Detecting similarity of rational space curves

Juan Gerardo Alcázar\textsuperscript{a,1}, Carlos Hermoso\textsuperscript{a}, Georg Muntingh\textsuperscript{b}

\textsuperscript{a}Departamento de Física y Matemáticas, Universidad de Alcalá, E-28871 Madrid, Spain
\textsuperscript{b}SINTEF ICT, PO Box 124 Blindern, 0314 Oslo, Norway

Abstract

We provide an algorithm to check whether two rational space curves are related by a similarity. The algorithm exploits the relationship between the curvatures and torsions of two similar curves, which is formulated in a computer algebra setting. Helical curves, where curvature and torsion are proportional, need to be distinguished as a special case. The algorithm is easy to implement, as it involves only standard computer algebra techniques, such as greatest common divisors and resultants, and Gröbner basis for the special case of helical curves. Details on the implementation and experimentation carried out on the computer algebra system Maple 18 are provided. Timings show that the algorithm is efficient for moderate degrees.

1. Introduction

Two objects are similar when one of them is the result of applying an isometry and scaling to the other. Therefore, two similar objects have the same shape, although their position or size can be different. Because of this, recognizing similar objects is important in the field of Pattern Recognition, where one typically has a database of objects and wants to compare, up to a similarity, a given object with all the elements in the database. If the objects in the database are rational space curves, similarity detection allows to identify a given space curve as, for instance, a twisted cubic, a Viviani curve, or an Archytas curve, to mention some examples of classical, well-known space curves.

Three-dimensional similarity detection is also important in Computer Graphics and Computer Vision, and therefore it has been addressed in a long list of papers. Following the introduction of \cite{7}, the methods proposed in these papers can be grouped into two different categories: shape-based and topology-based. In the first category, one picks feature descriptors for the objects to be checked,
giving rise to feature vectors that are later compared using appropriate metrics; see for instance the survey [6] or the papers [5,12,13]. In the second category, which has gained attention in recent years, a “skeleton” is computed from each object, which is later used for comparison purposes; see [11,16]. The aforementioned papers, and others that can be found in their bibliographies, focus on surfaces, upon which (almost) no structure is assumed. At most, some of these papers require the objects to be modeled by means of polyhedra, so that they are considered to be meshings of perhaps more complex shapes. Additionally, in these references similarity detection is considered only up to a certain tolerance, so that the criteria are approximate.

Our approach is different. First, we deal with exact one-dimensional objects with a strong structure, namely rational space curves defined by rational parametrizations. Furthermore, we exploit the structure of the space curves to check, in a deterministic fashion, whether they are similar, and to explicitly compute the similarities between both curves in the affirmative case. In order to do this, we build on previous work on similarities of plane curves [2] and symmetries of plane and space curves [3,4]. As in these papers, since the curves are rational, we reduce the problem to the parameter space. Analogously to the algorithm in [4], the algorithm in this paper is based on comparing curvatures and torsions. However, similarity has the additional substantial difficulty of determining the scaling. Interestingly, this forces us to distinguish as a special case the helical curves, i.e., space curves with proportional curvature and torsion.

The basic steps in the algorithm are as follows. If the two given rational space curves are similar then there exists a rational function relating the corresponding parameter spaces. Under the hypothesis that the parametrizations of the curves are proper, i.e., injective for almost all points, this rational function is a Möbius transformation. In our algorithm one first computes candidates for the scaling constants and then candidates for the Möbius transformations. After this, the similarities between the curves can be computed. If the input curves are non-helical, then we have two independent conditions involving the curvatures and torsions of the curves, and from these conditions the scaling constant can be found. If the input curves are helical, these two conditions are no longer independent, and a different approach based on a procedure in [4] is provided.

The structure of the paper is as follows. In Section 2 we provide some background on isometries, similarities, differential invariants and helical curves, and we prove some results that are needed later in the paper. In Section 3 we provide the algorithm to solve the problem, separately considering the case of non-helical and helical curves. In Section 4 we report on the experimentation with the algorithm, implemented in the computer algebra system Maple 18. Finally, conclusions and further work are presented in Section 5.
2. Background

2.1. Similarities and isometries of Euclidean space

A similarity of Euclidean space is a linear affine map from the space to itself that preserves ratios of distances. Equivalently, a map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a similarity if

$$f(x) = \lambda Q x + b, \quad 0 \neq \lambda \in \mathbb{R}, \quad b \in \mathbb{R}^3, \quad Q \in \mathbb{R}^{3 \times 3}, \quad Q^T Q = I, \quad \det(Q) = 1, \quad (1)$$

where the latter two conditions mean that $Q$ is a special orthogonal matrix, i.e., a rotation about a line. Equivalently, with $\|x\|$ the Euclidean norm of $x$ and $d(x, y) := \|x - y\|$ the Euclidean distance,

$$d(f(x), f(y)) = |\lambda| \cdot d(x, y), \quad x, y \in \mathbb{R}^3. \quad (2)$$

We refer to $\lambda$ as the (signed) ratio of the similarity. A similarity is said to preserve the orientation if $\lambda > 0$, and reverse the orientation if $\lambda < 0$. The identity map $f(x) = x$ is called the trivial similarity.

If $|\lambda| = 1$ then $f$ is an (affine) isometry, i.e., $f$ preserves distances. The classification of nontrivial isometries includes reflections (in a plane), rotations (about an axis), and translations, and these combine in commutative pairs to form twists, glide reflections, and rotatory reflections. A composition of three reflections in mutually perpendicular planes through a point $x$ yields a central inversion (with respect to the point $x$). The particular case of rotation by an angle $\pi$ is of special interest, and it is called a half-turn.

If $\lambda$ is not an eigenvalue of $Q$, then $f$ has a unique fixed point $c := (I - \lambda Q)^{-1} b$, called the center of the similarity. In particular any similarity that is not an isometry has a center, because $Q$, being orthogonal, has eigenvalues of modulus equal to 1. A dilatation is a special type of similarity, defined as a map that sends any line to a parallel line (which could be the original line). Any dilatation that is not a translation sends any point $x$ to $c + \lambda(x - c)$ and therefore takes the form $f(x) = \lambda I x + (1 - \lambda)c$. A dilative rotation is a composition of a dilatation with center $c$ with a rotation $Q$ about a line $\ell$ containing $c$, which takes the form

$$f(Qx) = Qf(x) = \lambda Q x + (1 - \lambda)c, \quad Qc = c. \quad (3)$$

We recall the following characterization of similarities from [8, p. 103].

**Theorem 1.** Any similarity is either an isometry or a dilative rotation.

Similarities form a group under composition, and isometries form a subgroup of this group.

2.2. Similarities and symmetries of rational space curves

Throughout the paper, we consider two nonplanar rational space curves $C_1, C_2 \subset \mathbb{R}^3$, neither lines nor circles. Such curves are irreducible and can be parametrized by rational maps

$$x_j : \mathbb{R} \rightarrow C_j \subset \mathbb{R}^3, \quad x_j(t) = (x_j(t), y_j(t), z_j(t)), \quad j = 1, 2. \quad (4)$$
The components $x_j, y_j, z_j$ of $x_j$ are rational functions of $t$ with real coefficients, therefore defined for all but a finite number of values of $t$. Note that rational curves are irreducible. We assume that the parametrizations are proper, i.e., birational or, equivalently, injective except for perhaps finitely many values of $t$. This can be assumed without loss of generality, since any rational curve can quickly be properly reparametrized. For these claims and other results on properness, the interested reader can consult [15] for plane curves and [1, §3.1] for space curves. We also assume that the numerators and denominators of the components of $x_j$ are relatively prime.

This paper concerns algebraic space curves that are similar, i.e., one is the image of the other under a similarity. We say that $C_1$ and $C_2$ are related by a similarity $f$ when $f(C_1) = C_2$. We first establish some basic properties.

Lemma 2 (See [4, Lemma 1]). A rational space curve different from a line cannot be invariant under a translation, glide reflection, or twist.

Therefore, reflections, rotations, and their combinations are the only isometries that leave a rational space curve different from a line invariant.

Lemma 3. Let $f$ be a nontrivial similarity that is not an isometry, leaving an algebraic space curve $C$ invariant. Then its center $c \in C$.

Proof. Since $f$ is not an isometry, $|\lambda| \neq 1$. If $|\lambda| > 1$ then $f^{-1}$ is a similarity with ratio $\lambda^{-1}$ satisfying $|\lambda^{-1}| < 1$, also leaving $C$ invariant. Therefore we can and will assume $|\lambda| < 1$. Let $x \in C$. Since $f(C) = C$, the entire orbit $\{x, f(x), f^2(x), \ldots\} \subset C$. Using [2], $|\lambda| < 1$, and $f(c) = c$, we have

$$\lim_{k \to \infty} d(c, f^k(x)) = \lim_{k \to \infty} |\lambda|^k d(c, x) = 0,$$

and $f^k(x)$ approaches $c$. Since $C \subset \mathbb{R}^3$ is closed, this limit $c \in C$.

