Critical behavior of two-dimensional random hopping fermions with $\pi$-flux

Takahiro Fukui

Institut für Theoretische Physik, Universität zu Köln, Zülpicher Strasse 77, 50937 Köln, Germany

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A two dimensional random hopping model with $N$-species and $\pi$-flux is studied. The field theory at the band center is shown to be in the universality class of $\text{GL}(4m,R)/O(4m)$ nonlinear sigma model. Vanishing beta function suggests delocalised states at the band center. Contrary to the similar universality class with broken time reversal symmetry, the present class is expected to have at least two fixed point. Large $N$-systems are shown to be in the weak-coupling fixed point, which is characterized by divergent density of state, while small $N$ systems may be in the strong-coupling fixed point.

I. INTRODUCTION

It is widely accepted that the metallic states are unstable in two dimensional (2D) disordered systems [1]. However, two typical exceptions have been known for a couple of decades. One is the system with strong spin-orbit coupling [2] and the other is the integer quantum Hall (IQH) system [3,4]. Especially, IQH transition has attracted much interest, because, as shown by Pruisken et al. [5], what is responsible for the delocalization is a topological term in the usual nonlinear sigma model (NLSM) description. Though his arguments clarified nonperturbative aspects of the delocalization phenomena in the IQH systems, it was not generally possible to calculate critical properties of the IQH transitions. In order to calculate them explicitly, a model of Dirac fermions with various types of disorder was studied extensively [6–10]. This model also intimate relationship with a statistical model [11–13] and with more physical systems such as $t$-$J$ model [14] or $d$-wave superconductor [15–17]. However, many problems remain to be explored, since generic IQH transition is in the strong-coupling limit [5], though some exact results can be obtained for models with rather limited realization of disorder, e.g. the model only with random vector potentials [18].

On the other hand, Gade found a new universality class having a random critical point [18]. It is due to what is called particle-hole symmetry, which enlarges the symmetry of the universality class to the general linear group. The renormalization group equations were calculated for the NLSM specifying such universality class with [19] or without [18] time-reversal symmetry. Recently, it was recognized that the random flux model belongs to this universality class [20] or to its supersymmetric version [21].

Recently, Hatsugai et al. (HWKM) [22] proposed an interesting 2D model with the particle-hole symmetry. This model describes random hopping fermions on the square-lattice with $\pi$-flux. It should have close relationship with various kinds of systems mentioned above, since it is described by the Dirac fermions near the band center. Numerical calculations by HWKM strongly suggest a random critical point at the band center. This model is actually in sharp contrast to the similar model but with on-site disorder [23] which has only localized states. Especially, what is interesting is the power-law behavior of the density of state (DOS) such as $\rho \sim E^\nu$ with $\nu$ dependent on the strength of the disorder. Their calculations show that the exponent $\nu$ keeps positive even for rather large randomness. This behavior of the DOS seems to be in contrast to the pioneering work on the two-sublattice model by Oppermann and Wegner [24], where the two-particle Green function has singularity at the band center. Even in one dimension, this symmetry plays a role in the existence of the critical point [25,26], which brings about a singularity of the DOS at the band center. Therefore, it is interesting to study the 2D model proposed by HWKM, especially paying attention to the behavior of DOS.

In this paper, we study the 2D random hopping model with $N$-species. The model with one species corresponds to the HWKM model. In order to take $\pi$-flux into account effectively, we firstly derive the Dirac Hamiltonian and next apply the NLSM method developed by Gade [18]. We show that the model can be described by the NLSM on the symmetric space $\text{GL}(4m,R)/O(4m)$, which is just the class $BDI$ by Zirnbauer [24]. We conclude that the band center is a random critical point. However, contrary to the class with broken time-reversal symmetry, the present model should have, at least, two fixed point, which are characterized by the scaling property of the DOS. It is shown that large $N$ systems are in the weak-coupling fixed point, where the DOS diverges. A discontinuity is, therefore, expected at zero disorder. It is also conjectured that the $N = 1$ model proposed by HWKM, or more generally, small $N$ systems are in the strong-coupling fixed point.

In Section II, we introduce the model and derive its continuum limit. In Section III, we calculate the generating functional of the Green functions and discuss the symmetry of the model. In Section IV, we use the trick of auxiliary field and give the saddle point solution. Section V is devoted to the derivation of the nonlinear sigma model describing the Goldstone mode. Section VI deals with the renormalization group. Summary is given in.
II. MODEL

We present a slightly generalized model of HWKM, introducing an additional internal degrees of freedom. The tight-binding Hamiltonian is defined by

$$H = \sum_{\langle i,j \rangle} \sum_{\sigma} c_{i\sigma}^\dagger t_{i\sigma,j\sigma} c_{j\sigma} + h.c.,$$

(2.1)

where $j = (j_x, j_y)$ and $\sigma = 1, 2, \ldots, N$ denote, respectively, a site on the square lattice and a species of fermions. The summation with respect to sites is over nearest-neighbor pairs. The hopping matrix elements are defined by

$$H_\sigma = 2t \sum_{-\pi \leq ak_x < \pi, \atop 0 \leq ak_y < \pi} \left( c_{\sigma k}^\dagger, c_{\sigma k+\mathbf{e}_0} \right) \left( \begin{array}{cc} \cos(ak_y) & \cos(ak_x) \\ \cos(ak_x) & -\cos(ak_y) \end{array} \right) \left( \begin{array}{c} c_{\sigma k} \\ c_{\sigma k+\mathbf{e}_0} \end{array} \right),$$

(2.5)

where $a$ is the lattice constant and $k_0 = (0, \pi/a)$. Eigenvalues are $E = \pm 2t \sqrt{\cos^2 ak_x + \cos^2 ak_y}$, and hence there exist two Fermi points given by $k_F = (\pm \pi/2a, \pi/2a)$. In order to investigate the model near the band center, it is convenient to linearize the dispersion at the two Fermi points and take a continuum limit \cite{23, 22}. This may be valid up to a certain energy scale specified by a cut-off $\Lambda$. The lattice operator takes the following form in the continuum limit,

$$c_{j\sigma}/a \sim \psi_{\sigma}(x) = i^{j_x+2j_y} \psi_1(x) + i^{j_x-j_y} \psi_2(x) + i^{-j_x+j_y} \psi_3(x) + i^{-j_x-j_y} \psi_4(x),$$

(2.6)

where $x = aj$.

