Low-dimensional modelling of dynamics via computer algebra

A.J. Roberts

22 March, 1996

Keywords: long-wave approximation, thin fluid film, low-dimensional modelling, centre manifold, computer algebra, bifurcation.

Abstract

I describe a method, particularly suitable to implementation by computer algebra, for the derivation of low-dimensional models of dynamical systems. The method is systematic and is based upon centre manifold theory. Computer code for the algorithm is relatively simple, robust and flexible. The method is applied to two examples: one a straightforward pitchfork bifurcation, and one being the dynamics of thin fluid films.

1 Introduction

Deterministic evolution equations, either ordinary differential equations or partial differential equations, are used to describe dynamics in the physical world. Often, these equations allow many modes of behaviour that are of little physical interest in particular applications, for example sound waves are neglected in many applications of fluid mechanics. The essential dynamical behaviour of the system is then determined by the evolution of a subset of the possible modes, the “critical” modes. Rapid oscillations or heavy damping characterise the modes that need to be eliminated from consideration. If the amplitudes of the modes are viewed as co-ordinates in a state space then the dynamical behaviour of the system corresponds to motion along some trajectory. When many of the modes are heavily damped, trajectories are rapidly attracted to some low-dimensional invariant manifold, which may be parameterised by the amplitudes of the critical modes. This geometric picture is the heart of the application of centre manifolds

*Dept. Mathematics & Computing, University of Southern Queensland, Toowoomba 4350, AUSTRALIA. E-mail: aroberts@usq.edu.au
to the rational construction of low-dimensional model systems by the elimination of physically uninteresting fast modes of behaviour. Such low-dimensional dynamical models are significantly easier to analyse, simulate and understand.

Applications of the techniques have ranged over, for example, triple convection [1], feedback control [3], economic theory [7], shear dispersion [17, 18], nonlinear oscillations [24], beam theory [23], flow reactors [3], and the dynamics of thin fluid films [24]. New insights given by the centre manifold picture enable one to not only derive the dynamical models, but also to provide accurate initial conditions [21, 22], boundary conditions [22], and the treatment of forcing [3].

In the application of centre manifold theory, the typical approach is to express the model explicitly in terms of asymptotic sums [8, 1, 26, 23, 2, 24, 16, e.g.]. To reduce the dynamics onto the centre manifold, one then has to substitute the asymptotic sums into the governing equations, reorder the summations, rearrange to extract dominant terms, and evaluate the expressions. While perfectly acceptable when done correctly, it leads to formidable working which obscures the construction of a model. Further, such asymptotic expansions, in common with the method of multiple scales, reinforce the notion that careful balancing of the “order” of small effects are necessary in the construction of a model rather than in its use in some situation. Instead, I propose in §2 an iterative method, based upon the residuals of the governing differential equations, for the construction of such low-dimensional, dynamical models. The evaluation of the residuals is a routine algebraic task which may be easily done using computer algebra, as seen in the examples of §§2 and 3, by simply coding the governing differential equations; it replaces the whole messy detail of the manipulation of asymptotic expansions (e.g. [8, §5.4]). The aim of this proposed approach is to minimise human time by using a novel algorithm which can be simply and reliably implemented in computer algebra with relatively small inefficiencies in the use of computer resources. After all: “It is unworthy of excellent persons to lose hours like slaves in the labour of calculation”... Gottfried Wilhelm von Leibniz.

The use of centre manifold theory in the construction of low-dimensional modes relies on three theorems. These specifically address dynamical systems written in the form:

\[
\begin{align*}
\dot{x} &= Ax + f(x, y), \\
\dot{y} &= By + g(x, y),
\end{align*}
\] (1)

where: the overdot denotes \(d/dt\); \(x(t)\) is \(m\)-dimensional; \(y(t)\) is \(n\)-dimensional (more generally, of infinite dimension); the eigenvalues of \(A\) have zero real-part; the eigenvalues of \(B\) have strictly negative real-parts bounded away from 0, \(\lambda_B < -\gamma \leq 0\); and the nonlinear functions \(f\) and \(g\) are smooth and are at least quadratic near the origin. Then [1, p4–5], [3, §2] or [4, p5&p35]:

**Existence** There exists a smooth \(m\)-dimensional centre manifold for (1) of the form \(y = h(x)\), tangent to \(y = 0\) at the origin. The dynamics on the centre
manifold are governed by
\[ \dot{a} = Aa + f(a, h(a)) . \] 

That is, provided it has a correct spectrum of eigenvalues, a smooth dynamical system possesses a centre manifold with low-dimensional, self-contained dynamics.

**Relevance** Let \((x(t), y(t))\) be a solution of the parent system \( (1) \) with initial point \((x(0), y(0))\) sufficiently small (in practice, sufficiently small may be quite wide), then provided the zero solution is stable, there exists a solution of \( (2) \) \(a(t)\) such that as \(t \to \infty\),
\[ x(t) = a(t) + O\left(e^{-\gamma t}\right) , \quad y(t) = h(a(t)) + O\left(e^{-\gamma t}\right) , \]

where \(\gamma > 0\) is some constant.

That is, from a wide range of initial conditions, all solutions tend exponentially quickly to a solution on the centre manifold, and hence the dynamical system \( (3) \), on the \(m\)-dimensional centre manifold, faithfully models the original.

**Approximation** Seeking an approximation to the centre manifold, \(y = \tilde{h}(x)\), just substitute into the governing equations \( (1) \), and use the chain rule to deduce that, given
\[ \mathcal{M}\tilde{h} = \frac{\partial \tilde{h}}{\partial x} \left[ Ax + f(x, \tilde{h}(x)) \right] - B\tilde{h} - g(x, \tilde{h}(x)) , \]

we wish to solve \(\mathcal{M}\tilde{h} = 0\). Suppose that as \(x \to 0\), \(\mathcal{M}\tilde{h} = O(|x|^q)\), then \(h(x) = \tilde{h}(x) + O(|x|^q)\). It is often convenient to appeal to a more general assertion that explicitly accounts for constant parameters, say \(\epsilon\): if \(\mathcal{M}\tilde{h} = O(|x|^q, |\epsilon|^{r'})\), then \(h(x, \epsilon) = \tilde{h}(x, \epsilon) + O(|x|^q, |\epsilon|^{r'})\). (This more general form is particularly relevant to unfolding bifurcations and to the long-wave, slowly-varying approximation \[20\].)

That is, provided we can satisfy the governing equations to some order of accuracy, then the centre manifold will have been found to the same order of accuracy.

