Polarization in Quantum Computations

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Abstract

We propose a realization of quantum computing using polarized photons. The information is coded in two polarization directions of the photons and two-qubit operations are done using conditional Faraday effect. We investigate the performance of the system as a computing device.

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After the early discussion of quantum computing [1,2], the field has attracted much attention because Shor [3] has shown that the famous factorization problem can, in principle, be speeded up considerably by quantum data manipulation techniques. The recent work on quantum computations has been reviewed by Bennett in Ref. [4]. Many realizations have been suggested, at present the most promising seem to be ions trapped electrodynamically [5] or in a cavity [6].

In the recent work [7], one of us considered the possible use of photon polarization states to carry quantum information. The advantage is that they provide a natural two-state basis with no additional Hilbert space components, such as the vacuum state, that may constitute losses of the coding. The single photon coding allows an easy detection, in contrast the vacuum state is hard to distinguish from a failed detection. The photon coding also allows long dephasing times and the possibility to transfer the information from one device to another through fibers. The purpose of this paper is to investigate how realistic this suggestion is by numerical integration of a semirealistic situation.

The elliptically polarized photon state \( (\alpha_+ a_+^\dagger + \alpha_- a_-^\dagger) |0\rangle \) can be manipulated by the Faraday effect induced by the presence of a second photon \( b_\pm^\dagger \). These are supposed to selectively transfer population from the ground state \( |0\rangle \) in Fig.1 to the levels \( |\pm 1\rangle \). Because each photon \( a_\pm^\dagger \) sees only one transition \( |\pm 1\rangle \rightarrow |2\rangle \) it becomes modified by the population transferred by the photons \( b_\pm^\dagger \). If we keep the transition \( |\pm 1\rangle \rightarrow |2\rangle \) off resonance, the atom acts as a dielectric only and hence the relative phases of \( a_\pm^\dagger \) are modified; this is a turning of the axis of the state \( (\alpha_+ a_+^\dagger + \alpha_- a_-^\dagger) |0\rangle \). It was shown in [7] that this allows the gated application of an arbitrary unitary transformation.

In this paper we are looking at two different cases. Case I corresponds to Fig.1 when \( \Delta_1 = 0 \). Here both transitions \( |0\rangle \rightarrow |\pm 1\rangle \) are in resonance and both photons \( a_\pm^\dagger \) experience a modified phase. The situation is symmetric: if \( b_\pm^\dagger \) is present alone we achieve a phase shift exactly opposite in sign to that caused by the presence of \( b_\pm^\dagger \) only. In Case II we detune one of the transitions, \( |0\rangle \rightarrow |+1\rangle \) say; see Fig.1. Then only the presence of the resonant photon \( b_+^\dagger \) affects the phase of the \( a_-^\dagger \)-photons. This corresponds to the gate where the presence of
the state $b_+|0\rangle$ does nothing. Most gates discussed earlier in the literature are of this type.

The four-level system shown in Fig.1 is described by the Hamiltonian

$$H = \Omega_2(a_+^\dagger a_+ + a_-^\dagger a_-) + \Omega_1(b_+^\dagger b_+ + b_-^\dagger b_-)$$

$$+ \omega_2|2\rangle\langle 2| + \omega_{1+}|1\rangle\langle +1| + \omega_{1-}|1\rangle\langle -1|$$

$$+ \omega_0|0\rangle\langle 0| + \lambda_1 (b_-|1\rangle\langle 0| + b_+|1\rangle\langle 0| + h.c.)$$

$$+ \lambda_2 (a_-|2\rangle\langle -1| + a_+|2\rangle\langle +1| + h.c.). \quad (1)$$

