A Solovay-like model for singular generalized descriptive set theory

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Abstract

Kunen’s proof of the non-existence of Reinhardt cardinals opened up the research on very large cardinals, i.e., hypotheses at the limit of inconsistency. One of these large cardinals, I0, proved to have descriptive-set-theoretical characteristics, similar to those implied by the Axiom of Determinacy: if \( \lambda \) witnesses I0, then there is a topology for \( V_{\lambda+1} \) that is completely metrizable and with weight \( \lambda \) (i.e., it is a \( \lambda \)-Polish space), and it turns out that all the subsets of \( V_{\lambda+1} \) in \( L(V_{\lambda+1}) \) have the \( \lambda \)-Perfect Set Property in such topology. In this paper, we find another generalized Polish space of singular weight \( \kappa \) of cofinality \( \omega \) such that all its subsets have the \( \kappa \)-Perfect Set Property, and in doing this, we are lowering the consistency strength of such property from I0 to \( \kappa \theta \)-supercompact, with \( \theta > \kappa \) inaccessible.

What is Kunen’s Theorem? Researchers in different areas will probably have different answers. In the field of very large cardinals, “Kunen’s Theorem” is the fact that there are no elementary embeddings from the universe to itself (if we assume the Axiom of Choice) [7]. This was a surprising result, the first really non-obvious proof that a large cardinal is inconsistent, but the consequences of such theorem were equally unexpected.

Kunen’s Theorem actually proves that there are no elementary embeddings from \( V_{\lambda+2} \) to itself, for any \( \lambda \). Looking for inconsistencies, a new breed of large cardinals was introduced, the rank-into-rank embeddings, starting from the existence of an elementary embedding from \( V_\lambda \) to itself. Woodin pushed this further, defining I0, i.e., the existence of an elementary embedding from \( L(V_{\lambda+1}) \) to itself, with critical point less than \( \lambda \). Instead of producing inconsistencies, I0 proved to be fruitful: it was initially used to prove the consistency of AD, and the more Woodin worked on it, the more similarities with \( R \) emerged, some of them with a descriptive set-theoretical “flavor”, especially mirroring the structure of \( L(R) \) under AD (more on this in [12], [3], [4]).

Taking cue from these descriptive aspects, with Luca Motto Ros we developed a generalized descriptive set theory on cardinals with countable cofinality (since if I0(\( \lambda \)) holds, then \( \text{cof}(\lambda) = \omega \)): A \( \lambda \)-Polish space is a completely metrizable topology with weight \( \leq \lambda \), and \( \lambda \)-Polish spaces share with Polish spaces many properties and characteristics.

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In this paper, we are interested in sets that have a tree structure. Cramer [2] proved that, under \( I^0(\lambda) \), all the subsets of \( V_{\lambda+1} \) in \( L(V_{\lambda+1}) \) have an analogue of the Perfect Set Property, the \( \lambda \)-Perfect Set Property. This is analogous to the fact that, if \( L(\mathbb{R}) \models AD \), then all the subsets of \( \mathbb{R} \) in \( L(\mathbb{R}) \) have the PSP.

Even more, in [5] it is described how this is a consequence of the fact that, under \( I^0(\lambda) \), all the sets have a property that is the generalization of being weakly homogeneously Souslin.

Looking at the classical case, a doubt emerge: while it is true that if \( L(\mathbb{R}) \models AD \), then all the subsets of \( \mathbb{R} \) in \( L(\mathbb{R}) \) have the PSP, this is not the best scenario, consistency-wise. Solovay proved that if there is an inaccessible card, \( V[G] \) is the generic extension via the Levy collapse of the inaccessible to \( \omega \), then in \( HOD_{\mathbb{R}^G} \) all the subsets of \( \mathbb{R} \) have the PSP, and an inaccessible cardinal is of consistency strength much lower than \( AD^{L(\mathbb{R})} \). This begs the question: can we do the same for the \( \lambda \)-PSP? Is \( I^0(\lambda) \) necessary for having an example of a \( \lambda \)-Polish space where all its subsets have the \( \lambda \)-PSP, or we can lower the consistency strength of the large cardinals involved?

This paper provides a positive answer for the second question. Starting with a \( \theta \)-supercompact cardinal \( \kappa \), with \( \theta \) inaccessible, if \( V[G] \) is a generic extension via supercompact Prikry forcing, then \( V[G] \) contains an inner model in which all the subsets of \( \omega \kappa \), the "\( \kappa \)-Baire space", have the \( \kappa \)-PSP, and we can extend this result to any \( \kappa \)-Polish space in the inner model. We are using the construction made by Kafkoulis in [6] of a generalized Solovay model: the construction is fairly similar to Solovay’s construction, but where Solovay uses the homogeneity of the collapse and its simple factorization, we need to investigate in detail the Prikry-like properties of the supercompact Prikry forcing and its quotients.

It is still an open problem the exact lower bound for the existence of a \( \lambda \)-Polish space, with \( \lambda > \omega \), with all the subsets with the \( \lambda \)-PSP, but surely it is a large cardinal.

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1 Preliminaries and notation

Forcing constructions In this subsection we collect some results about forcing constructions, with the further intent of fixing the notation. For more details, see for example the introductions of [11] or [1].

In the following, all forcing posets are separative.

If \( P \) is a forcing poset, and \( Q \subseteq P \) is a dense subset of \( P \), then \( (Q, \leq) \) is also a forcing poset and:

- If \( G \) is \( P \)-generic, then \( G \cap Q \) is \( Q \)-generic and \( V[G] = V[G \cap Q] \);
- If \( H \) is \( Q \)-generic, then \( H^P = \{ p \in P : \exists q \leq p \ q \in H \} \) is \( P \)-generic and \( V[H] = V[H^P] \);
- If \( \tau \) is a \( Q \)-name, then it is also a \( P \)-name, and if \( \tau \) is a \( P \)-name, then there is a \( Q \)-name \( \sigma \) such that \( \Vdash_P \tau = \sigma \);

1 Thanks to Gabriel Goldberg and Ralf Schindler for pointing out that it should be at least \( \omega \)-many Woodin cardinals: with the techniques in [11] it should be possible to prove, more in general, that if \( L(V_{\lambda+1}) \not\models AC \), then there is an inner model with \( \omega \)-many Woodin cardinals, and it is easy to construct, in ZFC, a set that has not the \( \lambda \)-PSP.
In such a case, if \( p, q \in P \) where \( ˇ \) is incompatible with \( \pi \) at most following connections between the two forcing notions:

- If \( G \) is \( P \)-generic, then \( H = \{ q \in Q : \exists p \in G \pi(p) \leq q \} \) is \( Q \)-generic and \( V[G] = V[H] \);
- If \( H \) is \( Q \)-generic, then \( G = \{ p \in P : \pi(p) \in H \} \) is \( P \)-generic and \( V[G] = V[H] \);
- \( \pi \) extends to \( P \)-names: if \( \tau \) is a \( P \)-name, define by induction on the rank \( \pi(\tau) = (\pi(\sigma), \pi(p)) : (\sigma, p) \in \tau \);
- For any \( p \in P, \tau P \)-name, \( p \Vdash P \varphi(\tau) \) iff \( \pi(p) \Vdash Q \varphi(\pi(\tau)) \).

Given \( P, R \) forcing posets we say that \( R \) projects into \( P \), \( P \leq R \), iff there is a \( \pi : R \to P \) such that:

- \( \pi \) is order-preserving and \( \pi(1_R) = 1_P \);
- For every \( q \in R \) and \( p' \in P \) such that \( p' \leq \pi(q) \) there is a \( q' \leq q \) in \( R \) such that \( \pi(q') = p' \).

