BORSUK–ULAM THEOREMS FOR ELEMENTARY ABELIAN 2-GROUPS

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Abstract. Let $G$ be a compact Lie group and let $U$ and $V$ be finite-dimensional real $G$-modules with $V^G = 0$. A theorem of Marzantowicz, de Mattos and dos Santos estimates the covering dimension of the zero-set of a $G$-map from the unit sphere in $U$ to $V$ when $G$ is an elementary abelian $p$-group for some prime $p$ or a torus. In this note, the classical Borsuk–Ulam theorem will be used to give a refinement of their result estimating the dimension of that part of the zero-set on which an elementary abelian $p$-group $G$ acts freely or a torus $G$ acts with finite isotropy groups. The methods also provide an easy answer to a question raised in [16].

Let $G$ be a compact Lie group and let $U$ and $V$ be finite-dimensional real $G$-modules (which we may assume to be equipped with a $G$-invariant Euclidean inner product) with the fixed subspace $V^G$ equal to zero. A theorem [15, Theorem 2.1] of Marzantowicz, de Mattos and dos Santos estimates the covering dimension of the zero-set of a $G$-map from $S(U)$, the unit sphere in $U$, to $V$ when $G$ is an elementary abelian $p$-group for some prime $p$ or a torus. In this note, the classical Borsuk–Ulam theorem will be used to give a refinement of their result which estimates the dimension of that part of the zero-set on which an elementary abelian $p$-group $G$ acts freely or a torus $G$ acts with only finite isotropy groups.

Section 1 reviews the relevant Borsuk–Ulam theorems. Elementary abelian $p$-groups are considered in Section 2; we deal only with the case of the prime $p = 2$, but the same method works for odd primes. The results for the case when $G$ has order 2 provide an easy answer to a question posed in [16, page 79]. Section 3 contains the analogous theory for actions of a torus and extends a Borsuk-Ulam for $S^1$ actions due to Fadell, Husseini and Rabinowitz [12, 11] to more general torus actions.

1. The Borsuk–Ulam theorem

We recall a version of the Borsuk–Ulam theorem.

Proposition 1.1. Let $G$ be a finite group. Suppose that $W$ is a compact, connected, smooth free $G$-manifold of dimension $n$ and that $V$ is a finite-dimensional real $G$-module of dimension $k$, with $V^G = 0$. Let $\zeta$ be the real vector bundle over the orbit space $W/G$ associated with the representation $V$.

Suppose that the mod 2 cohomology Euler class $e(\zeta)$ is non-zero. Then, for any continuous $G$-map $f : W \to V$, the compact subspace

$$\text{Zero}(f) = \{ x \in W \mid f(x) = 0 \}$$

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has covering dimension greater than or equal to \( n - k \).

**Proof.** We paste two copies of \( W \) together along the boundary \( \partial W \) to form a closed free \( G \)-manifold \( M = W \cup_{\partial W} W \), then form the orbit manifold \( \overline{M} = M/G \) and the associated vector bundle \( \xi = M \times_G V \) over \( \overline{M} \). This bundle is the pullback of the bundle \( \zeta \) over \( W/G \) through the folding map \( \pi : M \to W \).

Now the map \( f \) determines a section \( s \) of \( \xi = M \times_G V \): \( s([x]) = [x, f(\pi x)] \) for \( x \in M \). Since \( \overline{M} \) is a closed, connected \( n \)-manifold and \( e(\xi) \neq 0 \), by Poincaré duality there is a class \( a \in H^{n-k}(\overline{M}; \mathbb{F}_2) \) such that \( a \cdot e(\xi) = 1 \in \mathbb{F}_2 = H^n(\overline{M}; \mathbb{F}_2) \). By the classical Borsuk–Ulam theorem, as formulated for example in [8, Proposition 2.7], the cohomology group \( H^{n-k}(\overline{Z}; \mathbb{F}_2) \) of the zero-set \( \overline{Z} = \text{Zero}(s) \subseteq \overline{M} \) is non-zero. Hence the covering dimension of \( \overline{Z} \) is greater than or equal to \( n - k \). It follows that the inverse image \( Z \) of \( \overline{Z} \) under the projection \( M \to \overline{M} \) has covering dimension at least \( n - k \), because \( Z \to \overline{Z} \) is a finite cover. (See, for example, [14] or [7, Lemma 2.6].)

The set \( Z \) is a union of two copies of the compact space \( \text{Zero}(f) \) intersecting along \( \partial W \). So the covering dimension of \( \text{Zero}(f) \) is equal to the covering dimension of \( Z \) and is greater than or equal to \( n - k \). (Compare [9, Corollary 2.7].) \( \Box \)

There is a more general result for any compact Lie group \( G \) and an action that is not necessarily free. We give the formulation in rational cohomology.

