Random Function Iterations for Stochastic Fixed Point Problems

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Abstract

We study the convergence of random function iterations for finding an invariant measure of the corresponding Markov operator. We call the problem of finding such an invariant measure the stochastic fixed point problem. This generalizes earlier work studying the stochastic feasibility problem, namely, to find points that are, with probability 1, fixed points of the random functions [44]. When no such points exist, the stochastic feasibility problem is called inconsistent, but still under certain assumptions, the more general stochastic fixed point problem has a solution and the random function iterations converge to an invariant measure for the corresponding Markov operator. There are two major types of convergence: almost sure convergence of the iterates to a fixed point in the case of stochastic feasibility, and convergence in distribution more generally. We show how common structures in deterministic fixed point theory can be exploited to establish existence of invariant measures and convergence of the Markov chain. We show that weaker assumptions than are usually encountered in the analysis of Markov chains guarantee linear/geometric convergence. This framework specializes to many applications of current interest including, for instance, stochastic algorithms for large-scale distributed computation, and deterministic iterative procedures with computational error. The theory developed in this study provides a solid basis for describing the convergence of simple computational methods without the assumption of infinite precision arithmetic or vanishing computational errors.

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1. Introduction

Random function iterations (RFI) [32] generalize deterministic fixed point iterations, and are a useful framework for studying a number of important applications. An RFI is a stochastic process of the form $X_{k+1} := T_\xi X_k$ ($k = 0, 1, 2, \ldots$) initialized by a random variable $X_0$ with distribution $\mu_0$ and values on some set $G$. This of course includes initialization from a deterministic point $x_0 \in G$ via the $\delta$-distribution. Here $\xi_k$ ($k = 0, 1, 2, \ldots$) is an element of a sequence of i.i.d. random variables that map from a probability space into a measurable space of indices $I$ (not necessarily countable) and $T_i$ ($i \in I$) are self-mappings on $G$. The iterates $X_k$ form a Markov chain of random variables on the space $G$, which is, for our purposes, a Polish space. Deterministic fixed point iterations are included when the index set $I$ is just a singleton.

Our main motivation for the fundamental and abstract study pursued here is the very concrete application of X-ray free electron laser (X-FEL) imaging experiments [17, 27, 92]. We return to this specific application in Section 4. There are many more applications than one could reasonably list, but to reach the broadest possible audience, a simple example from first semester numerical analysis is illustrative. Consider the underdetermined linear system of equations

$$Ax = b, \quad A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, \quad m < n.$$

Equivalent to this problem is the problem of finding the intersection of the hyperplanes defined by the single equations $\langle a_j, x \rangle = b_j$ ($j = 1, 2, \ldots, m$) where $a_j$ is the $j$th row of the matrix $A$:

Find $\bar{x} \in \cap_{j=1}^m \{x \mid \langle a_j, x \rangle = b_j \}$. \hspace{1cm} (1)

An intuitive technique to solve this problem is the method of cyclic projections: Given an initial guess $x_0$, construct the sequence $(x_k)$ via

$$x_{k+1} = P_m P_{m-1} \cdots P_1 x_k,$$ \hspace{1cm} (2)

where $P_j$ is the orthogonal projection onto the $j$th hyperplane above. This method was proposed by von Neumann in [93] where he also proved that, without numerical error, the iterates converge to the projection of the initial point $x_0$ onto the intersection; Aronszajn showed that the rate of convergence is linear [5] (referred to as geometric or exponential in other communities).

The projectors have a closed form representation, and the algorithm is easily implemented. The results of one implementation for a randomly generated matrix $A$ and vector $b$ with $m = 50$ and $n = 60$ yields the following graph shown in Figure 1(a). As the figure shows, the method performs as predicted by the theory, up to the numerical precision of the implementation. After
that point, the iterates behave more like random variables with distribution indicated by the histogram shown in Figure 1(b). The theory developed in this study provides a solid basis for describing the convergence of simple computational methods without the assumption of infinite precision arithmetic or vanishing computational errors [80, 84]. This particular situation could be analyzed in the stability framework of perturbed convergent fixed point iterations with unique fixed points developed in [25]; our approach captures their results and opens the way to a much broader range of applications. An analysis of nonmonotone fixed point iterations with error can be found already in [48], though the precision is assumed to increase quickly to exact evaluation.

One of the main goals of the present study is to extend the approach established in [61] to the tomographic problem associated with X-FEL measurements in particular, and to noncontractive random function iterations more generally, accounting for randomness not only in the model and the algorithm, but also in the computers we use for implementations. The main object of interest is the Markov operator on a space of probability measures with the appropriate metric. We take for granted much of the basic theory of Markov chains, which interested readers can find, for instance, in [45] or [64]. We are indebted to the work of Butnariu and collaborators who studied stochastic iterative procedures for solving infinite dimensional linear operator equations in [22, 22–24]. Another important application motivating our analytical strategy involves stochastic implementations of deterministic algorithms for large-scale optimization problems [18, 29, 38, 67, 82]. Such stochastic algorithms are popular for distributed computation with applications in machine learning [8, 30, 33, 42, 78]. Here each $T_{\xi_k}$ represents a randomly selected, low-dimensional update mechanism in an iterative procedure. Our approach to the analysis of such algorithms allows for the first time expansive mappings and, in some cases, the analysis is simpler than current approaches.

We are concerned in this paper with (i) existence of invariant distributions of the Markov operators associated with the random function iterations, (ii) convergence of the Markov chain to an invariant distribution, and (iii) rates of convergence. As with classical fixed point iterations, the limit – or more accurately, limiting distribution – of the Markov chain, if it exists, will in general depend on the initialization. Uniqueness of invariant measures of the Markov operator is not a particular concern for feasibility problems where any feasible point will do. The notation and necessary background is developed in Section 2, which we conclude with the main statements of this study (Section 2.5). Section 3 contains the technical details, starting with existence theory in Section 3.1, general ergodic theory in Section 3.2 with gradually increasing regularity assumptions on the Markov operators, equicontinuity in Section 3.3 and finally Markov operators generated by nonexpansive mappings in Section 3.4. The assumptions on the mappings generating the Markov operators are commonly employed in the analysis of deterministic algorithms in continuous optimization. Our first main result, Theorem 2.17, establishes convergence for Markov chains that are generated from nonexpansive mappings in $\mathbb{R}^n$ and follows easily in Section 3.5 upon establishing tightness of the sequence of measures. Section 3.6 collects further facts needed for the second main result of this study, Theorem 2.18, which establishes convergence in the Prokhorov-Lévy metric of Markov chains to an invariant measure (assuming this exists) when the Markov operators are constructed from $\alpha$-firmly nonexpansive mappings in $\mathbb{R}^n$ (Definition 2.8). We conclude Section 3 with the proof in Section 3.8 of the last main result, Theorem 2.19, which provides for a quantification of convergence of the RFI when the underlying mappings are only almost $\alpha$-firmly nonexpansive in expectation (Definition 2.10) and when the discrepancy between a given measure and the set of invariant measures of the Markov operator, (29), is metrically subregular (Definition 2.15). We conclude this study with Section 4 where we focus on applications to optimization on measure spaces and (inconsistent) feasibility.
2. RFI and the Stochastic Fixed Point Problem

In this section we give a rigorous formulation of the RFI, then interpret this as a Markov chain and define the corresponding Markov operator. We then formulate modes of convergence of these Markov chains to invariant measures for the Markov operators and formulate the stochastic feasibility and stochastic fixed point problems. At the end of this section we present the main results of this article. The proofs of these results are developed in Section 3.

Our notation is standard. As usual, \( \mathbb{N} \) denotes the natural numbers including 0. For \( G \), an abstract topological space, \( \mathcal{B}(G) \) denotes the Borel \( \sigma \)-algebra and \( (G, \mathcal{B}(G)) \) is the corresponding measure space. We denote by \( \mathcal{P}(G) \) the set of all probability measures on \( G \). The support of the probability measure \( \mu \) is the smallest closed set \( A \), for which \( \mu(A) = 1 \) and is denoted by \( \text{supp} \mu \).

There is a lot of overlapping notation in probability theory. Where possible we will try to stick to the simplest conventions, but the context will make certain notation preferable. The notation \( \mu \) will be used.

Throughout, the pair \( (G, d) \) denotes a separable metric space with metric \( d \) and \( \mathcal{B}(x, r) \) is the open ball centered at \( x \in G \) with radius \( r > 0 \); the closure of the ball is denoted \( \overline{\mathcal{B}(x, r)} \). All of our results concerning existence of invariant measures, tightness of sequences and convergence will assume that \( (G, d) \) is Polish (i.e. also complete); in characterizing the regularity of the building blocks, completeness of the metric space is not required.

The distance of a point \( x \in G \) to a set \( A \subset G \) is denoted by \( d(x, A) := \inf_{w \in A} d(x, w) \). For the ball of radius \( r \) around a subset of points \( A \subset G \), we write \( \mathcal{B}(A, r) := \bigcup_{x \in A} \mathcal{B}(x, r) \). The 0-1-indicator function of a set \( A \) is given by

\[
\mathbb{1}_A(x) = \begin{cases} 
1 & \text{if } x \in A, \\
0 & \text{else.}
\end{cases}
\]

Continuing with the development initiated in the introduction, we will consider a collection of mappings \( T_i : G \to G \), \( i \in I \), on \( (G, d) \) (a separable complete metric space), where \( I \) is an arbitrary index set. The measure space of indexes is denoted by \( (I, \mathcal{I}) \), and \( I \) is an \( I \)-valued random variable on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). The pairwise independence of two random variables \( \xi \) and \( \eta \) is denoted \( \xi \perp \eta \). The random variables \( \xi_k \) in the sequence \( (\xi_k)_{k \in \mathbb{N}} \) (abbreviated \( (\xi_k) \)) are independent and identically distributed (i.i.d.) with \( \xi_k \) distributed as \( \xi \) (\( \xi_k \sim \xi \)). The method of random function iterations is formally presented in Algorithm 1.

**Algorithm 1**: Random Function Iterations (RFI)

1. **Initialization**: Set \( X_0 \sim \mu_0 \in \mathcal{P}(G), \xi_k \sim \xi \quad \forall k \in \mathbb{N} \).
2. for \( k = 0, 1, 2, \ldots \) do
   1. \( X_{k+1} = T_{\xi_k}X_k \)

We will use the notation

\[
X_k^X_0 := T_{\xi_{k-1}} \ldots T_{\xi_0} X_0 \tag{3}
\]

to denote the sequence of the RFI initialized with \( X_0 \sim \mu_0 \). When characterizing sequences initialized with the delta distribution of a point we use the notation \( X_k^\varepsilon \). The following assumptions will be employed throughout.
Assumption 2.1. (a) \(\xi_0, \xi_1, \ldots, \xi_k\) are i.i.d with values on \(I\) and \(\xi_k \sim \xi\). \(X_0\) is an random variable with values on \(G\), independent from \(\xi_k\).

(b) The function \(\Phi : G \times I \rightarrow G, (x, i) \mapsto T_i x\) is measurable.

2.1. RFI as a Markov chain

Markov chains are conveniently defined in terms of transition kernels. A transition kernel is a mapping \(p : G \times \mathcal{B}(G) \rightarrow [0, 1]\) that is measurable in the first argument and is a probability measure in the second argument; that is, \(p(\cdot, A)\) is measurable for all \(A \in \mathcal{B}(G)\) and \(p(x, \cdot)\) is a probability measure for all \(x \in G\).

**Definition 2.2** (Markov chain). A sequence of random variables \((X_k), X_k : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (G, \mathcal{B}(G))\) is called Markov chain with transition kernel \(p\) if for all \(k \in \mathbb{N}\) and \(A \in \mathcal{B}(G)\) \(\mathbb{P}\)-a.s. the following hold:

(i) \(\mathbb{P}(X_{k+1} \in A | X_0, X_1, \ldots, X_k) = \mathbb{P}(X_{k+1} \in A | X_k)\);

(ii) \(\mathbb{P}(X_{k+1} \in A | X_k) = p(X_k, A)\).

**Proposition 2.3.** Under Assumption 2.1, the sequence of random variables \((X_k)\) generated by Algorithm 1 is a Markov chain with transition kernel \(p\) given by

\[
(x \in G) (A \in \mathcal{B}(G)) \quad p(x, A) := \mathbb{P}(\Phi(x, \xi) \in A) = \mathbb{P}(T_\xi x \in A)
\]

for the measurable update function \(\Phi : G \times I \rightarrow G, (x, i) \mapsto T_i x\).

**Proof.** It follows from [51, Lemma 1.26] that the mapping \(p(\cdot, A)\) defined by (4) is measurable for all \(A \in \mathcal{B}(G)\), and it is immediate from the definition that \(p(x, \cdot)\) is a probability measure for all \(x \in G\). So \(p\) defined by (4) is a transition kernel. The remainder of the statement is an immediate consequence of the disintegration theorem (see, for example, [87]). □

The Markov operator \(\mathcal{P}\) is defined pointwise for a measurable function \(f : G \rightarrow \mathbb{R}\) via

\[
(x \in G) \quad \mathcal{P}f(x) := \int_G f(y)p(x, dy),
\]

when the integral exists. Note that

\[
\mathcal{P}f(x) = \int_G f(y)p^{\Phi(x, \xi)}(dy) = \int_\Omega f(T_\xi(\omega)x)\mathbb{P}(d\omega) = \int_I f(T_i x)\mathbb{P}^i(d\omega).
\]

The Markov operator \(\mathcal{P}\) is Feller if \(\mathcal{P}f \in C_b(G)\) whenever \(f \in C_b(G)\), where \(C_b(G)\) is the set of bounded and continuous functions from \(G\) to \(\mathbb{R}\). This property is central to the theory of existence of invariant measures introduced below. The next fundamental result establishes the relation of the Feller property of the Markov operator to the generating mappings \(T_i\).

**Proposition 2.4** (Theorem 4.22 in [11]). Under Assumption 2.1, if \(T_i\) is continuous for all \(i \in I\), then the Markov operator \(\mathcal{P}\) is Feller.

Let \(\mu \in \mathcal{P}(G)\). In a slight abuse of notation we denote the dual Markov operator \(\mathcal{P}^* : \mathcal{P}(G) \rightarrow \mathcal{P}(G)\) acting on a measure \(\mu\) by action on the right by \(\mathcal{P}\) via

\[
(A \in \mathcal{B}(G)) \quad (\mathcal{P}^* \mu)(A) := (\mu \mathcal{P})(A) := \int_G p(x, A)\mu(dx).
\]

This notation allows easy identification of the distribution of the \(k\)-th iterate of the Markov chain generated by Algorithm 1: \(\mathcal{L}(X_k) = \mu_0 \mathcal{P}^k\).
2.2. The Stochastic Fixed Point Problem

As studied in [44], the stochastic feasibility problem is stated as follows:

$$\text{Find } x^* \in C := \{ x \in G \mid P(x = T_i x) = 1 \}. \quad (5)$$

A point $x$ such that $x = T_i x$ is a fixed point of the operator $T_i$; the set of all such points is denoted by

$$\text{Fix } T_i = \{ x \in G \mid x = T_i x \}.$$ 

In [44] it was assumed that $C \neq \emptyset$. If $C = \emptyset$ we call this the inconsistent stochastic feasibility problem.

Inconsistent stochastic feasibility is far from exotic. Take, for example, the not unusual assumption of additive noise: define $T_\xi(x) := f(x) + \xi$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\xi$ is a measure without point masses. Then $P(T_\xi(x) = x) = P(\xi(x) = x - f(x)) = 0$. More concretely, let $f = \text{Id} - t \nabla F$ where $F : \mathbb{R}^n \to \mathbb{R}$ is a differentiable, strongly convex function and $t$ is some appropriately small stepsize. This yields the noisy gradient descent method

$$T_\xi(x) = x - t \nabla F(x) + \xi.$$ 

Though this has no fixed point, the additive noise $\xi$ can be constructed so that the resulting Markov chain is ergodic and its (unique!) invariant distribution concentrates around the unique global minimum of $F$. See Example 4.1 for more discussion.

The inconsistency of the problem formulation is an artifact of asking the wrong question. A fixed point of the (dual) Markov operator $P$ is called an invariant measure, i.e. $\pi \in P(G)$ is invariant whenever $\pi P = \pi$. The set of all invariant probability measures is denoted by $\text{inv } P$.

We are interested in the following generalization of (5):

$$\text{Find } \pi \in \text{inv } P. \quad (6)$$

We refer to this as the stochastic fixed point problem.

2.3. Modes of convergence

In [44], we considered almost sure convergence of the sequence $(X_k)$ to a random variable $X$:

$$X_k \to X \text{ a.s. as } k \to \infty.$$ 

Almost sure convergence is commonly encountered in the studies of stochastic algorithms in optimization, and can be guaranteed for consistent stochastic feasibility under most of the regularity assumptions on $T_i$ considered here (see [44, Theorem 3.8 and 3.9]) though this does not require the full power of the theory of general Markov processes. In fact, the next result shows that almost sure convergence is only possible for consistent stochastic feasibility.

The following statement first appeared in Lemma 3.2.1 of [43].

Proposition 2.5 (a.s. convergence implies consistency). Let $T_i : G \to G$ be continuous for all $i \in I$ with respect to the metric $d$. Let $\pi \in \text{inv } P \neq \emptyset$ and $X_0 \sim \pi$. Generate the sequence $(X_k^{X_0})_{k \in \mathbb{N}}$ via Algorithm 1 where $X_0 \perp \perp \xi_k$ for all $k$. If the sequence $(X_k^{X_0})$ converges almost surely, then the stochastic feasibility problem is consistent. Moreover, $\text{supp } \pi \subset C$.

Before proceeding to the proof, note that the measure remains the same for each iterate $X_k^{X_0}$ - the issue here is when the iterates converge (almost surely).
Proof. In preparation for our argument, which is by contradiction, choose any $x \in \text{supp} \pi$ where $\pi \in \text{inv} \mathcal{P}$, and define
\[
I^* := \{i \in I \mid d(T_i x, x) > \epsilon \}
\]
for $\epsilon \geq 0$. Note that $I^* \supseteq I^0$ whenever $\epsilon \leq \epsilon_0$. Define the set
\[
J^*_0 := \{i \in I \mid d(T_i x, T_i y) \leq \epsilon, \forall y \in \mathbb{B}(x, \delta) \}.
\]
These sets satisfy $J^*_1 \subset J^*_2$ whenever $\delta_1 \geq \delta_2$ and, since $T_i$ is continuous for all $i \in I$, we have that for each $\epsilon > 0$, $J^*_0 \uparrow I$ as $\delta \to 0$. A short argument shows that $I^*$ and $J^*_0$ are measurable for each $\delta$ and $\epsilon > 0$.

Suppose now, to the contrary, that $C = \emptyset$. Then $\mathbb{P}(T_\xi x = x) < 1$ and hence $\mathbb{P}(d(T_\xi x, x) > 0) > 0$. Since $I^* \supseteq I^0$ for $\epsilon \leq \epsilon_0$ we have $\mathbb{P}^\xi(I^0) \leq \mathbb{P}^\xi(I^*)$ whenever $\epsilon \leq \epsilon_0$. In particular, there must exist an $\epsilon_0$ such that $0 < \mathbb{P}^\xi(I^0)$. On the other hand, there is a constant $\delta > 0$ such that $\delta < \epsilon_0/2$ and $\mathbb{P}^\xi(K^0_\delta) > 0$ where $K^0_\delta := I^0 \cap J^0_\delta/2$. This construction then yields
\[
(\forall i \in K^0_\delta) \quad d(T_i y, x) \geq d(T_i x, x) - d(T_i y, T_i x) \geq \frac{\epsilon_0}{2} > \delta \quad \forall y \in \mathbb{B}(x, \delta).
\]

Next, we claim that $X_k^{X_0} \sim \pi$ for all $k \in \mathbb{N}$. Indeed, for any $Y \sim \pi$, if $\xi$ is independent of $Y$, then $T_\xi Y \sim \pi$. To see this, note that For $A \in \mathcal{B}(G)$ Fubini’s Theorem and disintegration yield
\[
\mathbb{P}(T_\xi Y \in A) = \mathbb{E}[\mathbb{E}[1_A(T_\xi Y) | \xi]] = \mathbb{E} \int 1_A(T_\xi y)\pi(dy) = \int \mathbb{E} 1_A(z)\mathbb{P}^{T_\xi y}(dz)\pi(dy)
= \int \int 1_A(z)p(y, dz)\pi(dy) = \pi(A) = \mathbb{P}(Y \in A).
\]
It follows that $\text{supp} \mathcal{L}(Y) = \text{supp} \mathcal{L}(T_\xi Y)$, and since $\xi_k$ are i.i.d., $X_k^{X_0} \sim \pi$ for all $k \in \mathbb{N}$. This establishes the claim.

The independence of $\xi_k$ and $X_0$ for all $k$ implies the independence of $\xi_k$ and $X_k^{X_0}$ for all $k$. Moreover, $\mathbb{P}(X_k^{X_0} \in B_\delta) = \pi(B_\delta) > 0$ for all $k \in \mathbb{N}$, where to avoid clutter we denote $B_\delta := \mathbb{B}(x, \delta)$. This yields
\[
(\forall k \in \mathbb{N}) \quad \mathbb{P}(X_k^{X_0} \in B_\delta, X_{k+1}^{X_0} \notin B_\delta) \geq \mathbb{P}(X_k^{X_0} \in B_\delta, \xi_k \in K^0_\delta) = \pi(B_\delta)\mathbb{P}^\xi(K^0_\delta) > 0.
\]
Thus, we also have
\[
(\forall k \in \mathbb{N}) \quad \mathbb{P}(X_k^{X_0} \in B_\delta, X_{k+1}^{X_0} \in B_\delta) = \mathbb{P}(X_k^{X_0} \in B_\delta) - \mathbb{P}(X_k^{X_0} \in B_\delta, X_{k+1}^{X_0} \notin B_\delta) \leq \pi(B_\delta) - \pi(B_\delta)\mathbb{P}^\xi(K^0_\delta) = \pi(B_\delta) \{1 - \mathbb{P}^\xi(K^0_\delta)\} < \pi(B_\delta).
\]
However, by assumption, $X_k^{X_0} \to X_\ast$ a.s. for some random variable $X_\ast$ with $\mathbb{P}(X_\ast \in B_\delta) = \pi(B_\delta) > 0$. If $X_\ast \in B_\delta$ then due to the a.s. convergence there exists a (random) $k_\ast$ such that $X_k^{X_0} \in B_\delta$ for all $k \geq k_\ast$. This implies that
\[
\mathbb{P}(X_k^{X_0} \in B_\delta, X_{k+1}^{X_0} \in B_\delta, X_\ast \in B_\delta) \to \mathbb{P}(X_\ast \in B_\delta) = \pi(B_\delta).
\]
which is a contradiction since by the above
\[
\mathbb{P}(X_k^{X_0} \in B_\delta, X_{k+1}^{X_0} \in B_\delta, X_\ast \in B_\delta) \leq \mathbb{P}(X_k^{X_0} \in B_\delta, X_{k+1}^{X_0} \in B_\delta) < \pi(B_\delta).
\]
So it must be true that $\mathbb{P}(d(T_\xi x, x) > 0) = 0$. In other words, $\mathbb{P}(T_\xi x = x) = 1$, hence $C \neq \emptyset$. Moreover, since the point $x$ was any arbitrary point in $\text{supp} \pi$, we conclude that $\text{supp} \pi \subset C$. \qed
For inconsistent feasibility, or more general stochastic fixed point problems that are the aim
of the RFI, Algorithm 1, we focus on convergence in distribution. Let \((\nu_k)\) be a sequence of
probability measures on \(G\). The sequence \((\nu_k)\) is said to converge in distribution to \(\nu\)
everywhere \(\nu \in \mathcal{P}(G)\) and for all \(f \in C_b(G)\) it holds that \(\nu_k f \to \nu f\) as \(k \to \infty\), where \(\nu f := \int f(x) \nu(dx)\).
Equivalently a sequence of random variables \((X_k)\) is said to converge in distribution if their
laws \((\mathcal{L}(X_k))_{k \in \mathbb{N}}\) do.

We now consider two modes of convergence in distribution for the corresponding sequence of
measures \((\mathcal{L}(X_k)))_{k \in \mathbb{N}}\) on \(\mathcal{P}(G)\):

1. convergence in distribution of the Cesàro averages of the sequence \((\mathcal{L}(X_k)))\) to a probability
measure \(\pi \in \mathcal{P}(G)\), i.e. for any \(f \in C_b(G)\)
   \[
   \nu_k f := \frac{1}{k} \sum_{j=1}^{k} \mathcal{L}(X_j)f = \mathbb{E} \left[ \frac{1}{k} \sum_{j=1}^{k} f(X_j) \right] \to \pi f, \quad \text{as } k \to \infty; \]

2. convergence in distribution of the sequence \((\mathcal{L}(X_k)))\) to a probability measure \(\pi \in \mathcal{P}(G)\),
i.e. for any \(f \in C_b(G)\)
   \[
   \mathcal{L}(X_k)f = \mathbb{E}[f(X_k)] \to \pi f, \quad \text{as } k \to \infty. \]

Clearly, the second mode of convergence implies the first. This is used in Section 3.5 and Section
3.7.

