Systematic construction of topological flat-band models by molecular-orbital representation

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On the basis of the “molecular-orbital” representation which describes generic flat-band models, we propose a systematic way to construct a class of flat-band models with finite-range hoppings that have topological natures. In these models, the topological natures are encoded not into the flat band itself but into the dispersive bands touched with the flat band. Such a band structure may become a source of exotic phenomena arising from the combination of flat bands, topology and correlations.

I. INTRODUCTION

Interplay among flat bands, topology, and electron-electron correlation gives rise to intriguing physics. A typical example is the fractional quantum Hall effect (FQHE) [1]. In the two-dimensional electron gas, formation of completely flat Landau levels occurs due to a strong external magnetic field, and the electron-electron correlation leads to the emergence of fractionalized quasiparticles composed of the electrons being attached to the flux [2].

The FQHE is also pursued in the lattice models with flat dispersion, which are called fractional Chern insulators (FCIs) [3–17]. Clearly, a key ingredient to realize FCIs is the exact or nearly flat bands with finite Chern number. Together with the theoretical developments, candidate materials for such phenomena have been intensively explored. Examples include metal organic frameworks (MOFs) with ions having strong spin-orbit coupling [18, 19], and twisted bilayer graphene [20–25].

To study exotic phases due to the combination of flat band, topology and correlations, simple tight-binding models having flat bands, such as a nearest-neighbor (NN) tight-binding models for a class of Lieb lattices [26, 27] and line graphs [28] and their variants [29], are expected to provide a good starting point. However, implementation of the topologically nontrivial structures to those well-known flat-band models, such as adding spin-orbit couplings, often leads to finite dispersion of flat bands [13, 16, 30–32]. For this reason, most of the theoretical studies of FCIs have been carried out on the models which have nearly flat bands with nontrivial topological natures. In those models, the flatness of the bands is controlled quantitatively.

In this paper, we introduce a different approach to construct “topological flat-band models”. Our models have finite-range hoppings and exact flat bands. In such models, it was proved on the basis of K-theory that flat bands must not have a finite Chern number [33]. However, it is possible to construct the models whose dispersive bands are topologically nontrivial and they have touching points with the flat bands. Such models will serve as another platform for studying the interplay among flat bands, topology, and electron correlations.

The key strategy is to construct such models is to make use of the “molecular-orbital” (MO) representation, which was developed in the prior works [34–36]. In this representation, we describe tight-binding models having flat bands using non-orthogonal basis composed of small number of atomic orbitals. For the line graphs, which we will consider in this paper, the MOs are usually defined on the dual lattice, e.g., a honeycomb lattice for a kagome lattice. Since there are variety of examples of topological models defined on a honeycomb lattice, one may simply implement such models for the molecular orbitals, and recast them into the original kagome lattice; see Fig. 1 for the schematic of the construction of such models. Then, the topologically nontrivial bands and the exact flat band coexist in the models thus obtained, as we will show.

The rest of this paper is organized as follows. In Sec. II, we explain a method of a systematic construction of topological flat-band models based on the MO representation. Then, our main results are illustrated in Sec. III, where we show three examples of topological flat-band models composed of MOs. In Sec. IV, we present a summary of this paper.

II. MOLECULAR-ORBITAL REPRESENTATION OF TOPOLOGICAL FLAT-BAND MODELS

In this section, we explain how to construct topological flat-band models by using the MO representation. Throughout this paper, we focus on a tight-binding models on a kagome lattice of spinless or spinful fermions. Application of the same strategy to other lattices is straightforward.

On a kagome lattice, each site is labeled by the position of the unit cell \( \mathbf{R} = r_1 a_1 + r_2 a_2 \), and the sublattice \( \eta = 1, 2, 3 \); we use the abbreviated form \( i = (\mathbf{R}, \eta) \). The annihilation and creation operators on \( i \) are represented by \( c_i, \sigma \), and \( c_i^\dagger, \sigma \), respectively. \( \sigma = \uparrow, \downarrow \) labels the spin degrees of freedom for the spinful systems; for the spinless...
systems, we simply omit this index.

