Late acceleration and \( w = -1 \) crossing in induced gravity

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We study the cosmological evolution on a brane with induced gravity within a bulk with arbitrary matter content. We consider a Friedmann-Robertson-Walker brane, invariantly characterized by a six-dimensional group of isometries. We derive the effective Friedmann and Raychaudhuri equations. We show that the Hubble expansion rate on the brane depends on the covariantly defined integrated mass in the bulk, which determines the energy density of the generalized dark radiation. The Friedmann equation has two branches, distinguished by the two possible values of the parameter \( \epsilon = \pm 1 \). The branch with \( \epsilon = 1 \) is characterized by an effective cosmological constant and accelerated expansion for low energy densities. Another remarkable feature is that the contribution from the generalized dark radiation appears with a negative sign. As a result, the presence of the bulk corresponds to an effective negative energy density on the brane, without violation of the weak energy condition. The transition from a period of domination of the matter energy density by non-relativistic brane matter to domination by the generalized dark radiation corresponds to a crossing of the phantom divide \( w = -1 \).

I. INTRODUCTION

A brane theory with an induced gravity term in the action, characterized by a length scale \( r_\ast \), has several novel features. In the simplest example, the DGP model \([1]\), the brane tension \( V \) and bulk cosmological constant \( \Lambda \) are neglected. Specific examples with an induced gravity term can be obtained in string theory, and are common in holographic descriptions \([2, 3, 4, 5]\). Despite possible problems at large distances \([6]\), the DGP model predicts an interesting cosmological evolution \([7]\). The Friedmann equation has two branches, distinguished by the two possible values of the parameter \( \epsilon = \pm 1 \). The branch with \( \epsilon = 1 \) is characterized by an effective cosmological constant and accelerated expansion for low energy density of the brane matter \([8]\). The generalization of the model for non-zero \( V \) and \( \Lambda \) displays the same two branches \([2]\). For \( \epsilon = -1 \), the limit \( r_\ast \rightarrow 0 \) reproduces the cosmological evolution of the Randall-Sundrum model \([9, 10]\), in which the effective cosmological constant is zero. The branch with \( \epsilon = 1 \) displays late-time accelerated expansion, in complete analogy to the DGP model. Various phenomenological \([11, 12]\) and observational \([13]\) consequences of the induced gravity terms have been considered.

A common feature of all brane cosmologies is the presence in the Friedmann equation of a contribution characterized as dark or Weyl or mirage radiation \([10, 14, 15, 16]\). In the absence of induced gravity on a Friedmann-Robertson-Walker (FRW) brane and for an AdS-Schwarzschild bulk, the size of this contribution is related to the mass of the black hole \([17]\). In the general case of an arbitrary bulk content, it is the total integrated mass in the bulk that determines the dark radiation \([18]\). The term in the Friedmann equation for the brane cosmological evolution does not, in general, scale \( \sim \ell^{-4} \), where \( \ell \) is the scale factor. For this reason, this term has been characterized as generalized dark radiation \([18, 19]\). Several examples of non-standard scaling are known \([20, 21, 22]\). Also, the energy exchange between the brane and the bulk may result in a complicated cosmological expansion \([23, 24, 25]\).

Of particular interest is the possibility of late time cosmological acceleration, with the parameter \( q \) varying continuously from values below 1 to values above it. This implies that the effective parameter \( w \) of the equation of state crosses the line \( w = -1 \) during the cosmological evolution. There are indications that the observational data may favor such a scenario \([26]\), and various models have been proposed in order to realize it \([27, 28, 29, 30, 31]\).

The purpose of the present work is to suggest an alternative simple mechanism within the context of the brane induced gravity. Assuming a FRW brane with induced gravity in the presence of a general bulk matter configuration, we determine the modified Friedmann and Raychaudhuri equations. This is achieved by implementing the Israel junction conditions \([32]\) within the covariant formalism \([33, 34, 35]\). As we shall show, the generalized dark radiation plays an important role in the cosmological evolution. In particular, in the branch with \( \epsilon = 1 \) the evolution at late times is dominated by an effective cosmological constant. Additional corrections arise from the brane matter and generalized dark radiation. The effective equation of state of the cosmological fluid takes a form such that the parameter \( w \) crosses the phantom divide \( w = -1 \) without violation of the weak energy condition.

In the next two sections we employ the covariant for-
malism in order to derive the Friedmann and Raychaudhuri equations for a brane with induced gravity in the presence of matter in the bulk. The role of the generalized dark radiation in the expansion is discussed in section IV. Our conclusions are given in section V. Throughout this work the following index conventions are used: bulk 5-dimensional indices are denoted by capital Latin letters $A, B, ... = 0, 1, 2, ..., 4$, and Greek letters denote spacetime indices $\alpha, \beta, ... = 0, 1, 2, 3$.

