ANALYTIC TORSION AND $L^2$-TORSION OF COMPACT LOCALLY
SYMMETRIC MANIFOLDS

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Abstract. In this paper we study the analytic torsion and the $L^2$-torsion of compact
locally symmetric manifolds. We consider the analytic torsion with respect to representa-
tions of the fundamental group which are obtained by restriction of irreducible represen-
tations of the group of isometries of the underlying symmetric space. The main purpose
is to study the asymptotic behavior of the analytic torsion with respect to sequences of
representations associated to rays of highest weights.

1. Introduction

Let $G$ be a real, connected, linear semisimple Lie group with finite center and of non-
compact type. Let $K \subset G$ be a maximal compact subgroup. Then $\tilde{X} = G/K$ is a
Riemannian symmetric space of the noncompact type. Let $\Gamma \subset G$ be a discrete, torsion
free, co-compact subgroup. Then $X = \Gamma \backslash \tilde{X}$ is a compact oriented locally symmetric
manifold. Let $d = \dim X$. Let $\tau$ be a finite-dimensional irreducible representation of $G$ on a
complex vector space $V_\tau$. Denote by $E_\tau$ the flat vector bundle over $X$ associated to the
representation $\tau|_\Gamma$ of $\Gamma$. By [MU, Lemma 3.1], $E_\tau$ can be equipped with a distinguished
Hermitian fiber metric, called admissible. Let $\Delta_p(\tau)$ be the Laplace operator acting on
$E_\tau$-valued $p$-forms on $X$. Denote by $\zeta_p(s; \tau)$ the zeta function of $\Delta_p(\tau)$ (see [Sh]). Then
the analytic torsion $T_X(\tau) \in \mathbb{R}^+$ is defined by

$$\log T_X(\tau) = \frac{1}{2} \sum_{p=0}^d (-1)^p p \frac{d}{ds} \zeta_p(s; \tau) \bigg|_{s=0}$$

(see [RS, Mu2]). Since we have chosen distinguished metrics, we don’t indicate the metric
dependence of $T_X(\tau)$. We also consider the $L^2$-torsion $T_X^{(2)}(\tau)$. Following Lott [Lo] and
Mathai [Mat], this torsion can be defined using the $\Gamma$-trace of the heat operators on $\tilde{X}$.

The main purpose of this paper is to study the asymptotic behavior of $T_X(\tau)$ and $T_X^{(2)}(\tau)$
for certain sequences of representations $\tau$ of $G$. This problem was first studied in [Mu3]
in the context of hyperbolic 3-manifolds. The method used in this paper was based on
the study of the twisted Ruelle zeta function. In [MP] we have developed a different
and simpler method which we used to extend the results of [Mu3] to compact hyperbolic
manifolds of any dimension. In the present paper, we generalize the results of the previous
papers to arbitrary compact locally symmetric spaces. Recently, Bismut, Ma, and Zhang \[BMZ1\], \[BMZ2\], studied the asymptotic behavior of the analytic torsion by a different method and in the more general context of analytic torsion forms on arbitrary compact manifolds. Furthermore, Bergeron and Venkatesh \[BV\] studied the asymptotic behavior of the analytic torsion if the flat bundle is kept fixed, but the discrete group varies in a tower of manifolds. Furthermore, Bergeron and Venkatesh \[BV\] studied the asymptotic behavior of the analytic torsion by a different method and in the more general context of analytic torsion forms on arbitrary compact \[BMZ1\], \[BMZ2\], studied the asymptotic behavior of the analytic torsion by a different method and in the more general context of analytic torsion forms on arbitrary compact manifolds. Recently, Bismut, Ma, and Zhang \[Mu3\] have been used to study the growth of the torsion in the cohomology of arithmetic groups in higher rank cases.

Now we explain our results in more detail. Let \(\delta(\tilde{X}) = \text{rank}_C(G) - \text{rank}_C(K)\). Occasionally we will denote this number by \(\delta(G)\). Let \(\mathfrak{g}\) be the Lie algebra of \(G\). Let \(\mathfrak{h} \subset \mathfrak{g}\) be a fundamental Cartan subalgebra. Let \(G_C\) denote the connected complex linear Lie group corresponding to the complexification \(\mathfrak{g}_C\) of \(\mathfrak{g}\) and let \(U\) be a compact real form of \(G_C\) such that \(\mathfrak{h}_C\) is the complexification of a Cartan subalgebra of \(U\). Then the irreducible finite dimensional complex representations of \(G\) can be identified with the irreducible finite dimensional complex representations of \(U\). Fix positive roots \(\Delta^+(\mathfrak{g}_C, \mathfrak{h}_C)\). Let \(\theta : \mathfrak{g} \to \mathfrak{g}\) be the Cartan involution. For a highest weight \(\lambda \in \mathfrak{h}_C^*\) which we always assume to be analytically integral with respect to \(U\) we let \(\tau_\lambda\) be the irreducible representation of \(G\) corresponding to the representation of \(U\) with highest weight \(\lambda\). We will also say that \(\tau_\lambda\) is the representation of \(G\) of highest weight \(\lambda\). Then we denote by \(\lambda_\theta \in \mathfrak{h}_C^*\) the highest weight of \(\tau_\lambda \circ \theta\), where we regard \(\theta\) as an involution on \(G\). Our main result is the following theorem.

**Theorem 1.1.** (i) Let \(\tilde{X}\) be even dimensional or let \(\delta(\tilde{X}) \neq 1\). Then \(T_X(\tau) = 1\) for all finite-dimensional representations \(\tau\) of \(G\).

(ii) Let \(\tilde{X}\) be odd-dimensional with \(\delta(\tilde{X}) = 1\). Let \(\lambda \in \mathfrak{h}_C^*\) be a highest weight with \(\lambda_\theta \neq \lambda\). For \(m \in \mathbb{N}\) let \(\tau_\lambda(m)\) be the irreducible representation of \(G\) with highest weight \(m\lambda\). There exist constants \(c > 0\) and \(C_{\tilde{X}} \neq 0\), which depends on \(\tilde{X}\), and a polynomial \(P_\lambda(m)\), which depends on \(\lambda\), such that

\[
\log T_X(\tau_\lambda(m)) = C_{\tilde{X}} \text{vol}(X) \cdot P_\lambda(m) + O(e^{-cm})
\]

as \(m \to \infty\). Furthermore, there is a constant \(C_\lambda > 0\) such that

\[
P_\lambda(m) = C_\lambda \cdot m \dim(\tau_\lambda(m)) + R_\lambda(m),
\]

where \(R_\lambda(m)\) is a polynomial whose degree equals the degree of the polynomial \(\dim(\tau_\lambda(m))\).

We note that (1.2) provides a complete asymptotic expansion for \(\log T_X(\tau_\lambda(m))\). If one is only interested in the leading term, one can use (1.3) which implies that there exists a constant \(C = C(\tilde{X}, \lambda) \neq 0\), which depends on \(\tilde{X}\) and \(\lambda\), such that

\[
\log T_X(\tau_\lambda(m)) = C \text{vol}(X) \cdot m \dim(\tau_\lambda(m)) + O(\dim(\tau_\lambda(m)))
\]
as \( m \to \infty \). Now the coefficient of the highest power can determined by Weyl’s dimension formula.

The condition \( \lambda \neq \lambda_\theta \) is essential for our method to work. It implies the existence of an increasing spectral gap for the corresponding Laplace operators (see Corollary 7.4). It is a challenging and very interesting problem to extend Theorem 1.1 to the case \( \lambda = \lambda_\theta \).

For hyperbolic manifolds, we proved the vanishing result (i) of Theorem 1.1 in [MP, Proposition 1.7]. In general it was first proved by Bismut, Ma, and Zhang [BMZ2]. It extends a result of Moscovici and Stanton [MS1] who showed that \( T_X(\rho) = 1 \), if \( \delta(X) \geq 2 \) and \( \rho \) is a unitary representation of \( \Gamma \). Our proof is different from the previous proofs and, as we believe, also simpler. It does not rely on the use of orbital integrals or the Fourier inversion formula.

Part (ii) is a consequence of the following two propositions. The first one shows that the asymptotic behavior of the analytic torsion with respect to the representations \( \tau_\lambda(m) \) is determined by the asymptotic behavior of the \( L^2 \)-torsion.

**Proposition 1.2.** Let \( \widetilde{X} \) be odd-dimensional with \( \delta(\widetilde{X}) = 1 \). Let \( \lambda \in \mathfrak{h}_\mathbb{C}^* \) be a highest weight. Assume that \( \lambda_\theta \neq \lambda \). For \( m \in \mathbb{N} \) let \( \tau_\lambda(m) \) be the irreducible representation of \( G \) with highest weight \( m \lambda \). Then there exists \( c > 0 \) such that

\[
(1.5) \quad \log T_X(\tau_\lambda(m)) = \log T_X^{(2)}(\tau_\lambda(m)) + O(e^{-cm})
\]

for all \( m \in \mathbb{N} \).

This result was first proved in [MP] for hyperbolic manifolds. It was also proved in [BMZ2] in the more general context of this paper (see Remark 7.8). Our method of proof of (1.5) follows the method developed in [MP].

The key result on which part (ii) of Theorem 1.1 relies is the computation of the \( L^2 \)-torsion. The computation is based on the Plancherel formula. It gives

**Proposition 1.3.** Let the assumptions be as in Proposition 1.2. There exists a constant \( C_{\widetilde{X}} \), which depends on \( \widetilde{X} \), and a polynomial \( P_\lambda(m) \), which depends on \( \lambda \), such that

\[
(1.6) \quad \log T_X^{(2)}(\tau_\lambda(m)) = C_{\widetilde{X}} \cdot \text{vol}(X) \cdot P_\lambda(m), \quad m \in \mathbb{N}.
\]

Moreover there is a constant \( C_\lambda > 0 \) such that

\[
(1.7) \quad P_\lambda(m) = C_\lambda \cdot m \cdot \dim(\tau_\lambda(m)) + O(\dim(\tau_\lambda(m)))
\]

as \( m \to \infty \).

Finally, we note that if one specializes the main result of [BMZ2], Theorem 1.1, to the case of analytic torsion of a locally symmetric space, one can also determine the leading term of the asymptotic expansion of (1.4). This has been carried out in [BMZ2] in the case of hyperbolic 3-manifolds.

If we consider one of the odd-dimensional irreducible symmetric spaces \( \widetilde{X} \) with \( \delta(\widetilde{X}) = 1 \) and choose \( \lambda \) to be (an integral multiple of) a fundamental weight, the statements can be made more explicit.
Let $G = \text{SO}^0(p, q)$, $K = \text{SO}(p) \times \text{SO}(q)$, $p > 1$, $p, q$ odd, $p \geq q$, and let $\tilde{X} := G/K$. Let $n := (p + q - 2)/2$. There are two fundamental weights $\tilde{\omega}^\pm_{f,n}$ which are not invariant under $\theta$ and we let $\omega^\pm_{f,n} := 2\tilde{\omega}^\pm_{f,n}$ (see (6.13)). One has $\omega^\pm_{f,n} = (\omega^\pm_{f,n})_\theta$. By equation (6.21), it suffices to consider the weight $\omega^+_{f,n}$. For $m \in \mathbb{N}$ let $\tau(m)$ be the representation with highest weight $m\omega^+_{f,n}$. By Weyl’s dimension formula there exists a constant $C > 0$ such that

$$\dim(\tau(m)) = Cm^{\frac{n(n+1)}{2}} + O\left(m^{\frac{n(n+1)}{2} - 1}\right)$$

as $m \to \infty$. Let $\tilde{X}_d$ be the compact dual of $\tilde{X}$. We let $\epsilon(q) := 0$ for $q = 1$ and $\epsilon(q) := 1$ for $q > 1$ and we let

$$C_{p,q} := \frac{(-1)^{\frac{p-1}{2}}2^{\epsilon(q)}\pi}{\text{vol}(\tilde{X}_d)}\left(\frac{n}{p-1}\right) .$$

**Corollary 1.4.** Let $p, q$ odd and let $\tilde{X} = \text{SO}^0(p, q)/\text{SO}(p) \times \text{SO}(q)$ and let $X = \Gamma \backslash \tilde{X}$. With respect to the above notation we have

$$\log T_X(\tau(m)) = C_{p,q} \text{vol}(X) \cdot m \dim(\tau(m)) + O\left(m^{\frac{n(n+1)}{2}}\right)$$

as $m \to \infty$.

The case $q = 1$ was treated in [MP] and the case $p = 3, q = 1$ in [Mu3]. In the latter case we have Spin$(3, 1) \cong \text{SL}(2, \mathbb{C})$. The irreducible representation of Spin$(3, 1)$ with highest weight $\frac{1}{2}(m, m)$ corresponds to the $m$-th symmetric power of the standard representation $\text{SL}(2, \mathbb{C})$ on $\mathbb{C}^2$ and we have

$$-\log T_X(\tau(m)) = \frac{1}{4\pi} \text{vol}(X)m^2 + O(m).$$

The remaining case is $\tilde{X} = \text{SL}(3, \mathbb{R})/\text{SO}(3)$. There are two fundamental weights $\omega_i, i = 1, 2$. Both are non-invariant under $\theta$. Let $\tau_i(m), i = 1, 2$, be the irreducible representation with highest weight $m\omega_i$. By Weyl’s dimension formula one has

$$\dim_{\tau_i}(m) = \frac{1}{2}m^2 + O(m),$$

as $m \to \infty$. Let $\tilde{X}_d$ be the compact dual of $\tilde{X}$.

**Corollary 1.5.** Let $\tilde{X} = \text{SL}(3, \mathbb{R})/\text{SO}(3)$ and $X = \Gamma \backslash \tilde{X}$. We have

$$\log T_X(\tau_i(m)) = \frac{4\pi \text{vol}(X)}{9 \text{vol}(\tilde{X}_d)}m \dim(\tau_i(m)) + O(m^2)$$

as $m \to \infty$.

Using the equality of analytic and Reidemeister torsion [Mu2], we obtain corresponding statements for the Reidemeister torsion $\tau_X(\tau_\lambda(m))$. Especially we have
Corollary 1.6. Let $X = \Gamma \backslash \tilde{X}$ be a compact odd-dimensional locally symmetric manifold with $\delta(\tilde{X}) = 1$. Let $\lambda \in \mathfrak{h}_c^*$ be a highest weight which satisfies $\lambda_0 \neq \lambda$. Let $\tau_X(\tau_\lambda(m))$ be the Reidemeister torsion of $X$ with respect to the representation $\tau_\lambda(m)$. Then $\text{vol}(X)$ is determined by the set $\{\tau_X(\tau_\lambda(m)) : m \in \mathbb{N}\}$.

Finally we note that Bergeron and Venkatesh [BV] proved results of a similar nature, but in a different aspect. Let $\delta(\tilde{X}) = 1$. Let $\Gamma \supset \Gamma_1 \supset \cdots \supset \Gamma_N \supset \cdots$ be a tower of subgroups of finite index with $\cap N \Gamma_N = \{e\}$. A representation $\tau$ of $G$ is called strongly acyclic, if the spectrum of the Laplacians $\Delta_p(\tau)$ on $\Gamma_N \backslash \tilde{X}$ stays uniformly bounded away from zero. Then for a strongly acyclic representation $\tau$ they show that there is a constant $c_{G,\tau} > 0$ such that

$$\lim_{N \to \infty} \log \frac{T_{\Gamma_N \backslash \tilde{X}}(\tau)}{|\Gamma : \Gamma_N|} = c_{G,\tau} \text{vol}(\Gamma \backslash \tilde{X}).$$

Next we explain our methods to prove Theorem 1.1. The first step is the proof of Proposition 1.2. We follow the proof used in [MP]. For an irreducible representation $\tau$ of $G$, and $t > 0$ put

$$K(t, \tau) := \sum_{p=0}^{d} (-1)^p p \text{Tr} \left( e^{-t \Delta_p(\tau)} \right).$$

Assume that $\tau|_\Gamma$ is acyclic, that is $H^*(X, E_\tau) = 0$. Then the analytic torsion is given by

$$\log T_X(\tau) := \frac{1}{2} \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(t, \tau) \, dt \right) \bigg|_{s=0}. \quad \text{(1.10)}$$

Now the key ingredient of the proof of Proposition 1.2 is the following lower bound for the spectrum of the Laplacians. For every highest weight $\lambda$ which satisfies $\lambda_0 \neq \lambda$, there exist $C_1, C_2 > 0$ such that

$$\Delta_p(\tau_\lambda(m)) \geq C_1 m^2 - C_2, \quad m \in \mathbb{N}, \quad \text{(1.11)}$$

(see Corollary 7.6). Since $\tau_\lambda(m)$ is acyclic and $\dim X$ is odd, $T_X(\tau_\lambda(m))$ is metric independent [Mu2]. Especially, it is invariant under rescaling of the metric. So we can replace $\Delta_p(\tau_\lambda(m))$ by $\frac{1}{m} \Delta_p(\tau_\lambda(m))$. Then

$$\log T_X(\tau(m)) = \frac{1}{2} \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} K \left( \frac{t}{m}, \tau(m) \right) \, dt \right) \bigg|_{s=0}$$

$$+ \frac{1}{2} \int_1^\infty t^{-1} K \left( \frac{t}{m}, \tau(m) \right) \, dt. \quad \text{(1.12)}$$

It follows from (1.11) and standard estimations of the heat kernel that the second term on the right is $O(e^{-\frac{C_1}{m^2}})$ as $m \to \infty$. To deal with the first term, we use a preliminary form of the Selberg trace formula. It turns out that the contribution of the nontrivial conjugacy classes to the trace formula is also exponentially decreasing in $m$. Finally, the identity contribution equals $\log T_X^2(\tau_\lambda(m))$ up to a term, which is exponentially decreasing in $m$. This implies Proposition 1.2.
To deal with the $L^2$-torsion, we recall that for any $\tau$, $\log T^{(2)}_{X}(\tau)$ it is defined in terms of the $\Gamma$-trace of the heat operators $e^{-t\Delta_p(\tau)}$ on the universal covering $\tilde{X}$. In our case, $e^{-t\tilde{\Delta}_p(\tau)}$ is a convolution operator and its $\Gamma$-trace equals the contribution of the identity to the spectral side of the Selberg trace formula applied to $e^{-t\Delta_p(\tau)}$. It follows that

$$\log T^{(2)}_{X}(\tau) = \text{vol}(X) \cdot t^{(2)}_{\tilde{X}}(\tau),$$

where $t^{(2)}_{\tilde{X}}(\tau)$ depends only on $\tilde{X}$ and $\tau$. To compute $t^{(2)}_{\tilde{X}}(\tau)$ we factorize $\tilde{X}$ as $\tilde{X} = \tilde{X}_0 \times \tilde{X}_1$, where $\delta(\tilde{X}_0) = 0$ and $\tilde{X}_1$ is irreducible with $\delta(\tilde{X}_1) = 1$. Let $\tau = \tau_0 \otimes \tau_1$ be the corresponding decomposition of $\tau$. Let $\tilde{X}_{0,d}$ be the compact dual symmetric space of $\tilde{X}_0$. Using a formula similar to [Lo, Proposition 11], we get

$$t^{(2)}_{\tilde{X}}(\tau) = (-1)^{\dim(\tilde{X}_0)/2} \frac{\chi(\tilde{X}_{0,d})}{\text{vol}(\tilde{X}_{0,d})} \dim(\tau_0) \cdot t^{(2)}_{\tilde{X}_1}(\tau_1).$$

This reduces the computation of $t^{(2)}_{\tilde{X}}(\tau)$ to the case of an irreducible symmetric space $\tilde{X}$ with $\delta(\tilde{X}) = 1$ which is odd-dimensional. From the classification of simple Lie groups it follows that the only possibilities for $\tilde{X}$ are $\tilde{X} = \text{SL}(3, \mathbb{R})/\text{SO}(3)$ or $\tilde{X} = \text{SO}^0(k, l)/\text{SO}(k) \times \text{SO}(l)$, $k, l$ odd. Using the Plancherel formula, $t^{(2)}_{\tilde{X}}(\tau)$ can be computed explicitly for these cases. Combined with Weyl’s dimension formula, it follows that $t^{(2)}_{\tilde{X}}(\tau_\lambda(m))$ is a polynomial in $m$. In this way we obtain our main result.

