Mesh stiffness calculation and vibration analysis of the spur gear pair with tooth crack, considering the misalignment between the base and root circles

Jingyu Hou | Shaopu Yang | Qiang Li | Yongqiang Liu | Jiujian Wang

Abstract
An improved variable cross-section cantilever beam model for evaluating the time-varying mesh stiffness (TVMS) of the perfect gear tooth is developed in which the tooth number of driving gear is less than 42 and that of driven is more than 42. The TVMS obtained by the proposed method is compared with the result without considering the misalignment between the base circle and gear root. Four types of root crack models and changes in TVMS of 13-crack levels are presented. The fault vibration characteristic of a single-stage spur gear reducer with root crack is analyzed and the correctness is qualitatively verified by the vibration signals of an experimental gearbox with crack or missing failure. The results presented in this paper are of great significance for a deep understanding of the possible causes of vibration and noise of gears and provide a theoretical foundation for the fault diagnosis of the gearbox.

KEYWORDS
backlash, gear, potential energy method, time-varying mesh stiffness (TVMS), tooth crack

1 | INTRODUCTION
Spur gear transmission system has been used in various mechanical fields, because of some characteristics, for example, simple structure, reliable operation, long service life, no axial forces, and so on during meshing. A great deal of research has been made on the dynamics of a pair of healthy spur gears. Amabili and Rivola investigated the steady-state response and stability of a single degree of freedom (DOF) model with time-varying mesh stiffness (TVMS) and damping coefficient. Later, Amabili and Fregolent proposed a new method for identifying the gear error and modal parameters, where the nonlinear meshing stiffness, mesh damping, and excitation caused by gear errors are taken into account. However, tooth crack, pitting, spalling may develop in gears for overload, harsh operating conditions, manufacturing errors, and so forth. Therefore, it is of great significance to study the mechanism and vibration characteristics of spur gear systems with faults for further understanding the causes of gear vibration and noise, as well as for gear fault diagnosis.
Tooth crack, as one of the common gear faults, is easy to cause other faults, such as tooth being broken, missing, and so on. In addition, the existence of tooth crack mainly affects the TVMS and leads to the change of vibration features of the whole gear system. As a result, it needs first to determine the mesh stiffness for prediction or evaluation of the gear dynamic behavior. At present, the potential energy principle proposed by Yang and Lin in 1987 and the improvements based on this method are the most frequently used analytical methods to solve the TVMS of healthy or unhealthy gears, where the gear tooth is modeled as a cantilever nonuniform beam. The vibrations of beams have been discussed by many scholars. In earlier studies, the gear tooth is simplified as a variable cross-section cantilever beam starting from the base circle. In fact, for a real gear tooth, the base circle and the root circle are not exactly coincident. For standard involute spur gears, the base circle and the root circle are equal when the number of teeth is 42; otherwise, the base circle may be bigger or smaller than the root circle. The gear tooth is regarded as a variable cross-section cantilever beam starting from the tooth circle by Liang et al., where the transition curve is expressed as a straight line if the base circle is bigger than the root circle, while the whole tooth profile is assumed to be an involute profile when the gear base circle is smaller than the root circle. It should be noted that the effect of fillet-foundation deflection on the stiffness was not analyzed in his study. Wan et al. put forward a method to improve the TVMS, considering the misalignment between the base and root circles. The multi-DOF gear dynamic model with tooth crack is simulated and verified by an experiment in which the number of teeth of both the pinion and gear is greater than 42, and only the case that the crack depth is above the central line of the tooth is considered. Lei et al. modeled the transition curve between base circle and root circle like an arc when the tooth number is less than 42, which makes the shape of the tooth profile closer to the real shape of the gear; the scheme in which the base circle is smaller than the root circle has not been investigated. Ma et al. proposed an improved TVMS model where the misalignment of the gear base circle, root circle, and the accurate transition curve is included and a parabolic curve is adopted to simulate the crack propagation, which lacks the influence of crack on the gear system. Additionally, the influence of root crack on the vibration response of the spur gear system has been discussed by many researchers. The dynamic simulation of a 6-DOF gear model with root crack was investigated, in which the crack on gear body deflection was taken into account. The effects of different crack sizes on the change in dynamic response and the natural frequencies related to the gear TVMS were studied by Mohammed and Rantatalo. Wu et al. reported the effect of tooth crack on the vibration response of a one-stage gearbox with spur gears. However, the gear dynamic models mentioned above are linear time-varying models where the clearance nonlinearity is ignored. Yang et al. worked out the TVMS, taking advantage of the method introduced by Chen and Shao, and analyzed the effect of tooth crack on 3-DOF spur gear with consideration of tooth backlash and bearing clearance nonlinearity. However, there is no detailed analysis of a broken fault on the condition that the crack depth reaches a maximum value.