In addition, note that $c$ is not an isolated point, since it is the limit of a sequence of points of $C$.

Lemma 4. Let $f$ be a similarity that is not an isometry. Then for any positive integer $n$, the $n$-fold composition $f^n$ is not an isometry either.

Proof. By [1], $f^n$ is a similarity of ratio $\lambda^n$, and $|\lambda| \neq 1$ implies $|\lambda^n| \neq 1$.

Lemma 5. Let $f$ be a similarity such that there exist distinct vectors $x, y$ with $d(f(x), f(y)) = d(x, y)$. Then $f$ is an isometry.

Proof. Since $d(x, y) = d(f(x), f(y)) \neq 0$, Equation [2] implies $|\lambda| = 1$.

Analogous to [2, Proposition 2] for algebraic plane curves, the following theorem states that a self-similarity of an algebraic space curve is an isometry.

Theorem 6. Let $f$ be a similarity that leaves an algebraic space curve $C$, which is not a union of (possibly complex) concurrent lines, invariant. Then $f$ is an isometry.
Proof. Suppose \( f \) is not an isometry. By Theorem 1 the similarity is a dilative rotation \( f(x) = \lambda Qx + (1 - \lambda)c \), with \(|\lambda| \neq 1\) and \( Q \) a rotation about a line \( \ell \) containing \( c \). Let \( \Pi \) be the plane through \( c \) normal to \( \ell \).

Since \( f \) maps lines through \( c \) to each other, we can assume without loss of generality that \( C \) has no such components. First consider the case that \( C \) has one or more planar irreducible components that are not a real or complex line. Since a similarity maps planes to planes, one of these components \( C' \subset C \) satisfies \( f^n(C') = C' \) for some integer \( n \geq 1 \). Since \( C' \) is not a line, it spans a plane \( \Pi' \) and \( f^n \) restricts to a plane similarity \( f' := f^n|_{\Pi'} \) with \( f'(C') = C' \), which is an isometry by [2, Proposition 2]. Hence \( f^n \) is an isometry by Lemma 5 and \( f \) is an isometry by Lemma 4.

It remains to show the case where \( C \) does not have any planar irreducible components besides lines. Supposing \( f \) is not an isometry, \( C \) cannot contain a line \( L \) parallel to \( \ell \), because then it would also contain any parallel line \( f^n(L), n \in \mathbb{N} \), of which there are infinitely many since each has a different distance to \( \ell \). Therefore the image \( C^\perp \) of the orthogonal projection \( p : C \rightarrow \Pi \) is a plane curve. Since \( C \) does not have any planar components, \( C^\perp \) does not have any lines. Moreover,

\[
f(C^\perp) = f \circ p(C) = p \circ f(C) = p \circ C = C^\perp,
\]

showing that the restriction \( f|_{\Pi} \) is a plane similarity that leaves \( C^\perp \) invariant. It follows that \( f|_{\Pi} \) is an isometry by [2] Proposition 2 and that \( f \) is an isometry by Lemma 5. \( \square \)

A nontrivial isometry \( f \) leaving an algebraic space curve \( C \) invariant is called a symmetry of \( C \). The curve \( C \) is called symmetric if it has a symmetry. For a background on symmetries of rational space curves, see [3, 4]. Analogously, two curves \( C_1, C_2 \) are said to be similar if there exists a similarity \( f \) such that \( f(C_1) = C_2 \).

Corollary 7. Let \( C_1, C_2 \) be similar irreducible algebraic space curves, neither lines nor circles. There are finitely many similarities \( f \) such that \( f(C_1) = C_2 \). Moreover, such a similarity \( f \) is unique if and only if \( C_1, C_2 \) are not symmetric.

Proof. Assume there are distinct similarities \( f_1, f_2 \) with \( f_1(C_1) = C_2 = f_2(C_1) \). Then \( f_1 \circ f_2^{-1} \) is a nontrivial similarity transforming \( C_1 \) into itself. By Theorem 6 \( f_1 \circ f_2^{-1} \) is a nontrivial isometry, and therefore a symmetry of \( C_1 \). Since the number of symmetries of a space curve different from a line or a circle is finite [3], the first part follows. As for the second part, if \( C_1 \) is not symmetric then \( f_1 \circ f_2^{-1} \) is the identity, and \( f_1 = f_2 \). Conversely, if \( C_1 \) has a symmetry \( f \), then \( f_1 \circ f \) is another similarity from \( C_1 \) to \( C_2 \). \( \square \)

Proposition 8. Let \( C_1, C_2 \) be irreducible algebraic space curves, not lines, for which there exist similarities \( f_i(x) = \lambda_i Q_i x + b_i \) such that \( f_i(C_1) = C_2 \), with \( i = 1, 2 \). Then \(|\lambda_1| = |\lambda_2|\).
Proof. One has \( f^{-1}_2(x) = \lambda_2^{-1}Q_2^{-1}(x - b_2) \). Then \((f_2^{-1} \circ f_1)(x) = \lambda Q x + b\), with
\[
\lambda := \frac{\lambda_1}{\lambda_2}, \quad Q := Q_2^T Q_1, \quad b := \frac{1}{\lambda_2} Q_2^T (b_1 - b_2),
\]
is a similarity since \(0 \neq \lambda \in \mathbb{R}\), \(\det(Q) = \det(Q_2^T) \det(Q_1) = 1\), and
\[
Q^T Q = Q_1^T Q_2 Q_2^T Q_1 = Q_1^T I Q_1 = I.
\]
Since \(f_2^{-1} \circ f_1\) leaves \(C_1\) invariant, Theorem 6 implies that \(f_2^{-1} \circ f_1\) is an isometry, implying \(|\lambda_1/\lambda_2| = 1\) and therefore \(|\lambda_1| = |\lambda_2|\). \(\square\)

It is well known that the birational functions on the line are the Möbius transformations \[15\], i.e., rational functions
\[
\varphi : \mathbb{R} \to \mathbb{R}, \quad \varphi(t) = \frac{at + b}{ct + d}, \quad \Delta := ad - bc \neq 0.
\]

The following result relates the similarity \(f\) in space to a Möbius transformation on the line. In [2] a proof was given for the case of plane curves, which generalizes \textit{mutatis mutandis} to the case of space curves.

**Theorem 9.** Let \(C_1, C_2 \subset \mathbb{R}^3\) be rational space curves with proper parametrizations \(x_1, x_2 : \mathbb{R} \to \mathbb{R}^3\). Then \(C_1\), \(C_2\) are similar if and only if there exists a similarity \(f\) and a Möbius transformation \(\varphi\) for which we have a commutative diagram
\[
\begin{array}{ccc}
C_1 & \xrightarrow{f} & C_2 \\
\uparrow x_1 & & \uparrow x_2 \\
\mathbb{R} & \xrightarrow{\varphi} & \mathbb{R}
\end{array}
\]  
(6)

Moreover, if \(C_1, C_2\) are related by a similarity \(f\), then there exists a unique Möbius transformation \(\varphi\) that makes the above diagram commute.

Since the Möbius transformation \(\varphi\) maps the real line to itself, its coefficients can always be assumed to be real by dividing by a common complex number if necessary \[4, Lemma 3\]. Notice that \(s = \varphi(t)\) provides the \(s\)-value generating the image, in \(C_2\), under the similarity \(f\), of the point generated by \(t\) in \(C_1\).

**Corollary 10.** Consider proper parametrizations \(x_j, j = 1, 2, \) as in \[1\], a similarity \(f\) as in \[1\], and a Möbius transformation \(\varphi\), related by \[6\]. Then
\[
|\lambda| \cdot \|x_1'(t)\| - \|(x_2 \circ \varphi)'(t)\| = 0.
\]
(7)

**Proof.** The commutative diagram \[6\] has the corresponding equation
\[
\lambda Q x_1(t) + b = (x_2 \circ \varphi)(t).
\]
Differentiating and taking norms yields \(\|\lambda Q x_1'(t)\| = \|(x_2 \circ \varphi)'(t)\|\), which, using the orthogonality of \(Q\), yields \[7\]. \(\square\)
2.3. Differential invariants

The remainder of the section concerns the effect of a similarity and Möbius transformation on the curvature $\kappa$ and torsion $\tau$ of a parametric curve $\mathbf{x}$, which are defined by

$$
\kappa = \kappa_{\mathbf{x}} := \frac{\| \mathbf{x}' \times \mathbf{x}'' \|}{\| \mathbf{x}' \|^3}, \quad \tau = \tau_{\mathbf{x}} := \frac{\langle \mathbf{x}' \times \mathbf{x}'' , \mathbf{x}''' \rangle}{\| \mathbf{x}' \times \mathbf{x}'' \|^2}
$$

Notice in particular that $\kappa \geq 0$, while $\tau$ can be positive, negative, or zero. Moreover, although $\tau$ and $\kappa^2$ are rational functions for any rational map $\mathbf{x}$, the curvature $\kappa$ is in general not rational.

