INFINITE ENERGY SOLUTIONS TO THE HOMOGENEOUS BOLTZMANN EQUATION

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Abstract. The goal of this work is to present an approach to the homogeneous Boltzmann equation for Maxwellian molecules with a physical collision kernel which allows us to construct unique solutions to the initial value problem in a space of probability measures defined via the Fourier transform. In that space, the second moment of a measure is not assumed to be finite, so infinite energy solutions are not a priori excluded from our considerations. Moreover, we study the large time asymptotics of solutions and, in a particular case, we give an elementary proof of the asymptotic stability of self-similar solutions obtained by A.V. Bobylev and C. Cercignani [J. Stat. Phys. 106 (2002), 1039–1071].

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1. Introduction

We consider the homogeneous Boltzmann equation in $\mathbb{R}^3$

\begin{equation}
\partial_t f(v,t) = Q(f,f)(v,t)
\end{equation}

with the bilinear form corresponding to a Maxwellian gas

\begin{equation}
Q(g,f)(v) = \int_{\mathbb{R}^3} \int_{S^2} B \left( \frac{v - v_s}{|v - v_s|} \cdot \sigma \right) \left( f(v')g(v'_s) - f(v)g(v_s) \right) d\sigma dv_s.
\end{equation}

Here, the unknown density $f = f(v,t)$ is independent of the space variable, moreover, we denote

\begin{equation}
v' = \frac{v + v_s}{2} + \frac{|v - v_s|}{2} \sigma, \quad v_s = \frac{v + v_s}{2} - \frac{|v - v_s|}{2} \sigma
\end{equation}

with $\sigma$ varying in the unit sphere $S^2$. Equation (1.1)–(1.2) is supplemented with a nonnegative initial datum

\begin{equation}
f(v,0) = f_0(v)
\end{equation}

which is assumed to be a density of a probability distribution (or, more generally, a probability measure).

The collision kernel $B$ in (1.2) is supposed to be a nonnegative function and, in the case of Maxwellian molecules, it depends only on the deviation angle $\theta$, defined by the equation $\cos \theta = \frac{v - v_s}{|v - v_s|} \cdot \sigma$. It is well-known that the physical collision kernel $B = B(s)$ has a nonintegrable singularity as $s \to 1$ of the form $(1-s)^{-5/4}$ (see e.g. [8, p. 1043], [24, Ch. 1.1] and references therein). By the method developed in this work can, we can handle this kind of non-integrability as well as other singular kernels $B$, see Remark 2.1 for more details.

In the study of the Boltzmann equation, it is natural to assume that the nonnegative initial datum satisfies

\begin{equation}
\int_{\mathbb{R}^3} f_0(v) dv = 1, \quad \int_{\mathbb{R}^3} f_0(v) v_i dv = 0 \ (i = 1, 2, 3), \quad \int_{\mathbb{R}^3} f_0(v) |v|^2 dv = 3,
\end{equation}

because these relations are interpreted as the unit mass, the zero mean value, and the unit temperature of the gas, respectively. The existence of a unique solution of the initial value problem (1.1)–(1.4) under assumptions (1.5) and for a large class on nonintegrable collision kernels is well-known, see e.g. [6, 21, 24] and the references therein. This solution satisfies $f \in C^1([0, \infty), L^1(\mathbb{R}^3))$ and

\begin{equation}
\int_{\mathbb{R}^3} f(v,t) dv = 1, \quad \int_{\mathbb{R}^3} f(v,t) v_i dv = 0 \ (i = 1, 2, 3), \quad \int_{\mathbb{R}^3} f(v,t) |v|^2 dv = 3
\end{equation}
for all $t > 0$. For more information about the Boltzmann equation and its physical meaning, we refer the reader to the book by Cercignani [13] and to the more recent review article by Villani [24].

In this work, we propose a method of studying properties of solutions to problem (1.1)–(1.4) under very weak assumptions on the collision kernel in which we do not need to assume that the second moment of the unknown is finite. Hence, solutions with infinite temperature (or infinite energy) are not excluded \textit{a priori} from our considerations. These solutions are important because, as described by Bobylev and Cercignani [8], they are connected to the shock-wave problem.

We limit ourselves to the study of the homogeneous Boltzmann equation for Maxwellian molecules. In fact, our reasoning will be based on the important observation by Bobylev [5, 6] showing that, in this case, the bilinear form (1.2) can be easily studied by the Fourier transform. More precisely, denoting

\begin{equation}
\varphi(\xi, t) \equiv \hat{f}(\xi, t) = \int_{\mathbb{R}^3} e^{-iv \cdot \xi} f(v, t) \, dv
\end{equation}

Bobylev was able to convert equation (1.1) into the following equation for the new unknown $\varphi = \varphi(\xi, t)$

\begin{equation}
\partial_t \varphi(\xi, t) = \int_{S^2} \mathcal{B} \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left( \varphi(\xi^+, t)\varphi(\xi^-, t) - \varphi(\xi, t)\varphi(0, t) \right) \, d\sigma
\end{equation}

where

\begin{equation}
\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2}
\end{equation}

and we recall that these two vectors $\xi^+$ and $\xi^-$ satisfy the well-known relations

\begin{equation}
\xi^+ + \xi^- = \xi \quad \text{and} \quad |\xi^+|^2 + |\xi^-|^2 = |\xi|^2,
\end{equation}

hence,

\begin{equation}
|\xi^+|^2 = |\xi|^2 \frac{1 + \frac{\xi \cdot \sigma}{|\xi|^2}}{2} \quad \text{and} \quad |\xi^-|^2 = |\xi|^2 \frac{1 - \frac{\xi \cdot \sigma}{|\xi|^2}}{2}.
\end{equation}

We also note that the formula for the Fourier transform of the bilinear operator $Q$ on the right-hand side of (1.8) is actually a particular case of a more general one which does not assume Maxwellian collision kernel, see [11, Appendix] for more details.

In the following, we study properties of solutions to equation (1.8) supplemented with an initial datum

\begin{equation}
\varphi(\xi, 0) = \varphi_0(\xi).
\end{equation}

All our results on solutions of (1.1)–(1.4) are formulated for the initial value problem in the Fourier variables (1.8)–(1.12) in the space of characteristic functions (see Definition 3.1).
Motivated by a series of papers by Toscani and coauthors [17, 12, 22], we study the problem (1.8)–(1.12) in the function space described, in the Fourier variables and for suitable values of the parameter $\alpha$, by the following pseudo-norm

\[ \| \varphi \|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi)|}{|\xi|^\alpha}. \]

For $\alpha = 0$ the quantity (1.13) defines the space $\mathcal{PM}$ of pseudo-measures, i.e. tempered distributions, whose Fourier transforms are bounded functions. Moreover, we notice that for positive $\alpha$ the quantity $\| \varphi \|_\alpha$ describes the behavior of $\varphi$ at zero (i.e. the moments of the inverse Fourier transform of $\varphi$) and for $\alpha$ negative $\| \varphi \|_\alpha$ characterizes the behavior of $\varphi$ at infinity (i.e. the regularity of the inverse Fourier transform of $\varphi$).

However, this quantity is not a norm, in general. For example, when $\alpha > 0$, the number $\| \varphi \|_\alpha$ is finite if the inverse Fourier transform of $\varphi$ has polynomial moments of (high enough) degree equal to zero. On the other hand, if $\alpha < 0$, then $\| \varphi \|_\alpha = 0$ for any tempered distribution $\varphi$, whose inverse Fourier transform is a polynomial of certain (not too high) degree (see e.g. [16, Ch. 4]). It is easy to verify that if we work modulo suitable equivalence classes, than $\| \varphi \|_\alpha$ is a norm and, as it was noticed in [11], this norm corresponds to the generalized homogeneous Besov space $\dot{B}^{-\alpha,\infty}_{p,\infty}$, based on the pseudo-measure space $\mathcal{PM}$ of tempered distributions.

For the Navier-Stokes and other parabolic equations, it is well-known that the regularity (in space) of the solution plays an important role, so that it is natural to consider negative values of $\alpha$ in this context. Le Jan and Sznitman [19] introduced the scaling invariant norm $\| \cdot \|_\alpha$ with $\alpha = -2$ for the Navier-Stokes equations. In [10], following this approach, we obtained the existence and the large time asymptotic of infinite energy solutions to the incompressible Navier-Stokes system in the space $\dot{B}^{-2,\infty}_{p,\infty}$. A similar approach was introduced in [3] for the study of a model of gravitating particles (see [9], for a review).

At variance with the Navier-Stokes system, in the case of the homogeneous Boltzmann equation (1.11), the space integrability of a solution plays a pivotal role. It means that we should take into account the behavior of the Fourier transform of a solution as $|\xi| \to 0$ and not when $|\xi| \to \infty$. In other words, if it is natural to take negative values of $\alpha$ for the Navier-Stokes equations and other parabolic systems, positive values of $\alpha$ should be considered for the Boltzmann equation.

In this direction, for $\alpha \geq 2$, Toscani and coauthors [17, 12, 22] were able to obtain several nice results for the homogeneous Boltzmann equation (see Villani [24], for a review). For example, in the case of $\alpha = 2$, Toscani and Villani [22] proved the uniqueness and the stability of solutions to the homogeneous Boltzmann equation for Maxwellian
gas with a physical nonintegrable collision kernel. Their proof required the energy of
the solution (i.e. the last equality in (1.6)) to be finite.

In this work, we treat the case $0 \leq \alpha \leq 2$ and we study the initial value problem
(1.8)–(1.12) in a larger space where infinite energy solutions are not excluded \textit{a priori}. In
this setting, our norms growth exponentially in time and the trend to equilibrium will
be described in self-similar variables. We do not impose the Grad cut-off assumption:
any collision kernel satisfying $(1 - s^2)^{\alpha_0/4}B(s) \in L^1(-1, 1)$ for some $\alpha_0 \in [0, 2]$ will be
included in our approach (see Remark 2.1 for more details).

2. Main results

We begin by defining the function set which plays the main role in our study of
properties of solution to the initial value problem (1.8)–(1.12). First, recall that any
solution $f = f(\cdot, t)$ of the homogeneous Boltzmann equation (1.1)–(1.2) is (after the well-
known normalization) a probability measure for every $t \geq 0$. Following the probabilistic
terminology, we are going to use the set $\mathcal{K}$ of “characteristic functions”, i.e. those
functions that are Fourier transforms of probability measures (cf. Definition 3.1). In the
next section, we will also introduce a more general set consisting of “positive definite
functions” (cf. Definition 3.2). The Bochner theorem (see Theorem 3.3) ensures that the
set of characteristic functions coincides with the set of positive definite functions that
are continuous.

The main interest in working with this more general framework of functions is that we
can easily derive a nice estimate of the quantity $\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)$ (see inequality (3.5))
that will be useful in the study of the collision term $Q$ with a non-integrable collision
kernel.

Inspired by the papers of Toscani and his coauthors, we introduce for each $\alpha \in [0, 2]$ the
space
\begin{equation}
\mathcal{K}_\alpha = \left\{ \varphi : \mathbb{R}^3 \to \mathbb{C} \text{ is a characteristic function such that } \|\varphi - 1\|_\alpha < \infty \right\},
\end{equation}
where
\begin{equation}
\|\varphi - 1\|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^{\alpha}}.
\end{equation}
The set $\mathcal{K}_\alpha$ endowed with the distance
\begin{equation}
\|\varphi - \varphi\|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \varphi(\xi)|}{|\xi|^{\alpha}}.
\end{equation}
is a complete metric space (see Proposition 3.10 below).

The definition of $\mathcal{K}_\alpha$ makes sense also for $\alpha > 2$, however, as we will see later, $\mathcal{K}_\alpha = \{1\}$
in this case. In fact, in order to have a non trivial function space in the case $\alpha > 2$,
higher order moments should be considered and a suitable Taylor polynomial should be subtracted from \( \varphi \) in the definition of the space given by eq. (2.1) as it was done in [12] (see also [24]). On the other hand, \( K^0 \) coincides with the set of all characteristic functions and the following imbeddings hold true

\[
\{1\} \subseteq K^\alpha \subseteq K^{\alpha_0} \subseteq K^0 \quad \text{for all } 2 \geq \alpha \geq \alpha_0 \geq 0,
\]

see Lemma 3.12 for the proofs of all these properties.

