While the Bekenstein-Hawking entropy is the unique notion of entropy that makes classical black hole thermodynamics consistent, alternative entropy notions (Rényi, Tsallis, and generalized constructs) abound in the literature. We explore conditions under which they are part of a consistent horizon thermodynamics for certain classes of modified gravity black holes. We provide examples in which black hole masses and temperatures going hand-in-hand with these alternative entropies coincide with their usual counterparts associated with the Bekenstein-Hawking entropy.

Keywords: black holes; entropy; black hole thermodynamics; modified gravity.

1. Introduction

Bekenstein’s discovery [1] of an entropy proportional to the area of a black hole event horizon was the first step to black hole thermodynamics, no doubt a very insightful development in physics. An essential complement to Bekenstein’s notion of entropy is Hawking’s discovery [2] that black holes radiate scalar field quanta with a blackbody spectrum at a well-defined temperature, now called Hawking temperature. Black hole entropy and temperature were crucial ingredients in building
In the following we use the notation of Ref. [6], in which the Lorentzian signature is $-+++$, $\kappa^2 \equiv 8\pi G$ where $G$ is Newton’s constant, and units are adopted in which the speed of light $c$ and the reduced Planck constant $\hbar$ are unity.

For the prototypical Schwarzschild black hole with geometry

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2d\Omega^2$$  \hspace{1cm} (1)$$

(where $M$ is the mass and $d\Omega^2 \equiv d\vartheta^2 + \sin^2\vartheta d\varphi^2$ denotes the line element of the two-sphere of unit radius), the Bekenstein-Hawking entropy is proportional to the area $A = 4\pi r_h^2$ of the event horizon with radius $r_h = 2GM$. It is only when the internal energy $E$ and the temperature are identified with the black hole mass $M$ and the Hawking temperature $T_H = \frac{1}{4\pi r_h} = \frac{1}{8\pi GM}$ \hspace{1cm} (2)$$

respectively, that the area law of the Bekenstein-Hawking entropy \hspace{1cm} (2)\hspace{1cm} is obtained.

Then, the thermodynamical definition of entropy $dE = TdS$, which here becomes a relation between $E, T,$ and $S$ mandatory for consistency, gives

$$dS = \frac{dE}{T} = 8\pi GMdM = d \left(4\pi GM^2\right)$$  \hspace{1cm} (3)$$

and

$$S = 4\pi GM^2 + S_0,$$  \hspace{1cm} (4)$$

with $S_0$ an integration constant. When there is no black hole $M$ and $S$ vanish simultaneously, which fixes the integration constant $S_0$ to zero, hence the resulting area law

$$S = \frac{A}{4G} = \frac{\pi r_h^2}{G}$$  \hspace{1cm} (5)$$

is a consequence of assuming $E = M$ and $T = T_H$.

As done in our previous work \hspace{1cm} [7], it is legitimate to ask whether it is necessary to assume $E = M$ and $T = T_H$. The first assumption is backed by a gedankenexperiment introduced in \hspace{1cm} [7] and based on an infalling spherical dust shell of mass $M$ and initially large radius. The Birkhoff theorem \hspace{1cm} [6] guarantees that the spacetime exterior to the shell is Schwarzschild, where the constant $M$ in the line element \hspace{1cm} (1)\hspace{1cm} is now the shell mass. By the same theorem, the interior metric is the Minkowski one. A black hole forms when the shrinking shell crosses its Schwarzschild radius $r_h$. As the spacetime geometry remains asymptotically flat during collapse, the mass $M$ in the line element \hspace{1cm} (1)\hspace{1cm} remains the shell mass and coincides with the system’s energy $E = M$. The latter is conserved during collapse because the exterior static geometry remains Schwarzschild with Schwarzschild mass $M$ (the initial mass of the shell). The spherical symmetry prevents the emission of gravitational waves (quadrupole waves in general relativity) carrying away energy.
Next, should one necessarily identify the temperature with the Hawking temperature? The calculation of the Hawking temperature, which neglects backreaction on the black hole, can only proceed after the horizon geometry is fixed, which allowed Hawking to compute $T_H$ as the quantity appearing in the thermal emission of (scalar) radiation. Suppose that the black hole is located in a thermal bath at temperature $T$: then, after a transient, thermal equilibrium is established between Hawking radiation and heat bath at a final common temperature $T = T_H$. The heat bath is the analogue of a thermometer measuring the black hole (Hawking) temperature.

Let us come now to possible variations. Different notions of entropy have appeared in recent literature, beginning with the Tsallis entropy [8,9,10] and the Rényi entropy [11,12,13,14,15]), and continuing with the Sharma-Mittal [16], Barrow [17], and Kaniadakis [18,19] entropies. Another entropy definition arises in the context of non-extensive statistical mechanics applied to Loop Quantum Gravity [20,21,22]. In spite of the differences, one recognizes four properties common to all these entropies: positivity, monotonicity, Bekenstein-Hawking limit, and generalized third law.

To begin with, these entropies are always positive, as is the Bekenstein-Hawking entropy $S$. In fact, $e^S$ is the number of states, or their volume, and $e^S > 1$ (indeed, $e^S \gg 1$). Second, all these entropies strictly increase with the Bekenstein-Hawking entropy $S$. Third, these alternative entropies admit a limit in which they become the Bekenstein-Hawking entropy $S$. Finally, these entropies go to zero when the Bekenstein-Hawking entropy $S$ is zero. The standard statistical mechanics of closed systems in thermal equilibrium interprets $e^S$ as the number of states and, accordingly, if the ground, or vacuum, state is unique $S$ vanishes when the temperature hits zero. The Bekenstein-Hawking entropy $S$ is very different in this sense because it diverges when $T \rightarrow 0$ and vanishes asymptotically at infinite temperatures. It seems natural that all generalized entropies should go to zero as $S \rightarrow 0$.

Adopting these four properties, in our previous work [23] we proposed two new definitions of generalized entropy containing, respectively, six or three parameters and reproducing, in appropriate limits, the generalized entropy notions of the literature mentioned above. Could these entropies replace the Bekenstein-Hawking entropy for non-Schwarzschild black hole solutions of theories of gravity alternative to general relativity? There is plenty of such alternative gravity theories, and of their solutions (recently reviewed in [24]), in the literature. This question is answered affirmatively in the following, where we restrict to static and spherical black holes.

The thermodynamical energy of such black holes in modified gravity is discussed in the next section, while their temperature going hand-in-hand with an entropy alternative to the Bekenstein-Hawking one is the subject of Sec. [3] Section [4] proposes specific models in featuring the Tsallis, Rényi, or other generalized entropy, while conclusions are drawn in Sec. [5].
2. Thermodynamic energy associated with alternative entropy

Consider static, spherical, and asymptotically flat spacetimes in alternative theories of gravity, described by the line element
\[
d s^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\Omega^2.\tag{6}
\]
Asymptotic flatness corresponds to \(\lim_{r\to+\infty} \lambda(r) = 0\) and, normalizing the time coordinate \(t\), to \(\lim_{r\to+\infty} \nu(r) = 0\).

