Accelerated Dual Averaging Methods for Decentralized Constrained Optimization

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Abstract—In this article, we study decentralized convex constrained optimization problems in networks. We focus on the dual averaging-based algorithmic framework that is well-documented to be superior in handling constraints and complex communication environments simultaneously. Two new decentralized dual averaging (DDA) algorithms are proposed. In the first one, a second-order dynamic average consensus protocol is tailored for DDA-type algorithms, which equips each agent with a provably more accurate estimate of the global dual variable than conventional schemes. We rigorously prove that the proposed algorithm attains $O(1/t)$ convergence for general convex and smooth problems, for which existing DDA methods were only known to converge at $O(1/\sqrt{t})$ prior to our work. In the second one, we use the extrapolation technique to accelerate the convergence of DDA. Compared to existing accelerated algorithms, where typically two different variables are exchanged among agents at each time, the proposed algorithm only seeks consensus on local gradients. Then, the extrapolation is performed based on two sequences of primal variables, which are determined by the accumulations of gradients at two consecutive time instants, respectively. The algorithm is proved to converge at $O(1)\left(\frac{1}{t^2} + \frac{1}{(1-\beta)^2}\right)$, where $\beta$ denotes the second largest singular value of the mixing matrix. We remark that the condition for the algorithmic parameter to guarantee convergence does not rely on the spectrum of the mixing matrix, making itself easy to satisfy in practice. Finally, numerical results are presented to demonstrate the efficiency of the proposed methods.

Index Terms—Acceleration, constrained optimization, decentralized optimization, dual averaging, multiagent system.

I. INTRODUCTION

CONSIDER a multiagent system consisting of $n$ agents. Each agent holds a private objective function. They are connected via a communication network in order to collaboratively solve the following optimization problem:

$$\min_{x \in \mathcal{X}} \left\{ f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right\}$$

(1)

where $f_i$ represents the local smooth objective function of agent $i$ and $\mathcal{X} \subseteq \mathbb{R}^m$ denotes the constraint set shared by all the agents. This problem is referred to as decentralized optimization in the literature and finds broad applications in optimal control of cyber-physical systems, sensor networks, and machine learning, to name a few. For an overview of decentralized optimization and its applications, please refer to [1], [2].

Over the last decade, many decentralized optimization algorithms have been proposed for solving Problem (1). For unconstrained problems, i.e., $\mathcal{X} = \mathbb{R}^m$, the authors in [3] and [4] developed decentralized gradient descent (DGD) methods with constant step sizes, where the local search performed by individual agents is guided by local gradients and a consensus protocol. However, because each individual gradient evaluated at the global optimum is not necessarily zero, the search directions induced by consensus-seeking and local gradient may conflict with each other, making it difficult to ascertain the exact solution to the problem. Several efforts have been made to overcome this drawback. For example, the authors in [5] proposed the EXTRA algorithm that adds a cumulative correction term to DGD to achieve consensus optimization. Alternatively, the additional gradient-tracking process based on dynamic average consensus scheme in [6] can be used. It is shown in [7] and [8] that, for unconstrained smooth optimization, the algorithms steered by the tracked gradient exactly converge at an $O(1/t)$ rate. Based on this idea, a decentralized Nesterov gradient descent was proposed in [9], where the rate of convergence is accelerated to $O(1/t^{1.4-\epsilon})$ for any $\epsilon \in (0, 1.4)$ at the expense of exchanging an additional variable among agents at each time instant. In [10], the authors proposed an accelerated decentralized algorithm with

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multiple consensus rounds at each time instant, and proved that after $t$ local iterations and $O(t \log t)$ communication rounds, the objective error is bounded by $O(1/t^2)$. By modeling Problem (1) as a linearly constrained optimization problem, centralized primal-dual paradigms, such as the augmented Lagrangian method, the alternating direction method of multipliers, and the dual ascent can also be used to design decentralized algorithms [11]–[13]. Based on the primal-dual reformulation, an accelerated primal-dual method was developed in [14]. The rate of convergence is improved to $O(1) \left( \frac{L}{t^2} + \frac{1}{t^\eta} \right)$, where $L$ denotes the smoothness parameter of each objective function and $\eta = \lambda_2(\mathcal{L})/\lambda_m(\mathcal{L})$ is the eigengap of the graph Laplacian $\mathcal{L}$. Notably, the authors established a lower bound for a class of decentralized primal-dual methods, suggesting that the developed algorithm therein is optimal in terms of gradient computations. The authors in [15] considered the Lagrangian dual formulation of the decentralized optimization problem and developed two algorithms based on dual accelerated methods. The algorithms are proved to be linearly convergent for strongly convex and smooth problems.

For constrained problems, the design and convergence analysis of decentralized optimization algorithms are more challenging [16]–[18]. The seminal work in [19] is based on the gossip protocol and projected subgradient method, where the step size was made decaying for convergence. The randomized smoothing technique and multiround consensus scheme are used to design a provably optimal decentralized algorithm for nonsmooth convex problems in [20]. To improve the performance using a constant step size, a variant of EXTRA (PG-EXTRA) was developed in [21], where the constraint is modeled as a nonsmooth indicator function and handled via the proximal operator. An $O(1/t)$ rate of convergence is proved for the squared norm of the difference of consecutive iterates. Recently the authors in [22] proposed an accelerated decentralized penalty (APM) method, where the constraint can be also treated as the nonsmooth part of the objective. Notably, there are some decentralized algorithms [23]–[29] available in the literature where the local search mechanism for individual agents is inspired by dual methods [30], e.g., mirror descent and dual averaging [31], [32]. Particularly, dual averaging is provably more efficient in exploiting sparsity than proximal gradient methods for $l_1$-regularized problems [32]. For example, the authors in [26] developed a decentralized dual averaging (DDA) algorithm where a linear model of the global objective function is gradually learned by each agent via gossip. Compared to other types of decentralized first-order methods, DDA has the advantage in simultaneously handling constraints and complex communication environments, e.g., directed networks [33], deterministic time-varying networks [29], and stochastic networks [26], [34]. From a technical perspective, this is because that DDA seeks consensus on linear models of the objective function rather than on the local projected iterates as in decentralized primal methods, e.g., DGD, therefore decoupling the consensus-seeking process from nonlinear projection and facilitating the rate analysis in complex communication environments. We present a more detailed comparison in Section III-A.

Although decentralized dual methods in the literature have demonstrated advantages over their primal counterparts in terms of constraint handling and analysis complexity, existing results focused on nonsmooth problems and can have an $O(1/\sqrt{t})$ rate of convergence. Considering this, a question naturally arises: If the objective functions exhibit some desired properties, e.g., smoothness, is it possible to accelerate the convergence rate of DDA beyond $O(1/\sqrt{t})$? We provide affirmative answer to this question in this article. The main results and contributions are summarized in the following.

1) First, we develop a new DDA algorithm, where a second-order dynamic average consensus protocol is deliberately designed to assist each agent in estimating the global dual variable. Compared to the conventional estimation scheme [26], the proposed method equips each agent with provably more accurate estimates. In particular, the estimation error accumulated over time is proved to admit an upper bound constituted by the successive difference of an auxiliary variable whose update uses the mean of local dual variables. Then, a rigorous investigation into the convergence of the auxiliary variable is carried out. Summarizing these two relations, we establish conditions for algorithm parameters such that the estimation error can be fully compensated, leading to an improved rate of convergence $O(1/t)$.

2) Second, we propose an accelerated DDA (ADDA) algorithm. Different from DDA, each agent employs a first-order dynamic average consensus protocol to estimate the mean of local gradients and accumulates the estimate over time to generate a local dual variable. By solving the convex conjugate of a 1-strongly convex function over this local dual variable, each agent produces a primal variable and uses it to construct another two sequences of primal variables in an iterative manner based on the extrapolation technique in [35] and the average consensus protocol. The rate of convergence is proved to be $O(1) \left( \frac{1}{t^2} + \frac{1}{t^{\eta(1-\beta)^2}} \right)$, where $\beta$ denotes the second largest singular value of the mixing matrix. Notably, the condition for the algorithm parameter to ensure convergence does not rely on the mixing matrix. Establishing such a condition that is independent on the mixing matrix offers the appealing advantage of convenient verification in practical applications.

3) Finally, the proposed algorithms are tested and compared with a few methods in the literature on decentralized LASSO problems characterized by synthetic and real datasets. The comparison results demonstrate the efficiency of the proposed methods.

Notation: We use $\mathbb{R}$ and $\mathbb{R}^n$ to denote the set of reals and the $n$-dimensional Euclidean space, respectively. Given a real number $a$, we denote by $\lceil a \rceil$ the ceiling function that maps $a$ to the least integer greater than or equal to $a$. Given a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its 2-norm. Given a matrix $P \in \mathbb{R}^{n \times n}$, its spectral radius is denoted by $\rho(P)$. Its eigenvalues and singular values are denoted by $\lambda_1(P) \geq \lambda_2(P) \geq \cdots \geq \lambda_n(P)$ and $\sigma_1(P) \geq \sigma_2(P) \geq \cdots \geq \sigma_n(P)$, respectively.
II. PRELIMINARIES

A. Basic Setup

We consider the finite-sum optimization problem (1), in which $\mathcal{X}$ is a convex and compact set, and $f_i$ satisfies the following assumptions for all $i = 1, \ldots, n$.

**Assumption 1:**

i) $f_i$ is continuously differentiable on $\mathcal{X}$.

ii) $f_i$ is convex on $\mathcal{X}$, i.e., for any $x, y \in \mathcal{X}$,
\[
    f_i(x) - f_i(y) - \langle \nabla f_i(y), x - y \rangle \geq 0. \tag{2}
\]

iii) $\nabla f_i$ is Lipschitz continuous on $\mathcal{X}$ with a constant $L > 0$, i.e., for any $x, y \in \mathcal{X}$,
\[
    \|\nabla f_i(x) - \nabla f_i(y)\| \leq L \|x - y\|. \tag{3}
\]

A direct consequence of Assumption 1(iii) is
\[
    f_i(x) - f_i(y) - \langle \nabla f_i(y), x - y \rangle \leq \frac{L}{2} \|x - y\|^2 \ \forall x, y \in \mathcal{X}. \tag{4}
\]

The above-mentioned assumptions are standard in this study of decentralized algorithms for convex optimization problems. Throughout this article, we denote by $x^*$ an optimal solution of Problem (1).