We consider the models written by the following non-orthogonal and unnormalized basis $C_{R,\sigma}^{\triangle}$ and $C_{R,\sigma}^{\nabla}$ which we call MOs [34, 36]:

$$C_{R,\sigma}^{\triangle} = \gamma_1^\triangle c_{R,1,\sigma} + \gamma_2^\triangle c_{R,2,\sigma} + \gamma_3^\triangle c_{R,3,\sigma},$$  \hspace{1cm} (1)

and

$$C_{R,\sigma}^{\nabla} = \gamma_1^\nabla c_{R+a_1,1,\sigma} + \gamma_2^\nabla c_{R,2,\sigma} + \gamma_3^\nabla c_{R+a_1-a_2,3,\sigma},$$  \hspace{1cm} (2)

with $\gamma_i^{\triangle/\nabla} \in \mathbb{C}$.

These MOs are defined on the triangles, and thus they are placed on a honeycomb lattice. Now, let us consider the tight-binding models for the MOs:

$$\mathcal{H} = \sum_{R,R'} C_{R,R'}^{\dagger} h_{R,R'} C_{R'},$$  \hspace{1cm} (3)

where $C_{R} = (C_{R,\uparrow}^{\triangle}, C_{R,\uparrow}^{\nabla}, C_{R,\downarrow}^{\triangle}, C_{R,\downarrow}^{\nabla})^T$, and $h_{R,R'}$ represents the Hamiltonian for the MOs which is defined on a honeycomb lattice. Using Eqs. (1) and (2), one can easily recast the model onto the original kagome lattice. The model thus obtained has an exact zero-energy flat band, since the projection from the original kagome sites onto the MOs causes the reduction of the degrees of freedom, and the kernel of the projection is enforced to have zero-energy [34, 36].

The Hamiltonian of Eq. (3) can be written in the momentum-space representation if $h_{R,R'}$ has a translational symmetry, i.e., $h_{R,R'}$ depends only on $R - R'$ and thus it can be written as $h_{R,R'} = h_{R-R'}$. In the original basis of a kagome lattice, it can be written as

$$\mathcal{H} = \sum_k c_k^\dagger \mathcal{H}_k c_k,$$  \hspace{1cm} (4)

with $c_k = (c_{1,k,\uparrow}, c_{2,k,\uparrow}, c_{3,k,\uparrow}, c_{1,k,\downarrow}, c_{2,k,\downarrow}, c_{3,k,\downarrow})^T$. The Hamiltonian matrix $\mathcal{H}_k$ has a form

$$\mathcal{H}_k = \Psi_k h_k \Psi_k^\dagger,$$  \hspace{1cm} (5)

where

$$h_k = \sum_R h_{R} e^{-i k \cdot R},$$  \hspace{1cm} (6)

is the Hamiltonian matrix for the MOs in the momentum-space representation, and

$$\Psi_k^\dagger = \begin{pmatrix}
\gamma_1^\triangle e^{i k \cdot a_1} & \gamma_2^\triangle & \gamma_3^\triangle e^{i k \cdot (a_1-a_2)} \\
\gamma_1^\nabla & \gamma_2^\nabla & \gamma_3^\nabla e^{i k \cdot (a_1-a_2)} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\gamma_1^\nabla e^{i k \cdot a_1} & \gamma_2^\nabla & \gamma_3^\nabla e^{i k \cdot (a_1-a_2)} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},$$

(7)

Henceforth, for simplicity, we set $(\gamma_1^\triangle, \gamma_2^\triangle, \gamma_3^\triangle, \gamma_1^\nabla, \gamma_2^\nabla, \gamma_3^\nabla) = (1, 1, 1, 1, 1, 1)$. For this choice, the quadratic band touching between the flat band and the dispersive band at $\Gamma$ point is enforced to occur, due to the reduction of the linear space spanned by the MOs [36, 37]. Namely, at $\Gamma$ point, $\Psi_k = 0$ is given

FIG. 1. Schematic figure for the construction of the tight-binding models considered in this paper.
As \(\gamma^\triangle/\nabla\) such that the vectors \((\gamma^\triangle_1, \gamma^\triangle_2, \gamma^\triangle_3)\) and \((\gamma'^\nabla_1, \gamma'^\nabla_2, \gamma'^\nabla_3)\) are linearly independent with each other, the band touching can be erased \([36, 38]\).

In Eq. (3), \(h_{RR', R'}\) can be a generic tight-binding Hamiltonian. In the previous works, rather simple forms of \(h_{RR', R'}\) are considered. For instance, the NN hopping model on a kagome and a breathing kagome lattices can be described by setting \(h_{RR', R'}\) as an “on-site potential” \([34, 36]\). Our strategy is to set \(h_{RR', R'}\) as well-known topological models defined on a honeycomb lattice, as we show in the next section. Although the models thus obtained have more or less complicated patterns of the hoppings on the original kagome lattice, they are finite-ranged and the topological natures of \(h_{RR', R'}\) are indeed succeeded to the dispersive bands.

