COMPLEX VARIETIES AND HIGHER INTEGRABILITY OF
Dir-MINIMIZING $Q$-VALUED FUNCTIONS

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Abstract. We provide new elementary proofs of the following two results: every complex variety is locally the graphs of a Dir-minimizing function, first proved by Almgren [1]; the gradients of Dir-minimizing functions, in principle square-summable, are $p$-integrable for some $p > 2$, proved by De Lellis and the author [4]. In the planar case, we prove that our integrability exponents are optimal.

0. Introduction

Almgren developed the theory of Dir-minimizing multi-valued functions in his big regularity paper [1] as a first step toward the regularity of area-minimizing currents in codimension bigger than 1. Following the pioneering ideas of De Giorgi, the starting point was the approximation of minimal currents via harmonic functions, which are the minimizers of the first non-constant term in the expansion of the area functional: the Dirichlet energy. However, due to the unavoidable phenomenon of branching points as, for example, in the area-minimizing currents induced by complex varieties, he needed to develop the theory of Dir-minimizing $Q$-valued functions, that are multi-valued functions minimizing a suitable Dirichlet energy.

In this paper, following the work in [3], we address two questions on Almgren’s $Q$-valued functions: we show that complex varieties are locally graphs of Dir-minimizing functions and prove the higher integrability of the gradient of a Dir-minimizing $Q$-function.

Theorem 0.1. Let $\mathcal{V} \subseteq \mathbb{C}^\mu \times \mathbb{C}^\nu \simeq \mathbb{R}^{2\mu} \times \mathbb{R}^{2\nu}$ be an irreducible holomorphic variety which is a $Q : 1$-cover of the ball $B_2 \subseteq \mathbb{C}^\mu$ under the orthogonal projection. Then, there exists a Dir-minimizing $Q$-valued function $f \in W^{1,2}(B_1, A_Q(\mathbb{R}^{2\nu}))$ such that $\text{graph}(f) = \mathcal{V} \cap (B_1 \times \mathbb{C}^\nu)$.

Theorem 0.2. There exists $p = p(n, m, Q) > 2$ such that, for every $\Omega \subseteq \mathbb{R}^m$ open and $u \in W^{1,2}(\Omega, A_Q(\mathbb{R}^n))$ Dir-minimizing, $|Du| \in L^p_{\text{loc}}(\Omega)$.

Theorem 0.1 provides many examples of Dir-minimizing functions and, in particular, shows that the Hölder continuity and the estimate of the singular set of a Dir-minimizer proved in [1] and [3] are optimal results. Theorem 0.1 has been proved by Almgren in his big regularity paper [1, Theorem 2.20] using a deep and complicated approximation theorem of minimal currents via graphs of Lipschitz $Q$-functions (see also [4]). Here we give a more elementary proof avoiding the approximation result by Almgren. Moreover, for the planar case we also provide an alternative argument which exploits the equality between the area and the energy of conformal maps. We hope that this approach can be extended to the study of regularity issues for more complicated calibrated geometries.
Theorem 0.2 has been first proved by De Lellis and the author in [4] in connection with a new higher integrability estimate for minimal currents and it plays a crucial role in the proof of Almgren’s approximation theorem given there. Here, we propose a different “intrinsic” proof, where “intrinsic” means based only on the metric theory of $Q$-valued functions as developed in [3]. In case $m = 2$, we can exploit the fact that Dir-minimizing functions have isolated singularities (proven in [3]) to find the optimal integrability. The optimality is indeed shown by the examples provided by complex varieties in the first part of the paper.

The paper is organized as follows. In Section 1 we collect some basic results and definitions on $Q$-valued functions and the rectifiable currents supported by their graphs. In Section 2 we identify complex varieties as graphs of Sobolev $Q$-valued functions and prove Theorem 0.1. Finally, Section 3 contains the proof of Theorem 0.2 which passes through a Caccioppoli and a reverse Hölder inequality for Dir-minimizing functions.

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1. $Q$-valued functions

In what follows, we adopt the notation and the approach introduced in [3], which differs from Almgren’s original one. For the definitions of the metric space of $Q$-points $(\mathcal{A}_Q, \mathcal{G})$, Sobolev $Q$-valued function and Dirichlet energy, we refer to [3]. We say that a function $f : \Omega \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ has a smooth local selection in $\Omega' \subseteq \Omega$ if, for every $x \in \Omega'$, there exist $r > 0$ and $f_i : B_r(x) \to \mathbb{R}^n$ smooth functions such that $f|_{B_r(x)} = \sum_{i=1}^Q [f_i]$. Note that, in this case, $|Df|^2 = \sum_{i=1}^Q |Df_i|^2$ is well defined on the whole $\Omega'$. We observe the following simple consequence of the definition, which for reader’s convenience we state as a lemma.

**Lemma 1.1.** Let $f : \Omega \subset \mathbb{R}^m \to \mathcal{A}_Q$ have a smooth local selection in $\Omega' \subseteq \Omega$. If $\dim_H(\Omega \setminus \Omega') \leq m - 2$ and $\int_{\Omega'} |Df|^2 < +\infty$, then $f$ belongs to $W^{1,2}(\Omega, \mathcal{A}_Q)$.

**Proof.** The proof follows from the characterization of classical Sobolev functions via the slice property. Indeed, for every $T \in \mathcal{A}_Q$, the function $x \mapsto \mathcal{G}(f(x), T)$ is smooth and satisfies $|D(\mathcal{G}(f(\cdot), T))| \leq |Df|$ in $\Omega'$ (cp. to [3, Proposition 2.17]). Therefore, since the projection of $\Omega \setminus \Omega'$ on each coordinate hyperplane is a set of $\mathcal{H}^{m-1}$ measure zero, for $\mathcal{H}^{m-1}$-a.e. line $l$ parallel to the axes, the restriction of $\mathcal{G}(f(\cdot), T)$ to $l$ belongs to $W^{1,2}$. Recalling [5, Section 4.9.2], it follows that $\mathcal{G}(f(\cdot), T) \in W^{1,2}(\Omega)$ with $|D(\mathcal{G}(f(\cdot), T))| \leq |Df|$ a.e. in $\Omega$. Hence, by the definition of Sobolev $Q$-functions [3, Definition 0.5], we conclude. \qed

We will need also a technical result about the lower semicontinuity of the $L^p$ norm of the gradient under weak convergence. Although this is a special case of the result in [2], we include here an elementary proof for the sake of completeness.

