Classification results for nonsingular Bernoulli crossed products

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Abstract

We prove rigidity and classification results for type III factors given by nonsingular Bernoulli actions of the free groups and more general free product groups. This includes a large family of nonisomorphic Bernoulli crossed products of type III\textsubscript{1} that cannot be distinguished by Connes $\tau$-invariant. These are the first such classification results beyond the well studied probability measure preserving case.

1 Introduction

In the past years, Popa’s deformation/rigidity theory has led to a broad range of rigidity theorems for probability measure preserving (pmp) Bernoulli actions $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$, see e.g. [Pop03, Pop04, Pop06, Ioa10, IPV10, PV21]. This includes numerous $W^*$-superrigidity results showing that both the group $\Gamma$ and its action $\Gamma \curvearrowright (X, \mu)$ can be entirely retrieved from the ambient $\mathrm{II}_{1}$ factor $L^\infty(X) \rtimes \Gamma$.

More recently, there has been a growing interest in nonsingular Bernoulli actions $\Gamma \curvearrowright (X, \mu) = \prod_{\gamma \in \Gamma} (X_0, \mu_\gamma)$, where the base measures $\mu_\gamma$ vary. For $\Gamma = \mathbb{Z}$, this provides under the appropriate assumptions a classical family of nonsingular ergodic transformations that have been widely studied, see e.g. [Ham81, Kos09, BKV19]. For nonamenable groups $\Gamma$, a first systematic study of nonsingular Bernoulli actions was made in [VW17]. In view of the wealth of rigidity theorems for pmp Bernoulli actions, this raises the natural problem to prove rigidity and classification theorems for the type III factors $L^\infty(X, \mu) \rtimes \Gamma$ associated with nonsingular Bernoulli actions.

There is a conceptual reason why obtaining such rigidity theorems is a hard problem. It was proven in [VW17, Theorem 3.1] that if a nonamenable group $\Gamma$ admits a nonsingular Bernoulli action of type III, then $\Gamma$ must have a nonzero first $L^2$-Betti number. In the pmp setting, all superrigidity theorems for Bernoulli actions are restricted to nonamenable groups with zero first $L^2$-Betti number! It has even been conjectured that a pmp Bernoulli action satisfies cocycle superrigidity (w.r.t. the appropriate target groups) if and only if $\Gamma$ is nonamenable with zero first $L^2$-Betti number.

Therefore, we only set out to prove strong rigidity theorems, providing partial classification results for natural families of type III Bernoulli crossed products $L^\infty(X, \mu) \rtimes \Gamma$. These are the first classification results for type III Bernoulli crossed products going beyond Connes $\tau$-invariant to distinguish between such type III factors. We specifically prove these results for the wide family of nonsingular Bernoulli actions of the free groups $\mathbb{F}_n$ that were introduced in [VW17, Section 7] and that we recall below. More generally, we consider such Bernoulli actions for arbitrary free product groups $\Gamma = \mathbb{Z} \ast \Lambda$ with $\Lambda$ being any nonamenable group.

Recall from [VW17, Section 7] that, given any standard Borel space $Y$ and equivalent probability measures $\nu \sim \eta$ on $Y$, and given any countably infinite group $\Lambda$, we can consider the

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nonsingular Bernoulli action of the free product $\Gamma = \mathbb{Z} \ast \Lambda$ given by

$$\Gamma \curvearrowright (X, \mu) = \prod_{h \in \Gamma} (Y, \mu_h) \quad \text{where} \quad \mu_h = \begin{cases} \nu & \text{if the last letter of } h \text{ belongs to } \mathbb{N} \subset \mathbb{Z}, \\ \eta & \text{otherwise,} \end{cases}$$

and $(g \cdot x)h = x_{g^{-1}h}$. \hspace{1cm} (1.1)

We always assume that $\nu$ and $\eta$ are not concentrated on a single atom, because otherwise $(X, \mu)$ consists of a single point. By [VW17, Proposition 7.1], the action $\Gamma \curvearrowright (X, \mu)$ is essentially free and ergodic, of type III whenever $\nu \neq \eta$. By [VW17, Section 7], this family of Bernoulli actions is rich: the crossed products can be of any possible type III$_\lambda$, $\lambda \in (0, 1]$, and they can have basically any possible Connes $\tau$-invariant (in the sense of [Con74]).

The main goal of this paper is to prove classification results for the crossed product factors $M = L^\infty(X) \rtimes \Gamma$ given by (1.1). We associate a measure class on the real line to the factor $M$ and prove that it is an isomorphism invariant for this family of type III factors. To formulate this first main result, we introduce some notation.

For every measure class $\mu$ on $\mathbb{R}$, we denote by $\tilde{\mu}$ the measure class defined by $\tilde{\mu}(U) = 0$ iff $\mu(-U) = 0$. We say that a measure class $\mu$ on $\mathbb{R}$ is stable if $\delta_0 \prec \mu$, $\tilde{\mu} \sim \mu$ and $\mu \ast \nu \sim \mu$. For every measure class $\mu$ on $\mathbb{R}$, there is a smallest stable measure class $\gamma$ such that $\mu \prec \gamma$. We denote this as $\gamma = \text{stc}(\mu)$. This measure class $\gamma$ can be defined as the join of the measure classes $\mu^{\ast n} \ast \tilde{\mu}^{\ast m}$, $n, m \geq 0$, where we use the convention that the 0'th convolution power is $\delta_0$. Given equivalent probability measures $\nu \sim \eta$ on a standard Borel space $Y$, we consider the stable measure class $\gamma = \text{stc}(\log d\nu/d\eta)_*(\nu)$. \hspace{1cm} (1.2)

**Theorem A.** For $i = 1, 2$, let $\Lambda_i$ be a nonamenable group and let $\nu_i \sim \eta_i$ be equivalent probability measures on standard Borel spaces $Y_i$. Consider the nonsingular Bernoulli actions of $\Gamma_i = \mathbb{Z} \ast \Lambda_i$ on $(X_i, \mu_i)$ given by (1.1). Denote by $M_i$ their crossed product von Neumann algebras and let $\gamma_i = \text{stc}(\log d\nu_i/d\eta_i)_*(\nu_i)$ be the stable measure class defined by (1.2). If $M_1 \cong M_2$, then $\gamma_1 \sim \gamma_2$.

We also analyze which conclusions can be drawn if $M_1$ merely embeds with expectation into $M_2$, meaning that there exists a faithful normal $\ast$-homomorphism $\pi : M_1 \rightarrow M_2$ and a faithful normal conditional expectation of $M_2$ onto $\pi(M_1)$. The following then provides large families of Bernoulli crossed products for which such embeddings with expectation do not exist. While a systematic study of embeddability between Bernoulli crossed products has been made in [PV21] in the probability measure preserving type II$_1$ setting, our Theorem B is the first such systematic nonembeddability result in the type III case.

To formulate this result, we provide the following canonical class of examples of (1.1). Define the set $\mathcal{P}$ of Borel probability measures on $\mathbb{R}$ by

$$\mathcal{P} = \{ \nu \mid \nu \text{ is a Borel probability measure on } \mathbb{R} \text{ with } \int_{\mathbb{R}} \exp(-x) \, d\nu(x) < +\infty \}.$$

Given $\nu \in \mathcal{P}$, there is a unique probability measure $\eta$ on $\mathbb{R}$ given by normalizing $\exp(-x) \, d\nu(x)$. By construction, the measure $(\log d\nu/d\eta)_*(\nu)$ is a translate of $\nu$. Then, (1.1) provides a nonsingular Bernoulli action for any free product $\mathbb{Z} \ast \Lambda$, with base space $Y = \mathbb{R}$.

Recall that a Borel set $K \subset \mathbb{R}$ is called independent if every set of $n$ distinct elements of $K$ generates a free abelian subgroup of $\mathbb{R}$ of rank $n$. 

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Theorem B. Let $K \subset \mathbb{R}$ be an independent Borel set. For $i = 1, 2$, let $\Lambda_i$ be nonamenable groups, put $\Gamma_i = \mathbb{Z} \ast \Lambda_i$ and let $\nu_i \in P$ be nonatomic measures supported on $K$. Consider the associated nonsingular Bernoulli actions with crossed product von Neumann algebra $M_i$.

If $M_1$ embeds with expectation into $M_2$, then $\nu_1 \prec \nu_2$. In particular, if $M_1 \cong M_2$, then $\nu_1 \sim \nu_2$.

So, Theorem B provides large classes of nonsingular Bernoulli crossed products that cannot be embedded with expectation one into the other. Moreover, the conclusions of Theorem B hold for further classes of probability measures $\nu_i \in P$, see Corollary 4.5 below. In Example 4.6, we use this result to provide mutually nonembeddable type III Bernoulli crossed products that cannot be distinguished by invariants from modular theory, like Connes $\tau$-invariant.

