Spacetimes as topological spaces, and the need to take methods of general topology more seriously

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Abstract

Why is the manifold topology in a spacetime taken for granted? Why do we prefer to use Riemann open balls as basic-open sets, while there also exists a Lorentz metric? Which topology is a best candidate for a spacetime; a topology sufficient for the description of spacetime singularities or a topology which incorporates the causal structure? Or both? Is it more preferable to consider a topology with as many physical properties as possible, whose description might be complicated and counterintuitive, or a topology which can be described via a countable basis but misses some important information? These are just a few from the questions that we ask in this Chapter, which serves as a critical review of the terrain and contains a survey with remarks, corrections and open questions.
1 Introduction.

1.1 The Manifold Topology vs. Finer or Incomparable Topologies.

In [1], the author supports that the manifold topology in a curved spacetime is the best possible and most natural choice, against the class of topologies that was suggested by Zeeman and Göbel (see [9] and [10], respectively). His main focus lies on topologies finer than the manifold one, but there is a misjudgement here: there are topologies in the class $\mathcal{Z}$ of Zeeman-Göbel topologies (as we shall see in paragraph 4), that are neither finer nor equal nor coarser than the manifold topology.

Instead of asking whether we need a finer topology for a sufficient mathematical description of a spacetime, we bring the topologisation question into a different level; why should one prefer a topology which describes spacetime singularities against a topology which hides singularities but incorporates the causal structure of the spacetime? As we shall see, the singularity theorems were proven under the frame of the manifold topology, while there are topologies in the class $\mathcal{Z}$ where the Limit Curve Theorem (LCT for abbreviation) fails to hold, and so sufficient conditions for the formation of singularities as we understand them in the presence of Riemannian basic-open balls fail as well.

Zeeman’s main arguments against the Euclidean $\mathbb{R}^4$ topology for Minkowski spacetime $M$ (extended by Göbel for curved spacetimes) can be summarised as follows:

1. The 4-dimensional Euclidean topology is locally homogeneous, whereas $M$ is not; every point has associated with it a light cone, separating space vectors from time vectors.

2. The group of all homeomorphisms of 4-dimensional Euclidean space is vast, and of no physical significance.

Heathcote’s antilogue belongs to (sic) a realist view of spacetime topology as against the instrumentalist position. A realist point of view divides the space into structural levels, such as metric tensor field, affine connection, conformal structure, differentiable manifold and topology. Heathcote highlights that the manifold topology is present as long as the structure of manifold is present, and there are two “untenable” possibilities for a replacement of the manifold topology, in both cases by finer topologies (see [1], page 255, for more details).
Heathcote’s arguments miss here that there are topologies in the class $\mathcal{Z}$ that are neither finer nor coarser than the manifold topology, as we shall see, but our disagreement does not lie only on this ground; we believe that the answer to the question “what comes first, the metric or the topology” cannot be a definite answer in favour of the metric (see paragraph 5). There is a Lorentz metric which is ignored by the Riemann open balls that serve as basic-open sets for the manifold topology. In addition, there are topologies different (and not finer, coarser or equal) than the manifold topology, which incorporate the causal structure of the spacetime and they could be considered as natural topologies for a spacetime, as well.

In recent articles (see [12] and [13] and paragraph 6 here) the authors talked about topologies in the class $\mathcal{Z}$, in the sense that general relativity generically leads to spacetime singularities where it breaks down as a physical theory; a particular topology in $\mathcal{Z}$ (that we have called $\mathcal{Z}$), different than the manifold one, was proven to be the most natural one for this frame. It is in those papers that the authors left as an open question a different approach to the topologisation of spacetime: the definition of a dynamical evolution of the spacetime given specific causal and topological conditions (see paragraph 7). It is conjectured that the challenges or even contradictions that arise in the study and understanding of spacetime geometry are due to the “static” nature of a topological structure; indeed, there is a specific fixed topology in the background and the topological properties arising from this topology affect the spacetime as a whole. This rigidity in the study of spacetime geometry might be cured if one develops a topological space with an evolving topology, which will incorporate quantum and relativistic frames within the spacetime and outside it (Planck time and length, singularities, etc.). One of the aims of this critical survey is to open such a discussion as well.

