Prolongation and stability of Zeno solutions to hybrid dynamical systems

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Abstract: The paper proposes a framework for the construction of solutions to a hybrid dynamical system that exhibit Zeno behavior. A new approach that enables solution to be prolonged after reaching its Zeno time is developed. It allows for a comprehensive stability analysis and asymptotic behavior characterization of such solutions. The results are applicable to a wide class of hybrid systems and match with practical experience of simulation of real-world phenomena. Moreover they are potentially useful for applications to interconnections of hybrid systems.

Keywords: hybrid dynamical system, Zeno behavior, asymptotic stability.

1. INTRODUCTION

Processes that combine continuous and discontinuous behavior naturally arise in a variety of real-world applications such as robotics, biological systems, chemical kinetics, logistics and networked control systems. The basic framework to model and analyse such a behavior is impulsive differential equations Samoilenko and Perestyuk (1987); Lakshmikantham et al. (1989); Samoilenko and Perestyuk (1995). Besides this theory we would like to mention the other more recent developments in the related fields like hybrid dynamical systems Van Der Schaft and Schumacher (2000), dynamic equations on time scales Bohner and Peterson (2012), discontinuous dynamical systems Akhmet (2010), switched systems Liberzon (2012) and hybrid automata Henzinger (2000).

Throughout the paper we will use one of the most recent and rapidly developing framework — a hybrid dynamical system proposed in Goebel et al. (2012). This framework is one of the most general and includes a majority of other classes of systems that model processes with continuous and discontinuous behavior. Moreover a variety of novel results are developed in Goebel et al. (2012) that are not available in the other frameworks. Also this framework appears well-adapted to the control-related problems. In particular the introduction of input-to-state stability (ISS) concept for hybrid dynamical systems gave a strong push and motivated a fast development of new methods for stability analysis of hybrid systems with exogenous input Cai and Teel (2009). The questions on robustness of ISS for hybrid systems were considered in Cai and Teel (2009). The questions on robustness of ISS for hybrid systems were considered in Cai and Teel (2009). The questions on robustness of ISS for hybrid systems were considered in Cai and Teel (2009). The questions on robustness of ISS for hybrid systems were considered in Cai and Teel (2009). The questions on robustness of ISS for hybrid systems were considered in Cai and Teel (2009).

The simplest example of unsolved problem is related to a bouncing ball modelled by hybrid dynamical system. The origin of the bouncing ball system is in some sense asymptotically stable (for a precise definition see Definition 2.8). There is a variety of Lyapunov-like theorems in Goebel et al. (2012) to verify this. However if we consider two such balls as one system (a so-called vacuous interconnection) then there are no methods to prove the asymptotic stability of the origin for the entire system. Moreover, this system is not asymptotically stable in the framework of Goebel et al. (2012). It seems that this framework is not suitable for modeling this rather simple mechanical system, however we claim that the stability problem can be resolved by a minor extension of the theory developed in Goebel et al. (2012). For more details of the just mentioned problem we refer to Section 3 and Figure 1. Here we only mention that this problem is caused by the Zeno solutions characterized by infinitely many impulsive jumps over a finite period of time. Such solutions are not defined after this time period.

Several approaches were proposed to cope with this problem. Some of these methods enable a solution to be prolonged beyond its Zeno time but only for certain classes of hybrid systems. In Johansson et al. (1999), a so-called regularization technique has been proposed and was illustrated for particular examples. It is based on perturbing the hybrid system in order to obtain non-Zeno solution, and then taking the limit as the perturbation goes to zero. A more formal procedure for obtaining generalized
solutions of Zeno hybrid system via regularization was presented in Goebel et al. (2004); Sanfelice et al. (2008). For a particular class of Lagrangian hybrid systems, a solution switches to a holonomically constrained dynamical system after the Zeno point is reached Ames et al. (2006), Or and Ames (2011). In the closely related class of switched systems Liberzon (2012), Shorten et al. (2007), a solution may converge to a switching surface in a finite time, along with increasingly fast switching events near this surface. This phenomenon is called chattering. In this case, the solutions can be extended by considering the set-valued Filippov solution Filippov and Arscott (1988), which involves sliding along the switching surface. In Cuijpers et al. (2001), a solution prolongation beyond Zeno which involves sliding along the switching surface. In this surface. This phenomenon is called chattering. In this

The ways to overcome these problems were proposed in simulators to ignore some events or looping indefinitely. The rest of the paper is organized as follows. In Section 3. A new approach for solution construction is presented in Section 4. In Section 5 we prove a series of propositions that enable stability analysis of solutions to a hybrid dynamical system with Zeno behavior. An illustrative example is given there. A short discussion on open problems in Section 6 completes the paper.