III. EXAMPLES OF TOPOLOGICAL FLAT-BAND MODELS ON A KAGOME LATTICE

A. Molecular-orbital Hofstadter model

The first model is the Hofstadter model \([39]\) for the molecular orbitals. Here we consider the spinless fermions. The Hamiltonian reads:

\[
\mathcal{H}(\phi) = \sum_{R} -t C_{R}^\dagger \left( C_{R'}^\nabla e^{2\pi i \phi r_1} C_{R-a_1}^\nabla + C_{R-a_1+a_2}^\nabla \right) + (\text{h.c.}),
\]

where \(\phi = p/q\) with \(p\) and \(q\) are relatively prime numbers. It should be emphasized that the model is different from the conventional Hofstadter model on a kagome lattice \([40]\) and its variants \([41, 42]\).

In Fig. 2, we show the energy spectrum as a function of \(\phi\). The diagram does not resemble neither the honeycomb Hofstadter model \([43, 44]\) nor the kagome Hofstadter model \([40]\). Remarkably, the zero-energy modes with macroscopic degeneracy remains for any \(\phi\). Note that the same behavior is also seen in the Hofstadter model on a Lieb lattice \([45]\) and a dice lattice \([46]\).

To study the topological properties, let us look at the band structure for the specific choice of \(p\) and \(q\); here we choose \(p = 1\), and \(q = 3\). The band structure in the magnetic Brillouin zone and the Chern numbers computed numerically by using the method of Ref. 47, are shown in Fig. 3(a). We see that the flat bands, which have three-fold degeneracy, are touched with the dispersive band at \(\Gamma\) point. Furthermore, looking at the band structure closely in the entire Brillouin zone, we also find Dirac cones formed the dispersive bands whose Dirac points degenerate with the flat band as well [Fig. 3(b)]. These Dirac cones originate from those of the Hofstadter model on a honeycomb lattice. Namely, the band touch-
B. Molecular-orbital Haldane model

Another representative model of Chern insulators on a honeycomb lattice is the Haldane model [48]. The MO-Haldane model reads

$$\mathcal{H} = \sum_{\mathbf{k}} \begin{bmatrix} C^\Delta_{\mathbf{k}} \hat{1}, C^\nabla_{\mathbf{k}} \hat{1} \end{bmatrix} h^{(H)}_{\mathbf{k}} \begin{bmatrix} C^\Delta_{\mathbf{k}} \\ C^\nabla_{\mathbf{k}} \end{bmatrix},$$

where

$$h^{(H)}_{\mathbf{k}} = \epsilon_{\mathbf{k}}\tau_0 + R_{\mathbf{k}} \cdot \tau + M\tau^z,$$

with $\tau_0$ being a $2 \times 2$ identity matrix, $\tau = (\tau^x, \tau^y, \tau^z)$ being the Pauli matrices,

$$\epsilon_{\mathbf{k}} = 2t' \cos \phi [\cos k \cdot a_1 + \cos k \cdot a_2 + \cos k \cdot (a_2 - a_1)],$$

and

$$R_{\mathbf{k}} - iR_{\mathbf{k}}^\dagger = t \sum_{p=1}^{3} e^{-ik \cdot \delta_p},$$

In Eq. (13), we have used $\delta_1 = a_1 - a_2$, $\delta_2 = 0$, and $\delta_3 = a_1$. The analytical expression of the dispersion relations of two dispersive bands can be obtained by mapping the original eigenvalue problem of the $3 \times 3$ matrix to that of the $2 \times 2$ matrix, $h^{(H)}_{\mathbf{k}} \Psi_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger$ [34, 36, 49, 50]. The dispersion relations thus obtained are

$$R_{\mathbf{k}} = 2t' \sin \phi [\sin k \cdot a_1 - \sin k \cdot a_2 + \sin k \cdot (a_2 - a_1)].$$

We plot the band structure and the Chern numbers for $(t, t', M, \phi) = (-1, -0.3, 0, 2/3)$ in Fig. 4(a). The zero-energy flat band is located between the upper and the lower dispersive bands. The lower dispersive band has the Chern number $-1$, thus the sum of the Chern numbers over the flat band and the upper dispersive band is 1. Therefore, we can again realize the flat band touched with topologically nontrivial band.

