Some Properties and Applications of Non-trivial Divisor Functions

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Abstract

The $j$th divisor function $d_j$, which counts the ordered factorisations of a positive integer into $j$ positive integer factors, is a very well-known arithmetic function; in particular, $d_2(n)$ gives the number of divisors of $n$. However, the $j$th non-trivial divisor function $c_j$, which counts the ordered proper factorisations of a positive integer into $j$ factors, each of which is greater than or equal to 2, is rather less well-studied. We also consider associated divisor functions $c_j^{(r)}$, whose definition is motivated by the sum-over divisors recurrence for $d_j$. After reviewing properties of $d_j$, we study analogous properties of $c_j$ and $c_j^{(r)}$, specifically regarding their Dirichlet series and generating functions, as well as representations in terms of binomial coefficient sums and hypergeometric series. We also express their ratios as binomial coefficient sums and hypergeometric series, and find explicit Dirichlet series and Euler products in some cases. As an illustrative application of the non-trivial and associated divisor functions, we show how they can be used to count principal reversible square matrices of the type considered by Ollerenshaw and Brée, and hence sum-and-distance systems of integers.

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1 Introduction

The $j$th divisor function $d_j$, which counts the ordered factorisations of a positive integer into $j$ positive integer factors, is a very well-known arithmetic function. In particular, $d_2(n)$ — sometimes called the divisor function — counts the number of ordered pairs of positive integers whose product is $n$, and therefore, considering only the first factor in each pair, also counts the number of divisors of $n$ (see papers 8 and 15 of [13] and p. 10 of [3]). The divisor function lies at the heart of a number of open number theoretical problems, e.g. the additive divisor problem of finding the asymptotic of

$$\sum_{n \leq x} d_j(n) d_j(n+h)$$

for large $x$, which is notoriously difficult if $j \geq 3$, see e.g. [10], [4], and, for $j = 3$, [9].

In the present paper, we consider the rather less well-studied $j$th non-trivial divisor function $c_j$, which counts the ordered proper factorisations of a positive integer into $j$ factors, each of which is greater than or equal to 2. While $d_j(n)$, for given $n$, is obviously monotone increasing in $j$, since factors of 1 can be freely introduced, $c_j(n)$ will shrink back to 0 as $j$ gets too large, and indeed $c_j(n) = 0$ if $n < 2^j$. 
Additionally we define the associated divisor function $c_j^{(r)}$, for $r \in \mathbb{N}_0$, by
\[
c_j^{(0)} = c_j, \quad c_j^{(r)}(n) = \sum_{m|n} c_j^{(r-1)}(m) \quad (n, r \in \mathbb{N}).\]

This definition is motivated by the sum-over divisors recurrence for $d_j$, see Lemma 1.

The paper is organised as follows. After reviewing properties of $d_j$ in Section 2, we proceed to study analogous properties of $c_j$ in Section 3, specifically regarding its associated Dirichlet series and generating function, and its representation in terms of binomial coefficient sums and hypergeometric series. A major complication in comparison to $d_j$ arises from the fact that $c_j$ is not multiplicative. We also provide formulae expressing $c_j$ in terms of $d_j$ and vice versa. In Section 4, we study the Dirichlet series and generating function for the associated divisor functions $c_j^{(r)}$. Noting general inequalities between the three types of divisor function in Section 5, we observe how their ratios can be expressed as binomial coefficient sums and hypergeometric series, and find explicit Dirichlet series and Euler products for some of these. As an illustrative application of the non-trivial and associated divisor functions, we show in Section 6 how they can be used to count principal reversible squares [12] and sum-distance systems of integers.

Throughout the paper we use the notations $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n$ having prime factorisation $n = p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}$, we also use the symbol $\Omega(n) = \sum_{k=1}^t a_k$.

## 2 The multiplicative arithmetic function $d_j(n)$

**Lemma 1.** Let $j, n \in \mathbb{N}$, $j \geq 2$. Then the $j$th divisor function $d_j(n)$ satisfies the sum-over-divisors recurrence relation
\[
d_j(n) = \sum_{m|n} d_{j-1}(m).
\]

**Proof.** In the general ordered factorisation in $j$ factors, $n = m_1m_2\cdots m_j$, $m = \frac{n}{m_j} = m_1\cdots m_{j-1}$ can be any divisor of $n$, and $m_1\cdots m_{j-1}$ is any ordered factorisation of $m$. Hence there are $\sum_{m|n} d_{j-1}(m)$ distinct ordered factorisations of $n$ in $j$ factors. \qed

**Lemma 2.** Let $p_1, \ldots, p_t$ be distinct primes, $t \in \mathbb{N}$. Then, for any $j \in \mathbb{N}$,
\[
d_j(p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}) = \prod_{k=1}^t \left(\frac{a_k + j - 1}{a_k}\right) \quad (a_1, \ldots, a_t \in \mathbb{N}_0).
\]

(2.1)

by induction on $j$. For $j = 1$ the formula is trivial. Suppose $j \in \mathbb{N}$ is such that (2.1) holds. Note that if $n = \prod_{i=1}^k p_i^{a_i}$, then $m|n$ if and only if $m = \prod_{i=1}^k p_i^{\tilde{a}_i}$ with $0 \leq \tilde{a}_i \leq a_i$ for all $i \in \{1, \ldots, k\}$. Using multiindex notation, we can write the latter condition in the form $0 \leq \tilde{a} \leq a$. Hence, by Lemma 1
\[
d_{j+1}(p_1^{a_1}\cdots p_k^{a_k}) = \sum_{0 \leq \tilde{a} \leq a} d_j(p_1^{\tilde{a}_1}\cdots p_k^{\tilde{a}_k}) = \sum_{0 \leq \tilde{a} \leq a} \prod_{i=1}^k \left(\frac{\tilde{a}_i + j - 1}{\tilde{a}_i}\right) = \prod_{i=1}^k \sum_{l=0}^{a_i} \left(\frac{l + j - 1}{l}\right)
\]
\[
= \prod_{i=1}^k \left(\frac{a_i + j}{a_i}\right),
\]
using combinatorial identity (1.49) of [5] in the last step. \qed
Corollary 1. For any \( j \in \mathbb{N} \), the \( j \)th divisor function \( d_j \) is a multiplicative arithmetic function, i.e. \( d_j(mn) = d_j(m)d_j(n) \), whenever \((m, n) = 1\).

This follows directly from (2.1), considering that \( m \) and \( n \) have no prime in common, so the right-hand side can be rearranged into two products with disjoint index sets. The arithmetic function \( d_j \) is not totally multiplicative. For example, \( d_3(20) = 18 \neq 27 = 3 \times 9 = d_3(2)d_3(10) \).