Now it is easy to derive the total continuum Hamiltonian including effects of the random hopping. The result is

$$H = \int d^2x \Psi^\dagger \mathcal{H}\Psi,$$

(2.7)

where $\Psi^\dagger = \left( \psi_1^\dagger, \ldots, \psi_4^\dagger \right)$ with $\psi_\sigma^\dagger = (\psi_{\sigma 1}, \psi_{\sigma 2}, \psi_{\sigma 3}, \psi_{\sigma 4})$, and

$$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_d,$$

$$(\mathcal{H}_p)_{\sigma\sigma'} = \varepsilon H_{\sigma\sigma'}; \quad \mathcal{H}_p = \alpha \omega i \partial_t, \quad \mathcal{H}_d = \frac{1}{2} \sum_{j=1}^4 \left( \left( V_{\sigma\sigma'} + V_{\sigma'\sigma} \right) \gamma_j + \left( U_{\sigma\sigma'} - U_{\sigma'\sigma} \right) i \alpha_j \right),$$

(2.8)

Here, $\mathcal{H}_p$ and $\mathcal{H}_d$ describe the pure and the disorder Hamiltonian, respectively, and $\mu = 1, 2$. $V_{\sigma\sigma'}$ and $U_{\sigma\sigma'}$ ($j = 1, \ldots, 4$) are independent random variables associated with the Fourier components of $\delta t_{x\sigma\sigma'}$ and $\delta t_{y\sigma\sigma'}$, assumed to obey the Gaussian distribution,

$$t_{j+x\sigma,j\sigma} = (-)^{j_y} t_{\delta t_{x\sigma\sigma}} + \delta t_{x\sigma\sigma},$$

(2.2)

$$t_{j+y\sigma,j\sigma} = t_{\delta t_{y\sigma\sigma}} + \delta t_{y\sigma\sigma},$$

(2.3)

with $\tilde{x} = (1, 0)$ and $\tilde{y} = (0, 1)$. Here, $\delta t_{x\sigma\sigma}$ and $\delta t_{y\sigma\sigma}$ are assumed to be independent random variables. The $N = 1$ model is just the one proposed by HWKM. As stressed in \cite{22}, the point is that the model has randomness in the hopping terms, which causes the particle-hole symmetry,

$$H \to -H \quad \text{for} \quad c_{j\sigma} \to (-)^{j_x+j_y} c_{j\sigma}.$$  

(2.4)

Accordingly, if there is an eigenstate of energy $e$, there also exists a conjugate pair with energy $-e$. The zero energy states is, however, special, because there is no partner for it.

Let us first consider the pure model without randomness. In this case, particles with different species do not couple each other. In the momentum space, the Hamiltonian for each species is

$$P[V_{\sigma\sigma'}] \propto \int \mathcal{D}[V] e^{-\frac{1}{\hbar} \int d^2x (V_{\sigma\sigma'})^2},$$

$$P[U_{\sigma\sigma'}] \propto \int \mathcal{D}[U] e^{-\frac{1}{\hbar} \int d^2x (U_{\sigma\sigma'})^2},$$

(2.9)

Matrices $\alpha$’s and $\gamma$’s are defined by

$$\alpha_1 = \sigma_1 \otimes \tau_3, \quad \alpha_2 = \sigma_3 \otimes 1, \quad \alpha_3 = 1 \otimes \tau_3, \quad \alpha_4 = \sigma_3 \otimes \tau_1,$$

$$\gamma_1 = 1 \otimes \tau_2, \quad \gamma_2 = \sigma_2 \otimes \tau_1, \quad \gamma_3 = \sigma_1 \otimes \tau_2, \quad \gamma_4 = \sigma_2 \otimes 1,$$

$$\gamma = \sigma_1 \otimes \tau_1.$$  

(2.10)

The particle-hole transformation Eq. (2.4) is expressed by $\gamma H \gamma = -H$. In the case of the on-site disorder \cite{23}, disorder potentials in the continuum limit commute each other and hence they are trivially diagonalized, while in the random hopping case, four $\gamma$’s are noncommutative. Moreover, there appear extra disorder terms.
described by matrices $\alpha$’s caused by multi-species. When $N = 1$, $\alpha$-terms disappear and this Hamiltonian reduces to the one derived by HWKM.

In order to apply the same techniques as Gade’s to the present model, it may be convenient to switch into the chiral basis, where $\gamma$ is diagonal. To this end, make a transformation, $\gamma \rightarrow U\gamma U$, where

$$ U = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 + i & 1 + i & 1 - i & 1 - i \\ 1 + i & -1 - i & 1 - i & 1 + i \\ -1 - i & 1 + i & 1 - i & 1 - i \\ -1 - i & -1 - i & 1 - i & 1 - i \end{pmatrix} .$$

(2.11)

The matrices defined in Eq. (2.10) are then converted to

$$\alpha_1 = \sigma_3 \otimes \tau_2, \quad \alpha_2 = \sigma_1 \otimes \tau_2, \quad \alpha_3 = 1 \otimes \tau_2, \quad \alpha_4 = -\sigma_2 \otimes \tau_1,$$

$$\gamma_1 = -\sigma_3 \otimes \tau_1, \quad \gamma_2 = -\sigma_1 \otimes \tau_1, \quad \gamma_3 = -1 \otimes \tau_1, \quad \gamma_4 = -\sigma_2 \otimes \tau_2,$$

$$\gamma = -1 \otimes \tau_3.$$

(2.12)

In the new basis, the Hamiltonian is real and symmetric due to the time-reversal symmetry. Moreover, the Hamiltonian has now only off-diagonal elements in $\tau$-space, as is expected for systems with the particle-hole symmetry. The $\tau$-space is thus associated with the $\pm$-sublattices in the lattice model [2.1].

III. REPLICA METHODS

In this section, we will firstly derive the connection of the Green functions between the lattice and the continuum theory [23], and secondly develop the continuum theory using the replica method.

A. Green function

Single-particle (e.g. retarded) Green function of the lattice system is given by

$$G_L(j\sigma, j\sigma'; z) = -i \int_0^\infty dt e^{iz}(0)|c_{j\sigma}(t)c^{\dagger}_{j\sigma}(0)|0\rangle, \quad (3.1)$$

where $z = E + i\omega$, $c_{j\sigma}(t) = e^{iHt}c_{j\sigma}e^{-iHt}$ and $|0\rangle$ denotes the fermion vacuum. This Green function is expressed in terms of the continuum fields as

$$G_L(j\sigma, j\sigma'; z) \sim e^{i\frac{z}{\pi}(x-x'+y-y')} \times \prod_{\alpha=1}^{4} e^{i\frac{\pi}{2}(m_{\alpha}(x-x') + m_{\alpha}(y-y'))} G_{\alpha\alpha}(x\sigma, x'\sigma'; z), \quad (3.2)$$

where constant vectors in oscillating phases factors are $\mathbf{n} = (0, 1, 0, 1)$ and $\mathbf{m} = (0, 0, 1, 1)$, and where $G_{\alpha\beta}(x\sigma, x'\sigma'; z) = -i \int_0^\infty dt e^{iz}(0)|\psi_{\alpha\sigma}(x, t)\psi^{\dagger}_{\alpha'\sigma'}(x', 0)|0\rangle = \langle x\sigma|z - \mathcal{H}^{-1}|x'\sigma'\beta\rangle$. In Eq. (3.2), terms breaking translational invariance, i.e., off-diagonal Green functions $G_{\alpha\beta}$ with $\alpha \neq \beta$ are neglected.