The different approaches in the study of spacetime geometry, and in the topologisation of a spacetime in our case, are due to different cultural backgrounds; Penrose states a similar argument in [14]. Those who come from QFT (quantum field theory), for example, and those from Einstein’s general relativity seem to view things in a different way (here we should add those who come from a purely mathematical background, as well). Those from QFT, according to Penrose, would tend to take renormalizability or, better, finiteness, as the primary aim of the union of relativity and quantum theories. Those having a relativistic background would take the deep conceptual conflicts (determinism, causality, background
independence) between the principles of quantum mechanics and those of general relativity to be the centrally important issues that needed to be resolved, and from whose resolution we should expect to move forward to a new physics of the future. Those from a purely (theoretical) mathematical background, coming straight from the Platonic world of Penrose, would love to see a spacetime as an integrated mathematical entity, a structure with physical properties coinciding harmonically with the mathematical formulation.

It should be said that the description of fluctuating topologies, or topological transitions, has become a debated topic in theoretical physics since the visionary introduction of the concept of Spacetime Foam, by John Wheeler in the Fifties [15]. In string theory these ideas have been explored in the early Nineties, among others, by Greene (see Refs. [16]), and innumerable have been the applications of the concept of spacetime foam to different problems (see e.g. Ref. [17]). In recent years, research lines emerged that aim to derive the concept of spacetime itself from quantum entanglement. The seminal paper of Raamsdonk [18] paved the way to the more recent works of Susskind and Maldacena [19] (for a readable review see New Scientist [20]). Authors who, by the way, are all building upon two fundamental, only apparently disconnected, papers written by Einstein in 1935, the so called E.R. and E.P.R. papers [21]. On the other hand, already in a model of spacetime as simple as a lattice (see, for example, Ref. [22]) we see how the actual topology of spacetime can deeply affect the formulation of the fundamental structures of physical theories (in that case, the definition of the fundamental commutator of QM is deformed by the lattice structure of the underlying spacetime).

The opinions that are presented in this chapter can be considered as opinions stemming from the family of pure mathematicians (plus a theoretical physicist) and it is expected that they will not easily drag the attention of a large number of physicists: it is in our beliefs though that a spacetime as an integrated mathematical entity, a spacetime studied as a topological space, would play a significant role to the to search for a theory of quantum gravity. In a few words, the methods of general topology should be taken more seriously from those working in QFT as well as those in general relativity, at least.
1.2 On Name-giving and Notation.

In the geometry of spacetime we introduce three relations: the chronological order $\ll$, the causal order $\prec$ and the relation horismos $\rightarrow$. These relations can be extended to any event space $(M, \ll, \prec, \rightarrow)$ having no metric (see [2] and [3]).

In particular, we say that $x$ chronologically precedes an event $y$ -written $x \ll y$- if $y$ lies inside the future null cone of $x$. $x$ causally precedes $y$ -written $x \prec y$- if $y$ lies inside or on the future null cone of $x$. Last, but not least, $x$ is at horismos with $y$ -written $x \rightarrow y$- if $y$ lies on the future null cone of $x$. The order $\ll$ is irreflexive, the order $\prec$ is reflexive and the relation $\rightarrow$ is reflexive, too.

In addition, the chronological future of an event $x$ is denoted by $I^+(x) = \{y \in M : x \ll y\}$ while its causal future by $J^+(x) = \{y \in M : x \prec y\}$ (with a minus instead of a plus sign, dually, for the pasts in each case, respectively). The future null cone of $x$ is denoted by $\mathcal{N}^+(x) \equiv \partial J^+(x) = \{y \in M : x \rightarrow y\}$ and, dually, we put a minus for the null past of $x$.

The chronological past and future of an event $x$ determine its time cone, its causal past and future its causal cone and its null past and future its light cone.

When physicists refer to the null cone of an event $x$ they actually mean the causal cone. Zeeman, as a working topologist, preferred to break down the definition of null cone into three parts, for working with the interior, closure, boundary and exterior of it (see paragraph 3, of [9]).

