A new family of one dimensional martingale couplings

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August 7, 2018

Abstract

In this paper, we exhibit a new family of martingale couplings between two one-dimensional probability measures $\mu$ and $\nu$ in the convex order. This family is parametrised by two dimensional probability measures on the unit square with respective marginal densities proportional to the positive and negative parts of the difference between the quantile functions of $\mu$ and $\nu$. It contains the inverse transform martingale coupling which is explicit in terms of the associated cumulative distribution functions. The integral of $|x-y|$ with respect to each of these couplings is smaller than twice the $W_1$ distance between $\mu$ and $\nu$. When $\mu$ and $\nu$ are in the decreasing (resp. increasing) convex order, the construction is generalised to exhibit super (resp. sub) martingale couplings.

Introduction

Let $\mathcal{P}(\mathbb{R})$ denote the set of probability measures on $\mathbb{R}$, and for all $\rho \geq 1$, let $\mathcal{P}_\rho(\mathbb{R}) = \{\mu \in \mathcal{P}(\mathbb{R}) \mid \int_{\mathbb{R}} |x|^\rho \mu(dx) < +\infty\}$ be the set of probability measures on $\mathbb{R}$ with finite $\rho$-th moment. We say that two probability measures $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ are in the convex order, and denote $\mu \leq_{cx} \nu$, if $\int_{\mathbb{R}} \varphi(x) \mu(dx) \leq \int_{\mathbb{R}} \varphi(x) \nu(dx)$ for any convex function $\varphi : \mathbb{R} \to \mathbb{R}$. We denote $\mu <_{cx} \nu$ if $\mu \leq_{cx} \nu$ and $\mu \neq \nu$. For $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$, let $\Pi(\mu, \nu)$ be the set of couplings between $\mu$ and $\nu$, that is $\Pi(\mu, \nu) = \{P \in \mathcal{P}(\mathbb{R} \times \mathbb{R}) \mid \forall A \in \mathcal{B}(\mathbb{R}), P(A \times \mathbb{R}) = \mu(A) \text{ and } P(\mathbb{R} \times A) = \nu(A)\}$.

In all the paper, a capital letter $M$ which denotes a coupling between $\mu$ and $\nu$ is associated to its small letter $m$ which denotes the regular conditional probability distribution of $M$ with respect to $\mu$, that is the ($\mu$-a.e.) unique Markov kernel such that $M(dx, dy) = \mu(dx) m(x, dy)$. Let $\Pi_M(\mu, \nu)$ be the set of martingale couplings between $\mu$ and $\nu$, that is $\Pi_M(\mu, \nu) = \{M \in \Pi(\mu, \nu) \mid (\mu(dx)-a.e., \int_{\mathbb{R}} |y| m(x, dy) < +\infty \text{ and } \int_{\mathbb{R}} y m(x, dy) = x)\}$. For all $\mu \in \mathcal{P}(\mathbb{R})$, we denote by $F_\mu(x) = \mu((\infty, x])$ the cumulative distribution function of $\mu$ and $F_\mu^{-1}$ its left-continuous generalised inverse, that is $F_\mu^{-1} : u \in (0, 1) \mapsto \inf\{x \in \mathbb{R} \mid F_\mu(x) \geq u\}$. For all $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ and $\rho \geq 1$, we define the Wasserstein distance with index $\rho$ by $W_\rho^\mu(\mu, \nu) = (\inf_{P \in \Pi(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x-y|^\rho P(dx, dy))^{1/\rho}$, which is also equal to $(\int_0^1 (F_\mu^{-1}(u) - F_\nu^{-1}(u))^\rho du)^{1/\rho}$ (see for instance Remark 2.19 (ii) Chapter 2 [10]).

Our main result is the following stability inequality which shows that if $\mu$ and $\nu$ are in the convex order and close to each other, then there exists a martingale coupling which expresses this proximity:

$$\forall \mu, \nu \in \mathcal{P}_1(\mathbb{R}) \text{ such that } \mu \leq_{cx} \nu, \inf_{P \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x-y| P(dx, dy) \leq 2W_1(\mu, \nu). \tag{0.1}$$

We prove this inequality by exhibiting a new family of martingale couplings $P$ such that $\int_{\mathbb{R} \times \mathbb{R}} |x-y| P(dx, dy) \leq 2W_1(\mu, \nu)$. We will see that the constant 2 is sharp in (0.1). However, we will also see that (0.1) cannot be generalised with $|x-y|$ and $W_1(\mu, \nu)$ replaced by $|x-y|^\rho$ and $W_\rho^\mu(\mu, \nu)$ for $\rho > 1$. The case $\rho = 2$ is easy, since for all $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ and $P \in \Pi_M(\mu, \nu)$, $\int_{\mathbb{R} \times \mathbb{R}} |x-y|^2 P(dx, dy) = \int_{\mathbb{R}} (|x|^2 \nu(dx) - \int_{\mathbb{R}} |x|^2 \mu(dx))$, which is independent from $P$. For all $n \in \mathbb{N}^*$, let $\mu_n = \mathcal{N}_1(0, n^2)$. Then $\inf_{P \in \Pi_M(\mu_n, \mu_{n+1})} |x-y|^2 P(dx, dy) = 2n + 1 \xrightarrow{n \to +\infty} +\infty$, whereas $W_2^2(\mu_n, \mu_{n+1}) = (\int_0^1 (F_{\mu_n}^{-1}(u) - F_{\mu_{n+1}}^{-1}(u))^2 du)^{1/2} = 1$, which

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makes the equivalent of (0.1) impossible to hold. Extension to the case \( \rho > 2 \) is immediate with the same example thanks to Hölder’s inequality which provides \(( \inf_{P \in \Pi^4(\mu_0, \mu_n)} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^\rho P(dx, dy) )^{1/\rho} \geq (\inf_{P \in \Pi^4(\mu_0, \mu_n)} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^2 P(dx, dy) )^{1/2} = \sqrt{2n + 1} \), whereas \( W_p(\mu_n, \mu_n+1) = \mathbb{E}[|G|^\rho] \) where \( G \sim N_1(0, 1) \).

Let us construct such a coupling in dimension 1 in a simple case. We say that a centered probability measure \( \mu \in \mathcal{P}_1(\mathbb{R}) \) is symmetric if \((-x) \mu(dx) = \mu(dx)\). Let then \( \mu \) and \( \nu \) be centered and symmetric probability measures on \( \mathbb{R} \) such that \( F_{\mu}^{-1}(u) \geq F_{\nu}^{-1}(u) \) for all \( u \in (0, 1/2) \) and \( F_{\mu}^{-1}(u) \leq F_{\nu}^{-1}(u) \) for all \( u \in (1/2, 1) \). Let \( U \) be a random variable uniformly distributed on \([0, 1]\). According to the inverse transform sampling, the probability distributions of \( F_{\mu}^{-1}(U) \) and \( F_{\nu}^{-1}(U) \) are respectively \( \mu \) and \( \nu \). Let \( Z \) be the random variable defined by

\[
Z = F_{\nu}^{-1}(U) \mathbb{I}_{\{F_{\nu}^{-1}(U) \neq 0, V \leq \frac{F_{\mu}^{-1}(U)+F_{\nu}^{-1}(U)}{2F_{\nu}^{-1}(U)}\}} - F_{\nu}^{-1}(U) \mathbb{I}_{\{F_{\nu}^{-1}(U) \neq 0, V > \frac{F_{\mu}^{-1}(U)+F_{\nu}^{-1}(U)}{2F_{\nu}^{-1}(U)}\}},
\]

where \( V \) is a random variable uniformly distributed on \([0, 1]\) independent from \( U \). Note that when \( F_{\nu}^{-1}(U) = 0 \), then \( Z = 0 \) and \( F_{\mu}^{-1}(U) = 0 \). We easily check that \( Z \) is distributed according to \( \nu \). Indeed, using that \((F_{\mu}^{-1}(U), F_{\nu}^{-1}(U))\) and \((-F_{\mu}^{-1}(U), -F_{\nu}^{-1}(U))\) are identically distributed (see Lemma 4.3 in Section 4), for all measurable and bounded function \( f : \mathbb{R} \to \mathbb{R} \), we have

\[
\mathbb{E}[f(Z)] = \mathbb{E}[f(F_{\nu}^{-1}(U)) \mathbb{I}_{\{F_{\nu}^{-1}(U) \neq 0, V \leq \frac{F_{\mu}^{-1}(U)+F_{\nu}^{-1}(U)}{2F_{\nu}^{-1}(U)}\}}] + \mathbb{E}[f(F_{\nu}^{-1}(U)) \mathbb{I}_{\{F_{\nu}^{-1}(U) \neq 0, V > \frac{F_{\mu}^{-1}(U)+F_{\nu}^{-1}(U)}{2F_{\nu}^{-1}(U)}\}}] + f(0) \mathbb{P}(F_{\nu}^{-1}(U) = 0) = \mathbb{E}[f(F_{\nu}^{-1}(U))].
\]

Moreover, we have \( F_{\nu}^{-1}(u) \leq F_{\mu}^{-1}(u) \leq 0 \) for all \( u \in (0, 1/2) \) and \( 0 \leq F_{\mu}^{-1}(u) \leq F_{\nu}^{-1}(u) \) for all \( u \in (1/2, 1) \), so that \( \frac{F_{\mu}^{-1}(u)+F_{\nu}^{-1}(u)}{2F_{\nu}^{-1}(u)} \in [0, 1] \) for all \( u \in (0, 1) \) such that \( F_{\nu}^{-1}(u) \neq 0 \). In addition to that, we have

\[
\frac{F_{\mu}^{-1}(u)+F_{\nu}^{-1}(u)}{2F_{\nu}^{-1}(u)} - F_{\nu}^{-1}(u)(1 - \frac{F_{\mu}^{-1}(u)+F_{\nu}^{-1}(u)}{2F_{\nu}^{-1}(u)}) = F_{\mu}^{-1}(u) \quad \text{for all } u \in (0, 1) \text{ such that } F_{\nu}^{-1}(u) \neq 0,
\]

and hence

\[
\mathbb{E}[Z|U] = F_{\mu}^{-1}(U) \mathbb{I}_{\{F_{\nu}^{-1}(U) \neq 0\}} = F_{\mu}^{-1}(U) \quad \text{since } F_{\nu}^{-1}(U) = 0 \implies F_{\mu}^{-1}(U) = 0, \text{ so } \mathbb{E}[Z|F_{\mu}^{-1}(U)] = F_{\mu}^{-1}(U).
\]

Therefore, the law of \((F_{\mu}^{-1}(U), Z)\) is an explicit martingale coupling between \( \mu \) and \( \nu \).

Furthermore, \( \mathbb{E}[|Z - F_{\nu}^{-1}(U)|] \leq \mathbb{E}[|Z - F_{\mu}^{-1}(U)|] + \mathbb{E}[|F_{\mu}^{-1}(U) - F_{\nu}^{-1}(U)|] \). Denoting \( \text{sg}(x) = 1_{\{x > 0\}} - 1_{\{x < 0\}} \) for \( x \in \mathbb{R} \), and remarking that \( \mathbb{P}\left(|Z - F_{\mu}^{-1}(U)| = (Z - F_{\nu}^{-1}(U))\text{sg}(F_{\nu}^{-1}(U) - F_{\mu}^{-1}(U))\right) = 1 \), we obtain that

\[
\mathbb{E}[|Z - F_{\nu}^{-1}(U)|] = \mathbb{E}[\text{sg}(F_{\nu}^{-1}(U) - F_{\mu}^{-1}(U))\text{sg}(Z - F_{\nu}^{-1}(U)|U)] = \mathbb{E}[|F_{\mu}^{-1}(U) - F_{\nu}^{-1}(U)|].
\]

Therefore, \( \mathbb{E}[|Z - F_{\nu}^{-1}(U)|] \leq 2\mathbb{E}[|F_{\nu}^{-1}(U) - F_{\mu}^{-1}(U)|] = 2W_1(\mu, \nu) \), so that (0.1) holds.

Let \( X \) and \( Y \) be two random variables with respective distributions given by \( \mathbb{P}(X = a) = \mathbb{P}(Y = b) = \mathbb{P}(X = -a) = \mathbb{P}(Y = -b) = 1/2 \) with 0 < a < b. In that case, there is a unique martingale coupling between \( \mu \) and \( \nu \). So we can compute \( \inf_{P \in \Pi^4(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^2 P(dx, dy) = \mathbb{E}[|Z - F_{\mu}^{-1}(U)|] = (b^2 - a^2)/b \) whereas \( W_1(\mu, \nu) = \mathbb{E}[|F_{\mu}^{-1}(1) - F_{\nu}^{-1}(1)|] = b - a \). The ratio between those two numbers is then \( (b + a)/b \), which tends to 2 when \( a \) increases towards \( b \), which shows that the constant 2 in (0.1) is sharp.

We develop in Section 1 an abstract construction of a new family of martingale couplings between two probability measures \( \mu \) and \( \nu \) on the real line with finite first moments and comparable in the convex order. This family is parametrized by two dimensional probability measures on the unit square with respective marginal densities proportional to the positive and the negative parts of the difference \( F_{\mu}^{-1} - F_{\nu}^{-1} \) between the quantile functions of \( \mu \) and \( \nu \). It contains the law of the above coupling \((F_{\mu}^{-1}(U), Z)\) when \( \mu \) and \( \nu \) are symmetric. Moreover, each coupling in the family is obtained as the image by \((u, y) \mapsto (F_{\mu}^{-1}(u), y)\) of \( 1_{(0,1)}(u)du \tilde{m}(u, dy) \) where \( \tilde{m} \) is a Markov kernel on \((0, 1) \times \mathbb{R} \) such that \( \int_{(0,1)} \tilde{m}(u, \{y \in \mathbb{R} \ | \ |y - F_{\mu}^{-1}(u)| = (y - F_{\mu}^{-1}(u))\text{sg}(F_{\nu}^{-1}(u) - F_{\mu}^{-1}(u))\})du = 1 \). Therefore, for \((U, Y)\) distributed according to \( 1_{(0,1)}(u)du \tilde{m}(u, dy), \mathbb{E}[|Y - F_{\mu}^{-1}(U)|] = \mathbb{E}[|F_{\mu}^{-1}(U) - F_{\nu}^{-1}(U)|] = W_1(\mu, \nu) \) so that \( \mathbb{E}[|Y - F_{\nu}^{-1}(U)|] \leq \mathbb{E}[|Y - F_{\nu}^{-1}(U)|] + \mathbb{E}[|F_{\nu}^{-1}(U) - F_{\mu}^{-1}(U)|] \).
\( F^{-1}_\mu(U) = 2W_1(\mu, \nu) \) which implies (0.1) as soon as the set of probability measures on the unit square in non empty.

