Optimal decay for the $n$-dimensional incompressible Oldroyd-B model without damping mechanism

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Abstract

By a new energy approach involved in the high frequencies and low frequencies decomposition in the Besov spaces, we obtain the optimal decay for the incompressible Oldroyd-B model without damping mechanism in $\mathbb{R}^n$ ($n \geq 2$). More precisely, let $(u, \tau)$ be the global small solutions constructed in [18], we prove for any $(u_0, \tau_0) \in \dot{B}_{2,1}^{-s}(\mathbb{R}^n)$ that

$$
\|\Lambda^\alpha (u, \Lambda^{-1} \text{div } \tau)\|_{L^q} \leq C (1 + t)^{-\frac{n}{4} - \frac{(s+1)q-n}{2q}}, \quad \Lambda \overset{\text{def}}{=} \sqrt{-\Delta},
$$

with $\frac{n}{2} - 1 < s < \frac{n}{p}$, $2 \leq p \leq \min(4, \frac{2n}{n-2})$, $p \neq 4$ if $n = 2$, and $p \leq q \leq \infty$, $\frac{n}{p} - \frac{n}{p} - s < \alpha \leq \frac{n}{q} - 1$. The proof relies heavily on the special dissipative structure of the equations and some commutator estimates and various interpolations between Besov type spaces. The method also works for other parabolic-hyperbolic systems in which the Fourier splitting technique is invalid.

Key Words: Oldroyd-B model; Time decay estimates; Besov space

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1. Introduction and the main result

The incompressible Oldroyd-B model without damping mechanism in $\mathbb{R}^n$ can be written as:

$$
\begin{align*}
\partial_t \tau + u \cdot \nabla \tau + F(\tau, \nabla u) &= D(u), \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= \text{div } \tau, \\
\text{div } u &= 0, \\
(u, \tau)|_{t=0} &= (u_0, \tau_0),
\end{align*}
$$

(1.1)

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where \( u = (u_1, u_2, \ldots, u_n) \) denotes the velocity, \( p \) is the scalar pressure of fluid. \( \tau = \tau_{i,j} \) is the non-Newtonian part of stress tensor which can be seen as a symmetric matrix here. \( D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \) is the symmetric part of \( \nabla u \) and \( F \) is a given bilinear form which can be chosen as

\[
F(\tau, \nabla u) = \tau \Omega(u) - \Omega(u) \tau + b(D(u) \tau + \tau D(u)),
\]

where \( b \) is a parameter in \([-1, 1]\), \( \Omega(u) = \frac{1}{2}(\nabla u - (\nabla u)^T) \) is the skew-symmetric part of \( \nabla u \).

The above Oldroyd-B model presents a typical constitutive law which does not obey the Newtonian law (a linear relationship between stress and the gradient of velocity in fluids). Such non-Newtonian property may arise from the memorability of some fluids. Formulations about viscoelastic flows of Oldroyd-B type are first introduced by Oldroyd [16] and are extensively discussed in [2]. One can find the derivation of (1.1) in [14], here we omit it.

The mathematical theory of Oldroyd-B model is an old subject, see [3]–[11], [13]–[16], [18]–[20]. Here, we only recall some results about Oldroyd-B model without damping mechanism. In fact, when neglecting the damping term in the stress tensor equation, (1.1) reduces to be a parabolic-hyperbolic system. Due to lack of smoothing effect of \( \tau \), it’s difficult to get the global solutions directly. Luckily, by exploiting the good structure of the system, we can obtain some hidden dissipation about \( \tau \). Based on the above analysis, by constructing the time-weighted energies, Zhu [19] obtained the global small solutions to (1.1) in \( \mathbb{R}^3 \). This result was extended by Chen and Hao [4] to the \( L^2 \) type Besov spaces in \( \mathbb{R}^n \). The first author of the present paper in [18] generalized the result of [4] to the \( L^p \) framework which the highly oscillating initial velocity are allowed.

Denote \( \mathcal{P} = \mathcal{I} - Q := \mathcal{I} - \nabla \Delta^{-1} \text{div} \) and

\[
f^\ell \overset{\text{def}}{=} S^j_{j_0+1} f \quad \text{and} \quad f^h \overset{\text{def}}{=} f - f^\ell
\]

for some fix integer \( j_0 \geq 0 \).

