Pairwise disjoint Moebius bands in space

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Abstract

V.V. Grushin and V.P. Palamodov proved in 1962 that it is impossible to place in $\mathbb{R}^3$ uncountably many pairwise disjoint polyhedra each homeomorphic to the Moebius band. We generalize this result in two directions. First, we give a generalization of this result to tame subsets in $\mathbb{R}^N$, $N \geq 3$. Second, we show that in case of $\mathbb{R}^3$ the theorem holds for arbitrarily topologically embedded (not necessarily tame) Moebius bands.

Keywords: Euclidean space, Moebius band, topological embedding, wild embedding.
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1 Introduction

Theory of wild embeddings comes back to papers of L. Antoine, P.S. Urysohn, J.W. Alexander from 1920’s. Let us mention the reviews [16], [17], [20], [22] and the books [23], [30], [36] which contain hundreds of references on the subject.
**Definition 1.1.** A subset $X \subseteq R^N$ is called a *polyhedron* if it is a union of a finite collection of simplices. A subset $X \subseteq R^N$ homeomorphic to a polyhedron is called *tame* if there is a homeomorphism $h$ of $R^N$ onto itself such that $h(X) \subseteq R^N$ is a polyhedron; otherwise, $X$ is called *wild*. An embedding $F : X \to R^N$ is called wild if its image $F(X)$ is a wild subset of $R^N$, and tame otherwise.

For cells and spheres, it is useful to compare this with the following definition:

**Definition 1.2.** A subset $X \subseteq R^N$ homeomorphic to $I^k$ or $S^k$ is called *flat* if there exists a homeomorphism $h$ of $R^N$ onto itself such that $h(X)$ is a $k$-simplex or the boundary of a $(k+1)$-simplex.

From classical results of A. Schoenflies and L. Antoine we know that each arc and each simple closed curve in $R^2$ is flat [30, II.4]. Antoine constructed first wild arcs in $R^3$ in [3, 54–58, p.65–70; 83, p. 97]. His first wild arc $ab$ [3, 54–58, p.65–70] consists of an arc $ap$ with a sequence of trefoils tied in it and convergent to the end point $p$, united with a straight line segment $pb$ (see the picture in [36, Exercise 2.7.3] or [23, Fig. 2.26]; this example is often attributed to R.L. Wilder because of the footnote on page 634 of [39]; let us underline that the footnote contains reference to Antoine’s Thesis [3]). Second wild arc [3, 83, p. 97] contains a wild Cantor set (now called Antoine necklace). Moreover, Antoine constructs an everywhere wild arc in $R^3$ [5]. Well-known Alexander horned sphere [1] and Antoine-Alexander sphere [4, p. 285], [2], [3, 21, p.283–284] provide examples of wild surfaces in $R^3$. For $N \geq 3$, each polyhedron $P \subseteq R^N$ with dim $P \geq 1$ can be wildly embedded in $R^N$ (see [29] and [11]; compare [3, 83, p. 97], [4, 5]). In $R^N$, $N \geq 3$, there exist everywhere wild cells and spheres of all dimensions $1, \ldots, N-1$ (see [8], [23, p.86–88, p.73–74], [36, Thm. 2.6.1]). It is known that for each $N$ and each $k \neq N-2$ any tame $k$-cell or $k$-sphere in $R^N$ is flat, refer e.g. to [23, Cor. 1.1.2, Thm. 1.1.3, Cor. 1.1.6]. For any $N$, each tame arc in $R^N$ is flat [30 p.125–126]. For $N \geq 4$ there exist tame but not locally flat $(N-2)$-cells and $(N-2)$-spheres in $R^N$ [23, Example 1.4.2].

“Being wild” should not be considered as a strange exception: in the sense of Baire category, most knots in $R^3$ are wild [33 Thm. 1] (see also [23, Prop. 6.3.8]); most arcs in $R^3$ are wild [13]; in contrast to this, most 2-spheres in $R^3$ are tame [13].
Classification of wild embeddings is a hard problem. By \cite[Thm.5.4]{26}, the classification problem of embeddings of the Cantor set $\mathcal{C}$ in $\mathbb{R}^3$ is at least as complicated as the classification of countable linear orders. Using methods of descriptive theory it is shown in \cite{31} that the classification problem for wild knots is strictly harder than that for countable structures.

