NODAL Vector solutions with clustered peaks for a nonlinear elliptic equations in $\mathbb{R}^3$*

Qihan He  
School of Mathematics and Statistics,  
Central China Normal University, Wuhan, 430079, P. R. China  
e-mail: heqihan277@163.com

Chunhua Wang‡  
School of Mathematics and Statistics,  
Central China Normal University, Wuhan, 430079, P. R. China  
e-mail: chunhuawang@mail.ccnu.edu.cn

June 1, 2021

Abstract
In this paper, we study the following coupled nonlinear Schrödinger system in $\mathbb{R}^3$
\begin{align*}
-\varepsilon^2 \Delta u + P(x)u &= \mu_1 u^3 + \beta v^2 u, \quad x \in \mathbb{R}^3, \\
-\varepsilon^2 \Delta v + Q(x)v &= \mu_2 v^3 + \beta u^2 v, \quad x \in \mathbb{R}^3,
\end{align*}
where $\mu_1 > 0, \mu_2 > 0$ and $\beta \in \mathbb{R}$ is a coupling constant. Whether the system is repulsive or attractive, we prove that it has nodal semi-classical segregated or synchronized bound states with clustered spikes for sufficiently small $\varepsilon$ under some additional conditions on $P(x), Q(x)$ and $\beta$. Moreover, the number of this type of solutions will go to infinity as $\varepsilon \to 0^+$. 

2000 Mathematics Subject Classification. 35J10, 35B99, 35J60.

Key words. Clustered peaks; NODAL Vector solutions; Nonlinear coupled.

*The authors sincerely thank Professor S.J. Peng for helpful discussions and suggestions. This work was partially supported by NSFC (No.11301204; No.11371159; No.11101171), the PhD specialized grant of the Ministry of Education of China(20110144110001).

‡Corresponding author
1 Introduction

In this paper, we consider the following nonlinear Schrödinger system in $\mathbb{R}^3$

\[
\begin{cases}
-\epsilon^2 \Delta u + P(x)u = \mu_1 u^3 + \beta v^2 u, & x \in \mathbb{R}^3, \\
-\epsilon^2 \Delta v + Q(x)v = \mu_2 v^3 + \beta u^2 v, & x \in \mathbb{R}^3,
\end{cases}
\]

(1.1)

where we assume that $P(x)$ and $Q(x)$ are continuous bounded radial functions, $\mu_1 > 0, \mu_2 > 0$ and $\beta \in \mathbb{R}$ is a coupling constant.

It motivates us to study problem (1.1) that we look for standing-wave solutions for the following time-dependent coupled nonlinear Schrödinger system:

\[
\begin{cases}
i \epsilon \frac{\partial \psi}{\partial t} = -\frac{\epsilon^2}{2m} \Delta_x \psi + P(x)\psi - \mu_1 |\psi|^2 \psi - \beta |\phi|^2 \psi, & x \in \mathbb{R}^3, t > 0, \\
i \epsilon \frac{\partial \phi}{\partial t} = -\frac{\epsilon^2}{2m} \Delta_x \phi + Q(x)\phi - \mu_2 |\phi|^2 \phi - \beta |\psi|^2 \phi, & x \in \mathbb{R}^3, t > 0, \\
\psi = \psi(x, t) \in C, \phi = \phi(x, t) \in C,
\end{cases}
\]

(1.2)

which models a binary mixture of Bose-Einstein condensates in two different hyperfine states (see [11, 12, 17, 37]), and where $\epsilon$ is the plank constant, $m$ is the atom mass, $P(x)$ and $Q(x)$ are the trapping potentials for two hyperfine states, respectively; the constants $\mu_1$ and $\mu_2$ represent the intraspecies scattering lengths and $\beta$ is the interspecies scattering length. The sign of the interspecies scattering length determines whether the interaction of states are repulsive or attractive. If $\beta > 0$, the interaction is attractive, and the components of a vector solution set to synchronize. On the other hand, if $\beta < 0$, the interaction is repulsive, leading to phase separations. These phenomena have been confirmed in experiments and in numeric simulations (see [12] [14] [17] [21] and references therein). Problem (1.2), also known as Gross-Pitaevskii equations, arises in many applications. For example, in some problems arising in nonlinear optics, in plasma physics and in the condensed matter physics. Physically, $\psi$ and $\phi$ are the corresponding condensed wave functions (see [2]).

This system (1.1) has been extensively investigated under various assumptions on $P(x), Q(x)$ and $\beta$ in recent years (see [1] [3]-[7], [9]-[11], [13]-[16], [18]-[33], [35, 36, 38, 39] and therein ). Here we want to mention some significant works. In [25], no matter the interspecies scattering length $\beta$ is positive or negative, Lin and Wei have obtained least energy solutions for the two coupled nonlinear Schrödinger system with the trap potentials by using Nehari’s manifold and derived their asymptotic behaviors by some techniques of singular perturbation problem. At the same time, Chen, Lin and Wei [15] have proved the existence of the positive solutions with any prescribed spikes by the reduction methods. In [1], Alves has been concerned with the existence and the concentration of positive solutions by the mountain pass theorem. Wan [38] used the category theory to study the multiplicity of positive solutions and their limiting behavior as $\epsilon \to 0^+$. Also in [39], Wan and Ávila utilized the category theory studying the relation between the number of positive standing waves solutions for the special system (1.1) with $P(x) = Q(x)$ and $\beta = 0$ in $\mathbb{R}^N$ and the topology of the set of minimum points of potentials. Pomponio in [33] also has
proved the existence of concentrating solutions for a general system with repulsive interaction of states and that how the location of the concentration points depends strictly on the potentials. In [7], Bartsch, Dancer and Wang considered the repulsive case and obtained segregated radial solutions by global bifurcation methods for the the general systems (1.1), establishing the existence of infinite branches of radial solutions with the property that $\sqrt{\mu_1 - \beta \psi} - \sqrt{\mu_2 - \beta \phi}$ has exactly $k$ nodal domains for solutions along the $k$th branch. Recently, Pi and Wang [32] have constructed multiple solutions with any prescribed spikes and proved that the spikes will approach the local maximum point of the trap potentials as $\epsilon \to 0^+$.

Here we should point out that in the results mentioned above, the solutions are positive. Although there is a wide literature studying existence, multiplicity and shape of positive solutions, there are few papers dealing with the case of nodal solutions, with the exception of the single Schrödinger equations for the one-dimensional case or the radial case [8] which allows methods, like the use of a natural constraint, which do not work in the nonradial setting considered here.

As far as we know, there are no results on the existence of nodal non-radial semi-classical bound states which have any prescribed nodal domain. In this paper, we will present some results which contributes to this respect.

In order to state our main results, first we assume that $\inf P(r) > 0$ and $\inf Q(r) > 0$ satisfy the following conditions:

( P ): There are constants $a \in \mathbb{R}$, $m > 1$ and $\theta > 0$, such that as $r \to 0^+$

$$ P(r) = 1 + ar^m + O(r^{m+\theta}). $$

( Q ): There are constants $b \in \mathbb{R}$, $n > 1$ and $\delta > 0$, such that as $r \to 0^+$

$$ Q(r) = 1 + br^n + O(r^{n+\delta}). $$

The main results of our paper are as follows.

**Theorem 1.1.** Let (P) and (Q) hold. Then for any fixed $k \in \mathbb{N}^+$, there exists a decreasing sequence $\{\beta_l\} \subset (-\sqrt{\mu_1 \mu_2}, 0)$ with $\beta_l \to -\sqrt{\mu_1 \mu_2}$ as $l \to \infty$ and $\epsilon_0 > 0$ such that for $\beta \in (-\sqrt{\mu_1 \mu_2}, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)$ and $\beta \neq \beta_l$, and $0 < \epsilon < \epsilon_0$, (1.1) has a vector solution $(u_\epsilon, v_\epsilon)$ with $k$ positive peaks and $k$ negative peaks, and the peaks of the solution approaching to the extremal point 0 of $P(x)$ and $Q(x)$ provided one of the following two conditions holds:

1. $m < n, a > 0$ and $b \in \mathbb{R}$; or $m > n, a \in \mathbb{R}$ and $b > 0$;
2. $m = n, aB + bC_0 > 0$, where $B$ and $C$ are defined in Proposition A.1.

Furthermore,

$$ \|\sqrt{\mu_1 - \beta}|u_\epsilon - \sqrt{\mu_2 - \beta}|v_\epsilon|\|_{H^1} + \|\sqrt{\mu_1 - \beta}|u_\epsilon - \sqrt{\mu_2 - \beta}|v_\epsilon|\|_{L^\infty} \to 0, \quad \text{as } \epsilon \to 0^+. $$

**Theorem 1.2.** Let (P) and (Q) hold. If $m = n, a > 0, b > 0$, then for any fixed $k \in \mathbb{N}^+$, there exist constants $\beta_0 > 0$ and $\epsilon_0 > 0$ such that for any $\beta < \beta_0$ and $0 < \epsilon < \epsilon_0$, (1.1) has a vector solution $(\bar{u}_\epsilon, \bar{v}_\epsilon)$ with $k$ positive peaks and $k$ negative
peaks which approach to the local minimum point 0 of \(P(x)\) and \(Q(x)\) as \(\epsilon \to 0^+\). Furthermore,

\[
\|\sqrt{\mu_2} \tilde{u}_\epsilon(\cdot) - \sqrt{\mu_1} \tilde{v}_\epsilon(T_\epsilon \cdot)\|_{H^1} + \|\sqrt{\mu_2} \tilde{u}_\epsilon(\cdot) - \sqrt{\mu_1} \tilde{v}_\epsilon(T_\epsilon \cdot)\|_{L^\infty} \to 0, \quad \text{as} \quad \epsilon \to 0^+.
\]

Here \(T_\epsilon \in SO(3)\) is the rotation on the \((x_1, x_2)\) plane of \(\tau_\epsilon\).

Next, we introduce some notations to be used in the proofs of the main results and formulate a version of the main results which give more precise descriptions about the segregated and synchronized character of the solutions. In doing so, we also outline the main idea and the approaches in the proofs of Theorems 1.1 and 1.2.

Define

\[
H_s = \left\{ u \in H^1(\mathbb{R}^3) : u \text{ is even in } y_h, h = 2, 3, \right. \\
\left. u(r \cos (\theta + \frac{\pi j}{k}), r \sin (\theta + \frac{\pi j}{k}), x_3) = (-1)^j u(r \cos \theta, r \sin \theta, x_3) \right\},
\]

where \(H^1(\mathbb{R}^3)\) is the usual Sobolev space with the norm for any bounded function \(K(x)\)

\[
\|u\|_{r,K}^2 = (u, u)_K = \int_{\mathbb{R}^3} (\epsilon^2 |\nabla u|^2 + K(x)|u|^2)dx,
\]

and define \(H = H_s \times H_s\) endowed with the following norm

\[
\|(u, v)\|_e^2 = \|u\|_{r,P}^2 + \|v\|_{r,Q}^2.
\]

Set

\[
w_{\mu, \epsilon}(x) = w\left(\frac{x - y}{\epsilon}\right)
\]

and

\[
S_\epsilon := \left\{ \frac{\min\{m, n\} - \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon}, \frac{\min\{m, n\} + \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right\},
\]

where \(\delta \in (0, \frac{1}{\tau + \sigma} \min\{n, m\})\), and \(\sigma\) will be defined in Proposition A.2. Denote

\[
x_j := \left( r \cos \left( \frac{(j - 1)\pi}{k} \right), r \sin \left( \frac{(j - 1)\pi}{k} \right), x_3 \right), \quad j = 1, 2, \ldots, 2k, \quad r \in S_\epsilon.
\]

It is well-known that the following problem has a unique radial solution denoted by \(w\)

\[
- \Delta u + u^3, \max_{x \in \mathbb{R}^3} u(x) = u(0), u > 0,
\]

and the solution \(w\) satisfies the following properties:

\[
w'(r) < 0, \quad \lim_{r \to \infty} r^{\frac{N - 1}{2}} e^r w(r) = C_0 > 0, \quad \lim_{r \to \infty} \frac{w'(r)}{w(r)} = -1.
\]
When \(-\sqrt{\mu_1\mu_2} < \beta < \min\{\mu_1, \mu_2\}\) or \(\beta > \max\{\mu_1, \mu_2\}\), \((U, V) := (\alpha w, \gamma w)\) is a solution of the following system:

\[
\begin{aligned}
-\Delta u + u &= \mu_1 u^3 + \beta v^2 u, & x \in \mathbb{R}^3, \\
-\Delta v + v &= \mu_2 v^3 + \beta u^2 v, & x \in \mathbb{R}^3,
\end{aligned}
\]  

(1.7)

where \(\alpha = \sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}}, \gamma = \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}}\).

We let

\[
U_r(x) = 2^k \sum_{j=1}^{2k} (-1)^{j-1} U_{x_j, \epsilon},
\]

\[
V_r(x) = 2^k \sum_{j=1}^{2k} (-1)^{j-1} V_{y_j, \epsilon}.
\]

We will verify Theorem 1.1 by proving the following result:

**Theorem 1.3.** Under the assumptions of Theorem 1.1, there exists a positive constant \(\epsilon_0 > 0\) such that for any \(0 < \epsilon < \epsilon_0\), (1.1) has a solution of the form

\[
(u_\epsilon, v_\epsilon) = (U_r(x) + \varphi(x), V_r(x) + \psi(x)),
\]

where \((\varphi(x), \psi(x)) \in H\) and

\[
\|(\varphi(x), \psi(x))\|_\epsilon = O\left(\epsilon^{\frac{2 + \min\{m, n\}}{2}} - \sigma\right), \quad |x^3| = O\left(\epsilon \ln \frac{1}{\epsilon}\right)
\]

for some small constant \(\sigma > 0\).

Let \(U_i\) be the unique radial solution of the following problem

\[
-\Delta u + u = \mu_i u^3, \quad \max_{x \in \mathbb{R}^3} u(x) = u(0), u > 0.
\]

It is well known that \(U_i\) is non-degenerate and \(U'_i(r) < 0, \lim_{r \to \infty} r^{N-1} e^r U_i(r) = C_0 > 0, \lim_{r \to \infty} \frac{U'_i(r)}{U_i(r)} = -1\).

We will use \((U_1, U_2)\) to build up the approximate solutions for (1.1).

Let \(x^j\) be defined in (1.5) and denote

\[
y^j := \left(\rho \cos \left(\frac{2j - 1}{2k}\right), \rho \sin \left(\frac{2j - 1}{2k}\right), x_3\right), j = 1, 2, \cdots, 2k,
\]

(1.8)

where \(\rho \in S_\epsilon\).

