Phase transition within deformed Ising model

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Deformation of Ising Hamiltonian by means of replacing a site spin $s_i$ by $s_i^q$ and statistics generalization with help of the substituting deformed probability $p_i^q$ instead of $p_i$ are studied jointly within mean–field scheme. Such deformed model is shown to be related to the phase transition of the second order with unusual set of critical indices depending essentially on the deformation parameter $q$. Scaling relations turn out to be invariant with respect to the deformation.

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I. INTRODUCTION

The Ising model is known to be a corner stone of the microscopic theory of phase transitions [1] – [3]. Evident simplicity of this model is based on a supposition that every lattice site $i$ has a spin taking only two magnitudes $s_i = \pm 1$. Although exact solution is not found for the three–dimension Ising model, its using allows one to explain main peculiarities of real phase transitions. Along this line, the qualitative picture becomes clear already within framework of the mean–field approach.

Recent considerations pay much attention to study complex systems which self–similarity derives to generalization of the Gibbs–Boltzmann statistics to Tsallis–type one (see Ref.[3] and references therein). Formally, such a generalization is performed by means of replacement of the probability $p_i$ by the so called deformed probability $p_i^q$ with a positive index $q \leq 1$. Here, we propose to complete such a procedure by the relevant deformation on the microscopic level. Namely, we deform the Ising Hamiltonian by means of replacing a site spin $s_i$ by $s_i^q$. Such a deformation is shown to uphold the second order phase transition, whereas the set of critical indices becomes depending on the parameter $q$ essentially.

The Letter is organized in the following manner. In Section 2 the $q$–deformed Hamiltonian is postulated and simplified within the mean–field approach. Section 3 deals with determination of the fractional average allowing us to obtain the definition of the order parameter. In Section 4 $q$–deformed distribution function is found to write a self–consistency equation for the order parameter. Section 5 devotes to the consideration of asymptotic solutions of the self–consistency equation. The last Section 6 contains a brief discussion concerning scaling relations.

II. MAIN STATEMENTS

We postulate the $q$–deformed Hamiltonian in the following form:

$$H = -\frac{1}{2} \sum_{i,j} J_{ij} s_i^q s_j^q - h \sum_i s_i^q. \quad (1)$$

Here, summation runs over $N$ lattice sites $i \neq j$, $J_{ij}$ is effective interaction potential, $h$ is external field, $s_i = \pm 1$ is a site spin, $q \leq 1$ is a deformation parameter.

In the simplest way, we use further the usual scheme of the mean–field approximation $[1] – [3]$. Within this approach, one follows to replace the multiplier $s_i^q$ by an average value $\langle s^q \rangle$. Moreover, we shall take into account the only interaction of nearest neighbor sites whose number equals $z$ and potential is reduced to the constant $J > 0$.

Then, the mean–field effective Hamiltonian is as follows:

$$H_{ef} = \sum_i \varepsilon_i; \quad \varepsilon_i = -h_q s_i^q, \quad h_q \equiv h + T_c \langle s^q \rangle \quad (2)$$

where a characteristic temperature $T_c \equiv zJ$ is introduced.

It is easy to foresee the postulated Hamiltonian will be meaningless if the index $q \leq 1$ is not chosen to satisfy the condition

$$(-1)^q = -1. \quad (3)$$

Then, the complex representation $-1 \equiv e^{i\pi(2n+1)}$, $n = 0, \pm 1, \ldots$ arrives at the set of rational numbers

$$q = \frac{2m + 1}{2n + 1}, \quad m \leq n \quad (4)$$

with integers $m,n = 0, \pm 1, \ldots$. Obviously, above reduction of the continuous set $q \in [0, 1]$ into the manifold of rational numbers $[3]$ is caused by the discrete symmetry of the Ising model being invariant with respect to the transformations $\{s_i\} \rightarrow \{-s_i\}, \ h \rightarrow -h$.

