Two Dimensional Kodaira-Spencer Theory
and
Three Dimensional Chern-Simons Gravity

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Abstract

Motivated by the six-dimensional formulation of Kodaira-Spencer theory for Calabi-Yau threefolds, we formulate a two-dimensional version and argue that this is the relevant field theory for the target space of local topological B-model with a geometry based on a Riemann surface. We show that the Ward identities of this quantum theory is equivalent to recursion relations recently proposed by Eynard and Orantin to solve the topological B model. Our derivation provides a conceptual explanation of this link and reveals a hidden affine $SL_2$ symmetry. Moreover we argue that our results provide the strongest evidence yet of the existence of topological M theory in one higher dimension, which in this case can be closely related to $SL_2$ Chern-Simons formulation of three dimensional gravity.

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1. Introduction

Topological strings have been solved in the context of local toric Calabi-Yau three-folds. In particular the topological vertex can be used to compute all genus amplitudes for topological A-model on these spaces \([1]\). On the other hand, using mirror symmetry, this construction can be interpreted as providing a full solution to the B-model topological string with a local Calabi-Yau geometry modelled on a Riemann surface (which we will refer to as the local B model).

There is however a more direct path to obtaining topological strings in the context of the local B model: Matrix models are conjectured to be equivalent to the topological B model on a local geometry \([2]\), where the Riemann surface is identified as the spectral curve of the matrix model. This gives another solution to the local B-models, namely the large \(N\) ’t Hooft expansion of the corresponding matrix models. There has been recent spectacular progress in solving these matrix models where it has been shown that the large \(N\) description of matrix model can be directly formulated \textit{intrinsically} on the Riemann surface in terms of certain recursion relations that essentially follow from the loop equations \([3]\). This new approach has the advantage that it applies to any local B-model, whether or not the spectral curve comes from a matrix model. This relation has been recently checked in the context a number of examples \([4]\). In the context of the B model the approach of \([3]\) has the remarkable feature that it automatically incorporates the holomorphic anomaly \([5]\): The partition function depends on the choice of A-cycles on the Riemann surface, and choosing different basis for A-cycles leads to a generalized Fourier transform of the partition function, as is expected on the basis of the general holomorphic anomaly equation of the topological string. Usually this fact is formulated as that the topological string partition function transforms as a wave function or a holomorphic block.

The aim of this note is to derive the recursion relations of \([3]\) directly in the B model using field theory techniques. We will demonstrate that these recursion relations are given in terms of Ward identities of the B-model field theory, which is the restriction of the Kodaira-Spencer theory on the Riemann surface. Quite surprisingly, while proving these recursion relations, we uncover an \(SL(2, \mathbb{R})\) current algebra. In this setup, the fact that the partition function becomes a wave function is directly related to the fact that one is dealing with a \textit{chiral} boson on the Riemann surface as the basic field of the gravity, and it is known that this partition function does depend on the choice of a basis for the A-cycles. In fact the best understanding of this phenomenon comes from the interpretation of the
wave function as a state for the three dimensional Chern-Simons theory. Here we also speculate about the existence of a topological M-theory, whose restriction to the local case suggests the existence of a three dimensional gravity theory, which leads to the topological strings as a quantum state. This could potentially explain the appearance of $SL(2, \mathbb{R})$ current algebra on the Riemann surface.

The organization of this paper is as follows: In section 2 we describe the basic setup and formulate the relevant 2d KS theory. In section 3 we show that the Ward identities of this theory are the same as those written in [3], and uncover an $SL(2, \mathbb{R})$ symmetry. In section 4 we speculate about embedding this symmetry in one higher dimension.

2. The Basic Setup

Kodaira-Spencer theory is the string field theory of the topological B-model on Calabi-Yau threefolds [6]. This theory can be considered as the quantization of the $\overline{\partial}$ operator on the Calabi-Yau manifold with a fixed complex structure, as captured by a holomorphic $(3, 0)$ form $\Omega$. More precisely, it is the quantization of the cohomologically trivial variations of $\overline{\partial}$, which do not change a fixed background complex structure. Thus the theory is defined in terms of a pair $(\overline{\partial}, \Omega)$. It is not known, at the present, how to use this formalism to solve all genus amplitudes for compact Calabi-Yau manifolds. One can compute, however, low genus amplitudes using this approach.

As we will see, the situation is much better for the local non-compact threefold modeled on a curve $\Sigma$. In the context of these local geometries it is natural to look for a reduction of this structure to the Riemann surface $\Sigma$ and directly quantize that system. This is the approach that we will follow in this note.