Let us begin with Einstein gravity and interior solutions inside matter, using the Tolman-Oppenheimer-Volkov (TOV) equation to discuss the mass. The time-time component of the Einstein equations reads
\[
-\kappa^2 \rho = \frac{1}{r^2} \left( r e^{-2\lambda} - r \right)', \tag{7}
\]
where \(\rho\) is the energy density and a prime denotes differentiation with respect to \(r\). The mass is identified by writing
\[
e^{-2\lambda} \equiv 1 - \frac{\kappa^2 m(r)}{4\pi r}, \tag{8}
\]
from which it follows that \(4\pi r^2 \rho = m'(r)\) and, integrating,
\[
m(r) = 4\pi \int_0^r r'^2 \rho(r') dr' + m_0, \tag{9}
\]
where \(m_0\) is an integration constant. In a (compact) star, the solution must be regular at the centre. Moreover, we impose
\[
\lambda \to 0, \quad \lambda'(r) = \frac{m - rm'}{r(r - 2m)} \to 0 \tag{10}
\]
as \(r \to 0\) to avoid a conical singularity, fixing \(m_0 = 0\) and
\[
m(r) = 4\pi \int_0^r r'^2 \rho(r') dr'. \tag{11}
\]
If the geometry is asymptotically Schwarzschild, the mass is
\[
M = m(r \to \infty) = 4\pi \int_0^\infty dr \ r^2 \rho(r). \tag{12}
\]
A different situation occurs if there is a central singularity, as in black holes: in this case the integration constant \(m_0\) is chosen so that
\[
M = m(r = \infty) = 4\pi \int_0^\infty dr \ r^2 \rho(r) + m_0 \tag{13}
\]
and now \(m(r = \infty)\) is not the total mass, which is instead defined by
\[
\bar{M} = \int d^3 x \sqrt{-g} \rho(r) = 4\pi \int_0^\infty \rho(r) r^2 e^{\lambda(r)} dr
= 4\pi \int_0^\infty \rho(r) r^2 \left[ 1 - \frac{2Gm(r)}{r} \right]^{-1/2} dr
= 4\pi \int_0^\infty dr \rho(r) r^2 \left[ 1 + \frac{Gm(r)}{r} - \frac{3G^2 m^2(r)}{r^2} + O(G^3) \right], \tag{14}
\]
where \( \gamma \) denotes the determinant of the three-dimensional Riemannian metric
\[
\gamma_{\ell m} \, dx^\ell \, dx^m = e^{2\lambda} \, dr^2 + r^2 \, d\Omega^2(2).
\]
(15)

Of course, we wish to compare this mass with the Newtonian mass. Let us use \( G \) instead of \( \kappa^2 \equiv 8\pi G \). The difference between the Schwarzschild mass \( M \) in (12) and the total mass \( \bar{M} \) is the gravitational binding energy of the spherical object \( E_B = M - \bar{M} \). We interpret the second term in the last line of Eq. (14) as the Newtonian gravitational potential energy
\[
-4\pi G \int_0^\infty dr \rho(r) \frac{m(r)}{r} = -\frac{G}{2} \int dV \int dV' \frac{\rho(r) \rho(r')}{|r - r'|},
\]
(16)
where \( dV \) and \( dV' \) are three-dimensional volume elements and the general-relativistic nonlinear corrections are identified by \( G^2 \) and higher powers of \( G \).

For a singular black hole, if the geometry is asymptotically Schwarzschild the integration constant \( m_0 \) in (9) can be fixed by imposing that \( m(r \to \infty) = M \). For the vacuum Schwarzschild black hole, the energy density \( \rho \) is identically zero and \( m_0 = M \), which can be regarded as the contribution from a Dirac delta centered at \( r = 0 \) \[25\]. As a result, the mass obtained here coincides with the usual ADM mass.

Moving from general relativity to modified gravity, one can still write the time-time component of the field equations as
\[
-\kappa^2 \rho_{\text{eff}} = \frac{1}{r^2} \left( r e^{-2\lambda} - r \right),
\]
(17)
but now \( \rho_{\text{eff}} \) is an effective energy density obtained by writing the field equations as effective Einstein equations containing an effective stress-energy tensor (made of the non-Einsteinian gravitational terms) in their right-hand sides. This procedure yields the effective mass
\[
m_{\text{eff}}(r) = 4\pi \int_0^r dr' r'^2 \rho_{\text{eff}}(r'),
\]
(18)
that we interpret as the mass acted upon by the attractive force at radius \( r \). For example, consider \( F(R) \) gravity with action
\[
S_{F(R)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \, F(R) + S^{(\text{matter})},
\]
(19)
where \( F(R) \) depends non-linearly on the Ricci scalar \( R \) and \( g \) is the determinant of the spacetime metric \( g_{\mu\nu} \), while \( S^{(\text{matter})} \) denotes the matter part of the action. Using the notation \( F(R) = R + f(R) \) and \( f_R(R) \equiv df(R)/dR \), the time-time field equation defines the total (i.e., matter plus effective) energy density \( \rho_{\text{eff}} = \rho + \rho_{F(R)} \), where
\[
\rho_{F(R)} \equiv \frac{1}{\kappa^2} \left\{ -\frac{f}{2} - e^{-2\lambda} \left[ \nu'' + (\nu' - \lambda') \nu' + \frac{2\nu'}{r} \right] f_R \\
+ e^{-2\lambda} \left[ f_R'' + \left( -\lambda' + \frac{2}{r} \right) f_R' \right] \right\},
\]
(20)
which produces the effective total mass $\bar{M}_{\text{eff}}$ of Eq. (14). The latter contains contributions from both matter and gravity since

$$
\bar{M}_{\text{eff}} = \int d^3x \sqrt{\gamma} \rho_{\text{eff}}(r) = \int d^3x \sqrt{\gamma} \left( \rho + \rho_{F(R)} \right). 
$$

(21)

The leading correction in the binding energy of Eq. (16) is

$$
E_{B,\text{eff}} = -G \int dV \int dV' \left[ \rho(r) + \rho_{F(R)}(r) \right] \left[ \rho(r') + \rho_{F(R)}(r') \right] \frac{|r - r'|}{|r - r'|} + \cdots
$$

(22)

and comprises contributions from the interactions between matter and gravitational energy in $F(R)$ gravity, between the gravitational energy densities of $F(R)$ gravity, and from the self-interaction of matter via the gravitational force. Accordingly, we interpret $M_{\text{eff}} \equiv m_{\text{eff}}(r \to \infty)$ as the total mass-energy of the system while $m_{\text{eff}}(r)$ is the mass-energy contained in a 2-sphere of radius $r$.

A black hole in modified gravity can have horizon radius $r_h$ different from $2GM_{\text{eff}} \equiv 2m_{\text{eff}}(r \to \infty)$. In general, if one decides to use $M_{\text{eff}}$ as the internal energy and $S = 4\pi r_h^2/4$ as the black hole entropy, then the temperature

$$
\frac{1}{T} = \frac{dS}{dM_{\text{eff}}}
$$

(23)

does not coincide with the Hawking temperature $T_H$. Alternatively, it is possible to do thermodynamics using the Hawking temperature in conjunction with the entropy

$$
dS = \frac{dM_{\text{eff}}}{T_H}
$$

(24)

instead of the Bekenstein-Hawking entropy. However, in general, $M_{\text{eff}} \neq m_{\text{eff}}(r_h)$ and the difference $M_{\text{eff}} - m_{\text{eff}}(r_h)$ could correspond to the energy outside of the horizon. Restricting to the thermodynamics of this black hole, one would identify $m_{\text{eff}}(r_h)$ with its internal energy and modify Eq. (24) as

$$
dS_{\text{bh}} = \frac{dm_{\text{eff}}(r_h)}{T_H}.
$$

(25)

