with (7)):

$$\|D_U \psi\|^{-2} \int_G d\nu_G(g) |U(g)\psi\rangle \langle U(g)\psi| = I. \quad (11)$$

Moreover, one can define the linear isometry

$$W_\psi: \mathcal{H} \ni \phi \mapsto \|D_U \psi\|^{-2} c_{\psi\phi} \in L^2(G, \nu_G; \mathbb{C}), \quad (12)$$

where $c_{\psi\phi}$ is the coefficient function (8). This map is called the generalized wavelet transform associated with $U$, with analyzing vector $\psi$ [13, 18, 28].

(v) The closed subspace $\text{Ran}(W_\psi) \subset L^2(G, \nu_G; \mathbb{C})$ is a reproducing kernel Hilbert space [13, 18, 23, 29], and $W_\psi$ intertwines the representation $U$ with the left regular $\gamma$-representation [13]. If $U$ is a unitary representation, the latter is nothing but left regular representation [6].

(vi) The square integrable representations of a semidirect product, with an abelian normal factor, can be classified [12]. The classical example is the one-dimensional affine group, which gives rise to the standard wavelet transform [12, 14, 16, 24]. In this case, the analyzing vector $\psi$, or mother wavelet, must belong to the domain of the (unbounded) Duflo-Moore operator.

(vii) In wavelet analysis, one often deals with discrete frames, instead of continuous ones [16]. The possibility of achieving discrete frames from group representations, and the relation between the existence of such frames and the square-integrability of the representations, is an important issue, the so-called discretization problem [18, 28, 30].

(viii) Square integrable unitary representations are also known as representations of the discrete series since they appear as discrete summands in the integral decomposition into irreducibles of the left regular representation of a locally compact group [6, 9, 19].

3. Quantum mechanics ‘on phase space’ and square integrable representations

In addition to the wavelet transform, which maps Hilbert space vectors into complex functions on a group, by means of a square integrable representation one can also define — following various possible approaches — a pair $(\mathcal{Q}, \mathcal{D})$ formed by a quantization and by a dequantization map [15, 17, 18, 23, 24]. These maps transform functions into operators and vice versa. Among the various approaches, we focus on the most natural group-theoretical generalization of the classical scheme proposed by Weyl, Wigner, Groenewold and Moyal [21, 31–33].

For the sake of simplicity, here we consider the case where the relevant group $G$ is unimodular and, in particular, the special case of the group of translations on phase space. Let $B_2(\mathcal{H})$ be the Hilbert space of Hilbert-Schmidt operators in $\mathcal{H}$, and $B_1(\mathcal{H}) \subset B_2(\mathcal{H})$ the Banach space of trace class operators. Given a square integrable projective representation $U: G \to \mathcal{U}(\mathcal{H})$ (with $G$ unimodular), we call dequantization map the linear isometry determined by [23, 24]

$$\mathcal{D}: B_2(\mathcal{H}) \to L^2(G) \equiv L^2(G, \nu_G; \mathbb{C}), \quad (\mathcal{D} A)(g) = d_U^{-1} \text{tr}(U(g)^* A), \quad \forall A \in B_1(\mathcal{H}), \quad (13)$$
where \(d_U\) is the positive constant appearing in Theorem 1, and we have exploited the fact that \(\mathcal{B}_e(\mathcal{H})\) is a dense linear subspace of the Hilbert space \(\mathcal{B}_2(\mathcal{H})\). The quantization map associated with \(U\) is nothing but the Hilbert space adjoint of the dequantization map; namely, the surjective partial isometry

\[
\Omega := \mathcal{D}^* : L^2_\gamma(G) \to \mathcal{B}_2(\mathcal{H}).
\]

Clearly, the linear map \(\Omega\) has, in general, a nontrivial kernel \(\ker(\Omega) = \mathrm{Ran}(\mathcal{D})\).

