GENERALIZED RANDOM SIMPLICIAL COMPLEXES

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Abstract

We consider a multi-parameter model for randomly constructing simplicial complexes. This model interpolates between random clique complexes [Kah09] and Linial-Meshulam random k-dimensional complexes [MW09], two models that have been extensively studied. While these models asymptotically exhibit nontrivial cohomology in only one or two dimensions, we show that in this generalized setting nontrivial cohomology can occur in several dimensions simultaneously. We establish upper and lower thresholds for the appearance of nontrivial cohomology in a particular dimension, and in some instances characterize the behavior at criticality.

1. Introduction

1.1. Background. The purpose of this work is to understand the topological behavior of a generalized model for random simplicial complexes, mentioned in [Kah14b] and recently explored in [CF14]. We define $X(n, p_1, p_2, \ldots)$ to be the probability distribution over simplicial complexes on the vertex set $[n] = \{1, \ldots, n\}$ where the distribution of the 1-skeleton agrees with $G(n, p_1)$, the Erdős-Rényi random graph model. The distributions of higher dimensional skeletons are constructed inductively: for any $k$, whenever the full boundary of a $k$-simplex is in the complex, that simplex is added with probability $p_k$.

The study of random topological spaces began with random graphs and $G(n, p)$, mentioned above, is perhaps the most common method for constructing them. For some $p = p(n)$, typically a function of $n$, we consider a graph $G$ on vertex set $[n]$ where every edge between two vertices of $G$ is added independently with probability $p$. This defines a probability measure on the set of all graphs on $n$ vertices and we say $G \sim G(n, p)$ to indicate $G$ is a random graph with law $G(n, p)$.

Most results in this field pertain to the asymptotic behavior of these models: what happens as we let the number of vertices $n$ tend to infinity. Letting some graph property $A$ define a subset of all the graphs on $n$ vertices, for $G \sim G(n, p)$ we say that $G \in A$ with high probability, or w.h.p., if

$$\lim_{n \to \infty} \mathbb{P}[G \in A] = 1.$$ 

In one of the major results in random graph theory, Erdős and Rényi established in [ER59] a sharp threshold of $p = \log n / n$, where on one side $G \sim G(n, p)$ was w.h.p. a connected graph, and on the other side w.h.p. disconnected.

Significant work has been done on the behavior of random graphs and, more recently, simplicial complexes since [ER59]. Frequently, results about a model pertain to its homological or cohomological behavior. Indeed, even the connectivity threshold of $G(n, p)$ is purely a statement about the 0-homology of these random graphs. For any graph $G$, if $m$ is the number of connected components of $G$, then $H_0(G, \mathbb{Z}) = \mathbb{Z}^m$.

One model for random complexes that has received attention is $Y_k(n, p)$, the Linial-Meshulam model for random $k$-dimensional simplicial complexes. This model looks at a complex on $n$ vertices with full $(k - 1)$-skeleton, adding every possible $k$-face independently with probability $p$. Linial and Meshulam first considered the case when $k = 2$ in [LM06],...
which the complex is w.h.p. k
with X
4 vertices that form a complete subgraph in X.

the central limit theorem for the distribution of Betti numbers β
X
as specific instances of a model. Several other results concerning the behavior of X
that primarily cohomology will w.h.p. be nontrivial in just one dimension. Kahle proved the first homological k-cycles to appear in these complexes and prove the threshold for p for which the complex is w.h.p. k-collapsible.

Another model of particular interest is the random clique complex model, X(n, p). Just as in our own model, the distribution of the 1-skeleton is identical to G(n, p), but in this case the edges present dictate the entire complex. Given some X ∼ X(n, p), for every set of k + 1 vertices that form a complete subgraph in X, called a (k + 1)-clique, X also contains the k-simplex spanned by those vertices. For a fixed dimension k, Kahle established in [Kah09] and [Kah14a] sharp thresholds for p for which there will be nontrivial k-th homology, determining that primarily cohomology will w.h.p. be nontrivial in just one dimension. Kahle proved several other results concerning the behavior of X(n, p), and in [KM+13] established a central limit theorem for the distribution of Betti numbers β
k
= \dim (H
k
(X, \mathbb{Q})) in this model.

We note all the previously mentioned models for random complexes can be realized as specific instances of X(n, p1, p2, . . .). The random graph model G(n, p) is identical to X(n, p, 0, . . .) while Yk(n, p) corresponds to X(n, 1, . . ., 1, pk = p, 0, . . .). Finally, clique complexes are the case X(n, p1, 1, . . .), and in fact many of our results are achieved through a reworking of frameworks laid down in [Kah09] and [Kah14a]. This appears to be the natural bridge between these models, and we show that often it exhibits unique cohomological behavior.

1.2. Statement of Results. Our theorems look at the (k − 1)-th homology or cohomology of X(n, p1, . . .). Since the (k − 1)-th (co)homology of a simplicial complex depends only on its k-skeleton, we need only focus our attention on the probabilities p1, p2, . . ., pk. As with X(n, p), if these probabilities are sufficiently small or large then our complex will have trivial (k − 1)-cohomology. Repeated application of our theorems for each k will often fully characterize the cohomology of our random complex in every dimension. An open problem regarding the (k − 1)-th homology of our complexes when pk = 1 is discussed after our statement of results.

We begin with a low-dimensional example to give some intuition for where the equations in the following theorems come from, as well as show how these results indicate the potential for non-trivial cohomology in multiple dimensions simultaneously.

Proposition 1. Let X ∼ X(n, p1, p2, . . .) with pk = n−α
k
for all i. If p2, p3 ̸= 1, 6α1 + 4α2 < 4, and 1 ≤ 2α1 + α2, then w.h.p. H1(X, \mathbb{Q}) ̸= 0 and H2(X, \mathbb{Q}) ̸= 0.

We show these bounds are tight in Theorems 2, 3 and 5.

Proof. Within this proof, and later in Section 5, we consider the appearance of a specific type of subcomplex in X. First, we establish the presence of triangles with unfilled 2-face and first edge, determined lexicographically, not contained in any 2-face of X. Our complex is defined on the vertex set [n], and for any j ∈ \binom{[n]_3}{3} we let Aj denote the event that the
vertex set corresponding to \( j \) forms such a subcomplex. We calculate its probability to be
\[
P[A_j] = p_3^3 (1 - p_2) (1 - p_1^2 p_2)^{n-3}.
\]
The first two terms are requiring the three edges of our triangle are in \( X \), while the 2-simplex itself is not present. The last term ensures our first edge does not form a 2-simplex with any of the other \( n - 3 \) vertices of \( X \).

Letting \( M_1 \) denote the number of such subcomplexes in \( X \), linearity of expectation tells us
\[
E[M_1] = \sum_{j \in \binom{n}{3}} P[A_j] = \binom{n}{3} p_3^3 (1 - p_2) (1 - p_1^2 p_2)^{n-3}.
\]
First moment methods tell us if \( p_2 \neq 1, \alpha_1 < 1, \) and \( 1 \leq 2\alpha_1 + \alpha_2 \), then \( E[M_1] \to \infty \). Second moment methods, detailed in Appendix A, then tell us that w.h.p. \( M_1 > 0 \).

We now wish show the existence of tetrahedrons with unfilled 3-face with the first triangle not contained in any 3-face. For each \( l \in \binom{n}{4} \), let \( B_l \) be the event that the vertices \( l \) forms such a subcomplex in \( X \). Then similar considerations tell us
\[
P[B_l] = p_1^6 p_4^4 (1 - p_3) (1 - p_1^3 p_2^3 p_3)^{n-4}.
\]
Letting \( M_2 \) denote the total number of such subcomplexes in \( X \), linearity of expectation shows
\[
E[M_2] = \sum_{l \in \binom{n}{4}} P[B_l] = \binom{n}{4} p_1^6 p_4^4 (1 - p_3) (1 - p_1^3 p_2^3 p_3)^{n-4}.
\]
First moment methods again tell us if \( p_3 \neq 1, 6\alpha_1 + 4\alpha_2 < 4, \) and \( 1 \leq 3\alpha_1 + 2\alpha_2 + \alpha_3 \), then \( E[M_2] \to \infty \). Second moment methods then establish that, for \( p_i \) satisfying these relations, w.h.p. \( M_2 > 0 \).

Combining the two sets of requirements on our \( p_i \) gives us that whenever \( p_2, p_3 \neq 1, 1 \leq 2\alpha_1 + \alpha_2, \) and \( 6\alpha_1 + 4\alpha_2 < 4, \) then with high probability \( M_1, M_2 > 0 \). Each of the individual subcomplexes that appears in \( X \) generates a non-trivial \( \mathbb{Z} \)-summand in the 1 or 2-homology, respectively. Thus w.h.p. \( H_1(X, \mathbb{Z}) \neq 0 \) and \( H_2(X, \mathbb{Z}) \neq 0 \), and our result follows by the Universal Coefficients Theorem, which we will cover in the next section. \( \square \)

Our first theorem establishes the sharp upper threshold for vanishing cohomology, or where the probabilities are sufficiently large that we have trivial homology.

**Theorem 2.** Let \( X \sim X(n, p_1, p_2, \ldots) \) with \( p_i = n^{-\alpha_i} \) and \( \alpha_i \geq 0 \) for all \( i \). If
\[
\sum_{i=1}^{k} \alpha_i \binom{k}{i} < 1
\]
then w.h.p. \( H^{k-1}(X, \mathbb{Q}) = 0 \).

The proof of this theorem is handled in Sections 3 and 4. This inequality is precisely what is required to ensure every \((k-1)\)-simplex of \( X \) is contained in a \( k \)-simplex, so no single face generates a non-trivial cocyle in \( H^{k-1}(X) \). With this condition satisfied we prove the result by applying [BS97 Theorem 2.1], a result connecting spectral gap theory and the homology of simplicial complexes and presented in Section 2. Most of the work lies in showing the various conditions of the theorem are met by our simplicial complexes, for which we use [HKPT12 Theorem 1.1], a tool for bounding the spectral gap of Erdős–Rényi random graphs.

Sections 5 and 6 look at the range of probabilities where, as in Proposition 1, our complexes will with high probability exhibit non-trivial rational homology.
**Theorem 3.** Let \( X \sim X(n, p_1, p_2, \ldots) \) with \( p_i = n^{-\alpha_i}, \alpha_i \geq 0 \) for all \( i \), and

\[
1 \leq \sum_{i=1}^{k} \alpha_i \binom{k}{i}.
\]

If

\[
\sum_{i=1}^{k-1} \alpha_i \binom{k-1}{i} < 1
\]

then w.h.p. \( H^{k-1}(X, \mathbb{Q}) \neq 0 \). Moreover, if \( p_k \neq 1 \) we can relax this bound to

\[
\sum_{i=1}^{k-1} \alpha_i \binom{k+1}{i+1} < k+1.
\]

The statement for the range given by (1) and (2) is shown by exhibiting that our complex will w.h.p. have far more \((k-1)\)-dimensional faces than faces of one dimension higher or lower, so the kernel of the coboundary map is necessarily large. In fact, the second moment argument used in the proof yields the stronger result that within this range of values our Betti number \( \beta^{k-1} \) will grow polynomially in \( n \). Meanwhile, the result for the range defined by (1) and (3) follows from model used in Proposition 1: showing our complex will w.h.p. contain certain subcomplexes that generate nontrivial homological cycles.

