Comments on Born–Infeld Theory

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Abstract

The low-energy effective action of supersymmetric D-brane systems consists of two terms, one of which is of the Born–Infeld type and one of which is of the Chern–Simons type. I briefly review the status of our understanding of these terms for both the Abelian and non-Abelian cases.

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1 Single BPS D-Brane

D-branes are surfaces on which open strings can end \[1\] \[2\] \[3\]. As such, their dynamics is described by open string field theory – e.g., of the type formulated by Witten \[4\]. However, this can be difficult to work with, so it is sometimes useful to consider the low energy effective action obtained by integrating out all the massive modes, keeping only the massless super-Maxwell multiplet. This is too difficult, however. The best one can do is to keep terms in which the massless fields are slowly varying at the string scale – keeping fields strengths, but not their derivatives. There is no restriction on the size of field strengths relative to the string scale, except that electric fields cannot exceed a certain critical value.

The basic structure is always the sum of two terms

\[ S = S_{DBI} + S_{CS} \]  

Here \( S_{DBI} \) is the Dirac–Born–Infeld term (the only one in the case of bosonic string theory), and \( S_{CS} \) is the Chern–Simons term. In the case of type II superstring theory, an \( N \) D-brane system is given by a \( U(N) \) gauge theory \[5\].

Let us begin by considering a single D-brane \((N = 1)\). In this case, ignoring fermi fields and taking a flat 10d background, the action for a D9-brane is

\[ S_{DBI} = T_9 \int d^{10} \sigma \sqrt{-\det(g_{\alpha \beta} + 2\pi \alpha' F_{\alpha \beta})}. \]  

Here \( T_9 \) is the D9-brane tension and \( g_{\alpha \beta} \) is the pullback of the (flat) spacetime metric \( \eta_{\mu \nu} \):

\[ g_{\alpha \beta} = \eta_{\mu \nu} \partial_{\alpha} X^\mu \partial_{\beta} X^\nu, \]  

where \( \sigma^\alpha (\alpha = 0, 1, \ldots, p) \) are the world-volume coordinates, and \( X^\mu(\sigma) (\mu = 0, 1, \ldots, 9) \) are the embedding functions.

The action \( S_{DBI} \) has world-volume diffeomorphism invariance. A natural gauge choice – called static gauge – is to identify the first \( p + 1 \) components of \( X^\mu \) with \( \sigma^\alpha \). In this gauge the D9-brane action becomes

\[ S_{DBI} = T_9 \int d^{10} \sigma \sqrt{-\det(\eta_{\alpha \beta} + 2\pi \alpha' F_{\alpha \beta})}. \]  

This formula was derived first for the bosonic string theory by Fradkin and Tseytlin by computing the disk partition function \[6\]. (For a review of Born–Infeld theory in the context of string theory see \[7\].)
The corresponding actions for Dp-branes with $p < 9$ can be deduced by using T duality. The result, which agrees with dimensional reduction, is

$$S_{DBI} = T_p \int d^{p+1} \sigma \sqrt{-\det(\eta_{\alpha\beta} + \partial_\alpha X^i \partial_\beta X^i + 2\pi \alpha' F_{\alpha\beta})}. \tag{5}$$

The index $i = p + 1, \ldots, 9$ labels the $9 - p$ directions transverse to the Dp-brane. The world-volume scalars $X^i$ can be regarded as Goldstone bosons associated to broken translational symmetries.

It is remarkable that the entire $\alpha'$ expansion of the low energy effective action of open strings – for slowly varying fields – can be encoded in such simple formulas. In particular, as observed by Bachas \cite{8}, for a D0-brane this gives

$$T_0 \int d\sigma^0 \sqrt{1 - \partial_0 X^i \partial_0 X^i}, \tag{6}$$

which is the standard action for a relativistic particle of mass $T_0$.

It is natural to wonder whether something analogous might be possible for an effective low energy theory of closed strings in terms of the gravity supermultiplet. No such formula is known, but if one could be found, it would be very interesting. It could be useful for exploring whether curvatures are bounded and whether some spacetime singularities are thereby evaded.