**Lemma 11.** For a similarity $f(\mathbf{x}) = \lambda \mathbf{Q} \mathbf{x} + \mathbf{b}$ and parametrization $\mathbf{x}$ as in (4),

$$
|\lambda| \cdot \kappa_{f \circ \mathbf{x}} = \kappa_{\mathbf{x}}, \quad \lambda \cdot \tau_{f \circ \mathbf{x}} = \tau_{\mathbf{x}}.
$$

**Proof.** A straightforward calculation yields, for any invertible matrix $\mathbf{M} \in \mathbb{R}^{3,3}$ and vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the identity

$$
(M \mathbf{u}) \times (M \mathbf{v}) = \det(M) \mathbf{M}^{-T} (\mathbf{u} \times \mathbf{v}).
$$

Using $(f \circ \mathbf{x})^{(n)} = \lambda \mathbf{Q} \mathbf{x}^{(n)}$ for $n = 1, 2, 3$ and $\det(\mathbf{Q}) = 1$ with $\mathbf{Q}$ orthogonal,

$$
|\lambda| \cdot \kappa_{f \circ \mathbf{x}} = \frac{\| (\mathbf{Q} \mathbf{x}' \times (\mathbf{Q} \mathbf{x}'')) \|}{\| \mathbf{Q} \mathbf{x}' \|^3} = \frac{\| \mathbf{x}' \times \mathbf{x}'' \|}{\| \mathbf{x}' \|^3} = \kappa_{\mathbf{x}},
$$

$$
\lambda \cdot \tau_{f \circ \mathbf{x}} = \frac{\langle (\mathbf{Q} \mathbf{x}') \times (\mathbf{Q} \mathbf{x}''), \mathbf{Q} \mathbf{x}''' \rangle}{\| (\mathbf{Q} \mathbf{x}') \times (\mathbf{Q} \mathbf{x}'') \|^2} = \frac{\langle \mathbf{x}' \times \mathbf{x}'' , \mathbf{x}''' \rangle}{\| \mathbf{x}' \times \mathbf{x}'' \|^2} = \tau_{\mathbf{x}}.
$$

Next we recall a lemma from [4], which describes the behavior of the curvature and torsion under reparametrization, for instance by a Möbius transformation.

**Lemma 12.** Let $\mathbf{x}$ be a rational parametrization (4) and let $\phi \in C^3(U)$, with $U \subset \mathbb{R}$ open. Then

$$
\kappa_{\mathbf{x} \circ \phi} = \kappa_{\mathbf{x}} \circ \phi, \quad \tau_{\mathbf{x} \circ \phi} = \tau_{\mathbf{x}} \circ \phi,
$$

whenever both sides are defined.

The following lemma relates the curvatures and torsions of similar curves.

**Lemma 13.** Suppose $\mathbf{x}_1, \mathbf{x}_2$ define curves $C_1, C_2$ with $f(C_1) = C_2$ for a similarity $f$ with ratio $\lambda$. Then there is a Möbius transformation $\varphi$ such that

$$
\kappa_{\mathbf{x}_2 \circ \varphi} = \kappa_{\mathbf{x}_2 \circ \varphi} = 1, \quad \kappa_{\mathbf{x}_2 \circ \varphi} = \kappa_{\mathbf{x}_1} = \frac{1}{\lambda}, \quad \tau_{\mathbf{x}_2 \circ \varphi} = \tau_{\mathbf{x}_2 \circ \varphi} = \frac{1}{\lambda} \tau_{\mathbf{x}_1}.
$$

**Proof.** By Theorem [3] there exist a Möbius transformation $\varphi$ such that $f \circ \mathbf{x}_1 = \mathbf{x}_2 \circ \varphi$. The statement follows from Lemmas 11 and 12.
2.4. Helical curves

Consider parametrizations $\mathbf{x}_i$, $i = 1, 2$, as in (4) defining non-planar curves. Then the torsion $\tau_{\mathbf{x}_i}$ is not identically zero, and we can consider the ratio

$$\mu_i := \frac{\kappa_{\mathbf{x}_i}}{\tau_{\mathbf{x}_i}}, \quad i = 1, 2.$$ 

Whenever this ratio is constant we refer to it as the proportionality constant. Such non-planar curves are called helical curves, generalizing the familiar circular helix in which case not only the quotient of the curvature and torsion, but also the curvature and torsion themselves are constant.

**Lemma 14.** Any rational helical curve $\mathbf{x}$ has proportionality constant $\mu \neq 0$.

**Proof.** Suppose $\mu = 0$. If $\mathbf{x}' \equiv 0$ or $\mathbf{x}'' \equiv 0$, then integrating would yield a point or a line, which are planar and therefore non-helical. Therefore, since $\kappa \equiv 0$, there exists a nonzero function $\nu$ such that $\mathbf{x}'' = \nu \mathbf{x}'$. Writing $\mathbf{x} = (x, y, z)$, integrating $x''/x' = \nu$, $y''/y' = \nu$, $z''/z' = \nu$ and taking exponentials yields $\mathbf{x}'(t) = \mathbf{x}_0 \cdot \exp \left( \int \nu(t) dt \right)$ for some constant vector $\mathbf{x}_0$. Therefore $\mathbf{x}$ is a line, contradicting that $\mathbf{x}$ is helical. We conclude $\mu \neq 0$.

**Proposition 15.** Suppose $\mathbf{x}_1, \mathbf{x}_2$ define helical curves $C_1, C_2$ with proportionality constants $\mu_1, \mu_2$ satisfying $f(C_1) = C_2$ for a similarity $f$ with ratio $\lambda$. Then $\mu_2 = \text{sgn}(\lambda) \cdot \mu_1$.

**Proof.** Taking the quotient in (10) yields

$$\mu_2 = \frac{\kappa_{\mathbf{x}_2} \circ \varphi}{\tau_{\mathbf{x}_2} \circ \varphi} = \frac{1}{\lambda} \cdot \frac{\kappa_{\mathbf{x}_1}}{\tau_{\mathbf{x}_1}} = \text{sgn}(\lambda) \cdot \mu_1.$$ 

This proposition provides a necessary condition for similarity of helical curves. The following example shows that the converse does not hold in general.

**Example 1.** The helical quintics $C_1, C_2$ parametrized by

$$\mathbf{x}_1(t) = \left( \frac{3}{4} t^5 + \frac{3}{8} t^4 + \frac{1}{4} t^3, \frac{5}{4} t^5 + t^4, -\frac{3}{5} t^5 + \frac{1}{2} t^4 + \frac{1}{3} t^3 \right),$$

$$\mathbf{x}_2(t) = \left( \frac{3}{2} t^5 + \frac{3}{4} t^4 + t^3, \frac{6}{5} t^5 + 3 t^4, -\frac{8}{5} t^5 + t^4 + \frac{4}{3} t^3 \right).$$

have proportionality constants $\mu_1 = \mu_2 = -4/3$. However, by applying Algorithm Similar3D in Section 3, one can show that $C_1$ and $C_2$ are not similar.

In order to check whether the condition in Proposition 15 is sufficient, we tried first several examples of helical cubics, following the method for constructing these curves presented in [10]. Interestingly, we could not find any counterexample with helical cubics, leaving us to conjecture that the converse of Proposition 15 holds for helical cubics.
Conjecture 16. Any two cubic rational helical space curves with proportionality constants of equal modulus are similar.

Helical rational curves with nonzero proportionality constants do exist, and the interested reader can consult [10, Chapter 23] or [14] for further information on helical curves.

3. Detecting and finding similarities of rational space curves

Let $C_1, C_2$ be curves with parametrizations $x_1, x_2$ as in (4). In this section we first present a criterion for whether $f(C_1) = C_2$ for a similarity $f$ with a given ratio $\lambda_0$. Next, to determine the potential ratios $\lambda_0$, we develop separate methods for helical and non-helical curves. The section concludes with a method for finding the similarities with a given ratio $\lambda_0$.

