SMOOTH MANIFOLDS WITH INFINITE FUNDAMENTAL GROUP ADMITTING NO REAL PROJECTIVE STRUCTURE

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Abstract. It is an important question whether it is possible to put a geometry on a given manifold or not. It is well known that any simply connected closed manifold admitting a real projective structure must be a sphere. Therefore, any simply connected manifold \( M \) which is not a sphere \((\dim M \geq 4)\) does not admit a real projective structure. Cooper and Goldman gave an example of a 3-dimensional manifold not admitting a real projective structure and this is the first known example. In this article, by generalizing their work we construct a manifold \( M^n \) with the infinite fundamental group \( \mathbb{Z}_2 * \mathbb{Z}_2 \), for any \( n \geq 4 \), admitting no real projective structure.

1. INTRODUCTION

As stated in Felix Klein’s Erlanger program of 1872, the classical \((X, G)\) geometry is the study of the properties of a space \( X \) which are invariant under a transitive action of a Lie group \( G \). Although this notion was introduced by Felix Klein, the study is initiated by Ehresmann [7]. The basic problem is to determine when one can put a certain kind geometric structure on a given manifold and classify such structures up to isomorphism. It is well known that every surface admits a real projective structure and the classification of these structures on surfaces is completely done ([4], [5]).

Thurston’s work, starting around the middle 1970’s, on geometrization of 3-manifolds is a significant contribution of geometric structures in low dimensional topology ([14]). A three manifold admitting one of Thurston’s geometries except the two of them, which are \( S^2 \times \mathbb{R} \) and \( H \times \mathbb{R} \), has a real projective structure determined uniquely by this structure. In the remaining two cases, the three manifold also has a real projective structure if the group acting on the manifold preserves the orientation on the \( \mathbb{R} \) direction ([6], [11]). On the other hand, there are some examples admitting a real projective structure, which is not obtained from Thurston’s eight geometries by Benoist’s work ([2]).

It was a conjecture that every three manifold admits a real projective structure. However, D. Cooper and W. Goldman showed that the connected sum of two copies of real projective three spaces does not admit a real projective structure ([6]).

It is well known that any simply connected manifold admitting a real projective structure is a sphere since the developing map (see p. [2]) must be a covering map. Since there are many examples of simply connected manifolds which are not spheres in dimension

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Date: November 26, 2018.
2010 Mathematics Subject Classification. 57N16, 57S25, 53A20.
Key words and phrases. real projective structure, developing map and holonomy.
bigger than 3 (e.g. $\mathbb{CP}^n$), there are many higher dimensional manifolds that do not admit a real projective structure.

The aim of this paper is to construct smooth $n$-dimensional manifolds with the infinite fundamental group $\mathbb{Z}_2 \ast \mathbb{Z}_2$ ($n \geq 4$), which do not admit a real projective structure by generalizing Cooper and Goldman’s work in [6].

ACKNOWLEDGEMENTS. I am grateful to my advisor Yıldıray Ozan for his support, comments and suggestions on this work.

2. PRELIMINARIES

First, we define an $(X, G)$ structure on a manifold $M$ following Ehresmann. Let $M$ be a real analytic manifold modelled on $X$ (there is a local isomorphism between $X$ and $M$) and $G$ be a Lie group acting on $X$ transitively. Then we say that $M$ has an $(X, G)$ structure or $M$ is an $(X, G)$ manifold. Therefore, an $(X, G)$ manifold has a canonical real analytic structure (see [8], [9], [10], [12] for more information about $(X, G)$ structures).

Let $M$ be any $(X, G)$ manifold and $\{(U_i, \phi_i)\}$ be an atlas on $M$ with transition maps

$$\gamma_{ij} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

such that

$$\gamma_{ij} \circ \phi_i = \phi_j.$$

Consider an analytic continuation of $\phi_1$ along a curve $\alpha$ in $M$ beginning in $U_1$. Inductively, on a component of $\alpha \cap U_i$, the analytic continuation of $\phi_1$ along $\alpha$ is of the form $\gamma \circ \phi_1$, where $\gamma \in G$. Therefore, $\phi_1$ can be analytically continued along every path to $\bigcup U_i$ on $M$. It follows that there is a global analytic continuation of $\phi_1$ on the universal cover $\widetilde{M}$ of $M$. Therefore, one can define a map

$$\text{dev} : \widetilde{M} \to X,$$

which is called a developing map. The map $\text{dev}$ is an immersion and is unique up to composition with elements of $G$. From the uniqueness property of $\text{dev}$, for any covering transformation $\Gamma_\alpha$ of $\widetilde{M}$ over $M$, there is an element $g_\alpha$ of $G$ such that

$$\text{dev} \circ \Gamma_\alpha = g_\alpha \circ \text{dev}.$$

Since

$$\text{dev} \circ \Gamma_\alpha \circ \Gamma_\beta = g_\alpha \circ \text{dev} \circ \Gamma_\beta = g_\alpha \circ g_\beta \circ \text{dev},$$

it follows that the map

$$\text{hol} : \pi_1(M) \to G,$$

$$\alpha \mapsto g_\alpha$$

is a homomorphism and called the holonomy of the geometric structure on $M$. For more details, see [14].
The pair \((\text{dev, hol})\) is called a developing pair for the geometric structure \((X, G)\). A real projective structure on \(M^n\) is then an \((\mathbb{RP}^n, PGL(n+1, \mathbb{R}))\) structure.

More precisely, \(M\) admits a real projective structure if there is a maximal atlas on \(M\) with projective coordinate changes. A covering \(\{U_i\}\) of \(M\) with a family of local diffeomorphisms \(\phi_i : U_i \rightarrow V_i \subset \mathbb{RP}^n\) is called a projective atlas if the local transformations \(\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)\) are projective (i.e. they are restrictions of some elements of the group \(PGL(n+1, \mathbb{R})\)).

![Figure 1. Projective Coordinate Charts](image)

**Remark 1.** Let \((\text{dev}_1, \text{hol}_1)\) and \((\text{dev}_2, \text{hol}_2)\) be two developing pairs for the same structure. Then they satisfy the identity \(\text{dev}_2 = g \circ \text{dev}_1\), for some \(g \in G\) and the holonomies are related as \(\text{hol}_2(\beta) = g \text{hol}_1(\beta) g^{-1}\), for any homotopy class \([\beta] \in \pi_1(M)\).

**Theorem 2.1** (Ehresmann-Weil-Thurston Principle). Let \(M\) be an \((X, G)\) manifold with holonomy representation \(\rho : \pi_1(M) \rightarrow G\). For \(\rho'\) sufficiently close to \(\rho\) in the space of representations \(\text{Hom}(\pi_1(M), G)\), there exists an \((X, G)\) structure on \(M\) with holonomy representation \(\rho'\).

The following well known observation is needed in the proof of Theorem 2.1.

**Lemma 2.2.** Let \(X\) and \(Y\) be Hausdorff spaces and \(f : X \rightarrow Y\) be a local homeomorphism. If \(X\) is compact and \(Y\) is connected then \(f\) is a finite sheeted covering map.

The following theorem will be needed to study some foliations and the leaf spaces induced by a real projective structure (page 19).

**Theorem 2.3.** Let \(F\) be a codimension one, \(C^1\), transversely oriented foliation of a compact manifold \(M^n\) with a compact leaf \(L\) such that \(H^1(L, \mathbb{R}) = 0\). Then all
leaves of $\mathcal{F}$ are diffeomorphic to $L$, and the leaves of $\mathcal{F}$ are the fibers of a fibration of $M^n$ over $S^1$ or $I$, which is an interval. We assume here that if $M^n$ has boundary, then the boundary of $M$ is a union of leaves of $\mathcal{F}$.

3. THE MAIN THEOREM

In this part, for any $n \geq 4$, we construct smooth $n$-dimensional manifolds with the fundamental group $\mathbb{Z}_2 \ast \mathbb{Z}_2$ admitting no real projective structure.

