Polynomial ring representations of endomorphisms of exterior powers

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Abstract
A polynomial ring with rational coefficients is an irreducible representation of Lie algebras of endomorphisms of exterior powers of an infinite countable dimensional $\mathbb{Q}$-vector space. We give an explicit description of it, using suitable vertex operators on exterior algebras, which mimick those occurring in the bosonic vertex representation of the Lie algebra $gl_{\infty}$, due to Date–Jimbo–Kashiwara and Miwa (DJKM).

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1 Introduction

1.1 The goal. Let $B_r := \mathbb{Q}[e_1, \ldots, e_r]$ be the polynomial ring in $r$ indeterminates $(e_1, \ldots, e_r)$. This paper supplies its explicit description as a module over the Lie algebras of endomorphisms of $k$-th exterior powers of a vector space $V$ of infinite countable dimension. The goal is achieved by means of certain vertex operators on the exterior algebra, defined by means of Schubert derivations.

The latter are distinguished Hasse-Schmidt derivations on exterior algebras, introduced in [12] and extensively treated in [14]; see also the survey [1] or [5, p. 116], for more related discussions. They have shown their versatility in applications to improve effectiveness in Schubert Calculus computations (see [3, 4]), to equivariant cohomology of Grassmannians (Cf. [18], but also [22, 23]), to generalise the Cayley-Hamilton theorem [13, 19] or, like in [16, 17] and in the present paper, to revisit the bosonic vertex representation of Lie algebras of endomorphisms as in [7] (see also [20] and [21, Propositions 5.2–5.3]), providing new methods and new insight.

The $\mathfrak{gl}(\bigwedge^k V)$–module structure of $B_r$, that we are going to describe, will be referred to as bosonic representation of $\bigwedge^k V$, by a possibly strong, but suggestive, abuse of terminology, due to the evident relationship with pioneering work by Date, Jimbo, Kashiwara and Miwa [7] (see also [20] and [21]) and a more general framework that, in [6], one refers to as DJKM (affine) Heisenberg algebra.

That $B_r$ is a representation of $\mathfrak{gl}(\bigwedge^k V)$ is easy to see in very special cases. For $k = 0$, it is just multiplication by rational numbers, as $\bigwedge^0 V = \mathbb{Q}$. For $k > r$, is the trivial null representation. For $r = k = 1$, it amounts to the well known general fact that any vector space is a module over the Lie algebra of its own endomorphisms. Thus, the linear extension of the set map $e_1^i \mapsto b_1$ is a vector space isomorphism $B_1 \to V$, making $B_1$ into a $\mathfrak{gl}(V)$-module, by pulling back that structure from $V$. For the general case see below, in the second part of this introduction, devoted to state precisely the main result. The third one will discuss, instead, background and motivation.

1.2 Statement of the main result. The ring $B_r$ possesses a $\mathbb{Q}$-basis formed by certain Schur determinants $\Delta_{\lambda}(H_r)$ (like in Section 2.2; see [11, ]), where $\lambda$ ranges over the set $\mathcal{P}_r$ of all the partitions of length at most $r$. Let $V := \bigoplus_{i \geq 0} \mathbb{Q} \cdot b_i$ be the $\mathbb{Q}$-vector space with basis $b := (b_i)_{i \geq 0}$. Then $\bigwedge^r V := \bigoplus_{\lambda \in \mathcal{P}_r} \mathbb{Q} \cdot [b]^r_\lambda$, where $[b]^r_\lambda := b_{r-1+\lambda_1} \wedge \cdots \wedge b_{\lambda_r}$.

It follows that the linear extension of the set map $\Delta_{\lambda}(H_r) \mapsto [b]^r_\lambda$ is a $\mathbb{Q}$-vector space isomorphism $B_r \to \bigwedge^r V$ sending $1 \mapsto [b]^r_0 := b_{r-1} \wedge \cdots \wedge b_0$. It can be phrased by saying that $\bigwedge^r V$ carries a structure of free $B_r$-module of rank 1 generated by $[b]^r_0$, such that $[b]^r_\lambda = \Delta_{\lambda}(H_r)[b]^r_0$ (Section 2.2).

The restricted dual of $V$ is $V^* := \bigoplus_{i \geq 0} \mathbb{Q} \cdot \beta_j$, where $\beta_j : V \to \mathbb{Q}$ is the unique linear form $\beta_j(b_i) = \delta_{ji}$. Then $\bigwedge^r V^* := \bigoplus_{\lambda \in \mathcal{P}_r} \mathbb{Q} \cdot [\beta]^r_\lambda$, where $[\beta]^r_\lambda := \delta_{\lambda, \mu}$. Let $\mathfrak{gl}(\bigwedge^k V)$ be the Lie algebra of the endomorphisms of $\bigwedge^k V$ vanishing at $[b]^k_\lambda$ for all partitions $\lambda \in \mathcal{P}_k$. 
but finitely many. If \( E_{\mu,\nu}^k := |b|_\mu^k \otimes |\beta|_\nu^k \), then:

\[
\mathfrak{gl}(\bigwedge^k V) = \bigwedge^k V \otimes \bigwedge^k V^* = \bigoplus_{\mu,\nu \in \mathcal{P}_k} \mathbb{Q} \cdot E_{\mu,\nu}^k.
\]

The \( B_r \) (bosonic) representation of \( \mathfrak{gl}(\bigwedge^k V) \), for all \( k, r \geq 0 \), is then naturally defined via the following equality:

\[
(E_{\mu,\nu}^k \Delta_\lambda(H_r))[b]^r_0 = [b]^k_\mu \wedge ([\beta]^k_{\nu,r}][b]^r_\lambda),
\]

where the contraction \([\beta]^k_{\nu,r} \) maps \( \bigwedge^r V \) to \( \bigwedge^{r-k} V \) (Section 2.4). To express the \( \mathfrak{gl}(\bigwedge^k V) \)-action (1) on \( B_r \) within a compact formula, a standard philosophy suggests to use generating functions. Let \( z_k := (z_1, \ldots, z_k) \) and \( w_k := (w_1, \ldots, w_k) \) be two sets of formal variables. The \( k \)-tuples of the formal inverses \((z_1^{-1}, \ldots, z_k^{-1})\) and \((w_1^{-1}, \ldots, w_k^{-1})\) will be denoted by \( z_k^{-1} \) and \( w_k^{-1} \) respectively. The standard notation \( s_{\mu}(z_k) \) and \( s_{\nu}(w_k^{-1}) \) stands for the symmetric Schur polynomials in the variables \( z_k \) and \( w_k^{-1} \) (See [11, p. 40]). Define

\[
\mathcal{E}(z_k, w_k^{-1}) = \sum_{\mu, \nu \in \mathcal{P}_k} E_{\mu,\nu}^k \cdot s_{\mu}(z_k) s_{\nu}(w_k^{-1}) : B_r \to B_r[z_k, w_k^{-1}]
\]

through

\[
\left( \mathcal{E}(z_k, w_k^{-1}) \Delta_\lambda(H_r) \right)[b]^r_0 = \sum_{\mu, \nu \in \mathcal{P}_k} s_{\mu}(z_k) s_{\nu}(w_k^{-1})[b]^k_\lambda \wedge ([\beta]^k_{\nu,r}][b]^r_\lambda).
\]

Our main result is:

**Theorem 7.5.** The equality below holds for all \( k, r \geq 0 \) and all \( \lambda \in \mathcal{P}_r \):

\[
(\mathcal{E}(z_k, w_k^{-1}) \Delta_\lambda(H_r))[b]^r_0 = \prod_{j=1}^{k} \left( \frac{z_j}{w_j} \right)^{r-k} \cdot \Gamma(z_k) \Gamma^*(w_k)[b]^r_\lambda,
\]

where the vertex operators \( \Gamma(z_k), \Gamma^*(w_k) : \bigwedge V \to \bigwedge[z_k, w_k^{-1}] V \) on the exterior algebra \( \bigwedge V \) of \( V \) are introduced in Definition 4.3 and studied in more details in Sections 5 and 6. They are merely defined as product of Schubert derivations. In case \( r \) is big with respect to the length of \( \lambda \), the vertex operators involved in formula (2) can be expressed as

\[
\Gamma(z_k) := \prod_{j=1}^{k} \frac{1}{E_r(z_j)} \exp \left( -\sum_{i \geq 1} \frac{1}{i} \delta(\omega_{i-1} p_i(z_k^{-1})) \right)
\]

and

\[
\Gamma^*(w_k) := \prod_{j=1}^{k} E_r(w_j) \exp \left( \sum_{i \geq 1} \frac{1}{i} \delta(\omega_{i-1} p_i(w_k^{-1})) \right),
\]

3
where $E_r(z)$ is the generic monic polynomial $1 - e_1 z + \cdots + (-1)^r e_r z^r$, the map $\sigma_{-1}$ is the locally nilpotent endomorphism of $V$ mapping $b_j \mapsto b_{j-1}$ if $j \geq 1$ and $b_0 \mapsto 0$, $\delta : \mathfrak{gl}(V) \mapsto \text{End}(\wedge V)$ is the natural representation of $\mathfrak{gl}(V)$ as a Lie algebra of (even) derivations of $\wedge V$ and, finally, $p_i(u_k)$ denotes the Newton power sum $u_1^i + \cdots + u_k^i$ of degree $i$.

In other words, the image of $\Delta_A(H_r)$ through $\mathcal{E}_k^{\mu, \nu}$ is the coefficient of $s_{\mu}(z_k)s_{\nu}(w_k^{-1})$ in the right hand side of (2). This may sounds tricky to evaluate, but it coincides with the coefficient of

$$z_1^{k-1+\mu_1} \cdots z_k^{\mu_k}, w_1^{-k+1-\nu_1} \cdots w_k^{-\nu_k}$$

of the second member of (2), multiplied by the Vandermonde determinants of $z_k$ and $w_k^{-1}$.

1.3 Background and Motivations. This paper is the first step towards the authors’ attempt to better understand a fundamental, although elementary, representation theoretical fact. Let $\mathcal{V} := \bigoplus_{j \in \mathbb{Z}} \mathbb{Q} \cdot b_j$ be a vector space with basis $(b_j)$ parameterized by the integers (one may think of $\mathcal{V}$ as being the vector space of the Laurent polynomials) and $\mathcal{V}^*$ its restricted dual with basis $(\beta_j)_{j \in \mathbb{Z}}$.

It is well known that $\mathcal{V} \oplus \mathcal{V}^*$ supports a canonical structure of Clifford algebra $\mathfrak{C} := \mathfrak{C}(\mathcal{V} \oplus \mathcal{V}^*)$ [8, p. 85] or [14] and that the Fermionic Fock space $F$ (also called the semi infinite wedge power $\wedge^{\infty/2} \mathcal{V}$, see [2]) is an irreducible representation of $\mathfrak{C}$. More precisely, $F$ is an invertible module over the Lie super-algebra $\mathfrak{C}$ generated by a distinguished vector $|0\rangle$, the vacuum, that in the formalism of the infinite wedge powers can be suggestively written as $b_0 \wedge b_{-1} \wedge b_{-2} \wedge \cdots$. The huge Clifford algebra $\mathfrak{C}$, whose elements are finite linear combinations of words of the form $b_{i_1} \cdots b_{i_k} \beta_{j_1} \cdots \beta_{j_k}$, contains in a natural way all, but not only, the Lie algebras $\mathfrak{gl}(\wedge^k \mathcal{V})$, for all $k \geq 0$. In particular, it turns out that $F$ is a $\mathfrak{gl}(\wedge^k \mathcal{V})$-module for all $k \geq 0$. Then, the bosonic Fock space $B := B_{\infty} := \mathbb{Q}[e_1, e_2, \ldots]$ gets a $\mathfrak{gl}(\wedge^k \mathcal{V})$-module structure, for all $k \geq 0$, pulling back that of $F$ via the boson-fermion correspondence, a natural module isomorphism $B \to F$ over the infinite dimensional Lie Heisenberg algebra.

