Differential Galois obstructions for non-commutative integrability

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Abstract

We show that if a holomorphic Hamiltonian system is holomorphically integrable
in the non-commutative sense in a neighbourhood of a non-equilibrium phase curve
which is located at a regular level of the first integrals, then the identity component
of the differential Galois group of the variational equations along the phase curve
is Abelian. Thus necessary conditions for the commutative and non-commutative
integrability given by the differential Galois approach are the same.

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1 Introduction

One of the main problems of Hamiltonian mechanics is to decide whether a given system is
integrable or not. There exist only few methods which give effective and rigorous necessary
conditions for integrability. One of them arose from a great idea of S. N. Kovalevskaya.
It relates integrability with the properties of solutions as a function of the complex time.
Too complicated branching of solutions is not compatible with the integrability. The idea
of Kovalevskaya was investigated for almost a century by leading mathematicians and
physicists of the epoch. Finally, at the beginning of eighties of the previous century, the
problem of mysterious relations between the integrability and branching of solutions was
explained by S. L. Ziglin in his elegant and powerfull theory formulated in [12]. In the
Ziglin theory the integrability is connected with the properties of the monodromy group
of the variational equations along a particular non-equilibrium solution of the considered
system. If the system is integrable, then the monodromy group cannot be too ‘big’. The
Ziglin theory was successfully applied to study the integrability of many Hamiltonian systems, see e.g. [3, 4, 5] and references therein. In the middle of nineties of the previous century, thanks to the works of A. Baider, R. C. Churchill, J. J. Morales, J.-P. Ramis, D. L. Rod, C. Simo and M. F. Singer the Ziglin theory was considerably developed, see [3, 6, 7, 8] and references therein. The main idea of this extension is to use the differential Galois group of the variational equations in order to obtain the necessary conditions for the integrability. We describe this approach shortly. For a more detailed description see the cited papers.

Let $(M, \omega)$ be a $2n$-dimensional complex connected analytic symplectic manifold. The symplectic form $\omega$ induces the Poisson bracket $\{\cdot, \cdot\}$ and the corresponding Poisson tensor $\Lambda$ on $M$. For a meromorphic function $H : M \to \mathbb{C}$, we denote by $X_H$ the Hamiltonian vector field generated by $H$. We have the well known identities

$$\{F, H\} = \omega(X_F, X_H) = \Lambda(dF, dH),$$

for arbitrary meromorphic functions $F$ and $H$ on $M$. Consider the Hamiltonian equations

$$\frac{d}{dt}x = X_H(x), \quad x \in M, \quad t \in \mathbb{C}.$$  

(1.2)

A meromorphic function $F : M \to \mathbb{C}$ is a first integral of system (1.2) iff $\{H, F\} = 0$. The Poisson bracket $\{F_1, F_2\}$ of two first integrals $F_1$ and $F_2$ of (1.2) is a first integral. Hence the set of all meromorphic first integrals of (1.2) is a Lie algebra with respect to the Poisson bracket.

A Hamiltonian system (1.2) is meromorphically integrable in the Liouville sense iff it admits $n$ meromorphic first integrals which are functionally independent on an open and dense subset of $M$.

Let $\varphi(t)$ be a non-equilibrium particular solution of (1.2). Here we consider $\varphi$ as a full analytic function, i.e., $\varphi$ is a maximal analytic continuation of a local solution. It defines a Riemann surface $\Gamma$ with $t$ as a local coordinate. The variational equations along $\Gamma$ have the form

$$\frac{d}{dt}y = A(t)y, \quad A(t) = \frac{\partial X_H}{\partial x}(\varphi(t)), \quad y \in T_\Gamma M.$$  

(1.3)

The coefficients of this equation belong to the field $M(\Gamma)$ of functions meromorphic on $\Gamma$. This is a differential field with the derivative with respect to the time as the derivation. Its subfield of constants is $\mathbb{C}$. Let $\mathcal{G}$ denote the differential Galois group of system (1.3). It is an algebraic subgroup of $\text{Sp}(2n, \mathbb{C})$.

