The second Hardy–Littlewood conjecture is true

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Abstract:
The second Hardy–Littlewood conjecture, that \( \pi(x) + \pi(y) \geq \pi(x + y) \) for integers \( x \) and \( y \) with \( \min\{x, y\} \geq 2 \), was formulated in 1923. It continues to attract attention to this day, almost 100 years later. In 1975 Udrescu proved that this conjecture holds for \((x, y)\) sufficiently large, but without an explicit effective bound on the region of validity. We shall revisit Udrescu’s result, modifying it to obtain explicit effective bounds, ultimately proving that the second Hardy–Littlewood conjecture is in fact unconditionally true. Furthermore we note that constraints on the prime counting function imply, (and are implied by), constraints on the location of the primes, and re-cast Segal’s 1962 equivalent reformulation of the second Hardy–Littlewood conjecture in the more symmetric (and perhaps clearer) form that for integers \( i \) and \( j \) with \( \min\{i, j\} \geq 2 \) one has \( p_{i+j-1} \geq p_i + p_j - 1 \).

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1 Introduction

The second Hardy–Littlewood conjecture [1] was formulated in 1923. It continues, now almost 100 years later, to attract considerable attention [2–5]. The conjecture states that, for integers $x$ and $y$ with $\min\{x,y\} \geq 2$, the prime counting function satisfies

$$\pi(x) + \pi(y) \geq \pi(x + y).$$

Many partial and related results along these lines are known [2–7]. The known partial results seem to suggest that proving the second Hardy–Littlewood conjecture would require techniques only slightly stronger than the prime number theorem. To justify this claim, and provide our proof of the second Hardy–Littlewood conjecture, we shall be particularly interested in extending and modifying Udrescu’s asymptotic result [7], by seeking and establishing effective and explicit bounds on the region where the quantity $\pi(x) + \pi(y) - \pi(x + y)$ is positive.

2 Udrescu’s asymptotic result

Udrescu [7] is generally credited with the result that:

**Theorem:** If $0 < \epsilon \leq 1$ and $x\epsilon \leq y \leq x$, then

$$\pi(x + y) < \pi(x) + \pi(y) \quad (2.1)$$

for $x$ and $y$ sufficiently large. See for instance page 237 of [8].

Let us rephrase this more symmetrically as follows:

**Theorem:** If $0 < \epsilon \leq 1$ and both $\frac{y}{x} \geq \epsilon$ and $\frac{x}{y} \geq \epsilon$, then

$$\pi(x + y) < \pi(x) + \pi(y) \quad (2.2)$$

for $x$ and $y$ sufficiently large.

We shall seek to improve this theorem to provide an effective and explicit statement regarding the region for which $\pi(x) + \pi(y) - \pi(x + y)$ is guaranteed positive.

3 General bounds on $\pi(x)$

There are a large number of known bounds on the prime counting function of the form:

$$\pi(x) > \frac{x}{\ln x - a}; \quad (x \geq n_a \geq \exp(a)); \quad (3.1)$$

$$\pi(x) < \frac{x}{\ln x - b}; \quad (x \geq n_b \geq \exp(b)). \quad (3.2)$$
Bounds of this type are basically just improvements on the prime number theorem. For now we shall keep the discussion general and let both \((a, n)\) and \((b, n)\) remain unspecified. The only truly general constraint is that \(b > a\). We find that it is useful to define \(\Delta = b - a > 0\). Eventually we shall make specific choices for \((a, n)\) and \((b, n)\), (and implicitly \(\Delta\)), to make our estimates explicit.

