Unifying Inference for Bayesian and Petri Nets*

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Abstract

Recent work by the authors equips Petri occurrence nets (PN) with probability distributions which fully replace nondeterminism. To avoid the so-called confusion problem, the construction imposes additional causal dependencies which restrict choices within certain subnets called structural branching cells (s-cells). Bayesian nets (BN) are usually structured as partial orders where nodes define conditional probability distributions. In the paper, we unify the two structures in terms of Symmetric Monoidal Categories (SMC), so that we can apply to PN ordinary analysis techniques developed for BN. Interestingly, it turns out that PN which cannot be SMC-decomposed are exactly s-cells. This result confirms the importance for Petri nets of both SMC and s-cells.

1 Introduction

At first sight, Bayesian nets (BN) and Petri Nets (PN) have very different purposes: efficient/intelligent analysis of probabilistic distributions for BN, a concurrent, nondeterministic model of computation for PN. But in fact BN and PN share a similar structure: a partial ordering representing incremental, local evolutions via concurrent firings for PN, the introduction of new variables with independent, conditional probabilities for BN.

A closer comparison can be carried on when equipping also PN with a suitable probability structure. A recent approach \cite{Bruni:2017} aims at fully replacing nondeterministic choices with probability distributions, while keeping concurrency expressiveness as much as possible. The problem here is the so-called confusion: in PN with confusion, a concurrent computation may exhibit non stable decision steps: delaying a choice may change the available options, due to the action of a concurrent transition.

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Figure 1: A PN with confusion

The simplest example of confusion is the Petri net in Fig. 1(a). Transitions $a$ and $b$ are enabled but in conflict, because they compete for the token in place 1: transition $c$ is also enabled and concurrent w.r.t. $a$ and $b$; however the firing of transition $a$ enables the transition $d$ that is in conflict with $c$. As a consequence, the concurrent run where $a$ and $c$ are executed puts in the same equivalence class two quite different traces, where different decisions are taken: (1) if $a$ is executed first, then two choices are taken ($a$ over $b$ and $c$ over $d$); (2) if $c$ is executed first, then only one choice is taken ($a$ over $b$). When choices are taken according to some probability distributions, this makes it impossible to assign a unique probability to the concurrent computation with $a$ and $c$.

The solution proposed by the authors in [4] is to translate the given PN into an equivalent confusionless net (CIPN). This is done by partitioning the net in structural branching cells ($s$-cells) where decisions must be resolved. S-cells are the equivalence classes of a preorder $\sqsubseteq$, that introduces some further causal dependencies. The preorder is obtained by closing transitively the relation including prime mutual exclusion and immediate causality. It follows that the preorder induces a partial order on $s$-cells, still denoted $\sqsubseteq$. In the example above there are two $s$-cells $C_1 \sqsubseteq C_2$, meaning that the choice between $a$ and $b$ must be resolved before the one between $c$ and $d$ (see Fig. 1(b)). $s$-cells can then be translated to a confusionless net, where the dependencies between $s$-cells are implemented by additional places in a way that corresponds to the execution strategy of [1].

To make confusionless a PN with confusion, it is necessary to delay non stable decisions until any two enabled transitions either do not share any precondition or they share all of them. Then such choice steps are equipped with probability distributions. In practice, our construction introduces a negation place $\overline{p}$ for every place $p$ of the original net, and adds suitable controls to make sure that whenever place $\overline{p}$ becomes inhabited, place $p$ is guaranteed never to become occupied. Thus when the present marking includes $\overline{p}$, all transitions requiring $p$ can be erased and the net simplified. The process is hierarchical, because each $s$-cell can be further decomposed in smaller $s$-cells under the assumption that some place $\overline{p}$ becomes inhabited.
The aim of this paper is to show that the partial order of s-cells induces a BN structure. The potential is to develop the countless applications of BN for inference and learning in the context of an expressive model like PN. We propose a strong formal connection between PN and BN via Symmetric Monoidal Categories (SMC).

On the side of BN, convenient categorical presentations have been recently proposed [11, 5, 6] which, in the discrete model, represent BN as string diagrams of a SMC \( \mathcal{K}(\mathcal{D}) \). Here, objects are natural numbers \( n \) which express that \( 2^n \) cases are possible, and arrows are rectangular matrices, where rows assign probability distributions on the output cases for every input case. An arrow \( f : X \to \mathcal{D}(Y) \) models a conditional probability distribution \( P(Y \mid X) \). Concurrent arrows of string diagrams represent independent probability distributions. Usual inference analysis of BN, like forward and backward inference, bayesian inversion and disintegration can be made explicit as standard categorical constructions [5].

A ClPN, and thus a PN, can also be mapped to an arrow of \( \mathcal{K}(\mathcal{D}) \), amenable to the same inference analysis techniques developed for BN. As for our translation PN-ClPN, this mapping is defined by well founded recursion on hierarchical branching cells. Here the effect of positive-negative information \( p/p \) is played by associating object 1 to a place (that is \( 2^1 = 2 \) cases), which represents explicitly the two options.

Translating a ClPN into a BN is more difficult. In fact, an s-cell may produce several nodes of the BN, since the presence of negative information may break down the cell into a full BN. Thus while in \( \mathcal{K}(\mathcal{D}) \) associativity of sequential composition takes care of the nested structure, in BN it will be necessary to introduce a nested version of BN, which, as far as we know, has not been proposed in the literature.

In Fig. 1(c) we show the BN derived from the PN in Fig. 1(a), represented as a string diagram. There, \( N_C \) is the subnet associated with the s-cell \( C \) and \( \delta \) is the family of probability distributions that rule the choices within \( C_1 \) (between \( a \) and \( b \)) and \( C_2 \) (between \( c \) and \( d \) when place 4 is marked, the trivial choice of \( c \) when 4 remains empty, i.e., they are conditional probabilities depending on the presence/absence of tokens in 4). Roughly, there is one node for each s-cell and wires are associated with places. The first node represents a variable that may take values \( 4/\overline{4}, \) i.e., it is the arrow

\[
\begin{array}{c|ccc}
  & \emptyset & \{4\} \\
\hline
\emptyset & p_b & p_a \\
\end{array}
: 0 \to 1
\]

where the probabilities \( p_a \) and \( p_b = 1 - p_a \) are of course determined by \( \delta \). The second node represents a variable that may take all combination of values \( 5/\overline{5} \) and \( 6/\overline{6} \), conditioned to the value of the first variable, i.e., it is the arrow

\[
\begin{array}{c|cccc}
  & \emptyset & \{5\} & \{6\} & \{5,6\} \\
\hline
\emptyset & 0 & 1 & 0 & 0 \\
\{4\} & 0 & p_c & p_d & 0 \\
\end{array}
: 1 \to 2
\]

where, again, the values \( p_c \) and \( p_d = 1 - p_c \) are drawn by \( \delta \). For instance, \( p_c \) is
the conditional probability that the place 5 is marked given that the place 4 is marked.

To define the arrow in $\mathcal{K}(\mathcal{D})$ that corresponds to a PN we exploit the monoidal category structure of nets and $\mathcal{K}(\mathcal{D})$: first each $N$ net is uniquely decomposed in a term $\langle N \rangle$ of an algebra whose constants are no further hierarchically decomposable s-cells, then the homomorphism $\llbracket \langle N \rangle, \delta \rrbracket$ returns the arrows in $\mathcal{K}(\mathcal{D})$.

It is interesting to compare the ClPN and the $\mathcal{K}(\mathcal{D})$ arrow for the same PN. The former model is much more informative in terms of concurrency and causality (see [2] for an event structure theory of persistent nets), while the latter is more straightforward in terms of structure and execution mode. It could be considered a fair algorithmic description of the execution style of [1][4] original model.

**Structure of the paper** In Section 2 we fix the notation, recall the basics of Petri nets and occurrence nets and explain the notion of s-cell from [4]. In Section 3 we provide a novel alternative characterisation of (the pre-order induced by) s-cells based on straightforward notion of parallel and sequential (de)composition of nets. This result further justifies the notion of s-cell as basic building block for occurrence nets. In Section 4 we define the mapping from PN to BN. To this aim, an intermediate term algebra is used that builds on the decomposition defined in Section 3 to break s-cells with non-empty initial interface into the hierarchical composition of other terms. Here some sort of case analysis is done: for each marking that can be provided to the s-cell we explore how it can be simplified (the absence of tokens allows for the removal of places and transitions). In Section 5 we show how the Bayesian structure can be exploited to reason about the marking of places of the original PN. Finally, in Section 6 we draw some concluding remarks and give pointers to related and future work.

In A we show the correspondence between PN decomposition and the approach by Abbes and Benveniste based on event structures, which justifies the assignment of probability distributions to s-cells.

We assume the reader is familiar with some basic concepts from Bayesian networks and category theory.

## 2 Background

### 2.1 Notation

We let $\mathbb{N}$ be the set of natural numbers and $2 = \{0, 1\}$. We write $U^S$ for the set of functions from $S$ to $U$: hence a subset of $S$ is an element of $2^S$, and a multiset $m$ over $S$ is an element of $\mathbb{N}^S$. A set can be seen as a multiset whose elements have unary multiplicity. Membership, union, difference and inclusion over sets and multisets are denoted by the (overloaded) symbols: $\in$, $\cup$, $\setminus$ and $\subseteq$, respectively.
Given a relation \( R \subseteq S \times S \), we let \( R^{-1} = \{ (y, x) \mid (x, y) \in R \} \) be its inverse relation, \( R^+ \) be its transitive closure and \( R^* \) be its reflexive and transitive closure. We say that \( R \) is acyclic if \( \forall s \in S, (s, s) \notin R^+ \).

### 2.2 Petri Nets

**Definition 1.** A Petri net \( N \) is a tuple \((P, T, F)\) where: \( P \) is the set of places, \( T \) is the set of transitions, and \( F \subseteq (P \times T) \cup (T \times P) \) is the flow relation.

For \( x \in P \cup T \), we denote by \( \bullet x = \{ y \mid (y, x) \in F \} \) and \( x^\bullet = \{ z \mid (x, z) \in F \} \) its pre-set and post-set, respectively. We assume that \( P \) and \( T \) are disjoint and non-empty and that \( \bullet t \) is non empty for every \( t \in T \). We write \( t : X \rightarrow Y \) for \( t \in T \) with \( X = \bullet t \) and \( Y = t^\bullet \). A marking is a multiset \( m \in \mathbb{N}^P \). A marking denotes a state of a Petri net. We say that the place \( p \in P \) is marked at \( m \) if \( p \in m \). We write \((N, m)\) for the net \( N \) marked by \( m \). In the following we write just \( N \) for the marked net \((N, \emptyset)\).

