SIXTEEN POINTS IN $\mathbb{P}^4$ AND THE INVERSE GALOIS PROBLEM FOR DEL PEZZO SURFACES OF DEGREE ONE

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ABSTRACT. A del Pezzo surface of degree one defined over the rationals has 240 exceptional curves. These curves are permuted by the action of the absolute Galois group. We show how a solution to the classical inverse Galois problem for a subgroup of the Weyl group of type $D_8$ gives rise to a solution of the inverse Galois problem for the action of this subgroup on the 240 exceptional curves. A del Pezzo surface of degree one with such a Galois action contains a Galois invariant sublattice of type $D_8$ within its Picard lattice; this can be characterized in terms of a certain set of sixteen points in $\mathbb{P}^4$.

1. INTRODUCTION

If $\tilde{X}_2$ is a del Pezzo surface of degree one, then it is the double cover of a quadric cone $X_2$ branched over a nonsingular genus four curve $X$ and the vertex of the cone. The quadric cone $X_2$ containing $X$ identifies a vanishing even theta characteristic $\theta_0$ of $X$. Let $\kappa$ denote the canonical class of $\tilde{X}_2$. The intersection pairing on $\tilde{X}_2$ gives $\text{Pic}(\tilde{X}_2)$ the structure of a lattice, which admits a decomposition of the form

$$\text{Pic}(\tilde{X}_2) = \langle \kappa \rangle \oplus \text{Pic}(\tilde{X}_2)^\perp$$

where $\text{Pic}(\tilde{X}_2)^\perp$ denotes the sublattice of divisor classes orthogonal to the canonical class under the intersection pairing. The sublattice $\text{Pic}(\tilde{X}_2)^\perp$ is a root lattice of type $E_8$, and such a lattice has 135 sublattices $\Lambda_{D_8} \subset \text{Pic}(\tilde{X}_2)^\perp$ of type $D_8$. In this article, we consider those del Pezzo surfaces of degree one whose Picard lattice has a Galois invariant sublattice of type $D_8$. The Galois action on $\text{Pic}(\tilde{X}_2)$ permutes the 240 exceptional curves of $\tilde{X}_2$ and acts on $\text{Pic}(\tilde{X}_2)^\perp$ though the Weyl group of type $E_8$, denoted $W_{E_8}$. As it turns out, the stabilizer in the Weyl group $W_{E_8}$ of a $D_8$-sublattice is of index 135, and in fact it is isomorphic to the Weyl group $W_{D_8}$. Further details about del Pezzo surfaces can be found in [Dol12, Chapter 8]. We show how to transmute the solution to the classical inverse Galois problem for subgroups of $W_{D_8}$ into a solution to the inverse Galois problem for del Pezzo surfaces of degree one with a Galois invariant sublattice $\Lambda_{D_8} \subset \text{Pic}(\tilde{X}_2)$ of type $D_8$.

Theorem 1.0.1. Let $\rho: \text{Gal}(\mathbb{Q}_{\text{sep}}/\mathbb{Q}) \rightarrow W_{D_8}$ be a continuous homomorphism. If $G := \text{Im}(\rho)$ is the Galois group of some irreducible polynomial over $\mathbb{Q}$, then there exists a nonsingular del Pezzo surface of degree one $\tilde{X}_2$ such that each $\sigma \in \text{Gal}(\mathbb{Q}_{\text{sep}}/\mathbb{Q})$ permutes the 240 exceptional curves of $\tilde{X}_2$ as described by $\rho(\sigma) \subset S_{240}$.

In particular, the subfield of $\mathbb{Q}_{\text{sep}}$ fixed by $\rho^{-1}(\text{id})$ is the splitting field of the 240 exceptional curves. The group $W_{D_8}$ admits a transitive permutation representation on a set of size 16 (see Section 2). If $\rho^\prime: W_{D_8} \rightarrow S_{16}$ is the associated morphism and if $G \subseteq W_{D_8}$ is a subgroup for which a solution to the inverse Galois problem is known, then there exists a (possibly reducible) polynomial $f$ of degree 16 such that $G \cong \text{Gal}(f)$ and each $\sigma \in G$ permutes the roots of $f$ as described by $\rho^\prime(\sigma)$. A solution to the inverse Galois problem is known for several subgroups of $W_{D_8}$, including $W_{D_8}$ itself.

Theorem 1.0.2. Let $G$ be a group which acts transitively on a set of size at most 16. Then there is an irreducible polynomial $f$ over $\mathbb{Q}$ such that $\text{Gal}(f) \cong G$.

