Wormholes in spacetime with torsion

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Analytical wormhole solutions in $U_4$ theory are presented. It is discussed whether the extremely short range repulsive forces, related to the spin angular momentum of matter, could be the “carrier” of the exoticity that threads the wormhole throat.

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I. INTRODUCTION

Wormhole physics has crept back into the literature since the analysis of classical traversable wormholes performed by Morris and Thorne [1]. Intuitively speaking such wormholes are tunnels linking widely separated regions of spacetime from where in-going causal curves can pass through and become out-going on the other side. The most striking feature of this unconventional arena is the possibility of constructing time machines. It has been shown that generic relative motions of the two wormhole’s mouths, or equivalently generic gravitational redshifts at the mouths due to external gravitational fields might produced closed timelike curves [2]. A fundamental difficulty associated with traversable wormholes, even in the absence of closed timelike curves, is the requirement of gravitational sources that violate the weak energy condition (WEC) in order to support them. Geometrically this could be understood as a breakdown of Hawking-Penrose singularity theorems [3]. Instead of becoming singular the timelike or null geodesic congruences diverge at the wormhole throat. Attempts to get around the violation of WEC for static wormholes in General Relativity (GR) have led some to look at wormholes in non-standard gravity theories, such as $R + R^2$ theories [4], Moffat’s non-symmetric theory [5], Einstein-Gauss-Bonnet theory [6], and Brans-Dicke (BD) theory [7]. Nevertheless, for static wormholes, the violation of WEC is a necessary consequence, no matter how many extra degrees of freedom one wishes to endow upon the theory. Thus, in the case of static BD wormholes, the BD scalar ends up as the “carrier” of exoticity. The bottom line is that WEC is violated for all static wormholes [8]. Within dynamical wormholes, it was shown that multiplying a static spherically symmetric line element by an overall time dependent conformal factor it is possible (depending on the explicit form chosen for the conformal factor) to either postpone the violation of WEC, to relegate it to the past, or else to restrict its violation to short intervals of time [9,10]. The bizarre gravitational interactions of Einstein-Cartan (EC) theory have proved to be useful in the prevention of singularities [11]. Thus, these interactions emerge as the most attractive mechanism to obtain the “flash” of “exotic” matter required to thread an evolving wormhole geometry. The present letter reports our findings in regard to spherically symmetric classical wormhole solutions in $U_4$ theory.

II. EVOLVING WORMHOLES ENDOVED WITH TORSION

The differentiable spacetime manifold of EC theory has a non-symmetric affine connection [1]. Its antisymmetric part, the torsion tensor $S_{\mu \nu \eta} \equiv \Gamma^{\alpha}_{\mu \nu \eta}$, is linked to the spin angular momentum of matter $\tau_{\mu \nu \eta}$. The field equations of $U_4$ theory are found to be [12],

\begin{align*}
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R &= 8 \pi \Sigma_{\mu \nu} \\
S_{\mu \nu \eta} + \delta^\eta_\mu S_{\nu \alpha} - \delta^\eta_\nu S_{\mu \alpha} &= 8 \pi \tau_{\mu \nu \eta}
\end{align*}

where $\delta^\eta_\mu$ is the Kronecker delta, and $g_{\mu \nu}$ the metric tensor with signature -2. $R_{\mu \nu} = R_{\eta \mu \nu}^{\eta}$, with $R_{\mu \nu \eta}^{\alpha}$ the curvature tensor of the Riemann-Cartan connection $\Gamma^{\alpha}_{\mu \nu} = \left\{\eta_{\mu \nu}\right\} + S_{\mu \nu} - S_{\nu}^\eta_{\mu} + S_{\mu \eta}^{\alpha}$, being $\left\{\eta_{\mu \nu}\right\}$ the Christoffel symbol of the metric. $\Sigma_{\mu \nu}$ is the stress-energy tensor.
where $\rho$ is the energy density, $\Pi = 1/3(p_r + 2p_\perp)$, the isotropic pressure, and $q$ is the energy flux. The anisotropic pressure, $\Delta \Pi$, is the difference between the local radial and lateral stresses $p_r, p_\perp$. $\mathbf{u}^\mu$ is the four velocity and $e^\mu$ is a space-like unit vector in the radial direction. If one substitutes Eq. (2) in Eq. (1), after a bit of algebra one arrives at the combined field equation,

