On the Renormalization of the Complex Scalar Free Field Theory

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Abstract

Polar coordinates are used for the complex scalar free field in $D = 4$ dimensions. The resulting non renormalizable theory is healed by using a recently proposed symmetric subtraction procedure. The existence of the coordinates transformation is proved by construction.

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1 Introduction

In power counting renormalizable theories there is a universally accepted rule, by which to every independent divergent one-particle-irreducible amplitude (1PI) one must associate a parameter in the tree-level action. This rule cannot be easily exported to any program of subtraction of infinities in nonrenormalizable theories. In fact, if this rule is used, the theory looses (in general) any predictivity and moreover the perturbative approach is unstable: for every new divergent 1PI amplitude emerging in the perturbative expansion, the whole series have to be updated from the beginning. Because of these two reasons we have proposed a new approach to the subtraction procedure for non renormalizable theories [1], [2]. To our opinion the removal of the infinities have to be considered as a pure mathematical problem that aims to give a meaning to undefined expressions. Obvious properties have to be maintained as locality of the counterterms and physical unitarity.

We have proposed a subtraction strategy where the symmetry properties of the path integral measure and the dynamics are imposed through Local Functional Equations (LFE’s) obeyed by the connected functional or by the vertex functional [1]. The action is not an adequate quantity in this procedure. This strategy of subtraction has been thoroughly analyzed [3] and successfully used for the nonlinear sigma model [4], for massive YM theory [5], and for the Electroweak Model [6].

Persistent objections from some experts in the field about this new strategy of ours has led me to consider a crystal clear (hopefully 3 ) example: free field theory. The example turned out to be much more interesting than I thought and therefore I decided to write it down. The state of the art is difficult to tell, since this problem is as old as quantum field theory and it is strictly connected to that of field-coordinate transformations. Therefore I apologize for the missed references.

The paper is self contained; however the proofs are only sketched, being present in previous works. I shall argue that standard polar coordinates

3 The presence of second derivatives in the interaction vertices (e.g. equations (8) and (9)) causes some problems: 1PI functions do not coincide with the Legendre transform of the connected amplitudes. In fact the equation of motion might eventually contract an internal line to a point. However this seems not to affect the subtraction procedure here presented. In particular one-particle-reducible graphs, promoted to 1PI by the equation of motion, enjoy the property of factorization and therefore they do not require extra subtractions.
cannot be used since the perturbative expansion (in loops) is around a vacuum where Spontaneous Breakdown of the U(1) Symmetry of the complex field occurs and a Goldstone boson appears. I modify the polar coordinates in order to meet the basic requirements of the equivalence theorem [7]-[12]. Then I show how the local symmetry transformations, associated to the path integral measure, can be implemented by using an infinite set of external sources, which eventually appear in the LFE’s. LFE’s are then used to prove the hierarchy structure of the vertex functional and finally to establish the subtraction procedure. It is amazing how the free field structure remains in such a complicated non renormalizable theory. Finally the sturdy Sections (10)-(12) are devoted to study the general structure of the counterterms, by solving the LFE’s at one-loop level, and to show why the textbook renormalization cannot manage the polar coordinates transformation. In Section 13 a solution of the LFE’s for the two-loop case is derived.

I leave to the conclusions a detailed discussion of the results.

## 2 Modified Polar Coordinates

I consider the action

\[ S = \Lambda^{D-4} \int d^D x [\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi]. \]  

(1)

If one uses the following parameterization for polar coordinates

\[ \phi = e^\chi, \]
\[ \chi = \psi + i\theta \]  

(2)

where \( \theta, \psi \in [-\infty, \infty] \), the action becomes

\[ S = \Lambda^{D-4} \int d^D x e^\chi e^{\chi^*} \left[ \partial^\mu \chi^* \partial_\mu \chi - m^2 \right] \]
\[ = \Lambda^{D-4} \int d^D x e^{2\psi} \left[ \partial^\mu \psi \partial_\mu \psi + \partial^\mu \theta \partial_\mu \theta - m^2 \right]. \]  

(3)

A series expansion of the exponential and a subsequent perturbative approach would take to a theory with a massless field \( \theta \). Since I want massive scalar field the parameterization in eq. (2) is not a good one. Instead I use

\[ e^\chi(x) = v^{-1} \phi(x) + 1 \]  

(4)
(where $v$ is a mass scale) so that the power expansion starts with

$$\chi(x) \sim v^{-1}\phi(x)$$

and therefore the equivalence theorem can be applied. The action in the chosen variables ($v$ might be absorbed by an overall rescaling of mass, coordinates and $\Lambda$)

$$S = \Lambda^{D-4} \int d^D x \left[ e^\chi e^{\chi^*} \partial^\mu \chi^* \partial_{\mu} \chi - m^2 (e^\chi - 1)(e^{\chi^*} - 1) \right]$$

which I can split into a free and interaction action

$$= \Lambda^{D-4} \int d^D x \left[ \partial^\mu \chi^* \partial_{\mu} \chi - m^2 \chi^* \chi \right]$$

$$+ \Lambda^{D-4} \int d^D x \left[ \sum_{n=1}^{\infty} \frac{1}{n!} \partial^\mu \chi^* \partial_{\mu} \chi (\chi^* + \chi)^n \right.$$

$$- m^2 \left( \sum_{n=2}^{\infty} \frac{1}{n!} \chi^* \chi^n + \sum_{n=2}^{\infty} \frac{1}{n!} \chi^* \chi^n + \left| \sum_{n=2}^{\infty} \frac{1}{n!} \chi^n \right|^2 \right) \right].$$

The perturbative expansion is in $\hbar$ (i.e. the number of loops, with some care on counting the $\hbar$ powers of the counterterms). I will try to give a meaning to the infinite number of divergent 1PI amplitudes by using dimensional regularization and eventually recover the free field theory in the variables $\phi$, $\phi^*$ at $D = 4$.

3 On the Conventional Approach

By proceeding in the conventional way, one meets a series of difficulties that have discouraged people to discuss the problem of coordinate transformations in quantum field theories. There has been some important progress in the use of field redefinition and its relation with the renormalization procedure [13], [14] and with the algebraic structure of the theory [15]–[24]. However, to my knowledge, no one has directly faced the task of taming the difficulties urging from the arbitrariness of the counterterms in conventional renormalization procedure.

The propagator in eq. (7) describes a complex scalar field. When one looks at the interaction part of the action, some appalling features immediately show up: the vertices are non-polynomial, they contain powers of the momentum (up to second power) and moreover they do not conserve
additively the charge suggested by the free field part. For instance the three legs vertices are

$$i\Lambda^{D-4} \int d^D x \left[ \partial^\mu \chi^* \partial_\mu \chi - \frac{1}{2} m^2 \chi^* \chi \right] (\chi^* + \chi)$$

(8)

while the four legs are

$$i\Lambda^{D-4} \int d^D x \left[ \partial^\mu \chi^* \partial_\mu \chi \frac{1}{2} (\chi^* + \chi)^2 - m^2 \left( \frac{1}{6} (\chi^* \chi^3 + \chi^* \chi) + \frac{1}{4} (\chi^* \chi)^2 \right) \right].$$

(9)

It is clear that already at one loop the number of independent divergent amplitudes is infinite. Since standard renormalization procedure requires that for any divergent 1PI independent amplitude one must introduce the corresponding local operator in the classical action, it is clear that the conventional approach takes to a dead-end.

In the sequel I use the method developed for the nonlinear sigma model, for massive Y-M theories and for the Electroweak Model. This amount to study the invariance properties of the path-integral measure, to derive the LFE’s associated to the invariance, to establish the hierarchy among the amplitudes, to fix the number of independent ancestor amplitudes (via Weak Power Counting (WPC) criterion) and finally to develop the subtraction strategy for the infinities. Then it is straightforward to check that the $\phi$-two-point function is that of a free field.