In such a case, if \( g \) is \( P \)-generic, we can define \( R/g = \{ r \in R : \pi(r) \in g \} \). Let \( \hat{Q} \) be the \( P \)-name for \( R/g \). Then the map \( \hat{i} : R \to P * \hat{Q} \), defined as \( \hat{i}(r) = (\pi(r), \hat{r}) \), where \( \hat{r} \) is the canonical \( P \)-name for \( r \), is a dense homomorphism. We have the following connections between the two forcing notions:

- If \( H \) is a \( R \)-generic filter over \( V \), then \( \pi''H \) is a \( P \)-generic filter over \( V \), and \( H \) is a \( R/\pi''H \)-generic filter over \( V[\pi''H] \); moreover, \( V[H] = V[\pi''H][H] \);
- If \( g \) is \( P \)-generic over \( V \) and \( h \) is \( \hat{Q} \)-generic over \( V[g] \), then \( \{ r \in R : \pi(r) \in g \wedge r \in h \} \) is \( R \)-generic over \( V \) and \( V[g][h] = V[h] \);
- If \( \tau \) is a \( R \)-name, then \( \pi(\tau) \) is a \( P \)-\( \hat{Q} \)-name, and for any \( g P \)-generic over \( V \) and \( h \) \( \hat{Q} \)-generic over \( V[g] \) \( \pi(\tau)_g = \pi(\tau) \);
- If \( g \) is \( P \)-generic, \( p \in \hat{Q} \) and \( \tau \) is a \( \hat{Q} \)-name, then \( p \Vdash \hat{Q} \varphi(\tau) \) iff \( p \Vdash R \varphi(\pi) \);
- For any \( p \in R, p \Vdash R \varphi(\pi) \) iff \( \pi(p) \Vdash P (\hat{p} \Vdash \hat{Q} \varphi(\pi)) \);
- For any \( p \in P, q \in R, p \Vdash P (q \in \hat{Q}) \) iff \( p \leq \pi(q) \).

\( \kappa \)-Polish spaces. In this subsection we collect some definitions and properties about \( \kappa \)-Polish spaces, as introduced in \([3]\), where the base theory is \( ZF + AC_\kappa \). Since we will work in a model of \( ZF + DC_\kappa \), everything in this section will apply.

A topological space \( X \) is \( \kappa \)-Polish if it is completely metrizable and has weight at most \( \kappa \). If \( \text{cof}(\kappa) = \omega \), then \( ^*\kappa \), as the product of the discrete topologies on \( \kappa \), is \( \kappa \)-Polish. If \( \{ \eta_n : n \in \omega \} \) is cofinal in \( \kappa \), then also \( C(\kappa) = \prod_{n \in \omega} \eta_n \), as the product of the discrete topologies on \( \eta_n \), is \( \kappa \)-Polish. In fact, \( ^*\kappa \cong C(\kappa) \).

The space \( ^*\omega \), with the bounded topology (i.e., the topology generated by the
basic open sets $N_\kappa = \{ x \in 2 : x \upharpoonright \lh(s) = s \}$, with $s \in 2^\kappa$ is a completele metrizable space with weight $2^\kappa$ therefore if $2^\kappa = \kappa$, then $^*2$ is $\kappa$-Polish and isomorphic to $\omega^\kappa$.

If $\text{cof}(\kappa) = \omega$, then the Woodin topology on $V_{\kappa+1}$ is the topology generated by the basic open sets $N_{(\alpha,a)} = \{ A \subseteq V_\kappa : A \cap V_\kappa = a \}$, with $\alpha < \kappa$ and $a \subseteq V_\alpha$. If $|V_\kappa| = \kappa$, i.e., if $\kappa$ is a fixed point of the beth-function (for example if $\kappa$ is limit of inaccessible cardinals), then the space $V_{\kappa+1}$ with the Woodin topology is a $\kappa$-Polish space, and it is isomorphic to $\omega^\kappa$.

Let $X$ be a $\kappa$-Polish space, with $\text{cof}(\kappa) = \omega$. Then there is a closed set $F \subseteq \omega^\kappa$ and a continuous bijection $f : F \to X$.

If $X$ is a $\kappa$-Polish space, we say that $A \subseteq X$ has the $\kappa$-Perfect Set Property, or $\kappa$-PSP, if either $|A| \leq \kappa$, or $^*2$ embeds into $A$ as a closed-in-$X$ set. If $2^\kappa = \kappa$, this is equivalent to $\omega^\kappa$, or $C(\kappa)$, embedding into $A$ as a closed-in-$X$ set. In fact, this is equivalent to $^*2$, $\omega^\kappa$, or $C(\kappa)$ embedding into $A$, disregarding the closure.

## 2 Supercompact Prikry forcing, basic notions

**Definition 2.1.** A cardinal $\kappa$ is $\delta$-supercompact, with $\delta \geq \kappa$, iff there exists a normal, fine ultrafilter $U_\delta$ over $\mathcal{P}_\kappa(\delta)$. Such ultrafilter is called a $\delta$-supercompactness measure for $\kappa$.

For $P,Q \in \mathcal{P}_\kappa(\delta)$, we say that $P \subseteq Q$ (strong inclusion) iff $P \subset Q$ and $|P| < |Q \cap \kappa|$.

For $B \subseteq \mathcal{P}_\kappa(\delta)$, we denote by $[B]^{[n]}$ the set of all $\subseteq$-increasing $n$-length sequences of elements of $B$, by $[B]^{[<\omega]}$ the set of all $\subseteq$-increasing finite sequences of elements of $B$, and by $[B]^{[\omega]}$ the set of all $\subseteq$-increasing $\omega$-length sequences of elements of $B$.

If $A \subseteq \mathcal{P}_\kappa(\delta)$ and $s \in [\mathcal{P}_\kappa(\delta)]^{[n]}$, then we call $A \setminus s = \{ P \in A : s(n-1) \not\subseteq P \}$. If $A \in U_\delta$, then $A \setminus s \in U_\delta$.

If $s,t \in [\mathcal{P}_\kappa(\delta)]^{[<\omega]}$, we say $s \subseteq t$ when $s(\lh(s) - 1) \subseteq t(0)$.

**Proposition 2.2.** Let $\kappa$ be $\delta$-supercompact, and let $U_\delta$ be a normal, fine ultrafilter over $\mathcal{P}_\kappa(\delta)$. Let $\{ A_P : P \in \mathcal{P}_\kappa(\delta) \}$ be a family of sets in $U_\delta$. Then

$$\Delta_{P \in \mathcal{P}_\kappa(\delta)} A_P = \{ Q \in \mathcal{P}_\kappa(\delta) : \forall P \subseteq Q \subseteq A_P \} \in U_\delta.$$ 

Moreover, if $F : [\mathcal{P}_\kappa(\delta)]^{[<\omega]} \to 2$, then there is an $A \in U_\delta$ such that for each $n \in \omega$, $F$ is constant on $[A]^{[n]}$.

**Definition 2.3.** Let $\kappa$ be a $\delta$-supercompact cardinal, and let $U_\delta$ be a $\delta$-supercompact measure for $\kappa$.

The elements of $\mathcal{P}_{U_\delta}$, the supercompact Prikry forcing for $U_\delta$, are all the sets of the form $p = (s,A)$, with $s = (P_0, \ldots, P_{n-1})$ a finite $\subseteq$-increasing sequence of elements of $\mathcal{P}_\kappa(\delta)$, and $A \in U_\delta$ such that for all $Q \in A$, $P_{i-1} \subseteq Q$. The sequence $s$ is called the stem of $p$, and we denote it as stem$(p)$. We call $\lh(p) = \lh(\text{stem}(p))$. The set $A$ is called the measure-one part of $p$. We say that $(t,B) \leq (s,A)$, with $(s,A),(t,B) \in \mathcal{P}_{U_\delta}$, iff $s \subseteq t$ (i.e., $\lh(t) \geq \lh(s)$ and for all $i < \lh(s)$ $t(i) = s(i)$), for all $\lh(s) \leq i < \lh(s)$ $t(i) \in A$, and $B \subseteq A$.

We say that $(t,B) \leq^* (s,A)$ iff $(t,B) \leq (s,A)$ and $s = t$. 


