Mesh filtration problems with a two-flow structure of phase velocities

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Abstract. This paper deals with the problem of filtration of a two-phase incompressible fluid within the Buckley-Leverett model. From a general point of view, a two-flow structure of conservation laws is investigated. In addition, since the solution in the Buckley-Leverett model is discontinuous, conservation laws are presented in the generalized integral form. A good illustration of the approach presented is the problem of gravitational segregation of oil and water in a porous medium. For this problem, a two-flow mesh structure of conservation laws is described.

1. Introduction
In [1, 2] for some problems of filtration of a two-phase incompressible immiscible fluid, multidimensional numerical models are proposed based on the IMPES method [3] with spatial approximations by the mixed finite element method [4] and the cell-centered finite volume method [5]. These problems are formulated within the Buckley-Leverett [6] model with discontinuous phase flows.

For the problem of gravity segregation [2], the phase flows were represented as the sum of two flows of different nature, one of which set the motion in the direction of the total flow, and the second one describe the flows oppositely, directed in different phases. For the first time, the idea of such a representation of phase flows was presented in [7], and was implemented with an upwind scheme. This approach is known as Hybrid Upwinding (HU). In this paper from a general point of view, the hybrid two-flow mechanism of formating the integral mass conservation laws is analyzed in the form that allows one to carry out calculations without highlighting the discontinuity surface of solutions. The results are illustrated by the problem of gravity segregation in a porous medium. From the point of view of applications, water and oil are considered. In Section 2, the initial conservation laws are presented with an analysis of the two-flow structure of phase velocities. In Section 3, this analysis is illustrated by the problem of gravity segregation within the explicit HU approach, described in [2].

2. The mass conservation law for a two-phase fluid
The Buckley-Leverett model is a system of conservation laws for mass and momentum for each phase. The mass conservation law can be written down in the following integral form: for an
arbitrary $t_1$ and $t_2$, $t_1 < t_2$

$$\langle \phi S_\alpha \rangle_\tau (t_2) - \langle \phi S_\alpha \rangle_\tau (t_1) + \frac{1}{|\tau|} \int_{t_1}^{t_2} \int_{\partial \tau} \mathbf{v}_\alpha \cdot \mathbf{n}_\tau \, d\gamma \, dt = \int_{t_1}^{t_2} \langle q_\alpha \rangle_\tau \, dt, \quad \alpha = w, o,$$  \hspace{1cm} (1)

where a cell $\tau$ is a subset of the computational domain $\Omega$ and $|\tau|$ is a volume of a cell $\tau$, the indices $w$ and $o$ correspond to the water and the oil phases, $\phi$ is the porosity independent of time, $S_w$ and $S_o$ are the water and the oil saturations, $\mathbf{v}_w$ and $\mathbf{v}_o$ are the velocities of water and oil flows, $\mathbf{n}_\tau$ is the unit vector of the external normal at the boundary $\partial \tau$, $q_w$ and $q_o$ are the sources of water and oil (injection and production wells), averaging

$$\langle \phi S_\alpha \rangle_\tau = \frac{1}{|\tau|} \int_{\tau} \phi S_\alpha \, d\mathbf{x}, \quad \langle q_\alpha \rangle_\tau = \frac{1}{|\tau|} \int_{\tau} q_\alpha \, d\mathbf{x}, \quad \alpha = w, o$$

are the functions of time. We assume that the entire pore reservoir is filled with a fluid. It means that for an arbitrary cell $\tau$

$$\langle \phi \rangle_{S_w} \tau (t_2) + \langle \phi \rangle_{S_o} \tau (t_1) = \langle \phi \rangle_\tau .$$  \hspace{1cm} (2)

Since $t_1$ and $t_2$ are arbitrary, it follows from equalities (1), (2) that at any fixed time

$$\int_{\partial \tau} \mathbf{v} \cdot \mathbf{n}_\tau \, d\gamma = \int_{\tau} (q_w + q_o) \, d\mathbf{x},$$  \hspace{1cm} (3)

where $\mathbf{v} = \mathbf{v}_w + \mathbf{v}_o$ is the total velocity of the flow of a two-phase fluid. We will assume that $\nabla \cdot \mathbf{v} \in L^2(\Omega)$, and therefore (3) can be rewritten as $\nabla \cdot \mathbf{v} = q_w + q_o$.