To overcome the shortcomings of the methods and research mentioned above, this paper takes the single-stage gear reducer as a subject. The gear tooth is assumed to be a cantilever beam with variable cross section starting from the root circle, where the whole tooth profile is regarded as an involute profile when the base circle is less than the root circle, and the transition curve between the root circle and the base circle is considered as an arc when the base circle is more than the root circle, as demonstrated in Figure 1A,B, respectively. The paper is organized as follows. In Section 2,
the TVMS for healthy and faulty teeth is obtained under the condition of misalignment between the base and root circles, based on the aforementioned method. To investigate the vibration response of tooth crack on the spur gear system in the time and frequency domains, the simulations and experiments are investigated in Section 3. At last, the main conclusions of this paper are made.

2 | CALCULATION OF TVMS

The elastic deformation of gear tooth changes periodically due to the alternate engagement of single and double tooth pair. Only a pair of gear teeth bear the load in the single-tooth engagement region, the elastic deformation is big and the corresponding mesh stiffness is small. On the contrary, the elastic deformation is small and the mesh stiffness is big in the double-tooth engagement region as the load is shared by two pairs of gear teeth. The TVMS, as one of the major contributors to the impact excitation, is formed on account of the alternation of single-tooth and double-tooth. The analysis of the variation of TVMS is indispensable in gear transmission. To describe the effective TVMS, the potential energy principle was adopted in this paper. The total potential energy stored by the deformation of the gear in the transmission is divided into Hertzian, bending, shear, axial compressive, and fillet-foundation energies. The values of Hertzian contact, bending, shear, axial compressive, and fillet-foundation stiffness can be extracted from their relationship to Hertzian, bending, shear, axial compressive, and fillet-foundation energies.

2.1 | TVMS calculation of a perfect gear tooth

2.1.1 | The base circle is equal to the root circle

Regardless of the transition curve between the base and the root circles, the potential energy principle is applied to calculate the TVMS of the gear tooth without fault when the base circle is equal to the root circle. The gear tooth is generally modeled as a non-uniform cantilever beam on the base circle, as shown in Figure 2. Under this condition, the scheme for calculating TVMS using the potential energy principle is defined as Method A, where the Hertzian, bending, shear, axial compressive, and fillet-foundation stiffness are all taken into account simultaneously. A rectangular coordinate xOz is established with the center of the gear O(0, 0) as the origin. d denotes the horizontal distance from the meshing point to the base circle. h represents the distance between the point on the tooth’s central line and the tooth’s curve where the horizontal distance from the base circle is d. h is the distance between the meshing point and the central line of the gear tooth where the horizontal distance from the base circle is x. The action force F can be decomposed into two orthogonal forces

\[ F_x = F \sin \alpha \]  
\[ F_y = F \cos \alpha \]

as shown in Figure 2. The calculations for the bending, shear, and axial compressive stiffness are expressed as

\[ \frac{1}{k_b} = \frac{2}{F^2} U_b, \]  \[ \frac{1}{k_s} = \frac{2}{F^2} U_s, \]  \[ \frac{1}{k_a} = \frac{2}{F^2} U_a. \]  

The action force F can be decomposed into two orthogonal forces

\[ F_x = F \sin \alpha \]  \[ F_y = F \cos \alpha \]

where \( U_b, U_s, U_a \) represent, respectively, the bending, shear, and axial compressive energies stored in a meshing gear tooth, which can be determined as

\[ U_b = \int_0^d \frac{1}{2EI} \left[ F_x(d - x) - F_y h \right]^2 dx, \]  \[ U_s = \int_0^d \frac{1}{2GA} F_y^2 dx, \]  \[ U_a = \int_0^d \frac{1}{2EA} F_x^2 dx. \]

shear modulus \( G \), cross-sectional area \( A \), and area moment of inertia \( I \) can be written as

\[ G = \frac{E}{2(1 + v)}, \]  \[ A = 2h_x L, \]  \[ I = \frac{1}{12} (2h_x)^2 L = \frac{2}{3} h_x^2 L. \]

where \( L \) is the tooth width.

