Is the Scaling of Supersonic Turbulence Universal?

Wolfram Schmidt

Lehrstuhl für Astronomie, Institut für Theoretische Physik und Astrophysik,
Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany

Christoph Federrath and Ralf Klessen

Institut für Theoretische Astrophysik, Universität Heidelberg,
Albert-Ueberle-Str. 2, D-69120 Heidelberg, Germany
(Dated: October 8, 2008)

The statistical properties of turbulence are considered to be universal at sufficiently small length scales, i.e., independent of boundary conditions and large-scale forces acting on the fluid. Analyzing data from numerical simulations of supersonic turbulent flow driven by external forcing, we demonstrate that this is not generally true for the two-point velocity statistics of compressible turbulence. However, a reformulation of the refined similarity hypothesis in terms of the mass-weighted velocity $p^{1/3}v$ yields scaling laws that are almost insensitive to the forcing. The results imply that the most intermittent dissipative structures are shocks closely following the scaling of Burgers turbulence.

The notion of universality is central to theoretical and observational accounts of turbulence. For incompressible turbulence, the mathematical analysis carried out by Kolmogorov led to the famous 2/3 law for the second order structure function of turbulent velocity fluctuations $\langle \delta v(x,t) \delta v(x+\ell,t) \rangle$. Remarkably, this scaling law was experimentally confirmed even if the premises of Kolmogorov’s theory—statistical equilibrium, homogeneity and isotropy—were not satisfied. It is commonly accepted that this is due to an inertial subrange of scales, where the dynamics of turbulence conditions the flow such that these premises are met asymptotically toward smaller length scales independent of the large-scale properties of the flow. Whereas terrestrial applications are mostly concerned with the incompressible regime, the idea that universal scaling also exists in the inertial subrange of the highly compressible, supersonic turbulence has become popular in astrophysics. Particularly, observational properties of star-forming clouds are explained by supersonic turbulent motion which is seen as an agent that controls the formation of stars besides gravity.

We tested the hypothesis of universality on data from numerical simulations of supersonic turbulence with the same root mean square (RMS) Mach numbers but different large-scale forcing. The result is that universality in the sense of the Kolmogorov theory is clearly violated even if intermittency corrections are applied, but scaling laws that are nearly independent of the forcing apply to the mass-weighted velocity $\bar{v} = p^{1/3}v$ introduced by Kritsuk et al. Moreover, the scaling exponents turn out to be consistent with log-Poisson models, if the most intermittent dissipative structures are shocks fulfilling the scaling of Burgers turbulence.

The fundamental relation for the scaling properties of incompressible turbulence is the refined similarity hypothesis,

$$S_p(\ell) = C_p \ell^{p/3}(\epsilon^p_3/3),$$

where the $C_p$ are constant dimensionless coefficients, $S_p(\ell) := \langle \delta v_\ell^p \rangle$ are the statistical moments of the velocity fluctuation $\delta v_\ell = |v(x,t) - v(x + \ell, t)|$ of order $p = 1, 2, 3, \ldots$ and $\epsilon_\ell$ is the rate of energy dissipation per unit mass averaged over a region of size $\ell$. The brackets $\langle \rangle$ denote the ensemble average. The factor $(\epsilon^p_3/3)^{p/3}$ is attributed to the intermittency of turbulence.

The scaling exponents of $S_p(\ell)$ in the inertial subrange are thus given by $\zeta_p = p/3 + \tau_p/3$.

Modeling turbulent energy dissipation by a random cascade obeying log-Poisson statistics, Dubrulle showed that the relative scaling exponents $Z_p$ have the general form

$$Z_p := \frac{\zeta_p}{\zeta_3} = (1 - \Delta)^{p/3} + \frac{\Delta}{1 - \beta} \left(1 - \beta^{p/3}\right).$$

The intermittency parameter $\beta$ is interpreted as a random cascade factor relating dissipative structures of different intensity, and $C = \Delta/(1 - \beta)$ is the co-dimension of the most intense dissipative structures. The co-dimension is related to the fractal dimension by $C = D - 3$. As argued by She and Lévêque, the scaling of these structures, $\ell^{-\Delta}$, is given by the inverse of the kinetic energy available for dissipation at the length scale $\ell$. For incompressible turbulence, the most intense dissipative structures are assumed to be vortex filaments, for which $C = 2$ and $\Delta = 2/3$.