3. Temperature associated with alternative entropy

Let us adopt the notation

$$
h(r) \equiv e^{2\nu(r)}, \quad h_1(r) \equiv e^{-2\lambda(r)},
$$

(26)

where the vanishing of $h(r)$ locates the black hole event horizon. We are going to show that, in general, if $h_1(r)$ does not vanish simultaneously with $h(r)$, the spacetime curvature diverges on the surface $h(r) = 0$. If, instead, $h_1(r)$ vanishes simultaneously with $h(r)$, the curvature remains finite and the surface $h_1(r) = h(r) = 0$ is an event horizon.
The proof of these statements hinges on the Kretschmann scalar, the square of the Ricci tensor, and the Ricci scalar

\[
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{1}{4h^4r^4} \left[ 4r^4h''^2h^2h_1^2 + 4r^4hh_1h''(h'_r h - h'h_1) + (h'^2h_1r^2)^2 
- 2r^4h'^3h_1h'_r + (rhh')^2 \left(h'^2r^2 + 8h_1\right)^2 
+ 8h^4 \left(r^2h'^2 + 2(1-h_1)^2\right) \right],
\tag{27}
\]

\[
R_{\mu\nu} R^{\mu\nu} = \frac{1}{8h^4r^4} \left[ 4r^4h''^2h^2h_1^2 + 4h \left[h(rh'_r + 2h_1)h' - rh'^2h_1 + 2h^2h'_r\right] r^3h_1h'' 
+ r^4h''^2h_1^2 + r^2h^2 \left(12h_1^2 + h'^2r^2\right) h'^2 - 2r^3hh_1(rh'_r + 2h_1)h'^3 
+ 4rh^3 \left(2h'_rh_1 - 4h_1 + 4h'^2 + h'^2r^2\right) h' 
+ 4h^4 \left(3h'^2r^2 + 4r(h_1 - 1)h'_r + 4(h_1 - 1)^2\right) \right],
\tag{28}
\]

\[
R = \frac{2h''h_1hr^2 - r^2h_1h'^2 + rh'(rh'_r + 4h_1) + 4h^2(h_1 + rh'_r - 1)}{2h^2r^2}.
\tag{29}
\]

The denominators of these algebraic curvature invariants contain positive powers of \(h(r)\) that make these invariants diverge in the limit \(h \to 0\). If \(h_1(r)\) vanishes simultaneously with \(h(r)\), the invariants \(27\) remain finite at the roots of \(h_1(r) = h(r) = 0\). In fact, if \(h_1(r) = h_2(r)h(r)\) and \(h_2 \neq 0\) and is regular at the roots of \(h(r) = 0\), substituting \(h_1 = h_2h\) in Eqs. \(27\) gives

\[
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = h''^2h_2^2 + h'h_2h'_r + \left(h'^2h'_r\right)^2 + \left(2h^2h'_r\right)^2 
+ 2 \frac{r^2(hh'_r + h'h_2)^2 + 2(h^2h'^2 - 1)^2}{r^4},
\tag{30}
\]

\[
R_{\mu\nu} R^{\mu\nu} = \frac{h''^2h_2^2}{2} + \frac{h''^2h'_r h_2^2}{2} + \frac{h'^2h'' h'_r h_2^2}{r} + \frac{h_2h'^2(hh'_r + h'_r)}{r} + \frac{3h_2^2h'^2r^2}{2r^2} 
+ \frac{h'^2h_2^2}{8} + \frac{h_2h'_r (hh'_r + h'_r)}{r^2} 
- \frac{2h'^2h'_r}{r^3} + \frac{2h_2h'^2r^2}{r^3} + \frac{h_2h'^2h'_r}{2r} + \frac{h_2h'^2h'_r}{r} 
\times \frac{3(hh'_r + h'_r)r^2 + 4r(hh'_r - 1)(hh'_r + h'_r) + 4(1-hh'_r)^2}{2r^3},
\tag{31}
\]
\[ R = 2h_2h'' + \frac{2rh_2h'}{r} + \frac{h'h'_2}{2} + \frac{2[h_1 + r(hh'_2 + h_2h') - 1]}{r^2}, \]  

(32)

an inspection shows that these invariants are finite when \( h(r) = 0 \).

Since both \( h_1(r) \) and \( h_2(r) \) vanish at the horizon, one can write \( h_1(r) = e^{-\lambda(r)} \) using \( m(r) \) and the horizon radius takes the form

\[ rh = \kappa \frac{m(r_h)}{4\pi} = 2Gm(r_h). \]  

(33)

The corresponding Hawking temperature is obtained geometrically. Near the horizon the radial coordinate is \( r \equiv rh + \delta r \) and

\[ e^{-2\lambda} = h_1 = hh_2 = C(r_h)(r - rh), \]  

(34)

\[ e^{2\nu} = h = h_1 = C(r_h)(r - rh), \]  

(35)

where \( C(r_h) \equiv 1 - m'(r_h) \). After a Wick rotation of the time coordinate \( t \to i\tau \), the line element (6) in the vicinity of the horizon becomes

\[ ds^2 \simeq C(r_h)\frac{dr}{C(r_h)}d\tau^2 + \kappa \rho^2 d\Omega^2. \]

(36)

This form can be simplified by passing to a new radius \( \rho \) defined in differential form by \( d\rho = d(\delta r)\sqrt{C(r_h)} \) or in finite form by

\[ \rho = 2\sqrt{\frac{rh}{C(r_h)}} \delta r, \]

(37)

and \( \delta r = \frac{C(r_h)\rho^2}{4rh} \).

Near the horizon, the line element (36) assumes the form

\[ ds^2 \simeq \frac{C(r_h)^2}{h_2^2(r_h)rh} \rho^2 d\tau^2 + d\rho^2 + rh^2 d\Omega^2. \]

(38)

It is necessary to impose that the Euclidean time coordinate \( \tau \) is periodic to prevent conical singularities near \( \rho = 0 \) in the Euclidean space generated by the Wick rotation,

\[ \frac{C(r_h)\tau}{2rh\sqrt{h_2'(r_h)}} \simeq \frac{C(r_h)\tau}{2rh\sqrt{h_2''(r_h)}} + 2\pi \]

(39)

and then the temperature corresponds to inverse of the period \( t_* \) of the Euclidean time. In finite temperature field theory, the Euclidean path integral formulation gives

\[ \int [D\phi] e^{i\phi L(\phi)} dt = Tr(e^{-t_*H}) = Tr(e^{-\frac{4\pi}{T}}) \]

(40)

and Schwarzschild black hole is endowed with the corresponding temperature

\[ T = \frac{C(r_h)}{4\pi rh\sqrt{h_2'(r_h)}} = \frac{C(r_h)}{8\pi Gm_{\text{eff}}(r_h)} = \frac{C(r_h)T_H}{\sqrt{h_2(r_h)}} \]

(41)
where the Hawking temperature is now
\[ T_H \equiv \frac{1}{8\pi G m_{\text{eff}}(r_h)}. \] (42)

In general, \( T \) deviates from the Hawking temperature by the factor
\[ \frac{C(r_h)}{\sqrt{h_2(r_h)}} \], which cannot be absorbed into a rescaling of time because (as mentioned after Eq. (6)) we have fixed the scale so that
\[ h(r \to \infty) = h_2(r \to \infty) h_1(r \to \infty) = e^{2\nu(r \to \infty)} = 1. \] (43)

Hawking radiation is obtained from the near-horizon geometry and the thermal distribution of the emitted radiation can correspond to the new temperature (41). Placing this black hole in a thermal bath in equilibrium at the temperature \( T \), the latter equals the temperature (41), which is then identified with the black hole temperature.