Proof. A list of examples can be obtained from the online database of Klüners and Malle [KM01, KM21].

Our techniques are related to those in the article by Elsenhans and Jahnel [EJ19], where they study plane quartics in terms of a set of eight points in $\mathbb{P}^3$ called a Cayley octad.

In [Cor07], Corn showed that the Brauer group of a del Pezzo surface is one of a finite number of explicit possibilities. Specifically, for a del Pezzo surface $X$ over $\mathbb{Q}$ there is a canonical isomorphism $\text{Br}X/\text{Br}\mathbb{Q} \cong \mathbb{Z}/2\mathbb{Z}$.

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allows us to resolve this question for the groups above.

Then the norm of an element \( \alpha \in K \) is the usual field norm \( N_{K/Q}(\alpha) \). Otherwise, we define the norm of \( \alpha \) by

\[
N_{K/Q}(\alpha) := \prod_j N_{K_j/Q}(\alpha_j).
\]

### 2. Background

Let \( K/Q \) be an étale algebra, and let \( K = \prod_j K_j \) be a decomposition into simple factors. If \( K \) is simple, then the norm of an element \( \alpha \in K \) is the usual field norm \( N_{K/Q} \). Otherwise, we define the norm of \( \alpha \) by

\[
N_{K/Q}(\alpha) := \prod_j N_{K_j/Q}(\alpha_j).
\]

#### 2.1. Étale algebras with a \( W_{D_8} \) action.

The (abstract) group \( W_{D_8} \) is isomorphic to an index 2 subgroup of the wreath product \( S_2 \wr S_8 \).

Specifically, it is the extension of \( S_8 \) by the subgroup of elements of \( \mu_2 \) whose entries multiply to 1. We have the exact sequence

\[
0 \longrightarrow \mu_2 \longrightarrow W_{D_8} \longrightarrow S_8 \longrightarrow 1.
\]
Essentially by definition, the wreath product $S_2 \wr S_8$ admits an action on a set of 16 elements $\Omega$ (the eight elements on which $S_8$ acts endowed with signs).

**Construction 2.1.1.** Given $\rho : \text{Gal}(\mathbb{Q}^\text{sep}/\mathbb{Q}) \to W_{\text{Dr}}$ a continuous homomorphism with image $G$, we describe how to construct an étale algebra $L$ of degree 16 such that $\text{Hom}(L, \mathbb{Q}^\text{sep})$ is isomorphic to $\Omega$ as a $G$-set. We may choose a set of orbit representatives $\beta_1, \ldots, \beta_r$ such that $\Omega = G \cdot \beta_1 \cup \cdots \cup G \cdot \beta_r$. Functionality between étale algebras and continuous Galois actions on a finite set allows us to identify an étale algebra $\Omega$.

The 16 elements of $\Omega$ are identified with the 16 homomorphisms $L \to \mathbb{Q}^\text{sep}$ via the $\text{Hom}$ functor. The natural quotient of $G$-sets $\Omega \to (\Omega/\pm)$ induces an inclusion of étale algebras $K \hookrightarrow L$, where the degree of $K$ is equal to 8. Up to isomorphism we may write $K = \prod_j K_j$ and

\[(\dagger) \quad L \cong \prod_j K_j[x]/(x^2 - \alpha_j)\]

for some finite field extensions $K_j/\mathbb{Q}$ and some $\alpha_j \in K_j$ with the property that $\prod_j N_{K_j/\mathbb{Q}}(\alpha_j) \in \mathbb{Q}^{\times 2}$. We call $K$ the distinguished subalgebra.

2.2. **Zariski density of generators.** We denote by $\text{Pol}^8$ the scheme whose set of $R$-valued points is the set of monic polynomials of degree 8 with coefficients in $R$, where $R$ is a commutative ring containing $\mathbb{Q}$. Of course, $\mathbb{A}^8_\mathbb{Q} \cong \text{Pol}^8$. If $K$ is an étale $\mathbb{Q}$-algebra of dimension 8, we may endow $K$ with the structure of an affine scheme, specifically the affine space $\mathbb{A}^8_K$. We denote this scheme by $\mathbb{A}^8_K$. The functor of points is given by $\mathbb{A}^8_K(R) = K \otimes_\mathbb{Q} R$. The scheme $\mathbb{A}^8_K$ is simply the Weil restriction $\text{Res}_{K/\mathbb{Q}} \mathbb{A}^1_K$. If $R$ is a commutative ring containing $\mathbb{Q}$, then we denote by $\mathbb{A}^8_K$ the scheme $\mathbb{A}^8_K \times_{\text{Spec} \mathbb{Q}} \text{Spec} R$. If $\beta \in \mathbb{A}^8_K(R)$, we denote by $[\beta]$ the endomorphism of $R \otimes_\mathbb{Q} K$ defined by multiplication-by-$\beta$. There is a canonical morphism of schemes $\chi^K : \mathbb{A}^8_K \to \text{Pol}^8$

