RATIONALITY OF MODULI OF ELLIPTIC FIBRATIONS WITH FIXED MONODROMY

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Abstract. We prove rationality results for moduli spaces of elliptic K3 surfaces and elliptic rational surfaces with fixed monodromy groups.

0. Introduction

Let $X$ be an algebraic variety of dimension $n$ over $\mathbb{C}$. One says that $X$ is rational if its function field $\mathbb{C}(X)$ is isomorphic to $\mathbb{C}(x_1,\ldots,x_n)$. The study of rationality properties of fields of invariants $\mathbb{C}(X)^G = \mathbb{C}(X/G)$ is a classical theme in algebraic geometry. For a finite group $G \subset \text{PGL}_n$ acting on $X = \mathbb{P}^{n-1}$ the problem is referred to as Noether’s problem (1916). It is still unsolved for $n = 4$. Another class of examples is provided by moduli spaces. Birationally, they are often representable as quotients of simple varieties, like projective spaces or Grassmannians, by actions of linear algebraic groups, like PGL$_2$. Rationality is known for each of the following moduli spaces:

- curves of genus $\leq 6$ [18], [22], [20], [21], [31];
- hyperelliptic curves [18], [4];
- plane curves of degrees $4n+1$ and $3n$ [19];
- Enriques surfaces [24];
- polarized K3 surfaces of degree 18 [32];
- stable vector bundles (with various numerical characteristics) on curves, Del Pezzo surfaces, $\mathbb{P}^3$, [23], [1], [11], [25], [29];

and in many other cases. For excellent surveys we refer to [12] and [33]. We will study rationality properties of moduli spaces of smooth non-isotrivial Jacobian elliptic fibrations over curves

$\pi : \mathcal{E} \to C$
with fixed global monodromy group $\tilde{\Gamma} = \tilde{\Gamma}(\mathcal{E}) \subset \text{SL}_2(\mathbb{Z})$. In [8] we developed techniques aimed at the classification of possible global monodromies $\tilde{\Gamma}$. The present paper gives a natural application of these techniques.

Let $\mathcal{B}$ be an irreducible algebraic family of Jacobian elliptic surfaces. Then the set of subgroups $\tilde{\Gamma} \subset \text{SL}_2(\mathbb{Z})$ such that $\tilde{\Gamma}$ is the (global) monodromy group of some $\mathcal{E}$ in this family is finite. Moreover, for every such group $\tilde{\Gamma}$ the subset of fibrations with this monodromy $\mathcal{B}_{\tilde{\Gamma}} := \{ b \in \mathcal{B} \mid \tilde{\Gamma}(\mathcal{E}_b) = \tilde{\Gamma}\}$ is an algebraic (not necessarily closed) subvariety of $\mathcal{B}$.

Generalizing this observation, we introduce (maximal) parameter spaces $\mathcal{F}_{\tilde{\Gamma}}$ of elliptic fibrations with fixed global monodromy $\tilde{\Gamma}$ (considered up to fiberwise birational transformations acting trivially on the base of the elliptic fibration). These parameter spaces can be represented as quotients of quasi-projective varieties by algebraic groups. In particular, we can consider irreducible connected components of the parameter space $\mathcal{F}_{\tilde{\Gamma}}$, which we call moduli spaces. Even though these moduli spaces need not be algebraic varieties, we can still make sense of their birational type.

**Theorem.** Let $\tilde{\Gamma} \subset \text{SL}_2(\mathbb{Z})$ be a proper subgroup of finite index. Then all moduli spaces of (Jacobian) elliptic rational or elliptic K3 surfaces with global monodromy $\tilde{\Gamma}$ are rational.

**Corollary.** For all $\tilde{\Gamma}$ with moduli $\mathcal{F}_{\tilde{\Gamma}}$ of dimension $> 0$ there exists a number field $K$ such that there are infinitely many nonisomorphic elliptic K3 surfaces over $K$ with global monodromy $\tilde{\Gamma}$.

**Remark 0.1.** Our method shows that many other classes of moduli of elliptic surfaces over $\mathbb{P}^1$ with fixed monodromy are rational or unirational. However, we cannot expect a similar result for all moduli spaces of elliptic surfaces over higher genus curves, since the moduli space of higher genus curves itself is not uniruled (by a result of Harris and Mumford [15]).

We proceed to give a more detailed description of our approach. First of all, we can work not with the monodromy group $\tilde{\Gamma}$ itself but rather with its image $\Gamma \subset \text{PSL}_2(\mathbb{Z})$ under the natural projection $\text{SL}_2(\mathbb{Z}) \rightarrow \text{PSL}_2(\mathbb{Z})$. Let $\mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$
be the upper half-plane and 
\[ \mathcal{H} = \mathcal{H} \cup \mathbb{Q} \cup \{ \infty \} . \]
The natural \( j \)-map 
\[ j : C \to \mathbb{P}^1 = \mathcal{H} / \text{PSL}_2(\mathbb{Z}) \]
decomposes as 
\[ j = j_{\Gamma} \circ j_{\mathcal{E}}, \]
where 
\[ j_{\mathcal{E}} : C \to M_{\Gamma} = \mathcal{H} / \Gamma \]
\[ j_{\Gamma} : M_{\Gamma} \to \mathbb{P}^1 = \mathcal{H} / \text{PSL}_2(\mathbb{Z}) . \]
Here \( M_{\Gamma} \) is the \( j \)-modular curve corresponding to \( \Gamma \); it is equipped with a special triangulation, obtained as the pullback of the standard triangulation of \( S^2 = \mathbb{P}^1(\mathbb{C}) \) (by two triangles with vertices at 0, 1 and \( \infty \)) under the map \( j_{\Gamma} \) (which ramifies only over 0, 1 and \( \infty \)). We call the obtained triangulation of \( M_{\Gamma} \) a \( j_{\Gamma} \)-triangulation. Let \( T_{\Gamma} \) be the preimage in \( M_{\Gamma} \) of the closed interval \([0, 1] \subset \mathbb{P}^1 \). The graph \( T_{\Gamma} \) is our main tool in the combinatorial analysis of \( \Gamma \).

Denote by \( \chi(\mathcal{E}) \) the Euler characteristic of \( \mathcal{E} \). It splits equivalence classes of Jacobian elliptic surfaces (modulo fiberwise birational transformations) into algebraic families. In particular, if \( C = \mathbb{P}^1 \) then the algebraic variety \( F_r \) parametrizing (equivalence classes of) Jacobian elliptic surfaces with given \( \chi(\mathcal{E}) \) is irreducible; here we put \( r = \chi(\mathcal{E})/12 \). Our goal is to analyze the birational type of (irreducible components)
\[ F_{r, \Gamma} \subset F_r \]
parametrizing fibrations with fixed monodromy group \( \Gamma \). It suffices to study parameter spaces \( F_{r, \Gamma} \) corresponding to \( \Gamma \subset \text{PSL}_2(\mathbb{Z}) \), since every irreducible component of \( F_{r, \Gamma} \) coincides with a component of \( F_{r, \Gamma} \).

From now on we assume that \( C = M_{\Gamma} = \mathbb{P}^1 \). Denote by \( \mathcal{R}_{d, \Gamma} \) the space of rational maps \( \mathbb{P}^1 \to \mathbb{P}^1 \) (of degree \( d \)) with prescribed ramification (encoded in \( T_{\Gamma} \)). The spaces \( F_{r, \Gamma} \) are quotients by the action of \( \text{PGL}_2 \times H_{\Gamma} \) of fibrations over \( \mathcal{R}_{d, \Gamma} \) with fibers (Zariski open subsets of) \( \text{Sym}^\ell(\mathbb{P}^1) \) (for appropriate \( d \) and \( \ell \)). Here \( \text{PGL}_2 \) acts (on the left) by changing the parameter on the base \( C = \mathbb{P}^1 \) and \( H_{\Gamma} \) is the group of automorphisms of \( M_{\Gamma} = \mathbb{P}^1 \) stabilizing the embedded graph \( T_{\Gamma} \) (acting on the right). The nontriviality of \( H_{\Gamma} \) means that there is a \( \Gamma' \subset \text{PSL}_2(\mathbb{Z}) \) containing \( \Gamma \) as a normal subgroup with \( H_{\Gamma} = \Gamma'/\Gamma \). So in most cases in order to prove rationality of \( F_{r, \Gamma} \) it is sufficient to establish it for \( \text{PGL}_2 \backslash \mathcal{R}_{d, \Gamma} \), which can be deduced from general rationality results for
PGL₂-quotients (see [3, 18]). To cover all cases we need to set up a rather extensive combinatorial analysis.

Here is a roadmap of the paper. In Section 1 we discuss finite covers \( M_\Gamma \to \mathbb{P}^1 \) in the spirit of Grothendieck’s “Dessins d’Enfants” program (see [28], [34] and the references therein) and introduce the invariants \( \text{GD}(\Gamma) \), \( \text{RD}(\Gamma) \) and \( \text{ET}(\Gamma) \). In “ideal” cases \( \text{ET}(\Gamma) \) coincides with the number of triangles in the \( j_\Gamma \)-triangulation of \( M_\Gamma \) (the notation \( \text{ET}(\Gamma) \) stands for “Effective Triangles”). In Section 2 we recall basic facts about elliptic fibrations and introduce the invariant \( \text{ET}(E) \). For an “ideal” elliptic fibration one has \( \text{ET}(\Gamma) = \text{ET}(E) \).

In Section 3 we discuss moduli of elliptic fibrations with fixed monodromy. In Sections 7 and 8 we formulate and prove several rationality results for PGL₂ and related quotients. In Section 5 we classify families of rational elliptic surfaces and elliptic K3 surfaces with different monodromy groups. In Section 4, we study relations between the combinatoric of the graph \( \Gamma \) and the topology of \( E \). And finally, in Section 10 we list (certain) relevant subgroups \( \Gamma \subset \text{PSL}_2(\mathbb{Z}) \) (represented by trivalent graphs \( T_\Gamma \)). There are too many monodromy groups of elliptic K3 surfaces to be drawn on paper, but we show how to obtain them from our list by simple operations.

1. Finite covers

Let \( \Gamma \) be a subgroup of finite index in \( \text{PSL}_2(\mathbb{Z}) \). The latter is isomorphic to a free product \( \mathbb{Z}/3 \ast \mathbb{Z}/2 \). Consider the map

\[
\overline{\mathcal{H}}/\Gamma = M_\Gamma \twoheadrightarrow \mathbb{P}^1 = \overline{\mathcal{H}}/\text{PSL}_2(\mathbb{Z}),
\]

ramified over the points 0, 1, \( \infty \in \mathbb{P}^1 \). Denote their preimages in \( M_\Gamma \) by \( A, B \) and \( I \), respectively. The possible ramification orders are 3 or 1 for \( A \)-points, 2 or 1 for \( B \)-points and arbitrary for \( I \)-points. The points 0, 1 and \( \infty \) subdivide the circle \( \mathbb{P}^1(\mathbb{R}) = \mathbb{S}^1 \) into three segments and, together with the upper and lower hemisphere, define a decomposition of \( \mathbb{P}^1(\mathbb{C}) = \mathbb{S}^2 \) into three triangles.

This induces a special triangulation of \( M_\Gamma \) with vertices in \( A, B \) and \( I \)-points which we call the \( j_\Gamma \)-triangulation. The preimage of the segment \([0, 1] \subset \mathbb{P}^1 \) defines a graph \( T_\Gamma \) which determines the \( j_\Gamma \)-triangulation uniquely. Interior vertices of \( T_\Gamma \) are marked by \( A_6 \) and ends are marked by either \( A_2 \) or \( B_2 \).

Notations 1.1. The graph datum \( \text{GD}(\Gamma) \) of \( T_\Gamma \) is the formal sum

\[
\text{GD}(\Gamma) := [a_6 A_6 + a_2 A_2 + b_2 B_2],
\]

where \( a_i \) (\( i = 6, 2 \)) is the number of \( A_i \) vertices and \( b_2 \) is the number of \( B_2 \)-vertices. Denote by \( \tau^0 = \tau^0(\Gamma) \) the number of vertices of \( T_\Gamma \) (including the ends), by \( \tau^1 = \tau^1(\Gamma) \) the number of edges and by \( \tau^2 = \tau^2(\Gamma) = \pi_0(M_\Gamma \setminus T_\Gamma) \).
Remark 1.2. For given \(a_2, b_2\) there is a unique group with 
\[ GD(\Gamma) = [A_6 + a_2A_2 + b_2B_2]. \]

Forgetting the markings of \(T\) we obtain a connected unmarked topological graph \(T_u\) with (possibly some) ends and all interior vertices of valency 3 — a trivalent graph.

Lemma 1.3. Let \(X\) be a compact orientable Riemann surface of genus \(g(X)\) and \(T^u \subset X\) an embedding of a connected trivalent graph such that
- the set \(X \setminus T^u\) is a disjoint union of topological cells;
- all interior vertices of \(T^u\) are trivalent;
- the ends of \(T^u\) are arbitrarily marked by two colors \(A_2\) and \(B_2\).

Then there exist a subgroup \(\Gamma \subset \text{PSL}_2(\mathbb{Z})\) and a unique complex structure on \(X\) such that \(X = M_\Gamma\) and \(T^u = T^u_\Gamma\).

Proof. Assume that we have an embedded graph \(T^u \subset X\) satisfying the conditions above. Mark by \(A\) all trivalent vertices and enlarge the graph \(T^u\) by putting a \(B\)-vertex in the middle of any edge bounded by two \(A\)-vertices. Put one \(I\)-vertex into every connected component of \(X \setminus T^u\) and connect all \(I\)-vertices with \(A\) and \(B\)-vertices at the boundary of the corresponding domain. By assumption, every connected component of \(X \setminus T^u\) is contractible. Consider the boundary of the individual cell. Every \(A\)-vertex of the boundary is connected by edges to \(B\)-vertices only. Similarly, the \(B\)-vertices are connected by edges only to \(A\)-vertices. Hence every triangle of the induced triangulation has vertices colored by three colors: \(A, B\) and \(I\). This gives a \(j_{\Gamma}\)-triangulation of \(X\). Following Alexander [1], we observe that a \(j_{\Gamma}\)-triangulation defines a map
\[ h : X \to \mathbb{P}^1 \]
which is cyclically ramified over \(A, B\) and \(I\) (see [8]). The trivalence of \(T^u\) implies that \(h\) has only 3 or 1-ramifications over 0 \(\in \mathbb{P}^1\) and only 2 or 1-ramifications over 1 \(\in \mathbb{P}^1\). Since \(\text{PSL}_2(\mathbb{Z}) = \mathbb{Z}/3 * \mathbb{Z}/2\) there is exactly one subgroup \(\Gamma \subset \text{PSL}_2(\mathbb{Z})\) (of finite index) which corresponds to the covering \(X \to \mathbb{P}^1\). Any graph \(T^u_\Gamma\) constructed via a subgroup \(\Gamma \subset \text{PSL}_2(\mathbb{Z})\) satisfies the conditions above. Indeed, we have already described the \(j_{\Gamma}\)-triangulation on \(M_\Gamma\). Triangles adjacent to a given \(I\)-vertex constitute a contractible cell and the division of \(M_\Gamma\) into neighborhoods of \(I\)-vertices is a cellular decomposition of \(M_\Gamma\). Hence after removing \(I\)-vertices with open edges from them we obtain the preimage of \([0, 1]\). If we forget the \(B\)-vertices which lie between two \(A\)-vertices we obtain the graph \(T^u_\Gamma\). Thus \(T^u_\Gamma \subset X = M_\Gamma\) is the boundary of
this cellular decomposition and $T_\Gamma$ is simply $T_\Gamma^u$ with an $A, B$-marking of the ends.

Remark 1.4. Graphs which are isotopic in $X$ (modulo diffeomorphisms of $X$ of degree 1) define conjugated subgroups of $\text{PSL}_2(\mathbb{Z})$.

Remark 1.5. Even if we omit the condition of compactness of $X$ we still get a bijection between conjugacy classes of subgroups of finite index of $\text{PSL}_2(\mathbb{Z})$ and embedded trivalent graphs with marked ends.

Remark 1.6. The topology of $X$ restricts the topology of $T_\Gamma^u$. The graph $T_\Gamma^u$ must contain some 1-skeleton of $X$. In particular, the map $\pi_1(T_\Gamma^u) \to \pi_1(X)$ is surjective. Hence $T_\Gamma^u$ can be a tree only if $X = S^2$.

For $X = \mathbb{P}^1$ the connectedness of $T^u$ guarantees that all the components of $X \setminus T^u$ are contractible. Hence we can classify graphs in $X = \mathbb{P}^1$ by drawing them on the plane. In general, connectedness of $T^u$ is necessary but not sufficient.

Definition 1.7. Define

$$\text{ET}(\Gamma) := 6\tau^0 = 6(a_6 + a_2 + b_2)$$
$$\Delta(\Gamma) := 6a_6 + 2a_2$$

Thus both $\text{ET}(\Gamma)$ and $\Delta(\Gamma)$ depend only on the marking of the ends but not on the embedding of the graph. Observe that $\Delta(\Gamma)$ is the number of triangles in the corresponding $j_\Gamma$-triangulation of $M_\Gamma$ and that

$$[\text{PSL}_2(\mathbb{Z}) : \Gamma] = \Delta(\Gamma)/2.$$
where the $v_{i,A}$ are local ramification data over $A$-vertices for $i = 1, \ldots, n$, (resp. $v_{i,B}$ for $B$-vertices, $i = n + 1, \ldots, n'$) and $\overline{v}_i$ are reduced local ramification data for unspecified other points in $M_\Gamma$ for $i > n'$ (distinct from $A$- and $B$-vertices of $M_\Gamma$).

For $f = j_\Gamma : M_\Gamma \to \mathbb{P}^1$ the distinguished (and the only) ramification points are $0, 1, \infty$. We write

$$\text{RD}(j_\Gamma) := [v_0, v_1, v_\infty]$$

for the global $j_\Gamma$-ramification datum.

**Example 1.10.** Assume that $\text{GD}(\Gamma) = [nA_6 + A_2 + B_2]$ is the graph datum of $T_\Gamma \subset M_\Gamma$ and let $j_\mathcal{E} : \mathbb{P}^1 \to M_\Gamma = \mathbb{P}^1$ be a finite cover. Then the $j_\mathcal{E}$-ramification datum

$$\text{RD}(j_\mathcal{E}) = [(2, 3)_A, (2, 2, 1)_B, (2), (2)]$$

means that $\deg(j_\mathcal{E}) = 5$, that $j_\mathcal{E}$ has ramification points of order 2 and 3 over one point $A_2 \in T_\Gamma$ and $(2, 2, 1)$ over one $B_2$-point and ramifications of order 2 over two other unspecified points in $M_\Gamma$.

## 2. Elliptic fibrations

In this section we briefly recall some basic facts of Kodaira’s theory \cite{23} of elliptic fibrations. For more details we refer to \cite{3}, \cite{14} and \cite{35}. Let

$$\pi : \mathcal{E} \to C$$

be a smooth non-isotrivial relatively minimal Jacobian elliptic fibration over a smooth projective curve $C$. This means that:

- $\mathcal{E}$ is a smooth compact complex projective surface and $\pi$ is a proper holomorphic map;
- the generic fiber of $\pi$ is a smooth curve of genus 1;
- the fibers of $\mathcal{E}$ do not contain exceptional curves of the first kind, i.e., rational curves $F$ such that $(F^2) = -1$ (relative minimality);
- there exists a (global) zero section $s : C \to \mathcal{E}$ (Jacobian elliptic fibration);
- the $j$-function which assigns to each smooth fiber $\pi^{-1}(p) = \mathcal{E}_p \subset \mathcal{E}$ its $j$-invariant is a non-constant rational function on $C$ (non-isotriviality).