4 The Complete Set of external Sources

The path integral measure is

$$\prod_x \mathcal{D}[\phi(x)] \mathcal{D}[\phi^*(x)] = \prod_x e^{\chi(x)} e^{\chi^*(x)} \mathcal{D}[\chi(x)] \mathcal{D}[\chi^*(x)]$$

(10)

and it is invariant under the local rotations

$$\delta_\alpha \phi(x) = i\alpha(x) \phi(x), \quad \alpha(x) \in \mathcal{R}$$

$$\delta_\alpha \chi(x) = i\frac{\alpha(x) \phi(x)}{1 + \phi(x)}$$

(11)

and local translations

$$\delta_\beta \phi(x) = \beta(x), \quad \delta_\beta \phi^*(x) = \beta^*(x), \quad \beta(x) \in \mathcal{C}$$

$$\delta_\beta \chi(x) = \frac{\beta(x)}{1 + \phi(x)}, \quad \delta_\beta \chi^*(x) = \frac{\beta^*(x)}{1 + \phi^*(x)}.$$
If one chooses to integrate over the variables $\chi, \chi^*$, the transformations in eqs. (11) and (12) are nonlinear. One needs a complete set of sources in order to handle the composite operators intervening in the whole algebra. By starting with

$$\Lambda^{D-4} \int d^D x [\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + J^* \phi + J \phi^*]$$

(13)

and the path integral external field-sources

$$\int d^D x [J^* \chi + J \chi^*],$$

(14)

under $\delta_\alpha$ one needs the extra sources $F^\mu, K$

$$\Lambda^{D-4} \int d^D x [F^\mu (i \phi^* \partial_\mu \phi - i \partial_\mu \phi^* \phi) + K \phi^* \phi].$$

(15)

Under $\delta_\beta$ one needs the sources

$$\Lambda^{D-4} \int d^D x [J^*_\beta (1 + \phi(x)) + J_\beta (1 + \phi^*(x)),]$$

(16)

and then

$$\delta_\alpha \frac{1}{(1 + \phi(x))^n} = -n \frac{1}{(1 + \phi(x))^{(n+1)}} i \phi \alpha(x) = -n i \frac{1}{(1 + \phi(x))^{n}} \alpha(x)$$

$$+ni \frac{1}{(1 + \phi(x))^{(n+1)}} \alpha(x)$$

$$\delta_\beta \frac{1}{(1 + \phi(x))^n} = -n \frac{1}{(1 + \phi(x))^{(n+1)}} \beta(x).$$

(17)

Thus the complete set of sources fixes the effective action at the tree level

$$\Gamma^{(0)} = \Lambda^{D-4} \int d^D x \left[ \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + J^* \phi + J \phi^* + F^\mu i (\phi^* \partial_\mu \phi - \partial_\mu \phi^* \phi) + K \phi^* \phi \right.$$  

$$+ \sum_{n=1}^{\infty} J^*_n (1 + \phi(x))^n + \sum_{n=1}^{\infty} J_n (1 + \phi^*(x))^n \right].$$

(18)

5 The Local Functional Equation for Rotations

Since the path integral measure is invariant under local rotations, the functional must be invariant under the change of coordinates (11). By standard
procedure, i.e. by expanding in \( \alpha \), one gets the LFE for the generating functional of the connected amplitudes

\[
\left\langle \partial^\mu i \left[ \phi^* \partial_\mu \phi - \partial_\mu \phi^* \phi \right] + i \left[ J^* \phi - J^\mu \phi^* \right] + 2 \partial^\mu (F^\mu \phi^* \phi) \\
+ i \Lambda^{-D+4} \left[ J^*_0 \left( \frac{1}{1 + \phi(x)} \right) - J^*_0 \left( \frac{1}{1 + \phi^*(x)} \right) \right] \\
+ \sum_{n=1}^\infty J^*_n \left[ -ni \frac{1}{(1 + \phi(x))^n} + ni \frac{1}{(1 + \phi(x))^{(n+1)}} \right] \\
+ \sum_{n=1}^\infty J_n \left[ ni \frac{1}{(1 + \phi^*(x))^n} - ni \frac{1}{(1 + \phi^*(x))^{(n+1)}} \right] \right\rangle = 0, \quad (19)
\]

where the brackets denote the weighted mean value over the paths. It is worth to introduce the notation

\[
\mathcal{R}[\alpha] W \equiv \int d^Dx \alpha(x) \left[ \partial^\mu \frac{\delta W}{\delta F^\mu} + i (J^\mu \frac{\delta W}{\delta J^\mu} - J^*_0 \frac{\delta W}{\delta J^*_0}) + 2 \partial^\mu (F^\mu \frac{\delta W}{\delta K}) \\
+ i J^*_0 (1 - \Lambda^{-D+4} \frac{\delta W}{\delta J^*_0}) - i J^*_0 (1 - \Lambda^{-D+4} \frac{\delta W}{\delta J^*_0}) \\
+ i \sum_{n=1}^\infty n J^*_n \left( \frac{\delta W}{\delta J^*_n} \right) + \frac{\delta W}{\delta J^*_{n+1}} \right) - i \sum_{n=1}^\infty n J_n \left( - \frac{\delta W}{\delta J_n} + \frac{\delta W}{\delta J_{n+1}} \right). \quad (20)
\]

Then eq. (19) for the generating functional \( W \) (which depends on \( F, K, J, J^*, J^*_0, \{J_n, n = 1 \cdots \} \)) of the connected amplitudes becomes

\[
\mathcal{R}[\alpha] W = 0. \quad (21)
\]

6 The Local Functional Equation forTranslations

Similarly one can obtain the LFE for the translations. The change of coordinates of eq. (12) generates the identity

\[
\left\langle -i (\Box + m^2) \phi^* + J^* - 2i F^\mu \partial_\mu \phi^* - i \partial_\mu F^\mu \phi^* + K \phi^* \\
+ \Lambda^{-D+4} J^*_0 \left[ \frac{1}{1 + \phi(x)} \right] - \sum_{n=1}^\infty n J^*_n \left[ \frac{1}{(1 + \phi(x))^{(n+1)}} \right] \right\rangle = 0
\]

and the corresponding Schwinger-Dyson equation are conceptually different from eqs. (23) and (24).
As before the operator is defined

\[
T[\beta]W \equiv \int d^D x \beta(x) \left[ (\Box + m^2) \frac{\delta W}{\delta J} + J^* - 2i F_\mu \partial_\mu \frac{\delta W}{\delta J} - i \partial_\mu F_\mu \frac{\delta W}{\delta J} + K \delta \Gamma_{\delta J} - \Lambda^D J_0 \frac{\delta W}{\delta J} - \sum_{n=1}^\infty n J^*_n \frac{\delta W}{\delta J^*_n} \right].
\] (23)

Eq. (22) becomes

\[
T[\beta] W = 0.
\] (24)

It is important to establish the algebra of \( R, T \). By a straightforward calculation

\[
[R[\alpha], T[\beta]] = -i T[\alpha \beta].
\] (25)

It should be noticed that both eqs. (21) and (24) are not aware of the choice one might operate for the integration variables: either \( \phi, \phi^* \) or \( \chi, \chi^* \). Moreover it is worth noticing that both equations are linear in \( W \).

7 Effective Action Functional

The situation becomes rather interesting when we derive the effective action functional \( \Gamma \), via Legendre transformations. This step is necessary in order to set up a strategy for the subtraction of the infinities of the perturbative expansion. The aim of this work is to formulate the field theory in terms of polar coordinates, then the Legendre transformation is done on the variables \( \chi, \chi^* \). \( \Gamma \) obeys the following LFE’s: for the rotations

\[
\partial_\mu \frac{\delta \Gamma}{\delta F_\mu} + i \left[ J^* \frac{\delta \Gamma}{\delta J^*} - J \frac{\delta \Gamma}{\delta J} \right] + 2 \partial_\mu (F_\mu \frac{\delta \Gamma}{\delta K})
\]
\[
- \partial_\chi \left( 1 - \Lambda^D J_0 \frac{\delta \Gamma}{\delta J_0} \right) + \partial_\chi \left( 1 - \Lambda^D J^*_0 \frac{\delta \Gamma}{\delta J^*_0} \right)
\]
\[
+ \sum_{n=1}^\infty n \left( J^*_n \left[ \frac{\delta \Gamma}{\delta J^*_n} + \frac{\delta \Gamma}{\delta J^*_n} \right] + J_n \left[ \frac{\delta \Gamma}{\delta J_n} - \frac{\delta \Gamma}{\delta J^*_n} \right] \right) = 0
\] (26)

and for the translations

\[
- (\Box + m^2) \frac{\delta \Gamma}{\delta J} + \Lambda^{D-4} J^* - 2i F_\mu \partial_\mu \frac{\delta \Gamma}{\delta J} - i \partial_\mu F_\mu \frac{\delta \Gamma}{\delta J} + K \frac{\delta \Gamma}{\delta J} - \Lambda^D J_0 \frac{\delta \Gamma}{\delta J_0} - \sum_{n=1}^\infty n J^*_n \frac{\delta \Gamma}{\delta J^*_n} = 0.
\] (27)
By using these equations and the tree-level effective action $\Gamma^{(0)}$ in eq. (18) it is possible to reconstruct the perturbative series in powers of $\hbar$ (loop-expansion).