The latter may well be interpreted as a sort of Poincaré duality for infinite dimensional Grassmannians. This case shall be analyzed in a forthcoming paper: although the formal framework looks the same, the case $r = \infty$ is not just a naive limit of our formula (2), as indicated by the presence of the factor $\prod(z_j/w_j)^{-r-k}$. Since the algebra of endomorphisms of the exterior algebra of $\mathcal{V} \cong \mathbb{Q}[X]$ is precisely the same Clifford algebra $\mathfrak{C}$ we alluded to above, we realised that was already relevant and interesting to give a first closer look to the $\mathfrak{gl}(\wedge \mathcal{V})$–structure of $\wedge \mathcal{V}$, certainly not treated in any literature we have consulted up to now. The task, however, does not look easy, at first sight, because $\mathfrak{gl}(\wedge \mathcal{V})$ also contains the vector spaces $\text{Hom}_B(\wedge^{k_1} \mathcal{V}, \wedge^{k_2} \mathcal{V}) \cong \wedge^{k_2} \mathcal{V} \otimes \wedge^{k_1} \mathcal{V}^*$, with $k_1 \neq k_2$. Thus, in this paper we are going to offer the description of the easiest case, namely the representation of homogeneous endomorphisms of $\wedge \mathcal{V}$ of degree 0 (with respect to the exterior algebra graduation).

The output is that the direct sum $\bigoplus_{k \geq 0} \mathfrak{gl}(\wedge^k \mathcal{V})$ is a Lie subalgebra of $\mathfrak{gl}(\wedge \mathcal{V})$, represented by $B_r$ for all $r \geq 0$. In the case of the fermionic Fock space, the $\mathfrak{gl}(\wedge \mathcal{V})$-structure
of $B_\infty$ is the DJKM one \cite{7, 20}. The reference \cite{17} already shows how the techniques of this paper are also suited to cope with the DJKM case and we know that the same methods will work as well for the DJKM representation of $\mathfrak{gl}(\wedge^k V)$. To our knowledge no such a computation is known for the case $k > 1$, and to gain feeling and experience we first coped with the more classical problem of describing the $\mathfrak{gl}(\wedge V)$-module structure of $\wedge V$. As a byproduct, examples of many computations with vertex operators occurring in the Heisenberg vertex algebra have been provided. Indeed, the references \cite{14, 15, 17} already show our vertex operators on Grassmann algebras tend, as $r \to \infty$ to the well known classical ones as in, e.g., \cite[Theorem 5.1]{21}.

1.4 Organisation of the paper. Most of preliminaries and basic notation are exposed in Section 2. To be as much self contained as possible, the first part recalls basics borrowed from the elementary theory of symmetric polynomials, such as, e.g. in \cite{11}, while a second part quickly accounts on the notion of Hasse-Schmidt derivation on Exterior algebras. The notion of inverse and transpose, essential for the sequel, are also discussed.

The very special case of HS derivations we will be concerned with are those that already in the very first reference \cite{12} was termed Schubert derivation. In a finite dimensional context, the Schubert derivation is nothing but the Chern polynomial of the universal quotient bundle over a Grassmannian: it acts as a HS–derivation on the exterior algebra of the homology of the projective space, which is the same as saying that to dealing with Schubert calculus for Grassmannians, Bézout theorem suffices.

The explicit expression of the Schubert derivations offered in Section 3, introduced for the first time in this paper, makes all evident their strict relationship with vertex operators. The Schubert derivations we consider are denoted by $\sigma_+(z), \overline{\sigma}_+(z), \sigma_-(w)$ and $\overline{\sigma}_-(w)$. Those with the same sign as subscripts commute in the algebra of endomorphisms of the exterior algebra. However, due to the fact that $\sigma_-(w)$ and $\overline{\sigma}_-(w)$ are locally nilpotent, they commute with $\sigma_+(z)$ and $\overline{\sigma}_+(z)$ just up to the multiplication by a rational function. To fully check our main Theorem 7.5, we devote Section 4 to state and prove some relevant commutation rules, some of which can be recognized within the phrasing of the categorical framework for the Boson-Fermion correspondence, depicted in \cite{9}. See also \cite{25} for a recent update.

Section 4 also contains an elegant definition of what we have proposed to name vertex operators on a Grassmann algebra, because its evident relationship with those occurring in the classical boson–fermion correspondence. They arise, however, in a definitely more elementary context that, in our opinion, would deserve to be further investigated, as we are currently doing.

Vertex operators in the sense of Definition 4.3 are homogeneous operators of the exterior algebra, one of positive and the other of negative degree. We devote one section to each one of them (Sections 5 and 6) to dig up their relationship with basic computations in multi-linear algebra, such as wedging and contracting. Of course this idea is already present in the infinite wedge power context (e.g. \cite[Chapter 5]{21}), but this paper, together with \cite{15, 16, 17}, is the first instance of applications of those techniques and ideas in finite dimensional landscapes. Finally, last Section 7 is concerned with the statement and the
proof of the main theorem together with some of its straightforward declinations in terms of familiar objects, like suitable deformations of the same Giambelli’s determinants occurring in classical Schubert Calculus – see Theorem 7.9. To achieve the proof of the main Theorem 7.5, we needed some preliminary Lemmas (such as 7.2 and 7.3) which, along with Theorems 5.5 and 6.3, we believe interesting in their own, as pieces of multilinear algebra properties addressed to wider general mathematical audiences.

2 Preliminaries and Notation

2.1 The main characters of this paper are polynomials rings \( B_r := \mathbb{Q}[e_1, \ldots, e_r] \) in the \( r \geq 1 \) indeterminates \((e_1, \ldots, e_r)\) and a \( \mathbb{Q}\)-vector space \( V := \bigoplus_{i \geq 0} Q \cdot b_i \) with basis \( b := (b_i)_{i \geq 0} \). The restricted dual of \( V \) is \( V^* := \bigoplus_{i \geq 0} \mathbb{Q} \cdot \beta_i \), where \( \beta_i(b_j) = \delta_{ij} \). We denote by \( b(z) \) and \( \beta(w^{-1}) \) the generating series of the basis elements of \( V \) and of \( V^* \), i.e.:

\[
b(z) := \sum_{i \geq 0} b_i z^i \quad \text{and} \quad \beta(w^{-1}) := \sum_{j \geq 0} \beta_j w^{-j}.
\] (3)

The exterior algebra of \( V \) is \( \wedge V := \bigoplus_{i \geq 0} \wedge^i V \), the direct sum of the exterior powers \( \wedge^i V \), where \( \wedge^0 V = \mathbb{Q} \) and \( \wedge^1 V = V \). The algebra structure is given by the \( \mathbb{Q}\)-linear extension of the juxtaposition. Given the generic polynomial \( E_r(z) := 1 - e_1 z + \cdots + (-1)^r e_r z^r \in B_r[z] \), one defines the sequence \( H_r := (h_j)_{j \in \mathbb{Z}} \) through the equality:

\[
\sum_{n \in \mathbb{Z}} h_n z^n := \frac{1}{E_r(z)}.
\] (4)

In particular \( h_j = 0 \) if \( j < 0 \) and \( h_0 = 1 \). Moreover for \( j \geq 0 \), \( h_j \) is an explicit polynomial homogeneous of degree \( j \) in \((e_1, \ldots, e_r)\), once one gives weight \( i \) to \( e_i \).

2.2 A partition is a monotonic non increasing sequence \( \lambda := (\lambda_1 \geq \lambda_2 \geq \ldots) \) of non negative integers, said to be its parts. Its length \( \ell(\lambda) \) is the number of its non zero parts, and \(|\lambda| = \sum_{i \geq 0} \lambda_i \) is its weight. Let \( \mathcal{P}_r \) the set of all partitions of length at most \( r \). To each \( \lambda \in \mathcal{P}_r \) we associate

\[
[b]_\lambda^r := b_{r-1+\lambda_1} \wedge b_{r-2+\lambda_2} \wedge \cdots \wedge b_{r-\lambda_r} \in \wedge^r V,
\] (5)

so that \(([b]_\lambda^r)_{\lambda \in \mathcal{P}_r} \) is a \( \mathbb{Q}\)-basis of \( \wedge^r V \), and the Schur determinant

\[
\Delta_\lambda(H_r) := \det(h_{\lambda_i-j+i})_{1 \leq i, j \leq r} = \begin{vmatrix}
\lambda_1 & \lambda_{r-1} & \cdots & h_{\lambda_r-r+1} \\
\lambda_{r+1} & \lambda_{r-2} & \cdots & h_{\lambda_r-r+2} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{r+r-1} & \lambda_{2+r-2} & \cdots & h_\lambda
\end{vmatrix}.
\] (6)
It is well known that
\[ B_r := \bigoplus_{\lambda \in \mathcal{P}_r} \mathbb{Q} \cdot \Delta_\lambda(H_r) \]  
(7)
i.e. the Schur determinants form a \( \mathbb{Q} \)-vector space basis of \( B_r \) parametrized by the partitions of length at most \( r \). It follows that \( B_r \) is naturally isomorphic to \( \wedge^r V \) via the \( \mathbb{Q} \)-linear extension of the sets map
\[ \Delta_\lambda(H_r) \mapsto [b]_\lambda^r. \]  
(8)

2.3 Especially in the last section we shall be concerned with Schur polynomials in a set of indeterminates. We recall them here. For each partition of length at most \( k \) and any set of \( k \) formal variables \( x_k := (x_1, \ldots, x_k) \), one defines
\[ \Delta_\lambda(x_k) = \det(x_j^{\lambda_i-i+1}) = \begin{vmatrix} x_1^{\lambda_1} & x_2^{\lambda_2} & \cdots & x_k^{\lambda_k} \\ x_1^{1+\lambda_k-1} & x_2^{1+\lambda_k-1} & \cdots & x_k^{1+\lambda_k-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k-1+\lambda_1} & x_2^{k-1+\lambda_1} & \cdots & x_k^{k-1+\lambda_1} \end{vmatrix}. \]

This is an skew symmetric polynomials in \( (x_1, \ldots, x_k) \) and therefore divisible by the Vandermonde determinant
\[ \Delta_0(x_k) := \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{k-1} & x_2^{k-1} & \cdots & x_k^{k-1} \end{vmatrix} = \prod_{i<j}(x_j - x_i). \]

The Schur polynomial associated to \( x_k \) and the partition \( \lambda \) is defined by the equality
\[ \Delta_\lambda(x_k) = s_\lambda(x_k) \cdot \Delta_0(x_k), \]
onlysaid to be the Jacobi-Trudy formula.