2. PRELIMINARY NOTION AND DEFINITIONS

The following notation and definitions are taken from Goebel et al. (2012):

\[
\begin{align*}
\dot{x} &= f(x), \quad x \in C, \\
x^+ &= g(x), \quad x \in D.
\end{align*}
\]

(H)

The state \( x \in \mathbb{R}^n \), \( n \in \mathbb{N} \) can change according to the differential equation \( \dot{x} = f(x) \) while \( x \in C \), and it can change according to the difference equation \( x^+ = g(x) \) while \( x \in D \). The sets \( C \subset \mathbb{R}^n \) and \( D \subset \mathbb{R}^n \) are called the flow and the jumps sets respectively, functions \( f : C \rightarrow \mathbb{R}^n \) and \( g : D \rightarrow \mathbb{R}^n \) are the flow and jump maps. The data of the hybrid system \( H \) is given by \( (C,f,D,g) \).

The parametrization of a solution to the hybrid system \( H \) is given by two parameters: \( t \in \mathbb{R}_{\geq 0} = [0,\infty) \), the amount of time passed, and \( j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), the number of jumps that have occurred. A certain subset of \( \mathbb{R}_{\geq 0} \times \mathbb{N}_0 \) can correspond to evolutions of hybrid systems. Such sets are called hybrid time domains.

\textbf{Definition 2.1.} (Hybrid time domain). Let \( t_0 \leq t_1 \leq t_2 \leq t_3 \leq \ldots \). A subset

\[
E = \bigcup_j \{[t_j, t_{j+1}], j \} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_0
\]

is a hybrid time domain if it is a union of a finite or infinite sequence of intervals \([t_j, t_{j+1}] \times \{j\}\), with the last interval (if existent) possibly of the form \([t_j, T)\) with \( T \) finite or \( T = \infty \).

Given a hybrid time domain \( E \) we denote:

\[
\text{sup}_E = \text{sup}_{\{t \in \mathbb{R}_{\geq 0} : \exists j \in \mathbb{N}_0 \text{ such that } (t, j) \in E\}}, \\
\text{sup}_j E = \text{sup}_{\{j \in \mathbb{N}_0 : \exists t \in \mathbb{R}_{\geq 0} \text{ such that } (t, j) \in E\}}.
\]

\textbf{Definition 2.2.} (Hybrid arc). A function \( \phi : E \rightarrow \mathbb{R}^n \) is a hybrid arc if \( E \) is a hybrid time domain and if for each \( j \in \mathbb{N} \), the function \( t \mapsto \phi(t, j) \) is locally absolutely continuous on the interval \( I^j = \{t : (t, j) \in E\} \).

Given a hybrid arc \( \phi \), the notation \( \text{dom}_t \phi \) represents its domain, which is a hybrid time domain.

\textbf{Definition 2.3.} (Complete hybrid arc). A hybrid arc \( \phi : E \rightarrow \mathbb{R}^n \) is called complete if \( \text{dom}_t \phi \) is unbounded, i.e., if \( \text{sup}_t \text{dom}_t \phi , E + \text{sup}_j E = \infty \).

\textbf{Definition 2.4.} (Zeno hybrid arc). A hybrid arc \( \phi : E \rightarrow \mathbb{R}^n \) is called Zeno if it is complete and \( \text{sup}_t \text{dom}_t \phi < \infty \).

The existence of a Zeno hybrid arc means that an infinite number of jumps occurs during a finite time. The time \( \tau = \text{sup}_t \text{dom}_t \phi \) is called the Zeno time.

\textbf{Definition 2.5.} (Solution to a hybrid system). A hybrid arc \( \phi \) is a solution to the hybrid system \( H \) if \( \phi(0,0) \in C \cup D \) and

(S1) for all \( j \in \mathbb{N} \) such that \( I^j = \{t : (t, j) \in \text{dom}_t \phi\} \) has nonempty interior.
\[ \phi(t, j) \in C \quad \text{for all} \quad t \in \mathbb{N}, \]
\[ \phi(t, j) = f(\phi(t, j)) \quad \text{for almost all} \quad t \in \mathbb{N}, \]
\[ (S2) \text{ for all } (t, j) \in \text{dom} \phi \text{ such that } (t, j + 1) \in \text{dom} \phi, \]
\[ \phi(t, j) \in D, \quad \phi(t, j + 1) = g(\phi(t, j)). \]

The properties of hybrid arcs (like completeness, Zeno, etc.) are automatically extended on the corresponding solutions.

Definition 2.6. (Maximal solution). A solution \( \phi \) to \( \mathcal{H} \) is maximal if there does not exist another solution \( \psi \) to \( \mathcal{H} \) such that \( \text{dom} \phi \) is a proper subset of \( \text{dom} \psi \) and \( \phi(t, j) = \psi(t, j) \) for all \( (t, j) \in \text{dom} \phi \).

Let \( S_2(\mathcal{A}) \) denote the set of all maximal solutions \( \phi \) to a hybrid system \( \mathcal{H} \) with \( \phi(0, 0) \in \mathcal{A} \).

Definition 2.7. (Strong forward-pre-invariance). A set \( \mathcal{A} \subset \mathbb{R}^n \) is called a strong forward-pre-invariant (SFPI) if for every \( \phi \in S_2(\mathcal{A}) \), \( \rho \in \mathcal{A} \), \( \rho(0) \in \mathcal{A} \), and \( \rho(0) \in \mathcal{A} \), such that \( \phi(t, j) \in \mathcal{A} \) for all \( (t, j) \in \text{dom} \phi \).