**Lemma 1.2** (Semicontinuity). Let $f_k, f \in W^{1,p}(\Omega, \mathcal{A}_Q)$, $p < \infty$, be such that $\lim_k \int_{\Omega} \mathcal{G}(f_k, f)^p = 0$ and $\sup_k \int_{\Omega} |Df_k|^p < \infty$. Then,

$$\int_{\Omega} |Df|^p \leq \liminf_{k \to +\infty} \int_{\Omega} |Df_k|^p.$$  \hspace{1cm} (1.1)
Proof. The proof of this result is very similar to the proof of the semicontinuity for the Dirichlet energy given in [3, Section 2.3.2]. Let \( \{ T_i \}_{i \in \mathbb{N}} \) be any dense subset of \( A_Q \) and recall that by [3, Proposition 4.2] \(|Df|\) is the monotone limit of \( h_N \) with

\[
h_N^2 = \max_{i_j \leq N} \sum_j (\partial_j G(f, T_{i_j}))^2.
\]

By the Monotone Convergence Theorem, \( \int |Df|^p = \sup_N \int h_N^p \). Therefore, denoting by \( P_{N=0} \) the collections \( P = \{ E_i \}_{i = \{l_1, \ldots, l_m\} \in N_m} \) of disjoint open subsets of \( \Omega \), as in [3] we conclude that

\[
\int_{\Omega} |Df|^p = \sup_N \int_{\Omega} h_N^p = \sup_N \sup_{P \in P_{N=0}} \sum_{E_i \in P} \int_{E_i} \left( \sum_j (\partial_j G(f, T_{i_j}))^2 \right)^{\frac{p}{2}}.
\]

(1.2)

It follows easily from the hypotheses that, for every \( i = \{l_1, \ldots, l_m\} \) and every open set \( E_i \), the vector-valued maps \( \partial_j G(f_k, T_{i_l}) \), \( \partial_m G(f_k, T_{i_m}) \) converge weakly in \( L^p(E_i) \) to \( \partial_j G(f, T_{i_l}), \partial_m G(f, T_{i_m}) \). Hence, by the semicontinuity of the norm,

\[
\int_{E_i} \left( \sum_j (\partial_j G(f, T_{i_j}))^2 \right)^{\frac{p}{2}} \leq \liminf_{k \to +\infty} \int_{E_i} \left( \sum_j (\partial_j G(f_k, T_{i_j}))^2 \right)^{\frac{p}{2}}.
\]

Summing in \( E_i \in P \), in view of (1.2), we achieve (1.1).

The main regularity results for Dir-minimizing \( Q \)-valued functions are collected in the following theorem (see [3, Theorems 0.9 and 0.11]). In order to state them, we recall the definition of regular and singular points.

Definition 1.3. A \( Q \)-valued function \( f \) is regular at a point \( x \in \Omega \) if there exist a neighborhood \( U \) of \( x \) and \( Q \) analytic functions \( f_i : U \to \mathbb{R}^n \) such that \( f|_U = \sum_i [f_i] \) and either \( f_i(y) \neq f_j(y) \) for every \( y \in U \) or \( f_i \equiv f_j \). The singular set \( \Sigma_f \) of \( f \) is the complement of the set of regular points.

Theorem 1.4. For every Dir-minimizing \( f \in W^{1, 2}(\Omega, A_Q) \) the following holds:

(i) there exists \( \alpha = \alpha(m, Q) > 0 \) \( (\alpha(2, Q) = 1/Q) \) such that \( f \in C^{0, \alpha}(\Omega') \) for every \( \Omega' \subset \subset \Omega \) and

\[
\text{Dir}(f, B_r(x)) \leq \left( \frac{1}{r^\alpha} \right) \text{Dist}(f, B_\rho(x)) \quad \forall r \leq \rho \text{ with } B_\rho(x) \subseteq \Omega;
\]

(1.3)

(ii) the Hausdorff dimension of \( \Sigma_f \) is at most \( m - 2 \) and, if \( m = 2 \), \( \Sigma_f \) consists of isolated points.

1.1. Push-forward of currents under \( Q \)-functions. We define now the integer rectifiable current associated to the graph of a \( Q \)-valued function.

Given a \( Q \)-valued function \( f : \mathbb{R}^m \to A_Q(\mathbb{R}^n) \), we set \( \bar{f} = \sum_i [f(x, f_i(x))] \), \( \bar{f} : \mathbb{R}^m \to A_Q(\mathbb{R}^{m+n}) \). If \( R \in \mathcal{G}_k(\mathbb{R}^m) \) is a rectifiable current associated to a \( k \)-rectifiable set \( M \) with multiplicity \( \theta \), \( R = \tau(M, \theta, \xi) \), where \( \xi \) is a borel simple \( k \)-vector field orienting \( M \) (we use the notation in [8]), and if \( f \) is a proper Lipschitz \( Q \)-valued function, we can define the push-forward of \( T \) under \( f \) as follows.
Definition 1.5. Given \( R = \tau(M, \theta, \xi) \in D_k(\mathbb{R}^m) \) and \( f \in \text{Lip}(\mathbb{R}^m, A_Q(\mathbb{R}^n)) \) as above, we denote by \( T_{f,R} \) the current in \( \mathbb{R}^{m+n} \) defined by

\[
\langle T_{f,R}, \omega \rangle = \int_M \theta \sum_i \langle \omega \circ f, D^M f_i \# \xi \rangle \, dH^k \quad \forall \omega \in D_k(\mathbb{R}^{m+n}),
\]

where \( \sum \| D^M f_i(x) \| \) is the differential of \( f \) restricted to \( M \).

Remark 1.6. Note that, by Rademacher’s theorem [3, Theorem 1.13] the derivative of a Lipschitz \( Q \)-function is defined a.e. on smooth manifolds and, hence, also on rectifiable sets.

