Thermodynamic Partition Function of Matrix Superstrings

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Abstract

We show that, in the limit of zero string coupling, $g_s \to 0$, the thermodynamic partition function of matrix string theory is identical to that of the finite temperature, discrete light-cone quantised (DLCQ) type IIA superstring. We discuss how the superstring is recovered in the decompactified $R^+ \to \infty$ limit.

1 Introduction and summary

M-theory is an eleven dimensional theory whose compactification on a circle is equivalent to ten dimensional IIA superstring theory with coupling constant $g_s$ related to the radius of the compact dimension. The higher Kaluza-Klein modes corresponding to M-theory degrees of freedom with non-zero momenta in the compact direction are the D0-branes of the IIA theory. According to the proposal of Banks, Fischler, Shenker and Susskind\footnote{1} the full dynamics of M-theory in the infinite momentum frame is described by the large $N$ limit of supersymmetric matrix quantum mechanics with gauge group U(N). There is a further conjecture that, at finite $N$, the matrix model describes discrete light-cone quantised (DLCQ) M-theory, the de-compactified limit occurring when $N$ and the light-cone compactification radius are taken to infinity in the appropriate way\footnote{2}.

Compactification of the matrix model on a spatial circle, to yield a description of IIA superstring theory by a two-dimensional super-Yang-Mills theory, was discussed by several authors\footnote{3}. Dijkgraaf, Verlinde and Verlinde\footnote{4} gave arguments to show how IIA superstrings emerge in the limit of weak
string coupling. In their scenario, strings are associated with the diagonal components of the $N \times N$ matrices which decouple from the off-diagonal components in the $g_s \to 0$ limit. The infrared dynamics of the diagonal elements are described by an $\mathcal{N} = 8$ supersymmetric sigma model with orbifold target space $(R^8)^N / S_N$. On the other hand, the non-interacting first-quantised superstring is described in the light-cone gauge by the supersymmetric sigma model with target space $R^8$. The idea of ref. is that, when $N$ is large, the quantum states of the conformal field theory with target space $(R^8)^N / S_N$ are identical to those of the second quantised superstring theory which lie in the appropriately symmetrised infinite direct product of Hilbert spaces of conformal field theories each with target space $R^8$. This idea is supported by the identification of elliptic genera on the orbifold $(M)^N / S_N$ for compact Kahler manifold $M$ and similar objects which are expected to arise from second quantisation of a superstring theory on $M \times S^1$, which was conjectured in and proven in.

In this Paper we shall present an alternative way of seeing how the matrix model produces the quantum states of the string theory. We shall compare the thermodynamic partition function of the matrix string theory in the limit $g_s \to 0$ with the thermodynamic partition function of the non-interacting discrete light-cone quantised (DLCQ) IIA superstring. We shall find that, at least in this non-interacting limit, the two are identical. This implies that, at zero string coupling (which means infinite Yang-Mills coupling), the spectrum of the matrix model is identical to the spectrum of all multi-string states of the non-interacting discrete-light-cone quantised IIA superstring theory. This identity requires neither taking the large $N$ limit nor the de-compactification of the light-cone, $R^+ \to \infty$.

The natural extension of our result to the interacting theory would be to show that the quantum states of the second quantised interacting DLCQ IIA string theory with total longitudinal momentum $N/R^+$ would be identical to the gauge invariant eigenstates of the Hamiltonian of $U(N)$ matrix string theory. This is the content of the conjecture in . It should be possible to check this idea by comparison of perturbative corrections to the thermodynamic partition functions of each model. Superstring perturbation theory has been discussed in the context of matrix string theory by several authors , , , , , , . It is likely that some of these proposals could be studied systematically within our approach.

The partition function of matrix string theory should be constructed by using the Hamiltonian of the appropriately compactified matrix theory,

$$\mathcal{H} = \frac{R^+}{2} \int_0^1 d\sigma_1 \text{Tr} \left[ \Pi_i^2 + \frac{1}{4\pi^2\alpha'^2}(D_1 X_i)^2 - \frac{1}{4\pi^2\alpha'^2 g_s^2} [X_i, X_i]^2 + \frac{E^2}{g_s^2\alpha'} \right. $$

$$- \frac{1}{2\pi\alpha'^{3/2} g_s} \psi^T \gamma_i [X^i, \psi] - \frac{i}{2\pi\alpha'} \psi^T \gamma \cdot D_1 \psi \right] \tag{1}$$

where the notation and conventions are explained in Section 3. This Hamiltonian is to be interpreted as the light-cone Hamiltonian of M-theory

$$\mathcal{H} = P^+ \tag{2}$$
where M-theory has two compactified dimensions, one of them spatial, in order that it describes the IIA superstring and the other the light-cone direction $X^+$ with radius $R^+$. The gauge invariant states of this Hamiltonian are those of M-theory in a sector of fixed longitudinal momentum

$$P^- = \frac{N}{R^+}$$  \hspace{1cm} (3)

which is discrete since momentum in a compact direction should be quantised in units of inverse radius. To obtain the theory on un-compactified ten dimensional space the limit $R^+ \to \infty$ would eventually have to be taken.\footnote{Note that in the matrix model description of M-theory, $N$ is the number of D0-branes which are the basic building blocks of all objects. In the matrix string theory on the other hand, because of the 9-11 flip which is used in (4), $N$ labels the quantum of momentum $P^+$.}

We use (2) and (3) to form the thermodynamic partition function,

$$Z[\beta] = \text{Tr} \left( e^{-\beta P^0} \right) = \text{Tr} \left( e^{-\frac{\sqrt{2}}{N}(P^+ + P^-)} \right)$$  \hspace{1cm} (4)

$$= \sum_{N=0}^{\infty} e^{-\beta N/\sqrt{2} R^+} \text{Tr}_N \left( e^{-\beta H/\sqrt{2}} \right)$$  \hspace{1cm} (5)

where in (4) the trace is to be taken over all eigenstates of both $P^+$ and $P^-$ and $\beta = 1/k_B T$ with $k_B$ the Boltzmann constant and $T$ the temperature. The trace over $P^-$ leads to the summation over $N$ in (5). Taking the trace over gauge invariant states of the D=2 super-Yang-Mills theory with fixed $N$ in (5) yields an expression for the partition function in terms of a Euclidean functional integral

$$Z[\beta] = \sum_{N=0}^{\infty} \int [dA][dX^i][d\psi] \exp \left( -\beta N/\sqrt{2} R^+ - S_E[A, X^i, \psi] \right)$$  \hspace{1cm} (6)

where the action $S_E$ is the continuation to two-dimensional Euclidean space of the action from which the Hamiltonian (4) can be obtained by canonical quantisation. The two dimensional integration in $S_E[A, X^i, \psi]$ is over a torus with spatial coordinate $\sigma_1 \in [0, 1)$ and temporal coordinate in $\sigma_2 \in [0, \beta/\sqrt{2})$ and the fields have periodic boundary conditions in the spatial directions, $X^i$ and $A_\alpha$ are periodic and $\psi$ are anti-periodic in Euclidean time. By suitable rescaling, the Euclidean time interval can be set to $[0, 1)$ and the temperature would then appear in coefficients in the action. Details are discussed in Section 3. Note that, as is usually the case, the thermal boundary conditions, which are different for Fermi and Bose degrees of freedom, break the supersymmetry of the theory. This reflects the different thermal populations of fermionic and bosonic states of the matrix model (as well as the IIA superstring) and is an expected feature of a supersymmetric theory at finite temperature. For a recent discussion of supersymmetry and other issues in matrix theory at finite temperature, see \cite{13, 14, 15}.
We then consider this partition function in the limit $g_s \to 0$. In that limit, it reduces to a model where the degrees of freedom are the diagonal elements of the matrix-valued fields. The gauge fields decouple and the scalar and fermion fields form a super-conformal field theory with target space the orbifold $(R^8)^N/S_N$. We shall find that, for this model, the thermodynamic free energy, defined by

$$F[\beta] = -\frac{1}{\beta} \ln Z[\beta]$$

is given by

$$F = -\frac{1}{\beta} \sum_{N=0}^{\infty} \sum_{r|N \text{ odd}} \sum_{s=0}^{N/r-1} \frac{1}{N} \exp \left(-\frac{N\beta}{\sqrt{2R^+}}\right) \cdot \int d\vec{X} d\psi \exp \left[-\frac{1}{4\pi\alpha'} \int \sqrt{g} \left(g^{\alpha\beta} \partial_\alpha \vec{X} \cdot \partial_\beta \vec{X} - i2\pi\alpha' \bar{\psi}^a e^a_\alpha \partial_\alpha \psi \right)\right]$$

where the odd integers $r$ are the odd divisors of $N$. The super-conformal field theory defined by the path integral has the light-cone gauge fixed Green-Schwarz superstring action. The worldsheet is a Euclidean torus with coordinates $0 \leq \sigma_1, \sigma_2 < 1$ and metric

$$g_{\alpha\beta} = \left(\frac{1}{\tau_1} \frac{\tau_2}{|\tau|^2}\right) = e^\alpha_a e^\beta_b$$

In the path integral in (7), the Bose fields $\vec{X}$ have periodic boundary conditions in both the worldsheet space and time directions whereas the Fermi fields $\psi$ have periodic and anti-periodic boundary conditions in the space and time directions, respectively. $\tau = \tau_1 + i\tau_2$ is the usual Teichmüller parameter of the torus. In (7) it takes the discrete values

$$\tau = \frac{s}{N/r} + i\frac{\beta R^+}{\sqrt{22\pi\alpha'} N/r}$$

We shall find that the IIA superstring partition function, using discrete light-cone quantisation, and where the compactification radius of $X^+$ is also $R^+$, is given by an expression identical to (7). In DLCQ of the IIA superstring, the Teichmüller parameter $\tau_1$, which is summed over the values

$$\tau_1 = 0, \frac{1}{N/r}, \frac{2}{N/r}, \ldots, \frac{N/r - 1}{N/r}$$

originates as a Lagrange multiplier which enforces the Virasoro constraint

$$\sum_{s=0}^{N/r-1} \exp \left(2\pi i \frac{s}{N/r} (h - \tilde{h})\right) = \begin{cases} 1 & \text{if } h - \tilde{h} = 0 \mod N/r \\ 0 & \text{otherwise} \end{cases}$$

which, we shall argue in Section 2, is the appropriate constraint for the DLCQ superstring. The other Teichmüller parameter $\tau_2$ arises from the summation
over eigenvalues of $P^+$. On the other hand, in the matrix string theory, these parameters originate from the sum over permutations which occur in the boundary conditions of the diagonal components of the matrices when they are the variables of the orbifold sigma model. Similar permutations have been argued to play a key role in the computation of the Witten index \[18, 19, 20, 21\] and in Yang-Mills theory partition functions on tori \[22, 23\]. They are also important in obtaining the spectrum of two dimensional Yang-Mills theory from the path integral \[24, 25\] and in the correspondence between two dimensional Yang-Mills theory and a random surface model \[26, 27, 28\].

