Gradient flows and instantons at a Lifshitz point

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Abstract. I provide a broad framework to embed gradient flow equations in non–relativistic field theory models that exhibit anisotropic scaling. The prime example is the heat equation arising from a Lifshitz scalar field theory; other examples include the Allen–Cahn equation that models the evolution of phase boundaries. Then, I review recent results reported in arXiv:1002.0062 describing instantons of Horava–Lifshitz gravity as eternal solutions of certain geometric flow equations on 3–manifolds. These instanton solutions are in general chiral when the anisotropic scaling exponent is $z = 3$. Some general connections with the Onsager–Machlup theory of non–equilibrium processes are also briefly discussed in this context. Thus, theories of Lifshitz type in $d + 1$ dimensions can be used as off–shell toy models for dynamical vacuum selection of relativistic field theories in $d$ dimensions.

1. Preliminaries: Lifshitz theories

Let us consider a point particle system with configuration space $Q$ and a system of local coordinates $q_I$ that describe its physical degrees of freedom. We further assume that $Q$ is endowed with a metric $O^{IJ}$ which is symmetric and non–degenerate so that the inverse metric $O_{IJ}$ defined through $O^{IK}O_{KJ} = \delta^I_J$ exists everywhere. The action comprises of two terms taking the difference of the kinetic and potential energy

$$S = \frac{1}{2} \int dt \sum_{I,J} \left( \frac{dq_I}{dt} O^{IJ} \frac{dq_J}{dt} - \frac{\partial W}{\partial q_I} O_{IJ} \frac{\partial W}{\partial q_J} \right),$$

(1)

where we also assume that the potential is derived from a superpotential $W$ that is a function on $Q$. This particular choice of potential is synonymous to having detailed balance condition in the system. Here, we will only consider systems with positive definite metric $O^{IJ}$ so that the inverse metric is manifestly positive definite and its minima coincide with the critical points of $W$, i.e., $\partial W/\partial q_I = 0, \forall I$. Then, the ground states of the system describe a particle sitting still at the minima of the potential and there can be more than one degenerate vacua depending upon the choice of $W$. It is often interesting to consider systems with configuration space having indefinite metric $O^{IJ}$, although some of the nice properties that are discussed in the following will be generally missing.

When the dimension of the configuration space $Q$ is finite, we have an ordinary point particle system, which is not particularly interesting by itself but it is a useful guide for some of the constructions described below. Here, we focus entirely on infinite dimensional systems determined by the action (1) and consider only those cases that $Q$ is the configuration space of a relativistic field theory in $d$ Euclidean space–time dimensions. The superpotential $W$ is
a functional over the field space, which for all practical purposes is taken to be the classical action of the relativistic field theory in \( d \) dimensions. This construction may look odd at first sight, as the corresponding action (1) describes a non–relativistic field theory in \( d + 1 \) space–time dimensions with \( t \) being the physical time coordinate, but, in fact, it provides our main framework. The resulting field theories are called models of Lifshitz type, [1], and they exhibit anisotropic scaling (typically it is \( z = 2 \) for theories with \( W \) that contains up to two derivative terms, but there can be more general examples as will be seen later).

The prime example of a Lifshitz theory in \( d + 1 \) space–time dimensions is provided by taking \( Q \) to be the configuration space of a relativistic free field theory in \( d \) Euclidean dimensions. For this it is appropriate to set \( q_I(t) = \varphi(t, x), q_J(t) = \varphi(t, x'), \mathcal{O}^{IJ} = \delta(x - x') = \mathcal{O}_{IJ} \) and choose

\[
W[\varphi] = \frac{1}{2} \int d^d x (\nabla \varphi)^2
\]

so that \( \frac{\partial W}{\partial q_I} = \delta W/\delta \varphi(x) = -\nabla^2 \varphi(x) \). Then, by integration over \( x' \) (equivalently, summation over \( J \)), the point particle action (1) takes the form

\[
S = \frac{1}{2} \int dt \, dt' d^d x [\left( \partial_t \varphi \right)^2 - (\nabla^2 \varphi)^2]
\]

and describes a non-relativistic field theory with anisotropic scaling \( z = 2 \) known as Lifshitz scalar theory. The resulting equations of motion exhibit scale invariance under the transformation \( x \rightarrow ax, t \rightarrow a^z t \) with exponent \( z = 2 \); it should be contrasted to the relativistic scalar free field theory in \( d+1 \) space–time dimensions whose Lagrangian density is \( (\partial_t \varphi)^2 - (\nabla \varphi)^2 \) and the corresponding scaling exponent is \( z = 1 \).

A simple variant of the model is obtained by considering the action of a self–interacting relativistic scalar field in \( d \) Euclidean dimensions as superpotential functional,

\[
W[\varphi] = \int d^d x \left[ \frac{1}{2} (\nabla \varphi)^2 + V(\varphi) \right],
\]

in which case the corresponding Lifshitz theory in \( d + 1 \) space–time dimensions takes the form

\[
S = \frac{1}{2} \int dt \, dt' d^d x \left[ \left( \partial_t \varphi \right)^2 - (\nabla^2 \varphi - V'(\varphi))^2 \right],
\]

where \( V'(\varphi) = \partial V/\partial \varphi \). This particular field theory will be discussed later in more detail as it connects with some interesting mathematical problems of current interest. One may also consider more complicated examples using multi–component scalar field models.