Since we are mainly interested in the DOS for the lattice system,

$$\rho(E) = \frac{1}{\pi} \lim_{\omega \rightarrow +i} \sum_{\sigma} \text{Im} G_L(j\sigma, j\sigma, E \pm i\omega), \quad (3.4)$$

let us define the following combinations of the Green functions for the continuum theory,

$$G^k(x) = \sum_{\sigma} \sum_{\alpha=1}^{4} G_{\alpha\alpha}(x\sigma, x\sigma, z_k),$$

$$K^{k\prime}(x, x') = \sum_{\sigma, \sigma', \alpha, \alpha'=1}^{4} \sum_{\sigma, \sigma', \alpha, \alpha'=1}^{4} G_{\alpha\alpha'}(x\sigma, x'\sigma', z_k) \times G_{\alpha'\alpha}(x'\sigma', x\sigma, z_{k'}), \quad (3.5)$$

where $k$ specify the retarded and advanced Green function. (See Eq. (3.8) for more precise definition.)

B. Generating functional of the Green functions

Since the Hamiltonian is real and symmetric in the chiral basis, we can apply the real bosonic replica method. Generating functional of the Green functions is introduced as

$$Z = \int \mathcal{D}[\Phi] e^{-S - S_s}, \quad (3.6)$$

where

$$S = \frac{1}{2} \int d^2x \sum_{a,k} s_k \langle \Phi^k_a(z_k - \mathcal{H})\Phi^k_a \rangle,$$

(3.7)

with $k = (p, q)$.

$$z_k = (-)^{q+1}E + (-)^{p+1}\omega = (-)^{q+1}E - s_k\omega,$$

$$s_k = -\text{sgn Im } z_k = (-)^{p}i. \quad (3.8)$$

Indices $a = 1, \ldots, m$ of scalar field $\Phi^k_a$ denotes the replica. In what follows, to simplify the notations, we sometimes use $\tilde{a} = (a, k)$ and $\tilde{a}' = (a', k')$, and hence, e.g., $\Phi^k_{\tilde{a}} = \Phi^k_a$. Fixing $\tilde{a}$, the field $\Phi_{\tilde{a}}$ is multi-component with respect to the species $\Phi^k_{\tilde{a}} \equiv (\Phi^1_{\tilde{a}}, \Phi^2_{\tilde{a}}, \ldots, \Phi^m_{\tilde{a}})$ and to Dirac indices $\Phi^k_{\tilde{a}} \equiv (\Phi^{\alpha}_{\tilde{a}})$ with $\alpha = 1, \ldots, 4$. $S_s$ is a source term which will be introduced momentarily. Green functions with quenched disorder is expressed by
\[ G^k(x) = \lim_{m \to 0} s_k \sum_{\sigma} \langle \Phi_{\sigma} a(x) \Phi_{\sigma} a(x) \rangle, \]

\[ K^{kk'}(x, x') = \lim_{m \to 0} s_k s_k' \sum_{\sigma, \sigma'} \sum_{a, a'} \langle \Phi_{\sigma'} a'(x) \Phi_{\sigma} a(x) \Phi_{\sigma'} a'(x) \Phi_{\sigma} a(x) \rangle, \quad (3.9) \]

where \( \langle \cdot, \cdot \rangle \) is the expectation value with respect to \( S \). Since the two components of the \( \tau \)-space in Eq. (2.12) play an important role in Gade's argument [8], let us introduce two fields \( \phi^+ \) and \( \phi^- \) explicitly, each of which has Dirac indices \( \alpha = 1, 2 \) of the \( \sigma \)-space, i.e., \( (\Phi_{\sigma} a, \Phi_{\sigma} b, \Phi_{\sigma} c, \Phi_{\sigma} d) = (\phi^+_{\sigma a}, \phi^-_{\sigma a}, \phi^+_{\sigma b}, \phi^-_{\sigma b}, \phi^+_{\sigma c}, \phi^-_{\sigma c}, \phi^+_{\sigma d}, \phi^-_{\sigma d}) \).

The two fields \( \phi^\pm \) reflect the particle-hole symmetry in the original lattice Hamiltonian, and are referred to as \( r = \pm \) fields. To be more concrete, let us write down the Lagrangian with respect to \( \phi^\pm \) fields,

\[ \mathcal{L} = \mathcal{L}_p + \mathcal{L}_b + \mathcal{L}_d + \mathcal{L}_s, \]

\[ \mathcal{L}_p + \mathcal{L}_b = \frac{1}{2} \sum_{a, a'} \sum_{\alpha} \left( \left( t \phi^+_{\sigma a} t \phi^-_{\sigma a} \right) \left( \begin{array}{l} \bar{z}_k \\
 z_k \end{array} \right) \left( \begin{array}{l} \phi^+_{\sigma a} \\
 \phi^-_{\sigma a} \end{array} \right) \right), \quad (3.10) \]

\[ \mathcal{L}_d = -\frac{1}{2} \sum_{a, a'} \sum_{\sigma, \sigma'} t \phi^+_{\sigma a} t \phi^-_{\sigma a} \left\{ -V^j_{\sigma a'} \left( \begin{array}{l} \phi^+_{\sigma a} \\
 \phi^-_{\sigma a} \end{array} \right) \left( \begin{array}{l} 0 \\
 -z_{\gamma j} \end{array} \right) \left( \begin{array}{l} \phi^+_{\sigma a} \\
 \phi^-_{\sigma a} \end{array} \right) \right\}, \quad (3.11) \]

\[ \mathcal{L}_s = -\sum_{a, a'} \sum_{\sigma, \sigma'} \left( t \phi^+_{\sigma a} t \phi^-_{\sigma a} \right) \left( \begin{array}{l} h_{a a'} \\
 0 \end{array} \right) \left( \begin{array}{l} \phi^+_{\sigma a'} \\
 0 \end{array} \right), \quad (3.12) \]

where

\[ h_0 = \tilde{\gamma}_0 \partial, \gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_1, \quad \gamma_3 = 1, \quad \gamma_4 = -i \sigma_2, \quad (3.13) \]

and \( \mathcal{L}_p, \mathcal{L}_b, \mathcal{L}_d, \) and \( \mathcal{L}_s \) describe pure-, breaking-, disorder-, and source-term, respectively. We set \( v = 1 \) without loss of generality. The parameters in the source terms are symmetric \( h_{a a'}(x) = h_{a a'}(x) \) and are, more explicitly, given by \( h_{a a'} \).

