Sum Formula of Multiple Hurwitz-Zeta Values

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Abstract

Let $s_1, \ldots, s_d$ be $d$ positive integers and define the multiple $t$-values of depth $d$ by

$$t(s_1, \ldots, s_d) = \sum_{n_1 > \cdots > n_d \geq 1} \frac{1}{(2n_1 - 1)^{s_1} \cdots (2n_d - 1)^{s_d}},$$

which is equal to the multiple Hurwitz-zeta value $2^{-w} \zeta(s_1, \ldots, s_d; -\frac{1}{2}, \ldots, -\frac{1}{2})$ where $w = s_1 + \cdots + s_d$ is called the weight. For $d \leq n$, let $T(2n, d)$ be the sum of all multiple $t$-values with even arguments whose weight is $2n$ and whose depth is $d$. Recently Shen and Cai gave formulas for $T(2n, d)$ for $d \leq 5$ in terms of $t(2n)$, $t(2)t(2n-2)$ and $t(4)t(2n-4)$. In this short note we generalize Shen-Cai’s results to arbitrary depth by using the theory of symmetric functions established by Hoffman.

1 Introduction

In recent years multiple zeta functions and many different variations and generalizations have been studied intensively due to their close relations to other objects in a lot of diverse branches of mathematics and physics. In particular, a large number of identities are established between their special values. In [4] Shen and Cai found a few very interesting equations which are similar in nature to Euler’s identity of double zeta values. They gave formulas of the sum $E(2n, d)$ of multiple zeta values at even arguments of fixed depth $d$ and weight $2n$, for $d \leq 4$. These have been generalized to arbitrary depth by Hoffman [1]. In [3] Shen and Cai turned to the following values

$$t(s_1, \ldots, s_d) = \sum_{n_1 > \cdots > n_d \geq 1} \frac{1}{(2n_1 - 1)^{s_1} \cdots (2n_d - 1)^{s_d}},$$

which we call multiple $t$-values of depth $d$ in this note. It is clear that this is equal to $2^{-w} \zeta(s_1, \ldots, s_d; -\frac{1}{2}, \ldots, -\frac{1}{2})$ where $w = s_1 + \cdots + s_d$ is called the weight. Put

$$T(2n, d) = \sum_{j_1 + \cdots + j_d = n \atop j_1, \ldots, j_d \geq 1} t(2j_1, \ldots, 2j_d).$$
Using similar but more complicated ideas from [4] Shen and Cai gave a few sum formulas for $T(2n, d)$ for $d \leq 5$ in [3]. For example,

$$T(2n, 5) = \frac{7}{128}t(2n) - \frac{3}{64}t(2)t(2n - 2) + \frac{1}{320}t(4)t(2n - 4).$$

(1)

In this note, we shall generalize these to arbitrary depth using ideas from [1] where Hoffman applied the theory of symmetric functions to study the generating function of $E(2n, d)$. It turns out that we need both Bernoulli numbers $B_j$ and Euler numbers $E_j$ defined by the following generating functions respectively:

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!}, \quad \sec x = \sum_{j=0}^{\infty} (-1)^j E_{2j} \frac{x^{2j}}{(2j)!},$$

(2)

and the Euler numbers $E_{2j+1} = 0$ for all $j \geq 0$.

Our main results are the following theorems.

**Theorem 1.1.** For $d \leq n$,

$$T(2n, d) = \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(-1)^j \pi^{2j}}{2^{2d-2}(2j)!} \binom{2d - 2 - 2j}{d - 1} t(2n - 2j),$$

where $t(2j) = 2^{-2j}(2^{2j} - 1)\zeta(2j)$. Or, equivalently,

$$T(2n, d) = \binom{2d - 2}{d - 1} \frac{t(2n)}{2^{2d-2}d} - \sum_{j=1}^{\lfloor \frac{d-1}{2} \rfloor} \binom{2d - 2 - 2j}{d - 1} \frac{t(2j)t(2n - 2j)}{2^{2d-3}(2^{2j} - 1)B_{2j}d}.$$

