Symmetry of Differential Equations and Quantum Theory

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Abstract. The symmetry study of main differential equations of mechanics and electrodynamics has shown, that differential equations, which are invariant under transformations of groups, which are symmetry groups of mathematical numbers (considered in the frame of the number theory) determine the mathematical nature of the quantities, incoming in given equations. It allowed to proof the main postulate of quantum mechanics, that to any mechanical quantity can be set up into the correspondence the Hermitian matrix by quantization. High symmetry of Maxwell equations allows to show, that to EM-field functions, incoming in given equations, can be set up into the correspondence the Quaternion (twice-Hermitian) matrices by their quantization.

1. Introduction

It is well known, that quantum theory is formulated by using of some postulates. For instance, the study of quantum mechanics in all textbooks (see, for example, [1], [2]) is beginning from the formulation of the main postulate, which is the following: "The linear self-adjoint operators are setting up into correspondence to observable physical quantities". It is the first main postulate of quantum mechanics. Given postulate is in fact containing in the first works on quantum mechanics, which are referred to be the birth of quantum mechanics, that is in the papers of Heisenberg, Born, Jordan, Dirac, Pauli.

In Heisenberg’s fundamental paper [3] and in the papers [4], [5],[7],[6] following it, dynamical variables were assumed to be represented by matrices. In paper [8], Dirac starts with a more general assumption. He supposes the quantum variables $x,y,...$ to be elements of an algebra, which means that sums $x + y$ and products $xy$ are defined which satisfy the ordinary laws of algebra, excluding the commutative law of multiplication. Among the elements of this algebra are the numbers of classical mathematics which were called $c$-numbers. All other elements were called $q$-numbers. It was in fact the step to the proof of the first main postulate, although the attempt to proof given postulate was not undertaken from the birth of quantum mechanics in 1925 up to now. We will show in present report, that given postulate can be strictly proved.

2. Algebra of the complex and hypercomplex numbers and its connection with the symmetry of quantum systems

Some useful results from algebra of the complex numbers were summarized in the paper [11]. They will be reproduced here for convenience of the readers and will be supplemented with some
results from algebra of the hypercomplex numbers.

The numbers 1 and \( i \) are usually used to be basis of the linear space of complex numbers over the field of real numbers. At the same time to any complex number \( a + ib \) can be set up in conformity the \([2 \times 2]\)-matrix according to bijective mapping

\[
f : a + ib \rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix},
\]

(1)

Bijectivity of mapping (1) indicates on the existence of inverse mapping, that is to any matrix, which has the structure, given by right side in relation (1), corresponds the complex number, determined by left side. The matrices

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}
\]

(2)

produce basis for complex numbers \( \{a + ib\}, a, b \in R \) in the linear space of \([2 \times 2]\)-matrices, defined over the field of real numbers.

It is convenient often to define the space of complex numbers over the group of real positive numbers, then the dimensionality of the matrices and basis has to be duplicated, since to two unities - positive 1 and negative \(-1\) can be set up in conformity the \([2 \times 2]\)-matrices according to bijective mapping

\[
\xi : 1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, -1 \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

(3)

which allow to recreate the operations with negative numbers without recourse of negative numbers themselves. Consequently, the following \([4 \times 4]\)-matrices, so called \([0,1]\)-matrices, can be basis of complex numbers

\[
\zeta : 1 \rightarrow [e_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\]

\[
i \rightarrow [e_2] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},
\]

\[
-1 \rightarrow [e_3] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
\]

\[
-i \rightarrow [e_4] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.
\]

(4)

The choise of basis is ambiguous. Any four \([4 \times 4]\) \([0,1]\)-matrices, which satisfy the rules of cyclic recurrence

\[
i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1
\]

(5)

can be basis of complex numbers. It is evident, that the system of complex numbers can be constructed by infinite number of the ways, at that cyclic basis can consist of \(m\) units, \(m \in N\),
starting from three. It is remarkable, that the conformity between complex numbers and matrices is realized by biective mappings. It means, that there is also existing the inverse mapping, by means of which to any square matrix, belonging to the linear space with a basis given by (4), or any other, satisfying the rules of cyclic recurrence like to (5) can be set up in conformity the complex number. In particular, to any Hermitian matrix $H$ can be set up in conformity the following complex number

$$
\zeta : H \rightarrow S + iA = \begin{bmatrix} S & -A \\ A & S \end{bmatrix},
$$

