INFLUENCE OF THERMAL FLUCTUATIONS ON AN UNDERDAMPED JOSEPHSON TUNNEL JUNCTION

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Abstract. Inspired by a recent experiment, we study the influence of thermal fluctuations on the $I$-$V$ characteristics of a Josephson junction, coupled to a strongly resistive environment. We obtain analytical results in the limit where the Josephson energy is larger than the charging energy and quasiparticles are absent.

1. Introduction

As it is well-known, the dynamics of a small Josephson tunnel junction is characterized by two energies: the Josephson coupling energy $E_J$ and the charging energy $E_C$. The behavior of the junction is determined by the competition between these two energy scales. If the Josephson coupling energy dominates, a superconducting state with a well-defined phase difference $\phi$ across the junction is possible. The junction will carry a Cooper pair current $I = I_J \sin \phi$ in the absence of an external voltage $V$. Here $I_J = 2eE_J/h$ is the Josephson critical current ($e$ is the electron charge). If on the other hand the charging energy dominates, an insulating state with a well-defined charge $Q$ on the junction is possible. This gives rise to a gap in the $I$-$V$ characteristics of the junction, associated to Coulomb blockade: the Cooper pair current $I = 0$ up to a voltage $V = 2E_C/e$. Here we defined $E_C = e^2/2C$, where $C$ is the capacitance of the junction.

The above statements are true under appropriate conditions on the impedance $Z(\omega)$ of the circuit connected to the junction, as we will discuss in some detail below. To be specific, we will consider two configurations. The voltage-biased set-up, in Fig. 1a, consists of an ideal voltage source (voltage $V_x$), connected to a series arrangement of a Josephson junction and a resistance $R$. In a current-biased set-up, see Fig. 1b, an ideal current source (bias current $I_x$) is connected...
to a parallel arrangement of a Josephson junction and a shunt resistance $R$. The two set-ups are equivalent if we choose $I_x = V_x / R$ \[3\]. The resistance $R$ plays a crucial role and determines the detailed form of the $I$-$V$ characteristics. In particular, the way in which the transition from superconducting to insulating behavior of the junction manifests itself in the current-voltage characteristic depends not only on the parameters $E_J$ and $E_C$, but also on $R$.

![Figure 1. Voltage biased (a) and current-biased (b) Josephson junction.](image)

The voltage biased set-up, Fig. 1a, has been analyzed long ago by Ivanchenko and Zil’berman \[4\] in the limit of strong charging effects, $E_C \to \infty$. More recently, a perturbative approach has been developed to obtain the current-voltage characteristic \[3, 5\]. This approach is valid for finite, but small, values of $E_J / E_C$. Specifically, in the zero temperature limit $T = 0$, a change of the position of the supercurrent peak is found upon increasing the ratio $R / R_Q$, where $R_Q = h / 4e^2$. For $R / R_Q < 1$, the supercurrent peak is centered at zero bias voltage; its width increases gradually as $R / R_Q$ is increased. If $R / R_Q \gg 1$, the supercurrent peak is centered around $eV = 2E_C$. This corresponds to the transition from superconducting behavior found for small values of the external resistance to a complete Coulomb blockade of Cooper pair tunneling in the presence of a resistive environment. The range of applicability of the perturbative analysis depends on $E_J$, $E_C$, and $R$. In the low resistance limit $R / R_Q < 1$, the approach breaks down at low energy (bias voltage or temperature) for any value of $E_J$. In the opposite limit $R > R_Q$, the condition $E_J / E_C \ll (R_Q / R)^{1/2}$ needs to be satisfied. A non-perturbative analysis was developed in \[6\], however its validity is limited to low energies, $eV, k_B T \ll E_C R_Q / R$.