4 Strategy I — implementing a two-term split

Let us start by defining the quantity

\[
HL_2 \equiv \pi(x) + \pi(y) - \pi(x+y).
\]  

(4.1)

We certainly have

\[
HL_2 > \frac{x}{\ln x - a} + \frac{y}{\ln x - a} - \frac{x+y}{\ln(x+y) - b}; \quad \text{(min\{x,y\} }\geq n_a; \ x+y \geq n_b).
\]  

(4.2)

That is, certainly

\[
HL_2 > x \left( \frac{1}{\ln x - a} - \frac{1}{\ln(x+y) - b} \right) + y \left( \frac{1}{\ln y - a} - \frac{1}{\ln(x+y) - b} \right)
\]  

whenever the condition \(C = \text{(min\{x,y\} }\geq n_a; \ x+y \geq n_b)\) holds. But then \(HL_2\) is certainly positive if both of these terms are individually positive. That is, certainly \(HL_2 > 0\) whenever both

\[
\left( \frac{1}{\ln x - a} - \frac{1}{\ln(x+y) - b} \right) > 0; \quad \text{and} \quad \left( \frac{1}{\ln y - a} - \frac{1}{\ln(x+y) - b} \right) > 0;
\]  

(4.4)

subject to condition \(C\). But then, since all three denominators are by construction positive, \(HL_2\) is certainly positive if both

\[
(\ln(x+y) - \ln x + a - b) > 0; \quad \text{and} \quad (\ln(x+y) - \ln y + a - b) > 0;
\]  

(4.5)

subject to condition \(C\). Rearrange, using \(\Delta = b - a\). Then a sufficient condition for positivity is

\[
\ln \left( 1 + \frac{y}{x} \right) > \Delta; \quad \text{and} \quad \ln \left( 1 + \frac{x}{y} \right) > \Delta;
\]  

(4.6)

subject to condition \(C\).

That is, now being fully explicit, \(HL_2 > 0\) is certainly positive whenever one satisfies the three conditions:

\[
\frac{y}{x} > e^\Delta - 1; \quad \frac{x}{y} > e^\Delta - 1; \quad \text{(min\{x,y\} }\geq n_a; \ x+y \geq n_b).
\]  

(4.7)
Proving Udrescu’s theorem using this two-term split then boils down to the fact that one can choose $\Delta = b - a > 0$ to be arbitrarily small by choosing $n_a$ and $n_b$ to be arbitrarily large.

For instance, the la Vallée Poussin proof of the prime number theorem implies that for every $\varepsilon > 0$ there is an $S > 0$ such that for all $x > S$

$$\frac{x}{\log x - (1 - \varepsilon)} < \pi(x) < \frac{x}{\log x - (1 + \varepsilon)}.$$  

(4.8)

(See for instance the discussion in reference [9].) That is, we can always take $b = 1 + \varepsilon$, and $a = 1 - \varepsilon$, and also $n_a = n_b = S$, with $\varepsilon$ sufficiently small and $S$ sufficiently large, so that $e^\Delta - 1 = \exp(2\varepsilon) - 1 = \epsilon$ can be made arbitrarily small.

But clearly this argument can be significantly strengthened in a number of places. For instance, we could (and should) weaken the condition that both terms in the sum (4.3) be individually positive; it would be sufficient if the combined sum (4.3) were positive. That is, the inequalities in equations (4.4)–(4.7) are sufficient for the positivity of $HL_2$, they are by no means necessary. (In fact the argument based on this two-term split is non-vacuous only if $\Delta < \ln 2$, which is a limitation that we shall carefully evade in the subsequent discussion.) Furthermore, we have not yet used the full flexibility in choosing $(a, n_a)$, $(b, n_b)$, and $\Delta$ to optimal effect.