### 7.1 Hierarchy

Eqs. (26) and (27) guarantee full hierarchy: every amplitude with at least one $\chi$ or $\chi^*$ leg (descendant) can be obtained from those (ancestors) without any of them (the elementary fields) [1], [2]. This is a great advantage since the number of independent counterterms for the ancestors is finite at every order in the loop expansion, as it will be discussed in Sections 9 and 10.

Hierarchy is here illustrated by an explicit example. By taking the derivative of (27) with respect to $J^*(y)$ and by putting all sources and fields to zero one gets

$$\frac{\delta^2 \Gamma}{\delta J(x)\delta J^*(y)} + \Lambda^{D-4} \delta(x - y) - \Lambda^{-D+4} \frac{\delta^2 \Gamma}{\delta \chi(x)\delta J^*(y)} \frac{\delta \Gamma}{\delta J^*_1} = 0.$$  

(28)

Thus the two-point function $\chi - J^*$ is known in terms of the two-point function $J - J^*$ and of the one-point $J^*_1$. In a perturbative approach one starts from $\Gamma^{(0)}$ which is a solution of both equations (26) and (27), by construction. Thus eq. (28) is realized at the tree level since

$$\frac{\delta^2 \Gamma^{(0)}}{\delta J(x)\delta J^*(y)} = 0$$

$$\frac{\delta^2 \Gamma^{(0)}}{\delta \chi(x)\delta J^*(y)} = \Lambda^{D-4} \delta(x - y)$$

$$\frac{\delta \Gamma^{(0)}}{\delta J^*_1} = \Lambda^{-D+4}.$$  

(29)

At the one-loop level one gets from eq. (28)

$$\Lambda^{-D+4} \frac{\delta \Gamma^{(0)}}{\delta J^*_1} \frac{\delta^2 \Gamma^{(1)}}{\delta \chi(x)\delta J^*(y)} = -\left(\Box + m^2\right) \frac{\delta^2 \Gamma^{(1)}}{\delta J(x)\delta J^*(y)}.$$  

(30)

In $D$ dimensions and by using the vertices in eqs. (8) and (9) the relevant quantities are

$$\frac{\delta^2 \Gamma^{(1)}}{\delta J(x)\delta J^*(y)} = \frac{i}{2} \left[\Delta_m(x - y)\right]^2$$

$$\frac{\delta^2 \Gamma^{(1)}}{\delta \chi(x)\delta J^*(y)} = \frac{i}{2} \left(\Box + m^2\right) \left[\Delta_m(x - y)\right]^2.$$  

(31)
where the free $\chi$ propagator is
\[ \Delta_m(x - y) \equiv \Lambda^{D-4}(0|T(\chi(x)\chi^*(y))|0). \] (32)
The derivative of the complex conjugate of eq. (27) with respect to $\chi$ yields a further descendant amplitude
\[ \left(\Box + m^2\right) \frac{\delta^2 \Gamma}{\delta J^*(x) \delta \chi(y)} + \Lambda^{-D+4} \frac{\delta^2 \Gamma}{\delta \chi^*(x) \delta \chi(y)} \frac{\delta \Gamma}{\delta J^*_1} = 0. \] (33)
If more $\chi$ and $\chi^*$ insertions are needed, further $J_n, J^*_m$ will intervene in increasing number. Of course one is very much interested to know if the two point function (connected) turns out correct in this formalism. Indeed one verifies that at one loop the necessary cancellation occurs and
\[ \frac{\delta^2 W^{(0)}}{\delta J^*(x) \delta J(y)} = \Delta_m(x - y) \]
\[ \frac{\delta^2 W^{(1)}}{\delta J^*(x) \delta J(y)} = 0, \] (34)
i.e. $\phi$ remains a free field. A further point of interest is whether the theory makes any sense at $D = 4$.

8 $D = 4$ Limit

The fundamental question is whether one can define a sensible theory at $D = 4$. If one succeeds then the existence of field-coordinate transformation is proven by construction. In this Section I use mostly heuristic arguments: the proofs have been given elsewhere \cite{2} and moreover the main points should not be masked by too many details. Let
\[ \hat{\Gamma} \equiv \Gamma^{(0)} + \sum_{n=1}^{\infty} \hat{\Gamma}^{(n)} \] (35)
where $\hat{\Gamma}^{(n)}$ are the local counterterms, that will be constructed on the ongoing. The generating functional of the Feynman amplitudes is given by
\[ Z = \exp iW \simeq \int e^{\chi + \chi^* \mathcal{D}[\chi^*]\mathcal{D}[\chi]} \exp \left( i \int d^D x (\chi J_0^* + \chi^* J_0) \right). \] (36)
By using the linear operator of eq. (20) for the rotations one gets
\[ \mathcal{R}[\alpha]Z = \left[ \partial^\mu \frac{\delta \hat{\Gamma}}{\delta F^\mu} + i \left[ J^* \frac{\delta \hat{\Gamma}}{\delta J} - J \frac{\delta \hat{\Gamma}^*_1}{\delta J^*_1} \right] + 2 \partial^\mu (F^\mu \delta \hat{\Gamma}) \right]. \]
\[-i \frac{\delta \hat{\Gamma}}{\delta \chi} \left(1 - \Lambda^{-D+1} \frac{\delta \hat{\Gamma}}{\delta J^*_1}\right) + i \frac{\delta \hat{\Gamma}}{\delta \chi^*} \left(1 - \Lambda^{-D+1} \frac{\delta \hat{\Gamma}}{\delta J_1}\right) \]
\[+ \sum_{n=1}^{\infty} n \left( J^*_n \left[-\frac{\delta \hat{\Gamma}}{\delta J^*_n} + \frac{\delta \hat{\Gamma}}{\delta J^*_{(n+1)}}\right] + J_n \left[\frac{\delta \hat{\Gamma}}{\delta J_n} - \frac{\delta \hat{\Gamma}}{\delta J_{(n+1)}}\right]\right) \cdot Z = 0, \quad (37)\]

where the dot indicates the insertion of the operator. Eq. (37) states that if \( \hat{\Gamma} \) (the counterterms) obeys the same equation (26) as the effective action, then the LFE is valid for the generating functionals. Similar result is valid for the translations in eq. (23).

\[T[\beta]Z = \left[- \left(\Box + m^2\right) \frac{\delta \hat{\Gamma}}{\delta J} + \Lambda^{D-4} J^* - 2i F^\mu \partial_\mu \frac{\delta \hat{\Gamma}}{\delta J} - i \partial_\mu F^\mu \frac{\delta \hat{\Gamma}}{\delta J} \right] \]
\[+ K \frac{\delta \hat{\Gamma}}{\delta J} - \Lambda^{D+4} \frac{\delta \hat{\Gamma}}{\delta \chi} \frac{\delta \hat{\Gamma}}{\delta J^*_1} \sum_{n=1}^{\infty} n \frac{J^*_n}{\delta J^*_{(n+1)}} \right] \cdot Z = 0. \quad (38)\]

Thus also for the translations, if \( \hat{\Gamma} \) satisfies the same equation as the effective action functional (27), then the LFE is valid for the generating functionals.

Eqs. (37) and (38) show that the perturbative series in \( D \) dimensions (without counterterms) satisfies both LFE’s. The problem now is to show that after the subtractions and the limit \( D = 4 \) the resulting finite theory still satisfies the same equations, that the subtraction procedure is achieved by local counterterms and that the two-point-function of the \( \phi \)-fields is that of a free theory.