Let us also emphasize that the Fourier transform of any probability measure with the finite moment of order \( \alpha \) belongs to \( K^\alpha \). This important feature, proved below in Lemma 3.15, allows us to transfer the properties stated in eq. (1.5) for the function \( f_0 \) into properties to be verified by the new variable \( \varphi_0 \), justifying in this way the choice of the functional setting \( K^\alpha \). We show, however, that the set \( K^\alpha \) is, in fact, bigger than the set of the Fourier transforms of probability measures with the finite moment of order \( \alpha \), see Remark 3.16.

In the next section, we present examples of functions from \( K^\alpha \) as well as some fundamental properties of the metric space \( K^\alpha \).

Next, for every \( \xi \in \mathbb{R}^3 \setminus \{0\} \), we define the quantity which appears systematically in our considerations:

\[
\lambda_\alpha \equiv \int_{S^2} B\left(\frac{\xi \cdot \sigma}{|\xi|}\right)\left(\frac{|\xi^-|^\alpha + |\xi^+|^\alpha}{|\xi|^\alpha} - 1\right) d\sigma.
\]

Note that, in view of relations (1.10), we have \( \lambda_2 = 0 \). In Corollary 4.2, we prove that \( \lambda_\alpha \) is finite, independent of \( \xi \), and positive for \( 0 < \alpha < 2 \), under the assumption \( (1 - s)^{\alpha/2}(1 + s)^{\alpha/2}B(s) \in L^1(-1, 1) \). However, to construct solutions to the initial-value problem (1.8)–(1.12), we have to impose the stronger assumption on the collision kernel, namely,

\[
(1 - s)^{\alpha_0/4}(1 + s)^{\alpha_0/4}B(s) \in L^1(-1, 1) \quad \text{for some } \alpha_0 \in [0, 2].
\]

Remark 2.1. We have already mentioned in the introduction that the physical collision kernel \( B = B(s) \) behaves at \( s = 1 \) as the function \((1 - s)^{-5/4}\). Hence, the assumption (2.6) holds true for this kind of singularity if \(-5/4 + \alpha_0/4 > -1\), that is for \( \alpha_0 > 1 \). More generally, as emphasized e.g. in [23] and in [24, Ch. 1.1], there are important collision kernels in physics and in modeling with the behavior \( B(s) \sim (1 - s)^{-1-\nu} \) as \( s \to 1 \) for some \( \nu > 0 \). We can deal with this kind of singularity if \( \nu < 1/2 \) provided \( \alpha_0 > 4\nu \).

We are now in a position to state our main result the existence of solutions to the initial value problem (1.8)–(1.12).
**Theorem 2.2** (Existence and uniqueness of solutions). Assume that $\mathcal{B}$ satisfies assumption (2.6) for some $\alpha_0 \in [0, 2]$. Then for each $\alpha \in [\alpha_0, 2]$ and every $\varphi_0 \in \mathcal{K}^\alpha$ there exists a classical solution $\varphi \in C([0, \infty), \mathcal{K}^\alpha)$ of problem (1.8)-(1.12). The solution is unique in the space $C([0, \infty), \mathcal{K}^{\alpha_0})$.

Notice that, for every initial datum $\varphi_0 \in \mathcal{K}^\alpha$ with $\alpha \in [\alpha_0, 2]$, the corresponding solution belongs to the space $C([0, \infty), \mathcal{K}^{\alpha_0})$ in view of imbedding (2.4).

**Remark 2.3.** Let us first explain that Theorem 2.2 generalizes known results on finite energy solutions to the initial value problem (1.1)-(1.4). Indeed, if $\varphi_0 \in \mathcal{K}^2$ is the Fourier transform of the function $f_0$ satisfying (1.5), then the corresponding solution $\varphi = \varphi(\xi, t)$ of problem (1.8)-(1.12), constructed in Theorem 2.2, is the Fourier transform of the solution $f = f(v, t)$ to the original initial value problem (1.1)-(1.4) which satisfies the important conservation laws from (1.6). To show this persistence property, it suffices to note that the existence of the solution to (1.1)-(1.4) satisfying (1.6) is well-known (see e.g. [21] where the same argument is valid for more general collision kernel satisfying (2.6)). By uniqueness, this solution agrees with our solution constructed in Theorem 2.2 in the space $C([0, \infty), \mathcal{K}^2)$.

**Remark 2.4.** In Theorem 2.2 we construct a large class of smooth solutions (and not only probability measures) to the original initial value problem (1.1)-(1.4). To see it, it suffices to apply the well-known regularization procedure based on the Bobylev identity

$$Q(g * M, f * M) = Q(g, f) * M,$$

where $Q$ is the Boltzmann operator (1.2) and $M$ denotes the Maxwellian probability distribution. Identity (2.7) results immediately after computing the Fourier transform of its both sides and using the Bobylev form of $\hat{Q}$ together with the equality $|\xi^+|^2 + |\xi^-|^2 = |\xi|^2$.

Now, let $\widehat{M}(\xi) = e^{-A|\xi|^2}$ for some $A > 0$ and $\varphi_0 = \widehat{\mu}_0 \in \mathcal{K}^\alpha$ for some probability measure $\mu_0$. Denote by $\varphi = \varphi(\xi, t)$ the solution to (1.8)-(1.12) with $\varphi_0$ as the initial datum. By the Bobylev identity (2.7) written in the Fourier variables, the function $\varphi(\xi, t)e^{-A|\xi|^2}$ is the solution of problem (1.8)-(1.12) corresponding to the initial datum $\varphi_0 \widehat{M} = (\mu_0 * M) \in \mathcal{K}^\alpha$. Computing the inverse Fourier transform of this rapidly decreasing in $\xi$ solution to (1.8)-(1.12), we obtain the smooth solution of the original problem (1.1)-(1.4) with the initial condition $\mu_0 * M$.

In this work, however, we do not address questions on regularity of solutions to the homogeneous Boltzmann equation for Maxwellian molecules. We refer the reader to the recent works [15, 23] and to references therein for proofs of smoothing properties.
of finite energy solutions (namely, those satisfying (1.6)) to (1.1)–(1.4) including the Gevrey smoothing and the Sobolev space regularity.

Next, we prove the stability inequality for solutions to problem (1.8)–(1.12) which were constructed in Theorem 2.2.

**Theorem 2.5 (Stability of solutions).** Assume that $B$ satisfies (2.6) for some $\alpha_0 \in [0, 2]$. Let $\alpha \in [\alpha_0, 2]$ and consider two solutions $\varphi, \tilde{\varphi} \in C([0, \infty), K^\alpha)$ of the problem (1.8)–(1.12) corresponding to the initial data $\varphi_0, \tilde{\varphi}_0 \in K^\alpha$, respectively. Then for every $t \geq 0$

\begin{equation}
\|\varphi(t) - \tilde{\varphi}(t)\|_{\alpha} \leq e^{\lambda_\alpha t} \|\varphi_0 - \tilde{\varphi}_0\|_{\alpha},
\end{equation}

where the constant $\lambda_\alpha \geq 0$ is defined in (2.7).

The exponential growth in time on the right-hand-side of inequality (2.8) is optimal, see Remark 6.5 below. We use this exponential estimate in our study of the asymptotic stability of solutions to problem (1.8)–(1.12).

To prove Theorems 2.2 and 2.5, we begin by imposing the cut-off assumption on the kernel $B$, namely, we assume $B \in L^1(-1, 1)$ (this is the condition (2.6) with $\alpha_0 = 0$). In this particular case, the results on the existence and the uniqueness of solutions to (1.8)–(1.12) are not new. It is well-known that, for integrable collision kernels, the solution of the initial value problem (1.8)–(1.12) has the explicit representation via the Wild sum (see (4.15)–(4.16), below) which is convergent under relatively weak assumptions imposed on the initial datum $\varphi_0$ (cf. e.g. [21, Thm. 2.1]). Here, for the completeness of the exposition, we prove that the Wild series converges in the space $C([0, \infty), K^\alpha)$. Moreover, we present another construction of solutions to (1.8)–(1.12) under the cut-off condition imposed on $B$, based on the Banach contraction principle, see Theorem 4.5 in Section 4.

The proofs of the existence, the uniqueness, and the stability of solutions to (1.8)–(1.12) in the space $C([0, \infty), K^\alpha)$ in the case of nonintegrable collision kernels $B$ satisfying (2.6) are our main contribution to this theory. We are able to remove the cut-off assumption and to complete the proofs of Theorems 2.2 and 2.5 by using a well-known approximation argument combined with suitable (and crucial for our reasoning) estimates for characteristic functions form the space $K^\alpha$, see Lemma 3.14.

Next, we study the large time behavior of solutions to the initial value problem (1.8)–(1.12). Here, the key role is played by self-similar solutions of equation (1.8) constructed by Bobylev and Cercignani [7, 8] in the following form

\begin{equation}
\varphi(\xi, t) = \Phi(\xi e^{\mu t}) \quad \text{for some} \quad \mu \in \mathbb{R}.
\end{equation}
Substituting the function $\varphi$ from (2.9) into equation (1.8), we obtain the equation for the profile $\Phi$ (here, $\eta = \xi e^{\lambda t}$)

$$\mu \eta \cdot \nabla \Phi(\eta) = \int_{S^2} B \left( \frac{\eta \cdot \sigma}{|\eta|} \right) \left( \Phi(\eta^+)\Phi(\eta^-) - \Phi(\eta)\Phi(0) \right) d\sigma,$$

where $\eta^+$ and $\eta^-$ are defined analogously as the vectors in (1.9).

Below, in Lemma 6.1, we recall an argument which allows us to calculate the scaling parameter $\mu$. We show that if a radial solution $\Phi \in K^\alpha$ of equation (2.10) satisfies

$$\lim_{|\eta| \to 0} \left( \Phi(\eta) - 1 \right)|\eta|^{-\alpha} = K$$

for some constant $K \neq 0$ then, necessarily,

$$\mu = \mu_\alpha = \frac{\lambda_\alpha}{\alpha},$$

where the constant $\lambda_\alpha$ is defined in (2.5). This is a well-founded argument because, for every $\alpha \in (0, 2)$ and $K \neq 0$, Bobylev and Cercignani [8] proved the existence of a solution $\Phi = \Phi_{\alpha,K}$ to equation (2.10) satisfying

$$\lim_{|\eta| \to 0} \frac{\Phi_{\alpha,K}(\eta) - 1}{|\eta|^{\alpha}} = K.$$

In Theorem 6.2 below, we sketch the Bobylev and Cercignani construction. Here, we only notice that the constant $K$ in (2.12) has to be nonpositive because every characteristic function $\Phi_{\alpha,K}$ satisfies $|\Phi_{\alpha,K}(\eta)| \leq 1$ for all $\eta \in \mathbb{R}^3$, see Remark 6.3 for more details.

Remark 2.6. If the solution $\varphi(\xi, t)$ and the self-similar profile $\Phi(\xi)$ from (2.9) are the Fourier transforms of functions $f = f(v, t)$ and $F = F(v)$, respectively, then we obtain the self-similar solution of the Boltzmann equation (1.1)–(1.2) in the original variables in the form

$$f(v, t) = e^{-3\mu t} F(v e^{-\mu t}).$$

Obviously, $\int_{\mathbb{R}^3} f(v, t) dv = \int_{\mathbb{R}^3} F(v) dv$ for all $t \in \mathbb{R}$. This solution, however, cannot have finite energy, because the condition $F \in L^1(\mathbb{R}^3, |v|^2 dv)$ leads immediately to the equality

$$\int_{\mathbb{R}^3} f(v, t)|v|^2 dv = e^{2\mu t} \int_{\mathbb{R}^3} F(v)|v|^2 dv$$

which contradicts (1.6) if $\mu \neq 0$, see [7, 8] for more detailed discussion.

