A STRICTLY DECREASING INVARIANT FOR RESOLUTION OF SINGULARITIES IN DIMENSION TWO

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Abstract. In this paper we construct a local invariant $\iota$ and prove that it strictly decreases in every step of the strategy for resolution of singularities in dimension two by Cossart, Jannsen and Saito.

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Introduction

Presently, the most general result on resolution of surface singularities is due to Jannsen, Saito and the first author \cite{CJS}. There it is shown that every Noetherian excellent scheme of dimension at most two admits a canonical resolution of singularities with boundaries via a finite sequence of permissible blow-ups. This result handles the non-embedded as well as the embedded case. In their strategy the centers of the upcoming blow-ups are purely determined by the geometry of the maximal singular locus plus the history of the preceding resolution process. This method is based on Hironaka’s work for hypersurfaces in \cite{H2}. Different to other approaches for resolution of singularities there is no invariant used to determine the centers. The proof of the finiteness of the constructed sequence of blow-ups is then done indirectly by deducing a contradiction from the hypothesis that the sequence is infinite.

The first results on resolution of singularities in positive characteristic are due to Abhyankar \cite{A}, Lipman \cite{L} and the result for hypersurfaces by Hironaka mentioned above. For a comparison of the different approaches see also \cite{CGO}. In \cite{Cu2} Cutkosky gave a simplified version of Abhyankar’s proof.

The result due to Lipman is valid for excellent 2-dimensional schemes, but in contrast to \cite{CJS} he obtains his resolution not only by blow-ups in regular centers but also by using normalizations. Thus is not clear how his proof extends to the embedded situation.

More recently, Benito and Villamayor \cite{BV} gave another proof for resolution of singularities in dimension two by using techniques of Rees Algebras and generic projections. Kawanoue and Matsuki applied and simplified these methods in \cite{KM} for their idealistic filtration program.

In characteristic zero the centers for the resolution are determined by a certain invariant which then strictly decreases after blowing up. Due to the lack of the existence of maximal contact the proof does not apply in positive characteristic. In \cite{Mo} Moh even proved that the candidate for the generalization to positive characteristic may increase under blow-ups in regular center. But he was also able to give a bound for the increase.

The improvement of the singularities in the above results is not given by an invariant which strictly decreases in every step. For example, \cite{KM} show that they come within

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finely many steps to the case which they call the monomial case and then they change the strategy in order to finish this special case.

For special hypersurfaces of the form $V(x^p - F(y, z))$, for some $F(y, z) \in K[[y, z]]$ and a field $K$ of characteristic $p$, Hauser and his student Wagner [HaW] were able to introduce a so called bonus in order to overcome the increase detected by Moh. But for the general case there exists up to now no strictly decreasing invariant for singularities in dimension two. In the present paper we fill this gap.

Emerging from a precise study of the strategy of [CJS] we develop a local invariant $\iota = \iota(X, Z, x)$, where $X$ is a reduced excellent scheme of dimension at most two embedded in a regular scheme $Z$ and $x \in X$. Since we want to obtain a local construction we can reduce the non-embedded case $x$ to the embedded on $X \subset Z$ by Cohen structure theory.

The invariant consists of three parts, $\iota = (\iota_0, \iota_{hs}, \iota_{poly})$, where

- $\iota_0 \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^3$ is a first rough measure for the complexity of the singularity,
- $\iota_{hs} \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^3 \times \mathbb{Q}_\infty^2$ ($\mathbb{Q}_\infty := \mathbb{Q} \cup \{\infty\}$) is taking control on the maximal singular locus coming from the beginning and
- $\iota_{poly} \in \mathbb{Q}^4$ is an invariant coming from Hironaka’s characteristic polyhedron and which takes care about singularities which arise during the resolution process.

The assumption that $X$ has dimension at most two is crucial in this considerations. Because of this condition the singularities which appear newly during the blow-ups have a very nice form. More precisely, newly arising components in the locus of maximal singularity are already regular and intersect the exceptional divisors of the preceding blow-ups (or more general, the boundary components) transversally. Therefore $\iota_{poly}$ is an effective tool to measure the singularities in dimension two.

The main result of this paper is

**Theorem A.** Let $X \subset Z$ and let $\pi_Z : Z' \to Z$ be a blow-up with center $D \subset X$ following the strategy of [CJS]. Let $x \in D$ and $x' \in \pi_Z^{-1}(x)$. Then we have

$$\iota(X', Z', x') < \iota(X, Z, x).$$

Here we use the lexicographical order on $\mathbb{N}^\mathbb{N} \times \mathbb{N}^3 \times \mathbb{N}^\mathbb{N} \times \mathbb{N}^3 \times \mathbb{Q}^2 \times \mathbb{Q}^4$, where we equip $\mathbb{N}^\mathbb{N}$ with the product order.

By the theorem $\iota$ measures locally an improvement for the singularities lying above the center of the blow-up in every step of [CJS].

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1. **First measure for the complexity of the singularity**

First of all, we have to recall some definitions and results of [CJS]. Using them we deduce the first rough measure $\iota_0 \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^3$ for the singularity.

Let $X$ be an excellent Noetherian scheme. Since we want to obtain a resolution of singularities only by using blow-ups, we may suppose that $X$ is connected.

The first entry for $\iota_0$ is the Hilbert-Samuel function $H_X : X \to \mathbb{N}^\mathbb{N}$ of $X$. In order to be able to compare the values in $\mathbb{N}^\mathbb{N}$ we equip it with the product order, i.e. for $\nu, \mu \in \mathbb{N}^\mathbb{N}$ we have $\nu \leq \mu$ if and only if $\nu(n) \leq \mu(n)$ for every $n \in \mathbb{N}$.

Before we recall the precise definition of $H_X$ let us mention the following: Since $X$ is excellent it is in particular catenary. Thus for any two irreducible closed subschemes $Y \subset Z$ of $X$ all maximal chains of irreducible closed subschemes $Y = Y_0 \subset Y_1 \subset \ldots \subset Y_t = Z$ have the same length $r$ denoted by $\text{codim}_X(Y)$. For three of them, $Y \subset Z \subset W$, one has $\text{codim}_W(Y) = \text{codim}_W(Z) + \text{codim}_Z(Y)$.
Remark 1.4. Roughly speaking, the strategy of [CJS] (see loc.cit. Remark 5.29) is as follows: First, they pick a maximal value \( \nu \) of the Hilbert-Samuel locus and consider the locus \( Y := X(\nu) \), where it is achieved. (In fact, they use the log-Hilbert-Samuel function, whose definition we recall later in Definition 1.16, but for the formulation of the strategy one could pick any reasonable first measure for the singularities). By induction on the dimension they obtain a finite sequence of blow-ups which resolve the strict transform \( Y' \) of \( Y \) and moreover, \( Y' \) intersect the exceptional divisors only transversally. At this point \( Y' \) is a good choice for a center and they blow it up.
During the blow-ups which provide \( Y' \) one might create new singularities in \( X'(\nu) \). In order to be able to handle them, the strategy assigns to the appearing irreducible components certain labels which are determined by the first occurrence of the singularities (see also Remark 2.3 below for more details on this).

No local constructions are involved. Thus the [CJS]-strategy of building this sequence of permissible blow-ups in order to resolve the singularities of \( X \) is of global nature. In our investigations we focus on the local situation and construct an invariant effectively measuring an improvement of the singularity at every point lying above the center.

In Strategy 2.2 we give a local variant of the process of [CJS] for schemes of dimension two. But we like to point out to the reader that the strategy in [CJS] Remark 5.29 is formulated for arbitrary dimensional schemes. It would be interesting to see if this constructed sequence of blow-ups is also finite in higher dimensions or if there exists a counterexample.

Let \( x \in X \). By passing to the completion of \( \mathcal{O}_{X,x} \) and Cohen structure theory we may assume that locally at \( x \) we are in an embedded situation, i.e. the singularity is given by an ideal \( J \subset R \) in a regular local excellent Noetherian ring \( (R, M, k(x) = R/M) \).

The next natural candidate for the study of singularities is the tangent cone of \( X \) at \( x \). This is the cone living in the graded ring \( \text{gr}_M(R) \) defined by the ideal

\[
\text{In}_M(J) := (\text{in}_M(f) \mid f \in J),
\]

where \( \text{in}_M(f) = f \mod M^{\nu+1} \in \text{gr}_M(R) \) denotes the initial form of \( f \) with respect to \( M \) and \( \nu := \nu(f) := \text{ord}_M(f) = \max \{ m \in \mathbb{N} \mid f \in M^m \} \), for \( f \neq 0 \), and \( \text{In}_M(0) = 0 \).

Going back to Hironaka we have the notion of the directrix of a cone. In our situation this means:

**Definition 1.5.** Let \( J \subset R \) be an ideal in \( R \) which describes the singularity \( X \) locally at \( x \).

1. A system of regular elements \( (y_1, \ldots, y_r) \) in \( R \) is said to determine the directrix \( \text{Dir}_x(X) \) of \( X \) at \( x \) if the generators of \( \text{In}_M(J) \subset \text{gr}_M(R) \) are contained in \( k(x)[Y] \), \( Y_j = y_j \mod M^2 \) for \( 1 \leq j \leq r \),

\[
(\text{In}_M(J) \cap k(x)[Y]) \text{ gr}_M(R) = \text{In}_M(J),
\]

and, moreover, \( (y) \) is required to be minimal with this property, i.e. the number of elements \( r \) has to be minimal.

2. We denote by \( e_x(X) := \dim_{k(x)}(\text{Dir}_x(X)) \leq \dim(X) \) the dimension of the directrix. Since we are in the embedded situation, we have \( e_x(X) = n - r \), where \( n \) is the dimension of \( R \).

3. Let \( K/k(x) \) be a field extension. Then \( e_x(X)_K \) denotes the dimension of the directrix associated to the homogeneous ideal \( \text{In}_M(J)_K \).

In [CJS] Definition 1.26 \( \text{Dir}_x(X) \) is defined without assuming to be in the embedded situation.

If \( X \subset Z \) for some regular Noetherian scheme \( Z \), then an alternative invariant to the Hilbert-Samuel-function \( H_X \) is Hironaka’s \( \nu^* \)-invariant. Locally one can define it as follows:

**Definition 1.6.** Let \( X \subset Z \) and \( x \in X \). Let \( J \subset R \) be the ideal in \( R = \mathcal{O}_{Z,x} \) determining the local situation at \( x \).

1. A system of elements \( (f) = (f_1, \ldots, f_m) \) in \( J \) is called a standard basis for \( J \) if the system of their initial forms \( (\text{in}_M(f)) = (\text{in}_M(f_1), \ldots, \text{in}_M(f_m)) \) is a standard basis for \( \text{In}_M(J) \subset \text{gr}_M(R) \), i.e.

   a. \( \text{In}_M(J) = (\text{in}_M(f_1), \ldots, \text{in}_M(f_m)) \),

   b. \( \text{in}_M(f_i) \notin (\text{in}_M(f_1), \ldots, \text{in}_M(f_{i-1})) \), for \( 2 \leq i \leq m \), and

   c. \( \nu_1 \leq \nu_2 \leq \ldots \leq \nu_r \), where \( \nu_i = \text{ord}_M(f_i) \).

2. Let \( (f) \) be a standard basis of \( J \). We define the \( \nu^* \)-invariant by

\[
\nu^*_Z(X,Z) := \nu^*(J,R) := (\nu_1, \nu_2, \ldots, \nu_r, \infty, \infty, \ldots) \in \mathbb{N}_\infty, \quad (\mathbb{N}_\infty := \mathbb{N} \cup \{ \infty \}).
\]
Actually, a standard basis of $J$ also generates the ideal $I$; see [H1] Corollary (2.21.d).

