Abstract. In this article we investigate the existence of \((2, 3)\)-cordial labelings of oriented hypercubes. In this investigation, we determine that there exists a \((2, 3)\)-cordial oriented hypercube for any dimension divisible by 3. Next, we provide examples of \((2, 3)\)-cordial oriented hypercubes of dimension not divisible by 3 and state a conjecture on existence for dimension \(3k+1\). We close by presenting the only 3D oriented hypercubes up to isomorphism that are not \((2, 3)\)-cordial.

1 Introduction.

Let \(G = (V, E)\) be an undirected graph with vertex set \(V\) and edge set \(E\), a convention we will use throughout this paper. A \((0, 1)\)-labeling of the vertex set is a mapping \(f : V \to \{0, 1\}\) and is said to be friendly if approximately one half of the vertices are labeled 0 and the others labeled 1. An induced labeling of the edge set is a mapping \(g : E \to \{0, 1\}\) where for an edge \(uv, g(uv) = \hat{g}(f(u), f(v))\) for some \(\hat{g} : \{0, 1\} \times \{0, 1\} \to \{0, 1\}\) and is said to be cordial if \(f\) is friendly and about one half the edges of \(G\) are labeled 0. A graph, \(G\), is called cordial if there exists a cordial induced labeling of the edge set of \(G\).\(^4\)

In this article we investigate a labeling of directed graphs that is not simply a cordial labeling of the underlying undirected graph. The labeling scheme we investigate here was introduced by L.B. Beasley in \(^2\). Let \(D = (V, A)\) be a directed graph with vertex set \(V\) and arc set \(A\) with a friendly vertex set mapping \(f\). Let \(g : A \to \{-1, 0, 1\}\) be the induced labeling of the arcs of \(D\) such that for...
any arc initiating at \( u \) and terminating at \( v \), \( \frac{\hat{f}(\overrightarrow{uv})}{\hat{g}(\overrightarrow{uv})} = f(v) - f(u) \). \( D \) is said to be \((2,3)\)-cordial if there exists a friendly vertex set mapping \( f \) such that \( g \) labels approximately one third of arcs 0, approximately one third of arcs 1, and approximately one third of arcs -1. Applications of balanced graph labelings can be found in the introduction of [5].

In [3], \((2,3)\)-cordial labelings are investigated on oriented trees, oriented paths, orientations of the Petersen graph, and complete graphs. In this article we consider \((2,3)\)-cordial labelings on oriented hypercubes. We confirm the existence of \((2,3)\)-cordial oriented hypercubes for every dimension \( 3k \) for \( k \in \mathbb{N} \). Additionally, we provide examples of \((2,3)\)-cordial oriented hypercubes for dimensions 4 and 7 and conjecture that there exists a \((2,3)\)-oriented hypercube of dimension \( 3k + 1 \) for every \( k \in \mathbb{N} \). We close by presenting the only 3D oriented hypercubes up to isomorphism that are not \((2,3)\)-cordial, that is we present the only two 3D oriented hypercube up to isomorphism that do not admit a \((2,3)\)-cordial labeling.

2 Preliminaries.

**Definition 1.** Let \( Z \) be a finite set and \( f : Z \rightarrow \{0,1\} \) be a mapping. The mapping \( f \) is a called a \((0,1)\)-labeling of the set \( Z \). If \(-1 \leq |f^{-1}(0)| - |f^{-1}(1)| \leq 1\), that is, the number of elements of \( Z \) labeled 0 and the number of elements of \( Z \) labeled 1 are about equal, then we say that the labeling \( f \) is friendly.

Let \( G = (V,E) \) be an undirected graph with vertex set \( V \) and edge set \( E \). Let \( f : V \rightarrow \{0,1\} \) be a labeling of \( V \). An induced labeling of the edge set is a mapping \( g : E \rightarrow \{0,1\} \) where for an edge \( uv, g(uv) = \hat{g}(f(u), f(v)) \) for some \( \hat{g} : \{0,1\} \times \{0,1\} \rightarrow \{0,1\} \) and is said to be cordial if \( f \) and \( g \) are both friendly labelings. A graph \( G \) is cordial if there exists a cordial induced labeling of the edge set of \( G \). In this article, as in [2], we define a cordial labeling of directed graphs that is not simply a cordial labeling of the underlying undirected graph.

