A DIFFERENT EXPRESSION OF THE WEIL-PETERSSON POTENTIAL ON THE QUASI-FUCHSIAN DEFORMATION SPACE

LEE-PENG TEO

Abstract. We extend a definition of the Weil-Petersson potential on the universal Teichmüller space to the quasi-Fuchsian deformation space. We prove that up to a constant, this function coincides with the Weil-Petersson potential on the quasi-Fuchsian deformation space. As a result, we prove a lower bound for the potential on the quasi-Fuchsian deformation space.

1. Introduction

In [TT03b], we defined a Hilbert manifold structure on the universal Teichmüller space $T(1)$. Under this structure, $T(1)$ is a disjoint union of uncountably many components. We denoted by $T_0(1)$ the component that contains the identity element. It can be characterized as the completion of the space $\text{Möb}(S^1)\setminus\text{Diff}^+(S^1)$ under the Weil-Petersson metric. Hence it is the largest submanifold of $T(1)$ where the Weil-Petersson metric can be defined.

The Weil-Petersson metric on $\text{Möb}(S^1)\setminus\text{Diff}^+(S^1)$, introduced by Kirillov [Kir87] via the orbit method, has been of interest to both mathematicians and physicists. It is a right-invariant Kähler metric, and hence it may play some role in the canonical quantization of the space $\text{Möb}(S^1)\setminus\text{Diff}^+(S^1)$.

In our subsequent work [TT04], we defined a Weil-Petersson potential on $T_0(1)$ in two different ways, and showed that they are equal. The first definition of the Weil-Petersson potential is given by $S_1 : T_0(1) \rightarrow \mathbb{R},$

$$S_1([\mu]) = \iint_{D^*} \left| \frac{(f_{\mu}^\prime)^2}{(f_{\mu}^\prime)^2} \right| d^2z + \iint_{D^*} \left| \frac{g_{\mu}^\prime}{g_{\mu}^\prime} \right|^2 d^2z - 4\pi \log |g_{\mu}(\infty)|,$$

where $w_{\mu} = g_{\mu}^{-1} \circ f_{\mu}$ is the conformal welding corresponding to $[\mu] \in T_0(1)$. The second definition of the Weil-Petersson potential comes from the study of the Grunsky operator $K_1$ of the univalent function $f_{\mu}$ associated to a point $[\mu] \in T(1)$. We proved that the Grunsky operator associated to $[\mu]$ is Hilbert-Schmidt if and only if $[\mu] \in T_0(1)$. Hence the function $S_2 : T_0(1) \rightarrow \mathbb{R}$ given by

$$S_2([\mu]) = \log \det(I - K_1 K_1^*)$$
is well-defined. We proved that

$$S_2 = -\frac{1}{12\pi_1}S_1.$$  

We call $S_1$ the universal Liouville action in our paper [TT04]. In fact, given $\Gamma$ a cocompact quasi-Fuchsian group, the quasi-Fuchsian deformation space $D(\Gamma)$ can be canonically mapped into the universal Teichmüller space, with the totally real submanifold — the space of Fuchsian groups — mapped to the point identity. The function $S_1$ bears a lot of resemblance to the critical value of the Liouville action functional $S_{cl}$ we constructed in [TT03a]. However, we haven’t established the precise relation between them. In this paper, we are going to extend the definition of the function $S_2$ to the quasi-Fuchsian deformation space of a quasi-Fuchsian group $\Gamma$, and prove that up to constants, it coincides with the critical Liouville action $S_{cl}$. As a result, we show that (see Corollary 4.4)

$$S_{cl}(\mu) \leq \frac{8\pi}{(2g-2)},$$

with equality appears if and only if $\mu$ corresponds to a Fuchsian group.

It is our intention to keep this paper concise. Hence we will not repeat the background material and conventions. We refer them to our previous papers [TT03a, TT03b, TT04].

2. Definition of the function $S_2$ on quasi-Fuchsian deformation space

2.1. The quasi-Fuchsian deformation space $D_g$.

2.1.1. A model of $D_g$. We fix a model for the quasi-Fuchsian deformation space of genus $g \geq 2$ in the following way. Let $\Gamma \in \text{PSU}(1,1)$ be a normalized cocompact Fuchsian group of genus $g$. Let $A^{-1,1}(\Gamma)$ be the space of bounded Beltrami differentials for $\Gamma$ and $B^{-1,1}(\Gamma)$ the unit ball of $A^{-1,1}(\Gamma)$ with respect to the sup-norm. For each $\mu \in B^{-1,1}(\Gamma)$, there exists a unique quasi-conformal (q.c.) mapping $w_\mu : \hat{C} \to \hat{C}$ satisfying the Beltrami equation

$$(w_\mu)z = \mu(w_\mu)\bar{z}$$

and fixing the points $-1, -i, 1$. The quasi-Fuchsian deformation space of the Fuchsian group $\Gamma$ is defined as

$$D(\Gamma) = B^{-1,1}(\Gamma) / \sim,$$

where $\mu \sim \nu$ if and only if $w_\mu|_{S_1} = w_\nu|_{S_1}$.

Given $[\mu] \in B^{-1,1}(\Gamma)$, let $\Gamma^\mu = w_\mu \circ \Gamma \circ w_\mu^{-1}$. By definition, $\Gamma^\mu$ is a normalized quasi-Fuchsian group and it is a Fuchsian group if and only if $\mu$ is symmetric, i.e.

$$\mu(z) = \mu\left(\frac{1}{\bar{z}}\right)\frac{z^2}{\bar{z}^2}.$$  

There is a canonical isomorphism $D(\Gamma) \cong D(\Gamma^\mu)$ given by $[\nu] \mapsto [\lambda]$, where $\lambda$ is the Beltrami differential of $w_\nu \circ w_\mu^{-1}$. We define $D_g$, the quasi-Fuchsian
deformation space of genus $g$ to be the space $\mathcal{D}(\Gamma)$, where $\Gamma$ is a Fuchsian group of genus $g$. It is unique up to isomorphism.

Given $[\mu] \in D(\Gamma)$, let $\Omega_1 = w_\mu(\mathbb{D})$ and $\Omega_2 = w_\mu(\mathbb{D}^*)$. Then $\Omega = \Omega_1 \sqcup \Omega_2$ is the domain of discontinuity of $\Gamma^\mu$ acting on $\mathbb{C}$ and $\mathcal{C} = w_\mu(S^1)$ is the quasi-circle separating the domains $\Omega_1$ and $\Omega_2$. There exists a unique q.c. mapping $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $f|_D$ is holomorphic, $f$ fixes the points $-1, -i, 1$ and $f(\mathbb{D}) = \Omega_1$. Similarly, there exists a unique q.c. mapping $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $g|_{\mathbb{D}^*}$ is holomorphic, $g$ fixes the points $-1, -i, 1$ and $g(\mathbb{D}^*) = \Omega_2$. By abusing notation, we also denote by $f$ and $g$ the univalent functions $f|_D$ and $g|_{\mathbb{D}^*}$. We say that $(f, g)$ is the pair of univalent functions associated to the point $[\mu] \in D_g$. In fact, $(f, g)$ is independent of the choice of the model $D(\Gamma)$ for $D_g$. It only depends on the quasi-Fuchsian group $\Gamma^\mu$. In case $\Gamma^\mu$ is a Fuchsian group, $\Omega_1 = \mathbb{D}$, $\Omega_2 = \mathbb{D}^*$ and hence $f = \text{id}$ and $g = \text{id}$. Using the biholomorphisms $f$ and $g$, we define the pair of Fuchsian groups $(\Gamma_1, \Gamma_2)$ associated to $\Gamma^\mu$ by $\Gamma_1 = f^{-1} \circ \Gamma^\mu \circ f$ and $\Gamma_2 = g^{-1} \circ \Gamma^\mu \circ g$.