We will use the following standard notions for multivariate polynomials $p \in \mathbb{R}[x_1, \ldots , x_n]$, viewed as a polynomial in $x_n$ with coefficients in $\mathbb{R}[x_1, \ldots , x_{n-1}]$. The leading term of $p$ with respect to $x_n$ is the monomial of $p$ with highest degree in $x_n$, and its coefficient is called the leading coefficient. Moreover, the content of $p$ with respect to $x_n$ is the greatest common divisor of its coefficients, viewed as elements of $\mathbb{R}[x_1, \ldots , x_{n-1}]$.

3.1. A criterion and algorithm for detecting similarity

Since $\kappa_{x_i}^2$ and $\tau_{x_i}$, with $i = 1, 2$, are rational, we can write

$$\kappa_{x_i}^2(t) = \frac{A_i(t)}{B_i(t)}, \quad \tau_{x_i}(t) = \frac{C_i(t)}{D_i(t)}, \quad i = 1, 2,$$

for coprime pairs $(A_i, B_i)$ and $(C_i, D_i)$, $i = 1, 2$, of polynomials. Let

$$K_\lambda(t, s) := A_1(t)B_2(s) - \lambda^2 \cdot A_2(s)B_1(t),$$

$$T_\lambda(t, s) := C_1(t)D_2(s) - \lambda \cdot C_2(s)D_1(t)$$

be the result of clearing denominators in the expressions $\kappa_{x_1}^2(t) - \lambda^2 \kappa_{x_2}^2(s) = 0$ and $\tau_{x_1}(t) - \lambda \tau_{x_2}(s) = 0$. Note that $K_{-\lambda} = K_\lambda$. For a fixed $\lambda$, we consider the bivariate greatest common divisor and $s$-resultant

$$G_\lambda := \gcd(K_\lambda, T_\lambda), \quad R_\lambda := \text{Res}_s(K_\lambda, T_\lambda).$$

To any Möbius transformation $\varphi$ as in (5), associate the Möbius-like polynomial

$$F(t, s) := (ct + d)s - (at + b), \quad ad - bc \neq 0,$$

as the result of clearing denominators in $s - \varphi(t) = 0$. Note that the Möbius-like polynomials are precisely the irreducible bilinear polynomials, since $ad - bc \neq 0$.

The following theorem provides a criterion for similarity of $C_1$ and $C_2$ with a given ratio.
Theorem 17. Let $x_1, x_2$ as in [4] define curves $C_1, C_2$. There exists a similarity $f(x) = \lambda_0 Qx + b$ such that $f(C_1) = C_2$ if and only if there exists a polynomial $F$ of type [13] dividing $G_{\lambda_0}$, associated with a Möbius transformation $\varphi$ satisfying (7) with $\lambda = \lambda_0$.

Proof. “$\Rightarrow$”: If $f(C_1) = C_2$ for some similarity $f(x) = \lambda_0 Qx + b$, by Theorem 9 there exists a Möbius transformation $\varphi$ such that $f \circ x_1 = x_2 \circ \varphi$. Let $F$ be the Möbius-like polynomial associated with $\varphi$. The points $(t, s)$ for which $K_{\lambda_0}(t, s) = T_{\lambda_0}(t, s) = 0$ are the points satisfying $\kappa_{x_1}(t) = |\lambda_0| \kappa_{x_2}(s)$ and $\tau_{x_1}(t) = \lambda_0 \tau_{x_2}(s)$. By Lemma 10, this includes the zero set $\{(t, s) : s = \varphi(t)\}$ of $F(t, s)$. Since $F$ is irreducible, Bézout’s theorem implies that $F$ divides $K_{\lambda_0}$ and $T_{\lambda_0}$, and therefore $G_{\lambda_0}$ as well. Moreover, since $Q$ is orthogonal, 

$$\|\varphi'(t)\|=\|(f \circ x_1)'\| = \|\lambda_0 Qx_1'\| = |\lambda_0| \cdot \|x_1'\|.$$ 

“$\Leftarrow$”: Let $\varphi$ be the transformation associated to $F$. Let $t_0 \in I \subset \mathbb{R}$ be such that $x_1(t)$ is a regular point on $C_1$ for every $t \in I$, and consider the arc length function

$$s = s(t) := \int_{t_0}^{t} \|x_1'(s)\| \, ds, \quad t \in I,$$

which (locally) has an infinitely differentiable inverse $t = t(s)$. For $\tilde{x}_2 := \lambda_0^{-1}x_2$,

$$\left\|\frac{d}{ds}(x_1 \circ t)\right\| = \left\|\frac{dx_1}{dt} \frac{dt}{ds}\right\| = 1 = \frac{1}{|\lambda_0|} \left\|\frac{d}{dt}(x_2 \circ \varphi) \frac{dt}{ds}\right\| = \left\|\frac{d}{ds}(\tilde{x}_2 \circ \varphi \circ t)\right\|,$$

by (7), so that both $x_1 \circ t$ and $\tilde{x}_2 \circ \varphi \circ t$ are parametrized by arc length.

Since $F$ divides $G_{\lambda_0}$, any zero $(t, \varphi(t))$ of $F$ is also a zero of $K_{\lambda_0}$ and $T_{\lambda_0}$, implying that $\kappa_{x_1} = |\lambda_0| \kappa_{x_2} \circ \varphi$ and $\tau_{x_1} = \lambda_0 \tau_{x_2} \circ \varphi$. Together with Lemmas 11 and 12 this yields

$$\kappa_{x_1 \circ t} = \kappa_{x_1} \circ t = |\lambda_0| \cdot \kappa_{x_2} \circ \varphi \circ t = \kappa_{\tilde{x}_2} \circ \varphi \circ t = \kappa_{\tilde{x}_2 \circ \varphi \circ t},$$

$$\tau_{x_1 \circ t} = \tau_{x_1} \circ t = \lambda_0 \cdot \tau_{x_2} \circ \varphi \circ t = \tau_{\tilde{x}_2} \circ \varphi \circ t = \tau_{\tilde{x}_2 \circ \varphi \circ t}.$$

The fundamental theorem of space curves [9] then implies that there exists an isometry $\tilde{f}(x) = Qx + b$, with $\det(Q) = 1$, such that $\tilde{f} \circ x_1 \circ t = \tilde{x}_2 \circ \varphi \circ t$ on $s(I)$. In terms of the similarity $f(x) := \lambda_0 \tilde{f}(x)$, it follows that

$$f(x_1(t)) = \lambda_0 \tilde{f}(x_1(t)) = \lambda_0 \tilde{x}_2(\varphi(t)) = x_2(\varphi(t)), \quad t \in I.$$

Therefore the irreducible algebraic curves $f(C_1)$ and $C_2$ have infinitely many points in common, implying $f(C_1) = C_2$. 

If some tentative values for $\lambda_0$ are known, similarity of curves can be detected quickly with this criterion, by checking if $G_{\lambda_0}$ has some Möbius-like factor. In
order to do this, and since $\lambda_0$ is in general an algebraic number\footnote{This is true if either $x_1$ or $x_2$ has coefficients in an algebraic field. However, it is unclear to us if $\lambda_0$ could be non-rational when the coefficients of both $x_1$ and $x_2$ are rational. In the planar case, the similarity ratio can certainly be non-rational even when $x_1$ and $x_2$ have rational coefficients. This is the case, for instance, when one considers the similarity $f(x) = \lambda_0 Q x$, $\lambda_0 = \sqrt{2}$, $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, with non-rational ratio $\lambda_0 = \sqrt{2}$, mapping rational points $(x, y)$ to rational points $f(x, y) = (x - y, x + y)$. Therefore, if $x_1$ has rational coefficients then $x_2 = f \circ x_1$ also has rational coefficients. However, we were unable to find an analogous example in $\mathbb{R}^3$.}, we can use efficient techniques for factoring bivariate polynomials with coefficients in an algebraic number field. For instance, the command \texttt{AFactor} in Maple 18 is fast and efficient: using this command, the factorization

$$(s^2 \sqrt{2}t - s^2 t^2 - \frac{1}{2} s^2 + t^2) \left(st - \frac{1}{3} \sqrt{3}\right) \left(st + \frac{1}{3} \sqrt{3}\right) \cdot p(t),$$

where $p(t)$ is a dense polynomial in $t, s$ of degree 9, is found in 0.015 seconds on a personal computer. Thus we arrive at Algorithm \texttt{Similar3D} for checking whether $C_1$ and $C_2$ are similar.

Note that by Theorem 6, any ‘self-similarity’ of an irreducible algebraic space curve that is not a line is a symmetry. Therefore, for $x_1 = x_2$ one has $|\lambda| = 1$, and Algorithm \texttt{Similar3D} reduces to the algorithm presented in \cite{4} for detecting symmetries of algebraic space curves.

