TODA SOLITON MASS CORRECTIONS
AND THE PARTICLE–SOLITON DUALITY CONJECTURE

G.W. DELIUS
Department of Mathematics
King’s College London
Strand, London WC2R 2LS, UK
e-mail: delius@mth.kcl.ac.uk

M.T. GRISARU
Brandeis University
Waltham, MA 02254, USA
e-mail: grisaru@binah.cc.brandeis.edu

ABSTRACT

We compute quantum corrections to soliton masses in affine Toda theories with imaginary exponentials based on the nonsimply-laced Lie algebras $c_n^{(1)}$. We find that the soliton mass ratios renormalize nontrivially, in the same manner as those of the fundamental particles of the theories with real exponentials based on the nonsimply-laced algebras $b_n^{(1)}$. This gives evidence that the conjectured relation between solitons in one Toda theory and fundamental particles in a dual Toda theory holds also at the quantum level. This duality can be seen as a toy model for S-duality.

*Supported by Habilitationsstipendium der Deutschen Forschungsgesellschaft
†On leave from Department of Physics, Bielefeld University, Germany
‡Work partially supported by the National Science Foundation under grant PHY-92-22318.
1 Introduction.

For the past few years affine Toda theories have provided an interesting arena for studying properties of two-dimensional integrable field theories. They are theories of $r$ real bosonic fields $\vec{\phi}$ coupled via exponential interactions, with lagrangians of the form

$$L = -\frac{1}{2} \vec{\phi} \cdot \Box \vec{\phi} - \frac{m^2}{\beta^2} \sum_i \tilde{n}_i \left( e^{\tilde{\beta} \vec{a}_i \cdot \vec{\phi}} - 1 \right)$$

(1.1)

where the $\vec{a}_i$ are the simple roots of a rank $r$ affine Lie algebra $\hat{\mathfrak{g}}$, the $n_i$ are Kač labels, $m$ sets the mass scale and $\tilde{\beta}$ is a coupling constant. For real $\tilde{\beta} = \beta$ the lagrangians describe systems of $r$ massive particles created by the fields $\vec{\phi}$. For purely imaginary $\tilde{\beta} = i\beta$ the theories possess soliton solutions.

The knowledge of the Toda theories is most complete in the real $\tilde{\beta}$ regime. Because the theories possess higher spin conserved charges [1, 2, 3, 4], even at the quantum level [5, 6, 7], the S-matrices for the scattering of the elementary particles are factorizable and have been determined exactly [8, 9]. One makes the remarkable observation [9] that the S-matrix of the Toda theory for one Kač-Moody algebra $\hat{\mathfrak{g}}$ at real coupling $\beta$ is equal to the S-matrix of the Toda theory for the dual Kač-Moody algebra $\hat{\mathfrak{g}}$ at coupling $4\pi/\beta$. Thus there is a strong coupling – weak coupling duality.

Many elegant results have also been obtained about the classical soliton solutions of the Toda theories with imaginary $\tilde{\beta}$ [10]–[20]. In particular their masses and more recently [21] the values of the higher spin conserved charges have been calculated. Again one makes another observation [11] of a different duality: the classical masses and charges of the solitons of the Toda theory based on an untwisted Kač-Moody algebra $\mathfrak{g}(1)$ are proportional to the unrenormalized masses and charges of the fundamental particles of the Toda theory based on a dual untwisted Kač-Moody algebra $\hat{\mathfrak{g}}(1)$.

These dualities are in close analogy to the conjectured duality between the magnetically charged monopoles and the electrically charged gauge particles in Yang-Mills theory [22, 23]. In its strong form this conjecture states that the magnetic monopoles of one theory can be reinterpreted as the gauge particles of another Yang-Mills theory with a dual gauge group and with the inverse coupling constant. This conjecture has recently become important because it is a consequence of S-duality in string theory [24, 25]. The relation in affine Toda theory between soliton and particle masses and the duality which relates strong and weak coupling while interchanging an algebra and its dual algebra [3] can be seen as a simplified laboratory for these ideas.

The Olive-Montonen duality conjecture [22, 23] was based on the classical mass-relation between the monopoles and the gauge particles. The conjecture was then further pursued only in cases when supersymmetry forbade any quantum corrections to the mass formulas [26, 27], so that the classical relation was guaranteed to persist. It was feared that any quantum corrections would spoil the duality; however, explicit calculations were too difficult to perform. In this paper we will perform calculations in the simpler Toda theories and see that here the relation between soliton and
particle masses persists even in the presence of non-trivial quantum corrections.

We will calculate the first quantum corrections to the masses of the solitons of the $c_n^{(1)}$ Toda theory. We will find that the mass ratios renormalize non-trivially. Indeed they renormalize in the same way as the mass ratios of the fundamental particles of the (dual) $b_n^{(1)}$ Toda theory found in [9]. Thus the duality relation is preserved, at least at the first order in quantum corrections.

In [28] Hollowood has calculated the first quantum correction to the masses of the solitons of (self-dual) $a_n^{(1)}$ Toda theory. He finds that in that case the mass ratios do not receive quantum corrections. This phenomenon of non-renormalization of mass-ratios holds also for the fundamental particles of all Toda theories based on self-dual Kač-Moody algebras. Thus Hollowood’s result is in agreement with the quantum duality conjecture.

In the case of $c_2^{(1)}$ Toda theory quantum corrections to the soliton masses have been computed by Watts [29] using essentially the same techniques. Our results are in disagreement with his when specialized to the $c_2^{(1)}$ case.

We use techniques similar to those employed by Hollowood [28], as borrowed from Dashen et al. [30]. The classical soliton states for the nonsimply-laced case, given by the Hirota solution, are constructed by folding from the simply-laced ones. Quantum corrections are obtained by looking at zero-point energies for oscillations around the classical solutions.

Our paper is organized as follows: in the next section we describe the classical soliton solutions for the $c_n^{(1)}$ theories. In section 3 we discuss the spectrum of perturbations around these classical solutions and compute the mass corrections, the result being given in (3.51) and (3.52). Section 4 contains a discussion.

## 2 Classical solutions in the $c_n^{(1)}$ theories

In this section, primarily to fix the notation, we review the Hirota construction of the classical solutions of Toda theories. The solutions for the nonsimply-laced algebras can be obtained by folding from those for the simply-laced algebras. In the present case we have constructed the soliton solutions of the $c_n^{(1)}$ theories from the solutions of the $a_{2n-1}^{(1)}$ theories described by Hollowood [10]. Other methods exist now for achieving the same goal [14].

