The **n**-homology of representations.

Tim Bratten  
Facultad de Ciencias Exactas, UNICEN. Tandil, Argentina.

Abstract  

The **n**-homology groups of a **g**-module provide a natural and fruitful extension of the concept of highest weight to the representation theory of a noncompact reductive Lie group. In this article we give an introduction to the **n**-homology groups and survey some developments, with a particular emphasis on results pertaining to the problem of calculating **n**-homology groups.

1 Introduction

The concept of a highest weight and its use to classify irreducible representations of compact Lie groups can be traced back nearly a century, to seminal work by E. Cartan and H. Weyl. For a compact, connected Lie group, the highest weight theory gives a tight parametrization of irreducible representations in terms of specific invariants associated to the group. If one tries to extend this concept to the representation theory of a noncompact, real reductive group one immediately encounters two problems. On the one hand, in the noncompact case, it turns out there are several conjugacy classes of complex Borel subalgebras, and what might be called a highest weight depends on the choice of a conjugacy class. On the other hand, it is quite common that what should be called a highest weight turns out to be zero for every choice of Borel subalgebra. This means, in the traditional sense, the highest weight does not exist for a great majority of irreducible representations.

Although there is no way to avoid the first problem, representation theorists have confronted the second problem by considering the highest weight to be a functorial construction and studying the related derived functors. This has proved to be especially fruitful, producing a strong and useful family of invariants associated to a representation. In this article we give a brief introduction to the **n**-homology (and **n**-cohomology) groups, followed by a survey of some results, focusing on developments related to the problem of calculating the **n**-homology of representations.

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2 \textbf{\textit{n}-homology and \textit{n}-cohomology}

In this section we introduce the \textit{n}-homology and \textit{n}-cohomology of \(g\)-modules (for more details see [10]).

Let \(g\) be a complex reductive Lie algebra. By definition, a Borel subalgebra of \(g\) is a maximal solvable subalgebra and a parabolic subalgebra of \(g\) is a subalgebra that contains a Borel subalgebra. If \(p \subseteq g\) is a parabolic subalgebra then the nilradical \(n\) of \(p\) is the largest solvable ideal in \([p, p]\). A Levi factor is a complementary subalgebra to \(n\) in \(p\). One knows that Levi factors exist and that they are exactly the subalgebras which are maximal with respect to being reductive in \(p\). When \(l\) is a Levi factor then

\[p = l \oplus n\]

is called a Levi decomposition.

Fix a parabolic subalgebra \(p\) with nilradical \(n\) and Levi factor \(l\). Let \(U(n)\) denote the enveloping algebra of \(n\) and let \(C\) be the 1-dimensional trivial module. If \(M\) is a \(g\)-module then the zero \(n\)-homology of \(M\) is the \(l\)-module

\[H_0(n, M) = C \otimes_{U(n)} M.\]

This \(l\)-module is sometimes referred to as the space of coinvariants, although it clearly depends on the choice of parabolic subalgebra. The definition of the zero homology determines a right exact functor from the category of \(g\)-modules to the category of \(l\)-modules. The \(n\)-homology groups of \(M\) are the \(l\)-modules obtained as the corresponding derived functors. There is a standard complex for calculating these homology groups, defined as follows. The right standard resolution of \(C\) is the complex of free right \(U(n)\)-modules given by

\[\cdots \to \Lambda^{p+1} n \otimes U(n) \to \Lambda^p n \otimes U(n) \to \cdots \to n \otimes U(n) \to U(n) \to 0.\]

Applying the functor

\[- \otimes_{U(n)} M\]

to the standard resolution we obtain a complex

\[\cdots \to \Lambda^{p+1} n \otimes M \to \Lambda^p n \otimes M \to \cdots \to n \otimes M \to M \to 0\]

of left \(l\)-modules called the standard \(n\)-homology complex. Here \(l\) acts via the tensor product of the adjoint action on \(\Lambda^p n\) with the given action on \(M\). Since \(U(g)\) is a
free $U(n)$-module, a routine homological argument identifies the pth homology of the standard complex with the pth $n$-homology group

$$H_p(n, M).$$

One can prove that the induced $l$-action on the homology groups of the standard complex is the correct one.