As a simple consequence of the Lipschitz decomposition in [3, Proposition 1.6], there exist \( \{ E_j \}_{j \in \mathbb{N}} \) closed subsets of \( \Omega \), positive integers \( k_j, l_j \in \mathbb{N} \) and Lipschitz functions \( f_{j,l} : E_j \to \mathbb{R}^n \), for \( l = 1, \ldots, L_j \), such that

\[
\mathcal{H}^k(M \setminus \bigcup_j E_j) = 0 \quad \text{and} \quad f|_{E_j} = \sum_{l=1}^{L_j} k_{j,l} \llbracket f_{j,l} \rrbracket.
\]

From the definition, \( T_{f,R} = \sum_{j,l} k_{j,l} f_{j,l} \# (R \llbracket E_j \rrbracket) \) is a sum of rectifiable currents defined by the push-forward under single-valued Lipschitz functions. Therefore, it follows that \( T_{f,R} \) is rectifiable and coincides with \( \tau(f(M), \theta_f, \bar{T}_f) \), where

\[
\theta_f(x, f_{j,l}(x)) = k_{j,l} \theta(x) \quad \text{and} \quad \bar{T}_f(x, f_{j,l}(x)) = \frac{D^M f_{j,l} \# \xi(x)}{|D^M f_{j,l} \# \xi(x)|} \quad \forall x \in E_j.
\]

By the standard area formula, using the above decomposition of \( T_{f,R} \), we get an explicit expression for the mass of \( T_{f,R} \):

\[
M(T_{f,R}) = \int_M |\theta| \sum_i \sqrt{\det (D^M f_i \cdot (D^M f_i)^T)} \, dH^k.
\]

Theorem 1.7. For every \( \Omega \) Lipschitz domain and \( f \in \text{Lip}(\Omega, A_Q) \), \( \partial T_{f,\Omega} = T_{f,\partial \Omega} \).

Up to now we have defined the push-forward under Lipschitz maps. Nevertheless, thanks to the approximate differentiability property of Sobolev \( Q \)-functions (see [3, Corollary 2.7]), for full dimensional current \( R = \llbracket \Omega \rrbracket \), the definition of \( T_{f,\Omega} \) in (1.4) makes sense for Sobolev functions as soon as the action is finite for every differential form \( \omega \in D^m(\mathbb{R}^{m+n}) \). It is easy to verify that this condition is satisfied if

\[
M(T_{f,\Omega}) = \int_\Omega \sum_i \sqrt{\det (D^M f_i \cdot (D^M f_i)^T)} < +\infty.
\]

For such functions, we have the following Taylor expansion of the mass of \( T_{f,\Omega} \).

Lemma 1.8. Let \( f \in W^{1,2}(\Omega, A_Q) \) such that \( M(T_{f,\Omega}) < +\infty \). Then,

\[
M(T_{f,\Omega}) = Q|\Omega| + \frac{\lambda^2}{2} \text{Dir}(f, \Omega) + o(\lambda^2) \quad \text{as} \quad \lambda \to 0.
\]
Proof. For every $\lambda > 0$, set $A_\lambda = \{|Df| \leq \lambda^{-\frac{2}{3}}\}$ and $B_\lambda = \{|Df| > \lambda^{-\frac{2}{3}}\}$. Since $f \in W^{1,2}(\Omega, AQ)$, for $\lambda \to 0$, we have that
\[
\text{Dir}(\lambda f, \Omega) = \text{Dir}(\lambda f, A_\lambda) + \lambda^2 \int_{B_\lambda} |Df|^2 = \text{Dir}(\lambda f, A_\lambda) + o(\lambda^2). \tag{1.8}
\]
Using the inequality $\sqrt{1 + x^2} \geq 1 + \frac{x^2}{2} - \frac{x^4}{4}$ for $|x| \leq 2$, since $\lambda |Df| \leq \sqrt{\lambda}$ in $A_\lambda$, for $\lambda \leq 4$ we infer that
\[
\mathcal{M}(T_{\lambda f, \Omega}) \geq \sum_i \int_{\Omega} \sqrt{1 + \lambda^2 |Df_i|^2} \geq Q |B_\lambda| + \int_{A_\lambda} \left(1 + \frac{\lambda^2 |Df|^2}{2} - C \lambda^4 |Df|^4\right)
\]
\[
\geq Q |\Omega| + \frac{\lambda^2}{2} \text{Dir}(f, A_\lambda) - \int_{A_\lambda} C \lambda^3 |Df|^2
\]
\[
= Q |\Omega| + \frac{\lambda^2}{2} \text{Dir}(f, A_\lambda) + o(\lambda^2). \tag{1.9}
\]
For what concerns the reversed inequality, we argue as follows. In $A_\lambda$, since for every multi index $\alpha$ with $|\alpha| \geq 2$ we have
\[
\lambda^{2|\alpha|}|M_{\alpha f_i}|^2 \leq C \lambda^{2|\alpha|} |Df_i|^{2|\alpha|} \leq C \lambda^3 |Df_i|^2,
\]
we use the inequality $\sqrt{1 + x^2} \leq 1 + \frac{x^2}{2}$ and get
\[
\mathcal{M}(T_{\lambda f, A_\lambda}) \leq \sum_i \int_{A_\lambda} \sqrt{1 + \lambda^2 |Df_i|^2 + C \lambda^3 |Df_i|^2}
\]
\[
= Q |A_\lambda| + \frac{\lambda^2}{2} \text{Dir}(f, A_\lambda) + o(\lambda^2). \tag{1.10}
\]
In $B_\lambda$, instead, we use the same inequality and the condition $\mathcal{M}(T_{f, \Omega}) < +\infty$ to infer
\[
\mathcal{M}(T_{\lambda f, B_\lambda}) \leq \sum_i \int_{B_\lambda} \sqrt{1 + \lambda^2 |Df_i|^2} + \sqrt{\sum_{|\alpha| \geq 2} \lambda^{2|\alpha|} M_{\alpha f_i}^2}
\]
\[
\leq Q |B_\lambda| + \frac{\lambda^2}{2} \text{Dir}(f, B_\lambda) + \sum_i \int_{B_\lambda} \lambda^2 \sqrt{\sum_{|\alpha| \geq 2} M_{\alpha f_i}^2}
\]
\[
\leq Q |B_\lambda| + o(\lambda^2) + \lambda^2 \mathcal{M}(T_{f, B_\lambda}) = Q |B_\lambda| + o(\lambda^2). \tag{1.11}
\]
From (1.9), (1.10) and (1.11), the proof follows. \hfill \square