The paper is organized as follows. In section 2 we collect some facts about representations of reductive Lie groups. Section 3 is concerned with Bochner-Laplace operators on locally symmetric spaces. The main result are estimations of the heat kernel of a Bochner-Laplace operator. In section 4 we consider the analytic torsion in general. The main result of this section is Proposition 4.2, which establishes part (i) of Theorem 1.1. Section 5 is devoted to the study of the $L^2$-torsion. We reduce the study of the $L^2$-torsion to the case of an irreducible symmetric space $\tilde{X}$ with $\delta(\tilde{X}) = 1$. This case is then treated in section 6. Especially we establish Proposition 5.3 in this case. In section 6 we prove a lower bound for the spectrum of the twisted Laplace operators. This is the key result for the proof of Proposition 1.2. In the final section 8 we prove our main result, Theorem 1.1.

2. Preliminaries

In this section we summarize some facts about representations of reductive Lie groups.

2.1. Let $G$ be a real reductive Lie group in the sense of [Kn2, p. 446]. Let $K \subset G$ be the associated maximal compact subgroup. Then $G$ has only finitely many connected components. Denote by $G^0$ the component of the identity. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G$ and $K$, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition.

We denote by $\hat{G}$ the unitary dual and by $\hat{G}_d$ the discrete series of $G$. By $\text{Rep}(G)$ we denote the equivalence classes of irreducible finite-dimensional representations of $G$. 

Let $Q$ be a standard parabolic subgroup of $G$ [Kn2, VII.7]. Then $Q$ has a Langlands decomposition $Q = MAN$, where $M$ is reductive and $A$ is abelian. $Q$ is called cuspidal if $\hat{M}_d \neq \emptyset$. Let $K_M = K \cap M$. Then $K_M$ is a maximal compact subgroup of $M$. Let $Q = MAN$ be cuspidal. For $(\xi, W_\xi) \in \hat{M}_d$ and $\nu \in a_\mathfrak{C}^*$, let

$$\pi_{\xi, \nu} = \text{Ind}_G^Q(\xi \otimes e^\nu \otimes \text{Id})$$

be the induced representation acting by the left regular representation on the Hilbert space

$$\mathcal{H}_{\xi, \nu} = \{ f: G \to W_\xi : f(g \text{man}) = e^{-(\nu + \rho_Q)(\log a)} \xi(m)^{-1} f(g), \forall m \in M, a \in A, n \in N, g \in G, f|_K \in L^2(K, W_\xi) \}$$

with norm given by

$$\|f\|^2 = \int_K |f(k)|^2_{W_\xi} dk.$$

If $\nu \in a^*$, then $\pi_{\xi, \nu}$ is unitarily induced. Denote by $\Theta_{\xi, \nu}$ the global character of $\pi_{\xi, \nu}$.

2.2. Next we recall some facts concerning the discrete series. Let $G$ be a linear semisimple connected Lie group with finite center. Let $K \subset G$ be a maximal compact subgroup. Assume that $\delta(G) = 0$. Then $G/K$ is even-dimensional. Let $n = \dim(G/K)/2$. Let $\mathfrak{t} \subset \mathfrak{k}$ be a compact Cartan subalgebra of $g$. Let $\Delta(\mathfrak{g}_C, \mathfrak{t}_C), \Delta(\mathfrak{t}_C, \mathfrak{t}_C)$ be the corresponding roots with Weyl-groups $W_G, W_K$. Then one can regard $W_K$ as a subgroup of $W_G$. Let $P$ be the weight lattice in $i\mathfrak{t}^*$. Let $\langle \cdot , \cdot \rangle$ be the inner product on $i\mathfrak{t}^*$ induced by the Killing form. Recall that $\Lambda \in P$ is called regular if $\langle \Lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{g}_C, \mathfrak{t}_C)$. Then $\hat{G}_d$ is parametrized by the $W_K$-orbits of the regular elements of $P$, where $W_K$ is the Weyl group of $\Delta(\mathfrak{t}_C, \mathfrak{t}_C)$, [Kn1, Theorem 12.20, Theorem 9.20]. If $\Lambda$ is a regular element of $P$, the corresponding discrete series will be denoted by $\omega_\Lambda$. For $\pi \in \hat{G}$ we denote by $\chi_\pi$ the infinitesimal character of $\pi$. For a regular element $\Lambda \in \mathfrak{h}_C^*$ let $\chi_\Lambda$ be the homomorphism of $\mathcal{Z}(\mathfrak{g}_C)$, defined by [Kn1] (8.32)]. By [Kn1, Theorem 9.20], the infinitesimal character of $\omega_\Lambda$ is given by $\chi_\Lambda$. Fix positive roots $\Delta^+(\mathfrak{g}_C, \mathfrak{t}_C)$ and let $P^+$ be the corresponding set of dominant weights. Let $\rho_G$ be the half sum of the elements of $\Delta^+(\mathfrak{g}_C, \mathfrak{t}_C)$ Then we have the following proposition.

**Proposition 2.1.** Let $\tau \in \text{Rep}(G)$. Then for $\pi \in \hat{G}_d$ one has

$$\dim (H^p(\mathfrak{g}, K; \mathcal{H}_{\pi, K} \otimes V_{\tau})) = \begin{cases} 1, & \chi_\pi = \chi_{\hat{\tau}}, p = n; \\ 0, & \text{else.} \end{cases}$$

Moreover, there are exactly $|W_G|/|W_K|$ distinct elements of $\hat{G}_d$ with infinitesimal character $\chi_{\hat{\tau}}$, where $\hat{\tau}$ is the contragredient representation of $\tau$.

**Proof.** Let $\Lambda(\hat{\tau}) \in P^+$ be the highest weight of $\tau$. Clearly $\Lambda(\hat{\tau}) + \rho_G$ is regular. Thus, since $W_G$ acts freely on the regular elements, the proposition follows from [BW, Theorem I.5.3] and the above remarks on infinitesimal characters.
2.3. Let $Q = MAN$ be a standard parabolic subgroup. In general, $M$ is neither semisimple nor connected. But $M$ is reductive in the sense of [Kn2, p. 466]. Let $K_M = K \cap M$, let $K^0_M$ be the component of the identity, and let $\mathfrak{k}_m := \mathfrak{k} \cap \mathfrak{m}$ be its Lie algebra. Assume that $\text{rank}(M) = \text{rank}(K_M)$. Then $M$ has a nonempty discrete series, which is defined as in [Kn1, XII, §8]. The explicit parametrization is given in [Kn1, Proposition 12.32], [Wa2, section 8.7.1].

3. Bochner Laplace operators

Let $G$ be a semisimple connected Lie group with finite center. Let $K \subset G$ be a maximal compact subgroup. Let $\tilde{X} = G/K$. Let $\Gamma$ be a torsion free, cocompact discrete subgroup of $G$ and let $X = \Gamma \backslash \tilde{X}$.

Let $\nu$ be a finite-dimensional unitary representation of $K$ on the space $(V_\nu, \langle \cdot, \cdot \rangle_\nu)$ let $\tilde{E}_\nu := G \times_\nu V_\nu$ be the associated homogeneous vector bundle over $\tilde{X}$. Let $R_g : \tilde{E}_\nu \to \tilde{E}_\nu$ be the action of $g \in G$. The inner product $\langle \cdot, \cdot \rangle_\nu$ induces a $G$-invariant fiber metric $\tilde{h}_\nu$ on $\tilde{E}_\nu$. Let $\nabla^\nu$ be the connection on $\tilde{E}_\nu$ induced by the canonical connection on the principal $K$-fiber bundle $G \to G/K$. Then $\nabla^\nu$ is $G$-invariant. Let $E_\nu := \Gamma \backslash \tilde{E}_\nu$

be the associated locally homogeneous bundle over $X$. Since $\tilde{h}_\nu$ and $\nabla^\nu$ are $G$-invariant, they can be pushed down to a metric $h_\nu$ and a connection $\nabla^\nu$ on $E_\nu$. Let $C^\infty(\tilde{X}, \tilde{E}_\nu)$ resp. $C^\infty(X, E_\nu)$ denote the space of smooth sections of $\tilde{E}_\nu$ resp. of $E_\nu$. Let

$$C^\infty(G, \nu) := \{ f : G \to V_\nu : f \in C^\infty, \quad f(gk) = \nu(k^{-1})f(g), \; \forall g \in G, \; \forall k \in K \}.$$ (3.1)

Let $L^2(G, \nu)$ be the corresponding $L^2$-space. There is a canonical isomorphism

$$A : C^\infty(\tilde{X}, \tilde{E}_\nu) \cong C^\infty(G, \nu)$$ (3.2)

which is defined by $Af(g) = R_g^{-1}(f(gK))$. It extends to an isometry

$$A : L^2(\tilde{X}, \tilde{E}_\nu) \cong L^2(G, \nu).$$ (3.3)

Let

$$C^\infty(\Gamma \backslash G, \nu) := \{ f \in C^\infty(G, \nu) : f(\gamma g) = f(g) \; \forall g \in G, \; \forall \gamma \in \Gamma \}$$ (3.4)

and let $L^2(\Gamma \backslash G, \nu)$ be the corresponding $L^2$-space. The isomorphisms (3.2) and (3.3) descend to isomorphisms

$$A : C^\infty(X, E_\nu) \cong C^\infty(\Gamma \backslash G, \nu), \quad L^2(X, E_\nu) \cong L^2(\Gamma \backslash G, \nu).$$ (3.5)

Let $\Delta_\nu = \nabla^\nu \star \nabla^\nu$ be the Bochner-Laplace operator of $\tilde{E}_\nu$. Since $\tilde{X}$ is complete, $\Delta_\nu$ with domain the space of smooth compactly supported sections is essentially self-adjoint [LM, p.
Its self-adjoint extension will be denoted by \( \tilde{\Delta}_\nu \) too. With respect to the isomorphism (3.2) one has

\[
\tilde{\Delta}_\nu = -R(\Omega) + \nu(\Omega_K),
\]

where \( R \) denotes the right regular representation of \( G \) on \( C^\infty(G, \nu) \) (see [Mi1, Proposition 1.1]). The heat operator

\[
e^{-t\tilde{\Delta}_\nu} : L^2(G, \nu) \to L^2(G, \nu)
\]

commutes with the action of \( G \). Therefore, it is of the form

\[
(e^{-t\tilde{\Delta}_\nu} \phi)(g) = \int_G H_t^\nu(g^{-1}g')(\phi(g')) \, dg'
\]

where

\[
H_t^\nu : G \to \text{End}(V_\nu)
\]

is in \( C^\infty \cap L^2 \) and satisfies the covariance property

\[
H_t^\nu(k^{-1}gk') = \nu(k)^{-1} \circ H_t^\nu(g) \circ \nu(k'), \forall k, k' \in K, \forall g \in G.
\]

It follows as in [BM, Proposition 2.4] that \( H_t^\nu \) belongs to all Harish-Chandra Schwartz spaces \( (C^q(G) \otimes \text{End}(V_\nu)), q > 0 \).

Now let \( \|H_t^\nu(g)\| \) be the sup-norm of \( H_t^\nu(g) \) in \( \text{End}(V_\nu) \). Let \( \tilde{\Delta}_0 \) be the Laplacian on functions on \( \tilde{X} \) and let \( H_t^0 \) be the associated heat kernel as above. We may use the principle of semigroup domination to bound \( \|H_t^\nu(g)\| \) by the scalar heat kernel. Indeed we have

**Proposition 3.1.** Let \( \nu \in \hat{K} \). Then we have

\[
\|H_t^\nu(g)\| \leq H_t^0(g)
\]

for all \( t \in \mathbb{R}^+ \) and \( g \in G \).

**Proof.** Let \( K_\nu(t, x, y) \) be the kernel of \( e^{-t\tilde{\Delta}_\nu} \), acting in \( L^2(\tilde{X}, \tilde{E}_\nu) \). Denote by \( |K_\nu(t, x, y)| \) the norm of the homomorphism

\[
K_\nu(t, x, y) \in \text{Hom}\left( (\tilde{E}_\nu)_y, (\tilde{E}_\nu)_x \right).
\]

It was proved in [Mu1, p. 325] that in the sense of distributions, one has

\[
\left( \frac{\partial}{\partial t} + \tilde{\Delta}_0 \right) |K_\nu(t, x, y)| \leq 0,
\]

where \( \tilde{\Delta}_0 \) acts in the \( x \)-variable. Using (3.15) in [Mu1] one can proceed as in the proof of Theorem 4.3 of [DL] to show that

\[
|K_\nu(t, x, y)| \leq K_0(t, x, y), \quad t \in \mathbb{R}^+, \, x, y \in \tilde{X},
\]

where \( K_0(t, x, y) \) is the kernel of \( e^{-t\tilde{\Delta}_0} \). See also [Gu, p. 7]. Now observe that

\[
H_t^\nu(g^{-1}g') = R^{-1}_g \circ K_\nu(t, gK, g'K) \circ R_{g'} \quad \text{and} \quad H_t^0(g^{-1}g') = K_0(t, gK, g'K).
\]
Since for each \( x \in \tilde{X} \), \( R_g : (\tilde{E}_\nu)_x \to (\tilde{E}_\nu)_{g(x)} \) is an isometry, the proposition follows from (3.9).

Now we pass to the quotient \( X = \Gamma \backslash \tilde{X} \). Let \( \Delta_\nu = \nabla^\nu \nabla^\nu \) be the Bochner-Laplace operator. It is essentially self-adjoint. Let \( R_\Gamma \) be the right regular representation of \( G \) on \( C^\infty(\Gamma \backslash G, \nu) \). By (3.6) it follows that with respect to the isomorphism (3.5) we have

(3.10) \[ \Delta_\nu = -R_\Gamma(\Omega) + \nu(\Omega_K). \]

Let \( e^{-t\Delta_\nu} \) be the heat semigroup of \( \Delta_\nu \), acting on \( L^2(\Gamma \backslash G, \nu) \). Then \( e^{-t\Delta_\nu} \) is represented by the smooth kernel

(3.11) \[ H_\nu(t, g, g') := \sum_{\gamma \in \Gamma} H_\nu^\gamma(t, g^\gamma g'). \]

The convergence of the series in (3.11) can be established, for example, using Proposition 3.1 and the methods from the proof of Proposition 3.2 below. Put

(3.12) \[ h_\nu^\gamma(t) := \text{tr} H_\nu^\gamma(t), \quad g \in G, \]

where \( \text{tr} : \text{End}(V_\nu) \to \mathbb{C} \) is the matrix trace. Then the trace of the heat operator \( e^{-t\Delta_\nu} \) is given by

(3.13) \[ \text{Tr}(e^{-t\Delta_\nu}) = \int_{\Gamma \backslash G} \text{tr} H_\nu(t, g, g) \, dg = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} h_\nu^\gamma(g^{-1}g) \, dg. \]

Using results of Donnelly we now prove an estimate for the heat kernel \( H_0^\nu \) of the Laplacian \( \tilde{\Delta}_0 \) acting on \( C^\infty(\tilde{X}) \).

**Proposition 3.2.** There exist constants \( C_0 \) and \( c_0 \) such that for every \( t \in (0, 1] \) and every \( g \in G \) one has

\[ \sum_{\gamma \in \Gamma} H_0^\nu(g^{-1}g) \leq C_0 e^{-c_0/t}. \]

**Proof.** For \( x, y \in \tilde{X} \) let \( \rho(x, y) \) denote the geodesic distance of \( x, y \). Since \( K(t, gK, g'K) = H_0^\nu(g^{-1}g') \) is the kernel of \( e^{-t\tilde{\Delta}_0} \), it follows from [Don1, Theorem 3.3] that there exists a constant \( C_1 \) such that for every \( g \in G \) and every \( t \in (0, 1] \) one has

(3.14) \[ H_0^\nu(g) \leq C_1 t^{-1} \exp \left( -\frac{\rho^2(gK, 1K)}{4t} \right). \]

Let \( x \in \tilde{X} \) and let \( B_R(x) \) be the metric ball around \( x \) of radius \( R \). Let \( h > 0 \) be the topological entropy of the geodesic flow of \( X \) (see [Man]). There exists \( C_2 > 0 \) such that

(3.15) \[ \text{vol} B_R(x) \leq C_2 e^{hR}, \quad R > 0 \]

[Man]. Since \( \Gamma \) is cocompact and torsion-free, there exists an \( \epsilon > 0 \) such that \( B_\epsilon(x) \cap \gamma B_\epsilon(x) = \emptyset \) for every \( \gamma \in \Gamma - \{1\} \) and every \( x \in \tilde{X} \). Thus for every \( x \in \tilde{X} \) the union over
all $\gamma B_t(x)$, where $\gamma \in \Gamma$ is such that $\rho(x, \gamma x) \leq R$ is disjoint and is contained in $B_{R+\epsilon}(x)$. Using (3.13) it follows that there exists a constant $C_3$ such that for every $x \in \tilde{X}$ one has

$$\# \{\gamma \in \Gamma : \rho(x, \gamma x) \leq R\} \leq C_3 e^{hR}.$$ 

Hence there exists a constant $C_4 > 0$ such that for every $x \in \tilde{X}$ one has

$$\sum_{\gamma \in \Gamma \atop \gamma \neq 1} e^{-\rho^2(\gamma x, x)/8} \leq C_4.$$ 

Now let

$$c_1 := \inf \{\rho(x, \gamma x) : \gamma \in \Gamma - \{1\}, x \in \tilde{X}\}.$$ 

We have $c_1 > 0$. Using (3.14) and (3.16), it follows that there are constants $c_0 > 0$ and $C_0 > 0$ such that for every $g \in G$ and $0 < t \leq 1$ we have

$$\sum_{\gamma \in \Gamma \atop \gamma \neq 1} H^0_t(g^{-1} \gamma gK, gK) \leq C_0 e^{-c_0/t}.$$ 

□

4. The analytic torsion

Let $\tau$ be an irreducible finite-dimensional representation of $G$ on $V_\tau$. Let $E_\tau$ be the flat vector bundle over $X$ associated to the restriction of $\tau$ to $\Gamma$. Let $\tilde{E}_\tau$ be the homogeneous vector bundle associated to $\tau|_K$ and let $E^\tau := \Gamma \backslash \tilde{E}_\tau$. There is a canonical isomorphism

$$E^\tau \cong E_\tau$$

[MtM, Proposition 3.1]. By [MtM, Lemma 3.1], there exists an inner product $\langle \cdot, \cdot \rangle$ on $V_\tau$ such that

1. $\langle \tau(Y)u, v \rangle = -\langle u, \tau(Y)v \rangle$ for all $Y \in \mathfrak{p}$, $u, v \in V_\tau$
2. $\langle \tau(Y)u, v \rangle = \langle u, \tau(Y)v \rangle$ for all $Y \in \mathfrak{p}$, $u, v \in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\tau|_K$ is unitary with respect to this inner product, it induces a metric on $E^\tau$, and by (4.1) on $E_\tau$, which we also call admissible. Let $\Lambda^p(E_\tau) = \Lambda^p T^*(X) \otimes E_\tau$. Let

$$\nu_p(\tau) := \Lambda^p \text{Ad}^* \otimes \tau : K \to \text{GL}(\Lambda^p \mathfrak{p}^* \otimes V_\tau).$$

Then there is a canonical isomorphism

$$\Lambda^p(E_\tau) \cong \Gamma \backslash (G \times \nu_p(\tau) \Lambda^p \mathfrak{p}^* \otimes V_\tau).$$

of locally homogeneous vector bundles. Let $\Lambda^p(X, E_\tau)$ be the space the smooth $E_\tau$-valued $p$-forms on $X$. The isomorphism (4.3) induces an isomorphism

$$\Lambda^p(X, E_\tau) \cong C^\infty(\Gamma \backslash G, \nu_p(\tau)).$$
where the latter space is defined as in (3.4). A corresponding isomorphism also holds for the spaces of $L^2$-sections. Let $\Delta_p(\tau)$ be the Hodge-Laplacian on $\Lambda^p(X, E_\tau)$ with respect to the admissible metric in $E_\tau$. Let $R_\Gamma$ denote the right regular representation of $G$ in $L^2(\Gamma \backslash G)$. By [MtM, (6.9)] it follows that with respect to the isomorphism (4.4) one has

$$\Delta_p(\tau)f = -R_\Gamma(\Omega)f + \tau(\Omega)\text{Id}f, \ f \in C^\infty(\Gamma \backslash G, \nu_p(\tau)).$$

