# The boundedness of intrinsic square functions on the weighted Herz spaces

Hua Wang *

Department of Mathematics, Zhejiang University, Hangzhou 310027, China

Abstract

In this paper, we will obtain the strong type and weak type estimates of intrinsic square functions including the Lusin area integral, Littlewood-Paley $g$-function and $g_{\lambda}^*$-function on the weighted Herz spaces $K_{q,p}^{\alpha,p}(w_1,w_2)$ ($K_{q,p}^{\alpha,p}(w_1,w_2)$) with general weights.

MSC(2010): 42B25; 42B35

Keywords: Intrinsic square functions; weighted Herz spaces; weighted weak Herz spaces; $A_p$ weights

1 Introduction and main results

Let $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$ and $\varphi_t(x) = t^{-n} \varphi(x/t)$. The classical square function (Lusin area integral) is a familiar object. If $u(x,t) = P_t * f(x)$ is the Poisson integral of $f$, where $P_t(x) = c_n t^{n/2} e^{t|x|^2}$ denotes the Poisson kernel in $\mathbb{R}^{n+1}$. Then we define the classical square function (Lusin area integral) $S(f)$ by (see [4] and [15])

$$S(f)(x) = \left( \int \int_{\Gamma(x)} |\nabla u(y,t)|^2 t^{1-n} \, dydt \right)^{1/2},$$

where $\Gamma(x)$ denotes the usual cone of aperture one:

$$\Gamma(x) = \{ (y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t \}$$

and

$$|\nabla u(y,t)|^2 = \left| \frac{\partial u}{\partial t} \right|^2 + \sum_{j=1}^{n} \left| \frac{\partial u}{\partial y_j} \right|^2.$$

Similarly, we can define a cone of aperture $\gamma$ for any $\gamma > 0$:

$$\Gamma_{\gamma}(x) = \{ (y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \gamma t \},$$

---

*E-mail address: wanghua@pku.edu.cn.
and corresponding square function

\[ S_\gamma(f)(x) = \left( \iint_{\Gamma_\gamma(x)} |\nabla u(y,t)|^2 t^{1-n} \, dy \, dt \right)^{1/2}. \]

The Littlewood-Paley \( g \)-function (could be viewed as a “zero-aperture” version of \( S(f) \)) and the \( g_\lambda^* \)-function (could be viewed as an “infinite aperture” version of \( S(f) \)) are defined respectively by (see, for example, [13] and [14])

\[ g(f)(x) = \left( \int_0^\infty |\nabla u(x,t)|^2 t \, dt \right)^{1/2} \]

and

\[ g_\lambda^*(f)(x) = \left( \iint_{\mathbb{R}^{n+1}} \left( \frac{t}{t + |x-y|} \right)^{\lambda n} |\nabla u(y,t)|^2 t^{1-n} \, dy \, dt \right)^{1/2}, \quad \lambda > 1. \]

The modern (real-variable) variant of \( S_\gamma(f) \) can be defined in the following way (here we drop the subscript \( \gamma \) if \( \gamma = 1 \)). Let \( \psi \in C^\infty(\mathbb{R}^n) \) be real, radial, have support contained in \( \{x : |x| \leq 1\} \), and \( \int_{\mathbb{R}^n} \psi(x) \, dx = 0 \). The continuous square function \( S_{\psi,\gamma}(f) \) is defined by (see, for example, [1] and [2])

\[ S_{\psi,\gamma}(f)(x) = \left( \iint_{\Gamma_\gamma(x)} |f \ast \psi_t(y)|^2 t^{1-n} \, dy \, dt \right)^{1/2}. \]

In 2007, Wilson [22] introduced a new square function called intrinsic square function which is universal in a sense (see also [23]). This function is independent of any particular kernel \( \psi \), and it dominates pointwise all the above-defined square functions. On the other hand, it is not essentially larger than any particular \( S_{\psi,\gamma}(f) \). For \( 0 < \beta \leq 1 \), let \( C_\beta \) be the family of functions \( \varphi \) defined on \( \mathbb{R}^n \) such that \( \varphi \) has support containing in \( \{x \in \mathbb{R}^n : |x| \leq 1\} \), \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 0 \), and for all \( x, x' \in \mathbb{R}^n \),

\[ |\varphi(x) - \varphi(x')| \leq |x - x'|^\beta. \]

For \( (y,t) \in \mathbb{R}^{n+1}_+ \) and \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), we set

\[ A_\beta(f)(y,t) = \sup_{\varphi \in C_\beta} |f \ast \varphi_t(y)| = \sup_{\varphi \in C_\beta} \left| \int_{\mathbb{R}^n} \varphi(y-z) f(z) \, dz \right|. \tag{1.1} \]

Then we define the intrinsic square function of \( f \) (of order \( \beta \)) by the formula

\[ S_\beta(f)(x) = \left( \iint_{\Gamma(x)} \left( A_\beta(f)(y,t) \right)^2 t^{1-n} \, dy \, dt \right)^{1/2}. \tag{1.2} \]

We can also define varying-aperture versions of \( S_\beta(f) \) by the formula

\[ S_{\beta,\gamma}(f)(x) = \left( \iint_{\Gamma_\gamma(x)} \left( A_\beta(f)(y,t) \right)^2 t^{1-n} \, dy \, dt \right)^{1/2}. \tag{1.3} \]
The intrinsic Littlewood-Paley $G$-function and the intrinsic $G_\lambda^*$-function will be given respectively by
\[
G_\beta(f)(x) = \left( \int_0^\infty \left( A_\beta(f)(x,t) \right)^2 \frac{dt}{t} \right)^{1/2}
\]
and
\[
G_{\Lambda,\beta}(f)(x) = \left( \int\int_{\mathbb{R}^{n+1}} \frac{t}{t+|x-y|} \lambda^n \left( A_\beta(f)(y,t) \right)^2 \frac{dydt}{tv^{n+1}} \right)^{1/2}, \lambda > 1. \tag{1.5}
\]

In [23], Wilson showed the following weighted $L^p$ boundedness of the intrinsic square functions.

**Theorem A.** Let $0 < \beta \leq 1, 1 < p < \infty$ and $w \in A_p$ (Muckenhoupt weight class). Then there exists a constant $C > 0$ independent of $f$ such that
\[
\|S_\beta(f)\|_{L^p_w} \leq C \|f\|_{L^p_w}.
\]

Moreover, in [7], Lerner obtained sharp $L^p_w$ norm inequalities for the intrinsic square functions in terms of the $A_p$ characteristic constant of $w$ for all $1 < p < \infty$. For further discussions about the boundedness of intrinsic square functions on various function spaces, we refer the readers to [5, 18, 19, 20, 21].

Before stating our main results, let us first recall some definitions about the weighted Herz and weak Herz spaces. For more information about these spaces, one can see [6, 8, 9, 11, 16] and the references therein. Let $B_k = B(0,2^k) = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for any $k \in \mathbb{Z}$. Denote $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{N}$ and $\tilde{\chi}_0 = \chi^z_{N_0}$, where $\chi_E$ is the characteristic function of the set $E$. For any given weight function $w$ on $\mathbb{R}^n$ and $0 < q < \infty$, we denote by $L^q_w(\mathbb{R}^n)$ the space of all functions $f$ satisfying
\[
\|f\|_{L^q_w} = \left( \int_{\mathbb{R}^n} |f(x)|^q w(x) \, dx \right)^{1/q} < \infty. \tag{1.6}
\]

**Definition 1.1** ([8]). Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$ and $w_1, w_2$ be two weight functions on $\mathbb{R}^n$.

(a) The homogeneous weighted Herz space $K^{\alpha,p}_q(w_1, w_2)$ is defined by
\[
K^{\alpha,p}_q(w_1, w_2) = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n) \setminus \{0\}, w_2 : \|f\|_{K^{\alpha,p}_q(w_1, w_2)} < \infty \right\},
\]
where
\[
\|f\|_{K^{\alpha,p}_q(w_1, w_2)} = \left( \sum_{k \in \mathbb{Z}} \left[ w_1(B_k) \right]^{\alpha p/n} \|f\chi_k\|_{L^q_{w_2}}^p \right)^{1/p}. \tag{1.7}
\]

(b) The non-homogeneous weighted Herz space $K^{\alpha,p}_q(w_1, w_2)$ is defined by
\[
K^{\alpha,p}_q(w_1, w_2) = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n), w_2 : \|f\|_{K^{\alpha,p}_q(w_1, w_2)} < \infty \right\}.
\]
A\nness of intrinsic square functions on weighted Herz spaces with \Lebesgue spaces. The main purpose of this paper is to consider the bounded-

\textbf{Theorem 1.2.}

Let \( \alpha < q < 1 \). Our main results in the paper are formulated as follows. In the extreme case, we will also prove that these operators are bounded from the weighted weak Herz space into the weighted weak \( \mathcal{S} \). Thus, weighted (weak) Herz spaces are generalizations of the weighted (weak) Lebesgue spaces. The main purpose of this paper is to consider the boundedness of intrinsic square functions on weighted Herz spaces with \( A_p \) weights. At the extreme case, we will also prove that these operators are bounded from the weighted Herz spaces to the weighted weak Herz spaces. Our main results in the paper are formulated as follows.