By the local case we shall mean a non-compact Calabi-Yau threefold defined by the hypersurface:

$$vw = H(x, y),$$

where $v, w$ belong to $\mathbb{C}$ and $x, y$ belong to $\mathbb{C}$ or $\mathbb{C}^*$. (To be precise, in the latter case the appropriate coordinates are $e^x, e^y \in \mathbb{C}^*$ or, equivalently, $x, y \in \mathbb{C}/2\pi i \mathbb{Z}$.) In these coordinates the holomorphic three-form $\Omega$ is given by

$$\Omega = \frac{dv}{v} \wedge dy \wedge dx.$$
The local Riemann surface $\Sigma$ is defined by the equation

$$H(x, y) = 0.$$ 

It is not difficult to see that the periods of the $(3, 0)$ form $\Omega$ on the local threefold can be reduced, upon integration to the $x$-$y$ plane, to the integral of the one-form

$$\omega = ydx$$

on the Riemann surface. We thus wish to define the quantum Kodaira-Spencer theory of the pair $(\overline{\partial}, \omega)$, on the Riemann surface $\Sigma$ given by $H(x, y) = 0$. More precisely, we wish to integrate over all deformation of $\overline{\partial}$ which do not affect the cohomology class of $\omega$, just as was the case for the 3-fold case.

The variation of the complex structure is captured locally by the deformation

$$\overline{\partial} \rightarrow \overline{\partial} - \mu \partial,$$

where the Beltrami differential $\mu$ is a tensor of type

$$\mu = \mu \overline{z} dz \otimes \partial_z.$$

For the variation to be globally trivial, it means that there is a diffeomorphism by a vector field $v = v z \partial z$ such that

$$\mu \overline{z} = \partial \overline{v} z.$$

We are interested in quantizing these deformations, while maintaining the cohomology class of $\omega$. As such, it is natural to formulate the variation of the $\overline{\partial}$ in terms of its action on $\omega$. The condition of not changing the cohomology class of $\omega = \omega_z dz$ means that

$$\delta \omega = d\phi,$$

for some function $\phi$. We will use the scalar $\phi$ as the basic field of our KS theory. In fact, we can re-express the vector field $v z$ in terms of $\phi$ as follows:

$$v z = \frac{z}{\omega_z}.$$

To see this, note that the Lie derivative of $\omega$ in terms of $v$ can be expressed as

$$\delta \omega = \mathcal{L}_v \omega = d(\iota_v \omega) - \iota_v d\omega = d(\iota_v \omega),$$
since \( \omega \) is closed. Note also that \( \iota_v \omega = \omega \cdot (\phi/\omega) = \phi \). This leads to the required relation \( \delta \omega = d\phi \).

In terms of the scalar field \( \phi \) the variation of the \( \overline{\partial} \) operator takes the form (using that \( \overline{\partial} v = \overline{\partial} \phi/\omega \))

\[
\overline{\partial} \rightarrow \overline{\partial} - \overline{\partial} \phi \omega \partial.
\]

Now before deformation the scalar field \( \phi \) should satisfy

\[
\partial \overline{\partial} \phi = 0.
\]

This one can see for example by making more explicit the dictionary from the general KS theory on the threefold with the reduction to the Riemann surface \( \Sigma \). The KS field \( A \) (the Beltrami differential) of [6] is identified as

\[
A \sim \overline{\partial} \phi/\omega.
\]

The dual form \( A' = \Omega \cdot A \), which is a \((2, 1)\) form in six dimensions, becomes in two dimensions a \((0, 1)\) form given by

\[
A' \sim \overline{\partial} \phi.
\]

Now the closed string field \( A' \) satisfies the gauge condition \( b_0 A' = 0 \). In the KS theory this becomes \( \partial A' = 0 \). With the above dictionary, this translates into \( \partial \overline{\partial} \phi = 0 \). (Closely related to this point of view, we can also think of \( \omega \) as the classical value of \( \partial \phi \). Since this is a holomorphic \((1,0)\) form, we have (again classically) the equation \( \overline{\partial} \partial \phi = 0 \).)

