Quantum limit on time measurement in a gravitational field

Supurna Sinha and Joseph Samuel
Raman Research Institute, Bangalore-560 080, India

E-mail: sam@rri.res.in

Received 13 October 2014
Accepted for publication 14 November 2014
Published 11 December 2014

Abstract
Good clocks are of importance both to fundamental physics and for applications in astronomy, metrology and global positioning systems. In a recent technological breakthrough, researchers at NIST have been able to achieve a stability of one part in $10^{18}$ using an ytterbium clock. This naturally raises the question of whether there are fundamental limits to time keeping. In this article we point out that gravity and quantum mechanics set a fundamental limit on the fractional frequency uncertainty of clocks. This limit comes from a combination of the uncertainty relation, the gravitational redshift and the relativistic time dilation effect. For example, a single ion aluminium clock in a terrestrial gravitational field cannot achieve a fractional frequency uncertainty better than one part in $10^{22}$. This fundamental limit explores the interaction between gravity and quantum mechanics on a laboratory scale.

Keywords: time keeping, quantum limit, gravitational field

(Some figures may appear in colour only in the online journal)

1. Introduction

Time is a key concept in physics. We use clocks to measure time. The history of timekeeping [1, 2] has seen the development of two kinds of clocks—terrestrial and celestial [9]. The earliest clocks were celestial: the Earth going around the Sun, the Moon around the Earth, and the Earth rotating on its axis, which measure a year, a month and a day, respectively. Early examples of terrestrial clocks are the clepsydra or water clock, the hour glass and the pendulum. Clocks are dynamical systems which exhibit a periodic (e.g. pendula) or decaying (e.g. radioactive decay) behavior. Recent examples of terrestrial clocks are quartz oscillators and atomic clocks. In the twentieth century, it was realized that the rotation of the Earth is not quite uniform and that astronomers and physicists need better clocks [3] than the spinning
Earth. Over the years terrestrial clocks have so improved that they now form the standard for timekeeping [4]. As first suggested by Kelvin [5] (Sir William Thomson) and later in the context of magnetic resonance by I I Rabi [6], atomic spectral lines provide a reliable standard for time and frequency. Due to giant strides in cold atom physics [13], we now have atomic clocks [7, 10, 15] which are stable to one part in $10^{18}$. Atomic clocks have led to innovations like global positioning systems, advanced communications, tests of general relativity and of variation of fundamental constants in nature.

Space and time provide the arena in which physics occurs. Traditionally, space intervals are determined by measuring rods and time intervals by clocks. Both rods and clocks are material objects, subject to physical laws. For example, it was earlier believed that measuring rods could be taken to be rigid, a belief which had to be revised with the advent of special relativity. With the realization that the speed of light is finite and a constant of nature, we could reduce length measurements to time measurements and eliminate measuring rods in favor of clocks. Since clocks also are material objects, subject to the laws of physics, we must ask whether there are limits on their time keeping from fundamental physics.

We would like clocks to be both accurate and stable. Accurate clocks will deliver a periodic signal close to a fiducial frequency. Stable clocks reliably maintain the same periodic signal and do not wander in frequency. Atomic clocks have now progressed to the point where they are both accurate and stable. Our focus is on the $Q$ factor, which is the inverse of the fractional frequency uncertainty 

$$Q^{-1} = \frac{\Delta \nu}{\nu}.$$ 

$Q$ is related to the stability $\sigma$ via $\sigma = \frac{\Delta \nu}{\nu} \sqrt{\frac{T_c}{N \tau}}$, $T_c$ being the time required for a single measurement cycle and $\tau$ the total averaging time and $N$ the number of atoms constituting the clock.