For a precise definition of stability we recall the definitions of standard function and distance to a closed set. A function \( \alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is called a class-\( \mathcal{K}_\alpha \) function if \( \alpha(0) = 0 \), continuous, strictly increasing, and unbounded. A function \( \rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is positive definite (\( \rho(s) > 0 \) for all \( s > 0 \) and \( \rho(0) = 0 \)). Given a vector \( x \in \mathbb{R}^n \) and a closed set \( \mathcal{A} \subset \mathbb{R}^n \), the distance of \( x \) to \( \mathcal{A} \) is defined by \( \|x\|_A^2 := \inf_{y \in \mathcal{A}} \|x - y\| \).

Definition 2.8. (Uniform global pre-asymptotic stability). Let \( \mathcal{A} \subset \mathbb{R}^n \) be closed. The set \( \mathcal{A} \) is said to be

- uniformly globally stable (UGS) if there exists a function \( \alpha \in \mathcal{K}_\infty \) such that any solution \( \phi \) to \( \mathcal{H} \) satisfies \( |\phi(t, j)|_A \leq \alpha(\|\phi(0, 0)\|_A) \) for all \( (t, j) \in \text{dom} \phi \);
- uniformly globally pre-attractive (UGpA) if for each \( \varepsilon > 0 \) and \( r > 0 \) there exists \( T > 0 \) such that, for any solution \( \phi \) to \( \mathcal{H} \) with \( \|\phi(0, 0)\|_A \leq r \), \( (t, j) \in \text{dom} \phi \) and \( t + j \geq T \) imply \( |\phi(t, j)|_A \leq \varepsilon \);
- uniformly globally pre-asymptotically stable (UGpAS) if it is both uniformly globally stable and uniformly globally attractive.

Definition 2.9. (\( \omega \)-limit set of a hybrid arc). The \( \omega \)-limit set of a hybrid arc \( \phi : \text{dom} \phi \rightarrow \mathbb{R}^n \), denoted \( \Omega(\phi) \), is the set of all points \( x \in \mathbb{R}^n \) for which there exists a sequence \( \{t_i, j_i\}_{i=1}^{\infty} \) of points \( (t_i, j_i) \in \text{dom} \phi \) with \( \lim_{i \rightarrow \infty} t_i + j_i = \infty \) and \( \lim_{i \rightarrow \infty} \phi(t_i, j_i) = x \). Every such point is an \( \omega \)-limit point of \( \phi \).

3. MOTIVATING EXAMPLE

Consider two hybrid dynamical systems \( \mathcal{H}_1 \) with states \( x_1 \in \mathbb{R}^{n_1} \) and inputs \( u_1 \in U_i \subset \mathbb{R}^{n_1} \), and \( \mathcal{H}_2 \) with states \( x_2 \in \mathbb{R}^{n_2} \) and inputs \( u_2 \in U_i \subset \mathbb{R}^{n_2} \),

\[
\begin{cases}
\dot{x}_1 = f_1(x_1, u_1), & (i, u_1) \in C_i, \\
\dot{x}_2 = g_1(x_1, u_1), & (i, u_1) \in D_i
\end{cases}
\]

where \( n_i, m_i \in \mathbb{N}, i = 1, 2 \). The sets \( C_i \subset \mathbb{R}^{n_1} \times U_i \) and \( D_i \subset \mathbb{R}^{n_2} \times U_i \) define the flow and the jumps sets respectively, functions \( f_i : C_i \rightarrow \mathbb{R}^{n_1} \) and \( g_i : D_i \rightarrow \mathbb{R}^{n_2} \) are the flow and jump maps. The data of the hybrid system \( \mathcal{H}_i \) is given by \( (C_i, f_i, D_i, g_i) \).

Let us interconnect these two systems with \( u_1 = h_1(x_1) \) and \( u_2 = h_2(x_2) \), where functions \( h_1 : \mathbb{R}^{n_1} \rightarrow U_2 \), \( h_2 : \mathbb{R}^{n_2} \rightarrow U_1 \). Then the entire interconnection can be represented as a single hybrid dynamical system \( \mathcal{H} \) with data \( (C, f, D, g) \), where its state is \( x := (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) and its flow map is \( f(x) := (f_1(x_1, h_2(x_2)), f_2(x_2, h_1(x_1))) \), its jump map is \( g(x) := (g_1(x_1, h_2(x_2)), g_2(x_2, h_1(x_1))) \).