The zero $n$-cohomology of a $g$-module $M$ is the $l$-module

$$H^0(n, M) = \text{Hom}_{U(n)}(\mathbb{C}, M).$$

This $l$-module is sometimes referred to as the space of invariants, and also clearly depends on the choice of parabolic subalgebra. The definition of the zero cohomology determines a left exact functor from the category of $g$-modules to the category of $l$-modules. By definition, the $n$-cohomology groups of $M$ are the $l$-modules obtained as the corresponding derived functors. These $l$-modules can be calculated by applying the functor

$$\text{Hom}_{U(n)}(-, M)$$

to the standard resolution of $\mathbb{C}$, this time by free left $U(n)$-modules. In a natural way, one obtains a complex of $l$-modules and the pth cohomology of this complex realizes the pth $n$-cohomology group

$$H^p(n, M).$$

It turns out that the structure of the $n$-cohomology is determined by the structure of the $n$-homology, in a simple way. Thus, it is often a matter of convenience whether one works with homology groups or cohomology groups. In this article, we will focus on results framed in terms of homology. The following proposition, whose proof is established by an analysis of standard complexes, can be used to translate results about $n$-homology into results about $n$-cohomology [9, Section 2].

**Proposition 2.1** Suppose $M$ is a $g$-module. Let $p \subseteq g$ be a parabolic subalgebra with nilradical $n$ and Levi factor $l$. Let $d$ denote the dimension of $n$. Then there are natural isomorphisms

$$H_p(n, M) \cong H^{d-p}(n, M) \otimes \Lambda^d n.$$
Harish-Chandra class. In particular, we assume the following setup. $G$ will denote a connected, complex reductive group. This means $G$ is a connected, complex Lie group with the property that the maximal compact subgroups are real forms of $G$. The group $G_0$ will denote a real form of $G$ and is assumed to have finitely many connected components. We call $G_0$ a linear reductive Lie group. The Lie algebras of $G$ and $G_0$ will be denoted $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively. For the remainder of this article we fix a maximal compact subgroup $K_0$ of $G_0$ and let $K \subseteq G$ be the complexification of $K_0$. In general, we write $K, L$ etc. to indicate complex subgroups of $G$ and denote the corresponding Lie algebras by $\mathfrak{k}, \mathfrak{l}$ etc. Subgroups of $G_0$ will be denoted by $K_0, L_0$ etc. with the corresponding real Lie algebras written as $\mathfrak{k}_0, \mathfrak{l}_0$ etc.

A representation of $G_0$ will mean a continuous linear action of $G_0$ in a complete, locally convex topological vector space. When we speak of irreducible or finite length representations, the corresponding definitions should be framed in terms of invariant closed subspaces. A vector $v$ in a representation $V$ is called smooth when
\[
\lim_{t \to 0} \frac{\exp(t\xi)v - v}{t}
\]
exists for each $\xi \in \mathfrak{g}_0$.

In order to define $n$-homology groups, we will be primarily interested in smooth representations. These are representations where every vector is smooth. In a natural way a smooth representation carries a compatible $\mathfrak{g}$-action. For a compact Lie group one can show that a finite length representation is finite-dimensional and therefore smooth.

We recall some basic results about the infinite-dimensional representations of reductive groups. In the 1950s, Harish-Chandra proved that an irreducible unitary representation $V$ has the property that each irreducible $K_0$-submodule has finite multiplicity in $V$. This led him to define and study admissible representations. This means that each irreducible $K_0$-submodule of the representation has finite multiplicity. Harish-Chandra then considered the subspace of $K_0$-finite vectors. By definition, a vector $v$ in a representation is called $K_0$-finite if the span of the $K_0$-orbit of $v$ is finite-dimensional. Although the subspace of $K_0$-finite vectors is not $G_0$-invariant, Harish-Chandra proved that $K_0$-finite vectors are smooth, and thus form a $(\mathfrak{g}, K_0)$-module called the underlying Harish-Chandra module.

On the other hand, it is possible to define abstractly the concept of a Harish-Chandra module. This is a $\mathfrak{g}$-module equipped with a compatible, locally finite $K_0$-action. Harish-Chandra proved that an irreducible Harish-Chandra module appears as the underlying $(\mathfrak{g}, K_0)$-module of $K_0$-finite vectors in an irreducible admissible Banach space representation for $G_0$ and W. Casselman proved that the same holds for any finite-length Harish-Chandra module. By now we know more. In particular, given a Harish-Chandra module $M$ we define a globalization $M_{\text{glob}}$ of $M$ to be an admissible representation for $G_0$ whose underlying Harish-Chandra is $M$. We assume our Harish-Chandra modules have finite-length. Then we can assert
that several canonical and functorial globalizations exist on the category of Harish-Chandra modules. These are: the smooth globalization of Casselman and Wallach \[5\], its dual (called: the distribution globalization), Schmid’s minimal globalization \[14\] and its dual (the maximal globalization). All four globalizations are smooth. We will let \(M_{\text{min}}\), \(M_{\text{max}}\), \(M_{\infty}\) and \(M_{\text{dis}}\) denote respectively, the minimal, the maximal, the smooth and the distribution globalizations of a Harish-Chandra module \(M\). If \(M_{\text{glob}}\) denotes a Banach globalization of \(M\), then there is a natural chain of inclusions

\[ M \subseteq M_{\text{min}} \subseteq M_{\infty} \subseteq M_{\text{glob}} \subseteq M_{\text{dis}} \subseteq M_{\text{max}}. \]