Following this in Section 7 we direct our attention to the boundary of \( \alpha_i \)-values between Theorems 2 and 3. Upon allowing the probabilities to be more varied functions of \( n \) we are able to establish several limit theorems.

**Theorem 4.** If \( X \sim X(n, p_1, p_2, \ldots) \) with \( p_i = n^{-\alpha_i} \) for \( i \geq 2 \) and \( p_1 = (\rho_1 \log n + \frac{k-1}{2} \log \log n + c)^{1/k} n^{-\alpha_1} \) such that \( \sum_{i}^{k} \alpha_i \binom{k}{i} = 1 \), \( \rho_1 = k - \sum_{i=1}^{k-1} \alpha_i \binom{k}{i+1} \), and \( c \in \mathbb{R} \), then the \((k-1)\)-th Betti number \( \beta^{k-1} \) approaches a Poisson distribution

\[
\beta^{k-1} \to \text{Poi}(\mu)
\]

with mean

\[
\mu = \frac{\rho_1^{k-1} e^{-c}}{k!}.
\]

Within this section we exhibit certain regimes of values of \( p_i \) where the probabilities that our complex will have trivial cohomology and nontrivial cohomology are both bounded away from 0.

In Section 8 we establish a lower bound for the lower threshold at which homology vanishes.

**Theorem 5.** Let \( X \sim X(n, p_1, p_2, \ldots) \) with \( p_i = n^{-\alpha_i} \) and \( \alpha_i \geq 0 \) for all \( i \). If

\[
k + 1 < \sum_{i=1}^{k-1} \alpha_i \binom{k+1}{i+1}
\]

then w.h.p. \( H_{k-1}(X, \mathbb{Z}) = 0 \).

This inequality is precisely the threshold for when our complex will w.h.p. not contain the boundary of a \( k \)-simplex. We prove our result by showing this subcomplex is the most likely type of homological cycle to appear in our complex. That is, when our complex w.h.p. no longer has these subcomplexes, it will have nothing contributing to \((k-1)\)-th homology.
1.3. Discussion. Primarily our results concern when \( p_i = n^{-\alpha_i} \) or \( p_i = 0 \) with \( \alpha_i \geq 0 \) for all \( i \) (in the latter case we say \( \alpha_i = \infty \)). We note that most of our work easily extends to when the \( p_i \) are more varied functions of \( n \). For example if \( p_i = \omega_i n^{-\alpha_i} \) with \( \omega_i = o(n^\epsilon) \) for all \( \epsilon > 0 \), then all our theorems concerning when the \( \alpha_i \) fall within certain regimes still hold provided the \( \alpha_i \) do not lie on the boundary between two thresholds.

We described our process for constructing random complexes as a bridge between \( X(n, p) \) and \( Y_k(n, p) \), and our results support this claim. These theorems imply the corresponding results for these models as well. The boundary between Theorems 2 and 3 is sharp when \( p_i = n^{-\alpha_i} \), and Theorem 4 establishes a sharp upper bound for the vanishing of cohomology in clique complexes. Moreover, our method for proving Theorem 4 can be adjusted to the case when \( p_i = 1 \) for all \( i < k \) to yield the correct threshold for Linial-Meshulam complexes.

While our upper bounds are seen to be sharp, we have not characterized the sharp lower threshold in our general case and clique complexes are one such exception. Kahle proved this bound in [Kah09], but we have been unable to generalize his arguments or find another method. For now we leave this as an open problem.

Open Problem What is the lower threshold for the vanishing of \( H^{k-1}(X, \mathbb{Q}) \) when \( p_k = 1 \)?

Noting \( p_k = 1 \) implies \( X \) cannot contain the unfilled boundary of a \( k \)-simplex, the question likely reduces to what subcomplex that generates a cohomological cycle has the lowest threshold for appearance in \( X \). We suspect the answer is determined, perhaps uniquely, by the largest \( l < k \) such that \( p_l \neq 1 \). The Linial-Meshulam model, meanwhile, lacks a lower threshold for vanishing cohomology, since \( p = 0 \) implies our complex will have full \((k-1)\)-skeleton with no \( k \)-simplices.

Both \( X(n, p) \) and \( Y_k(n, p) \) are cases of \( X(n, p_1, p_2, \ldots) \) but they do not fully characterize our model, there is potential for asymptotic behavior dramatically different from either one. We note the for any fixed integer \( l \) we can find some \( k \) such that the range of values for \( p_i \) defined by

\[
1 \leq \sum_{i=1}^{k} \alpha_i \binom{k}{i}
\]

and

\[
\sum_{i=1}^{k+l-2} \alpha_i \binom{k+l}{i+1} < k + l
\]

is nontrivial. By the monotonicity in \( k \) of the sums in (1) and (2), we may apply our theorems to exhibit cohomological behavior exemplifying how this model differs from previous ones.

Corollary 6. For any integer \( l \), there exists a \( k \) and a non-trivial range of probabilities \( p_i = n^{-\alpha_i} \) such that if \( X \sim X(n, p_1, p_2, \ldots) \) subject to these conditions, then w.h.p. \( H^k(X, \mathbb{Q}), \ldots, H^{k+l}(X, \mathbb{Q}) \neq 0 \).

2. Topological Preliminaries

2.1. Basic definitions. We wish to lay out the topological definitions fundamental to our work. For further reference, we direct the reader to [Hat02].

Essentially, the (co)homology of a topological space is a measure of the number of “holes” of a specific dimension in the space. Fixing some simplicial complex \( X \), for any \( k \geq 0 \) we define \( C^k(X) \) to be the vector space of linear \( \mathbb{Q} \)-valued functions on the \( k \)-simplices of \( X \). We call such functions \( k \)-cochains and it is not hard to see that \( C^k(X) \) is generated by the
characteristic functions of the $k$-faces of $X$. For some $(k+1)$-face $\sigma = [v_0, \ldots, v_{k+1}]$ in $X$, we define the $k$-faces $\sigma_i = [v_i, \ldots, v_{k+1}]$. We then define the $k$-th coboundary map $\delta^k : C^k(X) \rightarrow C^{k+1}(X)$ by, for some $\phi \in C^i(X)$,

$$
\delta^k(\phi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i).
$$

One can verify that $\delta^k \circ \delta^{k-1} = 0$. We call a $k$-cochain $\phi$ a coboundary if $\phi \in \text{Im}(\delta^{k-1})$ and a cocycle if $\phi \in \ker(\delta^k)$. With this we are able to define the $k$-th cohomology group of $X$ to be

$$
H^k(X, \mathbb{Q}) = \frac{\ker(\delta^k)}{\text{Im}(\delta^{k-1})},
$$

and the $k$-th Betti number $\beta^k := \dim(H^k(X, \mathbb{Q}))$.

The homology of $X$ is defined in a similar fashion. We fix $F$ to be $\mathbb{Z}$ or some field, typically $\mathbb{Q}$ or a finite field. Letting $C_k(X)$ be the $F$-vector space generated by the $k$-faces of $X$, we construct our $k$-th boundary map $\partial_k : C_k(X) \rightarrow C_{k-1}(X)$ by, for some $\sigma \in C_k(X)$,

$$
\partial_k(\sigma) = \sum_{i=0}^{k} (-1)^i \sigma_i.
$$

Then we define the integer $k$-th homology group of $X$ by

$$
H_k(X,F) = \frac{\ker(\partial_k)}{\text{Im}(\partial_{k+1})}.
$$

Another useful definition for our work is the link of a subcomplex. Given a simplicial complex $X$ and a $k$-dimensional simplex $\sigma$ in $X$, we define the link of $\sigma$ in $X$ (denoted $\text{lk}_X(\sigma)$) to be a new simplicial complex with vertex set corresponding to the vertices of $X$ that form an $(k+1)$-face with $\sigma$. We then construct the new simplicial complex by adding the $(l-1)$-face corresponding to a set of vertices $v_1, \ldots, v_l$ precisely when the vertices $\sigma \cup \{v_1, \ldots, v_l\}$ comprise a $(k+l)$-face in $X$.

A simplicial complex $X$ is pure $k$-dimensional if every face of $X$ is contained in a $k$-dimensional face.

Finally, let $G$ be some graph with ordered vertices, with $D$ and $A$ the associated degree and adjacency matrices of $G$, respectively. We then construct the normalized Laplacian of $G$, denoted $\mathcal{L}$, by

$$
\mathcal{L} = I - D^{-1/2}AD^{-1/2}.
$$

For our work we look at the spectral gap of $G$ (denoted $\lambda_2[G]$), which is the absolute value of the smallest non-zero eigenvalue of the normalized Laplacian of $G$.

2.2. Useful Theorems. There are several established theorems we use in our work.

The Universal Coefficients Theorem provides the link between the homology and cohomology of a simplicial complex $X$, telling us $H_{k-1}(X, \mathbb{Q}) \cong H^{k-1}(X, \mathbb{Q})$. So any statement about rational homology can be extended to cohomology, and vice versa. Moreover, one can see that any $\mathbb{Z}$-summand of $H_k(X, \mathbb{Z})$ corresponds to a $\mathbb{Q}$-summand of $H_k(X, \mathbb{Q})$. Typically, the result statement of a theorem simply corresponds to whichever group we worked with in our proof. We note that the vanishing of integer homology is a much stronger statement than the vanishing of rational homology or cohomology.

With the definitions established we introduce the first of the two theorems critical to our proof of Theorem 2. We use a special case of Theorem 2.1 in a paper by Ballmann and Świątkowski [BS97].
Cohomology Vanishing Theorem. [BS97, Theorem 2.1] Let $X$ be a pure $D$-dimensional finite simplicial complex such that for every $(D-2)$-dimensional face $\sigma$, the link $\text{lk}_X(\sigma)$ is connected and has spectral gap

$$\lambda_2[\text{lk}_X(\sigma)] > 1 - \frac{1}{D}$$

Then $H^{D-1}(X, \mathbb{Q}) = 0$.

We note that since $X$ is stipulated to be pure $D$-dimensional, the link of any $(D-2)$-face will be of dimension 1. The spectral gaps of these link complexes are therefore well-defined.

To produce the necessary estimates on these gaps we then need the help of the main result in [HKP12], established by Hoffman, Kahle, and Paquette. We present it in a concise statement sufficient for our needs, noting the actual result yields more general and precise results.

Spectral Gap Theorem. [HKP12, Theorem 1.1] Fix a $\delta > 0$ and let $G \sim G(n, p)$ with $p \geq (1 + \delta) \log n / n$. Then $G$ is connected and $\lambda_2(G) > 1 - o(1)$ with probability $1 - o(n^{-\delta})$.