2.4 The pairing
\[ (\beta_{i_1} \wedge \cdots \wedge \beta_{i_r})(v_1 \wedge \cdots \wedge v_r) = \begin{vmatrix} \beta_{i_1}(v_1) & \cdots & \beta_{i_1}(v_r) \\ \vdots & \ddots & \vdots \\ \beta_{i_r}(v_1) & \cdots & \beta_{i_r}(v_r) \end{vmatrix}, \]  
(9)
establishes a natural identification between \( \wedge^r V^* \) and \( (\wedge^r V)^* \). If one denotes by \( [\beta]^r_\mu \) the basis element
\[ \beta_{r-1+\mu_1} \wedge \cdots \wedge \beta_{\mu_r} \]  
(10)of \( \wedge^r V^* \), an easy check shows that \( [\beta]^r_\mu([b]_\lambda^r) = \delta_{\mu,\lambda}. \) The pairing (9) enables to attach to any \( \beta \in V^* \) a map \( \beta_\eta : \wedge V \to \wedge V \) of degree \(-1\) (with respect to the graduation of the exterior algebra) via the equality
\[ \eta(\beta_\eta u) = (\beta \wedge \eta)(u), \quad \forall(u, \eta) \in \wedge^r V \times \wedge^{r-1} V^*. \]  
(11)
2.5 Let $\mathbb{A}V[z]$ denote the formal power series in the indeterminate $z$ with coefficients in the exterior algebra $\mathbb{A}V$ of $V$. If $S$ is any set of indeterminates over $\mathbb{Q}$, denote by $\mathbb{Q}[S]$ the corresponding algebra of formal power series. The following is an extended reformulation of the main definition of the reference [12] (see also [14]).

2.6 Definition. By Hasse–Schmidt derivation on $\mathbb{A}V$ we mean any $\mathbb{Q}[S]$-linear extension of a $\mathbb{Q}$-linear map $D(z): \mathbb{A}V \to \mathbb{A}V[z]$ such that

$$D(z)(u \wedge v) = D(z)u \wedge D(z)v, \quad \forall u, v \in \mathbb{A}V,$$

(12)

which, by abuse of notation, will be denoted by the same symbol

$$D(z) : \mathbb{Q}[S] \otimes \mathbb{A}V \to \mathbb{Q}[S] \otimes \mathbb{A}V[z],$$

(instead of the more precise, but lengthier, $1_{\mathbb{Q}[S]} \otimes \mathbb{A}V D(z)$).

2.7 If $D_i \in \text{End}_{\mathbb{Q}}(\mathbb{A}V)$ are such that $\sum_{i \geq 0} D_i z^i := D(z)$, then (12) is equivalent to the system of relations holding for all $i \geq 0$

$$D_i(u \wedge v) = \sum_{j=0}^{i} D_j u \wedge D_{i-j} v,$$

By [14, Proposition 4.1.7, (iii)], if $D_0$ is invertible in $\text{End}_{\mathbb{Q}}(\mathbb{A}V)$, then $D(z)$ is invertible as a $\text{End}_{\mathbb{Q}}(\mathbb{A}V)$-valued formal power series and its inverse, $\overline{D}(z)$, is an HS-derivation as well. The integration by parts formulas follow for all $u, v \in \mathbb{A}V$:

$$D(z)(\overline{D}(z)u \wedge v) = u \wedge D(z)v,$$

(13)

$$\overline{D}(z)(D(z)u \wedge v) = u \wedge \overline{D}(z)v.$$

(14)

Formulas (13) and (14) are implicitly assuming the $\mathbb{Q}[[z]]$-linearity of $D(z)$ we alluded to in Definition 2.6. The extension of the linearity an HS-derivation over polynomials algebra will be assumed in the following without any further mention.

2.8 The notation

$$D(z)[b]^\lambda = [D(z)b]^\lambda$$

(15)

will be used as a shorthand for the equality

$$D(z)[b]^\lambda = D(z)(b_{r-1+\lambda_1} \wedge \cdots \wedge b_{r}) = D(z)b_{r-1+\lambda_1} \wedge \cdots \wedge D(z)b_{r},$$

meaning that $D(z)$ is a HS-derivation.

2.9 The transpose $D(z)^T : \mathbb{A}V^* \to \mathbb{A}V^*[z]$ of the HS derivation $D(z)$ is defined via its action on homogeneous elements. If $\eta \in \mathbb{A}V^*$, then one stipulates that $D(z)^T \eta(u) = \eta(D(z)u)$, for all $u \in \mathbb{A}V$. By [15, Proposition 2.8] $D(z)^T$ is a HS-derivation of $\mathbb{A}V^*$. 

8
3 Recap on Schubert Derivations

3.1 Recall that there is a natural representation \( \delta : \mathfrak{gl}(V) \rightarrow \text{End}(\bigwedge V) \) making any \( \phi \in \mathfrak{gl}(V) \) into an (even) derivation \( \delta(\phi) \) of \( \bigwedge V \). In other words \( \delta(\phi) \) is the unique \( \mathbb{Q} \)-vector space endomorphism of \( \bigwedge V \) such that
\[
\delta(\phi)(v \wedge w) = \delta(\phi)v \wedge w + v \wedge \delta(\phi)w
\]
for all \( v, w \in \bigwedge V \) together with the initial condition \( \delta(\phi)u = \phi(u) \) for all \( u \in V = \bigwedge^1 V \).

An easy check shows that
\[
D^{\phi}(z) = \exp \left( \sum_{i \geq 1} \frac{1}{i} \delta(\phi^i)z^i \right),
\]
is the unique HS derivation on \( \bigwedge V \) such that \( D^{\phi}(z)|_V = \sum_{i \geq 0} \phi^i z^i \).

Let now \( \sigma_1 : V \rightarrow V \) be such that \( \sigma_1 b_j = b_{j+1} \) and \( \sigma_{-1} : V \rightarrow V \) such that \( \sigma_{-1} b_j = b_{j-1} \), where by convention we put \( b_k = 0 \) if \( k < 0 \).

3.2 Definition. The Schubert derivations on \( \bigwedge V \) are the HS-derivations \( \sigma_+(z) : \bigwedge V \rightarrow \bigwedge V[z] \) and \( \sigma_-(z) : \bigwedge V \rightarrow \bigwedge V[z^{-1}] \) defined by
\[
\sigma_+(z) = \sum_{i \geq 0} \sigma_iz^i := \exp \left( \sum_{i \geq 1} \frac{1}{i} \delta(\sigma_1^i)z^i \right),
\]
\[
\sigma_-(z) = \sum_{i \geq 0} (-1)^i \sigma_{-i}z^{-i} := \exp \left( \sum_{i \geq 1} \frac{1}{i} \delta(\sigma_{-1}^i)z^{-i} \right),
\]
and their inverses in \( \text{End}_\mathbb{Q}(\bigwedge V)[z] \) and \( \text{End}_\mathbb{Q}(\bigwedge V)[z^{-1}] \) respectively:
\[
\overline{\sigma}_+(z) = \sum_{i \geq 0} (-1)^i \overline{\sigma}_iz^i := \exp \left( -\sum_{i \geq 1} \frac{1}{i} \delta(\sigma_1^i)z^i \right),
\]
\[
\overline{\sigma}_-(z) = \sum_{i \geq 0} (-1)^i \overline{\sigma}_{-i}z^{-i} := \exp \left( -\sum_{i \geq 1} \frac{1}{i} \delta(\sigma_{-1}^i)z^{-i} \right).
\]

In particular:
\[
\overline{\sigma}_\pm(z)u = u - \sigma_\pm z^\pm u \cdot z^\pm, \quad \forall u \in V = \bigwedge^1 V.
\]

3.3 It might be useful for the interested reader but not familiar with the main reference of the subject seeing explicitly the action of \( \sigma_\pm(z) \) and \( \overline{\sigma}_\pm(z) \) on basis elements. One has, for all \( j \geq 0 \)
\[
\sigma_+(z)b_j = \sum_{i \geq 0} b_{j+i}z^i \quad \text{and} \quad \overline{\sigma}_+(z)b_j = b_j - b_{j+i}z^i,
\]
\[
10
\]
\[ \sigma_-(z) b_j = \sum_{i \geq 0} \frac{b_{j-i}}{z^i}, \quad \text{and} \quad \overline{\sigma}_-(z) b_j = b_j - \frac{b_{j-1}}{z}, \quad (21) \]

putting \( b_i = 0 \) for \( i < 0 \).

3.4 We exploit the Schubert derivation \( \overline{\sigma}_+(z) \) or, equivalently, its inverse \( \sigma_+(z) \), to endow \( \bigwedge^r V \) with a \( B_r \)-module structure, by declaring that \( e_i u = \overline{\sigma}_i u \) or, equivalently, \( h_i u = \sigma_i u \), \( \forall u \in \bigwedge^r V \). In particular, for all \( u \in \bigwedge^r V \):

\[ \overline{\sigma}_+(z) u = \mathcal{E}_r(z) \cdot u \quad \text{and} \quad \sigma_+(z) u := \frac{1}{\mathcal{E}_r(z)} u, \quad \forall u \in \bigwedge^r V. \]

The fact that such a product structure is compatible with the natural vector space isomorphism \( B_r \to \bigwedge^r V \) given by (8) is a consequence of

3.5 Proposition. Giambelli’s formula for the Schubert derivation \( \sigma_+(z) \) holds:

\[ [b]_\lambda^r = \Delta_\lambda(\sigma_+(z)) := \det(\sigma_{\lambda_j-j+i})_{1 \leq i,j \leq r}[b]_0^r. \quad (22) \]

Hence \( \bigwedge^r V \) is a free \( B_r \)-module of rank 1 generated by \([b]_0^r\).

Proof. Formula (22) may be inferred as a particular case of the general determinantal formula for the exterior power of a polynomial ring due to Laksov and Thorup as in [24, Main Theorem 0.1]. It follows that \([b]_\lambda^r = \Delta_\lambda(\sigma_+(z))[b]_0^r = \Delta_\lambda(H_r)[b]_0^r\), proving the second part of the claim.

3.6 By virtue of 3.5, the map \( B_r \to \bigwedge^r V \) defined by \( \Delta_\lambda(H_r) \mapsto \Delta_\lambda(H_r)[b]_0^r \) is well defined and is an isomorphism as it maps the basis \((\Delta_\lambda(H_r))_{\lambda \in \mathcal{P}_r}\) of \( B_r \) to the basis \([b]_{\lambda}^r\) of \( \bigwedge^r V \).