For convenience, angular displacement is introduced to the calculation. If the angle \( \alpha \) is regarded as an independent variable, it can be known from Figure 2

\[ \frac{1}{k_b} = \frac{2}{F^2} U_b, \]  \[ \frac{1}{k_s} = \frac{2}{F^2} U_s, \]  \[ \frac{1}{k_a} = \frac{2}{F^2} U_a. \]  

\( U_b, U_s, U_a \) represent, respectively, the bending, shear, and axial compressive energies stored in a meshing gear tooth, which can be determined as

\[ U_b = \int_0^d \frac{1}{2EI} \left[ F_x(d - x) - F_y h \right]^2 dx, \]  \[ U_s = \int_0^d \frac{1.2G^2}{2GA} F_y^2 dx, \]  \[ U_a = \int_0^d \frac{F_x^2}{2EA} dx. \]

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where \( L \) is the tooth width.
\begin{align*}
x &= R_y [\cos \alpha + (a + b) \sin \alpha - \cos \alpha], \\
d &= R_y [\cos \alpha + (a + b) \sin \alpha - \cos \alpha], \\
h_s &= R_y [(a + b) \cos \alpha - \sin \alpha], \\
h &= R_y [(a + b) \cos \alpha - \sin \alpha].
\end{align*}

Substituting them into Equations (1)–(6) yields
\begin{align*}
\frac{1}{k_b} &= \int_{a_1}^{a_2} 3(1 + \cos \alpha) (a_2 - a) \sin \alpha \cos \alpha \left( a_2 - a \right) \cos \alpha \, da, \\
\frac{1}{k_s} &= \int_{a_1}^{a_2} \frac{1.2(1 + \nu)(a_2 - a) \cos \alpha \cos^2 \alpha \sin \alpha}{EL \sin \alpha + (a_2 - a) \cos \alpha} \, da, \\
\frac{1}{k_a} &= \int_{a_1}^{a_2} \frac{(a_2 - a) \cos \alpha \sin^2 \alpha}{2EL \sin \alpha + (a_2 - a) \cos \alpha} \, da.
\end{align*}

The Hertzian contact stiffness \( k_h \) and fillet-foundation stiffness \( k_f \) can be obtained by,\textsuperscript{24,25}
\begin{align*}
\frac{1}{k_h} &= \frac{4(1 - \nu^2)}{nEL}, \\
\frac{1}{k_f} &= \frac{\cos^2 \alpha \left( u_1 \right)^2 + M \left( u_1 \right)}{EL} + N \left( 1 + Q \tan^2 \alpha \right).
\end{align*}

Significantly, both Young’s modulus \( E \) and Poisson’s ratio \( \nu \) are mental nature properties. The Hertzian contact stiffness is dependent only on tooth width \( L \). Thus, \( k_h \) is a constant for normal gears during the meshing. Moreover, fillet-foundation stiffness is irrelevant to the position relationship between the base circle and the root circle, and the specific reason is that Equation (19) is always true whatever the position between the two circles.

\begin{equation}
\frac{1}{k_b} = \frac{4(1 - \nu^2)}{nEL},
\end{equation}

\begin{equation}
\frac{1}{k_s} = \frac{1.2(1 + \nu)(a_2 - a) \cos \alpha \cos^2 \alpha \sin \alpha}{EL \sin \alpha + (a_2 - a) \cos \alpha} \, da,
\end{equation}

\begin{equation}
\frac{1}{k_a} = \frac{(a_2 - a) \cos \alpha \sin^2 \alpha}{2EL \sin \alpha + (a_2 - a) \cos \alpha} \, da.
\end{equation}

\begin{equation}
\frac{1}{k_f} = \frac{\cos^2 \alpha \left( u_1 \right)^2 + M \left( u_1 \right)}{EL} + N \left( 1 + Q \tan^2 \alpha \right).
\end{equation}

\begin{equation}
\frac{1}{k_b} = \frac{4(1 - \nu^2)}{nEL},
\end{equation}

\begin{equation}
\frac{1}{k_s} = \frac{1.2(1 + \nu)(a_2 - a) \cos \alpha \cos^2 \alpha \sin \alpha}{EL \sin \alpha + (a_2 - a) \cos \alpha} \, da,
\end{equation}

\begin{equation}
\frac{1}{k_a} = \frac{(a_2 - a) \cos \alpha \sin^2 \alpha}{2EL \sin \alpha + (a_2 - a) \cos \alpha} \, da.
\end{equation}

\begin{equation}
\frac{1}{k_f} = \frac{\cos^2 \alpha \left( u_1 \right)^2 + M \left( u_1 \right)}{EL} + N \left( 1 + Q \tan^2 \alpha \right).
\end{equation}

2.1.2 | The base circle is bigger than the root circle

If the number of teeth is less than 42, that is to say, the base circle is bigger than the root circle, the tooth profile between the root circle and the base circle is described as an arc, as shown in Figure 3. \( O_1(x_0, z_0) \) and \( \rho \) are the center and radius of the transition curve, respectively, which satisfy the following equation
\begin{equation}
\begin{cases}
\frac{z_0 - R_y \sin \alpha_2}{x_0 - R_y \cos \alpha_2} = \tan \left( \frac{\pi}{2} + \alpha_2 \right) \\
\rho = \sqrt{(x_0 - R_y \cos \alpha_2)^2 + (z_0 - R_y \sin \alpha_2)^2} \\
\rho^2 + R_y^2 = (R_1 + \rho)^2
\end{cases}
\end{equation}