5 Strategy II — utilizing polar coordinates

To improve on Udrescu’s theorem we now find it useful to adopt polar coordinates in the $(x, y)$ plane. (We shall soon see that this allows us to in some sense “disentangle” $x$ and $y$.) Let us write $x = r \cos \theta$ and $y = r \sin \theta$. We certainly have

$$HL_2 > \frac{r \cos \theta}{\ln(r \cos \theta) - a} + \frac{r \sin \theta}{\ln(r \sin \theta) - a} - \frac{r \cos \theta + r \sin \theta}{\ln(r \cos \theta + r \sin \theta) - b};$$  

(5.1)

with this inequality being valid in the region

$$C \equiv (\min\{r \cos \theta, r \sin \theta\} \geq n_a; r(\cos \theta + \sin \theta) \geq n_b).$$  

(5.2)

This is the same region of validity as discussed previously, merely re-expressed in terms of polar coordinates. Rearrange the inequality as

$$HL_2 > r \left\{ \frac{\cos \theta}{\ln r + \ln \cos \theta - a} + \frac{\sin \theta}{\ln r + \ln \sin \theta - a} - \frac{\cos \theta + \sin \theta}{\ln r + \ln(\cos \theta + \sin \theta) - b} \right\}.$$  

(5.3)
Rationalize, noting that all three factors in the denominator are (by construction) positive, and focus on the numerator. Note that the coefficient of the naively possible \((\ln r)^2\) term is \(\cos \theta + \sin \theta - (\cos \theta + \sin \theta)\), that is, zero.

Then \(HL_2 > 0\) is guaranteed whenever
\[
P(\theta; a, b) - Q(\theta; \Delta) \ln r > 0
\]
where explicitly we have the rather messy formula
\[
Q(\theta; \Delta) = \Delta(\cos \theta + \sin \theta) - (\cos \theta + \sin \theta) \ln(\cos \theta + \sin \theta)
+ \cos \theta \ln \cos \theta - \sin \theta \ln \sin \theta,
\]
and the somewhat messier
\[
P(\theta; a, b) = -a(\cos \theta + \sin \theta)(a - b + \ln(\cos \theta + \sin \theta) - \ln \cos \theta - \ln \sin \theta)
- b(\cos \theta \ln \sin \theta + \sin \theta \ln \cos \theta)
+ (\cos \theta \ln \sin \theta + \sin \theta \ln \cos \theta) \ln(\cos \theta + \sin \theta)
- (\cos \theta + \sin \theta) \ln \cos \theta \ln \sin \theta.
\]

Then, still subject to the condition \(C\), we see that \(HL_2 > 0\) is guaranteed whenever
\[
r > r_0(\theta; a, b) \equiv \exp \left\{ \frac{P(\theta; a, b)}{Q(\theta; \Delta)} \right\}.
\]

This is our primary inequality, which we shall now simplify in various ways.

## 6 Simplifying the primary inequality

Now to keep \(r_0(\theta; a, b)\) finite, at least for \(\theta \in (0, \pi/2)\), excluding the end-points, it is necessary to avoid zeros in \(Q(\theta, \Delta)\). But \(Q(\theta, \Delta) = 0\) is equivalent to
\[
\Delta = \ln(\cos \theta + \sin \theta) - \frac{\cos \theta \ln \cos \theta + \sin \theta \ln \sin \theta}{\cos \theta + \sin \theta}.
\]

Let us consider the function
\[
f(\theta) = \ln(\cos \theta + \sin \theta) - \frac{\cos \theta \ln \cos \theta + \sin \theta \ln \sin \theta}{\cos \theta + \sin \theta}.
\]

This is positive in the first quadrant \(\theta \in (0, \pi/2)\), with minima at \(f(0) = 0 = f(\pi/2)\), and a maximum at \(f(\pi/4) = \ln 2\). See figure 1.
So to avoid zeros in $Q(\theta; \Delta)$, at least in the first quadrant $\theta \in (0, \pi/2)$, we need to impose $\Delta > \ln 2$, that is, $b > a + \ln 2$. This is now sufficient to guarantee that $r_0(\theta; \Delta)$ is finite on $(0, \pi/2)$, except at the endpoints.