Given $[\mu] \in D_g$, let $\mu_1, \ldots, \mu_d$ be a basis of $\Omega^{-1,1}(\Gamma^\nu)$. The Bers coordinates at the point $[\nu]$ is defined by the correspondence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \mapsto \Gamma^\varepsilon = w_\varepsilon \circ \Gamma^\nu \circ w_\varepsilon^{-1}$, where $w_\varepsilon : \mathbb{C} \to \mathbb{C}$ is the unique q.c. mapping with Beltrami differential $\varepsilon_1 \mu_1 + \ldots + \varepsilon_d \mu_d$ and fixing the points $-1, -i, 1$.

2.1.2. **The tangent and cotangent space of $D_g$**. The holomorphic tangent space at the point $[\nu] \in D_g$ is isomorphic to the space of harmonic Beltrami differentials $\Omega^{-1,1}(\Gamma^\nu)$ of $\Gamma^\nu$. The holomorphic cotangent space at the point $[\mu] \in D_g$ is isomorphic to the vector space $\Omega^{2,0}(\Gamma^\mu)$ of holomorphic quadratic differentials of $\Gamma^\mu$.

Given $\mu \in \Omega^{-1,1}(\Gamma^\nu)$, we denote by $\frac{\partial}{\partial \mu}$ and $\frac{\partial}{\partial \bar{\mu}}$ the holomorphic and anti-holomorphic vector fields in a neighbourhooed of $[\nu]$ defined using the Bers coordinates at the point $[\nu]$.

2.1.3. **The embedding $T_g \to D_g$ and the map $D_g \to T(1)$**. Let $T_g$ be the Teichmüller space of genus $g$ and $T(1)$ the universal Teichmüller space. $T_g$ can be realized as a complex submanifold of the quasi-Fuchsian deformation space $D_g = D(\Gamma)$. Namely

$$T_g = T(\Gamma) = \{ [\mu] \in D(\Gamma) : \mu(z) = 0 \text{ on } \mathbb{D} \cup S^1 \}.$$  

We denote by $i : T_g \hookrightarrow D_g$ the canonical inclusion map.

To define the map $\Xi : D(\Gamma) \to T(1)$, we use the following model for $T(1)$.

$$T(1) = L^{\infty}(\mathbb{D}^*)_1 / \sim,$$

where given $\mu \in L^{\infty}(\mathbb{D}^*)_1$, a Beltrami differential on $\mathbb{D}^*$ with sup-norm less than 1, we extend $\mu$ to be zero outside $\mathbb{D}^*$ and let $w^\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the unique q.c. mapping with Beltrami differential $\mu$ and fixing the points

---

1We can let $\Gamma$ to be any quasi-Fuchsian group of genus $g$. But for the description of some properties of $D(\Gamma)$ in terms of equivalence classes of Beltrami differentials, it will be convenient to assume that $\Gamma$ is a Fuchsian group.
−1, −i, 1. \(\mu \sim \nu\) if and only if \(w^\mu|_D = w^\nu|_D\). Given \([\mu] \in \mathcal{D}_g\), let \((f, g)\) be the pair of univalent functions associated to \([\mu]\). Extend \(f\) and \(g\) to be q.c. mappings. The map \(\Xi : \mathcal{D}(\Gamma) \to T(1)\) is defined so that \(\Xi([\mu])\) is the equivalence class of the Beltrami differential of \(g|_{D_\ast}\). It is independent of the choice of the representative \(\mu\) of \([\mu]\). It is not a one-to-one mapping. In particular, the subspace of Fuchsian groups in \(\mathcal{D}(\Gamma)\) is mapped to the point \([0] \in T(1)\). If \((f, g)\) is the pair of univalent functions associated to \([\mu] \in \mathcal{D}_g\), there exists a linear fractional transformation \(\lambda \in \text{PSL}(2, \mathbb{C})\) such that the functions \(f = \lambda \circ f\) and \(g = \lambda \circ g\) satisfies

\[
f(0) = 0, \quad f'(0) = 1, \quad g(\infty) = \infty.
\]

The pair of functions \((f, g)\) is then the functions in the conformal welding \(w = g^{-1} \circ f\) associated to \(\Xi([\mu]) \in T(1)\).

2.1.4. The inversion on \(\mathcal{D}_g\). There is an inversion map \(I : \mathcal{D}_g \to \mathcal{D}_g\) on the quasi-Fuchsian deformation space induced by the inversion \(\iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, z \mapsto 1/\bar{z}\). It is defined in the obvious way:

\[
I([\mu]) = [\iota^* \mu], \quad \text{for } [\mu] \in \mathcal{D}_g,
\]

where

\[
\iota^* \mu(z) = \mu\left(\frac{1}{\bar{z}}\right) \frac{z^2}{\bar{z}^2}.
\]

The space of Fuchsian groups is the set of fixed points of this map\(^2\). By uniqueness of q.c. mappings, it is easy to see that

\[
w_{1, \iota^* \mu}(z) = \frac{1}{w\mu(1/\bar{z})}, \quad f[\iota^* \mu](z) = \frac{1}{g[\mu](1/\bar{z})}, \quad g[\iota^* \mu](z) = \frac{1}{f[\mu](1/\bar{z})}.
\]

The relations between the quasi-Fuchsian groups and associated Fuchsian groups are given by

\[
\Gamma_{1, \iota^* \mu} = \iota \circ \Gamma^\mu \circ \iota \quad \Gamma_{1}[\iota^* \mu] = \iota \circ \Gamma_2[\mu] \circ \iota, \quad \Gamma_2[\iota^* \mu] = \iota \circ \Gamma_1[\mu] \circ \iota.
\]

2.2. The function \(S_2\).

2.2.1. Integral operators associated to \((f, g)\). Let

\[
A_1^1(\mathbb{D}) = \left\{ \psi \text{ holomorphic on } \mathbb{D} : \|\psi\|_2^2 = \int_{\mathbb{D}} |\psi(z)|^2 d^2z < \infty \right\},
\]

\[
A_1^2(\mathbb{D}^\ast) = \left\{ \psi \text{ holomorphic on } \mathbb{D}^\ast : \|\psi\|_2^2 = \int_{\mathbb{D}^\ast} |\psi(z)|^2 d^2z < \infty \right\}
\]