2. Complex varieties and Dir-minimizing functions

2.1. Complex varieties as minimal currents. In the following we consider irreducible holomorphic varieties $\mathcal{V} \subseteq \mathbb{C}^{\mu + \nu}$ of dimension $\mu$. Following Federer [6], we associate to $\mathcal{V}$ the integer rectifiable current of real dimension $2\mu$ denoted by $[\mathcal{V}]$ given by the integration over the manifold part of $\mathcal{V}$, $\mathcal{V}_{\text{reg}}$. Recall that the singular part $\mathcal{V}_{\text{sing}} = \mathcal{V} \setminus \mathcal{V}_{\text{reg}}$ is a complex variety of dimension at most $(\mu - 1)$. A well-known result by Federer asserts that $[\mathcal{V}]$ is a mass-minimizing cycle.

Theorem 2.1. Let $\mathcal{V}$ be an irreducible holomorphic variety. Then, the integer rectifiable current $[\mathcal{V}]$ has locally finite mass and is a locally mass-minimizing cycle, that means $\partial [\mathcal{V}] = 0$ and $\mathcal{M}([\mathcal{V}]) \leq \mathcal{M}(S)$ for every integer current $S$ with $\partial S = 0$ and supp $(S - [\mathcal{V}])$ compact.
We consider domains $\Omega \subseteq \mathbb{R}^{2\mu} \simeq \mathbb{C}^\mu$ with the usual identification $(x_l, y_l) \simeq z_l = (x_l + iy_l)$ for $l = 1, \ldots, \mu$. Moreover, $\mathcal{V} \subseteq \Omega \times \mathbb{R}^{2\nu} \subseteq \mathbb{R}^{2\mu+2\nu} \simeq \mathbb{C}^{\mu+\nu}$ is always supposed to be a $Q$ : 1-cover of $\Omega$ under the orthogonal projection $\pi$ onto $\Omega$, that is $\pi_{\#}[\mathcal{V}] = Q[\Omega]$. Clearly, under this hypothesis, there exists a $Q$-valued function $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^{2\nu})$ such that $\mathcal{V} = \text{graph}(f)$. From Definition 1.3, we readily deduce $\Sigma_f \subseteq \pi(\mathcal{V}_{\text{sing}})$, which in particular implies $\dim_\pi(\Sigma_f) \leq 2\mu - 2$. Therefore, locally in $\Omega \setminus \Sigma_f$, $\mathcal{V}$ is the superposition of graphs of holomorphic functions, that is, for every $w \in \Omega \setminus \Sigma_f$, there exist a radius $r$ and $Q$ holomorphic functions $f_i : B_r(w) \to \mathbb{C}^\nu$ such that $f|_{B_r(w)} = \sum_i [f_i]$. The following are the main properties of $f$.

**Proposition 2.2.** Let $\mathcal{V} \subseteq \Omega \times \mathbb{R}^{2\nu}$ be a holomorphic variety as above and $f$ the associated $Q$-valued function. Then, the following holds:

1. $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ and, for $\mu = 1$, $M([\mathcal{V}]|L\Omega) = Q + \frac{\text{Dir}(f, \Omega)}{2}$;
2. $[\mathcal{V}]|L\Omega = T_{f,\Omega}$ and $\partial([\mathcal{V}]|L\Omega, B_r(x)) = T_{f,\partial B_r(x)}$ for every $x$ and a.e. $r > 0$ with $B_r(x) \subseteq \Omega$.

**Proof.** Note that, for every smooth $h : \mathbb{R}^2 \to \mathbb{R}^{2\nu}$ and, as usual, $\bar{h}(w) = (w, h(w))$, 

$$\sqrt{\det(\bar{D}h \cdot \bar{D}h^T)} \leq 1 + \frac{|Dh|^2}{2},$$

with equality if and only if $h$ is conformal, i.e. $|\partial_x h| = |\partial_y h|$ and $\partial_x h \cdot \partial_y h = 0$. Indeed, (2.1) reads as

$$\det(\bar{D}h \cdot \bar{D}h^T) = \det \left(1 + \frac{|\partial_x h|^2}{2} \right) \leq \left(1 + \frac{|\partial_x h|^2 + |\partial_y h|^2}{2}\right)^2,$$

which in turn is equivalent to $0 \leq (|\partial_x h|^2 - |\partial_y h|^2)^2 + 4(\partial_x h \cdot \partial_y h)^2$.

In the case $\mu = 1$, applying (2.1) to the local holomorphic, hence conformal, selection of $f_i$, from (1.6) we get

$$M([\mathcal{V}]|L(\Omega \setminus \Sigma_f)) = Q + \frac{\text{Dir}(f, \Omega \setminus \Sigma_f)}{2}.$$  

In the case $\mu > 1$ and $g : \mathbb{R}^{2\mu} \to \mathbb{R}^{2\nu}$ smooth, (2.1) together with Binet–Cauchy’s formula (see [5, Section 3.2 Theorem 4]), for every $l = 1, \ldots, \mu$, we infer

$$\det(Dg \cdot Dg^T) = 1 + |Dg|^2 + \sum_{|\alpha| = |\beta| = 2} \sum_{M_{\alpha\beta}(Dg)^2}$$

$$\geq 1 + |\partial_{x_ig}^j|^2 + |\partial_{y_ig}^j|^2 + \sum_{i,j=1}^{2\nu} (\partial_{x_ig}^j \partial_{y_ig}^j - \partial_{x_ig}^j \partial_{y_ig}^j)$$

$$= \det(\nabla_i g \cdot \nabla_i g^T),$$

where $M_{\alpha\beta}$ stands for the $\alpha, \beta$ minors of a matrix and $\nabla_i$ denotes the derivative with respect to $x_i$ and $y_i$. Hence, if $f_i$ is a local holomorphic, consequently conformal, selection for $f : \Omega \subset \mathbb{R}^{2\mu} \to \mathcal{A}_Q$, we infer that

$$\mu Q + \frac{|Df|^2}{2} = \sum_{i=1}^{Q} \sum_{l=1}^{\mu} \left(1 + \frac{|\nabla_i f_i|^2}{2}\right)^2 \geq \mu \sum_{i=1}^{Q} \sqrt{\det(\nabla_i f_i \cdot \nabla_i f_i)}.$$
Integrating, we conclude, for $\mu > 1$,
\begin{equation}
M(\mu f) \Omega f, \Omega f, \Sigma f) \geq Q + \frac{\text{Dir}(f, \Omega f, \Sigma f)}{2\mu}.
\end{equation}
(2.4)

Now since the mass of $\mu f$ is finite, by (2.2) and (2.4) the energy of $f$ is finite in $\Omega f, \Sigma f$. Being $\dim H(S) \leq m - 2$, Lemma 1.1 gives (i).