### 8.1 Linearized Operators

The discussion of the perturbative construction of the coordinate-field transformations makes use of the linearized form of the operators acting on \( \Gamma \) as in eqs. (26) and (27). Rotations:

\[\rho(x) \equiv -\partial^\mu \frac{\delta \hat{\Gamma}^{(0)}}{\delta F^\mu} - i \left[ J^* \delta \frac{\delta \hat{\Gamma}^{(0)}}{\delta J^*} - J \delta \frac{\delta \hat{\Gamma}^{(0)}}{\delta J} \right] - 2\partial^\mu \left( F^\mu \frac{\delta \hat{\Gamma}^{(0)}}{\delta K} \right) \]
\[-i \frac{\delta \hat{\Gamma}^{(0)}}{\delta \chi} \frac{\delta \hat{\Gamma}^{(0)}}{\delta J^*_1} + i \frac{\delta \hat{\Gamma}^{(0)}}{\delta \chi^*} \frac{\delta \hat{\Gamma}^{(0)}}{\delta J_1} + i \left(1 - \frac{\delta \hat{\Gamma}^{(0)}}{\delta J^*_1}\right) \frac{\delta \hat{\Gamma}^{(0)}}{\delta \chi} - i \left(1 - \frac{\delta \hat{\Gamma}^{(0)}}{\delta J_1}\right) \frac{\delta \hat{\Gamma}^{(0)}}{\delta \chi^*} \]
\[-i \sum_{n=1}^{\infty} n \left( J^*_n \left[-\frac{\delta \hat{\Gamma}}{\delta J^*_n} + \frac{\delta \hat{\Gamma}}{\delta J^*_{(n+1)}}\right] + J_n \left[\frac{\delta \hat{\Gamma}}{\delta J_n} - \frac{\delta \hat{\Gamma}}{\delta J_{(n+1)}}\right]\right) \]

(39)
and translations:

\[ \tau(x) \equiv \left(\Box + m^2\right) \frac{\delta}{\delta J} + i2F^\mu \partial_\mu \frac{\delta}{\delta J} + i\partial_\mu F^\mu \frac{\delta}{\delta J} \]

\[ -K \frac{\delta}{\delta J} + \frac{\delta \Gamma^{(0)}}{\delta \chi} \frac{\delta}{\delta J_1^*} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta \chi} \frac{\delta}{\delta J_1^*} + \sum_{n=1}^{\infty} n J_n^* \frac{\delta}{\delta J_{(n+1)}}. \]  

(40)

8.2 Subtraction Strategy

The subtraction is performed by iteration, according to the forest formula [25]. The divergent part of the effective action is given by the pole part in the Laurent expansion of

\[ \frac{1}{\Lambda^{D-4}} \Gamma^{(n)} \bigg|_{\text{Pole Part}}. \quad (41) \]

The divergent part is removed by adding a counterterm to the effective action, i.e. the result of the eq. (41) is written as local counterterm in terms of variables in \( D \) dimensions and thus \( \hat{\Gamma}^{(n)} \) is generated. This procedure can be written in a symbolic way by

\[ \hat{\Gamma}^{(n)} = \Lambda^{D-4} \left( \frac{1}{\Lambda^{D-4}} \Gamma^{(n)} \right) \bigg|_{\text{Pole Part}}. \quad (42) \]

The subtraction strategy stands on two important results.

**Proposition 1.** Let the counterterms in \( \hat{\Gamma} \) satisfy the eqs. (26) and (27) then the subtraction up to order \( n - 1 \) breaks the LFE’s for the effective action \( \Gamma \) at order \( n \) by the local terms:

\[ \rho \Gamma^{(n)} - i\Lambda^{-D+4} \sum_{j=1}^{n-1} \left( \frac{\delta \Gamma^{(n-j)}}{\delta \chi} \frac{\delta \Gamma^{(j)}}{\delta J_1^*} - \frac{\delta \Gamma^{(n-j)}}{\delta \chi} \frac{\delta \Gamma^{(j)}}{\delta J_1} \right) \]

\[ = -i\Lambda^{-D+4} \sum_{j=1}^{n-1} \left( \frac{\delta \hat{\Gamma}^{(n-j)}}{\delta \chi} \frac{\delta \hat{\Gamma}^{(j)}}{\delta J_1^*} - \frac{\delta \hat{\Gamma}^{(n-j)}}{\delta \chi} \frac{\delta \hat{\Gamma}^{(j)}}{\delta J_1} \right) \]  

(43)

and for the translations,

\[ \tau \Gamma^{(n)} + \Lambda^{-D+4} \sum_{j=1}^{n-1} \frac{\delta \Gamma^{(n-j)}}{\delta \chi} \frac{\delta \Gamma^{(j)}}{\delta J_1^*} = \Lambda^{-D+4} \sum_{j=1}^{n-1} \frac{\delta \hat{\Gamma}^{(n-j)}}{\delta \chi} \frac{\delta \hat{\Gamma}^{(j)}}{\delta J_1^*}. \]  

(44)

It should be stressed that \( \Gamma^{(n)} \) can diverge in the limit \( D = 4 \) since the relevant poles have not been subtracted, while on the same ground \( \Gamma^{(n-j)}, j > 0, \) is finite.
A heuristic proof of the Proposition 1 goes as follows. $\hat{\Gamma}$ satisfies the LFE’s by assumption. Then if one adds $\rho \hat{\Gamma}^{(n)}$ and $\tau \hat{\Gamma}^{(n)}$ to both sides of eqs. (43) and (44) respectively, then the LFE’s become valid at order $n$ because of equations (37) and (38).

A proof can be given also by using the grading of $\Gamma^{(n)}$ according to the total power of $\hbar$ of the counterterms [2].

The second results can be derived directly from Proposition 1.

**Proposition 2** The subtraction rules consisting in the removal of the sole pole part in the Laurent expansion around $D = 4$ of the effective action $\Gamma$ (eq. (42)) yields a $\hat{\Gamma}$ that obeys the LFE’s. The counterterms are local.

According to eq. (42) the multiplication by $\Lambda^{-D+4}$ of both sides of eqs. (43) and (44) reduces the right hands sides to pure pole terms (no finite parts). Thus the removal of the sole pole part from $\Lambda^{-D+4} \Gamma$ reestablishes the validity of the LFE’s.

The whole subtraction procedure is invalidated by any finite renormalization, e.g. by on-shell renormalization. In fact it is true that, at a given order of the perturbation expansion, the equations (43) and (44) are still valid if one adds to the counterterm $\hat{\Gamma}^{(n)}$ any local solution $\mathcal{M}$ of the homogeneous equations

$$\rho \mathcal{M} = 0$$
$$\tau \mathcal{M} = 0.$$  \hspace{1cm} (45)

In doing so, however, the pole structure of the breaking terms in eqs. (43) and (44) is modified by finite (at $D = 4$) operators. Consequently there is no more a strategy (e.g. the pure pole subtraction) to avoid the increasing of the number of free parameters and the breakdown of the perturbative expansion. This unwanted feature of finite renormalizations does not appear in some special cases as, for instance, in a rescaling of $\Lambda$ (e.g. $\overline{\text{MS}}$).

**9 Weak Power Counting**

The LFE’s provide full hierarchy, as discussed in Section 7. In particular all the divergent amplitudes involving the elementary fields $\chi \chi^*$ are organized and controlled by a finite number of divergent ancestor amplitudes at any given order of the perturbative expansion. To establish which are the relevant ancestor amplitudes, the WPC criterion is very useful.
The degree of divergence of a graph \( G \) for an ancestor amplitude can be evaluated in the usual fashion. Let \( I \) be the number of internal \( \chi \) propagators, \( N_F \) the number of external \( F_\mu \) legs and \( N_X \) those of \( X \in \{ K, J, J^*, J_i, J_i^* | i = 1, \cdots \} \). \( V_{jk} \) denotes the number of vertices with \( k \) \( \chi \)-lines and \( j \) derivatives and \( N_{Xk} \) the number of \( X \) external sources with \( k \) internal lines of \( \chi \)'s attached. The superficial degree of divergence \( \delta(G) \) for a graph can be bounded by using standard arguments. One has (\( n \) number of loops)

\[
\delta(G) = D - 2I + \sum_{j,k} j V_{jk} + N_F
\]

\[
n = I - \sum_{j,k} V_{jk} - N_F - \sum_X N_X + 1
\] (46)

By removing \( I \) from these two equations one gets

\[
\delta(G) = D - 2n - \sum_{j,k} (2 - j) V_{jk} - N_F - 2 \sum_X N_X + 2.
\] (47)

The classical action \( 7 \) has vertices with \( j \leq 2 \), therefore it can be stated that

\[
\delta(G) \leq n(D - 2) + 2 - N_F - 2 \sum_X N_X.
\] (48)

By arguments similar to those used for eq. (48), one can prove that the subtraction procedure indicated by eq. (45) does not modify the upper bound on the superficial divergence (WPC theorem).