As seen in the previous section, identifying \( m_{\text{eff}}(r_h) \) with the black hole internal energy, Eq. (25) implies that
\[ S_{\text{bh}} = \int \frac{dm_{\text{eff}}(r_h)}{T}. \] (44)

Integrating the field equations of a certain gravitational theory, multiple integration constants \( c_i \) \( (i = 1, \cdots, N) \) appear. \( N \) is larger in theories with higher order field equations, corresponding to more degrees of freedom. For example, in general relativity the mass \( M \) of the Schwarzschild black hole appears in the metric coefficients \( e^{2\nu} = e^{-2\lambda} = 1 - 2M/r \) as an integration constant. The number \( N \) of integration constants depends on the theory and \( \lambda(r) \), \( \nu(r) \) (and, therefore, \( m(r) \), \( h(r) \), and \( h_{1,2}(r) \)) depend on the \( c_i \)’s. Equation (33) is solved for \( r_h(c_i) \) as a function of these integration constants. For the usual Schwarzschild black hole of general relativity, this relation gives the familiar Schwarzschild radius \( r_h = 2M \). Other quantities are obtained as functions of \( c_i \), such as \( h_2(r = r_h(c_i); c_i) \), etc. Equation (33) yields \( m(r_h) = m(r = r_h(c_i); c_i) = \frac{r_h(c_i)}{2G} \), which implies that the \( c_i \)’s can be parametrized using a single parameter \( \xi \), \( c_i = c_i(\xi) \). For example, for the Reissner-Nordström black hole one can fix the electric charge and choose the mass to be this single parameter \( \xi \), \( c_i = c_i(\xi) \). For example, for the Reissner-Nordström black hole one can fix the electric charge and choose the mass to be this single parameter \( \xi \) (which is equivalent to using the charge-to-mass ratio as a parameter). Proceeding in this way, (41) is used to turn Eq. (44) in the form
\[ S_{\text{bh}} = \frac{1}{2G} \int d\xi \frac{[4\pi r_h(c_i(\xi)) \sqrt{h_2(r = r_h(c_i(\xi)); c_i(\xi))}]}{1 - \frac{2m(r = r_h(c_i(\xi)); c_i(\xi))}{\partial r}} \sum_{i=1}^{N} \frac{\partial r_h(c_i)}{\partial c_i} \frac{\partial c_i}{\partial \xi}. \] (45)

\(^a\)Here we call Hawking temperature the quantity \( T_H \equiv \frac{1}{4\pi G r_h} \) without the factor \( \frac{C(r_h)}{\sqrt{h_2(r_h)}} \). The temperature (41) is given by the surface gravity \( \kappa \), \( T = \frac{\kappa}{2\pi} \), as in the standard formulation.
With the choice $\xi = r_h$, Eq. (45) reduces to
\[ S_{bh} = \frac{1}{2G} \int_0^{r_h} d\xi \frac{4\pi \xi \sqrt{h_2(r = \xi; c_i(\xi))}}{1 - \left. \frac{\partial m(r_c; c_i)}{\partial r} \right|_{r = \xi}}, \] (46)
where the integration constant is determined by the condition $S_{bh} = 0$ at $r_h = 0$.

For the Schwarzschild black hole with $h_2(x) = 1$, $m = M = \text{const.}$, one re-obtains the Bekenstein-Hawking entropy (5). If, instead, $h_2(r \rightarrow r_h)$ gives a non-trivial contribution, the entropy $S_{bh}$ can differ from the Bekenstein-Hawking entropy $S_{BH}$.

Equation (46) tells us that
\[ \left( 1 - \left. \frac{\partial m(r_c; c_i)}{\partial r} \right|_{r = r_h} \right)^2 = 16G^2 \left[ S'_{bh} (A) \right]^2; \] (47)
then, for certain expressions of the general entropies, we can find the corresponding form of
\[ \left( 1 - \left. \frac{\partial m(r_c; c_i)}{\partial r} \right|_{r = r_h} \right)^2. \] (48)

By now, several alternative notions of entropy have been introduced, with various motivations. The first was the Rényi entropy unrelated to statistics introduced in 1960 [11] to quantify the amount of information. It was used in many works, e.g., [12,13,14,15], it contains a single parameter $\alpha$, and is simply
\[ S_R = \frac{1}{\alpha} \ln (1 + \alpha S) \] (49)
where $S$ is the Bekenstein-Hawking entropy.

Another widely studied possibility is the Tsallis entropy [8,9,10]
\[ S_T = \frac{A_0}{4G} \left( \frac{A}{A_0} \right)^\delta \] (50)
originating in the non-extensive statistics of physical systems with long range interactions, where the Boltzmann-Gibbs entropy becomes inadequate because the partition function diverges. Here the constant $A_0$ has the dimensions of a length squared and the dimensionless parameter $\delta$ measures the non-extensivity. Clearly, as $\delta \rightarrow 1$, $S_T$ reduces to the Bekenstein-Hawking entropy.

The Tsallis entropy is used to define the more complicated Sharma-Mittal entropy [16]
\[ S_{SM} = \frac{1}{R} \left[ (1 + \delta S_T)^{R/\delta} - 1 \right], \] (51)
with two phenomenological parameters $R$ and $\delta$ to be determined by experiment. This entropy construct interpolates between the Rényi and the Tsallis entropies.
Another construct, the Kaniadakis entropy \cite{18,19} extends the familiar Boltzmann-Gibbs entropy to relativistic systems \cite{18,19},
\[ S_K = \frac{1}{K} \sinh (KS), \tag{52} \]
and it reduces to the Bekenstein-Hawking entropy as the parameter $K \to 0$.

Completely different motivations led to the introduction of the Barrow entropy, which was designed \cite{17} to describe spacetime foam in quantum gravity,
\[ S_B = \left( \frac{A}{A_{Pl}} \right)^{1+\Delta/2}, \tag{53} \]
The event horizon has area $A$, while $A_{Pl} \equiv 4G$ is the Planck area, and the exponent $\Delta$ embodies the quantum gravity deformation. Maximal quantum deformation corresponds to $\Delta = 1$, while $S_B$ becomes the usual Bekenstein-Hawking entropy if $\Delta \to 0$.

The last entropy that we consider, different from the previous ones, originated in non-extensive statistical mechanics applied to Loop Quantum Gravity \cite{20,12,21,22}. It is
\[ S_q = \frac{1}{1-q} \left[ e^{(1-q)\Lambda(\gamma_0)S - 1} \right], \tag{54} \]
where $q$ weights the probability of very frequent events differently than that of infrequent ones. Moreover,
\[ \Lambda(\gamma_0) = \frac{\ln 2}{\sqrt{3} \pi \gamma_0} \tag{55} \]
and $\gamma_0$ is the Barbero-Immirzi parameter. The use of different gauge groups attributes one of the two values $\frac{\ln 2}{\pi \sqrt{3}}$ or $\frac{\ln 3}{2\pi \sqrt{2}}$ to $\gamma_0$. In scale-invariant gravity $\gamma_0$ becomes a free parameter \cite{27,28,29}. The choice $\gamma_0 = \frac{\ln 2}{\pi \sqrt{3}}$ gives $\Lambda(\gamma_0) = 1$ and makes the Loop Quantum Gravity entropy \cite{54} reproduce extensive statistical mechanics. $S_q$ becomes the Bekenstein-Hawking entropy if $q = 1$. The entropy \cite{54} was used for black holes in \cite{20,12,21} and in cosmology in \cite{22}.

Two more generalizations of entropy were proposed recently in \cite{23}. The first is the six-parameter entropy
\[ S_G(\alpha_\pm, \beta_\pm, \gamma_\pm) = \frac{1}{\alpha_+ + \alpha_-} \left[ \left( 1 + \frac{\alpha_+}{\beta_+} S^{\gamma_+} \right)^{\beta_+} - \left( 1 + \frac{\alpha_-}{\beta_-} S^{\gamma_-} \right)^{-\beta_-} \right], \tag{56} \]
where all the parameters $(\alpha_\pm, \beta_\pm, \gamma_\pm)$ are positive. For suitable values of the parameters, this entropy reduces to the entropies \cite{50,49,51,53,52}, and \cite{54} previously reported. Taking $\alpha_+ = \alpha_- = 0$ and $\gamma_+ = \gamma_- = \gamma$, the values $\gamma = \delta$ or $\gamma = 1 + \Delta/2$ reproduce the Tsallis entropy \cite{50} and the Barrow entropy \cite{53}. If $\alpha_- = 0$ and we write $\alpha_+ = R$, $\beta_+ = R/\delta$, and $\gamma_+ = \delta$, the Sharma-Mittal entropy \cite{51} is obtained. If instead one takes the limit $\alpha_+ \to 0$ and $\beta_+ \to 0$ with $\alpha \equiv \alpha_+ / \beta_+$ finite, then by setting $\gamma_+ = 1$ one recovers the Rényi entropy \cite{49}. The
other limit $\beta \rightarrow 0$ of the general entropy (56) with $\gamma_\pm = 1$ and $\alpha_\pm = K$ reduces it to the Kaniadakis entropy (52). Finally, setting $\alpha_- = 0$ and $\gamma_+ = 1$ in (56), the limit $\beta_+ \rightarrow +\infty$ in conjunction with $\alpha = 1 - q$ gives back the Loop Quantum Gravity entropy (54) with $\Lambda(\gamma_0) = 1$.