Let $W$ be an $m$-dimensional ($m \geq 3$) smooth manifold with $\pi_1(W) \cong \mathbb{Z}_2$ and $S^1$ denote the unit circle in the complex plane $\mathbb{C}$. Now let $M = \tilde{W} \times S^1 / <\sigma>$, where the action is given by

$$\sigma : \tilde{W} \times S^1 \rightarrow \tilde{W} \times S^1$$

$$(p, z) \mapsto (\tau(p), \bar{z})$$

so that $<\tau> \cong \mathbb{Z}_2$ is the Deck transformation group of the universal cover $\tilde{W} \rightarrow W$ with $\tilde{W} / <\tau> \cong W$. Now the universal cover of $M$ is as follows:

$$\tilde{M} = \tilde{W} \times \mathbb{R} \rightarrow \tilde{W} \times S^1 \rightarrow \tilde{W} \times S^1 / <\sigma> \cong M.$$  

The induced homomorphism from $\sigma$ on the fundamental group is given as follows:

$$\sigma : \pi_1(\tilde{W} \times S^1) \rightarrow \pi_1(\tilde{W} \times S^1)$$

$$\mathbb{Z} \rightarrow \mathbb{Z}$$

$$1 \mapsto -1.$$  

With an easy observation, one can see the fundamental group of $M$ is as follows:

$$\pi_1(M) = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b | a^2 = 1, b^2 = 1 \rangle.$$  

By using the presentation of the fundamental group of $M$, we have a short exact sequence

$$1 \rightarrow \pi_1(\tilde{W} \times S^1) \cong \mathbb{Z} \rightarrow \pi_1(M) \rightarrow \mathbb{Z}_2 \rightarrow 1.$$  

The action of $\mathbb{Z}_2$ on the normal subgroup $\mathbb{Z}$ of $\pi_1(M)$ is given by multiplication with $-1$. Therefore, the fundamental group of $M$ has the following presentation:

$$\pi_1(M) \cong \mathbb{Z} \times \mathbb{Z}_2 = \langle c = ab, a | a^2 = 1, aca = c^{-1} \rangle.$$  

Here is the main result of this paper:

**Theorem 3.1.** Let $W$ be an $m$-dimensional ($m \geq 3$) smooth closed manifold with $\pi_1(W) \cong \mathbb{Z}_2$ and $M = \tilde{W} \times S^1 / <\sigma>$ as above. We assume that:

(i) Either $\tilde{W}$ is odd dimensional, or

(ii) $\tilde{W}$ is even dimensional and it is not the total space of a sphere bundle over a sphere, where both the base and the fiber are the sphere $S^{m/2}$.

Then the manifold $M$ does not admit a real projective structure.
Remark 2. Note that if $m = 2$ and $W$ is a closed surface with $\pi_1(W) \cong \mathbb{Z}_2$ then $W = \mathbb{R}P^2$. Thus, $\tilde{W} = S^2$ and

$$M = S^2 \times S^1 / < \sigma > \cong \mathbb{R}P^3 \# \mathbb{R}P^3.$$  

In other words, our construction does not yield any example other than $\mathbb{R}P^3 \# \mathbb{R}P^3$ in dimension 3.

Similarly, if $m = 3$ and $W$ is a closed 3-manifold with $\pi_1(W) \cong \mathbb{Z}_2$, then by the Elliptization Theorem (cf. Theorem 1.12 in [1]), $W = \mathbb{R}P^3$. Therefore, $\tilde{W} = S^3$ and thus

$$M = S^3 \times S^1 / < \sigma > \cong \mathbb{R}P^4 \# \mathbb{R}P^4.$$  

We follow Cooper and Goldman’s work closely. Therefore, we omit the proofs of several results, which are analogous to those in [6].

Let us take $m + 1 = n$ for simplicity. We prove Theorem 3.1 by contradiction. Therefore, we start with the assumption that $M$ admits a real projective structure. Hence, there exists a developing pair $(\text{dev}, \text{hol})$ for $M$. Before the proof of Theorem 3.1 we will prove the following lemma (compare with Lemma 4.1 in [6]).

**Lemma 3.2.** The map $\text{hol} : \pi_1(M) \to PGL(n+1, \mathbb{R})$ is injective.

**Proof.** Suppose not. Then the image of the holonomy is a proper quotient of the infinite dihedral group. This implies that it is finite ([15]). Let $H$ be the kernel of the homomorphism

$$\text{hol} : \pi_1(M) \to PGL(n+1, \mathbb{R}),$$

and $\tilde{M}' \to M$ be the covering space corresponding to the subgroup $H \leq \pi_1(M)$. Hence, the covering map $\tilde{M}' \to M$ is finite, whose total space is immersed into $\mathbb{R}P^n$ by the map $\varphi : \tilde{M}' \to \mathbb{R}P^n$. Here, the developing map descends to $\varphi$ and the map $\varphi$ is a covering map since $\tilde{M}'$ is compact.

Thus, $\tilde{M}'$ is a covering space of $\mathbb{R}P^n$. On the other hand, $\pi_1(\tilde{M}')$ is infinite because $\pi_1(M)$ is infinite and the covering map $\tilde{M}' \to M$ is finite. Therefore, this gives a contradiction since it is also isomorphic to a subgroup of $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$. $\square$
4. PROOF OF THE MAIN THEOREM

Proof of Theorem 3.1. We assume that $M$ admits a real projective structure and thus there exists a developing pair $(\text{dev}, \text{hol})$

$$\text{dev} : \tilde{M} \rightarrow \mathbb{RP}^n,$$

where $\tilde{M}$ is the universal cover of $M$ and

$$\text{hol} : \pi_1(M) \rightarrow \text{PGL}(n + 1, \mathbb{R}),$$

such that for all $\tilde{m} \in \tilde{M}$ and $g \in \pi_1(M)$, we have

$$\text{dev}(g \cdot \tilde{m}) = \text{hol}(g) \cdot \text{dev}(\tilde{m}).$$

Let $[A]$ and $[B]$ be the images of the generators of the fundamental group $\pi_1(M) = \mathbb{Z}_2 \ast \mathbb{Z}_2 = \langle a \rangle \ast \langle b \rangle$ under the holonomy map, meaning that

$$\text{hol} : \pi_1(M) \rightarrow \text{PGL}(n + 1, \mathbb{R}),$$

$$a \mapsto \text{hol}(a) = [A],$$

$$b \mapsto \text{hol}(b) = [B],$$

where $A, B \in \text{GL}(n + 1, \mathbb{R})$. Consider the exact sequence below.

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(M) \cong \mathbb{Z}_2 \ast \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow 1.$$  

Here, the infinite cyclic normal subgroup is generated by the product $c = ab$. For the subgroup of $\pi_1(M)$ generated by $a$ and $c^n$ there is an $n$-fold covering space $M^{(n)} \rightarrow M$ and the manifold $M^{(n)}$ is diffeomorphic to $M$.

Remark 3. If a manifold $M$ admits a real projective structure, then any covering space of $M$ admits a real projective structure. In other words, if a covering space of $M$ does not admit a real projective structure then $M$ can not admit a real projective structure.

Let us take $C = AB$. After passing to the double cover $M^{(2)}$ of $M$, we can assume that $M$ has a real projective structure, where $A$ and $B$ are conjugate, which will be explained in the proof of the lemma below.

Lemma 4.1. It is possible to arrange that $C$ is diagonalizable over $\mathbb{R}$ with positive eigenvalues.

Proof. First, let us observe that $A$ and $B$ are conjugate on the double cover $M^{(2)}$ of $M$: Let $a', b'$ and $c'$ be the elements of $M^{(2)}$ such that

$$c^2 = abab = c' = a'b',$$

where $a' = a$ and $b' = bab$. Therefore, the images of $a'$ and $b'$ are $A$ and $BAB$, which are clearly conjugate elements.

Since $a'^2 = 1$ and $\text{hol}$ is a homomorphism, $[A]^2 \in \text{PGL}(n + 1, \mathbb{R})$ is the identity. It follows that after rescaling $A$ we have $A^2 = \pm \text{Id}$, thus $A$ is diagonalizable over $\mathbb{C}$. If $A^2 = \text{Id}$ then the eigenvalues are $\pm 1$. Since we are only interested in $[A]$ we can multiply $A$ with $-1$ and arrange that the eigenvalue $-1$ has multiplicity at
most $\frac{n+1}{2}$ (if $n$ is odd) and $\frac{n}{2}$ (if $n$ is even). Otherwise, $A^2 = -Id$. Depending on the dimension, we have the cases below:

(i) If the dimension $n$ is odd, there exist $\frac{n+1}{2} + 1$ possible cases, up to conjugation, for the matrix $A$. In the first case $A^2 = -Id$ and the corresponding $(n+1) \times (n+1)$ matrix is

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
0 & 0 & -1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & -1 & 0
\end{bmatrix}
$$

In the remaining cases $A^2 = Id$ and there are $\frac{n+1}{2}$ possibilities. Along the diagonal there exist only $\pm 1$'s and all off-diagonal elements are $0$. The number of $-1$ eigenvalues of each $A_i$ is $i$ and the other eigenvalues are $1$, where $i \in \{1, 2, \ldots, \frac{n+1}{2}\}$. For example, $A_3$ is as follows:

$$
\begin{bmatrix}
-1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & -1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & -1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1
\end{bmatrix}
$$

(ii) If the dimension $n$ is even, there exist $\frac{n}{2}$ possible cases for $A$. Note that in this case $A^2 \neq -Id$ since $A$ has eigenvalues $\pm i$ and $n + 1$ is odd. Hence $A^2 = Id$ and thus the possible $A_i$ matrices are similar with the odd dimensional case.

Since $A$ and $B$ are conjugate, there is an element $P \in GL(n+1, \mathbb{R})$ such that $B = PAP^{-1}$. Then since $C = AB$, $C = APAP^{-1}$. Changing $P$ is a way to deform the holonomy in the sense of Theorem 2.1.