**Proposition 1.2.** Let \( G \) be a compact Lie group. Suppose that \( W \) is a compact, connected, smooth, \( n \)-dimensional \( G \)-manifold and that \( W \) admits an orientation which is invariant under the action of \( G \).

Let \( V \) be a finite-dimensional real \( G \)-module of dimension \( k \) admitting an orientation that is fixed by \( G \), and let \( f : W \to V \) be a continuous \( G \)-map.

Suppose that either (i) the image of the Borel cohomology Euler class \( e(V) \in H^n_B(*) ; \mathbb{Q} \) in \( H^n_B(W; \mathbb{Q}) \) is non-zero or (ii) the restriction of \( f \) to the boundary \( \partial W \) is nowhere zero and the associated relative Euler class in \( H^n_B(W, \partial W; \mathbb{Q}) \) is non-zero.

Then the compact subspace

\[
\text{Zero}(f) = \{ x \in W \mid f(x) = 0 \}
\]

has covering dimension greater than or equal to \( n - k \).

The Euler classes in the statement are determined by a choice of orientation for the vector space \( V \).

**Proof.** Choose a faithful representation \( G \subseteq \text{U}(\mathbb{C}^r) \) and write \( P = \text{U}(\mathbb{C}^r, \mathbb{C}^{r+N}) \) for the complex Stiefel manifold of isometric linear maps \( \mathbb{C}^r \hookrightarrow \mathbb{C}^{r+N} \). With the action of \( G \subseteq \text{U}(\mathbb{C}^r) \), \( P \) is a closed free \( G \)-manifold of dimension \( m + n \), say.

Again form the \( G \)-manifold \( M = W \cup_{\partial W} W \) as a union of two copies of \( W \). The connected manifold \( \overline{M} = (P \times M)/G \) of dimension \( m + n \) is orientable, and the map \( f \) determines a section \( s \) of the vector bundle \( \xi = (P \times M \times V)/G \) over \( \overline{M} \) with zero-set \( \overline{Z} = (P \times Z)/G \), where \( Z \) is the union of two copies of \( \text{Zero}(f) \).

If \( N \) is sufficiently large (so that \( P \) approximates the classifying space \( EG \) of \( G \)), then under the hypothesis (i) \( e(\xi) \in H^k(\overline{M}; \mathbb{Q}) \) is nonzero. We deduce from the Borsuk-Ulam argument that \( H^{m+n-k}((P \times Z)/G; \mathbb{Q}) \) is non-zero. Since the Grassmann manifold \( P/G \) has dimension \( m \), it follows that \( H^{n-k+i}(Z; \mathbb{Q}) \) is non-zero for some \( i \geq 0 \). Thus, \( Z \) and so also \( \text{Zero}(f) \) have covering dimension \( \geq n - k \).
For (ii), let \( W \) be the manifold \((P \times W)/G\) with boundary \( \partial W = (P \times \partial W)/G\), and let \( \xi \), now, be the vector bundle \((P \times W \times V)/G\) over \( W \). The section \( s \) of \( \xi \) given by \( f \) is non-zero on \( \partial W \) and the relative Euler class \( e(s; \partial W) \in H^k(W, \partial W; \mathbb{Q}) \) is non-zero if \( N \) is large. There is a dual class \( a \in H^{m+n-k}(W; \mathbb{Q}) \) such that \( a \cdot e(s; \partial W) = 1 \in \mathbb{Q} = H^{m+n}(W, \partial W; \mathbb{Q}) \). The restriction of \( a \) to the zero-set \( \text{Zero}(s) = (P \times \text{Zero}(f))/G \subseteq W - \partial W \) is non-zero. So again the covering dimension of \( \text{Zero}(f) \) must be at least \( n - k \). \( \square \)

2. Elementary abelian 2-groups

In this section we consider the case in which \( G \) is a non-trivial elementary abelian 2-group \( E \), considered as an \( F_2 \)-vector space of dimension \( l \geq 1 \).

Suppose that \( 0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_l = E \) is a flag in \( E \), with \( \dim_{F_2} E_i = i \). The dual vector space \( E^* \) parametrizes the 1-dimensional real representations of \( E \). For a real representation \( U \) of \( E \), we write \( U^{\alpha} \) for the \( \alpha \)-summand, so that \( U = \bigoplus_{\alpha \in E^*} U^{\alpha} \). The annihilator in \( E^* \) of a vector subspace \( F \subseteq E \) is denoted by \( F^0 \subseteq E^* \); it is isomorphic to \( (E/F)^* \). Set \( E^i = E_{l-i} \), so that \( 0 \leq E^0 \subseteq \cdots \subseteq E^1 \subseteq \cdots \subseteq E^l = E^* \) is a flag in \( E^* \). We introduce the subspaces

\[
U_i = \bigoplus_{\alpha \in E^*, \alpha \notin E^i} U^{\alpha} \quad \text{for} \quad 1 \leq i \leq l.
\]

Thus \( U = U^E \oplus \bigoplus_{i=1}^l U_i \), where \( U^E (= U^0) \) is the fixed subspace. This decomposition depends, of course, on the choice of the flag in \( E \).