3.7 The fact that \( \bigwedge^r V \) is a free \( B_r \)-module of rank 1 generated by \([b]_0^r\), as prescribed by equality (22), shows that the Schubert derivations \( \sigma_-(z), \overline{\sigma}_-(z) \) induces maps \( B_r \to B_r[z^{-1}] \) that, abusing notation, will be denoted in the same way. Their action on a basis element \( \Delta_\lambda(H_r) \) of \( B_r \) is defined through its action on \( \bigwedge^r V \):

\[ (\overline{\sigma}_-(z) \Delta_\lambda(H_r))[b]_0^r = \overline{\sigma}_-(z)[b]_{\lambda}^r, \quad (23) \]

\[ (\sigma_-(z) \Delta_\lambda(H_r))[b]_0^r = \sigma_-(z)[b]_{\lambda}^r. \quad (24) \]

Denote by \( \overline{\sigma}_-(z) H_r \) (respectively \( \sigma_-(z) H_r \)) the sequence \((\overline{\sigma}_-(z) h_j)_{j \in \mathbb{Z}}\) (respectively \((\sigma_-(z) h_j)_{j \in \mathbb{Z}}\)). Then the following statement gives a practical way to evaluate the image of \( \Delta_\lambda(H_r) \) through the maps \( \overline{\sigma}_-(z) \) and \( \sigma_-(z) \) defined by (24) and (23).

3.8 Proposition. For all \( r \geq 0 \) and all \( \lambda \in \mathcal{P}_r \)

\[ \sigma_-(z) h_j = \sum_{i \geq 0} \frac{h_{j-i}}{z^i} \quad \text{and} \quad \overline{\sigma}_-(z) h_j = h_j - \frac{h_{j-1}}{z}. \quad (25) \]
Moreover:

\[ \sigma_-(z)\Delta_\lambda(H_r) = \Delta_\lambda(\sigma_-(z)H_r) \quad \text{and} \quad \overline{\sigma}_-(z)\Delta_\lambda(H_r) = \Delta_\lambda(\overline{\sigma}_-(z)H_r). \] (26)

**Proof.** See [15, Theorem 5.7], by exploiting the Laksov & Thorup determinantal formula as in [24, Main Theorem 0.1]. \(\blacksquare\)

### 3.9 Remark.
It is important to notice that (26) only holds if \(\ell(\lambda) \leq r\). For example

\[ \Delta_{(1,1)}(\overline{\sigma}_-(z)H_1) = \begin{vmatrix} h_1 - \frac{1}{z} & 1 \\ h_2 & \frac{h_1}{z} - \frac{1}{z} \end{vmatrix} = -\frac{h_1}{z} + \frac{1}{z^2} \neq 0 = \overline{\sigma}_-(z)\Delta_{(1,1)}(H_1). \]

### 4 Commutation rules

#### 4.1 Let \(k \geq 1\) and let \(z_k := (z_1, \ldots, z_k)\) be one ordered \(k\)-tuple of formal variables. By \(z_k^{-1}\) we shall mean the \(k\)-tuple of the formal inverses \((z_1^{-1}, \ldots, z_k^{-1})\). Define maps \(\sigma_\pm(z_k), \overline{\sigma}_\pm(z_k) : \Lambda V \to \Lambda V[z_k, z_k^{-1}]\) respectively by

\[ \sigma_\pm(z_k) := \sigma_+(z_1) \cdots \sigma_+(z_k) \quad \text{and} \quad \overline{\sigma}_\pm(z_k) := \overline{\sigma}_+(z_1) \cdots \overline{\sigma}_+(z_k). \] (27)

The maps occurring in formulas (27) are multivariate HS derivations on \(\Lambda V\), in the sense that, for instance, \(\sigma_+(z_k)(u \wedge v) = \sigma_+(z_k)u \wedge \sigma_+(z_k)v\), as it is easy to check and adopting the linear extension of the Schubert derivation to polynomial coefficients as recalled in Definition 2.6. The same holds verbatim for \(\sigma_-(z_k)\) and \(\overline{\sigma}_\pm(z_k)\). It is an important point that the multivariate HS derivations in (27) are symmetric in the formal variables \(z_i\) and \(w_i\). This is a consequence of the first of the commutation rules of product of Schubert derivations we shall list in this section because needed in the sequel.

#### 4.2 Proposition. Let \(z, w\) be arbitrary formal variables. The equalities

\[ \overline{\sigma}_\pm(z)\overline{\sigma}_\pm(w) = \overline{\sigma}_\pm(w)\overline{\sigma}_\pm(z), \] (28)

\[ \sigma_\pm(z)\sigma_\pm(w) = \sigma_\pm(w)\sigma_\pm(z). \] (29)

hold in \(\text{End}_Q(\Lambda V)[[z^\pm, w^\pm]]\).

**Proof.** Equalities (28) and (29) hold if and only if are obvious consequences of the fact if \(i, j \geq 0\) then \(\sigma_{\pm i}\) and \(\sigma_{\pm j}\) are pairwise commuting. It is sufficient then to show that they commute when restricted to \(V\), because if they do, then

\[ \sigma_\pm(z)\sigma_\pm(w)[b]_\lambda = \left[ \sigma_\pm(z)\sigma_\pm(w)\right]^r_\lambda \sigma_\pm(z)[b]_\lambda^r, \]
with obvious meaning of the notation. But \( \sigma_{\pm i} \sigma_{\pm j} u = \sigma_{\pm i}^{\pm j} u = \sigma_{\pm j} \sigma_{\pm i} u \) for all \( u \in V \) and then the claim follows.

In order to give a compact expression of the \( \mathfrak{gl}(\wedge^k V) \)-module structure of \( B_r \), we shall need to introduce a generalisation of the classical vertex operators arising in the context of the so-called boson-fermion correspondence, like in e.g. [21], which we look at as a generalisation of the isomorphism \( B_r \rightarrow \wedge^r V \) recalled in Section 2.2, reaffirmed and refined in Section 3.6.

**4.3 Definition.** By vertex operators on \( \wedge V \) we mean the \( \mathbb{Q}[z_k, z_k^{-1}] \)-linear maps \( \Gamma(z_k), \Gamma^*(z_k) : \wedge V \rightarrow (\wedge V)[z_k, z_k^{-1}] \) of degree 1 and \(-1\), with respect to the exterior algebra graduation, given by:

\[
\Gamma(z_k)[b]_k^\lambda = \sigma_+(z_k)\overline{\sigma}_-(z_k)[b]_k^{\lambda+k}, \quad (30)
\]

\[
\Gamma^*(z_k)[b]_k^\lambda = \left( \overline{\sigma}_+(z_k)\Delta(\sigma_-(z_k)\mathcal{H}_{r-k}) \right)[b]_0^{\lambda-k}. \quad (31)
\]

**4.4 Proposition.** Proposition 4.2 guarantees that the vertex operators \( \Gamma(z_k) \) and \( \Gamma^*(z_k) \) are symmetric in the formal variables \( (z_1, \ldots, z_k) \). They will be studied in a more detailed way in Section 5 and 6, exploiting further commutation relations, for which we need the preliminary work exposed below. As a matter of fact, we notice that the commutativity of the product of Schubert derivations is granted only if they are of the same kind (both subscript “+” or both subscript “-”). In general, for \( i, j > 0 \), \( \sigma_i \) and \( \sigma_j \) do not commute, because \( \sigma_j \) is locally nilpotent. The simplest example is: \( \sigma_{-1} \sigma_1 b_0 = b_0 \neq 0 = \sigma_1 \sigma_{-1} b_0 \). The general pattern is that commutativity only holds up to the multiplication by a rational function. A first instance of non trivial commutation rule, needed in the sequel, is provided by the following:

**4.5 Proposition.**

i) If \( \lambda \in \mathcal{P}_r \setminus \mathcal{P}_{r-1} \) (i.e. \( \ell(\lambda) = r \)), then \( \overline{\sigma}_-(w) \) commutes with both \( \sigma_+(z) \) and \( \overline{\sigma}_+(z) \), i.e.

\[
\overline{\sigma}_-(w)\sigma_+(z) = \sigma_+(z)\overline{\sigma}_-(w), \quad (32)
\]

and

\[
\overline{\sigma}_-(w)\overline{\sigma}_+(z) = \overline{\sigma}_+(z)\overline{\sigma}_-(w). \quad (33)
\]

ii) if \( \lambda \in \mathcal{P}_{r-1} \) (i.e. \( [b]_k^\lambda = [b]_{k+1-1}^r \wedge b_0 \)):

\[
\overline{\sigma}_-(w)\sigma_+(z)[b]_k^\lambda = \left( 1 - \frac{z}{w} \right) \sigma_+(z)\overline{\sigma}_-(w)[b]_k^\lambda. \quad (34)
\]

**Proof.** As a matter of i), we observe that \( \overline{\sigma}_-(w)\sigma_+(z)b_\lambda = \sigma_+(z)\overline{\sigma}_-(z)b_\lambda \) if \( \lambda > 0 \). Indeed
\[ \overline{\sigma}_-(w) \sigma_+(z) b_\lambda = \overline{\sigma}_-(w) \left( \sum_{j \geq 0} b_{\lambda+j} z^j \right) \]  

(Definition of \( \sigma_+(z) b_\lambda \))

\[ = \sum_{j \geq 0} \left( b_{\lambda+j} - \frac{b_{\lambda+j-1}}{w} \right) z^j \]  

(Definition of \( \overline{\sigma}_-(w) \))

\[ = \sigma_+(z) \overline{\sigma}_-(w) b_\lambda. \]

Similarly

\[ \overline{\sigma}_-(w) \sigma_+(z) b_\lambda = \sigma_+(z) \overline{\sigma}_-(w) b_\lambda, \]

as a direct straightforward computation shows. Therefore, under the hypothesis \( \ell(\lambda) = r \):

\[ \overline{\sigma}_-(w) \sigma_+(z) [b]_\lambda^r = \overline{\sigma}_-(w) \sigma_+(z) b_{r-1+\lambda_1} \land \cdots \land \overline{\sigma}_-(w) \sigma_+(z) b_{r}, \]

\[ = \sigma_+(z) \overline{\sigma}_-(w) b_{r-1+\lambda_1} \land \cdots \land \sigma_+(z) \overline{\sigma}_-(w) b_{r} = \sigma_+(z) [\overline{\sigma}_-(w) [b]_\lambda^r, \]

and the same can be argued for the commutation of \( \sigma_+(z) \) and \( \overline{\sigma}_-(w) \).