Define the maps

$$f : GL(n+1, \mathbb{R}) \longrightarrow SL(n+1, \mathbb{R})$$

by

$$f(P) = APAP^{-1},$$

and

$$g : SL(n+1, \mathbb{R}) \longrightarrow \mathbb{R}^2$$
by
\[ g(Q) = (\text{trace}(Q), \text{trace}(Q^2)). \]

Note that these two maps are regular. Choosing an appropriate \( P \) depending on \( A \)
can be done as follows:

**Case 1:** If \( A \) has only one \(-1\) eigenvalue then the \(+1\) eigenspaces of \( A \)
and \( B \) intersect in a subspace of dimension at least \( n - 1 \) (if the manifold has dimension \( n \)) for every choice of \( P \). Since \( C = AB \), there is an \((n - 1)\)-dimensional subspace,
on which \( C \) is identity and thus \( C \) has eigenvalue 1 with multiplicity at least \((n - 1)\).
Moreover, by using below \( P_{txt} \) matrix one can see that \( \text{trace}(f) \) is nonconstant. If \( t \)
is odd
\[ \text{trace}(f) = t - 1 + \frac{x + 2t - 6}{x} \]
and if \( t \) is even
\[ \text{trace}(f) = \frac{t^2 - 6t + 8}{2} x \quad \frac{t^3 - 10t^2 + 28t - 32}{4} \]
\[ \frac{t - 2}{x} + \frac{t^2 - 6t + 4}{4}, \]
where \( x \in \mathbb{R} \). Since \( \text{trace}(f) \neq n + 1 \), there exist two more eigenvalues \( \lambda \) and \( \lambda^{-1} \)
of \( C \). Here we can assume \( \lambda \neq 1 \) by replacing \( C \) to \( C^2 \) if needed and then clearly 
\( \lambda^{-1} \neq 1 \). It follows that \( C \) has eigenvalues \( \lambda \), \( \lambda^{-1} \) and 1.

We may take \( P = (a_{ij})_{txt}, \ (t = n + 1) \) as follows:

- If \( t \) is even, let
  \[ a_{k1} = a_{1k} = \begin{cases} 1, & k \text{ is odd}, \\ 0, & k \text{ is even}, \end{cases} \]
  \[ a_{kt} = \begin{cases} 0, & k \text{ is odd}, \\ 1, & k \text{ is even}, \end{cases} \]
  \[ a_{tk} = \begin{cases} 0, & k \text{ is odd}, \\ 1, & k \text{ is even and } k \neq 2, \\ x, & k = 2, \end{cases} \]

and the core \((t - 2) \times (t - 2)\) matrix
\[
\begin{bmatrix}
  a_{22} & a_{23} & a_{24} & \cdots & a_{2(t-1)} \\
  a_{32} & a_{33} & a_{34} & \cdots & a_{3(t-1)} \\
  a_{42} & a_{43} & a_{44} & \cdots & a_{4(t-1)} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{(t-1)2} & a_{(t-1)3} & a_{(t-1)4} & \cdots & a_{(t-1)(t-1)}
\end{bmatrix}
\]
is the mirror image of the identity matrix
\[
\begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 1 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix}
\]

- If \( t \) is odd, let
  \[
a_{1k} = \begin{cases} 
0, & \text{if } k \geq 3 \text{ is odd or } k = 2, \\
1, & \text{if } k \text{ is even and } k \neq 2 \text{ or } k = 1,
\end{cases}
\]
  \[
a_{(k+1)1} = a_{kt} = \begin{cases} 
0, & \text{if } k \text{ is odd}, \\
1, & \text{if } k \text{ is even},
\end{cases}
\]
  \[
a_{tk} = \begin{cases} 
0, & \text{if } k \text{ is odd and } k \neq 1, \\
1, & \text{if } k \text{ is even and } k \neq 2, \\
x, & k = 2,
\end{cases}
\]

and the core \((t - 2) \times (t - 2)\) matrix is the identity matrix.

**Case 2:** \( A \) has two \(-1\) eigenvalues. Then we choose \( P \) as follows:
- If \( t \) is even, let \( k = t/2 \) and \( a_{k1} = y \). If \( k \neq t/2 \), let
  \[
a_{k1} = \begin{cases} 
1, & \text{if } k \text{ is odd}, \\
0, & \text{if } k \text{ is even}.
\end{cases}
\]

Also let \( a_{12} = y + x, \ a_{1(t-1)} = y, \ a_{1k} = 0 \) for \( 3 \leq k \leq t - 2 \). \( a_{t2} = x, \ a_{t(t-1)} = y - x, \ a_{tk} = 0 \) for \( 3 \leq k \leq t - 2 \). When \( k = (t/2) + 1 \) let \( a_{kt} = x \).

Otherwise, (i.e. \( k \neq (t/2) + 1 \))

\[
a_{kt} = \begin{cases} 
0, & \text{if } k \text{ is odd}, \\
1, & \text{if } k \text{ is even},
\end{cases}
\]

and the core matrix \((t - 2) \times (t - 2)\) is the identity matrix.

In this case,
\[
\text{trace}(Q) = t - 4 - \frac{2(-1 + x)}{1 - x - y + yx - y^2 + x^2} - \frac{(-y - x)(-y + x)}{2(y - 1)} - \frac{1 - x - y + yx - y^2 + x^2}{1 - x - y + yx - y^2 + x^2} - \frac{-x + xy - y^2 + x^2}{1 - x - y + yx - y^2 + x^2} - \frac{-y^2 - y + yx + x^2}{1 - x - y + yx - y^2 + x^2} - \frac{x(y - 1)}{1 - x - y + yx - y^2 + x^2} - \frac{(y - x)(y + x)}{1 - x - y + yx - y^2 + x^2},
\]

where \( Q = APAP^{-1} \).
Consider the composition below.

\[
\mathbb{R}^2 \longrightarrow GL(n + 1, \mathbb{R}) \longrightarrow SL(n + 1, \mathbb{R}) \longrightarrow \mathbb{R}^2
\]
given by

\[
(x, y) \longrightarrow P \longrightarrow f(P) = APAP^{-1} \longrightarrow g(Q) = (\text{trace}(Q), \text{trace}(Q^2)).
\]

Then the determinant of the Jacobian matrix of the composition at \((2, 3)\) is \(-128\).

For the other cases, we refer the reader to Appendix A.

For each case except Case 1, for a generic \(P\), the dimension of \(+1\) eigenspace of \(C\) is \(n - 2k\), where \(k\) is the number of \(-1\) eigenvalues of \(A\). We call a matrix \(P\) admissible if the number of distinct eigenvalues of \(C\) is \(2k + 1\), which are \(1\) and some pairs \(\lambda_1^{\pm 1}, \lambda_2^{\pm 1}, \ldots, \lambda_k^{\pm 1}\), \(\lambda_i \neq \lambda_j^{\pm 1}\) and \(\lambda_i \neq 1\). Let \(E\) denote the set of non-admissible matrices \(P\) in \(GL(n + 1, \mathbb{R})\). Below we will show that \(E\) is a proper algebraic set in \(GL(n + 1, \mathbb{R})\) and thus the set of admissible matrices constitutes an open dense subset in \(GL(n + 1, \mathbb{R})\).

Let \(T\) be the set of eigenvalues of \(C\). Since \(C\) is conjugate to \(C^{-1}\), there is an involution on \(T\). If we take \(P \in E\) then either some \(\lambda_i = 1\) or \(\lambda_i = \lambda_j^{\pm 1}\), for some \(i \neq j\). First, assume that some \(\lambda_i = 1\). Without loss of generality, let \(\lambda_k = 1\). Then

\[
\text{trace}(C) = m + \sum_{i=1}^{k-1} (\lambda_i + \lambda_i^{-1}).
\]

In this case, \(\text{trace}(C), \text{trace}(C^2), \ldots, \text{trace}(C^k)\) satisfy an algebraic relation. On the other hand, if some \(\lambda_i = \lambda_j^{\pm 1}\), for some \(i \neq j\), then again without loss of generality, we may assume that \(\lambda_{k-1} = \lambda_k\). Then

\[
\text{trace}(C) = m + \sum_{i=1}^{k-1} a_i (\lambda_i + \lambda_i^{-1}),
\]

where \(a_i = 1\), for \(1 \leq i \leq k - 2\) and \(a_{k-1} = 2\). Hence, again \(\text{trace}(C), \text{trace}(C^2), \ldots, \text{trace}(C^k)\) satisfy an algebraic relation, where \(1 \leq k \leq (n + 1)/2\) or \(1 \leq k \leq n/2\) and \(\dim [g \circ f(E)] = k - 1\). For example, if all eigenvalues are \(\lambda_1\) and \(\lambda_1^{-1}\) then

\[
\text{trace}(C) = \frac{n + 1}{2} (\lambda_1 + \lambda_1^{-1}) \text{ and } \text{trace}(C^2) = \frac{n + 1}{2} (\lambda_1^2 + \lambda_1^{-2}).
\]

Now, \(\text{trace}(C)\) and \(\text{trace}(C^2)\) satisfy the following algebraic relation

\[
(\text{trace}(C))^2 = ((n + 1)/2)(\text{trace}(C^2)) + (n + 1)^2/2.
\]

Since the determinant of the Jacobian of \(g \circ f\) is nonzero at some points, for example \((2, 3, \ldots, k + 1)\), the image of the map \(g \circ f\) contains an open set and thus \(E\) is a closed proper subset of \(GL(n + 1, \mathbb{R})\). It follows that \(GL(n + 1, \mathbb{R}) \setminus E\) is open and dense in the Euclidean topology. Therefore, it is possible to perturb \(P\) slightly and thus the map \(hol\), so that the matrix \(C\) is diagonalizable over complex numbers.

