GROWTH OF HOMOLOGY TORSION OF METABELIAN GROUPS

NIKOLAY NIKOLOV

Abstract. We study the growth of torsion of the abelianization of finite index subgroups in finitely generated metabelian groups. This complements the results in [3] which covered the finitely presented amenable groups.

1. Introduction

For subgroups $A, B$ of a group $G$ we denote by $[A, B]$ the subgroup of $G$ generated by all commutators $[a, b]$ with $a \in A, b \in B$. Set $G' = [G, G]$ and $G^{ab} = G/G' \simeq H_1(G, \mathbb{Z})$. Let $d(G)$ denote the minimal size of a generating set of a group $G$. Let $t(G)$ be the maximal size of a finite subgroup of $G$ (where we set $t(G) = \infty$ if there is no such maximum). Note that when $G$ is a finitely generated abelian group then $t(G)$ is the size of the torsion subgroup of $G$ and is always finite.

Given a finitely generated group $G$ and a subgroup $H$ of finite index in $G$ we are interested in the growth of $t(H^{ab})$ in terms of $[G : H]$. This topic is at the crossroads of group theory, operator algebras, geometry and number theory and has intriguing open questions, see [1], [2], [4]. It is easy to see (cf. [1], Lemma 27, restated as Lemma 7 below) that if $G$ is a finitely presented group then $t(H^{ab})$ is bounded above by an exponential function in $[G : H]$. In case $G$ is a finitely presented amenable group and $(H_i)$ is a Farber chain in $G$ (for example if $(H_i)$ are normal subgroups of $G$ with trivial intersection) it is proved in [3] that $t(H_i^{ab})$ grows subexponentially in $[G : H_i]$. By way of contrast [1] also proves that when $G$ is allowed to vary over all finitely generated solvable groups of derived length 3, then there is no single function $f$ in terms of $[G : H_i]$ which bounds $t(H_i^{ab})$ as $[G : H_i] \to \infty$. One is thus led to the question what happens for finitely generated metabelian groups which are not finitely presented. It is easy to see that some function bounding $t(H^{ab})$ exists in this class: There are only countably many finitely generated metabelian groups and a diagonal argument produces a function $f : \mathbb{N} \to \mathbb{N}$ with the following property: Given a finitely generated metabelian group $G$ there is $a = a(G) \in \mathbb{N}$ such that if $H \leq G$ with $a < [G : H] < \infty$, then $t(H^{ab}) < f([G : H])$. It is natural to expect that this non-constructive bound could be improved when we require that the coset space $G/H$ approximates $G$ sufficiently well, for example if $H$ is member of a chain of normal subgroups $(H_i)$ with trivial intersection.
Part 1 of Theorem 1 shows that the function $f$ above can be taken to be any superexponential function, e.g. $f(n) = n^n$.

**Theorem 1.** Let $G$ be a finitely generated metabelian group and let $A = G'$.

1. There is a constant $D$ depending on $G$ such that $t(H_{ab}^i) < D^{[G:H]}$ for every subgroup $H$ of finite index in $G$.

2. Let $(H_i)$ be a sequence of finite index subgroups in $G$ such that $[A : (A \cap H_i)] \to \infty$. Then

$$\lim_{i \to \infty} \frac{\log t(H_{ab}^i)}{[G : H_i]} = 0.$$ 

It is easy to see that the exponential bound in part 1 of Theorem 1 is sharp. Let us take $G = C_2 \wr \mathbb{Z}$ and for $n \in \mathbb{N}$ let $H_n = \pi^{-1}(n\mathbb{Z})$ where $\pi : G \to \mathbb{Z}$ is the homomorphism of $G$ onto the top group $\mathbb{Z}$. Then $t(H_n^ab) = 2^n$.

As a by-product of our method we can give a short proof of a special case of a theorem of Luck from [4].

**Theorem 2.** Let $G$ be a finitely presented group with an infinite abelian normal subgroup $A$. Let $(H_i)$ be a sequence of finite index subgroups of $G$ with $[A : (A \cap H_i)] \to \infty$. Then

$$\lim_{i \to \infty} \frac{\log t(H_{ab}^i)}{[G : H_i]} = 0.$$ 

Luck’s result proves subexponential growth of the torson of integral homology in all degrees (provided $G$ has type $F$) in the more general situation when $G$ has an infinite normal elementary amenable subgroup but under the stronger assumption that $(H_i)$ is a normal chain in $G$ with trivial intersection.