The following theorem states that the existence of a commutative $n$-dimensional Lie algebra of functionally independent first integrals implies the commutativity of the identity component of the differential Galois group of the variational equations. It was formulated by J. J. Morales and J. P. Ramis in [6, 8], see also [9, 10].

**Theorem 1.1.** If Hamiltonian system (1.2) possesses $n$ commuting functionally independent meromorphic first integrals in a connected neighbourhood of a non-equilibrium phase curve $\Gamma$, then the identity component of the differential Galois group of the variational equations along $\Gamma$ is Abelian.
Let us note that the first integrals are functionally independent in the neighbourhood of $\Gamma$ but not necessarily independent on $\Gamma$ itself.

The above theorem was successfully applied for proving the non-integrability of many systems, see e.g. [7, 11, 12, 13, 14, 15, 16]. Its strength lies in two facts. The differential Galois group is bigger than the monodromy group. Moreover, it is easier to determine the differential Galois group of given equations than their monodromy group.

In many cases Hamiltonian system (1.2) on a $2n$ dimensional manifold admits more than $n$ functionally independent but non-commuting first integrals, see e.g. [17, 18, 19].

Under certain conditions integration of such systems can be reduced to quadratures. Such examples gave motivation for introducing the notion of non-commutative integrability, see e.g. [20, 21, 22, 23] and references therein. Let us remind shortly the idea of the non-commutative integrability. As we work with complex Hamiltonian systems, we adopt basic definitions from [22] to this context.

Let $F_1 : M \to \mathbb{C}$ for $1 \leq i \leq k$, be functions holomorphic in a neighbourhood of a point $x \in M$. They define a natural map

$$F : U \to \mathbb{C}^k, \quad U \ni x \to F(x) = (F_1(x), \ldots, F_k(x)) \in \mathbb{C}^k. \quad (1.4)$$

In the cotangent space $T^*_x M$ we distinguish a linear subspace $F_x$ spanned by differentials $dF_i(x) : T_x M \to \mathbb{C}$, i.e.

$$F_x := \text{span}_{\mathbb{C}}\{dF_1(x), \ldots, dF_k(x)\}. \quad (1.5)$$

The Poisson tensor $\Lambda_x : \Lambda(x)$ at point $x$ is a bilinear form on $T^*_x M$ which induces a linear map $\Lambda^x_k : T^*_x M \to T_x M$, defined by

$$\Lambda_x(u, v) = \langle u, \Lambda^x_k(v) \rangle = u \cdot \Lambda^x_k(v) := u \left(\Lambda^x_k(v)\right), \quad \text{for all } u, v \in T^*_x M. \quad (1.6)$$

To simplify the notation we write $\Lambda_{i|F_x}$ and $\Lambda^x_k|F_x$ to denote the restriction of $\Lambda_x$ and $\Lambda^x_k$ to $F_x$, respectively.

**Definition 1.** We say that a holomorphic Hamiltonian system (1.2) is holomorphically integrable in the non-commutative sense iff there exist $k$ holomorphic first integrals $F_1, \ldots, F_k$ which are functionally independent on an open and dense subset $U$ of $M$, and satisfy

$$\{F_i, F_j\}(x) = a_{ij}(F_1(x), \ldots, F_k(x)), \quad \text{for } x \in U, \quad \text{and } 1 \leq i, j \leq k. \quad (1.7)$$

where $a_{ij} : \mathbb{C}^k \supset F^{-1}(U) \to \mathbb{C}$ are holomorphic functions; and, moreover, condition

$$\dim_{\mathbb{C}} F_x + \dim_{\mathbb{C}} \ker \Lambda^x_k|F_x = 2n \quad \text{for } x \in U. \quad (1.8)$$

is fulfilled.