Under the above assumptions, the bending stiffness, shear stiffness, and axial compressive stiffness when \( 0 < x_1 < x_0 \) can be derived as
\begin{align*}
\frac{1}{k_b} &= \frac{2}{F^2J_0} \int_{x_1}^{x_0 - x_1} \left( F_h - F_h \right)^2 \, dx_1, \\
\frac{1}{k_s} &= \frac{2}{F^2J_0} \int_{x_1}^{x_0 - x_1} \frac{1.2F^2}{2GA} \, dx_1, \\
\frac{1}{k_a} &= \frac{2}{F^2J_0} \int_{x_1}^{x_0 - x_1} \frac{F^2}{2EA} \, dx_1,
\end{align*}

where
\begin{align*}
l_{44} &= \frac{1}{12} \left( 2h_{44} \right)^2 L, \\
A_{44} &= 2h_{44} L, \\
h_{44} &= z_0 - \sqrt{\rho^2 - (x_0 - x_1 - x_0)^2}, \\
x_0 &= R_y \cos \alpha_2, \\
x_1 &= R_1 \cos \alpha_2,
\end{align*}

\begin{equation}
z_0 - \sqrt{\rho^2 - (x_0 - (R_y \cos \alpha_2 - R_1 \cos \alpha_2 - x_0)^2} = R_1 \sin \alpha_2. \tag{29}
\end{equation}

The calculations for bending, shear, and axial compressive stiffness are still Equations (14)–(16) when \( 0 < x < d \). The total bending, shear, and axial compressive stiffness from root circle to addendum circle can be derived from Equations (14)–(16) to (21)–(23). The algorithm is defined as Method B.

2.1.3 | The base circle is less than the root circle

The cantilever beam model of gear tooth starts from the root circle when the base circle is less than the root circle, which is plotted in Figure 4. The whole gear tooth profile follows the
involute, and the bending, shear, and axial compressive stiffness are expressed as

\[ \frac{1}{k_b} = \frac{2}{F^2} \int_{x_1}^{x_2} \left[ F_b(x - d) - F_x \right] \frac{F}{EI} \, dx, \]

\[ \frac{1}{k_s} = \frac{2}{F^2} \int_{x_1}^{x_2} \frac{1.2F^2}{2G} \, dx, \]

\[ \frac{1}{k_a} = \frac{1}{F^2} \int_{x_1}^{x_2} \frac{F^2}{2E} \, dx. \]

The integral with respect to \( x \) turns into the integral about \( \alpha \) in Equations (30)–(32), one could get

\[ \frac{1}{k_3} = \int_{\alpha_3}^{\alpha_4} \frac{3(1 + \cos \alpha_1[(\alpha_2 - \alpha) \sin \alpha - \cos \alpha]^2[\alpha_2 - \alpha] \cos \alpha) - \cos \alpha_1^3}{2EI \sin \alpha + (\alpha_2 - \alpha) \cos \alpha} \, d\alpha, \]

\[ \frac{1}{k_4} = \int_{\alpha_4}^{\alpha_3} \frac{1.2(1 + \nu)(\alpha_2 - \alpha) \cos \alpha \cos^2 \alpha_1 - \cos \alpha_1^3}{EI \sin \alpha + (\alpha_2 - \alpha) \cos \alpha} \, d\alpha, \]

\[ \frac{1}{k_5} = \int_{\alpha_3}^{\alpha_4} \frac{(\alpha_2 - \alpha) \cos \alpha \sin^2 \alpha_1 - \cos \alpha_1^3}{2EI \sin \alpha + (\alpha_2 - \alpha) \cos \alpha} \, d\alpha, \]

where \( \alpha_3 \) and \( \alpha_4 \) satisfy

\[ \begin{align*}
R_t \sin \alpha_4 &= R_0[(\alpha_3 + \alpha_2) \cos \alpha_3 - \sin \alpha_3] \\
R_t \cos \alpha_4 - R_0 \cos \alpha_2 &= R_0[\cos \alpha_3 + (\alpha_3 + \alpha_2) \sin \alpha_3 - \cos \alpha_2].
\end{align*} \]

The way to calculate the TVMS when the base circle is less than the root circle is defined as Method C.

