HÖRMANDER TYPE PSEUDODIFFERENTIAL CALCULUS ON HOMOGENEOUS GROUPS

SUSANA CORÉ AND DARYL GELLER

ABSTRACT. We produce, on general homogeneous groups, an analogue of the usual Hörmander pseudodifferential calculus on Euclidean space, at least as far as products and adjoints are concerned. In contrast to earlier works, we do not limit ourselves to analogues of classical symbols, nor to the Heisenberg group. The key technique is to understand “multipliers” of any given order \( j \), and the operators of convolution with their inverse Fourier transforms, which we here call convolution operators of order \( j \). (Here a “multiplier” is an analogue of a Hörmander-type symbol \( a(x, \xi) \), which is independent of \( x \).) Specifically, we characterize the space of inverse Fourier transforms of multipliers of any order \( j \), and use this characterization to show that the composition of convolution operators of order \( j_1 \) and \( j_2 \) is a convolution operator of order \( j_1 + j_2 \).

CONTENTS

1. INTRODUCTION
2. PRELIMINARIES
3. CHARACTERIZATION OF \( \tilde{M}^j(G) \)
4. CONVOLVING ELEMENTS OF THE \( \tilde{M}^j(G) \)
5. PSEUDODIFFERENTIAL CALCULUS ON \( G \)

References

1. INTRODUCTION

The goal of this article is to produce, on general homogeneous groups (in the sense of Folland-Stein [FS82]), an analogue of the usual pseudodifferential calculus on Euclidean space, at least as far as products and adjoints are concerned. Taylor [Tay84] and Christ-Geller-Głowacki-Polin [CGGP92] have developed analogues of classical pseudodifferential operators for homogeneous groups, but, as we shall explain, our analogue of the usual calculus allows one to study a useful wider class of operators.

The first author [Cor07] has already produced an analogue of the usual pseudodifferential calculus for the Heisenberg group. The arguments in the present article are simpler.
even in this special case. (Note, however, that in [Cor07], the first author completely characterized the kernels of the operators if they had negative order, even on general homogeneous groups \(G \neq \mathbb{R}\). This information on the kernels of the operators will be valuable in sequel articles.)

Before describing the methods we use, we make more precise the terms usual and classical pseudodifferential calculus used loosely in the previous paragraphs. We recall that the (Hörmander) symbol class of order \(m\), denoted by \(S^m_{1,0}\), consists of those functions \(a(x,\xi)\) in \(C^\infty(\mathbb{R}^n \times \mathbb{R}^n)\) such that for any pair of multiindices \(\alpha,\beta\), and any compact set \(B \subset \mathbb{R}^n\), there exists a constant \(C_{\alpha,\beta,B}\) such that
\[
\left|D^\alpha_x D^\beta_\xi a(x,\xi)\right| \leq C_{\alpha,\beta,B} (1 + \|\xi\|)^m - \|\alpha\| \quad \forall \ x \in B, \ \xi \in \mathbb{R}^n
\]
(1.1)
Given that Hörmander’s \(S^1_{1,0}\) symbols have become standard we shall refer to them as the usual symbols, and the associated calculus as the usual or Hörmander calculus of type \((1,0)\). The classical symbols of order \(m\) are those elements \(a(x,\xi)\) in \(S^m_{1,0}\) for which there are smooth functions \(a_{m-j}(x,\xi)\), homogeneous of degree \(m-j\) in \(\xi\), for \(\|\xi\| \geq 1\), such that
\[
a(x,\xi) \sim \sum_{j \geq 0} a_{m-j}(x,\xi)
\]
where the asymptotic condition means that for any \(N\)
\[
a(x,\xi) - \sum_{j=0}^{N} a_{m-j}(x,\xi) \in S^{m-N-1}_{1,0}
\]
Our approach to pseudodifferential operators on homogeneous groups favors the use of convolution operators and avoids the Fourier transform as much as possible. This idea goes back to the original work of Mikhlin [Mih48, Mih50] and of Calderón and Zygmund [CZ56, CZ57]. These pioneers studied operators of the form
\[
(Kf)(x) = \int K(x, x-y) f(y) \, dy
\]
so that
\[
(Kf)(x) = (K_x \ast f)(x)
\]
(1.2)
with \(K_x(z) = K(x, z)\) an integral kernel for each \(x\), singular only at the origin, and depending smoothly on \(x\). In the 1960’s Kohn and Nirenberg [KN65] introduced symbols, a different point of view, and the term pseudodifferential operator. They used the Fourier transform to rewrite the operator \(K\) in the form
\[
(Kf)(x) = \int e^{-2\pi i x \cdot \xi} a(x,\xi) \hat{f}(\xi) \, d\xi = \left( a_x \hat{f} \right)^\sim(x)
\]
where \(a\), the symbol of the operator, is the formal Fourier transform of \(K\) in the second variable, that is
\[
a_x(\xi) = \hat{K}_x(\xi) \quad \text{and} \quad a_x(\xi) = a(x,\xi)
\]
(1.3)
This last approach, which takes advantage of the properties of the Fourier transform, became prevalent. A major reason is that the Fourier transform converts convolution to a product, which is easier to handle. In particular, one can seek to use division to invert $K$. In problems where the Euclidean convolution structure is not relevant, these advantages are largely lost, and in many instances it is desirable to imitate the original definition of Mikhlin and Calderón-Zygmund.

For instance, if one is working on a Lie group, one can seek to define a class of pseudodifferential operators by (1.2), where now $*$ is group convolution. This idea originates in Folland-Stein [FS74] for the Heisenberg group, and was extended in Rothschild-Stein [RS76] to other settings. Calculi of such operators on homogeneous groups were developed by Taylor [Tay84] and studied in greater detail in [CGGP92]. (A related calculus, restricted to the Heisenberg group, and relying heavily on the Fourier transform, was developed in [BG88].) The second author further developed these ideas, to obtain a calculus in the real analytic setting on the Heisenberg group, in [Gel90]. All of these authors restricted themselves to analogues of the classical calculus.

The operators in the calculus of this article have the form (1.2) (with $*$ being group convolution), with $K_x$ being again given by (1.3), but where now $a(x,\xi)$ satisfies, for any compact set $B \subseteq \mathbb{R}^n$, the estimates

$$\left| D_\alpha \partial_\beta a(x,\xi) \right| \leq C_{\alpha,\beta,B} (1 + |\xi|)^{m-|\alpha|} \quad \forall x \in B, \xi \in \mathbb{R}^n \quad (1.4)$$

in analogy to (1.1). (Let us then say that $K$ has order $m$.) It is crucial to note that in this definition, and in what follows, $|\xi|$ denotes the homogeneous norm of $\xi$, and $|\alpha|$ is the “weighted” length of the multiindex $\alpha$.

Thus our $a(x,\xi)$ are not required to have an asymptotic expansion in homogeneous functions of $\xi$, as $\xi \to \infty$, as is required in the classical calculi.

We will show that there is a calculus for these operators, in that the adjoint of an operator of order $m$ is again an operator of order $m$, and that the composition of operators of order $m_1$ and $m_2$ is an operator of order $m_1 + m_2$. (When composing, one assumes that the “$a$” associated to the operator which is applied first, has compact support in $x$.) Further, there are asymptotic expansions, similar to the Kohn-Nirenberg formulas, for adjoints and compositions.

1.1. OUTLINE AND LIST OF RESULTS. This paper is organized as follows:

- Section 1 contains this introduction, and Section 2 is dedicated to establishing notation and basic terminology.
- In Section 3 we study the spaces $\mathcal{M}^\gamma(G)$, consisting of inverse Fourier transforms of “multipliers”. A “multiplier”, by definition, is a function $a$ as in (1.4) which is independent of $x$. Thus:

**Definition.** Suppose $m \in \mathbb{R}$. We shall say that $u \in C^\infty(G)$ is a *multiplier of order $m$* if for every multiindex $\alpha \in (\mathbb{Z}^+)^n$ there exists $C_\alpha > 0$ such that

$$\left| \partial^\alpha u(\xi) \right| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|} \quad \text{for all } \xi$$

$\mathcal{M}^m(G)$ will denote the space of multipliers of order $m$. 

If \( a \) satisfies (1.4), and \( a_x(\xi) = a(x, \xi) \), then for any \( x \), we surely have \( a_x \in \mathcal{M}^m \). The operator \( \mathcal{K} \) of (1.2), (1.3), is then of the form \( \mathcal{K}f(x) = [\hat{a}_x * f](x) \).

In this sense, our calculus of pseudodifferential operators will have, as its building blocks, operators of the form \( A f = \hat{u} * f \), where \( u \in \mathcal{M}^j(G) \) for some \( j \) and \( * \) is convolution on \( G \), and it is these convolution operators that will require our major focus. To avoid confusion, we caution the reader that these convolution operators will only be Fourier multiplier operators (in the ordinary sense) if \( * \) is Euclidean convolution. We nevertheless call elements of \( \mathcal{M}^j(G) \) “multipliers”, for lack of a better word.

The main result of Section 3 is the following characterization of the space of the inverse Fourier transform of multipliers, \( \hat{\mathcal{M}}^j(G) \). Here \( Q \) denotes the homogeneous dimension of \( G \), \( \mathcal{S}(G) \) denotes the space of Schwartz functions on \( G \) (which coincides with the usual space of Schwartz functions on \( \mathbb{R}^n \)), and \( \mathcal{S}_0(G) \) denotes the space of elements of \( \mathcal{S}(G) \), all of whose moments vanish.

**Proposition.** Say \( K \in \mathcal{S}'(G) \). Then \( K \in \hat{\mathcal{M}}^j(G) \) if and only if we may write
\[
K = \sum_{k=0}^{\infty} f_k,
\]
with convergence in \( \mathcal{S}' \), where
\[
f_k(x) = 2^{(j+Q)k} \varphi_k(2^k(x)) \quad (1.5)
\]
where \( \{ \varphi_k \}_k \subseteq \mathcal{S}(G) \) is a bounded sequence, and \( \varphi_k \in \mathcal{S}_0(G) \) for \( k \geq 1 \). Further, given such a sequence \( \{ \varphi_k \}_k \), if we define \( f_k \) by (1.5), then the series \( \sum_{k=0}^{\infty} f_k \) necessarily converges in \( \mathcal{S}'(G) \) to an element of \( K \) of \( \hat{\mathcal{M}}^j(G) \).

Related decompositions (with far weaker conditions on the \( \varphi_k \)) have occurred before in the literature on multipliers, when minimal smoothness of the multiplier was assumed. (In the situation of general dilations, but where ordinary Euclidean convolution is used, see [Ric04], pages 37-43.) It is our intention to show that this kind of decomposition is extremely helpful even when one assumes the full force of the \( \mathcal{M}^j \) conditions.

• In Section 4 the main result is the following key theorem:

**Theorem.** For any \( j_1, j_2 \),
\[
\hat{\mathcal{M}}^{j_1}(G) * \hat{\mathcal{M}}^{j_2}(G) \subseteq \hat{\mathcal{M}}^{j_1+j_2}(G).
\]

The proof uses the characterization of \( \hat{\mathcal{M}}^j(G) \) given in Section 3, an adaptation of Lemma 3.3 of Frazier-Jawerth [FJ85] to the homogeneous group setting, and some additional ideas.

Here is our lemma, adapted from Frazier-Jawerth [FJ85], who established the case in which \( G \) is Euclidean space. For \( \sigma \in \mathbb{Z}, I > 0 \), define
\[
\Phi^I_\sigma(x) = (1 + 2^\sigma |x|)^{-I}.
\]
Lemma. Say $J > 0$, and let $I = J + Q$. Then there exists $C > 0$ such that whenever $\sigma \geq \nu$,

$$\Phi^I_\sigma \ast \Phi^J_\nu \leq C 2^{-\sigma Q} \Phi^J_\nu.$$

Again, Frazier and Jawerth proved their lemma in order to study spaces of restricted smoothness (specifically, Besov spaces). But we intend to show that this lemma is also useful when one assumes the full force of the $\mathcal{M}^j$ conditions.

- In Section 5, we obtain our new calculus. At this point we simply refer to the general theory of Taylor [Tay84] to say that the desired calculus exists, as long as one has the key result of Section 4 (that $\tilde{\mathcal{M}}^{j_1} (G) \ast \tilde{\mathcal{M}}^{j_2} (G) \subseteq \tilde{\mathcal{M}}^{j_1 + j_2} (G)$), together with a few easily checked facts (which we do verify).