Boldyrev proposed $C = 1$ for supersonic turbulence (Kolmogorov-Burgers model), because he considered the most intense dissipative structures to be shocks, while keeping $\Delta = 2/3$ as in the She-Lévêque model for incompressible turbulence. However, the kinetic energy at
the length scale ℓ is proportional to ℓ for Burgers turbulence (dv / dt ∝ ℓ1/2). Following the arguments by She and Lévêque we propose that the most intense dissipative structures should obey the scaling law ℓ−1 rather than ℓ−2/3, i.e., Δ = 1. The scaling exponents obtained from equation (2) for Δ = 1 are markedly different from the prediction of the Kolmogorov-Burgers model.

In the following, we will determine the relative scaling exponents Zp := ζp/ζ3 from the relations S_p(ℓ) = S_3(ℓ)/ℓ^p, for simulations of supersonic isothermal turbulence with periodic large-scale stochastic forcing. In these simulations, the compressible (divergence-free) stochastic forcing was applied, in the other case the forcing was purely compressive (rotation-free). We use the term compressive synonymous to dilatational (rotation-free). In each case, the system was evolved over 10 auto-correlation time scales (subsequently denoted by T) of the force field at grid resolutions N = 256^3, 512^3 and 1024^3. Turbulence was found to be fully developed with a steady-state RMS Mach number ≈ 5.5 after about two autocorrelation time scales. Following previous numerical studies of supersonic turbulence, we computed transversal structure functions S_p(ℓ), i.e., the p-th moments of the velocity fluctuation projected perpendicular to the spatial separation ℓ. The structure functions were computed from a statistically converged sample in the interval 2 ≤ t/T ≤ 10 using a Monte Carlo algorithm.

The structure functions S_p(ℓ) averaged over the time interval 2 ≤ t/T ≤ 10 for N = 1024^3 are plotted as functions of the time-averaged third-order structure function S_3(ℓ) in Fig. 1. The exponents Z_p are given by the slope of log S_p vs. log S_3. For the determination of linear fit functions, fit_p = Z_p log S_3, we imposed the criterion ∀p ≤ 5 : err_p := |exp(log S_p) − S_p^Δ|/S_p^Δ < 0.01 in the fit range. This error criterion is fulfilled in the intervals 12.0 ≤ S_3^Δ ≤ 120 for solenoidal forcing and 25.0 ≤ S_3^Δ ≤ 150 for compressive forcing. One should note that the relations between S_p(ℓ) and S_3(ℓ) agree quite closely with the fit functions (err_p < 0.05) for large length scales corresponding to the maximal values of S_3.

The relative scaling exponents Z_p inferred from the time-averaged transversal structure functions are listed in Table I. In the case N = 1024^3, the standard errors of the parameters Z_p are of the order 10^{-3}. We estimate systematic errors to be of the order 10^{-2}. In Fig. 2 we compare these values to the instantaneous scaling exponents as functions of time for N = 1024^3. Most importantly, we see that the scaling exponents resulting from solenoidal forcing differ markedly from the case of compressive forcing. In each case, the variation of Z_1 and Z_2 over the time scale T clearly shows temporal correlation and there appears to be anti-correlation with the scaling exponents of order greater than three. This suggests that the instantaneous scaling exponents show an imprint of the stochastic variation of the large scale forcing rather than purely statistical scatter. Even for incompressible turbulence, an influence of the large scales on the minimal co-dimension C is about the number 1.5.