Note that

$$Q(\theta; \Delta) = (\cos \theta + \sin \theta)(\Delta - f(\theta));$$

while

$$\lim_{\theta \to \pi/4} Q(\theta; \Delta) = \lim_{\theta \to \pi/2} Q(\theta; \Delta) = \Delta;$$

$$Q(\pi/4; \Delta) = \sqrt{2} (\Delta - \ln 2).$$

and so

$$Q(\theta, \Delta) \geq \min\{\Delta, \sqrt{2} (\Delta - \ln 2)\}.$$  

Though in the previous (two-term) argument we were working in the region $\Delta \approx 0$, we are now no longer restricted $\Delta \approx 0$. Furthermore these current observations are still compatible with Udrescu’s theorem:

- For small $\Delta \approx 0$: Pick any angle $\theta$, (not exactly zero, not exactly $\pi/2$), such that $Q(\theta, \Delta) \neq 0$. Then going out far enough, $r > r_0(\theta; a, b)$, guarantees that the second Hardy–Littlewood conjecture is asymptotically satisfied.
• Similarly for large $\Delta \geq \ln(2)$: Pick any angle $\theta$, (not exactly zero, not exactly $\pi/2$). Then going out far enough, $r > r_0(\theta; a, b)$, guarantees that the second Hardy–Littlewood conjecture is asymptotically satisfied.

But now we have considerable extra information encoded in the function $r_0(\theta; a, b)$.

For instance it is easy to check that

$$\lim_{\theta \to 0} r_0(\theta; a, b) \sin \theta = \lim_{\theta \to \pi/2} r_0(\theta; a, b) \cos \theta = e^a; \quad (6.7)$$

and

$$r_0(\pi/4; a, b) \sin(\pi/4) = r_0(\pi/4; a, b) \cos(\pi/4) = e^a. \quad (6.8)$$

This strongly suggests that it would be useful to rewrite $r_0(\theta; a, b)$ as

$$r_0(\theta; a, b) = \frac{e^a}{\sin \theta \cos \theta} \exp \left\{ \frac{P(\theta; a, b) - aQ(\theta; \Delta) + (\ln \sin \theta + \ln \cos \theta)Q(\theta; \Delta)}{Q(\theta; \Delta)} \right\}. \quad (6.9)$$

If we now define

$$\tilde{P}(\theta; \Delta) = (\cos \theta \ln \cos \theta + \sin \theta \ln \sin \theta)\Delta$$

$$-(\cos \theta \ln \cos \theta + \sin \theta \ln \sin \theta) \ln(\cos \theta + \sin \theta)$$

$$+(\cos^2 \theta \ln \cos \theta + \sin^2 \theta \ln \sin \theta), \quad (6.10)$$

then it is easy to verify that

$$r_0(\theta; a, b) = \frac{e^a}{\sin \theta \cos \theta} \exp \left\{ \frac{\tilde{P}(\theta; \Delta)}{Q(\theta; \Delta)} \right\}. \quad (6.11)$$

As a consistency check it is worth verifying that

$$\lim_{\theta \to 0} \exp \left\{ \frac{\tilde{P}(\theta; \Delta)}{Q(\theta; \Delta)} \right\} = \lim_{\theta \to \pi/2} \exp \left\{ \frac{\tilde{P}(\theta; \Delta)}{Q(\theta; \Delta)} \right\} = 1, \quad (6.12)$$

and

$$\exp \left\{ \frac{\tilde{P}(\pi/4; \Delta)}{Q(\pi/4; \Delta)} \right\} = \frac{1}{\sqrt{2}}. \quad (6.13)$$

Now ask when $\tilde{P}(\theta; \Delta) = 0$. This happens when

$$\Delta = \ln(\cos \theta + \sin \theta) - \frac{\cos^2 \theta \ln \cos \theta + \sin^2 \theta \ln \sin \theta}{\cos \theta \ln \cos \theta + \sin \theta \ln \sin \theta}. \quad (6.14)$$
Figure 2. The functions $f(\theta)$ and $h(\theta)$ in the first quadrant $\theta \in (0, \pi/2)$.