We remark that in [MtM] it is not assumed that $G$ does not have compact factors, see the remark on page 372 of [MtM], and so we do not make this assumption either. Let

$$(4.5) \quad K(t, \tau) := \sum_{p=1}^d (-1)^p p \text{Tr}(e^{-t\Delta_p(\tau)}).$$

and

$$(4.6) \quad h(\tau) := \sum_{p=1}^d (-1)^p p \text{dim} \ H^p(X, E_\tau).$$

Then $K(t, \tau) - h(\tau)$ decays exponentially as $t \to \infty$ and it follows from (1.1) that

$$(4.7) \quad \log T_X(\tau) = \frac{1}{2} \left| \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(K(t, \tau) - h(\tau)) \ dt \right|_{s=0},$$

where the right hand side is defined near $s = 0$ by analytic continuation of the Mellin transform. Let $\tilde{E}_{\nu_p(\tau)} := G \times_{\nu_p(\tau)} \Lambda^p p^* \otimes V_\tau$ and let $\tilde{\Delta}_p(\tau)$ be the lift of $\Delta_p(\tau)$ to $C^\infty(\tilde{X}, \tilde{E}_{\nu_p(\tau)})$. Then again it follows from [MtM, (6.9)] that on $C^\infty(G, \nu_p(\tau))$ one has

$$(4.8) \quad \tilde{\Delta}_p(\tau) = -R_\Gamma(\Omega) + \tau(\Omega)\text{Id}.$$ 

Let $e^{-t\tilde{\Delta}_p(\tau)}$ be the corresponding heat semigroup on $L^2(G, \nu_p(\tau))$. It is a smoothing operator which commutes with the action of $G$. Therefore, it is of the form

$$\left(e^{-t\tilde{\Delta}_p(\tau)} \phi \right)(g) = \int_G H^{\tau, p}_t(g^{-1}g')\phi(g') \ dg', \ \phi \in (L^2(G, \nu_p(\tau)), \ g \in G),$$

where the kernel

$$(4.9) \quad H^{\tau, p}_t: G \to \text{End}(\Lambda^p p^* \otimes V_\tau)$$

belongs to $C^\infty \cap L^2$ and satisfies the covariance property

$$(4.10) \quad H^{\tau, p}_t(k^{-1}gk') = \nu_p(\tau)(k)^{-1}H^{\tau, p}_t(g)\nu_p(\tau)(k')$$

with respect to the representation (4.12). Moreover, for all $q > 0$ we have

$$(4.11) \quad H^{\tau, p}_t \in (C^q(G) \otimes \text{End}(\Lambda^p p^* \otimes V_\tau))^{K \times K},$$

where $C^q(G)$ denotes Harish-Chandra’s $L^q$-Schwartz space. The proof is similar to the proof of Proposition 2.4 in [BM]. Now we come to the heat kernel of $\Delta_p(\tau)$. First the integral kernel of $e^{-t\tilde{\Delta}_p(\tau)}$, regarded as an operator in $L^2(\Gamma \backslash G, \nu_p(\tau))$, is given by

$$(4.12) \quad H^{\tau, p}_t(t; g, g') := \sum_{\gamma \in \Gamma} H^{\tau, p}_t(g^{-1}\gamma g'),$$
As in section 3 this series converges absolutely and locally uniformly. Therefore the trace of the heat operator $e^{-t\Delta_p(\tau)}$ is given by

$$\text{Tr} \left( e^{-t\Delta_p(\tau)} \right) = \int_{\Gamma \setminus G} \text{tr} H^{\tau,p}(t; g, g) \, dg,$$

where $\text{tr}$ denotes the trace $\text{tr} : \text{End}(V) \to \mathbb{C}$. Let

(4.13) \[ h^{\tau,p}_t(g) := \text{tr} H^{\tau,p}_t(g). \]

Using (4.12) we obtain

(4.14) \[ \text{Tr} \left( e^{-t\Delta_p(\tau)} \right) = \int_{\Gamma \setminus G} \sum_{\gamma \in \Gamma} h^{\tau,p}_t(g^{-1}\gamma g) \, dg. \]

Put

(4.15) \[ k^\tau_t = \sum_{p=1}^d (-1)^p p h^{\tau,p}_t. \]

Then it follows that

(4.16) \[ K(t, \tau) = \int_{\Gamma \setminus G} \sum_{\gamma \in \Gamma} k^\tau_t(g^{-1}\gamma g) \, dg. \]

Let $R_\Gamma$ be the right regular representation of $G$ on $L^2(\Gamma \setminus G)$. Then (4.16) can be written as

(4.17) \[ K(t, \tau) = \text{Tr} R_\Gamma(k^\tau_t). \]

We shall now compute the Fourier transform of $k^\tau_t$. To begin with let $\pi$ be an admissible unitary representation of $G$ on a Hilbert space $\mathcal{H}_\pi$. Set

$$\tilde{\pi}(H^{\tau,p}_t) = \int_G \pi(g) \otimes H^{\tau,p}_t(g) \, dg.$$

This defines a bounded operator on $\mathcal{H}_\pi \otimes \Lambda^p p^* \otimes V_\tau$. As in [BM, pp. 160-161] it follows from (4.11) that relative to the splitting

$$\mathcal{H}_\pi \otimes \Lambda^p p^* \otimes V_\tau = (\mathcal{H}_\pi \otimes \Lambda^p p^* \otimes V_\tau)^K \oplus \left[ (\mathcal{H}_\pi \otimes \Lambda^p p^* \otimes V_\tau)^K \right]^\perp,$$

$\tilde{\pi}(H^{\tau,p}_t)$ has the form

$$\tilde{\pi}(H^{\tau,p}_t) = \begin{pmatrix} \pi(H^{\tau,p}_t) & 0 \\ 0 & 0 \end{pmatrix},$$

with $\pi(H^{\tau,p}_t)$ acting on $\left( \mathcal{H}_\pi \otimes \Lambda^p p^* \otimes V_\tau \right)^K$. Using (4.8) it follows as in [BM, Corollary 2.2] that

(4.18) \[ \pi(H^{\tau,p}_t) = e^{t(\pi(\Omega) - \tau(\Omega))} \text{Id} \]
on \((\mathcal{H}_\pi \otimes \Lambda^p p^* \otimes V_\tau)^K\). Let \(\{\xi_n\}_{n \in \mathbb{N}}\) and \(\{e_j\}_{j=1}^m\) be orthonormal bases of \(\mathcal{H}_\pi\) and \(\Lambda^p p^* \otimes V_\tau\), respectively. Then we have

\[
\text{Tr} \pi(H_{t_{\tau,p}}^p) = \sum_{n=1}^\infty \sum_{j=1}^m \langle \pi(H_{t_{\tau,p}}^p)(\xi_n \otimes e_j), (\xi_n \otimes e_j) \rangle \\
= \sum_{n=1}^\infty \sum_{j=1}^m \int_G \langle \pi(g)\xi_n, \xi_n \rangle \langle H_{t_{\tau,p}}^p(g)e_j, e_j \rangle \, dg \\
= \sum_{n=1}^\infty \int_G h_{t_{\tau,p}}^p(g) \langle \pi(g)\xi_n, \xi_n \rangle \, dg \\
= \text{Tr} \pi(h_{t_{\tau,p}}^p). 
\]

(4.19)

Let \(\pi \in \hat{G}\) and let \(\Theta_\pi\) denote its character. Then it follows from (4.15), (4.18) and (4.19) that

\[
\Theta_\pi(k_{\tau,t}) = e^{t(\pi(\Omega) - \tau(\Omega))} \sum_{p=1}^d (-1)^p p \cdot \dim(\mathcal{H}_\pi \otimes \Lambda^p p^* \otimes V_\tau)^K. 
\]

(4.20)

Now we consider the case of a principle series representation. Let \(Q\) be a standard cuspidal parabolic subgroup. Let \(Q = MAN\) be the Langlands decomposition of \(Q\). Denote by \(\mathfrak{a}\) the Lie algebra of \(A\). Let \(K_M = K \cap M\). Let \(\xi, W_\xi\) be a discrete series representation of \(M\) and let \(\nu \in \mathfrak{a}^*\). Let \(\pi_{\xi,\nu}\) be the induced representation and let \(\Theta_{\xi,\nu}\) be the character of \(\pi_{\xi,\nu}\) (see section 2).

**Proposition 4.1.** Let \(Y \in \mathfrak{a}\) be a unit vector and let \(p_Y^*\) be the orthogonal complement of \(Y\) in \(\mathfrak{p}\). Then

\[
\begin{align*}
(i) \quad & \Theta_{\xi,\nu}(k_{t_{\tau}}^\tau) = e^{t(\pi(\Omega) - \tau(\Omega))} \dim (W_\xi \otimes (\Lambda^{odd} p_Y^* - \Lambda^{ev} p_Y^*) \otimes V_\tau)^{K_M}, \\
(ii) \quad & \Theta_{\xi,\nu}(k_{t_{\tau}}^\tau) = 0 \text{ if } \dim \mathfrak{a}_q \geq 2.
\end{align*}
\]

**Proof.** By Frobenius reciprocity [Kn1, p. 208] and (4.20) we get

\[
\Theta_{\xi,\nu}(k_{t_{\tau}}^\tau) = e^{t(\pi(\Omega) - \tau(\Omega))} \sum_{p=1}^d (-1)^p p \dim (W_\xi \otimes \Lambda^p p^* \otimes V_\tau)^{K_M}.
\]

Now

\[
\mathfrak{p}^* = \mathbb{R}Y^* \oplus p_Y^*. 
\]
as $K_M$-module. Therefore, in the Grothendieck ring of $K_M$ we have

$$
\sum_{p=1}^{d} (-1)^p p\Lambda^p p^* = \sum_{p=1}^{d} (-1)^p p\Lambda^p p^*_Y \oplus \Lambda^{p-1} p^*_Y
$$

(4.21)

$$
= \sum_{p=1}^{d} (-1)^p p\Lambda^p p^*_Y + \sum_{p=0}^{d-1} (-1)^{p+1}(p+1)\Lambda^p p^*_Y
$$

$$
= \sum_{p=0}^{d} (-1)^{p+1} \Lambda^p p^*_Y.
$$

Tensoring with $W_\xi$ and $V_\tau$ and taking $K_M$-invariants, we obtain (i).

To prove (ii), suppose that there is a nonzero $H \in a \cap p_Y$. Since $M$ centralizes $H$, $\varepsilon(H) + i(H)$ is a $K_M$ intertwining operator between $\Lambda^\text{ev} p^*_Y$ and $\Lambda^\text{odd} p^*_Y$, and non-trivial since $H \neq 0$. Hence $\Lambda^\text{ev} p^*_Y$ and $\Lambda^\text{odd} p^*_Y$ are equivalent as $K_M$-modules and (ii) follows. □

Proposition 4.2. Assume that $\delta(\tilde{X}) \geq 2$ or assume that $\tilde{X}$ is even-dimensional. Then $T_X(\tau) = 1$ for all finite-dimensional irreducible representations $\tau$ of $G$.

Proof. Let

$$
R_\Gamma = \bigoplus_{\pi \in \hat{G}} m_\Gamma(\pi)\pi
$$

be the decomposition of the right regular representation $R_\Gamma$ of $G$ on $L^2(\Gamma \backslash G)$, see [Wa1, section 1]. Now insert $k^\tau_i$ on both sides and apply the trace. If we recall that $\Theta_\pi(k^\tau_i) = \text{tr} \pi(k^\tau_i)$ and use (4.17), we get

$$
K(t, \tau) = \sum_{\pi \in \hat{G}} m_\Gamma(\pi)\Theta_\pi(k^\tau_i).
$$

(4.22)

The series on the right hand side is absolutely convergent. First assume that $\delta(X) \geq 2$. By [D1, section 2.2] the Grothendieck group of all admissible representations of $G$ is generated by the representations $\pi_{\xi,\lambda}$, where $\pi_{\xi,\lambda}$ is associated to some standard cuspidal parabolic subgroup $Q$ of $G$ as in (2.2). Since $\delta(X) \geq 2$ one has $\Theta_{\xi,\lambda}(k^\tau_i) = 0$ for every such representation by Proposition 4.1. Thus one has $\Theta_\pi(k^\tau_i) = 0$ for every irreducible unitary representation of $G$. By (4.22) it follows that $K(t, \tau) = 0$. Let $h(\tau)$ be as in (4.4). Since $K(t, \tau) - h(\tau)$ decays exponentially as $t \to \infty$, it follows that $K(t, \tau) - h(\tau) = 0$ and using (4.7), the first statement follows.

Now assume that $d = \dim \tilde{X}$ is even. Note that as $K$-modules we have

$$
\Lambda^p p^* \cong \Lambda^{d-p} p^*, \quad p = 0, \ldots, d.
$$

Since $d$ is even, it follows that in the representation ring $R(K)$ we have the following equality

$$
\sum_{p=0}^{d} (-1)^p p\Lambda^p p^* = \frac{d}{2} \sum_{p=0}^{d} (-1)^p \Lambda^p p^*.
$$
Let \((\pi, \mathcal{H}_\pi) \in \hat{G}\). Then it follows from (4.20) that
\[
\Theta_\pi(k^\tau_t) = \frac{d}{2} e^{t(\pi(\Omega) - \tau(\Omega))} \sum_{p=0}^{d} (-1)^p \dim(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K.
\]

Let \(\mathcal{H}_{\pi,K}\) be the subspace of \(\mathcal{H}_\pi\) consisting of all smooth \(K\)-finite vectors. Then
\[
(\mathcal{H}_{\pi,K} \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K = (\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K.
\]
Thus the \((\mathfrak{g}, K)\)-cohomology \(H^* (\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau)\) is computed from the Lie algebra cohomology complex \([(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K, d]\) (see [BW]). Using the Poincaré principle we get
\[
\Theta_\pi(k^\tau_t) = \frac{d}{2} e^{t(\pi(\Omega) - \tau(\Omega))} \sum_{p=0}^{d} (-1)^p \dim H^p (\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau).
\]

Now by [BW, II.3.1, I.5.3] we have
\[
H^p (\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau) = \begin{cases} 
[\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau]^K, & \pi(\Omega) = \tau(\Omega); \\
0, & \pi(\Omega) \neq \tau(\Omega).
\end{cases}
\]
Hence for every \(\pi \in \hat{G}\) one has \(\Theta_\pi(k^\tau_t) \in \mathbb{Z}\) and \(\Theta_\pi(k^\tau_t)\) is independent of \(t > 0\). Since the series on the right hand side of (4.22) converges absolutely, there exist only finitely many \(\pi \in \hat{G}\) with \(m_\Gamma(\pi) \neq 0\) and \(\Theta_\pi(k^\tau_t) \neq 0\). Thus \(K(t, \tau)\) is independent of \(t > 0\). Let \(h(\tau)\) be defined by (4.16). Then \(K(t, \tau) - h(\tau) = O(e^{-ct})\) as \(t \to \infty\). Hence \(K(t, \tau) = h(\tau)\). By (4.7) it follows that \(T_X(\tau) = 1\).

5. \(L^2\)-torsion

In this section we study the \(L^2\)-torsion \(T^{(2)}_X(\tau)\). For its definition we refer to [Lo], [Mat]. Actually, in [Lo] and [Mat] only the case of the trivial representation \(\tau_0\) has been discussed. However the extension to a nontrivial \(\tau\) is straightforward. The definition is based on the \(\Gamma\)-trace of the heat operator \(e^{-t\Delta_p(\tau)}\) on the universal covering \(\tilde{X}\) (see [Lo], [Mat]). For our purposes, it suffices to introduce the \(L^2\)-torsion for representations \(\tau\) on \(X\) which satisfy \(\tau_0 \not\equiv \tau\).

Let \(h_t^{\tau,p}\) be the function defined by (1.13). By homogeneity it follows that in our case the \(\Gamma\)-trace is given by
\[
\text{Tr}_\Gamma(e^{-t\Delta_p(\tau)}) = \text{vol}(X)h_t^{\tau,p}(1).
\]
In order to define the \(L^2\)-torsion we need to know the asymptotic behavior of \(h_t^{\tau,p}(1)\) as \(t \to 0\) and \(t \to \infty\). First we consider the behavior as \(t \to 0\). Using (4.14) we have
\[
\text{vol}(X)h_t^{\tau,p}(1) = \text{Tr}(e^{-t\Delta_p(\tau)}) - \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma \setminus \{1\}} h_t^{\tau,p}(g^{-1}\gamma g) \, dg.
\]
To deal with the second term on the right, we consider the representation \( \nu_p(\tau) \) of \( K \) which is defined by (4.2), and for \( p = 0, \ldots, n \) we put

\[
E_p(\tau) := \tau(\Omega) \text{Id} - \nu_p(\tau)(\Omega_K),
\]

which we regard as endomorphism of \( \Lambda^p \mathfrak{p}^* \otimes V_\tau \). It defines an endomorphism of \( \Lambda^p T^*(X) \otimes E_\tau \). By (3.6) and (4.8) we have

\[
\Delta_p(\tau) = \Delta_{\nu_p}(\tau) + E_p(\tau).
\]

Let \( \nu_p(\tau) = \oplus_{\sigma \in \hat{K}} m(\sigma)\sigma \) be the decomposition of \( \nu_p(\tau) \) into irreducible representations. This induces a corresponding decomposition of the homogeneous vector bundle

\[
\tilde{E}_{\nu_p(\tau)} = \bigoplus_{\sigma \in \hat{K}} m(\sigma)\tilde{E}_\sigma.
\]

With respect to this decomposition we have

\[
E_p(\tau) = \bigoplus_{\sigma \in \hat{K}} m(\sigma) (\tau(\Omega) - \sigma(\Omega_K)) \text{Id}_{V_\sigma},
\]

where \( \sigma(\Omega_K) \) is the Casimir eigenvalue of \( \sigma \) and \( V_\sigma \) is the representation space of \( \sigma \), and

\[
\Delta_{\nu_p}(\tau) = \bigoplus_{\sigma \in \hat{K}} m(\sigma)\Delta_\sigma.
\]

This shows that \( \Delta_{\nu_p}(\tau) \) commutes with \( E_p(\tau) \). Let \( H^{t\nu_p}(\tau) \) be the kernel of \( e^{-t\tilde{\Delta}_{\nu_p}(\tau)} \) and let \( H^{t\nu_p}(\tau) \) be the kernel of \( e^{-t\tilde{\Delta}_p(\tau)} \). Using (5.4) we get

\[
H^{t\nu_p}(\tau) = \bigoplus_{\sigma \in \hat{K}} m(\sigma)\Delta_\sigma.
\]

Let \( c \in \mathbb{R} \) be such that \( E_p(\tau) \geq c \). By Proposition 3.1 it follows that

\[
\|H^{t\nu_p}(\tau)(g)\| \leq e^{-ct}H^0(\tau)(g), \quad t \in \mathbb{R}^+, \; g \in G.
\]

Taking the trace in \( \text{End}(\Lambda^p \mathfrak{p}^* \otimes V_\tau) \) we get

\[
\sum_{\gamma \in \Gamma \setminus \{1\}} |h^{t\nu_p}(g^{-1}\gamma g)| \leq \left( \frac{d}{p} \right) \dim(\tau)e^{-ct} \sum_{\gamma \in \Gamma \setminus \{1\}} H^0(g^{-1}\gamma g), \quad t \in \mathbb{R}^+, \; g \in G.
\]

By Proposition 3.2 there exist \( C_1, c_1 > 0 \) such that

\[
\int_{\Gamma \setminus G} \sum_{\gamma \in \Gamma \setminus \{1\}} |h^{t\nu_p}(g^{-1}\gamma g)| \, dg \leq C_1 e^{-c_1/t}
\]

for \( 0 < t \leq 1 \). Thus by (5.2)

\[
h^{t\nu_p}(1) = \frac{1}{\text{vol}(X)} \text{Tr} \left( e^{-t\Delta_p(\tau)} \right) + O(e^{-c_1/t})
\]
for $0 < t ≤ 1$. Using the asymptotic expansion of $\text{Tr} \left( e^{-t \Delta_p(\tau)} \right)$ (see [3]), it follows that there is an asymptotic expansion

\begin{equation}
(5.11)
    h_t^{\tau,p}(1) \sim \sum_{j=0}^{\infty} a_j t^{-d/2+j}
\end{equation}

as $t \to 0$. To study the behavior of $h_t^{\tau,p}(1)$ as $t \to \infty$, we use the Plancherel theorem, which can be applied since $h_t^{\tau,p}$ is a $K$-finite Schwarz function. Let $\pi$ be an admissible unitary representation of $G$ on a Hilbert space $\mathcal{H}_\pi$. It follows from (4.18) and (4.19) that

\[ \text{Tr} \pi(h_t^{\tau,p}) = e^{t(\pi(\Omega) - \tau(\Omega))} \dim (\mathcal{H}_\pi \otimes \Lambda^p p^* \otimes V_\tau)^K. \]

Let $Q = MAN$ be a standard parabolic subgroup of $G$. Let $(\xi, W_\xi)$ be a discrete series representation of $M$. Let $\langle \cdot, \cdot \rangle$ denote the inner product on the real vector space $a^*$ induced by the Killing form. Fix positive restricted roots of $a$ and let $\rho_a$ denote the corresponding half-sum of these roots. Define a constant $c(\xi)$ by

\begin{equation}
(5.12)
    c(\xi) := -\langle \rho_a, \rho_a \rangle + \xi(\Omega_M).
\end{equation}

Recall that for $\nu \in a^*$ one has

\begin{equation}
(5.13)
    \pi_{\xi,\nu}(\Omega) = -\langle \nu, \nu \rangle + c(\xi).
\end{equation}

Then by the Plancherel theorem, [HC, Theorem 3] and (5.13) we have

\begin{equation}
(5.14)
    h_t^{\tau,p}(1) = \sum_{Q} \sum_{\xi \in \hat{M}_d} e^{-t(\tau(\Omega) - c(\xi))} \int_{a^*} e^{-\|\nu\|^2} \dim (\mathcal{H}_{\xi,\nu} \otimes \Lambda^p p^* \otimes V_\tau)^K p_{\xi}(i\nu) d\nu.
\end{equation}

The exponents of the exponential factors in front of the integrals are controlled by the following lemma.