(6)

where $S$ and $A$ are symmetric and antisymmetric parts of Hermitian matrix. In the works [9], [10] was used the apparatus of hypercomplex n-numbers by calculation of the optical properties of 1D carbon zigzag shaped nanotubes, taking into account quantum nature of EM-field. Hypercomplex n-numbers were defined to be the elements of commutative ring $Z_n$, being to be the direct sum of $n$ fields $C$ of complex numbers, $n \in \mathbb{N}$, that is

$$
Z_n = C \oplus C \oplus \ldots \oplus C.
$$

(7)

It means, that any hypercomplex n-number $z \in Z_n$ is $n$-dimensional quantity with the components $k_\alpha \in C, \alpha = 0, n - 1$. It can be represented in row matrix form $z \in Z_n$

$$
z = [k_\alpha] = [k_0, k_1, \ldots, k_{n-1}].
$$

(8)

On the other hand, hypercomplex n-number $z \in Z_n$ can also be represented in the form

$$
z = \sum_{\alpha=0}^{n-1} k_\alpha \pi_\alpha,
$$

(9)

where $\pi_\alpha$ are basis elements of $Z_n$. They are the following

$$
\pi_0 = [1, 0, \ldots, 0, 0], \pi_1 = [0, 1, \ldots, 0, 0], \\
\ldots, \pi_{n-1} = [0, 0, \ldots, 0, 1].
$$

(10)

The simplest hypercomplex number is quaternion. Any quaternion number $x$ can be determined according to relation

$$
x = (a_1 + ia_2) e + (a_3 + ia_4) j,
$$

(11)

where \( \{a_m\} \in R, m = 1, 4 \) and $e, i, j, k$ produce basis, elements of which are satisfying the conditions

$$
(ij) = k, (ji) = -k, (ki) = j, (ik) = -j, \\
(ei) = (ie) = i, (ej) = (je) = j, (ek) = (ke) = k.
$$

(12)

Quaternion set \( \{Q\} \) can be considered to be the generalization of the set $Z_n$ for $n = 2$ in accordance with the relation

$$
Z_2^Q = Ce \oplus Cj,
$$

(13)

that is, it represents itself noncommutative ring. Geometrically set \( \{Q\} \) can be interpreted to be consisting of two mutually perpendicular planes in the space $R_4^Q$ determined by orthogonal one real axis and three imaginary axes, at that $i, j, k$ are imaginary unities along coordinate axes in three dimensional imaginary $Z_3^3$ space, being to be the subspace of $R_4^Q$. The result of their
multiplication is equivalent to $\pi/2$ rotation in the corresponding pure imaginary plane. It seems to be evident, that the space $\mathbb{R}^4$ is isomorphous to Minkowski space $\mathbb{R}^4$. Taking into account given interpretation, to any quaternion number $x = (a_1 + ia_2)e + (a_3 + ia_4)j$ can be set up in conformity the $[2 \times 2]$-matrix according to biective mapping

\[
q : (a_1 + ia_2)e + (a_3 + ia_4)j \rightarrow \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix} E + \begin{bmatrix} a_3 & -a_4 \\ a_4 & a_3 \end{bmatrix} J,
\]

where $E$ is $[2 \times 2]$ unity matrix, $J$ is the following $[2 \times 2]$ matrix

\[
J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

The matrices

\[
E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},
\]

\[
J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, K = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}
\]

produce basis for the set $\{x\}$ of quaternion numbers $\{x\} = \{(a_1 + ia_2)e + (a_3 + ia_4)j\}$ in the linear space of $[2 \times 2]$-matrices, defined over the field of real numbers. It is evident, that the matrices (16) produce anticommutative multiplicative group. It is also clear, that the system of quaternion numbers like to complex numbers can be constructed by infinite number of the ways. It is seen, taking into account the first main postulate of quantum mechanics, that for quantum mechanics and quantum theory at all, the relation (6) is especially significant. It allowed in the work [11] to formulate the following statement