Simple applications of these theorems may be found in the book by Carr\[4\]. Here I show how to use an algorithm, eminently suitable for computer algebra and based upon these theorems, to derive low-dimensional models of dynamical systems. In \(\S 2\), I develop the formalism in general and in a bifurcation example. In \(\S 3\) I show how straightforward it is to apply the techniques to a much more difficult problem, namely the flow of a thin film of fluid upon a solid substrate.
One important feature of the analysis is that we deal here with the problems in terms of the physical differential equations as given, and not in the abstract form \((I)\). A linear change of basis, such as \((x, y) = Pu\) for some linear transformation \(P\), is all that is needed to transform from a physical description, in terms of physical variables \(u\), into a form for a direct application of theory.

\section{Modelling a pitchfork bifurcation}

Consider the following variation to Burger’s equation featuring growth, \((1 + \epsilon)u\), nonlinearity, \(uu_x\), and dissipation, \(u_{xx}\):

\[
\frac{\partial u}{\partial t} = (1 + \epsilon)u + u \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}, \quad u(0) = u(\pi) = 0
\]

for some function \(u(x, t)\). View this as an infinite dimensional dynamical system, the state space being the set of all functions \(u(x)\) on \([0, \pi]\).

For all values of the parameter \(\epsilon\) there is a fixed point at the origin, that is, a trivial equilibrium state is \(u = 0\). A linearisation of the equation about this equilibrium, namely \(u_t = (1 + \epsilon)u + u_{xx}\), has modes \(\sin kx\) with associated eigenvalues \(\lambda_k = 1 - k^2 + \epsilon\) for wavenumbers \(k = 1, 2, \ldots\). Thus the \(k = 1\) mode, \(\sin x\), loses stability as \(\epsilon\) crosses zero, and the system undergoes a bifurcation.

To find the details of this pitchfork bifurcation is a simple task for low-dimensional modelling. Linearly, exactly at critical, \(\epsilon = 0\), all modes decay exponentially quickly except for the critical mode \(\sin x\); it has a zero decay rate and therefore is long lasting; by the first theorem we are assured that there exists a centre manifold. Nonlinearly, and for \(\epsilon\) and \(u(x)\) near 0, all modes decay exponentially except for the critical modes which have a slow evolution. Thus, exponentially quickly we can model the dynamics solely in terms of the evolution of the amplitude of the \(\sin x\) mode; I define \(a\) to be its amplitude. By the relevance theorem, the evolution of \(a\) in time forms an accurate one-dimensional model of the original infinite-dimensional dynamical system \((I)\).

\subsection{The iteration scheme}

I now proceed to simultaneously develop a novel and powerful algorithm to derive such a low-dimensional model while applying it to the specific example \((I)\). In general, I address dynamical systems in the form

\[
\dot{\mathbf{u}} = \mathcal{L}\mathbf{u} + \mathbf{f}(\mathbf{u}, \epsilon),
\]

where:

- \(\mathbf{u}(t)\) is the evolving state “vector” (corresponding to \((x, y)\) in \((I)\)), \(u(x, t)\) in the example;
• \( \mathcal{L} \) is a linear operator whose spectrum, as required by centre manifold theory, is discrete and separates into eigenvalues of zero real-part, the critical eigenvalues (corresponding to the modes \( \mathbf{x} \) in (1)), and eigenvalues with strictly negative real-part (corresponding to the modes \( \mathbf{y} \) in (1)); in the example \( \mathcal{L}u = u + u_{xx} \) (with implicit boundary conditions);

• \( \epsilon \) is a parameter, potentially a vector of parameters;

• and \( f \) is a function which is strictly nonlinear when considered as a function in \( u \) and \( \epsilon \) together, that is, \( f \) is quadratic or higher order in \( u \) and \( \epsilon \) as they tend to 0, \( f = \epsilon u + uu_x \) in the example.

For simplicity, I only treat the case where the critical eigenvalues of \( \mathcal{L} \) are exactly 0; let the multiplicity be \( m \). Cases where the critical eigenvalues have a non-zero imaginary part, as in a Hopf bifurcation, may be handled with the same ideas as described herein, but with more complicating detail. The aim is to find a low-dimensional model \( \dot{a} = g(a) \), such as (2), for the evolution of \( m \) “amplitudes” \( a \). These low-dimensional dynamics occur on the exponentially attractive centre manifold which may be described parametrically as \( u = v(a) \).

The first stage is to identify the \( m \) critical modes, that is, those associated with the eigenvalue zero; these are necessary in order to project the linear dynamics and nonlinear perturbations onto the slow modes of interest. They may be found from the nontrivial solutions, \( v_j \), of \( \mathcal{L}v = 0 \); in general we need the critical eigenspace and so may need to find all the generalised eigen-modes. Then, in terms of modal amplitudes \( a_j \), a linear approximation to the centre manifold and the evolution thereon is simply

\[
\mathbf{u}(t) \approx \sum_j v_j a_j = \mathbf{V}a \quad \text{such that} \quad \dot{\mathbf{a}} \approx \mathbf{G}a,
\]

where \( \mathbf{V} = [v_j] \) and where \( \mathbf{G} \) may be chosen in Jordan form in the case of generalised eigenvectors (in (1) the \( m \) critical modes are \( \mathbf{x} \), and the linear approximation is \( \mathbf{u} = (\mathbf{x}, \mathbf{0}) \) such that \( \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \)). In the example (3), the eigenvalue zero is of multiplicity \( m = 1 \) and so there is only the one critical mode, \( v(x) = \sin x \); hence the linear approximation is

\[
u(x, t) \approx a \sin x \quad \text{such that} \quad \dot{a} \approx 0.
\]

To model the nonlinear dynamics, this linear approximation needs to be modified by nonlinear terms; such modification would be equivalent to seeking the nonlinear shape of the centre manifold, \( \mathbf{y} = h(\mathbf{x}) \), in (1).