Lemma 2.4. Let \( \sigma \) be a generic \( \mathbb{P} \) that projects \( \sigma \) onto a dense, then for each \( m \), let \( s \) be such that for each \( \sigma \in [\mathbb{P}_\kappa(\delta)]^\omega \), we can define the filter in \( \mathbb{P}_{\delta} \) generated by \( \sigma \):

\[
\mathcal{F}_\sigma = \{(t, A) \in \mathbb{P}_\delta : t = \sigma \upharpoonright \operatorname{lh}(t), \forall i \geq \operatorname{lh}(t) \sigma(i) \in A\}.
\]

It is a maximal filter, but it is not necessarily generic.

**Lemma 2.4.** Let \( \mathbb{P}_{\delta} \) be the supercompact Prikry forcing for \( U_\delta \) and let \( G \) be \( \mathbb{P}_{\delta} \)-generic. Then \( \mathcal{F}_{\sigma_G} = G \).

**Proof.** Let \( p = (s, A) \in G, \operatorname{lh}(p) = n \). First of all, for each \( i < n \) \( \sigma_G(i) = s(i) \), therefore \( s = \sigma_G \upharpoonright n \). Since for every \( m \geq n \), \( D_m = \{p \in \mathbb{P}_{U_\delta} : \operatorname{lh}(p) > m\} \) is dense, then for each \( m \geq n \) there is a \( p_m \leq p, p_m \in G \) such that \( \operatorname{lh}(p_m) > m \).

So if \( n \leq i < m \) then \( p_m(i) = \sigma_G(i) \in A \), i.e., \( p \in \mathcal{F}_{\sigma_G} \).

We proved that \( G \subseteq \mathcal{F}_{\sigma_G} \). But since \( G \) is maximal, \( G = \mathcal{F}_{\sigma_G} \).

**Theorem 2.5.** Let \( \kappa \) be \( \delta \)-supercompact, with \( \operatorname{cof}(\delta) = \kappa \), and let \( U_\delta \) be a \( \delta \)-supercompactness measure for \( \kappa \). Let \( G \) be a generic set for \( \mathbb{P}_{U_\delta} \) over \( V \). Then in \( V[G] \) no new bounded subsets are added to \( \kappa \), every cardinal in the interval \([\kappa, \delta]\) of cofinality \( \geq \kappa \) in \( V \) has cofinality \( \omega \), and all the cardinals above \( \delta \) are preserved. Therefore, in \( V[G] \), \( |\delta| = \kappa \).

Because of Proposition 2.6, supercompact Prikry forcing enjoys the usual Prikry-like properties:

**Proposition 2.6** (Prikry condition). Let \( \kappa \) be \( \delta \)-supercompact, \( U_\delta \) be a \( \delta \)-supercompactness measure for \( \kappa \) and \( \mathbb{P}_{U_\delta} \) be the supercompact Prikry forcing for \( U_\delta \). Let \( \psi \) be a forcing statement and \( p \in \mathbb{P}_{U_\delta} \). Then there exists \( p' \leq^* p \) that decides \( \psi \), i.e., either \( p' \models \psi \) or \( p' \models \neg \psi \).

**Proposition 2.7** (Geometric condition). For all \( \sigma \in [\mathbb{P}_\kappa(\delta)]^\omega \), the filter \( \mathcal{F}_\sigma \) is generic iff for all \( A \in U_\delta \) there is an \( n \in \omega \) such that \( \sigma \upharpoonright [n, \omega] \subseteq A \).

## 3 Supercompact Prikry forcing, quotients

In this section, \( \kappa \) will always be a \( \delta \)-supercompact cardinal with \( \operatorname{cof}(\delta) = \kappa \), \( U_\delta \) will be a \( \delta \)-supercompactness measure, and \( \alpha \in [\kappa, \delta) \).

**Definition 3.1.** Let \( s \in [\mathbb{P}_\kappa(\delta)]^\{\omega\} \), and \( \alpha < \delta \). Then \( s \downharpoonright \alpha = (s(0) \cap \alpha, \ldots, s(n-1) \cap \alpha) \). Clearly, \( s \downharpoonright \alpha \in [\mathbb{P}_\kappa(\alpha)]^\{\omega\} \).

Given a \( \delta \)-supercompactness measure \( U_\delta \) for \( \kappa \), if \( A \in U_\delta \), then let \( A \downharpoonright \alpha = \{P \cap \alpha : P \in A\} \). Define also \( U_\alpha = \{A \downharpoonright \alpha : A \in U_\delta\} \). It is an \( \alpha \)-supercompactness measure for \( \kappa \). Moreover, if \( \alpha < \beta < \delta \) then \( U_\beta \downharpoonright \alpha = U_\alpha \).

Note that if \( B \in U_\alpha \), then \( \{P \in \mathbb{P}_\alpha : P \cap \alpha \in B\} \in U_\beta \), i.e., \( U_\beta \) projects into \( U_\alpha \).

**Definition 3.2.** Let \( \kappa \) be \( \delta \)-supercompact, \( U_\delta \) be a \( \delta \)-supercompactness measure for \( \kappa \) and \( \mathbb{P}_{U_\delta} \) be the supercompact Prikry forcing for \( U_\delta \).

If \( p = (s, A) \in \mathbb{P}_{U_\delta} \), then \( p \downharpoonright \alpha = (s \downharpoonright \alpha, A \downharpoonright \alpha) \in \mathbb{P}_{U_\alpha} \).

If \( G \) is \( \mathbb{P}_{U_\delta} \)-generic, then \( G \downharpoonright \alpha = \{p \downharpoonright \alpha : p \in G\} \).

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When $\alpha < \delta$, we would like to see $P_{U_\delta}$ as a projection of $P_{U_\delta}$, but this is not technically true. In fact, to have $P_{U_\delta} \preceq P_{U_\delta}$, we should have that for each $q \in P_{U_\delta}$ and for each $p \in P_{U_\delta}$ such that $p \preceq q \downarrow \alpha$ there exists a $q' \preceq q$ such that $q' \downarrow \alpha = p$, and this is not true for all the conditions of $P_{U_\delta}$. The key obstacle is that if $t \in [A \downarrow \alpha]^{<\omega}$, with $A \in U_\delta$, there is an $r \in [A]^{\omega}$ such that $r \downarrow \alpha$, but it is not necessary that $r$ is $\subseteq$-increasing. But if we fix $\alpha$, we can find a dense subforcing for which this is true.

**Definition 3.3.** Let $\alpha \in [\kappa, \delta)$. We say that $p = (s, A) \in P_{U_\delta}$ is $\alpha$-nice iff for all $t \in [A \downarrow \alpha]^{<\omega}$ there is an $r \in [A]^{\omega}$ such that $r \downarrow \alpha = t$.

**Lemma 3.4.** [Lemma 2.1.9] Let $\alpha \in [\kappa, \delta)$. Then for any $p \in P_{U_\delta}$ there is a $p' \leq^* p$ that is $\alpha$-nice. Therefore the set of $\alpha$-nice conditions is dense in $P_{U_\delta}$.

Call $P_{U_\delta}^\alpha = \{ p \in P_{U_\delta} : p \text{ is } \alpha\text{-nice} \}$. If $G$ is $P_{U_\delta}$-generic, then $G \cap P_{U_\delta}^\alpha$ is $P_{U_\delta}^\alpha$-generic, and $V[G] = V[G \cap P_{U_\delta}^\alpha]$. On the other hand, if $H$ is $P_{U_\delta}^\alpha$-generic, then $H^\alpha = \{ p \in P_{U_\delta} : \exists q \leq p \in H \} \in P_{U_\delta}$-generic and $V[H] = V[H^\alpha]$. Also, for any $P_{U_\delta}$-name $\tau$ there is a $P_{U_\delta}^\alpha$-name $\tau^\alpha$ such that $\Vdash_{P_{U_\delta}} \tau = \tau^\alpha$.

We defined $P_{U_\delta}^\alpha$, so that $P_{U_\delta} \preceq P_{U_\delta}^\alpha$, as witnessed by $\downarrow \alpha$. The first consequence is that if $G$ is a $P_{U_\delta}$-generic filter, then $H = G \cap P_{U_\delta}^\alpha$ is a $P_{U_\delta}^\alpha$-generic filter, and $g = H \downarrow \alpha$ is $P_{U_\delta}$-generic.