In the literature Dashkovskiy and Kosmykov (2013); Dashkovskiy et al. (2013) such choice of the flow set \( C \) and the jump set \( D \) is called natural. An important fact is that an interconnection of two hybrid systems \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) is a hybrid system of the form \( \mathcal{H} \). So one may use a variety of previously developed methods and techniques (for instance from Goebel et al. (2012)) to characterize solutions and the problem of a comprehensive analysis of interconnections seems to be solved. However, an essential problem appears in this context. It was discussed in Sanfelice (2011) and caused by the interconnection of a hybrid system with Zeno solution and a hybrid system with continuous complete solution. Such interconnection has a Zeno solution that is neither a part of solutions to every subsystem. Another good illustration of this problem is a vacuous interconnection of several bouncing balls when the balls start from different initial positions. The solution of such model may not allow all the balls to reach their own Zeno time as the original model of each bouncing ball does (see Figure 1). This leads to unnatural loss of asymptotic stability of the origin.

In this paper we propose a way to extend the hybrid framework Goebel et al. (2012) in order to cope with aforementioned problems.

4. HYBRID FRAMEWORK EXTENSION

The main source of the problems stated in the motivation section is that a solution to a hybrid system is not defined beyond its Zeno time. However, some experiments from real life like bouncing ball argue that a solution should be prolonged over its Zeno time. A bouncing ball after reaching the resting state continues to lie while time is counting further and further. This motivates us to allow solution to continue its evolution after reaching Zeno time. In our extended framework, Zeno solution continues its evolution from an \( \omega \)-limit point after reaching its Zeno time. It enables us to construct solutions that reflect real-world observations and to perform their stability analysis.
jump index in hybrid time domain increases by 1 so that a new position \(g\) coincides with the “classical” solution and therefore can be prolonged eventually discrete solution. If the jump map \(\Omega\) is open then the extended solution can continue its evolution from limit point \(x \in D\) and therefore should jump. This can lead to eventually discrete solution. If the jump map \(D\) is an open set then the extended solution can continue its evolution from the limit point \(x \in C\) and therefore can be prolonged beyond Zeno time of the ordinary time axis. From this viewpoint it is quite natural to model real processes with an open jump set \(D\) as it is done in the following example.

\(x(s, j+1) = g(x(s))\). In our settings we add one more rule to construct solution to a hybrid dynamical system:

- if for some fixed \(k \in \mathbb{N}_0\) hybrid arc \(\tilde{\phi}(t, j, k)\) is Zeno with non-empty \(\omega\)-limit set then a solution \(\tilde{\phi}\) to a hybrid system \(H\) is prolonged with initial condition \(\phi(\tau, 0, k + 1) \in \Omega(\phi)\), where \(\tau\) is the Zeno time for the hybrid arc \(\phi(\cdot, \cdot, k)\).

Our extended solution \(\tilde{\phi}\) is a concatenation of classical hybrid arcs \(\phi(t, j)\) with initial conditions \(\phi(0, 0) = \xi, \phi(\tau_1, 0) \in \Omega(\phi^0), \phi^0(\tau_2, 0) \in \Omega(\phi^1)\) and so on, where \(\tau_i\) is the Zeno time for the hybrid arc \(\phi^{i-1}\).

A new rule of extended solution’s construction leads to the following properties of the corresponding extended hybrid time domain \(E\) : if the point \((t_{0,k+1}, 0, k+1) \in E\) then there exist infinitely many points of the form \((t_{k+1}, \cdot, k) \in E\) such that \(\lim_{j \to \infty} t_{j,k} = t_{0,k+1}\).

In general, an \(\omega\)-limit set \(\Omega(\phi)\) may consist of several or infinitely many points. According to our new rule a single initial point can generate multiple solutions. Such situation appears, for example, in modelling of water tanks system (see Alur and Henzinger (1997) for details). This system has Zeno arcs with two \(\omega\)-limit points. Therefore two different extended solutions will be generated from a single initial point. This is quite natural since the considered physical process can evolve according to both of solutions in a real experiment.

**Remark 4.1.** For a given Zeno hybrid arc \(\varphi\) the \(\omega\)-limit point \(x \in \Omega(\varphi)\) is a limit of a sequence of points \(\{x_j\}\) of the state space such that \(x_i \in D, i = 1, 2, \ldots\). A behavior of the corresponding extended solutions now heavily depends on the properties of the jump set \(D\). If \(D\) is a closed set (it means that it contains all its limit points) then the extended solution continue its evolution from limit point \(x \in D\) and therefore should jump. This can lead to eventually discrete solution. If the jump map \(D\) is an open set then the extended solution can continue its evolution from the limit point \(x \in C\) and therefore can be prolonged beyond Zeno time of the ordinary time axis. From this viewpoint it is quite natural to model real processes with an open jump set \(D\) as it is done in the following example.