The second stage is to seek iterative improvements to a given level of description of the centre manifold and the low-dimensional evolution thereon. The aim is to find a low-dimensional description which satisfies the nonlinear dynamical equation (3). As in Newton’s method for finding the zero of a function, we use
the residual of the governing equations in order to guide corrections. The iteration scheme is successful as long as it ultimately drives the residual to zero to the desired order of accuracy—see the approximation theorem. Suppose that at one stage of the iteration we have the approximate model

\[ u \approx \tilde{v}(a) \quad \text{such that} \quad \dot{a} \approx \tilde{g}(a) ; \]

approximate because the residual of the governing differential equation (9)

\[ \dot{u} - \mathcal{L} u - f(u, \epsilon) = \frac{\partial \tilde{v}}{\partial a} \tilde{g} - \mathcal{L} \tilde{v} - f(\tilde{v}, \epsilon) = \mathcal{O}(a^q, \epsilon^r), \tag{9} \]

for some order of error, \( q \) and \( r \), and where \( a \) denotes \(|a|\). We seek to find “small” corrections, indicated by primes, so that

\[ u \approx \tilde{v}(a) + v'(a) \quad \text{such that} \quad \dot{a} \approx \tilde{g}(a) + g'(a), \]

is a better approximation to the centre manifold and the evolution thereon. The aim of each iteration is to improve the order of the errors \( (q, r) \) so that, by the approximation theorem, we improve the accuracy of the model. Substituting into the governing differential equation (9), and using the chain rule for time derivatives, leads to

\[ \left( \frac{\partial \tilde{v}}{\partial a} + \frac{\partial v'}{\partial a} \right) (\tilde{g} + g') = \mathcal{L} \tilde{v} + \mathcal{L} v' + f(\tilde{v} + v', \epsilon). \]

Given that it is impossible to solve this for the perfect corrections in one step, seek an approximate equation for the corrections of \( \mathcal{O}(a^q, \epsilon^r) \) by:

- ignoring products of corrections (primed quantities) because they will be small, \( \mathcal{O}(a^{2q} + \epsilon^{2r}) \), compared with the dominant effect of the linear correction terms;

- and replacing tilde quantities by their initial linear approximation wherever they are multiplied by a correction factor (introducing errors \( \mathcal{O}(a^{q+1}, \epsilon^{r+1}) \)—such approximation slows the iteration convergence to linear, as opposed to the quadratic convergence which would be otherwise obtained (if only it were practical).

Thus we wish to solve

\[ \frac{\partial \tilde{v}}{\partial a} \tilde{g} + V g' + \frac{\partial v'}{\partial a} G a = \mathcal{L} \tilde{v} + \mathcal{L} v' + f(\tilde{v} + v', \epsilon). \]

It is not obvious, but provided it is arranged so that \( G \) is in Jordan form, as is often physically appealing, we may significantly simplify the algorithm by also neglecting the term \( \frac{\partial v'}{\partial a} G a \) at a cost of increasing the number of iterations needed
by a factor no more than \( m \), the multiplicity of the zero eigenvalue of \( \mathcal{L} \). Thus, rearranging and recognising that \( \frac{\partial \tilde{v}}{\partial a} = \frac{\partial v}{\partial t} \) by the chain rule, we solve

\[
\mathcal{L}v' - Vg' = \frac{\partial \tilde{v}}{\partial t} - \mathcal{L}\tilde{v} - f(\tilde{v}, \epsilon),
\]

for the primed correction quantities. In the example, we seek to solve

\[
v' + \frac{\partial^2 v'}{\partial x^2} - g' \sin x = \frac{\partial \tilde{v}}{\partial t} - (1 + \epsilon)\tilde{v} - \frac{\partial \tilde{v}}{\partial x} - \frac{\partial^2 \tilde{v}}{\partial x^2},
\]

which in the first iteration from the linear approximation (8) is

\[
v' + \frac{\partial^2 v'}{\partial x^2} - g' \sin x = -\epsilon a \sin x - \frac{1}{2}a^2 \sin 2x.
\]

The great advantage of this approach is that the right-hand side is simply the residual of the governing equation (3) evaluated at the current, tilde, approximation; in essence, this residual is the quantity defined in (4). Thus at any iteration we just deal with physically meaningful expressions; all the complicated expansions and rearrangements of asymptotic expansions, as needed by the method of multiple scales (e.g. [15, §3.5]) or earlier methods to find the centre manifold (e.g. [8, §5.4]), are absent. There is, of course, a cost and that lies in evaluating the residual (which potentially has enormous algebraic detail), much of which is repeated at every iteration. However, with the advent of computer algebra, all this detail may be left to the computer to perform—such mindless repetition is ideal for a computer—whereas all a human need concern themselves with is setting up the solution of (10) and not at all with the detailed algebraic machinations of asymptotic expansions.

The main detail is to solve equations of the form

\[
\mathcal{L}v' - Vg' = r,
\]

for some given residual \( r \). Recognise that there are more unknowns than components in this equation; its solution is not unique. The freedom comes from the fact that we can parameterise the centre manifold via the amplitudes \( a \) in an almost arbitrary manner. The freedom can only be resolved by giving a precise meaning to the \( m \) amplitudes \( a \). Often one does define \( a \) to be precisely the modal amplitudes (as is done implicitly for (1) by seeking a centre manifold in the form \((x, h(x))\) in which case we seek corrections \( v' \) which are orthogonal to the generalised eigenvalues, \( z_j \), of the adjoint of \( \mathcal{L} \). In the example, \( \mathcal{L} \) is self adjoint under the inner product \( \langle u_1, u_2 \rangle = \frac{2}{\pi} \int_0^{\pi} u_1 u_2 dx \), and so the adjoint eigenvector is also simply \( z(x) = \sin x \) and so I require \( \langle \sin x, v' \rangle = 0 \). More general definitions, such as an energy related amplitude, give rise to similar considerations to those that follow. There are two approaches to solving (11).
1. Numerically, it is generally easiest to adjoin the amplitude condition to the equation and solve

\[
\begin{bmatrix}
L & -V \\
Z^T & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}' \\
\mathbf{g}'
\end{bmatrix} =
\begin{bmatrix}
\mathbf{r} \\
0
\end{bmatrix},
\]

where \( Z = [z_j] \).

2. However, algebraically it is usually more convenient to adopt the following procedure which is also familiar as part of other asymptotic methods. Rewrite (11) as \( L\mathbf{v}' = V\mathbf{g}' + \mathbf{r} \) and recognise that \( L \) is singular due to the zero eigenvalue of multiplicity \( m \). We choose the \( m \) components of \( \mathbf{g}' \) to place the right-hand side in the range of \( L \); this is achieved by taking the inner product of the equation with the adjoint eigenvalues \( z_j \) (this corresponds to considering just the \( x \) modes in (1)) and thus giving the set of \( m \) equations

\[
\langle Z, V \rangle \mathbf{g}' = -\langle Z, \mathbf{r} \rangle.
\]

In the example, \( g' = -\frac{2}{\pi} \int_0^\pi \sin x r(x) \, dx \) which in the first iteration gives \( g' = \epsilon a \). This equation is known as the solvability condition.