It is well known that $s^2 < 0$. We define

$$\text{ET}(\mathcal{E}) := -24s^2.$$

**Lemma 2.1.** We have

$$\text{ET}(\mathcal{E})/2 = -12s^2 = \chi(\mathcal{E}) = c_2(\mathcal{E}).$$
Proof. Well known, but we decided to include an argument. Since \( E \) is smooth and relatively minimal the canonical bundle \( K_E \) of \( E \) is induced from a one-dimensional bundle \( K \) on the base \( C \). The sheaf \( \pi^*K(C) \) is a subsheaf of \( K_E \). Since there are singular fibers we have the following equality

\[ h^0(E, \Omega^1) = h^1(E, \mathcal{O}) = g \]

where \( g \) is the genus of \( C \). By Riemann-Roch we obtain

\[ \chi(\mathcal{O}) = 1 - g + h^0(E, K_E) = \chi(E)/12. \]

We also know that \( s^2 + sK_E - 2g + 2 = 0 \) (genus formula). Therefore,

\[ 1 - g + h^0(E, K_E) = \deg(K) - 2g + 2 = \chi(E)/12 \]

since \( \deg(K) > 2g - 2 \) and hence \( h^1(C, K') = 0 \). Further,

\[ sK_E = \deg(K). \]

Thus \( s^2 + sK_E - 2g + 2 = 0 \) transforms to \( s^2 + \chi(E)/12 = 0. \)

Let \( C^{\text{sing}} = \{ p_1, \ldots, p_k \} \subset C \) be the set of points on the base corresponding to singular fibers. The topological Euler characteristic \( \chi(E) = c_2(E) \) is equal to the sum of Euler characteristics of the singular fibers \( E_{p_i} = \pi^{-1}(p_i) \) (since every generic fiber has Euler characteristic equal to 0). Therefore,

\[ \text{ET}(E) = \sum_{p_i \in C^{\text{sing}}} \text{ET}(E_{p_i}), \]

where the summation runs over all singular fibers of \( E \) and \( \text{ET}(E_{p_i}) \) is the contribution from the corresponding singular fiber. Since the fibration is Jacobian every singular fiber has a unique representative from Kodaira’s list and it is defined by the local monodromy. The possible types of singular fibers and their ET-contributions are:

| Fiber Type | ET | \( I^*_n \) | ET |
|------------|----|-------------|----|
| I \(_0\)  |    | \( I^*_0 \) | 12 |
| I \(_n\)  | 2n | \( I^*_n \) | 2n + 12 |
| II        | 4  | IV\(^*\)    | 16 |
| III       | 6  | III\(^*\)   | 18 |
| IV        | 8  | II\(^*\)    | 20 |

Here \( I_0 \) is a smooth fiber, \( I_n \) is a multiplicative fiber with \( n \)-irreducible components. The types II, III and IV correspond to the case of potentially good reduction. More precisely, the neighborhood of such a fiber is a (desingularization of a) quotient of a local fibration with smooth fibers by an automorphism of finite order. The corresponding order is 4 for the case III and 3 in the cases II, IV. The fibers of type \( I^*_0 \) (resp. \( I^*_n, II^*, III^*, IV^* \)) are obtained from fibers \( I_0 \).
(resp. I, IV, III, II) (after changing the local automorphism by the involution $x \mapsto -x$ in the local group structure of the fibration). We shall call them $\ast$-fibers in the sequel.

**Remark 2.2.** The local invariant $ET(E_p)$ has a monodromy interpretation. Namely, every element of a local monodromy at $p \in C^{\text{sing}}$ has a minimal representation as a product of elements conjugated to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $\text{SL}_2(\mathbb{Z})$. The length of this representation equals $ET(E_p)/2$. This explains the equality $ET(E_p^\ast) = ET(E_p) + 12$ — the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ is a product of 6 elements conjugated to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (elementary Dehn twists).

3. **Moduli spaces**

Every Jacobian elliptic fibration $E \to \mathbb{P}^1$ admits a Weierstrass model $\bar{E}$. Its geometric realization is given as follows: there exists a pair of section $s_{g_2} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r))$, $g_3 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6r))$ such that $E$ is given by
\begin{equation}
y^2z = 4x^3 - g_2xz^2 - g_3z^3,
\end{equation}
inside $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2r) \oplus \mathcal{O}_{\mathbb{P}^1}(3r))$, subject to conditions
- the discriminant $\Delta = g_2^3 - 27g_3^2$ is not identically 0;
- for every point $p \in \mathbb{P}^1$ we have
\begin{equation}
\min(3\nu_p(g_2), 2\nu_p(g_3)) < 12,
\end{equation}
where $\nu_p$ is the valuation corresponding to $p \in \mathbb{P}^1$ (see [14] or [13], Section 7).

Two pairs $(g_2, g_3)$ and $(g'_2, g'_3)$ define isomorphic Jacobian elliptic surfaces $(\mathcal{E}, s)$ and $(\mathcal{E}', s')$ if there exists an $h \in \text{GL}_2(\mathbb{C})$ transforming $(g_2, g_3)$ into $(g'_2, g'_3)$ under the natural action of $\text{GL}_2$ on (the $\text{GL}_2$-linearized) $\mathcal{O}_{\mathbb{P}^1}(r)$. We define $\mathcal{F}_r$ as the set of isomorphism classes of pairs $(g_2, g_3)$ subject to the conditions above.

The parameter space $\mathcal{F}_r$ has a natural structure of a (categorical) quotient of some open subvariety $U_r$ of the sum of two linear $\text{GL}_2$-representations
\[ H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6r)) \]
by the action of $\text{GL}_2$. Equivalently, $\mathcal{F}_r$ is a (categorical) quotient of the open subvariety $U'_r = U_r/\mathbb{G}_m$ of the weighted projective space
\[ \mathbb{P}_{4r,6r}(4r + 1, 6r + 1) \]
by the action of \( \text{PGL}_2 \).

**Lemma 3.1.** The variety \( U'_r \) is a disjoint union of locally closed subvarieties \( U'_{r, \tilde{\Gamma}} \), each preserved under the action of \( \text{PGL}_2 \), such that for every \( u \in U'_{r, \tilde{\Gamma}} \), one has \( \tilde{\Gamma}(\mathcal{E}_u) = \tilde{\Gamma} \).

**Proof.** The set of \((g_2, g_3)\) (of bounded degree) such that the \( j \)-map decomposes as \( j_{\Gamma} \circ j_{\mathcal{E}} \) is a closed algebraic variety (not necessarily irreducible). Its intersection with the open subvariety defined by the conditions \( \text{[2]} \) is also closed. It contains a \emph{finite} number of proper locally closed subvarieties corresponding to \emph{proper} finite index subgroups of \( \Gamma \) (their number is bounded as a function of \( r \)). The complement to these subvarieties consists of finitely many irreducible components, each preserved under the action of \( \text{PGL}_2 \). For \( u \) in any of these components the lift of \( \Gamma \) to the monodromy group of the corresponding fibration \( \mathcal{E}_u \) is constant. The number of such possible lifts is finite. This gives a finite decomposition of \( U'_r \) as claimed. \( \square \)

The unstable points of the \( \text{PGL}_2 \)-action on the weighted projective space correspond to sections \( g_2, g_3 \) with high order of vanishing at some point \( p \). Namely \( \nu_p(g_2) > 2r, \nu_p(g_3) > 3r \). However, the inequality (3.2) implies that \( 6r < 12 \). Thus, for \( r \geq 2 \), \( \mathcal{F}_r \) is a \( \text{PGL}_2 \)-quotient of some open subvariety of the semistable locus

\[
P_{4r,6r}(4r + 1, 6r + 1) \subset \mathbb{P}_{4r,6r}(4r + 1, 6r + 1).
\]

It follows that \( \mathcal{F}_r \) is a quasi-projective algebraic variety. This variety is clearly unirational and in fact rational by \([18]\).

Moreover, for \( r \geq 2 \) we can define a set of subvarieties \( \mathcal{F}_{r, \tilde{\Gamma}} \subset \mathcal{F}_r \) such that for every \( b \in \mathcal{F}_{r, \tilde{\Gamma}} \) the corresponding Jacobian elliptic surface \((\mathcal{E}_b, s)\) has global monodromy group \( \tilde{\Gamma} \).

**Remark 3.2.** Notice that the maps \( j_{\mathcal{E}} \) for elliptic fibrations corresponding to different points of the same irreducible component of \( \mathcal{F}_{r, \tilde{\Gamma}} \) can have different \( \text{RD}(j_{\mathcal{E}}) \), even over the \( A_2 \) or \( B_2 \)-ends of \( T_{\tilde{\Gamma}} \subset M_{\tilde{\Gamma}} \). Thus, for a given irreducible component, we have the notion of a \emph{generic} ramification datum \( \text{RD}(j_{\mathcal{E}}) \) and its \emph{degenerations}.

The case \( r = 1 \), corresponding to rational elliptic surfaces, is more subtle - the subvariety \( U'_1 \) contains unstable points. The quasi-projective locus of semistable points \( U'^{\text{sst}}_r \) is a disjoint union of locally closed \( \text{PGL}_2 \)-semistable subsets \( U'^{\text{sst}}_{r, \tilde{\Gamma}} \); taking quotients we obtain varieties \( \mathcal{F}_{1, \tilde{\Gamma}} \) parametrizing rational elliptic fibrations with global monodromy \( \tilde{\Gamma} \).
Let $W'_1 = U'_1 - U'^{\text{ss}}_1$ be the complement. It consists of pairs $(g_2, g_3)$ with
\[
  g_2 = l^3 f_2, \quad g_3 = l^4 f_3,
\]
where $l$ is a linear form (vanishing at a point $p$ and) coprime to $f_2, f_3$ and $\deg(f_2) = 1, \deg(f_3) = 2$. For $w \in W'_1$ we have $\deg(j) \leq 4$. The case of $\tilde{\Gamma} \neq \text{SL}_2(\mathbb{Z})$ corresponds to $\deg(j) \geq 2$. Thus we have to consider two cases:

- $\deg(j) = \deg(j_{\bar{E}}) = 2$;
- $\deg(j) \leq 4, \deg(j_{\bar{E}}) = 1$.

The first case does not occur since $j^{-1}(0)$ has ramification of type $(3, 1)$ (by the assumption that $f_2$ is coprime to $l$ and that $3\nu_p(g_2) < 12$). Thus the $j$-map cannot be decomposed even locally into a product of two maps. The second case leads to

**Lemma 3.3.** If $w \in W'_1$ and $\tilde{\Gamma}(E_w) \neq \text{SL}_2(\mathbb{Z})$ then $\deg(j_{\bar{E}}) = 1$ and one has one of the following graph and ramification data:

| $\text{GD}(\Gamma)$ | $\text{RD}(j_{\bar{E}})$ |
|----------------------|---------------------------|
| $[A_0 + A_2]$        | $[(3, 1)_0, (2, 2)_1, (3, 1)_\infty]$ |
| $[A_0 + A_2 + 2B_2]$ | $[(3, 1)_0, (2, 1, 1), (4)_\infty]$ |
| $[A_0 + 3B_2]$       | $[(3)_0, (1, 1, 1), (3)_\infty]$ |
| $[A_0 + B_2]$        | $[(3)_0, (2, 1)_1, (2, 1)_\infty]$ |

**Proof.** The formula $j = l f_2^3 / (l f_2^3 - f_3^2)$ shows that $j_{\bar{E}}$ has a point with local ramification datum $(3, 1)$ or $(3)$, corresponding to $[\text{PSL}_2(\mathbb{Z}) : \Gamma(E)] = 4$ or 3.

Since only two more branch points are allowed and one of them is 1 (with local ramifications 1 or 2), the Euler characteristic computation gives the ramification data listed in the statement plus one more:
\[
[(3, 1)_0, (2, 2)_1, (2, 2)_\infty].
\]

However, this datum is impossible for topological reasons (the only possible graph datum is $[A_0 + A_2]$ and there is a unique embedded graph $T_{\Gamma}$ with this datum).

If $\deg(j) = 3$ then one has a cyclic point of order 3, leading to the data above.

**Corollary 3.4.** Every irreducible component $W'_1, \tilde{\Gamma} \subset W'_1$ such that $\tilde{\Gamma}(E_w) \neq \text{SL}_2(\mathbb{Z})$ for $w \in W'_1, \tilde{\Gamma}$ is rational.
Consider an irreducible component $\mathcal{F}_{r,\tilde{\Gamma}}$ and the corresponding decomposition $j = j_{\Gamma} \circ j_{\mathcal{E}}$. Here

$$j_{\mathcal{E}} = (j_{\mathcal{E},2}, j_{\mathcal{E},3}) : \mathbb{P}^1 \to M_{\Gamma} = \mathbb{P}^1$$

is a pair of homogeneous polynomials in 2 variables. Let

$$j_{\mathcal{E}} = (j_{\mathcal{E},2}, j_{\mathcal{E},3}) : \mathbb{P}^1 \to M_{\Gamma} = \mathbb{P}^1$$

be the subset corresponding to smooth elliptic fibrations. Put

$$J_{\Gamma} := \{ j \mid \exists j_{\mathcal{E}} : \mathbb{P}^1 \to M_{\Gamma} \text{ s.t. } j = j_{\Gamma} \circ j_{\mathcal{E}} \}.$$ 

**Lemma 3.5.** If $j \in J_{\Gamma} \cap J_{\Gamma'}$ with $\Gamma \neq \Gamma'$ then there exist an $h \in \text{PSL}_2(\mathbb{Z})$, a group $\Gamma'' \subset \Gamma \cap h\Gamma'h^{-1}$ and a map $j''_{\mathcal{E}} : \mathbb{P}^1 \to M_{\Gamma''}$ such that $j = j_{\mathcal{E}}' \circ j_{\mathcal{E}}''$.

**Proof.** The monodromy group and its image in $\text{PSL}_2(\mathbb{Z})$ are uniquely determined by the smooth part of the elliptic fibration. Therefore, in any smooth family of elliptic surfaces

$$\Gamma(\text{generic fiber}) \supseteq \Gamma(\text{special fiber}).$$

Since $\Gamma$ is defined modulo conjugation by elements in $\text{SL}_2(\mathbb{Z})$ the claim follows. 

**Corollary 3.6.** We have a decomposition $\mathcal{G} = \bigsqcup_{\Gamma} \mathcal{G}_{\Gamma}$ into a finite (disjoint) union of algebraic $\text{GL}_2$-stable subvarieties such that for all $g = (g_2, g_3) \in \mathcal{G}_{\Gamma}$ the monodromy group $\tilde{\Gamma}(\mathcal{E}_g) \subset \text{SL}_2(\mathbb{Z})$ is a subgroup of a central $\mathbb{Z}/2$-extension of $\Gamma$.

**Remark 3.7.** For a given $g \in \mathcal{G}_{\Gamma}$ the map $j_{\mathcal{E}}$ is not unique. Let $j_{\mathcal{E}}$ and $j'_{\mathcal{E}}$ be two such maps. Then $j_{\mathcal{E}} = h_{\Gamma} \circ j'_{\mathcal{E}}$, where $h_{\Gamma} \in \text{Aut}(T_{\Gamma})$ is an automorphism of $M_{\Gamma}$, preserving $T_{\Gamma}$.

**Lemma 3.8.** We have a decomposition

$$\mathcal{G}_{\Gamma} = \bigsqcup_{k} \mathcal{G}_{\Gamma,k}$$

into a finite union of algebraic irreducible $\text{GL}_2$-stable subvarieties such that $\tilde{\Gamma}(\mathcal{E}_g) = \tilde{\Gamma}$ for all $g \in \mathcal{G}_{\Gamma,k}$.

**Proof.** Assume that some $g \in \mathcal{G}_{\Gamma}$ belongs to $\mathcal{G}_{\Gamma,1} \cap \mathcal{G}_{\Gamma,2}$, where $\mathcal{G}_{\Gamma,1}, \mathcal{G}_{\Gamma,2}$ are different (non-conjugated) lifts of $\Gamma$ into $\text{SL}_2(\mathbb{Z})$. Lemma 3.5 implies that there exists a proper subgroup $\Gamma'' \subset \Gamma$ such that $g$ belongs to $\mathcal{G}_{\Gamma''}$, contradiction. 

\[\Box\]
Let $\mathcal{G}_\Gamma = \mathcal{G}_{\Gamma,k}$ be an irreducible component of $\mathcal{G}_\Gamma$ as in Lemma 3.8 and $g \in \mathcal{G}_\Gamma$ its generic point. It determines a set of $\ast$-fibers on the base $\mathbb{P}^1$. We denote their number by $\ell$. Choose (one of) the $j_\varepsilon_g$, with ramification datum $\text{RD} = \text{RD}(j_\varepsilon_g)$. We get a map

$$\phi_U : U_g \to U_{j_\varepsilon_g} \times (\mathbb{P}^1)^\ell,$$

where $U_g \subset \mathcal{G}_\Gamma$ is a neighborhood of $g$ and $U_{j_\varepsilon_g} \subset \mathcal{R}(\text{RD})$ is a neighborhood of the map $j_\varepsilon_g = j_\varepsilon_g$ in the space

$$\mathcal{R}(\text{RD}) := \{ j : \mathbb{P}^1 \to \mathbb{P}^1 \mid \text{RD}(j) = \text{RD} \}$$

of rational maps with ramification datum $\text{RD}$.

**Lemma 3.9.** The map $\phi_U$ is a local (complex analytic) surjection.

**Proof.** First observe that on $M_\Gamma$ there is a projective local system $(\mathbb{Z} \oplus \mathbb{Z})/(\mathbb{Z}/2)$ which induces a projective local system on an open part of $\mathbb{P}^1$. The obstruction to the extension of this system to a linear $\mathbb{Z} \oplus \mathbb{Z}$-system is an integer modulo 2 which depends only on the topological type of the projective system. Therefore, it doesn’t change under a small variation of maps $j$ with fixed $\text{RD}$. Now it suffices to apply Kodaira’s main theorem which guarantees the existence and uniqueness of an elliptic fibration with a given linear $\mathbb{Z} \oplus \mathbb{Z}$-system.

**Corollary 3.10.** Let $\mathcal{F}'_{r,\Gamma} \subset \mathcal{F}_{r,\Gamma}$ be an (irreducible) component with generic ramification datum $\text{RD}$. Then $\mathcal{F}'_{r,\Gamma}$ surjects (rationally) onto the quotient of the variety of rational maps $\mathcal{R}(\text{RD})$ by $H_\Gamma$.

**Proof.** Since both $\mathcal{F}'_{r,\Gamma}$ and $\mathcal{R}(\text{RD})$ are algebraic varieties the local complex analytic surjection from Lemma 3.9 extends to an algebraic correspondence. Moreover, two decompositions of the map $j$ as $j = j_\Gamma \circ j_\varepsilon$ differ by an element in $H_\Gamma$. This gives a map to the quotient space, which is a (global) rational surjection.

**Proposition 3.11.** Every irreducible component $\mathcal{F}_{r,\Gamma}$ contains an open part $\mathcal{F}'_{r,\Gamma}$ with the following properties:

- $\mathcal{F}'_{r,\Gamma}$ is a quotient of an algebraic variety $U'_{r,\Gamma,\ell}$ by the (left) action of $\text{PGL}_2$ and (right) action of a subgroup $H_\Gamma$ of $\text{Aut}(T_\Gamma)$;
- $U'_{r,\Gamma,\ell}$ admits a fibration with fiber (an open subset of) $\text{Sym}^\ell(\mathbb{P}^1)$ and base the variety $\mathcal{R}_{r,\Gamma}$ of maps $f : \mathbb{P}^1 \to M_\Gamma$ with fixed local ramification data over $A_2$ and $B_2$-points of $T_\Gamma \subset M_\Gamma$;
- the action of $\text{PGL}_2$ on $U'_{r,\Gamma,\ell}$ is induced from the standard $\text{PGL}_2$-action on $\mathbb{P}^1$.
the group $\text{Aut}(T_\Gamma)$ is a subgroup of $\text{PGL}_2$ (acting on $M_\Gamma$).

