Automorphism group of plane curve computed by Galois points

Kei Miura · Akira Ohbuchi

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Abstract Let $C \subset \mathbb{P}^2$ be a smooth plane curve, and $P_1, \ldots, P_m$ be all inner and outer Galois points for $C$. Each Galois point $P_i$ determines a Galois group at $P_i$, say $G_{P_i}$. Then, by the definition of Galois point, an element of the Galois group $G_{P_i}$ induces a birational transformation of $C$. In fact, we see that it becomes an automorphism of $C$. We call this an automorphism belonging to the Galois point $P_i$. Then, we consider the group $G(C)$ generated by automorphisms belonging to all Galois points for $C$. In particular, we investigate the difference between $\text{Aut}(C)$ and $G(C)$, so that we determine the structure of $\text{Aut}(C)$.

Keywords Galois point · Plane curve · Birational transformation · Automorphism group

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1 Introduction

This article is a sequel to Kanazawa et al. (2001). We use the same notation and definitions as in Kanazawa et al. (2001) and Miura and Yoshihara (2000). Let us recall several definitions in brief.
Let $k$ be an algebraically closed field of characteristic zero. We fix it as a ground field of our discussions. Let $C$ be a smooth curve in $\mathbb{P}^2$ of degree $d$ ($d \geq 4$) and $K = k(C)$ be the function field of $C$. Let $P$ be a point of $\mathbb{P}^2$. We consider the morphism $\pi_P : C \to \mathbb{P}^1$, which is the restriction of the projection $\mathbb{P}^2 \to \mathbb{P}^1$ with the center $P$. Then we obtain the field extension induced by $\pi_P$, i.e., $\pi_P^* : k(\mathbb{P}^1) \hookrightarrow k(C)$. Putting $K_P = \pi_P^*(k(\mathbb{P}^1))$, we have the following definition.

**Definition 1** The point $P$ is called a Galois point for $C$ if the field extension $k(C)/K_P$ is Galois. Furthermore, we say that a Galois point $P$ is inner (resp. outer) if $P \in C$ (resp. $P \in \mathbb{P}^2 \setminus C$). Then we put $G_P = \text{Gal}(k(C)/K_P)$ and call a Galois group at $P$.

We denote by $\delta(C)$ (resp. $\delta'(C)$) the number of inner (resp. outer) Galois points for $C$.

Suppose that $P_1, P_2, \ldots, P_m$ are all Galois points for $C$. Let $\sigma_i$ be an element of $G_{P_i}$ ($1 \leq i \leq m$). Then, every $\sigma_i$ induces a birational transformation of $C$ over $\mathbb{P}^1$, we denote it by the same letter $\sigma_i$. We call $\sigma_i$ a birational transformation belonging to the Galois point $P_i$.

About the birational transformation belonging to Galois point, we recall the following theorem.

**Theorem A** (Yoshihara 2001) Let $C$ be a smooth curve in $\mathbb{P}^2$ of degree $d$ ($d \geq 4$). Then, every element $\sigma \in G_P$ is a restriction of a projective transformation of $\mathbb{P}^2$. That is, there exists $\tilde{\sigma} \in \text{Aut}(\mathbb{P}^2) = \text{PGL}(2, k)$ such that $\tilde{\sigma}|_C = \sigma$. Furthermore, $G_P$ is a cyclic group.

**Remark 1** If $C$ is not smooth or $d = 3$, then this theorem does not hold true. In the case where $d = 3$, we have studied about it in Miura (2013).

Our purpose of this paper is to investigate the Galois groups at all Galois points for $C$. More precisely, our problem is stated as follows.

**Definition 2** Let $G(C)$ denote the group generated by the automorphisms belonging to the Galois points for $C$. Namely,

$$G(C) := \langle G_{P_1}, G_{P_2}, \ldots, G_{P_m} \rangle \subset \text{Aut}(C).$$

Therefore, we examine the group $G(C)$, in particular, we shall investigate the difference between $\text{Aut}(C)$ and $G(C)$. For example, if $\text{Aut}(C) = G(C)$, then we may say that the Galois points play important roles for the automorphism group of $C$.

The group $G(C)$ was introduced by Kanazawa, Takahashi and Yoshihara in (2001). They defined $G(C)$ for only inner Galois points and studied the case where $C$ is the quartic curve with the maximal number of inner Galois points. We will recall their results in the following section. Our purpose of this paper is to extend the definition $G(C)$ for all (i.e. inner and outer) Galois points for $C$ and to investigate $G(C)$ in detail. Consequently, we shall determine the structure of $\text{Aut}(C)$.