We should now mention a few problems in name-giving that arise from when one corresponds order-theoretic and topological notions from the classical theory of ordered sets to a spacetime manifold. Following the construction of the interval topology (see [8]), it seems natural to say that a subset $A \subset X$ is a past set if $A = I^-(A)$ and a future set if $A = I^+(A)$. One then would expect that the future topology $\mathcal{T}^+$ is generated by the subbase $\mathcal{S}^+ = \{X \setminus I^-(x) : x \in X\}$ and the past topology $\mathcal{T}^-$ by $\mathcal{S}^- = \{X \setminus I^+(x) : x \in X\}$. Then, the interval topology $\mathcal{T}_\text{in}$ on $M$ would consist of basic sets which are finite intersections of subbasic-open sets of the past and the future topologies.

First of all, the names “future topology” and “past topology” are due to the lack of inspiration for other names for such topological analogues in a spacetime, but here we should have in mind that when one considers the chronological relation and identifies $\downarrow \{x\}$ with $I^-(x)$,
then obviously $x \notin I^-(x)$. On the contrary, things follow the pattern of the construction in [8] when one considers the causal order $\prec$. Furthermore, $M \setminus I^-(x)$ will not be a future set with $\ll$, according to the definition that a future set satisfies $X = \uparrow X$. All these are not real problems at all, when it comes to our target to describe particular topologies which incorporate the causal structure of a spacetime (see the section Topologies Different than the Manifold Topology, below, and the corresponding references in it); the problem is sort of corresponding more appropriate names to these topologies, as well as developing a more systematic and simplified notation. We believe that this is not a difficult task to achieve in the near future.

One more point, regarding the appropriateness of a name; the Minkowski space in particular (and spacetimes in general) is not up-complete, and a topology $\mathcal{T}_m$ for a spacetime belongs actually to a coarser topology than the interval topology of [8]. So we will treat the interval topology of [8] as a special case referring to up-complete sets, and our $\mathcal{T}_m$ spacetime topologies belonging to a more general case where up-completeness is not a necessary condition. It is worth mentioning though that for the particular case of 2-dimensional Minkowski spacetime, $\mathcal{T}_m$ under $\ll$ is the interval topology that one defines using [8].

Finally, we would also like to highlight the distinction between the interval topology $\mathcal{T}_m$ from the “interval topology” of A.P. Alexandrov (see [3], page 29 and the succeeding section here). $\mathcal{T}_m$ is of a more general nature, and it can be defined via any relation, while the Alexandrov topology is restricted to the chronological order. These two topologies are different in nature, as well as in definition, so we propose the use of “interval topology” for $\mathcal{T}_m$ exclusively, and not for the Alexandrov topology.

2 Topologies coarser than or equal to the manifold topology.

In the literature, starting from the first modern singularity theorem by Penrose (see [23]) till recent accounts on singularities such as [24], there is no explicit mentioning of the topology of a spacetime $M$, while Riemann metric and Riemann basic-open balls can be used whenever there is a need, for example for the proof of the Limit Curve Theorem (LCT) and the convergence of causal curves (for a detailed exposition see [25] and [26]). In addition to the
manifold topology $\mathcal{M}$, one can consider the Alexandrov topology $\mathcal{A}$ which has basic-open sets known as “diamonds” and are simply the intersections of future and past time-cones, of two distinct events respectively. This topology incorporates the causal structure of a spacetime, but equals the manifold topology only in the following case (see [3]).

**Theorem 2.1.** On a spacetime $M$, the following are equivalent:

1. $M$ is strongly causal.
2. $\mathcal{A}$ agrees with $\mathcal{M}$.
3. $\mathcal{A}$ is Hausdorff.

So, the main contribution of the topology $\mathcal{A}$ is a characterisation of strong causality, as soon as $\mathcal{A}$ is Hausdorff. Adding the fact that it incorporates the causality (in particular the chronology) of a spacetime by the construction of open diamonds, $\mathcal{A}$ looks like a great candidate for a spacetime topology when it is Hausdorff but, following Zeeman’s arguments, its group of homeomorphisms is vast and of no physical meaning, both in the Minkowski spacetime and in curved spacetimes.