An elementary fact from the theory of Markov chains (Proposition 3.1) is that, if the Markov
operator \(\mathcal{P}\) is Feller and \(\pi\) is a cluster point of \((\nu_k)\) with respect to convergence in distribution
then \(\pi\) is an invariant probability measure. Existence of invariant measures for a Markov
operator then amounts to verifying that the operator is Feller (by Proposition 2.4, automatic if
the \(T_i\) are continuous) and that cluster points exist (guaranteed by tightness – or compactness
with respect to the topology of convergence in distribution – of the sequence, see [14, Section
5]. In particular, this means that there exists a convergent subsequence \((\nu_{k_j})\) with

\[
(\forall f \in C_b(G)) \quad \nu_{k_j}f = \mathbb{E} \left[ \frac{1}{k_j} \sum_{i=1}^{k_j} f(X_i) \right] \to \pi f, \quad \text{as } j \to \infty. \]

Convergence of the whole sequence, i.e. \(\nu_k \to \pi\), amounts then to showing that \(\pi\) is the unique
cluster point of \((\nu_k)\) (see Proposition A.1).

Quantifying convergence is essential for establishing estimates for the distance of the iterates
to the limit point, when this exists.

**Definition 2.6** (R- and Q-linear convergence to points, Chapter 9 of [70]). Let \((x_k)\) be a
sequence in a metric space \((G,d)\).

1. \((x_k)_{k \in \mathbb{N}}\) is said to **converge R-linearly** to \(\bar{x}\) with rate \(c \in [0,1)\) if there is a constant \(\beta > 0\)
such that
   \[
   d(x_k, \bar{x}) \leq \beta c^k \quad \forall k \in \mathbb{N}. \quad (7)
   \]

2. \((x_k)_{k \in \mathbb{N}}\) is said to **converge Q-linearly** to \(\bar{x}\) with rate \(c \in [0,1)\) if
   \[
   d(x_{k+1}, \bar{x}) \leq cd(x_k, \bar{x}) \quad \forall k \in \mathbb{N}. \]
By definition, Q-linear convergence implies R-linear convergence with the same rate; the converse implication does not hold in general. Q-linear convergence is encountered with contractive fixed point mappings, and this leads to a priori and a posteriori error estimates on the sequence. This type of convergence is referred to as geometric or exponential convergence in different communities. The crucial distinction between R-linear and Q-linear convergence is that R-linear convergence permits neither a priori nor a posteriori error estimates.

Common metrics for spaces of measures are the Prokhorov-Lèvy distance and the Wasserstein metric.

**Definition 2.7** (Prokhorov-Lèvy&Wasserstein distance). Let \((G, d)\) be a separable complete metric space and let \(\mu, \nu \in \mathcal{P}(G)\).

(i) The Prokhorov-Lèvy distance, denoted by \(d_P\), is defined by

\[
d_P(\mu, \nu) = \inf \{\varepsilon > 0 \mid \mu(A) \leq \nu(B(A, \varepsilon)) + \varepsilon, \nu(A) \leq \mu(B(A, \varepsilon)) + \varepsilon \quad \forall A \in B(G)\}.
\]

(ii) For \(p \geq 1\) let

\[
\mathcal{P}_p(G) = \left\{\mu \in \mathcal{P}(G) \mid \exists x \in G : \int d^p(x,y)\mu(dy) < \infty\right\}.
\]

The Wasserstein \(p\)-metric on \(\mathcal{P}_p(G)\), denoted \(W_p\), is defined by

\[
W_p(\mu, \nu) := \left(\inf_{\gamma \in C(\mu, \nu)} \int_{G \times G} d^p(x,y)\gamma(dx,dy)\right)^{1/p} \quad (p \geq 1)
\]

where \(C(\mu, \nu)\) is the set of couplings of \(\mu\) and \(\nu\) (measures on the product space \(G \times G\) whose marginals are \(\mu\) and \(\nu\) respectively – see (80)).

### 2.4. Regularity

Our main results concern convergence of Markov chains under increasingly restrictive regularity assumptions on the mappings \(\{T_i\}\). The regularity of \(T_i\) is dictated by the application, and our primary interest is to follow this through to the regularity of the corresponding Markov operator. In [61] a framework was developed for a quantitative convergence analysis of set-valued mappings \(T_i\) that are calm (one-sided Lipschitz continuous – in the sense of set-valued-mappings – with Lipschitz constant greater than 1). A set-valued self-mapping on a metric space \((G, d)\) is denoted \(T : G \rightrightarrows G\). This setting includes, for instance, applications involving feasibility – consistent and inconsistent – as well as many randomized algorithms for large-scale optimization, convex and nonconvex. Studies concurrent with the present one define the regularity of fixed point mappings in \(p\)-uniformly convex spaces \((p \in (1, \infty))\) with parameter \(c > 0\) [13, 55]. These are uniquely geodesic metric spaces \((G, d)\) for which the following inequality holds [66]:

\[
(\forall t \in [0, 1])(\forall x, y, z \in G) \quad d(z, (1-t)x \oplus ty)^p \leq (1-t)d(z, x)^p + td(z, y)^p - \frac{c}{p}t(1-t)d(x, y)^p
\]

where \(w = (1-t)x \oplus ty\) for \(t \in (0, 1)\) denotes the point \(w\) on the geodesic connecting \(x\) and \(y\) such that \(d(w, x) = td(x, y)\). The constant \(c\) is tied to the curvature and the diameter of the space. When \(p = c = 2\), this inequality defines a CAT(0) space (Alexandrov [1] and Gromov [39]), the completion of which defines a Hadamard space. More generally, a CAT(\(\kappa\)) space is a geodesic metric space with sufficiently small triangles possessing comparison triangles with sides the same length as the geodesic triangle but for which the distance between points on the geodesic
triangle are less than or equal to the distance between corresponding points on the comparison triangle. CAT(κ) spaces are separable, but not complete, and locally 2-uniformly convex with parameter c approaching the value 2 from below as the diameter of the local neighborhood vanishes [68, Proposition 3.1].

To keep the notation simple, we will restrict ourselves to CAT(κ) spaces (that is, the case p = 2), though we note that the exponents in the definition below are a consequence of this choice of p.

**Definition 2.8** (pointwise almost (α-firmly) nonexpansive mappings in CAT(κ) metric spaces). Let \((G, d)\) be a CAT(κ) metric space and \(D \subset G\) and let \(F : D \rightrightarrows G\).

(i) The mapping \(F\) is said to be pointwise almost nonexpansive at \(x_0 \in D\) on \(D\), abbreviated pointwise ane, whenever

\[
\exists \epsilon \in [0, 1) : d(x^+, x_0^+) \leq \sqrt{1+\epsilon} d(x, x_0), \quad \forall x \in D, \forall x^+ \in Fx, x_0^+ \in Fx_0. \tag{12}
\]

The violation is a value of \(\epsilon\) for which (12) holds. When the above inequality holds for all \(x_0 \in D\) then \(F\) is said to be almost nonexpansive on \(D\) (ane). When \(\epsilon = 0\) the mapping \(F\) is said to be (pointwise) nonexpansive.

(ii) The mapping \(F\) is said to be pointwise almost α-firmly nonexpansive at \(x_0 \in D\) on \(D\), abbreviated pointwise α-fne whenever

\[
\exists \epsilon \in [0, 1)\) and \(\alpha \in (0, 1) : d^2(x^+, x_0^+) \leq (1 + \epsilon) d^2(x, x_0) - \frac{1-\alpha}{\alpha} \psi_c(x, x_0, x^+, x_0^+) \tag{13}
\]

for all \(x \in D\), \(\forall x^+ \in Fx, x_0^+ \in Fx_0\), where the transport discrepancy \(\psi_c\) of \(F\) at \(x, x_0, x^+ \in Fx\) and \(x_0^+ \in Fx_0\) is defined by

\[
\psi_c(x, x_0, x^+, x_0^+) := \frac{\epsilon}{2} \left( d^2(x^+, x) + d^2(x_0^+, x_0) + d^2(x^+, x_0^+) + d^2(x, x_0) - d^2(x^+, x_0) - d^2(x, x_0^+) \right). \tag{14}
\]

When the above inequality holds for all \(x_0 \in D\) then \(F\) is said to be almost α-firmly nonexpansive on \(D\), (α-fne). The violation is the constant \(\epsilon\) for which (13) holds. When \(\epsilon = 0\) the mapping \(F\) is said to be (pointwise) α-firmly nonexpansive, abbreviated (pointwise) α-fne.

The transport discrepancy \(\psi_c\) is a central object for characterizing the regularity of mappings in metric spaces and ties the regularity of the mapping to the geometry of the space. The parameter \(c\) is determined by the curvature and the diameter of the space, \((G, d)\). The following lemma is derived from [12, pp. 94].

**Lemma 2.9** (\(\psi_2\) is nonnegative in CAT(0) spaces). Let \((G, d)\) be a CAT(0) metric space and \(F : D \rightrightarrows G\) for \(D \subset G\). Then the transport discrepancy defined by (14) is nonnegative for all \(x, y \in D, x^+ \in Fx, y^+ \in Fy\). Moreover, if \(F\) is pointwise α-fne at \(x_0 \in D\) with violation \(\epsilon\) on \(D\), then \(F\) is pointwise ane at \(x_0\) on \(D\) with violation at most \(\epsilon\).

**Proof.** For any self-mapping on a CAT(0) space, the following four-point inequality holds at any points \(x, y, u, v\) [49, Theorem 2.3.1]:

\[
d^2(u, y) + d^2(x, v) - d^2(u, x) - d^2(v, y) \leq 2d(u, v)d(x, y). \tag{15}
\]

Thus \(\psi_2(x, x_0, x^+, x_0^+)\) defined by (14) (with \(c = 2\)) is nonnegative in this setting. It follows immediately from the definition (13), then, that in CAT(0) spaces pointwise α-fne mappings are pointwise ane with at most the same violation. \(\square\)
In CAT(κ) spaces the above statement does not hold. It is well known, for example, that in a CAT(κ) metric space the projector onto a convex set is α-fne with α = 1/2, but it is not nonexpansive [3].

The definition of pointwise α-fne mappings generalizes the same notions developed in [61] for Euclidean spaces. The notion of averaged mappings dates back to Mann, Krasnoselskii, and others [21, 35, 40, 52, 62], while the name “averaged” seemed to stick from [7]. We depart from this tradition because it does not fit with nonlinear spaces.

In normed linear spaces, Baillon and Bruck [6] showed that nonexpansive mappings whose orbits are bounded under convex relaxations are asymptotically regular with a universal rate constant. Precisely: let \( T : D \to D \) be nonexpansive, where \( D \) is a convex subset of a normed linear space, and define \( x_m \) recursively by \( x_m = T x_{m-1} := ((1 - \lambda) \text{Id} + \lambda T) x_{m-1} \) for \( \lambda \in (0, 1) \) and \( x_0 \in D \). If \( \|x - T T^k \| \leq 1 \) for any \( \lambda \in (0, 1) \) and for all \( 0 \leq k \leq m \), then [6, Main Result]

\[
\|x_m - T x_m\| < \frac{\text{diam } D}{\sqrt{\pi m} \lambda (1 - \lambda)}.
\]

Cominetti, Soto and Vaisman [31] recently confirmed a conjecture of Baillon and Bruck that a universal rate constant also holds for nonexpansive mappings with arbitrary relaxation in \((0, 1)\) chosen at each iteration; in particular, that

\[
\|x_m - T x_m\| \leq \frac{\text{diam } D}{\sqrt{\pi} \sum_{k=1}^{m} \lambda_k (1 - \lambda_k)},
\]

where \( x_m \) is defined recursively by \( x_m = T \lambda x_{m-1} := ((1 - \lambda_m) \text{Id} + \lambda_m T) x_{m-1} \) for \( \lambda_m \in (0, 1) \) \((m = 1, 2, \ldots)\). The operators \( T \lambda_m \) all have the same set of fixed points (namely \( \text{Fix } T \)), so these results are complementary to [44] where it was shown [44, Theorem 3.5] that sequences of random variables on compact metric spaces generated by Algorithm 1 with paracontractions such as \( T \lambda_m \) above converge almost surely to a random variable in \( \text{Fix } T \), assuming that this is nonempty (see Proposition 2.5 in this context). Necessary and sufficient conditions for linear convergence of the iterates were also determined in a more limited setting in [44, Theorems 3.11 and 3.15]. The results of [6, 31, 44], however, do not apply to inconsistent stochastic feasibility considered here.

Nonexpansive mappings have been explored in nonlinear metric spaces for instance in [37] and in Hadamard spaces recently in [77]. Our definition for \( \alpha = 1/2 \) is equivalent (after some algebra) to the definition of firmly contractive mappings given in [20, Definition 6] for Hilbert spaces (see (18) below). The notion of \( \lambda \)-firmly nonexpansive mappings was defined in [3] in the context of \( W \)-hyperbolic spaces. This nomenclature was appropriated in [13] where it was shown that \( \lambda \)-firmly nonexpansive mappings are \( \alpha \)-fne, though the converse does not hold in general [13, Proposition 4]. Rates of asymptotic regularity of compositions of firmly nonexpansive mappings on \( p \)-uniformly convex spaces have been established in [4, Theorem 3.2].

The violation \( \epsilon \) in (12) and (13) is a recently introduced feature in the analysis of fixed point mappings, first appearing in this form in [61]. It is interesting to note that in the same article [6] where Baillon and Bruck showed (16) for nonexpansive mappings, they also observed that mappings with Lipschitz constant greater than one also behave nicely, and conjectured that something similar was also possible for this case. Indeed, the analysis of [61] shows how this works, though something like a universal constant has not been explored. Many are familiar with mappings for which (12) holds with \( \epsilon < 0 \) at all \( x_0 \in G \), i.e. contraction mappings. In this case, the whole technology of pointwise \( \alpha \)-fne mappings is not required since an appropriate application of Banach’s fixed point theorem delivers existence of fixed points and convergence of fixed point iterations at a linear rate. We will have more to say about this later; for the moment
it suffices to note that the mappings associated with our target applications are expansive on all neighborhoods of fixed points and we will therefore require another property to guarantee convergence.

When $\|\cdot\|$ is the norm induced by the inner product and $d(x,y) = \|x - y\|$, the transport discrepancy $\psi_2$ defined by (14) has the representation

$$\psi_2(x,x_0,x_0^+,y) = \|(x - x^+) - (x_0 - x_0^+)\|^2. \quad (17)$$

This representation shows the connection between our definition and more classical notions. Indeed, in a Hilbert space setting $(G,\|\cdot\|)$, a set-valued mapping $F : D \rightrightarrows G$ ($D \subset G$) is pointwise $\alpha$-fne at $x_0$ with constant $\alpha$ and violation at most $\epsilon$ on $D$ if and only if

$$\|x^+ - x_0^+\|^2 \leq (1 + \epsilon)\|x - x_0\|^2 - \frac{1-\alpha}{\alpha}\|(x - x^+) - (x_0 - x_0^+)\|^2 \quad \forall x \in D, \forall x^+ \in Fx, \forall x_0^+ \in Fx_0. \quad (18)$$

In the stochastic setting, to ease the notation and avoid certain technicalities, we will consider single-valued mappings $T_i$ that are only almost $\alpha$-firmly nonexpansive in expectation. We can therefore write $x^+ = T_i x$ instead of always taking some selection $x^+ = T_i x$ and verifying the desired properties over all (measurable) selections, assuming these exist [94]. The next definition uses the update function $\Phi$ defined in Assumption 2.1(b).

**Definition 2.10** (pointwise almost ($\alpha$-firmly) nonexpansive in expectation). Let $(G,d)$ be a $p$-uniformly convex metric space with constant $c$, let $T_i : G \to G$ for $i \in I$, and let $\Phi : G \times I \to G$ be given by $\Phi(x,i) = T_i x$. Let $\psi_c$ be defined by (14) and let $\xi$ be an $I$-valued random variable.

(i) The mapping $\Phi$ is said to be **pointwise almost nonexpansive in expectation at $x_0 \in G$ on $G$**, abbreviated **pointwise one in expectation**, whenever

$$\exists \epsilon \in [0,1) : \quad E[d(\Phi(x,\xi),\Phi(x_0,\xi))] \leq \sqrt{1+\epsilon} d(x,x_0), \quad \forall x \in G. \quad (19)$$

When the above inequality holds for all $x_0 \in G$ then $\Phi$ is said to be **almost nonexpansive – one – in expectation on $G$**. As before, the violation is a value of $\epsilon$ for which (19) holds. When the violation is 0, the qualifier “almost” is dropped.

(ii) The mapping $\Phi$ is said to be **pointwise almost $\alpha$-firmly nonexpansive in expectation at $x_0 \in G$ on $G$**, abbreviated **pointwise $\alpha$-fne in expectation**, whenever

$$\exists \epsilon \in [0,1), \alpha \in (0,1) : \quad \forall x \in G, \quad \exists \epsilon \in [0,1) : \quad E\left[d^2(\Phi(x,\xi),\Phi(x_0,\xi))\right] \leq (1 + \epsilon)d^2(x,x_0) - \frac{1-\alpha}{\alpha} E[\psi_c(x,x_0,\Phi(x,\xi),\Phi(x_0,\xi))] \quad (20)$$

When the above inequality holds for all $x_0 \in G$ then $\Phi$ is said to be **almost $\alpha$-firmly nonexpansive ($\alpha$-fne) in expectation on $G$**. The violation is a value of $\epsilon$ for which (20) holds. When the violation is 0, the qualifier “almost” is dropped and the abbreviation $\alpha$-fne in expectation is used.

**Proposition 2.11.** Let $(G,d)$ be a CAT(0) space. The mapping $\Phi : G \times I \to G$ given by $\Phi(x,i) = T_i x$ is pointwise $\alpha$-fne in expectation at $y$ on $G$ with constant $\alpha$ and violation at most $\epsilon$ and pointwise one in expectation at $y$ on $G$ with violation at most $\epsilon$ whenever $T_i$ is pointwise $\alpha$-fne at $y$ on $G$ with constant $\alpha$ and violation no greater than $\epsilon$ for all $i$. 

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Proof. By Lemma 2.9, whenever \((G,d)\) is a CAT(0) space \((p\)-uniformly convex with \(p = 2\) and \(c = 2\)) \(\psi_2(x,y,\Phi(x,x),\Phi(y,y)) \geq 0\) for all \(i\) and for all \(x,y \in G\), so the expectation \(\mathbb{E}[\psi_2(x,y,\Phi(x,\xi),\Phi(y,\xi))]\) is well-defined and nonnegative for all \(x,y \in G\) \((\text{the value } +\infty \text{ can be attained})\). This implies that, for all \(i\), \(T_i\) is pointwise \(\alpha\)-fne at \(y\) on \(G\) with violation at most \(\epsilon\) on \(G\) whenever it is pointwise \(\alpha\)-fne at \(y\) with constant \(\alpha\) on \(G\) with violation at most \(\epsilon\) on \(G\) for all \(i\). It follows immediately from the definition, then, that \(\Phi\) is pointwise \(\alpha\)-fne in expectation at \(y\) with constant \(\alpha\) on \(G\) with violation at most \(\epsilon\) on \(G\), and also pointwise \(\alpha\)-fne in expectation at \(y\) on \(G\) with violation at most \(\epsilon\) on \(G\). \(\square\)

Lifting these notions of regularity to Markov operators yields an analogous definition on the space of measures which hinges on the update function \(\Phi\). To facilitate the discussion we denote the set of couplings where the distance \(W_2(\mu_1,\mu_2)\) is attained by

\[
C_*(\mu_1,\mu_2) := \left\{ \gamma \in C(\mu_1,\mu_2) \mid \int_{G \times G} d^2(x,y) \gamma(dx,dy) = W_2^2(\mu_1,\mu_2) \right\}.
\]

Note that by Lemma A.7(ii) this set is nonempty when \(W_2(\mu_1,\mu_2)\) is finite.

Definition 2.12 (pointwise almost \((\alpha\)-firmly) nonexpansive Markov operators). Let \((G,d)\) be a CAT\((\kappa)\) metric space, and let \(\mathcal{P}\) be a Markov operator with transition kernel

\[
(x \in G)(A \in \mathcal{B}(G)) \quad p(x,A) := \mathbb{P}(\Phi(x,\xi) \in A)
\]

where \(\xi\) is an \(I\)-valued random variable and \(\Phi : G \times I \to G\) is a measurable update function. Let \(\psi_\epsilon\) be defined by \((14)\).

(i) The Markov operator is said to be pointwise almost nonexpansive in measure at \(\mu_0 \in \mathcal{P}(G)\) on \(\mathcal{P}(G)\), abbreviated pointwise \(\epsilon\)-fne in measure, whenever

\[
\exists \epsilon \in [0,1] : \quad W_2(\mu \mathcal{P},\mu_0 \mathcal{P}) \leq \sqrt{1 + \epsilon} W_2(\mu,\mu_0), \quad \forall \mu \in \mathcal{P}(G). \tag{22}
\]

When the above inequality holds for all \(\mu_0 \in \mathcal{P}(G)\) then \(\mathcal{P}\) is said to be almost nonexpansive \((\text{ane})\) in measure on \(\mathcal{P}(G)\). As before, the violation is a value of \(\epsilon\) for which \((22)\) holds. When the violation is 0, the qualifier “almost” is dropped.

(ii) The Markov operator \(\mathcal{P}\) is said to be pointwise almost \(\alpha\)-firmly nonexpansive in measure at \(\mu_0 \in G\) on \(\mathcal{P}(G)\), abbreviated pointwise \(\alpha\)-fne in measure, whenever

\[
\exists \epsilon \in [0,1], \alpha \in (0,1) : \quad \forall \mu \in \mathcal{P}(G), \forall \gamma \in C_*(\mu,\mu_0)
\]

\[
W_2(\mu \mathcal{P},\mu_0 \mathcal{P})^2 \leq (1 + \epsilon)W_2(\mu,\mu_0)^2 - \frac{1-\alpha}{\alpha} \int_{G \times G} \mathbb{E}[\psi_\epsilon(x,y,\Phi(x,\xi),\Phi(y,\xi))] \gamma(dx,dy). \tag{23}
\]

When the above inequality holds for all \(\mu_0 \in \mathcal{P}(G)\) then \(\mathcal{P}\) is said to be \(\alpha\)-fne in measure on \(\mathcal{P}(G)\). The violation is a value of \(\epsilon\) for which \((23)\) holds. When the violation is 0, the qualifier “almost” is dropped and the abbreviation \(\alpha\)-fne in measure is employed.

Remark 2.13: By Lemma 2.9, when \((G,d)\) is a CAT(0) space the expectation on the right hand side of \((23)\) is nonnegative, and the corresponding Markov operator is pointwise \(\alpha\)-fne in measure at \(\mu_0\) whenever it is pointwise \(\alpha\)-fne in measure at \(\mu_0\) \((\text{Proposition 2.11})\). In particular, when \(\mu = \mu_0 \in \text{inv} \mathcal{P}\) the left hand side is zero and

\[
\int_{G \times G} \mathbb{E}[\psi_2(x,y,\xi x,\xi y)] \gamma(dx,dy) = 0.
\]

Here the optimal coupling is the diagonal of the product space \(G \times G\) and \(\psi_2(x,x,\xi x,\xi x) = 0\) for all \(x \in G\).
Proposition 2.14. Let \((G, d)\) be a separable complete CAT(\(\kappa\)) metric space and \(T_i : G \to G\) for \(i \in I\), let \(\Phi : G \times I \to G\) be given by \(\Phi(x, i) = T_i x\) and let \(\psi_c\) be defined by (14). Denote by \(P\) the Markov operator with update function \(\Phi\) and transition kernel \(p\) defined by (4). If \(\Phi\) is \(\alpha\)-fne in expectation on \(G\) with constant \(\alpha \in (0, 1)\) and violation \(\epsilon \in [0, 1)\), then the Markov operator \(P\) is \(\alpha\)-fne in measure on \(\mathcal{P}_2(G)\) with constant \(\alpha\) and violation at most \(\epsilon\), that is, \(P\) satisfies

\[
W_2^2(\mu_1 P, \mu_2 P) \leq (1 + \epsilon)W_2^2(\mu_1, \mu_2) - \frac{1 - \alpha}{\alpha} \int_{G \times G} \mathbb{E} [\psi_c(x, y, \Phi(x, \xi), \Phi(y, \xi))] \gamma(dx, dy)
\]

\(\forall \mu_2, \mu_1 \in \mathcal{P}_2(G), \forall \gamma \in C_\epsilon(\mu_1, \mu_2). \tag{24}\)

**Proof.** If \(W_2(\mu_1, \mu_2) = \infty\) the inequality holds trivially with the convention \(+ \infty - (+ \infty) = + \infty\). So consider the case where \(W_2(\mu_1, \mu_2)\) is finite. Since \((G, d)\) is a separable, complete metric space, by Lemma A.7(ii), the set of optimal couplings \(C_\epsilon(\mu_1, \mu_2)\) is nonempty. Since \(\Phi\) is \(\alpha\)-fne in expectation on \(G\) with constant \(\alpha\) and violation \(\epsilon\), we have

\[
\int_{G \times G} \mathbb{E} \left[ d^2(\Phi(x, \xi), \Phi(y, \xi)) \right] \gamma(dx, dy) \leq \int_{G \times G} \left( (1 + \epsilon )d^2(x, y) - \frac{1 - \alpha}{\alpha} \mathbb{E} [\psi_c(x, y, \Phi(x, \xi), \Phi(y, \xi))] \right) \gamma(dx, dy),
\]

where \(\gamma\) is any coupling in \(C_\epsilon(\mu_1, \mu_2)\), not necessarily optimal. In particular, since, for a random variable \(X \sim \mu_1\), we have \(\Phi(X, \xi) \sim \mu_1 P\), and for a random variable \(Y \sim \mu_2\), we have \(\Phi(Y, \xi) \sim \mu_2 P\), then, again for any optimal coupling \(\gamma \in C_\epsilon(\mu_1, \mu_2)\),

\[
W_2^2(\mu_1 P, \mu_2 P) \leq \int_{G \times G} \mathbb{E} \left[ d^2(\Phi(x, \xi), \Phi(y, \xi)) \right] \gamma(dx, dy)
\]

\leq \int_{G \times G} \left( (1 + \epsilon )d^2(x, y) - \frac{1 - \alpha}{\alpha} \mathbb{E} [\psi_c(x, y, \Phi(x, \xi), \Phi(y, \xi))] \right) \gamma(dx, dy)

= \left( 1 + \epsilon \right) W_2^2(\mu_1, \mu_2) - \int_{G \times G} \frac{1 - \alpha}{\alpha} \mathbb{E} [\psi_c(x, y, \Phi(x, \xi), \Phi(y, \xi))] \gamma(dx, dy).
\]

Since the measures \(\mu_2, \mu_1 \in \mathcal{P}_2(G)\) were arbitrary, as was the optimal coupling \(\gamma \in C_\epsilon(\mu_1, \mu_2)\), this completes the proof. \(\square\)

Contraction Markov operators have been studied in [50, 69] using the parallel notion of the **coarse Ricci curvature** \(\kappa(x, y)\) of the Markov operator \(\mathcal{P}\) between two points \(x\) and \(y\):

\[
\kappa(x, y) := 1 - \frac{W_1(\delta_x \mathcal{P}, \delta_y \mathcal{P})}{d(x, y)}.
\]

Generalizing this definition to \(W_p\) yields the coarse Ricci curvature with respect to \(W_p\):

\[
\kappa_p(x, y) := 1 - \frac{W_p^p(\delta_x \mathcal{P}, \delta_y \mathcal{P})}{d(x, y)^p}.
\]

A few steps lead from this object for the Markov operator \(\mathcal{P}\) with update function \(\Phi(\cdot, \xi) = T_\xi\) and transition kernel defined by (4) to the violation \(\epsilon\) in Proposition 2.14. Indeed, a formal adjustment of the proof of [69, Proposition 2] establishes that the property \(\kappa_2(x, y) \geq \kappa \in \mathbb{R}\) for all \(x, y \in G\) is equivalent to

\[
W_2(\mu \mathcal{P}, \mu' \mathcal{P}) \leq \sqrt{1 - \kappa} W_2(\mu, \mu') \quad \forall \mu, \mu' \in \mathcal{P}_2(G).
\]
When $\kappa > 0$, i.e. when the coarse Ricci curvature is bounded below by a positive number, this characterizes contractivity of the Markov operator. The negative of the violation in (22) is just a lower bound on the coarse Ricci curvature in $W_2$: $-\epsilon = \kappa \leq \kappa_2(x,y)$ for all $x,y \in G$. The consequences of Markov operators with Ricci curvature bounded below by a positive number have been extensively investigated. Our approach extends this to expansive mappings, which allows one to treat our target application of electron density reconstructions from X-FEL experiments (see Section 4).