Let us write

\[
e(U) = \prod_{\alpha \in E^*} \alpha^{\dim U^\alpha} \in S^*(E^*) = H^*_E(\ast; F_2) \quad \text{(and} \quad e(0) = 1)\]

in the symmetric algebra of \( E^* \) or the Borel cohomology ring of \( E \).

We shall need an elementary algebraic result.

**Lemma 2.1.** Let \( a \) be an ideal in a commutative ring \( A \), and let \( u(T), v(T) \in A[T] \) be non-zero polynomials with invertible leading coefficient.

(i). If \( a(T) \in A[T] \) is a polynomial such that \( a(T)u(T) \in a[T] \), then \( a(T) \in a[T] \).

(ii). Suppose that \( \deg(u(T)) > \deg(v(T)) \). Then, if \( a \in A \), but \( a \notin a, \) we have \( a \cdot v(T) \notin a[T] + (u(T)) \).

**Proof.** By passing to the quotient \( A/a \) one can reduce to the case in which \( a = 0 \). \( \square \)

**Lemma 2.2.** Suppose that \( U \) is a real representation of \( E \) such that \( U_i \neq 0 \) for \( i = 1, \ldots, l \). Then \( E \) acts freely on

\[
\tilde{X} = \prod_{i=1}^l S(U_i) \subseteq U = U^E \oplus \bigoplus_{i=1}^l U_i
\]

and the cohomology ring of the orbit space \( X = \tilde{X}/E \) is

\[
H^*(X; F_2) = S^*(E^*)/(e(U_1), \ldots, e(U_l)).
\]

Suppose, further, that \( V \) is an \( E \)-module with \( V^E = 0 \), and let \( \xi \) be the vector bundle \( \tilde{X} \times E V \) over \( X \). Then the \( F_2 \)-Euler class \( e(\xi) \) is non-zero if \( \dim U_i > \dim V_i \) for \( i = 1, \ldots, l \).
Proof. Notice that $U_i$ and $V_i$ for $i \leq j$ are $E/E_{l-j}$-modules.

Suppose that $v \in E$ and $x = (x_1, \ldots, x_l) \in X$ with $v \cdot x = x$. Now $E_{l-i+1}/E_{l-i}$ acts freely on $S(V_i)$ for $i = 1, \ldots, l$. So, since $v \cdot x_i = x_i$ if $v \in E_{l-i+1}$, we can conclude that $v \in E_{l-i}$. The deduction that $v = 0$ is achieved in $l$ steps.

The $\mathbb{F}_2$-cohomology ring $H^*(X) = \check{H}^*_E(X)$ is calculated step-by-step using the long exact sequences

$$
\cdots \to \check{H}^j_{E/E_{l-j}}\left( \prod_{i=1}^{j-1} S(U_i) \right) \xrightarrow{e(U_j)} \check{H}^j_{E/E_{l-j}}\left( \prod_{i=1}^{j-1} S(U_i) \right) \to \check{H}^j_{E/E_{l-j}}\left( \prod_{i=1}^j S(U_i) \right) \to \cdots
$$

in Borel cohomology.

At the $j$th step, starting with $j = 1$, we apply Lemma 2.1 with $A = S^*(E^{l-j+1})$ and $a = (e(U_1), \ldots, e(U_{j-1}))$. Choose $\alpha \in E^{l-j}$, $\alpha \notin E^{l-j-1}$, so that we can identify $S^*(E^{l-j})$ with the polynomial ring $A[T]$ on $T = \alpha$. Take $u(T) = e(U_j)$ and $v(T) = e(V_j)$.

Part (i) of Lemma 2.1 establishes the injectivity of multiplication by $e(U_j)$ in the exact sequence and so calculates $\check{H}^j(X; \mathbb{F}_2) = H^*(X; \mathbb{F}_2)$.

Part (ii), with $a = e(V_1) \cdots e(V_{j-1}) \in A$, so that $v(T) = a \cdot e(V_j) \in A[T]$, gives the non-vanishing of $e(\xi)$. \hfill \Box

Example 2.3. (Compare [13, Theorem 1.2] and [2, Section 2.7].) Suppose that $\varphi : \tilde{X} \to \mathbb{R}^m$ is a continuous map and that $\dim U_i > m2^{i-1}$ for $i = 1, \ldots, l$. Then $\varphi$ is constant on some $E$-orbit in $\tilde{X}$.