Let

\[
\tilde{U}_r = \sum_{j=1}^{2k} (-1)^{j-1} U_{1, x^j, \epsilon}, \quad \tilde{V}_\rho = \sum_{j=1}^{2k} (-1)^{j-1} U_{2, y^j, \epsilon}.
\]

(1.9)

To prove Theorem 1.2 we need to prove the following result.
Theorem 1.4. Under the assumptions of Theorem 1.2, there exists a positive constant \( \epsilon_0 \) such that for any \( 0 < \epsilon < \epsilon_0 \), (1.1) has a solution of the form

\[
(\bar{u}_e, \bar{v}_e) = (\tilde{U}_\rho(x) + \tilde{\varphi}(x), \tilde{V}_\rho(x) + \tilde{\psi}(x)),
\]

where \( (\tilde{\varphi}(x), \tilde{\psi}(x)) \in H \) and

\[
\| (\tilde{\varphi}(x), \tilde{\psi}(x)) \|_\epsilon = O\left( \epsilon^{\frac{3 + \min(m,n) - \sigma}{2}} \right), \quad |x^j| = O\left( \epsilon \ln \frac{1}{\epsilon} \right), \quad |y^j| = O\left( \epsilon \ln \frac{1}{\epsilon} \right)
\]

for some small constant \( \sigma > 0 \).

Remark 1.1. Radial symmetries can be replaced by the following weaker symmetrical assumptions: after suitably rotating the coordinate system,

(P') \( P(x) = P(x', x_3) = P(|x' - \bar{x}^3|, x_3 - \bar{x}_3) \) and \( P(x) \) has the following expansion:

\[
P(\bar{r}) = P(\bar{x}) + a|x - \bar{x}|^m + O(|x - \bar{x}|^{m+\theta}) \quad \text{as} \quad |x - \bar{x}| \to 0,
\]

where \( \bar{x} \in \mathbb{R}^3, a \in \mathbb{R}, m > 1, \theta > 0 \) and \( P(\bar{x}) > 0 \) are constants.

(Q') \( Q(x) = Q(x', x_3) = Q(|x' - \bar{x}'|, x_3 - \bar{x}_3) \) and \( Q(x) \) has the following expansion:

\[
Q(\bar{r}) = Q(\bar{x}) + b|x - \bar{x}|^n + O(|x - \bar{x}|^{n+\delta}) \quad \text{as} \quad |x - \bar{x}| \to 0,
\]

where \( \bar{x} \in \mathbb{R}^3, b \in \mathbb{R}, n > 1, \delta > 0 \) and \( Q(\bar{x}) > 0 \) are constants.

Remark 1.2. For \( N = 2 \), if we adjust the constants \( \delta, \tau_1, \tau_2 \) in (1.4), then both Lemma 2.4 and Proposition 2.1 still hold. In order to guarantee that Proposition 2.1 holds, we can find nodal synchronize solutions of (1.1) for the attractive case under the same assumptions. However, for the repulsive case, we can’t find nodal segregated solutions of (1.1), since Proposition 3.1 can not hold.

The proofs of our main result are based on the well-known Lyapunov-Schmidt reduction procedure. In particular, in order to deal with nodal clustered solutions, we perform the reduction in suitable symmetric settings in the spirit of [40] where infinitely many positive non-radial solutions for nonlinear Schrödinger equations were obtained. For the attractive case, we will construct nodal synchronize solutions approximately as \( \sum_{j=1}^{2k} (-1)^{j-1}U_{x_j, \epsilon}, \sum_{j=1}^{2k} (-1)^{j-1}V_{x_j, \epsilon} \) with the points \( x^j \) locating on and dividing equally the circle with radius \( C\epsilon \ln \frac{1}{\epsilon} \) into \( 2k \) parts. Since the distance between two neighbor peaks with the same sign is larger than that between two neighbor peaks with opposite sign, the interaction among peaks with opposite sign dominates that among peaks with the same sign. Hence, if the slower decaying functions between \( Q(x) \) and \( P(x) \) has local minimum at the center of the circle, we can easily conclude that the equilibrium is achieved for a suitable configuration of the points \( x^j \), which can be reduced to solve a minimization problem related to energy.
functional. Generally speaking, the key to construct nodal solutions by the reduction argument is to compare the influence between the interaction among the peaks with the same sign and that among the peaks with opposite sign, the idea in [40] can help us to construct a symmetric configuration space consisting of \( x^j \) \((j = 1, \cdots, 2k)\) and hence realize the key. For the repulsive case, we will construct nodal segregated solutions approximately as 
\[
2k \sum_{j=1}^{2k} (-1)^{j-1} U_{1,x^j,\epsilon} + \sum_{j=1}^{2k} (-1)^{j-1} U_{2,x^j,\epsilon}
\]
with the points \( x^j \) and \( y^j \) locating on and dividing equally the circles with radius \( C_1 \epsilon \ln 1/\epsilon \) and \( C_2 \epsilon \ln 1/\epsilon \) into 2\( k \) parts, respectively and vector \( \overrightarrow{oy}^j \) dividing equally angle \( \angle x^j ox^j + 1 \).

Then using the similar methods like the attractive case, we can construct nodal segregated solutions. This idea is also effective in finding infinitely many non-radial positive solutions for semilinear elliptic problems (see, [31]).

This paper is organized as follows. In section 2 we will study the finite-dimensional reduced problem and prove Theorem 1.3. We will put the study of the existence of segregated solutions for system (1.1) and the proof of the Theorem 1.4 into Section 3. Finally we will give all the technical calculations in the Appendix.

## 2 Synchronized Vector Solutions and the proof of Theorem 1.1

In this section we consider synchronized vector solutions and prove Theorem 1.1 by proving Theorem 1.3. The functional corresponding to (1.1) is
\[
I_\epsilon(u,v) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \epsilon^2 |\nabla u|^2 + P(x)u^2 + \epsilon^2 |\nabla v|^2 + Q(x)v^2 \right) dx 
- \frac{1}{4} \int_{\mathbb{R}^3} \left( \mu_1 |u|^4 + \mu_2 |v|^4 \right) dx - \frac{\beta}{2} \int_{\mathbb{R}^3} u^2 v^2 dx.
\]
Then \( I_\epsilon \in C^2(H) \) and its critical points correspond to the solutions of (1.1).

Define
\[
Y_j := \frac{\partial U_{x^j,\epsilon}}{\partial r}, Z_j := \frac{\partial V_{x^j,\epsilon}}{\partial r}, j = 1, 2, \cdots, 2k,
\]
where \( x^j \) is defined in (1.5) and define
\[
E = \left\{ (u, v) \in H : \sum_{j=1}^{2k} \int_{\mathbb{R}^3} (U_{x^j,\epsilon}^2 Y_j u + V_{x^j,\epsilon}^2 Z_j v) \ dx = 0 \right\}.
\]

Let
\[
J(\varphi, \psi) = I_\epsilon(U_r + \varphi, V_r + \psi), \ (\varphi, \psi) \in E.
\]

Expand \( J(\varphi, \psi) \) as follows:
\[
J(\varphi, \psi) = J(0, 0) + l(\varphi, \psi) + \frac{1}{2}Q(\varphi, \psi) + R(\varphi, \psi), \ (\varphi, \psi) \in E,
\]
where

\[ l(\varphi, \psi) \]

\[ = \sum_{j=1}^{2k} (-1)^{j-1} \int_{\mathbb{R}^3} (P(x) - 1)U_{x^j, \epsilon} \varphi - \mu_1 \int_{\mathbb{R}^3} \left( U_r^3 - \sum_{j=1}^{2k} (-1)^{j-1} U_{x^j, \epsilon}^3 \right) \varphi \]

\[ + \sum_{j=1}^{2k} (-1)^{j-1} \int_{\mathbb{R}^3} (Q(x) - 1)V_{x^j, \epsilon} \psi - \mu_2 \int_{\mathbb{R}^3} \left( V_r^3 - \sum_{j=1}^{2k} (-1)^{j-1} V_{x^j, \epsilon}^3 \right) \psi \]

\[ - \beta \int_{\mathbb{R}^3} \left( U_r V_r^2 - \sum_{j=1}^{2k} (-1)^{j-1} V_{x^j, \epsilon}^2 U_{x^j, \epsilon} \right) \varphi - \beta \int_{\mathbb{R}^3} \left( U_r^2 V_r - \sum_{j=1}^{2k} (-1)^{j-1} V_{x^j, \epsilon} U_{x^j, \epsilon}^2 \right) \psi, \]

\[ Q(\varphi, \psi) = \int_{\mathbb{R}^3} (e^2 |\nabla \varphi|^2 + P(x) \varphi^2 - 3\mu_1 U_{r}^2 \varphi^2) \]

\[ + \int_{\mathbb{R}^3} (e^2 |\nabla \psi|^2 + Q(x) \psi^2 - 3\mu_2 V_{r}^2 \psi^2) \]

\[ - \beta \int_{\mathbb{R}^3} (U_{r}^2 \psi^2 + 4U_r V_r \varphi \psi + V_{r}^2 \varphi^2) \]

and

\[ R(\varphi, \psi) = \int_{\mathbb{R}^3} \left( \mu_1 U_r \varphi^3 + \mu_2 V_r \psi^3 + \frac{\mu_1}{4} \varphi^4 + \frac{\mu_2}{4} \psi^4 \right) \]

\[ - \frac{\beta}{2} \int_{\mathbb{R}^3} \left[ (U_r + \varphi)^2 (V_r + \psi)^2 - U_{r}^2 V_{r}^2 - 2(U_r V_r^2 \varphi + U_r^2 \psi) \right. \]

\[ \left. - (U_{r}^2 \psi^2 + V_{r}^2 \varphi^2 + 4U_r V_r \varphi \psi) \right]. \]

In order to find a critical point \((\varphi, \psi) \in E\) for \(J(\varphi, \psi)\), we need to discuss each term in the expansion (2.3).

It is easy to check that

\[ \int_{\mathbb{R}^3} (e^2 \nabla u \nabla \varphi + P(x) u \varphi - 3\mu_1 U_{r}^2 u \varphi) + \int_{\mathbb{R}^3} (e^2 \nabla v \nabla \psi + Q(x) v \psi - 3\mu_2 V_{r}^2 v \psi) \]

\[ - \beta \int_{\mathbb{R}^3} (U_{r}^2 v \psi + V_{r}^2 u \varphi + 2U_r V_r u \psi + 2U_r V_r v \varphi) \]

is a bounded bi-linear functional in \(E\). Thus there exists a bounded linear operator \(L\) from \(E\) to \(E\) such that

\[ \langle L(u, v), (\varphi, \psi) \rangle \]

\[ = \int_{\mathbb{R}^3} (e^2 \nabla u \nabla \varphi + P(x) u \varphi - 3\mu_1 U_{r}^2 u \varphi) + \int_{\mathbb{R}^3} (e^2 \nabla v \nabla \psi + Q(x) v \psi - 3\mu_2 V_{r}^2 v \psi) \]

\[ - \beta \int_{\mathbb{R}^3} (U_{r}^2 v \psi + V_{r}^2 u \varphi + 2U_r V_r u \psi + 2U_r V_r v \varphi), \quad (u, v), (\varphi, \psi) \in E. \]
From the above analysis, we have the following lemma.

**Lemma 2.1.** There is a constant $C > 0$, independent of $\epsilon$, such that for any $r \in S$, 
\[ \|L(u, v)\| \leq C\|(u, v)\|, \quad (u, v) \in E. \]

Next, we discuss the invertibility of $L$.

**Lemma 2.2.** There exist constants $C_0 > 0$ and $\epsilon_0 > 0$, such that for any $0 < \epsilon < \epsilon_0$ and any $r \in S$, 
\[ \|L(u, v)\| \geq C_0\|(u, v)\|, \quad (u, v) \in E. \]

**Proof.** We argue by contradiction. Suppose that there exist $\epsilon_n \to 0^+, r_n \in S$, and $(u_n, v_n) \in E$ such that 
\[ \|L(u_n, v_n)\| = o_n(1)\|(u_n, v_n)\|_{\epsilon_n}. \]
Since $L$ is linear, we may as well assume that 
\[ \|(u_n, v_n)\|_{\epsilon_n}^2 = \epsilon_n^3 \]
and 
\[ \|L(u_n, v_n)\| = o_n(1)\epsilon_n^\frac{3}{2}. \tag{2.4} \]
Then 
\[ \langle L(u_n, v_n), (\varphi, \psi) \rangle = o_n(1)\|(\varphi, \psi)\|_{\epsilon_n}^\frac{3}{2}, \quad \forall (\varphi, \psi) \in E. \tag{2.5} \]
That is, 
\[ \int_{\mathbb{R}^3} (\epsilon_n^2|\nabla u_n|^2 + P(x)|u_n|^2 - 3\mu_2U_{r_n}^2u_n^3 + \frac{2}{9}\mu_1U_{r_n}^2u_n + Q(x)v_n\psi - 3\mu_2V_{r_n}^2v_n\psi) \]
\[ - \beta \int_{\mathbb{R}^3} (U_{r_n}^2v_n\psi + V_{r_n}^2u_n\varphi + 2U_{r_n}V_{r_n}u_n\psi + 2U_{r_n}V_{r_n}v_n\varphi) \]
\[ = o_n(1)\|(\varphi, \psi)\|_{\epsilon_n}^\frac{3}{2}, \quad \forall (\varphi, \psi) \in E. \tag{2.5} \]
In particular, we have 
\[ \int_{\mathbb{R}^3} (\epsilon_n^2|\nabla u_n|^2 + P(x)|u_n|^2 - 3\mu_1U_{r_n}^2u_n^3 + \frac{2}{9}\mu_1U_{r_n}^2u_n^3 + Q(x)|v_n|^2 - 3\mu_2V_{r_n}^2v_n^2) \]
\[ - \beta \int_{\mathbb{R}^3} (U_{r_n}^2u_n^2 + V_{r_n}^2u_n^2 + 4U_{r_n}V_{r_n}u_nv_n) \]
\[ = o_n(1)\epsilon_n^3. \tag{2.6} \]
We set $\tilde{u}_n(y) = u_n(\epsilon_ny + x^1)$ and $\tilde{v}_n(y) = v_n(\epsilon_ny + x^1)$. Then 
\[ \int_{\mathbb{R}^3} (|\nabla \tilde{u}_n|^2 + P(\epsilon_ny + x^1)\tilde{u}_n^2 + |\nabla \tilde{v}_n|^2 + Q(\epsilon_ny + x^1)\tilde{v}_n^2) = 1. \tag{2.7} \]
Therefore, there exist $u, v \in H^1(\mathbb{R}^3)$ such that $n \to \infty$,

$$
\tilde{u}_n \to u, \quad \text{weakly in } H^1_{loc}(\mathbb{R}^3), \quad \tilde{u}_n \to u, \quad \text{strongly in } L^2_{loc}(\mathbb{R}^3),
$$

$$
\tilde{v}_n \to v, \quad \text{weakly in } H^1_{loc}(\mathbb{R}^3), \quad \tilde{v}_n \to v, \quad \text{strongly in } L^2_{loc}(\mathbb{R}^3).
$$

Since $\tilde{u}_n$ and $\tilde{v}_n$ are even in $y_2$ and $y_3$, it is easy to see that $u$ and $v$ are even in $y_2$ and $y_3$.