The author in [18] obtained the following theorem:
Theorem 1.1. (see [18]) Let \( n \geq 2 \) and

\[
2 \leq p \leq \min(4, 2n/(n-2)) \quad \text{and, additionally, } p \neq 4 \text{ if } n = 2.
\]

For any \((u_0^h, \tau_0^h) \in \dot{B}_{2,1}^{n-1}(\mathbb{R}^n), u_0^h \in \dot{B}_{p,1}^{n-1}(\mathbb{R}^n), \tau_0^h \in \dot{B}_{p,1}^{n}(\mathbb{R}^n) \) with \( \text{div} \ u_0 = 0 \). If there exists a positive constant \( c_0 \) such that,

\[
\mathcal{X}_0 \overset{\text{def}}{=} \| (u_0, \tau_0) \|_{\dot{B}_{2,1}^{n-1}} + \| u_0 \|_{\dot{B}_{p,1}^{n-1}} + \| \tau_0 \|_{\dot{B}_{p,1}^{n}} \leq c_0,
\]

then the system (1.1) has a unique global solution \((u, \tau)\) so that for any \( T > 0 \)

\[
\begin{align*}
&u^\ell \in C_b([0, T]; \dot{B}_{2,1}^{n-1}(\mathbb{R}^n)) \cap L^1([0, T]; \dot{B}_{p,1}^{n+1}(\mathbb{R}^n)), \\
&\tau^\ell \in C_b([0, T]; \dot{B}_{2,1}^{n-1}(\mathbb{R}^n)), \quad (\Lambda^{-1} \mathbb{P} \text{div } \tau)^\ell \in L^1([0, T]; \dot{B}_{p,1}^{n+1}(\mathbb{R}^n)), \\
&u^h \in C_b([0, T]; \dot{B}_{p,1}^{n}(\mathbb{R}^n)) \cap L^1([0, T]; \dot{B}_{p,1}^{n+1}(\mathbb{R}^n)), \\
&\tau^h \in C_b([0, T]; \dot{B}_{p,1}^{n}(\mathbb{R}^n)), \quad (\Lambda^{-1} \mathbb{P} \text{div } \tau)^h \in L^1([0, T]; \dot{B}_{p,1}^{n}(\mathbb{R}^n)).
\end{align*}
\]

Moreover, there exists some constant \( C = C(p, n) \) such that

\[
\mathcal{X}(t) \leq C \mathcal{X}_0,
\]

with \( \mathcal{X}(t) \overset{\text{def}}{=} \| (u, \tau) \|^\ell_{L_t^\infty(\dot{B}_{2,1}^{n-1})} + \| u \|^h_{L_t^\infty(\dot{B}_{p,1}^{n-1})} + \| \tau \|^h_{L_t^\infty(\dot{B}_{p,1}^{n})} + \| u \|^h_{L_t^1(\dot{B}_{p,1}^{n+1})} \\
+ \| (u, (\Lambda^{-1} \mathbb{P} \text{div } \tau))^\ell_{L_t^1(\dot{B}_{2,1}^{n+1})} + \| \Lambda^{-1} \mathbb{P} \text{div } \tau \|^h_{L_t^1(\dot{B}_{p,1}^{n})} \).
\]

The natural next step is to look for a more accurate description of the long time behavior of the solutions. As there is no dissipation in the \( \tau \) equation, the usual Fourier splitting technique can not be used here. The spectral analysis for the linearized system may be valid. Here, we present another new pure energy method which motivated by [12], [17] to get the optimal decay of the solutions. Considering the linear system of (1.1), one can find \( u \) and \( \mathbb{P} \text{div } \tau \) satisfy the following damped wave equation:

\[
W_{tt} - \Delta W_t - \frac{1}{2} \Delta W = 0.
\]

Thus, we only expect to get the decay of \( u \) and the partial decay in \( \tau \), namely \( \mathbb{P} \text{div } \tau \).