The definition of $\nu^*_x(X, Z)$ as it is stated here looks as if it depends on a choice of a standard basis. By [CJS] Lemma 1.2 this is not the case and $\nu^*_x(X, Z)$ is really an invariant of the singularity.

A candidate for an invariant for the strategy in [CJS] should behave well under the blow-ups which are performed during their process. These are so called permissible blow-ups.

**Definition 1.7** ([CJS] Definition 2.1). Let $X$ be an excellent Noetherian scheme and $D \subset X$ a closed reduced scheme. Denote by $I_D \subset \mathcal{O}_X$ the ideal sheaf of $D$ in $X$, $\mathcal{O}_D = \mathcal{O}_X/I_D$ and $gr_{I_D}(\mathcal{O}_X) = \bigoplus_{i \geq 0} I_D^i/I_D^{i+1}$.

1. $X$ is normally flat along $D$ at $x \in D$ if $gr_{I_D}(\mathcal{O}_X)_x$ is a flat $\mathcal{O}_{D,x}$-module. $X$ is normally flat along $D$ if $X$ is normally flat along all points of $D$.

2. $D \subset X$ is permissible at $x \in D$ if $D$ is regular at $x$ and $X$ is normally flat along $D$ at $x$, and if $D$ contains no irreducible component of $X$ containing $x$. $D \subset X$ is permissible at all points of $D$.

3. The blow-up $\pi_D : Bl_D(X) \to X$ with a permissible center $D \subset X$ is called a permissible blow-up.

In order to give the reader a feeling for the notion of normal flatness, we recall the following two results.

**Theorem 1.8** ([CJS] Theorem 2.2(2)). Let $X$ and $D$ be as in the previous definition and suppose $X$ is a closed subscheme of a regular Noetherian scheme $Z$. Let $x \in D$. Denote by $(R = \mathcal{O}_{Z,x}, M, k(x))$ the local ring of $Z$ at $x$, and let $J \subset R$ (resp. $p \subset R$) be the defining ideal of $X \subset Z$ (resp. $D \subset Z$). Then the following are equivalent

1. $X$ is normally flat at $x$.

2. Let $u : gr_p(R) \otimes_R k(x) \to gr_M(R)$ be the natural map. Then $In_M(J)$ is generated in $gr_M(R)$ by $u(In_p(J))$.

3. There exists a standard basis $(f) = (f_1, \ldots, f_m)$ of $J$ such that $ord_M(f_i) = ord_p(f_i)$ for all $i \in \{1, \ldots, m\}$.

**Theorem 1.9** ([CJS] Theorem 2.3). Let $X$ and $D$ be as in the previous definition. Assume $D$ is regular. Let $x \in D$ and let $y \in D$ be the generic point of the irreducible component of $D$ containing $x$. Then the following are equivalent

1. $X$ is normally flat along $D$ at $x$.

2. $H^{(0)}_{O_{X,Y}} = H^{(\text{codim}_Y(x))}_{O_{X,Y}}$, where $Y = \overline{\{y\}}$ is the closure of $y$ in $X$.

3. $H_X = H_X(y)$.

Bennett, Hironaka and Singh showed the following result on the behavior of the $H_X$ under permissible blow-ups:

**Theorem 1.10** ([CJS] Theorem 2.10). Let $X$ be an excellent Noetherian scheme and $D \subset X$ a permissible closed subscheme, and let $\pi_X : X' = Bl_D(X) \to X$ be the blow-up with center $D$. Consider $x \in D$ and $x' \in \pi_X^{-1}(x)$ and set $\delta_{x'/x} = \text{trdeg}_{k(x)}(k(x'))$, where $k(x)$ (resp. $k(x')$) denotes the residue field of $x$ (resp. $x'$). Then

1. $H_X(x') \leq H_X(x)$.

2. If $H_X(x') = H_X(x)$ holds, then for any field extension $K/k(x')$ we have $e_{x'}(X')_K \leq e_x(X)_K - \delta_{x'/x}$.

Suppose $X$ is embedded in a regular scheme $Z$ and let $\pi_Z : Z' = Bl_D(Z) \to Z$ be the blow-up of $Z$ with center $D$. Then

3. $\nu^*_x(X', Z') \leq \nu^*_x(X, Z)$, and

4. $\nu^*_x(X', Z') = \nu^*_x(X, Z)$ holds if and only if $H_X(x') = H_X(x)$.
In [CJS] Theorem 1.15 it is shown that the set of all Hilbert-Samuel functions is well-founded, which means that every strictly descending sequence has to be finite. The proof for this relies on a result by Maclagan [Ma].

Hence by the previous theorem we only need to study those points closer where the Hilbert-Samuel function did not drop and, moreover, one can also use the \( \nu^s \)-invariant in order to detect them.

**Definition 1.11.** Let the situation be as in the previous theorem. Consider \( x \in D \) and \( x' \in \pi_X^{-1}(x) \).

1. If \( H_X(x') = H_X(x) \) then \( x' \) is called near to \( x \).
2. If \( x' \) is near to \( x \) and if we have additionally \( e_{x'}(X') + \delta_{x'/x} = e_x(X)k' = e_x(X) \), for \( k' = k(x') \), then \( x' \) is said to be very near to \( x \).

In order to achieve an improvement of the singularities one needs a precise knowledge of the locus of near points. The following result due to Hironaka (and later improved by Mizutani) gives some control on this locus.

**Theorem 1.12 ([CJS] Theorem 2.14).** Let \( x' \) be near to \( x \). Assume that either \( \text{char}(k(x)) = 0 \) or \( \text{char}(k(x)) \geq \dim(X)/2 + 1 \). Then \( x' \) lies in the projective space associated to the divisor space determined by the directrix and the tangent space of the center,

\[ x' \in \mathbb{P}(\text{Dir}_x(X)/T_x(D)) \subset \pi_X^{-1}(x). \]

If \( \dim(X) = 2 \), then \( \dim(X)/2 + 1 = 2 \) and the assumptions on the characteristic of the residue field is always fulfilled.

During the resolution process we have to take care of the arising exceptional divisor. This is important in order to obtain a canonical process to resolve singularities (e.g. try to find a canonical resolution, i.e. without doing any choices, of the singularity given by \( t^2 + xyz = 0 \)). Moreover, we only want to blow up in centers that have at most normal crossings (short n.c.) with the exceptional divisors. Also it might be important for applications not only to start with \( X \) but also a n.c. divisor on \( Z \). In [CJS] this is done via the notion of a boundaries.

In order to simplify the presentation we suppose in the following that \( X \subset Z \) is embedded in a regular scheme \( Z \). As we explained before this does not harm our aim to find a local invariant. But in fact, in [CJS] section 4 boundaries are even introduced in the non-embedded setting.

**Definition 1.13 ([CJS] Definition 3.3).** A boundary \( \mathcal{B} \) on \( Z \) is a collection of regular divisors on \( Z \), \( \mathcal{B} = \{B_1, \ldots, B_k\} \), such that they have at most n.c. singularities, i.e. each of them is regular and they are intersecting transversally.

Since the divisors in \( \mathcal{B} \) have n.c. we may choose the local coordinates \( (z_1, \ldots, z_n) \) at a point \( x \) such that \( (B_i)_x = V(z_i) \) for \( 1 \leq i \leq t \leq n \) and \( \mathcal{B}(x) := \{B \in \mathcal{B} \mid x \in B\} = \{B_1, \ldots, B_t\} \).

**Definition 1.14.** A closed subscheme \( D \subset X \) is called \( \mathcal{B} \)-permissible if it is permissible and has at most n.c. with the boundary \( \mathcal{B} \). The corresponding blow-up with center \( D \) is called a \( \mathcal{B} \)-permissible blow-up.

After a \( \mathcal{B} \)-permissible blow-up the transform \( \mathcal{B}' \) of a boundary \( \mathcal{B} \) is given by the strict transforms of the divisors of \( \mathcal{B} \) and further we add the exceptional divisor of the blow-up.

For an boundary it is important to have in mind at which step of the resolution process a boundary component occurred and how its relation to the maximal value of the Hilbert-Samuel function is. (We explain this more precisely after the next definition). In order to get hands on this one can use the following notion:

**Definition 1.15 ([CJS] Definition 3.6).** Let \( \mathcal{B} \) be a boundary on \( Z \). A history function for a boundary \( \mathcal{B} \) on \( X \) is a function

\[ O : X \to \{ \text{subsets of } \mathcal{B} \}; x \mapsto O(x) \]

which satisfies the following conditions:

1. For any \( x \in X \), \( O(x) \subset \mathcal{B}(x) \).
(2) For any \(x, y \in X\) with \(x \in \overline{\{y\}}\) and \(H_X(x) = H_X(y)\), we have \(O(y) \subset O(x)\).

(3) For any \(y \in X\), there exists a non-empty open subset \(U \subset \overline{\{y\}}\) such that \(O(x) = O(y)\) for all \(x \in U\) such that \(H_x(x) = H_X(y)\).

For such a function, we set for \(x \in X\),

\[ N(x) = B(x) \setminus O(x). \]

A component of \(B\) is called old for \(x\) if it is a component of \(O(x)\). On the other hand a component of \(N(x)\) is called new for \(x\).

Let us recall the history function which [CJS] use, see loc. cit. Lemma 3.7 and Lemma 3.14 and the part before: If we are the beginning of the resolution process or if the maximal value of the Hilbert-Samuel function just dropped to \(\nu\), then all boundary components are old, \(O(x) = B(x)\).

Let \(\pi: Z' \to Z\) be a blow-up with center \(D \subset X, x \in D\). Consider a point \(x' \in X'\) which is near to \(x\) \(\pi(x') = x\) and \(H_{X'}(x') = H_X(x) = \tilde{\nu}\). The boundary \(B'(x')\) on \(Z'\) consists of the strict transforms of the components in \(B(x)\) and the exceptional divisor of the last blow-up. The strict transforms of old components stay old, whereas the new exceptional divisor is defined to be new.

This continues for further blow-ups and near points; the strict transforms of old components remain to be old, and the new exceptional divisors and their strict transforms which arose after \(\tilde{\nu}\) appeared for the first time are defined to be new. As soon as the Hilbert-Samuel function drops, say to \(\nu' < \nu\), this process starts over and all the boundary components are old (since they appeared before \(\tilde{\nu}\) became a maximal value of \(H_X\)).

Note that in the notation of [CJS] Definition 3.15 we consider the complete transform \((B', O')\) of \((B, O)\).

**Definition 1.16.**

1. The log-Hilbert-Samuel function \(H^O_X: X \to \mathbb{N}^\times \mathbb{N}\) is defined by

\[ H^O_X(x) = (H_X(x), |O(x)|). \]

Here \(\mathbb{N}^\times \mathbb{N}\) is equipped with the lexicographical order (but still we use the product order on \(\mathbb{N}^\times \mathbb{N}\)).

2. For \(\tilde{\nu} \in \mathbb{N}^\times \mathbb{N}\) we define \(X(\geq \tilde{\nu}) = X^O(\geq \tilde{\nu}) := \{ x \in X \mid H^O_X(x) \geq \tilde{\nu}\}\) and \(X(\tilde{\nu}) = X^O(\tilde{\nu}) := \{ x \in X \mid H^O_X(x) = \tilde{\nu}\}\) and \(\Sigma^O_X := \{ H^O_X(x) \mid x \in X\}\) and \(\Sigma^o_{X,\text{max},0}\) denotes the set of maximal elements in \(\Sigma^O_X\) and \(X^O_{\text{max},0} := \bigcup_{\nu \in \Sigma^o_{X,\text{max},0}} X^O(\tilde{\nu})\).