II. THE EFFECT OF BULK MATTER ON A BRANE WITH INDUCED GRAVITY

The presence of an induced 4-dimensional curvature term on the brane, as well as matter in the bulk, leads to an effective action \[ S = \int d^4x \sqrt{-g} (\Lambda + M^3 R + \mathcal{L}_{\text{mat}}) + \int d^4x \sqrt{-g_4} (-V + \mathcal{L}_{\text{brane}} + \tau_c M^3 R_4), \] where $R$ is the curvature scalar of the 5-dimensional metric $g_{AB}$, $-\Lambda$ the bulk cosmological constant ($\Lambda > 0$), $V$ the brane tension, $g_{\alpha\beta}$ the induced 4-dimensional metric on the brane, $g_4$ its determinant, and $R_4$ the corresponding curvature scalar. The Einstein Equations (EE) take the form

\[ G^A_B = \frac{1}{2M^3} (T^A_B + \Lambda \delta^A_B), \]

where the energy-momentum (EM) tensor $T^A_B$ is

\[ T_{AB} = T_{AB}^{\text{bulk}} + \delta (s) \tau_{AB}. \]

The term $T_{AB}^{\text{bulk}}$ is the bulk matter contribution, while $\tau_{AB}$ is the contribution from the brane located at $s(x^A) = 0$.

In order to simplify the problem we assume a $Z_2$ symmetry around the brane. The modified 4-dimensional EE can be derived by employing Israel’s junction conditions and the Gauss equation for the extrinsic curvature of the surfaces $s(x^A) = 0$ (normal to the unit spacelike vector $n^A$) \[ \mathcal{S}_{\alpha\beta} = \frac{1}{4M^3} \mathcal{S}_{\alpha\beta} - \mathcal{E}_{\alpha\beta} + \frac{1}{3M^3} \mathcal{F}_{\alpha\beta} + \frac{\Lambda}{4M^3} g_{\alpha\beta}. \] The local quadratic corrections to the brane dynamics are represented by the tensor

\[ S_{\alpha\beta} = \frac{1}{12} \tau_{\alpha\beta} - \frac{1}{4} \tau_{\alpha\gamma} \tau_{\gamma\beta} + \frac{3 \tau_{\alpha\delta} \tau_{\gamma\beta} - \tau^2}{24} g_{\alpha\beta}, \]

where $\tau_{\alpha\beta}$ is defined according to

\[ \tau_{\alpha\beta} = T_{\alpha\beta}^{\text{brane}} - V g_{\alpha\beta} - 2 \tau_c M^3 G_{\alpha\beta}, \]

with $T_{\alpha\beta}^{\text{brane}}$ the brane matter contribution. The presence of the induced 4-dimensional curvature term results in a contribution to the tensor $\tau_{\alpha\beta}$ proportional to the Einstein tensor on the brane.

The 5-dimensional effects are encoded in the projected tensors

\[ F_{\alpha\beta} = T_{AB}^{\text{bulk}} g_A^B + \left( T_{AB}^{\text{bulk}} n_A n_B - \frac{T_{\text{bulk}}}{4} \right) g_{\alpha\beta}, \]

\[ E_{\alpha\beta} = E_{AB} g_A^B = C_{ACBD} n^C n^D g_{\alpha\beta}. \]

The tensors $F_{\alpha\beta}$ and $E_{\alpha\beta}$ are associated with the contributions from the matter and the free gravitational field of the 5-dimensional bulk. We note that, in the case of an empty bulk, the 5-dimensional contributions on the brane arise only through the non-local effects of the free gravitational field incorporated in $E_{\alpha\beta}$ (5D bulk gravitons).

For latter use, it is convenient to perform an 1+3 decomposition of the above tensors into irreducible parts with respect to the brane fluid velocity $\tilde{u}_\alpha = q^A_\alpha \tilde{u}_A$. The decomposition of the bulk EM tensor reads \[ T_{AB}^{\text{bulk}} = \tilde{\rho} \tilde{u}_A \tilde{u}_B + \tilde{\pi}_{AB} + 2 \tilde{q}_A \tilde{u}_B + \tilde{\pi}_{AB}, \]

where the energy density $\tilde{\rho}$, isotropic pressure $\tilde{\pi}$, energy flux vector $\tilde{q}_A$, and anisotropic pressure tensor $\tilde{\pi}_{AB}$ are the corresponding bulk dynamical quantities as measured by the brane observers $\tilde{u}_A$. Here $\tilde{h}_{AB} = g_{AB} + \tilde{u}_A \tilde{u}_B$ is the projection tensor normally to the prolonged brane velocity $\tilde{u}_A$. A straightforward calculation gives