In order to study the asymptotic stability of the self-similar solutions $\varphi(\xi, t) = \Phi_{\alpha,K}(\xi e^{\mu t})$ as well as the large time behavior of other solutions to system (1.8)–(1.12), it is more convenient to work in self-similar variables. Hence, given a solution $\varphi = \varphi(\xi, t)$ to equation (1.8) we consider the new function

$$\psi(\xi, t) = \varphi(\xi e^{-\mu t}, t) \quad \text{with} \quad \mu_\alpha = \frac{\lambda_\alpha}{\alpha},$$
which is the solution of the initial value problem

\begin{align}
\partial_t \psi + \mu_0 \xi \cdot \nabla \psi &= \int_{S^2} \mathcal{B} \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left( \psi(\xi^+, t)\psi(\xi^-, t) - \psi(\xi, t)\psi(0, t) \right) d\sigma, \\
\psi(\xi, 0) &= \psi_0(\xi) = \varphi_0(\xi).
\end{align}

Note that, in the new variables, the self-similar profiles $\Phi_{\alpha, K}$ are stationary solutions of equation (2.14) (cf. equation (2.10)).

Now, we are in a position to state our main result on the large time asymptotics of solutions to (1.8)–(1.12).

**Theorem 2.7 (Large time asymptotics of solutions).** *Assume that the collision kernel $\mathcal{B}$ satisfies the non cut-off condition (2.6) for some $\alpha_0 \in (0, 2)$. Let $\alpha \in [\alpha_0, 2)$. Suppose that $\psi_0, \tilde{\psi}_0 \in K^\alpha$ satisfy

\begin{equation}
\lim_{|\xi| \to 0} \frac{\psi_0(\xi) - \tilde{\psi}_0(\xi)}{|\xi|^{\alpha}} = 0.
\end{equation}

Then the corresponding solutions $\psi(\xi, t)$ and $\tilde{\psi}(\xi, t)$ of the rescaled problem (2.14)–(2.15) approach each other in the following sense

\begin{equation}
\lim_{t \to \infty} \|\psi(t) - \tilde{\psi}(t)\|_\alpha = 0.
\end{equation}

The proof of Theorem 2.7 (given in Section 7) is very simple and is an almost immediate consequence of the generalized version of the stability inequality (2.8) (see Lemma 4.8 and Corollary 5.2 below).

Combining Theorem 2.7 with the property of the self-similar profile stated in (2.12), we find the condition on the initial datum $\psi_0 = \varphi_0$ such the corresponding solution of (1.8)–(1.12) converges (in self-similar variables) toward the self-similar profile $\Phi_{\alpha, K}$. This particular case of Theorem 2.7 is stated in the following corollary.

**Corollary 2.8 (Self-similar asymptotics).** *Assume that the collision kernel $\mathcal{B}$ satisfies (2.6) for some $\alpha_0 \in (0, 2)$. Let $\alpha \in [\alpha_0, 2)$. Consider the initial datum $\psi_0 \in K^\alpha$ such that

\begin{equation}
\lim_{|\xi| \to 0} \frac{\psi_0(\xi) - 1}{|\xi|^{\alpha}} = K \quad \text{for some} \quad K \leq 0.
\end{equation}

Denote by $\Phi_{\alpha, K}$ the self-similar profile of Bobylev and Cercignani. Then the corresponding solution $\psi(\xi, t)$ of problem (2.14)–(2.15) satisfies

\begin{equation}
\lim_{t \to \infty} \|\psi(t) - \Phi_{\alpha, K}\|_\alpha = 0 \quad \text{if} \quad K < 0
\end{equation}

and

\begin{equation}
\lim_{t \to \infty} \|\psi(t) - 1\|_\alpha = 0 \quad \text{if} \quad K = 0.
\end{equation}
Remark 2.9. It is worth to reformulate the above asymptotic results for solutions to problem (1.8)–(1.12) before rescaling stated in (2.13). Under the assumptions of Theorem 2.7, for every initial conditions \( \varphi_0, \bar{\varphi}_0 \in K^\alpha \) such that

\[
\lim_{|\xi| \to 0} \frac{\varphi_0(\xi) - \bar{\varphi}_0(\xi)}{|\xi|^\alpha} = 0,
\]

the corresponding solutions \( \varphi = \varphi(\xi, t) \) anf \( \bar{\varphi} = \bar{\varphi}(\xi, t) \) of problem (1.8)–(1.12) satisfy

\[
\lim_{t \to \infty} e^{-\lambda_\alpha t} \|\varphi(t) - \bar{\varphi}(t)\|_\alpha = 0.
\]

This is the immediate consequence of the change of variables (2.13) leading to the following equalities

\[
\|\psi(t) - \tilde{\psi}(t)\|_\alpha = \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi e^{-\mu t}, t) - \bar{\varphi}(\xi e^{-\mu t}, t)|}{|\xi|^\alpha} = e^{-\lambda_\alpha t} \|\varphi(t) - \bar{\varphi}(t)\|_\alpha
\]
due to the identity \( \lambda_\alpha = \alpha \mu_\alpha \).

Remark 2.10. Let us discuss the large time behavior of solutions to (1.8)–(1.12) in the case \( \alpha = 2 \). Note first that \( \mu_2 = \lambda_2 = 0 \) which is the immediate consequence of the definitions (2.5) and (2.11) combined with the second equality in (1.10). The proof of Theorem 2.7 does not work for \( \alpha = 2 \) because the assumption \( \lambda_\alpha \neq 0 \) is essential in our reasoning. In particular, we are not able to adapt the proof of Theorem 2.7 to show the convergence of solutions to (1.8)–(1.12) from the space \( K^2 \) toward Maxwellians in the Fourier variables, \( \Phi_A(\eta) = e^{-A|\eta|^2} \) with \( A > 0 \), which are the solutions of equation (2.10) with \( \mu = 0 \).

The large time asymptotics of solutions to (1.8)–(1.12) in this limit case \( \alpha = 2 \) was studied by Toscani and Villani [22]. Their result on the convergence of solutions to (1.8)–(1.12) to Maxwellians \( \Phi_A(\eta) = e^{-A|\eta|^2} \) stated in [22] Cor. 5.3] can be formulated as follows. Assume that \( \varphi_0 \) is the Fourier transform of a mean zero probability measure with finite second moment (hence, \( \psi_0 \in K^2 \) by Lemma 3.15) which, moreover, satisfies (2.17) for some \( K = -A < 0 \). Denote by \( \psi = \psi(\xi, t) \) the corresponding solution. Then \( \|\psi(t) - \Phi_A\|_2 \) is decreasing to 0 as \( t \to \infty \).

In view of Remark 2.10, the asymptotics stated in Corollary 2.8 should be treated as the extension of the classical result on the convergence of solutions of the Boltzmann equation to Maxwellian. Roughly speaking, Corollary 2.8 says that solutions to original problem (1.1)–(1.4) with infinite energy (i.e. when the Fourier transform of their initial conditions satisfy (2.17) for some \( \alpha \neq 2 \) and \( K \neq 0 \) converge, in self-similar variables, toward the universal probability measure which is the Fourier transform of the self-similar profile constructed by Bobylev and Cercignani. This probability measure should be treated as the counterpart of Maxwellian which is the solution of (2.10) with \( \alpha = 2 \).
(recall that $\mu_2 = 0$). Such a convergence, in a pointwise sense and for very particular initial conditions (radially symmetric and in the form of a series) was proved in [8 Thm. 6.2]. The very simple proof of the convergence of solutions to (1.8)–(1.12) toward self-similar profile $\Phi_{\alpha,K}$ in the metric $\| \cdot \|_\alpha$ for every solution (not necessarily radial) corresponding to the initial datum $\psi_0 \in K^\alpha$ satisfying (2.17) is our main contribution to this theory.

3. Continuous positive definite functions

Since we deal with the Fourier transform of the Boltzmann equation and since this equation describes the time evolution of a probability measure (the unknown function $f(v,t)$) it is natural to begin our investigation recalling the classical definition of “characteristic functions”. These functions have been systematically used in the papers devoted to the study of the homogeneous Boltzmann equation (1.8)–(1.12) in Fourier variables (e.g. [5, 6, 7, 8, 12, 14, 15, 17, 22, 24]).

Definition 3.1. A function $\varphi : \mathbb{R}^N \to \mathbb{C}$ is called characteristic function if there is a probability measure $\mu$ (i.e. a Borel measure with $\int_{\mathbb{R}^N} \mu(dx) = 1$) such that we have the following identity $\varphi(\xi) = \hat{\mu}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} \mu(dx)$. The set of all characteristic functions $\varphi : \mathbb{R}^N \to \mathbb{C}$ we will be denoted by $K$.

In some estimates presented in this section, it is more convenient to introduced the more general setting provided by the definition of a positive definite function.

Definition 3.2. A function $\varphi : \mathbb{R}^N \to \mathbb{C}$ is called positive definite if for every $k \in \mathbb{N}$ and every vectors $\xi^1, ..., \xi^k \in \mathbb{R}^N$ the matrix $(\varphi(\xi^j - \xi^\ell))_{j,\ell=1,...,k}$ is positive Hermitian, i.e. for all $\lambda_1, ..., \lambda_k \in \mathbb{C}$ we have

\begin{equation}
\sum_{j,\ell=1}^{k} \varphi(\xi^j - \xi^\ell)\lambda_j\lambda_\ell \geq 0.
\end{equation}

The following celebrated theorem by Bochner plays a fundamental role in the theory of positive definite functions, since it states that the set of continuous positive definite functions coincides with the set of characteristic functions.

Theorem 3.3. A function $\varphi : \mathbb{R}^N \to \mathbb{C}$ is a characteristic function if and only if the following conditions are fulfilled

i. $\varphi$ is a continuous function on $\mathbb{R}^N$

ii. $\varphi(0) = 1$

iii. $\varphi$ is positive definite.
We refer the reader to the books either by Berg and Forst [2, Ch. I, §3] or by Jacob [18, Ch. 3] for proofs of properties of positive definite functions which will be listed below.

The reason why we prefer to introduce the larger set of positive definite functions (instead of simple characteristic functions) is that we can easily derive estimates on a certain product of positive definite functions (see inequality (3.5)) that will be useful for the study of the collision operator.

Before deriving such key estimates, we start with much simpler results that follow immediately from the definition of positive definite functions.

**Lemma 3.4.** Every positive definite function \( \varphi \) satisfies

\[
\varphi(\xi) = \varphi(-\xi) \quad \text{and} \quad \varphi(0) \geq 0
\]

and

\[
|\varphi(\xi)| \leq \varphi(0), \quad \text{hence} \quad \sup_{\xi \in \mathbb{R}^N} |\varphi(\xi)| = \varphi(0).
\]

**Lemma 3.5.** Any linear combination with positive coefficients of positive definite functions is a positive definite function. The set of positive definite functions is closed with respect to the pointwise convergence.

**Lemma 3.6.** The product of two positive definite functions is a positive definite function.

*Proof.* This is the immediate consequence of Definition 3.2 if we note that for every two positive Hermitian matrices \((a_{jk})_{j,k=1,...,N}\) and \((b_{jk})_{j,k=1,...,N}\), the matrix \((c_{jk})_{j,k=1,...,N}\) with elements \(c_{jk} = a_{jk}b_{jk}\) is positive Hermitian, see e.g. [18, Lemma 3.5.9].

**Lemma 3.7.** If \( \varphi \) is a positive definite function, so are \( \overline{\varphi} \) and \( \text{Re} \varphi \).

*Proof.* To show that \( \overline{\varphi} \) is a positive definite function it suffices to compute the complex conjugate of inequality (3.1). Using equality \( \text{Re} \varphi = (\varphi + \overline{\varphi})/2 \) we complete the proof by Lemma 3.5. □

Now, we state two important inequalities for positive definite functions which play the fundamental role in our reasoning when we deal with nonintegrable collision kernels. By this reason, for the completeness of the exposition, we sketch their proofs, see either [2, Ch. I, §3.4] or [18, Lemma 3.5.10] for more details.

**Lemma 3.8.** For any positive definite function \( \varphi = \varphi(\xi) \) such that \( \varphi(0) = 1 \) we have

\[
|\varphi(\xi) - \varphi(\eta)|^2 \leq 2(1 - \text{Re} \varphi(\xi - \eta))
\]

and

\[
|\varphi(\xi)\varphi(\eta) - \varphi(\xi + \eta)|^2 \leq (1 - |\varphi(\xi)|^2)(1 - |\varphi(\eta)|^2)
\]
for all $\xi, \eta \in \mathbb{R}^N$.

