The Covariant Superstring on K3

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Introduction

Covariant superstrings

Compactification

Dimensional reduction of the pure spinor

Conclusion
Introduction

- We want to use superstrings as a calculus for quantum field theories coupled to quantum gravity.
- The simplest definition of this calculus is as a free ten-dimensional string.
- To use this tool for lower-dimensional physics, we need to reduce from 10D.
- First method: Holographic (Brenno’s talk here last week)
- Second method: Compactification
- The compactification problem of relating the pure spinor in 10D to the RNS string in lower dimensions has resisted solution since 2002.
- We will solve this problem for compactification of the pure spinor string to six dimensions on a K3 surface (i.e. the four-dimensional Calabi-Yau manifold).
The Green-Schwarz Superstring

We start with the bosonic string \( x : \Sigma \to M \) with dynamics determined by

\[
S_{\text{bose}} \sim \int_{\Sigma} (\partial x)^2.
\]

Then we add space-time (\( M \)) supersymmetry

\[
\delta x^m = i\epsilon \gamma^m \theta, \quad \delta \theta = \epsilon.
\]

The Green-Schwarz variables are manifestly supersymmetric

\[
\Pi^m = dx^m - i\theta \gamma^m d\theta, \quad d\theta.
\]

And the Green-Schwarz superstring is defined by the action

\[
S_{\text{GS}} \sim \int_{\Sigma} \Pi^2 + S_{\text{WZ}}.
\]

Unfortunately the flat space limit of this string is only free in light-cone gauge so covariant quantization is not straightforward.
Siegel proposed the following modification:

\[ S_{GSS} \sim S_{GS} + \int_{\Sigma} d\theta, \]

where the Siegel constraint is given by

\[ d = p_\theta + i(\theta \gamma^m)\partial x_m + \theta^2 \partial \theta \text{-term}. \]

With this, the string in flat space is free

\[ S_{GSS} \sim \int_{\Sigma} [(\partial x)^2 + p_\theta \partial \theta], \]

So quantization gives the canonical operator product algebra

\[ x(z)x(0) \sim \log |z|^2, \quad p_\theta(z)\theta(0) \sim \frac{1}{z}. \]
Then it is easy to compute the operator product

\[ d(z) d(0) \sim \frac{1}{z} \Pi(0), \]

implying that (half of) the Siegel constraints are 2\textsuperscript{nd}-class constraints. There is also the 1\textsuperscript{st}-class Virasoro constraint

\[ T \sim :\Pi \Pi: + :d\partial \theta: \]

Problem: To use the Dirac procedure, we have to separate the constraints. Is there a Poincaré-covariant way to do this?
Berkovits’ formalism I: Hybrid

Consider the RNS string on a factorized background

\[ M \rightarrow M \times X \]
\[ \text{Hol}(\nabla_M) \rightarrow \text{Hol}(\nabla_M) \times \text{Hol}(\nabla_X) \]

If \( \text{Hol}(\nabla_X) \) is “special” then \( \exists \) map

\[
\begin{array}{ccc}
\text{“space-time”} & \text{“internal”} \\
\text{matter} & x^m, \psi^m & y^i, \bar{y}^\bar{i}, \psi^i, \bar{\psi}^\bar{i} \\
\text{ghosts} & b, c, \beta, \gamma & \downarrow \downarrow \\
\text{Berkovits variables} & x^m, p_\theta, \theta, \rho, \sigma & y^i, \bar{y}^\bar{i}, \psi^i, \bar{\psi}^\bar{i}
\end{array}
\]

with canonical operator products and a factorized algebra of constraints

\[ \text{SCA}_{10} \rightarrow \text{SCA}_{\text{space-time}} \oplus \text{SCA}_{\text{internal}}. \]
Berkovits’ formalism II: Pure spinor

The maximal number of supercharges realizable in this way is 8. Meanwhile, the minimal number of supersymmetries in $D = 10$ is 16.

Therefore, a new method is required to separate the ten-dimensional constraints

$$d(z)d(0) \sim \frac{1}{z} \Pi(0).$$

Borrow trick from harmonic superspace: Introduce bosonic “pure” spinor $\lambda$:

$$(\lambda \gamma^m \lambda) = 0 \quad m = 0, 1, \ldots, 9.$$  

(Together with their momenta, the pure spinors form a 22-dimensional cone over the space $SO(10)/U(5)$ of orthogonal complex structures on $\mathbb{R}^{10}$.) Then define Berkovits differential

$$Q = \oint \lambda d \quad : \quad Q^2 = 0.$$  

Proceed with BRST-type quantization!
Mysteries

- What is the pure spinor? Harmonic? Ghost? Twistor? Mix?
- What is the relation with the hybrid formalism?
- What is the relation (if any) with the pure spinors of generalized geometry?
- How to recover the correct spectrum in compactification with off-shell superspace?
- Many more questions...

