Dynamics and formation of localized states in flowing thin films: Bound states of solitary waves

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Abstract. The flow of a thin liquid film down an inclined heated substrate generates solitary waves in the form of traveling holes. These waves in turn form a variety of different bound states in which the constituent waves all travel with a common speed. The formation and properties of these states are investigated as function of the system parameters.

1. Introduction

The study of flow of thin liquid films down an inclined substrate has a long history \cite{12,4}. The problem has been studied both experimentally \cite{5} and numerically using a variety of different thin film equations \cite{15} as well as the primitive equations \cite{14}. These studies all reveal the formation and steepening of waves that may lead to shock-like structures or the formation of solitary waves. The latter result when the steeping of the wave due to nonlinearity is balanced by spatial dispersion. Similar phenomena take place on inclined heated substrates in which the temperature dependence of the surface tension is of paramount importance \cite{16}.

Heated substrates are of both practical and theoretical interest. On a horizontal substrate the primary instability as the Marangoni number increases is subcritical and leads to the breakup of the film into arrays of drops \cite{7,10,1}. These drops are separated by nominally dry regions, or by an ultrathin liquid film if the substrate is wetting. In the latter case the resulting multidrop state undergoes a slow coarsening process since the drops are able to communicate via the intervening film. In a recent paper Thiele and Knobloch described a general construction of multidrop solutions \cite{18} leaving aside their realizability via the solution of an initial value problem. In addition, they argued that on an inclined substrate the breakup of the liquid film may not proceed to completion owing to the presence of a dynamically generated thin film between the drops. This film is present even in the absence of an explicit disjoining pressure describing the wetting properties of the substrate, and the resulting solution is self-consistent in the sense that the resulting multidrop states all slide down the substrate unless trapped by wetting inhomogeneities or other types of surface roughness \cite{19}. Thiele and Knobloch \cite{18} examined sliding solutions of constant form subject to periodic boundary conditions together with their stability. The speed of these states solves a nonlinear eigenvalue problem, and increases with increasing inclination. In addition they identified the presence of intervals in the spatial period with no stable periodic solutions of constant form.
2. Formulation
In this paper we examine these intervals in order to identify the solutions present and describe their properties. We study the lubrication or longwave equation [18]

\[ h_t = -\left[ \frac{h^3}{3} (h_{xxx} + \alpha - h_x) - \frac{h^2}{2} \text{Ma} \ T_x \right]_x, \]  

(1)

where \( h(x,t) \) is the film thickness and \( T(x,t) \) is the surface temperature given by

\[ T_x = -\frac{\text{Bi} \ h_x}{(1 + \text{Bi} \ h)^2}. \]  

(2)

The equations have been nondimensionalized using the unperturbed film thickness \( h_0 \) for \( h(x,t) \) and the capillary length \( \ell \equiv \sqrt{\sigma_0/\rho g} \gg h_0 \) for the downstream coordinate \( x \), where \( \sigma_0 \) is the isothermal surface tension, \( \rho \) is the density and \( g \) is the acceleration due to gravity. Thus \( \text{Bi} \) and \( \text{Ma} \) are the Biot and Marangoni numbers that measure the importance of heat transfer to the ambient air and the temperature dependence of the surface tension, respectively. Owing to the choice of \( \ell \) the Bond number \( \text{Bo} = 1 \) [18]. Equations (1)-(2) describe a thin liquid film with a free upper surface on an inclined homogeneously heated substrate, with (scaled) inclination \( \alpha \). The corresponding physical inclination is sufficiently small that the film is primarily supported hydrostatically. Newton’s law of cooling is assumed at the liquid-gas interface with passive ambient air and the temperature dependence of the surface tension, respectively. Owing to the heat transfer to the downstream side of several smaller oscillations in the film height that are also visible in the figure (see also Fig. 2). In each case transients have been discarded. When \( L = 100 \) Fig. 1(a) reveals a single solitary wave of depression (hereafter a “hole”) that propagates through the periodic domain at constant speed. When \( L = 100 \) one finds a pair of such holes that form a bound state of two holes traveling at constant speed; this speed is almost identical to the speed of a.
single hole. When $L = 150$ one finds a bound state of three equally spaced holes traveling at constant speed. This state persists for $L = 200$ but is now embedded in a background of waves that travel faster: the waves grow in amplitude in the downstream direction and are absorbed by the bound state on the upstream side causing negligible perturbation. Solutions of this type are no longer “rotating” waves, i.e., waves that are steady in an appropriately moving reference frame. Figures 1(a,b,c) reveal the presence of $n$ holes in a domain of length $n\lambda$, where $\lambda$ is the wavelength of the pattern in Fig. 1(a). Thus these states only differ in the distribution of the three holes within the spatial period $3\lambda$. Since the speed of these states is almost the same the frequency is also almost the same, as can be seen from the time series $h(x,t)$ shown in Fig. 2. These time series moreover reveal the hole-like structure of the instantaneous profiles. In contrast, the frequency of the bound state in Fig. 2(d) is substantially lower.