With a proper choice of \(P\), it can be arranged that the arguments of complex eigenvalues \(\lambda_i\) of \(C\) are rational multiples of \(\pi\). Moreover, passing to a finite covering
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space $M^{(n)}$ of $M$ (see page 6), we can suppose all eigenvalues of $C$ are real and by passing to a further double cover these eigenvalues can be assumed to be positive.

This concludes the proof of Lemma 4.1. □

Hence, we have proved the following lemma for the case $A^2 = Id$.

**Lemma 4.2.** We can arrange that $C_i$ corresponding to $A_i$ so that its eigenvalues are $\{\lambda_{i}^{\pm}\}$ such that $\lambda_i > \lambda_{i-1} > \cdots > \lambda_1 > 1$, where $i \in \{1, \ldots, [(n + 1)/2]\}$ and the remaining eigenvalues of $C_i$ are all 1.

When $A^2 = -Id$ the corresponding matrix $C$ has eigenvalues $\lambda_1$ and $\lambda_{-1}$ with multiplicities both equal to $\frac{n + 1}{2}$.

When $A^2 = Id$, the possible $C_i$ matrices can be arranged depending on the number of $-1$ eigenvalues of $A_i$. Namely, the number of $-1$ eigenvalues of $A$ determine the number of different $\lambda_i$ eigenvalues of $C$. For example, if the number of $-1$ eigenvalues of $A$ is 2 then the corresponding $C$ is

$$
\begin{bmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \lambda_{-1} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \lambda_{1} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}.
$$

For each matrix $C_i$, the multiplicity of the eigenvalue $\lambda$ is the same as the multiplicity of $\lambda^{-1}$ since $C_i$ is conjugate to $C_i^{-1}$.

There is a 1-parameter diagonal subgroup $\rho : \mathbb{R} \rightarrow G \subset PGL(n + 1, \mathbb{R})$ such that $\rho(1) = [C]$. The group $G$ is identified with the unique one parameter subgroup containing the cyclic group $K$, which is generated by $C$ and thus each element of $G$ has real eigenvalues. $K$ is normal in $hol(\pi_1(M))$, so $G$ is normalized by $hol(\pi_1(M))$.

Let $N \rightarrow M$ be the double cover corresponding to the subgroup of $\pi_1(M)$ generated by $c = ab$. Clearly, $N \cong \tilde{W} \times S^1$ (see Section 3). Let $\pi : \tilde{N} \rightarrow N$ be the universal cover of $N$. Then $N$ has a real projective structure inheriting from $M$ with the same developing map $dev_M = dev_N$. The image of the holonomy for this projective structure on $\tilde{N}$ is generated by $[C]$.

Let $z \in gl(n + 1, \mathbb{R})$ be an infinitesimal generator of $G$ such that $G = exp(\mathbb{R} \cdot z)$. Consider the flow

$$
\Phi : \mathbb{RP}^n \times \mathbb{R} \rightarrow \mathbb{RP}^n
$$
on $\mathbb{RP}^n$ generated by $G$, which is given by

$$
\Phi(x, t) = exp(tz) \cdot x,
$$

for $x \in \mathbb{RP}^n$, $t \in \mathbb{R}$. 


Let $V$ be the vector field on $\mathbb{R}P^n$, the velocity of this flow. Since the vector field is preserved by this flow, $V$ is also preserved by $\text{hol}(\pi_1(N))$. Hence, $V$ pulls back via the developing map to a vector field $\text{dev}^{-1}(V) = \tilde{v}$ on $\tilde{N}$ and it is invariant under covering transformations thus covers a vector field $\pi(\text{dev}^{-1}(V)) = v$ of $N$.

In the paper [6], the following two lemmas are proved for the 3-dimensional case (Lemma 4.5 and Lemma 4.6). Moreover, the results are still valid in our case, for any dimension $n \geq 4$ and for any $C_i$ such that $1 \leq i \leq \frac{n+1}{2}$ (n is odd) or $1 \leq i \leq \frac{n}{2}$ (n is even).

**Lemma 4.3.** $\text{dev}(\tilde{N})$ does not contain any source or sink.

**Lemma 4.4.** The flow which is given by the vector field $v$ on $N$ is periodic and $N$ is fibered as a product $\tilde{W} \times S^1$ by the flowlines.

Let $X = \mathbb{R}P^n \setminus Z$, where $Z$ is the zero set of $V$. Then $X$ is foliated by flowlines. Let $\mathcal{L}$ be the leaf space of this foliation. Since $G$ is normalized by $\text{hol}(\pi_1(M))$ it follows that this group acts on $\mathcal{L}$. Since $\text{hol}(\pi_1(N)) \subset G$ the action of $\text{hol}(\pi_1(N))$ on $\mathcal{L}$ is trivial, the action $\text{hol}(\pi_1(M))$ on $\mathcal{L}$ is induced by the involution $\sigma$ by Section 3. Therefore, the holonomy gives an involution on $\mathcal{L}$.

Since $\text{dev}(\tilde{N}) \subset X$ there is a map from the leaf space of the induced foliation on $\tilde{N}$ into $\mathcal{L}$. The leaf space of $\tilde{N}$ is $\tilde{W}$ by Lemma 4.4. The induced map

$$h : \tilde{W} \rightarrow \mathcal{L}$$

is a local homeomorphism. Since $\text{dev}(\tilde{N}) \subset \mathbb{R}P^n$ is invariant under $\text{hol}(\pi_1(M))$ it follows that $h(\tilde{W}) \subset \mathcal{L}$ is invariant under involution.

After determining the possible generators $[C_i]$ of $\text{hol}$, we specify the orbit space $\mathcal{L}_i$, which corresponds to $C_i$ of $X = \mathbb{R}P^n \setminus Z$.

To determine the orbit space of $C_i$ we study the zero set of $C_i$ in the following cases.

**Case 1:** If the dimension is even $n = 2k$ then there are $\frac{n}{2} = k$ possible cases. Namely, the zero set $Z$ for $C_i$ is the disjoint union of $2i$ points and a linear subspace $\mathbb{R}P^{2k-2_i}$, where $1 \leq i \leq k$.

For $C_1$, the zero set consists of one source, one sink and a copy of $\mathbb{R}P^{2k-2}$. Call these elements as $p_1$: source, $p_2 : \mathbb{R}P^{2k-2}$ and $p_3$: sink. By taking the boundary of a tubular neighborhood of each element of $Z$ in $\mathbb{R}P^{2k}$, we determine each set of flowlines between any pair of $p_i$'s.

The corresponding flow for $C_1$ is given by

$$[x_0 \lambda_1^t : x_1 \lambda_1^{-t} : x_2 : x_3 : \ldots : x_n].$$

If $x_1 \neq 0$ then the flow can be written as

$$\left[\frac{x_0}{x_1} \lambda_1^t : 1 : \frac{x_2}{x_1} \lambda_1^t : \frac{x_3}{x_1} \lambda_1^t : \ldots : \frac{x_n}{x_1} \lambda_1^t \right].$$
Note that as $t \to -\infty$ the flow tends to its source, which is $[0 : 1 : 0 : 0 : \ldots : 0]$. When $x_0 \neq 0$, the flow can be written as

$$
\begin{bmatrix}
1 : \frac{x_1}{x_0} \lambda_1^{-2^r} : \frac{x_2}{x_0} \lambda_1^{-t} : \frac{x_3}{x_0} \lambda_1^{-t} : \ldots : \frac{x_n}{x_0} \lambda_1^{-t}
\end{bmatrix}.
$$

Similarly, as $t \to +\infty$ the flow tends to its sink, which is $[1 : 0 : 0 : \ldots : 0]$.

If we use Euclidean coordinates, there are $n$ parameters in $\mathbb{R}^n$, but one of them is not zero ($x_0 \neq 0$). Thus, the flowlines starting from $p_1$ and leaving $S_1$ from the northern (or the southern) hemisphere go to $p_3$ from the northern (or the southern) hemisphere of $S_2$, see Figure 2.

We consider the source coordinates in Euclidean coordinates as

$$
\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_n}{x_1}\right) \in \mathbb{R}^n
$$

and the sink coordinates in Euclidean coordinates as

$$
\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in \mathbb{R}^n.
$$

For the flowlines starting at $p_1$ and leaving $S_1$ from the northern hemisphere we assume $\frac{x_0}{x_1} > 0$, for $x_1 \neq 0$. Then $\frac{x_1}{x_0} > 0$, for $x_0 \neq 0$ and the flowlines go to $p_3$ from the northern hemisphere of $S_2$. Similarly, when $\frac{x_0}{x_1} < 0$, for $x_0 \neq 0$ and $x_1 \neq 0$ there is an identification between the southern hemispheres of $S_1$ and $S_2$.