In Section 2, we give an explicit example of such a probability measure on the unit square, which yields the inverse transform martingale coupling. This coupling is explicit in terms of the cumulative distribution functions of the above-mentioned densities and their left-continuous generalised inverses. It is therefore more explicit than the left-curtain and right-curtain couplings introduced by Beiglbock and Juillet. According to Henry-Labordère and Touzi [4], under the condition that \( \nu \) has no atoms and the set of local maximal values of \( F_\nu - F_\mu \) is finite, the left-curtain coupling can be explicit by solving two coupled ordinary differential equations starting from each right-most local maximiser. We also check that the inverse transform martingale coupling is stable with respect to its marginals \( \mu \) and \( \nu \) for the weak convergence topology. The building brick of the inverse transform martingale coupling is a martingale coupling between \( \rho \delta_{F^{-1}_\mu(u)} + (1 - \rho) \delta_{F^{-1}_\nu(v)} \) and \( p \delta_{F^{-1}_\mu(u)} + (1 - p) \delta_{F^{-1}_\nu(v)} \) with \( 0 < u < v < 1 \) such that \( F^{-1}_\mu(u) < F^{-1}_\mu(u) < F^{-1}_\nu(v) < F^{-1}_\nu(v) \) where we choose a common weight \( p \) (resp. \( 1 - p \)) for \( F^{-1}_\mu(u) \) and \( F^{-1}_\nu(v) \) (resp. \( F^{-1}_\nu(v) \) and \( F^{-1}_\nu(v) \)) to help ensuring that the second marginal is equal to \( \nu \) when the first is equal to \( \mu \). Then \( p \) is given by the equality of the means which is equivalent to the convex order: 

\[
\frac{1 - p}{p} = \frac{F^{-1}_\mu(u) - F^{-1}_\mu(u)}{F^{-1}_\nu(v) - F^{-1}_\nu(v)}
\]

The above symmetric coupling corresponds to the choice \( v = 1 - u \) for \( 0 < u < 1/2 \), which, by symmetry, leads to \( p = 1/2 \). Of course, in general, this choice is no longer possible. We instead rely on the necessary condition of Theorem 3.4.5 Chapter 3 [9]: \( \mu, \nu \in P_1(\mathbb{R}) \) are such that \( \mu \leq_{cx} \nu \) iff for all \( u \in [0, 1] \), \( \int_0^u F^{-1}_\mu(v) dv \geq \int_0^u F^{-1}_\nu(v) dv \) with equality for \( u = 1 \). This implies that for all \( u \in [0, 1] \), \( \Psi_+(u) := \int_0^u (F^{-1}_\mu - F^{-1}_\nu)^+(v) dv \geq \int_0^u (F^{-1}_\nu - F^{-1}_\mu)^-(v) dv := \Psi_-(v) \) where \( x^+ := \max(x, 0) \) and \( x^- := \max(-x, 0) \) respectively denote the positive and negative parts of a real number \( x \). We now choose \( v = \Psi^{-1}_-(\Psi_+(u)) \) where \( \Psi^{-1}_+ \) is the left-continuous generalised inverse of \( \Psi_+ \). Then \( d\Psi_+(u) \) a.e. \( u < v \) (consequence of \( \Psi_+ \leq \Psi_+^{-1} \) and \( F^{-1}_\mu(u) < F^{-1}_\mu(v) < F^{-1}_\nu(v) < F^{-1}_\nu(v) \) (consequence of the definitions of \( \Psi_+ \) and \( \Psi_- \)). Moreover the key equality \( \frac{d\Psi}{du} = \frac{p}{(F^{-1}_\nu(v) - F^{-1}_\nu(v))^+} = \frac{1 - p}{p} \) explains why the construction succeeds. More details are given in Section 2.

The construction is generalised in Section 3 to exhibit super (resp. sub) martingale couplings as long as \( \mu \) is smaller than \( \nu \) in the decreasing (resp. increasing) convex order. We recall that two probability measures \( \mu, \nu \in P_1(\mathbb{R}) \) are in the decreasing (resp. increasing) convex order and denote \( \mu \leq_{cx} \nu \) (resp. \( \mu \leq_{icx} \nu \)) if \( \int_\mathbb{R} \varphi(x) \mu(dx) \leq \int_\mathbb{R} \varphi(x) \nu(dx) \) for any decreasing (resp. increasing) convex function \( \varphi : \mathbb{R} \to \mathbb{R} \).

Acknowledgements : We thank Jean-François Delmas (CERMICS) for numerous and fruitful discussions.

1 A new family of martingale couplings

1.1 Definition

Let \( \mu \) and \( \nu \) be two probability measures on \( \mathbb{R} \) with finite first moment and such that \( \mu \leq_{cx} \nu \). Let \( U_- \) and \( U_+ \) be defined by

\[
U_- = \{ u \in (0, 1) \mid F^{-1}_\mu(u) > F^{-1}_\nu(u) \} \quad \text{and} \quad U_+ = \{ u \in (0, 1) \mid F^{-1}_\mu(u) < F^{-1}_\nu(u) \}.
\]

The probability measures \( \mu \) and \( \nu \) have finite first moments, so their mean are well defined and equal since they are in the convex order. So \( \gamma = \int_0^1 (F^{-1}_\mu(u) - F^{-1}_\nu(u))^+ du = \int_0^1 (F^{-1}_\nu(v) - F^{-1}_\nu(v))^+ du \in (0, +\infty) \).

We note \( Q \) the set of probability measures \( Q \) on \( (0, 1)^2 \) such that

(i) \( Q \) has first marginal \( \frac{1}{\gamma} (F^{-1}_\mu(u) - F^{-1}_\nu(u))^+ du \);

(ii) \( Q \) has second marginal \( \frac{1}{\gamma} (F^{-1}_\nu(v) - F^{-1}_\nu(v))^+ dv \);

(iii) \( Q \left\{ (u, v) \in (0, 1)^2 \mid u < v \right\} \) = 1.

Proposition 1.1. Let \( \mu, \nu \in P_1(\mathbb{R}) \) such that \( \mu \leq_{cx} \nu \). The set \( Q \) is non-empty.
This is direct consequence of Proposition 2.1 below. Let $Q$ be an element of $\mathcal{Q}$. Let $\pi_-$ and $\pi_+$ be two Markov kernels on $(0,1)$ such that

$$Q(du, dv) = \frac{1}{\gamma} (F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ du \pi_-(u, dv);$$

$$Q(du, dv) = \frac{1}{\gamma} (F_\mu^{-1}(v) - F_\nu^{-1}(v))^- dv \pi_+(v, du).$$

Let $(\tilde{m}(u, dy))_{u \in (0,1)}$ be the Markov kernel defined by

$$\begin{align*}
\left\{ \begin{array}{ll}
\int_{0 \in (0,1)} \frac{F_\mu^{-1}(u) - F_\nu^{-1}(u)}{F_\nu^{-1}(v) - F_\nu^{-1}(u)} \delta_{F_\nu^{-1}(v)}(dy) \pi_-(u, dv) + \\
\int_{0 \in (0,1)} \frac{F_\mu^{-1}(v) - F_\nu^{-1}(v)}{F_\nu^{-1}(v) - F_\nu^{-1}(u)} \pi_-(u, dv) \delta_{F_\nu^{-1}(u)}(dy) \\
\end{array} \right. \\
\text{if } u \in \mathcal{U}_- \text{ and } F_\nu^{-1}(v) > F_\mu^{-1}(u), \pi_-(u, dv)-a.e.;
\end{align*}$$

$$\begin{align*}
\left\{ \begin{array}{ll}
\int_{0 \in (0,1)} \frac{F_\mu^{-1}(u) - F_\nu^{-1}(u)}{F_\nu^{-1}(v) - F_\nu^{-1}(u)} \delta_{F_\nu^{-1}(v)}(dy) \pi_+(u, dv) + \\
\int_{0 \in (0,1)} \frac{F_\mu^{-1}(v) - F_\nu^{-1}(v)}{F_\nu^{-1}(v) - F_\nu^{-1}(u)} \pi_+(u, dv) \delta_{F_\nu^{-1}(u)}(dy) \\
\end{array} \right. \\
\text{if } u \in \mathcal{U}_+ \text{ and } F_\nu^{-1}(v) < F_\mu^{-1}(u), \pi_+(u, dv)-a.e.;
\end{align*}$$

$$\delta_{F_\nu^{-1}(u)}(dy) \quad \text{otherwise.}$$

Let $(m(x, dy))_{x \in \mathbb{R}}$ be the Markov kernel defined by

$$\begin{align*}
\left\{ \begin{array}{ll}
\delta_x(dy) & \text{if } F_\mu(x) = 0 \text{ or } F_\mu(x_-) = 1; \\
\frac{1}{\mu(\{x\})} \int_{F_\mu(x_-)} F_\mu(x) \tilde{m}(u, dy) du & \text{if } \mu(\{x\}) > 0; \\
\tilde{m}(F_\mu(x), dy) & \text{otherwise.}
\end{array} \right.
\end{align*}$$

(1.1)

For all $x \in \mathbb{R}$ such that $F_\mu(x) > 0$ and $F_\mu(x_-) < 1$, $m(x, dy)$ can be rewritten as

$$m(x, dy) = \int_0^1 \tilde{m}(F_\mu(x_-) + v(F_\mu(x) - F_\mu(x_-)), dy) dv.$$ 

(1.3)

**Proposition 1.2.** Let $Q \in \mathcal{Q}$ and $m$ given by (1.3). The probability measure $M(dx, dy) = \mu(dx) m(x, dy)$ is a martingale coupling between $\mu$ and $\nu$.

The proof of Proposition 1.2 relies on the two following lemmas.

**Lemma 1.3.** For $du$-almost all $u \in (0,1)$,

$$\begin{align*}
\left\{ \begin{array}{ll}
u \in \mathcal{U}_- & \Rightarrow \quad F_\nu^{-1}(v) > F_\mu^{-1}(u), \pi_-(u, dv)-a.e; \\
u \in \mathcal{U}_+ & \Rightarrow \quad F_\nu^{-1}(v) < F_\mu^{-1}(u), \pi_+(u, dv)-a.e.
\end{array} \right.
\end{align*}$$

**Proof.** We have

$$\begin{align*}
\int_{(0,1)} \left( \int_{(0,1)} 1\{F_\nu^{-1}(v) \leq F_\mu^{-1}(u)\} \pi_-(u, dv) \right) (F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ (u) du \\
= \gamma \int_{(0,1)^2} 1\{F_\nu^{-1}(v) \leq F_\mu^{-1}(u)\} Q(du, dv)
\end{align*}$$

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\[
\leq \gamma \int_{(0,1)^2} 1\{F^{-1}_\nu(v) \leq F^{-1}_\mu(v)\} Q(du, dv)
\]
\[
= \int_{(0,1)^2} 1\{F^{-1}_\mu(u) - F^{-1}_\nu(v) \geq 0\}(F^{-1}_\mu(v) - F^{-1}_\nu(v))^- dv \pi_+(v, du)
\]
\[
= 0,
\]
where we used for the inequality that \(u \leq v\), \(Q(du, dv)\)-almost everywhere and that \(F^{-1}_\nu\) is nondecreasing. So for \(du\)-almost all \(u \in \mathcal{U}_-, F^{-1}_\nu(v) > F^{-1}_\mu(u)\), \(\pi_-(u, dv)\)-almost everywhere. With a symmetric reasoning, we obtain that for \(du\)-almost all \(u \in \mathcal{U}_+, F^{-1}_\nu(v) < F^{-1}_\mu(u)\), \(\pi_+(u, dv)\)-almost everywhere. \(\square\)

Lemma 1.4. Let \(g : (0, 1) \times \mathbb{R} \ni (u, y) \mapsto (F^{-1}_\mu(u), y) \in \mathbb{R}^2\). Then
\[
\mu(dx) m(x, dy) = g du \bar{m}(u, dy).
\]

Proof. Let \(h : \mathbb{R}^2 \to \mathbb{R}\) be a measurable and nonnegative function. By Lemma 4.2, \(F_\mu(x) > 0\) and \(F_\mu(x-) < 1\), \(\mu(dx)\)-almost everywhere, so using (1.3), we have
\[
\int_{\mathbb{R} \times \mathbb{R}} h(x, y) \mu(dx) m(x, dy) = \int_{\mathbb{R} \times \mathbb{R}} h(x, y) \mathbf{1}_{\{0 < F_\mu(x), F_\mu(x-) < 1\}} \mu(dx) \int_0^1 \bar{m}(F_\mu(x-) + v(F_\mu(x) - F_\mu(x-)), dy) dv.
\]
Let \(\theta : (x, v) \mapsto F_\mu(x-) + v(F_\mu(x) - F_\mu(x-))\) so that \(\theta(x, v)(\mu(dx) \otimes dv) = \mathbf{1}_{(0,1)}(u) du\) by Lemma 4.4. By Lemma 4.4 again, \(x = F^{-1}_\mu(\theta(x, v))\), \(\mu(dx) \otimes dv\)-almost everywhere on \(\mathbb{R} \times (0, 1)\), so
\[
\int_{\mathbb{R} \times \mathbb{R}} h(x, y) \mu(dx) m(x, dy) = \int_{\mathbb{R} \times (0,1)} h(F^{-1}_\mu(\theta(x, v)), y) \mathbf{1}_{\{0 < F_\mu(F^{-1}_\mu(\theta(x, v))), F_\mu(F^{-1}_\mu(\theta(x, v))) < 1\}} \mu(dx) \bar{m}(\theta(x, v), dy) dv
\]
\[
= \int_{\mathbb{R}} \int_{(0,1)} h(F^{-1}_\mu(u), y) \mathbf{1}_{\{0 < F_\mu(F^{-1}_\mu(u)), F_\mu(F^{-1}_\mu(u)) < 1\}} \bar{m}(u, dy) du.
\]
By Lemma 4.2 and the inverse transform sampling, \(F_\mu(F^{-1}_\mu(u)) > 0\) and \(F_\mu(F^{-1}_\mu(u-)) < 1\) \(du\)-almost everywhere on \((0, 1)\), hence
\[
\int_{\mathbb{R} \times \mathbb{R}} h(x, y) \mu(dx) m(x, dy) = \int_{\mathbb{R}} \int_{(0,1)} h(F^{-1}_\mu(u), y) \bar{m}(u, dy) du.
\]
\(\square\)

Proof of Proposition 1.2. Let us show that \(M\) defines a coupling between \(\mu\) and \(\nu\). Let \(h : \mathbb{R} \to \mathbb{R}\) be a measurable and nonnegative function. We want to show that
\[
\int_{\mathbb{R} \times \mathbb{R}} h(y) \mu(dx) m(x, dy) = \int_{\mathbb{R}} h(y) \nu(dy),
\]
which by Lemma 1.4 and the inverse transform sampling is equivalent to
\[
\int_0^1 \int_{\mathbb{R}} h(y) \bar{m}(u, dy) du = \int_0^1 h(F^{-1}_\nu(u)) du.
\]
Thanks to Lemma 1.3, we get for \(du\)-almost all \(u \in (0, 1)\),
\[
\int_{\mathbb{R}} h(y) \bar{m}(u, dy)
\]
\[
= \int_{(0,1)} \left(1 - \frac{F^{-1}_\mu(u) - F^{-1}_\nu(u)}{F^{-1}_\nu(v) - F^{-1}_\mu(u)}\right) h(F^{-1}_\nu(u))(\pi_-(u, dv)\mathbf{1}_{\{F^{-1}_\nu(u) > F^{-1}_\mu(u)\}} + \pi_+(u, dv)\mathbf{1}_{\{F^{-1}_\nu(u) < F^{-1}_\mu(u)\}}
\]
+ \int_{(0,1)} \left( \frac{F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)}{F_{\nu}^{-1}(v) - F_{\mu}^{-1}(u)} \right) h(F_{\nu}^{-1}(v))(\pi_-(u, dv)\mathbb{1}_{\{F_{\nu}^{-1}(u) > F_{\mu}^{-1}(u)\}} + \pi_+(u, dv)\mathbb{1}_{\{F_{\nu}^{-1}(u) < F_{\mu}^{-1}(u)\}}) \\
+ h(F_{\mu}^{-1}(u))\mathbb{1}_{\{F_{\nu}^{-1}(u) = F_{\mu}^{-1}(u)\}) \\
= h(F_{\nu}^{-1}(u)) + \int_{(0,1)} \frac{(F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u))^+}{F_{\nu}^{-1}(v) - F_{\nu}^{-1}(u)}(h(F_{\nu}^{-1}(v)) - h(F_{\nu}^{-1}(u))) \pi_-(u, dv) \\
+ \int_{(0,1)} \frac{(F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u))^\text{-}}{F_{\nu}^{-1}(v) - F_{\nu}^{-1}(u)}(h(F_{\nu}^{-1}(v)) - h(F_{\nu}^{-1}(u))) \pi_+(u, dv). \tag{1.4} \label{eq:1.4}

