Nonequilibrium Statistical Mechanics and Thermodynamics from Darwinian Dynamics: a Primer

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We present here an exploration on the physical implications of the Darwinian dynamics. We first show that how the nonequilibrium statistical mechanics emerges naturally. We then show that the first three laws of the thermodynamics, the Zeroth Law, the First Law and the Second Law can be followed from the Darwinian dynamics, except the Third Law. The inability to derive the Third Law indicates that the Darwinian dynamics belongs to the "classical" domain. Specifically, the Second Law is proved from the dynamical point of view. Two types of current dynamical equalities are explicitly discussed in the paper: one is based on Feynman-Kac formula and one is a generalization of the Einstein relation. Both are directly accessible to experimental tests. Our demonstration indicates that the Darwinian dynamics is logically a simple and straightforward starting point to get into thermodynamics and is complementary to the conservative dynamics dominated in physics.

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One of the principle objects of theoretical research in any department of knowledge is to find the point of view from which the subject appears in its greatest simplicity.
Josiah Willard Gibbs (1839-1903)

I. INTRODUCTION

The theory proposed by Darwinian and Wallace\textsuperscript{1,2} on the evolution in biology has been the fundamental theoretical structure to understand biological phenomena for nearly one and half centuries, referred to as the Darwinian dynamics in the present paper. In its initial formulation, the theory was completely narrative. No single equation was used. There have been continuous effects to clarify its meaning and to make it into more quantitative hence more predictive\textsuperscript{3,4,5,6,7,8}. Tremendous progresses have made during past 100 years. Now the degree of its usage of mathematics is comparable to any other mathematically sophisticated natural science. From the physics point of view, this theory is a \textit{bona fide} nonequilibrium dynamical theory.

In physics there has been a sustained interest during past several decades in nonequilibrium processes\textsuperscript{9,10,11,12,13,14,15}. The important goals are to bridge its connection to equilibrium processes and to clarify the roles of entropy and the Second Law of thermodynamics. Thanks to recent progresses in experimental technologies, particularly the nanotechnology, many previous inaccessible regimes are now been actively explored. There have renewed interests in this field, ranging from physics\textsuperscript{16,17,18,19}, chemistry\textsuperscript{20}, material science\textsuperscript{21}, biology\textsuperscript{8}, and to many other fields\textsuperscript{22}. Quantitative experimental and theoretical studies find their ways into the cellular and molecular processes of life. There is a strong going interaction between physical and biological sciences. The purpose of the present paper is to look at the fundamental issues in statistical mechanics and thermodynamics from the point of view of the Darwinian dynamics and to gain a new insight.

There is even an active interest from philosophical point of view on the foundation of statistical mechanics and thermodynamics. Relevant to the present paper, following three fundamental but controversial problems have been formulated\textsuperscript{23}: 1) In what sense can thermodynamics said reduced to statistical mechanics? 2) How can one derive equations that are not time-reversal invariant from a time-reversal invariant dynamics? 3) How to provide a theoretical basis for the "approach to equilibrium" or irreversible processes?

The Darwinian dynamics can answer all three questions in its own way. For the first question, as long as the statistical mechanics is formulated according to the Boltzmann-Gibbs distribution, the main structures of statistical mechanics and thermodynamics are equivalent. For the second question, it is found that the thermodynamics is based on the energy conservation and on the Carnot heat engine. It deals with quantities at equilibrium or steady state without time. There is no direction of time. Hence, there is no conflict between the thermodynamics and the time-reversal dynamics. For the last and third question, Darwinian dynamics comes with an adaptive behavior\textsuperscript{1,2,3,4,6,7,8} and with a built-in direction of time. It naturally provides a framework to address the question of "approaching to equilibrium". If one would insist, the third question might be transformed into another one: What would be the...
implication that there is a mutual reduction between the Darwinian dynamics and the Newtonian dynamics

Answer to this last question will not be attempted in the present paper. The base to answer above three questions will be discussed in next few sections.

The rest of the paper is organized as follows. In section II the Darwinian dynamics will be summarized in the light of recent progress. In section III it will be shown that the statistical mechanics and canonical ensemble follows naturally from the Darwinian dynamics. In section IV the connection to thermodynamics is explored. There it will be shown that the Zeroth Law, the First Law, and the Second Law can follow from the Darwinian dynamics, not the Third Law. In section V two types of simple but seemly profound dynamical equalities discovered recently, one based on the Feynman-Kac formula and one a generalization of the Einstein relation, are discussed. In section VI the present demonstration is put into perspective. No mathematical rigor is pursued in the present paper, but the care has been taken to make the demonstrations as clear as possible. With the solid physical and biological foundations behind, a rigorous mathematical formulation is possible.

II. DARWINIAN DYNAMICS, ADAPTIVE LANDSCAPE, AND F-THEOREM

This section summarizes the recent results on the Darwinian dynamics.

A. Stochastic differential equation: the trajectory view

In the context of genetics the Darwin’s theory of evolution\(^1\) may be summarized verbally as that the evolution is a result of genetic variation and its ordering through elimination and selection. Both randomness and selection are equally important in this dynamical process. With an suitable time scale, the Darwinian dynamics may be represented by the following stochastic differential equation\(^2\)

\[
\frac{dq}{dt} = f(q) + N_f(q) \xi(t),
\]

where \(f\) and \(q\) are \(n\)-dimensional vectors and \(f\) a nonlinear function of \(q\). The genetic frequency of \(i\)-th trait is represented by \(q_i\). Nevertheless, in the present paper it will be treated as a generic real function of time \(t\). All quantities in this paper are dimensionless. They are assumed to be measured in their own proper units unless explicitly specified. The collection of all \(q\) forms a real \(n\)-dimensional phase space. The noise \(\xi\) is a standard Gaussian white noise with \(l\) independent components: \((\xi_i)_{\xi} = 0\), and

\[
(\xi_i(t) \xi_j(t'))_{\xi} = \theta \delta_{ij} \delta(t - t'),
\]

and \(i, j = 1, 2, \ldots, l\). Here \((...)_{\xi}\) denotes the average over the noise variable \(\{\xi_i(t)\}\), to be distinguished from the average over the distribution in phase space below. The positive numerical constant \(\theta\) describes the strength of noise.

A further description of the noise term in Eq. (1) is through the \(n \times n\) diffusion matrix \(D(q)\), which is defined by the following matrix equation

\[
N_f(q) N_f^T(q) = 2D(q),
\]

where \(N_f\) is an \(n \times l\) matrix, \(N_f^T\) is its the transpose, which describes how the system is coupled to the noisy source. This is the first type of the F-theorem, a generalization of Fisher’s fundamental theorem of natural selection in population genetics. According to Eq.(2) the \(n \times n\) diffusion matrix \(D\) is both symmetric and nonnegative. For the dynamics of state vector \(q\), all what is needed from the noisy term in Eq.(1) are the diffusion matrix \(D\) and the positive numerical parameter \(\theta\). Hence, it is not necessary to require the dimension of the stochastic vector \(\xi\) be the same as that of the state vector \(q\). This implies that in general \(l \neq n\).