In [61] a general quantitative analysis for iterations of expansive fixed point mappings is proposed consisting of two principle components: the constituent mappings are pointwise affine, and the transport discrepancy of the fixed point operator is $\Psi \in \mathbb{R}$, and the transport discrepancy of the fixed point operator is $\Psi : A \to B$. The inverse mapping $\Psi^{-1}(y) := \{ z \in A \mid \Psi(z) = y \}$, which clearly can be set-valued.

**Definition 2.15 (metric subregularity).** Let $(A,d_A)$ and $(B,d_B)$ be metric spaces and let $\Psi : A \to B$. The mapping $\Psi$ is called metrically subregular with respect to the metric $d_B$ for $y \in B$ relative to $A \subset A$ on $U \subset A$ with gauge $\rho$ whenever

$$\inf_{z \in \Psi^{-1}(y) \cap A} d_A(x,z) \leq \rho(d_B(y,\Psi(x))) \quad \forall x \in U \cap A. \quad (25)$$

Our definition is modelled after [34], where the case where the gauge is just a linear function $\rho(t) = \kappa t - \tau$ is developed. In this case, metric subregularity is one-sided Lipschitz continuity of the (set-valued) inverse mapping $\Psi^{-1}$. We will refer to the case when the gauge is linear to linear metric subregularity. For connections of this notion to the concept of transversality in differential geometry and its use in variational analysis see [47]. The main advantage of including the more general gauge function is to characterize sub-linear convergence rates of numerical methods. We apply metric regularity to the Markov operator on $\mathcal{P}(G)$ with the Wasserstein metric. In particular, the gauge of metric subregularity $\rho$ is constructed implicitly from another nonnegative function $\theta : [0,\infty) \to [0,\infty)$ satisfying

$$(i) \ \theta(0) = 0; \quad (ii) \ 0 < \theta(t) < t \ \forall t > 0; \quad (iii) \ \sum_{j=1}^{\infty} \theta^{(j)}(t) < \infty \ \forall t \geq 0. \quad (26)$$

For a CAT(0) space the operative gauge of metric subregularity satisfies

$$\rho \left( \left( \frac{\left( 1 + \epsilon \right)^2 - \left( \theta(t) \right)^2}{\tau} \right)^{1/2} \right) = t \quad \iff \quad \theta(t) = \left( (1 + \epsilon)^2 - \tau \left( \rho^{-1}(t) \right)^2 \right)^{1/2} \quad (27)$$

for $\tau > 0$ fixed and $\theta$ satisfying (26).

In the case of linear metric subregularity on a CAT(0) space this becomes

$$\rho(t) = \kappa t \quad \iff \quad \theta(t) = \left( (1 + \epsilon) - \frac{\tau}{\kappa^2} \right)^{1/2} t \quad (\kappa \geq \sqrt{\frac{\tau}{1+\epsilon}}).$$

The condition $\kappa \geq \sqrt{\frac{\tau}{1+\epsilon}}$ is not a real restriction since, if (25) is satisfied for some $\kappa' > 0$, then it is satisfied for all $\kappa \geq \kappa'$. The conditions in (26) in this case simplify to $\theta(t) = \gamma t$ where

$$0 < \gamma := 1 + \epsilon - \frac{\tau}{\kappa^2} < 1 \quad \iff \quad \sqrt{\frac{\tau}{1+\epsilon}} \leq \kappa \leq \sqrt{\frac{\tau}{\epsilon}}. \quad (28)$$

Metric subregularity plays a central role in the implicit function paradigm for solution mappings [16, 34]. Linear metric subregularity was shown in [44, Theorem 3.15] to be necessary and
sufficient for R-linear convergence in expectation of random function iterations for consistent stochastic feasibility. This result is a stochastic analog of the result [60, Theorem 2] in the deterministic setting.

We apply this to the Markov operator $\mathcal{P}$ on the metric space $(\mathcal{P}_2(G), W_2)$ in the following manner. Recall the transport discrepancy $\psi_c$ defined in (14). We construct the surrogate mapping $\Psi : \mathcal{P}(G) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$
\Psi(\mu) := \inf_{\pi \in \text{inv } \mathcal{P}} \inf_{\gamma \in \mathcal{C}_*(\mu, \pi)} \left( \int_{G \times G} \mathbb{E} [\psi_c(x, y, T_k x, T_k y)] \gamma(dx, dy) \right)^{1/2}.
$$

We call this the Markov transport discrepancy. It is not guaranteed that both $\text{inv } \mathcal{P}$ and $\mathcal{C}_*(\mu, \pi)$ are nonempty; when at least one of these is empty, we define $\Psi(\mu) := +\infty$. It is clear that $\Psi(\pi) = 0$ for any $\pi \in \text{inv } \mathcal{P}$. Whether $\Psi(\mu) = 0$ only when $\mu \in \text{inv } \mathcal{P}$ is a property of the space $(G, d)$. Indeed, as noted in the discussion after Lemma 2.9, in CAT($\kappa$) spaces with $\kappa > 0$ the transport discrepancy $\psi_c$ can be negative, and so by cancellation it could happen on such spaces that the Markov transport discrepancy $\Psi(\mu) = 0$ for $\mu \notin \text{inv } \mathcal{P}$. The regularity we require of $\mathcal{P}$ is that the Markov transport discrepancy $\Psi$ takes the value $0$ at $\mu$ if and only if $\mu \in \text{inv } \mathcal{P}$, and is metrically subregular for $0$ relative to $\mathcal{P}_2(G)$ on $\mathcal{P}_2(G)$ defined in (9).

Before moving to our main results, we put the more familiar contractive mappings into the present context. A survey of random function iterations for contractive mappings in expectation can be found in [85]. An immediate consequence of [85, Theorem 1] is the existence of a unique invariant measure and linear convergence in the Wasserstein metric from any initial distribution to the invariant measure. See also Example 4.3. Error estimates for Markov chain Monte Carlo methods under the assumption of positive Ricci curvature in $W_1$ (i.e. negative violation) are explored in [50]. Applications to waiting queues, the Ornstein–Uhlenbeck process on $\mathbb{R}^n$ and Brownian motion on positively curved manifolds, as well as demonstrations of how to verify the assumptions on the Ricci curvature are developed in [69]. The next result shows that update functions $\Phi$ that are contractions in expectation generate $\alpha$-fine Markov operators with metrically subregular Markov transport discrepancy.

**Theorem 2.16.** Let $(G, \| \cdot \|)$ be a Hilbert space, let $T_i : G \to G$ for $i \in I$ and let $\Phi : G \times I \to G$ be given by $\Phi(x, i) := T_i(x)$. Denote by $\mathcal{P}$ the Markov operator with update function $\Phi$ and transition kernel $p$ defined by (4). Suppose that $\Phi$ is a contraction in expectation with constant $r < 1$, i.e. $\mathbb{E} [\| \Phi(x, \xi) - \Phi(y, \xi) \|^2] \leq r^2 \| x - y \|^2$ for all $x, y \in G$. Suppose in addition that there exists $y \in G$ with $\mathbb{E} [\| \Phi(y, \xi) - y \|^2] < \infty$. Then the following hold.

(i) There exists a unique invariant measure $\pi \in \mathcal{P}_2(G)$ for $\mathcal{P}$ and

$$
W_2(\mu_0 \mathcal{P}^n, \pi) \leq r^n W_2(\mu_0, \pi)
$$

for all $\mu_0 \in \mathcal{P}_2(G)$; that is, the sequence $(\mu_k)$ defined by $\mu_{k+1} = \mu_k \mathcal{P}$ converges to $\pi$ Q-linearly (geometrically) from any initial measure $\mu_0 \in \mathcal{P}_2(G)$.

(ii) $\Phi$ is $\alpha$-fine in expectation with constant $\alpha = (1+r)/2$, and the Markov operator $\mathcal{P}$ is $\alpha$-fine on $\mathcal{P}_2(G)$; that is, $\mathcal{P}$ satisfies (24) with $\epsilon = 0$ and constant $\alpha = (1+r)/2$ on $\mathcal{P}_2(G)$.

(iii) If $\Psi$ defined by (29) satisfies

$$
\exists q > 0 : \quad \Psi(\mu) \geq q W_2(\mu \mathcal{P}, \mu) \quad \forall \mu \in \mathcal{P}_2(G),
$$

then $\Psi$ is linearly metrically subregular for $0$ relative to $\mathcal{P}_2(G)$ on $\mathcal{P}_2(G)$ with gauge $\rho(t) = (q(1-r))^{-1} t$. 

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Proof. Note that for any pair of distributions \( \mu_1, \mu_2 \in \mathcal{P}_2(G) \) and an optimal coupling \( \gamma \in C_*(\mu_1, \mu_2) \) (possible by Lemma A.7) it holds that
\[
W_2^2(\mu_1 \mathbb{P}, \mu_2 \mathbb{P}) \leq \int_{G \times G} \mathbb{E}[d^2(\Phi(x, \xi), \Phi(y, \xi))] \gamma(dx, dy) \\
\leq r^2 \int_{G \times G} d^2(x, y) \gamma(dx, dy) = r^2 W_2^2(\mu_1, \mu_2),
\]
where \( \xi \) is independent of \( \gamma \). Moreover, \( \mathbb{P} \) is a self-mapping on \( \mathcal{P}_2(G) \). To see this let \( \mu \in \mathcal{P}_2(G) \) independent of \( \xi \) and let \( y \) be a point in \( G \) where \( \mathbb{E}[\|\Phi(y, \xi) - y\|^2] < \infty \). Then by the triangle inequality and the contraction property
\[
\int_G \mathbb{E}[\|\Phi(x, \xi) - y\|^2] \mu(dx) \\
\leq 4 \left( \int_G \mathbb{E}[\|\Phi(x, \xi) - \Phi(y, \xi)\|^2] \mu(dx) + \mathbb{E}[\|\Phi(y, \xi) - y\|^2] \right) \\
\leq 4 \left( \int_{G \times G} \|x - y\|^2 \mu(dx) + \mathbb{E}[\|\Phi(y, \xi) - y\|^2] \right) < \infty.
\]
Therefore \( \mu \mathbb{P} \in \mathcal{P}_2(G) \). Altogether, this establishes that \( \mathbb{P} \) is a contraction on the separable complete metric space \( (\mathcal{P}_2(G), W_2) \) and hence Banach’s Fixed Point Theorem yields existence and uniqueness of \( \text{inv} \mathbb{P} \) and \( \text{Q-linear convergence of the fixed point sequence.} \)

To see (ii), note that, by (17),
\[
\mathbb{E}[\psi_2(x, y, T_x \xi, T_y \xi)] = \mathbb{E}\left[\|x - \Phi(x, \xi) - (y - \Phi(y, \xi))\|^2\right] \\
= \|x - y\|^2 + \mathbb{E}\left[\|\Phi(x, \xi) - \Phi(y, \xi)\|^2 - 2\langle x - y, \Phi(x, \xi) - \Phi(y, \xi) \rangle\right] \\
\leq (1 + r)^2\|x - y\|^2,
\]
where the last inequality follows from the Cauchy-Schwarz inequality and the fact that \( \Phi(\cdot, \xi) \) is a contraction in expectation. Again using the contraction property and (31) we have
\[
\mathbb{E}\left[\|\Phi(x, \xi) - \Phi(y, \xi)\|^2\right] \leq \|x - y\|^2 - (1 - r^2)\|x - y\|^2 \\
\leq \|x - y\|^2 - \frac{1 - r^2}{1 + r^2}\mathbb{E}[\psi_2(x, y, T_x \xi, T_y \xi)].
\]
The right hand side of this inequality is just the characterization (20) of mappings that are \( \alpha \)-fine in expectation with \( \alpha = (1 + r)/2 \). The rest of the statement follows from Proposition 2.14.

(iii) The proof is modeled after the proof of [13, Theorem 32]. By the triangle inequality and part (i) we have
\[
W_2(\mu_{k+1}, \mu_k) \geq W_2(\mu_k, \pi) - W_2(\mu_{k+1}, \pi) \\
\geq (1 - r)W_2(\mu_k, \pi) \quad \forall k \in \mathbb{N}.
\]

On the other hand, (30) implies that \( \Psi \) takes the value zero only at invariant measures so that by the uniqueness of invariant measures established in part (i)
\[
\Psi^{-1}(0) \cap \mathcal{P}_2(G) = \text{inv} \mathbb{P} \cap \mathcal{P}_2(G) = \{\pi\}.
\]
Combining this with (32) and (30) then yields for all \( k \in \mathbb{N} \)
\[
|\Psi(\mu_k) - 0| = \Psi(\mu_k) \geq qW_2(\mu_{k+1}, \mu_k) \\
\geq q(1 - r)W_2(\mu_k, \Psi^{-1}(0) \cap \mathcal{P}_2(G)).
\]

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In other words,
\[(q(1-r))^{-1} |\Psi(\mu_k) - 0| \geq W_2(\mu_k, \Psi^{-1}(0) \cap \mathcal{P}_2(G)) \quad \forall k \in \mathbb{N}. \quad (33)\]

Since this holds for any sequence \((\mu_k)_{k \in \mathbb{N}}\) initialized with \(\mu_0 \in \mathcal{P}_2(G)\), we conclude that \(\Psi\) is metrically subregular for 0 relative to \(\mathcal{P}_2(G)\) with gauge \(\rho(t) = (q(1-r))^{-1}t\) on \(\mathcal{P}_2(G)\), as claimed. \(\square\)

The simple example of a single Euclidean projector onto an affine subspace \((I = \{1\}, \text{ and } T_1 \text{ the orthogonal projection onto an affine subspace})\) shows that the statement of Theorem 2.16 fails without the assumption of contractivity.

### 2.5. Main Results

All of our main results concern Markov operators \(\mathcal{P}\) with update function \(\Phi(x, i) = T_i(x)\) and transition kernel \(p\) given by (4) for self mappings \(T_i : G \to G\). For any \(\mu_0 \in \mathcal{P}_2(G)\), we denote the distributions of the iterates of Algorithm 1 by \(\mu_k = \mu_0 \mathcal{P}^k = \mathcal{L}(X_k)\), and we denote \(d_{W_2}(\mu_k, \text{inv } \mathcal{P}) := \inf_{\pi' \in \text{inv } \mathcal{P}} W_2(\mu_k, \pi')\).

In most of our main results, it will be assumed that \(\text{inv } \mathcal{P} \neq \emptyset\). The existence theory is already well developed and is surveyed in Section 3.1 below. We show how existence is guaranteed when, for instance, the image is compact for some non-negligible collection of operators \(T_i\) (Proposition 3.2) or when the expectation of the random variables \(X_k\) is finite (Proposition 3.3).

The main convergence result for nonexpansive mappings follows from a fundamental result of Worm [96, Theorem 7.3.13].

**Theorem 2.17** (convergence of Cesàro average in \(\mathbb{R}^n\)). Let \(T_i : \mathbb{R}^n \to \mathbb{R}^n\) be nonexpansive \((i \in I)\) and assume \(\text{inv } \mathcal{P} \neq \emptyset\). Let \(\mu \in \mathcal{P}(\mathbb{R}^n)\) and \(\nu_k = \frac{1}{k} \sum_{j=1}^{k} \mu \mathcal{P}^j\), then this sequence converges in the Prokhorov-Lévy metric to an invariant probability measure for \(\mathcal{P}\), i.e. for each \(x \in \text{supp } \mu \subset \mathbb{R}^n\) the limit of the sequence \((\nu_k^x)\), denoted \(\pi^x\), exists and, more generally, \(\nu_k \to \pi^\mu\), an invariant measure, where
\[
\pi^\mu = \int_{\text{supp } \mu} \pi^x \mu(dx). \quad (34)
\]

When the mappings are \(\alpha\)-fne, we obtain the following stronger result. It is worth pointing interested readers to an analogous metric space result of [13, Theorem 27] in which it is shown that, on \(\rho\)-uniformly convex spaces, sequences generated by fixed point iterations of compositions of pointwise \(\alpha\)-fne mappings \(T_i\) converge in a weak sense whenever \(\cap_i \text{Fix } T_i\) is nonempty. When the composition is boundedly compact, then the fixed point iterations converge strongly to a fixed point.

**Theorem 2.18** (convergence for \(\alpha\)-firmly nonexpansive mappings on \(\mathbb{R}^n\)). Let \(T_i : \mathbb{R}^n \to \mathbb{R}^n\) be \(\alpha\)-fne with constant \(\alpha_i \leq \alpha < 1\) \((i \in I)\). Assume \(\text{inv } \mathcal{P} \neq \emptyset\). For any initial distribution \(\mu_0 \in \mathcal{P}(\mathbb{R}^n)\) the distributions \(\mu_k\) of the iterates generated by Algorithm 1 converge in the Prokhorov-Lévy metric to an invariant probability measure for \(\mathcal{P}\).

The proof of this result is very different than the strategy applied to the metric space result of [13, Theorem 27].

Trading the weaker assumption that the mappings \(T_i\) are only \(\alpha\)-fne against the assumption of metric subregularity of the Markov transport discrepancy \(\Psi\) yields rates of convergence by lifting the results of [61, Corollary 2.3] to the space of probability measures.
Theorem 2.19 (convergence rates). Let \((H,d)\) be a separable Hadamard space and let \(G \subset H\) be compact. Let \(T_i : G \to G\) be continuous for all \(i \in I\) and define \(\Psi : \mathcal{P}_2(G) \to \mathbb{R}_+ \cup \{+\infty\}\) by (29). Assume furthermore:

(a) there is at least one \(\pi \in \text{inv } \mathcal{P} \cap \mathcal{P}_2(G)\) where \(\mathcal{P}\) is the Markov operator with update function \(\Phi\) given by (4);

(b) \(\Phi\) is \(ac\)-fine in expectation with constant \(\alpha \in (0,1)\) and violation at most \(\epsilon\); and

(c) \(\Psi\) takes the value 0 only at points \(\pi \in \text{inv } \mathcal{P}\) and is metrically subregular (in the \(W_2\) metric) for 0 relative to \(\mathcal{P}_2(G)\) on \(\mathcal{P}_2(G)\) with gauge \(\rho\) given by (27) where \(\tau = (1 - \alpha)/\alpha\).

Then for any \(\mu_0 \in \mathcal{P}_2(G)\) the distributions \(\mu_k\) of the iterates of Algorithm 1 converge in the \(W_2\) metric to some \(\pi^{\mu_0} \in \text{inv } \mathcal{P} \cap \mathcal{P}_2(G)\) with rate characterized by

\[
d_{W_2} (\mu_{k+1}, \text{inv } \mathcal{P}) \leq \theta (d_{W_2} (\mu_k, \text{inv } \mathcal{P})) \quad \forall k \in \mathbb{N},
\]

where \(\theta\) given implicitly by (27) satisfies (26).

Remark 2.20: The compactness assumption on \(G\) can be dropped if \((H,d)\) is a Euclidean space.

An immediate corollary of this theorem is the following specialization to linear convergence.

Corollary 2.21 (linear convergence rates). Under the same assumptions as Theorem 2.19, if \(\Psi\) is linearly metrically subregular (i.e. with gauge \(\rho(t) = \kappa \cdot t\)) for 0 with constant \(\kappa\) satisfying

\[
\sqrt{\frac{1-\alpha}{1+\tau}} \leq \kappa < \sqrt{\frac{1 - \alpha}{1+\alpha}},
\]

then the sequence of iterates \((\mu_k)\) converges \(R\)-linearly to some \(\pi^{\mu_0} \in \text{inv } \mathcal{P} \cap \mathcal{P}_2(G)\):

\[
d_{W_2} (\mu_{k+1}, \text{inv } \mathcal{P}) \leq c d_{W_2} (\mu_k, \text{inv } \mathcal{P})
\]

where \(c := \sqrt{1 + \epsilon - \frac{(1-\alpha)^2}{\kappa^2 \alpha}} < 1\) and \(\kappa \geq \kappa'\) satisfies \(\kappa \geq \sqrt{(1-\alpha)/\alpha(1+\epsilon)}\). If \(\text{inv } \mathcal{P}\) consists of a single point then convergence is \(Q\)-linear.

3. Background Theory and Proofs

In this section we prepare tools to prove the main results from Section 2.5. We start by establishing convergence results on the supports of ergodic measures on a general Polish space \(G\), and then, for global convergence analysis, we restrict ourselves to \(\mathbb{R}^n\). We begin with existence of invariant measures. We then analyze properties of (and convergence of the RFI on) subsets of \(G\), called ergodic sets. Then we turn our attention to the global convergence analysis.

3.1. Existence of Invariant Measures

A sequence of probability measures \((\nu_k)\) is called \(tight\) if for any \(\epsilon > 0\) there exists a compact \(K \subset G\) with \(\nu_k(K) > 1 - \epsilon\) for all \(k \in \mathbb{N}\). By Prokhorov’s theorem (see, for instance, [14]), a sequence \((\nu_k) \subset \mathcal{P}(G)\), for \(G\) a Polish space, is tight if and only if \((\nu_k)\) is compact in \(\mathcal{P}(G)\), i.e. any subsequence of \((\nu_k)\) has a subsequence that converges in distribution (see, for instance [14]).

A basic building block is the existence of invariant measures proved by Lasota and T. Szarek [54, Proposition 3.1]. Based on this, we show how existence can be verified easily. But first, we show how to obtain existence constructively.
Proposition 3.1 (construction of an invariant measure). Let $\mu \in \mathcal{P}(G)$ and $\mathcal{P}$ be a Feller Markov operator. Let $(\mu^k)_{k \in \mathbb{N}}$ be a tight sequence of probability measures on a Polish space $G$, and let $\nu_k = \frac{1}{k} \sum_{j=1}^{k} \mu^j$. Any cluster point of the sequence $(\nu_k)_{k \in \mathbb{N}}$ is an invariant measure for $\mathcal{P}$.

Proof. Our proof follows [41, Theorem 1.10]. Tightness of the sequence $(\mu^k)$ implies tightness of the sequence $(\nu_k)$ and therefore by Prokhorov’s Theorem there exists a convergent subsequence $(\nu_{k_j})$ with limit $\pi \in \mathcal{P}(G)$. By the Feller property of $\mathcal{P}$ one has for any continuous and bounded $f : G \to \mathbb{R}$ that also $\mathcal{P}f$ is continuous and bounded, and hence

$$
| (\pi \mathcal{P} f - \pi f) | = | \pi (\mathcal{P} f) - \pi f | = \lim_j | \nu_{k_j} (\mathcal{P} f) - \nu_{k_j} f | = \lim_j \frac{1}{k_j} | \mu^{k_j + 1} f - \mu^k f | 
\leq \lim_j \frac{2 \| f \|_{\infty}}{k_j} = 0.
$$

Now, $\pi f = (\pi \mathcal{P} f)$ for all $f \in C_0(G)$ implies that $\pi = \pi \mathcal{P}$.