The second proposal advanced in [23] contains only three parameters:

$$S_G(\alpha, \beta, \gamma) = \gamma^{-1} \left[ (\frac{\alpha}{\beta} S + 1)^{\beta} - 1 \right].$$  \hfill (57)

Again, $\alpha, \beta,$ and $\gamma$ are positive. When $\gamma = \alpha$, $S_G$ coincides with the Sharma-Mittal entropy (51) with $S_T = S$ and $\delta = 1$. If we set $\gamma = (\alpha/\beta)^\beta$, then (57) reduces to the Tsallis entropy (50) if $\beta = \delta$ and to the Barrow entropy (53) if $\alpha \rightarrow \infty$. Finally, the limit $(\alpha, \beta) \rightarrow (0, 0)$ with $\alpha/\beta$ finite yields the Rényi entropy (49), provided that $\alpha/\beta$ is replaced by $\alpha$ and that $\gamma = \alpha$.

If spherical spacetimes are considered in conjunction with the Tsallis entropy (50), Eq. (47) becomes

$$h_2 \left( r = r_h; c_i(r_h) \right) = \delta^2 \left( \frac{4\pi r_h^2}{A_0} \right)^{2(\delta-1)}$$ \hfill (58)

while, for the same geometry, the Rényi entropy (49) gives

$$h_2 \left( r = r_h; c_i(r_h) \right) = \frac{1}{\left( 1 + \frac{\pi r_h^2}{G} \right)^2}.$$ \hfill (59)

The Kaniadakis entropy (52) leads to

$$h_2 \left( r = r_h; c_i(r_h) \right) = \cosh^2 \left( \frac{\pi r_h^2}{G} \right) G$$ \hfill (60)

and the six-parameter entropy (56) yields

$$h_2 \left( r = r_h; c_i(r_h) \right) = \frac{1}{\left( \alpha_+ + \alpha_- \right)^2} \left[ (\alpha_+ \gamma_+ \left( \frac{\pi r_h^2}{G} \right)^{\gamma_+ - 1} \left( 1 + \alpha_+ \frac{\pi r_h^2}{G} \right)^{\gamma_+ - 1} \right)^{\beta_+ - 1}$$

$$+ \alpha_- \gamma_- \left( \frac{\pi r_h^2}{G} \right)^{\gamma_- - 1} \left( 1 + \alpha_- \frac{\pi r_h^2}{G} \right)^{\gamma_- - 1} \right]^{\beta_- - 1}.$$ \hfill (61)

The simplified three-parameter entropy (57) gives

$$h_2 \left( r = r_h; c_i(r_h) \right) = \frac{\alpha^2}{\gamma^2} \left[ 1 + \left( \frac{\pi r_h^2}{\beta G} \right)^{2\beta - 2} \right].$$ \hfill (62)
These possibilities may seem rather abstract, therefore in the next section, we provide concrete models that realize these relations.

4. Spherically symmetric solutions of Einstein-two-scalar models

Our first model consists of general relativity with a scalar doublet \((\phi, \chi)\) as the matter source and with action

\[
S_{(GR\phi\chi)} = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{A(\phi, \chi)}{2} \partial_\alpha \phi \partial^\alpha \phi - B(\phi, \chi) \partial_\alpha \phi \partial_\alpha \chi - \frac{C(\phi, \chi)}{2} \partial_\alpha \chi \partial_\alpha \chi - V(\phi, \chi) \right],
\]

where \(V(\phi, \chi)\) is the two scalars interaction potential, while \(A, B,\) and \(C\) depend on both of them. The energy-momentum tensor is

\[
T^\mu_\nu(\phi\chi) = g_\mu^\nu \left[ -\frac{A(\phi, \chi)}{2} \partial_\rho \phi \partial^\rho \phi - \frac{C(\phi, \chi)}{2} \partial^\nu \chi \partial_\alpha \chi - V(\phi, \chi) \right] + A(\phi, \chi) \partial_\mu \phi \partial_\nu \phi + B(\phi, \chi) \left( \partial_\mu \phi \partial_\nu \chi + \partial_\nu \phi \partial_\mu \chi \right) + C(\phi, \chi) \partial_\mu \chi \partial_\nu \chi
\]

and there are two conservation equations stemming from the contracted Bianchi identities,

\[
\begin{align*}
&\frac{A_\phi}{2} \partial_\mu \phi \partial^\nu \phi + A_\chi \partial_\mu \phi \partial^\nu \chi + \left( B_\chi - \frac{1}{2} C_\phi \right) \partial_\mu \chi \partial^\nu \chi \\
&+ \left( -\frac{A_\phi}{2} + B_\phi \right) \partial_\mu \phi \partial^\nu \phi + B_\chi \partial_\mu \phi \partial^\nu \chi + C_\phi \partial_\mu \phi \partial^\nu \phi - V_\phi = 0, \tag{65} \\
&\left( -\frac{A_\chi}{2} + C_\phi \right) \partial_\mu \phi \partial^\nu \chi + B_\chi \partial_\mu \chi \partial^\nu \chi + C_\chi \partial_\mu \chi \partial^\nu \chi + C_\phi \partial_\mu \phi \partial^\nu \chi - V_\chi = 0, \tag{66}
\end{align*}
\]

where \(A_\phi \equiv \partial A(\phi, \chi)/\partial \phi.\) We use the identifications

\[
\phi = t, \quad \chi = r
\]

without loss of generality because, for general spherical solutions, \(\phi\) and \(\chi\) depend on \(t\) and \(r\) and spherical solutions of this model contains \(\phi(t, r), \chi(t, r)\) explicitly. One can then invert these functions in spacetime regions in which they are one-to-one, \(\partial_\mu \phi\) is timelike, and \(\partial_\mu \chi\) is spacelike. Then the scalar fields are redefined, trading \(\phi\) and \(\chi\) with \(t\) and \(r\). The latter now play the role of redefined scalar fields \(\tilde{\phi}\) and \(\tilde{\chi}\) (conversely, \(\phi(t, r) \rightarrow \phi(\tilde{\phi}, \tilde{\chi})\) and \(\chi(t, r) \rightarrow \chi(\tilde{\phi}, \tilde{\chi})\)). These new fields can be identified with \(t\) and \(r\) (see Eq. (67)). The variables change \((\phi, \chi) \rightarrow (\tilde{\phi}, \tilde{\chi})\) is now incorporated into redefinitions of the coefficients \(A, B,\) and \(C\) in the action integral (63). Thus, under mild assumptions, the ansatz (67) does not imply loss of generality.