Since the thermodynamic partition function encodes information about the energy levels and degeneracies of states, we conclude that the spectrum of the $g_s \to 0$ limit of matrix string theory is identical to the spectrum of the second-quantised non-interacting DLCQ IIA superstring.

It is interesting to study how the expression (7) will recover the superstring partition function in the limit $R^+ \to \infty$. The mechanism by which this occurs is clear. The only suppression of the magnitude of $N$ in the summation in (7) comes from the exponential which has $R^+$ in the denominator. When $R^+ \gg \beta$, the sum is dominated by very large $N$. When $N$ becomes very large, $\tau$ in (3) should correctly be defined as varying as the integers $s$ and $N/r$ vary. Then, when $N$ is very large, the increments in $\tau$ as we go between successive values of $s$ and $N/r$ are small and the summation over $s$, $N$ and $r$ become integrations over $\tau_1$ and $\tau_2$ and a summation over $r$. In fact, $\tau_1$ is varied by changing $s$ while holding $r$ and $N/r$ fixed, so that

\[
d\tau_1 = \frac{(s + 1)N/r - sN/r}{N/r} = \frac{1}{N/r}
\]

and $\tau_2$ is varied by changing $N/r$ but keeping $r$ and $s$ fixed:\[4\]

\[
\frac{d\tau_2}{\tau_2^2} = -d\left(\frac{1}{\tau_2}\right) = \frac{\sqrt{22\pi \alpha'}}{\beta R^+} \frac{1}{r} \left(\frac{N}{r} + 1 - \frac{N}{r}\right)
\]

so that

\[
\frac{1}{N} = \beta \frac{2\pi R^+}{4\pi^2 \alpha' \sqrt{2}} \frac{d\tau_1 d\tau_2}{\tau_2^2}
\]

With this identification, the continuum limit of the summations in (3) is

\[
\sum_{N=0}^{\infty} \sum_{r|N \text{ odd}}^{N/r-1} \sum_{s=0}^{N/r-1} e^{-N\frac{\beta}{2R^+}} \beta R^+ \frac{1}{\sqrt{22\pi \alpha'}} \int_0^1 d\tau_1 \int_0^\infty d\tau_2 \sum_{r=1}^{\infty} e^{-\frac{1}{16\pi^2 \tau_2^2} \tau_1 \tau_2^2} (11)
\]

\[4\]It is clear that this means changing $N$ to the next larger integer which is divisible by $r$
and the result is identical to the expression for the thermodynamic free energy of the superstring

$$F = -\frac{\sqrt{2\pi R}}{4\pi^2\alpha'} \int_0^1 d\tau_1 \int_0^\infty d\tau_2 \sum_{r=1}^\infty \frac{e^{-\frac{\pi^2\alpha'^2}{2\sqrt{2}\pi R}}}{2}.$$ 

$$\cdot \int d\bar{X} d\psi \exp \left[ -\frac{1}{4\pi\alpha'} \int \sqrt{g} \left( g^{\alpha\beta} \partial_\alpha \bar{X} \cdot \partial_\beta \bar{X} - i2\pi\alpha' \bar{\psi} \gamma^\alpha \epsilon^\alpha_a \partial_\alpha \psi \right) \right].$$ (12)

Integrating the remaining variables and taking into account that the space volume is given by $V = V_T \cdot \sqrt{2\pi R}$, with $V_T$ and $\sqrt{2\pi R}$ the volume of the transverse space and the length of the longitudinal direction, produces the well-known expression for the free energy density $[29, 30, 31, 32]$

$$\frac{F}{V} = -\int_{-1/2}^{1/2} \frac{2\pi^2 d\tau_1}{(4\pi^2\alpha')^5} \int_0^\infty \frac{d\tau_2}{\tau_2^9} |\theta_4(0, 2\tau)|^{-16}.$$ 

$$\cdot \left[ \theta_3 \left( 0, \frac{i\beta^2}{4\pi^2\alpha'\tau_2} \right) - \theta_4 \left( 0, \frac{i\beta^2}{4\pi^2\alpha'\tau_2} \right) \right].$$ (13)

In Section 2 we will review the derivation of the finite temperature partition function of the superstring. We use the relativistic particle and the bosonic string to illustrate the main idea in a simpler context and to derive some necessary formulae. Then we derive an expression for the free energy of the type IIA superstring using discrete light-cone quantisation, as well as the partition function of the superstring.

In Section 3 we discuss the thermodynamic partition function for the matrix string model. We argue that taking the $g_s \to 0$ limit reduces the model to one for the diagonal components of the matrices with boundary conditions that are (anti-)periodic up to permutations of the eigenvalues in both the space and compact Euclidean time directions. We then show how the combinatorics of permutations reduces to a sum over coverings of the torus, which produces a sum over the Teichmüller parameters in [7].

1.1 Notation

In this section, we summarise some of the notation. For a D-vector $x^\mu$, we shall use the convention for the Minkowski metric

$$ds^2 = dx^\mu g_{\mu\nu} dx^\nu \equiv (dx^0)^2 - (\bar{x}^2)^2 = 2dx^+ dx^- - (d\bar{x}_T)^2$$ (14)

where $\bar{x}$ is the vector made from the $D-1$ spatial components of $x^\mu$, the light-cone coordinates are

$$x^+ = \frac{1}{\sqrt{2}} (x^0 + x^{D-1}) \quad , \quad x^- = \frac{1}{\sqrt{2}} (x^0 - x^{D-1})$$ (15)

and $\bar{x}_T = (x^1, \ldots, x^{D-2})$. 

6
The Jacobi theta functions are

\[ \theta_1(\nu, \tau) = i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} z^{n-1/2}, \quad \theta_3(\nu, \tau) = \sum_{n=-\infty}^{\infty} q^{n^2/2} z^n \]

\[ \theta_2(\nu, \tau) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} z^{n-1/2}, \quad \theta_4(\nu, \tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} z^n \]

where \( q = \exp(2\pi i \tau), \quad z = \exp(2\pi i \nu) \)

We shall also denote by

\[ \tilde{\theta}_k(\nu, \tau) = \sum_{n=0, n \neq 0} \ldots \]

the theta function where the \( n = 0 \) term is absent from the sum. The modular transformation properties are

\[ \theta_1(\nu, \tau + 1) = e^{i\pi/4} \theta_1(\nu, \tau), \quad \theta_2(\nu, \tau + 1) = e^{i\pi/4} \theta_2(\nu, \tau) \]
\[ \theta_3(\nu, \tau + 1) = \theta_3(\nu, \tau), \quad \theta_4(\nu, \tau + 1) = \theta_3(\nu, \tau) \]

and

\[ \theta_1(\nu/\tau, -1/\tau) = -(-i\tau)^{1/2} e^{\pi i \nu^2/\tau} \theta_1(\nu, \tau) \]
\[ \theta_2(\nu/\tau, -1/\tau) = -(-i\tau)^{1/2} e^{\pi i \nu^2/\tau} \theta_4(\nu, \tau) \]
\[ \theta_3(\nu/\tau, -1/\tau) = -(-i\tau)^{1/2} e^{\pi i \nu^2/\tau} \theta_3(\nu, \tau) \]
\[ \theta_4(\nu/\tau, -1/\tau) = -(-i\tau)^{1/2} e^{\pi i \nu^2/\tau} \theta_2(\nu, \tau) \]

They obey Jacobi’s abstruse identity

\[ \theta_3^2(0, \tau) - \theta_4^2(0, \tau) - \theta_2^2(0, \tau) = 0 \]

and Jacobi’s triple product formulae are

\[ \theta_1(\nu, \tau) = 2q^{1/8} \sin(\pi \nu) \prod_{n=1}^{\infty} (1 - q^n) \left(1 - 2q^n \cos(2\pi \nu) + q^{2n}\right) \]
\[ \theta_2(\nu, \tau) = 2q^{1/8} \cos(\pi \nu) \prod_{n=1}^{\infty} (1 - q^n) \left(1 + 2q^n \cos(2\pi \nu) + q^{2n}\right) \]
\[ \theta_3(\nu, \tau) = \prod_{n=1}^{\infty} (1 - q^n) \left(1 - 2q^{n-1/2} \cos(2\pi \nu) + q^{2n-1}\right) \]
\[ \theta_4(\nu, \tau) = \prod_{n=1}^{\infty} (1 - q^n) \left(1 - 2q^{n-1/2} \cos(2\pi \nu) + q^{2n-1}\right) \]

The Dedekind eta function is

\[ \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \]

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and has the modular transformation properties
\[ \eta(\tau + 1) = e^{i\pi/12}\eta(\tau), \quad \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \] (23)

2 Thermodynamic free energy of the discrete light-cone quantised superstring

In this section, we will derive the thermodynamic free energy of the DLCQ IIA superstring. To illustrate the point, we begin with a review of the relativistic particle and the bosonic string. Then we discuss the DLCQ bosonic string at finite temperature. Finally, we derive the partition function of the DLCQ superstring.

2.1 Relativistic particle

To begin, we consider the free energy per unit volume of a relativistic particle, in \( D \) spacetime dimensions
\[ F_V = (-1)^f \frac{1}{\beta} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \ln \left( 1 - (-1)^f e^{-\beta\omega(p)} \right) \] (24)

\( f = 0 \) if the particle is a boson and \( f = 1 \) for a fermion. The energy is \( \omega(p) = \sqrt{p^2 + M^2} \) with \( M \) the mass.

To write the integral (24) in terms of light-cone momenta we use the identity
\[ \delta(p^0 - \omega(p\bar{p})) = 2p^0 \theta(p^0) \delta(p^2 - M^2) = \sqrt{2} (p^+ + p^-) \theta(p^+ + p^-) \delta(2p^+p^- - \bar{p}_T^2 - M^2) \] (25)
to write
\[ F_V = (-1)^f \frac{1}{\beta} \int \frac{dp^+ dp^- d^{D-2}p_T}{\sqrt{2\pi}(2\pi)^{D-2}} (p^+ + p^-) \theta(p^+ + p^-) \delta(2p^+p^- - \bar{p}_T^2 - M^2) \ln \left( 1 - (-1)^f e^{-\beta\sqrt{2}(\bar{p}_T^2 + M^2)} \right) \] (26)

which, upon integration over \( p^+ \), becomes
\[ F_V = (-1)^f \frac{1}{\beta} \int_0^\infty \frac{dp^-}{\sqrt{2\pi}} \int \frac{d^{D-2}p_T}{(2\pi)^{D-2}} \ln \left( 1 - (-1)^f e^{-\frac{\beta}{\sqrt{2}}\left( \frac{p_T^2 + M^2}{2p^-} + p^- \right)} \right) \] (27)

Here, in order to equate the two terms which arise from (26), we have made use of a symmetry of the integrand under an interchange of \( p^+ \) and \( p^- \).