To prove ii), equality (34), one observes that

\[ \overline{\sigma}_-(w) \sigma_+(z) b_0 = \overline{\sigma}_-(w) \sum_{j \geq 0} b_j z^j \]  

(Definition of \( \sigma_+(z) b_0 \))

\[ = b_0 + \sum_{j \geq 1} \left( b_j - \frac{b_{j-1}}{w} \right) z^j \]

\[ = b_0 + \sum_{j \geq 1} b_j z^j - \frac{z}{w} \sum_{j \geq 0} b_j z^j \]

\[ = \left( 1 - \frac{z}{w} \right) \sigma_+(z) b_0 \]

\[ = \left( 1 - \frac{z}{w} \right) \sigma_+(z) \overline{\sigma}_-(w) b_0 \]  

(35)

because, in general, \( \overline{\sigma}_-(w) \) acts on \([b]_0^r \) as the identity. So, if \( \ell(\lambda) < r \) (i.e. \( \lambda_r = 0 \)) one obtains:
Therefore:

\[ 4.6 \text{ Theorem.} \quad \text{For all } \beta \in \Lambda^r V, \text{ the following commutation rule holds:} \]
\[ \beta_0 \beta_+ (w) \sigma_+ (z) u = \left( 1 - \frac{z}{w} \right) \sigma_+ (z) \beta_0 \beta_+ (w) u. \quad (36) \]

4.7 Some preparation is needed to prove Theorem 4.6. First we introduce a piece of useful notation. If \( \beta \in V^* \), the contraction of the decomposable tensor \( u_1 \wedge \cdots \wedge u_r \) of \( \wedge^r V \) against \( \beta \) may be represented via the following diagram:

\[
\begin{vmatrix}
\beta \cdot u_1 & \beta \cdot u_2 & \cdots & \beta \cdot u_r \\
u_1 & u_2 & \cdots & u_r
\end{vmatrix}
= \begin{vmatrix}
\beta(u_1) & \beta(u_2) & \cdots & \beta(u_r) \\
u_1 & u_2 & \cdots & u_r
\end{vmatrix},
\]

which precisely means that \((-1)^{j+1} \beta \cdot u_j \) is the coefficient of the vector of \( \wedge^{r-1} V \) obtained by removing the wedge factor \( u_j \) from \( u_1 \wedge u_2 \wedge \cdots \wedge u_r \). For example

\[
\begin{vmatrix}
\beta \cdot u_1 & \beta \cdot u_2 & \beta \cdot u_3 \\
u_1 & u_2 & u_3
\end{vmatrix} = \beta(u_1) \cdot u_2 \wedge u_3 - \beta(u_2) u_1 \wedge u_3 + \beta(u_3) u_1 \wedge u_2,
\]

which is precisely the expanded expression of the contraction \( \beta \cdot (u_1 \wedge u_2 \wedge u_3) \).

4.8 Recall the generating function \( \beta(w^{-1}) := \sum_{j \geq 0} \beta_j w^{-j} \) introduced in formula (3). Since \( \beta_j \cdot b_i = \beta_j (b_i) = \delta_{ij} \), it clearly follows that \( \beta(w^{-1}) \cdot b_i = \sum_{i \geq 0} \beta_i (b_i) w^{-i} = w^{-i} \). Therefore:

\[
\beta(w^{-1}) \cdot b_i^\alpha = \begin{vmatrix}
\beta w^{-r+1-\lambda_1} & \beta w^{-r+2-\lambda_2} & \cdots & \beta w^{-\lambda_r} \\
b_{r-1+\lambda_1} & b_{r-2+\lambda_2} & \cdots & b_{\lambda_r}
\end{vmatrix}. \quad (38)
\]

4.9 Proposition. For all \( u \in \wedge^r V \):

\[
\beta(w^{-1}) \cdot \sigma_+ (z) u = \left( 1 - \frac{z}{w} \right) \sigma_+ (z) \beta(w^{-1}) \cdot u. \quad (39)
\]
Proof. Since each \( u \in \Lambda^r V \) is a finite linear combination of \([b]\Lambda_{\lambda}^r\), it is no harm to assume \( u = [b]\Lambda_{\lambda}^r \). Then we start to notice that

\[
\beta(w^{-1})z\bar{\sigma}_+(z)b_j = \beta(w^{-1})z(b_j - b_{j+1}z)
\]

(Definition of \( \bar{\sigma}_+(z) \))

\[
= \beta(w^{-1})z b_j - \beta(w^{-1})z b_{j+1}z
\]

(Action of \( z \))

\[
= \frac{1}{w^j} - \frac{z}{w^{j+1}} = \frac{1}{w^j} \left(1 - \frac{z}{w}\right).
\]

(40)

By using the expression of a contraction via diagram (37), one has:

\[
\beta(w^{-1})z\bar{\sigma}_+(z)[b]\Lambda_{\lambda}^r
\]

\[
= \begin{vmatrix}
\beta(w^{-1})z\bar{\sigma}_+(z)b_{r-1+\lambda_1} & \beta(w^{-1})z\bar{\sigma}_+(z)b_{r-2+\lambda_2} & \cdots & \beta(w^{-1})z\bar{\sigma}_+(z)b_{\lambda_r} \\
\bar{\sigma}_+(z)b_{r-1+\lambda_1} & \bar{\sigma}_+(z)b_{r-2+\lambda_2} & \cdots & \bar{\sigma}_+(z)b_{\lambda_r}
\end{vmatrix}.
\]

which by (40) is equal to:

\[
= \left(1 - \frac{z}{w}\right) \begin{vmatrix}
\frac{1}{w^{r-1+\lambda_1}} & \frac{1}{w^{r-2+\lambda_2}} & \cdots & \frac{1}{w^{\lambda_r}} \\
\bar{\sigma}_+(z)b_{r-1+\lambda_1} & \bar{\sigma}_+(z)b_{r-2+\lambda_2} & \cdots & \bar{\sigma}_+(z)b_{\lambda_r}
\end{vmatrix}.
\]

(41)

Since the determinant occurring in (41) is a linear combination of \([\bar{\sigma}_+(z)b\Lambda_{\lambda(0)}^{r-1}] = \bar{\sigma}_+(z)[b\Lambda_{\lambda(0)}^{r-1}]\)
(because \( \bar{\sigma}_+(z) \) is a HS derivation), where we denoted by \( \lambda^{(1)} \) the partition of length at most \( r - 1 \) obtained by omitting the \( j \)-th part, it follows that the action of \( \bar{\sigma}_+(z) \) can be factorized from the bottom row of (41), giving

\[
\left(1 - \frac{z}{w}\right)\bar{\sigma}_+(z) \begin{vmatrix}
\frac{1}{w^{r-1+\lambda_1}} & \frac{1}{w^{r-2+\lambda_2}} & \cdots & \frac{1}{w^{\lambda_r}} \\
b_{r-1+\lambda_1} & b_{r-2+\lambda_2} & \cdots & b_{\lambda_r}
\end{vmatrix} = \left(1 - \frac{z}{w}\right)\bar{\sigma}_+(z)\beta(w^{-1})[b\Lambda_{\lambda}^r],
\]

which ends the proof of the Proposition.
4.10 Lemma. For all \( u \in \land^r V \),
\[
\beta(w^{-1})\mathcal{L}u = \bar{\sigma}_-(w)(\beta_0 \sigma_-(w)u).
\]  
(42)

**Proof.** The proof is basically contained in [15, Proposition 4.3] but, because some mild deformity in the notation, we prefer to repeat it here. Recall the definition of transpose of a HS derivation on \( \land V \). We observe that \( \beta(w^{-1}) = \sigma_-(z)^T \beta_0 \). Then, for all \( \eta \in \land^{r-1} V \),
\[
\eta(\beta(w^{-1})\mathcal{L}u) = (\beta(w^{-1}) \land \eta)(u) \quad \text{(Definition 2.4 of contraction)}
\]
\[
= (\sigma_-(w)^T \beta_0 \land \eta)(u) \quad \text{(by the above observation)}
\]
\[
= \sigma_-(w)^T (\beta_0 \land \bar{\sigma}_-(w)^T \eta)(u) \quad \text{(Integration by parts)}
\]
\[
= \beta_0 \land \bar{\sigma}_-(w)^T \eta(\sigma_-(w)u) \quad \text{(Definition of transpose of \( \bar{\sigma}_-(w)^T \))}
\]
\[
= \bar{\sigma}_-(w)^T \eta(\beta_0 \sigma_-(w)u) \quad \text{(Again 2.4)}
\]
\[
= \eta(\bar{\sigma}_-(w)(\beta_0 \sigma_-(w)u)) \quad \text{(Definition of transpose)}.
\]

The last equality proves (42), due to the arbitrary choice of \( \eta \in \land^{r-1} V^* \approx (\land^{r-1} V)^* \). ■

4.11 Lemma. Taking \( \bar{\sigma}_+(z) \) commutes with taking contraction against \( \beta_0 \), i.e. for all \( \lambda \in \mathcal{P}_r \),
\[
\bar{\sigma}_+(z) = \sigma_-(w)^T \beta_0 \land \sigma_-(w) = \bar{\sigma}_+(z)(\beta_0 \sigma_-(w) = 1).
\]  
(43)

**Proof.** There are two cases. If \( \ell(\lambda) = r \), both members of (43) vanish. If \( \ell(\lambda) \leq r-1 \), then \( \beta_0 \bar{\sigma}_+(z)[b]_\lambda^r = (-1)^{r-\ell(\lambda)} \bar{\sigma}_-(w)[b]_{\lambda+1(r-1)}^r = \bar{\sigma}_+(z)(\beta_0 \sigma_-(w) = 1) \). ■

We are now in position to provide the

4.12 Proof of Theorem 4.6. We have:
\[
\beta_0 \sigma_-(w) \bar{\sigma}_+(z) u = \sigma_-(w) \bar{\sigma}_-(w)(\beta_0 \sigma_-(w) \bar{\sigma}_+(z) u)
\]
\[= \sigma_-(w)(\beta(w^{-1}) \land ...) \bar{\sigma}_+(z) u \quad \text{(Lemma 4.10)}
\]
\[= \left(1 - \frac{z}{w}\right) \sigma_-(w) \bar{\sigma}_+(z)(\beta(w^{-1}) \land ...) \bar{\sigma}_+(z) u \quad \text{(Proposition 4.9)}
\]
\[= \left(1 - \frac{z}{w}\right) \sigma_-(w) \bar{\sigma}_+(z)(\beta_0 \sigma_-(w) \bar{\sigma}_+(z) u) \quad \text{(again Lemma 4.10)}
\]
\[= \left(1 - \frac{z}{w}\right) \sigma_-(w) \bar{\sigma}_-(w) \bar{\sigma}_+(z)(\beta_0 \sigma_-(w) \bar{\sigma}_+(z) u) \quad \text{(formula (33) of Proposition 4.5)}
\]
\[= \left(1 - \frac{z}{w}\right) \bar{\sigma}_+(z)(\beta_0 \sigma_-(w) \bar{\sigma}_+(z) u) \quad \text{(Lemma 4.10)}
\]
\[= \left(1 - \frac{z}{w}\right) \bar{\sigma}_+(z) \bar{\sigma}_-(w) \bar{\sigma}_+(z) u \quad \text{(Lemma 4.10)}
\]
\[= \left(1 - \frac{z}{w}\right) \bar{\sigma}_+(z) u \quad \text{(Lemma 4.10)}
\]
\[= \left(1 - \frac{z}{w}\right) \bar{\sigma}_+(z)(\beta_0 \sigma_-(w) \bar{\sigma}_+(z) u) \quad \text{(Lemma 4.10)}
\]
\[= \left(1 - \frac{z}{w}\right) \bar{\sigma}_+(z)(\beta_0 \sigma_-(w) \bar{\sigma}_+(z) u) \quad \text{(Lemma 4.10)}
\]
\[\text{(44)}
\]
and Theorem 4.6 is thence proven.

5 The vertex operator $\Gamma(z_k)$

The main purpose of this section is to interpret the vertex operator $\Gamma(z_k)$, introduced in Definition 4.3, formula (30), in terms of wedging operation on the exterior algebra. This generalises [15, Proposition 4.2]. This will be done in Theorem 5.5 below and will be used in our main Theorem 7.5.