Let us represent the phase velocities as the sum of the two terms

$$\mathbf{v}_\alpha = \mathbf{v}_\alpha^{(1)} + \mathbf{v}_\alpha^{(2)}, \quad \alpha = w, o,$$

where the directions of the vectors $\mathbf{v}_\alpha^{(1)}$ coincide with the directions of the total vector $\mathbf{v}$:

$$\mathbf{v}_\alpha^{(1)} = \sigma_\alpha \mathbf{v}, \quad \sigma_\alpha \geq 0, \quad \alpha = w, o.$$  \hspace{1cm} (5)

Then according to equations (1) we have:

$$\int_{t_1}^{t_2} \int_{\partial \tau} \left( \mathbf{v}_w^{(2)} + \mathbf{v}_o^{(2)} \right) \cdot \mathbf{n}_\tau \, d\gamma \, dt = \int_{t_1}^{t_2} \langle q_w + q_o \rangle_\tau \, dt.$$

Let the equality hold

$$\sigma_w + \sigma_o = 1.$$  \hspace{1cm} (7)

Then from (4), (5) and (7) it immediately follows that

$$\mathbf{v}_w^{(2)} + \mathbf{v}_o^{(2)} = 0.$$  \hspace{1cm} (8)

This means that the phase velocity is presented as the sum of the term co-directional with respect to the total velocity and of the term in the opposite direction with respect to the second
term of the velocity of the other phase. The representation of the phase velocities structure is illustrated in figure 1. We use equalities (7), (8) in (6), and taking into account (3), we obtain

$$\langle \phi(S_w + S_o) \rangle (t_2) = \langle \phi(S_w + S_o) \rangle (t_1).$$

(9)

In fact, conditions (2) and (3) are equivalent, and based on condition (3) only, a mass conservation law was obtained in the form of (9).

Thus, in the above formulation, the mass conservation law is based on the two mechanisms corresponding to expansion (4). Firstly, it is equality (5) with allowance for equality (7). Secondly, it is the opposite direction of flows, given by equality (8). Both these mechanisms should be correctly taken into account when developing a numerical algorithm. In the next section, using an example of the gravity segregation problem, we comment on some results from [2] from the point of view of the above said.

3. The gravity segregation

In this section, we consider the mesh implementation of the mass conservation law using the gravity segregation problem as an example. The approach uses the results from [2], proposing an explicit implementation of the HU method (EHU), in contrast to the implicit IHU scheme from [7]. Our main objective is to obtain mesh analogies of conservation laws (9) and equality (8) within the framework of the EHU method.

Let us consider the law of conservation of momentum for each phase (the generalized Darcy law). Let $p$ be the pressure, and in the case when the pressure is differentiable with respect to spatial variable functions, the Darcy equation for each phase can be written down in the conventional differential form:

$$v_\alpha = -k\lambda_\alpha(S_\alpha)(\nabla p - \rho_\alpha g), \quad \alpha = w, o,$$

(10)
where \( k \) is the absolute permeability, \( \lambda_w(S_w) \) and \( \lambda_o(S_o) \) are the phase mobilities, \( \rho_w \) and \( \rho_o \) are the phase densities and \( \mathbf{g} \) is the vector of the gravity acceleration. For the sake of the numerical analysis, we consider an isotropic reservoir with constant porosity and unit absolute permeability tensor. It is important that the total mobility is positive:

\[
\lambda_w(S_w) + \lambda_o(S_o) > 0. \tag{11}
\]

In the case where pressure is a discontinuous function, one should use the approach proposed in [1], based on the transition to a generalized formulation for the Darcy law for the total velocity. For this, from the Darcy equations for the phases, taking into account inequality (11), we pass to the equation

\[
\frac{1}{k(\lambda_w + \lambda_o)} \mathbf{v} = -\nabla p + \frac{\lambda_w \rho_w + \lambda_o \rho_o}{\lambda_w + \lambda_o} \mathbf{g}, \tag{12}
\]