We then use a Taylor expansion of the logarithm to present the free energy per unit volume as
\[ F_V = -\sum_{n=1}^\infty (-1)^{(n+1)f} \frac{1}{n\beta} \int_0^\infty \frac{dp^-}{\sqrt{2\pi}} e^{-n\beta p^-} \int \frac{d^{D-2}p_T}{(2\pi)^{D-2}} e^{-\frac{n\beta}{\sqrt{2}}\left( \frac{p_T^2 + M^2}{2p^-} + p^- \right)} \] (28)
In this form, the free energy is a trace over the space of states of a density matrix which is made from the light-cone Hamiltonian,
\[ H_{l.c.} = \frac{\vec{p}^2 + M^2}{2p} \]  
(29)
as
\[ \text{Tr} \left( e^{-\frac{n\beta}{\sqrt{2}} H_{l.c.}} \right) = V_T \int \frac{d^{D-2}p_T}{(2\pi)^{D-2}} e^{-\frac{\sqrt{2}}{n\beta} \left( \frac{\vec{p}^2 + M^2}{2p} \right)} \]  
(30)
and which can be thought of as the trace of the light-cone time evolution operator over intervals of Euclidean time \( n\beta/\sqrt{2} \). The volume is \( V = LV_T \) with \( V_T \) and \( L \) the volume of the transverse space and length of the longitudinal direction, respectively.

The Gaussian integral over \( \vec{p} \) in (28) can be done explicitly to get
\[ F_V = -\sum_{n=1}^{\infty} \int_0^\infty dp^- \left( -1 \right)^{(n+1)f} \left( \frac{p^-}{\sqrt{2\pi n\beta}} \right)^{D/2} \exp \left[ -\frac{n\beta}{\sqrt{2}} \left( \frac{M^2}{2p^2} + p^- \right) \right] \]  
(31)
Finally, let us take note of the modification of the formula (31) which would be necessary in DLCQ. The light-cone direction \( X^+ \) is assumed to be compactified, so that we should identify \( X^+ \) and \( X^+ + 2\pi R^+ \). This identification gives the momentum conjugate to \( X^+ \), i.e. \( P^- \) a discrete spectrum. Thus, rather than the integral over \( p^- \) in (31) we should have a sum over the spectrum
\[ p^- = k/R^+ \]
\[ F_V = -\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{(n+1)f} \left( \frac{k}{\sqrt{2\pi n\beta R^+}} \right)^{D/2} \exp \left[ -\frac{n\beta}{\sqrt{2}} \left( \frac{R^+ M^2}{2k} + \frac{k}{R^+} \right) \right] \]  
(32)
This formula will be used to derive the path integral expression for the free energy of the DLCQ closed string. An expression similar to it was quoted in ref [34].

2.2 DLCQ Bosonic closed string

The equations of motion and constraints of the Bosonic closed string theory are
\[ \partial_+ \partial_- X^\mu = 0 \]  
(33)
\[ \partial_+ X^\mu \partial_+ X_\mu = 0 \]  
(34)
\[ \partial_- X^\mu \partial_- X_\mu = 0 \]  
(35)
\( X^+ \) is identified with \( X^+ + 2\pi R^+ \). In closed string theory, boundary conditions are periodic,
\[ X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) + \delta^\mu_+ 2\pi R^+ r \]  
(36)
where \( r \) is an integer which is the number of times the string world-sheet wraps the compact direction \( X^+ \).
The equations of motion have the light-cone gauge solution
\[ X^-(\tau, \sigma) = x^- + P^- \tau \]
\[ X^+(\tau, \sigma) = x^+ + P^+ \tau + rR^+ \sigma + \text{(oscillators)} \]
\[ \vec{X}_T(\tau, \sigma) = \vec{x}_T + \vec{P}_T \tau + \text{(oscillators)} \] (37)

Since the canonical commutation relations are
\[ [x^+, P^-] = -i \]
and \( x^+ \) is a compact variable, the momentum \( P^- \) is quantised as
\[ P^- = \frac{k}{R^+} \]

In this light-cone gauge, the Virasoro constraints can be solved to eliminate the light-cone components of the oscillators, \( \alpha^+_n \neq 0 \) and \( \tilde{\alpha}^+_n \neq 0 \). The remaining constraint which must be imposed is
\[ L_0 = 0 = \tilde{L}_0 \]

The sum of these gives the mass-shell condition,
\[ 0 = L_0 + \tilde{L}_0 \rightarrow P^\mu P_\mu = M^2 = \frac{2}{\alpha'} \left( h + \tilde{h} - 2 \right) \] (38)
and their difference gives the condition
\[ 0 = L_0 - \tilde{L}_0 \rightarrow P^- \cdot (rR^+) = kr = h - \tilde{h} \] (39)

where, the Hamiltonians are
\[ h = \sum_{i=1}^{\infty} \vec{\alpha}_m \cdot \vec{\alpha}_m \] (40)

and
\[ \tilde{h} = \sum_{i=1}^{\infty} \vec{\tilde{\alpha}}_m \cdot \vec{\tilde{\alpha}}_m \] (41)

The right- and left-moving light-cone gauge oscillators have the algebra
\[ [\alpha^i_m, \alpha^j_n] = m \delta_{m+n} \delta^{ij} \] (42)
\[ [\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = m \delta_{m+n} \delta^{ij} \] (43)

where \( i, j = 1, ..., 24 \). (For the bosonic string, we fix the spacetime dimension to be \( D=26 \).)

Now, we can examine the free energy of the bosonic string. At the one loop level, it is gotten by computing the sum of free energies of the particles in the string spectrum. This is obtained from (32) with \( f = 0 \), using the fact that for
the string $M^2$ are the eigenvalues of the operator given in (38) which also obey the constraint (39). The constraint can be enforced by a discrete sum,

$$\sum_{s=0}^{k-1} \frac{1}{k} \exp \left( 2\pi i \frac{s}{k} (\hat{h} - \bar{\hat{h}}) \right) = \begin{cases} 1 & \text{if } h - \bar{h} = 0 \text{ mod } k \\ 0 & \text{otherwise} \end{cases} \quad (44)$$

The trace over the string spectrum constrained by (39) can be written as

$$\frac{F}{V} = -\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{k}{\sqrt{2\pi n \beta R^+}} \right)^{13} \sum_{s=0}^{k-1} \frac{1}{k} \text{Tr} \exp \left( \frac{-n\beta R^+ M^2}{\sqrt{2}} \frac{2k}{\sqrt{2R^+}} + 2\pi i \frac{s}{k} (\hat{h} - \bar{h}) \right) \quad (45)$$

We introduce the notation

$$\tau(n, k, s) = \tau_1 + i \tau_2, \quad \tau_1(k, s) = \frac{s}{k}, \quad \tau_2(k, n) = \frac{n\beta R^+}{2\pi \alpha' \cdot \sqrt{2k}} \quad (46)$$

Later, we shall see that these parameters become the Teichmüller parameters of the worldsheet of the string which is a torus. We see that, for the DLCQ string, they take discrete values, which should merge to form a continuum of tori when the limit $R^+ \to \infty$ is taken. In terms of these parameters, the free energy of the Bosonic string is

$$\frac{F}{V} = -\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{s=0}^{k-1} \frac{1}{k^2} \left( \frac{1}{4\pi^2 \alpha' \tau_2} \right)^{13} e^{4\pi \tau_2 - n^2 \beta^2 / 4\pi \alpha' \tau_2} \text{Tr} \left( e^{-2\pi i \bar{\tau} h} e^{2\pi i \tau \bar{h}} \right) \quad (47)$$

The trace over the string states can be computed by noting that $\tilde{\alpha}_m$ are related to conventionally normalised harmonic oscillator operators by

$$\tilde{\alpha}_m = \sqrt{m} \tilde{a}_m, \quad \tilde{\alpha}_{-m} = \sqrt{m} \tilde{a}_m^\dagger \quad (48)$$

so that

$$\text{Tr} \left( x^{-\sum_{m=1}^{\infty} \frac{m \tilde{a}_m^\dagger \tilde{a}_m}{1 - \chi^m}} \right) = \left( \prod_{m=1}^{\infty} \frac{1}{1 - \chi^m} \right)^{24} \quad (49)$$

Then the free energy density reads

$$\frac{F}{V} = -\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{s=0}^{k-1} \frac{1}{k^2} \left( \frac{1}{4\pi^2 \alpha' \tau_2} \right)^{13} \exp \left( -\frac{n^2 \beta^2}{4\pi \alpha' \tau_2} \right) |\eta(\tau)|^{-48} \quad (50)$$

This is the free energy density of the DLCQ bosonic string. It is similar to the known free energy except that, rather than an integration over the Teichmüller parameters of the torus with the appropriate gauge fixed measure of the Polyakov path integral, there is a discrete summation with values of $\tau(n, k, s)$ given in (46).
Of course, it should recover the well known expression \[33\] for the partition function of the Bosonic closed string when the light-cone is de-compactified, i.e. in the limit \( R^+ \to \infty \). One can see from equation \[45\] that, in this limit, the sum is dominated by combinations of integers where \( nk \) is very large. The correct continuum limit is gotten by considering integration over all values of \( \tau_2 \) as the limit of doing the sum over the integers \( k \) while holding \( n \) and \( s \) fixed. Then

\[
-\tau_2d\left(\frac{1}{\tau_2}\right) = \tau_2(k, n) \left(\frac{1}{\tau_2(k, n)} - \frac{1}{\tau_2(k + 1, n)}\right) = \frac{1}{k}
\]

Furthermore, the integration over \( \tau_1 \) arises from the sum over \( s \), holding \( k \) and \( r \) fixed,

\[
d\tau_1(k, s) \equiv \tau_1(k, s + 1) - \tau_1(k, s) = \frac{s + 1}{k} - \frac{s}{k} = \frac{1}{k}
\]

so that

\[
\frac{d^2\tau}{\tau_2} = \frac{1}{k^2}
\]

accounts for the factor of \( 1/k^2 \) in \[50\]. In the limit where the summations over \( k \) and \( s \) become continuous integrals over \( \tau_1 \) and \( \tau_2 \) according to the formula

\[
\sum_{k=0}^{\infty} \sum_{s=0}^{k-1} \frac{1}{k^2} \rightarrow \int_{-1/2}^{1/2} d\tau_1 \int_{0}^{\infty} \frac{1}{\tau_2} d\tau_2
\]

and \( n \) is summed independently of these variables, the usual expression for the free energy density of the bosonic string is recovered,

\[
\frac{F}{V} = -\frac{1}{2(4\pi^2\alpha')^{1/2}} \int_{-1/2}^{1/2} d\tau_1 \int_{0}^{\infty} \frac{d\tau_2}{\tau_2^{1/2}} |\eta(\tau)|^{-48} \tilde{\theta}_3(0, i\beta^2/4\pi^2\alpha' \tau_2)
\]