This means that, at the level of the unperturbed equations, we are dealing with a free (chiral) boson quantum field theory with action (we will not be precise with normalizations)

\[
S = \int_{\Sigma} \partial \phi \overline{\partial} \phi.
\]

We can now capture the effect of the variation of \( \overline{\partial} \) in terms of an operator. Recall that for any Beltrami differential \( \mu \overline{\partial} \), the operator \( T(\mu) \) which implements this variation on the conformal field theory is given in terms of the holomorphic stress-tensor \( T_{zz} \) as

\[
T(\mu) = \int_{\Sigma} T_{zz} \mu \overline{\partial}.
\]

Given that we have a free boson system, we can write this very explicitly as

\[
T_{zz} = \frac{1}{2} \partial \phi \overline{\partial} \phi.
\]
Using the fact that $\mu = \overline{\partial}\phi/\omega$, we therefore have the interaction term

$$\int_{\Sigma} \partial\phi \overline{\partial}\phi \frac{\overline{\partial}\phi}{\omega}.$$  

Note that this operator can be written as total derivative

$$\int d \left( \partial\phi \overline{\partial}\phi \frac{\phi}{\omega} \right),$$

where we use that in perturbation theory $\partial\phi$ remains holomorphic. So, if $\omega$ has no zeroes, this interaction is trivial. Since for our case $\omega = ydx$, such zeroes can occur if either $y = 0$ or $dx = 0$. As we will show in the next section, the points where $y = 0$ do not contribute, but the locus where $dx = 0$ does. The points where $dx = 0$ correspond to branch points of the Riemann surface $H(x, y) = 0$ on the $x$-plane. We will thus arrive at the interaction operator

$$\sum_{\text{branch points}} \oint_P \partial\phi \overline{\partial}\phi \cdot \frac{\phi}{\omega},$$

which will be used in the next section to recover the recursion relations of [3].

In fact there is one additional term in the action that we will now explain: Note that the action we have thus far can be written as

$$S = \int \partial\phi (\overline{\partial} + \frac{\overline{\partial}\phi}{\omega} \overline{\partial})\phi$$

However, as we have explained $d\phi$ is the variation of $\omega$. In particular $\omega$, being a differential of type $(1, 0)$, can be viewed as the classical vev of $\partial\phi$. Motivated by this observation we view the first term in the above action as the full $\omega$ including the classical piece and arrive at the final form for the action

$$S = \int (\omega + \partial\phi)(\overline{\partial} + \frac{\overline{\partial}\phi}{\omega} \overline{\partial})\phi$$

Expanding this action, and introducing the topological string coupling constant $\lambda$ through the usual rescaling $\omega \rightarrow \omega/\lambda$ (a standard relation in KS theory), we obtain the field theory

$$S = \int \left[ \partial\phi \overline{\partial}\phi + \frac{1}{\lambda} \omega \overline{\partial}\phi + \frac{\lambda}{\omega} \overline{\partial}\phi (\partial\phi)^2 \right].$$  

(2.1)

The first term is the usual kinetic term. The second term can be interpreted as the coupling to a background holomorphic gauge field $A_z = \omega/\lambda$. Since in perturbation theory the field $\phi$ is chiral, this term will only influence the classical free energy, that scales as $\lambda^{-2}$. Finally,
the third term is capturing the perturbative corrections. (In fact, by shifting $\partial \phi$ with the background $\omega$ the whole action can be put into cubic form, reminiscent of the purely cubic forms encounter in open string field theory.)

In this action the cubic term is proportional to the string coupling $\lambda$. So, up to several subtleties related to the chiral nature of this quantum field theory, this model can be solved using trivalent Feynman diagrams. In the next section we will show how one can use this action to derive the recursion relations of [3].

3. The Recursion Relations

We will now discuss the quantum field theory based on the action (2.1) in terms of coordinate space perturbation theory. Following [3] we will consider not just the partition function, but general correlation function of operators $\partial \phi(z)$

$$ W(z_1, \ldots, z_n; \lambda) = \left\langle \partial \phi(z_1) \cdots \partial \phi(z_n) \right\rangle_{\text{con}}. $$

Here the subscript indicates that we only consider the connected correlators. We compute these correlators in the background of the interaction term

$$ \exp \int_{\Sigma} \lambda \frac{\bar{\phi} (\partial \phi)^2}{\omega} $$

Expanding in the coupling $\lambda$ brings down these interactions and this defines the perturbative correlators. The connected correlators have an expansion of the form

$$ W(z_1, \ldots, z_n; \lambda) = \sum_{g \geq 0} \lambda^{2g-2+n} W_g(z_1, \ldots, z_n). $$

3.1. The localisation to branch points

As noted in the previous section, the interaction can be written as a total derivative away from the zeroes of $\omega$. So we are left with contributions of the form

$$ \sum_{P} \int_{P} \frac{\phi(\partial \phi)^2}{\omega} $$

where $P \in \Sigma$ denote the positions of the zeroes of $\omega$. In the local coordinates the one-form $\omega$ is given by $\omega = ydx$, and such zeroes therefore occur either if $y(x) = 0$, or if the differential $dx$ vanishes. Let us consider these two cases separately.
If $y(x)$ has a zero at $x = x_0$, so that $y \sim c(x - x_0)$, the variable $z = x - x_0$ is a good local coordinate around this special point and we can expand the quantum field as

$$\partial \phi(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1}.$$ 

Here $\alpha_n$ are the usual creation and annihilation operators. Plugging this relation and $\omega \sim zdz$ into interaction vertex (3.1) we obtain the operator

$$O = \oint dz \frac{1}{z} \phi(\partial \phi)^2 \sim \sum_{k+m+n=-1} \frac{1}{k} \alpha_k \alpha_m \alpha_n.$$