Since in general $\lambda_0$ is an algebraic number, we might use techniques for factoring bivariate polynomials with coefficients in an algebraic number field. However, this is complicated and time-consuming. A much more efficient option is to directly seek M"obius-like factors using the method described in \cite{4, §3.2}.

By Theorem 17, whenever the associated M"obius transformation satisfies (7), the existence of such a factor is equivalent to $C_1$ and $C_2$ being similar.

It remains to compute the sets $S_0, S_1, S_2$ of tentative values for $\lambda_0$ in the next two sections, where it is necessary to distinguish between helical and non-helical curves.

### 3.2. Finding the ratio for non-helical curves

Assume $x_1, x_2$ define non-helical curves. By the following proposition, there are only finitely many nonzero $\lambda$ for which the resultant $R_\lambda$ is identically zero.

**Proposition 18.** The resultant $R_\lambda$ is identically zero if and only if $C_1, C_2$ are helical curves with proportionality constants $\mu_1, \mu_2$ satisfying $|\mu_1| = |\mu_2|$.

**Proof.** "$\Leftarrow$": Since the proportionality constants have the same absolute value $\mu := |\mu_1| = |\mu_2|$, $\frac{A_1(t)}{B_1(t)} = \kappa_{x_1}^2(t) = \mu^2 \cdot \tau_{x_1}^2(t) = \mu^2 \cdot \frac{C_1^2(t)}{D_1^2(t)}.$
Algorithm Similar3D

Require: Two proper parametrizations \( \mathbf{x}_1, \mathbf{x}_2 \) of two space curves \( C_1, C_2 \).
Ensure: Whether there exists a similarity \( f(\mathbf{x}) = \lambda Q \mathbf{x} + b \) with \( f(C_1) = C_2 \).

1. If \( C_1 \) and \( C_2 \) are both lines or both circles, return TRUE. Otherwise:
2. If \( C_1 \) or \( C_2 \) is a circle or a line, return FALSE.
3. Find the curvatures \( \kappa_{\mathbf{x}_1}, \kappa_{\mathbf{x}_2} \) and torsions \( \tau_{\mathbf{x}_1}, \tau_{\mathbf{x}_2} \) from [8].
4. Find the polynomials \( K_\lambda \) and \( T_\lambda \) from [11].
5. Find \( \mu_1 := \kappa_{\mathbf{x}_1}/\tau_{\mathbf{x}_1} \) and \( \mu_2 := \kappa_{\mathbf{x}_2}/\tau_{\mathbf{x}_2} \).
6. If precisely one of \( \mu_1, \mu_2 \) is constant, return FALSE.
7. If \( \mu_1, \mu_2 \) are both constant (helical case):
   7.1 If \( |\mu_1| \neq |\mu_2| \) return FALSE. Otherwise:
   7.2 Let \( G_\lambda := T_\lambda \).
   7.3 Choose \( t_0 \in \mathbb{Q} \) such that the leading coefficient of \( G_\lambda(t_0, s) \) with respect to \( s \) is not identically zero.
   7.4 Find the sets \( S_0, S_1, S_2 \) of tentative \( \lambda \) using the method in Section 3.3.
   7.5 For each \( \lambda \in S_0 \cup S_1 \cup S_2 \), check whether \( G_\lambda \) contains a Möbius-like factor \( F \) for which the associated Möbius transformation \( \varphi \) satisfies [7].
   7.6 If some \( \lambda \) succeeds, return TRUE, otherwise return FALSE.
8. If \( \mu_1, \mu_2 \) are not constant (non-helical case):
   8.1 Find the resultant \( R_\lambda = \text{Res}_s(K_\lambda, T_\lambda) \).
   8.2 Find the set \( S_0 \) of tentative \( \lambda \) using the method in Section 3.2.
   8.3 For each \( \lambda \in S_0 \), check whether \( G_\lambda \) contains a Möbius-like factor \( F \) for which the associated Möbius transformation \( \varphi \) satisfies [7].
   8.4 In the affirmative case, return TRUE, otherwise return FALSE.

with \( \mu \neq 0 \) because of Lemma [11]. Therefore

\[
K_\lambda(t, s) = \mu^2 \cdot \left( C_1^2(t)D_2^2(s) - \lambda^2C_2^2(s)D_1^2(t) \right) \\
= \mu^2 \cdot \left( C_1(t)D_2(s) - \lambda C_2(s)D_1(t) \right) \cdot \left( C_1(t)D_2(s) + \lambda C_2(s)D_1(t) \right) \\
= \mu^2 \cdot T_\lambda(t, s) \cdot T_{-\lambda}(t, s).
\]

Hence \( K_\lambda \) has a non-trivial factor, depending on \( s \), in common with both \( T_\lambda \) or \( T_{-\lambda} \), since \( K_\lambda = K_{-\lambda} \). It follows that \( R_\lambda \) is identically zero.

“\( \Rightarrow \)”: If \( R_\lambda \) is identically zero then \( K_\lambda, T_\lambda \) have nontrivial greatest common divisor \( G_\lambda \). Suppose \( T_\lambda \) has a factor \( S \) not depending on \( \lambda \). Then \( S \) divides both \( T_\lambda \) and \( T_0(t, s) = C_1(t)D_2(s) \), and therefore also \( C_2(s)D_1(t) \), contradicting that \( C_1, D_1 \) and \( C_2, D_2 \) are coprime. A similar argument shows that any nonconstant
factor of $K_\lambda$ depends on $\lambda$. It follows that $G_\lambda$ is a linear polynomial in $\lambda$ in constant proportion with $T_\lambda$. Since $G_\lambda, G_{-\lambda}$ both divide $K_\lambda = K_{-\lambda}$, which is a quadratic polynomial in $\lambda$, it follows that

$$K_\lambda = \nu \cdot T_\lambda \cdot T_{-\lambda}$$

for some nonzero constant $\nu$. Comparing coefficients it follows that

$$A_1(t)B_2(s) = \nu \cdot C_1^2(t)D_2^2(s), \quad A_2(s)B_1(t) = \nu \cdot C_2^2(s)D_1^2(t).$$

Dividing these equations yields

$$\frac{\kappa^2_{x_1}(t)}{\kappa^2_{x_2}(s)} = \frac{A_1(t)B_2(s)}{B_1(t)A_2(s)} = \frac{C_1^2(t)D_2^2(s)}{D_1^2(t)C_2^2(s)} = \frac{\tau_{x_1}^2(t)}{\tau_{x_2}^2(s)},$$

or equivalently

$$\frac{\kappa^2_{x_1}(t)}{\tau_{x_1}^2(t)} = \frac{\kappa^2_{x_2}(s)}{\tau_{x_2}^2(s)},$$

which must be constant. After taking square roots the statement follows. \[\square\]

Let $\Lambda^*(\lambda)$ be the content of the resultant $\text{Res}_s(K_\lambda, T_\lambda)$, viewed as a polynomial in $t$ with coefficients depending on $\lambda$. Let $\text{lc}_s(K_\lambda), \text{lc}_s(T_\lambda)$ be the leading coefficients with respect to $s$ of $K_\lambda, T_\lambda$. Notice that whenever $\text{lc}_s(K_\lambda), \text{lc}_s(T_\lambda)$ do not vanish identically and simultaneously for $\lambda = \lambda_0$, then $R_{\lambda_0}$ is the resultant of specializing $\text{Res}_s(K_\lambda, T_\lambda)$ at $\lambda = \lambda_0$ (see Lemma 4.3.1 of [17]). Let $\Lambda(\lambda)$ be the product of $\Lambda^*(\lambda)$ and the content of $\gcd(\text{lc}_s(K_\lambda), \text{lc}_s(T_\lambda))$ with respect to $t$. Let $S_0$ be the nonzero real roots of $\Lambda$.

**Proposition 19.** Let $x_1, x_2$ define non-helical curves $C_1, C_2$ satisfying $f(C_1) = C_2$ for some similarity $f(x) = \lambda_0 Qx + b$. Then $\lambda_0 \in S_0$.

**Proof.** By Lemma [13] there exists a Möbius transformation $\varphi$ such that

$$K_{\lambda_0}(t, \varphi(t)) = T_{\lambda_0}(t, \varphi(t)) = 0,$$

hold identically. Hence the Möbius-like polynomial $F$ associated to $\varphi$ divides $G_{\lambda_0}$. Since the polynomial ring $\mathbb{R}[t]$ is an integral domain, the bivariate polynomial $G_{\lambda_0}$ is non-constant precisely when the resultant $R_{\lambda_0}$ is identically zero, which implies $\Lambda(\lambda_0) = 0$. \[\square\]

**Example 2.** Let $C_1$ be the crunode parametrized by

$$x_1(t) = \left( \frac{t}{t^4+1}, \frac{t^2}{t^4+1}, \frac{t^3}{t^4+1} \right).$$
Figure 1: Left: The similar crunode curves from Example 2. Right: The family of helical curves $C_{\alpha}$, with $-1 \leq \alpha \leq 1$, from Example 3, with the curves $C_{-1}, C_0, C_1$ emphasized.