In the case where $d = 4$, we have obtained the results in Kanazawa et al. (2001) (see Example 1 below). Hereafter, we assume that $d \geq 5$. Our results are stated as follows.
Theorem 1 Suppose that $\delta(C) \geq 1$ or $\delta'(C) \geq 1$. Then we have the following.

(i) The group $G(C)$ is a normal subgroup of $\text{Aut}(C)$.
(ii) For general $C$, we have $G(C) = \text{Aut}(C)$.
(iii) If $G(C) \neq \text{Aut}(C)$, then the quotient group $\text{Aut}(C)/G(C)$ is isomorphic to one of the following: $C_n$, the cyclic group of order $n$; $D_{2n}$, the dihedral group of order $2n$; $A_4$, the alternating group on four letters; $S_4$, the symmetric group on four letters; or $A_5$, the alternating group on five letters.

2 Examples

Before starting to prove our main result, we mention some examples. First, we recall the result of Kanazawa, Takahashi and Yosihara in (2001).

Example 1 Let $C(4)$ be the curve defined by $YZ^3 + X^4 + Y^4 = 0$, where $(X : Y : Z)$ is the system of homogeneous coordinates of $\mathbb{P}^2$. It is known that $\delta(C) \leq 4$ for smooth plane curve $C$. Especially, the curve with $\delta(C) = 4$ is unique, it is projectively equivalent to $C(4)$ (cf. Yosihara (2001), Proposition 5).

In Kanazawa et al. (2001), they studied $G(C(4))$ for only inner Galois points. Here, we denote it by $G_{in}(C(4))$ to distinguish from our notation. Their result is stated as follows.

Theorem B (Kanazawa et al. 2001) There exist exact sequences of groups

$$1 \longrightarrow \langle \tau \rangle \longrightarrow G_{in}(C(4)) \longrightarrow A_4 \longrightarrow 1,$$

$$1 \longrightarrow \langle \rho \rangle \longrightarrow \text{Aut}(C(4)) \longrightarrow A_4 \longrightarrow 1,$$

where $\tau, \rho \in \text{Aut}(C(4))$ and $\tau = \rho^2, \rho^4 = 1$.

It is easy to check that $C(4)$ has an exactly one outer Galois point $Q = (1 : 0 : 0)$, and $G_Q$ is isomorphic to the cyclic group of order 4. In fact, we get $G_Q = \langle \rho \rangle$. Hence, our $G(C(4))$ can be obtained by adding the information of $G_Q$ to $G_{in}(C(4))$, that is, $G(C(4)) = \langle G_{in}(C(4)), \rho \rangle$. Referring to Klassen and Schaefer (1996), Kuribayashi and Komiya (1979), we have facts on $\text{Aut}(C(4))$, in particular, its order is 48. Since $G_{in}(C(4)) \subset \langle G_{in}(C(4)), \rho \rangle \subset \text{Aut}(C(4))$, by above exact sequences, we can check easily that $G(C(4))$ must be equal to $\text{Aut}(C(4))$.

Example 2 Next, we mention the Fermat curve $F(d) : X^d + Y^d + Z^d = 0$. We have known that $\delta'(C) \leq 3$ for smooth plane curve $C$. Especially, the curve with $\delta'(C) = 3$ is projectively equivalent to $F(d)$ (cf. Yosihara (2001), Proposition 5’). Referring to Tzermias (1995), we have the following fact on $\text{Aut}(F(d))$. The automorphism group of $F(d)$ is the semidirect product of the symmetric group on three letters $S_3$ and the direct sum of two copies of the cyclic group of order $d$. Indeed, we can view the elements of $S_3$ as permutation of three letters $X, Y, Z$. Also let $\mu(d)$ be the group of $d$-th root of unity and $\xi$ the generator of it. Then, we can view $\mu(d) \times \mu(d) \ni (\xi^i, \xi^j)$ as the automorphism given by $(X : Y : Z) \mapsto (\xi^iX : \xi^jY : Z), 0 \leq i, j \leq d - 1$. 
On the other hand, we have $\delta(F(d)) = 0$, $\delta'(F(d)) = 3$ (cf. Miura and Yoshihara 2000; Yoshihara 2001). Indeed, the three Galois points are $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, $P_3 = (0 : 0 : 1)$, and each Galois group $G_{P_i}$ is isomorphic to the cyclic group of order $d$. Let $\sigma_i$ be a generator of $G_{P_i}$. So, we obtain the representation as matrix as follows,

$$
\sigma_1 = \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \xi & 1 \\ 1 & \xi \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} \xi & \xi \\ \xi & 1 \end{pmatrix}.
$$