Now, we state the main result of the paper:
Theorem 1.2. Let \((u, \tau)\) be the global small solutions addressed by Theorem 1.1. If in addition 
\((u_0, \tau_0) \in \dot{B}_{2, 1}^{-s}(\mathbb{R}^n)\) with \(\frac{n}{2} - 1 < s < \frac{n}{p}\). For any \(p \leq q \leq \infty\) and \(\frac{n}{q} - \frac{n}{p} - s < \alpha \leq \frac{n}{q} - 1\), there holds
\[
\left\| \Lambda^\alpha (u, \Lambda^{-1} \text{Pdiv} \, \tau) \right\|_{L^q} \leq C \left(1 + t\right)^{-\frac{n}{q} - \frac{\alpha + s}{2q}}. \tag{1.2}
\]

Remark 1.3. Let \(p = q = 2\), one can deduce from (1.2) that
\[
\left\| \Lambda^\alpha (u, \Lambda^{-1} \text{Pdiv} \, \tau) \right\|_{L^2} \leq C \left(1 + t\right)^{-\frac{s}{2} - \frac{\alpha}{2}},
\]
which coincides with the heat flows, thus our decay rate is optimal in some sense.

2. Preliminaries

For readers’ convenience, in this section, we list some basic knowledge on Littlewood-Paley theory.

Definition 2.1. Let us consider a smooth function \(\varphi\) on \(\mathbb{R}\), the support of which is included in \([\frac{3}{4}, \frac{8}{3}]\) such that 
\[
\forall \tau > 0, \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \tau) = 1, \quad \text{and} \quad \chi(\tau) \overset{\text{def}}{=} 1 - \sum_{j \geq 0} \varphi(2^{-j} \tau) \in \mathcal{D}([0, 4/3]).
\]

Let us define \(\hat{\Delta}_j u = \mathcal{F}^{-1}(\varphi(2^{-j} |\xi|) \hat{u})\), and \(\hat{S}_j u = \mathcal{F}^{-1}(\chi(2^{-j} |\xi|) \hat{u})\).

Let \(p\) be in \([1, +\infty]\) and \(s\) in \(\mathbb{R}\), \(u \in \mathcal{S}'(\mathbb{R}^n)\). We define the Besov norm by
\[
\| u \|_{\dot{B}^s_{p, 1}} \overset{\text{def}}{=} \left\| (2^j s) \| \hat{\Delta}_j u \|_{L^p} \right\|_{\ell^1(\mathbb{Z})}.
\]

We then define the spaces \(\dot{B}^s_{p, 1} \overset{\text{def}}{=} \{ u \in \mathcal{S}'(\mathbb{R}^n), \| u \|_{\dot{B}^s_{p, 1}} < \infty\}\), where \(u \in \mathcal{S}'(\mathbb{R}^n)\) means that \(u \in \mathcal{S}'(\mathbb{R}^n)\) and \(\lim_{j \to -\infty} \| \hat{S}_j u \|_{L^\infty} = 0\) (see Definition 1.26 of [1]).

In this paper, it will be suitable to split tempered distributions \(u\) into low and high frequencies. For a fix integer \(j_0\) (the value of which will follow from the proof of the main theorem), we denote
\[
\| u \|_{\dot{B}^s_{p, 1}}^\ell \overset{\text{def}}{=} \sum_{j \leq j_0} 2^j s \| \hat{\Delta}_j u \|_{L^p} \quad \text{and} \quad \| u \|_{\dot{B}^s_{p, 1}}^h \overset{\text{def}}{=} \sum_{j \geq j_0 + 1} 2^j s \| \hat{\Delta}_j u \|_{L^p}.
\]

Let us now state some classical properties for the Besov spaces.
Lemma 2.2.  

- Let \( 1 \leq p \leq \infty \) and \( s_1, s_2 \in \mathbb{R} \) with \( s_1 > s_2 \), for any \( u \in \dot{B}_{p,1}^{s_1} \cap \dot{B}_{p,1}^{s_2}(\mathbb{R}^n) \), there holds

\[
\|u^\ell\|_{\dot{B}_{p,1}^{s_1}} \leq C\|u^\ell\|_{\dot{B}_{p,1}^{s_2}}, \quad \|u^h\|_{\dot{B}_{p,1}^{s_2}} \leq C\|u^h\|_{\dot{B}_{p,1}^{s_1}}.
\]

- If \( s_1 \neq s_2 \) and \( \theta \in (0, 1) \), \( \left[ \dot{B}_{p,1}^{s_1}, \dot{B}_{p,1}^{s_2} \right]_\theta = \dot{B}_{p,1}^{\theta s_1 + (1-\theta)s_2} \).