B. Communication Network

We consider solving Problem (1) in a decentralized fashion, that is, a pair of agents can exchange information only if they are connected in the communication network. To describe the network topology, an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is used, where $\mathcal{V} = \{1, \ldots, n\}$ denotes the set of $n$ agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the set of bidirectional channels, i.e., $(i, j) \in \mathcal{E}$ indicates that nodes $i$ and $j$ can send information to each other. Agent $j$ is said to be a neighbor of $i$ if there exists a link between them, and the set of $i$'s neighbors is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. For every pair $(i, j) \in \mathcal{E}$, a positive weight $p_{ij} > 0$ is assigned to $i$ and $j$ to weigh the information received from each other. Otherwise $p_{ij} = 0$ is considered. For the convergence of the algorithm, we make the following assumption for $P := [p_{ij}] \in [0, 1]^{n \times n}$.

**Assumption 2:**

i) $P1 = 1$ and $1^T P = 1^T$, where 1 denotes the all-one vector of dimension $n$.

ii) $P$ has a strictly positive diagonal, i.e., $p_{ii} > 0$.

Assumption 2 implies that $\sigma_2(P) < 1$ [28]. Given a connected network, the central edge weights and the Metropolis–Hastings algorithm [36] can be used to construct a weight matrix $P$ fulfilling Assumption 2.

C. Centralized Dual Averaging

Our algorithms are based on the dual averaging methods [31]. Before introducing them, we state the following definition.

**Definition 1:** A differentiable function $\psi$ is strongly convex with modulus $\mu > 0$ on $\mathcal{X}$, if
\[
    \psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle \geq \frac{\mu}{2} \|x - y\|^2 \ \forall x, y \in \mathcal{X}. \tag{5}
\]

Let $d$ be a strongly convex and differentiable function with modulus 1 on $\mathcal{X}$ such that $d(x) = \arg\min_{x \in \mathcal{X}} d(x)$ and $d(x^0) = 0$. To meet the condition in (5) for any $x^{(0)} \in \mathcal{X}$, one can choose
\[
    d(x) = \tilde{d}(x) - \langle \nabla \tilde{d}(x^{(0)}), x - x^{(0)} \rangle \tag{6}
\]
where $\tilde{d}$ is any strongly convex function with modulus 1, e.g., $d(x) = \|x\|^2/2$. The convex conjugate of $d$ is defined as
\[
    \nabla d^*(\cdot) = \text{argmax} \{\langle \cdot, x \rangle - d(x)\}.
\]

As a corollary of Danskin's Theorem, we have the following result [35].

**Lemma 1:** For all $x, y \in \mathbb{R}^m$, we have
\[
    \|\nabla d^*(x) - \nabla d^*(y)\| \leq \|x - y\|. \tag{7}
\]

**Dual averaging:** The dual averaging method can be applied to solving Problem (1) in a centralized manner. Starting with $x^{(0)}$, it generates a sequence of variables $\{x^{(t)}\}_{t \geq 0}$ iteratively according to
\[
    x^{(t)} = \nabla d^* \left(-\alpha_t z^{(t)}\right) \tag{8}
\]
where
\[
    z^{(t)} = \frac{\sum_{\tau=0}^{t-1} \nabla f(x^{(\tau)})}{t} \tag{9}
\]
and $\{\alpha_t\}_{t \geq 0}$ is a sequence of positive parameters that determines the rate of convergence. Let $\tilde{x}^{(t)} = t^{-1} \sum_{\tau=0}^{t-1} x^{(\tau)}$. It is proved in [31] that $f(\tilde{x}^{(t)}) - f(x^*) \leq O(1/\sqrt{t})$ when $\alpha_t = \Theta(1/\sqrt{t})$, that is, with order exactly $1/\sqrt{t}$. When the objective function is convex and smooth, a constant $\alpha_t = a$ can be used to achieve an ergodic $O(1/t)$ rate of convergence in terms of objective error [37].

**Accelerated dual averaging:** To speed up the rate of convergence, an accelerated dual averaging algorithm is developed in [35]. In particular, the variables are updated according to
\[
    u^{(t)} = \frac{A_{t-1}}{A_t} u^{(t-1)} + \frac{A_t}{A_{t-1}} w^{(t-1)} \tag{10a}
\]
\[
    v^{(t)} = \frac{A_{t-1}}{A_t} v^{(t-1)} + \frac{A_t}{A_{t-1}} w^{(t)} \tag{10b}
\]
where $a_t = a(t + 1)$ for some $a > 0$, $A_t = \sum_{\tau=1}^{t} a_\tau$ and
\[
    w^{(t)} = \nabla d^* \left(-\sum_{\tau=1}^{t} a_\tau \nabla f(u^{(\tau)})\right). \tag{11a}
\]

Note that $t \geq 2$ is considered for the abovementioned iteration, and the variables are initialized with $u^{(1)} = u^{(0)} = x^{(0)}$, $v^{(1)} = w^{(1)}$. For convex and smooth objective functions, it is proved that $f(v^{(t)}) - f(x^*) \leq O(1/t^2)$ [35].

III. ALGORITHMS AND CONVERGENCE RESULTS

In this section, we develop two new DDA algorithms that are provably more efficient than existing DDA-type algorithms.

A. Decentralized Dual Averaging

To solve Problem (1) in a decentralized manner, we propose a novel DDA algorithm. In particular, we employ the following dynamic average consensus protocol to estimate $z^{(t)}$ in (8)
\[
    z^{(t)}_i = \frac{1}{n} \sum_{j=1}^{n} p_{ij} \left(z^{(t-1)}_j + s^{(t-1)}_j\right). \tag{11a}
\]
Algorithm 1: Decentralized Dual Averaging.

Input: $a > 0$, $x^{(0)} \in \mathcal{X}$ and a strongly convex function $d$ with modulus 1 such that (5) holds

Initialize: $x_i^{(0)} = x^{(0)}$, $\hat{z}_i^{(0)} = 0$, and $s_i^{(0)} = \nabla f_i(x^{(0)})$ for all $i = 1, \ldots, n$

for $t = 1, 2, \ldots$ do

In parallel (task for agent $i$, $i = 1, \ldots, n$) collect $z_j^{(t-1)}$ and $s_j^{(t-1)}$ from all agents $j \in \mathcal{N}_i$

update $\hat{z}_i(t)$ and $s_i(t)$ by (11)

compute $x_i^{(t)}$ by (12)

broadcast $\hat{z}_i(t)$ and $s_i(t)$ to all agents $j \in \mathcal{N}_i$

end for

$$
\begin{align*}
\hat{z}_i^{(t)} &= \sum_{j=1}^{n} p_{ij} s_j^{(t-1)} + \nabla f_i\left(x_i^{(t)}\right) - \nabla f_i\left(x_i^{(t-1)}\right) \quad (11b)
\end{align*}
$$

where $z_i^{(t)}$ is a local estimate of $z^{(t)}$ generated by agent $i$, $s_i^{(t)}$ is a proxy of $\frac{1}{n} \sum_{j=1}^{n} \nabla f_i(x_j^{(t)})$, which aims to reduce the consensus error among variables $\left\{z_i^{(t)} : i = 1, \ldots, n\right\}_{t \geq 0}$.

Using it, each agent $i$ performs a local computation step to update its estimate of $x^{(t)}$

$$
\hat{x}_i^{(t+1)} = \sum_{j=1}^{n} p_{ij} x_j^{(t)} + \hat{x}_i - \sum_{j=1}^{n} p_{ij} x_j^{(t-1)} - a \left(\nabla f_i\left(x_i^{(t)}\right) - \nabla f_i\left(x_i^{(t-1)}\right)\right) \quad (15)
$$

where $a$ represents the step size and $p_{ij}$ denotes the $(i, j)$th entry of $P = (P + I)/2$. Notably, PG-EXTRA seeks consensus among variables $\left\{x_i^{(t)} : i = 1, \ldots, n\right\}$ at time $t + 1$ that are obtained via a projection operator at time $t$. In contrast, DDA manages to agree on $\left\{z_i^{(t)} : i = 1, \ldots, n\right\}$, which essentially decouples the consensus-seeking procedure from projection. After using the smoothness assumption in (3) to bound $\left\|\nabla f_i\left(x_i^{(t)}\right) - \nabla f_i\left(x_i^{(t-1)}\right)\right\|$ in (11b), the iteration in (11) can be kept almost linear, which greatly facilitates the rate analysis; see the proof of Lemma 5 for more details.

Next, we present the convergence result of Algorithm 1. Inspired by [26], we first establish the convergence property of an auxiliary sequence $\left\{y^{(t)}\right\}_{t \geq 0}$, which is instrumental in proving the convergence of $\left\{x_i^{(t)} : i = 1, \ldots, n\right\}_{t \geq 0}$. In particular, the update of $y^{(t)}$ obeys

$$
\nabla y^{(t)} = \nabla d^a\left(-a \bar{z}_i^{(t)}\right) \quad (16)
$$

where the initial vector $y^{(0)} = x^{(0)}$ and $\bar{z}_i^{(t)} = \frac{1}{n} \sum_{i=1}^{n} z_i^{(t)}$.

To proceed, we introduce the following $2 \times 2$ matrix:

$$
M = \begin{bmatrix} \beta & \beta \\ a \lambda (\beta + 1) & \beta (a \lambda + 1) \end{bmatrix} \quad (17)
$$

where $\beta = \sigma_2(P)$, and let $\rho(M)$ be the spectral radius of $M$.