Note that this coincides with the result given by Polchinski \[33\] and agrees with the one quoted by Atick and Witten \[29\] (see also \[30, 31, 32\]) when one uses the SL(2,Z) modular group to introduce a second integer in the thermal summation and restricts the integration in \( \tau \) over the fundamental domain of the torus

\[
-\frac{1}{2} < \tau_1 < \frac{1}{2} \, , \, \tau_2 > 0 \, , \, |\tau| > 1
\]

We further note that the free energies of both the DLCQ and de-compactified bosonic strings can be presented in a form of a path integral over coordinates transverse to the light-cone,

\[
\frac{F}{L} = -\sum_{n=1}^{\infty} \int_{-1/2}^{1/2} \frac{d\tau_1}{4\pi^2\alpha'} \int_{0}^{\infty} \frac{d\tau_2}{\tau_2^{2/3}} e^{-\frac{\phi^2}{4\pi^2\alpha'} \tau_2}
\]

\[
\cdot \int d\vec{X} \exp \left( -\frac{1}{4\pi\alpha'} \int d^2\sqrt{g} g^{\alpha\beta} \partial_\alpha \vec{X} \cdot \partial_\beta \vec{X} \right)
\]

The functional integral defines a conformal field theory on the transverse space \( R^{24} \) with worldsheet the torus with \( 0 \leq \sigma^1, \sigma^2 < 1 \) and metric \[8\).
2.3 DLCQ IIA superstring

Let us now consider the case of a closed type II superstring in ten dimensions. The superstring theory contains equal numbers of space-time bosons and space-time fermions which are treated differently in the formula (32). We can get the superstring partition function by using the average of the two contributions \( f = 0 \) and \( f = 1 \) in (32) and then summing over the total multiplicities of all allowed states of the superstring theory, weighted by the appropriate Boltzmann factor. This technique was originally used in [30]. The free energy is

\[
F = - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2}{k} \left( \frac{1}{4\pi^2 \alpha' \tau_2} \right)^5 e^{-n^2 \beta^2 / 4\pi \alpha' \tau_2} \sum_{M} e^{-\pi \alpha' \tau_2 M^2} \tag{55}
\]

with \( \tau_2 \) defined in (46). The mass spectrum for a closed superstring is given by the spectrum of the operator

\[
M^2 = \frac{2}{\alpha'} \sum_{i=1}^{8} \sum_{n=1}^{\infty} n_i \left( h^B_{n_i} + h^F_{n_i} + \tilde{h}^B_{n_i} + \tilde{h}^F_{n_i} \right) \tag{56}
\]

where \( h^B_{n_i}, h^F_{n_i}, \tilde{h}^B_{n_i}, \) and \( \tilde{h}^F_{n_i} \) are the number operators for each species of left- and right-moving worldsheet boson and fermion degree of freedom. The longitudinal momentum is given by

\[
P^- = \frac{k}{R^+} \tag{57}
\]

As in the case of the DLCQ bosonic string, the constraint \( L_0 - \tilde{L}_0 \approx 0 \) implies that the oscillators numbers obey

\[
\sum_{i=1}^{8} \sum_{n=1}^{\infty} n_i \left( h^B_{n_i} + h^F_{n_i} - \tilde{h}^B_{n_i} - \tilde{h}^F_{n_i} \right) = 0 \mod k \tag{58}
\]

The free energy for the superstring is obtained by introducing in (55), the trace over the string spectrum (56) and implementing the constraint (58) with a discrete Fourier sum. We use the formula for the trace over number operators

\[
\text{Tr} \left[ e^{-2\pi \tau_2 (h^B + h^F + \tilde{h}^B + \tilde{h}^F) + 2\pi i \tau_2 (h^B + h^F - \tilde{h}^B - \tilde{h}^F)} \right] = 2^8 \prod_{n=1}^{\infty} \left[ 1 + e^{2\pi i n} \right]^{16} \equiv 2^8 \theta_4(0, 2\tau)^{-16} \tag{59}
\]

where \( \tau \) is given in (46) and the factor \( 2^8 \) is due to the degeneracy of the fermionic ground states.

Inserting this result into (55), the free energy for the closed superstring then reads

\[
F = - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k^2} \left( \frac{1}{4\pi^2 \alpha' \tau_2} \right)^5 2^9 \theta_4(0, 2\tau)^{-16} e^{-n^2 \beta^2 / 4\pi \alpha' \tau_2} \tag{60}
\]
This is the free energy of the DCLQ type II superstring.

In the limit \( R^+ \to \infty \), again using the assumption that the summation over Teichmüller parameters goes to an integral over those parameters according to the formula

\[
\sum_{k=0}^{\infty} \sum_{s=0}^{k-1} \frac{1}{k^2} \to \int_{-1/2}^{1/2} d\tau_1 \int_0^{\infty} \frac{1}{\tau_2} d\tau_2
\]

(61)

this expression reduces to the free energy of the type II superstring,

\[
\frac{F}{V} = -\frac{2^7}{(4\pi^2\alpha')} \int_{-1/2}^{1/2} d\tau_1 \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} |\theta_4(0, 2\tau)|^{-16} \cdot [\theta_3(0, i\beta^2/4\pi^2\alpha'\tau_2) - \theta_4(0, i\beta^2/4\pi^2\alpha'\tau_2)]
\]

(62)

As in the case of the bosonic string, the trace in \((63)\) can be represented by an integral over the transverse coordinate of the string, when the world sheet is a torus. The Green-Schwarz action on a toroidal world-sheet is

\[
S_{\text{g.s.}} = -\frac{1}{4\pi\alpha'} \int \partial^2 \sigma \sqrt{g} g^{\alpha\beta} \left( \partial_\alpha \vec{X} \cdot \partial_\beta \vec{X} - i2\pi\alpha' \vec{\psi}^i \gamma_\alpha \partial_\beta \psi^i \right)
\]

(63)

where \( \gamma_\alpha = e_\alpha^a \gamma_a \), with \( e_\alpha^a \) the zweibein, and \( \gamma_a \) are two dimensional Dirac matrices. The world-sheet spinors \( \psi^i \) belong to the \( S_8 \) representation of \( SO(8) \) and \( g_{\alpha\beta} \) is given in \((6)\). The superstring free energy density then reads

\[
\frac{F}{L} = -\sum_{n=0}^{\infty} \int_{-1/2}^{1/2} \frac{d\tau_1}{4\pi^2\alpha'} \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} e^{-\frac{\beta^2n^2}{4\pi^2\alpha'\tau_2}} \cdot \int d\vec{X} d\psi^i \exp \left[ -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \left( \partial_\alpha \vec{X} \cdot \partial_\beta \vec{X} - i2\pi\alpha' \vec{\psi}^i \gamma_\alpha \partial_\beta \psi^i \right) \right]
\]

(64)

### 2.3.1 Equivalence to NSR formulation

The type II superstring free energy in the Neveu-Schwarz-Ramond formulation of superstring theory was discussed, for example, by Atick and Witten \((5)\), who obtained the free energy density

\[
\frac{F}{V} = \frac{1}{4} \left( \frac{1}{4\pi^2\alpha'} \right)^5 \int_{\mathcal{F}} d^2\sigma \frac{1}{\eta(\tau)} \left| \eta(\tau) \right|^{24} \sum_{n,m=-\infty}^{n+m=\infty} e^{-\beta^2(n^2+m^2|\tau|^2-2\tau_1\tau_2)/(4\pi^2\alpha'\tau_2)} \left[ (\theta_2^1 \bar{\theta}_2^1 + \theta_2^1 \bar{\theta}_2^1 + \theta_2^1 \bar{\theta}_2^1)(0, \tau) + e^{i\pi(n+m)} (\theta_2^1 \bar{\theta}_2^1 + \theta_2^1 \bar{\theta}_2^1)(0, \tau) \right. \\
\left. -e^{i\pi n} (\theta_2^1 \bar{\theta}_2^1 + \theta_2^1 \bar{\theta}_2^1)(0, \tau) - e^{i\pi m} (\theta_2^1 \bar{\theta}_2^1 + \theta_2^1 \bar{\theta}_2^1)(0, \tau) \right]
\]

(65)

where the Teichmüller parameters are integrated over the region \( \mathcal{F} \), which is the fundamental domain of the torus. In this expression the sum over \( m \) and \( n \) is the sum over maps of the torus in which the space and time coordinates, respectively, of the torus wrap the target space \( S^1 \) \( m \) and \( n \) times. One can use modular transformations to characterise these wrappings by a single integer. In
The free energy (66) then reads

\[ F = \frac{1}{4} \left( \frac{1}{4\pi^2\alpha'} \right)^5 \int_{-1/2}^{1/2} d\tau_1 \int_0^{\infty} \frac{d\tau_2}{\tau_2^5} \int_0^{\infty} \frac{d\tau_3}{\tau_3^5} \int_0^{\infty} \frac{d\tau_4}{\tau_4^5} \int_0^{\infty} \frac{d\tau_5}{\tau_5^5} e^{-\beta^2 n^2 / (4\pi^2\alpha')^2} C(n, \tau) \]

where \( C(n, \tau) \) is the following combination of thetas.

\[ C(n, \tau) = \left[ (\theta_2^4 \bar{\theta}_2^4 + \theta_4^4 \bar{\theta}_4^4 - \theta_3^4 \bar{\theta}_3^4 - \theta_3^4 \bar{\theta}_3^4) (0, \tau) \right] + e^{i\pi n} \left[ (\theta_2^4 \bar{\theta}_2^4 + \theta_4^4 \bar{\theta}_4^4 - \theta_3^4 \bar{\theta}_3^4 - \theta_3^4 \bar{\theta}_3^4) (0, \tau) \right] \]

Using Jacobi's identity (20) \( C(n, \tau) \) becomes

\[ C(n, \tau) = 2\theta_2^4 \bar{\theta}_2^4 (0, \tau) (1 - e^{i\pi n}) = 2 |\theta_2(0, \tau)|^8 (1 - e^{i\pi n}) \]

The free energy (66) then reads

\[ F = \left( \frac{1}{4\pi^2\alpha'} \right)^5 \int_{-1/2}^{1/2} d\tau_1 \int_0^{\infty} \frac{d\tau_2}{\tau_2^5} \int_0^{\infty} \frac{d\tau_3}{\tau_3^5} \int_0^{\infty} \frac{d\tau_4}{\tau_4^5} \int_0^{\infty} \frac{d\tau_5}{\tau_5^5} \int_0^{\infty} \frac{d\tau_6}{\tau_6^5} e^{-\beta^2 n^2 / (4\pi^2\alpha')^2} |\theta_2(0, \tau)|^8 \]

The product representation of \( \theta_2(0, \tau) \) in (21) implies that

\[ \left| \frac{1}{\eta(\tau)} \right|^{24} |\theta_2(0, \tau)|^8 = \left( \frac{1 + e^{2\pi i n \tau}}{1 - e^{2\pi i n \tau}} \right)^{16} = 2^8 |\theta_4(0, 2\tau)|^{-16} \]

and eq. (70) is identical with (24).