**Definition 2.** Let \( D = (V,A) \) be a directed graph with vertex set \( V \) and arc set \( A \). Let \( f : V \rightarrow \{0,1\} \) be a friendly vertex labeling and let \( g \) be the induced labeling of the arc set, \( g : A \rightarrow \{0,1,-1\} \) where for an arc \( u\overrightarrow{v}, g(u\overrightarrow{v}) = \hat{g}(f(u), f(v)) \). The labelings \( f \) and \( g \) are \((2,3)\)-cordial if \( f \) is friendly and about one third the arcs of \( D \) are labeled 1, one third are labeled -1 and one third labeled 0, that is, for any \( i, j \in \{0,1,-1\}, -1 \leq |g^{-1}(i)| - |g^{-1}(j)| \leq 1 \). A digraph, \( D \), is called \((2,3)\)-cordial if there exists \((2,3)\)-cordial labelings \( f \) of the vertex set and \( g \) of the arc set of \( D \). An undirected graph \( G \) is said to be \((2,3)\)-orientable if there is an orientation of the edges of \( G \) which is a \((2,3)\)-cordial digraph.

See [3] for an equivalent definition of \((2,3)\)-cordiality and \((2,3)\)-orientability beginning from the view of quasi-groups and quasi-group cordiality introduced in [7].

**Definition 3.** Let \( D_n \) be the set of all digraphs on \( n \) vertices. We will define \( T_n \) as the subset of \( D_n \) that consists of all digon-free digraphs, where a digon is a 2 cycle on a digraph.
Definition 4. Let $D = (V, A)$ be a digraph with vertex labeling $f : V \to \{0, 1\}$ and with induced arc labeling $g : A \to \{0, 1, -1\}$. Define $A_{f,g} : \mathcal{T}_n \to \mathbb{N}^3$ by $A_{f,g}(D) = (\alpha, \beta, \gamma)$ where $\alpha = |g^{-1}(1)|, \beta = |g^{-1}(-1)|$, and $\gamma = |g^{-1}(0)|$.

Let $D \in \mathcal{T}_n$ and let $D^R$ be the digraph such that every arc of $D$ is reversed, so that $\overrightarrow{uv}$ is an arc in $D^R$ if and only if $\overrightarrow{vu}$ is an arc in $D$. Let $f$ be a $(0,1)$-labeling of the vertices of $D$ and let $g(\overrightarrow{uv}) = f(v) - f(u)$ so that $g$ is a $(1,-1,0)$-labeling of the arcs of $D$. Let $\overline{f}$ be the complementary $(0,1)$-labeling of the vertices of $D$, so that $\overline{f}(v) = 0$ if and only if $f(v) = 1$. Let $\overline{g}$ be the corresponding induced arc labeling of $D$, $\overline{g}(\overrightarrow{uv}) = \overline{f}(v) - \overline{f}(u)$.

Lemma 1. Let $D \in \mathcal{T}_n$ with vertex labeling $f$ and induced arc labeling $g$. Let $A_{f,g}(D) = (\alpha, \beta, \gamma)$. Then

1. $A_{f,g}(D^R) = (\beta, \alpha, \gamma)$.
2. $A_{\overline{f}, \overline{g}}(D) = (\beta, \alpha, \gamma)$, and
3. $A_{\overline{f}, \overline{g}}(D^R) = A_{f,g}(D)$.

Proof. If an arc is labeled 1, -1, 0 respectively then reversing the labeling of the incident vertices gives a labeling of -1, 1, 0 respectively. If an arc $\overrightarrow{uv}$ is labeled 1, -1, 0 respectively, then $\overrightarrow{vu}$ would be labeled -1, 1, 0 respectively.

Example 1. Now, consider a graph, $\overline{III}_n$ in $\mathcal{G}_n$ consisting of three parallel edges and $n$-6 isolated vertices. Is $\overline{III}_n$ $(2,3)$-orientable? If $n = 6$, the answer is no, since any friendly labeling of the six vertices would have either no arcs labeled 0 or two arcs labeled 0. In either case, the orientation would never be $(2,3)$-cordial. That is $\overline{III}_6$ is not $(2,3)$-orientable, however with additional vertices like $\overline{III}_7$ the graph is $(2,3)$-orientable.

Thus, for our investigation here, we will use the convention that a graph, $G$, is $(2,3)$-orientable/(2,3)-cordial if and only if the subgraph of $G$ induced by its non-isolated vertices, $\tilde{G}$, is $(2,3)$-orientable/(2,3)-cordial.