The approach discussed so far is clearly appropriate if one is interested in a small junction with $E_J \leq E_C$, coupled to an environment which is not too resistive. An example is the recent experiment by Steinbach et al. \[7\], in good agreement with the theoretical predictions of Refs. \[4, 6\]. However, the perturbative approach is not applicable in the case of a Josephson junction with $E_J \geq E_C$, coupled to a strongly resistive environment, $R \gg R_Q$, usually referred to as the underdamped limit \[1\]. Nevertheless this is a relevant situation from an experimental point of view as well. Recently, Watanabe and Haviland measured the
In these experiments, $E_J \sim E_C$ and the environment could be tuned to large values of the resistance such that $R/R_Q \sim 10^2 - 10^4$. No evidence of a supercurrent peak was observed; rather a so-called Bloch nose [9] was found for small values of the current and the bias voltage. This corresponds to an onset of Coulomb blockade at low voltage and small current, followed by subsequent back-bending of the $I$-$V$ curve to smaller values of the voltage at higher currents which indicates a cross-over to superconducting behavior. The shape of the Bloch nose depended on the resistance $R$ as well as on temperature. The experimental findings could not be reconciled quantitatively with existing theories of Bloch nose [9, 10]. The main problem is that those theories are essentially based on the presence of dissipation due to quasiparticles. In the experiment [8], the tunable environment is a long array of superconducting junctions in the insulating regime [11], and quasiparticles are practically absent.

In this paper we analyze the finite temperature behavior of a Josephson junction with $E_J \gg E_C$, coupled to a very resistive environment, in the absence of quasiparticles. We obtain, to our knowledge for the first time, explicit formulae for the current-voltage characteristics at finite temperature for a current-biased configuration with large $R$, see Fig. 1b. We follow the approach pioneered in Ref. [9], which is based on the periodicity of the Josephson coupling energy as a function of the phase difference $\phi$ across the junction. As a result of this periodicity, the eigenstates of the junction are of the Bloch type, and the energy spectrum consists of bands of width of the order $E_C$, separated by gaps of the order $E_J$. In the limit $E_J \gg E_C$, the gaps are large and the junction stays in the lowest Bloch band at low temperature $k_B T \ll E_J$, as long as the bias current $I_x \ll I_J$. If $R = \infty$, Bloch oscillations [9] of the voltage $V$ occur, such that its time-averaged value vanishes: the junction is in the superconducting state. If $R$ is finite, relaxation within the lowest band modifies the dynamics of the junction. This results in the appearance of the Bloch nose: if $I_x$ is so small that relaxation prevents Bloch oscillations to occur, a finite voltage state develops with no current passing through the junction (Coulomb blockade), all current passes through the resistance. If $I_x$ is increased above a threshold value, Bloch oscillations develop and the a superconducting zero-voltage state is reached. Our approach is valid for a large temperature range and our results can be compared quantitatively with the experimental data of Ref. [8].

2. The model

The circuit of Fig. 1b can be modelled by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{\text{coul}} + \hat{H}_{\text{env}}$. Here,

$$\hat{H}_0 = \frac{\hat{Q}^2}{2C} - E_J \cos \hat{\phi}$$

(1)
describes the Josephson junction in the absence of the external circuit. The charge \( \hat{Q} \) and the phase \( \hat{\phi} \) are canonically conjugate operators, \( [\hat{Q}, \hat{\phi}] = 2i\hbar \). Since the Hamiltonian (1) is periodic in \( \phi \), the energy spectrum consists of bands \( \epsilon_n(q) \) where \( n \) is the band index and \( q \) the quasimomentum associated to \( \phi \), i.e., the quasicharge. The spectrum \( \epsilon_n(q) \) is a periodic function of \( q \) with period \( 2e \). We will limit ourselves to the first Brillouin zone, \( -e \leq q \leq e \). The corresponding eigenstates \( \psi_{nq}(\phi) \) are of the Bloch type. In the limit of interest here, \( E_J \gg E_C \), the bands are narrow compared to the gaps. Throughout this paper we will assume that all relevant energies are smaller than \( E_J \) and restrict ourselves to the lowest energy band, \( n = 0 \), with a dispersion 

\[
\epsilon_0(q) = \hbar \sqrt{8E_CE_J - U_0 \cos(\pi q/e)} \]

where the bandwidth is given by

\[
U_0 = 16\sqrt{\frac{2}{\pi}E_C \left[ \frac{E_J}{2E_C} \right]^{3/4}} e^{-\sqrt{8E_CE_J/E_C}}. \tag{2}
\]

The term

\[
\hat{H}_{coup} = -\frac{\hbar}{2e} \hat{I} \hat{\phi} \tag{3}
\]
couples \( \hat{\phi} \) to the total current \( \hat{I} = I_x + \delta \hat{I} \) which contains the constant bias current \( I_x \) and a fluctuating part \( \delta \hat{I} \). The fluctuations are induced by the environment (the resistance \( R \) in Fig. 1b) and the dynamics of \( \delta \hat{I} \) is governed by \( \hat{H}_{env} \).