Another characteristic class of Lifshitz models is provided by vector field theories with anisotropic scaling in space and time. As before, it is appropriate to use the action (1) as starting point and identify \( q_I \) with gauge fields \( A_i(x) \) that live in \( d \)-dimensional Euclidean space, letting \( i = 1, 2, \ldots, d \). The metric \( \mathcal{O}^{IJ} \) is provided by the standard Riemannian metric \( \text{Tr}(T_a T_b) \delta(x - x') \) in the space of all \( d \)-dimensional gauge field configurations \( A_i = A^{a}_i T_a \) with values in a Lie algebra \( [T_a, T_b] = i f_{abc} T_c \). Then, it is natural to define a Lifshitz gauge field theory in terms of the action, [2],

\[
S = \frac{1}{2} \int dt \, dt' d^d x \left[ \text{Tr}(E_i E^i) - \frac{1}{g^2} \text{Tr} \left( (\partial_i F^{ik}) (\partial_j F_{jk}) \right) \right],
\]

where

\[
E_i = \partial_t A_i, \quad F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]
\]
provide the electric and magnetic fields in $d + 1$ space–time dimensions, respectively, in the axial gauge $A_0(t, x) = 0$. This action follows from (1) using the superpotential functional

$$ W = \frac{1}{4g^2} \int d^d x \ Tr(F_{ij} F^{ij}) $$

(8)

associated to relativistic Yang–Mills theory in $d$ Euclidean dimensions, and, therefore, the resulting Lifshitz vector theory exhibits anisotropic scaling with exponent $z = 2$.

Finally, we consider Lifshitz theories of geometric type associated to tensors of rank 2, which were introduced in the literature as alternative theories of gravity and they became known as Hořava–Lifshitz gravities, [3, 4]. They are defined in arbitrary dimensions by assuming that space–time is of the form $M_{d+1} = R \times \Sigma_d$ and the metric admits the ADM (Arnowitt–Deser–Misner) decomposition, [5]

$$ ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). $$

(9)

The metric on the spatial slices $\Sigma_d$ is $g_{ij}$, whereas $N$ and $N^i$ are the lapse and shift functions, respectively, which depend on all space–time coordinates, in general. The infinite dimensional space of all Riemannian metrics $g_{ij}$ is called superspace and it is endowed with a metric

$$ G^{ijk\ell} = \frac{1}{2} \left( g^{ik} g^{j\ell} + g^{i\ell} g^{jk} \right) - \lambda g^{ij} g^{k\ell} $$

(10)

that generalizes the standard DeWitt metric, [6], using an arbitrary parameter $\lambda$ (other than 1). Here, for simplicity, we are suppressing the delta functions that are needed to integrate over space. The inverse metric is

$$ G_{ij} = \frac{1}{2} (g_{ik} g_{j\ell} + g_{i\ell} g_{jk}) - \frac{\lambda}{d\lambda - 1} g_{ij} g_{k\ell} $$

(11)

so that

$$ G^{ijk\ell} G_{k\ell mn} = \frac{1}{2} (\delta^i_m \delta^j_n + \delta^i_n \delta^j_m). $$

(12)

The metric in superspace is positive definite provided that $\lambda < 1/d$, but otherwise it is arbitrary. The action of Hořava–Lifshitz gravity in $d + 1$ dimensions is written as a sum of kinetic and potential terms. By identifying $q_I$ with $g_{ij}$ and $O^{IJ}$ with $G^{ijk\ell}$, the action (1) satisfying detailed balance takes the form, [3, 4],

$$ S = \frac{2}{\kappa^2} \int dt d^d x \sqrt{g} \ N \ K_{ij} G^{ijk\ell} K_{k\ell} - \frac{\kappa^2}{2} \int dt d^d x \sqrt{g} \ N \ E^{ij} G_{ij} E^{k\ell}, $$

(13)

where $K_{ij}$ is the second fundamental form measuring the extrinsic curvature of the spatial slices $\Sigma_d$ at constant $t$ (playing the role of momentum conjugate to the metric $g_{ij}$),

$$ K_{ij} = \frac{1}{2N} \left( \partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i \right) $$

(14)

and

$$ E^{ij} = -\frac{1}{2\sqrt{g}} \frac{\delta W[g]}{\delta g_{ij}}. $$

(15)

The gravitational coupling of the theory in $d + 1$ dimensions is $\kappa$.

Note that the kinetic term contains two time derivatives of the metric $g_{ij}$, and, as such, it is identical to ordinary general relativity in canonical form (though $\lambda$ is taken arbitrary here).
The potential term is different, however, as it is derived from a superpotential functional $W\{g\}$, which is typically chosen to be the Euclidean action of a relativistic gravitational theory in $d$ dimensions. If $W$ is the Einstein–Hilbert action, the resulting Hořava–Lifshitz theory will have anisotropic scaling $z = 2$, but if $W$ is the action of a covariant higher derivative gravitational theory in $d$ dimensions the scaling exponent will be $z > 2$. Examples of this kind will be discussed later. Finally, note that the action (13) is not generally covariant, since by construction it is only invariant under the restricted set of foliation preserving diffeomorphisms of the space–time $R \times \Sigma_d$. In the following, we restrict attention to the so called projectable version of Hořava-Lifshitz theory, meaning that the lapse function $N$ associated to the freedom of time reparametrization is restricted to be a function of $t$, whereas the shift functions $N_i$ associated to diffeomorphisms of $\Sigma_d$ can depend on all space-time coordinates. Also, in view of the applications that will be discussed next, we choose

$$N(t) = 1, \quad N_i(t, x) = 0$$

without great loss of generality. We will indicate later how the lapse and shift functions can be reinstated into the equations of motion.