**Example 4.1.** Consider a vacuous interconnection of two bouncing balls. Let \(x_1, x_3 \in \mathbb{R}_{\geq 0}\) stand for the heights of the balls and \(x_2, x_4 \in \mathbb{R}\) stand for the corresponding velocities. Then system has the form

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\gamma(x_1, x_2), \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -\gamma(x_3, x_4),
\end{align*}
\]

\(x_1^+ = x_1, \quad x_2^+ = \begin{cases} 
-\lambda x_2, & x \in D_1, \\
-x_2, & x \notin D_1,
\end{cases} \quad x_3^+ = x_3, \quad x_4^+ = \begin{cases} 
-\lambda x_1, & x \in D_2, \\
x_4, & x \notin D_2.
\end{cases}
\]

\(C_1 = \{x \in \mathbb{R}^4 : x_1 > 0 \text{ or } x_1 = 0, x_2 \geq 0\}, \quad D_1 = \{x \in \mathbb{R}^4 : x_1 = 0, x_2 < 0\}, \quad C_2 = \{x \in \mathbb{R}^4 : x_3 > 0 \text{ or } x_3 = 0, x_4 \geq 0\}, \quad D_2 = \{x \in \mathbb{R}^4 : x_3 = 0, x_4 < 0\}, \quad C = C_1 \cap C_2, \quad D = D_1 \cup D_2.
\]
where $\lambda \in (0, 1)$ is the restitution coefficient, $\gamma : \mathbb{R}^2 \to \mathbb{R}$ is given by
\[
\gamma(a, b) = \begin{cases} 0, & \text{if } a = b = 0, \\ 9.81, & \text{otherwise.} \end{cases}
\]

**Proof.** Suppose it is not true. Consider a solution $\phi^*$ to $\mathcal{H}$ with $|\phi^*(0, 0)\rangle_{\mathcal{A}} \leq r$ that has an $\omega$-limit point $\xi$ outside the set $\mathcal{A}$. It means that there exists a sequence $\{(t, j)_i\}_{i=1}^{\infty}$ of points $(t_i, j_i) \in \text{dom} \phi$ with $\lim_{i \to \infty} t_i + j_i = \infty$ and
\[
\lim_{i \to \infty} \phi^*(t_i, j_i) = \xi \notin \mathcal{A}.
\]
Then there exists $\delta > 0$ such that $|\xi|_{\mathcal{A}} = \delta$. From the UGpAS of the set $\mathcal{A}$ it follows that for $\varepsilon = \frac{\delta}{2}$ there exists $T > 0$ such that for every $(t, j) \in \text{dom} \phi$ with $t + j \geq T$ follows $|\phi(t, j)\rangle_{\mathcal{A}} \leq \frac{\delta}{2}$. However the existence of the limit (2) guarantees that for any $d > 0$ there exist a $d$-neighbourhood $U_d(\xi)$ and $(t^*, j^*) \in \text{dom} \phi$ such that $\phi(t^*, j^*) \in U_d(\xi)$. Choosing $d$ small enough to satisfy the conditions $t^* + j^* \geq T$ and $U_d(\xi) \cap U_d(\mathcal{A}) = \emptyset$ leads to $|\phi(t^*, j^*)\rangle_{\mathcal{A}} > \frac{\delta}{2}$, which contradicts the UGpAS of the set $\mathcal{A}$. This proves that the $\omega$-limit point $\xi \in \mathcal{A}$.

**Definition 5.1.** ($\mathcal{H} \cap \mathcal{A}$). If $\mathcal{A} \subset C \cup D$ is UGpAS for $\mathcal{H}$, then $\mathcal{A}$ can be considered as the state space for a new hybrid system with the new flow set $C \cap \mathcal{A}$ and the new jump set $D \cap \mathcal{A}$. We will denote this new system by $\mathcal{H} \cap \mathcal{A}$.

Indeed, from Lemma 5.1, UGpAS implies SFpI of the set $\mathcal{A}$ so every solution with initial condition in $\mathcal{A}$ will remain there for all $(t, j) \in \text{dom} \phi$. In the case when $\phi$ is a Zeno solution it will be prolonged from a point of its $\omega$-limit set $\Omega(\phi)$. From Lemma 5.2 follows that $\Omega(\phi) \subset \mathcal{A}$, so the solution will again remain in the set $\mathcal{A}$. It means that extended solutions of the system $\mathcal{H}$ with initial conditions in $\mathcal{A}$ will coincide with extended solutions of the system $\mathcal{H} \cap \mathcal{A}$ with the corresponding initial conditions. Since $\mathcal{A} \subset C \cup D$, no new solution will be generated.

For a comprehensive description of asymptotic behavior of extended solutions we introduce a new definition of stability over Zeno.

**Definition 5.2.** (UGpASoZ). Let $\mathcal{A} \in \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be

(i) uniformly globally stable over Zeno (UGSoZ) if there exists a function $\alpha \in \mathcal{K}\infty$ such that any solution $\phi \in \mathcal{H}$ satisfies $|\dot{\phi}(t, j, k)|_{\mathcal{A}} \leq \alpha(|\phi(0, 0, 0)\rangle_{\mathcal{A}})$ for all $(t, j, k) \in \text{dom} \phi$;

(ii) uniformly globally pre-attractive over Zeno (UGpAzo) if for each $\varepsilon > 0$ and $r > 0$ there exist $T > 0$ and $K > 0$ such that, for any solution $\phi \in \mathcal{H}$ with $|\phi(0, 0, 0)\rangle_{\mathcal{A}} \leq r$, from $(t, j, k) \in \text{dom} \phi$ with either $t + j \geq T$, $k = K$ or $k > K$ or $t + j \geq T$, $k = \sup_{\text{Zeno}} \text{dom} \phi$, $\sup_{\text{Zeno}} \text{dom} \phi < K$ it follows that $|\dot{\phi}(t, j, k)|_{\mathcal{A}} \leq \varepsilon$;

