\(\alpha\)-Symmetries, Coloured Dimensions and Gauge-String Correspondence

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**Abstract**

We propose a scenario of gauge-string correspondence by relating the SU(3) colour group to hidden space-time isometries originating from extra dimensions. These isometries (\(\alpha\)-symmetries) are the special symmetries of RNS superstring theories under global non-linear space-time transformations. The vertex operators for the octet of gluons are constructed by the procedure of “photon painting”, that is, with the SU(3) subgroup of the \(\alpha\)-symmetry generators acting on a regular open string photon, so the corresponding open string excitations are in the adjoint of SU(3). Remarkably, the operator algebra of these massless gluon vertices is closed and possesses the full zigzag symmetry, crucial for the isomorphism between open strings and QCD. As a result, the scattering amplitudes of the constructed open string vertex operators have a field-theoretic rather than a stringy structure, including the absence of standard tower of massive intermediate states. Our model also suggests that the total number of underlying hidden dimensions is three, with each extra dimension carrying its appropriate SU(3) colour and anticolour.

**PACS:** 04.50. + h;11.25.Mj.

June 2008

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Introduction

The gauge-string correspondence is a profound hypothesis and a promising approach to important long-standing problems in QCD (such as quark confinement), relating the observables (physical vertex operators) in string theory to local gauge invariant operators in QCD. In particular, such a correspondence identifies open strings with thin tubes of gluon field lines, connecting hadrons, so the Wilson loop’s expectation value \( < W(C) > \) on the QCD side is identified with the partition function \( Z(C) \) of the open string with the ends attached to the same contour \( C \). Once such an isomorphism holds, one could expect that correlation functions of massless vertex operators in open string theory are to reproduce QCD dynamics. Such a string-theoretic framework would be particularly efficient and natural to address the problem of confinement, as well as other non-perturbative QCD dynamics.

In practice, however, things are far more complicated. First of all, if an open string is to describe the gluon dynamics, its spectrum has to contain 8 massless vector bosons. It is well-known that a perturbative open string spectrum has only one massless physical excitation, a photon, which has no colour. Although this complication can be corrected (somewhat artificially) by introducing the appropriate Chan-Paton’s factors (such as Gell-Mann matrices), this is only the beginning of sorrows. The main and fundamental problem in identifying a QCD string, is that a spectrum of a normal open string contains an infinite tower of massive states, in addition to the photon. These states appear as intermediate poles in any Veneziano amplitude (including the scattering of massless gauge bosons), so there is no way to separate the gluon dynamics from massive string modes. For this reason, the Veneziano amplitude for the massless gauge bosons in superstring theory has little to do with the scattering amplitudes of gluons in QCD, since the latter do not, of course, have anything like an infinite set of intermediate massive states. This complication has an underlying geometrical reason. That is, the standard open string theory lacks an important symmetry that, however, is present on the gauge theory side. Namely, while the string theory is only invariant under reparametrizations with positive Jacobians (that do not change the worldsheet orientation), the Wilson loop is also invariant under the orientation change \( \square, \square, \square \). Technically, the loop equation satisfied by \( < W(C) > \), is the consequence of the zigzag symmetry. It is easy to check that Ward identities for perturbative string theory in flat space-time background do not reproduce the loop equation, indicating the absence of the zigzag symmetry. In other words, a stringy description of QCD requires the presence of closed subalgebra of massless vertex operators of gluons and, as a consequence,
the field-theoretic behaviour of their scattering amplitudes (including the absence of the intermediate poles in correlation functions and OPE).

In the supersymmetric case, this problem can be partially solved by the AdS/CFT correspondence, which is a special case of the gauge-string correspondence (see e.g. [3], [4], [5], [6]). However, the formalism developed in the AdS/CFT framework is incomplete. First of all, it mainly works in the large $N$ limit, which has little to do with realistic gauge group of QCD, such as $SU(3)$. In addition, the AdS/CFT correspondence only gives an isomorphism between closed string vertex operators (such as a dilaton) and gauge-invariant observables on the QCD side, such as $Tr(F^2)$. This correspondence is incomplete; for example, it can’t be used to calculate gluon scattering amplitudes in QCD since, in order to describe the emission of gluons in string theory, one needs to use the open string vertex operators coupled to the gauge potential $A_m$, rather than to the gauge-invariant fields. As we noted above, there seems to be no straightforward way to deduce such emission vertices from standard open string theory, or from the AdS/CFT approach. In this paper, we address the problem of the gauge-string correspondence, by constructing the massless vertex operators for the SU(3) octet of gluons and studying their scattering amplitudes. The key element of our construction is the set of very peculiar space-time symmetries (referred to as $\alpha$-symmetries) present in non-critical RNS superstring theories [7], [8]. These symmetries are realised non-linearly and are closely related to the presence of hidden space-time dimensions. Typically, they mix the matter and the ghost degrees of freedom in superstring theory; in particular, the variation of the matter part of the worldsheet RNS action is cancelled by that of the superconformal ghost part under the $\alpha$-symmetry transformations.

The $\alpha$-symmetry generators are the BRST-invariant picture-dependent primary fields which can be classified in terms of the ghost cohomologies $H_n \sim H_{-n-2}(n = 1, 2, \ldots)$ [7]. Typically, for each the cohomology $H_n \sim H_{-n-2}(n = 1, 2, \ldots)$ in $d$-dimensional non-critical string theory contains $n + 2$ generators, one $d$-vector and $n + 1$ scalars. For each given $n$ these operators induce the subset of $\alpha$ symmetries stemming from separate hidden space-time dimension, enhancing the space-time symmetry group and increasing the effective space-time dimensionality by one unit. The $\alpha$-symmetry generators from the first $n$ cohomologies $H_k \sim H_{-k-2}(1 \leq k \leq n)$, combined with the standard translation and rotation generators of the $SO(2,d-1)$ space-time isometry group (for $d$ space-time dimensions plus the Liouville direction) extend the symmetry group to $SO(2, d - 1 + n)$. In this paper, of special interest to us are the $\alpha$-symmetries that do not mix with the standard space-time Poincare generators, i.e. subgroup the $\alpha$-generators acting purely in extra dimensions.
These generators correspond to isometries of the hidden dimensions (other than those of the visible space-time) Remarkably, it turns out that these generators, applied to the Hilbert space of physical states of non-critical RNS superstring theory, form SU(3) subgroup which is translated into the gauge group of strong interaction in the gauge-string correspondence. The vertex operators for the gluon emission then emerge in the adjoint representation of the SU(3) subgroup of the extra dimensional α-generators, building the ground for the isomorphism between SU(3) QCD and giving rise special isolated zigzag-invariant sector of open string theory which is described in terms of nonzero ghost cohomologies and is detouched from the standard Hilbert space perturbative open string oscillations. So in our approach, the QCD string emerges not as a separate string theory in a certain space-time background (such as AdS), but as a special isolated sector of RNS superstring theory in flat background, related to isometries of hidden dimensions. Interestingly, the total number of hidden dimensions in our model turns out to be three, with each extra dimension carrying a corresponding colour or anticolour quantum number.

This paper is organized as follows. In the next section we review the classification of the α-symmetries in terms of the ghost cohomologies and their relation to the hidden space-time dimensions. In the section 3 we concentrate on the extra dimensional subset of the α-generators, not intertwining with the symmetry generators of visible space-time. We show this subset of the α-symmetries to form SU(3) subgroup and construct the octet of gluon vertex operators in the adjoint of SU(3) by acting with the appropriate α-generators on an open string photon. Next, we discuss the OPE structure of these vertex operators and the closedness of their operator algebra, implying the underlying zigzag symmetry. In particular, the absence of massive intermediate poles in their scattering amplitudes is explained by the special property of the α-symmetry generators to annihilate the massive states (typically, if an α-generator is applied to any string excitation other than massless, a BRST trivial operator is produced). In other words, the α-symmetry works like an “Occam’s razor”, shaving off the massive intermediate states and restoring the zigzag symmetry. In the section 4 we calculate some of the 4-point scattering amplitudes of the gluon vertex operators and show them to have a field-theoretic structure, consistent with the zigzag symmetry (no massive poles). In the concluding section we discuss the implications of our results and the directions for the future work.

2. α-Symmetries, Ghost Cohomologies and Hidden Dimensions

In our recent works Ref. we have shown that non-critical RNS superstring theories are invariant under the set of unusual non-linear space-time transformations, not at all
evident from the structure of their worldsheet actions. That is, consider the worldsheet action of $d$-dimensional RNS superstring theory given by

\[ S = \frac{1}{2\pi} \int d^2z \{ -\frac{1}{2} \partial X_m \bar{\partial} X^m - \frac{1}{2} \psi_m \bar{\partial} \psi^m - \frac{1}{2} \bar{\psi}_m \partial \bar{\psi}^m \} + S_{\text{ghost}} + S_{\text{Liouville}} \]

\[ S_{\text{ghost}} = \frac{1}{2\pi} \int d^2z \{ b \bar{\partial} c + \bar{b} \partial \bar{c} + \beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma} \} \tag{1} \]

\[ S_{\text{Liouville}} = \frac{1}{4\pi} \int d^2z \{ \partial \varphi \bar{\partial} \varphi + \lambda \bar{\partial} \lambda + \bar{\lambda} \partial \lambda - F^2 + 2 \mu_0 b e^{b \varphi} (ib\bar{\lambda} - F) \} \]

where $\varphi, \lambda, F$ are the components of the Liouville superfield, $X^m, m = 0, ..., d - 1$ are the space-time coordinates, $\psi^m, \bar{\psi}^m$ are their worldsheet superpartners; $b, c, \beta, \gamma$ are the fermionic and bosonic (super)reparametrization ghosts bosonized as

\[ b = e^{-\sigma}; c = e^\sigma, \]

\[ \beta = e^{\chi - \phi} \partial \chi \equiv \partial \xi e^{-\phi}; \gamma = e^{\phi - \chi} \tag{2} \]

This action is obviously invariant under two global $d$-dimensional space time symmetries - Lorenz rotations and translations. One can straightforwardly check, however, that in addition to these obvious symmetries the action (1) is also invariant under the following non-linear global transformations, mixing the matter and the ghost sectors of the theory [7]:

\[ \delta X^m = \epsilon \{ \partial (e^{\phi} \psi^m) + 2e^{\phi} \partial \psi^m \} \]

\[ \delta \psi^m = \epsilon \{ -e^{\phi} \partial^2 X^m - 2 \partial (e^{\phi} \partial X^m) \} \]

\[ \delta \gamma = e^{2\phi - \chi} \{ \psi_m \partial^2 X^m - 2 \partial \psi_m \partial X^m \} \]

\[ \delta \beta = \delta b = \delta c = 0 \tag{3} \]

with the generator of (3) given by

\[ T = \int \frac{dz}{2i\pi} e^{\phi} (\partial^2 X_m \psi^m - 2 \partial X_m \partial \psi^m) \tag{4} \]