2. Gradient flows as Euclidean solutions

Let us now consider the Euclidean form of the action (1) obtained by Wick rotation $t \rightarrow it$ and construct solutions that are applicable to all Lifshitz type theories. We have

$$S_{\text{Eucl}} = \frac{1}{2} \int dt \sum_{I, J} \left( \frac{dq_I}{dt} \partial_{IJ} \frac{dq_J}{dt} + \partial W \partial_{IJ} \frac{dq_I}{dq_J} \right),$$

(17)

which can be alternatively written as follows

$$S_{\text{Eucl}} = \frac{1}{2} \int dt \left( \frac{dq_I}{dt} \partial_{JK} \frac{\partial W}{\partial q_K} \right) O^{IJ} \left( \frac{dq_J}{dt} + O_{JL} \frac{\partial W}{\partial q_L} \right) \pm \int dt \frac{dq_I}{dt} \frac{\partial W}{\partial q_I},$$

(18)

by completing the square. Here, summation is implicitly assumed over repeated indices in order to simplify the presentation. For now and later use we also consider the action

$$S'_{\text{Eucl}} = \frac{1}{2} \int dt \left( \frac{dq_I}{dt} \partial_{JK} \frac{\partial W}{\partial q_K} \right) O^{IJ} \left( \frac{dq_J}{dt} + O_{JL} \frac{\partial W}{\partial q_L} \right),$$

(19)

which differs from (18) by boundary terms as

$$S_{\text{Eucl}} = S'_{\text{Eucl}} \pm \int dt \frac{dW}{dt}.$$

(20)

Since we are only considering positive definite metrics $O^{IJ}$, the action $S'_{\text{Eucl}}$ is bounded from below by zero. Thus, minima of the action are provided by special configurations that satisfy the system of first order equations

$$\frac{dq_I}{dt} = \pm O_{IJ} \frac{\partial W}{\partial q_I}.$$

(21)

These are gradient flow equations for the variables $q_I(t)$ derived from the superpotential $W\{q\}$. They also yield solutions of the equations of motion following from the action $S_{\text{Eucl}}$, since the difference is a total derivative term (20) that certainly cannot affect the classical equations of motion. However, this boundary term is needed to make the variational problem well-posed and
will be treated more carefully in the next section for, otherwise, the action $S_{\text{Eucl.}}$ may become infinite for solutions that become singular at a finite instant of time. In any case, the fixed points of the flow equations (21) represent static solutions, where the point particle is sitting still at the bottom of the potential. On the other hand, time dependent solutions of the flow equations are more interesting to consider in general, but they are complicated; we will present some examples that are worth studying in detail. It should also be noted that the superpotential $W$ changes monotonically along such flow lines when $O_{IJ}$ is positive definite, since

$$\frac{dW}{dt} = \frac{dq_I}{dt} \frac{\partial W}{\partial q_I} = \pm \frac{\partial W}{\partial q_I} O_{IJ} \frac{\partial W}{\partial q_J},$$

(22)

and, therefore, it provides an entropy functional for the corresponding evolution.

The simplest (but very instructive) example in the class of Lifshitz field theories is provided by the scalar field model (3) in the Euclidean domain. The corresponding gradient flow is linear and coincides with the forward or backward heat equation

$$\frac{\partial \varphi}{\partial t} = \pm \nabla^2 \varphi,$$

(23)

depending on the choice of sign. Then, solutions of the heat equation in $d$ dimensions are solutions of the Lifshitz scalar free field theory in $d + 1$ Euclidean dimensions. Note, however, that the fundamental solution of the heat equation, which is a Gaussian function with time dependent height $1/t^{d/2}$ and width $t^{1/2}$, cannot exist for ever since $\varphi(t, x)$ becomes singular at some finite instant of time, say $t = 0$. This simple example illustrates the difference between finite action solutions following from $S_{\text{Eucl.}}$ and $S'_{\text{Eucl.}}$, where the boundary terms (20) can in fact play important role. If we were only concerned with solutions of the flow equations that existed for a short time, the difference would have been irrelevant. The singularities are also preventing to uplift these solutions to Lifshitz theories that exist for all time.

These remarks are not only tailored for the Lifshitz scalar free field theory, but, in fact, they apply to all Lifshitz models. They also motivate the construction of instanton solutions of Lifshitz theories that will be discussed in the next section.

The next more complicated example is provided by a self–interacting Lifshitz scalar field as described by (5). According to the general framework, Euclidean solutions are provided by the non–linear flow equation

$$\frac{\partial \varphi}{\partial t} = \pm \left( \nabla^2 \varphi - V'(\varphi) \right),$$

(24)

which is called Allen–Cahn equation, [7]. This equation arose first in the literature as phenomenological model for the motion of phase boundaries by surface tension, but it was subsequently studied in mathematics very extensively. In this context, which is deeply connected to motion by mean curvature flow and the problem of minimal surfaces, [8, 9], it seems more appropriate to consider the closely related equation

$$\frac{\partial \varphi^\epsilon}{\partial t} = \pm \left( \nabla^2 \varphi^\epsilon - \frac{1}{\epsilon^2} V'(\varphi^\epsilon) \right),$$

(25)

which follows from above by implementing the anisotropic scaling transformation of the Lifshitz theory, $x \rightarrow x/\epsilon$ and $t \rightarrow t/\epsilon^2$, and investigate this in the limit of small $\epsilon$ when the potential has two equal wells, choosing, for example, $V(\varphi) = (\varphi^2 - \varphi_0^2)^2$. Solving the Allen–Cahn equation is not an easy task and it provides an active area of research in mathematics (see, for instance, [10], for some basic facts).