Is there a fundamental limit, set by the laws of quantum physics, to time keeping? This question was earlier raised by Salecker and Wigner [8, 11, 18], who derived theoretical limits on freely moving clocks. They consider the recoil momentum of clocks and derive lower bounds on the mass of a clock given a certain desired ‘accuracy’ of time keeping [11]. (In reference [11] the word ‘accuracy’ is used in the everyday sense and not in the precise sense now used by the time keeping community). Our analysis differs from theirs in two respects. First, we consider clocks in a trap as realized in present day laboratories and arrange the stiffness of the trap to eliminate the atomic recoil. The recoil is absorbed by the trap as a whole (Lamb–Dicke regime); second we consider the effect of an external gravitational field such as that of the Earth. We assume that the technological problems have been solved: stray electric fields have been eliminated, the clock has been cooled to absolute zero and so on. With these technological problems out of the way, we ask if there is a fundamental limit to the stability of clocks in a gravitational field. The answer it turns out is yes. The gravitational redshift and the uncertainty principle do impose such a limit.

We will see below that gravity and quantum mechanics combine to limit the $Q$ factor attainable by a clock of a given mass $m$. The limit stems from an unavoidable uncertainty in the position and motion of the center of mass of the clock. Collections of atoms emit spectral lines which are broadened by collisional effects [12]. Collisions can be eliminated by considering single atom or ion clocks. Atomic clocks are cooled to still their random thermal motion, which would otherwise cause variations in the apparent clock rate due to the Doppler effect. Even if a clock is cooled to absolute zero, quantum zero point motion still remains. This quantum relativistic broadening of the spectral line is the main subject of this letter. In an external gravitational field, there are uncertainties in the clock rate arising from both
position and motional uncertainty of the clock and this gives rise to fundamental limitations. Clocks currently developed in laboratories are far from the fundamental quantum limit. However, the existence of a quantum limit to timekeeping is important because of its fundamental significance. Recently there have been several experiments involving atomic clocks in a gravitational field [20] with a view to understanding the connection between the two fundamental pillars of physics, quantum theory and gravitation, since the unification program of these two theories lies at the forefront of present day research. Our fundamental limit reveals low energy connections between gravitation and quantum mechanics which could help understand the relation between the two theories. More recently there have been suggestions that such low energy gravitational effects can illuminate decoherence due to gravity [19, 21].

2. The uncertainty principle affects time keeping

Consider a clock of mass \( m \) placed in an uniform gravitational field \( g \). For definiteness we will suppose the clock to be a single atom (possibly an ion) in a trap, but our considerations are general. We suppose the \( Z \) axis of our coordinate system is chosen ‘vertical’ in the direction determined by the gravitational field as ‘up’. Let the quantum uncertainty in the vertical position of the atom be \( \Delta Z \) and the uncertainty in the vertical momentum be \( \Delta P_Z \). As is well known, the rate of clocks is affected by a gravitational field (see figure 1). If the location of the clock has an uncertainty \( \Delta z \) in the vertical direction, the clock rate \( \nu \) has an uncertainty \( \Delta \nu \) given by

\[
Q^{-1} = \left( \frac{\Delta \nu}{\nu} \right)_g = \left( \frac{\Delta T}{T} \right)_g \sim \frac{g \Delta Z}{c^2}.
\]  

(1)

This is a famous observation that emerged during the Bohr–Einstein debates at the Solvay conference [14]. See reference [14] for a thought experiment which is used to derive equation (1). It is convenient to introduce a natural length \( L_g = \frac{c^2}{g} \). For a terrestrial gravitational field of 10 m s\(^{-2}\), \( L_g \) works out to be \( 10^{16} \) m, which is about a lightyear. Thus we have

Figure 1. Atomic clock in a gravitational field \( g \). Figure shows an atom in a trap emitting from two positions differing in height. The signal received from the lower position appears red shifted relative to the higher position. The two positions shown are within the quantum uncertainty in vertical positions of the clock.
One might expect that by making ΔZ small, the stability of time measurement can be made arbitrarily high. However, because of Heisenberg’s uncertainty relation ΔZΔPz ≥ ℏ/2, confining the clock in a stiffer trap to reduce the position uncertainty results in a large momentum uncertainty in the vertical Z direction. The momentum uncertainty in the vertical direction can be viewed as a random motion of the clock. We can suppose that Pz ∼ ΔPz. Using the relations Pz = γmv, where γ is the Lorentz factor, which exceeds one by an amount of order (v/c)2, we can estimate the associated velocity of the random zero point vertical motion of the clock. The special relativistic time dilation effect due to the random motion causes uncertainty, which is of order v2/c2. The time dilation leads to an uncertainty in the clock rate (we have dropped a numerical factor of order 1):

\[ \frac{ΔT}{T} \sim \frac{ΔZ}{L_g}. \]  