As a result, the site energy in the effective Hamiltonian $H_{ef}$ takes the linear form

$$\varepsilon_i = -h_q s_i. \quad (5)$$

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III. CALCULATION OF THE FRACTIONAL AVERAGE

Now, we should express the fractional average \( \langle s^q \rangle \) by means of the order parameter \( \eta \equiv \langle s \rangle \). Obviously, such a problem will be quite correct only for self–similar systems where it is reduced to definition of an index \( p(q) \) in the relation

\[
\langle s^q \rangle \equiv \eta^{p(q)}.
\]

(6)

If relevant distribution varies very slightly near the origin \( x = 0 \) and decays abruptly in the limit \( x \to \infty \) (see Eqs. (11), (13) below), we have the following estimations:

\[
\langle x^q \rangle \sim \int x^q dx \sim x^{1+q}, \quad \langle x \rangle \sim \int x dx \sim x^2.
\]

(7)

These derive immediately to the principle relation

\[
\langle x^q \rangle \sim \langle x \rangle^{p(q)}, \quad p(q) = \frac{1 + q}{2}.
\]

(8)

Respectively, the effective field in Eq. (6) takes the form

\[
h_q = h + T c q \eta
\]

(9)

that differs from the usual one with change of the order parameter \( \eta \) by the power–law function \( q \eta \).

IV. STATISTICAL SCHEME

Along the line of using \( q \)–deformed quantities, we need to state on the Renyi entropy

\[
S_q \equiv (1 - q)^{-1} \ln \sum_{\{s_i\}} P^q\{s_i\}
\]

(10)

which is easy seen to take the Boltzmann form in the limit \( q \to 1 \). Then, the maximum entropy principle taken with accounting the normalization condition \( \sum_{\{s_i\}} P\{s_i\} = 1 \) and the definition of the \( q \)–deformed internal energy \( E_q \equiv \sum_{\{s_i\}} \varepsilon_i(s_i)P^q\{s_i\} \) derives to the distribution function

\[
P\{s_i\} = \prod_i p_i(s_i), \quad p_i = z_q^{-1} \exp_q (-\beta(1)_q \varepsilon_i),
\]

(11)

\[
(1)_q \equiv \sum_{\{s_i\}} P^q\{s_i\}
\]

(12)

where \( z_q \) is the specific partition function related to one site, \( \beta = \frac{1}{T} \) is inverse temperature measured in the energy units; \( q \)–deformed exponential is as follows:

\[
\exp_q(x) \equiv \left\{ \begin{array}{ll} [1 + (1 - q)x]^{\frac{1}{1-q}} & \text{at } 1 + (1 - q)x > 0, \\ 0 & \text{otherwise}. \end{array} \right.
\]

(13)

Such a form is known to generalize usual exponential related to the limit \( q \to 1 \). It allows one to generate in usual manner the set of \( q \)–deformed hyperbolic functions: \( \sinh_q(x), \cosh_q(x), \tanh_q(x) \) and \( \coth_q(x) \). We calculate now the site partition function

\[
z_q \equiv \sum_{s_i = \pm 1} \exp_q (-\beta(1)_q \varepsilon_i) = 2 \cosh_q(\beta(1)_q h_q)
\]

(14)

does that differs from the usual one with \( q \)–deformation and \( (1)_q \) factor appearance. According to Eqs. (11), (12), (14) the latter is determined by the equation

\[
(1)_q = \frac{[\exp_q (\beta(1)_q h_q)]^q + [\exp_q (-\beta(1)_q h_q)]^q}{[\exp_q (\beta(1)_q h_q) + \exp_q (-\beta(1)_q h_q)]^q}.
\]

(15)

With accounting Eq. (13), the order parameter \( \eta \equiv \langle s \rangle \) is defined through the self–consistency equation

\[
\eta^{\frac{1}{1-q}} = \tanh_q[\beta(1)_q h_q(\eta)]
\]

(16)

following from Eqs. (6), (11), (13) and condition (3).

Solutions of the system (15), (16) are shown in Fig. 1 for the external field \( h = 0 \) and different values of index \( q \). Main peculiarity of the temperature dependencies \( h(T) \), \( (1)_q(T) \) is in decreasing the order parameter \( 0 \leq \eta \leq 1 \) accompanied by the relevant increasing the parameter \( 1 \leq (1)_q \leq 2 \) at the temperature growth within the domain

\[
T_{cq} \leq T \leq T_q, \quad T_{cq} \equiv (1 - q) T, \quad T_q \equiv 2^{1-q} T
\]

(17)

(at \( T < T_{cq} \) one has \( \eta = (1)_q = 1 \), whereas at \( T > T_q \) there are \( \eta = 0, \ (1)_q = 2 \)). Thus, decreasing the deformation parameter \( 1 \geq q \geq 0 \) derives to monotonous growth of the order parameter that transforms smoothly falling down dependence \( h(T) \) into step-like one.