The integrand of (4) is a primary field of dimension 1, i.e. a physical generator. While it is not manifestly BRST-invariant (it doesn’t commute with the supercurrent terms of $Q_{\text{brst}}$) its BRST invariance can be restored by adding the appropriate $b$-$c$ ghost dependent terms according to the prescription described in [8]. The special property of this generator is that it is annihilated by $\Gamma^{-1}$ and has no analogues at higher pictures, such as 0, −1 and −2 (but has versions at higher positive pictures $+2, +3, ...$ which can be obtained by the
standard picture-changing) The physical operators with such a property are referred to as the positive ghost cohomology $H_1$, $[7]$, $[8]$. For the sake of completeness, we recall the definition and basic properties of the ghost cohomologies discussed in $[7]$, $[8]$. The positive number $n$ ghost cohomology $H_n$ ($n = 1, 2, ...$) consists of physical (BRST invariant and non-trivial) vertex operators violating the picture equivalence, existing at picture $n$ and above, that are annihilated at their minimal positive picture $n$ by the inverse picture changing operator $\Gamma^{-1} = c\partial \xi e^{-2\phi}$ (higher than $n$ pictures of such operators are related by the usual picture changing). Similarly, the negative number $-n$ ghost cohomology $H_{-n}$ ($n \geq 3$) consists of physical (BRST invariant and nontrivial) operators that exist at picture $-n$ or below, that are annihilated by the direct picture changing operator $\Gamma =: e^{\phi}G :$ at the minimal negative picture $-n$ (here $G$ is the full matter $+$ ghost worldsheet supercurrent). The operators of $H_{-n}$ at lower than $-n$ pictures are related by the usual picture-changing. 

There is an isomorphism between positive and negative ghost cohomologies $H_n \sim H_{-n}$ as any element of $H_{-n-2}$ (typically having the form $\sim e^{-(n+2)\phi} F_{\text{matter}}$ at the minimal negative picture) has a representation in $H_n$ obtained by replacing $e^{-(n+2)\phi} \rightarrow e^{n\phi}$ (with the matter part unchanged) and adding the $b - c$ ghost counterterms (by the prescription described in $[8]$) in order to protect the BRST invariance. The usual picture-independent observables, existing at all pictures, including picture 0 (at which the superconformal ghosts decouple) are by definition the elements of $H_0$. The cohomologies $H_{-1}$ and $H_{-2}$ are empty.

In particular, as the $\alpha$-symmetry generator (4) is the element of $H_1$, there is also a picture $-3$ version of this generator, with the manifest BRST invariance. This version can be obtained simply by replacing $e^{\phi} \rightarrow e^{-3\phi}$ in (4). Similarly to the picture $+1$-version, the picture $-3$ version is annihilated by $\Gamma$, so there are no versions of this operator at pictures $-2, -1$ and 0 while the versions at pictures below $-3$ can be obtained by straightforward inverse picture changing.

For this reason, the picture $-3$ version of (3) is an element of negative ghost cohomology $H_{-3}$.

It is not difficult to show that, just like the operator (3) (the element of $H_1$) generates the space-time symmetry transformations (2) of the RNS action (1), similarly the picture $-3$ generates the symmetry transformations identical to (2) (with $e^{\phi}$ replaced by $e^{-3\phi}$). For the critical ($d = 10$) uncompactified RNS superstrings the transformations (2) is the only additional (to translations and rotations) space-time symmetry generated by
$H_1 \sim H_{-3}$ currents. For non-critical strings ($d \neq 10$), however, there are $d + 1$ additional $\alpha$-symmetries, involving the Liouville sector. (one $d$-vector and one scalar).

The corresponding generators are given by

$$L^{m \alpha} = \oint \frac{dz}{2i\pi} e^{\phi} \{ \partial^2 \varphi \psi^m - 2 \partial \varphi \partial \psi^m + \partial^2 X^m \lambda - 2 \partial X^m \partial \lambda \}$$  \hspace{1cm} (5)$$

and

$$L^{\alpha -} = \oint \frac{dz}{2i\pi} e^{\phi} \{ \partial^2 \varphi \lambda - 2 \partial \varphi \partial \lambda \}$$ \hspace{1cm} (6)

For simplicity, the expressions for $L^{m \alpha}$ and $L^{\alpha -}$ are given in the limit of zero cosmological constant and zero dilaton field so one can ignore the effect of the background charge; the corresponding expressions accounting for the Liouville dressing, are somewhat longer and given in [7].

The appropriate space-time transformations are given by

$$\delta X_m = \epsilon_{m \alpha} \{ \partial (e^\varphi \lambda) + 2 e^\varphi \partial \lambda \}$$
$$\delta \lambda = - \epsilon_{m \alpha} \{ 2 \partial (e^\varphi \partial X^m) + e^\varphi \partial^2 X^m \}$$

$$\delta \gamma = \epsilon_{m \alpha} e^{2 \varphi - \chi} \{ \partial^2 X^m \lambda - 2 \partial X^m \partial \lambda \}$$
$$\delta \beta = \delta b = \delta c = \delta \varphi = \delta \psi^m = 0$$ \hspace{1cm} (7)

and

$$\delta \varphi = \epsilon_{- \alpha} \{ \partial (e^\varphi \lambda) + 2 e^\varphi \partial \lambda \}$$
$$\delta \lambda = - \epsilon_{- \alpha} \{ 2 \partial (e^\varphi \partial \varphi) + e^\varphi \partial^2 \varphi \}$$
$$\delta \gamma = \epsilon_{- \alpha} e^{2 \varphi - \chi} \{ \lambda \partial^2 \varphi - 2 \partial \varphi \partial \lambda \}$$

$$\delta \beta = \delta b = \delta c = \delta X^m = \delta \psi^m = 0$$ \hspace{1cm} (8)

The generators (5),(6) are the Virasoro primaries, annihilated by the inverse picture changing (or by the direct p.c., if they are taken in the $H_{-3}$ representation) They are BRST-invariant (upon adding the $b - c$ ghost correction terms which we have skipped) and therefore are the elements of $H_1 \sim H_{-3}$. As before, the $H_{-3}$ version of the generators (5),(6) with the manifest BRST-invariance (with no $b - c$ correction terms) can be obtained
simply by replacing $\phi \rightarrow -3\phi$ in (5) - (8). Combined with $\frac{(d+1)(d+2)}{2}$ dimension 1 Virasoro primaries:

\[
\begin{align*}
L^{mn} &= \oint \frac{dz}{2i\pi} \psi^m \psi^n \\
L^{+m} &= \oint \frac{dz}{2i\pi} e^{-\phi} \psi^m \\
L^{-m} &= \oint \frac{dz}{2i\pi} \psi^m \lambda \\
L^{+-} &= \oint \frac{dz}{2i\pi} e^{-\phi} \lambda
\end{align*}
\] (9)

inducing the $d + 1$ translations and $\frac{d(d+1)}{2}$ rotations in space-time (including the Liouville direction), the $d + 2$ currents (4) - (6) of $H_1 \sim H_{-3}$ enlarge the current algebra of the space-time symmetry generators from $SO(d, 2)$ to $SO(d + 1, 2)$, effectively bringing an extra dimension to the theory. Namely, introducing the $d + 2$-dimensional index $M = (m, +, -, \alpha); m = 0, \ldots, d - 2; \alpha = 1$ with the $(d, 2)$ metric $\eta^{MN}$ consisting of $\eta^{mn}, \eta^{+-} = -1, \eta^{-+} = \eta^{++} = 0, \eta^{\alpha\alpha} = 1$ and evaluating the commutators of the operators (4) - (6) and (9) it isn’t difficult to check that that

\[
[L^{M_1N_1}, L^{M_2N_2}] = \eta^{M_1M_2}L^{N_1N_2} + \eta^{N_1N_2}L^{M_1M_2} - \eta^{M_1N_2}L^{M_2N_1} - \eta^{M_2N_1}L^{M_1N_2}
\] (10)

Note that, as the $SO(d, 2)$ group of translations and rotations for non-critical RNS strings is identical to the isometry group of the $AdS_d$ space, the $H_1 \sim H_{-3}$ generators (4)-(6) from the first nonzero ghost cohomology are simply the stringy analogues of the off-shell symmetry generators from hidden space-time dimension, observed by Bars in the 2T physics approach \cite{7}, \cite{10}, \cite{11}.

It is straightforward to generalize the results to the case of the higher ghost cohomologies of the orders 2 and 3. The general rule is that each ghost cohomology $H_n \sim H_{n-2}; n = 1, 2, 3$ contains the space-time $\alpha$-symmetry generators that induce the global space-time transformations leaving the RNS action invariant, with the variation of the matter part cancelled by that of the ghost part. Remarkably, each $H_n \sim H_{n-2}$ cohomology pair contains $d + n + 1$ space-time symmetry generators, including one $d$-vector and $(n + 1)$ space-time scalars. Combining the collection of the $\alpha$-generators of of $H_k \sim H_{k-2}; 1 \leq k \leq n$ with the ordinary Poincare generators (including the Liouville direction), the space-time isometry group is increased from $SO(2, d)$ to $SO(2, d + n)$, i.e.
$n$ hidden dimensions are induced, with each $H_k$ associated with an extra dimension. The $H_2 \sim H_{-4}$ $\alpha$-generators (those giving rise to the second extra dimension) are given by:

$$
\begin{align*}
L^{\beta+} &= \oint \frac{dz}{2i\pi} e^{-4\phi} F_1(X, \psi) F_1(\varphi, \lambda)(z) \\
L^{\beta-} &= -\oint \frac{dz}{2i\pi} e^{-4\phi} F_1m(X, \lambda) F_1^m(\varphi, \psi)(z) \\
L^{\beta m} &= \oint \frac{dz}{2i\pi} e^{-4\phi} (F_1^m(X, \lambda) F_1(\varphi, \lambda) - F_1(X, \psi) F_1^m(\varphi, \psi))(z) \\
L^{\alpha \beta} &= \oint \frac{dz}{2i\pi} e^{-4\phi} \left( \frac{1}{2} F_2(\lambda, \varphi) + L_1(X, \psi) \partial L_1(\varphi, \lambda) - \partial L_1(X, \psi) L_1(\varphi, \lambda) \right)(z)
\end{align*}
$$