In the following, we denote by \( \zeta \) the Euler-Riemann zeta function (see e.g. [4] or [14]),

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\text{Re} \ s > 1).
\]

Lemma 3. For \( j \in \mathbb{N} \), the divisor function \( d_j \) has the Dirichlet series

\[
\sum_{n=1}^{\infty} \frac{d_j(n)}{n^s} = \zeta(s)^j.
\]

Proof. By repeated application of Lemma 1,

\[
d_j(n) = \sum_{n_1|n} \sum_{n_2|n_1} \ldots \sum_{n_{j-1}|n_{j-2}} d_1(n_{j-1}) = \sum_{(n_1, \ldots, n_j) \in \mathbb{N}^j : \prod_{k=1}^j n_k = n} 1.
\]

Hence the Dirichlet series for \( d_j \) takes the form

\[
\sum_{n=1}^{\infty} \frac{d_j(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{(n_1, \ldots, n_j) \in \mathbb{N}^j : \prod_{k=1}^j n_k = n} 1 = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \ldots \sum_{n_j=1}^{\infty} \left(\prod_{k=1}^j n_k\right)^{-1} = \prod_{k=1}^{j} \sum_{n_k=1}^{\infty} \frac{1}{n_k^s} = \zeta(s)^j.
\]

Corollary 2. For \( i, j \in \mathbb{N} \) and suitable \( s \in \mathbb{C} \),

\[
\left(\sum_{n=1}^{\infty} \frac{d_i(n)}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{d_j(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{d_{j+i}(n)}{n^s}.
\]

Remark. The statements of Lemma 3 and Corollary 2 extend to the divisor function \( d_0 \) defined as

\[
d_0(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad (n \in \mathbb{N}).
\]

There are other known Dirichlet series associated with \( d_j \) (see for example [7] Chapter XVII), such as

\[
\frac{\zeta(s)^3}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d_2(n^2)}{n^s}, \quad \text{and} \quad \frac{\zeta(s)^4}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{(d_2(n))^2}{n^s}.
\]
both of which can be obtained from the more general calculation (where we use the
Euler product for the zeta function and Newton’s inverse binomial series in the first
two steps, respectively),
\[
\frac{\zeta(s)^{r+2}}{\zeta(2s)} = \prod_p \frac{(1-p^{-s})(1+p^{-s})}{(1-p^{-s})(1-p^{-s})^{r+1}} = \prod_p (1-p^{-s}) \sum_{j=0}^{\infty} \frac{(j+r)!}{j!r!} p^{-sj}
\]
\[
= \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{(j+r)!}{j!r!} \frac{(j+r-1)!}{(j-1)!r!} p^{-sj}\right)
\]
\[
= \prod_p \sum_{j=0}^{\infty} \frac{(2j+r)(j+r-1)!}{j!r!} p^{-sj} = \sum_{n=0}^{\infty} f(n) \frac{1}{n^s},
\]
where \(f\) is the multiplicative arithmetic function
\[
f(p_1^{a_1}p_2^{a_2} \ldots p_t^{a_t}) = \prod_{j=1}^{t} \frac{(2a_j+r)(a_j+r-1)!}{r!a_j!}.
\]

If we define the generating function for the divisor function \(d_j\) as
\[
D_j(x) = \sum_{n=1}^{\infty} d_j(n) x^n \quad (j \in \mathbb{N})
\]
for values of \(x\) for which the power series converges, then we have the following recursive identity which reflects the sum-over-divisors recurrence of Lemma 1.

**Lemma 4.** Let \(j \in \mathbb{N}\). Then for all \(x \in \mathbb{C}\) for which the generating function \(D_j(x)\) is defined,
\[
D_j(x) = \sum_{k=1}^{\infty} D_{j-1}(x^k).
\]

**Proof.** We observe that
\[
\sum_{n=1}^{\infty} d_j(n) x^n = \sum_{k \in \mathbb{N}^j} x^{k_1 \ldots k_j} = \sum_{k_j=1}^{\infty} \sum_{k \in \mathbb{N}^{j-1}} x^{(k_1 \ldots k_j-1)k_j} = \sum_{k_j=1}^{\infty} \sum_{n=1}^{\infty} d_{j-1}(n)(x^{k_j})^n,
\]
and the stated identity follows. \(\Box\)

We remark that by a similar calculation,
\[
\sum_{n=1}^{\infty} d_j(n) x^n = \sum_{k \in \mathbb{N}^{j-1}} \sum_{k_j=1}^{\infty} (x^{k_1 \ldots k_{j-1}})^{k_j} = \sum_{k \in \mathbb{N}^{j-1}} \frac{x^{k_1 \ldots k_{j-1}}}{1-x^{k_1 \ldots k_{j-1}}} = \sum_{n=1}^{\infty} d_{j-1}(n) \frac{x^n}{1-x^n}
\]
provided \(x \neq 1\), however, the right-hand side is then not of the form of a generating function.

### 3 The non-trivial divisor function \(c_j\)

The arithmetic function \(c_j\) satisfies a sum-over-divisors recurrence with respect to \(j\) analogous to, but subtly different from, that given for \(d_j\) in Lemma 1.
Lemma 5. Let $j, n \in \mathbb{N}, j \geq 2$. Then the $j$th non-trivial divisor function satisfies the sum-over-divisors recurrence relation

$$c_j(n) = \sum_{m|n, m < n} c_{j-1}(m) = \sum_{m|n} c_{j-1}(m).$$

Proof. In the general ordered non-trivial factorisation of $n$ into $j$ factors, $n = m_1 \ldots m_j$, the factor $m_j$ can be any non-trivial divisor of $n$, so $m = \frac{n}{m_j} = m_1 \ldots m_{j-1}$ is any proper divisor of $n$, and $m_1 \ldots m_{j-1}$ is any non-trivial ordered factorisation of $m$. Thus there are $\sum_{m|n, m < n} c_{j-1}(m)$ distinct $j$-factor ordered non-trivial factorisations of $n$. The last identity follows as $c_j(1) = 0$ for any $j \in \mathbb{N}$. \qed

Lemma 6. For $j \in \mathbb{N}$, the non-trivial divisor function $c_j$ has the Dirichlet series

$$\sum_{n=1}^\infty \frac{c_j(n)}{n^s} = (\zeta(s) - 1)^j.$$

Proof. We have

$$\sum_{n=1}^\infty \frac{c_j(n)}{n^s} = \sum_{n=2}^\infty \frac{c_j(n)}{n^s} = \sum_{n=2}^\infty \sum_{m_1} \sum_{n_2} \cdots \sum_{n_j} \frac{1}{n^s} = \sum_{n_1=2}^\infty \sum_{n_2=2}^\infty \cdots \sum_{n_j=2}^\infty \frac{1}{n_1n_2 \cdots n_j} = \left(\sum_{n_1=2}^\infty \frac{1}{n_1^s}\right) \cdots \left(\sum_{n_j=2}^\infty \frac{1}{n_j^s}\right) = (\zeta(s) - 1) \cdots (\zeta(s) - 1) = (\zeta(s) - 1)^j.$$

We remark that, unlike $d_j$, $c_j$ is not a multiplicative arithmetic function. For example, $(2, 5) = 1$, and yet $c_2(10) = 2 \neq 0 \times 0 = c_2(2)c_2(5)$.