2.1.4 | Overall TVMS

On the one hand, the total effective mesh stiffness for single-tooth meshing duration can be derived as

\[ k_{ts} = \frac{1}{k_{b1}} + \frac{1}{k_{s1}} + \frac{1}{k_{a1}} + \frac{1}{k_{f1}}, \]

\[ k_{ts} = \frac{1}{k_{b2}} + \frac{1}{k_{s2}} + \frac{1}{k_{a2}} + \frac{1}{k_{f2}}, \]

\[ k_{ts} = \frac{1}{k_{b3}} + \frac{1}{k_{s3}} + \frac{1}{k_{a3}} + \frac{1}{k_{f3}}. \]

where \( k_{b1}, k_{s1}, k_{a1}, k_{f1} \) denote the bending stiffness, shear stiffness, axial compressive stiffness, and fillet-foundation stiffness of driving gear, respectively, and \( k_{b2}, k_{s2}, k_{a2}, k_{f2} \) are those of driven gear.

For the double-tooth-pair meshing duration, there are two pairs of gears meshing at the same time. The total effective TVMS can be expressed as

\[ k_{td} = \sum_{i=1}^{2} \left( \frac{1}{k_{b1}} + \frac{1}{k_{s1}} + \frac{1}{k_{a1}} + \frac{1}{k_{f1}} \right), \]

\[ k_{td} = \sum_{i=1}^{2} \left( \frac{1}{k_{b2}} + \frac{1}{k_{s2}} + \frac{1}{k_{a2}} + \frac{1}{k_{f2}} \right), \]

\[ k_{td} = \sum_{i=1}^{2} \left( \frac{1}{k_{b3}} + \frac{1}{k_{s3}} + \frac{1}{k_{a3}} + \frac{1}{k_{f3}} \right). \]

where \( i = 1 \) for the first pair and \( i = 2 \) for the second pair of meshing teeth.

2.1.5 | Comparison of different methods for solving TVMS

To verify the performance of Methods B and C, three standard involute spur gear pairs are chosen and their parameters are listed in Table 1: Young’s modulus \( E = 2.068 \times 10^{11} \, \text{Pa} \), Poisson’s ratio \( \nu = 0.3 \), and diametral pitch \( P = 8 \). Compared with Method B, the deformation energy of the gear tooth between the root circle and the base circle is ignored in Method A, which makes the TVMS larger than that in Method B. From Figure 5, it could be found that the result of Method A is always bigger than that of Method B. Figure 6 shows the comparison

**TABLE 1** Parameters of spur gear pairs

| Parameters          | Gear pair 1 | Gear pair 2 | Gear pair 3 |
|---------------------|------------|------------|------------|
| Number of teeth \( z \) | 19         | 48         | 19         | 48         | 19         | 48         |
| Young’s modulus \( E \) (GPa) | 206.8      | 206.8      | 206.8      | 206.8      | 206.8      | 206.8      |
| Poisson’s ratio \( \nu \) | 0.3        | 0.3        | 0.3        | 0.3        | 0.3        | 0.3        |
| Module \( m \) (mm) | 3.2        | 3.2        | 3.2        | 3.2        | 3.2        | 3.2        |
| Addendum coefficient \( h^* \) | 1          | 1          | 1          | 1          | 1          | 1          |
| Tip clearance coefficient \( c^* \) | 0.25       | 0.25       | 0.25       | 0.25       | 0.25       | 0.25       |
| Tooth width \( L \) (mm) | 16         | 16         | 16         | 16         | 16         | 16         |
| Pressure angle \( \alpha_0 \) (°) | 20         | 20         | 20         | 20         | 20         | 20         |
between Methods A and C. For gear pair 2, the length of the cantilever beam in Method A is larger than that in Method C; that is, extra deformation energy between the base circle and the root circle is taken into account, which will lead to the lower TVMS of the gear tooth obtained. The theoretical results are basically consistent with those in Figure 6.

If Methods B and C are used to calculate the TVMS of the pinion and gear, respectively, the results are compared with those procured by Method A, as shown in Figure 7. As can be seen from this figure, the mesh stiffness both in the single-tooth engagement region and in the double-tooth engagement region calculated by Methods B and C are less than those computed by using Method A.

### 2.2 TVMS of cracked gears

For spur gear reducer consisting of gear pair 3, compared with driven gear, the driving gear has a small number of teeth and each tooth engages in the meshing more frequently. As a consequence, the driving gear is easily prone to fatigue damages than the driven gear. In this section, we assume that there is a root crack in one tooth of the driving gear, while the other teeth are perfect. The path of crack propagation is considered as a line across the whole tooth width. According to the relationship between crack depth and the centerline of gear tooth, the crack models can be divided into two types: the type where the crack depth is below the tooth’s central line, and another type where the crack depth exceeds the centerline of the gear tooth. Under this situation, if the relationship between \( \alpha_1 \) and \( \alpha_g \) is also considered, the two kinds of crack models for driving gear can be further divided into four conditions, as shown in Figures 8-11. The bending and shear stiffness will be affected by the tooth crack for the change of the effective tooth thickness of the gear tooth. Method B is adopted to gain effective area moment of inertia and area of the section for the driving gear with crack. And the TVMS of driven gear will remain the same as that for a perfect tooth by Method C.