It is known that a large class of nonequilibrium processes can be described by such a stochastic differential equation\(3, 4, 6, 7\). There is a strong current interest on such stochastic and probability description ranging from physics\(8, 9\), chemistry\(10\), material science\(11\), biology\(12\), and other fields\(13, 14\).

The Darwinian dynamics was conceived graphically by Wright in 1932 as the motion of the system in an adaptive landscape\(1, 2\). Since then such a landscape has been known as the fitness landscape in some part of literature. However, there are a considerable amount of confusion on the definitions of fitness\(6, 8\). In this paper a more neutral term, the (Wright evolutionary) potential function, will be used to denote this landscape. The adaptive landscape connecting both the individual dynamics and its final destination is intuitively appealing. Nevertheless, it had been difficult to prove its existence in a general setting. The difficulty lies in the fact fact that typically the detailed balance condition does not hold in Darwinian dynamics, that is, \(D^{-1}(q)f(q)\) cannot be written as a gradient of scalar function\(9, 11, 12, 13, 15\).
Figure 1. Adaptive landscape with in potential contour representation. +: local basin; −: local peak; ×: pass (saddle point).

During the study of the robustness of the genetic switch in a living organism, a constructive method was discovered to overcome this difficulty: Eq. (1) can be transformed into the following form of stochastic differential equation,

\[ [R(q) + T(q)] \dot{q} = -\nabla \phi(q; \lambda) + N_{II}(q)\xi(t), \]  

(4)

where the noise \( \xi \) is from the same source as that in Eq. (1). The parameter \( \lambda \) denotes the influence of non-dynamical and external quantities. It should be pointed out that the potential function \( \phi \) may also implicitly depend on \( \theta \). The friction matrix \( R(q) \) is defined through the following matrix equation

\[ N_{II}(q)N_{II}^T(q) = 2R(q), \]  

(5)

which guarantees that \( R \) is both symmetric and nonnegative. This is the second type of the F-theorem. The F-theorem emphasizes the connection between the adaption and variation and is a reformulation of fluctuation-dissipation theorem in physics. For simplicity we will assume \( \det(R) \neq 0 \) in the rest of the paper. Hence \( \det(R + T) \neq 0 \). The breakdown of detailed balance condition or the time reversal symmetry is represented by the finiteness of the transverse matrix, \( T \neq 0 \). The usefulness of the formulation of Eq. (5) is already manifested in the successful solution of outstanding stable puzzle in gene regulatory dynamics and in a consistent formulation of the Darwinian dynamics.

The \( n \times n \) symmetric non-negative friction matrix \( R \) and the transverse matrix \( T \) are related to the diffusion matrix \( D \):

\[ R(q) + T(q) = \frac{1}{D(q) + A(q)}. \]

Here \( A \) is an antisymmetric matrix determined by both the diffusion matrix \( D(q) \) and the deterministic force \( f(q) \). One of more suggestive forms of above equation is

\[ [R(q) + T(q)]D[R(q) - T(q)] = R(q). \]  

(6)

This symmetric matrix equation implies \( n(n+1)/2 \) single equations from each of its element. The Wright evolutionary potential function \( \phi(q) \) is connected to the deterministic force \( f(q) \) by

\[ -\nabla \phi(q; \lambda) = [R(q) + T(q)]f(q). \]

Or its equivalent form,

\[ \nabla \times [[R(q) + T(q)]f(q)] = 0. \]  

(7)

Here the operation \( \nabla \times \) on an arbitrary \( n \)-dimensional vector \( v \) is a matrix generalization of the curl operation in lower dimensions \( (n = 2, 3) \): \( (\nabla \times v)_{i,j} = \nabla_i v_j - \nabla_j v_i \). Above matrix equation is hence antisymmetric and gives \( n(n-1)/2 \) single equations from each of its element. From Eq. (6) and (7) the friction matrix \( R \), the transverse matrix \( T \), and the potential function \( \phi \) can be constructed in terms of the diffusion matrix \( D \) and the deterministic force \( f \). The local construction was demonstrated in detail in Ref. 27. For a global construction an iterative method was outlined in Ref. 28.

In the case the stochastic drive may be ignored, that is, \( \theta = 0 \), the relationship between Eq. (1) and (4) remains unchanged. Furthermore, Eq. (4) becomes a deterministic equation

\[ [R(q) + T(q)]q = -\nabla \phi(q; \lambda). \]  

(8)

Because of the non-negativeness of the friction matrix, one obtains

\[ \frac{d}{dt} \phi(q; \lambda) = q \cdot \nabla \phi(q; \lambda) \]

\[ = -q^T[R(q) + T(q)]q \]

\[ = -q^T R(q) q \leq 0. \]  

(9)
It is immediately clear that the Wright evolutionary potential function $\phi(q; \lambda)$ is a Lyapunov function and the deterministic dynamics makes it non-increasing: the tendency to approach the nearby potential minimum to achieve the maximum probability. This is precisely what conceived by Wright. The adaptive dynamics has been actively exploring in biology.

The conservative Newtonian dynamics may be regarded as a further limit of zero friction matrix, $R = 0$. Hence, from Eq. (5), the Newtonian dynamics may be expressed as,

$$T(q) \dot{q} = -\nabla \phi(q; \lambda) .$$

Here the value of potential function is evidently conserved during the dynamics: $\dot{q} \cdot \nabla \phi(q; \lambda) = 0$, that is, the system moves along the equal potential contour in the adaptive landscape.

### B. Fokker-Planck equation: the ensemble view

It was heuristically argued that the steady state distribution $\rho(q)$ in the state space is, if exists,

$$\rho(q, t = \infty) \propto e^{-\beta \phi(q; \lambda)} .$$

Here $\beta = 1/\theta$. It takes the form of Boltzmann-Gibbs distribution function. Therefore, the potential function $\phi$ acquires both the dynamical meaning through Eq. (4) and the steady state meaning through Eq. (11).

It was further demonstrated that such a heuristical argument can be translated into an explicit procedure such that there is an explicit Fokker-Planck equation whose steady state solution is indeed given by Eq. (11). Starting for the the generalized Klein-Kramers equation, taking the limiting procedure of the zero mass limit, the desired Fokker-Planck equation corresponding to Eq. (4) is

$$\frac{\partial \rho(q, t)}{\partial t} = \nabla^\tau [D(q) + A(q)] [\theta \nabla + \nabla \phi(q; \lambda)] \rho(q, t) .$$

(12)

This equation is also a statement of conservation of probability. It can be rewritten as the probability continuity equation:

$$\frac{\partial \rho(q, t)}{\partial t} + \nabla \cdot j(q, t) = 0 ,$$

(13)

with the probability current density $j$

$$j(q, t) \equiv -[D(q) + A(q)] [\theta \nabla + \nabla \phi(q; \lambda)] \rho(q, t) .$$

(14)

The reduction of dynamical variables has often been done by the well-known Smoluchowski limit. In the above derivation we take the mass to be zero, keeping other parameters, including the friction and transverse matrices, to be finite. Nevertheless, in the Smoluchowski limit it is the friction matrix to be taken as infinite, keep all other parameters to be finite. Those two limits are in general not exchangeable.