Fig. 1. Temperature dependencies of both the renormalization parameter (upper panel) and the order parameter (low panel) for different indexes \( q \) (curves \( 1 \to 7 \) relate to \( q = 0, \ 1/17, \ 1/7, \ 3/11, \ 5/11, \ 5/7, \ 1, \) respectively).

As demonstrated in Fig. 2, the effect of the external field is similar to the usual case \( q = 1 \): high–temperature dependence \( \eta(h) \) has the monotonically growing form (dashed curves) that takes, at the low temperatures \( T < T_q \), the falling down domain located in the vicinity of the point \( h = \eta = 0 \) (solid curves). At fixed temperatures, decrease of the deformation parameter is seen to arrive at the more sharply defined non–monotone dependence. At critical field \( h_q = -T_c(1 - T/T_{cq}) \), the dependence \( \eta(h) \) undergoes a break, whereas in the limit of the small fields \( h \ll h_q \) one has

\[
\eta^{\frac{1}{1-q}} \sim \frac{h/T_c}{1 - T/T_q}.
\]

(18)
V. ASYMPTOTIC REGIMES

To obtain analytical description, we look over the possible asymptotic solutions of the system (15), (16) in limiting cases of both zero and non-zero external fields.

Zero field

For small order parameters ($\eta \ll 1$), one has the expansions
\[
\tanh_q(x) \simeq x - \frac{q(2-q)x^3}{3}, \quad \langle 1 \rangle_q \simeq 2^{1-q} \left[ 1 + 2^{(1-q)}(1 - q)^2 (\beta h_q)^2 \right].
\]

Then, the temperature dependence of the order parameter takes the power–law form
\[
\eta \simeq A^\alpha \left( 1 - \frac{T}{T_c} \right)^\beta,
\]
\[
A \equiv \frac{3}{(3 - 2q)(2q - 1)}, \quad \beta = \frac{1}{1+q}, \quad T_q \equiv 2^{1-q} T_c
\]
that corresponds to critical domains $T_q - T \ll T_q$ in Fig.1. At $q < 1$, the index $\beta$ exceeds the usual magnitude $\beta = 1/2$.

At the marginal magnitude $q = 0$, one has
\[
\tanh_0(x) = x, \quad \langle 1 \rangle_0 = 2
\]
and equation (16) arrives at the condition
\[
\sqrt{\eta} = \frac{T_q}{T} \sqrt{\eta}.
\]

It remains valid for arbitrary values $\eta$ at $T = T_q$. So, this case corresponds to the step–like curve 1 in Fig.1.

In the limit $q \ll 1$, there are the expansions
\[
\langle 1 \rangle_q \simeq 2^{1-q}, \quad \tanh_q(x) \simeq x \left( 1 - 2q \frac{x^2}{1 - x^2} \right).
\]

Respectively, the order parameter
\[
\eta \simeq \left( 1 + \frac{2q}{1 - T/T_q} \right)^{-1}
\]
decays very fast within tight window $1 - T/T_q \sim q \ll 1$ (see curve 2 in Fig.1).

Finally, we consider the form of the temperature dependence of the free energy $F \equiv -TN \ln(z_q)$ near the critical temperature $T_q$. Here, the hyperbolic cosine in Eq.(14) can be expanded over the argument $\beta(1/q)h_q \simeq \beta T_q \eta \tanh_q$ in series comprising of even terms only. In accordance with relevant temperature dependence (21), these terms have orders $(T_q - T)/T_q^n$ with the lowest index $n = 2$ because the first–order term ($n = 1$) is suppressed by self–action effects. Thus, in the limit $(T_q - T)/T_q \ll 1$ we obtain $F \sim T_q ((T_q - T)/T_q)^2$ and the specific heat $C \sim d^2F/dT^2$ does not vary with temperature. This results in the magnitude
\[
\alpha = 0
\]
of the critical index of the dependence $C \sim [(T_q - T)/T_q]^{-\alpha}$.