Euclidean solutions of gauge theories of Lifshitz type (6) are also interesting to consider as the corresponding gradient flow assumes the form

$$E_i \equiv \frac{\partial A_i}{\partial t} = \pm \frac{1}{g} \nabla_j F^j_i,$$

(26)
and coincides with the so-called Yang–Mills flow in $d$ dimensions. This is another very interesting system of equations that has been studied for some time in mathematics (e.g., [11, 12]), but many of its aspects remain open problems to this day. Of course, one may also consider further generalizations by combining the action functionals (4) and (8) into a $W$ that describes the action of a relativistic Yang–Mills–Higgs theory in $d$ Euclidean dimensions. Then, the gradient flow equations of the corresponding Lifshitz theory will be mixture of the Allen–Cahn equation and the Yang–Mills flow whose general properties remain to be studied (see, however, [13]).

Finally, we come to the Euclidean solutions of geometric Lifshitz theories, such as Hořava–Lifshitz gravity described by the action (13). In this case, the gradient flow equation becomes, [3, 4], [14],

$$K_{ij} \equiv \frac{1}{2} \frac{\partial g_{ij}}{\partial t} = \pm \frac{\kappa^2}{2} g_{ijkl} E^{k\ell}$$ (27)

and gives rise to geometric flows for the metric $g_{ij}$ on $\Sigma_d$ that depend on the choice of $W$. They can have second or higher order derivatives in space coordinates.

More precisely, we have the following evolution equation in the general case of Hořava–Lifshitz models

$$\frac{\partial g_{ij}}{\partial t} = \pm \frac{\kappa^2}{2\sqrt{g}} g_{ijkl} \frac{\delta W}{\delta g_{k\ell}},$$ (28)

which for $W$ given by the Einstein–Hilbert action in $d > 2$ dimensions,

$$W[g] = \frac{2}{\kappa^2} \int d^d x \sqrt{g} \ R,$$ (29)

it specializes to a variant of the celebrated Ricci flow equation (see, for instance, [15])

$$\frac{\partial g_{ij}}{\partial t} = \pm \frac{\kappa^2}{\kappa^2_w} \left( R_{ij} - \frac{2\lambda - 1}{2(d\lambda - 1)} R g_{ij} \right).$$ (30)

It runs forward or backward in time depending on the choice of the overall sign and the fixed points are Ricci flat metrics, $R_{ij} = 0$, independent of $\lambda$.

More general choices of $W[g]$ can also be made, depending on $d$, in which case the exponent $z$ of anisotropic scaling can become bigger than 2. Although the corresponding flow equations are mathematically much more complicated for $z > 2$, as they involve higher derivative terms, they are physically better behaved for various reasons. The simplest choice (29) that leads to the Ricci flow shares many similarities with the heat equation in that the geometry on $\Sigma_d$ becomes singular at some finite instant of time, [15]. This is actually a generic phenomenon that can render the action $S_{\text{Eucl}}$ (but not $S'_{\text{Eucl}}$) of $z = 2$ Hořava–Lifshitz gravity infinite along those flow lines ($W$ becomes infinite at the ultra–violet fixed point of type I ancient solutions and vanishes at the singularity in $d > 2$ dimensions). More importantly, the corresponding space–time metric on $R \times \Sigma_d$, which is supposed to exist for all time, will be incomplete by the presence of singularities, thus making such solutions totally unacceptable. In the next section we will provide simple criteria that circumvent this problem in Lifshitz type theories and obtain non–singular solutions, where it is appropriate.

Concluding this section, we mention that the geometric flow equations can be modified by allowing arbitrary reparametrizations on $\Sigma_d$ generated by a vector field $\xi_i(t,x)$ that may also depend on time. Then, the evolution (27) generalizes to

$$\frac{\partial g_{ij}}{\partial t} = \pm \frac{\kappa^2}{2\sqrt{g}} g_{ijkl} \frac{\delta W}{\delta g_{k\ell}} + \nabla_i \xi_j + \nabla_j \xi_i.$$ (31)

The projectable version Hořava–Lifshitz theory with lapse and shift functions $N(t)$ and $N_i(t,x)$ accommodates nicely this modification by choosing $\xi_i = N_i/N$. Of course, $N(t)$ can always be
set equal to 1 by appropriate redefinition of time, but $N_i(t, x)$ is left arbitrary, in general, since the theory is invariant under foliation preserving diffeomorphisms of the space–time manifold $R \times \Sigma_d$. The choice (16) with $N_i = 0$ amounts to taking $\xi_i = 0$ in the flow equations.

3. Instantons at a Lifshitz point
The definition of instantons in Lifshitz theories requires careful treatment of the boundary terms arising in the Euclidean action. Returning back to it we have

$$S_{\text{Eucl.}} = \frac{1}{2} \int dt \left( \frac{dq_I}{dt} \mp O_{IK} \frac{\partial W}{\partial q_K} \right) O^{IJ} \left( \frac{dq_J}{dt} \mp O_{JL} \frac{\partial W}{\partial q_I} \right) \pm \int dt \frac{dW}{dt} ,$$

which for theories with positive definite metric $O_{IJ}$ yields immediately

$$S_{\text{Eucl.}} \geq \pm \int dt \frac{dW}{dt} .$$

The lower bound is saturated for special configurations satisfying the gradient flow equation

$$\frac{dq_I}{dt} = \pm O_{IJ} \frac{\partial W}{\partial q_I} ,$$

which thus provide extrema of the Euclidean action and hence solutions to the classical equations of motion as for $S'_{\text{Eucl.}}$ that was considered before.