We next focus on solidity of nonsingular Bernoulli crossed products. Recall from [Oza03] that a II$_1$ factor $M$ is called solid if $A' \cap M$ is amenable for every diffuse von Neumann subalgebra $A \subset M$. In [Oza03], it is proven that the group von Neumann algebra $L(\Gamma)$ is solid for every word hyperbolic group $\Gamma$ and, more conceptually, for every biexact countable group $\Gamma$ (see [BO08, Chapter 15]). An arbitrary diffuse von Neumann algebra $M$ is called solid if $A' \cap M$ is amenable for every diffuse von Neumann subalgebra $A \subset M$ that is the range of a faithful normal conditional expectation (see [VV05]). One of the striking features of solid factors is that they are prime: they do not admit nontrivial tensor product decompositions. Also all their nonamenable subfactors with expectation are prime.

Solidity has a counterpart in ergodic theory, as discovered in [CI08]: an essentially free nonsingular action of a countable group $\Gamma$ on a standard nonatomic probability space $(X, \mu)$ is said to be a solid action if for every subequivalence relation $S$ of the orbit equivalence relation $R(\Gamma \curvearrowright X)$, there exists a partition of $(X, \mu)$ into $S$-invariant Borel sets $(X_n)_{n \geq 0}$ such that $S_{|_{X_n}}$ is amenable and $S_{|_{X_n}}$ is ergodic for every $n \geq 1$. Note that $\Gamma \curvearrowright (X, \mu)$ is a solid action if and only if for every diffuse von Neumann subalgebra $A \subset L^\infty(X)$, the relative commutant $A' \cap L^\infty(X) \rtimes \Gamma$ is amenable. In [CI08], it is proven that all pmp Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ are solid actions.

It remains an open question whether all nonsingular Bernoulli actions $\Gamma \curvearrowright \prod_{h \in \Gamma} (X_0, \mu_h)$ are solid actions. In [HIK20], it is proven that this is indeed the case when $X_0 = \{0, 1\}$ consists of two points and when the probability measures $(\mu_h)_{h \in \Gamma}$ have a stronger almost invariance property: for all $g \in \Gamma$, we have that $\mu_{gh} = \mu_h$ for all but finitely many $h \in \Gamma$. We prove that all nonsingular Bernoulli actions in (1.1) are solid actions. Note that our family mainly consists of Bernoulli actions with a diffuse base space, thus complementing the results of [HIK20].

Actually, our method to prove Theorems A and B is a “solidity method” that was introduced in [HSV16]. In [HSV16], it was proven that any faithful normal state $\psi$ on a free Araki-Woods factor $M$ with the property that the centralizer $M^\psi$ is nonamenable, must have a corner that is unitarily conjugate to a corner of the canonical free quasi-free state $\varphi$ on $M$. So in cases where the centralizer of the free quasi-free state $\varphi$ is a nonamenable II$_1$ factor, we can characterize $\varphi$ as the essentially unique state on $M$ having a nonamenable centralizer. As a consequence, the spectral measure class of the modular operator $\Delta_\varphi$, becomes an invariant of such von Neumann algebras $M$.

We thus introduce the following terminology: we say that a faithful normal state $\varphi$ on a von Neumann algebra $M$ is a solid state if every faithful normal state $\psi$ on $M$ with a nonamenable centralizer $M^\psi$ has a corner that is unitarily conjugate to a corner of $\varphi$ (see Definition 5.1). In particular, if $\varphi$ is a solid state on a type III factor and $M^\varphi$ is amenable, it follows that every faithful normal state on $M$ has an amenable centralizer. The main result of [HSV16] can then be reformulated as saying that the free quasi-free state on a free Araki-Woods factor is a solid state.
For our nonsingular Bernoulli actions in (1.1) with crossed product $M = L^\infty(X,\mu) \rtimes \Gamma$, it is in general not true that the crossed product state $\varphi_\mu$ is solid. Nevertheless, our proof of Theorems A and B is based on carefully analyzing which states on $M$ have a nonamenable centralizer. Under extra assumptions, we do find that $\varphi_\mu$ is a solid state. Our solidity results can then be summarized as follows.

**Theorem C.** Let $\Lambda$ be a nonamenable group and let $\nu \sim \eta$ be equivalent probability measures on a standard Borel space $Y$. Consider the nonsingular Bernoulli actions of $\Gamma = \mathbb{Z} \ast \Lambda$ on $(X,\mu)$ given by (1.1). Denote $M = L^\infty(X,\mu) \rtimes \Gamma$.

1. The nonsingular Bernoulli action $\Gamma \curvearrowright (X,\mu)$ is a solid action.

2. The factor $M$ is solid relative to $L(\Lambda)$ in the sense of [Mar16, Definition 3.2].

3. If $\Lambda$ is biexact, then $M$ is solid.

4. If $\Lambda$ is biexact and $(\log d\nu/d\eta)_* (\nu)$ is nonatomic, then the crossed product state $\varphi_\mu$ on $M$ is a solid state.

In [Oza04, Corollary 4.5], it was proven that for every pmp Bernoulli action $\Gamma \curvearrowright (X,\mu) = (X_0,\mu_0)^\Gamma$ of a biexact group $\Gamma$, the crossed product $M = L^\infty(X,\mu) \rtimes \Gamma$ is solid. It is an open problem whether the same holds for arbitrary nonsingular Bernoulli actions of biexact groups. By [HV12, Theorem C], this problem is equivalent to the open problem whether every nonsingular Bernoulli action of a biexact group is a solid action.

Since we expect that these open problems have a positive solution, it is tempting to believe that for any nonsingular Bernoulli action $\Gamma \curvearrowright (X,\mu)$ of a biexact group, the crossed product state $\varphi_\mu$ on $M = L^\infty(X,\mu) \rtimes \Gamma$ is a solid state. This is however not true, as we show in Example 5.4. In an attempt to give a more conceptual explanation for Theorem A, it is equally natural to try to prove the following statement: if $\Gamma \curvearrowright (X,\mu)$ is any nonsingular Bernoulli action with the property that the measure $\mu$ is $\Lambda$-invariant for a nonamenable subgroup $\Lambda < \Gamma$, then the spectral measure class of $\Delta_\varphi_\mu$ can be recovered as an invariant of $M$. But the same Example 5.4 shows that also this statement is false. This explains why our Theorem A is restricted to the natural family of actions introduced in (1.1).

We finally prove the following partial converse to Theorem A.

**Proposition D.** Let $\Lambda$ be a countable group and put $\Gamma = \mathbb{Z} \ast \Lambda$. For $i = 1, 2$, let $\nu_i \sim \eta_i$ be equivalent probability measures on the standard Borel spaces $Y_i$. Denote by $\Gamma \curvearrowright^{\alpha_i} (X_i,\mu_i)$ the associated nonsingular Bernoulli actions given by (1.1). Denote $\sigma_i = (\log d\nu_i/d\eta_i)_* (\nu_i)$.

If $\sigma_1 = \sigma_2$ and if the maps $\log d\nu_i/d\eta_i : (Y_i,\nu_i) \to \mathbb{R}$ are not essentially one-to-one, then there exists a measure preserving conjugacy between the actions $\Gamma \curvearrowright^{\alpha_i} (X_i,\mu_i)$. In particular, the crossed product factors $M_i = L^\infty(X_i,\mu_i) \rtimes_{\alpha_i} \Gamma$ are isomorphic.

It is clear that Proposition D is not an optimal result. One might for instance speculate that the assumption $\sigma_1 \sim \sigma_2$ should be sufficient to prove that the nonsingular Bernoulli actions $\alpha_i$ are orbit equivalent. Still, our result is nonempty: in Example 6.2, we provide examples where the hypotheses of Proposition D are satisfied with $Y_1$ being a finite set with atomic measures and $Y_2 = [0,1]$ with two measures $\nu_2 \sim \eta_2$ that are equivalent with the Lebesgue measure. In these examples, there is no obvious conjugacy between the nonsingular Bernoulli actions given by (1.1).
2 Preliminaries

A von Neumann subalgebra $B \subset N$ is said to be with expectation if there exists a faithful normal conditional expectation $E : N \to B$.

We start by recalling Popa’s theory of intertwining-by-bimodules, as introduced in [Popa03, Section 2]. We make use of the adaptations to the semifinite and infinite setting, which reached a final version in [HI15, Section 4]. So, let $M$ be any von Neumann algebra with separable predual and let $p, q \in M$ be nonzero projections. Let $A \subset pMp$ and $B \subset qMq$ be von Neumann subalgebras with expectation. We write $A \prec_M B$ if there exist projections $r \in A$, $s \in B$, a nonzero partial isometry $v \in rMs$ and a unital normal $*$-homomorphism $\theta : rAr \to sBs$ such that $av = v\theta(a)$ for all $a \in rAr$ and $\theta(rAr) \subset sBs$ is with expectation.

We write $A \prec_{f,M} B$ if for every nonzero projection $e \in A' \cap pMp$, we have that $Ae \prec_M B$.

When $A$ is finite, $B$ is semifinite, $E_B : qMq \to B$ is a faithful normal conditional expectation and $\Tr$ is a faithful normal semifinite trace on $B$, the following results are contained in [HI15, Theorem 4.3].

- $A \neq_M B$ if and only if there exists a sequence of unitaries $a_n \in \mathcal{U}(A)$ such that $\|E_B(x^*a_ny)\|_{L^2,\Tr} \to 0$ for all $x, y \in pMq$ with $\Tr(x^*x), \Tr(y^*y) < +\infty$.