- For any smooth homogeneous of degree \( m \in \mathbb{Z} \) function \( A \) on \( \mathbb{R}^n \setminus \{0\} \), the operator \( A(D) \) maps \( \dot{B}_p^s \) in \( \dot{B}_p^{s-m} \).

We are going to define the space of Chemin-Lerner (see \([1]\)) in which we will work, which is a refinement of the space \( L_p^\lambda(\dot{B}_p^s(\mathbb{R}^n)) \).

**Definition 2.3.** Let \((\lambda, p) \in [1, +\infty]^2 \) and \( T \in (0, +\infty] \). We define \( \overline{L}_T^\lambda(\dot{B}^{s}_{p,1}(\mathbb{R}^n)) \) as the completion of \( C([0, T]; \mathcal{S}(\mathbb{R}^n)) \) by the norm

\[
\|f\|_{\overline{L}_T^\lambda(\dot{B}^{s}_{p,1})} = \sum_{j \in \mathbb{Z}} 2^{js} \left( \int_0^T \|\Delta_j f(t)\|_{L_p}^\lambda dt \right)^{\frac{1}{\lambda}} < \infty.
\]

The following product estimates in Besov spaces play a key role in our analysis of the bilinear terms (see \([12]\)).

**Lemma 2.4.** Let \( 1 \leq p, q \leq \infty \), \( s_1 \leq \frac{n}{q}, s_2 \leq n \min\{\frac{1}{p}, \frac{1}{q}\} \) and \( s_1 + s_2 > \max\{0, \frac{1}{p} + \frac{1}{q} - 1\} \). For any \((u, v) \in \dot{B}_{q,1}^{s_1}(\mathbb{R}^n) \times \dot{B}_{p,1}^{s_2}(\mathbb{R}^n) \), we have

\[
\|uv\|_{\dot{B}_{p,1}^{s_1+s_2-\frac{n}{q}}} \leq C\|u\|_{\dot{B}_{q,1}^{s_1}}\|v\|_{\dot{B}_{p,1}^{s_2}}.
\]

Finally, we recall the following commutator’s estimate:

**Lemma 2.5.** (Lemma 2.100 from Bahouri et al. (2011)) Let \( \nabla u \in \dot{B}_{p,1}^{\frac{n}{p}}(\mathbb{R}^n) \) and \( v \in \dot{B}_{q,1}^{s}(\mathbb{R}^n) \) with \( 1 \leq p, q, r \leq \infty \). For any

\[
-1 - n \min\left\{\frac{1}{p}, 1 - \frac{1}{q}\right\} < s < \frac{n}{p}, \quad \text{if} \quad \text{div} u = 0,
\]

there holds

\[
\left\| (2^{|j|} \|[\Delta_j, u \cdot \nabla] v\|_{L^r})_j \right\|_{\ell^1(\mathbb{Z})} \leq C\|\nabla u\|_{\dot{B}_{p,1}^{\frac{n}{p}}}\|v\|_{\dot{B}_{q,1}^{s}}.
\]
3. Proof of the main theorem

In this section, we prove the main Theorem 1.2 by a pure energy method which is originated from the idea as in [12], [17].

Applying project operator $\Pi$ on both hand side of the first two equation in (1.1) gives

$$\left\{ \begin{align*}
\partial_t u + \Pi(u \cdot \nabla u) - \Delta u - \Pi \text{div } \tau &= 0, \\
\partial_t \Pi \text{div } \tau + \Pi \text{div } (u \cdot \nabla \tau) - \Delta u + \Pi \text{div } (F(\tau, \nabla u)) &= 0.
\end{align*} \right. \tag{3.1}$$

Denote

$$\phi \overset{\text{def}}{=} \Lambda^{-1} \Pi \text{div } \tau, \quad \text{with } \Lambda \overset{\text{def}}{=} \sqrt{-\Delta}.$$ 

A simple computation from (3.1) implies

$$\left\{ \begin{align*}
\partial_t u + u \cdot \nabla u - \Delta u - \Lambda \phi &= -[\Pi, u \cdot \nabla]u, \\
\partial_t \phi + u \cdot \nabla \phi + \Lambda u &= -[\Lambda^{-1} \Pi \text{div }, u \cdot \nabla ] \tau - \Lambda^{-1} \Pi \text{div } (F(\tau, \nabla u)).
\end{align*} \right. \tag{3.2}$$