Let us also divide the Green functions in Eq. (3.5) into two kinds of contributions from \( \phi^\pm \), and average them over disorder (2.3).

\[ G^k(x) = \sum_{r=\pm} G^k_r(x), \]

\[ K^{kk'}(x, x') = \sum_{r, r'=\pm} K^{kk'}_{rr'}(x, x'), \quad (3.14) \]

where

\[ G^k_r(x) = \lim_{m \to 0} s_k \sum_{\sigma} \langle \phi^+_{\sigma a}(x) \phi^-_{\sigma a}(x) \rangle, \]

\[ K^{kk'}_{rr'}(x, x') = \lim_{m \to 0} \frac{1}{4} \langle \phi^+_{\sigma a}(x) \phi^-_{\sigma a}(x) \phi^+_{\sigma a}(x) \phi^-_{\sigma a}(x) \rangle. \]

Overbars denote ensemble-average, defined by

\[ \langle O \rangle = \lim_{m \to 0} \int D[V] D[U] D[\Phi] O \]

\[ e^{-s \frac{1}{4} \int d^2 x \sum_{\sigma, \sigma'} \sum_1 \left[ (V_{\sigma a}^+)^2 + (V_{\sigma a}^-)^2 \right]}, \quad (3.16) \]

and hence, \( Z \) in Eq. (3.15) should be considered as including the disorder term in Eq. (3.16). Integration over \( V \) converts \( \mathcal{L}_d \) into the form of four-point interactions,

\[ \mathcal{L}_d = -\frac{g_0}{2N} \sum_{a, a'} \sum_{\sigma, \sigma'} t \phi^+_{\sigma a} t \phi^-_{\sigma a} \sum_{\sigma, \sigma'} t \phi^+_{\sigma a'} t \phi^-_{\sigma a'} \phi^+_{\sigma a'} \phi^-_{\sigma a'}. \]

(3.17)

As Gade discussed [8], this Lagrangian has unique symmetry property, which is caused by the particle-hole symmetry of the original model. Namely, it is easily verified from Eqs. (3.10), (3.12) and (3.17) that when \( z_k = 0 \), the Lagrangian is invariant under the transformation

\[ \phi^+_{\sigma a} \rightarrow g \phi^+_{\sigma a}, \quad \phi^-_{\sigma a} \rightarrow -s \partial g^{-1} \phi^-_{\sigma a}, \quad (3.18) \]

where \( g = g_{k k'} \in \text{GL}(4m, R) \), and \( s = s_{k k'} \) is a matrix defined by \( s = \delta_{a a'} s_k s_{k'} \). Since the symmetry-breaking terms are written as

\[ \mathcal{L}_b = \mathcal{L}_w + \mathcal{L}_e, \]

\[ \mathcal{L}_w = \frac{\omega}{2} \sum_{a} \sum_{\alpha=1} \left( t \phi^+_{\sigma a} \phi^+_{\sigma a} + t \phi^-_{\sigma a} \phi^-_{\sigma a} \right), \]

\[ \mathcal{L}_e = -\frac{i \tilde{E}}{2} \sum_{\sigma, \alpha=1} \left( t \phi^+_{\sigma a} \lambda \phi^+_{\sigma a} + t \phi^-_{\sigma a} \lambda \phi^-_{\sigma a} \right), \]

(3.19)

with \( \lambda = \delta_{a a'} (-)^{a + q} \delta_{k k'} \), GL(4m, R) is broken by \( \mathcal{L}_w \) up to \( O(4m) \). For non-zero energy, \( \mathcal{L}_E \) implies that the symmetry group is \( O(2m, 2m) \), which is broken to \( O(2m) \times O(2m) \) by the \( \mathcal{L}_w \). This suggests that all states with nonzero energies are localized.
IV. AUXILIARY MATRIX FIELD THEORY

A. Auxiliary fields

In order to apply the well-known trick of auxiliary fields, let us first introduce two kinds of fields,

\[ \rho_{aa'}^{kk'} = \frac{1}{2} \sum_{\sigma} \sum_{a=1}^{2} (\phi_{\sigma a a'}^{+} \phi_{\sigma a a'}^{-} + \phi_{\sigma a a'}^{-} \phi_{\sigma a a'}^{+}) , \]

\[ \sigma_{aa'}^{kk'} = \frac{i}{2} \sum_{\sigma} \sum_{a=1}^{2} (\phi_{\sigma a a'}^{+} \phi_{\sigma a a'}^{-} - \phi_{\sigma a a'}^{-} \phi_{\sigma a a'}^{+}) . \]  

(4.1)

By the use of these fields, we can express

\[ \mathcal{L}_\omega = -\omega \text{tr} s s^\dagger \rho s^\dagger , \]

\[ \mathcal{L}_s = -\text{tr} (h s^\dagger \rho s^\dagger - i\hbar s^\dagger \sigma s^\dagger) , \]

\[ \mathcal{L}_d = -\frac{g_0}{2N} \text{tr} \left[ (s^\dagger \rho s^\dagger)^2 + (s^\dagger \sigma s^\dagger)^2 \right] , \]

(4.2)

where \( Q \) and \( P \) are real and symmetric matrices, we have the following expression for the generating functional depending on auxiliary fields \( Q \) and \( P \) as well as \( \phi^\pm \),

\[ Z = \int \mathcal{D}[\phi^+] \mathcal{D}[\phi^-] \mathcal{D}[Q] \mathcal{D}[P] e^{-\int d^2x \mathcal{L}(\phi^\pm, Q, P)}, \]

\[ \mathcal{L} = \frac{N}{2g_0} \left[ \text{tr} (Q^2 + P^2) - \omega \text{tr} s(Q + iP) - \omega \text{tr} s(Q - iP) \right] + \mathcal{L}_p + \mathcal{L}_E - \frac{N}{2} \text{tr} \left( Q s^\dagger \rho s^\dagger + P s^\dagger \sigma s^\dagger \right) \]

\[ -\frac{N}{g_0} \left[ \text{tr} h^+(Q - iP) + \text{tr} h^-(Q + iP) - \omega \text{tr} s(h^+ + h^-) - 2 \text{tr} h^+ h^- \right] , \]

(4.5)

where we neglect \( \omega^2 \) term, which vanishes after the replica limit \( m \to 0 \). Integrating out the fields \( \phi^\pm \), we end up with the following action,

\[ Z = \int \mathcal{D}[Q] \mathcal{D}[P] e^{-S_a}, \]

\[ S_a = \int d^2x \frac{N}{2g_0} \left[ \text{tr} (Q^2 + P^2) - \omega \text{tr} s(Q + iP) - \omega \text{tr} s(Q - iP) \right] + \frac{N}{2} \text{tr} \text{Ln} C \]

\[ -\int d^2x \frac{N}{g_0} \left[ \text{tr} h^+(Q - iP) + \text{tr} h^-(Q + iP) - \omega \text{tr} s(h^+ + h^-) - 2 \text{tr} h^+ h^- \right] , \]

(4.6)

where \( \text{Tr} \) means the trace in \( x \)-space as well as \( \text{tr} \) and the trace in \( \alpha \)-space, and \( C \) is defined by

\[ C_{\alpha a' \bar{a} a''} = \left( \begin{array}{cc} E_k \delta_{aa'} \delta_{kk'} - (Q_{aa'}^{kk'} + iP_{aa'}^{kk'}) & \delta_{aa'} \\ h_{0aa'} \delta_{aa'} \delta_{kk'} & -h_{0aa'} \delta_{aa'} \delta_{kk'} \end{array} \right) \left( \begin{array}{cc} E_k \delta_{aa'} \delta_{kk'} - (Q_{aa'}^{kk'} + iP_{aa'}^{kk'}) & \delta_{aa'} \\ h_{0aa'} \delta_{aa'} \delta_{kk'} & -h_{0aa'} \delta_{aa'} \delta_{kk'} \end{array} \right) , \]