In Case I we assume that the states $|\pm 1\rangle$ are degenerate and that the transitions $|0\rangle \rightarrow |0\rangle$ are at resonance, $\Delta_1^I = \omega_{1\pm} - \omega_0 = \Omega_1 = 0$. The transitions $|\pm 1\rangle \rightarrow |2\rangle$ are assumed detuned, i.e. $\Delta_2^I = \omega_2 - \omega_0 - \Omega_1 - \Omega_2 = \omega_2 - \omega_{1\pm} - \Omega_2$ is nonzero. In Case II we lift the degeneracy of levels $|\pm 1\rangle$ by setting $\omega_{1+} = \omega_{1-}$. Then the transition $|0\rangle \rightarrow |1\rangle$ is taken at resonance $\omega_{1-} - \omega_0 = \Omega_1$ but the detunings $\Delta_1^{II} = \omega_{1+} - \omega_0 - \Omega_1$ and $\Delta_2^{II} = \omega_2 - \omega_0 - \Omega_1 - \Omega_2$ are nonzero. The transition $|+1\rangle \rightarrow |2\rangle$ is detuned by $\Delta_2 = \omega_2 - \omega_{1+} - \Omega_2 = \Delta_2^{II} - \Delta_1^{II}$; this is assumed well off resonance too.

The initial state is taken to be the disentangled form

$$|\Psi_{in}\rangle = (a_+^\dagger a_+ + a_-^\dagger a_-)(\beta_+ b_+^\dagger + \beta_- b_-^\dagger)|0\rangle, \quad (2)$$

where $|0\rangle$ denotes the vacuum of the fields. The coefficients are in general complex numbers normalized to unity. We propagate the state vector (2) to the time $t$ with the Hamiltonian (1) and write the final state as

$$|\Psi_{out}\rangle = e^{-iHt}|\Psi_{in}\rangle = \sum_{i=1}^{9} C_i|i\rangle, \quad (3)$$

where we have numbered the basis states according to the set

$$\{|1\rangle, |2\rangle, |3\rangle, \ldots, |9\rangle\} = \{|2\rangle, a_+^\dagger|1\rangle + 1\rangle, a_-^\dagger|1\rangle - 1\rangle, a_+^\dagger|+1\rangle, a_-^\dagger|-1\rangle, a_+^\dagger b_+^\dagger|0\rangle, a_-^\dagger b_-^\dagger|0\rangle, a_+^\dagger b_+^\dagger|0\rangle, a_-^\dagger b_-^\dagger|0\rangle\}. \quad (4)$$

Initially the coefficients $\{C_6, C_7, C_8, C_9\}$ are prepared. Of these, the Hamiltonian couples in Case I $C_6$ to $C_3$ and $C_9$ to $C_4$ only; in Case II $C_6$ to $C_3$ only. In these subspaces the system
can be solved exactly, and performing a rotating wave approximation with respect to the frequency \( \omega_0 + \Omega_1 + \Omega_2 \) we obtain in Case I

\[
C_9(t) = \cos(\lambda_1 t)C_9(0) + i \sin(\lambda_1 t)C_4(0)
\]

\[
C_6(t) = \cos(\lambda_1 t)C_6(0) + i \sin(\lambda_1 t)C_3(0);
\]

in Case II only (6) is valid. Choosing the interaction time such that \( \lambda_1 t = \pi \), we find that the probabilities are restored in these subspaces. We are now left in Case I with a 5 dimensional and in Case II a 7 dimensional subspace to consider numerically.

After the interaction, the state (3) is available for measurements. In the ideal situation, the initial photons would have been restored to the radiation field. This is desired because the information resides in these photons, and they should be available for subsequent computational operations. We can ensure that they have been returned by observing that the atom is back in its ground state \( |0\rangle \) by projecting the final state on this. After the interaction, the atom is available for inspection; a measurement on its state does no longer affect the outcome of the process. We write this state after an observation, \( |\Psi_0\rangle = |0\rangle \langle 0| |\Psi_{\text{out}}\rangle \), as

\[
|\Psi_0\rangle = \left( \frac{C_{++} e^{i\varphi_{++}} \alpha_+ a_+^\dagger + C_{+-} e^{i\varphi_{+-}} \alpha_- a_-^\dagger}{\alpha_+ \beta_+} \right) \beta_+ b_+^\dagger |0\rangle
+ \left( \frac{C_{++} e^{i\varphi_{++}} \alpha_+ a_+^\dagger + C_{+-} e^{i\varphi_{+-}} \alpha_- a_-^\dagger}{\alpha_- \beta_-} \right) \beta_- b_-^\dagger |0\rangle.
\]

We have written the amplitudes and phases of the new coefficients as \( C_{ij} e^{i\varphi_{ij}} (i, j \in \{-, +\}) \).