3 The thermodynamic partition function of matrix string theory

3.1 Matrix string Hamiltonian

In this section we derive the matrix string Hamiltonian starting from the matrix theory formalism [1]. In particular we wish to identify the dependence on the string coupling constant \( g_s \), on the string tension \( \alpha' \). The starting point is the matrix theory Hamiltonian, which describes a stack of \( N \) D0-branes. In terms of the ten dimensional superstring string tension \( \alpha' \) it has the form

\[ H = \frac{1}{2g^2} \text{Tr} \left( g^2 \alpha' \Pi_a^2 - \frac{1}{(2\pi\alpha')^2} [X^a, X^b]^2 - \frac{1}{2\pi\alpha'} \bar{\psi} \gamma_a [X^a, \psi] \right) \]
where \( a = 1, \ldots, 9 \). Each of the adjoint scalar matrices \( X^a \) is a hermitian \( N \times N \) matrix, where \( N \) is the number of 0-branes. The super-partners of the \( X \) fields are the 16 component spinors \( \psi \) which transform under the \( SO(9) \) Clifford algebra given by the \( 16 \times 16 \) matrices \( \gamma^a \). Conventionally \( M \)-theory is related to type-IIA string theory via the compactification of the 11-th direction, which relates the coupling constant \( g \) to the compactification radius of the 11-th dimension \( R_{11} \) through \( g \sqrt{\alpha'} = R_{11} \). The Hamiltonian \( (72) \) can also be expressed in eleven dimensional Planck units, by replacing \( g \sqrt{\alpha'} = R_{11} \) and \( \alpha' = l_p^2 g^{-2/3} \), with \( l_p \) the 11-dimensional Plank length

\[
H = \frac{R_{11}}{2} \text{Tr} \left( \Pi_a^2 - \frac{1}{4\pi 2^{16} l_p^3} \left[ X^a, X^b \right]^2 - \frac{1}{2\pi 2^{16} l_p^3} \psi^T \gamma_a \left[ X^a, \psi \right] \right). \tag{73}
\]

The mass-dimensions of \( H \) and of the fields are

\[
[H] = M, \quad [X^a] = M^{-1}, \quad [\psi] = M^0.
\]

We now compactify the 9th dimension on a circle of radius \( R_9 \). After the usual \( T \)-duality, we can identify \( X^9 \) with the covariant derivative

\[
X^9 \rightarrow iR_9 D_1 = iR_9 \left( \frac{\partial}{\partial \sigma_1} - iA_1 \right)
\]

where the coordinate \( \sigma_1 \in [0, 1] \). The momentum conjugate to \( X^9 \) will be identified with the electric field \( E \) via \( E = R_9 \Pi_9 \). This procedure \( [36] \) gives the Hamiltonian (where \( i = 1, \ldots, 8 \) now labels the transverse coordinates)

\[
H = \frac{R_{11}}{2} \int_0^1 d\sigma_1 \text{Tr} \left[ \Pi_i^2 + \frac{R_9^2}{4\pi 2^{16} l_p^3} (D_1 X_i)^2 - \frac{1}{4\pi 2^{16} l_p^3} \left[ X^i, X^j \right]^2 + \frac{E^2}{R_9^2}
\right.
\]

\[
\left. - \frac{1}{2\pi 2^{16} l_p^3} \psi^T \gamma_i \left[ X^i, \psi \right] - \frac{iR_9}{2\pi 2^{16} l_p^3} \psi^T \gamma_1 \cdot D_1 \psi \right]. \tag{74}
\]

To arrive at the matrix string point of view \( [3] \), we now interchange the 9-th and the 11-th direction by defining the string scale \( \sqrt{\alpha'} \) and the string coupling constant \( g_s \) in terms of \( R_9 \) and \( l_p \)

\[
R_9 = g_s \sqrt{\alpha'}, \quad l_p = g_s^{1/3} \sqrt{\alpha'}.
\]  \( \tag{75} \)

Note that the mass-dimension of \( g_s \) is 0. With this substitution we obtain the final result for the Hamiltonian of the matrix string model of ref. \( [\text{DVV}] \) (which we shall refer to as DVV),

\[
H = \frac{R_{11}}{2} \int_0^1 d\sigma_1 \text{Tr} \left[ \Pi_i^2 + \frac{1}{4\pi 2^{16} l_p^3} (D_1 X_i)^2 - \frac{1}{4\pi 2^{16} l_p^3} \left[ X^i, X^j \right]^2 + \frac{E^2}{g_s^2 \alpha'}
\right.
\]

\[
\left. - \frac{1}{2\pi 2^{16} l_p^3} \psi^T \gamma_i \left[ X^i, \psi \right] - \frac{i}{2\pi \alpha'} \psi^T \gamma_1 \cdot D_1 \psi \right]. \tag{76}
\]

Here, \( R_{11} \) is the radius of the dimension that must be compactified in order to obtain the matrix description of M-theory - in its original form the matrix model
describes M-theory in a reference frame which has infinite momentum in the 11'th direction. In the more sophisticated proposal of ref. [2], this compactified direction is the light-cone direction $R^+$. It was shown by Seiberg [37] that, with certain assumptions, a boost to the frame with infinite momentum of a theory compactified on a small circle $R_{11}$ is equivalent to one with the light cone compactified. For a more recent discussion of these issues see [38, 39, 40].

Here, anticipating that we are actually describing DLCQ M-theory, we shall make the replacement

$$R_{11} \rightarrow R^+$$

The hypothesis is that this model describes DLCQ M-theory with one dimension compactified (to describe the DLCQ IIA superstring). The canonical momenta which appear in (76) are normalised so that they have the conventional canonical commutators,

$$[X^i_{ab}, \Pi^j_{cd}(\sigma')] = i\varepsilon^{ij} \varepsilon_{cd} \varepsilon^{ai} \varepsilon^{bj}$$

$$[A_{ab}(\sigma), E_{cd}(\sigma')] = i\varepsilon^{ij} \varepsilon_{cd} \varepsilon^{ai} \varepsilon^{bj}$$

$$\{\psi_{ab}(\sigma), \psi_{cd}(\sigma')\} = \frac{1}{2} \varepsilon^{ij} \varepsilon_{cd} \varepsilon^{ai} \varepsilon^{bj} (\sigma - \sigma')$$

(77)

Note that $g_s^2 \alpha'$ plays the role of the inverse square of the Yang-Mills coupling constant: $g_s^2 \alpha' = g_{YM}^{-2}$.

### 3.2 Matrix string free energy

Using the Hamiltonian (76) we shall now construct the thermal partition function $Z$ of the matrix string theory in the limit $g_s \rightarrow 0$. The prescription discussed in the previous sections for the treatment of the thermal ensemble for systems in the light-cone frame, and the fact that $p^- = N/R^+$ lead us to write $Z$ as

$$Z = \text{Tr} e^{-\beta p^0} = \text{Tr} e^{-\beta (p^+ + p^-)/\sqrt{2}} = \sum_{N=0}^{\infty} e^{-N\beta/\sqrt{2} R^+} \text{Tr} \left\{ e^{-\beta H/\sqrt{2} \mathbf{R}_+} \right\}$$

(78)

where $P^+ = H$ is the matrix string theory Hamiltonian in (76). The trace of $\exp(-\beta H/\sqrt{2})$ is to be taken over gauge invariant states of the two-dimensional super-Yang-Mills theory. This trace has the standard path integral expression

$$Z[\beta] = \sum_{N=0}^{\infty} \int [dA_\alpha][dX^i][d\psi^a] \exp \left\{ -\beta N/\sqrt{2} R^+ - S_E[A, X^i, \psi] \right\}$$

(79)

where $S_E$ is the Euclidean action

$$S_E = \frac{1}{2} \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \text{Tr} \left\{ \frac{\beta R^+}{(2\pi\alpha')^2} D_1 X^i D_1 X^i + \frac{\sqrt{2}}{\beta R^+} D_2 X^i D_2 X^i + g_s^2 \alpha' \sqrt{2} F_{\mu\nu}^2 - \frac{\beta R^+}{(2\pi\alpha')^2} \sum_{i<j} [X^i, X^j]^2 - i \frac{\beta R^+}{\sqrt{2} 2\pi\alpha'} \psi^T \gamma D_1 \psi \right\}$$

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and the covariant derivative is $D_\mu = \partial_\mu - i[A_\mu,...]$. Here, we have rescaled the
time $\sigma_2$ so that the integration is over a box of area one, $0 \leq \sigma_\mu < 1$, and $\beta$
appears as a factor in various coupling constants. Boundary conditions in the
path integral are

\begin{align*}
A_\mu(\sigma_1 + 1, \sigma_2) &= A_\mu(\sigma_1, \sigma_2) \\
X^i(\sigma_1 + 1, \sigma_2) &= X^i(\sigma_1, \sigma_2) \\
\psi(\sigma_1 + 1, \sigma_2) &= \psi(\sigma_1, \sigma_2) \\
A_\mu(\sigma_1, \sigma_2 + 1) &= A_\mu(\sigma_1, \sigma_2) \\
X^i(\sigma_1, \sigma_2 + 1) &= X^i(\sigma_1, \sigma_2) \\
\psi(\sigma_1, \sigma_2 + 1) &= -\psi(\sigma_1, \sigma_2)
\end{align*}

The anti-periodicity of the fermion field in the Euclidean time comes from taking
the trace in (78).