Proof. Elliptic surfaces parametrized by a smooth irreducible variety have the same $\text{ET}(\mathcal{E})$, which depends on the number $\ell$ of $*$-fibers in $\mathcal{E}$, on the degree of $j_\mathcal{E}$ and on the ramification properties over the ends of $T_\Gamma$. Once $\ell$ is fixed, for any given $j_\mathcal{E}$, the $*$-mark can be placed over arbitrary $\ell$-points of $\mathbb{P}^1$. Their position defines a unique surface $\mathcal{E}$. This implies that $U'_{r,\tilde{\Gamma},\ell}$ is fibered with fibers (birationally) isomorphic to $\text{Sym}^\ell(\mathbb{P}^1) = \mathbb{P}^\ell$. The ramification properties of $j_\mathcal{E}$ remain the same on the open part of $U'_{r,\tilde{\Gamma},\ell}$ (since the number of $*$-fibers remains the same). Thus the base of the above fibration is the space of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1 = M_\Gamma$ with fixed ramification locus. Any such map defines an elliptic surface $\mathcal{E}$ with given $\Gamma$ (see [8]). The $\text{PGL}_2$-action on $U'_{r,\tilde{\Gamma},\ell}$ identifies points corresponding to isomorphic surfaces $\mathcal{E}$. Additional nontrivial isomorphisms correspond to exterior automorphisms of $\Gamma$, coming from the action on $M_\Gamma$, i.e., automorphisms of the graph $T_\Gamma$. \qed

Remark 3.12. If the $\text{PGL}_2 \times \text{Aut}(T_\Gamma)$-action on $U'_{r,\tilde{\Gamma},\ell}$ is almost free then the rationality of $\text{PGL}_2 \setminus U'_{r,\tilde{\Gamma},0} / \text{Aut}(T_\Gamma)$ implies the rationality the corresponding quotients for all $\ell$. In the other cases the degree of $j_\mathcal{E}$ is small and they are handled separately (see Section §).

Most of the graphs $T_\Gamma$ have trivial automorphisms. In particular, any nontrivial automorphism acts on the ends of the graph. In general, automorphisms of the pair $(M_\Gamma, T_\Gamma)$ correspond to elements of $\Gamma'/\Gamma$ where $\Gamma' \subset \text{PSL}_2(\mathbb{Z})$ is a maximal subgroup with the property that $\Gamma$ is a normal subgroup of $\Gamma'$.

Lemma 3.13. The group $\text{Aut}(T_\Gamma)$ acts freely on the set of ends and end-loops.

Proof. Consider $j_\Gamma : M_\Gamma \to \mathbb{P}^1$. Then $h \in \text{Aut}(T_\Gamma)$ is any element in $\text{PGL}_2(\mathbb{C})$ such that $j_\Gamma(hz) = j_\Gamma(z)$ for all $z \in T_\Gamma \subset M_\Gamma$. If $h$ stabilizes an end or an end-loop of $T_\Gamma$ then it stabilizes the unique adjacent vertex and its other end. Any element of $\text{PGL}_2(\mathbb{C})$ preserving a closed interval is the identity. \qed

Corollary 3.14. For $r \leq 2$, the only possible groups $\text{Aut}(T_\Gamma)$ are cyclic, dihedral or subgroups of $\mathfrak{S}_4$. More precisely, for graphs with one end $\text{Aut}(T_\Gamma) = 1$ and graphs with two ends $\text{Aut}(T_\Gamma)$ is a subgroup of $\mathbb{Z}/2$.

Lemma 3.15. Let

$$\mathcal{R} := \{f : \mathbb{P}^1 \to \mathbb{P}^1\}$$

be the space of rational maps with ramifications over exactly 0 and $\infty$. Then $\mathcal{R}$ is a $\mathbb{G}_m$-fibration over the product of symmetric spaces $\text{Sym}^{m_i}(\mathbb{P}^1)$.
Proof. Indeed any two cycles $c_1$ and $c_2$ of fixed degree are equivalent on $\mathbb{P}^1$. Therefore, there is a rational function $f$ on $\mathbb{P}^1$ with $c_1 = f^{-1}(0)$ and $c_2 = f^{-1}(\infty)$. If $c_1, c_2$ do not intersect then $\deg(f) = \deg(c_1) = \deg(c_2)$. The function $f$ is defined modulo multiplication by a constant. The space of cycles $c_1 = \sum_i n_i p_i$ is a product of symmetric powers $\text{Sym}^m(\mathbb{P}^1)$ where $m$ is the number of equal $n_i$. 

4. Combinatorics

In this section we investigate relations between $\text{ET}(\mathcal{E})$ and $\text{ET}(\Gamma)$. We keep the notations of the previous sections.

Lemma 4.1. Let $j : \mathcal{E} \to C$ be an elliptic fibration. Then

\begin{equation}
\text{ET}(\mathcal{E}) = \deg(j_{\mathcal{E}}) \Delta(\Gamma) + 8 \alpha_2 + 4 \alpha_1 + 6 \beta_1 + 12 \ell.
\end{equation}

Here $\alpha_1$ and $\alpha_2$ equal the number of points over $A_2$-ends of $T_\Gamma$ with ramification multiplicity $1 \pmod{3}$ and $2 \pmod{3}$, respectively, $\beta_1$ is the number of odd ramification points over the $B_2$-ends and $\ell$ is the number of $\ast$-fibers of $\mathcal{E}$.

Proof. The summand $\deg(j_{\mathcal{E}}) \Delta(\Gamma)$ corresponds to multiplicative fibers of $\mathcal{E}$. The next summands are the contributions of those singular fibers of $\mathcal{E}$ which are in the preimage of $A_2$ or $B_2$-ends of $T_\Gamma$. If the ramification order at a point $p$ over a $B_2$-end is even then the corresponding fiber with minimal ET is smooth and hence does not contribute to $\text{ET}(\mathcal{E})$. If it is odd then the fiber with minimal ET is of type III and we have to add $6 \beta_1$. Similarly, for the preimages of $A_2$-ends and $\ast$-twists. 

Corollary 4.2. In particular,

$$\text{ET}(\mathcal{E}) \leq \deg(j_{\mathcal{E}}) \text{ET}(\Gamma) + 12 \ell,$$

with equality if

$$2 \alpha_2 + \alpha_1 = a_2 \cdot \deg(j_{\mathcal{E}})$$

$$\beta_1 = b_2 \cdot \deg(j_{\mathcal{E}}).$$

Definition 4.3. We call $T_\Gamma$ saturated if all vertices of $T_\Gamma^u$ are trivalent and a tree if it is contractible.

Remark 4.4. For saturated graphs $\Delta(\Gamma) = 12 \text{rk} \pi_1(T_\Gamma)$, where

$$\text{rk} \pi_1(T_\Gamma) = \text{rk} H_1(T_\Gamma)$$

is the number of independent closed loops of $T_\Gamma \subset M_\Gamma$.

The following simple procedures produce new graphs:
If $T_1$ and $T_2$ are (unmarked) trivalent graphs we can join $T_1$ and $T_2$ along two edges. For the resulting graph $T'$ we have

$$ET(T') = ET(T_1) + ET(T_2) + 12.$$ 

If $T_i$ are marked and the marking of the ends of $T'$ is induced from the marking of the corresponding ends of $T_1$ and $T_2$ then

$$\Delta(T') = \Delta(T_1) + \Delta(T_2) + 12.$$ 

- We can glue an end $p$ of $T_1$ to an edge of $T_2$. In this case

$$ET(T') = ET(T_1) + ET(T_2).$$

The change of $\Delta$ depends on the marking of the end:

$$\Delta(T') = \begin{cases} 
\Delta(T_1) + \Delta(T_2) + 6 & \text{if } p = B_2 \\
\Delta(T_1) + \Delta(T_2) + 4 & \text{if } p = A_2.
\end{cases}$$

**Remark 4.5.** Any connected graph $T$ can be uniquely decomposed into a union of a saturated graph and a union of trees.

**Lemma 4.6.** $ET(\Gamma)$ is divisible by 12.

**Proof.** Every vertex of $T_\Gamma$ has either one or three incoming edges. Therefore, the number of edges

$$\tau^1 = \frac{1}{2}(\tau^0_1 + \tau^0_3),$$

($\tau^0_i$ is the number of vertices with $i$-edges). Thus $\tau^0 = \tau^0_1 + \tau^0_3$ is even and since $ET(\Gamma) = 6\tau^0$ we are done. \qed

**Example 4.7.** If $T_\Gamma$ is a tree with $k + 2$ vertices then

$$\begin{align*}
ET(\Gamma) &= 12k + 12 \\
\Delta(\Gamma) &= \begin{cases} 
6k & \text{if all ends are } B_2, \\
> 6k & \text{otherwise.}
\end{cases}
\end{align*}$$

**Lemma 4.8.** For all $\Gamma$ one has

$$\Delta(\Gamma) \geq ET(\Gamma)/2 + 6(\rk H_1(T_\Gamma) - 1).$$

**Proof.** A direct computation shows that for saturated graphs one has an equality. Suppose that $T_\Gamma$ is a concatenation of a saturated graph $T_{\text{sat}}$ and a tree $T_{\text{tree}}$. The number of ends drops by one and the number of $A_6$ vertices increases by 1. Thus the tree will add $12k + 12$ to $ET(\Gamma)$ but $\Delta(\Gamma)$ will change by $6k + 6$. Finally, the ratio $\Delta(\Gamma)/ET(\Gamma)$ only increases if we change $B_2$- to $A_2$-markings for some ends. Indeed, $\Delta(\Gamma)$ increases without changing $ET(\Gamma)$. \qed
Corollary 4.9. If $\Delta(\Gamma) = \text{ET}(\Gamma)/2$ then $T_\Gamma$ is a concatenation of a loop $L$ and some trees. Moreover, all the ends of $T_\Gamma$ are of type $B_2$.

Proposition 4.10. Let $\mathcal{E} \to C$ be an elliptic fibration with $\text{ET}(\mathcal{E}) < \text{ET}(\Gamma)$. Then:

- $M_\Gamma = \mathbb{P}^1$ and $T_\Gamma$ is a tree without $A_2$-ends and with $\text{ET}(\Gamma) > 24$;
- $\deg(j_\mathcal{E}) = 2$ and it is ramified in all $(B_2)$ ends of $T_\Gamma$ (and, possibly, some other points);
- $\mathcal{E}$ has 1 or 2 singular fibers of type $I_n$.

Proof. From 4.1 and 4.8 we conclude that $\text{rk}H_1(T_\Gamma) = 0$ which implies that $T_\Gamma$ is a tree and $M_\Gamma = \mathbb{P}^1$. By Lemma 2.1 and our assumption, $\text{ET}(\Gamma) > 24$, which implies that $\deg(j_\mathcal{E}) \leq 2$. If $\deg(j_\mathcal{E}) = 1$, we apply Corollary 4.2 and get a contradiction to the assumption. For $\deg(j_\mathcal{E}) = 2$ combine Definition 1.7 and (4.1):

$$\text{ET}(\mathcal{E}) = \text{ET}(\Gamma) + 4\alpha_2 + 4\alpha_1 + 8\alpha_2 + 6\beta_1 - 12.$$ 

Since $\alpha_1$, resp. $\beta_1$ is twice the number of unramified $A_2$, resp. $B_2$-ends, and $\alpha_2$ is the number of ramified $A_2$-ends we see that if at least one of them is not zero, then $\text{ET}(\mathcal{E}) \geq \text{ET}(\Gamma)$. The claim follows.

Corollary 4.11. For every elliptic fibration $\mathcal{E} \to \mathbb{P}^1$ one has

$$\text{ET}(\mathcal{E}) \geq \text{ET}(\Gamma).$$

Further, if $\deg(j_\mathcal{E}) = 2$ and $j_\mathcal{E}$ is ramified over only one $B_2$-point then

$$\text{ET}(\mathcal{E}) \geq 2 \text{ET}(\Gamma) - 12.$$ 

Proof. If $\deg(j_\mathcal{E}) = 2$ and $\mathbb{C} = \mathbb{P}^1$ then $j_\mathcal{E}$ ramifies in two points. If neither of these points is $B_2$ then, by Lemma 4.1, $\text{ET}(\mathcal{E}) \geq 2 \text{ET}(\Gamma)$. If both of these points are $B_2$-points then the covering $j_\mathcal{E}$ corresponds to a subgroup $\Gamma'$ of index 2 in $\Gamma$ and $C = M_{\Gamma'}$, contradiction. Otherwise, the claimed inequality follows from Lemma 4.1.

5. Elliptic K3 surfaces with $\deg(j_\mathcal{E}) > 1$

In this section we assume that $C = \mathbb{P}^1$, that $j_\mathcal{E} > 1$ and that $\Gamma$ is a proper subgroup of $\text{PSL}_2(\mathbb{Z})$. We consider

- general families : $\text{ET}(\mathcal{E}) - 12\ell = \deg(j_\mathcal{E}) \text{ET}(\Gamma)$,
- special families : $\text{ET}(\mathcal{E}) - 12\ell < \deg(j_\mathcal{E}) \text{ET}(\Gamma)$.

In Section 3 we showed that the main building block in the construction of moduli space of elliptic surfaces with fixed $\Gamma$ is the space of rational maps $j_\mathcal{E} : C \to M_\Gamma$ of fixed degree and ramification restrictions over certain points.
For a general family there are no such restrictions and the corresponding moduli spaces are rational by classical results of invariant theory for actions of PGL\(_2\) and its algebraic subgroups (see Section 7). For special families the corresponding space of rational maps is more complicated.

**Lemma 5.1.** There are no special families of elliptic K3 surfaces with 
\[ ET(\Gamma) = 48, 36. \]

**Proof.**
- If \( ET(\Gamma) = 48 \) then \( \Delta(\Gamma) \geq 18 \) and \( \deg(j_\mathcal{E}) \leq 2 \). However, \( \deg(j_\mathcal{E}) = 2 \) contradicts Corollary 4.11 (\( ET(\mathcal{E}) \geq 96 - 24 > 48 \)).
- If \( ET(\Gamma) = 36 \) and \( \Delta(\Gamma) > 16 \) then \( \deg(j_\mathcal{E}) = 2 \), contradicting to 4.11. We are left with \( \Delta(\Gamma) = 16, 14, 12 \) for \( \deg(j_\mathcal{E}) = 3 \) and \( \Delta(\Gamma) = 12 \) for \( \deg(j_\mathcal{E}) = 4 \).
- If \( \deg(j_\mathcal{E}) = 4 \) then \( T_\Gamma \) is a tree with \( GD(\Gamma) = [2A_6 + 4B_2] \). By Lemma 4.1, all ramifications over the \( B_2 \)-ends are even, which contradicts \( C = \mathbb{P}^1 \) (compute \( \chi(C) \)).
- If \( \deg(j_\mathcal{E}) = 3 \) then \( T_\Gamma \) is a tree (by 4.8) and 
\[
GD(\Gamma) = [2A_6 + a_2A_2 + (4 - a_2)B_2]
\]
with \( a_2 \leq 2 \). We have 
\[
48 \geq ET(\mathcal{E}) \geq 3(12 + 2a_2) + 4\alpha_1 + 8\alpha_2 + 6\beta_1,
\]
where \( \beta_1 \geq 2 \) (since \( \deg(j_\mathcal{E}) \) is odd there is odd ramification over some \( B_2 \)-end). Therefore, \( a_2 = 0 \) and consequently, \( \beta_1 \geq 4 \), contradiction.

\[ \square \]

**Lemma 5.2.** If \( T_\Gamma \) is not a tree and \( j_\mathcal{E} \) is special (and generic for the corresponding irreducible component of \( \mathcal{F}_{2,\Gamma} \)) then

| ET(\Gamma) | \( \deg(j_\mathcal{E}) \) | GD(\Gamma) | RD(\mathcal{E}) |
|------------|-----------------|-------------|---------------|
| 24         | 4               | \([2A_6 + 2B_2]\) | \([2, 2)_B, (2, 2)_B, (2), (2)\) |
| 24         | 3               | \([2A_6 + 2B_2]\) | \([2, 1)_B, (2, 1)_B, (2), (2)\) |
| 24         | 3               | \([2A_6 + A_2 + B_2]\) | \([3)_A, (2, 1)_B, (2)\) |
| 12         | 6               | \([A_6 + A_2]\) | \([3, 3)_A, (3, 3)_A, (2), (2)\) |
| 12         | 5               | \([A_6 + A_2]\) | \([3, 1, 1)_A\) |
| 12         | \(5 \leq d \leq 8\) | \([A_6 + B_2]\) | \([\beta = (\beta_i)_B, (2)^{d'}_B]\), |

where
\[
\beta_i \in \mathbb{N}, \quad \sum \beta_i = d, \quad \#\text{odd } \beta_i \leq 8 - d
\]
and
\[
d' = 2d - \#\text{nonzero } \beta_i.
\]
Proof. Follows from Lemma 4.1. First observe that $\Delta(\Gamma) \leq 16$, which implies that $a_6 = 2$ and $a_2 \leq 2$. If $a_2 = 2$ then $\Delta(\Gamma) = 16$ and

$$\alpha_1 = \alpha_2 = \beta_1 = 0.$$  

Hence both $A_2$-ends have a 3-cyclic ramification and the cover corresponds to a subgroup $\Gamma' \subset \Gamma$ of index 3. This excludes $GD(\Gamma) = [2A_6 + 2A_2]$. If $\deg(j_\mathcal{E}) = 4$ then $\Delta(\Gamma) = 12$ which implies that all preimages of $B_2$-ends have even ramification. The description of all other ramification data follows similarly from Lemma 4.1. Notice that the (omitted) possibilities

| ET($\Gamma$) | $\deg(j_\mathcal{E})$ | GD($\Gamma$) | RD($j_\mathcal{E}$) |
|-------------|-----------------|-------------|-----------------|
| 12          | 6               | $[A_6 + A_2]$ | $[(6)A, (3, 3)A]$ |
| 12          | 5               | $[A_6 + A_2]$ | $[(3)A]$         |

are degenerations of the listed cases (see Remark 3.2).\[ \Box \]

**Lemma 5.3.** If $T_\Gamma$ is a tree and $j_\mathcal{E}$ is special (and generic for the corresponding irreducible component of $\mathcal{F}_{2, \mathcal{E}}$) then

| $\deg(j_\mathcal{E})$ | GD($\Gamma$)                                    | RD($j_\mathcal{E}$) |
|-----------------------|------------------------------------------------|---------------------|
| $j_1$                 | $[A_6 + A_2 + 2B_2]$                            | $[(1, 1, 1)A, (2, 2)B, (2, 2)B, (2), (2)]$ |
| $j_2$                 | $[A_6 + A_2 + 2B_2]$                            | $[(3, 1)A, (2, 2)B, (2, 2)B] + *$ |
| $j_3$                 | $[A_6 + A_2 + 2B_2]$                            | $[(3, 1)A, (2, 2)B, (2, 1, 1)B, (2)]$ |
| $j_4$                 | $[A_6 + 2A_2 + B_2]$                            | $[(3)A, (1, 1, 1)A, (2, 1)B]$ |
| $j_5$                 | $[A_6 + A_2 + 2B_2]$                            | $[(1, 1, 1)A, (2, 1)B, (2, 1)B, (2), (2)]$ |
| $j_6$                 | $[A_6 + A_2 + 2B_2]$                            | $[(3)A, (1, 1)B, (2, 1)B, (2)]$ |

or $GD(\Gamma) = [A_6 + 3B_2]$ and

| $\deg(j_\mathcal{E})$ | RD($j_\mathcal{E}$) |
|-----------------------|---------------------|
| $j_7$                 | $[(2, 2, 2, 2)B, (2, 2, 2)B, (2, 2, 2)B, (2, 2, 2)B, (2), (2)]$ |
| $j_8$                 | $[(2, 2, 2)B, (2, 2, 2)B, (2, 2, 2)B, (2, 2, 2)B, (2), (2)]$ |
| $j_9$                 | $[(2, 2, 2)B, (2, 2, 2)B, (2, 2, 2)B, (2), (2)] + *$ |
| $j_{10}$              | $[(2, 2, 1)B, (2, 2, 1)B, (2, 2, 1)B, (2), (2)]$ |
| $j_{11}$              | $[(2, 1, 1)B, (2, 1, 1)B, (2, 1, 1)B, (2), (2)]$ |
| $j_{12}$              | $[(2, 1, 1)B, (2, 2)B, (2, 2)B, (2), (2)] + *$ |
| $j_{13}$              | $[(1, 1, 1)B, (2, 1)B, (2, 1)B]$ |
| $j_{14}$              | $[(2, 1)B, (2, 1)B, (2, 1)B, (2), (2)] + *$ |

or $ET(\Gamma) = 12$ and $GD(\Gamma) = [2A_6]$ with $\deg(j_\mathcal{E}) = 4 - 10, 12$.