The classical field equations of the Toda theories are given by

$$\Box \tilde{\phi} = -\frac{m^2}{i\beta} \sum_{j=0}^{r} n_j \alpha_j^j e^{i\beta \alpha_j^j \tilde{\phi}}$$

(2.1)

The Hirota solution starts with the Ansatz

$$\tilde{\phi} = \frac{i}{\beta} \sum_{j=0}^{r} \frac{2\tilde{\alpha}_j}{\alpha_j^j} \ln \tau_j$$

(2.2)
with the functions $\tau_j(x, t)$ to be determined. Since the set of roots $\vec{\alpha}_j$, $j = 0, 1, \ldots, r$ is overcomplete the Ansatz has an invariance which can be used to make a gauge choice

$$D^2 \tau_0 = m^2 \alpha_0^2 \left( \prod_{k=0}^{r} \tau_k^{1-K_{k0}+2\delta_{k0}} - \tau_0^2 \right) \tag{2.3}$$

where

$$D^2 \tau \equiv \dddot{\tau} - \dot{\tau}^2 - \tau'' + \tau'^2. \tag{2.4}$$

$K_{jk} \equiv 2\vec{\alpha}_j \cdot \vec{\alpha}_k/\alpha^2_j$ is the Cartan matrix, and dots and primes indicate time and space derivatives respectively. Substituting (2.2) and (2.3) into (2.1) gives the $\tau$ equations of motion

$$D^2 \tau_j = m^2 \alpha_j^2 \left( \prod_{k=0}^{r} \tau_k^{1-K_{kj}+2\delta_{kj}} - \tau_j^2 \right), \quad j = 1, \ldots, r \tag{2.5}$$

which therefore take the same form as the gauge condition in (2.3).

The $a_{2n-1}^{(1)}$ theory has rank $r = 2n - 1$, and roots of equal length $\alpha_j^2 = 2$ for all $j = 0, 1, \ldots, 2n - 1$. There exist $2n - 1$ single soliton solutions of the equations given by [10]

$$\tau_j^{(a)} = 1 + e^{\Omega_a \omega^ja}, \quad a = 1, \ldots, 2n - 1 \tag{2.6}$$

with

$$\Omega_a = \sigma_a(x - v_at) - \xi_a, \quad \omega = e^{\frac{2\pi}{\alpha}} \tag{2.7}$$

and

$$\sigma_a^2(1 - v_a^2) = 4m^2 \sin^2 \frac{a\pi}{2n} \equiv m_a^2 \tag{2.8}$$

$v_a$ is the velocity, $1/\sigma_a$ the width of the soliton and relation (2.8) expresses the Lorentz contraction. The real part of $\xi_a$ determines the center of mass position, while the imaginary part (which falls into $2n - 1$ equivalence classes) determines the asymptotic values of the soliton solution, and therefore its topological charge [10, 18, 19].

The Hirota Ansatz gives two-soliton solutions by a nonlinear superposition

$$\tau_j^{(ab)} = 1 + e^{\Omega_a \omega^ja} + e^{\Omega_b \omega^jb} + A_{ab}e^{\Omega_a+\Omega_b}j^{(a+b)} \tag{2.9}$$

with the “interaction” function

$$A_{ab} = -\frac{(\sigma_a - \sigma_b)^2 - (\sigma_a v_a - \sigma_b v_b)^2 - m_{a-b}^2}{(\sigma_a + \sigma_b)^2 - (\sigma_a v_a + \sigma_b v_b)^2 - m_{a+b}^2} \tag{2.10}$$

The single soliton solutions for the $c_n^{(1)}$ theory are constructed from these two-soliton solutions. We denote the roots of $c_n^{(1)}$ by $\vec{\alpha}$:

$$\tilde{\alpha}_0 = \vec{\alpha}_0, \quad \tilde{\alpha}_n = \vec{\alpha}_n$$

$$\tilde{\alpha}_j = \frac{1}{2}(\vec{\alpha}_j + \vec{\alpha}_{2n-j}) \tag{2.11}$$
It follows then from the general Ansatz (2.2) that for any Hirota solution of the $a_{2n-1}^{(1)}$ theory with $\tau_j = \tau_{2n-j}$ we obtain a solution of the $c_n^{(1)}$ theory

$$\tilde{\phi} = \sum_{j=0}^{n} \tilde{\alpha}_j \frac{2}{\tilde{\alpha}_j} \ln \tilde{\tau}_j$$

(2.12)

by setting $\tilde{\tau}_j = \tau_j$, so that $\tilde{\phi} = \tilde{\phi}$. Solutions $\tilde{\phi}$ of the $a_{2n-1}^{(1)}$ theory satisfying $\tau_j = \tau_{2n-j}$ are given by the two-soliton solutions (2.9) with soliton $a$ and soliton $2n - a$ moving together, i.e. with

$$b = 2n - a, \quad \Omega_b = \Omega_a$$

(2.13)

Thus, the $n$ one-soliton solutions of the $c_n^{(1)}$ theory are given by

$$\tilde{\tau}_j^{(a)} = 1 + e^{\Omega_a \delta_j^{(a)}} + e^{2\Omega_a A^{(a)}} , \quad a = 1, \ldots, n$$

(2.14)

with

$$\delta_j^{(a)} = \omega_j^a + \omega_j^{(2n-a)} = 2 \cos \left( \frac{\pi}{n} a j \right)$$

$$A^{(a)} = A_{a,2n-a} = \cos^2 \left( \frac{\pi}{2n} a \right)$$

(2.15)

and $\Omega_a$ given by (2.7). We note that for $a = n$ the solution for the $c_n^{(1)}$ theory is identical to that for the $a_{2n-1}^{(1)}$ theory (up to a shift in $\xi$):

$$\tilde{\tau}_j^{(n)} = 1 + 2e^{\Omega_n (-1)^j}$$

(2.16)

The two-soliton solutions $\tilde{\tau}_j^{(ab)}$ of the $c_n^{(1)}$ theory are similarly obtained from the four-soliton solutions of the $a_{2n-1}^{(1)}$ theory

$$\tilde{\tau}_j^{(ab)} = 1 + e^{\Omega_1} + e^{\Omega_2} + e^{\Omega_3} + e^{\Omega_4}$$

$$+ e^{\Omega_1 + \Omega_2} A_{12} + e^{\Omega_1 + \Omega_3} A_{13} + \cdots$$

$$+ e^{\Omega_1 + \Omega_2 + \Omega_3} A_{12} A_{13} A_{23} + \cdots$$

$$+ e^{\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4} A_{12} A_{13} A_{14} A_{23} A_{24} A_{34}$$

(2.17)

for solitons $a, 2n - a, b, 2n - b$, with

$$e^{\Omega_1} = e^{\Omega_a \omega^a} , \quad e^{\Omega_2} = e^{\Omega_a \omega^{-a}}$$

$$e^{\Omega_3} = e^{\Omega_b \omega^b} , \quad e^{\Omega_4} = e^{\Omega_b \omega^{-b}}$$

(2.18)

and

$$A_{12} = A_{a,2n-a} , \quad A_{13} = A_{ab} , \quad A_{14} = A_{a,2n-b} , \cdots$$

(2.19)