The limit $g_s \to 0$ of this theory was formulated by DVV [6]. In this limit,
the field configurations which have finite action are related to diagonal field
configurations by

\begin{align*}
X^i(\sigma) &= U(\sigma)X^i_D(\sigma)U^{-1}(\sigma) \\
\psi(\sigma) &= U(\sigma)\psi_D(\sigma)U^{-1}(\sigma) \\
A_\mu(\sigma) &= iU(\sigma)(\partial_\mu - iA^D_\mu(\sigma))U^{-1}(\sigma)
\end{align*}

where $X^i_D$, $\psi_D$ and $A^D_\mu$ are diagonal matrices. The fields $X^i$, $\psi$ and $A_\mu$
have the (anti-)periodic boundary conditions [3]. This implies that the diagonal
matrices have boundary conditions which are periodic up to permutations (and
gauge transformations generated by the Cartan sub-algebra),

\begin{align*}
(A^D_\mu)_a(\sigma_1 + 1, \sigma_2) &= (A^D_\mu)_b(\sigma_1, \sigma_2) + 2\pi(n_\mu)_a + \partial_\mu \theta_a \\
(X^i_D)_a(\sigma_1 + 1, \sigma_2) &= (X^i_D)_b(\sigma_1, \sigma_2) \\
(\psi_D)_a(\sigma_1 + 1, \sigma_2) &= (\psi_D)_b(\sigma_1, \sigma_2) \\
(A^i_D)_a(\sigma_1, \sigma_2 + 1) &= (A^i_D)_b(\sigma_1, \sigma_2) + 2\pi(m_\mu)_a + \partial_\mu \phi_a \\
(X^i_D)_a(\sigma_1, \sigma_2 + 1) &= (X^i_D)_b(\sigma_1, \sigma_2) \\
(\psi_D)_a(\sigma_1, \sigma_2 + 1) &= -(\psi_D)_b(\sigma_1, \sigma_2)
\end{align*}

where $a = 1,...,N$, $P(a)$ and $Q(a)$ are permutations and $n_\mu, m_\mu$ are integers.
Consistency requires that the two permutations commute,

$$PQ = QP$$

To compute the partition function, in the $g_s \to 0$ limit, we should now do the
path integral (79) over only the diagonal components of the matrix fields with
the action

$$S_{\text{diag}} = \frac{1}{2} \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \sum_{a=1}^N \left\{ \frac{\beta R^+}{(2\pi\alpha')^2} \partial_1 X^i_a \partial_1 X^i_a + \frac{\sqrt{2}}{\beta R^+} \partial_2 X^i_a \partial_2 X^i_a \right\}$$
\[ + \frac{g_s^2 \alpha'}{2} \sqrt{2} (\partial_1 A_{2a} - \partial_2 A_{1a})^2 - i \frac{\beta R^+}{\sqrt{22 \pi \alpha'}} \psi_a^T \gamma \partial_1 \psi_a - i \psi_a^T \partial_2 \psi_a \] \tag{85}

and with the boundary conditions (83). We should then sum over topologically distinct configurations, characterised by the permutations $P$ and $Q$ and by the integers $m_\mu, n_\mu$. We shall not discuss the validity of the assumptions leading to this starting point in the present Paper (see discussions in [9, 10]).

The partition function that we must compute thus decomposes into topological sectors as

\[ Z = \sum_{N=0}^{\infty} \frac{1}{N!} e^{-N \beta / \sqrt{2} R^+} \sum_{P, Q} Z(P, Q) , \tag{86} \]

(where we have suppressed the dependence on the integers in the boundary conditions for gauge fields). Here, for each $N$ we have divided by contribution by the volume of the Weyl group, $N!$, which reflects the fact that the eigenvalues are defined up to a global permutation. (From the M-theory point of view, this factor gives Boltzmann statistics to the D0-branes.) The possibility of introducing different weights in the sum over pairs of commuting permutation has been discussed in [22]. We shall show in what follows that the correct partition function for the type IIA string emerges from (86) only when all of the pairs $(P, Q)$ have the same weight.

### 3.2.1 Gauge fields are irrelevant

We first observe that, in all cases, when $g_s \to 0$ the gauge fields are irrelevant. Consider for the moment the QED partition function

\[ Z_{U(1)} = \int [dA_1] [dA_2] \exp \left[ - \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \left( \frac{g_s^2 \alpha'}{2} \beta R^+ (\partial_1 A_{2} - \partial_2 A_{1})^2 \right) \right] \tag{87} \]

where gauge fields are periodic up to integers and gauge transforms,

\[
\begin{align*}
A_\mu(\sigma_1, \sigma_2 + 1) &= A_\mu(\sigma_1, \sigma_2) + 2\pi m_\mu + \partial_\sigma \theta_a , \\
A_\mu(\sigma_1 + 1, \sigma_2) &= A_\mu(\sigma_1, \sigma_2) + 2\pi n_\mu + \partial_\sigma \phi_a .
\end{align*} \tag{88}
\]

The $N$’th power of this partition function would arise as the gauge field contribution to the full partition function in the topological sector where $P$ and $Q$ are the trivial permutation. (For a non-trivial permutation, though the details would be slightly different, the arguments below would still hold.)

We can un-twist this boundary conditions with a non-periodic gauge transformation to present the gauge field in terms of a periodic gauge field $\hat{A}_\mu$ according to

\[ A_\mu(\sigma_1, \sigma_2) = \hat{A}_\mu(\sigma_1, \sigma_2) + 2\pi n_\sigma_1 \delta_\mu^2 \tag{89} \]

The partition function becomes

\[ Z_{U(1)} = \sum_{n=-\infty}^{\infty} \exp \left[ - \frac{g_s^2 \alpha' n^2}{\sqrt{2} \beta R^+} \right] . \]
\[
\cdot \int [d\hat{A}_\mu] \exp \left[ - \int_0^1 \! d\sigma_1 \int_0^1 \! d\sigma_2 \left( \frac{g_s^2 \alpha'}{\sqrt{2\beta R^+}} (\partial_1 \hat{A}_2 - \partial_2 \hat{A}_1)^2 \right) \right]
\]  

(90)

Fixing the gauge \( \partial_2 \hat{A}_2 = 0 \) and taking into account the Faddev-Popov determinant, the integration on the gauge fields can be performed and gives

\[
Z_U(1) = \left( \frac{g_s^2 \alpha'}{\sqrt{2\pi \beta R^+}} \right)^{\frac{1}{2}} \sum_{n_2=-\infty}^{\infty} \exp \left[ - \frac{g_s^2 \alpha' (n_2)^2}{\sqrt{2\beta R^+}} \right]
\]  

(91)

Poisson re-summing

\[
Z_U(1) = \sum_{n_2=-\infty}^{\infty} \exp \left[ - \frac{\sqrt{2\pi \beta R^+} (n^2)^2}{g_s^2 \alpha'} \right]
\]  

(92)

In the \( g_s \to 0 \) limit only the \( n_2 = 0 \) sector survives and, for consistency with the approximations made above, the contribution of the Yang Mills term to the thermal partition function of the matrix string theory, should just be set to 1.

### 3.2.2 Generalities about permutations

A discussion on the permutations of \( N \) elements commuting with a given permutation \( P \) has been given in [22, 21]. It is useful at this point to review some of the salient points, for which we follow the appendix of [22].

Consider a fixed permutation \( P \) of a set of \( N \) elements. It can be decomposed into cycles which are subsets of the \( N \) elements such that the permutation interchanges elements cyclically inside each subset. For example, for \( N = 9 \), the permutation which can be denoted by

\[
\alpha = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
6 & 4 & 1 & 2 & 5 & 3 & 8 & 9 & 7
\end{pmatrix}
\]

has the three cycles \( \alpha(1) = 6, \alpha(6) = 3, \alpha(3) = 1 \) and \( \alpha(7) = 8, \alpha(8) = 9, \alpha(9) = 7 \), the two-cycle \( \alpha(2) = 4, \alpha(4) = 2 \) and the one-cycle \( \alpha(5) = 5 \). Thus it can be decomposed into the product of disjoint cycles

\[
\alpha = (163)(24)(5)(789)
\]

Consider such a decomposition of \( P \). Suppose that the number of cycles of length \( k \) is \( r_k \), so that \( N = \sum_k kr_k \). Clearly, the minimum value of \( k \) is one and the maximum is \( N \). (In the above example, \( r_1 = 1, r_2 = 1, r_3 = 2 \).)

Denote the elements of the set \( \{1, 2, ..., N\} \) by the three index notation based on how they transform under \( P \):

\[
a_{k,\alpha}^n \quad (k = 1, ..., N); (\alpha = 1, ..., r_k); (n = 1, ..., k)
\]  

(93)

This is the \( n \)th element of the \( \alpha \)’th cycle of length \( k \). The action of the permutation on this element is then

\[
P(a_{k,\alpha}^n) = a_{n+1 \mod k}^{k,\alpha}
\]  

(94)
Let $Q$ be a permutation which commutes with $P$. Consider its action on the cycle 
\[ \left( a_1^{k,\alpha}, \ldots, a_k^{k,\alpha} \right) \]
of $P$. Eq. (94) implies that
\[ Q \left( P \left( a_n^{k,\alpha} \right) \right) = Q \left( a_{n+1 \mod k}^{k,\alpha} \right) \]  
(95)
Since $PQ = QP$, this equation can be written as
\[ P \left( Q \left( a_n^{k,\alpha} \right) \right) = Q \left( a_{n+1 \mod k}^{k,\alpha} \right) \]  
(96)
This means that
\[ \left( Q \left( a_1^{k,\alpha} \right), \ldots, Q \left( a_k^{k,\alpha} \right) \right) \]
is a cycle of length $k$ in $P$, i.e. that
\[ \left( Q \left( a_1^{k,\alpha} \right), \ldots, Q \left( a_k^{k,\alpha} \right) \right) = \left( a_s^{k,\pi_k(\alpha)}, \ldots, a_{s+k-1 + k \mod k}^{k,\pi_k(\alpha)} \right) \]
Hence, there exists a permutation $\pi_k(\alpha)$ of the $r_k$ elements of the set of cycles of length $k$ which is induced by $Q$. Clearly, $Q$ commutes with $P$.

Any such permutation of the cycles of length $k$ for each $k$ corresponds to an acceptable element of $Q$. For each of these permutations, there is an additional piece of information - the ordering of a cycle is cyclic, so the permutation between two cycles of length $k$ can also involve an element of the cyclic group (or, alternatively, the first element of the cycle $\alpha$ can be mapped onto any element of the cycle $\pi_k(\alpha)$). This implies that there exist $r_k$ integers, $s(k,\alpha) \in \{1,2,\ldots,k\}$ (with $\alpha \in \{1,2,\ldots,r_k\}$) such that
\[ Q \left( a_n^{k,\alpha} \right) = a_{n+s(k,\alpha) \mod k}^{k,\pi_k(\alpha)} \]  
(97)
Thus, a permutation $Q$ which commutes with $P$ is completely determined by assigning for each value of $k$,

- A permutation $\pi_k \in S_k$ (where the permutation group acting on the $r_k$ cycles of order $k$ in $P$)
- A set of $r_k$ integers $s(k,\alpha) \in \{1,2,\ldots,k\}$

The permutation $Q$ is then defined by (97).