3 Existence.

We begin with examples of $(2,3)$-cordial oriented hypercubes for dimensions less than and equal to 3.

Example 2 (Dimension 1). Given in Figure 1a is a 1-dimensional oriented hypercube $C_1$ that is $(2,3)$-cordial as by the friendly vertex labeling $f$ shown, $A_{f,g}(C_1) = (1,0,0)$.

Example 3 (Dimension 2). Given in Figure 1b is a 2-dimensional oriented hypercube $C_2$ that is $(2,3)$-cordial as by the friendly vertex labeling $f$ shown, $A_{f,g}(C_2) = (1,1,2)$. 
Fig. 1: $(2, 3)$-cordial $k$-dimensional oriented hypercubes

Fig. 2: $(2, 3)$-cordial 3D oriented hypercube
Example 4 (Dimension 3). Given in Figure 2 is a 3-dimensional oriented hypercube $C_3$ that is (2,3)-cordial as by the friendly vertex labeling $f$ shown, $A_{f,g}(C_3) = (4, 4, 4)$.

In Examples 2, 3, and 4 we see for dimension less than or equal to 3, there exist (2,3)-cordial oriented hypercubes. The question of existence remains unanswered for dimension greater than 3. In the following theorem, this question is answered for the case in which dimension is a multiple of 3.

Theorem 1. Let $n$ be a multiple of 3, then there exists an $n$-dimensional oriented hypercube $C_n$ that is (2,3)-cordial.

Proof. We proceed by induction on the dimension $n$ in multiples of 3. Example 4 serves as a base case for $n = 3$. Suppose the claim is true for some $k$ that is a multiple of 3. Then there exists some oriented hypercube $Q_k = (V_k, A_k)$ of dimension $k$ that is (2,3)-cordial. That is, there exists a friendly labeling $f: V_k \to \{0, 1\}$ such that

$$A_{f,g} = \left(\frac{1}{3}|A_k|, \frac{1}{3}|A_k|, \frac{1}{3}|A_k|\right)$$

where $g$ is defined as in Definition 2. We aim to construct an oriented hypercube $Q_{k+3} = (V_{k+3}, A_{k+3})$ of dimension $k + 3$ that is (2,3)-cordial. We begin by constructing an oriented hypercube $Q_{k+1} = (V_{k+1}, A_{k+1})$ of dimension $k + 1$. Let $L_k$ denote the digraph $Q_k$ with vertex labeling $f$ and induced arc labeling $g$ applied and $\overline{L_k}$ denote the digraph $Q_k$ with vertex labeling $\overline{f}$ and induced arc labeling $\overline{g}$. Now, let us draw arcs from $L_k$ to $\overline{L_k}$ according to the trivial digraph isomorphism. That is, define an arc initiating at vertex $x$ in $L_k$ to vertex $y$ in $\overline{L_k}$ if and only if $x = y$. We then label each of these arcs as $\overline{f}(x) - f(x)$. The result is a labeled digraph, call it $L_{k+1}$. By construction, the underlying digraph of $L_{k+1}$ is an oriented hypercube of dimension $k + 1$, call it $Q_{k+1}$. Define $f_{k+1}$ and $g_{k+1}$ to be vertex and arc labelings of $Q_{k+1}$ respectively such that $Q_{k+1}$ with labelings $f_{k+1}$ and $g_{k+1}$ applied is the labeled oriented hypercube $L_{k+1}$. As $f_{k+1}$ applies friendly labelings $f$ and $\overline{f}$ to complementary subgraphs of $Q_{k+1}$, $f_{k+1}$ is a friendly labeling. Further, $g_{k+1}$ applies $g$ and $\overline{g}$ to complementary subgraphs of $Q_{k+1}$ and labels each arc $\overline{f}x$ from $L_k$ to $\overline{L_k}, f(x) - f(x)$. Then, as $f$ and $\overline{f}$ are friendly,

$$A_{f_{k+1},g_{k+1}} = \left(\frac{2}{3}|A_k| + 2^{k-1}, \frac{2}{3}|A_k| + 2^{k-1}, \frac{2}{3}|A_k|\right).$$