Let us make some remarks about the above definition.
Remark 1.1 As it is assumed that functions $F_1, \ldots, F_k$ are functionally independent at $x \in U$ we have $\dim_C F_x = k$. Hence differentials $dF_1(x), \ldots, dF_k(x)$ form a linear base in $T_x^* M$. Let $v^1_x, \ldots, v^k_x \in T_x^* M$ be the dual base. The matrix $A = [A_{ij}]$ of $\Lambda^x | F_x$ is given by

$$A_{ij} := \Lambda^x | F_x (dF_i(x), dF_j(x)) = \{F_i, F_j\}(x) = a_{ij}(F(x)) \quad (1.9)$$

Let $B = [B_{ij}]$ be the matrix of $\Lambda^x | F_x$ in the chosen bases, i.e.,

$$\Lambda^x | F_x (dF_j(x)) := \sum_{l=1}^k B_{lj} v^l_x. \quad (1.10)$$

Then, from (1.9) and (1.10) we obtain

$$A_{ij} = \Lambda^x | F_x (dF_i(x), dF_j(x)) = \langle dF_i(x), \Lambda^x | F_x (dF_j(x)) \rangle = \sum_{l=1}^k B_{lj} (dF_i(x), v^l_x) = B_{ij}. \quad (1.11)$$

This shows that $A = B = [a_{ij}(F(x))]$. Hence

$$\dim_C \ker \Lambda^x | F_x = k - \text{rank}[a_{ij}(F(x))]. \quad (1.12)$$

Remark 1.2 One can consider also the non-commutative integrability with meromorphic first integrals. We restrict ourselves to holomorphic first integrals to avoid technical difficulties in the proof of our main results.

Remark 1.3 The definition of the non-commutative integrability for real Hamiltonian systems can be obtained from the definition given above if we change $\mathbb{C}$ to $\mathbb{R}$. It should be mentioned, however, that usually for real Hamiltonian systems it is assumed that the first integrals are of class $C^\infty$.

Remark 1.4 If a real Hamiltonian system is integrable in the non-commutative sense with smooth first integrals, then the compact connected common levels of these first integrals are tori of dimension smaller than $n$.

The well known conjecture of A. T. Fomenko and A. S. Mishchenko says that if a system is integrable in the non-commutative sense with the first integrals of a given class, then it is integrable in the usual Liouville sense with the first integrals belonging to the same class of functions. Recently S. T. Sadetov [24] proved this conjecture for a case when the first integrals span a finite dimensional Lie algebra, i.e. when

$$\{F_i, F_j\} = \sum_{m=1}^k c_{ij}^m F_m,$$

where $c_{ij}^m$ are constant. Furthermore in [22] it was shown that if a real Hamiltonian system is integrable in the non-commutative sense, then it is integrable in the Liouville sense with the first integrals $F_1, \ldots, F_n \in C^\infty(M)$. This important result shows, on the
one hand, that the non-commutative integrability ‘means’ almost the commutative one. But, on the other hand, still there remains an open problem. In fact, let us assume that the considered system is integrable in the non-commutative sense with the first integrals which are real holomorphic. Does it imply that it is integrable in the Liouville sense with real holomorphic first integrals? There is a conjecture in [22] that the answer to this question is affirmative.

The purpose of this paper is to give necessary conditions for the non-commutative integrability of complex Hamiltonian systems. More precisely, our aim is to give necessary conditions in the spirit of the Morales-Ramis Theorem 1.1. The above mentioned results relating the non-commutative and commutative integrability suggest that these necessary conditions should be the same as for the commutative integrability. Our main result shows that in fact they are really the same.

**Theorem 1.2.** If Hamiltonian system (1.2) is integrable in the non-commutative sense with first integrals holomorphic in a connected neighborhood of a non-equilibrium phase curve \( \Gamma \), then the identity component of the differential Galois group of the variational equations along \( \Gamma \) is Abelian.

The rest of this paper contains a proof of the above theorem.

## 2 Proof of Theorem 1.2

Our proof is based on two facts. The first of them is the Lie-Cartan theorem (see page 126 in [25] and Section 3.2.2 of [26]) which we formulate in a form adequate for complex Hamiltonian systems.

