Permutation symmetry determines the discrete Wigner function

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The Wigner function provides a useful quasi-probability representation of quantum mechanics, with applications in various branches of physics. Many nice properties of the Wigner function are intimately connected with the high symmetry of the underlying operator basis composed of phase point operators: any pair of phase point operators can be transformed to any other pair by a unitary transformation. We show that, in the discrete scenario, such a highly symmetric representation can only appear in odd prime power dimensions except for dimensions 2 and 8. In addition, this permutation symmetry suffices to single out a unique discrete Wigner function not only among various discrete Wigner functions, but also among all possible quasi-probability representations of quantum mechanics. This special discrete Wigner function is also uniquely determined by covariance with respect to the Clifford group of the multipartite Heisenberg-Weyl group.

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The Wigner quasi-probability distribution in phase space, originally introduced for studying quantum correction to thermodynamics, has found numerous applications in various branches of physics, such as quantum optics, quantum chaos, and quantum computing. It also provides an alternative formulation of quantum mechanics, which is particularly suitable for studying quantum-classical correspondence [1,3]. The usefulness of the Wigner function is intimately connected to the high symmetry of the underlying operator basis composed of phase point operators. This basis is invariant under any linear canonical transformation, which is reminiscent of the symplectic geometry in classical phase space. Consequently, “no point, no direction, no scale is distinguished from any other in phase space”, as emphasized by Englert [4]. Mathematically, this means that the symmetry group of the basis acts doubly transitively on the phase point operators; that is, any pair of distinct phase point operators can be mapped to any other pair by a unitary transformation.

Recently, many discrete analogues of the Wigner function have been introduced and found various applications in quantum information science, such as quantum state tomography and quantum computation [5,7]. In addition, general quasi-probability representations have been found useful for studying quantum foundations [8]. For example, those representations based on symmetric informationally complete measurements (SICs) [9,12] play an important role in quantum Bayesianism [13]. All these quasi-probability representations can be formulated in terms of the mathematical theory of frames [14,15].

At this point, it is natural to reflect on the following questions: what is so special about the Wigner function? Is there a simple criteria that can single out a particular quasi-probability representation among all potential candidates? Although similar questions have been investigated extensively [1,5,7,16], no satisfactory answer is known, especially in the discrete scenario. For example, in an odd prime dimension, the discrete Wigner function introduced by Wootters [5] is uniquely determined by Clifford covariance [10]. However, such characterization has at least two limitations: in general, there are many different variants of the Clifford group, and any particular choice necessarily introduces a bias towards a particular representation. In addition, such an approach is not suitable for comparing Wigner functions with other quasi-probability functions.

In this paper we show that the operator basis underlying the Wootters discrete Wigner function is almost uniquely characterized by the permutation symmetry pertaining to the continuous analogue; that is, any pair of distinct phase point operators can be transformed to any other pair by a unitary transformation. Our criterion is motivated by symplectic geometry of the classical phase space and is based on intrinsic symmetry of phase point operators, so it does not suffer from the limitations mentioned in the previous paragraph. In particular, it applies equal well to Wigner functions and general quasi-probability functions, thereby facilitating their comparison. Our study establishes a deep connection between permutation symmetry and the discrete Wigner function as well as the Heisenberg-Weyl (HW) group. It demonstrates that the Wootters discrete Wigner function is the most symmetric quasi-probability representation of quantum mechanics. Our study also reveals the group theoretical root why such a highly symmetric representation can only exist in odd prime power dimensions except for dimensions 2 and 8. The exception in dimension 8 is tied with a special SIC known as Hoggar lines [17,19], which is of independent interest.