The star product [24,34] associated with the pair \((\Omega, \mathcal{D})\) is the binary operation defined by

\[
L^2_\gamma(G) \times L^2_\gamma(G) \ni (f_1, f_2) \mapsto f_1 \star f_2 := \mathcal{D}((\Omega f_1)(\Omega f_2)) \in L^2_\gamma(G).
\]

For functions living in \(\mathrm{Ran}(\mathcal{D})\) this operation can be regarded as the dequantized product of operators. The pair \((L^2_\gamma(G), \star)\) is a \(\mathbb{H}^*\)-algebra [24,35], whose annihilator ideal is \(\mathrm{Ran}(\mathcal{D})\).

For this group-theoretical star product we have, among others [24,34], the following result:

**Theorem 2** ([24]). Let \(U : G \to \mathcal{U}(\mathcal{H})\) be a square integrable projective representation, with multiplier \(\gamma : U(gh) = \gamma(g, h)U(g)U(h)\). For every \(f_1, f_2 \in L^2_\gamma(G)\), we have:

\[
(f_1 \star f_2)(g) = d_U^{-1} \int_G f_1(h) (P f_2(h^{-1}g) \gamma(h, h^{-1}g) \, d\nu_G(h)
\]

\[
= d_U^{-1} \int_G (P f_1)(h) f_2(h^{-1}g) \gamma(h, h^{-1}g) \, d\nu_G(h)
\]

\[
= d_U^{-1} \int_G (P f_1)(h) (P f_2)(h^{-1}g) \gamma(h, h^{-1}g) \, d\nu_G(h),
\]

where \(P\) is the orthogonal projector in \(L^2_\gamma(G)\) onto \(\mathrm{Ran}(\mathcal{D})\). Thus, for every \(f_1, f_2 \in \mathrm{Ran}(\mathcal{D})\),

\[
(f_1 \star f_2)(g) = d_U^{-1} \int_G f_1(h) f_2(h^{-1}g) \gamma(h, h^{-1}g) \, d\nu_G(h). \quad \text{(`\(\gamma\)-twisted convolution')} (17)
\]

In the case of the group of translations on phase space — \(G = \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{H} = L^2(\mathbb{R}^n)\), \(U\) is the Weyl system, \(\mathrm{Ran}(\mathcal{D}) = L^2_\gamma(G) \equiv L^2(\mathbb{R}^n \times \mathbb{R}^n, (2\pi)^{-n}d^q d^p; \mathbb{C})\) (i.e., the annihilator ideal is trivial) and \(d_U = 1\). Taking into account the fact that \(\gamma(q, p; q', p') = \exp((q' \cdot p - q \cdot p')/2)\), the \(\gamma\)-twisted convolution is nothing but the classical twisted convolution [15,24]. Note however that the function \(\gamma(q, p)\) is defined for \(\gamma(q, p) = \exp((q' \cdot p - q \cdot p')/2)\), the \(\gamma\)-twisted convolution is nothing but the classical twisted convolution [15,24]. Note however that the function \(\gamma(q, p)\) is defined for \(\gamma(q, p) = \exp((q' \cdot p - q \cdot p')/2)\).

Recall that endowing the Banach space \(L^1(G, \nu_G; \mathbb{C})\) (where, now, \(G\) is a generic locally compact group) with the convolution product and with a suitable involution — i.e., considering the triple

\[
(L^1(G, \nu_G; \mathbb{C}), \odot, 1 : \varphi \mapsto \varphi^*), \quad (\varphi_1 \odot \varphi_2)(g) := \int_G \varphi_1(h) \varphi_2(h^{-1}g) \, d\nu_G(h), \quad \varphi^*(g) := \Delta_G(g^{-1}) \varphi(g^{-1})
\]

we get a Banach \(*\)-algebra, the ‘group algebra’ [6,36,37]. A positive, bounded linear functional on the \(*\)-algebra \((L^1(G, \nu_G; \mathbb{C}), \odot, 1)\) — i.e., a suitable function in \(L^\infty(G)\) — is called a function of positive type on \(G\) [6]; namely, \(\chi \in L^\infty(G)\) is of positive type if

\[
\int_G \chi(g) (\varphi^* \odot \varphi)(g) \, d\nu_G(g) \geq 0, \quad \forall \varphi \in L^1(G). \quad \text{(PTF condition)} (18)
\]