**Lemma 5.1.** Let $(\tau, V_\tau) \in \text{Rep}(G)$. Assume that $\tau \not\sim \tau_\theta$. Let $P = MAN$ be a cuspidal parabolic subgroup of $G$. Let $\xi \in \hat{M}_d$ and assume that $\dim (W_\xi \otimes \Lambda^p p^* \otimes V_\tau)^{K_M} \neq 0$. Then one has

\[ \tau(\Omega) - c(\xi) > 0. \]
Proof. Assume that $\tau(\Omega) - c(\xi) \leq 0$. Then by (5.13) there exists a $\nu_0 \in a^*$ such that
$$\pi_{\xi,\nu_0}(\Omega) = \tau(\Omega).$$
Together with (5.14), our assumption and [BW, Proposition II.3.1] it follows that
$$\dim (H^p(\mathfrak{g}, K; H_{\xi,\nu_0,K} \otimes V_\tau)) \neq 0,$$
where $H_{\xi,\nu_0,K}$ are the $K$-finite vectors in $H_{\xi,\nu_0}$. Since $\tau \not\simeq \tau_\theta$, this is a contradiction to the first statement of [BW, Proposition II.6.12]. □

Let $\tau$ be an irreducible representation of $G$ which satisfies $\tau \not\simeq \tau_\theta$. It follows from (5.15) and Lemma 5.1 that there exists $c > 0$ such that (5.16)
$$h_\tau^{\tau,p}(1) = O \left( e^{-ct} \right)$$
as $t \to \infty$. Using (5.11) and (5.16) it follows from standard methods, see for example [Gi], that the Mellin transform
$$\int_0^\infty h_\tau^{\tau,p}(1) t^{s-1} dt$$
converges absolutely and uniformly on compact subsets of the half-plane $\text{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$ if $d = \dim(\tilde{X})$ is odd and has at most a simple pole at $s = 0$ for $d = \dim(\tilde{X})$ even. Thus we can define the $L^2$-torsion $T_X^{(2)}(\tau) \in \mathbb{R}^+$ by
$$\log T_X^{(2)}(\tau)$$
(5.17)
$$:= \frac{1}{2} \sum_{p=1}^d (-1)^p p \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty \text{Tr} \left( e^{-t\tilde{\Delta}_p(\tau)} \right) t^{s-1} dt \right) \bigg|_{s=0},$$
where the right hand side is defined near $s = 0$ by analytic continuation. For $t > 0$ let
$$K^{(2)}(t, \tau) := \sum_{p=1}^d (-1)^p p h_\tau^{\tau,p}(1).$$

Put
$$t_X^{(2)}(\tau) := \frac{1}{2} \sum_{p=1}^d (-1)^p p K^{(2)}(t, \tau) t^{s=1} dt \bigg|_{s=0}.$$
Then $t_X^{(2)}(\tau)$ depends only on the symmetric space $\tilde{X}$ and $\tau$, and we have
$$\log T_X^{(2)}(\tau) = \text{vol}(X) \cdot t_X^{(2)}(\tau).$$

Next we establish an auxiliary result concerning the twisted Euler characteristic. We let $\tau \in \text{Rep}(G)$ be arbitrary. Let $H^p(X, E_\tau) := \ker \Delta_p(\tau)$ be the space of $E_\tau$-valued harmonic $p$-forms. Let
$$\chi(X, E_\tau) := \sum_{p=0}^d (-1)^p \dim H^p(X, E_\tau)$$
be the twisted Euler characteristic. Furthermore, let \( \tilde{X}_d \) denote the compact dual of \( \tilde{X} \). The following proposition is a familiar consequence of the index theorem. For the convenience of the reader we include an independent proof.

**Proposition 5.2.** If \( \delta(\tilde{X}) \neq 0 \), we have \( \chi(X, E_\tau) = 0 \). If \( \delta(\tilde{X}) = 0 \), one has

\[
\chi(X, E_\tau) = (-1)^n \frac{\chi(\tilde{X}_d)}{\text{vol}(\tilde{X}_d)} \dim(\tau),
\]

where \( n = \dim(\tilde{X})/2 \).

**Proof.** Let \( \pi \in \hat{G} \). It follows from (4.18) and (4.20) that

\[
\sum_{p=0}^{d} (-1)^p \Theta_\pi(h^{P,\tau}_t) = e^{t(\pi(\Omega) - \tau(\Omega))} \sum_{p=0}^{d} (-1)^p \dim(\mathcal{H}_\pi \otimes \Lambda^p V^* \otimes V_\tau)^K.
\]

Using [BW, II.3.1] and the Poincaré principle as in the proof of Proposition 4.2, we get

\[
\sum_{p=0}^{d} (-1)^p \Theta_\pi(h^{P,\tau}_t) = \sum_{p=0}^{d} \dim H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau).
\]

Now by [BW, Theorem I.5.3] it follows that if \( H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau) \neq 0 \), then \( \chi_\pi = \chi_{\tilde{\tau}} \), where \( \tilde{\tau} \) is the contragredient representation of \( \tau \). By [Kn1, Corollary 10.37, Corollary 9.2] there are only finitely many representations \( \pi \in \hat{G} \) with a given infinitesimal character. Thus if \( Q = MAN \) is a fundamental parabolic subgroup with \( Q \neq G \) and if \( \xi \in \hat{M}_d \), it follows that there are only finitely many \( \lambda \in \mathfrak{a}^* \) such that

\[
\sum_{p=0}^{d} (-1)^p \Theta_{\xi,\lambda}(h^{P,\tau}_t) \neq 0.
\]

Hence by the Plancherel-Theorem, [HC, Theorem 3] and (5.22) we get

\[
\sum_{p=0}^{d} (-1)^p h^{P,\tau}_t(1) = \sum_{p=0}^{d} \sum_{\pi \in \hat{G}_d} d(\pi) \dim H^p(\mathfrak{g}, K; \mathcal{H}_{\pi,K} \otimes V_\tau),
\]

where \( \hat{G}_d \) denotes the discrete series of \( G \) and \( d(\pi) \) denotes the formal degree of \( \pi \). The sum is finite. Let

\[
b_p^{(2)}(X, E_\tau) := \lim_{t \to \infty} \text{Tr}_\tau \left( e^{-t\Delta_p(\tau)} \right)
\]

be the \( L^2 \)-Betti number. Using that (5.24) is independent of \( t \) and (5.1), we get

\[
\text{vol}(X) \sum_{p=0}^{d} (-1)^p h^{P,\tau}_t(1) = \sum_{p=0}^{d} (-1)^p b_p^{(2)}(X, E_\tau) = \chi^{(2)}(X, E_\tau).
\]
By the Γ-index theorem of Atiyah \cite{At} we have \(\chi(\tau) = \chi(X, \mathcal{E}_\tau)\). Hence by (5.24) and (5.25) we get

\[
(5.26) \quad \chi(X, \mathcal{E}_\tau) = \text{vol}(X) \cdot \sum_{p=0}^{d} (-1)^p \sum_{\pi \in \hat{G}_d} d(\pi) \dim H^p(\mathfrak{g}, \mathcal{H}_{\pi,K} \otimes V_\tau).
\]

If \(\delta(\widetilde{X}) \neq 0\) then \(\hat{G}_d\) is empty and hence, this sum equals zero, which proves the first statement. Now assume that \(\delta(\widetilde{X}) = 0\). Then \(\widetilde{X}\) is even-dimensional. Let \(\dim(\widetilde{X}) = 2n\). We keep the notation from section 2.2. By \cite[Corollary 5.2]{Ol} for \(\Lambda' = w(\Lambda(\tilde{\tau}) + \rho_G)\), \(w \in W_G/W_K\) one has

\[
d(\omega_{\Lambda'}) = \frac{\dim(\tau)}{\text{vol}(\widetilde{X}_d)}
\]

and so together with Proposition 2.1 we get

\[
(5.27) \quad \sum_{p=0}^{d} (-1)^p \sum_{\pi \in \hat{G}_d} d(\pi) \dim H^p(\mathfrak{g}, \mathcal{H}_{\pi,K} \otimes V_\tau)
\]

\[
= (-1)^n \frac{1}{\text{vol}(\widetilde{X}_d)} \#(W_G/W_K) \dim(\tau).
\]

Finally, by \cite[page 175]{Bo} one has

\[
\#(W_G/W_K) = \chi(\widetilde{X}_d).
\]

Applying equation (5.27), the proof of the Proposition follows. \(\square\)

**Remark 1.** We remark that if \(X\) is Hermitian and \(\tau\) is the trivial representation, then equation (5.21) reduces to Hirzebruch’s Proportionality principle.

Now we assume that \(\delta(\widetilde{X}) = 1\) and that \(\widetilde{X}\) is odd-dimensional. By the classification of simple Lie groups we have \(\widetilde{X} = \widetilde{X}_0 \times \widetilde{X}_1\), where \(\delta(\widetilde{X}_0) = 0\) and \(\widetilde{X}_1 = \text{SL}(3, \mathbb{R})/\text{SO}(3)\) or \(\widetilde{X}_1 = \text{SO}^0(p, q)/\text{SO}(p) \times \text{SO}(q)\), \(p, q\) odd. Let \(\widetilde{X}_0 = G_0/K_0\) and let \(G_1 = \text{SL}(3, \mathbb{R})\), \(K_1 = \text{SO}(3)\) or \(G_1 = \text{SO}^0(p, q)\), \(K_1 = \text{SO}(p) \times \text{SO}(q)\), \(p, q\) odd. Let \(G = G_0 \times G_1\). Let \(\tau\) be a finite-dimensional irreducible representation of \(G\) and assume that \(\tau \not\sim \tau_0\). Then \(\tau = \tau_0 \otimes \tau_1\), where \(\tau_i\) is an irreducible representation of \(G_i\), \(i = 0, 1\), and \(\tau_1 \not\sim \tau_{1,0}\).

**Proposition 5.3.** Let \(\delta(\widetilde{X}) = 1\) and let \(\widetilde{X}\) be odd-dimensional. Let \(\widetilde{X} = \widetilde{X}_0 \times \widetilde{X}_1\), where \(\widetilde{X}_1\) is an odd-dimensional irreducible symmetric space with \(\delta(\widetilde{X}_1) = 1\). Let \(\tau\) be a finite-dimensional irreducible representation of \(G\) with \(\tau \not\sim \tau_0\). Then

\[
i_\widetilde{X}^{(2)}(\tau) = (-1)^{\dim(\widetilde{X}_0)/2} \frac{\chi(\widetilde{X}_{0,d})}{\text{vol}(\widetilde{X}_{0,d})} \dim(\tau_0) \cdot i_{\widetilde{X}_1}^{(2)}(\tau_1).
\]
Proof. Let $\tilde{E} \to \tilde{X}$ be the homogeneous vector bundle associated to $\tau|_K$. Similarly, let $\tilde{E}_i \to \tilde{X}_i$ be the homogeneous vector bundle associated to $\tau_i|_{K_i}$, $i = 0, 1$. Then $\tilde{E} \cong \tilde{E}_1 \boxtimes \tilde{E}_2$ and

$$
\Lambda^k(\tilde{X}, \tilde{E}) \cong \bigoplus_{p+q=k} \left( \Lambda^p(\tilde{X}_0, \tilde{E}_0) \otimes \Lambda^q(\tilde{X}_1, \tilde{E}_1) \right).
$$

With respect to this decomposition we have

$$
\Delta_k(\tau) = \bigoplus_{p+q=k} \left( \Delta_p(\tau_0) \otimes \text{Id} + \text{Id} \otimes \Delta_q(\tau_1) \right).
$$

Let $H^{\tau,k}_t$ and $H^{\tau_i,p}_t$, $i = 0, 1$, be the corresponding heat kernels. Then it follows that $H^{\tau,k}_t = \bigoplus_{p+q=k} H^{\tau_i,p}_t \otimes H^{\tau_i,q}_t$. Hence for $h^{\tau,k}_t = \text{tr} H^{\tau,k}_t$ and $h^{\tau_i,p}_t = \text{tr} H^{\tau_i,p}_t$, $i = 0, 1$, we have

$$
h^{\tau,k}_t = \sum_{p+q=k} h^{\tau_i,p}_t \cdot h^{\tau_i,q}_t.
$$

Using this equality, we get

$$
\sum_{k=0}^d (-1)^k k h^{\tau,k}_t(1) = \sum_{p=0}^{d_1} \sum_{q=0}^{d_2} (-1)^{p+q}(p+q) h^{\tau_i,p}_t(1) \cdot h^{\tau_i,q}_t(1)
$$

(5.28)

$$
= \sum_{p=0}^{d_1} (-1)^p h^{\tau_i,p}_t(1) \cdot \sum_{q=0}^{d_2} (-1)^q h^{\tau_i,q}_t(1)
$$

$$
+ \sum_{q=0}^{d_2} (-1)^q h^{\tau_i,q}_t(1) \cdot \sum_{p=0}^{d_1} (-1)^p h^{\tau_i,p}_t(1).
$$

Let $\Gamma_i \subset G_i$, $i = 0, 1$, any cocompact, torsion free discrete subgroup. The existence of the $\Gamma_i$ follows from our assumptions on the $G_i$ stated in the introduction and from results of Borel [Bor]. Put $X_i = \Gamma_i \backslash \tilde{X}_i$ and $E_i = \Gamma \backslash \tilde{E}_i$. By (5.23) and the remark following it we have

$$
\sum_{p=0}^d (-1)^p h^{\tau_i,p}_t(1) = \frac{\chi(X_i)}{\text{vol}(X_i)}, \quad i = 0, 1.
$$

(5.29)

Taking the Mellin transform of (5.28) and using (5.29) and Proposition 5.2, the proposition follows.

\[\square\]

6. The Asymptotics of the $L^2$-torsion for $\delta(\tilde{X}) = 1$

In this section we study the asymptotic behaviour of the $L^2$-torsion of an odd-dimensional irreducible symmetric space $\tilde{X}$ with $\delta(\tilde{X}) = 1$. Then we can assume that $G = \SO^0(p, q)$, $p, q$ odd, and $K = \SO(p) \times \SO(q)$, or $G = \SL_3(\mathbb{R})$ and $K = \SO(3)$. To compute the $L^2$ torsion in these cases, we need some preparation. Let $P = MAN$ be a fundamental parabolic subgroup of $G$, i.e. we have $\dim(A) = 1$. Let $M^0$ be the identity component of $M$ and let $\mathfrak{m}$ be its Lie algebra. Then in our case $\mathfrak{m}$ is always semisimple. Let $K_M := K \cap M$,
let $K^0_M$ be the identity component of $K_M$ and let $\mathfrak{t}_m := \mathfrak{t} \cap \mathfrak{m}$ be its Lie algebra. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{t}_m$. Then $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{m}$ and of $\mathfrak{t}$. Moreover $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

Let $\Delta(\mathfrak{g}_C, \mathfrak{h}_C), \Delta(\mathfrak{m}_C, \mathfrak{t}_C), \Delta((\mathfrak{t}_m)_C, \mathfrak{t}_C)$ be the corresponding roots. Then there is a canonical inclusion $\Delta(\mathfrak{m}_C, \mathfrak{t}_C) \hookrightarrow \Delta(\mathfrak{g}_C, \mathfrak{h}_C)$. Fix a positive restricted root $e_1 \in \mathfrak{a}^*$ and fix positive roots $\Delta^+(\mathfrak{m}_C, \mathfrak{t}_C)$. In this way we obtain positive roots $\Delta^+(\mathfrak{g}_C, \mathfrak{h}_C)$. Let $\rho_G$ resp. $\rho_M$ be the half sums of the elements of $\Delta^+(\mathfrak{g}_C, \mathfrak{h}_C)$ and $\Delta^+(\mathfrak{m}_C, \mathfrak{t}_C)$, respectively. By our choices we have $\rho_G|_\mathfrak{m} = \rho_M$.

Let

$$T := \{ m \in K_M : \text{Ad}(m)|_\mathfrak{t} = \text{Id} \}.$$  

Then we have

$$T = \{ k \in K : \text{Ad}(k)|_\mathfrak{t} = \text{Id} \}.$$  

Thus $T$ is connected. Let $N_{K_M}$ and $N_{K_0}$ be the normalizers of $\mathfrak{t}$ in $K_M$ and $K^0_M$, respectively. Let $W_{K_M} := N_{K_M}/T$ and let $W_{\mathfrak{t}_m} = N_{K^0_M}/T$ be the Weyl group of $\Delta((\mathfrak{t}_m)_C, \mathfrak{t}_C)$. Moreover we let $W_\mathfrak{m}$ be the Weyl group of $\Delta(\mathfrak{m}_C, \mathfrak{t}_C)$. Finally we let

$$W(A) := \{ k \in K : \text{Ad}(k)a = a \}/K_M.$$  

The following lemma is certainly well-known and has already been used by Olbrich, [O1, page 15]. However, for the sake of completeness, we include a proof here.