Proof. Writing $\mathbb{R}[E]$ for the group ring of $E$, set $V = (\mathbb{R}[E]/\mathbb{R})^m$. Then $\dim V_i = m2^{i-1}$. The $E$-map $\tilde{X} \to V$: $x \mapsto [\sum_{e \in E} \varphi(ex)e]$ determines a section of $\check{X}$. Since $e(\xi) \neq 0$, the section must have a zero, and this zero in $X$ is the required $E$-orbit. \hfill \Box

As a first application of Lemma 2.1 we deduce some results about group actions on Stiefel manifolds.

Lemma 2.4. Let $P$ and $Q$ be finite dimensional Euclidean $E$-modules with $P^E = 0$, $\dim P_i = 1$ for $i = 1, \ldots, l$, and $Q^E = 0$. Then, for a given integer $n > l$, the group $E$ acts freely on the Stiefel manifold $\check{Y} = O(P, \mathbb{R}^n)$ of isometric linear maps $P \to \mathbb{R}^n$. Let $\eta$ be the real vector bundle over the orbit space $Y = \check{Y}/E$ associated with the representation $Q$.

If $\dim Q_i \leq n - i$ for each $i = 1, \ldots, l$, then the Euler class $e(\eta)$ is non-zero.

Proof. Set $U = \text{Hom}(P, \mathbb{R}^n)$, so that $U_i = \text{Hom}(P_i, \mathbb{R}^n)$ has dimension $n$. Let

$$
h = (h_{i,j}) : U \to R = \bigoplus_{1 \leq i < j \leq l} P_i^* \otimes P_j^*
$$

be given by the inner product on $\mathbb{R}^n$: $h_{i,j}(u_1, \ldots, u_l) = \langle u_i, u_j \rangle$. The zero-set of $h$ restricted to $\tilde{X}$ is exactly the Stiefel manifold $\check{Y} = O(P, \mathbb{R}^n)$. Notice that $R^E = 0$ and $\dim R_i = i - 1$.

Take $V = Q \oplus R$. Then $\xi = \alpha \oplus \beta$, where $\alpha$ and $\beta$ are the vector bundles over $X$ associated with $Q$ and $R$. By assumption, $\dim U_i = n > \dim V_i = \dim Q_i + i - 1$. So the Euler class $e(\xi) = e(\alpha) \cdot e(\beta)$ is non-zero.

Now the $E$-map $h$ determines a section of $\beta$ with zero-set equal to $Y$. By the Borsuk-Ulam theory [8, Proposition 2.7], the restriction, $e(\eta)$, of $e(\alpha)$ to $Y$ is non-zero. \hfill \Box
Proposition 2.5. (Compare [11, Theorem 5.4], [6, Theorem 1.1].) Suppose that $P$ and $Q$ are $E$-modules as in Lemma 2.4 such that $\dim Q_i \leq n-i$ for $i = 1, \ldots, l$. Let $f : O(P, \mathbb{R}^n) \to Q$ be a continuous $E$-equivariant map. Then the zero-set of $f$ is non-empty and has covering dimension greater than or equal to $ln - l(l - 1)/2 - \dim Q$.

In particular, we may take $Q_i = \mathbb{R}^{n-i} \otimes P_i$.

**Proof.** We can apply Proposition 1.1 to the manifold $W = O(P, \mathbb{R}^n)$ and the representation $V = Q$ using Lemma 2.4.

**Remark 2.6.** The space $Y = O(P, \mathbb{R}^n)/E$ in Lemma 2.4 can be identified with the space of flags $0 = D^0 \subset D^1 \subset \cdots \subset D^l \subset \mathbb{R}^n$, where $D^i$ is an $\mathbb{R}$-subspace of dimension $j$, in $\mathbb{R}^n$: send the orbit of $a \in O(P, \mathbb{R}^n)$ to the flag with $D^i = a(P_1 \oplus \cdots \oplus P_l)$. Let $\delta_i$ be the canonical $(n-i)$-dimensional real vector bundle over $B$ with fibre at the flag $(D^i)$ the orthogonal complement of $D^i$.

It is easy to deduce from the lemma that the product of the Euler classes

$$e(\delta_1) \cdots e(\delta_l) \in H^{ln-l(l+1)/2}(Y; \mathbb{F}_2) = \mathbb{F}_2$$

is non-zero.

**Proof.** By consideration of the projection from the space of flags of length $n$ to the space of flags of length $l$ we see that it is enough to deal with the case $l = n$.

So with $l = n$, let $\tau_i$ be the line bundle over $Y$ associated with the representation $P_i$. Then $\delta_i$ is the direct sum $\bigoplus_{j=i+1}^n \tau_j$. So $e(\delta_1) \cdots e(\delta_n) = t_0^i \cdots t_0^{i-1} \cdots t_0^{i-1}$, where $t_i = e(\tau_i)$.