Before introducing the Wootters discrete Wigner function, we need to introduce the (multipartite) HW group. In prime dimension $p$, the HW group $D$ is generated by the phase operator $Z$ and the cyclic-shift operator $X$,

$$Z|u\rangle = \omega^u |u\rangle, \quad X|u\rangle = |u+1\rangle, \quad (1)$$

where $\omega = e^{2\pi i/p}$, $u \in \mathbb{F}_p$, and $\mathbb{F}_p$ is the field of integers.
frames are interesting to us because they provide a unified mathematical framework for studying general quasi-probability representations of quantum mechanics, as explained in Refs. [14, 15]. Any operator frame in dimension \(d\) has at least \(d^2\) elements; those attaining the lower bound are called minimal. The symmetry group of an operator frame is composed of all unitary operators \(U\) that leave the frame invariant, that is, \(UF_jU^\dagger = F_{\sigma(j)}\), where \(\sigma\) is a permutation among the indices; by convention operators that differ only by overall phase factors are identified. This group is of paramount importance because the symmetry of the frame is responsible for the symmetry and usefulness of the derived quasi-probability representation, as manifested in the Wigner function. The frame \(\{F_j\}\) is super-symmetric if any pair of frame elements can be mapped to any other pair under its symmetry group. Similar terminology applies to operator bases and positive-operator-valued measures (POVMs).

A frame is called a Wigner-Wootters frame (or basis) if it has the form \(\{a + bV_\mu\}\) with \(V_\mu\) phase point operators and \(a, b\) real constants. According to the above discussion, any Wigner-Wootters frame is super-symmetric. Remarkably, the converse is also true except for frames constructed from two special SICs. Recall that a SIC in dimension \(d\) is composed of \(d^2\) subnormalized projectors onto pure states \(\Pi_j = |\psi_j\rangle \langle \psi_j|\) with equal pairwise fidelity \(|\langle \psi_j|\psi_k\rangle|^2 = (d\delta_{jk} + 1)/(d + 1)\). A frame of the form \(\{a + b\Pi_j\}\) is called a SIC frame (or basis). A SIC frame is super-symmetric if and only if the corresponding SIC is super-symmetric. According to a recent result of the author [19], the SIC in dimension 2, the Hesse SIC in dimension 3, and the set of Hoggar lines in dimension 8 are the only three super-symmetric SICs (super-SICs in short). The frames corresponding to the three SICs are referred to as tetrahedron frames, Hesse frames, and Hoggar frames, respectively. Similar terminology applies to operator bases and POVMs so constructed. Interestingly, a Hesse frame is also a Wigner-Wootters frame; note that the projectors onto one-dimensional eigenspaces of phase point operators in dimension 3 form the Hesse SIC. This observation provides a simple recipe for determining the Wigner function in dimension 3.

**Theorem 1.** Any super-symmetric operator frame is unitarily equivalent to a Wigner-Wootters frame except for tetrahedron frames in dimension 2 and Hoggar frames in dimension 8.

**Remark 1.** This theorem shows that super-symmetric operator frames can only exist in odd-prime power dimensions except for dimensions 2 and 8. In each odd-prime power dimension, the Wigner-Wootters frame and the Wootters discrete Wigner function are uniquely characterized by a simple permutation symmetry inherent in the classical phase space. The theorem does not assume that the frame is minimal; rather this condition is a con-
sequence of the permutation symmetry. To see this, note that any operator frame in dimension \(d\) has at least \(d^2\) elements. On the other hand, the Gram matrix of a super-symmetric frame \(\{F_j\}\) is a linear combination of the identity and the matrix with all entries equal to 1. Observing that \(\{F_j + c\}\) is also a super-symmetric frame for any real constant \(c\) as long as \(F_j + c\) are not traceless, we may assume that the Gram matrix is positive definite without loss of generality. The number of elements in the frame is equal to the rank of the Gram matrix, which is bounded from above by \(d^2\). Therefore, any super-symmetric operator frame is necessarily minimal and is thus an operator basis (not necessarily orthonormal).

To prove Theorem 2, we need to introduce several tools. A stepping stone is the following theorem, which establishes a surprising connection between permutation symmetry and the HW group. It is of independent interest in addition to proving Theorem 1.