Now, we can follow the proof of Section 3 in [18] or Lemma 4.1 and Lemma 4.2 in [17] to get (we omit the details)

$$\frac{d}{dt} \left( \| (u, \tau) \|_{B_{2,1}^{\frac{3}{2}}}^{\ell} + \| u \|_{B_{p,1}^{\frac{3}{p}}}^{h} + \| \phi \|_{B_{p,1}^{\frac{3}{p}}}^{h} \right) + \| (u, \tau) \|_{B_{2,1}^{\frac{3}{2}}}^{\ell} + \| u \|_{B_{p,1}^{\frac{3}{p}}}^{h} + \| \phi \|_{B_{p,1}^{\frac{3}{p}}}^{h} \leq C \left( \| (u, \tau) \|_{B_{2,1}^{\frac{3}{2}}}^{\ell} + \| u \|_{B_{p,1}^{\frac{3}{p}}}^{h} + \| \tau \|_{B_{p,1}^{\frac{3}{p}}}^{h} \right) \left( \| (u, \tau) \|_{B_{2,1}^{\frac{3}{2}}}^{\ell} + \| u \|_{B_{p,1}^{\frac{3}{p}}}^{h} + \| \phi \|_{B_{p,1}^{\frac{3}{p}}}^{h} \right). \tag{3.3} \right.$$ 

The following fact can be guaranteed by Theorem 1.1

$$\| (u, \tau) \|_{B_{2,1}^{\frac{3}{2}}}^{\ell} + \| u \|_{B_{p,1}^{\frac{3}{p}}}^{h} + \| \tau \|_{B_{p,1}^{\frac{3}{p}}}^{h} \leq \mathcal{X}(t) \leq \mathcal{X}_0 \ll 1 \quad \text{for all } t \geq 0. \tag{3.4} \right.$$ 

Thus absorbing all the terms in the right to left in (3.3) gives

$$\frac{d}{dt} \left( \| (u, \phi) \|_{B_{2,1}^{\frac{3}{2}}}^{\ell} + \| u \|_{B_{p,1}^{\frac{3}{p}}}^{h} + \| \phi \|_{B_{p,1}^{\frac{3}{p}}}^{h} \right) + \frac{1}{2} \left( \| (u, \phi) \|_{B_{2,1}^{\frac{3}{2}}}^{\ell} + \| u \|_{B_{p,1}^{\frac{3}{p}}}^{h} + \| \phi \|_{B_{p,1}^{\frac{3}{p}}}^{h} \right) \leq 0. \tag{3.5} \right.$$ 

Next, we want to use the interpolation inequality to get the Lyapunov-type inequality for the above energy norms.

According to (3.4) and Lemma 2.2 it’s obvious for any $\beta > 1$ that

$$\| u \|_{B_{p,1}^{h}}^{h} \geq C (\| u \|_{B_{p,1}^{\frac{3}{p}}}^{h})^\beta, \quad \| \phi \|_{B_{p,1}^{h}}^{h} \geq C (\| \phi \|_{B_{p,1}^{\frac{3}{p}}}^{h})^\beta. \tag{3.6} \right.$$ 

Thus, to get the Lyapunov-type inequality, we have to control \( \| (u, \phi) \|_{B_{2,1}^{s+1}}^\ell \) with \( (\| (u, \phi) \|_{B_{2,1}^{s}}^\ell)^{\eta} \) for some \( \eta > 1 \). This process can be obtained from the interpolation inequality, which implies that we must provide a low order estimates such as \( \| (u, \phi) \|_{B_{2,1}^{s-s'}}^\ell \) with \(-s < \frac{n}{2} - 1 \). However, only the incompressible part of stress tensor \((\mathbb{P} \text{div} \, \tau)\) have dissipation while the whole \( \tau \) itself don’t. Hence, it’s impossible to control \( \| (u, \phi) \|_{B_{2,1}^{s}}^\ell \) directly due to couple terms \( u \cdot \nabla \tau \) and \( F(\tau, \nabla u) \). To overcome this difficulty, we shall control \( \| (u, \tau) \|_{B_{2,1}^{s}}^\ell \) instead of \( \| (u, \phi) \|_{B_{2,1}^{s}}^\ell \). The price we have to pay is that the stronger condition imposed on \((u_0, \tau_0) \in B_{2,1}^{-s}(\mathbb{R}^n)\) instead of \((u_0, \phi_0) \in B_{2,1}^{s}(\mathbb{R}^n)\).