#### Condition 1: \( h_{c1} \geq h_i \) or \( h_{c1} < h_i \) and \( \alpha_1 \leq \alpha_g \)

\[
A_i = \begin{cases} 
(\bar{h}_{c1} + h_i) L, & 0 < x_1 < x_0 - x_r \\
2h_{c1} L, & \text{else}
\end{cases}
\]  

\[
l_i = \begin{cases} 
\frac{1}{12}(\bar{h}_{c1} + h_i)^2 L, & 0 < x_1 < x_0 - x_r \\
\frac{1}{12}(2h_{c1})^2 L, & \text{else}
\end{cases}
\]

where

\[
h_{c1} = -d_{crack} \sin \nu_{crack} + R_i \sin \alpha_g.
\]

#### Condition 2: \( h_{c1} \geq h_i \) and \( \alpha_1 > \alpha_g \)

\[
A_i = \begin{cases} 
(\bar{h}_{c1} + h_i) L, & 0 < x_1 < x_0 - x_r \\
(h_{c1} + h_i) L, & 0 < x < x_c \\
2h_{c1} L, & \text{else}
\end{cases}
\]

\[
l_i = \begin{cases} 
\frac{1}{12}(\bar{h}_{c1} + h_1)^2 L, & 0 < x_1 < x_0 - x_r \\
\frac{1}{12}(h_{c1} + h_1)^2 L, & 0 < x < x_c \\
\frac{1}{12}(2h_{c1})^2 L, & \text{else}
\end{cases}
\]

FIGURE 5 Mesh stiffness comparison between Methods A and B for gear 1

FIGURE 6 Mesh stiffness comparison between Methods A and C for gear 2

FIGURE 7 Mesh stiffness comparison between the proposed method (Methods B and C) and Method A for gear 3
\[ I_x = \begin{cases} 
\frac{1}{12}(h_{c1} + h_{c2})^3L, & 0 < x_1 < x_0 - x_r \\
\frac{1}{12}(h_{c1} + h_{c2})^3L, & 0 < x < g_c \\
\frac{1}{12}(2h_{c2})^3L, & \text{else}
\end{cases} \quad (43)
\]

\[ I_x = \begin{cases} 
\frac{1}{12}(h_{c1} - h_{c2})^3L, & 0 < x_1 < x_0 - x_r \\
\frac{1}{12}(h_{c1} - h_{c2})^3L, & 0 < x < d \\
\frac{1}{12}(2h_{c2})^3L, & \text{else}
\end{cases} \quad (45)
\]

Condition 3: \( h_{c2} \geq h_r \) or \( h_{c2} < h_r \) and \( \alpha_1 \leq \alpha_g \)

\[ A_x = \begin{cases} 
(h_{c1} - h_{c2})L, & 0 < x_1 < x_0 - x_r \\
(h_{c1} - h_{c2})L, & 0 < x < d \\
2h_{c2}L, & \text{else}
\end{cases} \quad (44)
\]

Condition 4: \( h_{c2} \geq h_r \) and \( \alpha_1 > \alpha_g \)