(4.7)

with \( E_k = (-)^{\text{Tr}+1} E \). The Green functions are given by

\[ G_{kk}^a(x) = \lim_{m \to 0} \frac{N}{g_0} \left[ \langle Q + iP \rangle_{aa}^{kk} \langle x \rangle - \omega s_k \right] , \]

\[ K_{rr'}^{kk'}(x, x') = \lim_{m \to 0} \left( \frac{N}{2g_0} \right)^2 \left( \langle Q - r' i P \rangle_{aa}^{kk'} \langle x \rangle \langle Q + r i P \rangle_{aa'}^{kk'}(x') \right) - \frac{N}{2g_0} (1 - \delta_{rr'}) \delta(x - x') . \]

(4.8)
B. Saddle point solution

Most dominant contribution to the action, which is assumed to be independent of the coordinate, should satisfy the saddle point equations,

$$\frac{1}{g_0} [(Q + i P) - \omega s]$$

$$= \frac{2 [E_k - (Q + i P)]}{[E_k - (Q - i P)]} + \nabla^2 (x, x), \quad (4.9)$$

where relation $h_0^2 = \nabla^2 \delta_{\alpha \beta}$ is used, and the factor 2 in the numerator comes from the summation with respect to $\alpha = 1, 2$. If $\omega = 0$, it is expected to be of the form

$$P = 0, \quad Q = (E_{0k} + s_k q) \delta_{\alpha \alpha'} \delta_{kk'} \quad (4.10)$$

Then, Eq. (4.9) reduces to one equation,

$$\frac{1}{g_0} Q = \frac{2(E_k - Q)}{(E_k - Q)^2 + \nabla^2 (x, x)}. \quad (4.11)$$

At zero energy this equation is easily solved as, setting $E_k = E_{0k} = 0$,

$$q = \Lambda (e^{\frac{2\pi}{\omega}} - 1)^{-\frac{1}{2}} \sim \Lambda e^{\frac{-\pi}{\omega}}, \quad (4.12)$$

where $\Lambda$ is a cut-off mentioned in Section II. Therefore, DOS is given by

$$\rho(E) = 0 \sim \frac{2\pi \Lambda}{\pi g_0} e^{-\frac{\pi}{\omega}}. \quad (4.13)$$

This equation shows that, for arbitrary coupling $g_0$, non-trivial solution exists, and the GL(4m,R) symmetry is spontaneously broken.

V. DERIVATION OF NONLINEAR SIGMA MODEL

In this section, we set $E = 0$ and concentrate on the band center. We will divide the fields into longitudinal and Goldstone modes and derive their transformation law under GL(4m,R) transformation (3.18). This will turn out to be of importance when we derive a gauge-independent effective action of the Goldstone mode integrating out the longitudinal mode.

A. Transformation properties

So far we have shown that the Lagrangian is invariant under the global GL(4m,R) transformation (3.18) when $\omega = 0$. Hence the action for the auxiliary fields (4.6) should keep the same invariance. To study the transformation property of the auxiliary fields, let us consider the Lagrangian (4.7). Due to the coupling term between $\phi^\pm$ and $Q, P$, the transformation (3.18) for the quadratic terms, $\phi^\pm_{\alpha \alpha'} \phi^\pm_{\alpha \alpha'} \rightarrow g \phi^\pm_{\alpha \alpha'} \phi^\pm_{\alpha \alpha'} T$, and $\phi^\pm_{\alpha \alpha} \phi^\pm_{\alpha \alpha} \rightarrow s^\dagger g^{-1} \phi^\pm_{\alpha \alpha} \phi^\pm_{\alpha \alpha} s g^{-1} T$ induce the transformation of $Q$ and $P$,

$$Q + i P \rightarrow s^\dagger g^{-1} s^\dagger (Q + i P) s g^{-1} s^\dagger,$$

$$Q - i P \rightarrow s^\dagger g s^\dagger (Q - i P) s g s^\dagger. \quad (5.1)$$

In the same way, the source terms should transform as

$$h^+ \rightarrow s^\dagger g^{-1} s^\dagger h^+ s^\dagger g^{-1} s^\dagger,$$

$$h^- \rightarrow s^\dagger g s^\dagger h^- s g s^\dagger. \quad (5.2)$$

Note that the saddle point solution is invariant under the transformation $o \in O(4m)$. Therefore, we can decompose the $Q$ and $P$ fields as follows

$$Q + i P = s^\dagger T s^\dagger L^+ T^{-1} s^\dagger,$$

$$Q - i P = s^\dagger T s^\dagger L^- T^{-1} s^\dagger, \quad (5.3)$$

where the field $T(x) \in GL(4m, R)/O(4m)$ describe the Goldstone mode, and longitudinal fields are parameterized as

$$L^+(x) = L_0^+ + s q, \quad L_0^+ = Q_L(x) \pm iP_L(x),$$

$$Q_L = \frac{1}{2}(R - s Rs), \quad P_L = \frac{1}{2}(R + s Rs), \quad (5.4)$$

with a real and symmetric matrix $R$. The constant imaginary shift $s q$ as well as the last two parameterizations ensure (3.3) that the integration over $\phi^\pm$ fields converges in Eq. (4.9). Moreover, it is easily shown that the degrees of freedom are equivalent: The two real and symmetric matrices $Q$ and $P$ are now converted to the two real and symmetric matrices $T$ ($^T T = T$ is one of possible gauges) and $R$. The following identities are hold;

$$s L_0^+ s = -L_0^+. \quad (5.5)$$

Next, examine the transformation of $T$ and $L$ fields. First, we have to consider the action of $g \in GL(4m, R)$ on $T$,

$$gT(x) = T'(x) o(T(x), g), \quad (5.6)$$

where $T'(x) \in GL(4m, R)/O(4m)$ and $o(T, g) \in O(4m)$. It should be noted that the field $o(T, g)$ is a nonlinear function of $T(x)$. Applying the transformation (4.6) to (5.3) and considering Eq. (5.4), we have the following transformation laws

$$T \rightarrow gT o^{-1}(T, g),$$

$$L^+ \rightarrow s^\dagger o(T, g) s^\dagger L^+ s^\dagger o^{-1}(T, g) s^\dagger,$$

$$L^- \rightarrow s^\dagger o(T, g) s^\dagger L^- s^\dagger o^{-1}(T, g) s^\dagger. \quad (5.7)$$
Under the change of variables \([5.3]\), the action \([4.4]\) yields,

\[
Z = \int \mathcal{D}[T] \mathcal{D}[L] I[L] e^{-S_{\text{a}}},
\]

\[
S_{\text{a}} = \int d^2x \frac{N}{2g_0} \left[ \text{tr} \ L^+ L^- - \omega \text{tr} \ (^{\dagger}TT)^{-1} s^\gamma L^+ s^\gamma + \omega \text{tr} \ (^{\dagger}TT) s^\gamma L^- s^\gamma \right] + \frac{N}{2} \text{Tr} \ln C
\]

\[
- \int d^2x \frac{N}{g_0} \left[ \text{tr} \ h^+ s^\gamma T s^\gamma L^+ s^\gamma + \text{tr} \ h^- s^\gamma T L^+ s^\gamma + \text{tr} \ h^- s^\gamma T h^+ s^\gamma - \omega \text{tr} \ s(h^+ + h^-) - 2\text{tr} \ h^+ h^- \right],
\]