Let \((M, \omega)\) be a 2n-dimensional complex connected analytic symplectic manifold. Assume that \(F_1, \ldots, F_k\) are functionally independent holomorphic functions defined in a non-empty connected open subset \(W \subset M\), and

\[
F : W \rightarrow \mathbb{C}^k, \quad W \ni x \mapsto (F_1(x), \ldots, F_k(x)) \in \mathbb{C}^k,
\]

is the momentum map. Moreover, we assume that there exist holomorphic functions \(a_{ij} : \mathbb{C}^k \rightarrow \mathbb{C}\) such that

\[
\{F_i, F_j\} = a_{ij} \circ F, \quad \text{i.e.}
\]

\[
\{F_i, F_j\}(x) = a_{ij}(F_1(x), \ldots, F_k(x)), \quad \text{for } i, j = 1, \ldots, k.
\]

**Theorem 2.1** (Lie-Cartan). Let \(c = F(p), \ p \in W\) and assume that the rank of matrix \([a_{ij}]\) is constant in a neighbourhood of \(c\). Then there exists a neighbourhood \(U \subset \mathbb{C}^k\) of point \(c\) and \(k\) functionally independent holomorphic functions \(g_i : U \rightarrow \mathbb{C}\), such that functions

\[
G_i = g_i \circ F : M \supset F^{-1}(U) \rightarrow \mathbb{C}, \quad i = 1, \ldots, k,
\]

satisfy

\[
\{G_{2i-1}, G_{2i}\} = 1 \quad \text{for } i = 1, \ldots, r,
\]

where \(2r\) is the rank of \(k \times k\) matrix \([a_{ij}]\). Moreover, all remaining Poisson brackets of functions \(G_i\) vanish.
Remark 2.1 To have an idea how to prove the above theorem, let us restrict $W$ in such a way that $\text{rank}[a_{ij}(y)] = 2r$ for all $y \in P = F(W)$. Then $P$ is an analytic submanifold of $\mathbb{C}^k$. We can define a Poisson bracket $\{\cdot, \cdot\}_A$ on $P$ demanding

$$\{y_i, y_j\}_A := a_{ij}(y_1, \ldots, y_k), \quad \text{for} \quad 1 \leq i, j \leq k.$$  \hspace{1cm} (2.2)

Thus, $P$ is a Poisson manifold. For a point $c \in P$ we can apply the local structure theorem for a Poisson bracket, see e.g., p. 348 in [27]. From this theorem it follows that there exists a neighbourhood $U$ of $c$, and holomorphic functions $g_i : U \rightarrow \mathbb{C}$, $1 \leq i \leq k$, such that

$$\{g_{2i-1}, g_{2i}\}_A = 1 \quad \text{for} \quad i = 1, \ldots, r,$$

and all the remaining brackets vanish. Now, for functions (2.1) we have

$$\{G_i, G_j\}(x) = a_{ij}(g_1(F(x)), \ldots, g_k(F(x))) = \{g_i, g_j\}_A(F(x)) \quad \text{for} \quad x \in F^{-1}(U). \hspace{1cm} (2.3)$$

Remark 2.2 Let point $c = (c_1, \ldots, c_k) \in \mathbb{C}^k$ and functions $G_i$, $1 \leq i \leq k$ be as in the above theorem. Moreover, we assume that $k = n + r$. The common level

$$\Sigma_c := \{x \in W \mid G_i(x) = c_i, \quad 1 \leq i \leq k\},$$

is an analytic submanifold of $M$ and $\dim \Sigma_c = 2n - k = n - r$. The tangent space $T_x \Sigma_c$ to this manifold at point $x$ is the intersection of kernels of differentials of all functions $G_i$ at point $x$, i.e.

$$T_x \Sigma_c = \bigcap_{i=1}^{k} \{v \in T_x M \mid dG_i(x) \cdot v = 0\} = \bigcap_{i=1}^{k} \ker dG_i(x). \hspace{1cm} (2.4)$$