Being $\mu f$ defined by the integration over $\mu f$, it follows straightforwardly that $T f, \Omega f$ is well-defined by (1.4) and coincides with $\mu f$. For the same reason, since also $\dim H(f) = 0$, $\partial(\mu f, B_r(x)) = T f, \partial B_r(x)$ for every $B_r(x) \subseteq \Omega f$ such that $f|_{\partial B_r(x)} \in W^{1,2}$ and $\mu f(\Omega f, B_r(x))$ is finite, that is for every $x$ and a.e. $r > 0$, thus concluding the proof of (ii).

2.2. Proof of Theorem 0.1. Now we are ready to prove the first main result of the paper. We divide the proof in two parts: in the first one we give an argument for the planar case which is particularly simple and exploits the equality between the area and the energy functionals; in the second part we give a proof valid in every dimension.

2.2.1. Planar case $\mu = 1$. In view of Proposition 2.2, we need only to show that $f$ is Dir-minimizing in $B_1$. Choose a radius $r \in [1, 2]$ such that $\partial B_r \cap \Sigma f = \emptyset$ and set $g = f|_{\partial B_r}$. Note that $g$ is Lipschitz continuous. For every $h \in \text{Lip}(B_r, A Q)$ with $h|_{\partial B_r} = g$, from the Taylor expansion of the mass and from (2.1), we infer that
\begin{equation}
M(T h, B_r) - Q \leq \frac{\text{Dir}(h, B_r)}{2}.
\end{equation}
(2.5)

By Theorem 1.7, $\partial T h, B_r = T h, \partial B_r = \partial(\mu f, L B_r)$. So, using Theorem 2.1 we infer
\begin{equation}
\text{Dir}(f, B_r) \leq 2(M(T h, B_r) - Q) \leq 2(M(T h, B_r) - Q) \leq \text{Dir}(h, B_r).
\end{equation}

Since the set of Lipschitz functions with trace $g$ is dense in $W^{1,2}(B_r, A Q)$ (see [3, Section 14]), this implies that $f$ is Dir-minimizing in $B_r$ and, a fortiori, in $B_1$. □

Remark 2.3. The planar result provides examples of Dir-minimizing functions with singular set of dimension $m - 2$ for every $m$, thus proving the optimality of the regularity Theorem 1.4. Indeed, if $g : B_1 \subseteq \mathbb{R}^2 \rightarrow A Q$ is Dir-minimizing and $\Sigma g \neq \emptyset$, then $f : B_1 \times \mathbb{R}^{m-2} \rightarrow A Q$ with $f(x_1, x_2, \ldots, x_m) = g(x_1, x_2)$ is also Dir-minimizing (see [3, Lemma 3.24]) and $\dim H(\Sigma g) = m - 2$.

2.2.2. General case $\mu \geq 1$. Here we exploit the expansion of the mass given in Lemma 1.8. The reason why this can be done without the strong approximation theory developed by Almgren in [1] and reproof with different methods in [4] is that, given as above a complex variety which is the graph of a multi-valued function, the rescaled current $L \mu f, [\mu f] = T f, L f$, where $L f : \mathbb{C}^{n+\nu} \rightarrow \mathbb{C}^{n+\nu}$ is given by $L f(x, y) = (x, y f, L f, B_1)$, is also a complex variety (being the $L f$’s linear complex maps), and, hence, it is also area-minimizing.

The proof is by contradiction. Assume $f$ is not Dir-minimizing in $B_1$. Then, there exists $u \in W^{1,2}(B_1, A Q)$ and $\eta > 0$ such that $\text{Dir}(u, B_1) \leq \text{Dir}(f, B_1) - \eta$ and $u|_{\partial B_1} = f|_{\partial B_1}$. Set
\begin{equation}
w = \begin{cases}
u & \text{in } B_1, \\
f & \text{in } B_2 \setminus B_1.
\end{cases}
\end{equation}
We want to use \( w \) in order to construct competitor currents for \( L_{\lambda} \# [\gamma] \). To this aim, consider for every \( \varepsilon > 0 \) the Lipschitz approximations \( w_\varepsilon \) given by (see [3, Proposition 4.4]). It enjoys the following properties:

(a) \( |E_\varepsilon| = o(\varepsilon^2) \) as \( \varepsilon \to 0 \), where \( E_\varepsilon = \{ w_\varepsilon \neq w \} \);
(b) \( \text{Lip}(w_\varepsilon) \leq \varepsilon^{-1} \);
(c) \( \|Dw_\varepsilon| - |Dw|\|_2 = o(1) \) as \( \varepsilon \to 0 \).