In order to study the less symmetric multiplicative properties of $c_j$, it is useful to express it in terms of its multiplicative cousin $d_j$. When $j = 2$, the non-trivial divisors for any $n$ are all the divisor except 1 and $n$, and hence $c_2(n) = d_2(n) - 2$ if $n \geq 2$, and $c_2(1) = d_2(1) - 1$. More generally, there is the following connection between the divisor function and the non-trivial divisor function.

Lemma 7. For $j, n \in \mathbb{N}$,

$$c_j(n) = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} d_i(n).$$

Proof. By Lemma 6 and Lemma 8,

$$\sum_{n=1}^\infty \frac{c_j(n)}{n^s} = (\zeta(s) - 1)^j = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \zeta^{j-i}(s) = \sum_{n=1}^\infty \frac{1}{n^s} \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} d_i(n),$$

and the claimed identity follows by the uniqueness of the coefficients of Dirichlet series (cf. 9). \qed
The formula in Lemma 7 extends to \( j = 0 \) if we define \( c_0(n) := d_0(n) \ (n \in \mathbb{N}) \). With this convention, we have the following inversion formula.

**Lemma 8.** For \( j, n \in \mathbb{N} \),

\[
d_j(n) = \sum_{i=0}^{j} \binom{j}{i} c_i(n).
\]

**Proof.** Considering the Dirichlet series for the right-hand side of the above equation, we find by Lemma 6

\[
\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{i=0}^{j} \binom{j}{i} c_i(n) = \sum_{i=0}^{j} \binom{j}{i} \sum_{n=1}^{\infty} \frac{1}{n^s} c_i(n) = \sum_{i=0}^{j} \binom{j}{i} (\zeta(s) - 1)^i
\]

\[
= (1 + (\zeta(s) - 1))^j = \zeta(s)^j = \sum_{n=1}^{\infty} \frac{d_j(n)}{n^s}
\]

by Lemma 3 and the result follows by the uniqueness of coefficients of Dirichlet series (cf. [6]).

For the generating function of \( c_j \),

\[
C_j(x) = \sum_{n=2^j}^{\infty} c_j(n) x^n,
\]

there is the following recursive identity.

**Lemma 9.** Let \( j \in \mathbb{N} \). Then for all \( x \in \mathbb{C} \) for which \( C_j(x) \) is defined,

\[
C_j(x) = \sum_{k=2}^{\infty} C_{j-1}(x^k).
\]

**Proof.**

\[
\sum_{n=2^j}^{\infty} c_j(n) x^n = \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \cdots \sum_{k_j=2}^{\infty} x^{k_1 k_2 \cdots k_j} = \sum_{k_2}^{\infty} \left( \sum_{k_3=2}^{\infty} \cdots \sum_{k_{j-1}=2}^{\infty} (x^{k_j})^{k_1 \cdots k_{j-1}} \right)
\]

\[
= \sum_{k_j=2}^{\infty} \sum_{n=2^{j-1}}^{\infty} c_{j-1}(n) (x^{k_j})^n
\]

and the statement follows.

We remark that grouping the terms in the calculation in the preceding proof differently yields the identity (for \( x \neq 1 \))

\[
\sum_{n=2^j}^{\infty} c_j(n) x^n = \sum_{k_1=2}^{\infty} \cdots \sum_{k_{j-1}=2}^{\infty} \sum_{k_j=2}^{\infty} (x^{k_1 \cdots k_{j-1}})^{k_j}
\]

\[
= \sum_{k_1=2}^{\infty} \cdots \sum_{k_{j-1}=2}^{\infty} \frac{x^{2k_1 \cdots k_{j-1}}}{1 - x^{k_1 \cdots k_{j-1}}} = \sum_{n=2^{j-1}}^{\infty} c_{j-1}(n) \frac{x^{2n}}{1 - x^n}.
\]

We conclude this section with the derivation of a hypergeometric series for \( c_j(n) \). The generalised hypergeometric series has the form

\[
_k F_n(a_1, a_2, \ldots, a_k; b_1, b_2, \ldots, b_n; z) = \sum_{m=0}^{\infty} \frac{a_1^m a_2^m \cdots a_k^m}{b_1^m b_2^m \cdots b_n^m} \frac{z^m}{m!},
\]
where $a^m$, with $m \in \mathbb{N}$, is the Pochhammer symbol (rising factorial)

$$a^m = \prod_{j=0}^{m-1} (a + j);$$

in particular, $1^m = m!$, $2^m = (m + 1)!$, $a^m = (a + m - 1)!/(a - 1)!$ if $a \in \mathbb{N}$ and, for negative $a$, $a^m = (-1)^m (-a)!/(-a - m)!$ if $-(a + m) \in \mathbb{N}_0$. By the usual convention on empty products, $a^0 = 1$.

**Theorem 1.** Let $j \in \mathbb{N}$ and suppose $n$ has the prime factorisation $n = p_1^{a_1} \cdots p_k^{a_k}$. Then the value of the non-trivial $j$th divisor function at $n$ has the hypergeometric form

$$c_j(p_1^{a_1} \cdots p_k^{a_k}) = (-1)^{1-j} j \, F_k\left((a_i+1)_{i=1}^k, (1-j); \{1\}_{i=1}^{k-1}, 2; 1\right).$$

**Proof.** Starting from the right-hand side expression, we find

$$(-1)^{1-j} \sum_{m=0}^\infty \frac{(\prod_{i=1}^k (a_i+1)^m)}{(1^m)^{k-1} 2^m m!} \left(1-j\right)^m$$

$$= (-1)^{1-j} \sum_{m=0}^{j-1} \frac{j (j-1)! (-1)^m \prod_{i=1}^k (a_i+m)!}{m+1! (j-m-1)! a! m!}$$

$$= \sum_{m=0}^{j-1} (-1)^{m-j+1} \binom{j}{m+1} \prod_{i=1}^k \binom{a_i+m}{m}$$

$$= \sum_{m=0}^{j-1} (-1)^{m-j+1} \binom{j}{m+1} d_{m+1}(p_1^{a_1} \cdots p_k^{a_k}) = c_j(p_1^{a_1} \cdots p_k^{a_k}),$$

by Lemmata 2 and 7. \hfill \square

In particular, for a prime power the above theorem gives

$$c_j(p^a) = (-1)^{1-j} j \, F_1(a+1, 1-j; 2; 1) = (-1)^{1-j} \sum_{m=0}^\infty \frac{(a+1)^m (1-j)^m \, 2^m m!}{2^m m!}.$$ 

Finally, we note the following multiplication rule for prime powers.