Then we can quotient $P_{U_\delta}^\alpha$ via $P_{U_\delta}$: if $g$ is $P_{U_\delta}$-generic, let $Q_{\alpha\delta}(g) = \{ q \in P_{U_\delta} : q \downarrow \alpha \in g \} = \{ q \in P_{U_\delta} : q \downarrow \alpha \text{ is } \alpha\text{-nice and } q \downarrow \alpha \in g \} = P_{U_\delta}^\alpha / g$. Let $\dot{Q}_{\alpha\delta}$ be the $P_{U_\delta}$-name for $Q_{\alpha\delta}(g)$. Then the map $i : P_{U_\delta}^\alpha \rightarrow P_{U_\delta} \ast \dot{Q}_{\alpha\delta}$, defined as $i(r) = (r \downarrow \alpha, r \uparrow)$, is a dense homomorphism. So if $H$ is a $P_{U_\delta}^\alpha$-generic filter, then it is also $Q_{\alpha\delta}(H \downarrow \alpha)$-generic over $V[H \downarrow \alpha]$, and $V[H] = V[H \downarrow \alpha][H]$. On the other hand, if $g$ is $P_{U_\delta}$-generic and $h$ is $Q_{\alpha\delta}(g)$-generic over $V[g]$, then $h$ is also $P_{U_\delta}^\alpha$-generic over $V$ and $V[g][h] = V[h]$.

With the above notation, the following is standard:

**Lemma 3.5.**

1. Let $\tau$ be a $P_{U_\delta}$-name. For each $p \in P_{U_\delta}^\alpha$ the following are equivalent:
   - $p \Vdash_{P_{U_\delta}} \varphi(\tau)$;
   - $p \Vdash_{P_{U_\delta}^\alpha} \varphi(\tau^\alpha)$;
   - $p \downarrow \alpha \Vdash_{P_{U_\delta}} (\dot{p} \Vdash_{Q_{\alpha\delta}} \varphi(i(\tau^\alpha)))$.

2. If $p \in Q_{\alpha\delta}(g)$ for some $g$ $P_{U_\delta}$-generic and $\tau$ is a $Q_{\alpha\delta}(g)$-name, then we can add
   - $p \Vdash_{Q_{\alpha\delta}(g)} \varphi(\tau)$.

**Proposition 3.6.** Let $g$ be $P_{U_\delta}$-generic over $V$, and let $\tau$ be a $Q_{\alpha\delta}(g)$-name for a function in $\omega^\kappa$. Let $p \in Q_{\alpha\delta}(g)$ be such that $p \Vdash \tau : \check{\omega} \rightarrow \check{\kappa}$. Consider $A_p = \{ \gamma < \kappa : 3m \in \omega \land q \Vdash \tau(m) = \check{\gamma} \}$. If $|A_p| < \kappa$ in $V[g]$, then for each $h$ $Q_{\alpha\delta}(g)$-generic such that $p \in h$, $\tau_h \in V[g]$.

**Proof.** (In this proof we are only using the forcing relation $\Vdash_{Q_{\alpha\delta}(g)}$, so we just write $\Vdash$.)

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Footnote: We thank the referee for having greatly improved the proof of this proposition.
Since $|A_p| < \kappa$ in $V[g]$, then also $B_p = \{(m, \gamma) : \exists q \leq p, q \in Q_{\alpha}(g) \wedge q \forces \tau(m) = \gamma\}$ has cardinality less than $\kappa$ in $V[g]$. Let $\eta < \kappa$ be such that $(|B_p| = \eta)V[g]$, and let $f \in V[g]$ be a bijection that witnesses it.

Let $b$ be $Q_{\alpha}(g)$-generic such that $p \in b$. Then $\tau_b \subseteq B_p$, and $f''\tau_b \subseteq \eta$. But then $f''\tau_b$ is a bounded subset of $\kappa$ in $V[g][h]$, that is a $\mathbb{P}_U$-generic extension, therefore by Theorem 3.8 $f''\tau_b \in V$. Since $f$ is a bijection, $\tau_b$ is definable from $f$ and $f''\tau_b$, so $\tau_b \in V[g]$. \hfill \qedsymbol

We are going now to approach the problem of homogeneity for the quotient forcing.

**Theorem 3.7.** [\ investigating] Each $f$ permutation of $\delta$ such that $f \mid \kappa = id$ induces an automorphism $\pi_f$ of $\mathbb{P}_U$ in the following way: If $s \in [\mathbb{P}_U(\delta)]^{<\omega}$, then $\pi_f(s) = (f''s(0), \ldots, f''s(lh(s) - 1))$, and if $p = (s, A) \in \mathbb{P}_U$, we define $\pi_f(p) = (\pi_f(s), \pi_f''A)$.

If $f$ is a permutation such that $f \mid \alpha = id$, then $\pi_f(p) \downarrow \alpha = p \downarrow \alpha$, and $f$ induces an automorphism of $Q_{\alpha}(g)$. This consideration permits us to prove that the forcing $Q_{\alpha}(g)$ is weakly homogeneous, as witnessed by the $\pi_f$:’s:

**Lemma 3.8.** [\ investigating] Let $\alpha < \delta$, $p \in \mathbb{P}_U$, $q \in \mathbb{P}_U$ be such that $p \mid \alpha = q \mid \alpha$. Then there is a permutation $f$ of $\delta$ such that $f \mid \alpha = id$, and for all $q_1 \leq q$ there is a $p_1 \leq p$ such that $p_1 \mid \alpha = q_1 \mid \alpha$ and $\pi_f(p_1)$ is compatible with $q_1$.

**Corollary 3.9.** Let $\alpha < \delta$, $g \mathbb{P}_U$-generic over $V$, let $\mathcal{A} = \{\pi_f : f$ is a permutation of $\delta$ such that $f \mid \kappa = id\}$. Then $Q_{\alpha}(g)$ is weakly homogeneous, and the homogeneity is witnessed by elements in $\mathcal{A}$. In particular, if $\tau$ is a $Q_{\alpha}(g)$-name that is invariant under members of $\mathcal{A}$ (for example, a canonical check-name for an element of $V[g]$), then for any formula $\phi$, if $q \forces_{Q_{\alpha}(g)} \phi(\tau)$ then $\forces_{Q_{\alpha}(g)} \phi(\tau)$.

**Proof.** Let $p, q \in Q_{\alpha}(g)$. Then $p \downarrow \alpha$ and $q \downarrow \alpha$ are in $g$, therefore there is an $r \in g$ such that $r \leq p \downarrow \alpha$, $q \downarrow \alpha$. Since $p, q \in \mathbb{P}_U$ and $\mathbb{P}_U \subseteq \mathbb{P}_U$, there are $p', q' \in \mathbb{P}_U$ such that $p' \leq p$, $q' \leq q$ and $p' \downarrow \alpha = r = q' \downarrow \alpha$. Since $r \in g$, then, $p', q' \in Q_{\alpha}(g)$. By Lemma 3.8 there exists a permutation $f$ of $\delta$ such that $f \mid \alpha = id$ and a $p'' \leq p'$ such that $\pi_f(p'')$ is compatible with $q'$. But then also $\pi_f(p)$ is compatible with $q$. \hfill \qedsymbol

4 The main construction

In this section, $\kappa$ will always be a $\theta$-supercompact cardinal, with $\theta > \kappa$ an inaccessible cardinal. To avoid cluttering, we are skipping $\theta$ in the notations, therefore $U$ will be a $\theta$-supercompactness measure, and if $\alpha \in [\kappa, \theta)$, then $Q_{\alpha}$ is the quotient of $\mathbb{P}_U^c$ over $\mathbb{P}_U^\alpha$

Let $G$ be $\mathbb{P}_U$-generic over $V$. Then we define, in $V[G]$, $H^G = \bigcup\{\mathcal{P}(\kappa) \cap V[G \downarrow \alpha] : \alpha < \theta\}$.