The steady state configuration solution of Eq. (12) is indeed given by Eq. (11). It would be interested to point out that the steady state distribution function, Eq. (11), is independent of both friction matrix $R$ and the transverse matrix $T$. Furthermore, we emphasize that no detailed balance condition is assumed in reaching this result. In addition, both the additive and multiplicative noises are treated here on equal footing.

Finally, it can be verified that above construction leading to Eq. (12) is valid and remains unchanged when there is an explicit time dependent in $R$, $T$, and/or $\phi$. In this case though there may not exist a steady state distribution if the Wright evolutionary potential function $\phi$ is time dependent.

### III. STATISTICAL MECHANICS

#### A. Central Relations in Statistical Mechanics

As discussed above, if treating the parameter $\theta$ as temperature, the steady state distribution function in phase space is indeed the familiar Boltzmann-Gibbs distribution, Eq. (11). The partition function, or the normalization constant, is then

$$Z_\theta(\lambda) = \int dq e^{-\beta \phi(q; \lambda)} .$$

(15)
The integral $\int dq$ denotes the summation over whole phase space. The normalized steady state distribution is

$$\rho_{\theta}(q) \equiv \frac{e^{-\beta \phi(q;\lambda)}}{Z_{\theta}} .$$

(16)

For a given observable quantity $O(q)$, its average or expectation value is

$$\langle O \rangle_q \equiv \int dq\ Rho(q) O(q)$$

$$= \frac{1}{Z_{\theta}} \int dq\ O(q) e^{-\beta \phi(q;\lambda)} .$$

(17)

The subscript $q$ denoted that the average is over phase space, not over the noise in Eq.(1) or (4). Eq.(17) is the summit of statistical mechanics.

**B. Stochastic process and canonical ensemble**

A main question is that for a given Fokker-Planck equation, can the corresponding stochastic differential equation in the form of Eq.(4) be recovered? The answer is affirmative and the procedure to carry it out is already contained in Eq.(12), which will be briefly demonstrated below.

A generic form for the Fokker-Planck equation may be expressed as follows:

$$\frac{\partial \rho(q, t)}{\partial t} = \nabla \cdot [\theta D(q) \nabla \rho(q, t)] .$$

(18)

Here $\overline{D}(q)$ is the diffusion matrix and $\overline{f}(q)$ the drift force. The main motivation to take such a form is simple: In the case detailed balance condition is satisfied, i.e., $A(q) = 0$ (and $\overline{T}(q) = 0$), the potential function $\overline{\phi}$ can be directly read from above equation: $\nabla \overline{\phi} = \overline{D}^{-1} \overline{f}$. It puts the diffusion effect in a very prominent position. Any other form of Fokker-Planck equations can be easily transformed into above form. This generic form of the Fokker-Planck equation is less tangible to additional complications such as the noise induced first order transitions caused by the $q$-dependent diffusion constant.

A potential function $\phi(q)$ can always be defined from the steady state distribution. There is an extensive mathematical literature addressing this problem\cite{30}. After this is done, though it can be a difficult mathematical problem, the procedure to relate the generic Fokker-Planck equation to Eq.(12) is particularly straightforward. Eq.(12) can be rewritten as

$$\frac{\partial \rho(q, t)}{\partial t} = \nabla \cdot [\theta \overline{D}(q) \nabla \phi(q)] + [D(q) + A(q)] \nabla \phi(q) \rho(q, t) .$$

(19)

The antisymmetric property of the matrix $A(q)$ has been used in reaching Eq.(19). Thus, comparing between Eq.(18) and (19), we have

$$D(q) = \overline{D}(q) ,$$

(20)

$$\phi(q) = \overline{\phi}(q) ,$$

(21)

$$f(q) = \overline{f}(q) + \theta \nabla \overline{A}(q) .$$

(22)

In reaching Eq.(22) we have used the relation

$$- [D(q) + A(q)] \nabla \phi(q) = f(q) .$$

The explicit equation for the anti-symmetric matrix $A(q)$ is

$$\theta \nabla \overline{A}(q) + [D(q) + A(q)] \nabla \phi(q;\lambda) = \overline{f}(q) ,$$

(23)

which is a first order linear inhomogeneous partial differential equation. The solution for $A$ can be formally written down

$$A(q) = \frac{1}{\theta} \int dq' [\overline{f}(q') - D(q') \nabla' \phi(q';\lambda)] e^{\beta (\phi(q';\lambda) - \phi(q';\lambda))} + A_0(q) e^{\beta \phi(q;\lambda)} .$$

(24)
Here $A_0(q)$ is a solution of the homogenous equation $\theta \nabla \cdot A(q) = 0$ and the two parallel vectors in the integrand, such as $dq \, T(q)$, forms a matrix. This completes our answer to the converse question.

It is interesting to note that the shift between the zero’s of the potential gradient and the drift is given by, from Eq. (22),

$$\Delta \bar{r} = \theta \nabla \cdot A(q),$$

that is, the extremals of the steady state distribution are not necessary determined by the zero’s of drift. This is the formula for such a shift shown extensively in numerical studies. This shift can occur even when $D = constant$.

Thus, the zero-mass limit approach to the stochastic differential equation is consistent in itself. The meaning of the potential $\phi$ is explicitly manifested in both local trajectory according to Eq. (31) and ensemble distribution according to Eq. (12). In particular, no detailed balance condition is assumed. There is no need to differentiate between the additive and multiplicative noises. This zero mass limit procedure which leads to Eq. (11) from Eq. (12) may be regarded as another prescription for the stochastic integration, in addition to those of Ito and Stratonovich. The connection to those methods of treating stochastic differential equation is suggested through Eqs. (15) and (12) (or Eq. (19)). The Ito’s method puts an emphasis on the martingale property of stochastic processes, which may be viewed as a prescription from mathematics. The Stronotovich method stresses the differentiability such that the usual differential chain-rule can be formally applied, which may be viewed as the prescription from engineering. The present approach emphasizes the role played by the potential function in both trajectory and ensemble descriptions. It may be regarded as the prescription from natural sciences.

We may conclude that the stochastic process, regardless of Ito, Stratonovich, the present method, or others, leads to the canonical ensemble with a temperature and a Boltzmann-Gibbs type distribution function.

C. Discrete stochastic dynamics

There is another kind of modelling predominant in population genetics and other fields which is discrete in phage space and/or time. Here we would not get into it in any detail, except quoting results when necessary. The reasons of being able to do so are: 1) It is known mathematically any discrete model can be represented by a continuous one exactly, though sometimes such a process may turn a finite dimension problem into an infinite dimension one; 2) By a coarse graining average the discrete dynamics in population genetics can often be simplified to continuous ones such as diffusion equations or Fokker-Planck equations. It is generally acknowledged in population genetics and in other fields that the diffusion approximation is a good start and usually accurate.