(iii) globally pre-attractive over Zeno (GpAzo) if for each $\varepsilon > 0$, $r > 0$, and for any solution $\phi \in \mathcal{H}$ with $|\phi(0, 0, 0)\rangle_{\mathcal{A}} \leq r$, there exist $T > 0$ and $K > 0$ such that from $(t, j, k) \in \text{dom} \phi$ with either $t + j \geq T$, $k = K$ or $k > K$ it follows that $|\dot{\phi}(t, j, k)|_{\mathcal{A}} \leq \varepsilon$;

(iv) uniformly globally pre-asymptotically stable over Zeno (UGpASoZ) if it is both UGSoZ and UGpAzo.

The conditions for the pre-attractivity actually mean that all solutions will reach the $\varepsilon$-neighbourhood of the set $\mathcal{A}$ no later than at the time $T$ after $K$-th Zeno occurrence. The uniformity means that $T$ and $K$ are the same for all

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A numerical simulation is presented on Figure 2. The arc of the blue ball the solution is now prolonged from this point and Zeno index is increased by 1. The extended solution exhibits a further Zeno behavior and its $\omega$-limit set is just the origin. At the Zeno time of the red ball, the solution is prolonged from its $\omega$-limit set $(0, 0, 0, 0) \in \mathbb{R}^4$ which is a single point again. The last arc of solution is trivial and purely continuous with $\sup_{\text{dom} \phi} \text{dom} \phi = \infty$. The concatenated solution corresponds to our experience.

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5. STABILITY ANALYSIS

In this section we introduce a new stability notion in order to describe asymptotic behavior of extended solutions to hybrid systems. Two auxiliary lemmas will be needed to justify stability characterization.

**Lemma 5.1.** UGS of a set $\mathcal{A}$ implies its strong forward-invariance.

**Proof.** Suppose it is not true. Let there exist a solution $\phi$ to $\mathcal{H}$ with $\phi(0, 0) \in \mathcal{A}$ and a point $x^* \in \text{ran} \phi(t, j)$ such that $x^* \notin \mathcal{A}$ for some $(t, j) \in \text{dom} \phi$. It means that there exists $\delta > 0$ such that $|\phi(t, j)|_{\mathcal{A}} = \delta$. Then from Definition 2.8 it follows directly that there is a function $\alpha \in \mathcal{K}\infty$ such that
\[
0 < \delta = |\phi(t, j)|_{\mathcal{A}} \leq \alpha(|\phi(0, 0)\rangle_{\mathcal{A}}) = \alpha(0) = 0.
\]
The contradiction proves that every solution $\phi$ starting in $\mathcal{A}$ remains in this set: $\phi(t, j) \in \mathcal{A}$ for all $(t, j) \in \text{dom} \phi$.

**Lemma 5.2.** Let $\mathcal{A}$ be UGpAS and every arc $\phi$ with initial condition in $\{C \cup D\} \setminus \mathcal{A}$ have a non-empty $\omega$-limit set, then $\Omega(\phi) \subset \mathcal{A}$. Fig. 2. Evolution of the height coordinates (top) and extended hybrid time domain (bottom) of two vacuously interconnected bouncing balls started with the height 3 (red) and 1 (blue) respectively.
solutions. If a solution does not undergo such number \((K)\) of Zeno occurrences then it should reach the corresponding \(\varepsilon\)-neighbourhood no later than time \(T\) after its last Zeno. 

**Theorem 1.** Let there exist a finite sequence \(A_0 \subset A_{n-1} \subset \ldots \subset A_i \subset A_0 = C \cup D\) such that \(A_i\) is UGpAS for the system \(\mathcal{H} \cap A_{i-1}, i = 1, \ldots, n\) and for all initial values \(\phi(0,0) \in A_{n-1} \setminus A_i, i = 1, \ldots, n-1\) the corresponding solutions \(\phi\) are Zeno with non-empty \(\omega\)-limit sets. Then \(A_n\) is UGpASoZ.

**Proof.** If \(n = 1\) then UGpAS of the set \(A_1\) implies its UGpASoZ with \(K = 0\). Let us consider the case \(n = 2\). First we prove stability of the set \(A_2\). From UGS of the sets \(A_1\) and \(A_2\) it follows that there exist functions \(\alpha_1, \alpha_2 \in \mathcal{K}_\infty\) such that

\[
|\varphi(t,j)|_{A_2} \leq \alpha_1(\varphi(0,0))_{A_1} \quad \forall \varphi(0,0) \in A_0,
\]

\[
|\varphi(t,j)|_{A_2} \leq \alpha_2(\varphi(0,0))_{A_2} \quad \forall \varphi(0,0) \in A_1
\]

and for all \((t,j) \in \text{dom}\phi\) the extended solution \(\tilde{\varphi}\) satisfies

\[
|\tilde{\varphi}(t,j,k)|_{A_2} \leq \alpha_2(\tilde{\varphi}(0,0,0))_{A_2} \quad \forall \varphi(0,0) \in A_0
\]

for all \((t,j,k) \in \text{dom}\phi\) with \(\alpha_2(s) = \max\{\alpha_1(s), \alpha_2(s)\}\). Hence \(A_2\) is UGSoZ.

