The Uniqueness of the Solution to Inverse Problems of Interpolation of Positive Operators in Banach Lattices

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Abstract. We prove that an interpolation pair of Banach lattices is uniquely determined by the collection of intermediate spaces with the property that these are interpolation spaces for positive operators. A correspondent result for exact interpolation is also presented.

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In [2] we have proved that an arbitrary Banach couple (= interpolation pair of Banach spaces) is determined, up to the equivalence of the norms, by the collection of all its interpolation spaces. By making use of this, we have proved in [3] that a Banach couple is determined by the collection of all its exact interpolation spaces up to the proportionality of the norms. (See also [4], where inverse interpolation problems were examined in a more general setting.) The goal of the present note is to show that similar assertions hold true when studying interpolation of positive operators in Banach lattices.

We first introduce notation and present some facts concerning interpolation of linear operators in Banach lattices. For basic definitions of the theory we refer the reader to [1].

For a Banach lattice $E$, we denote by $\| \cdot \|_E$ and by $E^+$ the norm in $E$ and the cone of positive elements, respectively; $|x|$ stands for the absolute value of $x \in E$. We denote by $\| \cdot \|_{E^*}$ the norm in the dual Banach lattice $E^*$ and use the notation $\|T\|_{E \to E}$ to denote the norm of a bounded linear operator $T$ in $E$. A linear operator $T$ in $E$ is called positive if $T$ maps $E^+$ into $E^+$. Let $L^+(E)$ stand for the cone of all such operators. Recall that a positive operator in a Banach lattice is automatically bounded.

Let $E$ and $F$ be two Banach lattices with $E \hookrightarrow F$. (Here and subsequently, the notation $E \hookrightarrow F$ for two Banach lattices $E$ and $F$ means that $E$ is an ideal in $F$ and there exists a constant $c > 0$ such that $\|x\|_E \leq c\|x\|_F$ for all $x \in E$.) If $\varphi \in F^*$ then, clearly, $\varphi|_E \in E^*$. We write $\|\varphi\|_E$ to denote $\|\varphi|_E\|_{E}$.

Two Banach lattices $X$ and $Y$ are said to form an interpolation pair $(X, Y)$ of Banach lattices if they both are ideals in a certain vector lattice $U$ which is a Hausdorff topological vector space such that the embeddings of $X$ and of $Y$ into $U$ are continuous. To each interpolation pair $(X, Y)$ of Banach lattices can

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be canonically associated two Banach lattices, the intersection $X \cap Y$ and the sum $X + Y$. As can be easily seen,

$$(X + Y)^+ = \{ x + y \mid x \in X^+, y \in Y^+ \}.$$  

We denote by $L^+((X, Y))$ the set of all linear operators $T$ in $X + Y$ with $T|_X \in L^+(X)$, $T|_Y \in L^+(Y)$. One can easily deduce from (1) that $L^+((X, Y)) \subset L^+(X + Y)$. Also, we denote by $L_1((X, Y))$ the set of rank one linear operators $T$ in $X + Y$ with the property that $T$ maps $X$ into $X$ and $Y$ into $Y$ continuously.

A Banach lattice $Z$ is called intermediate for an interpolation pair $(X, Y)$ of Banach lattices if $X \cap Y \hookrightarrow Z \hookrightarrow X + Y$. For a constant $c \geq 1$, we denote by $\text{P-Int}_c(X, Y)$ the collection of intermediate lattices $Z$ satisfying the following "interpolation property for positive operators": $T(Z) \subset Z$ and

$$||T||_{Z \to Z} \leq c \max\{||T||_{X \to X}, ||T||_{Y \to Y}\}$$  

for all $T \in L^+((X, Y))$. Set $\text{P-Int}_c(X, Y) = \bigcup_{c \geq 1} \text{P-Int}_c(X, Y)$. Similarly, we denote by $\text{R1-Int}_c(X, Y)$ the collection of intermediate lattices $Z$ satisfying the following condition: $T(Z) \subset Z$ and (2) holds true for all $T \in L_1((X, Y))$. Set $\text{R1-Int}_c(X, Y) = \bigcup_{c \geq 1} \text{R1-Int}_c(X, Y)$.