Therefore the group $G(F(d))$ is given by

$$
G(F(d)) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle = \begin{pmatrix} \xi^i \\ \xi^j \\ 1 \end{pmatrix}, \quad i, j = 0, 1, \ldots, d - 1.
$$

This is nothing but $\mu(d) \times \mu(d)$. Hence, we conclude that

$$
\text{Aut}(F(d))/G(F(d)) \cong S_3.
$$

**Remark 2** We conjecture that the similar result may hold true for higher dimensional Fermat variety.

**Example 3** Let $C(d)$ be the curve defined by $YZ^d - 1 + X^d + Y^d = 0$. This is a generalization of $C(4)$ as in Example 1. So, we assume $d \geq 5$. Referring to [Fukasawa (2010), Theorem 3], if $\delta(C) \geq 1$ and $\delta'(C) \geq 1$, then $C$ is projectively equivalent to $C(d)$. Indeed, we can check $\delta(C(d)) = 1$ and $\delta'(C(d)) = 1$. Then, $P = (0 : 0 : 1)$ is an inner Galois point, and $Q = (1 : 0 : 0)$ is an outer one, whose Galois groups are isomorphic to the cyclic group of order $d - 1$ and $d$, respectively. We put $G_P = \langle \sigma \rangle$ and $G_Q = \langle \tau \rangle$. Then, we obtain

$$
\sigma = \begin{pmatrix} \eta \\ \eta \\ 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} \xi^{d-1} \\ 1 \\ 1 \end{pmatrix},
$$

where $\eta$ (resp. $\xi$) is a primitive $(d - 1)$-th (resp. $d$-th) root of unity. So the group $G(C(d))$ is given by

$$
G(C(d)) = \langle \sigma, \tau \rangle = \{ \sigma^i \tau^j | i = 0, 1, \ldots, d - 2, \quad j = 0, 1, \ldots, d - 1 \}.
$$

In particular, $\sharp G(C(d)) = (d - 1)d$.

On the contrary, referring to Kontogeorgis (1998), we have $\sharp \text{Aut}(C(d)) = (d - 1)d$. Therefore, we conclude that

$$
\text{Aut}(C(d)) = G(C(d)).
$$

**3 Proofs**

Now, we prove our main theorem. Let $C$ be a smooth curve in $\mathbb{P}^2$ of degree $d$ ($d \geq 5$). First, we prove Theorem 1(i). Let $P_1, \ldots, P_l$ be Galois points for $C$. Then, for any
\( \tau \in \text{Aut}(C) \), we have that \( \tau(P_i) \) is also a Galois point, since every element of \( \text{Aut}(C) \) can be extended to a projective transformation (cf. Chang 1978). That is, \( \tau(P_i) = P_j \) for some \( j \). Therefore, we obtain that \( \tau G P_i \tau^{-1} = G P_j \). Hence we see that \( \tau \{G P_1, \ldots, G P_i\} \tau^{-1} = \{G P_1, \ldots, G P_i\} \). This proves our theorem (i).

Next, we note the following fact.

**Proposition C** (Yoshihara 2001) The point \( P \) is an inner (resp. outer) Galois point for \( C \) if and only if there exists a suitable coordinates change such that \( P = (0 : 0 : 1) \) and \( C \) is given by \( YZ^d-1 + f_d(X, Y) = 0 \) (resp. \( Z^d + f_d(X, Y) = 0 \)), where \( f_d(X, Y) \) is a homogeneous polynomial of \( X \) and \( Y \) of degree \( d \).

Suppose that \( P \) is an outer Galois point for \( C \). Then, we can assume that \( P = (0 : 0 : 1) \) and \( C \) is given by \( Z^d + f_d(X, Y) = 0 \). In the case where \( C \) has an additional Galois point, we have studied in Examples 2 and 3, hence, by suitable changes of coordinates, we assume \( f_d(X, Y) \neq X^d + Y^d, X^d + XY^{d-1}, Y^d + X^{d-1}Y \). Then, we see that \( \delta(C) = 0 \) and \( \delta'(C) = 1 \).

In the case where \( \delta(C) = 1 \) and \( \delta'(C) = 0 \), we can prove our theorem by the same argument below. So, we prove the case \( \delta(C) = 0 \) and \( \delta'(C) = 1 \).