In future articles, we will examine the calculus in more detail. Among other properties, we will seek to obtain, as in the classical situation discussed in detail in [CGGP92], explicit formulas for products and adjoints, criteria for existence of parametrices, and mapping properties of our pseudodifferential operators.

As we said, all of those properties are known for the analogues of classical pseudodifferential operators on $G$. Let us then motivate our present work by mentioning two expected applications of our new calculus, that the classical calculus cannot be expected to have.

- Classical multipliers are, as $\xi \to \infty$, asymptotic sums of series of terms which are homogeneous with respect to the (non-isotropic) dilations on $G$. Let us call these dilation $\delta_r$ (for $r > 0$). One does not always want to restrict oneself to functions which are homogeneous with respect to $\delta_r$ for all $r > 0$; in the theory of wavelets, for instance, one really only cares about dyadic dilations. It is easy to see that if a smooth function $u(\xi)$ is homogeneous of degree $j$ with respect to $\delta_2$ (for $\xi$ outside a compact set), then it is in $\mathcal{M}^j(G)$.

In [GM06], wavelet frames (for $L^p$, $1 < p < \infty$, and $H^1$) were constructed on stratified Lie groups $G$ with lattice subgroups, out of Schwartz functions $\psi$ of the form $f(L)\delta$, for a nonzero $f \in S(\mathbb{R}^+)$ with $f(0) = 0$; here $L$ is the sublaplacian on $G$. Part of the argument involved inverting a spectral multiplier (for the sublaplacian) which was homogeneous of degree zero with respect to dyadic dilations, and using the fact that it therefore satisfied standard multiplier conditions. We expect that, once we understand inversion for convolution operators of the kind considered in this article, we will be able to generalize these arguments to much more general Schwartz functions $\psi$ on $G$ (not just those of the special form $f(L)\delta$), and therefore be able to construct wavelet frames from much more general $\psi$.

- Let $(M, g)$ be a smooth, compact, oriented Riemannian manifold. One of the most important facts about analysis on $M$ is the following theorem of Strichartz [Str72]:

If $p(\xi) \in S^{m}_{1,\#}(\mathbb{R})$ then $p(\sqrt{\Delta}) \in OPS^{m}_{1,0}(M)$.

($S^{m}_{1,\#}$ denotes the space of symbols $a(x, \xi)$ in $S^{m}_{1,0}$ which are independent of $x$, or in other words, multipliers of order $m$ in the usual sense.) Here $\Delta$ denotes the Laplace-Beltrami operator on $M$, but one can replace $\sqrt{\Delta}$ by a general first
order positive elliptic pseudodifferential operator on $M$. As in the classical case [Tay84], [CGGP92], taking $G$ to be the Heisenberg group, we expect to be able to transplant our calculus to contact manifolds, and in particular, to smooth, compact strictly pseudoconvex CR manifolds $M_0$. We expect that one can show:

$$\text{If } p(\xi) \in S^{m,\#}_{1,\#}(\mathbb{R}), \text{ then } p(\sqrt{L}) \text{ is an operator of order } m \text{ in the new calculus on } M_0.$$ 

Here $L$ denotes the sublaplacian on $M_0$, and we expect to be able to replace $\sqrt{L}$ here by a general first-order operator in the new calculus on $M_0$, which is transversally elliptic and positive.

2. PRELIMINARIES

We present basic results for homogeneous groups, and introduce the notation to be used later. For more details see [FS82].

**Definition 1.** Let $V$ be a real vector space. A family $\{\delta_r\}_{r>0}$ of linear maps of $V$ to itself is called a set of **dilations on $V$**, if there are real numbers $\lambda_j > 0$ and subspaces $W_{\lambda_j}$ of $V$ such that $V$ is the direct sum of the $W_{\lambda_j}$ and

$$\delta_r|_{W_{\lambda_j}} = r^{\lambda_j} \text{ Id } \quad \forall j$$

**Definition 2.** A **homogeneous group** is a connected and simply connected nilpotent group $G$, with underlying manifold $\mathbb{R}^n$, for some $n$, and whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations $\{\delta_r\}_{r>0}$, which are automorphisms of $\mathfrak{g}$.

The dilations are of the form $\delta_r = \exp(A \log r)$, where $A$ is a diagonalizable linear operator on $\mathfrak{g}$ with positive eigenvalues. The group automorphisms $\exp \circ \delta_r \circ \exp^{-1} : G \to G$ will be called **dilations of the group** and will also be denoted by $\delta_r$. The group $G$ may be identified topologically with $\mathfrak{g}$ via the exponential map $\exp : \mathfrak{g} \to G$ and with such an identification

$$\delta_r : G \quad \mapsto \quad G$$

$$(x_1, \ldots, x_n) \quad \mapsto \quad (r^{a_1}x_1, \ldots, r^{a_n}x_n)$$

Henceforth the eigenvalues of the matrix $A$, listed as many times as their multiplicity, will always be denoted by $\{a_i\}_{i=1}^n$. Moreover, we shall assume without loss of generality that all the $a_i$ are nondecreasingly ordered and that the first is equal to 1, that is

$$1 = a_1 \leq \ldots \leq a_n.$$ 

**Definition 3.** The **homogeneous dimension** of the group $G$, denoted by $Q$, is the number

$$Q = \sum_{i=1}^n a_i.$$ 

Examples of such groups are $\mathbb{R}^n$, with the usual additive structure, the Heisenberg groups, and the upper triangular groups, consisting of triangular matrices with 1’s on the diagonal and dilations

$$\delta_r \left( \begin{bmatrix} a_{ij} \end{bmatrix} \right) = \begin{bmatrix} r^{j-i}a_{ij} \end{bmatrix}$$
Definition 4. Let $G$ be a homogeneous group with dilations $\{\delta_r\}$. A homogeneous norm on $G$, relative to the given dilations, is a continuous function $| \cdot | : G \longrightarrow [0, \infty)$, smooth away from the origin satisfying

\begin{enumerate}
\item $|x| = 0$ if and only if $x$ is the identity element,
\item $|x^{-1}| = |x|$ for every $x \in G$,
\item $|\delta_r(x)| = r|x|$ for every $x \in G$, and $r > 0$, i.e. the norm is homogeneous of degree 1.
\end{enumerate}

If $| \cdot |$ is a homogeneous norm on $G$ then there exists a constant $C \geq 1$ such that

$$|xy| \leq C(|x| + |y|) \quad \text{for every } x, y \in G$$

Homogeneous norms always exist. Moreover any two homogeneous norms $| \cdot |$ and $| \cdot |'$ on $G$ are always equivalent, i.e. there exist constants $C_1, C_2 > 0$ such that

$$C_1|x| \leq |x'| \leq C_2|x| \quad \forall x \in G$$

An example of a homogeneous norm is

$$|x| = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2n}}$$

where $A = \prod_{i=1}^{n} a_i$.

We observe that if the dilations are isotropic, i.e. all the weights $a_i$ are equal, and satisfy the condition of normalization $a_1 = 1$, then this homogeneous norm is simply the Euclidean norm $\| \cdot \|$.

If $x \in G$, $r > 0$, for a fixed homogeneous norm we define the ball centered at $x$ of radius $r$ as

$$B(x; r) = \{ y \in G : |x^{-1}y| < r \}$$

Note also that there exists $c > 0$ such that whenever $x, y \in G$, we have

$$|y| \leq |x|/2 \Rightarrow |y^{-1}x| \geq c|x|.$$  \hspace{1cm} (2.1)

Indeed, by use of dilations, we may assume that $|x| = 1$. We then let $c$ be the minimum value of $|y^{-1}x|$ for $(x, y)$ in the compact set $\{ x : |x| = 1 \} \times \{ y : |y| \leq 1/2 \} \subseteq G \times G$.

For $j = 1, \ldots, n$, we let $X_j$ (resp. $Y_j$) denote the left (resp. right) invariant vector field on $G$ which equals $\partial / \partial x_j$ at the origin. In this context we have

Proposition 5. a) We may write each left invariant vector field $X_j$ as

$$X_j = \partial / \partial x_j + \sum_{k=j+1}^{n} p_{j,k}(x) \partial / \partial x_k \quad j = 1, \ldots, n$$

where $p_{j,k}(x) = p_{j,k}(x_1, \ldots, x_{k-1})$ are homogeneous polynomials, with respect to the group dilations, of degree $a_k - a_j$. 
b) Any \( \frac{\partial}{\partial x_j} \) can be written as
\[
\frac{\partial}{\partial x_j} = X_j + \sum_{k=j+1}^{n} q_{j,k}(x)X_k \quad j = 1, \ldots, n
\]
where \( q_{j,k}(x) = q_{j,k}(x_1, \ldots, x_{k-1}) \) are homogeneous polynomials, with respect to the group dilations, of degree \( a_k - a_j \). (Note: by homogeneity considerations \( q_{j,k}(x) \) can only involve \( x_1, \ldots, x_{k-1} \), so by part a), multiplication by \( q_{j,k}(x) \) commutes with \( X_k \).

Entirely analogous formulas express each of the right invariant vector fields \( Y_j \) in terms of \( \left\{ \frac{\partial}{\partial x_k} \right\}_{k=j}^{n} \), as well as \( \frac{\partial}{\partial x_j} \) in terms of \( \{ Y_k \}_{k=j}^{n} \).

Proof. See [FS82]. \( \square \)

The Haar measure on \( G \) is simply the Lebesgue measure on \( \mathbb{R}^n \).

**Proposition 6.** Say \( p < -Q \). Then for some \( 0 < C_p < \infty \)
\[
\int_{|x|>r} |x|^p \, dx = C_p r^{p+Q}
\]
for every \( r > 0 \).

The **convolution** of two functions \( f, g \) on \( G \) is defined by
\[
(f \ast g)(x) = \int_G f(xy^{-1})g(y) \, dy = \int_G f(y)g(y^{-1}x) \, dy
\]
provided that the integrals converge.

\( \mathcal{S}(G) \) will denote the usual Schwartz space on \( G \), thought of as \( \mathbb{R}^n \), and \( \{ \| \cdot \|_{\mathcal{S}(G), N} \} \) an increasing family of norms topologizing \( \mathcal{S} \). We also let
\[
\mathcal{S}_o(G) = \left\{ f \in \mathcal{S}(G) : \int x^\alpha f(x) \, dx = 0 \text{ for every multiindex } \alpha \right\}.
\]

Thus
\[
\mathcal{S}_o(G) = \left\{ \psi \in \mathcal{S}(G) : \partial^\alpha \psi(0) = 0 \text{ for every multiindex } \alpha \right\}.
\]

Since the group law is polynomial, \( \mathcal{S}_o \) is invariant under (left or right) translations. Thus \( \mathcal{S} \ast \mathcal{S}_o \subset \mathcal{S}_o \), since if we write \( (\tau_y g)(x) = g(y^{-1}x) \), we have \( f \ast g = \int f(y) \tau_y(g) \, dy \).

Similarly, \( \mathcal{S}_o \ast \mathcal{S} \subset \mathcal{S}_o \).

For the multiindex \( \beta \in (\mathbb{Z}^+)^n \) we define
\[
|\beta| = \sum_{i=1}^{n} a_i \beta_i \quad \text{and} \quad \|\beta\| = \sum_{i=1}^{n} \beta_i
\]
Note that, if \( \alpha \) is a multiindex, there exists \( C > 0 \) such that \( |x^\alpha| \leq C |x|^{|\alpha|} \) for all \( x \in G \).

\footnote{It will be convenient to think of the Fourier transform on \( \mathbb{R}^n \) as a map from functions on \( G \) to functions on \( G \). To clarify this, on the Fourier transform side we will often use the dilations \( \delta_r \) and the homogeneous norm \( | \cdot | \), but never the group structure.}
Lemma 7. For every multiindex $\alpha$

$$a_1|\alpha| \leq |\alpha| \leq a_n|\alpha|$$

and for all $\xi \in \mathbb{R}^n$ there exist positive constants $c$ and $C$ such that

$$c(1 + |\xi|) \leq \langle \xi \rangle \leq C(1 + |\xi|)^{a_n}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

Definition 8. Suppose $j \in \mathbb{R}$. We say that $J \in C^\infty(G)$ is a multiplier of order $j$ if for every multiindex $\alpha \in (\mathbb{Z}^+)^n$ there exists $C_\alpha > 0$ such that

$$|\partial^\alpha J(\xi)| \leq C_\alpha (1 + |\xi|)^{-j-|\alpha|}$$

for all $\xi$. We denote the space of all multipliers of order $j$ by $\mathcal{M}^j(G)$.