As an example, if $K = \prod_{j=1}^8 \mathbb{Q}$ is the split étale $\mathbb{Q}$-algebra of dimension 8, then $\chi^K$ is simply the map defined by the elementary symmetric functions.

**Lemma 2.2.1.** The set $\chi^K(\mathbb{A}^8(\mathbb{Q}))$ is Zariski dense in $\text{Pol}^8$.

**Proof.** Let $\hat{K}$ be a splitting field for $K$. Since $\mathbb{A}^8$ is just affine space, $\mathbb{A}^8(\mathbb{Q})$ is dense in $\mathbb{A}^8_K$. On the other hand, if $A := \mathbb{Q}^8$ is the split étale $\hat{K}$ algebra, diagonalization gives an isomorphism $\psi : \mathbb{A}^8_K \to \mathbb{A}^8_K$. Diagonalization does not change the characteristic polynomial, so the diagram

\[\begin{array}{ccc}
\mathbb{A}^8_K & \xrightarrow{\psi} & \mathbb{A}^8_K \\
\chi^K & \downarrow & \chi^K \\
\text{Pol}^8 & \xrightarrow{\chi^K} & \mathbb{A}^8_K
\end{array}\]

of morphisms over $\hat{K}$ commutes. But $\chi^A$ is just the morphism defined by elementary symmetric functions, so it is surjective as a morphism of schemes, and thus so too is $\chi^K$. $\square$

**Corollary 2.2.2.** Given an étale algebra $L$ arising from Construction 2.1.1, with distinguished subalgebra $K$, the set

\[\left\{ (\alpha_1, \ldots, \alpha_r) \in \prod J \cap K_j \mid L \cong \prod J K_j[x]/(x^2 - \alpha_j), \ K_j \cong \mathbb{Q}(\alpha_j) \right\}\]

has a dense image under $\chi^K$ in $\text{Pol}^8$.

**Proof.** Choose a presentation for $L$ as in Equation $(\dagger)$. Since the set $\alpha \cdot K^{\times 2}$ is Zariski dense in $\mathbb{A}^8$ for any unit $\alpha \in K^{\times}$, and contains a dense subset of primitive elements, the result follows. $\square$
2.3. Tensors. We denote by $\text{Sym}_2 \mathbb{Q}^{m+1}$ the space of $(m+1) \times (m+1)$ symmetric matrices over $\mathbb{Q}$ and by $\text{Sym}_2 \mathbb{Q}^{n+1}$ the space of quadratic forms over $\mathbb{Q}$ in $n+1$ variables. Let $A \in \mathbb{Q}^{n+1} \otimes \text{Sym}_2 \mathbb{Q}^{m+1}$ be a tensor, symmetric in the last two entries. We view $A$ as an $(n+1) \times (m+1) \times (m+1)$ array of elements of $\mathbb{Q}$. We may think of such an array as being an ordered collection $A_0, \ldots, A_n$ of symmetric $(m+1) \times (m+1)$ matrices, the slices of $A$. We denote the contraction of $A$ along a vector $v \in \mathbb{Q}^{n+1}$ by $A(v, \cdot, \cdot)$. More generally, we will contract along an element of $R^m$, where $R$ is a $\mathbb{Q}$-algebra. Similarly, we will denote the contraction by an element $y \in R^{m+1}$ by $A(\cdot, y, \cdot)$ or $A(\cdot, \cdot, y)$, depending along which axis we contract.