In this chain the minimal globalization is known to coincide with the analytic vectors in \(M_{\text{glob}}\) while \(M_{\infty}\) coincides with the smooth vectors in \(M_{\text{glob}}\). In particular, one knows that a finite-length admissible Banach space representation for \(G_0\) is smooth if and only if it is finite dimensional. Later in this article we will review various results, often called comparison theorems, relating the \(\mathfrak{n}\)-homologies of a Harish-Chandra module to the \(\mathfrak{n}\)-homologies of a canonical globalization.

4 Some structural details

In this section we recall some structure theory and an important technical result about the decomposition of \(\mathfrak{n}\)-homology groups for certain \(\mathfrak{g}\)-modules, giving special emphasis on the case of a Borel subalgebra. Recall that a Cartan subalgebra \(\mathfrak{h} \subseteq \mathfrak{g}\) is a maximal abelian subalgebra whose elements are semisimple under the adjoint representation of \(\mathfrak{h}\) in \(\mathfrak{g}\). A nonzero eigenvalue \(\alpha \in \mathfrak{h}^*\) for the adjoint representation is called a root. \(\Sigma\) will denote the set of roots. Thus

\[ \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha \]

where \(\mathfrak{g}^\alpha\) is the eigenspace corresponding to root \(\alpha\). One knows that \(\alpha \in \Sigma\) if and only if \(-\alpha \in \Sigma\). If \(\mathfrak{b}\) is a Borel subalgebra of \(\mathfrak{g}\) containing \(\mathfrak{h}\) then the roots of \(\mathfrak{h}\) in \(\mathfrak{b}\) define a subset \(\Sigma^+ \subseteq \Sigma\) called the corresponding set of positive roots. When the sum of two positive roots is a root, then that sum is positive. One also knows that \(\Sigma\) is a disjoint union:

\[ \Sigma = \Sigma^+ \cup -\Sigma^+. \]

One can show there is a unique \(H_\alpha \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]\) such that \(\alpha(H_\alpha) = 2\). We use this element to define the value of the dual root. In particular, the dual root is given by

\[ \check{\alpha}(\mu) = \mu(H_\alpha) \text{ for } \mu \in \mathfrak{h}^*. \]

The linear reflection \(s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*\) corresponding to \(\alpha \in \Sigma\) is defined as

\[ s_\alpha(\mu) = \mu - \check{\alpha}(\mu)\alpha. \]
These reflections generate a finite subgroup of the general linear group of \( h^* \), denoted \( W \) and called the Weyl group of \( h \) in \( g \).

Let \( Z(g) \) denote the center of the enveloping algebra \( U(g) \) of \( g \). A \( g \)-infinitesimal character \( \Theta \) is a homomorphism of algebras

\[
\Theta : Z(g) \to \mathbb{C}.
\]

Since \( Z(g) \) acts on an irreducible Harish-Chandra module (and also any corresponding smooth globalization) by a scalar, the infinitesimal character is an important invariant associated to an irreducible, admissible representation. We now recall Harish-Chandra’s parametrization of infinitesimal characters. We choose a Borel subalgebra \( b \) containing \( h \). Thus

\[
b = h \oplus n \quad \text{where} \quad n = [b, b] \quad \text{is the nilradical of} \quad b.
\]

Then one knows that \( Z(g) \subseteq U(h) \oplus U(g)n \) and that the corresponding projection of \( Z(g) \) in \( U(h) \) defines an injective morphism of algebras called the unnormalized Harish-Chandra map. We can use this morphism to identify infinitesimal characters with Weyl group orbits in \( h^* \) in the following way. Let \( \rho \) denote one-half the sum of the positive roots and suppose \( \lambda \in h^* \). Then, via the unnormalized Harish-Chandra map, the composition

\[
\Theta : Z(g) \to U(h) \xrightarrow{\lambda + \rho} \mathbb{C}
\]

defines an infinitesimal character \( \Theta \). One knows that for \( w \in W \), the element \( w\lambda \in h^* \) defines the same infinitesimal character \( \Theta \). Abusing notation somewhat, we write \( \Theta = W \cdot \lambda \). The infinitesimal character is called regular when the only element of \( W \) fixing an element in the orbit \( W \cdot \lambda \), is the identity. This is equivalent to the condition that

\[
\check{\alpha}(\lambda) \neq 0 \quad \text{for each} \quad \alpha \in \Sigma.
\]

For a \( g \)-module \( M \) with regular infinitesimal character one has the following result. The notes by D. Milicic [13] contain a proof.