\(^2\)This is only true when we use a Fuchsian group \(\Gamma\) for the model \(\mathcal{D}_g = \mathcal{D}(\Gamma)\).
be Hilbert spaces of holomorphic functions on $\mathbb{D}$ and $\mathbb{D}^*$ and denote by $A^1_2(\mathbb{D})$ and $A^1_2(\mathbb{D}^*)$ the corresponding Hilbert-spaces of anti-holomorphic functions. Given a pair $f : \mathbb{D} \to \mathbb{C}$, $g : \mathbb{D}^* \to \hat{\mathbb{C}}$ of univalent functions such that $\hat{\mathbb{C}} \setminus (f(\mathbb{D}) \cup g(\mathbb{D}^*))$ has measure zero, define the kernel functions

$$K_1(z, w) = \frac{1}{\pi} \left( \frac{f'(z)}{(z - w)^2} - \frac{f'(w)}{(f(z) - f(w))^2} \right),$$

$$K_2(z, w) = \frac{1}{\pi} \frac{g'(z)g'(w)}{(f(z) - g(w))^2},$$

$$K_3(z, w) = \frac{1}{\pi} \frac{g'(z)f'(w)}{(g(z) - f(w))^2},$$

$$K_4(z, w) = \frac{1}{\pi} \left( \frac{1}{(z - w)^2} - \frac{g'(z)g'(w)}{(g(z) - g(w))^2} \right).$$

They define linear operators $K_l$, $l = 1, 2, 3, 4$ as follows,

$$K_1 : A^1_2(\mathbb{D}) \to A^1_2(\mathbb{D}), \quad (K_1\psi)(z) = \iint_{\mathbb{D}} K_1(z, w)\overline{\psi(w)}d^2w,$$

$$K_2 : A^1_2(\mathbb{D}^*) \to A^1_2(\mathbb{D}), \quad (K_2\psi)(z) = \iint_{\mathbb{D}^*} K_2(z, w)\overline{\psi(w)}d^2w,$$

$$K_3 : A^1_2(\mathbb{D}) \to A^1_2(\mathbb{D}^*), \quad (K_3\psi)(z) = \iint_{\mathbb{D}} K_3(z, w)\overline{\psi(w)}d^2w,$$

$$K_4 : A^1_2(\mathbb{D}^*) \to A^1_2(\mathbb{D}^*), \quad (K_4\psi)(z) = \iint_{\mathbb{D}^*} K_4(z, w)\overline{\psi(w)}d^2w.$$

The generalized Grunsky equality says that these operators satisfy the following relations (see e.g., [1101]):

$$K_1K_1^* + K_2K_2^* = I, \quad K_3K_3^* + K_4K_4^* = 0,$$

$$K_1K_3^* + K_2K_4^* = 0, \quad K_3K_1^* + K_4K_2^* = I.$$

$K_2, K_3$ are invertible operators and $K_1, K_4$ are operators of norm strictly less than one.

**Remark 2.1.** Our definition of the operators $K_l$ here can be viewed as the ‘pull–back’ of the corresponding definition on $T(1)$ via the map $\Xi : \mathbb{D}_g \to T(1)$.

**Remark 2.2.** If $\phi$ is a holomorphic function on $\mathbb{D}$, the principal–valued integral

$$\iint_{\mathbb{D}} \frac{\phi(w)}{(z - w)^2}d^2w$$
vanishes identically. Hence we can also represent operators $K_1$ and $K_4$ by
the singular kernels
\[
-\frac{1}{\pi} \frac{f'(z)f'(w)}{(f(z) - f(w))^2} \quad \text{and} \quad -\frac{1}{\pi} \frac{g'(z)g'(w)}{(g(z) - g(w))^2}.
\]

2.2.2. The definition of $S_2$. Let $K_l = K_lK_l^*$, $l = 1, 2, 3, 4$. In [TT04], we
define the function $S_2 : T_0(1) \to \mathbb{R}$, which up to a multiplicative constant is
a Weil-Petersson potential on $T_0(1)$, by
\[
S_2 = \log \det(I - K_1) = \log \det(I - K_4).
\]
This definition cannot be extended to $D_g$ since if $K_1$ and $K_4$ are defined
using the pair $(f, g)$ associated to a point on $D_g$, they are not trace-class
operators unless $K_1 = K_4 = 0$ (see the proof in [TT04]), which corresponds
to the case $\Gamma^h$ is a Fuchsian group. Nevertheless, motivated by the series
expansion
\[
\log \det(I - K) = -\text{Tr} \left( \sum_{n=1}^{\infty} \frac{K^n}{n} \right)
\]
valid when the operator $K$ has norm less than 1, we want to consider the
following operators
\[
\mathcal{D}_1 = \sum_{n=1}^{\infty} \frac{K_1^n}{n} \quad \text{and} \quad \mathcal{D}_2 = \sum_{n=1}^{\infty} \frac{K_4^n}{n}.
\]

Lemma 2.3. The operators $\mathcal{D}_1$ and $\mathcal{D}_2$ are well-defined operators with ker-
nels
\[
\mathcal{D}_1(z, w) = \sum_{n=1}^{\infty} \frac{K_{1,n}(z, w)}{n} \quad \text{and} \quad \mathcal{D}_2(z, w) = \sum_{n=1}^{\infty} \frac{K_{4,n}(z, w)}{n},
\]
which converge absolutely and uniformly on compact subsets of $D \times D$ and
$D^* \times D^*$ respectively. Here $K_{i,n}(z, w), i = 1, 4$ is the kernel of the operator
$K_i^n$.

Proof. It is sufficient to consider the operator $\mathcal{D}_1$. First, we notice that for
$n \geq 2$,
\[
K_{1,n}(z, w) = \int_{D} \int_{D} \int_{D} K_1(z, \zeta) \left((K_1^* K_1)^{n-1}\right)(\zeta, \eta) K_1^*(\eta, w) d^2\zeta d^2\eta
\]
\[
= \int_{D} \int_{D} \left((K_1^* K_1)^{n-1}\right)(\eta, \zeta) K_1(\zeta, z) \overline{K_1(\eta, w)} d^2\zeta d^2\eta
\]
\[
= \langle K_1^{n-1} v_z, v_w \rangle.
\]
Here we denote by $\langle \cdot, \cdot, \cdot \rangle$ the inner product on the Hilbert space $A_2^1(D)$, and
$v_z$ is the holomorphic function
\[
v_z(\zeta) = K_1(z, \zeta)
with norm
\[ \|v_z\|_2^2 = K_1(z, z). \]

By Cauchy-Schwarz inequality, we have
\[ |K_{1,n}(z, w)| \leq \|K_1^{n-1}v_z\|_2\|v_w\|_2 \leq \|K_1\|_\infty^{n-1}\|v_z\|_2\|v_w\|_2. \]

Hence
\[ \sum_{n=1}^\infty \frac{|K_{1,n}(z, w)|}{n} \leq \left( \sum_{n=1}^\infty \frac{\|K_1\|_\infty^{n-1}}{n} \right) \|v_z\|_2\|v_w\|_2, \]

which converges absolutely and uniformly on compact subsets of \( \mathbb{D} \times \mathbb{D} \) since \( \|K_1\|_\infty < 1 \) and
\[ \|v_z\|_2^2 \leq \frac{1}{\pi(1 - |z|^2)^2}. \]