- $A \prec_M B$ if and only if there exists an integer $n \in \mathbb{N}$, a finite projection $s \in M_n(\mathbb{C}) \otimes B$, a nonzero partial isometry $v \in (\mathbb{C}^n \otimes pM)s$ and a normal unital $*$-homomorphism $\theta : A \to s(M_n(\mathbb{C}) \otimes B)s$ such that $av = v\theta(a)$ for all $a \in A$.

Recall that given a von Neumann subalgebra $A \subset N$, one defines $\mathcal{N}_N(A) = \{u \in \mathcal{U}(N) \mid uAu^* = A\}$ and one calls $\mathcal{N}_N(A)^\prime\prime$ the normalizer of $A$ inside $N$. Note that $A' \cap N \subset \mathcal{N}_N(A)^\prime\prime$.

When $A \subset N$ is with expectation, also $\mathcal{N}_N(A)^\prime\prime \subset N$ is with expectation.

Assume again that $M$ is a von Neumann algebra with separable predual and that $A \subset pMp$ and $B \subset qMq$ are von Neumann subalgebras with expectation. When $e \in A' \cap pMp$ is a nonzero projection such that $Ae \prec_M B$, we can take a nonzero partial isometry $v$ as above, where $v \in rMs$ and $r \in A$. When $u \in \mathcal{N}_{pMp}(A)$, we can replace $v$ by $uv$ and replace $\theta$ by $\theta \circ \Ad u^*$. It follows that $Aueu^* \prec_M B$. We conclude from this argument that there exists a unique projection $z$ in the center $Z(\mathcal{N}_{pMp}(A)^\prime\prime)$ of the normalizer such that $Az \prec_{f,M} B$ and $A(p - z) \neq_M B$.

If $M$ is a von Neumann algebra with separable predual and if $B \subset M$ is a von Neumann subalgebra with expectation, then $M$ is solid relative to $B$ in the sense of [Mar16, Definition 3.2] if and only if every von Neumann subalgebra $Q \subset pMp$ with expectation and with diffuse center $Z(Q)$ satisfies at least one of the following properties: $Q$ is amenable or $Q \prec_M B$.

For every von Neumann algebra $M$ with separable predual, we denote by $c(M)$ its continuous core, which can be concretely realized as $M \rtimes_{\sigma,\varnothing} \mathbb{R}$ whenever $\varnothing$ is a faithful normal state on $M$ with modular automorphism group $(\sigma^\varnothing_t)_{t \in \mathbb{R}}$. We denote by $\lambda_\varnothing(t), t \in \mathbb{R}$, the canonical unitary operators in the crossed product $c(M) = M \rtimes_{\sigma,\varnothing} \mathbb{R}$, generating the von Neumann subalgebra $L_\varnothing(\mathbb{R}) \subset c(M)$. There is a canonical faithful normal semifinite trace $\Tr$ on $c(M)$. Both the inclusion $M \subset c(M)$ and the trace $\Tr$ are essentially independent of the choice of $\varnothing$, since Connes cocycle derivative theorem provides a trace preserving $*$-isomorphism $\theta : M \rtimes_{\sigma,\varnothing} \mathbb{R} \to M \rtimes_{\sigma,\varnothing} \mathbb{R}$ satisfying $\theta(a) = a$ for all $a \in M$ and $\theta(\lambda_\varnothing(t)) = [D\varnothing : Da]\lambda_\varnothing(t)$. The restriction of the trace $\Tr$ to $L_\varnothing(\mathbb{R})$ is semifinite. The unique trace preserving conditional expectation $E_{L_\varnothing(\mathbb{R})} : c(M) \to L_\varnothing(\mathbb{R})$ satisfies $E_{L_\varnothing(\mathbb{R})}(a) = \varnothing(a)1$ for all $a \in M$.

Whenever $P \subset M$ is a von Neumann subalgebra and $E : M \to P$ is a faithful normal conditional expectation, we obtain a canonical trace preserving embedding $c(P) \hookrightarrow c(M)$, which can be
concretely constructed by taking a faithful normal state \( \varphi \) on \( P \) and writing \( c(P) = P \rtimes_{\sigma, \varphi} \mathbb{R} \to M \rtimes_{\sigma, \varphi} \mathbb{R} = c(M) \). Note that this embedding depends on the choice of \( E \). In the trivial case where \( P = C1 \), we have that \( E(a) = \varphi(a)1 \) and the embedding corresponds to \( L_\varphi(\mathbb{R}) \subset c(M) \).

Given an action \( \Gamma \rtimes I \) of a countable group \( \Gamma \) on a countable set \( I \) and given a von Neumann algebra \( (P, \omega) \) equipped with a faithful normal state, we consider the generalized Bernoulli action \( \Gamma \rtimes (N, \omega) = (P, \omega)^I \). Here we use the notation \( (P, \omega)^I \) to denote the tensor product of copies of \( (P, \omega) \) indexed by \( I \). The action \( \Gamma \rtimes (N, \omega) \) is state preserving. We get a canonical action of \( \Gamma \) on the continuous core \( c(N) \) such that

\[
c(N \rtimes \Gamma) = c(N) \rtimes \Gamma.
\]

In [Pop03], Popa introduced his fundamental malleable deformation for probability measure preserving Bernoulli actions \( \Gamma \rtimes (X_0, \mu_0)^\Gamma \), which has been a cornerstone for deformation/rigidity theory. It has been extended in several directions. In [Ioa06], another malleable deformation was found, adapted to noncommutative Bernoulli actions \( \Gamma \rtimes (P, \tau)^\Gamma \), where \( (P, \tau) \) is a tracial von Neumann algebra. This can be adapted in a straightforward way to the nontracial case, i.e. for Bernoulli actions \( \Gamma \rtimes (P, \omega)^\Gamma \), where \( \omega \) is a faithful normal state on \( P \) (see e.g. [Mar16, Section 5]). Also Popa’s spectral gap rigidity for Bernoulli actions, as introduced in [Pop06], can be extended to the setting of generalized Bernoulli actions, i.e. for actions \( \Gamma \rtimes (X_0, \mu_0)^I \), where \( I \) is a countable set on which \( \Gamma \) is acting, see [IPV10, Section 4]. Putting all these generalizations together, we right away get the following variant of [IPV10, Corollary 4.3].

**Theorem 2.1.** Let \( (P, \omega) \) be an amenable von Neumann algebra with a faithful normal state. Let \( \Gamma \rtimes I \) be an action of a countable group \( \Gamma \) on a countable set \( I \). Assume that \( \text{Stab}(i) \) is amenable for every \( i \in I \) and assume that there exists a \( \kappa \in \mathbb{N} \) such that \( \text{Stab}(J) \) is finite whenever \( J \subset I \) and \( |J| \geq \kappa \). Denote, as above, \( (N, \omega) = (P, \omega)^I \) and let \( \Gamma \rtimes (N, \omega) \) be the generalized Bernoulli action. Write \( M = N \rtimes \Gamma \).

Let \( p \in c(M) \) be a projection of finite trace and \( A \subset p c(M)p \) a von Neumann subalgebra such that the relative commutant \( A' \cap p c(M)p \) has no amenable direct summand. Denote by \( Q = N_{pc(M)p}(A)'' \) the normalizer of \( A \). Let \( z \in Z(Q) \) be the maximal projection such that \( Az \prec_f L_\omega(\mathbb{R}) \). Put \( z' = p - z \). Then \( Qz' \prec_f L_\omega(\mathbb{R}) \vee L(\Gamma) \).

**Proof.** Write \( c(L(\Gamma)) = L_\omega(\mathbb{R}) \vee L(\Gamma) \). Replacing \( A \) by \( Az'' \) where \( z'' \) is an arbitrary projection in \( Z(Q)z' \), we may assume that \( A \neq L_\omega(\mathbb{R}) \) and we have to prove that \( Q \prec c(L(\Gamma)) \). Even though \( \omega \) is not necessarily tracial, the tensor length deformation makes sense in this context and the proof of [IPV10, Corollary 4.3] can be copied almost verbatim. The conclusion is that at least one of the following statements hold: \( Q \prec c(N) \times \text{Stab}i \) for some \( i \in I \), or \( Q \prec c(L(\Gamma)) \). The von Neumann algebra \( c(N) \rtimes \text{Stab}i \) is amenable. Since \( A' \cap p c(M)p \subset Q \), the von Neumann algebra \( Q \) has no amenable direct summand. Therefore, it is impossible that \( Q \prec c(N) \times \text{Stab}i \). This concludes the proof of the theorem.

Also the following result is an immediate noncommutative variant of known results for probability measure preserving Bernoulli actions. The method was introduced in [Pop03, Section 3] and the following version is a straightforward generalization of [Vae07, Lemma 4.2]. The same result still holds when replacing the ad hoc construction (2.1) by the quasinormalizer of \( B \) inside \( pMp \), but we only need this simpler version.

