THE PARTITION POSET COMPLEX AND THE GOODWILLIE
DERIVATIVES OF THE IDENTITY IN SPACES

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ABSTRACT. We produce a canonical highly homotopy-coherent operad structure
on the derivatives of the identity functor in spaces via a pairing of cosimplicial
objects, providing a new description of an operad structure on such objects first
described by Ching.

1. Introduction

The Goodwillie derivatives of the identity, denoted \( \partial_* \text{Id} \), on the category of
pointed spaces \( S_* \) play a central role to homotopy theory. As the coefficients of
the Taylor tower, the sequence controls a natural filtration of a space \( X \) which be-
gins with stabilization \( QX \) and converges to \( X \) whenever \( X \) is 1-connected.\(^1\) An
operad structure on \( \partial_* \text{Id} \) was first constructed by Ching in \([\text{Chi05}]\) and later Arone-
Ching \([\text{AC11}]\) showed that the Goodwillie derivatives of a reduced homotopy functor
\( F : S_* \to S_* \) naturally form a \( (\partial_* \text{Id}, \partial_* \text{Id}) \)-bimodule.

The author has recently shown in \([\text{Cla20}]\) that the Goodwillie derivatives of the
identity on the category of algebras over a (reduced) operad of spectra \( O \) forms a
natural “highly homotopy-coherent” operad, and moreover, that this operad struc-
ture is equivalent to that on \( O \). In this document, we show that similar techniques
may be utilized to provide an alternate construction for an operad structure on the
Goodwillie derivatives of the identity in pointed spaces, specifically we show the
following.

**Theorem 1.1.** The symmetric sequence \( \partial_* \text{Id} \) is a highly homotopy coherent operad,
expressed as an algebra over \( A \) (see \([\text{Cla20}, \text{Proposition 7.11}]\)).

It is natural to wonder whether the structure on \( \partial_* \text{Id} \) we describe in this docu-
ment agrees with the operad structure constructed by Ching in \([\text{Chi05}]\). The author
believes that the two should be equivalent, though is not aware of an explicit de-
scription between the two.

1.2. Outline of the argument. We make use of the models for \( \partial_* \text{Id} \) described
by Johnson \([\text{Joh95}]\) and Arone-Mahowald \([\text{AM99}]\), specifically, that the derivative
of the identity can be identified with the totalization of the cobar complex on the
commutative cooperad of spectra \( S \):

\[
\partial_* \text{Id} \simeq \text{holim}_\Delta \text{Cobar}(I, S, I).
\]

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\(^1\)In fact, this tower will converge for nilpotent spaces as well.
Let us write \( C_{id} \) for \( Cobar(I, S, I) \). In comparison to the cosimplicial object \( C(\mathcal{O}) \) of [Cla20], it is straightforward to show that \( C_{id} \) is a monoid with respect to the box product of cosimplicial objects \( \Box \) (see [Bat93]). Essentially, this structure is supplied by the canonical isomorphisms \( S^{op} \circ S^{op} \cong S^{op+q} \), and induces an \( A_\infty \)-pairing (with respect to the composition product of symmetric sequences) on totalizations. However, as with [Cla20], we run into issues of fibrancy. Careful inspection of the factors of \( C_{id} \) reveal that in order to provide a (levelwise) fibrant replacement \( C_{id} \xrightarrow{\sim} C_{id} \) we only need to replace the sphere spectrum \( S \) by a fibrant commutative monoid \( F_S \) in order to retain the monoidal structure with respect to \( \Box \). Thus, we show that \( \partial_* \text{Id} \) – modelled as \( \text{Tot} \ C_{id} \) – naturally carries the structure of an \( A_\infty \)-operad.

1.3. Future Work. The author expects that the constructions of this document should shed light onto describing more explicitly the structure possessed by algebras over the operad \( \partial_* \text{Id} \). Specifically, if \( Y \) is a commutative coalgebra in spectra then

\[
\mathbb{C}(Y) = \text{holim}_\Delta \text{Cobar}(I, S, Y)
\]

should come equipped with a left action of \( \partial_* \text{Id} \).

Let \( C(Y) \) denote \( \text{Cobar}(I, S, Y) \). There is an evident pairing \( C_{id} \Box C(Y) \to C(Y) \) which essentially provides this structure; however \( C(Y) \) is not fibrant in general and thus we cannot use our previous methods directly. A more specific example: for any space \( X \), the suspension spectrum \( \Sigma^\infty X \) is a commutative coalgebra\(^2\). The assignment \( X \mapsto \mathbb{C}(\Sigma^\infty X) \) supplies a hypothetical functor

\[
\Phi: S_* \to \text{Alg}_{\partial_* \text{Id}}.
\]

We are particularly interested in understanding the homotopical properties of such a functor, specifically, if \( \Phi \) is an equivalence of homotopy categories, perhaps after restricting to certain appropriate subcategories. Such a result would effectively give a homotopical descent property to a subcategory of \( S_* \) and also tie into recent work of Behrens-Rezk [BR20] regarding \( v_n \)-periodic homotopy (see also Kuhn [Kuh06]).