$$R^{\mu\nu}(\{} - \frac{1}{2} g^{\mu\nu} R_\eta^{\eta}(\{} ) = 8\pi \tilde{\sigma}^{\mu\nu}$$

with

$$\tilde{\sigma}^{\mu\nu} = \sigma^{\mu\nu} + 8\pi \left[ -4\tau^{\mu\nu}_{\eta} \xi_{\eta\alpha} - 2\tau^{\eta\alpha}_{\eta\eta} + \tau^{\eta\alpha}_{\eta\mu} \right] + 1/2 g^{\mu\nu} ( 4 \tau^{\eta}_{\eta} \xi_{\eta\alpha} + \tau^{\xi}_{\eta\alpha} \xi_{\eta\gamma} )$$

\{} means that the quantities have been computed for the Riemannian part, $\{ \eta_{\mu\nu} \}$, of the affine connection and are the same as in GR. $\sigma^{\mu\nu}$ is the metric energy momentum tensor,

$$\sigma^{\mu\nu} = \Sigma^{\mu\nu} - \nabla^\lambda (\tau^{\mu\nu\lambda} - \tau^{\mu\lambda\nu} + \tau^{\lambda\mu\nu})$$

with $\nabla^\lambda = \nabla + 2 S_{\lambda \rho}^\rho$, being $\nabla^\lambda$ the covariant derivative with respect to the affine connection $\Gamma^{\mu}_{\lambda \rho}$. The spin tensor is given by,

$$\tau_{\mu\nu}^{\eta} = s_{\mu\nu} u^\eta$$

with the constraint,

$$s_{\mu\nu} u^\nu = 0$$

where $s_{\mu\nu}$ is the spin density. The quantities $\sigma^{\mu\nu}$ and $\tau_{\mu\nu}^{\lambda}$ are microscopically fluctuating, thus, one has to compute an average of Eq. (3) in order to obtain an equation for bulk matter. In taking the average of a spherically symmetric, microscopically averaged quantities. The same as in GR.

Let the spacetime metric to assume the diagonal form,

$$ds^2 = \Omega(t) \{ e^{2\Phi(r)} \pm dt^2 - e^{2\Lambda(r)} \pm dr^2 - r^2 d\Omega_2^2 \}$$

with $\Omega(t)$ a conformal factor finite and positive defined throughout the domain of $t$, $\Phi$ the redshift function, $\Lambda$ the shape-like function, and $d\Omega_2^2$, the $S^2$ line element. The signs $+$, $-$, are used to represent quantities in the upper and lower universes respectively.

In the spirit of (13), we make the Ansatz $\Phi = -\alpha/r$, where $\alpha$ is a positive constant to be determined. This choice guarantees that the redshift function $\Phi$ is finite everywhere, and consequently there are no event horizons. The application of the field equations (4) leads to the following expressions,

$$\frac{3 \dot{\Omega}^2 e^{2\alpha/r}}{4 \Omega^3} + \frac{2 \dot{\alpha} e^{-2\alpha/r}}{r^3 \Omega} - \frac{1}{r^2 \Omega} - \frac{e^{-2\Lambda}}{r^2 \Omega} = 8\pi \rho - 16\pi^2 s^2 = 8\pi \tilde{\rho}$$