(In the above tables, $*+$ means that there exists a moduli space of elliptic surfaces with the same RD($j_\mathcal{E}$) and with an additional $*+$-fiber over an unspecified point.)
Proof. Assume that ET(Γ) = 24 and TΓ is a tree with

\[ GD(Γ) \neq [A_6 + 3B_2]. \]

First observe that \( \deg(j_Γ) \leq 6 \), since \( \Delta(Γ) \geq 8 \). If \( \deg(j_Γ) \geq 5 \) then, by [L1], \( GD(Γ) = [A_6 + A_2 + 2B_2] \). If \( \deg(j_Γ) = 6 \) then \( j_Γ \) has to be completely ramified over all ends and no other ramifications are allowed by Euler characteristic computation. Therefore, it is a group-covering and can’t be \( j_Γ \). If \( \deg(j_Γ) = 5 \) then there are two odd ramifications over \( B_2 \)-ends, and by [L1], \( ET(Γ) > 48 \).

We are left with

\[ GD(Γ) = [A_6 + 3A_2], \]
\[ = [A_6 + 2A_2 + B_2], \]
\[ = [A_6 + A_2 + 2B_2] \]

and \( 3 \leq \deg(j_Γ) \leq 4 \). If there are at least two \( A_2 \)-ends without 3-cyclic ramification points over them then \( ET(Γ) > 48 \) (see [L1]). The first case is impossible: \( \deg(j_Γ) = 4 \) does not occur (the degree is not divisible by 3), if \( \deg(j_Γ) = 3 \) and there is at most one 3-cyclic ramification over an \( A_2 \)-end then, by [L1], \( ET(Γ) > 48 \), contradiction. Consider the second case and \( \deg(j_Γ) = 4 \). Then \( \Delta(Γ) = 10 \) and \( 4\alpha_1 + 8\alpha_2 + 6\beta_1 \leq 16 \). Since \( \alpha_1 \geq 2 \) we have \( \alpha_2 = \beta_1 = 0 \) and \( \alpha_1 = 2 \). The only possible

\[ RD(j_Γ) = [(3, 1)_A, (3, 1)_A, (2, 2)_B], \]

which corresponds to a group covering, contradiction.

Similarly, if \( GD(Γ) = [A_6 + A_2 + 2B_2] \) and \( \deg(j_Γ) = 4 \) then \( \Delta(Γ) = 8 \) and \( 4\alpha_1 + 8\alpha_2 + 6\beta_1 \leq 16 \). We have \( \alpha_1 \geq 1 \) and \( 8\alpha_2 + 4\alpha_1 = 16 \) or 4. In the first case, both \( B_2 \)-ends are completely ramified, and we get \( j_1 \). The second case splits into subcases: \( \beta_1 = 0 \) or 2, leading to \( j_2 \), resp. \( j_3 \). If \( \deg(j_Γ) = 3 \), then if \( GD(Γ) = [A_6 + 2A_2 + B_2] \) then exactly one of the \( A_2 \)-ends has cyclic ramification. It follows that \( \beta_1 = 1 \), which leads to \( j_4 \). If \( GD(Γ) = [A_6 + A_2 + 2B_2] \) there are two subcases: there is cyclic ramification over the \( A_2 \)-end or not. In the first subcase, possible \( RD(j_Γ) \) include \([ (2, 1)_B, (2, 1)_B ]\), which is excluded as it gives a group covering. The other case leads to \( j_6 \). In the second subcase, we get \( j_5 \).

Consider the case \( TΓ = A_6 + 3B_2 \). Here \( \Delta(Γ) = 6 \) and

\[ ET(Γ) \geq 6 \deg(j_Γ) + 6n, \]

where \( n \) is a number of points with odd ramification over \( B_2 \)-vertices. It follows that

\[ 48 \geq 6 \deg(j_Γ) + 6\beta_1 \]

and \( \beta_1 \geq 3 \) if \( \deg(j_Γ) \) is odd and the number of odd ramifications over each \( B_2 \)-end is congruent to \( \deg(j_Γ) \) modulo 2.
If \( \text{deg}(j_E) = 8 \) then all preimages of \( B_2 \)-vertices are \( 2n \)-ramified. If \( \text{deg}(j_{\Gamma}) \) is odd then \( \text{ET}(\mathcal{E}) \geq 6 \text{deg}(j_{\Gamma}) + 18 \), which excludes \( \text{deg}(j_{\Gamma}) = 7 \). Now assume \( \text{deg}(j_{\Gamma}) = 6 \). The number of possible odd ramifications over any \( B_2 \)-end is even by 4.1 and it cannot exceed 2. There are two possibilities listed above. Assume that \( \text{deg}(j_E) = 5 \). The minimal possible ramifications are \( (2, 2), (2, 1), (1) \) over all \( B_2 \)-ends. Since \( 10 - 6 = 4 \) we can add two more points.

In \( \text{deg}(j_E) = 4 \) we could have further RD:

\[
\text{RD}(j_E) = [(2, 2), (2, 1, 1), (2, 2), (2, 2)] + *(2),
\]

but they are obtained as degenerations of \( j_{12} \) and \( j_{13} \).

The only GD(\( \Gamma \)) which allow \( \text{deg}(j_E) \geq 12 \) are \( [A_2 + B_2] \) and \( [2A_2] \). The first case corresponds to \( \text{PSL}_2(\mathbb{Z}) \) (which we don’t consider). The second case corresponds to subgroups \( \Gamma \subset \text{PSL}_2(\mathbb{Z}) \) of index 2. For a generic \( \mathcal{E} \) in each moduli space the ramification datum \( \text{RD}(j_E) \) is one of the following:

\[
\text{RD}(j_E) = [(3, \ldots, 3_n, 1, \ldots, 1)_{A}, (3, \ldots, 3_n, 1, \ldots, 1)_{A}, (2)^d] + *,
\]

where \( n_1, n_2, d \) are non-negative integers such that

\[
\text{deg}(j_E) - (n_1 + n_2) \leq 4,
\]

\[
3n_1, 3n_2 \leq \text{deg}(j_E) \text{ and, } d \leq 2\text{deg}(j_E - (n_1 + n_2 + 1)).
\]

(In particular, \( d \leq 4 \).)

6. Rational elliptic surfaces with \( \text{deg}(j_E) > 1 \)

**Lemma 6.1.** There are no special families of rational elliptic surfaces with \( \text{ET}(\Gamma) = 24 \).

**Proof.** If \( \text{deg}(j_E) = 2 \) then \( j_E \) cannot be ramified over more than one \( B_2 \)-end (otherwise it is a group covering). Therefore, we can apply Corollary [4.11] and get \( \text{ET}(\mathcal{E}) > 2 \cdot 24 - 12 > 24 \), contradiction (to [2.1]). Thus \( \text{deg}(j_E) = 3 \) or 4 and \( a_6 = 1 \). Moreover, \( \Delta(\Gamma) \leq 8 \). This leaves the cases:

\[
\text{GD}(\Gamma) = [A_6 + 3B_2],
\]

\[
= [A_6 + A_2 + 2B_2].
\]

In the first case \( \text{deg}(j_E) = 3 \) is impossible, and \( \text{deg}(j_E) = 4 \) leads to

\[
\text{RD}(j_E) = [(2, 2), (2, 2), (2, 2), (2, 2)] + *(2),
\]

which corresponds to a group covering. In the second case \( \text{deg}(j_E) \neq 4 \) (since \( \Delta(\Gamma) = 8 \)) and \( \text{deg}(j_E) = 3 \) implies that \( \beta_1 \geq 2 \) and \( \text{ET}(\mathcal{E}) \geq 36 \), contradiction. \( \square \)
Lemma 6.2. If \( j \varepsilon \) is special (and generic for the corresponding irreducible component of \( F_{1, \tilde{\Gamma}} \)) then

\[
\begin{array}{|c|c|c|}
\hline
j & \text{deg}(j) & \text{GD}(\Gamma) & \text{RD}(j) \\
\hline
j_{15} & 6 & 2A_2 & (3, 3)_A, (3, 3)_A, (2), (2) \\
\hline
j_{16} & 4 & 2A_2 & (3, 1)_A, (3, 1)_A, (2), (2) \\
\hline
j_{17} & 3 & 2A_2 & (3)_A, (1, 1, 1)_A, (2), (2) \\
\hline
j_{18} & 4 & [A_6 + B_2] & (2, 2)_B, (2), (2), (2) \\
\hline
j_{19} & 3 & [A_6 + B_2] & (2, 1)_B, (2), (2) \\
\hline
j_{20} & 3 & [A_6 + A_2] & (3)_A, (2), (2) \\
\hline
\end{array}
\]

Proof. If \( a_6 \geq 1 \) then \( \text{deg}(j \varepsilon) = 4 \) or \( 3 \). In the first case \( a_2 = 0 \) and \( \text{GD}(\Gamma) = [A_6 + B_2] \) and we have complete ramification over the \( B_2 \)-end. This gives \( j_{18} \). In the second case the ramification over \( B_2 \) is \( (2, 1)_B \) and we get \( j_{19} \). If \( \text{GD}(\Gamma) = [A_6 + A_2] \) then \( \text{deg}(j \varepsilon) = 3 \) and \( \alpha_1 = \alpha_2 = 0 \), leading \( j_{20} \).

It remains to consider \( \text{GD}(\Gamma) = [2A_2] \). We apply the same formulas as in the proof of Lemma 5.3, with the inequality

\[
\text{deg}(j \varepsilon) - n_1 - n_2 \leq 2.
\]

We have \( \text{deg}(j \varepsilon) \leq 6 \) and \( \alpha_1 = \alpha_2 = 0 \). Notice that \( \text{deg}(j \varepsilon) = 5 \) is impossible. \( \square \)

7. General rationality results

Notations 7.1. We will denote by \( \mathfrak{S}_n \) the symmetric group on \( n \) letters, by \( \mathfrak{A}_n \) the alternating group, by \( \mathfrak{D}_n \) the dihedral group and by \( \mathfrak{C}_n = \mathbb{Z}/n \) the cyclic group. In particular, \( \mathfrak{S}_2 = \mathfrak{C}_2 = \mathbb{Z}/2 \) and \( \mathfrak{D}_2 = \mathbb{Z}/2 \times \mathbb{Z}/2 \) (sometimes we prefer the notation \( \mathfrak{S}_2 \) over \( \mathfrak{C}_2 \) to stress that the action is by permutation). We write \( \text{Gr}(k, n) \) for the Grassmannian of \( k \)-planes in a vector space of dimension \( n \) and \( V_d \) for the space of binary forms of degree \( d \). We will denote by \( \text{GL}_2, \text{PGL}_2, \mathbb{G}_m \) etc. the corresponding complex algebraic groups. For a group \( G \), we denote by \( Z_g \) the centralizer of \( g \in G \) and by \( Z_G \) its center. We denote by \( M_2 = V_1 \oplus V_1 \) the space of \( 2 \times 2 \)-matrices. We write \( \mathcal{V} \rightarrow X \) or simply \( \mathcal{V} \rightarrow X \) for a locally trivial (in Zariski topology) fibration \( \mathcal{V} \) over \( X \) with generic fiber \( V \). We will often write \( G \)-map (etc.), instead of \( G \)-equivariant map.

We say that two algebraic varieties \( X \) and \( X' \) are birational, and write \( X \sim X' \), if \( \mathbb{C}(X) = \mathbb{C}(X') \). A variety \( X \) of dimension \( n \) is rational if \( X \sim \mathbb{A}^n \), k-stably rational if \( X \times \mathbb{A}^k \sim \mathbb{A}^{n+k} \) and stably rational if there exists such a \( k \in \mathbb{N} \). We say that \( X \) is unirational if \( X \) is dominated by \( \mathbb{A}^n \). The first basic result, a theorem of Castelnuovo from 1894, is:
**Theorem 7.2.** A unirational surface is rational.

Already in dimension three, one has strict inclusions

\[ \text{rational} \subset \text{stably rational} \subset \text{unirational} \]

(see the counterexamples in [10], [2], [11], [3]). There is a very extensive literature on rationality for various classes of varieties. We will use the following facts:

**Lemma 7.3.** Let \( S \to B \) be a ruled surface with base \( B \) and \( \pi : C \to S \) a conic bundle over \( S \). Assume that the restriction of \( \pi \) to a generic \( \mathbb{P}^1 \subset S \) is a conic bundle with at most three singular fibers. Then \( C \sim A^2 \times B \).

**Lemma 7.4.** Let \( \pi : C \to S \) be a conic bundle over an irreducible variety \( S \) and \( Y \subset C \) a subvariety such that the restriction of \( \pi \) to \( Y \) is a surjective finite map of odd degree. Then \( C \) has a section and \( C \sim S \times A^1 \).

Let \( G \) be an algebraic group. A (good) **rational action** of \( G \) is a homomorphism

\[ \rho_{\text{rat}} : G \to \text{Bir}(X) \]

such that there exists a birational model \( X' \) of \( X \) with the property that \( \rho_{\text{rat}} \) extends to a (regular) morphism \( G \times X' \to X' \). We consider only rational actions. We write \( X \sim_G Y \) for a \( G \)-birational (= \( G \)-equivariant birational) isomorphism between \( X \) and \( Y \). We will denote by \( G\setminus X \) a model for the field of invariants \( \mathbb{C}(X)^G \).

Let \( E \to X \) be a vector bundle. A **linear** action of \( G \) on \( E \) is a rational action which preserves the subspace of fiberwise linear functions on \( E \). In particular, there is a linear \( G \)-action on regular and rational sections of \( E \).

We are interested in rationality properties of quotient spaces for the actions of \( \text{PGL}_2 \), its subgroups and products of \( \text{PGL}_2 \) with finite groups. The finite subgroups of \( \text{PGL}_2 \) are

\[ \mathfrak{S}_n, \mathfrak{O}_n, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5. \]

We denote by \( \tilde{\mathfrak{S}}_n, \tilde{\mathfrak{O}}_n \) etc. their lifts to \( \text{GL}_2 \) (as central \( \mathfrak{S}_2 \)-extensions). We denote by

\[ B, T = \mathbb{C}^*, N_T \]
the upper-triangular group, the standard maximal torus and the normalizer of this torus in PGL$_2$ and by 
\[ \tilde{B}, \tilde{T}, N_{\tilde{T}} \]
the corresponding subgroups in GL$_2$ (or SL$_2$).

Let $V$ be an $n$-dimensional vector space, $\tilde{G} \subset \text{GL}(V)$ a subgroup and $G$ its projection to PGL$(V)$, acting naturally on $\mathbb{P}(V)$. Determining the rationality of quotients $G \backslash \mathbb{P}(V)$ (at least for finite groups) is known as Noether’s problem.

**Corollary 7.5** (of Theorem 7.2). For all $n \leq 3$ the space $G \backslash \mathbb{P}(V)$ is rational.

**Theorem 7.6.** \cite{29,36} A quotient of $\mathbb{P}(V)$ by a (projective) action of a connected solvable group, a torus or a finite abelian subgroup of a torus is rational.

A fundamental rationality result is the following theorem of Katsylo:

**Theorem 7.7.** \cite{17} For any representation $V$ of GL$_2$ or PGL$_2$ the quotient PGL$_2 \backslash \mathbb{P}(V)$ is rational.

In general, the quotients need not be rational (see Saltman’s counterexamples in \cite{30}). We now describe some partial results for $n = 4$, which we will use later on.

**Definition 7.8.** A finite group $\tilde{G} \subset \text{GL}_n = \text{GL}(V)$ is called imprimitive if there exists a decomposition $V = \oplus_{\alpha} V^{\alpha}$ such that for all $\alpha$ and $\tilde{g} \in \tilde{G}$ there is an $\alpha'$ with $\tilde{g}V^{\alpha} = V^{\alpha'}$. Otherwise, $G$ is called primitive.

**Remark 7.9.** There are 29 types of primitive subgroups of GL$_4$. For some of them, like $\mathfrak{A}_6, \mathfrak{A}_7, \text{PSL}_2(\mathbb{F}_7), \mathfrak{S}_6$, rationality of the quotient is still unknown.

**Theorem 7.10.** \cite{26} For every primitive solvable subgroup $G \subset \text{PGL}_4$ the quotient $G \backslash \mathbb{P}^3$ is rational.
Remark 7.11. In [26] it is shown that
\[ G^1 \mathbb{P}^3 \cong_G G' \backslash X_3, \]
where \( X_3 \) is the Segre cubic threefold and \( G' \) is a quotient of \( G \). The problem is then reduced to the (easy) case of imprimitive actions.

We will also need to consider quotients by nonlinear actions.

Lemma 7.12. The quotient of \( GL_2 \) (or \( PGL_2 \)) by the involution \( i : x \mapsto x^{-1} \) is rational.

Proof. The involution decomposes as a product \( i = i_1 \circ i_2 \), where
\[ i_1 : x := \begin{pmatrix} a & b \\ c & -a + d \end{pmatrix} \mapsto \begin{pmatrix} -a + d \\ -c \\ a \end{pmatrix} \]
and
\[ i_2 : y \mapsto y \cdot \det(y)^{-1}. \]
are two commuting involutions. Another set of independent generators of \( \mathbb{C}(a,b,c,d) \) is given by \( \{a, b, c, \det(x)\} \) (write \( d = (\det(x) + bc + a^2)/a \)). Now the involutions take the form
\[ i_1 : (a, b, c) \rightarrow (-a, -b, -c) \]
and
\[ i_2 : \det(x) \rightarrow \det(x)^{-1} \]
and we can write down independent generators of the field of invariants. If
\[ D := \frac{\det(x) + 1}{\det(x) - 1} \]
then
\[ i_2 : D \mapsto -D \]
\[ i_1 : (a, b, c, D) \mapsto (-a, -b, -c, -D). \]
This finishes the proof. \qed

A (rational) slice for the action of \( G \) is a subvariety \( S \subset X \) such that the general \( G \)-orbit intersects \( S \) in exactly one point. (The slice \( S \) need not be a rational variety. To avoid confusion, we will always refer to \( S \) as a slice.) A subvariety \( Y \subset X \) is called a \((G, H)\)-slice (where \( H \subset G \) is a subgroup) if \( G \cdot Y \sim X \) and \( gy \in Y \) implies that \( g \in H \). Clearly, \( G \backslash X \sim H \backslash Y \). Moreover, if \( f : X \rightarrow X' \) is a \( G \)-equivariant morphism and \( Y' \) is a \((G, H)\)-slice in \( X' \) then \( f^{-1}(Y') \) is a \((G, H)\)-slice in \( X \).
Notations 7.13. For a reductive group $G$ acting (rationally) on $X$ we denote by
\[ \text{St}_{\text{gen}} = \text{St}_{\text{gen}}(G, X) \]
the generic stabilizer (defined up to conjugacy). The action is called an *af*-action (*almost free*) if $\text{St}_{\text{gen}}$ is trivial.

We use a more precise version of Theorem 7.7:

**Theorem 7.14.** [17] Let $\rho : \text{PGL}_2 \to \text{PGL}(V)$ be a representation and $\tilde{\rho}$ a lifting of $\rho$ to a representation of $\text{GL}_2 \to \text{GL}(V)$. Let
\[ G' := \text{St}_{\text{gen}}(\text{GL}_2, V) \]
and $G := \text{GL}_2 / G'$. If the central $C_2 \not\subset G'$ then
\[ \mathbb{P}(V) \cong G \times S, \]
where $S$ is a rational variety (with trivial $G$-action).

If $C_2 \subset G'$ then
- either the $\text{PGL}_2$-action on $\mathbb{P}(V)$ has no slice and $G \setminus \mathbb{P}(V)$ is rational
- or
\[ \mathbb{P}(V) \cong G \times S, \]
where the slice $S$ is a rational variety (with trivial $G$-action).

We now explain some general techniques in the study of rationality of quotient varieties.

**Lemma 7.15.** Let $E \to X$ be a vector bundle of rank $r = \text{rk}(E)$. Let $G$ be an (affine) reductive group acting on $E$ such that the generic orbit of $G$ in $E$ projects isomorphically onto a generic orbit of $G$ in $X$. Then
\[ E \cong_G X \times \mathbb{A}^r \]
with trivial $G$-action on the affine space $\mathbb{A}^r$.