**Lemma 10.** Let $p$ be a prime and $j, a, b \in \mathbb{N}$. Then

$$c_j(p^{a+b}) = \sum_{k=0}^{j-1} (-1)^{k-j+1} \binom{j}{k+1} d_{k+1}(p^b) \frac{(b+k+1)^m}{(b+1)^m}.$$ 

**Proof.** By Lemma 2

$$\frac{d_j(p^{a+b})}{d_j(p^b)} = \frac{(a+b+j-1)! b! (j-1)!}{(a+b)! (j-1)! (b+j-1)!} = \frac{(b+j)^m}{(b+1)^m}.$$ 

The statement now follows by combining this result with Lemma 7. \hfill \square
4 The associated divisor function $c_j^{(r)}$

In analogy to the sum-over-divisors recurrence relation for the divisor function $d_j$ (Lemma 1), we define the $j$th associated divisor function $c_j^{(r)}$ by the following recurrence.

**Definition.** Let $j \in \mathbb{N}$. Then, for all non-negative integers $r$, the associated divisor function $c_j^{(r)}$ is defined recursively by

\[ c_j^{(0)}(n) = c_j(n), \quad c_j^{(r)}(n) = \sum_{m|n} c_j^{(r-1)}(m) \quad (n \in \mathbb{N}). \]

From the corresponding property of $c_j$, it follows that $c_j^{(r)}(n) = 0$ for all $r \in \mathbb{N}$ if $n < 2^j$.

**Lemma 11.** Let $j \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Then the associated divisor function $c_j^{(r)}$ can be expressed in terms of the divisor function $d_j$ as follows,

\[ c_j^{(r)}(n) = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} d_i n \quad (n \in \mathbb{N}). \]

by induction on $r$. For $r = 0$ this is the statement of Lemma 7. Now suppose that $r \in \mathbb{N}$ is such that

\[ c_j^{(r)} = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} d_i r \]

holds. Then by Lemma 11

\[ c_j^{(r+1)}(n) = \sum_{m|n} c_j^{(r)}(m) = \sum_{m|n} \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} d_i r(m) \]

\[ = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \sum_{m|n} d_i r(m), \]

and the claimed statement follows by Lemma 11.

**Lemma 12.** Let $j \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Then the associated divisor function $c_j^{(r)}$ satisfies

\[ c_j^{(r)} = \sum_{i=0}^r \binom{r}{i} c_{j+i}. \]

by induction on $r$. The identity is trivial for $r = 0$. Now assume $r \in \mathbb{N}$ is such that
the above identity holds. Then, using Lemma 5,

\[
c_j^{(r+1)}(n) = \sum_{m|n} c_j^{(r)}(m) = \sum_{i=0}^{r} \sum_{m|n} \binom{r}{i} c_{j+i}(m) = \sum_{i=0}^{r} \binom{r}{i} \sum_{m|n} c_{j+i}(m)
\]

\[
= \sum_{i=0}^{r} \binom{r}{i} \left( c_{j+i}(n) + \sum_{m|n, m \neq 1} c_{j+i}(m) \right) = \sum_{i=0}^{r} \binom{r}{i} (c_{j+i}(n) + c_{j+i+1}(n))
\]

\[
= \sum_{i=0}^{r} \left( \binom{r}{i} c_{j+i}(n) + \sum_{i=1}^{r+1} \left( \binom{r}{i-1} c_{j+i}(n) \right) \right) = c_j(n) + \sum_{i=1}^{r+1} \binom{r+1}{i} c_{j+i}(n)
\]

\[
= \sum_{j=0}^{r+1} \binom{r+1}{i} c_{j+i}(n) \quad (n \in \mathbb{N}).
\]

\[ \square \]

**Lemma 13.** For \( j \in \mathbb{N} \) and \( r \in \mathbb{N}_0 \), the Dirichlet series of the associated divisor function \( c_j^{(r)} \) is given by

\[
\sum_{n=1}^{\infty} \frac{c_j^{(r)}(n)}{n^s} = \zeta(s)^r(\zeta(s) - 1)^j.
\]

by induction on \( r \). The case \( r = 0 \) was shown in Lemma 5. For the induction step, note that, by a well-known result on Dirichlet convolution (see e.g. [14] p. 4), the identity \( c_j^{(r+1)}(n) = \sum_{m|n} c_j^{(r)}(m) \) \( (n \in \mathbb{N}) \) implies that

\[
\sum_{n=1}^{\infty} \frac{c_j^{(r+1)}(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{c_j^{(r)}(n)}{n^s}.
\]

\[ \square \]

The sum over divisors taken as the definition of \( c_j^{(r)} \) in terms of \( c_j^{(r-1)} \) gives rise to the following recursion formula for the generating functions

\[
C_j^{(r)}(x) = \sum_{n=2^j}^{\infty} c_j^{(r)}(n) x^n.
\]

**Lemma 14.** Let \( j, r \in \mathbb{N} \). Then for all \( x \in \mathbb{C} \) for which \( C_j^{(r)} \) is defined,

\[
C_j^{(r)}(x) = \sum_{k=1}^{\infty} C_j^{(r-1)}(x^k).
\]

**Proof.** We observe

\[
\sum_{n=2^j}^{\infty} c_j^{(r)}(n) x^n = \sum_{n=2^j}^{\infty} \sum_{m|n} c_j^{(r-1)}(m) x^n = \sum_{k=1}^{\infty} \sum_{m=2^j}^{\infty} c_j^{(r-1)}(m)(x^k)^m,
\]

and the statement follows. \[ \square \]
Again, we note that, if \( x \neq 1 \), then a different arrangement of terms in the above calculation gives the different identity

\[
\sum_{n=2}^{\infty} c_j^{(r)}(n)x^n = \sum_{m|n} \sum_{k=1}^{\infty} c_j^{(r-1)}(m)(x^m)^k = \sum_{n=2}^{\infty} c_j^{(r-1)}(n) \frac{x^n}{1-x^n}.
\]

To conclude this section, we prove a binomial form for the value of \( c_j^{(r)} \) at prime powers; this is somewhat analogous to Lemma 2, but note that the present function is not multiplicative.

**Lemma 15.** Let \( j, a \in \mathbb{N} \), \( r \in \mathbb{N}_0 \), and \( p \) a prime. Then

\[
c_j^{(r)}(p^a) = \binom{a + r - 1}{j + r - 1}.
\]

**Proof.** From Lemma 11 and Lemma 2 we find

\[
c_j^{(r)}(p^a) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} d_{i+r}(p^a) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \binom{a + i + r - 1}{a}
\]

\[
= \binom{a + r - 1}{j + r - 1} \binom{a + r - 1}{j - a}
\]

by combinatorial identity (3.47) of [5]. \( \square \)

## 5 Ratios of Divisor Functions

The divisor function, non-trivial divisor function and associated divisor functions are in the following ordering relation to each other.