3. In the local situation at a point \(x \in X(R = O_{Z,x}, J \subset R)\) defining ideal of \(X\) we set

\[ J^O := J \cdot I_{O(x)}, \]

where \(I_{O(x)}\) denotes the ideal defining the divisor given by the old components of \(B(x)\), i.e. \(I_{O(x)} = \langle \phi \rangle\) with \(\phi = \phi_1 \cdots \phi_d\) and the regular elements \(\phi_1, \ldots, \phi_d \in R\) are the generators of the old boundary components \(O(x)\).

We denote by \(\text{Dir}^O_x(X)\) the directrix associated to \(J^O\), by \(e^O_x(X)\) its dimension and by \(I(\text{Dir}^O_x(X))\) the corresponding ideal in the graded ring.

If we use the notation \(e(J) := e_x(X)\) in the local situation, then we have \(e(J^O) = e^O_x(X)\).

**Lemma 1.17.** Let the situation be as in the previous Definition part (3); \(J \subset R = O_{Z,x}\) locally defining \(X\) at \(x\) and \(\phi_1, \ldots, \phi_d \in R\) are the generators of the old boundary components \(O(x)\). Denote by \(M \subset R\) the maximal ideal of \(R\). Then we have

\[ I(\text{Dir}^O_x(X)) = I(\text{Dir}^O_x(X)) + \langle \text{in}_M(\phi_i) \mid 1 \leq i \leq d \rangle. \]

So \(\text{Dir}^O_x(X)\) (resp. \(e^O_x(X)\)) is exactly as the \(\text{Dir}^O_x(X)\) (resp. \(e^O_x(X)\)) as defined in [CJS] Definition 3.9.

**Proof.** Take any \(\Psi \in \text{In}_M(J^O)\) \((J^O = J \cdot I_{O(x)})\), then \(\Psi = \left( \prod_{1 \leq i \leq d} \Phi_i^{a(i)} \right) \cdot G\), with integers \(a(i) > 0\) and \(\Phi_i = \text{in}_M(\phi_i)\), for \(1 \leq i \leq d\), and \(G\) not divisible by any of the \(\Phi_i\). There exists a Hasse-Schmidt derivation \(D_i\) such that \(D_i(\Psi) = \Phi_i^{a(i)}\), so by a result of Giraud (see [G] Lemma 1.6 or [BHM] Corollary 2.3), \(\Phi_i^{a(i)} \in I(\text{Rid}^O_x(X))\), \(1 \leq i \leq d\). (Here \(\text{Rid}^O_x(X)\)
denotes the ridge associated to $J^O$, the latter is a generalization of the directrix, where the linear forms $(Y_1, \ldots, Y_s)$ in Definition 1.5 have to be replaced by additive polynomials $(\sigma_1, \ldots, \sigma_s)$, see also Remark 1.25). This leads to $\Phi_i \in I(\text{Dir}_x^O(X))$. The definition of the directrix ends the proof.

Lemma 1.18. Let $(R, M)$ be an excellent regular local Noetherian ring and $J \subset R$ a non-zero ideal. For an element $\varphi \in R$, we put $\mu_\varphi := \text{ord}_M(\varphi)$ and

$J^\varphi := J \cdot \langle \varphi \rangle$.

Let $(f) = (f_1, \ldots, f_m)$ be a standard basis of $J$ (Definition 1.6) and $\nu_i := \text{ord}_M(f_i)$, for $1 \leq i \leq m$. We have

1. $(\varphi \cdot f_1, \ldots, \varphi \cdot f_m)$ is a standard basis of $J^\varphi$, and
2. $\nu^*(J^\varphi, R) = (\mu_\varphi + \nu_1, \mu_\varphi + \nu_2, \ldots, \mu_\varphi + \nu_m, \infty, \infty, \ldots)$.

In the lemma $\varphi$ can be any element in $R$ and is not forced to be a monomial $\varphi_1 \cdots \varphi_d$ of regular elements in $R$.

If $\varphi$ is the monomial given by the old boundary components as above, then $J^\varphi = J^O$.

Proof. An element in $h \in J^\varphi$ can be written as

$h = A_1\varphi f_1 + \ldots + A_m\varphi f_m = \varphi \cdot (A_1f_1 + \ldots + A_mf_m)$

for certain $A_i \in R$, $1 \leq i \leq m$. Clearly, $A_1f_1 + \ldots + A_mf_m \in J$ and

$\text{in}_M(h) = \text{in}_M(\varphi) \cdot \text{in}_M(A_1f_1 + \ldots + A_mf_m) = \text{in}_M(J)^\varphi$

imply $\text{in}_M(J)^\varphi = (\text{in}_M(\varphi)) \cdot \text{in}_M(J) \subset \text{gr}_M(R)$. Now, we only have to go through the definitions of a standard basis and the $\nu^*$-invariant (both Definition 1.6) and obtain by using that $(f)$ is a standard basis of $J$ that $(\varphi \cdot f_1, \ldots, \varphi \cdot f_m)$ is a standard basis of $J^\varphi$ and $\nu^*(J^\varphi, R) = (\mu_\varphi + \nu_1, \mu_\varphi + \nu_2, \ldots, \mu_\varphi + \nu_m, \infty, \infty, \ldots)$. □

Remark 1.19. In fact, we even have that a center $D = V(p)$, with an ideal $p \subset R$, is permissible for $J^\varphi$ if and only if it is permissible for $J$ as well as $\varphi$.

Moreover, if the $\nu^*$-invariant of $J^\varphi$ drops after a permissible blow-up, then the $\nu^*$-invariant of $J$ or the one of $\varphi$ also drops, i.e. $\nu^*((J^\varphi')', R') < \nu^*(J^\varphi, R)$ implies $\nu^*(J', R') < \nu^*((\varphi')', R') < \nu^*(\varphi, R')$, where $R'$ is the local ring after the blow-up and $(\cdot)'$ denotes the strict transform of $(\cdot)$ in $R'$.

If $\varphi = \varphi_1 \cdots \varphi_d$ is product of regular elements $(\varphi_1, \ldots, \varphi_d)$ in $R$ which can be extended to a r.s.p. for $R$, then the system $(\varphi_1, \ldots, \varphi_d)$ can be extended to a system defining the directrix of $J^\varphi$.

This can all be seen by using an interpretation of these things in the language of idealistic exponents. (For an introduction to idealistic exponents we refer to the first sections of [Sc1], where also references to the original literature are given). The problem of lowering the $\nu^*$-invariant of $J$ (and thus the problem of lowering the Hilbert-Samuel function, see Theorem 1.10) is equivalent to the problem of resolving the idealistic exponent

$$E = (f_1, \nu_1) \cap \ldots \cap (f_m, \nu_m).$$

By passing from to $J^O = J \cdot I_{O(x)}$, $I_{O(x)} = \langle \varphi \rangle$, we consider the idealistic exponent

$$E^O = (\varphi \cdot f_1, \nu_1 + \mu_\varphi) \cap \ldots \cap (\varphi \cdot f_m, \nu_m + \mu_\varphi),$$

for which we have the following equivalence (roughly speaking, two idealistic exponents are equivalent if they undergo the same resolution process)

$$E^O \sim (f_1, \nu_1) \cap \ldots \cap (f_m, \nu_m) \cap (\varphi, \mu_\varphi) = E \cap (\varphi, \mu_\varphi).$$

From this one can deduce the first two statements of the remark.

If $\varphi = \varphi_1 \cdots \varphi_d$ with regular elements as above, then $(\varphi, \mu_\varphi) \sim (\varphi_1, 1) \cap \ldots \cap (\varphi_d, 1)$. (Note that $\mu_\varphi = \sum_{i=1}^d 1 = d$). This implies the third assertion.

The process of intersecting with old boundary components (or say exceptional divisors) is some well-known technique and is already used by others, e.g. in the proof for constructive resolution of singularities over fields of characteristic zero by Bierstone and Milman (see [Sc2] and in particular loc. cit. Construction 6.2, where this is explained in the setting of
idealistic exponents), or the proof for resolution of singularities in dimension three by Piltant and the first author in [CP2], or [CJS].

As we mentioned before we do not have a control on the old components; e.g. for $J = (y^p + u_1^p \cdot f(u_1, u_2))$, with some $f(u_1, u_2) \in R$, $D := V(y, u_1)$ is a permissible center, but $V(y + u_2^2)$ may appear as an old boundary component in which case $D$ is not $B$-permissible. Therefore we have also to consider the singularity defined by $J^O$. This corresponds to the process of transforming the Hilbert-Samuel locus of the beginning into a $B$-permissible center, i.e. in particular n.c. with the boundary. The aim is to lower the Hilbert-Samuel function or at least to make the number of appearing old boundary components strictly decrease. (In fact, this means we want to lower the $\nu'$-invariant of $J^O$). Thus the points to which have be considered are the following:

**Definition 1.20.** Let $\pi : Z' = Bl_D(Z) \to Z$ be a $B$-permissible blow-up of $Z$ with center $D \subset X$. Consider $x \in D$ and $x' \in \pi_Z^{-1}(x)$.

1. $x'$ is called $O$-near to $x$ if $H^O_X(x') = H^O_X(x)$. (Note that we denote the transform of the history function by $O$ and not by $O'$).

2. $x'$ is called very $O$-near to $x$ if it is $O$-near, very-near, and $e^O_{x'}(X') = e^O_X(X)_{k(x')} - \delta_{x'/x}$.

Clearly, $x'$ is $O$-near to $x$ if and only if $x'$ is near to $x$ and contained in the strict transform of all $B \in O(x)$.

The analogous results as in Theorem 1.12 and Theorem 1.10(1) and (2) are true if we replace all the definitions by log-definitions with the history function:

**Theorem 1.21 ([CJS] Theorem 3.18).** Let $x' \in \pi_Z^{-1}(x)$ be $O$-near to $x \in D$. Assume that we have either $\text{char}(k(x)) = 0$ or $\text{char}(k(x)) \geq \dim(X)/2 + 1$ ($k(x)$ denotes the residue field of $x$). Then

$$x' \in \mathbb{P}(\text{Div}^O_X(X)/T_x(D)) \subset \pi_Z^{-1}(x).$$

**Theorem 1.22 ([CJS] Theorem 3.23(1)).** Let $x \in D$ and $x' \in \pi_Z^{-1}(x)$. Suppose that either $\text{char}(k(x)) = 0$ or $\text{char}(k(x)) \geq \dim(X)/2 + 1$. If $x'$ is $O$-near and very near to $x$, then

$$e^O_{x'}(X') \leq e^O_X(X)_{k(x')} - \delta_{x'/x}.$$ 

After recalling the necessary definitions and result of [CJS] we can now define the first part of the invariant $t = (\iota_0, \iota_{\text{hs}}, \iota_{\text{poly}})$ and prove that it does not increase under permissible blow-ups:

**Definition 1.23 (First part of the invariant).** Let $x \in X \subset Z$ and $B$ be a boundary on $Z$. Then we define

$$\iota_0 := \iota_0(X, Z, x) := (H_X(x), |O_X(x)|, e_x(X), e^O_x(X)) = (H^O_X(x), e_x(X), e^O_x(X)).$$

Note that $\iota_0 \in \mathbb{N}^\mathbb{N} \times \{0, \ldots, \dim(Z)\} \times \{0, 1, \ldots, \dim(X)\} \times \{0, 1, \ldots, \dim(X)\} \subset \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$.