### 1.1 Supersymmetrization

The supersymmetrization of the D-brane action was worked out in 1996 by several different groups \cite{9} \cite{10} \cite{11}. The idea is to embed the D-brane in superspace $(X^\mu, \theta_1^a, \theta_2^a)$, where $(\theta_1, \theta_2)$ are MW spinors. The global $N = 2, D = 10$ supersymmetry is realized on superspace in the usual way ($\delta \theta = \epsilon, \delta X^\mu = \bar{\epsilon} \Gamma^\mu \theta$). The D-brane action is constructed out of the supersymmetry invariants

$$\Pi_{\alpha}^\mu = \partial_\alpha X^\mu - \bar{\theta} \Gamma^\mu \partial_\alpha \theta \tag{7}$$

and $\partial_\alpha \theta$. The $\theta$'s would give twice the number of desired fermions (16 instead of 8) except for the fact that half of them are compensated by a local fermionic symmetry: called kappa symmetry.

The addition of the $\theta$'s plus the requirements of global supersymmetry and local kappa symmetry determine the action. One finds

$$S_{DBI} = T_p \int d^{p+1} \sigma \sqrt{-\det(G_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta})} \tag{8}$$
\[ S_{CS} = \pm T_p \int \Omega_{p+1}, \]  
\tag{9} \]

where
\[ G_{\alpha\beta} = \eta_{\mu\nu} \Pi^\mu_\alpha \Pi^\nu_\beta \]  
\tag{10} \]
\[ F_{\alpha\beta} = F_{\alpha\beta} - B_{\alpha\beta} - b_{\alpha\beta}. \]  
\tag{11} \]

Here \( B \) is the pullback of NS-NS 2-form background field and \( b \) is a two-form involving the fermi fields, which in the IIA case is
\[ b = -\bar{\theta} \Gamma_{11} \Gamma_\mu d\theta (dX^\mu + \frac{1}{2} \bar{\theta} \Gamma^\mu d\theta). \]  
\tag{12} \]

The familiar result for \( \Omega_{p+1} \) (due to Douglas [12]) in the presence of R-R background fields is
\[ \Omega_{p+1} = \left( C e^{2\pi \alpha' F} \right)_{p+1}, \]  
\tag{13} \]
where \( C = \sum C^{(n)} \) is a formal sum of R-R \( n \)-form fields. \( (n \) is odd for IIA and even for IIB.) For \( C \) constant this is closed, but there is an additional piece involving the \( \theta \)’s that contributes to
\[ I_{p+2} = d\Omega_{p+1}. \]  
\tag{14} \]

It has the structure
\[ \left\{ e^{2\pi \alpha' F} f(\Pi^\mu, d\theta) \right\}_{p+2}. \]  
\tag{15} \]

### 1.2 Static Gauge

We will consider the gauge-fixed super D-brane action for the \( p = 9 \) case only. The formulas for \( p < 9 \) can be inferred by dimensional reduction. As before, the local diffeomorphism symmetry is used to identify the embedding functions \( X^\mu \) with the world volume coordinates \( \sigma^\alpha \). In addition, the local kappa symmetry is used to eliminate half of the \( \theta \) coordinates. A simple choice that preserves the manifest 10d covariance is to simply set one of the two \( \theta \)’s, \( \theta_2 \) say, to zero [13]. This has the remarkable consequence of completely eliminating the Chern–Simons term.

Renaming \( \theta_1 = \lambda \) and setting \( 2\pi \alpha' = 1 \) leaves the action
\[ \int d^{10} \sigma \sqrt{-\det(\eta_{\alpha\beta} + F_{\alpha\beta} - 2\lambda \Gamma^\alpha_\beta \partial_\beta \lambda + \bar{\lambda} \Gamma^\beta_\alpha \partial_\alpha \lambda \bar{\lambda} \Gamma^\alpha_\beta \partial_\beta \lambda)}. \]  
\tag{16} \]

This is the \( \mathcal{N} = 1, D = 10 \) super-Maxwell theory supplemented by higher-dimension interaction terms. The latter are very special, because in addition to the 16 linearly realized
supersymmetries of the free theory there are 16 additional non-linearly realized supersymmetries. That is why this formula is reminiscent of the Volkov-Akulov action \[ \lambda \] – \( \lambda \) can be interpreted as the Goldstone field for the broken supersymmetries. The formula is unique up to the freedom of field redefinitions. It would have been extremely difficult to discover if one had specialized to the static gauge before supersymmetrizing the action.