When a Feller Markov chain converges in distribution (i.e. $\mu^k \to \pi$), it does so to an invariant measure (since $\mu^k \to \pi \mathcal{P}$). A Markov operator need not possess a unique invariant probability measure or any invariant measure at all. Indeed, consider the normed space $(\mathbb{R}^n, \| \cdot \|)$ for the case that $T_i = P_i$, $i \in I$ is a projector onto a nonempty closed and convex set $C_i \subset \mathbb{R}^n$. A sufficient condition for the deterministic Alternating Projections Method to converge in the inconsistent case to a limit cycle for convex sets is that one of the sets is compact (this is an easy consequence of [28, Theorem 4]). Translating this into the present setting, a sufficient condition for the existence of an invariant measure for $\mathcal{P}$ is the existence of a compact set $K \subset \mathbb{R}^n$ and $\epsilon > 0$ such that $p(x, K) \geq \epsilon$ for all $x \in \mathbb{R}^n$. This holds, for instance, when there are only finitely many sets with one of them, say $C_\gamma$, compact and $\mathbb{P}(\xi = \gamma) = \epsilon$, since $p(x, C_\gamma) = \mathbb{P}(P_\gamma x \in C_\gamma) \geq \mathbb{P}(P_\gamma x \in C_\gamma, \xi = \gamma) = \mathbb{P}(\xi = \gamma) = \epsilon$ for all $x \in \mathbb{R}^n$. More generally, we have the following result.

Proposition 3.2 (existence of invariant measures for finite collections of continuous mappings). Let $G$ be a Polish space and let $T_i : G \to G$ be continuous for $i \in I$, where $I$ is a finite index set. If for one index $i \in I$ it holds that $\mathbb{P}(\xi = i) > 0$ and $T_i(G) \subset K$, where $K \subset G$ is compact, then there exists an invariant measure for $\mathcal{P}$.

Proof. We have from $T_i(G) \subset K$ that $\mathbb{P}(T_i x \in K) \geq \mathbb{P}(\xi = i)$ and hence for the sequence $(X_k)$ generated by Algorithm 1 for an arbitrary initial probability measure

$$
\mathbb{P}(X_{k+1} \in K) = \mathbb{E}[\mathbb{P}(T_{\xi_k} X_k \in K \mid X_k)] \geq \mathbb{P}(\xi = i) \quad \forall k \in \mathbb{N}.
$$

The assertion follows now immediately from [54, Proposition 3.1] since $\mathbb{P}(\xi = i) > 0$ and $\mathcal{P}$ is Feller by continuity of $T_j$ for all $j \in I$.

Next we mention an existence result which requires that the RFI sequence $(X_k)$ possess a uniformly bounded expectation.
Proposition 3.3 (existence in $\mathbb{R}^n$, RFI). Let $T_i : \mathbb{R}^n \to \mathbb{R}^n \ (i \in I)$ be continuous. Let $(X_k)$ be the RFI sequence (generated by Algorithm 1) for some initial measure. Suppose that for all $k \in \mathbb{N}$ it holds that $E[\|X_k\|] \leq M$ for some $M \geq 0$. Then there exists an invariant measure for the RFI Markov operator $P$ given by (4).

Proof. For any $\epsilon > M$ Markov’s inequality implies that

$$
\mathbb{P}(\|X_k\| \geq \epsilon) \leq \frac{E[\|X_k\|]}{\epsilon} \leq \frac{M}{\epsilon} < 1
$$

Hence,

$$
\limsup_{k \to \infty} \mathbb{P}(\|X_k\| \leq \epsilon) \geq \limsup_{k \to \infty} \mathbb{P}(\|X_k\| < \epsilon) \geq 1 - \frac{M}{\epsilon} > 0.
$$

Existence of an invariant measure then follows from [54, Proposition 3.1] since closed balls in $\mathbb{R}^n$ with finite radius are compact, $\mathbb{P}(X_k \in \cdot) = \mu P^k$ and continuity of $T_i$ yields the Feller property for $P$.

To conclude this section, we also establish that, for the setting considered here, the set of invariant measures is closed.

Lemma 3.4. Let $G$ be a Polish space and let $P$ be a Feller Markov operator, which is in particular the case under Assumption 2.1, if $T_i$ is continuous for all $i \in I$. Then the set of associated invariant measures $\text{inv} P$ is closed with respect to the topology of convergence in distribution.

Proof. Let $(\pi_n)_{n \in \mathbb{N}}$ be a sequence of measures in $\text{inv} P$ that converges in distribution to $\pi$. Thus we have $\pi_n P = \pi_n$ for all $n \in \mathbb{N}$ and we need to establish this also for the limiting measure $\pi$. For this it suffices to show that for all $f \in C_b(G)$,

$$
\int f(x)\pi(dx) = \int f(x)\pi P(dx) = \int f(x) \int p(y, dx)\pi(dy) = \int Pf(y)\pi(dy).
$$

This means that for $f \in C_b(G)$ we also have that $P f \in C_b(G)$. (By Proposition 2.4 we have that $P$ is Feller if Assumption 2.1 holds and $T_i$ is continuous for all $i \in I$.) Hence, for those $f$ we have that $\int f(x)\pi_n(dx) \to \int f(x)\pi(dx)$ as well as $\int Pf(x)\pi_n(dx) \to \int Pf(x)\pi(dx)$ which establishes the claim.

3.2. Ergodic theory of general Markov Operators

We understand here under ergodic theory the analysis of the properties of the RFI Markov chain when it is initialized by a distribution in the support of any ergodic measure for the Markov operator $P$. The convergence properties for these points can be much stronger than the convergence properties of Markov chains initialized by measures with support outside the support of the ergodic measures.

The consistent stochastic feasibility problem was analyzed in [44] without the need of the notion of convergence of measures since, as shown in Proposition 2.5, for consistent stochastic feasibility convergence of sequences defined by (3) is almost sure, if they converge at all. More general convergence of measures is more challenging as the next example illustrates.

Example 3.5 (nonexpansive mappings, negative result). For non-expansive mappings in general, one cannot expect that the sequence $(\mathcal{L}(X_k))_{k \in \mathbb{N}}$ converges to an invariant probability measure. Consider the nonexpansive operator $T := T_1 x := -x$ on $\mathbb{R}$ and set, in the RFI setup,
\( \xi = 1 \) and \( I = \{1\} \). Then \( X_{2k} = x \) and \( X_{2k+1} = -x \) for all \( k \in \mathbb{N} \), if \( X_0 \sim \delta_x \). This implies for \( x \neq 0 \) that \( (\mathcal{L}(X_k)) \) does not converge to the invariant distribution \( \pi_x = \frac{1}{2}(\delta_x + \delta_{-x}) \) (depending on \( x \)), since \( \mathbb{P}(X_{2k} \in B) = \delta_x(B) \) and \( \mathbb{P}(X_{2k+1} \in B) = \delta_{-x}(B) \) for \( B \in \mathcal{B}(\mathbb{R}) \). Nevertheless the Cesàro average \( \nu_k := \frac{1}{k} \sum_{j=1}^{k} \mathbb{P}X_j \) converges to \( \pi_x \).

As Example 3.5 shows, meaningful notions of ergodic convergence are possible (in our case, convergence of the Cesàro average) even when convergence in distribution can not be expected. We start by collecting several general results for Markov chains on Polish spaces. In the next section we restrict ourselves to equicontinuous and Feller Markov operators.

An invariant probability measure \( \pi \) of \( \mathcal{P} \) is called ergodic, if any \( p \)-invariant set, i.e. \( A \in \mathcal{B}(G) \) with \( p(x,A) = 1 \) for all \( x \in A \), has \( \pi \)-measure 0 or 1. Two measures \( \pi_1, \pi_2 \) are called mutually singular when there is \( A \in \mathcal{B}(G) \) with \( \pi_1(A^c) = \pi_2(A) = 0 \). The following decomposition theorem on Polish spaces is key to our development. For more detail see, for instance, [95].

**Proposition 3.6.** Denote by \( \mathcal{I} \) the set of all invariant probability measures for \( \mathcal{P} \) and by \( \mathcal{E} \subset \mathcal{I} \) the set of all those that are ergodic. Then, \( \mathcal{I} \) is convex and \( \mathcal{E} \) is precisely the set of its extremal points. Furthermore, for every invariant measure \( \pi \in \mathcal{I} \), there exists a probability measure \( q_\pi \) on \( \mathcal{E} \) such that

\[
\pi(A) = \int_{\mathcal{E}} \nu(A)q_\pi(d\nu).
\]

In other words, every invariant measure is a convex combination of ergodic invariant measures. Finally, any two distinct elements of \( \mathcal{E} \) are mutually singular.

**Remark 3.7:** If there exists only one invariant probability measure of \( \mathcal{P} \), we know by Proposition 3.6 that it is ergodic. If there exist more invariant probability measures, then there exist uncountably many invariant and at least two ergodic probability measures.

**Proposition 3.8.** Let \( \pi \) be an ergodic invariant probability measure for \( \mathcal{P} \), let \((G,\mathcal{G})\) be a measurable space, and let \( f : G \to \mathbb{R} \) be measurable, bounded and satisfy \( \pi|f|^p < \infty \) for \( p \in [1,\infty] \). Then

\[
\nu^\pi_k f := \frac{1}{k} \sum_{j=1}^{k} p^j(x,f) \to \pi f \quad \text{as } k \to \infty \quad \text{for } \pi \text{-a.e. } x \in G,
\]

where \( p^j(x,f) := \delta_x \mathcal{P}^j f = \mathbb{E}[f(X_j) | X_0 = x] \) for the sequence \((X_k)\) generated by Algorithm 1 with \( X_0 \sim \pi \).

**Proof.** This is a direct consequence of Birkhoff’s ergodic theorem, [51, Theorem 9.6].

For fixed \( x \) in Proposition 3.8, we want the assertion to be true for all \( f \in \mathcal{C}_b(G) \). This issue is addressed in the next section by restricting our attention to equicontinuous Markov operators. The results above do not require any explicit structure on the mappings \( T_i \) that generate the transition kernel \( p \) and hence the Markov operator \( \mathcal{P} \), however the assumption that the initial random variable \( X_0 \) has the same distribution as the invariant measure \( \pi \) is very strong. For Markov operators generated from discontinuous mappings \( T_i \), the support of an invariant measure may not be invariant under \( T_i \). To see this, let

\[
T_x := \begin{cases} 
  x, & x \in \mathbb{R} \setminus \mathbb{Q} \\
  -1, & x \in \mathbb{Q}
\end{cases}
\]
The transition kernel is then \( p(x, A) = \mathbb{1}_A(Tx) \) for \( x \in \mathbb{R} \) and \( A \in \mathcal{B}(\mathbb{R}) \). Let \( \mu \) be the uniform distribution on \([0, 1]\), then, since \( \lambda \)-a.s. \( T = \text{Id} \) (where \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \)), we have that \( \mu \mathcal{P}^k = \mu \) for all \( k \in \mathbb{N} \). Consequently, \( \pi = \mu \) is invariant and \( \text{supp} \pi = [0, 1] \), but \( T([0, 1]) = \{-1\} \cup [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q}) \), which is not contained in \([0, 1]\).

The next result shows, however, that invariance of the the support of invariant measures under continuous mappings \( T_i \) is guaranteed.

**Lemma 3.9** (invariance of the support of invariant measures). Let \( G \) be a Polish space and let \( T_i : G \to G \) be continuous for all \( i \in I \). For any invariant probability measure \( \pi \in \mathcal{P}(G) \) of \( \mathcal{P} \) it holds that \( T_\xi \text{supp} \pi \subset \text{supp} \pi \) a.s. where \( \text{supp} \pi := \text{supp} \pi \).

**Proof.** By Fubini’s Theorem, for any \( A \in \mathcal{B}(G) \) it holds that

\[
\pi(A) = \int_{\text{supp} \pi} p(x, A) \pi(dx) = \int_{\Omega} \int_{\text{supp} \pi} \mathbb{1}_A(T_\xi x) \pi(dx) d\mathcal{P}
= \int_{\Omega} \int_{\text{supp} \pi} \mathbb{1}_{T_\xi^{-1}A}(x) \pi(dx) d\mathcal{P}(\omega)
= \mathbb{E} \left[ \pi(T_\xi^{-1}A \cap \text{supp} \pi) \right] = \mathbb{E} \left[ \pi(T_\xi^{-1}A) \right].
\]

From \( 1 = \pi(\text{supp} \pi) = \mathbb{E} \left[ \pi(T_\xi^{-1}\text{supp} \pi) \right] \) and \( \pi(\cdot) \leq 1 \), it follows that \( \pi(T_\xi^{-1}\text{supp} \pi) = 1 \) a.s.

Note that \( T_\xi^{-1}\text{supp} \pi \) is closed for all \( i \in I \) due to continuity of \( T_i \) and closedness of \( \text{supp} \pi \). We show that \( \text{supp} \pi \subset T_\xi^{-1}\text{supp} \pi \) a.s. which then yields the claim. To see this, let \( S \subset G \) be any closed set with \( \pi(A \cap S) = \pi(A) \) for all \( A \in \mathcal{B}(G) \), and let \( x \in \text{supp} \pi \). Then \( \pi(\mathbb{B}(x, \epsilon) \cap S) > 0 \) for all \( \epsilon > 0 \), i.e. \( \mathbb{B}(x, \epsilon) \cap S \neq \emptyset \) for all \( \epsilon > 0 \). Now consider \( x_k \in \mathbb{B}(x, \epsilon_k) \cap S \), where \( \epsilon_k \to 0 \) as \( k \to \infty \). Then since \( S \) is closed, \( x_k \to x \in I \), from which we conclude that \( \text{supp} \pi \subset S \). Specifically, let \( S = T^{-1}\text{supp} \pi \) and note that \( T^{-1}\text{supp} \pi = D \setminus G \) for some \( D \) with \( \mathcal{P}(D) = 0 \). For any \( A \in \mathcal{B}(G) \) it holds that \( \pi(A \cap S) = \pi(A) - \pi(A \cap D) = \pi(A) \). From the argument above, we conclude that \( \text{supp} \pi \subset S = T^{-1}\text{supp} \pi \) as claimed.

The above result means that, if the random variable \( X_k \) enters \( \text{supp} \pi \) for some \( k \), then it will stay in \( \text{supp} \pi \) forever. This can be interpreted as a mode of convergence, i.e. convergence to the set \( \text{supp} \pi \), which is closed under application of \( T_\xi \) a.s. Equality \( T_\xi \text{supp} \pi = \text{supp} \pi \) a.s. cannot be expected in general. For example, let \( I = \{1, 2\} \), \( G = \mathbb{R} \) and \( T_1x = -1 \), \( T_2x = 1 \), \( x \in \mathbb{R} \) and \( \mathcal{P}(\xi = 1) = 0.5 = \mathcal{P}(\xi = 2) \), then \( \pi = \frac{1}{2}(\delta_{-1} + \delta_1) \) and \( \text{supp} \pi = \{-1, 1\} \). So \( T_1\text{supp} \pi = \{-1\} \) and \( T_2\text{supp} \pi = \{1\} \).

### 3.3. Ergodic convergence theory for equicontinuous Markov operators

As shown by Szarek [88] and Worm [96], equicontinuity of Markov operators and their generalizations give a nice structure to the set of ergodic measures. We collect some results here which will be used heavily in the subsequent analysis.

**Definition 3.10** (equicontinuity). A Markov operator is called equicontinuous, if \( \mathcal{P}^k f \) is equicontinuous for all bounded and Lipschitz continuous \( f : G \to \mathbb{R} \).

In the following we consider the union of supports of all ergodic measures defined by

\[
S := \bigcup_{\pi \in \mathcal{E}} \text{supp} \pi,
\]

where \( \mathcal{E} \subset \text{inv} \mathcal{P} \) denotes the set of ergodic measures.
**Proposition 3.11** (tightness of \((\delta_s P^k)\)). Let \(G\) be a Polish space. Let \(P\) be equicontinuous. Suppose there exists an invariant measure for \(P\). Then the sequence \((\delta_s P^k)_{k \in \mathbb{N}}\) is tight for all \(s \in S\) defined by (37).

**Proof.** In the proof of [88, Proposition 2.1] it is shown that, under the assumption that \(P\) is equicontinuous and \((\exists s, x \in G \text{ s.t. } \limsup_{k \to \infty} \nu_x^k(B(s, \epsilon)) > 0 \forall \epsilon > 0, (38)\)

then the sequence \((\delta_s P^k)\) is tight. It remains to demonstrate (38). To see this, let \(f = 1_{B(s, \epsilon)}\) for some \(s \in S\), \(\pi \in \mathcal{E}\) and \(\epsilon > 0\) in Proposition 3.8. Then for \(\pi\)-a.e. \(x \in G\) and \(\nu_x^k := \frac{1}{k} \sum_{j=1}^k \delta_x P^j\) we have

\[
\limsup_{k \to \infty} \nu_x^k(B(s, \epsilon)) = \lim_{k} \nu_x^k(B(s, \epsilon)) = \pi(B(s, \epsilon)) > 0.
\]

This completes the proof. \(\square\)

**Remark 3.12** (tightness of \((\nu_x^k)\)): Note that the sequence \((\nu_x^k)\) is tight for \(s \in S\), since by Proposition 3.11, for all \(\epsilon > 0\), there is a compact subset \(K \subset G\) such that \(p^k(s, K) > 1 - \epsilon\) for all \(k \in \mathbb{N}\), and hence also \(\nu_x^k(K) > 1 - \epsilon\) for all \(k \in \mathbb{N}\).

The next result due to Worm (Theorems 5.4.11 and 7.3.13 of [96]) concerns Cesàro averages for equicontinuous Markov operators.

**Proposition 3.13** (convergence of Cesàro averages [96]). Let \(P\) be Feller and equicontinuous, let \(G\) be a Polish space and let \(\mu \in \mathcal{P}(G)\). Then the sequence \((\nu_{\mu}^k)\) is tight \((\nu_{\mu}^k := \frac{1}{k} \sum_{j=1}^k \mu P^j)\) if and only if \((\nu_{\mu}^k)\) converges to a \(\pi_{\mu} \in \text{inv} P\). In this case

\[
\pi_{\mu} = \int_{\text{supp} \mu} \pi^x \mu(dx),
\]

where for each \(x \in \text{supp} \mu \subset G\) there exists the limit of \((\nu_{\mu}^k)\) and it is denoted by the invariant measure \(\pi^x\).

For the case the initial measure \(\mu\) is supported in \(\bigcup_{\pi \in \text{inv} P} \text{supp} \pi\), we have the following.

**Proposition 3.14** (ergodic decomposition). Let \(G\) be a Polish space and let \(P\) be Feller and equicontinuous. Then

\[
S = \bigcup_{\pi \in \text{inv} P} \text{supp} \pi,
\]

where \(S\) is defined in (37). Moreover \(S\) is closed, and for any \(\mu \in \mathcal{P}(S)\) it holds that \(\nu_{\mu}^k \to \pi_{\mu}\) as \(k \to \infty\) with

\[
\pi_{\mu} = \int_{S} \pi^x \mu(dx),
\]

where \(\pi^x\) is the unique ergodic measure with \(x \in \text{supp} \pi^x\).

**Proof.** This is a consequence of [96, Theorem 7.3.4] and [96, Theorem 5.4.11], Remark 3.12 and Proposition 3.13. \(\square\)

**Proposition 3.14** only establishes convergence of the Markov chain when it is initialized with a measure in the support of an invariant measure; moreover, it is only the average of the distributions of the iterates that converges.
Lemma 3.18. Let \( P \) (positive transition probability for ergodic measures) reached infinitely often starting from any other point in this support. Denote the Markov operator that is induced by the transition kernel in \( i \): The mapping \( f \) is nonexpansive, \( i \in I \). Let \( \pi \) be an ergodic invariant probability measure for \( \mathcal{P} \). Then for any \( s, \tilde{s} \in \text{supp} \pi \) it holds that

\[
\forall \epsilon > 0 \exists \delta > 0, \exists (k_j)_{j \in \mathbb{N}} \subseteq \mathbb{N} : p^{k_j}(s, \mathcal{B}(\tilde{s}, \epsilon)) \geq \delta, \forall j \in \mathbb{N}.
\]

Proof. Given \( \tilde{s} \in \text{supp} \pi \) and \( \epsilon > 0 \), find a continuous and bounded function \( f = f_{\tilde{s}, \epsilon} : G \to [0, 1] \) with the property that \( f = 1 \) on \( \mathcal{B}(\tilde{s}, \frac{\epsilon}{2}) \) and \( f = 0 \) outside \( \mathcal{B}(\tilde{s}, \epsilon) \). For \( s \in \text{supp} \pi \) let \( X_0 \sim \delta \) and \( (X_k) \) generated by Algorithm 1. By Remark 3.15 the sequence \( (\nu_k) \) converges to \( \pi \) as \( k \to \infty \), where \( \nu_k = \frac{1}{k} \sum_{j=1}^k p^j(s, \cdot) \). So in particular \( \nu_k f \to \pi f \geq \pi(\mathcal{B}(\tilde{s}, \frac{\epsilon}{2})) > 0 \) as \( k \to \infty \). Hence, for \( k \) large enough there is \( \delta > 0 \) with

\[
\nu_k f = \frac{1}{k} \sum_{j=1}^k p^j(s, f) \geq \delta.
\]

Now, we can extract a sequence \( (k_j) \subseteq \mathbb{N} \) with \( p^{k_j}(s, f) \geq \delta, j \in \mathbb{N} \) and hence

\[
p^{k_j}(s, \mathcal{B}(\tilde{s}, \epsilon)) \geq p^{k_j}(s, f) \geq \delta > 0. \]

\[\Box\]

3.4. Ergodic theory for nonexpansive mappings

We now specialize to the case that the family of mappings \( \{T_i\}_{i \in I} \) are nonexpansive operators.

Lemma 3.18. Let \( G \) be a Polish space. Let \( T_i : G \to G \) be nonexpansive, \( i \in I \) and let \( \mathcal{P} \) denote the Markov operator that is induced by the transition kernel in (4).

(i) \( \mathcal{P} \) is Feller.

(ii) \( \mathcal{P} \) is equicontinuous.

Proof. (i) The mapping \( T_i \) for \( i \in I \) is 1-Lipschitz continuous, so in particular it is continuous. Proposition 2.4 yields the assertion.

(ii) Let \( \epsilon > 0 \) and \( x, y \in G \) with \( d(x, y) < \epsilon/\|f\|_{\text{Lip}} \), then, using Jensen’s inequality, Lipschitz continuity of \( f \) and nonexpansivity of \( T_i \), we get

\[
|\delta_x p^k f - \delta_y p^k f| = |\mathbb{E}[f(X^x_k)] - \mathbb{E}[f(X^y_k)]| \\
\leq \mathbb{E}|[f(X^x_k) - f(X^y_k)]| \\
\leq \|f\|_{\text{Lip}} \mathbb{E}[d(X^x_k, X^y_k)] \\
\leq \|f\|_{\text{Lip}} \mathbb{E}[d(x, y)] < \epsilon
\]

for all \( k \in \mathbb{N} \). \[\Box\]
A very helpful fact used later on is that the distance between the supports of two ergodic measures is attained; moreover, any point in the support of the one ergodic measure has a nearest neighbor in the support of the other ergodic measure.

**Lemma 3.19** (distance of supports is attained). Let \( G \) be a Polish space and \( T_i : G \to G \) be nonexpansive, \( i \in I \). Suppose \( \pi, \tilde{\pi} \) are ergodic probability measures for \( \mathcal{P} \). Denote the support of a measure \( \pi \) by \( S_\pi := \text{supp} \pi \). Then for all \( s \in S_\pi \) there exists \( \tilde{s} \in S_{\tilde{\pi}} \) with \( d(s, \tilde{s}) = \text{dist}(s, S_{\tilde{\pi}}) = \text{dist}(S_\pi, S_{\tilde{\pi}}) \).

**Proof.** First we show, that \( \text{dist}(S_\pi, S_{\tilde{\pi}}) = \text{dist}(s, S_{\tilde{\pi}}) \) for all \( s \in S_\pi \). Therefore, recall the notation \( X_k^s = T_{\xi_{k-1}} \cdots T_{\xi_0} x \) and note that by nonexpansivity of \( T_i \), \( i \in I \) and Lemma 3.9 it holds a.s. that

\[
\text{dist}(X_{k+1}^s, S_\pi) \leq \text{dist}(X_k^s, T_{\xi_k} S_\pi) = \inf_{s \in S_\pi} d(T_{\xi_k} X_k^s, T_{\xi_k} s) \leq \text{dist}(X_k^s, S_{\tilde{\pi}})
\]

for all \( x \in G, \pi \in \text{inv} \mathcal{P} \) and \( k \in \mathbb{N} \). Suppose now there would exist an \( \tilde{s} \in S_{\tilde{\pi}} \) with \( \text{dist}(s, \tilde{\pi}) < \text{dist}(s, S_{\tilde{\pi}}) \). Then by Lemma 3.17 for all \( \epsilon > 0 \) there is a \( k \in \mathbb{N} \) with \( \mathbb{P}(X_k^s \in B(s, \epsilon)) > 0 \) and hence

\[
\text{dist}(s, S_{\tilde{\pi}}) \leq d(s, X_k^s) + \text{dist}(X_k^s, S_{\tilde{\pi}}) \leq \epsilon + \text{dist}(s, S_{\tilde{\pi}})
\]

with positive probability for all \( \epsilon > 0 \), which is a contradiction. So, it holds that \( \text{dist}(s, S_{\tilde{\pi}}) = \text{dist}(s, S_\pi) \) for all \( s, \tilde{s} \in S_\pi \).