The next three cases after (1) are

$$T(2n, 6) = \frac{21}{512}t(2n) - \frac{7}{192}t(2)t(2n - 2) + \frac{1}{256}t(4)t(2n - 4),$$

$$T(2n, 7) = \frac{33}{1024}\zeta(2n) - \frac{15}{512}t(2)t(2n - 2) + \frac{1}{256}t(4)t(2n - 4) - \frac{1}{21504}t(6)t(2n - 6),$$

$$T(2n, 8) = \frac{429}{16384}t(2n) - \frac{99}{4096}t(2)t(2n - 2) + \frac{15}{4096}t(4)t(2n - 4) - \frac{1}{12288}t(6)t(2n - 6).$$

As we mentioned in the above the proof of Theorem [4] utilizes the generating function of $T(2n, d)$ defined by

$$\Phi(u, v) = 1 + \sum_{n \geq d \geq 1} T(2n, d)u^nv^d$$

for which we have the following result.

**Theorem 1.2.** We have

$$\Phi(u, v) = \cos(\pi \sqrt{(1 - v)u/2}) \sec(\pi u/2).$$
The next theorem involves Euler numbers and is more useful computationally when the difference between $n$ and $d$ is small.

**Theorem 1.3.** For $d \leq n$ we have

$$T(2n, d) = \frac{(-1)^{n-d} \pi^{2n}}{4^n (2n)!} \sum_{\ell=0}^{n-d} \binom{n-\ell}{d} \binom{2n}{2\ell} E_{2\ell}. \quad (3)$$

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## 2 Proof of Theorem 1.2 and Theorem 1.3

We first recall some results on symmetric functions contained in [1, 2] with some slight modification. Let Sym be the subring of $\mathbb{Q}[x_1, x_2, \ldots]$ consisting of the formal power series of bounded degree that are invariant under permutations of the $x_j$. Define elements $e_j$, $h_j$, and $p_j$ in Sym by the generating functions

$$E(u) = \sum_{j=0}^{\infty} e_j u^j = \prod_{j=1}^{\infty} (1 + u x_j),$$

$$H(u) = \sum_{j=0}^{\infty} h_j u^j = \prod_{j=1}^{\infty} \frac{1}{1 - u x_j} = E(-u)^{-1},$$

$$P(u) = \sum_{j=1}^{\infty} p_j u^{j-1} = \sum_{j=1}^{\infty} \frac{x_j}{1 - u x_j} = \frac{H'(u)}{H(u)}.$$

Define a homomorphism $\mathfrak{T} : \text{Sym} \to \mathbb{R}$ such that $\mathfrak{T}(x_j) = 1/(2j - 1)^2$ for all $j \geq 1$. Hence for all $n \geq 1$

$$\mathfrak{T}(p_n) = t(2n) = \sum_{j \geq 1} \frac{1}{(2j - 1)^{2n}}.$$

First we need a simple lemma.

**Lemma 2.1.** For any positive integer $n$ let $\{2\}^n$ be the string $(2, \ldots, 2)$ with 2 repeated $n$ times. Then we have

$$t(\{2\}^n) = \frac{\pi^{2n}}{4^n (2n)!}. \quad (4)$$
Proof. It is easy to see that
\[
1 + \sum_{n=1}^{\infty} t\{2\}^n x^n = \prod_{j=1}^{\infty} \left( 1 + \frac{x}{(2j-1)^2} \right)
= \prod_{j=1}^{\infty} \left( 1 + \frac{x}{j^2} \right) / \prod_{j=1}^{\infty} \left( 1 + \frac{x}{(2j)^2} \right)
= \frac{\sinh(\pi \sqrt{x})}{\pi \sqrt{x}} \cdot \frac{\pi \sqrt{x}/2}{\sinh(\pi \sqrt{x}/2)}
= \cosh(\pi \sqrt{x}/2)
= \sum_{n=1}^{\infty} \frac{\pi^2 nx^n}{4^n(2n)!},
\]
This finishes the proof of the lemma. \(\square\)