(11)

with the matter+Liouville structures $L$ and $F$ ($L_1, F_1$ and $F_1^m$) being the primary fields of dimensions 2 and $\frac{5}{2}$:

$$
\begin{align*}
F_1(X, \psi) &= \psi m \partial^2 X^m - 2 \partial \psi m \partial X^m \\
F_1(\varphi, \lambda) &= \lambda \partial^2 \varphi - 2 \partial \lambda \partial \varphi \\
F_1^m(X, \lambda) &= \lambda \partial^2 X^m - 2 \partial \lambda \partial X^m \\
F_1^m(\varphi, \psi) &= \psi m \partial^2 \varphi - 2 \partial \psi m \partial \varphi \\
L_1(X, \psi) &= \partial X_m \partial X^m - 2 \partial \psi m \psi^m \\
L_1(\varphi, \lambda) &= (\partial \varphi)^2 - 2 \partial \lambda \lambda
\end{align*}
$$

(12)

and $F_2(\lambda, \varphi)$ being the primary field of dimension 5:

$$
F_2(\varphi, \lambda) = \frac{1}{4} (\partial \varphi)^5 - \frac{3}{4} \partial \varphi (\partial^2 \varphi)^2 + \frac{1}{4} (\partial \varphi)^2 \partial^3 \varphi \\
+ \lambda \partial \lambda (\partial^2 \varphi - (\partial \varphi)^3) - \frac{3}{2} \lambda \partial^2 \lambda \partial^2 \varphi + 3 \partial \lambda \partial^2 \lambda \partial \varphi \\ 
\equiv i : \left( \oint e^{-i\varphi} \lambda \right)^2 e^{3i\varphi} \lambda :
$$

. Combined with the matter + Liouville Poincare generators of $SO(2,d)$ and the $\alpha$-generators (4) - (6) of $H_1 \sim H_{-3}$, the $\alpha$-generators of $H_2 \sim H_{-4}$ satisfy the commutation relations (10) of $SO(2,n+2)$ with the $(M,N)$ space-time indices supplemented by yet another hidden dimension labelled by $\beta$: $M = (m, \pm, \alpha, \beta)$. Finally, the $\alpha$-generators at the level $H_3 \sim H_{-5}$ (bringing in the third hidden dimension labelled by $\gamma$) are constructed as
\[ L^{\gamma^+} = \int \frac{dz}{2i\pi} e^{-5\phi} \left\{ 2\partial F_1(X, \psi) - F_1(X, \psi) \partial F_2(\varphi, \lambda) \right\} \]
\[ L^{\gamma^m} = \int \frac{dz}{2i\pi} e^{-5\phi} \left\{ 2F^m_2(\psi, \lambda, \varphi) \partial F_1(X, \psi) - \partial F_2(\psi, \lambda, \varphi) F_1(X, \psi) \right\} + 2F_2(\varphi, \lambda) \partial F^m_1(X, \lambda) - \partial F_2(\varphi, \lambda) F^m_1(X, \lambda) \]
\[ L^{\gamma^-} = \int \frac{dz}{2i\pi} e^{-5\phi} \left\{ 2G_2(\psi, \lambda, \varphi) \partial F_1(X, \psi) - \partial G_2(\psi, \lambda, \varphi) F_1(X, \psi) \right\} + 3F_2m(\psi, \lambda, \varphi) \partial F^m_1(X, \lambda) - 2\partial F_2m(\psi, \lambda, \varphi) F^m_1(X, \lambda) - \partial F_2(\lambda, \varphi) F_1(X, \psi) \]
\[ L^{\gamma^\beta} = \int \frac{dz}{2i\pi} e^{-5\phi} \left\{ F_3(\varphi, \lambda) + \partial L_1(X, \psi) L_2(\varphi, \lambda) - \frac{4}{11} L_1(X, \psi) \partial L_2(\varphi, \lambda) \right\} \]
\[ L^{\gamma^\alpha} = \int \frac{dz}{2i\pi} e^{-5\phi} L_2m(\varphi, \psi) L^m_1(X, \lambda) \]

with the additional matter+Liouville blocks given by:
\[ F^m_2(\psi, \lambda, \varphi) = \partial^2 \psi^m \lambda \partial^2 \varphi - \psi^m \partial^2 \lambda \partial^2 \varphi + 3 \partial^2 \psi^m \partial \lambda \partial \varphi - 3 \partial \psi^m \partial^2 \lambda \partial \varphi \]
\[ G_2(\psi, \lambda, \varphi) = 4 \partial \psi_m \partial^2 \psi^m \partial \varphi - 2 \psi_m \partial^3 \psi^m \partial \varphi + (2d - 4)(\lambda \partial^3 \lambda \partial \varphi - 2 \partial \lambda \partial^2 \lambda \partial \varphi) \]
\[ L_2(\varphi, \lambda) = -\frac{5}{4} (\partial \varphi)^4 \partial \lambda + \frac{3}{4} (\partial^2 \varphi)^2 \partial \lambda + \frac{3}{2} \partial \varphi \partial^2 \varphi \partial^2 \lambda - \frac{5}{2} \partial \varphi \partial^3 \varphi \partial \lambda \]
\[ -\frac{1}{4} (\partial \varphi)^2 \partial^3 \lambda - 4 \partial \varphi \partial^2 \varphi \partial^2 \lambda + \partial^2 \varphi \partial^3 \varphi \lambda \]
\[ \boxed{L^m_2(\varphi, \psi) = -\frac{5}{4} (\partial \varphi)^4 \partial \psi^m + \frac{3}{4} (\partial^2 \varphi)^2 \partial \psi^m + \frac{3}{2} \partial \varphi \partial^2 \varphi \partial^2 \psi^m - \frac{5}{2} \partial \varphi \partial^3 \varphi \partial \psi^m} \]
\[ L^m_1(X, \lambda) = \partial^2 \lambda \psi^m + \lambda \partial^2 \psi^m \]
\[ F_3(\varphi, \lambda) = \left( \int e^{-i\varphi} \lambda \right)^4 e^{-5\phi + 4i\varphi} \lambda. \]

Combined with the space-time Poincare generators (9) and with the \( \alpha \)-generators of two lower ghost cohomologies, the \( \alpha \)-generators (14),(15) of \( H_{-3} \sim H_{-5} \) extend the space-time isometry group to \( SO(2, d + 3) \), unsealing the third hidden dimension. The problem of finding the BRST non-trivial \( \alpha \)-symmetry generators in the higher order ghost cohomologies is more complicated, as the expressions for the matter primaries of higher conformal dimensions become long and cumbersome. The only symmetry generators in the ghost cohomologies \( H_n \sim H_{n-2} \) for \( n > 3 \) are those mixing the ghost and the Liouville sectors, which can be obtained as the normal ordering of the operator \( \sim: \left( \int e^{-i\varphi} \lambda \right)^n e^{-n\phi + i(n-1)\varphi} \lambda: \) but such operators commute with the photon (thus producing no new gauge bosons) and
do not seem to have any straightforward analogues in the matter sector. At present, we do not know of any elegant and compact prescription to construct such operators and, even if such higher order $\alpha$-symmetries exist and can be found, there is no evidence they could be interpreted in terms of extra dimensions, like those of the first three cohomologies. On the contrary, if one is able to show that there are no $\alpha$-symmetries (pointing at extra space-time dimensions) in the ghost cohomologies $H_n \sim H_{-n-2}$ for $n > 3$, this may be an interesting interpretation of the $SU(3)$ strong interaction group in the language of extra dimensions. At present, at least, there is no compelling pro or contra evidence for the $\alpha$-symmetries (and related hidden space-time dimensions) originating from the ghost cohomologies at levels higher than 3.

3. Construction of gluon emission vertices

In this section, we concentrate on the subgroup of the the $\alpha$-symmetry generators (4) - (6), (11) - (15) from the first three ghost cohomologies $H_n \sim H_{-n-2}$ ($n = 1, 2, 3$) not mixing with Poincare transformations in visible space-time dimensions, i.e. the generators without the space-time index $m$. We shall use this subgroup of generators to construct the gluon emission vertices. There are altogether 9 generators: $L^{\pm \alpha}, L^{\pm \beta}, L^{\pm \gamma}$ and $L^{\alpha \beta}, L^{\alpha \gamma}, L^{\beta \gamma}$. Physically, these generators correspond to the $\alpha$-isometries of hidden space-time dimensions. Our goal now is to investigate how these $\alpha$-generators act on the massless open superstring state, i.e. the photon. Generically, by acting on a photon with 9 scalar $\alpha$-generators of the first three ghost cohomologies, one would expect to construct a multiplet of 9 vertex operators of massless vector gauge bosons of $H_n \sim H_{-n-2}$ ($n = 1, 2, 3$), related by the $\alpha$-transformations. Remarkably, however, one of the generators, $L^{\alpha -}$, drops out as it turns out to commute with the photon vertex (see the proof below). At the same time the remaining 8 generators do not commute with the photon, producing 8 new vertex operators (massless gauge bosons), with each of them inheriting the ghost cohomlogies of appropriate $\alpha$-transformations. Below we shall construct explicitly 8 linear combinations of the $\alpha$-generators generating $SU(3)$, so 8 massless gauge bosons, obtained by $\alpha$-transforming the photon operator, will be in the adjoint of $SU(3)$.

Since $L^{\alpha -}$ annihilates the photon, one can set

$$L^{\alpha -} \approx 0$$

as the subgroup of 9 scalar $\alpha$-generators is applied to the RNS Hilbert space. In particular this implies that any of the remaining 8 generators can be shifted by an operator proportional to $L^{\alpha -}$. Given the constraint (16) it is now straightforward to show
that SU(3) is induced by the following eight linear combinations of the \( \alpha \)-generators of \( H_n \sim H_{-n-2}; n = 1, 2, 3 \):

\[
F_+ = -\frac{1}{\sqrt{2}}(L^{\gamma^+} + L^{\gamma^-}) - \frac{i}{\sqrt{2}}(L^{\beta^+} + L^{\beta^-}) + L^{\alpha\beta} - iL^{\alpha\gamma} - \frac{i}{\sqrt{2}}(L^\alpha + L^\alpha^-) \\
F_- = -\frac{1}{\sqrt{2}}(L^{\gamma^+} + L^{\gamma^-}) - \frac{i}{\sqrt{2}}(L^{\beta^+} + L^{\beta^-}) - L^{\alpha\beta} + iL^{\alpha\gamma} - \frac{i}{\sqrt{2}}(L^\alpha + L^\alpha^-) \\
F_3 = -\frac{1}{\sqrt{2}}(L^{\gamma^+} - L^{\gamma^-}) - \frac{i}{\sqrt{2}}(L^{\beta^+} - L^{\beta^-})
\]

\[
L_1 = \frac{i}{2}L^{\beta\gamma} \\
L_2 = \frac{i}{\sqrt{2}}(L^\alpha + L^\alpha^-)
\]

\[
G_+ = -\frac{1}{\sqrt{2}}(L^{\gamma^+} + L^{\gamma^-}) + \frac{i}{\sqrt{2}}(L^{\beta^+} + L^{\beta^-}) + L^{\alpha\beta} + iL^{\alpha\gamma} + \frac{i}{\sqrt{2}}(L^\alpha + L^\alpha^-) \\
G_- = -\frac{1}{\sqrt{2}}(L^{\gamma^+} + L^{\gamma^-}) + \frac{i}{\sqrt{2}}(L^{\beta^+} + L^{\beta^-}) - L^{\alpha\beta} - iL^{\alpha\gamma} + \frac{i}{\sqrt{2}}(L^\alpha + L^\alpha^-) \\
G_3 = -\frac{1}{\sqrt{2}}(L^{\gamma^+} - L^{\gamma^-}) + \frac{i}{\sqrt{2}}(L^{\beta^+} - L^{\beta^-})
\]