More generally, given a cooperad \( Q \) in \( \text{Spt} \) and a \( Q \)-coalgebra \( Y \), we expect our techniques to underlie a proof of the following: (i) \( KQ \simeq \text{holim}_\Delta \text{Cobar}(I, Q, I) \) is an \( A_\infty \)-operad and (ii) \( \text{holim}_\Delta \text{Cobar}(I, Q, Y) \) is an algebra over \( KQ \). Such explicit constructions would also tie into bar-cobar duality (see, e.g. Ginzburg-Kapranov [GK94], Ching [Chi12a], or Francis-Gaitsgory [FG11]) by providing a rigid point-set model for \( KQ \) as an operad in \( \text{Spt} \), along with a functor \( \text{coAlg}_{Q}^{d,p} \to \text{Alg}_{KQ} \).

We also anticipate our constructions to provide an alternate description of a chain rule map

\[
\partial_* F \circ \partial_* G \to \partial_*(FG)
\]

for suitable functors \( F, G : S_* \to S_* \) induced by a box-product pairing of cosimplicial objects. The above are all matters of ongoing work and will not be further pursued in this document.

\(^2\)For a spectrum \( E \), let \( E^\vee = \text{hom}(E, S) \). For \( X \in S_* \), \( \mathbb{C}(\Sigma^\infty X) \simeq TQ(\Sigma^\infty X^\vee)^\vee \).
1.4. Conventions and notation. We make use of the category of pointed simplicial sets $S_*$ as our model for spaces, and use the theory of symmetric spectra developed by Hovey-Shipley-Smith [HSS99] as our model for a symmetric monoidal category $\text{Spt} = (\text{Sp}^\Sigma, \wedge, S)$ of spectra. In particular, we note that any suspension spectrum $\Sigma^\infty X$ is cofibrant. For simplicity, we denote by $\Omega^\infty$ the usual infinite loop space functor, $\text{Ev}_0$, precomposed with a fixed fibrant replacement monad $F$ on $\text{Spt}$: i.e. $\Omega^\infty = \text{Ev}_0 F$.

We denote by $\text{SymSeq}$ the category of symmetric sequences in spectra, i.e. $A \in \text{SymSeq}$ consists of objects $A[n] \in \text{Spt}$ with (right) action by $\Sigma_n$ for each $n \geq 0$. We will often assume that our symmetric sequences are reduced, that is, $A[0] = *$.

1.5. Organization of the paper. Section 2 provides the necessary background on functor calculus and the Goodwillie derivatives of the identity functor on $S_*$. In Section 3 we develop the necessary language of (op)lax (co)monoids to precisely describe the structure we work with in Section 4, which is dedicated to a proof of Theorem 1.1.

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2. Functor calculus and the derivatives of the identity

2.1. The Taylor tower. A central construction in functor calculus is that of the Taylor tower of $n$-excisive approximations associated to a functor $F: S_* \to S_*$ as follows

$$D_n F \quad \downarrow \quad F \quad \cdots \quad P_n F \quad P_{n-1} F \quad \cdots \quad P_0 F.$$ 

The functors $P_n F$ are called the $n$-th excisive approximation to $F$ and are initial in the homotopy category of $n$-excisive functors receiving a map from $F$. For simplicity, we base all of our approximations at the zero object $* \in S_*$. For $n \geq 1$, we set the $n$-th homogeneous layer, denoted $D_n F$, to be

$$D_n F := \text{hofib}(P_n F \to P_{n-1} F).$$

The following proposition summarizes the salient properties of $D_n F$ (see [Goo03]).

**Proposition 2.2.** Let $F: S_* \to S_*$ be a homotopy functor and $n \geq 1$. Then:

- $D_n F$ is $n$-homogeneous.
- $D_n F$ naturally factors through $\text{Spt}$ as $D_n F \simeq \Omega^\infty \circ D_n F \circ \Sigma^\infty$ such that $D_n F$ is $n$-homogeneous.

$^3$Recall a functor $G$ is $n$-homogeneous if $G$ is $n$-excisive and $P_{n-1} G \simeq *$. 
\[
D_n F \text{ is characterized by a spectrum with (right) } \Sigma_n \text{-action, } \partial_n F \text{ called the } \text{n-th derivative of } F, \text{ and there is an equivalence }^4
\]
\[D_n F(X) \simeq \Omega^\infty (\partial_n F \wedge_{\Sigma_n^s} (\Sigma^\infty X)^{\wedge n}).\]

- The spectrum \( \partial_n F \) may be calculated via cross effects \([\text{Goo03}] \S 3\) as
\[\partial_n F \simeq \text{cr}_n \mathbb{D}_n F(S, \ldots, S)\]
with \( \Sigma_n \text{-action given by permuting the inputs } S.\)

2.3. The Taylor tower of the identity functor on \( S_* \). The Taylor tower of \( \text{Id} \) is a central object of homotopy theory: it is not hard to show
\[
P_1 \text{Id}(X) \simeq D_1 \text{Id}(X) \simeq \Omega^\infty \Sigma^\infty (X)
\]
the stabilization of a space \( X \). The higher Blakers-Massey theorems \([\text{Goo92}] \S 2.1\) show that \( \text{Id} \) is 1-analytic and therefore the Taylor tower of the identity in \( S_* \) offers an interpolation between a simply connected space \( X \simeq \text{holim}_n P_n \text{Id}(X) \) and its stabilization \( \Omega^\infty \Sigma^\infty X.\)

The equivalences from \((2)\) show that \( \partial_1 \text{Id} \simeq S. \) Johnson \([\text{Joh95}]\) and later Arone-Mahowald \([\text{AM99}]\) later gave a description of the higher homogeneous layers and derivatives of \( \text{Id} \) in terms of the partition poset complex.