Now let \(N_{n,d}\) be the sum of all the monomial symmetric functions corresponding to partitions of \(n\) having length \(d\). Then clearly
\[
\mathfrak{T}(N_{n,d}) = T(2n, d).
\]
As in [1] we may define
\[
\mathcal{F}(u, v) = 1 + \sum_{n \geq 1, d \geq 1} N_{n,d} u^n v^d,
\]
then \(\mathfrak{T}\) sends \(\mathcal{F}(u, v)\) to the generating function
\[
\Phi(u, v) = 1 + \sum_{n \geq 1, d \geq 1} T(2n, d) u^n v^d.
\]
By Lemma 2.1 we have
\[
\mathfrak{T}(e_n) = t\{2\}^n = \frac{\pi^{2n}}{4^n(2n)!}.
\]
Hence
\[
\mathfrak{T}(E(u)) = \cosh(\pi \sqrt{u}/2),
\]
and
\[
\mathfrak{T}(H(u)) = \mathfrak{T}(E(-u)^{-1}) = 1 / \cosh(\pi \sqrt{-u}/2) = \sec(\pi \sqrt{-u}/2).
\]
Thus by [1] Lemma 1\(\] \(\mathcal{F}(u, v) = E((v - 1)u)H(u)\) and we get
\[
\Phi(u, v) = \mathfrak{T}(E((v - 1)u)H(u)) = \cosh(\pi \sqrt{(v - 1)u}/2) \sec(\pi \sqrt{u}/2)
= \cos(\pi \sqrt{(1 - v)u}/2) \sec(\pi \sqrt{u}/2).
\]
This proves Theorem \(\ref{thm:main} \)
Setting \( v = 1 \) in Theorem 1.2 we obtain

\[
\Phi(u, 1) = \sec(\pi \sqrt{u}/2).
\]

This yields immediately the following identity by \( \mathfrak{T} \)

\[
\mathfrak{T}(h_n) = \sum_{d=1}^{n} T(2n, d) = \frac{(-1)^n E_{2n} \pi^{2n}}{4^n (2n)!}.
\]

Now by [1, Lemma 2] we have

\[
N_{n,d} = \sum_{\ell=0}^{n-d} \binom{n}{\ell} (-1)^{n-d-\ell} h_{\ell} e_{n-\ell}.
\]

Applying the homomorphism \( \mathfrak{T} \) and using equation (4) and (6) we get Theorem 1.3 immediately.

### 3 Proof of Theorems 1.1 and a combinatorial identity

We now rewrite the generating function \( \Phi(4u, v) \) as follows using Theorem 1.2

\[
\Phi(4u, v) = \sum_{d \geq 0} v^d \tilde{G}_d(u) = \sec(\pi \sqrt{u}) \cos(\pi \sqrt{(1 - v)u}) = \sec(\pi \sqrt{u}) \sum_{j=0}^{\infty} \frac{\pi^{2j}}{(2j)!} (v - 1)^j u^j.
\]

Let \( D \) be the differential operator with respect to \( u \). Then

\[
\tilde{G}_d(u) = (-1)^d \sec(\pi \sqrt{u}) \sum_{j \geq d} \frac{(-1)^j \pi^{2j} u^j}{(2j)!} \binom{j}{d}
\]

\[
= \sec(\pi \sqrt{u}) \cdot \frac{(-u)^d}{d!} \cdot D^d \sum_{j \geq d} \frac{(-1)^j \pi^{2j} u^j}{(2j)!}
\]

\[
= \sec(\pi \sqrt{u}) \cdot \frac{(-u)^d}{d!} \cdot D^d \cos(\pi \sqrt{u})
\]

\[
= -\frac{\pi^2}{2} \sec(\pi \sqrt{u}) \cdot \frac{(-u)^d}{d!} \cdot D^{d-1} \frac{\sin(\pi \sqrt{u})}{\pi \sqrt{u}}
\]