So let us consider the function

$$h(\theta) = \ln(\cos \theta + \sin \theta) - \frac{\cos^2 \theta \ln \cos \theta + \sin^2 \theta \ln \sin \theta}{\cos \theta \ln \cos \theta + \sin \theta \ln \sin \theta}.$$  \hspace{1cm} (6.15)

This is positive in the first quadrant $\theta \in (0, \pi/2)$, with a minimum at $h(\pi/4) = \ln 2$, and (infinite) maxima, $h(\theta) = -\ln(\sin \theta \cos \theta) + O(\theta)$, at both endpoints, $\theta = 0$ and $\theta = \pi/2$. See figure 2.

Then

$$\tilde{P}(\theta, \Delta) = (\cos \theta \ln \cos \theta + \sin \theta \ln \sin \theta) \left(\Delta - h(\theta)\right).$$  \hspace{1cm} (6.16)

Observe that for $\Delta < \ln 2$ the function $\tilde{P}(\theta, \Delta)$ is always positive, while for $\Delta > \ln 2$ the function $Q(\theta, \Delta)$ is always positive.

7 Recapitulation

We have demonstrated that the quantity $HL_2$ is certainly positive in the region

$$r > r_0(\theta; a, b) = \frac{e^a}{\sin \theta \cos \theta} \exp \{L(\theta; \Delta)\},$$  \hspace{1cm} (7.1)
subject to the conditions $C \equiv (\min\{r \cos \theta, r \sin \theta\} \geq n_a; r(\cos \theta + \sin \theta) \geq n_b)$, with

$$L(\theta; \Delta) = \frac{(\cos \theta \ln \cos \theta + \sin \theta \ln \sin \theta)}{(\cos \theta + \sin \theta)} \frac{\Delta - h(\theta)}{\Delta - f(\theta)}. \quad (7.2)$$

and

$$h(\theta) = \ln(\cos \theta + \sin \theta) - \frac{\cos^2 \theta \ln \cos \theta + \sin^2 \theta \ln \sin \theta}{\cos \theta \ln \cos \theta + \sin \theta \ln \sin \theta}; \quad (7.3)$$

$$f(\theta) = \ln(\cos \theta + \sin \theta) - \frac{\cos \theta \ln \cos \theta + \sin \theta \ln \sin \theta}{\cos \theta + \sin \theta}. \quad (7.4)$$

But now these functions $L(\theta; \Delta)$, $h(\theta)$, and $f(\theta)$ are all sufficiently simple to permit considerable further analysis.

8 Effective bound on the region where $\pi(x) + \pi(y) > \pi(x + y)$

Observe that

$$\frac{\partial L(\theta; \Delta)}{\partial \Delta} = -\cos \theta \sin \theta (\ln \cos \theta - \ln \sin \theta)^2 \frac{Q(\theta; \Delta)^2}{Q(\theta; \Delta)^2}. \quad (8.1)$$

That is, the function $L(\theta; \Delta)$, and so also the function $r_0(\theta; a, b)$, is (in the first quadrant) monotone decreasing as a function of $\Delta$. If $\Delta > \ln 2$ then it is a global monotone decrease, while for $\Delta < \ln 2$ it is a piecewise monotone decrease. (For $\Delta < \ln 2$, as $\Delta$ increases for fixed $\theta$, the function $L(\theta, \Delta)$ first plunges to $-\infty$ as one encounters the pole arising due to the zero of $Q(\Delta, \theta)$, then jumps discontinuously to $+\infty$, and subsequently continues its monotone decrease.)