Our model of reference is going to be $L(H^G)$. 7
Proposition 4.1. ([6]) Lemma 2.1.11, Corollary 2.1.12, Corollary 2.1.13] In \( V[G] \), \( H^G = \mathcal{P}(\kappa) \cap L(H^G) \), so for each \( z \in \mathcal{P}(\kappa) \cap L(H^G) \) there is an \( \alpha < \theta \) such that \( z \in V[G \downarrow \alpha] \), and \( L(H^G) \models \theta = \kappa^+ \).

Lemma 4.2. ([6]) Lemma 2.1.11] Let \( \beta < \theta \). In \( V[G] \), let \( \dot{g}_\beta = \{ (\dot{P}, \dot{p}) : p = (P_1, \ldots, P_n, A) \in \mathbb{P}_U \wedge \exists i \in V = P_i \cap B \} \). Then \( \dot{g}_\beta \) is a \( \mathbb{P}_U \)-name for \( G \downarrow \beta \). Let \( A \) be the set of automorphisms generated by permutations \( f \) of \( \theta \) such that \( f \downarrow \kappa = \text{id} \). Then

\[
H^G = \bigcup \{ \mathcal{P}(\kappa) \cap V[\langle h(\dot{g}_\beta) \rangle] : h \in A \wedge \beta \in [\kappa, \theta) \}
\]

and therefore there is a \( \mathbb{P}_U \)-name for \( H^G \) that is invariant under members of \( A \).

Let \( \alpha < \theta \). Then \( |\text{tr}(\mathcal{P}_{U, \alpha})| \leq 2^{\omega_1^+} \). Let \( \eta = (2^{\omega_1^+})^V \). Then, since \( \theta \) is inaccessible, there is a \( \beta < \theta \) such that \( \beta > \eta \), and in \( V[G \downarrow \beta] \) we have that \( |\text{tr}(\mathcal{P}_{U, \alpha})| \leq \kappa \). But then it is possible to define, in \( V[G \downarrow \beta] \), a set \( E \subset \kappa \times \kappa \) such that \( (\mathcal{P}_{U, \alpha}, \beta) \in \mathcal{P}(\kappa) \) is isomorphic to \( (\kappa, E) \). So \( E \) is codeable as a member of \( \mathcal{P}(\kappa) \), therefore it is in \( \mathcal{P}(\kappa) \cap V[G \downarrow \beta] \) and so in \( H^G \), therefore \( \mathcal{P}_{U, \alpha} \in L(H^G) \). In fact, the same holds for any element of \( (H_{\kappa^+})^{V[G]_{\alpha}} \) for any \( \alpha < \theta \), so in particular, for all \( \alpha < \theta \) then \( \mathcal{P}_{U, \alpha} \in L(H^G) \), and for any \( \alpha < \beta < \kappa \), \( \mathcal{P}_{U, \alpha} \in L(H^G) \).

Notice also that, since \( (\text{cof}(\kappa) = \omega) \in V[G] \), then there is a \( \omega \)-sequence in \( \mathcal{P}(\kappa) \cap V[G \downarrow \kappa] \) that is cofinal in \( \kappa \), therefore such a sequence is in \( H^G \), and so \( (\text{cof}(\kappa) = \omega)^{L(H^G)} \).

Theorem 4.3. ([6]) Theorem 4.1.13] \( L(H^G) \equiv DC_{\kappa} \).

Therefore we can develop in \( L(H^G) \) a generalized descriptive set theory as in \([5, \omega, \kappa] \in \kappa \)-Polish space in \( L(H^G) \), and it makes sense to ask in \( L(H^G) \) which subsets of \( \omega \kappa \) have the \( \kappa \)-PSP. Note also that since \( \kappa \) is \( \theta \)-supercompact, \( |\mathcal{V}_{\omega, \kappa}| = \kappa \), and this is true also in \( L(H^G) \), therefore \( (\mathcal{V}_{\kappa + 1})^{L(H^G)} \) is a \( \kappa \)-Polish space in \( L(H^G) \), and \( L(H^G) \) homeomorphic to \( (\kappa, \kappa)^{L(H^G)} \), and in fact \( L(H^G) = (L(\mathcal{V}_{\kappa + 1}))^{L(H^G)} \).

Our first objective will be to find a way to identify and construct \( \mathcal{Q}_{\alpha\beta}(G \downarrow \alpha) \)-generics in \( L(H^G), \) for \( \kappa \leq \alpha < \beta < \theta \). The following is a standard fact of Prikry-like forcings, (see e.g. [6] Theorem 2.2.6), but we redo the proof because we need a slightly more precise statement. The proof uses the same ideas of Mathias’ proof of the geometric condition.

Proposition 4.4. Let \( \beta < \theta \). Then there is a \( \subset \)-descending sequence \( (A^*_n : n \in \omega) \in L(H^G) \) such that \( A^*_n \in U_\beta \), and for each \( \sigma \in [\mathcal{P}_\kappa(\beta)]^{\omega_1} \), \( \mathcal{F}_\sigma \) is \( \mathbb{P}_{U^\beta} \)-generic over \( V \) iff for all \( n \in \omega \) there is an \( m_n \in \omega \) such that for all \( i > m_n, \sigma(i) \in A^*_n \).

Proof. Since \( \theta \) is inaccessible, there exists a \( \gamma < \theta \) such that \( [U_\beta = \kappa]^V_{\omega_1} \), and by usual considerations (see e.g. [9], Corollary 2.16) there exists in \( V[G \downarrow \gamma] \) a collection of \( E_n \in V \) such that \( U_\beta = \bigcup_{n \in \omega} E_n \) and \( |E_n| < \kappa \). Since \( \mathcal{E}_n : n \in \omega \in (H_{\kappa + 1})^{V[G_{\omega_1}]}, \) then \( \mathcal{E}_n : n \in \omega \in L(H^G) \). Let \( E^*_n = \bigcup_{n \leq m} E_n \) and \( A^*_n = \cap E^*_m \). Since \( E^*_n \in V \), by the \( \kappa \)-completeness of \( U_\beta \), \( A^*_n \in U_\beta \), and \( (A^*_n : n \in \omega) \in L(H^G) \).

Let \( \sigma \in [\mathcal{P}_\kappa(\beta)]^{\omega_1} \). If \( \mathcal{F}_\sigma \) is \( \mathbb{P}_{U^\beta} \)-generic over \( V \), then by the Geometric Condition (Proposition 2.7) for any \( A \in U_\beta \), there is a \( n_A \in \omega \) such that for all \( i > n_A, \sigma(i) \in A \), and this is true also for the \( A^*_n \)'s. On the other hand, if for all \( n \in \omega \) there is an \( m_n \in \omega \) such that for all \( i > m_n, \sigma(i) \in A^*_n, \) let \( A \in U_\beta \).
and let $n \in \omega$ be such that $A \in E_n^\ast$. Then $A_n^\ast \subseteq A$, so there is an $m_n \in \omega$ such that for all $i > m_n$, $\sigma(i) \in A$, therefore by the Geometric Condition $F_\sigma$ is $P_{U_\beta}$-generic over $V$.

Now we introduce the main technique, that assures us that we can extend any $P_{U_\alpha}$-generic to a $\kappa$-perfect set of $P_{U_\beta}$-generics.

**Theorem 4.5.** Let $\alpha < \beta < \theta$, let $g = G \upharpoonright \alpha$. Let $\tau$ be a $Q_{\alpha, \beta}(g)$-name for an element of $^{\omega}\kappa$ that is not in $V[g]$. Let $p \in Q_{\alpha, \beta}(g)$ be such that $p \Vdash \tau : \omega \to \kappa$. Then $\{\tau_h : h \in Q_{\alpha, \beta}(g)\text{-generic}, p \in h\}$, as calculated in $L(H^G)$, contains a $\kappa$-perfect set.

**Proof.** (In this proof we use only the forcing relation $\Vdash_{Q_{\alpha, \beta}(g)}$, therefore to avoid cluttering we call it just $\Vdash$.)