Having put the right-hand side in the range of \( L \) we solve \( \mathbf{L}\mathbf{v}' = \hat{\mathbf{r}} = V\mathbf{g}' + \mathbf{r} \) for \( \mathbf{v}' \), making the solution unique by accounting for the definition of the amplitudes \( \mathbf{a} \). In the example, we solve the boundary value problem \( v' + v'' = \hat{r}(x) \) such that \( v'(0) = v'(?\pi) = 0 \) and that \( v \) has no \( \sin x \) component. In the first iteration, the problem \( v' + v'' = -\frac{1}{2}a^2 \sin 2x \) with the above conditions has the solution \( v' = \frac{1}{6}a^2 \sin 2x \).

Then the last step of each iteration is to update the approximations for the centre manifold shape and the evolution thereon. For the example, after the first iteration we deduce \( u \approx a \sin x + \frac{1}{6}a^2 \sin 2x \), which shows the nonlinear steepening/flattening of negative/positive slopes, and that the evolution is \( \dot{a} \approx \epsilon a \), which exhibits the loss of stability of the fixed point \( a = 0 \) as \( \epsilon \) becomes positive.

Further iterations in the example lead to the centre manifold being given by

\[
u = a \sin x + \frac{1-\frac{7}{12}}{6}a^2 \sin 2x + \frac{1-\frac{7}{12}}{32}a^3 \sin 3x + \mathcal{O}\left(a^4, \epsilon^2\right), \tag{12}\]

on which the system evolves according to

\[
\dot{a} = \epsilon a - \frac{1-\frac{7}{12}}{12}a^3 + \mathcal{O}\left(a^4, \epsilon^2\right). \tag{13}\]

The relevance theorem assures us that this 1-D model of the original infinite-dimensional dynamical system (5), is valid exponentially quickly in time. From the model (13), for example, we deduce the quantitative shape of the pitchfork bifurcation: there are stable fixed points at \( a \approx \sqrt{\epsilon/(1-\epsilon/3)} \). Physically, these fixed points represent a balance between the nonlinear steepening of the \( uu_x \) term, and the dissipation of \( u_{xx} \).
2.2 Computer algebra implementation

A principal reason for adopting this approach is because it is simply and reliably implemented in computer algebra. Based upon the above derivation, the general outline of the algorithm is:

1. preliminaries;
2. initial linear approximation;
3. repeat until residual is small enough;
   (a) compute residual,
   (b) find solvability condition,
   (c) compute correction to the centre manifold,
   (d) update approximations.

Complete details of a REDUCE program for the particular example follows. The reason for using REDUCE is that it has excellent pattern matching and replacement capabilities through its operator and let statements.

% Find pitchfork bifurcation in \( u_t = (1 + \varepsilon)u + uu_x + u_{xx} \)
% \( a(t) \) measures amplitude of \( \sin(x) \) component in \( u(x,t) \)
% on div; off allfac; on revpri; % improves appearance of output
let \( \sin(-x) \cos(-y) \Rightarrow (\sin(x+y) + \sin(x-y))/2; \) % a trig rule
% Define the inverse operator of \( u + u_{xx} \)
operator linv; linear linv;
let \( \text{linv}(\sin(-k*x),x) \Rightarrow \sin(k*x)/(1-k^2); \)
% Define inner product with \( \sin(x) \)
operator sindot; linear sindot;
let \{ sindot(\sin(x),x) \Rightarrow 1, sindot(\sin(-k*x),x) \Rightarrow 0 \};
% depend a,t; % asserts that \( a \) depends upon time \( t \)
let df(a,t) \Rightarrow g; % so \( da/dt \) is replaced by current \( g(a,\varepsilon) \)
% \( u := a*\sin(x); g := 0; \) % initial approximation
% iterate until PDE is satisfied (to requisite order)
let \{ \varepsilon^2 = 0, a^4 = 0 \}; % discard high-order terms in \( a \) & \( \varepsilon \)
repeat begin
write eqn:=df(u,t)-(1+eps)*u-u*df(u,x)-df(u,x,x);

gd:=-sindot(eqn,x);
write u:=u+linv(eqn+sin(x)*gd,x);
write g:=g+gd;
end until eqn=0;
%

Observe the how this implements the algorithm.

1. The preliminaries do the following.
   - ℓ4 improves the appearance of the printed output, whereas ℓ5 tells REDUCE that we wish to linearise products of trigonometric functions.
   - ℓ7–9 defines the operator \texttt{linv} to act as the inverse of \( \mathcal{L} \):
     - declaring it \texttt{linear} tells REDUCE to expand sums and products in the first argument and to only leave functions of the second argument inside the operator, for example, \( \texttt{linv}(\epsilon a \sin x+2a^2 \sin 2x,x) \) is expanded to \( \epsilon a \texttt{linv}(\sin x,x)+2a^2 \texttt{linv}(\sin 2x,x) \);
     - the \texttt{let} statement on ℓ9 defines the action of the operator as the solution to \( v' + v'_{xx} = \sin kx \), namely \( v' = \frac{1}{1-k^2} \sin kx \), the tilde before the \( k \) on the left-hand side matches any pattern (no action is defined for the singular case \( k = 1 \) because the pattern \( \sin(\sim k*x) \) does not match \( \sin(x) \), but any appearance of \( \texttt{linv}(\sin(x),x) \) usefully signals an error).
   - ℓ10–12 similarly defines \texttt{sindot} to be the inner product operator \( \langle \sin x, \cdot \rangle \), the \texttt{let} statement now being a list, enclosed within braces, of evaluation rules.
   - ℓ14–15 establishes that the variable \( a \) is to firstly depend upon time, as we use \( a \) as the time dependent amplitude in the model, and secondly that time derivatives of \( a \), \( df(a,t) \), are to be replaced by the value of \( g \), at the time of replacement, as \( g \) is to store the current approximate model evolution equation such as \( (13) \).

2. ℓ17 simply assigns the linear approximation \( (8) \) of the centre manifold to be the initial value of the variables \( u \) and \( g \).

3. ℓ20–26 performs the iterations.
   - ℓ20 controls the truncation of the asymptotic approximation. It gives a list of transformations which tell REDUCE to discard any factor in \( \epsilon^2 \) or higher and any factor in \( a^4 \) or higher; thus all expressions are computed to an error of \( \mathcal{O}(\epsilon^2,a^4) \).
3 Thin film fluid dynamics

We now turn to the modelling of an important physical problem, that of the flow of a thin film of viscous fluid upon a solid substrate. Examples include the flow of rainwater on a road or windscreen or other draining problems \cite{6}, paint and coating flows \cite{25, 27}, and the flow of many protective biological fluids \cite{12}. For simplicity we restrict attention to 2D fluid flow and seek a model for the evolution of the depth of the fluid film; because the film is thin expect that the vertical structure of the flow is relatively simple. I show how centre manifold theory can partially justify such a model, and how to derive the model using the algorithm developed in this paper.

