Matrix model generating function for quantum weighted Hurwitz numbers

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Abstract

The KP $\tau$-function of hypergeometric type serving as generating function for quantum weighted Hurwitz numbers is used to compute the Baker function and the corresponding adapted basis elements, expressed as absolutely convergent Laurent series in the spectral parameter. These are equivalently expressed as Mellin–Barnes integrals, analogously to Meijer $G$-functions, but with an infinite product of $\Gamma$-functions as integral kernel. A matrix model representation is derived for the $\tau$-function evaluated at trace invariants of an externally coupled matrix.

Keywords: quantum Hurwitz numbers, generating function, matrix integral, quantum Hurwitz numbers, KP $\tau$-function, hypergeometric $\tau$-function

1. Introduction

Hurwitz numbers enumerate branched coverings of the Riemann sphere, with specified ramification profiles at the branch points, and have been studied since the pioneering works of Hurwitz [27, 28]. Equivalently, they may be viewed as combinatorial invariants enumerating factorization of elements in the symmetric group $S_n$ as products of elements in specified conjugacy classes [16, 17].

There has been considerable interest in recent years in making use of $\tau$-functions, which are dynamical generating functions for solutions of classical integrable hierarchies, such as the KP or 2D Toda hierarchies, rather as generating functions in the combinatorial sense, for various types of enumerative geometrical and topological invariants related to Riemann surfaces. Pandharipande and Okounkov [34, 35] showed that special cases of $\tau$-functions of hypergeometric type [36, 37] may be viewed as generating functions for simple Hurwitz numbers (which enumerate branched coverings in which all but one, or two, of the branch
points have simple ramification profiles.) Several other instances of $\tau$-functions of hypergeometric type were shown to serve as generating functions for weighted single or double Hurwitz numbers $H^\omega_{\mu}(\mu)$, $H^\nu_{\mu}(\mu, \nu)$ [20–22, 26]. In many cases representations of such $\tau$-functions as random matrix integrals for various classes of measures were found. These include: simple Hurwitz numbers [11, 15, 33–35]; strongly monotonic Hurwitz numbers [19, 25]; weakly monotonic Hurwitz numbers and, more generally, polynomially or rationally weighted Hurwitz numbers [1–3, 7, 11, 14, 30, 33, 41].

Here, we derive a new matrix model representation for the KP $\tau$-function $\tau^{(H_{\mu}^{\omega})}(t)$ generating quantum weighted Hurwitz single numbers $H^\omega_{\mu}(\mu)$ [21–23]. The main result (theorem 5.1) is that, when the flow parameters $t = (t_1, t_2, \cdots)$ are set equal to the trace invariants

$$t_i = \frac{1}{i} \ln(x^i) = \frac{1}{i} \sum_{a=1}^{n} x^i_a,$$

(1.1)
of a given $n \times n$ matrix $X$, with eigenvalues $(x_1, \ldots, x_n)$, denoted $[X]$, the $\tau$-function $\tau^{(H_{\mu}^{\omega})}([X])$ is expressible as the product

$$\tau^{(H_{\mu}^{\omega})}([X]) = \beta^n(n-1)! \prod_{i=1}^{n} x^i_{a-1} \Delta(\ln x) \mathcal{Z}_{\mu}(\ln X)$$

(1.2)
of a simple explicit Vandermonde determinantal factor depending on the $x_a$’s and a generalized Brézin–Hikami [12] matrix integral of the form

$$\mathcal{Z}_{\mu}(Y) = \int_{M \in \text{Nor}_{\mu}^{n \times n}} d\mu_q(M) e^{iYM}. $$

(1.3)

Here

$$d\mu_q(M) = d\mu_0(M) \det(A_{H_{\mu}^{\omega}}(M))$$

(1.4)
is a conjugation invariant measure on the space $\text{Nor}_{\mu}^{n \times n}$ of $n \times n$ normal matrices with eigenvalues $\zeta_a \in C_n$ supported on a contour $C_n$ in the complex plane, surrounding simple poles of the integrand at the integers $-n, -n+1, \cdots$, $d\mu_0(M)$ is the Lebesgue measure on $\text{Nor}_{\mu}^{n \times n}$ and $A_{H_{\mu}^{\omega}}(z)$ is a convergent infinite product of Euler $\Gamma$-functions, as defined in equation (3.1) of section 3.

In section 2, the definition of weighted Hurwitz numbers, as introduced in [20–22, 26], is recalled and the KP $\tau$-function that serves as generating function for these is defined, focusing on the case $\tau^{(H_{\mu}^{\omega})}$ of quantum weighted generating functions. Following [4–6], we also define a natural basis $\{\phi_i^{H_{\mu}^{\omega}}\}_{i \in \mathbb{N}^+}$ for the element $W^{(H_{\mu}^{\omega})}$ of the infinite Sato–Segal–Wilson Grassmannian corresponding to this $\tau$-function, expressed as convergent Laurent series, and the recursion operators relating them.

In section 3, we derive a Mellin–Barnes integral representation for the $\phi_i^{H_{\mu}^{\omega}}$s, analogous to the one for the Meijer $G$-functions that appear in the case of rationally weight generating functions [14], but with integral kernels consisting of convergent infinite products of $\Gamma$-functions depending on the quantum parameter $q$. In section 4, the recursions relating the $\phi_i^{H_{\mu}^{\omega}}$s are applied to the finite determinantal expression arising when the KP flow parameters are equated to the trace invariants of a finite dimensional matrix, giving a Wronskian form for the $\tau$-function. This serves, in section 5, as the link to expressing it as a matrix integral of generalized Brézin–Hikami [12] type.
2. Generating functions for quantum Hurwitz numbers

2.1. Weighted Hurwitz numbers and weight generating functions

We recall the definition of pure Hurwitz numbers [16, 17, 27, 28, 31] and weighted Hurwitz numbers [20–22, 26].