**Lemma 16.** For any \( j \in \mathbb{N} \) and \( r \in \mathbb{N} \),

\[
c_j^{(r-1)}(n) \leq c_j^{(r)}(n);
\]

in particular,

\[
c_j(n) \leq c_j^{(r)}(n) \quad (n \in \mathbb{N}).
\]

Moreover,

\[
c_j^{(r)}(n) \leq d_{j+r}(n) \quad (n \in \mathbb{N}).
\]

**Proof.** For the first statement, it is sufficient to note that \( n|n \), so

\[
c_j^{(r)}(n) = \sum_{m|n} c_j^{(r-1)}(m) \geq c_j^{(r-1)}(n) \quad (n \in \mathbb{N}).
\]

We prove the final statement by induction on \( r \). As every non-trivial factorisation is a factorisation, it follows directly from their definitions that \( d_j(m) \geq c_j(m) \) for all \( m, j \in \mathbb{N} \). Hence

\[
d_{j+1}(n) = \sum_{m|n} d_j(m) \geq \sum_{m|n} c_j(m) = c_j^{(1)}(n),
\]

which gives the case \( r = 1 \). Now suppose \( r \in \mathbb{N} \) is such that the claimed statement holds. Then

\[
d_{j+r+1}(n) = \sum_{m|n} d_{j+r}(m) \geq \sum_{m|n} c_j^{(r)}(n) = c_j^{(r+1)}(n),
\]

which is the induction step. \( \square \)
Lemma \[\text{[16]}\] shows that, for any \( r \in \mathbb{N}_0 \), the normalised divisor ratio function \( c_j^{(r)} / d_j + r \) takes rational values between 0 and 1, with the zeros occurring exactly when \( j > \Omega(n) \). We have the following formulae for this function and the similar ratio \( c_j^{(r)} / d_r \).

**Theorem 2.** Let \( j \in \mathbb{N} \) and \( r \in \mathbb{N}_0 \), and suppose \( n \in \mathbb{N} \) has prime factorisation \( n = p_1^{a_1} \cdots p_t^{a_t} \). Then

\[
\frac{c_j^{(r)}(n)}{d_j + r(n)} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{(j + r - 1)^t_i}{\prod_{k=1}^{t_i} (a_k + j + r - 1)_i}
= t+1 F_t(\{1 - j - r\}_{i=1}^{t}, -j; \{1 - a_i - j - r\}_{i=1}^{t}; 1).
\tag{5.1}
\]

Also, for \( r \geq 1 \)

\[
\frac{c_j^{(r)}(n)}{d_r(n)} = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \frac{\prod_{k=1}^{t} (a_k + j + r - 1)_i}{(j + r - 1)_i}
= (-1)^j t+1 F_t(\{a_k + r\}_{k=1}^{t}, -j; \{r\}_{k=1}^{t}; 1).
\tag{5.2}
\]

**Proof.** By Lemma \[\text{[11]}\] and Lemma \[\text{[2]}\] we have that

\[
c_j^{(r)}(n) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} d_{j+r-1}(n) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \prod_{k=1}^{t} (a_k + j + r - 1)_i (j + r - 1)_i!
= \sum_{i=0}^{j} (-1)^i \binom{j}{i} \prod_{k=1}^{t} (a_k + j + r - 1)_i / (j + r - 1)_i!
= \left( \prod_{k=1}^{t} (a_k + j + r - 1)_i / (j + r - 1)_i \right) \sum_{i=0}^{j} (-1)^i \binom{j}{i} \prod_{k=1}^{t} (a_k + j + r - 1)_i / (j + r - 1)_i!
= d_{j+r}(n) \sum_{i=0}^{j} (-1)^i \binom{j}{i} \prod_{k=1}^{t} (a_k + j + r - 1)_i / (j + r - 1)_i !.
\tag{5.3}
\]

To obtain the hypergeometric form, we note that

\[
\frac{c_j^{(r)}(n)}{d_j + r(n)} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{(j + r - 1)_i !}{\prod_{k=1}^{t} (a_k + j + r - 1)_i / (j + r - 1)_i !}
= \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{(1 - j - r)_i !}{\prod_{k=1}^{t} (1 - j - r - a_k)_i !}
= t+1 F_t(\{1 - j - r\}_{k=1}^{t}, -j; \{1 - a_k - j - r\}_{k=1}^{t}; 1),
\]

establishing \( (5.1) \). For the proof of \( (5.2) \), we rewrite \( (5.3) \) in the form

\[
c_j^{(r)}(n) = \sum_{i=0}^{j} (-1)^{i-r} \binom{j}{i} \prod_{k=1}^{t} \frac{(a_k + i + r - 1)_i !}{(i + r - 1)_i ! a_k !}
= \left( \prod_{k=1}^{t} \frac{(a_k + r - 1)_i !}{a_k ! (r - 1)_i !} \right) \sum_{i=0}^{j} (-1)^{i-r} \binom{j}{i} \prod_{k=1}^{t} \frac{(a_k + r - 1)_i ! (r - 1)_i !}{(a_k + r - 1)_i !}\frac{(r - 1)_i !}{(i + r - 1)_i !}
= \prod_{k=1}^{t} \frac{(a_k + r - 1)_i !}{a_k ! (r - 1)_i !}
\]

and apply Lemma \[\text{[2]}\]. For the hypergeometric form, we rewrite the last expression as

\[
\frac{c_j^{(r)}(n)}{d_r(n)} = (-1)^j \sum_{i=0}^{j} \frac{(-j)_i !}{i !} \prod_{k=1}^{t} \frac{(a_k + r)_i !}{r !}
\]
and note as above that the sum can be extended to an infinite series since \((-j)^{j} = 0\) if \(i > j\).

Specifically for \(r = 0\), the formula (5.1) of Theorem 2 gives

\[
\frac{c_j(n)}{d_j(n)} = \sum_{i=0}^{j} (-1)^i \binom{j}{i} \prod_{k=1}^{l} (a_k-j)^{i_k-j_i}
\]

\[
= \prod_{i=1}^{l} (1-j_i; 1-t_i)
\]

if \(n\) has prime factorisation \(n = p_1^{a_1} \cdots p_t^{a_t}\).

Clearly these formulae simplify when \(n\) is a prime power. We note that in this case, Lemmata 2 and 15 give

\[
\frac{c_j(p^s)}{d_j(p^s)} = \frac{(a-1)_{j}}{a-j_{j}}.
\]

In the following, we give Dirichlet series for the ratio of divisor functions \(c_j/d_j\) for \(j \in \{1, 2, 3\}\), as well as corresponding Euler products. Note that the term \(n = 1\) can be omitted from the Dirichlet series, since \(c_j(1) = 0\) for all \(j \in \mathbb{N}\). For \(j = 1\),

\[
\sum_{n=2}^{\infty} \frac{c_1(n)}{d_1(n)} \frac{1}{n^s} = \sum_{n=2}^{\infty} \frac{1}{n^s} = \zeta(s) - 1.
\]