For $J \in \mathcal{M}^j(G)$, and $N \in \mathbb{Z}^+$, we define

$$\|J\|_{\mathcal{M}^j(G),N} = \sum_{|\alpha| \leq N} \| (1 + |\xi|)^{|\alpha|-j} \partial^\alpha J \|$$

where $\| \cdot \|$ denotes the supremum norm. The family $\{\| \cdot \|_{\mathcal{M}^j(G),N}\}_{N \in \mathbb{Z}^+}$ is a non-decreasing sequence of norms on $\mathcal{M}^j(G)$, which defines a Fréchet topology on $\mathcal{M}^j(G)$.

Suppose $J \in \mathcal{M}^j(G)$. We define

$$m_J : S(G) \rightarrow S'(G)$$

$$m_J(f) = \hat{J} * f$$

We close this section with some useful remarks about $S(G)$ and $S_o(G)$.

Remark 9. Note that for every $\psi \in \hat{S}_o$, we may find $\psi_1, \ldots, \psi_n \in \hat{S}_o$ with $\psi(\xi) = \sum_{j=1}^n \xi_j \psi_j(\xi)$. Indeed, we need only show this in two cases: (1) if $\psi$ vanishes in a neighborhood of $0$; and (2) if $\psi \in C^\infty_c(G)$. In case (1), it suffices to set $\psi_j(\xi) = \xi_j \psi(\xi)/\|\xi\|^2$, where $\|\xi\|$ is the Euclidean norm of $\xi$. In case (2), choose $\zeta \in C^\infty_c(G)$ with $\zeta \equiv 1$ in a neighborhood of $\text{supp} \psi$. Fix $\xi$ temporarily, and let $g(t) = \psi(t\xi)$. Then

$$\psi(\xi) = g(1) - g(0) = \int_0^1 g'(t)dt = \sum_{j=1}^n \xi_j \int_0^1 \partial_j \psi(t\xi)dt.$$ 

Noting that $\psi = \zeta \psi$, we see that in case (2) we may take $\psi_j(\xi) = \zeta(\xi) \int_0^1 \partial_j \psi(t\xi)dt$.

In fact, this construction shows the following uniformity:

- There is a linear map $T : \hat{S}_o \rightarrow (\hat{S}_o)^n$, such that
  - (i) if $T\psi = (\psi_1, \ldots, \psi_n) := (T_1 \psi, \ldots, T_n \psi)$, then $\psi(\xi) = \sum_{j=1}^n \xi_j \psi_j(\xi)$; and
  - (ii) for every $I$ there exist $C, J$ such that for $1 \leq j \leq n$, $\|T_j \psi\|_{S,I} \leq C\|\psi\|_{S,I}$. 

Remark 10. Taking inverse Fourier transforms in Remark 9 for every \( f \in S_o(G) \) we can find \( F_1, \ldots, F_n \) in \( S_o \) so that \( f(x) = \sum_{j=1}^n \partial_j F_j(\xi) \). By Proposition 8(b) we can find \( f_1, \ldots, f_n \) in \( S_o \), with \( f(x) = \sum_{j=1}^n X_j f_j(\xi) \). In fact, this construction shows the following uniformity: there exists a linear map \( S : S_o \to (S_o)^n \), such that

(i) if \( Sf = (f_1, \ldots, f_n) := (S_1 f, \ldots, S_n f) \), then \( f(x) = \sum_{j=1}^n X_j f_j(x) \); and

(ii) for every \( I \) there exist \( C, J \) such that for \( 1 \leq j \leq n, \)

\[
\|S_j f\|_{S,I} \leq C \|f\|_{S,J}.
\]

Remark 11. We can iterate the result of Remark 9 by applying this result to each \( \psi_j \) in place of \( \psi \), and so on. We then obtain the following uniformity. For each \( N \geq 1 \), let \( i(N) \) denote the number of multiindices \( \alpha \) with \( \|\alpha\| = N \). (Recall that the norm of a multiindex is equal to the sum of its coordinates, \( \|\alpha\| = \sum_{i=1}^n \alpha_i \).) Then there is a linear map \( T(N) : \hat{S}_o \to (\hat{S}_o)^{i(N)} \), such that

(i) if \( T(N)\psi = (\psi_\alpha)_{\|\alpha\|=N} := (T^{(N)}_\alpha \psi)_{\|\alpha\|=N} \), then \( \psi(\xi) = \sum_{\|\alpha\|=N} \xi^\alpha \psi_\alpha(\xi) \); and

(ii) for every \( I \) there exist \( C \) and \( J \) such that for all \( \alpha \) with \( \|\alpha\| = N \), we have

\[
\|T^{(N)}_\alpha \psi\|_{S,I} \leq C \|\psi\|_{S,J}.
\]

Remark 12. We can iterate the result of Remark 10 by applying this result to each \( f_j \) in place of \( f \), and so on. We then obtain the following uniformity. For each \( N \geq 1 \), let \( I_N \) denote the collection of \( N \)-tuples \( \beta = (\beta_1, \ldots, \beta_N) \) with \( 1 \leq \beta_j \leq n \) for all \( j \). For \( \beta \in I_N \), let \( X_\beta = X_{\beta_1} \cdots X_{\beta_N} \). Note \( \#I_N = n^N \). Then there is a linear map \( S^{(N)} : \hat{S}_o \to (\hat{S}_o)^{n^N} \), such that

(i) if \( S^{(N)} f = (f_\beta)_{\beta \in I_N} := (S^{(N)}_\beta f)_{\beta \in I_N} \), then \( f(x) = \sum_{\beta \in I_N} X_\beta f_\beta(x) \); and

(ii) for every \( I \) there exist \( C \) and \( J \) such that for all \( \beta \in I_N \), we have

\[
\|S^{(N)}_\beta f\|_{S,I} \leq C \|f\|_{S,J}.
\]

3. Characterization of \( \mathcal{M}\mathcal{U}(G) \)

Proposition 13. If \( m \in \mathcal{M}\mathcal{U}(G) \), there exists a bounded sequence \( \{\psi_k\}_{k=0}^\infty \subset S(G) \), with \( \psi_k \in \hat{S}_o \) for \( k \geq 1 \), such that, if we set

\[
m_k = 2^{jk} \psi_k \circ \delta_{2^{-k}},
\]

then \( m = \sum_{k=0}^\infty m_k \), with convergence both pointwise and in \( S' \).

Proof. Choose a sequence of functions \( \{\varphi_k\}_{k=0}^\infty \subset C^\infty_c(G) \), such that:

(i) \( \sum \varphi_k = 1 \);

(ii) \( 0 \leq \varphi_k \leq 1 \) for all \( k \);

(iii) \( \text{supp} \varphi_0 \subseteq \{\xi : |\xi| \leq 2\} \);

(iv) for all \( k \geq 1 \), \( \text{supp} \varphi_k \subseteq \{\xi : 2^{k-2} \leq |\xi| \leq 2^{k+1}\} \); and
(v) for all \( k \geq 1 \), \( \varphi_k = \varphi_1 \circ \delta_{2-k+1} \).

(For example, we could choose a smooth function \( \varphi_0 \) with \( 0 \leq \varphi_0 \leq 1 \), with \( \varphi_0 = 1 \) in \( \{ \xi : |\xi| \leq \frac{1}{2} \} \) and with \( \text{supp} \varphi_0 \subseteq \{ \xi : |\xi| \leq 2 \} \). Then we could let \( \varphi_1 = \varphi_0 \circ \delta_{1/2} - \varphi_0 \), \( \varphi_k = \varphi_1 \circ \delta_{2-k+1} \) for \( k > 1 \).)

Let \( \varphi = \varphi_1 \circ \delta_2 \), so that \( \varphi_k = \varphi \circ \delta_{2-k} \) for all \( k \geq 1 \), and so that \( \text{supp} \varphi \subseteq \{ \xi : \frac{1}{2} \leq |\xi| \leq 2 \} \).

We set \( m_k = \varphi_k m \) for \( k \geq 0 \), so that \( m = \sum_{k=0}^{\infty} m_k \), with pointwise convergence (at each point in \( \mathbb{R}^n \), only finitely many terms are nonzero.) (This is entirely analogous to the definition used on page 246 of [Ste93], for standard multipliers \( m \).) We also define \( \psi_k \) so that (3.1) (and hence (3.2)) holds; thus \( \psi_k = 2^{-j_k} m_k \circ \delta_{2^k} \), and we therefore have

\[
\psi_0 = \varphi_o (m \circ \delta_{2^k}) \\
\psi_k = 2^{-j_k} \varphi (m \circ \delta_{2^k}) \quad \text{for } k \geq 1.
\]

Since \( \psi_k \in C^\infty_\circ \), and vanishes in a neighborhood of the origin if \( k \geq 1 \), surely \( \check{\psi}_k \in \mathcal{S}(G) \) for all \( k \), and \( \check{\psi}_k \in \mathcal{S}_o(G) \) for \( k \geq 1 \).

Note also that \( \text{supp} \psi_k \subseteq \{ \xi : |\xi| \leq 2 \} \) for all \( k \). Thus, to show that \( \{ \psi_k \}_{k=0}^{\infty} \) is a bounded subset of \( \mathcal{S}(G) \), we need only show that for any multiindex \( \alpha \), \( \{ ||\partial^\alpha \psi_k||_\infty \}_{k=1}^{\infty} \) is a bounded sequence, where the sup norm is taken over \( \{ \xi : |\xi| \leq 2 \} \). But for \( k \geq 1 \), \( |\xi| \leq 2 \), we have by Leibniz’s rule that

\[
|\partial^\alpha \psi_k(\xi)| \leq C_1 \sum_{|\beta| \leq |\alpha|} 2^{-j_k}C_{|\beta|} 2^{|\beta|} |(\partial^\beta m)(\delta_{2^k}(\xi))|
\leq C_2 \sum_{|\beta| \leq |\alpha|} 2^{-j_k-j|\beta|} (1 + 2^k|\xi|)^{-j|\beta|}
= C_2 \sum_{|\beta| \leq |\alpha|} \left[ \frac{2^k}{(1 + 2^k|\xi|)} \right]^{-j|\beta|}
\leq C_\alpha
\]
as claimed.

Finally, note that, if \( k \geq 1 \), then for any \( \xi \in \mathbb{R}^n \), \( m_k(\xi) \neq 0 \) for at most 3 values of \( k \). Moreover \( |m_k(\xi)| = |\varphi_k(\xi)||m(\xi)| \leq C(1 + |\xi|)^j \). Thus \( \sum_k |m_k(\xi)| \leq C(1 + |\xi|)^j \) also. Since \( \sum_k m_k \) converges to \( m \) pointwise, it now follows that the convergence is in \( \mathcal{S}'(G) \) as well.

Next we show that the converse of the previous proposition also holds. We will also obtain more information about the senses in which \( \sum_k m_k \) converges to \( m \).

**Proposition 14.** (a) Suppose \( \{ \psi_k \}_{k=0}^{\infty} \subseteq \mathcal{S}(G) \) is a bounded sequence, with \( \psi_k \in \mathcal{S}_o \) for \( k \geq 1 \). Set

\[
m_k = 2^{jk} \psi_k \circ \delta_{2^{-k}}
\]

so that \( \check{m}_k = 2^{(j+Q)k} \check{\psi}_k \circ \delta_{2^k} \).
then $\sum_{k=0}^{\infty} m_k$ converges, in $C^\infty(G)$, and in $S'$, to an element $m$ of $M^j(G)$.