Note that $k - 2r = n - r$ functions $G_{2r+1}, \ldots, G_k$ commute with all other $G_1, \ldots, G_k$. Now, denote by $X_i$ the Hamiltonian vector fields generated by $G_i$ for $1 \leq i \leq k$. As functions $G_i$ are functionally independent, these vector fields are linearly independent. We have

$$0 = \{G_i, G_j\}(x) = dG_i(x) \cdot X_j(x) \quad \text{for} \quad 1 \leq i \leq k, \quad \text{and} \quad 2r < j \leq k.$$  \hspace{1cm} (2.5)

Hence $X_j(x) \in T_x \Sigma_c$ for $2r < j \leq k$, so these vector fields form a linear base in $T_x \Sigma_c$. As

$$0 = \{G_i, G_j\}(x) = \omega_x(X_i, X_j) \quad \text{for} \quad 2r < i, j \leq k,$$

$T_x \Sigma_c$ is an isotropic subspace of $T_x M$.

To formulate and prove the second fact we consider the symplectic Lie algebra $\text{sp}(V, \Omega)$ where $V$ is a complex symplectic vector space of dimension $2n$ with a symplectic form $\Omega$. By definition, elements of $\text{sp}(V, \Omega)$ are endomorphisms $A : V \rightarrow V$ such that $\Omega(x, Ay) = -\Omega(Ax, y)$ for all $x, y \in V$. For an element $A \in \text{sp}(V, \Omega)$ we define a function

$$H_A : V \rightarrow \mathbb{C}, \quad V \ni x \mapsto H_A(x) = \frac{1}{2} \Omega(Ax, x),$$

and a vector field

$$V \ni x \mapsto (x, v_A(x)) \in TV \quad \text{where} \quad v_A(x) = Ax.$$  

For a holomorphic function $F : V \rightarrow \mathbb{C}$ we denote by $X_F$ the corresponding Hamiltonian vector field.
**Proposition 2.1.** Let $A \in \text{sp}(V, \Omega)$. Then $v_A = X_{H_A}$.

For a proof see Proposition 2.5.1 on page 77 in [27].

**Proposition 2.2.** Let $A, B \in \text{sp}(V, \Omega)$. Then

$$A \circ B = B \circ A \iff \{H_A, H_B\} = 0 \iff [v_A, v_B] = 0.$$  \hspace{1cm} (2.7)

An easy proof of the above statement we leave to the reader. As a matter of fact, one can show that Lie algebra $\text{sp}(V, \Omega)$ is isomorphic to the Lie algebra of linear Hamiltonian vector fields with the commutator of vector fields as the Lie bracket, and it is also isomorphic to the Lie algebra of quadratic homogeneous Hamiltonian functions with the Poisson bracket as the Lie bracket, see Section 3.4 in [6].

Let $U$ be a non-empty open subset $V$. Functions holomorphic on $U$ form a ring denoted by $\mathcal{O}(U)$.

**Definition 2.** We say that Lie algebra $g \subset \text{sp}(V, \Omega)$ preserves a function $F \in \mathcal{O}(U)$ iff $v_A[F] = \{F, H_A\} = 0$ for all $A \in g$.

In other words, $g$ preserves $F$ iff $F$ is a common first integral of all elements of $g$ considered as linear Hamiltonian vector fields.

The lemma below is a generalisation of the so called Key Lemma, see [9, Lemma III.3.7, p.72].

**Lemma 2.1.** Assume that $F_1, \ldots, F_{n+r} \in \mathcal{O}(U)$, $0 \leq r \leq n$ are functionally independent on $U$ and

$$\{F_i, F_j\} = 0 \quad \text{for} \quad 1 \leq i \leq n-r \quad \text{and} \quad 1 \leq j \leq n+r.$$  \hspace{1cm} (2.8)

If a Lie algebra $g \subset \text{sp}(V, \Omega)$ preserves all $F_j$ for $1 \leq j \leq n+r$, then $g$ is Abelian.