This fluid flow problem has many nonlinearities, as shown in Figure 1: not only is the advection in the Navier-Stokes equation described by a nonlinear term, but also the thickness of the fluid film is to be found as part of the solution and its unknown location is another source of nonlinearity. When a fluid layer is thin then, as in dispersion in a pipe \cite{18}, the dominant dynamical processes occur along the thin film of fluid. Across the fluid film, viscosity acts quickly to damp almost all cross-film structure; the only long lasting dynamics are those of the relatively slow spread of the film along the substrate. In a nonlinear problem, this signature of interesting dynamics on a long time-scale along with uninteresting, quickly dissipated modes indicates that centre manifold theory may be used to create a low-dimensional model of the interesting dynamics.
Assuming no longitudinal variations in $x$, a linear analysis of the equations shows that there is one critical mode in the cross-film dynamics associated with conservation of the fluid, all others decay due to viscosity. Consequently, it is natural to express the low-dimensional model in terms of the film thickness $\eta(x,t)$. Seeking solutions which vary slowly along the film, that is, assuming $\partial/\partial x$ is a small parameter like $\epsilon$ used previously, a centre manifold analysis creates an effective model of the dynamics. Through the algorithm described herein, we find the long-term evolution of long-wavelength modes is approximately described by

$$\frac{\partial \eta}{\partial t} \approx - \frac{1}{3} \frac{\partial}{\partial x} \left( \eta^3 \eta'' \right)$$

(14)

$$- \frac{\partial}{\partial x} \left[ \frac{3}{5} \eta^5 \eta'' + 3 \eta^4 \eta' \eta'' + \eta^4 \eta'' \eta''' + \frac{11}{6} \eta^3 \eta' \eta''^2 - \eta^3 \eta' \eta''^2 \right],$$

where dashes or roman numerals on $\eta$ denote $\partial/\partial x$. The centre manifold theorems reasonably assure us that this model is indeed relevant to the long-term dynamics of thin films. However, this assurance is not yet rigorous because of deficiencies in the preconditions of current theorems.\(^2\)

The first line of the model (14) is the standard leading, or “lubrication,” approximation to thin film dynamics; the second line may be viewed either as correction terms, or as terms indicative of the error in the leading approximation. An interesting diversion is attributable to the fact that although this is a nonlinear problem, conservation of fluid applies no matter how thick the fluid layer (for example, the right-hand side of (14) can always be written as a gradient). Thus the analysis is valid for arbitrarily large variations in the thickness of the film! just so long as the variations are sufficiently slow. Theorems on the existence

\(^2\)The major constraint upon a rigorous application of the theorems by Gallay [1] is that we need to treat $\partial/\partial x$ as a small parameter, whereas in principle it is unbounded. However, provided we confine attention to functions slowly-varying in $x$, then such derivatives would indeed be small.
and approximation of such global (in $\eta$) centre manifolds are also given by Carr \[4\] pp31–32.

### 3.1 Governing equations

As shown in Figure 1 we solve the Navier-Stokes and continuity equations within the Newtonian fluid for the velocity field $\mathbf{q}$:

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{q},$$

$$\nabla \cdot \mathbf{q} = 0.$$  

For simplicity, restrict attention to two-dimensional flow taking place in the $xy$-plane: the $x$-axis is aligned along the solid bed of the flow; the $y$-axis is perpendicular. The viscous flow must stick to the solid bed to give the boundary condition

$$\mathbf{q} = 0 \quad \text{on} \ y = 0. \quad (15)$$

The surface of the fluid, $y = \eta(x, t)$, evolves with the flow. Because the free-surface is unknown we not only need two boundary conditions for the Navier-Stokes equations, we also need an extra boundary equation in order to be able to find $\eta$. The kinematic condition is that the fluid flow, as given by the velocity, $\mathbf{q}$, must follow the free-surface as given by $y = \eta(x, t)$:

$$\frac{\partial \eta}{\partial t} = v - u \frac{\partial \eta}{\partial x} \quad \text{on} \ y = \eta. \quad (16)$$

The other boundary conditions come from the forces acting across the free-surface. Above the fluid film we suppose there is a very light and essentially inviscid fluid such as air (one-thousandth the density of water).

- Since air is inviscid, it cannot sustain any tangential stress across the surface, thus

$$2\eta' \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) + \left(1 - \eta'^2\right) \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0 \quad \text{on} \ y = \eta. \quad (17)$$

- Since the density of air is very low, Bernoulli’s equation asserts that fluid stress exerted normally across the surface has to be constant, say $T_n = -p_a$, equal and opposite to air pressure which without loss of generality I take to be zero. However, the effect of surface tension is like that of an elastic membrane, it causes a pressure jump if the surface is curved: positive if the fluid surface is convex; negative if it is concave. For both the normal stress and surface tension to oppose atmospheric pressure

$$(1 + \eta'^2) p = 2\mu \left[ \frac{\partial v}{\partial y} + \eta'^2 \frac{\partial u}{\partial x} - \eta' \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] - \frac{\sigma \eta''}{\sqrt{1 + \eta'^2}} \quad \text{on} \ y = \eta,$$
where $\sigma$ is the coefficient of surface tension.

Now non-dimensionalise these equations, referring all scales to $\sigma$ as surface tension is taken to be the driving force. For a reference length, suppose that $h$ is a characteristic thickness of the thin film as shown schematically in Figure 1. Then non-dimensionalise by writing the equations with respect to: the reference length $h$; the reference time $\mu h/\sigma$; the reference velocity $U = \sigma/\mu$; and the reference pressure $\sigma/h$. With these choices, and in non-dimensional quantities, we solve the Navier-Stokes and continuity equations

\[ R \left( \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} \right) = -\nabla p + \nabla^2 \mathbf{q}, \quad (18) \]
\[ \nabla \cdot \mathbf{q} = 0, \quad (19) \]

where $R = \frac{ah}{\mu}$ is effectively a Reynolds number characterising the importance of the inertial terms—it may be written as $Uh/\nu$ for the above characteristic velocity. Observe that due to the absence of $R$ in the model (14), to its order of accuracy the inertial terms have no influence on the dynamics—it is creeping flow. The non-dimensional equations are subject to the bottom boundary condition (15), and on the surface the kinematic condition (16) and the two dynamic conditions, (17) and

\[ (1 + \eta'^2) p = 2 \left[ \frac{\partial v}{\partial y} + \eta'^2 \frac{\partial u}{\partial x} - \eta' \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] - \frac{\eta''}{\sqrt{1 + \eta'^2}} \quad \text{on } y = \eta. \quad (20) \]

In the Navier-Stokes equation I have neglected body forces. If the presence of gravity were to be acknowledged, then it would appear in the non-dimensional combination of the Bond number $b = g\rho h^2/\sigma$. This may be neglected if $b$ is extremely small ($h \ll 0.2\text{cm}$ for water). Including gravity, assuming $b$ is small but not negligible, is not a significant complication and I leave its inclusion as an exercise for the reader—perhaps the simplest interesting case is that of a fluid film on a vertical substrate.