Proof. We are going to use inequality (3.1) with suitable chosen vectors $\xi^j$ and constants $\lambda_j$. Indeed, for $\xi, \eta \in \mathbb{R}^N$ such that $\varphi(\xi) \neq \varphi(\eta)$ we consider the Hermitian matrix

$$
(3.6) \quad \begin{pmatrix}
\varphi(0) & \varphi(\xi) & \varphi(\eta) \\
\varphi(\xi) & \varphi(0) & \varphi(\xi - \eta) \\
\varphi(\eta) & \varphi(\xi - \eta) & \varphi(0)
\end{pmatrix},
$$

where $\varphi(0) = 1$. Next, with arbitrary and given $s \in \mathbb{R}$, we define

$$
\lambda_1 = s, \quad \lambda_2 = \frac{s|\varphi(\xi) - \varphi(\eta)|}{\varphi(\xi) - \varphi(\eta)}, \quad \lambda_3 = -\lambda_2.
$$

Hence, applying inequality (3.1), we find by a straightforward calculation

$$
1 + 2s^2 + 2s|\varphi(\xi) - \varphi(\eta)| - 2s^2 \text{Re} \varphi(\xi - \eta) \geq 0.
$$

This means that the discriminate of the quadratic form on the left-hand side (as the function of $s$) has to be nonpositive, hence,

$$
4|\varphi(\xi) - \varphi(\eta)|^2 \leq 4(2 - 2\text{Re} \varphi(\xi - \eta)).
$$

which completes the proof of (3.4).

On the other hand, inequality (3.5) is equivalent to the fact that the determinant of the Hermitian matrix (3.6) with $\varphi(0) = 1$ is non-negative. □

Let us now recall the definition of the function space

$$
(3.7) \quad \mathcal{K}^\alpha = \left\{ \varphi : \mathbb{R}^3 \to \mathbb{C} \text{ is a characteristic function such that } \|\varphi - 1\|_\alpha < \infty \right\},
$$

supplemented with the metric

$$
\|\varphi - \tilde{\varphi}\|_\alpha \equiv \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha}.
$$

First, we give examples of characteristic functions from the space $\mathcal{K}^\alpha$.

**Example 3.9.**

i. The function $\varphi = \varphi(\xi)$ satisfying $\varphi(0) = 1$ and $\varphi(\xi) = 0$ for $\xi$ different from zero is a positive definite function, however, it is not a characteristic function (since it is not continuous).

ii. The function $\varphi(\xi) = e^{-ib \cdot \xi}$, with fixed $b \in \mathbb{R}^3$, is the Fourier transform of the Dirac delta $\delta_b$ concentrated at $b$. It belongs to $\mathcal{K}^\alpha$ for every $\alpha \in [0, 1]$.

iii. Maxwellians in the Fourier variables, $\varphi(\xi) = e^{-A|\xi|^2}$ with fixed $A > 0$, belongs to $\mathcal{K}^\alpha$ for every $\alpha \in [0, 2]$. 

iv. The function \( \varphi_\alpha(\xi) = e^{-|\xi|^\alpha} \) is a characteristic function for each \( \alpha \in (0, 2] \) because this is the Fourier transform of the probability distribution of an \( \alpha \)-stable symmetric Lévy process, see e.g. [18, Examples 3.5.23 and 3.9.17] for more details. Hence, \( \varphi_\alpha \in \mathcal{K}_\beta \) for each \( \beta \in [0, \alpha] \).

**Proposition 3.10.** For every \( \alpha \in [0, 2] \), the set \( \mathcal{K}_\alpha \) endowed with the distance (2.3) is a complete metric space.

**Proof.** The proof is immediate because the set of characteristic functions is closed with respect to the pointwise convergence. \( \square \)

Next, we state without the proof simple properties of the space \( \mathcal{K}_\alpha \).

**Lemma 3.11.**

i. The space \( \mathcal{K}_\alpha \) is not a vector space (e.g. \( \varphi(\xi) \equiv 0 \) does not belong to \( \mathcal{K}_\alpha \)).

ii. \( \varphi \equiv 1 \in \mathcal{K}_\alpha \) for every \( \alpha \geq 0 \).

iii. For every \( \varphi \in \mathcal{K}_\alpha \) we have \( |\varphi(\xi)| \leq \varphi(0) = 1 \) (cf. (3.3)).

iv. For all \( \varphi, \tilde{\varphi} \in \mathcal{K}_\alpha \) their product satisfies \( \varphi \tilde{\varphi} \in \mathcal{K}_\alpha \).

v. Any linear and convex combination of functions from \( \mathcal{K}_\alpha \) belongs to \( \mathcal{K}_\alpha \) (cf. Lemma 3.5).

In the following lemma, we explain why we limit ourselves to \( \alpha \in [0, 2] \) in the definition of \( \mathcal{K}_\alpha \).

**Lemma 3.12.**

i. \( \mathcal{K}_0 = \mathcal{K} \)

ii. \( \mathcal{K}_{\alpha_1} \subseteq \mathcal{K}_{\alpha_2} \) if \( \alpha_2 \leq \alpha_1 \).

iii. \( \mathcal{K}_\alpha = \{1\} \) for every \( \alpha > 2 \).

**Proof.** In the case of i., it suffices to use (3.3) in order to see that any characteristic function \( \varphi \) is bounded, more precisely, it satisfies \( \sup_{\xi \in \mathbb{R}^N} |\varphi(\xi) - 1| \leq \varphi(0) + 1 \).

To show ii., for any \( \varphi \in \mathcal{K}_{\alpha_1} \), we proceed as follows

\[
\| \varphi - 1 \|_{\alpha_2} \leq \sup_{|\xi| \leq 1} \frac{|\varphi(\xi) - 1|}{|\xi|^{\alpha_2}} + \sup_{|\xi| > 1} \frac{|\varphi(\xi) - 1|}{|\xi|^{\alpha_2}}
\]

\[
\leq \sup_{|\xi| \leq 1} \frac{|\varphi(\xi) - 1|}{|\xi|^{\alpha_1}} + \sup_{|\xi| > 1} |\varphi(\xi) - 1|
\]

\[
\leq \| \varphi - 1 \|_{\alpha_1} + \varphi(0) + 1,
\]

since \( \alpha_2 \leq \alpha_1 \) and by using (3.3). Hence, \( \varphi \in \mathcal{K}_{\alpha_2} \).

Let us show iii. It follows immediately form eq. (2.2) that any \( \varphi \in \mathcal{K}_\alpha \) with \( \alpha > 2 \) satisfies

\[
|1 - \varphi(\xi)| \leq |\xi|^{\alpha-2} \| \varphi - 1 \|_{\alpha} \rightarrow 0 \quad \text{as} \quad |\xi| \rightarrow 0.
\]
Next, using inequality (3.4) we get for any unit vector \( \zeta \in \mathbb{R}^3 \) and all \( \xi \in \mathbb{R}^3 \)

\[
\left| \frac{\varphi(\xi + h\zeta) - \varphi(\xi)}{h} \right|^2 \leq 2 \left( 1 - \text{Re} \varphi(h\zeta) \right)^2 \leq 2 \left| \frac{1 - \varphi(h\zeta)}{h^2} \right|,
\]

thus, by (3.8), we have

\[
\lim_{h \to 0} \frac{\varphi(\xi + h\zeta) - \varphi(\xi)}{h} = 0.
\]

Hence, for all \( \zeta \in \mathbb{R}^3 \) the directional derivative \( \zeta \cdot \nabla \varphi(\xi) \) exists and is equal to zero, implying that \( \varphi \) is constant. \( \square \)

**Lemma 3.13.** Let \( \alpha \in [0, 2] \). Assume that \( \varphi \in \mathcal{K}^\alpha \). Then \( \text{Re} \varphi \in \mathcal{K}^\alpha \),

\[
\| \text{Re} \varphi - 1 \|_\alpha \leq \| \varphi - 1 \|_\alpha, \quad \text{and} \quad \sup_{\xi \in \mathbb{R}^3 \backslash \{0\}} \frac{|\text{Im} \varphi(\xi)|}{|\xi|^\alpha} \leq \| \varphi - 1 \|_\alpha.
\]

**Proof.** Let \( \varphi \in \mathcal{K}^\alpha \). It is well-known that \( \text{Re} \varphi \) is a characteristic function (e.g. it suffices to combine Lemma 3.7 with the Bochner Theorem 3.3). Now, by the Pythagoras theorem, we obtain

\[
|\varphi(\xi) - 1|^2 = |\text{Im} \varphi(\xi)|^2 + |\text{Re} \varphi(\xi) - 1|^2 \geq |\text{Re} \varphi(\xi) - 1|^2.
\]

Hence, we complete the proof of the first inequality in (3.9) dividing (3.10) by \( |\xi|^\alpha \) and computing the supremum with respect to \( \xi \in \mathbb{R}^3 \).

To show the second inequality in (3.9), we proceed analogously using the inequality \( |\varphi(\xi) - 1| \geq |\text{Im} \varphi(\xi)| \) resulting from (3.10). \( \square \)

Now, we are in a position to prove an inequality which implies (see the proof of Lemma 5.1) that the nonlinear term in equation (1.8) is well-defined for functions from \( \mathcal{K}^\alpha \) if we impose the condition (2.6) on the collision kernel.

**Lemma 3.14.** Let \( \alpha \in [0, 2] \). Assume that \( \varphi \in \mathcal{K}^\alpha \). For every \( \xi \in \mathbb{R}^3 \) define \( \xi^+ \) and \( \xi^- \) by equations (1.9) with some fixed \( n \in S^2 \). Then

\[
|\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)| \leq 4|\xi^+|^{\alpha/2}|\xi^-|^{\alpha/2}\|\varphi - 1\|_\alpha.
\]

**Proof.** First, recall that \( \varphi(0) = 1 \). We begin the elementary identity

\[
1 - |\varphi(\xi^+)|^2 = (1 - \varphi(\xi^+))(1 + \overline{\varphi(\xi^+)}) + 2 \text{Im} \varphi(\xi^+).
\]

Using the estimate \( |1 + \overline{\varphi(\xi^+)}| \leq 1 + |\varphi(\xi^+)| \leq 2 \) (cf. (3.3)) and second inequality in (3.9) we deduce from (3.12)

\[
0 \leq 1 - |\varphi(\xi^+)|^2 \leq 4|\xi^+|^{\alpha}\|\varphi - 1\|_\alpha.
\]
Obviously, an analogous inequality holds true if we replace $\xi^+$ by $\xi^-$. Now, applying inequality (3.5), we conclude

$$|\varphi(\xi^+)\varphi(\xi^-) - \varphi(\xi)| \leq \sqrt{(1 - |\varphi(\xi^+)|^2)(1 - |\varphi(\xi^-)|^2)}$$

$$\leq 4|\xi^+|^\alpha/2|\xi^-|^\alpha/2\|\varphi - 1\|_\alpha$$

for all $\xi \in \mathbb{R}^3$.

Lemma 3.15. Let $\alpha \in [0, 2]$. Assume that $\mu$ is a probability measure on $\mathbb{R}^3$ such that $\int_{\mathbb{R}^3} |v|^{\alpha} \mu(dv)$ is finite. If, moreover, $\alpha \in (1, 2]$, assume that $\int_{\mathbb{R}^3} v_i \mu(dv) = 0$ for $i \in \{1, 2, 3\}$. Then $\hat{\mu} \in K^\alpha$.