2.4. The partition poset complex. For \( n \geq 0 \) we denote by \( n \) the set \( \{1, \ldots, n\} \), note that \( 0 = \emptyset \). A partition \( \lambda \) of \( n \) is a decomposition \( n = \bigsqcup_{i \in I} T_i \) of nonempty subsets (here \( I \) is required to be a nonempty set). Given partition \( \lambda = \{T_i\}_{i \in I} \) and \( \lambda' = \{T'_j\}_{j \in J} \) of \( n \) we say that \( \lambda \leq \lambda' \) if there is a surjection \( f : J \to I \) such that \( T_i = \bigsqcup_{j \in f^{-1}(i)} T'_j \) for all \( i \in I.\)

Note that the set of partitions of \( n \) has a minimal element \( \text{min} \) consisting of only the trivial partition \( \{1, \ldots, n\} \), and a maximal element \( \text{max} \) consisting of the partition of \( n \) into singletons, i.e. \( \{\{1\}, \ldots, \{n\}\} \). The set of partitions of \( n \) then forms a poset with respect to \( \leq \), and so may be interpreted as a category. The partition poset as defined below is essentially (a quotient of) the nerve of this category.

**Definition 2.5.** Define the \( n \)-th partition poset complex \( \text{Par}(n) \) to be the pointed simplicial set with \( k \)-simplices given by chains
\[\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{k-1} \leq \lambda_k\]
of partitions of \( n \) such that any chain that does not satisfy \( \lambda_0 = \text{min} \) and \( \lambda_k = \text{max} \) is identified with the basepoint.

Face maps \( d_i : \text{Par}_k(n) \to \text{Par}_{k-1}(n) \) are given by removing the \( i \)-th entry \( \lambda_i \) and degeneracy maps \( s_j : \text{Par}_k(n) \to \text{Par}_{k+1}(n) \) are given by repeating the \( j \)-th entry \( \lambda_j \). Note, that the image of \( d_0 \) (resp. \( d_n \)) is only the basepoint if \( \lambda_1 \neq \lambda_0 \) (resp. \( \lambda_{k-1} \neq \lambda_k \)).

More generally, for a finite set \( T \) we define \( \text{Par}(T) \) analogously, e.g. by setting \( |T| = n \) and specifying a bijection \( T \cong n.\)

\(^4\)For \( X \) which have the homotopy type of a finite CW-complexes; or on an arbitrary space \( X \) if \( F \) commutes with filtered colimits (i.e. \( F \) is finitary).
Remark 2.6. Note that \( \text{Par}(n) \) inherits a natural action of \( \Sigma_n \) by permuting the elements of \( n \). A useful observation is that chains \( \alpha \in \text{Par}_k(n) \) are in bijective correspondence with isomorphism classes of planar, rooted trees with \( n \) labelled leaves and \( k \) levels, up to planar isomorphism.

Example 2.7. It is possible to calculate some low dimensional examples of partition poset complexes. For instance, \( \text{Par}(0) = \ast \), \( \text{Par}(1) \cong S^0 \) and \( \text{Par}(2) \cong S^1 \) with trivial \( \Sigma_2 \) action. Similarly, \( \text{Par}(3) \) may be identified with the 2-sphere with a disc glued-in at the equator, \( \Sigma_3 \) acts on \( \text{Par}(3) \) by permuting the three 2-discs (top hemisphere, bottom hemisphere and equator).

Moreover, it is known that there is a (nonequivaraint) equivalence (see Johnson \cite{Joh95} or Arone-Mahowald \cite{AM99})

\[
\text{Par}(n) \simeq (n-1)! \bigvee_{i=1}^{n-1} S^{n-1}.
\]

2.8. Relation to the derivatives of the identity. Work by Johnson \cite{Joh95} and later Arone-Mahowald \cite{AM99} characterizes the homogeneous layers \( D_n \text{Id} \) as mapping spectra out of the partition poset complex, i.e.

\[
D_n \text{Id}(X) \simeq \Omega^\infty \left( \text{hom}(\text{Par}(n), \Sigma_n X) \right)_{h\Sigma_n}
\]

and moreover shows there is an equivalence

\[
\partial_n \text{Id} \simeq \text{hom}(\text{Par}(n), S).
\]

2.9. Completion and higher stabilization. Associated to the stabilization adjunction \((\Sigma_\infty, \Omega_\infty)\) between \( S_* \) and \( \text{Spt} \), for any space \( X \), there is a coaugmented cosimplicial diagram \( X \to C(\Sigma^\infty X) \) such that

\[
C(\Sigma^\infty X) := \text{Cobar}(\Omega^\infty, \Sigma_\infty \Omega^\infty, \Sigma^\infty X)
\]

coaugmented by the unit map \( X \to \Omega^\infty \Sigma^\infty X \). \( C(\Sigma^\infty X) \) is functorial in \( X \) and provides a cosimplicial functor

\[
C(\Sigma^\infty -) = \left( \Omega^\infty \Sigma^\infty \xrightarrow{\Omega^\infty} (\Omega^\infty \Sigma^\infty)^2 \xrightarrow{\Omega^\infty} (\Omega^\infty \Sigma^\infty)^3 \ldots \right)
\]

whose coface maps are induced by inserting the unit map \( \text{id} \to Q := \Omega^\infty \Sigma^\infty \) and codegeneracy maps are induced by the counit map \( \Sigma^\infty \Omega^\infty =: K \to \text{id} \).