### 3.2 Linear picture

The construction of a centre manifold model rests upon an understanding of the dynamics linearised about some fixed point (the spectrum of the linear operator is crucial in theory and application). Here the fixed point is a film of constant thickness, non-dimensionally 1, and of zero velocity and pressure. The dynamics linearised about this fixed point are based on velocities and $\eta - 1$ being small. Thus, as well as neglecting products of small terms, boundary conditions on the free-surface are approximated by their evaluation on the approximate surface $y = 1$. The linearised equations are then

\[ R \frac{\partial \mathbf{q}}{\partial t} = -\nabla p + \nabla^2 \mathbf{q}, \]
\[ \nabla \cdot \mathbf{q} = 0, \]
\[ \mathbf{q} = \mathbf{0} \quad \text{on } y = 0, \]  
(21)
\[ \frac{\partial \eta}{\partial t} = v \quad \text{on } y = 1, \]
\[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \quad \text{on } y = 1, \]
\[ p = 2 \frac{\partial v}{\partial y} - \eta'' \quad \text{on } y = 1. \]

To investigate the dynamics of “long waves” on this thin film we treat \( \partial / \partial x \) as small, that is slowly-varying in \( x \), as justified formally in [20] and more rigorously for capillary-gravity waves in [13]. “Linearly” then we can base the construction of a centre manifold model on the limit when there are no longitudinal variations: \( \frac{\partial}{\partial x} = 0 \). Neglecting all \( x \) derivatives and seeking solutions proportional to \( e^{\lambda t} \) leads to the following. Firstly, \( \lambda_0 = 0 \) is an eigenvalue associated with the mode \( \eta = \text{const.} \) and \( u = v = p = 0 \). It is the presence of this eigenvalue which indicates the existence of a centre manifold model. Other modes exist, namely \( u_n = \sin \pi (n - \frac{1}{2}) y, v_n = p_n = \eta_n - 1 = 0 \) with eigenvalues \( \lambda_n = -\pi^2 (n - \frac{1}{2})^2 / \mathcal{R} \) for \( n = 1, 2, \ldots \). Thus for long-waves there exists one 0 eigenvalue of the dynamics, whereas all the rest of the eigenvalues are strictly negative, bounded above by \( -\gamma \approx -2.5 / \mathcal{R} \) (the smaller the Reynolds number, the more creeping the flow, and the faster the decay of the non-critical modes). Hence, by the relevance theorem, expect the dynamics of thin films to exponentially quickly approach a low-dimensional centre manifold based on the mode corresponding to the 0 eigenvalue, the film thickness \( \eta \), and characterised by slow variations in \( x \).

### 3.3 Iterative construction

The first task is to decide how to parameterise the centre manifold model. Since the critical mode is \( \eta = \text{const.} \), \( p = u = v = 0 \), it is appropriate to use the film thickness \( \eta \) as the parameter. This is especially useful since \( \eta \) has a direct physical meaning. Thus a linear description of the centre manifold is

\[ u = v = p = 0, \]  
(22)

and \( \eta \) free to vary according to \( \frac{\partial \eta}{\partial t} \approx 0 \).

Now we organise an iteration scheme to refine the description of the centre manifold. The order of error will be characterised by the number of spatial derivatives of \( \eta \); for a slowly-varying function higher-order derivatives are asymptotically smaller than lower-order derivatives. Suppose that the fields \( \tilde{\mathbf{q}}(\eta, y) \) and \( \tilde{p}(\eta, y) \) are an approximation to the centre manifold “shape,” with the associated approximate evolution \( \frac{\partial \eta}{\partial t} \approx \tilde{g}(\eta) \). Seek equations for corrections to this description by:
substituting into the governing equations: \[ q = \tilde{q}(\eta) + q'(\eta), \]
\[ p = \tilde{p}(\eta) + p'(\eta), \]
such that \( \frac{\partial \eta}{\partial t} = \tilde{g}(\eta) + g'(\eta) \);

• omit products of corrections;
• omit \( x \)-derivatives of corrections as both corrections and \( \frac{\partial}{\partial x} \) are small;
• approximate other terms involving corrections by replacing the current approximation, tilde quantities, with the initial linear approximation (here zero);
• rearrange the equations.

The upshot is that we solve equations
\[ \frac{\partial^2 u'}{\partial y^2} = R \left( \frac{\partial \tilde{u}}{\partial t} + \tilde{q} \cdot \nabla \tilde{u} \right) + \frac{\partial \tilde{p}}{\partial x} - \nabla^2 \tilde{u}, \quad (23) \]
\[ \frac{\partial^2 v'}{\partial y^2} - \frac{\partial p'}{\partial y} = R \left( \frac{\partial \tilde{v}}{\partial t} + \tilde{q} \cdot \nabla \tilde{v} \right) + \frac{\partial \tilde{p}}{\partial y} - \nabla^2 \tilde{v}, \quad (24) \]
\[ \frac{\partial v'}{\partial y} = -\nabla \cdot \tilde{q}, \quad (25) \]
with boundary conditions
\[ \frac{\partial u'}{\partial y} = - (1 - \eta^2) \left( \frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x} \right) - 2\eta' \left( \frac{\partial \tilde{v}}{\partial y} - \frac{\partial \tilde{u}}{\partial x} \right) \quad \text{on } y = \eta, \quad (26) \]
\[ p' - 2 \frac{\partial v'}{\partial y} = - (1 + \eta^2) \tilde{p} + 2 \left[ \frac{\partial \tilde{v}}{\partial y} + \eta^2 \frac{\partial \tilde{u}}{\partial x} - \eta' \left( \frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x} \right) \right] \]
\[ - \frac{\eta''}{\sqrt{1 + \eta^2}} \quad \text{on } y = \eta, \quad (27) \]
\[ q' = 0 \quad \text{on } y = 0, \quad (28) \]
\[ \tilde{g} + g' = \tilde{v} + v' - (\tilde{u} + u') \frac{\partial \eta}{\partial x} \quad \text{on } y = \eta. \quad (29) \]

