Two-weight inequalities for singular integral operators 
satisfying a variant of Hörmander’s condition

Vagif S. Guliyev

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Abstract. In this paper, we present some sufficient conditions for the 
boundedness of convolution operators that their kernel satisfies a certain version 
of Hörmander’s condition, in the weighted Lebesgue spaces $L_{p,\omega}(\mathbb{R}^n)$.

1. Introduction

Let $\mathbb{R}^n$ be $n$-dimensional Euclidean space, $x = (x_1, \ldots, x_n)$, $\xi = 
(\xi_1, \ldots, \xi_n)$ are vectors in $\mathbb{R}^n$, $x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$, $|x| = (x \cdot x)^{1/2}$, 
$\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\}$.

Suppose that $\omega$ be a positive, measurable, and real function defined 
in $\mathbb{R}^n$, i.e., is a weight function. By $L_{p,\omega}(\mathbb{R}^n)$ we denote the space of measurable functions $f(x)$ on $\mathbb{R}^n$ with finite norm

$$
\|f\|_{L_{p,\omega}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx \right)^{1/p}, \quad 1 \leq p < \infty.
$$
For \( \omega = 1 \), we obtain the nonweighted space \( L^p \), i.e., \( L^p,1(\mathbb{R}^n) = L^p(\mathbb{R}^n) \).

We write \( f \in L^p_{loc}(\mathbb{R}^n) \), \( 1 \leq p < \infty \), if \( f \) belongs to \( L^p(F) \) on any closed bounded set \( F \subset \mathbb{R}^n \).

Let \( K : \mathbb{R}_0^n \to \mathbb{R} \), \( K \in L^1_{loc}(\mathbb{R}_0^n) \), \( \mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\} \), be a function satisfying the following conditions:

1) \( K(tx) \equiv K(tx_1, \ldots, tx_n) = t^{-n} K(x) \) for any \( t > 0 \), \( x \in \mathbb{R}_0^n \);
2) \( \int_{|x|=1} K(x) \, d\sigma(x) = 0 \);
3) \( \int_0^1 \frac{w(t)}{t} \, dt < \infty \), where \( w(t) = \sup_{|\xi|=|\eta| \leq t} |K(\xi) - K(\eta)| \) for \( |\xi| = |\eta| = 1 \).

Let \( f \in L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), and consider the following singular integral

(1)

\[ Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) \, dy = \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^n : |x-y| > \varepsilon\}} K(x-y) f(y) \, dy. \]

In the following theorem Calderon and Zygmund [5] proved the boundedness of the operator \( T \).

**Theorem 1.** Suppose that the kernel \( K \) of the singular integral (1) satisfies conditions 1) – 3) and \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p < \infty \). Then the singular integral exists for \( x \in \mathbb{R}^n \) almost everywhere and the following inequalities holds

\[
\|Tf\|_{L^p(\mathbb{R}^n)} \leq C_1 \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty,
\]

\[
\int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}} dx \leq \frac{C_2}{\lambda} \int_{\mathbb{R}^n} |f(x)| \, dx,
\]

where \( C_1, C_2 > 0 \) is independent of \( f \).

Hörmander [13] imposed a weaker constraint on the kernel of the singular integral (1), namely,

(2)

\[
\int_{\{x \in \mathbb{R}^n : |x| > 2|y|\}} |K(x-y) - K(x)| \, dx \leq C,
\]

where \( K \in L^1_{loc}(\mathbb{R}_0^n) \) and \( C > 0 \) is a constant independent of \( y \). By replacing condition 3) with condition (2), under conditions 1), 2) he proved Theorem 1 for singular integrals with kernels satisfying condition (2). This condition is related to condition 3), and under this condition, inequality (2) holds (see [19]).
On the other hand, singular integrals whose kernels do not satisfy Hörmander’s condition (2) are widely considered, for example oscillatory and some other singular integrals (see [20]).

Suppose that \( K \in L_2(\mathbb{R}^n) \) is a function, satisfying the following conditions:

(K1) \( \| \hat{K} \|_\infty \leq C; \)

(K2) \( |K(x)| \leq \frac{C}{|x|^n}; \)

(K3) There exist functions \( A_1, \ldots, A_m \in L^1_{\text{loc}}(\mathbb{R}^n) \), and the finite family \( \Phi = \{ \phi_1, \ldots, \phi_m \} \) of essentially bounded functions in \( \mathbb{R}^n \) such that \( |\det [\phi_j(y_i)]|^2 \in RH_{\infty}(R^{nm}), \ y_i \in \mathbb{R}^n, \ i, j = 1, \ldots, m; \)

(K4) For a fixed \( \gamma > 0 \) and for any \( |x| > 2|y| > 0, \)

\[
\left| K(x - y) - \sum_{i=1}^{m} A_i(x) \phi_i(y) \right| \leq C \frac{|y|^{\gamma}}{|x - y|^{n + \gamma}},
\]

where \( C > 0 \) is a constant and \( \hat{K}(\xi) = \int_{\mathbb{R}^n} e^{-i(x, \xi)} K(x) dx \) is the Fourier transform of the function \( K \). In general, the functions \( A_i, \phi_i, \ i = 1, \ldots, m \) defined in \( \mathbb{R}^n \) are complex-valued.