Graphically, a Petri net is a directed bipartite graph whose nodes are the places (circles) and transitions (rectangles) and whose arcs are the elements of \( F \). The marking \( m \) is represented by inserting \( m(p) \) tokens (bullets) in each place \( p \in m \) (see Fig. 2(a)).

The operational semantics of a Petri net is defined by events called firings. A transition \( t \) is enabled at the marking \( m \), written \( m \rightharpoonup t \), if \( \bullet t \subseteq m \). The firing of a transition \( t \) enabled at \( m \) is written \( m \rightharpoonup m' \) with \( m' = (m \setminus \bullet t) \cup t^\bullet \). A firing sequence \( m \xrightarrow{t_1 \ldots t_n} m' \) from \( m \) to \( m' \) is a finite sequence of firings, sometimes abbreviated \( m \rightarrow^* m' \). Moreover, it is maximal if no transition is enabled at \( m' \). We say that \( m' \) is reachable from \( m \) if \( m \rightarrow^* m' \). The set of markings reachable from \( m \) is written \([m]\). A marked net \((N, m)\) is safe if each \( m' \in [m] \) is a set.

In the rest of the paper we only consider safe nets. More precisely we consider so-called occurrence nets.

### 2.3 Occurrence nets

We say that a net \((P, T, F)\) is acyclic if its flow relation \( F \) is so. Given an acyclic net we let \( \preceq = F^* \) be the (reflexive) causality relation and say that two transitions \( t_1 \) and \( t_2 \) are in immediate conflict, written \( t_1 \#_0 t_2 \) if \( t_1 \neq t_2 \land \bullet t_1 \cap \bullet t_2 \neq \emptyset \). The conflict relation \# is defined by letting \( x \# y \) if there are \( t_1, t_2 \in T \) such that \((t_1, x), (t_2, y) \in F^+ \) and \( t_1 \#_0 t_2 \).

**Definition 2** (Occurrence Net). A nondeterministic occurrence net (or just occurrence net) is an acyclic net \( O = (P, T, F) \) such that:

1. there are no backward conflicts (i.e., \( \forall p \in P, |\bullet p| \leq 1 \), and
2. there are no self-conflicts (i.e., \( \forall t \in T, \neg(t \# t) \)).

An occurrence net is deterministic if it does not have forward conflicts (i.e., \( \forall p \in P, |p^\bullet| \leq 1 \)).
Figure 2: A simple PN
A place \( p \) of an occurrence net \( O \) is called \textit{initial} if its pre-set is empty; it is called \textit{final} if its post-set is empty; it is called \textit{isolated} if it is both initial and final. We denote by \( ^O \) the set of its initial places and by \( O^\circ \) the set of its final places. The net \( N \) in Fig. 2(a) is an occurrence net. The sets of its initial and final places respectively are \( ^O N = \{1, 2, 3\} \) and \( N^\circ = \{5, 7, 8, 9, 10\} \).

Typically it is left implicit that all the initial places of an occurrence net are marked. Here we need to distinguish the cases in which only some initial places are marked.

**Definition 3 (Marked Occurrence Net).** A marked occurrence net \( M = (O, m) \) is an occurrence net \( O \) together with a subset \( m \) of initial, non-isolated places.

The idea is that:

- any initial place in \( m \) is already marked (by one token);
- any initial place not in \( m \) can receive a token from the context.

Given a marked occurrence net \( M = (O, m) \), we denote by \( {}^O M = {}^O O \setminus m \) the set of its initial (unmarked) places and by \( M^\circ = O^\circ \) the set of its final places. For the marked occurrence net \( (N, \{2, 3\}) \) in Fig. 2(a) we have \( {}^O (N, \{2, 3\}) = \{1\} \) and \( (N, \{2, 3\})^\circ = N^\circ = \{5, 7, 8, 9, 10\} \).

A \textit{deterministic nonsequential process} (or just \textit{process}) \( [9] \) represents the equivalence class of all firing sequences of a net that only differ in the order in which concurrent firings are executed. It is given as a mapping \( \theta : D \to N \) from a deterministic occurrence net \( D \) to \( N \) (preserving pre- and post-sets). The firing sequences of a processes \( D \) are its maximal firing sequences starting from the marking \( {}^O D \). A process of \( N \) is \textit{maximal} if its firing sequences are maximal in \( N \).

When \( N \) is an acyclic safe net, the mapping \( \theta \) is just an injective graph homomorphism: without loss of generality, we name the nodes in \( D \) as their images in \( N \) and let \( \theta \) be the identity.

### 2.4 Structural Branching Cells

In [4] we have proposed a solution for determining the smallest loci of decision within an acyclic finite net, called \textit{structural branching cells}: they are subnets where the decision of firing some transition is taken when it is guaranteed that no conflicting transition which is currently not enabled can become enabled in the future.

The construction in [4] takes a (finite) occurrence net as input, which can be, e.g., the (truncated) unfolding of any safe net and returns a partial order of structural branching cells.

To each transition \( t \) we assign a unique s-cell \([t]\). This is achieved by taking the equivalence class of \( t \) w.r.t. the equivalence relation \( \leftrightarrow \) induced by the least preorder \( \sqsubseteq \) that includes immediate conflict \#0 and causality \( \preceq \). Formally, we let \( \sqsubseteq \) be the transitive closure of the relation \( \#_0 \cup \preceq \cup \text{Pre}^{-1} \), where \( \text{Pre} = F \cap (P \times T) \). This way, each s-cell \([t]\) also includes the places in the pre-sets.
of the transitions in \([t]\). Since \(\#_0\) is subsumed by the transitive closure of the relation \(\leq \cup \text{Pre}^{-1}\), we equivalently set \(\subseteq = (\leq \cup \text{Pre}^{-1})^*\).

**Definition 4 (S-cells).** Let \(N = (P,T,F)\) be a finite occurrence net and \(\subseteq\) defined as above. Let \(\leftrightarrow = \{ (x,y) \mid x \subseteq y \wedge y \subseteq x \}\). The set \(\text{BC}(N)\) of s-cells is the set of equivalence classes of \(\leftrightarrow\), i.e., \(\text{BC}(N) = \{ [t]_{\leftrightarrow} \mid t \in T \}\).

We let \(C\) range over s-cells. It is immediate to note that s-cells are ordered by \(\subseteq\): we let \(C \subseteq C'\) if there are \(t \in C, t' \in C'\) with \(t \subseteq t'\).

For any s-cell \(C\), we denote by \(N_C\) the subnet of \(N\) whose elements are in \(\bigcup_{t \in C} t^*\), i.e., we include in \(N_C\) also all places in the post-set of some transition in \(C\).

Abusing the notation, we denote by \(^oC\) the set of all the initial places in \(N_C\) and by \(C^o\) the set of all the final places in \(N_C\). When the original net \((N,m)\) is marked we sometimes let its cells inherits the marking, i.e., we let the initial marking of \(N_C\) be \(m \cap \circ C\).

**Example 1.** The net in Fig. 2(a) has three s-cells, which are depicted in Fig. 2(b) \(C_1 = \{1,a,b\}\) concerning the choice between \(a\) and \(b\), and \(C_2 = \{2,c,d\}\) concerning the choice between \(c\) and \(d\), and \(C_3 = \{3,4,6,e,f,g,h\}\). The nets \(N_{C_1}, N_{C_2}\) and \(N_{C_3}\) are respectively shown in Fig. 2(c), 2(d) and 2(e). For \(C_1, \circ C_1 = \circ N_{C_1} = \{1\}\) and \(C_1^o = (N_{C_1})^o = \{4,5\}\). For \(C_2, \circ C_2 = \circ N_{C_2} \setminus \{2\} = \{2\} \setminus \{2\} = \emptyset\) and \(C_2^o = (N_{C_2})^o = \{6\}\).

The behaviour of a branching cell is characterised in terms of all its possible executions.

**Definition 5 (Transactions).** Let \(C \in \text{BC}(N)\) and \(m = \circ C\). Then, a transaction \(\theta\) of \(C\), written \(\theta : C\), is a maximal (deterministic) process of \((N_C,m)\). We denote by \(\Theta(C)\) the set of all the transactions of \(C\).

Since the set of transitions in a transaction \(\theta\) uniquely determines the corresponding process in \(N_C\), we write a transaction \(\theta\) simply as the set of its transitions. If \(i = \circ \theta\) is the set of initial places of \(\theta\) and \(o = \theta^o\) is the set of its final places, we write \(\theta : i \to o\). Note that in general, for \(\theta : i \to o \in \Theta(C)\), we have \(i \subseteq \circ C\) and \(o \subseteq C^o\). We write \(n(\theta)\) for the set of transitions and places of \(\theta\).

**Example 2.** Consider the net \(N_{C_3}\) in Fig. 2(e). It has the following three transactions: \(\theta_1 = \{f\}\), \(\theta_2 = \{e,g\}\) and \(\theta_3 = \{e,h\}\), with \(\theta_1 : \{3,4,6\} \to \{8\}\), \(\theta_2 : \{3,6\} \to \{7,9\}\) and \(\theta_3 : \{3,6\} \to \{7,10\}\).

### 3 Petri Nets Decomposition

We have already said that s-cells form a partial order. Here we show that it can be seen as a particular commutative monoidal category structure.

We proceed as follows:
1. we define set-theoretical parallel and sequential composition of nets;
2. we show that parallel and sequential composition, together with a suitable notion of identities, induce a commutative monoidal category structure over occurrence nets;
3. we show that s-cells are neither decomposable in parallel nor in series;
4. we show that each Petri net admits a unique maximal decomposition in terms of parallel and sequence (up to the axioms of commutative monoidal categories) and that such decomposition coincides with the partial order of s-cells.

This provides a new characterisation of s-cells as the building blocks of occurrence nets that supports our intuition about their relevance.

Intuitively, parallel composition takes two nets and put them side by side.