(b) In fact, we have the following uniformity. For every $I$ there exist $C$ and $J$ such that whenever $\{\psi_k\}_{k=0}^{\infty} \subset S(G)$ is a bounded sequence, with $\psi_k \in \hat{S}_o$ for $k \geq 1$, if we set $m_k = 2^{jk} \psi_k \circ \delta_{2^{-k}}$, then

$$\left\| \sum_{k=0}^{\infty} m_k \right\|_{M^j, I} \leq C \sup_k \|\psi_k\|_{S, J}.$$ 

(c) We also have the following uniformity. Say $j' > j$. Then, for every $I$ there exist $C$ and $J$ such that whenever $\{\psi_k\}_{k=0}^{\infty} \subset S(G)$ is a bounded sequence, with $\psi_k \in \hat{S}_o$ for $k \geq 1$, if we set $m_k = 2^{jk} \psi_k \circ \delta_{2^{-k}}$, then

$$\left\| \sum_{k=0}^{\infty} m_k \right\|_{M^{j'}, I} \leq C \sup_k 2^{-(j'-j)k} \|\psi_k\|_{S, J}.$$ 

(d) In (a), the series $\sum_{k=0}^{\infty} m_k$ converges to $m$ in $M^j(G)$, for any $j' > j$.

Proof. (a) Let us begin by showing that

$$\sum_k |m_k(\xi)| = \sum_k 2^{jk} |\psi_k(\delta_{2^{-k}}(\xi))| \leq C (1 + |\xi|)^j, \quad (*)$$

with uniform convergence on compact sets; from this, the convergence of $\sum_{k=0}^{\infty} m_k$, both pointwise and in $S'$, will be automatic.

First let us establish (II) for $|\xi| > 1$. In that case, we define $k_0 \geq 1$ to be the unique integer such that $2^{k_0-1} < |\xi| \leq 2^{k_0}$. We consider separately those terms in $\sum_k 2^{jk} |\psi_k(\delta_{2^{-k}}(\xi))|$ with $k < k_0$ and with $k \geq k_0$. Choose $N \in \mathbb{Z}^+$ such that $j + N > 0$. Since $\{\psi_k\}$ is a bounded subset of $S(G)$,

$$\sum_{k<k_0} 2^{jk} |\psi_k(\delta_{2^{-k}}(\xi))| \leq C_1 \sum_{k<k_0} 2^{jk} (2^{-k} |\xi|)^{-N}$$

$$= C_1 |\xi|^{-N} \sum_{k<k_0} 2^{(j+N)k}$$

$$\leq C_2 |\xi|^{-N} 2^{(j+N)k_0} \quad \text{since } j + N > 0$$

$$\leq C_3 |\xi|^j$$

$$\leq \tilde{C}_1 (1 + |\xi|)^j \quad \text{since } |\xi| > 1.$$ 

Here $\tilde{C}_1$ may be chosen independently of $\xi$. Note also that, for any fixed $k_0$, the convergence of $\sum_{k<k_0} 2^{jk} |\psi_k(\delta_{2^{-k}}(\xi))|$ is uniform for $2^{k_0-1} < |\xi| \leq 2^{k_0}$.

Next, by Remark [II] for every $k \geq 1$ and for every $N \in \mathbb{Z}^+$ we may write $\psi_k(\xi) = \sum_{\|\alpha\|=N} \xi^\alpha \psi_{k,\alpha}(\xi)$, with $\psi_{k,\alpha} \in S(G)$, so that for any $N$, $\{\psi_{k,\alpha} : k \geq 1, \|\alpha\| = N\}$ is a bounded subset of $S(G)$.  

Now choose $N > j$. Recall $k_0 \geq 1$. We have

$$\sum_{k \geq k_0} 2^j |\psi_k(\delta_{2^{-k}}(\xi))| \leq C_1 \sum_{k \geq k_0} 2^j \left( \sum_{|\alpha| = N} |\delta_{2^{-k}}(\xi)|^{|\alpha|} |\psi_{k, \alpha}(\delta_{2^{-k}}(\xi))| \right)$$

$$\leq C_2 \sum_{k \geq k_0} 2^j \left( \sum_{|\alpha| = N} 2^{-k|\alpha|} |\xi|^{|\alpha|} \right)$$

$$= C_3 \sum_{|\alpha| = N} |\xi|^{|\alpha|} \sum_{k \geq k_0} 2(j-|\alpha|)k$$

(3.4)

$$\leq C_4 \sum_{|\alpha| = N} |\xi|^{|\alpha|} 2(j-|\alpha|)k_0$$

since $|\alpha| = N \Rightarrow j - |\alpha| \leq j - N < 0$

$$\leq C_5 |\xi|^j$$

$$\leq C_2 (1 + |\xi|)^j \text{ since } |\xi| > 1.$$

Here again $C_2$ may be chosen independently of $\xi$. Note also that, for any fixed $k_0$, the convergence of $\sum_{k < k_0} 2^j |\psi_k(\delta_{2^{-k}}(\xi))|$ is uniform for $2^{k_0-1} < |\xi| \leq 2^{k_0}$. Therefore by (3.3) and (3.4), we have (3) for $|\xi| > 1$, with uniform convergence on bounded sets.

Now if $|\xi| \leq 1$, using a similar argument as in (3.4), and again choosing $N > j$, we have

$$\sum_{k} 2^j |\psi_k(\delta_{2^{-k}}(\xi))| = |\psi_0(\xi)| + \sum_{k=1}^{\infty} 2^j |\psi_k(\delta_{2^{-k}}(\xi))|$$

$$\leq C_1 + C_1 \sum_{k} 2^j \left( \sum_{|\alpha| = N} |\delta_{2^{-k}}(\xi)|^{|\alpha|} |\psi_{k, \alpha}(\delta_{2^{-k}}(\xi))| \right)$$

$$\leq C_1 + C_2 \sum_{k=1}^{\infty} 2^j \left( \sum_{|\alpha| = N} 2^{-k|\alpha|} |\xi|^{|\alpha|} \right)$$

$$= C_1 + C_2 \sum_{|\alpha| = N} \sum_{k=1}^{\infty} 2(j-|\alpha|)k$$

since $|\xi| \leq 1$

$$\leq C \text{ since } |\alpha| = N \Rightarrow j - |\alpha| \leq j - N < 0.$$

Here $C$ may be chosen independently of $\xi$ (for $|\xi| \leq 1$). Also the convergence is uniform for $|\xi| \leq 1$. This establishes (3), with uniform convergence on compact sets.

Finally, let $\alpha$ be a multiindex. Then

$$\partial^\alpha m_k = 2^j (j-|\alpha|)k (\partial^\alpha \psi_k) \circ \delta_{2^{-k}}.$$

Now, the sequence $\{\partial^\alpha \psi_k\}_{k=0}^\infty$ is a bounded subset of $\mathcal{S}(G)$, and $\partial^\alpha \psi_k \in \mathcal{S}_0$ for $k \geq 1$. Consequently, by (3), the series $\sum_k |\partial^\alpha m_k(\xi)|$ converges uniformly on compact sets,
and there exists a constant $C_\alpha$ such that

$$\sum_k |\partial^\alpha m_k(\xi)| \leq C_\alpha (1 + |\xi|^{j-|\alpha|})$$

for all $\xi$.

Accordingly, the series $\sum_k m_k$ converges in $C^\infty(G)$, and in $S'(G)$, to an element $m \in \tilde{\mathcal{M}}^j(G)$. This proves (a).

Part (b) follows from an examination of the proof of part (a) (and from the uniformity in Remark 11).

Part (c) follows at once from (b), if we set $\psi'_k = 2^{j''} \psi_k \circ \delta_{2^{-k}}$, note that $\{\psi'_k\}_{k=0}^\infty \subset S(G)$ is a bounded sequence, with $\psi'_k \in \hat{S}_o$ for $k \geq 1$, and also note that $m_k = 2^{j''} \psi'_k \circ \delta_{2^{-2k}}$.

Finally, part (d) follows at once from (c). \qed

We show now how the characterization of $\tilde{\mathcal{M}}^j(G)$ follows from the previous two propositions.

**Proposition 15.** Say $K \in S'(G)$. Then $K \in \tilde{\mathcal{M}}^j(G)$ if and only if we may write

$$K = \sum_{k=0}^\infty f_k,$$  \quad (3.5)

with convergence in $S'(G)$, where

$$f_k = 2^{(j+Q)k} \varphi_k \circ \delta_{2^k}$$  \quad (3.6)

where $\{\varphi_k\}_{k=0}^\infty \subset S(G)$ is a bounded sequence, and $\varphi_k \in S_o(G)$ for $k \geq 1$. Further, given such a sequence $\{\varphi_k\}_{k=0}^\infty$, if we define $f_k$ by (3.6), then the series $\sum_{k=0}^\infty f_k$ necessarily converges in $S'(G)$ to an element of $K$ of $\tilde{\mathcal{M}}^j(G)$.

Moreover, $\tilde{\mathcal{M}}^j(G) \subseteq E' + S(G)$, and any $K \in \tilde{\mathcal{M}}^j(G)$ is smooth away from the origin. Further, if $K \in \tilde{\mathcal{M}}^j(G)$ is as in (3.5), and if $\zeta \in C^\infty_c(G)$ equals 1 in a neighborhood of 0, then

$$(1 - \zeta)K = \sum_{k=0}^\infty (1 - \zeta)f_k,$$  \quad (3.7)

where the sum converges absolutely in $S(G)$.

**Proof.** The assertions of the first paragraph are immediate from the previous two propositions. Thus, if $K \in \tilde{\mathcal{M}}^j(G)$ is as in (3.5), we have (3.7), with convergence in $S'(G)$; we need to show the sum converges absolutely in $S(G)$. For this, by Leibniz’s rule, we only need show that for every $I, J \geq 0$, every multiindex $\alpha$, every $c > 0$, and every $r > 0$, there exists $C > 0$ such that

$$\sum_{k=0}^\infty 2^{Ik} |x|^I |\partial^\alpha \varphi_k(\delta_{2^k}(x))| \leq C$$

whenever $|x| > r$. 

Remark 17. One may think of Lemma 16 very crudely in the following manner: from the perspective of \( \Phi^j_\nu \), \( 2^{-\sigma Q} \Phi^j_\nu \) looks like a good approximation to the delta “function”, so convolving with it gives one back something akin to \( \Phi^j_\nu \).

4. Convolving Elements of the \( \mathcal{N}^j(G) \)

For \( \sigma \in \mathbb{Z}, I > 0 \), define

\[ \Phi^I_\sigma(x) = (1 + 2^\sigma |x|)^{-I}. \]

We then have the following key fact. The statement and proof have been adapted from Lemma 3.3 of [FJ85] where only the case of Euclidean space is dealt with.

**Lemma 16.** Say \( J > 0 \), and let \( I = J + Q \). Then there exists \( C > 0 \) such that whenever \( \sigma \geq \nu \),

\[ \Phi^I_\sigma * \Phi^J_\nu \leq C 2^{-\sigma Q} \Phi^J_\nu. \]

**Proof.** We note that

\[
\Phi^I_\sigma * \Phi^J_\nu(x) = \int_{|y| \leq |x|/2} \Phi^I_\sigma(y) \Phi^J_\nu(y^{-1}x) dy + \int_{|y| \geq |x|/2} \Phi^I_\sigma(y) \Phi^J_\nu(y^{-1}x) dy
\]

\[ := A + B. \]

In \( A \) we note that, by (3.1), \( \Phi^J_\nu(y^{-1}x) \leq C \Phi^J_\nu(x) \), so

\[ A \leq C \Phi^J_\nu(x) \int_G \Phi^I_\sigma(y) dy = C 2^{-\sigma Q} \Phi^J_\nu(x), \]

since \( I > Q \).

In \( B \), we just estimate \( \Phi^J_\nu(y^{-1}x) \leq 1 \). Consider first the case \( 2^\nu |x| \leq 1 \). Then

\[ B \leq \int_G \Phi^I_\sigma(y) dy = C 2^{-\sigma Q} \leq C 2^{-\sigma Q} \Phi^J_\nu(x). \]

Finally, if, instead, \( 2^\nu |x| \geq 1 \), we have

\[
B \leq \int_{|y| \geq |x|/2} \Phi^I_\sigma(y) dy \leq \int_{|y| \geq |x|/2} (2^\sigma |y|)^{-I} dy
\]

\[ = C 2^{-\sigma I} |x|^{-I+Q} = C 2^{-\sigma Q} (2^\sigma |x|)^{-J}. \]

Thus, since \( \sigma \geq \nu \),

\[ B \leq C 2^{-\sigma Q} (2^\sigma |x|)^{-J} \leq C 2^{-\sigma Q} (2^\nu |x|)^{-J} \leq C 2^{-\sigma Q} \Phi^J_\nu(x), \]

as desired. \( \square \)

**Remark 17.**
Lemma 18. Suppose \( L \geq 0 \) is an integer, and that \( \mathcal{B} \subseteq S(G) \) is bounded. Then there is a bounded subset \( \mathcal{B}' \subseteq S(G) \) as follows. Say \( k, l \in \mathbb{Z}, k \geq l \). Suppose that \( \varphi \in \mathcal{S}_o(G) \cap \mathcal{B} \), and that \( \psi \in \mathcal{B} \). Define \( w_{k,l} \in \mathcal{S}_o(G) \) by
\[
 w_{k,l} \circ \delta_2^l = 2^{kQ}(\varphi \circ \delta_2^k) * (\psi \circ \delta_2^l).
\] (4.1)

Then \( 2^{(k-l)L}w_{k,l} \in \mathcal{B}' \).