Eq. (48) restricts the set of divergent ancestor amplitudes to those with a finite number of external legs. It will be shown, in Section 10, that the number of counterterms becomes finite after the analysis of the local solutions of eqs (43) and (44).

At one loop and \( D = 4 \) the possible divergent ancestor amplitudes are: \( K \) one-point-functions, \( F_\mu - F_\nu \), \( K - F_\mu \), \( X - X' \) two-point-functions, \( (F_\mu - F_\nu) - X \) three-point-functions and \( F_\mu \) four-point-functions.

10 Local Solutions of the Linear LFE’s

The discussion on the divergent ancestor amplitudes has shown that, at any order of the perturbative expansion, the number of external legs must be
finite. However the number of external sources is infinite, thus the constraint
given by the WPC criterion is not enough to make finite the number of
counterterms. In this Section other constraints on the counterterms are
found. These constraints limit the counterterms to a finite number of local
invariant operators.

In the present Section the local solutions of the homogeneous eqs. (45)
are considered, since the aim is to illustrate the method only at the one loop
level. If higher loop are of interest, then it is necessary to solve the following
nonlinear equations and an example is briefly discussed in Section 13. For
the rotations
\[ \hat{\Gamma}^{(n)} = i \Lambda^{-D+4} \sum_{j=1}^{n-1} \left( \frac{\partial \hat{\Gamma}^{(n-j)}}{\partial \chi} \right) \delta \hat{\Gamma}^{(j)} \frac{\delta \hat{\Gamma}^{(j)}}{\delta j^1} \]  
(49)
and for the translations
\[ \tau \hat{\Gamma}^{(n)} = -\Lambda^{-D+4} \sum_{j=1}^{n-1} \frac{\partial \hat{\Gamma}^{(n-j)}}{\partial \chi} \delta \hat{\Gamma}^{(j)} \]  
(50)

In order to find the relevant local solutions of the linearized LFE's one
looks for suitable variables that have simple transformations under \( \rho \) and \( \tau \). This method is very similar to the bleaching technique introduced in
Ref. [3]. Moreover there are some helping directions in the search of local
invariants at one loop. For instance they need to be at most quadratic in
the sources \( X \) and quartic in \( F_{\mu} \).

The following notation is useful
\[ \rho[\alpha] \equiv \int d^D x \alpha(x) \rho(x) \]
\[ \tau[\beta] \equiv \int d^D x \beta(x) \tau(x) \]
(51)
\[ S[\phi, \phi^*, F, K] \equiv \int d^D z \left( \partial_{\mu} \phi^* \partial^\mu \phi - m^2 \phi^* \phi + J^* \phi + J \phi^* + F^\mu \phi^* \partial_{\mu} \phi + K \phi^* \phi \right) \]
(52)
One gets
\[ -J^*_0 \equiv \frac{\partial \Gamma^{(0)}}{\partial \chi} = (1 + \phi) \left[ \frac{\delta S}{\delta \phi} - \sum_{n=1}^{n} \frac{n J^*_n}{(1 + \phi)^{n+1}} \right] \]
\[ = (1 + \phi) \frac{\delta S}{\delta \phi} - \sum_{n=1}^{n} \frac{n J^*_n}{(1 + \phi)^n}. \]  
(53)
The rotation transformations generated by $\rho$ in eq. (39) are

\[ \rho[\alpha] \chi = i \alpha \frac{\phi}{1 + \phi}, \quad \rho[\alpha] \phi = i \alpha \phi \]
\[ \rho[\alpha] J = i \alpha J, \quad \rho[\alpha] F^\mu = \partial^\mu \alpha \]
\[ \rho[\alpha] K = 2 F^\mu \partial_\mu \alpha \]
\[ \rho[\alpha] J_1 = -i \alpha \left[ J_1 - \frac{\delta \Gamma^{(0)}}{\delta \chi} \right] \]
\[ n > 1 \]

and the translations generated by $\tau[\beta]$ in eq. (40) are

\[ \tau[\beta] \chi = \beta \frac{1}{1 + \phi}, \quad \tau[\beta] \phi = \beta \]
\[ \tau[\beta] K = 0 \]
\[ \tau[\beta] J = -\int d^D y \beta(y) \frac{\delta^2 S}{\delta \phi(y) \delta \phi(x)} \]
\[ \tau[\beta] J_1 = \beta \frac{\delta \Gamma^{(0)}}{\delta \chi}, \quad n > 1 \]

10.1 Properties of the rotations generator $\rho[\alpha]$

The local invariants under rotations are first investigated. $F_\mu$ plays the rôle of abelian gauge fields, thus it is convenient to define a covariant derivative

\[ D_\mu \equiv \partial_\mu - i F_\mu. \] (56)

Consider the following quantity

\[ \rho[\alpha] \frac{\delta S}{\delta \phi^*(x)} = \rho[\alpha] \left[ -(\Box + m^2)\phi + J + i F_\mu \partial^\mu \phi + i \partial^\mu (F_\mu \phi) + K \phi \right] \]
\[ = -i(\Box + m^2)(\alpha \phi) + i \alpha J - F_\mu \partial^\mu (\alpha \phi) + i \partial_\mu \alpha \partial^\mu \phi - \partial^\mu (F_\mu \alpha \phi) \]
\[ + i \partial^\mu (\partial^\mu \alpha \phi) + i K \alpha \phi + 2 F_\mu \partial^\mu \alpha \phi \]
\[ = i \alpha \frac{\delta S}{\delta \phi^*(x)}. \] (57)

One obtains similarly

\[ \rho[\alpha] \frac{\delta S}{\delta \phi(x)} = -i \alpha \frac{\delta S}{\delta \phi(x)}. \] (58)

From eqs. (57) and (58) one gets

\[ \rho[\alpha] \frac{\delta \Gamma^{(0)}}{\delta \chi(x)} = \rho[\alpha] \frac{\delta S}{\delta \chi(x)} - \rho[\alpha] \sum_{n=1} \frac{J^n_1}{(1 + \phi)^n} = \rho[\alpha] (1 + \phi) \frac{\delta S}{\delta \phi(x)} \]
\[ - i \alpha \left[ J_1^* - \frac{\delta \Gamma^{(0)}}{\delta \chi(x)} \right] \frac{1}{(1 + \phi)} + i \alpha(x) \frac{J_1^* \phi}{(1 + \phi)^2} \]
\[ - \sum_{n=2} \left[ i \alpha(x)(n J_n^*(x) - (n - 1) J_n^* - 1) J_{n-1}^* \right] \frac{n}{(1 + \phi)^n} - i \alpha(x) \frac{n^2 J_n \phi}{(1 + \phi)^{n+1}} \]
\[ = \rho[\alpha] \frac{\delta S}{\delta \phi(x)} + i\alpha \frac{\delta \Gamma^{(0)}}{\delta \phi(x)} - i\alpha(x) \frac{J^*_1}{(1 + \phi)^2} + i2\alpha(x) \frac{J^*_1}{(1 + \phi)^2} - \alpha(x) \sum_{n=2}^{n^2} \left[ J^*_n(x) \frac{n(1 + \phi)^{n+1}}{(1 + \phi)^{n+1}} - \frac{n^2 J^*_n(\phi)}{(1 + \phi)^{n+1}} \right] \]

\[ = -i\alpha(x) \frac{\delta S}{\delta \phi(x)} + i\alpha \frac{\delta \Gamma^{(0)}}{\delta \phi(x)} + i\alpha(x) \frac{J^*_1}{(1 + \phi)^2} + i\alpha(x) \sum_{n=2}^{n^2} \frac{J^*_n(x)}{(1 + \phi)^{n+1}} \]

\[ = -i\alpha(x) \frac{\delta \Gamma^{(0)}}{\delta \phi(x)} + i\alpha \frac{\delta \Gamma^{(0)}}{\delta \phi(x)} = 0. \] (59)

and

\[ \rho[\alpha] \frac{\delta \Gamma^{(0)}}{\delta \chi^*(x)} = 0. \] (60)