Continue the analysis of the Einstein equations sourced by this scalar doublet: their time-time, radius-radius, \((i, j)\), and time-radius components read (the remain-
such that the functions on \((\phi, \chi)\) are trivially satisfied because of the spherical symmetry) \(\frac{e^{2(\nu-\lambda)}}{\kappa^2} \left( \frac{2\nu'}{r^2} + \frac{e^{2\lambda} - 1}{r^2} \right) = -e^{2\nu} \left( \frac{A}{2} e^{-2\nu} - \frac{C}{2} e^{-2\lambda} - V \right)\), (68) \(\frac{1}{\kappa^2} \left( \frac{2\nu'}{r^2} - \frac{e^{2\lambda} - 1}{r^2} \right) = e^{2\lambda} \left( \frac{A}{2} e^{-2\nu} + \frac{C}{2} e^{-2\lambda} - V \right)\), (69) \(\frac{1}{\kappa^2} \left[ -e^{-2\nu} \left\{ \lambda + (\lambda - \nu) \frac{\lambda'}{\nu} \right\} + e^{-2\lambda} \left( r (\nu' - \lambda') + r^2 \nu'' + r^2 (\nu' - \lambda') \nu' \right) \right]
= r^2 \left( \frac{A}{2} e^{-2\nu} - \frac{C}{2} e^{-2\lambda} - V \right),
\hat{\lambda} = \frac{\kappa^2 r B}{2}. (70)

The solutions of Eqs. (68)-(71) are written as

\[ A(t, r) = \frac{1}{\kappa^2} \left[ - \left\{ \hat{\lambda} + \hat{\lambda} (\lambda - \nu) \right\} + e^{2(\nu-\lambda)} \left( \frac{e^{2\lambda} - 1}{r^2} + \frac{\nu'' + \nu' (\nu' - \lambda')}{\nu} \right) \right], \]
\[ B(t, r) = \frac{2\hat{\lambda}}{\kappa^2 r}, \]
\[ C(t, r) = \frac{1}{\kappa^2} \left[ \frac{e^{2(\nu-\lambda)}}{r^2} \left\{ \hat{\lambda} + (\hat{\lambda} - \nu) \frac{\lambda'}{\nu} \right\} - \frac{e^{2\lambda} - 1}{r^2} + \frac{\nu'' + \nu' (\lambda' - \nu')}{\nu} \right], \]
\[ V(t, r) = \frac{e^{-2\lambda}}{2\kappa^2} \left[ \frac{2 (\nu' - \lambda')}{r} + 2 \left( \frac{e^{2\lambda} - 1}{r^2} \right) \right]. \]

A, B, C, V depend on \((\phi, \chi)\) because they depend on \(t\) and \(r\) that, in turn, depend on \((\phi, \chi)\). Conversely, assigning A, B, C, V the model admits spherical solutions (6) such that the functions \(\nu\) and \(\lambda\) are arbitrary.

For static spacetime geometries, A, B, C, and V are time-independent or, equivalently, they only depend on \(\phi\) and Eqs. (72)-(75) reduce to

\[ A = \frac{1}{\kappa^2} e^{2(\nu-\lambda)} \left[ \frac{e^{2\lambda} - 1}{r^2} + \frac{\nu'}{\nu} - \nu' (\lambda' - \nu') + \nu'' \right], \]
\[ B = 0, \]
\[ C = \frac{1}{\kappa^2} \left[ 1 - \frac{e^{2\lambda}}{r^2} + \frac{\lambda' + \nu'}{\nu} + \nu' (\lambda' - \nu') - \nu'' \right], \]
\[ V = \frac{e^{-2\lambda}}{\kappa^2} \left[ \frac{e^{2\lambda} - 1}{r^2} - \frac{\nu' - \lambda'}{\nu} \right]. \]

Assuming

\[ e^{2\nu} = \frac{h}{h_2(r)} \left( 1 - \frac{r_h}{r} \right), \quad e^{-2\lambda} = h_1 = h_2 h = 1 - \frac{r_h}{r} \]
(80)
and replacing $r$ with $\phi$, the functions $A$, $B$, $C$, and $V$ become

$$A(\phi) = \frac{1}{\kappa^2 h_2(\phi)} \left(1 - \frac{r_h}{\phi}\right)^2 \left[-\frac{h'_2(\phi)}{4\phi h_2(\phi)} + \frac{3h'_2(\phi)}{4h_2(\phi)(\phi - r_h)} + \frac{h''_2(\phi)}{2h_2(\phi)}\right],$$

$$B(\phi) = 0,$$

$$C(\phi) = \frac{1}{\kappa^2} \left[\frac{5h'_2(\phi)}{4\phi h_2(\phi)} - \frac{3h'_2(\phi)}{4(\phi - r_h) h_2(\phi)} - \frac{h''_2(\phi)}{2h_2(\phi)} + \frac{1}{4} \left(\frac{h'_2(\phi)}{h_2(\phi)}\right)^2\right],$$

$$V(\phi) = \frac{1}{2\kappa^2} \left(1 - \frac{r_h}{\phi}\right) \frac{h'_2(\phi)}{h_2(\phi)},$$

It is important that $A$, $C$, and $V$ depend explicitly on the horizon radius $r_h$, hence the latter is fixed in this model. There should be other solutions in addition to those corresponding to Eq. (80), but they may be difficult to find explicitly. This problem is bypassed using the trick of Ref. [30] of adding to the Lagrangian the term $\mathcal{L}_{\rho\sigma} = \rho^\mu \partial_\mu \sigma$. Varying $\mathcal{L}_{\rho\sigma}$ with respect to $\rho^\mu$ yields constant $\sigma$,

$$\partial_\mu \sigma = 0,$$

We can identify $\sigma$ with the horizon radius $r_h$. Then, replacing $r_h$ with $\sigma$ in the functions $A$, $C$, and $V$ of Eqs. (81), (83), and (84), $r_h$ plays the role of an integration constant appearing in Eq. (85):

$$A(\phi, \sigma) = \frac{1}{\kappa^2 h_2(\phi, \sigma)} \left(\frac{\sigma}{\phi} - 1\right)^2 \left[-\frac{h_{2,\phi}}{4\phi h_2(\phi, \sigma)} + \frac{3h_{2,\phi}}{4(\phi - \sigma) h_2(\phi, \sigma)}\right],$$

$$B(\phi, \sigma) = 0,$$

$$C(\phi, \sigma) = \frac{1}{\kappa^2} \left[\frac{5h_{2,\phi}}{4\phi h_2(\phi, \sigma)} - \frac{3h_{2,\phi}}{4(\phi - \sigma) h_2(\phi, \sigma)} - \frac{h_{2,\phi}}{2h_2(\phi, \sigma)} + \left(\frac{h_2(\phi, \sigma)}{2h_2(\phi, \sigma)}\right)^2\right],$$

$$V(\phi, \sigma) = \frac{1}{2\kappa^2} \left(1 - \frac{\sigma}{\phi}\right) \frac{h_{2,\phi}}{h_2(\phi, \sigma)},$$

where $h_{2,\phi} = \partial h_2(\phi, \sigma)/\partial \phi$, $h_{2,\phi} = \partial^2 h_2(\phi, \sigma)/\partial \phi^2$.

While, in Einstein gravity, the Kerr geometry is the unique vacuum and asymptotically flat solution, in the present model the matter scalar doublet acts as hair. Hairy black holes are common in alternative theories of gravity, for example “first generation” scalar-tensor and more modern Horndeski theories\footnote{The Reissner-Nordström black hole already differs from the Schwarzschild one due to the Maxwell field sourcing the Einstein equations.}. 


Under the assumption (80), the effective mass (18) becomes the constant 
\[ m_{\text{eff}}(r) = \frac{r_c}{2G}, \] 
reduces to
\[ h_2(r = r_h; c_i(r_h)) = 16G^2 \left[ S_{bh}'(A) \right]^2 \] (90)
and choosing, as an example,
\[ h_2(\phi, \sigma) = 1 - \sigma \frac{\phi}{\phi} \left[ 1 - 16G^2 \left( S_{bh}'(4\pi\sigma^2) \right)^2 \right], \] (91)
we obtain
\[ h_2(\phi = r_h, \sigma = r_h) = 16G^2 \left[ S_{bh}'(A) \right]^2 \] (92)
as in (90). The condition
\[ h_2(\phi = r \to \infty, \sigma = r_h) = 1 \] (93)
is required by asymptotic flatness. As a conclusion, several of the generalized entropies in the last section can be realized with the choice (91).