Blomquist-Harper \cite{BH16} utilize the higher Blakers-Massey theorems of \cite{Goo92} to recover a classical result of Bousfield-Kan \cite{BK72}. That 1-connected spaces are equivalent to their completion with respect to stabilization, i.e. if \( X \in S_* \) is 1-connected, then

\[
X \xrightarrow{\sim} \text{holim}_\Delta C(X) \simeq X_{\Omega^\infty \Sigma^\infty}^{\wedge}.
\]

The key to their proof is to provide strong connectivity estimates of the following form.

Proposition 2.10. The comparison map \( X \to \text{holim}_{\Delta \leq n-1} C(X) \) is \((k(n+1)+1)\)-connected for \( X \in S_* \) \( k \)-connected.
In our situation we make use of the above connectivity estimates to show that $P_n \text{Id}$ may be recovered the totalization of $P_n Q^{\bullet+1}$ (see also Arone-Ching [AC11 §16]).

**Corollary 2.11.** There is an equivalence of functors $P_n \text{Id} \sim \text{holim}_\Delta P_n Q^{\bullet+1}$.

**Proof.** For $k \geq n$, the connectivity estimates from Proposition 2.10 are sufficient to induce an equivalence

$$P_n \text{Id} \sim P_n (\text{holim}_\Delta \leq k-1 C(-))$$

by [Goo03, Proposition 1.6]. Moreover, $\text{holim}_\Delta \leq k-1$ may be modeled as the homotopy limit over a punctured $n$-cube $\Delta^n$, and thus will commute with taking $P_n$ by [Goo03, Proposition 1.7].

2.12. The Snaith splitting. Let $\underline{S}$ denote the symmetric sequence in $\text{Spt}$ such that $\underline{S}[n] = S$ with trivial $\Sigma_n$ action. The Snaith splitting provides equivalences

$$K\Sigma^\infty X \simeq \bigsqcup_{k \geq 1} \Sigma^\infty X^\wedge_k \simeq \bigsqcup_{k \geq 1} \Sigma \wedge_k (\Sigma^\infty X)^\wedge_k \cong \underline{S} \circ (\Sigma^\infty X)$$

Said differently, the Taylor tower for $K$ splits as a coproduct of its homogeneous layers when evaluated on a suspension spectrum and that $\partial_* K \simeq \underline{S}$.

A result of Arone-Kankaanrinta [AK98] uses the above splittings to recover the model for $n$-th homogeneous layers and $n$-th derivatives of the identity in spaces in (3) and (4), respectively. The crux of their argument is that iterating the Snaith splitting provides equivalences

$$\partial_* Q^{k+1} \simeq \partial_* K^k \simeq S^0 k[n]$$

and that factors of $S^0 k[n]$ are in correspondence with elements of $\text{Par}_k(n)$.

Moreover, $\underline{S}$ possesses an inherent cooperad structure (see Section 3.11), with respect to which there there is further an equivalence

$$\partial_* \text{Id} \simeq \text{holim}_\Delta \text{Cobar}(I, \underline{S}, I).$$

The above equivalence is used by Ching [Chi05] to construct an operad structure on $\partial_* \text{Id}$ and will also be utilized in our constructions which follow.

3. **(op)Lax (co)monoids and $\bar{\partial}$-cooperads**

3.1. **Normal oplax monoids.** We refer to [Chi12b, Definition 1.1] for the definition of a normal oplax monoid structure on a category $C$. Essentially, we require $C$ to have “$k$-ary products”, i.e. functors $\otimes_k : C^k \to C$ which may fail to be strictly associative in that we may only have grouping maps of the form

$$\mu_{p_1,\ldots,p_n} : \otimes_k (X_1,\ldots,X_k) \to \otimes_n (\otimes_{p_1} (X_1,\ldots,X_{p_1}),\ldots,\otimes_{p_n} (X_{1+p_{n-1}},\ldots,X_{p_n}))$$

where $P_k := p_1 + \cdots + p_k$. Such maps $\mu$ are further required to satisfy associativity and unitality conditions (see, e.g. [Chi12b]).

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5Such diagrams are very finite, see e.g. Ching-Harper [CH19 §8]
We say an element $A$ is a **monoid** with respect to some normal oplax monoidal structure $(\mathcal{C}, \otimes, 1)$ if there are maps $m_k: \otimes_k(A, \ldots, A) \to A$ that the following associativity and unitality conditions

(8) \[
\otimes_k(A, \ldots, A) \xrightarrow{m_k} \otimes_n(\otimes_{p_1}(A, \ldots, A), \ldots, \otimes_{p_n}(A, \ldots, A)) \xrightarrow{m_n} \otimes_n(A, \ldots, A) \xrightarrow{m_n} A
\]

Similarly, given a monoid $A$ an **algebra** $X$ over $A$ is an element of $\mathcal{C}$ together with action maps $\omega_k: \otimes_k(A, \ldots, A, X) \to X$ which satisfy associativity and unitality in that the analogous diagrams to (8) commute with the last factor of $A$ replaced by $X$.