**Proposition 1.24.** Let $\pi : Z' = Bl_D(Z) \to Z$ be a $B$-permissible blow-up of $Z$ with center $D \subset X$. Suppose that either $\text{char}(k(x)) = 0$ or $\text{char}(k(x)) \geq \dim(X)/2 + 1$. Consider $x \in D$ and $x' \in \pi^{-1}(x)$. Then we have

$$\iota_0(X', Z', x') \leq \iota_0(X, Z, x).$$

**Proof.** This is an immediate consequence of Theorem 1.10 and Theorem 1.22. \qed

**Remark 1.25.** We remark that, in general, it might be important to take not only the dimension of the directrix but also the dimension of the ridge into account when constructing a first invariant for singularities. The latter is a generalization of the directrix such that Theorem 1.12 is true without the assumption on the characteristic of the residue field. The drawback of this is that the ridge is no more generated by linear forms but by additive polynomials. But in contrast to the dimension of the directrix the dimension of the ridge is upper semi-continuous, see Example 3.14.
2. Control on the old Hilbert-Samuel locus

Starting from this section we suppose that $X$ is reduced and of dimension at most two.

After a blow-up with center $D$ the interesting points above $x \in D$ are the $O$-near points, where the log-Hilbert-Samuel function did not drop. Since $\dim(X) = 2$ the condition of Theorem 1.21 on the characteristic of the residue field does always hold and the $O$-near points are characterized by this theorem. Therefore we get

**Observation 2.1** ([CJS] Proof of Theorem 5.28, Step 1 and Step 2). Let $X$ be an excellent scheme of dimension at most two embedded in a regular scheme $Z$ and let $\mathcal{B}$ be a boundary on $Z$. Let $\pi : Z' \to Z$ be a $\mathcal{B}$-permissible blow-up with center $D \subset X(\tilde{\nu})$, for some $\tilde{\nu} \in \Sigma^{\text{max},O}_X$. The center $D \subset X(\tilde{\nu})$ is either a closed point or $D$ is regular irreducible of dimension 1 and n.c. with $\mathcal{B}$.

If $D = x \in X$ is a closed point, then

$$D' := X'(\tilde{\nu}) \cap \pi^{-1}(x) \subset \mathbb{P}((\text{Dir}_x^O(X)) \cong \mathbb{P}^t_{k(x)},$$

for $t = e^O_x(X) - 1 \leq \dim(X) - 1 = 2 - 1 = 1$. If $t < 0$, then $\mathbb{P}^t_{k(x)} = \emptyset$ by convention. If $D'$ is empty, then there are no near points lying above $x$. If $D'$ is non-empty, then it is either a union of closed points or a projective line over $k(x)$. In the latter case it is the whole exceptional divisor of the blow-up and since the center was $\mathcal{B}$-permissible, we get that $D'$ is $\mathcal{B}'$-permissible.

Now suppose $D$ is regular irreducible of dimension 1 and n.c. with $\mathcal{B}$. Denote by $\eta$ its generic point and let $x \in D$ be a closed point. Then

$$X'(\tilde{\nu}) \cap \pi^{-1}(\eta) \subset \mathbb{P}((\text{Dir}_x^O(X)/T_x(D)) \cong \mathbb{P}^s_{k(\eta)},$$

for $s = e^O_x(X) - 2 \leq 0$. Moreover,

$$X'(\tilde{\nu}) \cap \pi^{-1}(\eta) \subset \mathbb{P}((\text{Dir}_x^O(X)) \cong \mathbb{P}^r_{k(\eta)},$$

for $r = e^O_x(X) - 1 \leq 0$. If $X'(\tilde{\nu}) \cap \pi^{-1}(\eta) \neq \emptyset$ then it consists of a unique point $\eta'$ with $k(\eta') \cong k(\eta)$. Thus $\pi$ induces an isomorphism

$$X'(\tilde{\nu}) \cap \pi^{-1}(D) =: D' \xrightarrow{\cong} D.$$

Therefore we get that $D'$ is either empty or a union of closed points or a $\mathcal{B}'$-permissible curve isomorphic to $D$.

Using this we can now state a local version of the strategy of [CJS]:

**Strategy 2.2.** Suppose we are at the beginning of our resolution process or the maximal value of the log-Hilbert-Samuel function just dropped after the last blow-up.

Let $x \in X$ such that $H^O_X(x) = \tilde{\nu} \in \Sigma^{\text{max},O}_X$ takes a maximal value of the log-Hilbert-Samuel function on $X$. Locally at $x$ the stratum $X(\tilde{\nu})$ is one of the following cases:

(I) a closed point $x \in X$: For trivial reason $x$ is $\mathcal{B}$-permissible. Since it is isolated the center of the upcoming blow-up $\pi$ will be $x$. If $X'(\tilde{\nu}) \cap \pi^{-1}(x) = \emptyset$, then the singularity improved locally and we start again with a new maximal value of $H^O_X$.

Suppose this is not the case. By the previous observation $X'(\tilde{\nu}) \cap \pi^{-1}(x)$ consists either of a union of closed points or it is a $\mathcal{B}$-permissible curve. This means locally there is a unique center for the next blow-up given. Either we are again in case (I) or we get to the case:

(II) a $\mathcal{B}$-permissible curve $C \subset X$: The curve $C$ is the biggest $\mathcal{B}$-permissible center, so we blow it up. If the singularity did not locally improve then there lies either a $\mathcal{B}'$-permissible curve $C'$ with $C' \cong C$ above $C$ in $X'(\tilde{\nu})$ or a union of closed points.

Again we come back to case (I) or (II).

(III) a family of irreducible curves $C_1, \ldots, C_t \subset X$ with $t \geq 1$ and if $t = 1$, then $C_1$ is not $\mathcal{B}$-permissible at $x$: Denote by $C$ the union of these curves. The aim is to make the strict transform of $C$ locally to a $\mathcal{B}$-permissible center. For this the choice of the center will be the closed point $x$ which is the intersection of all components of $C$. Since $\dim(C) = 1$, we may apply induction and after finitely many blow-ups of
closed points the strict transform of \( C \) becomes a \( B \)-permissible center and we blow it up. After that the strict transform of \( C \) is empty.

Nevertheless we are not done yet. The preparation of \( C \) may create new irreducible components in \( X'(\tilde{\nu}) \) which are not contained in the strict transform \( C' \) of \( C \). These are dealt with in:

\( (\text{III}^+) \) Newly created singularities: By Observation 2.1 the components in \( X'(\tilde{\nu}) \) which are created in \( (\text{III}) \) are either closed points or \( B \)-permissible curves. Thus the locus \( X'(\tilde{\nu}) \) of points in \( X' \) where the log-Hilbert-Samuel function attains the value \( \tilde{\nu} \) is locally either

(a) empty, which means the maximal value achieved by the log-Hilbert-Samuel function dropped and we start the strategy from the beginning,

(b) or \( X'(\tilde{\nu}) \) is locally a closed point which is treated as in \((I)\),

(c) or it is a family of finitely many \( B \)-permissible curves. If there is only one such curve we continue as in \((I)\).

But if there is more than one, we have to decide which to pick. By labelling the components in \( X'(\tilde{\nu}) \) by their year of birth [CJS] get a canonical candidate to choose for the next blow-up; namely, the \( B \)-permissible curve in \( X'(\tilde{\nu}) \) with the lowest label. In the next remark we precisely recall the procedure of labelling these components.

In \((\text{III})\) one first resolves the singularities of \( C \) and as soon as the strict transform of \( C \) is regular the following blow-ups aim to make it n.c. with the boundary components.

Remark 2.3 (Labels of the irreducible components of \( X(\tilde{\nu}) \)). In order to handle the new components in \( X(\tilde{\nu}) \) [CJS] label the irreducible component of \( X(\tilde{\nu}) \) as follows:

- At the beginning every irreducible component of \( C \) gets label zero.

- If the algorithm blows up an isolated closed point in \( X(\tilde{\nu}) \) with label \( j \in \mathbb{N} \), then all the irreducible components in \( X'(\tilde{\nu}) \) lying above \( x \) inherits the label \( j \) since they are dominating \( x \).

- If the center of the blow-up is a \( B \)-permissible curve \( C \) (as in \((\text{III})\)) with label \( j \in \mathbb{N} \), and suppose there exists a curve \( C' \subset X'(\tilde{\nu}) \) with \( C' \cong C \), then \( C' \) inherits label \( j \). If there is an isolated closed point lying above \( C \) in \( X(\tilde{\nu}) \), then the point can not dominate the whole irreducible component \( C \). Therefore we give these points a new label, namely the current year (number of steps) in the resolution process.

- In the preparation part \((\text{III})\) the center is strictly contained in a bigger irreducible component of \( X(\tilde{\nu}) \). Thus whatever is lying above the center it can not dominate a whole irreducible component of \( X(\tilde{\nu}) \) and the created components in \( X'(\tilde{\nu}) \) get in each step a new label as before.

This means once we have blown up the \( B \)-permissible strict transform of \( C \) the upcoming centers are uniquely determined by the lowest label. Note that these labels are local invariants. If we want to define them globally, then we have to look carefully at the global resolution process.

Note that in higher dimensions the components of lowest label do not necessarily give a \( B \)-permissible center. Already in dimension three there might appear singular curves in \( X'(\tilde{\nu}) \) after blowing up a closed point. Therefore the higher dimensional case is more delicate and so far it is not known if the procedure of [CJS] is finite or not.

Example 2.4. Consider the variety over a field \( k \) given by

\[
f = x^2 + y^5z^7 = 0.
\]

The maximal Hilbert-Samuel locus coincides with the locus of order two. There are two irreducible components \( V(x, y) \) and \( V(x, z) \). Since we are at the beginning of the resolution process both get label zero.

This is case \((\text{III})\) and we have to blow up the origin in order to separate these two components. In the \( Z \)-chart of the blow-up the strict transform is \( f' = x^2 + y^5z^{10} \). The
label of $V(x, y)$ is still zero and since $V(x, z)$ is lying on the exceptional divisor it gets the new label one. Hence the component of the maximal Hilbert-Samuel locus with minimal label and thus the center for the next blow-up is $V(x, y)$.

After blowing up we obtain in the $Y$-chart the singularity given by $f'' = x^2 + y^3z^{10}$. Again $V(x, y)$ lies in the singular locus and also on the exceptional divisor of the last blow-up. Moreover, it is dominating the center and thus has again label zero. So the center for the next blow-up following the strategy of [CJS] is $V(x, y)$.

If we skip the condition on the inheritance of the label if an irreducible component after a blow-up is dominating one before, then every new component in $X'(\tilde{v})$ would get a new label. This means in the previous example the center of the next blow-up would be $V(x, z)$ and not $V(x, y)$.

From a practical point of view this variant does not look nice, e.g. resolve the singularities of the variety given by $x^2 + y^3z^8 = 0$ defined over any field $k$ with each of the two variants of labelling.

**Remark 2.5 (CAUTION: Be careful with the different notions of old components!).** On the one hand, we have the old boundary components which do not necessarily have anything to do with the maximal Hilbert-Samuel locus. Their importance lies in the fact that the center has to be n.c. with the boundary.

On the other hand, we labelled the irreducible components of $X(\tilde{v})$. But as we have seen in the third step of the example: $V(x, y)$ is an irreducible component of the maximal Hilbert-Samuel locus with label zero but it is contained in the exceptional divisor of the last blow-up which is new.

Recall that we reduced our considerations from $x \in X$ to the local embedded situation $J \subset M \subset R$, where $(R, M, k(x) = R/M)$ is a (complete) regular local Noetherian excellent ring and $(0) \neq J \subset R$ the defining ideal of $X$ locally at $x$. Set $d = \dim(R)$.

Let $I_C \subset M \subset R$ be the ideal defining $C$. Since the dimension of $C$ is one its directrix $\text{Dir}_g(C)$ at a $x$ is at most of dimension one, $c_C(x) \geq 1$.