A measure of the efficiency of the process is the probability \( P_0 = |\langle 0| |\Psi_{\text{out}}\rangle|^2 \). A small value of \( P_0 \) makes the process inefficient, but once the state \( |0\rangle \) has been observed on the atom, the expressions in the brackets of (4) give the effect on the state \( (\alpha_+ a_+^\dagger + \alpha_- a_-^\dagger)|0\rangle \) conditioned on the presence of the photons \( b_\pm^\dagger \) on the lower transitions. These expressions contain the effect of the gating action of the system. In all cases investigated in this paper, however, \( P_0 \) has been found to deviate from unity by less than 1%. The process is efficient as given.

If the coefficients \( \eta_{ij} = C_{ij}/|\alpha_i \beta_j| \) in (4) are close to unity, the interaction only adds the phases \( \varphi_{ij} \); the polarization of the \( a^\dagger \)-field has been changed by the interaction. If we define
the initial phases $\varphi_\pm = \text{arg}(\alpha_\pm)$ and $\varphi^b_\pm = \text{arg}(\beta_\pm)$, we denote the phase changes by

$$\varphi_\pm = (\varphi_{++} - \varphi_{--} - \varphi^a_+ - \varphi^a_-)/2 - \varphi^b_\pm$$

$$\Delta\varphi_\pm = (\varphi_{++} - \varphi_{--} - \varphi^a_+ + \varphi^a_-)/2.$$  \hspace{1cm} (8) (9)

We now write the final state (7) in the form

$$|\Psi_0\rangle = \{e^{i\varphi_+} (\eta_{++} e^{i\Delta\varphi_+} \alpha_+ a_+^\dagger + \eta_{--} e^{-i\Delta\varphi_+} \alpha_- a_-^\dagger) \beta_+ b_+^\dagger + e^{i\varphi_-} (\eta_{+-} e^{i\Delta\varphi_-} \alpha_+ a_+^\dagger + \eta_{-+} e^{-i\Delta\varphi_-} \alpha_- a_-^\dagger) \beta_- b_-^\dagger \} |0\rangle.$$  \hspace{1cm} (10)

When we choose the initial coefficients $\alpha_\pm, \beta_\pm$ real, the phases (8)–(9) simplify; at the end of this paper we are going to discuss the influence of the phase on the gating performance.

In Case I, the symmetry requires that $\varphi_+ = \varphi_-$ and $\Delta\varphi_+ = -\Delta\varphi_- \equiv \Delta\varphi$. In Case II, we assert that $\varphi_+ \sim \varphi_- \approx 0$, which implies $\varphi_- \approx 0$ and $\Delta\varphi_- \approx 0$. We may consider the 4-dimensional subspace $\{a_+^\dagger b_+^\dagger |0\rangle, a_+^\dagger b_-^\dagger |0\rangle, a_-^\dagger b_+^\dagger |0\rangle, a_-^\dagger b_-^\dagger |0\rangle\}$. Assuming now that all coefficients $\eta_{ij}$ are unity, we obtain in the symmetric case the ideal transformation $U_I \sim e^{i\varphi} \text{Diag}\{e^{i\Delta\varphi}, e^{-i\Delta\varphi}, e^{-i\Delta\varphi}, e^{i\Delta\varphi}\}$. In the detuned Case II, we obtain $U_{II} \sim \text{Diag}\{1, 1, e^{i(\varphi_+ + \Delta\varphi_+)}, e^{i(\varphi_+ - \Delta\varphi_+)}\}$. This is a phase transformation of the bit denoted by $a_+^\dagger$ induced by the presence of the photon $b_+^\dagger$.