Figure 3 shows similar results for $\alpha = 0.5$. The holes move faster, but the bound states seen in Fig. 1 recur, although the small amplitude background traveling waves are present already when $L = 100$. The bound states now have a more complex internal structure. For example, when $L = 150$ the three-hole bound state has a visibly larger separation between the leading and middle holes than between the middle and trailing holes. When $L = 200$ we find a bound state of five holes, again with a monotonically increasing separation between adjacent holes in the downstream direction. A different bound state of five holes is stable when $L = 250$, and bound states of seven and eight holes are found when $L = 300$ and $L = 350$ (not shown), respectively. Figure 4 confirms the trend: the figure shows a stable bound state of 15 holes for $\alpha = 0.8$ and $L = 500$, starting from an unstable $P_1$ state. Figure 5 shows that the transitions between the different spatially periodic structures shown in Fig. 3 as $L$ increases occur via an interval of quasiperiodicity.

The situation is dramatically different for smaller inclinations $\alpha$ for which the film is closer to the horizontal film and hence closer to potential rupture. In Fig. 6 we show that this regime is characterized by the intermittent formation of deep holes. These evolve slowly from small
amplitude holes, last for a brief time, before subsiding into the same small amplitude state only to regrow again. These events are best thought of as attempts at film rupture that are thwarted by the flow of the film. The figure shows that this attempted rupture process is approximately periodic albeit with a characteristic frequency that is much lower than the frequency of the small amplitude holes. However, this frequency increases significantly with increasing $L$, as shown in Fig. 6, and this trend extends to $L = 28$ (Fig. 7) and $L = 53$ (not shown).

Figures 8 and 9 show the details of the transition at fixed $\alpha$ from a small amplitude state with relatively large downstream speed and a larger amplitude state characterized by deep holes and a lower speed as $Ma$ increases for modest spatial period ($L = 16$). Both states consist of a single hole per spatial period $L$. At $Ma = 3.6$ the solution takes the form of a periodic train of holes. At $Ma = 3.7$ this train of holes experiences periodic disruption which becomes more and more frequent as $Ma$ increases further and leads to the establishment of slugs of large amplitude holes within the original train of small amplitude holes. By the time $Ma = 3.86$ the deep holes predominate and the observed state resembles slugs of the small amplitude holes within a train of deep holes. The frequency of the appearance of the small amplitude holes decreases rapidly with increasing $Ma$ until at $Ma = 3.9$ the solution takes the form of a periodic train of deep holes; this state persists when $Ma$ is reduced to $Ma = 3.86$ indicating the presence

Figure 2. Time evolution of $h(\cdot, t)$ at an arbitrary point in $[0, L)$ for comparison with Fig. 1. Parameters are $\alpha = 0.4$, $Ma = 3.5$, $Bi = 0.5$ and (a) $L = 50$, (b) $L = 100$, (c) $L = 150$, and (d) $L = 200$. Note the presence of background traveling waves in (d).
Figure 3. Isovalues of $h(x,t)$ in the $(x,t)$ plane with $0 \leq x \leq L$ (horizontal axis) and $t \in [25000, 26250]$ (vertical axis). Parameters are $\alpha = 0.5$, $Ma = 3.5$, $Bi = 0.5$ and (a-f) $L = 50, 100, 150, 200, 250, 300$. Resolution is $N = 128$, starting from $h(x,0) = 1+0.1\sin(2\pi x/L)$.

of hysteresis. This type of behavior is another example of a quasiperiodic transition between two different spatially periodic states present because of the subcriticality of the $n = 1$ branch: these two states have quite different frequencies, with the amplitude of the small amplitude state dominating at smaller values of $Ma$ and the amplitude of the deep hole state dominating at larger values of $Ma$. No frequency locking takes place in consequence of translation invariance of the system: in a frame moving with the small amplitude holes the deep hole state is a periodic wave train that has nothing to lock to, and similarly in the frame moving with the deep hole state. States of this type correspond to secondary branches connecting different parts of the $n = 1$ branch and are known to occur at other parameter values [18]. The results shown in Figs. 8 and 9 indicate that these states may be stable.
4. Discussion and conclusions