**A question.** Let $G$ be an amenable group of type $F_{n+1}$. Corollary 2 of [3] proved subexponential growth of $t(H_n(M_i, \mathbb{Z}))$ for any Farber chain of finite index subgroups $(M_i)$ in $G$. In view of Theorem 1 we can ask whether in case $G$ is a metabelian group the conclusion holds under the weaker assumption that $G$ is of type $F_n$ or even $FP_n$.

**Question 3.** Let $G$ be a metabelian group of type $F_n$ and let $(M_i)$ be a chain of normal subgroups with trivial intersection in $G$. Is it true that

$$\lim_{i \to \infty} \frac{t(H_n(M_i, \mathbb{Z}))}{[G : M_i]} = 0?$$ 

Note that we need that $G$ is at least of type $FP_n$ in order to guarantee that $H_n(G, \mathbb{Z})$ is finitely generated.

2. **Proofs**

We begin with some elementary results.
Proposition 4. Let $N$ be a normal subgroup of a group $G$. Then
\[ t(N) \leq t(G) \leq t(N)t(G/N). \]
If $G/N$ is torsion-free then $t(N) = t(G)$. If $N$ is finite then $t(G) = |N|t(G/N)$.

Proof. This is clear. \qed

Lemma 5. Let $M$ be a right $\mathbb{Z}[G]$-module and let $L \leq M$ be a submodule of finite index. Then $[M(G-1) : L(G-1)] \leq [M : L]^d$ where $d = d(G)$

Proof. Let $g_1, \ldots, g_d$ be a generating set for $G$ of minimal size. Note that $M(G-1) = \sum_{i=1}^{d} M(g_i - 1)$. Therefore the map $f : M^d \to M(G-1)$ defined by $f(m_1, \ldots, m_d) = \sum_{i=1}^{d} m_i (g_i - 1)$ ($m_i \in M$) is surjective. Similarly $f(L^d) = L(G-1)$ and so $f$ induces an additive group homomorphism
\[ \tilde{f} : \left( \frac{M}{L} \right)^d \to \frac{M(G-1)}{L(G-1)} \]
which is surjective. The claim of the lemma follows. \qed

The following Lemma is well known (compare with Lemma 6 of [3]). For a vector $v = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ we denote by $|v|$ the $l_1$-norm of $v$, i.e. $|v| = \sum_{i=1}^{n} |x_i|$.

Lemma 6. Let $n \in \mathbb{N}$ and let $v_1, \ldots, v_k \in \mathbb{Z}^n$. Let $G = \mathbb{Z}^n/(\sum_{j=1}^{k} \mathbb{Z}v_j)$. Then $t(A) \leq \prod_{i=1}^{n} c_i$ where $c_1, \ldots, c_n$ are the $n$ largest values from the list $|v_1|, \ldots, |v_k|$.

Proof. Let $X$ be the $n \times k$ matrix with rows $v_1, \ldots, v_k$. Then $t(A)$ is the g.c.d of the non-zero minors of maximal rank in $X$. Any such minor has rank at most $n$ and so is at most $c_1 \cdots c_n$. \qed

Lemma 7 (Lemma 27 of [1]). Let $G$ be a finitely presented group. There is a constant $C$ depending on $G$ such that $t(H^{ab}) \leq C^{[G:H]}$ for any subgroup $H$ of finite index in $G$.

Proposition 8. Let $G$ be a group with a normal abelian subgroup $A$. Let $H$ be a subgroup of finite index in $G$ such that $HA = G$. Then
\[ t(H^{ab}) \leq [G : H]^{d(G/A)}t(G^{ab}). \]

Proof. Let $B = A \cap H$, this is a normal subgroup of $G$ with $[A : B] = [G : H] = n$ say. Considering $A$ and $B$ as $\mathbb{Z}[G/A]$-modules (with the action of $G/A$ on $A$ by conjugation) Lemma 5 gives $[[A,G] : [B,G]] \leq n^d$ where $d = d(G/A)$.

The group $A/[A,G]$ is central in $G/[A,G]$ and expanding $G' = [G,G] = [HA,HA]$ we obtain $G' = H'[A,G]$. Therefore $[G' : H'] = [[A,G] : ([A,G] \cap H')] \leq [A,G] : [B,G]$ since $B = H \cap A$ and so $H' \geq [B,H] = [B,G]$. Therefore $[G' : H'] \leq n^d$ and in particular $[(G' \cap H) : H'] \leq n^d$. 