**Lemma 6.1.** One has

$$\frac{|W_{K_M}|}{|W_{\mathfrak{t}_m}|} \cdot |W(A)| = 2.$$  

**Proof.** By [Kn2, Proposition 7.19 (b)], the order $\#(M/M^0)$ equals $\#(K_M/K^0_M)$. Let $k \in K_M$. Then $\text{Ad}(k)\mathfrak{t}$ is a maximal torus in $\mathfrak{t}_m$ and thus there exists a $k^0 \in K^0_M$ such that $\text{Ad}(k)\mathfrak{t} = \text{Ad}(k^0)\mathfrak{t}$. Hence every element of $K_M/K^0_M$ has a representative in $N_{K_M}$ and thus there are canonical isomorphisms $K_M/K^0_M \cong N_{K_M}/N_{K^0_M} \cong W_{K_M}/W_{\mathfrak{t}_m}$. In other words $|W_{K_M}|/|W_{\mathfrak{t}_m}|$ equals the number of components of $M$. Let $\mathfrak{a}_p$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$, let $\Delta_{\mathfrak{a}_p}$ be the corresponding restricted roots and let $W(\Delta_{\mathfrak{a}_p})$ be the corresponding Weyl-group. One has $W(\Delta_{\mathfrak{a}_p}) = N_K(\mathfrak{a}_p)/Z_K(\mathfrak{a}_p)$, where $N_K(\mathfrak{a}_p)$ resp. $Z_K(\mathfrak{a}_p)$ are the normalizer resp. centralizer of $\mathfrak{a}_p$ in $K$. Moreover by [Kn2, Proposition 8.85] each element of $W(A)$ has a representative in $N_K(\mathfrak{a}_p)$, i.e. can be extended to an element of $W(\Delta_{\mathfrak{a}_p})$ which fixes $\mathfrak{a}$. Now a case-by-case study easily implies that $W(\Delta_{\mathfrak{a}_p})$ contains such an element which is non-trivial if and only if $G = SO^0(p, 1)$. In this case $M$ is connected. In all other cases, $M$ has exactly two components. This proves the Lemma. \[ \square \]

Let $H_1 \in \mathfrak{a}$ with $e_1(H_1) = 1$. Then we normalize the Killing form $B$ by $1/B(H_1, H_1)$. We let $\|\cdot\|$ be the corresponding norm on the real vector-space $i\mathfrak{t}^* \oplus \mathfrak{a}^*$. Let $\Omega$ be the Casimir element with respect to the normalized Killing form. Then for $\tau \in \text{Rep}(G)$ with highest weight $\Lambda(\tau)$ we have

$$\tau(\Omega) = \|\Lambda(\tau) + \rho_G\|^2 - \|\rho_G\|^2.$$  

(6.1)
The restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{m}$ is non-degenerate and Ad-invariant. Let $\Omega_M$ be the corresponding Casimir element. For $\sigma \in \text{Rep}(M^0)$ with highest weight $\Lambda(\sigma) \in \mathfrak{i}^*$ we define
\begin{equation}
(6.2)
c(\sigma) := \|\Lambda(\sigma) + \rho_M\|^2 - \|\rho_G\|^2.
\end{equation}
Then one has $c(\sigma) = \chi_{\sigma}(\Omega_M) - \|\rho_G|_a\|^2$ and thus one has
\begin{equation}
(6.3)
c(\sigma) = c(\tilde{\sigma})
\end{equation}
for every $\sigma \in \text{Rep}(M^0)$. Let $\text{W}_g := \text{W}(\mathfrak{g}_C, \mathfrak{h}_C)$ be the Weyl group of $\Delta(\mathfrak{g}_C, \mathfrak{h}_C)$ and for $w \in \text{W}_g$ let $\ell(w)$ be its length with respect to the simple roots defined by $\Delta^+(\mathfrak{g}_C, \mathfrak{h}_C)$. Finally let
\[ W^1 := \{w \in \text{W}_g; w^{-1} \alpha > 0 \quad \forall \alpha \in \Delta^+(\mathfrak{m}_C, \mathfrak{t}_C)\}. \]
The subspace $\mathfrak{n}$ is even-dimensional and we write $\text{dim}(\mathfrak{n}) = 2n$. For $k = 0, \ldots, 2n$ let $H^k(\mathfrak{n}; V_\tau)$ be the Lie-algebra cohomology of $\mathfrak{n}$ with coefficients in $V_\tau$. Then the $H^k(\mathfrak{n}; V_\tau)$ are $M^0 A$-modules and their decomposition into irreducible $M^0 A$-components can be described by the following theorem of Kostant.

**Proposition 6.2.** In the sense of $M^0 A$-modules one has
\[ H^k(\mathfrak{n}; V_\tau) \cong \sum_{w \in W^1, \ell(w) = k} V_{\tau(w)}, \]
where $V_{\tau(w)}$ is the $M^0 A$ module with highest weight $w(\Lambda(\tau) + \rho_G) - \rho_G$.

**Proof.** See for example [Wr, Theorem 2.5.1.3].

**Corollary 6.3.** As $M^0 A$-modules we have
\[ \bigoplus_{k=0}^{2n} (-1)^k \Lambda^k \mathfrak{n}^* \otimes V_\tau = \bigoplus_{w \in W^1} (-1)^{\ell(w)} V_{\tau(w)}. \]

**Proof.** This follows from Proposition 6.2 and the Poincaré principle [Ko, (7.2.3)].

For $w \in W^1$ let $\sigma_{\tau,w} \in \text{Rep}(M^0)$ be the finite-dimensional irreducible representation of $M^0$ with highest weight
\begin{equation}
(6.4)
\Lambda(\sigma_{\tau,w}) := w(\Lambda(\tau) + \rho_G)|_t - \rho_M,
\end{equation}
and let $\lambda_{\tau,w} \in \mathbb{R}$ be such that
\begin{equation}
(6.5)
w(\Lambda(\tau) + \rho_G)|_a = \lambda_{\tau,w} e_1.
\end{equation}
Then we have the following corollary about the Casimir eigenvalue.

**Proposition 6.4.** For every $w \in W^1$ one has
\[ \tau(\Omega) = \lambda_{\tau,w}^2 + c(\sigma_{\tau,w}). \]
Proof. By (6.1) we have
\[ \tau(\Omega) = \|\Lambda(\tau) + \rho_G\|^2 - \|\rho_G\|^2 = \|w(\Lambda(\tau) + \rho_G)\|^2 - \|\rho_G\|^2 = \|\lambda_{\tau,w} a_1\|^2 + \|\Lambda(\sigma_{\tau,w}) + \rho_M\|^2 - \|\rho_G\|^2 = \lambda_{\tau,w}^2 + c(\sigma_{\tau,w}). \]

Let \( k^\tau \) be defined by (4.13). Our next goal is to compute the Fourier transform of \( k^\tau \). Note that, since \( T \) is connected, it follows from [Wa2, section 6.9, section 8.7.1] that for every discrete series representation \( \xi \) of \( M \) over \( W_\xi \) there exists a discrete series representation \( \xi^0 \) of \( M^0 \) over \( W_{\xi^0} \) such that \( \xi \) is induced from \( \xi^0 \). Moreover, since \( M^0 \) is semisimple, the discrete series of \( M^0 \) is parametrized as in section 2.2. By [Wa2, section 8.7.1], two discrete series representations \( \xi^0 \) and \( \xi^0_2 \) of \( M^0 \) with corresponding parameters \( \Lambda_{\xi^0} \), \( \Lambda_{\xi^0_2} \) as in section 2.2 induce the same discrete series representation of \( M \) if and only if \( \Lambda_{\xi^0} \) and \( \Lambda_{\xi^0_2} \) are \( W_{K^M} \)-conjugate. For \( \xi \in M_d \) and \( \lambda \in \mathbb{C} \) we let \( \pi_{\xi,\lambda} := \pi_{\xi,\lambda_{\xi^0}} \), \( \Theta_{\xi,\lambda} := \Theta_{\xi,\lambda_{\xi^0}} \).

**Proposition 6.5.** Let \( \xi \in M_d \) with infinitesimal character \( \chi(\xi) \). Let \( p_m := p \cap m \) and let \( v := \frac{1}{2} \dim p_m \). Then for \( \lambda \in \mathbb{C} \) one has
\[ \Theta_{\xi,\lambda}(k^\tau) = (-1)^v \sum_{w \in \mathbf{W}^{-1}} (-1)^{\ell(w)+1} e^{-t(\lambda^2 + \lambda_{\tau,w}^2)} \]

Proof. Let \( \xi^0 \), \( \Lambda_{\xi^0} \) be as above. Then one has
\[ \pi_{\xi,\lambda}(\Omega) = -\lambda^2 + \|\Lambda\|^2 - \|\rho_G\|^2. \]
Thus if \( \sigma \in \text{Rep}(M^0) \) is such that \( \chi_\sigma = \chi_\xi \) one has
\[ (6.6) \quad \pi_{\xi,\lambda}(\Omega) = -\lambda^2 + c(\sigma). \]
Moreover \( \xi_{|K_M} \) is induced from \( \xi^0_{|K^0_M} \), and so by Frobenius reciprocity one has
\[ [\Lambda^p p^* \otimes \mathcal{H}_{\xi} \otimes V_{\tau}]_{K^M} = [\Lambda^p p^* \otimes W_{\xi^0} \otimes V_{\tau}]_{K^M}. \]
Thus by (4.18) one has
\[ \Theta_{\xi,\lambda}(k^\tau) = e^{t(\pi_{\xi,\lambda}(\Omega) - \tau(\Omega))} \sum_{p=0}^{d} (-1)^p p [\Lambda^p p^* \otimes W_{\xi^0} \otimes V_{\tau}]_{K^M}. \]

Let \( p_Y \) be as in Proposition 4.1. Since \( \dim a = 1 \), it follows that as \( K^0_M \) modules \( p_Y \cong p_m \oplus n \). Using (4.2), it follows that as \( K^0_M \) modules we have
\[ \sum_{p=0}^{d} (-1)^p p [\Lambda^p p^*] = \sum_{p=0}^{d} (-1)^{p+1} \Lambda^p (p_m^* \oplus n^*) \]
\[ = \sum_{k=0}^{2n} (-1)^{k+1} (\Lambda^{2n} p_m^* - \Lambda^{2n+1} p_m^*) \otimes \Lambda^k n^*. \]
Thus together with Corollary 6.3 and the Poincaré principle one gets
\[
\sum_{p=0}^{d} (-1)^p p \left[ \Lambda^p p^* \otimes W_{\xi_0} \otimes V_{\tau} \right]^{K_0}_{M}
\]
\[
= \sum_{w \in W^1} (-1)^{\ell(w)+1} \left[ (\Lambda^{\text{ev}} p_m^* - \Lambda^{\text{odd}} p_m^*) \otimes W_{\xi_0} \otimes V_{\tau(w)} \right]^{K_0}_{M}
\]
\[
= \sum_{w \in W^1} (-1)^{\ell(w)+1} \chi(m, K_0; W_{\xi_0} \otimes V_{\tau(w)}),
\]

where \(\chi(m, K_0; W_{\xi_0} \otimes V_{\tau(w)})\) denotes the Euler-characteristic of the \((m, K_0)\)-cohomology with coefficients in the \(M^0\)-module \(V_{\tau(w)} \otimes W_{\xi_0}\). Thus the proposition follows from Proposition 2.1, Proposition 6.4, equation (6.7) and equation (6.3). \(\square\)

Next we come to the Plancherel measures. For \(\xi \in \hat{M}_d\) we let \(\xi_0 \in \hat{M}_d^0\) be as above. Fix a regular \(\Lambda_{\xi_0} \in \mathfrak{h}_C^*\) corresponding to \(\xi_0^0\) as in section 2.2 and let \(\Lambda_\xi := \Lambda_{\xi_0}\). Choose positive roots \(\Delta^+(m_C, t_C; \Lambda_\xi)\) such that \(\Lambda_\xi\) is dominant with respect to these roots. Let \(\Delta^+(g_C, h_C; \Lambda_\xi)\) be positive roots defined via \(\Delta^+(m_C, t_C; \Lambda_\xi)\) and \(e_1\) and let \(\rho_{G, \Lambda_\xi}\) be the half-sum of the elements of \(\Delta^+(m_C, t_C; \Lambda_\xi)\). For \(\lambda \in \mathbb{R}\) we let \(\mu_\xi(\lambda)\) be the Plancherel measure of \(\pi_{\xi, \lambda}\). Then there exists a polynomial \(P_\xi(z)\) such that one has

\[
\mu_\xi(\lambda) = P_\xi(i\lambda).
\]

The polynomial \(P_\xi(z)\) is given as follows. There exists a constant \(c_X\) which depends only on \(\hat{X}\) such that one has

\[
P_\xi(z) = (-1)^n c_X \prod_{\alpha \in \Delta^+(g_C, h_C; \Lambda_\xi)} \frac{\langle \alpha, \Lambda_\xi + ze_1 \rangle}{\langle \alpha, \rho_{G, \Lambda_\xi} \rangle}.
\]

[Kn1, Theorem 13.11], [Wa3, Theorem 13.5.1]. By [Ol, Lemma 5.1] and our normalizations one has

\[
c_X = \frac{1}{|W(A)| \text{vol}(X_d)}.
\]

Note that \(P_\xi(z)\) is an even polynomial in \(z\). Now let \(w \in W_m\). We regard \(W_m\) as a subgroup of \(W_g\). Then if we replace \(\Lambda_\xi\) by \(w\Lambda_\xi\), we have to replace \(\Delta^+(g_C, t_C; \Lambda_\xi)\) by \(w\Delta^+(g_C, t_C; \Lambda_\xi)\). This implies that \(P_\xi(z)\) depends only on the \(W_m\)-orbit of \(\Lambda_\xi\) or equivalently on the infinitesimal character \(\chi(\xi)\) of \(\xi\). Thus if for \(\sigma \in \text{Rep}(M^0)\) with highest weight \(\Lambda(\sigma)\) we let

\[
P_\sigma(z) := (-1)^n c_X \prod_{\alpha \in \Delta^+(g_C, h_C)} \frac{\langle \alpha, \Lambda(\sigma) + \rho_M + ze_1 \rangle}{\langle \alpha, \rho_G \rangle},
\]

where \(c_X\) is as in (6.8), it follows that \(P_\xi(\lambda) = P_\sigma(\lambda)\) if \(\chi(\sigma) = \chi(\xi)\). Putting everything together, we obtain the following corollary.
Proposition 6.6. Let $\tau \in \text{Rep}(G)$ and assume that $\tau \not\cong \tau_\theta$. Then one has
\[
\log T_X^{(2)}(\tau) = (-1)^v \pi \text{vol}(X) \frac{|W_m|}{|W_{KM}|} \sum_{w \in W^1} (-1)^{\ell(w)} \int_0^{[\lambda_{\tau,w}]} P_{\sigma_{\tau,w}}(t)dt.
\]

Proof. For a given regular and integral $\Lambda \in i t^*$ there are exactly $|W_m|/|W_{KM}|$ distinct elements of $\hat{M}_d$ with infinitesimal character $\chi_\Lambda$. Thus if one combines the Plancherel-Theorem with Proposition 4.1, Proposition 6.5, equation (6.7) and the previous remarks one obtains
\[
k^*_t(1) = (-1)^v \frac{|W_m|}{|W_{KM}|} \sum_{w \in W^1} (-1)^{\ell(w)+1} e^{-t\lambda^2_{\tau,w}} \int e^{-t\lambda^2} P_{\sigma_{\tau,w}}(i\lambda) d\lambda.
\]

We let
\[
I(t, \tau) := \text{vol}(X) k^*_t(1).
\]
By the computations below one has $|\lambda_{\tau,w}| > 0$ for every $w \in W^1$. Thus, since is $P_{\sigma}(\lambda)$ is an even polynomial of degree $2n$ for each $\sigma \in \hat{M}_d^0$, for $s \in C$ with $\text{Re}(s) > 2n + 1$ the integral
\[
\mathcal{M}I(s, \tau) := \int_0^{\infty} t^{s-1} I(t, \tau) dt
\]
exists. Moreover, by [Fr], Lemma 2 and Lemma 3, $\mathcal{M}I(s, \tau)$ has a meromorphic continuation to $C$ which is regular at $0$ and if $\mathcal{M}I(\tau)$ denotes its value at $0$ one has
\[
\mathcal{M}I(\tau) = 2\pi \text{vol}(X)(-1)^v \frac{|W_m|}{|W_{KM}|} \sum_{w \in W^1} (-1)^{\ell(w)} \int_0^{[\lambda_{\tau,w}]} P_{\sigma_{\tau,w}}(i\lambda) d\lambda.
\]
By definition one has
\[
\log T_X^{(2)}(\tau) = \frac{1}{2} \mathcal{M}I(\tau)
\]
and the proposition follows. \hfill \square

Now let $G = SO^0(p, q)$, $p > 1$, $p, q$ odd, $p \geq q$, $p = 2p_1 + 1$, $q = 2q_1 + 1$. Let $n := p_1 + q_1$. Let $K = SO(p) \times SO(q)$ and $\tilde{X} = G/K$. Then $\dim(\tilde{X}) = 2n + 1$. The normalized Killing form is given by
\[
\langle X, Y \rangle := \frac{1}{2n - 2} B(X, Y).
\]
We equip $\tilde{X}$ with the Riemannian metric defined by the restriction of $\langle \cdot, \cdot \rangle$ to $p$. We have $\mathfrak{m} \cong \mathfrak{so}(p - 1, q - 1)$. We realize the fundamental Cartan subalgebra as follows. Let
\[
(6.11) H_1 := E_{p,p+1} + E_{p+1,p}.
\]
Then we put
\[
\mathfrak{a} = \mathbb{R} H_1.
\]
Moreover we let

\[ H_i := \begin{cases} \sqrt{-1}(E_{2i-3,2i-2} - E_{2i-2,2i-3}), & 2 \leq i \leq p_1 + 1 \\ \sqrt{-1}(E_{2i-1,2i} - E_{2i,2i-1}) & p_1 + 1 < i \leq n + 1. \end{cases} \]

Then

\[ t := \bigoplus_{i=2}^{n+1} \sqrt{-1}H_i \]

is a Cartan subalgebra of \( m \) and

\[ h := a \oplus t \]

is a Cartan subalgebra of \( g \). Define \( e_i \in h_C^*, \) \( i = 1, \ldots, n + 1, \) by

\[ e_i(H_j) = \delta_{i,j}, \quad 1 \leq i, j \leq n + 1. \]

Then the sets of roots of \((g_C,h_C)\) and \((m_C,t_C)\) are given by

\[ \Delta(g_C,h_C) = \{ \pm e_i \pm e_j, 1 \leq i < j \leq n + 1 \} \]
\[ \Delta(m_C,t_C) = \{ \pm e_i \pm e_j, 2 \leq i < j \leq n + 1 \}. \]

We fix positive systems of roots by

\[ \Delta^+(g_C,h_C) := \{ e_i + e_j, i \neq j \} \cup \{ e_i - e_j, i < j \} \]
\[ \Delta^+(m_C,t_C) := \{ e_i + e_j, i \neq j, i,j \geq 2 \} \cup \{ e_i - e_j, 2 \leq i < j \}. \]

We parametrize the finite-dimensional irreducible representations \( \tau \) of \( G \) by their highest weights

\[ \Lambda(\tau) = k_1(\tau)e_1 + \cdots + k_{n+1}(\tau)e_{n+1}, \quad (k_1(\tau), \ldots, k_{n+1}(\tau)) \in \mathbb{Z}^{n+1}, \]

\[ k_1(\tau) \geq k_2(\tau) \geq \cdots \geq k_n(\tau) \geq |k_{n+1}(\tau)|. \]

If \( \Lambda \) is a weight as in (6.13), then

\[ \Lambda_\theta = k_1(\tau)e_1 + \cdots + k_n(\tau)e_n - k_{n+1}(\tau)e_{n+1}. \]

Now we let

\[ \omega_{f,n}^+ := \sum_{j=1}^{n+1} e_j; \quad \omega_{f,n}^- := (\omega_{f,n}^+)_\theta = \sum_{j=1}^{n} e_j - e_{n+1}. \]

Then \( \frac{1}{2}\omega_{f,n}^\pm \) are the fundamental weights of \( \Delta^+(g_C,h_C) \) which are not invariant under \( \theta \).

We parametrize the finite-dimensional irreducible representations \( \sigma \) of \( M^0 \) by their highest weights

\[ \Lambda(\sigma) = k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}, \quad (k_2(\sigma), \ldots, k_{n+1}(\sigma)) \in \mathbb{Z}^n \]
\[ k_2(\sigma) \geq k_3(\sigma) \geq \cdots \geq k_n(\sigma) \geq |k_{n+1}(\sigma)| \in \mathbb{Z}^n. \]
For $\sigma \in \text{Rep}(M^0)$ with highest weight $\Lambda(\sigma)$ as in (6.16) we let $w_0\sigma \in \text{Rep}(M^0)$ be the representation with highest weight
$$\Lambda(w_0\sigma) := k_2(\sigma)e_2 + \cdots + k_n(\sigma)e_n - k_{n+1}(\sigma)e_{n+1}.$$  
Then for every $\sigma \in \text{Rep}(M^0)$ one has $\bar{\sigma} = \sigma$ if $n$ is even and $\bar{\sigma} = w_0\sigma$ if $n$ is odd. Applying equation (6.10) this implies that
\begin{equation}
(6.17)
P_\sigma(\lambda) = P_{w_0\sigma}(\lambda) = P_{\bar{\sigma}}(\lambda)
\end{equation}
for every $\sigma \in \text{Rep}(M^0)$.