Definition 2.1 (Combinatorial). For a set of $k$ partitions $\{\mu^{(i)}\}_{i=1,...,k}$ of $N \in \mathbb{N}^+$, the pure Hurwitz number $H(\mu^{(1)},\ldots,\mu^{(k)})$ is $\frac{1}{N!}$ times the number of distinct ways that the identity element $I_N \in S_N$ in the symmetric group in $N$ elements can be expressed as a product

$$I_N = h_1 \cdots h_k$$

(2.1)
of $k$ elements $\{h_i \in S_N\}_{i=1,...,k}$ such that for each $i$, $h_i$ belongs to the conjugacy class $cyc(\mu^{(i)})$ whose cycle lengths are equal to the parts of $\mu^{(i)}$:

$$h_i \in cyc(\mu^{(i)}), \quad i = 1, \ldots, k.$$  

(2.2)

An equivalent definition involves the enumeration of branched coverings of the Riemann sphere.

Definition 2.2 (Geometric). For a set of partitions $\{\mu^{(i)}\}_{i=1,...,k}$ of weight $|\mu^{(i)}| = N$, the pure Hurwitz number $H(\mu^{(1)},\ldots,\mu^{(k)})$ is defined geometrically as the number of inequivalent $N$-fold branched coverings $C \to \mathbb{P}^1$ of the Riemann sphere with $k$ branch points $(Q^{(1)},\ldots,Q^{(k)})$, whose ramification profiles are given by the partitions $\{\mu^{(1)},\ldots,\mu^{(k)}\}$, normalized by the inverse $1/|\text{aut}(C)|$ of the order of the automorphism group of the covering.

The equivalence of the two follows from the monodromy homomorphism from the fundamental group of $\mathbb{P}^1/\{Q^{(1)},\ldots,Q^{(k)}\}$, the Riemann sphere punctured at the branch points, into $S_N$, obtained by lifting closed loops from the base to the covering [31].

To define weighted Hurwitz numbers [20–22, 26], we introduce a weight generating function $G(z)$, either as an infinite product

$$G(z) = \prod_{i=1}^{\infty} (1 + c_i z)$$

(2.3)
or an infinite sum

$$G(z) = 1 + \sum_{i=1}^{\infty} g_i z^i,$$  

(2.4)

either formally, or under suitable convergence conditions imposed upon the parameters. Alternatively, it may be chosen in the dual form

$$\tilde{G}(z) = \prod_{i=1}^{\infty} (1 - c_i z)^{-1},$$

(2.5)

which may also be developed as an infinite sum,

$$\tilde{G}(z) = 1 + \sum_{i=1}^{\infty} \tilde{g}_i z^i.$$  

(2.6)
The independent parameters determining the weighting may be viewed as either the Taylor coefficients \( \{g_i\}_{i \in \mathbb{N}} \), \( \{\tilde{g}_i\}_{i \in \mathbb{N}} \) or the parameters \( \{c_i\}_{i \in \mathbb{N}} \) appearing in the infinite product formulae (2.3) and (2.5). They are related by the fact that (2.3) and (2.5) are generating functions for elementary and complete symmetric functions, respectively,

\[
\tilde{g}_i = e_i(\mathbf{c}), \quad \tilde{g}_i = h_i(\mathbf{c}),
\]

in the parameters \( \mathbf{c} = (c_1, c_2, \ldots) \).

**Definition 2.3 (Weighted Hurwitz numbers).** For the case of weight generating functions of the form (2.3), choose a nonnegative integer \( d \) and a fixed partition \( \mu \) of weight \( |\mu| = N \). The weighted (single) Hurwitz number \( H^d_G(\mu) \) is then defined \([21, 26]\) as the weighted sum over all \( k \)-tuples \( (\mu^{(1)}, \ldots, \mu^{(k)}) \)

\[
H^d_G(\mu) := \sum_{k=1}^d \sum_{\mu^{(1)}} ^{\mu^{(k)}} \text{sgn}(\mu) \sum_{\lambda} \text{sgn}(\lambda) \text{sgn}(\mu^{(1)}) \cdots \text{sgn}(\mu^{(k)}) W_G(\mu^{(1)}, \ldots, \mu^{(k)}) \text{sgn}(\lambda) \text{sgn}(\mu^{(1)}) \cdots \text{sgn}(\mu^{(k)}),
\]

where

\[
\ell^*(\mu^{(i)}) := |\mu^{(i)}| - \ell(\mu^{(i)})
\]

is the colength of the partition \( \mu^{(i)} \), and the weight factor is defined to be

\[
\text{sgn}(\lambda) := \sum_{\sigma \in S_N} \prod_{i=1}^N \text{sgn}(\sigma(i) - \sigma(i+1))
\]

which, up to a normalization factor, is the monomial symmetric function \( m_\lambda(\mathbf{c}) \) \([32]\) in the variables \( \mathbf{c} = (c_1, c_2, \ldots) \) corresponding to the partitions \( \lambda \) of length \( k \) and weight \( d = \sum_{i=1}^k \ell^*(\mu^{(i)}) \) whose parts are equal to the colengths \( \{\ell^*(\mu^{(i)})\}_{i=1}^k \).