Non–zero field

Here, we start with the approximate equation
\[
\eta^{1+q} \simeq 2^{1-q} \beta h_q \left[ 1 - \frac{2^{(1-q)}}{A} (\beta h_q)^2 \right]^{1+q}
\]
following from Eqs. (16), (17) (notations $h_q$ and $A$ are given with Eqs.(9), (22)). In the limit $h \to 0$, differentiation of Eq.(28) with respect to the field $h$ yields the following expression for the susceptibility $\chi \equiv d\eta/dh$:
\[
T_c \chi = 2(1 + q)^{-1} \eta^{1-q} \left( T - T_q \right)^{1+q} \left( \frac{T_q}{T_q} + \frac{3 A}{\eta^{1+q}} \right)^{-1}
\]

Thus, in disordered state ($\eta = 0, \ T > T_q$) one has the susceptibility $\chi = 0$. Physically, this means a suppression of critical fluctuations in the disordered phase — quite differently from the usual picture.

In ordered phase ($\eta \neq 0, \ T < T_q$), inserting Eq.(21) into Eq.(29) we get the power–law dependence
\[
T_c \chi = \frac{A^{1+q} \eta^{1-q} T_q}{1 + q} \left( \frac{T_q}{T_q} \right)^{-\gamma}
\]
with the critical index
\[
\gamma = 1 - \frac{1}{2} \frac{1 - q}{1 + q}
\]

With decreasing the deformation parameter $1 \geq q \geq 0$, this index falls down monotonically from the usual value $\gamma = 1$ to $\gamma = 1/2$.

In the opposite limit $h \to \infty$, Eq.(28) related to the critical point $T = T_q$ derives to the power–law dependence
\[
\eta \simeq \left( A \frac{h}{T_c} \right)^{1/q}
\]
with the critical index
\[
\delta = \frac{3}{2} (1 + q)
\]
VI. DISCUSSION

Above mean–field consideration shows the deformation of Ising Hamiltonian by means of replacing a site spin \( s_i \) by a power function \( s_i^q \) derives to the phase transition of the second order with unusual set of critical indices \((27), (22), (31)\) and \((33)\). The zeroth magnitude of the first of them is caused apparently by the mean–field approach. It is easy to convince also in zeroing indexes \( \varepsilon, \zeta \) defined by the field dependence of the specific heat \( C \sim h^{-\varepsilon} \) and the space dependence of the correlation function \( G(r) \sim r^{-(d-2+\zeta)} \). Moreover, there are critical indexes \( \mu \) and \( \nu \) defining the field–temperature dependencies \( \xi \sim h^{-\mu} \) and \( \xi \sim (T_q - T)^{-\nu} \) of the correlation length \( \xi \). Among complete set of these indexes, only two are independent, whereas the rest are known to be determined by the scaling relations. Then, the question of fundamental importance arises how influences the deformation on these relations.

The first of them, Widom relation, takes the usual form \( \beta \delta = \beta + \gamma \) (34) that follows from comparison of two chains of definitions: \( \eta \sim \chi h \Rightarrow h \sim (T_q - T)^{\delta+\gamma} \) and \( h \sim \eta^\delta \sim (T_q - T)^{\beta \delta} \). Using the relation \( \xi \sim h^{-\mu} \) in the latter, one has

\[ \mu (\beta + \gamma) = \nu. \]  
(35)

Combination of Eqs. \((34), (35)\) yields the relation \( \beta \delta \mu = \nu \) that together with Eqs. \((22)\) and \((33)\) gets \( 3 \mu = 2 \nu \). Finally, using the fluctuation–dissipation theorem derives to the equality \( 2 \nu = \gamma \) (36) that defines the rest of indexes:

\[ \mu = \frac{1}{3} \left( 1 - \frac{1}{2} \frac{1-q}{1+q} \right), \quad \nu = \frac{1}{2} \left( 1 - \frac{1}{2} \frac{1-q}{1+q} \right). \]  
(37)

What about the Essam–Fisher equality, it takes the form

\[ \alpha + \beta + \gamma = \frac{3}{2} \]  
(38)

following from the consequence \( C(T_q - T)^2 \sim \eta \frac{1+q}{1-q} h \Rightarrow h \sim (T_q - T)^{2-\alpha} \) that differs from the usual one by replacing \( \eta \) by \( \eta \frac{1+q}{1-q} \). It is worthwhile to note all of equalities \((34) \rightarrow (36), (38)\) are invariant with respect to the deformation.

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Fig. 1
Fig. 2