The classical masses of the solitons in the $c_n^{(1)}$ theory are obtained from expressions for the energy and momentum of the Toda field theory by substituting in these expressions the soliton solutions [10]. This leads to

$$M = -2(1 - v^2)^{-\frac{1}{2}} \frac{m^2}{\beta^2} \int dx \sum_{j=0}^{n} n_j \left( e^{i\beta \tilde{\alpha}_j \tilde{\phi}} - 1 \right).$$

(2.20)
From the equations of motion (2.5) we obtain

\[
(e^{i\beta \vec{a}_j \cdot \vec{\phi}} - 1) = \frac{2}{m^2 n_j \alpha_j^2} \frac{D^2 \tau_j}{\tau_j^2}
\]  

(2.21)

For the \( c_n^{(1)} \) theory we have \( n_j \alpha_j^2 = 2 \) and

\[
D^2 \tau_j = \sigma^2 (v^2 - 1) \left( e^{\Omega \delta_j} + 4e^{2\Omega} A + e^{3\Omega} \delta_j A \right)
\]  

(2.22)

By a change of variables \( y = e^{\Omega} \) we can rewrite the integral in (2.20) as

\[
\int_{-\infty}^{\infty} dx \left( e^{i\beta \vec{a}_j \cdot \vec{\phi}} - 1 \right) = \int_{0}^{\infty} dy \frac{\sigma (v^2 - 1) \delta_j + 4Ay + \delta_j Ay^2}{m^2 (1 + \delta_j y + Ay^2)^2}
\]

\[
= \frac{\sigma (v^2 - 1)}{m^2} \left[ \frac{- (2 + \delta_j y)}{1 + \delta_j y + Ay^2} \right]_{0}^{\infty}
\]  

(2.23)

We note that for soliton \( n \), with \( A^{(n)} = 0 \), the square bracket evaluates to 1, whereas for the other solitons, with \( A^{(a)} \neq 0 \), it evaluates to 2. Using (2.8) and \( \sum_{j=0}^{n} n_j = 2n \) we find then the soliton rest masses for the \( c_n^{(1)} \) theory

\[
M_a = \frac{8n}{\beta^2} m_a = \frac{16n}{\beta^2} m \sin \frac{a\pi}{2n}, \quad a = 1, \ldots, n - 1
\]

\[
M_n = \frac{4n}{\beta^2} m_n = \frac{8n}{\beta^2} m
\]  

(2.24)

Up to an overall factor of \( 8n/\beta^2 \) these are the masses of the elementary particles in the real coupling \( b_n^{(1)} \) affine Toda theory. This reestablishes the, by now, well known fact that the classical masses of the solitons of the Toda theory based on an untwisted affine Kač-Moody algebra \( g^{(1)} \) are proportional to the classical masses of the fundamental particles of the Toda theory based on the dual untwisted Kač-Moody algebra \( \tilde{g}^{(1)} \). We want to extend this observation to the quantum level.

### 3 Quantum corrections to the soliton masses

In this section we compute the first quantum corrections to the masses of the solitons in the \( c_n^{(1)} \) theory. We follow the traditional procedure described for example in the textbook by Rajaraman [32] and used in ref. [28] for the case of \( a_n^{(1)} \) solitons, where quantum corrections are obtained from the zero-point fluctuations of the field around the classical solutions. We split the lagrangian into kinetic and potential terms

\[
\mathcal{L}[\vec{\phi}] = \frac{1}{2} \int \left( \frac{d\vec{\phi}}{dt} \right)^2 dx - V[\vec{\phi}]
\]

(3.1)

\[
V[\vec{\phi}] = \int dx \left( \frac{1}{2} \left( \frac{d\vec{\phi}}{dx} \right)^2 - \frac{m^2}{\beta^2} \sum_i n_i (e^{i\beta \vec{a}_i \cdot \vec{\phi}} - 1) \right)
\]  

(3.2)

5
and expand the potential $V[\phi]$ to second order in the small fluctuations $\vec{\eta}(x) = \vec{\phi}(x) - \vec{\phi}_s(x)$ around the static classical soliton solution $\vec{\phi}_s(x)$

$$V[\phi] = V[\vec{\phi}_s] + \int dx \frac{1}{2} \vec{\eta}^a(x) \mathcal{A}_{ab} \vec{\eta}^b(x) + \mathcal{O}(\vec{\eta}^3),$$  \hspace{1cm} (3.3)

$$\mathcal{A}_{ab} = -\delta_{ab} \frac{d^2}{dx^2} + m^2 \sum_i n_i \alpha_i^a \alpha_i^b e^{i \beta_i \vec{\alpha}_i \cdot \vec{\phi}_s}.$$  \hspace{1cm} (3.4)

We look for a complete set of orthogonal eigenfunctions $\vec{\eta}_k(x)$ of the operator $\mathcal{A}$

$$\mathcal{A} \vec{\eta}_k(x) = \nu_k^2 \vec{\eta}_k(x)$$  \hspace{1cm} (3.5)

Note that $\mathcal{A}$ is not hermitean, but it is symmetric and so its eigenfunctions are orthogonal without complex conjugation, i.e.,

$$\int dx \eta_k(x) \eta_{k'}(x) = \delta_{k,k'}$$  \hspace{1cm} (3.6)

We expand $\vec{\eta}(x,t) = \vec{\phi}(x,t) - \vec{\phi}_s(x,t) = \sum_k q_k(t) \vec{\eta}_k(x)$

$$\vec{\eta}(x,t) = \vec{\phi}(x,t) - \vec{\phi}_s(x,t) = \sum_k q_k(t) \vec{\eta}_k(x)$$  \hspace{1cm} (3.7)

to rewrite (3.1) as

$$L[\vec{\phi}] = L[\vec{\phi}_s] + \frac{1}{2} \sum_k (q_k(t))^2 - \frac{1}{2} \sum_k \nu_k^2 (q_k(t))^2 + \text{higher-order terms}.$$  \hspace{1cm} (3.8)

To quantize this, we promote the $q_k$ and $\dot{q}_k$ to canonical pairs of operators. Their zero-point fluctuations will give us the quantum corrections to the energy and thus the rest mass of the soliton $\vec{\phi}_s$. Also the vacuum energy obtains similar corrections and we will subtract them since only the energy difference to the vacuum energy is relevant.

$$M = M_{\text{classical}} + \left(\frac{1}{2} \hbar \sum_k \nu_k\right)_{\text{soliton}} - \left(\frac{1}{2} \hbar \sum_l \nu_l\right)_{\text{vacuum}} + \mathcal{O}(\hbar^2).$$  \hspace{1cm} (3.9)