In the absence of any current, \( \hat{H}_{coup} = 0 \) and quasicharge \( q \) is a well defined variable. In the limit of small bias current \( I_x \) and current fluctuations \( \delta I \), the effect of the coupling term (3) on \( q \) can be analyzed perturbatively. As it was shown in Refs. [9], a quantum Langevin equation for the quasicharge operator \( \hat{q} \) can be obtained, which reads

\[
\frac{d\hat{q}}{dt} = I_x - \frac{1}{R} \frac{d\epsilon_0}{d\hat{q}} + \delta \hat{I}. \tag{4}
\]

Here, \( I_x \) plays the role of a driving force and \( \delta \hat{I} \) of a random force. The statistical properties of the latter are determined by the Hamiltonian \( \hat{H}_{env} \). Specifically, for a resistive environment at equilibrium we have [13]

\[
\langle [\delta \hat{I}(t), \delta \hat{I}(t')]_{+} \rangle_{env, \omega} = 2(h\omega/R) \coth[h\omega/(2k_BT)], \tag{5}
\]

where \([\ldots, \ldots]_{+}\) denotes the anticommutator and \( \langle \ldots \rangle_{env} \) an average with respect to \( \hat{H}_{env} \). Finally, the derivative \( d\epsilon_0/d\hat{q} \) is the voltage operator \( \hat{V} \) across the junction, the corresponding term \( V/R \) acts as a damping term.

We will be particularly interested in the limit of small current fluctuations. In this limit, quasicharge remains a well defined quantity and we can replace operators by classical variables in Eq. (4). In the next Section, we will analyze the resulting classical Langevin equation in some detail, using well-known methods for stochastic equations [12]. We will see under which condition current fluctuations can be considered small and we will obtain the detailed, analytical form of \( I-V \) characteristics, as a function of the resistance \( R \) and of temperature \( T \).
3. Current-voltage characteristics

3.1. LINEARIZED LANGEVIN EQUATION

In the absence of fluctuations, $\delta \hat{I} = 0$, the Langevin equation (4) has a stationary solution when $I_x = (1/R)d\epsilon_0/dq \equiv V/R$, with $q = q_0 = (e/\pi) \arcsin(I_x/I_b)$. Here, $I_b = V_b/R$, with $V_b = \pi U_0/e$ the maximum voltage across the junction corresponding to the maximum slope of $\epsilon_0(q)$. The corresponding $I$-$V$ characteristic is linear, $I_x = V/R$. In other words, all the current passes through the resistance $R$, no current flows through the junction (Coulomb blockade) and the phase $\phi$ is completely undefined. This is consistent with the fact that we treat $q$ as a well-defined variable. Note that the stationary solution exists only as long as $I_x \leq I_b$; for larger current, $I_x > I_b$, the quasicharge becomes time-dependent. We will discuss that case in Sec. 3.2 below.

Let us now take into account small fluctuations $\delta \hat{I}$ such that the induced charge fluctuations $\delta q$ around the stationary solution $q = q_0$ are small. We consider the linear part of the average current-voltage characteristics and calculate

$$\langle V \rangle = \langle d\epsilon_0/dq \rangle = V_b \langle \sin(\pi q/e) \rangle.$$  \hspace{1cm} (6)

Here the symbol $\langle \ldots \rangle$ means averaging over the fluctuations of charge. In order to find the distribution of these fluctuations, we write $q = q_0 + \delta q$, such that the linearized, classical Langevin equation takes the form

$$\delta \dot{q} = -\frac{\delta q}{\tau} + \delta I,$$  \hspace{1cm} (7)

where we introduced the relaxation time $\tau$ such that $\tau^{-1} = (\pi/e)\sqrt{|I_b^2 - I_x^2|}$. The time $\tau$ defines the time scale characterizing the junction dynamics. Note that this is a relatively long time compared to the $RC$ time of the circuit: for $I_x = 0$, we find $\tau \sim e/I_b \sim R e^2/U_0 \gg RC$ as $U_0$ is exponentially small, see Eq. (2).