**Proof.** Denote by $O$ the $G$-orbit through a generic point in $E$. Shrinking (equivariantly) $X$, if necessary, we may assume that the map
\[ \pi : H^0(X, E) \to H^0(O, E|_O) \]
is surjective. With our assumptions, there exists a basis $s_1, \ldots, s_r$ such that for each $j$, the $G$-orbit of $s_j$ projects isomorphically onto its image in $X$ and generates a trivial 1-dimensional $G$-equivariant sub-bundle of the restriction $E|_O$ of $E$ to the orbit $O$. It follows that $E|_O = \oplus_{j=1}^r G \cdot s_j$. In particular, $H^0(O, E)$ contains the trivial $G$-module $M$ generated by $s_1, \ldots, s_r$. Moreover,
$M$ generates $H^0(O, E|_O)$ over every point of $O$. Since $\pi$ is a map of $G$-modules and $G$ is reductive $H^0(X, E)$ contains a submodule $M'$ such that $\pi(M') = M$ (as $G$-modules). The elements of $M'$ generate $E$ over a generic point of $X$. A basis $s'_1, ..., s'_r$ of $M'$ gives the desired splitting of the action. \hfill \Box

**Corollary 7.16.** Let $G$ be a reductive group and

$$E'' \rightarrow E' \rightarrow X$$

a $G$-equivariant sequence of vector bundles such that the generic $G$-orbit of $E'$ projects isomorphically onto its image. Choose a generic $G$-equivariant section $s'$ of $E' \rightarrow X$ and denote by $E''_{s'}$ the restriction of $E''$ to this section. Then

$$E'' \sim_G E''_{s'} \times \mathbb{A}^{r'}$$

(where $r' = \text{rk } E'$), with trivial $G$-action on $\mathbb{A}^{r'}$.

**Proposition 7.17.** Let $X$ be a variety with an action $\rho : G \rightarrow X$ of a linear algebraic group $G$. Let $E \rightarrow X$ be a vector bundle and $\tilde{\rho} : \tilde{G} \rightarrow E$ a $\tilde{G}$-action lifting $\rho$. Consider a generic orbit $G \cdot x \subset X$ and the linear action of $\tilde{G}$ on the space of sections $H^0(X, E)$.

Assume that $\tilde{G}$ is reductive and $V$ is a linear representation of $\tilde{G}$ which is contained in $H^0(X, E)$. Then there exists an affine open $X' \subset X$ such that the vector bundle $E \rightarrow X'$ admits a $\tilde{G}$-map onto a $\tilde{G}$-representation $V^*$.

If the action of $G$ on $X$ is almost free we may think of $X$ as being (birational to) a principal fibration over the quotient $G \backslash X$ with fiber $G$. If $G$ is affine we may assume that $X$ and $G \backslash X$ are also affine. Let us also recall a standard general construction of $G$-maps: if the ring $\mathbb{C}[X]$ is a direct sum of $G$-modules then any $G$-submodule $V \subset \mathbb{C}[X]$ defines a $G$-map $X \rightarrow \text{Spec}(V)$. We also have a vector bundle version of the above construction: let $E \rightarrow X$ be a $G$-vector bundle and $O$ a $G$-orbit through a generic point. Assume that $H^0(O, E)$ (the restriction of the space of sections to $O$) contains $V$ as a submodule. We obtain a $G$-map

$$v : H^0(O, E) \rightarrow V^*$$

(the dual module, considered as a vector bundle over a point).

**Lemma 7.18.** There exists a $G$-stable Zariski open $U \subset X$ and a rational $G$-map of $H^0(U, E) \rightarrow V^*$ extending $v$. 

Proof. A generic orbit $O$ has a $G$-equivariant neighborhood $U$, with $U/G$ affine, such that

$$H^0(U, E) \longrightarrow H^0(O, E).$$

The module $H^0(U, E)$ is a direct sum of finite dimensional irreducible $G$-modules. We can now take any submodule $V \subset H^0(U, E)$ which surjects isomorphically onto a submodule in $H^0(O, E)$.

**Lemma 7.19.** If $X$ has an $af$-action of $\text{PGL}_2$ then

$$X \times \mathbb{P}(V_{2d}) \sim_{\text{PGL}_2} X \times \mathbb{P}(V_{2d}),$$

with diagonal $\text{PGL}_2$-action on the left and trivial $\text{PGL}_2$-action on $\mathbb{P}(V_{2d})$ on the right.

**Proof.** We know that $\mathbb{C}[\text{PGL}_2]$, as a $\text{PGL}_2$-module, is sum of all even modules $V_{2d}$. This gives a $\text{PGL}_2$-map $s : X \rightarrow \mathbb{P}(V_{2d})$. The quotient

$$\text{PGL}_2 \backslash X \times \mathbb{P}(V_{2d})$$

is a projective bundle over the quotient $\text{PGL}_2 \backslash X$, with a section obtained from $s$. Therefore, it is birational to the product $(\text{PGL}_2 \backslash X) \times \mathbb{P}(V_{2d})$, which gives the claimed $\text{PGL}_2$-isomorphism.

**Corollary 7.20.** Let $X$ be a variety with an $af$-action of $\text{PGL}_2$. Then $X$ is a $(\text{PGL}_2, N_T)$-slice in

$$X \times \mathbb{P}(V_2)$$

(with diagonal $\text{PGL}_2$-action).

**Lemma 7.21.** Assume that $X$ has an $af$-action $\rho$ of $\text{PGL}_2$. Let $\mathcal{V} \rightarrow X$ be a vector bundle over $X$ with an action $\tilde{\rho}$ of $\text{GL}_2$ lifting $\rho$. Assume that $X$ contains a $\text{PGL}_2$-orbit $Y \sim \text{PGL}_2$ such that the $\text{GL}_2$-module $H^0(Y, \mathcal{V}_Y)$ contains $V_d$, for some odd $d$. Then

$$\mathbb{P}(\mathcal{V}) \sim_{\text{PGL}_2} \text{PGL}_2 \times S,$$

(with trivial $\text{PGL}_2$-action on $S$). Otherwise, $\mathcal{V}$ is induced from a $\text{GL}_2$-vector bundle on $\text{PGL}_2 \backslash X$.

**Proof.** Let $Y$ be an orbit such that $H^0(Y, \mathcal{V}_Y)$ contains $V_d$, for some odd $d$. Shrinking $X$, if necessary, gives a surjective map of $\text{GL}_2$-modules

$$H^0(X, \mathcal{V}) \longrightarrow H^0(Y, \mathcal{V}_Y).$$
Since $H^0(Y, \mathcal{V}_Y)$ is an algebra over $H^0(Y, \mathcal{O}_Y) = \bigoplus_{d \geq 0} V_{2d}$, it contains $V_1$ as a submodule. We obtain a $\text{PGL}_2$-equivariant surjective map
\[ \mathbb{P}(\mathcal{V}) \to \mathbb{P}(V_1) = \mathbb{P}^1. \]
Since the stabilizer of a point in $\mathbb{P}^1$ is solvable, we get a slice $S \subset \mathbb{P}(\mathcal{V})$, as claimed.

Assume that there is an orbit $Y \sim \text{PGL}_2$ such that $\mathcal{V}_Y$ contains only even weight $\text{GL}_2$-submodules. Then the central $\mathfrak{c}_2 \subset \text{GL}_2$ acts trivially on $\mathcal{V}_Y$. It follows that $\mathcal{V}_Y$ is a trivial $\text{PGL}_2$-bundle, and $H^0(Y, \mathcal{V}_Y)$ a trivial $\text{PGL}_2$-module. The semi-simplicity of the $\text{PGL}_2$-action implies that $H^0(Y, \mathcal{V}_Y)$ contains $H^0(Y, \mathcal{V}_Y)$ as a submodule. Shrinking $X$ if necessary, we can find linearly independent $\text{PGL}_2$-invariant sections, whose specializations to $Y$ generate $H^0(Y, \text{PGL}_2)$. Therefore, $\mathcal{V}$ is lifted from the quotient $\text{PGL}_2 \backslash X$.

**Lemma 7.22.** Let $V$ be a representation of $G$ of dimension $\geq 2$ (with $G$ acting on the left). Then $V \oplus V$ is a $G \times \text{GL}_2$-space (with right $\text{GL}_2$-action) and
\[ V \oplus V \sim_{G \times \text{GL}_2} \mathcal{V} \xrightarrow{M_2 = V_1 \oplus V_1} \text{Gr}(2, V), \]
a vector bundle with fibers $2 \times 2$-matrices (with right $\text{GL}_2$-action).

**Proof.** Consider the map
\[ V \oplus V \to \text{Gr}(2, V) \]
\[ (v, v') \mapsto \langle v, v' \rangle, \]
defined on the open, $G \times \text{GL}_2$-invariant subset of noncollinear pairs $(v, v') \in V \oplus V$ (with fibers consisting of pairs spanning the same 2-space). The $\text{GL}_2$-action on the fibers is the right multiplication on matrices:
\[ (v, v') \mapsto (av + bv', cv + dv'). \]

Assume that $G$ is reductive and denote by $G'' := \text{St}_{\text{gen}}(G, \text{Gr}(2, V))$ and by $G' := G/G''$ the quotient group of $G$ which acts effectively on $\text{Gr}(2, V)$.

**Corollary 7.23.** Assume that the action of $G'$ on $\text{Gr}(2, V)$ has a slice $S$ so that $\text{Gr}(2, V) \sim S \times G'$. Let $\mathcal{V}_S$ be the restriction of $\mathcal{V}$ to $S$ (this makes sense by Corollary 7.10). Then
\[ G \backslash \mathcal{V}/\text{GL}_2 \sim G' \backslash \mathcal{V}_S. \]
Remark 7.24. The group $G''$ acts as scalars on $V = V_1 \oplus V_1$ (it commutes with $GL_2$).

Lemma 7.25. Assume that we are in the situation of Corollary [7.23], $G = GL_2$ and $H \subset GL_2$ has finite image in $PGL_2$. Then $G \backslash V / H$ is rational.

Proof. By Corollary [7.24], the slice $S$ is 3-stably rational, since

$$S \times PGL_2 \sim Gr(2, V)$$

and $Gr(2, V)$ is rational. The quotient of $V_S$ by a fiberwise linear action is birational to $(M_2/H) \times S$ (every vector bundle admits an $H$-equivariant trivialization over an open subset of $S$). There is a left action of $GL_2 \subset V_1 \oplus V_1$ which commutes with $H$. Thus $M_2/H$ is (rationally) a three-dimensional variety with an $af$-action of $G_m$. The quotient (a surface) is unirational, hence rational (by Theorem [7.2]), and

$$G \backslash (V \oplus V) / H \sim S \times (V_1 \oplus V_1) / H \sim S \times C^3.$$

The group $PGL_2$ acts on $P(M_2)$ on both sides. We will need an explicit description of the action for some of its subgroups.

Lemma 7.26. We have

$$St_{\text{gen}}(N_T \times N_T, \mathbb{P}(M_2)) = C_2.$$ 

Proof. Indeed $N_T$ contains

$$G_m = \{t\} : (x, y) \mapsto (tx, t^{-1}y),$$

$$i : (x, y) \mapsto (y, x).$$

The corresponding actions on $\mathbb{P}(M_2)$ are

$$(a, b, c, d) \mapsto (t_1 t_2 a, t_1^{-1} t_2 b, t_1 t_2^{-1} c, t_1^{-1} t_2^{-1} d)$$

and

$$i_1 : a \to c, \ b \to d$$

$$i_2 : a \to b, \ c \to d,$$

respectively. A matrix $(a, b, c, d) \in M_2$ can be transformed to $(1, 1, 1, d)$ by a unique element of $G_m \times G_m$, the $G_2 \times G_2$-orbit of which consists of two elements (for $d, d^{-1}$). 

$\square$
Corollary 7.27. The group $N_T \times \mathbb{C}^*$ acts almost freely on $\mathbb{P}(M_2)$. There is an open, $N_T \times N_T$-stable subvariety $U \subset \mathbb{P}(M_2)$ such that

$$U \to \mathbb{C}^* \times \mathbb{C}^*$$

with a transitive action of $\mathbb{C}^* \times \mathbb{C}^* \subset N_T \times N_T$ on the fibers. The diagonal subgroup

$$\mathcal{G}_2^\Delta \subset \mathcal{G}_2 \times \mathcal{G}_2 = (N_T \times N_T)/(\mathbb{C}^* \times \mathbb{C}^*)$$

acts on each fiber as an involution $x \mapsto x^{-1}$. The factor $\mathcal{G}_2 = (\mathcal{G}_2 \times \mathcal{G}_2)/\mathcal{G}_2^\Delta$ acts on the base $\mathbb{C}^* \subset \mathbb{P}^1$ as an involution without fixed points, on the first factor in the fiber as $x \to x^{-1}$, and as identity on the second factor.

Corollary 7.28. Let $\mathcal{D} \subset N_T$ be a dihedral subgroup such that $\mathcal{D}/N_T = \mathbb{C}^*$. Then the $\mathbb{C}^*$-bundle

$$C = \mathcal{D}/\mathbb{P}(M_2) \to N_T/\mathbb{P}(M_2)$$

is induced from the $\mathbb{C}^*$-bundle

$$\mathcal{D}/\mathbb{P}(M_2)/N_T \to N_T/\mathbb{P}(M_2)/N_T = \mathbb{P}^1$$

and is hence birationally trivial.

Proof. Indeed, the left and the right actions of $N_T$ commute. By Lemma 7.26, $\text{St}_{\text{gen}}(N_T \times N_T, \mathbb{P}(M_2)) = \mathcal{C}_2$, which implies that the bundle is induced. \qed

Lemma 7.29. For every dihedral group $\mathcal{D}$ and every $H \subset N_T$ the conic bundle

$$C_H = \mathcal{D}/\mathbb{P}(M_2)/H \to N_T/\mathbb{P}(M_2)/H,$$

has a section.

Proof. The quotient $\mathcal{D}/U/H$ from Corollary 7.27 admits a fibration

$$\mathcal{D}/U/H \to \mathbb{P}^1/\mathcal{G}_2.$$
Here \( C_\Delta^* \times C_\Delta^* \) is the quotient of the fiber \( C^* \times C^* \) of \( U \to C^* \) by the intersection of \( D, H \) with the diagonal \( C_\Delta^* \subset C^* \times C^* \). Isomorphisms \( C_\Delta^* \to C^* \) and \( C_{\Delta D}^* \to C^* \) induce a birational fiberwise isomorphism

\[
C_H = D \setminus \mathbb{P}(M_2)/H \sim G_2 \setminus \mathbb{P}(M_2)/G_2 = C_0
\]

and it suffices to consider \( D = G_2, H = G_2 \). In this case, an alternative equivariant completion of \( U \) is given by

\[
U \subset \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3,
\]

with an action of \( G_2 \times G_2 \), where the first \( G_2 \) acts as an involution on the first two factors and identity on the base while the complementary \( G_2 \) acts only on the base. Thus the quotient is a conic bundle over the complement in

\[
\mathbb{P}_1 \times \mathbb{P}_2 / G_2 \times G_2 = \mathbb{P}_1 \times \mathbb{P}_1
\]

to the branch locus of the quotient map. Here the left (resp. right) \( G_2 \) acts as an involution on the left (resp. right) \( \mathbb{P}_1 \) and the branch locus is exactly the union of four lines. By Lemma 7.3, this conic bundle has a section (it is nonsingular on a pencil of lines minus at most two points). \( \square \)

**Lemma 7.30.** Let \( G \) be a subgroup of \( SL_2 \), not equal to \( \tilde{A}_5 \), and \( V \) a linear representation of \( G \). Then \( G \setminus \mathbb{P}(V) \) is rational.

**Proof.** For \( G = SL_2 \) this is a theorem of Katsylo [7]. We now consider proper subgroups \( G \subsetneq SL_2 \). If \( G \) is solvable and connected then rationality for the quotient follows from a theorem of Vinberg [30]. For compact \( G \) the proof is similar to the dihedral case described below. Assume now that \( G \) is finite and not equal to \( \tilde{A}_5 \). Then \( G \) is either

1. a finite subgroup of \( C^* \),
2. a dihedral group or
3. \( \tilde{A}_4, \tilde{S}_4 \).

The first case is easy. For dihedral groups all irreducible representations of \( G \) have dimension \( \leq 2 \) and the corresponding quotients are rational by Theorem 7.2. Let \( V \) be a faithful representation of a dihedral group \( D \) (otherwise,
we are reduced to a quotient group). Thus $V = W \oplus W'$, where $\dim W = 2$ and $\dim W' \geq 1$. Denote by $G' = G/\mathcal{C}'$ the quotient acting faithfully on $W'$ ($\mathcal{C}'$ is a cyclic group). We have $W \sim_{\mathcal{D}} \mathbb{C}^* \times \mathbb{P}^1$, with trivial action of $\mathcal{D}$ on $\mathbb{C}^*$ and trivial action of $\mathcal{C}'$ on $\mathbb{P}^1$. By Lemma 7.13,

$$\mathbb{C}^* \times \mathbb{P}^1 \times W' \sim_{G'} (\mathbb{C}^* \times \mathbb{P}^1) \times W',$$

with trivial action of $G'$ on $\mathbb{C}^* \times \mathbb{P}^1$. Thus

$$\mathcal{D}\backslash V \sim (\mathcal{D}'\backslash W') \times (\mathbb{C}^* \times \mathbb{P}^1)$$

and we can apply induction.

We turn to the last case. An irreducible representation of $\tilde{\mathfrak{A}}_4$ is either a character, or a faithful two-dimensional representation, or a three-dimensional representation, trivial on the center (a faithful representation of $\mathfrak{A}_4$). An irreducible representation of $\tilde{\mathfrak{S}}_4$ is either a faithful two-dimensional representation, a faithful four-dimensional representation $W := \text{Sym}^3(V_1)$ or a representation of $\mathfrak{S}_4$ (of dimension $\leq 3$).

For irreducible representations of dimension $\leq 3$ rationality for the quotient follows from Theorem 7.2. We turn to $W$. Recall that

$$W = \text{Sym}^3(V_1) = V_1^\chi \oplus V_1^{-\chi},$$

as a $\tilde{\mathfrak{A}}_4$-representation, where $V_1^\chi = V_1 \otimes \chi$, $V^{-\chi} = V_1 \otimes \chi^{-1}$ and

$$\chi : \mathfrak{A}_4 \to \mathbb{Z}/3 \subset \mathbb{C}^*$$

is the cubic character. A pair of (generic) points

$$p_\chi \in \mathbb{P}^1 = \mathbb{P}(V^\chi), p_{-\chi} \in \mathbb{P}^1 = \mathbb{P}(V^{-\chi})$$

defines a line $\mathbb{P}^1 \subset \mathbb{P}(W)$. This shows that

$$\mathbb{P}(W) \sim \tilde{\mathfrak{S}}_4 \left|_{\mathbb{P}^1 \times \mathbb{P}^1} \right.,$$

where $\mathfrak{S}_4$ acts on the base, $\mathfrak{A}_4$ acts linearly on the fiber $L$ and $\mathfrak{S}_2 = \tilde{\mathfrak{S}}_4/\mathfrak{S}_4$ acts as an involution on the fiber $L$. Thus $\tilde{\mathfrak{S}}_4 \backslash \mathbb{P}(W)$ is a conic bundle over the rational surface $\mathfrak{S}_4 \backslash (\mathbb{P}^1 \times \mathbb{P}^1)$. We now analyze the geometry of this bundle in more detail. Consider the action $\mathcal{D}_2 \subset \mathfrak{S}_4$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and on $\mathbb{P}^2 = \text{Sym}^2(\mathbb{P}^1)$. Every involution $i \in \mathcal{D}_2$ has two invariant points $x_i, y_i$. Consider the graphs $\mathbb{P}^1$ connecting the points $(x_i, y_i) - (y_i, x_i)$. Their set is equal to $\mathbb{P}^1$ and there is a graph:

$$l_i : (x_i, y_i) - (y_i, x_i) \subset \mathbb{P}^1 \times \mathbb{P}^1.$$
consisting of points \((x, i(x))\). The line \(l_i\) is exactly the subset of \(i\)-invariant points in \(\mathbb{P}^1 \times \mathbb{P}^1\). The action of \(D_2\) is free outside the three lines \(l_i, i \in D_2, i \neq 1\). There are exactly 6 points which are invariant under \(D_2\).