**Remark 1.** Any kernel satisfying condition (3) also satisfies the condition

\[
\int_{|x| > 2|y|} \left| K(x - y) - \sum_{i=1}^{m} A_i(x) \phi_i(y) \right| dx \leq C, \ |x| > 2|y|,
\]

Note that conditions (K1) – (K4) were imposed in [20] and condition (4) was studied in [10]. For example, for \( m = 1, A_1(x) = K(x), \phi_1(y) \equiv 1 \) condition (4) yields Hörmander’s condition (2). Note that, in this sense, condition (4) is a generalization of Hörmander’s condition (2).

There exist other conditions stronger than condition (2) (see [9, 21]). The function \( K(x) = (\sin x)/x \) satisfies conditions (K1) – (K4) and does not satisfy conditions 1), 2), and Hörmander’s condition (2) (see [3]).

**Definition 1.** [17] It is said that a locally integrable weight function \( \omega \) belongs to \( A_p(\mathbb{R}^n) \), where \( 1 < p < \infty \), if

\[
\sup_B \left( |B|^{-1} \int_B \omega(x) dx \right) \left( |B|^{-1} \int_B \omega(x)^{1-p'} dx \right)^{p-1} < \infty,
\]

where the supremum is taken over all balls \( B \subset \mathbb{R}^n \) and \( p' = \frac{p}{p-1} \).
For \( p = 1 \), we say \( \omega \in A_1(\mathbb{R}^n) \), if
\[
\sup_B \left( |B|^{-1} \int_B \omega(x) dx \right) \operatorname{ess} \sup_B \frac{1}{\omega(x)} < \infty,
\]
or
\[
|B|^{-1} \int_B \omega(x) dx \leq C \omega(x) \text{ a.e. } x \in B
\]
for any balls \( B \subset \mathbb{R}^n \).

Suppose that the function \( K \) satisfies conditions \((K1) - (K4)\). For \( f \in L_p(\mathbb{R}^n), 1 \leq p < \infty \) define the following convolution operator generated by the kernel \( K \) as
\[
Af(x) = \int_{\mathbb{R}^n} K(x - y) f(y) dy.
\]

For the convolution operator (5), the following theorem holds.

**Theorem 2.** [20] Suppose that \( w \in A_p(\mathbb{R}^n), 1 \leq p < \infty \), and the kernel of the convolution operator (5) satisfies conditions \((K1) - (K4)\). Then the following inequalities holds:

\[
\|Af\|_{L_{p,w}(\mathbb{R}^n)} \leq C_3 \|f\|_{L_{p,w}(\mathbb{R}^n)}, 1 < p < \infty,
\]

\[
\int_{\{x \in \mathbb{R}^n : |Af(x)| > \lambda\}} \omega(x) dx \leq \frac{C_4}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(x) dx,
\]

where \( C_3, C_4 > 0 \) is independent of \( f \).

Note that in the "nonweighted" case, when condition \((K2)\) is not imposed and condition (3) is replaced by condition (4), Theorem 2 was proved in [10].

**Lemma 1.** Suppose that \( 1 \leq p \leq q \leq \infty \) and \( u(t) \) and \( v(t) \) are positive functions defined on \((0, \infty)\).

(i) For the validity of the inequality

\[
\left( \int_0^\infty u(t) \left| \int_0^t \varphi(\tau) d\tau \right|^q dt \right)^{1/q} \leq K_1 \left( \int_0^\infty |\varphi(t)|^p v(t) dt \right)^{1/p}
\]

with a constant \( K_1 \), not depending on \( \varphi \), it is necessary and sufficient that

\[
\sup_{t > 0} \left( \int_t^\infty u(\tau) d\tau \right)^{p/q} \left( \int_0^t v(\tau)^{1-p'} d\tau \right)^{p-1} < \infty.
\]
(ii) For the validity of the inequality
\[
\left( \int_0^\infty u(t) \left( \int_t^\infty \varphi(\tau)d\tau \right)^q d\tau \right)^{1/q} \leq K_2 \left( \int_0^\infty |\varphi(t)|^p v(t) dt \right)^{1/p}
\]
with a constant \(K_2\), not depending on \(\varphi\), it is necessary and sufficient that
\[
\sup_{t>0} \left( \int_0^t u(\tau)d\tau \right)^{p/q} \left( \int_\tau^\infty v(\tau)^{1-p'}d\tau \right)^{p-1} < \infty.
\]

Lemma 1 was established by Muckenhoupt [18] for \(1 \leq p = q \leq \infty\) and J.S. Bradley [4], V.M. Kokilashvili [14], V.G. Maz'ya [16] for \(p < q\).