3. Calculating free faces

We call a $(k-1)$-face in a simplicial complex free if it is not part of any $k$-simplex. These play a significant role in the cohomology of this complex, at the very least they generate cocycles because they are contained in the kernel of our $(k-1)$-th coboundary map. We let $N_{k-1}$ denote the number of free $(k-1)$-faces. Our simplicial complex is defined on a vertex set $[n]$, and for any $j \in \binom{[n]}{k}$ we let $j$ correspond to the $(k-1)$-face on the associated vertices of $j$. We also let $C_j$ be the event that the vertex set corresponding to $j$ spans a free $(k-1)$-simplex. It follows that

$$N_{k-1} = \sum_{j \in \binom{[n]}{k}} 1_{C_j}.$$

We can then calculate it’s expectation to be

Lemma 7. If $j \in \binom{[n]}{k}$ be defined as above, then

$$\mathbb{P}[C_j] = \left( \prod_{i=1}^{k-1} p_i^{\binom{k}{i+1}} \right) \left( 1 - \prod_{i=1}^{k} p_i^{\binom{k}{i}} \right)^{n-k}.$$

Proof. The left parenthetical calculates the probability that $j$ is in our complex. Accordingly, for any $1 \leq i \leq k-1$ we need all the $\binom{k}{i+1}$ possible $i$-faces on the vertices of $j$ to be contained in our simplicial complex, which occurs with probability $p_i^{\binom{k}{i+1}}$. Meanwhile, the right parenthetical calculates the probability that these $k$ vertices do not form a $k$-simplex with any one of the $n-k$ other vertices. The only way this can happen, for a fixed vertex $v$, is if every possible face of dimension $1, \ldots, k$ that includes $v$ and vertices of $j$ is contained in our complex. This event that we wish to avoid occurs with probability $\prod_{i=1}^{k} p_i^{\binom{k}{i}}$ independently for each of the $n-k$ vertices, and our result follows. \qed
With this we can establish a threshold for which these subcomplexes don’t appear in our complex.

**Lemma 8.** Let $X \sim X(n, p_1, p_2, \ldots)$ with $p_i = n^{-\alpha_i}$ for all $i$ and let $N_{k-1}$ count the number of free $(k-1)$-faces in $X$. If $\sum_{i}^{k} \binom{n}{i} \alpha_i < 1$ then $N_{k-1} = 0$ w.h.p.

**Proof.** By (4) and linearity of expectation we have

$$E[N_{k-1}] = \sum_{j \in \binom{[n]}{k-1}} E[1_{C_j}] = \sum_{j \in \binom{[n]}{k-1}} \mathbb{P}[C_j]$$

$$= \sum_{j \in \binom{[n]}{k-1}} \left( \prod_{i=1}^{k-1} p_i^{\binom{n}{i}} \right) \left( 1 - \prod_{i=1}^{k} p_i^{\binom{n}{i}} \right)^{n-k}$$

$$= \binom{n}{k} \left( \prod_{i=1}^{k-1} p_i^{\binom{n}{i}} \right) \left( 1 - \prod_{i=1}^{k} p_i^{\binom{n}{i}} \right)^{n-k}$$

$$\leq \frac{n^k}{k!} \left( \prod_{i=1}^{k} n^{1- \sum_{i+1}^{\infty} \alpha_i(k)} \right) \left( e^{-(n-k) \prod_{i=1}^{k} \binom{n}{i}} \right)$$

$$\leq \frac{n^k}{k!} \left( \prod_{i=1}^{k} n^{1- \sum_{i+1}^{\infty} \alpha_i(k)} \right) \left( e^{-(n-\sum_{i=1}^{k} \alpha_i(k))} \right)$$

$$= \frac{1}{k!} \left( n^{k- \sum_{i=1}^{k} \alpha_i(k)} \right) \left( e^{-n \sum_{i=1}^{k} \alpha_i(k)} \right).$$

Our requirement $\sum_{i=1}^{k-1} \alpha_i(k) < 1$ implies that $\sum_{i=1}^{k-1} \alpha_i(k_{i+1}) < k$, and as a consequence we w.h.p. have $(k-1)$-faces in our complex. It also means the right parenthetical is $e^{-n^\epsilon}$ for some $\epsilon > 0$. This dominates the rest of the expression and we conclude that $E[N_{k-1}] \to 0$ exponentially. Markov’s inequality immediately tells us $N_{k-1} = 0$ w.h.p.. 

The fact that in this instance every $(k-1)$-face in our complex is with high probability contained in a $k$-simplex is necessary to utilize another powerful theorem that will ultimately enable us to conclude that $H^{k-1}(X, \mathbb{Q}) = 0$ w.h.p. within this regime of $\alpha_i$.

4. **Trivial Cohomology**

In this section we prove Theorem 2, the upper threshold for vanishing cohomology, for which [BS97] Theorem 2.1 and [HKP12] Theorem 1.1 are crucial to our argument.

We note that to understand the $(k-1)$-th cohomology of a complex we need only consider its $k$-skeleton, i.e. the subcomplex of $X$ induced by all its faces of dimension less than $k + 1$. We now let $X_k$ denote the $k$-skeleton of $X$, observing $H^k(X_k) = H^k(X)$ and $X_k$ is pure $k$-dimensional, one of the hypotheses of [BS97] Theorem 2.1. The following Lemma provides the first step to invoking the Cohomology Vanishing Theorem.

**Lemma 9.** Let $X \sim X(n, p_1, p_2, \ldots)$ such that $\sum_{i=1}^{k} \alpha_i(k) < 1$ and $X_k$ be its $k$-skeleton. Then $X_k$ is w.h.p. pure $k$-dimensional.

**Proof.** Since $\sum_{i}^{j} \alpha_i(k) \leq \sum_{i=1}^{k} \alpha_i(k)$ for any $j < k$, repeated applications of Lemma 8 tell us that every $i$-face of $X$ with $i < k$ is w.h.p. contained in some $k$-simplex. Our claim follows.
To establish trivial cohomology all that remains is to bound the spectral gaps of the links of $X_k$.

4.1. Using the Spectral Gap Theorem. We wish to understand the structure of the links of the $(k-2)$-faces in our complex. Given a $(k-2)$-face $\sigma \in X$, we let $L_\sigma$ denote the number of vertices in $\text{lk}_{X_k}(\sigma)$.

**Lemma 10.** For any $(k-2)$-face $\sigma \in X$, $L_\sigma$ has the same distribution as $\text{Bin}(n-k+1,p)$ and $\text{lk}_{X_k}(\sigma)$ the same distribution as $G(L_\sigma, p')$, where $p = \prod_{i=1}^{k-1} p_i^{(i-1)}$ and $p' = \prod_{i=1}^{k} p_i^{(i-1)}$.

**Proof.** We fix a $(k-2)$-face $\sigma$. For any vertex $v$ to be in $\text{lk}_{X_k}(\sigma)$, $X_k$ must contain every possible simplex that can be formed with $v$ and some subset of the vertices of $\sigma$. For every dimension $1 \leq i \leq k-1$, there are $\binom{k}{i-1}$ such simplices, each added with probability $p_i$. The events that two distinct vertices are contained in our link are statements about disjoint sets of simplices, so the events are independent with probability $p = \prod_{i=1}^{k-1} p_i^{(i-1)}$, and our statement about $L_\sigma$ follows. Similar calculations give us that the edge between any two vertices $u,v$ in $\text{lk}_{X_k}(\sigma)$ is included when every $i$-simplex involving $u,v$ and some subset of vertices of $\sigma$. This occurs with probability $p' = \prod_{i=1}^{k} p_i^{(i-1)}$. Again, the inclusion of any two edges are independent events and so $\text{lk}_{X_k}(\sigma)$ has the same distribution as $G(L_\sigma, p')$. □

Before we can use the Spectral Gap Theorem we must bound $L_\sigma$ from below.

**Lemma 11.** If $X \sim X(n,p_1,\ldots,p_k)$ with $\sum_{i}^{k} \alpha_i \binom{k}{i} < 1$ then w.h.p. $np/2 \leq L_\sigma$ for every $\sigma$.

**Proof.** For any specific $(k-2)$-face $\sigma$ and sufficiently large $n$, Chernoff bounds give us

\[
\mathbb{P}(L_\sigma < np/2) \leq \mathbb{P}(L_\sigma < 4\mu/7) \leq e^{-\frac{\mu^2}{16}},
\]

where $\mu = (n-k+1)p$. However, these probabilities are not independent. We let $J_\sigma$ denote the indicator random variable for the event that $L_\sigma < np/2$, then use Markov’s Inequality to see

\[
\mathbb{P}\left(\sum_{\sigma} J_\sigma \geq 1\right) \leq \mathbb{E}\left[\sum_{\sigma} J_\sigma\right] = \sum_{\sigma} \mathbb{E}[J_\sigma].
\]

There are at most $\binom{n}{k-1}$ $(k-2)$-faces in $X$ and by construction $\mathbb{E}[J_\sigma] = \mathbb{P}(L_\sigma < np/2)$, so

\[
\mathbb{P}\left(\sum_{\sigma} J_\sigma \geq 1\right) \leq \left(\frac{n}{k-1}\right) \mathbb{E}[J_\sigma] \text{ for a fixed } \sigma
\]

\[
\leq \left(\frac{n}{k-1}\right) e^{-\frac{\mu^2}{16}} \text{ (by (8))}
\]

\[
= \left(\frac{n}{k-1}\right) e^{-\frac{2(n-k+1)p}{7k}}
\]

\[
= \left(\frac{n}{k-1}\right) e^{-np \frac{2(n-k+1)}{7k}} e^{(k-1)p}.
\]

Since $\alpha_i \geq 0$ for all $i$, we know $\sum_{i=1}^{k-1} \alpha_i \binom{k-1}{i} < \sum_{i=1}^{k} \alpha_i \binom{k}{i} < 1$ and

\[
p = \prod_{i=1}^{k-1} p_i^{(i-1)} = n^{-\sum_{i=1}^{k-1} \alpha_i (k-1)} = n^{-1}.
\]
for some $\epsilon > 0$. Moreover, $\frac{(k-1)p}{98} \to 0$ so we may bound $e^{\frac{(k-1)p}{98}}$ from above by some constant $C$. We then have

\[
\mathbb{P}(\sum_{\sigma} J_{\sigma} \geq 1) \leq C \binom{n}{k-1} e^{-\frac{n}{100} \epsilon^4} \\
\leq C n^{k-1} e^{-\frac{99}{100} \epsilon^4} \\
= o(1).
\]

Thus with high probability $L_{\sigma} = 0$ for all $\sigma$, which completes our proof. \qed

We now prove one more Lemma before tackling the primary result of this Section.

**Lemma 12.** Fix a $\delta > 0$, if $X \sim X(n, p_1, p_2, \ldots)$ with $\sum_{i=1}^{k-1} \alpha_i \left( \binom{k-1}{i} \right) < 1$ then w.h.p.

\[
(1 + \delta) \frac{\log L_{\sigma}}{L_{\sigma}} \leq p'
\]

for all $(k-2)$-faces $\sigma$ in $X$.