**Proposition 2.2.** Make the same assumptions as in Theorem 2.1. Let \( p \in L(\Gamma) \) be a projection and \( B \subset pL(\Gamma)p \) a diffuse von Neumann subalgebra. Define

\[
D = \{ u \in pMp \mid \exists \beta \in \text{Aut}(B), \forall b \in B : ub = \beta(b)u \}''
\]

(2.1)
and note that $N_{pM}(B)'' \subset D$.

1. If $B \not\prec_{L(\Gamma)} L(\text{Stab } i)$ for every $i \in I$, then $D \subset pL(\Gamma)p$.

2. If $r Dr$ is nonamenable for every nonzero projection $r \in B' \cap pL(\Gamma)p$, then $D \subset pL(\Gamma)p$.

**Proof.** Since $L(\Gamma)$ lies in the centralizer of the state $\omega$ on $M$, all computations of [Vae07, Lemma 4.2] go through verbatim. So, if $B \not\prec_{L(\Gamma)} L(\text{Stab } i)$ for every $i \in I$, it follows from [Vae07, Lemma 4.2] that $D \subset pL(\Gamma)p$.

Next assume that there exists an $i \in I$ such that $B \not\prec_{L(\Gamma)} L(\text{Stab } i)$. It suffices to prove that $r Dr$ is amenable for some nonzero projection $r \in B' \cap pL(\Gamma)p$. By assumption, $\text{Stab } J$ is finite whenever $J \subset I$ and $|J| \geq \kappa$. Also, $B$ is diffuse. We thus find a finite nonempty subset $J \subset I$ such that $B \not\prec_{L(\Gamma)} L(\text{Stab } J)$ and $B \not\prec_{L(\Gamma)} L(\text{Stab } (J \cup \{ j \}))$ for every $j \in I \setminus J$.

As in [Vae07, Remark 3.8], we can take an integer $n \in \mathbb{N}$, a projection $q \in M_n(\mathbb{C}) \otimes L(\text{Stab } J)$, a nonzero partial isometry $v \in (C^n \otimes pL(\Gamma))q$ and a unital normal $\ast$-homomorphism $\theta : B \to q(M_n(\mathbb{C}) \otimes L(\Gamma))q$ such that $bv = v\theta(b)$ for all $b \in B$ and such that $\theta(B) \not\prec_{L(\text{Stab } J)} L(\text{Stab } (J \cup \{ j \}))$ for every $j \in I \setminus J$. When $u \in pMp$ and if $\beta \in \text{Aut}(B)$ such that $ub = \beta(b)u$ for all $b \in B$, it follows that $v^* uv \theta(b) = \theta(\beta(b)) v^* uv$ for all $b \in B$.

By [Vae07, Lemma 4.2], it follows that $v^* uv \in N \times \text{Norm } J$, where $\text{Norm } J = \{ g \in \Gamma \mid g \cdot J = J \}$. Writing $r = vv^*$ and $s = v^* v$, we get that $r$ is a projection in $B' \cap pL(\Gamma)p \subset D$, that $s$ is a projection in $N \times \text{Norm } J$ and that $v^* Dv \subset N \times \text{Norm } J$. Since $L(\Gamma) \subset N \times \Gamma$ is with expectation, also $B \subset pMp$ and thus $D \subset pMp$ are with expectation. It follows that $v^* Dv$ is with expectation in $s(N \times \text{Norm } J)s$. Since $J$ is finite and nonempty, the group $\text{Norm } J$ is amenable. We conclude that $v^* Dv$ is amenable, so that $r Dr$ is amenable. \(\square\)

### 3 Measure classes of faithful normal states

For any self-adjoint, possibly unbounded operator $T$, we denote by class$(T)$ its spectral measure class on $\mathbb{R}$. Note that a Borel set $U \subset \mathbb{R}$ has measure zero for class$(T)$ if and only if the spectral projection $1_U(T)$ equals 0.

Given a faithful normal state $\omega$ on a von Neumann algebra $M$, we define the measure class $\text{class}(\omega) := \text{class}(\log \Delta_\omega)$, where $\Delta_\omega$ is the modular operator of $\omega$. Of course, class$(\omega)$ highly depends on the choice of the state $\omega$ and hence, does not provide an invariant of the von Neumann algebra $M$. A key element of this paper is that certain von Neumann algebras, including many nonsingular Bernoulli crossed products, have a favorite state $\omega$ that can be essentially intrinsically characterized, so that class$(\omega)$ becomes an isomorphism invariant for this family of von Neumann algebras.

To establish these results, we rephrase [HSV16, Corollary 3.2] in the following way, also introducing the notation $\prec_f \omega$ for faithful normal states on a von Neumann algebras.

**Lemma 3.1** ([HSV16, Corollary 3.2]). Let $\varphi$ and $\omega$ be faithful normal states on a von Neumann algebra $M$, with corresponding canonical subalgebras $L_\varphi(\mathbb{R})$ and $L_\omega(\mathbb{R})$ of the continuous core $c(M)$. Then the following three statements are equivalent.

1. $L_\varphi(\mathbb{R}) \prec_f L_\omega(\mathbb{R})$.

2. There exist a nonzero partial isometry $v \in M$ and $\gamma > 0$ such that $\gamma^t [D \omega : D\varphi], \sigma^t_\varphi(v) = v$ for all $t \in \mathbb{R}$. 


3. There exists a nonzero partial isometry \( v \in M \) with \( e := v^*v \in M^\rho \), \( q := vv^* \in M^\omega \) and \( \varphi(e) \varphi(v^*xv) = \omega(q) \omega(x) \) for all \( x \in qMq \).

When these equivalent conditions hold, we write \( \varphi \prec \omega \). Note that by 3, we have \( \varphi \prec \omega \) iff \( \omega \prec \varphi \).

Also the following three statements are equivalent.

4. \( L_\varphi(\mathbb{R}) \prec_{f,c(M)} L_\omega(\mathbb{R}) \).

5. For every nonzero projection \( p \in M^\rho \), there exists \( \gamma > 0 \) and \( v \in M \) as in 2 with \( v^*v \leq p \).

6. For every nonzero projection \( p \in M^\rho \), there exists \( v \in M \) as in 3 with \( v^*v \leq p \).

When these equivalent conditions hold, we write \( \varphi \prec f \omega \).

In full generality, the relation \( \varphi \prec f \omega \) is not strong enough to conclude that \( \text{class}(\varphi) \prec \text{class}(\omega) \).

We nevertheless have the following partial results.

**Proposition 3.2.** Let \( \varphi \) and \( \omega \) be faithful normal states on a von Neumann algebra \( M \).

1. If \( \varphi \prec f \omega \), there exists an atomic probability measure \( \rho \) on \( \mathbb{R} \) such that \( \text{class}(\varphi) \prec \rho * \text{class}(\omega) \).

2. If \( \varphi \prec \omega \) and if \( M^\rho \) is a factor, then \( \text{class}(\varphi) \prec \text{class}(\omega) \).

**Proof.** 1. We apply point 5 of Lemma 3.1. We thus find a sequence of nonzero projections \( p_n \in M^\rho \), partial isometries \( v_n \in M \) and \( \gamma_n > 0 \) such that \( \sum_n p_n = 1 \), \( v_n^*v_n = p_n \) and

\[
\gamma_n^t [D_\omega : D_\varphi] \sigma_{t}^{\varphi}(v_n) = v_n \quad \text{for all } n \text{ and all } t \in \mathbb{R}.
\]

Define \( H = \ell^2(\mathbb{N}^2) \otimes L^2(M, \omega) \) and consider the unitary representation

\[
\theta : \mathbb{R} \to \mathcal{U}(H) : \theta(t)(\delta_{n,m} \otimes \xi) = (\gamma_n/\gamma_m)^t \delta_{n,m} \otimes \Delta_{n,m}^{\varphi} \xi.
\]

Then,

\[
V : L^2(M, \varphi) \to H : V(x) = \sum_{n,m} \gamma_n^{t/2} \delta_{n,m} \otimes v_nv_n^*
\]

is a well defined isometry satisfying \( \theta(t)V = V \Delta_{n,m}^{\varphi} \) for all \( t \in \mathbb{R} \). Hence, \( (\Delta_{n,m}^{\varphi})_{t \in \mathbb{R}} \) is unitarily equivalent with a subrepresentation of \( \theta \). Choosing an atomic probability measure \( \rho \) on \( \mathbb{R} \) with atoms in the points \( \log \gamma_n - \log \gamma_m \), it follows that \( \text{class}(\varphi) \prec \rho * \text{class}(\omega) \).

2. We apply point 2 of Lemma 3.1. Take a nonzero projections \( p \in M^\rho \), a partial isometry \( v \in M \) and \( \gamma > 0 \) such that \( v^*v = p \) and \( \gamma^t [D_\omega : D_\varphi] \sigma_{t}^{\varphi}(v) = v \) for all \( t \in \mathbb{R} \). Since \( M^\rho \) is a factor, we can choose partial isometries \( w_n \in M^\rho \) with \( w_n^*w_n \leq p \) and \( \sum_n w_n^*w_n = 1 \). We can then apply the proof of the first point to the partial isometries \( vw_n \), with \( \gamma_n = \gamma \) for all \( n \). The conclusion then becomes \( \text{class}(\varphi) \prec \text{class}(\omega) \).