By Proposition 2.2 and Lemma 1.8, for every open \( A \) such that \( E_\varepsilon \subseteq A \) and \( |A| \leq 2|E_\varepsilon| \),

\[
M \left( \frac{L_{\lambda} \# ([\gamma] \circ L_{\varepsilon} \times \mathbb{R}^{2n}))}{(T_{\lambda f, E_\varepsilon})} \leq M(T_{\lambda f, A}) \right.
\]

Using Fubini’s theorem and again Proposition 2.2, we can find radii \( r_{\lambda, \varepsilon} \) such that

\[
|E_\varepsilon \cap \partial B_{r_{\lambda, \varepsilon}}| = o(\varepsilon^2),
\]

(2.6)

\[
\partial (L_{\lambda} \# [\gamma] \circ B_{r}) = T_{\lambda f, \partial B_{r}}, \text{ and } M(T_{\lambda f, E_\varepsilon \cap \partial B_r}) = o(\varepsilon^2) + O(\lambda^2).
\]

(2.7)

Set \( S_{\varepsilon} = T_{\lambda f, \partial B_{r_{\lambda, \varepsilon}}} \setminus T_{\lambda w_\varepsilon, \partial B_{r_{\lambda, \varepsilon}}} \). Note that, by Theorem 1.7, being \( w_\varepsilon \) Lipschitz,

\[
\partial S_{\varepsilon} = \partial T_{\lambda f, \partial B_{r_{\lambda, \varepsilon}}} - \partial T_{\lambda w_\varepsilon, \partial B_{r_{\lambda, \varepsilon}}} \equiv \partial (L_{\lambda} \# [\gamma] \circ B_{r}) = 0.
\]

Moreover, since \( \text{Lip}(\lambda w_\varepsilon) \leq \lambda^{-1} \) and \( T_{\lambda f, \partial B_{r_{\lambda, \varepsilon}}} \setminus E_\varepsilon = T_{\lambda w_\varepsilon, \partial B_{r_{\lambda, \varepsilon}}} \setminus E_\varepsilon \), the mass of \( S_{\lambda, \varepsilon} \) can be estimated in the following way:

\[
M(S_{\lambda, \varepsilon}) = M(T_{\lambda f, E_\varepsilon \cap \partial B_{r_{\lambda, \varepsilon}}}) + M(T_{\lambda w_\varepsilon, E_\varepsilon \cap \partial B_{r_{\lambda, \varepsilon}}}) \leq o(\varepsilon^2) + O(\lambda^2) + O(\lambda^2) + o(\lambda \varepsilon).
\]

(2.8)

For \( \varepsilon = \lambda \), \( M(S_{\lambda, \lambda}) = O(\lambda^2) \) and, by the isoperimetric inequality [8, Theorem 30.1], there exists an integer rectifiable current \( R_\lambda \) such that

\[
\partial R_\lambda = S_{\lambda, \lambda} \text{ and } M(R_\lambda) \leq M(S_{\lambda, \lambda}) = O(\lambda^2).
\]

(2.9)

The current \( T_\lambda = T_{\lambda w_\lambda, B_{r_{\lambda}}} + R_\lambda \) contradicts now the minimality of the complex current \( L_{\lambda} \# ([\gamma] \circ B_{r_{\lambda}}) \). Indeed, it is easy to verify that \( \partial T_\lambda = \partial (L_{\lambda} \# [\gamma] \circ B_{r_{\lambda}}) \) and, for small \( \lambda \),

\[
M(T_\lambda) - M \left( L_{\lambda} \# [\gamma] \circ (B_{r_{\lambda}} \times \mathbb{R}^{2n}) \right) = Q|B_{r_{\lambda}}| + \frac{\lambda^2}{2} \text{Dir}(w_\lambda, B_{r_{\lambda}}) + \\
- Q|B_{r_{\lambda}}| - \frac{\lambda^2}{2} \text{Dir}(f, B_{r_{\lambda}}) + o(\lambda^2) \leq - \frac{\lambda^2 \eta}{4} + o(\lambda^2) < 0.
\]

3. Higher integrability of the gradients of Dir-minimizing functions

In this section we prove Theorem 0.2. As above, for the planar case we give a simple proof which in addition provides the optimal integrability exponent. This proof relies on the following proposition, because by Theorem 1.4 the singular points are isolated in dimension two.
Proposition 3.1. Let \( u \in W^{1,2}(B_2, A_Q) \) be Dir-minimizing and assume that \( \Sigma_u = \{0\} \). Then, \( |Du| \in L^p(B_1) \) for every \( p < \frac{2Q}{Q-1} \).

Proof. Let \( x \in B_1 \setminus \{0\} \) and set \( r = |x| \). Then, by \( \Sigma_u = \{0\} \), in \( B_r(x) \) there exists an analytic selection of \( u, u|_{B_r(x)} = \sum_i [u_i] \), where \( u_i : B_r(x) \rightarrow \mathbb{R}^n \) are harmonic functions. Using the mean value inequality for \( Du_i \), one infers that

\[
|Du_i(x)| \leq \int_{B_r(x)} |Du_i| \leq \frac{1}{\sqrt{\pi r}} \left( \int_{B_r(x)} |Du_i|^2 \right)^{\frac{1}{2}},
\]

from which

\[
|Du(x)|^2 = \sum_i |Du_i(x)|^2 \leq \frac{1}{\pi r^2} \sum_i \int_{B_r(x)} |Du_i|^2 = \frac{\text{Dir}(u, B_r(x))}{\pi r^2}.
\]

Using the decay estimate (1.3) with \( \rho = 1 \) together with (3.1), we deduce that

\[
|Du(x)| \leq \frac{\text{Dir}(u, B_2)}{\sqrt{\pi r^{1-\frac{2}{Q}}}},
\]

which in turn implies the conclusion,

\[
\int_{B_1} |Du|^p \leq C \int_{B_1} \frac{1}{|x|^{p-\frac{2}{Q}}} < +\infty, \quad \forall \; p < \frac{2Q}{Q-1}.
\]

Remark 3.2. The range \([2, 2Q(Q-1)^{-1}]\) for the integrability exponent is optimal. Consider, indeed, the complex variety \( \mathcal{Y}_Q = \{(z, w) : w^Q = z\} \subseteq \mathbb{C}^2 \). By Theorem 0.1, the \( Q \)-valued function \( u(z) = \sum_{w^Q = z} [w] \) is Dir-minimizing in \( B_2 \). Moreover, \( |Du(z)| = Q |z|^{\frac{1}{Q-1}} \). Hence, \( |Du| \in L^p \) for every \( p < \frac{2Q}{Q-1} \) and \( |Du| \notin L^p \).