Theorem 1: Suppose that Assumptions 1 and 2 are satisfied. If the constant $a$ in Algorithm 1 satisfies

$$
\frac{1}{a} > 2L \cdot \max \left\{ \frac{\beta}{(1-\beta)^2} \cdot 1 + \frac{8}{9 (1 - (\rho(M))^2)} \right\} \quad (18)
$$

3) Comparison with DGD algorithms: In existing DGD, a so-called gradient-tracking process similar to (11) is usually observed

$$
\hat{x}_i^{(t)} = \sum_{j=1}^{n} p_{ij} \left( x_j^{(t-1)} + a s_j^{(t-1)} \right) \\
\hat{s}_i^{(t)} = \sum_{j=1}^{n} p_{ij} s_j^{(t-1)} + \nabla f_i\left(x_i^{(t)}\right) - \nabla f_i\left(x_i^{(t-1)}\right) \quad (14)
$$
then, for all \( t \geq 1 \), it holds that
\[
 f(y^{(t)}) - f(x^*) \leq \frac{C}{at}
\]  
(19)
where \( y^{(t)} = t^{-1} \sum_{t=1}^{n} y^{(\tau)} \) with \( y^{(\tau)} \) defined in (16)
\[
 C := d(x^*) + \frac{8a\pi^2}{9nL (1 - (\rho(M))^2)}
\]
and
\[
 \pi^2 = \sum_{i=1}^{n} \left\| \nabla f_i(x^{(0)}) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x^{(0)}) \right\|^2
\]  
(20)
In addition, for all \( t \geq 1 \) and \( i = 1, \ldots, n \), we have
\[
 \| \tilde{x}^{(t)} - \tilde{y}^{(t)} \|^2 \leq \frac{D}{t}
\]  
(21)
where
\[
 \tilde{x}^{(t)} = t^{-1} \sum_{t=1}^{n} x^{(\tau)}
\]
\[
 D := \frac{8nC}{9\gamma (1 - (\rho(M))^2)} + \frac{8aL^2}{9(1 - (\rho(M))^2)}
\]
and
\[
 \gamma := \frac{1}{2} - aL - \frac{8aL}{9(1 - (\rho(M))^2)}
\]  
(22)
Proof: The proof is postponed to Appendix A. □
To obtain a more explicit version of (18), we identify the eigenvalues of \( M \) as \( \lambda_1 = (\xi_1 + \xi_2)/2 \) and \( \lambda_2 = (\xi_1 - \xi_2)/2 \), where
\[
 \xi_1 = \beta(2 + aL) > 0, \quad \xi_2 = \sqrt{\frac{1}{4} \beta^2 L^2 + 4aL\beta(\beta + 1)} > 0
\]  
(23)
Thus, we have \( |\lambda_1| > |\lambda_2| \) and \( \rho(M) = \lambda_1 > 0 \). By routine calculation, we can verify that \( \rho(M) \) and \( \nu(a) := \frac{1}{\max((1 - (\rho(M))^2)^2)} \) monotonically increase with \( a \). Due to
\[
 \nu \left( \frac{1}{2L} \right) < \frac{1}{\left( 1 - \frac{\beta}{2L} \right)^2}
\]
we have that as long as \( a \) satisfies
\[
 \frac{1}{a} > 2L \cdot \left\{ \beta \left( 1 - \beta \right)^2, 1 + \frac{1}{\left( 1 - \frac{\beta}{2L} \right)^2} \right\}
\]
(24)
then \( a \) also satisfies (18). Based on (24), we have
\[
 a = \Theta \left( \frac{1 - \beta^2}{L} \right)
\]
whose size is comparable to the DGD algorithms [8], [38], [39] in the literature.
Next, we consider an unconstrained version of Problem (1), i.e., \( \mathcal{X} = \mathbb{R}^m \), where the rate of convergence is stated for
\[
 f(\tilde{x}^{(t)}) - f(x^*) \leq \frac{C}{at}
\]  
(26)
where \( \tilde{x}^{(t)} = t^{-1} \sum_{t=1}^{n} x^{(\tau)} \) and \( \pi^2 \) is defined in (20).
Proof: The proof is given in Appendix B. □
B. Accelerated DDA
To further speed up the convergence, we develop a decentralized variant of the accelerated dual averaging method in (9) and (10). Different from Algorithm 1, we consider building consensus among variables \( \{v_i^{(t)}, i = 1, \ldots, n\} \) and propose the following iteration rule:
\[
 u_i^{(t)} = A_t^{-1} \sum_{j=1}^{n} p_{ij} u_j^{(t-1)} + \frac{at}{A_t} w_i^{(t-1)}
\]  
(27a)
\[
 v_i^{(t)} = \frac{A_t - 1}{A_t} \sum_{j=1}^{n} p_{ij} v_j^{(t-1)} + \frac{at}{A_t} w_i^{(t)}
\]  
(27b)
and
\[
 w_i^{(t)} = \nabla d^* \left( -\sum_{\tau=1}^{t} a_\tau q_i^{(\tau)} \right)
\]  
(28)
The overall algorithm is summarized in Algorithm 2. It is worth to mention that agents in Algorithm 2 consume the same communication resources as Algorithm 1 to achieve acceleration.
Assumption 3: For the problem in (1), the constraint set \( \mathcal{X} \) is bounded with the following diameter:
\[
 G = \max_{x, y \in \mathcal{X}} \| x - y \|
\]
Theorem 2: For Algorithm 2, if Assumptions 1, 2, and 3 are satisfied, and
\[
 a \leq \frac{1}{6L}
\]  
(30)
then, for all \( t \geq 1 \), it holds that
\[
f \left( \pi \left( t \right) \right) - f(x^*) \leq \frac{d(x^*)}{A_t} + \frac{t}{A_t} \left( 2G(LC_p + C_g) + \frac{6LC_p^2}{n} \right)
\]
where
\[
C_p := \left[ \frac{3}{1 - \beta} \right] \sqrt{nG}
\]
and
\[
C_g := 2L \left[ \frac{3}{1 - \beta} \right] \sqrt{nG + C_p}.
\]
In addition, for all \( t \geq 1 \) and \( i = 1, \ldots, n \), we have
\[
\left\| v_i^{(t)} - \pi^{(t)} \right\|_{2} \leq \frac{2aC_p}{A_t}.
\]

**Proof:** The proof is postponed to Appendix C.

For Algorithm 2 and Theorem 2, the following remarks are in order.

1) Comparison with existing accelerated algorithms: Accelerated methods for decentralized constrained optimization are rarely reported in the literature. Recently, the authors in [22] developed the APM algorithm, where the iteration rule reads
\[
y_i^{(t)} = x_i^{(t)} + \frac{\beta_1 (1 - \beta_{t-1})}{\theta_{t-1}} (x_i^{(t)} - x_i^{(t-1)}) \tag{33a}
\]
\[
s_i^{(t)} = \nabla f_i \left( y_i^{(t)} \right) + \frac{\beta_0}{\theta_t} \sum_{j=1}^{n} p_{ij} (y_i^{(t)} - y_j^{(t)}) \tag{33b}
\]
\[
x_i^{(t+1)} = \arg \min \left\{ x \in \mathcal{X} \right\} \left\| x - y_i^{(t)} + \frac{s_i^{(t)}}{L + \beta_0/\theta_t} \right\|_{2}^2 \tag{33c}
\]
where \( \beta_0 = L/\sqrt{1 - \lambda_2(P)} \) and \( \theta_k \) is a decreasing parameter satisfying \( \theta_k = \theta_{k-1}/(1 + \theta_{k-1}) \) with \( \theta_0 = 1 \). Letting \( s_i^{(1)} = \theta_k s_i^{(1)} \), we can equivalently rewrite (33b) and (33c) as
\[
s_i^{(t)} = \theta_t \nabla f_i \left( y_i^{(t)} \right) + \frac{\beta_0}{\theta_t} \sum_{j=1}^{n} p_{ij} (y_i^{(t)} - y_j^{(t)})
\]
\[
x_i^{(t+1)} = \arg \min \left\{ x \in \mathcal{X} \right\} \left\| x - y_i^{(t)} + \frac{s_i^{(t)}}{L \theta_t + \beta_0/\theta_t} \right\|_{2}^2
\]
from which we can see that new gradients are assigned with decreasing weights, whereas increasing weights are used for ADDA in (28). The reason for such different choices of parameters may be two-fold. First, parameter choices in (centralized) primal gradient descent and dual averaging methods are intrinsically different. Second, APM gradually increases the penalty parameter \( 1/\theta_t \) in order to enforce consensus, which essentially dilutes the weight for gradients, as shown earlier. We will show in simulation that decreasing weights over time slows down convergence. There are also a few other accelerated decentralized methods such as [9], [14], however, they do not apply to constrained problems.

2) Discussion about optimality: For ADDA, the rate of convergence is proved to be
\[
\mathcal{O} \left( \frac{1}{t^2} + \frac{1}{t(1 - \beta)^2} \right).
\]
In light of the lower bound in [14], it is not optimal in terms of the dependence on \( \beta \). In particular, the dominant term of the error in \( \mathcal{O}(1/(t(1 - \beta)^2)) \) becomes larger as \( \beta \) grows, i.e., the network becomes more sparsely connected. This is mainly because we consider a one-consensus-one-gradient update in the algorithm. However, extending the algorithm in [14] to handle constraints may require further investigation. In the simulation section, we demonstrate the superiority of ADDA over existing decentralized constrained optimization algorithms.

### IV. SIMULATION

In this section, we verify the proposed methods by applying them to solve the following constrained LASSO problems:
\[
\min_{x \in \mathbb{R}^m} \left\{ f(x) = \frac{1}{2n} \sum_{i=1}^{n} \| M_i x - c_i \|^2 \right\}, \text{s.t. } \| x \|_1 \leq R
\]
where \( M_i \in \mathbb{R}^{P_i \times m} \), \( c_i \in \mathbb{R}^{P_i} \), and \( R \) is a constant parameter that defines the constraint. In the simulation, each agent \( i \) has access to a local data tuple \((y_i, A_i)\) and \( R \). Two different problem instances characterized by both real and synthetic datasets are considered.

#### A. Case I: Real Dataset

In this setting, we use sparco [17], [40] to define the LASSO problem, and consider a cycle graph and a complete graph of \( n = 50 \) nodes. The corresponding weight matrix \( P \) is determined by following the Metropolis–Hastings rule [36]. Each local measurement matrix \( M_i \in \mathbb{R}^{12 \times 2560} \) and the local corrupted measurement \( c_i \in \mathbb{R}^{12} \). The constraint parameter is set as \( R = 1.1 \cdot \| x_g \|_1 \), where \( x_g \) with \( \| x_g \|_2 = 20 \) denotes the unknown variable to be recovered via solving LASSO. In this case, the simulation experiments were performed using MATLAB R2020b.