Another interesting local operator is given by the following expression

\[ \mathcal{N}_0 \equiv \left[ J^*_0 \ln(1 + \phi) + \sum_{n=2}^{n^2} \frac{1}{(1 + \phi)^n} \right] \bigg|_{J^*_0 = -\frac{\delta \Gamma^{(0)}}{\delta \chi^*(x)}}. \] (61)

By direct computation one gets

\[ \rho[\alpha] \mathcal{N}_0 = \rho[\alpha] J^*_0 \ln(1 + \phi) + i\alpha \frac{J^*_0 \phi}{(1 + \phi)} + i\alpha \frac{J^*_0 + J^*_1}{(1 + \phi)^2} - \alpha \sum_{n=2}^{n^2} \frac{J^*_n \phi}{(1 + \phi)^{n+1}} + i\alpha \sum_{n=2}^{n^2} \frac{J^*_n \phi}{(1 + \phi)^n} - i\alpha \sum_{n=1}^{n^2} \frac{n J^*_n}{(1 + \phi)^{n+1}} + i\alpha \sum_{n=1}^{n^2} \frac{n J^*_n \phi}{(1 + \phi)^n} - i\alpha \sum_{n=1}^{n^2} \frac{n J^*_n}{(1 + \phi)^{n+1}} \]

\[ = \rho[\alpha] J^*_0 \ln(1 + \phi) + i\alpha J^*_0 \phi + i\alpha \frac{J^*_0}{(1 + \phi)} + i\alpha \frac{J^*_1}{(1 + \phi)^2} - \alpha \sum_{n=1}^{n^2} \frac{n J^*_n \phi}{(1 + \phi)^{n+1}} + i\alpha \sum_{n=1}^{n^2} \frac{n J^*_n \phi}{(1 + \phi)^n} - i\alpha \sum_{n=1}^{n^2} \frac{n J^*_n}{(1 + \phi)^{n+1}} \]

\[ = \rho[\alpha] J^*_0 \ln(1 + \phi) + i\alpha J^*_0. \] (62)

By using eq. (59) one gets

\[ \rho[\alpha] \mathcal{N}_0 = -i\alpha \frac{\delta \Gamma^{(0)}}{\delta \chi(x)} \] (63)

\[ \rho[\alpha] \mathcal{N}_0^* = i\alpha \frac{\delta \Gamma^{(0)}}{\delta \chi^*(x)}. \] (64)

One can also consider

\[ \mathcal{N}_1 \equiv \left[ J^*_1 \ln(1 + \phi) + \sum_{n=2}^{n^2} \frac{n J^*_n}{(1 + \phi)^{n+1}} \right] \bigg|_{J^*_0 = -\frac{\delta \Gamma^{(0)}}{\delta \chi(x)}} \]

\[ = -\left[ \frac{\delta S}{\delta \phi} - \sum_{n=1}^{n^2} \frac{n J^*_n}{(1 + \phi)^{n+1}} \right] - \sum_{n=1}^{n^2} \frac{n J^*_n}{(1 + \phi)^{n+1}} = -\frac{\delta S}{\delta \phi}. \] (65)
From eq. (58) one gets
\[ \rho[\alpha]N_1 = -i\alpha N_1. \]  
(66)

Similarly one has \( j > 0 \)

\[ \mathcal{N}_j \equiv -J_0^* \frac{(j-1)!}{(1+\phi)^j} + \sum_{n=1} n(n+1) \cdots (n+j-1) J_n^* \frac{1}{(1+\phi)^{n+j}} \bigg|_{J_0^* = -\frac{\delta \rho(0)}{\delta \chi(x)}} \]

\[ = -\frac{(j-1)!}{(1+\phi)^j} \delta S + \sum_{n=1} J_n^* \frac{n(n+1) \cdots (n+j-1)}{(1+\phi)^{n+j}}. \]  
(67)

For instance

\[ \mathcal{N}_2 = -J_0^* \frac{1}{(1+\phi)^2} + \sum_{n=1} n(n+1) J_n^* \frac{1}{(1+\phi)^{n+2}} \bigg|_{J_0^* = -\frac{\delta \rho(0)}{\delta \chi(x)}} \]

\[ = \frac{1}{(1+\phi)^2} \delta S + \sum_{n=1} J_n^* \frac{n^2}{(1+\phi)^{n+2}}. \]  
(68)

and

\[ \mathcal{N}_3 = -J_0^* \frac{2}{(1+\phi)^3} + \sum_{n=1} n(n+1)(n+2) J_n^* \frac{1}{(1+\phi)^{n+3}} \bigg|_{J_0^* = -\frac{\delta \rho(0)}{\delta \chi(x)}} \]

\[ = \frac{2}{(1+\phi)^3} \delta S + \sum_{n=1} J_n^* \frac{n^2(n+3)}{(1+\phi)^{n+3}}. \]  
(69)

Notice also the relations \( j > 0 \)

\[ \frac{\delta \mathcal{N}_j(x)}{\delta \phi(y)} = \delta(x-y) \left[ -\mathcal{N}_{j+1} + \frac{(j-1)!}{(1+\phi)^{j-1}} \mathcal{N}_2 \right] \]

\[ \frac{\delta^2 \mathcal{N}_j(x)}{\delta \phi^*(y) \delta \phi(x)} = \frac{(j-1)!}{(1+\phi)^{j-1}} \delta^2 S \frac{\delta \phi^*(y) \delta \phi(x)}{\delta \phi^*(y) \delta \phi(x)}. \]  
(70)

Under rotations one has

\[ \rho[\alpha] \mathcal{N}_j = i\alpha j \frac{J_0^* \phi}{(1+\phi)^{j+1}} + i\alpha j \frac{J_0^* + J_1^*}{(1+\phi)^{j+1}} \]

\[ -i\alpha (j+1)! \frac{J_0^* \phi}{(1+\phi)^{j+2}} - i\alpha \sum_{n=2} n \cdots (n+j) \frac{J_n^* \phi}{(1+\phi)^{n+j+1}} \]

\[ + i\alpha \sum_{n=2} n^2 (n+1) \cdots (n+j-1) \frac{J_n^*}{(1+\phi)^{n+j+1}} - i\alpha \sum_{n=1} n \cdots (n+j) \frac{J_n^*}{(1+\phi)^{n+j+1}} \]

\[ = j \alpha \frac{J_0^*}{(1+\phi)^j} - i\alpha \sum_{n=1} n \cdots (n+j) \frac{J_n^*}{(1+\phi)^{n+j}} \]

\[ + i\alpha \sum_{n=1} n^2 (n+1) \cdots (n+j-1) \frac{J_n^* \phi}{(1+\phi)^{n+j}} = -i j \alpha \mathcal{N}_j. \]  
(71)
10.2 Properties of the translations generator $\tau[\beta]$

Consider now the properties under translations. Direct evaluation yields

\[
\tau[\beta] \frac{\delta \Gamma^{(0)}}{\delta \phi(x)} = \tau[\beta] \left[ J^* - \sum_{n=1}^{\infty} \frac{n J_n^*}{(1 + \phi)^{n+1}} \right]
\]

\[
= -\beta \frac{\delta \Gamma^{(0)}}{\delta \phi(x)} \frac{1}{1 + \phi} + 2\beta \frac{J_1^*}{1 + \phi} - \beta \sum_{n=2}^{\infty} \left[ n(n - 1) \frac{J_n^*}{(1 + \phi)^{n+1}} - n(n + 1) \frac{J_n^*}{(1 + \phi)^{n+2}} \right]
\]

\[
= -\beta \frac{\delta \Gamma^{(0)}}{\delta \phi(x)} \frac{1}{1 + \phi}
\]

or

\[
\tau[\beta] \frac{\delta \Gamma^{(0)}}{\delta \phi(x)} = 0.
\]  

(72)

By using

\[
\tau[\beta] \frac{\delta S}{\delta \phi^*(x)} = \tau[\beta] \left[ -(\Box + m^2)\phi + J + iF^\mu \partial_\mu \phi + i\partial^\mu (F_\mu \phi) + K \phi \right]
\]

\[
= -(\Box + m^2)\beta - \int d^D y \beta(y) \frac{\delta^2 S}{\delta \phi(y) \phi^*(x)} + iF^\mu \partial_\mu \beta + i\partial^\mu (F_\mu \beta)
\]

\[+ K \beta = 0,
\]