3.2. **Normal lax comonoids.** We will also use the dual notion of a normal lax monoidal structure. Formally a lax comonoid structure is simply an oplax monoid in the opposite category. In this case we will call the grouping maps $\mu$ **inclusion** maps.

Explicitly, we say that $Q$ is a **comonoid** with respect to a normal lax comonoidal structure $(\mathcal{C}, \amalg, 1)$ if there are comultiplication maps $\delta_k: Q \to \amalg_k(Q, \ldots, Q)$ that satisfy associativity (dual to diagram (8) q.v.).

3.3. **The box product of cosimplicial objects.** Given a monoidal category $(\mathcal{C}, \otimes, 1)$ and cosimplicial objects $X, Y \in \mathcal{C}^\Delta$, we define the cosimplicial object $X \boxtimes Y$ as follows.

(9) \[
(X \boxtimes Y)^n := \text{colim} \left( \bigoplus_{r+s=n-1} X^r \otimes Y^s \xrightarrow{\bigoplus_{p+q=n} X^p \otimes Y^q} \right)
\]

The maps in (9) are induced by $\text{id} \otimes d^0$ and $d^{r+1} \otimes \text{id}$, see e.g. [CH19, Definition 4.13] for the cosimplicial structure maps of $X \boxtimes Y$. When necessary to distinguish the underlying monoidal structure we write $\boxtimes$ for $\boxtimes$. Given $c \in \mathcal{C}$ we let $c$ denote the constant cosimplicial object on $c$.

**Remark 3.4.** A useful fact is that if $\mathcal{C}$ is closed symmetric monoidal then $(\mathcal{C}^\Delta, \boxtimes, 1)$ is a monoidal category. The box product has been used before to construct $A_\infty$-pairings on totalizations of $\boxtimes$-monoids in $\mathcal{C}^\Delta$: McClure-Smith [MS04] show that if $\mathcal{X}$ is a $\boxtimes$-monoid in $\mathcal{S}^\Delta$, then $\text{Tot} \mathcal{X}$ is an $A_\infty$-monoid. Similarly, Arone-Ching [AC16] and Ching-Harper [CH19] use the box product to construct an $A_\infty$-composition for coalgebra maps.

In our previous work [Cla20], the box product was used with respect to the composition product of symmetric sequences to produce an $A_\infty$ pairing on the totalization of certain cosimplicial symmetric sequences. As the monoidal category
(SymSeq, □, I) does not enjoy some particular formal properties, we do not get a strict monoidal structure on (SymSeqΔ, □ ◦, I). One useful way to parse what structure we do have is in terms of oplax monoids.

**Proposition 3.5.** The category of cosimplicial symmetric sequences has an oplax monoidal structure induced by the box product, □ ◦.

**Proof.** We set

\[ □_k^\circ (X_1, \ldots, X_k) = (\cdots ((X_1□_2X_2)□_3X_3)\cdots )□_k^\circ X_k \]

and note the grouping maps are obtained by the universal maps on colimits. □

3.6. **Reinterpreting the composition product.** We now adopt the notation of [Cla20] regarding Nlev-objects. Given a non-basepoint element \( \alpha \in \text{Par}_k(n) \) we let \( |\alpha| \) denote the corresponding profile in \( N^\circ k \) such that \( n \) is good for \( |\alpha| \). Moreover, we let \( \alpha_{j,i} \) be such that

\[ |\alpha| = (\alpha_1, 1, \ldots, \alpha_{2,1}, \ldots, (\alpha_{k,1}, \ldots, \alpha_{k,\alpha_{k-1}})) \]

where \( \alpha_j \) is inductively defined as \( \alpha_j := \alpha_{j,1} + \cdots + \alpha_{j,\alpha_{j-1}} \). Note \( n = \alpha^k \).

Said differently, \( \alpha_j \) is the number of partitions in \( \lambda_j \), and \( \alpha_{j,1}, \ldots, \alpha_{j,\alpha_{j-1}} \) is the size of the partitions appearing in \( \lambda_{j-1} \) for \( j = 1, \ldots, k \). Note that \( |\alpha| \) is not uniquely determined by \( \alpha \).

**Definition 3.7.** Let \( A_1, \ldots, A_k \) be reduced symmetric sequences in a symmetric monoidal category \((C, \otimes, 1)\). We define their composition product as follows.

\[ (A_1 \circ A_2 \circ \cdots \circ A_k)[n] = \bigvee_{\alpha \in \text{Par}_k(n)} (A_1 \otimes \cdots \otimes A_k)[\alpha] \]

where \( (A_1 \otimes \cdots \otimes A_k)[\alpha] := A_1[\alpha_1] \otimes \bigotimes_{i=1}^{\alpha_1} A_2[\alpha_{2,i}] \otimes \cdots \otimes \bigotimes_{i=1}^{\alpha_{k-1}} A_k[\alpha_{k,i}] \) as \( \Sigma^- \)-objects. Similarly, their dual composition product is defined as

\[ (A_1 \check\circ A_2 \check\circ \cdots \check\circ A_k)[n] = \prod_{\alpha \in \text{Par}_k(n)} (A_1 \otimes \cdots \otimes A_k)[\alpha]. \]