(5.8)

where \(I[L]\) is a measure and \(C\) is converted into, after the gauge-transformation

\[
C = \begin{pmatrix}
-L^+ & -s^\gamma \partial_\mu (v_\mu + a_\mu) + a_\mu & -s^\gamma \partial_\mu (v_\mu + a_\mu) - a_\mu \\
-s^\gamma \partial_\mu (v_\mu + a_\mu) + a_\mu & -L^- & -s^\gamma \partial_\mu (v_\mu + a_\mu) - a_\mu \\
-s^\gamma \partial_\mu (v_\mu + a_\mu) + a_\mu & -s^\gamma \partial_\mu (v_\mu + a_\mu) - a_\mu & -L^- \\
\end{pmatrix},
\]

(5.9)

with

\[
v_\mu \to o v_\mu o^{-1} + o \partial_\mu o^{-1}, \quad a_\mu \to o a_\mu o^{-1},
\]

(5.10)

which means that \(v\) is a gauge field associated with the hidden local \(O(4m)\) symmetry \([24]\). Actually, the hidden local \(O(4m)\) symmetry reflects the fact that the parameterization of the symmetric space \(\text{GL}(4m, \mathbb{R})/O(4m)\) is not unique. Another comment is that the breaking terms are not \(g\)-invariant, but are gauge-invariant. Therefore, the total Lagrangian including the breaking terms is gauge-invariant, as it should be.

### B. Derivative expansion

Integration over the massive mode yields an effective action for the Goldstone mode. To this end, let us divide the action into two parts,

\[
Z = \int \mathcal{D}[T] \mathcal{D}[L] e^{-S_L(L) - \delta S(T, L)},
\]

\[
S_L(L) = \int d^2x \frac{N}{2g_0} \text{tr} \ L^+ L^- + \frac{N}{2} \text{Tr} \ln A - \text{Ln} I[L],
\]

(5.12)

\[
\delta S(T, L) = \frac{N}{2} \text{Tr} \ln (1 + A^{-1} B) + \int d^2x \frac{N}{g_0} \left[ \text{tr} \ h^+ s^\gamma T s^\gamma L^+ s^\gamma + \text{tr} \ h^- s^\gamma T L^+ s^\gamma + \text{tr} \ h^- s^\gamma T h^+ s^\gamma - \omega \text{tr} \ s(h^+ + h^-) - 2\text{tr} \ h^+ h^- \right],
\]

(5.13)

where

\[
A = \begin{pmatrix}
-L^+ & -\gamma_\mu \partial_\mu \\
\gamma_\mu \partial_\mu & -L^- \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & -s^\gamma \gamma_\mu (v_\mu - a_\mu) s^\gamma \\
-s^\gamma \gamma_\mu (v_\mu + a_\mu) s^\gamma & 0 \\
\end{pmatrix}.
\]

(5.14)

It is reasonable to define effective Lagrangian for \(T\) field as \(S_T \equiv \langle \delta S(T, L) \rangle\), where \(\langle \cdot \cdot \cdot \rangle\) means the expectation value with respect to \(S_L\). Since we are now interested in long distance behavior of the Goldstone mode, we will expand these nonlocal Lagrangian up to quadratic order in fields, and will derive the effective NLSM for the Goldstone mode up to \(O(N^0)\) for large \(N\) system.

#### 1. Longitudinal mode

In this subsection, we derive the leading Lagrangian of the longitudinal mode. In order to get an \(O(N^0)\) NLSM
of the Goldstone mode, it is enough to derive the Lagrangian of the longitudinal mode up to $O(N)$. Hence, the measure term in Eq. (5.12) can be neglected. The first term in Eq. (5.12) is expanded as

$$\text{tr} \ L^+ L^- = \text{tr} \ L_0^+ L_0^- + q \text{tr} \ s(L_0^+ + L_0^-) - 4mq^2, \quad (5.15)$$

where the last term can be neglected, since it vanishes after the replica limit. The second term, $\text{Tr} \text{Ln} A$, can be expanded as

$$\text{Tr} \text{Ln} A = \text{Tr} \text{Ln} A_0 + \text{Tr} \text{Ln} (1 - A_0^{-1} L_0)$$

$$\sim \text{Tr} \text{Ln} A_0 - \text{Tr} A_0^{-1} L_0 - \frac{1}{2} \text{Tr} A_0^{-1} L_0 A_0^{-1} L_0, \quad (5.16)$$

where

$$A = A_0 - L_0,$$

$$A_0 = \begin{pmatrix} -sq & -\tilde{\gamma}_u \partial_\mu \\ \tilde{\gamma}_u \partial_\mu & -sq \end{pmatrix}, \quad L_0 = \begin{pmatrix} L_0^- & 0 \\ 0 & L_0^+ \end{pmatrix}. \quad (5.17)$$

The linear term with respect to $L_0$ in Eq. (5.16) can be neglected, as expected. Collecting the quadratic terms in Eqs. (5.15) and (5.16), we finally have

$$S_L(L) \sim \frac{1}{4d} \int d^2 x \left( \text{tr} \partial_\mu L_0^+ \partial_\nu L_0^- + h_L \text{tr} L_0^+ L_0^- \right), \quad (5.18)$$

where

$$\frac{1}{d} = \frac{N}{6\pi q^2}, \quad h_L = 12q^2. \quad (5.19)$$

The derivation of this action is outlined in Appendix A.

2. Goldstone mode

In order to derive the NLSM for the Goldstone mode, let us expand the action $\delta S$ in Eq. (5.13) up to second order with respect to the derivatives. In this leading order, $v$ does not appear because it is not gauge covariant: It enters the fourth order in the form of covariant derivative $\partial_\mu - v_\mu$. Thus, in the second order, we can simply neglect $v$.

![Diagram](attachment:fig1.png)

**FIG. 1.** Diagrams contributing (a) to order $N$ and (b), (c) to order $N^0$. Wavy line denotes $a_\mu$, while thin and thick lines denote, respectively, the propagator of $A_0$ and of $L$ field, summarized in Appendix A.