\textbf{Theorem 1.1.} Let \( 0 < \beta \leq 1, 0 < p < q < \infty, w_1 \in A_p, w_2 \in A_q. \) Then \( \mathcal{S}_\beta \) is bounded on \( \mathcal{K}_q^{\alpha,p}(w_1,w_2) \) provided that \( w_1 \) and \( w_2 \) satisfy either of the following:

\begin{enumerate}
  \item \( w_1 = w_2, 1 \leq q_1 = q_2 \leq q \) and \( -nq_1/q < \alpha q_1 < n(1 - q_2/q); \)
  \item \( w_1 \neq w_2, 1 \leq q_1 < \infty, 1 \leq q_2 \leq q \) and \( 0 < \alpha q_1 < n(1 - q_2/q). \)
\end{enumerate}

\textbf{Theorem 1.2.} Let \( 0 < \beta \leq 1, 0 < p \leq 1, 1 < q < \infty, w_1 \in A_p, w_2 \in A_q. \) If \( 1 \leq q_1 < \infty, 1 \leq q \leq q_2 \leq \alpha q_1 = n(1 - q_2/q), \) then \( \mathcal{S}_\beta \) is bounded from \( \mathcal{K}_q^{\alpha,p}(w_1,w_2) \) into \( \mathcal{W}_q^{\alpha,p}(w_1,w_2) \) and \( \mathcal{W}_q^{\alpha,p}(w_1,w_2) \).
Theorem 1.3. Let \(0 < \beta \leq 1, 0 < p < \infty, 1 < q < \infty, w_1 \in A_q, \) and \(w_2 \in A_q\). If \(\lambda > \max\{q_2, 3\}\), then \(G_{\alpha, \beta}^q\) is bounded on \(K_{\alpha, p}^q(w_1, w_2) (K_{\alpha, p}^q(w_1, w_2))\) provided that \(w_1\) and \(w_2\) satisfy either of the following

(i) \(w_1 = w_2, 1 \leq q_1 = q_2 \leq q\) and \(-aq_1/q < \alpha q_1 < n(1 - q_2/q)\);
(ii) \(w_1 \neq w_2, 1 \leq q_1 < \infty, 1 \leq q_2 \leq q\) and \(0 < \alpha q_1 < n(1 - q_2/q)\).

Theorem 1.4. Let \(0 < \beta \leq 1, 0 < p \leq 1, 1 < q < \infty, w_1 \in A_q, \) and \(w_2 \in A_q\). If \(1 \leq q_1 < \infty, 1 \leq q_2 \leq q, \alpha q_1 = n(1 - q_2/q)\) and \(\lambda > \max\{q_2, 3\}\), then \(G_{\alpha, \beta}^q\) is bounded from \(K_{\alpha, p}^q(w_1, w_2) (K_{\alpha, p}^q(w_1, w_2))\) into \(W K_{\alpha, p}^q(w_1, w_2) (W K_{\alpha, p}^q(w_1, w_2))\).

In [27], Wilson also showed that for any \(0 < \beta \leq 1,\) the functions \(S_{\alpha}(f)(x)\) and \(G_{\beta}(f)(x)\) are pointwise comparable, with comparability constants depending only on \(\beta\) and \(n\). Thus, as a direct consequence of Theorems 1.1 and 1.2, we obtain the following:

Corollary 1.5. Let \(0 < \beta \leq 1, 0 < p < \infty, 1 < q < \infty, w_1 \in A_q, \) and \(w_2 \in A_q\). Then \(G_{\alpha}^q\) is bounded on \(K_{\alpha, p}^q(w_1, w_2) (K_{\alpha, p}^q(w_1, w_2))\) provided that \(w_1\) and \(w_2\) satisfy either of the following

(i) \(w_1 = w_2, 1 \leq q_1 = q_2 \leq q\) and \(-aq_1/q < \alpha q_1 < n(1 - q_2/q)\);
(ii) \(w_1 \neq w_2, 1 \leq q_1 < \infty, 1 \leq q_2 \leq q\) and \(0 < \alpha q_1 < n(1 - q_2/q)\).

Corollary 1.6. Let \(0 < \beta \leq 1, 0 < p \leq 1, 1 < q < \infty, w_1 \in A_q, \) and \(w_2 \in A_q\). If \(1 \leq q_1 < \infty, 1 \leq q_2 \leq q, \alpha q_1 = n(1 - q_2/q)\), then \(G_{\beta}^q\) is bounded from \(K_{\alpha, p}^q(w_1, w_2) (K_{\alpha, p}^q(w_1, w_2))\) into \(W K_{\alpha, p}^q(w_1, w_2) (W K_{\alpha, p}^q(w_1, w_2))\).

2 \(A_p\) weights

The classical \(A_p\) weight theory was first introduced by Muckenhoupt in the study of weighted \(L^p\) boundedness of Hardy-Littlewood maximal functions in [12]. A weight \(w\) is a nonnegative, locally integrable function on \(\mathbb{R}^n, B = B(x_0, r_B)\) denotes the ball with the center \(x_0\) and radius \(r_B\). For any ball \(B\) and \(\lambda > 0, \) \(\lambda B\) denotes the ball concentric with \(B\) whose radius is \(\lambda\) times as long. For a given weight function \(w\) and a measurable set \(E,\) we also denote the Lebesgue measure of \(E\) by \(|E|\) and set weighted measure \(w(E) = \int_E w(x) dx\). We say that \(w\) is in the Muckenhoupt class \(A_p\) with \(1 < p < \infty,\) if there exists a constant \(C > 0\) such that for every ball \(B \subseteq \mathbb{R}^n,\)

\[
\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C. \tag{2.1}
\]

For the endpoint case \(p = 1, \) \(w \in A_1,\) if

\[
\frac{1}{|B|} \int_B w(x) dx \leq C \cdot \text{ess inf}_{x \in B} w(x) \quad \text{for every ball } B \subseteq \mathbb{R}^n, \tag{2.2}
\]
where $C$ is a positive constant which is independent of the choice of $B$. The smallest value of $C$ such that the above inequalities hold is called the $A_p$ characteristic constant of $w$ and denoted by $[w]_{A_p}$. If there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$
\left( \frac{1}{|B|} \int_B w(x)^r \, dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) \, dx \right) \quad \text{for every ball } B \subseteq \mathbb{R}^n, \tag{2.3}
$$

then we say that $w$ satisfies the reverse Hölder condition of order $r$ and write $w \in RH_r$.

The following properties for $A_p$ weights will be repeatedly used in this paper.

**Lemma 2.1** ([3]). Let $w \in A_p$ with $p \geq 1$. Then, for any ball $B$, there exists an absolute constant $C > 0$ such that

$$
w(2B) \leq C w(B). \tag{2.4}
$$

In general, for any $\lambda > 1$, we have

$$
w(\lambda B) \leq C \cdot \lambda^{np} w(B), \tag{2.5}
$$

where $C$ does not depend on $B$ nor on $\lambda$.

**Lemma 2.2** ([3, 4]). Let $w \in A_p \cap RH_r$, $p \geq 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that

$$
C_1 \left( \frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left( \frac{|E|}{|B|} \right)^{(r-1)/r} \tag{2.6}
$$

for any measurable subset $E$ of a ball $B$.

Throughout this article, $C$ always denotes a positive constant which is independent of the main parameters involved, but may vary from line to line.

## 3 Proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** We only need to show the theorem for the homogeneous case because the proof of the non-homogeneous result is similar and so is omitted here. Let $f \in K_q^{\alpha,p}(w_1, w_2)$. Following [10], for any $k \in \mathbb{Z}$, we decompose $f(x)$ as

$$
f(x) = f(x) \chi_{\{2^{k-2} < |x| \leq 2^{k+1}\}}(x) + f(x) \chi_{\{|x| \leq 2^{k-2}\}}(x) + f(x) \chi_{\{|x| > 2^{k+1}\}}(x) = f_1(x) + f_2(x) + f_3(x).
$$
For any \( \alpha_p \leq 0 \leq 1 \) is a sublinear operator, then we can write
\[
\|S_{\beta}(f)\|^p_{K_q^{\alpha_p}(w_1,w_2)} = \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha_p/n} \|S_{\beta}(f) \chi_k\|^p_{L^q_{w_2}}
\]
\[
\leq C \sum_{i=1}^3 \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha_p/n} \|S_{\beta}(f_i) \chi_k\|^p_{L^q_{w_2}}
\]
\[
= I_1 + I_2 + I_3.
\]

Since \( w_2 \in A_{q_2} \) and \( 1 \leq q_2 \leq q \), then \( w_2 \in A_q \). By Theorem A and Lemma 2.1, we have
\[
I_1 \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha_p/n} \|f_i\|^p_{L^q_{w_2}}
\]
\[
\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha_p/n} \|f \chi_k\|^p_{L^q_{w_2}}
\]
\[
\leq C \|f\|^p_{K_q^{\alpha_p}(w_1,w_2)}.
\]

For the term \( I_2 \), we first use Minkowski’s inequality to derive
\[
I_2 \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha_p/n} \left( \sum_{t=-\infty}^{k-2} \|S_{\beta}(f \chi_k) \chi_k\|_{L^q_{w_2}} \right)^p.
\]

For any \( \varphi \in C_{\beta} \), \( 0 < \beta \leq 1 \) and \( (y,t) \in \Gamma(x) \), we have
\[
|\langle f \chi_k \rangle \ast \varphi_t(y) | = \left| \int_{2^{t-1} < |z| \leq 2^t} \varphi_t(y - z) f(z) \, dz \right|
\]
\[
\leq C \cdot t^n \int_{\{2^{t-1} < |z| \leq 2^t\} \cap \{y - z \leq t\}} |f(z)| \, dz. \tag{3.1}
\]