Proof. Consider first $\alpha \in (0, 1]$. Using the definition of the Fourier transform of a probability measure $\mu(dv)$ we obtain

$$(3.13) \quad \frac{\hat{\mu}(\xi) - 1}{|\xi|^\alpha} \leq \int_{\mathbb{R}^3} \frac{|e^{-iv \cdot \xi} - 1|}{|\xi|^\alpha} \mu(dv).$$

Note now the by substituting $\xi = \eta/|v|$, we have

$$\sup_{\xi \in \mathbb{R}^3} \frac{|e^{-iv \cdot \xi} - 1|}{|\xi|^\alpha} = |v|^\alpha \sup_{\eta \in \mathbb{R}^3} \frac{|e^{-iv \cdot \eta/|v|} - 1|}{|\eta|^\alpha} \leq C |v|^\alpha,$$

where, in view of the elementary inequality $|e^{is} - 1| \leq |s|$ for all $s \in \mathbb{R}$, the constant $C = \sup_{s, \eta \in \mathbb{R}^3} |e^{-iv \cdot \eta/|v|} - 1||\eta|^{-\alpha}$ is finite for $\alpha \in (0, 1]$. Hence, we deduce from (3.13) that

$$\|\hat{\mu} - 1\|_\alpha \leq C \int_{\mathbb{R}^3} |v|^\alpha \mu(dv),$$

For $\alpha \in (1, 2]$, one should proceed analogously using the following counterpart of inequality (3.13)

$$(3.13) \quad \frac{\hat{\mu}(\xi) - 1}{|\xi|^\alpha} \leq \int_{\mathbb{R}^3} \left| \frac{e^{-iv \cdot \xi} + iv \cdot \xi - 1}{|\xi|^\alpha} \right| \mu(dv).$$

being the simple consequence of the additional assumption $\int_{\mathbb{R}^3} v_i \mu(dv) = 0$, for every $i \in \{1, 2, 3\}$. \hfill \Box

Remark 3.16. Let us provide a counterexample that the reverse implication in Lemma 3.15 for $\alpha \in (0, 2)$ is not true, in other words, we want to show that the space $K^\alpha$ is bigger than the space of of characteristic functions corresponding to probability measures with finite moments of order $\alpha$. It is well-known that the function $\varphi_\alpha(\xi) = e^{-|\xi|^\alpha}$, with $\alpha \in (0, 2)$, is the Fourier transform of the probability density $P_\alpha(x)$ of the $\alpha$-stable symmetric Lévy process, (see Example 3.9). Obviously, we have $\varphi_\alpha \in K^\alpha$. On the other hand, it is known that for every $\alpha \in (0, 2)$ the function $P_\alpha$ is smooth, nonnegative, and
satisfies the estimate \(0 < P_{\alpha}(x) \leq C(1 + |x|)^{-(\alpha + n)}\) for a constant \(C\) and all \(x \in \mathbb{R}^n\). Moreover,

\[
P_{\alpha}(x) \frac{1}{|x|^\alpha} \to c_0 \quad \text{when} \quad |x| \to \infty,
\]

where \(c_0 = \alpha^2 \pi^{-(n+2)/2} \sin(\alpha\pi/2)\Gamma\left(\frac{\alpha+n}{2}\right)\Gamma\left(\frac{\alpha}{2}\right).
\] We refer the reader to [4] for a proof of the formula (3.14) with the explicit constant \(c_0\).

In view of the limit relation (3.14), we have

\[
\int_{\mathbb{R}^3} P_{\alpha}(x)|x|^\alpha \, dx = \infty.
\]

4. Existence under cut-off assumption

In this section, we construct solutions of the initial value problem (1.8)–(1.12) and we study their stability in the space \(K^\alpha\) imposing the usual cut-off assumption on the collision kernel \(B\) (the pseudo-Maxwellian gas), say:

\[
\int_{S^2} B\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \, d\sigma \text{ is finite for all } \xi \in \mathbb{R}^3 \setminus \{0\}.\]

In the next section, we show how to relax this condition.

4.1. Technical results on the collision kernel. Let us first introduce parameters which appear systematically in our reasoning below.

**Lemma 4.1.** Let \(\alpha \in [0, 2]\) and \(B \in L^1(-1, 1)\). Then for all \(\xi \in \mathbb{R}^3 \setminus \{0\}\) the following quantity

\[
\gamma_{\alpha} \equiv \int_{S^2} B\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \frac{|\xi^-|^\alpha + |\xi^+|^\alpha}{|\xi|^\alpha} \, d\sigma
\]

is finite and independent of \(\xi\). Moreover,

\[
\gamma_{\alpha} > \gamma_2 = 2\pi \int_{-1}^{1} B(s) \, ds \quad \text{if} \quad 0 < \alpha < 2.
\]

**Proof.** Let \(\sigma = (\sigma_1, \sigma_2, \sigma_3) \in S^2\). Rotating \(\mathbb{R}^3\) (if necessary) and using the spherical coordinates we obtain the equalities

\[
\int_{S^2} g\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \, d\sigma = \int_{S^2} g(\sigma_3) \, d\sigma = 2\pi \int_{-1}^{1} g(s) \, ds,
\]

valid for every \(g \in L^1(-1, 1)\) and \(\xi \in \mathbb{R}^3 \setminus \{0\}\). Hence, by (4.4) with \(g = B\), recalling relations (1.11) we have \(\gamma_2 = 2\pi \int_{-1}^{1} B(s) \, ds\).

For \(0 < \alpha < 2\), we rewrite equalities (1.11) as follows

\[
|\xi^+|^\alpha = |\xi|^\alpha \left(\frac{1 + \xi \cdot \sigma}{2}\right)^{\alpha/2} \quad \text{and} \quad |\xi^-|^\alpha = |\xi|^\alpha \left(\frac{1 - \xi \cdot \sigma}{2}\right)^{\alpha/2}.
\]
and we use equality (4.4) with $g(s) = B(s) \left[ \left( \frac{1+s}{2} \right)^{\alpha/2} + \left( \frac{1-s}{2} \right)^{\alpha/2} \right]$ to obtain

$$\gamma_\alpha = 2\pi \int_{-1}^{1} B(s) \left[ \left( \frac{1+s}{2} \right)^{\alpha/2} + \left( \frac{1-s}{2} \right)^{\alpha/2} \right] ds.$$ 

The integral on the right hand side of (4.6) is finite because the function in the brackets is bounded for $s \in [-1, 1]$.

In order to show that $\gamma_\alpha > \gamma_2$, whenever $0 < \alpha < 2$, it suffices to use the elementary inequality

$$\left( \frac{1+s}{2} \right)^{\alpha/2} + \left( \frac{1-s}{2} \right)^{\alpha/2} > 1$$

which is valid for all $s \in (-1, 1)$. \hfill \Box

**Corollary 4.2.** Let $\alpha \in [0, 2]$. Assume that the function $(1-s^2)^{\alpha/2}B(s)$ is integrable on $[-1, 1]$. For every $\xi \in \mathbb{R}^3 \setminus \{0\}$ the following quantity

$$\lambda_\alpha \equiv \int_{s^2} B\left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left( \frac{|\xi^-|^\alpha + |\xi^+|^\alpha}{|\xi|^\alpha} - 1 \right) d\sigma$$

is finite, independent of $\xi$, and positive provided $0 < \alpha < 2$.

**Proof.** If $B \in L^1(-1, 1)$, this is the immediate consequence of Lemma 4.1. To handle more general $B$ it suffices to apply identities (4.5) together with the change of variables from (4.4) with the function $g = g(s)$ satisfying

$$0 \leq g(s) \equiv B(s) \left( \left( \frac{1+s}{2} \right)^{\alpha/2} + \left( \frac{1-s}{2} \right)^{\alpha/2} - 1 \right) \leq C B(s)(1-s^2)^{\alpha/2}$$

for every $\alpha \in (0, 2)$, a constant $C(\alpha) > 0$, and all $s \in [-1, 1]$ (see also Remark 4.3 below). \hfill \Box

**Remark 4.3.** We leave for the reader to check that

$$\lim_{s \to \pm 1} \left( \frac{1+s}{2} \right)^{\alpha/2} + \left( \frac{1-s}{2} \right)^{\alpha/2} - 1 = 1$$

provided $\alpha \in (0, 2)$. Hence, both functions in the numerator and the denominator are comparable in the sense that there are two positive constants $C_1, C_2$ such that

$$C_1(1-s^2)^{\alpha/2} \leq \left( \frac{1+s}{2} \right)^{\alpha/2} + \left( \frac{1-s}{2} \right)^{\alpha/2} - 1 \leq C_2(1-s^2)^{\alpha/2}.$$ 

By this reason, we prefer to use the estimate from (4.8) to keep the assumption on the collision kernel $B$ from Corollary 4.2 comparable with our standing assumption (2.6).

**Remark 4.4.** In the following, we systematically use the identity $\lambda_\alpha = \gamma_\alpha - \gamma_2$ valid for any collision kernel $B \in L^1(-1, 1)$. 

**HOMOGENEOUS BOLTZMANN EQUATION**
4.2. **Construction of solutions.** Now, we are going to construct solutions of problem (1.8)–(1.12) under the cut-off assumption (4.1) using the Banach contraction principle. The following theorem is a particular case of Theorem 2.2 under the assumption that $B$ satisfies (2.6) with $\alpha_0 = 0$.

**Theorem 4.5.** Let $\alpha \in [0, 2]$ and $B \in L^1(-1, 1)$. For every initial condition $\varphi_0 \in K^\alpha$ there exists a unique classical solution of problem (1.8)–(1.12) satisfying $\varphi \in X^\alpha \equiv C([0, \infty), K^\alpha)$.

In the proof of Theorem 4.5 we use the following nonlinear operator

\[ G(\varphi)(\xi) \equiv \int_{S^2} B \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \varphi(\xi^+) \varphi(\xi^-) d\sigma \]

where $\xi^+$ and $\xi^-$ are defined in (1.9). Hence, under the cut-off assumption (4.1), for the constant $\gamma_2 = 2\pi \int_{-1}^1 B(s) ds$ (cf. Lemma 4.1), and for $\varphi$ satisfying $\varphi(0, t) = 1$ for all $t \geq 0$, we write equation (1.8) in the following form

\[ \partial_t \varphi + \gamma_2 \varphi = G(\varphi). \]

Next, multiplying (4.10) by $e^{\gamma_2 t}$ and integrating with respect to $t$ we obtain the following equivalent formulation of problem (1.8)–(1.12)

\[ \varphi(\xi, t) = \varphi_0(\xi)e^{-\gamma_2 t} + \int_0^t e^{-\gamma_2 (t-\tau)} G(\varphi(\cdot, \tau))(\xi) d\tau. \]

**Lemma 4.6.** Let $\alpha \in [0, 2]$ and assume (4.1). For every $\varphi \in K^\alpha$, the function $G(\varphi)$ is continuous and positive definite. Moreover, for the constant $\gamma_\alpha$ defined in (4.2), we have

\[ |G(\varphi)(\xi) - G(\varphi)(\xi)| \leq \gamma_\alpha \|\varphi - \varphi\|_\alpha |\xi|\]

for all $\varphi, \varphi \in K^\alpha$ and all $\xi \in \mathbb{R}^3 \setminus \{0\}$.

**Proof.** Let $\varphi \in K^\alpha$. To show that $G(\varphi)$ is continuous and positive definite is suffices to follow the reasoning from [21, Lemma 2.1].

Hence, it suffices to show estimate (4.12) from all $\varphi, \varphi \in K^\alpha$. To do it, using inequalities $|\varphi(\xi^-)| \leq 1$, $|\varphi(\xi^+)| \leq 1$, and the definitions of the metric (2.3) as well as of the constant $\gamma_\alpha$ (see (4.2)), we obtain

\[ |G(\varphi)(\xi) - G(\varphi)(\xi)| = \int_{S^2} B \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left[ (\varphi(\xi^+) - \varphi(\xi^+)) \varphi(\xi^-) + \varphi(\xi^+) (\varphi(\xi^-) - \varphi(\xi^-)) \right] d\sigma \]

\[ \leq \int_{S^2} B \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left( \|\varphi - \varphi\|_{\alpha} |\xi^+|^\alpha + \|\varphi - \varphi\|_{\alpha} |\xi^-|^\alpha \right) d\sigma \]

\[ = \gamma_\alpha \|\varphi - \varphi\|_{\alpha} |\xi|^\alpha \]
for all $\xi \in \mathbb{R}^3$.

Now, we are in a position to prove the existence of solutions to (1.8)–(1.12) in the space $K^\alpha$.

**Proof of Theorem 4.5.** The solution to (1.8)–(1.12) is obtained as a fixed point of equation (4.11) via the Banach contraction principle applied to the nonlinear operator

$$
F(\varphi)(\xi,t) \equiv \varphi_0(\xi)e^{-\gamma_2 t} + \int_0^t e^{-\gamma_2 (t-\tau)} G(\varphi(\cdot,\tau))(\xi) \, d\tau
$$

(cf. equation (4.11)). We fix $\varphi_0 \in K^\alpha$ and, first, we show that the mapping $F$ is a contraction on the metric space $X^\alpha_T = C([0,T],K^\alpha)$ supplemented with the metric $\|\varphi - \bar{\varphi}\|_{X^\alpha_T} \equiv \sup_{\tau \in [0,T]} \|\varphi(\cdot,\tau) - \bar{\varphi}(\cdot,\tau)\|_\alpha$ provided $T > 0$ is sufficiently small.