It follows that the number of permutations $Q$ which commute with a given permutation $P$ is
\[ \prod_k r_k!k^{r_k} \]  
(98)
3.2.3 How permutations determine the world-sheet metric

Consider a set of diagonal components of fields obeying the boundary conditions

\[ \lambda_a(\sigma_1 + 1, \sigma_2) = \lambda_{P(a)}(\sigma_1, \sigma_2), \]
\[ \lambda_a(\sigma_1, \sigma_2 + 1) = (-1)^f \lambda_{Q(a)}(\sigma_1, \sigma_2). \]  

(99)

where \( f = 0 \) for a Bose field and \( f = 1 \) for a fermion. Consider those which occur in the cycles of length \( k \) of \( P \)

\[ \lambda_{a_{kn}}(\sigma_1 + 1, \sigma_2) = \lambda_{a_{n+1}}(\sigma_1, \sigma_2), \]
\[ \lambda_{a_{kn}}(\sigma_1, \sigma_2 + 1) = (-1)^f \lambda_{a_{n+1}}(\sigma_1, \sigma_2). \]  

(100)

For each fixed \( \alpha \), we fuse these fields together into a single function which has the property

\[ \lambda_{a_{kn}}(\sigma_1 + k, \sigma_2) = \lambda_{a_{n+1}}(\sigma_1, \sigma_2), \]
\[ \lambda_{a_{kn}}(\sigma_1, \sigma_2 + 1) = (-1)^f \lambda_{a_{n+1}}(\sigma_1 + s(k, \alpha), \sigma_2). \]  

(101)

Then, we consider a cycle of the permutation \( Q \), which must be a subset of the \( r_k \) k-cycles of \( P \). Say this cycle is of length \( r \) where \( 1 \leq r \leq r_k \). Then we fuse \( r \) of the above fields together to get the single field which has (anti-)periodic boundary conditions

\[ \lambda_{a^{k,r}}(\sigma_1 + k, \sigma_2) = \lambda_{a^{k,r}}(\sigma_1, \sigma_2), \]
\[ \lambda_{a^{k,r}}(\sigma_1, \sigma_2 + r) = (-1)^f \lambda_{a^{k,r}}(\sigma_1 + s, \sigma_2). \]  

(102)

where \( s = \sum_\alpha s(k, \alpha) \mod k \) is the accumulated shift for the \( r \) elements in the cycle of \( Q \).

The space on which coordinates which are arguments of the field in (102) take values is the torus depicted in fig. 1. The contribution to the path integral of this set of fields is denoted by

\[
Z(r, k, s) = \int [dX^i][d\psi] \exp \left[ -\frac{1}{2} \int_0^r d\sigma_2 \int_{\frac{r}{2}+\sigma_2}^{r+\sigma_2} d\sigma_1 \left( \frac{\beta R^+}{\sqrt{2(2\pi\alpha')}} (\partial_1 X^i)^2 + \frac{\beta}{\sqrt{22\pi\alpha'}} (\partial_2 X^i)^2 - i \frac{\beta R^+}{\sqrt{22\pi\alpha'}} \psi^T \gamma \cdot \partial_1 \psi - i \psi^T \partial_2 \psi \right) \right]
\]

(103)

where the integration region is the torus of fig.1. The boundary conditions for the Bose fields are

\[ X^i(\sigma_1 + k, \sigma_2) = X^i(\sigma_1, \sigma_2) \]
\[ X^i(\sigma_1, \sigma_2 + r) = X^i(\sigma_1 + s, \sigma_2) \]  

(104)

and for the Fermi fields are

\[ \psi(\sigma_1 + k, \sigma_2) = \psi(\sigma_1, \sigma_2) \]
where

\[ \psi(\sigma_1, \sigma_2 + r) = (-1)^r \psi(\sigma_1 + s, \sigma_2) \]  \hspace{1cm} (105)

It is possible to change the coordinates in the action (103) so that the integration region is the square torus \( (\sigma_1, \sigma_2) \in ([0, 1], [0, 1]) \). The appropriate coordinate transformation is

\[
\sigma'_1 = \frac{\sigma_1}{k} - \frac{s \sigma_2}{kr}, \\
\sigma'_2 = \frac{\sigma_2}{r}
\]  \hspace{1cm} (106)

Then, the action becomes

\[
S = \frac{1}{4\pi \alpha'} \int_0^1 \, d\sigma_2 \int_0^1 \, d\sigma_1 \sqrt{g} \left( g^{\alpha\beta} \partial_\alpha \vec{X} \cdot \partial_\beta \vec{X} - i2\pi \alpha' \bar{\psi}^i \gamma^\alpha c^\alpha_a \partial_\alpha \psi^i \right)
\]  \hspace{1cm} (107)

where

\[
g_{\alpha\beta} = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}
\]  \hspace{1cm} (108)

with

\[
\tau_1 = \frac{s}{k}, \quad \tau_2 = \frac{r \beta R^+}{2\sqrt{2k\pi \alpha'}}
\]  \hspace{1cm} (109)

and \( c^\alpha_a \) a zweibein which corresponds to the metric \( g_{\alpha\beta} \). Now, the boundary conditions are

\[
X^i(\sigma_1 + 1, \sigma_2) = X^i(\sigma_1, \sigma_2)
\]
\[
X^i(\sigma_1, \sigma_2 + 1) = X^i(\sigma_1, \sigma_2)
\]
\[
\psi(\sigma_1 + 1, \sigma_2) = \psi(\sigma_1, \sigma_2)
\]
\[ \psi(\sigma_1, \sigma_2 + 1) = (-1)^r \psi(\sigma_1, \sigma_2) \quad (110) \]

Note that the boundary condition for the Fermi field still depends on \( r \).

When \( r \) is even, the fermion and boson have the same boundary conditions. These sectors are supersymmetric. The mode expansion of both the fermions and bosons contain zero modes. Functional integration over bosonic zero modes produces a factor of the infinite volume of \( R^8 \) and integration over the fermionic zero mode produces a factor of zero. If we were computing the Witten index, this product of infinity times zero, suitably regulated would yield the number of zero energy states of the supersymmetric theory. However, here, we are computing an extensive thermodynamic variable - the free energy - from which we must extract a factor of the volume of the space in order to obtain the free energy density. In this case, sectors which contain fermion zero modes do not contribute.

On the other hand, when \( r \) is odd, the fermions have anti-periodic boundary conditions - supersymmetry is broken by this boundary condition - and there are no fermion zero modes in the mode expansion. These sectors will survive and contribute to the partition function. Thus,

\[ Z(k, r, s) = 0 \text{ when } r \text{ is even} \quad (111) \]

### 3.2.4 How the first few orders work

We shall now construct explicitly the first few terms contributing to the partition function (86). The first term is obtained by considering the trivial permutations of the \( N \) eigenvalues in both directions. The eigenvalues are then periodic in both directions and eq. (99) reads

\[ \lambda_i(\sigma_1 + 1, \sigma_2) = \lambda_i(\sigma_1, \sigma_2) , \]
\[ \lambda_i(\sigma_1, \sigma_2 + 1) = \lambda_i(\sigma_1, \sigma_2) . \quad (112) \]

The transverse partition function will then be given by the product of \( N \) partition function defined on the square torus of side 1.

\[ Z^{(1)} = \sum_{N=0}^{\infty} \frac{1}{N!} e^{-\frac{N^2 \beta}{2}} [Z(1, 1, 0)]^N. \quad (113) \]

The second order contribution to (86) is provided by summing on all the permutation that permute any two eigenvalues, and leave unchanged the other \( N - 2 \). We denote \( \lambda_1(\sigma_1, \sigma_2) \) and \( \lambda_2(\sigma_1, \sigma_2) \) the two permuted eigenvalues. There are of course 4 pairs of permutations \((P, Q)\) of two eigenvalues and all commuting. The contribution of the trivial permutations in both direction is already taken into account in \( Z^{(1)} \), this would give in fact a “disconnected” pair of permutations.

If \( P = (1, 2) \) and \( Q = (1)(2) \) the boundary conditions (99) are

\[ \lambda_i(\sigma_1 + 2, \sigma_2) = \lambda_i(\sigma_1, \sigma_2) , \]

24
\[ \lambda_i(\sigma_1, \sigma_2 + 1) = \lambda_i(\sigma_1, \sigma_2) \]  

for \( i = 1, 2 \), and as in (112) for the other \( N - 2 \) eigenvalues. Thus this pair of commuting permutation gives a contribution \( [Z(1, 1, 0)]^{N-2} \cdot Z(1, 2, 0) \). Analogously \( P = (1)(2) \) and \( Q = (1, 2) \) give the boundary conditions

\[
\begin{align*}
\lambda_i(\sigma_1 + 1, \sigma_2) &= \lambda_i(\sigma_1, \sigma_2) , \\
\lambda_i(\sigma_1, \sigma_2 + 2) &= \lambda_i(\sigma_1 + 1, \sigma_2)
\end{align*}
\]

(115)

for \( i = 1, 2 \) with a contribution to \( Z_T \) given by \( [Z(1, 1, 0)]^{N-2} \cdot Z(2, 1, 0) \). For \( P = (1, 2) \) \( Q = (1, 2) \) we have

\[
\begin{align*}
\lambda_i(\sigma_1 + 2, \sigma_2) &= \lambda_i(\sigma_1, \sigma_2) , \\
\lambda_i(\sigma_1, \sigma_2 + 1) &= \lambda_i(\sigma_1 + 1, \sigma_2)
\end{align*}
\]

(116)

so that \( r = 1, k = 2 \) and \( s = 1 \). By considering that there are \( \binom{N}{2} \) pairs and that is necessary to have at least \( N = 2 \) to have a permutation of two eigenvalues, the second order contribution to the partition function is thus given by

\[
Z^{(2)} = \sum_{N=2}^{\infty} \frac{1}{N!} \binom{N}{2} e^{-\frac{N \beta}{\sqrt{2}} [Z(1, 1, 0)]^{N-2} [Z(2, 1, 0) + Z(1, 2, 0) + Z(1, 2, 1)]}
\]

(117)

This term can be easily rearranged by shifting \( N \rightarrow N + 2 \) as

\[
Z^{(2)} = \sum_{N=0}^{\infty} \frac{1}{N!} e^{-\frac{N + 2 \beta}{\sqrt{2}} [Z(1, 1, 0)]^N} \sum_{r|2} \sum_{s=0}^{2/r-1} Z(r, \frac{2}{r}, s)
\]

(118)

where \( \sum_{r|N} \) means the sum over the divisors of \( N \). In an analogous way one can construct the third order contribution to the partition function \( Z_T \), this can be obtained by considering the pairs of connected commuting permutations of three eigenvalues. Taking the trivial permutation \( P = (1)(2)(3) \) in the \( \sigma_1 \) direction, the two permutations \( Q = (1, 2, 3), (1, 3, 2) \) in the \( \sigma_2 \) direction each one forms a connected commuting pair with \( P \). Any other permutation \( Q \) commute with \( P = (1)(2)(3) \) but the pair \( (P, Q) \) would be in this case a disconnected pair whose contribution to the partition function is already contained in the first two term of the expansion. It can be easily seen that the values of \( r, k \) and \( s \) corresponding to the two pairs \( (P = (1)(2)(3), Q = (1, 2, 3), (1, 3, 2)) \) are equal and given by \( r = 3, k = 1, s = 0 \). The other pairs of connected commuting permutations can be easily constructed to give Table 1.