Based on the reparametrization $\phi(t) = t + 1$ and similarity $f(x) = \lambda Q x + b$, $\lambda = 2$, $Q = \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, (14)

we define another crunode $C_2 := f(C_1)$ parametrized by $x_2 = f \circ x_1 \circ \phi$, i.e.,

$$x_2(t) = \left( \frac{2}{5} \left( \frac{t+1}{t+1} \right)^4 + 1, \frac{2}{5} \left( \frac{t+1}{t+1} \right)^4 + 1, \frac{2(t+1)^3}{(t+1)^4+1} + 2 \right).$$

The curves $C_1$ and $C_2$ are shown in Figure 1a, together with the invariant sets of their symmetries, i.e., two planes of reflection and an axis of rotation.

One verifies that $\kappa_{x_1}, \kappa_{x_2}$ are rational functions where the numerators and denominators have degree 36. Furthermore, the numerator and denominator of $\tau_{x_1}, \tau_{x_2}$ have degree 8. Therefore $K_\lambda(t,s)$ has bidegree (36, 36) and $T_\lambda(t,s)$ has bidegree (8, 8). After computing $R_\Lambda = \text{Res}_s(K_\lambda, T_\lambda)$, we get $\Lambda(\lambda) = \Lambda^*(\lambda) = \lambda^2 - 4$, so $S_0 = \{-2, 2\}$ contains the tentative values for $\lambda$. For $\lambda_0 = 2$ we obtain $G_2(t,s) = (s+t+1)(s-t+1)$ and two corresponding Möbius transformations satisfying (7), namely $\varphi_1(t) = -t - 1$ and $\varphi_2(t) = t - 1$. For $\lambda_0 = -2$ we obtain $G_{-2}(t,s) = (st-t+1)(st+t-1)$ and two corresponding Möbius transformations satisfying (7), namely $\varphi_3(t) = -(t+1)/t$ and $\varphi_4(t) = (-t+1)/t$. Therefore $C_1$ and $C_2$ are similar, and there are four different similarities mapping one to the other.

3.3. Finding the ratio for helical curves

Assume that $C_1, C_2$ are similar helical curves. Then their proportionality ratios are equal up to a sign by Proposition 15 and $\gcd(K_\lambda, T_\lambda) = T_\lambda$ for any $\lambda$ by the proof of Proposition 18. Therefore $R_\Lambda \equiv 0$, and we cannot use the method in
from Section 3.2 to find the potential ratios $\lambda$. However, since $G_\lambda = T_\lambda$ is known, we can directly apply Theorem 17 to find the $\lambda$ values for which $G_\lambda$ has a Möbius-like factor. In order to do this, we adapt the method in [4] 3.2, where the problem of directly computing the Möbius-like factors of a bivariate polynomial is efficiently solved. The idea is that, if $\lambda_0$ is the ratio we are seeking, the Möbius transformation $\varphi$ corresponding to the similarity is implicitly defined by $G_{\lambda_0}$, so that it can be reconstructed from its local data.

**Lemma 20.** Let either $c = 0$ or $t_0 \neq -d/c$, and consider the Taylor expansion

$$\varphi(t) = \frac{at + b}{ct + d} = s_0 + s_0'(t - t_0) + \frac{1}{2} s''_0(t - t_0)^2 + \cdots$$

Then, as homogeneous coordinates,

$$[a : b : c : d] = [2(s'_0)^2 - s_0 s''_0 : 2s_0 s'_0 + t_0 s_0 s''_0 - 2t_0(s'_0)^2 : -s''_0 : 2s'_0 + t_0 s''_0].$$

**Proof.** Differentiating (15) and evaluating at $t = t_0$ yields

$$s_0 = \varphi(t_0) = \frac{at_0 + b}{ct_0 + d}, \quad s'_0 = \varphi'(t_0) = \frac{\Delta}{(ct_0 + d)^2}, \quad s''_0 = \varphi''(t_0) = \frac{-2c\Delta}{(ct_0 + d)^3}.$$  

The statement follows from a straightforward calculation. \qed

Let $t_0 \in \mathbb{Q}$ be such that the leading coefficient $L(\lambda)$ of the polynomial $G_\lambda(t_0, s)$ with respect to $s$ is not identically zero. In order to detect Möbius-like factors of $G_\lambda$ using the implicit function theorem, we need to exclude any $\lambda$ from $\mathcal{S}_1 \cup \mathcal{S}_2$, with $\mathcal{S}_1 := \{0 \neq \lambda \in \mathbb{R} : L(\lambda) = 0\}$ and

$$\mathcal{S}_2 := \left\{ 0 \neq \lambda \in \mathbb{R} : G_\lambda(t_0, s) = 0, \frac{\partial G_\lambda}{\partial s}(t_0, s) = 0 \text{ for some } s \in \mathbb{C} \right\}.$$

The elements of $\mathcal{S}_2$ can be found by eliminating the variable $s$ from the bivariate polynomial system $G_\lambda(t_0, s) = \frac{\partial G_\lambda}{\partial s}(t_0, s) = 0$ in $\lambda, s$, for instance using the Sylvester resultant.

Suppose $\lambda \notin \mathcal{S}_1 \cup \mathcal{S}_2$. With the dependency on the variable $\lambda$ understood, write $G = G_\lambda$ and $G_t, G_s, G_{tu}, G_{ts}, G_{ss}$ for the first and second order partial derivatives of $G$. Suppose $G$ has a Möbius-like factor $F$, and let $s_0$ be a variable required to satisfy $F(t_0, s_0) = 0$. Since $G(t_0, s_0) = 0$ and $G_s(t_0, s_0) \neq 0$ one has $\frac{\partial F}{\partial s}(t_0, s_0) \neq 0$, and the equation $F(t, s) = 0$ implicitly defines a function $s = \varphi(t)$ in a neighborhood of $t_0$ with $s_0 = \varphi(t_0)$ as in (15).

In order to determine $F$, we find expressions for $s'_0, s''_0$ in terms of $\lambda, s_0$, using that $\varphi(t)$ is also implicitly defined by $G(t, s) = 0$, because $F$ divides $G$ and $G_s(t_0, s_0) \neq 0$. Differentiating once and twice the identity $G(t, \varphi(t)) = 0$ with respect to $t$, solving for $\varphi', \varphi''$, and evaluating at $t_0$ expresses

$$s'_0 = \varphi'(t_0) = -\frac{G_t}{G_s}(t_0, s_0),$$

$$s''_0 = \varphi''(t_0) = -\frac{G_t^2 G_{tu} - 2G_t G_s G_{ts} + G_s^2 G_{ss}}{G_s^3}(t_0, s_0).$$
in terms of the unknown \( s_0 \). Substituting these expressions into \([16]\) and multiplying by \(-G^2(t_0, s_0)\) yields polynomial expressions for the coefficients of \( \varphi \) in terms of \( s_0 \),

\[
\begin{align*}
a(s_0, \lambda) &= -(G^2_s G_{tt} - 2G_t G_s G_{ts} + G^2_t G_{ss}) s_0 - 2G^2_t G_s, \\
b(s_0, \lambda) &= + (G^2_s G_{tt} - 2G_t G_s G_{ts} + G^2_t G_{ss}) t_0 s_0 + 2s_0 G_t G^2_s + 2t_0 G^2_t G_s, \\
c(s_0, \lambda) &= -(G^2_s G_{tt} - 2G_t G_s G_{ts} + G^2_t G_{ss}), \\
d(s_0, \lambda) &= +(G^2_s G_{tt} - 2G_t G_s G_{ts} + G^2_t G_{ss}) t_0 + 2G_t G^2_s,
\end{align*}
\]

(19)

where these expressions are understood to be evaluated at \((t_0, s_0)\).