**Proof.** Let $X_j$ be the Hamiltonian vector field generated by $F_j$ for $1 \leq j \leq n+r$. From (2.8) we have

$$[X_j, X_i] = -X_{\{F_j, F_i\}} = 0 \quad \text{for} \quad 1 \leq i \leq n-r \quad \text{and} \quad 1 \leq j \leq n+r.$$  \hspace{1cm} (2.9)

Let $A \in g$ and $v_A$ be the corresponding linear Hamiltonian vector field with Hamiltonian function $H_A$. Then

$$[X_j, v_A] = -X_{\{F_j, H_A\}} = 0, \quad \text{for} \quad 1 \leq j \leq n+r,$$  \hspace{1cm} (2.10)

because $\{F_j, H_A\} = 0$ by assumption that $g$ preserves $F_j$ for $1 \leq j \leq n+r$.

As functions $F_1, \ldots, F_{n+r} \in \mathcal{O}(U)$ are functionally independent on $U$, their common level

$$\Sigma_c := \{x \in U \mid F_i(x) = c_i, \quad i = 1, \ldots, n+r\}, \quad c = (c_1, \ldots, c_k) \in \mathbb{C}^k,$$  \hspace{1cm} (2.11)

is a $(n-r)$-dimensional submanifold of $V$. The tangent space $T_x \Sigma_c$ to $\Sigma_c$ at point $x$ is the intersection

$$T_x \Sigma_c = \bigcap_{i=1}^{n+r} \{v \in T_x V \mid dF_i(x) \cdot v = 0\} = \bigcap_{i=1}^{n+r} \ker dF_i(x),$$  \hspace{1cm} (2.12)
Let us take another element \( \Sigma \). We show that \( X_1(x), \ldots, X_{n-r}(x) \in T_x \Sigma_c \). In fact, by (2.8), for arbitrary \( 1 \leq j \leq n + r \) and \( 1 \leq i \leq n - r \), equality

\[
dF_j(x) \cdot X_i(x) = \{F_j, F_i\}(x) = 0,
\]

holds. Vector fields \( X_1, \ldots, X_{n+r} \) are linearly independent at all points of \( \Sigma_c \). Hence we have \( n - r \) linearly independent vector fields \( X_1(x), \ldots, X_{n-r}(x) \in T_x \Sigma_c \). They form a linear base of \( T_x \Sigma_c \).

Now, let \( A \in \mathfrak{g} \). By assumption \( g \) preserves all functions \( F_j, 1 \leq j \leq n + r \), so vector field \( v_A(x) \) is tangent to \( \Sigma_c \) at \( x \). In fact, we have

\[
0 = v_A[F_j](x) = dF_j(x) \cdot v_A(x) = \{F_j, H_A\}(x), \quad \text{for} \quad 1 \leq j \leq n + r.
\]

Hence

\[
v_A(x) = \sum_{i=1}^{n-r} \lambda_i(x)X_i(x),
\]

because vector fields \( X_1(x), \ldots, X_{n-r}(x) \) span \( T_x \Sigma_c \). In the above formulae \( \lambda_i \) are holomorphic functions. Now, using (2.15) and (2.10), we obtain

\[
0 = [X_j, v_A](x) = [X_j, \sum_{i=1}^{n-r} \lambda_i X_i](x) = \sum_{i=1}^{n-r} (X_j[\lambda_i](x)X_i(x) + \lambda_i(x)[X_j, X_i](x)),
\]

for \( x \in \Sigma_c \). Taking into account (2.13) we achieve

\[
\sum_{i=1}^{n-r} X_j[\lambda_i](x)X_i(x) = 0 \quad \text{for} \quad 1 \leq j \leq n + r, \quad \text{and} \quad x \in \Sigma_c.
\]

This implies that \( X_j[\lambda_i](x) = d\lambda_i(x) \cdot X_j(x) = 0 \) for \( 1 \leq j \leq n + r \) and \( 1 \leq i \leq n - r \). In other words, functions \( \lambda_1, \ldots, \lambda_{n-r} \) are constant on \( \Sigma_c \). Summarising, if \( A \in \mathfrak{g} \), then on \( \Sigma_c \) we have

\[
v_A = \sum_{i=1}^{n-r} \lambda_i X_i, \quad \text{where} \quad \lambda_1, \ldots, \lambda_{n-r} \in \mathbb{C}.
\]