The quantization procedure outlined above is the standard technique for obtaining the quantum corrections to the energy of the quantum state arising from a static classical solution. It does however not explain why there is a quantum state associated to the classical solution. This is particularly unclear in light of the fact that the soliton solutions in affine Toda theory are complex, even though the fields $\phi$ in the Toda lagrangian (1.1) are required to be real. To see that a quantum state is associated to every classical solution, even complex ones, one has to use the path-integral techniques described by Dashen et al in ref. [30, 31]. One writes the Green’s function $t r (E - H)^{-1}$ in terms of a functional integral of the classical action over the real field configurations $\phi$ with periodic (in time) boundary conditions. One approximates this path-integral by saddle-point/steepest-descent techniques (which may necessitate excursions into the complex $\phi$-plane). One then extracts the quantum energy spectrum from the resulting expression and finds that a tower of energy levels arises from each saddle point of the action, i.e., from every solution of the classical equations of motion. In this point of view it is clear that, although the theory under consideration may be described by real fields, complex solutions of the classical equations contribute to the physical spectrum. The equation for the quantum masses of the solitons obtained from this approach is identical to (3.9).
3.1 The eigenmodes

We begin by finding the orthogonal eigenfunctions \( \vec{\eta}_k(x) \) of \( A \) using the following trick: if one has a second classical solution \( \vec{\phi}_{2s} \) to the field equations (2.1) then the difference \( \vec{\rho}(x, t) = \vec{\phi}_{2s}(x, t) - \vec{\phi}_s(x, t) \) satisfies the equation

\[
\Box \vec{\rho} = -m^2 \sum_{j=0}^{n} n_j \vec{\alpha}_j (e^{i\beta \vec{\alpha}_j \cdot (\vec{\phi}_s + \vec{\rho})} - e^{i\beta \vec{\alpha}_j \cdot \vec{\phi}_s}) \tag{3.10}
\]

If the time-dependence of \( \rho \) can be Fourier expanded in the form

\[
\vec{\rho}(x, t) = \sum_{l=1}^{\infty} e^{il\nu t} \vec{\rho}(l)(x) \tag{3.11}
\]

then the first Fourier component \( \vec{\rho}(1)(x) \) satisfies

\[
\left(-\nu^2 - \frac{d^2}{dx^2}\right) \vec{\rho}(1) = -m^2 \sum_i n_i \vec{\alpha}_i (\vec{\alpha}_i \cdot \vec{\rho}(1)) e^{i\beta \vec{\alpha}_i \cdot \vec{\phi}_s} \tag{3.12}
\]

because \( O(\rho^2) \) terms do not contribute. Thus \( \vec{\eta}(x) = \vec{\rho}(1)(x) \) is an eigenfunction of \( A \) with eigenvalue \( \nu \).

In our case we choose \( \vec{\phi}_{2s} \) to be a two-soliton solution where one of the solitons is described by \( \vec{\phi}_s \). Thus, for any soliton \( \vec{\phi}^{(a)} \) we consider the functions

\[
\vec{\rho} = \vec{\phi}^{(ab)} - \vec{\phi}^{(a)} \tag{3.13}
\]

The complete set of eigenfunctions of \( A \) is obtained by considering all values of \( b \), and in addition letting soliton \( b \) have all possible (real and imaginary) momenta, restricted only by the condition that \( \rho \) should be bounded.

We write the two-soliton solution as in (2.17), with

\[
\Omega_a = m_a x \\
\Omega_b = i(kx + \nu t) - \xi \tag{3.14}
\]

i.e. we put the soliton \( a \) at rest at the origin, while for soliton \( b \) the frequency of oscillation is, according to (2.7), (2.8),

\[
\nu^2 = k^2 + m_b^2. \tag{3.15}
\]

Here \( k \) can be real or imaginary, but \( \nu \) will turn out to be real. We find then from (2.10)

\[
A_{12} = \cos^2 \frac{a\pi}{2n} = A^{(a)} \quad , \quad A_{34} = \cos^2 \frac{b\pi}{2n} = A^{(b)} \\
A_{13} = A_{24} = A_+ \quad , \quad A_{14} = A_{23} = A_- \\
A_{\pm} = -\frac{m_a^2 + m_b^2 - m_{a+b}^2 - 2ikm_a}{m_a^2 + m_b^2 - m_{a+b}^2 + 2ikm_a} \tag{3.16}
\]
Defining $\delta \tau = \tau^{(ab)} - \tau^{(a)}$ we expand

$$\vec{\rho} = \frac{i}{\beta} \sum_{j=0}^{n} \frac{2}{\alpha_j^2} \vec{\alpha}_j \left( \frac{\delta \tau_j}{\tau_j} + \mathcal{O} \left( \frac{\left( \frac{\delta \tau_j}{\tau_j} \right)^2}{\tau_j} \right) \right)$$

(3.17)

Here, from the two-soliton solution,

$$\delta \tau_j = e^{i(kx + \nu t - \xi)} \left[ \delta_j^{(b)} + \left( \delta_j^{(a+b)} A_+ + \delta_j^{(a-b)} A_- \right) e^{m_a x} + \delta_j^{(b)} A_+ A_- A^{(a)} e^{2m_a x} \right]$$

+ $e^{2i(kx + \nu t - 2\xi)} A^{(b)} \left[ 1 + \delta_j^{(a)} A_+ A_- e^{m_a x} + A^2_+ A^2_- A^{(a)} e^{2m_a x} \right]$  \hspace{1cm} (3.18)

and

$$\tau_j = 1 + \delta_j^{(a)} e^{m_a x} + A^{(a)} e^{2m_a x}$$ \hspace{1cm} (3.19)

Thus we see that $\rho$ does indeed have an expansion of the form (3.11) and we obtain the eigenfunctions of $A$

$$\vec{\eta}_j^{(b)} = \vec{\rho}_{(1)} \sum_{j=0}^{n} \frac{2}{\alpha_j^2} \vec{\alpha}_j \tilde{\eta}_j$$ \hspace{1cm} (3.20)

$$\tilde{\eta}_j = \frac{\delta_j^{(b)} + \left( \delta_j^{(a+b)} A_+ + \delta_j^{(a-b)} A_- \right) e^{m_a x} + \delta_j^{(b)} A_+ A_- A^{(a)} e^{2m_a x}}{1 + \delta_j^{(a)} e^{m_a x} + A^{(a)} e^{2m_a x}} e^{ikx - \xi}$$ \hspace{1cm} (3.21)

The parameter $\xi$ determines only the normalization of the eigenmodes.

We have obtained all eigenfunctions of the second order differential operator $A$. This follows from the fact that we have $n$ solutions ($b = 1 \ldots n$) for any possible asymptotic behaviour at $x = -\infty$

$$\lim_{x \to -\infty} \tilde{\eta}_j = \delta_j^{(b)} e^{ikx - \xi}$$ \hspace{1cm} (3.22)

and the fact that the vectors $\sum_{j=0}^{n} \frac{2}{\alpha_j^2} \delta_j^{(b)} \vec{\alpha}_j$, $b = 1, \ldots, n$ are a complete basis of the root space.

We require that $\eta$ be bounded. This will be true for any real values of $k$ ("scattering states" to be treated in the next subsection) and for some discrete imaginary values of $k$ ("bound states" to be treated in subsection 3.3) determined by zeroes of $A_{\pm}(k)$.