Lemma 2. [15] Let \(u(t)\) and \(v(t)\) be positive functions on \((0, \infty)\).
(i) If the following condition is satisfied
\[
\sup_{t>0} \left( \int_t^\infty v(\tau)d\tau \right)^{p/q} \left( \int_\tau^\infty u(\tau) d\tau \right)^{p-1} < \infty,
\]
then the inequality
\[
\int_0^\infty v(t) \left( \int_0^t F(\tau)d\tau \right) dt \leq c \int_0^\infty u(t)|F(t)| dt
\]
holds, where the constant \(c > 0\) does not depend on \(F\).
(ii) If the following condition is satisfied
\[
\sup_{t>0} \left( \int_t^\infty v(\tau)d\tau \right)^{p/q} \left( \int_\tau^\infty u(\tau) d\tau \right)^{p-1} < \infty,
\]
then the inequality
\[
\int_0^\infty v(t) \left( \int_0^\infty F(\tau)d\tau \right) dt \leq c \int_0^\infty u(t)|F(t)| dt
\]
holds, where the constant \(c > 0\) does not depend on \(F\).

Lemma 3. [1, 6] Suppose that \(1 \leq p \leq q \leq \infty\) and \(u(x)\) and \(v(x)\) are positive functions defined on \(\mathbb{R}^n\).
(i) For the n-dimensional Hardy inequality
\[
\left( \int_{\mathbb{R}^n} \left( \int_{|y|<|x|/2} |f(y)| dy \right)^q \omega(x) dx \right)^{1/q} \leq C_5 \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p}
\]
with a constant $C_5$, independent on $f$, to hold, it is necessary and sufficient that the following condition be satisfied:

$$\sup_{R>0} \left( \int_{|x|>2R} \omega(x) \, dx \right)^{1/q} \left( \int_{|x|<R} \omega^{1-p'}(x) \, dx \right)^{1/p'} < \infty.$$  

(ii) For the $n$-dimensional (dual) Hardy inequality

$$\left( \int_{\mathbb{R}^n} \left( \int_{|y|>2|x|} |f(y)| \, dy \right)^q u(x) \, dx \right)^{1/q} \leq C_6 \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right)^{1/p}$$

with a constant $C_6$, independent on $f$, to hold, it is necessary and sufficient that the following condition be satisfied:

$$\sup_{R>0} \left( \int_{|x|<R} u(x) \, dx \right)^{1/q} \left( \int_{|x|>2R} \omega^{1-p'}(x) \, dx \right)^{1/p'} < \infty.$$  

Lemma 4. [8, 15] Suppose that $1 \leq p < \infty$, $\beta > 1$, $\varphi \in A_p(\mathbb{R}^n)$, and suppose that $u$, $u_1$ are positive increasing (decreasing) functions defined on $(0, \infty)$. Suppose that $\omega(x) = u(|x|)\varphi(x)$, $\omega_1(x) = u_1(|x|)\varphi(x)$ and the weighted pair $(\omega(x), \omega_1(x))$ satisfies the following condition:

(i) For $1 < p < \infty$, $A_p(\omega, \omega_1) < \infty$, where

$$A_p(\omega, \omega_1) := \sup_{r>0} \left( \int_{|x|>2r} \omega_1(x)|x|^{-np} \, dx \right) \left( \int_{|x|<r} \omega^{1-p'}(x) \, dx \right)^{p-1}$$

(ii) For $p = 1$, $A_1(\omega, \omega_1) < \infty$, where

$$A_1(\omega, \omega_1) := \sup_{r>0} \left( \int_{|x|>2r} \omega_1(x)|x|^{-n} \, dx \right) \text{ess sup} \frac{1}{\omega(x)}$$

(iii) For $1 < p < \infty$, $B_p(\omega, \omega_1) < \infty$, where

$$B_p(\omega, \omega_1) := \sup_{r>0} \left( \int_{|x|<r} \omega_1(x) \, dx \right) \left( \int_{|x|>2r} \omega^{1-p'}(x)|x|^{-np'} \, dx \right)^{p-1}$$

(iv) For $p = 1$, $B_1(\omega, \omega_1) < \infty$, where

$$B_1(\omega, \omega_1) := \sup_{r>0} \left( \int_{|x|<r} \omega_1(x) \, dx \right) \text{ess sup} \frac{1}{\omega(x)|x|^n}$$
Then there exists a positive constant $C$ depending only on $p$, $n$ such that, for any $t > 0$, the following inequality holds:

$$u_1(2t) \leq CA_p(\omega, \omega_1) u(t) \quad (u_1(t/2) \leq CB_p(\omega, \omega_1) u(t)).$$

In the case $\varphi = 1$ Lemma 4 was proved also in [11].