For any \( x \in C_k \), \( (y,t) \in \Gamma(x) \) and \( z \in \{2^{t-1} < |z| \leq 2^t\} \cap B(y,t) \) with \( \ell \leq k - 2 \), then by a direct computation, we can easily see that
\[
2t \geq |x - y| + |y - z| \geq |x - z| \geq |x| - |z| \geq |x|/2.
\]

Thus, by using the above inequality (3.1) and Minkowski’s inequality, we deduce
\[
|S_{\beta}(f \chi_k)(x)| = \left( \int_{\Gamma(x)} \left( \sup_{\varphi \in C_{\beta}} \left| \langle f \chi_k \rangle \ast \varphi_t(y) \right| \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}
\]
\[
\leq C \left( \int_{\mathbb{R}^n} \int_{|x - y| < t} \left| t^{-n} \int_{2^{t-1} < |z| \leq 2^t} |f(z)| \, dz \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}
\]
\[
\leq C \left( \int_{2^{t-1} < |z| \leq 2^t} |f(z)| \, dz \right) \left( \int_{\mathbb{R}^n} \frac{dt}{t^{2n+1}} \right)^{1/2}
\]
\[
\leq C \cdot \frac{1}{|x|^n} \left( \int_{2^{t-1} < |z| \leq 2^t} |f(z)| \, dz \right). \tag{3.2}
\]
Denote the conjugate exponent of $q > 1$ by $q' = q/(q - 1)$. Applying Hölder's inequality and the $A_q$ condition, we can deduce that

$$\int_{2^t - 1 < |z| \leq 2^t} |f(z)| \, dz \leq \left( \int_{2^t - 1 < |z| \leq 2^t} |f(z)|^q w_2(z) \, dz \right)^{1/q} \left( \int_{2^t - 1 < |z| \leq 2^t} w_2(z)^{-q'/q} \, dz \right)^{1/q'} \leq C \cdot |B_t| \left[ w_2(B_t) \right]^{-1/q} \|f\chi_t\|_{L^{q}_w} \cdot (3.3)$$

Substituting the above inequality (3.3) into (3.2), we thus obtain

$$I_2 \leq C \sum_{k \in \mathbb{Z}} \left[ w_1(B_k) \right]^{ap/n} \left( \sum_{\ell = -\infty}^{k-2} \left\{ \int_{2^{k-1} < |x| \leq 2^k} |S_\beta(f\chi_t)(x)|^q w_2(x) \, dx \right\}^{1/q} \right)^p \leq C \sum_{k \in \mathbb{Z}} \left[ w_1(B_k) \right]^{ap/n} \left( \sum_{\ell = -\infty}^{k-2} |B_t| \left[ w_2(B_t) \right]^{-1/q} \|f\chi_t\|_{L^{q}_w} \left\{ \int_{2^{k-1} < |x| \leq 2^k} w_2(x)^{1/q} \, dx \right\}^{1/q} \right)^p \leq C \sum_{k \in \mathbb{Z}} \left[ w_1(B_k) \right]^{ap/n} \left( \sum_{\ell = -\infty}^{k-2} \frac{|B_t|}{|B_k|} \cdot \frac{|w_2(B_k)|^{1/q}}{|w_2(B_t)|^{1/q}} \|f\chi_t\|_{L^{q}_w} \right)^p \ .$$

Here, we shall consider two cases. For the case of $0 < p \leq 1$, using the well-known inequality $(\sum_\ell |a_\ell|) \leq (\sum_\ell |a_\ell|)^p$ and changing the order of summation, we find that

$$I_2 \leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_\ell) \right]^{ap/n} \|f\chi_t\|_{L^{q}_w}^p \left( \sum_{k = \ell + 2}^{\infty} \frac{|B_k|^p}{|B_\ell|^p} \cdot \frac{|w_2(B_k)|^{p/q}}{|w_2(B_\ell)|^{p/q}} \cdot \frac{|w_1(B_k)|^{ap/n}}{|w_1(B_\ell)|^{ap/n}} \right) .$$

Moreover, it follows immediately from Lemma 2.1 that

$$I_2 \leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_\ell) \right]^{ap/n} \|f\chi_t\|_{L^{q}_w}^p \left( \sum_{k = \ell + 2}^{\infty} \frac{|B_k|^p}{|B_\ell|^p} \cdot \frac{|w_2(B_k)|^{p/q}}{|w_2(B_\ell)|^{p/q}} \cdot \frac{|w_1(B_k)|^{ap/n}}{|w_1(B_\ell+2)|^{ap/n}} \right) .$$

Since $B_k \supseteq B_{\ell+2}$ when $k \geq \ell + 2$ and $w_i \in A_{q_i}$ for $i = 1, 2$. Then by Lemma 2.2, we can get

$$\frac{w_i(B_k)}{w_i(B_{\ell+2})} \leq C \left( \frac{|B_k|}{|B_{\ell+2}|} \right)^{q_i} , \text{ for } i = 1 \text{ and } 2. \quad (3.4)$$

Therefore

$$I_2 \leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_\ell) \right]^{ap/n} \|f\chi_t\|_{L^{q}_w}^p \left( \sum_{k = \ell + 2}^{\infty} \left[ \frac{|B_k|}{|B_\ell|} \right]^{p - a q_1 p/n - q_2 p/q} \right) \leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_\ell) \right]^{ap/n} \|f\chi_t\|_{L^{q}_w}^p \left( \sum_{k = 0}^{\infty} 2^{-kn(p - a q_1 p/n - q_2 p/q)} \right) \leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_\ell) \right]^{ap/n} \|f\chi_t\|_{L^{q}_w}^p ,$$
where the last inequality holds since $\alpha q < n(1 - q_2/q)$. On the other hand, for the case of $1 < p < \infty$, we will use Hölder’s inequality to obtain

$$
\left( \sum_{k=-\infty}^{k-2} \frac{|B_k|}{|B_k|^2} \cdot \frac{|w_2(B_k)|^{1/q}}{|w_2(B_k)|^{1/q}} \cdot [w_1(B_k)]^{\alpha/n} \|f\chi_x\|_{L^q_{(w_2)}}^p \right)
\leq \left( \sum_{k=-\infty}^{k-2} [w_1(B_k)]^{\alpha/n} \|f\chi_x\|_{L^q_{(w_2)}}^p \frac{|B_k|^{p/2}}{|w_2(B_k)|^{p/2}} \cdot \frac{|w_2(B_k)|^{p/2q}}{|w_2(B_k)|^{p/2q}} \cdot \frac{|w_1(B_k)|^{\alpha p/2n}}{|w_1(B_k)|^{\alpha p/2n}} \right)^{p/p'}. 
$$

Using the same arguments as above, we can also prove the following estimates under the assumption that $\alpha q < n(1 - q_2/q)$.

$$
\sum_{k=\ell+2}^{\infty} \frac{|B_k|^{p/2}}{|B_k|^{p/2}} \cdot \frac{|w_2(B_k)|^{p/2q}}{|w_2(B_k)|^{p/2q}} \cdot \frac{|w_1(B_k)|^{\alpha p/2n}}{|w_1(B_k)|^{\alpha p/2n}} \leq C \tag{3.5}
$$

and

$$
\sum_{\ell=-\infty}^{k-2} \frac{|B_k|^{p/2}}{|B_k|^{p/2}} \cdot \frac{|w_2(B_k)|^{p/2q}}{|w_2(B_k)|^{p/2q}} \cdot \frac{|w_1(B_k)|^{\alpha p/2n}}{|w_1(B_k)|^{\alpha p/2n}} \leq C. \tag{3.6}
$$

Hence

$$
I_2 \leq C \sum_{k\in\mathbb{Z}} \left( \sum_{\ell=-\infty}^{k-2} \frac{|w_1(B_k)|^{\alpha p/n} \|f\chi_x\|_{L^q_{(w_2)}}^p |B_k|^{p/2}}{|w_2(B_k)|^{p/2}} \cdot \frac{|w_2(B_k)|^{p/2q}}{|w_2(B_k)|^{p/2q}} \cdot \frac{|w_1(B_k)|^{\alpha p/2n}}{|w_1(B_k)|^{\alpha p/2n}} \right)
\leq C \sum_{\ell\in\mathbb{Z}} \frac{|w_1(B_k)|^{\alpha p/n} \|f\chi_x\|_{L^q_{(w_2)}}^p \left( \sum_{k=\ell+2}^{\infty} \frac{|B_k|^{p/2}}{|B_k|^{p/2}} \cdot \frac{|w_2(B_k)|^{p/2q}}{|w_2(B_k)|^{p/2q}} \cdot \frac{|w_1(B_k)|^{\alpha p/2n}}{|w_1(B_k)|^{\alpha p/2n}} \right)}
\leq C \sum_{\ell\in\mathbb{Z}} \frac{|w_1(B_k)|^{\alpha p/n} \|f\chi_x\|_{L^q_{(w_2)}}^p}.
$$

Summarizing the above estimates for the term $I_2$, we obtain that for every $0 < p < \infty$,

$$
I_2 \leq C \sum_{\ell\in\mathbb{Z}} \frac{|w_1(B_k)|^{\alpha p/n} \|f\chi_x\|_{L^q_{(w_2)}}^p} \leq C \left\| f \right\|_{K^{\alpha p/n}_{\infty}(w_1, w_2)}^p.
$$