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Now we can apply Proposition 3 to $H/H'$ with a finite normal subgroup $(G' \cap H)/H'$ and we obtain
\[ t(H/H') = |(G' \cap H)/H'| \leq n^d t(H/((G' \cap H)). \]
On the other hand $H/((G' \cap H) \cong HG'/G' \leq G/G'$. Hence $t(H/((G' \cap H)) = t(HG'/G') \leq t(G/G')$ and the result follows. \(\square\)

**Proof of Theorem 2.** Let $G_i = AH_i$ and $m_i = [G : G_i], a_i = [G_i : H_i] = [A : (A \cap H_i)]$. Thus $[G : H_i] = m_i a_i$ with $a_i \to \infty$. Let $d = d(G)$. The Nielsen-Schreier theorem gives $d(G_i) \leq (d-1)m_i + 1 \leq dm_i$. By Lemma 7 there is a constant $C$ depending on $G$ such that $t(G_{i}^{ab}) \leq C^{m_i}$. Now we apply Proposition 8 to $G_i$ with normal subgroup $A$ and a finite index subgroup $H_i$ (which satisfies $H_i A = G_i$ by the definition of $G_i$). We obtain
\[ t(H_i^{ab}) \leq a_i^{d(G_i/A)} t(G_i^{ab}) \leq a_i^{dm_i} C^{m_i}. \]
Therefore
\[ \frac{\log t(H_i^{ab})}{[G : H_i]} \leq \frac{m_i (d \log a_i + \log C)}{a_i m_i} = \frac{d \log a_i + \log C}{a_i} \to 0, \]
since $a_i^{-1} \log a_i \to 0$ as $a_i \to \infty$. \(\square\)

**Proof of Theorem 4.** We need the following.

**Proposition 9.** Let $G$ be a finitely generated metabelian group. There is a constant $C$ depending on $G$ with the following property: Let $H$ be a subgroup of finite index in $G$ containing $G'$. Then $t(H^{ab}) \leq C^{[G : H]}$.

Let us postpone the proof of Proposition 9 for the moment and finish the proof of Theorem 1.

Define $G_i = AH_i$ and set $m_i = [G : G_i], a_i = [G_i : H_i] = [A : (A \cap H_i)]$ so that $[G : H_i] = a_i m_i$. By Proposition 3 we have $t(G_{i}^{ab}) \leq C^{m_i}$ for some constant $C$ depending only on $G$. On the other hand we can apply Proposition 8 to $G_i$ with a normal abelian subgroup $A$ and a finite index subgroup $H_i$ obtaining $t(H_i) \leq a_i^{d(G_i/A)} t(G_i^{ab})$. Since $G/A$ is abelian we deduce that $d(G_i/A) \leq d(G/A) = d$ say, and so $t(H_i^{ab}) \leq a_i^d C^{m_i}$. Part 1 follows since $a_i^d C^{m_i} \leq 2^d a_i^d C^{m_i} \leq (2^d C)^{[G : H_i]}$ and we can take $D = 2^d C$.

Part 2 of Theorem 1 easily follows from a computation similar to (1) using that $a_i \to \infty$. \(\square\)

It remains to prove Proposition 9.

**Proof of Proposition 9.** Let $g_1, \ldots, g_d$ be a generating set of $G$. Let $n = [G : H]$ and note that $H$ is normal in $G$ since $H \geq G'$. Let $L := G^{n}G'$, then $L \leq H$ and $[G : L] \leq n^d.$ Let $A := G'$, then $A$ is a $\mathbb{Z}[G^{ab}]$ module generated by $\{g_i g_j \mid 1 \leq i < j \leq d\}$. Let $W := [A, H] \leq A$. The quotient $H/W$ is a finitely generated nilpotent group of class at most 2 and $LW/W$ is a subgroup of finite index in $H/W$. Therefore $(LW/W)' = L'W/W$ has finite index in $(H/W)' = H'W$, i.e. the index $[H' : L'W]$ is finite. By Proposition
$t(H/LW) = t(H/H')|H'/LW|$ and in particular $t(H/H') \leq t(H/L'W)$. 
In turn $t(H/LW) \leq t(H/A)t(A/LW) \leq t(G^{ab})t(A/L'W)$. We will find a bound for $t(A/L'W)$. Note that $A/W$ is in the centre of $L/W$ and $L = \langle g_1^i, \ldots, g_d^j \rangle A$. Hence $L'W = \langle [g_i^j, g_j^i] \mid 1 \leq i < j \leq d \rangle W$.