Here \( L_1 \) and \( L_2 \) are the Cartan generators, three operators \( F_\pm \) and \( F_3 \) are in the lowering subalgebra of \( SU(3) \) while \( G_\pm \) and \( G_3 \) are in the raising subalgebra. The commutators of the operators (17) are straightforward to compute by using the commutation relations (10) and imposing the constraint (16) upon the computation. The next step is to analyse the transformations of the photon emission vertex by this octet of generators. Since all the generators (17) are the integrated space-time scalars of conformal dimension one, the result must involve the octet of dimension 1 vertex operators - the gauge vector bosons (provided there is no annihilation or BRST triviality). We start with the proof of the claim concerning the annihilation of a photon by \( L^{\alpha^-} \). The photon vertex operator taken at pictures 0 and \(-1\) is given by:

\[
V_{ph}^{(0)}(k) = A_m(k) \oint dz(\partial X^m + i(\bar{\psi}\gamma^m)\psi^m)e^{ik_nX^n}(z) \\
V_{ph}^{(-1)}(k) = A_m(k) \oint dz \bar{\psi}^m e^{ik_nX^n}(z); \\
m = 0, 1, 2, 3
\]

with the \( z \) integral taken over the worldsheet boundary.
If $L^\alpha^-$ (the element of $H_1 \sim H_{-3}$) taken at picture $-3$ or below (i.e. in the $H_{-3}$ representation), the annihilation of $V_{ph}$ is obvious since the former depends only on the Liouville and the ghost fields while $V_{ph}$ involves the matter. Things, however, are not as straightforward if $L^\alpha^-$ is taken at picture $+1$ or higher (i.e. in the $H_1$ representation), since the OPE’s involving the operators from nonzero cohomologies are generally picture-dependent \footnote{Still, it is easy to check that the annihilation of photon by $L^\alpha^-$ holds for the $H_1$ representation as well. The full BRST-invariant picture $+1$ representation of $L^\alpha^-$ can be obtained from its picture $-3$ version by changing $-3\phi \rightarrow \phi$ and adding the $b-c$ ghost dependent correction terms in order to protect its BRST invariance, using the prescription described in \footnote{so the complete expression for the picture $+1$ representation of $L^\alpha^-$ is}

$$L^\alpha^-_{+1} = \int \frac{dz}{2i\pi} e^\phi(\lambda \partial^2 \varphi - 2\partial \lambda \varphi)(z)$$

$$+ \frac{1}{8} \oint \frac{dz}{2i\pi} (z-w)^2 ce^x \{ (\psi_m \partial X^m)(\lambda \partial^2 \varphi - 2\partial \lambda \varphi)(z) + U_L(z) \} + ...$$

$$\equiv \int \frac{dz}{2i\pi} e^\phi(\lambda \partial^2 \varphi - 2\partial \lambda \varphi)(z) + \frac{1}{8} \oint \frac{dz}{2i\pi} (z-w)^2 \{ (\psi_m \partial X^m)F_L(z) + ce^x U_L(z) \} + ...$$

(19)

where the second contour integral in (19) is taken around an arbitrary point $w$, $U_L$ is the dimension 4 operator which depends only on the super Liouville fields $\varphi$ and $\lambda$ (and therefore the corresponding term does not contribute to the commutator of $L^\alpha^-_{+1}$ with $V_{ph}^{(0)}$) and we denoted the ghost-Liouville dependent operator $F_L(z) = ce^x(\lambda \partial^2 \varphi - 2\partial \lambda \varphi)(z)$ for the sake of brevity. We also have skipped the correction terms proportional to $\sim (z-w)^2 e^\phi(...)$ of $b-c$ ghost number zero, as these terms depend on Liouville and ghost fields only, automatically commuting with the picture zero photon.

The evaluation of the commutator of $L^\alpha^-$ with the photon at picture zero gives:

$$[L^\alpha^-_{+1}, V_{ph}^{(0)}] = \int du \oint \frac{dz}{2i\pi} (z-w)^2 G_L(z) \{ - \frac{1}{(z-u)^2} (\bar{A}\psi)e^{ik_mX^m} - \frac{1}{(z-u)} \partial((\bar{A}\psi)e^{ik_mX^m}) \}$$

$$= \int du \oint \frac{dz}{2i\pi} \frac{(z-w)^2}{(u-w)^2} \{ G_L(u) (\bar{A}\psi)e^{ik_mX^m}(u) - \frac{(z-w)^2}{(u-w)^2} \partial(G_L(\bar{A}\psi)e^{ik_mX^m}(u)) \}$$

$$\equiv \int du \{ 2(u-w)G_L(\bar{A}\psi)e^{ik_mX^m}(u) + (u-w)^2 \partial(G_L(\bar{A}\psi)e^{ik_mX^m}(u)) \}$$

$$= \int du \partial((u-z)^2 G_L(u)(\bar{A}\psi)e^{ik_mX^m}(u)) = 0$$
Finally, for the sake of completeness, we also note that the commutator of $V^{\alpha -}$ at the $H_{-3}$ representation with the photon at picture $-1$ (and subsequently all the pictures below) produces the BRST trivial result:

$$[L_{(-3)}^{\alpha -}, V_{ph}^{(-1)}] =: \Gamma^{-2}[Q_{brst}, \int du(z - u)^2 \partial \xi G_L(u)(\bar{A}_\tau^\alpha) e^{ikmX^m}(u)]:$$ (21)

This constitutes the proof of the annihilation of $V_{ph}$ by $L^{\alpha -}$.

On the contrary, the remaining eight scalar $\alpha$-generators acting on $V_{ph}$ produce new physical states of $H_n \sim H_{n-2}, n = 1, 2, 3$. Thus the $H_1 \sim H_{-3}$ $\alpha$-generator $L^{\alpha +}$ acts on the picture zero photon, producing the vertex operator $V^{\alpha +}$, the massless gauge vector boson of the same ghost cohomology:

$$[L_{(-3)}^{\alpha +}, V_{ph}^{(0)}(k)] = 3 \int du e^{-3\phi} \{(\bar{k}\partial \bar{X})(\bar{A}\partial X)\bar{\psi} + (\bar{k}\bar{\psi})(\bar{A}\partial \bar{\psi}) - (\bar{k}\partial \bar{X})^2(\bar{A}\bar{\psi})\} e^{ikX}$$ (22)

This operator is the element of $H_{-3}$. Its $H_1$ version is straightforward to obtain by replacing $-3\phi \to \phi$ and adding the $b - c$ correction term using the prescription of [8]. For the sake of completeness, the explicit expressions for some of the remaining gluon operators are given below:

$$V^{\beta +} = [L^{\beta +}, V_{ph}] = \int e^{-4\phi}\{\lambda \partial^2 \varphi - 2\partial \varphi \partial \lambda\}\{(\bar{k}\partial \bar{X})^2 + i(\bar{A}\partial \bar{\psi})(\bar{k}\bar{\psi})\} e^{ik\bar{X}} - i(\bar{k}\partial \bar{X})\partial W_{ph}(k)$$

$$V^{\beta -} = [L^{\beta -}, V_{ph}] = \int e^{-4\phi}\{2\lambda \partial \varphi\}(\bar{k}\partial \bar{X})^2(\bar{A}\bar{\psi}) + (\bar{k}\partial \bar{X})(\bar{A}\partial \bar{X})(\bar{A}\bar{\psi}) - i(\bar{k}\bar{\psi})(\bar{A}\partial X) - i(\bar{k}\partial \bar{X})(\bar{A}\bar{\psi}) - (\bar{k}\partial \bar{X})^2(\bar{A}\bar{\psi}) e^{ik\bar{X}} - i(\bar{k}\partial \bar{X})\partial W_{ph}(k)$$

$$V^{\alpha \beta} = [L^{\alpha \beta}, V_{ph}] = \int e^{-4\phi}\{(\partial L_1(\varphi, \lambda) + 2\partial \varphi L_1(\varphi, \lambda))(\bar{k}\bar{\psi})(\bar{A}\partial \bar{X})(\bar{A}\bar{\psi})\} e^{ik\bar{X}}$$

$$V^{\beta \gamma} = [L^{\beta \gamma}, V_{ph}] = \int e^{-5\phi}\{(\partial L_2(\varphi, \lambda) - \frac{11}{3}\partial \varphi L_2(\varphi, \lambda))(\bar{k}\bar{\psi})(\bar{A}\partial \bar{X})(\bar{A}\bar{\psi})\} e^{ik\bar{X}}$$

$$V^{\gamma +} = [L^{\gamma +}, V_{ph}] = \int e^{-5\phi}\{(\partial F_2(\lambda, \varphi) - \frac{10}{3}\partial \varphi F_2(\varphi, \lambda))(\bar{k}\bar{\psi})(\bar{A}\partial \bar{X})(\bar{A}\bar{\psi})\} e^{ik\bar{X}}$$

$$\times\{(\bar{k}\partial \bar{X})^2(\bar{A}\bar{\psi}) + i(\bar{A}\partial \bar{X})(\bar{k}\bar{\psi})\} e^{ik\bar{X}} - i(\bar{k}\partial \bar{X})\partial W_{ph}(k)$$ (23)
where $W_{ph}(k) = (\tilde{A} \partial X + i(\tilde{k}\bar{\psi})(\tilde{A}\bar{\psi}))e^{i\tilde{k}X}$ is the integrand of the photon vertex operator at picture zero. The explicit expressions for the vertex operators $V^{\gamma-}$ and $V^{\alpha\gamma}$ are quite lengthy (due to more complicated structure of the appropriate $\alpha$-generators) and we will skip them here for the sake of brevity, as we won’t use them in any further calculations in this paper. These expressions, if needed, are straightforward to obtain by applying $L^{\gamma-}$ and $L^{\alpha\gamma}$ to the photon, exactly as demonstrated above. The operators (22) are given in the negative cohomology representations, i.e. the elements of negative ghost cohomologies $H_{-n-2}(n = 1, 2, 3)$ Their versions in the corresponding isomorphic positive cohomology representations $H_n(n = 1, 2, 3)$ are straightforward to obtain either by the direct isomorphism construction described in [8] or (much easier) by replacing $(-n-2)\phi \rightarrow n$ and adding the correction terms using the formalism described ibide. In particular, some examples of manifest expressions for positively represented gluons will be considered in Section 5. In the following sections we will discuss the OPE properties and the structure constants of the gluon vertex operators (22), (23) and analyze their scattering amplitudes.