For comparison, the PG-EXTRA method in [21] and the APM method in [27] are simulated. For their algorithmic parameters, the step size for PG-EXTRA is set as \( 10^{-4} \), and the parameters for APM are set as \( L = 250 \) and \( \beta_0 = L/\sqrt{1 - \lambda_2(P)} \). For DDA and ADDA in this work, we use \( \alpha = 5 \times 10^{-4} \) and \( \alpha_i = (t + 1) \times 10^{-3} \), respectively, and \( \| x \|_2^2/2 \) as the prox-function. The projection onto an \( l_1 \) ball is carried out via the algorithm in [41]. All the methods are initialized with \( x^{(0)} = 0, \forall i \in \mathcal{V} \).

The performance of four algorithms are displayed in Figs. 1 and 2. In Fig. 1, the performance is evaluated in terms of the objective error \( f \left( \frac{1}{n} \sum_{i=1}^{n} x_i^{(t)} \right) - f(x^*) \), where \( x^* \) is identified using CVX [42]. It demonstrates that the DDA method outperforms other methods when the graph is a cycle. As the graph becomes denser, i.e., complete graph, the convergence of all algorithms becomes faster. Among them, the ADDA method demonstrates the most significant improvement. This is in line with Theorem 2, where the network connectivity impacts the convergence error in \( \mathcal{O}(1/t) \) as opposed to \( \mathcal{O}(1/t^2) \).

In Fig. 2, we compare the trajectories of consensus error, i.e., \( \sqrt{\sum_{i=1}^{n} \| x_i^{(t)} - x_{\text{avg}}^{(t)} \|^2} \), by all methods. When the graph is a cycle, the APM and PG-EXTRA have smaller consensus error than the developed methods, mainly because they build...
consensus directly among variables \( \{ x_i(t) : i = 1, \ldots, n \} \). When the graph is complete, the consensus error by the proposed DDA method vanishes because of the conservation property established in Lemma 3 and the complete graph structure.

### B. Case II: Synthetic Dataset

For the synthetic dataset, the parameters are set as \( n = 8 \), \( m = 30000 \), \( p_i = 2000 \), \( \forall i \in \mathcal{V} \), and the data is generated in the following way. First, each local measurement matrix \( M_i \) is randomly generated where each entry follows the normal distribution \( \mathcal{N}(0, 1) \). Next, each entry of the sparse vector \( x_g \) to be recovered via LASSO is randomly generated from the normal distribution \( \mathcal{N}(0, 1) \) with \( \| x_g \|_0 = 1500 \). Then, the corrupted measurement \( c_i \) is produced based on

\[
c_i = M_i x_g + b_i
\]

where \( b_i \) represents the Gaussian noise with zero mean and variance 0.01. The constraint parameter is set as \( R = 1.1 \cdot \| x_g \|_1 \). For this setting, we employed the message passing interface in Python 3.7.3 to simulate a network of eight nodes, where each node \( i \) is connected to a subset of nodes \( \{ 1 + i \mod 8, 1 + (i + 3) \mod 8, 1 + (i + 6) \mod 8 \} \).

For comparison, the proposed methods are compared to their centralized counterparts. The parameters for dual averaging and accelerated dual averaging are set as \( a = 1/(3 \times 10^5) \) and \( a_t = a(t + 1) \), respectively. Similarly, the function \( \| x \|_2^2/2 \) is used as a prox-function, and the algorithms are initialized with \( x_i(0) = 0, \forall i \in \mathcal{V} \).

The performance of the developed algorithms and their centralized counterparts is illustrated in Fig. 3. In particular, the performance is evaluated in terms of objective function value versus computing time. It demonstrates that the proposed methods outperform the corresponding centralized algorithms in the sense that the decentralized algorithms consume less computing time to reach the same degree of accuracy than their centralized counterparts.

### V. Conclusion

In this article, we have designed two DDA algorithms for solving decentralized constrained optimization problems with improved rates of convergence. In the first one, a novel second-order dynamic average consensus scheme is developed, with which each agent locally generates a more accurate estimate of the dual variable than existing methods under mild assumptions. This property enables each agent to use large constant weight in the local dual average step, and therefore, improves the rate of convergence. In the second algorithm, each agent retains the conventional first-order dynamic average consensus method to estimate the average of local gradients. Alternatively, the extrapolation technique together with the average consensus protocol is used to achieve acceleration over a decentralized network.

This article opens several avenues for future research. In this work, we focus on the basic setting with time-invariant bidirectional communication networks. We believe that the consensus-based dual averaging framework can be extended to tackle decentralized constrained optimization in complex networks, e.g., directed networks [43] and time-varying networks [38].
Furthermore, we expect that our approach, as demonstrated by its centralized counterpart, i.e., follow-the-regularized-leader, may deliver superb performance in the online optimization setting [44].

**APPENDIX A**

**ROADMAP FOR THE PROOFS**

Before proceeding to the proofs, we present Fig. 4 to illustrate how they relate to each other.

**APPENDIX B**

**PROOF OF THEOREM 1**

**A Preliminaries**

We introduce several notations to facilitate the presentation of the proof. Let

$$\begin{align*}
\begin{bmatrix}
    x_1^{(t)} \\
    \vdots \\
    x_n^{(t)}
\end{bmatrix}, \\
\begin{bmatrix}
    z_1^{(t)} \\
    \vdots \\
    z_n^{(t)}
\end{bmatrix}, \\
\begin{bmatrix}
    s_1^{(t)} \\
    \vdots \\
    s_n^{(t)}
\end{bmatrix}
\end{align*}$$

$$\begin{align*}
\begin{bmatrix}
    y_1^{(t)} \\
    \vdots \\
    y_n^{(t)}
\end{bmatrix}, \\
\begin{bmatrix}
    \nabla f_1(x_1^{(t)}) \\
    \vdots \\
    \nabla f_n(x_n^{(t)})
\end{bmatrix}
\end{align*}$$

$$\gamma^{(t)} = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_i^{(t)})$$

$$\sigma^{(t)} = \frac{1}{n} \sum_{i=1}^{n} s_i^{(t)}, \quad \tau^{(t)} = \frac{1}{n} \sum_{i=1}^{n} z_i^{(t)}.$$

Using these notations, we express (11) in the following compact form:

$$\begin{align*}
\zeta^{(t)} &= P \left( \zeta^{(t-1)} + s^{(t-1)} \right) \\
\sigma^{(t)} &= Ps^{(t-1)} + \nabla \zeta^{(t)} - \nabla \zeta^{(t-1)}
\end{align*}$$

where $P = P \otimes I$. Before proceeding to the proof of Theorem 1, we present several technical lemmas.

Recall a lemma from [39].

**Lemma 2:** Suppose that $\{\epsilon^{(t)}\}_{t \geq 0}$ and $\{\delta^{(t)}\}_{t \geq 0}$ are two sequences of positive scalars such that for all $t \geq 0$

$$\epsilon^{(t)} \leq \delta^t \epsilon^{(0)} + \sum_{\tau=0}^{t-1} \delta^{t-\tau-1} \epsilon^{(\tau)}$$

where $\delta \in (0, 1)$. Then, the following holds for all $t \geq 1$

$$\begin{align*}
\sum_{\tau=1}^{t} (\epsilon^{(\tau)})^2 &\leq \frac{2}{(1-\delta)^2} \sum_{\tau=0}^{t-1} (\epsilon^{(\tau)})^2 + \frac{2}{1-\delta^2} (\epsilon^{(0)})^2.
\end{align*}$$

**Lemma 3:** For Algorithm 1, we have that for any $t \geq 0$

$$\gamma^{(t)} + \sigma^{(t)} = \sum_{\tau=0}^{t} \gamma^{(\tau)}.$$

**Proof of Lemma 3:** We prove by induction. For $t = 0$, we readily have (35) satisfied since $s_i^{(0)} = \nabla f_i(x_i^{(0)})$ and $x_i^{(0)} = 0$ for all $i$. Suppose that (35) holds for $t - 1$. Using (34b)

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

and the double stochasticity of $P$, we have

$$\begin{align*}
\gamma^{(t)} &= \frac{1}{n} \left( 1^T \otimes I \right) \left( Ps^{(t-1)} + g^{(t)} - g^{(t-1)} \right) \\
&= \frac{1}{n} \left( 1^T \otimes I \right) \left( (P \otimes I)s^{(t-1)} + \nabla g^{(t)} - \nabla g^{(t-1)} \right) \\
&= \frac{1}{n} \left( (1^T P) \otimes I \right) s^{(t-1)} + g^{(t)} - g^{(t-1)} \\
&= \gamma^{(t-1)} + g^{(t)} - g^{(t-1)} = \gamma^{(t)}.
\end{align*}$$

Upon using a similar argument for $\sigma^{(t)}$, we have

$$\begin{align*}
\sigma^{(t)} &= \frac{1}{n} \left( 1^T \otimes I \right) \left( P \otimes I \right) \left( z^{(t)} + s^{(t)} \right) \\
&= \frac{1}{n} \left( 1^T \otimes I \right) \left( P \otimes I \right) \left( z^{(t)} + s^{(t)} \right) \\
\end{align*}$$

Using (35), we have

$$\gamma^{(t)} + \sigma^{(t)} = \sum_{\tau=0}^{t} \gamma^{(\tau)}.$$

Therefore, (35) holds for all $t$.

**Lemma 4:** If the parameter $a$ satisfies (18), then $\rho(M) < 1$, where $M$ is defined in (17).

**Proof of Lemma 4:** Recall the two real eigenvalues of $M$ in (23). Since $\xi_1 > 0$ and $\xi_2 > 0$, we have $\rho(M) = |\lambda_1| > |\lambda_2|$. Also, one can verify that $\rho(M)$ monotonically increases with $a$. The solution to the equation $|\lambda_1| = 1$ can be identified as $a = (1 - \beta^2)/(2\beta L)$. Therefore, we conclude that $\rho(M) < 1$ as long as (18) is satisfied.

The following lemma establishes the relation between sequences $\{x_i^{(t)}\}_{t \geq 0}$ and $\{y_i^{(t)}\}_{t \geq 0}$.