(74)

one gets

\[
\tau[\beta] \frac{\delta \Gamma^{(0)}}{\delta \chi^*(x)} = \tau[\beta] \left[ (1 + \phi^*) \frac{\delta S}{\delta \phi^*(x)} \right]
\]

\[
= (1 + \phi^*) \tau[\beta] \left[ \frac{\delta S}{\delta \phi^*(x)} \right] = 0.
\]

(75)

One gets also

\[
\tau[\beta] \frac{\delta S}{\delta \phi(x)} = 0.
\]

(76)

The transformation properties under translations of $N_0, N_1, N_j$ are

\[
\tau[\beta] N_0 = \tau[\beta] J_0^* \ln(1 + \phi) + \beta \frac{J_0^*}{1 + \phi} - \beta \frac{J_0^*}{(1 + \phi)^2}
\]

\[
- \beta \frac{J_1^*}{1 + \phi} + \beta \sum_{n=1}^{\infty} \frac{n J_n^*}{(1 + \phi)^{n+1}} - \beta \sum_{n=2}^{\infty} \frac{n J_n^*}{(1 + \phi)^{n+1}}
\]

\[
= -\tau[\beta] \frac{\delta \Gamma^{(0)}}{\delta \chi(x)} \ln(1 + \phi) = 0.
\]

(77)
Similarly one shows that

$$\tau[\beta]N_1 = 0$$  \hspace{1cm} (78)

and

$$\tau[\beta]N_j = 0.$$  \hspace{1cm} (79)

From the definitions (61) and (67) and with the use of the identity (75) it follows

$$\tau[\beta]N^*_j = 0, \ j = 0, \ldots.$$  \hspace{1cm} (80)

### 10.3 The algebra

By using the previous results, one can now prove the following relation

$$[\rho[\alpha], \tau[\beta]] = -i\tau[\alpha\beta].$$  \hspace{1cm} (81)

The most difficult terms are those involving $\delta\delta J^*$. Thus I consider only those

$$\left[\frac{\delta}{\delta J^*}, \frac{\delta}{\delta J^*}\right] \frac{\delta\Gamma(0)}{\delta \chi(y)} \frac{\delta}{\delta \chi(x)}.$$  \hspace{1cm} (82)

Now I use eqs. (55), (59) and (73) and I get

$$\left[\frac{\delta}{\delta J^*}, \frac{\delta}{\delta J^*}\right] = -i \int d^Dx \alpha(x)\tau[\beta][-i\frac{\delta\Gamma(0)}{\delta \chi(x)} + iJ^*_1] \frac{\delta}{\delta J^*_1(x)}.$$  \hspace{1cm} (83)

which is the right term as it appears in eq. (82).

### 11 One-loop counterterms

The study performed in Section 9 shows that at one-loop level the expected divergent ancestor amplitudes are

1. $K$-tadpole

$$\frac{\delta\Gamma^{(1)}}{\delta K(x)} = \Delta_m(0)$$  \hspace{1cm} (84)
2. $J - J^*$ two-point function
\[
\frac{\delta \Gamma^{(1)}}{\delta J(x) \delta J^*(y)} = i \frac{\Delta_m(x - y)^2}{2}
\]  
(85)

3. $J - J^*_n$ two-point function
\[
\frac{\delta \Gamma^{(1)}}{\delta J(x) \delta J^*_n(y)} = n^2 i \frac{\Delta_m(x - y)^2}{2}
\]  
(86)

4. $J_n - J^*_n$ two-point function
\[
\frac{\delta \Gamma^{(1)}}{\delta J_n(x) \delta J^*_n(y)} = n^2 n' i \frac{\Delta_m(x - y)^2}{2}
\]  
(87)

5. $K - K$ two-point function
\[
\frac{\delta \Gamma^{(1)}}{\delta K(x) \delta K(y)} = i \Delta_m(x - y)^2
\]  
(88)

6. $F_\mu - F_\nu$ two-point function
\[
\frac{\delta \Gamma^{(1)}}{\delta F^\mu(x) \delta F^\nu(y)} = i \Delta_m(x - y) \hat{\partial}_\mu \hat{\partial}_\nu \Delta_m(x - y).
\]  
(89)

By using the pole parts
\[
\Delta_m \simeq \frac{1}{(4\pi)^2} \frac{2}{D - 4} m^2
\]
\[
\Delta_m^2 \simeq i \frac{1}{(4\pi)^2} \frac{2}{D - 4}
\]  
(90)

and eq. (108) in Appendix A, one can write the counterterms at one loop
\[
\hat{\Gamma}^{(1)} = \Delta^{D-4} \frac{1}{(4\pi)^2} \frac{2}{D - 4} \left[ m^2 \mathcal{I}_1 - \frac{1}{6} \mathcal{I}_2 + \frac{1}{2} \mathcal{I}_3 + \frac{1}{2} \mathcal{I}_4 \right]
\]  
(91)

where
\[
\mathcal{I}_1 = \int d^D x (F^2 - K)
\]
\[
\mathcal{I}_2 = \int d^D x \left( \partial_\mu F_\nu - \partial_\nu F_\mu \right)^2
\]
\[
\mathcal{I}_3 = \int d^D x (F^2 - K)^2
\]
\[
\mathcal{I}_4 = \int d^D x N_2^* N_2.
\]  
(92)
Notice that the square modulus of $\mathcal{N}_2$ describes the counterterms for all the two-point functions $J - J^*$, $J - J^*_n$ and $J_n - J^*_n$. The rest of the divergent ancestor amplitudes $F - F - K$ and $F - F - F - F$ are described by the local invariant $\mathcal{I}_3$. It is amazing that by expanding $\mathcal{I}_4$ in powers of the fields $\chi$ and $\chi^*$ one gets all the counterterms of the infinitely many descendant amplitudes at one-loop.

12 Some more local invariants

The results of the analysis of Section 10 on the properties of $\rho(\alpha|\tau(\beta)$ suggests many local invariants (i.e. solutions of both $\rho(\alpha|\chi = 0$ and $\tau(\beta|\chi = 0$). Here is a partial list of them

\begin{align*}
\mathcal{I}_5 &= \int d^D x \frac{\delta S}{\delta \phi^*(x)} \frac{\delta S}{\delta \phi(x)} \\
\mathcal{I}_6 &= \int d^D x \frac{\delta \Gamma^{(0)}}{\delta \chi(x)} \frac{\delta \Gamma^{(0)}}{\delta \chi^*(x)} \\
\mathcal{I}_7 &= \int d^D x \left[ N_0^* \frac{\delta \Gamma^{(0)}}{\delta \chi^*(x)} + N_0 \frac{\delta \Gamma^{(0)}}{\delta \chi(x)} \right] \\
\mathcal{I}_8 &= \int d^D x \left[ \frac{\delta \Gamma^{(0)}}{\delta \chi^*(x)} + \frac{\delta \Gamma^{(0)}}{\delta \chi(x)} \right] \\
\mathcal{I}_9 &= \int d^D x D^\mu \frac{\delta S}{\delta \phi^*(x)} D^*_\mu \frac{\delta S}{\delta \phi(x)} \\
\mathcal{I}_{10} &= \int d^D x D^\mu N_1^* D^*_\mu N_1 \\
\mathcal{I}_{11} &= \int d^D x (\partial^\mu - 2iF^\mu)N_2^* (\partial_\mu + 2iF_\mu)N_2.
\end{align*}

The invariants in eq. (93) are excluded at the one-loop level for various reasons: either they are too high in dimensions (as $\mathcal{I}_9, \mathcal{I}_{10}$ and $\mathcal{I}_{11}$) or they contain source terms describing amplitudes that are not divergent (as $\mathcal{I}_8$) or the coefficients determined by the divergent ancestor amplitudes are just zero (as $\mathcal{I}_5, \mathcal{I}_6$ and $\mathcal{I}_7$).