Proof. This proof has three steps.

**STEP 1:** Say \( J > 0 \), and let \( I = J + Q \). Then by Lemma 16
\[
|w_{k,l} \circ \delta_2^l| \leq C2^{kQ}\Phi_k^I \Phi_l^J \leq C\Phi_0^I \circ \delta_2^l,
\]
so \( |w_{k,l}| \leq C\Phi_0^I \). In other words,
\[
(1 + |x|)^J w_{k,l}(x) \leq C.
\]

**STEP 2:** Notation as in Remark 12, say \( \beta \in \mathcal{I}_N \), and let \( r = \sum_{j=1}^{N} a_{\beta_j} \). Applying \( X_\beta \) to both sides of (4.1), we see that
\[
2^J(X_\beta w_{k,l}) \circ \delta_2^l = 2^{kQ}(\varphi \circ \delta_2^k) * [2^J(X_\beta \psi) \circ \delta_2^l],
\]
so that
\[
(X_\beta w_{k,l}) \circ \delta_2^l = 2^{kQ}(\varphi \circ \delta_2^k) * [(X_\beta \psi) \circ \delta_2^l].
\]
Note that \( X_\beta \psi \in X_\beta \mathcal{B} \), a bounded subset of \( S(G) \). Thus, by Step 1, for any \( J > 0 \) and any \( \beta \), we have
\[
(1 + |x|)^J (X_\beta w_{k,l})(x) \leq C_{J,\beta}.
\]
This proves the lemma in the case \( L = 0 \).

**STEP 3:** Finally, suppose \( L \geq 1 \). By Remark 12, we may assume that for some bounded subset \( \mathcal{B}_L \) of \( S(G) \) (depending only on \( \mathcal{B} \) and \( L \)), \( \varphi = X_X \varphi_1 \), where \( X_\beta \in \mathcal{I}_L \), and \( \varphi_1 \in \mathcal{S}_o(G) \cap \mathcal{B}_L \). Set \( Y_\beta = Y_{\beta_1} \cdots Y_{\beta_l} \). Also set \( r = \sum_{j=1}^{L} a_{\beta_j} \geq L \). Now
\[
\varphi \circ \delta_2^k = (X_\beta \varphi_1) \circ \delta_2^k = 2^{-rk}X_\beta(\varphi_1 \circ \delta_2^k),
\]
so
\[
w_{k,l} \circ \delta_2^l = 2^{kQ}[2^{-rk}X_\beta(\varphi_1 \circ \delta_2^k)] * (\psi \circ \delta_2^l)
\]
\[
= 2^{kQ}2^{-rk}(\varphi_1 \circ \delta_2^k) * Y_\beta(\psi \circ \delta_2^l)
\]
\[
= 2^{-(k-l)r}2^{kQ}(\varphi_1 \circ \delta_2^k) * [(Y_\beta \psi) \circ \delta_2^l].
\]
Since \( r \geq L \), we therefore have that
\[
(2^{(k-l)L}w_{k,l}) \circ \delta_2^l = c2^{kQ}(\varphi_1 \circ \delta_2^k) * [(Y_\beta \psi) \circ \delta_2^l],
\]
where \( 0 < c \leq 1 \). Note also that \( Y_\beta \psi \in Y_\beta \mathcal{B} \), a bounded subset of \( S(G) \). By the case \( L = 0 \) of the lemma, established in Step 2, we see that there is a bounded subset \( \mathcal{B}' \) of \( S(G) \), depending only on \( \mathcal{B} \) and \( L \), such that \( 2^{(k-l)L}w_{k,l} \in \mathcal{B}' \).

We now reformulate Lemma 18 in a manner that permits us to deal with the convolution of sums like that in (3.2).
Corollary 19. Suppose \( L \geq 0 \) is an integer, and that \( \mathcal{B} \subseteq \mathcal{S}(G) \) is bounded. Then there is a bounded subset \( \mathcal{B}' \subseteq \mathcal{S}(G) \) as follows. Say \( k, l \in \mathbb{Z}, k \geq l, \) and that \( j_1, j_2 \) are real numbers. Suppose that \( \varphi \in \mathcal{S}_0(G) \cap \mathcal{B}, \) and that \( \psi \in \mathcal{B} \). Define \( f \in \mathcal{S}_0(G), g \in \mathcal{S}(G), \eta_{k,l}, \eta'_{k,l} \in \mathcal{S}_0(G) \) by

\[
\begin{align*}
f &= 2^{(j_1+Q)k} \varphi \circ \delta_{2^k} \\
g &= 2^{(j_2+Q)l} \psi \circ \delta_{2^l} \\
f * g &= 2^{(j_1+j_2+Q)l} \eta_{k,l} \circ \delta_{2^l} \\
g * f &= 2^{(j_1+j_2+Q)l} \eta'_{k,l} \circ \delta_{2^l}
\end{align*}
\]

Then \( 2^{(k-l)(L-j_1)} \eta_{k,l} \in \mathcal{B}', \) and also \( 2^{(k-l)(L-j_1)} \eta'_{k,l} \in \mathcal{B}' \).

Proof. We have \( 2^{-(k-l)j_1} \eta_{k,l} = w_{k,l}, \) where \( w_{k,l} \) is as in (4.1). Thus the first statement follows at once from Lemma 18. For the second statement, we need only apply the first statement to \( \tilde{f}, \tilde{g} \) in place of \( f, g, \) and then use the fact that \( g * f = (f * \tilde{g})^\circ. \) \( \square \)

We are almost ready to prove our main result, that \( \mathcal{M}^{j_1} \ast \mathcal{M}^{j_2} \subseteq \mathcal{M}^{j_1+j_2} \). Before we begin the proof, it will be helpful to make some preliminary observations.

Say \( K_1 \in \mathcal{M}^{j_1}(G), K_2 \in \mathcal{M}^{j_2}(G). \) By Proposition 15, \( K_1, K_2 \in \mathcal{E}' + \mathcal{S}(G), \) which is a convolution algebra; thus it is possible to form \( K_1 \ast K_2. \) (Of course we cannot convolve two general elements of \( \mathcal{S}'(G). \)) To see that \( \mathcal{E}' + \mathcal{S}(G) \) is a convolution algebra, say \( F, G \in \mathcal{E}' + \mathcal{S}(G). \) To convolve them we simply write \( F = u + f, H = v + g, \) where \( u, v \in \mathcal{E}' \) and \( f, g \in \mathcal{S}(G) \); and then we define

\[
F \ast H = u \ast v + u \ast g + f \ast v + f \ast g.
\]

The first term on the right side is in \( \mathcal{E}' \) and the other terms are in \( \mathcal{S}(G), \) so \( F \ast H \) is in \( \mathcal{E}' + \mathcal{S}(G). \) It is easy to verify that this definition of \( F \ast H \) is independent of how one chooses to decompose \( F \) in the form \( u + f, \) or \( H \) in the form \( v + g. \)

It is also evident from (4.2) that if \( u_N \rightarrow u \) and \( v_N \rightarrow v \) in \( \mathcal{E}' \), while \( f_N \rightarrow f \) and \( g_N \rightarrow g \) in \( \mathcal{S}(G), \) then

\[
(u_N + f_N) \ast (v_N + g_N) \longrightarrow (u + f) \ast (v + g)
\]

in \( \mathcal{S}'(G). \)

We can now prove our main theorem:

Theorem 20. If \( j_1, j_2 \) are real numbers then \( \mathcal{M}^{j_1}(G) \ast \mathcal{M}^{j_2}(G) \subseteq \mathcal{M}^{j_1+j_2}(G) \)

Proof. Say \( K_1 \in \mathcal{M}^{j_1}(G), \) and \( K_2 \in \mathcal{M}^{j_2}(G). \) As in Proposition 13 we may write

\[
K_1 = \sum_{k_1=0}^{\infty} f_{k_1}, \quad K_2 = \sum_{k_2=0}^{\infty} g_{k_2},
\]

(convergence in \( \mathcal{S}'(G) \)), where

\[
f_{k_1} = 2^{(j_1+Q)k_1} \varphi_{k_1} \circ \delta_{2^{k_1}}, \quad g_{k_2} = 2^{(j_2+Q)k_2} \psi_{k_2} \circ \delta_{2^{k_2}};
\]

where \( \{ \varphi_k \} \) and \( \{ \psi_k \} \) are bounded sequences in \( \mathcal{S}(G), \) and \( \varphi_k, \psi_k \in \mathcal{S}_0(G) \) for \( k \geq 1. \)
We want to write $K_1 * K_2$ in the form $\sum_k 2^{j_1 + j_2 + Q} k \nu_k \circ \delta_{2k}$ (convergence in $S'(G)$), where $\{\nu_k\}$ is a bounded sequence in $S(G)$, and where $\nu_k \in S_o(G)$ for $k \geq 1$. Then we will know $K_1 * K_2 \in M^{j_1 + j_2}(G)$ by Proposition [14]

To this end we examine $f_{k_1} \ast g_{k_2} := h_{k_1,k_2}$. Formally, $K_1 * K_2 = \sum_{k_1,k_2} h_{k_1,k_2}$; we intend to write the double summation here as a sum of two double summations, one for $k_2 \geq k_1$, and one for $k_1 > k_2$. Note that, to say that $k_2 \geq k_1$, is to say that either $k_2 \geq \max(k_1,1)$, or $(k_1, k_2) = (0,0)$.

Pick $L > \max(j_1, j_2)$. By Corollary [19] we have

$$h_{k_1,k_2} = 2^{j_1 + j_2 + Q} \tau_{k_1,k_2} \circ \delta_{2k_1} \quad \text{for } k_2 \geq \max(k_1,1)$$

$$h_{k_1,k_2} = 2^{j_1 + j_2 + Q} \eta_{k_1,k_2} \circ \delta_{2k_2} \quad \text{for } k_1 > k_2,$$

where $\{2^{(k_2 - k_1)(L-j_2)} \tau_{k_1,k_2}\}_{k_2 \geq \max(k_1,1)}$ and $\{2^{(k_1 - k_2)(L-j_1)} \eta_{k_1,k_2}\}_{k_1 > k_2}$ are bounded subsets of $S_o(G)$.

Formally, then,

$$K_1 * K_2 = h_{o,o} + \sum_{k_2 \geq \max(k_1,1)} h_{k_1,k_2} + \sum_{k_1 > k_2} h_{k_1,k_2} \quad (4.4)$$

$$= h_{o,o} + \sum_{k_1=0}^{\infty} \sum_{k_2 = \max(k_1,1)}^{\infty} h_{k_1,k_2} + \sum_{k_2 = 0}^{\infty} \sum_{k_1 = k_2 + 1}^{\infty} h_{k_1,k_2} \quad (4.5)$$

$$= h_{o,o} + \sum_{k_1=0}^{\infty} u_{k_1} + \sum_{k_2=0}^{\infty} v_{k_2} \quad (4.6)$$

where

$$u_{k_1} = 2^{j_1 + j_2 + Q} k_1 \tau_{k_1} \circ \delta_{2k_1} \quad v_{k_2} = 2^{j_1 + j_2 + Q} k_2 \eta_{k_2} \circ \delta_{2k_2}$$

and where

$$\tau_{k_1} = \sum_{k_2 = \max(k_1,1)}^{\infty} \tau_{k_1,k_2} \quad \eta_{k_2} = \sum_{k_1 = k_2 + 1}^{\infty} \eta_{k_1,k_2} \quad (4.7)$$

The point is that these series converge absolutely in $S(G)$, and that $\{\tau_{k_1}\}$, $\{\eta_{k_2}\}$ are bounded sequences in $S_o(G)$. (Once this is verified, we will know, at least formally, that $K_1 * K_2 \in M^{j_1 + j_2}(G)$.) But this point is easily verified. Let $\|\cdot\|_{S,M}$ be any member of the family of norms defining the topology of $S(G)$. Then, for instance in the first series of (4.7) we have

$$\sum_{k_2 = \max(k_1,1)}^{\infty} \|\tau_{k_1,k_2}\|_{S,M} \leq C_M \sum_{k_2 = \max(k_1,1)}^{\infty} 2^{-(k_2 - k_1)(L-j_2)}$$

$$\leq C_M \sum_{k=0}^{\infty} 2^{-k(L-j_2)}$$

$$= C_M' \quad (4.8)$$

where \( C_M \) and \( C'_M \) are independent of \( k_1 \). Similar considerations apply to the other series. Of course the fact that \( \tau_{k_1}, \eta_{k_2} \in \mathcal{S}_0(G) \) is ensured by the absolute convergence of the series in \( \mathcal{S}(G) \) in (4.7).

