Simpson’s Theory and Superrigidity of Complex Hyperbolic Lattices

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Abstract We attack a conjecture of J. Rogawski: any cocompact lattice in $SU(2,1)$ for which the ball quotient $X = B^2/\Gamma$ satisfies $b_1(X) = 0$ and $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) \approx \mathbb{Q}$ is arithmetic. We prove the Archimedian superrigidity for representation of $\Gamma$ is $SL(3, \mathbb{C})$.

Théorie de Simpson et superrigidité des réseaux hyperboliques complexes

Résumé Soit $\Gamma \subset SU(2,1)$ un réseau cocompact et soit $X = B^2/\Gamma$. Nous preuves: si $b_1(X) = 0$ et $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) \approx \mathbb{Q}$ alors tous les representations $\rho$ de $\Gamma$ dans $SL(3, \mathbb{C})$ sont conjugué à la représentation naturelle ou la fermeture de Zariski de l’image $p(\Gamma)$ est compacte.

Version française abrégée - Le théorème classique de Margulis dit que tous les réseaux dans les groupes de Lie semi-simples sont superrigides. Ceci a été generalisé par Corlette [C] à la superrigidite des réseaux quaternioniques et de Cayley. D’autre part, Johnson et Millson ont montré qu’il existait des deformations des réseaux cocompact dans $SO(n,1)$ si on regarde $SO(n,1)$ comme plongé dans $SO(n+1,1)$.

C’est une question d’un intérêt fondamental de savoir si les réseaux hyperboliques complexes sont superrigides.

Dans cet article, nous considerons la question suivante de J. Rogawski.

Hypothèse Soit $X = B^2/\Gamma, \Gamma \subset SU(2,1)$ une surface hyperbolique complexe compacte. Supposons $b_1(X) = 0$ et $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) = \mathbb{Q}$. Allors $\Gamma$ est arithmetique et provient d’une algèbre avec division $E|\mathbb{Q}$ de rang 3 avec une involution.

Observons que pour tous les réseaux provenant d’algèbres avec division, on a effectivement $b_1(X) = 0$ et $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) = \mathbb{Q}$ [Rog].

Soit $\ell$ un fibré linéaire tautologie sur $X$ [Re]. La condition $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) = \mathbb{Q}$ dit que $[\ell] = k$: générateur dans $Pic(X)/tors \approx \mathbb{Z}$. 

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Le résultat principal de cet article prouve la superrigidité des représentations de \( \Gamma \) dans \( SL(3, \mathbb{C}) \) dans le cas \( k = 1 \).

**Théorème principal** Soit \( X = B^2/\Gamma \) et supposons que \( b_1(X) = 0 \) et \( H^{1,1}(X) \cap H^2(X, \mathbb{Q}) = \mathbb{Q} \). Soit \( [\ell] \) un générateur de \( Pic(X)/\text{tors} \approx \mathbb{Z} \). Si \( \rho \) est une représentation de \( \Gamma \) dans \( SL(3, \mathbb{C}) \) alors soit \( \rho \) est conjugué à la représentation naturelle de \( \Gamma \), soit la fermeture de Zariski de l’image \( \rho(\Gamma) \) est compacte.

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0 Main Theorem The classical theorem of Margulis establishes the superrigidity of lattices in semisimple Lie groups of rank $\geq 2$. The work of Corlette [C] extended this to (Archimedian) superrigidity of uniform quaternionic and Cayley lattices. On the other hand, by Johnson and Millson [JM] some uniform lattices in $SO(n,1)$ admit deformations as mapped to $SO(n + 1,1)$.

It is therefore of fundamental interest to study to what extent the complex hyperbolic lattices are superrigid. Since there are nontrivial holomorphic maps between different ball quotients [DM] one should confine oneself’s look to lattices (or manifolds) “minimal” in some sense.

The present note addresses the following conjecture of Jon Rogawski.

Conjecture. Let $X = B^2/\Gamma, \Gamma \subset SU(2,1)$ be a compact ball quotient. Suppose $b_1(X) = 0$ and $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) \approx \mathbb{Q}$. Then $\Gamma$ is arithmetic and comes from a division algebra $E|\mathbb{Q}$ of rank 3 with an involution.

Observe that for all lattices coming from division algebras, indeed $b_1(X) = 0$ and $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) \approx \mathbb{Q}$ [Rog].

Let $\ell$ be the tautological line bundle over $X$ [Re]. Since $Pic(X)/tors \approx \mathbb{Z}$, we have $[\ell] = k \cdot$ generator for some $k \in \mathbb{Z}$.

The main result of the paper establishes the superrigidity of representations of $\Gamma$ in $SL(3, \mathbb{C})$ for $\Gamma$ yielding $k = 1$ as follows.

Main Theorem. Let $X = B^2/\Gamma$ and suppose $b_1(X) = 0$ and $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) \approx \mathbb{Q}$. If $[\ell]$ generates $Pic(X)/tors \approx \mathbb{Z}$, then any representation of $\Gamma = \pi_1(X)$ in $SL(3, \mathbb{C})$ is either conjugate to the natural representation up to the twist by a character, or has a compact Zariski closure.

One hopes, that, applying methods of [GS] one is able to prove the $p$-adic superrigidity and to settle Rogawski’s conjecture.
I wish to thank Ron Livne, Jon Rogawski and Carlos Simpson for stimulating discussions.