We also compute the dispersions for the cylinder geometry, shown in Fig. 4(b). We see the chiral edge modes.
appear, due to the non-trivial Chern number and the bulk-edge correspondence \[51\]. Interestingly, the chiral edge modes cross with the bulk flat band at zero energy. Note that similar dispersion is found in a topological flat-band model on a Lieb lattice \[52, 53\].

In Fig. 4(c), we map the Chern number of the lowest band in the \(M-\phi\) space. We see that two topological phases with the Chern number 1 and \(-1\) are separated by the gapless region, where the lowest dispersive band overlaps with the flat band.

In the MO Haldane model, the flat band is located between two dispersive bands. For realization of topologically-nontrivial many-body states, on the other hands, it is often desirable to construct a model where flat band has the lowest energy \[5–8\]. In the present hands, it is often desirable to construct a model where topologically-nontrivial many-body states, on the other hand, is often desirable to construct a model where flat band has the lowest energy \[5–8\]. In the present

\[
\mathcal{H} = \sum_k \left[ C_{k,\uparrow}^{\Delta,\dagger}, C_{k,\uparrow}^{\nabla,\dagger} \right] \tilde{h}^{(H)}_{k} \left[ C_{k,\uparrow}^{\Delta}, C_{k,\uparrow}^{\nabla} \right],
\]

with

\[
\tilde{h}^{(H)}_{k} = h^{(H)}_{k} + M' \tau_0.
\]

The second term of (17) is an “on-site” term for the MOs, but it does not give rise to entire shift of energy in the original kagome model \[34, 36\]. In Fig. 5, we show the band structure for the representative choice of parameters. We see that the desirable band structure is obtained; namely the flat band, being touched with the dispersive band, has the lowest energy, and the total Chern number for these two bands is indeed finite.

C. Molecular-orbital Kane-Mele model

Finally, we present an example of a model having \(Z_2\) topology. To be concrete, we implement the Kane-Mele model \[54\] as the Hamiltonian of MOs. We now consider the spinful fermions. The Hamiltonian is given as

\[
\mathcal{H} = \sum_k \left[ C_{k,\uparrow}^{\Delta,\dagger}, C_{k,\uparrow}^{\nabla,\dagger}, C_{k,\downarrow}^{\Delta}, C_{k,\downarrow}^{\nabla} \right] h^{(KM)}_{k} \left[ C_{k,\uparrow}^{\Delta,\dagger}, C_{k,\uparrow}^{\nabla,\dagger}, C_{k,\downarrow}^{\Delta}, C_{k,\downarrow}^{\nabla} \right],
\]

with

\[
h^{(KM)}_{k} = \begin{pmatrix}
\epsilon_k(\phi) + R^z_k(\phi) + M & R^y_k(\phi) - i R^y_k(\phi) - M & 0 & 0 \\
R^y_k(\phi) + i R^y_k(\phi) - M & \epsilon_k(\phi) + R^z_k(\phi) - M & \tau'' \alpha_k & 0 \\
0 & \tau'' \alpha_k & 0 & 0 \\
0 & 0 & \epsilon_k(\phi) - R^z_k(\phi) - M & \epsilon_k(\phi) - R^z_k(\phi) - M
\end{pmatrix},
\]

the eigenstates of the entanglement Hamiltonian \(H_{\text{en}}(k)\):

\[
H_{\text{en}}(k)^T = \log \begin{pmatrix} 1 - P_1 P_\uparrow(k) P_1 \epsilon' \epsilon' \end{pmatrix},
\]

with

\[
P_\uparrow = \text{diag}(1, 1, 1, 0, 0, 0),
\]

and

\[
P_\downarrow = \sum_{n=1}^2 \psi_{k, n} \psi_{k, n}^\dagger.
\]

Here \(\psi_{k, n}\) denotes the \(n\)-th eigenvector of \(h_k \equiv \tilde{h}^{(KM)}_{k} \tilde{h}^{(KM)}_{k} \). In Fig. 6(c), we find three phases: the \(Z_2\) topological phase (enCh\(^\circ\) = 1), the trivial phase (enCh\(^\circ\) = 0), and the gapless phase where one of the lower dispersive bands overlaps with the flat bands.