**Proof.** We let $L = np/2$. Straightforward calculus shows the function $f(x) = \frac{(1+\delta)\log(x)}{x}$ is monotonically decreasing on $[\epsilon, \infty)$. Restricting our attention to $n$ large enough that $\epsilon < L$, we need only show $f(L) < p'$ to conclude $f(L_{\sigma}) < p'$ for all $\sigma$ with high probability. We let $\epsilon = 1 - \sum_{i=1}^{k-1} \alpha_i \left( \binom{k-1}{i} \right)$, noting $\epsilon > 0$ by our hypothesis, and calculate

\[
\frac{f(N)}{p'} = (1 + \delta) \frac{\log N}{N p'} \\
\leq (2 + 2\delta) \frac{\log n}{n p'} \\
= (2 + 2\delta) \frac{\log n}{n^{\sum_{i=1}^{k-1} \alpha_i \left( \binom{k-1}{i} \right) - \sum_{i=1}^{k-1} \alpha_i \left( \binom{k-1}{i} \right)}} \\
= (2 + 2\delta) \frac{\log n}{n^{\sum_{i=1}^{k-1} \binom{k-1}{i} \alpha_i}} \\
= (2 + 2\delta) \frac{\log n}{n^{\epsilon}} \\
= o(1).
\]

Thus w.h.p. $f(L_{\sigma}) < f(L) < p'$ for all $(k-2)$-faces $\sigma$. \qed

### 4.2. The Main Result.

We now have the machinery to prove one of our main results:

**Proof of Theorem 2** We begin by fixing the $\delta > 0$ we will use in applying [HKP12, Theorem 1.1]:

\[
\delta = \frac{k-2 - \sum_{i=1}^{k-1} \alpha_i \left( \binom{k-1}{i} \right)}{1 - \sum_{i=1}^{k-1} \alpha_i \left( \binom{k-1}{i} \right)}.
\]

A standard second moment technique tells us that if $f_{k-2}$ denotes the number of $(k-2)$-faces in $X$, or $X_k$, then w.h.p.

\[
f_{k-2} \leq (1 + o(1)) \binom{n}{k-1} \prod_{1}^{k-2} p_i^{\binom{k-1}{i+1}}.
\]

\[\text{We prove } f_{k-2} \sim \mathbb{E}[f_{k-2}] \text{ in Section 6.}\]
Meanwhile Lemma 12 tells us w.h.p.

\[ p' \geq \frac{(1+\delta)\log L_\sigma}{L_\sigma} \]

for all \((k-2)\)-faces \(\sigma\) of \(X\).

Recall each of these faces has the same distribution as \(G(L_\sigma, p')\), so by [HKP12] Theorem 1.1 and Lemma 12 the probability \(P_\sigma\) that \(\lambda_2[\text{lk}_X(\sigma)] < 1 - 1/k\) is \(o(N_\sigma^{-\delta})\). Letting \(P_X\) denote the probability there exists any \((k-2)\)-face \(\sigma\) whose link in \(X_k\) has a bad spectral gap, we use a union bound to see

\[ P_X \leq \sum_\sigma P_\sigma \]

\[ = \sum_\sigma o(L_\sigma^{-\delta}) \]

\[ \leq \sum_\sigma o((np/2)^{-\delta}). \]

The last line holds since w.h.p. \(L_\sigma > np/2\) so \(L_\sigma^{-\delta} < (np/2)^{-\delta}\). We then use 10 to conclude

\[ P_X \leq (1 + o(1)) \left( \frac{n}{k-1} \right) \left( \prod_{i=1}^{b_2-1} p_i \right) o \big(2^\delta(np)^{-\delta}\big) \]

\[ \leq (1 + o(1)) \frac{n^{k-1}}{(k-1)!} \left( \prod_{i=1}^{b_2-1} p_i \right) o \big(2^\delta(np)^{-\delta}\big) \]

\[ = O \left(2^\delta n^{k-1} - \sum_{i=1}^{k-2} \alpha_i(n-\sum_{i=1}^{k-1} \alpha_i(n))^{-\delta}\right) \]

\[ = O \left(2^\delta n^{k-1} - \sum_{i=1}^{k-2} \alpha_i(n-\sum_{i=1}^{k-1} \alpha_i(n))^{-\delta}\right). \]

But since

\[ \delta = \frac{k - \sum_{i=1}^{k-2} \alpha_i(n-\sum_{i=1}^{k-1} \alpha_i(n))^{-\delta}}{1 - \sum_{i=1}^{k-1} \alpha_i} \]

it follows

\[ P_X = O \left(C n^{k-1} - \sum_{i=1}^{k-2} \alpha_i(n-\sum_{i=1}^{k-1} \alpha_i(n))^{-\delta}\right) \]

\[ = O \big(C n^{-1}\big) \]

\[ = o(1). \]

Thus w.h.p. \(\lambda_2[\text{lk}_X(\sigma)] > 1 - \frac{1}{k}\) for every \((k-2)\)-face \(\sigma\) in \(X\). As we have already shown in Lemma 3 that w.h.p. the k-skeleton \(X_k\) of \(X \sim X(n, p_1, p_2, \ldots)\) is pure k-dimensional, we may apply [BS97] Theorem 2.1 on this subcomplex with identical \((k-1)\)-homology to conclude w.h.p. \(H^{k-1}(X_k, \mathbb{Q}) \cong H^{k-1}(X, \mathbb{Q}) = 0\).

5. **Nontrivial Homology: Boundaries of Simplices**

In this Section we will consider when \(\sum_{i=1}^{k} \alpha_i(k_i) > 1\), \(p_k \neq 1\) and \(\sum_{i=1}^{k-1} \alpha_i(k_{i+1}) < k + 1\) to prove the second half of Theorem 3.

Just as was shown for \(Y_k(n, p)\) in [ALLM13], the first type of homological \((k-1)\)-cycle to occur in, and last to disappear from, \(X(n, p_1, p_2, \ldots)\) is the boundary of a \(k\)-dimensional simplex that is not filled in, provided such a subcomplex is possible (ie. \(p_k \neq 1\)). If we have such a subcomplex with at least one of the \((k-1)\)-faces not contained in any \(k\)-face then we
know this is a nontrivial homological cycle, generating a \( \mathbb{Z} \)-summand in \( H_{k-1}(X, \mathbb{Z}) \). For any set of vertices \( j \in \binom{[n]}{k+1} \) we define \( A_j \) to be the event that \( j \) corresponds to the unfilled boundary of a simplex with the first \((k - 1)\)-face, determined by lexicographic order, not contained in any \( k \)-simplex. Letting \( M_{k-1} \) denote the total number of such faces in our simplicial complex, it follows that

\[
M_{k-1} = \sum_{j \in \binom{[n]}{k+1}} 1_{A_j}.
\]

We can then calculate the probability of \( A_j \).

**Lemma 13.** For \( j \in \binom{[n]}{k+1} \) defined as above,

\[
E[1_{A_j}] = P[A_j] = \left( \prod_{i=1}^{k-1} p_i^{(i+1)} \right) \left( 1 - p_k \right) \left( 1 - \prod_{i=1}^{k} p_i^{(i)} \right)^{n-k-1}. \tag{11}
\]

**Proof.** The first term calculates the probability that \( X \) contains the necessary \( i \)-faces for \( i < k \): we need every possible combination of \( i + 1 \) vertices out of the \( k + 1 \) in \( j \) to form \( i \)-simplices. The second term is the requirement that the associated \( k \)-simplex is not filled in. The last term is ensuring our first \((k - 1)\)-face does not form a \( k \)-simplex with any of the remaining \( n - k - 1 \) vertices, which occurs independently with probability \( \prod_{i=k+1}^{n} p_i^{(i)} \) for each vertex. \( \Box \)

We chose to consider precisely when the “first” \((k - 1)\)-face is free since narrowing our considerations simplifies the calculations without altering the relevant probability thresholds.

**Lemma 14.** Let \( X \sim X(n, p_1, p_2, \ldots) \) with \( p_i = n^{-\alpha_i} \) for all \( i \) and let \( M_{k-1} \) count the number of unfilled boundaries of \( k \)-simplices in \( X \) with their first \((k - 1)\)-face free of any \( k \)-simplex. If \( 1 \leq \sum_{i=1}^{k} \alpha_i(k) \) and \( \sum_{i=1}^{k-1} \alpha_i(k+1) < k + 1 \) then w.h.p. \( M_{k-1} > 0 \) and in fact \( M_{k-1} \sim E[M_{k-1}] \).

**Proof.** By linearity of expectation we have

\[
E[M_{k-1}] = \left( \frac{n}{k+1} \right) \left( \prod_{i=1}^{k-1} p_i^{(i+1)} \right) \left( 1 - p_k \right) \left( 1 - \prod_{i=1}^{k} p_i^{(i)} \right)^{n-k-1} \approx \left( \frac{1 - p_k}{k+1} \right)^{n-k-1} \left( \frac{\alpha_i(k)}{\sum_{i=1}^{k-1} \alpha_i(k+1)} \right). \tag{12}
\]

We see that our requirements \( p_k \neq 1, 1 < \sum_{i=1}^{k} \alpha_i(k), \) and \( \sum_{i=1}^{k-1} \alpha_i(k+1) < k + 1 \) imply

\[
E[M_{k-1}] = \frac{1 - o(1)}{(k+1)!} \left( \frac{n^{k+1} - \sum_{i=1}^{k-1} \alpha_i(k+1)}{k+1} \right). \tag{13}
\]

and so \( E[M_{k-1}] \to \infty. \)

The proof that this implies that \( M_{k-1} \sim E[M_{k-1}] \) is a straightforward and computationally tedious second moment argument, see e.g. [Kah14a], that can be found in Appendix A. \( \Box \)

**Proof of the second part of Theorem.** We now have that w.h.p. \( M_{k-1} > 0 \). But any such subcomplex must be a \( \mathbb{Z} \)-summand. They comprise the boundary of a \( k \)-simplex so a signed sum of the \((k - 1)\)-faces is in the kernel of our boundary map. Then the fact that one of
these faces, $\sigma$, is not contained in any $k$-simplex of $X$, no $(k-1)$-chain containing a non-zero coefficient of $\sigma$ is contained in the $(k-1)$-homological boundary of $X$. Thus we have found a non-trivial cycle and conclude that $H_{k-1}(X,\mathbb{Z})$ contains a $\mathbb{Z}$-summand and is nontrivial. By the Universal Coefficients Theorem we may conclude $H^{k-1}(X,\mathbb{Q}) \cong H_{k-1}(X,\mathbb{Q}) \neq 0$. $\square$

6. Nontrivial Cohomology: Betti Numbers Argument

Now we consider when $\sum_1^k \alpha_i^{(k)} > 1$ and $\sum_1^{k-1} \alpha_i^{(k-1)} < 1$, proving the other half of Theorem 3.

Proof of the first part of Theorem 3. For $X \sim X(n,p_1,p_2,\ldots)$, with the aforementioned conditions on $p_i$, we let $f_i$ denote the number of $i$-simplices in $X$ and $\beta_i = \dim (H^i(X,\mathbb{Q}))$.

Basic linear algebra tells us that

$$\sum_{1}^{k} \{ 1 \} = \sum_{1}^{k} \{ 1 \} \leq n - k + 1 \prod_{i=1}^{k} p_i^{(k)} \leq n \prod_{i=1}^{k} p_i^{(k)} = o(1),$$

because $\prod_{i=1}^{k} p_i^{(k)} < n^{-1}$ by our hypothesis. Since $\prod_{i=1}^{k-1} p_i^{(k-i)} = n^{c-1}$ for some $c > 0$ we have

$$\frac{\mathbb{E}[f_{k-2}]}{\mathbb{E}[f_{k-1}]} = \frac{k}{n - k + 1} \prod_{i=1}^{k-1} p_i^{(k-i)} \leq \frac{k}{n - k + 1} \prod_{i=1}^{k-1} p_i^{(k-i)} = \frac{k}{n - k + 1} n^{(1-c)} = o(1).$$

We write that $X \sim Y$ with high probability if for all $\epsilon > 0$, as $n \to \infty$ we have

$$\mathbb{P}(1 - \epsilon) \leq Y/X \leq (1 + \epsilon) \to 1.$$ 

Thus if we let $f_{k-1} := f_{k-1} - f_{k} - f_{k-2}$ it follows that

$$\mathbb{E}[f_{k-1}] \sim \mathbb{E}[\beta_{k-1}] \sim \mathbb{E}[f_{k-1}].$$

To make make more substantial conclusions about $\beta_{k-1}$ we again make use of Chebyshev’s Inequality\textsuperscript{2}. That is, if $\mathbb{E}[X] \to \infty$ and $\text{Var}[X] = o(\mathbb{E}[X]^2)$ then w.h.p. $X \sim \mathbb{E}[X]$.