**Theorem 4.1** Let \( M \) be a \( g \)-module with regular infinitesimal character \( \Theta \). Suppose \( b \) is a Borel subalgebra of \( g \) with Levi decomposition

\[
b = h \oplus n.
\]

Let \( \lambda \in h^* \) such that \( \Theta = W \cdot \lambda \) and let \( \rho \) be one half the sum of the positive roots. Then the Cartan subalgebra \( h \) acts semisimply on the \( n \)-homology groups \( H_p(n, M) \) with eigenvalues of the form \( w\lambda + \rho \) for \( w \in W \). In particular

\[
H_p(n, M) = \bigoplus_{w \in W} H_p(n, M)_{w\lambda + \rho}
\]

where

\[
H_p(n, M)_{w\lambda + \rho} = \{ v \in H_p(n, M) : \xi \cdot v = (w\lambda + \rho) (\xi)v \text{ for each} \xi \in h \}.
\]
A generalization of this result works for any parabolic subalgebra $p$ with Levi decomposition

$$p = l \oplus n.$$ 

In particular, if $M$ is a $g$-module $M$ with regular infinitesimal character $\Theta$ and if $Z(l)$ denotes the center of the enveloping algebra of $l$ then $H_p(n, M)$ is a semisimple $Z(l)$-module and decomposes into a direct sum of $Z(l)$-eigenspaces, where the associated $l$-infinitesimal characters that appear are related to $\Theta$ by an appropriately defined Harish-Chandra map.

5 Kostant’s theorem

When $G_0$ is a connected, compact Lie group, there is a result, called Kostant’s theorem, that calculates the $n$-homolgy groups of an irreducible representation. In this section we review that result, with special emphasis on the case of a Borel subalgebra.

Assume $G_0$ is a compact real form of $G$. Fix a Borel subalgebra $b$. Then the normalizer of $b$ in $G_0$ is a maximal torus $H_0$ and a real form for a Cartan subgroup $H$ of $G$. We let $h$ be the Lie algebra of $H$ and $n$ the nilradical of $b$. $\Sigma$ is the set of roots. The roots of $h$ in $b$ determine a set of positive roots $\Sigma^+ \subseteq \Sigma$. Let $\rho$ be one-half the sum of the positive roots. Suppose

$$\chi : H_0 \to \mathbb{C}$$

is a continuous character and let $\mu \in h^*$ denote the complexification of the derivative of $\chi$. To be consistent with the notation in Section 7 we use the shifted parameter

$$\lambda = \mu - \rho.$$

One knows that

$$\check{\alpha}(\lambda) \text{ is an integer for each } \alpha \in \Sigma.$$ 

The character $\chi$ is called antidominant and regular if

$$\check{\alpha}(\lambda) \notin \{0, 1, 2, 3, \ldots\} \text{ for each } \alpha \in \Sigma^+.$$ 

The Cartan-Weyl parametrization of irreducible representations is as follows.

**Theorem 5.1** Maintain the established notations.

(a) Suppose $M$ is an irreducible $G_0$-module. Then the space of coinvariants $H_0(n, M)$ is an irreducible $H_0$-module and the associated character $\chi$ is antidominant and regular. This character is called the lowest weight.

(b) If two irreducible representations have the same lowest weight then they are isomorphic.

(c) To each antidominant and regular character there is an irreducible $G_0$-module with the given character as its lowest weight.
We need to define the length function on the Weyl group. One knows that the Weyl group permutes the roots of \( h \) in \( g \). We can define the length of \( w \in W \) to be the number of roots in

\[-\Sigma^+ \cap w\Sigma^+.\]

Kostant’s theorem is the following:

**Theorem 5.2** Suppose \( M \) is the irreducible representation for \( G_0 \) with lowest weight \( \chi \) and let \( \lambda \in h^* \) be the shifted parameter. Then \( H_p(n, M) \) is a sum of irreducible \( H_0 \)-modules each having multiplicity one. The characters of \( H_0 \) that show up as eigenvalues in \( H_p(n, M) \) are exactly those whose derivative have the form \( w\lambda + \rho \) where the length of \( w \) is \( p \).