Note, however, that the lower bound of the action (33) need not be finite on general grounds. All solutions of the gradient flow equations that are taken over a finite time interval ($t_1$, $t_2$) will have finite action,

$$S_{\text{Eucl.}} = |W(t)|_{t_{12}} ,$$

but this does not necessarily extend smoothly to the entire line $-\infty < t < +\infty$. What we really need is to have eternal solutions of the flow equations, which exist for all time, and interpolate smoothly between different critical points of the superpotential $W$. Then, the bound will have topological meaning and the corresponding trajectories will be bona fide instantons with finite action

$$S_{\text{instanton}} = |\Delta W| \equiv |W(t = +\infty) - W(t = -\infty)| .$$

This requirement defines the notion of instantons in Lifshitz theories, [14], and places attention to those models that have degenerate vacua. Also, to avoid unnecessary complications that may arise when considering Lifshitz field theories on $R \times \Sigma_d$, we implicitly assume that $\Sigma_d$ is compact without boundaries (e.g., $S^d$) so that no additional space–boundary terms will come into play when putting lower bounds on the Euclidean action.

Let us elaborate more on the construction. First note that the lower bound of $S_{\text{Eucl.}}$ cannot be zero, since there are no periodic trajectories in Euclidean time; if that were the case, it would have been in contradiction with the monotonicity of $W(t)$ along the gradient flow lines. Thus, the lower bound is either positive and finite or infinite. Second, it is not at all guaranteed that the gradient flow equations will admit solutions that exist for sufficiently long time, as this is not easy to prove mathematically in general; if not, the Euclidean solutions we are considering will not be viable choices. Third, even if one can prove short time existence of the solutions, it is not clear whether these will extend to solutions for all time. Typically, the flow lines terminate at some finite instant of time by developing singularities and they cannot be continued further. Investigating the formation of singularities and their properties is a formidable task in general and special mathematical tools need to be developed in each case separately depending upon $W$. However, it is reasonable to expect that such singularities, if present, will be irremovable and that $W$ can become infinite along the corresponding flow lines; the simple examples of the
well studied heat equation and Ricci flow fully support this point. Thus, the only systematic way to avoid configurations with infinite Euclidean action is to consider eternal solutions of the flow equations. These are also natural from a physical point of view, since there is no a priori reason to have Lifshitz theories that make good sense only for finite time intervals.

In conclusion, we are only considering eternal solutions of the gradient flow equations as viable first order Euclidean time solutions of Lifshitz theories. Thus, to implement this program we do not need to prove the short time existence of the flow lines, in general, nor to investigate the possible formation of singularities in finite time. These may be interesting mathematical problems in their own right, but we can live without them when considering instantons. The only mathematical problems that have to be addressed in the present context is the existence, the construction and the classification of eternal solutions of gradient flows which are derived from a $W$ with at least two critical points. There are many examples of Lifshitz theories that one can study in this context, but we will confine our attention to non–relativistic gravitational models, following [14], as there has been considerable activity in this area in recent times.

4. Instantons of Hořava–Lifshitz gravity

In this section we concentrate for definiteness to geometric flows as Euclidean solutions of Hořava–Lifshitz gravity theories and provide explicit examples of instanton configurations. We will consider models in $3 + 1$ space–time dimensions and choose the action of three–dimensional topologically massive gravity as our superpotential functional, [16],

$$W = \frac{2}{\kappa^2} \int d^3 x \sqrt{g} (R - 2\Lambda) + \frac{1}{\omega} W_{CS}, \quad (37)$$

where

$$W_{CS} = \int d^3 x \sqrt{g} \varepsilon^{ijk} \Gamma^\ell_{im} \left( \partial_j \Gamma^m_{\ell k} + \frac{2}{3} \Gamma^m_{jn} \Gamma^n_{\ell k} \right) \quad (38)$$

is written in terms of the usual Levi–Civita connection of the three–dimensional metric $g$ and $\varepsilon^{ijk}$ is the fully anti-symmetric symbol with $\varepsilon^{123} = 1$. This choice yields a non–relativistic theory of gravity in $3 + 1$ dimensions with anisotropic scaling $z = 3$. As will be seen shortly, going beyond $z = 2$ is necessary in order to be able to produce examples of instanton configurations in Hořava–Lifshitz gravity.

Let us briefly describe some of the salient features of topologically massive gravity that will be needed in the following. The first term of $W$ is the usual Einstein–Hilbert term, which is also augmented with a three–dimensional cosmological constant $\Lambda$, whereas the second term is the so called gravitational Chern–Simons action. The latter flips sign under orientation reversing transformations and, as a result, the theory of topologically massive gravity is not invariant under parity. The classical equations of motion that follow from $W$ by varying the metric read as

$$R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} + \frac{\kappa^2}{\omega} C_{ij} = 0, \quad (39)$$

where $C_{ij}$ is the Cotton tensor of the metric $g$, which is defined as follows

$$C_{ij} = \frac{\varepsilon^{k\ell}}{\sqrt{g}} \nabla_k (R_{j\ell} - \frac{1}{4} R g_{j\ell}) \quad (40)$$

and it is a traceless and covariantly conserved symmetric tensor. Thus, the classical equations of motion contain second and third order derivative terms, in general, explaining the scaling exponent $z = 3$ in the associated $(3 + 1)$–dimensional Hořava–Lifshitz gravity.
According to the general theory, Euclidean solutions of the Lifshitz theory on \( R \times \Sigma_3 \) are provided by trajectories of the corresponding gradient flow equation which takes the form, [14],

\[
\dot{g}_{ij} = \pm \frac{\kappa^2}{R_{\infty}} \left( R_{ij} - \frac{2\lambda - 1}{2(3\lambda - 1)} R g_{ij} - \frac{\Lambda}{3\lambda - 1} g_{ij} \right) \mp \frac{\kappa^2}{\omega} C_{ij}
\]  

(41)

keeping both sign options for completeness. This is called Ricci–Cotton flow for the metric \( g \) on \( \Sigma_3 \). Its fixed points are independent of the parameter \( \lambda \) and coincide with the classical solutions of topologically massive gravity given by (39). Instanton solutions exist provided that there are more than one fixed points and they correspond to trajectories that interpolate smoothly between them as \( t \) varies from \(-\infty\) to \(+\infty\). It is certainly very difficult to derive general mathematical results about the properties of the Ricci–Cotton flow because it is a third order equation and the standard tools based on maximum principle do not apply. Nevertheless, it is possible to make appropriate ansatz that reduce the flow equation into a simpler system of ordinary differential equations, which in turn can be studied in detail and yield explicit configurations.