For the Tsallis entropy (50), we find
\[ h_2(\phi, \sigma) = 1 - \sigma \frac{\phi}{\phi} \left[ 1 - \delta^2 \left( 4\pi\sigma^2 \right)^{2(\delta-1)} \right] \] (94)
while, to include other examples, the Rényi entropy (49) gives
\[ h_2(\phi, \sigma) = 1 - \sigma \frac{\phi}{\phi} \left[ 1 - \frac{1}{\left( 1 + \frac{\alpha\sigma^2}{G} \right)^2} \right], \] (95)
and the Kaniadakis entropy (52) leads to
\[ h_2(\phi, \sigma) = 1 - \sigma \frac{\phi}{\phi} \left[ 1 - \cosh \left( \frac{\pi K \sigma^2}{G} \right) \right]. \] (96)
The six-parameter entropy (50) and the three-parameter entropy (57) give, respectively,
\[ h_2(\phi, \sigma) = 1 - \sigma \frac{\phi}{\phi} \left\{ 1 - \frac{1}{\left( \alpha_+ - \alpha_- \right)^2} \left[ \alpha_+ \gamma_+ \left( \frac{\pi \sigma^2}{G} \right)^{\gamma_+ - 1} \left( 1 + \frac{\alpha_+ \gamma_+}{\beta_+} \left( \frac{\pi \sigma^2}{G} \right) \right) \right] \right. \]
\[ + \alpha_- \gamma_- \left( \frac{\pi \sigma^2}{G} \right)^{\gamma_- - 1} \left( 1 + \frac{\alpha_- \gamma_-}{\beta_-} \left( \frac{\pi \sigma^2}{G} \right) \right) \left( \beta_- - 1 \right)^2 \left\} \] (97)
and
\[ h_2(\phi, \sigma) = 1 - \sigma \frac{\phi}{\phi} \left[ 1 - \frac{\alpha^2}{\gamma^2} \left( 1 + \left( \frac{\pi \alpha \sigma^2}{\beta G} \right) \right)^{2\beta-2} \right]. \] (98)
4.1. Thermodynamics of the model (98)

Let us examine the thermodynamics of the model (98) and compare it with the one obtained with the Bekenstein-Hawking entropy (5) for the solution (80), (98).

For the solution $\phi = r$ and $\sigma = r_h$, Eq. (46) gives, at the horizon $r = r_h$,

$$S_{bh} = \frac{1}{2G} \int_0^{r_h} d\xi 4\pi\xi \left\{ \frac{\alpha}{\gamma} \left[ 1 + \left( \frac{\pi\alpha\xi^2}{\beta G} \right) \right]^{\beta-1} \right\}. \quad (99)$$

Now $m(r) = M = r_h/(2G)$ is constant and, because the thermodynamical energy is $E = M = r_h/(2G)$, it is

$$T = \frac{C (r_h) T_H}{\sqrt{\hbar^2 (r_h)}} = \frac{\gamma T_H}{\alpha \left[ 1 + \left( \frac{\pi\alpha r_h^2}{\beta G} \right) \right]^{\beta-1}}, \quad (101)$$

where the last equation follows from Eq. (41). Here $C (r_h) = 1$ because $m(r) = M$ is constant and, since

$$T_H = \frac{1}{8\pi G m_{\text{eff}} (r_h)} = \frac{1}{8\pi G m (r_h)} = \frac{1}{8\pi G M} = \frac{1}{4\pi r_h}, \quad (102)$$

the expressions (100) and (101) coincide. The integral (99) is computed explicitly giving the three-parameter entropy (57) as

$$S_{bh} = \gamma^{-1} \left\{ \left[ 1 + \left( \frac{\pi\alpha r_h^2}{\beta G} \right) \right]^{\beta-1} - 1 \right\} = \gamma^{-1} \left[ \left( 1 + \frac{\alpha}{\beta S} \right)^{\beta-1} - 1 \right]. \quad (103)$$

When we use the Bekenstein-Hawking entropy (5), we need to specify the energy or the temperature. Postulating that the internal energy $E$ is the mass function (18), for the solution (80), (98) one finds $E = M$ because $m_{\text{eff}}(r) = M = \text{const.}$ and the thermodynamic relation $\frac{1}{T} = \frac{dS}{dE}$ gives the standard Hawking temperature $T_H$, which is different from the temperature (101). If, instead, (101) is adopted as the temperature,

$$T = \frac{T_H}{\alpha \left[ 1 + \left( \frac{\pi\alpha r_h^2}{\beta G} \right) \right]^{\beta-1}} = \frac{\gamma}{4\pi\alpha r_h \left[ 1 + \left( \frac{\pi\alpha r_h^2}{\beta G} \right) \right]^{\beta-1}}, \quad (104)$$

then

$$dE = \frac{dS}{T} = \frac{4\pi^2\alpha r_h^3}{\gamma G} \left[ 1 + \left( \frac{\pi\alpha r_h^2}{\beta G} \right) \right]^{\beta-1} d r_h = \frac{4\pi\beta}{\gamma} \left\{ \left[ 1 + \left( \frac{\pi\alpha r_h^2}{\beta G} \right) \right]^{\beta} - \left[ 1 + \left( \frac{\pi\alpha r_h^2}{\beta G} \right) \right]^{\beta-1} \right\} r_h dr_h. \quad (105)$$
and the internal energy is

$$E = \frac{2\beta^2 G}{\alpha\gamma} \left\{ \frac{1}{\beta + 1} \left[ 1 + \left( \frac{\pi \alpha r_h^2}{\beta G} \right)^\beta \right]^{\beta + 1} - \frac{1}{\beta} \left[ 1 + \left( \frac{\pi \alpha r_h^2}{\beta G} \right)^\beta \right]^\beta \right\} + E_0, \quad (106)$$

where $E_0$ is an integration constant. The condition $E(r_h = 0) = 0$ determines this integration constant,

$$E_0 = -\frac{2\beta^2 G}{\alpha\gamma} \left( \frac{1}{\beta + 1} - \frac{1}{\beta} \right). \quad (107)$$

In any case, the expression of $E$ obtained is not $M = r_h/(2G)$ as in Eq. (90).

5. Conclusions

When applied to black holes, entropy notions alternative to the Bekenstein-Hawking entropy usually lead to inconsistencies in the thermodynamics because the relation $TdS = dE$ linking entropy, energy, and temperature is violated if temperature and internal energy are the Hawking temperature and the black hole mass. However, consistency can be achieved for non-Schwarzschild black holes in modified gravity if the horizon radius (consequently, the size of its area appearing in Bekenstein’s area law), are modified. This is precisely what we have explored in the previous sections. Indeed, consistency of new entropy proposals with Hawking temperature and area law is possible for certain analytical black hole solutions, reported above.

While it is interesting that the consistency problem of black hole thermodynamics can actually be cured in this way, the examples shown here are not a panacea and there are certain limitations. We have restricted the scope of this investigation to spherical, static, and asymptotically flat black holes of modified gravity. Strictly speaking, modified gravity theories that are interesting to model the current expansion of the universe without dark energy have a built-in, time-varying, cosmological “constant” and are not asymptotically flat. However, it is true that (possibly with the exception of some primordial black holes), the cosmological dynamics can be safely neglected near the black hole horizon, the spacetime region where we set our discussion. Therefore, on physical grounds, this is not a fundamental limitation.

The exploration of specific theories of gravity, for example Horndeski or Degenerate Higher Order (DHOST) theories involves plenty of details and this is the reason why we have not committed to any specific theory here, but we have provided a general, although perhaps preliminary, discussion. We have focused on the possibility of changing the black hole mass, but the strength of the gravitational coupling $G_{\text{eff}}$ varies as well in many modified gravities and can be used to one’s advantage to search for viable thermodynamics with generalized entropy, area law, black hole radii deviating from the Schwarzschild radius, and non-standard gravitational couplings (this is not an option, for example, in the Einstein-scalar doublet model of Sec. 4). For example, in Horndeski gravity the effective gravitational coupling becomes a function of both the gravitational scalar $\phi$ of the theory and of
\n\n$$\n\n$$

Specific theories, their known black hole solutions, and the possibility of consistent black hole thermodynamics will be analyzed elsewhere.