In the more general case of a parabolic subalgebra \( p \), let \( L_0 \) be the normalizer of \( p \) in \( G_0 \) and let \( l \) be the complexified Lie algebra of \( L_0 \). One knows that \( L_0 \) is connected and that \( l \) is a Levi factor of \( p \). Indeed, if \( L \) is the connected subgroup of \( G \) with Lie algebra \( l \) then \( L_0 \) is the compact real form of \( L \). Suppose \( M \) is an irreducible representation for \( G_0 \) and let \( n \) be the nilradical of \( p \). Then Kostant’s Theorem describes the structure of the \( p \)th homology group \( H_p(n, M) \) as an \( L_0 \)-module. In particular, the theorem states that an irreducible representation \( V \) of \( L_0 \) has, at most, multiplicity one in \( H_p(n, M) \) and gives a precise condition when \( V \) appears, in terms of the degree \( p \), the lowest weight of \( M \) and the lowest weight of \( V \). We refer the reader to [10, Chapter IV, Section 9] for more details.

## 6 Flag manifolds and comparison theorems

As we mentioned before, when \( G_0 \) is noncompact, there are several conjugacy classes of Borel subalgebras and the structure of the \( n \)-homology groups of a representation can depend on the choice of \( G_0 \)-conjugacy class. On the other hand, when \( M \) is a Harish-Chandra module, then the locally finite \( K_0 \)-action on \( M \) extends naturally to a locally holomorphic \( K \)-action, and it turns out that the \( n \)-homology groups of \( M \) depend on the \( K \)-conjugacy classes of Borel subalgebras. In order to compare the \( n \)-homology groups of \( M \) with the \( n \)-homology groups of a smooth globalization, we therefore need to know something about the relationship between \( G_0 \)-conjugacy classes and \( K \)-conjugacy classes. There is an elegant geometric result, referred to as **Matsuki duality**, that gives us the needed information. We now review that result.

One knows that the group \( G \) acts transitively on the set of Borel subalgebras of \( g \). The corresponding \( G \)-homogeneous complex manifold \( X \) is called the full flag space. In general, if \( p \) is a parabolic subalgebra of \( g \) then the normalizer of \( p \) in \( G \) is the connected subgroup \( P \) with Lie algebra \( p \) and the corresponding quotient

\[ Y = G/P \]
is called a flag manifold. The points in $Y$ are naturally identified with the $G$-conjugates to $\mathfrak{p}$.

Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be the complexification of a Cartan involution of $\mathfrak{g}_0$ corresponding to the maximal compact subgroup $K_0$. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called stable if $\mathfrak{g}_0 \cap \mathfrak{h}$ is a real form and if $\theta(\mathfrak{h}) = \mathfrak{h}$. A Borel subalgebra is called very special if it contains a stable Cartan subalgebra. A stable Cartan subalgebra of a Borel subalgebra is unique (when it exists). A point in the full flag space is called very special if the corresponding Borel subalgebra is.

Matsuki has established the following \[11\].

**Theorem 6.1** Let $X$ be the full flag space. Then
(a) The subset of very special points in a $G_0$-orbit is a nonempty $K_0$-orbit.
(b) The subset of very special points in a $K$-orbit is a nonempty $K_0$-orbit.

It follows that the very special points give a one-to-one correspondence between the $G_0$-orbits and the $K$-orbits on $X$, defined by the following duality. A $G_0$-orbit $S$ is said to be dual to a $K$-orbit $Q$ when $S \cap Q$ contains a special point. In this duality, open $G_0$-orbits correspond to closed $K$-orbits and the (unique) closed $G_0$-orbit corresponds to the (unique) open $K$-orbit. We note that Matsuki has established a similar result for any flag manifold \[12\].

**Example 6.2** Suppose $G = SL(2, \mathbb{C})$, the group of $2 \times 2$ complex matrices with determinant 1 and let $G_0 = SL(2, \mathbb{R})$. Then the full flag space $X$ is isomorphic to the Riemann sphere. $G_0$ has three orbits on $X$. The closed $G_0$-orbit can be identified with an equatorial circle and the other two orbits are the corresponding open hemispheres. It turns out every point in the closed orbit is very special, independent of the choice of $K_0$ (this is true in general for the closed orbit). Put $K_0 = SO(2, \mathbb{R})$. Thus $K = SO(2, \mathbb{C})$. Then the three $K$-orbits on $X$ are a punctured plane, containing the closed $G_0$-orbit, and two fixed points, which can be identified with the respective poles in each of the open hemispheres. These two poles are the other very special points.