Now, we repeat our procedure, constructing an oriented hypercube $Q_{k+2} = (V_{k+2}, A_{k+2})$ of dimension $k + 2$. We draw arcs from $L_{k+1}$ and $\overline{L_{k+1}}$. Just as in the previous case, we define an arc from a vertex $x$ in $L_{k+1}$ to vertex $y$ in $\overline{L_{k+1}}$ if and only if $x = y$, and we label this arc $f_{k+1}(x) - f_{k+1}(x)$. The result, as in the previous step, is a labeled digraph, call it $L_{k+2}$. The underlying digraph of $L_{k+2}$ is again an oriented hypercube, now of dimension $k + 2$, call it $Q_{k+2}$. As before, define $f_{k+2}$ and $g_{k+2}$ to be vertex and arc labelings of $Q_{k+2}$ respectively such
that when applied to $Q_{k+2}$ yield the labeled oriented hypercube $L_{k+2}$. As before, $f_{k+2}$ applies friendly labelings $f_{k+1}$ and $\bar{f}_{k+1}$ to complementary subgraphs, thus $f_{k+2}$ is friendly. Also, $g_{k+2}$ applies $g_{k+1}$ and $\bar{g}_{k+1}$ to complementary subgraphs of $Q_{k+2}$ and labels each arc $\overline{xy}$ from $L_{k+1}$ to $\overline{f_{k+1}}(x) \ominus f_{k+1}(x)$ and $\bar{f}$. As $f_{k+1}$ and $\overline{f}_{k+1}$ are friendly,

$$A_{f_{k+2}, g_{k+2}} \left( \left( \frac{4}{3} |A_k| + 2^k \right) + 2^k, \left( \frac{4}{3} |A_k| + 2^k \right) + 2^k, \frac{4}{3} |A_k| \right).$$

In our final step, we construct an oriented hypercube $Q_{k+3} = (V_{k+3}, A_{k+3})$ of dimension $k + 3$ by drawing edges between two identically labeled cubes $L_{k+2}$. We draw an arc from vertex $x$ in the first $L_{k+2}$ to vertex $y$ in the second $L_{k+2}$ if and only if $x = y$ and we label this arc $f_{k+2}(x) - f_{k+2}(x) = 0$. The result is a labeled digraph, call it $L_{k+3}$. The underlying digraph of $L_{k+3}$ is an oriented hypercube of dimension $k + 3$, call it $Q_{k+3}$. Finally, we define $f_{k+3}$ and $g_{k+3}$ to be vertex and arc labelings of $Q_{k+3}$ respectively such that when applied to $Q_{k+3}$ yield the labeled oriented hypercube $L_{k+3}$. Then $f_{k+3}$ simply labels each complementary subgraph $Q_{k+2}$ according to $f_{k+2}$ and $g_{k+3}$ labels each complementary subgraph $Q_{k+2}$ according to $g_{k+2}$ and the newly drawn $2^{k+2}$ edges are labeled 0. Let $\omega = \frac{1}{3}|A_k| + 2^{k+1}$. Then

$$A_{f_{k+3}, g_{k+3}} (Q_{k+3}) = \left( 2\omega, 2\omega, \frac{8}{3}|A_k| + 2^{k+2} \right).$$

Simplifying, we have

$$A_{f_{k+3}, g_{k+3}} (Q_{k+3}) = \left( \frac{1}{3}(k + 3)2^{k+2}, \frac{1}{3}(k + 3)2^{k+2}, \frac{1}{3}(k + 3)2^{k+2} \right).$$

As $f_{k+3}$ is constructed to be a friendly labeling, the above implies $Q_{k+3}$ is $(2, 3)$-cordial.

### 3.1 A Conjecture on Existence for Dimension $3k + 1$.

We have now answered the question of existence of $(2, 3)$-cordial oriented hypercubes for dimension less than and equal to 3 and all dimensions which are a multiple of 3. In this section, we now consider the existence of $(2, 3)$-cordial oriented hypercubes with dimension $3k + 1$ for $k \in \mathbb{N}$.

**Example 5 (Tesseract, Dimension 4).** Given in Figures 5a and 5b are two 3D oriented hypercubes, $A$ and $B$, that are $(2, 3)$-cordial as demonstrated by the friendly vertex labelings and induced arc labelings shown. In Figure 4 edges are drawn between the vertices of the oriented cube $B$ (outer) of Figure 4a and the vertices of oriented cube $A$ (inner) of Figure 5a. By the induced arc labeling scheme $g$, 2 of these 8 edges (red) receive an induced label of 0 regardless of their orientation, and the remaining 6 edges (dashed) can be oriented such that 3 receive label 1 and 3 receive label −1, yielding a 4D oriented hypercube. As the outer and inner cubes of Figure 4 have $(2, 3)$-cordial labelings applied, this is to say the dashed arcs in Figure 4 can be oriented such that the result is a $(2, 3)$-cordial 4D oriented hypercube.
Fig. 3: (2, 3)-cordial 3D oriented hypercubes, A and B

