Partial Compensation of Thermal Noise in the Fundamental Mode of an Optical Cavity

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Abstract—Thermal noise of optical cavities limits the accuracy of many experiments on precision laser spectroscopy and interferometry. The study of the physical properties of this noise opens up opportunities for creating more stable cavities, reducing phase noise of optical radiation, and performing accurate optical studies. The paper proposes a method for recording thermal noise of TEM₀₀ mode of a Fabry–Perot cavity using two higher-order “probe” modes, which allows partial compensation of noise in the fundamental mode. Mathematical modeling is performed, which confirms the method efficiency.

Keywords: Fabry, Perot cavity, thermal noise of mirrors, Laguerre–Gaussian modes

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1. INTRODUCTION

High-stability optical cavities are widely used in many experiments associated with precision optical measurements: gravitational wave recording [1], search for dark matter [2, 3], high-resolution spectroscopy, optical atomic clock [4, 5], and other basic research [6, 7]. Laser frequency stabilization by reference monolithic high-Q Fabry–Perot cavities using the Pound–Drever–Hall method [8] makes it possible to narrow the radiation spectral linewidth to less than 100 mHz [9–11].

The instability of the cavity mode frequency is related to its length as

\[ \frac{\Delta v}{v} = -\frac{\Delta L}{L}. \]  

(1)

The fundamental stability limit \( L \) is defined by Brownian thermal fluctuations of the cavity, which are conventionally referred to as the thermal noise. Its physical properties can be described by the fluctuation—dissipation theorem [12, 13]. Random vibrations of this nature are inherent to any heated body. The thermal noise of the optical cavity can be presented as random vibrations of the mirror surface, resulting in fluctuations of its length averaged over the radiation mode spot on each mirror. Currently, the maximum achievable frequency stability of the best laser systems with reference cavities is limited by thermal noise. The sensitivity of gravitation-wave LIGO detectors in the frequency range of 30–200 Hz is also limited by thermal fluctuations of interferometer mirrors [15]. The efficient method for calculating the thermal noise of mirrors is proposed in [14].

Significant progress in reducing the thermal noise can be achieved by lengthening the cavity, decreasing its operating temperature, and using materials with high mechanical Q-factor (quartz, single-crystal silicon) for producing mirrors [16]. The thermal noise limit can also be approximately twofold reduced when using other Gaussian higher-order modes instead of the fundamental mode TEM₀₀ [17]. This effect is achieved due to larger transverse sizes of higher-order modes, which leads to better averaging of surface thermal fluctuations.

In the presence of several ultrastable laser systems with close stability parameters, it is possible to average their frequencies, thus overcoming the thermal noise limit for one system [18].

In this study, we propose a method for studying the thermal noise of a chosen “fundamental” mode of a cavity by exciting two additional “probe” modes in it, which makes it possible to generate a radio-frequency signal whose noise contains information about the thermal noise of the “fundamental” mode. Then, this signal can be used to partially compensate for laser radiation noises in the fundamental mode.
by analogy with the method for compensating vibrations, described in [19]. The presence of the frequency response of cavity mirrors to the change in the power of radiation introduced into it [9, 20] offers opportunities of active suppression of thermal noises. Furthermore, measurements and analysis of thermal fluctuations will allow studying mechanical properties of multilayer Bragg mirrors [21].

2. MEASUREMENTS OF THE THERMAL NOISE OF THE CAVITY MODE

Since the laser mode on cavity mirrors has finite sizes, the contribution to its thermal noise is made only by reflecting surface areas onto which radiation is incident. It can be shown that the phase shift of the beam with a characteristic intensity profile \( g(\vec{r}) \) normalized to unity, reflected from the mirror surface perturbed by the thermal noise \( \vec{u}(\vec{r}, t) \) (see Fig. 1) is written as [22]

\[
\Delta \phi(t) = \int_S g(\vec{r}) \cdot (\vec{k}, \vec{u}(\vec{r}, t)) \cdot d^2 r.
\] (2)

In this case, integration is performed over the mirror surface \( S \), \( \vec{r} \) is the radius vector of the point on the mirror surface, \( \vec{u}(\vec{r}, t) \) is the shift of the mirror point with position vector \( \vec{r} \) with respect to the unper- turbated position, and \( \vec{k} \) is the wave vector of incident radiation, perpendicular to the surface. Using the phase shift \( \Delta \phi(t) \), we can introduce the “effective” shift of the mirror surface along the \( x \) axis,

\[
X(t) = \frac{\Delta \phi(t)}{|\vec{k}|} = \int_S g(\vec{r}) \cdot u_x(\vec{r}, t) \cdot d^2 r.
\] (3)

Any cavity eigenmode can be presented as a superposition of Hermite—Gaussian or Laguerre—Gaussian functions [23]. Due to more efficient averaging of reflecting surface vibrations, a poorer sensitivity to thermal noises is inherent to modes whose intensity is distributed over a larger mirror area. The intensity distribution \( g(\vec{r}) \) for each mode has a specific shape with the result that thermal noises for various modes can be controlled by thermal fluctuations in not intersecting mirror regions. This fact underlies the method we proposed for studying the thermal noise.