For \( s \in S_\pi \) let \( (\tilde{s}_m) \subset S_{\tilde{\pi}} \) be a minimizing sequence for \( \text{dist}(s, S_{\tilde{\pi}}) \), i.e. \( \lim_m d(s, \tilde{s}_m) = \text{dist}(s, S_{\tilde{\pi}}) \). Now define a probability measure \( \gamma_k^m \) on \( G \times G \) via

\[
\gamma_k^m f := \mathbb{E} \left[ \frac{1}{k} \sum_{j=1}^{k} f(X_j^s, X_j^{\tilde{s}_m}) \right]
\]

for measurable \( f : G \times G \to \mathbb{R} \). Then \( \gamma_k^m \in C(\nu_k^s, \nu_k^{\tilde{s}_m}) \) where \( C(\nu_k^s, \nu_k^{\tilde{s}_m}) \) is the set of all couplings for \( \nu_k^s \) and \( \nu_k^{\tilde{s}_m} \) (see (80)). Also, by Lemma A.5 and Remark 3.15 the sequence \( (\gamma_k^m)_{k \in \mathbb{N}} \) is tight for fixed \( m \in \mathbb{N} \) and there exists a cluster point \( \gamma^m \in C(\pi, \tilde{\pi}) \). The sequence \( (\gamma^m) \subset C(\pi, \tilde{\pi}) \) is again tight by Lemma A.5. Thus for any cluster point \( \gamma \in C(\pi, \tilde{\pi}) \) and the bounded and continuous function \( (x, y) \mapsto f^M(x, y) = \min(M, d(x, y)) \) this yields

\[
\gamma_k^m d = \gamma_k^m f^M \searrow \gamma^m f^M \quad \text{as } k \to \infty
\]

for all \( M \geq d(s, \tilde{s}_m), m \in \mathbb{N} \). Since by the Monotone Convergence Theorem \( \gamma^m f^M \nearrow \gamma^m d \) as \( m \to \infty \), it follows that \( \gamma^m f^M = \gamma^m d \) for all \( M \geq d(s, \tilde{s}_m) \). The same argument holds for \( M \geq d(s, \tilde{s}_j) \) and a subsequence \( (\gamma_{m_j}^m) \) with limit \( \gamma \) such that \( \gamma d = \gamma f^M \). Hence,

\[
\gamma d = \gamma f^M = \lim_j \gamma_{m_j}^m f^M = \lim_j \gamma_{m_j}^m d \leq \lim_j d(s, \tilde{s}_{m_j}) = \text{dist}(s, S_{\tilde{\pi}}).
\]

In particular for \( \gamma \)-a.e. \( (x, y) \in S_\pi \times S_{\tilde{\pi}} \) it holds that \( d(x, y) = \text{dist}(S_\pi, S_{\tilde{\pi}}) \), because \( d(x, y) \geq \text{dist}(S_\pi, S_{\tilde{\pi}}) \) on \( S_\pi \times S_{\tilde{\pi}} \). Taking the closure of these \( (x, y) \) in \( G \times G \), we see that for any \( s \in S_\pi \) there is \( \tilde{s} \in S_{\tilde{\pi}} \) with \( d(s, \tilde{s}) = \text{dist}(S_\pi, S_{\tilde{\pi}}) \) by Lemma A.4. \( \square \)

### 3.5. Convergence for nonexpansive mappings in \( \mathbb{R}^n \)

By Proposition 3.13 tightness of a sequence of Cesàro averages is equivalent to convergence of said sequence. So our focus is on tightness in the Euclidean space setting.

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Lemma 3.20 (tightness of $(\mu^P_k)$ in $\mathbb{R}^n$). On the Euclidean space $(\mathbb{R}^n, \|\cdot\|)$ let $T_i : \mathbb{R}^n \to \mathbb{R}^n$ be nonexpansive for all $i \in I$, and let $\text{inv } P \neq \emptyset$ for the corresponding Markov operator. The sequence $(\mu^P_k)_{k \in \mathbb{N}}$ is tight for any $\mu \in \mathcal{P}(\mathbb{R}^n)$.

Proof. First, let $\mu = \delta_x$ for $x \in \mathbb{R}^n$. We know that the sequence $(\delta_x P_k)$ is tight for $s \in S$ by Proposition 3.11. So for $\epsilon > 0$ there is a compact $K \subset \mathbb{R}^n$ with $p^k(s, K) \geq 1 - \epsilon$ for all $k \in \mathbb{N}$. Recall the definition of $X^x_k$ in (3). Since a.s. $\|X^x_k - X^y_k\| \leq \|x - y\|$, we have that $p^k(x, \mathbb{B}(K, \|x - s\|)) = P(X^x_k \in \mathbb{B}(K, \|x - s\|)) \geq p^k(s, K) \geq 1 - \epsilon$ for all $k \in \mathbb{N}$. Hence $(\delta_x P_k)$ is tight.

Now consider the initial random variable $X_0 \sim \mu$ for any $\mu \in \mathcal{P}(\mathbb{R}^n)$. For given $\epsilon > 0$ there is a compact $K^\mu \subset \mathbb{R}^n$ with $\mu(K^\mu) > 1 - \epsilon$. From the special case established above, there exists a compact $K_\epsilon \subset \mathbb{R}^n$ with $p^k(0, K_\epsilon) > 1 - \epsilon$ for all $k \in \mathbb{N}$. Let $M > 0$ such that $K^\mu_\epsilon \subset \mathbb{B}(0, M)$ and let $x \in \mathbb{B}(0, M)$. We have that $p^k(x, \mathbb{B}(K_\epsilon, M)) > 1 - \epsilon$ for all $x \in \mathbb{B}(0, M)$, since $\|X^x_k - X^y_k\| \leq \|x\| \leq M$. Hence $\mu P^k(\mathbb{B}(K_\epsilon, M)) > (1 - \epsilon)^2$, which implies tightness of the sequence $(\mu P_k)$.

□

Remark 3.21 (tightness of $(\nu^\mu_k)$ in $\mathbb{R}^n$): The tightness of the sequence $(\nu^\mu_k)$ for any $\mu \in \mathcal{P}(\mathbb{R}^n)$ follows immediately from tightness of $(\mu P_k)$ as in Remark 3.12.

We are now in a position to prove the first main result.

Proof of Theorem 2.17. By Lemma 3.18 the Markov operator $P$ is Feller and equicontinuous. By Lemma 3.20 the sequence $(\mu^P_k)$ is tight, and so the sequence of of Cesàro averages $(\nu^\mu_k)$ is also tight (see Remark 3.21). Hence by Proposition 3.13 $\nu^\mu_k \to \pi^\mu$ with $\pi^\mu$ given by (34). □

3.6. More properties of the RFI for nonexpansive mappings

This section is devoted to the preparation of some tools used in Section 3.7 to prove convergence of the distributions of the iterates of the RFI. When the Markov chain is initialized with a point not supported in $S$, i.e. when $\text{supp } \mu \setminus S \neq \emptyset$, the convergence results on general Polish spaces are much weaker than for the ergodic case in the previous section. One problem is that the sequences $(\nu^\mu_k)_{k \in \mathbb{N}}$ for $x \in G \setminus S$ need not be tight anymore. The right-shift operator $R$ on $L^2$, for example, with the initial distribution $\delta_{e_1}$, generates the sequence $R^k e_1 = e_k$, $k = 1, 2, \ldots$. Examples of spaces on which we can always guarantee tightness are, of course, Euclidean spaces as seen in the previous section, and compact metric spaces – since then $(\mathcal{P}(G), d_P)$ is compact.

For the case that the sequence of Cesàro averages does not necessarily converge, we have the following result.

Lemma 3.22 (convergence of with nonexpansive mappings). Let $(G, d)$ be a separable complete metric space and let $T_i : G \to G$ be nonexpansive for all $i \in I$. Suppose $\text{inv } P \neq \emptyset$. Let $X_0 \sim \mu \in \mathcal{P}(G)$ and let $(X_k)$ be the sequence generated by Algorithm 1. Denote the support of any measure $\mu$ by $S_\mu$, and denote $\nu_k := \frac{1}{k} \sum_{j=1}^k \mu P^j$.

(i) $\forall \pi \in \text{inv } P, \ \text{dist}(X_{k+1}, S_\pi) \leq \text{dist}(X_k, S_\pi)$ a.s. $\forall k \in \mathbb{N}$.

(ii) If the sequence $(\nu_k)$ has a cluster point $\pi \in \text{inv } P$, then,

(a) $\text{dist}(X_k, S_\pi) \to 0$ a.s. as $k \to \infty$;

(b) all cluster points of the sequence $(\nu_k)$ have the same support;

(c) cluster points of the sequence $(\mu P^k)$ have support in $S_\pi$ (if they exist).
Proof. (i). By Lemma 3.9, the sets on which $T_{\xi_k} S_\pi$ is not a subset of $S_\pi$ are $\mathbb{P}$-null sets and their union is also a $\mathbb{P}$-null set. This yields

$$\text{(\forall s \in S_\pi) \ dist(X_{k+1}, S_\pi) \leq d(X_{k+1}, T_{\xi_k} s) = d(T_{\xi_k} X_k, T_{\xi_k} s) \leq d(X_k, s) \ a.s.,}$$

and hence

$$\text{dist}(X_{k+1}, S_\pi) \leq \text{dist}(X_k, S_\pi) \ a.s.}$$

(iia) Define the function $f = \min(M, \text{dist}(\cdot, S_\pi))$ for some $M > 0$. Since this is bounded and continuous, we have for a subsequence $(\nu_{k_j})$ converging to $\pi$, that $\nu_{k_j} f = \frac{1}{k_j} \sum_{n=1}^{k_j} \mu P^n f \rightarrow \pi f = 0$ as $j \rightarrow \infty$. Now part (i) and the identity

$$\mu P^{n+1} f = \mathbb{E}[\min(M, \text{dist}(X_{n+1}, S_\pi))] \leq \mathbb{E}[\min(M, \text{dist}(X_n, S_\pi))] = \mu P^n f$$

yield $\mu P^n f = \mathbb{E}[\min(M, \text{dist}(X_n, S_\pi))] \rightarrow 0$ as $n \rightarrow \infty$. Again by part (i)

$$Y := \lim_{n \rightarrow \infty} \min(M, \text{dist}(X_n, S_\pi))$$

exists and is nonnegative; so by Lebesgue’s dominated convergence theorem it follows that $Y = 0$ a.s., since otherwise $\mathbb{E}[Y] > 0 = \lim_{n \rightarrow \infty} \mu P^n f$ would yield a contradiction.

(iib) Let $\pi_1, \pi_2$ be two cluster points of $(\nu_k)$ with support $S_1, S_2$ respectively, then these probability measures are invariant for $\mathbb{P}$ by Proposition 3.1. By Corollary 3.16 the intersection $S_1 \cap S_2$ must be nonempty. Suppose now w.l.o.g. $\exists y \in S_1 \backslash S_2$. Then there is an $\epsilon > 0$ with $\mathbb{B}(y, 2\epsilon) \cap S_2 = \emptyset$. Let $f: G \rightarrow [0, 1]$ be a continuous function that takes the value 1 on $\mathbb{B}(y, \frac{\epsilon}{2})$ and 0 outside of $\mathbb{B}(y, \epsilon)$. Then $\pi_1 f > 0$ and $\pi_2 f = 0$. But there are two subsequences of $(\nu_k)$ with $\nu_{k_j} f \rightarrow \pi_1 f$ and $\nu_{k_j} f \rightarrow \pi_2 f$ as $j \rightarrow \infty$. For the former sequence we have, for $j$ large enough,

$$\exists \delta > 0: \frac{1}{k_j} \sum_{n=1}^{k_j} \mu P^n f \geq \delta > 0.$$ 

So, one can from this extract a sequence $(m_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ with $\mu P^{m_k} f \geq \delta$, $k \in \mathbb{N}$. Note that $\mathbb{P}(X_{m_k} \in \mathbb{B}(y, \epsilon)) \geq \mu P^{m_k} f \geq \delta > 0$. This implies $\text{dist}(X_{m_k}, S_2) \geq \epsilon$ with $\mathbb{P} \geq \delta$ and hence $\mathbb{E}[\text{dist}(X_{m_k}, S_2)] \geq \delta \epsilon$, in contradiction to (iia). So there cannot be such $y$ which yields $S_1 = S_2$, as claimed.

(iic) Let $\nu$ be a cluster point of the sequence $(\mu P^k)$, which is assumed to exist, and assume there is $s \in \text{supp } \nu \backslash S_\pi$ and $\epsilon > 0$ such that $\text{dist}(s, S_\pi) > 2\epsilon$. Let $f: G \rightarrow [0, 1]$ be a continuous function, that takes the value 1 on $\mathbb{B}(s, \frac{\epsilon}{2})$ and 0 outside of $\mathbb{B}(s, \epsilon)$. With (iia) we find, that

$$0 < \nu f = \lim_{j} \mathbb{P}^{X_{k_j}} f \leq \lim_{j} \mathbb{P}(X_{k_j} \in \mathbb{B}(s, \epsilon)) = 0.$$ 

Were $\mathbb{P}(X_{k_j} \in \mathbb{B}(s, \epsilon)) \geq \delta > 0$ for $j$ large enough, then this would imply that

$$\mathbb{E}[\text{dist}(X_{k_j}, S_\pi)] \geq \delta \epsilon$$

for $j$ large enough, which is a contradiction. We conclude that there is no such $s$, which completes the proof. 

We now prepare some tools to handle convergence of the distributions of the iterates of the RFI for $\alpha$-finite mappings in Section 3.7. We restrict ourselves to Polish spaces with finite dimensional metric (see Definition 3.25) in order to apply a differentiation theorem. We begin with the next technical fact.

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Lemma 3.23 (characterization of balls in $(\mathcal{E}, d_P)$). Let $(G, d)$ be a separable complete metric space and $T_i : G \to G$ be nonexpansive, $i \in I$. Let $\mathcal{E}$ denote the the (convex) set of ergodic measures associated to the Markov operator $\mathcal{P}$, which is induced by the family of mappings $\{T_i\}^i \in I$ and the marginal probability law of the random variables $\xi_k$. Let $\pi, \tilde{\pi} \in \mathcal{E}$ and denote the support of the measure $\pi$ by $S_\pi$ (and similarly for $\tilde{\pi}$). Then

$$\tilde{\pi} \in \mathcal{F}(\pi, \epsilon) \iff S_\pi \subset \mathcal{F}(S_\pi, \epsilon)$$

for $\epsilon \in (0, 1)$, where $\mathcal{F}(\pi, \epsilon)$ is the closed $\epsilon$-ball with respect to the Prokhorov-Lévy metric $d_P$.

Proof. By Lemma 3.19 there exist $s \in S_\pi$ and $\tilde{s} \in S_{\tilde{\pi}}$ such that $d(s, \tilde{s}) = \text{dist}(S_\pi, S_{\tilde{\pi}})$. First note that, if $\pi \neq \tilde{\pi}$, then $S_\pi \cap S_{\tilde{\pi}} = \emptyset$ by Corollary 3.16, and hence $d(s, \tilde{s}) = \text{dist}(S_\pi, S_{\tilde{\pi}}) > 0$. Recall the notation $X^s_k := T_{k-1} \cdots T_0 x$ for $x \in G$ and note that by Lemma A.4(i) and Lemma 3.9, supp $\mathcal{L}(X^s_k) \subset S_\pi$ and supp $\mathcal{L}(X^\tilde{s}_k) \subset S_{\tilde{\pi}}$. So it holds that $d(X^s_{k}, X^\tilde{s}_k) \geq \text{dist}(S_\pi, S_{\tilde{\pi}})$ a.s. for all $k \in \mathbb{N}$. Since $T_i$ is nonexpansive, we have that $d(X^s_{k}, X^\tilde{s}_k) \leq d(s, \tilde{s})$ a.s. for all $k \in \mathbb{N}$. So, both inequalities together imply the equality

$$d(X^s_{k}, X^\tilde{s}_k) = d(s, \tilde{s}) \quad \text{a.s.} \quad \forall k \in \mathbb{N}. \quad (39)$$

Now, letting $c := \min(1, d(s, \tilde{s}))$, we show that $d_P(\pi, \tilde{\pi}) = c$, where $d_P$ denotes the Prokhorov-Lévy metric (see Lemma A.6). Indeed, take $(X, Y) \in C(\mathcal{L}(X^s_k), \mathcal{L}(X^\tilde{s}_k))$. Again, by Lemma A.4(i) and Lemma 3.9 supp $\mathcal{L}(X^\pi_s) \subset S_\pi$ and supp $\mathcal{L}(X^\tilde{\pi}_s) \subset S_{\tilde{\pi}}$ and hence $d(x, Y) \geq \text{dist}(S_\pi, S_{\tilde{\pi}}) = d(s, \tilde{s})$ a.s. We have, thus

$$\mathbb{P}(d(X, Y) > c - \delta) \geq \mathbb{P}(d(X, Y) > d(s, \tilde{s}) - \delta) = 1 \quad \forall \delta > 0,$$

which implies $d_P(\mathcal{L}(X^s_k), \mathcal{L}(X^\tilde{s}_k)) \geq c$ by Lemma A.6(i). In particular, for $c = 1$ it follows that $d_P(\mathcal{L}(X^s_k), \mathcal{L}(X^\tilde{s}_k)) = 1$, since $d_P$ is bounded by 1. Now, let $c < 1$, i.e. $c = d(s, \tilde{s}) < 1$. We have by (39)

$$\inf_{(X,Y)\in C(\mathcal{L}(X^s_k),\mathcal{L}(X^\tilde{s}_k))} \mathbb{P}(d(X, Y) > c) \leq \mathbb{P}(d(X^s_k, X^\tilde{s}_k) > c) = 0 \leq c.$$

Altogether we find that $d_P(\mathcal{L}(X^s_k), \mathcal{L}(X^\tilde{s}_k)) = c$, again by Lemma A.6(i). Since also supp $\nu^s_k \subset S_\pi$ and supp $\nu^\tilde{s}_k \subset S_{\tilde{\pi}}$, where $\nu^s_k = \frac{1}{k} \sum_{j=1}^k \mathcal{L}(X^s_j)$ for any $x \in G$, it follows that

$$c \leq d_P(\nu^s_k, \nu^\tilde{s}_k) \leq \max_{j=1,\ldots,k} d_P(\mathcal{L}(X^s_j), \mathcal{L}(X^\tilde{s}_j)) = c \quad (40)$$

by Lemma A.6(v). Now taking the limit $k \to \infty$ of (40) and using Remark 3.15, it follows that $d_P(\pi, \tilde{\pi}) = c$. This proves the assertion. \[\square\]

Definition 3.24 (Besicovitch family). A family $\mathcal{B}$ of closed balls $B = \mathcal{B}(x_B, \epsilon_B)$ with $x_B \in G$ and $\epsilon_B > 0$ on the metric space $(G, d)$ is called a Besicovitch family of balls if

(i) for every $B \in \mathcal{B}$ one has $x_B \notin B' \in \mathcal{B}$ for all $B' \neq B$, and

(ii) $\bigcap_{B \in \mathcal{B}} B \neq \emptyset$.

Definition 3.25 ($\sigma$-finite dimensional metric). Let $(G, d)$ be a metric space. We say that $d$ is finite dimensional on a subset $D \subset G$ if there exist constants $K \geq 1$ and $0 < r < \infty$ such that Card $\mathcal{B} \leq K$ for every Besicovitch family $\mathcal{B}$ of balls in $(G, d)$ centered on $D$ with radius $< r$. We say that $d$ is $\sigma$-finite dimensional if $G$ can be written as a countable union of subsets on which $d$ is finite dimensional.

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Proposition 3.26 (differentiation theorem, [74]). Let \((G, d)\) be a separable complete metric space. For every locally finite Borel regular measure \(\lambda\) over \((G, d)\), it holds that
\[
\lim_{r \to 0} \frac{1}{\lambda(B(x, r))} \int_{B(x, r)} f(y) \lambda(dy) = f(x) \quad \text{for } \lambda\text{-a.e. } x \in G, \forall f \in L^1_{\text{loc}}(G, \lambda) \tag{41}
\]
if and only if \(d\) is \(\sigma\)-finite dimensional.

Proposition 3.27 (Besicovitch covering property in \(\mathcal{E}\)). Let \((G, d)\) be separable complete metric space with finite dimensional metric \(d\) and let \(T_i : G \to G\) be nonexpansive, \(i \in I\). The cardinality of any Besicovitch family of balls in \((\mathcal{E}, d_F)\) is bounded by the same constant that bounds the cardinality of Besicovitch families in \(G\).

Proof. Let \(\mathcal{B}\) be a Besicovitch family of closed balls \(B = \mathbb{B}(\pi_B, \epsilon_B)\) in \((\mathcal{E}, d_F)\), where \(\pi_B \in \mathcal{E}\) and \(\epsilon_B > 0\). Note that if \(\epsilon_B \geq 1\), then \(|B| = 1\), since in that case \(B = \mathcal{E}\) since \(d_F\) is bounded by 1. So let \(|B| > 1\), that implies \(\epsilon_B < 1\) for all \(B \in \mathcal{B}\).

The defining properties of a Besicovitch family translate then with help of Lemma 3.23 into
\[
\pi_B \notin B', \ \forall B' \in B \setminus \{B\} \iff S_{\pi_B} \cap \overline{\mathcal{B}(S_{\pi_B}, \epsilon_B')} = \emptyset, \ \forall B' \in B \setminus \{B\}, \tag{42}
\]
and
\[
\bigcap_{B \in \mathcal{B}} B \neq \emptyset \iff \bigcap_{B \in \mathcal{B}} \overline{\mathcal{B}(S_{\pi_B}, \epsilon_B)} \neq \emptyset. \tag{43}
\]
Now fix \(\pi\) in the latter intersection in (43) and let \(s \in S_{\pi}\). Also fix for each \(B \in \mathcal{B}\) a point \(s_B \in S_{\pi_B}\) with the property that \(s_B \in \text{argmin}_{s \in S_{\pi_B}} d(s, \tilde{s})\) (possible by Lemma 3.19). Then the family \(\mathcal{C}\) of balls \(\overline{\mathbb{B}(s_B, \epsilon_B)} \subset G, B \in \mathcal{B}\) is also a Besicovitch family: We have \(s_B \notin B'\) for \(B \neq B'\) due to (42) and by the choice of \(s_B\) one has \(s \in \bigcap_{B \in \mathcal{C}} B\). Since the cardinality of any Besicovitch family in \(G\) is bounded by a uniform constant, it follows, that also the cardinality of \(\mathcal{B}\) is uniformly bounded. \(\square\)

Remark 3.28 (Euclidean metric on \(\mathbb{R}^n\) is finite dimensional): The cardinality of any Besicovitch family in \(\mathbb{R}^n\) is uniformly bounded depending on \(n\) [63, Lemma 2.6].

Lemma 3.29 (equality around support of ergodic measures implies equality of measures). Let \((G, d)\) be a separable complete metric space with the finite dimensional metric \(d\) and let \(T_i : G \to G\) be nonexpansive \((i \in I)\). If \(\pi_1, \pi_2 \in \text{inv} \mathcal{P}\) satisfy
\[
\pi_1(\mathcal{B}(S_\pi, \epsilon)) = \pi_2(\mathcal{B}(S_\pi, \epsilon)) \tag{44}
\]
for all \(\epsilon > 0\) and all \(\pi \in \mathcal{E}\), then \(\pi_1 = \pi_2\).

Proof. From Proposition 3.6 follows the existence of probability measures \(q_1, q_2\) on the set \(\mathcal{E}\) of ergodic measures for \(\mathcal{P}\) such that one has
\[
\pi_j(A) = \int_{\mathcal{E}} \pi(A) q_j(d\pi), \quad A \in \mathcal{B}(G), \ j = 1, 2.
\]
If we set \(q = \frac{1}{2}(q_1 + q_2)\), then by the Radon-Nikodym theorem, there are densities \(f_1, f_2 \geq 0\) on \(\mathcal{E}\) with \(q_j = f_j \cdot q\) and hence
\[
\pi_j(A) = \int_{\mathcal{E}} \pi(A) f_j(\pi) q(d\pi), \quad A \in \mathcal{B}(G), \ j = 1, 2.
\]
For $q$-measurable subsets $E \subset \mathcal{E}$, one can define a probability measure on $\mathcal{E}$ via

$$
\tilde{\pi}_j(E) := \int_{\mathcal{E}} \mathbb{I}_E(\pi) f_j(\pi) q(\text{d}\pi), \quad j = 1, 2. \tag{45}
$$

One then has for $\epsilon > 0$ and $\pi \in \mathcal{E}$ that

$$
\pi_j(\mathbb{E}(S_\pi, \epsilon)) = \tilde{\pi}_j(\mathbb{E}(\pi, \epsilon)), \quad j = 1, 2, \tag{46}
$$

where $\mathbb{E}(\pi, \epsilon) := \{ \tilde{\pi} \in \mathcal{E} \mid d_P(\tilde{\pi}, \pi) \leq \epsilon \}$. This is due to Lemma 3.23, from which follows

$$
\tilde{\pi}(\mathbb{E}(S_\pi, \epsilon)) = \begin{cases} 1, & \tilde{\pi} \in \mathbb{E}(\pi, \epsilon) \\ 0, & \text{else} \end{cases}.
$$

With the above characterizations of $\pi_j$ and $\tilde{\pi}_j$, we can use Proposition 3.26 to show that $f_1 = f_2$ $q$-a.s., which, together with (45), would imply that $\pi_1 = \pi_2$, as claimed. To apply Proposition 3.26 we require that $d_P$ is finite dimensional. But this follows from Proposition 3.27. So Proposition 3.26 applied to $\tilde{\pi}_j$ with respect to $q$ then gives $q$-a.s.

$$
\lim_{\epsilon \to 0} \frac{\tilde{\pi}_j(\mathbb{E}(\pi, \epsilon))}{f_j(\pi)} = f_j(\pi), \quad j = 1, 2. \tag{47}
$$

And since $\tilde{\pi}_1(\mathbb{E}(\pi, \epsilon)) = \tilde{\pi}_2(\mathbb{E}(\pi, \epsilon))$ by (46) and assumption (44), we have $f_1 = f_2$ $q$-a.s., which completes the proof.