So if we choose any $\Delta \geq \ln 2$ then certainly

$$r_0(\theta; a, b) < \frac{e^a}{\sin \theta \cos \theta} \exp \{L(\theta; \ln 2)\}. \quad (8.2)$$

But the function $L(\theta; \ln 2)$ is maximized at $\theta = \pi/4$, see figure 3, where

$$\lim_{\theta \to \pi/4} \{L(\theta; \ln 2)\} = 2 - \frac{\ln 2}{2}; \quad \lim_{\theta \to \pi/4} \exp \{L(\theta; \ln 2)\} = \frac{e^2}{\sqrt{2}}. \quad (8.3)$$

Hence, as long as $\Delta = b - a \geq \ln 2$,

$$r_0(\theta; a, b) < \frac{e^{2+a}}{\sqrt{2} \sin \theta \cos \theta}. \quad (8.4)$$
Consequently for \( \Delta \geq \ln 2 \) the quantity \( HL_2 \) is certainly positive in the region
\[
r > r_0(\theta; a) \equiv \frac{e^{2+a}}{\sqrt{2} \sin \theta \cos \theta},
\]
still subject to the conditions \( C \equiv (\min\{r \cos \theta, r \sin \theta\} \geq n_a; r(\cos \theta + \sin \theta) \geq n_b) \).

It now remains to choose \((a, n_a), (b, n_b)\), and \(\Delta = b - a\) appropriately.

9 Explicit bounds on \( \pi(x) \)

9.1 Upper bound

We know (Dusart, 2010) [10]
\[
\pi(x) < \frac{x}{\ln x - 1.1}; \quad (x \geq 60184).
\]

Now we can greatly improve the range of validity of the at the cost of slightly weakening the strength of the bound. Rearranging,
\[
\frac{x}{\pi(x)} - \ln x > -1.1; \quad (x \geq 60184).
\]
Therefore the function $\frac{x}{\pi(x)} - \ln x$ has an absolute minimum for $x \in \mathbb{R}$ and that minimum occurs somewhere in the range $x \in [2, 60184]$. A quick search finds that the minimum occurs at $x = 24137$, where $\pi(24137) = 2688$ and

$$\frac{24137}{\pi(24137)} - \ln(24137) = \frac{24137}{2688} - \ln(24137) = -1.1119625213904091138... \quad (9.3)$$

So let us define

$$b_* = -\frac{24137}{\pi(24137)} + \ln(24137) = -\frac{24137}{2688} + \ln(24137) = 1.1119625213904091138... , \quad (9.4)$$

and note that

$$\exp(b_*) = 3.0403192349162205549... < 4. \quad (9.5)$$

Then for all $\epsilon_* > 0$

$$\pi(x) < \frac{x}{\ln x - b_* - \epsilon_*}; \quad (x \geq 4). \quad (9.6)$$

Note especially that the range of validity of this upper bound is now greatly improved.

### 9.2 Lower bound

We know (Rosser & Schonfeld 1962) [11]

$$\pi(x) > \frac{x}{\ln x - \frac{1}{2}}; \quad (x \geq 67). \quad (9.7)$$

Let us again weaken this slightly, and mildly improve the range of validity. From previous discussion, we had seen the utility of setting $\Delta > \ln 2$. This suggests we define $a_* = b_* - \ln 2 = 0.41881534083046380438...$ since then $\Delta = (b_* + \epsilon_*) - (a_* - \epsilon_*) = \ln 2 + 2\epsilon_* > \ln 2$. It is now easy to check that for all $\epsilon_* > 0$

$$\pi(x) > \frac{x}{\ln x - a_* + \epsilon_*}; \quad (x \geq 59). \quad (9.8)$$

Note especially that the range of validity of this lowe bound is now mildly improved.