For the case of dual generating functions of the form (2.5), the weighted (single) Hurwitz number \( H^d_G(\mu) \) is defined \([21, 26]\) as the weighted sum

\[
H^d_G(\mu) := \sum_{k=1}^d \sum_{\mu^{(1)}} ^{\mu^{(k)}} \text{sgn}(\mu) \sum_{\lambda} \text{sgn}(\lambda) \text{sgn}(\mu^{(1)}) \cdots \text{sgn}(\mu^{(k)}) W_G(\mu^{(1)}, \ldots, \mu^{(k)}) \text{sgn}(\lambda) \text{sgn}(\mu^{(1)}) \cdots \text{sgn}(\mu^{(k)}),
\]

where the weight factor is defined as

\[
\text{sgn}(\lambda) := \sum_{\sigma \in S_N} \prod_{i=1}^N \text{sgn}(\sigma(i) - \sigma(i+1))
\]

which, again up to a normalization factor, is the forgotten symmetric function \( f_\lambda(\mathbf{c}) \) \([32]\) in the variables \( \mathbf{c} = (c_1, c_2, \ldots) \) corresponding to the partition \( \lambda \) of length \( k \) and weight \( d = \sum_{i=1}^k \ell^*(\mu^{(i)}) \) with parts equal to the colengths \( \{\ell^*(\mu^{(i)})\}_{i=1}^k \).

### 2.2. Quantum weighted Hurwitz numbers

In the following, we consider a variant of the case of quantum weighted Hurwitz numbers \([21, 22]\), with weight generating function of dual type (2.5), with the constants \( c_i \) chosen to be...
for some parameter $q$ with $0 < |q| < 1$, so that
\[
\widetilde{G}(z) = H_q(z) := \prod_{i=0}^{\infty} (1 - q^i z)^{-1} = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n},
\]
(2.14)
where
\[
(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1})
\]
is the $q$-Pochhammer symbol.

**Remark 2.1.** This may be viewed as the scaled $q$-exponential function [9], since
\[
\lim_{q \to 1} H_q((1 - q)z) = e^z.
\]
(2.16)

Specializing (2.11) and (2.12) to this case gives:

**Definition 2.4 (Quantum weighted Hurwitz numbers).** For weight generating function (2.14), choosing a nonnegative integer $d$ and a fixed partition $\mu$ of $N$, the quantum weighted (single) Hurwitz number $H_d^\mu (\mu)$ is the weighted sum over all $k$-tuples $(\mu(1), \ldots, \mu(k))$
\[
H_d^\mu (\mu) := \sum_{k=1}^{d} \sum_{\mu(1), \ldots, \mu(k) : |\mu(i)| = N \sum_{i=1}^{k} \ell^*(\mu(i)) = d} \widetilde{W}_{d^\mu}(\mu(1), \ldots, \mu(k)) H(\mu(1), \ldots, \mu(k), \mu)
\]
(2.17)
where
\[
\ell^*(\mu(i)) := |\mu(i)| - \ell(\mu(i))
\]
is the colength of the partition $\mu(i)$, and the weight factor is
\[
\widetilde{W}_{d^\mu}(\mu(1), \ldots, \mu(k)) := \frac{(-1)^{d-k}}{k!} \sum_{\sigma \in S_k} \sum_{1 \leq b_1 \leq \cdots \leq b_k} q^{(b_{\sigma(i)}-1)\ell^*(\mu(i)) - 1} \cdots q^{(b_{\sigma(k)}-1)\ell^*(\mu(k))}
\]
\[
= \frac{(-1)^{d-k}}{k!} \prod_{\sigma \in S_k} \frac{1}{(1 - q^{\sum_{i=1}^{k} \ell^*(\mu(\sigma(i)))})}.
\]
(2.19)

**Remark 2.2 (Riemann–Hurwitz formula).** If $d$ is defined as in the constraint on the weighted sums (2.17),
\[
d = \sum_{i=1}^{k} \ell^*(\mu(i)),
\]
(2.20)
the Riemann–Hurwitz formula gives the Euler characteristic $\chi$ of the branched cover $C \to P^1$ with $k + 1$ branch points $(Q^{(1)}, \ldots, Q^{(k)}, Q)$ with ramification profiles $(\mu^{(1)}, \ldots, \mu^{(k)}, \mu)$ as
\[
\chi = N + \ell(\mu) - d.
\]
(2.21)
If $C$ is connected and $g$ is the genus of $C$, we have
\[
\chi = 2 - 2g.
\]
(2.22)
2.3. Hypergeometric \(\tau\)-functions

We next recall the definition \[21, 22, 26\] of the KP \(\tau\)-function of hypergeometric type \[36, 37\] that serves as generating function for the quantum weighted Hurwitz numbers \(H^d_{Hq}(\mu, \nu)\). For the weight generating function \(H^q(z)\) and a nonzero small parameter \(\beta\), we define two doubly infinite sequences of numbers \(\{r_i^{(H_q, \beta)}, \rho_i\}_{i \in \mathbb{Z}}\), labeled by the integers

\[
r_i^{(H_q, \beta)} := \beta H_q(i \beta), \quad i \in \mathbb{Z}, \quad \rho_0 = 1,
\]

related by

\[
r_i^{(H_q, \beta)} = \rho_i^{(H_q, \beta)} / \rho_{i-1}^{(H_q, \beta)},
\]

where \(\beta\) is chosen such that the \(H_q(i \beta)\) does not vanish for any integer \(i \in \mathbb{Z}\). For each partition \(\lambda\) of \(N\), we define the associated content product coefficient \[21, 22, 26\]

\[
r^{(H_q, \beta)}_{\lambda} := \prod_{(i,j) \in \lambda} r_{j-i}^{(H_q, \beta)}.
\]

The KP \(\tau\)-function of hypergeometric type associated to these parameters is then defined as the Schur function series

\[
\tau^{(H_q, \beta)}(t) := \sum_{N=0}^{\infty} \sum_{|\lambda|=N} \frac{d_{\lambda} \prod_{i} r_i^{(H_q, \beta)}}{N!} s_{\lambda}(t),
\]

where \(d_{\lambda}\) is the dimension of the irreducible representation of the symmetric group \(S_N\) and \(t = (t_1, t_2, \ldots)\) are the infinite sequence of KP flow parameters, equated to the infinite sequence of normalized power sum symmetric functions \((p_1, p_2, \ldots)\) \[32\] in some auxiliary infinite sequence of variables \((x_1, x_2, \ldots)\).