TABLE I: Relative scaling exponents Z_p from fits of time-averaged structure functions S_p vs. S_3.

| N     | Z_1   | Z_2   | Z_3   | Z_4   | Z_5   |
|-------|-------|-------|-------|-------|-------|
| 256^3 | 0.603 | 0.627 | 1.149 | 1.072 | 1.126 |
| 512^3 | 0.628 | 0.627 | 1.105 | 1.097 | 1.095 |

In the case N = 1024^3, the standard errors of the parameters Z_p are of the order 10^{-3}. We estimate systematic errors to be of the order 10^{-2}. In Fig. 2 we compare these values to the instantaneous scaling exponents as functions of time for N = 1024^3. Most importantly, we see that the scaling exponents resulting from solenoidal forcing differ markedly from the case of compressive forcing. In each case, the variation of Z_1 and Z_2 over the time scale T clearly shows temporal correlation and there appears to be anti-correlation with the scaling exponents of order greater than three. This suggests that the instantaneous scaling exponents show an imprint of the stochastic variation of the large scale forcing rather than purely statistical scatter. Even for incompressible turbulence, an influence of the large scales on the minimal co-dimension C is about the number 1.5.

The Kolmogorov-Burgers model implied by equation (2) for Δ = 2/3 and β = 1 − Δ/C = 1/3 is plotted as dotted line together with our data in Fig. 3. Clearly, there are large deviations both for solenoidal and for compressive forcing. One reason is that the Kolmogorov-Burgers model only applies in the hypersonic limit, whereas the most intense dissipative structures are constituted by varying fractions of vortex filaments and shocks depending on the RMS Mach number. Then one would expect 1 ≤ C ≤ 2. Moreover, fitting the log-Poisson model with Δ = 2/3, we find C_{sol} ≈ 0.76 and C_{comp} ≈ 0.67, which is about the minimal co-dimension C = Δ = 2/3, for which the Z_p are real. In the case of compressive forcing, no closely matching fit function exists. On the other hand, fitting the
one-parameter family of models with $\Delta = 1$, we obtain co-dimensions $C_{\text{sol}} \approx 1.5$ and $C_{\text{comp}} \approx 1.1$. These models match the time-averaged relative scaling exponents very well. We cannot fully discriminate other families of models though. For instance, assuming $C = 1$ as in the Kolmogorov-Burgers model and varying $\Delta$ as fit parameter, yields $\Delta_{\text{sol}} = 0.79$ and $\Delta_{\text{comp}} \approx 0.94$. Nevertheless, in the case of compressive forcing, the match with the numerically computed scaling exponents is much better in comparison to the $\Delta = 2/3$ models.

To carry over the log-Poisson models to compressible turbulence, the refined similarity hypothesis might be applied in the form \( \tilde{S}_p(\ell) = \tilde{C}_p \ell^{\tilde{\beta} p/3} \langle (\rho \epsilon)_{\ell}^{p/3} \rangle_{\ell} \), where $\tilde{S}_p(\ell) := \langle \delta (\rho^{1/3} \nabla)^p \rangle_{\ell}$, leads to the relative scalings $\tilde{S}_p(\ell) = \tilde{S}_3(\ell) \ell^{\tilde{Z}_p}$ for any $\tilde{\beta}_3 > 0$. As proposed by Kritsuk et al. [4], the exponents $\tilde{Z}_p$ associated with the two-point statistics of $\tilde{v} := \rho^{1/3} \nabla$, are then given by an expression analogous to (2) with parameters $\tilde{\beta}$ and $\tilde{\Delta}$.

The mass-weighted structure functions $\tilde{S}_p(\ell)$ vs. smooths out fluctuations at length scales $\lesssim \ell$, $\rho_{\ell} := \langle \rho \rangle_{\ell}$ and $\epsilon(x, t)$ is the local rate of energy dissipation per unit mass. On the other hand, one might consider the rate of dissipation per unit volume, $\tilde{\epsilon}_\ell := \rho_{\ell} \epsilon_{\ell}$, as the variable from which the hierarchy should be constructed [10]. This is also suggested by the alternative formulation of the log-Poisson model by She and Waymire [3]. Following this proposition, the refined similarity hypothesis with mass-weighing,
TABLE II: Relative scaling exponents $\tilde{Z}_p$ from fits of time-averaged mass-weighted structure functions $\tilde{S}_p^u$ vs. $\tilde{S}_3^\perp$ with mass weighing $\tilde{v} = \rho^{1/3} v$.