The existence of a Lorentz metric in a spacetime is enough to make us conclude that neither the manifold topology $\mathcal{M}$ nor the Alexandrov topology $\mathcal{A}$ “in its best”, that is when Theorem 2.1 is satisfied, can fully describe a spacetime topologically. The manifold topology is a natural topology for a manifold, but not such a natural one for a spacetime manifold!

### 3 The class $\mathcal{Z}$ of Zeeman-Göbel topologies.

The class $\mathcal{Z}$ of Zeeman topologies on a spacetime manifold $M$ consists of topologies which have the property that they induce the 1-dimensional manifold topology on every time axis and the 3-dimensional manifold topology on every space axes. This class was first introduced in [9], in the special case of Minkowski spacetime, and it was generalised in [10] for any curved spacetime. In particular, paper [9] is the natural continuation of [11], where Zeeman proved that causality in Minkowski spacetime implies the Lorentz group. He then showed that the group of all homeomorphisms of the finest topology in $\mathcal{Z}$, which is coarser than the discrete topology, is generated by the inhomogeneous Lorentz group and dilatations. In addition,
unlike the topology of $\mathbb{R}^4$, this fine topology $F$ is not locally homogeneous and the light cone through any point can be deduced by $F$. There is also a quite interesting lemma; the topology on a light ray induced from $F$ is discrete. Here one should not confuse topological discreteness (every set is open) with discreteness in the sense of (finite or infinite) countability. Apart from the group of homeomorphisms of $M$ under $F$ and its physical interpretation, the topological boundary of the null cone has the maximum number of open sets: there is definitely a connection here with the maximum speed, that of light.

Zeeman mentioned three other alternative topologies in $3$ different than $F$, that we will consider in section 4, as well as their analogues for curved spacetimes.

Göbel found that the analogue of $F$ in a curved spacetime has the property that the group of all homeomorphisms under this topology is isomorphic to the group of all homothetic transformations. In a few words, under the relativistic analogue of $F$, a homeomorphism is an isometry.

A problem, that was noticed first by Zeeman himself, is that $F$ is technically difficult, as it does not admit a countable base and so it is not the best tool for a working physicist. This was one of the arguments of the authors of [28] and [27] as well, but we object that this is not an attractive reason for avoiding a topology which is much more natural in a spacetime from the manifold topology. Natural in the sense that it incorporates the differential, causal and conformal structures and the group of homeomorphisms of the spacetime is not vast and it has physical meaning. So, the argument that $F$ has “too many open sets” and does not admit a countable base should be reconsidered. Since we are dealing with both the Lorentz metric as well as the Riemann metric in a spacetime manifold, a natural topology which will describe the properties of the spacetime should be compatible with every possible structure which is defined on the spacetime. $F$ is such a topology.

For some reason the supporters of the manifold topology, like Heathcote, believed that all the topologies in $3$ are strictly finer than $M$, but actually this is not true. The three alternative topologies that Zeeman introduces in [9] are linked in their construction to three topologies that we mention in the next paragraph, each of which belongs to the class $3$ but is incomparable to $M$ (see [26], [29] and [30]). Göbel ([10] page 297, (C)) actually states that there are other topologies in $3$, but without a clear reference that there are topologies that are not necessarily finer or coarser or equal to the manifold topology. This is important,
since the criticism against the class $3$ bases many of its arguments against the term finer topology. Let us now look at a sample of three topologies in $3$, which are not finer than the manifold topology $\mathcal{M}$.

## 4 Topologies different than the manifold topology.

In [35] we remark that the Path Topology $\mathcal{P}$ of Hawking-King-McCarthy (see [28]) is the general relativistic analogue of the topology introduced in Example 1 of [9] (page 169). Low showed in [27] that under this topology $\mathcal{P}$ (that we name $Z^T$ for consistency of notation) the Limit Curve Theorem (LCT) fails to hold. In [35] we introduced three (among others) more topologies that the LCT fails to hold, all incorporating the differential, causal and conformal structure of the spacetime manifold. In particular, we showed the following theorem.

**Theorem 4.1.** There are three distinct topologies in a spacetime manifold which admit a countable basis, they incorporate the causal and conformal structures and the LCT fails with each one of them respectively. These are the interval topologies $T^{\rightarrow}_{in}$, $T^{\leq}_{in}$ and $T^{<\sim}_{in}$, which are all in the class $3$.