When $M$ is a Harish-Chandra module and $\mathfrak{n}$ is the nilradical of a Borel subalgebra then one knows that the homology groups $H_p(\mathfrak{n}, M)$ are finite-dimensional, so it may seem reasonable to ask when $H_p(\mathfrak{n}, M)$ coincides with the $\mathfrak{n}$-homolgy groups of a smooth globalization. It turns out this not only depends on the choice of Borel subalgebra, but also in the the choice of smooth globalization. When $\mathfrak{n}$ is the nilradical of a very special Borel subalgebra, $M$ is a Harish-Chandra module, and $M_{\text{min}}$ is the minimal globalization, then H. Hecht and J. Taylor have shown \[8\] that the natural map

$$M \to M_{\text{min}}$$

induces isomorphisms $H_p(\mathfrak{n}, M) \to H_p(\mathfrak{n}, M_{\text{min}})$. 

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On the other hand, for the maximal globalization, there are counterexamples to this result.

The result of Hecht and Taylor has been generalized in the following form. A Levi factor $\mathfrak{l}$ of a parabolic subalgebra $\mathfrak{p}$ is called stable if $\mathfrak{l} \cap \mathfrak{g}_0$ is a real form $\mathfrak{l}_0$ of $\mathfrak{l}$ and if $\theta(\mathfrak{l}) = \mathfrak{l}$. The parabolic subalgebra $\mathfrak{p}$ is called very special if it contains a stable Levi factor. Such a Levi factor is unique. Unlike the case of the full flag space, there may be parabolic subalgebras which are not $G_0$-conjugate to a very special parabolic subalgebra, so we are not considering all orbits on every flag manifold. However, suppose $\mathfrak{p}$ is very special and $\mathfrak{l}$ is the stable Levi factor. Define $L_0$ to be the subgroup of $G_0$ that normalizes $\mathfrak{p}$ and normalizes $\mathfrak{l}$. Then $L_0$ is a linear reductive Lie group with complexified Lie algebra $\mathfrak{l}$ and maximal compact subgroup $L_0 \cap K_0$, called the associated real Levi subgroup. We have the following result [9, Proposition 2.24].

**Proposition 6.3** Suppose $\mathfrak{p}$ is a very special parabolic subalgebra with $L_0$ and $\mathfrak{l}$ defined as above. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{p}$ and suppose $M$ is a Harish-Chandra module for $(\mathfrak{g}, K_0)$. Then the $\mathfrak{n}$-homology groups are Harish-Chandra modules for $(\mathfrak{l}, K_0 \cap L_0)$.

For the minimal globalization, we have the following [2].

**Theorem 6.4** Maintain the hypothesis of the previous proposition. Then the standard complex induces a Hausdorff topology on $H_p(\mathfrak{n}, M_{\text{min}})$ and the natural map $M \to M_{\text{min}}$ induces isomorphisms

$$H_p(\mathfrak{n}, M_{\text{min}}) \cong H_p(\mathfrak{n}, M_{\text{min}}).$$

One might conjecture that above theorem works for the smooth globalization, and W. Casselman has informed the author that he has proven something along these lines, although details are unclear. Two partial comparison theorems about smooth globalizations have been published by other mathematicians. H. Hecht and J. Taylor have shown the result for minimal parabolic subgroups of $G_0$ [6], while U. Bunke and M. Olbrich have shown the result for any real parabolic subgroup [4].

D. Vogan has conjectured that all four canonical globalizations commute with the $\mathfrak{n}$-homology groups of a very special parabolic subalgebra when the corresponding $G_0$-orbit on the flag manifold is open [15]. We remark that it has recently been shown that Vogan’s conjecture is true for one globalization if and only it’s true for the dual [3]. Thus the conjecture is proven for both the minimal and maximal globalization.
7 The $n$-homology of standard modules

In the noncompact case, the problem of calculating $n$-homology groups can be quite complicated and there seems to be little hope of just writing down a formula that generalizes Kostant's theorem for all irreducible representations. However, there are certain representations, called standard modules, whose $n$-homology groups are a bit more predictable. These standard modules are generically irreducible, coincide with irreducibles when $G_0$ is compact, and can be used to classify the irreducible representations. In this section we define the standard representations and consider their $n$-homology groups, focusing on the case of the full flag space.