1. Any quantumphysical Hermitian operator in Hilbert space, which is set up in conformity to corresponding classical physical quantity, defines two real sets of observables, being to be connected between themselves and belonging to the field of real numbers (amplitude and phase sets or the sets of average values and their dispersion), that is the set of complex quantities is determined by Hermitian operator in general case, which allows correctly describe the real noninstantaneous processes. Practical consequence of the statement 1 for quantumphysical tasks is the following. In particular, by the solution of main quantummechanical equation - Schroedinger equation, the eigenvalues $E_n$, $n \in \mathbb{N}$, of Hamilton operator, have to be represented in complex form. In the application to atom physics $ReE_n$ represents itself the energy of nth atomic level [more precisely its mathematical expectation], $ImE_n$ is its dispersion, that is the width of given level. Additional gauge invariance of complex relativistic fields was studied in [12] and in [11]. It has been found, that conserving quantity, corresponding to invariance of generalized relativistic equations under the operations of additional gauge symmetry group - multiplicative group $\mathfrak{R}$ of all real numbers (without zero) - is purely imaginary charge. So, it was shown, that complex fields are characterized by complex charges. Let us to remember, that generalized relativistic equations, that is Lagrange equations for any complex relativistic field can be represented in the form of one matrix relativistic differential equation of the first order in partial derivatives. So, the equation for the set $u(x)$ of field functions has the following form

\[
(\alpha_\mu \partial_\mu + \kappa \alpha_0) u(x) = 0.
\]

Matrices $\alpha_\mu, \alpha_0$ in equations (17) are matrices with constant numerical elements. They have an order, which coincides with the dimension of corresponding space of Lorentz group...
representation. In particular, they are \([n \times n]\)-matrices, if \(\{u_i(x)\}, i = 1, n\) are scalar functions. Total gauge transformation, established in \([12]\) and in \([11]\) is the following
\[
 u'(x) = \beta \exp(i \alpha) u(x),
\]  
(18)

where \(\alpha, \beta \in \mathbb{R}\), \(u(x)\) is the set of field functions. It differs from known gauge transformation by presence of factor \(\beta \in \mathbb{R}\), which is substantial, since it recognizes clearly the additional gauge symmetry of complex relativistic fields. The following statements were proved

A. Conserving quantity - complex charge, which is invariant under total gauge transformations, corresponds to any complex relativistic field (scalar, vector, spinor).

B. Conserving quantity - purely imaginary charge, which is invariant under total gauge transformations, corresponds to any real relativistic field (scalar, vector, spinor). The statement B is especially interesting. It means in fact, that in complete set of characteristic field functions of any relativistic field (complex or real) enters at least one complex (or pure imaginary) function. It is understandable, that transformation of field functions by relation (18) is equivalent to multiplication of field functions by arbitrary complex number. The attention was drawn in \([11]\), that the relation (18) gives automorphism of the space of field functions. It is known, that automorphism of any linear space leads to a number useful properties of the objects, which belong to given space. Both the statements give the key for correct generalization of relativistic field equations, in particular, electrodynamics equations. Given results can be generalized, if to take into consideration, that differential equations (17) are invariant under the same transformation (18). Therefore, it is seen, that the correspondence between the symmetry of differential equations and the mathematical nature [in the concept of the number theory] of the quantities, incoming in given equations seems to be taking place. So, we come to the statement:

2. Differential equations, which are invariant under transformations of groups, which are symmetry groups of mathematical numbers (considered in the frame of the number theory) determine the mathematical nature of the quantities, incoming in given equations. In the case of invariance of differential equations under transformation (18), that is in the case of invariance under transformations of multiplicative group of complex numbers, the proof seems to be evident. Really, the multiplication, for instance, of full set of field function on complex numbers means, that the functions themselves have to be complex. It seems to be essential that differential equations for dynamics of nonrelativistic classical mechanics are invariant under transformation (18) too. Really, the dynamics of classical mechanics systems is described by Lagrange equations or by equivalent canonical Hamilton equations. It is known, that, for instance, Hamilton equations are invariant under contact transformations of variables, that is under the transformation of linear elements - positions and directions, but not points. The transformation (18) is referred to given class. It means, that by quantization all physical quantities, which determine the dynamics of classical mechanics systems have to be represented by quantum-mechanical description by one of the variants of the representation of complex numbers, in particular, taking into account (6), by Hermitian matrices. Therefore, we have the proof of the statement, being to be the proof of the main postulate of quantum mechanics.