Note, if \( C \) is stable, i.e. \( C = \text{Spt} \) the category of symmetric spectra, then finite coproducts and products are equivalent and hence the natural comparison

\[ A_1 \circ \cdots \circ A_k \sim A_1 \check\circ \cdots \check\circ A_k \]

is a weak equivalence of symmetric sequences. Moreover, given \( p \in N^\circ k \) with \( n \) good for \( p \) (see [Cla20] Definition 6.4) and reduced symmetric sequences \( A_1, \ldots, A_k \) there

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\[ ^6 \text{For instance, } - \circ - \text{ is not symmetric and will only commute with colimits in the right-hand variable.} \]

\[ ^7 \text{We denote by } \Sigma_\alpha \leq \Sigma_\alpha \text{ the subgroup of permutations } \sigma \text{ which fix the partition } \alpha. \]
are isomorphisms:
\[
(A_1 \circ \cdots \circ A_k)[p] \cong \bigvee_{\alpha \in \text{Par}_k(n), |\alpha| = p} (A_1 \otimes \cdots \otimes A_k)[\alpha]
\]
\[
(A_1 \tilde{\circ} \cdots \tilde{\circ} A_k)[p] \cong \prod_{\alpha \in \text{Par}_k(n), |\alpha| = p} (A_1 \otimes \cdots \otimes A_k)[\alpha]
\]

Remark 3.8. If the underlying product is closed (more accurately, if \(\otimes\) commutes with coproducts) then \(\circ\) is associative and defines a monoidal product for symmetric sequences in \(\mathcal{C}\) with unit \(I\). We say a symmetric sequence \(A\) is a (co)operad if it is a (co)monoid with respect to \(\circ\).

For our purposes we will need to work with the dual composition product, which is rarely strictly associative, and therefore it does not make sense to speak of strict (co)monoids with respect to \(\tilde{\circ}\). However, the universal map on products makes \(\tilde{\circ}\) lax comonoidal which is sufficient to discuss comonoids and cobar resolutions. For the remainder of this document, we denote the inclusion maps for the lax comonoidal structure \(\tilde{\circ}\) by \(\mu\).

Definition 3.9. Let \(Q\) be a reduced symmetric sequence in \(\mathcal{C}\). We say that \(Q\) is a \(\tilde{\circ}\)-cooperad if \(Q\) is a lax comonoid with respect to \(\tilde{\circ}\). Following standard arguments, a \(\tilde{\circ}\)-cooperad consists of

- A cocomposition map \(\delta: Q \to Q \tilde{\circ} Q\)
- A counit map \(\epsilon: Q \to I\)
- The cocomposition is required to be coassociative and counital in that the following diagrams must commute

\[
\begin{align*}
Q & \xrightarrow{\delta} Q \tilde{\circ} Q \\
\downarrow \delta & \quad \quad \downarrow \delta \\
Q \tilde{\circ} Q & \quad (Q \tilde{\circ} Q) \tilde{\circ} Q \\
\downarrow \text{id} \tilde{\circ} \delta & \quad \quad \downarrow \mu_{2,1} \\
Q \tilde{\circ} (Q \tilde{\circ} Q) & \xrightarrow{\mu_{1,2}} Q \tilde{\circ} Q \tilde{\circ} Q
\end{align*}
\]

and

\[
\begin{align*}
Q \tilde{\circ} I & \xrightarrow{\text{id} \tilde{\circ} \epsilon} Q \tilde{\circ} Q \\
\downarrow \cong & \quad \quad \downarrow \cong \\
Q & \quad Q \tilde{\circ} Q \\
\downarrow \delta & \quad \quad \downarrow \delta \\
Q \tilde{\circ} Q & \xrightarrow{\delta} Q \tilde{\circ} Q
\end{align*}
\]

Remark 3.10. It is possible to define (divided power) coalgebras with respect to a \(\tilde{\circ}\)-cooperad \(Q\) as being certain lax comonads with respect to the functor
\[
Y \mapsto Q \tilde{\circ}(Y) = \prod_{n \geq 0} (Q[n] \wedge Y^\wedge n)_{\Sigma_n},
\]
however we will not require such descriptions for our purposes.
3.11. The commutative cooperad of spectra. We will now show that $S$ satisfies the criteria necessary to be a cooperad of spectra; hence rightfully referring to its coalgebras as commutative coalgebras of spectra. First, we will require a technical lemma.

**Proposition 3.12.** There is a $\Sigma_k$-equivariant map

$$\Psi_{k,n} : \text{Par}_p(n) \times \Sigma_n \left( \prod_{T_1, \ldots, \Pi T_n = k} \text{Par}_q(T_1) \times \cdots \times \text{Par}_q(T_n) \right) \rightarrow \text{Par}_{p+q}(k).$$