R.H. Bing proved that it is impossible to place in $\mathbb{R}^3$ uncountably many pairwise disjoint wild closed surfaces. Scheme of proof is given in \cite{9}, and the full proof needs results of \cite{7} and \cite{10}; also, proof of Bing is explained in \cite{16}, Thm. 3.6.1. (The assumption of wildness is essential: concentric spheres of all positive radii is an uncountable family of pairwise disjoint tame surfaces.) J.R. Stallings constructed \cite{38} a family of pairwise disjoint wild 2-disks in $\mathbb{R}^3$ which has a cardinality of continuum. J. Martin showed that all except countably many disks in such a collection must be locally tame at their interior points \cite{32}, (this result was predicted in \cite{10} where important facts needed for Martin’s arguments are proved). R.B. Sher modified Stallings’ construction so that no two disks of the family are ambiently homeomorphic, that is, no self-homeomorphism of $\mathbb{R}^3$ can map one disk onto another \cite{37}.

For higher dimensions, Bing non-embedding result has at the moment only partial generalizations, see \cite[Thm. 1, 2]{15}, \cite[Thm. 10.5]{17}, \cite[p.383, Thm. 3C.2]{22}. The results of Stallings and Sher were generalized by the author to the case of arbitrary $\mathbb{R}^N$, $N \geq 3$, and arbitrary perfect compact space embeddable in a hyperplane $\mathbb{R}^{N-1} \subset \mathbb{R}^N$ \cite{25}, \cite{24}. During my Conference talk \cite{24} I was asked by Professor Scott Carter if it is possible to place uncountably many pairwise disjoint Moebius bands in $\mathbb{R}^3$. He informed that in prior years he knew the impossibility proof from Bob Williams (University of Texas). I became very interested in this question. I found a paper of V.V. Grushin and V.P. Palamodov containing a partial answer \cite[Thm. 3]{27}:

Let $P$ be a 2-dimensional connected polyhedron which can be represented as a (finite) union of 2-simplices. In $\mathbb{R}^3$ one can place uncountably many pairwise disjoint polyhedra homeomorphic to $P$ if and only if $P$ is orientable and each point $p \in P$ has a planar neighborhood. Let us underline that for the “$\Rightarrow$” direction, each copy of $P$ is assumed to be a polyhedron in $\mathbb{R}^3$. The proof given in \cite{27} essentially exploits this strong assumption. In his Zentralblatt review to Grushin-Palamodov paper, H.G. Bothe states their result \cite[Thm. 3]{27} in a more general form, without assumption that $P$ is a union of 2-simplices \cite{12}.

Below, I generalize result of V.V. Grushin and V.P. Palamodov in two directions: 1) I replace $\mathbb{R}^3$ by arbitrary $\mathbb{R}^N$ but assume that the subsets are...
tame (Corollary 2.2); and 2) I stay in $\mathbb{R}^3$ but allow the subsets to be wild (Proposition 3.2, Corollary 3.6). Corollary 3.6 improves Bothe’s assertion [12].

**Remark 1.3.** In [28], results of Grushin and Palamodov are represented in a much more stronger form than originally given in [27]. Unfortunately, this leads to several mistakes.

1) “Simple continua” in terminology of V.K. Ionin and Yu.G. Nikonorov need not be tame; for example, wild arc of Antoine-Wilder [3, 54–58, p.65–70], [36, Exercise 2.7.3] is a “simple continuum”.

2) Theorem 1 in [28] is false; wild Antoine-Wilder arc can serve as a counterexample. (Its “spaciality” in terms of [28] can be derived from [3, 56–57, p.66–69]; and it is not hard to show that it is “thin” in terminology of [28]. I am planning to explain this in detail in another paper.)

3) Let us analyze Theorem 2 of [28]. Suppose that $K$ is a wild $(N-1)$-sphere in $\mathbb{R}^N$, where $N \neq 4$. In terminology of [28], $K$ is “thick” by [9, 12, Thm. 10.5], [22, p.383, Thm. 3C.2]. Assuming that [28, Thm. 2] is true, we conclude that $K$ can not be represented as a union of a finite family of tame cells. This hard statement is of course not contained in [27]. (An $(N-1)$-sphere in $\mathbb{R}^N$ which is the union of interiors of tame $(N-1)$-cells is itself locally flat and hence flat by [14]. If the interiors of the cells do not cover the sphere, the statement is proved under strong additional conditions in [18, Thm. 6.4] using A.V. Chernavskii’s result [19, Thm. 2], [21].)

1.1 **Conventions and Notation**

By a map we always mean a continuous map.

For any metric compact space $M$, the space $C(M, \mathbb{R}^N)$ of all maps $M \to \mathbb{R}^N$ endowed with the uniform distance is known to be a complete separable metric space.

By $\text{conv}(S)$ we denote the convex hull of a set $S \subset \mathbb{R}^N$.