For the steady state distribution, all one needs to know is the Wright evolutionary potential function $\phi$ and the positive numerical constant $\theta$ which in many cases can be set to be unity: $\theta = 1$. Hence, discrete or continuous representation is not a physically or biologically relevant point.

IV. THERMODYNAMICS

Given the Boltzmann-Gibbs distribution, the partition function can be evaluated according to Eq. (15). Hence, at the steady state, all observable quantities are in principle known according to Eq. (17). One may wonder then what would be the value of thermodynamics. First, there is a practical reason. In many cases the calculation of the partition function is a hard problem, if possible. It would be desirable if there are alternatives. Thermodynamics gives us a set of useful relations between observable quantities based on general properties of the system such as symmetries. Useful and precise information on one phenomenon can be inferred from the information on other quantities. Second, there is a theoretical reason. The thermodynamics has a scope far more general than most other fields in physics. It is the only field in classical physics whose foundation and structure not only have survived quantum mechanics and relativity shakeups, but become stronger. Furthermore, thermodynamics has a formal elegance which is exceedingly satisfying aesthetically. Its influence is far beyond physical sciences.

There exists already numerous excellent books exposing the thermodynamics from statistical mechanics point of view. A thorough treatment can be found in Callen. Concise and elementary treatments from thermodynamics point of view were given by Pippard and by Reiss. In the light of those superb expositions, the present discussion may appear incomplete as well as arbitrary. For a systematic discussion on thermodynamics the reader is sincerely encouraged to consult those books and/or any of her/his favorites not listed here. The main objective here is to show that the Darwinian dynamics indeed implies the main structure of thermodynamics, though at a first glance it seems to have no connection. The Darwinian dynamics is at the extreme end of nonequilibrium processes.
Even given a limited scope of presentation there are already excellent and recent papers. The paper by Oono and Panconi\textsuperscript{38} is such an example. It gave a comprehensive review on the problems from the point of view steady state thermodynamics. There are overlaps at various places. Nevertheless, there is one main difference: The role of "temperature" is emphasized here, instead. The paper by Sekimoto\textsuperscript{39} gave a detailed discussion on the connection between thermodynamics and Langevin dynamics. The main difference is that in the present paper the detailed balance condition is not needed.

### A. Zeroth Law

From the Darwinian dynamics, the steady state distribution is given by a Boltzmann-Gibbs type distribution, Eq.(11), determined by the Wright evolutionary potential function $\phi$ of the system and a positive parameter $\theta$ of the noise strength. Hence, the analogy of the Zeroth Law of thermodynamics is implied by the Darwinian dynamics: There exists a temperature-like quantity, represented by the positive parameter $\theta$. This "temperature" $\theta$ is absolute in that it does not depend on the system’s material details.

### B. First Law

From the partition function $Z_\theta$, we may define a quantity

$$F_\theta \equiv -\theta \ln Z_\theta . \quad (26)$$

We may also define the average Wright evolutionary potential function,

$$U_\theta \equiv \int dq \, \phi(q; \lambda) \, \rho_\theta(q) . \quad (27)$$

From the distribution function we may further define a positive quantity

$$S_\theta \equiv -\int dq \, \rho_\theta(q) \ln \rho_\theta(q) . \quad (28)$$

It is then straightforward to verify that

$$F_\theta = U_\theta - \theta S_\theta , \quad (29)$$

precisely the fundamental relation in thermodynamics satisfied by free energy, internal energy, and entropy. Hence we have the free energy $F_\theta$, the internal energy $U_\theta$, and the entropy $S_\theta$. The subscript $\theta$ emphasizes the steady state nature of those quantities. Due to the finite strength of stochasticity, that is, $\theta > 0$, not all $U_\theta$ is readily usable: $F_\theta$ is always smaller than $U_\theta$. A part of $\theta S_\theta$ called "heat" cannot be utilized.

It can also be verified from definitions that if the system consists of several non-interacting parts, $F_\theta$, $U_\theta$, and $S_\theta$ are sum of those corresponding parts. Hence, they are extensive quantities. No attention is paid here to the fine difference between additive and extensive properties. Instead, the "temperature" $\theta$ is an intensive quantity: it must be the same for all those parts because they are contacting the same noisy source. Therefore, we conclude that the analogy of the First Law of thermodynamics is implied in the Darwinian dynamics.

The fundamental relation for the free energy, Eq.(26), as well as the internal energy, Eq.(27), may be expressed in their differential forms. Considering an infinitesimal process which causes changes in both the Wright evolutionary potential function via parameter $\lambda$ and in the steady state distribution function, the change in the internal energy according to Eq.(27) is

$$dU_\theta = \int dq \, \phi(q; \lambda) \frac{\partial \rho_\theta(q)}{\partial \lambda} \, d\lambda + \int dq \, \phi(q; \lambda) \, d\rho_\theta(q)$$

$$= \mu \, d\lambda + \theta dS_\theta . \quad (30)$$

This is the differential form for the internal energy. Here the steady state entropy definition of Eq.(28) has been used, along with $\int dq \, d\rho_\theta(q) = 0$, and

$$\mu \equiv \frac{\partial U_\theta}{\partial \lambda} \bigg|_\theta .$$

(31)
Eq. (30) can be written in the usual form in thermodynamics:
\[
dU = \bar{d}W + \bar{d}Q.
\]

The part corresponding to the change in entropy is the "heat" exchange: \(\bar{d}Q = \theta \, dS\) and the part corresponding to
the change in the Wright evolutionary potential function is the "work" \(\bar{d}W = \mu \, d\lambda\). The conservation of "energy" is
most clearly represented by Eq. (30). For the free energy,
\[
dF = dU - \theta \, S - \theta dS.
\]
(32)

Eq. (30) and (32) may be useful in some applications. For example, the "temperature" can be found via Eq. (30):
\[
\theta = \frac{\partial U}{\partial S} \bigg|_{\lambda}.
\]
(33)

This relation may be used to find the "temperature" in a nonequilibrium process if it is not obvious to identify a priori.

The convexity of a thermodynamical quantity is naturally incorporated by the Boltzmann-Gibbs distribution. There
is no restriction on the size of the system. Even for a finite system, phase transitions can occur, because singular
behaviors can be built into the potential function, and controlled by external quantities.

C. Second Law

First, we remind the reader of a few more important definitions.
A reversible process is such a process that all the relation between quantities and parameters in question is defined
through the Boltzmann-Gibbs distribution, Eq. (11). From the Darwinian dynamics point of view, the reversible
process in reality is necessarily a slow or quasi-static process in order to ensure the relevancy of steady state distribution
for its any practical realization.
An isothermal process is a reversible process in which "temperature" \(\theta\) remains unchanged, \(\theta = \text{constant}\). No confusion
with the thermostated processes, which are in general nonequilibrium dynamical processes, should arise.
An adiabatic process is a reversible process in which the coupling between the system and the noise source is switched
off and the system vary in such a way the distribution function remains unchanged along the dynamics trajectory
when following each point in phase space. This implies that the entropy remains unchanged, \(S = \text{constant}\). The
adiabatic process has often been used in irreversible processes in that there is no heat exchange between the system
and the noisy environment, hence \(S = \text{constant}\) (c.f. Eq. (42)).