Now, we determine the type of matrix of an element of \( \text{Aut}(C) \). For any \( \varphi \in \text{Aut}(C) \), we see that \( \varphi \) must fix \( P \), otherwise \( \varphi(P) \) is also Galois point, this contradicts \( \delta'(C) = 1 \). Furthermore, \( \varphi \) must satisfy

\[
(Z^d + f_d(X, Y))^\varphi = \lambda(Z^d + f_d(X, Y)),
\]

for some \( \lambda \in k \setminus \{0\} \). Hence, \( \varphi \) can be represented as a matrix \( A_\varphi \) as follows,

\[
A_\varphi = \begin{pmatrix}
  a & b & 0 \\
  c & d & 0 \\
  0 & 0 & 1
\end{pmatrix}.
\]

Noting that \( \pi_P \) is given by \( \pi_P : (X : Y : Z) \mapsto (X : Y) \), we have

\[
(\pi_P)_*\varphi = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}.
\]

So, we obtain \( (\pi_P)_*\varphi \in \text{Aut}(\mathbb{P}^1) \). Especially, on the line \( \{Z = 0\} \cong \mathbb{P}^1 \), we have

\[
f_d(X, Y)^\varphi = \lambda f_d(X, Y),
\]

for some \( \lambda \in k \setminus \{0\} \). Let \( (\alpha_1 : \beta_1), \ldots, (\alpha_d : \beta_d) \) be the roots of \( f_d(X, Y) = 0 \). We put \( p_i := (\alpha_i : \beta_i) \).

By considering the action of \( (\pi_P)_*\varphi \) on the set \( \{p_1, p_2, \ldots, p_d\} \), we obtain the following.

(i) If \( p_1, p_2, \ldots, p_d \in \mathbb{P}^1 \) are general points, then \( (\pi_P)_*\varphi \) does not act on \( \{p_1, p_2, \ldots, p_d\} \). That is, there exists no such \( \varphi \), we conclude that \( G(C) = \text{Aut}(C) \).
(ii) If \((\pi_P)_*\varphi\) acts on \(\{p_1, p_2, \ldots, p_d\}\), then we can take orbits \(O_1, O_2, \ldots, O_s\), where \(O_i = \{(\alpha_1^{(i)} : \beta_1^{(i)}), \ldots, (\alpha_s^{(i)} : \beta_s^{(i)})\}\). Then, by putting \(f_d(X, Y) = \prod_{i=1}^s \prod_{j=1}^s (\alpha_j^{(i)} X - \beta_j^{(i)} Y)\), we get the curve \(C\) having the automorphism \(\varphi\).

Hence, we infer that there exists an exact sequence,

\[ 1 \longrightarrow G(C) \longrightarrow \text{Aut}(C) \overset{(\pi_P)_*}{\longrightarrow} G' \longrightarrow 1, \]

where \(G' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \subset \text{Aut}(\mathbb{P}^1)\). In particular, this is split, i.e., \(\text{Aut}(C)/G(C) \cong G'\). It is well known that the finite subgroup of \(\text{Aut}(\mathbb{P}^1)\) is isomorphic to one of the five types as in Theorem 1. Thus we complete the proof.

4 Example

We shall construct \(C\) where \(\text{Aut}(C)/G(C)\) is isomorphic to five groups as in Theorem 1. Concretely, we present a defining equation of \(C\) and the group \(G'\).

Example 4 \(C_n\): the cyclic group of order \(n\).

Putting

\[ f_6(X, Y) = (X - Y)(X - \omega Y)(X - \omega^2 Y)(X - 2Y)(X - 2\omega Y)(X - 2\omega^2 Y), \]

we define \(C\) by \(YZ^5 + f_6(X, Y) = 0\), where \(\omega^2 + \omega + 1 = 0\). The genus of \(C\) is 10. Then, \(P = (0 : 0 : 1)\) is an inner Galois point with \(G_P \cong C_5\). Furthermore, we put \(\varphi\) as

\[ \varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & e \end{pmatrix}, \]

where \(e^5 = \omega^2\). Then, we see that \(\varphi \in \text{Aut}(C)\). Here, we have

\[ G' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \right\} \cong C_3. \]

So, we conclude

\[ \text{Aut}(C)/G(C) \cong C_3. \]

Example 5 \(D_{2n}\): the dihedral group of order \(2n\).