**Theorem 2.** Every super-symmetric operator frame is covariant with respect to a multipartite HW group; its symmetry group is a subgroup of the Clifford group.

**Proof.** The proof is similar to the proof of Theorem 2 (concerning super-SICs) in Ref. [19]. Suppose \(\{F_j\}\) is a super-symmetric operator frame in dimension \(d\) with symmetry group \(G\). Then \(\{F_j\}\) has \(d^2\) elements, and \(G\) acts \(2\)-transitively on the frame elements. According to Burnside’s theorem on \(2\)-transitive permutation groups [19, 23, 24], \(G\) has a unique minimal normal subgroup, say \(N\), which is either regular elementary abelian or primitive nonabelian simple. In either case, \(N\) is irreducible according to Lemma 1 below. In the former case, \(N\) is a faithful irreducible projective representation of an elementary abelian group, so it is necessarily projectively equivalent to the HW group in a prime power dimension (see Lemma 11 in Ref. [19]). Since \(N\) is normal in \(G\), it follows that \(G\) is a subgroup of the Clifford group. To complete the proof, it remains to exclude the latter possibility. Suppose on the contrary that \(N\) is primitive nonabelian simple, then \(G\) is an almost simple \(2\)-transitive permutation group. Consequently, \(N\) satisfies one of the three conditions specified in Lemma 9 of Ref. [19]. All these possibilities can be excluded by inspecting the minimal degree of nontrivial irreducible projective representations of \(N\). Note that the minimal degree for the alternating group \(A_k\) is \(k - 1\) when \(k \geq 8\) [24].

**Lemma 1.** Suppose \(G\) is a subgroup of the symmetry group of an operator basis. Then the number of orbits of \(G\) on the basis elements is equal to the sum of squared multiplicities of all the inequivalent irreducible components of \(G\). In particular, \(G\) acts transitively on the basis if and only if it is irreducible.

**Proof.** The proof is almost the same to the proof of Lemma 7.2 in the author’s thesis [18].

**Lemma 2.** The stabilizer of each basis element of a super-symmetric operator basis has two irreducible components, which are inequivalent.

**Proof.** The stabilizer of a given basis element acts transitively on the remaining basis elements, so it has two orbits and thus two inequivalent irreducible components according to Lemma 1.

Let \(\{F_j\}\) be a super-symmetric operator frame in dimension \(q = p^n\) and \(S\) the stabilizer of one of the frame elements, say \(F_1\). According to Theorem 2, \(S\) can be identified with a transitive subgroup of the Clifford group, where “transitive” means that \(S\) acts (by conjugation) transitively on nontrivial displacement operators. In addition, \(S\) has trivial intersection with the HW group, so it is isomorphic to a transitive subgroup of \(Sp(2n, p)\), where a linear group on \(\mathbb{F}_p^{2n}\) is transitive if it acts transitively on nonzero vectors. To establish our main result, it is therefore crucial to figure out transitive subgroups of \(Sp(2n, p)\). Fortunately, transitive linear groups have already been classified by Hering [26]; see also Table 7.3 in Ref. [24].

**Lemma 3.** Any transitive subgroup \(H\) of \(GL(2n, p)\) with \(n \geq 2\) satisfies one of the following conditions:

1. \(Sp(2m, p^k) \leq H\) with \(1 \leq m \leq n\) and \(mk = n\).
2. \(SL(m, p^k) \leq H \leq \Gamma L(m, p^k)\) with \(m \geq 3\) and \(mk = 2n\), where \(\Gamma L(m, p^k)\) is the general semilinear group.
3. \(H \leq \Gamma L(1, p^{2n})\).
4. \(p = 2, n = 3k, \) and \(G_2(p^k) \leq H, \) where \(G_2(p^k)\) is an exceptional group of Lie type.
5. \(p = 2, n = 2, \) and \(H \simeq A_6\) or \(H \simeq A_7, \) where \(A_6\) and \(A_7\) are alternating groups of degrees 6 and 7.
6. \(p = 2, n = 3, \) and \(H \simeq PSU(3, 3).\)
7. \(p = 3, n = 2, \) and \(2^{1+4} \leq H, \) where \(2^{1+4}\) denotes an extraspecial 2-group of this order.
8. \(p = 3, n = 3, \) and \(H \simeq SL(2, 13).\)