Moreover, denote by $I_{OC} \subset R$ the ideal defining the divisor given by the boundary components which are old with respect to the Hilbert-Samuel function $H_C$ of $C$. Then the ideal $I_C^O := I_C \cdot I_{OC}$ is also of interest for us. More precisely, it will appear when we have to consider the question if $C$ is intersecting the boundary transversally.

More generally, let $I \subset R$ be any non-zero ideal in $R$ such that its corresponding directrix has dimension $e$ at most one, for example $I = I_C$ or $I = I_C^O$. Let $(v, z)$ be a regular system of parameters (short r.s.p.) for $R$ such that the system $(z) = (z_1, \ldots, z_c), c = d - e$, yields the directrix of $I$ at the maximal ideal $M$. Note that $(v)$ is either empty or consists of only one element, which we also denote by $v$ in this case.

For $h \in R$ we have an expansion in a finite sum

\[ h = \sum_{(A, B) \in \mathbb{Z}_+^d} D_{A, B} v^A z^B \]

with coefficients $D_{A, B} \in R^\times \cup \{0\}$. Due to [HI] at the beginning of §2 there exist finite expansions such that the set of appearing exponents $\{(A, B) \mid D_{A, B} \neq 0\}$ is minimal and unique.

**Definition 2.6.**

1. For $h \in R$ with an expansion (2.1) we set $n = \ord_M(h)$ and define

\[ \delta(h; v; z) := \begin{cases} \min \left\{ \frac{A}{n - |B|} \mid D_{A, B} \neq 0 \wedge |B| < n \right\} \in \frac{1}{n!} \cdot \mathbb{N}_\infty, & \text{if } c = d - 1, \\ \infty, & \text{if } c = d. \end{cases} \]

(Here we use the notation $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ and $\frac{1}{n!} \cdot \mathbb{N} = \left\{ \frac{1}{n!} \cdot a \mid a \in \mathbb{N} \right\}$.)

2. Let $(g) = (g_1, \ldots, g_m)$ be a $(v)$-standard basis for $I$ (Definition 1.6). Then we define

\[ \delta(g; v; z) := \min \{ \delta(g; v; z) \mid 1 \leq i \leq m \} \]

\[ \delta(I; v) := \max_{(g)v} \{ \delta(g; v; z) \}, \]
where the maximum is taken over all \((v)\)-standard bases \((g)\) of \(I\) and all systems \((z)\) which determine the directrix of \(I\) at \(M\).

We use the notation \(\delta_C(x) := \delta(I_C; v)\) and \(\delta_C^2(x) := \delta(I^2_C; v)\) (in each case with an appropriate system \((v)\)).

By a theorem of Hironaka, which we recall in the next section (Theorem 3.2), the maximum \(\delta(I; v)\) exists and can be achieved by certain good choice for \((g)\) and \((z)\) in \(\tilde{R}\). This implies \(\delta(I; v) \in \frac{1}{t} \cdot \mathbb{N}_\infty\) where \(t\) is the maximal order at \(M\) achieved by an element of \((g)\).

In fact, if \((v)\) is non-empty, then the number \(\delta(I; v)\) is connected to Hironaka’s characteristic polyhedron and it is an invariant of the singularity, thus independent of the embedding, see Theorem 3.6.

**Notation 2.7.** Let \(\tilde{\nu} = H^O_X(x) \in \Sigma_{X}^{\text{max}, O}\). Suppose we are in the following situation of the resolution process:

\[
X' \xrightarrow{\pi} X \to \cdots \to X_\ast \xrightarrow{\pi} X_{\ast-1} \to \cdots \to X_0,
\]

where \(X_0\) is the scheme with which the resolution problem initially started. Let \(x \in X\), \(x' \in X'\), \(x = \pi(x')\) and suppose \(x\) is lying above \(x_\ast \in X_\ast\), \(x_{\ast-1} = \pi_\ast(x_\ast) \in X_{\ast-1}\) and

\[
H^O_{X'}(x') = H^O_X(x) = H^O_{X_\ast}(x_\ast) = \tilde{\nu} < H^O_{X_{\ast-1}}(x_{\ast-1}).
\]

(Note that the decrease may in particular happen if we the strict transform of an old boundary component is empty at \(x_\ast\)). This means \(\tilde{\nu}\) locally first appeared at \(x_\ast\) as a maximal value of the log-Hilbert-Samuel function.

Denote \(C_\ast = X_\ast(\tilde{\nu}) \subset X_\ast\) and let \(C\) (resp. \(C'\)) be its strict transform in \(X\) (resp. \(X'\)).

By definition every component in the boundary \(B_\ast\) of \(X_\ast\) is old and all the cases (I) to (III) of Strategy 2.2 may appear for \(C_\ast\). By Observation 2.1 the irreducible components appearing in the Hilbert Samuel locus \(X(\tilde{\nu})\) which arose due to blow-ups after \(X_\ast\) are already of a good shape. More precisely, they are \(B\)-permissible and thus any of them could be used as a center without any further preparation. (In general, this is not true if the dimension of \(X\) is greater or equal to three!). Moreover, if two of them intersect, then the labels which \([CJS]\) assign to them, can distinguish them. Before measuring the improvement in these good cases (which we do in the next section), we have to get a control for case (III), where \(C_\ast\) is a family of curves or only one curve, but which is not \(B\)-permissible. For this we introduce the second part of our invariant \(\iota\).

**Definition 2.8 (Control on the old Hilbert-Samuel locus).** Let \(x \in X \subset Z\) and \(B\) be a boundary on \(Z\). Suppose \(x\) takes a maximal value \(\tilde{\nu} \in \Sigma_{X}^{\text{max}, O}\) of the log-Hilbert-Samuel function on \(X\), \(H^O_{X}(x) = \tilde{\nu}\). Let \(C \subset X(\tilde{\nu})\) be as in Notation 2.7 (i.e. \(C\) is the strict transform of the Hilbert-Samuel locus \(C_\ast = X_\ast(\tilde{\nu})\) when the maximal value \(\nu\) first appeared) and suppose \(x \in C\).

If we are in case (II) or (III), i.e. \(C\) has dimension one, then we define

\[
\iota_{hs} := \iota_{hs}(X, Z, x) := (H_C(x), |O_C(x)|, e_C(x), e^C_C(x), \delta_C(x), \delta^2_C(x)) = (H^O_C(x), e_C(x), e^O_C(x), \delta_C(x), \delta^2_C(x) = (\iota_{hs}(C, Z, x), \delta_C(x), \delta^2_C(x)).
\]

Note that \(\iota_{hs} \in \mathbb{N}^3 \times \{0, \ldots, n\} \times \{0, 1\} \times \{0, 1\} \times \frac{1}{t} \cdot \mathbb{N}_\infty \times \frac{1}{t'} \cdot \mathbb{N}_\infty \subset \mathbb{N}^3 \times \mathbb{N}^3 \times \mathbb{Q}^2_\infty\) for \(n = \dim(Z)\) and certain fixed integers \(t, t' \in \mathbb{N}\) and \(\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}\).

Otherwise (case (I) or (II’)), we set

\[
\iota_{hs}(X, Z, x) := (0, \ldots, 0, 0, 0, 0, 0, 0) \in \mathbb{N}^3 \times \mathbb{N}^3 \times \mathbb{Q}^2_\infty.
\]

**Remark 2.9.** (1) In fact, in the second part of the definition we could have simply set \(\iota_{hs}(X, Z, x) = 0\), because all it telling us is that we are not in case (II) or (III). But in order to make things comparable we need to have an element in \(\mathbb{N}^3 \times \mathbb{N}^3 \times \mathbb{Q}^2_\infty\).

(2) In \(\iota_{hs}(X, Z, x)\) the number \(\delta_C(x)\) measures how far \(C\) is away from being regular at \(x\) and, as soon as this the case, \(\delta^2_C(x)\) measures how far \(C\) is away from being transversal to the old boundary components.
(3) In general, it might happen that $\delta_C(x) = \delta_O^2(x) = \infty$. But, in fact, these are good cases. As we explained before there exist a $(u)$-standard basis $(g) = (g_1, \ldots, g_m)$ of $I_C = I_C \cdot \hat{R}$ and regular elements $(z) = (z_1, \ldots, z_\nu)$ in $\hat{R}$ such that the $\delta_C(x)$ is achieved with these data. Then $\delta_C(x) = \infty$ implies $g_i \in (z)^{\nu_i}$, $n_i = ord_M(g_i)$, for every $i \in \{1, \ldots, m\}$. Thus $V(z) \subset X(\hat{\nu})$ and it is a permissible center.

Since we are in dimension two the irreducible components appearing in $X(\hat{\nu})$ are either contained in $V(z)$ or tangent to it. In the latter case one can show that we must have $\delta_C(x) < \infty$. Thus $V(z) = X(\hat{\nu})$. (By [CJS] Lemma 1.37(2) we obtain then that there exist regular elements $(y) = (y_1, \ldots, y_\nu)$ in $R$ such that $(y) : \hat{R} = (z)$ and thus we can choose $(z) \in R$).

But $V(z)$ is maybe not $B$-permissible (e.g. if $V(z + v^2)$ is an old boundary component). Then we have to consider $\delta_O^2(x)$. If $\delta_C(x) = \infty$, then we have as above $X^O(\hat{\nu}) = V(z^O)$, where $(z^O) = (z_1^O, \ldots, z_\nu^O)$ correspond to the elements giving the directrix of $I_C^O := I_C \cdot O_C$, where $I_{O_C} \subset R$ the ideal of the divisor defined by the boundary components which are old with respect to $H_C$. Moreover, $V(z^O)$ is a $B$-permissible center and thus we can not be in case (III). In particular, in the definition of $u_h(X, Z, x)$ for (III) it then follows that $\delta_C(x) = \delta_O^2(x) = \infty$ never occurs.

If $\delta_C(x) = \delta_O^2(x) = \infty$, then [CJS] choose $V(z^O)$ as the next center and after the blow-up $H_O^2$ strictly decreases at every point lying above the center.

**Proposition 2.10.** Let $\pi : Z' \to Z$ be a $B$-permissible blow-up with center $D \subset X$ following the [CJS]-strategy. Consider $x \in D$ and $x' \in \pi^{-1}(x)$ and assume $i_0(X', Z', x') = i_0(X, Z, x)$. Then

1. $\; \; u_{h^O}(X', Z', x') \leq u_h(X, Z, x)$, and
2. $\; \; \text{the inequality is strict if } u_h(X, Z, x) \neq (0, \ldots, 0, 0, 0, 0)$.

**Proof.** Recall Notation 2.7. In case (I) and (III$^+$) the assertion is trivial because by definition $\; \; u_{h^O}(X', Z', x') = u_h(X, Z, x) = (0, \ldots, 0, 0, 0, 0)$. Moreover, in case (II) we have that $C$ is $B$-permissible and thus $D = C$. This implies $C' = \emptyset$, i.e. we are in case (III$^+$), and $u_{h^O}(X', Z', x') = (0, \ldots, 0, 0, 0, 0) < u_h(X, Z, x)$.

Therefore we may assume that $C$ is a family of irreducible curves or a single curve which is not $B$-permissible. By Proposition 1.24 we have

$\; \; i_0(C', Z', x') = (H_{C'}(x'), [O_{C'}(x')], e_{C'}(x'), e_{O^2}(x')) \leq i_0(C, Z, x)$. If the inequality is strict, then we are done. Hence suppose $i_0(C'(X', Z', x')) = i_0(C, Z, x)$. Let $(v, z)$ be a r.s.p. for $R = O_{Z, x}$ such that the system $(z) = (z_1, \ldots, z_\nu), c = d - e$, yields the directrix of $C$ at $x$ where $d = \dim(R)$ and $e = e_C(x) \in \{0, 1\}$. Denote by $I_{C'} \subset R$ the ideal defining $C$ locally at $x$.