Now we pass to the proof of Theorem 0.2 for \( m \geq 3 \). The first step is a Caccioppoli’s inequality for Dir-minimizing functions. For \( P \in \mathbb{R}^n \), we denote by \( \tau_P \) the following map: \( \tau_P : A_Q(\mathbb{R}^n) \rightarrow A_Q(\mathbb{R}^n) \),

\[
\tau_P(T) := \sum_i [T_i - P], \quad \text{for every} \; T = \sum_i [T_i].
\]

Lemma 3.3 (Caccioppoli’s inequality). Let \( u \in W^{1,2}(\Omega, A_Q) \) be Dir-minimizing. Then, for every \( P \in \mathbb{R}^n \) and every \( \eta \in C_c^\infty(\Omega) \),

\[
\int_\Omega |Du|^2 \eta^2 \leq \int_\Omega |\tau_P u|^2 |D\eta|^2.
\]

In particular, in the case \( \Omega = B_{2r} \),

\[
\int_{B_{4r}} |Du|^2 \leq \frac{4}{r^2} \int_{B_{2r}} |\tau_P u|^2.
\]

Proof. Recall the outer variation [3, Proposition 3.1] for Dir-minimizing functions,

\[
0 = \int \sum_i \langle Df_i(x) : D_x \psi(x, f_i(x)) \rangle \, dx + \int \sum_i \langle Df_i(x) : D_y \psi(x, f_i(x)) \cdot Df_i(x) \rangle \, dx,
\]
and apply it to $\psi(x, y) = \eta(x)^2 (y - P)$, where $P$ and $\eta$ are as in the statement. Since $D_x \psi(x, y) = 2 \eta(x) D\eta(x) \otimes (y - P)$ and $D_y \psi(x, y) = \eta(x)^2 \text{Id}_n$, this leads to
\[
0 = \int_{\Omega} \sum_i \langle Du_i(x) : 2 \eta D\eta \otimes (u_i - P) \rangle + \int_{\Omega} \sum_i \langle Du_i(x) : \eta^2 Du_i(x) \rangle.
\tag{3.4}
\]
Applying Hölder’s inequality in (3.4), we conclude (3.2):
\[
\int_{\Omega} \eta^2 |Du|^2 = - \sum_i \int_{\Omega} \langle Du_i \cdot (u_i - P), \eta D\eta \rangle \leq \int_{\Omega} \sum_i |Du_i| |u_i - P| |\eta| |D\eta|
\leq \int_{\Omega} \left( \sum_i |Du_i|^2 |\eta|^2 \right)^{\frac{s}{2}} \left( \sum_i |u_i - P| |D\eta|^2 \right)^{\frac{2}{s}}
\leq \left( \int_{\Omega} \eta^2 |Du|^2 \right)^{\frac{s}{2}} \left( \int_{\Omega} |\tau_P(u)|^2 |D\eta|^2 \right)^{\frac{2}{s}}.
\]
The last conclusion of the lemma follows from (3.2) choosing $\eta \equiv 1$ in $B_{3r/2}$ and $|D\eta| \leq \frac{2}{s}$. □

The following reverse Hölder inequality is the basic estimate for the higher integrability.

**Proposition 3.4.** Let $\frac{2m}{m+2} < s < 2$. Then, there exists $C > 0$ such that, for every $u : \Omega \to A_Q$ Dir-minimizing, $x \in \Omega$ and $r < \min \{1, \text{dist}(x, \partial \Omega)/2\}$,
\[
\left( \int_{B_r(x)} |Du|^2 \right)^{\frac{s}{2}} \leq C \left( \int_{B_{2r}(x)} |Du|^s \right)^{\frac{s}{2}}.
\tag{3.5}
\]

**Proof.** The proof is divided into two steps.

**Step 1:** we assume that $u$ has average 0, $\eta \circ u = \sum_{Q \in Q} \frac{u_i}{Q} = 0$.

The proof is by induction on the number of values $Q$. The basic step $Q = 1$ is clear: indeed, in this case $\eta \circ u = u = 0$. Now, we assume that (3.5) holds for every $Q' < Q$ and, by contradiction, it does not hold for $Q$.

Then, up to translations and dilations of the domain, there exists a sequence $(u_i)_l \subset W^{1,2}(B_{4}, A_Q)$ of Dir-minimizing functions such that $\eta \circ u_l = 0$ and
\[
\left( \int_{B_4} |Du_l|^2 \right)^{\frac{s}{2}} \leq \left( \int_{B_2} |Du_l|^s \right)^{\frac{s}{2}}.
\tag{3.6}
\]
Moreover, without loss of generality, we may also assume that $\int_{B_4} |u_l|^2 = 1$. Using Caccioppoli’s inequality (3.3), we have that $\text{Dir}(u_l, B_4) \leq 4$, which in turn, by (3.6), implies
\[
\text{dist}(u_l, Q [0])_{W^{1,s}(B_4)} \leq C < +\infty.
\]
Since $s^* > 2$, we can apply the compact Sobolev embedding (see [3, Proposition 2.11]) to deduce that there exists a subsequence (not relabeled) $u_l$ converging to some $u$ in $L^2(B_4)$. From (3.6) and Lemma 1.2, we deduce that
\[
\int_{B_4} |u_l|^2 = 1 \quad \text{and} \quad \int_{B_4} |Du_l|^s = 0,
\tag{3.7}
\]
which implies that \( u \) is constant, \( u \equiv T \in \mathcal{A}_Q \). Since by Theorem 1.4 the \( u_i \)'s are equi-bounded and equi-Hölder in \( B_2 \), always up to a subsequence (again not relabeled), the \( u_i \)'s converge uniformly to \( T \) in \( B_2 \). This implies, in particular, that

\[
\eta \circ T = \lim_{l \to +\infty} \eta \circ u_l = 0.
\]