**Remark 2.** Here \(Sp(2m, p^k), SL(m, p^k), \Gamma L(m, p^k), \) and \(G_2(p^k)\) are extension-field-type subgroups of \(GL(2n, p)\). Complete classification is also available in the case \(n = 1\) but is not necessary here.

**Lemma 4.** Any transitive subgroup \(H\) of \(Sp(2n, p)\) for odd prime \(p\) contains the central involution of \(Sp(2n, p)\).

Before proving this lemma, we need to introduce several additional concepts. A Singer cyclic group of a classical group over a finite field is an irreducible cyclic subgroup of maximal order [27, 28]. Any generator of a Singer cyclic group is called a Singer cycle. Singer cyclic groups of \(GL(n, p), SL(n, p), \) and \(Sp(2n, p)\) have orders \(p^n-1, (p^n-1)/(p-1), \) and \(p^n+1,\) respectively. In all these cases, all Singer cyclic groups of a given group are conjugated to each other. We are only concerned with Singer...
Proof of Lemma [4]. When \( n = 1 \), the group \( \text{Sp}(2n, p) \approx \text{SL}(2, p) \) has a unique involution \([22]\). Since \( H \) has even order, it must contain the involution.

When \( n \geq 2 \), the group \( H \) must satisfy one of the conditions 1, 2, 3, 7, 8 in Lemma \([3]\). We shall show that \( H \) contains the central involution in cases 1, 7, and 8, while cases 2 and 3 cannot happen.

In case 1, the center of \( \text{Sp}(2m, p^k) \) coincides with that of \( \text{Sp}(2n, p) \), so \( H \) contains the central involution.

Case 2 cannot happen because \( \text{Sp}(2n, p) \) cannot contain \( \text{SL}(m, p^k) \) with \( 3 \leq m \leq 2n \) and \( mk = 2n \). Note that each Singer cycle of \( \text{SL}(m, p^k) \) has order \( (p^m - 1)/(p^k - 1) \), but \( \text{Sp}(2n, p) \) has no such element \([27, 28]\).

Case 3 can be ruled as follows. The group \( \Gamma L(1, p^{2n}) \) is a semidirect product \( \text{Gal}(F_{p^{2n}}/F_p) \rtimes \text{GL}(1, p^{2n}) \), where \( \text{GL}(1, p^{2n}) \) can be identified as a Singer cyclic group of \( \text{GL}(2n, p) \), and \( \text{Gal}(F_{p^{2n}}/F_p) \) is the Galois group of the field \( F_{p^{2n}} \) over the prime field \( F_p \), which is cyclic of order \( 2n \) \([23]\). The intersection of any Singer cyclic group of \( \text{GL}(2n, p) \) with \( \text{Sp}(2n, p) \) has order at most \( p^{n+1} \) \([28]\). The order of \( H \) is at most \( 2n(p^{n+1} - 1) \) cannot be divisible by \( p^{n+1} - 1 \); so \( H \) cannot be transitive.

If case 7 holds, let \( P \) be the normal subgroup of \( H \) that is isomorphic to \( 2^{1+4} \). Then the center of \( P \) has order 2 and is contained in the center of \( H \). On the other hand, \( H \) has order divisible by a Zsigmondy prime of \( 3^4 - 1 \) so it contains a Zsigmondy cycle. The involution in the center of \( P \) commutes with the Zsigmondy cycle, so it must coincide with the central involution of \( \text{Sp}(2n, p) \).