Denote $[n] := \text{Proj} \left[ \mathbb{Q}[x_0, \ldots, x_n] \right]$, $\mathbb{P}^n := \text{Proj} \left[ \mathbb{Q}[y_0, \ldots, y_m] \right]$, and denote the dual projective space of $\mathbb{P}^n$ by $\mathbb{P}^m$. We will also denote $x := (x_0, \ldots, x_n)$ and $y := (y_0, \ldots, y_m)$. If $x \in \mathbb{Q}^{n+1}$, then the contraction $A(x, y, y)$ is the quadratic form

$$y^T A_0 x + \cdots + x_n A_n y.$$

If $x \in \mathbb{P}^n(\mathbb{Q})$ is a point, then the contraction $A(x, \cdot, \cdot)$ is a symmetric $(m+1) \times (m+1)$ matrix with entries in $\mathbb{Q}$, well-defined up to scaling. Write $(q_0, \ldots, q_n)$ for the quadratics defined by the $n+1$ slices of $A$. The $q_i$ define the rational map

$$\psi: \mathbb{P}^m \dasharrow \mathbb{P}^n, \quad y \mapsto A(\cdot, y, y) = (q_0(y), \ldots, q_n(y)).$$

3. Proof of the main result

3.1. Construction of 16 points in $\mathbb{P}^4$. Given a hyperelliptic genus 3 curve $Y$ of the form

$$Y: y^2 = f(z_0, z_1)$$

such that $f(z_0, z_1)$ is homogeneous, square-free, and $f(0, 1), f(1, 0) \in \mathbb{Q}[z]$, we describe a method to construct 16 points in $\mathbb{P}^4$ based on the construction in [KV21, Section 6].

Define $K := \mathbb{Q}[z]/(f(z, 1))$ and $L := \mathbb{Q}[z]/(f(z^2, 1))$. An elementary calculation shows that the conditions on $f(z_0, z_1)$ allow us to write

$$f(z_0, z_1) = b(z_0, z_1)^2 - z_0 z_1 c(z_0, z_1) = -\det \begin{bmatrix} z_0 z_1^3 & b(z_0, z_1) \\ b(z_0, z_1) & c(z_0, z_1) \end{bmatrix}$$

for some homogeneous polynomials $b(z_0, z_1), c(z_0, z_1)$ of degree 4. If $\alpha_1, \ldots, \alpha_8$ are the roots of $f(z, 1)$, then the set

$$\mathcal{C} := \{ (\pm \sqrt{\alpha_1}, \pm \sqrt{\alpha_2}, \frac{b(\alpha, 1)}{\alpha + \alpha^2} : 1 \leq j \leq 8 \}$$

of points in $\mathbb{P}^4$ is split over $L$, the Galois closure of $L/\mathbb{Q}$. One can check that when the $\alpha_j$ are distinct, the Vandermone matrix of degree 2 forms evaluated at the points of $\mathcal{C}$ has corank 5, so $\mathcal{C}$ is contained in the intersection of 4 quadrics. Generically, $\mathcal{C}$ is a complete intersection of 4 quadrics in $\mathbb{P}^4$. If $L$ is presented as in Equation (1) such that $\alpha_1, \ldots, \alpha_8$ are primitive elements of $K_1, \ldots, K_8$ (respectively), then we have that

$$\{ \sqrt{\alpha_j} : 1 \leq j \leq 8 \} = \text{Hom}_2(L, \text{Q}^{\text{sep}}).$$

Thus, the permutation action of $\text{Gal}(\text{Q}^{\text{sep}}/\mathbb{Q})$ on the 16 points of $\mathcal{C}$ is identical to the action of $\text{Gal}(\text{Q}^{\text{sep}}/\mathbb{Q})$ on $\Omega$.

We can explicitly describe the linear space of quadrics containing $\mathcal{C}$. Write $\mathbb{P}^4 := \text{Proj}(\mathbb{Q}[y_0, y_1, y_2, y_3, y_4])$, which admits a natural projection to $\mathbb{P}^2 := \text{Proj}(\mathbb{Q}[y_2, y_3, y_4])$. Under the Veronese embedding

$$\nu_2: \mathbb{P}^1 \dasharrow \text{Proj}(\mathbb{Q}[y_2, y_3, y_4]), \quad (z_0 : z_1) \mapsto (z_0^2 : z_0 z_1 : z_1^2)$$

the quartic forms $z_0 z_1^3, b(z_0, z_1), c(z_0, z_1)$ can be identified with the three quadratic forms $y_1 y_4, b(y_2, y_3, y_4), c(y_2, y_3, y_4)$; the identified quadratic forms are unique modulo $y_1^2 - y_2 y_4$. The determinantal representation for $f(z_0, z_1)$ becomes a quadratic determinantal representation, i.e., is a tensor in $\text{Sym}^2 \mathbb{Q}^3 \otimes \text{Sym}_2 \mathbb{Q}^2$.