Let us now turn to estimate the last term $I_3$. In this case, for any $x \in C_k$, $(y, t) \in \Gamma(x)$ and $z \in \{2^{\ell-1} < |z| \leq 2^\ell\} \cap B(y, t)$ with $\ell \geq k + 2$, it is easy to check that

$$
2t \geq |x - y| + |y - z| \geq |x - z| \geq |z - |z|| \geq \frac{|z|}{2}.
$$

9
Then it follows from the inequality (3.1) and Minkowski’s inequality that

\[ |S_\beta(f\chi)(x)| \leq C \left( \int_{\mathbb{R}^d} \int_{|y-x|<\ell} t^n \int_{|z|<2^\ell} \left| f(z) \right| dz \right)^{1/2} \left( \int_{\mathbb{R}^d} \frac{dy}{t^{n+1}} \right)^{1/2} \]

\[ \leq C \left( \int_{|z|<2^\ell} \frac{|f(z)|}{|z|^n} dz \right)^{1/2} \left( \int_{|z|<2^\ell} \frac{dt}{t^{2n+1}} \right)^{1/2} \]

This estimate together with (3.3) implies

\[ |S_\beta(f\chi)(x)| \leq C \cdot \frac{1}{|B_\ell|} \left( \int_{2^\ell-1<|z|\leq2^\ell} |f(z)| dz \right) \leq C \cdot \left[ w_2(B_\ell) \right]^{-1/q} \| f\chi \|_{L^q_{\omega_1}}. \]  \tag{3.7}

Hence

\[ I_3 \leq C \sum_{k \in \mathbb{Z}} \left[ w_1(B_k) \right]^{op/n} \left( \sum_{\ell=k+2}^\infty \left\{ \int_{2^{k-1}<|x|\leq2^k} \left| S_\beta(f\chi)(x) \right|^q w_2(x) dx \right\}^{1/q} \right)^p \]

\[ \leq C \sum_{k \in \mathbb{Z}} \left[ w_1(B_k) \right]^{op/n} \left( \sum_{\ell=k+2}^\infty \left[ w_2(B_\ell) \right]^{-1/q} \| f\chi \|_{L^q_{\omega_1}} \left\{ \int_{2^{k-1}<|x|\leq2^k} w_2(x) dx \right\}^{1/q} \right)^p \]

\[ \leq C \sum_{k \in \mathbb{Z}} \left[ w_1(B_k) \right]^{op/n} \left( \sum_{\ell=k+2}^\infty \left[ w_2(B_\ell) \right]^{1/q} \| f\chi \|_{L^q_{\omega_1}} \right)^p. \]

Now we will consider the following two cases again. For the case of $0 < p \leq 1$, by using the inequality \( \left( \sum_\ell |a_\ell|^p \right) \leq \sum_\ell |a_\ell|^p \) and changing the order of summation, we obtain

\[ I_3 \leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_\ell) \right]^{op/n} \| f\chi \|_{L^q_{\omega_1}}^p \left( \sum_{k=-\infty}^{\ell-2} \left[ w_2(B_k) \right]^{p/q} \left\{ \sum_{k=-\infty}^{\ell-2} \left[ w_2(B_k) \right]^{p/q} \left[ w_1(B_\ell) \right]^{op/n} \right\} \right)^p \]

\[ \leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_\ell) \right]^{op/n} \| f\chi \|_{L^q_{\omega_1}}^p \left( \sum_{k=-\infty}^{\ell-2} \left[ w_2(B_k) \right]^{p/q} \left[ w_2(B_{\ell-2}) \right]^{p/q} \left[ w_1(B_\ell) \right]^{op/n} \right)^p. \]

Since $w_i \in A_{\eta_i}$, then there exist $r_i > 1$ such that $w_i \in RH_{r_i}$ for $i = 1, 2$. Thus by Lemma 2.2 again, we can get

\[ \frac{w_2(B_k)}{w_1(B_{\ell-2})} \leq C \left( \frac{|B_k|}{|B_{\ell-2}|} \right)^{\delta_i}, \quad \text{for } i = 1 \text{ and } 2, \]  \tag{3.9}
where $\delta_i = (r_i - 1)/r_i > 0$. Therefore, we have
\[
I_3 \leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_{\ell}) \right]^{\alpha p/n} \| f \ell \|_{L^p_{w_2}} \left( \sum_{k = -\infty}^{\ell-2} \left| \frac{B_k}{|B_{\ell-2}|} \right|^\alpha \delta_i^{p/n + \delta_2 p/q} \right)
\]
\[
\leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_{\ell}) \right]^{\alpha p/n} \| f \ell \|_{L^p_{w_2}} \left( \sum_{k = -\infty}^{0} 2^{kn(\alpha \delta_i^{p/n + \delta_2 p/q})} \right)
\]
\[
\leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_{\ell}) \right]^{\alpha p/n} \| f \ell \|_{L^p_{w_2}},
\]
where in the last inequality we have used the fact that $\alpha \delta_i^{p/n + \delta_2 p/q} > 0$ under our assumption (i) or (ii). On the other hand, for the case of $1 < p < \infty$, an application of Hölder's inequality gives us that
\[
\left( \sum_{\ell = k+2}^{\infty} \left[ \frac{w_2(B_{\ell})}{w_2(B_{k})} \right]^{1/p} \left[ w_1(B_{\ell}) \right]^{\alpha p/n} \| f \ell \|_{L^q_{w_2}} \right)^p
\]
\[
\leq \left( \sum_{\ell = k+2}^{\infty} \left[ \frac{w_2(B_{\ell})}{w_2(B_{k})} \right]^{p/2q} \| f \ell \|_{L^q_{w_2}} \right)^{p/p'} \left( \sum_{\ell = k+2}^{\infty} \left[ \frac{w_2(B_{\ell})}{w_1(B_{\ell})} \right]^{\alpha p/2n} \right)^{p/p'}.
\]
By using the same arguments as for $I_3$, we are able to prove that the following two series is bounded by an absolute constant under the assumption (i) or (ii).
\[
\sum_{k = -\infty}^{\ell-2} \left[ \frac{w_2(B_{k})}{w_2(B_{\ell})} \right]^{p/2q} \left[ \frac{w_1(B_{\ell})}{w_1(B_{k})} \right]^{\alpha p/2n} \leq C. \quad (3.10)
\]
and
\[
\sum_{\ell = k+2}^{\infty} \left[ \frac{w_2(B_{\ell})}{w_2(B_{k})} \right]^{p/2q} \left[ \frac{w_1(B_{\ell})}{w_1(B_{k})} \right]^{\alpha p/2n} \leq C. \quad (3.11)
\]
Consequently
\[
I_3 \leq C \sum_{\ell \in \mathbb{Z}} \left( \sum_{\ell = k+2}^{\infty} \left[ w_1(B_{\ell}) \right]^{\alpha p/n} \| f \ell \|_{L^q_{w_2}} \right)^{p/p'} \left( \sum_{k = -\infty}^{\ell-2} \left[ \frac{w_2(B_{k})}{w_2(B_{\ell})} \right]^{p/2q} \left[ \frac{w_1(B_{\ell})}{w_1(B_{k})} \right]^{\alpha p/2n} \right)^{p/p'}
\]
\[
\leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_{\ell}) \right]^{\alpha p/n} \| f \ell \|_{L^q_{w_2}} \left( \sum_{k = -\infty}^{\ell-2} \left[ \frac{w_2(B_{k})}{w_2(B_{\ell})} \right]^{p/2q} \left[ \frac{w_1(B_{\ell})}{w_1(B_{k})} \right]^{\alpha p/2n} \right)^{p/p'}
\]
\[
\leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_{\ell}) \right]^{\alpha p/n} \| f \ell \|_{L^q_{w_2}}.
\]
From the above discussions for the term $I_3$, we know that for any $0 < p < \infty$,
\[
I_3 \leq C \sum_{\ell \in \mathbb{Z}} \left[ w_1(B_{\ell}) \right]^{\alpha p/n} \| f \ell \|_{L^q_{w_2}} \leq C \| f \|_{K_i^{n, \alpha}(w_1, w_2)}^{p/n}.
\]
Summing up the above estimates for $I_1$, $I_2$ and $I_3$, we complete the proof of Theorem 1.1.

Proof of Theorem 1.2. Let $f \in \dot{K}^{\alpha,p}(w_1,w_2)$. For any $k \in \mathbb{Z}$, as in the proof of Theorem 1.1, we will split $f(x)$ into three parts

$$f(x) = f(x) \chi_{\{2^{k-2} < |x| \leq 2^{k+1}\}}(x) + f(x) \chi_{\{|x| \leq 2^{k-2}\}}(x) + f(x) \chi_{\{|x| > 2^{k+1}\}}(x)$$

$$= f_1(x) + f_2(x) + f_3(x).$$

Then for any given $\lambda > 0$, we have

$$\lambda^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} w_2 \left( \{ x \in C_k : |S_\beta(f)(x)| > \lambda \} \right)^{p/q} \leq \sum_{i=1}^{3} \lambda^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} w_2 \left( \{ x \in C_k : |S_\beta(f_i)(x)| > \lambda/3 \} \right)^{p/q} = I'_1 + I'_2 + I'_3.$$

Applying Chebyshev’s inequality, Theorem A and Lemma 2.1, we obtain

$$I'_1 \leq \lambda^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left( \frac{3^q}{\lambda^q} \left\| S_\beta(f_1) \right\|_{L^q_{w_2}} \right)^{p/q} \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left\| f_1 \right\|_{L^p_{w_2}}^p \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left\| f \chi k \right\|_{L^p_{w_2}}^p \leq C \left\| f \right\|_{\dot{K}^{\alpha,p}(w_1,w_2)}^p.$$