We then use the Schur character formula \[18, 32\]

\[
s_{\lambda} = \sum_{\mu, | \mu| = |\lambda|} \frac{\chi_{\lambda}(\mu)}{\zeta_{\mu}} p_{\mu},
\]

where \(\chi_{\lambda}(\mu)\) is the irreducible character of the \(S_N\) representation determined by \(\lambda\) evaluated on the conjugacy class \(\text{cyc}(\mu)\) consisting of elements with cycle lengths equal to the parts of \(\mu\), and

\[
\zeta_{\mu} = \ell(\mu) \prod_{i=1}^{\ell(\mu)} m_i(\mu)!^{m_i(\mu)}
\]

is the order of the stabilizer of the elements of this conjugacy class, re-express the Schur function series \(2.27\) as an expansion in the basis \(\{p_{\mu}\}\) of power sum symmetric functions, where

\[
p_{\mu} := \prod_{i=1}^{\ell(\mu)} p_{\mu_i}, \quad p_{j} = j^t, \quad j \in \mathbb{N}.
\]
Theorem 2.1 ([20–23, 26]). The \( \tau \)-function \( \tau^{H_\alpha, \beta}(t) \) may equivalently be expressed as
\[
\tau^{H_\alpha, \beta}(t) = \sum_{\mu} \sum_{|\mu|=d} \beta^d H_{H_\alpha}^{d}(\mu) p_{\mu}(t)
\]
(2.31)
and is thus a generating function for the weighted Hurwitz numbers \( H_{H_\alpha}^{d}(\mu) \).

2.4. The (dual) Baker function and adapted basis

By Sato’s formula [38–40], the dual Baker–Akhiezer function \( \Psi^{\ast}_{(G, \beta)}(z, t) \) corresponding to the KP \( \tau \)-function equation (2.27) is
\[
\Psi^{\ast}_{(H_\alpha, \beta)}(z, t) = e^{-\sum_{i=1}^{\infty} t_i \tau^{H_\alpha, \beta}(t + [z^{-1}])},
\]
(2.32)
where
\[
[z^{-1}] := \left( \frac{1}{z}, \frac{1}{2z^2}, \ldots, \frac{1}{nz^n}, \ldots \right).
\]
(2.33)
Evaluating at \( t = 0 \) and setting
\[
x := \frac{1}{z},
\]
(2.34)
we define
\[
\phi^{(H_\alpha, \beta)}_{1}(x) := \Psi^{\ast}_{H_\alpha, \beta}(1/z, 0) = \tau^{H_\alpha, \beta}([x]).
\]
(2.35)
More generally, following [4–6], we introduce a sequence of functions \( \{ \phi^{(H_\alpha, \beta)}_{k}(x) \}_{k \in \mathbb{N}} \), defined as contour integrals around a loop centred at the origin (or as formal residues)
\[
\phi^{(H_\alpha, \beta)}_{k}(x) = \frac{\beta}{2\pi i x^{k-1}} \oint_{\zeta=0} \rho^{(H_\alpha, \beta)}(\zeta) e^{-x\zeta} \frac{d\zeta}{\zeta},
\]
(2.36)
where \( \rho^{(H_\alpha, \beta)}(\zeta) \) is the Fourier series
\[
\rho^{(H_\alpha, \beta)}(\zeta) := \sum_{i \in \mathbb{Z}} \rho^{(H_\alpha, \beta)}_{i} \zeta^{-i-1},
\]
(2.37)
with the \( \rho^{(H_\alpha, \beta)}_{i} \)'s given by equations (2.23) and (2.24). Then \( \{ \phi^{(H_\alpha, \beta)}_{k}(1/z) \}_{k \in \mathbb{N}^+} \) forms a basis for the element \( W^{(H_\alpha, \beta)} \) of the infinite Sato–Segal–Wilson Grassmannian corresponding to the \( \tau \)-function \( \tau^{(H_\alpha, \beta)}(t) \).

The \( \phi^{(H_\alpha, \beta)}_{k}(x) \)'s may alternatively be expanded as Laurent series by evaluating the integrals as a sum of residues at the origin,
\[
\phi^{(H_\alpha, \beta)}_{k}(x) = \beta x^{k-1} \sum_{j=0}^{\infty} \frac{\rho^{(H_\alpha, \beta)}_{i} H^{(H_\alpha, \beta)}_{-j} \left( \frac{x}{\beta} \right)^{j}}{j!}.
\]
(2.38)
It follows that these satisfy the recursive sequence of equations
\[
\beta(D + k - 1) \phi^{(H_\alpha, \beta)}_{k} = \phi^{(H_\alpha, \beta)}_{k-1}, \quad k \in \mathbb{Z}.
\]
(2.39)
We then have the following results regarding convergence of the Taylor series (2.14) and (2.38) and the asymptotic form of $H_q(z)$ for large $|z|$, which are all proved in the appendix.

**Lemma 2.2.** For any $\beta < 0$, $\ j \geq 1$ the radius of convergence of the Taylor series of the function $\log H_q(\beta(j + z))$ is greater than $\frac{1}{2}$.