Let $\tau \in \text{Rep}(G)$ with highest weight $\tau_1e_1 + \cdots + \tau_{n+1}e_{n+1}$. For $k = 0, \ldots, n$ let
\begin{equation}
(6.18)
\lambda_{\tau,k} = \tau_{k+1} + n - k
\end{equation}
and let $\sigma_{\tau,k}$ be the representation of $G$ with highest weight
\begin{equation}
(6.19)
\Lambda_{\sigma_{\tau,k}} := (\tau_1 + 1)e_2 + \cdots + (\tau_k + 1)e_{k+1} + \tau_{k+2}e_{k+2} + \cdots + \tau_{n+1}e_{n+1}.
\end{equation}
Then as in [MP, section 2.7] one has
\begin{equation}
(6.20)
\{(\lambda_{\tau,w},\sigma_{\tau,w},l(w)) : w \in W^1\} = \{((\lambda_{\tau,k},\sigma_{\tau,k},k) : k = 0, \ldots, n\}
\sqcup \{(-\lambda_{\tau,k},w_0\sigma_{\tau,k},2n-k) : k = 0, \ldots, n\}.
\end{equation}
Combining (6.14), (6.17) and (6.20) and Proposition 6.6 it follows that
\begin{equation}
(6.21)
T_X^{(2)}(\tau) = T_X^{(2)}(\tau_0)
\end{equation}
for each $\tau \in \text{Rep}(G)$. Now for $p, q \in \mathbb{N}$ we let $\epsilon(q) := 0$ for $q = 1$ and $\epsilon(q) := 1$ for $q > 1$ and we let
\begin{equation}
(6.22)
C_{p,q} := \frac{(-1)^{p+1}2^{q2\pi}}{\text{vol}(X_d)} \left(\frac{p+q-2}{p+q-2}\right).
\end{equation}
Then we have

**Proposition 6.7.** For $p, q$, odd, $p \geq q$ let $\tilde{X} = \text{SO}^0(p,q)/\text{SO}(p) \times \text{SO}(q)$ and let $X = \Gamma\backslash \tilde{X}$. Let $\Lambda \in \mathfrak{h}_C^+$ be a highest weight as in (6.13) and assume that $\Lambda_\theta \neq \Lambda$. For $m \in \mathbb{N}$ let $\tau_\Lambda(m)$ be the irreducible representation of $\text{SO}^0(p,q)$ with highest weight $m\Lambda$. There exists a polynomial $P_\Lambda(m)$ whose coefficients depend only on $\Lambda$, such that for all $m \in \mathbb{N}$ we have
$$\log T_X^{(2)}(\tau_\Lambda(m)) = C_{p,q} \text{vol}(X) P_\Lambda(m).$$
Moreover there is a constant $C_\Lambda > 0$, which depends on $\Lambda$, such that
\begin{equation}
(6.23)
P_\Lambda(m) = C_\Lambda \cdot m \dim(\tau_\Lambda(m)) + O(\dim(\tau_\Lambda(m)))
\end{equation}
as $m \to \infty$. If $\Lambda = \omega_{f,n}^\pm$, where $\omega_{f,n}^\pm$ are as in (6.13), then $C_\Lambda = 1$.

**Proof.** Let $\Lambda = \tau_1e_1 + \cdots + \tau_{n+1}e_{n+1}$. By (6.14) and (6.21) we may assume that $\tau_{n+1} > 0$. Put $\tau(m) := \tau_{\Lambda}(m)$. Then
\begin{equation}
(6.24)
\lambda_{\tau(m),k} = m\tau_{k+1} + n - k, \quad k = 0, \ldots, n,
\end{equation}
and by Proposition 6.6, (6.20) and (6.17) we have
\[
\log T_X^{(2)}(\tau(m)) = 2\pi \log(X) (-1)^v \frac{|W_m|}{|W_{KM}|} \sum_{k=0}^n (-1)^k \int_0^{\lambda_{r(m),k}} P_{\sigma_{r(m),k}}(t) dt.
\]
In the hyperbolic case the term \((-1)^v |W_m|/|W_{KM}| \) equals 1. Therefore this equation agrees with [MP, (5.16), (5.17)]. Note that \(n = \dim n\). Let \(c_\tilde{X}\) be defined by (6.9) and put
\[
(6.25) 
\]
Then it follows from (6.10) and (6.20) that \(P_{\Lambda}(m) := (-1)^n \sum_{k=0}^n (-1)^k \int_0^{\lambda_{r(m),k}} P_{\sigma_{r(m),k}}(t) dt \).

So it remains to compute the constant. By (6.9) and Lemma 6.3 one has
\[
\frac{|W_m|}{|W_{KM}|} c_\tilde{X} = \frac{1}{2 \vol(\tilde{X}_d)}.
\]
Recall that \(m_C \cong \mathfrak{so}(2n, \mathbb{C}), (\mathfrak{g}_m)_C \cong \mathfrak{so}(2p_1, \mathbb{C}) \oplus \mathfrak{so}(2q_1, \mathbb{C})\) and so by [Kn2, page 685] one has \(|W_m| = n!2^{n-1}, |W_{tn}| = p_1!q_1!2^{n-1-\epsilon(q)}\), where \(\epsilon(q)\) is as above. Thus as in [Ol, Proposition 1.3] one has
\[
\frac{|W_m|}{|W_{tn}|} = 2^{\epsilon(q)} \left( \frac{p+q-2}{p-1} \right)^2.
\]
Furthermore one has \(v = \frac{\dim p_m}{2} = \frac{(p-1)(q-1)}{2}\) and thus we get \(v + n = \frac{pq-1}{2}\). This proves the first part of the proposition.

To determine the highest order term of the polynomial \(P_{\Lambda}(m)\), we proceed as in [MP, Lemma 5.4] to show that
\[
P_{\sigma_{r(m),k}}(t) = (-1)^{n+k} c_\tilde{X} \dim(\tau(m)) \prod_{j=0}^n \frac{t^2 - \lambda_{r(m),j}^2}{\tau_{r(m),k} - \lambda_{r(m),j}^2}.
\]
Denote the product on the right by \(\Pi_k(t; m)\). Then it follows from (6.23) that
\[
(6.26) 
P_{\Lambda}(m) = \dim(\tau(m)) \cdot \sum_{k=0}^n \int_0^{\lambda_{r(m),k}} \Pi_k(t; m) dt.
\]
To deal with the sum, we follow [BV, 5.9.1]. Put \(\lambda_{r(m),n+1} = 0\). Then \(\lambda_{r(m),k}, k = 0, \ldots, n + 1\) is a strictly decreasing sequence. For \(k = 0, \ldots, n\) set
\[
Q_k(t; m) := \sum_{j=0}^k \Pi_j(t; m).
\]
Then $Q_k(t; m)$ is the unique even polynomial of degree $\leq 2n$ which satisfies

$$Q_k(\pm \lambda_{\tau(m), j}) = \begin{cases} 1, & \text{if } j \leq k, \\ 0, & \text{if } n \leq j > k. \end{cases}$$

(6.27)

Moreover we have

$$\sum_{k=0}^{n} \int_{0}^{\lambda_{\tau(m), k}} \Pi_k(t; m) \, dt = \sum_{k=0}^{n} \int_{0}^{\lambda_{\tau(m), k}} Q_k(t; m) \, dt.$$  

(6.28)

As proved in [BV, Sect. 5.9.1], each integral on the right is positive. This can be seen as follows. By (6.27), the polynomial $Q'_k$ has a root in each interval $[\lambda_{\tau(m), j+1}, \lambda_{\tau(m), j}]$, $[-\lambda_{\tau(m), j}, -\lambda_{\tau(m), j+1}]$ for $1 \leq j < n$, $j \neq k$ and a root in $[-\lambda_{\tau(m), n}, \lambda_{\tau(m), n}]$. Since $Q'_k$ is of degree $\leq 2n - 1$, it follows that $Q_k$ is either constant or strictly increasing on $[\lambda_{\tau(m), k+1}, \lambda_{\tau(m), k}]$. Furthermore, $Q_n(t; m)$ is a polynomial of degree $2n$, which is equal to 1 at $2n + 2$ pairwise distinct points. Hence $Q_n \equiv 1$. Thus by (6.27) and (6.28) we get

$$(n + 1)(m \tau_n + n) = (n + 1)\lambda_{\tau(m), 0} \geq \sum_{k=0}^{n} (\lambda_{\tau(m), k} - \lambda_{\tau(m), k+1})$$

(6.29)

Since $P_{\Lambda}(m)$ is a polynomial in $m$, it follows that there exists $C_{\Lambda} > 0$ such that (6.23) holds. If $\Lambda$ is one of the fundamental weight $\omega_{f, n}^{\pm}$, defined by (6.13), then it follows as in [MP, Section 5] that $C_{\Lambda} = 1$. This proves the second part of the proposition. □

Finally we turn to the case $G = SL_3(\mathbb{R})$, $K = SO(3)$. We define our fundamental Cartan subalgebra as follows. Let

$$H_1 := \text{diag}(1, 1, -2); \quad a := \mathbb{R}H_1.$$  

Then we have $m = \mathfrak{sl}_2(\mathbb{R})$, if $\mathfrak{sl}_2(\mathbb{R})$ is embedded into $\mathfrak{g}$ as an upper left block. Let

$$H_2 := \left( \begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array} \right); \quad t := \mathbb{R}T_1$$

embedded into $\mathfrak{g}$ as an upper left block. Then $t$ is a Cartan subalgebra of $m$ and

$$\mathfrak{h} := a \oplus t$$

is a $\theta$-stable fundamental Cartan subalgebra of $\mathfrak{g}$. Note that $\mathfrak{h}$ is different from the usual Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which consist of all diagonal matrices of trace 0. Define $f_1 \in a^*$ and $f_2 \in it^*$ by

$$f_1(H_1) = 3; \quad f_2(H_2) = i.$$  

We fix $f_1$ as a positive restricted root of $a$. Then we can define positive roots by

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) := \{f_1 - f_2, f_1 + f_2, 2f_2\}; \quad \Delta^+(\mathfrak{m}, \mathfrak{t}) = \{2f_2\}.$$
Under our normalization one has

\begin{equation}
\langle f_1, f_1 \rangle = 1; \quad \langle f_2, f_2 \rangle = \frac{1}{3}; \quad \langle f_1, f_2 \rangle = 0.
\end{equation}

One easily sees that \(\dim \mathfrak{n} = 2\), hence \(n = 1\). Moreover by \([Kn2, \text{ page 485}]\) one has \(|W(A)| = 1\). For \(k \in \mathbb{N}\) let \(\sigma_k \in \text{Rep}(M^0)\) be of highest weight \(kf_2\). Then it follows from (6.10) and (6.9) that

\begin{equation}
P_{\sigma_k}(z) = -\frac{9}{8 \text{vol}(\tilde{X}_d)} (k + 1) \left( z^2 - \left( \frac{k + 1}{3} \right)^2 \right).
\end{equation}

Define \(e_i \in \tilde{\mathfrak{h}}_C^*\) by \(e_i(\text{diag}(t_1, t_2, t_3)) = \sum_j \delta_{i,j} t_j\). Then one can choose positive roots

\begin{equation}
\Delta^+(\mathfrak{g}_C, \tilde{\mathfrak{h}}_C) := \{ e_1 - e_2, e_1 - e_3, e_2 - e_3 \}
\end{equation}

and there is a standard inner-automorphism \(\Phi\) of \(\mathfrak{g}_C\) which sends \(\mathfrak{h}_C\) to \(\tilde{\mathfrak{h}}_C\) and which satisfies

\begin{equation}
\Phi^*(e_1 - e_2) = 2f_2; \quad \Phi^*(e_1 - e_3) = f_1 + f_2; \quad \Phi^*(e_2 - e_3) = f_1 - f_2.
\end{equation}

The fundamental weights \(\tilde{\omega}_1, \tilde{\omega}_2 \in \tilde{\mathfrak{h}}_C^*\) are given by

\[\tilde{\omega}_1 = \frac{2}{3}(e_1 - e_2) + \frac{1}{3}(e_2 - e_3)\]

and

\[\tilde{\omega}_2 = \frac{1}{3}(e_1 - e_2) + \frac{2}{3}(e_2 - e_3)\]

Thus the fundamental weights \(\omega_1, \omega_2 \in \mathfrak{h}_C^*\) are given by

\begin{equation}
\omega_1 := \Phi^*(\tilde{\omega}_1) = \frac{1}{3}f_1 + f_2; \quad \omega_2 := \Phi^*(\tilde{\omega}_2) = \frac{2}{3}f_1.
\end{equation}

Let \(N_0 := \mathbb{N} \cup \{0\}\). If \(\Lambda\) is a weight, \(\Lambda = \tau_1 \omega_1 + \tau_2 \omega_2\), \(\tau_1, \tau_2 \in N_0\), then a standard computation shows that

\begin{equation}
\Lambda_\theta = \tau_2 \omega_1 + \tau_1 \omega_2.
\end{equation}

Now we fix \(\tau_1, \tau_2 \in N_0\), \(\tau_1 + \tau_2 > 0\) and for \(m \in \mathbb{N}\) we let \(\tau(m)\) be the representation of \(G\) with highest weight

\begin{equation}
\Lambda(\tau(m)) := m\tau_1 \omega_1 + m\tau_2 \omega_2.
\end{equation}

We let \(\tilde{W}_\tilde{g}\) be the Weyl-group of \(\Delta(\mathfrak{g}_C, \tilde{\mathfrak{h}}_C)\). Then \(\tilde{W}_\tilde{g}\) consists of all permutations of \(e_1, e_2, e_3\). Let

\[\tilde{W}^1 := (\Phi^*)^{-1}W^1 = \{ w \in \tilde{W}_\tilde{g} : w^{-1}(e_1 - e_2) > 0 \}.
\]
Then one has
\[
\{(w, \ell(w)); w \in \tilde{W}^1\} \\
= \left\{ (\text{Id}, 0); \left( \begin{array}{ccc} e_1 & e_2 & e_3 \\ e_1 & e_3 & e_2 \end{array} \right), 1 \right\}; \left( \begin{array}{ccc} e_1 & e_2 & e_3 \\ e_3 & e_1 & e_2 \end{array} \right), 2 \right\}.
\]
By a direct computation we get
\[
\{w(\Lambda(\tau(m)) + \tilde{\rho}_G), \ell(w); w \in \tilde{W}^1\} \\
= \left\{ \left( \frac{2m\tau_1 + m\tau_2 + 3}{3} (e_1 - e_2) + \frac{m\tau_1 + 2m\tau_2 + 3}{3} (e_2 - e_3); 0 \right), \right. \\
\left. \left( \frac{2m\tau_1 + m\tau_2 + 3}{3} (e_1 - e_2) + \frac{m\tau_1 - m\tau_2}{3} (e_2 - e_3); 1 \right), \right. \\
\left. \left( \frac{-m\tau_1 + m\tau_2}{3} (e_1 - e_2) + \frac{-2m\tau_1 - m\tau_2 - 3}{3} (e_2 - e_3); 2 \right) \right\}.
\]
As in [BV, 5.9.2] we introduce the following constants
\[
A_1(\tau(m)) := \frac{m\tau_1 + 1}{2}; \quad A_2(\tau(m)) := \frac{m\tau_1 + m\tau_2 + 2}{2}; \\
A_3(\tau(m)) := \frac{m\tau_2 + 1}{2}
\]
and
\[
C_1(\tau(m)) := \frac{m\tau_1 + 2m\tau_3 + 3}{3}; \quad C_2(\tau) := \frac{m\tau_1 - m\tau_2}{3}; \\
C_3(\tau) := \frac{2m\tau_1 + m\tau_2 + 3}{3}.
\]
Note that on \( \tilde{h}^*_\mathbb{C} \) one has \( \tilde{\omega}_1 = e_1 \); \( \tilde{\omega}_2 = e_1 + e_2 \), since the matrices in \( \tilde{h}^*_\mathbb{C} \) have trace 0. Then, combining (3.34) and (6.38), we get
\[
\{(\Lambda(\sigma_{\tau(m)}, w), \lambda_{\tau(m)}, w, \ell(w)); w \in W^1\} = \\
\{((2A_1(\tau(m)) - 1)f_2, C_1(\tau(m)), 0), ((2A_2(\tau(m)) - 1)f_2, C_2(\tau(m)), 1), \\
((2A_3(\tau(m)) - 1)f_2, -C_3(\tau(m)), 2) \}.
\]
Thus if we apply (3.32) we obtain
\[
\sum_{w \in W^1} (-1)^{\ell(w)} \int_{\lambda_{\tau(m), w}}^{\lambda_{\tau(m), w}} P_{\tau(m), w}(t) dt \\
= -\frac{C_{\text{Stk}(\mathbb{R})}}{\text{vol}(X_d)} \sum_{k=1}^{3} (-1)^{k+1} A_k(\tau(m)) \int_{C_k(\tau(m))}^{C_k(\tau(m))} \left( \frac{9}{4} t^2 - A_k(\tau(m))^2 \right) dt \\
= -\frac{3}{4 \text{vol}(X_d)} \sum_{k=1}^{3} (-1)^{k+1} A_k(\tau(m)) [C_k(\tau(m))]^2 (3C_k(\tau(m))^2 - 4A_k(\tau(m))^2).
We can now prove our main result about the $L^2$-torsion for the case $G = SL_3(\mathbb{R})$.

**Proposition 6.8.** Let $\tilde{X} = SL(3, \mathbb{R})/SO(3)$ and $X = \Gamma \backslash \tilde{X}$. Let $\Lambda \in \mathfrak{h}^*_G$ be a highest weight with $\Lambda_0 \neq \Lambda$. For $m \in \mathbb{N}$ let $\tau_\Lambda(m)$ be the irreducible representation of $SL(3, \mathbb{R})$ with highest weight $m\Lambda$. There exists a polynomial $P_\Lambda$ whose coefficients depend only on $\Lambda$ such that

$$\log T^{(2)}_X(\tau_\Lambda(m)) = \frac{\pi \text{vol}(X)}{\text{vol}(\tilde{X}_d)} P_\Lambda(m).$$

Moreover, there exists a constant $C(\Lambda) > 0$ depending only on $\Lambda$ such that

$$P_\Lambda(m) = C(\Lambda) m \dim(\tau_\Lambda(m)) + O(\dim(\tau_\Lambda(m))),$$

as $m \to \infty$. If $\Lambda$ equals one of the fundamental weights $\omega_{f,i}$ then $C(\Lambda) = 4/9$.