\[ F_{\alpha\beta} = \frac{3 \tilde{\rho} + 4 (\tilde{p} - \tilde{p}_\parallel)}{4} \tilde{u}_\alpha \tilde{u}_\beta + \frac{3 \tilde{\rho} + 4 (\tilde{p} - 2 \tilde{p}_\parallel)}{12} \tilde{h}_{\alpha\beta} + 2 \tilde{q}_\alpha \tilde{u}_\beta + \tilde{\pi}_{\alpha\beta}, \]

where $\tilde{p}_\parallel = T_{AB}^{\text{bulk}} n_A n_B$ is the bulk pressure in the direction perpendicular to the brane as measured by a brane observer $\tilde{u}_A$. The traceless tensor $\tilde{\pi}_{\alpha\beta}$ is the contribution from the bulk anisotropic pressure to the effective 4-dimensional gravitational field.

The Weyl “electric” part tensor $E_{\alpha\beta}$ is decomposed as \[ E_{\alpha\beta} = \mathcal{E} \left( \tilde{u}_\alpha \tilde{u}_\beta + \frac{1}{3} \tilde{h}_{\alpha\beta} \right) + 2 \mathcal{Q} \tilde{u}_\alpha \tilde{u}_\beta + \mathcal{P}_{\alpha\beta}, \]

where $\mathcal{E}$, $\mathcal{Q}$, and $\mathcal{P}_{\alpha\beta}$ correspond to dynamical quantities of the 5-dimensional bulk spacetime.

With these identifications the divergence of the EM tensor of the brane matter, derived also from the Codacci equation for the extrinsic curvature, implies \[ \tau_{\alpha\beta} : \beta = T_{\alpha\beta}^{\text{brane} : \beta} = - \left( \tilde{q}_C n^C \right) \tilde{u}_\alpha + \tilde{\pi}_{AB} n_B g_{\alpha A}. \] This shows that, in general, the brane matter is not conserved. There can be energy exchange (outflow or inflow)
between the brane and the bulk, depending on the character of the vector field \((\tilde{q}_C n^C)\) \(\tilde{u}_\alpha + \tilde{\pi}_{AB} u^B \tilde{g}_\alpha^A\), that includes the energy flux vector \(\tilde{q}_A\) and the bulk anisotropic stress vector \(\tilde{\pi}_{AB} u^B\).

It is important to point out that the exact form or, at least, the set of equations that governs the evolution of the above bulk dynamical quantities as well as the rate of the energy exchange, cannot be deduced from on brane considerations and one has to incorporate the full EE \((\ref{EE})\).

### III. Generalized Friedmann and Raychaudhuri Equations on a FRW Brane

In the presence of bulk matter, the basic tool for the covariant description of the brane dynamics is the irreducible decomposition \((\ref{reddec})\) and \((\ref{reddec2})\). These equations represent the most general form of the tensors \(\mathcal{F}_{\alpha\beta}\) and \(\mathcal{E}_{\alpha\beta}\) and hold for any geometric brane background. We shall restrict our considerations to a FRW brane, for which the 3-dimensional hypersurfaces \(D\) normal to the prolonged cosmological observers \(\tilde{u}_A\) are invariant under a six-dimensional group of isometries. It follows that the surfaces \(D\) have constant curvature, parametrized by the constant \(k_c = 0, \pm 1\).

The assumption of maximal symmetry of the spatial hypersurfaces \(D\) implies that \((\ref{Reddec})\)

\[
\mathcal{E}_{\alpha\beta} = \mathcal{E} \left(\tilde{u}_\alpha \tilde{u}_\beta + \frac{1}{3} \tilde{h}_{\alpha\beta}\right). \tag{13}
\]

In addition we have \(\tilde{q}_\alpha = 0 = \tilde{\pi}_{\alpha\beta}\) and \(4\tilde{\rho} = \tilde{\rho}_\parallel + 3\rho_\perp\), where \(\rho_\perp\) is the isotropic pressure contribution of the bulk fluid. Consequently, equation \((\ref{Reddec2})\) becomes

\[
\mathcal{F}_{\alpha\beta} = \tilde{\rho} \tilde{u}_\alpha \tilde{u}_\beta + \rho_\parallel + \tilde{\rho} \tilde{h}_{\alpha\beta}. \tag{14}
\]

We observe that both the bulk corrections \((\ref{Reddec})\) and \((\ref{Reddec2})\) have a perfect fluid form. In particular the “dark fluid” component \(\mathcal{E}_{\alpha\beta}\) has a radiation equation of state that justifies, in the case of an AdS-Schwarzschild bulk, the use of the term dark or Weyl radiation for its contribution to the effective Friedmann equation.