**Lemma 2.3.** The series (2.38) is absolutely convergent for all $x$ provided $\beta < 0$.

**Lemma 2.4.** The asymptotic form of $H_q(z)$ for large $|z|$ in the left half plane is given by

$$\log H_q(z) \sim \frac{\log^2(-z)}{2\log q} - \frac{\log(-z)}{2\log q} - S(-z) - C_q + o(1)$$

(2.40)

with

$$S(z) = \sum_{k=1}^{\infty} \frac{\cos \left( \frac{2\pi k \log z}{\log q} \right)}{\sinh \left( \frac{2\pi k \log q}{2\log q} \right)} , \quad C_q = -\frac{\pi^2}{6} \left( \frac{1}{\log q} + \frac{\log q}{2\pi^2} \right).$$

(2.41)

Following [4–6], we introduce the recursion operator

$$R := \beta x H_q(\beta D),$$

(2.42)

where $D$ is the Euler operator

$$D := x \frac{d}{dx},$$

(2.43)

and verify that the $\phi^k_{(H_q,\beta)}$ also satisfy the recursion relations

$$R(\phi^k_{(H_q,\beta)}) = \phi^{(H_q,\beta)}_{k-1}, \quad k \in \mathbb{Z},$$

(2.44)

and the $k = 1$ value $\phi^1_{(H_q,\beta)}(x)$ coincides with (2.35).

### 3. Mellin–Barnes integral representation of $\phi_k(x)$

As in the case of rational weight generating functions [13], we can equivalently represent the function $\phi^k_{(H_q,\beta)}$ in the form of a Mellin–Barnes integral, provided $\beta < 0$. Define

$$A_{H_q,\beta}(z) := (-\beta)^{-k-1} \Gamma(1-k-z) \prod_{m=0}^{\infty} \left( (-\beta q^{-m})^{-z} \frac{\Gamma(-\beta^{-1}q^{-m})}{\Gamma(z-\beta^{-1}q^{-m})} \right).$$

(3.1)

**Theorem 3.1.** The following integral representation of $\phi^k_{(H_q,\beta)}(x)$ is valid for all $x \in \mathbb{C}$,

$$\phi^k_{(H_q,\beta)} = \frac{1}{2\pi i} \int_{C_k} A_{H_q,\beta}(s) x^s ds,$$

(3.2)

where the contour $C_k$ starts at $+\infty$ immediately above the real axis, proceeds to the left above the axis, winds around the poles at the integers $s = -k, -k + 1 \ldots$ in a counterclockwise sense and returns, just below the axis, to $\infty$.

The proof of this theorem depends on the asymptotic behaviour of $A_{H_q,\beta}(s)$ as $s \to \infty$ in the right half plane, which follows from the next two lemmas, whose proofs are given in the appendix.
Lemma 3.2. The asymptotic form of the function $A_{H,q,k}$ for large $|z|$, $\Re(z) > 0$ and $\beta < 0$, is given by

$$\log A_{H,q,k}(z) \sim \frac{z \log^2(-\beta z)}{2 \log q} + O(z \log z), \quad \Re(z) > 0.$$  \hfill (3.3)

Lemma 3.3. The asymptotic form of the sum

$$\hat{S}(z) = \sum_{k=0}^{\infty} \left[ \beta^{-1} q^{-m} \log(1 - \beta z q^m) + z \right],$$  \hfill (3.4)

for large $|z|$, $\Re(z) > 0$ and $\beta < 0$, is given by

$$\hat{S}(z) \sim -\frac{z \log(-\beta z)}{\log q} + O(z).$$  \hfill (3.5)

We now proceed to the proof of theorem 3.1.

Proof of theorem 3.1. The asymptotic formula (3.3) ensures that the integral in (3.2) is convergent. The poles are simple and located at the integers $\{i-k\}_{i \in \mathbb{N}}$... and the residue at $s = i-k$ is

$$\frac{\beta}{(i-1)!} \left( \frac{x}{\beta} \right)^{i-k} \rho^{(H,q)}_{i-k-1},$$  \hfill (3.6)

where $\rho^{(H,q)}_{j}$ is defined in (2.24). Evaluating the integral as the sum over residues at the poles $\{s = i-k\}_{i \in \mathbb{N}}$ thus gives equation (2.38).

4. The $\tau$-function $\tau^{(H,q,\beta)}(t)$ evaluated on power sums

As detailed in [4–6], $\tau^{(H,q,\beta)}(t)$ is the KP $\tau$-function corresponding to the Grassmannian element $W^{(H,q,\beta)}$ spanned by the basis elements $\{\phi^{(H,q,\beta)}(1/z)\}_{i \in \mathbb{N}^+}$ obtained from the monomial basis $\{z^i\}_{i \in \mathbb{N}}$ by applying a suitable group element $g$.

$$W^{(H,q,\beta)} = \text{span}\{\phi^{(H,q,\beta)}(1/z) := g^{(H,q,\beta)}(z^{-1})\},$$  \hfill (4.1)

where

$$(g^{(H,q,\beta)} f)(z) := \text{res}_{\zeta = 0} \left( \rho^{(H,q,\beta)}(\zeta) f(z/\zeta) e^{\frac{\zeta}{\beta}} \right)$$  \hfill (4.2)

and

$$\rho^{(H,q,\beta)}(z) := \sum_{n \in \mathbb{Z}} \rho^{(H,q,\beta)}_{n-1} z^n.$$  \hfill (4.3)

If $\tau^{(H,q,\beta)}(t)$ is evaluated at the trace invariants

$$t = [X], \quad t_i = \frac{1}{i} \text{tr} X^i$$  \hfill (4.4)
of a diagonal $n \times n$ matrix

$$X := \text{diag}(x_1, \ldots, x_n),$$

(4.5)

it follows from the Cauchy–Binet [32] identity that it is expressible as the ratio of $n \times n$ determinants [24]

$$\tau(\eta, \beta)([X]) = \frac{\prod_{i=1}^{n} x_i^{n-1} \det \left( \phi_i^{(\eta, \beta)}(x_j) \right)_{1 \leq i, j \leq n}}{\Delta(x)},$$

(4.6)

where

$$\Delta(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

(4.7)

is the Vandermonde determinant.