Squeezing the atom in the vertical direction to minimize the gravitational uncertainty in the clock rate results in increased uncertainty due to the time dilation effect. There is thus an optimal value for the width of the trap and a corresponding limit to the Q factor of the clock. These ideas can be made slightly more precise for the case of a harmonic trap.

3. Harmonic trap

An atom clock of mass m, in a gravitational field g is trapped in a harmonic trap with angular frequency Ωt. If the operating frequency of the clock is ν = ωc/(2π), where ωc is the angular frequency, the Doppler recoil energy of the clock is

\[ E_r = \frac{1}{2m} \left( \frac{ℏω_c}{c} \right)^2. \]

To prevent this recoil from affecting the observed frequency of the clock we choose Ωt rather larger than ΩLD = ℏc, so that we are in the Lamb–Dicke regime. The recoil momentum is then taken not by the atom but by the whole trap. This is very similar to the Mössbauer effect [16].

For an atom whose centre of mass motion is in the quantum ground state of a harmonic trap, we have an expression for the variance in the height got by equating the mean kinetic (or potential) energy to half the zero point energy

\[ \frac{1}{2} m Ω_t^2 (ΔZ)^2 = \frac{ℏΩ_t}{4}, \]

which gives

\[ (ΔZ)^2 = \frac{ℏ}{2mΩ_t}. \]

This quantum uncertainty in the vertical position of the clock gives an uncertainty in the clock rate. Let us introduce Ωs = c/ν, which has the dimensions of frequency and write

\[ \frac{Δv}{ν} \sim \frac{gΔZ}{c^2} = \frac{g}{c} \sqrt{\frac{ℏ}{2mc^2Ω_t}} \sim \frac{Ω_s}{\sqrt{2ω_cΩ_t}}, \]
where \( \omega_m = \frac{mc^2}{\lambda} \) is the mass of the clock converted into frequency units. (Note that \( \omega_m \) is not the operating angular frequency of the clock, which we denote by \( \omega = 2\pi \nu \)).

It is evident that the gravitational contribution is made smaller by stiffening the trap. However, this leads to larger accelerations within the trap. A typical acceleration is \( a = \Omega^2 \Delta Z \) which gives rise to a time dilation redshift of

\[
\delta \nu/\nu \sim \frac{\Omega^2 \Delta Z}{2 \omega_m \Omega^2}.
\]

As we know from the principle of equivalence, acceleration has the same effect as gravity; hence the similarity between the first equalities of (4) and (5).

Adding the variances of the two effects, we find

\[
\left( \frac{\delta \nu}{\nu} \right)_{\text{Total}} \sim \left( \frac{\delta \nu}{\nu} \right)_\ell + \left( \frac{\delta \nu}{\nu} \right)_g \sim \frac{\Omega^2}{2 \omega_m \Omega^2} + \frac{\Omega^2}{4 \omega_m^2}.
\]

The total fractional uncertainty in the clock rate has a minimum at \( \Omega = \frac{\omega_m^2}{\Omega^2} \). The corresponding value for \( Q^{-1} \) is

\[
Q^{-1} = \frac{\delta \nu}{\nu} \geq \frac{\Omega^*}{\omega_m} \sim \sqrt{\frac{3}{2}} \left( \frac{\Omega^*}{\omega_m} \right)^{1/3} = \sqrt{\frac{3}{2}} \left( \frac{\lambda_c}{L_g} \right)^{1/3} = \sqrt{\frac{3}{2}} \left( \frac{\hbar g}{mc^3} \right)^{1/3},
\]

where \( \lambda_c = \frac{\hbar}{2mc} \) is the Compton wavelength of the clock. The main result can be stated as:

\[
Q \leq \frac{2}{\sqrt{3}} \left( \frac{mc^3}{\hbar g} \right)^{1/3}.
\]

We have thus derived a theoretical limit for the stability of clocks in a gravitational field. As the gravitational field tends to 0, so does \( \Omega^* \) and we leave the Lamb–Dicke regime and the regime of validity of our bound. The conditions of the Salecker–Wigner bound (equation (6) of reference [11]) apply, since the clock is not in a trap.