\( \square \)

For later purposes, we prove the following rather specific and technical variant of the second point in Proposition 3.2.

**Lemma 3.3.** Let \( N \) be a von Neumann algebra with von Neumann subalgebra \( M \subset N \) and faithful normal conditional expectation \( E : N \to M \). Let \( \omega_0 \) and \( \varphi_0 \) be faithful normal states on \( M \), write \( \omega = \omega_0 \circ E \) and \( \varphi = \varphi_0 \circ E \). Assume that there exists a subset \( \mathcal{G} \subset \mathcal{U}(N) \) such that \( u^*\sigma_t^\varphi(u) \in M \) for all \( t \in \mathbb{R} \), \( u \in \mathcal{G} \) and such that the linear span of \( \mathcal{G}M \) is dense in \( L^2(N, \omega) \).

If \( \varphi \prec \omega \) and if \( M^{\varphi_0} \) is a factor, there exists \( u \in \mathcal{G} \) such that

\[
\text{class}(\varphi_0) \prec \text{class}((\omega \circ \text{Ad } u)|_M) \prec \text{class}(\omega).
\]

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Proof. By Lemma 3.1, we find a nonzero element \( v \in N \) and \( \gamma > 0 \) such that
\[
\gamma^t \sigma_t^\gamma(v) [D\omega : D\varphi_t] = \gamma^t [D\omega : D\varphi_t] \sigma_t^\gamma(v) = v \quad \text{for all } t \in \mathbb{R}.
\] (3.1)
Since the linear span of \( GM \) is dense in \( L^2(N, \omega) \), we can choose \( u \in G \) such that \( E(u^*v) \neq 0 \).

Denote \( a_t := u^*\sigma_t^\gamma(u) \in \mathcal{U}(M) \). Define the faithful normal state \( \psi = \omega \circ \text{Ad} u \) on \( N \). By construction, \( [D\psi : D\omega_t] = a_t \). Since \( a_t \in M \), we get that \( \psi = \psi_0 \circ E \), where \( \psi_0 \) is defined as the restriction of \( \psi \) to \( M \). We have to prove that \( \text{class}(\varphi_0) \prec \text{class}(\psi_0) \).

For every \( x \in N \) and \( t \in \mathbb{R} \), we have
\[
\sigma_t^\psi(x) [D\psi : D\varphi_t] = u^* \sigma_t^\psi(uux^*)u [D\psi : D\varphi_t] = a_t \sigma_t^\psi(x) [D\omega : D\psi]_t [D\psi : D\varphi_t] = a_t \sigma_t^\psi(x) [D\omega : D\varphi_t].
\]
Applying this to \( x = E(u^*v) \), using (3.1) and using that \( [D\psi : D\varphi_t] = [D\psi_0 : D\varphi_0]_t \in M \) and \( [D\omega : D\varphi] = [D\omega_0 : D\varphi_0] \in M \), we get that
\[
\gamma^t \sigma_t^\psi_0(x) [D\psi_0 : D\varphi_0]_t = \gamma^t a_t \sigma_t^\psi(E(u^*v)) [D\omega : D\varphi]_t = a_t E(\sigma_t^\psi(u^*) \gamma^t \sigma_t^\psi(v)) [D\omega : D\varphi]_t = a_t E(a_t^* u^* v) = E(u^* v) = x.
\]

Defining the partial isometry \( w \in M \) as the polar part of \( x \), we still have
\[
\gamma^t \sigma_t^\psi_0(w) [D\psi_0 : D\varphi_0]_t = w.
\]
Since \( M_\varphi \) is a factor, it then follows from the second point of Proposition 3.2 that \( \text{class}(\varphi_0) \prec \text{class}(\psi_0) \). Since \( \psi = \psi_0 \circ E \), we have that \( \text{class}(\varphi_0) \prec \text{class}(\psi) \). Since \( \psi = \omega \circ \text{Ad} u \), the modular operators of \( \psi \) and \( \omega \) are unitarily equivalent, so that \( \text{class}(\psi) = \text{class}(\omega) \).

## 4 Proof of Theorems A and B

We deduce Theorems A and B from a more general rigidity result for certain nonsingular coinduced actions.

Let \( G \) be a countable amenable group and let \( G \bowtie^\alpha (Z, \zeta) \) be any nonsingular action on a nontrivial standard probability space. Given any countable infinite group \( \Lambda \), we consider the free product \( \Gamma = G * \Lambda \) and the countable set \( \mathcal{I} = (G * \Lambda)/G \) with base element \( i_0 = eG \). We then define \( (X, \mu) = (Z, \zeta)^i \) and the nonsingular action \( \Gamma \bowtie (X, \mu) \) by
\[
(g \cdot x)_i = x_{g^{-1} \cdot i} \quad \text{when } g \in \Lambda,
\]
\[
(g \cdot x)_i = x_{g^{-1} \cdot i} \quad \text{when } i \neq i_0 \text{ and } g \in G, \quad \text{and} \quad (g \cdot x)_{i_0} = g \cdot x_{i_0} \quad \text{when } g \in G.
\] (4.1)
So, \( \Lambda \) acts as a generalized Bernoulli action on \( X = Z^\mathcal{I} \), while the action of \( G \) is the diagonal product of a generalized Bernoulli action on \( Z^{\mathcal{I} \setminus \{i_0\}} \) and the given action \( G \bowtie^\alpha Z \), viewed as the \( i_0 \)-coordinate.

Note that the nonsingular Bernoulli action in (1.1) arises as a special case of (4.1) by taking \( G = Z \) and \( Z \bowtie (Z, \zeta) = \prod_{n \in \mathbb{Z}} (Y, \mu_n) \) by Bernoulli shift.

When \( i \neq j \) are distinct elements, the stabilizer \( \text{Stab}\{i, j\} \) is trivial. It follows that \( \Gamma \bowtie (X, \mu) \) is essentially free. The action of \( \Lambda \) is measure preserving and ergodic. It follows that \( \Gamma \bowtie (X, \mu) \) is ergodic, with the Krieger type being determined in the following way: if \( \zeta \) is \( G \)-invariant, then \( \Gamma \bowtie (X, \mu) \) is of type \( \Pi_1 \). In all other cases, \( \Gamma \bowtie (X, \mu) \) is of type \( \Pi_3 \).

We associate to \( G \bowtie^\alpha (Z, \zeta) \) the stable measure class \( \text{stc}(\alpha) \) defined as the smallest stable measure class such that \( (\log d(g \cdot \zeta)/d\zeta)_g(\zeta) \prec \text{stc}(\alpha) \) for all \( g \in G \). Note that \( \Gamma \bowtie (X, \mu) \) is of
type $\mathrm{III}_1$ if and only if $\text{stc}(\alpha)$ is equivalent with the counting measure on $\mathbb{Z}\log \lambda$. Otherwise, $\Gamma \actson (X, \mu)$ is of type $\mathrm{III}_1$.

Theorems A and B will be deduced from the following more general result.

**Theorem 4.1.** For $i \in \{1, 2\}$, let $G_i$ be countable amenable groups with nonsingular actions $G_i \actson (Z_i, \zeta_i)$ on nontrivial standard probability spaces. Let $\Lambda_i$ be nonamenable groups. Put $\Gamma_i = G_i \ast \Lambda_i$ and define $\Gamma = G \ast (X_i, \mu_i)$ by (4.1). Write $M_i = L^\infty(X_i, \mu_i) \rtimes \Gamma_i$.

1. If $M_1 \cong M_2$, then $\text{stc}(\alpha_1) \sim \text{stc}(\alpha_2)$.

2. If $M_1$ embeds with expectation into $M_2$, there exists an atomic probability measure $\rho$ on $\mathbb{R}$ such that $\text{stc}(\alpha_1) \prec \rho * \text{stc}(\alpha_2)$.

We prove Theorem 4.1 by combining several ingredients: we first prove how the crossed product of an action of the form (4.1) can be embedded in a state preserving way into a generalized Bernoulli crossed product as studied in Theorem 2.1. We then combine Theorem 2.1 with the results of Section 3 to reach the conclusions of Theorem 4.1.

**Lemma 4.2.** Let $\Gamma \actson (X, \mu)$ be defined by (4.1). Define $P = L^\infty(Z, \zeta) \rtimes G$ and denote by $\omega$ the canonical crossed product state on $P$. There is a canonical state preserving embedding $\psi$ of $L^\infty(X, \mu) \rtimes \Gamma$ into the generalized Bernoulli crossed product $(P, \omega)^\Gamma \rtimes \Gamma$, and there is a state preserving conditional expectation of $(P, \omega)^\Gamma \rtimes \Gamma$ onto the image of $\psi$.

**Proof.** We denote by $(u_g)_{g \in G}$ the canonical unitary operators in the crossed products $L^\infty(X) \rtimes \Gamma$ and $(P, \omega)^\Gamma \rtimes \Gamma$. We denote by $(v_g)_{g \in G}$ the canonical unitary operators in the crossed product $L^\infty(Z) \rtimes G = P$. We denote by $\pi_0: (P, \omega) \to (P, \omega)^\Gamma$ the canonical embedding in coordinate $i_0 \in I$. Note that $\pi_0(P)$ commutes with $(u_g)_{g \in G}$ inside $(P, \omega)^\Gamma \rtimes \Gamma$. We denote by $\varphi$ the crossed product state on $L^\infty(X, \mu) \rtimes \Gamma$. We still denote by $\omega$ the natural state on $(P, \omega)^\Gamma \rtimes \Gamma$.

