The Properties and Application of Poisson Distribution

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Abstract. Poisson distribution is not only a very important distribution in probability theory, but also a useful tool to study the random events. Firstly, this paper gives a simple introduction of Poisson distribution and some characteristics of the Poisson distribution. Secondly, some properties of Poisson distribution are introduced and proved. In the end, some application of Poisson distribution are introduced, such as the application in the rare event and in daily life. We also do some analysis about the application of Poisson distribution.

1. Introduction

1.1 The definition of the Poisson distribution
Poisson distribution¹[1] is a common and important discrete probability distribution. In independent trials, \( P_n \) represents the probability of the event occurring in the experiment A, which is related to the total number of trials. If \( nP_n \rightarrow \lambda \), then when \( n \rightarrow \infty \),

\[
\lim_{\lambda \to \infty} \frac{n^k e^{-\lambda}}{k!} \rightarrow \frac{\lambda^k}{k!} e^{-\lambda}.
\]

There is only one parameter \( \lambda \) in the Poisson distribution \( P(\lambda) \), which is both the mean of the Poisson distribution and the variance of the Poisson distribution. In the real example, when a random event occurs randomly and independently at a fixed average instantaneous rate, for example, a call received by a telephone exchange, the passengers at a bus stop, particles emitted by a radioactive substance, white blood cells certain areas under the microscope, then the number of times the event occurs per unit time (area or volume) follows the Poisson distribution approximately.

1.2 The characteristics of the Poisson distribution
(1) The Poisson distribution is a probability distribution that describes and analyzes rare events. To observe such event, the sample size \( n \) must be large.

(2) \( \lambda \) is the only parameter that Poisson distribution depends on. The smaller \( \lambda \) is, more biased the distribution is. The distribution tends to be symmetric, as it get larger.

(3) The distribution tends to be normal distribution when \( \lambda=20 \).

2. Poisson distribution properties

2.1 Simple properties Poisson distribution
Theorem 1¹[2] If \( \lambda \) is a random vector of the Poisson distribution \( P(\lambda) \), then:

\[
\lim_{\lambda \to \infty} P\left( \frac{(p\lambda - \lambda)}{\sqrt{\lambda}} < x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt
\]
Prove It is known that the characteristic function of $P_{\lambda}$ is $\phi_{\lambda} = e^{i\lambda t}$, then the characteristic function of

$$\eta_{\lambda} = (p_{\lambda} - \lambda) / \sqrt{\lambda}$$

is $g_{\lambda}(t) = \phi_{\lambda}(\frac{t}{\sqrt{\lambda}}) = e^{\lambda (e^{it/\sqrt{\lambda}} - 1)}$,

for any t, we have

$$e^{\frac{it}{\lambda}} = 1 + \frac{it}{\lambda} - \frac{t^2}{2\lambda} + O\left(\frac{1}{\lambda}\right) (\lambda \to \infty),$$

then

$$\lambda (e^{\frac{it}{\lambda}} - 1) - i\sqrt{\lambda}t = -\frac{t^2}{2} + \lambda \cdot o(1) \to -\frac{t^2}{2} (\lambda \to \infty).$$

Then for any $\lambda_n \to \infty$, $\lim_{\lambda_n \to \infty} g_{\lambda_n}(t) = e^{\frac{-t^2}{2}}$.

But $e^{\frac{-t^2}{2}}$ is the characteristic function of $N(0,1)$. It is known that the sufficient and necessary condition of $\{F_n(x)\}$ converges to $F(x)$ is their corresponding characteristic functions converge. So

$$\lim_{\lambda_n \to \infty} P((p_{\lambda_n} - \lambda) / \sqrt{\lambda_n} < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt.$$ 

Because $\lambda_n$ is a random value, which means that

$$\lim_{\lambda_n \to \infty} P((p_{\lambda_n} - \lambda) / \sqrt{\lambda} < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt.$$ (4)

2.2 The unbiasedness of the Poisson distribution

The parameter $\lambda$ of the Poisson distribution has obvious significance, and the research on its estimation methods and properties is an important subject of modern statistical research. Many researchers have done research about this, but the most common one is to find the unbiased estimation of parameters $\lambda$. Different parameter distribution solve different claims problems, the Poisson distribution model of claims involves not only unbiased estimation of parameters $\lambda$, but also unbiased estimation of parametric functions of $\lambda$.

When $X_1, X_2, \ldots X_n$ is a simple random sample, and follows Poisson distribution:

$$P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}, x = 0,1,2,3\ldots,$$

then it can be proved that sample mean and sample variance $\bar{X}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ are unbiased estimates of population parameter $\lambda$. Generalize to the general case, for any $\alpha$, $0 \leq a \leq 1, a\bar{X} + (1-a)s^2$ are all unbiased estimations of parameter $\lambda$. There are some literatures referring to estimation of parameters in Poisson distribution, such as [3].