\textsuperscript{2}We relegated our first second moment argument to the appendix, but this one is relatively brief.
Now
\[
\text{Var}[f_{k-1}] = \text{E}[f_{k-1}^2] - \text{E}[f_{k-1}]^2
\]
\[
= \text{E}[f_{k-1}^2] - \frac{n}{k} \left( \prod_{i=1}^{k-1} p_i \right)^2.
\]

To calculate \( \text{E}[f_{k-1}^2] \), we consider \( j, l \in \binom{[n]}{k} \) and let \( A_j \) (resp. \( A_l \)) be the event that the subset of vertices \( j \) (resp. \( l \)) spans a \((k-1)\)-face in \( X(n, p_1, \ldots, p_k) \). Then
\[
\text{E}[f_{k-1}^2] = \sum_{j, l \in \binom{[n]}{k}} \text{P}[A_j \cap A_l] = \binom{n}{k} \sum_{l \in \binom{[n]}{k}} \text{P}[A_j \cap A_l],
\]
the second equality following by symmetry and letting \( j \) denote a fixed set of vertices, say \( \{1, \ldots, k\} \). We proceed by grouping the \( l \) according to the size of their intersections with \( j \). Through this approach we see
\[
\text{E}[f_{k-1}^2] = \binom{n}{k} \sum_{l \in \binom{[n]}{k}} \text{P}[A_j \cap A_l]
\]
\[
= \binom{n}{k} \prod_{i=1}^{k-1} p_i \left( \sum_{m=0}^{k-1} \binom{k}{m} \frac{n-k}{k-m} \prod_{i=1}^{m-1} p_i \right)
\]
\[
= \binom{n}{k} \prod_{i=1}^{k-1} p_i \left( \sum_{m=0}^{k-1} \binom{k}{m} \frac{n-k}{k-m} \prod_{i=1}^{m-1} p_i \right).
\]
We may pull the \( m = 0 \) case out of the summation and use \( \binom{n-k}{k} < \binom{n}{k} \) to get
\[
\text{E}[f_{k-1}^2] \leq \text{E}[f_{k-1}]^2 + \binom{n}{k} \prod_{i=1}^{k-1} p_i \left( \sum_{m=1}^{k-1} \binom{k}{m} \frac{n-k}{k-m} \prod_{i=1}^{m-1} p_i \right).
\]
This tells us
\[
\frac{\text{Var}[f_{k-1}]}{\text{E}[f_{k-1}]^2} \leq \frac{\binom{n}{k} \prod_{i=1}^{k-1} p_i \left( \sum_{m=1}^{k-1} \binom{k}{m} \frac{n-k}{k-m} \prod_{i=1}^{m-1} p_i \right)}{\binom{k}{2} \left( \prod_{i=1}^{k-1} p_i \right)^2}
\]
\[
\leq \frac{\sum_{m=1}^{k} \binom{k}{m} \frac{n-k}{k-m} \prod_{i=1}^{m-1} p_i}{\binom{n}{k}}
\]
\[
= \sum_{m=1}^{k} \frac{\binom{n}{k}}{1} \prod_{i=1}^{m-1} p_i \frac{\binom{k}{m} \frac{n-k}{k-m}}{\binom{n}{k}}
\]
\[
= o(1).
\]
The final line holds form our hypothesis \( \sum_{i=1}^{n-1} \alpha_i \frac{\binom{m}{i+1}}{m} \leq \frac{m}{k} \sum_{i=1}^{k-1} \alpha_i \frac{\binom{k}{i+1}}{k} < \frac{m}{k} k = m \), so
\[
\prod_{i=1}^{m-1} p_i \frac{\binom{m}{i+1}}{\binom{n}{k}} = n \sum_{i=1}^{m-1} \alpha_i \frac{\binom{m}{i+1}}{\binom{n}{k}} = o(n^m). \]
We conclude \( f_{k-1} \sim \text{E}[f_{k-1}] \).

As nothing in the above argument was unique to the case of \( f_{k-1} \), we may conclude that w.h.p. \( -f_{k-2} \sim \text{E}[-f_{k-2}] \) and \( -f_k \sim \text{E}[-f_k] \). By linearity of expectation \( \bar{f}_{k-1} \sim \text{E}[\bar{f}_{k-1}] \), then from (14) and (14) we conclude that w.h.p. \( \beta_{k-1} \sim \text{E}[\beta_{k-1}] \sim f_{k-1} \). Thus with high probability \( \beta_{k-1} = \dim(H^{k-1}(X, \mathbb{Q})) \neq 0 \), which completes our proof. \( \square \)
It is worth noting that the requirement that \( \sum_{i=1}^{k-1} \alpha_i(k-1)_i < 1 \) is precisely demanding that w.h.p. there are no free \((k-2)\)-faces in our complex. In fact under these conditions we proved a stronger result than nontrivial homology.

**Lemma 15.** If \( X \sim X(n, p_1, p_2, \ldots) \) with \( p_i = n^{-\alpha_i} \) and \( \alpha_i \geq 0 \) for all \( i \). Let \( f_{k-1} \) denote the number of \((k-1)\)-faces of \( X \) and \( \beta_{k-1} \) the \((k-1)\)-th Betti number. If

\[
\sum_{i=1}^{k-1} \alpha_i(k-1)_i < 1 < \sum_{i=1}^{k} \alpha_i(k)_i,
\]

then w.h.p. \( f_{k-1} \sim \beta_{k-1} \).

We note our proof implies allowing \( \sum_{i=1}^{k} \alpha_i(k)_i = 1 \) still ensures w.h.p. nontrivial cohomology.

**Lemma 16.** If \( \sum_{i=1}^{k} \alpha_i(k)_i = 1 \), then \( H^{k-1}(X, \mathbb{Q}) \neq 0 \) and in fact \( \beta_{k-1} \geq \left(\frac{k}{k+1}\right) f_{k-1} \) w.h.p.

**Proof.** We first calculate

\[
\frac{\mathbb{E}[f_k]}{\mathbb{E}[f_{k-1}]} = \frac{n-k}{k+1} \prod_{i=1}^{k} p_i = \frac{n-k}{n(k+1)} \approx \frac{1}{k+1}.
\]

The machinery established in the previous section then does most of the legwork for us. Since \( \beta_{k-1} \) is bounded between \((f_{k-1} - f_{k} - f_{k-2})\) and \( f_{k-1} \), while \( f_{k-1} \sim \mathbb{E}[f_{k-1}] \) and \((f_{k-1} - f_{k} - f_{k-2}) \sim \mathbb{E}[(f_{k-1} - f_{k} - f_{k-2})] \sim \left(\frac{k}{k+1}\right) \mathbb{E}[f_{k-1}] \). \( \Box \)

7. **Behavior at the Boundary**

In this section we explore the behavior of the \((k-1)\)-th cohomology of \( X(n, p_1, p_2, \ldots) \) at the upper threshold line. Specifically, we refine our face probabilities to elicit some more interesting behavior.

7.1. **Free faces.** To get to the threshold for free faces, and thus trivial cohomology, we have to refine our model slightly. Unfortunately there is no nice way to articulate the sets of \( p_i \) that inhabit this threshold, so we will provide several explicit examples, proving one in detail, to provide some intuition for what the probabilities at this critical threshold look like.

We consider the case where \( p_i = n^{-\alpha_i} \) for \( i \geq 2 \) and \( p_1 = (\rho_1 \log n + \rho_2 \log \log n + c)^{1/k} n^{-\alpha_1} \) for some constants \( \rho_1, \rho_2, c \). Under the assumption that \( \sum_{i=1}^{k} \alpha_i(k)_i = 1 \) it follows that

\[
\mathbb{E}[N_{k-1}] = \frac{n^k}{k!} \prod_{i=1}^{k-1} n^{-\alpha_i(k)_i} \left( e^{-n \left( \sum_{i=1}^{k} p_i(k)_i \right)} \right)
= \frac{n^k}{k!} \sum_{i=1}^{k-1} \alpha_i(k)_i \left( (\rho_1 + o(1)) \log n \right)^{\frac{k+1}{k}} e^{-(\rho_1 \log n + \rho_2 \log \log n + c)}
= \frac{n^k}{k!} \sum_{i=1}^{k-1} \alpha_i(k)_i \left( (\rho_1 + o(1)) \log n \right)^{\frac{k+1}{k}} n^{-\rho_1 (\log n)^{-\rho_2} e^{-c}}.
\]
Thus if we set \( \rho_1 = k - \sum_{i=1}^{k-1} \alpha_i(k_{i+1}) \) and \( \rho_2 = \frac{k-1}{2} \), then

\[
\mathbb{E}[N_{k-1}] \to \frac{\rho_1^{k-1} e^{-c}}{k!},
\]
as \( n \to \infty \). A factorial moment argument gives rise to the following result.

**Lemma 17.** If \( p_i = n^{-\alpha_i} \) for \( i \geq 2 \) and \( p_1 = (\rho_1 \log n + \frac{k-1}{2} \log \log n + c)^{1/k} n^{-\alpha_1} \) such that \( \sum_i^k \alpha_i(k_i) = 1 \), \( \rho_1 = k - \sum_{i=1}^{k-1} \alpha_i(k_{i+1}) \), and \( c \in \mathbb{R} \), then the number \( N_{k-1} \) of free \((k-1)\)-faces approaches a Poisson distribution

\[
N_{k-1} \to \text{Poi}(\mu)
\]
with mean

\[
\mu = \frac{\rho_1^{k-1} e^{-c}}{k!}.
\]

**Proof.** This factorial moment proof is tedious and fairly standard, so we will hide it in the appendix. \( \square \)

We chose only to alter \( p_1 \) but a similar technique could be used for any of the \( p_i \), or even all of them. Near identical proofs will hold for any such approach and we present one more result.

**Lemma 18.** If \( p_i = (\rho_1 \log n + \rho_2 \log \log n + c)^{1/k} n^{-\alpha_i} \) such that \( \sum_i^k \alpha_i(k_i) = 1 \), \( \rho_1 = k - \sum_{i=1}^{k-1} \alpha_i(k_{i+1}) \), \( \rho_2 = \frac{2^k-k-1}{k^2} \), and \( c \in \mathbb{R} \), then the number \( N_{k-1} \) of free \((k-1)\)-faces approaches a Poisson distribution

\[
N_{k-1} \to \text{Poi}(\mu)
\]
with mean

\[
\mu = \frac{\rho_1^{k-1} e^{-c}}{k!}.
\]

In either of these to cases, if we condition on the event that \( N_{k-1} = 0 \) then slight modifications to our original Vanishing Cohomology proof allow us to still conclude that \( H^{k-1}(X, \mathbb{Q}) = 0 \) w.h.p.. More specifically we may conclude Theorem 4 presented in the opening sections.