The corresponding action on \(\mathbb{P}^2\) can be described as follows. There are three points corresponding to \((x_i, y_i)\) which are stable under \(D_2\) and three lines (images of \(l_i\)) so that the action is free on the torus \(\mathbb{C}^* \times \mathbb{C}^*\) (the complement in \(\mathbb{P}^2\) to the union of \(l_i\)). The group \(D_2\) acts on \(\mathbb{C}^* \times \mathbb{C}^*\) as a translation by the subgroup of points of order 2.

The quotient \(\mathbb{P}^2_q := D_2 \setminus \mathbb{P}^2\) is a nonsingular variety isomorphic to \(\mathbb{P}^2\) (indeed the only possible singularities come from the three \(D_2\)-invariant points in \(\mathbb{P}^2\) but the quotient by the local action is nonsingular). The diagonal \(\mathbb{P}^1_\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1\) projects onto a conic \(C \subset \mathbb{P}^2\), which is invariant under \(D_2\). The conic \(C\) intersects the “vertical” and “horizontal” subgroups in \(\mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{P}^2\) in two points and does not intersect the line at infinity.

Thus in \(\mathbb{P}^2_q = D_2 \setminus \mathbb{P}^2\), the image of \(\mathbb{P}^1_\Delta\) intersects \(\mathbb{C}^*\) in one point. Therefore, the images of \(\mathbb{P}^1_\Delta\) and of \(l_i\) are lines (since pairwise intersections of the \(l_i\) are equal to 1) and the \((C_2)^3\)-covering \(\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2_q\) is ramified exactly over a union of four lines. If suffices to observe that every conic bundle over \(\mathbb{P}^2_q\) has a section. Indeed, let \(p\) be the intersection point of two lines \(l_i\) and \(l_i'\), and consider the pencil of lines in \(\mathbb{P}^2_q\) through \(p\). Each line in this pencil intersects the ramification locus in at most three points and we can apply Lemma 7.3.

Now we turn to reducible representations \(V = \oplus_{\alpha \in A} V^\alpha\) of \(\tilde{A}_4\). If \(V\) is faithful for \(\tilde{A}_4\) then there is an \(\alpha_0 \in A\) such that \(V^{\alpha_0}\) is a three-dimensional irreducible faithful representation of \(\tilde{A}_4\) and

\[
V \sim \tilde{A}_4 V^{\alpha_0} \times (\oplus_{\alpha \neq \alpha_0} V^\alpha)
\]

with trivial action of \(\tilde{A}_4\) on \(\oplus_{\alpha \neq \alpha_0} V^\alpha\) (by Lemma 7.13). If \(V\) is faithful for \(A_4\) then \(V\) contains a faithful irreducible three-dimensional representation of \(A_4\) and we can apply the same argument. In all other cases \(V\) is a sum of one-dimensional representations and we are reduced to Case 1.

Finally, consider reducible representations \(V\) of \(\tilde{S}_4\). If \(V\) is faithful then it contains either a faithful irreducible two-dimensional representation or the faithful representation \(W\). Again, we apply Lemma 7.13 as before. If \(V\) is faithful for \(S_4\) then it contains a faithful irreducible representation of dimension \(\leq 3\) and we conclude as above. In all other cases \(V\) is a sum of one-dimensional representations.
Lemma 7.31. Let $V$ be a representation of $G \subseteq \text{SL}_2$, with $G \neq \tilde{A}_5$. Then $G \backslash \text{Gr}(2, V)$ is rational.

Proof. The relevant groups $G$ can be subdivided as follows:

1. $G$ is a subgroup of the normalizer of a maximal torus;
2. $G$ an infinite subgroup of a Borel subgroup;
3. $G = A_4, \tilde{A}_4$;
4. $G = S_4, \tilde{S}_4$.

Let $V = W \oplus W'$ be a reducible representation of $G$. Then (birationally)

$$
\begin{array}{c}
\text{Gr}(2, V) \\
\downarrow \text{Hom}(C^2_x, W) \\
\text{Gr}(2, W)
\end{array}
$$

(where $x$ is a point on the base). In particular, if $\dim W \leq 2$ then

$$
\text{Gr}(2, V) \sim^G \text{Hom}(W', W),
$$

with linear $G$-action on $\text{Hom}(W', W)$. This reduction suffices for the relevant infinite groups (for example, for connected solvable $G$ we can apply Lemma 7.3). Further,

- if $\text{St}_{\text{gen}}(G, \text{Gr}(2, V)) = 1$ then (birationally)
  $$
  G \backslash \text{Gr}(2, V) \rightarrow G \backslash \text{Gr}(2, W),
  $$

  a vector bundle.

- if $\text{St}_{\text{gen}}(G, \text{Gr}(2, V)) = \mathcal{C} \subset Z_G$ (a cyclic subgroup) then (birationally)
  $$
  G' \backslash \text{Gr}(2, V) \\
  \downarrow \mathcal{C} \backslash \text{Hom}(C^2_x, W) \\
  G' \backslash \text{Gr}(2, W).
  $$

We now consider $A_4, \tilde{A}_4, S_4$. The rationality of $G \backslash \text{Gr}(2, V)$ for irreducible representations of these groups follows from the fact that all of them have dimension $\leq 3$. Assume now that $V = W \oplus W'$, with $W$ irreducible of dimension 3. The classification of these representations implies that the action of the center must be trivial. Then, birationally,

$$
\begin{array}{c}
\text{Gr}(2, V) \\
\downarrow \text{Hom}(C^2_x, W') \\
\mathbb{P}^2 = \mathbb{P}(W').
\end{array}
$$
The G-action is equivalent to a G-action on a vector bundle

\[
\begin{array}{ccc}
G \setminus \text{Gr}(2, V) \\
\downarrow \\
G \setminus \text{Gr}(2, W) = G \setminus \mathbb{P}^2.
\end{array}
\]

Finally, let us consider the case of \(\tilde{S}_4\). Let \(W\) be its unique irreducible representation of dimension four (as in Lemma 7.30). We claim that \(\tilde{S}_4 \setminus \text{Gr}(2, W)\) is rational. Indeed, as \(\mathfrak{A}_4\)-modules, we have

\[
W = W^\chi \oplus W^{-\chi},
\]

where \(W^\chi, W^{-\chi}\) are two copies of the standard representation of \(\mathfrak{A}_4\) of dimension 2 and \(\chi\) (resp. \(-\chi\)) indicates the eigenspace decomposition for the nontrivial character

\[
\chi : \mathfrak{A}_4 \to \mathbb{Z}/3 \subset \mathbb{C}^*.
\]

Further,

\[
\text{Gr}(2, W) \sim \text{Hom}(W^\chi, W^{-\chi}),
\]

with a linear \(\mathfrak{A}_4\)-action (since the center acts trivially) and a permutation \(\mathfrak{S}_2\) inverting the map \(w \in \text{Hom}(W^\chi, W^{-\chi})\). More precisely, \(W^{-\chi} = (W^\chi)^*\) and

\[
\text{Hom}(W^\chi, W^{-\chi}) = \text{Sym}^2(W^{-\chi}) \oplus C_1,
\]

where \(C_1\) corresponds to skew symmetric maps and \(\mathfrak{A}_4\) acts on \(C_1\) by \(\chi\). The involution \(\mathfrak{S}_2 = \mathfrak{S}_4/\mathfrak{A}_4\) acts on \(C_1\) and on \(\text{Sym}^2(W^{-\chi})\) as \(t \mapsto t^{-1}\). In particular, if \(\mathbb{C}^* \times \mathbb{C}^*\) is the diagonal group acting on \(\text{Sym}^2(W^{-\chi}) \oplus C_1\) then \(\mathfrak{S}_2\) acts as

\[
X \to s^{-1}X,
\]

where \(s \in \mathbb{C}^* \times \mathbb{C}^*\) and \(X \in \text{Sym}^2(W^{-\chi}) \oplus C_1\). There is an equivariant map

\[
f : \text{Hom}(W^\chi, W^{-\chi}) \to C_1, \\
s \mapsto (x, s(y)) - (s(x), y),
\]

with an effective action of \(\mathfrak{S}_3 = \mathfrak{S}_4/\mathfrak{D}_2\) on the target \(C_1\), which to a subspace \(s \in \mathbb{C}^2 \subset W^\chi \oplus W^{-\chi}\) assigns the value of the 2-form \((x, s(y)) - (s(x), y)\). The fiber of \(f\) is \(\mathfrak{D}_2\)-birational to \(\text{Sym}^2(W^\chi) = \mathbb{P}^2\). We have already seen in the proof of Lemma 7.30 that \(\mathfrak{D}_2 \setminus \mathbb{P}^2 = \mathbb{P}^2\). Thus \(\tilde{S}_4 \setminus \text{Gr}(2, W)\) is a \(\mathbb{C}^*\)-bundle over a \(\mathbb{P}^2\)-fibration over \(\mathfrak{S}_2 \setminus C_1\). It is clear that this \(\mathbb{P}^2\)-fibration is trivial. The quotient conic bundle is nondegenerate over a product of \(\mathbb{P}^2\) with an open subvariety in \(C_1/\mathfrak{S}_3\). Hence it has a section. Rationality of \(\tilde{S}_4 \setminus \text{Gr}(2, W)\), and more generally, \(\tilde{S}_4 \setminus \text{Gr}(2, W \oplus \cdots \oplus W)\), follows (the latter is a vector bundle over the former).
Assume now that $V = nW \oplus V'$, where $\dim V' \geq 1$, and $n \in \mathbb{N}$. Since the $\mathfrak{S}_4$-action on $\text{Gr}(2, nW)$ is a f there is a $\tilde{\mathfrak{S}}_4$-equivariant homogeneous rational map $f : \text{Gr}(2, nW) \to V'$ sending the generic $\tilde{\mathfrak{S}}_4$-orbit in $W$ to the generic $\tilde{\mathfrak{S}}_4$-orbit in $V'$. Notice that the center $\mathfrak{C}_2$ acts as a scalar on $\text{Hom}(W, V')$. We have (birationally)

\begin{equation}
\begin{array}{c}
\tilde{\mathfrak{S}}_4 \backslash \text{Gr}(2, V) \\
\downarrow \\
\tilde{\mathfrak{S}}_4 \backslash \text{Gr}(2, nW)
\end{array}
\sim
\begin{array}{c}
\mathbb{C}^* \\
\downarrow \\
\mathbb{C}^* \times
\end{array}
\begin{array}{c}
\mathbb{C}^* \\
\downarrow \\
\mathbb{C}^* \times
\end{array}
\begin{array}{c}
\text{Gr}(2, nW) \\
\downarrow \\
\text{Gr}(2, nW)
\end{array}
\end{equation}

(7.1)

(with rational bases). The projective bundle on the right has a section. Indeed, (7.2)

\begin{equation}
\begin{array}{c}
\text{Gr}(2, nW) \\
\downarrow \\
\text{Gr}(2, nW)
\end{array}
\sim
\begin{array}{c}
\text{Hom}(\mathfrak{C}_2^2, V')
\end{array}
\end{equation}

is an equivariant quotient bundle of the trivial bundle with fiber $\text{Hom}(W, V')$. The map $f$ defines an $\tilde{\mathfrak{S}}_4$-equivariant section $s(f)$ in the projective bundle in (7.1). The (equivariant) linear projection

$\text{Hom}(W, V') \to \text{Hom}(\mathfrak{C}_2^2, V')$

maps $s(f)$ to an equivariant section of the bundle in (7.2). Thus $s(f)$ projects onto a section of the bundle on the right in (7.2), making it birationally trivial.

Let $V$ be a faithful $\text{SL}_2$-representation of dimension $\geq 3$. Then

| $V$       | $\text{St}_{\text{gen}}$ |
|-----------|-------------------------|
| $\dim \geq 5$ | 1                       |
| $V_4$     | $\mathfrak{C}_2$       |
| $V_3$     | $\mathfrak{D}_2$       |
| $V_2 \oplus V_0$ | $\mathfrak{C}_2$   |
| $V_2$     | $\mathbb{N}_T$         |
| $V_1 \oplus V_1$ | $\mathbb{C}^*$    |
| $V_1 \oplus V_0$ | $\mathbb{B}$     |
Further,

- $\text{Gr}(2, V_4)$ has a $(\text{PGL}_2, N_T)$-slice $S = \text{Sym}^2(\mathbb{P}^2)$ with an $af$-action of $N_T/C_2$, (where $C_2$ is the center of $N_T$);
- $\text{Gr}(2, V_3)$ has a $(\text{PGL}_2, \mathbb{A}_4)$-slice birational to $\mathbb{P}^1$, with $\mathbb{A}_4$ acting on $\mathbb{P}^1$ as $C_3$.

Proof. Consider first irreducible representations $V = V_d = \text{Sym}^d(V_1)$ and assume that the stabilizer of a generic line $\mathbb{P}^1 \subset \mathbb{P}(V)$ contains a nontrivial cyclic group $C$. Then $C$ fixes at least two points in this $\mathbb{P}^1$. Any orbit of $C$ on $\mathbb{P}^1$ is a union of a zero-cycle $C \cdot x$ and a zero-cycle supported in the fixed points. In particular, the subvariety of points in $\mathbb{P}(V_d)$ which are fixed by $C$ has dimension $\leq d/|C|$. The dimension of the variety of $C$-fixed lines in $\mathbb{P}(V)$ is therefore $\leq 2d/|C|$. The subvariety of distinct cyclic subgroups $C \subset \text{PGL}_2$ has dimension 2 and $\dim \text{Gr}(2, V_d) = 2d - 2$. Since $d/|C| \leq d/2$ the inequalities $2d - 4 > 2d/2$ and $d - 4 > 0$ imply the result.

Assume that $V = \oplus_{j \in J} V_{d_j}, |J| \geq 2$ and that $\text{St}_{\text{gen}} \neq 1$. Then $d_j \leq 2$, for all $j \in J$. Indeed, the stabilizer of a generic $\mathbb{P}^1$ through a generic point $p \in \mathbb{P}(V_d)$ is a subgroup of the stabilizer of $p$, which stabilizes some generic line in the tangent space at $p$. This group is trivial for $d > 2$ and equal to $C_2$ for $d = 2$.

If $V = V_2 \oplus V'$, with $\dim V' > 2$, then $\text{Gr}(2, V)$ is (birationally) a fibration over $\text{Gr}(2, V_2)$, with fibers $\text{Hom}(C^2, V')$ so that $\text{St}_{\text{gen}} = 1$ if $\dim V' > 3$. If $V = V_2 \oplus V_1$ then $\text{St}_{\text{gen}}$ is the same as the (generic) stabilizer of the $N_T$-action on $\text{Hom}(V', V_1), V' \in \text{Gr}(2, V_2) = \mathbb{P}^2$, hence trivial. For $V = V_2 \oplus V_0$, $\text{St}_{\text{gen}} = C_2$.

In the remaining cases $d_j = 0$ or 1, for all $j \in J$. If $V$ contains at least three copies of $V_1$ then the argument above shows that the action is $af$. Similarly, if $V = V_1 \oplus V_1$ then $\text{St}_{\text{gen}} = C^* \subset \mathbb{A}_4$ and if $V = V_1 \oplus V_1 \oplus V_0$ then $\text{St}_{\text{gen}} = 1$. For $V_1 \oplus 3V_0$, the generic stabilizer is the same as for three linear functionals - which is zero.

\section*{Lemma 7.33.} The quotient $\text{PGL}_2 \setminus \text{Gr}(2, V)$ is 2-stably rational.

Remark 7.34. For even $d \geq 10$, $\text{PGL}_2 \setminus \text{Gr}(2, V_d)$ is rational by [32].

Proof. By Lemma 7.32, if $\dim V \geq 5$ then the $\text{St}_{\text{gen}} = 1$ and we can apply Lemma 7.13 and Corollary 7.20 to conclude that $\text{PGL}_2 \setminus \text{Gr}(2, V) \times C^2 \sim_G N_T \setminus \text{Gr}(2, V)$. The claim follows from Lemma 7.31. It remains to consider:
1. Gr(2, V₄),
2. Gr(2, V₃),
3. reducible V.

In the first case, Stₙ₉₆(PGL₂, Gr(2, V₄)) = G₂, with normalizer N₇ ⊂ PGL₂. We claim that the subset X ⊂ Gr(2, V₄) of G₂-invariant points is a (PGL₂, N₇)-slice. Indeed, there is a Zariski open subset U ⊂ X such that the stabilizer of each point in U is exactly G₂. In particular, g·U intersects U only if g ∈ N₇. Consider the P² ⊂ P(V₄) consisting of G₂-invariant subschemes containing 4 points. Any line in U joins a pair of points in this P². Therefore, we have a (birational) N₇-isomorphism of U and Sym²(P²). The stabilizer of a generic point in X is a central subgroup in N₇ whose action on P² is equivalent to a linear action on C². (Indeed, Sym²(V₁) = C ⊕ W₂, where C is the trivial representation - the invariant symmetric form - and W₂ is a faithful two-dimensional representation of N₇/G₂). Thus instead of X with the N₇-action we can consider C² × C² with the (N₇/G₂) × G₂-action (where the second G₂ interchanges the factors). In particular, (by linearity)

\[ N₇ \backslash X \sim C^* \times N₇ \backslash P³, \]

and is hence rational.

In the second case, Gr(2, V₄) has a surjection of degree 2 onto P(V₄). The connected component of the preimage of the (PGL₂, G₄)-slice P¹ in P(V₄) is a (PGL₂, A₄)-slice, isomorphic to P¹. The quotient is rational.

If V is reducible and the PGL₂-action on the Grassmannian has nontrivial stabilizer then dim V < 5. Rationality follows since dim Gr(2, V) ≤ 4 and the generic orbit has dimension at least 2.

**Proposition 7.35.** Let G, H be finite solvable subgroups of PGL₂. Then

\[ G \backslash PGL₂ / H \]

is rational.

*Proof.* The action is birational to the (projective) action of G × H on P(M₂), where G acts on the right and H on the left. The groups G, H are either:

- cyclic;
- dihedral or
- A₄, G₄.

The case of *primitive* solvable groups is covered by Theorem 7.10 [26]. If V is reducible then there is a nontrivial action of C* on G \ P(V)/H, leading to rationality. This covers the case when either G or H is cyclic.
We claim that if \( V \) is irreducible and imprimitive (for the \( G \times H \)-action) then either \( G \) or \( H \) is dihedral. By definition, \( V := M_2 = \bigoplus \alpha V^\alpha \), such that \( \tilde{g}V^\alpha = V^{\alpha'} \) for all \( \tilde{g} \in G \times H \). Moreover, by irreducibility, all \( V^\alpha \) must have the same dimension, \( = 1 \) or \( 2 \). Notice that imprimitivity for an action of a group \( G' \) implies imprimitivity for the induced action of every subgroup \( G'' \subseteq G' \) (with the same decomposition of \( V \)). We now claim that the actions of \( A_4 \times A_4 \), and consequently of \( A_4 \times S_4 \) and \( S_4 \times S_4 \) are primitive. Indeed, \( A_4 \times A_4 \) contains \( D_2 \times D_2 \) as a normal subgroup, for which the imprimitive structure is either a sum of two subspaces of dimension 2 or four subspaces of dimension 1, corresponding to the choice of a subgroup \( S_2 \subset D_2 \). The first possible imprimitive structure for \( D_2 \times D_2 \) does not extend to one for \( A_4 \times A_4 \) (which has no index 2 subgroups). The second structure is also impossible: \( A_4 \) rotates the subgroups \( S_2 \subset D_2 \), hence there is no \( A_4 \)-invariant imprimitive structures for \( D_2 \times D_2 \).

It remains to consider the case when both \( G \) and \( H \) are dihedral. On \( V_1 \) there is a unique imprimitive structure, corresponding to the eigenspaces \( C_1, C_2 \) of the elements of \( G \). In particular, there is an imprimitive structure on

\[
M_2 = V_1 \oplus V_1' = (C_1 \oplus C_1') \oplus (C_2 \oplus C_2').
\]

We claim that (birationally)

\[
\begin{array}{c}
G \backslash \mathbb{P}(M_2)/H \\
\downarrow
\\
\mathbb{P}^2 = G \backslash \text{Sym}^2(\mathbb{P}^1)
\end{array}
\]

is a conic bundle degenerating precisely over the image of the diagonal and the subvarieties in \( \mathbb{P}^2 \) with nontrivial stabilizers.