To conclude the Section it is worth noticing that, according to textbook renormalization, one is expected to introduce in the classical action all the local invariants $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ and $\mathcal{I}_4$ with arbitrary parameters. However, in doing so, the very starting point given by the classical action (18) is deeply changed. The perturbative expansion is modified by a whole set of new vertices. The WPC is not anymore valid, due to the presence of four derivatives.
interactions.

\section{Higher-order-loop counterterms}

It is outside the scope of the present paper to perform higher-loop calculations. However in this Section some features of the two-loop amplitudes are briefly discussed in order to have a glance on the whole strategy.

The two-loop invariant local counterterms must obey the inhomogeneous equations where the given term is (e.g. in eq. (51))

\begin{equation}
- \Lambda^{-D+4} \frac{\delta \tilde{\Gamma}^{(1)}}{\delta \chi(x)} \frac{\delta \tilde{\Gamma}^{(1)}}{\delta J_1^*(x)}.
\end{equation}

Only $I_4$ contributes to the inhomogeneous term in eq. (94). The others in eq. (91) do not depend on $\chi$ and $J_1^*$. One gets

\begin{equation}
- \Lambda^{-D+4} \frac{\delta \tilde{\Gamma}^{(1)}}{\delta \chi(x)} \frac{\delta \tilde{\Gamma}^{(1)}}{\delta J_1^*(x)} = \Lambda D - 4 \left( \frac{4 \pi}{D - 4} \right)^2 N_2^*(x) \frac{\delta}{\delta \phi(x)} \int d^D y N_2(y) N_2^*(y) \right].
\end{equation}

To find a solution of inhomogeneous equation one needs some more algebra. It useful to define the operator

\begin{equation}
\mathcal{D} \equiv \rho + i(\tau - \tau^*).
\end{equation}

Eqs. (49) and (50) imply that any counterterm $\tilde{\Gamma}^{(n)}$ must obey the homogeneous equation

\begin{equation}
\mathcal{D} \tilde{\Gamma}^{(n)} = 0.
\end{equation}

The following relations can be easily derived

\begin{align*}
\mathcal{D}(y) N_2^*(x) &= i 2 \delta(x - y) N_2^*(x) \\
\mathcal{D}(y)(1 + \phi(x))^k &= i k \delta(x - y) (1 + \phi(x))^k.
\end{align*}

Moreover by using

\begin{equation}
[\mathcal{D}(y), \frac{\delta}{\delta \phi(x)}] = -i \delta(x - y) \frac{\delta}{\delta \phi(x)}
\end{equation}

one can show that

\begin{equation}
\mathcal{D}(y) \frac{\delta \mathcal{X}}{\delta \phi(x)} = -i \delta(x - y) \frac{\delta \mathcal{X}}{\delta \phi(x)}, \quad \forall \mathcal{X} : \mathcal{D} \mathcal{X} = 0.
\end{equation}
Equations (98), (99) and (100) give

$$D(x) \int d^D y \frac{N_2^*(y)}{1 + \phi(y)} \frac{\delta}{\delta \phi(y)} \left[ \int d^D z N_2(z) N_2^*(z) \right] = 0$$

$$D(x) \int d^D y \frac{N_2^*(y)}{(1 + \phi(y))^2} N_2(y) N_2^*(y) = 0. \quad (101)$$

With a further identity

$$\left[ \tau(x), \frac{\delta}{\delta \phi(y)} \right] = -\delta(x-y)(1 + \phi(x)) N_2(x) \frac{\delta}{\delta \bar{J}_1(x)} , \quad (102)$$

it is straightforward to verify that

$$\tau(x) \int d^D y \left\{ \frac{N_2^*(y)}{1 + \phi(y)} \frac{\delta}{\delta \phi(y)} \left[ \int d^D z N_2(z) N_2^*(z) \right] - \frac{N_2^*(y) N_2(y)}{2(1 + \phi(y))^2} N_2^*(y) \right\}$$

$$= -\frac{N_2^*(x)}{(1 + \phi(x))^2} \frac{\delta}{\delta \phi(x)} \left[ \int d^D z N_2(z) N_2^*(z) \right] - \frac{N_2^*(x)}{(1 + \phi(x))^3} N_2(x) N_2^*(x) + \frac{N_2^*(x)}{(1 + \phi(x))^3} N_2(x) N_2^*(x)$$

$$= -\frac{N_2^*(x)}{(1 + \phi(x))^2} \frac{\delta}{\delta \phi(x)} \left[ \int d^D z N_2(z) N_2^*(z) \right], \quad (103)$$

i.e. a solution of eq. (50) has been found. Eq. (101) guarantees that also eq. (49) is satisfied.

To complete the two-loop analysis one needs to evaluate the divergent parts of a (finite) set of ancestor amplitudes, enough to fix the coefficient of the solution in eq. (103) and the homogeneous part of $\hat{\Gamma}^{(2)}$.

### 14 Conclusions

The use of the LFE’s, derived from the invariance properties of the path integral measure, of the hierarchy and of the WPC allows a complete classification of the divergent amplitudes for the scalar complex free field theory in polar coordinates. Minimal subtraction in dimensional regularization, enforced by the prescription of pure pole subtraction, agrees with this structure and thus provides a perfect procedure for the symmetric subtraction (i.e. preserving the LFE’s) of all the infinities. Thus the theory can be made finite at $D = 4$.

In this paper the LFE’s are derived, the hierarchy is proven and discussed, the WPC is illustrated as an important tool for the analysis of the
divergent ancestor amplitudes. Finally the local solutions of the LFE’s at
one loop are discussed in some details and the counterterms \( \hat{\Gamma}^{(1)} \) are es-
tablished. It is argued that conventional renormalization procedure is not a
viable method for the use of polar coordinates, while the approach presented
in this paper yields a perturbative expansion which is finite and consistent.
It is briefly illustrated (at the one-loop-level) that the original field in cartes-
ian coordinates \((\phi)\) has the correct two-point-function. Moreover a solution
of the inhomogeneous LFE’s is derived for the two-loop counterterms. One
can conclude with good confidence that the existence of the coordinates
transformation for \(D = 4\) is proven by construction.

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A \( F_\mu - F_\nu \) two-point-function

The one loop contribution to the two-point function \( F_\mu - F_\nu \) is as in scalar
QED

\[
\Gamma^{(1)}_{F_\mu F_\nu}(p) = -i \int \frac{d^Dk}{(2\pi)^D} \left[ \frac{(2k+p)_\mu(2k+p)_\nu}{(k^2 - m^2)[(p+k)^2 - m^2]} - 2 \frac{g_{\mu\nu}}{k^2 - m^2} \right]
\]

\[
= -i 4(p_\mu p_\nu - p^2 g_{\mu\nu}) \int \frac{d^Dk}{(2\pi)^D} \int \frac{1}{2} dy \frac{y^2}{[k^2 - C]^2} - 2g_{\mu\nu}\Delta_m(0) \quad (104)
\]

where

\[
C \equiv m^2 - p^2 \left( \frac{1}{4} - y^2 \right)
\]

\[
\Delta_m(0) = i \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 - m^2} \quad (105)
\]

and use has been made of the identity

\[
\int d^Dk \frac{k^2}{[k^2 - C]^2} = \frac{D}{D-2} \int d^Dk \frac{C}{[k^2 - C]^2}. \quad (106)
\]
The integration on the momentum gives

\[
\Gamma^{(1)}_{F_\mu F_\nu}(p) = 4(p_\mu p_\nu - p^2 g_{\mu\nu}) \int_{-\frac{1}{2}}^{\frac{1}{2}} dy y^2 \frac{1}{(4\pi)^D} \frac{\Gamma(2 - \frac{D}{2})}{\Gamma(1)} (\frac{D}{2})^{D-2} - 2g_{\mu\nu} \frac{1}{(4\pi)^D} \frac{\Gamma(2 - \frac{D}{2})}{(1 - \frac{D}{2})\Gamma(1)} [m^2]^{\frac{D}{2}-1} \tag{107}
\]

The pole part is then

\[
\Lambda^{-D+4}\Gamma^{(1)}_{F_\mu F_\nu}(p)\bigg|_{\text{pole part}} = \frac{2}{3} \frac{1}{(4\pi)^2} \frac{1}{D-4} (p^2 g_{\mu\nu} - p_\mu p_\nu) - 4 \frac{1}{(4\pi)^2} \frac{1}{D-4} m^2 g_{\mu\nu}. \tag{108}
\]

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