**Definition 6** (Parallel composition). Let \((P_1, T_1, F_1, m_1)\) and \((P_2, T_2, F_2, m_2)\) be two Petri nets whose nodes are disjoint (i.e., with \((P_1 \cup T_1) \cap (P_2 \cup T_2) = \emptyset\)). Their parallel composition is given by the element-wise union of their components:

\[(P_1, T_1, F_1, m_1) \oplus (P_2, T_2, F_2, m_2) = (P_1 \cup P_2, T_1 \cup T_2, F_1 \cup F_2, m_1 \cup m_2)\]

Sequential composition is defined over (marked) occurrence nets only.

**Definition 7** (Sequential composition). Let \(M_1 = (O_1, m_1)\) and \(M_2 = (O_2, m_2)\) be two marked occurrence nets, with \(O_j = (P_j, T_j, F_j)\) for \(j = 1, 2\), whose nodes are disjoint except for the final places of \(M_1\) that are identical to the unmarked initial places of \(M_2\) (i.e., with \(M_1^\circ = (P_1 \cup T_1) \cap (P_2 \cup T_2) = {}^\circ M_2\)). Their sequential composition is given by the element-wise union of their components (but note that the places in \({}^\circ M_2\) are shared):

\[(P_1, T_1, F_1, m_1); (P_2, T_2, F_2, m_2) = (P_1 \cup P_2, T_1 \cup T_2, F_1 \cup F_2, m_1 \cup m_2)\]

Let us write \(M : i \rightarrow o\) for a marked occurrence net with \(i = {}^\circ M\) and \(o = M^\circ\). Then we note that for \(M_j : i_j \rightarrow o_j\) for \(j \in [1, 4]\):

- \(M_1 \oplus M_2 : i_1 \cup i_2 \rightarrow o_1 \cup o_2\), when the parallel composition is defined;
- \(M_1; M_2 : i_1 \rightarrow o_2\), when the sequential composition is defined;
- parallel composition is commutative and associative and has the empty net \(\emptyset = (\emptyset, \emptyset, \emptyset) : \emptyset \rightarrow \emptyset\) as neutral element, i.e. it forms a commutative monoid;
- sequential composition is associative;
- for each set of places \(i\) the identity net \(I_i = (i, \emptyset, \emptyset, \emptyset) : i \rightarrow i\) consisting just of (unmarked) isolated places \(i\) behaves as the identity w.r.t. composition;
the monoid of parallel composition is functorial: \( I_0 = 0, I_{i_1 \cup i_2} = I_{i_1} \oplus I_{i_2} \) and \((\mathcal{M}_1; \mathcal{M}_2) \oplus (\mathcal{M}_3; \mathcal{M}_4) = (\mathcal{M}_1 \oplus \mathcal{M}_3); (\mathcal{M}_2 \oplus \mathcal{M}_4)\).

In the following, we assume \( \oplus \) has higher precedence over \( ; \), e.g. we write \( \mathcal{M}_1 \oplus \mathcal{M}_2; \mathcal{M}_3 \) instead of \((\mathcal{M}_1 \oplus \mathcal{M}_2); \mathcal{M}_3\).

From the above we get that marked occurrence nets form the arrows of a commutative (strict) monoidal pre-category (it is not a monoidal category because parallel and sequential composition are defined on concrete nets and impose some disjointness requirements on their places and transitions).

**Example 3.** Consider the marked occurrence nets \( N_{C_1} : \{1\} \rightarrow \{4,5\}, (N_{C_2}; \{2\}) : \emptyset \rightarrow \{6\}, \) and \((N_{C_3}; \{3\}) : \{4,6\} \rightarrow \{7,8,9,10\}\) in Fig. 2(c), 2(d) and 2(e).

Note that the parallel composition of \( N_{C_1} \) and \( N_{C_2} \) is defined because the nets neither share places nor transitions. The resulting net \( N_{C_1} \oplus (N_{C_2}; \{2\}) : \{1\} \rightarrow \{4,5,6\}\) is shown in Fig. 2(f). We remark that neither \( N_{C_1} \oplus (N_{C_1}; \{3\}) \) nor \((N_{C_2}; \{2\}) \oplus (N_{C_3}; \{3\})\) are defined because \( N_{C_3}\) shares the place 4 with \( N_{C_1}\) and the place 6 with \( N_{C_2}\). Similarly, note that none of the considered occurrence nets can be composed sequentially, because their interfaces do not match.

For instance, the final place 5 of \( N_{C_1} \oplus (N_{C_2}; \{2\}) : \{1\} \rightarrow \{4,5,6\}\) does not appear as an initial place of \((N_{C_3}; \{3\}) : \{4,6\} \rightarrow \{7,8,9,10\}\). We can fix this mismatch by considering the net \( I_{\{5\}} : \{5\} \rightarrow \{5\}\) and noting that \((N_{C_1}; \{3\}) \oplus I_{\{5\}} : \{4,6,5\} \rightarrow \{7,8,9,10\}\) is well defined. Then,

\[
N_{C_1} \oplus (N_{C_2}; \{2\}); (N_{C_3}; \{3\}) \oplus I_{\{5\}} : \{1\} \rightarrow \{5,7,8,9,10\}
\]

stands for the net \( N \) in Fig. 2(a).

A marked occurrence net is called **trivial** if it has no transitions.

We say a marked occurrence net \( \mathcal{M} \) is decomposable in parallel if there exists two non-trivial marked occurrence nets \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) such that \( \mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \). Similarly, we say that it is decomposable in series if there exists two non-trivial marked occurrence nets \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) such that \( \mathcal{M} = \mathcal{M}_1; \mathcal{M}_2 \).

**Lemma 1.** Any s-cell \( N_{C} \) cannot be decomposed in series and in parallel.

*Proof.* By contraposition, it is immediate to prove that the sequential/parallel composition of two non-trivial nets is not an s-cell. \( \square \)

**Proposition 1.** Any marked occurrence net can be uniquely decomposed as the parallel and sequential composition of its s-cells (and identities), up to the axioms of commutative monoidal pre-categories.

*Proof.* For the existence, the partial order of s-cell (is unique and it) induces a decomposition of the net. For instance this can be done by stratifying the s-cells in layers \( L_1, \ldots, L_n \) where each layer \( L_j \) is the (largest) parallel composition of some identity \( I_{s_j} \) with all s-cells whose predecessors are in layers \( L_1, \ldots, L_{j-1} \) and then taking their sequential composition \( L_1; \ldots; L_n \).

For uniqueness, suppose two different decompositions can be found, then they must have the same s-cells (because s-cells are not decomposable) ordered in the same way (because the ordering is induced by the places they share), hence they coincide. \( \square \)
The behaviour of the decomposition below, already discussed in Example 3: or both of them are marked (i.e., $N$ can be explained by considering all the possible ways in which its initial places are guaranteed to stay empty (i.e., they are dead). In fact it can happen that the removal of the places in $s$ and of the transitions and places that causally depend on them$^{4}$ will allow to further decompose the s-cell.

We let $N_C \oplus s$ be the net obtained by removing all dead nodes as explained above. Additionally, isolated places are also removed. The cancellation of some transitions can break the equivalence class induced by $\subseteq$, which explains why $N_C \oplus s$ is not necessarily an s-cell. Also note that some of the final places of $N_C$ can become dead and canceled. The final dead places can be computed by taking $N_C^\oplus \ominus (N_C \ominus s)^\ominus$. Thus in general we have $N_C \ominus s : i' \rightarrow o'$ for some $i' \subseteq i \setminus s$ and $o' \subseteq o$. We write $N_C \ominus m$ for the marked net $(N_C \ominus s, (N_C \ominus s)) : \emptyset \rightarrow o'$, where $N_C : i \rightarrow o$ and $s = i \setminus m$, i.e., for the net $N_C \ominus s$ whose initial places are all marked.

To some extent the behaviour of an s-cell is determined by considering its behaviour under all possible initial markings. Consequently we can further explore the behaviour of $N_C : i \rightarrow o$ by considering $N_C \ominus m$ for all $m \subseteq i$.

Example 5. Consider the s-cell $(N_{C_3}, \{3\}) : \{4, 6\} \rightarrow \{7, 8, 9, 10\}$ in Fig. 2(c). The behaviour of $(N_{C_3}, \{3\})$ can be explained by considering all the possible ways in which its initial places 4 and 6 can be marked: none of them is marked (i.e., $N_{C_3} \ominus \{3\}$), just one of them is marked (i.e., either $N_{C_3} \ominus \{3, 4\}$ or $N_{C_3} \ominus \{3, 6\}$), or both of them are marked (i.e., $N_{C_3} \ominus \{3, 4, 6\}$). Net $N_{C_3} \ominus \{3\}$ depicted in Fig. 2(g) is obtained by removing from $N_{C_3}$ the initial places 4 and 6, and all the elements that causally depends on them, i.e., the transitions $f$, $g$ and $h$ and the places 7, 8, 9 and 10. The remaining nets are in Fig. 2(h), 2(f). It is worth noticing that in $N_{C_3} \ominus \{3, 4\}$ the place 4 is also removed from $N_{C_3} \ominus \{6\}$ because, after removing the place 6 and thus the transition $f$, the place 4 remains isolated.

### 3.1 Place Removal

Given a possibly marked s-cell $N_C : i \rightarrow o$ (with $i \neq \emptyset$), we are interested in studying what happens under the hypothesis that some tokens arrive in a subset of places $m \subseteq i$ while the places in $s = i \setminus m$ are guaranteed to stay empty (i.e., they are dead). In fact it can happen that the removal of the places in $s$ and of the transitions and places that causally depend on them$^{4}$ will allow to further decompose the s-cell.

We let $N_C \oplus s$ be the net obtained by removing all dead nodes as explained above. Additionally, isolated places are also removed. The cancellation of some transitions can break the equivalence class induced by $\subseteq$, which explains why $N_C \oplus s$ is not necessarily an s-cell. Also note that some of the final places of $N_C$ can become dead and canceled. The final dead places can be computed by taking $N_C^\oplus \ominus (N_C \ominus s)^\ominus$. Thus in general we have $N_C \ominus s : i' \rightarrow o'$ for some $i' \subseteq i \setminus s$ and $o' \subseteq o$. We write $N_C \ominus m$ for the marked net $(N_C \ominus s, (N_C \ominus s)) : \emptyset \rightarrow o'$, where $N_C : i \rightarrow o$ and $s = i \setminus m$, i.e., for the net $N_C \ominus s$ whose initial places are all marked.