Taking $Q_i = \mathbb{R}^{n-i} \otimes P_i$ in Lemma 2.4 we deduce that $e(\eta) = t_0^{i-1} \cdots t_0^{n-i} \cdots t_0^n$ is non-zero.

The result follows from the $S_n$-symmetry of $P = P_1 \oplus \cdots \oplus P_n$.

**Example 2.7.** It follows that the $l$ sections $s_i$ of $\delta_i$, $i = 1, \ldots, l$, over the flag manifold $Y$ have a common zero. One can write down an example with exactly one zero. Choose linearly independent $v_1, \ldots, v_l$ in $\mathbb{R}^n$ and define the value of $s_i$ at $(D^j)$ to be component of $v_i$ in the orthogonal complement of $D^j$ in the decomposition $\mathbb{R}^n = D^j \oplus (D^j)^\perp$. Then $\bigcap_i$ Zero($s_i$) is precisely the flag $(D^j)$ with $D^j = \mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_j$.

**Remark 2.8.** (Compare [3, Section 3].) Similar methods can be used to describe the cohomology ring of the flag manifold as

$$H^*(O(P, \mathbb{R}^n)/E; \mathbb{F}_2) = \mathbb{F}_2[t_1, \ldots, t_l]/(e_1, \ldots, e_l),$$

where $e_i = \sum_{r_1 + \cdots + r_i = n-i+1} t_1^{r_1} \cdots t_l^{r_l}$.

More symmetrically, we can replace the classes $e_i$ by

$$\bar{e}_i = \sum_{r_1 + \cdots + r_i = n-i+1} t_1^{r_1} \cdots t_i^{r_i} = e_i + \sum_{i < j \leq l} a_{i,j} e_j,$$

where $a_{i,j} = \sum_{r_j + \cdots + r_i = n-i+1} t_j^{r_j} \cdots t_i^{r_i}$.

The top-dimensional class is represented by $\prod_{i=1}^l t_i^{n-i}$.

**Proof.** Consider the sphere-bundles

$$S(P_i^* \otimes \zeta_i) \to O(P_1 \oplus \cdots \oplus P_i, \mathbb{R}^n) \to O(P_1 \oplus \cdots \oplus P_{i-1}, \mathbb{R}^n),$$

for $1 \leq i \leq l$, where $\zeta_i$ is an $E$-vector bundle of dimension $n-i+1$ with $P_1 \oplus \cdots \oplus P_{i-1} \oplus \zeta_i = \mathbb{R}^n$. The computation is effected by induction using the $H^*_E$-Gysin
sequences of the sphere-bundles and Lemma 2.1(i). The generator $e_i$ corresponds to the Euler class $e(P_i \otimes \zeta_i) = w_{n-i+1}(-(P_1 \oplus \cdots \oplus P_i))$

Write, for $j \geq i$ and $j \leq l$,

$$e_i[j] = \sum_{r_1+\cdots+r_j=n-i+1} t_1^{r_1} \cdots t_j^{r_j},$$

so that $e_i = e_i[i]$ and $e_i[j+1] = e_i[j] + t_{j+1} e_{i+1}[j+1]$. One easily shows by induction on $k$ that

$$e_i[i+k] = \sum_{i \leq j \leq i+k} \left( \sum_{s_j+\cdots+s_{i+k}=j-i} t_j^{s_j} \cdots t_{i+k}^{s_{i+k}} \right) e_j.$$

This then establishes the formula for $\eta_i$.

The generator of $H^{n-l(l-1)/2}(Y; \mathbb{F}_2)$ was identified as $e(\eta)$ in Remark 2.9 above.

**Remark 2.9.** (Compare [6] Corollary 3.4.) At the cost of losing the $S_l$-symmetry, suppose given integers $1 \leq n_1 \leq \cdots \leq n_i \leq \cdots \leq n_l \leq n$ such that $n_i \geq i$. Consider the submanifold

$$\tilde{Y}' = \{ a \in O(P, \mathbb{R}^n) \mid D^j = a(P_1 \oplus \cdots \oplus P_j) \subseteq \mathbb{R}^n_j \subseteq \mathbb{R}^n, \ j = 1, \ldots, l \}$$

of $\tilde{Y} = O(P, \mathbb{R}^n)$ and the quotient $Y' = \tilde{Y}'(\mathbb{F}_2) \subseteq Y$. Let $\eta'$ denote the restriction of $\eta$ to $Y'$. Then the arguments used in Remark 2.8 and Lemma 2.4 show that

$$H^*(Y'; \mathbb{F}_2) = \mathbb{F}_2[t_1, \ldots, t_l]/(e'_1, \ldots, e'_l),$$

where $e'_i = \sum_{r_1+\cdots+r_j=n_i-i+1} t_1^{r_1} \cdots t_j^{r_j}$, and that the Euler class $e(\eta')$ is non-zero if $\dim Q_1 \leq n_i - i$ for $i = 1, \ldots, l$.