**Lemma 5:** Suppose that Assumptions 1, 2 hold and the parameter $a$ in Algorithm 1 satisfies (18). Then, for all $t \geq 0$, it holds that

$$\begin{align*}
\sum_{\tau=0}^{t} \left\| x^{(\tau)} - y^{(\tau)} \right\|^2 &\leq \frac{8}{9 (1-\rho(M))^2} \sum_{\tau=0}^{t-1} \left\| y^{(\tau+1)} - y^{(\tau)} \right\|^2 + \frac{8\pi^2}{9L^2 (1-\rho(M))^2} \\
\end{align*}$$

where

$$\pi^2 = \sum_{i=1}^{n} \left\| \nabla f_i(x_i^{(0)}) - \nabla g^{(0)} \right\|^2.$$

**Proof of Lemma 5:** From Lemma 3, we have

$$\gamma^{(t)} = \gamma^{(t-1)} + \gamma^{(t-1)}.$$
Define
\[ \bar{s}^{(t)} = s^{(t)} - 1 \otimes \bar{s}^{(t)}, \bar{z}^{(t)} = z^{(t)} - 1 \otimes \bar{s}^{(t)}. \]  
(38)
These in conjunction with (34a) lead to
\[ \bar{z}^{(r)} = Pz^{(r-1)} - 1 \otimes \bar{s}^{(r-1)} + Ps^{(r-1)} - 1 \otimes s^{(r-1)}. \]  
(39)
Because of \(1 \otimes \bar{z}^{(r-1)} = (1 \otimes I)\bar{z}^{(r-1)}\) and (36), we have
\[ \bar{z}^{(r-1)} = \frac{1}{n}(1 \otimes I)(1^T \otimes I)z^{(r-1)} = \left( \frac{11^T}{n} \otimes I \right)z^{(r-1)}. \]  
(40)
In addition, we have
\[ \begin{align*}
\bar{z}^{(r)} - 1 \otimes \bar{z}^{(r-1)} & = (P \otimes I)z^{(r-1)} - \left( \frac{11^T}{n} \otimes I \right)z^{(r-1)} \\
& = \left( \frac{11^T}{n} \otimes I \right)z^{(r-1)} - \left( \frac{11^T}{n} \otimes I \right)z^{(r-1)} \\
& = \left( \frac{11^T}{n} \otimes I \right)z^{(r-1)} - \left( \frac{11^T}{n} \otimes I \right)z^{(r-1)} \\
& = \left( \frac{11^T}{n} \otimes I \right)z^{(r-1)} \\
\end{align*} \]
where the last equality is due to the double stochasticity of \(P\).
Using the same arguments as mentioned above for \(Ps^{(r-1)} = 1 \otimes s^{(r-1)}\) and Assumption 2, we have
\[ \begin{align*}
\| \bar{z}^{(r)} \| & \leq \| Pz^{(r-1)} - 1 \otimes \bar{z}^{(r-1)} \| + \| Ps^{(r-1)} - 1 \otimes s^{(r-1)} \| \\
& \leq \beta \| \bar{z}^{(r-1)} \| + \beta \| \bar{s}^{(r-1)} \|. \\
\end{align*} \]  
(41)
Similarly, from Lemma 3, we obtain
\[ \begin{align*}
\| \bar{s}^{(r)} \| & \leq \| Ps^{(r-1)} - (1 \otimes I)\bar{s}^{(r-1)} + \nabla^T(\tau) - \nabla(\tau) \| \\
& \leq \beta \| \bar{s}^{(r-1)} \| + \| \nabla^T(\tau) - \nabla(\tau) \| \\
& \leq \beta \| \bar{s}^{(r-1)} \| + L \| x^{(r)} - x^{(r-1)} \| \\
\end{align*} \]  
(42)
where the last inequality is due to Assumption 1. Using
\[ \| x^{(r)} - x^{(r-1)} \| \leq \| x^{(r)} - y^{(r)} \| + \| x^{(r-1)} - y^{(r-1)} \| + \| y^{(r)} - y^{(r-1)} \| \]
and Lemma 1, we obtain
\[ \begin{align*}
\| \bar{s}^{(r)} \| & \leq \beta (aL + 1) \| \bar{s}^{(r-1)} \| + aL(\beta + 1) \| \bar{s}^{(r-1)} \| + \| y^{(r)} - y^{(r-1)} \| \\
& \leq a(\beta + 1) \| \bar{s}^{(r-1)} \| + a\beta \| \bar{s}^{(r-1)} \| + \| y^{(r)} - y^{(r-1)} \|. \\
\end{align*} \]
Upon substituting the abovementioned inequality into (42), we obtain
\[ \begin{align*}
\| \bar{z}^{(r)} \| & \leq \beta aL(\beta + 1) \| \bar{s}^{(r-1)} \| + aL(\beta + 1) \| \bar{s}^{(r-1)} \| + \| y^{(r)} - y^{(r-1)} \| \\
& \leq L \| y^{(r)} - y^{(r-1)} \|. \\
\end{align*} \]  
(43)
By combining (41) and (43), we establish the following inequality:
\[ \begin{align*}
\left\| \bar{z}^{(r)} \right\| & \leq M \left[ \left\| \bar{z}^{(r-1)} \right\| + L \left\| y^{(r)} - y^{(r-1)} \right\| \right] \\
\left\| \bar{s}^{(r)} \right\| & \leq M \left[ \left\| \bar{s}^{(r-1)} \right\| + L \left\| y^{(r)} - y^{(r-1)} \right\| \right] \\
\end{align*} \]  
where \(M\) is defined in (17). By iterating the above inequality and using
\[ \| \bar{z}^{(0)} \| = 0, \| \bar{s}^{(0)} \| = \sqrt{\sum_{i=1}^{n} \left\| \nabla f_i(x^{(0)}) - g^{(0)} \right\|^2} := \pi \]
we obtain
\[ \| \bar{z}^{(r)} \| \leq L^{t-1} \sum_{r=0}^{t-1} \left\| y^{(r+1)} - y^{(r)} \right\| + M^t \left[ 0 \right] \]
Recall the eigenvalues of matrix \(M\) in (23). Then, an analytical form can be presented for the \(n\)th power of \(M\) (see, e.g., [45])
\[ M^n = \lambda^n_1 \left( \frac{M - \lambda_2 I}{\lambda_1 - \lambda_2} \right) - \lambda^n_2 \left( \frac{M - \lambda_1 I}{\lambda_1 - \lambda_2} \right). \]
Therefore,
\[ \| \bar{z}^{(r)} \| \]
\[ \leq \frac{2\beta L}{\xi_2} \sum_{r=0}^{t-1} \left( \rho(M) \right)^{t-r-1} \| y^{(r+1)} - y^{(r)} \| + \frac{2\beta}{\xi_2} \left( \rho(M) \right)^t \pi \]
Due to the assumption that \(1/a > 2\beta L/(1 - \beta)^2\) and \(\beta \in (0, 1)\), we have
\[ \xi_2 = \beta a L \left( 1 + 4(\beta + 1)/\beta a L \right) > \beta a L \left( 1 + 8(\beta + 1)/(1 - \beta)^2 \right) > 3\beta a L. \]
Therefore
\[ \| \bar{z}^{(r)} \| \]
\[ \leq \frac{2\beta L}{3a} \sum_{r=0}^{t-1} \left( \rho(M) \right)^{t-r-1} \| y^{(r+1)} - y^{(r)} \| + \frac{2\beta L}{3a} \left( \rho(M) \right)^t \pi. \]
Using Lemma 1, we further obtain
\[ \| x^{(r)} - y^{(r)} \| \leq a \| \bar{z}^{(r)} \| \]
\[ \leq \frac{2\beta L}{3a} \sum_{r=0}^{t-1} \left( \rho(M) \right)^{t-r-1} \| y^{(r+1)} - y^{(r)} \| + \frac{2\beta L}{3a} \left( \rho(M) \right)^t \pi. \]
Further using the abovementioned relation and Lemma 2, inequality (37) follows as desired.

Then, a lemma is stated for the prox-mapping.

**Lemma 6:** Given a sequence of variables \(\{ \xi^{(r)} \}_{r \geq 0}\) and a positive sequence \(\{ a_t \}_{t \geq 0}\) generated by
\[ \nu^{(t)} = \nabla d^t \left( - \sum_{r=1}^{t} a_r \xi^{(r)} \right) \]
where \(\nu^{(0)} = x^{(0)}\) in (5), it holds that
\[ \sum_{r=1}^{t} a_r \left\langle \xi^{(r)}, \nu^{(r)} - x^* \right\rangle \leq d(x^*) - \sum_{r=1}^{t} \frac{1}{2} \left\| \nu^{(r)} - \nu^{(r-1)} \right\|^2. \]

**Proof of Lemma 6:** Define
\[ m_t(x) = \sum_{r=1}^{t} a_r \left\langle \xi^{(r)}, x \right\rangle + d(x) \]
where $m_0(x) = d(x)$. Since
\[
\nu^{(\tau-1)} = \arg\min_{x \in \mathcal{X}} m_{\tau-1}(x)
\]
ad $m_{\tau-1}(x)$ is strongly convex with modulus 1, we have
\[
m_{\tau-1}(x) - m_{\tau-1}(\nu^{(\tau-1)}) \geq \frac{1}{2} \|x - \nu^{(\tau-1)}\|^2 \quad \forall x \in \mathcal{X}.
\]
Upon taking $x = \nu^{(\tau)}$ in the abovementioned inequality and using
\[
m_{\tau}(x) = m_{\tau-1}(x) + a_{\tau} \langle \zeta^{(\tau)}, x \rangle
\]
we obtain
\[
0 \leq m_{\tau-1}(\nu^{(\tau)}) - m_{\tau-1}(\nu^{(\tau-1)}) - \frac{1}{2} \|\nu^{(\tau)} - \nu^{(\tau-1)}\|^2
\]
\[
= m_{\tau}(\nu^{(\tau)}) - a_{\tau} \langle \zeta^{(\tau)}, \nu^{(\tau)} \rangle - m_{\tau-1}(\nu^{(\tau-1)})
\]
\[
- \frac{1}{2} \|\nu^{(\tau)} - \nu^{(\tau-1)}\|^2
\]
which is equivalent to
\[
a_{\tau} \langle \zeta^{(\tau)}, \nu^{(\tau)} \rangle \leq m_{\tau}(\nu^{(\tau)}) - m_{\tau-1}(\nu^{(\tau-1)})
\]
\[
- \frac{1}{2} \|\nu^{(\tau)} - \nu^{(\tau-1)}\|^2.
\]
Iterating the abovementioned equation from $\tau = 1$ to $\tau = t$ yields
\[
\sum_{\tau=1}^{t} a_{\tau} \langle \zeta^{(\tau)}, -x^{*} \rangle
\]
\[
\leq m_{t}(\nu^{(t)}) - m_{0}(\nu^{(0)}) - \frac{t}{2} \|\nu^{(t)} - \nu^{(t-1)}\|^2
\]
\[
= m_{t}(\nu^{(t)}) - \frac{t}{2} \|\nu^{(t)} - \nu^{(t-1)}\|^2
\]
(47)