We can then define the state preserving embedding $\psi: L^\infty(X) \rtimes \Gamma \to (P, \omega)^\Gamma \rtimes \Gamma$ such that the restriction of $\psi$ to $L^\infty(X)$ is the canonical embedding of $L^\infty(X, \mu) = L^\infty((Z, \zeta)^\Gamma)$ into $(P, \omega)^\Gamma$ and such that

$$\psi(u_g) = u_g \text{ for all } g \in \Lambda, \text{ and } \psi(u_h) = \pi_0(v_h) u_h \text{ for all } h \in G.$$.

By construction, $\psi \circ \sigma^*_g = \sigma^*_g \circ \psi$. There thus exists a state preserving conditional expectation of $(P, \omega)^\Gamma \rtimes \Gamma$ onto the image of $\psi$.

By Lemma 4.2 any embedding with expectation of a crossed product von Neumann algebra $N$ into $L^\infty(X, \mu) \rtimes \Gamma$ will induce an embedding with expectation of $N$ into $(P, \omega)^\Gamma \rtimes \Gamma$. As a preparation to prove Theorem 4.1, we thus prove a general rigidity result for such embeddings into generalized Bernoulli crossed products $(P, \omega)^\Gamma \rtimes \Gamma$. We actually prove a very general result of this kind, which is of independent interest.

Let $\mathcal{R}$ be a nonsingular countable equivalence relation on a standard probability space $(X, \mu)$. In the context of the discussion above, $\mathcal{R}$ would be the orbit equivalence relation of a nonsingular action of the form (4.1), but we prove results for arbitrary equivalence relations $\mathcal{R}$. We denote by $\varphi_\mu$ the canonical faithful normal state on the von Neumann algebra $L(\mathcal{R})$. We assume that the centralizer $L(\mathcal{R})^{\varphi_\mu}$ is large, in the sense that it has no amenable direct summand. We prove that if $L(\mathcal{R})$ embeds with expectation $E$ into a noncommutative Bernoulli crossed product $(M, \omega)$ satisfying the appropriate conditions, then automatically $\varphi_\mu \circ E \prec \omega$, using the notation of Lemma 3.1.
Recall that a countable nonsingular equivalence relation $\mathcal{R}$ on a standard probability space $(X,\mu)$ is said to be purely infinite if for every nonnegligible Borel set $U \subset X$, the restriction $\mathcal{R}|_U$ does not admit an $\mathcal{R}$-invariant probability measure that is equivalent with $\mu$. When $\mathcal{R}$ is ergodic, this is equivalent to saying that $\mathcal{R}$ is of type III.

**Proposition 4.3.** Let $\mathcal{R}$ be a countable nonsingular equivalence relation on the standard probability space $(X,\mu)$. Assume that $\mathcal{R}$ is purely infinite and that the centralizer $L(\mathcal{R})^{\pi_\mu}$ has no amenable direct summand.

Let $(M,\omega)$ be the noncommutative generalized Bernoulli crossed product $M = (P,\omega)^I \rtimes \Gamma$, where $P$ is amenable, $\omega$ is a faithful normal state on $P$, $\Gamma$ is any countable group and the action $\Gamma \curvearrowright I$ has the properties that $\text{Stab}(i)$ is amenable for all $i \in I$ and that there exists $\kappa \in \mathbb{N}$ such that $\text{Stab}(J)$ is finite whenever $J \subset I$ satisfies $|J| \geq \kappa$.

If $\pi : L(\mathcal{R}) \to M$ is an embedding of $L(\mathcal{R})$ as a von Neumann subalgebra of $M$ admitting a faithful normal conditional expectation $E : M \to \pi(L(\mathcal{R}))$, then $\varphi_\mu \circ \pi^{-1} \circ E \prec_f \omega$.

**Proof.** We write $\varphi = \varphi_\mu$ and $N = L(\mathcal{R})$. We view $N$ as a von Neumann subalgebra of $M$ with the faithful normal conditional expectation $E : M \to N$. We still denote by $\varphi$ the faithful normal state $\varphi \circ E$ on $M$. By Lemma 3.1, we have to prove that $L_{\varphi}(\mathbb{R}) \prec_f L_\omega(\mathbb{R})$ inside the continuous core $c(M)$.

Take an arbitrary nonzero projection $p \in L_{\varphi}(\mathbb{R})$ of finite trace and write $A = L_{\varphi}(\mathbb{R})p$. We prove that $A \prec_f L_\omega(\mathbb{R})$. Note that $N_{\varphi}p$ commutes with $A$ and has no amenable direct summand. Denote $Q = A' \cap c(M)p$ and let $z \in Z(Q)$ be the maximal projection such that $Az \prec_f L_\omega(\mathbb{R})$. Put $z' = p - z$. Assume that $z' \neq 0$. We derive a contradiction. Write $c(L(\Gamma)) = L_\omega(\mathbb{R}) \vee L(\Gamma)$. By Theorem 2.1, $Q \prec c(L(\Gamma))$.

Write $B = (L^\infty(X) \vee L_{\varphi}(\mathbb{R}))p$. Since $L^\infty(X)$ commutes with $L_{\varphi}(\mathbb{R})$, we have that $B \subset Q$. Thus, $B \prec c(L(\Gamma))$. We prove that $L^\infty(X) \prec_M L(\Gamma)$. Assume the contrary. Denote by $E_{L(\Gamma)} : M \to L(\Gamma)$ the unique $\omega$-preserving conditional expectation. Since $L^\infty(X) \not\prec_M L(\Gamma)$, we can take a sequence of unitaries $w_n \in \mathcal{U}(L^\infty(X))$ such that $E_{L(\Gamma)}(x^* w_n y) \to 0$ $*$-strongly for all $x, y \in M$. We claim that

$$\|E_{c(L(\Gamma))}(x^* w_n y)\|_{2,\text{Tr}} \to 0$$

for all $x, y \in c(M)$ with $\text{Tr}(x^* x) < +\infty$ and $\text{Tr}(y^* y) < +\infty$.

By density, it suffices to prove this claim for $x = x_1 x_2$ and $y = y_1 y_2$ with $x_1, y_1 \in M$ and $x_2, y_2 \in L_\omega(\mathbb{R})$ with $\text{Tr}(x_2^* x_2) < +\infty$ and $\text{Tr}(y_2^* y_2) < +\infty$. But then,

$$E_{c(L(\Gamma))}(x^* w_n y) = x_2^* E_{L(\Gamma)}(x_1^* w_n y_1) y_2,$$

so that the claim follows. We find in particular that $\|E_{c(L(\Gamma))}(x^* w_n p y)\|_{2,\text{Tr}} \to 0$ for all $x, y \in c(M)$. Since $w_n p$ is a sequence of unitaries in $B$, this implies that $B \not\prec c(L(\Gamma))$, which is a contradiction. So, we have proven that $L^\infty(X) \not\prec_M L(\Gamma)$.

Take projections $e \in L^\infty(X)$, $q \in L(\Gamma)$, a nonzero partial isometry $v \in eMq$ and a normal unital $*$-homomorphism $\theta : L^\infty(X)e \to qL(\Gamma)q$ such that $av = v\theta(a)$ for all $a \in L^\infty(X)e$. Write $B_1 = L^\infty(X)e$. Since $v^* v$ commutes with $\theta(B_1)$, also the support projection of $E_{L(\Gamma)}(v^* v)$ commutes with $\theta(B_1)$. We may cut down with this projection and thus assume that the support projection of $E_{L(\Gamma)}(v^* v)$ equals $q$. Next, we may also replace $e$ by the support of the homomorphism $\theta$ and assume that $\theta$ is faithful.

Define $B_2 = \theta(B_1)$ and

$$D_2 = \{ u \in qMq \mid \exists \beta \in \text{Aut}(B_2), \forall b \in B_2 : ub = \beta(b)u \}^\prime.$$
Define $D_1 = N_{eM_1}(B_1)''$. Note that $eL(R)e \subset D_1$. Whenever $u \in U(eMe)$ normalizes $B_1$, we get that $v^*uv \in D_2$. Write $s = v^*v$. Thus, $s \in D_2$ and $v^*D_1v \subset sD_2s$. Also, $vv^* \in D_1$. Since $L(R)$ has no amenable direct summand and $L(R) \subset M$ is with expectation, we conclude that $sD_2s$ has no amenable direct summand. Let $z \in Z(D_2)$ be the central support of $s$ in $D_2$. Then $D_2z$ has no amenable direct summand. When $r \in B_2 \cap qL(\Gamma)q \subset D_2$ is a nonzero projection, since $q$ is equal to the support projection of $E_{L(\Gamma)}(s)$, we find that $rs \neq 0$. Thus, $rz \not\in 0$, so that $rD_2r$ is nonamenable. Since this holds for every choice of $r$, it follows from Proposition 2.2 that $D_2 \subset qL(\Gamma)q$. In particular, $L(R) \prec_M L(\Gamma)$. It follows that $L(R)$ has a direct summand that is finite, contradicting our assumption that $R$ is purely infinite. This final contradiction shows that $z' = 0$. So, $L(\varphi(R)) \prec_I L(\omega(R))$ and the proposition is proven.