Strategy: Compactify on K3 which has the advantages that it
- is simple,
- has a hybrid description,
- has harmonics in its superspace,
- can be compactified further to 4 dimensions on $T^2$ to make K3-fibered Calabi-Yau 3-fold.
K3 orbifold

Start with a torus (very special holonomy)

\[
T^4 \cong \frac{\mathbb{C} \times \mathbb{C}}{\mathbb{Z} \times \mathbb{Z}},
\]

and gauge \(\mathbb{Z}_2 \subset SU(2)_- \hookrightarrow SU(2)_+ \times SU(2)_- \cong Spin(4)\) subgroup of the four-dimensional Lorentz (i.e. rotation) group.

\[
\Rightarrow \quad \text{K3 orbifold } X = T^4/\sim,
\]

with \(2^4 = 16\) fixed points which can be blown up to Eguchi-Hanson spaces \(\mathcal{O}_{\mathbb{P}^1}(-2) \cong T^*S^2\) (gravitational instantons).
Orbifold identification gives new massless “twisted” string states at each orbifold point.

The Casimir energies of the worldsheet fields is given by

\[ E = (-)^F \left[ \frac{1}{48} - \frac{1}{16}(2\theta - 1)^2 \right] \begin{cases} -\frac{1}{24} & \text{real periodic boson} \\ \frac{1}{48} & \text{real anti-periodic boson} \end{cases} \]

Contribution with \( n \) untwisted pure spinors:

\[ E = 6(-\frac{1}{24}) + 4(\frac{1}{48}) + 8(\frac{1}{12}) + 8(-\frac{1}{24}) + n(-\frac{1}{12}) + (11 - n)(\frac{1}{24}) \]

\[ = -\frac{1}{8}(n - 5). \]

Spectrum matches iff \( n = 5 \) \( \Rightarrow \) 6 of the original 11 pure spinors are \( \mathbb{Z}_2 \)-twisted.
Breaking $Spin(10) \rightarrow U(5)$, we can solve the pure spinor constraint

$$\lambda \rightarrow \gamma \quad \gamma u_{[ab]} \quad \gamma u_{[ab]u_{cd]}$$

$$16 = 1 \oplus \overline{10} \oplus 5$$

Dimensionally reducing further $SU(5) \rightarrow SU(3) \times SU(2)$, we find that

$$\overline{10} = (3, 1) \oplus (\bar{3}, 2) \oplus (1, 1)$$

$$u_{[ab]} = u^I, \quad u_{Ia'}, \quad u.$$

This can be rearranged into a six-dimensional spinor $(\lambda^A) = \left( \begin{array}{c} \gamma \\ \gamma u^I \end{array} \right)$, a

“harmonic” iso-spinor: $(u^a) = \left( \begin{array}{c} 1 \\ u \end{array} \right)$, and junk: $(u_{Ia'}) \leftrightarrow 6.$
The Berkovits differential splits

\[ Q = \oint \left( \lambda^A u^a d_{Aa} + \lambda_{Aa'} d^{Aa'} + \Lambda^A d_{A2} \right) = Q_0 + Q_1 + Q_2 \]

were \( Q_m \) has \( m \) factors of 6 in it.

The Hilbert space of string fields forms a chain complex with differential \( Q \) graded by number of 6s

\[ \cdots \xrightarrow{Q} C^\bullet \xrightarrow{Q} C^\bullet \xrightarrow{Q} C^\bullet \xrightarrow{Q} \cdots \]

Claim:

\[ H_Q(C^\bullet) = H_{Q_0}^0(C^\bullet). \]

In words: The physical states are described by the cohomology of a short differential on a small Hilbert space without 6-variables.
The reduced differential factorizes the cohomology further:

\[ Q_0 = q + \delta \]

were

\[ q = \int \lambda^A d_A^+ \quad \text{and} \quad \delta = \int \psi_{aa'} \partial x^{aa'} \]

are both differentials. Here, \( d_A^+ = u^a \left( p_{Aa} + i \theta^B_a \partial x_{AB} + \epsilon_{ABCD} \theta^B_a \theta^C_b \partial \theta^D_b \right) \)
and \( \psi_{aa'} = \gamma u_a \theta_{1a'} \).

By Dolbeault’s theorem, the cohomology of the double complex becomes the relative cohomology

\[ H_{Q_0} \approx H_\delta(H_q). \]
We have found that the spectrum of the pure spinor formalism on K3 is computed by the relative cohomology $H_{\delta}(H_q)$.

We will review below that $H_q$ is the space of projective superfields.

Meanwhile, $H_{\delta}$ is the ring of vertex operators of the form $\omega_{aa' bb'} \ldots \psi^{aa'} \psi^{bb'} \ldots$ which are annihilated by $\delta = \psi^{aa'} \partial_{aa'}$ and not $\delta$-exact.

In other words, $H_{\delta} \approx H_{dR}$ is just the de Rham cohomology of K3.