### 9.3 Summary

We shall now see that these two specific and fully explicit bounds,

$$\pi(x) < \frac{x}{\ln x - b_* - \epsilon_*}; \quad (x \geq 4); \quad (9.9)$$

$$\pi(x) > \frac{x}{\ln x - a_* + \epsilon_*}; \quad (x \geq 59); \quad (9.10)$$

for arbitrary $\epsilon_* > 0$ will be good enough for our purposes. (These bounds are not necessarily “optimal”, but they are certainly “good enough”.) The use of $\epsilon_*$ is just an aide to Maple to prevent annoying numerical problems. In many situations it can be quietly ignored.
10 Explicit bound on the region where $\pi(x) + \pi(y) > \pi(x + y)$

From the above discussion, we now see that $HL_2 > 0$ is guaranteed whenever

$$r > \tilde{r}_0(\theta; a_*) = \frac{\sqrt{2^{a_*}}}{\sin \theta \cos \theta},$$

(10.1)

subject to the conditions $C \equiv (\min\{r \cos \theta, r \sin \theta\} \geq 59; r(\cos \theta + \sin \theta) \geq 4)$. Now if both $r \cos \theta \geq 59$ and $r \sin \theta \geq 59$, then certainly $r(\cos \theta + \sin \theta) \geq 118 \geq 4$ and we can discard that last condition. So we now have that $HL_2 > 0$ is guaranteed whenever

$$r > \tilde{r}_0(\theta; a_*) = \frac{7.942608522081797465...}{\sin \theta \cos \theta},$$

(10.2)

subject to the condition $C_{\text{explicit}} \equiv \min\{r \cos \theta, r \sin \theta\} \geq 59$. This is certainly true if

$$r > \frac{8}{\sin \theta \cos \theta}.$$

(10.3)

Figure 4. Polar plot of the boundary of the region $r > 8/(\sin \theta \cos \theta)$ in the first quadrant $\theta \in (0, \pi/2)$.
Now perform a polar plot of the boundary of the region $r > \frac{8}{\sin \theta \cos \theta}$. See figure 4. By inspection almost all of the $(r, \theta)$ plane satisfies $HL_2 > 0$, except for small regions near $\theta = 0$ and $\theta = \pi/2$.

Figure 5. Parametric plot of $(x, y)_\theta = \left( \frac{8}{\sin \theta}, \frac{8}{\cos \theta} \right)$ as a function of $\theta$ in the range $\theta \in (0, \pi/2)$. Note that this accurately reproduces the polar plot (as it should).

The boundary of this region can also be recast parametrically in terms of the original $x$ and $y$ variables as

$$(x, y)_\theta = (r_\theta \cos \theta, r_\theta \sin \theta) = \left( \frac{8}{\sin \theta}, \frac{8}{\cos \theta} \right).$$

(10.4)

See figure 5. Consequently, the inequality $HL_2 > 0$ is certainly satisfied in the region $\min\{x, y\} > 8\sqrt{2} = 11.313708498984760390\ldots$. So the inequality $HL_2 > 0$ is certainly satisfied in the region $\min\{x, y\} > 12$. But this condition gives you no extra constraint beyond what was already needed for the initial inequalities on $\pi(x)$ and $\pi(y)$ to hold, namely $C_{\text{explicit}} = \min\{r \cos \theta, r \sin \theta\} = \min\{x, y\} > 59$.

That is, by craftily choosing $(a, n_a)$, $(b, n_b)$, and $\Delta$, we have established that $HL_2 > 0$ in the entire region $\min\{x, y\} > 59$. The specific choice of parameters $(a, n_a)$, $(b, n_b)$, and $\Delta$ is by no means unique, and there are many different choices leading to the same or very closely related bounds.
11 Wrap up

But we already know that $HL_2 \geq 0$ in the region $2 \leq \min\{x, y\} \leq 1731$.

Indeed, a weaker bound along these lines was first established for $2 \leq \min\{x, y\} \leq 41$, see reference [12], which is however not quite good enough for our current purposes. But subsequent stronger bounds, that $HL_2 \geq 0$ in the region $2 \leq \min\{x, y\} \leq 132$, see reference [13], or the region $2 \leq \min\{x, y\} \leq 146$, see reference [14], or the region $2 \leq \min\{x, y\} \leq 1731$, see reference [15], are more than sufficient to complete the job. Therefore $HL_2 \geq 0$ in the entire region $2 \leq \min\{x, y\}$, and we are done.