Therefore we only need verify that \( K_1 \ast K_2 \) is in fact equal to (4.6), as elements of \( \mathcal{S}'(G) \). By Proposition 15, we do know that (4.6) converges in \( \mathcal{S}'(G) \).

Let us first show that
\[
\left( \sum_{k_1=0}^{N} f_{k_1} \right) \ast \left( \sum_{k_2=0}^{N} g_{k_2} \right) \longrightarrow K_1 \ast K_2
\]
in \( \mathcal{S}'(G) \), as \( N \to \infty \). Indeed, choose \( \zeta \in C_c^\infty (G) \) with \( \zeta = 1 \) near 0. We may write
\[
\sum_{k_1=0}^{N} f_{k_1} = \zeta \left( \sum_{k_1=0}^{N} f_{k_1} \right) + (1 - \zeta) \left( \sum_{k_1=0}^{N} f_{k_1} \right).
\]
As \( N \to \infty \), we have that
\[
(1 - \zeta) \left( \sum_{k_1=0}^{N} f_{k_1} \right) \to (1 - \zeta) K_1
\]
in \( \mathcal{S}(G) \). Similar considerations apply to \( \sum_{k_2=0}^{N} g_{k_2} \). Thus (4.9) follows from (4.3).

Thus we need only show that \( \left( \sum_{k_1=0}^{N} f_{k_1} \right) \ast \left( \sum_{k_2=0}^{N} g_{k_2} \right) \) approaches (4.6) in \( \mathcal{S}'(G) \), as \( N \to \infty \).

Let us make the convention that any sum of the form \( \sum_{k=i}^{j} \) is zero if \( j < i \). In place of (4.5) and (4.6), we then rigorously have that
\[
\left( \sum_{k_1=0}^{N} f_{k_1} \right) \ast \left( \sum_{k_2=0}^{N} g_{k_2} \right) = h_{a,o} + \sum_{k_1=0}^{N} h_{k_1,k_2} + \sum_{k_2=0}^{N} h_{k_1,k_2}
\]
\[
= h_{a,o} + \sum_{k_1=0}^{N} u_{k_1} + \sum_{k_2=0}^{N} v_{k_2},
\]
where
\[
u_{k_1} = 2(j_1+j_2+Q)k_1 \tau_{k_1} \circ \delta_{2k_1} \quad \nu_{k_2} = 2(j_1+j_2+Q)k_2 \eta_{k_2} \circ \delta_{2k_2}
\]
and where
\[
\tau_{k_1} = \sum_{k_2=\max(k_1,1)}^{N} \tau_{k_1,k_2} \quad \eta_{k_2} = \sum_{k_1=\max(k_1,1)}^{N} \eta_{k_1,k_2}.
\]

The argument of (4.8) shows that \( \tau_{k_1} N \) and \( \eta_{k_2} N \) are bounded subsets of \( \mathcal{S}_0(G) \), and moreover that \( \tau_{k_1} N \to \tau_{k_1} \) in \( \mathcal{S}(G) \) for each \( k_1 \), and \( \eta_{k_2} N \to \eta_{k_2} \) in \( \mathcal{S}(G) \) for each \( k_2 \), as \( N \to \infty \).
To complete the proof, we need to show that
\[ \sum_{k_1=0}^{N} u_{k_1} - \to \infty \sum_{k_1=0}^{\infty} u_{k_1} \]
in \( S'(G) \) and that
\[ \sum_{k_2=0}^{N} \psi_{k_2} - \to \infty \sum_{k_2=0}^{\infty} \psi_{k_2} \]
in \( S'(G) \).

Let us prove the first of these; the proof of the second is virtually identical.

We work on the Fourier transform side, and we write \( k \) in place of \( k_1 \) for simplicity.

Set \( m_k = \hat{u}_k \) and \( m_k^N = \hat{u}_k^N \). We need to show that
\[ \sum_{k=0}^{N} m_k^N - \to \infty \sum_{k=0}^{\infty} m_k \]
in \( S'(G) \).

Set \( \psi_k = \hat{\tau}_k \) and \( \psi_k^N = \hat{\tau}_k^N \). We have that
\[ m_k^N = 2(j_1 + j_2) k \psi_k^N \circ \delta_{2-k} \]
\[ m_k = 2(j_1 + j_2) k \psi_k \circ \delta_{2-k} \]
that \( \{ \psi_k^N \}_{k,N} \) and \( \{ \psi_k \}_k \) are bounded subsets of \( \hat{S}_\alpha \), and moreover that \( \psi_k^N \to \psi_k \) in \( \mathcal{S}(G) \) for each \( k \), as \( N \to \infty \).

Proposition 14 (c) now implies at once that
\[ \sum_{k=0}^{N} m_k^N - \to \infty \sum_{k=0}^{\infty} m_k \]
in \( \mathcal{M}'(G) \) for any \( j' > j_1 + j_2 \). In particular, \( \sum_{k=0}^{N} m_k^N - \to \infty \sum_{k=0}^{\infty} m_k \)
in \( S'(G) \), as desired. \( \square \)

5. PSEUDODIFFERENTIAL CALCULUS ON \( G \)

5.1. INTRODUCTION.

Definition 21. We say that an operator \( A \) in \( \mathcal{S}(\mathbb{R}^n) \) is a Fourier multiplier operator if for a suitable function \( a \)
\[ [Af](x) = \int_{\mathbb{R}^n} e^{-2i\pi x \cdot \xi} a(\xi) \hat{f}(\xi) d\xi \] (5.1)
for \( f \in \mathcal{S}(\mathbb{R}^n) \). The function \( a(\xi) \) is called the multiplier.

Notice that Fourier multiplier operators can be written as convolution operators
\[ [Af](x) = (a \hat{f}) \ast (x) = (\hat{a} * f)(x) \]
where * is here the usual convolution on \( \mathbb{R}^n \). \( S_{\rho,\#}^m \) will be used to denote the space of those Fourier multipliers which are in \( S_{\rho,0}^m \). For \( p \in S_{\rho,\#}^m \), we define a family of seminorms by
\[ \|p\|_{\alpha,m,\rho} = \sup_{\xi} \left\{ \langle \xi \rangle^{-m+\rho} \|D^\alpha_{\xi} p(\xi)\| \right\} \]
where \( \langle \xi \rangle = (1 + \|\xi\|^2)^{1/2} \). Equipped with these seminorms \( S_{\rho,\#}^m \) is a Fréchet space.
Recall that given $J \in \mathcal{M}^j(\mathbb{R}^n)$ the multiplier operator $m_J$ is defined by

$$S' \xrightarrow{m_J} S' \quad f \xrightarrow{J} \hat{f} \ast f$$

Note $\mathcal{M}^j(\mathbb{R}^n) = S^j_{1,\#}$.

Suppose $a(x, D)$ is in $\text{Op} \left( S^m_{\rho,0} \right)$ with symbol $a(x, \xi)$. For any $x \in \mathbb{R}^n$ we define $a_x(\xi) = a(x, \xi)$. If $A(y)$ is the operator of Fourier multiplication by $a_y(\xi)$, then for any $f \in S(\mathbb{R}^n)$

$$\left[ a(x, D)f \right](x) = \int e^{-2\pi i x \cdot \xi} a(x, \xi) \hat{f}(\xi) \, d\xi$$

$$= \int e^{-2\pi i x \cdot \xi} a_x(\xi) \hat{f}(\xi) \, d\xi$$

$$= \left[ A(x)f \right](x) \quad (5.2)$$

Therefore, loosely speaking, one can say that a pseudodifferential operator is a multiplier operator where the multiplier depends smoothly upon the point at which we are.

Since $\left[ A(x)f \right](x) = (\hat{a}_x \ast f)(x)$ then

$$\left[ a(x, D)f \right](x) = (\hat{a}_x \ast f)(x).$$

Therefore, locally one can always represent a pseudodifferential operator on $\mathbb{R}^n$ with symbol in $S_{\rho,0}$, by a smooth family of convolution operators, where one convolves with an element $\hat{a}_x \in (S^m_{\rho,\#})'$. This point of view is useful when working with pseudodifferential operators on homogeneous groups $G$, where the group Fourier transform is cumbersome to use. Taylor shows in [Tay84] that smooth families of convolution operators can also be used to construct certain classes of pseudodifferential operators on $G$. If $G$ is a homogeneous group, instead of Fourier multiplier operators as in (5.1), one considers convolution operators on $G$, defined by

$$Af = a \ast f$$

where $\ast$ is now group convolution. One requires that $a$ belong to some Fréchet space $\mathcal{X}$ of smooth functions on $\mathbb{R}^n$. One assumes that $\mathcal{X} \subseteq S^m_{\rho,\#}$ for some $m \in \mathbb{R}$ and $\rho \in (0, 1]$, and says that $A \in \text{Op}(\mathcal{X})$.

Say now that, instead, $a(y, \xi) \in C^\infty(U \times \mathbb{R}^n)$ where $U \subseteq G$ is open. Set $a_y(\xi) = a(y, \xi)$ and suppose $a_y$ is a smooth function of $y$, taking values in $\mathcal{X}$, for $y \in U$. For $y \in U$, one defines an operator $A(y) : C^\infty_c(G) \rightarrow C^\infty(G)$ by

$$A(y)f = a_y \ast f \quad (5.3)$$

Then for $y \in U$ one defines

$$[\mathfrak{A}f](y) = [A(y)f](y) \quad (5.4)$$
Notice the analogy with (5.2). One denotes the set of such operators by \( \text{Op} (\tilde{X}) \).

When \( a(y, z) \) has compact support in \( y \), we denote the collection of such operators \( \mathcal{A} \) by \( \text{Op}_c (\tilde{X}) \).

The following theorem contains results from Taylor \cite{Tay84}.

**Theorem 22.** Suppose \( G \) is a Lie group, and \( \{X^m\}_{m \in \mathbb{R}} \) is a nested family of Fréchet spaces satisfying the following conditions

a) If \( m \geq 0 \) then \( X^m \subset S^m_{\rho, \#} \) for some \( \rho \in (0, 1] \).

b) If \( m < 0 \) then \( X^m \subset S^m_{\sigma, \#} \) for some \( \sigma \in (0, 1] \).

c) If \( A \in \text{Op} (X^{m_1}) \), and \( B \in \text{Op} (X^{m_2}) \), then \( AB \in \text{Op} (X^{m_1 + m_2}) \), the product being continuous.

d) If \( p(\xi) \in X^m \), then \( D^\alpha_x p(\xi) \in X^{m - \tau \| \alpha \|} \) for some \( \tau \in (0, 1] \).

e) If \( K_j \in X^{m - \tau j} \), then there exists \( K \in X^m \) such that, for any \( M \), if \( N \) is sufficiently large,

\[
K - (K_0 + \cdots + K_N) \in S^{-M}_{\rho, \#}
\]

f) If \( p(\xi) \in X^m \) then \( \bar{p}(\xi) \in X^m \).