Notice that for every $\varphi \in X^\alpha_T$ and for every $t \in [0,T]$, the function $F(\varphi)(t)$ is continuous and positive definite. This is the immediate consequence of Lemma 3.5 if one approximates the integral on the right-hand side of (4.14) by finite sums with positive coefficients. Here, one should remember that $G(\varphi)(\tau)$ is continuous and positive definite for every $\tau \in [0,t]$ by Lemma 4.6.

Next, for every $\varphi \in X^\alpha_T$, we can rewrite equation (4.14) as follows

$$
F(\varphi)(\xi,t) - 1 = (\varphi_0(\xi) - 1)e^{-\gamma_2 t} + \int_0^t e^{-\gamma_2 (t-\tau)} (G(\varphi(\cdot,\tau))(\xi) - \gamma_2) \, d\tau
$$

Hence, using Lemma 4.6 with $\bar{\varphi} \equiv 1$, recalling the definition of the constant $\gamma_2 = G(1)$ from (1.2), and estimating $e^{-\gamma_2 (t-\tau)} \leq 1$ for every $\tau \in [0,t]$, we obtain

$$
|F(\varphi)(\xi,t) - 1| \leq \|\varphi_0 - 1\|_\alpha |\xi|^\alpha + \gamma_2 \alpha \int_0^t \|\varphi(\xi,\tau) - 1\|_\alpha d\tau |\xi|^\alpha.
$$

Dividing this inequality by $|\xi|^\alpha$ and computing the supremum with respect to $\xi \in \mathbb{R}^3$ and $t \in [0,T]$ we obtain that $F : X^\alpha_T \to X^\alpha_T$ together with the estimate

$$
\|F(\varphi) - 1\|_{X^\alpha_T} \leq \|\varphi_0 - 1\|_\alpha + \gamma_2 \alpha T \|\varphi - 1\|_{X^\alpha_T}.
$$

In a similar way, using Lemma 4.6 for every $\varphi, \bar{\varphi} \in X^\alpha_T$, we get

$$
|F(\varphi)(\xi,t) - F(\bar{\varphi})(\xi,t)| \leq \gamma_2 \alpha T \|\varphi - \bar{\varphi}\|_{X^\alpha_T} |\xi|^\alpha,
$$

and consequently,

$$
\|F(\varphi) - F(\bar{\varphi})\|_{X^\alpha_T} \leq \gamma_2 \alpha T \|\varphi - \bar{\varphi}\|_{X^\alpha_T}.
$$

Hence, the Banach contraction principle provides the unique solution (the fixed point) of equation (4.11) in the space $X^\alpha_T$ provided $T < 1/\gamma_2$.

Note finally that we have constructed the unique solution on $[0,T]$ where $T$ is independent of the initial condition. Hence, choosing $\varphi(\xi,T)$ as the initial datum we obtain
the unique solution on \([T, 2T]\). Consequently, repeating this procedure, we construct the unique solution on any finite time interval.

4.3. Remark on Wild’s sum. Under the cut-off assumption (4.1), for every characteristic function \(\varphi_0\) as an initial datum and for \(\gamma_2 = 1\) in (4.2) (which can be always normalized by a suitable time rescaling of equation (1.8), cf. [21 Sec. 2]), it is possible to derive the following explicit representation of a classical solution to (1.8)–(1.12)

\[\varphi(\xi, t) = e^{-t} \sum_{n=0}^{\infty} \varphi^{(n)}(\xi)(1 - e^{-t})^n,\]

where

\[\varphi^{(0)}(\xi) = \varphi_0(\xi)\]

\[\varphi^{(n+1)}(\xi) = \frac{1}{n+1} \sum_{j=0}^{n} \tilde{G}(\varphi^{(j)}, \varphi^{(n-j)})(\xi)\]

with the bilinear operator \(\tilde{G}\) of the following form

\[\tilde{G}(\varphi, \tilde{\varphi})(\xi) \equiv \int_{S^2} B \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \varphi(\xi^+) \tilde{\varphi}(\xi^-) d\sigma.\]

Notice that we have \(\tilde{G}(\varphi, \varphi) = G(\varphi)\) for every \(\varphi\), where \(G\) is defined in (4.9). This series is called Wild’s sum [25]. The proof that it converges toward the unique classical solution of problem (1.8)–(1.12) can be found e.g. either in [21 Thm 2.1] or in [8 Sect. 4]. Here, we show that the series converges in the space \(K^\alpha\).

**Theorem 4.7.** Let \(\alpha \in (0, 2]\) and \(\varphi_0 \in K^\alpha\). Assume that \(\gamma_2 = 1\) in (4.2). Then the series defined in (4.15)–(4.16) converges toward a solution to (1.8)–(1.12) which belongs to the space \(C([0, \infty), K^\alpha)\).

**Proof.** By inspection of the proof of Lemma 4.6 with \(\tilde{\varphi} \equiv 1\), we immediately obtain the inequality

\[\|\tilde{G}(\varphi, \tilde{\varphi}) - 1\|_\alpha \leq \gamma_+^\alpha \|\varphi - 1\|_\alpha + \gamma_-^\alpha \|\tilde{\varphi} - 1\|_\alpha\]

for all \(\varphi, \tilde{\varphi} \in K^\alpha\), where \(\gamma_+^\alpha \equiv \int_{S^2} B((\xi \cdot \sigma)/|\xi|)|\xi^+|^\alpha/|\xi|\| d\sigma\) are finite, independent of \(\xi\), and satisfy \(\gamma_+^\alpha + \gamma_-^\alpha = \gamma_2^\alpha\).

Now, we proceed by induction to show the estimate

\[\|\varphi^{(n)} - 1\|_\alpha \leq \gamma_+^\alpha \|\varphi_0 - 1\|_\alpha\]

for every \(n \in \mathbb{N}\), where \(\varphi^{(n)}\) and defined in (4.16) and the constant \(\gamma_+^\alpha \geq \gamma_2 = 1\) appers in (4.2). Inequality (4.18) reduces to an obvious equality if \(n = 0\). For \(n \geq 1\), using definition (4.16), the
estimate of the bilinear form (4.17), and the inductive argument, we obtain

\[ \| \varphi^{(n+1)} - 1 \|_{\alpha} \leq \frac{1}{n+1} \sum_{j=0}^{n} \| \tilde{G} \left( \varphi^{(j)}, \varphi^{(n-j)} \right) - 1 \|_{\alpha} \]

\[ \leq \frac{1}{n+1} \sum_{j=0}^{n} \gamma_{\alpha}^+ \| \varphi^{(j)} - 1 \|_{\alpha} + \gamma_{\alpha}^- \| \varphi^{(n-j)} - 1 \|_{\alpha} \]

\[ = \frac{1}{n+1} \sum_{j=0}^{n} \gamma_{\alpha} \| \varphi^{(j)} - 1 \|_{\alpha} \]

\[ \leq \| \varphi_0 - 1 \|_{\alpha} \left( \frac{1}{n+1} \sum_{j=0}^{n} \gamma_{\alpha}^{1+j} \right). \]

(4.19)

Recall now that \( \gamma_{\alpha} = \gamma_{\alpha}^+ + \gamma_{\alpha}^- \geq \gamma_2 \) (\( = 1 \)) by Lemma 4.1; hence, \( \gamma_{\alpha}^{1+j} \leq \gamma_{\alpha}^{n+1} \) for each \( j \in \{0, ..., n\} \). Using these inequalities to estimate the right-hand side of (4.19) we complete the proof of (4.18).

Coming back to the function \( \varphi(\xi, t) \) given by the series (4.15) and applying (4.18), we obtain

\[ \| \varphi(t) - 1 \|_{\alpha} \leq e^{-t} \| \varphi_0 - 1 \|_{\alpha} \sum_{n=0}^{\infty} (1 - e^{-T})^n \gamma_{\alpha}^n. \]

Chosing \( T > 0 \) so small to have \( (1 - e^{-T}) \gamma_{\alpha} < 1 \) we obtain the convergence of the series on the right-hand side for any \( t \in [0, T] \).

However, since \( T \) from the proof of Theorem 4.7 is independent of the initial condition, we may choose \( \varphi(\xi, T) \) as the initial datum and show the convergence of the corresponding Wild series on \( [T, 2T] \). Consequently, repeating this procedure, we show the converges of series (4.15)–(4.16) toward a solution from \( C([0, T], K^\alpha) \) for any \( T > 0 \).

**4.4. Stability and uniqueness of solutions.** For each \( R \in (0, \infty] \), we define the quasi-metric for any \( \varphi, \tilde{\varphi} \in K^\alpha \) by the following formula

\[ \| \varphi - \tilde{\varphi} \|_{\alpha, R} \equiv \sup_{|\xi| \leq R} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha}. \]

(4.20)

The following stability lemma, shown here in the case of the integrable collision kernel, will be generalized in Section 5 (see Corollary 5.2) for solutions of the initial value problem (1.8)–(1.12) with any nonintegrable kernel satisfying (2.6).

**Lemma 4.8.** Assume that \( \alpha \in [0, 2] \) and \( B \in L^1(-1, 1) \). Consider two solutions \( \varphi, \tilde{\varphi} \in C([0, \infty), K^\alpha) \) of problem (1.8)–(1.12) corresponding to the initial data \( \varphi_0, \tilde{\varphi}_0 \in K^\alpha \), respectively. Then for every \( t \geq 0 \) and \( R \in (0, \infty] \)

\[ \| \varphi(t) - \tilde{\varphi}(t) \|_{\alpha, R} \leq e^{\lambda \alpha t} \| \varphi_0 - \tilde{\varphi}_0 \|_{\alpha, R}. \]

(4.21)
where the constant \( \lambda = \gamma - \gamma_2 \geq 0 \) is defined in (2.7).

Proof. It follows from equation (1.8) that the function

\[
h(\xi, t) = \frac{\varphi(\xi, t) - \bar{\varphi}(\xi, t)}{|\xi|^{\alpha}}
\]

satisfies

\[
(4.22) \quad \partial_t h(\xi, t) = \int_{S^2} \mathcal{B} \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left( \frac{\varphi(\xi^+) \varphi(\xi^-) - \bar{\varphi}(\xi^+) \bar{\varphi}(\xi^-)}{|\xi|^{\alpha}} - h(\xi, t) \right) d\sigma.
\]

Now, for \( |\xi^+| \leq |\xi| \leq R \) and \( |\xi^-| \leq |\xi| \leq R \), we use the following inequalities

\[
|\varphi(\xi^+) \varphi(\xi^-) - \bar{\varphi}(\xi^+) \bar{\varphi}(\xi^-)| \\
\leq |\varphi(\xi^+) - \bar{\varphi}(\xi^+)| |\varphi(\xi^-)| + |\varphi(\xi^-) - \psi(\xi^-)||\bar{\varphi}(\xi^+)| \\
\leq \|\varphi(t) - \bar{\varphi}(t)\|_{\alpha, R} (|\xi^+|^\alpha + |\xi^-|^\alpha)
\]

to deduce from equation (4.22) that the function \( h = h(\xi, t) \) satisfies

\[
(4.23) \quad |\partial_t h(\xi, t) + \gamma_2 h(\xi, t)| \leq \gamma_\alpha \|\varphi(t) - \bar{\varphi}(t)\|_{\alpha, R}
\]

with the constants \( \gamma_\alpha \) and \( \gamma_2 \) defined in (4.2). It follows from inequality (4.23)

\[
|\partial_t (e^{\gamma_2 t} h(\xi, t))| \leq \gamma_\alpha e^{\gamma_2 t} \|\varphi(t) - \bar{\varphi}(t)\|_{\alpha, R}
\]

for every \( t > 0 \), hence,

\[
|e^{\gamma_2 t} h(\xi, t)| \leq |h(\xi, 0)| + \int_0^t |\partial_s (e^{\gamma_2 s} h(\xi, s))| \, ds \\
\leq |h(\xi, 0)| + \gamma_\alpha \int_0^t e^{\gamma_2 s} \|\varphi(s) - \bar{\varphi}(s)\|_{\alpha, R} \, ds.
\]

Finally, we compute the supremum with respect to \( |\xi| \leq R \) and we apply the Gronwall lemma to obtain

\[
\|\varphi(t) - \bar{\varphi}(t)\|_{\alpha, R} \leq \|\varphi_0 - \bar{\varphi}_0\|_{\alpha, R} e^{t(\gamma_\alpha - \gamma_2)}.
\]

The proof is complete because \( \gamma_\alpha - \gamma_2 = \lambda \) by (4.7).

\[
\square
\]

5. Nonintegrable collision kernels – existence, uniqueness, and stability of solutions

In this section, we complete the proofs of Theorems 2.2 and 2.5 on the existence and the stability of solutions to (1.8)-(1.12) with no cut-off assumption imposed on the collision kernel. More precisely, we assume that \( \mathcal{B} \) satisfies (2.6) for some \( \alpha_0 \in [0, 2] \).