If $c = d$, then by Theorem 1.12 there are no near points which means $H_{C'}(x') < H_C(x)$. Therefore we may assume $c = d - 1$ and $(v)$ is a non-empty system consisting of one element, also denoted by $v$. Theorem 1.12 implies that the only point where the equality may happen is the origin $x'$ of the $V$-chart. Note that if $\delta_C(x')$ (resp. $\delta_C^2(x')$) is maximal for given generators and parameters, then $\delta_C(x')$ (resp. $\delta_C^2(x')$) is automatically maximal for the transforms of the given data since $x'$ is the origin of a chart.

Let $h \in I_C$ and set $n = ord_M(h)$. Consider an expansion as in (2.1),

$$h = \sum_{(A, B) \in \mathbb{Z}_{\geq 0}^d} D_{A, B} v^A z^B.$$  

Since the center of the blow-up is the origin and since $x'$ corresponds to the origin of the $V$-chart, we get that the strict transform of $h$ is given by

$$h' = \sum_{(A, B) \in \mathbb{Z}_{\geq 0}^d} D_{A, B} (v')^{A+B-n} (z')^B,$$

where $(v', z')$ is an r.s.p. of $O_{Z', x'}$. We observe that $\frac{A+B-n}{n-|B|} = \frac{A}{n-|B|} - 1$ and from this we obtain

$$\delta(h'; v'; z') = \delta(h; v; z) - 1.$$
In particular, this implies $\delta_C(x') = \delta_C(x) - 1$ if $\delta_C(x) \geq 2$. If $\delta_C(x) < 2$, then $\delta_C(x) - 1 < 1$ which means we must have $H'_C(x') < H_C(x)$. (A reference for the last implication on the decrease of the Hilbert-Samuel function is [CJS] Theorem 8.6). In both cases $t_{hs}(X', Z', x') < t_{hs}(X, Z, x)$.

Hence we may assume without loss of generality that $C$ is already regular. Since we are in case (III) of Strategy 2.2 $C$ is not B-permissible (otherwise we are in case (II)). This means $C$ is not intersecting the boundary transversally (e.g. $C = V(z)$ and $B = \{V(z + v')\}$). This only can happen for boundary components which are old with respect to the value $H_C(x)$.

By repeating the above argument for the ideal $I^2_{O} = I_C \cdot I_{O_C} \subset R$ instead of $I_C$ ($I_{O_C} \subset R$ the ideal of the divisor given by the boundary components which are old with respect to $H_C$) implies $\delta_{C}^{O}(x') = \delta_{C}^{O}(x) - 1$ if $\delta_{C}^{O}(x) \geq 2$, or if $\delta_{C}^{O}(x) < 2$ then $H_{C}^{O}(x') < H_{C}^{O}(x)$. Moreover, recall that we explained in the previous remark that $\delta_C(x) = \delta_{C}^{O}(x) = \infty$ can not occur in (III). This finally concludes the proof of the proposition.

\[\square\]

**Remark 2.11.**

1. Suppose $\dim(X) = 1$. Then $t_{hs}(X, Z, x)$ is a local invariant for $X$ which strictly decreases in every step of the strategy of [CJS].

2. In principle $t_{hs}$ is defined in any dimension but then we lose the good control on what is above the center of the blow-up.

Moreover, note that in dimension bigger than two the assumptions on the residue field characteristic of Theorem 1.12, Theorem 1.21 and Proposition 1.24 are not necessarily valid. Thus a closer study of the locus of near points has to be done unless one is working in the situation over a perfect field.

In order to get in dimension two control on the newly created irreducible components in $X(\bar{\nu})$ we use in next section Hironaka’s characteristic polyhedron and define $t_{poly}$.

### 3. Refinement via Hironaka’s Characteristic Polyhedron

By passing to the completion of the local ring and using Cohen structure theory we can reduce to the following situation:

Let $(R, M, k = R/M)$ be a (complete) regular local ring with maximal ideal $M$ and let $(0) \neq I \subset R$ be an ideal.

For this situation Hironaka introduced characteristic polyhedra which reflect the more refined nature of the singularities [H1]. (See also [CJS] section 7 or [CSc]).

The characteristic polyhedron in which we are interested is the one associated to the ideal $I = J^{O} = J \cdot I_{O}$, where $J \subset R$ is the defining ideal of $X$ at some point $x$ and $I_{O} := I_{O(x)}$ denotes the ideal defining the divisor given by the old boundary components of $B(x)$.

Let $(u, y) = (u_1, \ldots, u_d; y_1, \ldots, y_r)$ a r.s.p. of $R$. In fact, Hironaka defined the characteristic polyhedron in a slightly more general setting, but for us the interesting case is if the system $(y)$ is chosen such that their initial forms $Y_{j} := \text{in}_{M}(y_{j}) = y_{j}$ mod $M_{2}$ generate the ideal of the directrix $\text{Dir}_{M}(I)$ of $I$ at $M$ (Definition 1.5). Therefore we restrict our attention to this case.

**Definition 3.1.**

1. A $F$-subset of $\mathbb{R}_{\geq 0}$ is a closed convex subset $\Delta \subset \mathbb{R}_{\geq 0}$ such that $v \in \Delta$ implies $v + w \in \Delta$ for every $w \in \mathbb{R}_{\geq 0}$.

2. Let $g \in R$ be an element in $R$ with $g \notin \langle u \rangle$. Then we can expand $g$ as in (2.1) into a finite sum

\[g = \sum_{(A, B) \in \mathbb{Z}_{\geq 0}^{d+r}} C_{A,B} u^{A} y^{B}\]

with coefficients $C_{A,B} \in R^{x} \cup \{0\}$. Denote by $n = \text{ord}_{I_{M}}(g)$. The polyhedron associated to $(g, u, y)$ is then defined as the smallest $F$-subset $\Delta(g; u; y)$ containing the points

\[\bigg\{ \frac{A}{n - |B|} \bigg| C_{A,B} \neq 0 \land |B| < n \bigg\}.\]
(3) Let \((f) = (f_1, \ldots, f_m)\) be a system of elements in \(R\) with \(f_i \notin \langle u \rangle\). Then the polyhedron \(\Delta(f; u; y)\) associated to \((f, u, y)\) is defined to be the smallest \(F\)-subset containing \(\bigcup_{i=1}^m \Delta(f_i; u; y)\).

(4) For an ideal \(I \subset R\) we set:
\[
\Delta(I; u; y) := \bigcap \Delta(f; u; y),
\]
where the intersection runs over all possible standard bases \((f) = (f_1, \ldots, f_m)\) of \(I\) (Definition 1.6).

Finally, the characteristic polyhedron of \((I; u)\) is defined by
\[
\Delta(I; u) := \bigcap \Delta(I; u; y),
\]
where the intersection ranges over all systems \((y)\) extending \((u)\) to a r.s.p. of \(R\) and such that their initial forms generate the directrix \(\text{Diff}_M(I)\).

In general, one has to consider so-called \((u)\)-standard bases ([CJS] Definition 6.7) but since we are only dealing with the directrix of \(I\) any standard-basis is already a \((u)\)-standard basis.

By introducing the procedure of vertex preparation Hironaka was able to prove the following result. (Again this is also valid in the general case, but we state it here in our special case).

**Theorem 3.2 ([HI] Theorem (4.8)).** Let \(I \subset R\) be a non-zero ideal and \((u, y)\) be a r.s.p. for \(R\) such that \((y)\) yields the ideal generating the directrix of \(I\).

Then there exists a standard basis \((\tilde{f}) = (\tilde{f}_1, \ldots, \tilde{f}_m)\) of \(\tilde{I} = I \cdot \tilde{R}\) and a system of elements \((\tilde{y}) = (\tilde{y}_1, \ldots, \tilde{y}_r)\) in \(\tilde{R}\) such that \((u, \tilde{y})\) is a r.s.p. of \(\tilde{R}\), \((\tilde{y})\) determines the directrix of \(I\), and
\[
\Delta(\tilde{f}; u; \tilde{y}) = \Delta(I; u).
\]

In [CPI] it is shown that in the case \(R\) a \(G\)-ring, \(m = 1\), and \(r = 1\) one can find coordinates in \(R\) such that the characteristic polyhedron is achieved \(\Delta(f; u; y) = \Delta(I; u)\), \(I = (f)\). Further the authors studied under which conditions this result can be generalized to arbitrary ideals \(I\) [CSc].

In [Sc1] the second author introduced a variant of the characteristic exponent for idealistic exponents and used them in [Sc2] to show that the invariant of Bierstone and Milman for resolution of singularities in characteristic zero can be purely determined by these polyhedra.

**Observation 3.3 (The case \(e^O_{\infty}(X) \leq 1\)).** Suppose \(e^O_{\infty}(X) \leq 1\). Then the situation is similar to the case of curves. First, let us note that \(e^O_{\infty}(X) \leq e_x(X)\).

If \(e^O_{\infty}(X) = 0\), then Theorem 1.21 implies that there are no \(O\)-near points, i.e. \(H^O_X(x') < H^O_X(x)\) for every point \(x' \in \pi^{-1}(x)\) lying above \(x \in D\), where \(\pi : Z' \to Z\) is a blow-up with center \(D \subset X\) following the strategy of [CJS].

If \(e^O_{\infty}(X) = 1\), then the characteristic polyhedron of \(J^O = J \cdot I_O\) is 1-dimensional and as in the previous section for \(C\) we consider
\[
\delta^O_{\infty}(x) := \delta(J^O; u) \quad \text{(Definition 2.6)}.
\]

Recall that \(\delta^O_{\infty}(x) \in \frac{1}{t} \cdot \mathbb{N}_\infty\) for some fixed \(t \in \mathbb{N}\). By Remark 2.9(3) \(\delta^O_{\infty}(x) = \infty\) implies that there is a \(B\)-permissible curve which is also the center of the next blow-up and as in the case \(e^O_{\infty}(X) = 0\) it follows that there are no \(O\)-near points.

As in the proof of Proposition 2.10 we obtain \(\delta^O_{\infty}(x') = \delta^O_{\infty}(x) - 1\) for every point \(x'\) which is \(O\)-near to \(x\). Therefore we set in this case
\[
\epsilon_{\text{poly}} := \epsilon_{\text{poly}}(X, Z, x) := \begin{cases} 
(0, 0, 0, 0), & \text{if } e^O_{\infty}(X) = 0, \\
(0, 0, 0, \delta^O_{\infty}(x)), & \text{if } e^O_{\infty}(X) = 1.
\end{cases}
\]

(In fact, it would suffice to take the values 0 resp. \(\delta^O_{\infty}(x)\) instead of \((0, 0, 0, 0)\) resp. \((0, 0, 0, \delta^O_{\infty}(x))\), but in order to make the invariant comparable with the one in the case \(e^O_{\infty}(X) = 2\) we need to define an element in \(\mathbb{Q}^4_{\infty}\)).
The previous argument together with Proposition 1.24 and Proposition 2.10 imply Theorem A in the case $e^O_2(X) \leq 1$. Hence it remains to consider the case $e^O_2(X) = e_2(X) = 2$.