Moreover, the flowlines starting from the equator of $S_1$ go to $p_2 = \mathbb{R}P^{2k-2}$ and the flowlines starting from $p_2$ go to the equator of $S_2$. Therefore, the leaf space can be thought as a sphere $S^{2k-1}$ with two disjoint equators. We simply say that $S^{2k-1}$ has a double equator.
The table below describes the subspaces of $L$ consisting of the flowlines starting at $p_i$ and ending at $p_j$, for the matrix $C_1$. The symbol ‘∅’ shows that there is no flowline. In the table, we label source points in the upper horizontal line and in the vertical line we label sink points.

| source/sink | $p_1$ | $p_2$ | $p_3$ |
|-------------|-------|-------|-------|
| $p_1$       | ∅     | ∅     | ∅     |
| $p_2$       | $S^{2k-2}$ | ∅     | ∅     |
| $p_3$       | $S^{2k-1}$ | $S^{2k-2}$ | ∅     |

Table 1. The subspaces of the leaf space $L_1$ for even dimensional case.

Note that the table above implies that the leaf space $L_1$ consists of a copy of $S^{2k-1}$ with a double equator. If a sphere has a double equator, we will denote the sphere as $S$. Therefore, the leaf space is $L_1 = S^{2k-1}$.

For $C_2$, the zero set consists of $p_1, p_2, p_3 = \mathbb{R}P^{2k-4}, p_4, p_5$ and the leaf space is $L_2 = S^{2k-1} \cup S^{2k-3}$. The corresponding flow is given by

$$[x_0 \lambda_1^t : x_1 \lambda_1^{-t} : x_2 \lambda_2^t : x_3 \lambda_2^{-t} : x_4 : x_5 : \cdots : x_{2k}].$$

Similarly, the table below gives a list of subspaces of the space of flowlines for $C_2$ that starts at each $p_i$ and ends at each $p_j$, $i, j = 1, 2, ..., 5$.

| source/sink | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|-------------|-------|-------|-------|-------|-------|
| $p_1$       | ∅     | ∅     | ∅     | ∅     | ∅     |
| $p_2$       | $S^{2k-3} - S^0$ | ∅     | ∅     | ∅     | ∅     |
| $p_3$       | $S^{2k-2} - S^{2k-3}$ | $S^{2k-4}$ | ∅     | ∅     | ∅     |
| $p_4$       | $S^{2k-1} - S^{2k-2}$ | $S^{2k-3} - S^{2k-4}$ | $S^{2k-4}$ | ∅     | ∅     |
| $p_5$       | $S^{2k-1} - S^{2k-2}$ | $S^{2k-2} - S^{2k-3}$ | $S^{2k-3} - S^0$ | $S^0$ | ∅     |

Table 2. The subspaces of the leaf space $L_2$ in even dimensional case.

In all cases, the subspaces above the diagonal in each table are empty and the nonempty spheres on the antidiagonal have a double equator.

To understand the topology of the space $L$ clearly, we give an example in dimension 6. Consider the matrix $C_3$ with different eigenvalues

$$[\lambda_1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \lambda_1^{-1} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \lambda_2 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \lambda_2^{-1} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \lambda_3 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \lambda_3^{-1} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$$
and thus the corresponding leaf space is \( L_3 = S^5 \cup S^3 \cup S^1 \), where \( S^5 = S^5_1, \ S^3 = S^3_2, \ S^1 = S^1_3 \) in Table 3. Moreover, the involution \( \tau \) interchanges the spheres symmetric with respect to the vertical line through \( S^5_1 \) in Table 3. Indeed, \( \tau(S^0_j) = S^0_{7-j} \) and \( \tau(S^i_l) = S^i_{7-l} \).

**Table 3.** The leaf space \( L_3 = S^5 \cup S^3 \cup S^1 \). Note that all the spheres in the diagram except the ones in the last row have a double equator.

Now, consider the matrix \( C_2 \) in dimension 6

\[
\begin{bmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_2^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

such that the zero set consists of \( p_1, \ p_2, \ p_3 = \mathbb{RP}^2, \ p_4, \ p_5 \) and the corresponding leaf space is \( L_2 = S^5 \cup S^3 \).
Moreover, we have a similar table for each $C_i$ matrix. In general, for $C_i$ the leaf space becomes $\mathcal{L}_i = S^{2k-1} \cup S^{2k-3} \cup \ldots \cup S^{2k-1-2(i-1)}$.

**Case 2:** The dimension is odd, let us say $n = 2k-1$. Then there are \( \frac{n+1}{2} + 1 = k+1 \) possible cases.

We get leaf spaces similar to Case 1, for $1 \leq i \leq k$. For example, for $C_2$ the zero set consists of four points and a copy of $\mathbb{RP}^{2k-5}$, call them $p_1, p_2, p_3 = \mathbb{RP}^{2k-5}, p_4$ and $p_5$.

| source/sink | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|-------------|-------|-------|-------|-------|-------|
| $p_1$       | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $p_2$       | $S^{0}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $p_3$       | $S^{2k-4} - S^{0}$ | $S^{2k-5}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $p_4$       | $S^{2k-3} - S^{2k-4}$ | $S^{2k-4} - S^{2k-5}$ | $S^{2k-5}$ | $\emptyset$ | $\emptyset$ |
| $p_5$       | $S^{2k-2} - S^{2k-3}$ | $S^{2k-3} - S^{2k-4}$ | $S^{2k-4} - S^{0}$ | $S^{0}$ | $\emptyset$ |

**Table 5.** The subspaces of the leaf space $\mathcal{L}_2$ in odd dimensional case.

The leaf space is $\mathcal{L}_2 = S^{2k-2} \cup S^{2k-4}$, see Table 5.

In general, for $C_i$ the zero set consists of $p_1, p_2, \ldots, p_{i+1} = \mathbb{RP}^{2k-1-2i}, \ldots, p_{2i+1}$ and the leaf space $\mathcal{L}_i = S^{2k-2} \cup S^{2k-4} \cup \ldots \cup S^{2k-2i}$, for $1 \leq i < k$ and if $i = k$, $\mathcal{L}_i = S^{2k-2} \cup S^{2k-4} \cup \ldots \cup S^{2} \cup S^{0}$.

We have an immersion induced by the developing map (see page 12)

\[ h : \tilde{W}^{n-1} \longrightarrow \mathcal{L}^{n-1}. \]

The decomposition of $\mathcal{L}$ contains two $(n-1)$-dimensional open discs, which are $D^0_{n-1}$ and $D^1_{n-1}$, see Figure 3.

Now, we give some auxiliary lemmas for the proof of Theorem 3.1. For more detail see [3].
Lemma 4.5. Assume that \( D_n^{a-1} \) and the map \( h \) are as above. Let \( K \subseteq D_n^{a-1} \) or \( K \subseteq D_n^{a-1} \) be a closed disc and assume that \( h^{-1}(K) = H \) is not empty. Then the restriction of \( h \) to the subspace \( H \), \( h': H \rightarrow K \) is onto and it is a finite sheeted covering.

Proof. Since \( h': H \rightarrow K \) is a submersion, \( H \) is a submanifold in \( \tilde{W} \) with boundary. Then the induced map \( h': H \rightarrow K \) is a local homeomorphism. Since local homeomorphisms are open maps, \( h' \) is open. The map \( h' \) is also closed. To see this, take a closed subset \( Y \) of \( H \). Since \( H \) is compact, the subset \( Y \) is also compact. The image of \( Y \) is compact because \( h' \) is continuous. Finally, since \( K \) is Hausdorff, \( h'(Y) \) is closed. Therefore, the map \( h' \) is both open and closed. Then \( h'(H) = K \) since \( K \) is connected. Since \( H \) is compact and \( h': H \rightarrow K \) is a local homeomorphism, where both \( H \) and \( K \) are Hausdorff, by Lemma 2.2, \( h' \) is a covering projection. Moreover, it is finite sheeted since \( H \) is compact. \( \square \)

Lemma 4.6. The image \( h(\tilde{W}) \) contains the top dimensional open discs \( D_n^{a-1} \) and \( D_n^{a-1} \) in \( \mathcal{L} \). Moreover, when we restrict \( h \) to the preimages of these discs, the map \( h^{-1}(D_n^{a-1}) \rightarrow D_n^{a-1} \) is a finite sheeted covering space.