Now we discuss the analogy of Carnot cycle on which the Carnot heat engine is based. The Carnot cycle consists
of four reversible processes: two isothermal processes and two adiabatic processes (Fig. 1.a,b). The efficiency \(\nu\) of the
Carnot heat engine is defined as the ratio of the total net work performed over the heat absorbed at high temperature:
\[
\nu \equiv \frac{\Delta W_{\text{total}}}{\Delta Q_{12}}.
\]
(34)

(a) (b)

Figure 2. Carnot cycle. (a). The \(\mu - \lambda\) representation. (b). The \(\theta - S\) representation. In this temperature-entropy
representation, the Carnot cycle is a rectangular.

The total net work done by the system is represented by the shaded area enclosed by the cycle. For the heat
absorbed at the high isothermal process \(1 \rightarrow 2\),
\[
\Delta Q_{12} = \theta_{\text{high}} \Delta S_{\theta,12}.
\]
(35)

For the adiabatic process \(2 \rightarrow 3\), an external constraint represented by \(\lambda\) is released (or applied),
\[
\Delta S_{\theta,23} = 0, \quad \Delta Q_{23} = 0.
\]
(36)
For the heat absorbed (rather, released) at the low isothermal process $3 \to 4$,
\[ \Delta Q_{34} = \theta_{low} \Delta S_{\theta,34} = -\Delta Q_{43}. \]  
(37)

For the adiabatic $4 \to 1$, an external constraint is applied (released),
\[ \Delta S_{\theta,41} = 0, \quad \Delta Q_{41} = 0. \]  
(38)

Using the First Law, Eq. (29) and the fact that the free energy is a state function
\[ \Delta F_{\text{total}} = \Delta Q_{\text{total}} - \Delta W_{\text{total}} = 0. \]  
(39)

The minus sign in front of the total work represents that it is the work done by the system, not to the system. The total heat absorbed by the system is
\[ \Delta Q_{\text{total}} = \Delta Q_{12} + \Delta Q_{34} = \Delta Q_{12} - \Delta Q_{43} = \Delta W_{\text{total}}. \]

We further have
\[ \Delta S_{\theta,12} = \Delta S_{\theta,43}. \]  
(40)

From Eq. ( ), ( ) and ( ) the Carnot heat engine efficiency is then
\[ \nu = 1 - \frac{\Delta Q_{43}}{\Delta Q_{12}} = 1 - \frac{\theta_{low}}{\theta_{high}}, \]  
(41)

precisely the form in thermodynamics. The beauty of Carnot heat engine is that its efficiency is completely independent of any material details. It brings out the most fundamental property of thermodynamics and is a direct consequence of the Boltzmann-Gibbs distribution function and the First Law. It reveals a property of Nature which may not be contained in a conservative dynamics, at least it is still not obviously to many people from the Newtonian dynamics point after more than 150 years of intensive studies. The Second Law of thermodynamics may be stated as that for all heat engines operating between two temperatures, Carnot heat engine is the most efficient. The Second Law is implied in the Darwinian dynamics.

There are many other versions of the Second Law, on which the reader is suggested to consult the books listed at the beginning of this section. Here we mentioned two equivalent versions from the stability point of view, which frame following discussions.

Minimum free energy statement: For given the potential function and the temperature, the free energy achieves its lowest possible value if the distribution is the Boltzmann-Gibbs distribution.

Maximum entropy statement: For given potential function and its average, the entropy attains its maximum value when the distribution is the Boltzmann-Gibbs distribution. This version of the Second Law is the most influential. Its inverse statement, the so-called maximum entropy principle, has been extensively employed in the probability inference both within and beyond physical and biological sciences.

It is attempting to generalize the entropy definition to the arbitrary time dependent distribution in analogy to Eq. (28):
\[ S(t) \equiv - \int dq \rho(q,t) \ln \rho(q,t). \]  
(42)

There are two apparent drawbacks for such definition, however. First, even if the evolution of the distribution function $\rho(q,t)$ is governed by the Fokker-Planck equation, Eq. (12), in general the sign of its time derivative, $dS(t)/dt = \dot{S}(t)$, cannot be determined, whether or not it is close to the steady state distribution. Though $\dot{S}(t)$ might indeed be divided into an always positive part and the rest, such a partition is arbitrary. More seriously, in general $S(t)$ can be either larger or smaller than $S_0$, which makes such a definition lose its appealing in the view of the maximum entropy statement of the Second Law. We will return to $S(t)$ later.

Nevertheless, if taking the lesson from the potential function that only the relative value is important, we may introduce a reference point in the functional space into a general entropy definition. One definition for the referenced entropy is
\[ S_r(t) \equiv - \int dq \rho(q,t) \ln \frac{\rho(q,t)}{\rho_{\theta}(q)} + S_{\theta}. \]  
(43)
With the aid of inequality $\ln(1 + x) \leq x$ and the normalization condition $\int dq \ \rho(q, t) = \int dq \ \rho_0(q) = 1$, it can be immediately verified that

$$S_r(t) = \int dq \ \rho(q, t) \ \ln \left(1 + \frac{\rho_0(q) - \rho(q, t)}{\rho(q, t)}\right) + S_\theta,$$

$$\leq \int dq \ \left(\rho_0(q) - \rho(q, t)\right) + S_\theta,$$

$$\leq S_\theta. \quad (44)$$

The equality holds when $\rho(q, t) = \rho_0(q)$. This inequality is independent of the details of the dynamics and is evidently a maximum entropy statement. Furthermore, with the aid of the Fokker-Planck equation, Eq. (12), the time derivative of this referenced entropy, $dS_r(t)/dt = \dot{S}_r(t)$ is always non-negative:

$$\dot{S}_r(t) = -\int dq \ \frac{\partial \rho}{\partial t}(q, t) \ \ln \frac{\rho(q, t)}{\rho_0(q)}$$

$$= -\int dq \ \left(\nabla \ln \frac{\rho(q, t)}{\rho_0(q)}\right)^T [D(q) + A(q)][\theta \nabla + \nabla \phi(q; \lambda)]\rho(q, t)$$

$$\geq 0. \quad (45)$$

Hence, this referenced entropy $S_r(t)$ has all the desired properties for the maximum entropy statement.