We should emphasized that the isotropy of the brane implies the existence of a preferred spacelike direction \(e^A\) representing the (local) axis of symmetry with respect to which all the geometrical, kinematical and dynamical quantities are invariant. Although the preferred spatial direction can be chosen in two different ways, it is natural to select \(e^A\) to be the normal to the timelike congruence generated by the bulk observers \(u^A\), i.e. \(u^A e_A = 0\). In this case, the radius \(\ell\) represents the average length scale for distances between any pair of brane observers and essentially corresponds to the scale factor of the FRW brane. It is defined according to \((\ref{Reddec3})\)

\[
3 (\ln \ell)_A e^A = (g^{AB} + u^A u^B) e_{A;B}. \tag{15}
\]

As we have noted in section II, the dark radiation component \(\mathcal{E}\) cannot be determined from on brane considerations because it depends on the bulk degrees of freedom. It follows that we must solve the 5-dimensional \(\text{EE} (\ref{EE})\) in order to find the exact form of \(\mathcal{E}\). This has been done in \((\ref{Reddec4})\) for an arbitrary bulk matter configuration. In particular we can express the dark radiation \(\mathcal{E}\) in terms of the integrated mass of the bulk fluid within radius \(\ell\):

\[
\mathcal{E} = -\frac{1}{2M^3} \left[ \frac{1}{2} \left( \mathcal{C}_{\text{BULK}} - 4\rho_\perp \right) + \frac{\mathcal{M}}{\pi^2 \ell^4} \right], \tag{16}
\]

where

\[
\mathcal{M} = \int_{\ell_0}^{\ell} 2\pi^2 \rho^3 d\ell + \mathcal{M}_0 \tag{17}
\]

is the generalized comoving mass of the bulk fluid within a spherical shell with radii \(\ell_0\), \(\ell\), and \(\rho = \mathcal{C}_{\text{AB}} u^A u^B\) is the energy density of the bulk fluid as measured by the bulk observers \(u^A\). This interpretation is strictly correct only for \(k_c = 1\). However, we shall refer to \(\mathcal{M}\) as the integrated mass for all geometries of the hypersurfaces \(D\). The integration constant \(\mathcal{M}_0\) in equation \((\ref{Reddec4})\) can be interpreted as the mass of a black hole at \(\ell_0 = 0\).

Assuming a perfect fluid matter configuration on the brane, the energy-momentum tensor can be written in terms of the energy density \(\tilde{\rho}\) and the isotropic pressure \(\tilde{p}\) as

\[
\mathcal{T}^{\text{BRANE}} = \tilde{\rho} \tilde{u}_\alpha \tilde{u}_\beta + \tilde{p} \tilde{h}_{\alpha\beta}. \tag{18}
\]

The divergence of the brane EM tensor \((\ref{Reddec5})\) can be split along and normally to \(\tilde{u}^\alpha\). In this way we obtain the equation that describes the transfer of energy between the bulk and the brane

\[
\tilde{\rho} + 3H (\tilde{\rho} + \tilde{p}) = 2\tilde{q}_C n^C. \tag{19}
\]

The brane evolution can be studied by determining the generalized Friedmann and Raychaudhuri equations on the brane in the presence of bulk matter. These follow from the Gauss-Codazzi equations and the timelike part of the trace of the Ricci identities applied to the (irrotational, geodesic and shear-free) timelike congruence \(\tilde{u}_\alpha\). They have the form

\[
H^2 = \frac{1}{3} R_{\alpha\beta} \tilde{u}^\alpha \tilde{u}^\beta - \frac{R_3}{6} + \frac{R_4}{6}, \tag{20}
\]

\[
\dot{H} = -H^2 - \frac{1}{3} R_{\alpha\beta} \tilde{u}^\alpha \tilde{u}^\beta, \tag{21}
\]

where \(R_{\alpha\beta} = g_{\alpha\beta} - \frac{2}{\ell^2} g_{\alpha\beta}\) is the modified Ricci tensor of the brane, \(3\dot{H} = \tilde{u}^\alpha\alpha\) the Hubble parameter, \(R_3\) the scalar curvature of the 3-dimensional hypersurfaces \(D\), and a dot denotes differentiation with respect to \(\tilde{t}\), i.e. \(\dot{H} = H_{\alpha} \tilde{u}^\alpha\).
Using equations (14, 6) and (13, 18) we obtain

$$H^2 \equiv \left( \frac{\dot{r}}{r} \right)^2 = \frac{G_{\alpha\beta}\vec{u}^\alpha\vec{u}^\beta}{3} - \frac{R_3}{6} = -\frac{R_3}{6}$$