To some extent the behaviour of an s-cell is determined by considering its behaviour under all possible initial markings. Consequently we can further explore the behaviour of $N_C : i \rightarrow o$ by considering $N_C \ominus m$ for all $m \subseteq i$.

Example 4. The canonical form of $(N, \{2, 3\})$ in Fig. 2(a) is given by the decomposition below, already discussed in Example 3:

$$N_{C_1} \oplus (N_{C_2}, \{2\}); (N_{C_3}, \{3\}) \oplus I(3) : \{1\} \rightarrow \{5, 7, 8, 9, 10\}$$

### 4 Compiling nets

In this section we associate each finite occurrence net with an arrow in the Kleisli category $K\ell(D)$ of discrete probability distributions. This is achieved in

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$^4$In such cases, all the transitions that depend on some place in $s$ cannot be fired and the places in their post-set are also dead.
two steps. We first introduce a language for representing occurrence nets and show how the s-cell decomposition can be used to associate each occurrence net with a particular term. Then, we map terms into arrows in $K\ell(D)$.

4.1 Language of nets

The decomposition of a net in branching cells can be described by terms generated by the following grammar, where $m, s$ are sets of places and $\Theta$ is a set of transactions:

$$T ::= I_s \mid \bot \mid \sum_{m \subseteq s} m \triangleright T \mid T \circ T' \mid \Theta(C) \mid \sum_{m \subseteq s} m \triangleright T$$

Here the idea is that $\Theta(C)$ denotes a basic building block consisting of the set of transactions of an s-cell whose initial places are all marked. The case of an s-cell $C$ with a set of unmarked initial places $s$ is represented as the formal sum $\sum_{m \subseteq s} m \triangleright T$, where all the possible $\binom{|s|}{2}$ initial markings $m$ are considered, each paired with the encoding of $N_C \oplus m$. The term $I_s$ denotes the identity net, consisting just of a set of unmarked places with no transitions (i.e., all places are initial and final). The term $\bot_s$ denote a net with no initial places and no transitions, whose only final places are $s$ (i.e., the places $s$ are dead). The terms $T \circ T$ and $T; T'$ denote respectively the composition in parallel and in series.

The terms of the algebra are taken up to the axioms of commutative monoidal (pre-)categories, where additionally we have $\bot_\emptyset = I_\emptyset$ and $\bot_{s_1 \cup s_2} = \bot_{s_1} \oplus \bot_{s_2}$.

4.1.1 Typing

Not all terms are valid though. We introduce a type system to discard ill-formed terms. Our types are triples of the form $(i, s, o)$ where $i$ is the set of initial unmarked places, $s$ is the set of all places and transitions appearing in a term and $o$ is the set of final places.

We write $T : i \rightarrow o$ for $T : (i, s, o)$. The typing rules are in Fig. 3. The rules for $I_s$ and $\bot_s$ are self-explanatory. The rule for $\oplus$ states that a term is well-typed when its subterms are well-typed and do not share place nor transitions (i.e., $s \cap s' = \emptyset$). The case of sequential composition $T; T'$ additionally requires that the set of final places of $T$ coincides with the set of the initial unmarked places of $T'$. The rule for $\sum_{m \subseteq s} m \triangleright T_m$ requires all subterms $T_m$ to have the same sets of initial and final places (respectively, $\emptyset$ and $o$), which captures the idea that a sum represents the execution of a s-cell under all possible markings. The rule for $\Theta(C)$ follows immediately.

Lemma 2. If $T : i \rightarrow o$ then $i \cup o \subseteq s$.

Proof. The proof is by rule induction. \qed

Typing is unique, as stated by the following result.

Lemma 3. If $T : i \rightarrow o$ and $T' : i' \rightarrow o'$ then $i = i'$, $o = o'$, $s = s'$.
\[
I_s : s \xrightarrow{s} s \quad \downarrow_s : \emptyset \xrightarrow{s} s \quad T : i \xrightarrow{s} o \quad T' : i' \xrightarrow{s'} o' \quad s \cap s' = \emptyset
\]

\[
T \oplus T' : i \cup i' \xrightarrow{s \cup s'} o \cup o' \quad \forall m \subseteq i. T_m : \emptyset \xrightarrow{s_m} o \quad s = \bigcup_{m \subseteq i} s_m
\]

\[
T; T' : i \xrightarrow{s \cup s'} o
\]

\[
o = \bigcup_{\theta \in \Theta} \theta^o \quad s = \bigcup_{\theta \in \Theta} \theta \n(\theta)
\]

\[
C(\Theta) : \emptyset \xrightarrow{s} o
\]

Figure 3: Type system

**Proof.** The proof is by rule induction. \qed

Hereafter we assume terms to be well-typed.

### 4.2 From Nets to Terms

In this section we introduce a mapping from occurrence nets to terms.

**Definition 9.** Let $\mathcal{M}$ be a marked occurrence net. The corresponding term $\llbracket \mathcal{M} \rrbracket$ is given by the homomorphic extension (w.r.t. identities, parallel and sequential composition)\(^2\) of the encoding defined below over s-cells.

\[
\llbracket N_C, i \rrbracket \left\{ \begin{array}{ll}
\sum_{m \subseteq^o (N_C, i)} m \triangleright (\downarrow_m \oplus T_m) & \text{if } \circ N_C = i \\
N_m = N_C \otimes i \cup m & \text{otherwise}
\end{array} \right.
\]

where:

\[
\begin{align*}
T_m &= \llbracket \text{can}(N_m) \rrbracket \\
d_m &= N_C^0 \setminus N_m^0
\end{align*}
\]

The encoding of a marked s-cell $C$ considers two cases: (i) all initial places of the s-cell are marked (Eq. 1a); and (ii) some initial tokens are unmarked. In the first case, a completely marked s-cell is mapped to the term $C(\Theta(N_C))$ that describes all the possible executions of $N_C$, i.e., its transactions. Differently, when some initial places are unmarked, the corresponding term is obtained by composing the behaviour of the s-cell under each possible marking $m \subseteq^o (N_C, i)$.

\(^2\)This just means that $\llbracket I_s \rrbracket = I_s$, $\llbracket \mathcal{M}_1 \oplus \mathcal{M}_2 \rrbracket = \llbracket \mathcal{M}_1 \rrbracket \oplus \llbracket \mathcal{M}_2 \rrbracket$ and $\llbracket \mathcal{M}_1 ; \mathcal{M}_2 \rrbracket = \llbracket \mathcal{M}_1 \rrbracket ; \llbracket \mathcal{M}_2 \rrbracket$. 

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The term \( m \triangleright (⊥_{d_m} \oplus T_m) \) describes the behaviour of \( C \) when all places in \( i \cup m \) are marked and the remaining initial places are dead. For this reason, \( ⊥_{d_m} \) and \( T_m \) are defined in terms of the net \( N_m = N_C @ i \cup m \). The term \( ⊥_{d_m} \) stands for the final places that are dead when the initial marking is \( i \cup m \). The term \( T_m \) encodes the net \( N_C @ i \cup m \); we just remark here, as already mentioned, that we need to compute the canonical form of \( N_m \), because removing elements from \( C \) may originate a complex net an not an s-cell (as for \( N_{c, \{3, 6\}} \) in Fig. 2(i)).

**Lemma 4.** For any finite occurrence net \( N \) and marking \( m \subseteq ^{\circ}N \), \( \langle N, m \rangle \) is defined, unique (up-to the structure of commutative monoidal pre-categories) and well-typed.

**Example 6.** Consider the marked occurrence net \( (N, \{2, 3\}) \) in Fig. 2(a) whose canonical form is in Example 4:

\[
(N, \{2, 3\}) = N_{C_1} \oplus (N_{C_2}, \{2\}); (N_{C_3}, \{3\}) \oplus I_{\{5\}}
\]

Then, the corresponding term is obtained by

\[
\langle N, \{2, 3\} \rangle = \langle N_{C_1} \rangle \oplus \langle N_{C_2}, \{2\} \rangle; \langle N_{C_3}, \{3\} \rangle \oplus \langle I_{\{5\}} \rangle
\]  

(2)

The term \( \langle N_{C_1} \rangle \) is obtained by applying Eq. 1b, because \( i = \emptyset \) and \( ^{\circ}N_{C_1} = \{1\} \neq \emptyset \) (see \( N_{C_1} \) in Fig. 2(c)). Then,

\[
\langle N_{C_1} \rangle = \emptyset \triangleright (⊥_{d_0} \oplus T_0) + \{1\} \triangleright (⊥_{d_{11}} \oplus T_{11})
\]  

(3)

Note that \( N_0 = N_{C_1} @ \emptyset \) is obtained from \( N_{C_1} \) by removing all elements that depends on the unique unmarked initial place 1. Hence, \( N_0 = N_{C_1} @ \emptyset = 0 = I_0 \). Consequently, \( T_0 = \langle N_0 \rangle = I_0 \). Moreover \( d_0 = \{4, 5\} \).

For the marking \( \{1\} \), we have \( N_{\{1\}} = N_{C_1} @ \{1\} = (N_{C_1}, \{1\}) \). Since \( N_{C_1} \) is an s-cell, \( \text{can}(N_{C_1} @ \{1\}) = (N_{C_1}, \{1\}) \). Therefore, \( T_{\{1\}} = \langle N_{C_1}, \{1\} \rangle \), which is obtained by using Eq. 1a. The net \( N_{C_1} \) has two transactions, one for each transition, i.e., \( \Theta(N_{C_1}) = \{\{a\}, \{b\}\} \). Then, \( T_{\{1\}} = C(\{\{a\}, \{b\}\}) \). Moreover, \( d_{\{1\}} = \emptyset \) because \( (N_{\{1\}})^{\circ} = (N_{C_1}, \{1\})^{\circ} = N_{C_1}^{\circ} \). Consequently,

\[
\langle N_{C_1} \rangle = \emptyset \triangleright (⊥_{d_0} \oplus \emptyset) + \{1\} \triangleright (⊥_{d_0} \oplus C(\{\{a\}, \{b\}\}))
\]

\[= \emptyset \triangleright (⊥_{d_0} + \{1\} \triangleright C(\{\{a\}, \{b\}\}))
\]  

(4)

Intuitively, the term \( \emptyset \triangleright (⊥_{d_0} + \{1\}) \) states that the s-cell \( C_1 \) does not generate any token in its final places when the initial place 1 remains unmarked. Differently, \( \{1\} \triangleright C(\{\{a\}, \{b\}\}) \) describes the behaviour of \( C_1 \) when its initial place is marked. In this case, the behaviour corresponds to the non-deterministic choice of the transactions \( \{a\} \) and \( \{b\} \).