(see [TT04]). In fact, as a function of \( w \),
\[ \|K_{1,n}(z, w)\|_2^2 = K_{1,2n}(z, z) \leq \|K_1\|_\infty^{2n-1}\|v_z\|_2^2, \]

which implies that as a function of \( w \),
\[ \lim_{k \to \infty} \sum_{n=1}^k \frac{K_{1,n}(z, w)}{n} = \sum_{n=1}^\infty \frac{K_{1,n}(z, w)}{n} \]
in \( A_2^1(\mathbb{D}) \). Hence it follows that if \( \psi \in A_2^1(\mathbb{D}) \),
\[ (\Omega_1 \psi)(z) = \lim_{k \to \infty} \sum_{n=1}^k \frac{(K_1^n \psi)(z)}{n} = \lim_{k \to \infty} \iint_{\mathbb{D}} \left( \sum_{n=1}^k \frac{K_{1,n}(z, w)}{n} \right) \psi(w)d^2w \]
\[ = \iint_{\mathbb{D}} \left( \sum_{n=1}^\infty \frac{K_{1,n}(z, w)}{n} \right) \psi(w)d^2w, \]

which proves that \( \sum_{n=1}^\infty \frac{1}{n}K_{1,n}(z, w) \) is the kernel for \( \Omega_1 \).

\( \square \)

**Corollary 2.4.** Let \((f, g)\) and \((\Gamma_1, \Gamma_2)\) be the pairs of univalent functions and Fuchsian groups associated to a point on \( \mathbb{D}_g \). The functions \( \Omega_1(z, z) \) and \( \Omega_2(z, z) \) defined using \((f, g)\) are nonnegative real valued continuous functions on \( \mathbb{D} \) and \( \mathbb{D}^* \) that are automorphic \((1, 1)\) forms with respect to \( \Gamma_1 \) and \( \Gamma_2 \) respectively.

**Proof.** Again, it suffices to consider \( \Omega_1 \). It follows from the proof of the previous lemma that
\[ \Omega_1(z, z) = \sum_{n=1}^\infty \frac{K_{1,n}(z, z)}{n} \]
converges absolutely and uniformly on compact subsets of \( \mathbb{D} \). Hence it is continuous. Moreover, since \( K_1 \) is a positive self-adjoint operator,
\[ K_{1,n}(z, z) = \langle K_1^{n-1}v_z, v_z \rangle \geq 0 \]
for all $n$. Hence $\Omega_1(z, z) \geq 0$. Now for every $\gamma \in \Gamma_1$, there exists $\tilde{\gamma} \in \Gamma^\mu$ such that

$$f \circ \gamma = \tilde{\gamma} \circ f.$$ 

Hence it is easy to check from the definition of $K_1(z, w)$ that

$$K_1(\gamma z, \gamma w)\gamma'(z)\gamma'(w) = K_1(z, w) \quad \forall \gamma \in \Gamma_1$$

and consequently

$$K_{1,n}(\gamma z, \gamma w)\gamma'(z)\gamma'(w) = K_{1,n}(z, w) \quad \forall \gamma \in \Gamma_1.$$ 

Therefore, $\Omega_1(z, z)$ is an automorphic $(1,1)$-form with respect to $\Gamma_1$. \hfill \Box

Now we can define the function $S_2 : \Omega \to \mathbb{R}$.

**Definition 2.5.** The function $S_2 : \Omega \to \mathbb{R}$ is defined as follows:

$$S_2([\mu]) = \int \int_{\Gamma_1 \setminus \mathbb{D}} \Omega_1(z, z)d^2z = \sum_{n=1}^{\infty} \frac{1}{n} \int \int_{\Gamma_1 \setminus \mathbb{D}} K_{1,n}(z, z)d^2z,$$

where $\Omega_1(z, w)$ is defined using the univalent function $f$ associated to $[\mu]$.

**Remark 2.6.** The function $S_2$ can be considered as the regularized trace of the operator $-\log \det(I - K_1)$ on $A^1_2(\mathbb{D})$.

2.2.3. **Behavior of $S_2$ under inversion.** The relations (2.2) give us the following relations for the kernels $K_l$ associated to $[\mu]$ and $[\iota^* \mu]$ on $\Omega$:

$$K_l[\iota^* \mu](z, w) = K_{4-l}[\mu]\left(\frac{1}{z}, \frac{1}{w}\right)\frac{1}{z^2 w^2}, \quad l = 1, 2, 3, 4.$$ 

In particular,

$$\Omega_4[\mu](z, z) = \Omega_1[\iota^* \mu]\left(\frac{1}{z}, \frac{1}{z}\right)\frac{1}{|z|^4}.$$ 

Using this, we have the following result.

**Proposition 2.7.** The function $S_2$ is invariant under inversion, i.e. $S_2 \circ \iota = S_2$.

**Proof.** Given a point $[\mu]$ on $\Omega$ with the associated univalent functions $(f, g)$ and Fuchsian groups $(\Gamma_1, \Gamma_2)$, we are going to prove that

$$\int \int_{\Gamma_1 \setminus \mathbb{D}} K_{1,n}(z, z)d^2z = \int \int_{\Gamma_2 \setminus \mathbb{D}^*} K_{4,n}(z, z)d^2z$$

for all $n$. From the relations (2.2) we have

$$K_1(z, w) + K_2(z, w) = I_1(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}, \quad z, w \in \mathbb{D};$$

$$K_3(z, w) + K_4(z, w) = I_2(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}, \quad z, w \in \mathbb{D}^*.$$
where $I_1$ and $I_2$ are the identity operators on $A^1_2(\mathbb{D})$ and $A^1_2(\mathbb{D}^*)$ respectively. On the other hand,

$$K_{1,n}(z, w) = (I_1 - K_2^n)(z, w) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} K_2^k (z, w),$$

$$K_{4,n}(z, w) = (I_2 - K_3^n)(z, w) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} K_3^k (z, w).$$

Now for $k \geq 1$,

$$\int \int \left( K_2^k \right)(z, z) d^2 z = \int \int \int \left( K_2(z, \zeta) \left( (K_2^* K_2)^{k-1} K_2^* \right) (\zeta, z) \right) d^2 \zeta d^2 z$$

$$= \sum_{\gamma_2 \in \Gamma_2} \int \int \int K_2(z, \gamma_2 \zeta) \left( (K_2^* K_2)^{k-1} K_2^* \right) (\gamma_2 \zeta, z) |\gamma_2^2(\zeta)|^2 d^2 \zeta d^2 z.$$

For every $\gamma_2 \in \Gamma_2$, there exists $\gamma \in \Gamma_1$ such that

$$\gamma \circ g = g \circ \gamma_2$$

and $\gamma_1 \in \Gamma_1$ such that

$$\gamma \circ f = f \circ \gamma_1.$$

Hence

$$K_2(\gamma_1 z, \gamma_2 \zeta) \gamma_1'(z) \gamma_2^2(\zeta) = K_2(z, \zeta)$$

whenever the pair of elements $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2$ are associated to the same element $\gamma \in \Gamma$. Then it is easy to show that

$$\left( (K_2^* K_2)^{k-1} K_2^* \right)(\gamma_2 \zeta, z) \gamma_2^2(\zeta) = \left( (K_2^* K_2)^{k-1} K_2^* \right)(\zeta, \gamma_1^{-1} z) |(\gamma_1^{-1})'(z)| d^2 \zeta d^2 z.$$