Hence, $H_{\delta}(H_q)$ is the cohomology algebra of K3 vertex operators with projective superfield wave functions.
Projective superspace

The space-time part of the hybrid formalism on K3 lives in projective superspace

\[ \mathbb{R}^{6|8} \times \mathbb{C}P^1 \approx \mathbb{R}^{6|8} \times \frac{(\text{Spin}(6) \times \text{SU}_R(2))}{\text{Spin}(6) \times U(1)_R}. \]

Vertex operators have superspace wave functions \( \Phi(x, \theta, u) \), which can be expanded in harmonics

\[
\Phi(x, \theta, u) = \sum_{n=-\infty}^{\infty} u^n \varphi_n(x, \theta)
\]

(infinite number of ordinary \( \mathcal{N} = (1, 0) \) superfields).

The Siegel constraints act, by the operator product, as superspace derivatives

\[
d_{\alpha i}(z) \Phi(x, \theta, u) = \frac{1}{z} D_{\alpha i} \Phi(x, \theta, u).
\]
The flat, six-dimensional superspace derivatives satisfy
\[ \{D_{\alpha i}, D_{\beta j}\} = 2i\varepsilon_{ij}(\gamma^a)_{\alpha\beta}\partial_a. \]

Introduce zwei-beine \(\{u^\pm_i\}_{i=1}^2\) of \(S^2\) and define \(D^\pm_\alpha = u^{\pm i}D_{\alpha i}\). Then
\[ \{D^+, D^+\} = 0 \]
gives the integrability condition for the “analytic subspace” \(\text{ker}D^+\).

The space parameterized by \((x, \theta^\pm, u^\pm)\) is called “harmonic superspace”.
The analytic subspace parameterized by \((x, \theta^+, u^+)\) and in \(\text{ker}D^+\) is called “projective superspace”.

Returning to the cohomology, the differential
\[ q = \oint \lambda d^+ \]
acts as \(D^+\), implying that string wave functions are projective superfields.
Solving the off-shell problem

Projective superspace is off-shell: Equations of motion not imposed by supersymmetry.

In fact, $H_q = \{0\}$ if the superfield $\Phi$ is not entire on $\mathbb{C}P^1$.

Restriction of $\Phi$ to be an entire function truncates the harmonic expansion and this puts the component superfields on-shell.

To impose this in string theory, we need to bosonize the projective parameter:

$$u = e^{-\rho - i\sigma}$$

in terms of chiral bosons

$$\rho(z)\rho(0) \sim -\log z, \quad \sigma(z)\sigma(0) \sim -\log z.$$ 

To ensure only $\rho/\sigma$-dependence in the combination $u$, vertex operators must have no poles with the constraint $J = \partial(\rho + i\sigma)$ (which generates shifts $(\rho, \sigma) \mapsto (\rho + a, \sigma + ia)$).
Emergence of worldsheet superconformal symmetry

The condition that the harmonic expansion of the vertex has no poles is implied by the constraint

\[ G^- = :e^{-i\sigma} :. \]

Together with the Virasoro constraint

\[ T = :\Pi^2: + :d^+ \partial \theta^- : + \ldots, \]

the algebra closes if we also define

\[ G^+ = -\frac{1}{4!}\epsilon^{ABCD} :d^+_A (d^+_B (d^+_C (d^+_D (e^{2\rho+3i\sigma})))) :. \]

This terminates the expansion and puts the wave function on-shell. The constraints \( \{J, G^\pm, T\} \) generate the desired \( N = 2 \) worldsheet superconformal algebra of the hybrid (a.k.a. RNS) superstring on K3.
Résumé

- We started with the pure spinor superstring in ten flat dimensions.
- We solved the pure spinor condition and rewrote all worldsheet fields in \((6 + 2 + 2^*)\)-variables appropriate to compactification on K3.
- We proved a theorem that the cohomology reduces to a small Hilbert space which removes 6 of the original 11 pure spinor degrees of freedom.
- We interpreted the remaining 4 + 1 pure spinor variables as an unconstrained \(Spin(5, 1)\) spinor and a holomorphic projective variable \(u\) parameterizing \(\mathbb{C}P^1 \approx SU(2)/U(1)\).
- We bosonized \(u\) and defined the constraints \(J\) and \(G^-\) which simplify the description of the physical state conditions.
- We closed the constraint algebra by introducing \(G^+\), thereby recovering the physical state conditions.
- Thus, we succeed in deriving the hybrid string on K3 from the compactification of the ten-dimensional pure spinor string.
Thank you!
The work reported here was done in collaboration with Osvaldo Chandía and Brenno Carlini Vallilo in the following papers:

- O. Chandia, Wm D. Linch III and B. C. Vallilo, “Compactification of the Heterotic Pure Spinor Superstring I,” JHEP 0910, 060 (2009) [arXiv:0907.2247 [hep-th]].
- O. Chandia, Wm D. Linch III, B. C. Vallilo, “Compactification of the Heterotic Pure Spinor Superstring II,” JHEP 1110, 098 (2011). [arXiv:1108.3555 [hep-th]].
- O. Chandia, Wm D. Linch III, B. C. Vallilo, “The Covariant Superstring on K3,” [arXiv:1109.3200 [hep-th]].