Indeed, since \( H \subset N_T \) (a \( C_2 \)-extension of \( C^* \)), (birationally)

\[
\begin{array}{c}
G \backslash \mathbb{P}(M_2)/H \\
\downarrow
\\
C^* = N_T/H
\\
N_T \backslash \mathbb{P}(M_2)/H.
\end{array}
\]

The quotient \( C^* \backslash \mathbb{P}(M_2) \) is (birationally) a fibration over \( \mathbb{P}^1 \times \mathbb{P}^1 \), with \( S_2 \) acting by permutation, where the coordinate \( \mathbb{P}^1 \)'s are the projectivizations of
the two-dimensional eigenspaces for the $\mathbb{C}^*$-action on $M_2$. Thus
\[
\mathbb{P}(M_2)/H \xrightarrow{\text{}} \mathbb{P}^2 = \mathbb{P}^1 \times \mathbb{P}^1/\mathcal{G}_2
\]
is a conic bundle nondegenerate outside a conic (the image of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$). The $G$-action commutes with the $N_T$-action and is effective on the base. This proves the claim.

We have $G \subset N_T$ and
\[
G/\mathbb{P}^2 \rightarrow N_T/\mathbb{P}^2
\]
is a conic bundle. Since the left and right actions of $N_T$ commute, $G/\mathbb{P}^2$ contains an open subvariety $U \times \mathbb{C}^*$ where the restriction of the conic bundle is nondegenerate. Here $\mathbb{C}^* = G \backslash N_T$ and $U$ is a subset of $\mathbb{P}^1 = N_T/\mathbb{P}^2$. Therefore the conic bundle has at most 2 singular fibers on any completion of the fiber $\mathbb{C}^* \subset U \times \mathbb{C}^*$. Rationality follows.

We can now describe some open subvariety in the quotient $G/\mathbb{P}(M_2)/H$ explicitly. Consider the action of $\mathbb{C}^* \subset N_T$ on both sides $\mathbb{C}^* \backslash \mathbb{P}(M_2)/\mathbb{C}^*$. With respect to this action $\mathbb{P}(M_2)$ is birationally equivalent to a trivial $\mathbb{C}^* \times \mathbb{C}^*$-fibration over $\mathbb{P}^1$. Now we add the action of $\mathcal{G}_2$ on both sides. The product $\mathcal{G}_2 \times \mathcal{G}_2$ acts on the base $\mathbb{P}^1$. The group $\mathcal{G}_4$ contains a normal subgroup $\mathcal{D}_2 \subset N_T$ and the action of each $\mathcal{G}_2 \subset \mathcal{D}_2$ inverts the respective $\mathbb{C}^*$ action. Thus (birationally)
\[
N_T \backslash \mathbb{P}(M_2)/NT \xrightarrow{\text{}} \mathbb{P}^1 - 3 \text{ pts},
\]
where the deleted points are the ramification points of the map $\mathbb{P}^1 \rightarrow \mathbb{P}^1/\mathcal{D}_2$.

In particular, there is an open $U$ such that
\[
G/\mathbb{P}(M_2)/H \xrightarrow{\text{}} \mathbb{C}^* \xrightarrow{\text{}} U \xrightarrow{\text{}} \mathbb{C}^* \xrightarrow{\text{}} \mathbb{P}^1 - 3 \text{ pts}.
\]
By Lemma 7.3, the conic bundles are trivial.
Finally, the conic bundles on $\mathbb{P}^2/\mathfrak{S}_4$ and $\mathbb{P}^2/\mathfrak{A}_4$ have sections. Indeed, both $\mathfrak{A}_4$ and $\mathfrak{S}_4$ contain dihedral subgroups of index 3 ($\mathfrak{D}_2$, resp. $\mathfrak{D}_4$). The image of the section in the conic bundle over $\mathfrak{D}_2\backslash\mathbb{P}^2$ (resp. $\mathfrak{D}_4\backslash\mathbb{P}^2$), has odd degree in the conic bundles over $\mathfrak{A}_4\backslash\mathbb{P}^2$ and $\mathfrak{S}_4\backslash\mathbb{P}^2$, respectively. We apply Lemma 7.4.

**Proposition 7.36.** Let $V$ be an irreducible $\text{GL}_2$-representation and $H \subset \text{SL}_2$ a finite group, not equal to $\mathfrak{A}_5$. Then

$$\text{GL}_2\backslash(V \oplus V)/H$$

is rational.

**Proof.** First of all, $V_1 \oplus V_1/H$ is rational. Next, by Lemma 7.22

$$V \oplus V \sim_{\text{GL}_2 \times \text{GL}_2} V$$

$$\downarrow_{M_2 = V_1 \oplus V_1}$$

$$\text{Gr}(2, V).$$

First we assume that $V$ has odd weight. The Grassmannian $\text{Gr}(2, V)$ carries the action of $\text{PGL}_2$. If we restrict the bundle $V$ to a generic $\text{PGL}_2$-orbit $O$ in $\text{Gr}(2, V)$ then the corresponding module $H^0(O, V_O)$ contains $V_1$ as a submodule. By Lemma 7.18, this gives an equivariant map

$$V \to V_1 \oplus V_1$$

with a 1-transitive action of $\text{GL}_2$ on the target. Thus

$$(7.3) \quad \text{GL}_2\backslash V/H \sim H\backslash \text{Gr}(2, V)$$

(with the same subgroup $H \subset \text{GL}_2$ appearing on the left). Indeed, $\text{GL}_2 \subset (V_1 \oplus V_1) = M_2$ and multiplication by $H$ on the right gives an orbit $x \cdot H$. This orbit is a $(\text{GL}_2 \times H, \text{H}^x \times H)$-slice (with $\text{H}^x = xHx^{-1}$) and it is stabilized exactly by $\text{H}^x \times H$, acting doubly transitively on the set $\text{H}^x \cdot x$. It follows that every point $x' \in x \cdot H$ is a $(\text{H}^x \times H, \text{H}^x)$-slice of the orbit $x \cdot H$. The quotient $H\backslash \text{Gr}(2, V)$ is rational by (7.3).

Assume that $V$ has even weight. If the $\text{PGL}_2$-action is $af$ then

$$\text{GL}_2\backslash V/H \sim (\text{PGL}_2\backslash \text{Gr}(2, V)) \times (\mathbb{C}^*/(V_1 \oplus V_1))/H).$$

If it is not $af$, then, by Lemma 7.32, $V = V_4$ or $V_2$.

For $V = V_4$ we have the $(\text{PGL}_2, N_T)$-slice $X = \text{Sym}^2(\mathbb{P}^2)$ with the $N_T$-action which we can replace by $\mathbb{C}^2 \times \mathbb{C}^2$ with a $(N_T/\mathfrak{C}_2) \times \mathfrak{C}_2$-linear action. In particular, we identify the quotient with a quotient of $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus V_1 \oplus V_1$.
by a linear action of $N\tilde{T} \times S_2 \times H$ (where $N\tilde{T} \subset GL_2$). The action of $N\tilde{T} \times H$ on $V_1$ is transitive with stabilizer $C_2 \times H$. Hence it is equivalent to the action of $D_2 \times H$ on $C^2 \oplus C^2 \oplus V_1$, which is a $C^2$-vector bundle (permutation of the anti-invariant part of $S_2$-action) over $C^2 \times V_1$, with $D_2 \times H$ action. The latter quotient is rational. For $V = V_2$ the action is transitive on $Gr(2, V) = \mathbb{P}^2$ and the quotient has dimension 2 - rationality follows.

We will also need a more general result for $H = S_2$.

**Proposition 7.37.** Let

$$X \xrightarrow{L} Y = \prod_{j \in J} \mathbb{P}(V_{d_j})$$

be a $GL_2$-homogeneous line bundle. If at least one $d_j \neq 2$ then $GL_2 \backslash X \times X / S_2$ is rational.

**Proof.** **Case 1.** $|J| = 1$. If $d = d_1$ is even or if $d$ is odd and the line bundle has odd degree on $\mathbb{P}(V_d)$ then

$$X \times X \sim_{GL_2 \times S_2} V_d \oplus V_d$$

and we apply Proposition 7.36. If the line bundle has even degree then it is trivial and $GL_2$ acts as $PGL_2 \times \mathbb{C}^*$. If the $PGL_2$-action on $\mathbb{P}(V_d)$ is $af$ we have

$$\mathbb{P}(V_d) \sim_{PGL_2} S \times PGL_2,$$

for a rational slice $S$ (with trivial $PGL_2$-action). We have a $PGL_2 \times \mathbb{C}^* \times S_2$-action on

$$\mathbb{C} \times PGL_2 \times S \times \mathbb{C} \times PGL_2 \times S.$$

The quotient variety is a vector bundle over $PGL_2 \backslash PGL_2 \times PGL_2 / S_2$ (rational by Lemma [7.12]). The claim follows. If the $PGL_2$-action is not $af$, then $V = V_3$ or $V_1$. For $V_1$ the quotient is rational by dimensional reasons. For $V_3$ we have a projection

$$\mathbb{C} \times \mathbb{P}^3 \times \mathbb{C} \times \mathbb{P}^3$$

$$\downarrow_{P^1 \times P^1}$$

$$Gr(2, V_3)$$

 commuting with both actions. Recall that $Gr(2, V_3)$ has $P^1$ as a $(PGL_2, \mathcal{A}_4)$-slice, with $\mathcal{A}_4$ effectively acting as a cyclic group $C_3 = \mathcal{A}_4 / D_2$ on $P^1$ (the group $D_2$ acts trivially on the $(PGL_2, \mathcal{A}_4)$-slice $P^1 \subset P^4$ and similarly for $Gr(2, V_3)$,
see Lemma 7.32). Thus the quotient is the same as for the bundle

$$\mathbb{P}^{1} \times \mathbb{C} \times \mathbb{P}^{1} \times \mathbb{C}$$

under the action of $\mathfrak{A}_4 \times \mathfrak{S}_2$. In particular, it is a vector bundle over a $\mathbb{P}^2 = \mathcal{D}_2 \backslash \mathbb{P}^1 \times \mathbb{P}^1 / \mathfrak{S}_2$-fibration over $\mathbb{P}^1 = \mathbb{P}^1 / \mathfrak{C}_3$, hence is rational.

**Case 2.** $|J| \geq 2$. If at least one $d_j$ is odd and $> 1$ or if all $d_j = 1$ and $|J| > 2$, then there is a slice $S$ and the $\text{PGL}_2$-action is $af$. We can write $Y$ as (the total space of the) line bundle:

$$X \xrightarrow{L} S \times \text{PGL}_2$$

and, using Lemma 7.21, reduce to either a vector bundle over

$$\text{PGL}_2 \backslash \text{PGL}_2 \times \text{PGL}_2 / \mathfrak{S}_2,$$

when $L$ is trivial on $\text{PGL}_2$, or to

$$\text{GL}_2 \backslash \text{GL}_2 \times \text{GL}_2 / \mathfrak{S}_2$$

otherwise. In both cases the base is rational by Lemma 7.12.

If $d_j = 1$ for every $j \in J$ and $|J| = 2$ then there is a map

$$(\mathbb{P}^1)^4 \rightarrow \mathbb{P}(V_4) = \text{Sym}^4(\mathbb{P}^1) = \mathbb{P}^4$$

(of degree 24, mapping 4 points to a form of degree 4). The preimage in $(\mathbb{P}^1)^4$ of the $(\text{PGL}_2, \mathfrak{S}_4)$-slice $\mathbb{P}_s = \mathbb{P}^1$ of $\mathbb{P}^4$, will be a set of six lines $\mathbb{P}^1_{g,h}$, labeled by a pair of generators $g, h \in \mathcal{D}_2$ (which act trivially on $\mathbb{P}^1_s \subset \mathbb{P}^4$). More precisely, the line $\mathbb{P}^1_{g,h}$ is the set given by $(x : gx : hx : ghx) \in (\mathbb{P}^1)^4$, for $x \in \mathbb{P}^1$. The map $\mathbb{P}^1_{g,h} \rightarrow \mathbb{P}^1_s = \mathbb{P}^1_{t,s} / \mathcal{D}_2$ has degree 4. Thus $\mathbb{P}^1_{g,h}$ is a $(\text{PGL}_2, \mathcal{D}_2)$-slice of $(\mathbb{P}^1)^4$ and the quotient of a vector bundle $\xrightarrow{L \equiv L} \mathbb{P}^1$ by a linear action of $\mathcal{D}_2$ is rational.

Assume that all $d_i$ are even. Then $L$ is (birationally) trivial. Unless $|J| = 2$ and $d_1 = d_2 = 2$, there is a decomposition of

$$Y \times Y = \mathbb{P}(V_d) \times Y' \times \mathbb{P}(V_d) \times Y'$$

such that the $\text{PGL}_2$-action is $af$ and

$$\mathbb{P}(V_d) \times Y' \times \mathbb{P}(V_d) \times Y' \sim_{\text{PGL}_2 \times \mathfrak{S}_2} (Y' \times Y') \times (\mathbb{P}(V_d) \times \mathbb{P}(V_d))$$
(with trivial PGL$_2$-action on $\mathbb{P}(V_4)$), by Lemma 7.19. The quotient is birational to a vector bundle over $\text{PGL}_2 \times \mathbb{C}^* \setminus X \times X'/\mathcal{G}_2$, where $X'$ is the trivial line bundle over $Y'$.

We have reduced to $|J| = 1$ treated in Case 1 or to $|J| = 2$ and $d_1 = d_2 = 2$, treated in Lemma 7.38. □

**Lemma 7.38.** The quotient
\[
X := \text{PGL}_2 \setminus (\mathbb{P}_1(V_2) \times \mathbb{P}_2(V_2) \times \mathbb{P}_1(V_2) \times \mathbb{P}_2(V_2))/\mathcal{G}_2
\]
is rational, where $\mathbb{P}_1(V_2)$ and $\mathbb{P}_2(V_2)$ are different copies of $\mathbb{P}^2 = \mathbb{P}(V_2)$ and $\mathcal{G}_2$ acts by permutation.

**Proof.** Consider the projection
\[
X \to \text{PGL}_2 \setminus \mathbb{P}_1(V_2) \times \mathbb{P}_1(V_2)/\mathcal{G}_2
\]
and the PGL$_2 \times \mathcal{G}_2$-equivariant map of degree 6
\[
pr : \mathbb{P}(V_2) \times \mathbb{P}(V_2) \to \mathbb{P}(V_4)
\]
\[
(\Omega_1, \Omega_2) \mapsto \Omega_1 \cdot \Omega_2.
\]
The space $\mathbb{P}(V_4)$ has a (PGL$_2$, $\mathcal{G}_4$)-slice $\mathbb{P}^1_s$ (the $\mathcal{D}_2$-invariant polynomials). The zeroes of a (polynomial) $p \in \mathbb{P}^1_s$ form an orbit under $\mathcal{D}_2$. The preimage $pr^{-1}(\mathbb{P}^1_s) \subset \mathbb{P}^2 \times \mathbb{P}^2$ consists of 3 lines, each invariant under $\mathcal{D}_2$. Indeed, the ordered pair $(\Omega_1, \Omega_2)$ corresponds to a choice of a generator $g \in \mathcal{D}_2$ such that $x, g(x)$ are zeroes of $\Omega_1$ and $h(x), hg(x)$ are zeroes of $\Omega_2$. Thus the line $\mathbb{P}^1_s \subset \mathbb{P}^2 \times \mathbb{P}^2$ consists of tupels \{(x, gx), (hx, ghx)\}, where $x$ is an arbitrary point in $\mathbb{P}^1$ and $(x, gx) = \Omega_1, (hx, ghx) = \Omega_2$. The map $\mathbb{P}^1_s \to \mathbb{P}^1_s$ has degree two and its fibers coincide with orbits of $h$ (since $g$ acts trivially on $\mathbb{P}^1_s$). The action of $h$ is given by
\[
h : \{(x, gx), (hx, ghx)\} \mapsto \{(hx, ghx), (x, gx)\}.
\]
Thus $h(\Omega_1, \Omega_2) = (\Omega_2, \Omega_1)$ and the action of $h$ coincides with the restriction of the permutation action on $\mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^1_s$. The line $\mathbb{P}^1_g$ is invariant under $\mathcal{D}_4 \times \mathcal{G}_2$ (considered as a subgroup of (PGL$_2 \times \mathcal{G}_2$)). The group $\mathcal{G}_4$ permutes the lines in $pr^{-1}(\mathbb{P}^1_s)$. Each $\mathbb{P}^1_g$ is a (PGL$_2 \times \mathcal{G}_2$, $\mathcal{D}_4 \times \mathcal{G}_2$)-slice of $\mathbb{P}^2 \times \mathbb{P}^2$. Therefore,
\[
X \sim \mathcal{D}_4 \setminus \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2/\mathcal{G}_2.
\]
The space $\mathbb{P}^2 \times \mathbb{P}^2$ contains a subspace $\mathbb{C}^2 \times \mathbb{C}^2$ with a linear action of $\mathcal{D}_4 \times \mathcal{G}_2$. Indeed, the action of $\mathcal{D}_4$ on $\mathbb{P}^1$ corresponds to the irreducible representation of $\mathcal{D}_4$ on $\mathbb{C}^2 = V$. Under the $\mathcal{D}_4$-action, one has a decomposition $\text{Sym}^2(V) = V' \oplus V''$, where $\dim V' = 2$, $\dim V'' = 1$ and the action of $\mathcal{D}_4$ on $\mathbb{P}^2$ is equivalent
to the linear action on $V'$. The additional $S_2$ permutes the $\mathbb{P}^2$ and hence acts by permutation on $V' \oplus V'$. Thus

$$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \sim_{\mathcal{D}_4 \times S_2} V' \oplus V'$$

(a vector bundle).

Consider the effective action of (the nonabelian group) $\mathcal{D}_4 \times S_2$ on $\mathbb{P}^1$. It has a normal subgroup $\mathcal{D}_2 \times S_2$ with generators $g, h, k$ and an element $i, i^2 = 1$ which commutes with $g, k$ and acts on $h$ as $ihi = gh$. The stabilizer of a generic point on $\mathbb{P}^1_g$ is a normal abelian subgroup generated by $g, hk$. Thus $\mathcal{D}_4 \times S_2$ acts on $\mathbb{P}^1_g$ effectively through the quotient $\mathcal{D}_4 / \langle g, hk \rangle = \mathcal{D}_2$. The action of this $\mathcal{D}_2$ on $\mathbb{P}^1_g$ is almost free. Indeed, the action of $k$ coincides with the action of $h$ and permutes $Q_1, Q_2$. Thus the orbits of $h$ and $k$ on $\mathbb{P}^1_g$ coincide with fibers of the map $\mathbb{P}^1_g \to \mathbb{P}^1$. On the other hand, $i$ acts nontrivially on $\mathbb{P}^1_g$. We claim that

$$\mathcal{D}_4 \backslash (V' \oplus V') \times \mathbb{P}^1 / S_2$$

$$\mathcal{D}_4 \backslash (V' \times \mathbb{P}^1)$$

is a vector bundle. Indeed, consider the subspace $V'_{inv} \subset V' \oplus V'$ of invariant vectors (under the permutation). The action of $\mathcal{D}_4 \times S_2$ on $((V' \oplus V') / V'_{inv}) \times \mathbb{P}^1$ is almost free. Hence

$$\mathcal{D}_4 \backslash (V' \oplus V') \times \mathbb{P}^1 / S_2$$

$$\mathcal{D}_4 \backslash ((V' \oplus V') / V'_{inv}) \times \mathbb{P}^1 / S_2$$

is a vector bundle with base a quotient of the vector bundle $(V' \oplus V'/V'_{inv}) \to \mathbb{P}^1_g$ by $\mathcal{D}_4 \times S_2$. The variety $(V' \oplus V'/V'_{inv}) \times \mathbb{P}^1$ has a fiberwise (scalar) $\mathbb{C}^*$-action commuting with the $\mathcal{D}_4 \times S_2$-action. Since every $\mathbb{C}^*$-action has a slice,

$$X' := \mathcal{D}_4 \backslash ((V' \oplus V') / V'_{inv}) \times \mathbb{P}^1 / S_2,$$

is rational by dimensional reasons: $X'/\mathbb{C}^*$ is a unirational, therefore, rational surface and

$$X' \sim (X'/\mathbb{C}^*) \times \mathbb{C}^*.$$

□


**Proposition 7.39.** Let $X$ be $V \oplus V$, where $V = V_d$ is an irreducible $GL_2$-representation, $\ell > 0$ and $H \subset SL_2$ with $H \neq \mathfrak{A}_5$. Then

$$GL_2 \backslash X \times \mathbb{P}(V_{\ell})/H$$

is rational (where $H$ acts trivially on $\mathbb{P}(V_{\ell})$).