For any $x \in C_k$, it follows from the inequalities (3.2) and (3.3) that

$$|S_\beta(f_2)(x)| \leq \sum_{\ell=-\infty}^{k-2} |S_\beta(f\chi \ell)(x)| \leq C \sum_{\ell=-\infty}^{k-2} \frac{1}{|x|^n} \left( \int_{2^{\ell-1} < |z| \leq 2^{\ell}} |f(z)| \, dz \right) \leq C \sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} [w_2(B_\ell)]^{-1/q} \left\| f \chi \ell \right\|_{L^q_{w_2}}.$$

By using Lemma 2.1, the inequality (3.4) and the fact that $\alpha q_1 = n(1 - q_2/q)$,
we deduce that
\[
|S_\beta(f_2)(x)| \leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n}[w_2(B_k)]^{1/q}} \times \sum_{\ell=-\infty}^{k-2} \left[ \frac{1}{[w_1(B_{\ell})]^{\alpha/n}} \left\| f \chi_{\ell} \right\|_{L_{2}^{q}} \right]^{1/2} \cdot \frac{|B_\ell|}{|B_k|} \cdot \frac{[w_2(B_{\ell})]^{1/q}}{[w_2(B_{\ell+2})]^{1/q}} \cdot \frac{[w_1(B_k)]^{\alpha/n}}{[w_1(B_{\ell+2})]^{\alpha/n}}
\]
\[
\leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n}[w_2(B_k)]^{1/q}} \times \sum_{\ell=-\infty}^{k-2} \left[ \frac{1}{[w_1(B_{\ell})]^{\alpha/n}} \left\| f \chi_{\ell} \right\|_{L_{2}^{q}} \right]^{1/2} \cdot \left( \frac{|B_{\ell+2}|}{|B_k|} \right)^{1-\alpha q/n-\alpha q/2} \cdot \frac{[w_1(B_{\ell})]^{\alpha/n}}{[w_1(B_{\ell+2})]^{\alpha/n}}.
\]

Moreover, since \(0 < p \leq 1\), then we have that for any \(x \in C_k\),
\[
|S_\beta(f_2)(x)| \leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n}[w_2(B_k)]^{1/q}} \left( \sum_{\ell=-\infty}^{k-2} \left[ \frac{1}{[w_1(B_{\ell})]^{\alpha/n}} \left\| f \chi_{\ell} \right\|_{L_{2}^{q}} \right]^{p} \right)^{1/p} \leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n}[w_2(B_k)]^{1/q}} \left\| f \right\|_{K_{q}^{\alpha,p}(w_1,w_2)}^{p}.
\]  

Set \(A_k = [w_1(B_k)]^{-\alpha/n}[w_2(B_k)]^{-1/q}\). If \(\{ x \in C_k : |S_\beta(f_2)(x)| > \lambda/3 \} = \emptyset\), then the inequality
\[
I_2' \leq C \cdot |f|_{K_{q}^{\alpha,p}(w_1,w_2)}^{p}
\]  
holds trivially. Now we suppose that \(\{ x \in C_k : |S_\beta(f_2)(x)| > \lambda/3 \} \neq \emptyset\). First it is easy to verify that \(\lim_{k \to \infty} A_k = 0\). Then for any fixed \(\lambda > 0\), we are able to find a maximal positive integer \(k_\lambda\) such that
\[
\lambda/3 \leq C \cdot A_{k_\lambda} \left\| f \right\|_{K_{q}^{\alpha,p}(w_1,w_2)}^{p}.
\]  

Hence
\[
I_2' \leq \lambda^{p} \sum_{k=-\infty}^{k_\lambda} \left[ w_1(B_k) \right]^{\alpha p/n} \left[ w_2(B_k) \right]^{p/q} \leq C \left\| f \right\|_{K_{q}^{\alpha,p}(w_1,w_2)}^{p} \sum_{k=-\infty}^{k_\lambda} \left[ w_1(B_k) \right]^{\alpha p/n} \left[ w_2(B_k) \right]^{p/q}.
\]

Because \(B_k \subseteq B_{k_\lambda}\), then by Lemma 2.2 with the same notations \(\delta_i\) as in (3.9), we can get
\[
\frac{w_i(B_k)}{w_i(B_{k_\lambda})} \leq C \left( \frac{|B_k|}{|B_{k_\lambda}|} \right)^{\delta_i}, \text{ for } i = 1 \text{ and } 2.
\]
Therefore

\[ I'_2 \leq C \left\| f \right\|^p_{K^\alpha p_{(w_1 w_2)}} \sum_{k=-\infty}^{k_n} \left( \frac{|B_k|}{|B_{k_n}|} \right)^{\alpha \delta_1 p \alpha + \alpha \delta_2 p \alpha q} \]

\[ \leq C \left\| f \right\|^p_{K^\alpha p_{(w_1 w_2)}}. \]

On the other hand, it follows from the inequalities (3.3) and (3.7) that

\[ |S_\beta(f_3)(x)| \leq \sum_{\ell=k+2}^{\infty} |S_\beta(f \chi)(x)| \]

\[ \leq C \sum_{\ell=k+2}^{\infty} \int_{2^{\ell-1} < |z| \leq 2^{\ell}} \frac{|f(z)|}{|z|} dz \]

\[ \leq C \sum_{\ell=k+2}^{\infty} [w_2(B_{\ell})]^{-1/q} \left\| f \chi \right\|_{L^q_{\alpha}}. \]

In the present situation, since \( B_k \subseteq B_{\ell-2} \) with \( \ell \geq k + 2 \), then it follows from the inequality (3.9) that

\[ |S_\beta(f_3)(x)| \leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n}[w_2(B_k)]^{1/q}} \]

\[ \times \sum_{\ell=k+2}^{\infty} [w_1(B_\ell)]^{\alpha/n} \left\| f \chi \right\|_{L^q_{\alpha}} \cdot \frac{[w_2(B_k)]^{1/q}}{[w_2(B_{\ell-2})]^{1/q}} \cdot \frac{[w_1(B_k)]^{\alpha/n}}{[w_1(B_{\ell-2})]^{\alpha/n}} \]

\[ \leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n}[w_2(B_k)]^{1/q}} \sum_{\ell=k+2}^{\infty} [w_1(B_\ell)]^{\alpha/n} \left\| f \chi \right\|_{L^q_{\alpha}}. \]

Furthermore, recall that \( 0 < p \leq 1 \), then for any \( x \in C_k \), we have

\[ |S_\beta(f_3)(x)| \leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n}[w_2(B_k)]^{1/q}} \left( \sum_{\ell=k+2}^{\infty} [w_1(B_\ell)]^{\alpha p/n} \left\| f \chi \right\|^p_{L^q_{\alpha}} \right)^{1/p} \]

\[ \leq C \cdot \frac{1}{[w_1(B_k)]^{\alpha/n}[w_2(B_k)]^{1/q}} \left\| f \right\|^p_{K^\alpha p_{(w_1 w_2)}}. \]

Repeating the arguments used for the term \( I'_2 \), we can also obtain

\[ I'_3 \leq C \left\| f \right\|^p_{K^\alpha p_{(w_1 w_2)}}. \]

Combining the above estimates for \( I'_1 \), \( I'_2 \) and \( I'_3 \), and then taking the supremum over all \( \lambda > 0 \), we finish the proof of Theorem 1.2.
4 Proofs of Theorems 1.3 and 1.4

In order to prove the main theorems of this section, let us first establish the following results.

**Proposition 4.1.** Let $0 < \beta \leq 1$, $q = 2$ and $w \in A_{q_2}$ with $1 \leq q_2 \leq q$. Then for any $j \in \mathbb{Z}_+$, we have
\[
\|S_{\beta,2^j}(f)\|_{L^q_w} \leq C \cdot 2^{jnq_2/2} \|S_{\beta}(f)\|_{L^q_w}.
\]

**Proof.** Since $w \in A_{q_2}$, then by Lemma 2.1, we know that for any $(y,t) \in \mathbb{R}^{n+1}$,
\[
w(B(y,2^jt)) = w(2^jB(y,t)) \leq C \cdot 2^{jnq_2} w(B(y,t)) \quad j = 1,2,\ldots.
\]

Therefore
\[
\|S_{\beta,2^j}(f)\|_{L^q_w}^2 = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n+1}_+} \left( A_{\beta}(f)(y,t) \right)^2 \chi_{|x-y| < 2^j} \frac{dydt}{t^{n+1}} \right) w(x) \, dx
\]
\[
= \int_{\mathbb{R}^{n+1}_+} \left( \int_{|x-y| < 2^j} w(x) \, dx \right) \left( A_{\beta}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}}
\]
\[
\leq C \cdot 2^{jnq_2} \int_{\mathbb{R}^{n+1}_+} \left( \int_{|x-y| < t} w(x) \, dx \right) \left( A_{\beta}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}}
\]
\[
= C \cdot 2^{jnq_2} \|S_{\beta}(f)\|_{L^q_w}^2.
\]

Taking square-roots on both sides of the above inequality, we are done. \(\square\)

**Proposition 4.2.** Let $0 < \beta \leq 1$, $2 < q < \infty$ and $w \in A_{q_2}$ with $1 \leq q_2 \leq q$. Then for any $j \in \mathbb{Z}_+$, we have
\[
\|S_{\beta,2^j}(f)\|_{L^q_w} \leq C \cdot 2^{jnq_2/2} \|S_{\beta}(f)\|_{L^q_w}.
\]

**Proof.** For any $j \in \mathbb{Z}_+$ and $0 < \beta \leq 1$, it is easy to see that
\[
\|S_{\beta,2^j}(f)\|_{L^q_w}^2 = \|S_{\beta,2^j}(f)^2\|_{L^q_w}.
\] (4.1)