3. To any mechanical quantity can be set up in the correspondence the Hermitian matrix by quantization. It is in fact the consequence of the statement 2. The choose of construction of mathematical apparatus of quantum mechanics on the base of Hermitian matrices is convenient, however, it is the only one variant from the infinity of variants of the representations of quantum mechanical quantities by complex numbers. We have to remark, that the description of the processes in classical mechanics by means of complex number is also correct. But in very many practical cases, for instance, for mechanical tasks, described by Newton equation, force and impulse can be characterized by the same phase factor (that is, it can not be taking into consideration). Further will be considered the case of invariance of differential equations.
under transformation of anticommutative group of quaternion numbers. It is the case of electrodynamics.

4. To any electrodynamics quantity can be set up in the correspondence the Quaternion (that is twice-Hermitian) matrix by quantization of EM-field. The symmetry studies of Maxwell equations gave new insight on the nature of electromagnetic field. It has in general case quaternion structure, consisting of four independent field constituents, which differ each other by the parities under space inversion and time reversal [11]. The presence along with vector quaternion characteristics the independent scalar quaternion characteristics of EM-field allows to describe EM-field instead of unobservable vector and scalar potentials by observable electric field 4-vector-function with the components \( E_\alpha(\vec{r}, t) = \{E_x(\vec{r}, t), E_y(\vec{r}, t), E_z(\vec{r}, t), i\frac{\epsilon_\mu}{\lambda}(\vec{r}, t)\} \) and (or in the case of free EM-field) by means of magnetic field 4-vector-function \( H_\mu(\vec{r}, t) = \{H_x(\vec{r}, t), H_y(\vec{r}, t), H_z(\vec{r}, t), i\frac{\epsilon_\nu}{\lambda}(\vec{r}, t)\} \), where \( \epsilon_\mu(\vec{r}, t) \), \( \epsilon_\nu(\vec{r}, t) \) are the \( j_\mu(\vec{r}, t) \)-component of 4-current density, corresponding to contribution of electric and magnetic component of charge densities correspondingly, \( \lambda \) is conductivity, which for the case of EM-field propagation in vacuum is \( \lambda = \frac{1}{1207} \) (Ohm)\(^{-1} \). Then electric \( j_e(\vec{r}, t) \) (and magnetic \( j_g(\vec{r}, t) \) in general case) current densities in the right side of Maxwell equations can be represented by well known relations \( j_e(\vec{r}, t) = \rho_e(\vec{r}, t)v_\lambda(\vec{r}, t) \) and analogous relation (in general case) for magnetic \( j_g(\vec{r}, t) \) current density. Now the invariance of Maxwell equations under transformations (18) becomes to be evident. Therefore, take into account the statement 4, we obtain the independent proof, that all EM-field quantities have to be considered minimum being to be complex quantities (for correct description of electromagnetic phenomena). Given picture is practically always used both in the theory and in practical applications. However, even given representation in some cases is insufficient, it concerns, for instance, the dynamics of optical transitions [15]. At the same time, it is easily to see, that Maxwell equations are invariant under transformations of quaternion non-abelian multiplicative group. In its turn, it leads to conclusion, that really to EM-field functions can be set up in the correspondence the Quaternion (that is twice-Hermitian) matrices by the quantization of EM-field. It is in fact the consequence of the presence along with symmetry of Maxwell equations under transformations, given by (18), the Rainich [13] dual symmetry and additional dual symmetry, established in [11]. Let us remark for comparison, that the equations of the dynamics of mechanical systems are not invariant under transformations of quaternion multiplicative group. It is the consequence of non-abelian character of given group.

3. Conclusions

The symmetry study of main differential equations of mechanics and electrodynamics has shown, that differential equations, which are invariant under transformations of groups, which are symmetry groups of mathematical numbers (considered in the frame of the number theory) determine the mathematical nature of the quantities, incoming in given equations. The main postulate of quantum mechanics, consisting in that, that to any mechanical quantity can be set up into the correspondence the Hermitian matrix by quantization was proved. High symmetry of Maxwell equations, consisting in the presence along with gauge symmetry under transformations, given by (18), the Rainich dual symmetry and additional hyperbolic dual symmetry, established in [11], allowed to show, that to EM-field functions, incoming in given equations, can be set up into the correspondence the Quaternion (twice-Hermitian) matrices by their quantization.

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