However, the unknown location of the free surface of the film is a major technical difficulty. One way to proceed is to scale the vertical coordinate, \( \zeta = y/\eta \), so that the free surface corresponds to \( \zeta = 1 \) precisely. Because \( \eta \) varies with \( x \) and \( t \), this scaling of \( y \) affects space-time derivatives and so plays havoc

\[ ^3 \text{Be careful not to confuse corrections, indicated by primes, and derivatives of } \eta \text{ with respect to } x, \text{ also indicated by primes.} \]
with details of the governing equations. Under the change of coordinates from 
\((x, y, t)\) to
\[
\xi = x, \quad \zeta = y \eta(x, t), \quad \tau = t,
\]
the chain rule shows that derivatives transform according to
\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \zeta \frac{\eta'}{\eta} \frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \zeta \frac{\dot{\eta}}{\eta} \frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial y} = \frac{1}{\eta} \frac{\partial}{\partial \zeta}.
\] (30)

Fortunately we may implement the analysis in computer algebra and thus relegate 
virtually all such details to the computer.

The only places where we need to explicitly consider these rules are in the 
terms in the equations for the corrections, \(u', v',\) and \(p'.\) However, these only 
involve \(y\) derivatives, see (23–27), which simply transform \(\frac{\partial}{\partial y} \rightarrow \frac{1}{\eta} \frac{\partial}{\partial \zeta}.\) Thus, we 
multiply the residuals on the right-hand sides of these equations by the appropriate 
power of \(\eta,\) as seen in lines 40, 47 and 53 of the following REDUCE program.

```reduce
% Construct slowly-varying centre manifold of thin film fluids.
% Allows for large changes in film thickness.
% on div; off allfac; factor d,h,re; % improves printing
% use stretched coordinates: ys=y/h(x,t), xs=x, ts=t
depend xs,x,y,t;
depend ys,x,y,t;
depend ts,x,y,t;
let { df(~a,x) => df(a,xs)-ys*h(1)/h(0)*df(a,ys)
  , df(~a,t) => df(a,ts)-ys*g/h(0)*df(a,ys)
  , df(~a,y) => df(a,ys)/h(0)
  , df(~a,x,2) => df(df(a,x),x) }
% solves -df(p,ys)=rhs s.t. sub(ys=1,p)=0
operator psolv; linear psolv;
let {psolv(ys^-~n,ys) => (1-ys^(n+1))/(n+1)
  ,psolv(ys,ys) => (1-ys^2)/2
  ,psolv(1,ys) => (1-ys) }
% solves df(u,ys,2)=rhs s.t. sub(ys=0,u)=0 & sub(ys=1,df(u,y))=0
operator usolv; linear usolv;
let {usolv(ys^-~n,ys) => (ys^(n+2)/(n+2)-ys)/(n+1)
  ,usolv(ys,ys) => (ys^3/3-ys)/2
  ,usolv(1,ys) => (ys^2/2-ys) }
% use operator h(m) to denote df(h,x,m)
operator h;
depend h,xs,ts;
let { df(h(~m),xs) => h(m+1)
  ,df(h(~m),xs,2) => h(m+2)
```

\[ \]
3: Thin film fluid dynamics

$$\text{df}(\overset{\sim}{m},\tau) \Rightarrow \text{df}(\xi,\eta,\zeta)$$

\%

\% Linear solution

\% Iteration. Use d to count the number of derivatives of x, and throw away this order or higher in \(d/dx\)

\%

\% Continuity & bed

\% Vertical momentum & normal stress

\% Horizontal momentum & bed & tangential stress

\%

1. Preliminaries

- The change of coordinate rules (30) are implemented in \(\ell 5–12\), where \(\xi, \eta, \zeta\), and \(\tau\). Then wherever in the algebraic details we need these rules they are automatically invoked by \textsc{reduce}. For example, in the evaluation of the residuals of the nonlinear equations and in boundary conditions within the iteration.

- \(\ell 13–17\) defines the operator \textsc{psolv} which is used to find the pressure correction through (24). For each term in \(\zeta^n\) in the right-hand side, it integrates with the correct integration constant to give a contribution to the pressure.
19

• $\ell18–22$ defines the operator $\text{usolv}$ which is used to find the horizontal velocity correction through $(23)$. For each term of the form $\zeta^n$, it solves an ODE with the appropriate bottom and linearised free-surface boundary condition.

• $\ell23–28$ For compactness of the output, it is convenient to represent the film thickness and its derivatives via an operator $h$, $\ell24$. $h$ is to depend upon $x$ and $t$, $\ell25$, and $h(m)$ denotes $\partial^m\eta/\partial x^m$, for example $h(0)$ and $h(1)$ denote $\eta$ and $\eta'$ respectively, and so $x$ derivatives operate according to the transformations on $\ell26–27$. Whereas $\ell28$ encodes the fact that

$$\frac{\partial}{\partial t} \left( \frac{\partial^m\eta}{\partial x^m} \right) = \frac{\partial^m}{\partial x^m} \left( \frac{\partial\eta}{\partial t} \right) = \frac{\partial^m\eta}{\partial x^m},$$

where the dependence of $\eta$ upon $x$ is recognised in the spatial derivatives of $g$.

Note that there is no need to define a specific operator to compute the correction to the vertical velocity, from $(25)$, because it is obtained simply by integrating the right-hand side as implemented by the standard $\text{int}$ operator in REDUCE.

2. The linear approximation is specified in $\ell31$.

3. Iteration. The iteration is carried out until the errors are $\mathcal{O}(\partial^7/\partial x^7)$, $\ell35$, whether in the form $\eta^{iv}$ or as $\eta''\eta'$ or as some other such nonlinear combination of derivatives of the film thickness $\eta$. This is achieved by carrying a dummy variable $d$ whose exponent counts the number of $x$ derivatives, and discarding any term of order 7 or more in $d$.

$\ell36$ establishes the expansion for $\eta''/\sqrt{1 + \eta'^2}$ in a series in small slope $\eta'$. Note the use of $d$ to count the total number of $x$ derivatives in each term.