If case 8 holds, then \( H \) contains a Zsigmondy cycle; the unique involution in \( H \) commutes with the Zsigmondy cycle and is thus the central involution of \( \text{Sp}(2n, p) \).

Proof of Theorem \([4]\). Let \( \{ F_j \} \) be a super-symmetric operator frame in dimension \( d \). Then the number of frame elements is equal to \( d^2 \). When \( d = 2 \), each frame element can be written as \( F_j = a \Pi_j + b \), where \( \Pi_j \) is a rank-1 projector, and \( a, b \) are real constants independent of \( j \). The symmetry requirement on the frame implies that the projectors \( \Pi_j \) form a SIC.

In general, according to Theorem \([2]\), \( d \) is equal to a prime power \( q = p^n \), the frame \( \{ F_j \} \) is covariant with respect to the HW group, and its symmetry group is a subgroup of the Clifford group that acts 2-transitively on frame elements. Let \( S \) be the stabilizer (within the Clifford group) of the frame element \( F_1 \), then \( S \) is transitive and it induces a transitive subgroup \( H \) of \( \text{Sp}(2n, p) \). In addition, \( S \) has two irreducible components according to Lemma \([2]\).

When \( p \) is odd, according to Lemma \([1]\), \( H \) contains the central involution of \( \text{Sp}(2n, p) \), so that \( S \) contains a principal involution \( U \) in its center. With a suitable choice of the phase factor, \( U \) is a phase point operator. Since \( U \) has nonzero trace, it follows that all elements of \( S \) commute with \( U \). Therefore, the two irreducible components of \( S \) correspond to the two eigenspaces of \( U \). Accordingly, \( F_j \) is a linear combination of the projectors onto the two eigenspaces or, equivalently, a linear combination of \( U \) and the identity; so \( \{ F_j \} \) is a Wigner-Wootters frame.

When \( p = 2 \) and \( n \geq 2 \), let \( h \) be the smaller degree of the two irreducible components of \( S \). If \( h = 1 \), then the projector onto the irreducible component of degree 1 is a fiducial state for a super-SIC. According to Ref. \([19]\), the only super-SIC in even prime power dimension \( q > 2 \) is the set of Hoggar lines, so \( \{ F_j \} \) is a Hoggar frame. Incidentally, the stabilizer of each fiducial state of the set of Hoggar lines is isomorphic to \( \text{PSU}(3, 3) \) \([18, 19]\). In view of this observation, it remains to consider the scenario \( 2 \leq h \leq 2^n/2 \). According to Lemma \([3]\), \( H \) satisfies one of the conditions 1 to 6 in the lemma. Cases 2 and 3 cannot happen according to the same reasoning as in the proof of Lemma \([4]\). The other cases can be excluded by the observation that the minimal degree of nontrivial irreducible projective representations of \( H \) in each case is always larger than \( 2^n/2 \) except when \( n = 2 \) and \( H \) is isomorphic to \( \text{Sp}(2, 2^2) \approx \text{SL}(2, 2^2) \); see the appendix for more details. In the exceptional case, calculation shows that the Clifford group has two types of subgroups that are isomorphic to \( \text{SL}(2, 2^2) \): the first type is transitive and irreducible, while the other one is non-transitive and reducible.

Theorem \([1]\) has several important consequences.

**Theorem 3.** In any odd prime power dimension, an operator basis is covariant with respect to the Clifford group if and only if it is a Wigner-Wootters basis. The Wootters discrete Wigner function is the unique Clifford covariant Wigner function. In any even prime power dimension, no operator basis or discrete Wigner function is covariant with respect to the Clifford group.

This theorem follows from Theorem \([1]\) and the observation that any operator basis covariant with respect to the Clifford group (or even the restricted Clifford group discussed in Refs. \([16, 33]\)) is necessarily super-symmetric. In addition, tetrahedron bases and Hoggar bases are not
Clifford covariant. Interestingly, an operator basis is super-symmetric if and only if it is Clifford covariant, except for the two special cases. Theorem 3 generalizes the result of Gross concerning Clifford covariant Wigner functions in odd prime dimensions [10].

