DEFINABLE CLOSURE IN RANDOMIZATIONS

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Abstract. The randomization of a complete first order theory $T$ is the complete continuous theory $T^R$ with two sorts, a sort for random elements of models of $T$, and a sort for events in an underlying probability space. We give necessary and sufficient conditions for an element to be definable over a set of parameters in a model of $T^R$.

1. Introduction

A randomization of a first order structure $M$, as introduced by Keisler [Kei1] and formalized as a metric structure by Ben Yaacov and Keisler [BK], is a continuous structure $N$ with two sorts, a sort for random elements of $M$, and a sort for events in an underlying atomless probability space. Given a complete first order theory $T$, the theory $T^R$ of randomizations of models of $T$ forms a complete theory in continuous logic, which is called the randomization of $T$. In a model $N$ of $T^R$, for each $n$-tuple $\vec{a}$ of random elements and each first order formula $\varphi(\vec{v})$, the set of points in the underlying probability space where $\varphi(\vec{a})$ is true is an event denoted by $\llbracket \varphi(\vec{a}) \rrbracket$.

In a first order structure $M$, an element $b$ is definable over a set $A$ of elements of $M$ (called parameters) if there is a tuple $\vec{a}$ in $A$ and a formula $\varphi(u, \vec{a})$ such that

$$M \models (\forall u)(\varphi(u, \vec{a}) \leftrightarrow u = b).$$

In a general metric structure $N$, an element $b$ is said to be definable over a set of parameters $A$ if there is a sequence of tuples $\vec{a}_n$ in $A$ and continuous formulas $\Phi_n(x, \vec{a}_n)$ whose truth values converge uniformly to the distance from $x$ to $b$. In this paper we give necessary and sufficient conditions for definability in a model of the randomization theory $T^R$. These conditions can be stated in terms of sequences of first order formulas. The results in this paper will be applied in a forthcoming paper about independence relations in randomizations.

In Theorem 3.1.2, we show that an event $E$ is definable over a set $A$ of parameters if and only if it is the limit of a sequence of events of the form $\llbracket \varphi_n(\vec{a}_n) \rrbracket$, where each $\varphi_n$ is a first order formula and each $\vec{a}_n$ is a tuple from $A$.

In Theorem 3.3.6, we show that a random element $b$ is definable over a set $A$ of parameters if and only if $b$ is the limit of a sequence of random elements.

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such that for each \( n \),
\[
\llbracket (\forall u)(\varphi_n(u, \vec{a}_n) \leftrightarrow u = b_n) \rrbracket
\]
has probability one for some first order formula \( \varphi_n(u, \vec{v}) \) and a tuple \( \vec{a}_n \) from \( A \). In Section 4 we give some consequences in the special case that the underlying first order theory \( T \) is \( \aleph_0 \)-categorical.

Continuous model theory in its current form is developed in the papers [BBHU] and [BU]. The papers [Go1], [Go2], [Go3] deal with definability questions in metric structures. Randomizations of models are treated in [AK], [Be], [BK], [EG], [GL], [Ke1], and [Ke2].

2. Preliminaries

We refer to [BBHU] and [BU] for background in continuous model theory, and follow the notation of [BK]. We assume familiarity with the basic notions about continuous model theory as developed in [BBHU], including the notions of a theory, structure, pre-structure, model of a theory, elementary extension, isomorphism, and \( \kappa \)-saturated structure. In particular, the universe of a pre-structure is a pseudo-metric space, the universe of a structure is a complete metric space, and every pre-structure has a unique completion.

In continuous logic, formulas have truth values in the unit interval \([0,1]\) with 0 meaning true, the connectives are continuous functions from \([0,1]\) into \([0,1]\), and the quantifiers are sup and inf. A tuple is a finite sequence, and \( A^{< \aleph_0} \) is the set of all tuples of elements of \( A \).

2.1. The theory \( T^R \). We assume throughout that \( L \) is a finite or countable first order signature, and that \( T \) is a complete theory for \( L \) whose models have at least two elements.

The randomization signature \( L^R \) is the two-sorted continuous signature with sorts \( K \) (for random elements) and \( B \) (for events), an \( n \)-ary function symbol \( \llbracket \varphi(\cdot) \rrbracket \) of sort \( K^n \to B \) for each first order formula \( \varphi \) of \( L \) with \( n \) free variables, a \([0,1]\)-valued unary predicate symbol \( \mu \) of sort \( B \) for probability, and the Boolean operations \( \top, \bot, \land, \lor, \neg \) of sort \( B \). The signature \( L^R \) also has distance predicates \( d_B \) of sort \( B \) and \( d_K \) of sort \( K \). In \( L^R \), we use \( B, C, \ldots \) for variables or parameters of sort \( B \). \( B \models C \) means \( d_B(B, C) = 0 \), and \( B \subseteq C \) means \( B = B \cap C \).

A pre-structure for \( T^R \) will be a pair \( \mathcal{P} = (\mathcal{K}, \mathcal{B}) \) where \( \mathcal{K} \) is the part of sort \( K \) and \( \mathcal{B} \) is the part of sort \( B \). The reduction of \( \mathcal{P} \) is the pre-structure \( \mathcal{N} = (\hat{\mathcal{K}}, \hat{\mathcal{B}}) \) obtained from \( \mathcal{P} \) by identifying elements at distance zero, and the associated mapping from \( \mathcal{P} \) onto \( \mathcal{N} \) is called the reduction map. The completion of \( \mathcal{P} \) is the structure obtained by completing the metrics in the reduction of \( \mathcal{P} \). A pre-structure \( \mathcal{P} \) is called pre-complete if the reduction of \( \mathcal{P} \) is already the completion of \( \mathcal{P} \).

In [BK], the randomization theory \( T^R \) is defined by listing a set of axioms. We will not repeat these axioms here, because it is simpler to give the following model-theoretic characterization of \( T^R \).
Definition 2.1.1. Given a model $M$ of $T$, a nice randomization of $M$ is a pre-complete structure $(K, B)$ for $L^R$ equipped with an atomless probability space $(\Omega, B, \mu)$ such that:

1. $B$ is a $\sigma$-algebra with $\top, \bot, \cap, \cup, \neg$ interpreted by $\Omega, \emptyset, \cap, \cup, \setminus$.
2. $K$ is a set of functions $a : \Omega \to M$.
3. For each formula $\psi(\vec{x})$ of $L$ and tuple $\vec{a}$ in $K$, we have 
   $$[\psi(\vec{a})] = \{\omega \in \Omega : M \models \psi(\vec{a}(\omega))\} \in B.$$ 
4. $B$ is equal to the set of all events $[\psi(\vec{a})]$ where $\psi(\vec{v})$ is a formula of $L$ and $\vec{a}$ is a tuple in $K$.
5. For each formula $\theta(u, \vec{v})$ of $L$ and tuple $\vec{b}$ in $K$, there exists $a \in K$ such that 
   $$[\theta(a, \vec{b})] = [(\exists u \theta)(\vec{b})].$$
6. On $K$, the distance predicate $d_K$ defines the pseudo-metric 
   $$d_K(a, b) = \mu([a \neq b]).$$
7. On $B$, the distance predicate $d_B$ defines the pseudo-metric 
   $$d_B(B, C) = \mu(B \triangle C).$$

Definition 2.1.2. For each first order theory $T$, the randomization theory $T^R$ is the set of sentences that are true in all nice randomizations of models of $T$.

It follows that for each first order sentence $\varphi$, if $T \models \varphi$ then $T^R \models [\varphi] = \top$. The following basic facts are from [BK], Theorem 2.1 and Proposition 2.2, Example 3.4 (ii), Proposition 2.7, and Theorem 2.9.

Fact 2.1.3. For every complete first order theory $T$, the randomization theory $T^R$ is complete.

Fact 2.1.4. Every model $M$ of $T$ has nice randomizations.

Fact 2.1.5. (Fullness) Every pre-complete model $P = (K, B)$ of $T^R$ has perfect witnesses, i.e.,

1. For each first order formula $\theta(u, \vec{v})$ and each $\vec{b}$ in $K^n$ there exists $a \in K$ such that
   $$[\theta(a, \vec{b})] = [(\exists u \theta)(\vec{b})];$$
2. For each $B \in B$ there exist $a, b \in K$ such that
   $$B = [a = b].$$

Corollary 2.1.6. Every model $N$ of $T^R$ has a pair of elements $c, d$ such that 
$$[c \neq d] = \top.$$ 

Proof. Every model of $T$ has at least two elements, so $T \models (\exists u)(\exists v)u \neq v$. The result follows by applying Fullness twice. \qed

Fact 2.1.7. (Strong quantifier elimination) Every formula $\Phi$ in the continuous language $L^R$ is $T^R$-equivalent to a formula with the same free variables and no quantifiers of sort $K$ or $B$. 

Lemma 2.1.8. Let $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ be a pre-complete model of $T^R$ and let $a, b \in \mathcal{K}$ and $\mathcal{B} \in \mathcal{B}$. Then there is an element $c \in \mathcal{K}$ that agrees with $a$ on $\mathcal{B}$ and agrees with $b$ on $-\mathcal{B}$, that is, $\mathcal{B} \subseteq [c = a]$ and $(-\mathcal{B}) \subseteq [c = b]$.

Definition 2.1.9. In Lemma 2.1.8 we will call $c$ a characteristic function of $\mathcal{B}$ with respect to $a, b$.

Note that the distance between any two characteristic functions of an event $\mathcal{B}$ with respect to elements $a, b$ is zero. In particular, in a model of $T^R$, the characteristic function is unique.