Proof. Since the map \( h \) is a local homeomorphism, \( h(\tilde{W}) \) contains at least one point in one of the \((n - 1)\)-dimensional open discs in \( \mathcal{L} \). By Lemma 4.5 if \( h(\tilde{W}) \) contains one point of an open disc, it is onto that open disc. Without loss of generality, let us say \( h(\tilde{W}) \) contains \( D_n^{a-1} \). Assume that \( h(\tilde{W}) \) does not contain any point in \( D_n^{a-1} \). Then \( h(\tilde{W}) \) can not contain a point from the equators \( S_n^{a-2} \) of \( S_n^{a-1} \). Because if the image \( h(\tilde{W}) \) contained a point from one of the equators \( S_n^{a-2} \) then the neighborhood of that point would have some points from \( D_n^{a-1} \). Then in this case, \( h : \tilde{W} \rightarrow D_n^{a-1} \) would be a covering map. Hence, \( \tilde{W} \) would be a disjoint union of open discs, which is a contradiction. In addition, \( h_1 : h^{-1}(D_n^{a-1}) \rightarrow D_n^{a-1} \) is a finite sheeted covering space by the above lemma. \( \square \)

In fact the above lemma implies the following corollary.

Corollary 4.7. \( \tilde{W} \setminus h^{-1}(D_n^{a-1}) \) is a nonempty \((n - 2)\)-dimensional manifold.
We will use the following well known fact repeatedly.

**Lemma 4.8.** Let \( L \) be an \( n \)-dimensional connected and simply connected manifold and \( U \subset L \) be an open ball, where \( n \geq 3 \). Then \( L \setminus U \) is connected and simply connected.

To proceed further, we consider the three cases of the leaf space \( \mathcal{L} \).

**Case 1:** Consider the immersion \( h : \tilde{W} \to \mathcal{L}_i = \mathcal{S}^{n-1} \cup \mathcal{S}^{n-3} \cup \ldots \cup \mathcal{S}^{n+1-2(i-1)} \), for \( n + 1 - 2i \geq 2 \).

Now, we remove the top dimensional open discs namely, \( D_n^{-1} \) and \( D_n^{-1} \) from the leaf space \( \mathcal{L} \) and their preimages from \( \tilde{W} \). Then the remaining \( (n - 2) \)-dimensional manifold \( \mathcal{G}^{n-2} = \tilde{W} \setminus h^{-1}(D_n^{-1}) \) is a closed connected manifold by Lemma [4.8] and Corollary [4.7] and the map

\[
\mathcal{G}^{n-2} \to \mathcal{S}^{n-2} \cup \mathcal{S}^{n-3} \cup \ldots \cup \mathcal{S}^{n+1-2(i-1)}
\]

is still an immersion. Next, the \( (n - 2) \)-dimensional open discs \( D_n^{-2} \)'s are removed from \( \mathcal{L} \) and their preimages from \( \mathcal{G}^{n-2} \) and we get an immersion as follows

\[
\mathcal{G}^{n-3} \to \mathcal{S}^{n-3} \cup \mathcal{S}^{n-3} \cup \ldots \cup \mathcal{S}^{n+1-2(i-1)}.
\]

Here, \( \mathcal{G}^{n-3} \) is an \( (n - 3) \)-dimensional manifold since the image of \( \mathcal{G}^{n-2} \) should contain points from the equators of \( \mathcal{S}^{n-2} \)'s.

We continue removing the top dimensional open discs from the leaf space \( \mathcal{L} \) and their preimages from the remaining part of \( \tilde{W} \) until we get

\[
\mathcal{G}^{n+1-2i} \to \mathcal{L}^{n+1-2i} = \mathcal{S}^{n+1-2i} \cup \mathcal{S}^{n+1-2i} \cup \ldots \cup \mathcal{S}^{n+1-2i} \cup \mathcal{S}^{n+1-2i}.
\]

By Lemma [4.8], \( \mathcal{G}^{n+1-2i} \) is still connected and simply connected as long as \( n + 1 - 2i \geq 2 \).

**Case 2:** If the dimension of the manifold is \( 2k \), for some \( k \in \mathbb{Z} \) and \( i = k \) then removing cells as above we finally obtain the following immersion

\[
\mathcal{G}^{2} \to \mathcal{L}^{2} = \mathcal{S}^{2} \cup \mathcal{S}^{2} \cup \ldots \cup \mathcal{S}^{2} \cup \mathcal{S}^{1}.
\]

Next, we remove small open discs containing the north and south poles of the 2-dimensional spheres and one of the equators of \( \mathcal{S}^{1} \) then foliate the complement with circles.

**Case 3:** If the dimension of the manifold is \( 2k - 1 \), for some \( k \in \mathbb{Z} \) and \( i = k \) then the immersion analogously will be

\[
\mathcal{G}^{2} \to \mathcal{L}^{2} = \mathcal{S}^{2} \cup \mathcal{S}^{2} \cup \ldots \cup \mathcal{S}^{2} \cup \mathcal{S}^{0}.
\]

Then we remove small open discs containing the north and south poles of the 2-dimensional spheres and \( \mathcal{S}^{0} \) then foliate the complement with circles.

Note that in all cases above there are foliations on \( \mathcal{L}^{n+1-2i} \) with the spheres \( \mathcal{S}^{r} \)'s \( (r = n - 2i \) in Case 1 and \( r = 1 \) in Case 2 and 3) after removing the small open discs containing the north and south poles of each \( (n + 1 - 2i) \)-sphere in \( \mathcal{L}^{n+1-2i} \). The number of the preimages of these open discs in \( \mathcal{G}^{n+1-2i} \) is finite and we remove these open discs from \( \mathcal{G}^{n+1-2i} \). Hence, there is also a foliation on \( \mathcal{G}^{n+1-2i} \) with \( r \)-dimensional manifolds \( \mathcal{J}^{r} \). These \( r \)-manifolds must be sphere since the sphere \( \mathcal{S}^{r} \) is closed in the leaf space, its
preimage is also closed. Furthermore, the preimage of $S^r$ is compact because $G_{n+1-2i}$ is compact. Hence, the map is a covering by Lemma 2.2, $J^r \rightarrow S^r$.

For $r = 1$, after removing the preimages of small open discs containing the north and south poles of each sphere in $L$, the remaining part of $\tilde{W}$ is foliated by 1-dimensional manifolds. These 1-dimensional manifolds are circles since $G_{n+1-2i}$ is compact and $J^1 \rightarrow S^1$ is a covering map. Hence, the remaining part of $\tilde{W}$ is foliated by circles and hence it is an annulus.

By Theorem 2.3 for $r > 1$, the foliation on $G$ is $S^r \times I$, where $I = [-1, 1]$. Hence, the leaf space of this foliation on $G_{n+1-2i}$ is $I = [-1, 1]$.

On the other hand, the quotient space of the leaf space $L_{n+1-2i}$ is a non-Hausdorff space which is a union of intervals with one extra origin $I^* = I_1 \cup I_2 \cup \ldots \cup I_{2i-1} \cup \{0\}$. The involution interchanges these intervals with each other except the one, which represents the sphere $S_{n+1-2i}$ and on that sphere it changes the double origins with each other. There is still an immersion $\tilde{h} : I \rightarrow I^*$ induced from the immersion $h$ such that $\tilde{h}(\pm 1)$ are the end points of some of the intervals in $I^*$. Such an immersion is an embedding, whose image contains one interval with only one copy of the origin. Therefore, this gives a contradiction since in this case the immersed image of $\tilde{W}$ in $L$ can not be invariant under involution.

Finally, we consider the case, where $C_0$ is the matrix corresponding to $A$, such that $A^2 = -Id$. The zero set is the disjoint union of two copies of $\mathbb{R}P^{k-1}$ in $\mathbb{R}P^{2k-1}$, $Z = l_1 \cup l_2$, where $l_i = \mathbb{R}P^{k-1}, i = 1, 2$. Consider the following diagram, where $\pi$ is the universal covering map.

\[
\tilde{N} = \tilde{W} \times \mathbb{R} \xrightarrow{\text{dev}} \mathbb{R}P^{2k-1} \\
\downarrow \pi \\
N = \tilde{W} \times S^1
\]

Then $\text{dev}^{-1}(l_i)$ is invariant under $\pi_1(N) \cong \mathbb{Z}$-action and $\alpha_i$ is an $(k-1)$-dimensional submanifold of $N$, where $\alpha_i = \text{dev}^{-1}(l_i)/\mathbb{Z} = \pi(\text{dev}^{-1}(l_i)) \subseteq N$. Therefore,

\[
\alpha_i \rightarrow l_i
\]

is a covering map. Now, we have two cases:

(i) $\alpha_1 \cup \alpha_2 = \emptyset$. It means that $\text{dev}(\tilde{N})$ is empty. Therefore, we can use Lemma 4.3 and Lemma 4.4 in this case also.

Note that each flowline starts at $l_1$ and ends at $l_2$. We consider the boundary of a tubular neighborhood of $\mathbb{R}P^{k-1}$ in $\mathbb{R}P^{2k-1}$, which is the total space of an $S^{k-1}$ bundle over $\mathbb{R}P^{k-1}$. Since there exists a unique flowline passing through any point of the total space of this bundle, the leaf space $L$ is that total space. (Note that if $(k - 1)$ is even, $\mathbb{R}P^{k-1}$ is nonorientable and thus the bundle is nontrivial.) The immersion $h = \tilde{W} \rightarrow L$, induced from the developing map $\text{dev} : \tilde{M} \rightarrow \mathbb{R}P^n$, is a covering map. Now consider the diagram below.
Since $k \geq 3$, $\pi_1(q^*(L)) = 0$, so $\tilde{W}$ and $q^*(L)$ are two simply connected coverings of $L$. Hence, we have a homeomorphism $q^*(L) \cong \tilde{W}$.