To do this we apply \( \Delta_j \) to the first two equations in (1.1) and use a standard commutator’s process to get

\[
\begin{align*}
\partial_t \Delta_j u + u \cdot \nabla \Delta_j u + \Delta_j \nabla p - \Delta \Delta_j u - \Delta_j \text{div} \, \tau &= [u \cdot \nabla, \Delta_j] u, \\
\partial_t \Delta_j \tau + u \cdot \nabla \Delta_j \tau + \Delta_j F(\tau, \nabla u) - \Delta_j D(u) &= [u \cdot \nabla, \Delta_j] \tau.
\end{align*}
\]

Taking \( L^2 \) inner product with \( \Delta_j u, \Delta_j \tau \), respectively and using the following cancellations

\[
\begin{align*}
\int_{\mathbb{R}^n} \Delta_j \text{div} \, \tau \cdot \Delta_j u \, dx + \int_{\mathbb{R}^n} \Delta_j D(u) \cdot \Delta_j \tau \, dx &= 0, \\
\int_{\mathbb{R}^n} u \cdot \nabla \Delta_j u \cdot \Delta_j u \, dx + \int_{\mathbb{R}^n} \Delta_j \nabla p \cdot \Delta_j u \, dx &= \int_{\mathbb{R}^n} u \cdot \nabla \Delta_j \tau \cdot \Delta_j \tau \, dx = 0,
\end{align*}
\]

we have

\[
\frac{1}{2} \frac{d}{dt} (\| \Delta_j u \|_{L^2}^2 + \| \Delta_j \tau \|_{L^2}^2) \leq C \| [u \cdot \nabla, \Delta_j] u \|_{L^2} \| \Delta_j u \|_{L^2} + \| \Delta_j F(\tau, \nabla u) \|_{L^2} \| \Delta_j \tau \|_{L^2} + C \| [u \cdot \nabla, \Delta_j] \tau \|_{L^2} \| \Delta_j \tau \|_{L^2},
\]

which implies that

\[
\frac{d}{dt} (\| \Delta_j u, \Delta_j \tau \|_{L^2}^2) \leq C (\| [u \cdot \nabla, \Delta_j] u \|_{L^2}^2 + \| [u \cdot \nabla, \Delta_j] \tau \|_{L^2}^2 + \| \Delta_j F(\tau, \nabla u) \|_{L^2}) \quad (3.7)
\]

Integrating the above inequality from 0 to \( t \), and multiplying by \( 2^{-js} \), we get by summing up about \( j \in \mathbb{Z} \) that

\[
\begin{align*}
\| (u, \tau)(t, \cdot) \|_{B_{2,1}^{s}} &\leq \| (u_0, \tau_0) \|_{B_{2,1}^{s}} + C \int_0^t \| F(\tau, \nabla u) \|_{B_{2,1}^{s}} \, dt' \\
&+ C \int_0^t (\sum_{j \in \mathbb{Z}} 2^{-js} \| [\Delta_j u \cdot \nabla] u \|_{L^2} + \sum_{j \in \mathbb{Z}} 2^{-js} \| [\Delta_j u \cdot \nabla] \tau \|_{L^2}) \, dt'. \quad (3.8)
\end{align*}
\]
For any $-\frac{n}{p} \leq s < \frac{n}{p}$, from Lemma 2.4 and Lemma 2.5 one has

$$\|F(\tau, \nabla u)\|_{B_{2,1}^{−s}} \leq C \|\nabla u\|_{B_{p,1}^{−\frac{n}{p}}} \|\tau\|_{B_{2,1}^{−s}} \leq C(\|u\|_{B_{2,1}^{\frac{n}{p} + 1}} + \|u\|_{B_{p,1}^{\frac{n}{p} + 1}})\|\tau\|_{B_{2,1}^{−s}}, \quad (3.9)$$

$$\sum_{j \in \mathbb{Z}} 2^{-j\|}[\hat{\Delta}_j, u \cdot \nabla]u\|_{L^2} + \sum_{j \in \mathbb{Z}} 2^{-j\|}[\hat{\Delta}_j, u \cdot \nabla]\tau\|_{L^2} \leq C\|\nabla u\|_{B_{p,1}^\frac{n}{p}} \|u\|_{B_{2,1}^{−s}} + \|\nabla u\|_{B_{p,1}^\frac{n}{p}} \|\tau\|_{B_{2,1}^{−s}} \leq C(\|u\|_{B_{2,1}^{\frac{n}{p} + 1}} + \|u\|_{B_{p,1}^{\frac{n}{p} + 1}})(u, \tau)_{B_{2,1}^{−s}}. \quad (3.10)$$