An $N$-manifold is a separable metric space each point of which has a neighborhood homeomorphic to $\mathbb{R}^N$. An $N$-manifold-with-boundary is a separable metric space each point of which has a neighborhood whose closure is homeomorphic to an $N$-simplex. (So, any $N$-manifold is an $N$-manifold-with-boundary.) For an $N$-manifold-with-boundary $M$, its boundary is denoted by $\partial M$ and its interior $M - \partial M$ by $\text{Int} M$.

By $\overline{Y}$ we denote the closure of a set $Y$ in a given ambient space.
2 Families of tame \((N-1)\)-surfaces in \(R^N\)

In their proof of [27, Thm. 2], Grushin and Palamodov show that if two polyhedral Moebius bands in \(R^3\) consisting of the same number of triangles could be “sufficiently close” to each other, than they would be orientable. Our next Proposition is based on the same idea. We make a more careful investigation; and only the “limit” Moebius band is assumed to be tame.

In contrast to Grushin and Palamodov who measure “nearness” of two polyhedral Moebius bands by considering broken lines with vertices in centers of the faces, we use properties of space of maps \(C(M, R^N)\) (inspired by the Bing idea, see [16, Thm. 3.6.1], [9]); we thus obtain Corollary 2.2.

**Proposition 2.1.** Let \(M\) be a compact triangulable \((N-1)\)-manifold-with-boundary, and let \(f_1, f_2, \ldots, F\) be its embeddings into \(R^N\) such that: \(F(M)\) is tame; the sequence \(\{f_n\}\) converges to \(F\) in \(C(M, R^N)\); and \(f_n(M) \cap F(M) = \emptyset\) for each \(n\). Then \(M\) is orientable.

**Proof.** Since \(F(M)\) is a tame subset of \(R^N\), there exists a homeomorphism \(h : R^N \cong R^N\) such that \(h(F(M))\) is a polyhedron. Using uniform continuity of \(h\) on a large compact ball \(B \supset \text{Int} B \supset F(M)\) we see that the sequence \(\{h \circ f_n\}\) converges to \(h \circ F\) in \(C(M, R^N)\). Therefore we will omit the homeomorphism \(h\); we will assume that \(F(M)\) is already a polyhedron in \(R^N\), and the sequence \(\{f_n\}\) converges to \(F\).

Let us assume that \(M\) is non-orientable. Fix any triangulation \(\tau\) of the polyhedron \(F(M)\). There exists a sequence of \((N-1)\)-simplices \(\Delta_1, \ldots, \Delta_m\) of \(\tau\) such that each pair of adjacent simplices (and also the last and the first one) have a common \((N-2)\)-face; and this sequence does not possess coherent orientations of its simplices. We may also assume that \(\Delta_i \neq \Delta_j\) for \(i \neq j\).

For each \(i = 1, \ldots, m\) let \(A_i\) be the center of the simplex \(\Delta_i\), and \(A_{i,i+1}\) be the center of the \((N-2)\)-simplex \(\Delta_i \cap \Delta_{i+1}\). (By \(\Delta_{m+1}\) we mean \(\Delta_1\); in particular, \(A_{m+1} = A_1\), and \(A_{m,m+1}\) is the center of the \((N-2)\)-simplex \(\Delta_m \cap \Delta_1\).) Let \(A_i A_{i,i+1} A_{i+1}\) be the broken line defined as the union of two segments: \(A_i A_{i,i+1}\) and \(A_{i,i+1} A_{i+1}\). Let us denote \(P_i = F^{-1}(A_i)\) and \(q_i = F^{-1}(A_i A_{i,i+1} A_{i+1})\) for each \(i = 1, \ldots, m\).

Take a positive number \(\delta\) such that

\[
\delta < \min_{i=1,\ldots,m} \{d(A_i, \partial \Delta_i); \ d(A_i A_{i,i+1} A_{i+1}, \partial (\Delta_i \cup \Delta_{i+1}))\}.
\]
Take an integer $K$ such that $d(f_K, F) < \delta$.

Fix an orientation of $R^N$. For each $i = 1, \ldots, m$ we have

$$d(f_K(P_i), A_i) = d(f_K(P_i), F(P_i)) < \delta < d(A_i, \partial \Delta_i);$$

moreover, $f_K(P_i) \notin \Delta_i \subset F(M)$. Therefore the point $f_K(P_i)$ defines the direction of a normal vector for the hyperplane containing the simplex $\Delta_i$. Moreover, for each $i = 1, \ldots, m$ we have $d(f_K(q_i), A_i A_{i,i+1} A_{i+1}) = d(f_K(q_i), F(q_i)) < \delta$ and $f_K(q_i) \cap (\Delta_i \cup \Delta_{i+1}) \subset f_K(M) \cap F(M) = \emptyset$; hence the orientations of $\Delta_i$ and $\Delta_{i+1}$ thus defined are coherent. This contradiction finishes the proof. 