### 7.2. Betti Numbers

We can use any such boundary result to prove an identical result about \( \beta^{k-1} \).

**Proof of Theorem 4** From Lemma 17 we know that in this instance \( N_{k-1} \to \text{Pois}(\mu) \). We suppose \( N_{k-1} = m \) for some \( m \in \mathbb{Z} \). Then the characteristic functions of these \( m \) free faces are cocycles by virtue of them not being contained in any \( k \)-faces. We will show these are not coboundaries, and in fact are the only cohomological cocycles.

Let these faces be denoted \( \sigma_1, \ldots, \sigma_m \), and their respective characteristic functions by \( \phi_1, \ldots, \phi_m \). Letting \( H_{k-2} \) denote the number of \((k-2)\) faces of \( X \) that are contained in \( m \) or fewer \((k-1)\)-faces, then

\[
\mathbb{E}[H_{k-2}] = \binom{n}{k-1} \prod_{i=1}^{k-2} p_i^{(k-1)} \left( \sum_{j=0}^{m} \binom{n-k+1}{j} \left( \prod_{i=1}^{k-1} p_i^{(k-1)} \right)^j \left( 1 - \prod_{i=1}^{k-1} p_i^{(k-1)} \right)^{n-k+1-j} \right)
\]

\[
= o(e^{-n^\epsilon}) \quad \text{for some } \epsilon > 0.
\]
Therefore w.h.p. no such \((k - 1)\)-face exists, specifically our complex contains no faces contained solely in some combination of our \(\sigma_i\). We now suppose there exists some \((k - 2)\)-cochain \(\lambda\) such that \(\delta^{d - 2}(\lambda) = \sum_{i=1}^{n} a_i \phi_i\) with \(a_i \neq 0\) for some \(i\). It follows that our \(\lambda\) is not a \((k - 2)\)-coboundary. However if we consider the subgraph \(X' = X - \{\sigma_1, \ldots, \sigma_m\}\) then since \(H_{k - 2} = 0\) w.h.p., \(X'\) w.h.p. has no free \((k - 2)\)-faces but \(\delta^{d - 2}(\lambda) = 0\) in \(X'\). Since \(\sum_{i=1}^{k - 1} \alpha_i(k - 1) < 1\), it follows from a mildly modified version of the argument in Theorem 2 that w.h.p. \(H^{d - 2}(X', \mathbb{Q}) = 0\), a contradiction since \(\lambda\) was not a coboundary in \(X\), and thus also in \(X'\). Therefore no such \(\lambda\) exists and we determine each \(\phi_i\) generates a nontrivial cocycle in \(H^{d - 1}(X)\).

Showing that these cochains are the only contributors to cohomology involves again directing our attention to \(X'\). By construction \(X'\) has no free \(k - 1\)-faces, then a more careful version of our spectral gap argument allows us to again invoke Garland’s Theorem to determine \(H^{d - 1}(X', \mathbb{Q}) = 0\). Our result then follows. \(\square\)

7.3. Boundaries of a simplex. An identical approach as with the free faces will yield thresholds where the number of boundaries of simplices in our complex approaches a Poisson distribution.

Lemma 19. If \(p_i = n^{-\alpha_i}\) for \(i \geq 2\) and \(\rho_1 = (\gamma_1 \log n + \frac{k+1}{2} \log \log n + c)^1/k n^{-\alpha_1}\) such that \(p_k \neq 1\), \(\sum_{i=1}^{k} \alpha_i(k) = 1\), \(\rho_1 = k - \sum_{i=1}^{k - 1} \alpha_i(k_{i+1})\), and \(c \in \mathbb{R}\), then the number \(M_{k - 1}\) of boundaries \(k\)-simplices that contribute a \(\mathbb{Z}\)-summand to integer homology approaches a Poisson distribution

\[M_{k - 1} \rightarrow \text{Poi}(\mu)\]

with mean

\[\mu = \frac{\rho_1^{k+1} e^{-c}}{(k + 1)!} (1 - p_k).\]

8. Trivial Homology: A Lower Bound

In this section we prove Theorem 3. We note this is exactly the requirement that the expectation of unfilled boundaries of \(k\)-simplices in our complex is \(o(1)\). Logic dictates that this would be most likely type of homological cycle, so its threshold for appearing in our complexes would be the lowest. We proceed by verifying this intuition and showing that minimal homological cycles are supported on a bounded number. After establishing these points we may apply a union bound to conclude our result.

8.1. Cycles of small vertex support. We begin with a few definitions. For a \((k - 1)\)-chain \(C\) the support of \(C\) is the union of \((k - 1)\)-faces in \(C\) with non-zero coefficients, while the vertex support is the underlying vertex set of the support. A pure \((k - 1)\)-dimensional subcomplex \(K\) is strongly connected if every pair of \((k - 1)\)-faces \(\sigma, \tau \in K^{k - 1}\) can be connected by a sequence of faces \(\sigma = \sigma_0, \sigma_1, \ldots, \sigma_j = \tau\) such that \(\dim(\sigma_i \cap \sigma_{i+1}) = k - 2\) for \(0 \leq i < j - 1\). Every \((k - 1)\)-cycle is a linear combination of \((k - 1)\)-cycles with strongly connected support.

Lemma 20. Let \(\sum_{i=1}^{k - 1} \alpha_i(k - 1) > 1\), and \(N\) such that \(\frac{k - \sum_{i=1}^{k - 1} \alpha_i(k - 1)}{\sum_{i=1}^{k - 1} \alpha_i(k - 1) - 1} < N\). Then there are a.a. no strongly connected pure \((k - 1)\)-dimensional subcomplexes of \(X(n, p_1, p_2, \ldots)\) with a vertex support of \(N + k\) or more vertices.
Proof. We begin by ordering the faces of our subcomplex, $K$: $f_1, f_2, \ldots, f_m$ such that each face $f_i$ has $(k-2)$-dimensional intersection with at least one $f_j$ for $j < i$. This is possible by our requirement that $K$ is strongly connected. This then gives us an ordering of the supporting vertices $v_1, \ldots, v_n$ induced by the ordering of faces, where we look at the vertex supports of $f_1, f_1 \cup f_2, f_1 \cup f_2 \cup f_3, \ldots$. In this fashion each vertex after $v_k$ corresponds to the adding of at least one $(k-1)$-face, and the $(k^{-1})$ $i$-dimensional faces that we know weren’t in the sub complex before, since they contain this new vertex.

Now suppose $K$ has $N + k$ vertices, it follows that we have at least $(\binom{k}{i+1}) + N^{(k-1)}$ $i$-dimensional faces for each $1 \leq i \leq k-1$. Now by our hypothesis $\prod_{i=1}^{k-1} p_i^{(\binom{k}{i+1})} = \prod_{i=1}^{k-1} n^{-\alpha(i+1)} = n^{-k+\beta}$ and $\prod_{i=1}^{k-1} p_i^{(k^{-1})} = \prod_{i=1}^{k-1} n^{-(k-1)} = n^{-1-\epsilon}$ for some $\beta, \epsilon > 0$. We then pick $N$ such that $\beta < N\epsilon$. We apply a union bound on the probability of there being a subcomplex isomorphic to $K$ in $\mathcal{X}(n, p_1, p_2, \ldots)$:

$$\mathbb{P}(\exists \text{ subcomplex}) \leq (N+k)! \left(\frac{n}{N+k}\right) \prod_{i=1}^{k-1} p_i^{(\binom{k}{i+1})} \prod_{i=1}^{k-1} n^{-(N+k)+(\beta-N\epsilon)} \leq n^{N+k} n^{-(N+k)} n^{\beta-N\epsilon} \leq n^{\beta-N\epsilon} \leq O(n^{-\delta}) \text{ for some } \delta > 0.$$

The last line holds by our choice of $N$. Since there are finitely many isomorphism classes of strongly connected $(k-1)$-complexes on $N + k$ vertices and we have that w.h.p none of them are sub complexes of $\mathcal{X}(n, p_1, p_2, \ldots)$ by a union bound. Finally, any such complex on more vertices will necessarily contain a strongly connected subcomplex on $N + k$ vertices (e.g. the one exhibited via the face ordering above), so we are done. \qed

8.2. The threshold for a simplex boundary. Here we prove our lower threshold for vanishing homology, which is sharp when $p_k \neq 1$.

Proof of Theorem \[\square\] Let $\gamma$ denote some non-trivial $(k-1)$-cycle supported on less than $N + k$ vertices; we restrict our attention to the subcomplex $K$ induced by the faces of $\gamma$. As in our proof in the above Lemma we impose an ordering on the vertex support of $K$: $v_1, \ldots, v_{k+m}$ for some $m < N$. We then prove our result by removing a vertex at a time from $K$ and counting the corresponding faces removed since they contain that vertex.

Since we have a non-trivial cycle every vertex is contained in at least $k$ $(k-1)$-simplices. Thus, when we begin by removing $v_{k+m}$ the best case scenario, in terms of minimizing faces we must remove, is when $v_{k+m}$ is contained in exactly $k$ $(k-1)$-simplices. In this case we then remove $\binom{k}{i}$ $i$-dimensional faces. We then remove vertices $v_{k+m-1}, \ldots, v_{k+1}$, and by construction each one was contained in a $(k-1)$-face comprised exclusively of vertices before it, so at each removal step we remove at least that simplex. Thus at each removal we account for at least $\binom{k-1}{i}$ $i$-faces. Then finally the last $k$ vertices correspond to our initial $(k-1)$-simplex. Putting this all together we get a lower bound on the probability of a subcomplex isomorphic to $K$ appearing:
The last line holds since

$$\sum \text{we introduce some useful notation, defining } \eta \text{ which we know from previous sections to be the probability that a specific (which is the probability that our complex contains the unfilled boundary of a specific}$$

Thus if we can show $\text{Var}[\ldots]$ By Chebyshev’s inequality,

$$-k \text{−face forms } M_{k-1} \text{−complexes on less than } N + k \text{ vertices we apply this argument to each of them and apply a union bound to conclude that w.h.p. none of them are subcomplexes of } X(n, p_1, p_2, \ldots) \text{. Thus we w.h.p. have no non-trivial } (k - 1)\text{−cycles and so } H_{k-1}(X, Z) = 0.$$

**Appendix A. Boundaries of Simplices**

We consider the case where $1 \leq \sum_i^k (i) \alpha_i$, where from [13] we have that $E[M_{k-1}] \to \infty$. By Chebyshev’s inequality,

$$\mathbb{P}[|M_{k-1} - E[M_{k-1}]| \geq E[M_{k-1}]] \leq \frac{\text{Var}[M_{k-1}]}{E[M_{k-1}]^2}.$$  

Thus if we can show $\text{Var}[M_{k-1}] = o(\mathbb{E}[M_{k-1}]^2)$, then we may conclude

$$\mathbb{P}[M_{k-1} > 0] \to 1.$$  

Considering $M_{k-1}$ as a sum of indicator random variables, then

$$\text{Var}[M_{k-1}] \leq E[M_{k-1}] + \sum_{i,j \in \binom{[n]}{2}} \text{Cov}[Y_i, Y_j]$$

$$= E[M_{k-1}] + \sum_{i,j \in \binom{[n]}{2}} (P[A_i \cap A_j] - P[A_i]P[A_j]).$$

Clearly $E[M_{k-1}] = o(\mathbb{E}[M_{k-1}]^2)$, to handle the sums we consider pairs $i, j \in \binom{[n]}{k+1}$ and break them into 3 cases depending on $I = |i \cap j|$. To make the calculations more readable we introduce some useful notation, defining $\eta_k$ by

$$\eta_k = (1 - p_k) \prod_{i=1}^{k-1} p_{i+1}^{(i+1)},$$

which is the probability that our complex contains the unfilled boundary of a specific $k$-simplex. We define $\gamma_k$ by

$$\gamma_k = \prod_{i=1}^k p_i^{(i)},$$

which we know from previous sections to be the probability that a specific $(k - 1)$-face forms a $k$-face with some vertex.
A.1. $I = 0$. We begin by calculating $\mathbb{P}[A_i \cap A_j]$. The probability that both boundaries are in our complex but unfilled is $\eta_k^2$. By inclusion-exclusion principles the probability that neither $\sigma_i$ nor $\sigma_j$, the associated first $(k-1)$-faces of these subcomplexes, form a $k$-simplex with a vertex outside of $i \cup j$ is $1 - 2\gamma_k + \gamma_k^2$, and there are $n - 2k - 2$ such vertices. Finally, we need to consider the probability that no $k$-face is formed between one of those two and a single vertex of the other vertex set. While this probability can be explicitly calculated, every term that isn’t 1 will contain at least a copy of $\gamma_k$, so we conclude this probability is $1 - O(\gamma_k)$.