As can be seen from Table 1 there are 4 groups of 2 pairs of \( (P, Q) \) with the same values of \( (r, k, s) \), to which correspond the same integration region in the action in (103) or the same metric in (107).

There are \( \binom{N}{3} \) choices for three eigenvalues and is necessary to have at least \( N = 3 \) to have a permutation of three eigenvalues, the third order contribution
Table 1: Pairs of commuting connected permutations of three eigenvalues.

| P           | Q           | r | k | s |
|-------------|-------------|---|---|---|
| (1)(2)(3)   | (1,2,3)     | 3 | 1 | 0 |
| (1)(2)(3)   | (1,3,2)     | 3 | 1 | 0 |
| (1,2,3)     | (1)(2)(3)   | 1 | 3 | 0 |
| (1,2,3)     | (1,3,2)     | 1 | 3 | 1 |
| (1,2,3)     | (1,2,3)     | 1 | 3 | 2 |
| (1,3,2)     | (1,2,3)     | 1 | 3 | 2 |

to the partition function then reads

\[ Z^{(3)} = \sum_{N=3}^{\infty} \frac{1}{N!} \left( \frac{N}{3} \right) e^{-\frac{N \beta}{\sqrt{2R}}} \left[ Z(1, 1, 0) \right]^{N-3} \cdot 2 \left[ Z(3, 1, 0) + Z(1, 3, 0) + Z(1, 3, 1) + Z(1, 3, 2) \right]. \] (119)

By shifting \( N \to N + 3 \) we get

\[ Z^{(3)} = \sum_{N=0}^{\infty} \frac{1}{N!} e^{-\frac{(N+3) \beta}{\sqrt{2R}}} \left[ Z(1, 1, 0) \right]^{N} \sum_{r|3} \sum_{s=0}^{\frac{3}{r}-1} Z(r, \frac{3}{r}, s). \] (120)

Let us compute now the fourth order. We should consider at this order both the pair of commuting connected permutations of four eigenvalues and the permutations of two pairs of two eigenvalues. Following the procedure illustrated above, for the commuting connected permutations of four eigenvalues we can construct the Table 2.

As can be seen from Table 2 there are 7 groups of 6 pairs of \((P, Q)\) with the same values of \((r, k, s)\), to which correspond the same integration region in the action in (103) or the same metric in (107). Taking into account that there are \( \binom{N}{4} \) groups of 4 eigenvalues and \( \frac{N(N-1)(N-2)(N-3)}{8} \) pairs of pairs of eigenvalues, after the shift \( N \to 4 \) the fourth order contribution to the partition function reads

\[ Z^{(4)} = \sum_{N=0}^{\infty} \frac{1}{N!} e^{-\frac{(N+4) \beta}{\sqrt{2R}}} \left[ Z(1, 1, 0) \right]^{N} \cdot \sum_{r|4} \sum_{s=0}^{\frac{4}{r}-1} Z(r, \frac{4}{r}, s). \]

\[ \cdot \sum_{N=0}^{\infty} \frac{1}{N!} e^{-\frac{(N+4) \beta}{\sqrt{2R}}} \left[ Z(1, 1, 0) \right]^{N} \left[ \frac{1}{2} \sum_{r|2} \sum_{s=0}^{\frac{2}{r}-1} Z(r, \frac{2}{r}, s) \right]^2. \] (121)
Table 2: Pairs of commuting connected permutations of three eigenvalues.

|   |   |   |   |
|---|---|---|---|
| P | Q | r | k |
| (1)(2)(3)(4) | (1,2,3,4) | 4 | 1 |
| (1)(2)(3)(4) | (1,2,4,3) | 4 | 1 |
| (1)(2)(3)(4) | (1,3,2,4) | 4 | 1 |
| (1)(2)(3)(4) | (1,3,4,2) | 4 | 1 |
| (1)(2)(3)(4) | (1,4,3,2) | 4 | 1 |
| (1)(2)(3)(4) | (1,4,2,3) | 4 | 1 |
| (1)(2)(3)(4) | (1,2,3,4) | 4 | 1 |
| (1,2,3,4) | (1)(2)(3)(4) | 1 | 4 |
| (1,2,4,3) | (1)(2)(3)(4) | 1 | 4 |
| (1,3,2,4) | (1)(2)(3)(4) | 1 | 4 |
| (1,3,4,2) | (1)(2)(3)(4) | 1 | 4 |
| (1,4,3,2) | (1)(2)(3)(4) | 1 | 4 |
| (1,4,2,3) | (1)(2)(3)(4) | 1 | 4 |
| (1,2,3,4) | (1,2,3,4) | 1 | 4 |
| (1,2,4,3) | (1,2,4,3) | 1 | 4 |
| (1,3,2,4) | (1,3,2,4) | 1 | 4 |
| (1,3,4,2) | (1,3,4,2) | 1 | 4 |
| (1,4,3,2) | (1,4,3,2) | 1 | 4 |
| (1,4,2,3) | (1,4,2,3) | 1 | 4 |
| (1,2,3,4) | (1,2,3,4) | 1 | 4 |
| (1,2,4,3) | (1,2,4,3) | 1 | 4 |
| (1,3,2,4) | (1,3,2,4) | 1 | 4 |
| (1,3,4,2) | (1,3,4,2) | 1 | 4 |
| (1,4,3,2) | (1,4,3,2) | 1 | 4 |
| (1,4,2,3) | (1,4,2,3) | 1 | 4 |
| (1,2,3,4) | (1,3,2,4) | 2 | 2 |
| (1,2,4,3) | (1,4,2,3) | 2 | 2 |
| (1,3,2,4) | (1,2,3,4) | 2 | 2 |
| (1,3,4,2) | (1,2,4,3) | 2 | 2 |
| (1,4,3,2) | (1,4,2,3) | 2 | 2 |
| (1,4,2,3) | (1,3,2,4) | 2 | 2 |
| (1,2,3,4) | (1,3,2,4) | 2 | 2 |
| (1,2,4,3) | (1,4,2,3) | 2 | 2 |
| (1,3,2,4) | (1,3,2,4) | 2 | 2 |
| (1,4,2,3) | (1,3,2,4) | 2 | 2 |
Up to the fourth order \( Z \) then reads

\[
Z = \sum_{N=0}^{\infty} \frac{1}{N!} e^{-\frac{N \beta}{\sqrt{2} R}} [Z(1, 1, 0)]^N.
\]

\[
= \left\{ 1 + e^{-\frac{\sqrt{2} \beta}{R}} \sum_{r|2, s=0}^{2/r-1} Z(r, \frac{2}{r}, s) + \frac{1}{2} \left[ e^{-\frac{\sqrt{2} \beta}{R}} \sum_{r|2, s=0}^{2/r-1} Z(r, \frac{2}{r}, s) \right]^2 \right. \\
+ \left. \sum_{r|3, s=0}^{3/r-1} Z(r, \frac{3}{r}, s) + \sum_{r|4, s=0}^{4/r-1} Z(r, \frac{4}{r}, s) \right\}(122)
\]

The pattern is quite clear, keeping into account also the higher orders, each term in the square brackets containing sums over connected tori with the same area \((rk)\), is exponentiated with a weight given by

\[
e^{-\frac{\sqrt{2} \beta}{rk}}.
\]

3.2.5 General solution

It is clear from the discussion of the previous subsection that the free energy of the matrix string theory is given by the expression

\[
F = -\frac{1}{\beta} \log Z = -\frac{1}{\beta} \sum_{N=0}^{\infty} \frac{1}{N!} e^{-\frac{N \beta}{\sqrt{2} R}} \sum_{r|N, \text{ odd}}^{N/r-1} \sum_{s=0}^{N/r} Z\left(r, \frac{N}{r}, s\right)(123)
\]

where \( Z(r, k, s) \) is given in eq.(103). Note that \( r \) gives the height of the torus, in the \( \sigma_2 \) direction and, taking into account eqn. (111) and the discussion before it, \( r \) must be odd.

With appropriate re-labeling of the integers, and evaluating the functional integral (103), this is identical to (60), the expression for the free energy of the DLCQ type II superstring. Note that, in comparing this with (60), rather than \( 1/k^2 \) appearing in the summand in (123), there is \( 1/N = 1/kr \) where we label \( k = N/r \). This difference of a factor of \( k/r \) is absorbed in a factor of \( 1/\tau_2 \) in (60).

4 Speculations

We must caution the reader that one must be careful in interpreting thermodynamic quantities in a theory such as string theory which intrinsically contains gravity. Because of gravitational effects, one would not expect an exact, equilibrium thermal ensemble to exist. However, to the extent that, at the basic level, both superstring theory and M-theory have some degree of space-time symmetry such as time translation invariance, it is sensible to consider the energy of
quantum states and the thermodynamic partition function which we consider simply as a spectral function which encodes the energies and degeneracies of stationary quantum states.

One interesting feature of string theory is the existence of a limiting temperature [42], which could be interpreted as evidence of a phase transition [29, 43, 44]. M-theory should also contain this feature. Relevant to understanding it in M-theory is to consider the Hagedorn transition in the discrete light cone quantisation of the superstring, that is, in the strongly coupled matrix model. The relevant partition function is (60) which re-display below:

\[
\frac{F}{V} = - \sum_{n=\infty}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}} \left( \frac{1}{4\pi^{2}\alpha'\tau_{2}} \right)^{\frac{5}{2}} 2^{8} |\theta_{4}(0, \tau)|^{-16} e^{-n^{2}\beta^{2}/4\pi\alpha'\tau_{2}}
\]

High temperature occurs for small values of \(\beta\). When \(\beta\) is small compared to \(\alpha'\), the region of summation with small values of \(\tau_{2}\) are important in the summand which behaves as

\[
\frac{1}{nk} \exp(-n^{2}\beta^{2}/4\pi\alpha'\tau_{2}) \exp(2\pi/\tau_{2})
\]

\[\propto \frac{1}{nk} \exp(-nk\beta\sqrt{2}/2R^{+}) \exp(4\pi^{2}\alpha'\sqrt{2}k/n\beta R^{+})\]

The summation over \(k\) diverges at \(T_H = 1/k_B\sqrt{8\pi^{2}\alpha'}\), the Hagedorn temperature, independent of the compactification radius \(R^{+}\). Since \(k\) (see fig.1) is the spatial length of the string, this means that near the Hagedorn temperature, the partition sum is dominated by very long strings.

As was mentioned in the introductory section, the approach that we have followed in this Paper could be generalized to an expansion of the partition function in powers of the string coupling constant, \(g_s\). What would have to be shown is that the order \(g_s^{2}\) correction to the free energy is given by a summation over worldsheets with genus \(g\). This singular perturbation theory is similar to that encountered in the localization formulae for path integrals describing integrable models [45]. Development of this perturbation theory could well shed light on some aspects of the correspondence between gauge theory and M-theory supergravity contained in the Maldacena conjecture (for recent reviews see [46, 47]).

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