From (2.38) and (2.39), it also follows that each $\phi_k^{(\eta, \beta)}(x)$ may be expressed as a finite lower triangular linear combination of the powers of the Euler operator $D$ applied to $\phi_n(x)$:

$$\phi_k^{(\eta, \beta)}(x) = \beta^{n-k} D^{n-k} \phi_n^{(\eta, \beta)}(x) + \sum_{j=0}^{n-k-1} \Gamma_{kj} \phi_n^{(\eta, \beta)}(x), \quad k = 1, \ldots, n,$$

(4.8)

where the constant coefficients $\Gamma_{kj}$ are easily determined from (2.39). Therefore, by elementary column operations, we have

$$\tau(\eta, \beta)([X]) = \kappa_n^{(\eta, \beta)} \left( \prod_{i=1}^{n} x_i^{n-1} \det \left( D^{j-i} \phi_n^{(\eta, \beta)}(x_j) \right)_{1 \leq i, j \leq n} \right),$$

(4.9)

where

$$\kappa_n^{(\eta, \beta)} := \frac{\beta^{n(n-1)}}{\prod_{i=1}^{n} \rho_i^{(\eta, \beta)}},$$

(4.10)

Now define the diagonal matrix

$$Y = \text{diag}(y_1, \ldots, y_n)$$

(4.11)

by

$$X = e^Y, \quad Y = \ln(X), \quad x_i = e^{y_i}, \quad i = 1, \ldots, n,$$

(4.12)

and let

$$f_n^{(\eta, \beta)}(y) := \phi_n^{(\eta, \beta)}(e^y) = \int_{C_n} A_{\eta, \beta}(s) e^{ys} \, ds.$$
where
\[
\Delta(e^y) := \prod_{1 \leq i < j \leq n} (e^{y_i} - e^{y_j}).
\] (4.15)

5. Matrix integral representation of \( \tau^{(H_q, \beta)} ([X]) \)

We now state and prove our main result.

Consider the generalized Brézin–Hikami [12] matrix integral
\[
Z_{d\mu_q}(Y) = \int_{M \in \text{Nor}_{C_n}^{p\times n}} d\mu_q(M) e^{Y M},
\] (5.1)
where
\[
d\mu_q(M) = d\mu_0(M) \det(A_{H_q, n}(M))
\] (5.2)
is a conjugation invariant measure on the space \( \text{Nor}_{C_n}^{p\times n} \) of \( n \times n \) normal matrices (i.e. unitarily diagonalizable)
\[
M = UZU^\dagger \in \text{Nor}_{C_n}^{p\times n}, \quad U \in U(n), \quad Z = \text{diag}(\zeta_1, \ldots, \zeta_n),
\] (5.3)
with eigenvalues \( \zeta_i \in \mathbb{C}_n \) supported on the contour \( \mathcal{C}_n \), and \( d\mu_0(M) \) is the Lebesgue measure on \( \text{Nor}_{C_n}^{p\times n} \).

**Theorem 5.1.** The KP \( \tau \)-function \( \tau^{(H, \beta)}(t) \) has the following matrix integral representation when restricted to the trace invariants \([X]\) of an externally coupled matrix:
\[
\tau^{(H_q, \beta)} ([X]) \sim \frac{\beta^{n(n-1)} (\prod_{i=1}^{n-1} x_i^{n-1}) \Delta (\ln(x))}{\prod_{i=1}^{n} \Delta (\ln(x))} Z_{d\mu_q} (\ln(X)). \tag{5.4}
\]

**Proof.** Using the Harish–Chandra–Itzykson–Zuber integral [25, 29]
\[
\int_{U \in U(n)} d\mu_H(U) e^{Y U U^\dagger} = \frac{\prod_{i=1}^{n-1} \det (e^{y} \zeta_i)}{\Delta(y) \Delta(\zeta)} \tag{5.5}
\]
where \( d\mu_H(U) \) is the Haar measure on \( U(n) \), to evaluate the angular integral gives
\[
Z_{d\mu_H}(Y) = \frac{\prod_{i=1}^{n} \det (e^{y_i} \zeta_i)}{\Delta(y) \Delta(\zeta)} \int_{C_n} d\zeta A_{H_q, n}(\zeta) \Delta(\zeta) \det (e^{y_i} \zeta_i)_{1 \leq i, j \leq n}
\]
\[
= \frac{\prod_{i=1}^{n} \det (e^{y_i} \zeta_i)}{\Delta(y) \Delta(\zeta)} \prod_{i=1}^{n} \det (f_{n}^{(H_q, \beta)}(\zeta_i)_{1 \leq i, j \leq n}. \tag{5.6}
\]
where we have used the Andréiev identity [8] in the second line. By equation (4.14), we therefore have the matrix integral representation (5.4) of \( \tau^{(H_q, \beta)} ([X]) \). \( \square \)
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Appendix. Proofs of lemmas 2.2, 2.3, 3.2 and 3.3

We here provide the proofs of the lemmas that were omitted from the body of the paper.