2. Main results

Theorem 3. Suppose that the kernel $K$ of the convolution operator (5) satisfies the conditions $(K1) - (K4)$ and $\phi \in A_p(\mathbb{R}^n)$, $1 \leq p < \infty$. If $\omega(x) = u(x)\phi(x)$ and $\omega_1(x) = u_1(x)\phi(x)$ are weight functions on $\mathbb{R}^n$, satisfies the conditions

$$A_p(\omega, \omega_1) < \infty, \quad B_p(\omega, \omega_1) < \infty,$$

and there exist $b > 0$ such that

$$\sup_{|x|/4 < |y| \leq 4|x|} u_1(y) \leq b u(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{6}$$

Then there exists a $C_7 > 0$ such that, for any $f \in L_{p,\omega}(\mathbb{R}^n)$, $1 < p < \infty$ the following inequality holds

$$\int_{\mathbb{R}^n} |Af(x)|^p \omega_1(x) \, dx \leq C_7 \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx. \tag{7}$$

Moreover, the condition (6) can be replaced by the condition: there exist $b > 0$ such that

$$u_1(x)\left(\sup_{|x|/4 \leq |y| \leq |x|} \frac{1}{u(y)}\right) \leq b \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Proof. For $k \in \mathbb{Z}$ we define $E_k = \{x \in \mathbb{R}^n : 2^k < |x| \leq 2^{k+1}\}$, $E_{k,1} = \{x \in \mathbb{R}^n : |x| \leq 2^{k-1}\}$, $E_{k,2} = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^{k+1}\}$, $E_{k,3} = \{x \in \mathbb{R}^n : |x| > 2^{k+2}\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k \in \mathbb{Z}}$ is equal to 3.
Let \( 1 < p < \infty \). Given \( f \in L_{p, \omega}(\mathbb{R}^n) \), we write

\[
|Af(x)| = \sum_{k \in \mathbb{Z}} |Af(x)| \chi_{E_k}(x) 
\leq \sum_{k \in \mathbb{Z}} |Af_{k,1}(x)| \chi_{E_k}(x) + \sum_{k \in \mathbb{Z}} |Af_{k,2}(x)| \chi_{E_k}(x) 
+ \sum_{k \in \mathbb{Z}} |Af_{k,3}(x)| \chi_{E_k}(x) 
\]

(8)

\[= A_1 f(x) + A_2 f(x) + A_3 f(x), \]

where \( \chi_{E_k} \) is the characteristic function of the set \( E_k \), \( f_{k,i} = f \chi_{E_k,i} \), \( i = 1, 2, 3 \).

First we shall estimate \( \|A_1 f\|_{L_{p, \omega}} \). Note that for \( x \in E_k, \ y \in E_{k,1} \) we have \( |y| \leq 2^{k-1} \leq |x|/2 \). Moreover, \( E_k \cap \text{supp} f_{k,1} = \emptyset \) and \( |x - y| \geq |x|/2 \). Hence by condition (K2)

\[
A_1 f(x) \leq C \sum_{k \in \mathbb{Z}} \left( \int_{\mathbb{R}^n} \frac{|f_{k,1}(y)|}{|x - y|^n} \, dy \right) \chi_{E_k} 
\leq C \int_{|y| \leq |x|/2} |x - y|^{-n} |f(y)| \, dy \leq 2^n C |x|^{-n} \int_{|y| \leq |x|/2} |f(y)| \, dy
\]

for any \( x \in E_k \). Hence we have

\[
\int_{\mathbb{R}^n} |A_1 f(x)|^p \omega_1(x) \, dx \leq (2^n C)^p \int_{\mathbb{R}^n} \left( \int_{|y| < |x|/2} |f(y)| \, dy \right)^p |x|^{-np} \omega_1(x) \, dx.
\]

Since \( A_p(\omega, \omega_1) < \infty \), the Hardy inequality

\[
\int_{\mathbb{R}^n} \omega_1(x) |x|^{-np} \left( \int_{|y| < |x|/2} |f(y)| \, dy \right)^p \, dx \leq C_9 \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx
\]

holds and \( C_9 \leq c' A_p(\omega, \omega_1) \), where \( c' \) depends only on \( n \) and \( p \). In fact the condition \( A_p(\omega, \omega_1) < \infty \) is necessary and sufficient for the validity of this inequality (see [1], [6]). Hence, we obtain

(9)

\[
\int_{\mathbb{R}^n} |A_1 f(x)|^p \omega_1(x) \, dx \leq C_8 \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx.
\]

where \( C_9 \) is independent of \( f \).