Then on \( \text{Op} (\tilde{X}) \) we have a pseudodifferential calculus, of the usual type containing products and adjoints, more specifically

i) If \( \mathfrak{A} \in \text{Op} (X^{m_1}) \) (as in (5.3), (5.4)), and \( \mathfrak{B} \in \text{Op}_c (X^{m_2}) \),

then \( \mathfrak{A} \mathfrak{B} \in \text{Op} (X^{m_1 + m_2}) \), and it has an asymptotic expansion

\[
[\mathfrak{A} \mathfrak{B} f](x) \sim \sum_{\gamma \geq 0} [A^{[\gamma]}(x)B_{[\gamma]}(x)f](x), \tag{5.5}
\]

in the sense that the operator \( \mathfrak{A} \mathfrak{B} = \sum_{\| \gamma \| \leq N} A^{[\gamma]}(x)B_{[\gamma]}(x) \) becomes arbitrarily highly smoothing as \( N \to \infty \). Here the operators \( A^{[\gamma]}(x) \), \( B_{[\gamma]}(x) \) are of the form

\[
A^{[\gamma]}(x)g = \hat{a}^{[\gamma]}_x * g \quad \quad \quad B_{[\gamma]}(x)g = \hat{b}^{[\gamma]}_{x,x} * g
\]

where

\[
a^{[\gamma]}_x (\xi) = (D^\gamma_\xi a)(x, \xi)
\]

and

\[
b^{[\gamma]}_{x,x}(\xi) := b^{[\gamma]}_x (x, \xi) \in C^\infty (\mathbb{R}^n \times \mathbb{R}^n) \text{ and is compactly supported in } x,
\]

and moreover

\[
b^{[\gamma]}_{x,x} \text{ is a smooth function of } x, \text{ with values in } X^{m_2}.
\]

Then by hypothesis \( \mathcal{A} \) of this theorem, \( A^{[\gamma]}(x)B_{[\gamma]}(x) \in \text{Op} (X^{m_1 + m_2 - \tau \| \gamma \|}) \), so that (5.5) is an asymptotic expansion within the \( \text{Op} (\tilde{X}^\mu) \) spaces.
ii) If $\mathfrak{A} \in \text{Op} \left( \tilde{X}^m \right)$ then the adjoint $\mathfrak{A}^* \in \text{Op} \left( \tilde{X}^m \right)$ and it has the following asymptotic expansion

$$[\mathfrak{A}^* f](x) \sim \sum_{\gamma \geq 0} [A^{(\gamma)}(x)f](x)$$

(5.6)

in the same sense as in i). Here the operators $A^{(\gamma)}(x)$ are of the form

$$A^{(\gamma)}(x) = \tilde{a}_x^{(\gamma)} * g$$

where

$$a^{(\gamma)}_x(\xi) = D_\xi^{\gamma} c^{(\gamma)}(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n),$$

and moreover $c^{(\gamma)}_x(\xi)$ is a smooth function of $x$, with values in $\tilde{X}^m$.

Then by hypothesis d) of this theorem, $A^{(\gamma)}(x) \in \text{Op} \left( \tilde{X}^{m - \|\gamma\|} \right)$, so that (5.6) is an asymptotic expansion within the $\text{Op} \left( \tilde{X}^\mu \right)$ spaces.

Proof. The proof can be found in [Tay84].

5.2. PSEUDODIFFERENTIAL CALCULUS ON A HOMOGENEOUS GROUP $G$.

Drawing from results of previous sections, the following theorem shows that in the case of a homogeneous group $G$ the family of spaces $\{M_j^t(G)\}_{j \in \mathbb{R}}$, as in Definition 9, does in fact generate a pseudodifferential calculus, fully analogous to the usual pseudodifferential calculus on Euclidean space, insofar as products and adjoints are concerned. Explicitly, elements of $\text{Op} \left( \tilde{M}^j \right)$ are operators of the form $\mathfrak{A}$, where

$$[\mathfrak{A} f](y) = [\tilde{a}_y * f](y),$$

where $a(x, \xi)$ satisfies, for any compact set $B \subseteq G$, the estimates

$$|D_\xi^\beta D_x^\alpha a(x, \xi)| \leq C_{\alpha, \beta, B} (1 + |\xi|)^{j - |\alpha|} \quad \forall x \in B, \xi \in G.$$

**Theorem 23.** Suppose $G$ is a homogeneous group, with weights $a_1, \ldots, a_n$. Then the family of multiplier spaces $\{M^j(G)\}_{j \in \mathbb{R}}$ satisfies the following properties

a) $\{M^j(G)\}_{j \in \mathbb{R}}$ is a nested family of Fréchet spaces.

b) If $j \geq 0$ then $M^j(G) \subseteq S^m_{\rho, \#}$ for $\rho = \frac{a_1}{a_n}$.

c) If $j < 0$ then $M^j(G) \subseteq S^m_{\rho, \#}$ for $\rho = \frac{a_1}{a_n}$ and $\sigma = \frac{1}{a_n}$.

d) $M^{j_1}(G) * M^{j_2}(G) \subseteq M^{j_1 + j_2}(G)$ and the product is continuous.

e) If $J \in M^j(G)$, and $\alpha$ is a multiindex, then $D^\alpha J \in M^{j - |\alpha|}(G)$. Consequently $D^\alpha J \in M^{-\|\alpha\|}(G)$. 
24 SUSANA CORÉ AND DARYL GELLER

f) Let \( J_i \in \mathcal{M}^{j_i}(G) \), for \( i = 0, 1, 2, \ldots \). Then there exists a \( J \in \mathcal{M}^j(G) \) such that, for any \( M \), if \( N \) is sufficiently large

\[
\left( J - \sum_{i=0}^{N} J_i \right) \in S_{\rho,\#}^{-M} \quad \text{for} \quad \rho = \frac{a_1}{a_n},
\]

g) If \( J \in \mathcal{M}^j(G) \) then \( \tilde{J} \in \mathcal{M}^j(G) \).

And therefore we have on \( G \) a pseudodifferential calculus including products and adjoints, analogous to Hörmander’s \( S_{1,0} \)-pseudodifferential calculus on \( \mathbb{R}^n \). In fact (by e)), in the situation of (5.5), \( A^{[\gamma]}(x)B^{[\gamma]}(x) \in \text{Op} \left( \tilde{\mathcal{M}}^{m_1+m_2-|\gamma|} \right) \), and in the situation of (5.6), \( A^{(\gamma)}(x) \in \text{Op} \left( \tilde{\mathcal{M}}^{m-|\gamma|} \right) \).

Proof. a) This is clear. We also note that if \( j_1 \leq j_2 \), the inclusion map \( \mathcal{M}^{j_1}(G) \subseteq \mathcal{M}^{j_2}(G) \) is continuous.

b) If \( J \in \mathcal{M}^j(G) \), and \( \alpha \in (\mathbb{Z}^+)^n \) then there exists a positive constant \( C_{\alpha} \) such that for all \( \xi \)

\[
|\partial^\alpha J(\xi)| \leq C_{\alpha}(1 + |\xi|)^{j-|\alpha|}
\]

\[
\leq C_{\alpha} \frac{(1 + |\xi|)^j}{(1 + |\xi|^{\|\alpha\|\alpha_1})}
\]

\[
\leq C'_{\alpha} \frac{(1 + |\xi|)^j}{\langle \xi \rangle^{\frac{a_1}{a_n}} \|\alpha\|}
\]

\[
\leq C''_{\alpha} \langle \xi \rangle^j \frac{a_1}{a_n} \|\alpha\|
\]

where for the second and third inequalities we have made use of Lemma 7. Then \( J \in S_{\rho,\#}^{j_{\frac{a_1}{a_n}}} \). Therefore \( \mathcal{M}^j(G) \subseteq S_{\rho,\#}^j \) with \( \rho = \frac{a_1}{a_n} \in (0, 1] \).

c) If \( J \in \mathcal{M}^j(G) \), and \( \alpha \in (\mathbb{Z}^+)^n \), then there exists a positive constant \( C_{\alpha} \) such that for all \( \xi \)

\[
|\partial^\alpha J(\xi)| \leq C_{\alpha}(1 + |\xi|)^{j-|\alpha|}
\]

\[
\leq C'_{\alpha} \frac{(1 + |\xi|)^j}{\langle \xi \rangle^{\|\alpha\|\alpha_1}}
\]

\[
\leq C''_{\alpha} \langle \xi \rangle^{j \frac{1}{a_n}} \frac{a_1}{a_n} \|\alpha\|
\]

where for the second and last inequalities we have made use of Lemma 7. Then \( J \in S_{\rho,\#}^{j_{\frac{1}{a_n}}} \). Therefore \( \mathcal{M}^j(G) \subseteq S_{\rho,\#}^{j_\sigma} \), with \( \sigma = \frac{1}{a_n} \) and \( \rho = \frac{a_1}{a_n} \).

d) In the previous section we established that \( \tilde{\mathcal{M}}^{j_1}(G) + \tilde{\mathcal{M}}^{j_2}(G) \subseteq \tilde{\mathcal{M}}^{j_1+j_2}(G) \) for all \( j_1, j_2 \in \mathbb{R} \).
To prove the continuity of the product we define the following bilinear map. For a fixed pair \( j_1, j_2 \in \mathbb{R} \), we set

\[
T : \mathcal{M}^{j_1}(G) \times \mathcal{M}^{j_2}(G) \rightarrow \mathcal{M}^{j_1+j_2}(G)
\]

\[
(J_1, J_2) \mapsto (\hat{J}_1 \ast \hat{J}_2)\
\]

and will show that \( T \) is continuous.

We consider the following mappings, where “double” arrows are used to denote mapping which are sequentially continuous or separately sequentially continuous (for elementary reasons that we will be discussed soon)

\[
\mathcal{M}^{j_1}(G) \times \mathcal{M}^{j_2}(G) \xrightarrow{T} \mathcal{M}^{j_1+j_2}(G)
\]

Here \( i \) denotes the inclusion. \( \tilde{T} \) is the same map as \( T \) but it is thought as a map from \( \mathcal{M}^{j_1}(G) \times \mathcal{M}^{j_2}(G) \) to \( \mathcal{S}'(G) \). Notice that the spaces, \( \mathcal{M}^{j_1+j_2}(G) \), and \( \mathcal{M}^{j_1}(G) \times \mathcal{M}^{j_2}(G) \), are Fréchet spaces.

To prove the continuity of \( T \) it suffices to prove that the diagram is accurate: \( i \) is sequentially continuous, and \( \tilde{T} \) is separately sequentially continuous. Then by the Closed Graph Theorem, we will know that \( T \) is separately sequentially continuous, hence continuous. It is clear that \( i \) is sequentially continuous.

Now we want to prove that \( \tilde{T} \) is separately sequentially continuous, which means that we want to show that for each fixed \( J_2 \in \mathcal{M}^{j_2}(G) \) and \( J_1 \in \mathcal{M}^{j_1}(G) \) the operators

\[
\mathcal{M}^{j_1}(G) \xrightarrow{T_{J_2}} \mathcal{S}'(G)
\]

\[
J_1 \mapsto (\hat{J}_1 \ast \hat{J}_2)\
\]

\[
\mathcal{M}^{j_2}(G) \xrightarrow{T_{J_1}} \mathcal{S}'(G)
\]

\[
J_2 \mapsto (\hat{J}_1 \ast \hat{J}_2)\
\]

are sequentially continuous.

In order to show that the mapping \( T_{J_2} \) is sequentially continuous, we shall see that it can be written as the composition of sequentially continuous mappings \( F, H \) defined as follows, and the Fourier transform.

\[
\mathcal{M}^{j_1}(G) \xrightarrow{F} \mathcal{E}' \oplus \mathcal{S} \xrightarrow{H} \mathcal{S}'(G) \xrightarrow{\hat{\cdot}} \mathcal{S}'(G)
\]

\[
J_1 \mapsto (\zeta \hat{J}_1, (1-\zeta) \hat{J}_1) \mapsto (\zeta \hat{J}_1) \ast J_2 + [(1-\zeta) \hat{J}_1] \ast J_2 \mapsto \left[ (\zeta \hat{J}_1) \ast J_2 + [(1-\zeta) \hat{J}_1] \ast J_2 \right]\
\]

where \( \zeta \in \mathcal{C}_c^\infty(G) \), and \( \zeta = 1 \) in a neighborhood of zero.

\( H \) is sequentially continuous, since convolution with \( J_2 \in \mathcal{S}' \) is continuous from \( \mathcal{E}' \) to \( \mathcal{S}' \), or from \( \mathcal{S}' \) to \( \mathcal{S}' \).