The polynomial \( F \) divides \( G \) if and only if the resultant \( \text{Res}_s(F, G) \) is identically zero, or equivalently precisely when

\[
0 = G(t, \varphi(t)) = G\left(t, \frac{a(s_0, \lambda)t + b(s_0, \lambda)}{c(s_0, \lambda)t + d(s_0, \lambda)}\right)
\]

(20)

holds identically. Clearing denominators yields a polynomial \( P(t) \), whose coefficients are polynomials \( P_i(s_0, \lambda) \). Then \( F \) divides \( G \) if and only if there exist \( s_0 \in \mathbb{R} \) and \( 0 \neq \lambda \in \mathbb{R} \) for which the \( P_i \) are simultaneously zero, i.e., when there is such a point \((s_0, \lambda)\) on the real variety generated by the ideal \( \langle P_i \rangle_i \subset \mathbb{R}[s, \lambda] \). Using Gröbner bases, one eliminates the variable \( s \) from this ideal, resulting in a principal ideal \( \langle \Lambda \rangle \subset \mathbb{R}[\lambda] \). Let \( S_0 := \{ 0 \neq \lambda \in \mathbb{R} : \Lambda(\lambda) = 0 \} \). We have shown:

**Theorem 21.** Suppose \( f(C_1) = C_2 \) for a similarity \( f \) with ratio \( \lambda \). Let \( t_0 \) be such that \( \text{lc}(G_\lambda(t_0, s)) \) does not vanish identically. Then \( \lambda \in S_0 \cup S_1 \cup S_2 \).

Therefore, \( f(C_1) = C_2 \) for a similarity \( f \) with ratio \( \lambda_0 \) if and only if

(i) \( \lambda_0 \in S_1 \cup S_2 \) and \( G_{\lambda_0} \) has a Möbius-like factor, or
(ii) \( \lambda_0 \in S_0 \) and the polynomials \( P_i \), after substituting \( \lambda_0 \), have a common real root \( s_0 \),

for which, in either case, the corresponding Möbius transformation satisfies \([7]\).

**Example 3.** Consider the family of curves \( \{ C_\alpha \}_\alpha \) defined by the parametrizations

\[
x_\alpha(t) = \left(-\frac{1}{3} t^3 + \frac{2}{3} t^3 + \alpha t^2, \frac{2}{3} t^3 + \alpha t^2, \frac{2}{3} t^3 - \alpha^2\right), \quad \alpha \in \mathbb{R}.
\]

These curves are shown in Figure [16] for parameters \(-1 \leq \alpha \leq 1\), with the curves \( C_{-1}, C_0, C_1 \) emphasized. Except for the line \( C_0 \), each curve \( C_\alpha \) is a cubic helical curve with proportionality constant \( \mu_\alpha \) satisfying \( |\mu_\alpha| = \sqrt{2} \), since

\[
\kappa_{x_\alpha} = \frac{2|\alpha|\sqrt{2}}{(\alpha^2 + 3t^2)^2}, \quad \tau_{x_\alpha} = \frac{2\alpha}{(\alpha^2 + 3t^2)^2}.
\]

In order to determine if the curves \( C_1, C_{-1} \) are similar, we compute

\[
G_\lambda(t, s) = 9\lambda t^4 + 9s^4 + 6\lambda t^2 + 6s^2 + \lambda + 1.
\]
Letting \( t_0 = 1 \), one has \( G_\lambda(t_0, s) = 9s^4 + 6s^2 + 16\lambda + 1 \) with constant leading coefficient \( L(\lambda) = 9 \), implying \( S_1 = \emptyset \). Moreover, \( \frac{2G_\lambda}{\partial s}(t_0, s) = 36s^3 + 12s \), so \( S_2 = \{-1/16\} \). Since

\[
\frac{s'}{s_0(3s_0^2 + 1)} = -\frac{4\lambda}{s_0(3s_0^2 + 1)}, \quad \frac{s''}{s_0^3(3s_0^2 + 1)^3} = -\frac{2\lambda(45s_0^6 + 30s_0^4 + 72\lambda s_0^2 + 5s_0^2 + 8\lambda)}{s_0^3(3s_0^2 + 1)^3},
\]

we have, after scaling by a common factor,

\[
\begin{align*}
a(s_0, \lambda) &= -s_0(45s_0^6 + 30s_0^4 + 120\lambda s_0^2 + 5s_0^2 + 24\lambda), \\
b(s_0, \lambda) &= 3s_0(27s_0^6 + 18s_0^4 + 40\lambda s_0^2 + 3s_0^2 + 8\lambda), \\
c(s_0, \lambda) &= -(45s_0^6 + 30s_0^4 + 72\lambda s_0^2 + 5s_0^2 + 8\lambda), \\
d(s_0, \lambda) &= 81s_0^6 + 54s_0^4 + 72\lambda s_0^2 + 9s_0^2 + 8\lambda.
\end{align*}
\]

Substituting (21) into (20) and clearing denominators, we get a polynomial

\[
G_{-1}(t, s) = 3(s - t)(s + t)(3s^2 + 3t^2 + 2),
\]

and the corresponding Möbius transformations \( \varphi_1(t) = t \) and \( \varphi_2(t) = -t \) satisfy (23). Since \( \lambda = -1 \) succeeds, one does not need to try \( \lambda = -1/16 \in S_2 \) by Proposition 8. We conclude that \( C_1 \) and \( C_2 \) are similar under two similarities.

### 3.4. Finding the similarities

Suppose that using Algorithm Similar3D we have determined that \( f(C_1) = C_2 \) for a similarity \( f(x) = \lambda_0 Qx + b \). Then we have computed the associated Möbius transformation \( \varphi \) and the ratio \( \lambda_0 \), and we would like to find \( Q \) and \( b \). For this purpose, we adapt to our problem the discussion in [4, §4]. By Theorem 9

\[
\lambda_0 \cdot Qx_1(t) + b = x_2(\varphi(t)).
\] \hspace{1cm} (22)

Once \( Q \) is determined, one finds \( b \) by evaluating (22) at \( t = 0 \).

Without loss of generality, we assume that \( x_1(t) \), and therefore any of its derivatives, is well defined at \( t = 0 \) and that \( x'_1(0), x''_1(0) \) are well defined, nonzero, and not parallel. This is equivalent to requiring that the curvature \( \kappa_{x_1}(0) \) is well defined and nonzero, which can always be achieved by a reparametrization of type \( t \mapsto t + \alpha \).

To determine \( Q \), we consider separately the cases when the coefficient \( d \) of the Möbius transformation \( \varphi \) satisfies \( d \neq 0 \) or \( d = 0 \). If \( d = 0 \), then \( 0 \neq \Delta = -bc \) implies \( c \neq 0 \), and Equation (22) becomes

\[
\lambda_0 \cdot Qx_1(t) + b = x_2(\varphi(t)) = x_2(\tilde{a}/t + \tilde{b}), \quad \tilde{a} := \frac{b}{c}, \quad \tilde{b} := \frac{a}{c}.
\]

Writing \( \tilde{x}_2(t) := x_2(1/t) \), we obtain

\[
\lambda_0 \cdot Qx_1(t) + b = \tilde{x}_2(\tilde{a}t + \tilde{b}).
\] \hspace{1cm} (23)
Evaluating (23) at \( t = 0 \) yields
\[
\lambda_0 \cdot Q \mathbf{x}_1(0) + b = \tilde{x}_2(\tilde{b}),
\]
while differentiating (23) once and twice and evaluating at \( t = 0 \) yields
\[
\lambda_0 \cdot Q \mathbf{x}'_1(0) = \tilde{x}'_2(\tilde{b}) \cdot \tilde{a}, \quad \lambda_0 \cdot Q \mathbf{x}''_1(0) = \tilde{x}''_2(\tilde{b}) \cdot \tilde{a}^2.
\]

Taking the cross product in (25) and using (9) with \( Q \) orthogonal,
\[
\lambda_0^2 \cdot Q (\mathbf{x}'_1(0) \times \mathbf{x}''_1(0)) = \tilde{x}'_2(\tilde{b}) \times \tilde{x}''_2(\tilde{b}) \cdot \tilde{a}^3.
\]
Combining (25) and (26), with
\[
B := [\lambda_0 \cdot \mathbf{x}'_1(0), \lambda_0 \cdot \mathbf{x}''_1(0), \lambda_0^2 \cdot \mathbf{x}'_1(0) \times \mathbf{x}''_1(0)],
\]
yields
\[
QB = C := [\tilde{x}'_2(\tilde{b}) \cdot \tilde{a}, \tilde{x}'_2(\tilde{b}) \cdot \tilde{a}^2, \tilde{x}'_2(\tilde{b}) \times \tilde{x}''_2(\tilde{b}) \cdot \tilde{a}^3],
\]
and \( Q = CB^{-1} \).