Proof of lemma 2.2.

\[
\log H_q(\beta(j + z)) = \log H_q(\beta j) - \sum_{k=0}^{\infty} \log \left( 1 - \frac{z \beta q^k}{1 - j \beta q^k} \right). \tag{A.1}
\]

Since we are only interested in \(|z| \leq \frac{1}{2}\), the following inequality holds:

\[
\left| \frac{z \beta q^k}{1 - j \beta q^k} \right| < \left| \frac{z}{j} \right| \leq \frac{1}{2}. \tag{A.2}
\]

Then each logarithm can be expanded into convergent Taylor series:

\[
\log H_q(\beta(j + z)) = \log H_q(\beta j) - \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} \frac{1}{i} \left( \frac{z \beta q^k}{1 - j \beta q^k} \right)^i. \tag{A.3}
\]

Let us denote by \(k_j\) the largest integer such that \(-j \beta q^k_j > 1\). In order to show that the order of summation can be changed without losing the convergence we split the sum as follows:

\[
\left| \sum_{i=1}^{\infty} \frac{1}{i} \left( \frac{z}{j} \right)^i \sum_{k=0}^{k_j} \left( \frac{j \beta q^k}{1 - j \beta q^k} \right)^i \right| < \sum_{i=1}^{\infty} \frac{1}{i} \left( \left| \frac{z}{j} \right| \right)^i \frac{\log(-j \beta)}{\log q} - 1, \tag{A.4}
\]

\[
\left| \sum_{l=k_j+1}^{\infty} \frac{1}{i} \left( \frac{z}{j} \right)^i \sum_{k=0}^{\infty} \left( \frac{j \beta q^k}{1 - j \beta q^k} \right)^i \right| < \sum_{i=1}^{\infty} \frac{1}{i} \left( \left| \frac{z}{j} \right| \right)^i \frac{1}{1-q} < \sum_{i=1}^{\infty} \frac{1}{i} \left( \left| \frac{z}{j} \right| \right)^i \frac{1}{1-q} \tag{A.5}
\]

Proof of lemma 2.3.

Consider the logarithm of \(\rho_m^{(H_q,\beta)}\):

\[
\log \rho_m^{(H_q,\beta)} = j \log \beta + \sum_{i=1}^{j} \log H_q(i \beta). \tag{A.6}
\]

Using lemma 2.2 we can replace the sum with an integral as follows:

\[
\sum_{i=1}^{j} \log H_q(i \beta) = \int_{\frac{1}{2}}^{j + \frac{1}{2}} \log H_q(z \beta) dz + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{4^i (2i + 1)!} \left. \frac{d^i \log H_q(z \beta)}{dz^{2i}} \right|_{z=i}. \tag{A.7}
\]

Derivatives of the leading term of the expansion (2.40) are given by

\[
\frac{d^n \log^2(-z)}{dz^n} = \frac{2(n - 1)!}{z^n} \left( \log(-z) - \sum_{k=1}^{n-1} \frac{1}{k} \right). \tag{A.8}
\]
Consequently, there exist positive constants $M_1$ and $M_2$ such that
\[
\left| \frac{d^n \log^2(\beta(l+z))}{n! dz^n} \right|_{z=0} < \frac{M_1 \log l + M_2 \log n}{n}, \quad n \geq 2.
\] (A.9)

The correction from higher derivatives can be estimated as
\[
\sum_{l=1}^{j} \sum_{i=1}^{\infty} \frac{1}{(2i+1)!} \left| \frac{d^{2i} \log H_q(z\beta)}{dz^{2i}} \right|_{z=l} < \sum_{l=1}^{\infty} \frac{2(M_1 + M_2) \log l}{l} = O(1)
\] (A.10)
and is therefore bounded, which results in
\[
\sum_{l=1}^{j} \log H_q(l\beta) = \int_{j+\frac{1}{2}}^{j+\frac{1}{2}} \log H_q(z\beta)dz + O(1).
\] (A.11)

Substituting the asymptotic expansion (2.40) into the last formula one gets
\[
\log \rho^l(H_q,\beta) = \frac{j \log^2(j)}{2 \log q} + O(j \log(j)).
\] (A.12)

Since $\log q < 0$
\[
\lim_{m \to \infty} \rho^m(H_q,\beta) = 0,
\] (A.13)
so the series (2.38) is absolutely convergent.

**Proof of lemma 2.4.** The ‘counting’ function
\[
\eta(\zeta) = \log(1 - e^{-2\pi i\zeta})
\] (A.14)
has the following useful properties:
\[
\text{res}_{\zeta=n} \frac{d\eta(\zeta)}{d\zeta} = 1, \quad n \in \mathbb{Z},
\] (A.15)
\[
|\eta(ze^{-i\omega})| = \exp(-2\pi \zeta \sin(\omega)) + O\left(e^{-4\pi \zeta \sin(\omega)}\right), \quad \omega \in (0, \pi), \zeta \in \mathbb{R}, \zeta \to \infty
\] (A.16)
\[
\eta(-\zeta^*) = (\eta(\zeta))^*.
\] (A.17)

It is therefore straightforward to write down the following integral representation for $\log H_q(z)$:
\[
\log H_q(z) = -\sum_{m=0}^{\infty} \log(1 - zq^m) - \int_{0}^{\infty} \log(1 - zq^\zeta)d\zeta
\]
\[
-\frac{\omega}{\pi} \log(1 - z) - i \sum_{\sigma = \pm 1} \sigma \int_{0}^{\infty} \left[ \log(1 - zq^{e^{i\sigma \omega}}) \frac{\partial \eta(q^{e^{i\sigma \omega}})}{\partial \zeta} \right] d\zeta,
\] (A.18)
where $0 < \omega < \arctan\left(-\frac{\pi - |\arg(z)|}{\log(q)}\right)$. Integrating by parts and moving the contour of integration for the last integral through the singularities of $\log(1 - zq^\zeta)$ to $\omega = \frac{\pi}{2}$, one gets another integral representation
\[
\sum_{m=0}^{\infty} \log(1 - zq^m) = \int_0^\infty \log(1 - z\beta q^\zeta) \, d\zeta \\
+ \frac{1}{2} \log(1 - z) - \int_0^\infty \sum_{\sigma = \pm 1} \log q \left( \frac{1}{1 - zq^{-1}e^{2\pi \zeta}} \log(1 - e^{-2\pi \zeta}) \right) \, d\zeta + S(-z),
\]
where the term \(S(-z)\) arises as a sum of the residues at the points
\[
\zeta_k = \frac{i\pi(2k + 1) + \log(z)}{\log q}.
\]