| $N$   | $\tilde{Z}_1$ | $\tilde{Z}_2$ | $\tilde{Z}_3$ | $\tilde{Z}_4$ | $\tilde{Z}_5$ |
|-------|--------------|--------------|--------------|--------------|--------------|
| sol.  | 256$^3$      | 0.546        | 0.839        | 1.094        | 1.150        |
|       | 512$^3$      | 0.550        | 0.845        | 1.082        | 1.122        |
|       | 1024$^3$     | 0.539        | 0.840        | 1.080        | 1.112        |
| comp. | 256$^3$      | 0.635        | 0.893        | 1.034        | 1.026        |
|       | 512$^3$      | 0.634        | 0.887        | 1.050        | 1.068        |
|       | 1024$^3$     | 0.605        | 0.869        | 1.066        | 1.100        |

In summary, the scaling of turbulent supersonic velocity fields is characterized by power laws that vary substantially with the large-scale forcing. This finding disproves the conventional notion of universality in the supersonic regime and bears consequences on the theory of turbulence-regulated star formation. On the other hand, calculating two-point statistics of the mass-weighted velocity $\tilde{v} = \rho^{1/3} v$, we found that the influence of the forcing was considerably reduced. Based on the corresponding formulation of the refined similarity hypothesis (3), the scaling exponents are very well described by log-Poisson models (2), for which the parameter $\Delta$ is close to unity. This is the expected value for the most intense dissipative structures in supersonic turbulence.

[1] U. Frisch, *Turbulence* (Cambridge University Press, 1995).
[2] A. Kolmogorov, Akademiia Nauk SSSR Doklady 30, 301 (1941).
[3] M.-M. Mac Low and R. S. Klessen, Reviews of Modern Physics 76, 125 (2004).
[4] A. G. Kritsuk, P. Padoan, R. Wagner, and M. L. Norman, in *AIP Conference Proceedings: Turbulence and Nonlinear Processes in Astrophysical Plasmas* (2007), vol. 706.
[5] A. N. Kolmogorov, J. Fluid Mech. 13, 82 (1962).
[6] B. Dubrulle, Phys. Rev. Lett. 73, 959 (1994).
[7] Z.-S. She and E. C. Waymire, Phys. Rev. Lett. 74, 262 (1995).
[8] Z.-S. She and E. Leveque, Phys. Rev. Lett. 72, 336 (1994).
[9] S. Boldyrev, Astrophys. J. 569, 841 (2002).
[10] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, F. Massaioli, and S. Succi, Phys. Rev. E 48, 29 (1993).
[11] C. Federrath, R. Klessen, and W. Schmidt (2008), in preparation.
[12] P. Colella and P. R. Woodward, J. Comp. Phys. 54, 174 (1984).
[13] B. Fryxell, K. Olson, P. Ricker, F. X. Timmes, M. Zingale, D. Q. Lamb, P. MacNeice, R. Rosner, J. W. Truran, and H. Tufo, Astrophys. J. S. 131, 273 (2000).
[14] V. Eswaran and S. B. Pope, Comp. Fluids. 16, 257 (1988).
[15] W. Schmidt, W. Hillebrandt, and J. C. Niemeyer, Comp. Fluids. 35, 353 (2006).
[16] P. Padoan, R. Jimenez, A. Nordlund, and S. Boldyrev, Phys. Rev. Lett. 92, 191102 (2004).
[17] A. G. Kritsuk, M. L. Norman, P. Padoan, and R. Wagner, Astrophys. J. 665, 416 (2007).
[18] A. Alexakis, P. D. Mininni, and A. Pouquet, Phys. Rev. Lett. 95, 264503 (2005).
[19] R. C. Fleck, Jr., Astrophys. J. 458, 739 (1996).