$$+ \left( \frac{\dot{\rho} + V - 2r_cM^3G_{\alpha\beta}\vec{u}^\alpha\vec{u}^\beta}{144M^6} \right)^2$$

$$+ \frac{\mathcal{M}}{6\pi^2M^3\ell^4} - \frac{\Lambda}{12M^3},$$

(22)

$$\dot{H} = -H^2 - \frac{\Lambda}{12M^3} - \frac{2\left( \dot{\rho} + V - 2r_cM^3G_{\alpha\beta}\vec{u}^\alpha\vec{u}^\beta \right)}{144M^6} -$$

$$- \frac{3\left( \dot{\rho} - V - 2r_cM^3G_{\alpha\beta}\vec{u}^\alpha\vec{u}^\beta \right)}{144M^6} \left( \ddot{\rho} + 2V - 2r_cM^3G_{\alpha\beta}\vec{u}^\alpha\vec{u}^\beta \right)$$

(23)

where $x^\alpha$ is a unit spacelike vector field that lies on the spatial hypersurfaces $\mathcal{D}$ and is normal to the brane fluid velocity vector ($x^\alpha x_\alpha = 1$, $x^\alpha \dot{x}_\alpha = 0$).

Equation (22) is quadratic with respect to the timelike eigenvalue of the induced Einstein tensor on the brane, and can be solved to obtain

$$G_{\alpha\beta}\vec{u}^\alpha\vec{u}^\beta = \frac{V + \dot{\rho}}{2M^3r_c} + \frac{6}{r_c^2}$$

$$+ \frac{\epsilon\sqrt{3}}{2} \left[ \frac{12}{r_c^4}\frac{(V + \dot{\rho})}{M^3r_c^3} - \frac{2\mathcal{M}}{\pi^2M^3\ell^4r_c^2} + \frac{\Lambda}{64r_c^2} \right]^{1/2},$$

(24)

with $\epsilon = \pm 1$. The generalized Friedmann equation becomes

$$\frac{r_c^2}{2} \left( H^2 + \frac{R_3}{6} \right) = 1 + \frac{r_c(V + \dot{\rho})}{12M^3} +$$

$$+ \epsilon \left[ 1 + \frac{r_c(V + \dot{\rho})}{6M^3} + \frac{r_c^2\Lambda}{12M^3} - \frac{r_c^2\mathcal{M}}{6\pi^2M^3\ell^4} \right]^{1/2}.$$ (25)

From equations (4, 6) and (13, 17) we find

$$G_{\alpha\beta}x^\alpha x^\beta = \frac{\tau_{\alpha\beta}\vec{u}^\alpha\vec{u}^\beta}{48M^6} +$$

$$\frac{\Lambda}{4M^3} + \frac{\mathcal{M}}{6\pi^2M^3\ell^4} + \frac{\bar{p}_\parallel}{3M^3}.$$ (26)

This implies

$$G_{\alpha\beta}x^\alpha x^\beta = \frac{1}{2r_c^2M^3G_{\alpha\beta}\vec{u}^\alpha\vec{u}^\beta + 12M^3 - r_c(\dot{\rho} + V)} \times$$

$$\left[ \frac{2\mathcal{M}}{\pi^2\ell^4} + \frac{\bar{p}_\parallel^2 - 2\bar{p}\dot{\bar{p}}}{4M^3} + 4\bar{p}_\parallel + 3\Lambda \right]$$

$$+ (G_{\alpha\beta}\vec{u}^\alpha\vec{u}^\beta)^2 M^3r_c^2 - (G_{\alpha\beta}\vec{u}^\alpha\vec{u}^\beta)(2V - \bar{p} + \dot{\rho})r_c$$

$$+ \frac{3V^2 - 2V(\ddot{\rho} - 2\dot{\rho})}{4M^3}. $$

The r.h.s. of eq. (19) accounts for the possible energy exchange between the bulk and the brane. Because of the $Z_2$ symmetry that we have assumed around the brane, the energy flows in or out of both sides of the brane with equal rates. Several explicit cases of such behavior have been studied in the past. In ref. 22, an example of a non-static bulk populated by non-relativistic matter is given. The bulk matter is pressureless, but has an initial outgoing velocity in the radial direction. The bulk metric is assumed to have the AdS-Tolman-Bondi form. Bulk matter can flow into the brane and modify the cosmological expansion. An interesting possibility is to identify this type of matter with dark matter, whose density would then scale differently from $\ell^{-3}$. Another interesting case is discussed in refs. 21. The bulk metric is assumed to have the AdS-Vaidya form, and the bulk energy-momentum tensor corresponds to a radiation field. The resulting cosmological solution describes a brane Universe that exchanges (emits or absorbs) relativistic matter with the bulk. In a particular application, the energy loss from the brane has been matched to the rate of production of Kaluza-Klein gravitons during the collisions in a thermal bath of brane particles 13, 21.