Since $z$ is good local coordinate at this point, the field $\phi(z)$ has no singularities. In the operator formalism the operator $O$ should therefore be regarded to act on the vacuum $|0\rangle$. This state satisfies $\alpha_n|0\rangle = 0$ for $n \geq 0$. Because of the condition that $k + m + n = -1$, we see that necessarily the mode expansion of $O$ will have to contain annihilation operators $\alpha_{+n}$ that vanish on this vacuum state. Therefore we obtain the relation

$$O|0\rangle = 0,$$

and the action of the interaction at these zeroes is trivial and can be ignored.

As we mentioned, a second source of zeroes of $\omega = ydx$ are the points where $dx$ vanishes. If we think of the curve $H(x, y) = 0$ as an orbit in the phase space of Hamiltonian mechanics, these are the turning points. In complex geometry these zeroes are branch points of the algebraic curve. Generically, these are simple branch points. At such a point the curve is locally described by

$$(y - y_0)^2 = x - x_0.$$ 

A good local coordinate is therefore

$$z = y - y_0 = (x - x_0)^{1/2},$$

where we clearly see that we are dealing with a branch point. In terms of the variable $z$ we have $dx \sim zdz$ and therefore also $\omega \sim zdz$ (since $y$ attains the regular value $y_0$ at this point). So $\omega$ has indeed a (single) zero at the ramification point $z = 0$. Now in terms of
the coordinate \( x \) the interaction vertex \( \mathcal{O} \) does not act not on the regular vacuum state \(|0\rangle\), but on a twisted state \(|\sigma\rangle\), around which the field \( \partial \phi(x) \) has a half-integer mode expansion

\[
\partial \phi(x) \sim \sum_{n \in \mathbb{Z}} \alpha_{n - \frac{1}{2}} (x - x_0)^{-n - \frac{1}{2}}.
\]

Equivalently, in terms of “good” coordinate \( z \) on the double cover the scalar field satisfies the condition

\[
\phi(-z) = -\phi(z). \tag{3.2}
\]

This behaviour can be understood from the fact that the perturbed one-form \( \omega + d\phi \) should have the same behaviour as \( \omega \), which around the ramification point takes the form \( zdz \). This fixes the boundary condition on the field \( \phi(z) \) and determines in turn the nature of the boundary state \(|\sigma\rangle\). Working out the decomposition of \( \mathcal{O} \) in these twisted modes we easily see that in this case \( \mathcal{O}|\sigma\rangle \neq 0 \). Therefore the branch points give non-vanishing contributions to the interaction and we are left with

\[
\sum_{\text{branch points}} \oint_P \frac{\phi(\partial \phi)^2}{\omega} \tag{3.3}
\]

We will now have to contract the insertions \( \partial \phi(z_i) \) with the operators appearing in the interaction vertex. If \( z \) is the local coordinate around the branch point \( P \) this gives terms of the form

\[
\left\langle \partial \phi(w) \oint_z \frac{\phi(z) \partial \phi(z) \partial \phi(z)}{\omega(z)} \ldots \right\rangle
\]

To evaluate these kinds of expressions we need to determine the chiral correlator

\[
B(z, w) = \langle \partial \phi(z) \partial \phi(w) \rangle
\]

for a free boson on the surface \( \Sigma \). This two-point function is well-know to be given by the Bergmann kernel. To define it uniquely, we have to fix the loop momenta through a set of \( A \)-cycles

\[
\oint_{A_I} \partial \phi = p_I.
\]

The standard kernel \( B(z, w) \) takes these \( p_I = 0 \). Note that this prescription already breaks the modular invariance, since \( Sp(2g, \mathbb{Z}) \) transformations, that relate one set of homology cycles to another, will act via generalized Fourier transformation on the correlators of the chiral boson.
Now first, up to total derivatives, we can consider the contraction of \( \partial \phi(w) \) with \( \phi(z) \). This will be given by the primitive of the Bergmann kernel that we will denote as \( G(z, w) \)

\[
\langle \phi(z) \partial \phi(w) \rangle = G(z, w) := \int^z B(v, w) dv.
\]