**Proof.** There exist $\tau_1, \tau_2 \in \mathbb{N}_0$, $\tau_1 \neq \tau_2$, such that $\Lambda = \tau_1 \omega_1 + \tau_2 \omega_2$. Put $\tau(m) := \tau_\Lambda(m)$. Then by Proposition 6.6, equation (6.39), (6.40) and (6.41), the first statement is proved and it remains to consider the asymptotic behavior of the polynomial $P_\Lambda$. We differentiate two cases. First we assume that $\tau_1 \tau_2 \neq 0$. Then if we put

$$\alpha_4(\tau) := \begin{cases} \frac{\tau^4}{18} + \frac{2\tau^3}{9} + \frac{2\tau^2}{3}; & \tau_1 \geq \tau_2 \\ \frac{\tau^4}{18} + \frac{2\tau^2}{9} + \frac{2\tau_2}{3}; & \tau_2 \geq \tau_1, \end{cases}$$

an explicit computation using equation (6.39), (6.40) and (6.41) shows that

$$\sum_{w \in W^1} (-1)^{\ell(w)} \int_{0}^{\lambda_{\tau(m),w}} P_{\sigma_{\tau(m),w}}(t) dt = -\frac{\alpha_4(\tau)}{\text{vol}(\tilde{X}_d)} m^4 + O(m^3),$$

as $m \to \infty$. Note that $\alpha_4(\tau) > 0$ by our assumption on $\tau_1$ and $\tau_2$. Now we assume that $\tau_1 \tau_2 = 0$. Then if we define

$$\alpha_3(\tau) := \frac{2(\tau_1^3 + \tau_2^3)}{9},$$

an explicit computation using equation (6.39), (6.40) and (6.41) gives

$$\sum_{w \in W^1} (-1)^{\ell(w)} \int_{0}^{\lambda_{\tau(m),w}} P_{\sigma_{\tau(m),w}}(t) dt = -\frac{\alpha_3(\tau)}{\text{vol}(\tilde{X}_d)} m^3 + O(m^2),$$

as $m \to \infty$. For $SL_3(\mathbb{R})$ one has $v = 1$ and using Lemma 6.1 one gets $\frac{|W_{K_M}|}{|W|} = 1$. Moreover, every element of $\text{Rep}(M^0)$ is self-dual. Thus using Proposition 5.6 we obtain

$$\log T^{(2)}_X(\tau(m)) = \text{vol}(X) \frac{\pi \alpha_4(\tau)}{\text{vol}(\tilde{X}_d)} m^4 + O(m^3)$$

as $m \to \infty$, if $\tau_1 \tau_2 \neq 0$, and

$$\log T^{(2)}_X(\tau(m)) = \text{vol}(X) \frac{\pi \alpha_3(\tau)}{\text{vol}(\tilde{X}_d)} m^3 + O(m^2),$$

as $m \to \infty$, if $\tau_1 \tau_2 = 0$. 


as \( m \to \infty \), if \( \tau_1 \tau_2 = 0 \). Now we define constants
\[
d_3(\tau) := \frac{\tau_1^2 \tau_2 + \tau_2^2 \tau_1}{2}; \quad d_2(\tau) := \left( \frac{4\tau_1 \tau_2 + \tau_1^2 + \tau_2^2}{2} \right).
\]
Then by Weyl’s dimension formula one has
\[
\dim \tau(m) = d_3(\tau)m^3 + d_2(\tau)m^2 + O(m),
\]
as \( m \to \infty \). Note that \( d_3(\tau) > 0 \) for \( \tau_1 \tau_2 \neq 0 \) and that \( d_3(\tau) = 0, d_2(\tau) > 0 \) for \( \tau_1 \tau_2 = 0 \). This completes the proof of the proposition. \( \square \)

7. Lower bounds of the spectrum

In this section we assume that \( \widetilde{X} \) is odd-dimensional and that \( \delta(\widetilde{X}) = 1 \). Our goal is to establish the lower bound (7.1) for the spectrum of the Laplace operators \( \Delta_\mu(m) \).

To this end we use (5.4), which reduces the problem to the estimation from below of the endomorphism \( E_\mu(\tau_\lambda(m)) \).

First we introduce some notation. Let \( \widetilde{X} = G/K \). There is a decomposition \( \widetilde{X} = \widetilde{X}_0 \times \widetilde{X}_1 \) with \( \delta(\widetilde{X}_0) = 0 \) and \( \widetilde{X}_1 \) is an irreducible symmetric space with \( \delta(\widetilde{X}_1) = 1 \). Since \( \widetilde{X}_0 \) is even-dimensional, the dimension of \( \widetilde{X}_1 \) is odd. Let \( G = G_0 \times G_1 \) be the corresponding decomposition of \( G \). Then \( \delta(G_0) = 0 \) and by the classification of simple Lie groups, \( G_1 = SO^0(p,q) \), \( p,q \) odd, or \( G_1 = SL(3,\mathbb{R}) \). Let \( \mathfrak{g}_i, i = 0,1 \) be the Lie algebra of \( G_i \). Let \( \mathfrak{t}_0 \subset \mathfrak{g}_0 \) be a compact Cartan subalgebra and let \( \mathfrak{h}_1 \subset \mathfrak{g}_1 \) be a fundamental Cartan subalgebra. Then \( \mathfrak{h}_1 \) is of split rank one. Put
\[
\mathfrak{h} := \mathfrak{t}_0 \oplus \mathfrak{h}_1.
\]
Then \( \mathfrak{h} \) is a Cartan subalgebra of split rank one. Let \( (\tau, V_\tau) \in \text{Rep}(G) \) with highest weight \( \lambda \in \mathfrak{h}_1^* \). Then \( \lambda = \lambda_0 + \lambda_1 \), where \( \lambda_0 \in \mathfrak{t}_0^* \mathbb{C} \) and \( \lambda_1 \in \mathfrak{h}_1^* \mathbb{C} \) are highest weights. Let \( \theta: \mathfrak{g} \to \mathfrak{g} \) be the Cartan involution. Assume that \( \lambda_\theta \neq \lambda \). Then \( \lambda_1 \) satisfies \( (\lambda_1)_\theta \neq \lambda_1 \). Let \( (\tau_i, V_{\tau_i}) \in \text{Rep}(G_i), i = 0,1 \), be the representations with highest weight \( \lambda_i \). Then \( \tau \cong \tau_0 \otimes \tau_1 \).

Let \( \mathfrak{g}_i = \mathfrak{t}_i \oplus \mathfrak{p}_i \)

be the Cartan decomposition of \( \mathfrak{g}_i, i = 0,1 \). We may choose \( \mathfrak{p} \) such that \( \mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \). Then we have
\[
\Lambda^p \mathfrak{p}^* \otimes V_\tau \cong \bigoplus_{r+s=p} (\Lambda^r \mathfrak{p}_0^* \otimes V_{\tau_0}) \otimes (\Lambda^s \mathfrak{p}_1^* \otimes V_{\tau_1})
\]
Let \( \Omega_i \in \mathcal{Z}(\mathfrak{g}_i, \mathbb{C}), i = 1, 2 \), be the Casimir operator of \( \mathfrak{g}_i \). Then \( \Omega = \Omega_0 \otimes \text{Id} + \text{Id} \otimes \Omega_1 \).
Similarly, we have \( \Omega_K = \Omega_{0,K} \otimes \text{Id} + \text{Id} \otimes \Omega_{1,K} \). Set
\[
\nu_{s,\mu}(\tau_i) := \Lambda^p \text{Ad}_{\mathfrak{p}_i}^* \otimes \tau_i: K_i \to \text{GL}(\Lambda^p \mathfrak{p}_i^* \otimes V_{\tau_i}), \quad i = 0, 1.
\]

Let
\[
E_{s,\mu}(\tau_i) := \tau_i(\Omega_i) \text{Id} - \nu_{p}(\tau_i)(\Omega_{i,K}), \quad i = 0, 1.
\]
be the corresponding endomorphisms acting in $\Lambda^p \mathfrak{p}_i^* \otimes V_{\tau}$. Then it follows that

$$ (7.2) \quad E_p(\tau) = \bigoplus_{r+s=p} \left( E_{0,r}(\tau_0) \otimes \text{Id} + \text{Id} \otimes E_{1,s}(\tau_1) \right). $$

Therefore it suffices to estimate $E_{i,p}(\tau_i), i = 0, 1$.

Let us first recall the general formula for the Casimir eigenvalues. We let $\mathfrak{g}$ be a semisimple real Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{b}$, $\mathfrak{b} \subset \mathfrak{p}$ be a $\theta$-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Let the associated groups $G$ and $K$ be as in the introduction. Let $\| \cdot \|$ denote the norm induced by the (suitably normalized) Killing form on the real vector space $\mathfrak{b}^* \oplus i\mathfrak{t}^*$. Fix positive roots $\Delta^+(\mathfrak{g}_C, \mathfrak{h}_C)$, $\Delta^+(\mathfrak{t}_C, \mathfrak{t}_C)$ and let $\rho_G$ resp. $\rho_K$ be the half sum of the positive roots.

Let $\tau$ be an irreducible finite-dimensional complex representation of $G$ with highest weight $\Lambda(\tau) \in \mathfrak{b}^* \oplus i\mathfrak{t}^*$ and let $\nu$ be an irreducible unitary representation of $K$ with highest weight $\Lambda(\nu) \in i\mathfrak{t}^*$. Then we have

$$ (7.3) \quad \tau(\Omega) = \| \Lambda(\tau) + \rho_G \|^2 - \| \rho_G \|^2; \quad \nu(\Omega_K) = \| \Lambda(\nu) + \rho_K \|^2 - \| \rho_K \|^2. $$

Now we prove the following general bound, which we use to deal with $E_{0,p}(\tau_0)$.

**Lemma 7.1.** Let $\lambda \in \mathfrak{h}_C^*$ be a highest weight. Given $m \in \mathbb{N}$, let $\tau_{\lambda}(m)$ be the irreducible representation with highest weight $m\lambda$. There exists $C > 0$ such that

$$ E_p(\tau_{\lambda}(m)) \geq -Cm $$

for all $p = 0, \ldots, d$ and $m \in \mathbb{N}$.

**Proof.** Let $\tau \in \text{Rep}(G)$ be of highest weight $\Lambda(\tau)$. Let $\nu' \in \hat{K}$ with highest weight $\Lambda(\nu') \in i\mathfrak{t}^*$. Assume that $[\tau|_K : \nu'] \neq 0$. We claim that there is a weight $\lambda$ of $\tau$ such that $\Lambda(\nu') = \lambda|_\mathfrak{t}$. To see this, let $V_\tau$ be the space of the representation $\tau$ and let $V_\tau(\Lambda(\nu'))$ be the eigenspace of $\mathfrak{t}$ with eigenvalue $\Lambda(\nu')$. Then $V_\tau(\Lambda(\nu'))$ is invariant under $\mathfrak{h}$. So it decomposes into joint eigenspaces of $\mathfrak{h}$. Let $\lambda$ be the weight of one of these eigenspaces. Then $\lambda|_\mathfrak{t} = \Lambda(\nu')$.

Now we note that as a weight of $\tau$, $\lambda$ belongs to the convex hull of the Weyl group orbit of $\Lambda(\tau)$ (see [Kn2, Theorem 7.41]). Thus we get

$$ (7.4) \quad \| \Lambda(\tau) \| \geq \| \lambda \| \geq \| \lambda|_\mathfrak{t} \| = \| \Lambda(\nu') \|. $$

Now let $\nu \in \hat{K}$ with $[\nu_\tau(p) : \nu] \neq 0$. Then by [Kn2, Proposition 9.72] there exists $\nu' \in \hat{K}$ with $[\tau|_K : \nu'] \neq 0$ of highest weight $\Lambda(\nu') \in i\mathfrak{t}^*$ and $\mu \in i\mathfrak{t}^*$ which is a weight of $\nu_\tau(p)$ such that the highest weight $\Lambda(\nu)$ of $\nu$ is given by $\mu + \Lambda(\nu')$. Since $\Lambda(\tau)$ is dominant we have

$$ \| \Lambda(\tau) + \rho_G \|^2 \geq \| \Lambda(\tau) \|^2. $$

Thus by (7.4) we get

$$ \| \Lambda(\tau) + \rho_G \|^2 - \| \Lambda(\nu) + \rho_K \|^2 \geq \| \Lambda(\tau) \|^2 - \| \Lambda(\nu') \|^2 - 2\| \mu + \rho_K \| \cdot \| \Lambda(\nu') \| - \| \mu + \rho_K \|^2 \geq -2\| \mu + \rho_K \| \cdot \| \Lambda(\tau) \| - \| \mu + \rho_K \|^2. $$
There is $C > 0$ such that $\|\mu + \rho_K\| \leq C$ for all weights $\mu$ of $\nu_p$. Hence there is $C_1 > 0$ such that for all $\tau \in \text{Rep}(G)$ one has
\begin{equation}
(7.5) \quad \|\Lambda(\tau) + \rho_G\|^2 - \|\Lambda(\nu) + \rho_K\|^2 \geq -C_1(\|\Lambda(\tau)\| + 1)
\end{equation}
for all $\nu \in \hat{K}$ with $[\nu_p(\tau): \nu] \neq 0$. Now we apply this to $\tau_\lambda(m)$. By definition of $\tau_\lambda(m)$ we have $\Lambda(\tau_\lambda(m)) = m\lambda$. Using \eqref{eq:7.3}, \eqref{eq:7.5}, the lemma follows. \hfill $\square$

Now we turn to the estimation of $E_{1,p}(\tau_1)$. In this case we have either $G_1 = SO^0(p, q)$, $p, q$ odd, or $G = SL(3, \mathbb{R})$. We deal with these cases separately.

### 7.1. The case $G = SO^0(p, q)$

Let $p = 2p_1 + 1$, $q = 2q_1 + 1$. Let $n := p_1 + q_1$. Let $K = SO(p) \times SO(q)$ and $\hat{X} = G/K$. We let $t$ and $\mathfrak{h}$ be as in section 3. Also the Killing form will be normalized as in this section. Then we have the following lemma.

**Lemma 7.2.** Let $\Lambda \in \mathfrak{h}_C^*$ be given as $\Lambda = k_1 e_1 + \cdots + k_{n+1} e_{n+1}$, $k_1 \geq k_2 \geq \cdots \geq k_{n+1} \geq 0$. Let $\Lambda' \in \mathfrak{h}_C^*$ belong to the convex hull of the set $\{w\Lambda, w \in W_G\}$ and let $\lambda \in \mathfrak{t}^*$ be given by $\lambda := \Lambda'|_1$. Then one has
\[
\|\lambda\|^2 \leq \sum_{i=1}^n k_i^2.
\]

**Proof.** Recall that the Weyl group $W_G$ consist of permutations and even sign changes of the $e_1, \ldots, e_{n+1}$. Thus there exist $\alpha_1, \ldots, \alpha_m \in (0, 1), \sum_{j=1}^m \alpha_j = 1$, and for each $j = 1, \ldots, m$ a $\sigma_j \in S^{n+1}$, the symmetric group, and a sequence $\epsilon_{j,1}, \ldots, \epsilon_{j,n+1} \in \{\pm 1\}$ such that
\[
\Lambda' = \sum_{j=1}^m \alpha_j \left( \sum_{i=1}^{n+1} \epsilon_{j,i} k_i e_{\sigma_j(i)} \right).
\]

Thus one has
\[
\lambda = \sum_{j=1}^m \alpha_j \left( \sum_{\sigma_j(i) \neq p_1+1}^{n+1} \epsilon_{j,i} k_i e_{\sigma_j(i)} \right)
\]
and so one gets
\[
\|\lambda\| \leq \sum_{j=1}^m \alpha_j \left( \sum_{\sigma_j(i) \neq p_1+1}^{n+1} \epsilon_{j,i} k_i e_{\sigma_j(i)} \right) = \sum_{j=1}^m \alpha_j \left( \sum_{i=1}^{n+1} k_i \right) \leq \sum_{j=1}^m \alpha_j \left( \sum_{i=1}^{n+1} k_i^2 \right) = \sum_{i=1}^{n+1} k_i^2.
\]
\hfill $\square$
Now we let $\Lambda(\tau) \in \mathfrak{h}_C^*$ be given by
\[
\Lambda(\tau) := \tau_1 e_1 + \cdots + \tau_{n+1} e_{n+1}, \quad \tau_1 \geq \tau_2 \geq \cdots \geq \tau_{n+1} > 0.
\]
For $m \in \mathbb{N}$ we let $\tau(m)$ be the representation of $G$ with highest weight
\[
\Lambda(\tau(m)) := m \Lambda(\tau).
\]
Then we have the following proposition.

**Proposition 7.3.** There exists a constant $C$ such that
\[
E_p(\tau(m)) \geq m^2 \tau_{n+1} - Cm
\]
for all $m$.

**Proof.** Recall that $\nu_p(\tau(m)) = \tau(m)|_K \otimes \nu_p$. Let $\nu \in \hat{K}$ be such that $[\nu_p(\tau(m)) : \nu] \neq 0$. By [Kn2, Proposition 9.72], there exists a $\nu' \in \hat{K}$ with $[\tau(m) : \nu'] \neq 0$ of highest weight $\lambda(\nu') \in \mathfrak{h}_C^*$ and a $\mu \in \mathfrak{b}_C^*$ which is a weight of $\nu_p$ such that the highest weight $\lambda(\nu)$ of $\nu$ is given by $\mu + \lambda(\nu')$. Now there is a $\tilde{\Lambda} \in \mathfrak{h}_C^*$ which is a weight of $\tau(m)$ such that $\lambda(\nu') = \tilde{\Lambda}|_K$.

By [Ha, Theorem 7.41], $\tilde{\Lambda}$ belongs to the convex hull of the Weyl group orbit of $\Lambda(\tau(m))$. Thus applying (7.3) and Lemma 7.2, we obtain constants $C_{1,2}$ which are independent of $m$ such that
\[
\nu(\Omega_K) = \|\lambda(\nu) + \rho_K\|^2 - \|\rho_K\|^2 \leq \|\lambda(\nu')\|^2 + C_1(1 + \|\lambda(\nu')\|)
\]
\[
\leq m^2 \left( \sum_{j=1}^n \tau_j^2 \right) + C_2m.
\]

One the other hand by (7.3) we have
\[
\tau(m)(\Omega) = \|\Lambda(\tau(m)) + \rho_G\|^2 - \|\rho_G\|^2
\]
\[
= \sum_{j=1}^{n+1} (m \tau_j + n + 1 - j)^2 - \sum_{j=1}^{n+1} (n + 1 - j)^2 \geq m^2 \sum_{j=1}^{n+1} \tau_j^2.
\]
This implies the proposition. \qed

7.2. The case $G = \text{SL}(3, \mathbb{R})$. We use the notation of section 3. We choose the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, which is defined by (6.30). The fundamental weights $\omega_i \in \mathfrak{h}_C^*$, $i = 1, 2$, are given by (6.35). Let $\Lambda \in \mathfrak{h}_C^*$ be a highest weight. For $m \in \mathbb{N}$ let $\tau_\Lambda(m)$ be the irreducible representation with highest weight $m \Lambda$.

**Proposition 7.4.** Assume that $\Lambda$ satisfies $\Lambda_0 \neq \Lambda$. Then there exists $C_\Lambda > 0$ such that
\[
E_p(\tau_\Lambda(m)) \geq \frac{1}{9} m^2 - C_\Lambda m
\]
for all $m \in \mathbb{N}$ and $p = 0, \ldots, 5$. 
Proof. There exist $\tau_1, \tau_2 \in \mathbb{N}_0$ such that $\Lambda = \tau_1 \omega_1 + \tau_2 \omega_2$. Note that by (6.33) and (5.34) one has $\rho_G = f_1 + f_2$. Then by (5.33) and (6.31) we get
\[
\tau_\Lambda(m)(\Omega) = \|m\Lambda + \rho_G\|^2 - \|\rho_G\|^2
\]
\[
= \frac{4(\tau_1^2 + \tau_2^2)m^2 + 4(\tau_1 + \tau_2)}{9}.\]
Next recall that there is a natural isomorphism $\mathfrak{t}_C \cong \mathfrak{su}(2)_C = \mathfrak{sl}(2, \mathbb{C})$ (see [Ha, Sect. 4.9]). Furthermore if we embed $\mathfrak{sl}(2, \mathbb{C})$ into $\mathfrak{g}_C$ as an upper left block then $\mathfrak{t}_C$ is isomorphic to a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. For $j \in \mathbb{N}$ we let $\nu_j$ denote the representation of $\mathfrak{t}_C$ with highest weight $jf_2$. Then we deduce from the branching law from $\text{GL}_3(\mathbb{C})$ to $\text{GL}_2(\mathbb{C})$, [GW, Theorem 8.1.1] that
\[
\tau_\Lambda(m)|_{\mathfrak{t}_C} = \bigoplus_{j=0}^m \bigoplus_{k=0}^m \nu_{j+k}.
\]
If we use
\[
\nu_j(\Omega_K) = \frac{j^2}{3} + \frac{2}{3}j.
\]
and argue as in the proof of Proposition 7.3, we obtain a constant $C$ which is independent of $\tau_1, \tau_2$ and $m$ such that for every $\nu \in K$ with $[\nu_p(\tau(m)) : \nu] \neq 0$ for some $p$ one has
\[
\nu(\Omega_K) \leq \frac{(m(\tau_1 + \tau_2) + C)^2}{3} + \frac{2(m(\tau_1 + \tau_2) + C)}{3}.
\]
Thus we obtain a constant $C_\Lambda$ such that for every $m$ and every $p$ one has
\[
E_p(\tau_\Lambda(m)) \geq \frac{(\tau_1 - \tau_2)^2}{9}m^2 - C_\Lambda m.
\]
By (6.36) the condition $\Lambda \theta \neq \Lambda$ is equivalent to $\tau_1 \neq \tau_2$. This proves the Proposition. \qed

Now we can summarize our results.