As a standard practice, we consider the increasing sequence of bounded collision kernels

\[
(5.1) \quad \mathcal{B}_n(s) \equiv \min \{ \mathcal{B}(s), n \} \leq \mathcal{B}(s), \quad n \in \mathbb{N},
\]
and, for each \( \alpha \in [\alpha_0, 2] \), the sequence \( \varphi_n \in C([0, \infty), K^\alpha) \) of the corresponding solutions to problem (1.8)-(1.12) (constructed in Theorem 4.5) with the kernels \( B_n \) and with the same initial datum \( \varphi_0 \in K^\alpha \). Note that, under the assumption (2.6) (cf. Corollary 4.2), we have

\[
\lambda_{\alpha,n} \equiv \int_{S^2} B_n \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left( \frac{|\xi^-|^{\alpha} + |\xi^+|^{\alpha}}{|\xi|^\alpha} - 1 \right) d\sigma \leq \lambda_{\alpha}
\]

for every \( n \in \mathbb{N} \), hence by the stability lemma 4.8 with \( R = \infty \), it follow

\[
\| \varphi_n(t) - 1 \|_\alpha \leq e^{\lambda_{\alpha,n} t} \| \varphi_0 - 1 \|_\alpha \leq e^{\lambda_{\alpha} t} \| \varphi_0 - 1 \|_\alpha
\]

for all \( t \geq 0 \).

**Lemma 5.1.** Assume that the collision kernel satisfies (2.6) with some \( \alpha_0 \in [0, 2] \). Let \( \alpha \in [\alpha_0, 2] \). The sequence of solutions \( \{\varphi_n\}_{n=1}^\infty \subset C([0, \infty), K^\alpha) \) is bounded in \( C(R^3 \times [0, \infty)) \) and equicontinuous.

**Proof.**

1. **Uniform bound.** Since \( \varphi_n(\cdot, t) \) is a characteristic function for every \( t \geq 0 \), by (3.3), we have \( |\varphi_n(\xi, t)| \leq \varphi_n(0, t) = 1 \) for all \( \xi \in \mathbb{R}^3 \) and \( t \geq 0 \).

2. **Modulus of continuity in time.** We use the equation satisfied by \( \varphi_n \) and inequalities (3.11) and (5.3) as follows (remember that \( \varphi_n(0, t) = 1 \))

\[
|\partial_t \varphi_n(\xi, t)| \leq \int_{S^2} B_n \left( \frac{\xi \cdot \sigma}{|\xi|} \right) |\varphi_n(\xi^+, t)\varphi_n(\xi^-, t) - \varphi_n(\xi, t)\varphi_n(0, t)| d\sigma
\]

\[
\leq 4 \| \varphi_n(t) - 1 \|_\alpha \int_{S^2} B_n \left( \frac{\xi \cdot \sigma}{|\xi|} \right) |\xi^+|^{\alpha/2} |\xi^-|^{\alpha/2} d\sigma
\]

\[
\leq 4 \beta_\alpha e^{\lambda_{\alpha} t} \| \varphi_0 - 1 \|_\alpha |\xi|^\alpha
\]

for all \( \xi \in \mathbb{R}^3 \) and \( t \geq 0 \). Here, \( \beta_\alpha \) denotes the finite and independent of \( \xi \) number which, by identities (4.5) and by the change of variables from (4.4), satisfies

\[
\beta_\alpha \equiv \int_{S^2} B \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \frac{|\xi^+|^{\alpha/2} |\xi^-|^{\alpha/2}}{|\xi|^\alpha} d\sigma
\]

\[
= \int_{S^2} B \left( \frac{\xi \cdot \sigma}{|\xi|} \right) \left( \frac{1 + \frac{\xi \cdot \sigma}{|\xi|}}{2} \right)^{\alpha/4} \left( \frac{1 - \frac{\xi \cdot \sigma}{|\xi|}}{2} \right)^{\alpha/4} d\sigma
\]

\[
= 2\pi \int_{-1}^1 B(s) \left( \frac{1 + s}{2} \right)^{\alpha/4} \left( \frac{1 - s}{2} \right)^{\alpha/4} ds.
\]

Since \( \alpha \in [\alpha_0, 2] \), it is now clear that the constant \( \beta_\alpha \) is finite for any collision kernel \( B \) satisfying (2.6).
Step 3. Modulus of continuity in space. It suffices to apply inequality (3.4) combined with (3.9) and (5.3) to obtain the estimate
\[
|\varphi_n(\xi, t) - \varphi_n(\eta, t)| \leq \sqrt{2}(1 - \text{Re} \varphi_n(\xi - \eta, t)) \leq \sqrt{2} |\xi - \eta|^{\alpha/2} \left\| \varphi_n(t) - 1 \right\|_{\alpha/2}^{1/2} \leq \sqrt{2} |\xi - \eta|^{\alpha/2} e^{\lambda \alpha t/2} \left\| \varphi_0 - 1 \right\|_{\alpha/2},
\]
for all \( t \geq 0 \), where the right-hand side is independent of \( n \).

Now, we are in a position to construct a solution to (1.8)-(1.12). By Lemma 5.1, the Ascoli-Arzelà theorem, and the Cantor diagonal argument, we deduce that there exists a subsequence of solutions \( \{\varphi_{n_k}\}_{n_k} \) converging uniformly in every compact set of \( \mathbb{R}^3 \times [0, \infty) \). We are going prove that the function
\[
(5.4) \quad \varphi(\xi, t) = \lim_{n_k \to \infty} \varphi_{n_k}(\xi, t)
\]
is a solution of problem (1.8)-(1.12) with the singular kernel \( B \) satisfying (2.6). Note here that \( \varphi(\cdot, t) \) is a characteristic function for every \( t \geq 0 \) as the pointwise limit of characteristic functions.

Here, we are allowed to use the Lebesgue dominated convergence theorem to pass to the limit \( n_k \to \infty \) in the Boltzmann operator
\[
(5.5) \quad \int_{S^2} B_{n_k} \left( \frac{\xi \cdot \sigma}{|\xi|} \right) (\varphi_{n_k}(\xi^+, t)\varphi_{n_k}(\xi^-, t) - \varphi_{n_k}(\xi, t)\varphi_{n_k}(0, t)) \, d\sigma
\]
(where, as usual \( \varphi_{n_k}(0, t) = 1 \)) because, following the calculations from Step 2 of the proof of Lemma 5.1 its integrand can be majorized by the integrable (on \( S^2 \)) function
\[
4e^{\lambda \alpha t}\|\varphi_0 - 1\|_\alpha B \left( \frac{\xi \cdot \sigma}{|\xi|} \right) |\xi^+|^{\alpha/2} |\xi^-|^{\alpha/2}.
\]
Since, the Boltzmann operator form (5.5) converges uniformly on every compact subset of \( \mathbb{R}^3 \times [0, \infty) \), there exists a continuous function \( \zeta = \zeta(\xi, t) \) such that \( \partial_t \varphi_{n_k} \to \zeta \). By the limit relation (5.4), we immediately conclude that \( \zeta = \partial_t \varphi \). Hence, \( \varphi \) is a solution to the initial-value problem (1.8)-(1.12).

To show the \( \varphi(\cdot, t) \in K^\alpha \), it suffices to pass to the pointwise limit \( n_k \to \infty \) in inequality (5.3) written in the following equivalent way
\[
\frac{|\varphi_{n_k}(\xi, t) - 1|}{|\xi|^\alpha} \leq e^{\lambda \alpha t}\|\varphi_0 - 1\|_\alpha
\]
for all \( \xi \in \mathbb{R}^3 \setminus \{0\} \) and \( t \geq 0 \).

In order to prove stability inequality form Theorem 2.5, it suffices to consider two sequences of solutions \( \{\varphi_n\}_{n \in \mathbb{N}} \) and \( \{\tilde{\varphi}_n\}_{n \in \mathbb{N}} \) to equation (1.8) with the truncated kernel...
\( B_n \) and corresponding to the initial conditions \( \varphi_0 \) and \( \tilde{\varphi}_0 \), respectively. By the compactness argument from Lemma 5.1, there exists a subsequence \( n_k \to \infty \) and solutions \( \varphi, \tilde{\varphi} \) to equation (1.8) such that
\[
\varphi(\xi, t) = \lim_{n_k \to \infty} \varphi_{n_k}(\xi, t) \quad \text{and} \quad \tilde{\varphi}(\xi, t) = \lim_{n_k \to \infty} \tilde{\varphi}_{n_k}(\xi, t).
\]
Using the stability lemma 4.8 and estimate (5.2), we obtain
\[
\frac{|\varphi_{n_k}(\xi, t) - \tilde{\varphi}_{n_k}(\xi, t)|}{|\xi|^\alpha} \leq e^{\lambda \alpha t} \|\varphi_0 - \tilde{\varphi}_0\|_\alpha
\]
for all \( \xi \in \mathbb{R}^3 \setminus \{0\} \) and \( t \geq 0 \). Passing to the limit \( n_k \to \infty \), we complete the proof of stability inequality (2.8) which, in particular, implies the uniqueness of solutions to (1.8)-(1.12) in the space \( C([0, \infty), K^\alpha) \).

An analogous argument allows us to remove the cut-off assumption from the stability lemma 4.8.

**Corollary 5.2.** Assume that \( B \) satisfies the non cut-off condition (2.6) for some \( \alpha_0 \in [0, 2] \). For every \( \alpha \in [\alpha_0, 2] \) and \( R > 0 \), the stability estimates (4.21) from Lemma 4.8 hold true for solutions to problem (1.8)-(1.12) with the kernel \( B \).

### 6. Self-similar solutions by Bobylev and Cercignani

In this section, we are going to formulate (in a way the most suitable for our applications) results by Bobylev and Cercignani [3] on solutions \((\mu, \Phi)\) to equation
\[
\mu \eta \cdot \nabla \Phi(\eta) = \int_{S^2} B \left( \frac{\eta \cdot \sigma}{|\eta|} \right) \left( \Phi(\eta^+)\Phi(\eta^-) - \Phi(\eta)\Phi(0) \right) d\sigma.
\]
Recall that, in this case, the function \( \varphi(\xi, t) = \Phi(\xi e^{\mu t}) \) is the self-similar solution of equation (1.8).

Let us first compute the scaling parameter \( \mu \) in equations (6.1) for any collision kernel \( B \) satisfying the weaker assumption
\[
(1 - s)^{\alpha/2}(1 + s)^{\alpha/2} B(s) \in L^1(-1, 1) \quad \text{for some } \alpha \in [0, 2].
\]

**Lemma 6.1.** Assume that the collision kernel satisfies the assumption (6.2) for some \( \alpha \in [0, 2] \). Let \( \Phi \) be a \( C^1 \)-solution of (6.1) with the following properties
\[
\Phi(\eta) = \Phi(|\eta|) \quad \text{and} \quad \lim_{|\eta| \to 0} \frac{\Phi(|\eta|) - 1}{|\eta|^\alpha} = K
\]
for some \( K \neq 0 \). Then
\[
\mu = \frac{1}{\alpha} \int_{S^2} B \left( \frac{\eta \cdot \sigma}{|\eta|} \right) \left( \frac{|\eta^-|^\alpha + |\eta^+|^\alpha}{|\eta|^\alpha} - 1 \right) d\sigma
\]
Proof. Since, $\Phi$ is radially symmetric, by the Hospital rule, we obtain

$$
\lim_{|\eta|\to 0} \frac{\eta \cdot \nabla \Phi(\eta)}{|\eta|^\alpha} = \lim_{|\eta|\to 0} \frac{\Phi'(|\eta|)}{|\eta|^{\alpha-1}} = \alpha \lim_{|\eta|\to 0} \frac{\Phi(|\eta|) - 1}{|\eta|^\alpha} = \alpha K.
$$

On the other hand, since $|\eta^\pm| \leq |\eta|$ and $\Phi(0) = 1$, by assumption (6.3), we get

$$
\frac{\Phi(\eta^+)\Phi(\eta^-) - \Phi(\eta)^2}{|\eta^-|^\alpha + |\eta^+|^\alpha - |\eta|^\alpha} = \frac{\Phi(\eta^+) - \Phi(\eta)}{|\eta|^\alpha} \to K
$$

as $|\eta| \to 0$. Hence, dividing equation (6.1) by $|\eta|^\alpha$, passing to the limit $|\eta| \to 0$ by the Lebesgue dominated convergence theorem in the integral on the right-hand side, and using relation (6.5), we obtain equality (6.4).