The encoding of \( (N_{C_2}, \{2\}) \) is obtained by using Eq. 1a,

\[
\langle N_{C_2}, \{2\} \rangle = C(\{\{c\}, \{d\}\})
\]  

(5)

For \( (N_{C_3}, \{3\}) \), we obtain the following term by analogous calculations

\[
\langle N_{C_3}, \{3\} \rangle = \emptyset \triangleright (⊥_{d_0} \oplus C(\{\{e\}\})) + \{4\} \triangleright (⊥_{d_0} \oplus C(\{\{e\}\})) + \{6\} \triangleright (⊥_{d_0} \oplus C(\{\{e\}\}) \oplus C(\{\{g\}, \{h\}\})) + \{4, 6\} \triangleright C(\{\{f\}, \{e, g\}, \{e, h\}\})
\]  

(6)
which describes the behaviour of $\mathbb{C}_3$ for every possible initial marking of its initial places (i.e., $\emptyset$, $\{4\}$, $\{6\}$, and $\{4, 6\}$). The most interesting case is the subterm $\{6\} \triangleright (\perp_{\{8\}} \oplus \mathbb{C}(\{\{e\}\}) \oplus \mathbb{C}(\{\{g\}, \{h\}\}))$ obtained from $\{6\} \triangleright (\perp_{\{8\}} \oplus \mathbb{T}_{\{6\}})$. Consider the net $N_{\{6\}} = (N_{\mathbb{C}_3} \oplus \mathbb{T}_{\{6\}})$ in Fig. 2(1) which contains two $s$-cells. Consequently, its canonical form is given by the parallel composition of two $s$-cells, which are respectively encoded as $\mathbb{C}(\{\{e\}\})$ and $\mathbb{C}(\{\{g\}, \{h\}\})$.

Finally,

$$\langle I_{\{5\}} \rangle = I_{\{5\}} \tag{7}$$

To show that the term $\langle N, m \rangle$ is a good representative of the probabilistic semantics of $N$, we prove that it characterises the configurations allowed by the semantics of Abbes and Benveniste. The interested reader can find all technical details in the Appendix.

### 4.3 From Terms to $\mathcal{K}\ell(\mathcal{D})$

Given a set $X$, a discrete probability distribution with finite support over $X$ is a function $\omega : X \to [0, 1]$ such that $\sum_{x \in X} \omega(x) = 1$ and $\text{supp}(\omega) = \{x \in X \mid \omega(x) > 0\}$ is a finite set. The function $\omega$ can be sometimes written as the formal convex combination\footnote{The ‘ket’ notation $r|x\rangle$ has no particular meaning: it is just syntactic sugar.}

$$\omega = r_1|x_1\rangle + \ldots + r_n|x_n\rangle$$

where $\text{supp}(\omega) = \{x_1, \ldots , x_n\}$ and $r_j = \omega(x_j)$ for $j \in [1, n]$. We let $\mathcal{D}(X)$ be the set of discrete probability distributions $\omega$ over $X$ and write $\mathcal{D}$ for the discrete probability monad over the category $\text{Set}$ of sets (as objects) and functions (as arrows). The category $\mathcal{K}\ell(\mathcal{D})$ is the Kleisli category of the monad $\mathcal{D}$: its objects are sets, its arrows $f : X \to Y$ are functions $f : X \to \mathcal{D}(Y)$. It has been shown in [11] that $\mathcal{K}\ell(\mathcal{D})$ forms a symmetric monoidal category and that Bayesian networks can be seen as special kinds of arrows in $\mathcal{K}\ell(\mathcal{D})$ that can be represented as string diagrams using wire-and-box notation. According to this view, a diagram from $n$ to $k$ represents an arrow from $2^n$ to $2^k$ in $\mathcal{K}\ell(\mathcal{D})$.

We next show how to interpret Petri nets as Bayesian networks by exploiting $\mathcal{K}\ell(\mathcal{D})$. To this aim we need to map the arrows of a commutative pre-monoidal pre-category to those of a symmetric monoidal category: in the first case the objects are sets of places, while in the latter they are natural numbers representing a totally ordered set of ports. Therefore the mapping is defined parametrically on some arbitrarily chosen total orders of initial and final places.

Given a set of places $s$, we let $\pi_s$ denote a bijective function $\pi_s : s \to |s|$ that assigns a position to each element of $s$. We write $\pi$ when the set $s$ is implicit. Overloading the notation, we let $\pi$ also denote the string such that the place $p \in s$ appears in position $\pi(p)$. Note that $\pi$ is without repetitions: each $p \in s$ appears exactly once in $\pi$. We let $\epsilon$ denote the empty string (over the empty set of places). For $p \in s$ and $m \subseteq s$, we also write $p \in \pi$ and $m \subseteq \pi$ when $\pi$ is a linearization of $s$. 
Given π and π’ two such strings over s, we let \( \chi^\pi_{\rho} : |s| \to |s| \) denote the unique permutation that swaps π into π’, i.e. such that for any \( p \in s \) we have \( \chi^\pi_{\rho}(\pi(p)) = \pi'(p) \). By coherence of symmetries we have, e.g., \( \chi^\pi_{\rho} ; \chi^\pi'_{\rho'} = \chi^\pi_{\rho} \).

Given two strings π over s and π’ over s’ with \( s \cap s' = \emptyset \) we use juxtaposition to denote the string \( \pi \pi' \) over \( s \cup s' \) such that \( (\pi \pi')(p) = \pi(p) \) if \( p \in s \) and \( (\pi \pi')(p) = |s| + \pi'(p) \) if \( p \in s' \).

As a matter of notation, we assume that a string π over s implicitly defines an ordering over \( 2^s \), e.g., a subset of s can be seen as a binary string of length \( |s| \), which are then ordered lexicographically. Correspondingly, the permutation \( \chi^\pi_{\rho} : |s| \to |s| \) induces an isomorphism on \( 2^s \), that we denote with the same name \( \chi^\pi_{\rho} \).

In the following we assume a function \( \delta \) is given that associates every constant \( C(\Theta) \) with a finite discrete probability distribution over the elements in \( \Theta \). To ease readability, we write \( \delta_{C(\Theta)} \) for the probability distribution \( \delta(C(\Theta)) \) over \( \Theta \).

**Definition 10.** Let \( T : i \to o \) be a well-typed term, \( \pi \) a string over i, \( \rho \) a string over o. Then, \([T, \delta]_\rho\) stands for an arrow \( 2^{|i|} \to 2^{|o|} \) in \( K(\mathcal{D}) \) (i.e., a diagram from \( |i| \) to \( |o| \)) defined by structural induction as follows:

\[
[I_s, \delta]_\rho^\pi = \chi^\pi_{\rho} \tag{8}
\]
\[
[\bot_s, \delta]_\rho^\pi = \delta^{|s|}_0 \tag{9}
\]
\[
[T_1 \oplus T_2, \delta]_\rho^\pi = \chi^\pi_{\rho_1 \rho_2}(\![T_1, \delta]_{\rho_1}^{\pi_1} \otimes \![T_2, \delta]_{\rho_2}^{\pi_2}) ; \chi^\rho_{\rho_1 \rho_2} \tag{10}
\]
\[
[T_1 ; T_2, \delta]_\rho^\pi = \![T_1, \delta]_\rho^\pi ; \![T_2, \delta]_\rho^\pi \tag{11}
\]
\[
[C(\Theta), \delta]_\rho^\pi = \lambda m \cdot \sum_{\theta : \emptyset \to m \subseteq \Theta} \delta_{C(\Theta)}(\theta) \tag{12}
\]
\[
[\sum_{m \subseteq i} \langle m \triangleright T_m, \delta \rangle]_\rho^\pi = \![\sum_{m \subseteq i} \langle T_{\pi^{-1}(1)}, \delta \rangle]_\rho^\pi ; \ldots ; \![\sum_{m \subseteq i} \langle T_{\pi^{-1}(|i|)}, \delta \rangle]_\rho^\pi \tag{13}
\]

where in Eq. (9) the probability distribution \( \delta^{|s|}_0 \) assigns probability 1 to the case \( \emptyset \) and \( 0 \) to all the remaining \( 2^{|s|} - 1 \) cases and in Eq. (13) the arrows is obtained as the copairing of each \( T_m \) for all \( m \subseteq i \).

The cases in Eqs. (8) and (9) are straightforward. The cases in Eqs. (10) and (11) just exploit the monoidal category structure. It is worth noting that while the operation \( \oplus \) is commutative, this is not the case for the monoidal operation of the Kleisli category, hence denoted with a different symbol \( \otimes \). The case in Eq. (12) is the most interesting: \( [C(\Theta), \delta]_\rho \) must assign a probability distribution to the elements in the powerset of the places in \( \rho \): given \( m \subseteq \rho \) its probability is computed by taking the sum of the probabilities assigned by \( \delta \) to all processes \( \theta \) whose final places are exactly \( m \). This is correct as any two such processes are mutually exclusive alternatives. Finally, the case in Eq. (13) is

\footnote{It is important to mention that in Eq. (13) the order of the arrows in the copairing is the one induced by \( \pi \): remember that \( \pi \) induces an order on \( 2^i \), then \( \pi^{-1}(k) \) denotes the \( k \)-th subset \( m \subseteq i \) according to the order in \( \pi \).}
the most complex, as it exploits the hierarchical decomposition of s-cells. Here we take each $T_m$ and compute $2^{|i|}$ arrows $[T_m, \delta]_\rho^\pi : 2^0 \to 2^{|i|}$. Then, via co-pairing we get an arrow from $2^{|i|}$ to $2^{|i|}$. The order of the arrows in the co-pair expression is important to associate them to the right element $m \subseteq i$ (according to the order induced by $\pi$).

**Proposition 2.** $\llbracket T, \delta \rrbracket_\rho^\pi = \chi_\rho^\pi; \llbracket T, \delta \rrbracket_\rho^\pi; \chi_\rho^\pi$.

**Proof.** The proof is by structural induction on $T$.

For the case $T = \bot_s$, we have $\chi_\rho^\pi; [\bot_s, \delta]_\rho^\pi; \chi_\rho^\pi = [\bot_s, \delta]_\rho^\pi; \chi_\rho^\pi = \delta_0^{|s|}; \chi_\rho^\pi = \delta_0^{|s|}$.