Let

$$X := \bigoplus_{1 \leq i < j \leq d} \mathbb{Z}[G^{ab}]$$

be the free $\mathbb{Z}[G^{ab}]$-module with basis $\{e_{i,j} \mid 1 \leq i < j \leq d\}$ and let $f : X \to A$ be the surjective $\mathbb{Z}[G^{ab}]$-module homomorphism such that $f(e_{i,j}) = [g_i, g_j]$. Since $X$ is a Noetherian module $\ker f$ is generated (as a $\mathbb{Z}[G^{ab}]$-module) by finitely many elements, say $\{r_1, \ldots, r_k\} \subset X$.

Let $Y := X/(X(H - 1) = \bigoplus_{i < j} e_{i,j} \mathbb{Z}[G/H]$ and denote by $\pi : X \to Y$ the natural quotient map such that $\pi(e_{i,j} \cdot aG') = e_{i,j} \cdot aH$ for each $e_{i,j}$ and each $a \in G$. Again there is a unique $\mathbb{Z}[G/H]$-module homomorphism $h : Y \to A/W = A/[A, H]$ such that $h(e_{i,j}) = [g_i, g_j]W$. We have $f = h \circ \pi$.

Using the commutator identities $[ab, c] = [a, c]^b[b, c]$ and $[c, ab] = [c, b][c, a]^b$, we can write $[g_i^n, g_j^m] = \prod_{s=1}^{\min(n, m)} [g_i, g_j]^{z_{i,j,s}}$ for some elements $z_{i,j,s} \in G$. Define $u_{i,j} = e_{i,j} \cdot \sum_{s=1}^{\min(n, m)} z_{i,j,s}G'$. We have $f(u_{i,j}) = [g_i^n, g_j^m]$.

We have

$$L'W = \langle [g_i^n, g_j^m] \mid 1 \leq i < j \leq d \rangle W = f(X(H - 1)) + f(\sum_{1 \leq i < j \leq n} \mathbb{Z}u_{i,j})$$

and from $f = h \circ \pi$ it follows that

$$\frac{A}{L'W} \cong \frac{Y}{\pi(\ker f) + \sum_{i<j} \mathbb{Z}\pi(u_{i,j})}.$$

We consider the $l^1$ norm $\| \cdot \|_X$ on the free $\mathbb{Z}$-module $X$ (respectively the $l^1$ norm $\| \cdot \|_Y$ on $Y$) as the sum of the absolute values of coordinates computed with respect to the standard $\mathbb{Z}$-basis $\{e_{i,j} \mid i < j, \tilde{g} \in G^{ab}\}$ of $X$ (respectively the $\mathbb{Z}$-basis $\{e_{i,j} \tilde{g} \mid i < j, \tilde{g} \in G/H\}$ of $Y$). Note that for any element $v \in X$ we have $|v|_X \geq \|\pi(v)\|_Y$ and also $|v|_X = |v \cdot \tilde{g}|_X$ for any $\tilde{g} \in G^{ab}$.

Let $c = \max\{\|r_1\|_X, \ldots, \|r_k\|_X\}$ where $r_1, \ldots, r_k$ are the $\mathbb{Z}[G^{ab}]$ module generators of $\ker f$. Then $\pi(\ker f)$ is generated as $\mathbb{Z}$-module by the set $T := \{\pi(r_s) \tilde{g} \mid s = 1, \ldots, k, \tilde{g} \in G/H\}$. Observe that $|\pi(u_{i,j})|_Y \leq |u_{i,j}|_X \leq n^2$ for each $1 \leq i < j \leq d$, while for each $w = \pi(r_s) \tilde{g}$ we have $|w|_Y \leq |r_s|_X \leq c$.

The group $(Y, +)$ is a free $\mathbb{Z}$-module of rank $nd(d - 1)/2$ and $A/L'W \cong Y/Z$ where

$$Z = \pi(\ker f) + \sum_{1 \leq i < j \leq d} \mathbb{Z}\pi(u_{i,j}) = \sum_{w \in T} \mathbb{Z}w + \sum_{1 \leq i < j \leq d} \mathbb{Z}\pi(u_{i,j}).$$

Lemma 5 applied to $Y/Z$ gives $t(A/L'W) \leq (n^2)^{d(d-1)/2}c^{nd(d-1)/2}$ and so

$$t(H/H') \leq t(G^{ab})t(A/L'W) \leq t(G^{ab})(n^2)^{d(d-1)/2}c^{nd(d-1)/2}.$$
Since \( t(G^{ab}) \) does not depend on \( H \) and \( n^2 < 3^n \) we may take \( C = (t(G^{ab})3c)^{d^2/2} \), giving \( t(H/H') < C^n \) as required.

\[ \square \]

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