We define \(C\) by \(X^{2n} + X^n Y^n + Y^{2n} + Z^{2n} = 0\). The genus of \(C\) is \((2n - 1)(n - 1)\). Then, \(P = (0 : 0 : 1)\) is an outer Galois point with \(G_P \cong C_{2n}\). Further, we can take
\( \sigma, \tau \in \text{Aut}(C) \setminus G_P, \)
\[
\sigma = \begin{pmatrix}
\xi & 0 & 0 \\
0 & \xi^{n-1} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \tau = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
where \( \xi^n = 1. \) Then we can check that \( \langle (\pi_P)_* \sigma, (\pi_P)_* \tau \rangle \cong D_{2n}. \) That is,
\[
\langle \begin{pmatrix}
\xi & 0 \\
0 & \xi^{n-1}
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \rangle \cong D_{2n}.
\]
We conclude that
\[
G' \cong D_{2n}, \quad \text{Aut}(C) / G(C) \cong D_{2n}.
\]

On the following examples (Examples 6–8), we construct the defining equation of \( C, \) by referring to Klein (1956).

**Example 6** \( A_4 : \) the alternating group on four letters.
We define \( C \) by \( 2X^{12} - 66X^8Y^4 - 66X^4Y^8 + 2Y^{12} + Z^{12} = 0. \) The genus of \( C \) is 55. Then, \( P = (0 : 0 : 1) \) is an outer Galois point with \( G_P \cong C_{12}. \) Further, we can take \( \sigma, \tau \in \text{Aut}(C) \setminus G_P, \)
\[
\sigma = \begin{pmatrix}
1 & i & 0 \\
\frac{1}{\sqrt{2i}} & \frac{i}{\sqrt{2i}} & 0 \\
0 & \frac{1}{\sqrt{2i}} & 1
\end{pmatrix}, \quad \tau = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
where \( i^2 = -1. \) Then, we can check \( \langle (\pi_P)_* \sigma, (\pi_P)_* \tau \rangle \cong A_4. \) Hence, we have
\[
G' \cong A_4, \quad \text{Aut}(C) / G(C) \cong A_4.
\]

**Example 7** \( S_4 : \) the symmetric group on four letters.
We define \( C \) by \( X^{24} + 150X^{20}Y^4 + 159X^{16}Y^8 + 3476X^{12}Y^{12} + 159X^8Y^{16} + 150X^4Y^{20} + Y^{24} + Z^{24} = 0. \) The genus of \( C \) is 253. Then, \( P = (0 : 0 : 1) \) is an outer Galois point with \( G_P \cong C_{24}. \) Further, we can take \( \sigma, \tau \in \text{Aut}(C) \setminus G_P, \)
\[
\sigma = \begin{pmatrix}
1 & i & 0 \\
\frac{1}{\sqrt{2i}} & \frac{i}{\sqrt{2i}} & 0 \\
0 & \frac{1}{\sqrt{2i}} & 1
\end{pmatrix}, \quad \tau = \begin{pmatrix}
i & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Then, we can check \( \langle (\pi_P)_* \sigma, (\pi_P)_* \tau \rangle \cong S_4. \) Therefore, we have
\[
G' \cong S_4, \quad \text{Aut}(C) / G(C) \cong S_4.
\]
Example 8  $A_5$ : the alternating group on five letters.
First, we define homogeneous polynomials $\theta_i(X, Y)$ ($i = 1, 2$) as follows,
\[
\theta_1(X, Y) = XY(X^{10} + 11X^5Y^5 - Y^{10}),
\]
\[
\theta_2(X, Y) = X^{20} - 228X^{15}Y^5 + 494X^{10}Y^{10} + 228X^5Y^{15} + Y^{20}.
\]
Then, we put $C$ as $\theta_1(X, Y)^5 + \theta_2(X, Y)^3 + Z^{60} = 0$. Note that the genus of $C$ is 1711. We see that $P = (0 : 0 : 1)$ is an outer Galois point with $G \cong C_{60}$. Further, we can take $\sigma, \tau \in \text{Aut}(C) \setminus G_P$,
\[
\sigma = \begin{pmatrix}
\xi - \xi^4 & \xi^3 - \xi^2 & 0 \\
\xi^3 - \xi^2 & -\xi + \xi^4 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \tau = \begin{pmatrix}
\xi & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
where $\xi$ is a primitive 5-th root of unity. Finally, we conclude that
\[
G' \cong A_5, \quad \text{Aut}(C)/G(C) \cong A_5.
\]

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