For the case $T = I_s$, we have $\chi_\rho^\pi; [I_s, \delta]_\rho^\pi; \chi_\rho^\pi = \chi_\rho^\pi; \chi_\rho^\pi; \chi_\rho^\pi = \chi_\rho^\pi$ by coherence of symmetries.

For the case $T = T_1 \oplus T_2$, we have

$$\chi_\rho^\pi; [T_1 \oplus T_2, \delta]_\rho^\pi; \chi_\rho^\pi = \chi_\rho^\pi; (\chi_\rho^\pi; [T_1, \delta]_\rho^\pi; \chi_\rho^\pi)\chi_\rho^\pi = \chi_\rho^\pi; [T_1, \delta]_\rho^\pi; \chi_\rho^\pi \oplus \chi_\rho^\pi$$

by coherence of symmetries.

For the case $T = T_1; T_2$, let us assume that $[T_1, \delta]_\rho^\pi = \chi_\rho^\pi; [T_1, \delta]_\rho^\mu; \chi_\rho^\mu$ and $[T_2, \delta]_\rho^\nu = \chi_\rho^\nu; [T_2, \delta]_\rho^\rho; \chi_\rho^\rho$, so that, as a particular case we have $[T_1, \delta]_\rho^\pi = \chi_\rho^\pi; [T_1, \delta]_\rho^\pi$ and $[T_2, \delta]_\rho^\mu = [T_2, \delta]_\rho^\rho; \chi_\rho^\rho$ (because $\chi_\rho^\rho = I_{|\pi|}$). Then we have

$$\chi_\rho^\pi; [T_1 \oplus T_2, \delta]_\rho^\pi; \chi_\rho^\pi = \chi_\rho^\pi; [T_1, \delta]_\rho^\pi; [T_2, \delta]_\rho^\pi; \chi_\rho^\pi$$

$$= [T_1, \delta]_\rho^\pi; [T_2, \delta]_\rho^\pi; \chi_\rho^\pi$$

$$= [T_1; T_2, \delta]_\rho^\pi$$

For the case $T = C(\Theta)$, likewise the case for $\bot_s$, the definition is purely functional.

For the case $T = \sum_{m \subseteq i} m \triangleright T_m$, let us assume that for any $m \subseteq i$ we have $[T_m, \delta]_\rho^\pi = \chi_\rho^\pi; [T_m, \delta]_\rho^\pi; \chi_\rho^\pi = [T_m, \delta]_\rho^\pi; \chi_\rho^\pi$. Then, we have

$$\chi_\rho^\pi; [\sum_{m \subseteq i} m \triangleright T_m, \delta]_\rho^\pi; \chi_\rho^\pi = \chi_\rho^\pi; (\sum_{m \subseteq i} [T_m, \delta]_\rho^\pi; \chi_\rho^\pi)\chi_\rho^\pi = \chi_\rho^\pi; [T_m, \delta]_\rho^\pi; \chi_\rho^\pi$$

$$= \chi_\rho^\pi; [\sum_{m \subseteq i} [T_m, \delta]_\rho^\pi; \chi_\rho^\pi; \chi_\rho^\pi; \chi_\rho^\pi]$$

$$= \chi_\rho^\pi; [\sum_{m \subseteq i} [T_m, \delta]_\rho^\pi; \chi_\rho^\pi; \chi_\rho^\pi; \chi_\rho^\pi]$$

$$= [T_{\sum_{m \subseteq i} \delta}^\pi, \delta]_\rho^\pi$$

$$= [\sum_{m \subseteq i} m \triangleright T_m, \delta]_\rho^\pi$$

**Proposition 3.** The definition of $\llbracket T, \delta \rrbracket_\rho^\pi$ is well given.
Proof. We must show that: (1) the typing is consistent with the definition, (2) that the choice of \( \pi_1, \rho_1, \pi_2, \rho_2 \) in Eq. (10) and of \( \gamma \) in Eq. (11) is inessential for the result, and (3) that \([T_1 \oplus T_2, \delta]_{\rho}^\pi = [T_2 \oplus T_1, \delta]_{\rho}^\pi\).

For (1), we must prove that if \( T : i \rightarrow o, \pi \) is a string over \( i \) and \( \rho \) is a string over \( o \), then \([T, \delta]_{\rho}^\pi : 2^{\{i\}} \rightarrow 2^{\{o\}}\). The proof is a straightforward rule induction.

For (2), we just exploit Proposition 2. In the case of Eq. (10), we have

\[
[T_1 \oplus T_2, \delta]_{\rho}^\pi = \chi_{\pi_1 \pi_2} \cdot ([T_1, \delta]_{\rho_1}^\pi \otimes [T_2, \delta]_{\rho_2}^\pi) ; \chi_{\rho_1 \rho_2}^i
\]

\[
= \chi_{\pi_1 \pi_2} \cdot [(\chi_{\pi_1} \cdot [T_1, \delta]_{\rho_1}^\pi) \otimes (\chi_{\pi_2} \cdot [T_2, \delta]_{\rho_2}^\pi) ; \chi_{\rho_1 \rho_2}^i]
\]

\[
= \chi_{\pi_1 \pi_2} \cdot [(T_1, \delta]_{\rho_1}^\pi \otimes [T_2, \delta]_{\rho_2}^\pi) ; \chi_{\rho_1 \rho_2}^i
\]

\[
= \chi_{\pi_1 \pi_2} \cdot ([T_1, \delta]_{\rho_1}^\pi \otimes [T_2, \delta]_{\rho_2}^\pi) ; \chi_{\rho_1 \rho_2}^i
\]

In the case of Eq. (11), we have

\[
[T_1; T_2, \delta]_{\rho}^\pi = [T_1, \delta]_{\rho_1}^\pi ; [T_2, \delta]_{\rho_1}^\pi
\]

\[
= [T_1, \delta]_{\rho_1}^\pi ; \chi_{\rho_1}^\gamma ; [T_2, \delta]_{\rho_1}^\pi
\]

\[
= [T_1, \delta]_{\rho_1}^\pi ; [T_2, \delta]_{\rho_1}^\pi
\]

Finally, for (3), we have:

\[
[T_1 \oplus T_2, \delta]_{\rho}^\pi = \chi_{\pi_1 \pi_2} \cdot ([T_1, \delta]_{\rho_1}^\pi \otimes [T_2, \delta]_{\rho_2}^\pi) ; \chi_{\rho_1 \rho_2}^i
\]

\[
= \chi_{\pi_1 \pi_2} \cdot [(T_2, \delta]_{\rho_2}^\pi \otimes [T_1, \delta]_{\rho_1}^\pi) ; \chi_{\rho_1 \rho_2}^i
\]

\[
= \chi_{\pi_1 \pi_2} \cdot ([T_2, \delta]_{\rho_2}^\pi \otimes [T_1, \delta]_{\rho_1}^\pi) ; \chi_{\rho_1 \rho_2}^i
\]

\[
= [T_1; T_2, \delta]_{\rho}^\pi
\]

Example 7. Consider the net depicted in Fig. 2(a) and the corresponding term calculated in Example 6. We show the encoding of the net by considering a generic distribution \( \delta \) and use lexicographic order of places. We start from Eq. 3

\[
\langle N, \{2, 3\} \rangle = \langle N_{\text{C}_1} \rangle \oplus \langle N_{\text{C}_2}, \{2\} \rangle ; \langle N_{\text{C}_3}, \{3\} \rangle \oplus \langle I_{\{5\}} \rangle
\]

Then, the string diagram for \([\langle N, \{2, 3\} \rangle, \delta]_{5,7,8,9,10}^1\] is shown in Fig. 3 and can be computed as follows.

\[
\begin{align*}
\langle\langle N, \{2, 3\} \rangle, \delta \rangle_1^5 &\quad \text{by def.} \\
\langle\langle N_{\text{C}_1} \rangle \oplus \langle N_{\text{C}_2}, \{2\} \rangle; \langle N_{\text{C}_3}, \{3\} \rangle \oplus \langle I_{\{5\}} \rangle, \delta \rangle_5 &\quad \text{by def.} \\
\langle\langle N_{\text{C}_1} \rangle \oplus \langle N_{\text{C}_2}, \{2\} \rangle; \langle N_{\text{C}_3}, \{3\} \rangle \oplus \langle I_{\{5\}} \rangle, \delta \rangle_5 &\quad \text{by def.} \\
\chi_{\text{C}_1}^1 \cdot \langle\langle N_{\text{C}_1} \rangle, \delta \rangle_1^5 \otimes \langle\langle N_{\text{C}_2}, \{2\} \rangle, \delta \rangle_5^5; \chi_{\text{C}_2}^4.5.6 \quad \text{by def.} \\
\chi_{\text{C}_2}^4.5.6 \cdot \langle\langle N_{\text{C}_3}, \{3\} \rangle, \delta \rangle_5^4.6 \otimes \langle\langle I_{\{5\}} \rangle, \delta \rangle_5^5 &\quad \text{by def.}
\end{align*}
\]

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We now show the calculation for each of the boxes in Fig. 4. To ease readability, in the following we let

\[ C_a = C\left(\{\{a\}, \{b\}\}\right) \]
\[ C_c = C\left(\{\{c\}\}\right) \]
\[ C_r = C\left(\{\{d\}\}\right) \]
\[ C_g = C\left(\{\{g\}, \{h\}\}\right) \]
\[ C_f = C\left(\{\{f\}, \{e, g\}, \{e, h\}\}\right) \]

For \( \llbracket (N_1), \{2\}, \delta \rrbracket \), we start from Eq. (4), i.e.,

\[ \llbracket (N_1) \rrbracket = \emptyset \triangleright \bot_{\{4,5\}} + \{1\} \triangleright C_a \]

By Eq. (14),

\[ \llbracket (N_1), \delta \rrbracket^{1}_{2,5} = \begin{bmatrix} \emptyset \triangleright \bot_{\{4,5\}} \triangleright_{\{1,5\}} & \{4\} & \{5\} & \{4,5\} \\ \emptyset \triangleright \bot_{\{4,5\}} \triangleright_{\{1,5\}} & 1 \triangleright_{\{1,5\}} & 0 \triangleright_{\{1,5\}} & 0 \triangleright_{\{1,5\}} \\ \{1\} \triangleright_{\{1,5\}} & 0 \triangleright_{\{1,5\}} & p_a \triangleright_{\{1,5\}} & 1 - p_a \triangleright_{\{1,5\}} \end{bmatrix} \]

where the first row in the table corresponds to \( c_{0}^{\{4,5\}} \), as prescribed by Eq. (9).