Plugging the above two estimates into (3.8) implies

$$\|(u, \tau)(t, \cdot)\|_{B_{2,1}^{−s}} \leq \|(u_0, \tau_0)\|_{B_{2,1}^{−s}} + C \int_0^t \|(u\|_{B_{2,1}^{\frac{n}{p} + 1}} + \|u\|_{B_{p,1}^{\frac{n}{p} + 1}})(u, \tau)_{B_{2,1}^{−s}} dt'. \quad (3.11)$$

It is easy to deduce from the definition of $\mathcal{X}'(t)$ in Theorem 1.1 that

$$\int_0^t \|(u\|_{B_{2,1}^{\frac{n}{p} + 1}} + \|u\|_{B_{p,1}^{\frac{n}{p} + 1}}) dt' \leq \mathcal{X}_0.$$ 

Hence, by the Gronwall inequality, one can get from (3.11), for any $-\frac{n}{p} \leq s < \frac{n}{p}$, that

$$\|(u, \tau)(t, \cdot)\|_{B_{2,1}^{−s}} \leq C_0 \quad (3.12)$$

for all $t \geq 0$, where $C_0 > 0$ depends on the norm of $\|(u_0, \tau_0)\|_{B_{2,1}^{−s}}$ and $\mathcal{X}_0$.

For any $s > 1 - \frac{n}{2}$, it follows from interpolation inequality in Lemma 2.2 that

$$\|(u, \phi)\|_{B_{2,1}^{\frac{n}{p} + 1}} \leq C(\|(u, \phi)\|_{B_{2,1}^{\frac{n}{p} + 1}})^{\theta_1}(\|(u, \phi)\|_{B_{2,1}^{\frac{n}{p} + 1}})^{1-\theta_1} \leq C(\|(u, \tau)\|_{B_{2,1}^{−s}})^{\theta_1}(\|(u, \phi)\|_{B_{2,1}^{\frac{n}{p} + 1}})^{1-\theta_1}, \quad \theta_1 = \frac{4}{n + 2s + 2} \in (0, 1),$$

this together with (3.12) implies that

$$\|(u, \phi)\|_{B_{2,1}^{\frac{n}{p} + 1}} \geq C(\|(u, \phi)\|_{B_{2,1}^{\frac{n}{p} + 1}})^{1-\theta_1}. \quad (3.13)$$