4. Gluon Vertices: Structure Constants and Closeness of the Operator Algebra

In this section we analyze the structure constants (3-point functions) of the operators (22), (23). In particular, the remarkable property of the operators (22), (23), which we will prove below, is the closeness of their operator algebra, related to their underlying zigzag symmetry, reflecting the relevance of these operators to the gauge theory dynamics and gauge-string correspondence. Using the explicit expressions for the operators (23), it is in principle straightforward to calculate their structure constants. There is no need, however, in the direct calculation (not hard but somewhat long), as the form of the constants is actually fixed by the $\alpha$-symmetry. Consider the 3-point correlation function of the standard open string photons:

$$<V_{ph}(k)V_{ph}(p)V_{ph}(q)> = A_l(k)A_m(p)A_n(q) <cW_l(z_1)cW_m(z_2)cW_n(z_3) > = i\{(\tilde{k}\tilde{A}(\tilde{q}))\tilde{A}(\tilde{p})\tilde{A}(\tilde{q}) - (\tilde{q}\tilde{A}(\tilde{p}))\tilde{A}(\tilde{k})\tilde{A}(\tilde{q}) + (\tilde{p}\tilde{A}(\tilde{k}))\tilde{A}(\tilde{p})\tilde{A}(\tilde{q})\}$$

(24)

where, for the certainty, $V_{ph}$ taken in unintegrated form and $W_m$ are the dimension 1 integrands of the integrated photon operator: $W_n = e^{-\phi}\psi_n e^{i\tilde{k}X}$ at picture $-1$ and $W_n = (\partial X_n + i(\tilde{k}\bar{\psi})\psi_n)e^{i\tilde{k}X}$ at picture 0 (note that the extra term $\sim \gamma\psi_n e^{i\tilde{k}X}$ in the picture zero expression for the unintegrated photon is ignored here since it doesn’t contribute to the 3-point function; two photon operators in the correlator (24) must be taken at picture...
With the correlation function (24), the OPE of two dimension 1 photon operators is

\[ W_l(k; z_1) W_m(p; z_2) = \frac{C_{lm}^n(k, p) W_n(q; \frac{z_1 + z_2}{2})}{z_1 - z_2} \]

\[ + \sum_{N=0}^{\infty} (z_1 - z_2)^{N+\frac{d-2}{2}} C^{(N)}(k, p) W^{(N)}(q) \]

\[ C_{lm}^n(k, p) = i(k^n \eta_{lm} - q_m \eta_l^n + p_l \eta_m^n) \]

\[ \vec{k} + \vec{p} + \vec{q} = 0 \]  

(25)

where we have skipped BRST trivial tachyonic term of the order of \((z - w)^{-2}\). The \(W^{(N)}(q)\)-operators appearing in the higher order terms of the OPE (25) correspond to the massive poles in the Veneziano amplitude (after the appropriate on-shell conditions on \(q\) are imposed). Structurally, these operators have the form

\[ W^{(N)}(q) = U_{N+2}(X, \psi)e^{iqX} \]

(26)

where \(U_{N+2}\) is some polynomial in \(X\) and \(\psi\) of conformal dimension \(N + 2\) and the on-shell condition for \(W^{(N)}\) (corresponding to the mass of the appropriate physical pole in the Veneziano amplitude) is given by

\[ (\vec{q})^2 = -2N - 2 \]

(27)

Note that, apart from the color related factor, the structure constants \(C_{lm}^n\) in front of the photon simply reproduce the 3-gluon vertex of QCD. The next step is to apply \(\alpha\)-transform to both sides of the OPE (25). For the brevity, it is convenient to write

\[ [L^i, L^j] = D^{ij}_k L^k (i = 1, ..., 8); \]

\[ V_m^i \equiv [L^i, W_m] \]

(28)

where \(L^i\) stands for any of eight SU(3) \(\alpha\)-generators (17) - \(F_\varphi, G_\pm, F_3, G_3\) and \(L_{1,2}\), \(V^i\) are the corresponding gluon vertices and \(T^{ij}_k\) are the SU(3) structure constants. Consider the OPE of two gluon integrands of dimension 1:

\[ W^i_l(k; z_1) W^j_m(p; z_2) = \oint_{z_1} \frac{dw_1}{2i\pi} \oint_{z_2} \frac{dw_2}{2i\pi} T^i(w_1) T^j(w_2) W_l(k; z_1) W_m(p; z_2) \]

(29)

with \(k\) and \(p\) being the on-shell momenta of 2 photons and \(T^i, T^j\) are the integrands of \(L_i, L_j\). Firstly, let’s concentrate on the simple pole of the OPE. Since \(T^i\) and \(T^j\) are the
dimension one primaries and the physical operators, the only BRST non-trivial operator in their full OPE will be that of the dimension one, i.e. the one appearing in the simple pole term $\sim (z_1 - z_2)^{-1}$. This operator is simply the integrand of the r.h.s. of the commutator of $L_i$ and $L_j$, i.e. $D_{k}^{ij} T^k(z)$ where $T^k$ is the integrand of $L^k$. Therefore the simple pole in the OPE of $W_i^j$ and $W_m^j$ is given by

$$W_i^j(k, z_1) W_m^j(p, z_2) \sim \frac{D_k^{ij} C_{lm}^m(k, p, q)}{z_1 - z_2} \int_{z_1 + z_2} \frac{dw}{2i\pi} [T^k(w), W_n(q; \frac{z_1 + z_2}{2})]$$

(30)

in particular, this indicates that the 3-point gluon vertex is reproduced by the OPE (30). But what about the full OPE of two gluons, that is, the higher order terms? In the standard case of two photons, these terms (with the on-shell condition (27) imposed) give rise to the massive intermediate states in any $n$-point amplitude with $n \geq 4$, killing the “naive” gauge-string correspondence. How about gluons? Remarkably, it turns out that all the intermediate massive states in the OPE of two gluons (30) are BRST-trivial; for this reason they do not lead to any massive poles (destroying the zigzag invariance) and thus the gauge-string correspondence is protected. This occurs due to very special property of the SU(3) $\alpha$-generators (proof given below): they produce new physical states only when applied to massless vertex operators; however, the $\alpha$-transform of any massive superstring mode gives BRST-trivial result! Here is the proof. Repeating the arguments we used to fix the 3-gluon vertex (30) and using the full OPE (25) of two photons, it is easy to check that the term of the order of $(z_1 - z_2)^n$ in the OPE of two gluons $W_i^j(k)$ and $W_m^j(p)$ is given by

$$\sim C^{(N)}(k, p, q) D_{k}^{ij} [L^k, W^{(N)}(q)]$$

Our goal is thus to show that the operator $[L^k, W^{(N)}(q)]$ is BRST-trivial for any $(\vec{q})^2 \neq 0$. For brevity, below we will give the proof for the case $L^{(k)} = L^{\alpha+}$; the proof is totally analogous for all other SU(3) $\alpha$-generators.

To simplify things further, it is more convenient to take the massive operators of the right-hand side of (25) at the unintegrated $b - c$-picture, that is,

$$V^{(N)}(q) = cW^{(N)}(q) = cU_{N+2}(X, \psi)e^{i\vec{q}\vec{X}}$$

(31)

(since the unintegrated and integrated vertices are related by the BRST-invariant $Z$-transformation [12], [8], any (non)triviality statement proven for an unintegrated vertex, applies to the integrated one as well) The BRST-invariance condition for the vertex operator (31) implies

$$\{Q_{brst}, cU_{N+2}\} e^{i\vec{q}\vec{X}} = cU_{N+2}[Q_{brst}, e^{i\vec{q}\vec{X}}].$$

(32)
\[ [Q_{\text{brst}}, e^{iqX}] = \frac{1}{2}(q)^2 \partial c e^{iqX} + c \partial (e^{iqX}) + \frac{i}{2} \gamma(k\bar{\psi})e^{iqX} \]  \hspace{1cm} (33)

Writing

\[ U_{N+2}^\alpha \equiv [L^\alpha, U_{N+2}] \]
\[ [L^\alpha, e^{iqX}] = Z^\alpha e^{iqX} \]
\[ V_{N+2}^\alpha(q) \equiv [L^\alpha, V_{N+2}(q)] \]
\[ = (U_{N+2}^\alpha + U_{N+2}Z^\alpha)e^{iqX} \]  \hspace{1cm} (34)

where

\[ Z^\alpha = i\partial(e^{-3\phi})(k\bar{\psi}) + 3ie^{-3\phi}(k\partial\bar{\psi}), \]  \hspace{1cm} (35)

we apply the \( \alpha \)-transformation by \( L^\alpha \) to the both sides of (33) obtaining

\[ \{Q_{\text{brst}}, cU_{N+2}^\alpha\} e^{iqX} + \{Q_{\text{brst}}, cU_{N+2}\} Z^\alpha e^{iqX} \]
\[ \hspace{1cm} = \frac{1}{2}(q)^2 c\partial U_{N+2}^\alpha e^{iqX} + \frac{1}{2}(q)^2 c\partial U_{N+2}Z^\alpha e^{iqX} = -\frac{1}{2}(q)^2 \partial cV_N^\alpha(q) \]  \hspace{1cm} (36)

where we used the commutation relations (34), as well as

\[ : \gamma Z^\alpha := :\gamma e^{-3\phi} := :c^2 := 0. \]  \hspace{1cm} (37)

Substituting the identity (32) we cast the equation (36) as

\[ \{Q_{\text{brst}}, cU_{N+2}^\alpha\} e^{iqX} + cU_{N+2}Z^\alpha [Q_{\text{brst}}, e^{iqX}] = -\frac{1}{2}(q)^2 \partial cV_N^\alpha(q). \]  \hspace{1cm} (38)