Therefore, $\tilde{W}$ is the total space of an $S^{k-1}$ bundle over $S^{k-1}$, see Figure 4.

Since by assumption $\tilde{W}$ is not the total space of an $S^{k-1}$ bundle over $S^{k-1}$ (in the statement of Theorem 3.1), $h$ is not a covering map and this is a contradiction.

(ii) $\alpha_1 \cup \alpha_2 \neq \emptyset$. In this case, Lemma 4.3 does not hold and we present an alternative argument as follows:

Without loss of generality, assume that $\alpha_1$ is nonempty. Let $\phi$ be the closure of a flowline of $v$ with one endpoint on $\alpha_1$. $\phi$ is a compact 1-submanifold of $N$ because its preimage in $\tilde{N}$ maps into a closed invariant interval in $\mathbb{RP}^{2k-1}$ with one endpoint in each $l_i$. Hence, the other endpoint of $\phi$ is in $\alpha_2$, which is also necessarily nonempty.

To show that $\alpha_1$ is connected, take a component $\gamma$ of $\alpha_1$. Let $U$ be the tubular neighborhood of $\gamma$ in $N$. $\text{dev}(\pi^{-1}(\gamma)) \subset l_1$ and actually they are equal. Hence, $\text{dev}(\pi^{-1}(U))$ contains a neighborhood of $l_1$. Thus, $U$ contains the total space $\Upsilon$ of an $S^{k-1}$ bundle over $\mathbb{RP}^{k-1}$ transverse to the flow and bounds a small neighborhood of $\gamma$. Since $U$ is preserved by the flow it follows that $U = \Upsilon \times \mathbb{R}$. The boundary of $U$ in $N$ is contained in $\alpha_1 \cup \alpha_2$. Therefore, $\alpha_1$ and $\alpha_2$ are both connected and $N = \alpha_1 \cup U \cup \alpha_2$ (this argument is analogous to the one in [6], p.8).

Since $\alpha_i \rightarrow l_i$, for $i = 1, 2$ is a covering map, there are two possibilities for $\alpha_i$, which are $S^{k-1}$ and $\mathbb{RP}^{k-1}$.

If $\alpha_i = S^{k-1}$ then the boundary of the neighborhood of $\alpha_i$ is $\partial \nu(\alpha_i) = S^{k-1} \times S^{k-1}$, which is the total space of a sphere bundle over a sphere. Since $k \geq 3$, the homotopy exact sequence implies that $\pi_1(\nu(\alpha_i))$ is trivial. $N$ can be written as $N = \nu(\alpha_1) \cup \nu(\alpha_2)$, where the two neighborhoods are glued along their boundaries via a diffeomorphism. Finally, by Van Kampen’s theorem

$$\pi_1(N) \cong \pi_1(\nu(\alpha_1)) \ast \pi_1(\nu(\alpha_2))/L,$$

where $L$ is the normal subgroup corresponding to the kernel of the homomorphism $\Phi : \pi_1(\nu(\alpha_1)) \ast \pi_1(\nu(\alpha_2)) \rightarrow \pi_1(N)$. 

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node (L) at (0,0) {$L$};
    \node (S) at (4,0) {$S^{k-1}$};
    \node (RP) at (0,-2) {$\mathbb{RP}^{k-1}$};
    \node (S1) at (4,-2) {$S^{k-1}$};
    \node (qL) at (2,0) {$q^*(L)$};
    \node (S2) at (2,-2) {$S^{k-1}$};
    \draw[->, bend right=15] (L) to node[anchor=west] {$\text{covering}$} (qL);
    \draw[->, bend right=15] (S) to node[anchor=west] {$\text{covering}$} (S1);
    \draw[->, bend right=15] (RP) to node[anchor=west] {$q$} (S2);
\end{tikzpicture}
\caption{$q^*(L)$ is an $S^{k-1}$ bundle over $S^{k-1}$.}
\end{figure}
However, this gives a contradiction because $\pi_1(N) \cong \mathbb{Z}$.

If $\alpha_i = \mathbb{R}P^{k-1}$ then $\partial\nu(\alpha_i) = S^{k-1} \times \mathbb{R}P^{k-1}$. Similarly,

$$\pi_1(N) \cong \pi_1(\nu(\alpha_1)) \ast \pi_1(\nu(\alpha_2))/L$$

and $\pi_1(\nu(\alpha_i)) \cong \mathbb{Z}_2$. Therefore, we get

$$\mathbb{Z} \cong \mathbb{Z}_2 \ast \mathbb{Z}_2/L.$$ 

However, this is not possible since it is a well known fact that $\mathbb{Z}_2 \ast \mathbb{Z}_2$ has no normal subgroup whose quotient is equal to $\mathbb{Z}$. Therefore, this gives a contradiction.

This finishes the proof. \[\square\]

5. An Obstruction To The Existence Of Real Projective Structures

In this section, we will give an obstruction to obtain examples of manifolds with the infinite fundamental group $\mathbb{Z}$ admitting no real projective structure.

General properties of Pontryagin classes give the following theorem.

**Theorem 5.1.** If there is an immersion $M^{n-1} \rightarrow \mathbb{R}^n$, where $M$ is an orientable manifold then the Pontryagin classes $p_i(M^{n-1})$ are all two torsion, for $i \geq 1$.

**Theorem 5.2.** Let $M^n$ be a simply connected manifold which does not admit any immersion into $\mathbb{R}^{n+1}$. Then $M \times S^1$ does not have any real projective structure.

**Proof.** Assume that $M \times S^1$ admits a real projective structure. Then there exists a developing map such that

$$M \times \mathbb{R} \xrightarrow{\text{dev}} \mathbb{R}P^{n+1}$$

$$M \times S^1$$

Consider the following diagram.

$$\begin{array}{c}
\mathcal{S}^{n+1} \\
M^n \xrightarrow{\text{dev}} \mathbb{R}P^{n+1}
\end{array}$$

Since the map $M \rightarrow \mathbb{R}P^{n+1}$ is an immersion and the double cover $S^{n+1} \rightarrow \mathbb{R}P^{n+1}$ is a local diffeomorphism, $M \rightarrow S^{n+1}$ is also an immersion. Moreover,

$$M \rightarrow S^{n+1} \setminus \{p\} = \mathbb{R}^{n+1}$$

is an immersion where $p$ is a point in $S^{n+1}$, which is not in the image of $M$. However, this yields a contradiction. \[\square\]
Example: Let $M = \mathbb{C}P^2$. The first Pontryagin class of $\mathbb{C}P^2$ is $p_1 = c_1^2 - 2c_2$, where $c_i$'s are Chern classes, for $i = 1, 2$. Then

$$p_1 = c_1^2 - 2c_2 = 9 - 2 \cdot 3 = 3.$$ 

Hence $p_1$ is not a torsion class. By Theorem 5.1 there is no immersion $\mathbb{C}P^2 \to \mathbb{R}^5$ and it contradicts to the existence of the developing map. Therefore, $\mathbb{C}P^2 \times S^1$ does not have a real projective structure.

Theorem 5.3. Assume that $W^{n-1}$ and $M$ as in Theorem 3.1. Assume further that the universal cover $\tilde{W}$ of $W$ does not admit an immersion into $\mathbb{R}^n$. Then $M$ has no real projective structure.

Proof. Assume on the contrary that $M$ has a real projective structure. Then the universal cover $\tilde{W} \times \mathbb{R}$ of $M$ has a real projective structure and thus the developing map $\text{dev}: \tilde{W} \times \mathbb{R} \to \mathbb{RP}^n$ provides an immersion of $\tilde{W}$ into $\mathbb{R}^n$. This finishes the proof. \hfill \Box

Remark 4. Note that since $S^{n-1}$ has an immersion into $\mathbb{R}^n$, the above theorem does not imply that $\mathbb{RP}^n \# \mathbb{RP}^n$ can not have a real projective structure.

Appendix A.