Consequently,

$$\int \int \left( K_2^k \right)(z, z) d^2 z$$

$$= \sum_{\gamma_1, \gamma_2} \int \int \int K_2(\gamma_1^{-1} z, \zeta) \left( (K_2^* K_2)^{k-1} K_2^* \right)(\zeta, \gamma_1^{-1} z) |(\gamma_1^{-1})'(z)| d^2 \zeta d^2 z$$

$$= \int \int \int K_2(z, \zeta) \left( (K_2^* K_2)^{k-1} K_2^* \right)(\zeta, z) d^2 z d^2 \zeta$$

$$= \int \int \int K_3(\zeta, z) \left( (K_3^* K_3)^{k-1} K_3^* \right)(z, \zeta) d^2 z d^2 \zeta = \int \int \left( K_3^k \right)(z, z) d^2 z.$$
We have used the equality $K_2(z, w) = K_3(w, z)\forall z \in \mathbb{D}, w \in \mathbb{D}^*$ in the last line. Finally, from (2.3) we have
\[
\iint_{\Gamma_1 \setminus \mathbb{D}} K_{1,n}(z, z)d^2z = \frac{1}{4\pi} \text{Area}(\Gamma_1 \setminus \mathbb{D}) + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \iint_{\Gamma_1 \setminus \mathbb{D}} (K_k^2)(z, z)d^2z
\]
\[
= \frac{1}{4\pi} \text{Area}(\Gamma_1 \setminus \mathbb{D}^*) + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \iint_{\Gamma_1 \setminus \mathbb{D}^*} (K_k^3)(z, z)d^2z
\]
\[
= \iint_{\Gamma_2 \setminus \mathbb{D}^*} K_{4,n}(z, z)d^2z.
\]

It follows from the definition that
\[
\iint_{\Gamma_1 \setminus \mathbb{D}} D_1[\mu](z, z)d^2z = \iint_{\Gamma_2 \setminus \mathbb{D}^*} D_4[\mu](z, z)d^2z = \iint_{\Gamma_1 \setminus \mathbb{D}^*} D_1[\mu^*(\mu)](z, z)d^2z.
\]

3. THE FIRST VARIATION OF THE FUNCTION $S_2$

Given $\mu \in \Omega^{-1,1}(\Gamma^\nu)$ a tangent vector at the point $[\nu]$, we define
\[
\mu_1 = f^*(\mu|_{\Omega_1}) \quad \text{and} \quad \mu_2 = g^*(\mu|_{\Omega_2}).
\]

We separate the computation of the variation of $S_2$ into a few lemmas.

**Lemma 3.1.** Given $[\nu] \in \mathcal{D}_g$, let $\mu \in \Omega^{-1,1}(\Gamma^\nu)$ be such that $\mu$ has support on $\Omega_2$. Let $(f^\varepsilon, g_\varepsilon)$ be the univalent functions associated to $\Gamma^\varepsilon = w_{\epsilon\mu} \circ \Gamma^\nu \circ w_{\varepsilon\mu}^{-1}$. At the point $[\nu]$, the variation of the kernel $K_1$ in the direction $\mu$ is given by
\[
\frac{\partial}{\partial \varepsilon} \lim_{\varepsilon \to 0} K_1^\varepsilon(z, w) = - \iint_{\mathbb{D}^*} \iint_{\mathbb{D}^*} \mu_2(u)K_2(z, u)K_4(u, \xi)K_2^2(\xi, w)d^2ud^2\xi
\]

**Proof.** See the proof of Lemma 2.7 and Theorem 3.1 in [1910].

**Lemma 3.2.** Given $[\nu] \in \mathcal{D}_g$, let $\mu \in \Omega^{-1,1}(\Gamma^\nu)$ be such that $\mu$ has support on $\Omega_2$. Let $(f^\varepsilon, g_\varepsilon)$ and $(\Gamma_1^\nu, \Gamma_2^\nu)$ be the univalent functions and Fuchsian groups associated to $\Gamma^\varepsilon = w_{\epsilon\mu} \circ \Gamma^\nu \circ w_{\varepsilon\mu}^{-1}$. Then for all $n \geq 1$,
\[
\frac{\partial}{\partial \varepsilon} \lim_{\varepsilon \to 0} \iint_{\Gamma_1 \setminus \mathbb{D}} K_{1,n}(z, z)d^2z = n \iint_{\Gamma_1 \setminus \mathbb{D}} \iint_{\mathbb{D}} \left(\frac{\partial}{\partial \varepsilon} \lim_{\varepsilon \to 0} K_1^\varepsilon(z, \xi)\right)K_{1,n-1}(\xi, z)d^2\xi d^2z.
\]

**Proof.** Since $f^\varepsilon = w_\varepsilon \circ f$, the group $\Gamma_1^\varepsilon = (f^\varepsilon)^{-1} \circ \Gamma^\varepsilon \circ f^\varepsilon$ is a constant with respect to $\varepsilon$, i.e. $\Gamma_1^\varepsilon = \Gamma_1^0 = \Gamma_1$. From Lemma 3.1, it is easy to check that for all $\gamma_1 \in \Gamma_1$,
\[
\left(\frac{\partial}{\partial \varepsilon} \lim_{\varepsilon \to 0} K_1^\varepsilon\right)(\gamma_1z, \gamma_1w)\gamma_1'(z)\gamma_1'(w) = \frac{\partial}{\partial \varepsilon} \lim_{\varepsilon \to 0} K_1^\varepsilon(z, w).
\]
We find that
\[
\frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \iint_{\Gamma'_1 \setminus \mathbb{D}} \mathcal{K}^\varepsilon_{1,n}(z, z) d^2 z
\]
\[
= \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \iint_{\Gamma'_1 \setminus \mathbb{D}} \ldots \iint_{\mathbb{D}} \mathcal{K}_{1}^\varepsilon(z_1, z_1) \mathcal{K}_{1}^\varepsilon(z_2, z_2) \ldots \mathcal{K}_{1}^\varepsilon(z_n-1, z_n) d^2 z_1 \ldots d^2 z_n d^2 z
\]
\[
= \iint_{\Gamma'_1 \setminus \mathbb{D}} \ldots \iint_{\mathbb{D}} \left( \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \mathcal{K}_{1}^\varepsilon(z_1, z_1) \right) \mathcal{K}_{1}(z_1, z_2) \ldots \mathcal{K}_{1}(z_n-1, z_n) d^2 z_1 \ldots d^2 z_n d^2 z
\]
\[
+ \ldots
\]
\[
+ \iint_{\Gamma'_1 \setminus \mathbb{D}} \ldots \iint_{\mathbb{D}} \mathcal{K}_{1}(z_1, z_1) \mathcal{K}_{1}(z_2, z_2) \ldots \left( \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \mathcal{K}_{1}^\varepsilon(z_n-1, z_n) \right) d^2 z_1 \ldots d^2 z_n d^2 z
\]
\[
= n \iint_{\Gamma'_1 \setminus \mathbb{D}} \left( \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \mathcal{K}_{1}^\varepsilon(z, z) \right) \mathcal{K}_{1,n-1}(\zeta, z) d^2 \zeta d^2 z.
\]

\[\Box\]