### 3.2 Scattering states contributions

The sum over contributions from real values of $k$ is done in the same manner as in section 3.2 of ref. [28] by Hollowood, with minor modifications for the case of the $c_n^{(1)}$ theory. We summarize the procedure for completeness and refer the reader to that reference for further details.

All real values of $k$ give acceptable periodic solutions, with the frequency determined by (3.15). However, it is convenient to discretize the sum over zero point energies (and thus eliminate an infrared divergence ), by putting the soliton system
in a box of size $L$ and imposing suitable boundary conditions on the solutions. For this purpose it is simpler to place the center-of-mass of soliton $\phi^{(a)}$ at $x = L/2$, thus shifting in (3.21) $m_a x \to m_a (x - L/2)$, and impose boundary conditions that the solutions vanish at $x = 0$ and $x = L$.

Acceptable solutions that vanish at $x = 0$ are given by

$$\tilde{\eta}(k, x) - \tilde{\eta}(-k, x)$$

Requiring that the solutions vanish also at $x = L$ (in the limit of large $L$) restricts the values of $k$ according to

$$k_p L + \rho_b(k) = \pi p , \quad p \in \mathbb{Z}$$

where

$$\rho_b(k) = -\frac{i}{4} \epsilon_a \ln \frac{A_+(k)A_-(k)}{A_+(-k)A_-(k)}$$

where $\epsilon_a = 2 - \delta_{a,n}$.

As discussed in ref. [28] the sum over the modes diverges and two renormalizations are required. The first one, as written in (3.9), corresponds to subtracting from the zero-point energies of the modes around the soliton solution the zero point energies of the modes around the vacuum (corresponding to $\rho_b(k) = 0$ in (3.24)). This removes a quadratic divergence. Thus, the scattering states contributions to the soliton mass are given by the sum

$$\Delta M_a = \frac{1}{2} \sum_{b=1}^n \sum_{k_p \geq 0} \left[ \sqrt{k_p^2 + m_b^2} - \sqrt{[k_p + \rho_b(k)]^2 + m_b^2} \right]$$

Following Hollowood we have taken $L \to \infty$ and replaced the sum over $p$ by an integral over $k$

$$\sum_{p \geq 0} = L \left[ \int_{-\infty}^{\infty} \frac{dk}{2\pi} + O(L^{-1}) \right]$$

with $\epsilon_b(k) = \sqrt{k^2 + m_b^2}$.

A second renormalization is required in order to remove a logarithmic divergence in the integral. As discussed by Hollowood it is achieved by normal-ordering the original Toda Hamiltonian. We arrive at the following expression for the corrections from scattering states to the mass of soliton $a$ (corresponding to eq. (3.25) in [28])

$$\Delta^{\text{scattering}} M_a = m_a \epsilon_a \left( -\frac{n + 1}{2\pi} + \sum_{b=1}^n \Omega_b \right)$$

(3.28)
where

\[
\Omega_b = \int_{-\infty}^{\infty} \frac{dk}{4\pi} \left( \frac{1}{2} \left( \frac{m_b}{m} \right)^2 \frac{1}{\sqrt{k^2 + m_b^2}} \right.
\]
\[
- \sqrt{k^2 + m_b^2} \left[ \frac{2(m_a^2 + m_b^2 - m_{a+b}^2)}{(m_a^2 + m_b^2 - m_{a+b}^2)^2 + 4m_b^2k^2} + \frac{2(m_a^2 + m_b^2 - m_{a-b}^2)}{(m_a^2 + m_b^2 - m_{a-b}^2)^2 + 4m_b^2k^2} \right] \right)
\]

The first term in the integrand is the normal-ordering counterterm.

The integral is straightforward to perform. Substituting the explicit expressions in (2.8) for the masses and making the change of variable

\[
k = m_b \frac{u}{\sqrt{1 - u^2}}
\]

we obtain

\[
\Omega_b = \frac{1}{\pi} \sin \frac{2\pi}{2n} \cot \left( \frac{(a+b)\pi}{2n} \right) \tan^{-1} \frac{1}{\cot \left( \frac{(a+b)\pi}{2n} \right)} + \Omega_b^\infty
\]

where the inverse tangent is evaluated on the principal branch between 0 and \(\pi/2\) and

\[
\Omega_b^\infty = \frac{1}{2\pi} \cot \frac{a\pi}{n} \sin \frac{2b\pi}{n} \int_{-1}^{1} \frac{du}{1 - u^2}
\]

In the sum of (3.28) the terms coming from \(\Omega_b^\infty\) cancel, and a tedious calculation yields then

\[
\Delta^{scattering} M_a = m_a \epsilon_a \left( -\frac{n + 1}{2\pi} - \frac{1}{8} \cot \frac{\pi}{2n} + \frac{1}{2} \cot \frac{a\pi}{2n} \left( \frac{a}{2n} - \frac{1}{2} \right) \right)
\]

### 3.3 Bound states contributions

Additional contributions to the mass corrections come from imaginary values of \(k\) which give bounded modes. We consider first the case \(a \neq n\). For imaginary values of \(k\) the asymptotic values of the \(\tilde{\eta}_j\) of (3.21) can remain finite at both \(x = \pm\infty\) if either \(A_+(k) = 0\) or \(A_-(k) = 0\). Writing

\[
A_\pm(k) = -\frac{\alpha_\pm - ik}{\alpha_\pm + ik}
\]

with

\[
\alpha_\pm = \frac{m_a^2 + m_b^2 - m_{a\mp b}^2}{2m_a} = \pm m_b \cos \frac{\pi(a \mp b)}{2n}
\]

we must have \(k = -i\alpha_\pm\). At zeroes of \(A_+(k)\) the asymptotic forms of the modes are

\[
\lim_{x \to \infty} \tilde{\eta}_j(k = -i\alpha_+) = e^{(m_b \cos \frac{\pi(a-b)}{2n} - m_a)x} \delta_j^{(a-b)} \frac{A_-}{A(a)} \]

\[
\lim_{x \to -\infty} \tilde{\eta}_j(k = -i\alpha_+) = e^{m_b \cos \frac{\pi(a-b)}{2n} x} \delta_j^{(b)}
\]
Thus, the modes will be bounded if

\[(i)\quad \sin \frac{\pi b}{2n} \cos \frac{\pi (a - b)}{2n} - \sin \frac{\pi a}{2n} \leq 0 \quad (3.38)\]

\[(ii)\quad \sin \frac{\pi b}{2n} \cos \frac{\pi (a - b)}{2n} \geq 0 \quad (3.39)\]

i.e. if

\[b \leq a \quad \text{or} \quad b = n \quad (3.40)\]

Actually the case \(b = a\) corresponds to the translational zero mode and should not be included here. The case \(b = n\) is special because \(A_+ = A_- = 0\) so that its asymptotic behaviour at \(x \to \infty\) is given by

\[\lim_{x \to \infty} \tilde{\eta}_j(k = -i\alpha) = e^{(m_b \cos \frac{\pi (a-b)}{2n} - 2m_a)x} \frac{\delta_j^{(b)}}{A^{(a)}} \quad (3.41)\]

rather than (3.37).