We are now going to consider the performance qualities of the model system as a gated bit transformation. The input to the calculation is the initial state (2). To begin we choose the "classical" case when only one of the input states is present. In the symmetric Case I, the choice of state is not important, c.f. $U_I$, but for the Case II, we need to look at the states $a_-^\dagger b_+^\dagger |0\rangle$ and $a_+^\dagger b_-^\dagger |0\rangle$. First we choose to discuss the single input state $a_-^\dagger b_+^\dagger |0\rangle$ with $\alpha_- = \beta_+ = 1$.

As stated above, the interaction time is chosen such that $t = \pi/\lambda_1$; in the calculations we choose $\lambda_1 = 1$. For large detunings ($\omega_2 \to \infty$) $\eta_{++}$ approaches unity, but the phase shift $\Delta\varphi$ goes to zero. In Case I, the numerical investigations show that we can retain $\eta_{++}^2 > 0.9$ if we choose $\Delta\varphi_I > 5$. For $\Delta\varphi_I = 5$ we find $\Delta\varphi_+ \approx 10^\circ$. This is achieved with $\lambda_2 = 1$; larger phases can be achieved by increasing $\lambda_2$, but the restoring of the population suffers. For $\lambda_2 \leq 1.5$ we
can achieve $\Delta \varphi_+ \geq 15^\circ$ and $\eta^2_+ > 0.75$. The results can be illustrated in a graph plotting $\Delta \varphi_+$ as a function of $\eta^2_+$ with the detuning as a parameter. For the symmetric Case I, this is done in Fig.2a. As we can see, for $\Delta_2^I > 5$, no dependence on detuning is seen. The corresponding results for Case II are shown in Fig.2b. Here the dependence on detuning is much stronger; however for large values of detuning, $\Delta_2^II = 15$ and $\Delta_2^II = 30$, we can reach $\Delta \varphi_+ \geq 43^\circ$ with $\eta^2_+ \geq 0.9$. Thus the operation of this gate is much more efficient as is to be expected. For larger values of $\Delta_2^II$ the results tend to become independent of the detuning.

We now choose to look at the case $\lambda_2 = 2.5$ and $\Delta_2 = 30$. For the Case I this gives $\Delta \varphi_+ \simeq 10^\circ$ and $\eta^2_+ \simeq 0.90$. In Case II it gives $\Delta \varphi_+ \simeq 10^\circ$ and $\eta^2_+ \simeq 0.99$. In order to see where the missing population goes in Case I, we plot the population of the states $a^\dagger_- b^\dagger_+ |0\rangle$, $a^\dagger_+ |0\rangle$, $a^\dagger_+ b^\dagger_- |0\rangle$ and $|2\rangle$ in Fig.3. At time $t = \pi$, the population of $a^\dagger_+ b^\dagger_- |0\rangle$ is restored to 90% but the missing population is on the level $a^\dagger_+ |1\rangle$. This is mediated through the off-resonant transition $| - 1 \rangle \rightarrow |2\rangle \rightarrow | + 1 \rangle$ which proceeds at the effective Rabi rate $(\lambda_2^2/\Delta_2^I) \sim 6.25/30$. With time, this increases the population of the state $a^\dagger_+ | + 1 \rangle$ as can be seen in Fig.3; this increase is modulated at the rate $\lambda_1$ by the population pulsations on level $a^\dagger_- | - 1 \rangle$. This effect can be decreased by increasing $\Delta_2 \gg \lambda_2^2$. In Case II, the population of the level $a^\dagger_+ b^\dagger_- |0\rangle$ is restored to better than 99% and the population on states $a^\dagger_+ | + 1 \rangle$ and $a^\dagger_+ b^\dagger_- |0\rangle$ remain below $10^{-3}$.

After having described the ”classical” inputs, where each 2-bit pure state has been treated separately, we now turn to consider the genuine quantum situation described by the input state $|\psi\rangle$. The performance of the system acting on this state is, of course, essential for its usefulness as a quantum computing device.