**Proof of Theorem 4.1.** If $G_1 \curlyvee (Z_1, \zeta_1)$ is measure preserving, then stc$(\alpha_1) = \delta_0$ and there is nothing to prove. We may thus assume that $G_1 \curlyvee (Z_1, \zeta_1)$ is not measure preserving. As explained above, it follows that $\Gamma \curlyvee (X_1, \mu_1)$ is ergodic and of type III.

Denote by $\varphi_1$ the canonical crossed product state on $M_1$. Assume that $\pi : M_1 \to M_2$ is any embedding with expectation. By Lemma 4.2, we view $M_2$ as a von Neumann subalgebra of a generalized Bernoulli crossed product $N_2 = (P_2, \omega_2)^{\{i,j\}} \rtimes \Gamma_2$, where $I_2 = \Gamma_2/G_2$ and $P_2 = L^\infty(Z_2, \zeta_2) \rtimes G_2$, with crossed product state $\omega_2$ on $P_2$. We still denote by $\omega_2$ the natural state on $N_2$. There is a unique faithful normal conditional expectation $E : N_2 \to M_2$ satisfying $\omega_2 = \varphi_2 \circ E$. The action $\Gamma_2 \curvearrowright I_2 = \Gamma_2/G_2$ has amenable stabilizers and has the property that Stab$\{i, j\} = \{e\}$ when $i \neq j$.

We apply Proposition 4.3 to the orbit equivalence relation $R_1 = R(\Gamma_1 \curvearrowright X_1)$, which is ergodic and of type III. Note that $L(R_1)$ is canonically isomorphic with $M_1$ and the state $\varphi_{\mu_1}$ in Proposition 4.3 is equal to the canonical state $\varphi_1$ on the crossed product $M_1 = L^\infty(X_1, \mu_1) \rtimes \Gamma_1$. The centralizer $M^{\varphi_1}_1$ contains $L^\infty(X_1, \mu_1) \rtimes \Lambda_1$, which has trivial relative commutant in $M_1$. So, $M^{\varphi_1}_1$ is a nonamenable factor. By Proposition 4.3, we find that $\varphi_1 \circ \pi^{-1} \circ E \prec f \omega_2$. Using Proposition 3.2, we find an atomic probability measure $\rho$ such that

$$\text{class}(\varphi_1 \circ \pi^{-1} \circ E) \prec \rho \ast \text{class}(\omega_2).$$

Since $\text{class}(\varphi_1) \prec \text{class}(\varphi_1 \circ \pi^{-1} \circ E)$, we get that

$$\text{class}(\varphi_1) \prec \rho \ast \text{class}(\omega_2). \quad (4.2)$$

We now prove that $\text{class}(\varphi_1) = \text{stc}(\alpha_1)$ and $\text{class}(\omega_2) = \text{stc}(\alpha_2)$. By construction, for every $g \in \Gamma_1$, the measure $g \cdot \mu_1$ is of the form $g \cdot \mu_1 = \prod_{i \in \mathbb{F}} (g_i \cdot \zeta_1)$ with $g_i \in G_1$ and with all but finitely many $g_i$ equal to $e$. Moreover, any collection of such $g_i \in G_1$ can be realized by the appropriate choice of $g \in \Gamma_1$. Since class$(\varphi_1)$ is the join of the measure classes $(\log d(g \cdot \mu_1)/d\mu_1)_{\ast}(\mu_1)$, $g \in \Gamma_1$, it follows that class$(\varphi_1)$ is the join of all convolution products of $(\log d(g \cdot \zeta_1)/d\zeta_1)_{\ast}(\zeta_1)$, $g \in G_1$. This is precisely stc$(\alpha_1)$.

Secondly, class$(\omega_2)$ is the join of all convolution powers of class$(\omega_2|\mathcal{P}_2)$. Since class$(\omega_2|\mathcal{P}_2)$ is the join of the measure classes $(\log d(g \cdot \zeta_2)/d\zeta_2)_{\ast}(\zeta_2)$, $g \in G_2$, it follows that class$(\omega_2) = \text{stc}(\alpha_2)$. The second statement of the theorem thus follows from (4.2).

To prove the first statement of the theorem, assume that $\pi$ is a $\ast$-isomorphism between $M_1$ and $M_2$. By symmetry, it suffices to prove that stc$(\alpha_1) \prec \text{stc}(\alpha_2)$. We apply Lemma 3.3. Since $P_2 = L^\infty(Z_2) \rtimes G_2$, we may view

$$N_2 = (P_2, \omega_2)^{\{i,j\}} \rtimes \Gamma_2 = L^\infty((Z_2, \zeta_2)^{\{i,j\}}) \rtimes G_2,$$

where $G_2 = G_2|_{\Gamma_2} \Gamma_2 = G_2^{(\{i,j\})} \rtimes \Gamma_2$ is the generalized wreath product group that acts naturally on $(Z_2, \zeta_2)^{\{i,j\}} = (X_2, \mu_2)$. For every $g \in G_2$, we have that $u_g^* \sigma^\omega_2(u_g) \in L^\infty(X_2, \mu_2) \subset M_2 = \pi(M_1)$.  

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Since the linear span of \( u_g L^\infty(X,\mu) \), \( g \in G \), is dense in \( L^2(N,\omega) \), we certainly have the density of the linear span of \( u_g M_2 \), \( g \in G \).

Since \( \varphi_1 \circ \pi^{-1} \circ E < \omega_2 \), since \( \omega_2 = \varphi_2 \circ E \) and since \( M_1^{\pi} \) is a factor, it follows from Lemma 3.3 that \( \text{class}(\varphi_1) < \text{class}(\omega_2) \). We have proven above that \( \text{class}(\varphi_1) = \text{stc}(\alpha_1) \) and class \( \omega_2 = \text{stc}(\alpha_2) \). So the theorem is proven. \( \square \)

Theorem A is an immediate consequence of Theorem 4.1, as we show now.

**Proof of Theorem A.** Given equivalent probability measures \( \nu \sim \eta \) on a standard Borel space \( Y \) and given a nonnambale group \( \Lambda \), the nonsingular Bernoulli action of \( \Gamma = \mathbb{Z} \rtimes \Lambda \) given by (1.1) is isomorphic with the action defined by (4.1) associated with \( G = \mathbb{Z} \rtimes \alpha (Z,\zeta) = \prod_{n \in \mathbb{Z}} (Y, \mu_n) \), where \( \mu_n = \nu \) if \( n \in \mathbb{N} \) and \( \mu_n = \eta \) if \( n \in \mathbb{Z} \setminus \mathbb{N} \). Define \( \gamma = \text{stc}(\log d\nu/d\eta)_*(\nu) \).

By Theorem 4.1, it suffices to prove that \( \gamma \sim \text{stc}(\alpha) \). By definition, \( \text{stc}(\alpha) \) is the smallest stable measure class satisfying \( (\log d(n \cdot \zeta)/d\zeta)_*(\zeta) < \text{stc}(\alpha) \) for all \( n \in \mathbb{Z} \). The measure \( (\log d(n \cdot \zeta)/d\zeta)_*(\zeta) \) is equivalent with the \( |n| \)-fold convolution power of \( (\log d\nu/d\eta)_*(\nu) \) or its opposite, depending on the sign of \( n \). So, \( \text{stc}(\alpha) \sim \text{stc}(\log d\nu/d\eta)_*(\nu) = \gamma \). \( \square \)

To prove Theorem B, we use the following lemma about the relation between independent Borel sets and convolution products. This result is essentially contained in [LP97, Section 3], but we provide an elementary proof for convenience.

As mentioned in the introduction, recall that a Borel set \( K \subset \mathbb{R} \) is said to be independent if every finite subset \( F \subset K \) generates a free abelian subgroup of \( \mathbb{R} \) of rank \( |F| \). In other words, a Borel set \( K \subset \mathbb{R} \) is independent if and only if for every \( n \)-tuple of distinct elements \( x_1, \ldots, x_n \in K \), the homomorphism \( \mathbb{Z}^n \to \mathbb{R} : \lambda \mapsto \sum_{k=1}^n \lambda_k x_k \) is injective.

Also recall that given a measure class \( \mu \) on \( \mathbb{R} \), we define \( \bar{\mu} \) by \( \bar{\mu}(U) = \mu(-U) \) and we denote by \( \text{stc}(\mu) \) the join of the measure classes \( \mu^m \cdot \mu^m \), \( n, m \geq 0 \).

**Lemma 4.4.** Let \( K \subset \mathbb{R} \) be an independent Borel set. We decompose any probability measure \( \eta \) on \( \mathbb{R} \) as the sum \( \eta = \eta_a + \eta_c \) of an atomic and a nonatomic measure.