Fig. 4: 4D (2, 3)-Cordial Oriented Hypercube constructed from cubes A and B
**Definition 5.** Let $D_1$ and $D_2$ be directed graphs with same sized vertex sets and friendly vertex labelings $f_1$ and $f_2$ respectively. Let $\beta: V(D_1) \rightarrow V(D_2)$ be a bijection on the vertex sets of $D_1$ and $D_2$ respectively. Then, let $h: V(D_1) \rightarrow \{0, 1\}$ such that $h(v_1) = |f_1(v) - f_2(\beta(v))|$ for all $v \in V(D_1)$. Then define $\Phi_\beta(D_1, D_2) = |h^{-1}(0)|$. In contexts where the bijection $\beta$ is clear, we write $\Phi(D_1, D_2).

**Remark 1.** In the context of the previous definition, given arcs are drawn between vertices of digraphs $D_1$ and $D_2$ according to the bijection $\beta$, $\Phi(D_1, D_2)$ is simply the count of arcs shared by $D_1$ and $D_2$ that receive induced label 0 by $g$. In the following example, we work within such a context, and therefore, interpret $\Phi(D_1, D_2)$ this way.

**Example 6 (Dimension 7).** We have introduced 3 3D oriented hypercubes in Figures 2, 3a, and 3b, each with a (2, 3)-cordial labeling. Let us denote the labeled oriented cube in Figure 2 as $C$. For this example, we adopt the convention that $A$, $B$, and $C$ refer to labeled digraphs rather than the underlying unlabeled digraphs. We seek to construct a (2, 3)-cordial 7D oriented hypercube from these three cubes, $A$, $B$, and $C$. As given in Figure 4 cubes $A$ and $B$ can be combined to form a 4D oriented cube such that only 2 of the arcs they share receive label 0. That is by the bijection between $V(A)$ and $V(B)$ defined by the edges drawn in Figure 4 $\Phi(A, B) = 2$. In Figure 5 we construct 2 individual 4D oriented cubes, 1 from cubes $A$ and $C$, and 1 from cubes $B$ and $C$. As in Figure 4 arcs drawn between distinct cubes define bijections between distinct vertex sets. With respect to these bijections, in Figure 5 we see $\Phi(A, C) = \Phi(B, C) = 4$.

Now, for $D \in \{A, B, C\}$, given $f$ is the friendly vertex labeling of $D$ and $g$ is the induced arc labeling of $D$, define $\overline{D}$ to be the underlying digraph $D$ labeled instead by $\overline{f}$ and $\overline{g}$. Recall, such a labeling is (2, 3)-cordial by Lemma 1. Then, for all $D_1, D_2 \in \{A, B, C\}$, $\Phi(\overline{D_1}, \overline{D_2}) = \Phi(D_1, D_2)$ and $\Phi(\overline{\overline{D_1}}, \overline{D_2}) = \Phi(D_1, D_2) = 8 - \Phi(D_1, D_2)$. Then, for all $D_1 \neq D_2$, taking $\Phi(D_1, D_2)$ to be with respect to the appropriate bijection between $V(D_1)$ and $V(D_2)$ defined in either Figure 4, 5a, or 5b we have $\Phi(\overline{\overline{A}}, B) = 6$ and $\Phi(\overline{\overline{A}}, C) = \Phi(\overline{A}, C) = 4$. Lastly, note we can construct a 4D oriented hypercube between 2 identical cubes $D$ by drawing arcs between like vertices. According to such a bijection, $\Phi(D, D) = 8$. Now, define $\gamma = \{\overline{A}, A, \overline{B}, B, \overline{C}, C\}$. Then for all $Q_1, Q_2 \in \gamma$, $\Phi(Q_1, Q_2)$ with reference to the appropriate aforementioned bijections between $V(Q_1)$ and $V(Q_2)$ are given below in Table 1. As $\Phi$ is commutative by definition, the lower diagonal of Table 1 is left empty.