Its implementation requires the choice of three modes: the “fundamental” one \( M_1 \), i.e., that whose thermal noise should be partially characterized, and “probe” modes \( M_2 \) and \( M_3 \). Let all three laser beams corresponding to modes be related to the cavity by frequency, and the stability of each is limited by thermal noises. To avoid interference effects, all modes should have different frequencies, which can be implemented using a single laser, detuning the frequency using modulators. Let us divide the mirror surface into two conditional areas: \( O_1 \) where the fundamental mode \( M_1 \) intensity is concentrated and \( O_2 \) where the probe mode \( M_2 \) intensity is concentrated. In this case, the mode \( M_2 \) should be chosen so that it should be as weak as possible superimposed with mode \( M_1 \). In turn, the mode \( M_3 \) should be strongly superimposed with modes \( M_1 \) and \( M_2 \). Examples of three modes satisfying these requirements are the Laguerre—Gaussian \( LG_{00} \) (\( M_1 \)), \( LG_{03} \) (\( M_2 \)), \( LG_{10} \) (\( M_3 \)) shown in Fig. 2. \( M_2 \) belongs to helical Laguerre—Gaussian modes [17].

When heterodyning the optical fields of modes \( M_2 \) and \( M_3 \), the thermal noises in the area \( O_2 \) will be correlated and partially compensated in the beat signal (radio-frequency range). In this case, the thermal noise caused by mirror surface vibrations in the area \( O_1 \) spanned by the fundamental mode \( M_1 \) remains in the signal. Thus, the obtained signal frequency fluctuations can be subtracted from the mode \( M_1 \) radiation frequency, e.g., using an acousto—optic modulator (Fig. 3).

Fig. 1. Gausses beam of the TEM\(_{00}\) mode incident on the mirror surface distorted by thermal noise.
This reduces the contribution of thermal noises to the instability of the laser radiation frequency in the fundamental mode $M_1$. In this case, residual frequency fluctuations $\delta \vartheta_{123}$ are given by

$$\delta \vartheta_{123} (t) = - \frac{\dot{\vartheta}_1}{L} \left[ g_1 (\vec{r}) - (g_3 (\vec{r}) - g_2 (\vec{r})) \right] \cdot u_x (\vec{r}, t) \cdot d^2 r.$$  \hspace{0.5cm} (4)

3. MODELING

The above-described method was analyzed using mathematical modeling. To this end, the surface $u_x (\vec{r})$ was multiply randomly created, which corresponds to Brownian thermal fluctuations; then, according to formula (3), integral surface shifts were calculated for the intensity profile of each mode. For simplicity, the calculation considered only thermal noises of one reflecting surface in the static case. The role of the mirror perturbed by thermal noises was played by the fractal surface resulted from the Brownian process. The method for modeling such surfaces is described in [24]. The resulting functions $u_x (\vec{r})$ are described using two parameters: the lacunarity $r$ and Hurst exponent $h$. The latter parameter to a greater extent is responsible for the surface “noisiness.” To model the thermal noise being the $1/f$-noise, the values of $h$ close to unity are most appropriate [25]. The lacunarity parameter was taken equal to 0.5 for all calculations.
In the calculations, the Brownian surfaces of various noise levels with Hurst exponents in the range of 0.5–1 were used (see Table 1). For each value, 15 random surfaces were generated and $X_i$ were calculated by formula (3) for each of three modes $i = 1, 2, 3$ (Fig. 5).

The obtained values of $X_i$ composed 15-term samples for which standard deviations $\sigma_{X_i}$ were calculated. Root-mean-square surfaces $\sigma_{X_{133}}$ were also calculated for combinations $X_{133} = X_1 - (X_3 - X_2)$, which make it possible to judge the efficiency of the proposed method for determining the thermal noise. The data obtained are listed in Table 1. By virtue of the fact that characteristic values of humps and dips of the Brownian surfaces used in the calculation are unrelated to physical properties of actual mirrors, not magnitudes of calculated values, but their ratios are significant. As may be noted in the last column of Table 1, the ratio $\sigma_{X_{133}}/\sigma_{X_1}$ varies in the vicinity of 0.7, which suggests that the subtraction of the difference of probe mode noises from the sought mode results in that its noise decreases. This decrease appears relatively small due to the insufficiently good matching of mode profiles (Fig. 4a).