**Lemma 1.** Let $(X, Y)$ be an interpolation pair of Banach lattices and $c \geq 1$. Then $\text{P-Int}_c(X, Y) \subset \text{R1-Int}_c(X, Y)$.

**Proof.** First, consider an intermediate Banach lattice $Z$ and a continuous linear operator $T$ in $X + Y$ of rank one. Necessary, $T$ is of the form $T(\cdot) = \varphi(\cdot)x$ with $\varphi \in (X + Y)^*$, $x \in X + Y$. Define $|T|$ by $|T|(\cdot) = |\varphi|(\cdot)|x|$. We claim that $|T|$ maps $Z$ into itself if and only if $T$ does, and $||T||_{Z \to Z} = ||T||_{Z \to Z}$ when $T(Z) \subset Z$. Really, the following two cases are possible: $x \in Z$ and $x \notin Z$. In the first case we have

$$||T||_{Z \to Z} = ||\varphi||_{Z}||x||_{Z} = ||\varphi||_{Z}||x||_{Z} = ||T||_{Z \to Z}.$$  

In the second one,

$$T(Z) \subset Z \iff \varphi|_{Z} = 0 \iff |\varphi|_{Z} = 0 \iff |T|(Z) \subset Z$$  

and

$$T(Z) \subset Z \implies ||T|| = ||T|| = 0.$$  

Thus, the claim is proved.

Now, let $Z \in \text{P-Int}_c(X, Y)$. Consider an operator $T \in L_1((X, Y))$. $T$ is clearly continuous in $X + Y$ and we can define $|T|$ as above. Then $|T| \in L^+((X, Y))$. Since $Z \in \text{P-Int}_c(X, Y)$, we have $|T|_{|Z} \in L^+(Z)$. The latter implies that $T$ maps $Z$ into itself. Moreover,

$$||T||_{Z \to Z} = ||T||_{Z \to Z} \leq c \max\{||T||_{X \to X}, ||T||_{Y \to Y}\} \leq c \max\{||T||_{X \to X}, ||T||_{Y \to Y}\}. $$
Thus, $Z \in \text{R1-Int}_c(X,Y)$.

For two Banach lattices $E$ and $F$, let the notation $E \simeq F$ mean that $E$ and $F$ coincide as vector lattices and their norms are equivalent, the notation $E \cong F$ mean that the norms are proportional.

For two interpolation pairs of Banach lattices $(X,Y)$ and $(V,W)$, we write $(X,Y) \simeq (V,W)$ if $X \simeq V$ and $Y \simeq W$ or $X \simeq W$ and $Y \simeq V$. We write $(X,Y) \cong (V,W)$ if $X \cong V$ and $Y \cong W$ or $X \cong W$ and $Y \cong V$.

**Theorem 1.** Let $(X,Y)$ and $(V,W)$ be two interpolation pairs of Banach lattices with

$$\text{P-Int}(X,Y) = \text{P-Int}(V,W).$$

Then $(X,Y) \simeq (V,W)$.

**Proof.** Since $X, Y \in \text{P-Int}(X,Y)$ and $V, W \in \text{P-Int}(V,W)$, we obtain $X,Y \in \text{R1-Int}(V,W)$ and $V,W \in \text{R1-Int}(X,Y)$ by making use of Lemma 1. So it remains to prove

$$X, Y \in \text{R1-Int}(V,W) \text{ and } V, W \in \text{R1-Int}(X,Y) \Rightarrow (X,Y) \simeq (V,W).$$

By analyzing proofs in [2] one can verify that this implications was actually proved there. Note also that from [4, Proposition 1.4 and Theorem 2.8] the latter implication follows immediately.

**Theorem 2.** Let $(X,Y)$ and $(V,W)$ be two interpolation pairs of Banach lattices with

$$\text{P-Int}_1(X,Y) = \text{P-Int}_1(V,W).$$

Then $(X,Y) \cong (V,W)$.

**Proof.** We can proceed similarly to the proof of Theorem 1, but here we need to refer to [3] and to [4, Theorem 3.7].

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