Our primary application of Lemma 2.2 concerns the covering dimension of the zero-set of an $E$-map from $U$ to $V$.

**Proposition 2.10.** Suppose that $U$ and $V$ are $E$-modules as in Lemma 2.2 with $\dim U_i > \dim V_i$ for $i = 1, \ldots, l$. Let $f : U \to V$ be a continuous $E$-map. Then the zero-set of $f$ contains a compact free $E$-subspace with covering dimension greater than or equal to $\dim U - \dim V$.

**Proof.** We can apply Proposition 1.1 to an equivariant tubular neighbourhood $W = X \times D(U(E) \oplus \mathbb{R}^l)$ of $X$ in $U$, using Lemma 2.2.

This implies the following result established in [15] Theorem 2.1.

**Corollary 2.11.** Suppose that $U$ and $V$ are $E$-modules with $V = 0$ and $\dim U - \dim V > \dim U^E$, and suppose that $F \leq E$ is a maximal subgroup such that $\dim U^F - \dim V^F > \dim U - \dim V$.

Let $f : U \to V$ be a continuous $E$-map. Then the zero-set of $f$ restricted to $U^F$ contains a compact free $E/F$-subspace with covering dimension greater than or equal to $\dim U - \dim V$.

**Proof.** Consider the action of $E/F$ on $U^F$ and $V^F$ and the restriction of $f$ to the fixed-points. We have $\dim U^F - \dim V^F > \dim(U^F)E = \dim U^E$. To simplify the notation, let us assume that $F = 0$ and make the abbreviation $d^\alpha = \dim U^\alpha - \dim V^\alpha$. We show that there is a flag $(E_i)$ such that $\dim U_i > \dim V_i$ for $i = 1, \ldots, l$. The result will then follow from Proposition 2.10.
Suppose that $E_i$ have been constructed for $j = l, \ldots, l - i + 1$ for some $i: 1 \leq i < l$.
By assumption $\dim U - \dim V > \dim U^{E_{l-i+1}} - \dim V^{E_{l-i+1}}$, that is,
$$\sum_{\alpha \in E^*_l : \alpha \notin E_{l-1}} d^\alpha = \sum_{E_{l-1} \subseteq \alpha \in E^*_l : \dim E' = i} (\sum_{\alpha \in E' : \alpha \notin E'_{l-1}} d^\alpha) > 0.$$  
Choose a subspace $E'$ such that $\sum_{\alpha \in E' : \alpha \notin E'_{l-1}} d^\alpha > 0$ and take $E_{l-i}$ to be the annihilator of $E' = E''$.

Corollary 2.12. Suppose that $V$ is a Euclidean $E$-module with $V^E = 0$ and that $M$ is a closed $E$-manifold of dimension $n$, such that the fixed submanifold $M^E$ is non-empty and has some component of codimension strictly greater than $\dim V$. Then the zero-set of any $E$-equivariant map $f : M \to V$ contains a compact subspace disjoint from $M^E$ and of covering dimension at least $n - \dim V$.

Proof. Choose a point $x \in M^E$ in a component with codimension greater than $\dim V$. Take $U$ to be the tangent space $\tau_x M$ at $x$ embedded as an open $E$-subspace $U \hookrightarrow M$ using the exponential map given by an $E$-equivariant Riemannian metric on $M$ (mapping $v \in U$ to $exp_x(\epsilon v/\sqrt{1 + \|v\|^2})$ for small $\epsilon > 0$). Now apply Corollary 2.11.

We finish this discussion of elementary abelian 2-groups with an example involving an infinite dimensional mapping space.

Proposition 2.13. Let $U$ and $V$ be finite-dimensional Euclidean $E$-modules such that $U \neq 0$, $\bigcap_{\alpha : U=\neq 0} \ker \alpha = 0$ and $V^E = 0$.
Consider the $E$-space $\operatorname{map}_*(S(\mathbb{R} \oplus U), \mathbb{R})$ of real-valued functions on the sphere $S(\mathbb{R} \oplus U)$ that are zero at the basepoint $(1, 0)$. Suppose that $f : \operatorname{map}_*(S(\mathbb{R} \oplus U), \mathbb{R}) \to V$ is an $E$-equivariant map.

Then, for any $d \geq 0$, the zero-set of $f$ contains a compact free $E$-subspace with covering dimension greater than or equal to $d$.