Now we will prove the pre-attractivity. For this purpose we will show that each extended solution \(\tilde{\phi}\) to the hybrid system \(\mathcal{H}\) satisfies the uniform pre-attractivity conditions \((ii)\) of Definition 5.2 with respect to the set \(A_2\). Note that since \(A_2\) is UGpAS for the system \(\mathcal{H} \cap A_1\) every solution initiated from \(A_1\) satisfies the pre-attractivity conditions of the Definition 2.8: \(\forall x, r > 0\) there exists \(T_1 > 0\) such that any solution \(\varphi\) to \(\mathcal{H} \cap A_1\) such that \(\varphi(0,0)|_{A_2} \leq r, \varphi(0,0) \in A_1\) satisfies \(\varphi(t,j)|_{A_2} \leq \varepsilon\) for all \((t,j) \in \text{dom}\phi\) with \(t + j \geq T_1\). It means that the corresponding extended solution satisfies the uniform pre-attractivity conditions \((ii)\) of Definition 5.2 with \(T = T_1\) and \(K = 0\).

It remains to show that the conditions \((ii)\) of Definition 5.2 are also satisfied for solutions starting outside the set \(A_1\). Let \(\tilde{\varphi}(0,0,0)|_{A_2} \leq r, \tilde{\varphi}(0,0,0) \in A_0 \setminus A_1\) and let the arc \(\phi = \tilde{\varphi}(t,j,0)\) be Zeno. From the conditions of Theorem 1 its \(\omega\)-limit set is non-empty, hence this solution is being prolonged from the set \(\Omega(\phi)\).

From Lemma 5.2 it follows that \(\Omega(\phi) \subset A_1\) and from Definition 5.1 it follows that the set \(A_1\) can be considered as a new state space for the system \(\mathcal{H} \cap A_1\). Since \(A_2\) is UGpAS for the system \(\mathcal{H} \cap A_1\) and \(\Omega(\phi) \subset A_1\) it follows that an extended solution \(\tilde{\phi}\) issued from \(A_0 \setminus A_1\) satisfies the pre-attractivity conditions \((ii)\) of Definition 5.2 with \(T = T_1\) an \(K = 1\).

Since there are no other types of solutions to the system \(\mathcal{H}\) starting outside \(A_1\), the set \(A_2\) is UGpAoZ with \(T = T_1 + T_2(\phi)\). Let \(A_2\) be UGSoZ. From Proposition 5.1 (Goebel et al. (2012)) let \(A \subset \mathbb{R}^n\) be closed. If \(V(x)\) is a Lyapunov function candidate for \(\mathcal{H}\) and there exist \(\alpha_1, \alpha_2 \in \mathcal{K}_\infty\), and a continuous \(\rho \in \mathcal{P}\) such that

\[
\alpha_1(|x|_A) \leq V(x) \leq \alpha_2(|x|_A) \quad \forall x \in C \cup D \cup g(D)
\]

\[
\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_A) \quad \forall x \in C
\]

\[
V(g(x)) - V(x) \leq -\rho(|x|_A) \quad \forall x \in D
\]

then \(A\) is UGpAS.

**Example 5.1.** Consider the following system with state \(x = (x_1, x_2, x_3) \in \mathbb{R}^3\)

\[
\dot{x}_1 = x_2,
\]

\[
\dot{x}_2 = -\gamma(x_1, x_2), x \in C, x_2^+ = -\lambda x_2, x \in D
\]

\[
\dot{x}_3 = -x_3,
\]

and with flow and jumps set given by

\[
C = \{x \in \mathbb{R}^3 : x_1 > 0 \text{ or } x_1 = 0, x_2 \geq 0\},
\]

\[
D = \{x \in \mathbb{R}^3 : x_1 = 0, x_2 < 0\},
\]

where \(\lambda \in (0, 1)\) and \(\gamma : \mathbb{R}^2 \to \mathbb{R}\) is given by (1).