In order to solve the stochastic equation (7), we should specify the correlation function $\langle \delta I(t)\delta I(t') \rangle$, see Eq. (5). The frequencies $\omega$ of interest here are small, $\omega \sim 1/\tau$, determined by the long time scale $\tau$ characterizing the dynamics of quasicharge. For finite temperatures $k_B T \gg h/\tau$, the correlation function $\langle \delta I(t)\delta I(t') \rangle \sim 2k_B T/R$, independent of frequency and thus $\langle \delta I(t)\delta I(t') \rangle = (2k_B T/R)\delta(t - t')$. This means that the relevant current fluctuations are classical and completely uncorrelated on the long time scale $\tau$ characterizing the quasicharge dynamics.

The stochastic equation (7) can now be solved, using standard methods developed for the analysis of Brownian motion [12]. As a result, one obtains the probability distribution $W(\delta q, t; \delta q_0)$ to find the fluctuation $\delta q$ at time $t$ given that $\delta q = \delta q_0$ at time $t = 0$. In the long time limit, $t \to \infty$, this probability
distribution does not depend on time and is independent of $\delta q_0$. It is given by the Gaussian distribution

$$W(\delta q, \tau) = \sqrt{\frac{1}{\pi \gamma \tau}} \exp \left( -\frac{(\delta q)^2}{\gamma \tau} \right),$$

where we introduced the parameter $\gamma = \frac{2k_B T}{R}$.

Using Eq. (8) we immediately obtain the average square of the charge fluctuations, $\langle (\delta q)^2 \rangle = k_B T \tau / R$. Quasicharge can be considered well-defined as long as these fluctuations are small, $\langle (\delta q)^2 \rangle \ll e^2$. This condition gives an additional restriction on temperature, $k_B T \tau / h \ll R / R_Q$. Together with the condition $k_B T \gg \hbar / \tau$ found above we obtain the temperature window

$$1 \ll k_B T \tau / h \ll R / R_Q.$$  

This means in particular that $R / R_Q \gg 1$ for the analysis presented here to be correct. If this condition is verified, quasicharge is a well-defined variable, and quantum fluctuations can be neglected. For small $I_x$ the energy scale $\hbar / \tau$ is of the order of $eV_b R_Q / R$, thus the condition (9) reads $eV_b R_Q / R \ll k_B T \ll eV_b$. This has a transparent physical interpretation: the temperature has to be much smaller than $eV_b$ in order not to smear the Bloch nose, but it must be larger than $eV_b (R_Q / R) \ll eV_b$ in order to justify the neglect of quantum fluctuations. The experiment of Ref. [8] was performed at very large values of $R / R_Q$, such that the condition (9) was verified.

With the help of Eqs. (6) and (8) we readily obtain the average voltage for the linear part of the average $I$-$V$ characteristic,

$$\langle V \rangle = I_x R \left[ 1 - \frac{\pi}{2} \frac{k_B T}{eR} \frac{1}{\sqrt{I_b^2 - I_x^2}} \right].$$  

Equation (10) represents the main result of this section. The first term on the right hand side of Eq. (10) coincides with the linear $I$-$V$ curve which was obtained above without taking into account current fluctuations. The second term is entirely due to the thermal fluctuations, which suppress the resistance. In other words, the Coulomb blockade is smeared due to thermal activation.

In this subsection we have considered the limit of small enough bias current, $I_x < I_b$, such that the Langevin equation could be linearized around a well-defined, stationary solution. For larger values of $I_x$, no stationary solution exists: quasicharge is an oscillating function of time and the full non-linearity of the Langevin equation should be taken into account. In Section 3.2 we will use a Fokker-Planck approach to obtain the complete current-voltage characteristics.
3.2. FOKKER-PLANCK APPROACH

If the applied bias current $I_x$ exceeds $I_b$, quasicharge is a dynamical variable even in the absence of fluctuations. The corresponding non-linear equation of motion can be integrated, though, and $q(t)$ can be found. As a result, the voltage $V$ becomes an oscillating function of time, with a period given by $\tau$. Its time-averaged value $\overline{V}$ over one period of the oscillation is given by

$$\overline{V} = I_b R \left[ I_x/I_b - \sqrt{(I_x/I_b)^2 - 1} \right].$$

In other words, on measurement time scales much longer than the period $\tau$, the voltage oscillations average out and a stationary situation is reached.