Construction 3.1.1. Let $\mathbb{P}^2 := \text{Proj}(\mathbb{K}[y_2, y_3, y_4])$. Given a conic $C \subset \mathbb{P}^2$ and a tensor $B \in \text{Sym}_2 \mathbb{K}^2 \otimes \text{Sym}_2 \mathbb{K}^2$ representing a $2 \times 2$ matrix of quadratic forms, we can construct a tensor $A \in \mathbb{K}^4 \otimes (\text{Sym}_2 \mathbb{K}^2 \otimes \text{Sym}_2 \mathbb{K}^2)$ as follows: Writing $B(y, \cdot) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, we have that

$$a(y) = \sum_{2 \leq i,j \leq 4} a_{ij} y_i y_j, \quad b(y) = \sum_{2 \leq i,j \leq 4} b_{ij} y_i y_j, \quad c(y) = \sum_{2 \leq i,j \leq 4} c_{ij} y_i y_j.$$
We may write the defining equation for $C$ as $\sum_{2 \leq i, j \leq 4} a_{ij}y_iy_j$. (The terms with $i \neq j$ appear twice.) Define the slices of $A$ to be
\[
A_0 := \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{22} & a_{23} & a_{24} \\
0 & 0 & a_{32} & a_{33} & a_{34} \\
0 & 0 & a_{42} & a_{43} & a_{44}
\end{bmatrix}, \quad A_1 := \begin{bmatrix}
0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & b_{22} & b_{23} & b_{24} \\
0 & 0 & b_{32} & b_{33} & b_{34} \\
0 & 0 & b_{42} & b_{43} & b_{44}
\end{bmatrix},
\]
\[
A_2 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & c_{22} & c_{23} & c_{24} \\
0 & 0 & c_{32} & c_{33} & c_{34} \\
0 & 0 & c_{42} & c_{43} & c_{44}
\end{bmatrix}, \quad A_3 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{22} & d_{23} & d_{24} \\
0 & 0 & d_{32} & d_{33} & d_{34} \\
0 & 0 & d_{42} & d_{43} & d_{44}
\end{bmatrix}.
\]
The tensor $A$ generically defines a genus 4 curve via an intersection of the two symmetric determinantal varieties $X_2 := Z(4x_0x_2 - x_1^2)$ and $X_3 := Z(\det(A^{(2)}(x, \cdot, \cdot)))$, where
\[
A^{(2)}(x, \cdot, \cdot) := x_0 \begin{bmatrix}
a_{22} & a_{23} & a_{24} \\
a_{32} & a_{33} & a_{34} \\
a_{42} & a_{43} & a_{44}
\end{bmatrix} + x_1 \begin{bmatrix}
b_{22} & b_{23} & b_{24} \\
b_{32} & b_{33} & b_{34} \\
b_{42} & b_{43} & b_{44}
\end{bmatrix} + x_2 \begin{bmatrix}
c_{22} & c_{23} & c_{24} \\
c_{32} & c_{33} & c_{34} \\
c_{42} & c_{43} & c_{44}
\end{bmatrix} + x_3 \begin{bmatrix}
d_{22} & d_{23} & d_{24} \\
d_{32} & d_{33} & d_{34} \\
d_{42} & d_{43} & d_{44}
\end{bmatrix}.
\]
From the four symmetric matrices $A_0, \ldots, A_3$ we obtain an intersection of four quadrics in $\mathbb{P}^4$.

**Example 3.1.2.** Let
\[
f(z_0, z_1) := (z_0^2 - z_1^2)(z_0^2 + 4z_1^2)(z_0^2 - 9z_1^2)(z_0^2 - 16z_1^2) \in \mathbb{P}^1_{101}[z_0, z_1].
\]
Choosing
\[
b(z_0, z_1) := z_0^4 - 86z_0^2z_1^2 + 24z_1^4 \sim y_2^2 + 86y_3^2 + 24y_4^2, \quad c(z_0, z_1) := -z_0z_1^3 - y_3y_4,
\]
we have that $f(z_0, z_1) = b(z_0, z_1)^2 - z_0z_1^3c(z_0, z_1)$. The tensor given by Construction 3.1.1, represented as a matrix of linear forms, is
\[
\begin{pmatrix}
-x_0 & -51x_1 & 0 & 0 & 0 \\
-51x_1 & -x_2 & 0 & 0 & 0 \\
0 & 0 & x_1 & 0 & -51x_3 \\
0 & 0 & 0 & 86x_1 + x_3 & 51(x_0 - x_2) \\
0 & 0 & -51x_3 & 51(x_0 - x_2) & 24x_1
\end{pmatrix}
\]
and the resulting genus 4 curve in $\mathbb{P}^3$ is defined by the common vanishing locus of
\[
g(x) := x_0x_2 - x_1^2,
\]
\[
g(x) := 25x_0^2x_1 + 51x_0x_1x_2 + 44x_1^3 + 24x_1^3x_3 + 25x_1x_2^2 + 29x_1x_3^2 + 25x_3^3.
\]
Computer algebra can be used to verify that the genus 4 curve is nonsingular. (See subsection 1.1.)