Profile matching of chosen modes can be improved by selecting the emission wavelength $\lambda$, since the Gaussian beam radius $\omega$ is proportional to $\sqrt{\lambda}$. As the mode M1 wavelength threefold decreases, it is possible to achieve a sufficiently accurate identity of its contour and the central spot contour of the mode M3, as shown in Fig. 4b. In this case, the signal of probe mode beats more accurately describes the thermal noise of the central mirror area in which the mode M1 intensity is concentrated. The results of corresponding calculations are listed in Table 2.
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was calculated for the quantity \( \sigma_{132}^* = X_1 - 3(X_1 - X_2) \) which accounts for the intensity ratio of modes \( M_3 \) and \( M_1 \) at the mirror center, equal to 3. In the experiment, the threefold increase in the \( M_3 \) and \( M_2 \) beat frequency signal can be provided using a frequency multiplier. The obtained values of \( \sigma_{132}^*/\sigma_{X_1}^* \) in the vicinity of 0.5 suggest that an improvement in the geometric congruence results in a more efficient noise compensation.

4. CONCLUSIONS

The method of partial compensation for thermal fluctuations of optical cavities in the fundamental radiation mode proposed in this study is based on the fact that the thermal noise of each individual mode is controlled by the noise of that mirror surface area on which its intensity is concentrated. Mathematical modeling of Brownian surfaces made it possible to calculate averaged mirror shifts due to surface Brown-

\begin{table}[h]
\centering
\caption{Simulation results for the case where all modes have the same wavelength}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( h \) & \( \sigma_{X_1} \) & \( \sigma_{X_2} \) & \( \sigma_{X_3} \) & \( \sigma_{X_{132}} \) & \( \sigma_{X_{132}}/\sigma_{X_1} \) \\
\hline
0.5 & 11.47 & 7.52 & 9.19 & 9.35 & 0.82 \\
0.6 & 11.29 & 6.13 & 6.95 & 8.40 & 0.75 \\
0.7 & 9.27 & 4.46 & 5.66 & 6.74 & 0.73 \\
0.8 & 4.01 & 2.56 & 2.68 & 3.14 & 0.79 \\
0.9 & 3.15 & 1.58 & 1.27 & 1.72 & 0.55 \\
0.94 & 3.68 & 1.40 & 2.25 & 2.54 & 0.69 \\
0.95 & 2.63 & 0.86 & 1.33 & 1.79 & 0.68 \\
0.98 & 1.52 & 0.70 & 0.90 & 1.12 & 0.74 \\
\hline
\end{tabular}
\end{table}

Fig. 5. Illustration of thermal noises for three modes. (a) Spatial intensity profiles of modes (top—down) LG_{00} (\( M_1 \)), LG_{03} (\( M_2 \)), LG_{10} (\( M_3 \)) and Brownian surface \( u_x(r) \) with \( h = 0.8 \). (b) Functions obtained by multiplying corresponding intensity profiles by \( u_x(r) \).
ian motion for various modes and to confirm that the mode beat LG_{10} and LG_{03} signal partially contains the thermal noise of the mode LG_{00}. The choice of the fundamental mode LG_{00} is not accidental: most stabilized lasers and gravitational detectors operate at this mode, since this case provides the best coupling with the laser mode, hence, a maximum power. In this case, high-order probe modes having a lower level of thermal noises are auxiliary and allow for compensation for fluctuations in the fundamental mode.

In the case where all modes have the same wavelength, the method allows compensation for \( \sim 30\% \) of the thermal noise of the fundamental mode. If a mode has a trebled frequency, the coincidence of intensity profiles appears to be better and subtraction allows suppression of \( \sim 50\% \) of noise. In this case, the typical standard deviations \( \sigma_{X_{132}} \) of the mirror surface for the fundamental mode even after subtracting a part of thermal noises appear to be higher than \( \sigma_{X} \) corresponding to probe modes LG_{10} and LG_{03}. It is difficult to efficiently excite high-order modes; therefore, the method can be used to increase stability of the radiation frequency corresponding to the chosen operating mode. Three modes for which the analysis was performed were chosen to illustrate the method, and most likely do not represent an optimum combination. A more optimal choice of modes featuring better matching of intensity profiles will make it possible to more efficiently compensate for the noise in the fundamental mode.

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