Proof. We can choose $\alpha_1, \ldots, \alpha_l \in E^*$, one at a time, such that the intersection $\bigcap_{i=1}^l \ker \alpha_i$ has dimension $l - i$ for $i = 1, \ldots, l$ and then form the flag $(E_i)$ in $E$ such that $E_i^* = \bigcap_{j=1}^i \ker \alpha_j$. With respect to this flag, which we now fix, each $U_i$ is non-zero.

It is easy to see that, for $j \geq 1$, the space $P[j] = S^{2j-1}(U^*)$ of homogeneous polynomial functions $U \hookrightarrow \mathbb{R}$ of odd degree $2j - 1$ satisfies $\dim P[j]_i \geq \dim U_i$ for $i = 1, \ldots, l$.

For $k \geq 1$, we can embed the $E$-module $U[k] = \bigoplus_{j=1}^k P[j]$ in the mapping space as a space of homogeneous polynomials (restricted to $S(\mathbb{R} \oplus U) \subseteq \mathbb{R} \oplus U$)

$$\sum_{j=1}^k \ell^{2k-2j} P[j] \subseteq \operatorname{map}_*(S(\mathbb{R} \oplus U), \mathbb{R})$$
of degree $2k - 1$, where $t$ is the coordinate function on $\mathbb{R}$.

Since $U[k]_i \geq k \dim U_i$, the assertion follows by applying Proposition 2.10 for $k$ sufficiently large, to the module $U[k]$ instead of $U$. 

Remark 2.14. In the special case that $l = 1$, so that $E$ has order 2, and $U$ is the non-trivial 1-dimensional $E$-module $\mathbb{R}$ with the involution $-1$, $\operatorname{map}_*(S(\mathbb{R} \oplus U), \mathbb{R})$ is the loop space $\Omega(\mathbb{R}, 0)$ with the involution that reverses loops. Since we can include $\Omega(\mathbb{R}, 0)$ in $\Omega(S^n, *)$ for any $n \geq 1$, Proposition 2.13 answers a question posed at the
end of [10] (although the method contradicts the assertion in Proposition 1 of that paper).

3. Tori

Let $L$ be a free abelian group of dimension $l \geq 1$, and write $T = (\mathbb{R} \otimes L)/L$ for the associated homomorphism $t$.

Proof.

(See [4, Remark 4.4].) A $T^l$-equivariant and Zero($f$) map $f : T \times \mathbb{R}^l \to \mathbb{R}^l$ such that $f(x) = (e^{2\pi i a x}, e^{2\pi i b y})$ and $V = C \otimes C$ with the action $(x, y) \mapsto (e^{2\pi i a x}, e^{2\pi i b y})$. Choose $a', b' \geq 1$ with $aa' - bb' = 1$. Then $f : T \to V$

is $T$-equivariant and Zero($f$) = $\{0\}$.

We write

$$e(U) = \prod_{\alpha \in L^*} \alpha^{\dim U^\alpha} \in S^*(E^*) = H_T^*(\ast; \mathbb{Q})$$

for the Euler class in $T$-equivariant Borel cohomology.

Proposition 3.3. (Compare [12] Theorem (2.3)). Suppose that $U^T = 0$, $V^T = 0$, $\dim U > \dim V$ and $\varphi : U \to \mathbb{R}$ is a continuous $T$-map such that $\varphi(0) < 0$ and $\varphi(x) > 0$ for $\|x\|$ sufficiently large.

If $U : V \to T$ is a $T$-map, then the intersection Zero($f$) \cap Zero($\varphi$) has covering dimension greater than or equal to $2(\dim C U - \dim C V) - 1$.

Proof. (Compare [5] Section 5). Choose $v \in L$ such that $\alpha(v) \neq 0$ for all $\alpha \in L^* - \{0\}$ such that $U^\alpha \neq 0$ or $V^\alpha \neq 0$. Let $\rho : T = \mathbb{R}/\mathbb{Z} \to T = (\mathbb{R} \otimes L)/L$ be the associated homomorphism $t + \mathbb{Z} \mapsto tv + L$. Then $U^\rho(T) = 0$ and $V^\rho(T) = 0$. 


Choose radii $r$ and $R$, $0 < r < R$, such that $\varphi(x) < 0$ if $\|x\| = r$ and $\varphi(x) > 0$ if $\|x\| = R$. The annulus $W = \{x \in U \mid r \leq \|x\| \leq R\}$ is a compact manifold of dimension $n = 2 \dim U$ which is $T$-equivariantly diffeomorphic to $D(\mathbb{R}) \times SU$.

Write $k = 2 \dim V + 1$. Now apply Proposition 1.2 with condition (ii) and $G = T$ to the map $(\varphi, f) : W \to \mathbb{R} \oplus V$. The relative Euler class in $H^k_T(W, \partial W; \mathbb{Q}) = H^{k-1}_T(S(U); \mathbb{Q}) = \mathbb{Q}$ is the image of $e(V)$ and is non-zero (because $k-1 = 2 \dim V \leq 2(\dim U - 1)$).