In all the above cases the distribution of matter in the bulk and the brane is assumed to be consistent with the six-dimensional group of isometries that permits the embedding of a Friedmann-Robertson-Walker brane. As a result, the possible presence of inhomogeneities in the distribution of brane matter is neglected. We also note that the full description of the brane dynamics (e.g. the form of the effective equation of state of the “mirage” component or the rate in which bulk matter is transformed into “mirage” matter) requires explicit input about the form of the bulk matter and its interaction with the brane matter (i.e. the use of the full 5D field equations). Nevertheless, we shall see in the next section that the cosmological evolution can be efficiently described through the generalized dark radiation term (generalized comoving mass) without explicit reference to the bulk.

IV. ACCELERATED EXPANSION AND $w = -1$ CROSSING

The Friedmann equation (20) is a generalization of the well known equation for an AdS-Schwarzschild bulk 7. In the standard case there is a black hole located at $\ell_0 = 0$, so that the integrated mass is constant: $\mathcal{M} = \mathcal{M}_0$. For this value our result (20) contains the known contribution characterized as dark radiation. For a general bulk content, the dark radiation term is replaced by the contribution $\mathcal{M}_0/\left(6\pi^2M^3\ell^4\right)$. A non-trivial bulk matter configuration leads to an integrated mass $\mathcal{M}$ that is a function of the scale factor $\ell$. As a result, the generalized dark radiation term does not scale $\sim \ell^{-4}$. For this reason, this term has been characterized as generalized dark radiation 13, 19. In references 20, 21, 22 several examples were given, with the radiation term scal-
ing $\sim \ell^{-n}$ with $n = 0, 2, 3$, or having more complicated behavior.

The two values of $\epsilon$ in equation (20) correspond to two disconnected branches of solutions. The nature of the predicted expansion is clearer in the limit of low energy density. Let us consider first the Randall-Sundrum case [3], in which the bulk cosmological constant $-\Lambda$ and the brane tension $V$ are related through $\Lambda = V^2/(12M^3)$. We define the energy scale $k = V/(12M^3) = [\Lambda/(12M^3)]^{1/2}$. The Friedmann equation can be written as

$$
\frac{r_c^2}{2} \left( H^2 + \frac{k_c}{\ell^2} \right) = 1 + k r_c + k r_c \tilde{\rho},
$$

$$
+ \epsilon \left[ (1 + kr_c)^2 + 2kr_c \tilde{\rho} - 2(kr_c)^2 \tilde{\rho}_d \right]^{1/2} \left( 1 + kr_c \right)/6(M^3/k),
$$

where $\tilde{\rho}_d = M(\ell)/\left(k \pi^2 \ell^4\right)$ is the effective energy density of the generalized dark radiation and the constant $k_c = 0, \pm 1$ parametrizes the spatial curvature of the brane.

For $\tilde{\rho}, \tilde{\rho}_d \ll V$, keeping only the terms linear in $\tilde{\rho}, \tilde{\rho}_d$, we find

$$
H^2 = \frac{2(1 + \epsilon)(1 + kr_c)}{r_c^2} + 1 + \epsilon + kr_c \frac{\tilde{\rho}}{kr_c(1 + kr_c)} 6(M^3/k)
$$

$$
-\frac{\epsilon}{1 + kr_c} \frac{\tilde{\rho}_d}{6(M^3/k)} - \frac{k_c}{\ell^2}.
$$

For $\epsilon = -1$ we have

$$
H^2 = \frac{1}{6M^2_\Pi} (\tilde{\rho} + \tilde{\rho}_d) - \frac{k_c}{\ell^2},
$$

with $M^2_\Pi = M^3(r_c + 1/k)$. The expansion is conventional, apart from the presence of the energy density of the generalized dark radiation.

For $\epsilon = 1$ and $kr_c \ll 1$ we obtain

$$
H^2 = \frac{4}{r_c^2} + \frac{1}{6M^2_\Pi} \left( \tilde{\rho} - \frac{kr_c}{2} \tilde{\rho}_d \right) - \frac{k_c}{\ell^2},
$$

with $M^2_\Pi = M^3r_c/2$. For $\epsilon = 1$ and $kr_c \gg 1$ we have

$$
H^2 = \frac{4k}{r_c} + \frac{1}{6M^2_\Pi} (\tilde{\rho} - \tilde{\rho}_d) - \frac{k_c}{\ell^2},
$$

with $M^2_\Pi = M^3r_c$. In both cases an effective cosmological constant appears, despite the fine tuning of the bulk cosmological constant and the brane tension. The other striking feature is the negative sign of the contribution proportional to the energy density of the dark radiation.