The function \(S(z)\) is periodic in \(\log(|z|)\). The sum in (2.41) is uniformly convergent for \(\arg(z) \in [-\frac{\pi}{2}, \frac{\pi}{2}]\) and therefore the function \(S(-z)\) is bounded in the left half plane. Since \(|q^K| = 1\) the last integral can be expanded into Taylor series in \(z^{-1}\) that is convergent for \(|z| > 1\). Rewriting the first integral in (A.18) as
\[
- \log q \int_0^\infty \log(1 - zq^{\zeta}) \, d\zeta = \int_1^\infty \log(1 + \lambda^{-1})\lambda^{-1} \, d\lambda + \int_0^1 \log(1 + \lambda)\lambda^{-1} \, d\lambda
\]
\[
- \int_{-\infty}^{-z^{-1}} \log(1 + \lambda)\lambda^{-1} \, d\lambda - \frac{1}{2} \int_{-z^{-1}}^1 \frac{d \log^2 \lambda}{d\zeta},
\]
we can expand it in \(z\) to get the statement of the lemma.
\[\quad\]
\textbf{Proof of lemma 3.2.} Using Binet’s integral representation [10] for \(\Gamma(z)\) and noticing that
\[
- \sum_{m=0}^{\infty} \left( z - \frac{1}{2} \right) \log(1 - z\beta q^m) = \left( z - \frac{1}{2} \right) \log H_q(\beta z),
\]
one gets, upon cancellation of like terms,
\[
\log A_{H_q,k}(z) = \log \left( \Gamma(1 - k - z)H_q^{-1}(\beta z) \right) + \hat{S}(z)
\]
\[
- \sum_{m=0}^{\infty} \int_0^\infty \frac{1}{\lambda} \left( \frac{1}{2} - \frac{1}{\lambda} + \frac{1}{e^\lambda - 1} \right) e^{\beta^{-1}q^{-m}}(e^{-\zeta} - 1) \, d\zeta.
\]
The factor \(\left( \frac{1}{2} - \frac{1}{\lambda} + \frac{1}{e^\lambda - 1} \right)\) is bounded by a positive function:
\[
0 < \left( \frac{1}{2} - \frac{1}{\lambda} + \frac{1}{e^\lambda - 1} \right) \leq \frac{t}{12}.
\]
Therefore, provided \(\Re(z) > 0\),
\[
\left| \int_0^\infty \left( \frac{1}{2} - \frac{1}{\lambda} + \frac{1}{e^\lambda - 1} \right) e^{\theta z q^{-1} - 1} \, d\zeta \right| < \int_0^\infty \left| e^{-\theta z q^{-1} - 1} \right| \frac{dt}{12} = \frac{1}{12(\Re(z) - q^{-K}\beta^{-1})}.
\]
Furthermore, since
\[
\sum_{k=0}^{\infty} \frac{1}{(z - q^{-k}\beta^{-1})} = - \frac{d \log(H_q(\beta z))}{dz},
\]
taking into account the asymptotic behaviour of $H_q(2.40)$ gives
\[ \sum_{k=1}^{\infty} \int_0^\infty \left( \frac{1}{t^2} - \frac{1}{e^t - 1} \right) e^{q^{-1} \beta^{-1} - t} \, dt = O \left( \frac{\log \beta}{\log q} \right) \tag{A.28} \]

Combining (A.24), (A.28) and using lemmas 2.4 and 3.3 we obtain
\[ \log A_{H_q, k}(z) = z \log(H_q(\beta z)) + O(z \log z) = z \log^2(-\beta z) + O(z \log z). \tag{A.29} \]

**Proof of lemma 3.3.** We repeat the steps leading to (A.19) in the proof of the lemma 2.4 in order to get the following integral representation:
\[ \hat{S}(z) = \int_0^\infty \left[ \beta^{-1} q^{-\zeta} \log(1 - z \beta q^\zeta) + z \right] \, d\zeta \]
\[ + \frac{\log q}{2\pi} \sum_{\sigma = \pm 1} \int_0^\infty h(i\sigma \zeta, z) \log \left( 1 - e^{-2\pi \zeta} \right) \, d\zeta \]
\[ + \beta^{-1} \tilde{S}(-\beta z), \tag{A.30} \]

where
\[ h(\theta, z) = -\frac{1}{\beta q^{-\theta} \log(1 - z \beta q^\theta)} - \frac{z}{1 - z \beta q^\theta} \tag{A.31} \]

and
\[ \tilde{S}(z) = 2\pi z \sum_{k=1}^{\infty} \frac{\sin(2\pi k \log(z)) - 2\pi k \log q \cos(2\pi k \log(z))}{\sinh(2\pi k \log q)(1 + 4\pi^2 k^2 \log q \pi)}. \tag{A.32} \]

The sum $\tilde{S}$ is bounded by a linear function. The last integral in (A.30) grows like $\log(z)$. The leading contribution therefore comes from the first integral, which can be computed analytically:
\[ \int_0^\infty \left[ \beta^{-1} q^{-\zeta} \log(1 - z \beta q^\zeta) + z \right] \, d\zeta = \frac{z}{\log q} \left( 1 - \frac{1}{\beta z} \log(1 - \beta z) \right). \tag{A.33} \]

Substituting (A.32) and (A.33) into (A.30) yields the statement of the lemma. \[ \square \]

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