(10)

$$\frac{3 \dot{\Omega}^2 e^{2\alpha/r}}{4 \Omega^3} + \frac{2 \dot{\alpha} e^{-2\alpha/r}}{r^3 \Omega} - \frac{\dot{\Omega} e^{2\alpha/r}}{\Omega^2} - \frac{1 - \alpha e^{-2\alpha/r}}{r^2 \Omega} = 8\pi p_r - 16\pi^2 s^2 = 8\pi \tilde{p}_r$$

(11)

$$\frac{3 \dot{\Omega}^2 e^{2\alpha/r}}{4 \Omega^3} - \frac{\dot{\Omega} e^{2\alpha/r}}{\Omega^2} - \frac{e^{-2\Lambda} \alpha}{r^3 \Omega} + \frac{\dot{\alpha}^2 e^{-2\alpha/r}}{r^4 \Omega} - \frac{\alpha \dot{\alpha} e^{-2\alpha/r}}{r^2 \Omega} = 8\pi p_\perp - 16\pi^2 s^2 = 8\pi \tilde{p}_\perp$$

(12)

$$\frac{\dot{\Omega} \alpha}{r^2 \Omega^2} = 8\pi q e^{\Lambda} e^{\alpha/r}$$

(13)
with $s^2 = 2g_{\mu\nu}s^{\mu\nu}$ the square of the spin density. In the expressions given above, dashes denote derivatives with respect to $r$ while dots, derivatives with respect to $t$. Following the procedure presented in [10], we shall look for analytical wormhole solutions independent of the conformal factor $\Omega(t)$. In order to do so, let us impose the constraint, 

$$\rho - \Pi = \frac{\Upsilon(t) e^{2\alpha/r}}{\Omega(t)}$$

(14)

or equivalently,

$$\frac{\bar{\Omega} e^{2\alpha/r}}{\Omega^2} + \frac{e^{-2\Lambda}}{3\Omega} \left[ \Lambda' \left( \frac{4}{r} + \frac{\alpha}{r^2} \right) + \frac{2e^{2\Lambda}}{r^2} - \frac{2}{r^4} - \frac{\alpha^2}{r^4} \right] = \Upsilon(t) \frac{e^{2\alpha/r}}{\Omega(t)}.$$  

(15)

Actually, the definition of $\Upsilon$ is needed so as to obtain the complete behavior of the equation of state. With this degree of freedom, it is straightforward to obtain the desired temporal evolution of the metric [10]. Therefore, in what follows, we shall analyze the stationary part of Eq. (15). After separating variables, the radial part of equation (15) can be rewritten in the compact form,

$$\frac{2e^{2\Lambda}}{r^2} + \Lambda' \left( \frac{4}{r} + \frac{\alpha}{r^2} \right) = \left( \frac{2}{r^2} + \frac{\alpha^2}{r^4} \right).$$  

(16)

The change of variables $\Lambda = \ln \chi$ transforms Eq. (16) into a Bernoulli equation, which integrates straightforwardly to,

$$e^{-2\Lambda} = \frac{c_6 \zeta^2 + c_7 \zeta^3 + c_8 \zeta^4 + c_9 \zeta^5 + c_{10} \zeta^6 + c_{11} \zeta^7 + c_{12} \zeta^8 + c_{13} \zeta^9 + e^{2\zeta^2} K}{(4\zeta + 1)^9}$$

(17)

with $\zeta = r/\alpha$, $K$ an integration constant, and the coefficients $c_i$ are listed in Table I. Since the mouths of the wormhole, for a fixed value of $t$, must connect two asymptotically flat spacetimes the geometry at the wormhole’s throat is severely constrained. The so-called “flaring out” condition asserts that the inverse of the embedding function $r(z)$, must satisfy $d^2r/dz^2 > 0$ at or near the throat [1]. Stated mathematically,

$$-\frac{\Lambda' e^{-2\Lambda}}{(1 - e^{-2\Lambda})^2} > 0.$$  

(18)