A function of positive type \(\chi \in L^\infty(G)\) agrees \(\nu_G\)-almost everywhere with a bounded continuous function [6], the ‘continuous version’ of \(\chi\), and

\[
\|\chi\|_\infty = \chi(e) \quad (\chi(e) \equiv \text{value at the identity} \ e \in G \text{ of the ‘continuous version’ of} \ \chi). \quad (19)
\]

Moreover, for a bounded continuous function \(\chi : G \to \mathbb{C}\) the following facts are equivalent [6,36]:
P1 $\chi$ is of positive type;

P2 $\chi$ is such that
\[
\int_{G} \int_{G} \chi(g^{-1}h) \overline{\varphi(g)} \varphi(h) \, dv_{G}(g) \, dv_{G}(h) \geq 0, \quad \forall \varphi \in C_{c}(G);
\]  
(20)

P3 for every finite set $\{g_{1}, \ldots, g_{m}\} \subset G$ and arbitrary numbers $c_{1}, \ldots, c_{m} \in \mathbb{C}$,
\[
\sum_{j,k} \chi(g_{j}^{-1}g_{k}) \overline{c_{j}} c_{k} \geq 0, \quad (\chi \text{ is a ‘positive definite function’} [6]).
\]  
(21)

Condition (21) defining a positive definite function may be regarded as a ‘discretization’ of (20).

Let us now assume that $G$ is abelian. By Bochner’s theorem [6], denoting by $CM(\hat{G})$ the Banach space of complex Radon measures on $\hat{G}$ — the dual group of $G$ — we can add a further item to the previous list:

P4 $\chi$ is the Fourier transform of a positive measure $\mu$ in $CM(\hat{G})$.

The physical relevance of functions of positive type becomes immediately clear once we set $G = \mathbb{R}^{n} \times \mathbb{R}^{n}$ ($\Rightarrow G = \hat{G}$) and we recall that a classical state is a normalized positive functional on the commutative $C^*$-algebra of classical observables. This is the algebra $C_{0}(\mathbb{R}^{n} \times \mathbb{R}^{n})$ of continuous complex functions vanishing at infinity, endowed with the point-wise product [36,37]. The Banach space dual of $C_{0}(\mathbb{R}^{n} \times \mathbb{R}^{n})$ is $CM(\mathbb{R}^{n} \times \mathbb{R}^{n})$ and the associated states are the Borel probability measures on $\mathbb{R}^{n} \times \mathbb{R}^{n}$. The vector-covector pairing $(f, \mu) \mapsto \langle f \rangle_{\mu}$, provides the expectation value of the observable $f$ in the state $\mu$. The latter — a set function — can conveniently be replaced with an ordinary function, its symplectic Fourier transform:

\[
\langle f \rangle_{\mu} = \int_{\mathbb{R}^{n} \times \mathbb{R}^{n}} f(q,p) \, d\mu(q,p), \quad \text{with: } f = \mathcal{F} \text{ and } \mu \text{ probability measure},
\]  
(22)

The characteristic function $\chi \equiv \hat{\mu}$ of $\mu$ is a continuous function of positive type on $\mathbb{R}^{n} \times \mathbb{R}^{n}$. All functions of the latter kind — that may be called functions of classical positive type — form a convex cone $P_{n}$. The probability measure normalization condition $\mu(\mathbb{R}^{n} \times \mathbb{R}^{n}) = 1$ corresponds to the normalization of $\chi$ as a functional: $\chi(0) = \| \chi \|_{\infty} = 1$. Thus, the convex set $\hat{P}_{n} \subset P_{n}$ of normalized functions of classical positive type coincides with the set of all functions associated to quantum mechanics.

Passing to the quantum setting, recall that in the phase-space approach to quantum mechanics a pure state $\hat{\rho}_{\psi} = \langle \psi | \psi \rangle$ is replaced with its Wigner function $\rho_{\psi} \in L^{2}(\mathbb{R}^{n} \times \mathbb{R}^{n})$ [15,17,23,24,31],

\[
\rho_{\psi}(q,p) := \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^{n}} e^{-ip \cdot x} \psi(q - \frac{x}{2}) \overline{\psi(q + \frac{x}{2})} \, d^{n}x, \quad \psi \in L^{2}(\mathbb{R}^{n}), \quad \| \psi \| = 1.
\]  
(24)