**Definition 3.4 (Key ingredient for $t_{\text{poly}}$ in the case $e^O_2(X) = e_2(X) = 2$).** Let $\Delta \subset \mathbb{R}^2_{\geq 0}$ be a $F$-subset with finitely many vertices 
$$v^{(1)}, v^{(2)}, \ldots, v^{(t)} \in \mathbb{R}^2_{\geq 0}.$$ 
Suppose the vertices are ordered by the lexicographical order of the coordinates $v = (v_1, v_2)$. (In fact, since $\Delta$ is living in dimension two it suffices to order them with respect to their first entry, i.e. $v_1^{(1)} < v_1^{(2)} < \ldots < v_1^{(t)}$; this yields the same ordering). Then we define

1. $\delta(\Delta) := \inf\{v_1 + v_2 \mid v = (v_1, v_2) \in \Delta\}$
2. $\alpha_1(\Delta) := \inf\{v_1 \mid v = (v_1, v_2) \in \Delta\}$
3. $\beta_1(\Delta) := \inf\{v_2 \mid v = (\alpha_1(\Delta), v_2) \in \Delta\}$
4. $\gamma_1(\Delta) := \sup\{v_2 \mid (\delta(\Delta) - v_2, v_2) \in \Delta\}$

Note that $(\alpha_1(\Delta), \beta_1(\Delta)) = v^{(1)}$ and $(\delta(\Delta) - \gamma_1(\Delta), \gamma_1(\Delta))$ are vertices of $\Delta$. Pick $i \in \{1, \ldots, t\}$ such that $v^{(i)} = (\delta(\Delta) - \gamma_1(\Delta), \gamma_1(\Delta))$.

If $i = t$ then we put $s_1(\Delta) := \infty$.

Analogously we define $\alpha_2(\Delta), \beta_2(\Delta), \gamma_2(\Delta)$, and $s_2(\Delta)$ by interchanging the role of $v_1$ and $v_2$.

For $\Delta = \Delta(I; u)$ one sees easily that $s_1(\Delta(I; u))$ depends on the choice of the coordinates. In particular, it depends on the choice of $u_2$ (e.g. $I = (y^2 + u_1^2 u_2 - y^2 + u_1^2 v_2 - u_1^2)$ for $v_2 = u_2 + u_1^2$). Therefore we sometimes also write $s_1(\Delta(I; u))_{(u_1, u_2)}$ or $s_1(\Delta(I; u))_{(u, y)}$ if we want to indicate this dependence.

Before defining $t_{\text{poly}}$ for the remaining cases we show some nice property of the above numbers if $\Delta = \Delta(I; u)$ is Hironaka’s characteristic polyhedron. In this case all the vertices have rational coordinates and thus $\delta(\Delta(I; u))$, $\alpha_1(\Delta(I; u))$, $\beta_1(\Delta(I; u))$, $\gamma_1(\Delta(I; u))$, $s_1(\Delta(I; u)) \in \mathbb{Q}_\infty$ are rational numbers or infinity, where the latter case may only happen if the characteristic polyhedron has a very special shape (e.g. if $\delta(\Delta(I; u)) = \infty$, then $\Delta(I; u) = \emptyset$). In order to state the theorem we need the following notion:

**Definition 3.5.** Let $I \subset R$ be a non-zero ideal in an excellent Noetherian regular local ring $R$ and let $(u_1, y) = (u_1, u_2; y_1, \ldots, y_r)$ be a r.s.p. for $R$ such that $(y)$ yields the directrix of $I$.

Set $\delta := \delta(\Delta(I, u))$. For $\lambda \in \mathbb{Q}_{\geq 0}$ we define

$$\mathcal{I}_\lambda := \left\{ u^A y^B \mid |B| + \frac{|A|}{\delta} \geq \lambda \right\},$$

which yields a graded ring which we denote by $grs(R)$. (Note that $\delta$ is a positive number). If we denote the images of $u_i$ and $y_j$ by the corresponding capital letters, then we have $grs(R) \cong k[U, Y]$, where $k = R/M$ is the residue field of $R$.

Moreover, we define, for $g = \sum C_{A,B} u^A y^B$ (finite sum with $C_{A,B} \in R^X \cup \{0\}$ as in (2.1)),

$$v_\delta(g) := \min\{\lambda \in \mathbb{Q} \mid g \in \mathcal{I}_\lambda\},$$

$$in_\delta(g) := \sum_{|B| + \frac{|A|}{\delta} = v_\delta(g)} C_{A,B} U^A Y^B,$$

$$In_\delta(I) := \{in_\delta(g) \mid g \in I\},$$

where the sum in the second line ranges over those $(A, B) \in \mathbb{Z}_{\geq 0}^2 \times \mathbb{Z}_{\geq 0}$ fulfilling $|B| + \frac{|A|}{\delta} = v_\delta(g)$ and $C_{A,B} = C_{A,B} \mod M \in k$.

**Theorem 3.6.** Let $I \subset R$ be a non-zero ideal in an excellent Noetherian regular local ring $R$ and let $(u_1, y) = (u_1, u_2; y_1, \ldots, y_r)$ be a r.s.p. for $R$ such that $(y)$ yields the directrix of $I$. Suppose $V(u_1)$ defines a boundary component (i.e. the variable $u_1$ is fixed).
(1) The number $\delta(\Delta(I,u)) > 1$ is an invariant of the singularity $R/I$. Therefore the same is true for the quasi-homogeneous ideal $In\sigma(I) \subset gr_{\delta}(R)$.

(2) Moreover, also the numbers $\alpha(I,\Delta(I,u)), \beta(I,\Delta(I,u))$ and $\gamma(I,\Delta(I,u))$ are invariants of the singularity $R/I$.

In particular, these notions are independent of the embedding.

The result for $\alpha(I,\Delta(I,u))$ and $\beta(I,\Delta(I,u))$ is an unpublished result due to Jannsen and the first author.

Proof. Set $\delta := \delta(I;u)$, $\alpha := \alpha(I;u)$, $\beta := \beta(I;u)$ and $\gamma := \gamma(I;u)$. Let $(y)$ be such that $\Delta(I;u,y) = \Delta(I;u)$. Since $(y)$ yields the directrix of $I$ we get $\delta > 1$.

The invariance of $\delta$ can be seen by using Hironaka’s trick: Let $a,b \in \mathbb{N}$ such that $(a,b) = 1$ and $\delta = \frac{a}{b}$. Since $\delta > 1$, we have $a > b$. We introduce a new variable, say $t$, and blow-up $b$ times the origin, where we consider each time the origin of the $T$-chart. By computing the transformation of an element in $I$ under these blow-ups, we see that $V(y,t)$ becomes permissible. Moreover, we obtain from this that we can blow up $a-b > 0$ times with center $V(y,t)$ and look each time at the origin of the $T$-charts. After that $V(y,t)$ is no more permissible. Hence we have constructed a sequence of permissible blow-ups from which we can recover $a$ and $b$ and thus $\delta$ if $\delta < \infty$. In particular, the sequence is finite if $\delta < \infty$. Thus if we can construct the sequence such that its second part is not finite, then this characterizes the case $\delta = \infty$. Therefore $\delta$ is independent of the embedding and an invariant of the singularity itself. This implies the same for $In\sigma(I) \subset gr_{\delta}(R)$.

If $\alpha \geq 1$, then $V(u_1,y)$ is a permissible center and we denote its generic point by $\eta$. For $\alpha > 1$, we have $e\eta(I) = 1$ and

$$\alpha = \delta(\Delta(I_0;u)),$$

(This is not true if $V(u_1,y)$ is not permissible, for example consider $I = (y^n + u^n_1 - u^3_2)$ for an integer $n$.) Thus $\alpha$ is an invariant of the singularity.

In the case $0 < \alpha \leq 1$ we introduce a new variable $T$ and pass from $R$ to

$$S := (R[T]/(T^d-u_1))_{(u,y,T)},$$

where we choose $d \in \mathbb{Z}_{>0}$ large enough such that $\alpha_S := \alpha(\Delta(I_S;T,u_2)) > 1$, $I_S = I \cdot S$. Note that a point $(v_1,v_2) \in \Delta(I,u)$ corresponds to $(d \cdot v_1,v_2) \in \Delta(I_S;T,u_2)$.

Thus $\alpha_S = d \cdot \alpha > 1$ and the previous result implies that $\alpha$ is an invariant of the singularity.

Since the non-negative number $\alpha$ is an invariant whenever $\alpha > 0$ it also has to be an invariant of the singularity if $\alpha = 0$.

Let us now come to the invariance of $\beta$: If $\alpha > 1$, then set $I := In\sigma(I) \subset R/(y,u_1)[Y,U_1]$. (Recall that $\alpha = \delta(I_0;u)$.) Let $S'$ be the localization of $R/(y,u_1)[U_1,Y]$ at the ideal $(\overline{u_2},U_1,Y)$, where $\overline{u_2}$ denotes the image of $u_2$ in $R/(y,u_1)$. For $I' = I \cdot S'$, we have

$$\alpha + \beta = \delta(\Delta(I';\overline{u_2},U_1))$$

and the invariance of $\alpha$ and $\delta$ implies that of $\beta$.

In the case $0 < \alpha \leq 1$ we pass to $S$ as in (3.2) for $d$ large enough such that $\alpha_S > 1$. From the previous we can then deduce that $\beta$ is an invariant of the singularity.

If $\alpha = 0$, then we must have $\beta > 1$ (otherwise $(y)$ does not determine the directrix of $I$). We set $\tilde{R} = R/(u_1)$, $\tilde{I} = I \cdot \tilde{R}$ and denote by $\overline{u_2}$ the image of $u_2$ in $\tilde{R}$. We get

$$\beta = \delta(\Delta(\tilde{I};\overline{u_2})).$$

This implies the invariance of $\beta$ in the remaining case.

For $\gamma$, consider $I_0 := In\sigma(I)$ (Definition 3.5). We set $S_0 := gr_{\delta}(R,U,Y)$ and $I_0 := I \cdot S_0$. We obtain that $\gamma$ is also an invariant of the singularity since

$$\gamma = \beta(\Delta(I_0;U)).$$

$\square$
Remark 3.7. (1) Since the numbers $\delta(R/I) := \delta(\Delta(I, u))$, $\alpha_1(R/I) := \alpha_1(\Delta(I, u))$, $\beta_1(R/I) := \beta_1(\Delta(I, u))$ and $\gamma_1(R/I) := \gamma_1(\Delta(I, u))$ are invariants of the singularity $R/I$, any process of maximizing or minimizing them with respect to the choice of $(u)$ can be skipped.

(2) The ideal $I_{\delta}(I) \subset gr(\delta(R))$ defines a quasi-homogeneous tangent cone which is an invariant of the singularity $R/I$ and which is a refinement of the usual tangent cone given by the initial forms with respect to the maximal ideal. As we have seen in the proof above this carries crucial information on the singularity. Moreover, it appears also in other works, e.g. it is used in the proof for resolution of threefold singularities by Piltant and the first author [CP2], and it is also hidden in the constructive proof for resolution of singularities over fields of characteristic zero, see [Sc2] Remark 5.6.

Definition 3.8 (Refinement for $e^O_\infty(X) = e_\infty(X) = 2$). Let $x \in X \subset Z$ and $B$ be a boundary on $Z$. The definition splits into three cases. In the following we abbreviate $\star^O = \star(\Delta(J^O; u))$.

(1) If there is no boundary component, $|B(x)| = 0$ (i.e. we are at the beginning of the process), then

$$t_{\text{poly}} := t_{\text{poly}}(X, Z, x) := (\infty, \infty, \infty, \infty).$$

(2) If there is one boundary component, $|B(x)| = 1$, given by $V(u_1)$ then

$$t_{\text{poly}} := t_{\text{poly}}(X, Z, x) := (\beta^O_1, \gamma^O_1, u_1 O_{s_1(u_1)}, \alpha^O_1),$$

where the supremum in the third entry it taken over all possible choices for $u_2$.