**Remark 3.30:** In the assertion of Lemma 3.29, it is enough to claim the existence of a sequence $(\epsilon_k^\pi)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ with $\epsilon_k^\pi \to 0$ as $k \to \infty$ satisfying

$$
\pi_1(\mathbb{E}(S_\pi, \epsilon_k^\pi)) = \pi_2(\mathbb{E}(S_\pi, \epsilon_k^\pi)) \quad \forall \pi \in \mathcal{E}, \forall k \in \mathbb{N},
$$

because from Proposition 3.26 one has the existence of the limit in (47) $q$-a.s.

### 3.7. Convergence theory for $\alpha$-firmly nonexpansive mappings

Continuing the development of the convergence theory under greater regularity assumptions on the mappings $T_i$ ($i \in I$), in this section we examine what is achievable under the assumption that the mappings $T_i$ are $\alpha$-fne (Definition 2.8). We restrict ourselves to the Euclidean space $(\mathbb{R}^n, \|\cdot\|)$, and begin with a technical lemma that describes properties of sequences whose relative expected distances are invariant under $T_\xi$.

**Lemma 3.31** (constant expected separation). Let $T_i : \mathbb{R}^n \to \mathbb{R}^n$ be $\alpha$-fne with $\alpha_i \leq \alpha < 1$, $i \in I$. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ and $X \sim \mu$, $Y \sim \nu$ independent of $(\xi_k)$ satisfy

$$
\mathbb{E}\left[\|X^X_k - X^Y_k\|^2\right] = \mathbb{E}\left[\|X - Y\|^2\right] \quad \forall k \in \mathbb{N},
$$

where $X^x_k := T_{\xi_{k-1}} \cdots T_{\xi_0} x$ for $x \in \mathbb{R}^n$ is the RFI sequence started at $x$. Then for $\mathbb{P}^{(X,Y)}$-a.e. $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ we have $X^x_k - X^y_k = x - y$ $\mathbb{P}$-a.s. for all $k \in \mathbb{N}$. Moreover, if there exists an invariant measure for $\mathcal{P}$, then

$$
\pi^x(\cdot) = \pi^y(\cdot - (x - y)) \quad \mathbb{P}^{(X,Y)}$-a.s.
$$

for the limiting invariant measures $\pi^x$ of the Cesàro average of $(\delta_x \mathbb{P}^k)$ and $\pi^y$ of the Cesàro average of $(\delta_y \mathbb{P}^k)$. 

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Proof. By the Hilbert space characterization of α-fne mappings (18), one has
\[ E \left[ \| X - Y \|^2 \right] \geq E \left[ \| T_{\xi_0} X - T_{\xi_0} Y \|^2 \right] + \frac{1 - \alpha}{\alpha} \left( E \left[ \| (X - T_{\xi_0} X) - (Y - T_{\xi_0} Y) \|^2 \right] \right) \]
\[ \geq \ldots \]
\[ \geq E \left[ \| T_{\xi_{k-1}} \cdots T_{\xi_0} X - T_{\xi_{k-1}} \cdots T_{\xi_0} Y \|^2 \right] \]
\[ + \frac{1 - \alpha}{\alpha} \sum_{j=0}^{k-1} E \left[ \| (T_{\xi_j} \cdots T_{\xi_0} X - T_{\xi_j} \cdots T_{\xi_0} Y) - (T_{\xi_j} \cdots T_{\xi_{k-1}} Y - T_{\xi_j} \cdots T_{\xi_0} Y) \|^2 \right], \]
where we used \( T_{\xi_{k-1}} := Id \) for a simpler representation of the sum. We will denote \( X_k^X = T_{\xi_{k-1}} \cdots T_{\xi_0} x \). The assumption \( E \left[ \| X_k^X - X_k^Y \|^2 \right] = E \left[ \| X - Y \|^2 \right] \) for all \( k \in \mathbb{N} \) then implies, that for \( j = 1, \ldots, k \) \( \mathbb{P} \)-a.s.
\[ X_k^X - X_k^Y = X_k^Y - X_{k-1}^Y \quad (k \in \mathbb{N}), \]
and hence by induction
\[ X_k^X - X_k^Y = X - Y. \]
By disintegrating and using \( (X, Y) \perp (\xi_k) \) we have \( \mathbb{P} \)-a.s.
\[ 0 = E \left[ \left\| (X - X_k^X) - (Y - X_k^Y) \right\|^2 \right| X, Y \]
\[ = \int_{I_{k+1}} \left\| (X - T_{i_k} \cdots T_{i_0} X) - (Y - T_{i_k} \cdots T_{i_0} Y) \right\|^2 p^\xi(di_k) \cdots p^\xi(di_0). \]
Consequently, for \( \mathbb{P}(X,Y) \)-a.e. \( (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \), we have
\[ X_k^x - X_k^y = x - y \quad \forall k \in \mathbb{N} \quad \mathbb{P} \text{-a.s.} \]
So in particular for any \( A \in \mathcal{B}(\mathbb{R}^n) \)
\[ p_k^A(x, A) = \mathbb{P}(X_k^x \in A) = \mathbb{P}(X_k^y \in A - (x - y)) = p_k^A(y, A - (x - y)) \]
and hence, denoting \( f_k = f(\cdot + h) \) and \( \nu_k^y = \frac{1}{k} \sum_{j=1}^k p^j(x, \cdot) \), one also has for \( f \in C_b(\mathbb{R}^n) \) by Theorem 2.17
\[ \nu_k^y f_{x-y} \to \pi_y^y f_{x-y} = \pi_y^x f_{x-y} \quad \text{and} \quad \nu_k^y f \to \pi_y^x f \quad \text{as} \quad k \to \infty, \]
where \( \pi_y^x := \pi^y(\cdot - (y - x)) \). So from \( \nu_k^y f_{x-y} = \nu_k^y f \) for any \( f \in C_b(\mathbb{R}^n) \) and \( k \in \mathbb{N} \) it follows that \( \pi_y^x \). \( \square \)

We can now give the proof of the second main result. For a given \( h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) we will define sequences of functions \( (\overline{h}_k) \) on \( \mathbb{R}^n \times \mathbb{R}^n \) via
\[ \overline{h}_k(x, y) := E \left[ h(X_k^x, X_k^y) \right], \quad X_k^x := T_{\xi_{k-1}} \cdots T_{\xi_0} z \quad \text{for any} \quad z \in \mathbb{R}^n \quad (k \in \mathbb{N}). \]
Note that, by continuity of \( T_i, \ i \in I \) and Lebesgue’s dominated convergence theorem, \( \overline{h}_k \in C_b(\mathbb{R}^n \times \mathbb{R}^n) \) for all \( k \in \mathbb{N} \) whenever \( h \in C_b(\mathbb{R}^n \times \mathbb{R}^n) \).

**Proof of Theorem 2.18.** Let \( x, y \in \mathbb{R}^n \), define \( F(x, y) := \| x - y \|^2 \) and the corresponding sequence of functions
\[ F_k(x, y) := E \left[ F(X_k^x, X_k^y) \right], \quad X_k^x := T_{\xi_{k-1}} \cdots T_{\xi_0} z \quad \text{for any} \quad z \in \mathbb{R}^n \quad (k \in \mathbb{N}). \]
By the remarks preceding this proof, $\mathcal{F}_k \in C_b(\mathbb{R}^n \times \mathbb{R}^n)$ for all $k \in \mathbb{N}$. From the regularity of $T_i$, $i \in I$ and the characterization (18), we get that a.s. for all $k \in \mathbb{N}$

$$\|X^x_k - X^y_k\|^2 \geq \|X^x_{k+1} - X^y_{k+1}\|^2 + \frac{1-\alpha}{\alpha}\|(X^x_k - X^y_k) - (X^x_{k+1} - X^y_{k+1})\|^2.$$  \hfill (48)

After computing the expectation, this is the same as

$$\mathcal{F}_k(x, y) \geq \mathcal{F}_{k+1}(x, y) + \frac{1-\alpha}{\alpha}\mathbb{E}\left[\|(X^x_k - X^y_k) - (X^x_{k+1} - X^y_{k+1})\|^2\right].$$

We conclude that $(\mathcal{F}_k(x, y))$ is a monotonically nonincreasing sequence for any $x, y \in G$.

Recall the notation $\mathcal{S}_\pi := \text{supp } \pi$ for some measure $\pi$. Let $s, \tilde{s} \in \mathcal{S}_\pi$ for the ergodic invariant measure $\pi \in \mathcal{E}$ and define the sequence of measures $\gamma_k$ by

$$\gamma_k f := \mathbb{E}\left[f(X^x_k, X^y_k)\right]$$

for any measurable function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Note that due to nonexansiveness the pair $(X^x_k, X^y_k)$ a.s. takes values in $G_r := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : ||x - y||^2 \leq r\}$ for $r = ||s - \tilde{s}||^2$, so that $\gamma_k$ is concentrated on this set. Since $(X^x_k)$ is a tight sequence by Lemma 3.20, and likewise for $(X^y_k)$, we know from Lemma A.5 that the sequence $(\gamma_k)$ is tight as well. Let $\gamma$ be a cluster point of $(\gamma_k)$, which is again concentrated on $G_{||s-\tilde{s}||^2}$ and consider a subsequence $(\gamma_{k_j})$ such that $\gamma_{k_j} \to \gamma$. By Lemma A.5 we also know that $\gamma \in C(\nu_1, \nu_2)$ where $\nu_1$ and $\nu_2$ are the distributions of the limit in convergence in distribution of $(X^x_{k_j})$ and $(X^y_{k_j})$. For any $f \in C_b(\mathbb{R}^n \times \mathbb{R}^n)$ we have $\gamma_{k_j} f \to f$. So consider the case $f = F^M$ where $F^M := \min(M, F)$ for $M \in \mathbb{R}$. Since $\|x - y\|^2 = F(x, y) = F^M(x, y)$ almost surely (with respect to $\gamma_{k_j}$ and $\gamma$) for $M \geq ||s - \tilde{s}||^2$, we have

$$\gamma_{k_j} F = \gamma_{k_j} F^M \to \gamma F^M = \gamma F.$$

However, by the monotonicity in (48) we now also obtain convergence for the entire sequence:

$$\gamma_k F = \gamma_k F^M \to \gamma F^M = \gamma F.$$

Let $(X, Y) \sim \gamma$ and $(\tilde{\xi}_k) \perp (\xi_k)$ be another i.i.d. sequence with $(X, Y) \perp (\tilde{\xi}_k), (\xi_k)$. We use the notation $\bar{X}^x_k := T_{\tilde{\xi}_k-1} \cdots T_{\tilde{\xi}_0}x, x \in \mathbb{R}^n$. Define the sequence of functions

$$\mathcal{F}^M_k(x, y) := \mathbb{E}\left[F^M(\bar{X}^x_k, \bar{X}^y_k)\right] \quad (k \in \mathbb{N}),$$

and note that $\mathcal{F}^M_k \in C_b(\mathbb{R}^n \times \mathbb{R}^n)$. When $M \geq ||s - \tilde{s}||^2$ this yields

$$\gamma \mathcal{F}_k = \gamma \mathcal{F}^M_k = \mathbb{E}\left[\min \left(M, \|\bar{X}^x_k - \bar{X}^y_k\|^2\right)\right] = \lim_{j \to \infty} \gamma_{k_j} \mathcal{F}^M_k$$

$$= \lim_{j \to \infty} \mathbb{E}\left[\min \left(M, \|\bar{X}^x_{k_j} - \bar{X}^y_{k_j}\|^2\right)\right]$$

$$= \lim_{j \to \infty} \mathbb{E}\left[\min \left(M, \|X^x_{k+k_j} - X^y_{k+k_j}\|^2\right)\right]$$

$$= \lim_{j \to \infty} \gamma_{k+k_j} \mathcal{F}^M = \gamma \mathcal{F}^M = \gamma F.$$
This means that for all $k \in \mathbb{N}$,  
\[ \mathbb{E} \left[ \left\| X_k^Y - X_k^Y \right\|^2 \right] = \mathbb{E} \left[ \left\| X - Y \right\|^2 \right]. \]

For $\mathbb{P}(X,Y)$ a.e. $(x,y)$ we have $x,y \in S_\pi$ and thus $\pi^x = \pi^y = \pi$ where $\pi^x$ is the unique ergodic measure with $x \in S_{\pi^x}$ (see Remark 3.15). An application of Lemma 3.31 then yields $\pi(\cdot) = \pi(\cdot - (x - y))$, i.e. $x = y$. Hence $X = Y$ a.s. implying $\nu_1 = \nu_2 =: \nu$ and $\gamma F = 0$. That means  
\[ \gamma_k F = \mathbb{E} \left[ \left\| X_k^s - X_k^s \right\|^2 \right] \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \]

Now Lemma A.6 yields  
\[ \mathbb{P} \left( \left\| X_k^s - X_k^s \right\| > \epsilon \right) \leq \frac{\mathbb{E} \left[ \left\| X_k^s - X_k^s \right\| \right]}{\epsilon} \leq \frac{\mathbb{E} \left[ \sqrt{\left\| X_k^s - X_k^s \right\|^2} \right]}{\epsilon} \rightarrow 0 \]

as $k \rightarrow \infty$ for any $\epsilon > 0$; this yields convergence of the corresponding probability measures $\delta_s \mathcal{P}^k$ and $\delta_s \mathcal{P}^k$ in the Prokhorov metric:  
\[ d_p(\delta_s \mathcal{P}^k, \delta_s \mathcal{P}^k) \rightarrow 0. \]

By the triangle inequality, therefore, if $\delta_s \mathcal{P}^k \rightarrow \nu$, then also $\delta_s \mathcal{P}^k \rightarrow \nu$ for any $s \in S_\pi$. Hence  
\[ d_p(\delta_s \mathcal{P}^k, \nu) \leq d_p(\delta_s \mathcal{P}^k, \delta_s \mathcal{P}^h) + d_p(\delta_s \mathcal{P}^h, \nu) \rightarrow 0, \quad \text{as} \quad j \rightarrow \infty. \]

By Lebesgue’s dominated convergence theorem we conclude that, for any $f \in C_b(\mathbb{R}^n)$ and $\mu \in \mathcal{P}(S_\pi)$,  
\[ \mu \mathcal{P}^k f = \int_{S_\pi} \delta_s \mathcal{P}^k f \mu(ds) \rightarrow \nu f, \quad \text{as} \quad j \rightarrow \infty. \]

In particular, $\mu \mathcal{P}^k \rightarrow \nu$ and taking $\mu = \pi$ yields $\nu = \pi$. Thus, all cluster points of $\delta_s \mathcal{P}^k$ for all $s \in S_\pi$ have the same distribution $\pi$ and hence, because the sequence is tight, $\delta_s \mathcal{P}^k = p^k(x, \cdot) \rightarrow \pi$.

Now, let $\mu \in \mathcal{P}(S)$, where $S = \bigcup_{s \in S_\pi} S_s$. By what we have just shown we have for $x \in \text{supp} \mu$, that $p^k(x, \cdot) \rightarrow \pi^x$, where $\pi^x$ is unique ergodic measure with $x \in S_{\pi^x}$. Then, again by Lebesgue’s dominated convergence theorem, one has for any $f \in C_b(\mathbb{R}^n)$,  
\[ \mu \mathcal{P}^k f = \int f(y)p^k(x,dy)\mu(dx) \rightarrow \int f(y)\pi^x(dy)\mu(dx) =: \pi^\mu f \quad \text{as} \quad k \rightarrow \infty, \quad (49) \]

and the measure $\pi^\mu$ is again invariant for $\mathcal{P}$ by invariance of $\pi^x$ for all $x \in S$. Now, let $\mu = \delta_x$, $x \in \mathbb{R}^n \setminus S$. We obtain the tightness of $\delta_x \mathcal{P}^k$ from the tightness of $\delta_s \mathcal{P}^k$ for $s \in S$. Indeed, for $\epsilon > 0$ there exists a compact $K_\epsilon \subset \mathbb{R}_s$ with $p^k(s, K_\epsilon) > 1 - \epsilon$ for all $k \in \mathbb{N}$. This together with the fact that $T_i$, $i \in I$ is nonexpansive implies that $\left\| X_k^s - X_k^s \right\| \leq \| x - s \|$ for all $k \in \mathbb{N}$ hence $p^k(x, \mathbb{E}(K_\epsilon, \| x - s \|)) > 1 - \epsilon$, where $p$ is the transition kernel defined by (4). Tightness implies the existence of a cluster point $\nu$ of the sequence $\delta_x \mathcal{P}^k$. From Theorem 2.17 we know that $\nu_k = \frac{1}{k} \sum_{j=1}^k \delta_s \mathcal{P}^j \rightarrow \pi^x$ for some $\pi^x \in \text{inv} \mathcal{P}$ with $S_{\pi^x} \subset S$. Furthermore, we have $\nu \in \mathcal{P}(S_{\pi^x}) \subset \mathcal{P}(S)$ by Lemma 3.22(iiic). So by (49) there exists $\pi^\nu \in \text{inv} \mathcal{P}$ with $\nu \mathcal{P}^k \rightarrow \pi^\nu$.

In order to complete the proof we have to show that $\nu = \pi^x$, i.e. $\pi^x$ is the unique cluster point of $\delta_s \mathcal{P}^k$ and hence convergence follows by Proposition A.1. It suffices to show that $\pi^\nu = \pi^x$, since then, as $k \rightarrow \infty$  
\[ d_p(\nu, \pi^x) = \lim_k d_p(\delta_x \mathcal{P}^k, \pi^x) = \lim_k d_p(\delta_s \mathcal{P}^k, \pi^x) = d_p(\nu \mathcal{P}^j, \pi^x) = d_p(\nu \mathcal{P}^j, \pi^\nu) \rightarrow 0. \]
To begin, fix \( \pi \in \text{inv} \, \mathcal{P} \). For any \( \epsilon > 0 \) let \( A_k := \{ X_k^x \in \mathbb{H}(S_{\pi}, \epsilon) \} \). By nonexpansivity \( A_k \subset A_{k+1} \) for \( k \in \mathbb{N} \), since we have by Lemma 3.9 a.s.

\[
\text{dist}(X_{k+1}^x, S_{\pi}) \leq \text{dist}(X_k^x, T_{k} S_{\pi}) \leq \text{dist}(X_k^x, S_{\pi}).
\]

Hence \( (p^k(x, \mathbb{H}(S_{\pi}, \epsilon))) = (\mathbb{P}(A_k)) \) is a monotonically increasing sequence and bounded from above and therefore the sequence converges to some \( b_{\epsilon}^x \in [0, 1] \) as \( k \to \infty \). It follows

\[
b_{\epsilon}^x = \lim_{k} p^k(x, \mathbb{H}(S_{\pi}, \epsilon)) = \lim_{k} \frac{1}{k} \sum_{j=1}^{k} p^j(x, \mathbb{H}(S_{\pi}, \epsilon)).
\]

and thus \( \nu(\mathbb{H}(S_{\pi}, \epsilon)) = \pi^x(\mathbb{H}(S_{\pi}, \epsilon)) \) for all \( \epsilon \), which make \( \mathbb{H}(S_{\pi}, \epsilon) \) both \( \nu \)- and \( \pi^x \)-continuous. Note that there are at most countably many \( \epsilon > 0 \) for which this may fail, see [53, Chapter 3, Example 1.3]).

With the same argument used for (50) we also obtain for any \( k \in \mathbb{N} \) that \( \nu \mathcal{P}^k(\mathbb{H}(S_{\pi}, \epsilon)) = \pi^x(\mathbb{H}(S_{\pi}, \epsilon)) \) with only countably many \( \epsilon \) excluded, and so

\[
\pi^x(\mathbb{H}(S_{\pi}, \epsilon)) = \pi^x(\mathbb{H}(S_{\pi}, \epsilon))
\]

also needs to hold for all except countably many \( \epsilon \). Since \( \pi^x \in \text{inv} \, \mathcal{P} \), this implies that \( \pi^x = \pi^x \) by Lemma 3.29 combined with Remark 3.30. For a general initial measure \( \mu_0 \in \mathcal{P}(\mathbb{R}^n) \), one has, yet again by Lebesgue’s dominated convergence theorem, that

\[
\mu_0 \mathcal{P}^k f = \int f(y) p^k(x, dy) \mu_0(dx) \to \int f(y) \pi^x(dy) \mu_0(dx) =: \pi^{\mu_0} f,
\]

where \( \pi^x \) denotes the limit of \( (\delta_x \mathcal{P}^k) \) and the measure \( \pi^{\mu_0} \) is again invariant for \( \mathcal{P} \). This completes the proof. \( \square \)

**Remark 3.32** (a.s.

\[
\gamma_k F \to \gamma F = 0,
\]

by monotonicity of \( (\gamma_k F) \).

### 3.7.1. Structure of ergodic measures for \( \alpha \)-firmly nonexpansive mappings

**Proposition 3.33** (structure of ergodic measures). Let \( T_i : \mathbb{R}^n \to \mathbb{R}^n \) be \( \alpha \)-fne with constant \( \alpha_i \leq \alpha < 1 \) (\( i \in I \)) and assume there exists an invariant probability distribution for \( \mathcal{P} \). Any two ergodic measures \( \pi, \bar{\pi} \) are shifted versions of each other, i.e. there exist \( s \in \text{supp} \, \pi \) and \( \bar{s} \in \text{supp} \, \bar{\pi} \) with \( \pi = \bar{\pi}(-s - \bar{s}) \).

**Proof.** Denote \( \text{supp} \, \pi = S_{\pi} \). Since we can find for any \( s \in S_{\pi} \) a closest point \( \bar{s} \in S_{\bar{\pi}} \), i.e. \( \text{dist}(S_{\pi}, S_{\bar{\pi}}) = d(s, \bar{s}) \), by Lemma 3.19, the assertion follows from

\[
dist(S_{\pi}, S_{\bar{\pi}}) \leq \sqrt{\mathbb{E}[|X_k^x - X_k^\bar{s}|^2]} \leq ||s - \bar{s}|| \quad \forall k \in \mathbb{N},
\]

where we also used that \( \text{supp} \, \mathcal{L}(X_k^x) \subset S_{\pi} \), and \( \text{supp} \, \mathcal{L}(X_k^\bar{s}) \subset S_{\bar{\pi}} \). \( \square \)

**Proposition 3.34** (specialization to projectors). Let the mappings \( T_i = P_i : \mathbb{R}^n \to \mathbb{R}^n \) be projectors onto nonempty closed and convex sets (\( i \in I \)). If there exist two ergodic measures \( \pi_1, \pi_2 \), then there exist infinitely many ergodic measures \( \pi_\lambda \) with \( \pi_\lambda := \pi_2(-\lambda a) \) for all \( \lambda \in [0, 1] \), where \( a \) is the shift such that \( \pi_1 = \pi_2(-a) \).
Proof. For any pair \((s_1, s_2)\) \(\in\) \(\text{supp}\,\pi_1 \times \text{supp}\,\pi_2\) of closest neighbors it holds that \(a = s_1 - s_2\) by Proposition 3.33. Lemma 3.31 and (51) yield \(P_k s_1 = P_k s_2 + a\) a.s. Hence, \(a \perp (s_i - P_k s_i)\), \(i = 1, 2\), and then \(P_k (s_2 + \lambda a) = P_k s_2 + \lambda a\) for \(\lambda \in [0, 1]\). Hence \(X_k^{s_2 + \lambda a} = X_k^{s_2} + \lambda a\) and \(\lim_k L(X_k^{s_2 + \lambda a}) = \pi_2(\cdot - \lambda a)\). Note, that if \(P\) is Feller and the sequence \((\mu P^k)\) converges for some \(\mu \in \mathbb{R}\), then the limit is also an invariant measure. \(\square\)

3.8. Rates of Convergence

We now prove the third main result of this paper.

Proof of Theorem 2.19. First note that since \(G\) is compact and \(\text{inv}\,P\) is nonempty, there is at least one \(\pi \in \text{inv}\,P \cap \mathcal{P}_2(G)\) and one \(\mu \in \mathcal{P}_2(G)\) with \(W_2(\mu, \pi) < \infty\) and \(\mu P \in \mathcal{P}_2(G)\). The Markov operator \(P\) is therefore a self-mapping on \(\mathcal{P}_2(G)\), hence \(W_2(\mu, \mu P) < \infty\), and for any \(\mu_1, \mu_2 \in \mathcal{P}_2(G)\) the set of optimal couplings \(C_*(\mu_1, \mu_2)\) is nonempty (see Lemma A.7).