Proof of Lemma 2.1.8. By Fact 2.1.3 (2), there exist $d, e \in \mathcal{K}$ such that $\mathcal{B} \vdash [d = e]$. The first order sentence

$$(\forall u)(\forall v)(\forall x)(\forall y)(\exists z)[(x = y \rightarrow z = u) \land (x \neq y \rightarrow z = v)]$$

is logically valid, so we must have

$\llbracket (\exists z)[(d = e \rightarrow z = a) \land (d \neq e \rightarrow z = b)] \rrbracket \vdash \top$.

By Fact 2.1.3 (1) there exists $c \in \mathcal{K}$ such that

$\llbracket d = e \rightarrow c = a \rrbracket \vdash \top$, $\llbracket d \neq e \rightarrow c = b \rrbracket \vdash \top$, so $\llbracket d = e \rrbracket \subseteq [c = a]$ and $\llbracket d \neq e \rrbracket \subseteq [c = b]$.

We will need the following result, which is a consequence of Theorem 3.11 of [Be]. Since the setting in [Be] is quite different from the present paper, we give a direct proof here.

Proposition 2.1.10. Every model of $T^R$ is isomorphic to the reduction of a nice randomization of a model of $T$.

Proof. Let $\mathcal{N} = (\hat{\mathcal{K}}, \hat{\mathcal{B}})$ be a model of $T^R$ of cardinality $\kappa$. Let $\Omega$ be the Stone space of the Boolean algebra $\hat{\mathcal{B}} = (\hat{\mathcal{B}}, \top, \bot, \cap, \cup, \neg)$. Thus $\Omega$ is a compact topological space, the points of $\Omega$ are ultrafilters, we may identify $\hat{\mathcal{B}}$ with the Boolean algebra of clopen sets of $\Omega$, and $\mu^\mathcal{N}$ is a finitely additive probability measure on $\hat{\mathcal{B}}$.

We next show that $\mu$ is $\sigma$-additive on $\hat{\mathcal{B}}$. To do this, we assume that $A_0 \supseteq A_1 \supseteq \cdots$ in $\hat{\mathcal{B}}$ and $C = \bigcap_{n \in \mathbb{N}} A_n \in \hat{\mathcal{B}}$, and prove that $\mu(C) = \lim_{n \to \infty} \mu(A_n)$. Indeed, the family $\{C \cup (\Omega \setminus A_n) : n \in \mathbb{N}\}$ is an open covering of $\Omega$, so by the topological compactness of $\Omega$, we have $\Omega = \bigcup_{k=0}^{m} (C \cup (\Omega \setminus A_k))$ for some $n \in \mathbb{N}$. Then $C = A_n$, so $\mu(C) = \mu(A_n) = \lim_{n \to \infty} \mu(A_n)$.

By the Caratheodory theorem, there is a complete probability space $(\Omega, \mathcal{B}, \mu)$ such that $\mathcal{B} \supseteq \hat{\mathcal{B}}$, $\mu$ agrees with $\mu^\mathcal{N}$ on $\hat{\mathcal{B}}$, and for each $\mathcal{B} \in \mathcal{B}$ and $m > 0$ there is a countable sequence $A_{m0} \subseteq A_{m1} \subseteq \cdots$ in $\hat{\mathcal{B}}$ such that

$$B \subseteq \bigcup_{n} A_{mn} \text{ and } \mu\left(\bigcup_{n} A_{mn}\right) \leq \mu(B) + 1/m.$$
Note that since the probability space \((\Omega, \mathcal{B}, \mu)\) is complete, every subset of \(\Omega\) that contains a set in \(\mathcal{B}\) of measure one also belongs to \(\mathcal{B}\) and has measure one.

We claim that for each \(\mathcal{B} \in \mathcal{B}\) there is a unique event \(f(\mathcal{B}) \in \hat{\mathcal{B}}\) such that \(\mu(f(\mathcal{B}) \triangle \mathcal{B}) = 0\). The uniqueness of \(f(\mathcal{B})\) follows from the fact that the distance function \(d_\mathcal{B}(C, D) = \mu(C \triangle D)\) is a metric on \(\hat{\mathcal{B}}\). To show the existence of \(f(\mathcal{B})\), for each \(m > 0\) let \(A_{m0} \subseteq A_{m1} \subseteq \cdots\) be as in [211]. Note that \((A_{m0}, A_{m1}, \ldots)\) is a Cauchy sequence of events in the model \(\mathcal{N}\), so there is an event \(C_m \in \hat{\mathcal{B}}\) such that \(C_m = \lim_{n \to \infty} A_{mn}\). Hence \(\lim_{n \to \infty} \mu(A_{mn} \triangle C_m) = 0\), so \(\mu((\bigcup_n A_{mn}) \triangle C_m) = 0\). Then \((C_1, C_2, \ldots)\) is a Cauchy sequence, so there is an event \(f(\mathcal{B}) = \lim_{m \to \infty} C_m\) in \(\hat{\mathcal{B}}\) with \(\mu(f(\mathcal{B}) \triangle \mathcal{B}) = 0\).

We make some observations about the mapping \(f: \mathcal{B} \to \hat{\mathcal{B}}\). If \(\mathcal{B}, \mathcal{C} \in \mathcal{B}\) and \(d_\mathcal{B}(\mathcal{B}, \mathcal{C}) = 0\), then \(f(\mathcal{B}) = f(\mathcal{C})\). For each \(\mathcal{B}, \mathcal{C} \in \mathcal{B}\), we have
\[
f(\mathcal{B} \cup \mathcal{C}) = f(\mathcal{B}) \cup f(\mathcal{C}), \quad f(\mathcal{B} \cap \mathcal{C}) = f(\mathcal{B}) \cap f(\mathcal{C}),
\]
\[
\Omega \setminus f(\mathcal{B}) = f(\Omega \setminus \mathcal{B}), \quad \mu(\mathcal{B}) = \mu(f(\mathcal{B})).
\]
Moreover, the mapping \(f\) sends \(\mathcal{B}\) onto \(\hat{\mathcal{B}}\), because if \(\mathcal{C} \in \hat{\mathcal{B}}\) then \(\mathcal{C} \in \mathcal{B}\) and \(f(\mathcal{C}) = \mathcal{C}\). Therefore the mapping \(\hat{f}\) that sends the equivalence class of each \(\mathcal{B} \in \mathcal{B}\) under \(d_\mathcal{B}\) to \(f(\mathcal{B})\) is well defined and is an isomorphism from the reduction of the pre-structure \((\hat{\mathcal{B}}, \bigcup, \bigcap, \neg, \top, \bot, \mu)\) onto the measured algebra \((\hat{\mathcal{B}}, \bigcup, \bigcap, \neg, \top, \bot, \mu)\).

A model \(\mathcal{M}\) of \(T\) is \(\kappa^+\)-universal if every model of \(T\) of cardinality \(\leq \kappa\) is elementarily embeddable in \(\mathcal{M}\). By Theorem 5.1.12 in [CK], every \(\kappa\)-saturated model of \(T\) is \(\kappa^+\)-universal, so \(\kappa^+\)-universal models of \(T\) exist. We now assume that \(\mathcal{M}\) is a \(\kappa^+\)-universal model of \(T\), and prove that \(\mathcal{N}\) is isomorphic to the reduction of a nice randomization of \(\mathcal{M}\) with the underlying probability space \((\Omega, \mathcal{B}, \mu)\).

In the following paragraphs, we will use boldface letters \(\mathbf{b}, \mathbf{d}, \ldots\) for elements of \(\hat{\mathcal{K}}\). Let \(L_{\hat{\mathcal{K}}}\) be the first order signature formed by adding a constant symbol for each element \(\mathbf{b} \in \hat{\mathcal{K}}\). For each \(\omega \in \Omega\), the set of \(L_{\hat{\mathcal{K}}^{-}}\)-sentences
\[
U(\omega) = \{ \psi(\mathbf{b}) : \omega \in [\psi(\mathbf{b})] \}
\]
is consistent with \(T\) and has cardinality \(\leq \kappa\). By the Compactness and Löwenheim-Skolem theorems, each \(U(\omega)\) has a model \((\mathcal{M}_\omega, \mathbf{b}_\omega)_{\mathbf{b} \in \hat{\mathcal{K}}}\) of cardinality \(\leq \kappa\). Since \(\mathcal{M}\) is \(\kappa^+\)-universal, for each \(\omega \in \Omega\) we may choose an elementary embedding \(h_\omega: \mathcal{M}_\omega \prec \mathcal{M}\). Then \((\mathcal{M}, h_\omega(\mathbf{b}_\omega))_{\mathbf{b} \in \hat{\mathcal{K}}} \models U(\omega)\) for every \(\omega \in \Omega\). It follows that for each formula \(\psi(\mathbf{v})\) of \(L\) and each tuple \(\mathbf{b} \in \hat{\mathcal{K}}^{<\mathbb{N}}\),
\[
[\psi(\mathbf{b})] = \{ \omega \in \Omega : \mathcal{M}_\omega \models \psi(\mathbf{b}_\omega) \} = \{ \omega \in \Omega : \mathcal{M} \models \psi(h_\omega(\mathbf{b}_\omega)) \} \in \hat{\mathcal{B}}.
\]
For each formula \(\psi(\mathbf{v})\) of \(L\) and tuple \(\mathbf{c}\) of functions in \(M^\Omega\), define
\[
[\psi(\mathbf{c})] := \{ \omega \in \Omega : \mathcal{M} \models \psi(\mathbf{c}(\omega)) \}.
\]
Let $\mathcal{K}$ be the set of all functions $a : \Omega \to M$ such that for some element $b \in \widehat{\mathcal{K}}$, we have
\[
\mu(\{\omega \in \Omega : a(\omega) = h_\omega(b_\omega)\}) = 1.
\]
We claim that for each $a \in \mathcal{K}$ there is a unique element $f(a) \in \widehat{\mathcal{K}}$ such that
\[
\mu(\{\omega \in \Omega : a(\omega) = h_\omega(f(a)_\omega)\}) = 1.
\]
The existence of $f(a)$ is guaranteed by the definition of $\mathcal{K}$. To prove uniqueness, suppose $b, d \in \widehat{\mathcal{K}}$ and
\[
\mu(\{\omega \in \Omega : a(\omega) = h_\omega(b_\omega)\}) = \mu(\{\omega \in \Omega : a(\omega) = h_\omega(d_\omega)\}) = 1.
\]
Then
\[
\mu(\{\omega \in \Omega : h_\omega(b_\omega) = h_\omega(d_\omega)\}) = 1,
\]
so
\[
\mu([b = d]) = \mu(\{\omega \in \Omega : b_\omega = d_\omega\}) = 1,
\]
and hence $d_\mathcal{K}(b, d) = 0$. Since $d_\mathcal{K}$ is a metric on $\widehat{\mathcal{K}}$, it follows that $b = d$.