We turn to consider
\[
\sum_{\tau=1}^{t} a_{\tau} \langle \zeta^{(\tau)}, x^{*} \rangle
\]
\[
\leq \max_{x \in \mathcal{X}} \left\{ \sum_{\tau=1}^{t} a_{\tau} \langle \zeta^{(\tau)}, x \rangle - d(x) \right\} + d(x^{*})
\]
\[
= - \min_{x \in \mathcal{X}} \left\{ \sum_{\tau=1}^{t} a_{\tau} \langle \zeta^{(\tau)}, x \rangle + d(x) \right\} + d(x^{*})
\]
\[
= - m_{t}(\nu^{(t)}) + d(x^{*})
\]
which together with (47) leads to inequality in (46), thereby concluding the proof.

\[\Box\]

**B Proof of Theorem 1**

**Proof of Theorem 1:** For all $\tau \geq 1$, we have
\[
\frac{1}{n} \sum_{i=1}^{n} a \left( f_i \left( \nu^{(\tau)} \right) - f_i(x^{*}) \right)
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} a \left( f_i \left( x_i^{(\tau-1)} \right) - f_i(x^{*}) + L \left\| \nu^{(\tau)} - x_i^{(\tau-1)} \right\|^2
\]
\[
+ \left\langle \nabla f_i \left( x_i^{(\tau-1)} \right), \nu^{(\tau)} - x_i^{(\tau-1)} \right\rangle \right)
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} a \left( \frac{L}{2} \left\| \nu^{(\tau)} - x_i^{(\tau-1)} \right\|^2 + \left\langle \nabla f_i \left( x_i^{(\tau-1)} \right), \nu^{(\tau)} - x_i^{(\tau-1)} \right\rangle \right)
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} a \left( \frac{L}{2} \left\| \nu^{(\tau)} - x_i^{(\tau-1)} \right\|^2 + a \left\langle \tilde{g}^{(\tau-1)}, \nu^{(\tau)} - x_i^{(\tau-1)} \right\rangle \right)
\]
(48)

where the two inequalities are due to Assumption 1. By iterating (48) from $\tau = 1$ to $\tau = t$ and using Lemma 6 ($a_{\tau} = a, \zeta^{(\tau)} = \tilde{g}^{(\tau-1)},$ and $\nu^{(\tau)} = \nu^{(\tau)}$), we obtain
\[
\sum_{\tau=1}^{t} a \left( f \left( \nu^{(\tau)} \right) - f(x^{*}) \right)
\]
\[
\leq \frac{1}{n} \sum_{\tau=1}^{t} \sum_{i=1}^{n} a \left( \frac{L}{2} \left\| \nu^{(\tau)} - x_i^{(\tau-1)} \right\|^2 \right)
\]
\[
- \frac{1}{2} \sum_{\tau=1}^{t} \sum_{i=1}^{n} \left\| \nu^{(\tau)} - y^{(\tau)} \right\|^2 + d(x^{*})
\]
\[
= \frac{aL}{2n} \sum_{\tau=1}^{t} \left\| \nu^{(\tau)} - x^{(\tau-1)} \right\|^2 - \frac{1}{2} \sum_{\tau=1}^{t} \sum_{i=1}^{n} \left\| \nu^{(\tau)} - y^{(\tau)} \right\|^2 + d(x^{*}).
\]

Then, we use the inequality
\[
\left\| \nu^{(\tau)} - x^{(\tau-1)} \right\|^2 \leq 2 \left\| \nu^{(\tau)} - y^{(\tau)} \right\|^2 + 2 \left\| y^{(\tau)} - x^{(\tau-1)} \right\|^2
\]
and the convexity of $f$ to get
\[
at \left( f \left( \tilde{y}^{(t)} \right) - f(x^{*}) \right)
\]
\[
\leq \frac{1}{n} \sum_{\tau=1}^{t} \left( aL - \frac{1}{2} \right) \left\| \nu^{(\tau)} - y^{(\tau)} \right\|^2
\]
\[
+ \frac{aL}{n} \sum_{\tau=1}^{t} \left\| x^{(\tau-1)} - y^{(\tau)} \right\|^2 + d(x^{*}).
\]

Upon using Lemma 5, one has
\[
at \left( f \left( \tilde{y}^{(t)} \right) - f(x^{*}) \right)
\]
\[
\leq d(x^{*}) + \frac{8a\pi^2}{9nL(1 - (\rho(M))^2)} = C
\]
(49)

where $\gamma > 0$ is defined in (22). Therefore, we have (19) as desired.

From (49) and $f \left( y^{(t)} \right) - f(x^{*}) \geq 0$, we have
\[
\sum_{\tau=1}^{t} \left\| \nu^{(\tau)} - y^{(\tau-1)} \right\|^2 \leq \frac{nC}{\gamma}
\]

By using the abovementioned inequality, the convexity of $\| \cdot \|^2$, Jensen’s Inequality, and Lemma 5, we arrive at
\[
t \left\| \nu^{(t)} - \tilde{y}^{(t)} \right\|^2 \leq \sum_{\tau=1}^{t} \left\| y^{(\tau)} - y^{(\tau)} \right\|^2
\]
\[
\leq \frac{8}{9(1 - (\rho(M))^2)} \sum_{\tau=0}^{t-1} \left\| y^{(\tau+1)} - y^{(\tau)} \right\|^2 + \frac{8\pi^2}{9L^2(1 - (\rho(M))^2)}
\]
\[
\frac{8nC}{9\gamma (1-\rho(M))^2} + \frac{8\pi^2}{9L^2 (1-\rho(M))^2} = D
\]
which implies (21) as desired.

**APPENDIX C: PROOF OF COROLLARY 1**

**Proof of Corollary 1:** We consider
\[
f \left( x_i^{(\tau)} \right) - f \left( y^{(\tau)} \right)
\]
\[
\leq \frac{1}{n} \sum_{j=1}^{n} \left( f_j \left( x_i^{(\tau)} \right) - \langle \nabla f_j \left( x_j^{(\tau)} \right), y^{(\tau)} - x_j^{(\tau)} \rangle - f_j \left( x_j^{(\tau)} \right) \right)
\]
\[
\leq \frac{1}{n} \sum_{j=1}^{n} \left( \langle \nabla f_j \left( x_j^{(\tau)} \right), x_i^{(\tau)} - x_j^{(\tau)} \rangle - \langle \nabla f_j \left( x_j^{(\tau)} \right), y^{(\tau)} - x_j^{(\tau)} \rangle + \frac{L}{2} \|x_i^{(\tau)} - y^{(\tau)} + y^{(\tau)} - x_j^{(\tau)}\|^2 \right)
\]
\[
= \frac{1}{n} \sum_{j=1}^{n} \left( \langle \nabla f_j \left( x_j^{(\tau)} \right), x_i^{(\tau)} - y^{(\tau)} \rangle + L \|x_i^{(\tau)} - y^{(\tau)}\|^2 + \frac{L}{n} \|y^{(\tau)} - x^{(\tau)}\|^2 \right)
\]
(50)
where the two inequalities follow from Assumption 1. When \( X = \mathbb{R}^m \), the closed-form solutions for (12) and (16) can be identified as
\[
x_i^{(\tau)} = -a x_i^{(\tau)}, y^{(\tau)} = -a x^{(\tau)}
\]
implying that \( y^{(\tau)} = \frac{1}{n} \sum_{i=1}^{n} x_i^{(\tau)} \). Upon summating up (50) from \( i = 1 \) to \( i = n \), we obtain
\[
\sum_{i=1}^{n} \left( f \left( x_i^{(\tau)} \right) - f \left( y^{(\tau)} \right) \right) \leq 2L \|y^{(\tau)} - x^{(\tau)}\|^2.
\]
(51)

Then, we iterate (51) from \( \tau = 1 \) to \( \tau = t \) and use the convexity of \( f \) to get
\[
t \sum_{i=1}^{n} \left( f \left( \bar{x}_i^{(t)} \right) - f \left( y^{(\tau)} \right) \right) \leq \sum_{\tau=1}^{t} \sum_{i=1}^{n} \left( f \left( x_i^{(\tau)} \right) - f \left( y^{(\tau)} \right) \right)
\]
\[
\leq 2L \sum_{\tau=1}^{t} \|y^{(\tau)} - x^{(\tau)}\|^2
\]
(52)
where \( \bar{x}_i^{(t)} = t^{-1} \sum_{\tau=1}^{t} x_i^{(\tau)} \). Using Lemma 5, we obtain
\[
t \sum_{i=1}^{n} \left( f \left( \bar{x}_i^{(t)} \right) - f \left( y^{(\tau)} \right) \right)
\]
\[
\leq \frac{16L}{9 (1-\rho(M))^2} \sum_{\tau=0}^{t-1} \|y^{(\tau+1)} - y^{(\tau)}\|^2 + \frac{16\pi^2}{9L (1-\rho(M))^2}.
\]
(53)

Recall (49), we have
\[
\alpha t \left( f \left( y^{(t)} \right) - f \left( x^* \right) \right) \leq d(x^*) + \frac{8\pi^2}{9nL (1-\rho(M))^2}
\]
\[
- \left( \frac{1}{2} - \alpha L - \frac{8\pi^2}{9 (1-\rho(M))^2} \right) \sum_{\tau=1}^{t} \|y^{(\tau)} - y^{(\tau-1)}\|^2.
\]
(54)
By multiplying \( n/\alpha > 0 \) on both sides of (54) and adding the resultant inequality to (53), we get
\[
t \left( f \left( \bar{x}_i^{(t)} \right) - f \left( x^* \right) \right) \leq t \sum_{i=1}^{n} \left( f \left( \bar{x}_i^{(t)} \right) - f \left( x^* \right) \right)
\]
\[
\leq \frac{1}{n} \sum_{\tau=1}^{t} \|y^{(\tau)} - y^{(\tau-1)}\|^2
\]
\[
+ \frac{n}{2\alpha} \|x^*\|^2 + \frac{8\pi^2}{3L (1-\rho(M))^2}.
\]
(55)
Upon using condition in (25), we arrive at (26) as desired.