One variation in the iteration is that the loop, $\ell37–55$, uses new information as it becomes available:

(a) first, the $v$ correction from continuity, $\ell38–40$;
(b) second, the pressure correction is found, $\ell40–47$, from the vertical momentum residual and the normal stress across the free-surface (the term in $v'$ on the left-hand side of $(24)$ is included in the right-hand side of $\ell43$ because we use the latest approximation to $v$);
(c) third, the $u$ correction is found, $\ell48–53$, from the horizontal momentum residual and the tangential stress across the free-surface;
(d) and lastly, the latest version of the model evolution $\frac{\partial\eta}{\partial t} = g$ from the kinematic free surface condition, $\ell54$. 
Upon executing this program, I find not only the evolution equation (14), but also the velocity and pressure fields. Specifically, to errors of fifth-order in $\partial/\partial x$,

$$
\begin{align*}
  u & \approx \left( \zeta - \frac{1}{2} \zeta^2 \right) \eta^2 \eta''', \\
  v & \approx -\frac{1}{2} \zeta^2 \eta^2 \eta' \eta'', \\
  p & \approx -\eta'' + \frac{3}{2} \eta^2 \eta'' - (1 + \zeta) \eta \eta' \eta''' - \left( \frac{1}{2} + \zeta - \frac{1}{2} \zeta^2 \right) \eta^2 \eta'''.
\end{align*}
$$

Expressions such as these inform us of the dominant details of the flow modelled by (14); higher-order terms were computed but I have not recorded them here.

3.4 Aside

Although this model describes the long-term dynamics of thin films, it is limited in its usefulness (even with gravitational effects included). For example, in the linearised problem (21) at finite wavenumber, the leading branch of the spectrum merges with the next branch, the gravest shear mode $u_1 = \sin \pi \zeta / 2$, at a wavenumber $k \approx 2$ to form a pair of oscillatory decaying modes. Such oscillations are the remnants, under the strong viscous dissipation in thin films, of the waves which surface tension can support. In many applications, such decaying waves seem important, for example see the review by Chang[6]. However, the model (14) cannot describe the necessary oscillations because it has only one component and is only first-order in time. This appears at first sight to be a strong limitation to the practical usefulness of centre manifold techniques in applications. But with some imagination we can modify the governing equations so that a centre manifold model is formed based on the two leading branches of the spectrum [24]. Such a model has much wider applicability because it is a much improved description at finite wavenumber and it resolves shorter transients in time.

4 Conclusion

In summary, the proposed algorithm for the computer algebra derivation of low-dimensional models of dynamical systems is relatively:

- simple to implement, because the computation of the residual is via a direct coding of the governing differential equations;
- reliable, because it relies upon the actual residual going to zero—any error is picked up by a lack of convergence;
flexible, because it removes the explicit tyranny of primitive approaches, such as the method of multiple scales, in forcing highly specific scalings upon the parameters and variables in the model—instead centre manifold theory assures the model is accurate to the order of accuracy of the residual.

Acknowledgements: I thank Valéry Roy for stimulating discussions during the preparation of this work.

References

[1] A. Arneodo, P.H. Coullet, and E.A. Spiegel. The dynamics of triple convection. *Geophys. Astro. Fluid Dyn.*, 31:1–48, 1985.

[2] V. Balakotaiah and H.C. Chang. Dispersion of chemical solutes in chromatographs and reactors. *Phil Trans R Soc Lond A*, 351:39–75, 1995.

[3] E. Boe and H.C. Chang. Dynamics of delayed systems under feedback control. *Chem. Eng. Sci.*, 44:1281–1294, 1989.

[4] J. Carr. Applications of centre manifold theory. *Appl Math Sci*, 35, 1981.

[5] J. Carr and R.G. Muncaster. The application of centre manifold theory to amplitude expansions. ii. infinite dimensional problems. *J. Diff. Eqns*, 50:280–288, 1983.

[6] H.C. Chang. Wave evolution on a falling film. *Annu. Rev. Fluid Mech.*, 26:103–136, 1994.

[7] Chiarella. *The Elements Of A Nonlinear Theory Of Economic Dynamics*, volume 343. Lect Notes In Eco And Math Systems, 1990.

[8] P.H. Coullet and E.A. Spiegel. Amplitude equations for systems with competing instabilities. *SIAM J. Appl. Math.*, 43:776–821, 1983.

[9] S.M. Cox and A.J. Roberts. Centre manifolds of forced dynamical systems. *J. Austral. Math. Soc. B*, 32:401–436, 1991.

[10] S.M. Cox and A.J. Roberts. Initial conditions for models of dynamical systems. *Physica D*, 85:126–141, 1995.

[11] Th. Gallay. A center-stable manifold theorem for differential equations in banach spaces. *Commun. Math. Phys*, 152:249–268, 1993.

[12] J.B. Grotberg. Pulmonary flow and transport phenomena. *Annu. Rev. Fluid Mech.*, 26:529–571, 1994.
§4: Conclusion

[13] M. Hărăguş. Model equations for water waves in the presence of surface tension. preprint, 1995.

[14] G. Iooss and M. Adelmeyer. Topics in Bifurcation Theory. World Sci, 1992.

[15] A. Jeffrey and T. Kawahara. Asymptotic methods in nonlinear wave theory. Applicable Mathematics Series. Pitman, 1982.

[16] Z. Mei and A.J. Roberts. Modelling turbulent flood waves. In preparation, 1995.

[17] G.N. Mercer and A.J. Roberts. A centre manifold description of contaminant dispersion in channels with varying flow properties. SIAM J. Appl. Math., 50:1547–1565, 1990.

[18] G.N. Mercer and A.J. Roberts. A complete model of shear dispersion in pipes. Jap. J. Indust. Appl. Math., 11:499–521, 1994.

[19] R.H. Rand and D. Armbruster. Perturbation methods, bifurcation theory and computer algebra, volume 65 of Applied Mathematical Sciences. Springer-Verlag, 1987.

[20] A.J. Roberts. The application of centre manifold theory to the evolution of systems which vary slowly in space. J. Austral. Math. Soc. B, 29:480–500, 1988.

[21] A.J. Roberts. Appropriate initial conditions for asymptotic descriptions of the long term evolution of dynamical systems. J. Austral. Math. Soc. B, 31:48–75, 1989.

[22] A.J. Roberts. Boundary conditions for approximate differential equations. J. Austral. Math. Soc. B, 34:54–80, 1992.

[23] A.J. Roberts. The invariant manifold of beam deformations, part 1:the simple circular rod. J. Elas., 30:1–54, 1993.

[24] A.J. Roberts. Low-dimensional models of thin film fluid dynamics. Phys. Letts. A, 212:63–72, 1996.

[25] K.J. Ruschak. Coating flows. Annu. Rev. Fluid Mech., 17:65–89, 1985.

[26] S.W. Shaw and C. Pierre. Normal modes for non-linear vibratory systems. J Sound Vibration, 164(1):85–124, 1993.

[27] E.O. Tuck and L.W. Schwartz. A numerical and asymptotic study of some third-order odes relevant to draining and coating flows. Siam Review, 32:453–469, 1990.