Let us prove that the origin is UGpASoZ. As one may notice, flow and jump sets of system (3) do not depend on \(x_3\). This system can be interpreted as an interconnection of a bouncing ball with state \((x_1, x_2)\) and some other process with state \(x_3\). Each time when the ball bounces at the floor the state variable \(x_3\) changes its sign.
Let function $V$ be defined by

$$V(x) = (1 + \theta \arctan x_2) \left( \frac{x_2^2}{2} + \gamma x_1 \right)$$

with

$$\theta = \frac{1 - \lambda^2}{\pi (1 + \lambda^2)} \quad \text{and} \quad \gamma = 9.81.$$

The set $A_1 = \{ (x \in \mathbb{R}^3 : x_1 = x_2 = 0) \}$ is UGpAS since $V$ satisfies Proposition 5.1 with respect to the origin for a single bouncing ball Goebel et al. (2012) and the distance from a point $(x_1, x_2, x_3) \in \mathbb{R}^3$ to the set $A_1$ coincides with the Euclidean norm $\| \cdot \|$ of the corresponding vector $(x_1, x_2) \in \mathbb{R}^2$:

$$\|(x_1, x_2, x_3)\|_{A_1} = \|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}.$$

Then we arrive to a system of the form (3) with the new state space $A_1$, the new flow set $C \cap A_1 = \{ x \in A_1 : x_1 > 0 \cup x_1 = 0, x_2 \geq 0 \} = A_1$ and the new jump set $D \cap A_1 := \{ x \in A_1 : x_1 = 0, x_2 < 0 \} = \emptyset$. This system is purely continuous and the origin $(0, 0, 0)$ is UGpAS. Since all the arcs issued outside the set $A_1$ are Zeno, from Theorem 1 it follows that the origin is UGpASoZ. □

Theorem 1 proposes a sequential narrowing of the state space of a hybrid system. For the last example this process can be described with the following sequence of sets:

$$A_0 = \text{whole state space} \quad A_1 = \{0\} \times \mathbb{R} \quad A_2 = \text{origin}$$

Theorem 1 has been used to prove UGpASoZ of the origin without constructing solutions to the system (3) and their prolongation explicitly. One may check that for a particular initial data the corresponding solutions to the system (3) have two $\omega$-limit points. If $\phi(0, 0, 0) = (a, b, c) \in \mathbb{R}^3$ with $\sqrt{a^2 + b^2} \neq 0$, $c \neq 0$ then system (3) has Zeno hybrid arc with Zeno time $\tau = \frac{1}{2} \left( b + \frac{\sqrt{b^2 + 2ca}}{2} \right)$ and the $\omega$-limit set consisting of two points $(0, 0, \pm c \cdot e^{-\gamma})$. Hence the initial point $(a, b, c)$ generates two solutions. Despite such complex situation we were able to use Theorem 1 to verify UGpASoZ without knowing the exact number of $\omega$-limit points of Zeno arcs. Moreover, Theorem 1 along with Lemma 5.2 can be used for $\omega$-limit points localization. If one finds a function $V$ satisfying Proposition 5.1 for some set $A$, then following Lemma 5.2, $\omega$-limit points of solutions are contained in the set $A$.

6. DISCUSSION AND OPEN QUESTIONS

The results presented here are beneficial for construction and stability analysis of solutions to hybrid dynamical systems that exhibit Zeno behavior. The main contributions of the paper are the following. First, we have introduced the extended hybrid time domain and new prolonged solution concept that heavily relies on the axiomatics and notation of Goebel et al. (2012). These extended solutions helped us to avoid such undesired effects as freezing of solutions. Second, we propose a generalisation of the attractivity concept and prove theorems that provide Lyapunov-like sufficient conditions for stability without knowing the explicit solution and $\omega$-limit points. However, in order to apply these theorems one should be able to verify whether all hybrid arcs are either Zeno or intersect an appropriate set $A_i$.

We believe that the proposed way of solution’s prolongation from its $\omega$-limit points can also be achieved without the introduction of a new 3-dimensional hybrid time domain. However it would cause a significant redefining of the basic concepts of hybrid dynamical systems framework. An important advantage of the proposed approach is the ability to utilize a wide range of previously developed results on UGpAS, e.g. from Goebel et al. (2012), for stability analysis of extended solutions.

Several problems have no answers yet and are very exciting to be solved. The first one is an extension of the results to infinite dimensional setting. This can be described by a vacuous interconnection of infinitely many bouncing balls. One can easily check that this system has a qualitatively different behavior depending on the initial conditions. Let the balls be enumerated by index $n$ from 1 to $\infty$ and each ball starts its way with zero velocity and vertical position equals to $n$. Then the Zeno index $k$ of the hybrid time domain for this case tends to infinity while the ordinary time $t \to \infty$. However if every ball starts with the position $\pm \frac{1}{n}$ then the Zeno index $k$ reaches infinity by a finite ordinary time $t$. If we interconnect each of these systems with a purely continuous process that tends to zero (like $\frac{dx}{dt} = -x$) then a solution of the entire interconnection will tend to the origin in the first case and will "freeze" away from the origin in the second one. This situation gives an intuition that such kind of systems can be treated using some analogues of a local stability concept and needs a deeper investigation for a comprehensive analysis of its behavior.

Another challenging issue is an interconnection of a completely continuous and a completely discrete system. The resulting flow and jump sets obtained in “natural” manner lead to a system with only discrete time domain. The examples of such processes are for instance sample-and-hold control where a discrete-time algorithm measures the state of a continuous time system and updates it. In this case an entire interconnected system will have a solution with only discrete time and we just lose the continuous process.

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