We now make some observations about the mapping $f : \mathcal{K} \to \widehat{\mathcal{K}}$. This mapping sends $\mathcal{K}$ onto $\widehat{\mathcal{K}}$, because for each $b \in \widehat{\mathcal{K}}$, we have $f(a) = b$ where $a$ is the element of $\mathcal{K}$ such that $a(\omega) = h_\omega(b_\omega)$ for all $\omega \in \Omega$. Suppose $\bar{c} \in \mathcal{K}^{< \mathbb{N}}$ and $\bar{d} = f(\bar{c})$. We have $\bar{d} \in \widehat{\mathcal{K}}^{< \mathbb{N}}$ and
\[
[\psi(\bar{d})] = \{\omega \in \Omega : \mathcal{M} \models \psi(h_\omega(\bar{d}_\omega))\} = \{\omega \in \Omega : \mathcal{M} \models \psi(\bar{c}(\omega))\} = [\psi(\bar{c})].
\]
Since the probability space $(\Omega, \mathcal{B}, \mu)$ is complete, $[\psi(\bar{d})] \in \bar{\mathcal{B}} \subseteq \mathcal{B}$, and $[\psi(\bar{d})] = [\psi(\bar{c})]$, we have $[\psi(\bar{c})] \in \mathcal{B}$ and $[\psi(\bar{d})] = f([\psi(\bar{c})])$. Therefore, if $a, c \in \mathcal{K}$ and $d_\mathcal{K}(a, c) = 0$, then $d_\mathcal{K}(f(a), f(c)) = 0$, and hence $f(a) = f(c)$. This shows that $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ is a well-defined pre-complete structure for $L^\mathcal{K}$, and that the mapping $f$ that sends the equivalence class of each $\mathcal{B} \in \mathcal{B}$ to $f(\mathcal{B})$, and the equivalence class of each $a \in \mathcal{K}$ to $f(a)$, is an isomorphism from the reduction of $\mathcal{P}$ to $\mathcal{N}$.

It remains to show that $\mathcal{P}$ is a nice randomization of $\mathcal{M}$. It is clear that $\mathcal{P}$ satisfies conditions (1)-(3) in Definition 2.1.1.

Proof of (4): We have already shown that $[\psi(\bar{c})] \in \mathcal{B}$ for each formula $\psi(\bar{v})$ of $L$ and each tuple $\bar{c} \in \mathcal{K}$. For the other direction, let $\mathcal{B} \in \mathcal{B}$. By Corollary 2.1.6 there exist $a, e \in \mathcal{K}$ such that $[a \neq e] \equiv \Omega$. We may choose a function $b \in M^\Omega$ such that $b(\omega) = e(\omega)$ whenever $a(\omega) \neq e(\omega)$, and $b(\omega) \neq a(\omega)$ for all $\omega \in \Omega$. Then $b \in \mathcal{K}$ and $[a \neq b] \equiv \Omega$. By Lemma 2.1.8 there exists $e \in \mathcal{K}$ which is a characteristic function of $\mathcal{B}$ with respect to $a, b$. Then $[e = a] \equiv \mathcal{B}$. Let $d \in M^\Omega$ be the function such that $d(\omega) = a(\omega)$ for $\omega \in \mathcal{B}$, and $d(\omega) = b(\omega)$ for $\omega \in \neg \mathcal{B}$. Then $\mu([e = d]) = 1$, so $d \in \mathcal{K}$, and $[a = d] = \mathcal{B}$. Thus (4) holds with $\psi$ being the sentence $a = d$.

Proof of (5): Consider a formula $\theta(u, \bar{v})$ of $L$ and a tuple $\bar{b} \in \mathcal{K}$. By Fullness, there exists $c \in \mathcal{K}$ such that
\[
[\theta(c, \bar{b})] \equiv [\exists u \theta(u, \bar{b})].
\]
We may choose a function \( a \in M^\Omega \) such that for all \( \omega \in \Omega \),
\[
\mathcal{M} \models [\theta(c(\omega), \vec{b}(\omega)) \leftrightarrow (\exists u)\theta(u, \vec{b})] \text{ implies } a(\omega) = c(\omega),
\]
and
\[
\mathcal{M} \models [(\exists u)\theta(u, \vec{b}(\omega)) \rightarrow \theta(a(\omega), \vec{b}(\omega))].
\]
Then \( \mu([a = c]) = 1 \), so \( a \in \mathcal{K} \) and
\[
[\theta(a, \vec{b})] = [(\exists u)\theta(u, \vec{b})],
\]
as required.

Proof of (6) and (7): By Fact 2.1.4, the properties
\[
(\forall x)(\forall y)d_\mathbb{R}(x, y) = \mu(\llbracket x \neq y \rrbracket), \quad (\forall U)(\forall V)d_\mathbb{R}(U, V) = \mu(U \triangle V)
\]
hold in some model of \( T^R \). By Fact 2.1.3, these properties hold in all models of \( T^R \), and thus in \( \mathcal{N} \). Therefore (6) and (7) hold for \( \mathcal{P} \). \( \square \)

2.2. Types and Definability. For a first order structure \( \mathcal{M} \) and a set \( A \) of elements of \( \mathcal{M} \), \( \mathcal{M}_A \) denotes the structure formed by adding a new constant symbol to \( \mathcal{M} \) for each \( a \in A \). The type realized by a tuple \( \vec{b} \) over the parameter set \( A \) in \( \mathcal{M} \) is the set \( \text{tp}^\mathcal{M}(\vec{b}/A) \) of formulas \( \varphi(\vec{u}, \vec{a}) \) with \( \vec{a} \in A^{<\mathbb{N}} \) satisfied by \( \vec{b} \) in \( \mathcal{M}_A \). We call \( \text{tp}^\mathcal{M}(\vec{b}/A) \) an \( n \)-type if \( n = |\vec{b}| \).

In the following, let \( \mathcal{N} \) be a continuous structure and let \( A \) be a set of elements of \( \mathcal{N} \). \( \mathcal{N}_A \) denotes the structure formed by adding a new constant symbol to \( \mathcal{N} \) for each \( a \in A \). The type \( \text{tp}^\mathcal{N}(\vec{b}/A) \) realized by \( \vec{b} \) over the parameter set \( A \) in \( \mathcal{N} \) is the function \( p \) from formulas to \([0, 1]\) such that for each formula \( \Phi(\vec{x}, \vec{a}) \) with \( \vec{a} \in A^{<\mathbb{N}} \), we have \( \Phi(\vec{x}, \vec{a})^p = \Phi(\vec{b}, \vec{a})^\mathcal{N} \).

We now call the notions of definable element and algebraic element from [BBHU]. An element \( b \) is definable over \( A \) in \( \mathcal{N} \), in symbols \( b \in \text{dcl}^\mathcal{N}(A) \), if there is a sequence of formulas \( \langle \Phi_k(x, \vec{a}_k) \rangle \) with \( \vec{a}_k \in A^{<\mathbb{N}} \) such that the sequence of functions \( \langle \Phi_k(x, \vec{a}_k)^\mathcal{N} \rangle \) converges uniformly in \( x \) to the distance function \( d(x, b)^\mathcal{N} \) of the corresponding sort. \( b \) is algebraic over \( A \) in \( \mathcal{N} \), in symbols \( b \in \text{acl}^\mathcal{N}(A) \), if there is a compact set \( C \) and a sequence of formulas \( \langle \Phi_k(x, \vec{a}_k) \rangle \) with \( \vec{a}_k \in A^{<\mathbb{N}} \) such that \( b \in C \) and the sequence of functions \( \langle \Phi_k(x, \vec{a}_k)^\mathcal{N} \rangle \) converges uniformly in \( x \) to the distance function \( d(x, C)^\mathcal{N} \) of the corresponding sort.

If the structure \( \mathcal{N} \) is clear from the context, we will sometimes drop the superscript and write \( \text{tp}, \text{dcl}, \text{acl} \) instead of \( \text{tp}^\mathcal{N}, \text{dcl}^\mathcal{N}, \text{acl}^\mathcal{N} \).

Fact 2.2.1. ([BBHU], Exercises 10.7 and 10.10) For each element \( b \) of \( \mathcal{N} \), the following are equivalent, where \( p = \text{tp}^\mathcal{N}(b/A) \):

(1) \( b \) is definable over \( A \) in \( \mathcal{N} \);
(2) in each model \( \mathcal{N}' \succ \mathcal{N} \), \( b \) is the a unique element that realizes \( p \) over \( A \);
(3) \( b \) is definable over some countable subset of \( A \) in \( \mathcal{N} \).