Since $q/2 > 1$, then by duality, we have
\[
\|S_{\beta,2^j}(f)^2\|_{L^{q/2}_w} = \sup_{\|b\|_{L^{q/2}_w} \leq 1} \left| \int_{\mathbb{R}^n} S_{\beta,2^j}(f)(x)^2 b(x) w(x) \, dx \right|
\]
\[
= \sup_{\|b\|_{L^{q/2}_w} \leq 1} \left| \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^{n+1}_+} \left( A_{\beta}(f)(y,t) \right)^2 \chi_{|x-y| < 2^j} \frac{dydt}{t^{n+1}} \right) b(x) w(x) \, dx \right|
\]
\[
= \sup_{\|b\|_{L^{q/2}_w} \leq 1} \left| \int_{\mathbb{R}^{n+1}_+} \left( \int_{|x-y| < 2^j} b(x) w(x) \, dx \right) \left( A_{\beta}(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \right|.
\] (4.2)
For $w \in A_{q_2}$, we denote the weighted maximal operator by $M_w$; that is

$$M_w(f)(x) = \sup_{x \in B} \frac{1}{w(B)} \int_B |f(y)|w(y) \, dy,$$

where the supremum is taken over all balls $B$ which contain $x$. Then, by Lemma 2.1, we can get

$$\int_{|x-y| < 2^t} b(x)w(x) \, dx \leq C \cdot 2^{jnq_2} w(B(y,t)) \cdot \frac{1}{w(B(y,2t))} \int_{B(y,2t)} b(x)w(x) \, dx$$

$$\leq C \cdot 2^{jnq_2} w(B(y,t)) \inf_{x \in B(y,2t)} M_w(b)(x)$$

Substituting the above inequality (4.3) into (4.2) and then using Hölder’s inequality together with the $L_w^{(q/2)'}$ boundedness of $M_w$, we thus obtain

$$\|S_{\beta,2'}(f)^2\|_{L_w^{(q/2)}} \leq C \cdot 2^{jnq_2} \sup_{\|b\|_{L_w^{(q/2)'}}, \leq 1} \left| \int_{\mathbb{R}^n} S_{\beta}(f)(x)^2 M_w(b)(x)w(x) \, dx \right|$$

$$\leq C \cdot 2^{jnq_2} \|S_{\beta}(f)^2\|_{L_w^{(q/2)}} \sup_{\|b\|_{L_w^{(q/2)'}}, \leq 1} \|M_w(b)\|_{L_w^{(q/2)'}}$$

$$\leq C \cdot 2^{jnq_2} \|S_{\beta}(f)^2\|_{L_w^{(q/2)}}$$

$$= C \cdot 2^{jnq_2} \|S_{\beta}(f)^2\|_{L_w^{(q/2)}}^2.$$

This estimate together with (4.1) implies the desired result. \qed

**Proposition 4.3.** Let $0 < \beta \leq 1$, $1 < q < 2$ and $w \in A_{q_2}$ with $1 \leq q_2 \leq q$. Then for any $j \in \mathbb{Z}_+$, we have

$$\|S_{\beta,2'}(f)\|_{L_w^q} \leq C \cdot 2^{jnq_2/q} \|S_{\beta}(f)\|_{L_w^q}.$$

**Proof.** We will adopt the same method given in [17]. For any $j \in \mathbb{Z}_+$, set

$$\Omega_{\lambda} = \{ x \in \mathbb{R}^n : S_{\beta}(f)(x) > \lambda \} \quad \text{and} \quad \Omega_{\lambda,j} = \{ x \in \mathbb{R}^n : S_{\beta,2'}(f)(x) > \lambda \}.$$

We also set

$$\Omega_{\lambda}^* = \{ x \in \mathbb{R}^n : M_w(\chi_{\Omega_{\lambda}})(x) > \frac{1}{2^{j(nq_2+1) \cdot [w]_{A_{q_2}}}} \}.$$

Observe that $w(\Omega_{\lambda,j}) \leq w(\Omega_{\lambda}^*) + w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_{\lambda}^*))$. Thus, for any $j \in \mathbb{Z}_+$,

$$\|S_{\beta,2'}(f)\|_{L_w^q}^q = \int_0^\infty q\lambda^{q-1} w(\Omega_{\lambda,j}) \, d\lambda$$

$$\leq \int_0^\infty q\lambda^{q-1} w(\Omega_{\lambda}^*) \, d\lambda + \int_0^\infty q\lambda^{q-1} w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_{\lambda}^*)) \, d\lambda$$

$$= 1 + II.$$
The weighted weak type estimate of $M_w$ yields

$$I \leq C \cdot 2^{jnq_2} \int_0^\infty q \lambda^{q-1} w(\Omega_\lambda) \, d\lambda \leq C \cdot 2^{jnq_2} \|S_\beta(f)\|_{L^q_w}^q. \tag{4.4}$$

To estimate II, we now claim that the following inequality holds.

$$\int_{\mathbb{R}^n \setminus \Omega_\lambda^c} S_{\beta,2\gamma}(f)(x)^2 w(x) \, dx \leq C \cdot 2^{jnq_2} \int_{\mathbb{R}^n \setminus \Omega_\lambda} S_\beta(f)(x)^2 w(x) \, dx. \tag{4.5}$$

Assuming the claim for the moment, then it follows from Chebyshev’s inequality and the inequality (4.5) that

$$w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_\lambda^c)) \leq \lambda^{-2} \int_{\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_\lambda^c)} S_{\beta,2\gamma}(f)(x)^2 w(x) \, dx \leq \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_\lambda^c} S_{\beta,2\gamma}(f)(x)^2 w(x) \, dx \leq C \cdot 2^{jnq_2} \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_\lambda} S_\beta(f)(x)^2 w(x) \, dx.$$

Hence

$$II \leq C \cdot 2^{jnq_2} \int_0^\infty q \lambda^{q-1} \left( \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_\lambda} S_\beta(f)(x)^2 w(x) \, dx \right) \, d\lambda.$$

Changing the order of integration yields

$$II \leq C \cdot 2^{jnq_2} \int_{\mathbb{R}^n} S_\beta(f)(x)^2 \left( \int_0^\infty q \lambda^{q-3} \, d\lambda \right) w(x) \, dx \leq C \cdot 2^{jnq_2} \frac{q}{2-q} \cdot \|S_\beta(f)\|_{L^q_w}^q. \tag{4.6}$$

Combining the above estimate (4.6) with (4.4) and taking $q$-th root on both sides, we are done. So it remains to prove the inequality (4.5). Set $\Gamma_{2\gamma}(\mathbb{R}^n \setminus \Omega_\lambda^c) = \bigcup_{x \in \mathbb{R}^n \setminus \Omega_\lambda} \Gamma_{2\gamma}(x)$ and $\Gamma(\mathbb{R}^n \setminus \Omega_\lambda) = \bigcup_{x \in \mathbb{R}^n \setminus \Omega_\lambda} \Gamma(x).$ For each given $(y,t) \in \Gamma_{2\gamma}(\mathbb{R}^n \setminus \Omega_\lambda^c),$ by Lemma 2.1, we thus have

$$w\left( B(y,2^gt) \cap (\mathbb{R}^n \setminus \Omega_\lambda^c) \right) \leq C \cdot 2^{jnq_2} w\left( B(y,t) \right).$$

It is not difficult to check that $w\left( B(y,t) \cap \Omega_\lambda \right) \leq \frac{w(B(y,t))}{2}$ and $\Gamma_{2\gamma}(\mathbb{R}^n \setminus \Omega_\lambda^c) \subseteq \Gamma(\mathbb{R}^n \setminus \Omega_\lambda).$ In fact, for any $(y,t) \in \Gamma_{2\gamma}(\mathbb{R}^n \setminus \Omega_\lambda^c),$ there exists a point $x \in \mathbb{R}^n \setminus \Omega_\lambda^c$ such that $(y,t) \in \Gamma_{2\gamma}(x).$ Then we can deduce

$$\begin{align*}
\int_{B(y,2^gt)} \chi_{\Omega_\lambda}(z) w(z) \, dz &= \int_{B(y,2^gt)} \chi_{\Omega_\lambda}(z) \cdot \frac{1}{w(B(y,2^gt))} \int_{B(y,2^gt)} \chi_{\Omega_\lambda}(z) w(z) \, dz.
\end{align*}$$

17
Therefore

\[ \Gamma \]

The above inequality implies in particular that there is a point \( z \) which is equivalent to

From the definition of \( \Gamma \)

Proof of Theorem 1.3.