For \(j = 2\), we have \(c_2(n) = d_2(n) - 2\) for \(n \geq 2\), so the Dirichlet series for the ratio of divisor functions can be written as

\[
\sum_{n=2}^{\infty} \frac{c_2(n)}{d_2(n)} \frac{1}{n^s} = \sum_{n=2}^{\infty} \frac{1}{n^s} - 2 \sum_{n=2}^{\infty} \frac{1}{d_2(n)n^s} = 1 + \zeta(s) - 2 \sum_{n=1}^{\infty} \frac{1}{d_2(n)n^s};
\]

we note the Euler product for the Dirichlet series in the last term,

\[
\prod_{p \text{ prime}} \sum_{j=0}^{\infty} \frac{1}{d_2(p^j)n^s} = \prod_{p \text{ prime}} \sum_{j=0}^{\infty} \frac{1}{(j+1)p^{sj}}
\]

\[
= \prod_{p \text{ prime}} \left( \frac{p^s}{log\left(\frac{1}{1-p^s}\right)}\right).
\]

For \(j = 3\), we have \(c_3(n) = d_3(n) - 3d_2(n) + 3\) for \(n \geq 2\), which gives the Dirichlet series for the ratio of divisor functions

\[
\sum_{n=2}^{\infty} \frac{c_3(n)}{d_3(n)} \frac{1}{n^s} = \zeta(s) - 1 - 3 \sum_{n=1}^{\infty} \frac{d_2(n)}{d_3(n)} \frac{1}{n^s} + 3 \sum_{n=1}^{\infty} \frac{1}{d_3(n)n^s};
\]

noting

\[
\frac{d_2(p^j)}{d_3(p^j)} = \frac{2}{j+2} \frac{1}{d_3(p^j)} = \frac{2}{(j+1)(j+2)} = \frac{2}{j+1} - \frac{2}{j+2},
\]

we find the Euler products

\[
\prod_{p \text{ prime}} \sum_{j=0}^{\infty} \frac{2}{(j+2)p^{sj}} = \prod_{p \text{ prime}} \sum_{j=2}^{\infty} \frac{2p^{2s}}{j^{p^sj}}
\]

\[
= \prod_{p \text{ prime}} \left( 2p^{2s} \left( log\left(\frac{1}{1-p^s}\right) - \frac{1}{p^s}\right) \right).
and
\[\sum_{n=1}^{\infty} \frac{1}{d_3(n)} n^s = \prod_{p \text{ prime}} \sum_{j=0}^{\infty} \left( \frac{2}{j+1} - \frac{2}{j+2} \right) \frac{1}{p^{sj}} = \prod_{p \text{ prime}} \left( \sum_{j=1}^{\infty} \frac{2p^s}{jp^{sj}} - \sum_{j=2}^{\infty} \frac{2p^{2s}}{jp^{sj}} \right)\]
\[= \prod_{p \text{ prime}} \left( 2p^s \log \frac{1}{1 - \frac{1}{p^s}} - 2p^{2s} \left( \log \frac{1}{1 - \frac{1}{p^s}} - \frac{1}{p^s} \right) \right)\]
\[= \prod_{p \text{ prime}} 2p^{2s} \left( \frac{1}{p^s} - \left( 1 - \frac{1}{p^s} \right) \log \frac{1}{1 - \frac{1}{p^s}} \right).\]

6 Counting Principal Reversible Squares

As an illustration for the use of the non-trivial and associated divisor functions, we show how they can be used to count the different principal reversible squares of a given size.

A reversible square matrix \( M = (M_{i,j})_{i,j \in \mathbb{Z}_n} \in \mathbb{R}^{n \times n} \) is an \( n \times n \) matrix with the following symmetry properties (cf. [12], [11]),

(R) the row and column reversal symmetry
\[M_{i,j} + M_{i,n+1-j} = M_{i,k} + M_{i,n+1-k},\]
\[M_{i,j} + M_{n+1-i,j} = M_{k,j} + M_{n+1-k,j} \quad (i, j, k \in \mathbb{Z}_n),\]

(V) the vertex cross sum property
\[M_{i,j} + M_{k,l} = M_{i,l} + M_{k,j} \quad (i, j, k, l \in \mathbb{Z}_n).\]

Note that the index calculations are performed in the cyclic ring \( \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} \), and the top left corner of the matrix has indices \((1, 1) \in \mathbb{Z}_n^2\).

An \( n \times n \) principal reversible square is a reversible square matrix \( M \) such that \( \{M_{i,j} \mid i, j \in \mathbb{Z}_n\} = \{1, 2, \ldots, n^2\} \), the entries in each row and each column appear in increasing order, and \( M_{1,j} = j \ (j \in \{1, 2\}) \).

**Definition.** Let \( n, \alpha \in \mathbb{N} \). The pair of tuples
\[
((i_1, i_2, \ldots, i_{\alpha-1}, i_{\alpha}), \ (j_1, j_2, \ldots, j_{\alpha-1}, j_{\alpha})) \in (\mathbb{N}^\alpha)^2
\]
is called a divisor path set for \( n \) (of length \( \alpha \)) if
\[i_1 | i_2 | \ldots | i_{\alpha-1} | i_{\alpha}, \quad 1 < i_1 < i_2 < \ldots < i_{\alpha-1} < i_{\alpha} = n,
\]
and
\[j_1 | j_2 | \ldots | j_{\alpha-1} | j_{\alpha} | n, \quad 1 < j_1 < j_2 < \ldots < j_{\alpha-1} < j_{\alpha} \leq n.
\]

**Theorem 3.** Let \( n \in \mathbb{N} \). Then from any divisor path set for \( n \), a unique \( n \times n \) principal reversible square can be constructed. Conversely, every \( n \times n \) principal reversible square arises from a unique divisor path set.

For the details of the construction and proof of Theorem 3, we refer the reader to Chapter 3 of [12]. In that book, a principal reversible square constructed from a divisor path set of length \( \alpha \) is said to have \( \alpha - 1 \) progressive factors.

Alternatively, Theorem 3 and the construction can be obtained as a special case, for a two-dimensional square array, of Theorem 9 of [8]; note that the ratios of consecutive divisors in the divisor path set correspond to the factors appearing in the joint ordered
factorisation defined in [8]. Specifically, the above divisor path set corresponds to the joint ordered factorisation

\[(2, j_1), (1, i_1), (2, j_2/j_1), (1, i_2/i_1), (2, j_3/j_2), \ldots, (2, j_\alpha/j_\alpha-1), (1, i_\alpha/i_\alpha-1), (2, n/j_\alpha)\]

of \((n, n)\), with the last entry omitted if \(j_\alpha = n\).

Using the bijection between divisor path sets and principal reversible squares given by Theorem 3, we can count the number of different principal reversible squares of size \(n \times n\) in terms of the non-trivial and associated divisor functions of \(n\) as follows.

**Theorem 4.** Let \(n \in \mathbb{N}\). The number of different \(n \times n\) principal reversible squares is given by

\[
N_n = \sum_{j=1}^{\Omega(n)} c_j(n) (c_j(n) + c_{j+1}(n)) = \sum_{j=1}^{\Omega(n)} c_j^{(0)}(n)c_j^{(1)}(n).
\]

**Proof.** By Theorem 3, it is sufficient to count the number of different divisor path sets for \(n\).

Suppose \(((i_1, \ldots, i_\alpha), (j_1, \ldots, j_\alpha))\) is a divisor path set for \(n\) of length \(\alpha\). Then the left-hand tuple gives an ordered factorisation of \(n\) into \(\alpha\) factors,

\[
\frac{i_1}{i_1} \frac{i_2}{i_2} \frac{i_3}{i_3} \cdots \frac{i_\alpha}{i_\alpha-1} = n
\]

with all factors > 1; there are \(c_\alpha(n)\) such non-trivial factorisations.