Using

$$\text{Tr} \text{Ln} (1 + A^{-1} B) \sim \text{Tr} A^{-1} B - \frac{1}{2} \text{Tr} A^{-1} BA^{-1} B,$$

as well as

$$A^{-1} \sim A_0^{-1} + A_0^{-1} L_0 A_0^{-1} + A_0^{-1} L_0 A_0^{-1} L_0 A_0^{-1}, \quad (5.20)$$

it turns out that diagrams summarized in Fig. 1 contribute to $O(N^0)$. Accordingly, we reach

$$S_T = \int d^2 x \left\{ \frac{1}{b} \text{tr} a_\mu^2 - \frac{1}{c} \text{tr} a_\mu^2 + \frac{N}{2g_0} \omega q \text{tr} \left[ (^4TT)^{-1} + (^4TT) \right] \right\}$$

$$- \int d^2 x \frac{N}{g_0} \left[ q \text{tr} h^+ s^{\frac{2}{3}} (T^\dagger T) s^{\frac{2}{3}} - q \text{tr} h^- s^{\frac{2}{3}} (T^\dagger T)^{-1} s^{\frac{2}{3}} - \omega \text{tr} s h^+ + h^- - 2 \text{tr} h^+ h^- \right], \quad (5.22)$$

where coupling constants are given by

$$\frac{1}{b} = \frac{N}{4\pi} - \frac{d_b}{4\pi}, \quad \frac{1}{c} = \frac{d_c}{4\pi} + O(N^{-1}),$$

$$d_b \sim 0.4593, \quad d_c \sim 0.1453. \quad (5.23)$$

The contribution from the diagrams (b) and (c) is denoted by $d_b$ and $d_c$, respectively. An outline of the calculation is summarized in Appendix A. Some comments may be in order. Firstly, what is characteristic in this Lagrangian is that there appears $\text{tr} a_\mu^2$-term as well as the principal term. As we shall see later, this term plays an important role in the scaling property of DOS. Secondly, at the leading order of the large $N$ expansion, the coupling constant $1/c$
is 0, and in the next leading order, it is a finite positive values. This sign is quite relevant to the renormalization group flow, as we shall see momentarily. Lastly, this Lagrangian is manifestly gauge-invariant. If we fix the gauge as \( T = T \), the action becomes

\[
S_T = \int d^2 x \left[ -\frac{1}{4b} \text{tr} \partial_\mu T^{-2} \partial_\mu T^2 - \frac{1}{4c} \text{tr}^2 T^{-2} \partial_\mu T^2 + \frac{N}{2g_0} \omega q \text{tr} (T^{-2} + T^2) \right]
- \int d^2 x \frac{N}{g_0} \left[ q \text{tr} h^+ s^2 T^2 s^2 - q \text{tr} h^- s^2 T^{-2} s^{-2} - \omega \text{tr} s(h^+ + h^-) - 2t \text{tr} h^+ h^- \right].
\]  

(5.24)

This is an action of GL(4m,R)/O(4m) NLSM, similar to the one for the two-sublattice model with broken time-reversal symmetry derived by Gade. The Green function is

\[
\overline{G}^k_{\pm}(x) = \lim_{m \to 0} \frac{N}{g_0} g_{kk} \left[ \langle (T^{\pm2})^{kk} \rangle - \omega \right],
\]

\[
\overline{K}^{kk'}_{rr'}(x, x') = \lim_{m \to 0} \left( \frac{N}{2g_0} \right)^2 \left\langle \left( s^2 T^2 s^2 \right)^r \left( s^2 T^2 s^2 \right)^{r'} \right\rangle - \frac{N}{2g_0} (1 - \delta_{rr'}) \delta(x - x').
\]

(5.25)

VI. RENORMALIZATION GROUP

By the use of the relation between the noncompact symmetric space GL(4m,R)/O(4m) and the compact symmetric space U(4m)/SO(4m), it is easy to write down the beta functions of the renormalization group

\[
\beta_b(b, c) = \frac{1}{2} n b^2 - \frac{1}{4} \left( \frac{1}{2} n^2 + n \right) b^3 + \frac{1}{8} \left( \frac{3}{8} n^3 + \frac{5}{4} n^2 + n \right) b^4
\]

\[
- \frac{1}{16} \left[ \frac{19}{48} b^4 + \frac{43}{24} \left( \frac{3}{16} \zeta(3) \right) n^3 + \left( \frac{9}{4} + \frac{3}{8} \zeta(3) \right) n^2 + \frac{1}{2} n \right] b^5 - O(b^6),
\]

\[
\beta_c(b, c) = \frac{c^2 \beta_b(b, c)}{n b^2},
\]

(6.1)

where \( n = 4m \) and \( \zeta \) is the Riemann zeta function. Here we changed the sign of \( b \) of beta functions in Ref. [33], because the present symmetric space is noncompact. We also took the normalization of \( b \) and \( c \) into account by calculating the renormalization constants explicitly up to two-loop order. The scaling equations are, therefore, given by, in the replica limit \( m \to 0 \),

\[
\frac{db}{dl} = 0,
\]

\[
\frac{dc}{dl} = -\frac{1}{2} c^2 \left[ 1 - b + \left( \frac{b}{2} \right)^2 - \frac{1}{2} \left( \frac{b}{2} \right)^3 + O(b^4) \right],
\]

(6.2)

where \( l = \ln L \) with \( L \) being the length of the system and coupling constants are rescaled as \( b, c \to \frac{b}{2\pi}, \frac{c}{2\pi} \), whose initial values are given by

\[
b_0 = \frac{2}{N - (d_b - d_c)} \sim \frac{2}{N - 0.314},
\]

\[
c_0 = \frac{2}{d_b} \sim 4.
\]

(6.3)

Let us now discuss the flow of the coupling constants. First of all, vanishing beta function for \( b \) implies that \( E = 0 \) state is delocalized [18]. For \( E \neq 0 \), the universality class \( O(2m, 2m)/O(2m) \times O(2m) \) suggests that such states are localized, as discussed in Section III. Therefore, only the band center is just on the critical point for the present model. Actually, this is in good agreement with the numerical calculation of HWKM for the \( N = 1 \) model.

To clarify the flow of \( c \), let us first consider the case where \( N \) is large enough to ignore the \( O(N^{-1}) \) in Eq. (5.23). In this case, \( b \) is quite small while \( c \) is positive. The scaling equation (5.2) then tells us that \( c \) scales to 0. Therefore, large \( N \) systems behave similar to the class with broken time reversal symmetry [18]. Namely, since the \( \zeta \) function is given by [18]

\[
\zeta = \frac{n + 1}{2} b + \frac{b^2}{2c - nb} - \frac{3}{64} \left( 4 - 3n^2 - n^3 \right) b^4
\]

\[
\to \frac{b}{2} + \frac{b^2}{2c} - \frac{3}{16} b^3 \quad (m \to 0),
\]

(6.4)

the DOS at the zero energy diverges under the change of the length scale;

\[
\rho \propto \exp \left( \int_{-\infty}^{\infty} \zeta dl \right) \to \infty.
\]

(6.5)
On the one hand, this may be likely, since Oppermann and Wegner already derived a singular behavior of the Green function of the two-sublattice model such as $1/N(d-2)$, using the large $N$ expansion. On the other hand, the present divergence of the DOS suggests a discontinuity with respect to $g_0$ at $g_0 = 0$, since DOS is exactly zero at the band center when $g_0 = 0$.

However, the numerical calculation for the $N = 1$ system by HWKM suggests the convergence of the DOS even for rather large $g_0$. Therefore, small $N$ systems, at least $N = 1$ system, should belong to a different fixed point. Actually in Eq. (6.3), there is a nontrivial zero for $\beta_c$ at $b = b_c \sim 3.087$. Accordingly, if $b_0 < b_c$, $c$ flows to 0, while if $b_0 > b_c$, $c$ flows to infinity, provided that $\epsilon_0 > 0$. The scaling equation (6.2) up to four loop order shows, therefore, the existence of a strong-coupling fixed point in addition to the weak-coupling fixed point $c = 0$ mentioned above. Since $c$ diverges at a certain length scale $l = l_c$, the DOS in the strong-coupling limit may scale as

$$\rho \propto q \exp \left( \int_{l_0}^{l_c} \zeta dl \right) \propto q. \quad (6.6)$$

Namely, DOS is expected to be convergent in the strong-coupling phase. The initial value of the present calculation is $b_0 = 2.92$ for $N = 1$, which is still in the weak-coupling regime, to be sure, but lies quite near to the zero of the beta function.