IV. SUMMARY AND OUTLOOK

In summary, we have introduced a systematic method to construct topological flat-band models with finite-range hoppings. The existence of flat bands is guaranteed.
FIG. 6. (a) The band structure of the MO Kane-Mele model for \((t, t', t'', \phi, M) = (-1, -0.06, -0.05, \pi, -0.1)\), (b) the same parameter on a cylinder geometry. (c) The \(t''-M\) phase diagram of the MO Kane-Mele model at \(t' = -0.06\).

since the model is constructed by the MOs. Although flat bands themselves are not allowed to have non-trivial topological numbers, the dispersive bands being touched with the flat bands can have the topologically nontrivial nature. Such models will serve as a platform to look for intriguing phenomena arise from the topology and the correlation effects. In this light, studying the many-body effects in these models will be an interesting future problem.

Throughout this paper, we consider the models on a two-dimensional kagome lattice, but it is straightforward to apply our method to other lattices, including ones in three dimensions. For instance, if we consider the model on a pyrochlore lattice, the MOs are defined on a diamond lattice. Thus, to find the \(\mathbb{Z}_2\) topological model, the Fu-Kane-Mele model [59] can be used as a MO Hamiltonian. If we consider the many-body effects in the model thus obtained, the flat band will lead to the ferromagnetism, while the \(\mathbb{Z}_2\) topological nature leads to the topological surface states. Then, the interplay between these two may leads to the quantum anomalous Hall effect, as in the case of magnetic topological insulators [60, 61].

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[1] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. 48, 1559 (1982).
[2] J. K. Jain, Phys. Rev. Lett. 63, 199 (1989).
[3] H. Katsura, I. Maruyama, A. Tanaka, and H. Tasaki, Europhys. Lett. 91, 57007 (2010).
[4] D. Green, L. Santoz, and C. Chamon, Phys. Rev. B 82, 075104 (2010).
[5] E. Tang, J. W. Mei, and X. G. Wen, Phys. Rev. Lett. 106, 236802 (2011).
[6] K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, Phys. Rev. Lett. 106, 236803 (2011).
[7] T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett. 106, 236804 (2011).
[8] D. N. Sheng, Z. C. Gu, K. Sun, and L. Sheng, Nat. Commun. 2, 389 (2011).
[9] N. Regnault and B. A. Bernevig, Phys. Rev. X 1, 021014 (2011).
[10] XL. Qi, Phys. Rev. Lett. 107, 126803 (2011).
[11] Z. Liu, E. J. Bergholtz, H. Fan, and A. M. Lauchli, Phys. Rev. Lett. 109, 186805 (2012).
[12] S. Takayoshi, H. Katsura, N. Watanabe, and H. Aoki, Phys. Rev. A 88, 063613 (2013).
[13] E. J. Bergholtz and Z. Liu, Int. J. Mod. Phys. B 27, 1330017 (2013).
[14] C. H. Lee and XL. Qi, Phys. Rev. B 90, 085103 (2014).
[15] M. Udagawa and E. J. Bergholtz, J. Stat. Mech. (2014) P10012.
[16] E. J. Bergholtz, Z. Liu, M. Trescher, R. Moessner, and M. Udagawa, Phys. Rev. Lett. 114, 016806 (2015).
[17] J. Behrmann, Z. Liu, and E. J. Bergholtz, Phys. Rev. Lett. 116, 216802 (2016).
[18] Z. Liu, Z.-F. Wang, J.-W. Mei, Y.-S. Wu, and F. Liu, Phys. Rev. Lett. 110, 106804 (2013).
[19] M. G. Yamada, T. Soejima, N. Tsuji, D. Hirai, M. Dincă, and H. Aoki, Phys. Rev. B 94, 081102(R) (2016).
[20] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and F. Jarillo-Herrero, Nature (London) 556, 43 (2018).
[21] Y. Cao, V. Fatemi, A. Demir, S. Fang, S. L. Tomarken, J. Y. Luo, J. D. Sanchez-Yamagishi, K. Watanabe, T. Taniguchi, E. Kaxiras, R. C. Ashoori, and P. Jarillo-Herrero, Nature (London) 556, 80 (2018).
[22] M. Koshino, N. F. Q. Yuan, T. Koretsune, M. Ochi, K. Kuroki, and L. Fu, Phys. Rev. X 8, 031087 (2018).
[23] E. M. Spanton, A. A. Zibrov, H. Zhou, T. Taniguchi, K. Watanabe, M. P. Zaletel, and A. F. Young, Science 360, 62 (2018).
[24] K. Hejazi, C. Liu, H. Shapourian, X. Chen, and L. Balents, Phys. Rev. B 99, 035111 (2019).
[25] K. Hejazi, C. Liu, and L. Balents, Phys. Rev. B 100, 035115 (2019).
[26] E. H. Lieb, Phys. Rev. Lett. 62, 1201 (1989).
[27] B. Sutherland, Phys. Rev. B 34, 5208 (1986).
[28] A. Mielke, J. Phys. A: Math. Gen. 24 L73 (1991); J. Phys. A: Math. Gen. 24 3311 (1991).