**Proof.** If $\ell$ is even and the action of $GL_2$ or a quotient of $GL_2$ by a central subgroup is $af$ then we apply Lemma 7.19 combined with Proposition 7.36, resp. 7.37.

If $\ell$ is odd and the action is $af$ then there exists a slice, which is a rational variety, by Lemma 7.31 resp. 7.30. Rationality follows.

Now we assume that the action is not $af$. This means that $d \leq 4$. The subcases with $d \leq 2$ are trivial since the action on the corresponding Grassmannian is transitive. If $\ell$ is odd, then the $PGL_2$-action on $Gr(2, V) \times \mathbb{P}(V_{\ell})$ has a rational slice and our claim follows.

If $d = 3$, the action of $PGL_2$ on $Gr(2, V_3)$ has a $(PGL_2, \mathfrak{A}_4)$-slice $\mathbb{P}^1$. For even $\ell > 0$ the action of $\mathfrak{A}_4$ on $\mathbb{P}^\ell$ is faithful and it lifts to a linear representation of $\mathfrak{A}_4$. Further, $\mathfrak{A}_4$-acts on $\mathbb{P}^1$ is through a cyclic quotient. Thus

$$(\mathbb{P}^1 \times \mathbb{P}(V_{\ell})) \sim_{\mathfrak{A}_4} \mathbb{P}^1 \times \mathbb{P}(V_{\ell})$$

with trivial $\mathfrak{A}_4$-action on the $\mathbb{P}^1$ on the right. This implies that the quotient is equivalent to

$$\mathbb{P}^1 \times (\mathbb{P}^\ell/\mathfrak{A}_4) \times (V_1 \oplus V_1)/\mathbb{C}^* \times H,$$

a product of rational varieties.

If $d = 4$, the action of $PGL_2$ on $Gr(2, V_4)$ has a $(PGL_2, N_T)$-slice $X'$. The action of $N_T$ on $\mathbb{P}(V_{\ell})$ is linear and the quotient of $X \times \mathbb{P}^\ell$ is a vector bundle over the quotient of $X$, which is rational. \hfill $\square$

**Proposition 7.40.** Let $X = (\overset{L}{\rightarrow} Y)^2$, where $Y = \prod_{j \in J} \mathbb{P}(V_{d_j})$ and $\ell > 0$. Then

$$GL_2 \backslash X \times \mathbb{P}(V_{\ell})/\mathfrak{S}_2$$

is rational (where $\mathfrak{S}_2$ acts trivially on $\mathbb{P}(V_{\ell})$ and by permutation on $X$).

**Proof.** The same argument as in the proof of Proposition 7.39 shows that it suffices to assume that the action on $X$ is not $af$. This happens only if $Y = \mathbb{P}^2$ or $\mathbb{P}^1$. The case $Y = \mathbb{P}^2$ reduces to Proposition 7.39 (Grassmannian). If $Y = \mathbb{P}^1$ then the action of $PGL_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is transitive and

$$GL_2 \backslash X \times \mathbb{P}(V_{\ell})/\mathfrak{S}_2 \sim (\mathbb{C}^*/\mathbb{P}(V_{\ell})) \times (\mathbb{C}^2/\mathbb{C}^* \times \mathfrak{S}_2),$$

a rational variety. \hfill $\square$
8. Special rationality results

In this section we collect rationality results for spaces of rational maps \( \mathbb{P}^1 \to \mathbb{P}^1 \) with prescribed (special) ramification over exactly three distinguished points \((0, 1, \infty)\) and unspecified ramifications over other points.

Let \( \mathcal{R}(r_0, r_1, r_\infty) \) be the space of rational maps \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) with local ramification data (vectors) \( r_0, r_1, r_\infty \) over the points \( 0, 1, \infty \).

**Proposition 8.1.** Assume that \( (r_0, r_1, r_\infty) \) satisfies one of the following:

- all entries of the vectors \( r_0, r_\infty \) are even and some fixed number of entries of \( r_1 \) is even;
- all entries of the vectors \( r_0, r_\infty \) are even and a fixed number of entries of \( r_1 \) is divisible by \( 3 \);
- all entries of the vectors \( r_0, r_\infty \) are divisible by \( 3 \) and all entries of \( r_1 \) are even.

Then \( \mathcal{R}(r_0, r_1, r_\infty) \) is a finite union of irreducible rational varieties.

**Proof.** In these cases the map \( f = f_0/f_\infty \) is given by coprime polynomials satisfying the equations:

\[
\begin{align*}
    f_0^2 - f_\infty^2 &= g_1^2 g'_1; \\
    f_0^2 - f_\infty^2 &= g_2^3 g'_1; \\
    f_0^3 - f_\infty^3 &= g_1^2 g'_1,
\end{align*}
\]

where \( g'_1 \) is an arbitrary polynomial. The first equation leads to

\[
(f_0 - f_\infty)(f_0 + f_\infty) = g_1^2 g'_1
\]

and, by coprimality, to

\[
\begin{align*}
    f_0 - f_\infty &= g_{11}^2 g'_{11}, \\
    f_0 + f_\infty &= g_{12}^2 g'_{12},
\end{align*}
\]

with arbitrary \( g_{11}, g'_{11}, g_{12}, g'_{12} \) (satisfying the obvious degree conditions) — a union of rational varieties.

The second case is analogous. Consider the third case: since \( f_0^3 - f_\infty^3 \) is a square we obtain

\[
\begin{align*}
    f_0 - f_\infty &= g_1^2, \\
    f_0 - \zeta f_\infty &= g_2^3, \\
    f_0 - \zeta^2 f_\infty &= g_3^2
\end{align*}
\]

(where \( \zeta^3 = 1 \)) and we need to solve

\[
\frac{2\zeta}{1 + \zeta} g_1^2 + \frac{1 - \zeta}{1 + \zeta} g_2^3 = g_3^2.
\]

Now we apply the parametrization as above. \( \Box \)
Corollary 8.2. Let $\mathcal{R}(r_0, r_1, r_\infty)$ be as in 8.1. Then

$$\text{PGL}_2 \backslash \mathcal{R}(r_0, r_1, r_\infty)$$

is rational.

Proof. We have established an explicit parametrization of $\mathcal{R}(r_0, r_1, r_\infty)$ as a direct sum of spaces of polynomials (with different weights as irreducible $\text{GL}_2$-representations). By the theorem of Katsylo 7.14, the corresponding quotients are rational.

Remark 8.3. Only the first case with $g'_1 = 1$ can admit a nontrivial action of $H_\Gamma$ (which necessarily is $\mathbb{Z}/3$). But even in this case the action of $\mathbb{Z}/3$ is linear and it commutes with the action of $\text{GL}_2$ on pairs of polynomials. Lemma 7.31 implies rationality.

Lemma 8.4. Every irreducible component of the variety $\mathcal{R}$ of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 5 and prescribed global ramification datum

$$\text{RD}(f) = [(2, 2, 1)_0, (2, 2, 1)_1, (2, 2, 1)_\infty, (2), (2)]$$

is rational.

Proof. Changing the variables (fixing two ramification points over $1 \in \mathbb{P}^1$ as 0, $\infty$), we can write $f = F_1/F_2$ where

$$F_1(x) = \hat{f}_1(x)^2 \hat{a}_1(x)^2 \hat{b}_1(x)$$
$$F_2(x) = \hat{f}_2(x)^2 \hat{a}_2(x)^2 \hat{b}_2(x)$$

where $\hat{f}_1, \hat{f}_2, \hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2$ are linear forms in $x$. Since the leading coefficients of $F_1$ and $F_2$ are equal we can assume that they are both equal to 1 and write $\hat{f}_1(x) = x + f_1, \ldots, \hat{f}_2(x) = x + b_2$, with some nonzero constants $f_1, \ldots, b_2$. Since we have one free parameter (under the action of $\text{PGL}_2$) we can assume that $b_1 = 1$. Thus

$$\hat{f}_1(x)^2 \hat{a}_1(x)^2 \hat{a}_2(x) - \hat{f}_2(x)^2 \hat{b}_1(x)^2 \hat{b}_2(x) = \sum g_i x^i = c_1 x^2(x + c_2)$$

with arbitrary constants $c_1, c_2$. We get a system of equations on the coefficients $g_i$:

$$g_4 = 0, g_1 = 0, g_0 = 0.$$

Remark that the coefficients of $g$ are symmetric functions on pairs $(f_1, a_1)$ and $(f_2, a_2)$. To parametrize $\mathcal{R}$ we introduce the following variables:

$$X_1 = a_1 + f_1, Y_1 = a_1 f_1, X_2 = f_2 + a_2, Y_2 = f_2 a_2, b_1, b_2.$$
Write the equations on the coefficients $g_j$ as
\begin{align*}
2X_1 + b_1 &= 2X_2 + b_2 \\
Y_1^2 b_1 &= Y_2^2 b_2 \\
Y_1^2 + 2X_1Y_1 b_1 &= Y_2^2 + 2X_2Y_2 b_2.
\end{align*}
Since $b_1 = 1$, for a fixed $b_2$ we get
\begin{align*}
2X_1 + 1 &= 2X_2 + b_2 \\
Y_1 &= \pm \sqrt{b_2}Y_2 \\
b_2Y_2 + 2\sqrt{b_2}X_1 &= Y_2 + 2X_2 b_2.
\end{align*}
This is a union of two (affine) lines. After a rational covering ($\sqrt{b_2}$) our surface is (rationally) a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, a rational surface.

**Lemma 8.5.** Every irreducible component of the variety $\mathcal{R}$ of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 and ramification datum
\[
\text{RD}(f) = [(2, 2)_0, (2, 1, 1)_1, (2, 1, 1)_{\infty}]
\]
is a rational surface.

**Proof.** Using the $\text{PGL}_2$-action on the preimage $\mathbb{P}^1$ we can assume that the points $(2, 2)$ are $+1, -1$, respectively, and that the point of degree 2 (in the local ramification datum $(2, 1, 1)$) over 0 is $\infty$. Thus we can write
\[
(x^2 - 1)^2 - c(x + c_1)(x + c_2)(x + c_3)^2 = g_2(x),
\]
where $g_2$ is an arbitrary polynomial of degree 2 and $c$ is some constant. We get two equations
\[
c = 1,
\]
\[
c_1 + c_2 + 2c_3 = 0.
\]
Thus we have a (rational) surjection of $\mathbb{P}^2$ onto $\mathcal{R}$.

**Lemma 8.6.** Every irreducible component of the variety $\mathcal{R}$ of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 with ramification datum
\[
\text{RD}(f) = [(2, 2)_0, (3, 1)_1, (2, 1, 1)_{\infty}, (2), (2)]
\]
is a rational curve.

**Proof.** A generic map with this ramification datum is given by the equation $f = f_1/f_2$, where
\[
f_1 = (x^2 - 1)^2, \quad f_2 = (x + c_1)(x + c_2)(x + c_3)^2
\]
and
\[
f_1 - f_2 = (x^2 - 1)^2 - c(x + c_1)(x + c_2)(x + c_3)^2 = g_1(x),
\]
where \( g_1(x) \) is linear. Thus \( c = 1 \) and
\[
\begin{align*}
c_1 + c_2 + 2c_3 &= 0, \\
c_1c_2 + 2c_1c_3 + 2c_2c_3 + c_3^2 &= 0,
\end{align*}
\]
clearly rational. \( \square \)

Lemma 8.7. The irreducible component of the variety \( \mathcal{R} \) of rational maps \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree 3 with ramification datum
\[
\text{RD}(f) = [(2, 1)_0, (2, 1)_1, (2, 1)_\infty, (2)]
\]
is a rational curve.

Proof. Reduces easily to the rationality of a cuspidal cubic curve. \( \square \)

9. Rationality of moduli

Theorem 9.1. Any connected component of a moduli space of rational or K3 elliptic surfaces with fixed monodromy group is rational.

Proof. In Proposition 3.11 we have identified (Zariski open subsets of) the corresponding moduli spaces \( \mathcal{F}_{r, \Gamma} \) as quotients (by the left \( \text{PGL}_2 \) and right \( H_\Gamma \)-action)
\[
\text{PGL}_2 \backslash \mathcal{U}_{r, \Gamma, \ell}/H_\Gamma.
\]
Here
\[
\mathcal{U}_{r, \Gamma, \ell} \sim \text{PGL}_2 \times H_\Gamma \times \text{Sym}^\ell(\mathbb{P}^1) \times \mathcal{R}_\Gamma
\]
and
\[
\mathcal{R}_\Gamma = \{ f : \mathbb{P}^1 \to \mathbb{P}^1 \}
\]
is the space of rational maps (with prescribed ramification). For elliptic rational or K3 surfaces \( \ell \leq 3 \) and \( H_\Gamma \) is either trivial, cyclic, dihedral or a subgroup of \( \mathcal{S}_4 \) (see Corollary 3.14). The actions of \( \text{PGL}_2 \) and \( H_\Gamma \) commute and \( H_\Gamma \) acts only on \( \mathcal{R}_\Gamma \).

First we consider general families:
\[
\text{ET}(\mathcal{E}) - 12\ell = \deg(j_\mathcal{E}) \text{ET}(\Gamma).
\]
For \( \mathbf{d} = (d_1, ..., d_k) \in \mathbb{N}^k \) we put
\[
\mathbb{P}^\mathbf{d} := \prod_{j=1}^k \mathbb{P}(V_{d_j}).
\]
Recall that \( \mathcal{R}_\Gamma \) is (birationally) the total space of a line bundle over the space
\[
\mathbb{P}^\mathbf{d} \times \mathbb{P}^\mathbf{d'},
\]
where $\sum_{j=1}^{k} d_j = \sum_{j=1}^{k'} d'_j$.

**Case 1.** $d \neq d'$. Then, by $3.14$, $H_{\Gamma} = 1$ and rationality of $\text{PGL}_2 \backslash \mathcal{R}_{\Gamma}$ (in all cases) follows from the rationality of $\text{PGL}_2 \backslash \mathbb{P}^d \times \mathbb{P}^{d'}$, which is the theorem of Katsylo $7.14$.

**Case 2.** $d = d'$ and $k \geq 2$. By Corollary $3.14$, $H_{\Gamma} = S_2$ (permutation of the factors). This case is covered by Proposition $7.37$.

**Case 3.** $d = d' = (d)$. This case is covered by Proposition $7.36$.

Now we discuss the *special families*:

$$\text{ET}(\mathcal{E}) - 12\ell < \deg(j_\mathcal{E}) \text{ET}(\Gamma).$$

We use the classification of these families established in Section 5. All families listed in Lemma 5.2 are covered by Propositions 7.37 and the Theorem 7.14. Consider the families listed in Lemma 5.3: Lemma 7.30 covers the cases $j_1, j_4, j_5, j_6, j_{13}$. The case $j_2, j_8$ and $j_{12}$ are covered by Proposition 8.1, $j_3$ by Lemma 8.4, $j_7, j_9, j_{10}$ by 7.1 and 8.3, $j_{11}$ by Lemma 8.5. The case $j_{14}$ is covered by Lemma 8.7. Finally, the families $j_{15}$ and $j_{16}$ (listed in Lemma 6.2) are covered by Proposition 7.37 and the remaining families $j_{17} - j_{20}$ by Theorem 7.14.

**Remark 9.2.** Our methods extend to some moduli spaces of elliptic surfaces with higher Euler characteristic. In particular, the results of Section 8 imply that any moduli space of Jacobian elliptic surfaces over $\mathbb{P}^1$ such that a generic surface in this space has only singular fibers of multiplicative type is rational. However, we expect that there are nonrational moduli spaces already for Euler characteristic 36.

10. Pictures

In this section we give a combinatorial description of monodromy groups of elliptic K3 surfaces. More precisely, we describe a simple procedure which allows to enumerate all the possible graphs $\Gamma$ with given $\text{ET}(\Gamma)$. Let $\mathcal{E} \to \mathbb{P}^1$ be an elliptic K3 surface. We have shown in Section 4 that

$$48 = \text{ET}(\mathcal{E}) \geq \text{ET}(\Gamma)$$
and that $\text{ET}(\Gamma)$ is divisible by 12. Thus $\text{ET}(\Gamma)$ equals 12, 24, 36 or 48 and all possible $\Gamma \subset \text{PSL}_2(\mathbb{Z})$ are described by connected trivalent graphs $T_\Gamma$ with $\leq 8$ edges embedded into $S^2$, with an arbitrary bicoloring of the ends.

**Case** $\text{ET}(\Gamma) = 12$ : There is only one tree $T_{12}$ with $\text{ET}(T_{12}) = 12$

**Figure 1.** The tree $T_{12}$.

The ends of $T_{12}$ can be either $A$ or $B$-vertices. To obtain all possible graphs $T_\Gamma$ with $\text{ET}(\Gamma) = 12$ we just need to attach to $T_{12}$ a single loop $L$.

**Figure 2.** The loop $L$.

This gives the following list of graphs:

**Figure 3.** The case $\text{ET}(\Gamma) = 12$.

There is only one saturated graph from the list above which has no outer loops (Figure 4).

**Figure 4.**

This graph will be a basic building block in the construction of graphs with $\text{ET}(\Gamma) > 12$ - we will attach trees and loops to its edges. The edges are numbered to simplify the count of all possible outcomes.
Case $\text{ET}(\Gamma) = 24$: Again, we have only one topological tree $T_{24}$ with $\text{ET}(T) = 24$:

\begin{center}
\includegraphics[width=0.2\textwidth]{t24.png}
\end{center}

\textbf{Figure 5.} The tree $T_{24}$.

Case $\text{ET}(\Gamma) = 36$: There are only 3 saturated graphs without end-loops (modulo equivalent embedding into the sphere):

\begin{center}
\includegraphics[width=0.5\textwidth]{t36.png}
\end{center}

\textbf{Figure 6.} The case $\text{ET}(\Gamma) = 36$.

Any other graph is either a tree or a sum of a saturated graph $T'$ with $\text{ET}(T') = 0, 12, 24$ with trees (with complementary ET). There is only one topological tree $T_{36}$ with $\text{ET}(T_{36}) = 36$.

\begin{center}
\includegraphics[width=0.2\textwidth]{t36.png}
\end{center}

\textbf{Figure 7.}

The number of possible markings of the tree or loops at the ends is 81 but due to the symmetry of the graph the actual number of graphs $T_\Gamma$ corresponding to different placement of loops at the end and markings is smaller: there are 34 different $T_\Gamma$ of this type.

The number of markings of $T_{36}$ is 16 but due to its symmetry the number of different graphs $T_\Gamma$ is 7. (Recall that two graphs $T_\Gamma$ give the same $\Gamma$ modulo conjugation if they are isotopic in a $S^3$).

The graphs of tree type with one end loop are topologically equivalent to:
There are 8 possible markings of the above graph and they all give different $T_\Gamma$ with $ET(\Gamma) = 36$. We have 12 different $T_\Gamma$ with 2 end-loops, 6 with 3 end-loops and one with 4 end-loops.

All topological graphs which are sums of a loop and a tree can be obtained by placing a loop into a tree. Thus there are two types:

![Figure 8.](image)

This gives 8 graphs $T_\Gamma$ in the first case and 4 in the second case.

**Case** $ET(\Gamma) = 48$: We have one tree $T_{48}$ with $ET(\Gamma) = 48$:

![Figure 9. The tree $T_{48}$.](image)

Here is the list of all saturated graphs with $ET(\Gamma) = 48$.

![Figure 10. Saturated graphs in the case $ET(\Gamma) = 48$.](image)

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