5.1 Lemma. For all $j \geq 0$ and all $k \geq 1$ one has

$$\sigma_+(z_k) b_j = b_j + \sum_{i=1}^{k} (-1)^i e_i(z_k) b_{j+i}$$

and

$$\sigma_+(z_k) b_j = b_j + \sum_{i \geq 1} h_i(z_k) b_{j+i},$$

where $e_i(z_k)$ and $h_i(z_k)$ are, respectively, the elementary and complete symmetric polynomial of degree $i$ in the indeterminates $z_k := (z_1, \ldots, z_k)$.

Proof. Formula (44) is the content of [17, Lemma 5.7] to which we refer to. Formula (45) is a consequence of (44), keeping into account that $\sigma_+(z_k)$ and $\sigma_+(z_k)$ are one the inverse of the other in $\text{End}_Q(\bigwedge V)[z_k]$.

5.2 Lemma. One has:

$$\sigma_-(z_k) b_{j+k} = b_{j+k} + \sum_{i=1}^{k} (-1)^i e_i(z_k^{-1}) b_{j+k-i},$$

where $e_i(z_k^{-1}) = e_i(z_1^{-1}, \ldots, z_k^{-1})$ is the elementary symmetric polynomial of degree $i$ in $(z_1^{-1}, \ldots, z_k^{-1})$.

Proof. The proof works the same as in [17, Lemma 6.7]. The formula holds for $k = 1$ is true, because

$$\sigma_-(z_1) b_{j+1} = b_{j+1} - \frac{b_j}{z_1}.$$

By induction, suppose that (46) holds for $k - 1 \geq 0$. Then it holds for $k$. Indeed

$$\sigma_-(z_1) \sigma_-(z_2) \cdots \sigma_-(z_k) b_{j+k}$$

$$= \sigma_-(z_1) \left[ b_{j+k} - e_1 \left( \frac{1}{z_2}, \ldots, \frac{1}{z_k} \right) b_{j+k-1} + \cdots + (-1)^{k-1} e_{k-1} \left( \frac{1}{z_2}, \ldots, \frac{1}{z_k} \right) b_{j+1} \right]$$

$$= \sigma_-(z_1) \sigma_-(z_2) \cdots \sigma_-(z_k) b_{j+k}$$

$$= \sigma_-(z_1) \sigma_-(z_2) \cdots \sigma_-(z_k) b_{j+k}.$$
\[ b_{j+k} = b_{j+k-1} \frac{1}{z_1} - e_1 \left( \frac{1}{z_2}, \ldots, \frac{1}{z_k} \right) \left( b_{j+k-1} - \frac{b_{j+k-2}}{z_1} \right) + \ldots \]
\[ + (-1)^{k-1} e_{k-1} \left( \frac{1}{z_2}, \ldots, \frac{1}{z_k} \right) \left( b_{j+1} - \frac{b_j}{z_1} \right) \quad \text{(definition of } \sigma_-(z_1)) \]
\[ = b_{j+k} + \sum_{i=1}^{k} (-1)^i e_i \left( \frac{1}{z_1}, \ldots, \frac{1}{z_k} \right) b_{j+i}, \]
as desired.

5.3 Lemma. The following equality holds for all \( 1 \leq i \leq k \):
\[
\frac{e_i(z_k)}{z_1 \cdots z_k} = e_{k-i} \left( \frac{1}{z_1}, \ldots, \frac{1}{z_k} \right). \tag{47}
\]

Proof. Recall the following definition of the elementary symmetric polynomials in \( k \) indeterminates through generating functions:
\[
\sum_{i=0}^{k} e_i(z_k) t^i = \prod_{i=1}^{k} (1 + z_i t). \tag{48}
\]
By dividing both sides of (48) by \( e_k(z_k) = z_1 \cdots z_k \) we get
\[
\sum_{i=0}^{k} \frac{e_i(z_k)}{z_1 \cdots z_k} t^i = \prod_{i=1}^{k} \left( \frac{1}{z_i} + t \right). \tag{49}
\]
The claim then follows by comparing the coefficient of \( t^i \) in the two sides (49).

5.4 Lemma. For all \( k \geq 1 \), \( r \geq 0 \) and \( \lambda \in P_r \):
\[
[b]_0^k \wedge \sigma_+(z_k)[b]_\lambda^r = e_k(z_k)^r \sigma_-(z_k)[b]_\lambda^{r+k}. \tag{50}
\]

Proof. Equality (50) holds for \( r = 1 \):
\[ [b]_0^k \land \overline{\sigma}_+(z_k)[b]_\lambda^k = [b]_0^k \land \overline{\sigma}_+(z_k)b_\lambda \]

\[ = [b]_0^k \land (b_\lambda - e_1(z_k)b_{\lambda+1} + \cdots + (-1)^k e_k(z_k)b_{\lambda+k}) \quad \text{(By 44)} \]

\[ = [b]_0^k \land (-1)^k e_k(z_k) \left[ b_{\lambda+k} - \frac{e_{k-1}(z_k)}{e_k(z_k)}b_{\lambda+k-1} + \cdots + (-1)^k \frac{1}{e_k(z_k)}b_\lambda \right] \quad \text{(By Factorization)} \]

\[ = e_k(z_k)\overline{\sigma}_-(z_k)b_{\lambda+k} \land [b]_0^k \quad \text{(By lemma (5.2))} \]

\[ = e_k(z_k)\overline{\sigma}_-(z_k)(b_{\lambda+k} \land \overline{\sigma}_-(z_k)[b]_0^k) \quad \text{(Integration by parts)} \]

\[ = e_k(z_k)\overline{\sigma}_-(z_k) (b_{\lambda+k} \land b_{k-1} \land \cdots \land b_0) \quad \text{([\overline{\sigma}_-(z_k)][b]_0^k = [b]_0^k)} \]

\[ = e_k(z_k)\overline{\sigma}_-(z_k)[b]_\lambda^{k+1}. \]

Therefore the property is true for \( r = 1 \). Assume now (50) holds true for \( r - 1 \geq 0 \). Then

\[ [b]_0^k \land \overline{\sigma}_+(z_k)[b]_\lambda^k = [b]_0^k \land \overline{\sigma}_+(z_k)b_{\lambda+1} \land \cdots \land \overline{\sigma}_+(z_k)b_\lambda \]

\[ = [b]_0^k \land (-1)^k e_k(z_k)\overline{\sigma}_-(z_k)b_{\lambda+1} \land \cdots \land (-1)^k e_k(z_k)\overline{\sigma}_-(z_k)b_{\lambda+k} \]

\[ = e_k(z_k)^r [\overline{\sigma}_-(z_k)b_{\lambda+1} \land \cdots \land \overline{\sigma}_-(z_k)b_{\lambda+k} \land \overline{\sigma}_-(z_k)[b]_0^k] \]

\[ = e_k(z_k)^r\overline{\sigma}_-(z_k) (b_{\lambda+1} \land b_{\lambda+k} \land b_{\lambda+2} \land \cdots b_0) \]

\[ = e_k(z_k)^r\overline{\sigma}_-(z_k)[b]_\lambda^{r+k}, \]

as claimed.

**5.5 Theorem.** For all \( u \in \bigwedge^k \mathbb{V}[w_k, w_k^{-1}] \) we have:

\[ \sigma_+(z_1, \ldots, z_k)[b]_0^k \land u = \prod_{j=1}^r z_j^r \cdot \Gamma(z_k)u, \quad (51) \]

the equality holding in \( \bigwedge^k \mathbb{V}[z_k, w_k][w_k^{-1}] \).

**Proof.** Recall that we consider all the Schubert derivations extended by linearity over rings of formal power series with rational coefficients. See definition 2.6. Then our arbitrary \( u \) is intended as a possibly infinite linear combination of \( [b]_\lambda^k \) with coefficients being polynomials. Then we can assume with no harm that \( u = [b]_\lambda^k \), a basis element of \( \bigwedge^k \mathbb{V} \). Keeping the same notation as in 2, we first apply integration by parts. Then
\[
\sigma_+(z_k)[b]_0^k \land [b]_{\lambda}^r = \sigma_+(z_k)\left([b]_0^k \land \sigma_+(z_k)[b]_{\lambda}^r\right) \quad \text{(By integration by parts (13))}
\]
\[
= \sigma_+(z_k)e_k(z_k)^r\sigma_-(z_k)[b]_{\lambda}^{r+k} \quad \text{(By Lemma 5.4)}
\]
\[
= e_k(z_k)^r\sigma_+(z_k)\sigma_-(z_k)[b]_{\lambda}^{r+k}
\]
\[
= \prod_{j=1}^{k} z_j^r\Gamma(z_k)[b]_{\lambda}^{r} \quad \text{(Definition of } \Gamma(z_k)\text{)}
\]
as desired.

If \( k = 1 \), and \( z = z_1 \), one obtains
\[
\sigma_+(z)b_0 \land [b]_{\lambda}^r = z^r\Gamma(z)[b]_{\lambda}^{r} = z^r\sigma_+(z)\sigma_-(z)[b]_{\lambda}^{r+1},
\]
which is precisely [15, Proposition 5.4] or [16, Proposition 3.2]. They shape looks more involved because we use here better notation.

6 The vertex operator \( \Gamma^*(w_k) \)

In the same vein of Section 5, this section will be devoted to interpret in terms of contraction operations the action of the vertex operator \( \Gamma^*(w_k) \) on \( \land V \). The output will be Theorem 6.3, stated at the end of the section, which will be another building block of the main Theorem 7.5. We begin with some preparation.

6.1 Lemma. The following equality holds for all \( r \geq 1 \) and all \( \lambda \in P_r \):
\[
\sigma_-(w_0 \sigma_-(w))[b]_{\lambda}^{r} = \Delta_{\lambda}(\sigma_-(w)H_{r-1})[b]_0^{r-1}. \tag{52}
\]
Proof. This is [15, Lemma 5.8].

6.2 Lemma.
\[
\beta(w^{-1})_\mu [b]_{\lambda}^r = w^{-r+1}\sigma_+(w)\Delta_{\lambda}(\sigma_-(w)H_{r-1})[b]_0^{r-1} \tag{53}
\]
Proof. Invoking Lemma 4.10,
\[
\beta(w^{-1})_\mu [b]_{\lambda}^r = \sigma_-(w)(\beta_0 \sigma_-(w))[b]_{\lambda}^r. \tag{54}
\]
Since \( \beta_0 \sigma_-(w)[b]_{\lambda}^r \) is a linear combination of \( [b]_{\mu}^{r-1} \) with \( \ell(\mu) = r - 1 \) (i.e. no \( b_0 \) occurs in the monomial), then by [15, Proposition 4.3]
\[
\sigma_-(w)(\beta_0 \sigma_-(w))[b]_{\lambda}^r = w^{-r+1}\sigma_+(w)\sigma_{-r+1}(\beta_0 \sigma_-(w)[b]_{\lambda}^r).
\]
Using Lemma 6.1 one obtains (53).
6.3 Theorem. The following equality holds:

\[(\beta(w_k^{-1}) \land \beta(w_{k-1}^{-1}) \land \cdots \land \beta(w_1^{-1})) \cup [b]_{k}^{\sigma} = \frac{\Delta_0(w_k)}{(w_1 \cdots w_k)^{r-1}} \Gamma^s(w_k)[b]_{k}^{\sigma}. \tag{55}\]

where \(\Delta_0(w_k)\) denotes the Vandermonde determinant \(\prod_{1 \leq i < j \leq k}(w_j - w_i)\).