This (one-to-one) correspondence extends to all trace class operators [23,24,36,37], yielding a dense subspace $LW_{n}$ of $L^{2}(\mathbb{R}^{n} \times \mathbb{R}^{n})$, containing the convex cone $W_{n}$ of all functions associated with positive trace class operators. $W_{n}$ contains the convex set $\hat{W}_{n}$ of all Wigner functions:

\[
\rho \in \hat{W}_{n} \iff \rho \in W_{n}, \quad \lim_{r \to +\infty} \int_{|q|^{2} + |p|^{2} \leq r^{2}} \rho(q,p) \, d^{n}q \, d^{n}p = \text{tr}(\hat{\rho}) = 1.
\]  
(25)

Here, $\hat{\rho}$ is the density operator in $L^{2}(\mathbb{R}^{n})$ corresponding to the Wigner function $\rho \in \hat{W}_{n}$.
One can further replace a Wigner function $\rho$ with its symplectic Fourier-Plancherel transform
\[
(F_w \rho)(q,p) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \rho(q',p') e^{i(qq' - pp')} \, dq' dp',
\]
where, in general, the rhs integral should be regarded as a suitable Hilbert space limit [14]. The subspace $L^2_{\rho}$ is mapped by $F_w$ — a self-adjoint, unitary operator in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ — onto another dense subspace of $L^2(\mathbb{R}^n \times \mathbb{R}^n)$:
\[
L_{\rho} := \{ F_w \rho : \rho \in L^2_{\rho} \}.
\]
The convex cone $L_{\rho}$ is mapped by $F_w$ onto a convex cone $Q_{\rho} \subset L_{\rho}$. By analogy with the classical setting, one may call
\[
\tilde{\rho} := (2\pi)^n F_w \rho, \quad \text{with } q \in L_{\rho}.
\]
the quantum characteristic function of the quasi-probability distribution [22] $\rho$. Similarly to the classical setting, a quantum characteristic function $\tilde{\rho} \in Q_{\rho}$ is characterized by the normalization condition $\tilde{\rho}(0) = 1$. These characteristic functions form a convex subset $Q_{\rho}$ of $L_{\rho}$. Moreover, one finds out that [18,23,24]
\[
\tilde{\rho}(q,p) = \text{tr}(U(\rho,\rho)^*) \hat{\rho} = (\mathcal{D} \hat{\rho})(q,p),
\]
where $U$ is the Weyl system; so the previously described group-theoretical dequantization scheme yields a direct generalization of the characteristic functions rather than of the Wigner functions.

Now, the following problem arises: Is it possible to characterize intrinsically the set $W_{\rho}$ of all Wigner functions or the set $Q_{\rho}$ of quantum characteristic functions? A rigorous analysis of this problem requires a notion of function of quantum positive type, which, as in the classical setting, relies on a suitable $*$-algebra of functions and its positive functionals. As previously observed, the Hilbert space $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ becomes a $C^*$-algebra if endowed with the twisted convolution
\[
(A_1 \ast A_2)(q,p) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} A_1(q',p') A_2(q-q',p-p') e^{i(qq' - pp')} \, dq' dp',
\]
and with a suitable involution $J: A \mapsto A^*, A^*(q,p) := \overline{A(-q,-p)}$. Also recall that the twisted convolution is nothing but the canonical star product associated with the Weyl system. A function of quantum positive type is a positive, bounded linear functional on the $C^*$-algebra $(L^2(\mathbb{R}^n \times \mathbb{R}^n), \otimes, J)$ [36,37]. Therefore, $Q \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is of quantum positive type if
\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} Q(q,p) (A^* \ast A)(q,p) \, dq dp \geq 0, \quad \forall A \in L^2(\mathbb{R}^n \times \mathbb{R}^n). \quad \text{(QPTF condition)}
\]

The functions of quantum positive type — more precisely, the continuous ones — enjoy properties reminiscent of the main properties of their classical counterparts (the functions in $P_n$; i.e., the positive multiples of classical phase-space characteristic functions) [36,37]. E.g., every continuous function of quantum positive type $Q$ is bounded and $\|Q\|_\infty = Q(0)$ (recall (19)). Moreover, for a continuous function $Q: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ the following facts are equivalent:

**Q1** $Q$ is of quantum positive type;

**Q2** $Q$ is such that $(z \equiv (q,p) \in \mathbb{R}^n \times \mathbb{R}^n, d^{2n}z \equiv d^{2n}q \, d^{2n}p, \omega(z,z') \equiv \dot{q} \cdot \dot{p} - p \cdot \dot{q}' )$:
\[
\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} Q(z - z') A(z) e^{i\omega(z,z')/2} \, d^{2n}z \, d^{2n}z' \geq 0, \quad \forall A \in C_c(\mathbb{R}^n \times \mathbb{R}^n);
\]

**Q3** for every finite set $\{z_1, \ldots, z_m\}$ in phase space and arbitrary numbers $c_1, \ldots, c_m \in \mathbb{C}$,
\[
\sum_{j,k} Q(z_k - z_j) e^{i\omega(z_k,z_j)/2} c_j c_k \geq 0; \quad (Q \text{ is a ‘quantum positive definite function’})
\]

**Q4** $Q \in Q_{\rho}$; namely, $Q$ is — up to a normalization factor — the (symplectic) Fourier-Plancherel transform of a Wigner distribution.
4. The physics behind a mathematical \textit{divertissement}

We will now describe an intriguing interplay between functions of classical and quantum positive type, and its physical interpretation in the light of the theory of open quantum systems. Once again, the backbone of this construction is group-theoretical. Recall that the convolution product \(\mu_1 \otimes \mu_2\) of two probability measures \(\mu_1, \mu_2 \in \text{CM}(G)\) [6], is a probability measure too, and indeed the set of all Radon probability measures on \(G\), endowed with this product, becomes a semigroup with identity (the unit point mass measure \(\delta_e\) at the identity \(e\) of \(G\)). If \(G\) is abelian, the point-wise product \(\chi_1 \chi_2\) of two (continuous) functions of positive type on \(G\) is a (continuous) function of positive type too, because the Fourier transform maps the convolution of probability measures into the point-wise multiplication of characteristic functions. Thus, taking \(G = \mathbb{R}^n \times \mathbb{R}^n\), the set \(P_n \subset P_n\) of all normalized functions of ‘classical’ positive type, endowed with the point-wise product, is a semigroup, with the identity \(\chi \equiv 1\). Now, what is the result of multiplying a function of classical positive type by a continuous function of quantum positive type?

\textbf{Theorem 3} ([36,37]). The function \(\chi \mathcal{Q}\), which is obtained by performing the point-wise product of any \(\chi \in P_n\) by any \(\mathcal{Q} \in Q_n\), belongs to \(Q_n\); in particular, if \(\chi\) and \(\mathcal{Q}\) are normalized, to the set \(Q_n\) of quantum characteristic functions.

Consider, then, a multiplication semigroup of functions of classical positive type

\[
\{\chi_t : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}\}_{t \in \mathbb{R}^+} \subset \hat{P}_n, \quad \chi_t \chi_s = \chi_{t+s}, \ t, s \geq 0, \ \chi_0 \equiv 1.
\]

Such semigroup — assumed to be continuous wrt the topology of uniform convergence on compact sets on \(\hat{P}_n\) [6] — can be classified, since they are images, via the Fourier transform, of convolution semigroups of probability measures, ruled by the Lévy-Kintchine formula [38,39].

The fact that \(\chi_t\) is a bounded continuous function allows us to define the \textit{bounded operator}

\[
\mathcal{R}_t : L^2(\mathbb{R}^n \times \mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n \times \mathbb{R}^n), \quad (\mathcal{R}_t f)(q, p) := \chi_t(q, p) f(q, p), \quad t \geq 0.
\]