Since

\begin{align*}
\int_{(0,1)} \int_{(0,1)} \frac{(F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u))^+}{F_{\nu}^{-1}(v) - F_{\nu}^{-1}(u)}(h(F_{\nu}^{-1}(v)) - h(F_{\nu}^{-1}(u))) \pi_-(u, dv) du \\
= \gamma \int_{(0,1)^2} \frac{h(F_{\nu}^{-1}(v)) - h(F_{\nu}^{-1}(u))}{F_{\nu}^{-1}(v) - F_{\nu}^{-1}(u)}Q(du, dv) \\
= \int_{(0,1)} \int_{(0,1)} \frac{h(F_{\nu}^{-1}(v)) - h(F_{\nu}^{-1}(u))}{F_{\nu}^{-1}(v) - F_{\nu}^{-1}(u)}(F_{\nu}^{-1}(v) - F_{\nu}^{-1}(u))^\text{-}(v)dv \pi_+(v, du) \\
= - \int_{(0,1)} \int_{(0,1)} \frac{(F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u))^\text{-}}{F_{\nu}^{-1}(v) - F_{\nu}^{-1}(u)}(h(F_{\nu}^{-1}(u)) - h(F_{\nu}^{-1}(v))) \pi_+(u, dv),
\end{align*}

we deduce that \( \int_{0}^{1} \int_{\mathbb{R}} h(y) \tilde{m}(u, dy) du = \int_{0}^{1} h(F_{\nu}^{-1}(u)) \) du, hence \( M \) is a martingale coupling between \( \mu \) and \( \nu \). In particular with \( h: y \mapsto |y| \), using the inverse transform sampling, we have

\[ \int_{0}^{1} \int_{\mathbb{R}} |y| \tilde{m}(u, dy) du = \int_{0}^{1} |F_{\nu}^{-1}(u)| du = \int_{\mathbb{R}} |y| \nu(\text{dy}) < +\infty, \]

so \( \int_{\mathbb{R}} y \tilde{m}(u, dy) \) is well defined \( du \)-almost everywhere on \((0, 1)\). Let us show now that \( M \) defines a martingale coupling between \( \mu \) and \( \nu \). By Lemma 1.3, for \( du \)-almost all \( u \in U_- \),

\[ \int_{\mathbb{R}} y \tilde{m}(u, dy) = \int_{(0,1)} \left( 1 - \frac{F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)}{F_{\nu}^{-1}(v) - F_{\mu}^{-1}(u)} \right) F_{\nu}^{-1}(u) \pi_-(u, dv) \\
+ \int_{(0,1)} \frac{(F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u))}{F_{\nu}^{-1}(v) - F_{\nu}^{-1}(u)} F_{\nu}^{-1}(v) \pi_-(u, dv) \\
= \int_{(0,1)} (F_{\nu}^{-1}(u) + F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u)) \pi_-(u, dv) \\
= F_{\mu}^{-1}(u). \]

In the same way, for \( du \)-almost all \( u \in U_+ \),

\[ \int_{\mathbb{R}} y \tilde{m}(u, dy) = F_{\mu}^{-1}(u). \]

Else if \( u \in (0, 1) \setminus (U_- \cup U_+) \), then by definition of \( \tilde{m}(u, dy) \),

\[ \int_{\mathbb{R}} y \tilde{m}(u, dy) = F_{\mu}^{-1}(u), \]

so for \( du \)-almost all \( u \in [0, 1] \), \( \int_{\mathbb{R}} y \tilde{m}(u, dy) = F_{\mu}^{-1}(u) \).
Let \( x \in \mathbb{R} \). If \( \mu(\{x\}) > 0 \), then
\[
\int y \, m(x, dy) = \frac{1}{\mu(\{x\})} \int y \int_{F_\mu(x)}^{F_\nu(x)} \tilde{m}(u, dy) \, du
\]
\[
= \frac{1}{\mu(\{x\})} \int_{F_\mu(x)}^{F_\nu(x)} F_\mu^{-1}(u) \, du
\]
\[
= \frac{1}{\mu(\{x\})} \int_{F_\mu(x)}^{F_\nu(x)} x \, du = x. \]

By Lemma 4.2, for \( \mu(dx) \)-almost all \( x \in \mathbb{R} \), \( 0 < F_\mu(x), F_\mu(x^-) > 1 \), and if \( \mu(\{x\}) = 0 \), then
\[
\int y \, m(x, dy) = \int y \tilde{m}(F_\mu(x), dy) = F_\mu^{-1}(F_\mu(x)).
\]

Since by Lemma 4.1, \( F_\mu^{-1}(F_\mu(x)) = x \), \( \mu(dx) \)-almost everywhere on \( \mathbb{R} \), we conclude that \( \int y \, m(x, dy) = x \) for \( \mu(dx) \)-almost all \( x \in \mathbb{R} \).

So \( \mu(dx) m(x, dy) \) is a martingale coupling between \( \mu \) and \( \nu \).

Let us complete this section with simple examples. Let \( \mu, \nu \in \mathcal{P}_1(\mathbb{R}) \) be such that \( \mu <_{cx} \nu \). Suppose that the difference of the quantile functions change sign only once, that is there exists \( p \in (0,1) \) such that \( u \mapsto \int_0^u (F_\mu^{-1}(v) - F_\nu^{-1}(v)) \, dv \) is nondecreasing on \([0,p]\) and nonincreasing on \([p,1]\). Then one can easily see that any probability measure \( Q \) defined on \((0,1)\) satisfying properties (i) and (ii) of the definition of \( Q \) is concentrated on \((0,p) \times (p,1)\) and therefore satisfies (iii). In particular, the probability measure \( Q_1 \) defined on \((0,1)^2\) by
\[
Q_1(du, dv) = \frac{1}{\gamma^2}(F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \, du \, (F_\mu^{-1}(v) - F_\nu^{-1}(v))^- \, dv
\] (1.5)
is an element of \( Q \).

Suppose now that \( \mu \) and \( \nu \) are symmetric with common mean \( \alpha \in \mathbb{R} \) and that their respective quantile functions satisfy \( F_\mu^{-1} \geq F_\nu^{-1} \) on \((0,1/2)\] and \( F_\mu^{-1} \leq F_\nu^{-1} \) on \([1/2,1)\). We introduce in the introduction an explicit coupling between \( \mu \) and \( \nu \) in the case \( \alpha = 0 \). Let us show here that this coupling is in fact associated to a particular element of \( Q \). According to Lemma 4.3, \( F_\mu^{-1}(u) - \alpha = \alpha - F_\mu^{-1}(1-u) \) and \( F_\nu^{-1}(u) = 2\alpha - F_\nu^{-1}(1-u) \) for \( du \)-almost all \( u \in (0,1) \), which is helpful in order to see that the probability measure \( Q_2 \) defined on \((0,1)^2\) by
\[
Q_2(du, dv) = \frac{1}{\gamma^2}(F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \, du \delta_{1-u}(dv). \] (1.6)
is an element of \( Q \). For that element \( Q_2 \), using Lemma 1.3 and Lemma 4.3, we have for \( du \)-almost all \( u \in (0,1) \),
\[
\tilde{m}(u, dy) = \frac{F_\nu^{-1}(u) - F_\mu^{-1}(u)}{2(F_\nu^{-1}(u) - \alpha)} \delta_{2\alpha - F_\nu^{-1}(u)}(dy) + \frac{F_\nu^{-1}(u) + F_\mu^{-1}(u) - 2\alpha}{2(F_\nu^{-1}(u) - \alpha)} \delta_{F_\nu^{-1}(u)}(dy)
\] (1.7)
if \( F_\mu^{-1}(u) \neq F_\nu^{-1}(u) \), and \( \tilde{m}(u, dy) = \delta_{F_\nu^{-1}(u)}(dy) \) otherwise. Let \( u \in (0,1) \). If \( F_\mu^{-1}(u) = F_\nu^{-1}(u) \neq \alpha \), then \( \delta_{F_\nu^{-1}(u)}(dy) \) coincides with the right-hand side of (1.7). Furthermore if \( F_\nu^{-1}(u) = \alpha \), since \( \alpha \leq F_\mu^{-1}(u) \geq F_\nu^{-1}(u) \) or \( \alpha \leq F_\mu^{-1}(u) \leq F_\nu^{-1}(u) \), then \( F_\mu^{-1}(u) = \alpha \). Therefore (1.7) holds for \( du \)-almost all \( u \in (0,1) \) such that \( F_\nu^{-1}(u) \neq \alpha \) and \( \tilde{m}(u, dy) = \delta_{F_\nu^{-1}(u)}(dy) \) for \( du \)-almost all \( u \in (0,1) \) such that \( F_\nu^{-1}(u) = \alpha \).

Let \( U \) and \( V \) be two independent random variables uniformly distributed on \([0,1]\) and let \( Z \) the random variable defined as in the introduction with the mean \( \alpha \) taken into account, that is
\[
Z = F_\nu^{-1}(U)1_{\{F_\nu^{-1}(U) \neq \alpha \}}, V \leq \frac{F_\nu^{-1}(U) + F_\nu^{-1}(U) - 2\alpha}{2(F_\nu^{-1}(U) - \alpha)}, + (2\alpha - F_\nu^{-1}(U))1_{\{F_\nu^{-1}(U) \neq \alpha \}}, V > \frac{F_\nu^{-1}(U) + F_\nu^{-1}(U) - 2\alpha}{2(F_\nu^{-1}(U) - \alpha)}, + \alpha1_{\{F_\nu^{-1}(U) = \alpha \}}.
\]
Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and measurable function. Then

$$
\mathbb{E}[f(Z)|U] = \left( \frac{F^{-1}_\mu(U) + F^{-1}_\nu(U) - 2\alpha}{2(F^{-1}_\mu(U) - \alpha)} f(F^{-1}_\mu(U)) + \frac{F^{-1}_\nu(U) - F^{-1}_\mu(U)}{2(F^{-1}_\nu(U) - \alpha)} f(2\alpha - F^{-1}_\nu(U)) \right) \mathbb{1}_{\{F^{-1}_\nu(U) \neq \alpha\}} + f(\alpha) \mathbb{1}_{\{F^{-1}_\nu(U) = \alpha\}}.
$$

Let $W$ be a random variable independent from $U$ and uniformly distributed on $[0,1]$, and let $T = F_\mu(F^{-1}_\mu(U)_- + W(F_\mu(F^{-1}_\mu(U)) - F_\mu(F^{-1}_\mu(U)_-))$, which is uniformly distributed on $[0,1]$ and satisfies $F^{-1}_\mu(T) = F^{-1}_\mu(U)$ P-a.s. according to Lemma 4.4. Then for any measurable and bounded function $h : \mathbb{R} \to \mathbb{R}$, denoting $\theta(x,w) = F_\mu(x_-) + w(F_\mu(x) - F_\mu(x_-))$ and using the inverse transform sampling, we have

$$
\mathbb{E} \left[ \mathbb{1}_{\{F^{-1}_\nu(U) \neq \alpha\}} \frac{F^{-1}_\mu(U) + F^{-1}_\nu(U) - 2\alpha}{2(F^{-1}_\mu(U) - \alpha)} f(F^{-1}_\mu(U)) h(F^{-1}_\mu(U)) \right] = \mathbb{E} \left[ \mathbb{1}_{\{F^{-1}(T) \neq \alpha\}} \frac{F^{-1}_\mu(T) + F^{-1}_\nu(T) - 2\alpha}{2(F^{-1}_\nu(T) - \alpha)} f(F^{-1}_\nu(T)) h(F^{-1}_\nu(U)) \right] = \int_0^1 \int_0^1 \mathbb{1}_{\{F^{-1}_\nu(\theta(x,u)) \neq \alpha\}} \frac{F^{-1}_\mu(\theta(x,u)) + F^{-1}_\nu(\theta(x,u)) - 2\alpha}{2(F^{-1}_\nu(\theta(x,u)) - \alpha)} f(F^{-1}_\nu(\theta(x,u))) h(x) d\mu(dx) du.
$$

In the same way, we have

$$
\mathbb{E} \left[ \mathbb{1}_{\{F^{-1}_\nu(U) \neq \alpha\}} \frac{F^{-1}_\nu(U) - F^{-1}_\mu(U)}{2(F^{-1}_\nu(U) - \alpha)} f(2\alpha - F^{-1}_\nu(U)) h(F^{-1}_\mu(U)) \right] = \mathbb{E} \left[ \mathbb{1}_{\{F^{-1}(x) \neq \alpha\}} \frac{F^{-1}_\mu(x) - F^{-1}_\nu(x)}{2(F^{-1}_\nu(x) - \alpha)} f(2\alpha - F^{-1}_\nu(x)) h(x) d\mu(dx) ,
$$

and

$$
\mathbb{E}[f(\alpha) \mathbb{1}_{\{F^{-1}_\nu(U) = \alpha\}} h(F^{-1}_\mu(U))] = f(\alpha) \int_\mathbb{R} \mathbb{1}_{\{F^{-1}_\nu(\theta(x,u)) = \alpha\}} h(x) d\mu(dx). 
$$

Therefore,

$$
\mathbb{E}[f(Z)h(F^{-1}_\mu(U))] = \mathbb{E}[\mathbb{E}[f(Z)|U]h(F^{-1}_\mu(U))] = \int_{\mathbb{R}} \int_0^1 f(y) \mathbb{E}[h(\theta(x,u), dy)] d\mu(dx) = \int_{\mathbb{R}} \int_0^1 f(y) h(x) m(dx, dy) d\mu(dx),
$$

where we used (1.3) for the last equality. So the law of $(F^{-1}_\mu(U), Z)$ is $\mu(dx) m(x, dy)$.

### 1.2 Stability Inequality

**Theorem 1.5.** For all $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ such that $\mu \leq_{cs} \nu$,

$$
\inf_{P \in \Pi^0(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y| P(dx, dy) \leq 2W_1(\mu, \nu),
$$

(1.8)

where the constant 2 is sharp. Moreover, provided that $\mu \neq \nu$, for all $Q \in \mathcal{Q}$, the martingale coupling $M$ is such that

$$
\int_{\mathbb{R} \times \mathbb{R}} |y - x| M(dx, dy) \leq 2W_1(\mu, \nu),
$$

(1.9)
The proof of Theorem 1.5 relies on Proposition 1.6 below. Note that since $\Pi^M(\mu, \nu) \subset \Pi(\mu, \nu)$, we always have $W_1(\mu, \nu) \leq \inf_{P \in \Pi^M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y| \, P(dx, dy)$.

The optimal martingale coupling $P \in \Pi^M(\mu, \nu)$ which minimises $\int_{\mathbb{R} \times \mathbb{R}} |x - y| \, P(dx, dy)$ was actually exposed by Hobson and Klümmel [5] under the dispersion assumption that there exists an interval $E$ of positive length such that $(\mu - \nu)^+(E) = (\nu - \mu)^+(E) = 0$. They show that the optimal coupling $\Pi^HK$ is unique. Moreover, in the simpler case where $\mu \wedge \nu = 0$, if $a < b$ denote the endpoints of $E$, then there exist two nonincreasing functions $R : (0, 1) \to (-\infty, a)$ and $S : (0, 1) \to [b, +\infty)$ such that for all $u \in (0, 1)$, denoting $\tilde{\Pi}^HK(u, dy) = \Pi^HK(F^{-1}_\mu(u), dy)$, one has

$$
\tilde{\Pi}^HK(u, dy) = \frac{S(u) - F^{-1}_\mu(u)}{S(u) - R(u)} \delta_{R(u)}(dy) + \frac{F^{-1}_\mu(u) - R(u)}{S(u) - R(u)} \delta_{S(u)}(dy).
$$

If $\mu$ and $\nu$ are reduced to two atoms each and are such that $\mu \leq c \leq \nu$, then there exists a unique martingale coupling between $\mu$ and $\nu$, so $\Pi^HK$ derives of course from $\mathcal{Q}$. However, for any $Q \in \mathcal{Q}$, we can see thanks to Lemma 1.3 that $\tilde{m}(u, \{F^{-1}_\mu(u)\}) > 0$ for $du$-almost all $u \in (0, 1)$. Since $R$ and $S$ are nonincreasing and $F^{-1}_\nu$ is nondecreasing, we necessarily have $\Pi^{HK} \neq \tilde{m}$ as long as $F^{-1}_\nu$ takes at least three different values on $(0, 1)$, so $\Pi^HK$ does not derive from an element of $\mathcal{Q}$ in that case. For example, if $\mu(dx) = \frac{1}{2}1_{[-1, 1]}(x) \, dx$ and $\nu(dx) = \frac{1}{2}(1_{[-2, -1]} + 1_{[1, 2]})(x) \, dx$, then

$$
\Pi^HK(x, dy) = \left(\frac{1}{2} - \frac{3x}{2\sqrt{12} - 3x}\right) \delta_{\frac{1}{2}(x + \sqrt{12} - 3x)}(dy) + \left(\frac{1}{2} + \frac{3x}{2\sqrt{12} - 3x}\right) \delta_{\frac{1}{2}(x - \sqrt{12} - 3x)}(dy).
$$

which satisfies $\Pi^HK(x, \{F^{-1}_\mu(F(x))\}) > 0$ if $x \in \{(3 - \sqrt{33})/6, (\sqrt{33} - 3)/6\}$, whereas $m(x, \{F^{-1}_\nu(F(x))\}) > 0$ for $dx$-almost all $x \in (-1, 1)$, for all $Q \in \mathcal{Q}$.