Since $g \in L(H^G)$, then $(\text{cof}(\kappa) = \omega)_{L(H^G)}$. Fix $(\eta_n : n \in \omega) \in L(H^G)$, a sequence cofinal in $\kappa$. Fix also a sequence $(A_n^\ast : n \in \omega) \in L(H^G)$ for $P_{U_\beta}$ as in Proposition 4.4.

The idea is the following: for each $s \in \bigcup_{n \in \omega} \prod_{i < \omega} \eta_i$, we are going to define in $L(H^G)$ a filter $p_s, q_s, r_s \in Q_{\alpha, \beta}(g)$, so that $p_s \leq r_s \leq q_s$ and if $t = s^\frown (\xi)$, then $q_t \leq q_s$, in the following way:

- if $\text{lh}(s) = n$, for each $\xi_1, \xi_2 \in \eta_n$, if $\xi_1 \neq \xi_2$ then $q_{s^\frown (\xi_1)}$ and $q_{s^\frown (\xi_2)}$ are going to force a different behavior for $\tau$, and for this we will use Proposition 3.6.

- if $x \in \prod_{i < \omega} \eta_i$, then $(p_x\upharpoonright n : n \in \omega)$ generates a $Q_{\alpha, \beta}(g)$-generic filter, and for this we will use Proposition 4.4.

- the condition $r_s$ decides the behavior of $\tau$ completely up to a certain point.

At the end of the construction, for each $x \in \prod_{i < \omega} \eta_i$, $(q_x\upharpoonright n : n \in \omega)$ generates a $Q_{\alpha, \beta}(g)$-generic filter $G_x$ (because of the $p_x\upharpoonright n$’s). Then the function that to each $x \in \prod_{i < \omega} \eta_i$ associates the interpretation of $\tau$ under $G_x$ is injective (because of the way we have chosen the $q$’s) and continuous (because of the way we have chosen the $r$’s).

Let $p = p_\emptyset \in Q_{\alpha, \beta}(g)$. By Proposition 3.6,

$$\{|\gamma < \kappa : \exists q \leq p_\emptyset \exists m \in \omega \wedge g \Vdash \tau(m) = \gamma\}| = \kappa.$$  

in $V[g]$. Let

$$\Gamma_m(p_\emptyset) = \{\gamma < \kappa : \exists q \leq p_\emptyset q \Vdash \tau(m) = \gamma\}.$$

Then there must exist an $m_\emptyset$ such that $|\Gamma_{m_\emptyset}(p_\emptyset)| \geq \theta_0$. Let $\{\xi_n : \xi < \eta_0\}$ be an enumeration of a subset of $\Gamma_{m_\emptyset}(p_\emptyset)$. Then for each $\xi < \eta_0$ choose $q_{\xi}(\xi) \leq p_\emptyset$ such that $q_{\xi}(\xi) \Vdash \tau(\eta_\emptyset) = \tau_\xi$. The sequence $(q_{\xi}(\xi) : \xi < \eta_0)$ has been constructed in $V[g]$, but by the remark after Lemma 1.2 such sequence is also in $L(H^G)$.

Extend $q_{\xi}(\xi)$ to $r_{\xi}(\xi)$ so that for each $m < m_\emptyset$, $r_{\xi}(\xi)$ decides the value of $\tau(m)$.

So let $r_{\xi}(\xi) = (s, A)$. Since $r_{\xi}(\xi) \in Q_{\alpha, \beta}(g)$, $s \upharpoonright \alpha = \sigma_\xi \upharpoonright \text{lh}(s)$, and $\sigma_\xi \upharpoonright \text{lh}(s, \omega) \subseteq A_\xi \upharpoonright \alpha$. We would like to use Proposition 4.4 to find a $p = (t, B) \leq r_{\xi}(\xi)$ such that $B \subseteq A_\xi^\ast$, but we cannot just take $(s, A \cap A_\emptyset^\ast)$, since it is possible that this condition is not in $Q_{\alpha, \beta}(g)$. We need to find in $L(H^G)$ a $Q_{\alpha, \beta}(g)$-generic set to use as a guide to define $p$.  

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Define $T$ in $L(H^G)$ in the following way.

The tree $T$ is generated by the sequences $\langle P_0, \ldots, P_m \rangle \in [\mathcal{P}_n(\beta)]^{<\omega}$ such that $\langle P_0, \ldots, P_m \rangle \upharpoonright \text{lh}(s) = s$, for all $\text{lh}(s) \leq i \leq n$ $P_i \in A$, there exist $n \in \omega$, $i_0, \ldots, i_k$ such that if $j > i_k$ then $P_j \in A_{i_k}$, $P_m \in A^*_n$, and $\langle P_0, \ldots, P_m \rangle \upharpoonright \alpha = \sigma_p \upharpoonright m$. Since the definition uses only the sequence $\langle A^n : n \in \omega \rangle$ and $g$, it is clear that $T \in L(H^G)$, and if $\sigma$ is a branch in $T$, then $s \subseteq \sigma$, $\sigma \upharpoonright \alpha = \sigma_g$, for all $i \geq \text{lh}(s) \sigma(i) \in A$, and for all $n \in \omega$ there is an $m_n \in \omega$ such that for all $i \geq m_n$, $\sigma(i) \in A^*_n$.

Let $h$ be an $\mathbb{Q}_{\alpha\beta}(g)$-generic over $L(H^G)$ such that $r(\xi) \in h$. Then $h$ is $\mathbb{P}_{U^\beta}$-generic over $L(H^G)$. All dense subsets of $\mathbb{P}_{U^\beta}$ in $V$ are in $(H_{\alpha^+})^{V[G]\upharpoonright s}$ for some $\gamma < \theta$, therefore $h$ is also $\mathbb{P}_{U^\beta}$-generic over $V$. Then, by Proposition $4.3$, $\sigma_h$ is a branch of $T$ in $L(H^G)[h]$. But then, by absoluteness of well-foundedness, there is a $\alpha' \in L(H^G)$, branch of $T$, and by Proposition $4.3$, $F_{\alpha'}$ is $\mathbb{P}_{U^\beta}$-generic over $V$.

Consider $m_0 \geq \text{lh}(s)$ such that for all $i \geq m_0$, $\sigma'(i) \in A^*_n$. Then we have that $\alpha' \upharpoonright [m_0, \omega) \subseteq A \cap A^*_n$, so if we define $A = (A \cap A^*_n) \setminus (\sigma' \upharpoonright m_0)$, then $p = (\alpha' \upharpoonright m_0, A) \upharpoonright r(\xi)$ and every sequence compatible with $p$ is going to have a tail inside $A^*_n$. But this is still not enough, since it could be that $p$ is not $\alpha$-nice, i.e., $p \notin \mathbb{P}_{U^\beta}$-$U^\beta$. By definition, though, $p \in F_{\alpha'}$, that is $\mathbb{P}_{U^\beta}$-generic over $V$, and therefore by Lemma $3.4$, there exists a $p(\xi) \leq p$ such that $p(\xi) \in \mathbb{P}_{U^\beta} \cap F_{\alpha'}$.

Since $p(\xi) \in F_{\alpha'}$, $p(\xi) \downarrow \alpha \in F_{\sigma[-n]} = F_{\sigma_g} = g$, therefore $p(\xi) \in \mathbb{Q}_{\alpha\beta}(g)$, and since $p(\xi) \leq p \leq r(\xi)$, $p(\xi)$ decides the value of $\tau(m)$ with $m \leq m_g$.

We continue by induction: for any $s \in \bigcup_{n \in \omega} \prod_{i < n} \eta_i$, say $\text{lh}(s) = n$, suppose that $p_s \in \mathbb{Q}_{\alpha\beta}(g)$ is defined. Then there must exist an $m_s$ such that $|\Gamma_m(p_s)| \geq \eta_n$, so we can find in $\mathbb{Q}_{\alpha\beta}(g)$ at least $\eta_n$-many extensions of $p_s$ whose interpretations of $\tau$ differ on $m_s$. Choose $q_{\sigma(\xi)}$ to be the $\xi$-th of them, so that $q_{\sigma(\xi)} \upharpoonright [m_s, \omega) \upharpoonright \gamma_{\sigma(\xi)}(\xi)$ for some $\gamma_{\sigma(\xi)}(\xi)$, so that if $\xi \neq \xi_2$ then $\gamma_{\sigma(\xi)}(\xi_2) \neq \gamma_{\sigma(\xi)}(\xi_2)$. Extend $q_{\sigma(\xi)}$ to $r_{\sigma(\xi)}$ so that $r_{\sigma(\xi)}$ can compute $\tau$ up until $m_s$ and $n + 1$.