On the other hand, from the definition of $E$, we know that $(u, v)$ satisfies

$$
\int_{\mathbb{R}^3} \left( U^2 \frac{\partial U}{\partial x_1} u + V^2 \frac{\partial V}{\partial x_1} v \right) = 0. \tag{2.8}
$$

Now we claim that $(u, v)$ satisfies

$$
\begin{aligned}
-\Delta u + u - 3\mu_1 U^2 u - \beta V^2 u - 2\beta U V v &= 0, \quad x \in \mathbb{R}^3, \\
-\Delta v + v - 3\mu_2 V^2 v - \beta U^2 v - 2\beta U U v &= 0, \quad x \in \mathbb{R}^3.
\end{aligned} \tag{2.9}
$$

Define

$$
\hat{E} = \left\{ (\varphi, \psi) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left( U^2 \frac{\partial U}{\partial x_1} u + V^2 \frac{\partial V}{\partial x_1} v \right) = 0 \right\}.
$$

For any $R > 0$, let $(\varphi, \psi) \in C^\infty_0(B_R(0)) \times C^\infty_0(B_R(0)) \cap \hat{E}$ and be even in $y_2$ and $y_3$. Then $(\varphi_n(y), \psi_n(y)) := (\varphi(\frac{y}{\epsilon}, \frac{y_2}{\epsilon}), \psi(\frac{y}{\epsilon}, \frac{y_3}{\epsilon})) \in C^\infty_0(B_{Rn}(x^1)) \times C^\infty_0(B_{Rn}(x^1))$. Inserting $(\varphi_n(y), \psi_n(y))$ into (2.5), we find that

$$
\begin{aligned}
\int_{\mathbb{R}^3} (\nabla u \nabla \varphi + w \varphi - 3\mu_1 U^2 u \varphi) + \int_{\mathbb{R}^3} (\nabla v \nabla \psi + v \psi - 3\mu_2 V^2 v \psi) \\
-\beta \int_{\mathbb{R}^3} (U^2 v \psi + V^2 w \varphi + 2UV w \varphi + 2UV v \varphi) &= 0. \tag{2.10}
\end{aligned}
$$

However, since $u$ and $v$ are even in $y_2$ and $y_3$, (2.10) holds for any function $(\varphi, \psi) \in C^\infty_0(B_R(0)) \times C^\infty_0(B_R(0))$, which is odd in $y_2$ or $y_3$. Therefore, (2.10) holds for any $(\varphi, \psi) \in C^\infty_0(B_R(0)) \times C^\infty_0(B_R(0)) \cap \hat{E}$. By the density of $C^\infty_0(B_R(0)) \times C^\infty_0(B_R(0))$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, we obtain that

$$
\begin{aligned}
\int_{\mathbb{R}^3} (\nabla u \nabla \varphi + u \varphi - 3\mu_1 U^2 u \varphi) + \int_{\mathbb{R}^3} (\nabla v \nabla \psi + v \psi - 3\mu_2 V^2 v \psi) \\
-\beta \int_{\mathbb{R}^3} (U^2 v \psi + V^2 w \varphi + 2UV w \varphi + 2UV v \varphi) &= 0, \quad \forall (\varphi, \psi) \in \hat{E}. \tag{2.11}
\end{aligned}
$$

Noting that $(U, V) = (\alpha w, \gamma w)$ and $w$ is a solution of (1.4), we can show that (2.10) holds for $(\varphi, \psi) = (\frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_1})$. Thus (2.10) is true for any $(\varphi, \psi) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. Therefore, we have verified (2.10).
From Proposition 2.3 of \[31\], we can know that \((U, V)\) is non-degenerate. Since we work in the space of functions which are even in \(y_2\) and \(y_3\), the kernel of \((U, V)\) is given by the one dimensional \((\theta(\beta) \frac{\partial w}{\partial x_1}, \frac{\partial v}{\partial x_1})\). So, we get \((u, v) = c\left(\frac{\partial U}{\partial x_1}, \frac{\partial V}{\partial x_1}\right)\) for some constant \(c\). From (2.8) we can see \((u, v) = (0, 0)\).

As a result,
\[
\int_{B_R(-x_1)} (u_n^2 + v_n^2) = o_n(1)e^3, \forall R > 0.
\]

By direct calculation, we get
\[
\int_{\mathbb{R}^3} (U_{r_n}^2 u_n^2 + V_{r_n}^2 v_n^2) = o_n(1)e_n^3 + o_R(1)e_n^3.
\]

As a result,
\[
o_n(1)e_n^3
= \int_{\mathbb{R}^3} (\epsilon_n^2 |\nabla u_n|^2 + P(x)|u_n|^2 - 3\mu_1 U_{r_n}^2 u_n^2) + \int_{\mathbb{R}^3} (\epsilon_n^2 |\nabla v_n|^2 + Q(x)|v_n|^2 - 3\mu_2 V_{r_n}^2 v_n^2)
- \beta \int_{\mathbb{R}^3} (U_{r_n}^2 v_n^2 + V_{r_n}^2 u_n^2 + 4U_{r_n} V_{r_n} u_n v_n)
= (1 + o_n(1) + o_R(1))e_n^3.
\]

(2.12)

This is a contradiction. So we complete the proof.

**Lemma 2.3.** For any \((\varphi, \psi) \in E\), we have
\[
\|R(\varphi, \psi)\| = O(e^{-\frac{3}{2}}\|\varphi\|_e^3 + e^{-3}\|\varphi\|_e^4) + (\varphi, \psi)\|_e^4),
\]

\[
\|R'(\varphi, \psi)\| = O(e^{-\frac{3}{2}}\|\varphi\|_e^3 + e^{-3}\|\varphi\|_e^4) + (\varphi, \psi)\|_e^3)
\]

and
\[
\|R''(\varphi, \psi)\| = O(e^{-\frac{3}{2}}\|\varphi\|_e + e^{-3}\|\varphi\|_e^2).\]
Proof. By direct calculation, we have, for any \((u_1, v_1), (u_2, v_2) \in E\)
\[
|R(\varphi, \psi)| = \left| \int_{\mathbb{R}^3} (\mu_1 u_r \varphi^3 + \mu_2 V_r \psi^3 + \frac{\mu_1}{4} \varphi^4 + \frac{\mu_2}{4} \psi^4) \right.
\]
\[
- \frac{\beta}{2} \int_{\mathbb{R}^3} [(U_r + \varphi)^2 (V_r + \psi)^2 - U_r^2 V_r^2 - 2(U_r V_r^2 \varphi + U_r^2 V_r \psi) \right.
\]
\[
-(U_r^2 \varphi^2 + V_r^2 \psi^2 + 4U_r V_r \varphi \psi)]
\]
\[
= \left| \int_{\mathbb{R}^3} (\mu_1 u_r \varphi^3 + \mu_2 V_r \psi^3 + \frac{\mu_1}{4} \varphi^4 + \frac{\mu_2}{4} \psi^4) \right.
\]
\[
- \frac{\beta}{2} \int_{\mathbb{R}^3} (\varphi^2 \psi^2 + 2U_r \varphi \psi^2 + 2V_r \varphi^2 \psi)
\]
\[
\leq C \int_{\mathbb{R}^3} \left( \sum_{j=1}^{2k} U_{x_j, \epsilon} |\varphi|^3 + \varphi^4 + \sum_{j=1}^{2k} V_{x_j, \epsilon} |\psi|^3 + \psi^4 \right)
\]
\[
\leq C(\epsilon^{-\frac{\beta}{2}} \|(\varphi, \psi)\|_2^2 + \epsilon^{-4} \|(\varphi, \psi)\|_4^4)
\]
and
\[
|\langle R'(\varphi, \psi), (u_1, v_1) \rangle |
\]
\[
= \left| \int_{\mathbb{R}^3} (3\mu_1 u_r \varphi^2 u_1 + 3\mu_2 V_r \psi^2 v_1 + \mu_1 \varphi^3 u_1 + \mu_2 \psi^3 v_1) \right.
\]
\[
+ \beta \int_{\mathbb{R}^3} (\varphi^2 \psi u_1 + \varphi \psi^2 v_1 + 2U_r \varphi \psi u_1 + 2U_r \varphi \psi v_1 + 2V_r \varphi \psi u_1 + 2V_r \varphi \psi v_1)
\]
\[
\leq C \int_{\mathbb{R}^3} \left[ \left( \sum_{j=1}^{2k} U_{x_j, \epsilon} + \sum_{j=1}^{2k} V_{x_j, \epsilon} \right) (\varphi^2 + \psi^2)(|u_1| + |v_1|) + (|\varphi|^3 + |\psi|^3)(|v_1| + |u_1|) \right]
\]
\[
\leq C(\epsilon^{-\frac{\beta}{2}} \|(\varphi, \psi)\|_2^2 + \epsilon^{-4} \|(\varphi, \psi)\|_3^3) \|(u_1, v_1)\|_\epsilon.
\]

And by similar calculation, we get that
\[
|\langle R''(\varphi, \psi)(u_1, v_1), (u_2, v_2) \rangle | \leq C(\epsilon^{-\frac{\beta}{2}} \|(\varphi, \psi)\|_2^2 + \epsilon^{-4} \|(\varphi, \psi)\|_3^3) \|(u_1, v_1)\|_\epsilon \|(u_2, v_2)\|_\epsilon.
\]

So we complete the proof of this lemma.

**Lemma 2.4.** There exists a small constant \(\tau \in D\) such that
\[
\|l\| = O(\rho^\min\{n, m\} + \epsilon^{-\frac{3(1-\tau)}{2\tau}} + \epsilon^{-\frac{2\min\{n, m\}}{2\tau}})\epsilon^{\frac{3}{2}},
\]
where \(D = \{ x \in (0, \frac{1}{3}) : (1 - x)(2 - x) \geq \frac{11\sqrt{2}}{10} \}. \)
Proof. By direct computations, we have

\[
\sum_{j=1}^{2k} (-1)^{j-1} \int_{\mathbb{R}^3} (P(x) - 1)U_{x,j}^1 \varphi + \sum_{j=1}^{2k} (-1)^{j-1} \int_{\mathbb{R}^3} (Q(x) - 1)V_{x,j}^1 \psi
\]
\[
\leq \sum_{j=1}^{2k} \left( \int_{\mathbb{R}^3} \|P(x) - 1\|^2 U_{x,j}^2 \varphi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \varphi^2 \right)^{\frac{1}{2}} + \sum_{j=1}^{2k} \left( \int_{\mathbb{R}^3} \|Q(x) - 1\|^2 V_{x,j}^2 \psi \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} \psi^2 \right)^{\frac{1}{2}}
\]
\[
\leq C \varepsilon^2 (r^m + e^{\frac{-3(1-r)^{\alpha}}{r}}) \|\varphi\|_e + C \varepsilon^2 (r^m + e^{\frac{-3(1-r)^{\alpha}}{r}}) \|\psi\|_e,
\]
\[
\leq C(r^{\min\{m,n\}} + e^{\frac{-3(1-r)^{\alpha}}{r}}) \varepsilon^2 \|\varphi, \psi\|_e.
\]

(2.13)

and

\[
\beta \int_{\mathbb{R}^3} \left( \sum_{j=1}^{2k} (-1)^{j-1} V_{x,j}^2 \varphi - U_{x,j}^2 \right) \varphi + \beta \int_{\mathbb{R}^3} \left( \sum_{j=1}^{2k} (-1)^{j-1} V_{x,j}^2 \psi - U_{x,j}^2 \right) \psi
\]
\[
\leq C \varepsilon^2 e^{-\frac{|x^1|^2}{\varepsilon^2}} \|\varphi, \psi\|_e.
\]

(2.14)

(2.15)

Combining (2.13), (2.14) and the definition of \( l \), we can deduce that

\[
\|l\| = O(r^{\min\{m,n\}} + e^{\frac{-3(1-r)^{\alpha}}{r}} + e^{\frac{-2r\sin \theta^2}{\varepsilon}}) \varepsilon^2.
\]

Proposition 2.1. For \( \varepsilon \) sufficiently small, there exists a \( C^1 \)-map \((\varphi, \psi)\) from \( S_\varepsilon \) to \( H\): \( (\varphi, \psi) := (\varphi(r), \psi(r)) \), \( r = |x| \) satisfying \((\varphi, \psi) \in E \) and

\[
\left\langle \frac{\partial J(\varphi, \psi)}{\partial (\varphi, \psi)}, (g, h) \right\rangle = 0, \forall (g, h) \in E.
\]

Moreover, there exists a small constant \( 0 < \tau_2 < \min\{1, \frac{\min\{n,m\} - 1-\sigma}{\min\{n,m\}}\} \) such that

\[
\|l(\varphi, \psi)\|_e \leq (r^{(1-\tau_2)\min\{m,n\}} + e^{\frac{-3(1-r)^{\alpha}}{r}} + e^{\frac{-2r\sin \theta^2}{\varepsilon}}) \varepsilon^2.
\]

(2.16)

Proof. It follows from Lemma 2.4 that \( l \) is a bounded linear functional in \( E \). Thus there exists an \( l' \in E \) such that \( l(\varphi, \psi) = \langle l', (\varphi, \psi) \rangle \). Thus finding a critical point for \( J(\varphi, \psi) \) is equivalent to solving

\[
l' + L(\varphi, \psi) + R'(\varphi, \psi) = 0.
\]

(2.16)
By Lemma 2.2, \(L\) is invertible. Hence (2.16) can be written as

\[
(\varphi, \psi) = A(\varphi, \psi) := -L^{-1}l' - L^{-1}R'(\varphi, \psi).
\] (2.17)

We choose a small constant \(0 < \tau_2 < \min\{\frac{1}{5}, \frac{\min\{n, m\} - \sigma}{\min\{n, m\}}\}\) and set

\[
S = \left\{ (\varphi, \psi) \in E : \| (\varphi, \psi) \|_{\epsilon} \leq \epsilon^\frac{2}{3} \left( r^{(1 - \tau_2) \min\{m, n\}} + e^{-\frac{2(1 - \tau_2)(1 - \tau_1)r}{r}} + e^{-\frac{(1 - \tau_2)2r\sin\frac{\pi}{2}}{r}} \right) \right\}.
\]

For \(\epsilon\) sufficiently small, we have

\[
\|A(\varphi, \psi)\| \leq C\|l'\| + C\|R'(\varphi, \psi)\|
\]

\[
\leq C\epsilon^\frac{2}{3} \left( r^{\min\{m, n\}} + e^{-\frac{2(1 - \tau_2)(1 - \tau_1)r}{r}} \right)
\]

\[
+ C(\epsilon^{-\frac{2}{3}} \| (\varphi, \psi) \|_{\epsilon}^2 + e^{-4\| (\varphi, \psi) \|_{\epsilon}})
\]

\[
\leq \epsilon^\frac{2}{3} \left( r^{(1 - \tau_2) \min\{m, n\}} + e^{-\frac{2(1 - \tau_2)(1 - \tau_1)r}{r}} + e^{-\frac{(1 - \tau_2)2r\sin\frac{\pi}{2}}{r}} \right), \quad \forall (\varphi, \psi) \in S,
\]

which implies that \(A\) is a map from \(S\) to \(S\).