But

\[ \{Q_{\text{brst}}, cU_{N+2}^\alpha\} = -\{Q_{\text{brst}}, cU_{N+2}Z^\alpha\} \]  \hspace{1cm} (39)

since the operator : \( U_{N+2}^\alpha + U_{N+2}Z^\alpha : \) is BRST-invariant. In fact, this point deserves separate clarification. That is, the BRST-invariance of : \( U_{N+2}^\alpha + U_{N+2}Z^\alpha : \) is guaranteed since the structure of this operator is inherited from the appropriate term of the \( \alpha \)-transformed OPE (25) of two photons. Since the photons of the OPE (25) are on-shell, the BRST commutator with each term in their full OPE has to vanish separately. If the \( \alpha \)-transformed term of the order of \( (z_1 - z_2)^N \) in the OPE (25) is given by \( C^{(N)}[L^\alpha, W^{(N)}(q)] \), the corresponding operator in the OPE of two gluons is given by

\[ cW^{(N)}(q) = [L^\alpha, cW^{(N)}(q)] = c(U_{N+2}^\alpha + U_{N+2}Z^\alpha)e^{iqX} \]  at the unintegrated \( b - c \) picture. This operator is invariant for all values \( q \), both on and off-shell (as a matter of
fact, it is exact for all \((\vec{q})^2 \neq 2N + 2\). In the special off-shell case \((\vec{q})^2 = 0\) using (37) along with the commutator (33) gives

\[
0 = \{Q_{brst}, e^{W^{\alpha+,N}(q)}\} = \{Q_{brst}, c(U_{N+2}^{\alpha+} + U_{N+2}Z^{\alpha+})\} e^{i\vec{q}\vec{X}} - c(U_{N+2}^{\alpha+} + U_{N+2}Z^{\alpha+})[Q_{brst}, e^{i\vec{q}\vec{X}}] = \{Q_{brst}, c(U_{N+2}^{\alpha+} + U_{N+2}Z^{\alpha+})\} e^{i\vec{q}\vec{X}}
\]

(40)
as the second term in the commutator (40) vanishes due to (33) and (37). Thus : \(U_{N+2}^{\alpha+} + U_{N+2}Z^{\alpha+}\) is BRST-invariant and therefore, in view of (40), the identity (36) can be written as

\[
-\{Q_{brst}, U_{N+2}Z^{\alpha+}\} e^{i\vec{q}\vec{X}} + cU_{N+2}Z^{\alpha+}[Q_{brst}, e^{i\vec{q}\vec{X}}] = -\{Q_{brst}, cU_{N+2}Z^{\alpha+} e^{i\vec{q}\vec{X}}\} = -\frac{1}{2}(\vec{q})^2 \partial cV_{\alpha+}(q)
\]

(41)
Now, since the unintegrated vertex operator \(V_{\alpha+}^{\alpha+}(q)\) is dimension zero primary field, it is annihilated by the zero mode of the full stress tensor: \([T_0, V_{\alpha+}^{\alpha+}(q)] = 0\) where \(T_0 = \oint \frac{dz}{2i\pi} zT(z)\). Therefore, as \(T_0 = \{Q_{brst}, b_0\}\), the identity (41) implies:

\[
V_{\alpha+}^{\alpha+}(q) = \frac{2}{(\vec{q})^2} [Q_{brst}, b_0 cU_{N+2}Z^{\alpha+} e^{i\vec{q}\vec{X}}]
\]

(42)
This concludes the proof that the \(\alpha\)-transformation of any massive physical operator by \(L^{\alpha+}\) is BRST-exact. The proof is completely analogous for any other \(\alpha\)-generator \(L^k; k = 1, ..., 8\), so the generalization of (42) for an arbitrary \(\alpha\)-generator transforming a massive physical operator, is

\[
V_{\alpha+}^k(q) \equiv [L^k, V_{\alpha+}(q)] = \frac{2}{(\vec{q})^2} [Q_{brst}, b_0 cU_{N+2}Z^k e^{i\vec{q}\vec{X}}]
\]

(43)
with the operators \(Z^k(q)\) defined according to

\[
[L^k, e^{i\vec{q}\vec{X}}] = Z^k(q)e^{i\vec{q}\vec{X}}
\]

(44)
This constitutes the proof of the closeness (and, accordingly, of the underlying zigzag symmetry) of the gluon operator algebra (30). Speaking metaphorically, the \(\alpha\)-symmetry transform, applied to the OPE of photons, acts like an Occam’s razor: it shaves off the infinite tower of the higher order on the right-hand side (leading to massive poles in scattering amplitudes) protecting the zigzag symmetry of the photons turned gluons. In the following sections we will use this “lex parsimoniae” to directly compute the 4-point scattering amplitude involving the gluons.
5. Zigzag-Invariant vs. Veneziano amplitudes: technical remarks

Using the results of the previous section (zigzag symmetry of the constructed gluon vertex operators and the closeness of their OPE (30)) it is now relatively easy to calculate N-point scattering amplitudes of gluons with higher number of vertices \((N > 3)\) from string theory, despite a complicated formal structure of the vertex operators (22),(23).

Since all the massive intermediate states appearing in the operator product of any two gluon operators are BRST-trivial, they do not contribute to worldsheet correlations, so only the massless gluon vertices appearing on the right hand side of the OPE (multiplied by the 3-gluon structure constants) are relevant. Therefore we can use the simplicity of the zigzag-invariant operator algebra of gluons to compute their correlation functions by bootstrap. This calculation (which can be generalized to higher number of scattering gluons as well as to include the loop corrections) will be performed in the section 6 of this paper. Before we proceed with the calculations, however, it is instructive to make few technical comments concerning the structure of the 4-point amplitude.

As our principal calculation of the 4-point amplitude is almost independent on the arguments of this section, a reader, not interested in the technical details, can skip it and go directly to the Section 6, where this calculation is presented.

As we already noted above, we expect the 4-point amplitude of gluon vertex operators (22),(23) to have field-theoretic behaviour, implying the absence of massive resonances. Such a behaviour immediately follows from with the OPE (30) for the gluons and the underlying zigzag symmetry; however, it may seem a bit of a surprise from technical point of view. Indeed, the structure of 4-point Veneziano amplitude in “orthodox” string theory is well-known; it has an infinite sequence of massive poles, corresponding to integer values of the Mandelstam parameters. The question is - what is so special about the gluon operators (22),(23) that distinguishes them from usual vertex operators (such as photons) leading to radically different pole structure? Not quite surprisingly, the answer lies in the ghost number structure of the gluon operators and their “non-standard” ghost numbers. To answer the question, it is useful to recall how the “standard” Veneziano amplitude emerges in open string theory. Typically, the 4-point amplitude of open string theory involves 3 unintegrated vertex operators and one integrated, i.e. it has the form \(\int_0^1 dz \, <W(z)cW_1(z_1)cW_2(z_2)cW_3(z_3)>\) with the points \(z_1, z_2, z_3\) fixed and the remaining operator \(W(z)\) integrated over the worldsheet boundary. Such a structure is dictated by the \(b - c\) ghost number anomaly which is equal to \(-3\), so to cancel it, one needs three unintegrated vertices, each of them carrying the \(b - c\) ghost number +1. Here the \(b - c\) ghost factor...
< c(z_1)c(z_2)c(z_3) > particularly leads to the standard Koba-Nielsen measure given by
\sim (z_1 - z_2)(z_1 - z_3)(z_2 - z_3). Using the $SL(2, C)$ invariance, one fixes $z_1 = 0, z_2 = 1, z_3 = \infty$
so the remaining $z$ integral is proportional to $\sim \int_0^1 dzz^{-\frac{4}{3}}(1-z)^{-\frac{4}{3}}$ which is just the Euler’s
beta-function of the Mandelstam variables, leading to the Veneziano amplitude. Such is
the standard case situation. With the gluon operators (22), (23) things are more subtle.
Since these operators are the elements of nonzero ghost cohomologies, they do not exist
at arbitrary ghost numbers and one has to be careful to ensure the correct balance of
ghost numbers to cancel both superconformal ghost number anomalies (equal to +2 for
the $\phi$-field and $-1$ for the $\chi$-field), in addition to the $b-c$ anomaly. And this is where the
difference strikes.

Consider the 4-point functions of the gluon operators (22), (23) with two of them be-
ing in the positive and two in negative ghost cohomology representations (for the reasons
explained in \[8\] any correlation function involving the picture-dependent operators always
has to involve the operators from cohomologies of opposite signs). The peculiar property
of operators in positive cohomologies is that they only exist in the integrated form, as the
$b-c$ correction terms that ensure the overall BRST invariance of the positive cohomology
elements can only be constructed in the integrated case \[8\]. This is related to the fact that
integrated and unintegrated forms of vertex operators correspond to their representations
in two adjacent $b-c$ pictures, which are the fermionic analogues of the usual superconformal
ghost pictures, with the BRST-invariant $Z$-transformation being the analogue of picture-
changing operator $\Gamma$. \[12\], \[8\] Not surprisingly, the inequivalence in the superconformal
pictures (resulting in the emergence of ghost cohomologies) is related to inequivalence in
the fermionic $b-c$-pictures, in view of the supersymmetry. In fact, there is an underly-
ing geometrical principle relating the bosonic and fermionic ghost cohomologies, which is
not yet understood completely and, by itself, it is an interesting direction for the future
research. Therefore, contrary to the case of picture-independent generators which 4-point
function contains only one integrated vertex, leading to “standard” Veneziano amplitude,
the 4-point function of gluons has to contain at least 2 integrated operators, corresponding
to insertions from positive cohomologies. At the same time, the presence of two integrated
vertices in the 4-point function does not violate the $b-c$ ghost number balance since, as
we know, the integrated operators of positive cohomologies involve the correction terms
proportional to the $c$-ghost field. These correction terms now play a significant role; it is
because of them that one can have a 4-point correlator with two integrated vertices but
with the total $b-c$ ghost number of the correlator still equal to 3 (needed to cancel the
terms are any calculations in this paper. These expressions can be obtained straightforwardly by complicated explicit form but fortunately their manifest expressions are not needed for $c$, and are given by the integrands of the appropriate expressions of (22), (23) multiplied by $c$, while the positively represented operators of $H_3$, including the necessary $b - c$ correction terms are

\[
V^{\beta\gamma} = [L^{\beta\gamma}, V_{ph}] = \oint dz e^{3\phi} \{ \partial L_2(\varphi, \lambda) - \frac{11}{3} \partial \phi L_2(\varphi, \lambda) \}(\bar{k}\psi)(\bar{k}\partial X)(\bar{A}\psi)e^{i\vec{k} \cdot \vec{X}}(z)
\]

\[
-\frac{22}{15} \oint u \frac{(u - z)^5}{u} \{ ce^{2\phi + \chi} P^{(3)}_{\phi - \chi}(\bar{k}\psi)(\bar{k}\partial X)(\bar{A}\psi)e^{i\vec{k} \cdot \vec{X}}(z) + e^{3\phi} G^{(1)}_{\bar{A}}(L_2, X, \psi, \phi, \chi, \sigma) \}
\]

\[
V^{\gamma+} = [L^{\gamma+}, V_{ph}] = \oint dz e^{3\phi} \{ (\partial F_2(\lambda, \varphi) - \frac{10}{3} \partial \phi F_2(\varphi, \lambda))(z)
\]

\[
\times ((-i(\bar{k}\partial X)^2(\bar{A}\psi) + i(\bar{A}\partial^2 X)(\bar{k}\psi))e^{i\vec{k} \cdot \vec{X}} - i(\bar{k}\psi)\partial W_{ph}(z, \bar{k}))(z) \}
\]

\[
-\frac{2}{3} \oint u \frac{(u - z)^5}{u} \{ ce^{2\phi + \chi} P^{(3)}_{\phi - \chi}(\varphi, \lambda)(z)(-i(\bar{k}\partial X)^2(\bar{A}\psi) + i(\bar{k}\psi)(\bar{k}\partial \psi)(\bar{A}\psi) + (\bar{A}\partial X)(\bar{k}\partial X)(\bar{k}\psi))(z)
\]

\[
+ e^{3\phi} G^{(2)}_{\bar{A}}(F_2, X, \psi, \phi, \chi, \sigma) \}
\]