\[
= \frac{\pi^2 u \tan(\pi \sqrt{u})}{2d} \frac{2}{\pi \sqrt{u}} G_{d-1}(u)
\]
by [1] (12) (the definition of $G_k$ is defined on page 9). By [1] Lemma 3 we have

$$
\tilde{G}_d(u) = -\frac{\pi^2 u}{2d} \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \frac{(-4\pi^2 u)^j}{2^{2d-3}(2j+1)!} \left( \frac{2d - 2j - 3}{d - 1} \right) 
$$

(7)

$$
+ \frac{\pi \sqrt{u}}{2d} \tan(\pi \sqrt{u}) \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(-4\pi^2 u)^j}{2^{2d-2}(2j)!} \left( \frac{2d - 2j - 2}{d - 1} \right) 
$$

(8)

$$
= \frac{\pi \sqrt{u}}{2d} \tan(\pi \sqrt{u}) \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(-4\pi^2 u)^j}{2^{2d-2}(2j)!} \left( \frac{2d - 2j - 2}{d - 1} \right) + \text{terms of degree } < d.
$$

It is well-known that

$$
\tan x = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} 2^{2m}(2^{2m} - 1)B_{2m}x^{2m-1}}{(2m)!}.
$$

Hence

$$
\frac{\pi \sqrt{u}}{2} \tan(\pi \sqrt{u}) = \sum_{m=1}^{\infty} 4^m t(2m)u^m.
$$

Therefore $T(2n, d)$ is the coefficient of $u^n$ in

$$
\tilde{G}_d(u/4) = \frac{1}{d} \sum_{m=2}^{\infty} t(2m)u^m \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \frac{(-\pi^2 u)^j}{2^{2d-2}(2j)!} \left( \frac{2d - 2j - 2}{d - 1} \right).
$$

This implies Theorem 1.1 immediately. Notice that by comparing Theorem 1.1 and Theorem 1.3 we get the following identity of between Bernoulli numbers and Euler numbers.

**Theorem 3.1.** For all $d \leq n$

$$
\sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(22n-2j - 1)B_{2n-2j}}{2^{2d-1}d} \left( \frac{2d - 2j - 2}{d - 1} \right) \left( \frac{2n}{2j} \right) = \frac{(-1)^{n-d}\pi^{2n}}{4^n(2n)!} \sum_{\ell=0}^{n-d} \binom{n - \ell}{d} \binom{2n}{2\ell} E_{2\ell}.
$$

Further we have

$$
\sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(22n-2j - 1)B_{2n-2j}}{2^{2d-1}d} \left( \frac{2d - 2j - 2}{d - 1} \right) \left( \frac{2n}{2j} \right)
$$

$$
= \begin{cases} 
0, & \text{if } n < d < 2n; \\
\frac{n}{2^{2d-1}d} \left( \frac{2d - 2n - 1}{d - 1} \right), & \text{if } d \geq 2n.
\end{cases}
$$
Proof. We only need to show the second identity. Notice that when \( d > n \) the coefficient of \( u^n v^d \) is 0 in \( \Phi(u, v) \). Thus the coefficient of \( u^n \) in \( \tilde{G}_d(u/4) \) is zero. By (7) and (8) we have

\[
\sum_{j=0}^{\left\lfloor \frac{d-1}{2} \right\rfloor} \frac{(2^{2n-2j} - 1)B_{2n-2j}}{2^{2d-1}d} \binom{2d - 2j - 2}{d - 1} \binom{2n}{2j} = (-1)^n (2n)! (2\pi)^{2n} \times \text{Coeff. of } u^n \text{ of (7) (i.e. } j = n - 1) \]

\[
= \begin{cases} 
0, & n < d < 2n; \\
\frac{n}{2^{2d-1}d} (2d - 2n - 1), & d \geq 2n,
\end{cases}
\]

as desired. \( \square \)

References

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