Fact 2.2.2. ([BBHU], Exercise 10.8 and 10.11) For each element \( b \) of \( \mathcal{N} \), the following are equivalent, where \( p = \text{tp}^\mathcal{N}(b/A) \):

...
(1) \( b \) is algebraic over \( A \) in \( \mathcal{N} \);
(2) in each model \( \mathcal{N}' \supset \mathcal{N} \), the set of elements \( b \) that realize \( p \) over \( A \) in \( \mathcal{N}' \) is compact.
(3) \( b \) is algebraic over some countable subset of \( A \) in \( \mathcal{N} \).

**Fact 2.2.3.** (Definable Closure, Exercises 10.10 and 10.11 in [BBHU])

1. If \( A \subseteq \mathcal{N} \) then \( \text{dcl}(A) = \text{dcl}(\text{dcl}(A)) \) and \( \text{acl}(A) = \text{acl}(\text{acl}(A)) \).
2. If \( A \) is a dense subset of \( B \) and \( B \subseteq \mathcal{N} \), then \( \text{dcl}(A) = \text{dcl}(B) \) and \( \text{acl}(A) = \text{acl}(B) \).

It follows that for any \( A \subseteq \mathcal{N} \), \( \text{dcl}(A) \) and \( \text{acl}(A) \) are closed with respect to the metric in \( \mathcal{N} \).

We now turn to the case where \( \mathcal{N} \) is a model of \( T^R \). In that case, a set of elements of \( \mathcal{N} \) may contain elements of both sorts \( K, B \). But as we will now explain, we need only consider definability over sets of parameters of sort \( K \).

**Remark 2.2.4.** Let \( \mathcal{N} = (\hat{\mathcal{K}}, \hat{\mathcal{B}}) \) be a model of \( T^R \). Since every model of \( T \) has at least two elements, \( \mathcal{N} \) has a pair of elements \( a, b \) of sort \( K \) such that \( \mathcal{N} \models [a = b] = \perp \). For each event \( D \in \hat{B} \), let \( 1_D \) be the characteristic function of \( D \) with respect to \( a, b \). Then in the model \( \mathcal{N} \), \( D \) is definable over \( \{a, b, 1_D\} \), and \( 1_D \) is definable over \( \{a, b, D\} \).

**Proof.** By Fact 2.2.1.

In view of Remark 2.2.4 and Fact 2.2.3, if \( C \) is a set of parameters in \( \mathcal{N} \) of both sorts, and there are elements \( a, b \in C \) such that \( \mathcal{N} \models [a = b] = \perp \), then an element of either sort is definable over \( C \) if and only if it is definable over the set of parameters of sort \( K \) obtained by replacing each element of \( C \) of sort \( B \) by its characteristic function with respect to \( a, b \). For this reason, in a model \( \mathcal{N} \) of \( T^R \) we will only consider definability over sets of parameters of sort \( K \). We write \( \text{dcl}_K(A) \) for the set of elements of sort \( B \) that are definable over \( A \) in \( \mathcal{N} \), and write \( \text{dcl}(A) \) for the set of elements of sort \( K \) that are definable over \( A \) in \( \mathcal{N} \). Similarly for \( \text{acl}_B(A) \) and \( \text{acl}(A) \).

### 2.3. Conventions and Notation

We will assume hereafter that \( \mathcal{N} = (\hat{\mathcal{K}}, \hat{\mathcal{B}}) \) is a model of \( T^R \), \( \mathcal{P} = (\mathcal{K}, \mathcal{B}) \) is a nice randomization of a model \( \mathcal{M} \models T \) with probability space \( (\Omega, \mathcal{B}, \mu) \), and \( \mathcal{N} \) is the reduction of \( \mathcal{P} \). The existence of \( \mathcal{P} \) is guaranteed by Proposition 2.1.10.

We will use boldfaced letters \( \mathbf{a}, \mathbf{b}, \ldots \) for elements of \( \hat{\mathcal{K}} \). For each element \( \mathbf{a} \in \hat{\mathcal{K}} \), we will choose once and for all an element \( a \in \mathcal{K} \) such that the image of \( a \) under the reduction map is \( \mathbf{a} \). It follows that for each first order formula \( \varphi(\vec{v}) \), \( [\varphi(\vec{a})] \) is the image of \( [\varphi(\vec{a})] \) under the reduction map. For any countable set \( A \subseteq \hat{\mathcal{K}} \) and each \( \omega \in \Omega \), we define

\[
A(\omega) = \{a(\omega) : \mathbf{a} \in A\}.
\]

When \( A \subseteq \hat{\mathcal{K}} \), \( \text{cl}(A) \) denotes the closure of \( A \) in the metric \( d_{\mathcal{K}} \). When \( B \subseteq \hat{\mathcal{B}} \), \( \text{cl}(B) \) denotes the closure of \( B \) in the metric \( d_{\mathcal{B}} \), and \( \sigma(B) \) denotes the smallest \( \sigma \)-subalgebra of \( \hat{\mathcal{B}} \) containing \( B \).
3. Randomizations of Arbitrary Theories

3.1. Definability in Sort \( \mathbb{B} \). We characterize the set of elements of \( \hat{B} \) that are definable in \( \mathcal{N} \) over a set of parameters \( A \subseteq \hat{K} \).

**Definition 3.1.1.** For each \( A \subseteq \hat{K} \), we say that an event \( E \) is first order definable over \( A \), in symbols \( E \in \text{fdcl}_B(A) \), if \( E = \llbracket \varphi(\bar{a}) \rrbracket \) for some first order formula \( \varphi(\bar{v}) \) and tuple \( \bar{a} \) in \( A^{<\mathbb{N}} \).

**Theorem 3.1.2.** For each \( A \subseteq \hat{K} \), dcl\(_B\)(\( A \)) = cl(fdcl\(_B\)(\( A \))) = \sigma(fdcl\(_B\)(\( A \))).

**Proof.** By quantifier elimination (Fact 2.1.7), in any elementary extension \( \mathcal{N}' \succ \mathcal{N} \), two events have the same type over \( A \) if and only if they have the same type over fdcl\(_B\)(\( A \)). Then by Fact 2.2.1, dcl\(_B\)(\( A \)) = dcl\(_B\)(fdcl\(_B\)(\( A \))). Moreover, fdcl\(_B\)(fdcl\(_B\)(\( A \))) is equal to the definable closure of fdcl\(_B\)(\( A \)) in the pure measured algebra \( (\hat{B}, \mu) \). By Observation 16.7 in [BBHU], the definable closure of fdcl\(_B\)(\( A \)) in \( (\hat{B}, \mu) \) is equal to \( \sigma(fdcl\(_B\)(\( A \))) \), so dcl\(_B\)(\( A \)) = \( \sigma(fdcl\(_B\)(\( A \))) \). Since fdcl\(_B\)(\( A \)) is a Boolean subalgebra of \( \hat{B} \), cl(fdcl\(_B\)(\( A \))) is a Boolean subalgebra of \( \hat{B} \). By metric completeness, cl(fdcl\(_B\)(\( A \))) is a \( \sigma \)-algebra and \( \sigma(fdcl\(_B\)(\( A \))) \) is closed, so cl(fdcl\(_B\)(\( A \))) = \( \sigma(fdcl\(_B\)(\( A \))) \). □

**Corollary 3.1.3.** The only events that are definable without parameters in \( \mathcal{N} \) are \( \top \) and \( \bot \).

**Proof.** For every first order sentence \( \varphi \), either \( T \models \varphi \) and \( T^R \models \llbracket \varphi \rrbracket = \top \), or \( T \models \neg \varphi \) and \( T^R \models \llbracket \varphi \rrbracket = \bot \). So fdcl\(_B\)(\( \emptyset \)) = \{ \top, \bot \}. □

3.2. First Order and Pointwise Definability. To prepare the way for a characterization of the definable elements of sort \( \hat{K} \), we introduce two auxiliary notions, one that is stronger than definability in sort \( K \) and one that is weaker than definability in sort \( K \). We will work in the nice randomization \( \mathcal{P} = (\mathcal{K}, \mathcal{B}) \) of \( \mathcal{M} \), and let \( A \) be a subset of \( \hat{K} \) and \( b \) be an element of \( \hat{K} \).

**Definition 3.2.1.** A first order formula \( \varphi(u, \bar{v}) \) is functional if

\[ T \models (\forall \bar{v})(\exists^{\leq 1} u) \varphi(u, \bar{v}). \]

We say that \( b \) is first order definable on \( E \) over \( A \) if there is a functional formula \( \varphi(u, \bar{v}) \) and a tuple \( \bar{a} \in A^{<\mathbb{N}} \) such that \( E = \llbracket \varphi(b, \bar{a}) \rrbracket \).

We say that \( b \) is first order definable over \( A \), in symbols \( b \in \text{fdcl}(A) \), if \( b \) is first order definable on \( \top \) over \( A \).

**Remarks 3.2.2.** \( b \) is first order definable over \( A \) if and only if there is a first order formula \( \varphi(u, \bar{v}) \) and a tuple \( \bar{a} \) from \( A \) such that

\[ \mu(\llbracket (\forall u)(\varphi(u, \bar{a}) \leftrightarrow u = b) \rrbracket) = 1. \]

First order definability has finite character, that is, \( b \) is first order definable over \( A \) if and only if \( b \) is first order definable over some finite subset of \( A \).