**Lemma 3.3.** Given \([\nu] \in \mathcal{D}_g\), let \(\mu \in \Omega^{-1,1}(\Gamma^\nu)\) be such that \(\mu\) has support on \(\Omega_2\). Let \((f^\varepsilon, g^\varepsilon)\) and \((\Gamma'_1, \Gamma'_2)\) be the univalent functions and Fuchsian groups associated to \(\Gamma^\varepsilon = w^\mu \circ \Gamma^\nu \circ w^{-1}_\varepsilon\). Then there exists an \(r > 0\) such that the series

\[
\sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial \varepsilon} \iint_{\Gamma'_1 \setminus \mathbb{D}} \mathcal{K}_{1,n}^\varepsilon(z, z) d^2 z,
\]

converges uniformly to

\[
- \iint_{\mathbb{D}^*} \ldots \iint_{\mathbb{D}} \mu_2^\varepsilon(u) \mathcal{K}_2^\varepsilon(z, u) \mathcal{K}_1^\varepsilon(u, \eta) (\mathcal{K}_2^\varepsilon)^*(\eta, \zeta) (I - \mathcal{K}_{1}^\varepsilon)^{-1}(\zeta, z) d^2 \eta d^2 \zeta d^2 z d^2 u
\]

in the ball \(\{\varepsilon \in \mathbb{C} : |\varepsilon| < r\}\). Here \(\mu_2^\varepsilon\) is the Beltrami differential

\[
g_\varepsilon^\nu \left( \left( \frac{\mu}{1 - |\varepsilon| \mu^2 (w_{\varepsilon \mu})^2} \right) \circ (w_{\varepsilon \mu})^{-1} \right).
\]

**Proof.** By shifting the origin of differentiation, it is easy to see from Lemma 3.1 and Lemma 3.2 that

\[
\frac{\partial}{\partial \varepsilon} \iint_{\Gamma'_1 \setminus \mathbb{D}} \mathcal{K}_{1,n}^\varepsilon(z, z) d^2 z
\]
\[
= - n \iint_{\Gamma'_1 \setminus \mathbb{D}} \ldots \iint_{\mathbb{D}^*} \mu_2^\varepsilon(u) \mathcal{K}_2^\varepsilon(z, u) \mathcal{K}_1^\varepsilon(u, \eta) (\mathcal{K}_2^\varepsilon)^*(\eta, \zeta) \mathcal{K}_{1,n-1}^\varepsilon(\zeta, z) d^2 u d^2 \eta d^2 \zeta d^2 z.
\]
Choose \( C_T \), the canonical complex analytic embedding \( H \) hence the \( \ell^\text{2} \)-norm of the function

\[
u^\varepsilon_w(\zeta) = \iint_{\mathbb{D}^2} \mu_2^\varepsilon(u)K_2^\varepsilon(w,u)K_4^\varepsilon(u,\eta)(K_2^\varepsilon)^*(\eta,\zeta)d^2u d^2\eta
\]
satisfies the inequality

\[
\|\nu^\varepsilon_w\|_2^2 \leq \iint_{\mathbb{D}^2} \left| \iint_{\mathbb{D}^2} \mu_2^\varepsilon(u)K_2^\varepsilon(w,u)K_4^\varepsilon(u,\eta)(K_2^\varepsilon)^*(\eta,\zeta)d^2u \right| d^2\zeta
\]

\[
\leq \iint_{\mathbb{D}^2} |\mu_2^\varepsilon(\zeta)|^2 |K_2^\varepsilon(w,\zeta)|^2d^2\zeta
\]

\[
\leq \|\mu_2^\varepsilon\|_\infty^2 K_2^\varepsilon(w,w) \leq \frac{\|\mu_2^\varepsilon\|_\infty^2}{\pi(1-|w|^2)^2}.
\]

Hence, the \( \ell^\text{2} \)-norm of the function

\[
v_{w,n}^\varepsilon(z) = \iint_{\mathbb{D}^2} \iint_{\mathbb{D}^2} \mu_2^\varepsilon(u)K_2^\varepsilon(w,u)K_4^\varepsilon(u,\eta)(K_2^\varepsilon)^*(\eta,\zeta)d^2u d^2\eta K_1^\varepsilon_n^{-1}(\zeta, z)d^2\zeta
\]

\[(K_1^\varepsilon K_1^\varepsilon)^{n-1}\nu^\varepsilon_w(z)\]
satisfies

\[
\|v_{w,n}^\varepsilon\|_2 \leq \|(K_1^\varepsilon K_1^\varepsilon)^{n-1}\nu^\varepsilon_w\|_2 \leq \frac{\|K_1^\varepsilon\|_\infty^{2n-2}\|\mu_2^\varepsilon\|_\infty}{\sqrt{\pi}(1-|w|^2)^2}.
\]

Therefore, by Lemma 2.3 in [TT04], we have

\[|v_{w,n}^\varepsilon(z)| \leq \frac{\|v_{w,n}^\varepsilon\|_2}{\sqrt{\pi}(1-|z|^2)} \leq \frac{\|K_1^\varepsilon\|_\infty^{2n-2}\|\mu_2^\varepsilon\|_\infty}{\pi(1-|z|^2)(1-|w|^2)}\]

Choose \( C_1 \) and \( C_2 \) such that \( \|K_1^\varepsilon\|_\infty < C_1 < 1 \) and \( \|\mu_2^\varepsilon\|_\infty < C_2 < 1 \). By the continuity of the map \( \mathcal{H} : T(1) \to \mathcal{B}(\ell^2) \) proved in the Appendix A of [TT03], the canonical complex analytic embedding \( T(\Gamma_1) \to T(1) \) and the smooth dependence of \( \mu \) on \( \varepsilon \), we can find a number \( r > 0 \), such that for all \( \varepsilon \) in the ball \( \{|\varepsilon| < r\} \), we have \( \|K_1^\varepsilon\|_\infty < C_1 \) and \( \|\mu_2^\varepsilon\|_\infty < C_2 \). Hence for \( |\varepsilon| < r \),

\[
\left| \frac{\partial}{\partial \varepsilon} \iint_{\Gamma_1 \setminus \mathbb{D}} \frac{K_{1,n}(z, z)}{n} d^2 z \right| = \left| \iint_{\Gamma_1 \setminus \mathbb{D}} v_{z,n}^\varepsilon(z) d^2 z \right| \leq \frac{C_1^{2n-2} C_2}{\pi} \iint_{\Gamma_1 \setminus \mathbb{D}} \frac{d^2 z}{(1-|z|^2)^2} \leq \frac{C_1^{2n-2} C_2}{4\pi} \text{Area}(\Gamma_1 \setminus \mathbb{D}).
\]