From (3.7) and (3.8), one infers that \( T \) is not a point of multiplicity \( Q \). Therefore, since \( u_l \to T \) uniformly in \( B_2 \), for \( l \) large enough the \( u_n \)'s must split in the sum of two Dir-minimizing functions \( u_l = [v_l] + [w_l] \), where the \( v_l \)'s are \( Q_1 \)-valued functions and the \( w_l \)'s are \( Q_2 \)-valued, with \( Q_1, Q_2 \) positive and \( Q_1 + Q_2 = Q \). Applying now the inductive hypothesis to \( v_l \) and \( w_l \) we contradict (3.6) for \( l \) large enough,

\[
\left( \int_{B_z(x)} |Du|^2 \right)^{\frac{1}{2}} \leq \left( \int_{B_z(x)} |Dv|^2 \right)^{\frac{1}{2}} + \left( \int_{B_z(x)} |Dw|^2 \right)^{\frac{1}{2}} \\
\leq C \left( \int_{B_z(x)} |Dv|^s \right)^{\frac{1}{s}} + C \left( \int_{B_z(x)} |Dw|^s \right)^{\frac{1}{s}} \\
\leq 2 C \left( \int_{B_z(x)} |Du|^s \right)^{\frac{1}{s}}.
\]

**Step 2: generic Dir-minimizing function \( u \).**

Let \( u \) be Dir-minimizing and \( \varphi = \eta \circ u \); then, by [3, Lemma 3.23], \( \varphi : \Omega \to \mathbb{R}^n \) is harmonic and \( D\varphi = \sum_i D u_i \), from which

\[
|D\varphi|^2 \leq Q \sum_i |Du_i|^2 = Q |Du|^2.
\]

Moreover, again by [3, Lemma 3.23], the \( Q \)-valued function \( v = \sum_i [u_i - \varphi] \) is Dir-minimizing as well. Note that

\[
|Du|^2 \leq 2 |Dv|^2 + 2 Q |D\varphi|^2 \quad \text{and} \quad |Dv|^2 \leq 2 |Du|^2 + 2 Q |D\varphi|^2.
\]

Using the inequality \( \sum_j a_j \leq \sqrt{\sum_j a_j^2} \) for positive \( a_j \), we deduce

\[
\left( \int_{B_z(x)} |Du|^2 \right)^{\frac{1}{2}} \leq \left( \int_{B_z(x)} 2 |Dv|^2 + 2 Q |D\varphi|^2 \right)^{\frac{1}{2}} \\
\leq 2 \left( \int_{B_z(x)} |Dv|^2 \right)^{\frac{1}{2}} + 2 Q \left( \int_{B_z(x)} |D\varphi|^2 \right)^{\frac{1}{2}}.
\]

For the first term in the right hand side of (3.11), we use Step 1, since \( \eta \circ v = 0 \), to get

\[
\left( \int_{B_z(x)} |Dv|^2 \right)^{\frac{1}{2}} \leq C \left( \int_{B_{2r}(x)} |Dv|^s \right)^{\frac{1}{s}} \leq C \left( \int_{B_{2r}(x)} (2 |Du|^2 + 2 Q |D\varphi|^2)^{\frac{1}{2}} \right)^{\frac{1}{s}} \\
\leq C \left( \int_{B_{2r}(x)} 2 |Du|^s + 2 Q |D\varphi|^s \right)^{\frac{1}{s}} \leq C \left( \int_{B_{2r}(x)} |Du|^s \right)^{\frac{1}{s}}.
\]
For the remaining term in (3.11), we use the standard estimate for harmonic functions,
\[ |D\varphi(x)| \leq C \frac{r^n}{r^n} \|D\varphi\|_{L^1(B_{2r})} \quad \forall \, x \in B_r, \]  
(3.13)
and infer
\[ \left( \frac{1}{r^n} \int_{B_r(x)} |D\varphi|^2 \right)^{\frac{1}{2}} \leq \frac{C}{r^n} \|D\varphi\|_{L^1(B_{2r})} \leq \frac{C}{r^n} \left( \int_{B_{2r}(x)} |D\varphi|^s \right)^{\frac{1}{s}} r^n(1-\frac{1}{s}) \]
\[ \leq C \left( \frac{1}{r^n} \int_{B_{2r}(x)} |D\varphi|^s \right)^{\frac{1}{s}} \leq C \left( \frac{1}{r^n} \int_{B_{2r}(x)} |D\varphi|^s \right)^{\frac{1}{s}} r^n(1-\frac{1}{s}) \]
\[ \leq C \left( \frac{1}{r^n} \int_{B_{2r}(x)} |D\varphi|^s \right)^{\frac{1}{s}} \leq C \left( \frac{1}{r^n} \int_{B_{2r}(x)} |D\varphi|^s \right)^{\frac{1}{s}} \]
(3.14)
Clearly, (3.11), (3.12) and (3.14) finish the proof. □

The proof of Theorem 0.2 is now an easy consequence of the following reverse Hölder inequality with increasing supports proved by Giaquinta and Modica in [7, Proposition 5.1].

**Theorem 3.5** (Reversed Hölder inequality). Let \( \Omega \subseteq \mathbb{R}^m \) be open and \( g \in L^{q}_{\text{loc}}(\Omega) \), with \( q > 1 \) and \( g \geq 0 \). Assume that there exist positive constants \( b \) and \( R \) such that
\[ \left( \frac{1}{r^n} \int_{B_r(x)} g^q \right)^{\frac{1}{q}} \leq b \int_{B_{2r}(x)} g, \quad \forall \, x \in \Omega, \forall \, r < \min \left\{ R, \text{dist}(x, \partial \Omega)/2 \right\}. \]
(3.15)
Then, there exist \( p = r(q,b) > q \) and \( c = c(m,q,b) \) such that \( g \in L^p_{\text{loc}}(\Omega) \) and
\[ \left( \frac{1}{r^n} \int_{B_r(x)} g^p \right)^{\frac{1}{p}} \leq c \left( \frac{1}{r^n} \int_{B_{2r}(x)} g^q \right)^{\frac{1}{q}}, \quad \forall \, x \in \Omega, \forall \, r < \min \left\{ R, \text{dist}(x, \partial \Omega)/2 \right\}. \]
Proof of Theorem 0.2. Consider the function \( g = |Du|^s \), where \( s < 2 \) is the exponent in Proposition 3.4. Estimate (3.5) implies that hypothesis (3.15) of Theorem 3.5 is satisfied with \( q = \frac{2}{s} > 1 \). Hence, there exists an exponent \( p' > q \), such that \( g \) belongs to \( L^{p'}_{\text{loc}}(\Omega) \), i.e. \( |Du| \in L^p_{\text{loc}}(\Omega) \) for \( p = p' \cdot s > 2 \). □

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