The second row is obtained by Eq. (12), by assuming that \( \delta_{C_a}(\{a\}) = p_a \) and \( \delta_{C_a}(\{b\}) = 1 - p_a \).

For \( \llbracket (N_2), \{2\}, \delta \rrbracket^{6}_{6} \), we start from Eq. (13), i.e.,

\[ \llbracket (N_2), \{2\} \rrbracket = C_c \]

Then,

\[ \llbracket (N_2), \{2\}, \delta \rrbracket^{6}_{6} = \begin{bmatrix} \emptyset \triangleright \{6\} \\ \emptyset \triangleright_{\{6\}} & 1 - p_c \triangleright_{\{6\}} & p_c \triangleright_{\{6\}} \end{bmatrix} \]

where \( \delta_{C_c}(\{c\}) = p_c \) and \( \delta_{C_c}(\{d\}) = 1 - p_c \).
For $\mathbb{B}[\{N_{C_3}, \{3\}\}, \delta]_{7,8,9,10}$, we start from Eq. (14), i.e.,

$$\mathbb{B}[\{N_{C_3}, \{3\}\}] = \emptyset \triangleright (\perp_{X_{9,10}} \oplus C_e)$$

$$+ \{4\} \triangleright (\perp_{X_{9,10}} \oplus C_e)$$

$$+ \{6\} \triangleright (\perp_{X} \oplus C_e \oplus C_g)$$

$$+ \{4, 6\} \triangleright C_f$$

$$\mathbb{B}[\{N_{C_3}, \{3\}\}, \delta]_{7,8,9,10} = \left[\begin{array}{c}
\perp_{X_{8,10}} \oplus C_e, \delta]_{7,8,9,10} \\
\perp_{X_{8,10}} \oplus C_e, \delta]_{7,8,9,10} \\
\perp_{X} \oplus C_e \oplus C_g, \delta]_{7,8,9,10}
\end{array}\right]$$

where the last column (i.e., the one tagged with dots) represents all the remaining nine (inessential) cases. The first two rows are obtained as follows:

$$\perp_{X_{8,10}} \oplus C_e, \delta]_{7,8,9,10} = \left[\begin{array}{c}
\perp_{X_{8,10}}, \delta]_{7,8,9,10} \\
\perp_{X_{8,10}}, \delta]_{7,8,9,10}
\end{array}\right]$$

The third row is obtained analogously after fixing $\delta_{C_e}(\{g\}) = p_g$ and $\delta_{C_e}(\{h\}) = 1 - p_g$. The last row is obtained by Eq. (13) and taking $\delta_{C_f}(\{f\}) = p_f$, $\delta_{C_f}(\{e, g\}) = p'_g$, and $\delta_{C_f}(\{e, h\}) = 1 - p_f - p'_g$.

5 Forward and Backward Inference and Disintegration

In this section we illustrate how to perform bayesian reasoning over Petri nets by following the approach presented in [3]. We first recall some notions, which will be used in our reasoning. **Marginalisation** is an operation $\Pi_1 : X \otimes Y \rightarrow X$ that projects a joint distribution $P(x, y)$ on $X \otimes Y$ to the marginal distribution on $X$ computed as $P(x) = \sum_y P(x, y)$. Similarly, we have $\Pi_2 : X \otimes Y \rightarrow Y$ for the projection of $P(x, y)$ over $Y$ defined as $P(y) = \sum_x P(x, y)$. Consider the arrow $\mathbb{B}[\{N, \{2, 3\}\}] : 2^2 \rightarrow 2^3$ in Fig. 4 and suppose we are interested in reasoning about the probability of producing a token in the place
Figure 5: Simplified string diagram for $[[N, \{2, 3\}], \delta]$

7. In such case, marginalisation can be used to obtain an arrow $f : 2^1 \to 2^1$ that discards the wires corresponding to the places 5, 8, 9 and 10, as shown in Fig. 5. The wire diagram corresponds to the term:

$$
(\llbracket N, 1 \rrbracket^4, 5 \rrbracket 1 \llbracket N, 2, 2 \rrbracket^6, 6 \llbracket N, 3, 3 \rrbracket^4, 7, 8, 9, 10 \llbracket \Pi_1 \otimes \Pi_1 \Pi_1)
$$

From Eq. (14), we obtain

$$
\alpha = \llbracket N, 1 \rrbracket^1_4, 5 ; \Pi_1 = \\
\begin{array}{c|c|c|c}
\emptyset & \{4\} \\
\hline
\emptyset & 1 & 0 \\
\{1\} & 1 - p_a & p_a \\
\end{array}
$$

(17)

Analogously, from Eq. (16)

$$
\gamma = \llbracket N, 3 \rrbracket^1_7, 8, 9, 10 ; \Pi_1 \otimes \Pi_1 \Pi_1 = \\
\begin{array}{c|c|c|c|c|c|c|c|c}
\emptyset & \{7\} \\
\hline
\emptyset & 0 & 1 \\
\{4\} & 0 & 1 \\
\{6\} & 0 & 1 \\
\{4, 6\} & p_f & 1 - p_f \\
\end{array}
$$

(18)

We write $\beta$ for $[[N, 2], \delta]^6_6$ in Eq. (16).

Then, $\alpha \otimes \beta$ is obtained as

$$
\alpha \otimes \beta = \\
\begin{array}{c|c|c|c|c|c|c|c|c}
\emptyset & \{4\} & \{6\} & \{4, 6\} \\
\hline
\emptyset & 1 - p_c & 0 & p_c & 0 \\
\{1\} & (1 - p_a) (1 - p_c) & p_a (1 - p_c) & (1 - p_a) p_c & p_a p_c \\
\end{array}
$$

(19)

Finally,

$$
\psi = \alpha \otimes \beta ; \gamma = \\
\begin{array}{c|c|c|c|c|c|c|c|c}
\emptyset & \{7\} \\
\hline
\emptyset & 0 & 1 \\
\{1\} & p_a p_c p_f & 1 - p_a p_c p_f \\
\end{array}
$$

(20)

This means that, given that a token appears in place 1 with probability 1, the place 7 will be marked with probability $1 - p_a p_c p_f$. Using the notation
in [11], this value is computed by precomposing the state \(\omega = \{1\} \rangle\) with the arrow \(\psi\), i.e., by letting \(\psi_\ast(\omega) = \omega; \psi = p_a p_c p_f \{\emptyset\} + (1 - p_a p_c p_f)\{\{1\}\}\).

As an example of backward reasoning, given the a priori probability \(\frac{1}{2}\) that a token can appear in place 1, we can compute the probability that place 1 is marked given that a token appears in place 7, which is

\[
\frac{1 - p_a p_c p_f}{1 + (1 - p_a p_c p_f)} = \frac{1 - p_a p_c p_f}{2 - p_a p_c p_f}
\]

Using the notation in [11], this value is computed by setting (for \(\psi : X \rightarrow D(Y)\) and \(q\) a predicate on \(Y\))

\[
\psi^\ast(q)(x) = \sum_{y \in Y} \psi(x)(y) \cdot q(y)
\]

\[
= \psi(x)(\emptyset) \cdot q(\emptyset) + \psi(x)(\{7\}) \cdot q(\{7\})
\]

\[
= \psi(x)(\{7\})
\]

where \(q\) is the predicate such that \(q(\{7\}) = 1\) (and \(q(\emptyset) = 0\)) and then computing

\[
\omega|_{\psi^\ast(q)} = \sum_{x \in X} \frac{\omega(x) \cdot \psi^\ast(q)(x)}{\omega \models \psi^\ast(q)(x)}
\]

\[
= \frac{\omega(\emptyset) \cdot \psi^\ast(q)(\emptyset)}{\omega \models \psi^\ast(q)(\emptyset)} + \frac{\omega(\{1\}) \cdot \psi^\ast(q)(\{1\})}{\omega \models \psi^\ast(q)(\{1\})}
\]

\[
= \frac{\frac{1}{2} \cdot 1}{\omega \models \psi^\ast(q)(\emptyset)} + \frac{\frac{1}{2} \cdot (1 - p_a p_c p_f)}{\omega \models \psi^\ast(q)(\{1\})}
\]

\[
= \frac{\frac{1}{2} \cdot 1}{\omega \models \psi^\ast(q)(\emptyset)} + \frac{1 - p_a p_c p_f}{\omega \models \psi^\ast(q)(\{1\})}
\]

where

\[
\omega \models \psi^\ast(q) = \sum_{x \in X} \omega(x) \cdot \psi^\ast(q)(x)
\]

\[
= \omega(\emptyset) \cdot \psi^\ast(q)(\emptyset) + \omega(\{1\}) \cdot \psi^\ast(q)(\{1\})
\]

\[
= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1 - p_a p_c p_f)
\]

\[
= \frac{2 - p_a p_c p_f}{2}
\]

6 Conclusion

In this paper we have shown how to derive a Bayesian network from a probabilistic Petri net in the style of [11][12]. The construction is computed via an intermediate representation of a PN as a term in a monoidal (pre-)category structure, exploiting the string diagram representation of BN outlined in [11]. As shown in Section 5, the BN representation can then be exploited to reason
about conditional probabilities of marking reachability, via forward and backward inference. Notably, when transitions have non-empty post-sets then each marking corresponds to a unique deterministic process (i.e., a unique configuration of the underlying event structure) and thus the inference can be transferred to processes as well.

There are many ways in which PN have been enriched with probabilistic behaviour \cite{7, 15, 10, 8, 13, 10, 3, 12}. To avoid confusion, most of them replace nondeterminism with probability only in part, or focus on interleaved computations, or introduce time dependent stochastic distributions. The approach considered here differs from the others in the literature because: (1) it is purely probabilistic, (2) it deals well with concurrent computations, (3) it addresses confusion.

In the literature, there are very few papers investigating the connections between PN and BN. In \cite{14} the relation is drawn in the opposite direction, i.e., PN are used to encode the reasoning of BN. The connection established in this paper provides two views for the same model: on the one side, the standard token game of the PN view (suitable extended with probabilistic choices) gives a concrete, probabilistic computational model. On the other side, the BN semantics allows us to reason about the properties of the computations of the underlying concrete model.