**Proof.** Let $\beta_j \in \text{Par}_q(T_j)$ be given by $\mu^j_0 \leq \cdots \leq \mu^j_q$ for $j = 1, \ldots, n$ and via $i \mapsto \alpha_i$, identify $n$ with the set $A = \{\alpha_1, \ldots, \alpha_n\}$. Pick $\Lambda \in \text{Par}_p(A)$ of the form $\lambda_0 \leq \cdots \leq \lambda_p$ and let $\lambda'_j$ denote the partition obtained by replacing a set $\{\alpha_s\}_{s \in S} \in \lambda_j$ by $\prod_{s \in S} T_s$. The image $\Psi_{k,n}(\Lambda; \beta_1, \ldots, \beta_n) \in \text{Par}_{p+q}(k)$ is given by the chain

$$\lambda_0 \leq \cdots \leq \lambda'_{p-1} \leq \lambda'_p \cong \prod_{i=1}^n \mu^i_0 \leq \prod_{i=1}^n \mu^i_1 \leq \cdots \leq \prod_{i=1}^n \mu^i_q.$$

Note further that the middle isomorphism is an isomorphism of $\Sigma_n$ objects. \qed

**Proposition 3.13.** The symmetric sequence $S$ is a $\delta$-cooperad.

**Proof.** Note that $S[\alpha] \cong S$ for any $\alpha \in \text{Par}_k(n)$ and so we obtain

$$\delta_k : S[k] \cong S \rightarrow \prod_{\text{Par}_2(k)} S \cong (S \delta S)[k]$$

as the induced map on products. The counit map is given by the collapsing map $S \rightarrow *$ above level 1.

To show that $\delta$ satisfies associativity we show that there are well-defined inclusion maps

$$\mu_{p,q} : (S^{\delta p}) \delta (S^{\delta q}) \rightarrow S^{\delta p+q}.$$ It follows that $\mu_{p,q}$ must be given by the following composite

$$\begin{align*}
(S^{\delta p}) \delta (S^{\delta q})[k] &= \prod_{\alpha \in \text{Par}_2(k)} \left( \prod_{\text{Par}_p(\alpha_1)} S \right) \land_{\Sigma_n} \left( \prod_{i=1}^n \left( \prod_{\text{Par}_q(\alpha_{2,i})} S \right) \right) \\
&\rightarrow \prod_{n \geq 1} \left( \prod_{\text{Par}_p(n)} S \right) \land_{\Sigma_n} \left( \prod_{T_1, \ldots, \Pi T_n = k} \text{Par}_q(T_1) \times \cdots \times \text{Par}_q(T_n) \right) \left( \prod_{T_1, \ldots, \Pi T_n = k} S \land \cdots \land S \right) \\
&\rightarrow \prod_{n \geq 1} \left( \prod_{\text{Par}_p(n) \times \Sigma_n} \left( \prod_{\text{Par}_q(T_1) \times \cdots \times \text{Par}_q(T_n)} S \land S \land \cdots \land S \right) \right) \\
&\rightarrow \prod_{n \geq 1} \Psi_{k,n} \left( \prod_{\text{Par}_p(n) \times \Sigma_n} \left( \prod_{\text{Par}_q(T_1) \times \cdots \times \text{Par}_q(T_n)} S \land S \land \cdots \land S \right) \right) \cong (S^{\delta p+q})[k].
\end{align*}$$
In general, \( \mu_{p_1, \ldots, p_k} \) is given by repeated applications of the above composite from the left to the right.

Remark 3.14. Note that any space \( X \) yields an algebra over the \( \delta \)-cooperad \( \mathcal{S} \) as the canonical diagonal maps \( X \to X \wedge \cdots \wedge X \) provide morphisms
\[
\Sigma^n X \to \Sigma^n (X \wedge \cdots \wedge X)_{\Sigma_n} \cong (\Sigma X \wedge \cdots \wedge \Sigma X)_{\Sigma_n}.
\]

3.15. The cobar complex. Let \( Q \) be a \( \delta \)-cooperad with left \( Q \)-comodule \( M \) and right \( Q \)-comodule \( N \), we denote by \( \text{Cobar}(N, Q, M) \) the following cosimplicial object (showing only coface maps)
\[
\begin{array}{c}
N \delta M \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
N \delta Q \delta M \\
N \delta Q \delta Q \delta M \\
N \delta Q \delta Q \delta Q \delta M \\
N \delta Q \delta Q \delta Q \delta Q \delta M \\
N \delta Q \delta Q \delta Q \delta Q \delta Q \delta M \\
N \delta Q \delta Q \delta Q \delta Q \delta Q \delta Q \delta M \\
\end{array}
\]
Coface maps \( d^i \) are obtained by inserting the cocomposition on \( Q \) at the \( i \)-th position (if \( i = 0, n \) we use the coaction maps on \( N \) and \( M \) resp.) and then using the lax monoidal structure; codegeneracy maps \( s^j \) are induced by the counit \( Q \to I \) at the \( j \)-th spot.

4. \( A_\infty \)-operad structure on \( \partial_* \text{Id} \)

Let \( C_{\text{Id}} \) denote the cobar complex on the reduced cooperad \( \mathcal{S} \), i.e.
\[
C_{\text{Id}} := \text{Cobar}(I, \mathcal{S}, I).
\]
It is not hard to see that \( C_{\text{Id}} \) is an (oplax) monoid with respect to \( \square^\circ \), however, \( C_{\text{Id}} \) is not fibrant enough for our purposes. In order to run the machinery used before in [Cla20], we must replace \( C_{\text{Id}} \) by an objectwise fibrant diagram \( \mathfrak{C}_{\text{Id}} \) which retains the \( \square^\circ \)-monoid structure. Our method is to first replace \( C_{\text{Id}} \) by \( \text{Cobar}^\delta(I, \mathcal{S}, I) \) and use the special structure of \( \mathcal{S} \) to provide an (objectwise) fibrant replacement.