If \( b \) is first order definable on \( E \) over \( A \), then \( E \) is first order definable over \( A \cup \{ b \} \).
If \( b \) is first order definable on \( D \) over \( A \), and \( E \) is first order definable over \( A \cup \{ b \} \), then \( b \) is first order definable on \( D \cap E \) over \( A \).

**Lemma 3.2.3.** If \( b \) is first order definable over \( A \) then \( b \) is definable over \( A \) in \( N \). Thus \( \text{fdcl}(A) \subseteq \text{dcl}(A) \).

**Proof.** Let \( N' > N \) and suppose that \( \text{tp}^{N'}(b) = \text{tp}^{N'}(d) \). Then
\[
[\varphi(b, \bar{a})] = [\varphi(d, \bar{a})] = \top.
\]
Since \( \varphi \) is functional,
\[
[(\forall t)(\forall u)(\varphi(t, \bar{a}) \land \varphi(u, \bar{a}) \rightarrow t = u)] = \top.
\]
Then \([b = d] = \top\), so \( b = d \), and by Fact 2.2.1, \( b \in \text{dcl}(A) \). \( \square \)

**Definition 3.2.4.** When \( A \) is countable, we define
\[
[b \in \text{dcl}^M(A)] := \{ \omega \in \Omega : b(\omega) \in \text{dcl}^M(A(\omega)) \}.
\]

**Lemma 3.2.5.** If \( A \) is countable, then
\[
[b \in \text{dcl}^M(A)] = \bigcup\{ [\theta(b, \bar{a})] : \theta(u, v) \text{ functional, } \bar{a} \in A^{<\omega} \},
\]
and \([b \in \text{dcl}^M(A)] \in \mathcal{B} \).

**Proof.** Note that for every first order formula \( \theta(u, v) \), the formula
\[
\theta(u, v) \land (\exists^{\leq 1} u) \theta(u, v)
\]
is functional. Therefore \( \omega \in [b \in \text{dcl}^M(A)] \) if and only if \( b(\omega) \in \text{dcl}^M(A(\omega)) \), and this holds if and only if there is a functional formula \( \theta(u, v) \) and a tuple \( \bar{a} \in A^{<\omega} \) such that \( M \models \theta(b(\omega), \bar{a}(\omega)) \). Since \( A \) and \( L \) are countable, \([b \in \text{dcl}^M(A)] \) is the union of countably many events in \( \mathcal{B} \), and thus belongs to \( \mathcal{B} \). \( \square \)

**Definition 3.2.6.** When \( A \) is countable, we say that \( b \) is pointwise definable over \( A \) if
\[
\mu([b \in \text{dcl}^M(A)]) = 1.
\]

**Corollary 3.2.7.** If \( A \) is countable, then \( b \) is pointwise definable over \( A \) if and only if there is a function \( f \) on \( \Omega \) such that:

1. For each \( \omega \in \Omega \), \( f(\omega) \) is a pair \( (\theta_\omega(u, v), \bar{a}_\omega) \) where \( \theta_\omega(u, v) \) is functional and \( \bar{a}_\omega \in A^{[\omega]} \);
2. \( f \) is \( \sigma(\text{fdcl}_B(A)) \)-measurable (i.e., the inverse image of each point belongs to \( \sigma(\text{fdcl}_B(A)) \));
3. \( M \models \theta_\omega(b(\omega), \bar{a}_\omega(\omega)) \) for almost every \( \omega \in \Omega \).

**Proof.** If \( \omega \in [b \in \text{dcl}^M(A)] \), let \( f(\omega) \) be the first pair \( (\theta_\omega, \bar{a}_\omega) \) such that \( \theta_\omega(u, v) \) is functional, \( \bar{a}_\omega \in A^{[\omega]} \), and \( M \models \theta_\omega(b(\omega), \bar{a}_\omega(\omega)) \). Otherwise let \( f(\omega) = (\bot, \emptyset) \). The result then follows from Lemma 3.2.6. \( \square \)

**Lemma 3.2.8.** If \( b \) is definable over \( A \) in \( N \), then \( b \) is pointwise definable over some countable subset of \( A \).
Proof. By Fact 2.2.1 (3), we may assume that $A$ is countable. By Lemma 3.2.8 the measure $r := \mu(\{b \in \text{dcl}(A)\})$ exists. Suppose $b$ is not pointwise definable over $A$. Then $r < 1$. For each finite collection $\chi_1(u, v), \ldots, \chi_n(u, v)$ of first order formulas, each tuple $\vec{a} \in A^{\infty \mathbb{N}}$, and each $\omega \in \Omega \setminus [b \in \text{dcl}(A)]$, the sentence

$$(\exists u)[u \neq b(\omega) \land \bigwedge_{i=1}^{n} [\chi_i(b(\omega), \vec{a}(\omega)) \leftrightarrow \chi_i(u, \vec{a}(\omega))]$$

holds in $\mathcal{M}$, because $b(\omega)$ is not definable over $A(\omega)$. Therefore in $\mathcal{P}$ we have

$$\mu([\exists u][u \neq b(\omega) \land \bigwedge_{i=1}^{n} [\chi_i(b, \vec{a}) \leftrightarrow \chi_i(u, \vec{a})]]) \geq 1 - r.$$ 

By condition 2.1.1 (5), there is an element $d \in \hat{\mathcal{K}}$ such that

$$\mu([d \neq b \land \bigwedge_{i=1}^{n} [\chi_i(b, \vec{a}) \leftrightarrow \chi_i(d, \vec{a})]]) \geq 1 - r.$$ 

It follows that $\mu([d \neq b]) \geq 1 - r$, and $[\chi_i(b, \vec{a})] \models [\chi_i(d, \vec{a})]$ for each $i \leq n$. By compactness, in some elementary extension of $\mathcal{N}$ there is an element $d$ such that $\mu([d \neq b]) \geq 1 - r$, and $[\chi(b, \vec{a})] = [\chi(d, \vec{a})]$ for each first order formula $\chi(u, v)$. Then $d \neq b$, and by quantifier elimination, $\text{tp}(d/A) = \text{tp}(b/A)$. Hence by Fact 2.2.1 (2), $b \notin \text{dcl}(A)$. \hfill $\square$

The following example shows that the converse of Lemma 3.2.8 fails badly.

Example 3.2.9. Let $\mathcal{M}$ be a finite structure with a constant symbol for every element. Then every element of $\mathcal{K}$ is pointwise definable without parameters, but the only elements of $\hat{\mathcal{K}}$ that are definable without parameters are the equivalence classes of constant functions $b: \Omega \to \mathcal{M}$.

3.3. Definability in Sort $\mathbb{K}$. We will now give necessary and sufficient conditions for an element of $b \in \hat{\mathcal{K}}$ to be definable over a parameter set $A \subseteq \hat{\mathcal{K}}$ in $\mathcal{N}$.

Theorem 3.3.1. $b$ is definable over $A$ if and only if there exist pairwise disjoint events $\{E_n: n \in \mathbb{N}\}$ such that $\sum_{n \in \mathbb{N}} \mu(E_n) = 1$, and for each $n$, $E_n$ is definable over $A$, and $b$ is first order definable on $E_n$ over $A$.

Proof. ($\Rightarrow$): Suppose $b \in \text{dcl}(A)$. By Lemma 3.2.8 $b$ is pointwise definable over some countable subset $A_0$ of $A$. The set of all events $C$ such that $b$ is first order definable on $C$ over $A_0$ is countable, and may be arranged in a list $\{C_n: n \in \mathbb{N}\}$. Let $E_0 = C_0$, and

$$E_{n+1} = C_{n+1} \cap \neg(C_0 \cup \cdots \cup C_n).$$

The events $E_n$ are pairwise disjoint, and for each $n$ we have

$$E_0 \cup \cdots \cup E_n = C_0 \cup \cdots \cup C_n.$$
By Remarks 3.2.2 for each $n$, $b$ is first order definable on $E_n$ over $A$. By Lemma 3.2.3 and pointwise definability,
\[ \sum_{n \in \mathbb{N}} \mu(E_n) = \lim_{n \to \infty} \mu(C_0 \sqcup \cdots \sqcup C_n) = \mu(\operatorname{dcl}^{\forall \exists}(A_0)) = 1. \]

By Remarks 3.2.2 $E_n$ is definable over $A \cup \{b\}$, and since $b$ is definable over $A$, $E_n$ is definable over $A$ by Fact 2.2.3.

$(\Leftarrow)$: Let $E_n$ be as in the theorem. For each $n$, we have $E_n = [\theta_n(b, \bar{a}_n)]$ for some functional formula $\theta_n$ and tuple $\bar{a}_n \in A^{<\mathbb{N}}$. Since $E_n$ is definable over $A$, by Theorem 3.1.2 there is a sequence of formulas $\psi_k(\bar{v})$ and tuples $\bar{a}_k \in A^{<\mathbb{N}}$ such that
\[ \lim_{k \to \infty} d_{\mathbb{B}}([\psi_k(\bar{a}_k)], [\theta_n(b, \bar{a})]) = 0. \]

Suppose $d$ has the same type over $A$ as $b$ in some elementary extension $N'$ of $\mathcal{N}$. Then
\[ \lim_{k \to \infty} d_{\mathbb{B}}([\psi_k(\bar{a}_k)], [\theta_n(d, \bar{a})]) = 0. \]

Hence
\[ [\theta_n(d, \bar{a}_n)] = [\theta_n(b, \bar{a}_n)] = E_n \]
in $N'$. Since $\theta_n(u, \bar{v})$ is functional, we have $[\theta_n(b, \bar{a})] \sqsubseteq [d = b]$ for each $n$. Then
\[ \mu([d = b]) \geq \sum_{n \in \mathbb{N}} \mu(E_n) = 1, \]
so $d = b$. Then by Fact 2.2.1 $b \in \operatorname{dcl}(A)$.

\[ \square \]

**Corollary 3.3.2.** An element $b \in \widehat{K}$ is definable without parameters if and only if $b$ is first order definable without parameters. Thus $\operatorname{dcl}(\emptyset) = \operatorname{fdcl}(\emptyset)$.