In order to investigate the influence of small current fluctuations $\delta I$ on the $I$-$V$ characteristics, the full non-linear stochastic problem needs to be solved. This can be conveniently done using a Fokker-Planck approach. In this approach, the probability distribution $W(q, t)$ to find quasicharge $q$ at time $t$ is found, as a solution of the Fokker-Planck equation

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial q} \left( I_x - I_b \sin \left( \frac{\pi q}{e} \right) \right) W + \frac{\gamma^2}{2} \frac{\partial^2 W}{\partial q^2}.$$  

In writing this equation, we limited ourselves again to the case of classical fluctuations only, $k_B T \gg \hbar/\tau$. In addition, we note that this equation can be used only as long as the fluctuations are $\delta$-correlated on the time scale defined by $\tau$. Hence, we need to restrict ourselves again to the temperature interval.

We are particularly interested in the stationary solution of $W(q)$, reached in the long time limit $t \gg \tau$. The average voltage is then given by

$$\langle V \rangle = I_b R \int_{-e}^{+e} dq \sin(\pi q/e) W(q),$$

where $W(q)$ is normalized, $\int_{-e}^{+e} W(q) dq = 1$. In the long time limit $t \gg \tau$, this corresponds to the measured, time-averaged voltage, $\langle V \rangle = \overline{V}$.

Let us first analyze the solution of the problem in the absence of charge fluctuations [parameter $\gamma$ equals to zero in Eq. (12)]. Solving Eq. (12) and using the normalization condition for the distribution function we obtain

$$W(q) = \begin{cases} 
\delta(q - q_0), & I_x \leq I_b \\
\frac{1}{2e \sqrt{(I_x/I_b)^2 - 1}} \sin(\pi q/e), & I_x > I_b,
\end{cases}$$

where $q_0$ is the stationary solution found in Section 3.1. Substituting Eq. (14) into Eq. (13) for the current-voltage characteristics we immediately obtain the linear...
Figure 2. Calculated $I-V$ characteristics for an underdamped junction. Left panel: bias current $I_x/I_b$ as a function of voltage $\langle V \rangle/V_b$ across the junction, for various values of the parameter $k_B T/eV_b$. Curves from right to left correspond to $k_B T/eV_b = 0, 0.004, 0.02,$ and $0.1$. Right panel: current $I/I_b$ through the junction as a function of $\langle V \rangle/V_b$. Curves from right to left correspond to the same values of $k_B T/eV_b$.

result $\langle V \rangle = I_x R$ for $I < I_b$; for the opposite case $I_x > I_b$ we reproduce the result (11). The resulting $I-V$ characteristic, a so-called Bloch nose, is shown in Fig. 2 as the curve corresponding to the parameter $k_B T/eV_b = 0$.

Next we consider the current-voltage characteristics in the presence of fluctuations. To find the distribution function, $W(q)$, we use a Fourier representation

$$W(q) = \sum_{n=-\infty}^{+\infty} W_n e^{i\pi n q/e}. \quad (15)$$

Substituting the right hand side of Eq. (15) into Eq. (12) and using the periodicity condition, $W(q + 2\pi) = W(q)$, we obtain the following recurrence relation for the distribution function

$$2(n + i\eta_x) W_n = \eta_b (W_{n-1} - W_{n+1}). \quad (16)$$

Here we introduced the short hand notation $\eta_j = 2eI_j/(\pi\gamma)$ for $j = x, b$. To find the distribution function $W_n$ in Eq. (16) we use the analogy with the recurrence relation for the modified Bessel function \[ 2\nu I\nu(\xi) = \xi[I\nu-1(\xi) - I\nu+1(\xi)] \]. Here $\nu$ is the index and $\xi$ is the argument of the modified Bessel function. A comparison with (16) yields the distribution function $W_{\nu-\eta_x} = CI\nu(\eta_b)$, where $C = (2eI_{\eta_b}(\eta_b))^{-1}$ is the normalization constant. Substituting this result into Eq. (15) and using the fact that the distribution function is real, $W_{-n} = W^*_n$, we obtain, upon integration over $q$, the following result for the $I-V$ characteristics

$$\langle V \rangle = I_x R - \frac{k_B T \sinh(\pi \eta_x)}{e |I_{\eta_x}(\eta_b)|^2} \quad (17)$$