Proof. For \(k = 1\) the property

\[\beta(w_1^{-1}) \cup [b]_{1}^{\sigma} = w_1^{-1} \sigma_+(w_1) \Delta_{\lambda}(\sigma_-(w_1) H_{r-1})[b]_0^{r-1}\]

is just Lemma 6.2. Arguing by induction, let us assume the claim holding true for \(0 \leq k - 1 \leq r - 1\) and let us show it holds for \(k\). We have

\[\beta(w_k^{-1}) \land \beta(w_{k-1}^{-1}) \land \cdots \land \beta(w_1^{-1}) \cup [b]_{k}^{\sigma} = \beta(w_k^{-1}) \cup (\beta(w_{k-1}^{-1}) \land \cdots \land \beta(w_1^{-1}) \cup [b]_{k}^{\sigma}).\]

Using the inductive hypothesis:

\[= \beta(w_k^{-1}) \cup \frac{\Delta_0(w_{k-1})}{(w_1 \cdots w_{k-1})^{r-1}} \sigma_+(w_{k-1}) \Delta_{\lambda}(\sigma_-(w_{k-1}) H_{r-k+1})[b]_0^{r-k+1}.\]

By applying Lemma 6.2, one gets:

\[= w_k^{-r+k+1} \sigma_+(w_k) [\beta_0 \sigma_-(w_k) \sigma_+(w_{k-1}) \Delta_{\lambda}(\sigma_-(w_{k-1}) H_{r-k+1})[b]_0^{r-k+1}] \]

\[= \frac{\Delta_0(w_{k-1})}{(w_1 \cdots w_{k-1})^{r-1}} \sigma_+(w_k) \sigma_+(w_{k-1}) \left[(\beta_0 \sigma_-(w_k) \Delta_{\lambda}(\sigma_-(w_{k-1}) H_{r-k+1})[b]_0^{r-k+1})\right]\]

Now we use the commutation rules prescribed by Theorem 4.6 and Lemma 4.11:

\[= \frac{\Delta_0(w_{k-1})}{(w_k - w_{k-1})^{r-k}} \frac{\Delta_0(w_{k-1})}{(w_1 \cdots w_{k-1})^{r-1}} \sigma_+(w_k) \Delta_{\lambda}(\sigma_-(w_k) H_{r-k})[b]_0^{r-k}\]

\[= \frac{\Delta_0(w_k)}{(w_1 \cdots w_{k-1} \cdot w_k)^{r-1}} \sigma_+(w_k) \Delta_{\lambda}(\sigma_-(w_k) H_{r-k})[b]_0^{r-k}\]

\[= \frac{\Delta_0(w_k)}{(w_1 \cdots w_{k-1} \cdot w_k)^{r-1}} \Gamma^s(w_k)[b]_{k}^{\sigma},\]

as claimed.
7 The main Theorem and its declinations

In this section we shall be concerned with the several declinations of the main theorem describing the $B_r$ representation of $\mathfrak{gl}(\bigwedge^k V)$.

7.1 Let $\mathfrak{gl}(\bigwedge^k V) := \bigwedge^k V \otimes \bigwedge^k V^*$ be the restricted Lie algebra of endomorphisms of $\bigwedge^k V$, with respect to the natural commutator. With the same notation as in, a basis of $\bigwedge^k V \otimes \bigwedge^k V^*$ is $(\mu, \nu) \in \mathbb{P}_k$, i.e.

$$\bigwedge^k V \otimes \bigwedge^k V^* = \bigoplus_{\mu, \nu \in \mathbb{P}_k} \mathbb{Q} \cdot [b]_\mu \otimes [\beta]_\nu^k,$$

where $[\beta]_\nu^k([b]_\mu^k) = \delta_{\mu, \nu}$.

Then the $\mathfrak{gl}(\bigwedge^k V)$-module structure of $B_r$ is defined through the following equality holding in $\bigwedge^r V$:

$$(b)_{\mu}^k \otimes [\beta]_\nu^k \star \Delta_\lambda(H_r)) [b]_0^r = [b]_{\mu}^k \wedge ([\beta]_\nu^k \cup [b]_{\lambda}^k). \quad (56)$$

This action is very easy to describe in the case $k = r$, but it becomes trickier when $r-k > 0$. To describe it we shall consider the generating function

$$\mathcal{E}(z_k, w_k^{-1}) := \sum_{\mu, \nu} [b]_\mu^k \otimes [\beta]_\nu^k \cdot s_\mu(z_k) s_\nu(w_k^{-1}) : B_r \to B_r[[z_k, w_k]][z_k^{-1}, w_k^{-1}].$$

Our main result will consist in the explicit description of $\mathcal{E}(z_k, w_k^{-1})\Delta_\lambda(H_r)$ in case $k \leq r$ (because otherwise one would obtain the trivial null action), where $\mathcal{E}(z_k, w_k^{-1})\Delta_\lambda(H_r)$ is such that

$$\left(\mathcal{E}(z_k, w_k^{-1})\Delta_\lambda(H_r)\right) [b]_0^r = \sum_{\mu \in \mathbb{P}_k} s_\mu(z_k) [b]_\mu^k \wedge (s_\nu(w_k^{-1}) [\beta]_\nu^k \cup [b]_{\lambda}^k), \quad (57)$$

and where $s_\mu(z_k)$ and $s_\nu(w_k^{-1})$ denote the Schur symmetric polynomials labeled by the respective partitions with respect to the variables $z_k$ and $w_k^{-1}$ respectively.

7.2 Lemma. The generating function of the basis $(\mu, \nu) \in \mathbb{P}_k$ of $\bigwedge^k V$ is:

$$\sum_{\mu \in \mathbb{P}_k} s_\mu(z_k) [b]_\mu^k = \sigma_+(z_k) [b]_0^k.$$

Proof. By exploiting the definition of the $B_k$-module structure of $\bigwedge^k V$, we have
\[
\begin{align*}
\sum_{\mu \in \mathcal{P}_k} s_\mu(z_k)[b]_\mu^k = & \quad \sum_{\mu \in \mathcal{P}_k} s_\mu(z_k)\Delta_\mu(H_k)[b]_0^k \\
= & \quad \left( \sum_{\mu \in \mathcal{P}_k} s_\mu(z_k)\Delta_\mu(H_k) \right)[b]_0^k \\
= & \quad \prod_{j=1}^k \left( 1 + h_1 z_j + h_2 z_j^2 + \cdots \right)[b]_0^k \\
= & \quad \frac{1}{E_k(z_1)} \cdot \frac{1}{E_k(z_2)} \cdots \frac{1}{E_k(z_k)}[b]_0^k \quad \text{(By Cauchy formula as in [10, Proposition 2, (iii)])} \\
= & \quad \sigma_+(z_k)[b]_0^k.
\end{align*}
\]

7.3 Lemma. The generating function of the basis elements \(\wedge^k V^*\) is:
\[
\sum_{\nu \in \mathcal{P}_k} s_\nu(w_{k}^{-1})[\beta]_\nu^k = \frac{\prod_{j=1}^k w_{k}^{j-1}}{\Delta_0(w_{k})} \cdot \beta(w_{k}^{-1}) \wedge \cdots \wedge \beta(w_{1}^{-1}). \quad (58)
\]

Proof. The one we propose consists in expanding the wedge product of the generating series of the basis \((\beta_j)_{j \geq 0}\) of \(V^*\):
\[
\beta(w_{k}^{-1}) \wedge \cdots \wedge \beta(w_{1}^{-1}) \quad (59)
\]
\[
= \sum_{\nu \in \mathcal{P}_k} \sum_{\tau \in S_k} \sgn(\tau) \beta_{k-\tau(1)+\nu(1)} \wedge \beta_{k-\tau(2)+\nu(2)} \cdots \wedge \beta_{k-\tau(k)+\nu(k)}; \\
= \sum_{\nu \in \mathcal{P}_k} \Delta_\nu(w_{k}^{-1}) \sum_{\nu \in \mathcal{P}_k} s_\nu(w_{k}^{-1})[\beta]_\nu^k \cdot \frac{\Delta_0(w_{k})}{w_{k}^{k-1} \cdots w_{1}^{k-1}}, \quad (60)
\]
whence the claim, obtained by multiplying both (58) and (59) by \((\prod_{j=1}^k w_{j}^k)/\Delta_0(w_{k})\).

7.4 Lemma. For all \(u \in \wedge^r V[[w_k, w_{k}^{-1}]]\)
\[
\sum_{\mu} (\Delta_\mu(z_k))[b]_\mu^k \wedge u = \sigma_+(z_k)[b]_0^k \wedge u = \prod_{j=1}^k z_j^r \Gamma(z_k) u. \quad (61)
\]
**Proof.** We specified that the equality holds in $\bigwedge^r \mathbb{V}[w_k,w_k^{-1}]$ to emphasize the supposed $\mathbb{Q}[w_k,w_k^{-1}]$ linearity of the Schubert derivation. This said, it is not restrictive to assume that $u$ is a basis element $[b]_\lambda$ of $\bigwedge^r \mathbb{V}$. The basic remark is that

$$\frac{1}{E_k(z_1)} \cdot \frac{1}{E_k(z_2)} \cdots \frac{1}{E_k(z_k)} = \prod_{j=1}^{k} (1 + h_1 z_j + h_2 z_j^2 + h_3 z_j^3 + \cdots) = \sum_{\mu \in \mathcal{P}_k} s_\mu(z_k) \Delta_\lambda(H_k),$$

where in the last equality we used one of the declination of the celebrated Cauchy formula, as in [10, Proposition 2, (iii)], already used in the proof of Lemma 7.2. Therefore:

$$\sum \mu \in \mathcal{P}_k [b]_\mu^k s_\mu(z_k) = \sum \mu \in \mathcal{P}_k (\Delta_\mu(H_k)s_\mu(z_k))[b]_0^k = \frac{1}{E_k(z_1) \cdots E_k(z_k)}[b]_0^k = \sigma_+(z_k)[b]_0^k,$$

where in the last equality we have repeatedly used the module structure of $\bigwedge^r \mathbb{V}$ over $B_r$. Then:

$$\sum \mu \in \mathcal{P}_k [b]_\mu^k s_\mu(z_k) \wedge u = \sigma_+(z_k)[b]_0^k \wedge u,$$

and the result now follows from Theorem 5.5. 

We can finally express the action of the generating function $E_{\mu,\nu}(z_k,w_k^{-1})$ on a basis element of $B_r$.