**Proof.** $(\Rightarrow)$: Suppose $b \in \operatorname{dcl}(\emptyset)$. By Theorem 3.3.1 there is an event $E$ such that $\mu(E) > 0$, $E$ is definable without parameters, and $b$ is first order definable on $E$ without parameters. By Corollary 3.1.3 we have $E = \top$, so $b$ is first order definable without parameters.

$(\Leftarrow)$: By Lemma 3.2.3.

\[ \square \]

**Corollary 3.3.3.** If $\operatorname{fdcl}_{\mathbb{B}}(A)$ is finite, then $\operatorname{dcl}_{\mathbb{B}}(A) = \operatorname{fdcl}_{\mathbb{B}}(A)$ and $\operatorname{dcl}(A) = \operatorname{fdcl}(A)$.

**Proof.** $\operatorname{dcl}_{\mathbb{B}}(A) = \operatorname{fdcl}_{\mathbb{B}}(A)$ follows from Theorem 3.1.2. Lemma 3.2.3 gives $\operatorname{dcl}(A) \supseteq \operatorname{fdcl}(A)$. For the other inclusion, suppose $b \in \operatorname{dcl}(A)$. By Theorem 3.3.1 there is a finite partition $E_0, \ldots, E_k$ of $\top$, a tuple $\bar{a} \in A^{<\mathbb{N}}$, and first order formulas $\psi_i(\bar{v})$ such that $E_i = [\psi_i(\bar{a})]$ and $b$ is first order definable on $E_i$. Then there are functional formulas $\varphi_i(u, \bar{v})$ such that $E_i = [\varphi_i(b, \bar{a})]$.

We may take the formulas $\psi_i(\bar{v})$ to be pairwise inconsistent and such that
3.2.3) and the set \( d \) and that \( D \) assume that disjoint. We have \( E \).

By (2), \( n \).

Proof. (\( \Rightarrow \)): Suppose \( b \in \text{dcl}(A) \). Then (1) holds by Lemma \( 3.2.8 \). \( \lbrack \varphi(b, \bar{a}) \rbrack \) is obviously definable over \( A \cup \{ b \} \), so \( \lbrack \varphi(b, \bar{a}) \rbrack \) is definable over \( A \) by Fact 2.2.3 and thus (2) holds.

(\( \Leftarrow \)): Assume conditions (1) and (2). By (1) and Lemma \( 3.2.5 \) there is a sequence of functional formulas \( \theta_n(u, \bar{v}) \) and tuples \( \bar{a}_n \in A^{<\mathbb{N}} \) such that

\[
\lbrack [b \in \text{dcl}^{\mathbb{M}}(A)] \rbrack = \bigcup_{n \in \mathbb{N}} \lbrack \theta_n(b, \bar{a}_n) \rbrack = \Omega.
\]

Let \( E_n = [\theta_n(b, \bar{a}_n)] \), so \( b \) is first order definable on \( E_n \) over \( A \). By Remark 3.2.2 we may take the \( E_n \) to be pairwise disjoint, and thus \( \sum_{n \in \mathbb{N}} \mu(E_n) = 1 \). By (2), \( E_n \) is definable over \( A \) for each \( n \). Then by Theorem 3.3.1 \( b \in \text{dcl}(A) \).

Corollary 3.3.5. \( b \) is definable over \( A \) if and only if:

(1) \( b \) is pointwise definable over some countable subset of \( A \);

(2) for each functional formula \( \varphi(u, \bar{v}) \) and tuple \( \bar{a} \in A^{<\mathbb{N}} \), \( \lbrack \varphi(b, \bar{a}) \rbrack \) is definable over \( A \).

Theorem 3.3.6. \( b \) is definable over \( A \) if and only if \( b = \lim_{m \to \infty} b_m \), where each \( b_m \) is first-order definable over \( A \). Thus \( \text{dcl}(A) = \text{cl}(\text{fdcl}(A)) \).

Proof. (\( \Rightarrow \)): Suppose that \( b \in \text{dcl}(A) \). If \( A \) is empty, then \( b \) is already first order definable from \( A \) by Corollary 3.3.2. Assume \( A \) is not empty and let \( c \in A \). Let \( \{ E_n : n \in \mathbb{N} \} \) be as in Theorem 3.3.1 and fix an \( \varepsilon > 0 \). Then for some \( n \), \( \sum_{k=0}^{n} \mu(E_k) > 1 - \varepsilon \). For each \( k \), \( E_k \) is definable over \( A \), so by Theorem 3.1.2 there is an event \( D_k \in \text{fdcl}(A) \) such that \( \mu(D_k \triangle E_k) < \varepsilon/n \). Since the events \( E_k \) are pairwise disjoint, we may also take the events \( D_k \) to be pairwise disjoint. We have \( E_k = [\theta_k(b, \bar{a}_k)] \) for some functional \( \theta_k(u, \bar{v}) \), so we may assume that \( D_k \) has the additional properties that \( D_k \subseteq \lbrack (\exists u) \theta_k(u, \bar{a}_k) \rbrack \), and that \( D_k = [\psi_k(\bar{a}_k)] \) for some formula \( \psi_k(\bar{v}) \). Then there is a unique element \( d \in \hat{K} \) such that

\[
\begin{align*}
\{ M \models \theta_k(d(\omega), \bar{a}_k(\omega)) \} & \quad \text{if } k \leq n \text{ and } \omega \in [\psi_k(\bar{a}_k)], \\
\{ d(\omega) = c(\omega) \} & \quad \text{if } \omega \in \Omega \setminus \bigcup_{k=0}^{n} [\psi_k(\bar{a}_k)].
\end{align*}
\]

Then \( d \) is first order definable over \( A \), and \( d \notin b \).

(\( \Leftarrow \)): This follows because first order definability implies definability (Lemma 3.2.3) and the set \( \text{dcl}(A) \) is metrically closed (Fact 2.2.3 (2)).
The following result was proved in [Be] by an indirect argument using Lascar types. We give a simple direct proof here.

**Proposition 3.3.7.** For any model $\mathcal{N} = (\hat{\mathcal{K}}, \hat{\mathcal{B}})$ of $\mathcal{T}^R$ and set $A \subseteq \hat{\mathcal{K}}$, $\text{acl}_{\mathcal{B}}(A) = \text{dcl}_{\mathcal{B}}(A)$ and $\text{acl}(A) = \text{dcl}(A)$.

**Proof.** By Facts 2.2.1 and 2.2.2, we may assume $\mathcal{N}$ is $\mathcal{N}_1$-saturated and $A$ is countable. Suppose an event $E \in \hat{\mathcal{B}}$ is not definable over $A$. By Fact 2.2.1 and $\mathcal{N}_1$-saturation there exists $D \in \hat{\mathcal{B}}$ such that $\text{tp}(D/A) = \text{tp}(E/A)$ but $d_{\mathcal{B}}(D, E) > 0$. By $\mathcal{N}_1$-saturation again, there is a countable sequence of events $\langle F_n : n \in \mathbb{N} \rangle$ in $\hat{\mathcal{B}}$ such that

$$\mu(C \cap F_n) = \mu(C \setminus F_n) = \frac{\mu(C)}{2}$$

for each $n$ and each event $C$ in the Boolean algebra generated by $\text{fdcl}_{\mathcal{B}}(A) \cup \{D, E\} \cup \{F_k : k < n\}$.

For each $n$, let

$$D_n = (D \cap F_n) \cup (E \setminus F_n).$$

Then for each $C \in \text{fdcl}_{\mathcal{B}}(A)$ and $n \in \mathbb{N}$, we have

$$\mu(D_n \cap C) = \mu(D \cap C)/2 + \mu(E \cap C)/2 = \mu(E \cap C).$$

By quantifier elimination, $\text{tp}(D_n/A) = \text{tp}(E/A)$ for each $n \in \mathbb{N}$. Moreover, whenever $k < n$ we have

$$D_n \setminus D_k = ((D \setminus D_k) \cap F_n) \cup ((E \setminus D_k) \setminus F_n),$$

so

$$\mu(D_n \setminus D_k) = \mu(D \setminus D_k)/2 + \mu(E \setminus D_k)/2.$$ 

Note that whenever $\text{tp}(D'/A) = \text{tp}(D''/A)$, we have $\mu(D') = \mu(D'')$, and hence

$$\mu(D' \setminus D'') = \mu(D'' \setminus D') = d_{\mathcal{B}}(D', D'')/2.$$ 

Therefore

$$d_{\mathcal{B}}(D_n, D_k) = d_{\mathcal{B}}(D, D_k)/2 + d_{\mathcal{B}}(E, D_k)/2 \geq d_{\mathcal{B}}(D, E)/2.$$ 

It follows that the set of realizations of $\text{tp}(E/A)$ is not compact, and $E$ is not algebraic over $A$. This shows that $\text{acl}_{\mathcal{B}}(A) = \text{dcl}_{\mathcal{B}}(A)$.