We remark that by the probability current density definition of Eq. (14), $j$ is zero at the steady state. This may differ from the usual probability current density definition which may be based on Eq. (18) and takes the form $j(q, t) \equiv -[\theta D \nabla - f(q)]\rho(q, t)$, which is not zero at the steady state. Instead, $\nabla \cdot j = 0$ at the steady state. Though the general definition of entropy of Eq. (12) may not be appealing, a general definition of free energy is consistent with the Second Law. We demonstrate it here. First, a general definition for the internal energy may be:

$$U(t) \equiv \int dq \ \phi(q; \lambda) \ \rho(q, t). \quad (46)$$

Given the distribution and the potential function, quantities defined in Eq. (12) and (16) can be evaluated. Following the form of Eq. (20), a general definition of free energy would be, with the "temperature" $\theta$,

$$F(t) \equiv U(t) - \theta \ S(t). \quad (47)$$

It can be verified that $F(t) \geq F_\theta$ and its time derivative is always non-positive, $\dot{F}(t) \leq 0$. So defined time dependent free energy indeed satisfies the minimum free energy statement of the Second Law. It differs from the referenced entropy $S_r(t)$ by a minus sign and by a constant:

$$F(t) = -\theta S_r(t) + U_\theta.$$

The generalized entropy $S(t)$ has one desired property regarding to the adiabatic processes (either reversible or irreversible) in that $D = 0$ during the adiabatic process. Hence,

$$\dot{S}(t) = -\int dq \ \frac{\partial \rho}{\partial t}(q, t) \ \ln \rho(q, t)$$

$$= -\int dq \ \left(\nabla \ln A(q) \ \nabla \phi(q; \lambda)\rho(q, t)\right) \ \ln \rho(q, t)$$

$$= -\int dq \ \left[(A(q) \ \nabla \phi(q; \lambda)) \cdot \nabla\right] \int^\rho(q, t) \ dp \ \ln p$$

$$= 0. \quad (48)$$
This is the known result in conservative Newtonian dynamics that the entropy remains unchanged. In deriving above equation we have used two properties: 1) the no-coupling to the noisy environment has been translated into the fact that the terms associated with the diffusion matrix $D$ and "temperature" $\theta$ are set to be zero in Eq.(12), because they are related to the noisy source whose information is not available during an adiabatic process; and 2) the incompressible condition of $\nabla \cdot [A(q) \nabla \phi(q; \lambda)] = 0$, which is typically satisfied in the Newtonian dynamics. In this conservative case, it can be verified that $\dot{S}_r(t) = 0$, too, for any adiabatic process.

It may be worthwhile to mention another referenced entropy $S_{r2}(t)$ which approaches the steady state entropy $S_\theta$ from above. It’s form is simple:

$$S_{r2}(t) \equiv - \int dq \, \rho_\theta(q) \ln \rho(q, t).$$

(49)

It can be verified that $S_{r2}(t) \geq S_\theta$ and $\dot{S}_{r2}(t) \leq 0$.

D. Third Law

Now we consider the behavior near zero "temperature", $\theta \to 0$. To be specific we assume the system is dominated by a stable fixed point. As suggested by the Boltzmann-Gibbs distribution, Eq.(11), only the regime of phase space near this stable fixed point will be important. Hence the Wright evolutionary potential function can be expanded around this point, taking as $q = 0$:

$$\phi(q; \lambda) = \phi(0; \lambda) + \frac{1}{2} \sum_{j=1}^{n} k_j(\lambda)q_j^2.$$  

(50)

Here we have also assumed that the number of independent modes is the same as the dimension of the phase space, though it may not necessary be so. This assumption will not affect our conclusion below. Those independent modes are represented by $q_j$ without loss of generality. The "spring coefficients" $\{k_j\}$ are functions of external parameters represented by $\lambda$.

The partition function according to Eq.(15) can be readily evaluated in this situation:

$$Z_\theta = e^{-\beta \phi(0; \lambda)} \prod_{j=1}^{n} \sqrt{\frac{2\pi \theta}{k_j}}.$$  

(51)

So is the entropy according according to Eq.(28):

$$S_\theta = n \left[ \theta - \frac{1}{2} \ln \theta \right] + \frac{1}{2} \sum_{j=1}^{n} \frac{k_j}{2\pi}.$$  

(52)

The first term does not depend on external parameters, but the second term does. This suggests that the entropy depends on control process in a finite manner at low enough temperature. Hence, the Darwinian dynamics does not imply the Third Law in which it states that in the limit of zero temperature the difference in entropy between different processes is zero.

One should not be surprised by above conclusion, because the Darwinian dynamics is essentially a classically dynamics. Same a conclusion could also be reached from classical physics. With quantum mechanics, the agreement to the Third Law is found and a stronger conclusion has been reached: Not only the difference in entropy should be zero, the entropy itself is zero at zero temperature.

We may conclude that a complete neglecting noise is not viable choice in general. When noise is small enough, new phenomena would happen. Phrasing differently, there appears to exist a bottom near which there is something.

To summarize, in this section we have shown that except the Third Law, all other Laws of thermodynamics would follow from Darwinian dynamics. The concern on which stochastic integration method, Ito, Stratonovitch, or others, is consistent with the Second Law is dissolved: Any of them can be made to be consistent with the Second Law. We also note that based on the thermodynamical relations, the fundamental relation of Eq.(28), the conservation of energy of Eq.(30), the universal heat engine efficiency of Eq.(41), supplemented by the additive of extensive quantities and the temperature of Eq.(33), the Boltzmann-Gibbs distribution is implied. In this sense the statistical mechanics and the thermodynamics are equivalent.

Thermodynamics deals with the steady state properties. The key property is determined by the Boltzmann-Gibbs distribution of Eq.(11) which only depends on the Wright evolutionary potential function $\phi$ and the "temperature"
\( \theta \). The rest relations are determined by the various symmetries of the system. No dynamical information can be inferred from them. In particular, there is no way to recover the information on two quantities determine the local time scales, the friction matrix \( R \) and the transverse matrix \( T \), from thermodynamics. In this sense the time is lost in thermodynamics. With this consideration, it is evident that thermodynamics contains no direction of time and hence is consistent with the time-reversal conservative Newtonian dynamics.

V. DYNAMICAL EQUALITIES

We have explored the steady state consequences of the Darwinian dynamics in statistical mechanics and in thermodynamics. In this section we explore its general dynamical consequences. Two types of recently found dynamical equalities will be discussed: one based on the Feynman-Kac formula and other a generalization of the Einstein relation.

A. Feynman-Kac formula

Previous discussions demonstrate that the Boltzmann-Gibbs distribution plays a dominant role. It is naturally to work in a representation in which Boltzmann-Gibbs distribution appears in a most straightforward manner, or, as close as possible. The standard approach in this spirit is as follows. First, choose the dominant part of evolution operator \( L \). The remaining part is denoted as \( \delta L \). In this subsection a general methodology to carry out this procedure is summarized.