**APPENDIX D: PROOF OF THEOREM 2**

**A Preliminaries**

For Algorithm 2, we define
\[
\mathbf{u}^{(t)} = \left[ \begin{array}{c} u_1^{(t)} \\ \vdots \\ u_n^{(t)} \end{array} \right], \quad \mathbf{v}^{(t)} = \left[ \begin{array}{c} v_1^{(t)} \\ \vdots \\ v_n^{(t)} \end{array} \right], \quad \mathbf{w}^{(t)} = \left[ \begin{array}{c} w_1^{(t)} \\ \vdots \\ w_n^{(t)} \end{array} \right]
\]
\[
\mathbf{q}^{(t)} = \left[ \begin{array}{c} q_1^{(t)} \\ \vdots \\ q_n^{(t)} \end{array} \right], \quad \nabla \mathbf{v}^{(t)} = \left[ \begin{array}{c} \nabla f_1(u_1^{(t)}) \\ \vdots \\ \nabla f_n(u_n^{(t)}) \end{array} \right]
\]
\[
\mathbf{v}^{(t)} = \frac{1}{n} \sum_{i=1}^{n} u_i^{(t)}, \quad \mathbf{w}^{(t)} = \frac{1}{n} \sum_{i=1}^{n} v_i^{(t)}, \quad \mathbf{w}^{(t)} = \frac{1}{n} \sum_{i=1}^{n} w_i^{(t)}
\]
\[
\bar{u}^{(t)} = \mathbf{u}^{(t)} - 1 \otimes \mathbf{p}^{(t)}, \quad \bar{v}^{(t)} = \mathbf{v}^{(t)} - 1 \otimes \mathbf{p}^{(t)}, \quad \bar{q}^{(t)} = \mathbf{q}^{(t)} - 1 \otimes \mathbf{p}^{(t)}.
\]

Based on these notations, we present the steps in (27) and (29) in the following compact form:
\[
\mathbf{u}^{(t)} = \frac{A_{t-1}}{A_t} \left( P \mathbf{v}^{(t-1)} \right) + \frac{a_t}{A_t} \mathbf{w}^{(t-1)}
\]
(56a)
\[
\mathbf{v}^{(t)} = \frac{A_{t-1}}{A_t} \left( P \mathbf{v}^{(t-1)} \right) + \frac{a_t}{A_t} \mathbf{w}^{(t-1)}
\]
(56b)
\[
\mathbf{q}^{(t)} = P \mathbf{q}^{(t-1)} + \nabla \mathbf{v}^{(t)} - \nabla \mathbf{v}^{(t-1)}
\]
(56c)
where $\mathbf{P} = \mathbf{P} \otimes I$. According to (27), we have
\[
\bar{v}(t) = \frac{A_{t-1}}{A_t} \bar{v}(t-1) + \frac{a_t}{A_t} p(t-1) \tag{57a}
\]
\[
\bar{\pi}(t) = \frac{A_{t-1}}{A_t} \pi(t-1) + \frac{a_t}{A_t} \pi(t) \tag{57b}
\]

Before proving Theorem 2, we present Lemma 7 that establishes upper bounds for consensus error vectors $\hat{u}^{(t)}$ and $\hat{v}^{(t)}$.

**Lemma 7:** For Algorithm 2, if Assumptions 1–3 are satisfied, then
\[
\|\hat{v}^{(t)}\| \leq \frac{a_t}{A_t} C_p, \|\hat{u}^{(t)}\| \leq \frac{a_t}{A_t} C_p \tag{58}
\]
for all $t \geq 1$, where $C_p = \left[\frac{3}{\sqrt{\beta}}\right] \sqrt{n} G$, and $\beta = \sigma_2(P)$.

**Proof of Lemma 7:** Since both $u_i(t), v_i(t), i = 1, \ldots, n$, $\pi(t)$ and $\bar{\pi}(t)$ are within the constraint set, we readily have
\[
\|u_i(t) - 1 \otimes \bar{v}(t)\| \leq \sqrt{n} G \quad \text{and} \quad \|v_i(t) - 1 \otimes \bar{\pi}(t)\| \leq \sqrt{n} G
\]
by Assumption 3. Using
\[
a_t \leq \frac{2(t+1)}{t(t+3)} \geq 1 \quad \forall t \geq 1
\]
and the definition of $C_p = \left[\frac{3}{\sqrt{\beta}}\right] \sqrt{n} G$, we have that (58) holds for $1 \leq t \leq \left[\frac{3}{\sqrt{1-\beta}}\right]$.

When $t \geq \left[\frac{3}{\sqrt{1-\beta}}\right]$, we prove by an induction argument. Suppose that (58) holds for some $t \geq \left[\frac{3}{\sqrt{1-\beta}}\right]$. Next, we examine the upper bounds for $\|\hat{v}^{(t+1)}\|$ and $\|\hat{u}^{(t+1)}\|$, respectively.

**i) Upper bound for $\|\hat{v}^{(t+1)}\|$:** Using
\[
\mathbf{P} \bar{v}(t) - 1 \otimes \bar{\pi}(t) = \left(\mathbf{P} - \frac{11T}{n} \otimes I\right) \hat{v}(t)
\]
(56b) and (57b), we obtain
\[
\hat{v}^{(t+1)} = \frac{A_t}{A_{t+1}} \left(\mathbf{P} - \frac{11T}{n} \otimes I\right) \hat{v}(t) + \frac{a_{t+1}}{A_{t+1}} \hat{w}^{(t+1)}.
\]
Evaluating the norm of both sides of the above-mentioned equality yields
\[
\|\hat{v}^{(t+1)}\| = \left\| \frac{A_t}{A_{t+1}} \left(\mathbf{P} - \frac{11T}{n} \otimes I\right) \hat{v}(t) + \frac{a_{t+1}}{A_{t+1}} \hat{w}^{(t+1)} \right\|
\leq \beta \|\hat{v}(t)\| + \frac{a_{t+1}}{A_{t+1}} \|\hat{w}^{(t+1)}\|
\leq \frac{a_t}{A_t} \beta C_p + \frac{a_{t+1}}{A_{t+1}} \sqrt{n} C_p
\]
where the last inequality follows from the hypothesis that $\|\hat{v}(t)\| \leq a_t C_p / A_t$ and Assumption 3. Since $a_t / A_t$ monotonically decreases with $t$, we have
\[
\|\hat{v}^{(t+1)}\| \leq \frac{a_t}{A_t} (\beta C_p + \sqrt{n} G) \leq \frac{a_t}{A_t} C_p \left(\beta + \frac{1}{\left[\frac{3}{1-\beta}\right]}\right)
\]
where the last inequality is due to $\sqrt{n} G = \left[\frac{C_p}{\sqrt{\beta}}\right]$. It then remains to prove
\[
\left(\beta + \frac{1}{\left[\frac{3}{1-\beta}\right]}\right) \leq \frac{A_t}{a_t} \cdot \frac{a_{t+1}}{A_{t+1}} \quad \forall t \geq \left[\frac{3}{1-\beta}\right] \tag{59}
\]
to obtain the bound for $\|\hat{v}^{(t+1)}\|$ as desired. To prove (59), we let
\[
t_0 = \left[\frac{3}{1-\beta}\right]
\]
which implies
\[
\frac{3}{t_0} \leq 1 - \beta.
\]

Based on the above relation, we further obtain
\[
\beta + \frac{1}{t_0} \leq t_0 - 2 \cdot \frac{t_0}{t_0 + 4} \leq t_0 + 2 \cdot \frac{t_0}{t_0 + 4}.
\]
(60)
This in conjunction with
\[
t(t + 3) \geq 1 \forall t \geq 1
\]
and the definitions of $a_t$ and $A_t$ yields
\[
\beta + \frac{1}{t_0} \leq t_0 + 2 \cdot \frac{t_0}{t_0 + 4} \leq \frac{A_{t_0}}{a_{t_0}} \cdot \frac{a_{t_0+1}}{A_{t_0+1}}.
\]
Since $A_t a_{t+1} / (A_t A_{t+1})$ monotonically increases with $t$, we have (59) satisfied.

**ii) Upper bound for $\|\hat{u}^{(t+1)}\|$:** Using the same arguments as mentioned above, we have
\[
\|\hat{u}^{(t+1)}\| = \left\| \frac{A_t}{A_{t+1}} \left(\mathbf{P} - \frac{11T}{n} \otimes I\right) \hat{v}(t) + \frac{a_{t+1}}{A_{t+1}} \hat{w}^{(t+1)} \right\|
\leq \beta \|\hat{v}(t)\| + \frac{a_{t+1}}{A_{t+1}} \|\hat{w}^{(t+1)}\|
\leq \frac{a_t}{A_t} \beta C_p + \frac{a_{t+1}}{A_{t+1}} \sqrt{n} C_p.
\]
By following the same line of reasoning as in the first part, we are able to obtain
\[
\|\hat{u}^{(t+1)}\| \leq \frac{a_{t+1}}{A_{t+1}} C_p.
\]
Summarizing the above bounds, the proof is completed. □

**Lemma 8:** Suppose Assumptions 1–3 are satisfied. For Algorithm 2, we have
\[
\bar{q}^{(t)} = \bar{\bar{q}}^{(t)} \tag{61}
\]
and
\[
\|\bar{q}^{(t)}\| \leq \frac{a_t}{A_t} C_g \tag{62}
\]
for all $t \geq 1$, where $C_g = \left[\frac{3}{\sqrt{1-\beta}}\right] L(2\sqrt{n} G + 2 C_p) / (1 - \beta)$, and $\beta = \sigma_2(P)$.