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A Correctness of mapping to terms

The remaining of this section is devoted to establish a correspondence between the semantics of Abbes and Benveniste for a marked net \((N, m)\) and the corresponding term \(\langle N, m \rangle\).

A.1 Prime Event Structures

A prime event structure (also PES) \([17,18]\) is a triple \(\mathcal{E} = (E, \preceq, \#)\) where: \(E\) is the set of events; the causality relation \(\preceq\) is a partial order on events; the conflict relation \(\#\) is a symmetric, irreflexive relation on events such that conflicts are inherited by causality, i.e., \(\forall e_1, e_2, e_3 \in E. e_1 \# e_2 \preceq e_3 \Rightarrow e_1 \# e_3\).

The PES \(\mathcal{E}_N\) associated with a net \(N\) can be formalised using category theory as a chain of universal constructions, called coreflections. Hence, for each PES \(\mathcal{E}\), there is a standard, unique (up to isomorphism) nondeterministic occurrence net \(N_\mathcal{E}\) that yields \(\mathcal{E}\) and thus we can freely move from one setting to the other.

Given an event \(e\), its downward closure \(\lfloor e \rfloor = \{ e' \in E \mid e' \preceq e \}\) is the set of causes of \(e\). As usual, we assume that \(\lfloor e \rfloor\) is finite for any \(e\). Given \(B \subseteq E\), we say that \(B\) is downward closed if \(\lfloor e \rfloor \subseteq B\) and that \(B\) is conflict-free if \(\forall e, e' \in B. \neg (e \# e')\). We let the immediate conflict relation \(\#_0\) be defined on events by letting \(e \#_0 e'\) iff \(\lfloor e \rfloor \times \lfloor e' \rfloor \cap \# = \{(e, e')\}\), i.e., two events are in immediate conflict if they are in conflict but their causes are compatible.

A.2 Abbes and Benveniste’s Branching Cells

In the following we assume that a finite PES \(\mathcal{E} = (E, \preceq, \#)\) is given. A prefix \(B \subseteq E\) is any downward-closed set of events (possibly with conflicts). Any prefix \(B\) induces an event structure \(\mathcal{E}_B = (B, \preceq_B, \#_B)\) where \(\preceq_B\) and \(\#_B\) are the restrictions of \(\preceq\) and \(\#\) to the events in \(B\). A stopping prefix is a prefix \(B\) that is closed under immediate conflicts, i.e., \(\forall e \in B, e' \in E. e \#_0 e' \Rightarrow e' \in B\). Intuitively, a stopping prefix is a prefix whose (immediate) choices are all available. It is initial if the only stopping prefix strictly included in \(B\) is \(\emptyset\).

A configuration \(v \subseteq \mathcal{E}\) is any set of events that is downward closed and conflict-free. Intuitively, a configuration represents (the state reached after executing) a concurrent but deterministic computation of \(\mathcal{E}\). Configurations are ordered by inclusion and we denote by \(\mathcal{V}_\mathcal{E}\) the poset of configurations of \(\mathcal{E}\) and by \(\Omega_\mathcal{E}\) the poset of maximal configurations of \(\mathcal{E}\).

The future of a configuration \(v\), written \(E^v\), is the set of events that can be executed after \(v\), i.e., \(E^v = \{ e \in E \mid \forall e' \in v. \neg (e \# e') \}\). We write \(\mathcal{E}^v\) for the event structure induced by \(E^v\).

A configuration \(v\) is stopped if there is a stopping prefix \(B\) with \(v \in \Omega_B\) and \(v\) is recursively stopped (or r-stopped) if there is a sequence of configurations \(\emptyset = v_0 \subset \ldots \subset v_n = v\) such that for any \(i \in [0, n]\) the set \(v_{i+1} \setminus v_i\) is a stopped configuration of \(\mathcal{E}^{v_i}\) for \(v_i\) in \(\mathcal{E}\).

A branching cell is any initial stopping prefix of the future \(E^v\) of a recursively stopped configuration \(v\). Intuitively, a branching cell is a minimal subset...
of events closed under immediate conflict. We remark that branching cells are determined by considering the whole (future of the) event structure $E$ and they are recursively computed as $E$ is executed. Remarkably, every maximal configuration has a branching cell decomposition.

Example 8. Consider the PES $E_N$ in Fig. A.6(a) and its maximal configuration $v = \{a,c,e,g\}$. We show that $v$ is recursively stopped by exhibiting a branching cell decomposition. The initial stopping prefixes of $E_N = E_N^{\emptyset}$ are shown in Fig. A.6(b). There are two possibilities for choosing $v_1 \subseteq v$ and $v_1$ recursively stopped: either $v_1 = \{a\}$ or $v_1 = \{c\}$. When $v_1 = \{a\}$, the choices for $v_2$ are determined by the stopping prefixes of $E_N^{\{a\}}$ (see Fig. A.6(c)) and the only possibility is $v_2 = \{a,c\}$. From $E_N^{\{a,c\}}$ in Fig. A.6(d) we take $v_3 = v$. Note that $\{a,c,e\}$ is not recursively stopped because $\{e\}$ is not maximal in the stopping prefix of $E_N^{\{a,c\}}$ (see Fig. A.6(e)). Finally, note that the branching cells of $E_N^{\{a\}}$ (Fig. A.6(f)) and $E_N^{\{b\}}$ (Fig. A.6(g)) correspond to different choices in $E_N^\emptyset$ and thus have different stopping prefixes.

A.3 AB’s decomposition and terms

The recursively stopped configurations of a marked net $(N,m)$ characterise all the allowed executions of $N$ under the marking $m$. Hence, we formally link the recursively stopped configurations of $E_{(N,m)}$ with the deterministic processes associated with $\langle N,m \rangle$. We start by introducing the notion of configurations associated to a term.

Definition 11. Given a term $T : i \rightarrow o$ and a marking $m \subseteq i$, the set of config-
urations of $T$ under $m$, written $\text{Conf}(T, m)$, is defined inductively as follows.

\begin{align*}
\text{Conf}(I_s, m) &= \{\emptyset\} \\
\text{Conf}(\bot_s, \emptyset) &= \{\emptyset\} \\
\text{Conf}(T_1 \oplus T_2, m) &= \{v_1 \cup v_2 \mid \forall j = 1, 2. T_j : i_j \xrightarrow{\sigma_j} o_j \wedge v_j \in \text{Conf}(T_j, m \cap i_j)\} \\
\text{Conf}(T_1; T_2, m) &= \{v_1 \cup v_2 \mid v_1 \in \text{Conf}(T_1, m) \wedge T_2 : i_2 \xrightarrow{\sigma_2} o_2 \wedge v_2 \in \text{Conf}(T_2, v_1^2 \cap i_2)\} \\
\text{Conf}(C(\Theta), \emptyset) &= \emptyset \\
\text{Conf}(\sum_{m \subseteq i} m \triangleright T_m, m_j) &= \text{Conf}(T_j, \emptyset)
\end{align*}

**Proposition 4.** Let $(N, m) : i \to o$ be a finite marked occurrence net and $T = \langle N, m \rangle$. Then, for $j \subseteq i$, $v$ is a maximal r-stopped configuration of $\mathcal{E}_{(N, m \cup j)}$ iff $v \in \text{Conf}(T, j)$.

**Proof.** The proof follows by structural induction on $T$.

- $T = I_s$. For all $j \subseteq i$, we have $\text{Conf}(I_s, j) = \{\emptyset\}$. Consequently, $v \in \text{Conf}(I_s, j)$ implies $v = \emptyset$. Since $\langle N, m \rangle = I_s$, $(N, m) = I_s$. Then, $s = i$ and $m = \emptyset$. Therefore, $\mathcal{E}_{(N, m \cup j)} = \emptyset$. Consequently, $v \in \mathcal{E}_{(N, m \cup j)}$ implies $v = \emptyset$.

- $T = \bot_s$. It holds trivially because there is no $(N, m)$ such that $\langle N, m \rangle = \bot_s$.

- $T = T_1 \oplus T_2$. Then, $(N, m) = (N_1, m_1) \oplus (N_2, m_2)$, $T_1 = \langle N_1, m_1 \rangle T_2 = \langle N_2, m_2 \rangle$. By inductive hypothesis, $v_i \in \text{Conf}(T_i, j_i)$ iff $v_i$ is an r-stopped configuration of $\mathcal{E}_{(N_i, m_i \cup j_i)}$. The proof follows by noting that the union of two disjoint r-stopped configurations is an r-stopped configuration.

- $T = T_1; T_2$. Then, $(N, m) = (N_1, m_1); (N_2, m_2)$, $T_1 = \langle N_1, m_1 \rangle T_2 = \langle N_2, m_2 \rangle$. By inductive hypothesis, $v_i \in \text{Conf}(T_i, j_i)$ iff $v_i$ is an r-stopped configuration of $\mathcal{E}_{(N_i, m_i \cup j_i)}$. The proof follows by noting that $v_1$ is an r-stopped configuration of $\mathcal{E}_{(N, m \cup j)}$ and $v_2$ is an r-stopped configuration of $\mathcal{E}_{(N, m \cup j)}$. Consequently, $v = v_1 \cup v_2$ is an r-stopped configuration of $\mathcal{E}_{(N, m \cup j)}$.

- $T = C(\Theta(N_C))$. Then, $N = N_C$ and $m = \circ C$. Moreover, $v \in \mathcal{E}_{(C, \circ C)}$ implies that $v$ is a maximal deterministic process of $\langle C, \circ C \rangle$, i.e., a transaction. Hence, $v \in \Theta(N_C)$ and $v \in \text{Conf}(T, \emptyset)$.

- $T = \sum_{j \subseteq i} j \triangleright \bot_s \oplus T_j$ with $T_j = \langle \text{can}(N_C \oplus m \cup j) \rangle$. Then, $v \in \text{Conf}(T, j)$ iff $v \in \text{Conf}(T_j, \emptyset)$. By inductive hypothesis, $v$ is a maximal r-stopped configuration of $\mathcal{E}_{N_C \oplus m \cup j}$. The proof is completed by noting that $\mathcal{E}_{N_C \oplus m \cup j} = \mathcal{E}_{(N_C, m \cup j)}$.

\[\square\]