In a similar manner, at zeros of \(A_-(k)\) we have

\[\lim_{x \to \infty} \tilde{\eta}_j(k = -i\alpha_-) = e^{-m_b \cos \frac{\pi (a-b)}{2n} - 2m_a)x} \frac{A_+}{A^{(a)}} \quad (3.42)\]

\[\lim_{x \to -\infty} \tilde{\eta}_j(k = -i\alpha_-) = e^{-m_b \cos \frac{\pi (a+b)}{2n} x} \delta_j^{(b)} \quad (3.43)\]

giving the conditions

\[(i)\quad \sin \frac{\pi b}{2n} \cos \frac{\pi (a + b)}{2n} + \sin \frac{\pi a}{2n} \geq 0 \quad (3.44)\]

\[(ii)\quad \sin \frac{\pi b}{2n} \cos \frac{\pi (a + b)}{2n} \leq 0 \quad (3.45)\]

i.e.

\[b \geq n - a \quad (3.46)\]

The case \(b = n\) leads to the same state as the \(b = n\) case in (3.40) and we must count it only once. One might expect that further bounded solutions may arise from (3.21) at poles of \(A_{\pm}\) by choosing \(\exp \xi = A_{\pm}\). Repeating the preceding analysis one finds however that the state obtained from a pole of \(A_{\pm}\) in the perturbation by soliton \(b\) is equal to the state obtained from a zero of \(A_{\pm}\) in the perturbation by soliton \(n - b\). Thus we obtain

\[\Delta^\text{bound} M_{a} = \frac{1}{2} \sum_{b=1}^{a-1} \nu_{+}^{(b)} + \frac{1}{2} \sum_{b=a-n}^{n} \nu_{-}^{(b)} , \quad a = 1, \ldots, n-1 \quad (3.47)\]

where the frequencies are obtained from (3.15)

\[\nu_{\pm}^{(b)} = 2m \sin \frac{b\pi}{2n} \left| \sin \frac{(a \mp b)\pi}{2n} \right| \quad (3.48)\]

Performing the sum gives the result

\[\Delta^\text{bound} M_{a} = \frac{1}{2} m_a \left( \cot \frac{\pi}{2n} + \cot \frac{a\pi}{2n} \right) , \quad a = 1, \ldots, n-1 \quad (3.49)\]

The case \( a = n \) has to be treated separately because \( A^{(n)} = 0 \). By similar considerations as the above one finds that

\[
\Delta^{\text{bound}} M_n = \sum_{b=1}^{n} \nu_{+}^{(b)} = \frac{1}{4} m_n \cot \frac{\pi}{2n}.
\]  

(3.50)

3.4 Result

Combining the classical mass in (2.24), the corrections from the scattering states in (3.33) and those from the bound states in (3.49),(3.50) we obtain the quantum soliton mass to first order

\[
M_{a}^{\text{soliton}} = \epsilon_a m \sin \frac{a \pi}{2n} \left( \frac{8n}{\beta^2} - \frac{n + 1}{\pi} + \frac{1}{4} \cot \frac{\pi}{2n} + \frac{a}{2n} \cot \frac{a \pi}{2n} \right) + \mathcal{O}(\beta^2),
\]

(3.51)

where \( \epsilon_a = 2 - \delta_{a,n} \). Thus the mass ratios receive the following quantum corrections

\[
\Delta \left( \frac{M_a}{M_b} \right) = \frac{m_a \epsilon_a \beta^2}{m_b \epsilon_b 16n^2} \left( a \cot \frac{a \pi}{2n} - b \cot \frac{b \pi}{2n} \right) + \mathcal{O}(\beta^4).
\]

(3.52)

4 Discussion

Classically it has been known for some time that the soliton masses in the \( c_{n}^{(1)} \) theories are proportional to the masses of the fundamental particles of the \( b_{n}^{(1)} \) theories. We can now check this soliton – particle duality at the quantum level.

For the \( c_{n}^{(1)} \) Toda theory with imaginary exponentials

\[
\mathcal{L} = -\frac{1}{2} \vec{\phi} \cdot \square \vec{\phi} + \frac{m^2}{\beta^2} \sum_i n_i \left( e^{i\beta \vec{a}_i \cdot \vec{\phi}} - 1 \right)
\]

(4.1)

we have found the mass corrections, which we rewrite to the order of our approximation as

\[
M_{a}^{\text{soliton}} = \epsilon_a m \left( \frac{8n}{\beta^2} - \frac{n + 1}{\pi} + \frac{1}{4} \cot \frac{\pi}{2n} \right) \sin \frac{a \pi}{2n} \left( 1 + \frac{a}{2n} \cot \frac{a \pi}{2n} \right) + \mathcal{O}(\beta^2),
\]

(4.2)

On the other hand, for the \( b_{n}^{(1)} \) Toda theory with real exponentials

\[
\mathcal{L} = -\frac{1}{2} \vec{\phi} \cdot \square \vec{\phi} - \frac{\tilde{m}^2}{\beta^2} \sum_i \tilde{n}_i \left( e^{i\beta \tilde{a}_i \cdot \vec{\phi}} - 1 \right)
\]

(4.3)

the full quantum masses of the fundamental particles have been found in [9] to be

\[
M_{a}^{\text{particle}}(b_{n}^{(1)}, \tilde{m}, \beta^2) = \epsilon_a \tilde{m} \sin \left( \frac{a \pi}{2n} \left( 1 + \frac{a}{2n} \cot \frac{a \pi}{2n} \right) \right) + \mathcal{O}(\beta^2),
\]

(4.4)
Comparing this with our result for the soliton masses in the $c_n^{(1)}$ theories we find the relationship
\[ M_{a \text{ particle}}^{(b_n^{(1)}, \tilde{m}, \beta^2)} = M_{a \text{ soliton}}^{(c_n^{(1)}, m, \beta^2)} + \mathcal{O}(\beta^2) \] (4.5)
provided the mass scales are related by
\[ \tilde{m} = m \left( \frac{8n}{\beta^2} - \frac{n + 1}{\pi} + \frac{1}{4} \cot \frac{\pi}{2n} \right). \] (4.6)

In particular for the mass ratios
\[ \frac{M_{a \text{ particle}}^{(b_n^{(1)}, \beta^2)}}{M_{b \text{ particle}}^{(b_n^{(1)}, \beta^2)}} = \frac{M_{a \text{ soliton}}^{(c_n^{(1)}, \beta^2)}}{M_{b \text{ soliton}}^{(c_n^{(1)}, \beta^2)}} + \mathcal{O}(\beta^4). \] (4.7)

Thus the mass ratios of the solitons of the imaginary coupling $c_n^{(1)}$ Toda theories are equal to the mass ratios of the fundamental particles of the real coupling $b_n^{(1)}$ Toda theory, even after taking the first quantum corrections into account. We expect this relation to hold at the full quantum level, to all order in $\beta^2$.