We are led to consider locally homogeneous geometries on \( \Sigma_3 \), which for simplicity it is assumed to have the topology of \( S^3 \), and make the Bianchi IX ansatz for the three–dimensional metrics

\[
ds^2 = \gamma_1(t)\sigma_1^2 + \gamma_2(t)\sigma_2^2 + \gamma_3(t)\sigma_3^2
\]  

(42)

so that the isometry group is \( SU(2) \). Here, \( \sigma_i \) are the left–invariant one–forms of \( SU(2) \) satisfying the defining relations \( 2d\sigma_i + \varepsilon^{ijk}\sigma_j \wedge \sigma_k = 0 \). Remarkably, this class of metrics provides consistent truncation of the Ricci–Cotton flow, as for all other homogeneous model three–geometries that arise in Bianchi classification. Sticking to this mini–superspace model, we will be able to describe all instanton solutions of \( z = 3 \) Hořava–Lifshitz gravity with \( SU(2) \) isometry, following [14]. Since the instantons of Lifshitz theories correspond to eternal solutions of the gradient flow equations, their space–time interpretation is straightforward as they give rise to complete and regular metrics on \( R \times S^3 \). Here, due to space limitation, we only present in words the construction of these instantons and refer the reader to our published work for the details. However, we are permitted to give one simple example later to illustrate our general results on \( SU(2) \) gravitational instantons.

First, we need to characterize all fixed points, i.e., homogeneous vacuum solutions of three–dimensional topologically massive gravity, [17, 18, 19] (but see also [20] for an overview). It turns out that apart from the maximally symmetric vacuum with \( \gamma_1 = \gamma_2 = \gamma_3 \), which always exists for all values of parameters, there can be additional fixed points that are formed by balancing the effect of the Einstein and Cotton terms to geometry. Clearly the sign of the Chern–Simons coupling \( \omega \) plays important role in this problem as it distinguishes the case that the Einstein and Cotton tensors compete against each other from the case that they work together at the fixed points. Additional fixed points are present only in the first case and they correspond to axially symmetric metrics on \( S^3 \). The exact shape of the squashed or elongated spheres that can arise depends on the actual values of the couplings. When \( \Lambda = 0 \), it is also possible to have totally anisotropic fixed points beyond a critical value of \( \omega \). Then, the instanton solutions are configurations interpolating smoothly between these fixed points and describe eternal solutions of the Ricci–Cotton flow. Depending on the values of parameters, there are axially symmetric deformations of the metric that connect the fixed points, in which case the instantons have enhanced \( SU(2) \times U(1) \) group of isometries, or there can be more general instantons with \( SU(2) \) isometry alone, which are, however, difficult to describe in closed form. In any case, it is possible to classify all such instanton solutions, compute their action and find their moduli, as it is explained in the original work [14] when \( \lambda < 1/3 \) and \( \Lambda \geq 0 \).

An important property of these instantons is their chirality. Since the Cotton tensor is odd under parity, orientation reversing transformations on \( S^3 \) flip the sign of the coupling constant.
This has dramatic effect on the formation of fixed points since the Einstein and Cotton tensor will work together and not against each other. Thus, the instantons of $z = 3$ Hořava–Lifshitz gravity will cease to exist as they rely on the existence of more than one fixed point to support themselves, which is possible only for one sign of the coupling $\omega$ (assuming that $\kappa_w$ is finite). Chiral instantons are not common to field theories, but here we have a model provided by Lifshitz theories that make them possible. Another important point is the absence of instantons in $z = 2$ Hořava–Lifshitz gravity, which explains the need to resort to models with higher anisotropy scaling exponent. In that case, the gradient flow equation is the Ricci flow, which admits only one fixed point on $S^3$, [15], as consequence of the Poincaré conjecture. Thus, there can be no gravitational instantons of the kind we are considering here. Finally, one may also consider instantons of Hořava–Lifshitz gravity with higher scaling exponent, say $z = 4$, by choosing $W$ to be the action of three–dimensional new massive gravity (and generalizations thereof), [21]. Homogeneous vacua with $SU(2)$ symmetry have been systematically classified in this case, [22], and, therefore, gravitational instantons with $SU(2)$ isometry can be made available by considering eternal solutions of a certain fourth order geometric flow equation for Bianchi IX metrics. The details are lying beyond the scope of this presentation.

In the remaining part of this section, we illustrate the results given in [14], by presenting the simplest possible example of an instanton solution of $z = 3$ Hořava–Lifshitz gravity. We choose $W$ to be the gravitational Chern–Simons action by dropping the Einstein–Hilbert term in the limit $\kappa_w \to \infty$. Then, the underlying three–dimensional theory is the conformal rather than the topologically massive gravity, [23]. The fixed points obey $C_{ij} = 0$ and correspond to conformally flat metrics on $\Sigma^3$. The associated Lifshitz theory is pure Cotton and its Euclidean solutions are simply described by trajectories of the Cotton flow, which was also introduced in [24],

$$\partial_t g_{ij} = \mp \frac{\kappa^2}{\omega} C_{ij},$$

and it is independent of $\lambda$. Since the Cotton tensor is traceless, this deformation of the metric preserves the volume of space and the calculations simplify a lot.