Note that the maximisation problem $\sup_{P \in \Pi^M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y| \, P(dx, dy)$ is discussed by Hobson and Neuberger [6].

Proposition 1.6. $(\tilde{m}(u, dy))_{u \in (0, 1)}$ minimises

$$
\int_0^1 \int_{\mathbb{R}} |F^{-1}_\nu(\tilde{F}_\mu(u)) - y| \, du \tilde{p}(u, dy)
$$

among all Markov kernels $(\tilde{p}(u, dy))_{u \in (0, 1)}$ such that

$$
\left\{ \begin{array}{l}
\int_{u \in (0, 1)} du \tilde{p}(u, dy) = \nu(dy) \\
\int_{x \in (-1, 1)} dx y \tilde{p}(u, dy) = F^{-1}_\mu(u), \ du\text{-almost everywhere on } (0, 1)
\end{array} \right. \quad (1.9)
$$

Moreover, $\int_0^1 \int_{\mathbb{R}} |F^{-1}_\nu(\tilde{F}_\mu(u)) - y| \, du \tilde{m}(dy, du) = W_1(\mu, \nu)$.

Remark 1.7. Given a Markov kernel $(\tilde{p}(u, dy))_{u \in (0, 1)}$ which satisfies (1.9), we can define $(p(x, dy))_{x \in \mathbb{R}}$ as in (1.2) with $\tilde{p}(u, dy)$ instead of $m(u, dy)$ so that $\mu(dx) \, p(x, dy)$ is a martingale coupling between $\mu$ and $\nu$.

Conversely, given a martingale coupling $\mu(dx) \, p(x, dy)$ between $\mu$ and $\nu$, we can define $\tilde{p}(u, dy) = p(F^{-1}_\mu(u), dy)$ so that $(\tilde{p}(u, dy))_{u \in (0, 1)}$ satisfies (1.9).

Proof of Proposition 1.6. Let $\tilde{p}$ be a Markov kernel as described in the statement. By Jensen’s inequality, for $du$-almost every $u \in (0, 1),

$$
|F^{-1}_\nu(u) - F^{-1}_\mu(u)| \leq \int_{\mathbb{R}} |F^{-1}_\nu(u) - y| \tilde{p}(u, dy) \leq \int_{\mathbb{R}} |F^{-1}_\nu(u) - y| \tilde{p}(u, dy).
$$

So $\int_0^1 |F^{-1}_\nu(u) - F^{-1}_\mu(u)| \, du \leq \int_0^1 \int_{\mathbb{R}} |F^{-1}_\nu(u) - y| \, du \tilde{p}(u, dy)$.

Therefore, it is sufficient to prove that $\int_{\mathbb{R}} |F^{-1}_\nu(u) - y| \tilde{m}(u, dy) = |F^{-1}_\nu(u) - F^{-1}_\mu(u)|$, $du$-almost everywhere on $(0, 1)$. Applying (1.4) to the measurable and nonnegative function $h : y \mapsto |F^{-1}_\nu(u) - y|$ yields

$$
\int_{\mathbb{R}} |F^{-1}_\nu(u) - y| \tilde{m}(u, dy) = \int_{(0, 1)} \frac{F^{-1}_\mu(u) - F^{-1}_\nu(u)}{F^{-1}_\nu(v) - F^{-1}_\mu(u)} |F^{-1}_\nu(u) - F^{-1}_\nu(v)| \pi_-(u, dv).
$$
Using Lemma 1.3, we deduce that
\[
\int_{\mathbb{R}} |F_{\nu}^{-1}(u) - y| \tilde{m}(u, dy) = \int_{(0,1)} (F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u))^{+} \pi_{+}(u, dv) + \int_{(0,1)} (F_{\mu}^{-1}(u) - F_{\nu}^{-1}(u))^{-} \pi_{+}(u, dv) = |F_{\nu}^{-1}(u) - F_{\mu}^{-1}(u)|. 
\]

Proof of Theorem 1.5. By Lemma 1.4 and Proposition 1.6,
\[
\int_{\mathbb{R} \times \mathbb{R}} |y - x| \mu(dx) m(x, dy) = \int_{0}^{1} \int_{\mathbb{R}} |y - F_{\mu}^{-1}(u)| du \tilde{m}(u, dy) 
\leq \int_{0}^{1} \int_{\mathbb{R}} |y - F_{\nu}^{-1}(u)| du \tilde{m}(u, dy) + \int_{0}^{1} \int_{\mathbb{R}} |F_{\nu}^{-1}(u) - F_{\mu}^{-1}(u)| du \tilde{m}(u, dy) = 2W_{1}(\mu, \nu).
\]
Since \(\mu(dx) m(x, dy)\) is a martingale coupling between \(\mu\) and \(\nu\) (Proposition 1.2), we get (1.8). Let us show now that the constant 2 in sharp, that is
\[
\sup_{\mu, \nu \in \mathcal{P}_{1}(\mathbb{R}), \mu <_{\text{cv}} \nu} \left( \inf_{p \in \Pi_{1}^{\mu}(\mu, \nu)} \frac{\int_{\mathbb{R} \times \mathbb{R}} |x - y| P(dx, dy)}{W_{1}(\mu, \nu)} \right) = 2.
\]
Let \(a, b \in \mathbb{R}\) such that \(0 < a < b\). Let \(\mu = \frac{1}{2} \delta_{a} + \frac{1}{2} \delta_{-a}\) and \(\nu = \frac{1}{2} \delta_{b} + \frac{1}{2} \delta_{-b}\). Since \(\mu\) and \(\nu\) are two probability measures with equal means such that \(\mu\) is concentrated on \([-a, a]\) and \(\nu\) on \(\mathbb{R}\setminus[-a, a]\), then \(\mu <_{\text{cv}} \nu\). Any coupling \(H\) between \(\mu\) and \(\nu\) is of the form
\[
H = p\delta_{(a,b)} + p'\delta_{(a,-b)} + r\delta_{(-a,b)} + r'\delta_{(-a,-b)},
\]
where \(p, p', r, r' \geq 0\) and \(p + p' = r + r' = p + r' = p' + r = 1/2\). One can easily see that \(H\) is a martingale coupling iff \(b(p - p') = a/2\) and \(b(r' - r) = -a/2\), that is
\[
H = \frac{(b + a)}{4b} \delta_{(a,b)} + \frac{(b - a)}{4b} \delta_{(a,-b)} + \frac{(b + a)}{4b} \delta_{(-a,b)} + \frac{(b - a)}{4b} \delta_{(-a,-b)}. 
\]
Since there is only one martingale coupling, we trivially have
\[
\inf_{p \in \Pi_{1}^{\mu}(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y| P(dx, dy) = \int_{\mathbb{R} \times \mathbb{R}} |x - y| H(dx, dy) = \frac{b^{2} - a^{2}}{b}.
\]
On the other hand, since \(W_{1}(\mu, \nu) = \int_{\mathbb{R}} |F_{\mu}(t) - F_{\nu}(t)| dt\) (see for instance Remark 2.19 (iii) Chapter 2 [10]),
\[
W_{1}(\mu, \nu) = \int_{-\infty}^{-b} 0 dt + \int_{-b}^{-a} \frac{1}{2} dt + \int_{-a}^{a} 0 dt + \int_{a}^{b} \frac{1}{2} dt + \int_{b}^{\infty} 0 dt = b - a. 
\]
So, we have
\[
\inf_{p \in \Pi^d(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| P(dx, dy) / W_1(\mu, \nu) = 1 + \frac{a}{b},
\]
which tends to 2 as \(b\) tends to \(a\).

Also, the stability inequality (1.8) does not generalise for a cost function of the form \((x, y) \mapsto |x - y|^\rho\) with \(W_\rho^p(\mu, \nu)\) replacing \(W_1(\mu, \nu)\) for \(\rho > 1\), as shown in the next proposition in general dimension.

**Proposition 1.8.** Let \(d \geq 1\) and \(\rho > 1\). Then
\[
\sup_{\substack{\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d) \\ \mu < \nu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d)}} \left( \inf_{p \in \Pi^d(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\rho P(dx, dy) / W_\rho^p(\mu, \nu) \right) = +\infty.
\]

The proof of Proposition 1.8 will need the following lemma for the case \(1 < \rho < 2\).

**Lemma 1.9.** Let \(d \geq 1\) and \(\rho \in (1, 2)\). There exists \(C_\rho > 0\) such that
\[
\forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |x - y|^\rho \geq C_\rho \left( |x|^\rho - \frac{\rho}{\rho - 1} |x|^{\rho - 2} (x, y)_{\mathbb{R}^d} + \frac{1}{\rho - 1} |y|^\rho \right),
\]
where, by convention, for all \(y \in \mathbb{R}^d\) and for \(x = 0\) we choose \(|x|^{\rho - 2} (x, y)_{\mathbb{R}^d}\) equal to its limit 0 as \(x \to 0\).

Note that when \(\rho = 2\), both sides of the inequality are equal with \(C_2 = 1\).

**Proof.** If \(x = 0\), any \(C_\rho \leq \rho - 1\) suits. Else, dividing by \(|x|^\rho\) and using that \(y/|x|\) explores \(\mathbb{R}^d\) when \(y\) explores \(\mathbb{R}^d\), we see that the statement reduces to show that for all \(x, y \in \mathbb{R}^d\) such that \(|x| = 1\),
\[
|x - y|^\rho \geq C_\rho \left( 1 - \frac{\rho}{\rho - 1} (x, y)_{\mathbb{R}^d} + \frac{1}{\rho - 1} |y|^\rho \right).
\]

For all \(x, y \in \mathbb{R}^d\) such that \(|x| = 1\), there exist \(y_1, y_2 \in \mathbb{R}\) such that \(y = y_1 x + y_2 x^\perp\), where \(x^\perp\) is an element of \(\text{span}(x)^\perp\) such that \(|x^\perp| = 1\). The inequality to prove becomes
\[
((1 - y_1^2 + y_2^2)^{\rho/2} \geq C_\rho \left( 1 - \frac{\rho}{\rho - 1} y_1 + \frac{1}{\rho - 1} (y_1^2 + y_2^2)^{\rho/2} \right).
\]
Let \(L : (y_1, y_2) \mapsto ((1 - y_1^2 + y_2^2)^{\rho/2}\) and \(R : (y_1, y_2) \mapsto 1 - \frac{\rho}{\rho - 1} y_1 + \frac{1}{\rho - 1} (y_1^2 + y_2^2)^{\rho/2}\). When \((y_1, y_2) \to (1, 0)\), we have
\[
R(y_1, y_2) = \frac{1}{\rho - 1} \left( (\rho - 1 - (y_1 - 1 + 1) + (1 + 2(y_1 - 1) + (y_1 - 1)^2 + y_2^2)^{\rho/2} \right)
= \frac{1}{\rho - 1} \left( (\rho - 1 - (y_1 - 1) + 1 + (y_1 - 1)^2 + y_2^2 + \frac{\rho}{2} (y_1 - 1)^2 + \frac{\rho}{2} y_2^2 \right)
= \frac{1}{\rho - 1} \left( \rho (y_1 - 1)^2 + \frac{\rho}{2} y_2^2 - \rho (1 - \frac{\rho}{2}) (y_1 - 1)^2 + o((y_1 - 1)^2 + y_2^2) \right).
\]
Since \(\rho < 2\), then \(L(y_1, y_2) \geq (1 - y_1^2 + y_2^2)\) for any \((y_1, y_2)\) in the ball centered at \((1, 0)\) with radius 1. So
\[
\limsup_{(y_1, y_2) \to (1, 0)} \frac{R(y_1, y_2)}{L(y_1, y_2)} \leq \frac{\rho}{2(\rho - 1)}.
\]
On the other hand, when \( y_1^2 + y_2^2 \to +\infty \),
\[
\frac{R(y_1, y_2)}{L(y_1, y_2)} \sim \frac{(y_1^2 + y_2^2)^{\rho/2}}{(\rho - 1)(y_1^2 + y_2^2)^{\rho/2}} = \frac{1}{\rho - 1}.
\]

So \((y_1, y_2) \mapsto R(y_1, y_2)/L(y_1, y_2)\) is defined and continuous on \((\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(1, 0)\}\), bounded in the ball centered at \((1, 0)\) with radius 1 and has a finite limit when the norm of \((y_1, y_2)\) tends to \(+\infty\). Therefore this function is bounded on \((\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(1, 0)\}\) by a certain constant \(K > 0\), which proves the inequality with \(C_\rho = \min(\rho - 1, 1/K)\). 

\[\square\]

**Proof of Proposition 1.8.** The case \(\rho > 2\) was done in the introduction in the one dimensional case. Its extension to dimension \(d\) is immediate. We now consider the case \(1 < \rho < 2\). Let \(\mu, \nu \in \mathcal{P}_\rho(\mathbb{R}^d)\) be such that \(\mu <_{cv} \nu\), and let \(P\) be a martingale coupling between \(\mu\) and \(\nu\). Thanks to Lemma 1.9, there exists \(C_\rho > 0\) such that
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|\rho P(dx, dy) \geq C_\rho \left( \int_{\mathbb{R}^d} |x|^{\rho} \mu(dx) - \frac{\rho}{\rho - 1} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x|^{\rho - 2}(x, y)_{\mathbb{R}^d} P(dx, dy) \right)
\]
\[
+ \frac{1}{\rho - 1} \int_{\mathbb{R}^d} |y|\rho \nu(dx).
\]

Since \(P\) is a martingale coupling, we have for \(\mu(dx)\)-almost all \(x \in \mathbb{R}^d\), \(\int_{\mathbb{R}^d} |x|^{\rho - 2}(x, y)_{\mathbb{R}^d} \pi(x, dy) = |x|^{\rho}\), where both sides are equal to 0 when \(x = 0\), so
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|\rho P(dx, dy) \geq C_\rho \left( \int_{\mathbb{R}^d} |y|\rho \nu(dx) - \int_{\mathbb{R}^d} |x|^{\rho} \mu(dx) \right).
\]

Let us now specify a choice of \(\mu\) and \(\nu\). Let \(X\) be a non-constant \(d\)-dimensional random variable such that \(\mathbb{E}[|X|^\rho] < +\infty\). Then the mapping which associates to \(t \in \mathbb{R}_+\) the law of \(\mathbb{E}[X] + t(X - \mathbb{E}[X])\) is increasing in the convex order. Indeed, for all \(0 \leq s < t\) and for all convex functions \(\varphi: \mathbb{R}^d \to \mathbb{R}\),
\[
\varphi(\mathbb{E}[X] + s(X - \mathbb{E}[X])) = \varphi \left( \frac{(t - s)}{t} \mathbb{E}[X] + \frac{s}{t} \mathbb{E}[X] + t(X - \mathbb{E}[X]) \right)
\]
\[
\leq \frac{(t - s)}{t} \varphi(\mathbb{E}[X]) + \frac{s}{t} \varphi(\mathbb{E}[X] + t(X - \mathbb{E}[X]))
\]
\[
= \frac{(t - s)}{t} \varphi(\mathbb{E}[X] + t(X - \mathbb{E}[X])) + \frac{s}{t} \varphi(\mathbb{E}[X] + t(X - \mathbb{E}[X]))
\]
\[
\leq \frac{(t - s)}{t} \mathbb{E}[\varphi(\mathbb{E}[X] + t(X - \mathbb{E}[X]))] + \frac{s}{t} \varphi(\mathbb{E}[X] + t(X - \mathbb{E}[X]))
\]
where we used Jensen’s inequality in the last inequality. So, by taking the expected value, we obtain
\[
\mathbb{E}[\varphi(\mathbb{E}[X] + s(X - \mathbb{E}[X]))] \leq \mathbb{E}[\varphi(\mathbb{E}[X] + t(X - \mathbb{E}[X]))].
\]