On the other hand, for \(\epsilon\) sufficiently small, we have

\[
|A(\varphi_1, \psi_1) - A(\varphi_2, \psi_2)|
\]

\[
\leq C\|R'(\varphi_1, \psi_1) - R'(\varphi_2, \psi_2)\|
\]

\[
\leq C\|R''(\lambda(\varphi_1, \psi_1) + (1 - \lambda)(\varphi_2, \psi_2))\| (\varphi_1, \psi_1) - (\varphi_2, \psi_2))\|_{\epsilon}
\]

\[
\leq \frac{1}{2}\| (\varphi_1, \psi_1) - (\varphi_2, \psi_2) \|_{\epsilon}.
\]

Thus for \(\epsilon\) sufficiently small, \(A\) is a contraction map. Therefore we have proved that when \(\epsilon\) is sufficiently small, \(A\) is a contraction map from \(S\) to \(S\). So the results follow from the contraction mapping theorem. This completes the proof.

Now we are ready to prove Theorem 1.1. Let \((\varphi, \psi) = (\varphi(r), \psi(r))\) be the map obtained in Proposition 2.1. Define

\[
F(r) = I_\epsilon(U_r + \varphi_r, V_r + \psi_r), \quad r \in S_\epsilon.
\]

With the same argument as in [13, 34], we can easily check that if \(r\) is a critical point of \(F(r)\), then \((U_r + \varphi_r, V_r + \psi_r)\) is a critical point of \(I_\epsilon\).

**Proof of Theorem 1.3** It follows from Lemmas 2.1 and 2.3 that

\[
\|L(\varphi_r, \psi_r)\| \leq C\| (\varphi_r, \psi_r) \|_{\epsilon}, \quad \|R(\varphi, \psi)\| \leq C(\epsilon^{-\frac{2}{3}} \| (\varphi, \psi) \|_{\epsilon}^3 + \epsilon^{-4} \| (\varphi, \psi) \|_{\epsilon}^4).
\]
So from Lemma 2.4 and Proposition A.2 we obtain that

\[ F(r) = 2k\epsilon^3 \left[ A + aBr^m + C \left( \frac{\mu\gamma^4}{2} + \frac{\mu^2\gamma^4}{2} + \beta\alpha^2\gamma^2 \right) e^{-\frac{2r\sin \frac{\pi}{2}}{k\epsilon}} + O(r^{m-1}) \right]. \]

Without loss of generality, we may as well assume that \( n > m \). Therefore

\[ F(r) = 2k\epsilon^3 \left[ A + aBr^m + Ce^{-\frac{2r\sin \frac{\pi}{2}}{k\epsilon}} + O(r^{m-1}) \right], \]

where \( A, B, C \) are fixed positive constant.

Consider \( \min \{ F(r) : r \in S_\epsilon \} \), where \( S_\epsilon \) is defined in (1.4).

Let

\[ f(r) := aBr^m + Ce^{-\frac{2r\sin \frac{\pi}{2}}{k\epsilon}}. \]

By the assumption, we know that \( a > 0 \). So by direct calculation, we can get that \( f(r) \) has a local minimum point

\[ \bar{r} = \frac{m + o_\epsilon(1)}{2\sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon}. \]

So there exists \( \epsilon_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \), there is \( r_0 \in S_\epsilon \) such that \( f'(r_0) = 0 \).

By direct computation, we can obtain that

\[ F(\bar{r}) = 2k\epsilon^3 \left[ A + \frac{m + o_\epsilon(1)}{2\sin \frac{\pi}{2k}} \right]^m aB \left( \epsilon \ln \frac{1}{\epsilon} \right)^m + \frac{maB}{2\sin \frac{\pi}{2k}} r^{m-1} + O \left( r^{m-1} \epsilon \right) \]

\[ = 2k\epsilon^3 \left[ A + \left( aB \left( \frac{m + o_\epsilon(1)}{2\sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m \right]. \]

On the other hand, we also have

\[ F \left( \frac{m - \delta}{2\sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right) = 2k\epsilon^3 \left[ A + aB \left( \frac{m - \delta}{2\sin \frac{\pi}{2k}} \right)^m \left( \epsilon \ln \frac{1}{\epsilon} \right)^m + C e^{m-\delta} + O(r^{m-1}) \right] \]

\[ \geq 2k\epsilon^3 (A + C e^{m-\delta}) \]

and

\[ F \left( \frac{m + \delta}{2\sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right) = 2k\epsilon^3 \left[ A + aB \left( \frac{m + \delta}{2\sin \frac{\pi}{2k}} \right)^m \left( \epsilon \ln \frac{1}{\epsilon} \right)^m + C e^{m+\delta} + O(r^{m-1}) \right] \]

\[ = 2k\epsilon^3 \left[ A + \left( aB \left( \frac{m + \delta}{2\sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m \right]. \]

Hence, \( F(r) \) has a local minimum point \( r_\epsilon \) in \( S_\epsilon \), and \( r_\epsilon \) is an interior point of \( S_\epsilon \). Thus \( r_\epsilon \) is a critical point of \( F(r) \). As a result, \( (U_{r_\epsilon} + \varphi_{r_\epsilon}, V_{r_\epsilon} + \psi_{r_\epsilon}) \) is a solution of (1.1).

For the case \( m > n \), the same method can be used to prove the result.
For the case $m = n$, then
\[
F(r) = 2ke^3\left[A + (aB + bC_0)r^m + Ce^{\frac{2r\sin \frac{k}{r}}{r}} + O(r^{m-1}e)\right].
\]

And let
\[
f(r) = (aB + bC_0)r^m + Ce^{\frac{2r\sin \frac{k}{r}}{r}}.
\]

Using the above methods, we can prove the result. This completes the proof.

3 Segregated Vector Solutions and the proof of Theorem 1.2

In this section we consider segregated vector solutions and prove Theorem 1.2 by proving Theorem 1.4. Let
\[
\tilde{Y}_j = \partial U_{1,x_j,\epsilon} / \partial r, \quad \tilde{Z}_j = \partial U_{2,y_j,\epsilon} / \partial \rho, \quad j = 1, 2, \ldots, 2k,
\]
where $x_j$ and $y_j$ are defined in (1.5) and (1.8) respectively.

For simplicity of notation, in the sequel we use $U_{1,x_j,\epsilon}$ and $U_{2,y_j,\epsilon}$ to replace $U_{x_j,\epsilon}$ and $V_{x_j,\epsilon}$ respectively. In this section, we assume $(r, \rho) \in S_{\epsilon} \times S_{\epsilon}$. (3.1)

Define
\[
\tilde{E} = \left\{(\varphi, \psi) \in H : \sum_{j=1}^{2k} \int_{\mathbb{R}^3} U_{1,x_j,\epsilon}^2 \tilde{Y}_j \varphi = 0, \sum_{j=1}^{2k} \int_{\mathbb{R}^3} U_{2,y_j,\epsilon}^2 \tilde{Z}_j \psi = 0 \right\}. \quad (3.2)
\]

Let
\[
\tilde{J}(\tilde{\varphi}, \tilde{\psi}) = \tilde{J}(0, 0) + \tilde{I}(\tilde{\varphi}, \tilde{\psi}) + \frac{1}{2} \tilde{Q}(\tilde{\varphi}, \tilde{\psi}) + \tilde{R}(\tilde{\varphi}, \tilde{\psi}), \quad (\tilde{\varphi}, \tilde{\psi}) \in \tilde{E}.
\]

Then, similar to (2.3), $\tilde{J}(\tilde{\varphi}, \tilde{\psi})$ has the following expansion:
\[
\tilde{J}(\tilde{\varphi}, \tilde{\psi}) = \tilde{J}(0, 0) + \tilde{I}(\tilde{\varphi}, \tilde{\psi}) + \frac{1}{2} \tilde{Q}(\tilde{\varphi}, \tilde{\psi}) + \tilde{R}(\tilde{\varphi}, \tilde{\psi}), \quad (\tilde{\varphi}, \tilde{\psi}) \in \tilde{E},
\]

where $\tilde{Q}(\tilde{\varphi}, \tilde{\psi})$ and $\tilde{R}(\tilde{\varphi}, \tilde{\psi})$ are the same as $Q(\varphi, \psi)$ and $R(\varphi, \psi)$ in section 2 if $U_{x_j,\epsilon}, V_{x_j,\epsilon}, \varphi$, and $\psi$ are replaced by $U_{1,x_j,\epsilon}, U_{2,y_j,\epsilon}, \tilde{\varphi}$, and $\tilde{\psi}$ respectively. We note that there exists a bounded linear operator $\tilde{B}_\epsilon : \tilde{E} \to \tilde{E}$ corresponding to $\tilde{Q}(\tilde{\varphi}, \tilde{\psi})$. 

Note that $\tilde{l}(\tilde{\varphi}, \tilde{\psi})$ has the following form

$$
\tilde{l}(\tilde{\varphi}, \tilde{\psi}) = 2k \sum_{j=1}^{2k} \frac{(-1)^{j-1}}{j} \int_{\mathbb{R}^3} \left( P(|x|) - 1 \right) U_{1,x^j,\epsilon} \tilde{\varphi} - \mu_1 \int_{\mathbb{R}^3} \left( \tilde{U}_r^3 - \sum_{j=1}^{2k} (-1)^{j-1} U_{1,x^j,\epsilon}^3 \right) \tilde{\varphi}
$$

$$
+ 2k \sum_{j=1}^{2k} \frac{(-1)^{j-1}}{j} \int_{\mathbb{R}^3} \left( Q(|x|) - 1 \right) U_{2,y^j,\epsilon} \tilde{\psi} - \mu_2 \int_{\mathbb{R}^3} \left( \tilde{V}_r^3 - \sum_{j=1}^{2k} (-1)^{j-1} U_{2,y^j,\epsilon}^3 \right) \tilde{\psi}
$$

$$
- \beta \int_{\mathbb{R}^3} (\tilde{U}_r \tilde{V}_r^2 \tilde{\varphi} + \tilde{U}_r^2 \tilde{V}_r \tilde{\psi}).
$$

From the above analysis, we have the following lemma:

**Lemma 3.1.** There exists a constant $C > 0$, independent of $\epsilon$, such that for any $(r, \rho) \in S_\epsilon \times S_\epsilon$

$$
\|\tilde{B}_c(\varphi, \psi)\| \leq C \|(\varphi, \psi)\|_\epsilon, \quad (\varphi, \psi) \in \tilde{E}.
$$

**Lemma 3.2.** There exist $\epsilon_0 > 0$, $\beta_0 > 0$ and $C_0 > 0$ such that for any $\beta < \beta_0$ and any $\epsilon \in (0, \epsilon_0)$, $(r, \rho) \in S_\epsilon \times S_\epsilon$, we have

$$
\|\tilde{B}_c(\varphi, \psi)\| \geq C_0 \|(\varphi, \psi)\|_\epsilon, \quad (\varphi, \psi) \in \tilde{E}.
$$

**Proof.** The argument is similar to Lemma 2.2. We argue by contradiction. Suppose that there are $\epsilon_n \to 0^+$, $(r_n, \rho_n) \in S_{\epsilon_n} \times S_{\epsilon_n}$ and $(\varphi_n, \psi_n) \in \tilde{E}$ with $\|(\varphi_n, \psi_n)\|_{\epsilon_n}^2 = \epsilon_n^6$ satisfying

$$
\langle \tilde{B}_c(\varphi_n, \psi_n), (g, h) \rangle = o_n(1) \|(\varphi_n, \psi_n)\|_{\epsilon_n} \|(g, h)\|_{\epsilon_n}, \quad \forall (g, h) \in \tilde{E}. \quad (3.3)
$$

That is,

$$
\int_{\mathbb{R}^3} (\epsilon_n^2 \nabla \varphi_n \nabla g + P(x) \varphi_n g - 3 \mu_1 \tilde{U}_r^2 \varphi_n) g
$$

$$
+ \int_{\mathbb{R}^3} (\epsilon_n^2 \nabla \psi_n \nabla h + Q(x) \psi_n h - 3 \mu_2 \tilde{V}_r^2 \psi_n) h
$$

$$
- \beta \int_{\mathbb{R}^3} (\tilde{U}_r^2 \psi_n h + \tilde{V}_r^2 \varphi_n g + 2 \tilde{U}_r \tilde{V}_r \varphi_n h + 2 \tilde{U}_r \tilde{V}_r \psi_n g)
$$

$$
= o_n(1) \|(\varphi_n, \psi_n)\|_{\epsilon_n} \|(g, h)\|_{\epsilon_n}, \quad \forall (g, h) \in \tilde{E}. \quad (3.4)
$$
In particular, we have

\[
\int_{\mathbb{R}^3} (\epsilon_n^2 |\nabla \varphi_n|^2 + P(x)|\varphi_n|^2 - 3\mu_1 \bar{U}_r^2 \varphi_n^2) \\
+ \int_{\mathbb{R}^3} (\epsilon_n^2 |\nabla \psi_n|^2 + Q(x)|\psi_n|^2 - 3\mu_2 \bar{V}_\rho^2 \psi_n^2) \\
- \beta \int_{\mathbb{R}^3} (\bar{U}_r^2 \psi_n^2 + \bar{V}_\rho^2 \varphi_n^2 + 4\bar{U}_r \bar{V}_\rho \varphi_n \psi_n) \\
= o_n(1)\epsilon_n^3
\]  

(3.5)

and

\[
\int_{\mathbb{R}^3} (\epsilon_n^2 |\nabla \varphi_n|^2 + P(x)|\varphi_n|^2 + \epsilon_n^2 |\nabla \psi_n|^2 + Q(x)|\psi_n|^2) = \epsilon_n^3.
\]

We set \( \tilde{u}_n(x) = \varphi_n(x_n x + x^1) \), \( \tilde{u}_n(x) = \psi_n(x_n x + y^1) \). Then we have

\[
\int_{\mathbb{R}^3} (|\nabla \tilde{u}_n(x)|^2 + P(\epsilon_n x + x^1)|\tilde{u}_n(x)|^2 + |\nabla \tilde{v}_n(x)|^2 + Q(\epsilon_n x + y^1)|\tilde{v}_n(x)|^2) = 1.
\]

Upon passing to a subsequence, we may as well assume that there exist \( u, v \in H^1(\mathbb{R}^3) \) such that as \( n \to +\infty \)

\[
\tilde{u}_n(x) \to u \text{ weakly in } H^1_{loc}(\mathbb{R}^3), \quad \tilde{u}_n(x) \to u \text{ strongly in } L^2_{loc}(\mathbb{R}^3),
\]

\[
\tilde{v}_n(x) \to v \text{ weakly in } H^1_{loc}(\mathbb{R}^3), \quad \tilde{v}_n(x) \to v \text{ strongly in } L^2_{loc}(\mathbb{R}^3).
\]

Moreover, \( u \) and \( v \) satisfy

\[
\int_{\mathbb{R}^3} \left( \nabla \frac{\partial U_1}{\partial x_1} \nabla u + \frac{\partial U_1}{\partial x_1} u \right) = 0, \quad \int_{\mathbb{R}^3} \left( \nabla \frac{\partial U_2}{\partial x_1} \nabla v + \frac{\partial U_2}{\partial x_1} v \right) = 0.
\]

We claim that \( u \) and \( v \) satisfy

\[
-\Delta u + u - 3\mu_1 U_1^2 u = 0, \quad -\Delta v + v - 3\mu_2 U_2^2 v = 0.
\]

Let \( \hat{\varphi}(x) \in C_0^\infty(B_R(0)) \) and be even in \( y_2 \) and \( y_3 \). Define \( \hat{\varphi}_n(x) := \hat{\varphi}(\frac{x-x^1}{\epsilon_n}) \in C_0^\infty(B_{\epsilon_n R}(x^1)) \). Then inserting \( (\hat{\varphi}_n(x), 0) \) into \( (3.4) \) and preceding as we have done in Lemma 2.2 we can see that \( u \) satisfies

\[
-\Delta u + u - 3\mu_1 U_1^2 u = 0 \quad \text{in } \mathbb{R}^3.
\]

Also, by the non-degeneracy of \( U_1, \) we find that \( u = 0 \). In the same way, we also find that \( v = 0 \).