Can one beat this fundamental limit by replicating the clocks \( N \) times (distributed in an optical lattice of traps) and then taking the mean clock rate? The answer is no, as the following argument shows. If one clock in a trap has a rate uncertainty \( \Delta \), one can consider creating an ensemble of \( N \) identical clocks and averaging the readings. We would expect the rate uncertainty to decrease as \( \Delta_N = \Delta / \sqrt{N} \). Our bound \( B_N \) for \( N \) clocks (equation (7)) scales with \( N \) as \( N^{-2/3} \) and so decreases faster with \( N \) and \( \Delta_N \geq B_N \) for all \( N \). Thus the bound cannot be violated by considering an ensemble of clocks.

4. Numerical estimates

Here we briefly describe possible systems and the corresponding quantum limits to time-keeping. The systems in which the effects we describe are largest are the lighter clocks. For example the Hydrogen clock could be compared with an ytterbium clock. We can make an estimate of the uncertainty in clock rates \( \pm \) by inserting the values of the relevant parameters for clocks of different types in gravitational fields ranging from terrestrial to that of a Neutron star. While the terrestrial examples are within experimentally achievable domain, the celestrial examples have been quoted just to make a point of principle to illustrate what may happen in an astrophysically relevant gravitational field. The results are displayed in table 1. Our
analysis shows that there is an ultimate achievable limit set by gravity. The limit clearly
depends on the gravitational field. The higher the gravitational field, the lower the attainable
$Q$ value.

5. Conclusion

We have shown that there is a fundamental quantum limit to the fractional frequency
uncertainty achievable by a clock of mass $m$ in a gravitational field. This limit comes from
restrictions from the uncertainty principle on one hand and the gravitational redshift on the
other hand.

We note that it is a fundamental quantum limit and cannot be violated by any clock in a
gravitational field. It gives a limit to the $Q$ value of timekeeping because of gravitational
effects. In some cases the well known quantum limit set by the width (or inverse lifetime) of
the excited state may be more stringent than our limit (7) in a terrestrial gravitational field. But
in a sufficiently strong gravitational field the reverse will be true. At a conceptual level, Time
is fundamental to general relativity as uncertainty is to quantum mechanics. There is con-
siderable current effort in combining the two theories into a larger framework. Much of
the effort focuses on high energy physics and attempts to unravel the microscopic structure
of space–time [17, 22, 25]. In contrast, our effort here is to understand low energy gravita-
tional quantum physics. There are recent papers [19, 21, 24] which also address low
energy gravitational quantum physics. They go on to suggest relations between gravity
and decoherence [23]. Such questions are of interest for further research but beyond the
scope of the present letter. Given the enormous current interest in unifying gravity and
quantum mechanics, such low energy effects which may be testable in a laboratory are of
great interest.

Table 1. Fundamental limits on $Q^{-1}$.

| Clock type       | $\nu$ Hz | Atomic mass | Gravitational field | Limit on $Q^{-1}$ |
|------------------|----------|-------------|---------------------|-------------------|
| Aluminium        | $10^{15}$| 27          | Terrestrial         | $10^{-22}$        |
| Aluminium        | $10^{15}$| 27          | White dwarf         | $10^{-18}$        |
| Aluminium        | $10^{15}$| 27          | Neutron star        | $10^{-14}$        |
| Hydrogen         | $10^9$   | 1           | Terrestrial         | $10^{-21}$        |
| Hydrogen         | $10^9$   | 1           | White dwarf         | $10^{-17}$        |
| Hydrogen         | $10^9$   | 1           | Neutron star        | $10^{-13}$        |
| Electron spin    | $10^{13}$| 1/1836      | Terrestrial         | $10^{-19}$        |
| Precession       | $10^{13}$| 1/1836      | White dwarf         | $10^{-15}$        |
| Precession       | $10^{13}$| 1/1836      | Neutron star        | $10^{-15}$        |
| Electron spin    | $10^{13}$| 1/1836      | Neutron star        | $10^{-11}$        |
| Precession       | $10^{10}$| 1           | Terrestrial         | $10^{-21}$        |
| Neutron spin     | $10^{10}$| 1           | Neutron star        | $10^{-13}$        |