In this preliminary account of our time-stepping results we have provided evidence for the stability of secondary branches that bifurcate from the primary branches $P_n$. There are two types of such secondary branches: branches that connect branches with different values of $n$ (as in Fig. 5) and branches that connect different parts of the same branch (as in Fig. 9). The work of Thiele and Knobloch [18] shows that such secondary branches are common in this problem. We believe, moreover, that the bound states consisting of $n$ holes also originate in secondary bifurcations from the branches $P_n$. Plots of $h(\xi)$ as a function of $\xi$ and of $h(\xi)$ vs $h'(\xi)$ are useful for understanding the evolution of the solution profile along the secondary branches originating
Figure 5. Quasiperiodic transition between two spatially periodic states, showing $h(\cdot, t)$ for $L = 40, 42, 44, 46, 48, 50$ from top to bottom over two different time intervals. Parameters are $\alpha = 0.5$, $Ma = 3.5$, $Bi = 0.5$. Resolution is $N = 128$, starting from $h(x, 0) = 1 + 0.1 \sin(2\pi x / L)$. 
Figure 6. Time evolution of $h(\cdot,t)$ together with $\max_x(h(x,t))$ and $\min_x(h(x,t))$ for $x$ in $[0,L)$. Parameters are $\alpha = 0.3$, $Ma = 3.5$, $Bi = 0.5$ and from top to bottom $L = 25.1, 25.2, 25.3, 25.4, 25.5, 25.6$. Resolution is $N = 128$, starting from $h(x,0) = 1 + 0.1 \sin(2\pi x/L)$. 
in these bifurcations. Such plots for $n = 2$ show that the primary $P_2$ branch consists of a pair of equidistant structures that spread apart as $L$ increases and begin to take on the form of holes. The secondary bifurcations from $P_2$ lead to nonequidistant structures that start out equidistant (on $P_2$) but become increasingly nonequidistant as one follows the secondary branch. A typical example is provided in Fig. 17 of [18]: among five secondary branches that bifurcate from $P_2$ of this type three extend to large $L$ (Figs. 17(b,c,f) of [18]) and hence differ from the branches responsible for the quasiperiodic structures described above. Of these branch (b) remains at small amplitude and takes the form of a flowing, slightly undulated film. In contrast, branches (c,f) develop into well-defined bound states of holes. Thiele and Knobloch did not compute the stability of these secondary branches although naive eigenvalue counting suggests that both of these branches can acquire stability at the final saddle-node bifurcation at which these branches turn towards large $L$. Moreover, phase space projections indicate that the bound
Figure 8. Isovalues of $h(x,t)$ in the $(x,t)$ plane with $0 \leq x \leq 16$ (horizontal axis) and $t \in [1800,2000]$ (vertical axis) for Ma = 3.6, 3.7, 3.86 and 3.9, showing the details of the transition from the small amplitude hole state to the large amplitude hole state with increasing Ma. Parameters are $\alpha = 0.3$, $Bi = 0.5$. Resolution is $N = 128$.

states initially form via spatial period-doubling followed by, with increasing $L$, an approach to a homoclinic orbit. In the limit of large $L$ this orbit starts from the flat state $h = 1$, executes a pair of oscillations, before returning to the flat state. This type of solution is quite distinct from the 2-pulse solutions that are associated with the Shilnikov bifurcation. In the latter the solution undergoes an excursion from the flat state, returns to the vicinity of the flat state, before undergoing a second excursion from the flat state and then returning to it. We believe that this new type of homoclinic orbit is associated with the $n : 1$ resonance, cf. [9], since the period-doubled states are triggered by the excitation of an $n = 1$ perturbation of the primary $n = 2$ periodic oscillation. Evidently, the same mechanism is involved in generating bound states involving $n$ holes ($n > 2$). However, since the state $h = 1$ is unstable at these parameter values these solutions must lose stability with increasing $L$ (i.e., as the holes move further apart), resulting in a succession of transitions to bound states of ever increasing complexity, in qualitative correspondence with the time-stepping results presented here. The details of these transitions are not fully understood but should also involve quasiperiodic structures since the states involved are still periodic, albeit with a larger period. Similar transitions in systems with reflection symmetry typically involve intervals of chaos [3, 6]. However, there is a further possibility arising from the frequent return of the solution to the special state $h = 1$. This state plays an essential role in the states shown in Figs. 2 and 6 despite the lack of resemblance between our hole states and the solitary elevation waves computed for a related problem in Ref. [8]. A clear example is shown in Fig. 10. A possible mechanism generating complex near-homoclinic structures arising from the close approach to the $h = 1$ state in a system with the same symmetry properties as the present system is identified in [17] and may play a role here as well.