(45)

where the $z$-integrals in the correction terms are taken around some point $u$ of the world-sheet boundary (which can be fixed to zero by the $SL(2, C)$ symmetry) and $P^{(3)}_{\phi - \chi}$ is conformal dimension 3 polynomial in the derivatives of $\phi$ and $\chi$ defined according to

\[
P^{(n)}_{L(\phi_1(z), \ldots, \phi_n(z))} = e^{-L(\phi_1(z), \ldots, \phi_n(z))} \frac{d^n}{dz^n} e^{L(\phi_1(z), \ldots, \phi_n(z))} \text{ with } L \text{ being an arbitrary given function of arbitrary } n \text{ fields } \phi_1(z), \ldots, \phi_n(z) \text{ (} L = \phi - \chi \text{ in our case). The conformal dimension } \frac{27}{2} \text{ operators } G^{(1)}_{\bar{A}}(L_2, X, \psi, \phi, \chi, \sigma) \text{ and } G^{(2)}_{\bar{A}}(F_2, X, \psi, \phi, \chi, \sigma) \text{ are the polynomials in the worldsheet fields } F_2(\varphi, \lambda), L_2(\varphi, \lambda), X, \psi, \phi, \chi, \sigma \text{ and their derivatives. They have complicated explicit form but fortunately their manifest expressions are not needed for any calculations in this paper. These expressions can be obtained straightforwardly by}
evaluating the fifth order non-singular terms in the operator products:

\[
: cc^2\chi^2 - 2\phi :: (z) : \text{be}^5\phi^2 - 2\chi \cdot \left\{ \frac{1}{120} P^{(5)}_{2\phi - 2\chi - \sigma} (\partial L_2 - \frac{11}{3} \partial \phi L_2) \right. \\
\left. - \frac{11}{1080} P^{(6)}_{2\phi - 2\chi - \sigma} L_2 \right\} R_1(X, \psi) : (w) \\
\sim \ldots + (z - w)^5 : e^{3\phi} G^{(1)}_{\frac{3\pi}{4}} : \left( \frac{z + w}{2} \right) + \ldots
\]

(46)

\[
: cc^2\chi^2 - 2\phi :: (z) : \text{be}^5\phi^2 - 2\chi \cdot \left\{ \frac{1}{120} P^{(5)}_{2\phi - 2\chi - \sigma} (\partial F_2 - \frac{10}{3} \partial \phi F_2) \right. \\
\left. - \frac{1}{1080} P^{(6)}_{2\phi - 2\chi - \sigma} \right\} R_2(X, \psi) : (w) \\
\sim \ldots + (z - w)^5 : e^{3\phi} G^{(2)}_{\frac{3\pi}{4}} : \left( \frac{z + w}{2} \right) + \ldots
\]

where

\[
R_1(X, \psi) = (\vec{k} \vec{\psi})(\vec{k} \partial \vec{X})(\vec{A} \vec{\psi}) e^{i\vec{k} \vec{X}}(z)
\]

and

\[
R_2(X, \psi) = - (\vec{k} \partial \vec{X})^2 (\vec{A} \vec{\psi}) + i(\vec{A} \partial^2 \vec{X})(\vec{k} \vec{\psi}) e^{i\vec{k} \vec{X}} - i(\vec{k} \vec{\psi}) \partial W_{ph}(z, \vec{k})
\]

The ghost number anomaly cancellation condition then determines that the 4-point correlator is given by the crossterm contribution of the basic term (proportional to \(e^{3\phi}\)) in one of two positively represented operators (\(V^{\gamma +}\) or \(V^{\beta \gamma}\)) and the correction term in the second (proportional to \(ce^{2\phi + \chi}\)) as the ghost factors of the negatively represented unintegrated operators, \(V^{\alpha \beta}(z_3)\) and \(V^{\alpha +}(z_4)\) are given by to \(ce^{-4\phi}\) and \(ce^{-3\phi}\) respectively (with \(z_{3,4}\) being the locations of the operators). Using the \(SL(2, C)\) symmetry, we can fix \(z_1 = 0, z_3 = 1, z_4 = \infty\), so the resulting double integral for the 4-point function consists of the terms with the structure

\[
A \sim \int_0^1 dz \int_0^1 du u^5 (z - u)^a (u - 1)^b (z - 1)^c
\]

where \(a, b\) are linear in the Mandelstam parameters. (e.g. typically, \(a(s) = \frac{s}{2} + m, b(t) = \frac{t}{2} + n, c = -s - t + p\) where \(m, n\) and \(p\) are integer numbers depending on the particular contraction) and the \(u^5\)-factor originates from non-local \(c\)-dependent correction terms of positively represented gluon vertices.

The integration in \(z\) can be done analytically if \(Re\{c\} \geq -1\) and the answer is given by

\[
A \sim (1 + c)^{-1} u^{5+a} (u - 1)^b F_1(1, -a, 2 + c; \frac{1}{u}).
\]

22
The subsequent integration in $u$ is also possible, if $Re\{b\} > 1$ leading to long and cumbersome combination of terms with the structure

\[
A \sim \sum_{m_1=1}^{6} \frac{P_{m_1}(a, b) \, _2F_1(m_1, m_1 + 1 - c, m_1 + 7 + a + b; 1)}{(1 + c) \Gamma(1 - a)\Gamma(a + b + 13)} - \frac{120\pi \text{Csc}(\pi a)\Gamma(2 + c)\, _2F_1(-6, -1 - a - c, 7 + b; 1)}{(1 + c)\prod_{j=1}^{6} (j + b)\Gamma(-a)\Gamma(2 + a + c)} \tag{47}
\]

where $m_1$ are integers running from 1 to 6, $P_{m_1}(a, b)$ are polynomials in $a$ and $b$ (lengthy and different for each $m_1$).

The overall number of terms turns out to be annoyingly huge, as the already cumbersome manifest expressions for the gluon vertices (22), (23), (45) lead to hudge number of contraction with each contraction itself producing lengthy combination of terms with different values of $m, n, p$ and $m_1$. Despite such a complicated full expression for the amplitude obtained by the direct integration, its structure described above is already sufficient to understand qualitatively the absence of stringy pattern involving infinite number massive poles. For each given $b$ and $c$ (recall that the integration is performed for $Re\{b, c\} \geq -1$) it is the presence of two $\gamma$-functions in the denominator that screens off the massive poles corresponding to large negative integer values of $a$, or the tachyonic poles related to integer positive $a$ values, if $|a|$ becomes significantly larger than $m_1$.

Fortunately the cumbersome expression (47) emerging as a result of $z$ and $u$ integrations simplifies radically in the on-shell limit and the bootstrap calculation, demonstrated in the next section indicates that in the end all this multitude of terms must conspire to converge to quite an elegant answer with only the massless pole remaining. Nevertheless the expression (47) is still of some interest as it illustrates (qualitatively at least) how the non-standard ghost coupling of the gluon vertices (elements of nonzero ghost cohomologies) modifies the Venezino amplitude, reducing its structure from stringy to field-theoretic and truncating the infinite tower of the massive modes. In the next section we will demonstrate the direct computation of the 4-point gluon amplitude, based on the zigzag invariance of the OPE (30).

6. Computation of the 4-point Amplitude

In this section, we compute the 4-point gluon scattering amplitude:

\[
A^{i_1\ldots i_4}(p_{\bar{1}}, p_{\bar{2}}, p_{\bar{3}}, p_{\bar{4}}) = < V^i_1(p_{\bar{1}}) V^{i_2}(p_{\bar{2}}) V^{i_3}(p_{\bar{3}}) V^{i_4}(p_{\bar{4}}) > \tag{48}
\]

using the closeness of the OPE (30). We have
where \( W^m \) are the dimension 1 integrands of photon vertex operators and the \( z_a \) subscripts in the contour integrals refer to the points around which the contour integrals are taken (two photon operators have to be at the integrated \( b-c \) picture and the remaining two are to be taken unintegrated, as we explained in the previous section). As previously, since \( \oint T^i \) are the physical operators of zero momentum, the only BRST non-trivial terms in the full operator product of any two \( \oint T^i(z) \) and \( \oint T^j(w) \) are those with conformal dimension 1, i.e. of the order \((z-w)^{-1}\), given by \( D_{ij}^{kr} \oint T^k \). Furthermore, in the bootstrap of any two photon operators (25) it is sufficient to retain the massless terms since all the massive operators become BRST-trivial after the appropriate \( \alpha \)-transform by \( \oint T^k \), as a consequence of the zigzag symmetry of the gluon OPE (30). To elucidate the pole structure of the amplitude (48), it is convenient to take the photon operators slightly off-shell first, imposing the on-shell condition \((\vec{p}_i \vec{p}_j) = 0; i = 1, \ldots, 4 \) upon the calculation.