**7.5 Theorem (First Version).** For all $\lambda \in \mathcal{P}_r$:

$$(E(z_k,w_k^{-1})\Delta_\lambda(H_r))[b]_0^r = \prod_{j=1}^{k} \left( \frac{z_j}{w_j} \right)^{r-k} \Gamma(z_k)^r \Gamma^*(w_k)[b]_\lambda^r. \quad (62)$$

**Proof.** We have

$$E(z_k,w_k^{-1})[b]_\lambda^r = \sum_{\mu \in \mathcal{P}_k} s_\mu(z_k)[b]_\mu^k \wedge \sum_{\nu \in \mathcal{P}_k} s_\nu(w_k) \left( [\beta]_{\nu,\mu}^k[b]_\lambda^r \right) \quad \text{(definition of} \ E(z_k,w_k^{-1}))$$

$$= \sigma_+(z_k)[b]_0^k \wedge \frac{1}{\Delta_0(w_k)} \cdot \beta(w_k^{-1}) \wedge \cdots \wedge \beta(w_k^{-1}) \cdot [b]_\lambda^r \quad \text{(Lemmas} \ 7.2 \text{and} \ 7.3)$$

$$= \sigma_+(z_k)[b]_0^k \wedge \frac{1}{\prod_{j=1}^{k} w_j^{r-k}} \Gamma^*(w_k)[b]_\lambda^r. \quad \text{(Theorem} \ 6.3)$$
Now we use the fact that $\Gamma^{*}(w_k)[b]_\lambda^r$ is a $\mathbb{Q}[w_k, w_k^{-1}]$-linear combination of $[b]_{\mu}^{r-k}$ and then, by invoking Theorem 5.5, applied to last equality above, becomes:

$$\prod_{j=1}^{k} \left( \frac{z_j}{w_j} \right)^{r-k} \Gamma(z_k) \Gamma^{*}(w_k)[b]_\lambda^r,$$

as announced. □

7.6 Corollary. If $r-k \geq \ell(\lambda)$ then

$$\Gamma(z_k) \Gamma^{*}(w_k)[b]_\lambda^r = \sigma_+(z_k) \overline{\sigma}-(w_k) \sigma_-(w_k)[b]_\lambda^r.$$

Proof. First of all notice that for every $\lambda \in \mathcal{P}_r$, it turns out that

$$\overline{\sigma}+(w_k) \sigma_-(w_k)[b]_\lambda^{r-k} = \sum_{\mu \in \mathcal{P}_r} a_{\mu}(w_k, w_k^{-1}) [b]_\mu^{r-k},$$

where $a_{\mu}(w_k, w_k^{-1}) \in \mathbb{Q}[w_k, w_k^{-1}]$. Then we have:

$$\Gamma(z_k) \Gamma^{*}(w_k)[b]_\lambda^r = \Gamma(z_k) \overline{\sigma}+(w_k) \sigma_-(w_k)[b]_\lambda^{r-k}$$

$$= \sum_{\mu \in \mathcal{P}_r} a_{\mu}(w_k, w_k^{-1}) \Gamma(z_k)[b]_\mu^{r-k}$$

$$= \sum_{\mu \in \mathcal{P}_r} a_{\mu}(w_k, w_k^{-1}) [b]_\lambda^r$$

$$= \sigma_+(z_k) \overline{\sigma}-(w_k) \sum_{\mu \in \mathcal{P}_r} a_{\mu}(w_k, w_k^{-1}) [b]_\mu^{r}$$

$$= \sigma_+(z_k) \overline{\sigma}-(w_k) \sigma_+(z_k) \overline{\sigma}-(w_k)[b]_\lambda^r.$$

7.7 Remark. If $\ell(\lambda) > r-k$, Corollary 7.6 fails. We have however the following uniform way to compute $\Gamma^{*}(w_k)[b]_\lambda^r$.

7.8 Proposition. For all $\lambda \in \mathcal{P}_r$ and $k, r \geq 0$:

$$\Gamma^{*}(w_k)[b]_\lambda^r = \begin{bmatrix}
1 & \frac{1}{w_1^{r-1+\lambda_1}} & \cdots & \frac{1}{w_1^{r+\lambda_r}} \\
\vdots & \ddots & \ddots & \ddots \\
\frac{1}{w_k^{r-1+\lambda_1}} & \cdots & \frac{1}{w_k^{r+\lambda_r}} \\
b_{r-1+\lambda_1} & \cdots & b_{\lambda_r}
\end{bmatrix} \in \bigwedge^{r-k} V,$$
This enables to state a second version of 4.1.

**Proof.** By Theorem 6.3

\[
\Gamma^r(w_k)[b]_\lambda^r = \frac{(z_1 \cdots z_k)^{r-1}}{\Delta_0(w_k)} (\beta(w_k^{-1}) \wedge \beta(w_{k-1}^{-1}) \wedge \cdots \wedge \beta(w_1^{-1}))_\lambda[\beta(w_k^{-1}) \wedge \cdots \wedge \beta(w_1^{-1})]_\lambda[\beta(w_k^{-1}) \wedge \cdots \wedge \beta(w_1^{-1})]_\lambda.
\]

\[
= \frac{(z_1 \cdots z_k)^{r-1}}{\Delta_0(w_k)} \begin{vmatrix}
1 & 1 & 1 \\
\frac{w^{-1}_{i-1+\lambda_1}}{w^{-1}_{r-1}} & \frac{w^{-1}_{i-2+\lambda_2}}{w^{-1}_{r-2}} & \cdots & \frac{w^{-1}_{i-r+\lambda_r}}{w^{-1}_{r-r}} \\
\vdots & \vdots & \ddots & \\
\frac{w^{-1}_{k-1+\lambda_1}}{w^{-1}_{r-1}} & \frac{w^{-1}_{k-2+\lambda_2}}{w^{-1}_{r-2}} & \cdots & \frac{w^{-1}_{k-r+\lambda_r}}{w^{-1}_{r-r}} \\
b_{r-1+\lambda_1} & b_{r-2+\lambda_2} & \cdots & b_{\lambda_r}
\end{vmatrix}.
\]

(63)

Let us agree that

\[
w_k^{r-k} \cdots w_2^{r-2} w_1^{r-1} \Delta_\lambda(w_k^{-1}, H_{r-k})[\beta]_0 = \beta(w_k^{-1}) \wedge \cdots \wedge \beta(w_1^{-1})]_\lambda[\beta(w_k^{-1}) \wedge \cdots \wedge \beta(w_1^{-1})]_\lambda,
\]

defines \(\Delta_\lambda(w_k, H_{r-k}) \in B_r[w^{-1}]\). The expansion of (63) as a linear combinations of basis elements of \(\bigwedge^{r-k} V\), Giambelli’s formula (22) and the expansion rule of a determinant, easily imply that

\[
\Delta_\lambda(w_k^{-1}, H_{r-k}) = \begin{vmatrix}
\frac{1}{w_1^{\lambda_1}} & \frac{1}{w_1^{\lambda_2}} & \cdots & \frac{1}{w_1^{\lambda_r-r+1}} \\
\vdots & \vdots & \ddots & \\
\frac{1}{w_k^{\lambda_1+k-1}} & \frac{1}{w_k^{\lambda_2+k-2}} & \cdots & \frac{1}{w_k^{\lambda_r+k-r}} \\
h_{\lambda_1+k} & h_{\lambda_2+k+1} & \cdots & h_{\lambda_r+k+r-1} \\
\vdots & \vdots & \ddots & \\
h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \cdots & h_{\lambda_r}
\end{vmatrix}.
\]

(64)

This enables to state a second version of 7.5, which works well for practical purposes and generalises [16, Main Theorem 4.3]:

**7.9 Theorem (second Version).** The following equality holds:

\[
\mathcal{E}(z_k, w_k^{-1}) \Delta_\lambda(H_r) = \prod_{j=1}^{k} \frac{z_j}{w_j}^{r-k} \frac{1}{\mathcal{E}_r(z_j)^{r-k}} \Delta_\lambda(w_k^{-1}, \sigma_-(z) H_r).
\]

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\[
\prod_{j=1}^{k} \left( \frac{z_j}{w_j} \right)^{r-k} \frac{1}{E_r(z_j)} \quad \begin{array}{cccc}
\frac{1}{w_1^{\lambda_1}} & \frac{1}{w_1^{\lambda_2-1}} & \cdots & \frac{1}{w_1^{\lambda_{r+1}}} \\
\vdots & \vdots & \ddots & \\
\frac{1}{w_k^{\lambda_1+k-1}} & \frac{1}{w_k^{\lambda_2+k-2}} & \cdots & \frac{1}{w_k^{\lambda_{r+k-1}}} \\
\end{array}
\]

where

\[
\sigma_-(z_k)h_{\lambda_1+k} \quad \sigma_-(z_k)h_{\lambda_2+k+1} \quad \cdots \quad \sigma_-(z_k)h_{\lambda_{r+k+r-1}} \\
\vdots \quad \vdots \quad \ddots \quad \\
\sigma_-(z_k)h_{\lambda_1+r-1} \quad \sigma_-(z_k)h_{\lambda_2+r-2} \quad \cdots \quad \sigma_-(z_k)h_{\lambda_r} \\
\]

Proof. By Theorem 7.5 we have:

\[
\mathcal{E}(z_k, w_k^{-1}) \Delta_\lambda(H_r) = \prod_{j=1}^{k} \left( \frac{z_j}{w_j} \right)^{r-k} \Gamma(z_k) \Gamma^*(w_k) \Delta_\lambda(H_r) 
\]

(27) (Definition 4.3–(30) of \( \Gamma(z_k) \))

\[
= \prod_{j=1}^{k} \left( \frac{z_j}{w_j} \right)^{r-k} \sigma_+(z_k) \sigma_-(z_k) \Delta_\lambda(w_k, H_r) 
\]

(Definition 3.4 of the \( B_r \)-module structure of \( \bigwedge^r V \) and Proposition 3.8)

as desired. Expression (66) for \( \sigma_-(z_k)h_j \) is Lemma 5.2.

7.10 Finally, let us define, as it is customary, new formal variable \((x_j)_{j \geq 1}\) through the equality:

\[
\exp \left( \sum_{j \geq 0} x_j z^j \right) = \frac{1}{E_r(z)}.
\]
In this case one can write
\[
\prod_{j=1}^{k} \frac{1}{E_r(z_j)} = \exp \left( \sum_{j=0}^{r-k} x_j p_j(z_k) \right),
\]
where \( p_j(z_k) = z_1^j + \cdots + z_k^j \) is the \( j \)-th power sum symmetric polynomial in \((z_1, \ldots, z_k)\) and where \( x_i \) is precisely the \( i \)-th degree power sum in the \( r \) universal roots \((y_1, \ldots, y_r)\) of the polynomial \( E_r(z) \), i.e. \( E_r(z) = \prod_{i=1}^{r} (1 - y_i z) \) in the universal splitting \( \mathbb{Q} \)-algebra for the polynomial \( E_r(z) \in \mathbb{Q}[z] \). This allows to shape our result in the form

7.11 Corollary. We have:

\[
\mathcal{E}(z_k, w_k^{-1}) \Delta_{\lambda}(H_r) = \prod_{j=1}^{k} \left( \frac{z_j}{w_j} \right)^{r-k} \exp \left( \sum_{j=0}^{r-k} x_j p_j(z_k) \right) \Delta_{\lambda}(w_k, \overline{z}_-(z_k)H_{r-k}). \tag{67}
\]

Formula (67) is easy to use for practical computations of the \( \mathfrak{gl}((\wedge^k V) \)-representation of \( B_r \), for those special case of \( k \) and \( r \) that everybody may possibly need.

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