In particular, we use the construction of minimal globalizations given in [7]. Let $X$ be the full flag space and, since we need to keep track of points, introduce the following notation. For $x \in X$ we let $b_x$ be the corresponding Borel subalgebra and let $n_x$ denote the nilradical of $b_x$. When we are interested in calculating the $n_x$-homology of Harish-Chandra modules, we can assume $b_x$ is a very special Borel subalgebra. In that case, $h_x$ denotes the stable Cartan subalgebra of $b_x$ and $H_0 \subseteq G_0$ is the corresponding real Cartan subgroup (thus $H_0$ is the associated Levi subgroup). By our linear assumptions on $G_0$, it follows that $H_0$ is abelian, so that an irreducible, admissible representation of $H_0$ is a continuous character

$$\chi: H_0 \rightarrow \mathbb{C}.$$ 

Let $S \subseteq X$ be the $G_0$-orbit of $x$. In a natural way, $\chi$ extends to a character of the normalizer of $b_x$ in $G_0$ (we note that $H_0$ and the normalizer of $b_x$ coincide exactly when $S$ is open). Thus $\chi$ determines a $G_0$-homogeneous analytic line bundle over $S$. One can then define the concept of a polarized section [7, Section 8]. When $S$ is open, the polarized sections are holomorphic sections, and in general the polarized sections are locally isomorphic with the restricted holomorphic functions. Let $\mathcal{A}(x, \chi)$ denote the sheaf of polarized sections on $S$ and, for $p=0,1,2,3,\ldots$ let

$$H^p_c(S, \mathcal{A}(x, \chi))$$

denote the corresponding compactly supported sheaf cohomology group. Suppose $\mu \in h^*_x$ is the complexified differential of $\chi$, let $\rho$ be one-half the sum of the positive roots for $h_x$ in $b_x$ and let $\lambda \in h^*_x$ denote the corresponding shifted parameter. Thus

$$\mu = \lambda + \rho.$$ 

We have the following theorem [7].

**Theorem 7.1** Maintain the previously defined notations. Let $q$ be the codimension of the $K$-orbit of $x$ in $X$.

(a) $H^p_c(S, \mathcal{A}(x, \chi))$ carries a natural topology and a continuous $G_0$-action, so that
the resulting representation is a minimal globalization.

(b) $H^p_c(S, A(x, \chi))$ has infinitesimal character $\Theta = W \cdot \lambda$.

(c) When $\lambda$ is antidominant then $H^p_c(S, A(x, \chi)) = 0$ when $p \neq q$.

(d) When $\lambda$ is antidominant and $\Theta$ is regular then $H^p_c(S, A(x, \chi))$ contains a unique irreducible submodule. In particular, $H^q_c(S, A(x, \chi)) \neq 0$.

When $\lambda$ is antidominant and $\Theta$ is regular, we call $H^p_c(S, A(x, \chi))$ a regular standard module. These modules can be used to parametrize irreducible representations with regular infinitesimal character. For the remainder of this article we will make some remarks about how to calculate the $n$-homology of regular standard modules.

But we first note that, in the case of a singular infinitesimal character, the definition of standard module is more subtle, and the calculation of $n$-homology is more elusive.

To state results we will need to differentiate points where we calculate $n$-homology and the corresponding parameters for eigenvalues of a Cartan subalgebra (see Theorem 4.1). In particular, we fix a very special point $x \in X$ as a base point. For $\lambda \in h^*_x$ we put $\lambda(x) = \lambda$. When $b_y$ is a very special Borel subalgebra and $h_y \subseteq b_y$ is the stable Cartan subalgebra, then there exists $g \in G$ such that

$$g b_x g^{-1} = b_y \quad \text{and} \quad g h_x g^{-1} = h_y.$$

Thus

$$(h^*_x)^g = h_y$$

This isomorphism is independent of the choice of $g \in G$. For $\lambda \in h^*_x$ put

$$\lambda(y) = \lambda^g \in h^*_y.$$

We note that $\alpha \in h^*_x$ is a root of $h_x$ in $g$ if and only if $\alpha(y)$ is a root of $h_y$ and that $\alpha$ is positive at $x$ if and only if $\alpha(y)$ is positive at $y$. In particular, $\rho(y)$ is one-half the sum of the positive roots for $h_y$ in $b_y$.

The circle of ideas utilized in [7] depend on an identification of the derived functor of $n$-homology, in a certain weight (Theorem 4.1), with the geometric fiber applied to a certain, corresponding localization functor. These ideas originate in the elegant generalization of Casselman’s submodule theorem, given by A. Beilinson and J. Bernstein in [1]. This identification, together with some functorial rigmarole, immediately leads to the following result.