Hence

\[ w(B(y,t) \cap \Omega) \leq \frac{w(B(y,t))}{2}. \]

Hence

\[ w(B(y,t)) = w(B(y,t) \cap \Omega) + w(B(y,t) \cap (\mathbb{R}^n \setminus \Omega)) \]

which is equivalent to

\[ w(B(y,t)) \leq 2 \cdot w(B(y,t) \cap (\mathbb{R}^n \setminus \Omega)). \]

The above inequality implies in particular that there is a point \( z \in B(y,t) \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset \). In this case, we have \( (y,t) \in \Gamma(z) \) with \( z \in \mathbb{R}^n \setminus \Omega \), which implies \( \Gamma_{2^n}(\mathbb{R}^n \setminus \Omega) \subseteq \Gamma(\mathbb{R}^n \setminus \Omega) \). Thus we obtain

\[ w(B(y,2^n t) \cap (\mathbb{R}^n \setminus \Omega)) \leq C \cdot 2^{j n \Omega(x)} w(B(y,t) \cap (\mathbb{R}^n \setminus \Omega)). \]

Therefore

\[
\int_{\mathbb{R}^n \setminus \Omega} S_{\beta,2^n}(f)(x)^2 w(x) dx
= \int_{\mathbb{R}^n \setminus \Omega} \left( \int_{\Gamma_{2^n}(x)} (A_\beta(f)(y,t))^2 \frac{dydt}{2^{n+1}} \right) w(x) dx
\leq \int_{\Gamma_{2^n}(\mathbb{R}^n \setminus \Omega)} \left( \int_{B(y,2^n t) \cap (\mathbb{R}^n \setminus \Omega)} w(x) dx \right) (A_\beta(f)(y,t))^2 \frac{dydt}{2^{n+1}}
\leq C \cdot 2^{j n \Omega(x)} \int_{\Gamma(\mathbb{R}^n \setminus \Omega)} \left( \int_{B(y,t) \cap (\mathbb{R}^n \setminus \Omega)} w(x) dx \right) (A_\beta(f)(y,t))^2 \frac{dydt}{2^{n+1}}
\leq C \cdot 2^{j n \Omega(x)} \int_{\mathbb{R}^n \setminus \Omega} S_\beta(f)(x)^2 w(x) dx,
\]

which is exactly what we want. This completes the proof of Proposition 4.3. \( \square \)

We are now in a position to give the proofs of the main theorems.

Proof of Theorem 1.3. From the definition of \( \mathcal{G}_{\lambda,\beta} \), we readily see that

\[
|\mathcal{G}_{\lambda,\beta}(f)(x)|^2 = \int_{\mathbb{R}^{n+1}} \left( \frac{t}{t + |x-y|} \right)^{\lambda n} (A_\beta(f)(y,t))^2 \frac{dydt}{2^{n+1}}
= \int_0^\infty \int_{|x-y|<t} \left( \frac{t}{t + |x-y|} \right)^{\lambda n} (A_\beta(f)(y,t))^2 \frac{dydt}{2^{n+1}}
+ \sum_{j=1}^\infty \int_0^{\infty} \int_{2^{j-1} \leq |x-y| < 2^j} \left( \frac{t}{t + |x-y|} \right)^{\lambda n} (A_\beta(f)(y,t))^2 \frac{dydt}{2^{n+1}}
\leq C \left[ S_\beta(f)(x)^2 + \sum_{j=1}^\infty 2^{-j \lambda n} S_{\beta,2^n}(f)(x)^2 \right]. \quad (4.7)
\]
Let \( f \in \dot{H}^{\alpha,p}_q(w_1, w_2) \). We decompose \( f(x) = f_1(x) + f_2(x) + f_3(x) \) as in Theorem 1.1, then we have

\[
\|G^{\alpha,\beta}_\lambda(f)\|_{K^{\alpha,p}_q(w_1, w_2)}^p \leq C \sum_{i=1}^{3} \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|G^{\alpha,\beta}_\lambda(f_i)\|_{L^q_{w_2}}^p
\]

\[
= J_1 + J_2 + J_3.
\]

Note that \( \lambda > \max\{q_2, 3\} \geq \max\{q_2, 2q_2/q\} \) when \( q_2 \leq q \). Since \( w_2 \in A_{q_2} \) and \( 1 \leq q_2 \leq q \), then \( w_2 \in A_q \). Applying Propositions 4.1–4.3, Theorem A and the above inequality (4.7), we obtain

\[
\|G^{\alpha,\beta}_\lambda(f_1)\|_{L^q_{w_2}} \leq C \left( \|S^{\alpha,\beta}_{\lambda}(f_1)\|_{L^q_{w_2}} + \sum_{j=1}^{\infty} 2^{-j^{\lambda n/2}} \|S^{\alpha,2j}_{\lambda}(f_1)\|_{L^q_{w_2}} \right)
\]

\[
\leq C \|f_1\|_{L^q_{w_2}} \left( 1 + \sum_{j=1}^{\infty} 2^{-j^{\lambda n/2}} [2^{jnq_2/2} + 2^{jnq_2/q}] \right)
\]

\[
\leq C \|f_1\|_{L^q_{w_2}}. \quad (4.8)
\]

From the above estimate (4.8) and Lemma 2.1, it follows that

\[
J_1 \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|G^{\alpha,\beta}_\lambda(f_1)\|_{L^q_{w_2}}^p
\]

\[
\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f_1\|_{L^q_{w_2}}^p
\]

\[
\leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f\chi_k\|_{L^q_{w_2}}^p
\]

\[
\leq C \|f\|_{K^{\alpha,p}_q(w_1, w_2)}^p.
\]

For any \( j \in \mathbb{Z}_+, x \in C_k, (y, t) \in \Gamma_{2^j}(x) \) and \( z \in \{2^{j-1} < |z| \leq 2^j\} \cap B(y, t) \) with \( \ell \leq k - 2 \), then by a simple calculation, we can easily deduce

\[
t + 2^j t \geq |x - y| + |y - z| \geq |x - z| \geq |x| - |z| \geq \frac{|x|}{2}.
\]

Thus, by the previous inequality (3.1) and Minkowski’s inequality, we get

\[
|S^{\alpha,2j}_{\lambda}(f\chi_t)(x)| = \left( \int_{\Gamma_{2^j}(x)} \left( \sup_{\varphi \in \mathcal{C}_\delta} |(f\chi_t) * \varphi_t(y)| \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2}
\]

\[
\leq C \left( \int_{\frac{|x-y|}{2^j t}} \int_{|x-y| < 2^j t} \left| t^{-n} \int_{2^{j-1} < |z| \leq 2^j} |f(z)| \, dz \right| \frac{dydt}{t^{n+1}} \right)^{1/2}
\]

\[
\leq C \left( \int_{2^{j-1} < |z| \leq 2^j} |f(z)| \, dz \right) \left( \int_{\frac{1}{2^j}} \int_{2^{j-1} < |z| \leq 2^j} 2^{jn} \frac{dt}{t^{2n+1}} \right)^{1/2}
\]

\[
\leq C \cdot 2^{3jn/2} \frac{1}{|x|^n} \left( \int_{2^{j-1} < |z| \leq 2^j} |f(z)| \, dz \right). \quad (4.9)
\]
Moreover, by using Minkowski’s inequality, (3.3) and (4.9), we obtain

\[
\|S_{\beta,2'}(f_2)\chi_k\|_{L^q_{\nu_2}} \leq \sum_{\ell=-\infty}^{k-2} \|S_{\beta,2'}(f\chi)\chi_k\|_{L^q_{\nu_2}} \\
\leq C \cdot 2^{3jn/2} \sum_{\ell=-\infty}^{k-2} \left( \int_{2^{\ell-1}<|z|\leq 2^\ell} |f(z)| \, dz \right) \left( \int_{2^{\ell-1}<|x|\leq 2^\ell} \frac{w_2(x)}{|x|^qn} \, dx \right)^{1/q} \\
\leq C \cdot 2^{3jn/2} \sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} \cdot \frac{w_2(B_k)^{1/q}}{w_2(B_\ell)^{1/q}} \|f\chi\|_{L^p_{\nu_2}}.
\]

Consequently

\[
J_2 \leq C \sum_{k\in\mathbb{Z}} \left[ w_1(B_k) \right]^{\alpha/p} \left( \|S_{\beta}(f_2)\chi_k\|_{L^q_{\nu_2}} + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \|S_{\beta,2'}(f_2)\chi_k\|_{L^q_{\nu_2}} \right)^p \\
\leq C \sum_{k\in\mathbb{Z}} \left[ w_1(B_k) \right]^{\alpha/p} \left( \sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} \cdot \frac{w_2(B_k)^{1/q}}{w_2(B_\ell)^{1/q}} \|f\chi\|_{L^p_{\nu_2}} \right)^p \times \left( 1 + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \cdot 2^{3jn/2} \right)^p \\
\leq C \sum_{k\in\mathbb{Z}} \left[ w_1(B_k) \right]^{\alpha/p} \left( \sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} \cdot \frac{w_2(B_k)^{1/q}}{w_2(B_\ell)^{1/q}} \|f\chi\|_{L^p_{\nu_2}} \right)^p,
\]

where the last inequality holds under our assumption \( \lambda > 3 \). On the other hand, for any \( j \in \mathbb{Z}_+, \ x \in C_k, \ (y, t) \in \Gamma_{2^j}(x) \) and \( z \in \{ 2^j < |z| \leq 2^j \} \cap B(y, t) \) with \( \ell \geq k + 2, \) it is easy to verify that

\[
t + 2^j t \geq |x - y| + |y - z| \geq |x - z| \geq |z| - |x| \geq \frac{|z|}{2}.
\]

Then it follows from the inequality (3.1) and Minkowski’s inequality that

\[
|S_{\beta,2'}(f\chi)(x)| \leq C \left( \int_{\frac{|z|}{2^j+1}}^{\infty} \int_{|x-y|<2^j} \left| t^{-n} \int_{2^{j-1}<|z|\leq 2^j} |f(z)| \, dz \right|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2} \\
\leq C \left( \int_{2^{j-1}<|z|\leq 2^j} |f(z)| \, dz \right) \left( \int_{\frac{|z|}{2^j+1}}^{\infty} 2^j t^{-n} \frac{dt}{t^{2n+1}} \right)^{1/2} \\
\leq C \cdot 2^{3jn/2} \left( \int_{2^{j-1}<|z|\leq 2^j} \frac{|f(z)|}{|z|^n} \, dz \right). \tag{4.10}
\]