Consider $T$ the tree of all generic sequences compatible with $r(\xi)$; there is a branch in a generic extension, and therefore there is one in $L(H^G)$, and there must exist an $m_n$ such that its tail after $m_n$ is contained in $A^*_n$. From this, we can find $m^+_n$ and $p_{\sigma(\xi)} \in \mathbb{Q}_{\alpha\beta}(g)$ such that its stem has length $m^+_n$, and its measure one part is inside $A^*_n$.

For each $x \in \prod_{i < \omega} \eta_i$, let $\sigma_x = \bigcup_{n < \omega} \text{stem}(p(\xi)|_n)$. Note that for each $n$, $p(\xi)|_n, q(\xi)|_n, \tau(\xi)|_n \in F_{\sigma_x}$. Let $F_{\sigma_x} = F_{\sigma_g} \cap \mathbb{P}_{U^\beta}$. Because of our construction:

- for each $x \in \prod_{i < \omega} \eta_i$, $F_{\sigma_x}$ is $\mathbb{Q}_{\alpha\beta}(g)$-generic over $V[g]$: for each $n \in \omega$, we have that the measure-one part of $p(\xi)|_n$ is inside $A^*_n$, therefore for each $i > m^+_n = \text{lh}(p(\xi)|_n) \sigma_x(i) \in A^*_n$, so by Proposition $4.3$, $F_{\sigma_x}$ is $\mathbb{P}_{U^\beta}$-generic over $V$ and $F_{\sigma_x}$ is $\mathbb{P}_{U^\beta}$-generic over $V$. Since $p(\xi)|_n \in \mathbb{Q}_{\alpha\beta}(g)$, then $p(\xi)|_n \downarrow \alpha \in g$, so for each $n \in \omega$ $\text{stem}(p(\xi)|_n) \downarrow \alpha \subseteq \sigma_g$. But then $\sigma_x \downarrow \alpha = \sigma_g$, so $F_{\sigma_x} \downarrow \alpha = F_{\sigma_g} = g$, and $F_{\sigma_x}$ is $\mathbb{Q}_{\alpha\beta}(g)$-generic over $V[g]$;

- for each $x, y \in \prod_{i < \omega} \eta_i$, if $x \neq y$ then $\tau_{\sigma_x} \neq \tau_{\sigma_y}$: let $n$ be the maximal such that $x \upharpoonright n = y \upharpoonright n$. Then $q_y(\xi)|_{n+1} \upharpoonright \tau(\xi)|_{n+1} = \gamma_y|_{n+1}$, with $\gamma_y|_{n+1} \neq \gamma|_{n+1}$;

- for each $x, y \in \prod_{i < \omega} \eta_i$, if $n > 0$ is such that $x \upharpoonright n = y \upharpoonright n$, then there is an $m > n$ such that $\tau_{\sigma_x} \upharpoonright m = \tau_{\sigma_y} \upharpoonright m$: consider $r(\xi)|_n$; it is both in

\(^{3}\)The construction of $T$ is as in the proof of [8] Lemma 2.2.7
\[ F_{\sigma_{\nu}} \] and in \( F_{\sigma_{\nu}} \), and forces a value for \( \tau(i) \) for every \( i < m_{x}(n-1), n+1 \), therefore for every \( i < m_{x}(n-1), n+1 \) \( \tau_{F_{\sigma_{\nu}}}(i) = \tau_{F_{\sigma_{\nu}}}(i) \).



The function \( x \mapsto \tau_{F_{\sigma_{\nu}}} \) is therefore a continuous function from \( C(\kappa) = \prod_{\eta \in \omega} \eta \) to \( \{ \eta : h \in Q(\kappa) \} \), whose image is closed-in-\( \omega \kappa \), and therefore the theorem is proved.

We are going to prove now that in \( L(H^{G}) \) all subsets of \( \omega \kappa \) have the \( \kappa \)-PSP. The proof is similar to the original proof by Solovay of the consistency of "All subsets of \( \omega \) have the PSP": Solovay split the forcing \( Col(\omega, \kappa) \), with \( \kappa \) inaccessible, in \( Col(\omega, \kappa) \) \( \times \) \( Col(\alpha, \beta, \kappa) \times Col(\beta, \kappa) \), arguing that if \( A \) is an uncountable and ordinal-definable set of reals in \( V[G] \), a generic extension via \( Col(\omega, \kappa) \), then by the homogeneity of \( Col(\alpha, \beta, \kappa) \) there is a perfect set of \( Col(\alpha, \beta, \kappa) \)-generics \( H \), each of the elements of the perfect set \( x \) satisfy the formula that defines \( A \) in \( V[G] \). But Solovay proved that there is an \( H \) such that \( V[G] = V[G] / \alpha \| x \| H \), so the perfect set is inside \( A \).

In our case, we are going to split the forcing \( P_{U} \) in \( P_{U} \times Q_{\alpha} \times Q_{\beta} \) (it is not the same forcing, but it is equivalent), and we are going to use Theorem 4.5 in the first instance, and Corollary 3.3 in the second instance. It remains to show an analogue of the existence of an \( H \) such that \( V[G] = V[G] / \alpha \| x \| H \). This is what we are going to use:

**Proposition 4.6.** \[ 0 \] Proposition 3.1.7, Proposition 3.1.8] Let \( \alpha < \beta < \kappa \).

Then for each \( h \in L(H^{G}) \) \( Q_{\alpha}(\beta) \)-generic over \( G[V \uparrow \alpha] \), there exists \( G^{*} \) \( \mathbb{P}_{U} \)-generic over \( V \) such that \( \sigma_{G^{*}} \downarrow \beta = \sigma_{h} \) and \( H^{G^{*}} = H^{G} \).

So even if we have not proved that for each \( G \) \( \mathbb{P}_{U} \)-generic, and for each \( h \) \( Q_{\alpha}(\beta) \)-generic, we can find a generic \( H \) such that \( V[G \uparrow \alpha] / h \| H = V[G] \), anyway there is a \( G^{*} \) such that

\[ L(V_{\kappa+1})^{V[G \uparrow \alpha] / h \| G^{*}} = L(V_{\kappa+1})^{V[G^{*}} = L(V_{\kappa+1})^{V[G]} , \]

and this is enough.

**Theorem 4.7.** Each subset of \( \omega \kappa \) in \( L(H^{G}) \) has the \( \kappa \)-PSP.

**Proof.** Let \( A \subseteq \omega \kappa \), \( A \in L(H^{G}) \). Suppose that \( (|A| \leq \kappa)^{L(H^{G})} \). Since \( A \in L(H^{G}) \), \( A \) is definable in \( L(H^{G}) \) with parameters in \( \text{Ord} \cup H^{G} \cup \{ H^{G} \} \).

Moreover, \( L(H^{G}) \) is a definable class in \( V[G] \) with parameter \( H^{G} \), therefore there are a formula \( \varphi \) and \( x_{0}, \ldots, x_{n} \in \text{Ord} \cup H^{G} \) such that \( x \in A \) iff \( V[G] \models \varphi(x, x_{0}, \ldots, x_{n}, H^{G}) \). Let \( \alpha < \kappa \) be such that \( x_{0}, \ldots, x_{n} \in V \uparrow \alpha \), that exists by Proposition 4.3.

The set \( (\omega \kappa)^{V[G \uparrow \alpha]} \) has cardinality \( \eta = 2^{\kappa} \) in \( V[G \uparrow \alpha] \). Since \( \kappa \) is inaccessible, there is a \( \gamma < \kappa \) such that \( \gamma \geq \eta \), and \( (|\gamma| = \kappa)^{V[G \uparrow \gamma]} \) by Theorem 2.3.