Next we estimate \( \|A_3 f\|_{L_{p, \omega}} \). It is easy to verify, for \( x \in E_k, \ y \in E_{k,3} \) we have \( |y| > 2|x| \) and \( |x - y| \geq |y|/2 \). Since \( E_k \cap \text{supp} f_{k,3} = \emptyset \), for \( x \in E_k \)
by condition (K2) we obtain
\[ A_3 f(x) \leq C \int_{|y|>2|x|} \frac{|f(y)|}{|x-y|^n} \, dy \leq 2^n C \int_{|y|>2|x|} |f(y)| |y|^{-n} \, dy. \]

Hence we have
\[ \int_{\mathbb{R}^n} |A_3 f(x)|^p \omega_1(x) \, dx \leq (2^n C)^p \int_{\mathbb{R}^n} \left( \int_{|y|>2|x|} |f(y)| |y|^{-n} \, dy \right)^p \omega_1(x) \, dx. \]

Since \( B_p(\omega, \omega_1) < \infty \), the Hardy inequality
\[ \int_{\mathbb{R}^n} \left( \int_{|y|>2|x|} |f(y)| |y|^{-n} \, dy \right)^p \omega_1(x) \, dx \leq C_6 \int_{\mathbb{R}^n} |f(x)|^p \omega_1(x) \, dx \]
holds and \( C_6 \leq c'' B_p(\omega, \omega_1) \), where \( c'' \) depends only on \( n \) and \( p \). In fact the condition \( B_p(\omega, \omega_1) < \infty \) is necessary and sufficient for the validity of this inequality (see [1], [6]). Hence, we obtain
\[ \int_{\mathbb{R}^n} |A_3 f(x)|^p \omega_1(x) \, dx \leq C_9 \int_{\mathbb{R}^n} |f(x)|^p \omega_1(x) \, dx, \]
where \( C_9 \) is independent of \( f \).

Finally, we estimate \( \|A_2 f\|_{L^p,\omega_1} \). From the \( L^p,\phi(\mathbb{R}^n) \) boundedness of \( T \) and condition (6) we have
\[ \int_{\mathbb{R}^n} |A_2 f(x)|^p \omega_1(x) \, dx = \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z}} |A f_k,2(x) \chi_{E_k}(x) \right)^p \omega_1(x) \, dx \]
\[ = \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z}} |A f_k,2(x) \chi_{E_k}(x) \right)^p \omega_1(x) \, dx \]
\[ = \sum_{k \in \mathbb{Z}} \int_{E_k} |A f_k,2(x)\chi_{E_k}(x)|^p \omega_1(x) \, dx \]
\[ \leq \sum_{k \in \mathbb{Z}} \sup_{x \in E_k} u_1(x) \int_{\mathbb{R}^n} |A f_k,2(x)|^p \omega_1(x) \, dx \]
\[ \leq \|A\|^p_p \sum_{k \in \mathbb{Z}} \sup_{x \in E_k} u_1(x) \int_{\mathbb{R}^n} |f_k,2(x)|^p \omega_1(x) \, dx \]
\[ = \|A\|^p_p \sum_{k \in \mathbb{Z}} \sup_{y \in E_k} u_1(y) \int_{E_k,2} |f(x)|^p \omega_1(x) \, dx, \]
where $\|A\|_\phi = \|A\|_{L^p,\phi}$. Since $2^{k-1} < |x| \leq 2^{k+2}$, we have by condition (a)

$$\sup_{y \in E_k} u_1(y) = \sup_{2^{k-1} < |y| \leq 2^{k+2}} u_1(y) \leq \sup_{|x|/4 < |y| \leq 4|x|} u_1(y) \leq b u(x)$$

for almost all $x \in E_{k,2}$. Therefore we get

$$\int_{\mathbb{R}^n} |A_2 f(x)|^p \omega_1(x) dx \leq \|A\|^p_{p,\phi} b \sum_{k \in Z} \int_{E_{k,2}} |f(x)|^p u(x) \phi(x) dx$$

(11)

$$\leq C_{10} \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx$$

since the multiplicity of covering $\{E_{k,2}\}_{k \in Z}$ is equal to 3, where $C_{10} = 3\|A\|^p_{p,\phi}$.

Inequalities (8), (9), (10), (11) imply (7) which completes the proof. □

Analogously proved the following theorem.

**Theorem 4.** Suppose that the kernel $K$ of the convolution operator (5) satisfies the conditions (K1) – (K4), and $\omega(x) = u(x)\phi(x)$, $\omega_1(x) = u_1(x)\phi(x)$ are weight functions on $\mathbb{R}^n$, $\phi \in A_1(\mathbb{R}^n)$. If the weighted pair $(\omega(x), \omega_1(x))$ satisfies condition (6) and

$$\mathcal{A}_1(\omega, \omega_1) \equiv \sup_{r > 0} \left( \int_{|x| > 2r} \omega_1(x)|x|^{-n} dx \right) \text{ess sup } \frac{1}{\omega(x)} < \infty,$$

$$\mathcal{B}_1(\omega, \omega_1) \equiv \sup_{r > 0} \left( \int_{|x| < r} \omega_1(x) dx \right) \text{ess sup } \frac{1}{\omega(x)|x|^n} < \infty,$$

Then there exists a $C_{11} > 0$ such that, for any $f \in L^1(\mathbb{R}^n)$, the following inequality holds

$$\int_{\{x \in \mathbb{R}^n : |A_1 f(x)| > \lambda\}} \omega_1(x) dx \leq \frac{C_{11}}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(x) dx.$$  