1. Computations of Higgs bundles. We admit a knowledge of Simpson’s theory [S1]. Let $X$ be as above and let $\rho_0 : \Gamma \to PSU(n,1)$ be the natural representation. Then the corresponding Higgs bundle is as follows [Re]. Take $E = TX \otimes \ell \oplus \ell$ as a holomorphis bundle and define $\theta \in H^0(\Omega^1 \otimes End(E))$ by \[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
\]

In view of the Simpson’s theory, for proving the Main Theorem one needs to show that any complex variation of Hodge structure [S1] of type (2,1) over $X$ is as above. Indeed, any representation is deformable to one, corresponding to a variation of Hodge structure [S2], and the natural representation is rigid [W].

So let $F = (\xi \oplus \eta, \theta)$ be a variation of complex Hodge structure, rank $\xi = 2$, rank $\eta = 1$, $\theta \in H^0(\Omega^1(X) \oplus \text{Hom}(\eta, \xi)) \approx H^0(\text{Hom}(TX \otimes \xi, \eta))$.

1.2. Lemma. Let $\lambda, \mu$ be rank two bundles over $X$ and let $f \in H^0(\text{Hom}(\lambda, \mu)), f \neq 0$. Then either rank $f \leq 1$ everywhere or

\[ (c_1(\mu), [\omega]) \geq (c_1(\lambda), [\omega]) \]

with the equality iff $\lambda \approx \mu$, and rank $f = 2$ everywhere. Here $[\omega]$ is the Kähler class.

Proof: Consider $\wedge^2 f : \wedge^2 \lambda \to \wedge^2 \mu$. If $\wedge^2 f \neq 0$, then $\wedge^2 \mu \oplus (\wedge^2 \lambda)^{-1}$ has a nontrivial holomorphic section, whose zero locus is an effective divisor, so $(c_1(\wedge^2 \mu \otimes (\wedge^2(\lambda)^{-1}), [\omega]) \geq 0$ and the equality implies $\wedge^2 f$ is an isomorphism.

2. Proof of the Main Theorem:

Let $F = (\xi \oplus \eta, \theta)$ be as above.

Case 1 Rank $\theta = 2$ somewhere.

Applying the lemma, we get

\[ (c_1(TX \otimes \xi), [\omega]) \leq (c_1(\eta), [\omega]) \]

Now, $[\omega] \sim [\ell]$ since $X$ is hyperbolic, and $c_1(TX) = -3[\ell]$ in $H^2(X, \mathbb{R})$, so

\[ (c_1(\eta) - 2c_1(\xi), [\ell]) \leq 3[\ell]^2. \]
On the other hand, since $\xi \oplus \eta$ is a deformation of the flat bundle, $c_1(\xi \oplus \eta) = 0$, i.e. $c_1(\xi) = -c_1(\eta)$, so

\[(*) \quad (c_1(\xi), [\ell]) \leq [\ell]^2.\]

Since $F$ is $\theta$-stable [S1], $(c_1(\eta), [\ell]) < 0$, so $(c_1(\eta), [\ell]) > 0$. This leaves the only possibility $(c_1(\xi), [\ell]) = [\ell]^2$, because $[\ell]$ generates $Pic(X) / tors$. So $\xi = \ell \otimes \alpha$, where $\alpha$ is a linear unital flat bundle, corresponding to $Pic(X) / tors \approx H^1_{tors}(X, \mathbb{Z})$. (recall that $b_1(X) = 0$). Moreover, since $(*)$ becomes an equality, we get by lemma above $\eta \approx TX \otimes \xi$ and $\theta$ takes the form $\left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$. Hence $F \approx (TX \otimes \ell \oplus \ell) \otimes \alpha$ and the proof is complete in this case.

Case 2 Rank $\theta \leq 1$ everywhere on $X$. There exists a collection of points $(p_1, \cdots, p_k)$ such that $\text{Ker} \theta$ extends to a rank one subbundle of $TX \otimes \xi$, say $\alpha \otimes \eta$. Since $H^2(X - \{p_1, \cdots, p_k\}) \approx H^2(X \setminus \{p_1, \cdots, p_k\}, \mathcal{O})$, so from the exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1$ and the five-lemma we deduce that $H^1(X, \mathcal{O}^*) \approx H^1(X \setminus \{p_1, \cdots, p_k\}, \mathcal{O}^*)$, so $c_1(\alpha)$ is in the image of $Pic(X)$ in $H^2(X, \mathbb{Z})$. Let $C$ be an irreducible curve of sufficiently high degree, which does not meet $p_1, \cdots, p_k$. Since $TX \otimes \ell \oplus \ell$ remains $\theta$-stable on $C$ [S1] we get $(c_1(\alpha \otimes \ell)|_C, [C]) < 0$. In view of $H^{1,1}(X) \cap H^2(X_1, \mathbb{Q}) \approx \mathbb{Q}$ we can rewrite this as $(c_1(\alpha), [\ell]) < -[\ell]^2$. Since $[\ell]$ generates $Pic(X) / tors$, this actually means $(c_1(\alpha), [\ell]) \leq -2[\ell]^2$. Now, $c_1(TX) = -3[\ell]$, so $(c_1(TX / \alpha), [\ell]) \geq -[\ell]^2$. On $C$ we have an isomorphism

$$\theta|_C : TX|_C \otimes \xi \to \text{Im} \theta \subset \eta|_C.$$ 

Hence $(c_1(\text{Im} \theta), [C]) = (c_1(TX / \alpha) + c_1(\xi), [C]) \geq (c_1(\xi) - [\ell], [C])$. Since, again, $[\ell]$ generates $Pic(X) / tors$ and $(c_1(\xi), [\ell]) > 0$ we get $(c_1(\xi) - [\ell], [C]) \geq 0$. This contradicts the $\theta$-stability of $\xi \oplus \eta|_C$, because $\text{Im} \theta|_C$ is $\theta$-invariant. The proof is complete.

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