As a result,

\[
\int_{B_R(-x^1)} \varphi_n^2 = o_n(1)\epsilon_n^3, \quad \int_{B_R(-y^i)} \psi_n^2 = o_n(1)\epsilon_n^3, \quad \forall R > 0.
\]

18
Thus, it follows from \(3.3\) and Lemma \([A.1]\) that

\[
o_n(1)\epsilon_n^3 = \int_{\mathbb{R}^3} (\epsilon_n^2 |\nabla \varphi_n|^2 + P(x)|\varphi_n|^2 - 3\mu_1 \tilde{U}_r^2 \varphi_n^2) + \int_{\mathbb{R}^3} (\epsilon_n^2 |\nabla \psi_n|^2 + Q(x)\psi_n^2 - 3\mu_2 \tilde{V}_\rho^2 \psi_n^2) - \beta \int_{\mathbb{R}^3} (\tilde{U}_r^2 \psi_n^2 + \tilde{V}_\rho^2 \varphi_n^2 + 4\tilde{U}_r \tilde{V}_\rho \varphi_n \psi_n) \geq \|((\tilde{\varphi}_n, \tilde{\psi}_n))\|_{r_n}^2 - C\beta \|((\tilde{\varphi}_n, \tilde{\psi}_n))\|_{r_n}^2 + \epsilon_n^3(o_n(1) + o_R(1)). \tag{3.6}
\]

If \(\beta < \beta_0 := \frac{1}{\epsilon_0}\), and for large \(n\) and large \(R\), we get a contradiction. So the result in this Lemma is true. This completes the proof.

From \([2.13]\), \([2.14]\) and Lemma \([A.1]\) we can get the following Lemma.

**Lemma 3.3.** There exists a small constant \(\tilde{\tau}_1 \in D\) such that

\[
\|\tilde{l}\| = O(\epsilon^m + \rho^n + e^{-\frac{3(1-\epsilon)\tau_1}{3}} + e^{-\frac{3(1-\epsilon)\tau_2}{3}} + e^{-\frac{2\epsilon_n \sin \frac{\pi \tau_1}{6}}{\epsilon_n^3}} + e^{-\frac{2\epsilon_n \sin \frac{\pi \tau_2}{6}}{\epsilon_n^3}} + e^{-\frac{\beta}{(\ln \frac{1}{\epsilon})^3} e^{-\sqrt{\frac{r_n \cos \frac{\pi \tau_1}{6} + r \sin \frac{\pi \tau_2}{6}}}}}
\]

where \(D\) has been defined in Lemma \([2.4]\).

**Proposition 3.1.** For \(\epsilon > 0\) sufficiently small, there exists a \(C^1\)-map \((\tilde{\varphi}, \tilde{\psi})\) from \(S_x \times S_x\) to \(H\) such that \((\tilde{\varphi}(r, \rho), \tilde{\psi}(r, \rho), r = |x^1|, \rho = |y^1|\), satisfying \((\tilde{\varphi}, \tilde{\psi}) \in E\), and

\[
\left\langle \frac{\partial \tilde{J}(\tilde{\varphi}, \tilde{\psi})}{\partial (\tilde{\varphi}, \tilde{\psi})}, (g, h) \right\rangle = 0, \ \forall (g, h) \in \tilde{E}
\]

Moreover, there exists a constant \(0 < \tilde{\tau}_2 < \min\{\frac{1}{\epsilon}, \frac{\min\{n, m\} - 1 - \sigma}{\min\{n, m\}}\}\) and a constant \(\tilde{C}\) such that

\[
\|((\tilde{\varphi}, \tilde{\psi}))\| \leq e^{\frac{1}{\epsilon}} (r^{(1-\epsilon_0)m} + \rho^{(1-\epsilon_0)n} + e^{-\frac{3(1-\epsilon)(1-\epsilon_0)\tau_1}{3}} + e^{-\frac{3(1-\epsilon)(1-\epsilon_0)\tau_2}{3}} + e^{-\frac{\tau_1 \sin \frac{\pi \tau_1}{6}}{r^{\epsilon_n}}} + e^{-\frac{\tau_2 \sin \frac{\pi \tau_2}{6}}{r^{\epsilon_n}}} + \tilde{C} e^{-\frac{\beta}{(\ln \frac{1}{\epsilon})^3} e^{-\sqrt{\frac{r_n \cos \frac{\pi \tau_1}{6} + r \sin \frac{\pi \tau_2}{6}}}}})
\]

**Proof.** From the definition of \(\tilde{l}(\tilde{\varphi}, \tilde{\psi})\), we know that \(\tilde{l}(\tilde{\varphi}, \tilde{\psi})\) is a bounded linear functional in \(\tilde{E}\). Thus it follows from Reisz Representation Theorem that there is an \(\tilde{\ell} \in \tilde{E}\) such that

\[
\tilde{l}(\tilde{\varphi}, \tilde{\psi}) = \langle \tilde{\varphi}, (\tilde{\varphi}, \tilde{\psi}) \rangle.
\]

So finding a critical point of \(\tilde{J}(\tilde{\varphi}, \tilde{\psi})\) is equivalent to solving

\[
\tilde{\ell} + \tilde{B}_\varepsilon(\tilde{\varphi}, \tilde{\psi}) + \tilde{R}'(\tilde{\varphi}, \tilde{\psi}) = 0. \tag{3.7}
\]
By Lemma 3.2, \( \tilde{B} \) is invertible. Hence (3.7) can be written as

\[
(\tilde{\varphi}, \tilde{\psi}) = \tilde{A}(\tilde{\varphi}, \tilde{\psi}) := -\tilde{B}_c^{-1}\tilde{\Psi} - \tilde{B}_c^{-1}\tilde{R}'(\tilde{\varphi}, \tilde{\psi}).
\]

We choose a small constant \( 0 < \tilde{\tau}_2 < \min\{\frac{1}{5}, \min\{n, m\} - 1 - \psi\} \) and a sufficiently large constant \( \tilde{C} \), and set

\[
\tilde{S} = \left\{ (\tilde{\varphi}, \tilde{\psi}) \in \tilde{E} : \| (\tilde{\varphi}, \tilde{\psi}) \| \leq \varepsilon \frac{2}{\rho} \left( r(1 - \tilde{\tau}_2)m + \rho(1 - \tilde{\tau}_2)\nu + e^{-\frac{3(1 - \tilde{\tau}_2)(1 - \tilde{\tau}_1)\rho}{\varepsilon}} + e^{-\frac{(1 - \tilde{\tau}_2)2r\sin \frac{\theta}{\rho}}{\varepsilon}} + e^{-\frac{(1 - \tilde{\tau}_2)2\rho\sin \frac{\theta}{\rho}}{\varepsilon}} + e^{-\frac{\tilde{C} \beta}{(\ln 4)^\frac{1}{n}}} e^{-\frac{\sqrt{(\rho - r \cos \frac{\theta}{\rho})^2 + (r \sin \frac{\theta}{\rho})^2}}{\varepsilon}} \right) \right\}.
\]

For \( \varepsilon \) sufficiently small, we have

\[
\| \tilde{A}(\tilde{\varphi}, \tilde{\psi}) \| \\
\leq C\| \tilde{I}_k \| + C\| \tilde{R}'(\tilde{\varphi}, \tilde{\psi}) \| \\
\leq C\left( r^m + \rho^n + e^{-\frac{3(1 - \tilde{\tau}_1)\rho}{\varepsilon}} + e^{-\frac{3(1 - \tilde{\tau}_2)\rho}{\varepsilon}} + e^{-\frac{2r\sin \frac{\theta}{\rho}}{\varepsilon}} + e^{-\frac{2\rho\sin \frac{\theta}{\rho}}{\varepsilon}} + e^{-\frac{\tilde{C} \beta}{(\ln 4)^\frac{1}{n}}} e^{-\frac{\sqrt{(\rho - r \cos \frac{\theta}{\rho})^2 + (r \sin \frac{\theta}{\rho})^2}}{\varepsilon}} \right) \varepsilon^{\frac{2}{\rho}} + C(\varepsilon^{-\frac{3}{2}} \| (\tilde{\varphi}, \tilde{\psi}) \|)^2 + \varepsilon^{-\frac{1}{2}} \| (\tilde{\varphi}, \tilde{\psi}) \|_2^2
\]

which implies that \( \tilde{A} \) is a map from \( \tilde{S} \) to \( \tilde{S} \).

On the other hand, for \( \varepsilon \) sufficiently small, we get

\[
\| \tilde{A}(\tilde{\varphi}_1, \tilde{\psi}_1) - \tilde{A}(\tilde{\varphi}_2, \tilde{\psi}_2) \| \\
\leq C\| \tilde{R}'(\tilde{\varphi}_1, \tilde{\psi}_1) - \tilde{R}'(\tilde{\varphi}_2, \tilde{\psi}_2) \| \\
\leq C\| \tilde{R}''(\lambda(\tilde{\varphi}_1, \tilde{\psi}_1) + (1 - \lambda)(\tilde{\varphi}_2, \tilde{\psi}_2)) \| \| (\tilde{\varphi}_1, \tilde{\psi}_1) - (\tilde{\varphi}_2, \tilde{\psi}_2) \|_e
\]

\[
\leq C \left( e^{-\frac{3}{\rho}} \| (\tilde{\varphi}_1, \tilde{\psi}_1) \|_e + \| (\tilde{\varphi}_2, \tilde{\psi}_2) \|_e + \varepsilon^{-\frac{3}{2}} \| (\tilde{\varphi}_1, \tilde{\psi}_1) \|_2^2 + \| (\tilde{\varphi}_2, \tilde{\psi}_2) \|_2^2 \right) \| (\tilde{\varphi}_1, \tilde{\psi}_1) - (\tilde{\varphi}_2, \tilde{\psi}_2) \|_e
\]

\[
\leq \frac{1}{2} \| (\tilde{\varphi}_1, \tilde{\psi}_1) - (\tilde{\varphi}_2, \tilde{\psi}_2) \|_e.
\]
Thus for $\epsilon$ sufficiently small, $A$ is a contraction map. Therefore we have proved that when $\epsilon$ is sufficiently small, $A$ is a contraction map from $\hat{S}$ to $\hat{S}$. So the results follow from the contraction mapping theorem. This completes the proof.

Now we are ready to prove Theorem 1.2. Let $(\tilde{\varphi}(r, \rho), \tilde{\psi}(r, \rho))$ be the map obtained in Proposition 3.1. Define

$$
\tilde{F}(r, \rho) = I(\tilde{U}_r + \tilde{\varphi}(r, \rho), \tilde{V}_\rho + \tilde{\psi}(r, \rho)), \quad (r, \rho) \in S_\epsilon \times S_\epsilon.
$$

We can check that for $\epsilon$ sufficiently small, if $(r, \rho)$ is a critical point of $\tilde{F}(r, \rho)$, then $(\tilde{U}_r + \tilde{\varphi}(r, \rho), \tilde{V}_\rho + \tilde{\psi}(r, \rho))$ is a critical point of $I$.

**Proof of Theorem 1.4** From Lemma 2.3, 3.3, and Proposition 3.1, we have

$$
\tilde{F}(r, \rho) = 2k^3 \left[ \hat{A} + a\tilde{B}r^m + b\hat{C}\rho^n + B_1e^{-2r\sin \frac{\pi}{2k}} + B_2e^{-2\rho\sin \frac{\pi}{2k}} + o_\epsilon(1) \right].
$$

Consider the minimization problem

$$
\min \{ \tilde{F}(r, \rho) : (r, \rho) \in S_\epsilon \times S_\epsilon \}.
$$

Since $\tilde{F}(r, \rho)$ is defined in a closed domain, the minimization can be attained. So we may assume that

$$
\tilde{F}(r_1, \rho_1) = \min \{ \tilde{F}(r, \rho) : (r, \rho) \in S_\epsilon \times S_\epsilon \}.
$$

We claim that $(r_1, \rho_1)$ is an interior point of $S_\epsilon \times S_\epsilon$.