When this D9-brane action is dimensionally reduced to give the Dp-brane action:

- 16 supersymmetries and \( p + 1 \) translation symmetries are linearly realized.
- 16 supersymmetries and \( 9 - p \) translation symmetries are non-linearly realized and correspond to Goldstone modes on the world volume.

It would be very interesting to rederive this formula by computing the superstring disk partition function in the presence of the appropriate boundary interactions. This ought to be a tractable extension of the calculations described in recent papers [15] [16].

2 Non-Abelian Generalizations

When one has \( N \) coincident type II Dp-branes the world-volume theory is a \( U(N) \) gauge theory. As such, it should be given by a non-Abelian generalization of the formulas of the preceding section. The explicit construction of such an action is a difficult problem that has been studied extensively (starting with [17]), but is not yet completely settled.

Tseytlin proposed a specific recipe for generalizing Abelian formulas to non-Abelian ones [18]. His proposal – referred to as the symmetrized trace prescription – works as follows. An expression in the Abelian theory, such as \( \sqrt{-\det(\eta_{\alpha\beta} + F_{\alpha\beta})} \), has an expansion of the form

\[
1 + \frac{1}{4} F^2 - \frac{1}{8} (F^4 - (F^2)^2) + \ldots, \tag{17}
\]

where \( F^2 = F_{\alpha\beta}F^{\beta\alpha} \), etc. In the non-Abelian case, \( F \) is also a hermitian \( N \times N \) matrix. Tseytlin’s proposal is to take the trace of each term in the expansion, and to resolve the ordering ambiguities by averaging over all possible choices. The result is denoted

\[
\text{Str} \sqrt{-\det(\eta_{\alpha\beta} + F_{\alpha\beta})}. \tag{18}
\]

Studies by Hashimoto and Taylor [19] and others suggest that this is a correct rule through terms of order \( F^4 \), but that it fails at higher orders.
2.1 The Chern–Simons Term

It has been clear since the work of Douglas [12] that D-branes can carry R-R charges associated with lower dimensional branes. More recently, it has been realized that in the non-Abelian case they can also carry charges associated with higher dimension D-branes [20] [21].

Myers discovered an interesting part of the the answer by exploring consistency with T duality [21]. He focused on the dependence on the bosonic fields $A_\alpha$ and $X^i$, each of which are now $N \times N$ matrices in the static gauge with all fermi fields set to zero. He included the dependence on $B$ and $C$ background fields. For the Chern–Simons term he obtained the result

$$S_{CS} = T_p \int \text{Str} \left( P[e^{iX^iX}Ce^B]e^F \right).$$ (19)

This is a subtle formula that requires some explanation. First of all, $C = \sum C^{(n)}$, as before. $P[\ldots]$ means the pullback to the world volume, since $B$ and $C$ are bulk fields. $X$ refers to the $9-p$ scalars $X^i$, which are now $N \times N$ matrices. The operation $I_X I_X$ acting on an $n$-form gives an $n-2$-form. For example,

$$I_X I_X C^{(2)} = X^j X^i C^{(2)}_{ij} = \frac{1}{2} C^{(2)}_{ij} [X^j, X^i]$$ (20)

Moreover, in the pullback of a function $f(x^\alpha, x^i)$, the matrices $X^i$ need to substituted for the bulk coordinates $x^i$. This requires an ordering prescription, since $[X^i, X^j] \neq 0$. The proposed formula is

$$P[f] = \exp(X^i \frac{\partial}{\partial x^i}) f(\sigma^\alpha, x^i) |_{x^i=0}.$$ (21)

These rules are sufficiently subtle that it is extremely hard to check whether or not $S_{CS}$ is invariant under gauge transformations of the type $C \rightarrow C + d\Lambda$. I would guess that this fails at some point, but I am not sure. It does work at low orders.

A crucial feature of this formula, for which there is a lot of evidence by now, is that multi D-brane systems can be sources of higher D-brane charge as well as lower D-brane charge, since all the R-R fields appear. This is to be contrasted with the Abelian case where $(Ce^F)_{p+1}$ only depends on $C^{(p+1)}, C^{(p-1)}, \ldots$.