Now, we construct a (2, 3)-cordial 7D oriented hypercube by drawing edges between cubes in the set $\gamma$ according to the previously defined vertex set bijections. Given in Figure 6 are 2 6D oriented hypercubes constructed from cubes in $\gamma$ where for all $Q_1, Q_2 \in \gamma$, an edge between cube $Q_1$ and $Q_2$ signifies 8 edges between cubes $Q_1$ and $Q_2$ drawn according to the appropriate bijection between $V(Q_1)$ and $V(Q_2)$. Note, in Figure 6 an edge between $Q_1$ and $Q_2$ is labeled $\Phi(Q_1, Q_2)$. Observe for each cube in Figure 6 the edge label sum is equal to 32. By Remark 1 this is to say a total of 32 edges shared by distinct cubes in $\gamma$ receive induced label 0 by $g$ regardless of orientation. As each 6D cube in Figure
Fig. 5: 4D oriented cubes constructed from cubes $A, B, C$
Table 1: $\Phi(Q_1, Q_2)$ for all $Q_1, Q_2 \in \gamma$

|   | $A$ | $B$ | $C$ |
|---|-----|-----|-----|
| $A$ | 8   | 2   | 6   |
| $A$ | 8   | 6   | 2   |
| $B$ | 8   | 0   | 4   |
| $B$ | 8   | 4   | 4   |
| $C$ | 8   | 0   |

(a) has a total of $12 \cdot 8 = 96$ edges drawn between cubes in $\gamma$, and $32 = 96/3$, the remaining edges not labeled 0 in each 6D cube can be oriented such that half are labeled 1 and half are labeled $-1$ making each 6D cube $(2, 3)$-cordial. Now, in Figure 7 a 7D cube is constructed from these $(2, 3)$-cordial 6D oriented cubes. In Figure 7 as in Figure 6 an edge between cube $Q_1$ and $Q_2$ signifies 8 edges between cubes $Q_1$ and $Q_2$ drawn according to the appropriate bijection between $V(Q_1)$ and $V(Q_2)$, and each edge between $Q_1$ and $Q_2$ is labeled $\Phi(Q_1, Q_2)$.

![Fig. 6: (2, 3)-cordial 6D oriented hypercubes](image)

In Figure 7 the edge label sum is 22. Similar to before, by Remark 1 this is to say a total of 22 of the 64 edges drawn between vertices of the inner 6D cube and the outer 6D cube receive label 0. The remaining 42 edges can be oriented such that 21 receive a label of 1 and 21 receive a label of $-1$ by $g$. Because each 6D cube is $(2, 3)$-cordial, such a choice yields a $(2, 3)$-cordial 7D oriented hypercube.
Fig. 7: 7D (2, 3)-cordial oriented hypercube constructed from $\gamma$ cubes
In the previous two examples we have confirmed there exist $(2, 3)$-cordial oriented hypercubes of dimension $3k + 1$ for $k = 1, 2$. We now state the following conjecture.

**Conjecture 1.** Let $n$ be a multiple of 3, then there exists an $(n + 1)$-dimensional oriented hypercube $C_{n+1}$ that is $(2, 3)$-cordial.

4 Non-(2, 3)-Cordial Oriented Cubes.

In the previous section, we demonstrated the existence of $(2, 3)$-cordial oriented hypercubes of varying dimension including dimension 3. Now, we demonstrate the existence of oriented cubes that are not $(2, 3)$-cordial, that is, we demonstrate there exist oriented cubes that do not admit $(2, 3)$-cordial labelings.

Fig. 8: Oriented cube $V$, 3 vertices of out-degree 3 (labeled $b_i$), and 2 vertices of in-degree 3 (labeled $v_i$)

**Theorem 2.** The oriented cube $V$ in Figure 8 is not $(2, 3)$-cordial.

**Proof.** There are \( \binom{8}{4} \) possible friendly vertex labelings for the oriented cube $V$. By a brute force algorithm, it can be shown that none of these vertex labelings induces a $(2, 3)$-cordial labeling.

**Corollary 1.** The oriented cube $V^R$ for $V$ in Figure 8 is not $(2, 3)$-cordial.

**Proof.** By Lemma 1.1, $A_{f,g}(V) = A_{f,g}(V^R)$ for any vertex-arc labeling $f, g$. Thus, given $V$ does not admit a $(2, 3)$-cordial labeling by Theorem 2, neither does $V^R$. Equivalently, $V^R$ is not $(2, 3)$-cordial.

**Theorem 3.** The cubes $V$ and $V^R$ are the only oriented cubes up to isomorphism that are not $(2, 3)$-cordial.

**Proof.** This can be shown by a simple brute force algorithm.
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