Then using (25) and (28), the 4-point correlator (49) is written as

\[
A^{i_1 \cdots i_4}(p_1^i, \ldots, p_4^i) = \prod_{j=1}^4 A^{m_j}(p_j^i) \\
< \oint_{z_1} T^{i_1} \cdots \oint_{z_4} T^{i_4} \int_0^1 dz_1 W^{m_1}(z_1; p_1^i) \int_0^1 dz_2 W^{m_2}(z_2; p_2^i) c W^{m_3}(z_3; p_3^i) c W^{m_4}(z_4; p_4^i) > \\
= \frac{4}{\sqrt{z_1}} \prod_{j=1}^4 A^{m_j}(p_j^i) \int_0^1 dz_1 \int_0^1 dz_2 \{ (z_1 - z_2)^{-1} C^{m_1 m_2}_{m_3 m_4} (p_1^i, p_2^i) \}
\]

where the structure constants \( C^{m_1 m_2}_{m_3 m_4} (p_1^i, p_2^i) \) are given by the photon 3-vertex (25)

and the \( \alpha \)-generator insertion (related to the colour group theoretic factor) is transformed to
\[ R^{i_1 \ldots i_4}(z_1, \ldots, z_4) = (D_j^{i_1 i_2} \oint_{z_2} T^j \oint_{z_3} T^{i_3} \oint_{z_4} T^{i_4} + D_j^{i_1 i_3} \oint_{z_2} T^j \oint_{z_3} T^{i_2} \oint_{z_4} T^{i_4} + D_j^{i_1 i_4} \oint_{z_2} T^j \oint_{z_3} T^{i_2} \oint_{z_4} T^{i_3}) \]

\[ + (D_j^{i_2 i_3} \oint_{z_1} T^j \oint_{z_3} T^{i_1} \oint_{z_4} T^{i_4} + D_j^{i_2 i_4} \oint_{z_1} T^j \oint_{z_3} T^{i_1} \oint_{z_4} T^{i_3} + D_j^{i_2 i_3} \oint_{z_1} T^j \oint_{z_2} T^{i_1} \oint_{z_4} T^{i_4} + D_j^{i_2 i_4} \oint_{z_1} T^j \oint_{z_2} T^{i_1} \oint_{z_4} T^{i_3}) \]

, so we have expressed the 4-point amplitude in terms of 3-point gluon vertices given in (30). Keeping in mind that the momenta are still slightly off-shell, the three-point worldsheet correlators of gluon integrands (30) are given by

\[ <W^{i_1}_{n_1}(z_2; \tilde{k})W^{i_2}_{n_2}(z_3; \tilde{p})W^{i_3}_{n_3}(z_4; \tilde{q})> = (z_2 - z_3)^{(\tilde{k}\tilde{p})-1} (z_2 - z_4)^{(\tilde{k}\tilde{q})-1} (z_3 - z_4)^{(\tilde{p}\tilde{q})-1} D^{ijk}C^{n_1 n_2 n_3}_{n_1 n_2 n_3}(\tilde{k}, \tilde{p}, \tilde{q}) \delta(\tilde{k} + \tilde{p} + \tilde{q}) \]

Next, using the \( SL(2, C) \) symmetry we can fix \( z_3 = 1, z_4 = \infty, u = 0 \) where \( u \) is again the arbitrary point in the \( C \)-dependent correction term, emerging in the BRST-invariant expression of one of two integrated positively represented gluon operators in the (as we explained in the previous section). Using (50) and (52) along with the momentum conservation \( \sum_{a=1}^{4} \vec{p}_a = 0 \), the 4-point function is given by

\[ A^{i_1 \ldots i_4}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = \prod_{j=1}^{4} A^{m_j}(\vec{p}_j) \int_{0}^{1} dz_1 \int_{0}^{1} dz_2 \{ \]

\[ (z_1 - z_2)^{(\vec{p}_1 \vec{p}_2)-1} (z_2 - 1)^{-1-(\vec{p}_3 \vec{p}_4)} c^{n}_{m_1 m_2}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \]

\[ \times (D_j^{i_1 i_2} D_j^{i_3 i_4} + D_j^{i_1 i_3} D_j^{i_2 i_4} + D_j^{i_1 i_4} D_j^{i_2 i_3}) + (z_1 - 1)^{(\vec{p}_1 \vec{p}_3)-1} (z_2 - 1)^{-1-(\vec{p}_2 \vec{p}_4)} c^{n}_{m_1 m_4}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \]

\[ \times (D_j^{i_1 i_2} D_j^{i_3 i_4} + D_j^{i_1 i_3} D_j^{i_2 i_4} + D_j^{i_1 i_4} D_j^{i_2 i_3} + (z_1 - 1)^{(\vec{p}_1 \vec{p}_4)-1} (z_2 - 1)^{-1-(\vec{p}_2 \vec{p}_3)} c^{n}_{m_1 m_3}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \]

\[ \times (D_j^{i_1 i_2} D_j^{i_3 i_4} + D_j^{i_1 i_3} D_j^{i_2 i_4} + D_j^{i_1 i_4} D_j^{i_2 i_3}) \}

(53)

Imposing the on-shell limit \( (\vec{p}_i \vec{p}_j) \rightarrow 0; i, j = 1, \ldots, 4 \) it is straightforward to evaluate the integrals in \( z_1 \) and \( z_2 \), so the final answer for the on-shell 4-point tree amplitude is

\[ A^{i_1 \ldots i_4}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = \prod_{j=1}^{4} A^{m_j}(\vec{p}_j) \{ D_j^{i_1 i_2} D_j^{i_3 i_4} + D_j^{i_1 i_3} D_j^{i_2 i_4} + D_j^{i_1 i_4} D_j^{i_2 i_3} \}

\[ \times \left\{ \frac{c^{n}_{m_1 m_2}(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)}{(\vec{p}_1 \vec{p}_2)(\vec{p}_3 \vec{p}_4)} + \frac{c^{n}_{m_1 m_3}(\vec{p}_1, \vec{p}_3, \vec{p}_2, \vec{p}_4)}{(\vec{p}_1 \vec{p}_3)(\vec{p}_2 \vec{p}_4)} \right\} \delta(\sum_{a=1}^{4} \vec{p}_a) \}

(54)
This concludes the calculation of the 4-point amplitude of the gluon-vertex operators. This amplitude is manifestly cross-symmetric and the factors in numerators (quadratic in the structure constants and hence quadratic in the momenta), along with the transversality constraints on the polarization vectors, protect it from the double poles. The group-theoretic factor is easily recognized as \( \sim Tr(t^{i_1}...t^{i_4}) \) (with \( t^{i_k} \) being the SU(3) generators), as one would expect for QCD amplitudes. The bootstrap calculation performed in this section, based on the zigzag invariance of the OPE of the gluons, can be generalized to include higher number of points and, in principle, the loop corrections as well.

**Conclusions**

In this paper we have considered a new example of gauge-string correspondence, constructing eight SU(3) gluon vertex operators in RNS superstring theory and showing them to reproduce perturbative QCD amplitudes. Remarkably, the constructed 8 vertex operators possess full zigzag symmetry and their OPE is closed. This leads to the absence of intermediate massive states in their scattering amplitudes which therefore have field-theoretic (rather than a stringy) structure, reproducing the QCD dynamics.

It should be stressed here that the gauge-string isomorphism discussed in this paper is not an analogue of AdS/CFT duality, since the latter is the correspondence between strongly coupled region of QCD and perturbative region of string theory in AdS background. The example of gauge-string correspondence constructed in this paper is the one between perturbative QCD amplitudes and perturbative amplitudes in open string theory, similar to the correspondence between D-instanton expansion in twistor string theory and perturbative expansion in \( N = 4 \) super Yang-Mills theory, observed by Witten \[13\] and elaborated in \[14\], \[15\] and other works. It would be interesting to understand the connection between these two examples of gauge-string isomorphism, as well as the relation between twistor superstrings and RNS model. Interestingly, twistor superstring theory with \( SU(2,2|4) \) global symmetry emerges naturally in the context of 2T physics \[16\], \[17\] and extra dimensions, where Berkovits-Witten theory results as one of the holographic pictures of 4 + 2-dimensional string theory, obtained as a result of 2T gauge fixing. We already have mentioned that the first order \( \alpha \)-generators of \( H_1 \sim H_{-3} \) (related to the first hidden space-time dimension) are in one-to-one correspondence to the off-shell space-time symmetries observed in the 2T approach \[5\], \[6\]. This appears to be an interesting project where many intriguing connections can be anticipated.

The SU(3) colour group stems naturally from SU(3) subgroup of the \( \alpha \)-generators, inducing global non-linear space-time isometries in hidden extra dimensions. The gluon
vertex operators are obtained as a result of “photon painting”, i.e. by applying SU(3) α-generators to photon vertex operators. The zigzag invariance of gluon OPE is ensured by the special property of the α-symmetry, proven in this paper: the α-transform of any massive vertex operator is trivial, while the α-generators applied to photon produce new physical states, corresponding to coloured QCD gluons.

This particularly suggests that each α-generator of SU(3) carries an associate colour-anticolour quantum number. Since the α-generators that paint a photon can be classified in terms of ghost cohomologies $H_n \sim H_{-n-2}$ while each cohomology is associated with hidden space-time dimension, this naturally implies that each hidden dimension has its own associate colour-anticolour contributing to the photon painting. Attributing a colour-anticolour to a hidden dimension would nicely match the classification of gluons in terms of ghost cohomologies. Indeed, suppose the extra dimension associated with $n = 1$ carries red and antired colours, the one related to $n = 2$ paints with green and antigreen, the one of $n = 3$ is blue-antiblue. Let us start with $H_1 \sim H_{-3}(n = 1)$. The only SU(3) α-generator from this cohomology is $L^{α+}$, so we identify the corresponding vertex operator $V^{α+}$ with the $r\bar{r}$ (red-antired) gluon. The $n = 2$ cohomology, associated with the second hidden dimension has 3 generators, $L^{αβ} L^{β+}$ and $L^{β−}$. The β-index adds the green (and anti-green) colour to the painting, so the corresponding gluons are associated with the colour pairs $g\bar{g}, g\bar{r}$ and $gr$. Finally, the $n = 3$ cohomology contains 4 vertex operators $V^{γβ}, V^{γα}, V^{γ+}$ and $V^{γ−}$, giving rise to 4 gluons with the colour combinations involving the blue (and antiblue) colour: $b\bar{g}, b\bar{g}, br$ and $br$ (the $b\bar{b}$-pairing must be skipped since it is the linear combination of the previously listed gluons $r\bar{r}, g\bar{g}$ and the non-existing “ninth” gluon of white colour $r\bar{r} + g\bar{g} + bb$.

The construction discussed in this paper has so far involved the open string operators only. The role of the ghost cohomologies, α-symmetries and their analogues in the closed string sector is yet to be understood. Such an understanding would be particularly important since beyond the tree approximation the Yang-Mills theory, with all the loops included, is described by the closed string amplitudes. Clarifying the geometrical meaning of closed string ghost cohomologies could particularly connect our formalism and AdS/CFT approach. Interestingly, the AdS/CFT correspondence generally does not fix a gauge group which depends on various parameters on string theory side (such as the number $N$ of RR-flux units in case of duality between $SU(N) \times SU(N)$ gauge theory and type IIB strings in warped resolved conifold backgrounds [6]). Our model, however, suggests that the $SU(3)$ case is special, as it is related to the structure of extra dimensional
α-symmetries classified the first three cohomologies. It is not clear at present if the there are α-symmetries in ghost cohomologies of orders higher than 3 and (if yes) if they have any geometrical meaning like those of SU(3). If one proves that the set of α-symmetries contained in the first three cohomologies is complete, this may indicate that we have certain new specific form of gauge-string duality in the case of $N = 3$. We hope to address these questions (along with many others) in future works.
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