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References

[1] J. D. Bekenstein, “Black holes and entropy,” Phys. Rev. D 7 (1973), 2333-2346 doi:10.1103/PhysRevD.7.2333

[2] S. W. Hawking, “Particle Creation by Black Holes,” Commun. Math. Phys. 43 (1975), 199-220 [erratum: Commun. Math. Phys. 46 (1976), 206] doi:10.1007/BF02345020

[3] J. M. Bardeen, B. Carter and S. W. Hawking, “The Four laws of black hole mechanics,” Commun. Math. Phys. 31, 161-170 (1973) doi:10.1007/BF01645742

[4] R. M. Wald, “The thermodynamics of black holes,” Living Rev. Rel. 4 (2001), 6 doi:10.12942/lrr-2001-6 [arXiv:gr-qc/9912119 [gr-qc]].

[5] S. Carlip, “Black Hole Thermodynamics,” Int. J. Mod. Phys. D 23, 1430023 (2014) doi:10.1142/S0218271814300237 [arXiv:1410.1486 [gr-qc]].

[6] R. M. Wald, General Relativity (Chicago University Press, Chicago, 1984).

[7] S. Nojiri, S. D. Odintsov and V. Faraoni, “Area-law versus Rényi and Tsallis black hole entropies,” Phys. Rev. D 104 (2021) no.8, 084030 doi:10.1103/PhysRevD.104.084030 [arXiv:2109.05315 [gr-qc]].

[8] C. Tsallis, C, “Possible generalization of Boltzmann-Gibbs statistics”. Journal of Statistical Physics. 52 (1-2) (1988), 479-487 doi:10.1007/BF01016429

[9] J. Ren, “Analytic critical points of charged Rényi entropies from hyperbolic JHEP 05 (2021), 080 doi:10.1007/JHEP05(2021)080 [arXiv:2012.12802 [hep-th]].

[10] S. Nojiri, S. D. Odintsov and E. N. Saridakis, “Modified cosmology from extended entropy with varying exponent,” Eur. Phys. J. C 79 (2019) no.3, 242 doi:10.1140/epjc/s10052-019-6740-5 [arXiv:1903.03098 [gr-qc]].

[11] A. Rényi, “On measures of information and entropy” Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics and Probability, University of California Press (1960), 547-56

[12] V. G. Czinner and H. Iguchi, “Rényi Entropy and the Thermodynamic Stability of Black Holes,” Phys. Lett. B 752 (2016), 306-310 doi:10.1016/j.physletb.2015.11.061 [arXiv:1511.06963 [gr-qc]].

[13] L. Tannukij, P. Wongjun, E. Hirunrisisawat, T. Deesuwan and C. Promsiri, “Thermodynamics and phase transition of spherically symmetric black hole in de Sitter space from Rényi statistics,” Eur. Phys. J. Plus 135 (2020) no.6, 500 doi:10.1140/epjp/s13360-020-00517-2 [arXiv:2002.00377 [gr-qc]].

[14] C. Promsiri, E. Hirunrisisawat and W. Liewrian, “Thermodynamics and Van der Waals phase transition of charged black holes in flat space-time via Rényi statistics,” Phys. Rev. D 102 (2020) no.6, 064014 doi:10.1103/PhysRevD.102.064014 [arXiv:2003.12986 [hep-th]].

[15] D. Samart and P. Channuie, “AdS to dS phase transition mediated by thermalon in Einstein-Gauss-Bonnet gravity from Rényi statistics,” [arXiv:2012.14828 [hep-th]].
[16] A. Sayahian Jahromi, S. A. Moosavi, H. Moradpour, J. P. Morais Graça, I. P. Lobo, I. G. Salako and A. Jawad, “Generalized entropy formalism and a new holographic dark energy model,” Phys. Lett. B 780 (2018), 21-24 doi:10.1016/j.physletb.2018.02.052 [arXiv:1802.07722 [gr-qc]].

[17] J. D. Barrow, “The Area of a Rough Black Hole,” Phys. Lett. B 808 (2020), 135643 doi:10.1016/j.physletb.2020.135643 [arXiv:2004.09444 [gr-qc]].

[18] G. Kaniadakis, “Statistical mechanics in the context of special relativity. II.,” Phys. Rev. E 72 (2005), 036108 doi:10.1103/PhysRevE.72.036108 [arXiv:cond-mat/0507311 [cond-mat]].

[19] N. Drepanou, A. Lympiris, E. N. Saridakis and K. Yesmakhanova, “Kaniadakis holographic dark energy,” [arXiv:2109.09181 [gr-qc]].

[20] A. Majhi, “Non-extensive Statistical Mechanics and Black Hole Entropy From Quantum Geometry,” Phys. Lett. B 775, 32-36 (2017) doi:10.1016/j.physletb.2017.10.043 [arXiv:1703.09355 [gr-qc]].

[21] K. Mejrhit and S. E. Ennadifi, “Thermodynamics, stability and Hawking–Page transition of black holes from non-extensive statistical mechanics in quantum geometry,” Phys. Lett. B 794, 45-49 (2019) doi:10.1016/j.physletb.2019.03.055.

[22] Y. Liu, “Non-extensive Statistical Mechanics and the Thermodynamic Stability of FRW Universe,” doi:10.1299/0295-5075/ac352 [arXiv:2112.15077 [gr-qc]].

[23] S. Nojiri, S. D. Odintsov and V. Faraoni, “From nonextensive statistics and black hole entropy to the holographic dark universe,” Phys. Rev. D 105 (2022) no.4, 044042 doi:10.1103/PhysRevD.105.044042 [arXiv:2201.02424 [gr-qc]].

[24] V. Faraoni, A. Giusti and B. H. Fahim, “Spherical inhomogeneous solutions of Einstein and scalar–tensor gravity: A map of the land,” Phys. Rept. 925, 1-58 (2021) doi:10.1016/j.physrep.2021.04.003 [arXiv:2101.00266 [gr-qc]].

[25] J. M. Heinzle and R. Steinbauer, “Remarks on the distributional Schwarzschild geometry,” J. Math. Phys. 43, 1493-1508 (2002) doi:10.1063/1.1448684 [arXiv:gr-qc/0112047 [gr-qc]].

[26] R. Steinbauer and J. A. Vickers, “The Use of generalised functions and distributions in general relativity,” Class. Quant. Grav. 23, R91-R114 (2006) doi:10.1088/0264-9381/23/10/R01 [arXiv:gr-qc/0603078 [gr-qc]].

[27] O. J. Veraguth and C. H. T. Wang, “Immirzi parameter without Immirzi ambiguity: Conformal loop quantization of scalar-tensor gravity,” Phys. Rev. D 96, no.8, 084011 (2017) doi:10.1103/PhysRevD.96.084011 [arXiv:1705.09141 [gr-qc]].

[28] C. H. T. Wang and D. P. F. Rodrigues, “Closing the gaps in quantum space and time: Conformally augmented gauge structure of gravitation,” Phys. Rev. D 98, no.12, 124041 (2018) doi:10.1103/PhysRevD.98.124041 [arXiv:1810.01252 [gr-qc]].

[29] C. H. T. Wang and M. Stankiewicz, “Quantization of time and the big bang via scale-invariant loop gravity,” Phys. Lett. B 800, 135106 (2020) doi:10.1016/j.physletb.2019.135106 [arXiv:1910.03300 [gr-qc]].

[30] S. Nojiri and S. D. Odintsov, “Regular multihorizon black holes in modified gravity with nonlinear electrodynamics,” Phys. Rev. D 96 (2017) no.10, 104008 doi:10.1103/PhysRevD.96.104008 [arXiv:1708.05226 [hep-th]].