We assume that

$$
\tilde{g}_1(r) = a\tilde{B}r^m + B_1e^{-2r\sin \frac{\pi}{2k}} \quad \text{and} \quad \tilde{h}_1(\rho) = b\hat{C}\rho^n + B_2e^{-2\rho\sin \frac{\pi}{2k}}.
$$

By direct computation, we see that $\tilde{g}_1(r)$ attain the local minimization at

$$
\tilde{r} = \frac{m + o(1)}{2\sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon}.
$$

We have that

$$
\tilde{g}_1(\tilde{r}) = \left( a\tilde{B} \left( \frac{m}{2\sin \frac{\pi}{2k}} \right)^m + o(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m,
$$

$$
\tilde{g}_1 \left( \frac{m - \delta}{2\sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right) = C\epsilon^{m-\delta},
$$

and

$$
\tilde{g}_1 \left( \frac{m + \delta}{2\sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right) = \left( a\tilde{B} \left( \frac{m + \delta}{2\sin \frac{\pi}{2k}} \right)^m + o(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m.
$$
Similarly, $\tilde{h}_1(\rho)$ also attains the local minimization at

$$\tilde{\rho} = \frac{m + o_\epsilon(1)}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon}.$$ 

And we also have

$$\tilde{h}_1(\tilde{\rho}) = \left( b \tilde{C} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m,$$

$$\tilde{h}_1 \left( \frac{m - \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right) = C \epsilon^{m - \delta},$$

and

$$\tilde{h}_1 \left( \frac{m + \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right) = \left( b \tilde{C} \left( \frac{m + \delta}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m.$$

And we may assume that

$$\tilde{g}_2(r) = a \tilde{B} r^m + (B_1 + o_\epsilon(1)) e^{-\frac{2r \sin \frac{\pi}{2k}}{2k}} \quad \text{and} \quad \tilde{h}_2(\rho) = b \tilde{C} \rho^m + B_2 e^{-\frac{2\rho \sin \frac{\pi}{2k}}{2k}}.$$ 

By direct computation, we see that $\tilde{g}_2(r)$ attains the local minimization at

$$\bar{r} = \frac{m + o_\epsilon(1)}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon}.$$ 

We have that

$$\tilde{g}_2(\bar{r}) = \left( a \tilde{B} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m,$$

$$\tilde{g}_2 \left( \frac{m - \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right) = C \epsilon^{m - \delta}$$

and

$$\tilde{g}_2 \left( \frac{m + \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right) = \left( a \tilde{B} \left( \frac{m + \delta}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m.$$

Similarly, $\tilde{h}_2(\rho)$ also attains the local minimization at

$$\tilde{\rho} = \frac{m + o_\epsilon(1)}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon}.$$ 

And we also have

$$\tilde{h}_2(\tilde{\rho}) = \left( b \tilde{C} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m,$$

$$\tilde{h}_2 \left( \frac{m - \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right) = C \epsilon^{m - \delta}.$$
and 
\[ \tilde{h}_2 \left( \frac{m + \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon} \right) = \left( b\tilde{C} \left( \frac{m + \delta}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m. \]

If \( o_\epsilon(1) > 0 \), then
\[ \tilde{F}(r_1, \rho_1) \leq 2k\epsilon^3 \left[ A + \min_{(r, \rho) \in S \times S} \{ \tilde{g}_2(r) + \tilde{h}_2(\rho) + O(\epsilon^{m-1}\epsilon + \rho^{n-1}\epsilon) \} \right] \]
\[ \leq 2k\epsilon^3 \left[ A + \left( a\tilde{B} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + b\tilde{C} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m. \]

If \( o_\epsilon(1) \leq 0 \), then
\[ \tilde{F}(r_1, \rho_1) \leq 2k\epsilon^3 \left[ A + \left( a\tilde{B} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + b\tilde{C} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m. \]

Thus we get
\[ \tilde{F}(r_1, \rho_1) \leq 2k\epsilon^3 \left[ A + \left( a\tilde{B} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + b\tilde{C} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m. \] (3.8)

For convenience, we denote \( r_i := \frac{m - \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon}, r_e := \frac{m + \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon}, \rho_i := \frac{m - \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon}, \rho_e := \frac{m + \delta}{2 \sin \frac{\pi}{2k}} \epsilon \ln \frac{1}{\epsilon}. \)

If \( o_\epsilon(1) > 0 \), then
\[ \tilde{F}(r_i, \rho) \geq 2k\epsilon^3 \left[ A + \tilde{g}_1(r_i) + \tilde{h}_1(\rho) + O(\epsilon^{m-1}\epsilon + \rho^{n-1}\epsilon) \right] \]
\[ \geq 2k\epsilon^3 \left[ A + \epsilon^{m-\delta} + \left( b\tilde{C} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m. \]

If \( o_\epsilon(1) \leq 0 \), then
\[ \tilde{F}(r_i, \rho) = 2k\epsilon^3 \left[ A + \tilde{g}_2(r_i) + \tilde{h}_2(\rho) + O(\epsilon^{m-1}\epsilon + \rho^{n-1}\epsilon) \right] \]
\[ \geq 2k\epsilon^3 \left[ A + \epsilon^{m-\delta} + \left( b\tilde{C} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m. \]

Therefore, we have
\[ \tilde{F}(r_i, \rho) \geq 2k\epsilon^3 \left[ A + \epsilon^{m-\delta} + \left( b\tilde{C} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m. \] (3.9)
Similarly, we also have

\[ \tilde{F}(r, \rho) \geq 2k e^3 \left[ \tilde{A} + \left( a \tilde{B} \left( \frac{m + \delta}{2 \sin \frac{\pi}{2k}} \right)^m + b \tilde{C} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m \right], \tag{3.10} \]

\[ \tilde{F}(r, \rho) \geq 2k e^3 \left[ \tilde{A} + C \epsilon^{m-\delta} + \left( a \tilde{B} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m \right] \tag{3.11} \]

and

\[ \tilde{F}(r, \rho) \geq 2k e^3 \left[ \tilde{A} + \left( a \tilde{B} \left( \frac{m}{2 \sin \frac{\pi}{2k}} \right)^m + b \tilde{C} \left( \frac{m + \delta}{2 \sin \frac{\pi}{2k}} \right)^m + o_\epsilon(1) \right) \left( \epsilon \ln \frac{1}{\epsilon} \right)^m \right]. \tag{3.12} \]

From (3.8) to (3.12), we can see that when \( \epsilon \) is sufficiently small, the local minimization of \( \tilde{F}(r, \rho) \) can’t be obtained at the boundary of \( S_r \times S_r \). That is, \((r_1, \rho_1)\) is an interior point of \( S_r \times S_r \). Thus \((r_1, \rho_1)\) is a critical point of \( \tilde{F}(r, \rho) \). So \((\tilde{U}_{r_1}, \tilde{\varphi}(r_1, \rho_1), \tilde{V}_{\rho_1} + \tilde{\psi}(r_1, \rho_1))\) is a solution of (1.1). This completes the proof.

## A Energy estimate

In this section, we will give out some energy estimates of the approximate solutions. Recall that

\[ x^j := \left( r \cos \left( \frac{(j-1)\pi}{k} \right), r \sin \left( \frac{(j-1)\pi}{k} \right), x_3 \right), \quad j = 1, 2, \ldots, 2k, \]

\[ y^j := \left( \rho \cos \left( \frac{(2j-1)\pi}{2k} \right), \rho \sin \left( \frac{(2j-1)\pi}{2k} \right), x_3 \right), \quad j = 1, 2, \ldots, 2k, \]

\[ U_r(x) = \sum_{j=1}^{2k} (-1)^{j-1} U_{x^j, \epsilon}, \quad V_r(x) = \sum_{j=1}^{2k} (-1)^{j-1} V_{y^j, \epsilon}, \]

\[ \tilde{U}_r = \sum_{j=1}^{2k} (-1)^{j-1} U_{x^j, \epsilon}, \quad \tilde{V}_\rho = \sum_{j=1}^{2k} (-1)^{j-1} U_{y^j, \epsilon} \]

and

\[ I_\epsilon(u, v) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \epsilon^2 |\nabla u|^2 + P(x) u^2 + \epsilon^2 |\nabla v|^2 + Q(x) v^2 \right) - \frac{1}{4} \int_{\mathbb{R}^3} \left( \mu_1 |u|^4 + \mu_2 |v|^4 \right) \]

\[ = \frac{\beta}{2} \int_{\mathbb{R}^3} u^2 v^2. \]

**Proposition A.1.** Assume that (P) and (Q) hold. Then we get the following energy estimate:

\[ I_\epsilon(U_{x^j, \epsilon}, V_{y^j, \epsilon}) = \epsilon^3 \left[ A + a Br^m + bC_0 r^n + O(r^{m-1} \epsilon + r^{n-1} \epsilon + e^{-(2-\gamma)(1-\epsilon)\gamma}) \right], \]

where \( a, b \) is given in (P) and (Q), \( \tau \) is determined in Lemma 2.4, \( A = \frac{1}{4} (\mu_1 \alpha^4 + \mu_2 \gamma^4 + 2\beta \alpha^2 \gamma^2) \int_{\mathbb{R}^3} w^4 \, dx, \quad B = \frac{1}{8} \alpha^2 \int_{\mathbb{R}^3} w^2 \, dx, \quad \text{and} \quad C_0 = \frac{1}{2} \gamma^2 \int_{\mathbb{R}^3} w^2 \, dx. \]
Proof. By direct computation, we have

\[
I_{e}(U_{x^{j},e}, V_{x^{j},e}) = \frac{1}{2} \int_{\mathbb{R}^{3}} \left( e^{2} |\nabla U_{x^{j},e}|^{2} + U_{x^{j},e}^{2} + e^{2} |\nabla V_{x^{j},e}|^{2} + V_{x^{j},e}^{2} \right) \\
- \frac{1}{4} \int_{\mathbb{R}^{3}} \left( \mu_{1} |U_{x^{j},e}|^{4} + \mu_{2} |V_{x^{j},e}|^{4} \right) - \frac{\beta}{2} \int_{\mathbb{R}^{3}} U_{x^{j},e}^{2} V_{x^{j},e}^{2} \\
+ \frac{1}{2} \int_{\mathbb{R}^{3}} \left[ (P(x) - 1)U_{x^{j},e}^{2} + (Q(x) - 1)V_{x^{j},e}^{2} \right] \\
= \frac{1}{4} \int_{\mathbb{R}^{3}} \left( \mu_{1} |U_{x^{j},e}|^{4} + \mu_{2} |V_{x^{j},e}|^{4} \right) + \frac{\beta}{2} \int_{\mathbb{R}^{3}} U_{x^{j},e}^{2} V_{x^{j},e}^{2} \\
+ \frac{1}{2} \int_{\mathbb{R}^{3}} \left[ (P(x) - 1)U_{x^{j},e}^{2} + (Q(x) - 1)V_{x^{j},e}^{2} \right].
\] (A.1)

But

\[
\frac{1}{4} \int_{\mathbb{R}^{3}} \left( \mu_{1} |U_{x^{j},e}|^{4} + \mu_{2} |V_{x^{j},e}|^{4} \right) = \frac{\epsilon^{3}}{4} \int_{\mathbb{R}^{3}} \left( \mu_{1} U^{4} + \mu_{2} V^{4} \right) \\
= \frac{\epsilon^{3}}{4} \left( \mu_{1} \alpha^{4} + \mu_{2} \gamma^{4} \right) \int_{\mathbb{R}^{3}} w^{4}
\] (A.2)

and

\[
\frac{\beta}{2} \int_{\mathbb{R}^{3}} U_{x^{j},e}^{2} V_{x^{j},e}^{2} = \frac{\beta}{2} \epsilon^{3} \int_{\mathbb{R}^{3}} U^{2} V^{2} = \frac{\beta}{2} \epsilon^{3} \alpha^{2} \gamma^{2} \int_{\mathbb{R}^{3}} w^{4}.
\] (A.3)

For any \( m > 1 \) and any \( 0 < d < 1 \), we have

\[
|e^{y} + x^{j}|^{m} = |x^{j}|^{m} \left( 1 + O \left( \frac{|e^{y}|}{|x^{j}|} \right) \right), y \in B_{\frac{1}{4}r}(0).
\]

Since

\[
P(r) = 1 + ar^{m} + O(r^{m+\theta}) \quad \text{as } r \to 0^{+},
\]

25
we get
\[
\frac{1}{2} \int_{\mathbb{R}^3} (P(x) - 1)U^2_{x^j, \epsilon}
= \frac{1}{2} \epsilon^3 \int_{\mathbb{R}^3} (P(\epsilon y + x^j) - 1)U^2 \\
= \frac{1}{2} \epsilon^3 \left[ \int_{B_{(1-\tau)/\epsilon}(0)} (P(\epsilon y + x^j) - 1)U^2 + \int_{B_{r_{(1-\tau)/\epsilon}}(0)} (P(\epsilon y + x^j) - 1)U^2 \right] \\
= \frac{1}{2} \epsilon^3 \left[ \int_{B_{(1-\tau)/\epsilon}(0)} (\epsilon |y + x^j|^m + O(|\epsilon y + x^j|^{m+\theta}))U^2 + O(e^{-\frac{(2-\tau)(1-\tau)r}{\epsilon}}) \right] \\
= \frac{1}{2} \epsilon^3 \left[ \int_{B_{(1-\tau)/\epsilon}(0)} a|\epsilon y + x^j|^m \left( 1 + O\left( \frac{|y|}{|x^j|} \right) \right) + O(|\epsilon y + x^j|^{m+\theta}) \right] U^2 \\
+ O(e^{-\frac{(2-\tau)(1-\tau)r}{\epsilon}}) \\
= \frac{1}{2} \epsilon^3 \left[ \int_{B_{(1-\tau)/\epsilon}(0)} a|\epsilon y + x^j|^m \left( 1 + O\left( \frac{|y|}{|x^j|} \right) \right) \right] U^2 \\
+ O(e^{-\frac{(2-\tau)(1-\tau)r}{\epsilon}}) \\
= \epsilon^3 \left[ aBr^m + O(r^{m-1} \epsilon) + O(e^{-\frac{(2-\tau)(1-\tau)r}{\epsilon}}) \right],
\] (A.4)

where \( \tau \) is a small positive constant. Noting that
\[
Q(r) = 1 + br^m + O(r^{n+\delta}) \quad \text{as} \ r \to 0^+,
\]
by the same argument as above, we can get
\[
\frac{1}{2} \int_{\mathbb{R}^3} (Q(x) - 1)V^2_{x^j, \epsilon} = \epsilon^3 \left[ bCr^m + O(r^{m-1} \epsilon) + O(e^{-\frac{(2-\tau)(1-\tau)r}{\epsilon}}) \right].
\] (A.5)

So combining (A.4) - (A.5), we get
\[
I_\epsilon(U_{x^j, \epsilon}, V_{x^j, \epsilon}) = \epsilon^3 \left[ A + aBr^m + bCr^m + O(r^{m-1} \epsilon + r^{n-1} \epsilon + e^{-\frac{(2-\tau)(1-\tau)r}{\epsilon}}) \right].
\]