Equation (17) is the main result of this section. It describes the complete current-voltage characteristics in the presence of small fluctuations. The $I-V$ characteristics of Eq. (17) are shown in the left panel of Fig. 2 for various values of the
parameter $k_B T / e V_b$. For comparison we plot the current $I$ through the junction as a function of $\langle V \rangle$ in the right panel of Fig.2 for the same values of $k_B T / e V_b$. This would correspond to a measurement in the voltage-biased set-up, see Fig.1. We see that the $I$-$V$ curve corresponding to the limit without fluctuations is asymptotically reached by the finite temperature $I$-$V$ curves, both for $I_x / I_b \ll 1$ and for $I_x / I_b \gg 1$. Mathematically, this is a direct result of the asymptotics of the modified Bessel functions $[14]$. Physically, this can be easily understood from the fact that the point $I_x = I_b$ is unstable, separating a stationary quasicharge solution from an oscillating one. In the vicinity of $I_b$, any small current fluctuation will drastically change the nature of the dynamics. Thus the $I$-$V$ characteristic is strongly affected by fluctuations for $I_x \sim I_b$.

We finally note that the analytical form of the result (17) resembles the well-known result $[15]$ for the $I$-$V$ characteristic of an overdamped junction with $R \ll R_Q$ and $E_J \gg E_C$. In fact, these two cases can be related by duality arguments. Here, we studied the temperature driven diffusion of quasicharge $q$, governed by Eq. (12), and found the time-averaged voltage across the junction as the average $\langle \sin(\pi q / e) \rangle$ over the fluctuations of $q$. In an overdamped junction, it is the dual phase variable $\phi$ that diffuses, governed by an equation similar to (12) and one is interested in the time averaged current $\langle \sin \phi \rangle$ over the fluctuations of $\phi$. This is why the resulting $I$-$V$ curves can be related to each other, essentially by exchanging the role of $I$ and $V$.

4. Discussion

In this paper we have considered the influence of thermal fluctuations on the current-voltage characteristics of an underdamped Josephson junction with $R \gg R_Q$ and $E_J \gg E_C$, in the absence of quasiparticles. To obtain analytical results for the $I$-$V$ characteristics, we have analyzed the appropriate Langevin equation using a Fokker-Planck approach. The resulting $I$-$V$ characteristic is essentially a Bloch nose $[9]$, smeared by thermal fluctuations. Our results are valid as long as the inequality (9) is satisfied. Since we restricted our analysis to the lowest energy band we need to impose in addition $k_B T \ll E_J$ and $I_x \ll I_J$.

The present work is motivated by recent experiments $[8]$ on a small, voltage biased Josephson junction coupled to a tunable environment. This environment consisted of a long array with a large number ($\sim 10^2$) Josephson junctions. Each junction is a small SQUID-loop, threaded by a magnetic flux $\Phi$. Thus the Josephson coupling energy $E_J^{\text{array}}(\Phi)$ of the array is flux-dependent and the ratio $E_J^{\text{array}} / E_C^{\text{array}}$ could be changed during the experiment. When $\Phi$ is tuned close to $\Phi_0 / 2$, where $\Phi_0$ is the superconducting flux quantum, the ratio $E_J^{\text{array}} / E_C^{\text{array}} \ll 1$ and the array is in the insulating regime.

We ignore quasiparticles: in order for them to reach the small Josephson junction, they have to tunnel through the entire array; the corresponding probability is
negligibly small. The frequency-dependent impedance \(Z(\omega)\) of the array, as seen by the small junction, is then entirely due to the dynamics of Cooper pairs. In the stationary limit of interest here, we need to know the zero-frequency component \(Z(\omega = 0)\). Calculations by Efetov [16] in the limit \(E_{J}^{\text{array}} / E_{C}^{\text{array}} \ll 1\) show that the real part \(R\) of \(Z(\omega = 0)\) is strictly infinite at zero temperature; at finite \(T\), thermal activation gives rise to a finite, large value of \(R\), which depends on the ratio \(E_{J}^{\text{array}} / E_{C}^{\text{array}}\), and thus on \(\Phi\). Note that any residual quasiparticle contribution would be insensitive to the Josephson coupling and hence independent of \(\Phi\). Once \(R(\Phi)\) and the parameters of the small junction are known, our results can be compared in principle with the experimental data of [8].

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