The right-hand tuple gives an ordered factorisation of \(n\) into \(\alpha + 1\) factors,

\[
\frac{j_1}{j_1} \frac{j_2}{j_2} \frac{j_3}{j_3} \cdots \frac{j_\alpha}{j_\alpha-1} \frac{n}{j_\alpha} = n,
\]

where the last factor is > 1 if and only if \(j_\alpha < n\), and all other factors are > 1. As the two cases are mutually exclusive, the total number of different factorisations is then given by \(c_\alpha(n) + c_{\alpha+1}(n) = c_\alpha^{(1)}(n)\), by Lemma 12. The statement of the theorem follows by summing over \(\alpha \in \mathbb{N}\), noting that \(c_\alpha(n) = 0\) if \(\alpha > \Omega(n)\).

**Remark.** Combining the formula of Theorem 4 with Lemma 12 the count \(N_n\) can be expressed in terms of the (multiplicative) divisor functions,

\[
N_n = \sum_{j=1}^{\Omega(n)} \sum_{l=1}^{j} \sum_{m=0}^{j} (-1)^{l+m} \binom{j}{l} \binom{j}{m} d_l(n)d_{m+1}(n). \quad (6.1)
\]

Using the prime factorisation \(n = \prod_{k=1}^{t} p_k^{a_k}\), this takes the form

\[
N_n = \sum_{j=1}^{\Omega(n)} \sum_{l=1}^{j} \sum_{m=0}^{j} (-1)^{l+m} \binom{j}{l} \binom{j}{m} \prod_{k=1}^{t} \left( \frac{a_k + l - 1}{l - 1} \right) \left( \frac{a_k + m}{m} \right).
\]

We note that the terms of the sum in (6.1) bear some similarity to the sum underlying the additive divisor problem (1.1).

**Corollary 3.** Let \(n \in \mathbb{N}\). Then \(N_n = 1\) if and only if \(n\) is prime.
Proof. If \( n \) is prime, then \( c_1(n) = 1 \) and \( c_j(n) = 0 \) for all \( j \geq 2 \), and it follows that \( N_n = c_1(n)(c_1(n) + c_2(n)) = 1 \). Conversely, suppose \( n \geq 2 \) is an integer such that \( N_n = 1 \). By Theorem \( 4 \)

\[
N_n = \sum_{j=1}^{\Omega(n)} c_j(n)^2 + \sum_{j=1}^{\Omega(n)} c_j(n)c_{j+1}(n).
\]

As all terms of these sums are non-negative and \( c_1(n) = 1 \), the total can equal 1 only if \( c_j(n) = 0 \) \((j \geq 2)\), which implies that \( n \) is prime. \( \square \)

Corollary 4. Let \( n = p^a \) with \( a \in \mathbb{N} \) and prime \( p \). Then

\[
N_n = \binom{2a-1}{a}.
\]

Proof. Theorem \( 4 \) and Lemma \( 15 \) give

\[
N_{p^a} = \sum_{j=1}^{a} \binom{a-1}{j-1} \binom{a}{j} = \sum_{j=0}^{a-1} \binom{a-1}{j} \binom{a}{j+1} = \binom{2a-1}{a}
\]

by combinatorial identity (3.20) of \cite{5}. \( \square \)

Counting principal reversible squares is of interest not only in view of their bijection to most perfect squares \cite{12}, but also because of their relationship with sum-and-distance systems. In the present context, these are composed of two finite component sets, of equal cardinality, of natural numbers, such that the numbers formed by considering all sums and all absolute differences of all pairs of numbers, each taken from one of the component sets, with or without inclusion of the component sets themselves, combine to an arithmetic progression without repetitions. Such systems arise naturally from the question of constructing a certain type of rank 2 traditional magic squares using the formulae given in \cite{11}. We refer the reader to \cite{8} for further details and for the extension of the following definitions and of Theorem \( 5 \) to any finite number of component sets of arbitrary finite cardinality.

Definition. Let \( m \in \mathbb{N} \) and consider positive integers \( a_j, b_j \in \mathbb{N} \) \((j \in \{1, \ldots, m\})\) such that

\[
a_1 < a_2 < \cdots < a_m, \quad b_1 < b_2 < \cdots < b_m.
\]

We call \( \{\{a_j : j \in \{1, \ldots, m\}\}, \{b_j : j \in \{1, \ldots, m\}\}\} \)

a) an \( m + m \) \( (\)non-inclusive\( ) \) sum-and-distance system if

\[
\{a_j + b_k, |a_j - b_k| : j, k \in \{1, \ldots, m\}\} = \{1, 3, 5, \ldots, 4m^2 - 1\};
\]

b) an \( m + m \) \( (\)inclusive\( ) \) sum-and-distance system if

\[
\{a_j + b_k, a_j + b_k, |a_j - b_k| : j, k \in \{1, \ldots, m\}\} = \{1, 2, 3, \ldots, 2m(m + 1)\}.
\]

It is easy to see that the conditions in a) and b) above are equivalent to

\[
\{\pm a_j : j \in \{1, \ldots, m\}\} + \{\pm b_k : k \in \{1, \ldots, m\}\} = \{-4m^2 + 1, -4m^2 + 3, \ldots, 4m^2 - 3, 4m^2 - 1\} \quad (6.2)
\]

and

\[
\{0, \pm a_j : j \in \{1, \ldots, m\}\} + \{0, \pm b_k : k \in \{1, \ldots, m\}\} = \{-2m(m + 1), -2m(m + 1) + 1, \ldots, 2m(m + 1)\}, \quad (6.3)
\]

respectively, where we use the usual sum of sets \( A + B = \{x + y : x \in A, y \in B\}\).
Example. For $m = 3$ there are the seven $3 + 3$ (non-inclusive) sum-and-distance systems

\[
\begin{align*}
\{1,3,5\}, \{6,18,30\}, & \quad \{1,7,9\}, \{2,22,26\}, \quad \{1,11,13\}, \{14,18,22\}, \\
\{1,23,25\}, \{2,6,10\}, & \quad \{3,9,15\}, \{16,18,20\}, \quad \{3,21,27\}, \{4,6,8\}, \\
\{7,9,11\}, \{12,18,24\}, &
\end{align*}
\]

but just the one inclusive $3 + 3$ sum-and-distance system $\{1,2,3\}, \{7,14,21\}$.