**Proposition A.2.** Assume that \( P \) and \( Q \) hold. Then there exists a small constant \( 0 < \sigma < \min\{\frac{1}{10}, \min\{m, n\} - 1\} \) and a positive constant \( C \) such that
\[
I_\epsilon(U_r, V_r) = 2k\epsilon^3 \left[ A + aBr^m + bCr^m + C\left( \frac{a\alpha}{2} + \frac{\beta\gamma^4}{2} + \beta\alpha^2\gamma^4 \right) + O(r^{m-1} \epsilon + r^{n-1} \epsilon + e^{-\frac{(1-\sigma)(2-\tau)r}{\epsilon}}) \right],
\]
where \( \tau \) is defined in Lemma 2.4.
Proof. We know that

\[ I_x(U_r, V_r) = \sum_{j=1}^{2k} I_x(U_{x_j}, V_{x_j}) \]

\[-\frac{\mu_1}{4} \int_{\mathbb{R}^3} \left[ U_r^4 - \sum_{j=1}^{2k} U_{x_j}^4 - 2 \sum_{i \neq j} (-1)^{i+j} U_{x_j}^3 V_{x_j} V_{x_i} \right] \]

\[-\frac{\mu_2}{4} \int_{\mathbb{R}^3} \left[ V_r^4 - \sum_{j=1}^{2k} V_{x_j}^4 - 2 \sum_{i \neq j} (-1)^{i+j} V_{x_j}^3 V_{x_i} \right] \]

\[-\frac{\beta}{2} \int_{\mathbb{R}^3} \left[ U_r^2 V_r^2 - \sum_{j=1}^{2k} U_{x_j}^2 V_{x_j}^2 - \sum_{i \neq j} (-1)^{i+j} V_{x_j} U_{x_i} V_{x_i} V_{x_i} - \sum_{i \neq j} (-1)^{i+j} V_{x_j}^2 V_{x_i} V_{x_i} \right] \]

\[+ \frac{1}{2} \sum_{i \neq j} (-1)^{i+j} \int_{\mathbb{R}^3} \left[ (P(x) - 1) U_{x_j} V_{x_i} + (Q(x) - 1) V_{x_j} V_{x_i} \right]. \]

(A.6)

But there exists a small positive \(0 < \sigma < \min\{\frac{1}{10}, \min\{m, n\} - 1\}\) such that

\[-\frac{\mu_1}{4} \int_{\mathbb{R}^3} \left[ U_r^4 - \sum_{j=1}^{2k} U_{x_j}^4 - 2 \sum_{i \neq j} (-1)^{i+j} U_{x_j}^3 U_{x_i} \right] \]

\[= \frac{\mu_1}{2} \int_{\mathbb{R}^3} \left[ \sum_{|i-j|=1 \text{or } 2k-1} U_{x_j} U_{x_i} + O\left( \sum_{1<|i-j|<2k-1} U_{x_j} U_{x_i} + \sum_{i \neq j} U_{x_j}^2 U_{x_i} \right) \right] \]

\[= \frac{\mu_1 \alpha^4}{2} \int_{\mathbb{R}^3} \sum_{|i-j|=1 \text{or } 2k-1} w_{x_j} w_{x_i} + O\left( e^{- (1+\sigma) |x| - x^2} \right) \]

\[= \varepsilon^3 \left( C \frac{\mu_1 \alpha^4}{2} e^{- \frac{2 \pi \sin \phi}{\varepsilon}} + O\left( e^{- (1+\sigma) |x| - x^2} \right) \right) \]

(A.7)

Similarly, we have

\[-\frac{\mu_2}{4} \int_{\mathbb{R}^3} \left[ V_r^4 - \sum_{j=1}^{2k} V_{x_j}^4 - 2 \sum_{i \neq j} (-1)^{i+j} V_{x_j}^3 V_{x_i} \right] \]

\[= \frac{\mu_2 \alpha^4}{2} \int_{\mathbb{R}^3} \sum_{|i-j|=1 \text{or } 2k-1} w_{x_j} w_{x_i} + O\left( e^{- (1+\sigma) |x| - x^2} \right), \]

(A.8)

\[= \varepsilon^3 \left( C \frac{\mu_2 \alpha^4}{2} e^{- \frac{2 \pi \sin \phi}{\varepsilon}} + O\left( e^{- (1+\sigma) |x| - x^2} \right) \right) \]

27
Lemma A.1.  

Then we have

\[ -\frac{\beta}{2} \int_{\mathbb{R}^3} \left[ U_r v_x^2 - 2^k \frac{U_{x_i}}{e} v_{x_i}^2 - \sum_{i \neq j} \frac{1}{(i+j)^{\frac{1}{2}}} U_{x_j} v_{x_j}^2 \right] \]

\[ = \beta \alpha^2 \gamma^2 \int_{\mathbb{R}^3} \sum_{|i-j|=1} w_{x_i, x_j}^3 w_{x_i, e}^3 + e^3 O(e^{-\frac{2r \sin \varphi}{\varphi}}) \]

\[ = e^3 (C \beta \alpha^2 \gamma^2 e^{\frac{2r \sin \varphi}{\varphi}} + O(e^{-\frac{(1+\varepsilon)|x^1-x^2|}{\varepsilon}})) \]

(A.9)

Combining (A.6)–(A.9) and Proposition A.1 we can get

\[ I_\varepsilon(U_r, V_r) = 2ke^3 \left[ A + aB r^m + bC_0 r^n + C\left( \frac{\mu_1 \alpha^4}{2} + \frac{\mu_2 \gamma^4}{2} + \beta \alpha^2 \gamma^2 \right) e^{-\frac{2r \sin \varphi}{\varphi}} + O\left( r^{m-1} e + r^{n-1} e + e^{-\frac{(1+\varepsilon)(2r \sin \varphi)}{\varepsilon}} \right) \right] . \]

(A.10)

This completes the proof.

Lemma A.1.

\[ \int_{\mathbb{R}^3} U_{x_i, x_i}^2 U_{x_j, x_j}^2 = e^3 \alpha_i (1) e^{-\frac{2|x^i-x^j|}{\varphi}}. \]

Proof. Denote

\[ \Omega_1 = \{ x \in \mathbb{R}^3 : |x - y^i| \geq |x - x^i|, \ \Omega_2 = \{ x \in \mathbb{R}^3 : |x - y^i| \leq |x - x^i|, \}

\[ \omega_1 = \{ x \in \mathbb{R}^3 : |x^i - y^i| \geq |x^j - y^j|, \ \omega_2 = \{ x \in \mathbb{R}^3 : |x^i - y^i| \leq |x^j - y^j| \}

and

\[ \omega'_1 = \{ x \in \omega_1 : |x - y^i| \leq \varepsilon \left( \frac{\ln 1}{\varepsilon} \right)^{\frac{1}{4}}, \ \omega'_2 = \{ x \in \omega_1 : |x - y^i| \geq \varepsilon \left( \frac{\ln 1}{\varepsilon} \right)^{\frac{1}{4}} \}. \]

Then we have

\[ \int_{\mathbb{R}^3} U_{x_i, x_i}^2 U_{x_j, x_j}^2 = \int_{\Omega_1} U_{x_i, x_i}^2 U_{x_j, x_j}^2 + \int_{\Omega_2} U_{x_i, x_i}^2 U_{x_j, x_j}^2 . \]

Since we can estimate this term \( \int_{\Omega_1} U_{x_i, x_i}^2 U_{x_j, x_j}^2 \) similarly, here we only estimate

\[ \int_{\Omega_2} U_{x_i, x_i}^2 U_{x_j, x_j}^2 . \]

By the definition of \( \Omega_2 \), we can conclude that

\[ |x - x^i| \geq \frac{1}{2} |x^i - y^j|, \ \forall x \in \Omega_2. \]

28
Then we have

\[
\int_{\Omega_2 \cap \omega_2} U_{1,x', \epsilon}^2 U_{2,y', \epsilon}^2 \leq C e^{-\frac{2|x - x'|}{\epsilon}} \int_{\Omega_2 \cap \omega_2} e^{-\frac{|x - y'|}{\epsilon}} \\
\leq C e^{-\frac{2|x - x'|}{\epsilon}} \int_{\Omega_2 \cap \omega_2} e^{-\frac{|x - y'|}{2\epsilon}} e^{-\frac{|x - y'|}{2\epsilon}}. \tag{A.11}
\]

\[
\int_{\Omega_2' \cap \omega'_1} U_{1,x', \epsilon}^2 U_{2,y', \epsilon}^2 \leq \int_{\Omega_2' \cap \omega'_1} U_{1,x', \epsilon}^2 U_{2,y', \epsilon}^2 + \int_{\Omega_2' \cap \omega'_1} U_{1,x', \epsilon}^2 U_{2,y', \epsilon}^2,
\]

\[
\int_{\Omega_2' \cap \omega'_1} U_{1,x', \epsilon}^2 U_{2,y', \epsilon}^2 \leq C e^{-\frac{2|x - x'|}{\epsilon}} \int_{\Omega_2' \cap \omega'_1} e^{-\frac{2(|x - x'| \pm |x - y'| - |x - y'|)}{\epsilon}} \\
\leq C e^{-\frac{2|x - x'|}{\epsilon}} e^{3} \int_{|x| \geq (\ln \frac{1}{\epsilon})^\alpha} \frac{1}{|x|^2} \\
\leq C e^{-\frac{2|x - x'|}{\epsilon}} e^{3} \left( \ln \frac{1}{\epsilon} \right)^\alpha \\
\leq C e^{-\frac{2|x - x'|}{\epsilon}} e^{3} \left( \ln \frac{1}{\epsilon} \right)^\alpha. \tag{A.12}
\]

and

\[
\int_{\Omega_2' \cap \omega'_1} U_{1,x', \epsilon}^2 U_{2,y', \epsilon}^2 \leq C e^{-\frac{2|x - x'|}{\epsilon}} \int_{\Omega_2' \cap \omega'_1} e^{-\frac{2(|x - x'| \pm |x - y'| - |x - y'|)}{\epsilon}} \\
\leq C e^{-\frac{2|x - x'|}{\epsilon}} e^{3} \int_{|x| \geq (\ln \frac{1}{\epsilon})^\alpha} \frac{1}{|x|^4} \\
\leq C e^{-\frac{2|x - x'|}{\epsilon}} e^{3} \left( \ln \frac{1}{\epsilon} \right)^\alpha. \tag{A.13}
\]

From (A.12) and (A.13), we can easily get

\[
\int_{\Omega_2' \cap \omega'_1} U_{1,x', \epsilon}^2 U_{2,y', \epsilon}^2 = o(1) e^{-\frac{2|x - x'|}{\epsilon}} e^{3}. \tag{A.14}
\]
Combining (A.11) and (A.14), we can get
\[ \int_{\Omega_2} U_{1,x',1}^2 U_{2,y',1}^2 = a(1) e^{\frac{2|x_{1} - y_{1}|}{\epsilon}} \epsilon^3. \]

With the same method, we can also obtain that
\[ \int_{\Omega_1} U_{1,x',1}^2 U_{2,y',2}^2 = a(1) e^{\frac{2|x_{1} - y_{1}|}{\epsilon}} \epsilon^3. \]

So
\[ \int_{\mathbb{R}^3} U_{1,x',1}^2 U_{2,y',2}^2 = a(1) e^{\frac{2|x_{1} - y_{1}|}{\epsilon}} \epsilon^3. \]

This completes the proof.

Using Lemma [A.1] similar to Proposition [A.1] we can get the following Proposition.

**Proposition A.3.** Assume that (P) and (Q) hold. Then we get the following energy estimate:

\[ I(\epsilon U_{1,x',1}, U_{2,y',1}) = \frac{1}{2} \int_{\mathbb{R}^3} e^3 \left[ \tilde{A} + a\tilde{B} \rho^n + b\tilde{C} \rho^n - a(1) e^{\frac{2\sqrt{(\rho - r \cos \pi \tau) + (\rho - r \sin \pi \tau)}}{\epsilon}} \right], \]

where \( a, b \) is given in (P) and (Q), \( \tilde{\tau}_1 \) has been determined in Lemma 3.3, \( \tilde{A} = \frac{1}{2} \int_{\mathbb{R}^3} \mu_1 U_{1,1}^2 dx, \tilde{B} = \frac{1}{2} \int_{\mathbb{R}^3} U_{1,1}^2 dx, \) and \( \tilde{C} = \frac{1}{2} \int_{\mathbb{R}^3} U_{2,2}^2 dx. \)

**Proof.** We know that

\[ I(\epsilon U_{1,x',1}, U_{2,y',1}) = \frac{1}{2} \int_{\mathbb{R}^3} e^3 \left[ \tilde{A} + a\tilde{B} \rho^n + b\tilde{C} \rho^n - a(1) e^{\frac{2\sqrt{(\rho - r \cos \pi \tau) + (\rho - r \sin \pi \tau)}}{\epsilon}} \right], \]

Since \( U_i \) is the unique radial solutions of the following problem

\[ -\Delta u + u = \mu_i u^3, \max_{x \in \mathbb{R}^3} u = u(0), u > 0, \]
we have

\[ \frac{1}{2} \int_{\mathbb{R}^3} (\epsilon^2 |\nabla U_{1,x,j,\epsilon}|^2 + U_{1,x,j,\epsilon}^2 + \epsilon^2 |\nabla U_{2,y,j,\epsilon}|^2 + U_{2,y,j,\epsilon}^2) \]

\[ - \frac{1}{4} \int_{\mathbb{R}^3} (\mu_1 |U_{1,x,j,\epsilon}|^4 + \mu_2 |U_{2,y,j,\epsilon}|^4) \]

\[ = \frac{1}{4} \int_{\mathbb{R}^3} (\mu_1 |U_{1,x,j,\epsilon}|^4 + \mu_2 |U_{2,y,j,\epsilon}|^4) \]

\[ = \frac{1}{4} \epsilon^3 \int_{\mathbb{R}^3} (\mu_1 U_1^4 + \mu_2 U_2^4). \]

Similar to (A.4), noting that

\[ P(r) = 1 + ar^m + O(r^{m+\delta}) \quad \text{as } r \rightarrow 0^+ \]

and

\[ Q(r) = 1 + br^n + O(r^{n+\delta}) \quad \text{as } r \rightarrow 0^+ , \]

we can get that

\[ \frac{1}{2} \int_{\mathbb{R}^3} (P(x) - 1) U_{1,x,j,\epsilon}^2 \, dx = e^{3} \left[ a \tilde{B} r^m + O(r^{m-1}) + O(e^{-2(\tau_1 - \tau_2) \frac{1}{2} |x_1 - x_2|}) \right] \] (A.15)

and

\[ \frac{1}{2} \int_{\mathbb{R}^3} (Q(x) - 1) U_{2,y,j,\epsilon}^2 \, dx = e^{3} \left[ b \tilde{C} \rho^n + O(\rho^{n-1}) + O(e^{-2(\tau_1 - \tau_2) \frac{1}{2} |x_1 - x_2|}) \right] , \] (A.16)

where \( \tilde{\tau}_1 > 0 \) is a constant.