Taking $\beta = \frac{1}{1-\theta_1}$ in (3.6) gives

$$\|u\|_{B_{p,1}^{\frac{n}{p} + 1}} \geq C(\|u\|_{B_{p,1}^{\frac{n}{p} + 1}})^{1-\theta_1}, \quad \|\phi\|_{B_{p,1}^{\frac{n}{p} + 1}} \geq C(\|\phi\|_{B_{p,1}^{\frac{n}{p} + 1}})^{1-\theta_1}. \quad (3.14)$$
Thus, inserting (3.13) and (3.14) into (3.5) yields
\[
\frac{d}{dt} \left( \|(u, \phi)\|_{B^{\gamma}_{2,1}}^{\ell} + \|u\|_{B^{\gamma}_{p,1}}^h + \|\phi\|_{B^{\gamma}_{p,1}}^h \right) + \bar{c} \left( \|u\|_{L^{n+2s-2}_{x,t}}^{\ell} + \|u\|_{L^{n+2s-2}_{x,t}}^h + \|\phi\|_{L^{n+2s-2}_{x,t}}^h \right) \leq 0.
\]
Solving this differential inequality directly, we obtain
\[
\|u\|_{B^{\gamma}_{p,1}}^{\ell} + \|u\|_{B^{\gamma}_{p,1}}^h + \|\phi\|_{B^{\gamma}_{p,1}}^h \leq C \left( X_0^{\gamma - \frac{4}{n+2s-2}} + \frac{4\bar{c}}{n+2s-2} t \right)^{-\frac{n+2s-2}{4}} 
\leq C(1 + t)^{-\frac{n+2s-2}{4}}.
\]
Moreover, from Lemma 2.2 we further get
\[
\|u\|_{B^{\gamma}_{p,1}}^{\ell} \leq C \left( \|u\|_{B^{\gamma}_{p,1}}^{\ell} \right)^{\frac{\theta_2}{2}} \|u\|_{B^{\gamma}_{p,1}}^{h} \left( \|u\|_{B^{\gamma}_{p,1}}^{\ell} \right)^{1-\theta_2}, \quad \theta_2 = \frac{n+1-\gamma}{n+2s-1} \in (0, 1),
\]
which combines (3.12) with (3.15) gives
\[
\|u\|_{B^{\gamma}_{p,1}}^{\ell} \leq C(1 + t)^{-\frac{(n+1-\gamma)\theta_2}{2}} = C(1 + t)^{-\frac{n+1-\gamma}{2}}.
\]
For any \( \frac{n}{p} - \frac{n}{2} - s < \gamma < \frac{n}{p} - 1 \), by the interpolation inequality we have
\[
\|u\|_{B^{\gamma}_{p,1}}^{\ell} \leq C \left( \|u\|_{B^{\gamma}_{p,1}}^{\ell} \right)^{\frac{\theta_2}{2}} \|u\|_{B^{\gamma}_{p,1}}^{h} \left( \|u\|_{B^{\gamma}_{p,1}}^{\ell} \right)^{1-\theta_2}, \quad \theta_2 = \frac{n+1-\gamma}{n+2s-1} \in (0, 1),
\]
In the light of \( \frac{n}{p} - \frac{n}{2} - s < \gamma < \frac{n}{p} - 1 \), we see that
\[
\|u\|_{B^{\gamma}_{p,1}}^{\ell} \leq C \left( \|u\|_{B^{\gamma}_{p,1}}^{\ell} \right)^{\frac{\theta_2}{2}} \|u\|_{B^{\gamma}_{p,1}}^{h} \left( \|u\|_{B^{\gamma}_{p,1}}^{\ell} \right)^{1-\theta_2}, \quad \theta_2 = \frac{n+1-\gamma}{n+2s-1} \in (0, 1),
\]
from which and (3.16) gives
\[
\|u\|_{B^{\gamma}_{p,1}}^{\ell} \leq C \left( \|u\|_{B^{\gamma}_{p,1}}^{\ell} \right)^{\frac{\theta_2}{2}} \|u\|_{B^{\gamma}_{p,1}}^{h} \left( \|u\|_{B^{\gamma}_{p,1}}^{\ell} \right)^{1-\theta_2}, \quad \theta_2 = \frac{n+1-\gamma}{n+2s-1} \in (0, 1),
\]
Thanks to the embedding relation \( \dot{B}^{0}_{p,1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \), one infer that
\[
\|\Lambda^{\gamma}(u, \phi)\|_{L^p} \leq C(1 + t)^{-\frac{n}{2} - \frac{1}{p} - \frac{s+\gamma}{2}}.
\]
For any $p \leq q \leq \infty$ and $\frac{n}{q} - \frac{n}{p} - s < \alpha \leq \frac{n}{q} - 1$, by the Gagliardo-Nirenberg type interpolation inequality, which can be found in the Chap. 2 of [1], taking

$$k\theta_3 + m(1-\theta_3) = \alpha + n\left(\frac{1}{p} - \frac{1}{q}\right), \quad m = \frac{n}{p} - 1,$$

we get

$$\|\Lambda^\alpha(u, \phi)\|_{L^q} \leq C\|\Lambda^m(u, \phi)\|_{L^p}^{\frac{1-\theta_3}{\theta_3}}\|\Lambda^k(u, \phi)\|_{L^p}^{\frac{\theta_3}{\theta_3}}$$

$$\leq C\left\{(1+t)^{-\frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{m+s}{2}}\right\}^{1-\theta_3}\left\{(1+t)^{-\frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{k+s}{2}}\right\}^{\theta_3}$$

$$= C(1+t)^{-\frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{s + \alpha}{2}}.$$

Consequently, we have completed the proof of our theorem. □

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