Of course, precise studies of this strong-coupling regime are beyond our scope, since we are based on the perturbations: The initial values $b_0$ in Eq. (6.3) is valid only for large $N$ cases. The zero of the beta function depends, moreover, on the order of the loop expansion. Namely, $b_c$ is computed as $b_c = 2$ and 3.087 for two- and four-loop order, respectively, but no $b_c$ for one- and three-loop order. However, numerical calculations by HWKM for the $N = 1$ system suggests the convergence of the DOS, which tells that such model cannot be in the weak-coupling fixed point.

Therefore, we conjecture the existence of a critical $N$ which separate the strong- and the weak-coupling phases. It is quite interesting to observe the weak-coupling one which is suggested in this paper for the first time, and to determine the phase diagram more precisely in the $g_0$-$N$ plane by nonperturbative methods or by numerical calculations.

VII. SUMMARY

We have investigated the 2D random hopping fermion model proposed by HWKM, using field theoretical treatments developed by Gade. Starting from the tight-binding Hamiltonian, we have firstly derived the continuum Dirac Hamiltonian which effectively describes the band center of the lattice model, and next constructed the generating functional of the Green functions averaged-over the disorder. It has been shown that the symmetry group is $\text{GL}(4\mathbb{R})$, which is spontaneously broken to $O(4\mathbb{R})$. Integrating out the massive modes, we have derived a nonlinear sigma model on the symmetric space $\text{GL}(4\mathbb{R})/O(4\mathbb{R})$, describing the Goldstone mode. The beta functions of the renormalization group show that the band center is a random critical point, where the density of state diverges for large $N$ systems. This corresponds to a weak-coupling fixed point, which may share basic properties with the two sublattice model studied by Oppermann and Wegner.

However, due to a nontrivial zero of the beta function, it is likely that the small $N$ system is in the strong-coupling limit. This fixed point is still critical but presumably with convergent density of state. This fixed point may occur at a large value of the coupling constant, so that it is beyond our perturbative theory. It is, therefore, quite interesting to study the phase diagram more precisely by nonperturbative methods or by numerical calculations.

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APPENDIX A: CALCULATIONS OF THE LEADING ORDER ACTIONS FOR THE LONGITUDINAL AND THE GOLDSTONE MODES

The basic propagator $A_{\alpha\alpha'}^{-1}$ is

$$A_{\alpha\alpha';\alpha'\alpha}(x,x') = \delta_{\alpha\alpha'}\delta_{kk'}\left( \delta_{\alpha\alpha'}s_k A^{-1}_d(x,x') - A^{-1}_{dd\alpha\alpha'}(x,x') \delta_{\alpha\alpha'}s_k A^{-1}_d(x,x') \right), \quad (A1)$$

where,
The last equation follows from Eq. (5.5). By using these formulas, we have the following expressions contributing to the mass term in the above equation as well as the first one in Eq. (5.15) give the mass term in Eq. (5.18), while the second tr term becomes the kinetic term.

In order to calculate the diagrams in Fig. 1 for the Goldstone mode in Section V B 2, we need the propagator of the longitudinal mode, given by

\[
\begin{align*}
\langle L^\pm_{0,aa}(p)L^\mp_{0,bb}(−p) \rangle &= \left( \delta_{ab}^\pm \delta_{a'b'}^k + \delta_{ab}^k \delta_{a'b'}^l \right) \frac{d}{p^2 + h_L^2}, \\
\langle L^\pm_{0,aa}(p)L^\mp_{0,bb}(−p) \rangle &= -s^k s^{k'} \left( \delta_{ab}^\pm \delta_{a'b'}^k + \delta_{ab}^k \delta_{a'b'}^l \right) \frac{d}{p^2 + h_L^2}.
\end{align*}
\]

The last equation follows from Eq. (5.5). By using these formulas, we have the following expressions contributing to the action (5.22) from each diagram in Fig. 1.

\[
\begin{align*}
(a) &= N \int \frac{d^2p}{(2\pi)^2} \frac{q^2}{(p^2 + q^2)^2} \int d^2x \text{tr} a^2, \\
(b) &= -Nd \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2p'}{(2\pi)^2} \frac{(p'p')^2 - 2q^2p'p + q^4}{(p^2 + q^2)^2((p - p')^2 + h_L^2)} \int d^2x \left( \text{tr} a^2_\mu + \text{tr} a^2_\mu \right), \\
(c) &= 2Nd(1 + 4m) \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2p'}{(2\pi)^2} \frac{q^2(2p'p + p^2) - q^4}{(p^2 + q^2)^3((p - p')^2 + h_L^2)} \int d^2x \text{tr} a^2_\mu.
\end{align*}
\]

Coefficients of these terms (integrals over \( p \)) are in principle dependent on \( g_0 \). To illustrate this, let us consider the first one, for example: Since it is convergent, we can integrate it over the whole 2D momentum space, giving the value \( N/4\pi \). However, for the present model, the cut-off is a physical parameter, denoting the scale under which the linearization procedure of the dispersion relation is valid. Accordingly, if we introduce the cut-off to the (convergent) integral, it turns out to be dependent on \( g_0 \) through the saddle point solution \( q \) such as \( \frac{N}{4\pi} \left( 1 - \frac{q^2}{\Lambda^2 + q^2} \right) = \frac{N}{4\pi}(1 - e^{-\frac{N}{\pi}}) \). However, for quite small \( g_0 \), the exponential in this expression can be neglected again.

* Email: fukui@thp.uni-koeln.de

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[33] Consider only $\phi^+$ sector, because similar arguments hold in the other sector. Coupling term yields $-i\phi^+ s^\frac{1}{2} \left( Q + iP \right) s^\frac{1}{2} \phi^+ = -i\phi^+ s^T T^{-1} \left[ q - s^\frac{1}{2} \left( Q_L + iP_L \right) s^\frac{1}{2} \right] T^{-1} \phi^+$. In this equation, $q$ basically ensures the convergence of the integration over $\phi^+$. For the integration to be really convergent, $s^\frac{1}{2} \left( Q_L + iP_L \right) s^\frac{1}{2}$ needs being purely imaginary. Note that $s^\frac{1}{2} \left( x y \ y z \right) s^\frac{1}{2} = \left( -x \ y \ y -iz \right)$, where $x, y$ and $z$ are $2m \times 2m$ matrices with the same $p$ sector. Therefore, $Q$ and $P$ should have only diagonal and only off-diagonal matrix elements in $k$ sector, respectively. Using $s \left( x y \ y z \right) s = \left( -x \ y \ y -z \right)$, they can be written as Eq. (5.4).
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