(12)

**Theorem 5.** Suppose that the kernel $K$ of the convolution operator (5) satisfies the conditions (K1) – (K4), and $\varphi \in A_1(\mathbb{R}^n)$. Let $u$ and $u_1$ be positive increasing functions on $(0, \infty)$, such that the weights functions $\omega(x) = u(|x|)\varphi(x)$ and $\omega_1(x) = u_1(|x|)\varphi(x)$ satisfy the condition

$$\mathcal{A}_1(\omega, \omega_1) < \infty$$

Then inequality (12) is valid.
Proof. Suppose that $f \in L_{1, \omega}(\mathbb{R}^n)$. Let $u_1$ are positive increasing functions on $(0, \infty)$ and $A_1(\omega, \omega_1) < \infty$.

Without loss of generality we can suppose that $u_1$ may be represented by

$$u_1(t) = u_1(0+) + \int_0^t \psi(\tau) d\tau,$$

where $u_1(0+) = \lim_{t \to 0} u_1(t)$ and $u_1(t) \geq 0$ on $(0, \infty)$. In fact there exists a sequence of increasing absolutely continuous functions $\varpi_n$ such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim_{n \to \infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0, \infty)$ (see [2, 11, 7, 8, 12] for details).

We have

$$\int_{\{x \in \mathbb{R}^n : |Af(x)| > \lambda\}} \omega_1(x) dx = u_1(0+) \int_{\{x \in \mathbb{R}^n : |Af(x)| > \lambda\}} \phi(x) dx + \int_{\{x \in \mathbb{R}^n : |Af(x)| > \lambda\}} \left( \int_0^{|x|} \psi(\tau) d\tau \right) \phi(x) dx = J_1 + J_2.$$

If $u_1(0+) = 0$, then $J_1 = 0$. If $u_1(0+) \neq 0$ by the weak $L_1$ boundedness of $A$, $\phi \in A_1(\mathbb{R}^n)$ thanks to Lemma 4

$$J_1 \leq \frac{1}{\lambda} |A_\phi| u_1(0+) \int_{\mathbb{R}^n} |f(x)| \phi(x) dx \leq \frac{1}{\lambda} |A_\phi| \int_{\mathbb{R}^n} |f(x)| u_1(|x|) \phi(x) dx \leq \frac{b}{\lambda} |A_\phi| \int_{\mathbb{R}^n} |f(x)| \omega(x) dx.$$

After changing the order of integration in $J_2$ we have

$$J_2 = \int_0^\infty \psi(t) \left( \int_{|x| > t} \chi \{x : |Af(x)| > \lambda\} \phi(x) dx \right) dt \leq \int_0^\infty \psi(t) \left( \int_{|x| > t} \chi \{x : |A(f \chi_{|y| > t/2})(x)| > \lambda\} \phi(x) dx 
+ \int_{|x| > t} \chi \{x : |A(f \chi_{|y| \leq t/2})(x)| > \lambda\} \phi(x) dx \right) dt = J_{21} + J_{22}.$$
Using the weak $L_1$ boundedness of $A$ and Lemma 4 we have

$$J_{21} \leq \frac{\|A\|}{\lambda} \int_0^\infty \psi(t) \left( \int_{|y| > t/2} |f(y)| |\phi(y)dy| \right) dt$$

$$= \frac{\|A\|}{\lambda} \int_{\mathbb{R}^n} |f(y)| \left( \int_0^{2|y|} \psi(t) dt \right) \phi(y) dy$$

$$\leq \frac{\|A\|}{\lambda} \int_{\mathbb{R}^n} |f(y)| u_1(2|y|) \phi(y) dy$$

$$\leq b \frac{\|A\|}{\lambda} \int_{\mathbb{R}^n} |f(y)| \omega(y) dy.$$

Let us estimate $J_{22}$. For $|x| > t$ and $|y| \leq t/2$ we have $|x|/2 \leq |x-y| \leq 3|x|/2$, and so

$$J_{22} \leq c_4 \int_0^\infty \psi(t) \left( \int_{|x| > t} \chi \left\{ y : \int_{|y| \leq t/2} |f(y)| |x-y|^{-n} dy > \lambda \right\} \phi(x) dx \right) dt$$

$$\leq c_5 \int_0^\infty \psi(t) \left( \int_{|y| \leq t/2} |f(y)| |y|^{-n} dy > \lambda \right) \left( \int_{|x| > t} \phi(x) |x|^{-n} dx \right) dt$$

$$= \frac{c_6}{\lambda} \int_0^\infty \psi(t) \left( \int_{|y| \leq t/2} |f(y)| |y|^{-n} dy \right) \left( \int_{|x| \leq t/2} |f(y)| dy \right) dt.$$