**Corollary 2.2.** It is impossible to place in $R^N$, $N \geq 3$ uncountably many pairwise disjoint tame homeomorphic images of a compact triangulable non-orientable $(N - 1)$-manifold-with-boundary.

**Proof.** Suppose the contrary. Let $\mathcal{H} = \{h_\alpha : M \rightarrow R^N, \alpha \in A\}$ be an uncountable family of embeddings such that $h_\alpha(M) \cap h_\beta(M) = \emptyset$ for each $\alpha \neq \beta$, where $M$ is a compact triangulable non-orientable $(N - 1)$-manifold-with-boundary.

The space $C(M, R^N)$ is separable. Hence its uncountable subset $\mathcal{H}$ has a converging subsequence of distinct elements $h_{\alpha_m}$ (the limit map $h = \lim_{m \rightarrow \infty} h_{\alpha_m}$ also belongs to $\mathcal{H}$). To finish the proof, apply Proposition 2.1.

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3 **Families of topological Moebius bands in $R^3$**

**Definition 3.1.** A set $X \subset R^N$ is called **locally tame** at $x \in X$ if there is a neighborhood $U$ of $x$ in $R^N$ and a homeomorphism $h$ of $\overline{U}$ into $R^N$ such that $h(\overline{U} \cap X)$ is a polyhedron.

By [6] or [34], closed locally tame subsets of $R^3$ are tame.

**Proposition 3.2.** Let $M$ be a compact non-orientable 2-manifold-with-boundary. It is impossible to place uncountably many pairwise disjoint homeomorphic images of $M$ in $R^3$. 

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Proof. It suffices to consider the case when \( M \) is a (compact) Moebius band
\[ \mu = I \times I/(0,t) \sim (1,1-t). \]
Let us assume that \( \{M_\alpha, \alpha \in A\} \) is an uncountable family of pairwise disjoint subsets of \( \mathbb{R}^3 \), and \( h_\alpha : \mu \cong M_\alpha \) is a homeomorphism for each \( \alpha \in A \).

Let \( \pi : I \times I \to \mu \) be the factorization map. Represent \( \mu \) as a union of two disks:
\[ \mu = D_1 \cup D_2, \]
where
\[ D_1 = \pi \left( \left[ 0; \frac{3}{4} \right] \times I \right); \quad D_2 = \pi \left( \left( \left[ 0; \frac{1}{4} \right] \cup \left[ \frac{1}{2}; 1 \right] \right) \times I \right). \]

Apply [32, Lemma 2] to the uncountable family of 2-disks \( \{h_\alpha(D_1), \alpha \in A\} \). There exists an uncountable subset \( A_1 \subset A \) such that each disk \( h_\alpha(D_1), \alpha \in A_1 \), is locally tame at each \( x \in \text{Int} \, h_\alpha(D_1) \). Again, apply [32, Lemma 2] to the uncountable family of 2-disks \( \{h_\alpha(D_2), \alpha \in A_1\} \); there exists an uncountable subset \( A_2 \subset A_1 \) such that each \( h_\alpha(D_2), \alpha \in A_2 \), is locally tame at each of its interior points.

We obtain an uncountable set \( A_2 \) such that each disk \( h_\alpha(D_i), \alpha \in A_2, \)
\( i = 1, 2 \), is locally tame at each of its interior points. Hence each Moebius band \( h_\alpha(\mu), \alpha \in A_2 \), is locally tame at each of its interior points. Replace \( \mu \) by a smaller Moebius band \( \tilde{\mu} = \pi (I \times \left[ \frac{1}{4}; \frac{3}{4} \right]) \subset \mu \); we obtain an uncountable family of pairwise disjoint locally tame Moebius bands \( \{h_\alpha(\tilde{\mu}), \alpha \in A_2\} \). By [6] or [34], each band \( h_\alpha(\tilde{\mu}), \alpha \in A_2 \) is tame. Application of Corollary 2.2 finishes the proof.

Recall that a topological space \( X \) is called \emph{locally planar} if each point \( x \in X \) has a neighborhood \( U \) which can be embedded in plane.

**Definition 3.3.** [27, p.165] Let \( P \) be a locally planar 2-dimensional polyhedron (in some \( \mathbb{R}^N \)). Take any of its triangulations. \( P \) is called \emph{orientable} if one can give coherent orientations to all its 2-simplices.