However we should also take into account the boundary condition (3.2) at the branch point. The field \( \partial \phi \) should be anti-periodic around the twist field insertion. This we can enforce by inserting an explicit projector to the odd part of the propagator, as is customary in the computation of twist field correlation functions in orbifold models. So we get (in the local coordinate \( z \), close to the branch point)

\[
\langle \phi(z) \partial \phi(w) \rangle_{\text{twist}} = \frac{1}{2} \int^z B(v, w) dv.
\]

Note in particular that at the branch point \( z = 0 \) this twisted propagator vanishes, which is forced by the anti-periodicity. However, in this case there is a matching zero in the denominator, because also \( \omega \) vanishes at the branch point. We can therefore apply l’Hôpital’s rule, and consider instead the limit

\[
\lim_{z \to 0} \frac{\langle \phi(z) \partial \phi(w) \rangle_{\text{twist}}}{\omega(z)} = \lim_{z \to 0} \frac{\int^z B(v, w) dv}{\omega(z) - \omega(\tilde{z})}
\]

where \( z \) and \( \tilde{z} \) are the two points on the two branches of \( \Sigma \) that project to the same image in the \( x \)-plane (so that \( \tilde{z} \sim -z \) close to \( z = 0 \)).

Finally we also have to deal with the fact that the interaction consists of a cubic term that has to be normal ordered. Now recall that this term originated from the stress-tensor insertion

\[
\oint v^z T_{zz}, \tag{3.4}
\]

where the vector field was taken to be \( v^z = \phi/\omega \). The self-interactions in the stress-tensor are usually defined by point-splitting regularization

\[
T(z) = \lim_{\tilde{z} \to z} \frac{1}{2} \left[ \partial \phi(z) \partial \phi(\tilde{z}) - \frac{1}{(z - \tilde{z})^2} \right]
\]

To have a consist perturbation theory we have to insert this definition into equation (3.4) with \( v^z = \phi/\omega \).

Summarizing all this, and up to some further subtleties that we discuss in section 3.3, we obtain the recursion relation of [3]. This relation can be considered as the Schwinger-Dyson equations for the interacting boson field theory. Expanding the recursion relation
gives a graphical representation in terms of trivalent Feynman diagrams. There will be both connected and disconnected contributions. For the connected diagrams with $n + 1$ external legs the recursion relation takes the form

$$W_g(w, z_1, z_2, \ldots, z_n) = \sum_{P} \text{Res}_{z=P} \int_{\Sigma} B(v, w) \frac{\rho(z)}{\omega(z) - \omega(z')} \left[ W_{g-1}(z, z; z_2, \ldots, z_n) \right]$$

$$+ \sum_{h=0}^{g} \sum_{Z=Z' \cup Z''} W_h(z, z_1', \ldots, z_m') W_{g-h}(z, z''_1, \ldots, z''_{n-m})$$

(3.5)

Here $P$ is summed over all branch points of the spectral curve $\Sigma$. The set $Z$ denotes the collection of “free” marked points $Z = \{z_1, \ldots, z_n\}$, and in the disconnected piece there is a sum over all splittings of the set $Z$ into two disjoint (and possibly empty) subsets $Z'$ and $Z''$ of order $m$ and $n - m$.

3.2. The partition function

We now turn to the partition function itself, which has an expansion

$$Z = \exp F, \quad F = \sum_{g \geq 0} \lambda^{2g-2} F_g.$$ 

To derive the final recursion relation of [3] we will employ a rescaling symmetry. Starting point will be again the complete action (2.2) that we recall here for convenience

$$S = \int \partial \overline{\phi} \partial \phi + \frac{1}{\lambda} \omega \partial \phi + \frac{\lambda}{\omega} (\partial \phi)^2 \overline{\partial \phi}.$$ 

Consider now the action of the vector field $\lambda \partial \overline{\phi}$ on the free energy $F$. On the one hand we clearly have

$$\lambda \frac{\partial F}{\partial \lambda} = \sum_{g \geq 0} (2g - 2) F_g \lambda^{2g-2}.$$ 

(3.6)

On the other hand the effect of this rescaling on the action has the effect of bringing down two possible terms. One of them is the cubic interaction term we have focused thus far. In this case of the vacuum amplitude, following the logic of the previous derivation, this term does not contribute, as there are no other $\partial \phi$ observables inserted anywhere. Only the second term in the action will contribute and gives the insertion

$$\lambda \frac{\partial Z}{\partial \lambda} = -\frac{1}{\lambda} \langle \int_{\Sigma} \omega \partial \phi \rangle.$$
Just as for the cubic interaction term, it is convenient to write this as a total derivative \( \frac{1}{\lambda} \int d(\omega \phi) \), which gives zero, except when \( \phi \) has poles. This happens when \( \phi \) is near the branch points \( P \). So we can write this term as

\[
\frac{1}{\lambda} \sum_{\text{branch points}} \langle \oint_P \omega \phi \rangle.
\]