(3) If there are two boundary components, $|B(x)| = 2$, given by $V(u_1)$ and $V(u_2)$, then

$$t_{\text{poly}} := t_{\text{poly}}(X, Z, x) := \inf_{\geq t_{\text{hs}}} \{ (\beta^O_1, \gamma^O_1, s_1^{O_{s_1(u_1, u_2)}}, \alpha^O_1) \mid i \in \{1, 2\} \}.$$
Suppose there exists no boundary component passing through $x$, i.e. $|B(x)| = 0$. Then $t_{\text{poly}}(X', x') < t_{\text{poly}}(X, x) = (\infty, \infty, \infty, \infty)$ since $x'$ is contained in the exceptional divisor, $|B(x')| = 1$. This means that without loss of generality $|B(x)| \geq 1$ and hence we may assume that $V(u)$ is a boundary component.

Further, we already know by Proposition 2.10(2) that $t_{\text{ls}}$ drops strictly as long as the strict transform of the old part $C$ of the Hilbert–Samuel locus is not $B$-permissible. Therefore it remains to consider the cases that the center of the blow-up is a whole irreducible component of the Hilbert–Samuel locus, i.e. it is either an irreducible curve or an isolated closed point.

Since $\dim(X) = 2$ Theorem 1.21 implies that the near points are contained in the projective space associated to the directrix. Thus they do not lie in the $Y$-charts of the blow-up.

Note that if the associated polyhedron is minimal (i.e. coincides with the characteristic polyhedron $\Delta(J^O;u)$) for given generators and parameters, then at the origin of a chart of the blow-up the same is true for the polyhedron in the transformed data.

If the center is an irreducible curve, say $V(y, u_1)$ (resp. $V(y, u_2)$), then the $O$-near points can only be the origin of the $U_1$-chart (resp. the origin of the $U_2$-chart). An easy computation shows that the point $(v_1, v_2) \in \Delta(J^O;u)$ is translated to $(v_1 - 1, v_2)$ under the blow-up in $V(u_1, y)$ (resp. to $(v_1, v_2 - 1)$ under the blow-up in $V(u_2, y)$). In the first case we have $\beta^O_1(X', w') = \beta^O_1(X, x)$, $\gamma^O_1(X', x') = \gamma^O_1(X, x)$, $s^O_1(X', x') = s^O_1(X, x)$, and

$$\alpha^O_1(X', x') = \alpha^O_1(X, x) - 1 < \alpha^O_1(X, x),$$

and in the second $\beta^O_1(X', x') < \beta^O_1(X, x)$. This implies $t_{\text{poly}}(X', x') < t_{\text{poly}}(X, x)$ if the center of the blow-up is an irreducible curve.

After finitely many of these blow-ups we get $\alpha^O_1(X', x') < 1$ and $\alpha^O_1(X', x') < 1$ and the center of the next blow-up is an isolated closed point. Hence we may suppose that the center of the blow-up is an isolated closed point in the Hilbert–Samuel stratum. If $x'$ is the origin of the $U_2$-chart, then

$$\beta^O_1(X', x') = \beta^O_1(X, x) + \alpha^O_1(X, x) - 1 < \beta^O_1(X, x),$$

since by assumption $\alpha^O_1(X, x) < 1$. Otherwise the curve $V(y, u_1)$ would be permissible.

Let us consider the $U_1$-chart. If $x'$ is its origin, then the computation of the transform shows that we have $\beta^O_1(X', x') = \gamma^O_1(X, x)$. Hence if $\gamma^O_1(X, x) < \beta^O_1(X, x)$ then we get a decrease.

Suppose $\gamma^O_1(X, x) = \beta^O_1(X, x)$. Recall that by definition $s^O_1(X, x) \geq 1$ and $s^O_1(X', x') \geq 1$. If $s^O_1(X, x) = 1$ then $\beta^O_1(X', x') < \beta^O_1(X, x)$, again this follows by computing the transform. For $s^O_1(X, x) \geq 2$, one can deduce from the behaviour of the points in the polyhedron that $s^O_1(X', x') = s^O_1(X, x) - 1 < s^O_1(X, x)$. Again we obtain a decrease. If $1 < s^O_1(X, x) < 2$ then we have $\beta^O_1(X', x') = \beta^O_1(X, x)$ and $\gamma^O_1(X', x') < \gamma^O_1(X, x)$.

It remains the case $\gamma^O_1(X, x) = \beta^O_1(X, x)$ and $s^O_1(X, x) = \infty$. This is a very special situation. This means the characteristic polyhedron consists of only one vertex, namely $(\alpha^O_1(X, x), \beta^O_1(X, x))$ and $\beta^O_1(X, x) = \alpha^O_1(X, x) < 1$. Thus we have $\beta^O_1(X', x') = \beta^O_1(X, x)$ and $\gamma^O_1(X', x') = \gamma^O_1(X, x)$ and $s^O_1(X', x') = \infty$ and

$$\alpha^O_1(X', x') = \alpha^O_1(X, x) + \alpha^O_1(X, x) - 1 < \alpha^O_1(X, x).$$

This shows that $(\beta^O_1, \gamma^O_1, s^O_1, \alpha^O_1)$ strictly decreases at the origin of the $U_1$-chart of a point blow-up.

For the of the decrease of $(\beta^O_1, \gamma^O_1, s^O_1_{(u_2)}, \alpha^O_1_{(u_1)})$ in the case that $x'$ is a point in the $U_1$-chart but not the origin we refer to [Cul] Theorem 7.31 (if the residue field extension is trivial), resp. to [C] Theorem 26, or [CJS] Theorem 13.4 (if the residue field extension is non-trivial). In the latter case it shown that $\beta^O_1(X', x') < \beta^O_1(X, x)$. If this decrease is not the case for a trivial residue field extension then the local coordinates for $x$ are changed in such a way that $x'$ becomes the origin of the $U_1$-chart and the previous arguments are used to deduce a decrease. But one has to be careful, within this the maximizing of the slope $s^O_1_{(u_2)}(X, x)$ has to be taken into account.

Putting everything together we obtain $t_{\text{poly}}(X', Z', x') < t_{\text{poly}}(X, Z, x).$
Remark 3.10. (1) As one sees from the proof, the role of \( \gamma^O_1(X, x) \) in the invariant is only to indicate if we have \( \beta^O(X, x) > \gamma^O_1(X, x) \) or \( \beta^O(X, x) = \gamma^O_1(X, x) \). Therefore one could alternatively replace \( \gamma^O_1(X, x) \) by

\[
\tilde{\gamma}^O_1(X, x) := \begin{cases} 
1, & \text{if } \beta^O(X, x) = \gamma^O_1(X, x), \\
0, & \text{if } \beta^O(X, x) > \gamma^O_1(X, x).
\end{cases}
\]

(2) Moreover, one could overcome \( s_1(X, x) = \infty \) in the cases in which it is necessary to consider the slope. Namely, replace the third entry of \( t_{\text{poly}} \) by

\[
\sigma_1(X, x) := \begin{cases} 
s_1(X, x)(u_2) & \text{if } \beta_1(X, x) \geq 1, \\
\delta(X, x) & \text{if } \beta_1(X, x) < 1.
\end{cases}
\]

We only look at the slope if there is no permissible curve in the Hilbert-Samuel locus. If \( \beta_1(X, x) \geq 1 \), then \( 1 \leq s_1(X, x)(u_2) < \infty \) for any choice of \( u_2 \) (and this can be achieved by some \( u_2 \in R \)). Suppose \( s_1(X, x)(u_2) = \infty \) (possibly for \( u_2 \in \hat{R} \)), then the curve \( V(y, u_2) \) is permissible but this contradicts the condition that there is no permissible curve in the Hilbert-Samuel locus.

On the other hand, if \( \beta_1(X, x) < 1 \) then we may have \( s_1(X, x)(u_2) = \infty \), for some \( u_2 \) (possibly in \( \hat{R} \)). But if we look back into the proof we see that \( \alpha_1(X, x) < 1 \) and the characteristic polyhedron has only one vertex \( (\alpha_1(X, x), \beta_1(X, x)) \). This means that the invariant \( \delta(X, x) = \alpha_1(X, x) + \beta_1(X, x) \) strictly decreases if \( \beta_1(X, x) \) does not. In fact, we could even replace \( \delta(X, x) \) by 0 in the definition of \( \sigma_1(X, x) \), because \( \alpha_1(X, x) \) decreases if \( \beta_1(X, x) \) does not.

Since we are in dimension at most two [CSc] Theorem A implies that the characteristic polyhedron can be computed without passing to the completion. In particular, the part of \( t_{\text{poly}} \), which has to be considered for the decrease, is determined in \( R \) and one needs not to pass to the completion \( \hat{R} \).

Finally, let us mention some example where interesting things happen. First of all, the strategy of [CJS] differs in characteristic zero from the usual one.

Example 3.11. Consider the variety \( X = V(x^2 - y^2 z) \) over a field of characteristic zero and with no boundary given. Then [CJS] chooses \( V(x, y) \) as the center of the first blow-up. On the other hand, \( V(x) \) has maximal contact with \( X \) and the coefficient ideal is given defined by the monomial \( y^2 z \) and following the strategy of Bierstone and Milman (see [Sc2]) we have to blow up with center \( V(x, y, z) \).

The [CJS] strategy can also be formulated if the dimension is arbitrary, see [CJS] Remark 5.29. But then it is not clear if the strategy will work. Nevertheless the difference to the known characteristic zero strategies is getting bigger.

Example 3.12. Consider the threefold \( X = V(t^2 + x^4 + y^2 z^5 + x^2 z^3 + y^7 z) \) over a field of characteristic zero and with no boundary given. Show that the singular locus is given by \( V(t, x, y) \). This is then the center of the [CJS]-strategy. The invariant introduced by Bierstone and Milman (see for example [Sc2]) at the origin is \( (2, 0; 2, 0; \ldots) \).

After the blow-up with center \( V(t, x, y) \) we obtain at the origin of the \( Y \)-chart that the strict transform is \( X' = V(t^2 + x^4 y^2 + z^5 + x^2 z^3 + y^7 z) \) and thus the Bierstone-Milman invariant is \( (2, 0; \hat{2}, \ldots) \geq_{\text{lex}} (2, 0; 2, \ldots) \).

Where is the improvement hidden?

The constructed local invariant \( \iota(X, Z, x) \) is not upper semi-continuous. More precisely, the following example due to Piltant and the first author show that \( \beta^O(X, x) \) is not upper semi-continuous and a variant of it yields that also the dimension of the directrix is not upper semi-continuous.

Example 3.13. Let \( R \) be a ring as before and add a variable \( T \). Consider \( X = V(f) \) with \( f = y^p + T u_1^p \) and an empty boundary. Along \( x \in V(y, u_1) \) the values of \( \beta^O(X, x) = \beta(X, x) \) changes depending if \( T \) is a \( p \)-power or not.
Example 3.14. Let $R$ be a ring as before and add a variable $T$. Consider $X = V(f)$ with $f = y^p + Tu_1^p$. Along $x \in V(y,u_1)$ the dimension of the directrix changes depending if $T$ is a $p$-power or not.

In the situation over a field $k$ the directrix and the characteristic polyhedron may change if we pass to an extension of $k$ and thus all the data obtained from it may change.

Example 3.15. Let $k$ be a non-perfect field of characteristic $p \neq 2$ and $\lambda \in k \setminus k^p$. Consider $f = x^p + \lambda y^p + u_1^{p+4} + yu_2^{p+2} + y^{p+1}u_1$. Observe how the directrix and the characteristic polyhedron change by passing to $k' = k[T]/(T^p - \lambda)$.

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