The Hardy inequality

$$\int_0^\infty \psi(t) \left( \int_{|y| \leq t/2} |f(y)| dy \right) dt \leq C \int_{\mathbb{R}^n} |f(y)| \omega(|y|) dy$$

for $p = 1$ is characterized by the condition $C \leq c' A'_1$ (see [4], [14]), where

$$A'_1 = \sup_{r > 0} \left( \int_{2r}^\infty \left( \int_{|x| > t} \phi(x) |x|^{-n} dx \right) \psi(t) dt \right) \text{ess sup}_{|x| < r} \frac{1}{\omega(x)}$$

$$= \sup_{r > 0} \left( \int_{|x| > 2r} \phi(x) |x|^{-n} \left( \int_{|x|} \psi(t) dt \right) dx \right) \text{ess sup}_{|x| < r} \frac{1}{\omega(x)}$$

$$\leq \sup_{r > 0} \left( \int_{|x| > 2r} \phi(x) |x|^{-n} u_1(|x|) dx \right) \text{ess sup}_{|x| < r} \frac{1}{\omega(x)}$$

$$= \sup_{r > 0} \left( \int_{|x| > 2r} \omega_1(|x|) |x|^{-n} dx \right) \text{ess sup}_{|x| < r} \frac{1}{\omega(x)} = A_1(\omega, \omega_1) < \infty.$$
Hence, applying the Hardy inequality, we obtain
\[ J_{22} \leq \frac{C_{12}}{\lambda} \int_{\mathbb{R}^n} |f(x)|\omega(|x|)dx. \]

Combining the estimates of \( J_1 \) and \( J_2 \), we get (12) for \( \omega_1(t) = \omega_1(0+) + \int_0^t \psi(\tau)d\tau \). By Fatou’s theorem on passing to the limit under the Lebesgue integral sign, this implies (12). The theorem is proved. \( \square \)

Analogously proved the following theorem.

**Theorem 6.** Suppose that \( 1 < p < \infty \), the kernel \( K \) of the convolution operator (5) satisfies the conditions (K1) – (K4) and \( \varphi \in A_p(\mathbb{R}^n) \). Let \( u, u_1 \) are positive increasing functions on \((0, \infty)\), \( \omega(x) = u(|x|)\varphi(x) \), \( \omega_1(x) = u_1(|x|)\varphi(x) \) and \( A_p(\omega, \omega_1) < \infty \). Then inequality (7) is valid.

**Theorem 7.** Suppose that the kernel \( K \) of the convolution operator (5) satisfies the conditions (K1) – (K4) and \( \varphi \in A_1(\mathbb{R}^n) \). Let \( u \) and \( u_1 \) are positive decreasing functions on \((0, \infty)\), such that the weights functions \( \omega(x) = u(|x|)\varphi(x) \) and \( \omega_1(x) = u_1(|x|)\varphi(x) \) satisfy the condition
\[ B_1(\omega, \omega_1) < \infty \]
Then inequality (12) is valid.

**Proof.** Without loss of generality we can suppose that \( \omega_1 \) may be represented by
\[ \omega_1(t) = \omega_1(+\infty) + \int_{t}^{+\infty} \psi(\tau)d\tau, \]
where \( \omega_1(+\infty) = \lim_{t \to +\infty} \omega_1(t) \) and \( \omega_1(t) \geq 0 \) on \((0, \infty)\). In fact there exists a sequence of decreasing absolutely continuous functions \( \omega_n \) such that \( \omega_n(t) \leq \omega_1(t) \) and \( \lim_{n \to \infty} \omega_n(t) = \omega_1(t) \) for any \( t \in (0, \infty) \) (see [2, 11, 7, 8, 12] for details).

We have
\[
\int_{\{x \in \mathbb{R}^n: |Af(x)| > \lambda\}} \omega_1(x)dx = u_1(+\infty) \int_{\{x \in \mathbb{R}^n: |Af(x)| > \lambda\}} \phi(x)dx \\
+ \int_{\{x \in \mathbb{R}^n: |Af(x)| > \lambda\}} \left( \int_{|x|}^{+\infty} \psi(\tau)d\tau \right) \phi(x)dx \\
= I_1 + I_2.
\]
If \( u_1(+\infty) = 0 \), then \( I_1 = 0 \). If \( u_1(+\infty) \neq 0 \), by the weak \( L_1 \)
boundedness of \( A, \phi \in A_1(\mathbb{R}^n) \) thanks to Lemma 4

\[
J_1 \leq \frac{1}{\lambda} \| A \| \| u_1(+) \| \int_{\mathbb{R}^n} |f(x)| \phi(x) dx \\
\leq \frac{1}{\lambda} \| A \| \int_{\mathbb{R}^n} |f(x)| u_1(|x|) \phi(x) dx \\
\leq \frac{b}{\lambda} \| A \| \int_{\mathbb{R}^n} |f(x)| \omega(|x|) dx.
\]