If we specialize our result to the case of $c_2^{(1)}$ we can compare to the calculation of Watts [29]. It would appear that for the $n = 2$ theory Watts omitted, in the mass corrections for soliton $a = 1$, the contribution from the $b = n - a = 1$ state in the second sum of (3.47). It is an accident of the $n = 2$ case that omission of this contribution still leads to reasonable results (except that the masses of the solitons turn out to be proportional to those of the fundamental particles in the $b_2^{(1)}$ theory with imaginary coupling rather than the theory with real coupling). However, in the general $n$ case such an omission would lead to completely unreasonable results for the soliton mass corrections, which would not be related to the masses of fundamental particles of any theory.

The equality between the soliton masses $M_{a \text{ soliton}}^{(c_n^{(1)}, m, \beta^2)}$ and the particle masses $M_{a \text{ particle}}^{(b_n^{(1)}, \tilde{m}, \beta^2)}$ does not imply that the solitons of the $c_n^{(1)}$ theory can be reinterpreted as the fundamental particles of the $b_n^{(1)}$ theory. One feature that stands in the way of such an interpretation is the fact that the solitons have a multiplet structure (i.e. there are several solitons of the same mass) whereas the fundamental particles do not. The corresponding problem was noted already in the original paper [22] on the monopole – gauge particle duality. In that case the gauge particles have a multiplet structure but the monopoles do not. In sine-Gordon theory ($a_1^{(1)}$ Toda theory) it is however known that a breather state (soliton – antisoliton bound state) can be identified with the fundamental particle of that theory. This would make it interesting to extend our calculations to the breathers of Toda theory.

Using the strong coupling – weak coupling duality discovered in [9] we can extend
the relationship \( (4.5) \) to the following squares of relations

\[
\begin{align*}
\frac{M^{\text{particle}}}{M^{\text{particle}}} (b_1^{(1)}, \beta^2) &= \frac{M^{\text{soliton}}}{M^{\text{soliton}}} (c_1^{(1)}, \beta^2) \\
\frac{M^{\text{particle}}}{M^{\text{particle}}} (a_{2n-1}^{(2)}, (4\pi/\beta)^2) &= \frac{M^{\text{soliton}}}{M^{\text{soliton}}} (?a_{2n-1}^{(2)}, (4\pi/\beta)^2) \\
\frac{M^{\text{particle}}}{M^{\text{particle}}} (c_1^{(1)}, \beta^2) &= \frac{M^{\text{soliton}}}{M^{\text{soliton}}} (?b_1^{(1)}, \beta^2) \\
\frac{M^{\text{particle}}}{M^{\text{particle}}} (d_{n+1}^{(2)}, (4\pi/\beta)^2) &= \frac{M^{\text{soliton}}}{M^{\text{soliton}}} (?d_{n+1}^{(2)}, (4\pi/\beta)^2)
\end{align*}
\]

(4.8)

The algebras which are preceded by a question mark in the above relations have been identified by classical calculation only. It would clearly be desirable to have quantum calculations for the solitons of all affine algebras to confirm this diagram and similar ones involving the remaining Toda theories.

Looking at the relations (4.8) one wonders what really is the full algebra which underlies the \( c_n^{(1)} \) Toda theory at the quantum level. It seems to be some \( \beta \)-dependent amalgamate of \( c_n^{(1)}, b_n^{(1)}, a_{2n-1}^{(2)} \) and \( d_{n+1}^{(2)} \). It would be very interesting to unearth this structure from the quantization of affine Toda theory.

An interesting part of this algebraic structure has been described by Bernard and LeClair \[33\]. Applying their formalism one observes that the \( c_n^{(1)} \) affine Toda theory has a \( U_q(d_{n+1}^{(2)}) \) symmetry. Here \( U_q(d_{n+1}^{(2)}) \) is the quantum deformation \[34\] of the enveloping algebra of the affine Kac-Moody algebra \( d_{n+1}^{(2)} \). The deformation parameter \( q \) is related to the coupling constant by \( q = \exp(8i\pi^2/\beta^2) \). The quantum solitons are expected to transform in some finite dimensional representation under this quantum affine algebra. This implies that the soliton scattering matrices, which also have to respect this symmetry, are proportional to the corresponding R-matrices of the quantum affine algebra. The soliton S-matrices for \( a_n^{(1)} \) Toda theory have been obtained this way \[35\].

However, this can not be the whole story. In the present work we have shown that the mass ratios of the \( c_n^{(1)} \) quantum solitons receive quantum corrections. But it is known that any S-matrices obtained from R-matrices for the finite dimensional representations of quantum affine algebras predict \( \beta \)-independent (classical) mass ratios; these ratios are determined by pole locations, and the location of poles of R-matrices is fixed by the values of the Casimir numbers of the algebra in the finite dimensional representations \[36\].

Recently R-matrices with moving poles have been discovered \[37\]. They arise from an enlargement of quantum affine algebras, namely from type I quantum affine superalgebras \[38\]. This suggests that the problem of finding the S-matrices for the solitons with \( \beta \)-dependent mass ratios might find its solution through the identification of some, as yet unknown, larger quantum symmetry algebra. This algebra should at the same time explain the relations (4.8).

While finding the soliton S-matrices for non-selfdual algebras is still an open problem, the corresponding problem for the fundamental particles has been solved.
In all affine Toda theories based on non-selfdual Kac-Moody algebras, the mass ratios of the fundamental particles renormalize. Therefore the S-matrix formulas obtained in [3], which have been given an elegant Lie algebraic description in [39], do not apply. The correct formulas [3, 40] contain a $\beta$-dependent Coxeter number interpolating between the integer Coxeter numbers of dual Kac-Moody algebras. In our case it interpolates between $c_n^{(1)}$ and $d_{n+1}^{(2)}$. This should be part of the algebraic structure to be discovered.

There are clearly many lessons to be learned from the quantum theory of affine Toda theory. We have two goals in mind in pursuing these studies: we hope to learn from these integrable theories how to handle soliton effects in other physical theories and we hope to discover the new algebraic structure describing the quantum symmetry of these theories.

We thank Mike Freeman for discussions.

Note added: Watts has explained to us why he omits the $b = n - a = 1$ state in his calculation [29] of the $c_n^{(1)}$ soliton masses: At exactly the location at which $A_-$ has the zero which leads to this bound state, $A_+$ has a pole, leading to a problem in (3.42). We argue that this coincidence of the zero of $A_+$ with a pole of $A_-$ is an artifact of the classical approximation and does not hold in the quantum theory. In the quantum theory the poles and zeros of the interaction functions $A_{ab}$ become the poles and zeros of the soliton $a$–soliton $b$ S-matrix. Their location is determined by the quantum soliton and breather mass ratios. As we have seen in this paper, these soliton mass ratios receive quantum corrections and thus the poles and zeros move. It will be interesting to analyze this in detail.