An input consisting of a pair of two-level systems contains 4 degrees of freedom: the 4 complex numbers involved lose two parameters to the over-all phase and two to the normalization conditions. It is still difficult to display the results of a 4 parameter input space, and hence we start by considering only real coefficients in Eq. (2). The influence of the phases $\varphi^a_\pm, \varphi^b_\pm$ will be discussed below. We are thus left with two real parameters, one for each input bit. We choose to display our results as functions of $\alpha^2_- = 1 - \alpha^2_+$ for the two
\[ |\beta_1\rangle = \frac{1}{\sqrt{2}} \left( b_+^\dagger + b_-^\dagger \right) |0\rangle, \quad |\beta_2\rangle = \left( \frac{\sqrt{3}}{2} b_+^\dagger + \frac{1}{2} b_-^\dagger \right) |0\rangle. \tag{10} \]

We want to introduce a quality factor for the use of a system like this in computations. The performance is close to ideal, when the parameter \( \eta_{ij} \simeq 1 \). However, when either one of the input parameters \( \alpha_i, \beta_j \) becomes close to zero, any minute value in the corresponding coefficient \( C_{ij} \) is likely to cause a large value \( \eta_{ij} \). Thus we want to consider the retention of that product \( \alpha_i \beta_j \) which is the largest. A value close to unity here signals a good performance. To test this idea we consider the variables

\[
\eta_{-+}^2 \quad (\alpha_-^2 \geq 0.5) \quad ; \quad \eta_{+-}^2 \quad (\alpha_-^2 \leq 0.5). \tag{11} \]

Another measure of the efficiency of the process can be given by the retention of the ratio between the two components \( b_\pm^\dagger \) in Eq.\( (7) \). This starts from \( |\beta_+ / \beta_-|^2 \) and if retained the parameter

\[
R = \left( \frac{|C_{++}|^2 + |C_{--}|^2}{|C_{+-}|^2 + |C_{-+}|^2} \right) \left( \frac{\beta_-}{\beta_+} \right)^2 \tag{12} \]

should be close to unity. The retention parameter \( R \) for Case I and the inputs \( |\beta_1\rangle \) and \( |\beta_2\rangle \) is shown in Fig.\( 4a \) together with the corresponding quality factor in Eq.\( (11) \). In Fig.\( 4b \) the same parameters are shown for the asymmetric Case II. As we can see, the retention parameter \( R \) is at its worst about 70%; in Case II it is better than 90%. In Case I, the quality factor \( (11) \) is good to within 90% and in the asymmetric Case II to better than 95%.

Finally we want to return to the question of the influence of the initial phases. These do affect the outcome, but their influence seems to be smaller than the influence of the magnitudes. We consider the achieved phase shifts as functions of the superposition coefficients \( \alpha \) and \( \beta \). In Fig.5 we plot the phase shifts \( \Delta \varphi_\pm \) against \( \alpha_\pm^2 \) in the asymmetric Case II shown for \( |\beta_1\rangle \) and \( |\beta_2\rangle \). For \( |\beta_1\rangle \), we also consider the case when the initial phase \( \varphi_+^a \) is set to the value \( \varphi_+^a = \pi / 4 \). The behaviour is close to ideal; in the range \( \alpha_-^2 \in (0.1, 0.9) \), nearly ideal behaviour is observed, \( \Delta \varphi_+ \simeq 9.5^\circ \) and \( |\Delta \varphi_-| < 0.4^\circ \). The effect of the initial phase
is small. In the symmetric Case I, the behaviour was found to be less optimal: we saw only a small difference for the two $\beta$-states, but for $\alpha^2$ in the range $(0.1, 0.9)$ the phase shift changed from $30^\circ$ to $10^\circ$. Thus in Case I, the magnitude of the angle remains considerable but it does depend on the value of $\alpha$. We have not carried out a systematic investigation of the influence of the phase factors; the results reported here indicate that they cause no drastic changes. If needed, their effects can easily be evaluated using the method presented here.