Consequently, by Weierstrass-M-test, the series

\[
\sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial}{\partial \varepsilon} \iint_{\Gamma_1 \setminus \mathbb{D}} K_{1,n}^\varepsilon(z, z) d^2 z
\]
converges uniformly and absolutely on the set \( \{ \varepsilon \in \mathbb{C} : |\varepsilon| < r \} \). The same proof above shows that as a function of \( z \), the series
\[
\sum_{n=1}^{\infty} v_{z,n}^\varepsilon(z)
\]
converges uniformly on any compact subset of \( \mathbb{D} \); in particular, on a fundamental domain of \( \Gamma_1 \) on \( \mathbb{D} \). Therefore,
\[
\sum_{n=1}^{\infty} \frac{1}{n} \int_{\Gamma_1 \setminus \mathbb{D}} \frac{\partial}{\partial \varepsilon} K_{1,n}^\varepsilon(z,z) d^2 z = -\sum_{n=1}^{\infty} \int_{\Gamma_1 \setminus \mathbb{D}}  v_{z,n}^\varepsilon(z) d^2 z = -\int_{\Gamma_1 \setminus \mathbb{D}} \sum_{n=1}^{\infty}  v_{z,n}^\varepsilon(z) d^2 z
\]
\[
= -\int_{\Gamma_1 \setminus \mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}^*} \int_{\mathbb{D}^*} \mu_2^\varepsilon(u) K_{2}^\varepsilon(z,u) K_{1}^\varepsilon(u,\eta) (K_{2}^\varepsilon)^*(\eta,\zeta) \left( \sum_{n=1}^{\infty} K_{1,n-1}^\varepsilon(\zeta,z) \right) d^2 u d^2 \eta d^2 \zeta d^2 z.
\]
The conclusion of the lemma then follows from the standard operator theory that
\[
\sum_{n=1}^{\infty} K_{1,n-1}^\varepsilon(\zeta,z) = (I - K_1^\varepsilon)^{-1}(\zeta,z).
\]
\[\square\]

Now we state a lemma we need from elementary analysis:

**Lemma 3.4.** Let \( \mathcal{O} \) be a ball with center at the origin of \( \mathbb{C} \) and let \( h_n : \mathcal{O} \to \mathbb{R} \) be a sequence of differentiable real-valued functions on \( \mathcal{O} \) that converges to the function \( h : \mathcal{O} \to \mathbb{R} \). If \( \frac{\partial h}{\partial z} : \mathcal{O} \to \mathbb{C} \) converges uniformly to \( k : \mathcal{O} \to \mathbb{C} \), then
\[
\frac{\partial h}{\partial z} = k.
\]

Given \([\mu] \in \mathcal{D}_g\), let \( \vartheta([\mu]) \in \Omega^{2,0}(\Gamma^\mu) \) be the quadratic differential defined by
\[
\vartheta([\mu])(z) = \begin{cases} S(f^{-1})(z), & \text{if } z \in \Omega_1, \\ S(g^{-1})(z), & \text{if } z \in \Omega_2. \end{cases}
\]
Here
\[
S(h) = \left( \frac{h''}{h'} \right)' - \frac{1}{2} \left( \frac{h''}{h'} \right)^2
\]
is the Schwarzian derivative of the function \( h \). We have

**Theorem 3.5.** The real-valued function \( S_2 : \mathcal{D}_g \to \mathbb{R} \) is a differentiable function. At the point \([\nu] \in \mathcal{D}_g\), its variation along the direction \( \mu \in \mathcal{D}_g \).
\( \Omega^{-1,1}(\Gamma^\nu) \) is given by
\[
\frac{\partial S_2}{\partial \varepsilon^\mu}([\nu]) = -\frac{1}{6\pi} \int_{\Omega^{\nu}\setminus\Omega} \partial([\nu])\mu.
\]

**Proof.** First, we assume \( \mu \) has support on \( \Omega_2 \). From Lemma 3.4 and Lemma 3.3, we have
\[
\frac{\partial S_2}{\partial \varepsilon^\mu}([\nu]) = -\int_{\Gamma_2 \setminus \Gamma_1 \setminus \mathbb{D}} \int_{\mathbb{D}^*} \mu_2(u)K_2(z,u)K_4(u,\eta)
\]
\[
K_2^*(\eta,\zeta)(I - K_1)^{-1}(\zeta,z)d^2\eta d^2\zeta d^2u.
\]
Define \( R_2 : \mathbb{D}^* \times \mathbb{D} \rightarrow \mathbb{C} \) and \( R_2^* : \mathbb{D} \times \mathbb{D}^* \rightarrow \mathbb{C} \) be as in the proof of Theorem 3.1 in [TT04]. Then
\[
\frac{\partial S_2}{\partial \varepsilon^\mu}([\nu]) = -\int_{\Gamma_2 \setminus \Gamma_1 \setminus \mathbb{D}} \int_{\mathbb{D}^*} \mu_2(u)K_2(z,u)K_4(u,\eta)
\]
\[
K_2^*(\eta,\zeta)R_2^*(\zeta,v)R_2(v,z)d^2vd^2\eta d^2\zeta d^2u.
\]
By the transformation property of the functions \( K_l, l = 1, 2, 3, 4 \) with respect to the groups \( \Gamma_1 \) and \( \Gamma_2 \), we can transform the integral into
\[
-\int_{\Gamma_2 \setminus \mathbb{D}} \int_{\mathbb{D}^*} \mu_2(u)K_2(z,u)K_4(u,\eta)
\]
\[
K_2(\eta,\zeta)R_2(\zeta,v)R_2(v,z)d^2vd^2\eta d^2\zeta d^2u,
\]
which can be further be manipulated as in the proof of Theorem 3.1 in [TT04] to get
\[
\frac{\partial S_2}{\partial \varepsilon^\mu}([\nu]) = \frac{1}{6\pi} \int_{\mathbb{D}^*} S(g)\mu_2 = -\frac{1}{6\pi} \int_{\Omega^{\nu}\setminus\Omega_2} S(g^{-1})\mu.
\]
Now if \( \mu \) has support on \( \Omega_1 \), we let \( \kappa = \iota^*\nu \) and \( \eta = \iota^*\mu \). Then \( \eta \in \Omega^{-1,1}(\Gamma^\nu) \) and it has support on \( \Omega_2[\kappa] \). By Theorem 2.7 and what we have proved above
\[
\frac{\partial S_2}{\partial \varepsilon^\mu}([\nu]) = \frac{\partial S_2}{\partial \varepsilon^\eta}([\kappa]) = -\frac{1}{6\pi} \int_{\Gamma^{\nu}\setminus\Omega_2[\kappa]} S(g[\kappa]^{-1})\eta = -\frac{1}{6\pi} \int_{\Gamma^{\nu}\setminus\Omega_1} S(f[\nu]^{-1})\mu.
\]
Finally, for general \( \mu \in \Omega^{-1,1}(\Gamma^\nu) \), we write \( \mu = \alpha + \beta \), where \( \alpha \) has support on \( \Omega_1 \) and \( \beta \) has support on \( \Omega_2 \). Then
\[
\frac{\partial S_2}{\partial \varepsilon^\mu}([\nu]) = \frac{\partial S_2}{\partial \varepsilon^\alpha}([\nu]) + \frac{\partial S_2}{\partial \varepsilon^\beta}([\nu]),
\]
so the result of the theorem follows. \( \square \)
4. THE CLASSICAL LIOUVILLE ACTION

Let $\Gamma'$ be a normalized cocompact Fuchsian group of genus $g$ realized as a subgroup of $\text{PSL}(2, \mathbb{R})$ and let $\alpha(z) = (z - i)/(z + i)$ be the linear fractional transformation that maps the upper half plane $U$ to the unit disc $\mathbb{D}$. The Fuchsian group $\Gamma = \alpha \circ \Gamma' \circ \alpha^{-1}$ is then a subgroup of $\text{PSL}(2, \mathbb{R})$. $\alpha$ induces a complex analytic isomorphism $\mathcal{G} : \mathcal{D}(\Gamma') \to \mathcal{D}(\Gamma)$ by

$$\left[\mu\right] \in \mathcal{D}(\Gamma') \mapsto \mathcal{G}(\left[\mu\right]) = \left[\left(\alpha^{-1}\right)^*\mu\right].$$

We define the function $S_2 : \mathcal{D}(\Gamma') \to \mathbb{R}$ by $S_2 = S_2 \circ \mathcal{G}$.