Since \((H, d)\) is a Hadamard space and \(G \subset H\), the function \(\Psi(\mu)\) defined by (29) is extended real-valued, nonnegative (see Lemma 2.9), and finite since \(C_*(\mu, \pi)\) and \(\text{inv}\,P\) are nonempty. Moreover, by assumption (c) and the definition of metric subregularity (Definition 2.15) this satisfies \(\Psi(\pi) = 0 \iff \pi \in \text{inv}\,P\), hence \(\Psi^{-1}(0) = \text{inv}\,P\) and \(\Psi(\pi) = 0\) for all \(\pi \in \text{inv}\,P\), and for all \(\mu \in \mathcal{P}_2(G)\)

\[
\inf_{\pi \in \text{inv}\,P} W_2^2(\mu, \pi) = \inf_{\pi \in \Psi^{-1}(0)} W_2^2(\mu, \pi) \leq (\rho(d_\mathcal{P}(0, \Psi(\mu))))^2 = (\rho(\Psi(\mu)))^2.
\]

Rewriting this for the next step yields

\[
\frac{1-\alpha}{\alpha} \left(\rho^{-1} \left(\inf_{\pi \in \text{inv}\,P} W_2(\mu, \pi)\right)\right)^2 \leq \frac{1-\alpha}{\alpha} \Psi^2(\mu). \quad (52)
\]

On the other hand, by assumption (b) and Proposition 2.14 (which applies because we are on a separable Hadamard space) we have

\[
\frac{1-\alpha}{\alpha} \Psi^2(\mu) \leq \int_{G \times G} \mathbb{E}[\psi_2(x, y, T_k x, T_k y)] \gamma(dx, dy) \leq (1 + \epsilon)W_2^2(\mu, \pi) - W_2^2(\mu P, \pi) \quad \forall \pi \in \text{inv}\,P, \forall \mu \in \mathcal{P}_2(G). \quad (53)
\]

Incorporating (52) into (53) and rearranging the inequality yields

\[
W_2^2(\mu P, \pi) \leq (1 + \epsilon)W_2^2(\mu, \pi) - \frac{1-\alpha}{\alpha} \left(\rho^{-1} \left(\inf_{\pi' \in \text{inv}\,P} W_2(\mu, \pi')\right)\right)^2 \quad \forall \pi \in \text{inv}\,P, \forall \mu \in \mathcal{P}_2(G).
\]

Since this holds at any \(\mu \in \mathcal{P}_2(G)\), it certainly holds at the iterates \(\mu_k\) with initial distribution \(\mu_0 \in \mathcal{P}_2(G)\) since \(P\) is a self-mapping on \(\mathcal{P}_2(G)\). Therefore

\[
W_2^2(\mu_{k+1}, \pi) \leq (1 + \epsilon)W_2^2(\mu_k, \pi) - \frac{1-\alpha}{\alpha} \left(\rho^{-1} \left(\inf_{\pi' \in \text{inv}\,P} W_2(\mu_k, \pi')\right)\right)^2 \quad \forall \pi \in \text{inv}\,P, \forall k \in \mathbb{N}. \quad (54)
\]

Equation (54) simplifies. Indeed, by Lemma 3.4, \(\text{inv}\,P\) is closed with respect to convergence in distribution. Moreover, since \(G\) is assumed to be compact, \(\mathcal{P}_2(G)\) is locally compact ([2, Remark 7.19] so, for every \(k \in \mathbb{N}\) the infimum in (54) is attained at some \(\pi_k\). This yields

\[
W_2^2(\mu_{k+1}, \pi_{k+1}) \leq W_2^2(\mu_{k+1}, \pi_k) \leq (1 + \epsilon)W_2^2(\mu_k, \pi_k) - \frac{1-\alpha}{\alpha} \left(\rho^{-1} (W_2(\mu_k, \pi_k))\right)^2 \quad \forall k \in \mathbb{N}. \quad (55)
\]
Taking the square root and recalling (26) and (27) yields (35).

To obtain convergence, note that for \( \mu_0 \in \mathcal{P}_2(G) \) satisfying \( W_2(\mu_0, \pi) < \infty \) and \( \mu_0 \mathcal{P} \in \mathcal{P}_2(G) \) (exists by compactness of \( G \) ), the triangle inequality and (55) yield

\[
W_2(\mu_{k+1}, \mu_k) \leq W_2(\mu_{k+1}, \pi_k) + W_2(\mu_k, \pi_k) \\
\leq \theta (W_2(\mu_k, \pi_k)) + W_2(\mu_k, \pi_k).
\]

Using (35) and continuing by backwards induction yields

\[
W_2(\mu_{k+1}, \mu_k) \leq \theta^{k+1} (d_0) + \theta^k (d_0)
\]

where \( d_0 := \inf_{\pi \in \text{inv } \mathcal{P}} W_2(\mu_0, \pi) \). Repeating this argument, for any \( k < m \)

\[
W_2(\mu_m, \mu_k) \leq \theta^m (d_0) + 2 \sum_{j=k+1}^{m-1} \theta^j (d_0) + \theta^k (d_0).
\]

By assumption, \( \theta \) satisfies (26), so for any \( \delta > 0 \)

\[
W_2(\mu_m, \mu_k) \leq \theta^m (d_0) + 2 \sum_{j=k+1}^{\infty} \theta^j (d_0) + \theta^k (d_0) < \delta
\]

for all \( k, m \) large enough; that is the sequence \( (\mu_k)_{k \in \mathbb{N}} \) is a Cauchy sequence in \( (\mathcal{P}_2(G), W_2) \) – a separable complete metric space (Lemma A.7 (iii)) – and therefore convergent to some probability measure \( \pi^{\mu_0} \in \mathcal{P}_2(G) \). By Proposition 2.4 the Markov operator \( \mathcal{P} \) is Feller since \( \mathcal{T}_t \) is continuous, and by Proposition 3.1 when a Feller Markov chain converges in distribution, it does so to an invariant measure: \( \pi^{\mu_0} \in \text{inv } \mathcal{P} \). □

**Proof of Corollary 2.21.** In the case that the gauge \( \rho \) is linear with constant \( \kappa' \), then \( \theta(t) \) is linear with constant

\[
c = \sqrt{1 + \epsilon - \frac{1 - \alpha}{\kappa^2}} < 1,
\]

where \( \kappa \geq \kappa' \) satisfies \( \kappa^2 \geq (1 - \alpha)/\alpha(1 + \epsilon) \). Specializing the argument in the proof above to this particular \( \theta \) shows that, for any \( k \) and \( m \) with \( k < m \), we have

\[
W_2(\mu_m, \mu_k) \leq d_0 c^m + 2d_0 \sum_{j=k+1}^{m-1} c^j + d_0 c^k. \tag{56}
\]

Letting \( m \to \infty \) in (56) yields R-linear convergence (Definition 2.6) with rate \( c \) given above and leading constant \( \beta = \frac{1 + \epsilon}{d_0} \).

If, in addition, \( \text{inv } \mathcal{P} \) is a singleton, then \( \{ \pi^{\mu_0} \} = \text{inv } \mathcal{P} \) in the above and convergence is actually Q-linear, which completes the proof. □

### 4. Examples: Stochastic Optimization and Inconsistent Nonconvex Feasibility

To fix our attention we focus on the following optimization problem

\[
\min_{\mu \in \mathcal{P}_2(\mathbb{R}^n)} \int_{\mathbb{R}^n} \mathbb{E}_\xi[f_{\xi}(x) + g_{\psi}(x)] \mu(dx). \tag{57}
\]
It is assumed throughout that \( f_i : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable for all \( i \in I_f \) and that \( g_i : \mathbb{R}^n \to \mathbb{R} \) is proper and lower semi-continuous for all \( i \in I_g \). The random variable with values on \( I_f \times I_g \) will be denoted \( \xi = (\xi^f, \xi^g) \). This model covers deterministic composite optimization as a special case: \( I_f \) and \( I_g \) consist of single elements and the measure \( \mu \) is a point mass.

The algorithms reviewed in this section rely on resolvents of the functions \( f_i \) and \( g_i \), denoted \( J_{f_i} \) and \( J_{g_i} \). The resolvent of a subdifferentially regular function \( f : G \subset \mathbb{R}^n \to \mathbb{R} \) (the epigraph of \( f \) is Clarke regular [81]) is defined by \( J_f(x) := (\text{Id} + \partial f)^{-1}(x) := \{ z \in G | x = z + \partial f(z) \} \). For proper, lower semicontinuous convex functions \( f : \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \} \), this is equivalent to the proximal mapping [65] defined by

\[
\text{prox}_f(x) := \underset{y}{\text{argmin}} \{ f(y) + \frac{1}{2} \| y - x \|_2^2 \}. \tag{58}
\]

In general one has

\[
\text{prox}_f(x) \subset J_f(x) \tag{59}
\]

whenever the subdifferential is defined.

### 4.1. Stochastic (nonconvex) forward-backward splitting

We begin with a general prescription of the forward-backward splitting algorithm together with abstract properties of the corresponding fixed point mapping, and then specialize this to more concrete instances.

**Algorithm 2:** Stochastic Forward-Backward Splitting

**Initialization:** Set \( X_0 \sim \mu_0 \in \mathcal{P}_2(G) \), \( X_0 \sim \mu \), \( t > 0 \), and \( (\xi_k)_{k \in \mathbb{N}} \) another i.i.d. sequence with values on \( I_f \times I_g \) and \( X_0 \perp \perp (\xi_k) \).

\[
\text{for } k = 0, 1, 2, \ldots \text{ do}
\]

\[
X_{k+1} = T^{FB}_{\xi_k} X_k := J_{g_{\xi_k}} \left( X_k - t \nabla f_{\xi_k}(X_k) \right) \tag{60}
\]

When \( f_{\xi^f}(x) = f(x) + \xi^f \cdot x \) and \( g_{\xi^g} \) is the zero function, then this is just steepest descents with linear noise discussed in Section 2.2. More generally, (60) with \( g_{\xi^g} \) the zero function models stochastic gradient descents, which is a central algorithmic template in many applications. We show how the approach developed above opens the door to an analysis of this basic algorithmic paradigm for nonconvex problems.

**Proposition 4.1.** On the Euclidean space \((\mathbb{R}^n, \| \cdot \|)\) suppose the following hold:

(a) for all \( i \in I_f \), \( \nabla f_i \) is Lipschitz continuous with constant \( L \) on \( G \subset \mathbb{R}^n \) and hypomonotone on \( G \) with violation \( \tau_f > 0 \) on \( G \subset \mathbb{R}^n \):

\[
- \tau_f \| x - y \|^2 \leq \langle \nabla f_i(x) - \nabla f_i(y), x - y \rangle \quad \forall x, y \in G. \tag{61}
\]

(b) there is a \( \tau_g \) such that for all \( i \in I_g \), the (limiting) subdifferential \( \partial g_i \) satisfies

\[
- \frac{\tau_g}{2} \left\| (x^+ + z) - (y^+ + w) \right\|^2 \leq \langle z - w, x^+ - y^+ \rangle. \tag{62}
\]

at all points \((x^+, z) \in \text{gph} \partial g_i \) and \((y^+, w) \in \text{gph} \partial g_i \) where \( z = x - x^+ \) for \( \{ x^+ \} = J_{g_i}(x) \) for any \( x \in \bigcup_{i \in I_f} (\text{Id} - t \nabla f_i)(G) \) and where \( w = y - y^+ \) for \( \{ y^+ \} = J_{g_i}(y) \) for any \( y \in \bigcup_{i \in I_f} (\text{Id} - t \nabla f_i)(G) \).
(c) $T_i^{FB}$ is a self-mapping on $G \subset \mathbb{R}^n$ for all $i$.

Then the following hold.

(i) $T_i^{FB}$ is $\alpha$-fne on $G$ with constant $\alpha = 2/3$ and violation at most

$$
\epsilon = \max\{0, (1 + 2\tau_g) \left(1 + t(2\tau_f + 2tL^2)\right) - 1\}
$$

for all $i \in I$.

(ii) $\Phi(x, i) := T_i x$ is $\alpha$-fne in expectation on $G$ with constant $\alpha = 2/3$ and violation at most $\epsilon$ given in (63).

(iii) The Markov operator $P$ corresponding to (60) is $\alpha$-fne in measure on $\mathcal{P}_2(G)$ with constant $\alpha = 2/3$ and violation no greater than $\epsilon$ given in (63), i.e. it satisfies (24).

(iv) Suppose that assumption (a) holds with condition (61) being satisfied for $\tau_f < 0$ (that is, $\nabla f_i$ is strongly monotone for all $i$), and that condition (62) holds with $\tau_g = 0$ (for instance, when $g_i$ is convex). Then, whenever there exists an invariant measure for the Markov operator $P$ corresponding to (60) for all step lengths $t \in \left(0, \frac{|\tau_f|}{L}\right]$, the distributions of the sequences of random variables converge to an invariant measure in the Prokhorov-Lévy metric.

(v) Let $G$ be compact and $G \cap \text{inv } P \neq \emptyset$. If $\Psi$ given by (29) takes the value 0 only at points in $\text{inv } P$ and is metrically subregular for 0 on $\mathcal{P}_2(G)$ with gauge $\rho$ given by (27) with $\tau = 1/2$, $\epsilon$ satisfying (63), and $\theta$ satisfying (26), then the Markov chain converges to an invariant distribution with rate given by (35).

Before proving the statement, some background for conditions (61) and (62) might be helpful. The inequality (61) is satisfied by functions $f$ that are prox-regular [73]. This traces back to Federer’s study of curvature measures [36] where such functions would be called functions whose epigraphs have positive reach. Inequality (62) is equivalent to the property that $J g_i$ is $\alpha$-fne with constant $\alpha_i = 1/2$ and violation $\tau_g$ on $G$ [61, Proposition 2.3]. Any differentiable function $g_i$ with gradient satisfying (61) with constant $\tau_g/(2(1 + \tau_g))$ will satisfy (62) with constant $\tau_g$. In the present setting, if $g_i$ is prox-regular on $G$, then $\partial g_i$ is hypomonotone on $G$ and therefore satisfies (62) [61]. Of course, convex functions are trivially hypomonotone with constant $\tau = 0$.

Proof. (i). This is [61, Proposition 3.7].

(ii). This follows immediately from Part (i) above and Proposition 2.11.

(iii). This follows immediately from Part (ii) above and Proposition 2.14.

(iv). Inserting the assumptions with their corresponding constants into the expression for the violation (63) shows that the $\epsilon$ is zero for all step-lengths $t \in \left(0, \frac{|\tau_f|}{L}\right]$. So by part (i) we have that, for all step lengths small enough, the mappings $T_i^{FB}$ are $\alpha$-fne with constant $\alpha = 2/3$. (Steps sizes up to twice the upper bound considered here can also be taken, but then the constant $\alpha$ approaches 1.) If the corresponding Markov operator possesses invariant measures, then convergence in distribution of the corresponding Markov chain follows from Theorem 2.18.

(v). This follows from Part (iii) and Theorem 2.19.

The assumptions of Proposition 4.1(iv) are not unusual. What is new is the generality of global convergence. The compactness assumption on $G$ in part (v) is just to permit the application of Theorem 2.19. As noted in Remark 2.20 this assumption can be dropped for mappings $T_i$
on Euclidian space. The result narrows the work of proving convergence of stochastic forward-backward algorithms to verifying existence of \( \text{inv } P \). The next corollary shows how this is done for the special case of stochastic gradient descent.

**Corollary 4.2** (stochastic gradient descent). In problem (57) let \( g_i(x) := 0 \) for all \( i \) at each \( x \). In addition to assumptions of Proposition 4.1, assume that

(i) the expectation \( \mathbb{E}[f_\xi(x)] \) attains a minimum at \( \bar{x} \in G \subset \mathbb{R}^n \) with value \( \mathbb{E}[f_\xi(\bar{x})] = \bar{p} \);

(ii) \( \mathbb{E}\|X_0 - \bar{x}\|^2 \) exists where \( X_0 \) is a random variable with distribution \( \mu_0 \in \mathcal{P}_2(G) \);

(iii) \( \nabla f_i \) is strongly monotone with constant \( |\tau_f| \) for all \( i \) (that is condition (61) is satisfied with \( \tau_f < 0 \)).

Then Algorithm 2 is the stochastic gradient descent algorithm and, for a fixed step length \( t \in (0, \frac{\tau_f}{2L}] \), when initialized with \( \mu_0 \), the distributions of the iterates converge in the Prokhorov-Lèvy metric to an invariant measure of the corresponding Markov operator. Moreover, whenever \( \Psi \) given by (29) takes the value 0 only at points in \( \text{inv } P \) and is metrically subregular for 0 on \( \mathcal{P}_2(G) \) with gauge \( \rho \) given by (27) with \( \tau = 1/2, \epsilon_i \leq \epsilon \) for all \( i \) with \( \epsilon \) given by (63) and \( \theta \) satisfying (26), then the Markov chain converges to a point in \( \text{inv } P \) with rate given by (35).

**Proof.** In this case \( T_i^{FB} := \text{Id} - t\nabla f_i \). If we can show that the corresponding Markov operator possesses invariant measures, then the statement follows from Proposition 4.1.

To establish existence of invariant distributions, note that

\[
\|X_{k+1} - \bar{x}\|^2 = \|X_k - t\nabla f_\xi(X_k) - \bar{x}\|^2 - \|X_{k+1} - X_k - t\nabla f_\xi(X_k)\|^2 = \|X_k - \bar{x}\|^2 - \|X_{k+1} - X_k\|^2 - 2t(\nabla f_\xi(X_k),X_{k+1} - X_k + X_k - \bar{x}).
\]

For functions with Lipschitz continuous gradients the following growth condition holds

\[
(\nabla f_\xi(X_k),X_{k+1} - X_k) \geq f_\xi(X_{k+1}) - f_\xi(X_k) - \frac{L}{2}\|X_{k+1} - X_k\|^2.
\]

Interchanging the gradient and the expectation in \( \mathbb{E}[\nabla f_\xi(x)] \) together with strong monotonicity of the gradients \( \nabla f_i \) yields

\[
\langle \nabla \mathbb{E}[f_\xi(X_k)],X_k - \bar{x}\rangle \geq \mathbb{E}[f_\xi(X_k)] - \bar{p} + \frac{|\tau_f|}{2}\|X_k - \bar{x}\|^2.
\]

It follows that

\[
\mathbb{E}\|X_{k+1} - \bar{x}\|^2 \leq (1 - t|\tau_f|)\mathbb{E}\|X_k - \bar{x}\|^2 - (1 - tL)\mathbb{E}\|X_{k+1} - X_k\|^2 - 2t(\mathbb{E}[f_\xi(X_{k+1})] - \bar{p}) \leq (1 - t|\tau_f|)\mathbb{E}\|X_k - \bar{x}\|^2 + 2\bar{p}t\bar{p}.
\]

This yields

\[
\mathbb{E}\|X_k - \bar{x}\|^2 \leq (1 - t|\tau_f|)^k\mathbb{E}\|X_0 - \bar{x}\|^2 + 2|\tau_f|\bar{p}\sum_{i=0}^{k-1}(1 - t|\tau_f|)^i \leq \mathbb{E}\|X_0 - \bar{x}\|^2 + \frac{2\bar{p}}{|\tau_f|}.
\]

So the sequence \( \left( \mathbb{E}\|X_k - \bar{x}\|^2 \right)_{k \in \mathbb{N}} \) is bounded if \( \mathbb{E}\|X_0 - \bar{x}\|^2 \) exists and hence the existence of an invariant measure follows from Theorem 3.3. \( \square \)

Note that the step size in the stochastic gradient could be large enough that the gradient descent mapping is expansive, even though the gradient is assumed to be strongly monotone. This explains the frequent observation that descent methods perform well even for step sizes larger than the usual analysis recommends.
4.2. Stochastic Douglas-Rachford and X-FEL Imaging

Another prevalent algorithm for nonconvex problems is the Douglas-Rachford algorithm [56]. This is based on compositions of reflected resolvents:

\[ R_f := 2J_f - \text{Id}. \]  \hspace{1cm} (64)

Algorithm 3: Stochastic Douglas-Rachford Splitting

**Initialization:** Set \( X_0 \sim \mu_0 \in \mathcal{P}_2(G), X_0 \sim \mu \), and \((\xi_k)_{k \in \mathbb{N}}\) another i.i.d. sequence with \( \xi_k = (\xi_k^I, \xi_k^J) \) taking values on \( I_f \times I_g \) and \( X_0 \perp \perp (\xi_k) \).

\[ \text{for } k = 0, 1, 2, \ldots \text{ do} \]

\[ X_{k+1} = T_{\xi_k}^{DR}X_k := \frac{1}{2} \left( R_{\xi_k^I} \circ R_{\xi_k^J} + \text{Id} \right) (X_k) \] \hspace{1cm} (65)

Algorithm 3 has been studied for solving large-scale, convex optimization and monotone inclusions (see for example [19, 26]). The result below opens the analysis to nonconvex, nonmonotone problems.

**Proposition 4.3.** On the Euclidean space \((\mathbb{R}^n, \| \cdot \|)\), suppose the following hold:

(a) there is a \( \tau_g \) such that for all \( i \in I_g \), the (limiting) subdifferential \( \partial g_i \) satisfies

\[ -\frac{\tau_g}{2} \| (x^+ + z) - (y^+ + w) \|^2 \leq \langle z - w, x^+ - y^+ \rangle. \] \hspace{1cm} (66)

at all points \((x^+, z) \in \text{gph} \partial g_i \) and \((y^+, w) \in \text{gph} \partial g_i \) where \( z = x - x^+ \) for \( \{x^+\} = J_{g_i}(x) \) for any \( x \in G \subset \mathbb{R}^n \) and where \( w = y - y^+ \) for \( \{y^+\} = J_{g_i}(y) \) for any \( y \in G \).

(b) there is a \( \tau_f \) such that for all \( i \in I_f \), the (limiting) subdifferential \( \partial f_i \) satisfies

\[ -\frac{\tau_f}{2} \| (x^+ + z) - (y^+ + w) \|^2 \leq \langle z - w, x^+ - y^+ \rangle. \] \hspace{1cm} (67)

at all points \((x^+, z) \in \text{gph} \partial f_i \) and \((y^+, w) \in \text{gph} \partial f_i \) where \( z = x - x^+ \) for \( \{x^+\} = J_{f_i}(x) \) for any \( x \in \bigcup_{j \in I_g} \{J_{g_j}(G)\} \) and where \( w = y - y^+ \) for \( \{y^+\} = J_{f_i}(y) \) for any \( y \in \bigcup_{j \in I_g} J_{g_j}(G) \).

(c) \( T_i^{DR} \) is a self-mapping on \( G \subset \mathbb{R}^n \) for all \( i \).

Then the following hold.

(i) For all \( i \in I_f \times I_g \) the mapping \( T_i^{DR} \) defined by (65) is \( \alpha \)-a.e. on \( G \) with constant \( \alpha = 1/2 \) and violation at most

\[ \epsilon = \frac{1}{2} ((1 + 2\tau_g)(1 + 2\tau_f) - 1) \] \hspace{1cm} (68)

on \( G \).

(ii) \( \Phi(x, i) := T_i^{DR}x \) is \( \alpha \)-a.e. in expectation with constant \( \alpha = 1/2 \) and violation at most \( \epsilon \) given by (68).

(iii) The Markov operator \( \mathcal{P} \) corresponding to (65) is \( \alpha \)-a.e. in measure with constant \( \alpha = 1/2 \) and violation no greater than \( \epsilon \) given by (68), i.e. it satisfies (24).
(iv) Suppose that assumptions (a) and (b) hold with conditions (66) and (67) being satisfied for \( \tau_g = \tau_f = 0 \) (i.e., when \( f_i \) and \( g_i \) are convex for all \( i \)). Then, whenever there exists an invariant measure for the Markov operator \( \mathcal{P} \) corresponding to (65), the distributions of the sequences of random variables converge to an invariant measure in the Prokhorov-Lévy metric.

(v) Let \( G \) be compact and \( G \cap \text{inv} \mathcal{P} \neq \emptyset \). If \( \Psi \) given by (29) takes the value 0 only at points in \( \text{inv} \mathcal{P} \) and is metrically subregular for 0 on \( \mathcal{P}_2(G) \) with gauge \( \rho \) given by (27) with \( \tau = 1/2, \epsilon \) satisfying (68), and \( \theta \) satisfying (26), then the Markov chain converges to an invariant distribution with rate given by (35).

Proof. (i). By [61, Proposition 3.7] for all \( j \in I_g \), \( J_{g_j} \) is \( \alpha \)-fne with constant \( \alpha = 1/2 \) and violation \( \epsilon_g = 2\tau_g \) on \( G \). Likewise, for all \( i \in I_f \), \( J_{f_i} \) is \( \alpha \)-fne with constant \( \alpha = 1/2 \) and violation \( \epsilon_f = 2\tau_f \) on \( \bigcup_{j \in I_g} (J_{g_j}(G)) \). By [61, Propositions 2.3-2.4], for all \( i \in I_f \times I_g \) the Douglas-Rachford mapping \( T_i^{\text{DR}} \) is therefore \( \alpha \)-fne with constant \( \alpha = 1/2 \) and violation at most \( \frac{1}{2} ((1 + 2\tau_g)(1 + 2\tau_f) - 1) \) on \( G \).

(ii) - (v) follow in the same way as their counterparts in Proposition 4.1.

4.2.1. Application to X-FEL Imaging

For nonconvex problems, the Douglas-Rachford algorithm is popular because its set of fixed points is often smaller than other popular algorithms [58, Theorem 3.13]. We briefly discuss its application to the problem of X-ray free electron laser imaging, for which the analytical framework establish here is intended.