Theorem 2 Let $g_1(\theta) = g_1(\lambda) = e^{\lambda}$, then it can be proved that the unbiased estimator of $g_1(\lambda)$ is $2^{-\lambda}$, not $g_1(\theta) = e^{\bar{X}}$.

Prove: From theorem 1 we have
\[ E(g_1(\theta)) = E(e^{X}) = E(e^{nX}) = \sum_{z=0}^{\infty} e^{nz} \frac{(n\lambda)^z}{z!} e^{-n\lambda} \neq e^{\lambda} \] \tag{6}

\[ E[g_1(\lambda)] = E(2^X) = \sum_{x=0}^{\infty} 2^x P(X_i = x) = \sum_{x=0}^{\infty} 2^x \frac{\lambda^x}{x!} e^{-\lambda} \]

\[ = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(2\lambda)^x}{x!} = e^{-\lambda} e^{2\lambda} = e^{\lambda}. \] \tag{7}

2.3 The additivity of the Poisson distribution
Let \((\Omega, F, P)\) be the complete probability space, \(X, Y\) are mutually independent and follow the Poisson distribution with the parameter \(\lambda_1, \lambda_2\) separately, then \(X + Y\) follows the \(P(\lambda_1 + \lambda_2)\).

3. The application of Poisson distribution

3.1 Application of Poisson distribution in rare events

The probability of an accident per vehicle passing through an intersection is \(p = 0.0001\). If there are 1000 cars through this intersection at some given time, try to find the probability distribution of the number of accidents \(X\) and the probability of more than 2 accidents.

**Answer:** First, accidents within a certain period are rare events. Observing whether 1000 cars passing through the intersection have accidents or not, can be regarded as \(n = 1000\) Bernoulli trial, the probability of an accident is \(p = 0.0001\). Therefore, it is a binomial distribution, that is \(X \sim B(1000, 0.0001)\).

\[ Y = p(x \geq 2) = 1 - p(x = 0) - p(x = 1) = 1 - 0.9999^{1000} - 1000 \times 0.0001 \times 0.9999^{999} \]

Because \(p = 0.0001\) is very small, \(n = 1000\), it can be approximated as a Poisson distribution, we can use:

\[ p(v_n = m) = C_n^m p^m (1-p)^{n-m} \approx \frac{np^m}{m!} e^{-np} \quad (m = 0, 1, \ldots, n). \] \tag{8}

Because \(np = 1000 \times 0.0001 = 0.1\),

So \(p(x \geq 2) = 1 - \frac{1^0}{0!} e^{-0.1} - \frac{0.1}{1!} e^{-0.1} = 0.0045\). \tag{9}

3.2 The application of Poisson distribution in real life

In order to monitor the contamination of drinking water, the number of bacteria per milliliter of drinking water in a community was tested and 400 records were obtained as shown in the table below:

| Bacteria number in ml of water | 0 | 1 | 2 | ≥ 3 | Sum |
|------------------------------|---|---|---|-----|-----|
| Occurrence number \(f\)       | 243 | 120 | 31 | 6    | 400 |

Try to analyze whether the number of bacteria in drinking water obeys Poisson distribution.

**Answer:** The average number of bacteria per milliliter of water can be calculated and it is \(\bar{x} = 0.5\), the variance \(s^2 = 0.496\), they are very close. Therefore, it can be considered that the number of bacteria per milliliter approximately obeys Poisson distribution. Let \(\bar{x} = 0.5\) replace \(\lambda\) in the formula and we get:
\[ p(x = k) = \frac{0.5^k}{k!} e^{-0.5} \quad (k = 0, 1, 2, \cdots) \]  

The calculated results are shown in the following table:

| Bacteria number in 1ml of water | 0 | 1 | 2 | ≥3 | Total |
|-------------------------------|---|---|---|----|------|
| Occurrence number f           | 243 | 120 | 31 | 6  | 400  |
| Frequency                    | 0.6075 | 0.3000 | 0.0775 | 0.0150 | 1   |
| Probability                  | 0.6065 | 0.3033 | 0.0758 | 0.0144 | 1   |
| Theoretical frequency        | 242.60 | 121.32 | 30.32 | 5.76 | 400  |

**Analyze:** It can be seen from table 2 that the frequency distribution of the number of bacteria is quite consistent with the Poisson distribution of \( \lambda = 0.5 \). Further, it is appropriate to use Poisson distribution to describe the distribution of bacteria per unit volume (or area).

### 4. Conclusion

By introducing several properties of Poisson distribution and studying its application in many aspects, we can clearly understand the importance of Poisson distribution properties and the universality of its application. Because the Poisson distribution has so many useful properties, its use is so widespread that it can be applied to rare events, financial insurance and even many aspects of our daily lives. The research on Poisson distribution property and its application can be naturally applied to the theory of Poisson distribution for random processes with small probability of success and large number of trials. The probability expression in Poisson distribution contains only one parameter \( \lambda \), which reduces the workload of parameter determination and modification, and the model construction is relatively simple, which is of great practical significance.

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