The Fokker-Planck equation, Eq. (12), can be rewritten as

\[
\frac{\partial}{\partial t} \rho(q,t) = L(\nabla, q; \lambda) \rho(q,t), \tag{53}
\]

with \( L = \nabla^\prime [D(q) + A(q)] [\theta \nabla + \nabla \phi(q)] \). It’s solution can be expressed in various ways. The most suggestive form in the present context is that given by Feynman’s path integral\(^{14}\). If at time \( t’ \) the system is at \( q’ \), the probability for system at time \( t \) and at \( q \) is given by summation of all trajectories allowed by Eq. (14) connection those two points:

\[
\pi(q, t; q’, t’) = \sum_{\text{trajectories}} \left\{ q(t) = q; q(t’)= q’ \right\}. \tag{54}
\]

In terms of the summation over the trajectories, the solution to Eq. (53) (and Eq. (12)) may be expressed as

\[
\rho(q, t) = \int dq’ \pi(q, t; q’, t’) \rho(q, t=0) = \langle \delta(q(t)-q) \rangle|_{\text{trajectory}} . \tag{55}
\]

The delta function \( \delta(q(t)-q) \) is used to explicitly specify the end point. There is a summation over initial points \( q’ \) weighted by the initial distribution function \( \rho(q’, t=0) \).

Now, considering that the system is perturbed by \( \delta L(q; \lambda) \), represented, for example, by a change in control parameter \( \lambda \). The new evolution equation is

\[
\frac{\partial}{\partial t} \rho_{\text{new}}(q,t) = [L(\nabla, q; \lambda) + \delta L(q; \lambda)] \rho_{\text{new}}(q,t). \tag{56}
\]

The perturbation may act as a source or sink for the probability distribution. The probability is no longer conserved: in general \( \int dq \rho_{\text{new}}(q,t) \neq \int dq \rho_{\text{new}}(q,t=0) \). According to the Feynman-Kac formula\(^{22}\), its solution to this new equation can be expressed as

\[
\rho_{\text{new}}(q, t) = \left\langle \delta(q(t)-q) e^{\int_0^t dt’ \delta(L(q(t’)))} \right\rangle_{\text{trajectory}}, \tag{57}
\]

with \( \rho_{\text{new}}(q’, t=0) = \rho(q’, t=0) \) and the trajectories following the dynamics of Eq. (14), the same as that in Eq. (55). Thus, the evolution of the new density can be expressed by the evolution of the original dynamics. The corresponding procedure in quantum mechanics is that in the interaction picture\(^{22}\). Eq. (57) is a powerful equality. Various dynamical equalities can be obtained starting from Eq. (57). Indeed, its direct and indirect consequences have been extensively explored\(^{43,44}\).
B. Dynamical work and free energy difference

We have noticed the special role played by the Boltzmann-Gibbs distribution, Eq. (11). In particular, it is independent of the friction and transverse matrices $R, T$. Evidently the instantaneous Boltzmann-Gibbs distribution with $\lambda = \lambda(t)$ is

$$
\rho_\theta(q; \lambda(t)) = \frac{e^{-\beta \phi(q; \lambda(t))}}{Z_\theta(\lambda(0))}. \tag{58}
$$

Here we have explicitly indicated that the parameter is time-dependent. This distribution function is no longer the solution of the Fokker-Planck equation of Eq. (12). There will be transitions out of this instantaneous Boltzmann-Gibbs distribution function due to the time-dependence of the parameter $\lambda$. While such transitions may be hard to conceive in classical mechanics, they can be easily identified in quantum mechanics, because of discreteness of states. One of such well studied models is the dissipative Landau-Zener transition.

The interesting question is that whether the transitions can be reversed such that the instantaneous distribution is indeed an explicit solution for another but closely related evolution equation. This means that the original Fokker-Planck equation has to be modified in a special way to become a new equation. Indeed, this modified evolution equation can be found for any function $\bar{\rho}(q, t)$, which reads,

$$
\frac{\partial}{\partial t} \rho_{\text{new}}(q, t) = \left[ L(\nabla, q, t) - \frac{1}{\bar{\rho}(q, t)} L(\nabla, q, t) \bar{\rho}(q, t) + \left( \frac{\partial \ln |\bar{\rho}(q, t)|}{\partial t} \right) \right] \rho_{\text{new}}(q, t). \tag{59}
$$

It can be verified $\rho_{\text{new}}(q, t) = \bar{\rho}(q, t)$ is indeed a solution of above equation. Treating

$$
\delta L = -\frac{1}{\bar{\rho}(q, t)} L(\nabla, q, t) \bar{\rho}(q, t) + \frac{\partial \ln |\bar{\rho}(q, t)|}{\partial t}
$$

and the Feynman-Kac formula Eq. (57) may be applied. The analogous procedure is well studied on the transitions during adiabatic processes in interaction picture of quantum mechanics and of statistical mechanics.

Now, let $\bar{\rho}$ be the instantaneous Boltzmann-Gibbs distribution of Eq. (58): $\bar{\rho} = \rho_\theta(q; \lambda(t))$. We have

$$
\delta L = -\beta \lambda \frac{\partial \phi(q; \lambda)}{\partial \lambda}. \tag{61}
$$

Eq. (61) can be solved by summing over all trajectories using the Feynman-Kac formula, Eq. (57). At the same time, we know the instantaneous Boltzmann-Gibbs distribution of Eq. (58) is its solution. Hence equal those two solutions to the same equation, we have following equality

$$
\frac{e^{-\beta \phi(q; \lambda(t))}}{\int dq e^{-\beta \phi(q; \lambda(0))}} = \left. \left\langle \delta(q - q(t)) e^{-\beta \int_0^t dt' \lambda(t') \frac{\partial \phi(q(t'); \lambda(t'))}{\partial \lambda} \right\rangle \right|_{\text{trajectory}} \tag{60}
$$

Following Jarzynski we define the dynamical work

$$
W_t = \int_0^t dt' \lambda(t') \frac{\partial \phi(q(t'); \lambda(t'))}{\partial \lambda} \tag{61}
$$

The equality between the free energy difference $\Delta F_\theta = F_\theta(t) - F_\theta(0)$ and the dynamical work $W_t$ is, after summation over all final points of the trajectories in Eq. (60),

$$
e^{-\beta \Delta F_\theta} = \left. \langle e^{-\beta W_t} \rangle \right|_{\text{trajectory}} \tag{62}
$$

This elegant equality connects the steady state quantities $\Delta F_\theta$ to the work done in a dynamical process. It was first discovered by Jarzynski. It should be emphasized that there is no assumption of steady state at time $t$ for the system governed by Eq. (12). In fact, it is known, for example, in case of the Landau-Zener transition that it is not. This equality has been discussed and extended by various authors from various perspectives. The connection of this equality to the Feynman-Kac formula was first explicitly pointed out in Ref. 48. There have been experimental verifications of this equality.