After changing the order of integration in \( J_2 \) we have

\[
J_2 = \int_0^\infty \psi(t) \left( \int_{\{|x| < t\}} \chi \{ x : |Af(x)| > \lambda \} \phi(x) dx \right) dt \\
\leq \int_0^\infty \psi(t) \left( \int_{\{|x| < 2t\}} \chi \{ x : |Af(\chi_{\{|y| > t/2\}}(x))| > \lambda \} \phi(x) dx \\
+ \int_{\{|x| < t\}} \chi \{ x : |Af(\chi_{\{|y| \leq 2t\}}(x))| > \lambda \} \phi(x) dx \right) dt \\
= I_{21} + I_{22}.
\]

Using the weak \( L_1 \) boundedness of \( A \) and Lemma 4 we obtain

\[
I_{21} \leq \| A \| \int_0^\infty \psi(t) \left( \int_{\{|x| < 2t\}} |f(x)| \phi(x) dx \right) dt \\
= \| A \| \int_{\mathbb{R}^n} |f(x)| \phi(x) \left( \int_{\{|x|/2\}}^\infty \psi(t) dt \right) dx \\
\leq \| A \| \int_{\mathbb{R}^n} |f(x)| u_1(|x|/2) \phi(x) dx \\
\leq b \| A \| \int_{\mathbb{R}^n} |f(x)| u(|x|) \phi(x) dx \\
= b \| A \| \int_{\mathbb{R}^n} |f(x)| \omega(x) dx.
\]
Let us estimate $J_{22}$. For $|x| < t$ and $|y| \geq 2t$ we have $|y|/2 \leq |x - y| \leq 3|y|/2$, and so

$$I_{22} \leq c_8 \int_0^\infty \psi(t) \left( \int_{|x| < t} \chi \left\{ y : \int_{|y| \geq 2t} |f(y)| |x - y|^{-n} dy > \lambda \right\} \phi(x) dx \right) dt$$

$$\leq c_9 \int_0^\infty \psi(t) \left( \int_{|x| < t} \phi(x) dx \right) \left( \int_{|y| \geq 2t} |f(y)| |y|^{-n} dy \right) dt$$

$$\leq \frac{c_9}{\lambda} \int_0^\infty \psi(t) \left( \int_{|x| < t} \phi(x) dx \right) \left( \int_{|y| \geq 2t} |f(y)| |y|^{-n} dy \right) dt.$$

The Hardy inequality

$$\int_0^\infty \psi(t) \left( \int_{|y| \geq 2t} |y|^{-n} |f(y)| dy \right) dt \leq C \int_{\mathbb{R}^n} |f(y)| \omega(|y|) dy$$

for $p = 1$ is characterized by the condition $C \leq c' B'_1$ (see [4], [14]), where

$$B'_1 = \sup_{r > 0} \left( \int_0^r \left( \int_{|x| < t} \phi(x) dx \right) \psi(t) dt \right) \ess \sup_{|x| > 2r} \frac{1}{\omega(x)}$$

$$= \sup_{r > 0} \left( \int_{|x| < r} \phi(x) \left( \int_{|x| < r} \psi(t) dt \right) dx \right) \ess \sup_{|x| > 2r} \frac{1}{\omega(x)}$$

$$\leq \sup_{r > 0} \left( \int_{|x| < r} \phi(r) \omega_1(|x|) dx \right) \ess \sup_{|x| > 2r} \frac{1}{\omega(x)}$$

$$= \sup_{r > 0} \left( \int_{|x| < r} \omega_1(|x|) dx \right) \ess \sup_{|x| > 2r} \frac{1}{\omega(x)} < \infty.$$ 

Condition (c') of the theorem guarantees that $B' \leq \mathcal{B} < \infty$. Hence, applying the Hardy inequality, we obtain

$$I_{22} \leq \frac{C_{13}}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(|x|) dx.$$ 

Combining the estimates of $I_1$ and $I_2$, we get (12) for $\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(t) dt$. By Fatou’s theorem on passing to the limit under the Lebesgue integral sign, this implies (12). The theorem is proved. \hfill \Box

Analogously proved the following theorem.

**Theorem 8.** Suppose that $1 < p < \infty$, the kernel $K$ of the convolution operator (5) satisfies the conditions (K1)–(K4) and $\varphi \in A_p(\mathbb{R}^n)$. Suppose
that $u, u_1$ are positive decreasing functions on $(0, \infty)$, $\omega(x) = u(|x|)\varphi(x)$, $\omega_1(x) = u_1(|x|)\varphi(x)$ and $B_p(\omega, \omega_1) < \infty$. Then inequality (7) is valid.

**Remark 2.** Note that for the case in which $u = u_1 = 1$, Theorem 3 was proved in [20] by using different methods. Further, in the case $1 < p < \infty$ Theorems 6 and 8 was proved in [3].

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Department of Mathematical Analysis
Baku State University
Institute of Mathematics and Mechanics, Baku
Azerbaijan
(E-mail: vagif@guliyev.com)

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