The scalar product of this equality by an arbitrary vector function with square-summable components, with vanishing normal component on the boundary of the domain \( \Omega \), leads to the following integral identity

\[
\int_{\Omega} \frac{1}{k(\lambda_w + \lambda_o)} \mathbf{v} \cdot \mathbf{u} \, d\mathbf{x} = \int_{\Omega} p \nabla \cdot \mathbf{u} \, d\mathbf{x} + \int_{\Omega} \frac{\lambda_w \rho_w + \lambda_o \rho_o}{\lambda_w + \lambda_o} \mathbf{g} \cdot \mathbf{u} \, d\mathbf{x}. \tag{13}
\]

In this equality, the discontinuous pressure is allowed, and it is this equality that we consider to be the original definition of the generalized Darcy law for the total flow. Then for the differentiable pressure according to (10), (12), equality (4) with allowance for (5) can be written down in the form:

\[-k \lambda_o (\nabla p - \rho_o \mathbf{g}_o) = -k \sigma_\alpha [(\lambda_w + \lambda_o) \nabla p - (\lambda_w \rho_w + \lambda_o \rho_o) \mathbf{g}] + \mathbf{v}_\alpha^{(2)}, \quad \alpha = w, o,\]

whence it follows that \( \sigma_\alpha = \lambda_o/(\lambda_w + \lambda_o) \) and

\[
\mathbf{v}_w^{(2)} = k \frac{\lambda_w}{\lambda_w + \lambda_o} (\rho_w - \rho_o) \mathbf{g}, \quad \mathbf{v}_o^{(2)} = k \frac{\lambda_o}{\lambda_w + \lambda_o} (\rho_o - \rho_w) \mathbf{g}. \tag{14}
\]

Thus, equalities (7), (8) are valid, and they were not obtained as an assumption, but as a result of using the generalized Darcy law in expansion (4).

Following [2], we use a mixed finite element method for the spatial approximation of velocity by the Raviart-Thomas elements of the least degree [4] and the pressure by piecewise constant functions. Moreover, for the spatial approximation of saturations, a cell-centered finite volume method [5] is applied. This means that the mesh saturations similar to the pressure are represented by piecewise constant functions.

Let us denote

\[
\lambda^+_\alpha = \lambda_\alpha(S^+_\alpha), \quad \lambda^-_\alpha = \lambda_\alpha(S^-_\alpha), \quad \sigma^+_\alpha = \frac{\lambda^+_\alpha}{\lambda^+_\alpha + \lambda^-_\alpha}, \quad \sigma^-_\alpha = \frac{\lambda^-_\alpha}{\lambda^+_\alpha + \lambda^-_\alpha}, \quad \alpha = w, o, \tag{15}
\]

where \( S^+_\alpha \) and \( S^-_\alpha \) are the saturations in the cell \( \tau \) and outside of \( \tau \), respectively. Moreover, let us distinguish the following parts of the set \( \partial \tau \):

\[
F^+_\tau = \{ \mathbf{x} \in \partial \tau : \mathbf{v} \cdot \mathbf{n}_\tau > 0 \}, \quad F^-_\tau = \{ \mathbf{x} \in \partial \tau : \mathbf{v} \cdot \mathbf{n}_\tau < 0 \},
\]

\[
G^+_\tau = \{ \mathbf{x} \in \partial \tau : (\rho_w - \rho_o) \mathbf{g} \cdot \mathbf{n}_\tau > 0 \}, \quad G^-_\tau = \{ \mathbf{x} \in \partial \tau : (\rho_w - \rho_o) \mathbf{g} \cdot \mathbf{n}_\tau < 0 \}.
\]
Then the mesh mass conservation law for each phase within the EHU approach may be written down as follows:

\[
\phi|_\tau^n \frac{S_{w,\tau}^{n+1} - S_{w,\tau}^n}{\Delta t} + \int_{F^+_w} \sigma^+_w \mathbf{v} \cdot \mathbf{n}_\tau \, d\gamma + \int_{F^-_w} \sigma^-_w \mathbf{v} \cdot \mathbf{n}_\tau \, d\gamma + k(\rho_w - \rho_o) \int_{G^+_w} \frac{\lambda^+_w \lambda^-_o}{\lambda_w + \lambda_o} \mathbf{g} \cdot \mathbf{n}_\tau \, d\gamma = \int_{\tau} q_w, \tag{16}
\]