MacKay and Watts have published independently from us on the same day a paper [41] calculating the first mass corrections for the solitons of all affine Toda theories. For the non-selfdual Toda theories they find that the soliton mass ratios do not renormalize like the fundamental particles of the dual theories. They are thus in contradiction to our result in this paper for the $c_n^{(1)}$ Toda theory. We believe that this discrepancy may be due to the same $b=n-a$ bound state mentioned above.

References

[1] G. Wilson, *The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras*, Ergod. Th. and Dynam. Sys. 1 (1981) 361.

[2] D. Olive, N. Turok, *Algebraic structure of Toda systems*, Nucl. Phys. B220 (1983) 491.

[3] D. Olive, N. Turok, *Local conserved densities and zero-curvature conditions for Toda lattice field theories*, Nucl. Phys. B257 (1985) 227.

[4] D. Olive, N. Turok, *The Toda lattice field theory hierarchies and zero-curvature conditions in Kac-Moody algebras*, Nucl. Phys. B265 (1986) 469.
[5] G.W. Delius, M.T. Grisaru, D. Zanon, Quantum conserved currents in affine Toda theories, Nucl. Phys. B385 (1992) 307.

[6] B. Feigin, E. Frenkel, Free field resolutions in affine Toda field theories, Phys. Lett. B276 (1992) 79.

[7] M.R. Niedermaier, The quantum spectrum of the conserved charges in affine Toda theories, Nucl. Phys. B424 (1994) 184.

[8] H.W. Braden, E. Corrigan, P.E. Dorey, Affine Toda field theory and exact S-matrices, Nucl. Phys. B338 (1990) 689.

[9] G.W. Delius, M.T. Grisaru, D. Zanon, Exact S-Matrices for nonsimply laced affine Toda theories, Nucl. Phys. B382 (1992) 365.

[10] T. Hollowood, Solitons in affine Toda field theories, Nucl. Phys. B384 (1992) 523.

[11] D.I. Olive, N. Turok, J.W.R. Underwood, Solitons and the energy-momentum tensor for affine Toda theory, Nucl. Phys. B401 (1993) 663.

[12] D.I. Olive, N. Turok, W.R. Underwood, Affine Toda Solitons and Vertex Operators, Nucl. Phys. B409 (1993) 509.

[13] M.A.C. Kneipp, D.I. Olive, Solitons and Vertex Operators in Twisted Affine Toda Field Theories, hep-th/9404030 (1994).

[14] A. Fring, P.R. Johnson, M.A.C. Kneipp, D.I. Olive, Vertex Operators and Soliton Time Delays in Affine Toda Field Theory, hep-th/9405034 (1994).

[15] M.A.C. Kneipp, D.I. Olive, Crossing and Antisolitons in Affine Toda Theories, Nucl. Phys. B408 (1993) 565.

[16] H. Aratyn, C.P. Constantinidis, L.A. Ferreira, Hirota’s Solitons in the Affine and the Conformal Affine Toda Models, Nucl. Phys. B406 (1993) 727.

[17] N.J. MacKay, W.A. McGhee, Affine Toda solitons and automorphisms of Dynkin diagrams, Int. J. Mod. Phys. A8 (1993) 2791.

[18] W.A. McGhee, The Topological Charges of the $a^{(1)}_n$ Affine Toda Solitons, Int. J. Mod. Phys. A9 (1994) 2645.

[19] J. Underwood, On the Topological Charges of Affine Toda Solitons, hep-th/9306031 (1993).

[20] Z. Zhu, D.G. Caldi, Multi-Soliton Solutions of Affine Toda Models, UB-TH-0193 (1993).

[21] M. Freeman, Conserved charges and soliton solutions in affine Toda theory, hep-th/9408092 (1994).

[22] P. Goddard, J. Nuyts, D. Olive, Gauge theories and magnetic charge, Nucl. Phys. B125 (1977) 1.
[23] C. Montonen, D. Olive, Magnetic monopoles as gauge particles, Phys. Lett. 72B (1977) 117.

[24] A. Sen, Strong-Weak Coupling Duality in Four Dimensional String Theory, hep-th/9402002 (1994).

[25] C. Vafa, E. Witten, A strong coupling test of S-duality, hep-th/9408074 (1994).

[26] E. Witten, D. Olive, Supersymmetry algebras that include topological charges, Phys. Lett. 78B (1978) 97.

[27] H. Osborn, Topological charges for $N = 4$ supersymmetric gauge theories and monopoles of spin 1, Phys. Lett. 83B (1979) 321.

[28] T. Hollowood, Quantum Soliton Mass Corrections in SL(n) Affine Toda Field Theory, hep-th/9209024, Phys. Lett. B300 (1993) 73.

[29] G.M.T. Watts, Quantum mass corrections for $C_{2}^{(1)}$ affine Toda theory solitons, Nucl. Phys. B338 (1994) 40.

[30] R.F. Dashen, B. Hasslacher, A. Neveu, Phys. Rev. D10 (1974) 4130.

[31] R.F. Dashen, B. Hasslacher, A. Neveu, Particle spectrum in model field theories from semiclassical functional integral techniques, Phys. Rev. D11 (1975) 3424.

[32] R. Rajaraman, Solitons and Instantons, North-Holland (1982).

[33] D. Bernard, A. LeClair, Quantum Group Symmetries and Non-Local Currents in 2D QFT, Commun. Math. Phys. 142 (1991) 99.

[34] V.G. Drinfel’d, Quantum Groups, Proc. Int. Congr. Math., Berkeley (1986) 798.

[35] T. Hollowood, Quantizing SL(N) Solitons and the Hecke Algebra, Int. J. Mod. Phys. A8 (1993) 947.

[36] G.W. Delius, M.D. Gould, Y.-Z. Zhang, On the construction of trigonometric solutions of the Yang-Baxter equation, hep-th/9405030, Nucl. Phys. B (in print) (1994).

[37] A.J. Bracken, G.W. Delius, M.D. Gould, Y.-Z. Zhang, Solutions of the quantum Yang-Baxter equation with extra non-additive parameters, hep-th/9405138, J. Phys. A (in print) (1994).

[38] G.W. Delius, M.D. Gould, J.R. Links, Y.-Z. Zhang, On Type-I Quantum Affine Superalgebras, hep-th/9408008 (1994).

[39] P.E. Dorey, Roots systems and purely elastic S-matrices II, Nucl. Phys. B374 (1992) 741.

[40] E. Corrigan, P.E. Dorey, R. Sasaki, On a generalized bootstrap principle, Nucl. Phys. B408 (1993) 579.
[41] N.J. MacKay, G.M.T. Watts, Quantum mass corrections for affine Toda solitons, hep-th/9411169.