Now suppose $b \in \text{acl}(A) \setminus \text{dcl}(A)$. There is an element $c \in \hat{\mathcal{K}}$ such that $\text{tp}(b/A) = \text{tp}(c/A)$ but $d_{\mathcal{K}}(b, c) > 0$. For each first order formula $\psi(u, \vec{v})$ and $\vec{a} \in A^{<\mathbb{N}}$, $[\psi(b, \vec{a})] \subseteq \text{acl}_{\mathcal{K}}(\{b\} \cup A) \subseteq \text{acl}_{\mathcal{B}}(\text{acl}(A))$. By Fact 2.2.3, $[\psi(b, \vec{a})] \subseteq \text{acl}_{\mathcal{B}}(A)$. By the preceding paragraph, $[\psi(b, \vec{a})] \subseteq \text{acl}_{\mathcal{B}}(A)$.

Since $\text{tp}(b/A) = \text{tp}(c/A)$, we have $\text{tp}(\psi(b, \vec{a})/A) = \text{tp}(\psi(c, \vec{a})/A)$. By Fact 2.2.1 it follows that $[\psi(b, \vec{a})] = [\psi(c, \vec{a})]$ for every first order formula $\psi(u, \vec{v})$. Then $\text{tp}(b(\omega)/A(\omega)) = \text{tp}(c(\omega)/A(\omega))$ for $\mu$-almost all $\omega$. By $\mathcal{N}_1$-saturation, there are countably many independent events $D_n \in \hat{\mathcal{B}}$ such that $D_n \subseteq [b \neq c]$ and $\mu(D_n) = d_{\mathcal{K}}(b, c)/2$. Let $c_n$ agree with $c$ on $D_n$ and agree with $b$ elsewhere. We have $\text{tp}(c_n/A) = \text{tp}(b/A)$ for every $n \in \mathbb{N}$, and
$d_{\mathfrak{K}}(c_i, c_k) = d_{\mathfrak{K}}(b, c)/2$ whenever $k < n$. Thus the set of realizations of $\text{tp}(b/A)$ is not compact, contradicting the fact that $b \in \text{acl}(A)$.

\[\square\]

4. A Special Case: $\aleph_0$-categorical theories

4.1. Definability and $\aleph_0$-Categoricity. We use our preceding results to characterize $\aleph_0$-categorical theories in terms of definability in randomizations.

**Theorem 4.1.1.** The following are equivalent:

1. $T$ is $\aleph_0$-categorical;
2. $\text{fdcl}_\mathcal{B}(A)$ is finite for every finite $A$;
3. $\text{dcl}_\mathcal{B}(A)$ is finite for every finite $A$;
4. $\text{fdcl}_\mathcal{B}(A) = \text{dcl}_\mathcal{B}(A)$ for every finite $A$;
5. $\text{fdcl}(A)$ is finite for every finite $A$;
6. $\text{dcl}(A)$ is finite for every finite $A$;
7. $\text{dcl}(A) = \text{acl}(A)$ for every finite $A$;

**Proof.** By the Ryll-Nardzewski Theorem (see [CK], Theorem 2.3.13), (1) is equivalent to

(0) For each $n$ there are only finitely many formulas in $n$ variables up to $T$-equivalence.

Assume (0) and let $A \subseteq \hat{\mathcal{K}}$ be finite. Then (2) holds. Moreover, there are only finitely many functional formulas in $|A| + 1$ variables, so (5) holds.

Then by Corollary 3.3.3 (3), (4), (6), and (7) hold.

Now assume that (0) fails.

**Proof that (2) and (3) fail:** For some $n$ there are infinitely many formulas in $n$ variables that are not $T$-equivalent. Hence there is an $n$-type $p$ in $T$ without parameters that is not isolated. So there are formulas $\varphi_1(\vec{v}), \varphi_2(\vec{v}), \ldots$ in $p$ such that for each $k > 0$, $T \models \varphi_k \rightarrow \varphi_{k+1}$ but the formula $\theta_k = \varphi_k \land \neg \varphi_{k+1}$ is consistent with $T$. The formulas $\theta_k$ are consistent but pairwise inconsistent. By Fullness, for each $k > 0$ there exists an $n$-tuple $\vec{b}_k \in \hat{\mathcal{K}}^n$ such that $[\theta_k(\vec{b}_k)] = T$. Since the measured algebra $(\hat{\mathcal{B}}, \mu)$ is atomless, there are pairwise disjoint events $E_1, E_2, \ldots$ in $\hat{\mathcal{B}}$ such that $\mu(E_k) = 2^{-k}$ for each $k > 0$. By applying Lemma 2.1.8 $k$ times, we see that for each $k > 0$ there is an $n$-tuple $\vec{a}_k \in \hat{\mathcal{K}}^n$ that agrees with $\vec{b}_i$ on $E_i$ whenever $0 < i \leq k$. Whenever $0 < k \leq j$, we have $\mu([\vec{a}_k = \vec{a}_j]) \geq 1 - 2^{-k}$. So $\langle \vec{a}_1, \vec{a}_2, \ldots \rangle$ is a Cauchy sequence, and by metric completeness the limit $\vec{a} = \lim_{k \to \infty} \vec{a}_k$ exists in $\hat{\mathcal{K}}^n$. Let $A = \text{range}(\vec{a})$. For each $k > 0$ we have $E_k = [\vec{a} = \vec{b}_k] = [\theta_k(\vec{a})]$, so $E_k \in \text{fdcl}_\mathcal{B}(A)$. Thus $\text{fdcl}_\mathcal{B}(A)$ is infinite, so (2) fails and (3) fails.

**Proof that (4) fails:** Let $E_k$ be as in the preceding paragraph. The set $\text{fdcl}_\mathcal{B}(A)$ is countable. But the closure $\text{cl}(\text{fdcl}_\mathcal{B}(A))$ is uncountable, because for each set $S \subseteq \mathbb{N} \setminus \{0\}$, the supremum $\bigcup_{k \in S} E_k$ belongs to $\text{cl}(\text{fdcl}_\mathcal{B}(A))$. Thus by Theorem 3.1.2

$\text{dcl}_\mathcal{B}(A) = \text{cl}(\text{fdcl}_\mathcal{B}(A)) \neq \text{fdcl}_\mathcal{B}(A)$,
and (4) fails.

Proof that (5), (6), and (7) fail: By Corollary 2.1.6 there exist \( c, d \in K \) such that \([c \neq d]\) = \( \top \). Let \( C \) be the finite set \( C = A \cup \{c, d\} \). By Remark 2.2.3 for any event \( D \in \text{fdcl}_B(A) \), the characteristic function \( 1_D \) of \( D \) with respect to \( c, d \) is definable over \( C \). Moreover, we always have \( d_\mathbb{K}(1_D, 1_E) = d_\mathbb{E}(D, E) \). It follows that \( \text{fdcl}(C) \) is infinite, so (5) and (6) fail. To show that (7) fails, we take an event \( D \in \text{dcl}(A) \setminus \text{fdcl}_B(A) \). By Theorem 3.1.2 we have \( D \in \text{cl}(\text{fdcl}(A)) \). It follows that \( 1_D \in \text{cl}(\text{fdcl}(C)) \), so by Theorem 3.3.6 \( 1_D \in \text{dcl}(C) \). Hence \( \text{dcl}(C) \) is uncountable. But \( \text{fdcl}(C) \) is countable, so (7) fails.

By the Ryll-Nardzewski Theorem, if \( T \) is \( \aleph_0 \)-categorical then for each \( n \), \( T \) has finitely many \( n \)-types; so each type \( p \) in the variables \( (u, \bar{v}) \) has an isolating formula, that is, a formula \( \varphi(u, \bar{v}) \) such that \( T \models \varphi(u, \bar{v}) \leftrightarrow \bigwedge p \).

We now characterize the definable closure of a finite set \( A \subseteq \hat{K} \) in the case that \( T \) is \( \aleph_0 \)-categorical. Hereafter, when \( A \) is a finite subset of \( \hat{K} \), \( \bar{a} \) will denote a finite tuple whose range is \( A \).

**Corollary 4.1.2.** Suppose that \( T \) is \( \aleph_0 \)-categorical, \( b \in \hat{K} \), and \( A \) is a finite subset of \( \hat{K} \). Then \( b \in \text{dcl}(A) \) if and only if:

1. \( b \) is pointwise definable over \( A \);
2. for every isolating formula \( \varphi(u, \bar{v}) \), if \( \mu(\llbracket \varphi(b, \bar{a}) \rrbracket) > 0 \) then \( \llbracket \varphi(b, \bar{a}) \rrbracket = \llbracket (\exists u) \varphi(u, \bar{a}) \rrbracket \).

**Proof.** (\( \Rightarrow \)): Suppose \( b \in \text{dcl}(A) \). (1) holds by Lemma 3.2.8. Suppose \( \varphi(u, \bar{v}) \) is isolating and \( \mu(\llbracket \varphi(b, \bar{a}) \rrbracket) > 0 \). We have \([\varphi(b, \bar{a})] \in \text{fdcl}_B(\{b\} \cup A) \), so by Corollary 3.3.5 \([\varphi(b, \bar{a})] \in \text{dcl}(A) \). By Theorem 4.1.1 \([\varphi(b, \bar{a})] \in \text{fdcl}_B(A) \). We note that \( (\exists u) \varphi(u, \bar{v}) \) is an isolating formula, so \([\exists u) \varphi(u, \bar{a}) \rrbracket \) is an atom of \( \text{fdcl}_B(A) \). Therefore (2) holds.

(\( \Leftarrow \)): Assume (1) and (2). By (2), for every isolating formula \( \varphi(u, \bar{v}) \) such that \( \mu(\llbracket \varphi(b, \bar{a}) \rrbracket) > 0 \), we have \( \llbracket \varphi(b, \bar{a}) \rrbracket \in \text{fdcl}_B(A) \).