Moreover, the precise definition of the wormhole’s throat (wormhole’s minimum radius) entails a vertical slope of the embedding surface,

$$\lim_{r \to r_{th}^-} \frac{dz}{dr} = \lim_{r \to r_{th}^+} \pm \sqrt{e^{2\Lambda} - 1} = \infty.$$  

(19)

Equations (18) and (19) will be satisfied if and only if

$$\lim_{r \to r_{th}^+} e^{-2\Lambda} = 0^+,$$  

(20)

thus, in order to fix the constant $K$, we must select a value for the dimensionless radius ($\zeta_{th} > 0$) such that Eq. (20) is satisfied. As an example, let us impose $\zeta_{th} = 1$, which sets $K = -798517.5 e^{-2}$ and $r_{th} = \alpha$. It is easily seen that, as $\zeta \to \infty$, $e^{2\Lambda \zeta} \to 1$. In this way the nonmonotonic coordinate $\zeta$ (dimensionless radius) decreases from $+\infty$ to a minimum value $\zeta_{th}$, representing the location of the throat of the wormhole and then increases from $\zeta_{th}$ to $+\infty$. Because of the ill behavior of the radial coordinate $r$ near the throat, the spatial geometry is better studied by introducing the proper radial coordinate, $dl = ± e^{\Lambda} dr$, which is well behaved throughout the spacetime ranging from $-\infty$ to $+\infty$ with $l = 0$ at the throat. In this way, if $l \to +\infty$, $dz/dr \to 0$, i.e. far from the throat in both radial directions the space become asymptotically flat, while if $l \to 0$, $dz/dr \to \infty$, yielding a vertical slope of the embedding surface. The aforementioned properties of $\Lambda$, together with the definition of $\Phi$, bear out that the metric tensor describes two asymptotically flat spacetimes joined by a throat for each fixed value of $t$.

Concerning the status of WEC’s violations, extensions of Hawking-Penrose singularity theorems for EC theory have been already performed [14]. The inequality,

$$\left( \tilde{\sigma}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \tilde{\sigma}_{\eta} \eta \right) \xi_{\mu} \xi_{\nu} \geq 0$$

(21)
that must hold for all timelike unit vectors $\xi^\mu$, generalizes the strong energy condition in EC theory. The corresponding generalization for WEC is given by the inequality,
\[
\tilde{\sigma}^{\mu\nu} \xi_\mu \xi_\nu \geq 0
\]
that must hold for all timelike unit vectors $\xi^\mu$. It is easily seen following the procedure sketched in [10] that the dynamics of the geometry lets one move the energy condition violating region around in time leading to a temporary suspension of the need for the WEC violation. This hardly solves the problem since struts of “exotic” material are still required at least in an infinitesimal time interval. It is possible to address this problem in $U_4$ theory with a distribution of matter with square spin effects overwhelming the mass terms. Hehl et al [12] have estimated the required critical density $\rho_c$ for such a situation. In the case of a spin fluid of neutrons with isotropic pressure, $\rho_c \approx 10^{54}$ g cm$^{-3}$, which is orders of magnitude larger than the density at the center of the most massive neutron stars. In the case of electrons, the estimated value is $\rho_c \approx 10^{47}$ g cm$^{-3}$.

III. OUTLOOK

It was shown that EC theory admits evolving wormhole geometries which could hide a “flash” of exoticity in the torsion tensor. In spite of the very restrictive conditions upon the matter density, the very early universe might provide a fruitful scenario for wormholes to rise. Unfortunately, the functional forms of the conformal factor $\Omega(t)$ associated with realistic descriptions of the expansion of the universe cannot be used for the enlargement of these wormholes (see the discussion about spherically symmetric time-dependent wormholes of ref. [13]).

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| $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ | $c_7$ | $c_8$ | $c_9$ |
|------|------|------|------|------|------|------|------|
| 2    | 70   | 1071 | 9310 | 49805| 164493| 311622.5| 262144|

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