Near the branch points we write \( \omega = \partial \phi_{cl} \), where \( \phi_{cl}(z) \) can be interpreted as the classical value of the field \( \phi \) and so this contour term can be written (using integration by parts) as

\[
-\frac{1}{\lambda} \sum_P \langle \oint_P \phi_{cl} \partial \phi \rangle = -\frac{1}{\lambda} \sum_P \oint_P \phi_{cl} \langle \partial \phi \rangle = -\frac{1}{\lambda} \sum_P \text{Res}_{z=P} \phi_{cl}(z) W(z)
\]

Now the normalized (or connected) one-point function \( W(z) \) has a perturbative expansion

\[
W(z) = \sum_{g \geq 0} \lambda^{2g-1} W_g(z)
\]

Inserting this expansion into (3.7) and comparing this with (3.6) we thus find the recursion relation of [3]

\[
\mathcal{F}_g = \frac{1}{2 - 2g} \sum_P \text{Res}_{z=P} \phi_{cl}(z) W_g(z).
\]

### 3.3. The chiral projection and a hidden affine \( SL(2, \mathbb{R}) \) symmetry

Up to this point the relation of the action (2.2) to the manipulations leading to the recursion relations has not been very precise. Indeed the authors of [3] have remarked that their recursion relations and the corresponding graphical solution cannot follow from a straightforward Feynman expansion, since certain contractions and diagrams are missing, and there is a special order in which the vertices are connected. This last point is due to the fact that the interactions have been rewritten as contour integrals, which we can think of as a (local) Hamiltonian formalism. These operators in general do not commute, and the specific time ordering prescription breaks the general covariance. Exactly the same point was met in [8].

From our point of view another source of subtleties arise because we are dealing with a chiral field theory and up to now we have not consistently implemented the chiral projection on the scalar field \( \phi \). As written in action (2.2) the anti-holomorphic component of \( \phi \) is a propagating field. This gives unwanted contributions in the contractions of \( \phi \) with itself. We will now rectify this point.

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At the level of the free theory the projection onto the chiral modes can be done by adding a multiplier (1,0) form $\gamma$ to the action

$$\int \left( \frac{1}{2} \partial \phi \overline{\partial} \phi - \gamma \overline{\partial} \phi \right).$$

Integrating out $\gamma$ enforces the chiral condition $\overline{\partial} \phi = 0$. On the other hand, integrating out $\phi$ expresses $\gamma$ as

$$\gamma = \partial \phi.$$ 

In fact, it is now suggestive to relabel

$$\phi = \beta,$$

to make clear that we are dealing with a bosonic $\beta$-$\gamma$ system with spins 0 and 1. After a partial integration, this $\beta\gamma$ system has the action

$$\int_\Sigma \left( \frac{1}{2} \partial \bar{\beta} \overline{\partial} \beta + \beta \overline{\partial} \gamma \right) + \oint_{\infty} \beta \gamma,$$

where by $\oint_{\infty}$ here and in the following we mean integration over the boundaries of the Riemann surface if there are any (including the branch points). With this unconventional action we get the following Green’s functions for the fields $\beta$ and $\gamma$

$$\langle \beta(z) \beta(w) \rangle = 0,$$

$$\langle \beta(z) \gamma(w) \rangle = G(z, w),$$

$$\langle \gamma(z) \gamma(w) \rangle = B(z, w).$$

(3.8)

Here, as before, $\partial_z G(z, w) = B(z, w)$.

Using this formalism we can now write the interaction term in an elegant fashion as

$$\oint_{\infty} \left( \frac{\omega}{\lambda} \beta + \beta \gamma + \frac{\lambda}{\omega} \beta \gamma^2 \right).$$

(3.9)

Here the middle term is just added for suggestive reasons; it does not contribute as long as the total momentum flowing in or out of the boundary of $\Sigma$ is zero.

The main advantage of writing the action like this is that it automatically reproduces the diagrammatics of $\Box$. The correlators that are considered are of the form

$$\langle \gamma(z_1) \ldots \gamma(z_n) \rangle.$$
The string loop interactions now come from the $\beta\gamma^2$ term. Since $\beta$ only contracts with $\gamma$ each $\beta$ in the interaction term $\beta\gamma^2$ gets paired up with a $\gamma(z_i)$ leaving two additional $\gamma$’s. At the end when all the $\beta$’s have been contracted, we use the $\gamma$ correlations and thus recover the recursion relations as well as the boundary conditions needed to solve them. This reformulation makes our derivation of the recursion relation precise.