Furthermore, by Minkowski’s inequality, (3.3) and (4.10), we have

\[
\|S_{\beta,2'}(f_3)\chi_k\|_{L^q_{\nu_2}} \leq \sum_{\ell=k+2}^{\infty} \|S_{\beta,2'}(f\chi)\chi_k\|_{L^q_{\nu_2}} \\
\leq C \cdot 2^{3jn/2} \sum_{\ell=k+2}^{\infty} \frac{|w_2(B_k)|^{1/q}}{|w_2(B_\ell)|^{1/q}} \|f\chi\|_{L^p_{\nu_2}}.
\]

20
Therefore
\[ J_3 \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left( \|S_\beta(f_3)\|_{L^{q_2}} + \sum_{j=1}^{\infty} 2^{-j\lambda_n/2} \|S_{\beta,2j}(f_3)\|_{L^{q_2}} \right)^p \]
\[ \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left( \sum_{t=k+2}^{\infty} \frac{|w_2(B_k)|^{1/q}}{|w_2(B_t)|^{1/q}} \|f\chi_t\|_{L^{q_2}} \right)^p \times \left( 1 + \sum_{j=1}^{\infty} 2^{-j\lambda_n/2} \cdot 2^{3jn/2} \right)^p \]
\[ \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left( \sum_{t=k+2}^{\infty} \frac{|w_2(B_k)|^{1/q}}{|w_2(B_t)|^{1/q}} \|f\chi_t\|_{L^{q_2}} \right)^p, \]
where the last inequality also holds since \( \lambda > 3 \). Following along the same lines as in Theorem 1.1, we can also show that
\[ J_2 \leq C \|f\|^p_{K^{p,q}(w_1,w_2)} \]
and
\[ J_3 \leq C \|f\|^p_{K^{p,q}(w_1,w_2)}. \]
Summing up the above estimates for \( J_1, J_2 \) and \( J_3 \), we complete the proof of Theorem 1.3.

**Proof of Theorem 1.4.** Let \( f \in \dot{K}_q^{p,q}(w_1,w_2) \). We set \( f(x) = f_1(x) + f_2(x) + f_3(x) \) as in Theorem 1.2, then for any given \( \sigma > 0 \), we can write
\[ \sigma^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} w_2 \left( \left\{ x \in C_k : |G_{\lambda,\beta}^{*}(f)(x) | > \sigma \right\} \right)^{p/q} \]
\[ \leq \sum_{i=1}^{3} \sigma^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} w_2 \left( \left\{ x \in C_k : |G_{\lambda,\beta}^{*}(f_i)(x) | > \sigma/3 \right\} \right)^{p/q} \]
\[ = J_1' + J_2' + J_3'. \]
Since \( \lambda > \max\{q_2,3\} \geq \max\{q_2,2q_2/q\} \) when \( q_2 \leq q \). Applying Chebyshev’s inequality, Lemma 2.1 and (4.8), we obtain
\[ J_1' \leq \sigma^p \cdot \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \left( \frac{3q}{\sigma^q} \|G_{\lambda,\beta}^{*}(f_1)\|_{L^{q_2}} \right)^{p/q} \]
\[ \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f_1\|_{L^{q_2}} \]
\[ \leq C \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{\alpha p/n} \|f\chi_k\|_{L^{q_2}} \]
\[ \leq C \|f\|^p_{K^{p,q}(w_1,w_2)}. \]
For the term \( J_2' \), when \( x \in C_k \), then it follows from (4.7), (4.9), (3.3) and the
fact $\lambda > 3$ that

$$|G^*_{\lambda,\beta}(f_2)(x)| \leq \sum_{\ell=-\infty}^{k-2} |G^*_{\lambda,\beta}(f\chi_\ell)(x)|$$

$$\leq C \sum_{\ell=-\infty}^{k-2} \left( |S_\beta(f\chi_\ell)(x)| + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} |S_{\beta,2^j}(f\chi_\ell)(x)| \right)$$

$$\leq C \left( \sum_{\ell=-\infty}^{k-2} \int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| \, dz \right) \left( 1 + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \cdot 2^{3j n/2} \right)$$

$$\leq C \sum_{\ell=-\infty}^{k-2} \frac{1}{|x|^n} \left( \int_{2^{\ell-1} < |z| \leq 2^\ell} |f(z)| \, dz \right)$$

$$\leq C \sum_{\ell=-\infty}^{k-2} \frac{|B_\ell|}{|B_k|} [w_2(B_\ell)]^{1-1/q} \|f\chi_\ell\|_{L_{w_2}^q}.$$  

For the last term $J'_3$, when $x \in C_k$, by using (4.7), (4.10), (3.3) and the fact that $\lambda > 3$, we get

$$|G^*_{\lambda,\beta}(f_3)(x)| \leq \sum_{\ell=\infty}^{\infty} |G^*_{\lambda,\beta}(f\chi_\ell)(x)|$$

$$\leq C \sum_{\ell=\infty}^{\infty} \left( |S_\beta(f\chi_\ell)(x)| + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} |S_{\beta,2^j}(f\chi_\ell)(x)| \right)$$

$$\leq C \left( \sum_{\ell=\infty}^{\infty} \int_{2^{\ell-1} < |z| \leq 2^\ell} \frac{|f(z)|}{|z|^n} \, dz \right) \left( 1 + \sum_{j=1}^{\infty} 2^{-j\lambda n/2} \cdot 2^{3j n/2} \right)$$

$$\leq C \sum_{\ell=\infty}^{\infty} \int_{2^{\ell-1} < |z| \leq 2^\ell} \frac{|f(z)|}{|z|^n} \, dz$$

$$\leq C \sum_{\ell=\infty}^{\infty} [w_2(B_\ell)]^{1-1/q} \|f\chi_\ell\|_{L_{w_2}^q}.$$  

The rest of the proof is exactly the same as that of Theorem 1.2, and we finally obtain

$$J'_2 \leq C \|f\|_{K^{p,\alpha}_{\lambda}(w_1,w_2)}^p$$

and

$$J'_3 \leq C \|f\|_{K^{p,\alpha}_{\lambda}(w_1,w_2)}^p.$$  

Combining the above estimates for $J'_1$, $J'_2$ and $J'_3$, and then taking the supremum over all $\sigma > 0$, we conclude the proof of Theorem 1.4. \hfill \Box
References

[1] S. Y. A. Chang, J. M. Wilson and T. H. Wolff, Some weighted norm inequalities concerning the Schrödinger operators, Comment. Math. Helv, 60(1985), 217–246.

[2] S. Chanillo and R. L. Wheeden, Some weighted norm inequalities for the area integral, Indiana Univ. Math. J, 36(1987), 277–294.

[3] J. Garcia-Cuerva and J. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland, Amsterdam, 1985.

[4] R. F. Gundy and R. L. Wheeden, Weighted integral inequalities for non-tangential maximal function, Lusin area integral, and Walsh-Paley series, Studia Math, 49(1974), 107–124.

[5] J. Z. Huang and Y. Liu, Some characterizations of weighted Hardy spaces, J. Math. Anal. Appl, 363(2010), 121–127.

[6] Y. Komori and K. Matsuoka, Boundedness of several operators on weighted Herz spaces, J. Funct. Spaces Appl, 7(2009), 1–12.

[7] A. K. Lerner, Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals, Adv. Math., 226(2011), 3912–3926.

[8] S. Z. Lu and D. C. Yang, The decomposition of the weighted Herz spaces and its applications, Sci. China (Ser. A), 38(1995), 147–158.

[9] S. Z. Lu and D. C. Yang, Hardy-Littlewood-Sobolev theorems of fractional integration on Herz-type spaces and its applications, Canad. J. Math, 48(1996), 363–380.

[10] S. Z. Lu, D. C. Yang and G. E. Hu, Herz Type Spaces and Their Applications, Science Press, Beijing, 2008.

[11] S. Z. Lu, K. Yabuta and D. C. Yang, Boundedness of some sublinear operators in weighted Herz-type spaces, Kodai Math. J, 23(2000), 391–410.

[12] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc, 165(1972), 207–226.

[13] B. Muckenhoupt and R. L. Wheeden, Norm inequalities for the Littlewood-Paley function $g_A^*$, Trans. Amer. Math. Soc, 191(1974), 95–111.

[14] E. M. Stein, On some functions of Littlewood-Paley and Zygmund, Bull. Amer. Math. Soc, 67(1961), 99–101.

[15] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, New Jersey, 1970.

[16] L. Tang and D. C. Yang, Boundedness of vector-valued operators on weighted Herz spaces, Approx. Theory Appl, 16(2000), 58–70.
[17] A. Torchinsky, Real-Variable Methods in Harmonic Analysis, Academic Press, New York, 1986.

[18] H. Wang, Intrinsic square functions on the weighted Morrey spaces, J. Math. Anal. Appl, 396(2012), 302–314.

[19] H. Wang, Boundedness of intrinsic square functions on the weighted weak Hardy spaces, Integr. Equ. Oper. Theory, to appear.

[20] H. Wang and H. P. Liu, The intrinsic square function characterizations of weighted Hardy spaces, Illinois J. Math, to appear.

[21] H. Wang and H. P. Liu, Weak type estimates of intrinsic square functions on the weighted Hardy spaces, Arch. Math., 97(2011), 49–59.

[22] M. Wilson, The intrinsic square function, Rev. Mat. Iberoamericana, 23(2007), 771–791.

[23] M. Wilson, Weighted Littlewood-Paley Theory and Exponential-Square Integrability, Lecture Notes in Math, Vol 1924, Springer-Verlag, 2007.