**Lemma 3.1.3.** Let $L/\mathbb{Q}$ be an étale algebra arising from Construction 2.1.1, let $K$ be the distinguished subalgebra, and let
\[
U := \left\{ (\alpha_1, \ldots, \alpha_r) \in \prod_j K_j : L \cong \prod_j K_j[x]/(x^2 - \alpha_j), \quad K_j \cong \mathbb{Q}(\alpha_j) \right\}.
\]
Let $V \subseteq U$ be the subset such that for any $\alpha \in V$, Construction 3.1.1 applied to the characteristic polynomial of $\alpha$ produces a nonsingular genus 4 curve and a base locus $\mathcal{C}$ of dimension 0 and degree 16 of the linear space of quadrics. Then $V$ is non-empty.

**Proof.** Example 3.1.2 shows that there is a non-empty open subscheme of $\text{Pol}^8$ where the constructed curve is nonsingular. The result follows from Corollary 2.2.2.
3.2. Cycles on double covers. In [Rei72], Reid showed that the algebraic cycles on the complete intersection of 3 quadrics correspond to algebraic cycles within the Prym variety of the natural double cover of the degeneracy locus. An alternative reference for this construction is [Tyu75]. In this section, we show how a similar construction allows us to identify pairs of points on the intersection of 4 quadrics in \( \mathbb{P}^4 \) with the 112 exceptional curves on a del Pezzo surface of degree one.

The following lemma contains some results we will freely use about quadrics.

**Lemma 3.2.1.** Let \( k \) be a field of characteristic not equal to 2, let \( A \in \text{Sym}_2 k^{m+1} \) be of rank \( r > 0 \), and let \( Q := Z(y^T A y) \subset \mathbb{P}^m_k \) be the associated quadric. Then:

(a) Any singular quadric is a cone over a non-singular quadric.

(b) The singular locus of \( Q \) is contained in every maximal isotropic subspace. In other words, every maximal isotropic subspace is a cone over an isotropic subspace of the nonsingular part. Furthermore, the singular locus of \( Q \) is the linear subspace \( \mathbb{P}(\ker A) \subset \mathbb{P}^m \).

(c) Over \( \bar{k} \), the dimension of a maximal isotropic subspace \( \ell = \left\lfloor \frac{m}{2} \right\rfloor + \dim(\ker A) - 1 \).

(d) If \( r \) is even, there are two distinct families of maximal isotropic subspaces of \( Q \). If \( r \) is odd, then there is a unique family of maximal isotropic subspaces.

(e) If \( r \) is even, and \( V, W \) are maximal, then \( V, W \) are contained in the same family if and only if \( \dim(V) - \dim(V \cap W) \equiv 0 \mod 2 \).

**Proof.** See [GH94]. \( \square \)

If \( \mathcal{L} \) is a maximal isotropic subspace of a quadric \( Q \), we will denote by \( |\mathcal{L}| \) the family of maximal isotropic subspaces containing \( \mathcal{L} \).

**Proposition 3.2.2.** Let \( S^m_r \) be the space of quadrics of even rank \( r \). Let \( x \in S^m_r \) and \( |\mathcal{L}| \) denote a family of maximal isotropic subspaces of \( x \). Then the choice of generator \( (x, |\mathcal{L}|) \mapsto x \) defines a nontrivial double cover branched over \( S^m_{r-1} \).

**Proof.** See [Tyu75, Section 5]. \( \square \)

**Definition 3.2.3.** The generator double cover is the morphism \( \text{gen}: (x, |\mathcal{L}|) \mapsto x \) given by Proposition 3.2.2.