Theorem 1 also implies that the Hesse SIC is the unique SIC covariant with respect to the Clifford group. In addition, it generalizes a recent result of the author on super-SICs [19] to POVMs that are not necessarily rank-1.

**Theorem 4.** Any super-symmetric POVM is a Wigner-Wootters POVM except for tetrahedron POVMs in dimension 2 and Hoggar POVMs in dimension 8.

In summary, we have established a simple connection between permutation symmetry and the discrete Wigner function as well as the HW group. In particular, the Wootters discrete Wigner function is almost uniquely characterized by the permutation symmetry appearing in the continuous analogue. It is the unique discrete Wigner function that is covariant with respect to the Clifford group. We also settle the question of why such a highly symmetric quasi-probability representation can only appear in odd prime power dimensions. In addition, our study provides valuable insight on various discrete symmetric structures behind finite-state quantum mechanics, such as symmetric POVMs and symmetric operator bases. An interesting problem left open is whether similar results hold in the continuous scenario.

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**Appendix**

In this appendix, we show that the minimal degree of nontrivial irreducible projective representations of any transitive subgroup \( H \) of \( \text{Sp}(2n, 2) \) with \( n \geq 2 \) is always larger than \( 2^n/2 \) except when \( n = 2 \) and \( H \) is isomorphic to \( \text{Sp}(2, 2^2) \). Recall that \( H \) satisfies one of the conditions 1 to 6 in Lemma 3 of the main text, and that cases 2 and 3 cannot happen.

In case 1, according to Table II in Ref. [34], when \( n \geq 3 \), the minimal degree of nontrivial irreducible projective representations of \( \text{Sp}(2m, 2^k) \) with \( mk = n \) is

\[
\begin{align*}
\begin{cases}
2^n - 1, & k = n; \\
\frac{(2^n-1)(2^n-2^k)}{2(2^n+1)}, & k < n;
\end{cases}
\end{align*}
\]

which is always larger than \( 2^n/2 \). It remains to consider the cases \( n = 2 \) and \( k = 1, 2 \). The two groups \( \text{Sp}(4, 2) \) and \( \text{Sp}(2, 2^2) \simeq \text{SL}(2, 2^2) \) are isomorphic to the symmetric group of degree 6 and alternating group of degree 5, respectively. The minimal degree of nontrivial irreducible projective representations is 4 for \( \text{Sp}(4, 2) \) and 2 for \( \text{Sp}(2, 2^2) \). Incidentally, according to theorem 7 in Ref. [21], the HW group in dimension \( 2^n \) with \( n \geq 2 \) is not complemented in the Clifford group \( C \), so the Clifford group has no subgroup isomorphic to \( \text{Sp}(2n, 2) \).

In case 4, according to Sec. 5.3 in Ref. [35], the minimal degree of nontrivial irreducible projective representations of \( G_2(2^k) \) (assuming \( n = 3k \)) is

\[
\begin{align*}
\begin{cases}
6, & k = 1; \\
2^n - 1, & 2 \nmid k, k \geq 3; \\
2^n + 1, & 2 \nmid k, k \geq 2;
\end{cases}
\end{align*}
\]

which is always larger than \( 2^n/2 \).

In case 5, \( \text{Sp}(4, 2) \) cannot contain \( A_7 \) (which has larger order than \( \text{Sp}(4, 2) \)). The minimal degree of nontrivial irreducible projective representations of \( A_6 \) is 3. Incidentally, any subgroup of the Clifford group in dimension 4 that is isomorphic to \( A_6 \) is irreducible.

In case 6, the two minimal degrees of nontrivial irreducible projective representations of \( \text{PSU}(3, 3) \) are 6 and 7 according to Table V in Ref. [34].

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