**Proposition 7.2** Suppose $V = H^q_c(S, A(x, \chi))$ is a regular standard module. Maintain the previously introduced notations. Let $C_\chi$ denote the 1-dimensional representation of $H_0$ corresponding to $\chi$ and let $n_x$ be the nilradical of $b_x$. Then we have the following.

(a) $H_p(n_x, V)_{\lambda + \rho} = 0$ for $p \neq q$ and $H_q(n_x, V)_{\lambda + \rho} = C_\chi$. 

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(b) If $b_y$ is a very special Borel subalgebra with nilradical $n_y$ and $y \notin S$ then

\[ H_p(n_y, V)_{\lambda(y)+\rho(y)} = 0 \quad \text{for each } p. \]

According to Theorem 4.1, the problem of calculating of the $n_y$-homology groups of $V$, at a special point $y \in X$, reduces to the problem of calculating the values in the weights $(w\lambda + \rho)(y)$ for $w \in W$. In geometric terms, this means calculating the geometric fibers of certain localizations of $V$ or, equivalently, calculating the result of the so called *intertwining functor*. We briefly consider this problem.

In general, a positive root is called *simple* if it cannot be decomposed into a nontrivial sum of positive roots. Let $\Sigma^+_y$ be the positive roots associated to $h_y$ in $b_y$. For a simple root $\alpha \in \Sigma^+_y$, the problem of calculating the values of the $n_y$-homology groups in the weight $(s_\alpha \lambda + \rho)(y)$ can be geometrically reduced to specific calculations for certain real subgroups of $SL(2, \mathbb{C})$. To a large extent, this idea is already exploited and explained in [7] and some of the necessary calculations are dealt with there.

We finish with an example where, using these ideas, a general formula, like Kostant’s, can be actually written down. Assume $G_0$ is a connected, complex reductive Lie group. Fix a very special Borel subalgebra $b_x$ with stable Cartan subalgebra $h_x$ and let $\Sigma^+_x$ denote the corresponding positive roots. For each $w \in W$, the set

\[ w(\Sigma^+_x) = \Sigma^+_{w\cdot x} \]

defines a new set of positive roots and thus a corresponding Borel subalgebra $b_{w\cdot x}$ of $g$ containing the stable Cartan subalgebra $h_x$. Thus the point $w \cdot x \in X$ is very special. Because $G_0$ is a complex reductive group, one knows that each Borel subalgebra of $g$ is $G_0$-conjugate to a Borel subalgebra of the form $b_{w\cdot x}$. Suppose $H$ is the Cartan subgroup of $G$ with Lie algebra $h_x$. Then each $\alpha \in \Sigma$ defines a holomorphic character of $H$ and by restriction, a corresponding character of $H_0$. We write $\chi_\alpha$ for this character of $H_0$. If we let $\lambda$ be the shifted parameter and put $y = w \cdot x$ then $(w\lambda)(y) = \lambda$.

Using the above ideas, one can deduce the following.

**Theorem 7.3** Let $G_0$ be a connected, complex reductive group. Suppose $V = H^0_0(S, \mathcal{A}(x, \chi))$ is the previously defined regular standard module and assume the $G_0$-orbit of $x$ is open in $X$. We define a chain $\alpha_1, \ldots, \alpha_k$ of simple roots to be a finite sequence of roots of $h_x$ such that for each $j$, $\alpha_{j+1}$ is simple for the set of positive roots defined by

\[ s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_j}(\Sigma^+_x). \]

Suppose $\alpha_1, \ldots, \alpha_k$ is a chain of simple roots and let $w \in W$ be the ordered product of reflections given by this chain. Let $\chi_w$ be the character of $H_0$ defined by

\[ \chi_w = \chi_{\alpha_1}^{-1} \chi_{\alpha_2}^{-1} \cdots \chi_{\alpha_k}^{-1} \]
and let $C_\chi \otimes C_{\chi_w}$ be the 1-dimensional representation of $H_0$ corresponding to the character $\chi \cdot \chi_w$. Let $q^w$ denote the codimension of the $K$-orbit of $y = w \cdot x$ in $X$. Then

$$H_p(n_y, V) = 0 \text{ for } p \neq q^w \text{ and } H_{q^w}(n_y, V) = C_\chi \otimes C_{\chi_w}.$$ 

We note that the hypothesis of the theorem implies that the representation $V$ is irreducible, and also remark that any attempt to write down a similar result for other orbits (even in the case of a connected, complex reductive group), when the standard module is reducible, is considerably more complicated.

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