Sum-and-distance systems of the non-inclusive and inclusive variety are intimately connected with principal reversible squares. Indeed, let $n \in \mathbb{N}$ and consider a principal reversible square $(M_{j,k})_{j,k \in \mathbb{Z}_n}$; for all $j \in \{1, \ldots, n\}$, define $\alpha_j = M_{1,j} - 1$ and $\beta_j = M_{j,1} - 1$. From property (V) of the reversible square and the fact that $M_{1,1} = 1$, it then follows that

\[
M_{j,k} = M_{j,1} + M_{1,k} - 1 = \alpha_j + \beta_k + 1 \quad (j, k \in \{1, \ldots, n\}); \quad (6.4)
\]

note that $\alpha_1 = \beta_1 = 0$. By property (R),

\[
\alpha_j + \alpha_{n+1-j} = \alpha_n, \quad \beta_j + \beta_{n+1-j} = \beta_n \quad (j \in \{1, \ldots, n\}). \quad (6.5)
\]

Now suppose $n$ is even, $n = 2m$. Then setting

\[
a_j = \alpha_{m+j} - \alpha_{m+1-j}, \quad b_j = \beta_{m+j} - \beta_{m+1-j} \quad (j \in \{1, \ldots, m\})
\]

defines an $m + m$ non-inclusive sum-and-distance system. Indeed, it follows from (6.5) that

\[
a_j = 2\alpha_{m+j} - \alpha_n, \quad -a_j = 2\alpha_{m+1-j} - \alpha_n \quad (j \in \{1, \ldots, m\})
\]

and correspondingly for $\pm b_k$ ($k \in \{1, \ldots, m\}$), so the equality (6.5) can be verified using the facts that $\alpha_n + \beta_n = M_{n,n} - 1 = n^2 - 1$ and that

\[
\{\alpha_j + \beta_k : j, k \in \{1, \ldots, n\}\} = M_{j,k} - 1 : j, k \in \{1, \ldots, n\} = \{0, 1, \ldots, n^2 - 1\}.
\]

Conversely, given an $m + m$ non-inclusive sum-and-distance system and setting

\[
\alpha_{m+j} = \frac{a_m + a_j}{2}, \quad \alpha_{m+1-j} = \frac{a_m - a_j}{2} \quad (j \in \{1, \ldots, m\}), \\
\beta_{m+k} = \frac{b_m + b_k}{2}, \quad \beta_{m+1-k} = \frac{b_m - b_k}{2} \quad (k \in \{1, \ldots, m\}),
\]
equation (6.4) defines a principal reversible square (or its transpose).

Similarly, if $n = 2m + 1$ is odd, then an $m + m$ inclusive sum-and-distance system can be obtained from any $n \times n$ principal reversible square by setting

\[
a_j = \alpha_{m+1+j} - \alpha_{m+1-j}, \quad b_j = \beta_{m+1+j} - \beta_{m+1-j} \quad (j \in \{1, \ldots, m\});
\]

now by (6.5),

\[
a_j = \alpha_{m+1+j} - \frac{\alpha_n}{2}, \quad -a_j = \alpha_{m+1-j} - \frac{\alpha_n}{2} \quad (j \in \{1, \ldots, m\}),
\]

and similarly for $b_k$ ($k \in \{1, \ldots, m\}$), and hence equality (6.3) follows in analogy to the above. Conversely, setting

\[
\alpha_{m+1+j} = a_m + a_j, \quad \alpha_{m+1-j} = a_m - a_j \quad (j \in \{1, \ldots, m\}), \\
\beta_{m+1+k} = b_m + b_k, \quad \beta_{m+1-k} = b_m - b_k \quad (k \in \{1, \ldots, m\}),
\]

and $\alpha_{m+1} = a_m, \beta_{m+1} = b_m$, we obtain a principal reversible square (or its transpose), via (6.3), from an inclusive sum-and-distance system.

Thus we have proven the following statement.
Theorem 5. Let \( m \in \mathbb{N} \). Then there is a bijection between the \( m + m \) non-inclusive sum-and-distance systems and the \( 2m \times 2m \) principal reversible squares, and there is a bijection between the \( m + m \) inclusive sum-and-distance systems and the \( (2m+1) \times (2m+1) \) principal reversible squares.

In conjunction with Theorem 4, this gives the following counting of non-inclusive and inclusive sum-and-distance systems.

Corollary 5. Let \( m \in \mathbb{N} \). Then there are

\[
N_{2m} = \sum_{j=1}^{\Omega(2m)} c_j^{(0)}(2m) c_j^{(1)}(2m)
\]
different \( m + m \) non-inclusive sum-and-distance systems, and

\[
N_{2m+1} = \sum_{j=1}^{\Omega(2m+1)} c_j^{(0)}(2m+1) c_j^{(1)}(2m+1)
\]
different \( m + m \) inclusive sum-and-distance systems.

To conclude, we briefly note that \( m + m \) sum-and-distance systems of either variety have the general property that the sum of squares of all entries of their component sets is invariant, determined only by the size \( m \).

Theorem 6. Let \( m \in \mathbb{N} \) and \( \{a_1, \ldots, a_m\}, \{b_1, \ldots, b_m\} \) a (non-inclusive or inclusive) sum-and-distance system. Then

\[
\sum_{j=1}^{m} (a_j^2 + b_j^2) = \begin{cases} 
\frac{1}{3!} (2m)((2m)^4 - 1) & \text{in the non-inclusive case,} \\
\frac{1}{4!} (2m+1)((2m+1)^4 - 1) & \text{in the inclusive case.}
\end{cases}
\]

Proof. In the non-inclusive case we use the formula

\[
\sum_{j=1}^{n} (2j - 1)^2 = \frac{n(4n^2 - 1)}{3} \quad (n \in \mathbb{N})
\]
to find

\[
2m \sum_{j=1}^{m} (a_j^2 + b_j^2) = \sum_{j=1}^{m} \sum_{k=1}^{m} ((a_j + b_k)^2 + (a_j - b_k)^2)
\]

\[
= \sum_{j=1}^{2m^2} (2j - 1)^2 = \frac{1}{6} 4m^2(16m^4 - 1).
\]

In the inclusive case, we similarly use the formula

\[
\sum_{j=1}^{n} j^2 = \frac{n(n + 1)(2n + 1)}{6} = \frac{(2n + 1 - 1)(2n + 1 + 1)(2n + 1)}{24}
\]

\[
= \frac{(2n + 1)^2 - 1)(2n + 1)}{24} \quad (n \in \mathbb{N})
\]
and the identity for \( n = 2m(m + 1) \)

\[
2n + 1 = 4m^2 + 4m + 1 = (2m + 1)^2
\]
to calculate

\[(2m + 1) \sum_{j=1}^{m} (a_j^2 + b_j^2) = \sum_{j=1}^{m} \sum_{k=1}^{m} k = 1^m((a_j + b_j)^2 + (a_j - b_j)^2) + \sum_{j=1}^{m} a_j^2 + \sum_{j=1}^{m} b_j^2 \]

\[= \sum_{j=1}^{2m(m+1)} j^2 = \frac{1}{24} ((2m + 1)^4 - 1)(2m + 1)^2. \]

\[\square\]

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