Let us take another element \( B \in \mathfrak{g} \). On \( \Sigma_c \) we can write

\[
v_B = \sum_{i=1}^{n-r} \gamma_i X_i, \quad \text{where} \quad \gamma_1, \ldots, \gamma_{n-r} \in \mathbb{C}.
\]

Let us calculate the Lie bracket of \( v_A \) and \( v_B \) on \( \Sigma_c \). We have

\[
[v_A, v_B] = \sum_{i=1}^{n-r} \lambda_i X_i \sum_{k=1}^{n-r} \gamma_k X_k = \sum_{i,k=1}^{n-r} \lambda_i \gamma_k [X_i, X_k] = 0,
\]

because \( X_1, \ldots, X_{n-r} \) are commuting vector fields. Thus, by Proposition 2.2, \( A \) and \( B \) commute. In this way we showed that \( g \) is Abelian. \( \square \)
Now we proceed to the proof of Theorem 1.2. Assume that the system is integrable in the non-commutative sense in an open connected neighbourhood $U$ of $\Gamma$ with the holomorphic independent first integrals $F_1, \ldots, F_k$, and let $2r = \text{rank}[a_{ij}(y)]$ where $y = F(\varphi(t))$. Condition (1.8) implies that $r = k - n$.

If $F$ is a holomorphic first integral of the considered system, then the variational equations (1.3) have a polynomial first integral $f$. It is the first non-vanishing term in the Taylor expansion of $F$ around $\Gamma$. More precisely, for $p \in \Gamma$ first integral $f$ is a polynomial function in $T_pM$ which depends holomorphically on $p$. Thus, the first integrals $F_1, \ldots, F_k$ give rise to first integrals $f_1, \ldots, f_k$ of the variational equations. By the Ziglin Lemma, see section 1.3 in [3] or Lemma III.1.7 in [9], we can assume that $f_1, \ldots, f_k$ are functionally independent. Moreover, let us fix $p \in \Gamma$ and let $V = T_pM$, $\Omega = \omega(p)$. Equations (1.7) imply that on $(V, \Omega)$ we have

$$\{f_i, f_j\}(x) = \alpha_{ij}(f_1(x), \ldots, f_k(x)) \quad \text{for} \quad 1 \leq i, j \leq k, \quad \text{and} \quad x \in V,$$

for certain polynomial functions $\alpha_{ij}$ in $k$ variables. To see this, it is enough to expand both sides of equations (1.7) around point $p$ and compare the lowest order terms. Note also that $\text{rank}[\alpha_{ij}] \leq 2r$.

From [8, 6] we know that the differential Galois group $\mathcal{G}$ of the variational equations (1.3) is a subgroup of $\text{Sp}(2n, \mathbb{C})$ and that $f_1, \ldots, f_k$ are invariants of this group, see also Lemmas III.2.3 and III.1.13 in [9].

Applying the Lie-Cartan theorem to polynomial functions $f_1, \ldots, f_k$ we obtain new independent holomorphic functions $h_1, \ldots, h_k$ that are also invariants of the differential Galois group of the variational equations (1.3). Thus, among them there is at least $n - r$ functions, let us say $h_1, \ldots, h_{n-r}$, which commute with all $h_i$ for $1 \leq i \leq k$.

Thus, the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ preserves all $h_1, \ldots, h_k$. Hence, by Lemma 2.1, $\mathfrak{g}$ is Abelian and this means that the identity component $\mathcal{G}^0$ of $\mathcal{G}$ is Abelian. This finishes our proof of Theorem 1.2.

**Remark 2.3** Let us note that using Ziglin lemma from the holomorphic first integrals $h_1, \ldots, h_k$ it is possible to construct $k$ algebraically independent polynomial invariants of the Galois group and this fact can be important in applications. Furthermore, from this it is possible to give an alternative proof of Lemma 2.1 in the lines of the Morales-Ramis original papers (using Proposition 8 of [8] or Proposition 3.4 of [9]) by assuming that the first integrals are polynomials (instead of holomorphic) and algebraically independent (instead of functionally independent, although in this case both concepts are equivalent [3]).

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