Thus

$$\mathbb{P}[A_i \cap A_j] = \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-2k-2}(1 - O(\gamma_k)),$$

and by (11) in Section 5 we know

$$\mathbb{P}[A_i] \mathbb{P}[A_j] = \left(\eta_k (1 - \gamma_k)^{n-k-1}\right)^2$$

$$= \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-k-1}$$

$$= \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-2k-2}(1 - 2\gamma_k + \gamma_k^2)^{k+1}$$

$$= \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-2k-2}(1 - O(\gamma_k)).$$

So $\mathbb{P}[A_i \cap A_j] - \mathbb{P}[A_i] \mathbb{P}[A_j] = \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-2k-2}O(\gamma_k)$ and there are $O\left(n^{2k+2}\right)$ such pairs $i, j$. The overall contribution of such pairs is seen to be

$$S_0 = O\left(n^{2k+2}\eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-2k-2}\gamma_k\right)$$

$$= O\left(n^{2k+2}\eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-k-1}\gamma_k\right).$$

The second equality holds by restricting our consideration to $n > k$, then by assumption $\gamma_k \leq n^{-1} < k^{-1}$. It follows that we can find some constant $C > 0$ such that

$$1 > (1 - 2\gamma_k + \gamma_k^2)^{k+1} > (1 - 2\gamma_k)^{k+1} > (1 - 2k^{-1})^k > C,$$

so this term does not affect our big-$O$ calculations.

Now we know

$$\mathbb{E}[M_{k-1}]^2 = \left(\frac{n}{k+1}\right)^2 \eta_k^2 (1 - \gamma_k)^{2(n-k-1)} = O\left(n^{2k+2}\eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-k-1}\right),$$

so since $\gamma_k \to 0$ we conclude

$$\frac{S_0}{\mathbb{E}[M_{k-1}]^2} = O(\gamma_k) = o(1).$$

Hence the contribution of these pairs to the variance is seen to be $o\left(\mathbb{E}[M_{k-1}]^2\right)$.

A.2. $I = 1$. In this case the probability of both $i$ and $j$ being in our complex is again $\eta_k^2$ since the two don’t share a simplex of dimension greater than 0. We again use inclusion-exclusion to calculate the probability that $\sigma_i$ and $\sigma_j$ don’t form $k$-simplices with another vertex. However, these faces may or may not both contain the shared vertex: if they don’t then the calculations are identical to above, so we assume the alternative. In this case when calculating the probability that both form $k$-faces with some new vertex we now observe that these two faces would share a common edge. So the probability is $(1 - 2\gamma_k + \gamma_k^2)^{n-2k-1}$ and there are $n - 2k - 1$ such vertices to account for. Similarly, the probability we don’t
have a k-face consisting of \(\sigma_i\) or \(\sigma_j\) and a vertex in \(i \triangle j\) is \(1 - \Omega(\gamma_k p_1^{-1})\). We then calculate \(\mathbb{P}[A_i \cap A_j]\) to be

\[
\mathbb{P}[A_i \cap A_j] = \eta_k^2 (1 - 2\gamma_k + \gamma_k^2 p_1^{-1})^{n-2k-1} (1 - \Omega(\gamma_k p_1^{-1})) .
\]

Now we wish to calculate \(\mathbb{P}[A_i \cap A_j] - \mathbb{P}[A_i] \mathbb{P}[A_j]\). We begin by observing

\[
1 - 2\gamma_k + \gamma_k^2 = (1 - 2\gamma_k + \gamma_k^2 p_1^{-1}) \frac{1 - 2\gamma_k + \gamma_k^2}{1 - 2\gamma_k + \gamma_k^2 p_1^{-1}}
\]

\[
= (1 - 2\gamma_k + \gamma_k^2 p_1^{-1}) \left(1 - \frac{\gamma_k^2(p_1^{-1} - 1)}{1 - 2\gamma_k + \gamma_k^2 p_1^{-1}}\right)
\]

\[
= (1 - 2\gamma_k + \gamma_k^2 p_1^{-1}) (1 - \Omega(\gamma_k p_1^{-1})).
\]

The last equality holds by an identical argument to the one in the first case: we can bound \(1 - 2\gamma_k + \gamma_k^2 p_1^{-1}\), and consequently its inverse, from above and below by constants. We use this to calculate

\[
\mathbb{P}[A_i] \mathbb{P}[A_j] = \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-k-1}
\]

\[
= \eta_k^2 [(1 - 2\gamma_k + \gamma_k^2 p_1^{-1}) (1 - \Omega(\gamma_k p_1^{-1}))]^{n-k-1}
\]

\[
= \eta_k^2 (1 - 2\gamma_k + \gamma_k^2 p_1^{-1})^{n-k-1} (1 - \Omega(\gamma_k p_1^{-1}))^{n-k-1}.
\]

But since \(\gamma_k < n^{-1}\) we have

\[
(1 - \Omega(\gamma_k p_1^{-1}))^{n-k-1} = 1 - \Omega(n\gamma_k^2 p_1^{-1})
\]

\[
= 1 - \Omega(\gamma_k p_1^{-1}).
\]

We calculate

\[
\mathbb{P}[A_i] \mathbb{P}[A_j] = \eta_k^2 (1 - 2\gamma_k + \gamma_k^2 p_1^{-1})^{n-k-1} (1 - \Omega(\gamma_k p_1^{-1}))
\]

\[
= \eta_k^2 (1 - 2\gamma_k + \gamma_k^2 p_1^{-1})^{n-2k-1} (1 - 2\gamma_k + \gamma_k^2 p_1^{-1}) (1 - \Omega(\gamma_k p_1^{-1}))
\]

\[
= \eta_k^2 (1 - 2\gamma_k + \gamma_k^2 p_1^{-1})^{n-2k-1} (1 - \Omega(\gamma_k)) (1 - \Omega(\gamma_k p_1^{-1}))
\]

\[
= \eta_k^2 (1 - 2\gamma_k + \gamma_k^2 p_1^{-1})^{n-2k-1} (1 - \Omega(\gamma_k p_1^{-1})).
\]

Thus \(\mathbb{P}[A_i \cap A_j] - \mathbb{P}[A_i] \mathbb{P}[A_j] = \eta_k^2 (1 - 2\gamma_k + \gamma_k^2 p_1^{-1})^{n-2k-1} \Omega(\gamma_k p_1^{-1})\) and there are \(O(n^{2k+1})\) such pairs \(i, j\). We can bound the total contribution of these pairs to the variance as

\[
S_1 = \Omega(n^{2k-1}\eta_k^2(1 - 2\gamma_k + \gamma_k^2 p_1^{-1})^{n-2k+1} \gamma_k p_1^{-1}))
\]

\[
= \Omega(n^{2k-1}\eta_k^2(1 - 2\gamma_k + \gamma_k^2 p_1^{-1})^{n-k-1} \gamma_k p_1^{-1})).
\]

Just as before, the second equality holds from bounding \((1 - 2\gamma_k + \gamma_k^2 p_1^{-1})^{k-1}\) on by constants on either side.
Since $E[M_{k-1}]^2 = O(n^{2k+2}\eta_k^2(1 - 2\gamma_k + \gamma_k^2)^{n-k-1})$, it follows that

\[
\frac{S_1}{E[M_{k-1}]^2} = O\left(\frac{(1 - 2\gamma_k + \gamma_k^2)^{n-k-1}k}{n(1 - 2\gamma_k + \gamma_k^2)^{n-k-1}}\right)
= O\left(\frac{\gamma_k p_1^{-1}}{n}\left(1 + \frac{\gamma_k^2 p_1^{-1} - 1}{1 - 2\gamma_k + \gamma_k^2}\right)^{n-k-1}\right)
= O\left(\frac{\gamma_k p_1^{-1}}{n}\left(1 + \frac{\gamma_k^2 p_1^{-1} - 1}{1 - 2\gamma_k + \gamma_k^2}\right)^{n-k-1}\right).
\]

We proceed by bounding the right term by a constant.

\[
\left(1 + \frac{\gamma_k^2 p_1^{-1}}{1 - 2\gamma_k}\right)^{n-k-1} \leq \left(1 + \frac{\gamma_k^2 p_1^{-1}}{1 - \frac{2}{n}}\right)^{n-k-1}
\leq \sum_{j=0}^{n-k-1} \left(\frac{n - k - 1}{j}\right)\left(\frac{\gamma_k^2 p_1^{-1}}{1 - \frac{2}{n}}\right)^j
\leq \sum_{j=0}^{n-k-1} \eta^j\left(\frac{\gamma_k^2 p_1^{-1}}{1 - \frac{2}{n}}\right)^j
\leq \sum_{j=0}^{n-k-1} \eta^j\left(\frac{\gamma_k p_1^{-1}}{1 - \frac{2}{n}}\right)^j
= O(1).
\]

Then since $\gamma_k p_1^{-1} \to 0$,

\[
\frac{S_1}{E[M_{k-1}]^2} = O\left(\frac{\gamma_k p_1^{-1}}{n}\right)
= O\left(\frac{1}{n}\right)
= o(1).
\]

Thus the contribution of these pairs is also $o(E[N_{k-1}]^2)$, as desired.

A.3. $2 \leq I \leq k$. In this final case the probability of the two subcomplexes being contained is $\eta_k^2\eta_I^{-1}$ where $\eta_I := \prod_I^{I-1} p_I^{\binom{I}{i}}$. The $\eta_I^{-1}$ is accounting for all the faces common two $i$ and $j$, which we would otherwise have counted twice. Similarly, the probability that neither will form a $k$-simplex with some other vertex is $(1 - 2\gamma_k + \gamma_k^2\gamma_I^{-1})^{n-2k-2+I}$ with $\gamma_I := \prod_I^{I} p_I^{\binom{I}{i}}$. We note $\sigma_i$ and $\sigma_j$ share between $I-2$ and $I$ vertices, and the above probability corresponded to an intersection is $I$ vertices. Assuming this case provides an upper bound on $P[A_i \cap A_j]$. The probability of one not forming a $k$-simplex with one vertex of the other is $1 - O(\gamma_k\gamma_I^{-1})$.