For all \(n \in \mathbb{N}\), let \(\mu_n\) and \(\mu_{n+1}\) be respectively the laws of \(\mathbb{E}[X] + n(X - \mathbb{E}[X])\) and \(\mathbb{E}[X] + (n+1)(X - \mathbb{E}[X])\). To set this idea on a simple example, one could choose \(X \sim \mathcal{N}_d(0, I_d)\) so that \(\mu_n = \mathcal{N}_d(0, n^2 I_d)\) and \(\mu_{n+1} = \mathcal{N}_d(0, (n + 1)^2 I_d)\). So, if \(X\) is non-constant, centered and such that \(\mathbb{E}[|X|^\rho] < +\infty\), then for all \(n \in \mathbb{N}\),
\[
\inf_{P \in \Pi^d(\mu_n, \mu_{n+1})} \mathbb{E}[\frac{|x - y|^{\rho} P(dx, dy)}{W^p_\rho(\mu_n, \mu_{n+1})}] \geq \frac{C_\rho}{\rho - 1} \frac{((n+1)^{\rho} - n^{\rho})C_\rho}{\rho - 1} \sim_{n \to +\infty} n^{\rho - 1} \to +\infty.
\]

\[\square\]
2 The inverse transform martingale coupling

2.1 Definition of the inverse transform martingale coupling

Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ such that $\mu \leq_{xc} \nu$. Let $\Psi_+ : u \in [0,1] \mapsto \int_0^u (F^{-1}_\mu - F^{-1}_\nu)(v) \, dv$, $\Psi_- : u \in [0,1] \mapsto \int_0^u (F^{-1}_\mu - F^{-1}_\nu)^-(v) \, dv$, $\varphi = \Psi_+ \circ \Psi_-$ and $\tilde{\varphi} = \Psi_+ \circ \Psi_-$ where $\Psi^{-1}_-$ (or $\Psi^{-1}_+$) denotes the left continuous generalised inverse of $\Psi_-$ (or $\Psi_+$), that is

\[
\varphi : u \in [0,1] \mapsto \inf\{r \in [0,1] \mid \Psi_-(r) \geq \Psi_+(u)\};
\]

\[
\tilde{\varphi} : u \in [0,1] \mapsto \inf\{r \in [0,1] \mid \Psi_+(r) \geq \Psi_-(u)\},
\]

which are well defined thanks to the equality $\Psi_-(1) = \Psi_+(1)$, consequence of the equality of the means.

Let $Q$ be the probability measure defined on $(0,1)^2$ by

\[
Q(du, dv) = \frac{1}{\gamma}(F^{-1}_\mu(u) - F^{-1}_\nu(u))^+ \, du \, \pi_-(u, dv),
\]

where $\pi_-(u, dv) = I_{(0,\varphi(u)<1)} \, \delta_{\varphi(u)}(dv) + I_{(\varphi(u)\in(0,1))} \, dv$ and $\gamma = \Psi_-(1) = \Psi_+(1)$.

**Proposition 2.1.** The probability measure $Q$ is an element of $\mathcal{Q}$ as defined in Section 1. Moreover,

\[
Q(du, dv) = \frac{1}{\gamma}(F^{-1}_\mu(v) - F^{-1}_\nu(v))^- \, dv \, \pi_+(v, du),
\]

where $\pi_+(v, du) = I_{(0,\tilde{\varphi}(v)<1)} \, \delta_{\tilde{\varphi}(v)}(du) + I_{(\tilde{\varphi}(v)\in(0,1))} \, du$.

For this particular $Q \in \mathcal{Q}$, Definition (1.1) of the Markov kernel $(\tilde{m}(u, dy))_{u \in (0,1)}$ becomes

\[
\begin{cases}
(1 - \frac{F^{-1}_\mu(u) - F^{-1}_\nu(u)}{F^{-1}_\mu(\varphi(u)) - F^{-1}_\nu(\varphi(u))}) \delta_{F^{-1}_\mu(u)}(dy) + \frac{F^{-1}_\mu(u) - F^{-1}_\nu(u)}{F^{-1}_\mu(\varphi(u)) - F^{-1}_\nu(\varphi(u))} \delta_{F^{-1}_\nu(\varphi(u))}(dy) \\
\text{if } F^{-1}_\nu(\varphi(u)) > F^{-1}_\mu(u) > F^{-1}_\nu(1) \text{ and } \varphi(u) < 1;
\end{cases}
\]

\[
\begin{cases}
(1 - \frac{F^{-1}_\mu(u) - F^{-1}_\nu(u)}{F^{-1}_\mu(\tilde{\varphi}(u)) - F^{-1}_\nu(\tilde{\varphi}(u))}) \delta_{F^{-1}_\mu(u)}(dy) + \frac{F^{-1}_\mu(u) - F^{-1}_\nu(u)}{F^{-1}_\mu(\tilde{\varphi}(u)) - F^{-1}_\nu(\tilde{\varphi}(u))} \delta_{F^{-1}_\nu(\tilde{\varphi}(u))}(dy) \\
\text{if } F^{-1}_\nu(\tilde{\varphi}(u)) < F^{-1}_\mu(u) < F^{-1}_\nu(1) \text{ and } \tilde{\varphi}(u) < 1;
\end{cases}
\]

\[
\delta_{F^{-1}_\nu(\varphi(u))}(dy) \quad \text{otherwise.}
\]

Note that if $\varphi(u) = 0$, then $\Psi_+(u) = 0$ so for all $v \in (0,1)$, $F^{-1}_\mu(v) \leq F^{-1}_\nu(v)$ and $F^{-1}_\mu(u) \leq F^{-1}_\nu(u)$ by left-continuity. So $F^{-1}_\nu(u) > F^{-1}_\mu(u)$ implies that $\varphi(u) > 0$ so that with the condition $\varphi(u) < 1$, $F^{-1}_\nu(\varphi(u))$ makes sense. For similar reasons, if $F^{-1}_\mu(u) < F^{-1}_\nu(u)$ then $F^{-1}_\nu(\tilde{\varphi}(u))$ makes sense.

We define the Markov kernel $(m(x, dy))_{x \in \mathbb{R}}$ as in Definition (1.2). Then by Proposition 1.2, $M(dx, dy) = \mu(dx) m(x, dy)$ is a martingale coupling, called the inverse transform martingale coupling.

**Remark 2.2.** Let us mention that the inverse transform martingale coupling permits to give the following short constructive proof of Strassen’s theorem. Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ be such that $\mu \leq_{xc} \nu$. For $t \in \mathbb{R}$, $\int_{(t-x)^+}^{\infty} \mu(dx) \leq \int_{(t-x)^+}^{\infty} \nu(dx)$ by convexity of $x \in \mathbb{R} \mapsto (t-x)^+$. By the Fubini-Tonelli theorem, $\int_{(t-x)^+}^{\infty} \mu(dx) = \int_{-\infty}^{\infty} F_\mu(x) \, dx$. Hence $\varphi_\mu(t) = \int_{-\infty}^{\infty} F_\mu(x) \, dx \leq \varphi_\nu(t) = \int_{-\infty}^{\infty} F_\nu(x) \, dx$ for all $t \in \mathbb{R}$. Hence the respective Fenchel-Legendre transforms $\varphi^*_\mu$ and $\varphi^*_\nu$ of $\varphi_\mu$ and $\varphi_\nu$ satisfy $\varphi^*_\mu \geq \varphi^*_\nu$. For all $u \in (0,1)$ and for all $t \in \mathbb{R}$, $F^{-1}_\mu(u) \leq t \iff u \leq F_\mu(t)$, so

\[
sup_{q \in [0,1]} \left( qt - \int_0^q F^{-1}_\mu(u) \, du \right) = \int_0^q F_\mu(t) \, (t - F^{-1}_\mu(u)) \, du = \int_0^1 (t - F^{-1}_\mu(u))^+ \, du = \varphi_\mu(t).
\]
Since \( q \mapsto (\int_0^q F_\mu^{-1}(u) \, du) \) is convex on \([0,1]\), we get the well known fact (see for instance Lemma A.22 in [3]) that for all \( q \in \mathbb{R} \), \( \varphi_\mu^*(q) = (\int_0^q F_\mu^{-1}(u) \, du) \mathbb{I}_{[0,1]}(q) + (+\infty) \mathbb{I}_{[0,1]}(q) \). The inequality \( \int_0^q F_\mu^{-1}(u) \, du \geq \int_0^q F_\nu^{-1}(u) \, du \) for all \( q \in [0,1] \) which is an equality for \( q = 1 \) implies the existence of the inverse transform martingale coupling \( M \) between \( \mu \) and \( \nu \).

**Proof of Proposition 2.1.** Let \( h : (0,1)^2 \to \mathbb{R} \) be a measurable and bounded function.

\[
\int_{(0,1)^2} h(u,v) Q(du, dv) = \frac{1}{\gamma} \int_{(0,1)^2} h(u,v)(F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \mathbb{1}_{\{\varphi(u) < 1\}} \, du \delta_\varphi(u)(dv)
\]

\[
+ \frac{1}{\gamma} \int_{(0,1)^2} h(u,v)(F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \mathbb{1}_{\{\varphi(u) \in \{0,1\}\}} \, du dv
\]

\[
= \frac{1}{\gamma} \int_{(0,1)} h(u,\varphi(u))(F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \mathbb{1}_{\{\varphi(u) < 1\}} \, du
\]

\[
+ \frac{1}{\gamma} \int_{(0,1)^2} h(u,v)(F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \mathbb{1}_{\{\varphi(u) \in \{0,1\}\}} \, du dv.
\]

Since \( \Psi_- \) is continuous, one has \( \Psi_-(\varphi^{-1}(u)) = u \) for all \( u \in (0,1) \). By Lemma 4.1, we deduce that \( \varphi^{-1}(\varphi(u)) = u \), \( (F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \) du-almost everywhere on \((0,1)\). Therefore, on the one hand, by Lemma 4.5,

\[
\int_{(0,1)} h(u,\varphi(u))(F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \mathbb{1}_{\{\varphi(u) < 1\}} \, du
\]

\[
= \int_{(0,1)} h(\varphi(u), \varphi(u))(F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \mathbb{1}_{\{\varphi(u) < 1\}} \mathbb{1}_{\{\varphi(\varphi(u)) < 1\}} \, du
\]

\[
= \int_{(0,1)} h(\varphi(v), v)(F_\mu^{-1}(v) - F_\nu^{-1}(v))^+ \mathbb{1}_{\{\varphi(v) < 1\}} \mathbb{1}_{\{\varphi(\varphi(v)) < 1\}} \, dv
\]

\[
= \int_{(0,1)^2} h(u,v)(F_\mu^{-1}(v) - F_\nu^{-1}(v))^+ \mathbb{1}_{\{\varphi(v) < 1\}} \, dv \delta_\varphi(v)(du).
\]

On the other hand, still by Lemma 4.5,

\[
\int_{(0,1)^2} h(u,v)(F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \mathbb{1}_{\{\varphi(u) \in \{0,1\}\}} \, du dv
\]

\[
= \int_{(0,1)} \left( \int_{(0,1)} h(\varphi(u), v) \, dv \right) (F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \mathbb{1}_{\{\varphi(u) \in \{0,1\}\}} \, du
\]

\[
= \int_{(0,1)} \left( \int_{(0,1)} h(\varphi(u), v) \, dv \right) (F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \mathbb{1}_{\{\varphi(u) \in \{0,1\}\}} \, du
\]

\[
= 0
\]

(2.3)

With the same reasoning, one can show that

\[
\int_{(0,1)^2} h(u,v)(F_\mu^{-1}(v) - F_\nu^{-1}(v))^+ \mathbb{1}_{\{\varphi(v) \in \{0,1\}\}} \, du dv = 0.
\]

So

\[
\int_{(0,1)^2} h(u,v) Q(du, dv) = \frac{1}{\gamma} \int_{(0,1)^2} h(u,v)(F_\mu^{-1}(v) - F_\nu^{-1}(v))^+ \mathbb{1}_{\{\varphi(v) < 1\}} dv \delta_\varphi(v)(du)
\]

\[
+ \frac{1}{\gamma} \int_{(0,1)^2} h(u,v)(F_\mu^{-1}(v) - F_\nu^{-1}(v))^+ \mathbb{1}_{\{\varphi(v) \in \{0,1\}\}} dv,
\]

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hence \( Q(du, dv) = \frac{1}{r}(F_\mu^{-1}(v) - F_\nu^{-1}(v))^- dv \pi_+(v, du) \), where \( \pi_+(v, du) = \mathbb{1}_{\{0 < \varphi(v) < 1\}} \delta_{\varphi(v)}(du) + \mathbb{1}_{(\varphi(v) \in \{0, 1\})} du \).

Therefore, it is clear that \( Q \) has first marginal \( \frac{1}{r}(F_\mu^{-1}(u) - F_\nu^{-1}(u))^- du \) and second marginal \( \frac{1}{r}(F_\mu^{-1}(v) - F_\nu^{-1}(v))^- dv \). In particular for \( h : (u, v) \mapsto \mathbb{1}_{\{u < v\}} \),

\[
Q \left( \{(u, v) \in (0, 1)^2 \mid u < v\} \right) = \frac{1}{\gamma} \int_0^1 (F_\mu^{-1}(u) - F_\nu^{-1}(u))^- \mathbb{1}_{\{u < \varphi(u) < 1\}} du
\]

Let us show that \( u < \varphi(u), (F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ du \)-almost everywhere on \((0, 1)\). For all \( u \in (0, 1) \), \( \varphi(u) \leq u \iff \Psi_-(\varphi(u)) \leq u \iff \Psi_-(u) \leq \Psi_-(\varphi(u)) \leq \Psi_+(u) \leq \Psi_+(u) \). Suppose \( (F_\mu^{-1}(u) > F_\nu^{-1}(u)) \). Since \( F_\mu^{-1} \) and \( F_\nu^{-1} \) are left continuous, this implies \( F_\mu^{-1}(u - h) > F_\nu^{-1}(u - h) \) for all \( h > 0 \) small enough. So, for all \( h > 0 \) small enough, \( \Psi_-(u) = \Psi_-(u - h) \leq \Psi_+(u - h) < \Psi_+(u) \), which implies \( u < \varphi(u) \), \((F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ du \)-almost everywhere on \((0, 1)\). So, by Lemma 4.5,

\[
Q \left( \{(u, v) \in (0, 1)^2 \mid u < v\} \right) = \frac{1}{\gamma} \int_0^1 (F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \mathbb{1}_{\{u < \varphi(u) < 1\}} du
= \frac{1}{\gamma} \int_0^1 (F_\mu^{-1}(u) - F_\nu^{-1}(u))^- \mathbb{1}_{\{0 < u < 1\}} du
= 1.
\]

We saw in Section 1.1 a concrete example of an element \( Q_1 \in Q \) (see (1.5)) when \( \mu, \nu \in P_1(\mathbb{R}) \) are such that \( \mu \ll \nu \) and there exists \( p \in (0, 1) \) such that \( u \mapsto \int_0^u (F_\mu^{-1}(v) - F_\nu^{-1}(v)) dv \) is nondecreasing on \([0, p]\) and nonincreasing on \([p, 1]\). Any probability measure \( Q \) defined on \((0, 1)\) satisfying properties (i) and (ii) of the definition of \( Q \) is concentrated on \((0, p) \times (p, 1)\) and therefore satisfies (iii). The probability measure \( Q_1 \) is the simplest example that comes to mind, but it is associated with a not very straightforward martingale coupling between \( \mu \) and \( \nu \). Of course, the inverse transform martingale coupling presented in this section is a valid example, but we show here that it inspires another coupling which is sort of the nonincreasing twin of the inverse transform martingale coupling. Let \( \chi_- : u \in [0, 1] \mapsto \int_0^u (F_\mu^{-1} - F_\nu^{-1})^{-}(v) dv \) and \( \Gamma = \chi_- \circ \chi_+ \) where \( \chi_- \) denotes the left continuous generalised inverse of \( \chi_+ \), that is

\[
\Gamma : u \in [0, 1] \mapsto \inf \{ r \in [0, 1] \mid \chi_-(r) \geq \chi_+(u) \},
\]

which is well defined since \( \chi_+(1) = \chi_+(1 - p) = \gamma \), consequence of the equality of the means. Let \( Q \) be the probability measure defined on \((0, 1)^2\) by

\[
Q(du, dv) = \frac{1}{\gamma} (F_\mu^{-1} - F_\nu^{-1})^+(u) du \pi_-(u, dv),
\]

where \( \pi_-(u, dv) = \mathbb{1}_{\{\Gamma(u) > 0\}} \delta_{1-\Gamma(u)}(dv) + \mathbb{1}_{\{\Gamma(u) = 0\}} dv \). In the symmetric case, that is when \( \mu \) and \( \nu \) are symmetric and \( p = 1/2 \), we have \( \Gamma(u) = u \), so we get \( Q_2 \) (see (1.6)), hence \( Q \) is a generalisation of the symmetric coupling.