So in \( V[G \uparrow \gamma] \) there is a bijection between \( (\omega \kappa)^{V[G \uparrow \alpha]} \) and \( \kappa \), and this is codeable inside \( P(\kappa) \cap V[G \uparrow \gamma]^{\mathbf{P}_{G}} \), therefore this is true also in \( L(H^{G}) \). But then

\footnote{This result is announced at the beginning of Section 3.1, yet Proposition 3.1.8 is proved without taking into consideration \( h \). The trick to reach the full result is in the proof of \[ 0 \] Theorem 3.2.1. In short, Kafkoulis defines a forcing notion \( Q \), and proves that each \( Q \)-generic generates a \( \mathbb{P}_{U} \)-generic \( G^{*} \) with the desired properties. If we take the \( Q \)-generic that contains the condition \( \sigma_{h} \), \( \alpha, \beta, \mathbf{P}_{h}(\theta) \in Q \), then \( G^{*} \) is as we wanted.

\footnote{e.g. \( \{ (\alpha, \eta, \beta) \mid f(\alpha)(\eta) = \beta \} \), where \( f \) is the bijection.}
\[(\forall \kappa) \mathcal{V}[\mathcal{G}^{\|\|\kappa\|}] = \kappa^{L(H)}\]. But since \((\forall \kappa) \mathcal{V}[\mathcal{G}^{\|\|\kappa\|}] \subseteq \kappa^{L(H)}\), it must be that there is an \(x \in A\) such that \(x \notin (\forall \kappa) \mathcal{V}[\mathcal{G}^{\|\|\kappa\|}]\). In particular \(\mathcal{V}[\mathcal{G}] \models \varphi(x, x_0, \ldots, x_n, H^G)\).

Let \(\beta < \theta\) be such that \(x \in \mathcal{V}[\mathcal{G} \downarrow \beta]\), that exists by Proposition 4.1 and fix a \(\mathcal{P}_{\mathcal{U}}\)-name \(\mathcal{F}\) for \(H^G\) that is invariant under any \(\pi_f\) as in Lemma 4.2. Then

\[\mathcal{V}[\mathcal{G} \downarrow \beta][\mathcal{P}_{\mathcal{U}}^\beta] \models \varphi(x, x_0, \ldots, x_n, H^G),\]

so there is a \(p \in Q_\beta(\mathcal{G} \downarrow \beta) \cap \mathcal{G}\) such that \(p \vDash Q_\beta(\mathcal{G} \downarrow \beta) \varphi(\bar{x}, \bar{x}_0, \ldots, \bar{x}_n, \bar{i}(F^\beta)_{\mathcal{G} \downarrow \beta})\), where \(F^\beta\) is the \(\mathcal{P}_{\mathcal{U}}\)-name equivalent to \(\mathcal{F}\) and \(\bar{i}\) is the dense embedding between \(\mathcal{P}_{\mathcal{U}}^\beta\) and \(\mathcal{P}_{\mathcal{U}} \ast Q_\beta\). Therefore, by Corollary 3.9 \(\mathcal{V}[\mathcal{G} \downarrow \beta] \vDash \varphi(\bar{x}, \bar{x}_0, \ldots, \bar{x}_n, \bar{i}(F^\beta)_{\mathcal{G} \downarrow \beta})\) is true in \(\mathcal{V}[\mathcal{G} \downarrow \beta]\).

Since \(\mathcal{V}[\mathcal{G} \downarrow \beta] = \mathcal{V}[\mathcal{G} \downarrow \alpha][\mathcal{G} \downarrow \beta] \cap \mathcal{P}_{\mathcal{U}}^\beta\), there is a \(Q_{\alpha\beta}(\mathcal{G} \downarrow \alpha)\)-name \(\tau\) for \(x\), and there is a \(p \in Q_{\alpha\beta}(\mathcal{G} \downarrow \alpha)\) that forces the above, so that for any \(h \in Q_{\alpha\beta}(\mathcal{G} \downarrow \alpha)\)-generic over \(\mathcal{V}[\mathcal{G} \downarrow \alpha]\) such that \(p \vDash \tau\), we can find in \(L(H^G)\) a set of \(Q_{\alpha\beta}(\mathcal{G} \downarrow \alpha)\)-generics \(\{h_x : x \in \prod_{n \in \omega} \eta_n\}\) such that \(\{h_x : x \in \prod_{n \in \omega} \eta_n\}\) is a \(\kappa\)-perfect subset of \(\kappa\). Also, for each \(x \in \prod_{n \in \omega} \eta_n\), \(p \in h_x\), therefore

\[\mathcal{V}[\mathcal{G} \downarrow \alpha][h_x] \vDash (\exists \tau) \mathcal{V}[\mathcal{G} \downarrow \alpha] \varphi(\bar{x}, \bar{x}_0, \ldots, \bar{x}_n, \bar{i}(F^\beta)_{h_x})\].

Fix \(x \in \prod_{n \in \omega} \eta_n\). Then by Proposition 4.6 there is a \(\sigma^*\) such that \(\sigma_{\mathcal{G}, \downarrow \beta} = \sigma_{h_x}\), and \(H^G = H^\mathcal{G}\). Then \(\mathcal{V}[\mathcal{G}^*] \models \varphi(\tau_{h_x}, x_0, \ldots, x_n, H^G)\), and so \(\mathcal{V}[\mathcal{G}^*] \models \varphi(\tau_{h_x}, x_0, \ldots, x_n, H^G)\). But \(\varphi(\tau_{h_x}, x_0, \ldots, x_n, H^G)\) says: \(\tau_{h_x}\) is an element of \(\kappa\) in \(L(H^G)\) that satisfies in \(L(H^G)\) the formula that witnesses the membership in \(A\) using parameters \(x_0, \ldots, x_n\), and this is absolute among models that contain \(H^G\) and \(x_0, \ldots, x_n\). Therefore \(\mathcal{V}[\mathcal{G}] \models \varphi(\tau_{h_x}, x_0, \ldots, x_n, H^G)\), and therefore \(\tau_{h_x} \in A\) for all \(x \in \prod_{n \in \omega} \eta_n\). The map \(x \mapsto \tau_{h_x}\) then witnesses that \(A\) satisfies the \(\kappa\)-PSP.

**Corollary 4.8.** \(\text{Con}(\text{ZFC} + \exists \kappa, \theta < \theta, \kappa \text{ is inaccessible and } \kappa \text{ is a } \theta\text{-supercompact cardinal})\)

\[\rightarrow\]

\(\text{Con}(\text{ZF} + \exists \kappa > \omega(\text{DC}_\kappa + \text{there is a } \kappa\text{-Polish space } X \text{ such that all the subsets of } X \text{ have the } \kappa\text{-PSP})) (P)\).

Note that if \(V\) is a model of ZFC, then \(L(V_{\lambda+1})\) is a model of ZF + DC\(\lambda\) (see e.g. [3] Lemma 4.10), therefore the previously known upperbound for the consistency strength of \((P)\) was \(\text{IO}\).

In fact, we can substitute “there is a \(\kappa\)-Polish space” in \((P)\) with “For all \(\kappa\)-Polish spaces” thanks to the following Corollary:

**Corollary 4.9.** Let \(X\) be a \(\kappa\)-Polish space in \(L(H^G)\). Then all its subsets in \(L(H^G)\) have the \(\kappa\)-PSP.
Proof. Let $X$ be a $\kappa$-Polish space in $L(H^G)$ and $A \subseteq X$. Since $X$ is $\kappa$-Polish, there is a closed set $F \subseteq ^\omega \kappa$ and a continuous bijection $f : F \rightarrow X$. Consider $B = f^{-1}(A)$. Then $B \subseteq ^\omega \kappa$ and $B \in L(H^G)$, therefore by Theorem 4.7 $B$ has the $\kappa$-PSP. If $|B| \leq \kappa$, then $|A| \leq \kappa$, and if $^{\omega} 2$ embeds into $B$, then $^{\omega} 2$ embeds into $A$. \hfill \Box

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