Given $[\nu] \in \mathcal{D}(\Gamma')$, we let $w_\nu$ be the unique q.c. mapping with Beltrami differential $\nu$ and fixing the points 0, 1, $\infty$. Let $\kappa = (\alpha^{-1})^*\nu$, then $w_\nu = \alpha^{-1} \circ w_\kappa \circ \alpha$. Let $J_1 = \alpha^{-1} \circ f \circ \alpha$, $J_2 = \alpha^{-1} \circ g \circ \alpha$, where $(f, g)$ is the pair of univalent functions associated to $[\kappa] \in \mathcal{D}(\Gamma)$.

4.1. Relation between classical Liouville action and the function $S_2$. Given $[\mu] \in \mathcal{D}(\Gamma')$, let $\hat{\vartheta}(\left[\mu\right]) \in \Omega^{2,0}(\Gamma'^\nu)$ be the quadratic differential defined by

$$\hat{\vartheta}(\left[\mu\right])(z) = \begin{cases} S(J_1^{-1})(z), & \text{if } z \in w_\mu(U), \\ S(J_2^{-1})(z), & \text{if } z \in w_\mu(L). \end{cases}$$

Here $L$ is the lower half plane. If $\kappa = (\alpha^{-1})^*\mu$, then $\vartheta([\kappa]) = (\alpha^{-1})^*\hat{\vartheta}([\mu])$.

It follows from Theorem 3.5 that

**Theorem 4.1.** Given a subgroup $\Gamma'$ of $\text{PSL}(2, \mathbb{R})$, which is a normalized cocompact Fuchsian group of genus $g$, the real-valued function $S_2 : \mathcal{D}(\Gamma') \to \mathbb{R}$ is a differentiable function. At the point $[\nu] \in \mathcal{D}(\Gamma')$, its variation along the direction $\mu \in \Omega^{-1,1}(\Gamma'^\nu)$ is given by

$$\frac{\partial S_2}{\partial \epsilon\mu}([\nu]) = -\frac{1}{6\pi} \int_{\Gamma'^\nu \setminus \Omega'} \hat{\vartheta}(\nu)\mu.$$

Here $\Omega' = w_\nu(U) \sqcup w_\nu(L)$ is the set of discontinuity of the group $(\Gamma')^\nu$.

**Proof.** We let $\kappa = (\alpha^{-1})^*\nu$, $\eta = (\alpha^{-1})^*\mu$. Then by Theorem 3.5

$$\frac{\partial S_2}{\partial \epsilon\mu}([\nu]) = \frac{\partial S_2}{\partial \epsilon\eta}([\kappa]) = -\frac{1}{6\pi} \int_{\Gamma'^\nu \setminus \Omega'} \vartheta([\kappa])\eta = -\frac{1}{6\pi} \int_{\Gamma'^\nu \setminus \Omega'} \hat{\vartheta}(\nu)\mu.$$

\[\square\]

In [TT03a], we define the classical Liouville action $S_{cl} : \mathcal{D}(\Gamma') \to \mathbb{R}$ and prove that $-S_{cl}$ is a Weil-Petersson potential on $\mathcal{D}(\Gamma')$. Theorem 4.2 in [TT03a] says that
Theorem 4.2. Let $\Gamma'$ be as in Theorem 4.1. At the point $[\nu] \in \mathfrak{D}(\Gamma')$, the variation of the function $S_{cl}$ along the direction $\mu \in \Omega^{-1,1}((\Gamma')^{\nu})$ is given by

$$\frac{\partial S_{cl}}{\partial \epsilon_{\mu}}([\nu]) = 2 \int_{(\Gamma')^{\nu} \setminus \Omega^{\nu}} \tilde{\vartheta}([\nu]) \mu.$$ 

Theorem 4.1 and Theorem 4.2 give a relation between $S_{cl}$ and $S_2$.

Theorem 4.3. Let $\Gamma'$ be as in Theorem 4.1. On the deformation space $\mathfrak{D}(\Gamma')$, we have

$$S_{cl} = -12\pi S_2 + 8\pi (2g - 2).$$

Proof. Since $\mathfrak{D}(\Gamma')$ is connected, from Theorem 4.1 and Theorem 4.2 we have

$$S_{cl} = -12\pi S_2 + C,$$

where $C$ is a constant. Now at the origin $[0]$ of $\mathfrak{D}(\Gamma')$, $S_{cl}([0]) = 8\pi (2g - 2)$ and $S_2([0]) = 0$, hence $C = 8\pi (2g - 2)$.

This gives us the following inequality for the classical Liouville action.

Corollary 4.4. The classical Liouville action $S_{cl} : \mathfrak{D}(\Gamma') \to \mathbb{R}$ satisfies the following inequality

$$S_{cl} \leq 8\pi (2g - 2).$$

It attains its maximum value along the subspace of Fuchsian groups.

Proof. This follows from the theorem above and the nonnegativity of $S_2$ established in Corollary 2.4.

Remark 4.5. It follows that the normalized potential $-S_{cl} + 8\pi (2g - 2)$ on the quasi-Fuchsian deformation space $\mathfrak{D}(\Gamma')$ is a nonnegative function.

REFERENCES

[Kir87] A. A. Kirillov, Kähler structure on the $K$-orbits of a group of diffeomorphisms of the circle, Funktsional. Anal. i Prilozhen. 21 (1987), no. 2, 42–45.

[TT03a] Leon A. Takhtajan and Lee-Peng Teo, Liouville action and Weil-Petersson metric on deformation spaces, global Kleinian reciprocity and holography, Comm. Math. Phys. 239 (2003), no. 1-2, 183–240.

[TT03b] , Weil-Petersson metric on the universal Teichmüller space I: Curvature properties and Chern forms, Preprint arXiv: math.CV/0312172 (2003).

[TT04] , Weil-Petersson metric on the universal Teichmüller space II: Kähler potential and period mapping, Preprint arXiv: math.CV/0406408 (2004).

Faculty of Information Technology, Multimedia University, Jalan Multimedia, Cyberjaya, 63100, Selangor, Malaysia

E-mail address: lpteo@mmu.edu.my