Suppose that $0 = E_0 \subseteq \cdots \subseteq E_i \subseteq \cdots \subseteq E_l = E$, with $\dim_{\mathbb{Q}} E_i = i$, is a flag in the $\mathbb{Q}$-vector space $E$. Put $L_i = L \cap E_i$ and $T_i = (\mathbb{R} \otimes L_i)/L_i \leq T$. Writing $E^i = E^i_{l-i} \subseteq E^*$ for the annihilator of $E_{l-i}$, we define

$$U_i = \bigoplus_{\lambda \in P_0(E^i), \lambda \notin P_0(E^{i-1})} U \lambda .$$

So $U = U^T \oplus \bigoplus_{i=1}^l U_i$.

**Lemma 3.4.** Suppose that $U$ is a complex representation of $T$ such that $U_i \neq 0$ for all $i$. Then $T$ acts with finite isotropy groups on

$$\tilde{X} = \prod_{i=1}^l S(U_i) \subseteq U = U^T \oplus \bigoplus_{i=1}^l U_i$$

and $H^*_T(\tilde{X}; \mathbb{Q}) = S^*(E^*)/(e(U_1), \ldots, e(U_l))$.

If $V$ is a complex $G$-module with $V^T = 0$ and $\dim U_i > \dim V_i$ for $i = 1, \ldots, l$, then the image of $e(V)$ in $H^*_T(\tilde{X}; \mathbb{Q})$ is non-zero.

**Proof.** Consider $x = (x_1, \ldots, x_l) \in \tilde{X}$ fixed by $v + L \in T$, where $v \in \mathbb{R} \otimes L$. For each $i$, there is some $\alpha_i \in L^*$ with $\alpha_i \in E_i = E^*_{l-i}$, $\alpha_i \notin E^*_{i-1} = E_{l-i+1}^*$ such that the component of $x_i$ in $U^{\alpha_i}$ is non-zero. Since $v + L$ fixes $x_i$, we have $\alpha_i(v) \in \mathbb{Z}$. Because the $\alpha_i$ form a $\mathbb{Q}$-basis of $E^*$, the vector $v \in \mathbb{R} \otimes L$ lies in $E = \mathbb{Q} \otimes L$ and $v + L \in T$ has finite order. Thus the isotropy group of $x$ is finite.

The rest of the proof follows mutatis mutandis the argument in Lemma 2.2 using long exact sequences

$$\cdots \to H^*_T(T_{T_i}) \bigoplus_{i=1}^{j-1} S(U_i) \xrightarrow{e(U_j)} H^*_T(T_{T_{j-1}}) \bigoplus_{i=1}^{j-1} S(U_i) \to H^*_T(T_{T_{j-1}}) \bigoplus_{i=1}^j S(U_i) \to \cdots$$

in Borel cohomology. \hfill $\square$

There are complex analogues of Propositions 2.3 and 2.10.

**Proposition 3.5.** Let $P$ and $Q$ be finite dimensional Hermitian $T$-modules with $P^T = 0$, $Q^T = 0$, and $\dim_{\mathbb{C}} P_i = 1$ for $i = 1, \ldots, l$. Suppose that, for some $n > l$, $f : U(P, \mathbb{C}^n) \to Q$ is a $T$-equivariant map from the complex Stiefel manifold of isometric $\mathbb{C}$-linear maps $P \hookrightarrow \mathbb{C}^n$ to $Q$ and that $\dim_{\mathbb{C}} Q_i \leq n - i$ for each $i = 1, \ldots, l$. Then the zero-set of $f$ is a non-empty free $T$-space with covering dimension at least $2ln - l^2 - 2\dim_{\mathbb{C}} Q$.

**Proof.** This can be established by using Proposition 1.2 with condition (i) taking $G = T$, $W = U(P, \mathbb{C}^n)$ and $V = Q$.
Proposition 3.6. Suppose that $U$ and $V$ are $T$-modules as in Lemma 3.4 with $\dim U_i > \dim V_i$ for $i = 1, \ldots, l$. Let $f : U \to V$ be a continuous $T$-map. Then the zero-set of $f$ contains a compact $T$-subspace with finite isotropy groups and covering dimension greater than or equal to $2(\dim C_U - \dim C_V)$.

Proof. We can apply Proposition 1.2 with condition (i) to an equivariant tubular neighbourhood $W = \tilde{X} \times D(U^E \oplus \mathbb{R}^l)$ of $\tilde{X}$ in $U$, using Lemma 3.4, with $k = 2\dim C_V$ and $n = 2\dim C_U$. The orientations, being determined by complex structures, are invariant under the action of $G = T$. □

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