Within the mini–superspace model of Bianchi IX metrics on $S^3$, we are led to consider axially symmetric configurations with fixed volume $2\pi^2 L^3$ setting, in particular,

$$\gamma_1 = \gamma_2 \equiv x_1 L^2 \frac{L^2}{4}, \quad \gamma_3 = \frac{L^2}{4x^2}.$$  \hspace{1cm} (44)

Then, the Cotton flow equation reduces consistently to a single equation for the variable $x(t)$ that parametrizes the shape of space (squashing) and reads as

$$\frac{dx}{dt} = \pm \frac{4\kappa^2}{\omega L^3} \frac{x^3 - 1}{x^5}.$$  \hspace{1cm} (45)

Clearly, there are two fixed points, one with $x = 1$ and another with $x = \infty$. They both describe conformally flat metrics on $S^3$ as $dx/dt$ vanishes there. The first one corresponds to the round metric on $S^3$ having $\gamma_1 = \gamma_2 = \gamma_3$, whereas the second one arises in the correlated limit $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = 0$ with the volume held fixed. As such it looks degenerate, since $S^3$ is completely flattened out, but it is non–singular (it can be explicitly verified that it has no curvature singularities). These are the two bona fide fixed points that can and will support an instanton.

Simple integration of equation (45) yields a branch with $x \geq 1$ that exists for all time and it is explicitly given by

$$\pm \frac{t}{\tau} = x^3 - 1 + \log(x^3 - 1),$$  \hspace{1cm} (46)
Figure 1. Plot of the potential \( C_{ij}C^{ij} \) for metrics with \( SU(2) \times U(1) \) isometry

where \( \tau = |\omega|L^3/12\kappa^2 \) is the characteristic time scale of the problem. As \( t \) ranges from \(-\infty\) to \(+\infty\), \( x(t) \) interpolates smoothly between the values 1 and \( \infty \) that describe the location of the two fixed points. As noted before, the choice of sign distinguishes instantons from anti-instantons. The instanton we have obtained in this fashion has finite Euclidean action, which turns out to be \( S_{\text{inst.}} = 4\pi^2/|\omega| \), and it has no modulus other than \( L \). It can also be seen without much effort that the solution (46) is the only instanton with \( SU(2) \) isometry (up to permutations of the three principal axes of \( S^3 \)) of the pure Cotton theory. In this case, the instanton and the anti-instanton are simply interrelated by parity transformations on \( S^3 \).

The figure above serves to illustrate the situation. It is a plot of the potential term \( V = C_{ij}C^{ij} \) of pure Cotton \( z = 3 \) Hořava–Lifshitz gravity for axially symmetric Bianchi IX metrics on \( S^3 \), which turns out to be proportional to \( (e^{-6\beta_+}-e^{-3\beta_+})^2 \). The horizontal axis is the shape modulus of the sphere, which is conveniently described here by the variable \( \beta_+ = \log x \); the notation is borrowed from mixmaster dynamics where this parametrization is widely used, [25], [26]. Thus, the two fixed points are located at the origin and at infinity and clearly they are degenerate since the potential vanishes there. There is also a small bump in between the two fixed points which is peaked at \( \beta_+ = (\log 2)/3 \), where \( \gamma_1 = \gamma_2 = 2\gamma_3 \).

The instanton interpolates between the two degenerate vacua, as in ordinary particle systems. This analogy is made exact by noting that the Euclidean theory of Hořava–Lifshitz gravity reduces in this case to the dynamics of a single mode \( \beta_+(t) \) derived from the action

\[
S_{\text{eff.}} = \frac{3\pi^2 L^3}{\kappa^2} \int dt \left[ \left( \frac{d\beta_+}{dt} \right)^2 + \frac{16\kappa^4}{\omega^2 L^8} \left( e^{-6\beta_+} - e^{-3\beta_+} \right)^2 \right].
\]

(47)

The change of variable \( \beta_+ = \log x \) is necessary to cast the kinetic term in canonical form. The instanton solutions of this effective point particle model satisfy equation (45) written in terms of the variable \( \beta_+ \). Also, for these configurations, the effective action equals \( 4\pi^2/|\omega| \), as required for consistency of the interpretation. The calculation is easily done by noting that the potential term in (47) is itself derived from a superpotential, as

\[
\left( e^{-6\beta_+} - e^{-3\beta_+} \right)^2 = \left( \frac{dW}{d\beta_+} \right)^2 ; \quad W(\beta_+) = \frac{1}{3}e^{-3\beta_+} - \frac{1}{6}e^{-6\beta_+}
\]

(48)

so that

\[
S_{\text{eff.}}^{\text{inst.}} = 2 \frac{3\pi^2 L^3}{\kappa^2} \frac{4\kappa^2}{|\omega| L^3} |W(\infty) - W(0)| = \frac{4\pi^2}{|\omega|}.
\]