Bound states of $n$ pulses have been observed in a reaction-diffusion system involving both two [11] and three species [20]. Both systems are excitable in the sense that finite amplitude perturbations produce a finite amplitude traveling wave (TW) resembling a spatially periodic train of steadily moving pulses. Since these systems are invariant under spatial reflection the direction of propagation is selected by the initial condition. In the regime of interest these wavetrains are unstable to perturbations with wavelength $n\lambda$, where $\lambda$ is the TW wavelength and $n > 1$, that evolve on a timescale that is long compared to the time on which the initial structure forms. In systems in which the pattern-forming instability is supercritical these Eckhaus instabilities usually lead to wavelength change mediated by the formation of defects at which the wave amplitude drops to zero. However, in excitable systems the periodic pulse train coexists with a stable homogeneous state. As a result the evolution of the Eckhaus instability conserves phase and the instability leads to clumping, i.e., the formation of a bound state of $n$
Figure 9. Time evolution of \( h(\cdot, t) \) at a fixed point in \([0, L]\) with \( L = 16 \) and \( \text{Ma} = 3.6, 3.7, 3.86 \) and \( 3.9 \), showing the details of the transition from the small amplitude hole state to the large amplitude hole state with increasing \( \text{Ma} \). Parameters are \( \alpha = 0.3, B_i = 0.5 \). Resolution is \( N = 128 \).

pulses whenever the domain period \( L = n\lambda \). Similar bound states occur in activator-inhibitor systems in the presence of advection [21]; in these systems the advection selects the direction of drift, much as the inclination selects the direction of drift of the holes in the present system. In [11] the formation of these structures is related to the nonmonotonic behavior of the dispersion relation \( c(d) \) that determines the speed \( c \) of a periodic train of pulses as a function of their separation \( d \).

It is tempting to try to interpret the results of the present work in terms of a similar picture. However, the bound states shown in Fig. 1 and elsewhere are not found in the excitable regime for our system, i.e., the regime in which the primary instability is subcritical. On the contrary, the bound states in Fig. 1 are found for values of \( L \) much beyond the threshold for primary instability; moreover, computations in Ref. [18] do not reveal evidence for significant nonmonotonicity in the dispersion relation along the relevant secondary branches that bifurcate from \( P_n \). The possibility remains therefore that the bound states of holes found here are created by a distinct mechanism, perhaps one that involves slow drift along a branch of traveling waves followed by a fast jump to the homogeneous state \( h = 1 \) and back again. This mechanism is characteristic of slow-
Figure 10. Time evolution of $h(\cdot, t)$ at a fixed point in $[0, L)$. Parameters are $\alpha = 0.135$, $Ma = 3.5$, $Bi = 0.5$ and $L = 42.5$. Observe also the appearance of structure arising from the close approach of the solution to $h = 0$; for $L > 42.5$ we observe rupture of the film (negative values of $h$), at least with the resolution $N = 256$ used here.

fast systems, and is also capable of generating finite groups of pulses; these may be arranged periodically or otherwise, much as found in [6].

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References
[1] Boos W and Thess A 1999 Phys. Fluids 11 1484
[2] Burelbach J P, Bankoff S G and Davis S H 1988 J. Fluid Mech. 195 463
[3] Coombes S and Osbaldestin A H 2000 Phys. Rev. E 62 4057
[4] Craster R V and Matar O K 2009 Rev. Mod. Phys. 81 1131
[5] Liu J, Paul J D and Gollub J P 1993 J. Fluid Mech. 250 69
[6] Higuera M, Knobloch E and Vega J M 2005 Physica D 201 83
[7] Joo S W, Davis S H and Bankoff S G 1991 J. Fluid Mech. 230 117
[8] Kalliadasis S, Demekhin E A, Ruyer-Quil C and Velarde M G 2003 J. Fluid Mech. 492 303
[9] Kent P and Elgin J 1992 Nonlinearity 5 899
[10] Krishnamoorthy S, Ramaswamy B and Joo S W 1995 Phys. Fluids 7 2291
[11] Or-Guil M, Kevrekidis I G and Bär M 2000 Physica D 135 154
[12] Oron A, Davis S H and Bankoff S G 1997 Rev. Mod. Phys. 69 931
[13] Oron A and Rosenau P 1992 J. Phys. II France 2 131
[14] Ramaswamy B, Chippada S and Joo S W 1996 J. Fluid Mech. 325 163
[15] Ruyer-Quil C and Manneville P 2000 Eur. Phys. J. B 15 357
[16] Ruyer-Quil C, Scheil B, Kalliadasis S, Velarde M G and Zeytounian R Kh 2005 J. Fluid Mech. 538 199
[17] Rodriguez J. D. and Schell M 1990 Phys. Lett. A 146 25
[18] Thiele U and Knobloch E 2004 Physica D 190 213
[19] Thiele U and Knobloch E 2006 Phys. Rev. Lett. 97 204501
[20] Yochelis A, Knobloch E, Xie Y, Qu Z and Garfinkel A 2008 Europhys. Lett. 83 64005
[21] Yochelis A and Sheintuch M 2009 Phys. Chem. Chem. Phys. 11 9210