We see

\[
P[A_i \cap A_j] = \eta_k^2\eta_I^{-1}(1 - 2\gamma_k + \gamma_k^2\gamma_I^{-1})^{n-2k-2+I}(1 - O(\gamma_k\gamma_I^{-1}))
\]

Just as in the previous case,

\[
1 - 2\gamma_k + \gamma_k^2 = (1 - 2\gamma_k + \gamma_k^2\gamma_I^{-1})(1 - O(\gamma_k^2\gamma_I^{-1}))
\]
We now calculate
\[ \mathbb{P}[A_i] \mathbb{P}[A_j] = \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-k-1} \]
\[ = \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{1-1} (1 - O(\gamma_k^{-1}))^{n-k-1} \]
\[ = \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-2k-2} (1 - O(\gamma_k^{-1})) . \]

It then follows that
\[ \frac{\mathbb{P}[A_i] \mathbb{P}[A_j]}{\mathbb{P}[A_i \cap A_j]} = \frac{\eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-2k-2} (1 - O(\gamma_k^{-1}))}{\eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-2k-2} (1 - O(\gamma_k^{-1}))} \]
\[ = O(\eta_k) . \]

Thus if \( \eta_k \neq 1 \) then \( \mathbb{P}[A_i \cap A_j] - \mathbb{P}[A_j] \mathbb{P}[A_j] = (1 - o(1)) \mathbb{P}[A_i \cap A_j] \), and otherwise \( \mathbb{P}[A_i \cap A_j] - \mathbb{P}[A_i] \mathbb{P}[A_j] = \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-2k-1} O(\gamma_k^{-1}) \). There are \( O(n^{2k+2-I}) \) such pairs, so their total contribution to the variance is either
\[ S_I = O\left(n^{2k+2-I} \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-2k-2} (1 - O(\gamma_k^{-1}))\right) \]
\[ = O\left(n^{2k+2-I} \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-k} \right) , \]
or
\[ S_I = O\left(n^{2k+2-I} \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-k-1} \right) . \]

In the first case we have
\[ \frac{S_I}{\mathbb{E}[M_{k-1}]^2} = O\left(\frac{n^{2k+2-I} \eta_k^2 (1 - 2\gamma_k + \gamma_k^2)^{n-k} \right) \]
\[ = O\left(\frac{\eta_k^2}{n^I} \right) \]
\[ = O\left(n^{-I+\Sigma_{l=1}^{I-1} \alpha_I(l+1)} \right) \]
\[ = o(1) . \]

In the second case we have
\[ \frac{S_I}{\mathbb{E}[M_{k-1}]^2} = O\left(\frac{\gamma_k \gamma_{I-1}}{n^I} \left( \frac{1 - 2\gamma_k + \gamma_k^2 \gamma_{I-1}}{1 - 2\gamma_k + \gamma_k^2} \right)^{n-k-1} \right) \]
\[ = O\left(\frac{\gamma_k \gamma_{I-1}}{n^I} \right) \]
\[ = O\left(n^{-I} \right) \]
\[ = o(1) . \]

Thus \( S_I = o(\mathbb{E}[M_{k-1}]^2) \) for \( 2 \leq I \leq k \). We therefore have that \( \mathbb{E}[M_{k-1}^2] = o(\mathbb{E}[M_{k-1}]^2) \) and use Chebyshev’s Inequality to conclude that \( M_{k-1} \sim \mathbb{E}[M_{k-1}] \).
Appendix B. Factorial Moments of Free Faces

Similar to previous second moment calculations:
\[
E[N_{k-1}^2] = \left( \binom{n}{k} \sum_{m=0}^{k} \binom{k}{m} \binom{n-k}{k-m} \left( \prod_{i=1}^{k-1} p_i^{2^{(i+1)}-\binom{m+1}{i}} \right) \left( 1 - 2 \prod_{i=1}^{k-1} p_i^{\binom{k}{i}} + \prod_{i=1}^{k} p_i^{2^{(i)}-\binom{m}{i}} \right) \right)^{n-2k+m} (1 - o(1))
\]
\[
\approx \left( \binom{n}{k} \left( \prod_{i=1}^{k-1} p_i^{2^{(i+1)}} \right) \sum_{m=0}^{k} \binom{k}{m} \binom{n-k}{k-m} \left( \prod_{i=1}^{m-1} p_i^{\binom{m+1}{i}} \right) \left( 1 - 2 \prod_{i=1}^{k-1} p_i^{\binom{k}{i}} + \prod_{i=1}^{k} p_i^{2^{(i)}-\binom{m}{i}} \right) \right)^{n-2k+m}
\]

Pulling out the \( m = 0 \) summand, asymptotically it looks like
\[
\left( \binom{n}{k} \binom{k-1}{k} \left( \prod_{i=1}^{k} p_i^{\binom{k}{i}} \right) \left( 1 - \prod_{i=1}^{k} p_i^{\binom{k}{i}} \right) \right)^{2(n-2k)} \approx \left( \left( \binom{n}{k} \left( \prod_{i=1}^{k-1} p_i^{\binom{k}{i}} \right) \left( 1 - \prod_{i=1}^{k} p_i^{\binom{k}{i}} \right) \right)^{n-k} \right)^2 = E[N_{k-1}]^2.
\]

Meanwhile, the \( m = k \) term is seen to be \( E[N_{k-1}] \). We claim the \( k - 1 \) other summands do not contribute in the limit. For a fixed \( m = 1, \ldots, k - 1 \) let \( d_m < 1 \) be some constant value such that
\[
d_m > \max \left\{ 1 - \frac{m(m-1)}{k(k-1)}, 1 - \frac{m - \sum_{i=1}^{m-1} \alpha_i (i+1)}{k - \sum_{i=1}^{m-1} \alpha_i i} \right\}.
\]

Both possible minimum values are seen to be between 0 and 1, so this construction makes sense. Then, for sufficiently large \( n \), we have
\[
1 - 2 \prod_{i=1}^{k} p_i^{\binom{k}{i}} + \prod_{i=1}^{k} p_i^{2^{(i)}-\binom{m}{i}} = 1 - \left( 2 - \prod_{i=1}^{k} p_i^{\binom{k}{i}} - \binom{m}{i} \right) \prod_{i=1}^{k} p_i^{\binom{k}{i}} \\
\leq 1 - (1 + d_m) \prod_{i=1}^{k} p_i^{\binom{k}{i}}.
\]

Thus there exists a constant \( D \) such that
\[
\left( 1 - 2 \prod_{i=1}^{k} p_i^{\binom{k}{i}} + \prod_{i=1}^{k} p_i^{2^{(i)}-\binom{m}{i}} \right)^{n-2k+m} \leq \left( 1 - (1 + d_m) \prod_{i=1}^{k} p_i^{\binom{k}{i}} \right)^{n-2k+m} \leq D e^{-n(1 + d_m) \left( \prod_{i=1}^{k} p_i^{\binom{k}{i}} \right)} = D e^{-(1 + d_m) \rho_1 \log n + k^{-1} \log \log n + \log n} = D n^{-(1 + d_m) \rho_1 \log n} = D e^{-(1 + d_m) c}.
\]

Then our construction of \( d_m \) ensures the corresponding summand approaches 0 as \( n \to \infty \). It then follows that
\[
E[(N_{k-1})_2] = E[N_{k-1}^2] - E[N_{k-1}] = E[N_{k-1}]^2 (1 - o(1)) \to E[N_{k-1}]^2
\]
as \( n \to \infty \). We now need to establish a similar result for each factorial moment.

We now direct our attention to the factorial moments of \( N_{k-1} \) of arbitrary degree \( l \) and assume that \( E[(N_{k-1})_j] \to E[N_{k-1}]^j \) for all \( j < l \). Using the notation of a previous section
we have
\[ \mathbb{E}[N_{k-1}^l] = \mathbb{E} \left[ \left( \sum_{\sigma \in \binom{\mathbb{N}}{k}} Y_{\sigma} \right)^l \right] = \sum_{\sigma_1, \ldots, \sigma_l \in \binom{\mathbb{N}}{k}} \mathbb{P}[A_{\sigma_1} \cap \cdots \cap A_{\sigma_l}]. \]

We break up this sum into two parts: where no two \( \sigma_i \)’s share the exact same vertex set and where such two faces exist. Considering the first case, an identical argument as above tells us that the only summand that contributes asymptotically corresponds to when no two faces share any vertices, and this converges to \( \mathbb{E}[N_{k-1}] \).

Directing our attention to the second case, we let \( s(l, j) \) and \( S(l, j) \) to denote Stirling numbers of the first and second kind, respectively. There are \( S(l, j) \) ways to break our \( \sigma_i \) up into \( j \) groups where each group shares the same underlying vertex set. Moreover, fixing such a \( j \) and configuration, the corresponding contribution to \( \mathbb{E}[N_{k-1}^l] \) would be \( \mathbb{E}[N_{k-1}^1] \).

We begin by pulling out \( S(l, l-1) = -s(l, l-1) \) copies of \( \mathbb{E}[N_{k-1}^{l-1}] \). We observe the number of groupings of the \( \sigma_i \) into \( k-2 \) groups has been overcounted: there should only be \( S(l, l-2) \) such combinations but we have counted \( -s(l, l-1)S(l-1, l-2) \) of them previously, so we add \( S(l, l-2) + s(l, l-1)S(l-1, l-2) = -s(l, l-2) \) copies of \( \mathbb{E}[N_{k-1}^{l-2}] \).

We now assume that for a fixed \( j \) attaching a coefficient of \(-s(l, m)\) to \( \mathbb{E}[N_{k-1}^m] \) for all \( m > j \) ensures every partition of the \( \sigma_i \) into \( j+1 \ldots l-1 \) sets is properly counted. Then for each \( m > j \), the \(-s(l, m)\) copies of \( \mathbb{E}[N_{k-1}^m] \) counts \(-s(l, m)S(m, j)\) partitions into just \( j \) groups. Meanwhile we know there are actually only \( S(l, j) \) distinct partitions, so we must add:

\[
S(l, j) + \sum_{m=j+1}^{l-1} s(l, m)S(m, j) = \sum_{m=j+1}^{l} s(l, m)S(m, j)
= \sum_{m=j}^{l} s(l, m)S(m, j) - s(l, j)S(j, j)
= \delta_{l, j} - s(l, j) = -s(l, j).
\]

The last line follows from a well known Stirling number identity. We may now use induction to conclude that \( \mathbb{E}[N_{k-1}^l] \to \mathbb{E}[N_{k-1}] - \sum_{j=1}^{l-1} s(l, j)\mathbb{E}[N_{k-1}^j] \) and thus \( \mathbb{E}[(N_{k-1})_j] \to \mathbb{E}[N_{k-1}] \) for any fixed \( l \). The theorem follows immediately.

Acknowledgements

We would like to thank Chris Hoffman, Matt Junge, Matt Kahle, Yogeshwaran D., and Gugan Thoppe for helpful conversations on this subject and readings of early versions of this manuscript.

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