Since \( \hat{\cdot} \) is the Fourier transform from \( \mathcal{S}' \) to \( \mathcal{S}' \), it is continuous.
Finally in order to show that $F$ is continuous, for a fixed $\zeta \in C^\infty_c(G)$, we shall show that the mappings

$$
\begin{align*}
&\mathcal{M}^{i_1}(G) \xrightarrow{R_1} \mathcal{E}' \\
&J_1 \xrightarrow{} \zeta J_1
\end{align*}
$$

$$
\begin{align*}
&\mathcal{M}^{i_1}(G) \xrightarrow{R_2} \mathcal{S} \\
&J_1 \xrightarrow{} (1 - \zeta) J_1
\end{align*}
$$

are continuous.

Consider the following mappings

$$
\begin{align*}
&\mathcal{M}^{i_1}(G) \xrightarrow{R_2} \mathcal{S}(G) \\
&J_1 \xrightarrow{} J_1
\end{align*}
$$

The sequential continuity of $\tilde{R}_2$ follows from $i : \mathcal{M}^{i_1}(G) \hookrightarrow \mathcal{S}'(G)$ being sequentially continuous. The continuity of $R_2$ now follows from the the Closed Graph Theorem.

The continuity of $R_1$ also follows from the inclusion $i : \mathcal{M}^{i_1}(G) \longrightarrow \mathcal{S}'(G)$ being continuous.

The sequential continuity of $T_{J_1}$ follows in a completely analogous fashion. Hence $\tilde{T}$ is separately sequentially continuous, and therefore $T$ is continuous.

e) Assume $\alpha$ is any multiindex. Since $J \in \mathcal{M}^{i_1}(G)$, then $D^\alpha J \in \mathcal{M}^{i_1-\|\alpha\|}(G)$.

f) Let $\psi : \mathbb{R}^n \longrightarrow [0, 1]$ be a $C^\infty$ function such that $\psi = 0$ for $|\xi| \leq 1$, and $\psi = 1$ for $|\xi| \geq 2$. Let $\{t_i\}_i$ be a positive decreasing sequence such that all $t_i \leq 1$ and $\lim_{i \to 0^+} t_i = 0$. These $t_i$ will be specified later. Define

$$
J(\xi) = \sum_{i=0}^{\infty} \psi(\delta_{t_i}(\xi)) J_i(\xi)
$$

Since $t_i \to 0$ as $i \to 0^+$, then for any fixed $\xi$, $\psi(\delta_{t_i}(\xi)) = 0$ for all, except for a finite number of $i$; so there are only finitely many non zero terms in the previous sum. Consequently this sum is well defined, and it follows that $J \in C^\infty(\mathbb{R}^n)$. In order to prove part f), it suffices to show that $J - \sum_{i=0}^{N} J_i \in \mathcal{M}^{i_1-(N+1)}(G)$ for any $N \in \mathbb{Z}^+$, since then by parts (b) and (c) we shall have

$$
J - \sum_{i=0}^{N} J_i \in \mathcal{M}^{i_1-(N+1)}(G) \subseteq S^{[j-(N+1)]\sigma}_{\left(\frac{a_{j-N}}{a_n}\right),\#} \quad \text{for } \sigma = \frac{1}{a_n}.
$$

Hence if we choose $N$ sufficiently large so that $[j-(N+1)]\sigma < -M$

$$
J - \sum_{i=0}^{N} J_i \in S^{-[N]}_{\left(\frac{a_i}{a_n}\right),\#}
$$

as desired.
For $|\beta| \neq 0$, $(\partial^\beta \psi)(\delta_t(\xi)) = 0$, when $|\delta_t(\xi)| = t|\xi| \leq 1$, or when $|\delta_t(\xi)| = t|\xi| \geq 2$. Thus for $|\beta| \neq 0$, $(\partial^\beta \psi)(\delta_t(\xi)) \neq 0$ implies that $1 < |\delta_t(\xi)| = t|\xi| < 2$, i.e. $t^{-1} < |\xi| < 2t^{-1}$. This implies that if $0 < t \leq 1$ there exists some positive constant $C'$ such that $t < C'(1 + |\xi|)^{-1}$. Therefore if $0 < t \leq 1$, we have

$$|\partial^\beta [(\psi(\delta_t(\xi)))]| = t|\beta| |(\partial^\beta \psi)(\delta_t(\xi))| \leq C' |\beta| t|\xi| \leq C'(1 + |\xi|)^{-|\beta|}.$$  

Therefore $\left\{ \psi \circ \delta_t \right\}_{0 < t \leq 1}$ is a bounded subset of $\mathcal{M}'(G)$.

From Leibniz’s rule and since $J_i \in \mathcal{M}^{j-i}(G)$, it easily follows that

$$\left| \partial^\alpha \left[ \psi(\delta_t(\xi)) J_i(\xi) \right] \right| \leq C_i \gamma(1 + |\xi|)^{(j-i)-|\alpha|}.$$  

In particular $\left\{ (\psi \circ \delta_t) J_i \right\}_{0 < t \leq 1} \subset \mathcal{M}^{j-i}(G)$.

We set $C_i = \max \left\{ C_i, \alpha : |\alpha| \leq i \right\}$. Since $\psi(\xi) = 0$ if $|\xi| \leq 1$ we have $\psi(\delta_t(\xi)) \neq 0$ implies $|\delta_t(\xi)| = t|\xi| > 1$. We select $t_i > 0$ such that $t_i < t_{i-1}$, and $C_i t_i \leq 2^{-i}$. Then $\psi(\delta_{t_i}(\xi)) = 0$ if $t_i (1 + |\xi|) \leq 1$. Therefore for any multiindex $\alpha$ such that $|\alpha| \leq i$ we have

$$\left| \partial^\alpha \left[ \psi(\delta_{t_i}(\xi)) J_i \right] \right| \leq C_i (1 + |\xi|)^{j-i-|\alpha|}$$

$$\leq C_i t_i (1 + |\xi|)^{j-i+1-|\alpha|}$$

$$\leq 2^{-i} (1 + |\xi|)^{j-i+1-|\alpha|}$$

(5.7)

For any multiindex $\beta$ we choose $i_0$ such that $i_0 \geq |\beta|$, and we express $J$ as

$$J = \sum_{i=0}^{i_0} (\psi \circ \delta_{t_i}) J_i + \sum_{i=i_0+1}^{\infty} (\psi \circ \delta_{t_i}) J_i$$

(5.8)

Since $\sum_{i=0}^{i_0} (\psi \circ \delta_{t_i}) J_i$ is a finite sum and $(\psi \circ \delta_{t_i}) J_i \in \mathcal{M}^{j-i}(G) \subseteq \mathcal{M}'(G)$; we have $\sum_{i=0}^{i_0} (\psi \circ \delta_{t_i}) J_i \in \mathcal{M}'(G)$. Therefore there exists a positive constant $C$ such that for all $\xi$

$$\left| \partial^\beta \sum_{i=0}^{i_0} \psi(\delta_{t_i}(\xi)) J_i(\xi) \right| \leq C (1 + |\xi|)^{j-|\beta|}$$

(5.9)

By (5.7),

$$\left| \partial^\beta \sum_{i=i_0+1}^{\infty} \psi(\delta_{t_i}(\xi)) J_i(\xi) \right| \leq \sum_{i=i_0+1}^{\infty} 2^{-i} (1 + |\xi|)^{j-i+1-|\beta|} \leq (1 + |\xi|)^{j-|\beta|}.$$  

(5.10)

From (5.8), (5.9), and (5.10), $J \in \mathcal{M}'(G)$.

Moreover, for $N \in \mathbb{Z}^+$, writing

$$J - \sum_{i=0}^{N} J_i = \sum_{i=0}^{N} \left[ (\psi \circ \delta_{t_i}) - 1 \right] J_i + \sum_{i=N+1}^{\infty} (\psi \circ \delta_{t_i}) J_i$$

(5.11)
and working in the same fashion as before, we obtain
\[
\left( \sum_{i=N+1}^{\infty} (\psi \circ \delta_{t_i}) J_i \right) \in \mathcal{M}^{j-(N+1)}(G).
\]
On the other hand, since \( \psi(\xi) = 1 \) for \( |\xi| \geq 2 \), \( \psi(\delta_{t_i}(\xi)) - 1 = 0 \) for \( |\delta_{t_i}(\xi)| = t_i |\xi| \geq 2 \). So, if \( 0 \leq i \leq N \), \( \psi(\delta_{t_i}(\xi)) - 1 = 0 \) for \( |\xi| \geq 2 t_i^{-1} \). Then \( \sum_{i=0}^{N} [(\psi \circ \delta_{t_i}) - 1] J_i \in \mathcal{M}^{-\infty}(G) = \mathcal{S}(G) \). Consequently for any \( N \)
\[
\left( J - \sum_{i=0}^{N} J_i \right) \in \mathcal{M}^{j-(N+1)}(G)
\]
i.e. \( J \sim \sum_{i=0}^{\infty} J_i \).
g) This is evident. \( \Box \)

REFERENCES

[BG88] Richard Beals and Peter Greiner, Calculus on Heisenberg manifolds, Annals of Mathematics Studies, vol. 119, Princeton University Press, Princeton, NJ, 1988. MR 953082 (89m:32223)
[CZ56] A. P. Calderón and A. Zygmund, On singular integrals, Amer. J. Math. 78 (1956), 289–309. MR 0084633 (18,894a)
[CZ57] A.-P. Calderón and A. Zygmund, Singular integral operators and differential equations, Amer. J. Math. 79 (1957), 901–921. MR 0100768 (20 #7196)
[CGGP92] Michael Christ, Daryl Geller, Paweł Głowacki, and Larry Polin, Pseudodifferential operators on groups with dilations, Duke Math. J. 68 (1992), no. 4, 777–799. MR 1185817 (94b:35316)
[Cor07] Susana Coré, Hörmander pseudodifferential calculus of type \((1,0)\) on the Heisenberg group (2007), available at http://www.math.smith.edu/~score/psd.calculus.pdf
[FS74] G. B. Folland and E. M. Stein, Estimates for the \( \bar{\partial}_b \) complex and analysis on the Heisenberg group, Comm. Pure Appl. Math. 27 (1974), 429–522. MR 0367477 (51 #3719)
[FS82] G. B. Folland and Elias M. Stein, Hardy spaces on homogeneous groups, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J., 1982. MR 657581 (84h:43027)
[FJ85] Michael Frazier and Björn Jawerth, Decomposition of Besov spaces, Indiana Univ. Math. J. 34 (1985), no. 4, 777–799. MR 808825 (87b:46083)
[Gel83] Daryl Geller, Liouville’s theorem for homogeneous groups, Comm. Partial Differential Equations 8 (1983), no. 8, 1665–1677. MR 729197 (85f:58109)
[Gel90] Daryl Geller, Analytic pseudodifferential operators for the Heisenberg group and local solvability, Mathematical Notes, vol. 37, Princeton University Press, Princeton, NJ, 1990. MR 1030277 (91d:58243)
[GM06] Daryl Geller and Azita Mayeli, Continuous wavelets and frames on stratified Lie groups. I, J. Fourier Anal. Appl. 12 (2006), no. 5, 543–579. MR 2267634 (2007g:42055)
[KN65] J. J. Kohn and L. Nirenberg, An algebra of pseudo-differential operators, Comm. Pure Appl. Math. 18 (1965), 269–305. MR 0176362 (31 #636)
[Mih50] S. G. Mihlin, Singular integral equations, Amer. Math. Soc. Translation 1950 (1950), no. 24, 116. MR 0036434 (12,107d)
[Mih48] S. G. Mihlin, Singular integral equations, Uspehi Matem. Nauk (N.S.) 3 (1948), no. 3(25), 29–112 (Russian). MR 0027429 (10,305a)
[Ric04] Fulvio Ricci, Fourier and spectral multipliers in \( \mathbb{R} \) and in the Heisenberg group (2004), available at http://homepage.sns.it/fricci/papers/multipliers.pdf
Linda Preiss Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), no. 3-4, 247–320. MR 0436223 (55 #9171)

Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. MR **1232192** (95c:42002)

Robert S. Strichartz, *A functional calculus for elliptic pseudo-differential operators*, Amer. J. Math. **94** (1972), 711–722. MR 0310713 (46 #9811)

Michael E. Taylor, *Noncommutative microlocal analysis. I*, Mem. Amer. Math. Soc. **52** (1984), no. 313, iv+182. MR **764508** (86f:58156)

E-mail address: score@math.smith.edu

Department of Mathematics, Smith College, Northampton, MA 01063

E-mail address: daryl@math.sunysb.edu

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794-3651