Note that \( \Gamma(1) = 1 - p \), hence \( \Gamma(u) < 1 \) for all \( u \in (0, 1) \). It is clear that \( Q \) satisfies property (i) of the definition of \( Q \). By Lemma 4.5 applied with the functions \( f_1 : u \in (0, 1) \mapsto (F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ \) and \( f_2 : u \in (0, 1) \mapsto (F_\mu^{-1}(1 - u) - F_\nu^{-1}(1 - u))^+ \), we have

\[
\frac{1}{\gamma} \int_0^1 (F_\mu^{-1} - F_\nu^{-1})^+(u)h(1 - \Gamma(u))1_{\{\Gamma(u) > 0\}} du = \frac{1}{\gamma} \int_0^1 (F_\mu^{-1} - F_\nu^{-1})^-(1 - v)h(1 - v) dv
= \frac{1}{\gamma} \int_0^1 (F_\mu^{-1} - F_\nu^{-1})^-(v)h(v) dv;
\]
\[
\frac{1}{\gamma} \int_{(0,1)^2} (F_\mu^{-1} - F_\nu^{-1})^+ (u) h(v) \mathbb{1}_{\{\Gamma(u)=0\}} \, du \, dv = 0,
\]
for any measurable and bounded function \( h : (0, 1) \rightarrow \mathbb{R} \). So \( Q \) satisfies (ii) as well, and therefore (iii).

We saw with Proposition 1.6 that for all \( Q \in \mathcal{Q} \), \( \int_0^1 \int_{\mathbb{R}} |F_\nu^{-1}(u) - y| \, du \, \tilde{m}(u, dy) = W_1(\mu, \nu) \). The next proposition shows that the inverse transform martingale coupling and its nonincreasing twin, when it exists, play particular roles among the martingale couplings which derive from \( Q \) when 1 is replaced by \( \rho > 1 \).

**Proposition 2.3.** Let \((\bar{m}^{IT}(u, dy))_{u \in (0,1)}\) be defined as in (2.2), and \((\bar{m}^{NIT}(u, dy))_{u \in (0,1)}\) its nonincreasing twin deriving from (2.4) when it exists. Then, for all \( \rho > 2 \), \((\bar{m}^{IT}(u, dy))_{u \in (0,1)}\) and \((\bar{m}^{NIT}(u, dy))_{u \in (0,1)}\) respectively minimises and maximises

\[
C_\rho(\bar{m}) = \int_0^1 \int_\mathbb{R} |F_\nu^{-1}(u) - y|^\rho \, du \, \bar{m}(u, dy),
\]

among all Markov kernels \((\bar{m}(u, dy))_{u \in (0,1)}\) defined as in (1.1) and parametrised by \( \mathcal{Q} \). For all \( 1 < \rho < 2 \), \((\bar{m}^{IT}(u, dy))_{u \in (0,1)}\) and \((\bar{m}^{NIT}(u, dy))_{u \in (0,1)}\) respectively maximises and minimises \( C_\rho(\bar{m}) \) among the same family of Markov kernels. If \( \rho = 2 \), then all the Markov kernels \((\bar{m}(u, dy))_{u \in (0,1)}\) parametrised by \( \mathcal{Q} \) share the same value of \( C_2(\bar{m}) \).

**Proof.** Let \( \rho > 1 \). Let \( Q \in \mathcal{Q} \) and \((\bar{m}(u, dy))_{u \in (0,1)}\) derived from \( Q \) according to (1.1). Applying (1.4) to the measurable and nonnegative function \( h : y \mapsto |F_\nu^{-1}(u) - y|^\rho \) yields

\[
\int_\mathbb{R} |F_\nu^{-1}(u) - y|^\rho \, \bar{m}(u, dy) = \int_{(0,1)} \frac{(F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ |F_\nu^{-1}(u) - F_\nu^{-1}(v)|^{\rho - 1}}{F_\nu^{-1}(v) - F_\nu^{-1}(u)} \, du \, \pi_-(u, dv) + \int_{(0,1)} \frac{(F_\mu^{-1}(u) - F_\nu^{-1}(u))^- |F_\nu^{-1}(u) - F_\nu^{-1}(v)|^{\rho - 1}}{F_\nu^{-1}(v) - F_\nu^{-1}(u)} \, du \, \pi_+(u, dv).
\]

Using Lemma 1.3, we deduce that

\[
C_\rho(\bar{m}) = \int_{(0,1)^2} (F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ |F_\nu^{-1}(u) - F_\nu^{-1}(v)|^{\rho - 1} \, du \, \pi_-(u, dv) + \int_{(0,1)^2} (F_\mu^{-1}(u) - F_\nu^{-1}(u))^- |F_\nu^{-1}(u) - F_\nu^{-1}(v)|^{\rho - 1} \, du \, \pi_+(u, dv) = 2\gamma \int_{(0,1)^2} |F_\nu^{-1}(u) - F_\nu^{-1}(v)|^{\rho - 1} \, Q(du, dv).
\]

Let \( c : (0, 1)^2 \rightarrow \mathbb{R} \) be the right-continuous function defined for all \((u, v) \in (0, 1)^2\) by \( c(u, v) = |F_\nu^{-1}(u_+) - F_\nu^{-1}(v_+)|^{\rho - 1} \). Since the set of discontinuities of \( F_\nu^{-1} \) is at most countable and since the marginals of \( Q \) have densities, \( C_\rho(\bar{m}) = 2\gamma \int_{(0,1)^2} c(u, v) \, Q(du, dv) \). Let us show that \( C_\rho(\bar{m}^{IT}) = 2\gamma \int_0^1 c(\Psi_+^{-1}(\gamma u), \Psi_+^{-1}(\gamma u)) \, du \). By Lemma 4.1, \( \Psi_+^{-1}(\Psi_+(u)) = u \), \( d\Psi_+(u) \)-almost everywhere on \((0, 1)\), so using (2.3) then Proposition 4.10 Chapter 0 [8] and the fact that \( 0 < \Psi_+^{-1}(u) < 1 \) for all \( u \in (0, \gamma) \), we have

\[
C_\rho(\bar{m}^{IT}) = 2\gamma \int_0^1 (F_\mu^{-1}(u) - F_\nu^{-1}(u))^+ |F_\nu^{-1}(u) - F_\nu^{-1}(\varphi(u))|^{\rho - 1} \mathbb{1}_{\{0 < \varphi(u) < 1\}} \, du = 2\gamma \int_0^1 |F_\nu^{-1}(\Psi_+(u)) - F_\nu^{-1}(\Psi_+^{-1}(\Psi_+(u)))|^{\rho - 1} \mathbb{1}_{\{0 < \Psi_+^{-1}(\Psi_+(u)) < 1\}} \, d\Psi_+(u)
\]

\[
= 2\gamma \int_0^1 |F_\nu^{-1}(\Psi_+(u)) - F_\nu^{-1}(\Psi_+(u)))|^{\rho - 1} \mathbb{1}_{\{0 < \Psi_+^{-1}(\Psi_+(u)) < 1\}} \, d\Psi_+(u)
\]

\[
= 2\gamma \int_0^1 |F_\nu^{-1}(\Psi_+^{-1}(\gamma u)) - F_\nu^{-1}(\Psi_+^{-1}(\gamma u)))|^{\rho - 1} \, du.
\]
Since the set of discontinuities of $(\Psi_+ \circ F_\nu)^{-1}$ and $(\Psi_- \circ F_\nu)^{-1}$ are at most countable, we get $\mathcal{C}_\rho(\tilde{m}^{IT}) = 2\gamma \int_0^1 c(\Psi_+^{-1}(\gamma u), \Psi_-^{-1}(\gamma u)) \, du$. On the other hand, when $(\tilde{m}^{NIT}(u, dv))_{v \in (0,1)}$ exists, for all $u \in (0,1), \chi_+(u) = \chi_+(u)$ and $\chi_-(u) = \gamma_+ - \chi_+(1 - u)$. If $Y$ is a random variable with distribution function $\Psi_-/\gamma$, we see that the distribution function of $1-Y$ is $\chi_-/\gamma$. So if $U$ is a random variable uniformly distributed on $[0,1]$, then $1 - \Psi_+^{-1}(\gamma(1 - u))$ has distribution $dx_{\chi_-/\gamma}$. Since $u \mapsto 1 - \Psi_+^{-1}(\gamma(1 - u))$ is nondecreasing, it is shown in [1] (see the proof of Lemma 4.3 for more details) that $1 - \Psi_+^{-1}(\gamma(1 - u)) = \chi_-(\gamma u), du$-almost everywhere on $(0, 1)$. So we show with similar arguments than above that $\mathcal{C}_\rho(\tilde{m}^{NIT}) = \int_0^1 c(\Psi_+^{-1}(\gamma u), \Psi_-^{-1}(\gamma(1 - u))) \, du$.

If $\rho \geq 2$, then $c$ satisfies the Monge condition, that is for all $u, u', v, v' \in (0,1)$ such that $u \leq u'$ and $v \leq v'$,

$$c(u', v') - c(u, v') - c(u', v) + c(u, v) \leq 0,$$

which follows from the monotonicity of $F_\nu^{-1}$ and the fact that $(x, y) \mapsto |x - y|^{\rho-1}$ is convex and therefore satisfies the Monge condition. Since $Q$ has marginals $d\Psi_+ / \gamma$ and $d\Psi_- / \gamma$, by Theorem 3.1.2 Chapter 3 [7], we have

$$\int_0^1 c(\Psi_+^{-1}(\gamma u), \Psi_-^{-1}(\gamma u)) \, du \leq \int_{(0,1)^2} |F_\nu^{-1}(u) - F_\nu'^{-1}(v)|^{\rho-1} Q(du, dv) \leq \int_0^1 c(\Psi_+^{-1}(\gamma u), \Psi_-^{-1}(\gamma(1 - u))) \, du.$$

If $1 < \rho \leq 2$, then $(x, y) \mapsto |x - y|^{\rho-1}$ is concave so $-c$ satisfies the Monge conditions and a symmetric reasoning shows that the inverse transform martingale coupling is a maximiser, whereas its nonincreasing twin is a minimiser when it exists. For $\rho = 2$, the inverse transform martingale coupling is therefore both a minimiser and a maximiser, hence the constant value of $\mathcal{C}_\rho(\tilde{m})$ among all Markov kernels $(\tilde{m}(u, dv))_{u \in (0,1)}$ parametrised by $Q$.

### 2.2 Stability of the inverse transform martingale coupling with respect to the marginal laws

Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ such that $\mu <_{ct} \nu$. Let $(\mu_n)_{n \in \mathbb{N}}$ and $(\nu_n)_{n \in \mathbb{N}}$ be two sequences of probability measures on $\mathbb{R}$ with finite first moments such that for all $n \in \mathbb{N}, \mu_n <_{ct} \nu_n$. For all $n \in \mathbb{N}$, let $M_n$ be the inverse transform martingale coupling between $\mu_n$ and $\nu_n$.

**Proposition 2.4.** If $W_1(\mu_n, \mu) \xrightarrow{n \to +\infty} 0$ and $W_1(\nu_n, \nu) \xrightarrow{n \to +\infty} 0$, then

$$W_1(M_n, M) \xrightarrow{n \to +\infty} 0,$$

where $M$ is the inverse transform martingale coupling between $\mu$ and $\nu$.

**Proof.** For all $n \in \mathbb{N}$, let $\Psi_{n+} : u \in [0,1] \mapsto \int_0^u (F_{\mu_n}^{-1} - F_\nu^{-1})^{-}(v) \, dv, \Psi_{n-} : u \in [0,1] \mapsto \int_0^u (F_{\mu_n}^{-1} - F_\nu^{-1})^{+}(v) \, dv, \varphi_n = \Psi_{n+}^{-1} \circ \Psi_{n+}$ and $\check{\varphi}_n = \Psi_{n-}^{-1} \circ \Psi_{n-}$. Let $(\tilde{m}_n(u, dy))_{u \in (0,1)}$ be the Markov kernel defined as in (2.2) with $F_{\mu_n}^{-1}, F_\nu^{-1}, \varphi$ and $\check{\varphi}$ respectively replaced by $F_{\mu_n}^{-1}, F_\nu^{-1}, \varphi_n$ and $\check{\varphi}_n$. According to (1.4), for any measurable and bounded function $h : \mathbb{R} \to \mathbb{R}$ and for $du$-almost all $u \in (0, 1)$,

$$\int_{\mathbb{R}} h(y) \tilde{m}_n(u, dy) = h(F_{\nu_n}^{-1}(u)) + \int_{(0,1)} \frac{(F_{\mu_n}^{-1}(u) - F_{\nu_n}^{-1}(v))^{+}(h(F_{\nu_n}^{-1}(v)) - h(F_{\nu_n}^{-1}(u)))}{F_{\nu_n}^{-1}(v) - F_{\nu_n}^{-1}(u)} \, \pi_{n-}(u, dv)$$

$$+ \int_{(0,1)} \frac{(F_{\mu_n}^{-1}(u) - F_{\nu_n}^{-1}(v))^{-}(h(F_{\nu_n}^{-1}(v)) - h(F_{\nu_n}^{-1}(u)))}{F_{\mu_n}^{-1}(v) - F_{\nu_n}^{-1}(u)} \, \pi_{n+}(u, dv),$$

where $\pi_{n-}(u, dv) = 1_{(0, \varphi_n(u)) < 1} \delta_{\varphi_n(u)}(dv) + 1_{(\varphi_n(u), \infty)} dv$ and $\pi_{n+}(u, dv) = 1_{(0, \check{\varphi}_n(u)) < 1} \delta_{\check{\varphi}_n(u)}(dv) + 1_{(\check{\varphi}_n(u), \infty)} dv$, and where all the integrands are well defined thanks to Lemma 1.3. We saw in the proof of Proposition 2.1 that $0 < \varphi_n(u) < 1, (F_{\mu_n}^{-1}(u) - F_{\nu_n}^{-1}(u))^{+} du$-almost everywhere and $0 < \check{\varphi}_n(u) < 1,$
Then, $(F^{-1}_{\mu}(u) - F^{-1}_{\nu}(u))^{-} du$ almost-everywhere. Let $h : \mathbb{R}^{2} \to \mathbb{R}$ be a bounded and Lipschitz continuous function. Then,

$$
\int_{\mathbb{R} \times \mathbb{R}} h(x, y) \mu_{n}(dx) m_{n}(x, dy) = \int_{(0,1)} \int_{\mathbb{R}} h(F^{-1}_{\mu}(u), y) \tilde{m}_{n}(u, dy) du
$$

$$
= \int_{(0,1)} h(F^{-1}_{\mu}(u), F^{-1}_{\nu}(u)) du
$$

$$
+ \int_{(0,1)} \frac{(F^{-1}_{\mu}(u) - F^{-1}_{\nu}(u))^{+}}{F^{-1}_{\nu}(\varphi_{n}(u)) - F^{-1}_{\nu}(u)} (h(F^{-1}_{\mu}(u), F^{-1}_{\nu}(\varphi_{n}(u))) - h(F^{-1}_{\mu}(u), F^{-1}_{\nu}(u))) du
$$

$$
+ \int_{(0,1)} \frac{(F^{-1}_{\mu}(u) - F^{-1}_{\nu}(u))^{-}}{F^{-1}_{\nu}(\varphi_{n}(u)) - F^{-1}_{\nu}(u)} (h(F^{-1}_{\mu}(u), F^{-1}_{\nu}(\varphi_{n}(u))) - h(F^{-1}_{\mu}(u), F^{-1}_{\nu}(u))) du.
$$