According to Lemma 6.1 and Corollary 4.2, for each $\alpha \in [0,2]$ we introduce the constant

$$
\mu_\alpha = \frac{\lambda_\alpha}{\alpha} = \frac{1}{\alpha} \int_{S^2} B \left( \frac{\eta \cdot \sigma}{|\eta|} \right) \left( \frac{|\eta^-|^\alpha + |\eta^+|^\alpha}{|\eta|^\alpha} - 1 \right) d\sigma
$$

which is finite and independent of $\eta$ if $B$ satisfies assumption (6.2).

**Theorem 6.2** (Bobylev & Cercignani [8]). Assume that the collision kernel $B$ satisfies the weaker non cut-off assumption (6.2) for some $\alpha \in (0,2)$. For every $K < 0$ and for $\mu = \mu_\alpha$ defined in (6.6) there exists a radially symmetric solution $\Phi = \Phi_{\alpha,K} \in K^\alpha$ of equation (6.1) satisfying

$$
\lim_{|\eta|\to 0} \frac{\Phi_{\alpha,K}(\eta) - 1}{|\eta|^\alpha} = K.
$$

**Sketch of proof.** This result was shown in [8, Thm. 6.2]. Let us sketch that proof for the reader convenience and for the completeness of our exposition.

The authors of [8] look for radially symmetric solutions of equation (6.1). Hence, introducing the function

$$
\phi(x) = \Phi(\eta) \quad \text{where} \quad x = \frac{|\eta|^2}{2}
$$

and using identities (1.11) combined with the change of variables from (4.4), they reduce equation (6.1) to

$$
2\mu \partial_x \phi(x) = \int_0^1 G(s) \left( \phi(sx)\phi((1-s)x) - \phi(0)\phi(x) \right) ds,
$$

where

$$
G(s) = 4\pi B(1 - 2s) \quad \text{for} \quad s \in (0,1).
$$
Now, to keep our notation consistent with that used in [8], we have to introduce the parameter

\[ \tilde{\alpha} = \frac{\alpha}{2} \in (0, 1). \]  

A solution to equation (6.9) is obtained in the form of the series

\[ \phi(x) = \phi_{\tilde{\alpha}, K}(x) = \sum_{n=0}^{\infty} \frac{u_n x^{n\tilde{\alpha}}}{\Gamma(n\tilde{\alpha} + 1)} \]

with the coefficients defined by the recurrence formula

\[ u_0 = 1, \quad u_1 = \sqrt{2}K, \quad u_n = \frac{1}{\gamma(\tilde{\alpha}, n)} \sum_{j=1}^{n-1} B_{\tilde{\alpha}}(j, n - j) u_j u_{n-j} \quad \text{for } n \geq 2. \]

Here,

\[ \gamma(\tilde{\alpha}, n) = n\lambda(\tilde{\alpha}) - \lambda(n\tilde{\alpha}) \]

\[ \lambda(p) = \int_{0}^{1} G(s)(s^p + (1-s)^p - 1) \, ds \]

\[ B_{\tilde{\alpha}}(j, \ell) = \frac{\Gamma(n\tilde{\alpha} + 1)}{\Gamma(j\tilde{\alpha} + 1)\Gamma(\ell\tilde{\alpha} + 1)} \int_{0}^{1} G(s)s^{j\tilde{\alpha}}(1-s)^{\ell\tilde{\alpha}} \, ds. \]

Next, the reasoning from [8] consists in showing that the series (6.12)-(6.13) converges toward a solution of (6.9). The proof that \( \phi(|x|^2/2) \) is a characteristic function is written in [8, p. 1054].

Coming back to our original notation, we obtain a solution to (6.1) in the form of the series

\[ \Phi_{\alpha, K}(\eta) = \sum_{n=0}^{\infty} \frac{u_n 2^{-n/2}(|\eta|^{\alpha})^n}{\Gamma(n\alpha/2 + 1)}, \]

where \( u_n \) is defined in (6.13). Obviously, this limit function belongs to \( K_{\alpha} \) and satisfies relation (6.7) by the definition of first two elements of the sequence \( u_n \) from (6.13).

**Remark 6.3.** Bobylev and Cercignani, in their proof of Theorem 6.2, show the convergence of the series (6.17) without any sign condition imposed on the constant \( K \) (in fact, complex \( K \) is also allowed). Let us explain that if we limit ourselves to characteristic functions satisfying (6.7), then necessarily \( K \leq 0 \). Indeed, first notice that \( K \) has to be a real number because, by using the identity \( \Phi(-\eta) = \overline{\Phi(\eta)} \) we have

\[ K = \lim_{|\eta| \to 0} \frac{\Phi(\eta) - 1}{|\eta|^\alpha} = \lim_{|\eta| \to 0} \frac{\Phi(-\eta) - 1}{|\eta|^\alpha} = \lim_{|\eta| \to 0} \frac{\Phi(\eta) - 1}{|\eta|^\alpha} = \overline{\Phi(0)}. \]

This means that, in particular, we have

\[ K = \lim_{|\eta| \to 0} \frac{\text{Re} \, \Phi(\eta) - 1}{|\eta|^\alpha}. \]
The right-hand side of inequality (6.18) is nonpositive because of Lemma 3.7 and inequality (3.3).

**Remark 6.4.** Theorem 6.2 is shown under the assumption that the function from (6.10) satisfies the estimate $0 \le G(s) \le Cs^{-(1+\gamma)}$ for some constants $\gamma \in (0, 1)$, $C > 0$ and for all $s \in (0, 1)$, see [8, Assump. (A) on p. 1052]. It is clear, however, from the proof by Bobylev and Cercignani that their reasoning holds true provided the quantities in (6.14)–(6.16) are finite. Hence, one can assume, for example, that $(s(1-s))^{\gamma}G(s) \in L^1(0,1)$. Using the formula (6.10) we discover our assumption (6.2) with $\gamma = \alpha/2$.

**Remark 6.5.** Now, it is clear that the estimate of the growth of the quantity $\| \varphi(t) - \tilde{\varphi}(t) \|_\alpha$ expressed by inequality (2.8) is optimal. Indeed, this can be easily seen when we substitute in (2.8) the self-similar solution $\varphi(\xi, t) = \Phi_{\alpha,K}(\xi e^{\mu t})$ and $\tilde{\varphi} \equiv 1$. In this special case, since $\mu = \lambda_\alpha/\alpha$, we have

$$
\| \varphi(t) - 1 \|_\alpha = e^{\lambda_\alpha t} \sup_{\xi \in \mathbb{R}^3} \frac{|\Phi_{\alpha,K}(\xi e^{\mu t}) - 1|}{|\xi e^{\mu t}|^\alpha} = e^{\lambda_\alpha t} \| \Phi_{\alpha,K} - 1 \|_\alpha
$$

for all $t > 0$.

### 7. Asymptotic Stability of Solutions

Now, we are ready to prove our main result on the large time asymptotics of solutions to (1.8)-(1.12) with a nonintegrable collision kernel satisfying (2.6).

**Proof of Theorem 2.7.** First, we apply the stability inequality (4.21) with some $R \in (0, \infty)$ which is now valid, by Corollary 5.2, in the case of any kernel satisfying (2.6).

Using relation (2.13) we substitute $\varphi(\xi, t) = \psi(\xi e^{\mu t}, t)$ and $\tilde{\varphi}(\xi, t) = \tilde{\psi}(\xi e^{\mu t}, t)$ into inequality (4.21) to obtain

$$
(7.1) \sup_{|\xi| \le R} \frac{|\psi(\xi e^{\mu t}, t) - \tilde{\psi}(\xi e^{\mu t}, t)|}{|\xi|^\alpha} \le e^{\lambda_\alpha t} \sup_{|\xi| \le R} \frac{|\psi_0(\xi) - \tilde{\psi}_0(\xi)|}{|\xi|^\alpha}
$$

for all $t > 0$ and each $R \in (0, \infty)$. Next, it follows from equality $\alpha \mu = \lambda_\alpha$ that

$$
\sup_{|\xi| \le R} \frac{|\psi(\xi e^{\mu t}, t) - \tilde{\psi}(\xi e^{\mu t}, t)|}{|\xi|^\alpha} = e^{\lambda_\alpha t} \sup_{|\xi| \le R e^{\mu t}} \frac{|\psi(\xi, t) - \tilde{\psi}(\xi, t)|}{|\xi|^\alpha},
$$

hence, by (7.1),

$$
(7.2) \sup_{|\xi| \le R e^{\mu t}} \frac{|\psi(\xi, t) - \tilde{\psi}(\xi, t)|}{|\xi|^\alpha} \le \sup_{|\xi| \le R} \frac{|\psi_0(\xi) - \tilde{\psi}_0(\xi)|}{|\xi|^\alpha}
$$
for all $t > 0$ and each $R \in (0, \infty]$. Since $R$ is arbitrary, we are allowed to substitute $R = S e^{-\mu t}$ in (7.2) (when $S$ will be chosen later on) to obtain

\[
\sup_{|\xi| \leq S} \frac{|\psi(\xi, t) - \tilde{\psi}(\xi, t)|}{|\xi|^\alpha} \leq \sup_{|\xi| \leq Se^{-\mu t}} \frac{|\psi_0(\xi) - \tilde{\psi}_0(\xi)|}{|\xi|^\alpha}
\]

(7.3)

Now, we are in a position to complete the proof. Recall that $|\psi(\xi, t)| \leq 1$ and $|\tilde{\psi}(\xi, t)| \leq 1$. Hence, for every $\varepsilon > 0$ there exists $S > 0$ such that

\[
\sup_{|\xi| > S} \frac{|\psi(\xi, t) - \tilde{\psi}(\xi, t)|}{|\xi|^\alpha} \leq \frac{2}{R^\alpha} \leq \varepsilon.
\]

Consequently, with this choice of $S$, by (7.3) and (7.4), we have

\[
\|\psi(t) - \tilde{\psi}(t)\|_\alpha \leq \sup_{|\xi| \leq S} \frac{|\psi(\xi, t) - \tilde{\psi}(\xi, t)|}{|\xi|^\alpha} + \sup_{|\xi| > S} \frac{|\psi(\xi, t) - \tilde{\psi}(\xi, t)|}{|\xi|^\alpha}
\]

(7.5)

\[
\leq \sup_{|\xi| \leq Se^{-\mu t}} \frac{|\psi_0(\xi) - \tilde{\psi}_0(\xi)|}{|\xi|^\alpha} + \varepsilon.
\]

By the assumption on $\psi_0$ and $\tilde{\psi}_0$ (see (2.10)), we immediately obtain that the first term on the right hand side of (7.5) tends to zero as $t \to \infty$. Since, $\varepsilon > 0$ can be arbitrary small we complete the proof of Theorem 2.7.

\[\Box\]

Remark 7.1. Corollary 2.8 implies that solutions the original problem (1.1)-(1.4), which converge toward self-similar profile by Bobylev and Cercignani (in the sense stated in Corollary 2.8), cannot have finite energy. Indeed, by the Toscani and Villani result recalled in Remark 2.10, finite energy solutions to (1.1)–(1.4) have to converge in the metric $\| \cdot \|_2$ toward Maxwellian. This fact is in contrast with a result by Mischler and Wennberg [20] who showed that any solution of the homogenous Boltzmann equation which satisfies certain bounds on moments of order $\alpha < 2$ must necessarily have also bounded energy. However, they consider the equation with so-called hard potential. As we have explained, such a phenomenon cannot be true for Maxwellian molecules.

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