[29] S. Miyahara, K. Kubo, H. Ono, Y. Shimomura, and N. Furukawa, J. Phys. Soc. Jpn. 74 1918 (2005).

[30] H. M. Guo and M. Franz, Phys. Rev. B 80, 113102 (2009).

[31] M. Kurita, Y. Yamaji, and M. Imada, J. Phys. Soc. Jpn. 80, 044708 (2010).

[32] A. Bolens and N. Nagaosa, Phys. Rev. B 99, 165141 (2019).

[33] L. Chen, T. Mazaheri, A. Seidel, and X. Tang, J. Phys. A: Math. Theor. 47, 152001 (2014).

[34] Y. Hatsugai and I. Maruyama, Europhys. Lett. 95 20003 (2011).

[35] Y. Hatsugai, K. Shiraishi, and H. Aoki, N. J. Phys. 17, 025009 (2015).

[36] T. Mizoguchi and Y. Hatsugai, Europhys. Lett. 127, 47001 (2019).

[37] D. L. Bergman, C. Wu, and L. Balents, Phys. Rev. B 78, 125104 (2008).

[38] D. Billetowski and R. Moessner, Phys. Rev. B 98 235109 (2018).

[39] D. R. Hofstadter, Phys. Rev. B 14, 2239 (1976).

[40] T. Kimura, H. Tamura, K. Shiraishi, and H. Takanayagi, Phys. Rev. B 65, 081307(R) (2002).

[41] K. Ohgushi, S. Murakami, and N. Nagaosa, Phys. Rev. B 62, R6065 (2000).

[42] S. Maiti and T. Sedrakyan, Phys. Rev. B 99, 174418 (2019).

[43] R. Rammal, J. Phys. Paris 46, 1345 (1985).

[44] Y. Hatsugai, T. Fukui, and H. Aoki, Phys. Rev. B 74, 205414 (2006).

[45] H. Aoki, M. Ando, and H. Matsumura, Phys. Rev. B 54, R17296(R) (1996).

[46] J. Vidal, R. Mosseri, and B. Doucet, Phys. Rev. Lett. 81, 5888 (1998).

[47] T. Fukui, Y. Hatsugai, and H. Suzuki, J. Phys. Soc. Jpn. 74, 1674 (2005).

[48] F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988).

[49] T. Mizoguchi, L. D. C. Jaubert, R. Moessner, and M. Udagawa, Phys. Rev. B 98, 144446 (2018).

[50] T. Mizoguchi and M. Udagawa, Phys. Rev. B 99, 235118 (2019).

[51] Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993); Phys. Rev. B 48, 11851 (1993).

[52] C. Weeks and M. Franz, Phys. Rev. B 82, 085310 (2010).

[53] S. Jana, A. Saha, and A. Mukherjee, Phys. Rev. B 100, 045420 (2019).

[54] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005); Phys. Rev. Lett. 95, 226801 (2005).

[55] T. Fukui and Y. Hatsugai, J. Phys. Soc. Jpn. 83, 113705 (2014).

[56] H. Araki, T. Kariyado, T. Fukui, and Y. Hatsugai, J. Phys. Soc. Jpn. 85, 043706 (2016).

[57] T. Fukui and Y. Hatsugai, J. Phys. Soc. Jpn. 85, 083703 (2016).

[58] H. Araki, T. Fukui, and Y. Hatsugai, Phys. Rev. B 96, 165139 (2017).

[59] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. 98, 106803 (2007).

[60] C.-Z. Chang and M. Li, J. Phys.: Condens. Matter 28, 123002 (2016).

[61] Y. Tokura, K. Yasuda, and A. Tsukazaki, Nat. Rev. Phys. 1, 126 (2019).