The Jarzynski equality places the Boltzmann-Gibbs distribution hence the canonical ensemble in a central position. They are simply natural consequences from the Darwinian dynamics. However, if starting from the conservative Newtonian dynamics, the appropriate ensemble is the micro-canonical ensemble. Any distribution function which
is a function of the potential function or Hamiltonian would be the solution of the Liouville equation. From this point of view the Boltzmann-Gibbs distribution and the associated temperature appear arbitrary: It is just one among infinite possibilities. This concern has been raised in literature regarding to the generality of the equality of Eq. (62). No satisfactory treatment of this concern within Newtonian dynamics has been given. Rather, it has been an "experimental attitude": If one does this and makes sure the procedure is correct one gets that, and it works. Instead, the Darwinian dynamics provides one a priori reason to justify the use of Boltzmann-Gibbs distribution in the derivation of the Jarzynski equality.

C. Generalized Einstein relation

In deriving the Boltzmann-Gibbs distribution from the Darwinian dynamics, a generalization of the Einstein relation, Eq. (6):

\[ [R(q) + T(q)] D(q) [R(q) - T(q)] = R(q) , \]

has been used\textsuperscript{28}. This is another general and simple dynamical equality. In the presence of detailed balance condition, that is, \( T = 0 \), this relation reduces to \( RD = 1 \), which was discovered a century ago by Einstein\textsuperscript{55} and since known as the Einstein relation. Variants of the Einstein relation in different settings were obtained earlier and independently by Nernst\textsuperscript{56}, Townsend\textsuperscript{57}, Sutherland\textsuperscript{58}. Similar to the Jarzynski equality, the generalized Einstein relation is connected to the Boltzmann-Gibbs distribution.

Experimentally, all those quantities in Eq. (6) can be measured. Hence, this generalized Einstein relation should be subjected to experimental tests in the absence of detailed balance, that is, when \( T \neq 0 \).

For simplicity, we consider a situation realizable with current technology: a charged nanoparticle or macromolecule, an electron or a proton, with charge denoted by \( e \), in the presence a strong uniform magnetic field \( B \) and emersed in a viscous liquid with friction coefficient \( \eta \). We restrict our attention to two dimensional case (\( n = 2 \)). The corresponding Darwinian dynamical equation of Eq. (4) in this case is the Langevin equation with the Lorentz force for a "massless" charged particle\textsuperscript{59}:

\[ \eta \dot{q} + \frac{e}{c} B \hat{z} \times \dot{q} = -\nabla \phi(q) + N_{\text{f}} \xi(t) \]  \( (63) \)

The friction matrix is

\[ S = \eta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \] \( (64) \)

The transverse matrix is

\[ T = \frac{e}{c} B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \] \( (65) \)

and the "temperature" is \( \theta = k_B T_{BG} \), the Boltzmann constant and the thermal equilibrium temperature. The corresponding Fokker-Planck equation following Eq. (12) is

\[ \frac{\partial \rho(q,t)}{\partial t} = \nabla \left[ D \theta \nabla + [D + A] \nabla \phi(q) \right] \rho(q,t) . \] \( (66) \)

This is a precisely a diffusion equation with diffusion matrix \( D \). Both \( D \) and \( A \) can be obtained via the generalized Einstein relation, Eq. (6):

\[ D = \frac{\eta}{\eta^2 + \left( \frac{e}{c} B \right)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \] \( (67) \)

\[ A = \frac{\frac{e}{c} B}{\eta^2 + \left( \frac{e}{c} B \right)^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \] \( (68) \)

In a typical classical situation, though all quantities can be measured experimentally, the friction coefficient is likely less sensitive to the magnetic field. Then the experimentally one may need to focus on the diffusion in the present
of magnetic field without any potential field. In this case the evolution of distribution is governed by the standard diffusion equation:

$$\frac{\partial \rho(q, t)}{\partial t} = \theta d_B \nabla^2 \rho(q, t),$$

(69)

with

$$d_B = \frac{\eta}{\eta^2 + (eB)^2}.$$  

The solution to Eq. (69) with $\rho(q, t = 0) = \delta q(t = 0) - q$ is standard (two dimension, $n = 2$):

$$\rho(q, t) = \frac{1}{2\pi t} \exp\left\{-\frac{q^2}{2d_B \theta t}\right\}.$$  

Averaging over trajectories governed by Eq. (69), $\langle (q(t) - q(t = 0))\rangle_{trajectory} = 0$ and

$$\langle (q(t) - q(t = 0))^2\rangle_{trajectory} = 4d_B \theta t.$$  

The readily experimental system may be that by injection of electrons into a semiconductor one measures their diffusion in the presence of a magnetic field. Every quantity in the generalized Einstein relation of Eq. (6) can be measured and controlled experimentally. Such experiments may has already been done (????). Another experimental system may be on ionized hydrogen or deuterium. For charged macromolecules and nano-particles, the friction coefficient may be too large to allow a measurable magnetic field effect accessible by current magnets. As a numerical example, for the zero magnetic field diffusion constant of $d_B = 0 k_B T_{BG} \sim 10^4 \text{cm}^2/\text{sec}$, which amounts to diffuse about 100 cm in 1 second, the friction coefficient is $\eta = 1/d_B = 4 \times 10^{-16} \text{dyne}/(\text{cm/sec})$ at temperature $T_{BG} = 300K$. Assuming one net electron charge, for magnetic field $B = 1 \text{Telsa}$, we have $eB/c \sim 1.6 \times 10^{-16} \text{dyne}/(\text{cm/sec})$, comparable to the friction coefficient.

VI. PROSPECT

In the present paper we have presented the statistical mechanics and thermodynamics as natural consequences of the Darwinian dynamics. Two types of recently found general dynamical equalities have been explored. Both can be directly tested experimentally. Everything appears in its right place except one: From the physics point of view it is the conservative dynamics from which we should start, not that of the Darwinian. This physics view has indeed tremendous of experimental supports. Remarkable progresses have been made along this line of reasoning during past 150 years. It is still the subject of current intensive research focus$^{16,17,54}$. The physics effort may be condensed to one question. The natural consequence of the conservative dynamics is the micro-canonical ensemble, from which the canonical ensemble just appears to be one of its infinite possibilities. How and why does Nature choose the canonical ensemble and the Second Law? There is no consensus yet on the answer.

The difficulty in reaching the Second Law from the conservative dynamics may give a boost to consider the Darwinian dynamics. There is, however, a genuine and compelling reason to do so: the Darwinian dynamics is the most fundamental and successful dynamical theory in biological sciences. Furthermore, as having demonstrated above, from it the Second Law and other nonequilibrium properties follow naturally. Logically it provides a simple starting point. It must contain an element of truth.

The conservative dynamics and the Darwinian dynamics appear to occupy the two opposite ends of the theoretical description of Nature. Both have been extremely successful. In many aspects they appear to be complementary to each other. Wether or not there is a hidden reason such that they are truly related to each other is not known presently. It waits to be discovered by further experimental and theoretical studies. The present deliberation may provide a certain utility for this endeavor.

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Biases and prejudices are unavoidable. I apologize to those whose important works are not mentioned here, likely the result of my own oversight. I would appreciate the reader’s effort very much to bring her/his or other’s important works to my attention (e-mail: aoping@u.washington.edu). This work was supported in part by USA NIH grant under HG002894.

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