\[ A_x = \begin{cases} 
(h_{c1} - h_{c2})L, & 0 < x_1 < x_0 - x_r \\
(h_{c1} - h_{c2})L, & 0 < x < g_c \\
2h_{c2}L, & \text{else}
\end{cases} \quad (46)
\]
The 13-crack levels corresponding to different crack depths or crack angles are presented in Table 2. In Figure 12, the detailed effects of crack depth and crack angle on TVMS are illustrated in which Figure 12B,D are the local enlargement of Figure 12A,C, respectively. From Figure 12, it could be found that only the TVMS of 1.5 mesh periods may be affected in our model during one revolution of the driving gear, where one mesh period is defined as an angular displacement of driving gear experiencing a double-tooth-pair meshing duration and a single-tooth-pair meshing duration. In Figure 12AB, the effects of crack depth on TVMS are presented when \( \nu_{\text{crack}} \) is fixed at 45°, where Case 1 to Case 3 denotes the case of the crack depth is above the central line of the tooth. With the aggravation of gear tooth fault, the crack depth \( q_{\text{crack}} \) will reach the maximum of 0.65, and pass through the central line of the tooth and the crack depth is denoted as \( q_{\text{crack}} \). The variation of TVMS with crack depth \( q_{\text{crack}} \) is shown in Cases 4–7 in Figure 12A. It can be seen that the TVMS will decrease with the increase of crack depth, especially when it reaches the central line of the gear tooth. If it continues to expand, the reduction of TVMS may be more obvious. For Case 7, the mesh stiffness is zero in single-tooth-pair meshing duration, while the mesh stiffness in double-tooth-pair meshing duration decreases sharply, which is caused by the
The TVMS of the gear tooth is presented in Figure 12C,D, where \( v_{\text{crack}} \) is changing from 30° to 60° and the crack depth is selected as \( q_{1\text{crack}} = 4 \text{ mm} \) or \( q_{2\text{crack}} = 2 \text{ mm} \). The maximum values of \( q_{1\text{crack}} \) and \( q_{2\text{crack}} \) are different with different crack angles, such as \( q_{1\text{crackmax}} = 9.2 \text{ mm} \) for \( v_{\text{crack}} = 30^\circ \), \( q_{1\text{crackmax}} = 6.5 \text{ mm} \) for \( v_{\text{crack}} = 45^\circ \), and \( q_{1\text{crackmax}} = 5.3 \text{ mm} \) for \( v_{\text{crack}} = 60^\circ \). Similar to the change of crack depth, the TVMS of the gear tooth will gradually decrease with the increase of crack angle.

3 | DYNAMIC ANALYSIS OF A SPUR GEAR SYSTEM WITH ROOT CRACK

3.1 | Modeling of a spur gear system

To pay attention to the effects of crack growth on the vibration response of spur gear system, a 4-DOF spur gear dynamic model with root crack fault is presented in Figure 13, where both the torsional
vibration of the gear pair around the axle and the translation vibration of the supporting system (bearing, shaft, gearbox) in the radial direction are included. The equations of the torsional motion could be written as

\[
\begin{align*}
I & \frac{d^2 \theta_1}{dt^2} = -W_0R_0 + T_1, \\
I & \frac{d^2 \theta_2}{dt^2} = W_0R_2 - T_2.
\end{align*}
\]

The vertical motion equations, that is, vibrations in the y direction of driving and driven gears are

\[
\begin{align*}
\begin{cases}
-m_1 \frac{d^2 y_1}{dt^2} + c_1 \frac{dy_1}{dt} + k_1 f_1(y_1) &= -F_1 - W_0, \\
m_2 \frac{d^2 y_2}{dt^2} + c_2 \frac{dy_2}{dt} + k_2 f_2(y_2) &= -F_2 + W_0
\end{cases}
\end{align*}
\]

where

\[
W_0 = c_0 \left( R_1 \frac{d \theta_1}{dt} + R_2 \frac{d \theta_2}{dt} + \frac{dy_1}{dt} - \frac{dy_2}{dt} - \frac{de}{dt} \right) + k_0 (R_1 \theta_1 + R_2 \theta_2 + y_1 - y_2 - e).
\]

**FIGURE 13** Dynamical model of the spur gear system

**FIGURE 14** Time histories of the spur gear system for different crack cases: (A) Healthy case, (B) Case 1, (C) Case 2, (D) Case 3, (E) Case 4, (F) Case 5, (G) Case 6, (H) Case 7
\[ e = F \cos(\omega_0 t + \phi_0), \]  

and \( f_i \) denotes the nonlinear clearance function of supporting system, \( f_h \) is nonlinear displacement function of tooth backlash, \( k_h(t) \) is the TVMS.

To eliminate the influence of torsional displacement of gear system, the relative angular displacement \( y_3 = R_{012} \theta_1 + R_{02} \theta_2 + y_1 - y_2 - e \) is introduced into Equations (48) and (49), and they could be rewritten as

\[
\begin{align*}
    m_1 \frac{d^2 y_1}{dt^2} + c_1 \frac{dy_1}{dt} + k_1 f_1(y_1) + c_h \frac{dy_2}{dt} + k_h(t) f_h(y_3) &= -F_1 \\
    m_2 \frac{d^2 y_2}{dt^2} + c_2 \frac{dy_2}{dt} + k_2 f_2(y_2) - c_h \frac{dy_2}{dt} - k_h(t) f_h(y_3) &= -F_2 \\
    m_3 \frac{d^2 y_3}{dt^2} + c_3 \frac{dy_3}{dt} + k_h(t) f_h(y_3) - m_0 \frac{d^2 y_1}{dt^2} - m_n \frac{d^2 y_2}{dt^2} - m_m \frac{d^2 e}{dt^2} &= F_m
\end{align*}
\]  