Class. Quantum Grav. 32 (2015) 015018 S Sinha and J Samuel
Acknowledgments

It is a pleasure to thank John Barrow, Miguel Campiglia, Brükner Caslav, Fabio Costa, Avinash Deshpande, Sourav Dutta, Nico Giulini, Bei-Lok Hu, H S Mani, Igor Pikovski, Hema Ramachandran, Sadiq Rangwala, R Sorkin, A M Srivastava and Magdalena Zych for discussions.

References

[1] Sobel D 2007 Longitude: The True Story of a Lone Genius Who Solved the Greatest Scientific Problem of His Time (Walker and Company) reprint edn (October 30) (Washington: National Institute of Standards and Technology)
[2] Ramsey N F 1983 History of atomic clocks J. Res. Natl. Bur. Stand. 88 301
[3] Winkler G M R and Van Flandern T C 1977 Astron. J. 82 84
[4] Lorimer D R and Kramer M 2005 Handbook of Pulsar Astronomy (Cambridge: Cambridge University Press)
[5] Thomson W and Tait P G 1879 Treatise on Natural Philosophy (Cambridge: Cambridge University Press)
[6] Kusch P, Millman S and Rabi I I 1940 The radiofrequency spectra of atoms hyperfine structure and Zeeman effect in the ground state of Li+, Li−, K39 and K41 Phys. Rev. 57 765–80
[7] Chou C, Hume D, Rosenbland T and Wineland D 2010 Optical clocks and relativity Science 329 1630
[8] Barrow J D 1996 Wigner inequalities for a black hole Phys. Rev. D 54 6563–4
[9] McNamara G 2008 Clocks in the Sky: The Story of Pulsars (Berlin: Springer)
[10] Hinkley N et al 2013 Science 341 1215
[11] Salecker H and Wigner E P 1958 Quantum limitations of the measurement of space–time distances Phys. Rev. 109 571
[12] Dicke R H 1953 The effect of collisions upon the doppler width of spectral lines Phys. Rev. 89 472–3
[13] Cohen-Tannoudji C and Guéry-Odelin D 2011 Advances in Atomic Physics: An Overview (Singapore: World Scientific)
[14] Schilpp P 1970 Albert Einstein: Philosopher-Scientist (US: Open Court Publishing Co)
[15] Campbell C J et al 2012 Single-ion nuclear clock for metrology at the XIX decimal place Phys. Rev. Lett. 108 120802
[16] Möschbauer R L 1958 Kernresonanzfluoreszenz von Gammastrahlung in Ir191 Z. Phys. 151 124–43
[17] Mukhi S 2011 String theory: a perspective over the last 25 years Class. Quantum Grav. 28 153001
[18] Wigner E P 1957 Rev. Mod. Phys. 29 255–68
[19] Zych M, Costa F, Pikovski I and Brukner C 2011 Quantum interferometric visibility as a witness of general relativistic proper time Nat. Commun. 2 505
[20] Müller H, Peters A and Chu S 2010 Nature 463 926
[21] Pikovski I, Zych M, Costa F and Brukner C 2013 Universal decoherence due to gravitational time dilation 2 arXiv:1311.1095
[22] Rovelli C 2011 Loop quantum gravity: the first 25 years Class. Quantum Grav. 28 153002
[23] Sinha S 1997 Decoherence at absolute zero Phys. Lett. A 228 1
[24] Sinha S and Samuel J 2011 Atom interferometry and the gravitational redshift Class. Quantum Grav. 28 145018
[25] Sorkin R D 2009 Light, links and causal sets J. Phys. Conf. Ser. 174 012018