Choosing An Appropriate $P$ Depending On The Matrix $A$

In this section, we continue choosing an appropriate $P$ for $A$ and calculate $\text{trace}(Q)$ to say that the determinant of the Jacobian matrix at some points is nonzero by considering the following composition:

$$\mathbb{R}^2 \to GL(n+1, \mathbb{R}) \to SL(n+1, \mathbb{R}) \to \mathbb{R}^2$$

given by

$$(x, y) \mapsto P \mapsto f(P) = APAP^{-1} \mapsto g(Q) = (\text{trace}(Q), \text{trace}(Q^2)).$$

Case 2: $A$ has two $-1$ eigenvalues. Then we choose $P$ as follows:

- If $t$ is odd, set $k = (t - 1)/2$ and $a_{k1} = y$. If $k \neq (t - 1)/2$, let $a_{k1} = \begin{cases} 1, & k \text{ is odd}, \\ 0, & k \text{ is even}. \end{cases}$

$a_{t2} = x$, $a_{t(t-1)} = y - x$, $a_{tk} = 0$, for $3 \leq k \leq t - 2$, $a_{12} = y + x$, $a_{1(t-1)} = y$. If $t \neq 5$, take $a_{1((t+3)/2)} = a_{1((t-1)/2)} = 1$; otherwise, $a_{1k} = 0$, for $3 \leq k \leq t - 2$ and if $t = 5$ then $a_{13} = 1$. When $k = ((t+1)/2) + 1$ let $a_{kt} = x$. Otherwise, (i.e. $k \neq ((t+1)/2) + 1$)

$$a_{kt} = \begin{cases} 0, & k \text{ is odd}, \\ 1, & k \text{ is even}. \end{cases}$$

and the core matrix $(t - 2) \times (t - 2)$ is the identity matrix.
If \( A \) has two \(-1\) eigenvalues and \( t = 9 \) then we choose \( P_{t \times t} \) as below.

\[
\begin{bmatrix}
1 & y + x & 0 & 1 & 0 & 1 & 0 & y & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
y & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & x \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & x & 0 & 0 & 0 & 0 & 0 & y & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y - x
\end{bmatrix}.
\]

If \((t - 1)/2\) is even then

\[
\text{trace}(Q) = t - 6 - \frac{2y}{1 + y + 2x + y^2} - \frac{-y - x}{1 + y + 2x + y^2} - \frac{1 + y^2}{1 + y + 2x + y^2} - \frac{-1 - 2x - y^2 + yx}{1 + y + 2x + y^2} - \frac{y(1 + x)}{1 + y + 2x + y^2} - \frac{-y^2}{1 + y + 2x + y^2} + \frac{y + 2x}{1 + y + 2x + y^2} + \frac{1}{1 + y + 2x + y^2} + \frac{1 + y + x + y^2}{1 + y + 2x + y^2} + \frac{1 + y + x + yx}{1 + y + 2x + y^2} - \frac{-x - y^2 + yx}{1 + y + 2x + y^2} - \frac{x(-1 + y)}{1 + y + 2x + y^2} + \frac{(y - x)(-1 + y)}{1 + y + 2x + y^2},
\]

where \( Q = APAP^{-1} \).

Considering the same map with the case \( t \) is even, we get the determinant of the Jacobian matrix at \((2, 3)\) is \(-1792/4913\).

If \((t - 1)/2\) is odd then

\[
\text{trace}(Q) = t - 6 - \frac{3y}{y^2 + 2y + 2x} - \frac{-y - x}{y^2 + 2y + 2x} - \frac{2y^2}{y^2 + 2y + 2x} - \frac{y + x}{y^2 + 2y + 2x} - \frac{-y^2 + yx - y - x}{y^2 + 2y + 2x} - \frac{y^2 + yx + x}{y^2 + 2y + 2x} + \frac{y^2 + yx + 2y}{y^2 + 2y + 2x} + \frac{x}{y^2 + 2y + 2x} + \frac{y(x - y - 1)}{y^2 + 2y + 2x} - \frac{xy}{y^2 + 2y + 2x} + \frac{y(y - x)}{y^2 + 2y + 2x} + \frac{y(y - x)}{y^2 + 2y + 2x},
\]

and the determinant of the Jacobian matrix at \((2, 3)\) is \(-768/6859\).

In each case the determinant of the Jacobian is nonzero and thus the image of the map \( f \circ g \) contains an open set.

**Case 3:** If \( A \) has more than two \(-1\) eigenvalues, we take \( P \) as below.

First, consider the following composition.

\[\mathbb{R}^k \to GL(n+1, \mathbb{R}) \to SL(n+1, \mathbb{R}) \to \mathbb{R}^k,\]
given by

$$(x_1, x_2, ..., x_k) \mapsto P \mapsto f(P) = APAP^{-1} = Q \mapsto g(Q),$$

where $g(Q) = (\text{trace}(Q), \text{trace}(Q^2), ..., \text{trace}(Q^k))$ and $k$ is the number of $-1$ eigenvalues of $A$. The Jacobian matrix is given by

$$J = \begin{bmatrix}
\frac{\partial \text{trace}(Q)}{\partial x_1} & \frac{\partial \text{trace}(Q)}{\partial x_2} & \cdots & \frac{\partial \text{trace}(Q)}{\partial x_k} \\
\frac{\partial \text{trace}(Q^2)}{\partial x_1} & \frac{\partial \text{trace}(Q^2)}{\partial x_2} & \cdots & \frac{\partial \text{trace}(Q^2)}{\partial x_k} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \text{trace}(Q^k)}{\partial x_1} & \frac{\partial \text{trace}(Q^k)}{\partial x_2} & \cdots & \frac{\partial \text{trace}(Q^k)}{\partial x_k}
\end{bmatrix}.$$ 

- If $t$ is even,

  let $a_{12} = x_2$, $a_{1(t/2)} = a_{1(t+2)/2} = x_3$, $a_{1(t-1)} = x_1$, $a_{2(t-2)} = x_3$,

  $a_{(t/2)1} = x_2$, $a_{(t+2)/2} = x_3$, $a_{(t/2)t} = x_3$, $a_{((t+2)/2)t} = x_1$,

  $a_{(t-1)1} = 1$, $a_{42} = x_3$, $a_{(t-1)} = x_2$, and all the diagonal elements are 1.

  According to the number of $-1$ eigenvalues of $A$, we determine the number of different variables $x_i \in \mathbb{R}$, where $3 \leq i \leq k$ and $k = t/2$. In the core matrix, on the antidiagonal there are only $x_i$'s (except $x_3$) as a pair, which are symmetric with respect to the diagonal. Moreover, the number of some $x_i$'s are more than two conforming to the dimension. In addition, other entries of $P$ are all 0.

  For example, if $A$ has six $-1$ eigenvalues and $t = 14$ then $P$ is as below.

  $$P = \begin{bmatrix}
1 & x_2 & 0 & 0 & 0 & 0 & x_3 & x_3 & 0 & 0 & 0 & 0 & x_1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & x_2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & x_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & 0 & 0 & 0 & 0 & x_3 & \cdots \\
x_3 & 0 & 0 & 0 & 0 & 0 & x_1 & 1 & 0 & 0 & 0 & 0 & x_1 & \cdots \\
0 & 0 & 0 & 0 & x_4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 1 & 0
\end{bmatrix}.$$ 

  At the point $(2, 3, 4, 5, 6, 7)$ the determinant of the Jacobian is

  $$\begin{align*}
&3203652023 \\
&129225403018523774123952000
\end{align*}.$$ 

- If $t$ is odd,

  let $a_{12} = x_2$, $a_{1(t+1)/2} = x_3$, $a_{1(t-1)} = x_1$, $a_{2(t-2)} = x_3$,

  $a_{(t-1)/2} = x_2$, $a_{((t+1)/2)} = 1$, $a_{((t+3)/2)} = x_3$, $a_{(t-1)} = 1$, 

  1 eigen-values and $t = 14$ then $P$ is as above.
\[ a_{(t-1)/2} = x_3, \quad a_{(t+3)/2} = x_1, \quad a_{t2} = x_3, \quad a_{(t-1)} = x_2 \] and the diagonal elements are all 1.

In the core matrix, on the antidiagonal there are only \( x_i \)'s (except \( x_3 \)) as a pair, which are symmetric with respect to the diagonal. Moreover, the number of some \( x_i \)'s are more than two conforming to the dimension. In addition, other entries of \( P \) are all 0.

For example, if \( A \) has five \(-1\) eigenvalues and \( t = 13 \) then \( P \) is as follows:

\[
P = \begin{pmatrix}
1 & x_2 & 0 & 0 & 0 & x_3 & 0 & 0 & 0 & 0 & x_1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & x_2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 \\
& x_2 & 0 & 0 & 0 & 0 & 1 & 0 & x_1 & 0 & 0 & 0 & 0 & x_3 \\
& 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
& x_3 & 0 & 0 & 0 & 0 & x_1 & 0 & 1 & 0 & 0 & 0 & 0 & x_1 \\
& 0 & 0 & 0 & 0 & x_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0, \\
& 0 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
& 0 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
& 1 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
& 0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 1 \\
\end{pmatrix}
\]

At the point \((2, 3, 4, 5, 6)\) the determinant of the Jacobian is

\[
\frac{74929536}{42961619719375}.
\]

**Case 4:** If \( A \) has eigenvalues \( \pm i \) then both \( +i \) eigenspace and \( -i \) eigenspace of \( A \) are \( \frac{n+1}{2} \) dimensional. Now, we choose \( P \) as in Case 3 with \( k = \frac{n+1}{2} \) variables.

Note that the calculations above are done with the program Maple.

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