From Lemma [A.1] we have that

\[ \frac{1}{2} \int_{\mathbb{R}^3} U_{1,x,j,\epsilon}^2 U_{2,y,j,\epsilon}^2 \, dx = \beta e^{3} a_c (1) e^{-\frac{2|x_1 - x_2|}{\epsilon}}. \]

Therefore, we have

\[ I_\epsilon(U_{1,x,j,\epsilon}, U_{2,y,j,\epsilon}) = e^{3} \left[ \tilde{A} + a \tilde{B} r^m + b \tilde{C} \rho^n - a_c (1) e^{-2(\tau_1 - \tau_2) \frac{1}{2} |x_1 - x_2|} \right] \]

\[ + O(e^{-\frac{1(\tau_1 - \tau_2) r}{\epsilon}} + e^{-\frac{1(\tau_1 - \tau_2) \rho}{\epsilon}} + \rho^{n-1} \epsilon + r^{m-1} \epsilon) . \]

We complete the proof.
Proposition A.4. Assume that (P) and (Q) hold. Then there exist positive constants $B_1$ and $B_2$ such that

$$I_\varepsilon \tilde{U}_r, \tilde{V}_\rho = 2k \varepsilon^3 \left[ A + a \tilde{B} \tilde{r}^m + b \tilde{C} \tilde{\rho}^n + B_1 \varepsilon^{-2r \sin \frac{k}{\varepsilon}} + B_2 \varepsilon^{-2p \sin \frac{k}{\varepsilon}} + o_\varepsilon(1) \varepsilon^{-\frac{1}{2r} \sin \frac{k}{\varepsilon}^2 \pi^2} + O \left( e^{-\frac{(1-\varepsilon^2)(2r-1)r}{\varepsilon^2} \sin \frac{k}{\varepsilon}^2} + e^{-\frac{(1-\varepsilon^2)(2p-1)p}{\varepsilon^2} \sin \frac{k}{\varepsilon}^2} \right) \right],$$

where $\sigma$ has been determined in Proposition A.2.

Proof. We can obtain that

$$I_\varepsilon \tilde{U}_r, \tilde{V}_\rho = \sum_{j=1}^{2k} I_\varepsilon(U_{1,x_j}, U_{2,y_j})$$

$$= \frac{\mu_1}{4} \int_{\mathbb{R}^3} \left( \left| \tilde{U}_r \right|^4 - \sum_{j=1}^{2k} U_{1,x_j}^4 - 2 \sum_{i \neq j} (-1)^{i+j} U_{1,x_i}^3 U_{1,x_j}^1 \right)$$

$$- \frac{\mu_2}{4} \int_{\mathbb{R}^3} \left( \left| \tilde{V}_\rho \right|^4 - \sum_{j=1}^{2k} U_{2,y_j}^4 - 2 \sum_{i \neq j} (-1)^{i+j} U_{2,y_i}^3 U_{2,y_j}^1 \right)$$

$$- \frac{\beta}{2} \int_{\mathbb{R}^3} \left( \left| \tilde{U}_r \right|^2 \left| \tilde{V}_\rho \right|^2 - \sum_{j=1}^{2k} U_{1,x_j}^2 U_{2,y_j}^2 \right)$$

$$+ \frac{1}{2} \sum_{i \neq j} (-1)^{i+j} \int_{\mathbb{R}^3} \left[ (P(x) - 1) U_{1,x_i} U_{1,x_j} + (Q(x) - 1) U_{2,y_i} U_{2,y_j} \right].$$

(A.17)

Similar to (A.7), we can get that there exist positive constants $B_1$ and $B_2$ such that

$$- \frac{\mu_1}{4} \int_{\mathbb{R}^3} \left( \left| \tilde{U}_r \right|^4 - \sum_{j=1}^{2k} U_{1,x_j}^4 - 2 \sum_{i \neq j} (-1)^{i+j} U_{1,x_i}^3 U_{1,x_j}^1 \right)$$

$$= B_1 \varepsilon^{-2r \sin \frac{k}{\varepsilon}} e^3 + O \left( e^{-\frac{(1+\varepsilon^2)(2r-1)r}{\varepsilon^2} \sin \frac{k}{\varepsilon}^2} \right) e^3$$

(A.18)

and

$$- \frac{\mu_2}{4} \int_{\mathbb{R}^3} \left( \left| \tilde{V}_\rho \right|^4 - \sum_{j=1}^{2k} U_{2,y_j}^4 - 2 \sum_{i \neq j} (-1)^{i+j} U_{2,y_i}^3 U_{2,y_j}^1 \right)$$

$$= B_2 \varepsilon^{-2p \sin \frac{k}{\varepsilon}} e^3 + O \left( e^{-\frac{(1+\varepsilon^2)(2p-1)p}{\varepsilon^2} \sin \frac{k}{\varepsilon}^2} \right) e^3.$$
On the other hand, we have

$$\frac{1}{2} \int_{\mathbb{R}^3} (P(x) - 1) U_{1,x^i,e} U_{1,x^j,e}$$

$$= \frac{1}{2} \int_{B^m(0)} (P(x) - 1) U_{1,x^i,e} U_{1,x^j,e} + \frac{1}{2} \int_{B^m(0)} (P(x) - 1) U_{1,x^i,e} U_{1,x^j,e}$$

$$\leq C r m \int_{\mathbb{R}^3} U_{1,x^i,e} U_{1,x^j,e} + C \frac{1}{2} \int_{B^m(0)} (U_{2,x^i,e}^2 + U_{1,x^j,e}^2)$$

$$\leq C e^3 (r^m e^{-\frac{2r \sin \frac{\pi}{2} k}{\epsilon}} + e^{-\frac{3(2-\tilde{\tau})^2}{C \epsilon}})$$

$$= O e^3 (r^{2m} + e^{-\frac{4r \sin \frac{\pi}{2} k}{\epsilon}} + e^{-\frac{3(2-\tilde{\tau})^2}{C \epsilon}}).$$

Similarly, we have

$$\frac{1}{2} \int_{\mathbb{R}^3} (Q(x) - 1) U_{2,y^i,e} U_{2,y^j,e} = O e^3 \left( \rho^2 + e^{-\frac{2\rho \sin \frac{\pi}{2} k}{\epsilon}} + e^{-\frac{3(2-\tilde{\tau})^2}{C \epsilon}} \right).$$

Then

$$\left| \int_{\mathbb{R}^3} \left( |\hat{U}_r|^2 |\hat{V}_\rho|^2 - \sum_{j=1}^{2k} U_{1,x^i,e}^2 U_{2,y^j,e}^2 \right) \right| \leq C \sum_{j=1}^{2k} \int_{\mathbb{R}^3} U_{1,x^i,e}^2 U_{2,y^j,e}^2$$

$$= e^3 \alpha (1) e^{-\frac{2\sqrt{(\rho - r \cos \frac{\pi}{2} k)^2 + (r \sin \frac{\pi}{2} k)^2}}{\epsilon}}.$$

From (A.17)–(A.22) and Proposition A.3 we can easily have that

$$I_{\epsilon} (\hat{U}_r, \hat{V}_\rho) = 2k e^3 \left[ A + a B r m + b C \rho^n + B_1 e^{-\frac{2r \sin \frac{\pi}{2} k}{\epsilon}} + B_2 e^{-\frac{2\rho \sin \frac{\pi}{2} k}{\epsilon}} \right.$$

$$+ \alpha (1) e^{-\frac{2\sqrt{(\rho - r \cos \frac{\pi}{2} k)^2 + (r \sin \frac{\pi}{2} k)^2}}{\epsilon}}$$

$$+ O \left( e^{-\frac{(1 - \tilde{\tau}) (2 - \tilde{\tau})}{C \epsilon}} + e^{-\frac{(1 - \tilde{\tau}) (2 - \tilde{\tau})}{C \epsilon}} \right)$$

$$+ \rho^{n-1} \epsilon + r^{m-1} \epsilon + e^{-\frac{(1 + \sigma) r \sin \frac{\pi}{2} k}{\epsilon}} + e^{-\frac{(1 + \sigma) \rho \sin \frac{\pi}{2} k}{\epsilon}} \right].$$

This completes the proof.

References

[1] C. Alves, Local mountain pass for a class of elliptic system, J. Math. Anal. Appl., 335 (2007), 135-150.

[2] A. Ankiewicz and N. Akhmediev, Partially coherent solitons on a finite background, Phys. Rev. Lett., 82 (1999) 2661-2664.
[3] A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, C. R. Math. Acad. Sci. Paris, 342 (2006), 453-458.

[4] A. Ambrosetti and E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, J. Lond. Math. Soc., 75 (2007), 67-82.

[5] A. Ambrosetti, G. Cerami and D. Ruiz, Solitons of linearly coupled systems of semilinear non-autonomous equations on R^n, J. Funct. Anal., 254 (2008), 2816-2845.

[6] A. Ambrosetti, E. Colorado and D. Ruiz, Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations, Calc. Var. Partial Differential Equations, 30 (2007), 85-112.

[7] T. Bartsch, N. Dancer and Z. Wang, A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, J. Funct. Anal., 254 (2008), 2816-2845.

[8] T. Bartsch and M. Willem, Infinitely many radial solutions of a semilinear elliptic problem on R^N, Arch. Rational Mech. Anal., 124 (1993), 261-276.

[9] T. Bartsch and Z. Wang, Note on ground states of nonlinear Schrödinger systems, J. Partial Differential Equations, 19 (2006), 200-207.

[10] T. Bartsch, Z. Wang and J. Wei, Bound states for a coupled Schrödinger system, J. Fixed Point Theory Appl., 2 (2007), 353-367.

[11] J. Bohn, J. Bruke, B. Esry and C. Greene, Hartree-fock theory for double condensates, Phys. Rev. Lett., 78 (1997), 3594-3597.

[12] E. Burt, E. Cornell, R. Ghrist, C. Myatt and C. Wieman, Production of two overlapping Bose-Einstein condensates by sympathetic cooling, Phys. Rev. Lett., 78 (1997), 586-589.

[13] D. Cao, E. Noussair and S. Yan, Solutions with multiple peaks for nonlinear elliptic equations, Proc. Roy. Soc. Edinburgh Sect. A, 129 (1999), 235-264.

[14] S. Chang, C. Lin, T. Lin and W. Lin, Segregated nodal domains of two-dimensional multispecies Bose-Einstein condensates, Phys. D, 196 (2004), 341-361.

[15] X. Chen, T. Lin and J. Wei, Blow up and solitary wave solutions with ring profiles of two-component nonlinear Schrödinger systems, Phys. D, 239 (2010), 613-626.

[16] M. Conti, S. Terracini and G. Verzini, Nehari’s problem and competing species systems, Ann. Inst. H. Poincaré Anal. Non Linéaire, 19 (2002), 871-888.

[17] E. Cornell, J. Ensher, D. Hall, R. Matthews and C. Wieman, Dynamics of component separation in a binary mixture of Bose-Einstein condensates, Phys. Rev. Lett., 81 (1998), 1539-1542.
[18] E. Dancer and J. Wei, Spike solutions in coupled nonlinear Schrödinger equations with attractive interaction, Trans. Amer. Math. Soc., 361 (2009), 1189-1208.

[19] E. Dancer, J. Wei and T. Weth, A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system, Ann. Inst. H. Poincare Anal. Non Linéaire, 27 (2010), 953-969.

[20] D. De Figueiredo and O. Lopes, Solitary waves for some nonlinear Schrödinger systems, Ann. Inst. H. Poincare Anal. Non Linéaire, 25 (2008), 149-161.

[21] K. Dieckmann, S. Gupta, Z. Hadzibabic, E. Kempen, W. Ketterle, C.Schunck, C. Stan, B. Verhaar and M. Zwierlein, Radio-frequency spectroscopy of ultracold fermions, Science, 300 (2003), 1723-1726.

[22] S. Kim, O. Kwon and Y.Lee, Solutions with a prescribed number of zeros for nonlinear Schrödinger systems, Nonlinear Anal., 86 (2013), 74-88.

[23] T. Lin and J. Wei, Ground state of N coupled nonlinear Schrödinger equations in \( \mathbb{R}^n; n \leq 3 \), Commun. Math. Phys., 255 (2005), 629-653.

[24] T. Lin and J. Wei, Spikes in two coupled nonlinear Schrödinger equations, Ann. Inst. H. Poincare Anal. Non Linéaire, 22 (2005), 403-439.

[25] T. Lin and J. Wei, Spikes in two-component systems of nonlinear Schrödinger equations with trapping potentials. J. Differential Equations, 229 (2006), 538-569.

[26] Z. Liu and Z. Wang, Multiple bound states of nonlinear Schrödinger systems, Comm. Math. Phys., 282 (2008), 721-731.

[27] Z. Liu and Z. Wang, Ground states and bound states of a nonlinear Schrödinger system, Advanced Nonlinear Studies, 10 (2010), 175-193.

[28] L. Maia, E. Montefusco and B. Pellacci, Positive solutions for weakly coupled nonlinear Schrödinger system, J. Differential Equations, 229 (2006), 743-767.

[29] E. Montefusco, B. Pellacci and M. Squassina, Semiclassical states for weakly coupled nonlinear Schrödinger systems, J. Eur. Math. Soc., 10 (2008), 47-71.

[30] B. Noris, H. Tavares, S. Terracini and G. Verzini, Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition, Comm. Pure Appl. Math., 63 (2010), 267-302.

[31] S. Peng and Z. Wang, Segregated and synchronized vector solutions for nonlinear Schrödinger systems, Arch. Rat. Mech. Anal., 208 (2013), 305-339.

[32] H. Pi and C. Wang, Multi-bump solutions for nonlinear Schrödinger equations with electromagnetic fields, ESAIM Control Optim. Calc. Var., 19 (2013), 91-111.
[33] A. Pomponio, Coupled nonlinear Schrödinger systems with potentials, J. Differential Equations, 227 (2006), 258-281.

[34] O. Rey, The role of the Green’s function in a nonlinear elliptic equation involving the critical Sobolev exponent, J. Funct. Anal., 89(1990), 1-52.

[35] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^N$, Comm. Math. Phys., 271 (2007), 199-221.

[36] S. Terracini and G. Verzini, Multipulse phase in k-mixtures of Bose-Einstein condensates, Arch. Ration. Mech. Anal., 194 (2009), 717-741.

[37] E. Timmermans, Phase separation of Bose-Einstein condensates, Phys. Rev. Lett., 81 (1998), 5718-5721.

[38] Y. Wan, Multiple solutions and their limiting behavior of coupled nonlinear Schrödinger systems, Acta Math. Sci. Ser. B Engl. Ed., 30 (2010), 1199-1218.

[39] Y. Wan, A. Ávila, Multiple solutions of a coupled nonlinear Schrödinger system, J. Math. Anal. Appl., 334 (2007), 1308-1325.

[40] J. Wei and T. Weth, On the number of nodal solutions to a singularly perturbed Neumann problem, Manuscripta Math., 117 (2005), 333-344.