Interestingly, this way of expressing the action suggests a hidden $SL(2, \mathbf{R})$ symmetry. Let us first recall the Wakimoto representation of the $SL(2, \mathbf{R})_k$ current algebra [9]. This conformal field theory consists of another $\beta\gamma$ system, that we will write as $(\hat{\beta}, \hat{\gamma})$ and that now has spins 1 and 0, together with an extra scalar field $\chi$. In terms of these variables the $SL(2, \mathbf{R})$ currents, all of spin 1 of course, are expressed as

$$J_+(z) = \hat{\beta},$$
$$J_3(z) = \hat{\beta}\hat{\gamma} + \frac{1}{2} \alpha_+ \partial \chi,$$
$$J_-(z) = \hat{\beta}\hat{\gamma}^2 + \alpha_+ \hat{\gamma} \partial \chi + k \partial \hat{\gamma}. \tag{3.10}$$

Here $\alpha_+^2 = 2k - 4$ and the central charge is given by $c = \frac{k}{k-2}$. The scalar field $\chi$ has a background charge $1/\alpha_+$. Furthermore, in order to compute correlation functions one needs to add various insertions of the screening charge

$$S_+ = \oint_{\infty} \hat{\beta} e^{-2\chi/\alpha_+}. \tag{3.11}$$

In the application we have in mind $\chi$ does not appear and thus it is natural to view this as the special limit of $k = 2$. In this so-called critical limit, where the central charge $c \to \infty$ and $\alpha_+ \to 0$, the contribution of the scalar $\chi$ decouples and can be ignored. In fact the $\beta\gamma$ system by itself carries a representation of the current algebra $SL(2, \mathbf{R})$ at level $k = 2$. (In the analytic continuation to the case of $SU(2)$ this critical level corresponds to the value $k = -2$.) However, this representation is far from irreducible. For example, there is a very large center spanned by the modes of the limiting case of the rescaled stress-tensor

$$u(z) = (k - 2)T(z) = \sum_a J_a^2,$$

(instead of just the identity operator). One way to describe the representations in this critical limit, is that the combination $v(z) = \alpha_+ \partial \chi(z)$ becomes a classical, non-dynamical scalar field that parametrizes the affine $SL(2, \mathbf{R})$ representations. This classical scalar field
can be traded with the expectation values of the rescaled stress-tensor \( u(z) \). All of this is intimately connected to the theory of integrable systems, see \([10]\) for more on this.

There is a traditional topological twisting of this model related to the KPZ model of 2d gravity \([11]\), where the spins of the \( \beta-\gamma \) system are changed from \((1,0)\) to \((0,1)\). In this twist the triplet of affine \( SL(2,\mathbb{R}) \) currents \((J_+, J_3, J_-)\) change its spins into \((0,1,2)\). This twisting needs a section of the canonical bundle \( K_\Sigma \) on the Riemann surface, \( i.e. \), we have to pick a meromorphic \((1,0)\) form \( \omega \). The twisted fields \((\hat{\beta}, \hat{\gamma})\) are related to the untwisted fields as

\[
\hat{\beta} = \omega \beta, \quad \hat{\gamma} = \frac{1}{\omega} \gamma.
\]

Ignoring the scalar \( \chi \) and total derivatives we so find that the triplet of currents can be expressed as

\[
J_+(z) = \omega \beta, \\
J_3(z) = \beta \gamma, \\
J_-(z) = \frac{1}{\omega} \beta \gamma^2.
\]

Using this notation we can take the interaction of the KS theory, that we have managed to put in the form

\[
\oint_{\infty} \left( \frac{\omega}{\lambda} \beta + \beta \gamma + \frac{\lambda}{\omega} \beta \gamma^2 \right),
\]

and rewrite it in a more suggestive way as

\[
\oint_{\infty} J_+(z, \lambda),
\]

where we introduced a one-parameter family of currents

\[
J_+(z, \lambda) = \frac{1}{\lambda} J_+(z) + J_3(z) + \lambda J_-(z).
\]

Here the variable \( \lambda \) can be seen as a spectral parameter that picks a \( U(1) \) inside \( SL(2,\mathbb{R}) \), or more geometrically a null plane inside \( \mathbb{R}^2 \). It therefore takes it values on the twistor “sphere”, or perhaps more correctly the twistor upper-half plane (the appropriate real structure is not quite obvious)

\[
\lambda \in H = \frac{SL(2,\mathbb{R})}{U(1)}.
\]

The group \( SL(2,\mathbb{R}) \) acts on this parameter in the usual fashion by fractional linear transformations

\[
\lambda \to \frac{a\lambda + b}{c\lambda + d}, \quad ad - bc = 1.
\]
Possibly one can think of the interaction (3.13) as a (singular) $k = 2$ limit of the screening charge $S_+ \text{ given in (3.11)}$. Anyway, there is in the critical case $k = 2$ a huge algebra that commutes with the interaction vertex $J_+ (\lambda)$, namely the full classical Virasoro algebra. It is not clear what the precise role of this structure is and to which extend it is related to other models of two-dimensional gravity. In the next section we discuss the potential meaning of the $SL(2, \mathbb{R})$ structure from a three-dimensional perspective.