The generator double cover allows us to identify the secants between 16 points in \( \mathbb{P}^4 \) with exceptional curves of a del Pezzo surface of degree 1. In our particular case of a block-diagonal tensor \( A \in k^4 \otimes (\text{Sym}_2 k^2 \oplus \text{Sym}_2 k^3) \), the locus of quadrics of rank at most 4 is \( X_5 := Z(\det A(x, \cdot, \cdot)) \), which is the union of the quadric cone \( X_2 \) where the first block degenerates and the symmetric cubic \( X_3 \) where the second block degenerates. The generator double cover \( \text{gen}: \tilde{X}_5 \to X_5 \) is branched over the locus of quadrics of rank at most 3, which consists of the genus four curve \( X = X_2 \cap X_3 \) as well as the singularities of \( X_2 \) and \( X_3 \). Generically, the web of quadrics defined by \( A \) does not contain any quadrics of rank 2. We obtain by restriction a double cover \( \text{gen}: \tilde{X}_2 \to X_2 \) branched along a genus 4 curve and the vertex of the cone; in other words, \( \tilde{X}_2 \) is a del Pezzo surface of degree one.

Let \( \mathfrak{C} \) be the intersection of all the quadrics in the web defined by \( A \), which we saw before is generically a complete intersection of four quadrics in \( \mathbb{P}^4 \). If \( \ell \) is a secant of \( \mathfrak{C} \) and \( V_\ell \subset k^5 \) is the affine cone over \( \ell \), then we define a subscheme of \( \tilde{X}_2 \) by

\[
\tau(\ell) := \left\{ (x, |\mathcal{L}|) \in \tilde{X}_2 : \mathcal{L} := V_\ell + \ker A(x, \cdot, \cdot) \text{ is a maximal isotropic subspace of } A(x, y, y) \right\}.
\]

Observe that \( \tau(\ell) \) is well-defined; the kernel of a quadric \( Q \) of rank 4 generically does not meet \( V_\ell \), so the space \( V_\ell + \ker Q \) is a maximal isotropic subspace of \( Q \). Notice that the automorphism \( \eta: \mathbb{P}^4 \to \mathbb{P}^4 \) defined by \( \eta: (y_0 : y_1 : y_2 : y_3 : y_4) \mapsto (-y_0 : -y_1 : y_2 : y_3 : y_4) \) acts invariantly on all of the quadratic forms in the web defined by \( A \).

If \( A(x, y, y) \) is a quadratic form of rank 4 vanishing on \( \ell \), then it must also vanish on \( \eta(\ell) \). Furthermore, the intersection of the maximal isotropic subspaces \( V_\ell + \ker A(x, \cdot, \cdot) \) and \( \eta(V_\ell) + \ker A(x, \cdot, \cdot) \) is generically \( \ker A(x, \cdot, \cdot) \), and thus the two maximal isotropic subspaces lie in opposite families. In other words, the curves \( \tau(\ell), \tau(\eta(\ell)) \) on \( \tilde{X}_2 \) only intersect along the branch locus and are exchanged by \( \text{Aut}(\tilde{X}_2/X_2) \).

The last result we need is a theorem from [KV21]. To clarify the statement of the theorem, any nonhyperelliptic genus 4 curve \( X \) with a vanishing even theta characteristic \( \theta_0 \) has a unique vanishing even theta characteristic. This allows us to partition the 2-torsion classes of the Jacobian variety into one of two types:
• (odd) \( \epsilon \in \text{Jac}(X)[2] \) is of the form \([\theta - \theta_0]\) for some odd theta characteristic of \( X \).
• (even) \( \epsilon \in \text{Jac}(X)[2] \) is not even.

There are 120 odd 2-torsion classes and 135 nontrivial even 2-torsion classes.

**Theorem 3.2.4 ([KV21, Theorem 1.1.3]).** Let \( k \) be a field of characteristic not 2 or 3. Then:

(a) There is a canonical bijection between:

\[
\begin{align*}
&\text{k-isomorphism classes of tuples } (X, \epsilon, \theta_0), \\
&\text{where } X \text{ is a smooth genus 4 curve with vanishing even theta characteristic } \theta_0 \text{ with a rational divisor class defined over } k, \\
&\text{and } \epsilon \text{ is a nontrivial even 2-torsion class} \\
\leftrightarrow \\
&\text{nondegenerate orbit classes of } \mathbb{Z}^4 \oplus (\text{Sym}_2 \mathbb{Z}^2 \oplus \text{Sym}_2 \mathbb{Z}^3) \\
&\text{under the action of } \text{GL}_4(k) \times \text{GL}_2(k) \times \text{GL}_3(k)
\end{align*}
\]

Let \( A \in k^4 \otimes (\text{Sym}_2 \mathbb{Z}^2 \oplus \text{Sym}_2 \mathbb{Z}^3) \) be a nondegenerate tensor and let \( \theta_0 \) and \( \epsilon \) be the associated line bundles on \( X \). Then:

(b) The images of the 120 secants of \( \mathcal{E} \) under \( \psi \) define 56 + 8 tritangent planes of \( X \). Viewing \( X \cap H \) as a divisor of \( X \), eight of these tritangents satisfy \( X \cap H = 2D \) where \( D \in [\theta_0] \). The other 56 tritangents satisfy \( X \cap H = 2D \), where \( D \) is the effective representative of an odd theta characteristic of \( X \).