Since $\mu_{n}$ converges weakly towards $\mu$, then $F^{-1}_{\mu_{n}}(u)$ converges towards $F^{-1}_{\mu}(u)$ $du$-almost everywhere on $(0,1)$. Since $h$ is bounded, by the dominated convergence theorem,

$$
\int_{(0,1)} h(F^{-1}_{\mu}(u), F^{-1}_{\nu}(u)) du \to_{n \to \infty} \int_{(0,1)} h(F^{-1}_{\mu}(u), F^{-1}_{\nu}(u)) du.
$$

On the other hand, using Lemma 4.1 then Proposition 4.10 Chapter 0 [8] and the equality $\Psi_{n+}(1) = \Psi_{n-}(1)$, we have

$$
\int_{(0,\Psi_{n+}(1))} \frac{h(F^{-1}_{\mu}(\Psi_{n+}(u)), F^{-1}_{\nu}(\Psi_{n+}(u))) - h(F^{-1}_{\mu}(\Psi_{n+}(u)), F^{-1}_{\nu}(\Psi_{n+}(u)))}{F^{-1}_{\nu}(\Psi_{n+}(u)) - F^{-1}_{\nu}(\Psi_{n+}(u))} du
$$

$$
= \int_{(0,\Psi_{n+}(1))} h(F^{-1}_{\mu}(\Psi_{n+}(1) v), F^{-1}_{\nu}(\Psi_{n+}(1) v)) - h(F^{-1}_{\mu}(\Psi_{n+}(1) v), F^{-1}_{\nu}(\Psi_{n+}(1) v)) dv.
$$

Since $h$ is Lipschitz continuous (in particular with respect to its second variable), then the integrand above is bounded. Since for all $u \in [0,1]$, $x \mapsto x^{+}$ is Lipschitz continuous with constant 1,

$$
|\Psi_{n+}(u) - \Psi_{+}(u)| \leq \int_{0}^{u} |(F^{-1}_{\mu}(v) - F^{-1}_{\nu}(v))^{+} - (F^{-1}_{\mu}(v) - F^{-1}_{\nu}(v))^{+}| dv
$$

$$
\leq \int_{0}^{u} |F^{-1}_{\nu}(v) - F^{-1}_{\nu}(v)| dv + \int_{0}^{u} |F^{-1}_{\nu}(v) - F^{-1}_{\nu}(v)| dv
$$

$$
\leq W(\mu_{n}, \mu) + W(\nu_{n}, \nu),
$$

so $\Psi_{n+}$ converges uniformly to $\Psi_{+}$ on $[0,1]$. Moreover, for all $x \in \mathbb{R}$, $|\Psi_{n+}(F_{\mu_{n}}(x)) - \Psi_{+}(F_{\mu}(x))| \leq \sup_{[0,1]} |\Psi_{n+} - \Psi_{+}| + |\Psi_{+}(F_{\mu_{n}}(x)) - \Psi_{+}(F_{\mu}(x))|$, so $\Psi_{n+}(F_{\mu_{n}}(x))/\Psi_{n+}(1) \to \Psi_{+}(F_{\mu}(x))/\Psi_{+}(1)$ for all $x \in \mathbb{R}$ up to the at most countable set of discontinuities of $F_{\mu}$. This implies that $d(\Psi_{n+}(F_{\mu_{n}}(x))/\Psi_{n+}(1))$ converges to $d(\Psi_{+}(F_{\mu}(x))/\Psi_{+}(1))$ for the weak convergence topology. This implies the pointwise convergence of the left continuous pseudo-inverses $du$-almost everywhere on $(0,1)$, that is $F^{-1}_{\mu_{n}}(\Psi_{n+}(1) u) \to F^{-1}_{\mu}(\Psi_{n+}(1) u)$ for $du$-almost all $u \in (0,1)$. In the same way, $F^{-1}_{\nu}(\Psi_{n+}(1) u) \to F^{-1}_{\nu}(\Psi_{n+}(1) u)$ for $du$-almost all $u \in (0,1)$. In the same way, $F^{-1}_{\nu}(\Psi_{n+}(1) u) \to F^{-1}_{\nu}(\Psi_{n+}(1) u)$ for $du$-almost all $u \in (0,1).$
and $F^{-1}_p(\Psi^{-1}_n(\Psi_n(1)u)) \to F^{-1}_p(\Psi^{-1}_n(\Psi(1)u))$ for $du$-almost all $u \in (0, 1)$. Therefore, by the dominated convergence theorem,

$$
\int_{(0,1)} \frac{(F^{-1}_p(u) - F^{-1}_p(u))^+}{F^{-1}_p(\varphi_n(u)) - F^{-1}_p(\varphi_n(u))} (h(F^{-1}_p(u), F^{-1}_p(\varphi_n(u))) - h(F^{-1}_p(u), F^{-1}_p(u))) \, du
$$

$$
\xrightarrow{n \to +\infty} \int_{(0,1)} \frac{h(F^{-1}_p(\Psi^{-1}_n(\Psi_n(1)v)), F^{-1}_p(\Psi^{-1}_n(\Psi_n(1)v))) - h(F^{-1}_p(\Psi^{-1}_n(\Psi_n(1)v)), F^{-1}_p(\Psi^{-1}_n(\Psi_n(1)v))))}{\Psi^{-1}_n(\Psi_n(1)v)) - F^{-1}_p(\Psi^{-1}_n(\Psi_n(1)v)))} \, dv
$$

$$
= \int_{(0,1)} \frac{(F^{-1}_p(u) - F^{-1}_p(u))^+}{F^{-1}_p(\varphi(u)) - F^{-1}_p(\varphi(u))} (h(F^{-1}_p(u), F^{-1}_p(\varphi(u))) - h(F^{-1}_p(u), F^{-1}_p(u))) \, du.
$$

We can show in the same way that

$$
\int_{(0,1)} \frac{(F^{-1}_p(u) - F^{-1}_p(u))^-}{F^{-1}_p(\varphi_n(u)) - F^{-1}_p(\varphi_n(u))} (h(F^{-1}_p(u), F^{-1}_p(\varphi_n(u))) - h(F^{-1}_p(u), F^{-1}_p(u))) \, du
$$

$$
\xrightarrow{n \to +\infty} \int_{(0,1)} \frac{h(F^{-1}_p(u), F^{-1}_p(\varphi(u))) - h(F^{-1}_p(u), F^{-1}_p(u)))}{F^{-1}_p(\varphi(u)) - F^{-1}_p(\varphi(u))} \, du.
$$

Finally, we showed that

$$
\int_{\mathbb{R} \times \mathbb{R}} h(x, y) \mu_n(dx) m_n(x, dy) \xrightarrow{n \to +\infty} \int_{\mathbb{R} \times \mathbb{R}} h(x, y) \mu(dx) m(dy),
$$

for any bounded and Lipschitz continuous function $h : \mathbb{R} \to \mathbb{R}$, that is $M_n \xrightarrow{n \to +\infty} M$ for the weak convergence topology. Since the convergence for the Wasserstein distance topology is equivalent to the convergence for the weak convergence topology and the convergence of the first order moments (see for instance Theorem 4.6 Chapter 6 [11]), $\int_{\mathbb{R}} |x| \mu_n(dx) \xrightarrow{n \to +\infty} \int_{\mathbb{R}} |x| \mu(dx)$ and $\int_{\mathbb{R}} \nu_n(dy) \xrightarrow{n \to +\infty} \int_{\mathbb{R}} \nu(dy)$. Therefore, $\int_{\mathbb{R} \times \mathbb{R}} |(x, y)| M_n(dx, dy) \xrightarrow{n \to +\infty} \int_{\mathbb{R} \times \mathbb{R}} |(x, y)| M(dx, dy)$, hence $W_1(M_n, M) \xrightarrow{n \to +\infty} 0$. \qed

3 Corresponding super and submartingale couplings

Let us begin with $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ such that $\mu \leq_{d_{\mathbb{R}}} \nu$. According to Theorem 4.4.3 Chapter 4 [9], $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ are such that $\mu \leq_{d_{\mathbb{R}}} \nu$ iff for all $u \in [0, 1]$, $\int_{0}^{u} F^{-1}_p(v) \, dv \geq \int_{0}^{u} F^{-1}_p(v) \, dv$. This implies that for all $u \in [0, 1], \Psi_+(u) \geq \Psi_-(u)$. Let then $u_0 = \tilde{\Psi}(1) = \Psi^{-1}_n(\Psi(1))$. Since $\Psi_+$ is continuous, $\Psi_+(u_0) = \Psi_-(1)$. If $u_0 = 1$, then $\Psi_+(1) = \Psi_-(1)$ so $\mu$ and $\nu$ have equal means and $\mu \leq_{d_{\mathbb{R}}} \nu$, so we refer to Section 1 for the construction of a family of martingale couplings. If $u_0 = 0$, then $\Psi_+(1) = 0$, which implies that for all $u \in (0, 1), F^{-1}_p(u) \geq F^{-1}_p(u)$. So by the inverse transform sampling, $(F^{-1}_p(U), F^{-1}_p(V))$ is a supermartingale coupling between $\mu$ and $\nu$, where $U$ is a random variable uniformly distributed on $(0, 1)$. We suppose then $u_0 \in (0, 1)$. Let $\gamma = \int_{0}^{u_0} (F^{-1}_p(u) - F^{-1}_p(u))^+ \, du = \int_{0}^{u_0} (F^{-1}_p(u) - F^{-1}_p(u))^- (u) \, du$. We note $Q^d$ the set of probability measures $Q^d$ on $(0, u_0) \times (0, 1)$ such that

(i) $Q^d$ has first marginal $\frac{1}{\gamma^d}(F^{-1}_p(u) - F^{-1}_p(u))^+ \, du$;

(ii) $Q^d$ has second marginal $\frac{1}{\gamma^d}(F^{-1}_p(u) - F^{-1}_p(u))^+ \, du$;

(iii) $Q^d \{(u, v) \in (0, u_0) \times (0, 1) \mid u < v\} = 1$.

The existence of the inverse transform supermartingale coupling introduced below implies that $Q^d$ is non-empty. Let $Q^d$ be an element of $Q^d$. Let $\pi^d$ and $\pi^d_+$ be two Markov kernels such that

$$
Q(du, dv) = \frac{1}{\gamma^d}(F^{-1}_p(u) - F^{-1}_p(u))^+ \, du \pi^d (u, dv) = \frac{1}{\gamma^d}(F^{-1}_p(u) - F^{-1}_p(u))^+ \, du \pi^d_+(v, du).
$$

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Let \((\tilde{m}^d(u, dy))_{u \in (0,1)}\) be the Markov kernel defined by

\[
\begin{align*}
&\left\{ \begin{array}{ll}
\int_{(0,1)} \frac{F_\mu^{-1}(u) - F_\nu^{-1}(u)}{F_\mu^{-1}(v) - F_\nu^{-1}(u)} \delta_{F_\mu^{-1}(u)}(dy) \pi_\mu^d(u, dv) + 
\int_{(0,1)} \frac{F_\nu^{-1}(v) - F_\mu^{-1}(u)}{F_\mu^{-1}(v) - F_\nu^{-1}(u)} \pi_\nu^d(u, dv) \delta_{F_\mu^{-1}(u)}(dy) \\
\end{array} \right.
\end{align*}
\]

if \(u < u_0\), \(u \in \mathcal{U}_-\) and \(F_\nu^{-1}(v) > F_\mu^{-1}(u)\), \(\pi_\mu^d(u, dv)\)-a.e.;

\[
\int_{(0,u_0)} \frac{F_\mu^{-1}(u) - F_\nu^{-1}(u)}{F_\mu^{-1}(v) - F_\nu^{-1}(u)} \delta_{F_\mu^{-1}(u)}(dy) \pi_\mu^d(u, dv) + 
\int_{(u_0, 1)} \frac{F_\nu^{-1}(v) - F_\mu^{-1}(u)}{F_\mu^{-1}(v) - F_\nu^{-1}(u)} \pi_\nu^d(u, dv) \delta_{F_\mu^{-1}(u)}(dy)
\]

if \(u \in \mathcal{U}_+\) and \(F_\nu^{-1}(v) < F_\mu^{-1}(u)\), \(\pi_\mu^d(u, dv)\)-a.e.;

\[
\delta_{F_\mu^{-1}(u)}(dy)
\]

otherwise.

Let \((m^d(x, dy))_{x \in \mathbb{R}}\) be the Markov kernel defined as in (1.2) with \((\tilde{m}(u, dy))_{u \in (0,1)}\) replaced by \((\tilde{m}^d(u, dy))_{u \in (0,1)}\).

With the very same arguments that in Section 1, we show that \(M^d(dx, dy) = \mu(dx) m^d(x, dy)\) is a coupling between \(\mu\) and \(\nu\). Moreover, for \(du\)-almost all \(u \in (0, u_0) \cap \mathcal{U}_-\) and for \(du\)-almost all \(u \in \mathcal{U}_+\), \(\int_{\mathbb{R}} y \tilde{m}^d(u, dy) = F_\mu^{-1}(u)\). Otherwise, for \(du\)-almost all \(u \in ((0, u_0) \cap \mathcal{U}_-) \cup \mathcal{U}_+\), \(\int_{\mathbb{R}} y \tilde{m}^d(u, dy) = F_\mu^{-1}(u) \leq F_\nu^{-1}(u)\). So for \(du\)-almost all \(u \in (0,1)\), \(\int_{\mathbb{R}} y \tilde{m}^d(u, dy) \leq F_\mu^{-1}(u)\), hence \(\int_{\mathbb{R}} y \tilde{m}(x, dy) \leq x\), \(\mu(dx)\)-almost everywhere on \(\mathbb{R}\).

Therefore, for all \(Q^d \in \mathcal{Q}^d\), \(M^d(dx, dy)\) is a supermartingale coupling between \(\mu\) and \(\nu\).

In particular, let \(Q^d\) be the probability measure defined on \((0, u_0) \times (0,1)\) by

\[
Q^d(du, dv) = \frac{1}{\gamma^d}(F_\nu^{-1}(v) - F_\mu^{-1}(u)) + du \pi_\mu^d(u, dv),
\]

where \(\pi_\mu^d(u, dv) = \mathbb{1}_{(0, \varphi(u) < u_0)} \delta_{\varphi(u)}(dv) + \mathbb{1}_{\varphi(u) \in (0,1)} dv\). Using Lemma 4.5, we have \(\int_0^{u_0} h(\varphi(u)) d\Psi_+(u) = \int_0^1 h(v) d\Psi_+(v)\) for any measurable and bounded function \(h : [0,1] \to \mathbb{R}\), which is the key property to show that \(Q^d \in \mathcal{Q}^d\). Moreover, \(Q^d(du, dv) = \frac{1}{\gamma^d}(F_\mu^{-1}(u) - F_\nu^{-1}(v)) + dv \pi_\mu^d(v, du)\), where \(\pi_\mu^d(v, du) = \mathbb{1}_{(0, \varphi(v) < u_0)} \delta_{\varphi(v)}(du) + \mathbb{1}_{\varphi(v) \in (0, u_0)} du\). For this particular \(Q^d \in \mathcal{Q}^d\), Definition (3.1) of the Markov kernel \((\tilde{m}^d(u, dy))_{u \in (0,1)}\) becomes

\[
\begin{align*}
&\left\{ \begin{array}{ll}
\left(1 - \frac{F_\mu^{-1}(u) - F_\nu^{-1}(u)}{F_\mu^{-1}(\varphi(u)) - F_\nu^{-1}(u)}\right) \delta_{F_\mu^{-1}(u)}(dy) + 
\frac{F_\mu^{-1}(u) - F_\nu^{-1}(u)}{F_\mu^{-1}(\varphi(u)) - F_\nu^{-1}(u)} \delta_{F_\mu^{-1}(\varphi(u))}(dy) \\
\end{array} \right.
\end{align*}
\]

if \(u < u_0\), \(F_\mu^{-1}(\varphi(u)) > F_\mu^{-1}(u) > F_\nu^{-1}(u)\) and \(\varphi(u) < 1\);

\[
\begin{align*}
&\left\{ \begin{array}{ll}
\left(1 - \frac{F_\nu^{-1}(u) - F_\mu^{-1}(u)}{F_\nu^{-1}(\varphi(u)) - F_\mu^{-1}(u)}\right) \delta_{F_\nu^{-1}(u)}(dy) + 
\frac{F_\nu^{-1}(u) - F_\mu^{-1}(u)}{F_\nu^{-1}(\varphi(u)) - F_\mu^{-1}(u)} \delta_{F_\nu^{-1}(\varphi(u))}(dy) \\
\end{array} \right.
\end{align*}
\]

if \(F_\nu^{-1}(\varphi(u)) < F_\mu^{-1}(u) < F_\nu^{-1}(u)\) and \(\varphi(u) < u_0\);

\[
\delta_{F_\mu^{-1}(u)}(dy)
\]

otherwise.