(52)

where

\[ f_i(y) = \begin{cases} 
    y_i - d_i, & y_i > d_i, \\
    0, & |y_i| \leq d_i, \\
    y_i + d_i, & y_i < -d_i 
\end{cases} \]

and

\[ f_h(y_3) = \begin{cases} 
    y_3 - d_3, & y_3 > d_3, \\
    0, |y_3| \leq d_3, \\
    y_3 + d_3, & y_3 < -d_3 
\end{cases} \]

3.2 | Vibration response of simulated signals

By substituting the proposed analytical TVMS models for gear pair 3 into the single-stage spur gear reducer established in Section 3.1, the time domain and frequency domain responses about \( y_3 \) can be derived for the...
perfect gear and cracked gear with increasing deterioration levels, as shown in Figure 14. The main parameters of the gear system are consistent with those in Hou et al., and the shaft frequency of driving is 40 Hz. From Figure 14, the fault influences are not very obvious in Case 1. The obvious changes happened in the gear vibration signals as the crack level increased. The existence of root crack leads to periodic impulses appearing in the time domain. The period of the impulse is the same as the rotation period of the pinion. The more serious root crack depth will result in a greater increment of the amplitude of periodic impulses fluctuation. Similarly, from Figure 15, it could be found that a large number of sidebands appear near the mesh frequency and its multiplication in the vibration signal spectrum with the occurrence of root crack. Besides, not only the amplitude of the mesh frequency and its multiplication will increase with the crack depth, but also the amplitude of sidebands may also increase. The gear tooth may suffer a sudden breakage when crack depth \( q_{\text{crack}} \) reaches the theoretical maximum of 3.9 mm for the fixed angle between the crack line and the tooth central line. In the time domain, it is represented as Case 7, that is, the amplitude of periodic impulses fluctuation will sharply increase. At the same time, in the frequency domain, the amplitudes of the mesh frequency and its multiplication also increase sharply, especially the amplitude of their sidebands.

### 3.3 Vibration response of experimental signals

To qualitatively examine the correctness and reliability of theoretical results, a test rig for drivetrain dynamics simulator is used in this section, where the gear transmission system is composed of a single-stage reduction gearbox. The gears with root cracks or missing faults are shown in Figure 16.

The experimental results in the vertical direction are illustrated in Figure 17, in which all the teeth of driven are perfect, whereas the driving gear contains three states, namely, perfect or one of teeth is cracked or missing. The rotation speed of the driving gear is 40 Hz \( f_{n1} \) and the gear meshing frequency is 1440 Hz \( f_m \). The amplitudes of mesh frequency \( f_m \), its frequency-doubling \( 2f_m \), and their sidebands \( f_m \pm f_{n1} \) under different health conditions are shown in Table 3. The impulses fluctuation caused by the fault is not obvious because of the existence of disturbances during the experiment, as shown in Figure 17A–C. In the frequency domain, Figure 17D–F, one can clearly find the amplitudes of sidebands of gears with cracks or missing faults that are bigger than those of perfect, starting in the resonance region (1440 and 2880 Hz). The vibration trend of simulated signals basically coincides with that of the experiment.

### 4 CONCLUSIONS

In this paper, a modified analytical algorithm is proposed to solve the TVMS of gears by considering the misalignment between the base and the root circles. The effects of crack level on TVMS and vibration features of single-stage spur gear reducer are described in detail. The investigation yields the following conclusions:

1. The gear tooth is assumed to be a variable cross-section cantilever beam starting from the root circle instead of the
base circle. It will be closer to the real tooth shape. Three kinds of situations are compared according to the relation that who is big and who is small on the base circle and the root circle in the process of solving the TVMS. The non-coincidence between the root circle and the base circle may bring a large error to the calculation result of mesh stiffness, which can be improved by employing the method proposed in this paper.

(2) The existence of tooth crack mainly affects the area and the area moment of inertia of the tooth section, and the TVMS of pinion with root crack is calculated when the number of teeth is considered. The TVMS will reduce as the size of the crack grows. The occurrence of broken tooth failure, namely the crack depth reaches the maximum, makes the TVMS in the single-tooth engagement region decrease to zero, and that of in double-tooth engagement region also decreases sharply.

(3) In terms of the system response, the periodic pulse fluctuation appears in the time domain for the existence of tooth crack. And the greater the crack depth, the more obvious the sidebands. The sidebands appear near the mesh frequency and its multiplication in the vibration signal spectrum by the appearance of tooth crack. And the greater the crack depth, the more obvious the sidebands.

(4) The experiment of single-stage reduction gearbox with defects was designed to analyze the failure features, which agrees qualitatively with the theoretical results.

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CONFLICT OF INTEREST
The authors declare that there are no conflict of interest.

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Research data are not shared.

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