4. The 3d Gravity Lift

An important aspect of the formulation of topological B model in the setup of [3] is the fact that it in a natural way explains [5] why the partition function satisfies the holomorphic anomaly equation of [6]. In particular to even formulate the partition function, one has to choose a set of A-cycles on the Riemann surface, such that the period of the Bergmann kernel vanishes over them. If we chose a different set of A-cycles, the partition function transforms as a wave function as is the interpretation of holomorphic anomaly equation [12], see also [13].

In our setup this comes about because we have a chiral boson $\phi$ and to define it quantum mechanically we need to specify its periods over only half the cycles, which in the present case are the A-cycles, where $\int_A \partial \phi = 0$. It is known from the study of 2d CFT’s [14] that this is indeed consistent with how the chiral blocks are defined, and how they transform if we choose a different symplectic basis for them. For example, the chiral blocks transform according to Fourier transform, if we switch the A-cycles and B-cycles. This phenomenon for 2d CFT’s is best described by considering the associated three dimensional Chern-Simons theory [15]: Consider the abelian $U(1)$ Chern-Simons theory in three dimension, with the action

$$\int A \wedge dA$$

where $A$ denotes the $U(1)$ connection. In particular in the Coulomb gauge we have the term $\int A_x \dot{A}_y$, which means $A_x$ and $A_y$ are conjugate variables. Viewing the three dimensional manifold as Riemann surface times time, and identifying the chiral fields of 2d with the restriction of $A = d\phi$ on the Riemann surface, we see the fact that the periods of $d\phi$ over the A-cycles and B-cycles form conjugate variables and do not commute.

It is very natural to view this as the natural motivation for us as well. For the general topological string this suggests the existence of a topological M-theory in one
higher dimension, namely 7 for the Calabi-Yau threefold, or 3 for the current case (where the CY is based on a Riemann surface). This in fact was one of the motivations for the introduction of topological M theory [16,17].

However the abelian Chern-Simons theory in 3d cannot be the whole story here for two reasons: First of all we have in addition to the free term the interactions in 2d, and they should lead to some deformations of the 3d theory. Secondly the 2d theory is a gravity theory, and not a current algebra, and so we expect that the 3d theory to also be a gravity theory. It is natural to ask whether we can relate our theory to the \( SL(2,\mathbb{R}) \) Chern-Simons formulation of 3d gravity [18] in the geometry \( \Sigma \times \mathbb{R} \). Such a 3d theory would give rise to chiral \( SL(2,\mathbb{R}) \) current algebra living on \( \Sigma \). But this is surprisingly what we have found in the last section!

We are thus led to believe that 3d gravitational Chern-Simons theory underlies what we have found in connection with the Kodaira-Spencer theory on the Riemann surface. However we need to better understand the meaning of the insertion terms

\[
\oint_{\infty} J_+(z, \lambda).
\]

First of all, what is the twistorial meaning of \( \lambda \)? In other words, why should the coupling constant of topological string parameterize a twistorial sphere and what is the role of the \( SL_2 \) transformations acting on the coupling constant? Here it should be stressed that the relevant object is the local form \( \omega/\lambda \), which can be viewed as a varying point on the twistor sphere. This choice of background is what distinguishes the 3d theory from ordinary (chiral) \( SL(2,\mathbb{R}) \) gravity.

Secondly, why would one insert this interaction term at the boundaries and branch points of the surface? Is it related to the screening operators at the critical level? One interpretation may be that this term shifts the gravitational background so that the connection is not flat and would correspond to the background \( \Sigma \times \mathbb{R} \). If this is the right interpretation one would need to better understand why this particular insertion creates this gravitational background.

Recalling that we have been discussing only the reduction of Kodaira-Spencer theory to two dimensions, we could lift our findings back to six dimensions. In this context we would be led to the seven dimensional M-theory formulation, mentioned above. We feel we have found a first concrete evidence for the existence of topological M-theory. It would
be very important to deepen our understanding of the 3d lift of Kodaira-Spencer theory, and further extend it to the topological M-theory in seven dimensions.

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