Now let us address the case \( d \neq 0 \). Differentiating (22) once and twice, yields
\[
\lambda_0 \cdot Q \mathbf{x}'_1(t) = \mathbf{x}'_2(\varphi(t)) \cdot \varphi'(t) = \mathbf{x}'_2 \left( \frac{at + b}{ct + d} \right) \frac{\Delta}{(ct + d)^2},
\]
\[
\lambda_0 \cdot Q \mathbf{x}''_1(t) = \mathbf{x}''_2(\varphi(t))(\varphi'(t))^2 + \mathbf{x}'_2(\varphi(t)) \varphi''(t)
\]
\[
= \mathbf{x}_2'' \left( \frac{at + b}{ct + d} \right) \frac{\Delta^2}{(ct + d)^4} - 2 \mathbf{x}_2' \left( \frac{at + b}{ct + d} \right) \frac{c \cdot \Delta}{(ct + d)^3},
\]
where \( \Delta = ad - bc \). Evaluating (29) and (30) at \( t = 0 \) yields
\[
\lambda_0 \cdot Q \mathbf{x}'_1(0) = \mathbf{x}'_2(b/d) \cdot \Delta/d^2,
\]
\[
\lambda_0 \cdot Q \mathbf{x}''_1(0) = \mathbf{x}''_2(b/d) \cdot \Delta^2/d^4 - 2 \mathbf{x}'_2(b/d) \cdot c \cdot \Delta/d^3.
\]

Taking the cross product and using (9) with \( \lambda_0^2 \cdot Q \) orthogonal yields
\[
\lambda_0^2 \cdot Q (\mathbf{x}'_1(0) \times \mathbf{x}''_1(0)) = (\Delta^3/d^6) \cdot (\mathbf{x}'_2(b/d) \times \mathbf{x}''_2(b/d)).
\]

Since \( \lambda_0 \) and \( \varphi \) are known, the matrix \( Q \) can again be determined from its action on \( \mathbf{x}'_1(0), \mathbf{x}''_1(0) \), and \( \mathbf{x}'_1(0) \times \mathbf{x}''_1(0) \), which is given by Equations (31)–(33).

**Example 4.** Let us find the similarity between the crunode curves \( C_1, C_2 \) in Example 2 corresponding to \( \lambda_0 = 2 \), \( \varphi(t) = t - 1 \). Then \( \varphi \) has coefficients \( a = 1, b = -1, c = 0, d = 1 \), and therefore \( \Delta = 1 \). From (27), (28) one obtains
\[
B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 6/5 & 16/5 & 0 \\ -8/5 & 12/5 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad Q = CB^{-1} = \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Substituting \( t = 0 \) in (22) yields \( b = x_2(-1) - 2Qx_1(0) = [0, 0, 2]^T \), consistent with Example 2.
### Table 1: CPU time $t$ (seconds) for random rational parametrizations of various degrees $m$ and coefficients with bitsize bounded by $\tau$.

| $m$ | $\tau = 4$   | $\tau = 8$   | $\tau = 16$  | $\tau = 32$  |
|-----|---------------|---------------|--------------|--------------|
| 3   | 0.327         | 0.375         | 0.577        | 0.842        |
| 4   | 0.655         | 1.170         | 1.497        | 3.120        |
| 5   | 1.263         | 1.700         | 3.292        | 7.098        |
| 6   | 1.716         | 3.900         | 6.880        | 15.288       |
| 7   | 4.336         | 6.896         | 14.290       | 27.659       |
| 8   | 8.253         | 12.683        | 22.168       | 35.927       |
| 9   | 11.762        | 10.998        | 21.466       | 57.424       |
| 10  | 12.340        | 23.509        | 46.519       | 90.746       |

4. **Experimentation and practical performance**

Algorithm Similar3D was implemented in the computer algebra system Maple 18, and was tested on an Intel Core i7 laptop, with 2.9 GHz processor and 8 GB RAM.

We present tables with timings corresponding to different groups of examples. Table 1 lists timings for random rational non-helical parametrizations with various degrees $m$ and coefficients with bitsizes at most $\tau$. The degree of the parametrization corresponds to the highest degree in the numerators and denominators of the components. Similarly, the bitsize of the parametrization corresponds to the largest bitsize of the coefficients of the numerators and denominators of the components. In order to generate these examples, we randomly created curves $C$ with given degree $m$ and bitsize $\tau$, and we ran the algorithm with $C_1 \equiv C$ and $C_2 \equiv f(C)$, with $f$ the similarity of $[14]$. We observed that in practice almost all the time was consumed computing the tentative values of $\lambda$.

Additionally, Table 2 lists the timings for a family of daisies of increasing degree $m = 4j + 4$, parametrically given by

$$x(t) = \left(u \sum_{i=0}^{j} (-1)^i \left(\frac{2j}{2i}\right) u^{2j-2i} v^{2i} \cdot \frac{1}{1+t^{4j+4}} \right),$$

where

$$u = \frac{1-t^2}{1+t^2}, \quad v = \frac{2t}{1+t^2}, \quad j = 0, 1, \ldots$$

In each case we tested Algorithm Similar3D with $C_1 \equiv C$ and $C_2 \equiv f(C)$, with $f$ again the similarity of $[14]$.

The curves in Table 1 and Table 2 have rational coefficients. Algorithm Similar3D can also be applied to curves with non-rational coefficients. However, in this case the performance is poor. To illustrate this, we consider the similarity

$$g(x) = \lambda_0 Q x, \quad \lambda_0 = \sqrt{2}, \quad Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
Table 2: Average CPU time (seconds) of Algorithm Similiar3D applied to daisies of various degrees.

| Degree | CPU time |
|--------|----------|
| 4      | 0.858    |
| 8      | 3.182    |
| 12     | 19.603   |
| 16     | 72.166   |
| 20     | 209.541  |

Table 3: CPU time $t$ (seconds) for random rational parametrizations, one of them with non-rational coefficients, of various degrees $m$ and coefficients with bitsize bounded by $\tau$.

| $t$ | $\tau = 4$ | $\tau = 8$ | $\tau = 16$ |
|-----|------------|------------|-------------|
| $m = 3$ | 1.623   | 3.245     | 5.960      |
| $m = 4$ | 13.884  | 22.557    | 37.409     |
| $m = 5$ | 58.298  | 77.673    |             |

so that the bottom-right element of $\lambda_0 Q$ is non-rational, namely $\sqrt{2}$. Now we pick a random parametrization with rational coefficients, defining a curve $C_1$, and we let $C_2 = g(C_1)$. Notice that the parametrization of $C_2$ has non-rational coefficients. Table 3 shows the timings for Algorithm Similiar3D for small coefficient bitsizes and low degrees (the machine resources are exhausted for degree 6 or higher).

For non-helical rational curves, the bottleneck of Algorithm Similiar3D is the computation of the resultant $R_\lambda = \text{Res}_s(K_\lambda, T_\lambda)$. In fact, we avoided the direct computation of this resultant. Instead, we computed for various values of $t_0$ the specialized bivariate resultants $\text{Res}_s(K_\lambda(s, t_0), T_\lambda(s, t_0))$ and computed their greatest common divisor; this yields a finite list of tentative values of $\lambda$. However, even the computation of these bivariate resultants is time-consuming as the degrees or bitsizes of the coefficients grow.

We tested Algorithm Similiar3D for several helical curves. We created these examples (including Example 1) by using the results on the generation of cubic and quintic polynomial helices in [10, §23], as well as the algorithm in [14] for generating general rational helices of any degree. The timings corresponding to these examples are shown in Table 4. These curves are polynomial helices of degree at most 7. Two different examples with degree 5 are provided: the first one corresponds to Example 3 with two helical space curves with the same proportionality constant, which nevertheless are not similar; the second one corresponds to two similar helical space curves. Although the method in [14] can produce rational, non-polynomial helices, Algorithm Similiar3D took a long
time for even the easiest examples of those. In the case of helical curves, we observed that the bottleneck of Algorithm Similar3D is the use of Gröbner bases for eliminating the variable $s$ from the ideal $\langle P_i \rangle_i \subset \mathbb{R}[s, \lambda]$.

5. Conclusion

We have presented a deterministic algorithm for deciding whether any two rational space curves are related by a similarity, and for determining the similarity in this case. The algorithm exploits the relationship between the curvature and torsion of two similar space curves and extends the results of [4], where the problem of detecting the symmetries of rational space curves was addressed. Interestingly, unlike for symmetry detection, it is necessary to distinguish the cases of non-helical and helical curves. In the first case, the experimentation performed so far shows that the algorithm is useful for curves of medium degrees or bitsizes. In the second case, the algorithm is useful for polynomial helices of low degree.

The bottleneck of the algorithm, both for non-helical and helical curves, is the computation of the similarity ratio. This operation depends on the computation of certain resultants in the non-helical case, and on Gröbner bases elimination in the helical case, which becomes time-consuming as the degrees or bitsizes grow.

Future work includes seeking alternatives for finding the similarity ratio, as well as detecting symmetries and similarities of implicitly defined algebraic space curves. Finally, it would be interesting to find alternatives for the special, but important, case of bounded space curves.

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