All these topologies are not finer (neither equal nor coarser) than the manifold topology and singularity theorems, under each one of them respectively, cannot be formed in the way that are described via the manifold topology. These three topologies, together with the manifold topology, give the intersection topologies $Z$, $Z^T$, $Z^S$, which are finer than the manifold topology, where $Z$ is coarser than the Fine topology $F$ and $Z^T$ (the path topology of [28]) and $Z^S$ are incomparable to $F$.

Low, in [27], supports in his conclusion that LCT failing in the $\mathcal{P}$ (which also fails in the extra five topologies that we suggest in Theorem 4.1) makes the manifold topology remaining both technically easier to work with and fruitful. We have some objections. All the six topologies of [35] and in particular those in Theorem 4.1 are technically easy to work with (they each have a countable base of open sets) and they are fruitful, as they belong to $3$ and are all behaving like order topologies, in the sense that they satisfy the orderability problem (or weaker versions of it, referring to non-linear orders; see [4], [5], [7] and [6]). Each one of them is induced either from the causal or chronological orders or from (the irreflexive) horismos, with the exception of $Z^S$ which is induced by a particular spacelike non-causal
order that we describe in [30]. More specifically, $Z$, $Z^T$ and $Z^S$ have open sets bounded by Riemann open balls centered at an event $x$, intersected with the timecone union spacecone of $x$ in the case of $Z$, the timecone of $x$ in the case of $Z^T$ and the spacecone of $x$ in the case of $Z^S$, respectively. The rest three topologies have unbounded open sets which are timecone union spacecone in the case of $T^{\rightarrow}_{in}$, timecone in the case of $T^{<}_{in}$ and, spacecone in the case of $T^{<,\rightarrow}_{in}$, respectively, at an event $x$. For a more detailed treatment we refer to [35].

On the other hand, the manifold topology misses the Lorentz metric and so the causal structure of the spacetime as well, so we conclude the following.

**Corollary 4.1.** The manifold topology $\mathcal{M}$, on a spacetime $\mathcal{M}$, is based on the Riemann metric and is sufficient for describing spacetime singularities, but does not incorporate the Lorentz metric, while each of the topologies in Theorem 4.1 fail to describe singularities that appear under $\mathcal{M}$, but incorporate the Lorentz metric.

**Corollary 4.2.** The Fine Topology $F$ is the best possible candidate for a spacetime $\mathcal{M}$, as it is strictly finer than $\mathcal{M}$, strictly coarser than the discrete topology and, simultaneously, finer than many of the topologies introduced in Theorem 4.1 (with the certain exception of the path topology $Z^T$ and $Z^S$). In addition, the group of homeomorphisms of $\mathcal{M}$ under $F$ is isomorphic to the Lorentz group and dilatations, in the case of special relativity, and to the group of homothetic symmetries in the case of general relativity, while under the manifold topology the group of homeomorphisms of $\mathcal{M}$ is vast and of no physical significance.

### 5 In the beginning was the metric...or the topology?

This is a more important question as it seems to be. Speaking about spacetime manifolds as mathematical objects, it is vital that a natural topology will incorporate all the mathematical structures appearing in the manifold, including the Lorentz metric as well as the Riemann metric. In this sense (a “Platonic mathematical” sense in the view of Penrose, which is projected to the physical world [14]) the manifold topology is not a natural topology in a spacetime manifold, even if it is defined via the Riemann metric. The metric tensor field, the affine connection and the conformal structure, the differentiable manifold with its topology, are all important constituents of the spacetime manifold, but what about the Lorentz metric and the structure of the null cone? Having mentioned this, we believe that
“in the beginning was the topology”, in a spacetime manifold. A topology like $F$, where the group of homeomorphisms of $M$ under $F$ has a physical meaning and which incorporates all the metric structures in the manifold.