We first note that
\[
\text{Cobar}^\delta(I, \mathcal{S}, I)[n]^k \cong \prod_{\mathbb{P}a(n)} S
\]
and moreover that a coface map \( d^i \) (resp. codegeneracy map \( s^j \)) of \( \text{Cobar}(I, \mathcal{S}, I)[n] \) is induced by the corresponding face map \( d_i \) (resp. degeneracy map \( s_j \)) of \( \mathbb{P}a(n) \).

The benefit of working with \( \text{Cobar}^\delta(I, \mathcal{S}, I) \) is that we only need to fibrantly replace \( S \) as a commutative monoid. For now, we fix a simplicial fibrant replacement monad \( FF \to F \) on \( \text{Alg}_{\text{Com}} \) with unit \( \nu: id \to F \) (see Blumberg-Riehl [BR14]).

Definition 4.1. Let \( \mathfrak{C}_{\text{Id}} \) be the cosimplicial symmetric sequence given by
\[
\mathfrak{C}_{\text{Id}}[n] := \prod_{\mathbb{P}a(n)} FS
\]
with coface and codegeneracy maps induced from the simplicial structure of \( \mathbb{P}a(n) \).

There is then an equivalence of cosimplicial objects \( \text{Cobar}^\delta(I, \mathcal{S}, I) \xrightarrow{\sim} \mathfrak{C}_{\text{Id}} \) induced by the map \( S \xrightarrow{\sim} FS \). In particular, \( \mathfrak{C}_{\text{Id}} \) is objectwise fibrant and so we define
\[
\partial_* \text{Id} := \text{Tot} \mathfrak{C}(\mathcal{S}) \cong \text{hom}_\Delta(\Sigma \cdot \Delta, \mathfrak{C}(\mathcal{S}))^{\mathbb{S}}.
\]
Since \( \text{Par}_k(n) \) is a finite set for each \( n, k \), the comparison maps \( \boxtimes \) induce an equivalence of cosimplicial objects \( C_{\text{Id}} \sim \text{Cobar}^\delta(I, S, I) \) and so \( \text{Tot} C_{\text{Id}} \simeq \text{holim}_\Delta C_{\text{Id}} \).

**Proposition 4.2.** The fattened up cosimplicial symmetric sequence \( C_{\text{Id}} \) is a normal oplax monoid with respect to \( \boxtimes^\delta \).

**Proof.** First we construct maps \( \mu_{p,q} : C_{\text{Id}} \boxtimes^\delta C_{\text{Id}} \rightarrow C_{\text{Id}}^{p+q} \), similar to the proof of Proposition \[\text{[3,13]}\] We set \( \mu_{p,q} \) to be the composite

\[
(\text{C}_{\text{Id}} \boxtimes^\delta \text{C}_{\text{Id}})[k] \rightarrow \prod_{n \geq 1} \left( \prod_{\text{Par}_p(n)} \text{FS} \right) \wedge \Sigma_n \left( \prod_{T_1 \sqcup \cdots \sqcup T_n \Rightarrow k} \text{Par}_q(T_1) \times \cdots \times \text{Par}_q(T_n) \right) \rightarrow \prod_{n \geq 1} \left( \prod_{\text{Par}_p(n) \times \Sigma_n} \left( \prod_{T_1 \sqcup \cdots \sqcup T_n \Rightarrow k} \text{FS} \wedge \text{FS} \wedge \cdots \wedge \text{FS} \right) \right) \rightarrow \prod_{\text{Par}_{p+q}(k)} \text{FS} \wedge \cdots \wedge \text{FS} \rightarrow \text{C}_{\text{Id}}^{p+q}[k]
\]

where \( \uparrow \) is induced by the monoidal structure on \( \text{FS} \).

The comparison map \( C_{\text{Id}} \boxtimes^\delta C_{\text{Id}} \rightarrow C_{\text{Id}} \) is then supplied at cosimplicial degree \( n + 1 \) by the following commuting diagrams for each \( p + q = n \)

\[
\begin{array}{ccc}
(S^p \circ S^q) & \rightarrow & (S^p) \circ (S^q) \\
\sim & \downarrow \text{id} \circ d^0 & \sim \\
(S^p \circ (S^q+1)) & \rightarrow & (S^p) \circ (S^q+1) \\
\mu_{p+1,q} & \downarrow & \mu_{p+1,q}
\end{array}
\]

Note the unit map \( I \rightarrow C_{\text{Id}} \) is induced by the unit \( S \rightarrow \text{FS} \) on factors of the form \( \text{Par}_p(1) \) and the trivial map \( S \rightarrow \ast \) otherwise.

**Corollary 4.3** (Proof of Theorem \[\text{[4,11]}\]. \( \partial_{\text{Id}} \) is an \( A_\infty \)-operad.

**Proof.** Since \( \partial_{\text{Id}} = \text{Tot} C_{\text{Id}} \cong \text{hom}_\Delta (\Sigma^\Delta, C_{\text{Id}})^\Sigma \), this follows from a straightforward adaptation of the proof of \[\text{[C20, Theorem 8.3]}\].

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