Here, a high-energy X-ray pulse illuminates molecules suspended in fluid. A two dimensional, low-count diffraction image is recorded for each pulse. The goal is to reconstruct the three-dimensional electron density of the target molecules from the observed two-dimensional diffraction images (on the order of \( 10^9 \)). This is a stochastic tomography problem with a nonlinear model for the data - stochastic because the molecule orientations are random, and uniformly distributed on SO(3). Computed tomography with random orientations has been studied for more than two decades [9, 10] and been successfully applied for inverting the Radon transform (a linear operator) with unknown orientations [71, 83]. The model for the data in X-FEL imaging is nonlinear and nonconvex: Fraunhoffer diffraction with missing phase [15, 91]. The problem of recovering a 2-dimensional slice of the object from diffraction intensity data is the optical phase retrieval problem [76, 79]. The most successful and widely applied methods for solving this problem are fixed point algorithms where the fixed point mappings consist of compositions and averages of projection mappings onto nonconvex sets [59]. A theoretical framework for unifying and extending the first proofs of local convergence of these methods, with rates, was established in [61]. This analysis accommodates mappings that not only are not contractions, but are actually expansive. Moreover, unlike many other approaches, the framework does not require that the constituent mappings have common fixed points. This has been applied to prove, for the first time, local linear convergence of a wide variety of fundamental algorithms for phase retrieval [46, 58, 61, 89].

The goal of X-FEL imaging is to determine the electron density of a molecule from experimental samples of its scattering probability distribution. The physical model for the experiment is

\[
\|(F(\rho))_i\| = \phi_i, \quad \forall \ i = 1, 2, \ldots, n.
\]
Here $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a unitary linear operator accounting for the propagation of an electromagnetic wave, $\rho \in \mathbb{C}^n$ is the unknown electron density that interacts with the wave at one end (the pupil or object plane) of the instrument, and $\phi_i \in \mathbb{R}_+$ is the probability of observing a scattered photon in the $i$'th voxel ($i = 1, 2, \ldots, n$) of the imaging volume. The problem is to determine $\rho$ from $\phi_i$. The set of possible vectors satisfying such measurements is given by

$$C := \{ \rho \in \mathbb{C}^n | \| (F(\rho))_i \| = \phi_i, \ \forall \ i = 1, 2, \ldots, n \}. \quad (70)$$

Although this set is nonconvex, it is prox-regular [57]. To give an idea of the size of this problem, in a typical experiment $n = O(10^9)$.

For this model, we have measurements $Y_\xi$ where $\xi$ is a uniformly distributed random variable which takes values on $SO(3)$. In an X-FEL experiment, $Y_\xi$ is a two-dimensional measurement of photon counts (so, real and nonnegative) on a plane $H_\xi$ passing through the origin in the domain of $\phi$ with orientation $\xi$. The value of $\xi$ is in fact not observable, however, by observing three-electron correlations, this can be estimated [91, 92]. The set $C$ above is then replaced by the random set

$$C(\xi) := \{ \rho \in \mathbb{C}^n | \| (F(\rho))_i \| = (Y_\xi)_i, \ \text{at voxels} \ i \ \text{intersecting} \ H_\xi \}. \quad (71)$$

In addition to the random sets generated by the data, there are certain a priori qualitative constraints that can (and should) be added depending on the type of experiment that has been conducted. Often these are support constraints, or real-valuedness, or nonnegativity. All of these are convex constraints for which we reserve the set $C_0$ for the qualitative constraints.

The problem is a specialization of (57) where $I_f = \{ 1 \}$, $I_g = SO(3)$, $\xi_k^f = 1$ for all $k$, $\xi_k^g$ is a uniformly distributed random variable on $SO(3)$ for all $k$, and

$$|\forall k | \quad f_{\xi_k^f}(\rho) := \frac{\lambda}{2(1-x)} \text{dist}^2(\rho, C_0)$$

$$g_{\xi_k^g}(\rho) := 0 \quad \text{if} \ \rho \in C(\xi_k^g)$$

$$+\infty \quad \text{otherwise.} \quad (72)$$

The algorithm we propose for this problem is Algorithm 3. Assumptions (a) and (b) of Proposition 4.3 are easily verified, and in fact, for this application $\tau_f = 0$ since $C_0$ is convex. In [58] the fixed points of the deterministic version of Algorithm 3 for the simpler, two-dimensional phase retrieval problem have been characterized, and metric subregularity of the transport discrepancy (14) has been determined for geometries applicable to cone and sphere problems [59] such as this. So for a majority of relevant instances, there is good reason to expect that Proposition 4.3 can be applied provably to X-FEL measurements. The determination of the domain $G$ in condition (c) of Proposition 4.3 is therefore key. There are some unresolved cases, however, that are relevant for optical phase retrieval (see [58, Example 5.4]), and this needs further study.

4.3. Inconsistent set feasibility

We conclude this study with our explanation for the numerical behavior observed in Fig. 1. This is an affine feasibility problem:

$$\text{Find} \quad x \in L := \cap_{j \in I} \{ x | \langle a_j, x \rangle = b_j \}. \quad (73)$$

When the intersection is empty we say that the problem is inconsistent. Consistent or not, we apply the method of cyclic projections (2). Even though the projectors onto the corresponding
problems have an analytic expression, this representation can only be evaluated to finite precision. The trick here is to view the algorithm not as inexact cyclic projections onto deterministic hyperplanes, but rather as exact projections onto randomly selected hyperplanes.

Indeed, consider the following generalized affine noise model for a single affine subspace: $H_\tilde{T}(\xi,\zeta) = \{ x \in \mathbb{R}^n \mid \langle a + \xi, x - \tilde{T} \rangle = \zeta \}$, where $a \in \mathbb{R}^n$ and $\tilde{T}$ satisfies $A\tilde{T} = b$ for a given $b \in \mathbb{R}$ and noise $(\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}$ is independent. The key conceptual distinction is that the analysis proceeds with exact projections onto randomly selected hyperplanes $H_\tilde{T}(\xi,\zeta)$, rather than working with inexact projections onto deterministic hyperplanes.

**Proposition 4.4.** Given $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, define the hyperplane $H = \{ y \mid \langle a, y \rangle = b \}$ and fix $\tilde{T} \in H$. Define the random mapping $T(\xi,\zeta) : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(\xi,\zeta)x := P_{H(\xi,\zeta)}x = x - \frac{\langle a + \xi, x - \tilde{T} \rangle - \zeta}{\|a + \xi\|^2}(a + \xi)$$

where $(\xi, \zeta) \in \mathbb{R}^n \times \mathbb{R}$ is a vector of independent random variables satisfying

$$d := E \left[ \frac{\|b + \zeta\|^2}{\|a + \xi\|^2} \right] < \infty, \quad (74a)$$

$$c := \inf_{z \in S} E \left[ \frac{(a + \xi, z)^2}{\|a + \xi\|^2} \right] > 0 \quad (74b)$$

where $S$ is the set of unit vectors in $\mathbb{R}^n$. Algorithm (1) with this random function initialized with any $\mathbb{R}^n$-valued random variable $X_0$ with distribution $\mu^0 \in \mathcal{P}(\mathbb{R}^n)$ converges Q-linearly to a unique invariant distribution.

**Proof.** Each mapping $T(\xi,\zeta)$ is the orthogonal projector onto the hyperplane $H(\xi,\zeta)$, and so is $\omega$-finite with constant $\alpha = 1/2$ (no violation). It follows immediately from the definition, then, that this is both nonexpansive in expectation and $\omega$-finite in expectation with $\alpha = 1/2$. By Proposition 2.14 the corresponding Markov operator $\mathcal{P}$ satisfies (24), provided $\text{inv} \mathcal{P} \neq \emptyset$. We will show below, that there do indeed exist invariant measures, but for the moment, let us just assume this holds.

Since the randomly selected projectors are mappings on a Hilbert space, using the identity (17) the surrogate function $\Psi$ on the space of measures $\mathcal{P}(\mathbb{R}^n)$ defined by (29) can be written as

$$\Psi(\mu) = \inf_{\pi \in \text{inv} \mathcal{P}} \inf_{\gamma \in C^*_\mu(\mu, \pi)} \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} E(\xi,\zeta) \left[ \||x - T(\xi,\zeta)x| - (y - T(\xi,\zeta)y)||^2 \right] \gamma(dx, dy) \right)^{1/2}.$$  

where, recall, $C^*_\mu(\mu, \pi)$ is the set of optimal couplings, that is the set of couplings where $W_2(\mu, \pi)$ is attained. An elementary calculation shows that

$$\| (x - T(\xi,\zeta)x) - (y - T(\xi,\zeta)y) \|^2 = \left( \frac{x - y}{\|x - y\|} \cdot \frac{a + \xi}{\|a + \xi\|} \right)^2 \|x - y\|^2.$$  

Now, we use the assumptions on the random variables $\xi$ and $\zeta$ in (74). Condition (74b) is satisfied for example when $\xi$ is isotropic or radially symmetric; condition (74a) when $\zeta$ has bounded variance and $\|\xi\|$ is bounded away from 1. (Physically, you would interpret this as the noise being bounded away from the signal in energy.) Taking the expectation and using the
assumption on the noise yields

$$
\Psi^2(\mu) = \int_{\mathbb{R}^n \times \mathbb{R}^n} E_{(\xi, \zeta)} \left[ \left\| (x - T_{(\xi, \zeta)}x) - (y - T_{(\xi, \zeta)}y) \right\|^2 \right] \gamma(dx, dy) \quad \gamma \in C_*(\mu, \pi_\mu)
$$

\[ \geq c \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^2 \gamma(dx, dy) \quad \gamma \in C_*(\mu, \pi_\mu)
\]

\[ = c W_2^2(\mu, \pi_\mu). \quad (76) \]

With this, we have shown that $\Psi$ is linearly metrically subregular for 0 on $\mathcal{P}(\mathbb{R}^n)$ with constant $1/\sqrt{c}$ where $c$ is given by (74b). By Remark 2.20, we can then apply Corollary 2.21 to conclude that, if an invariant measure exists, the random function iteration of repeatedly projecting onto the randomly selected affine subspaces converges R-linearly with respect to the Wasserstein metric to an invariant measure.

It remains to show that an invariant measure exists and is unique. This follows from the observation that a much stronger property holds, namely that $\Phi$ is a contraction in expectation. Indeed, the same calculation behind (75) shows that

$$
\|T_{(\xi, \zeta)}x - T_{(\xi, \zeta)}y\|^2 = \left( 1 - \cos^2 \left( \frac{a + \xi}{\|a + \xi\|}, \frac{x - y}{\|x - y\|} \right) \right) \|x - y\|^2.
$$

Again, taking the expectation over $(\xi, \zeta)$ yields

$$
\mathbb{E} \left[ \|T_{(\xi, \zeta)}x - T_{(\xi, \zeta)}y\|^2 \right] \leq (1 - c)\|x - y\|^2.
$$

From Theorem 2.16 we get that there exists a unique invariant measure $\pi_0$ for $\mathcal{P}$ (even $\pi_0 \in \mathcal{P}_2$) and that it satisfies

$$
W_2^2(\mu \mathcal{P}^k, \pi_0) \leq (1 - c)^k W_2^2(\mu, \pi_0).
$$

Convergence is therefore Q-linear, not just R-linear. \qed

Note that the noise satisfying (74) depends implicitly on the point $x$, which will determine the concentration of the invariant distribution of the Markov operator. This corresponds to the fact that the exact projection, while unique, depends on the point being projected. One would expect the invariant distribution to be concentrated on the exact projection.

Extending this model to finitely many distorted affine subspaces as illustrated in Fig. 1 (i.e. we are given $m$ normal vectors $a_1, \ldots, a_m \in \mathbb{R}^n$ and displacement vectors $b_1, \ldots, b_m$) yields a stochastic version of cyclic projections (2) which converges Q-linearly (geometrically) in the Wasserstein metric to a unique invariant measure for the given noise model.

Indeed, for a collection of (not necessarily distinct) points $x_j \in H_j := \{ y \mid \langle a_j, y \rangle = b_j \} \quad (j = 1, 2, \ldots, m)$ denote by $P^j_{(\xi_j, \zeta_j)}$ the exact projection onto the $j$-th random affine subspace centered on $x_j$, i.e.

$$
P^j_{(\xi_j, \zeta_j)}x = x - \frac{\langle a_j + \xi_j, x - x_j \rangle - \zeta_j (a_j + \xi_j)}{\|a_j + \xi_j\|^2} (a_j + \xi_j),
$$

where $(\xi_i)_{i=1}^m$ and $(\zeta_i)_{i=1}^m$ are i.i.d. and $(\xi_i) \perp \perp (\zeta_i)$. The stochastic cyclic projection mapping is

$$
T_{(\xi, \zeta)}x = P^m_{(\xi_m, \zeta_m)} \circ \ldots \circ P^1_{(\xi_1, \zeta_1)}x, \quad x \in \mathbb{R}^n
$$

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Lemma A.3. because then every subsequence has a convergent subsequence.

Remark A.2: In a compact metric space, it is enough, that all cluster points are the same, where \( (\text{Proposition 4.4} \text{ we see that} \) Proposition A.1 A. Appendix W \( ) \) exists a sequence \( (\xi_m, \zeta_m) \to (\xi_1, \zeta_1) \) with limit \( x \). Moreover, separability of \( \mathcal{B}(G) \) with the property (78), then for \( \text{(78)} \) has the same open \( G \) if and only \( \sigma \)-algebra \( \mathcal{B}(G) \) is generated by the family \( \mathcal{A} := \{A_1 \times A_2 | A_1, A_2 \subset G \text{ closed} \} \). One has \( \mathcal{B}(G \times G) \supset \mathcal{B}(G) \otimes \mathcal{B}(G) \).

For the other direction, note that any metric \( d_\infty \) with the property (78) has the same open and closed sets. If \( A \subset G \times G \) satisfying (78), then for \( (a_k, b_k) \in A \) with \( (a_k, b_k) \to (a, b) \in G \times G \text{ w.r.t. } d_\infty \) it holds that \( d_\infty((a_k, b_k), (a, b)) \to 0 \) as \( k \to \infty \), i.e. \( (a, b) \in A \), so \( A \subset G \times G, d_\infty \). It follows that all open sets in \( G \times G, d_\infty \) are the same for any metric that satisfies (78). So, without loss of generality, let

\[
d_\infty ((\frac{x_1}{y_1}, \frac{z_1}{y_1}), (\frac{x_2}{y_2}, \frac{z_2}{y_2})) = \max(d(x_1, x_2), d(y_1, y_2)).
\]

Moreover, separability of \( G \times G \) yields that any open set is the countable union of balls: there exists a sequence \( (u_k) \) on \( U \) that is dense for \( U \subset G \times G \) open. We can find a sequence of constants \( \epsilon_k > 0 \) with \( \bigcup_k \mathcal{B}(u_k, \epsilon_k) \subset U \). If there exists \( x \in U \), which is not covered by any ball, then we may enlarge a ball, so that \( x \) is covered: since there exists \( \epsilon > 0 \) with \( \mathcal{B}(x, \epsilon) \subset U \) and there exists \( m \in \mathbb{N} \) with \( d(x, u_m) < \epsilon/2 \) by denseness, we may set \( \epsilon_m = \epsilon/2 \) to get \( x \in \mathcal{B}(u_m, \epsilon_m) \subset \mathcal{B}(x, \epsilon) \subset U \). Now to continue the proof, let \( d_\infty \) be given by (79). Then for any
open $U \subset G \times G$ there exists a sequence $(u_k)$ on $U$ and a corresponding sequence of positive constants $(\epsilon_k)$ such that $U = \bigcup_k \mathbb{B}(u_k, \epsilon_k)$. This together with the fact that

$$\mathbb{B}(u_k, \epsilon_k) = \mathbb{B}(u_{k,1}, \epsilon_k) \times \mathbb{B}(u_{k,2}, \epsilon_k) \in \mathcal{B}(G) \otimes \mathcal{B}(G) \quad (u_k = (u_{k,1}, u_{k,2}) \in G \times G)$$

yields $\mathcal{B}(G \times G) \subset \mathcal{B}(G) \otimes \mathcal{B}(G)$, which establishes equality of the $\sigma$-algebras. \hfill $\Box$

Lemma A.4 (couplings). Let $G$ be a Polish space and let $\mu, \nu \in \mathcal{P}(G)$. Let $\gamma \in C(\mu, \nu)$, where

$$C(\mu, \nu) := \{ \gamma \in \mathcal{P}(G \times G) \mid \gamma(A \times G) = \mu(A), \gamma(G \times A) = \nu(A) \quad \forall A \in \mathcal{B}(G) \}, \quad (80)$$

then

(i) $\text{supp} \gamma \subset \text{supp} \mu \times \text{supp} \nu$,

(ii) $\{ x \mid (x, y) \in \text{supp} \gamma \} = \text{supp} \mu$.

Proof. We let the product space be equipped with the metric in (79) (constituting a separable complete metric space since $G$ is Polish).

(i) Suppose $(x, y) \in \text{supp} \gamma$ and let $\epsilon > 0$, then

$$\mu(\mathbb{B}(x, \epsilon)) = \gamma(\mathbb{B}(x, \epsilon) \times G) \geq \gamma(\mathbb{B}(x, \epsilon) \times \mathbb{B}(y, \epsilon)) = \gamma(\mathbb{B}((x, y), \epsilon)) > 0.$$

Analogously, we have $\nu(\mathbb{B}(y, \epsilon)) > 0$. So $(x, y) \in \text{supp} \mu \times \text{supp} \nu$.

(ii) Suppose $x \in \text{supp} \mu$, then $\gamma(\mathbb{B}(x, \epsilon) \times G) > 0$ for all $\epsilon > 0$. Since $G$ is Polish, the support of the measure is nonempty whenever the measure is nonzero, and (again, since $G$ is Polish) the support of the measure is closed, there either exists $y \in G$ with $(x, y) \in \text{supp} \gamma$ or there exists a sequence $(x_k, y_k)$ on $\text{supp} \gamma$ with $x_k \to x$ as $k \to \infty$. Hence the assertion follows. \hfill $\Box$

Lemma A.5 (convergence in product space). Let $G$ be a Polish space and suppose $(\mu_k), (\nu_k) \subset \mathcal{P}(G)$ are tight sequences. Let $X_k \sim \mu_k$ and $Y_k \sim \nu_k$ and denote by $\gamma_k = \mathcal{L}((X_k, Y_k))$ the joint law of $X_k$ and $Y_k$. Then $(\gamma_k)$ is tight.

If furthermore, $\mu_k \to \mu \in \mathcal{P}(G)$ and $\nu_k \to \nu \in \mathcal{P}(G)$, then cluster points of $(\gamma_k)$ are in $C(\mu, \nu)$, where the set of couplings $C(\mu, \nu)$ is defined in (80) in Lemma A.4.

Proof. By tightness of $(\mu_k)$ and $(\nu_k)$, there exists for any $\epsilon > 0$ a compact set $K \subset G$ with $\mu_k(G \setminus K) < \epsilon/2$ and $\nu_k(G \setminus K) < \epsilon/2$ for all $n \in \mathbb{N}$, so also

$$\gamma_k(G \times G \setminus K \times K) \leq \gamma_k((G \setminus K) \times G) + \gamma_k(G \times (G \setminus K))$$

$$= \mu_k(G \setminus K) + \nu_k(G \setminus K)$$

$$< \epsilon$$

for all $k \in \mathbb{N}$, implying tightness of $(\gamma_k)$. By Prokhorov’s Theorem, every subsequence of $(\gamma_k)$ has a convergent subsequence $\gamma_{k_j} \to \gamma$ as $j \to \infty$ where $\gamma \in \mathcal{P}(G \times G)$.

It remains to show that $\gamma \in C(\mu, \nu)$. Indeed, since for every $f \in C_b(G \times G)$ we have $\gamma_{n_k} f \to \gamma f$, we can choose $f(x, y) = g(x) \mathbb{1}_G(y)$ with $g \in C_b(G)$. Also,

$$\mu g \leftarrow \mu_{n_k} g = \gamma_{n_k} f \to \gamma f = \gamma(\cdot \times G) g,$$

which implies the equality $\mu = \gamma(\cdot \times G)$. Similarly $\nu = \gamma(G \times \cdot)$ and hence $\gamma \in C(\mu, \nu)$. \hfill $\Box$
Lemma A.6 (properties of the Prokhorov-Lévy distance). Let \((G,d)\) be a separable complete metric space.

(i) The Prokhorov-Lévy distance (Definition 2.7) has the representation

\[
d_P(\mu, \nu) = \inf \left\{ \epsilon > 0 \left| \inf_{(X,Y) \in C(\mu,\nu)} \mathbb{P}(d(X,Y) > \epsilon) \leq \epsilon \right. \right\},
\]

where the set of couplings \(C(\mu, \nu)\) is defined in \((80)\) in Lemma A.4. Furthermore, the inner infimum for fixed \(\epsilon > 0\) is attained and the outer infimum is also attained.

(ii) \(d_P(\mu, \nu) \in [0,1]\).

(iii) \(d_P\) metrizes convergence in distribution, i.e. for \(\mu_k, \mu \in \mathcal{P}(G), k \in \mathbb{N}\) the sequence \(\mu_k\) converges to \(\mu\) in distribution if and only if \(d_P(\mu_k, \mu) \to 0\) as \(k \to \infty\).

(iv) \((\mathcal{P}(G), d_P)\) is a separable complete metric space.

(v) For \(\mu_j, \nu_j \in \mathcal{P}(G)\) and \(\lambda_j \in [0,1], j = 1, \ldots, m\) with \(\sum_{j=1}^m \lambda_j = 1\) we have

\[
d_P\left( \sum_j \lambda_j \mu_j, \sum_j \lambda_j \nu_j \right) \leq \max_j d_P(\mu_j, \nu_j).
\]

Proof. (i) See [86, Corollary to Theorem 11] for the first assertion. To see that the infimum is attained, let \(\gamma_k \in C(\mu, \nu)\) be a minimizing sequence, i.e. for \((X_k, Y_k) \sim \gamma_k\) it holds that \(\mathbb{P}(d(X_k, Y_k) > \epsilon) = \gamma_k(U_\epsilon) \to \inf_{(X,Y) \in C(\mu,\nu)} \mathbb{P}(d(X,Y) > \epsilon)\), where \(U_\epsilon := \{(x,y) | d(x,y) > \epsilon\} \subset G \times G\) is open. The sequence \((\gamma_k)\) is tight and for a cluster point \(\gamma\) we have \(\gamma \in C(\mu, \nu)\) by Lemma A.5. From [72, Theorem 36.1] it follows that 

\[
\gamma(U_\epsilon) \leq \lim \inf_j \gamma_{\epsilon_j}(U_\epsilon).
\]

To see, that the outer infimum is attained, let \((\epsilon_k)\) be a minimizing sequence, chosen to be monotonically nonincreasing with limit \(\epsilon \geq 0\). One has that \(U_\epsilon = \bigcup_k U_{\epsilon_k}\) where \(U_{\epsilon_k} \supset U_{\epsilon_{k+1}}\) and hence \(\gamma(U_\epsilon) = \lim_k \gamma(U_{\epsilon_k}) \leq \lim_k \epsilon_k = \epsilon\).

(ii) Clear by (i).

(iii) See [14].

(iv) See [75, Lemma 1.4].

(v) If \(\epsilon > 0\) is such that \(\mu_j(A) \leq \nu_j(\mathbb{B}(A, \epsilon)) + \epsilon\) and \(\nu_j(A) \leq \mu_j(\mathbb{B}(A, \epsilon)) + \epsilon\) for all \(j = 1, \ldots, m\) and all \(A \in \mathcal{B}(G)\), then also \(\sum_j \lambda_j \mu_j(A) \leq \sum_j \lambda_j \nu_j(\mathbb{B}(A, \epsilon)) + \epsilon\) as well as \(\sum_j \lambda_j \nu_j(A) \leq \sum_j \lambda_j \mu_j(\mathbb{B}(A, \epsilon)) + \epsilon\).

\[
\square
\]

Lemma A.7 (properties of the Wasserstein metric). Recall \(\mathcal{P}_p(G)\) and \(W_p\) from Definition 2.7.

(i) The representation of \(\mathcal{P}_p(G)\) is independent of \(x\) and for \(\mu, \nu \in \mathcal{P}_p(G)\) the distance \(W_p(\mu, \nu)\) is finite.

(ii) The distance \(W_p(\mu, \nu)\) is attained when it is finite.

(iii) The metric space \((\mathcal{P}_p(G), W_p(G))\) is complete and separable.
(iv) If $W_p(\mu_k, \mu) \to 0$ as $k \to \infty$ for the sequence $(\mu_k)$ on $\mathcal{P}(G)$, then $\mu_k \to \mu$ as $k \to \infty$.

Proof.  
(i) See [90, Remark after Definition 6.4].

(ii) From Lemma A.5 we know that a minimizing sequence $(\gamma_k)$ for $W_p(\mu, \nu)$ is tight and hence there is a cluster point $\gamma \in C(\mu, \nu)$. By continuity of the metric $d$ it follows that $d$ is lower semi-continuous and bounded from below and from [87, Theorem 9.1.5] it follows that $\gamma d \leq \lim \inf_j \gamma_k d = W_p(\mu, \nu)$.

(iii) See [90, Theorem 6.9].

(iv) See [90, Theorem 6.18].

Note that the converse to Lemma A.7 (iv) does not hold.

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