12 Segal’s reformulation of Hardy–Littlewood 2

In 1962 Segal derived a strictly equivalent reformulation of the second Hardy–Littlewood conjecture in terms of a bound on the location of the $n^{th}$ prime [16]. We find it useful to further reformulate Segal’s result in the following more symmetric (and hopefully clearer) form: For integers $i$ and $j$:

$$p_{i+j-1} \geq p_i + p_j - 1 \quad (\min\{i, j\} \geq 2).$$  

(12.1)

12.1 Hardy–Littlewood 2 $\Rightarrow$ Segal

To check that the second Hardy–Littlewood conjecture implies the Segal conjecture: Take $x = p_i - 1$ and $y = p_j - 1$ then the second Hardy–Littlewood conjecture implies

$$\pi(p_i - 1) + \pi(p_j - 1) \geq \pi(p_i + p_j - 2); \quad (i, j \geq 2).$$  

(12.2)

Thence

$$(i - 1) + (j - 1) \geq \pi(p_i + p_j - 2); \quad (i, j \geq 2).$$  

(12.3)

So

$$i + j - 1 \geq \pi(p_i + p_j - 2) + 1; \quad (i, j \geq 2);$$  

(12.4)

whence

$$\pi(p_{i+j-1}) \geq \pi(p_i + p_j - 2) + 1; \quad (i, j \geq 2).$$  

(12.5)

Now converting this into a strict inequality

$$\pi(p_{i+j-1}) > \pi(p_i + p_j - 2); \quad (i, j \geq 2);$$  

(12.6)
whence
\[ p_{i+j-1} > p_i + p_j - 2; \quad (i, j \geq 2). \]  \hfill (12.7)

Now weakening the strict equality
\[ p_{i+j-1} \geq p_i + p_j - 1; \quad (i, j \geq 2). \]  \hfill (12.8)

QED.

12.2 Segal \implies Hardy–Littlewood 2

To check that the Segal conjecture implies the second Hardy–Littlewood conjecture:

Let \( x \) and \( y \) be integers \( \geq 2 \) and define
\[ i = \pi(x); \quad j = \pi(y); \]  \hfill (12.9)

so that
\[ p_i \leq x \leq p_{i+1} - 1; \quad p_j \leq y \leq p_{j+1} - 1. \]  \hfill (12.10)

Then
\[ p_i + p_j \leq x + y \leq p_{i+1} + p_{j+1} - 2. \]  \hfill (12.11)

Then, employing the (reformulated) Segal condition in the second inequality below,
\[ \pi(x + y) \leq \pi(p_{i+1} + p_{j+1} - 2) \leq \pi(p_{i+j+1} - 1) = i + j = \pi(x) + \pi(y) \]  \hfill (12.12)

That is
\[ \pi(x + y) \leq \pi(x) + \pi(y). \]  \hfill (12.13)

QED.

13 Discussion

We have established the truth of the second Hardy–Littlewood conjecture using slight modifications of well-known bounds and some rather delicate algebra and analysis. In many ways this matches currently known partial results [2–6], where the strategy has typically involved working with explicit bounds on the prime counting function \( \pi(x) \) that improve on the asymptotic estimate given by the prime number theorem. In particular we demonstrated that Udrescu’s theorem is intimately related to the prime number theorem.
Perhaps the key step in this explicit verification of the second Hardy–Littlewood conjecture was the choice of adopting polar coordinates \((x, y) = (r \cos \theta, r \sin \theta)\). Doing so allowed us to “disentangle” \((x, y)\) by picking a specific direction \(\theta\) and then obtaining effective and ultimately explicit bounds \(r_0(\theta; a, b)\), and subsequently \(\tilde{r}_0(\theta; a)\), on the region of guaranteed validity of the second Hardy–Littlewood conjecture.

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