As a conclusion, we discuss how well a quantum gate can be realized in our model. We choose to look at the Controlled-NOT gate, which changes the value of the target bit whenever the control bit has the value one. Based on the considerations above, we conclude that the asymmetric Case II is better suited to work as a gate. Its performance can easily be improved from the results presented above by increasing $\Delta_{II}^2$, $\Delta_{II}^1$ and $\lambda_2$ in a suitable way. Here we use the parameters $\Delta_{II}^2 = 70$, $\Delta_{II}^1 = 65$, $\lambda_2 = 6.85$, $\lambda_1 = 2$, and $t = \pi$: this enables us to approximate the transformation $U_{II}$ to the accuracy $10^{-3}$ with a phase shift of $60^\circ$. This has to be applied three times in sequence in order to get a phase shift of $180^\circ$, which is needed for the Controlled-NOT gate. After performing suitable transformations between the circular and linear bases (see [7]), we obtain as the final result the Controlled-NOT transformation $C_N$:

$$C_N = \begin{bmatrix} 0.995e^{-i33^\circ} & O(10^{-3}) & O(10^{-2}) & O(10^{-2}) \\ O(10^{-3}) & 0.995e^{-i33^\circ} & O(10^{-2}) & O(10^{-2}) \\ O(10^{-2}) & O(10^{-2}) & O(10^{-3}) & -0.997 \\ O(10^{-2}) & O(10^{-2}) & -0.997 & O(10^{-3}) \end{bmatrix}.$$  

The overall phases $e^{-i33^\circ}$ and $-1$ are irrelevant. We see that the Controlled-NOT gate can be realized in this case to the accuracy $10^{-2}$.

The present scheme has been found to perform reasonably well as a computing device. It is naturally not good enough to be an element of a computer network of realistic size, but there seems to be no suggestion in the literature which satisfies this criterion. The
performance of our scheme can be improved by sequential application of the $b^{\dagger}$- and $a^{\dagger}$-photons, with final restoration of the $b^{\dagger}$-state by a third pulse. Such a scheme seems to require perfectly controlled pulses, which we regard as even more unrealistic than the model we have investigated. To implement our method in a multi-step computation we assume all initial information to be coded in a set of field modes residing uncoupled in the same cavity. During their coherence time, we shoot through the cavity volume a sequence of suitably chosen atoms which couple the photon pairs, i.e. perform the two-qubit operations. To affect all possible unitary transformations, the cavity has to be rather complicated, containing a suitable arrangement of $\lambda$-plates to give access to all desired polarization states. Also the atoms have to be able to couple just the desired modes at each stage of the calculation. This and the restrictions imposed by loss rates and decoherence times pose extremely strict limitations on the computations possible. If several cavities are necessary, the dissipative effects on photons transferred between them raise further problems. However, such difficulties seem to afflict other schemes suggested too. Which one can be optimized the most remains an experimental challenge.
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FIG. 1. The 4-level system used in the gated Faraday effect
FIG. 2. The phase shift $\Delta \varphi_+$ as a function of $\eta_{-+}^2$ for several values of the detuning $\Delta_2^I$, $\Delta_2^{II}$; some values of $\lambda_2$ used are marked. In all figures Fig.#a corresponds to Case I, Fig.#b to Case II where $\Delta_1^{II} = \Delta_2^{II}/2$. 
FIG. 3. The populations of the basis states as functions of time (Case I)
FIG. 4. The retention $R$ and the quality factor (20) as functions of $\alpha^2$, for $|\beta_1\rangle$ (solid lines) and $|\beta_2\rangle$ (dotted lines).
FIG. 5. The phase shifts $\Delta \varphi_{\pm}$ as functions of $\alpha_{-}^{2}$, for $|\beta_{1}\rangle$ (solid lines) and $|\beta_{2}\rangle$ (dotted lines).

The shift $\Delta \varphi_{+}$ is shown also for the case of a non-zero initial phase $\varphi_{+}^{a}$. (Case II)