Note that orientability implies local planarity; but we prefer to write “locally planar and orientable” in order to avoid occasional misunderstanding.

**Remark 3.4.** For a locally planar 2-dimensional polyhedron \( P \), the following are equivalent to the above definition.

1) Let \( \hat{P} \) be the union of all 2-dimensional simplices of \( P \). Note that \( \hat{P} \) is a homogeneous non-branching 2-dimensional complex in the sense of [35, p.314, p.462]. By [35, 43.24] \( \hat{P} = \cup P_i \) where each \( P_i \) is a 2-dimensional
pseudomanifold and \( \dim(P_i \cap P_j) \leq 0 \) for each \( i \neq j \). We say that the polyhedron \( P \) is orientable if each pseudomanifold \( P_i \) is orientable.

2) \( P \) is called orientable if it does not contain a homeomorphic copy of the Moebius band (this is taken as a definition in [12]).

Corollary 3.6 extends [27, Thm. 3] and its generalization stated in [12]. First, let us prove a lemma (compare [27, p. 167, Lemma] where the case of polyhedra all whose maximal simplices are 2-dimensional is considered).

**Lemma 3.5.** Let \( P \) be a locally planar 2-dimensional polyhedron (in some \( \mathbb{R}^N \)). Moreover, suppose that \( P \) is orientable. Then there exists an orientable 2-manifold-with-boundary \( M \) such that \( P \subset M \subset \mathbb{R}^N \) and \( M \) is a polyhedron in \( \mathbb{R}^N \).

**Proof.** By [27, p.167–168, Lemma] it suffices to find a locally planar orientable 2-dimensional polyhedron \( \tilde{P} \) with \( P \subset \tilde{P} \subset \mathbb{R}^N \) such that each maximal simplex of \( \tilde{P} \) is 2-dimensional.

Fix any triangulation \( \tau \) of \( P \). For each isolated point \( A \) of \( P \) (0-dimensional maximal simplex of \( \tau \)), replace \( A \) with a small triangle \( \Delta \) such that \( A \) is among its vertices and \( \Delta \cap P = \{A\} \). Similarly, if \( \sigma \) is a maximal 1-dimensional simplex of \( \tau \), replace it with a narrow triangle \( \Delta = \text{conv}(\sigma, B) \), where \( B \notin \sigma \) is a point sufficiently close to the midpoint of \( \sigma \) so that \( \Delta \cap P = \sigma \cap P \). It can be easily seen that the polyhedron \( \tilde{P} \) obtained after all replacements satisfies the conditions listed above.

**Corollary 3.6.** Let \( P \) be a 2-dimensional polyhedron (in some \( \mathbb{R}^N \)). The following are equivalent:

(a) in \( \mathbb{R}^3 \) one can place uncountably many pairwise disjoint polyhedra homeomorphic to \( P \);
(b) in \( \mathbb{R}^3 \) one can place uncountably many pairwise disjoint subsets homeomorphic to \( P \);
(c) \( P \) is locally planar and orientable.

**Proof.** (a) \( \Rightarrow \) (b) is evident.

Let us prove (b) \( \Rightarrow \) (c).

1) Suppose that a point \( p \in P \) has no planar neighborhood. There exist a 2-simplex \( \Delta \subset P \) and an arc \( \alpha \subset P \) such that \( p \in \text{Int} \Delta \cap \partial \alpha \). By [40], it is impossible to place uncountably many disjoint homeomorphic copies of the “umbrella” \( \Delta \cup \alpha \) (hence also of \( P \)) in \( \mathbb{R}^3 \). (V.V. Grushin and V.P. Palamodov seem to be unfamiliar with [40] while writing [27], and they included the
proof of impossibility to place in $R^3$ uncountably many disjoint polyhedra homeomorphic to the umbrella.)

2) Assume that $P$ is non-orientable. Then it contains a homeomorphic image of the Moebius band $\mu$. By Proposition 3.2, it is impossible to place uncountably many disjoint homeomorphic copies of $\mu$ (hence also of $P$) in $R^3$.

Finally, $(c) \Rightarrow (a)$ can be proved similarly to [27], using Lemma 3.5. Namely, there exists an orientable polyhedral 2-manifold-with-boundary $M$ with $P \subset M \subset R^N$. It is known that $M$ is homeomorphic to a sphere with handles and holes; clearly we can place in $R^3$ uncountably many pairwise disjoint polyhedra each PL-homeomorphic to $M$. The inclusion $P \subset M$ can be interpreted as a PL embedding; thus we obtain uncountably many disjoint homeomorphic polyhedral copies of $P$ in $R^3$.

\[ \square \]

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