The above features also appear in the DGP model [1], characterized by $\Lambda = V = 0$. The Friedmann equation now reads

$$
\frac{r_c^2}{2} \left( H^2 + \frac{k_c}{\ell^2} \right) = 1 + r_c \tilde{\rho}/12M^3
$$

$$
+ \epsilon \left[ 1 + \frac{r_c}{6M^3} (\tilde{\rho} - \tilde{\rho}_d) \right]^{1/2},
$$

with $\tilde{\rho}_d = r_c M(\ell)/(\pi^2 \ell^4)$. For $\epsilon = 1$ and $\tilde{\rho} \ll M^3/r_c$ we have

$$
H^2 = \frac{4}{r_c^2} + \frac{1}{6M^2_\Pi} \left( \tilde{\rho} - \frac{1}{2} \tilde{\rho}_d \right) - \frac{k_c}{\ell^2},
$$

with $M^2_\Pi = M^3r_c/2$.

It is obvious from the above that the brane cosmological expansion in the branch with $\epsilon = 1$ has novel properties arising from: a) an effective cosmological constant, and b) an effective negative energy density associated with the generalized dark radiation. The second feature is not a consequence of a violation of the weak energy condition, as the energy density is assumed positive both in the bulk and on the brane.

In the case of an AdS-Schwarzschild bulk we have $\tilde{\rho}_d \sim \ell^{-4}$. At late times the contribution from the dark radiation is subleading to the contribution from thebrane matter $\tilde{\rho} \sim \ell^{-3}$. On the other hand, if there is a nontrivial matter configuration in the bulk so that $\tilde{\rho}_d \sim \ell^{-n}$ with $n < 3$, the cosmological constant and the effective negative energy density can be the leading effects.

An example with $\tilde{\rho}_d \sim \ell^{-2}$ is given in reference [22]. The bulk is assumed to contain a scalar field in a global monopole (hedgehog) configuration. The field also interacts with the brane through a localized quadratic potential, so that the brane can be embedded in the bulk spacetime. There is no significant energy exchange between the brane and the bulk at low energy densities. For large $\ell$, the dominant contribution to the integrated mass arises from the field kinetic term, so that $\mathcal{M} \sim \ell^2$ and $\tilde{\rho}_d \sim \ell^{-2}$. The contribution from the brane potential is $\sim \ell^{-4}$ and, therefore, negligible for large $\ell$.

The evolution in the presence of a cosmological constant $\lambda_{\text{eff}}$ and a matter contribution $\tilde{\rho} = \tilde{\rho}_0(\ell/\ell_0)^{-n}$ is determined by

$$
\frac{1}{\ell} \frac{d\ell}{dt} = \left( 1 + \delta c^2/\ell^2 \right)^{1/2},
$$

where $\ell = \ell/\ell_0$, $\ell = (\lambda_{\text{eff}}/6M^2)^{1/2}$, $c^2 = \tilde{\rho}_0/\lambda_{\text{eff}}$ and $\delta = \pm 1$. For $\delta = 1$ the solution is

$$
\tilde{\ell}^{n/2} = c \sinh \left( \frac{n}{2} \ell + d^c_a \right),
$$

with $d^c_a = \sinh^{-1}(1/c)$, while for $\delta = -1$ it is

$$
\tilde{\ell}^{n/2} = c \cosh \left( \frac{n}{2} \ell + d^c_c \right),
$$

with $d^c_c = \cosh^{-1}(1/c)$. The acceleration parameter $q = \dot{\tilde{\ell}}^2/(\tilde{\ell} \ell)^2$ for $\delta = 1$ is

$$
q = 1 - \frac{n}{2} \left[ \cosh \left( \frac{n}{2} \ell + d^c_a \right) \right]^2,
$$

while for $\delta = -1$ it is

$$
q = 1 + \frac{n}{2} \left[ \sinh \left( \frac{n}{2} \ell + d^c_c \right) \right]^2.
$$
The Friedmann equation for the branch with $\epsilon = 1$ can be written in the form \[ \ddot{a} + \frac{3}{a} \dot{a}^2 = -\frac{8\pi G}{3} \rho - \frac{4\pi G}{3} p \] when one of the energy densities $\tilde{\rho}_\text{m}, \tilde{\rho}_\text{dm}$ is negligible. If the dominant matter contribution comes from the non-relativistic matter on the brane, we have $\delta = 1$, $n = 3$ and $q < 1$. If the dominant matter contribution comes from the generalized dark radiation with $n < 3$, we have $\delta = -1$ and $q > 1$.