Every formula \( \Theta(u, \bar{v}) \) in \( T \)-equivalent to a finite disjunction of isolating formulas in the variables \( (u, \bar{v}) \). It follows that \( \text{fdcl}_B(A \cup \{b\}) \subseteq \text{fdcl}_B(A) \). Therefore by Corollary 3.3.5 \( b \in \text{dcl}(A) \). □

**Corollary 4.1.3.** Suppose that \( T \) is \( \aleph_0 \)-categorical, \( b \in \hat{K} \), and \( A \) is a finite subset of \( \hat{K} \). Then \( b \in \text{dcl}(A) \) if and only if for every isolating formula \( \psi(\bar{v}) \) there is a functional formula \( \varphi(u, \bar{v}) \) such that \( \llbracket \psi(\bar{a}) \rrbracket \subseteq \llbracket \varphi(b, \bar{a}) \rrbracket \).

**Proof.** (\( \Rightarrow \)): Suppose \( b \in \text{dcl}(A) \). By Theorem 4.1.1 \( b \) is first order definable over \( \bar{a} \), so there is a functional formula \( \varphi(u, \bar{v}) \) such that \( \llbracket \varphi(b, \bar{a}) \rrbracket = \top \). Then for every isolating \( \psi(\bar{v}) \) we have \( \llbracket \psi(\bar{a}) \rrbracket \subseteq \llbracket \varphi(b, \bar{a}) \rrbracket \).

(\( \Leftarrow \)): There is a finite set \( \{\psi_0(\bar{v}), \ldots, \psi_k(\bar{v})\} \) that contains exactly one isolating formula for each \( |\bar{a}| \)-type of \( T \). By hypothesis, for each \( i \leq k \) there is a functional formula \( \varphi_i(u, \bar{v}) \) such that \( \llbracket \psi_i(\bar{a}) \rrbracket \subseteq \llbracket \varphi_i(b, \bar{a}) \rrbracket \). Since the
formulas \( \psi_i(\overline{v}) \) are pairwise inconsistent, the formula \( \bigvee_{i=0}^{k} (\psi_i(\overline{v}) \land \varphi_i(\overline{u}, \overline{v})) \) is functional, and
\[
\llbracket \bigvee_{i=0}^{k} (\psi_i(\overline{a}) \land \varphi_i(\overline{b}, \overline{a})) \rrbracket = \top.
\]
Hence \( b \) is first order definable over \( \overline{a} \), so by Lemma \ref{lemma:isolating} we have \( b \in \text{dcl}(A) \). \qed

4.2. The Theory \( \text{DLO}^R \). We will use Corollary \ref{corollary:universal} to give a more natural characterization of the definable closure of a finite parameter set in a model of \( \text{DLO}^R \), where \( \text{DLO} \) is the theory of dense linear order without endpoints.

Note that in \( \text{DLO} \), every type in \( (v_1, \ldots, v_n) \) has an isolating formula of the form \( \bigwedge_{i=1}^{n-1} u_i \alpha_i u_{i+1} \) where \( \{u_1, \ldots, u_n\} = \{v_1, \ldots, v_n\} \) and each \( \alpha_i \in \{<,=\} \).

(This formula linearly orders the equality-equivalence classes).

**Corollary 4.2.1.** Let \( T = \text{DLO}, b \in \hat{K}, \) and \( A \) be a finite subset of \( \hat{K} \). Then \( b \in \text{dcl}(A) \) if and only if for every isolating formula \( \psi(v_1, \ldots, v_n) \) there is an \( i \in \{1, \ldots, n\} \) such that \( \llbracket \psi(\overline{a}) \rrbracket \subseteq \llbracket b = a \rrbracket \).

**Proof.** For any \( M \models \text{DLO} \) and parameter set \( A \), we have \( \text{dcl}^M(A) = A \). Therefore for every isolating formula \( \psi(v_1, \ldots, v_n) \) and functional formula \( \varphi(u, v_1, \ldots, v_n) \) there exists \( i \in \{1, \ldots, n\} \) such that
\[
\text{DLO} \models (\psi(v_1, \ldots, v_n) \land \varphi(u, v_1, \ldots, v_n)) \rightarrow u = v_i.
\]
The result now follows from Corollary \ref{corollary:universal} \( \square \).

In the theory \( \text{DLO} \), we define \( \min(u, v) \) and \( \max(u, v) \) in the usual way. For \( a, b \in \hat{K} \), we let \( \min(a, b) \) be the unique element \( e \in \hat{K} \) such that
\[
\llbracket e = \min(a, b) \rrbracket = \top,
\]
and similarly for max. For finite subsets \( A \) of \( \hat{K} \), \( \min(A) \) and \( \max(A) \) are defined by repeating the two-variable functions \( \min \) and \( \max \) in the natural way.

We next show that in \( \text{DLO}^R \), the definable closure of a finite set can be characterized as the closure under a “choosing function” of four variables.

**Definition 4.2.2.** In the theory \( \text{DLO} \), let \( \ell \) be the function of four variables defined by the condition
\[
\ell(u, v, x, y) = x \text{ if } u < v, \text{ and } \ell(u, v, x, y) = y \text{ if not } u < v.
\]
For \( a, b, c, d \in \hat{K} \), let \( \ell(a, b, c, d) \) be the unique element \( e \in \hat{K} \) such that \( \llbracket e = \ell(a, b, c, d) \rrbracket = \top \). Given a set \( A \subseteq \hat{K} \), let \( \text{lcl}(A) \) be the closure of \( A \) under the function \( \ell \).

Note that in \( \text{DLO} \), the function \( \ell \) is definable without parameters. In both \( \text{DLO} \) and \( \text{DLO}^R \), \( \min(u, v) = \ell(u, v, u, v) \), and \( \max(u, v) = \ell(u, v, v, u) \).

**Proposition 4.2.3.** Let \( T = \text{DLO} \). Then for every finite subset \( A \) of \( \hat{K} \), \( \text{dcl}(A) = \text{lcl}(A) \).
Proof. It is clear that \( \text{lcl}(A) \subseteq \text{dcl}(A) \).

We prove the other inclusion. If \( A \) is empty, the result is trivial, so we assume \( A \) is non-empty. Let \( 0 = \min(A), 1 = \max(A) \). We have \( 0, 1 \in \text{lcl}(A) \). Let \( \Omega_0 = [0 < 1] \). Note that \( \Omega \setminus \Omega_0 = [0 = 1] \). If \( \mu(\Omega_0) = 0 \), then \( A \) is a singleton, and we trivially have \( \text{lcl}(A) = \text{dcl}(A) = A \). We may therefore assume that \( \mu(\Omega_0) > 0 \). To simplify notation we will instead assume that \( \Omega_0 = \Omega \); the argument in the general case is similar.

In the following, all characteristic functions are understood to be with respect to \( 0, 1 \). Note that \( \ell(\bar{a}, \bar{b}, 0, 1) \) is the characteristic function of the event \([a < b]\). If \( \bar{d} \) is the characteristic function of an event \( D \) and \( \bar{e} \) is the characteristic function of an event \( E \), then \( \ell(\bar{d}, 1, 1, 0) \) is the characteristic function of \( \neg D \), \( \min(\bar{d}, \bar{e}) \) is the characteristic function of \( D \cap E \), and \( \max(\bar{d}, \bar{e}) \) is the characteristic function of \( D \cup E \). It follows that for every quantifier-free first order formula \( \varphi(\bar{v}) \) of DLO with \( |\bar{v}| = |\bar{a}| \), the characteristic function of the event \([\varphi(\bar{a})]\) belongs to \( \text{lcl}(A) \). Since DLO admits quantifier elimination, the characteristic function of every event that is first order definable over \( A \) belongs to \( \text{lcl}(A) \). Hence by Theorem 4.1.1, the characteristic function of every event in \( \text{dcl}_2(A) \) belongs to \( \text{lcl}(A) \). Moreover, for every \( c \in A \) and event \( D \subseteq \text{dcl}_2(A) \) with characteristic function \( \bar{d}, \bar{c} \upharpoonright D := \ell(\bar{d}, 1, 0, c) \) is the element that agrees with \( c \) on \( D \) and agrees with \( 0 \) on the complement of \( D \), so \( c \upharpoonright D \) belongs to \( \text{lcl}(A) \). Let \( \{D_1, \ldots, D_n\} \) be the set of atoms of \( \text{dcl}_2(A) \) (which is finite because DLO is \( \aleph_0 \)-categorical). By Corollary 4.2.1, every element of \( \text{dcl}(A) \) has the form

\[
\max(c_1 \upharpoonright D_1, \ldots, c_n \upharpoonright D_n)
\]

for some \( c_1, \ldots, c_n \in A \). Therefore \( \text{dcl}(A) \subseteq \text{lcl}(A) \). \qed

**Example 4.2.4.** In this example we show that the exchange property fails for DLO\(^R\), even though it holds for DLO. Thus the exchange property is not preserved under randomizations. Let \( T = \text{DLO} \). By Fullness, there exist elements \( a, b \in \hat{\mathcal{K}} \) such that \( \max(a, b) \notin \{a, b\} \). Let \( c = \max(a, b), d = \min(a, b) \). It is easy to check that

\[
\text{dcl}(\{a, b\}) = \{a, b, c, d\}, \quad \text{dcl}(\{a, c\}) = \{a, c\}, \quad \text{dcl}(\{a\}) = \{a\}.
\]

Thus \( c \in \text{dcl}(\{a, b\}) \setminus \text{dcl}(\{a\}) \) but \( b \notin \text{dcl}(\{a, c\}) \).

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