(b) Let \( e_2 \) denote the Weil pairing on \( \text{Jac}(X)[2] \). Then the 56 distinct odd theta characteristics constructed from the secants of \( \mathcal{E} \) are precisely the odd theta characteristics \( \theta \) such that \( e_2(\theta \otimes \theta') \epsilon = 0 \).

The inclusion \( X \hookrightarrow \tilde{X}_2 \) induces a restriction morphism of divisor classes. We denote by \( \text{Pic}(\tilde{X}_2)^{\perp} \) the divisor classes orthogonal to the canonical class under the intersection pairing. There is an exact sequence

\[
0 \to 2 \text{Pic}(\tilde{X}_2)^{\perp} \to \text{Pic}(\tilde{X}_2)^{\perp} \to \text{Pic}(X)[2] \to 0.
\]

Additionally, the anti-canonical class of \( \tilde{X}_2 \) restricts to the unique vanishing even theta characteristic \( \theta_0 \) of \( X \). If \( e \in \text{Pic}(\tilde{X}_2)^{\perp} \) is an exceptional curve and \( \kappa \) is the canonical class, then \( e + \kappa \in \text{Pic}(\tilde{X}_2)^{\perp} \), the restriction of \( e \) to \( X \) defines an odd theta characteristic, and the sublattice generated by

\[
R_{D_8} := \{ e + \kappa \in \text{Pic}(\tilde{X}_2)^{\perp} : e^2 = -1, \ e_2((e + \kappa)_{|X}, \epsilon) = 0 \}
\]

is a root lattice of type \( D_8 \); the set of roots of this lattice is precisely \( R_{D_8} \).

**Proposition 3.2.5.** Let \( \ell \) be a secant of \( \mathcal{E} \). Then the curve \( \tau(\ell) \) is one of the 112 exceptional curves of \( \tilde{X}_2 \) with the property that

\[
e_2((\tau(\ell) + \kappa)_{|X}, \epsilon) = 0.
\]

Furthermore, each of the 112 exceptional curves of this type is of the form \( \tau(\ell) \) for some secant of \( \mathcal{E} \).

**Proof.** Follows immediately from Theorem 3.2.4 and the discussion above. \( \square \)

The automorphism \( \eta \) restricts to an automorphism of \( \mathcal{E} \). A pair of geometric points of \( \mathcal{E} \) is either:

• Type 8: a pair of the form \( \{ p, \eta(p) \} \), or
• Type 112: a pair not of the form \( \{ p, \eta(p) \} \).

We see that any \( G \)-action preserves the type of a pair \( \{ p, q \} \). In particular, Proposition 3.2.5 shows that the \( G \)-set of pairs of type 112 is isomorphic (as a \( G \)-set) to the 112 roots of the \( D_8 \) lattice identified in Proposition 3.2.5.

3.3. **Proof of the main theorem.** Let \( \rho : \text{Gal}(\overline{\mathbb{Q}}^{\text{sep}}/\mathbb{Q}) \to W_{D_8} \) be a continuous homomorphism with image \( G \). By Construction 2.1.1, we construct an étale algebra \( L \) of degree 16 on which Galois acts via \( G \), as well as its distinguished subalgebra \( K \) of degree 8.

We may choose an element \( (\alpha_1, \ldots, \alpha_r) \in V \subset K \) as in Lemma 3.1.3 and let \( f \) be its characteristic polynomial over \( \mathbb{Q} \). Construction 3.1.1 allows us to construct a locus of 16 points with a \( G \)-action from \( f \), as well as a nonsingular genus 4 curve \( X \) contained in a quadric cone \( X_2 \). The domain of the generator double cover \( \text{gen} : \tilde{X}_2 \to X_2 \) is a del Pezzo surface of degree one, and the Galois action on the 112 roots of the \( D_8 \) sublattice from Proposition 3.2.5 has the Galois action prescribed by \( \rho \).
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