For a cosmological fluid with an equation of state $p_{\text{eff}} = w \rho_{\text{eff}}$ and an effective state parameter $w$, the acceleration parameter is $q = -(1 + 3w)/2$. As a result, $q > 1$ implies $w < -1$, while $q < 1$ implies $w > -1$. It is apparent from our discussion above that the branch of equation \[ \ell = 4 \] with $\epsilon = 1$ can lead to an evolution during which the line $w = -1$ is crossed. This happens if the cosmological constant is the leading contribution, and the brane dark matter dominates over the generalized dark radiation at early times, while the reverse takes place at later times.

V. SUMMARY AND CONCLUSIONS

The main results of this work are the Friedmann equation \[ \ell = 2 \] or \[ \ell = 3 \], and the Raychaudhuri equation \[ \ell = 5 \]. The role of the bulk matter in the Friedmann equation takes a very simple form. The total integrated mass in the bulk up to the location of the brane, divided by $\ell^4$, determines the energy density of the generalized dark radiation up to a numerical factor. The quantity $\ell$ is the scale factor on the brane in the Gauss normal frame, which can be interpreted as the location of the brane in the bulk frame \[ \ell = 1 \]. Equation \[ \ell = 2 \] is a generalization of the Friedmann equation for an AdS-Schwarzschild bulk \[ \ell = 2 \], and is consistent with the respective equation in the absence of the induced gravity term \[ \ell = 2 \]. The Raychaudhuri equation is significantly more complicated than the respective one in the absence of induced gravity. However, it demonstrates that the influence of the bulk is encoded in the bulk integrated mass and the pressure perpendicularly to the brane as measured by a brane observer.

The generalized dark radiation can play an important role in the cosmological evolution on the brane if it scales more slowly than $\sim \ell^{-3}$ and there is no significant energy exchange between the brane and bulk. In such a case, at late times the dark radiation would become more important than the brane matter, that is assumed to be non-relativistic and scale $\sim \ell^{-3}$. In section IV an example was discussed in which the generalized dark radiation scales $\sim \ell^{-2}$ \[ \ell = 2 \] and the energy exchange between the brane and the bulk is negligible.

The cosmological evolution in the $\epsilon = 1$ branch of equation \[ \ell = 4 \] is dominated by an effective cosmological constant at late times. The brane matter and the generalized dark radiation modify the leading exponential expansion. A remarkable feature is that the contribution from the dark radiation in the Friedmann equation appears with a negative sign in the $\epsilon = 1$ branch. As a result, the presence of the bulk corresponds to an effective negative energy density on the brane. This feature is not a consequence of a violation of the weak energy condition, as the energy density is assumed positive both in the bulk and on the brane.

An effective negative energy density accelerates the expansion on the brane beyond the rate induced by the cosmological constant. Therefore, the acceleration parameter $q$ becomes larger than 1 when the generalized dark radiation dominates over the non-relativistic brane matter. This was shown explicitly in section IV. If the brane matter dominates, the acceleration parameter is smaller than 1. It is then obvious that the value $q = 1$ is obtained at the point of comparable size of the two contributions. This corresponds to the crossing of the phantom divide $w = -1$ for the effective equation of state of the cosmological fluid.

The $w = -1$ crossing is difficult to realize in dark energy models without the inclusion of higher derivative terms or multiple fields \[ \ell = 3 \]. In this respect, a brane Universe with induced gravity provides in a simple way two desirable features: a) an effective cosmological constant, and b) an effective negative energy density, that can trigger the $w = -1$ crossing when it becomes larger than the energy density of the ordinary non-relativistic matter.

Before concluding we must point out that a number of recent works has shown that the self-accelerating branch of the induced gravity models contains a ghost mode that renders it unstable \[ \ell = 3 \]. This is a serious problem for the physical relevance of cosmologies within this branch. It may still be possible to construct a viable model if the instability is not rapid \[ \ell = 3 \]. An alternative possibility is that the ghost modes are eliminated by an additional mechanism, such as the compactification of the extra dimension \[ \ell = 3 \]. For example, this may be feasible in the context of the Randall-Sundrum model, if the effective compactification radius $1/k$ is chosen appropriately within the range $1/k < r_c$. The resulting cosmological evolution is described by eq. \[ \ell = 3 \], in which an effective cosmological constant and a generalized radiation term with a negative sign appear. As remarked in \[ \ell = 3 \], an arbitrariness in the fine tuning of the cosmological constant is implicit in such a model. In the DGP model \[ \ell = 1 \] the brane tension and the bulk cosmological constant are taken equal to zero, while in our example they are chosen similarly to the Randall-Sundrum model, so as to give contributions that cancel each other \[ \ell = 1 \].

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