Then \(M^d(dx, dy) = \mu(dx) m^d(x, dy)\) is a supermartingale coupling, called the inverse transform supermartingale coupling.

If \(\mu, \nu \in \mathcal{P}_1(\mathbb{R})\) are such that \(\mu \preceq_{\text{lex}} \nu\), denoting by \(\overline{\mu}\) and \(\overline{\nu}\) the respective images of \(\mu\) and \(\nu\) by \(x \mapsto -x\), one can easily see that \(\mu \preceq_{\text{lex}} \nu\) if \(\overline{\mu} \preceq_{\text{dcs}} \overline{\nu}\). So if \(M^d(dx, dy)\) denotes a supermartingale coupling between \(\overline{\mu}\) and \(\overline{\nu}\), then the image of \(M^d(dx, dy)\) by \((x, y) \mapsto (-x, -y)\) is a supermartingale coupling between \(\mu\) and \(\nu\). In particular, the image of the inverse transform supermartingale coupling by \((x, y) \mapsto (-x, -y)\) is a supermartingale coupling between \(\mu\) and \(\nu\).
4 Appendix

**Lemma 4.1.** Let $I \subset \mathbb{R}$ be an interval, $F : I \to \mathbb{R}$ a bounded and nondecreasing càdlàg function and $F^{-1}$ its left continuous pseudo-inverse, that is

$$F^{-1} : u \in F(I) \mapsto \inf\{r \in I \mid F(r) \geq u\}$$

Then $F^{-1}(F(x)) = x$, $dF(x)$-almost everywhere on $I$.

**Proof.** Let $a = \inf F(I)$ and $b = \sup F(I)$. If $a = b$, then $dF(x)$ is the trivial measure on $I$ so the statement is straightforward. Else, let $G : I \to [0,1]$ defined for all $x \in I$ by $G(x) = (F(x) - a)/(b - a)$. It is well known that for all $u \in (0,1)$, $G^{-1}(G(G^{-1}(u))) = G^{-1}(u)$. So $G^{-1}(G(G^{-1}(U))) = G^{-1}(U)$, where $U$ is a random variable uniformly distributed on $[0,1]$. By the inverse transform sampling, it implies that $G^{-1}(G(x)) = x$, $dG(x)$-almost everywhere on $I$. For all $u \in F(I)$, we have $F^{-1}(u) = G^{-1}((u - a)/(b - a))$, hence $F^{-1}(F(x)) = G^{-1}(G(x)) = x$, $dG(x)$-almost everywhere on $I$. Since $dG(x) = \frac{1}{b-a}dF(x)$, $dF(x)$ and $dF(x)$ are equivalent, so $F^{-1}(F(x)) = x$, $dF(x)$-almost everywhere on $I$. 

**Lemma 4.2.** Let $\mu \in \mathcal{P}(\mathbb{R})$. Then $F_\mu(x) > 0$ and $F_\mu(x_-) < 1$, $\mu(dx)$-almost everywhere on $\mathbb{R}$.

**Proof.** If $\{x \in \mathbb{R} \mid F_\mu(x) = 0\}$ is nonempty, then it is an interval of the form $(-\infty,a]$ or $(-\infty,a)$, depending on whether $F_\mu(a) = 0$ or not. If $F_\mu(a) = 0$, then $\mu(\{x \in \mathbb{R} \mid F_\mu(x) = 0\}) = \mu((-\infty,a)) = F_\mu(a) = 0$. Else, since for all $x < a$, $F_\mu(x) = 0$, then $\mu(\{x \in \mathbb{R} \mid F_\mu(x) = 0\}) = \mu((-\infty,a)) = F_\mu(a) = 0$.

If $\{x \in \mathbb{R} \mid F_\mu(x_-) = 1\}$ is nonempty, then it is an interval of the form $[a,\infty)$ or $(a,\infty)$, depending whether $F_\mu(a_-) = 1$ or not. If $F_\mu(a_-) = 1$, then $\mu(\{x \in \mathbb{R} \mid F_\mu(x_-) = 1\}) = \mu([a,\infty)) = 1 - F_\mu(a_-) = 0$. Else, since for all $x > a$, $F_\mu(x_-) = 1$, then $\mu(\{x \in \mathbb{R} \mid F_\mu(x_-) = 1\}) = \mu((a,\infty)) = 1 - F_\mu(a) = 1 - \lim_{x \to a, x > a} F_\mu(x_-) = 0$, by right continuity of $F_\mu$. 

**Lemma 4.3.** Let $\mu \in \mathcal{P}_1(\mathbb{R})$. Then $\mu$ is symmetric with mean $\alpha \in \mathbb{R}$ iff

$$F_\mu^{-1}(u_+) = 2\alpha - F_\mu^{-1}(1 - u),$$

for all $u \in (0,1)$. In that case, $F_\mu^{-1}(u) = 2\alpha - F_\mu^{-1}(1 - u)$ for $u \in (0,1)$ up to the at most countable set of discontinuities of $F_\mu^{-1}$.

**Proof.** Let $U$ be a random variable uniformly distributed on $[0,1]$. Then, by the inverse transform sampling, $F_\mu^{-1}(1 - U) \sim \mu$, so $2\alpha - F_\mu^{-1}(1 - U) \sim \mu$ since $\mu$ is symmetric with mean $\alpha$. Since $u \mapsto 2\alpha - F_\mu^{-1}(1 - u)$ is nondecreasing, then one can show that $2\alpha - F_\mu^{-1}(1 - u) = F_\mu^{-1}(u)$, $du$-almost everywhere on $(0,1)$. Indeed, as shown in [1], for all $u,q \in (0,1)$ such that $q < u$, if $F_\mu^{-1}(u) < 2\alpha - F_\mu^{-1}(1 - q)$, then

$$\mathbb{P}(2\alpha - F_\mu^{-1}(1 - U) \leq F_\mu^{-1}(u)) \leq q < u \leq \mathbb{P}(F_\mu^{-1}(U) \leq F_\mu^{-1}(u)) = \mathbb{P}(2\alpha - F_\mu^{-1}(1 - U) \leq F_\mu^{-1}(u)),$$

which is contradictory, so $F_\mu^{-1}(u) \geq \sup_{q \in (0,u)} (2\alpha - F_\mu^{-1}(1 - q))$. By symmetry, $2\alpha - F_\mu^{-1}(1 - u) \geq \sup_{q \in (0,u)} F_\mu^{-1}(q) = F_\mu^{-1}(u)$ by left-continuity and monotonicity of $F_\mu^{-1}$. Since $F_\mu^{-1}$ has an at most countable set of discontinuities, then for $du$-almost all $u \in (0,1)$, $2\alpha - F_\mu^{-1}(1 - u) = \sup_{q \in (0,u)} (2\alpha - F_\mu^{-1}(1 - u)) \leq F_\mu^{-1}(u) \leq 2\alpha - F_\mu^{-1}(1 - u)$. Therefore, $2\alpha - F_\mu^{-1}(1 - u) = F_\mu^{-1}(u_+)$, $du$-almost everywhere on $(0,1)$ and even everywhere on $[0,1)$ since both sides are right-continuous. 

**Lemma 4.4.** Let $\mu \in \mathcal{P}(\mathbb{R})$, let $X : \Omega \to \mathbb{R}$ be a random variable with distribution $\mu$ and let $V$ be a random variable independent from $X$ and uniformly distributed on $[0,1]$. Let $W : \Omega \to \mathbb{R}$ be the random variable defined by

$$W = F_\mu(X_-) + V(F_\mu(X) - F_\mu(X_-)).$$

Then $W$ is uniformly distributed on $[0,1]$, and $F_\mu^{-1}(W) = X$, $\mathbb{P}$-almost surely.
Proof. Let \( f: \mathbb{R} \to \mathbb{R} \) be a measurable and bounded function. Then

\[
\mathbb{E}[f(W)] = \mathbb{E}[f(F_\mu(X_-) + V(F_\mu(X) - F_\mu(X_-)))] = \int_0^1 \int_{\mathbb{R}} f(F_\mu(x_-) + v(F_\mu(x) - F_\mu(x_-))) \mu(dx) \, dv
\]

\[
= \int_0^1 \int_{\mathbb{R}} \mathbb{1}_{\{\mu(x) = 0\}} f(F_\mu(x_-) + v(F_\mu(x) - F_\mu(x_-))) \mu(dx) \, dv
\]

\[
\quad + \int_0^1 \int_{\mathbb{R}} \mathbb{1}_{\{\mu(x) > 0\}} f(F_\mu(x_-) + v(F_\mu(x) - F_\mu(x_-))) \mu(dx) \, dv
\]

\[
= \int_0^1 \int_{\mathbb{R}} \mathbb{1}_{\{\mu(x) = 0\}} f(F_\mu(x)) \mu(dx) + \sum_{x \in \mathbb{R}, \mu(x) > 0} \int_{F_\mu(x)}^1 f(v) \, dv
\]

\[
= \int_0^1 \int_{\mathbb{R}} \mathbb{1}_{\{\mu(F^{-1}_\mu(u)) = 0\}} f(F_\mu(F^{-1}_\mu(u))) \, du + \sum_{x \in \mathbb{R}, \mu(x) > 0} \int_{F_\mu(x)}^1 f(v) \, dv
\]

where we used for the last but one equality the inverse transform sampling, and for the last equality the fact that \( F_\mu(F^{-1}_\mu(u)) = u \) if \( F_\mu \) is continuous in \( F^{-1}_\mu(u) \). One can easily see that for all \( x \in \mathbb{R} \) and \( u \in (0,1) \),

\[
F_\mu(x_-) < u \leq F_\mu(x) \implies x = F^{-1}_\mu(u) \implies F_\mu(x_-) \leq u \leq F_\mu(x),
\]

which implies

\[
\bigcup_{x \in \mathbb{R}, \mu(x) > 0} [F_\mu(x_-), F_\mu(x)] \subset \{ u \in [0,1] \mid \mu(\{F^{-1}_\mu(u)\}) > 0 \} \subset \bigcup_{x \in \mathbb{R}, \mu(x) > 0} [F_\mu(x_-), F_\mu(x)],
\]

so

\[
\sum_{x \in \mathbb{R}, \mu(x) > 0} \int_{F_\mu(x)}^1 f(v) \, dv = \int_0^1 \mathbb{1}_{\{\mu(F^{-1}_\mu(u)) = 0\}} f(u) \, du.
\]

Therefore, \( \mathbb{E}[f(W)] = \int_0^1 \mathbb{1}_{\{\mu(F^{-1}_\mu(u)) = 0\}} f(u) + \int_0^1 \mathbb{1}_{\{\mu(F^{-1}_\mu(u)) > 0\}} f(u) = \int_0^1 f(u) \, du \), so \( W \) is uniformly distributed on \([0,1]\). Moreover, if \( F_\mu(X_-) = F_\mu(X) \), then \( W = F_\mu(X) \) and by Lemma 4.1, \( F^{-1}_\mu(W) = X \) \( \mathbb{P} \)-almost surely. Since \( \mathbb{P} \)-a.s., \( V > 0 \), if \( F_\mu(X_-) < F_\mu(X) \), then \( \mathbb{P} \)-a.s., \( F_\mu(X_-) < W \leq F_\mu(X) \) so \( F^{-1}_\mu(W) = X \).

\[\square\]

**Lemma 4.5.** Let \( f_1, f_2 : (0,1) \to \mathbb{R} \) be two nonnegative and integrable functions such that \( \int_0^1 f_1(u) \, du = \int_0^1 f_2(u) \, du \). Let \( \Psi_1 : [0,1] \ni u \mapsto \int_0^u f_1(v) \, dv \), \( \Psi_2 : [0,1] \ni u \mapsto \int_0^u f_2(v) \, dv \) and \( \Gamma = \Psi_2^{-1} \circ \Psi_1 \) where \( \Psi_2^{-1} \) denotes the càg pseudo-inverse of \( \Psi_2 \). Then for any measurable and bounded function \( h : [0,1] \to \mathbb{R} \),

\[
\int_0^1 h(\Gamma(u)) f_1(u) \, du = \int_0^1 h(v) f_2(v) \, dv.
\]

**Proof.** Let \( h : [0,1] \to \mathbb{R} \) be a measurable and bounded function. Since \( \Psi_1 \) is nondecreasing and continuous, using Proposition 4.10 Chapter 0 [8], we have

\[
\int_0^1 h(\Gamma(u)) f_1(u) \, du = \int_0^1 h(\Psi_2^{-1}(\Psi_1(u))) \, d\Psi_1(u) = \int_0^{\Psi_1(1)} h(\Psi_2^{-1}(u)) \, du.
\]

Since \( \int_0^1 f_1(u) \, du = \int_0^1 f_2(u) \, du \), we have \( \Psi_1(1) = \Psi_2(1) \), and since \( \Psi_2 \) is nondecreasing and continuous, using once again Proposition 4.10 Chapter 0 [8], we have

\[
\int_0^{\Psi(1)} h(\Psi_2^{-1}(w)) \, dw = \int_0^{\Psi(1)} h(\Psi_2^{-1}(w)) \, dw = \int_0^1 h(\Psi_2^{-1}(\Psi_2(v))) \, d\Psi_2(v),
\]

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By Lemma 4.1, \( \Psi^{-1}_2(\Psi_2(v)) = v \), \( d\Psi_2(v) \)-almost everywhere on \((0,1)\), so
\[
\int_0^1 h(\Psi^{-1}_2(\Psi_2(v))) d\Psi_2(v) = \int_0^1 h(v) f_2(v) dv.
\]

References

[1] A. Alfonsi, J. Corbetta, and B. Jourdain. Sampling of probability measures in the convex order and approximation of Martingale Optimal Transport problems. ArXiv e-prints:1709.05287, Sept. 2017.

[2] M. Beiglböck and N. Juillet. On a problem of optimal transport under marginal martingale constraints. ArXiv e-prints:1208.1509, Aug. 2012.

[3] H. Föllmer and A. Schied. Stochastic Finance: An Introduction in Discrete Time. De Gruyter, 2011.

[4] P. Henry-Labordère and N. Touzi. An explicit martingale version of the one-dimensional Brenier theorem. Finance and Stochastics, 20(3):635–668, Jul 2016.

[5] D. Hobson and M. Klimmek. Robust price bounds for the forward starting straddle. Finance and Stochastics, 19(1):189–214, Jan 2015.

[6] D. Hobson and A. Neuberger. Robust bounds for forward start options. Mathematical Finance, 22(1):31–56.

[7] S. T. Rachev and L. Rüschendorf. Mass Transportation Problems, Volume I: Theory. Springer, 1998.

[8] D. Revuz and M. Yor. Continuous Martingales and Brownian Motion. Springer, 1999.

[9] M. Shaked and J. G. Shanthikumar. Stochastic Orders. Springer, 2007.

[10] C. Villani. Topics in Optimal Transportation. American Mathematical Society, 2003.

[11] C. Villani. Optimal Transport, Old and New. Springer, 2009.