6 Ambient cosmology: a failure due to a topological misconception.

In [12] we described the motivation for a 5-dimensional “ambient space”, where our 4-dimensional spacetime is its conformally related ambient boundary at infinity, by linking it to the singularity problem in general relativity. In cosmology the infinities that are inherent in the spacetime metric according to the singularity theorems indicate the necessity of a conformal geometry of metrics to absorb them, not a breakdown of general relativity. The construction of this model in ambient cosmology can be found in [31], [32], [33] and [34], where the authors started from the construction of the metric, leaving the topological problem for the end. As we observed in [13], it is the topology succeeding (and, unfortunately, not preceding or at least being constructed simultaneously with) the metric that showed a failure in the construction and in results concerning the convergence of causal curves; it is the topology $Z$ (see [29]) where the LCT fails and not in $F$. Furthermore, why should one bother to add an extra dimension while a 4-dimensional spacetime under the topology $Z$ has already the properties of the ambient boundary, and while the structure of the ambient boundary is totally unknown to us (we lack knowledge even for basic results on causality: see, for example, [29] for an important correction on [12]).

We conjecture that there is an analogous problem with the study of spacetime manifolds, when considering the manifold topology and ignore the topologies in $Z$, especially $F$; we see things from a restricted perspective and, automatically, we are subjected to conclusions that might be either false or have a weak theoretical support.
7 Towards an evolving topology and a Quantum Theory of Gravity.

If the main problem for a working physicist is that $F$ is not an easy topology to work with, due to the lack of a countable basis of open sets, or if topologies like $\mathcal{M}$ and those six mentioned in paragraph 4 are missing something important from the spacetime structure, then we believe that there is something deeper behind all this and this certainly is of a topological nature. We have already expressed in [29] an idea of an evolving topology with respect to the class $\mathbb{Z}$, so that different topologies of this class are assigned to each stage of the evolution as well as where the spacetime itself is subjected to singularities. It could be, for example, that the interval topology from horismos $\rightarrow$ (see [29]) could give a sufficient description of the planck time and objects like black holes, while other topologies (where the LCT theorem holds for example) could explain the phase transition from locality to non-locality. Topologies like $\mathbb{Z}^T$ are linked to a discrete space while $\mathbb{Z}^S$ to a discrete time, while $\mathbb{Z}$ to a discrete light (these are actually remarks of Zeeman in [9], for their special relativistic analogues). By evolution we do not necessarily mean (and this is not our desire at all) to consider kinematically that the spacetimes of our interest are foliated manifolds where leaves of foliation have open sets which vary over time. The question is different: how does a spacetime manifold appear from a functional space? An answer to such a question which refers to the transition from nonlocality to locality seems to need a richer topological background; a backgrould that the class $\mathbb{Z}$ could possibly provide.

8 The need to take methods of general topology more seriously.

We believe that the concerns against a “finer” topology, as expressed in [1], are reasonable. Reasonable are similar concerns expressed in [27]; when we restrict ourselves to the validity of general relativity. The problem is that eventhough the manifold topology $\mathcal{M}$ has somehow worked nicely in the last century or so, it is problematic in describing fully properties of a spacetime in a sufficient way; it lacks important information, as we have seen in the previous paragraphs. $F$ is a finer topology which resolves, at least in a mathematical way, all such
issues, and -at the moment- there is no other candidate topology to compete with.

Criticism (in oral communication with physicists) against $F$, and against topologies like those mentioned in paragraph 4, highlight that there is a value of considering these alternative topologies in $\mathcal{Z}$ since they may, for example, lead to a new physical theory; or they may allow one to extract new, physically interesting, predictions from the old theory. But it seems, according to the critics, that there is a point of diminishing returns; that, eventually, further treatment of these topologies, in the abstract, can no longer be justified. At some point, there is a burden to extract from these topologies, some concrete result of genuine physical interest; no such result is in sight and, therefore, that we have reached that point.

The problem of such a criticism is that the main points of [9] and [10] have not been understood, and this is quite disappointing. There is a prejudice against general topology; only the reference to it is enough to discourage working mathematical physicists and theoretical physicists to read carefully a related article. The labyrinth that we seem to be when talking about string theory and quantum theory of gravity, for example, is not only related to the need for an extra physical input, but for an extra mathematical input as well. The authors wish that this Chapter contributes to the reopening of a discussion in this serious and fascinating subject.

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