Myers discovered a dielectric effect in which an R-R field strength can cause the brane to expand into new dimensions. For example, a system of $N$ D0-branes in the presence of an electric $F^{(4)} = dC^{(3)}$ becomes a fuzzy two-sphere with $[X^i, X^j] \sim N \epsilon^{ijk} X^k$. (See [22] for earlier related work.) For large $N$ this describes an ordinary $S^2$ with radius proportional
to $N$. This can be interpreted as a spherical D2-brane with $N$ D0-branes bound to it. The Myers effect is relevant to the appearance of “giant gravitons” on the AdS side of the AdS/CFT correspondence [23].

### 2.2 Supersymmetrization

Part of the rationale for Tseytlin’s symmetrized trace prescription is that a field strength commutator $[F_{ij}, F_{kl}]$ is proportional to $[D_i, D_j]F_{kl}$ and thus can be regarded as being higher-order in derivatives. This reflects an inherent ambiguity in the meaning of “slowly varying fields” in the non-Abelian case. This might be resolved by requiring that the action have all the desired symmetries: supersymmetry, kappa symmetry, etc.

In the case of $N$ coincident D-branes the supersymmetric $U(N)$ world-volume theory should again have as its physical field content gauge fields $A_\alpha$, transverse scalars $X^i$, and fermi fields $\lambda$ – this time all in the adjoint of the $U(N)$ Lie algebra. An interesting question is how to generalize the $U(1)$ formulation with local diffeomorphism invariance and local kappa symmetry to $N > 1$. It is natural to suppose that one should start with world volume fields $A_\alpha(\sigma), X^\mu(\sigma), \theta(\sigma)$, each of which is $U(N)$ valued. Then, in order to end up with the right physical degrees of freedom, one would need $U(N)$ generalizations of the diffeomorphism and kappa symmetries. These would allow us to choose a gauge that would restrict $X^\mu \rightarrow X^i$ and $\theta \rightarrow \lambda$.

The case of D9-branes is somewhat special in that there are no transverse directions $X^i$. Thus, in static gauge, the only world-volume fields are the gauge fields $A_\alpha$ and the fermi fields $\theta$. Still this is quite general, because the results for $p < 9$ can be deduced by dimensional reduction. One still needs kappa transformations in the adjoint of $U(N)$ so that a gauge choice can reduce $\theta$ to $\lambda$.

This kind of a set-up has been explored recently by Bergshoeff, de Roo, and Sevrin [24]. They carried out an iterative analysis that allowed them to deduce the action up to a certain order. Specifically, they determined terms in $S_{CS}$ with the structures $\theta D\theta, \theta D\theta F$, and $\theta D\theta F^2$, and in $S_{DBI}$ with the structures $1, F^2, \theta D\theta, \theta D\theta F$, and $\theta D\theta F^2$. Up to this order they succeeded in obtaining unique results with all the desired properties. They also gave the formulas in the gauge $\theta_2 = 0, \theta_1 = \lambda$. As in the Abelian case, $S_{CS}$ does not contribute in this gauge. They observed, in particular, that the $\lambda D\lambda F^2$ terms cannot be be expressed in terms of symmetrized traces.
The iterative analysis of Bergshoeff, de Roo, and Sevrin is technically difficult and cannot be pushed much further. It seems to me that the best hope for complete results, generalizing those of the Abelian case, would use an approach that does not invoke the static gauge at the outset. This would seem to require a matrix generalization of diffeomorphism invariance, but I doubt that such a thing is possible.

3 Conclusion

In conclusion, we have presented a review of the structure of low energy effective actions for D-branes. We saw that D-brane world volume actions are always given as the sum of a Dirac–Born–Infeld term and a Chern–Simons term and each term contains a lot of important information.

It would be desirable to have explicit exact results for the non-Abelian case so that one could explore the non-Abelian generalization of various effects that have been studied in the Abelian case. These include classical solutions that describe various sorts of solitons and brane configurations, as well as physical effects associated with electric fields approaching limiting values.

A powerful approach that has received a great deal of attention lately is BSFT: boundary string field theory or background independent string field theory [25]. This provides the logical basis for deriving D-brane effective actions in terms of disk partition functions with appropriate boundary interactions. The BSFT approach allows one to formulate unstable D-brane systems and to test some of Sen’s conjectures regarding tachyon condensation [26]. This has been done with notable success in recent works [27] [28]. It also provides another approach to studying the formation of various sorts of solitons and to formulating the non-Abelian Born–Infeld problem. Unfortunately, the relevant path integrals may not be amenable to analytic evaluation.

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