**Proposition 7.5.** Let $\delta(\bar{X}) = 1$, $\bar{X}$ odd-dimensional. Let $\lambda \in \mathfrak{b}_C^*$ be a highest weight with $\lambda_\theta \neq \lambda$. For $m \in \mathbb{N}$ let $\tau_\lambda(m)$ be the irreducible representation of $G$ with highest weight $m\lambda$. There exist $C_1, C_2 > 0$ such that
\[
E_p(\tau_\lambda(m)) \geq C_1m^2 - C_2
\]
for all $p = 0, \ldots, d$ and $m \in \mathbb{N}$.

**Proof.** Let $\lambda = \lambda_0 + \lambda_1$ with $\lambda_0 \in \mathfrak{t}_{0,C}^*$ and $\lambda_1 \in \mathfrak{b}_{1,C}^*$ highest weights, and $(\lambda_1)_{\theta} \neq \lambda_1$. Let $\tau_i(m)$, $i = 0, 1$, be the irreducible representations of $G_i$ with highest weight $m\lambda_i$. Then $\tau(m) = \tau_0(m) \otimes \tau_1(m)$. Let $E_{0,p}(\tau_0(m))$ and $E_{1,p}(\tau_1(m))$ be defined by (7.1). By Lemma 7.1 there exists $C > 0$ such that
\[
E_{0,p}(\tau_0(m)) \geq -Cm
\]
for all \( p = 0, \ldots, d \) and \( m \in \mathbb{N} \). Furthermore, by Proposition 7.3 and Proposition 7.4 there exist \( C_3, C_4 > 0 \) such that
\[
E_{1,p}(\tau_1(m)) \geq C_3 m^2 - C_4
\]
for all \( p = 0, \ldots, d \) and \( m \in \mathbb{N} \). Combined with (7.2) the proof follows.

**Corollary 7.6.** Let the assumptions be as in Proposition 7.5. There exist constants \( C_1, C_2 > 0 \) such that
\[
\Delta_p(\tau_\lambda(m)) \geq C_1 m^2 - C_2
\]
for all \( p = 0, \ldots, d \) and \( m \in \mathbb{N} \).

**Proof.** Recall that the Bochner-Laplace operator satisfies \( \Delta_p(\tau_\lambda(m)) \geq 0 \). Hence the corollary follows from (5.4) and Proposition 7.5. \( \square \)

### 8. Proof of the main results

First assume that \( \delta(\tilde{X}) \neq 0 \). Note that \( \delta(\tilde{X}) = 0 \) implies that \( \dim \tilde{X} \) is even. Hence, it follows from Proposition 4.2 that \( T_X(\tau) = 1 \) for all finite-dimensional irreducible representations of \( G \), which proves part (i) of Theorem 1.1.

Now assume that \( \delta(\tilde{X}) = 1 \). Let \( \mathfrak{h} \subset \mathfrak{g} \) be a fundamental Cartan subalgebra. Let \( \lambda \in \mathfrak{h}_\mathbb{C}^\ast \) be a highest weight with \( \theta \neq \lambda \). For \( m \in \mathbb{N} \) let \( \tau(m) \) be the irreducible representation of \( G \) with highest weight \( m\lambda \). Then \( \tau(m) \not\sim \tau(m)\theta \) for all \( m \in \mathbb{N} \). Hence by [BW, Chapter VII, Theorem 6.7] we have \( H^p(X, E_{\tau(m)}) = 0 \) for all \( p = 0, \ldots, d \). Then by (4.7) we have

\[
\log T_X(\tau(m)) = \frac{1}{2} \frac{d}{ds} \left. \left( \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(t, \tau(m)) \, dt \right) \right|_{s=0}.
\]

Since \( \tau(m) \) is acyclic and \( \dim X \) is odd, \( T_X(\tau(m)) \) is metric independent [Mu2, Corollary 2.7]. Especially we can rescale the metric by \( \sqrt{m} \) without changing \( T_X(\tau(m)) \). Equivalently we can replace \( \Delta_p(\tau(m)) \) by \( \frac{1}{m} \Delta_p(\tau(m)) \). Using (8.1) we get

\[
\log T_X(\tau(m)) = \frac{1}{2} \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(t, \tau(m)) \, dt \right) \bigg|_{s=0}.
\]

To continue, we split the \( t \)-integral into the integral over \([0, 1]\) and the integral over \([1, \infty)\). This leads to

\[
\log T_X(\tau(m)) = \frac{1}{2} \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} K\left( \frac{t}{m}, \tau(m) \right) \, dt \right) \bigg|_{s=0} + \frac{1}{2} \int_1^\infty t^{-1} K\left( \frac{t}{m}, \tau(m) \right) \, dt.
\]

We first consider the second term on the right hand side. To this end we need the following lemma.
Lemma 8.1. Let \( h_t^{\tau(m),p} \) be defined by (4.13) and let \( H_t^0 \) be the heat kernel of the Laplacian \( \tilde{\Delta}_0 \) on \( C^\infty(\tilde{X}) \). There exist \( m_0 \in \mathbb{N} \) and \( C > 0 \) such that for all \( m \geq m_0, g \in G, t \in (0, \infty) \) and \( p \in \{0, \ldots, d\} \) one has

\[
\left| h_t^{\tau(m),p}(g) \right| \leq C \dim(\tau(m)) e^{-\frac{t\tau}{2}} H_t^0(g).
\]

Proof. Let \( p \in \{0, \ldots, n\} \). Let \( H_t^{\nu(\tau(m))} \) be the kernel of \( e^{-t\tilde{\Delta}_0} \) and let \( H_t^{\tau(m),p} \) be the kernel of \( e^{-t\Delta_0} \). By (5.7) we have

\[
H_t^{\tau(m),p}(g) = e^{-tE_0(\tau(m))} \circ H_t^0(g).
\]

Thus by proposition 3.1 and Proposition 7.4 there exists an \( m_0 \) such that for \( m \geq m_0 \) one has

\[
(8.3) \quad \left\| H_t^{\tau(m),p}(g) \right\| \leq e^{-\frac{t\tau}{2}} H_t^0(g).
\]

Taking the trace in \( \text{End}(\Lambda^p p^* \otimes V_{\tau(m)}) \) for every \( p \in \{0, \ldots, d\} \), the lemma follows. \( \square \)

Using (4.16), (4.15) and Lemma 8.1, we obtain

\[
\left| K \left( \frac{t}{m}, \tau(m) \right) \right| \leq C e^{-\frac{t\tau}{2}} \dim(\tau(m)) \int_{\Gamma \setminus G} \sum_{\gamma \in \Gamma} H_t^{0}_{\gamma/m}(g^{-1}\gamma g) \, dg
\]

\[
= C e^{-\frac{t\tau}{2}} \dim(\tau(m)) \text{Tr}(e^{-\frac{t}{m}\Delta_0}).
\]

Furthermore, by the heat asymptotic [Gi] we have

\[
\text{Tr}(e^{-\frac{1}{m}\Delta_0}) = C_d \text{vol}(X)m^{d/2} + O(m^{(d-1)/2})
\]

as \( m \to \infty \). Hence there exists \( C_1 > 0 \) such that

\[
\left| K \left( \frac{t}{m}, \tau(m) \right) \right| \leq C_1 m^{d/2} \dim(\tau(m)) e^{-\frac{t\tau}{4}}, \quad t \geq 1.
\]

Thus we obtain

\[
\left| \int_1^\infty t^{-1} K \left( \frac{t}{m}, \tau(m) \right) \, dt \right| \leq C_1 m^{d/2} \dim(\tau(m)) e^{-m/4} \int_1^\infty t^{-1} e^{-\frac{t\tau}{4}} \, dt.
\]

Using Weyl’s dimension formula, it follows that

\[
(8.4) \quad \int_1^\infty t^{-1} K \left( \frac{t}{m}, \tau(m) \right) \, dt = O(e^{-m/8}).
\]

Now we turn to the first term on the right hand side of (8.2). We need to estimate \( K(t, \tau(m)) \) for \( 0 < t \leq 1 \). To this end we use (4.16) to decompose \( K(t, \tau(m)) \) into the sum of two terms: The contribution of the identity

\[
(8.5) \quad I(t, \tau(m)) := \text{vol}(X)k_t^{\tau(m)}(1),
\]
where \( k_t^{\tau(m)} \) is defined by (4.13), and the remaining term

\[
H(t, \tau(m)) := \int_{\Gamma \setminus \mathbb{G}} \sum_{\gamma \in \Gamma, \gamma \neq 1} k_t^{\tau(m)}(g^{-1} \gamma g) \, dg
\]

First we consider \( H(t, \tau(m)) \). Using Proposition 8.1 and Proposition 3.2, it follows that for every \( m \geq m_0 \) and every \( t \in (0, 1] \) we have

\[
\sum_{\gamma \in \Gamma, \gamma \neq 1} k_t^{\tau(m)}(g^{-1} \gamma g) \leq Ce^{-tm^2/2} \dim(\tau(m)) \sum_{\gamma \in \Gamma, \gamma \neq 1} H_t^0(g^{-1} \gamma g) \leq C_1 \dim(\tau(m)) e^{-tm^2/2} e^{-c_0/t}.
\]

Hence using Weyl’s dimension formula we get

\[
\left| k_t^{\tau(m)}(1) \right| \leq C_2 e^{-cm} e^{-c_1/t}, \quad 0 < t \leq 1.
\]

This implies that there is \( c_2 > 0 \) such that

\[
\left. \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} H \left( \frac{t}{m}, \tau(m) \right) \, dt \right) \right|_{s=0} = \int_0^1 t^{-1} H \left( \frac{t}{m}, \tau(m) \right) \, dt = O(e^{-c_2 m})
\]

as \( m \to \infty \).

It remains to consider the contribution of the identity \( I(t, \tau(m)) \). By Lemma 8.1 there exists \( C > 0 \) such that for all \( m \geq m_0 \) and \( p = 0, \ldots, d \) we have

\[
|H_t^{\tau(m),p}(1)| \leq C \dim(\tau(m)) e^{-tm^2/2} H_t^0(1).
\]

Next we estimate \( H_t^0(1) \) using the Plancherel-Theorem. Since the function \( H_t^0(1) \) is \( K \)-biinvariant, the Plancherel-Theorem for \( H_t^0(1) \) reduces to the spherical Plancherel theorem [He, Theorem 7.5]. Thus if \( Q = MAN \) is a fixed minimal standard parabolic subgroup, it follows from (5.13) that

\[
H_t^0(1) = e^{-t\|\rho_a\|^2} \int_{a^*} e^{-t\|\nu\|^2} \beta(\nu) d\nu,
\]

where \( \beta(\nu) \) is the spherical Plancherel-density. Thus there exists \( C_1 > 0 \) such that \( |H_t^0(1)| \leq C_1 \) for \( t \geq 1 \). Hence, by (4.15) we get

\[
|k_t^{\tau(m)}(1)| \leq C_2 \dim(\tau(m)) e^{-tm^2/2}
\]

for \( t \geq 1 \) and \( m \geq m_0 \). By (8.5) and Weyl’s dimension formula it follows that there exist \( C, c > 0 \) such that

\[
|I \left( \frac{t}{m}, \tau(m) \right) | \leq Ce^{-cm} e^{-ct}
\]
for \( t \geq 1 \) and \( m \geq m_0 \). Hence we get

\[
\left. \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} I \left( \frac{t}{m}, \tau(m) \right) \, dt \right) \right|_{s=0} = \left. \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} I \left( \frac{t}{m}, \tau(m) \right) \, dt \right) \right|_{s=0} + O(e^{-cm})
\]

(8.8)

for \( m \geq m_0 \). Since we are assuming that \( \delta(\tilde{X}) = 1 \), \( \dim(X) \) is odd. Then it follows from (5.14) and the definition of \( k_t^{\tau(m)} \) by (4.13), that \( k_t^{\tau(m)}(1) \) has an asymptotic expansion of the form

\[
k_t^{\tau(m)}(1) \sim \sum_{j=0}^\infty c_j t^{-d/2+j}.
\]

Since \( d \) is odd, the expansion has no constant term. This implies that

\[
\int_0^\infty t^{s-1} I(t, \tau(m)) \, dt
\]

is holomorphic at \( s = 0 \). Therefore we get

\[
\left. \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} I \left( \frac{t}{m}, \tau(m) \right) \, dt \right) \right|_{s=0}.
\]

By definition, the right hand side equals \( \log T_X^{(2)}(\tau(m)), T_X^{(2)}(\tau(m)) \) the \( L^2 \)-torsion. Combined with (8.2), (8.4) and (8.6) we obtain

\[
(8.9) \quad \log T_X(\tau(m)) = \log T_X^{(2)}(\tau(m)) + O(e^{-cm}).
\]

This proves Proposition 1.2.

Combining Proposition 5.3 with Proposition 6.7 and Proposition 6.8, we obtain Proposition 1.3. Together with Proposition 1.2 we obtain part (ii) of Theorem 1.1.

Corollary 1.4 follows from Proposition 6.7 and Corollary 1.5 follows from Proposition 6.8.

References

[At] M.F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*. Colloque "Analyse et Topologie" en l’Honneur de Henri Cartan (Orsay, 1974), pp. 437-2. Asterisque, No. 32-33, Soc. Math. France, Paris, 1976, MR0420729, Zbl 0323.58015.

[BM] D. Barbasch, H. Moscovici, *\( L^2 \)-index and the trace formula*, J. Funct. An. 53 (1983), no.2, 151-201., MR0722507, Zbl 0537.58039.

[BMZ1] J.-M. Bismut, X. Ma, W. Zhang, *Operateurs de Toeplitz et torsion analytique asymptotique*. C. R. Math. Acad. Sci. Paris 349 (2011), no. 17-18, 977-981, MR2838248, Zbl 1227.58010.

[BMZ2] J.-M. Bismut, X. Ma, W. Zhang, *Asymptotic torsion and Toeplitz operators*, Preprint, 2011.

[Bor] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology 2 (1963), 111-122., MR0447474, Zbl 0116.38603.
[Bo] R. Bott, *The index theorem for homogeneous differential operators*, 1965 Differential and Combinatorial Topology, (A Symposium in Honor of Marston Morse) pp. 167–186 Princeton Univ. Press, Princeton, N.J., MR0182022, Zbl 0173.26001.

[BV] N. Bergeron, A. Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, http://arxiv.org/abs/1004.1083 (2010).

[BW] A. Borel, N. Wallach *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Princeton University Press, Princeton, 1980, MR0554917, Zbl 0443.22010.

[De] P. Delorme, *Formules limites et formules asymptotiques pour les multiplicités dans $L^2(G/Γ)$*, Duke Math. J. 53 (1986), no. 3, 691-731, MR0860667, Zbl 0623.22012.

[DL] H. Donnelly, P. Li, *Lower bounds for the eigenvalues of Riemannian manifolds*, Michigan math. J. 29 (1982), 149 - 161, MR0654476, Zbl 0488.58022.

[Do1] H. Donnelly, *Asymptotic expansions for the compact quotients of properly discontinuous group actions*, Illinois J. Math. 23 (1979), no. 3, 485-496, MR0537526, Zbl 0411.53033.

[Fr] D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. Math. 84 (1986), no. 3, 523-540, MR0837526, Zbl 0411.53033.

[Gi] P.B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*. Second edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995, MR1396308, Zbl 0856.58001.

[GW] R. Goodman, N. Wallach *Representations and invariants of the classical groups*, Encyclopedia of Mathematics and its Applications, 68. Cambridge University Press, Cambridge, 1998, MR1606831, Zbl 0901.22001.

[Gu] B. Guneysu, *Kato’s inequality and form boundedness of Kato potentials on arbitrary Riemannian manifolds*, 2011, arXiv:1105.0532v3.

[Ha] Brian C. Hall, *Lie groups, Lie algebras, and representations*. Graduate Texts in Mathematics, 222, Springer-Verlag, Berlin, New York, 2003, MR1997306, Zbl 1026.22001.

[He] S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions*, Corrected reprint of the 1984 original Mathematical Surveys and Monographs, 83. American Mathematical Society, Providence, RI, 2000, MR1790156, Zbl 0965.43007.

[HC] Harish-Chandra *Harmonic analysis on real reductive groups. III. The Maass-Selberg relations and the Plancherel formula*. Ann. of Math. (2) 104 (1976), no. 1, 117-201, MR0439994, Zbl 0331.22007.

[Kn1] A. Knapp, *Representation theory of semisimple groups*, Princeton University Press, Princeton, 2001, MR1880691, Zbl 0993.22001.

[Kn2] A. Knapp, *Lie groups beyond an introduction*, Second Edition, Progress in Mathematics, 140. Birkhäuser Boston, Inc., Boston, MA, 2002, MR1920389, Zbl 1075.22501.

[Ko] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Annals of Math. (2) 74 (1961), 329-378, MR0142696, Zbl 0134.03501.

[Lo] J. Lott, *Heat kernels on covering spaces and topological invariants*. J. Differential Geom. 35 (1992), no. 2, 471-510, MR1158347, Zbl 0770.58040.

[LM] H.B. Lawson, M.-L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989, MR1031992, Zbl 0688.57001.

[Man] A. Manning, *Topological entropy for geodesic flows*. Ann. of Math. (2) 110 (1979), no. 3, 567-573, MR0554385, Zbl 0426.58016.

[MaM] Y. Matsushima, S. Murakami, *On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds*, Ann. of Math. (2) 78 (1963), 365-416, MR0153028, Zbl 0125.10702.
R. J. Miatello, *The Minakshisundaram-Pleijel coefficients for the vector-valued heat kernel on compact locally symmetric spaces of negative curvature*. Trans. Amer. Math. Soc. **260** (1980), no. 1, 1 - 33, MR0570777, Zbl 0444.58015.

J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426, MR0196736, Zbl 0147.23104.

H. Moscovici, R. Stanton, *Eta invariants of Dirac operators on locally symmetric manifolds*, Inv. Math. **95** (1989), 629–666, MR0979370, Zbl 0672.58043.

H. Moscovici, R. Stanton, *R-torsion and zeta functions for locally symmetric manifolds*, Inv. Math. **105** (1991), 185–216, MR1109626, Zbl 0789.58073.

W. Müller, *The trace class conjecture in the theory of automorphic forms. II*. Geom. Funct. Anal. **8** (1998), no. 2, 315-355, MR1616155, Zbl 1073.1151.

W. Müller, *Analytic torsion and R-torsion for unimodular representations*, J. Amer. Math. Soc. **6** (1993), 721-753, MR1189689, Zbl 0789.58071.

W. Müller, *The asymptotics of the Ray-Singer analytic torsion of hyperbolic 3 manifolds*, Preprint 2010, arXiv:1003.5168, to appear in: “Metric and Differential Geometry”, The Jeff Cheeger Anniversary Volume, Progress in Math. Vol. 297, pp. 317 - 352, Birkhäuser, 2012.

W. Müller, J. Pfaff, *The asymptotics of the Ray-Singer analytic torsion for hyperbolic manifolds*, Intern. Math. Research Notices 2012, doi: 10.1093/imrn/rns130.

M. Olbrich, *L²-invariants of locally symmetric spaces*. Doc. Math. **7** (2002), 219–237, MR1938121, Zbl 1029.58019.

D.B. Ray, I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*. Advances in Math. **7** (1971), 145-210, MR0295381, Zbl 0239.58014.

M.A. Shubin, *Pseudodifferential operators and spectral theory*. Second edition. Springer-Verlag, Berlin, 2001, MR1852334, Zbl 0980.35180.

N. Wallach, *On the Selberg trace formula in the case of compact quotient* Bull. Amer. Math. Soc. **82** (1976), no. 2, 171-195, MR0404533, Zbl 0351.22008.

N. Wallach, *Real reductive groups. I*, Pure and Applied Mathematics, 132. Academic Press, Inc., Boston, MA, 1988, MR0929683, Zbl 0666.22002.

N. Wallach, *Real reductive groups. II*, Pure and Applied Mathematics, 132-II. Academic Press, Inc., Boston, MA, 1992, MR1170566, Zbl 0785.22001.

G. Warner, *Harmonic analysis on semi-simple Lie groups. I*, Die Grundlehren der mathematischen Wissenschaften, Band 188. Springer-Verlag, New York-Heidelberg, 1972, MR0498999, Zbl 0265.22020.