**Proof of Lemma 8:** The proof of (61) directly follows from the proof of Lemma 3, and is omitted here for brevity.

For (62), we subtract $1 \otimes \bar{q}^{(t)}$ from both sides of (56c) to get
\[
\bar{q}^{(t)} - 1 \otimes \bar{q}^{(t)} = \mathbf{P} \bar{q}^{(t-1)} - 1 \otimes \bar{q}^{(t-1)} + \hat{v}^{(t)} - \hat{v}^{(t-1)} - 1 \otimes (\bar{q}^{(t)} - \bar{q}^{(t-1)}).
\]
(63)
Using the same procedure in (40) leads to
\[
\|\bar{q}^{(t)}\| \leq \beta \|\bar{q}^{(t-1)}\| + \|\hat{v}^{(t)} - \hat{v}^{(t-1)} - 1 \otimes (\bar{q}^{(t)} - \bar{q}^{(t-1)})\|.
\]
(64)
Since the objective is smooth, we obtain
\[
\|\hat{v}^{(t)} - \hat{v}^{(t-1)} - 1 \otimes (\hat{q}^{(t)} - \hat{q}^{(t-1)})\| \\
= \|\hat{v}^{(t)} - \hat{v}^{(t-1)} - 1 \otimes (\hat{g}^{(t)} - \hat{g}^{(t-1)})\| \\
= \|\hat{v}^{(t)} - \hat{v}^{(t-1)} - \left(\frac{11}{n} \otimes I\right)(\hat{v}^{(t)} - \hat{v}^{(t-1)})\| \\
\leq \|\hat{v}^{(t)} - \hat{v}^{(t-1)}\| \leq L \|u^{(t)} - u^{(t-1)}\|.
\]
To bound \(\|u^{(t)} - u^{(t-1)}\|\), we consider
\[
u^{(t)} - u^{(t-1)} = \frac{A_{t-1}}{A_t} P v^{(t-1)} + \frac{a_t}{A_t} w^{(t-1)} - u^{(t-1)}
\]
\[
= \frac{A_{t-1}}{A_t} P (v^{(t-1)} - u^{(t-1)}) + \frac{A_{t-1}}{A_t} (P - I \otimes I) u^{(t-1)} \\
+ \frac{a_t}{A_t} (w^{(t-1)} - u^{(t-1)})
\]
where the first equality is due to (56a). From (56a) and (56b), we have \(v^{(t-1)} - u^{(t-1)} = \frac{a_t}{A_{t-1}} (w^{(t-1)} - w^{(t-2)})\). In addition, we have \((P - I \otimes I) u^{(t-1)} = (P - I \otimes I) (u^{(t-1)} - 1 \otimes \pi^{(t-1)})\). Therefore, it holds that
\[
\|u^{(t)} - u^{(t-1)}\| \\
\leq \frac{A_{t-1}}{A_t} \frac{a_t}{A_{t-1}} \|w^{(t-1)} - w^{(t-2)}\| + 2 \frac{A_{t-1}}{A_t} \|u^{(t-1)}\| \\
+ \frac{a_t}{A_t} \|w^{(t-1)} - u^{(t-1)}\| \\
\leq \frac{a_t}{A_t} \sqrt{nG} + \frac{2 a_t}{A_t} C_p + \frac{a_t}{A_t} \sqrt{nG} \\
= \frac{a_t}{A_t} (2 \sqrt{nG} + 2 C_p)
\]
(65)
where Lemma 7 and Assumption 3 are used to get the second inequality. By substituting (65) into (64), we obtain
\[
\|q^{(t)}\| \leq \beta \|q^{(t-1)}\| + \frac{a_t}{A_t} L (2 \sqrt{nG} + 2 C_p).
\]
(66)
By initialization, we have \(q^{(0)} = 0\) and therefore
\[
\|q^{(t)}\| \leq L (2 \sqrt{nG} + 2 C_p) \sum_{t=1}^{t_0} \beta^{t_0 - t} \frac{a_t}{A_t} \leq L (2 \sqrt{nG} + 2 C_p)
\]
implying that (62) is valid for \(1 \leq t < \left\lceil \frac{3}{1 - \beta} \right\rceil\). Next, we prove that (62) also holds for \(t \geq \left\lceil \frac{3}{1 - \beta} \right\rceil\) by mathematical induction.
Suppose that (62) holds true for some \(t \geq \left\lceil \frac{3}{1 - \beta} \right\rceil\). Using this hypothesis and (66), we obtain
\[
\|q^{(t+1)}\| \leq \frac{a_t}{A_t} \beta C_g + \frac{A_{t+1}}{A_t} L (2 \sqrt{nG} + 2 C_p) \\
\leq \frac{a_t}{A_t} \beta C_g + \frac{1}{1 - \beta} \left(\frac{3}{1 - \beta}\right)
\]
Finally, using the same argument with (59) and (60) in the proof of Lemma 7, we arrive at (62) as desired.

\[\square \]

\[\textbf{B Proof of Theorem 2}\]

\[\textbf{Proof of Theorem 2}: Using } A_{t-1} = A_r - a_r, \text{ we have } \]
\[A_t \left(f \left(\bar{v}^{(t)}\right) - f(x^*)\right) \]
\[= \sum_{t=1}^{t} \left(\begin{array}{c}
A_{t} f \left(\bar{v}^{(t)}\right) - A_{t-1} f \left(\bar{v}^{(t-1)}\right)
\end{array}\right) - \sum_{t=1}^{t} a_t f(x^*) \\
= \sum_{t=1}^{t} \left(\begin{array}{c}
A_{t} f \left(\bar{v}^{(t)}\right) - f(x^*) + a_t f \left(\bar{v}^{(t)}\right) - f(x^*)
\end{array}\right) \\
+ A_{t-1} f \left(\bar{v}^{(t-1)}\right) - f (\bar{v}^{(t-1)})\right)
\]
Upon using the convexity of \(f\), we obtain
\[A_t \left(f \left(\bar{v}^{(t)}\right) - f(x^*)\right) \leq \sum_{t=1}^{t} \left(\begin{array}{c}
A_{t} f \left(\bar{v}^{(t)}\right) - f(x^*)
\end{array}\right) \\
+ a_t \left\langle \nabla f \left(\bar{v}^{(t)}\right), \bar{v}^{(t)} - x^*\right\rangle \\
+ A_{t-1} \left\langle \nabla f \left(\bar{v}^{(t-1)}\right), \bar{v}^{(t-1)} - x^*\right\rangle
\]
By (57b), we obtain
\[A_t \left(f \left(\bar{v}^{(t)}\right) - f(x^*)\right) \leq \sum_{t=1}^{t} \left(\begin{array}{c}
A_{t} f \left(\bar{v}^{(t)}\right) - f \left(\bar{v}^{(t)}\right)
\end{array}\right) \\
+ \sum_{t=1}^{t} a_t \left\langle \nabla f \left(\bar{v}^{(t)}\right), \bar{v}^{(t)} - x^*\right\rangle \\
+ \left\langle \sum_{t=1}^{t} a_t \left\langle \nabla f \left(\bar{v}^{(t)}\right), \bar{v}^{(t)} - x^*\right\rangle \\
+ \frac{1}{n} \sum_{t=1}^{t} \sum_{i=1}^{n} a_t \left\langle \nabla f \left(\bar{v}^{(t)}\right) - q_i^{(t)}, w_i^{(t)} - x^*\right\rangle
\]
(67)
where the last inequality is due to the smoothness of \(f\). To bound (I), we consider
\[
\|\bar{v}^{(t)} - \bar{v}^{(t)}\|^2 \\
= \frac{1}{n} \sum_{i=1}^{n} \left(\|\bar{v}^{(t)} - u_i^{(t)} + u_i^{(t)} - v_i^{(t)} + v_i^{(t)} - \bar{v}^{(t)}\|^2 \\
\leq \frac{1}{n} \sum_{i=1}^{n} \left(3 \left\langle \bar{v}^{(t)} - u_i^{(t)}\right\rangle + 3 \left\langle u_i^{(t)} - v_i^{(t)}\right\rangle + 3 \left\langle v_i^{(t)} - \bar{v}^{(t)}\right\rangle\right)
\]
+ 3 \left\langle \bar{v}^{(t)} - \bar{v}^{(t)}\right\rangle\right)
\]
where the first inequality follows from $\|x + y + z\|^2 \leq 3\|x\|^2 + 3\|y\|^2 + 3\|z\|^2$, and the last inequality is due to Lemma 7 and (57). For (II), by letting $\zeta^{(r)} = q^{(r)}$ and $\nu^{(r)} = w^{(r)}$ in Lemma 6, we have

$$\sum_{t=1}^{T} a_{\tau} \langle \nabla f \left( \pi^{(r)} \right) - q^{(r)}(x), w^{(r)} - x \rangle \leq d(x^*) - \sum_{\tau=1}^{T} \frac{1}{2} \|w^{(r)} - w^{(r-1)}\|^2. \tag{69}$$

To bound (III), we use (61) to get

$$a_{\tau} \left\| \nabla f \left( \pi^{(r)} \right) - \bar{g}^{(r)} \right\| \leq \frac{1}{n} \sum_{i=1}^{n} \left\| \nabla f_i \left( \pi^{(r)} \right) - \nabla f_i \left( u^{(r)} \right) \right\| + \left\| \nabla f_i \left( u^{(r)} \right) - \bar{g}^{(r)} \right\| \leq \frac{L}{n} \sum_{i=1}^{n} \left\| w_i - u_i^{(r)} \right\| \leq \frac{L}{\sqrt{n}} \sqrt{C_{\tau}} \frac{LC_p \sqrt{n}}{\sqrt{\tau}}.$$

Recall Lemma 8 that $\left\| \nabla f \left( \pi^{(r)} \right) - q^{(r)} \right\| \leq C_{\tau} a_{\tau} / (\sqrt{n} A_{\tau})$. Therefore,

$$\sum_{i=1}^{n} a_{\tau} \left\langle \nabla f \left( \pi^{(r)} \right) - q_i^{(r)}, w_i^{(r)} - x \right\rangle \leq \left( \frac{a_{\tau}^2}{A_{\tau}} \right) \sqrt{n} G(LC_p + C_g). \tag{70}$$

Finally, by collectively substituting (68), (69), and (70) into (67), we get

$$A_{\tau} \left( f \left( \pi^{(r)} \right) - f(x^*) \right) \leq \frac{G(LC_p + C_g)}{\sqrt{n}} + \frac{3LC_p^2}{n} \sum_{\tau=1}^{T} \frac{a_{\tau}^2}{A_{\tau}} + d(x^*) + \frac{1}{2n} \sum_{\tau=1}^{T} \left( \frac{3LC_p^2}{A_{\tau}} - 1 \right) \left\| \bar{w}^{(r)} - \bar{w}^{(r-1)} \right\|^2.$$

Based on the condition in (30) and the fact that $a_{\tau}^2 / A_{\tau} \leq 2a$, we obtain (31) as desired.

The inequality in (32) directly follows from Lemma 7. \qed

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