\[
k(\rho_o - \rho_w) \int_{G^+_w} \frac{\lambda^+_w \lambda^-_o}{\lambda_w + \lambda_o} \mathbf{g} \cdot \mathbf{n}_\tau \, d\gamma + \int_{G^-_w} \frac{\lambda^-_w \lambda^+_o}{\lambda_w + \lambda_o} \mathbf{g} \cdot \mathbf{n}_\tau \, d\gamma = \int_{\tau} q_o. \tag{17}
\]

Here \(\Delta t\) is the time step. In the notations of equations (1) \(\Delta t = t_2 - t_1\). Since according to (15) \(\sigma^+_w + \sigma^+_o \equiv 1\) and \(\sigma^-_w + \sigma^-_o \equiv 1\), from (16), (17) with taking into account (3) it follows that for any \(\tau\)

\[
S_{w,\tau}^{n+1} + S_{o,\tau}^{n+1} = S_{w,\tau}^n + S_{o,\tau}^n. \tag{18}
\]

Let us note that condition (3) is automatically valid for the mesh solution obtained by the mixed finite element method.

Now consider the mesh analogue of equality (8). The velocities \(\mathbf{v}_\alpha^{(2)}\) are given by the following equalities:

\[
\int_{G^+_w} \mathbf{v}_w^{(2)} \cdot \mathbf{n}_\tau \, d\gamma = - \int_{G^+_o} \mathbf{v}_o^{(2)} \cdot \mathbf{n}_\tau \, d\gamma = k(\rho_w - \rho_o) \int_{G^+_w} \frac{\lambda^+_w \lambda^-_o}{\lambda_w + \lambda_o} \mathbf{g} \cdot \mathbf{n}_\tau \, d\gamma,
\]

\[
\int_{G^-_w} \mathbf{v}_w^{(2)} \cdot \mathbf{n}_\tau \, d\gamma = - \int_{G^-_o} \mathbf{v}_o^{(2)} \cdot \mathbf{n}_\tau \, d\gamma = k(\rho_w - \rho_o) \int_{G^-_w} \frac{\lambda^-_w \lambda^+_o}{\lambda_w + \lambda_o} \mathbf{g} \cdot \mathbf{n}_\tau \, d\gamma,
\]

and (8) is identically fulfilled.

It is very important that the solution of equations (16), (17) satisfies the inequalities

\[
0 \leq S_{\alpha,\tau}^n \leq 1, \quad \alpha = w, o. \tag{19}
\]

This property is the main result of the EHU approach. Below is one numerical example borrowed from paper [2]. We speak about the problem discussed in [8]. A reservoir with 50m \(\times\) 50m \(\times\) 10m dimensions is considered. Initially, the upper half of the reservoir is filled with water, and the lower half with light oil \((\rho_w < \rho_o)\). At all boundaries of the reservoir, the requirement of the absence of the fluid outflow is determined by the homogeneous Neumann condition for the normal components of the phase velocities. In fact, a one-dimensional problem is considered, although all calculations for the 2D problem are carried out. Figure 2 shows the dependence of water saturation on the coordinate \(z\) and on time in the section \(y = 25m\). We have quite a complicated process consisting of a number of stages. This is the decomposition of the initial discontinuity the two discontinuities moving in opposite directions, the appearance of reflected waves in the form of two discontinuities moving towards each other, and their merging into a discontinuity, which determines the final segregation of the two-phase liquid. The qualitative picture of the process presented in [8] is exactly reproduced. These stages are illustrated in figure 3. Figures 2 and 3 are borrowed from [2].
4. Conclusion
For the filtration problems of a two-phase fluid, it is proposed to select the total velocity as the required function, and to represent the phase velocities as a sum of components, one of which coincides in the direction with the total velocity. In this case, simple conditions that ensure the fulfillment of the mass balance are obtained. Using a meaningful example, a method for constructing the mesh equations satisfying the conditions obtained is demonstrated.

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