(49)
Similar considerations apply to all other instanton solutions arising from consistent truncation of the Ricci–Cotton flow to the Bianchi IX mini–superspace sector. Such effective point particle systems, which are also commonly used in mixmaster dynamics, prove useful for comparing $SU(2)$ gravitational instantons of Hořava–Lifshitz gravity with those of general relativity. Recall that in $(3+1)$–dimensional Einstein gravity, the instantons are defined as regular geometries with self–dual Riemann (or more generally Weyl) curvature tensor. This property is not shared by the instantons of Hořava–Lifshitz gravity. Yet the $SU(2)$ instantons of Einstein gravity admit an alternative description as trajectories of a point particle moving under the influence of an effective potential, [27]. The difference is that the metric in the space of truncated degrees of freedom is indefinite (inherited by the choice $\lambda = 1$ in the Einsteinian DeWitt superspace metric) and the effective potential does not exhibit degenerate vacua. In that case, the $SU(2)$ instantons (such as the self–dual Taub–NUT solution) correspond to special trajectories of the particle extending from a local maximum to a local minimum of the effective potential. In terms of the four–dimensional geometry, they are supported by removable (nut) singularities at one end, leading to complete and everywhere regular space–time metrics. On the other hand, the singularities that the effective point particle may encounter in Hořava–Lifshitz gravity are not removable, which explains the need to have interpolating trajectories between degenerate vacua to account for instantons in non–relativistic gravitational theories. An important consequence of this difference is in the asymptotic structure of the corresponding space–time metrics: asymptotically locally flat metrics are only possible in Einstein gravity.

5. Concluding remarks

The framework we have presented here is quite broad and can be used to provide a toy model for dynamical vacuum selection in relativistic field theories associated to an action $W$ in $d$ Euclidean dimensions. The instanton solutions of the associated Lifshitz theory in $d + 1$ space–time dimensions describe off-shell transitions among the many different vacua that populate the landscape of the $d$–dimensional theory. This alternative interpretation of Lifshitz models is in the spirit of the Onsager–Machlup theory for non-equilibrium processes in thermodynamics, [28], which is also based on the action (1) (or better to say $S^\text{Eucl.}_d$) and it is often called Onsager–Machlup functional in the literature. In this general context, $W$ is the entropy function that changes monotonically in time and it is proportional to the logarithm of the probability of a given fluctuation. The gradient of $W$ is the thermodynamic force measuring the tendency of a system to seek equilibrium. Linearization of the flow equations around the fixed points describe small fluctuations away from equilibrium states, whereas the instanton solutions incorporate non-linear effects for large transitions between different equilibrium states of the system. It will be interesting to strengthen the analogies between non-equilibrium processes and gradient flow equations in the future.

Some aspects of our work are also reminiscent of renormalization group equations, in particular for geometric theories, and we would like to understand them better. Recall that transitions among vacua of string theory are often described by running solutions of the beta–function equations. These off–shell processes are themselves gradient flows derived from an effective gravitational action coupled to a dilaton and possibly other massless fields of string theory. Thus, they can be alternatively viewed as Euclidean solutions of a Lifshitz theory in one dimension higher by identifying the renormalization group time with the extra Euclidean time dimension. In this context, the off–shell dilaton field is treated as a Lifshitz scalar field, and, likewise, all other off-shell massless fields are treated as Lifshitz tensor fields. However, there is an important technical difference that prevent us from taking immediate advantage of the results we described above. Here, we have been working under the assumption that the metric in field space is positive definite (the same also applies to Onsager–Machlup theory) in order to obtain instantons as extrema of the Euclidean action and interpret $W$ as entropy functional.
On the other hand, the renormalization group equations used in the off–shell formulation of string theory follow from an indefinite metric by choosing, in particular, \( \lambda = 1/2 \) in DeWitt’s superspace metric (it is the same choice that turns (30) into the ordinary Ricci flow equation for the pure metric sector of the theory). Of course, one can adjust the value of \( \lambda \) accordingly to make this metric positive definite, thus taking advantage of all goodies that come with it, and then use the resulting equations as toy model for dynamical vacuum selection in string theory. This modification does not affect the structure of the fixed points, whose defining relations are inert to \( \lambda \), but the flow lines will not be trajectories of the renormalization group equations. It remains to be seen what else can be learned from this analogy.

Another aspect of the present work that is potentially interesting for mathematics is the unifying framework that Lifshitz theories provide to all gradient flow equations. Our results on Hořava–Lifshitz theories in \( 3 + 1 \) dimensions also provide a good reason to study higher order geometric flows more systematically. The Cotton flow is the simplest example of this kind, being an equation of third order, but one should also study more systematically equations of mixed order such as the Ricci–Cotton flow. These systems possess an abundance of fixed points, and, hence, they have much richer structure than the ordinary second order Ricci flow allowing for eternal (instanton) solutions. The main technical problem is to develop methods to estimate curvature bounds and obtain general criteria for the formation of singularities along such flow lines. These issues remain largely unexplored at this time and they call for immediate attention by the experts.

Finally, we comment on the possible use of our instanton solutions to Hořava–Lifshitz theory when viewed as an alternative theory of gravitation at short distances. It will be interesting to obtain a Euclidean path integral formulation of such a non–relativistic theory of gravitation, where our instanton configurations will contribute substantially, since they are weighted by the exponential of (minus) their action. Also, mini–superspace models of quantum cosmology will be interesting to investigate in this context by focusing, in particular, to the mixmaster universe based on the homogeneous three–geometries, [26]. Comparison with ordinary gravity may provide some valuable lessons. Of course, it still remains to settle some open questions regarding the general validity of Hořava’s theory, which appears to contain an unphysical scalar graviton mode, and its consistency with large scale gravitational physics. Also, the condition of detailed balance on which our instanton constructions were based appears to be rather restrictive on physical grounds, as it does not seem to allow for asymptotically flat solutions. A generalization of the original theory was recently made to circumvent some of these problems, [29], and it might still take time to settle the situation one way or another.

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