The set \(\{\mathcal{R}_t\}_{t \in \mathbb{R}^+}\) is a semigroup of operators [38]. It is natural to consider a suitable restriction of the family of operators \(\{\mathcal{R}_t\}_{t \in \mathbb{R}^+}\). Indeed, the complex linear span generated by the convex cone \(Q_n\) of continuous functions of quantum positive type is the dense subspace \(L^2(\mathbb{R}^n \times \mathbb{R}^n)\), and a semigroup of operators \(\{\mathcal{C}_t\}_{t \in \mathbb{R}^+}\) in \(L^2(\mathbb{R}^n \times \mathbb{R}^n)\) is defined as follows. By Theorem 3, it is consistent to set \[36,37\]

\[
\mathcal{C}_t : LQ_n \rightarrow LQ_n, \quad (\mathcal{C}_t \mathcal{Q})(q, p) := \chi_t(q, p) \mathcal{Q}(q, p),
\]

and we have: \(\mathcal{C}_t Q_n \subset Q_n, \mathcal{C}_t \hat{Q}_n \subset \hat{Q}_n, \{\mathcal{C}_t\}_{t \in \mathbb{R}^+}\) is called a \textit{classical-quantum semigroup} [40].

A classical-quantum semigroup is not only a mathematical \textit{divertissement}. The Weyl system \(U\) induces a symmetry action of the group \(\mathbb{R}^n \times \mathbb{R}^n\) on the space \(B_r(\mathcal{H})\) (with \(\mathcal{H} = L^2(\mathbb{R}^n)\)), where the quantum states live; i.e., the \textit{isometric representation} \(U \vee U\), where \(U \vee U(q, p) A := U(q, p) A U(q, p)^*\). Given a convolution semigroup \(\{\mu_t\}_{t \in \mathbb{R}^+}\) of probability measures on \(\mathbb{R}^n \times \mathbb{R}^n\), setting

\[
\mu_t [U] : B_r(\mathcal{H}) \rightarrow B_r(\mathcal{H}), \quad \mu_t [U] A := \int_{\mathbb{R}^n \times \mathbb{R}^n} \left(U \vee U(q, p) A\right) d\mu_t(q, p),
\]

one obtains a semigroup of operators \(\{\mu_t [U]\}_{t \in \mathbb{R}^+}\) in \(B_r(\mathcal{H})\), a \textit{twirling semigroup} [38–43].

\textbf{Theorem 4} ([36–38, 40, 41, 43, 44]). Let \(\{\chi_t\}_{t \in \mathbb{R}^+}\) be the multiplication semigroup of functions of classical positive type such that \(\chi_t(q, p) = \int e^{i(q q' - p p')} d\mu_t(q', p')\), and let \(\{\mathcal{C}_t\}_{t \in \mathbb{R}^+}\) be the associated classical-quantum semigroup. Then, the quantization map \(\mathcal{Q} : B_r(\mathbb{R}^n \times \mathbb{R}^n) \rightarrow B_r(\mathcal{H})\), generated by the Weyl system \(U\), intertwines \(\{\mathcal{C}_t\}_{t \in \mathbb{R}^+}\) with the twirling semigroup \(\{\mu_t [U]\}_{t \in \mathbb{R}^+}\), namely,

\[
\mathcal{Q} \left(\mathcal{C}_t \mathcal{Q}\right) = \mu_t [U] (\mathcal{Q} \mathcal{Q}), \quad \mathcal{Q} \in LQ_n. \quad \mathcal{Q} \in LQ_n \ (\Leftrightarrow \mathcal{Q} \mathcal{Q} \in B_r(\mathcal{H})), \quad t \geq 0.
\]

\(\{\mu_t [U]\}_{t \in \mathbb{R}^+}\) is a quantum dynamical semigroup — i.e., a semigroup of completely positive, trace-preserving linear maps — thus it describes the temporal evolution of an open quantum system. Moreover, it does not decrease the von Neumann entropy of a quantum state.
5. State-preserving products from square integrable representations

Quantum observables are embedded in a C*-algebra, the bounded operators, with the algebraic structure provided by the product (composition) of operators. States — density operators: unit trace, positive trace class operators — are only ‘indirectly’ involved in this structure, as positive functionals on this algebra. The product of two states is not, in general, itself a state. Endowing the space of trace class operators with, e.g., the Jordan product \((A, B) \mapsto A \circ B := (AB + BA)/2\), or, say, with the Lie product \((A, B) \mapsto A \cdot B := (AB - BA)/2\), one gets algebraic structures preserving selfadjointness. But the Jordan product is not associative, and the composition or both pure, but do not coincide. This proves the first assertion. To prove the second assertion, Precisely, let \(\hat{\rho}, \hat{\sigma}\) be density operators in a separable complex Hilbert space \(\mathcal{H}\); then:

(i) \(\hat{\rho} \hat{\sigma}\) is a density operator if and only if \(\hat{\rho} = \hat{\sigma} \equiv \hat{\pi}\), where \(\hat{\pi}\) is a pure state;

(ii) \(\hat{\rho} \circ \hat{\sigma}\) is a density operator if and only if \(\hat{\rho} = \hat{\sigma} \equiv \hat{\pi}\), where \(\hat{\pi}\) is a pure state;

(iii) \(\hat{\rho} \circ \hat{\sigma}\) is not a density operator.

The “if part” of (i), (ii) is obvious. Moreover, denoting by \(\langle \cdot, \cdot \rangle_{\text{HS}}\) the Hilbert-Schmidt product,

\[
\text{tr}(\hat{\rho} \hat{\sigma}) = \langle \hat{\rho}, \hat{\sigma} \rangle_{\text{HS}} \leq \sqrt{\text{tr}(\hat{\rho}^2)} \sqrt{\text{tr}(\hat{\sigma}^2)}, \quad \text{(Cauchy-Schwarz inequality)}
\]

and \(\text{tr}(\hat{\rho}^2) < 1\) if \(\hat{\rho}\) is not pure. Hence, \(\text{tr}(\hat{\rho} \hat{\sigma}) < 1\) if either \(\hat{\rho}\) or \(\hat{\sigma}\) is not pure; or if they are both pure, but do not coincide. This proves the first assertion. To prove the second assertion, just notice that \(\text{tr}(\hat{\rho} \circ \hat{\sigma}) = \text{tr}(\hat{\rho} \hat{\sigma})\). Hence, by the previous point, \(\text{tr}(\hat{\rho} \circ \hat{\sigma}) = 1\) if and only if \(\hat{\rho}, \hat{\sigma}\) are pure states and \(\hat{\rho} = \hat{\sigma}\). Finally, \(\hat{\rho} \circ \hat{\sigma}\) cannot be a density operator because \(\text{tr}(\hat{\rho} \hat{\sigma} - \hat{\sigma} \hat{\rho}) = 0\).

Now, one can ask: Is it possible to endow the Banach space \(\mathcal{B}_1(\mathcal{H})\) with a binary operation giving rise to an associative algebra structure and such that the product of two states is a state too? The answer is positive, and to construct such a product we need the following ingredients:

- A square integrable projective representation \(U\) of a locally compact group \(G\) in \(\mathcal{H}\). For simplicity, we assume that \(G\) is unimodular (recall the positive constant \(d_G\) in Theorem 1).
- A fiducial density operator \(\hat{\nu}\) in \(\mathcal{B}_1(\mathcal{H})\). The choice of \((U)\) and \(\hat{\nu}\) characterizes the product.

Then, we set \((\nu_G\) Haar measure, \(U \cup U(g) \hat{\nu} := U(g) \hat{\nu} U(g)^*\) and Bochner integral on the rhs):

\[
A \boxdot B := d_U^{-2} \int_G d\nu_G(g) \text{tr}(A(U \cup U(g)) \hat{\nu}) (U \cup U(g) B) \in \mathcal{B}_1(\mathcal{H}), \quad \text{('twirled product')} \quad (40)
\]

for all \(A, B \in \mathcal{B}_1(\mathcal{H})\). One can prove that \((\mathcal{B}_1(\mathcal{H}), \boxdot)\) is an associative algebra, and the twirled product \(\boxdot\) is state-preserving (if \(\hat{\rho}\) and \(\hat{\sigma}\) are density operators, \(\hat{\rho} \boxdot \hat{\sigma}\) is a density operator too).

The square-integrability of the representation \(U\) is an essential ingredient here, because with this condition \(d_U^{-2} \text{tr}(A(U \cup U(g)) \hat{\nu}) \nu_G(g)\) is a complex measure (actually, a probability measure if \(A \equiv \hat{\rho}\) is a density operator) — see Proposition 7 of [23] — so that definition (40) is consistent.

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