1. If \( \eta \) is a nonatomic probability measure on \( \mathbb{R} \), the set \( C = \{ x \in \mathbb{R} \mid \eta(x + K) > 0 \} \) is countable. For any probability measure \( \eta \) on \( \mathbb{R} \), we define the measure class \( \pi_K(\eta) \) on \( K \) by \( \pi_K(\eta) := \bigvee_{x \in C}(\delta_{x} * \eta)\|_{K} \).

2. Let \( \mu \) be a nonatomic probability measure on \( \mathbb{R} \) and let \( \eta \) be any probability measure on \( \mathbb{R} \). Denote by \( \eta_a \) the atomic part of \( \eta \). Then, \( \pi_K(\eta + \mu) = \pi_K(\eta_a + \mu) \). In particular, if also \( \eta \) is nonatomic, then \( \pi_K(\eta + \mu) = 0 \). If \( \eta_a \neq 0 \), we conclude that \( \pi_K(\eta + \mu) \sim \pi_K(\mu) \).

3. For every probability measure \( \mu \) on \( \mathbb{R} \) and every atomic probability measure \( \rho \) on \( \mathbb{R} \), we have that \( \pi_K(\rho * \text{stc}(\mu)) = \pi_K(\mu) \vee \pi_K(\bar{\mu}) \).

4. If \( x \in \mathbb{R} \) and \( \mu \) is a probability measure with \( \mu(\mathbb{R} - K) = 0 \), then \( \pi_K(\delta_x * \mu) = \mu_c \) and \( \pi_K(\delta_x * \bar{\mu}) = 0 \).

**Proof.** We first prove the following two claims.

(i) If \( x \in \mathbb{R} \setminus \{0\} \), then \( (x + K) \cap K \) has at most one element.

(ii) If \( x \in \mathbb{R} \), then \( (x - K) \cap K \) has at most two elements.
To prove (i), if \((x + K) \cap K \) is nonempty and \(x \neq 0\), we can write \(x = a - b\) with \(a, b \in K\) and \(a \neq b\). Since \(K\) is independent, one deduces that \((x + K) \cap K = \{a\}\). To prove (ii), if \((x - K) \cap K \) is nonempty, we can write \(x = a + b\) with \(a, b \in K\). Since \(K\) is independent, one deduces that \((x - K) \cap K = \{a, b\}\).

1. Fix a nonatomic probability measure \(\eta\) on \(\mathbb{R}\). If \(x \neq y\), it follows from (i) that \(\eta((x + K) \cap (y + K)) = 0\). Since \(\eta\) is a finite measure, the set \(C = \{x \in \mathbb{R} \mid \eta(x + K) > 0\}\) is countable.

2. Fix \(x \in \mathbb{R}\). Then, \((\eta \ast \mu)(x + K) = \int_{\mathbb{R}} \mu(-y + x + K) \, d\eta(y)\). There are only countable many \(y \in \mathbb{R}\) such that \(\mu(-y + x + K) > 0\). Thus,

\[
(\eta \ast \mu)(x + K) = \int_{\mathbb{R}} \mu(-y + x + K) \, d\eta(y) = 0.
\]

This holds for every \(x \in \mathbb{R}\), so that \(\pi_K(\eta \ast \mu) = 0\). It follows that \(\pi_K(\eta \ast \mu) \sim \pi_K(\eta_h \ast \mu)\).

By definition, \(\pi_K(\delta_x \ast \mu) \sim \pi_K(\mu)\) for every \(x \in \mathbb{R}\). Then also \(\pi_K(\eta_h \ast \mu) \sim \pi_K(\mu)\), whenever \(\eta_h \neq 0\).

3. Write \(\rho_1 = \rho \ast \text{stc}(\mu_a)\). Then, \(\rho \ast \text{stc}(\mu) \sim \rho_1 \ast \text{stc}(\mu_c)\). By 2, we know that \(\pi_K(\rho_1 \ast \mu_c^m \ast \tilde{\mu}_c^m) = 0\) when \(n + m \geq 2\). Since \(\rho_1\) is atomic, also \(\pi_K(\rho_1 \ast \delta_0) = 0\). Therefore,

\[
\pi_K(\rho \ast \text{stc}(\mu)) \sim \pi_K(\rho_1 \ast \text{stc}(\mu_c)) \sim \pi_K(\rho_1 \ast \mu_c) \vee \pi_K(\rho_1 \ast \tilde{\mu}_c) \sim \pi_K(\mu_c) \vee \pi_K(\tilde{\mu}_c).
\]

4. By definition, \(\pi_K(\delta_x \ast \mu) \sim \pi_K(\delta_x \ast \mu_c)\). When \(y \neq x\), it follows from (i) that \((\delta_x \ast \mu_c)(y + K) = 0\). Also, \(\delta_x \ast \mu_c\) is supported on \(x + K\). Thus, \(\pi_K(\delta_x \ast \mu) \sim \delta_x \ast \delta_x \ast \mu_c = \mu_c\).

We also have \(\pi_K(\delta_x \ast \tilde{\mu}_c) \sim \pi_K(\delta_x \ast \tilde{\mu}_c)\). By (ii), for every \(y \in \mathbb{R}\), we have

\[
(\delta_x \ast \tilde{\mu}_c)(y + K) = \tilde{\mu}_c(-x + y + K) = \mu_c(x - y - K) = \mu_c(K \cap (x - y - K)) = 0.
\]

This holds for all \(y \in \mathbb{R}\) and thus, \(\pi_K(\delta_x \ast \tilde{\mu}_c) = 0\).

We can then deduce the following consequence of Theorem 4.1.

**Corollary 4.5.** For \(i \in \{1, 2\}\), let \(\nu_i \sim \eta_i\) be equivalent probability measures on a standard Borel space \(Y_i\). Let \(\Lambda_i\) be nonatomic groups. Denote \(\Gamma_i = \mathbb{Z} \rtimes \Lambda_i\) and consider the nonsingular Bernoulli action \(\Gamma_i \rtimes (X_i, \mu)\) given by (1.1). Write \(\sigma_i = (\log d\nu_i/d\eta_i)_*(\nu_i)\). Denote \(M_i = L^\infty(X_i) \rtimes \Gamma_i\). Let \(K \subset \mathbb{R}\) be an independent Borel set and use the notation \(\pi_K\) introduced in Lemma 4.4.

1. If \(M_1 \cong M_2\), then \(\pi_K(\sigma_1) \vee \pi_K(\tilde{\sigma}_1) \sim \pi_K(\sigma_2) \vee \pi_K(\tilde{\sigma}_2)\).

2. If \(M_1\) admits an embedding with expectation into \(M_2\), then \(\pi_K(\sigma_1) \vee \pi_K(\tilde{\sigma}_1) \prec \pi_K(\sigma_2) \vee \pi_K(\tilde{\sigma}_2)\).

**Proof.** Considering \(G_i = \mathbb{Z} \rtimes \alpha_i\) \((Z_i, \zeta_i) = \prod_{n \in \mathbb{Z}} (Y_i, \mu_{i,n})\), where \(\mu_{i,n} = \nu_i\) if \(n \in \mathbb{N}\) and \(\mu_{i,n} = \eta_i\) if \(n \in \mathbb{Z} \setminus \mathbb{N}\), we have seen in the proof of Theorem A that \(\text{stc}(\alpha_i) \sim \text{stc}(\sigma_i)\).

If \(M_1\) admits an embedding with expectation into \(M_2\), Theorem 4.1 provides an atomic probability measure \(\rho\) such that \(\text{stc}(\alpha_1) \prec \rho \ast \text{stc}(\sigma_2)\). Thus, \(\text{stc}(\sigma_1) \prec \rho \ast \text{stc}(\sigma_2)\). Applying \(\pi_K\) and using Lemma 4.4.3, we conclude that \(\pi_K(\sigma_1) \vee \pi_K(\tilde{\sigma}_1) \prec \pi_K(\sigma_2) \vee \pi_K(\tilde{\sigma}_2)\). If \(M_1 \cong M_2\), also the converse absolute continuity holds.

**Proof of Theorem B.** In the context of Theorem B, the measure \(\sigma_i = (\log d\nu_i/d\eta_i)_*(\nu_i)\) is a translate of \(\nu_i\) and \(\nu_i\) is supported on \(K\). By Lemma 4.4.4, we get that \(\pi_K(\sigma_1) \sim \nu_i\) and \(\pi_K(\tilde{\sigma}_1) = 0\). The result thus follows from Corollary 4.5. 

\[
\square
\]
Example 4.6. Fix a compact independent set $K \subset \mathbb{R}$ such that $K$ is homeomorphic to a Cantor set (see e.g. [Rud62, Theorems 5.1.4 and 5.2.2]). Fix a countable nonamenable group $\Lambda$ and put $\Gamma = \mathbb{Z} \rtimes \Lambda$. Put $Y = [0, 1) \cup K$.

Given any nonatomic probability measure $\rho$ on $K$, define the probability measure $\nu_\rho$ on $Y$ as $(\lambda + \rho)/2$, where $\lambda$ is the Lebesgue measure on $[0, 1]$. Define the probability measure $\eta_\rho$ on $Y$ by normalizing $\exp(-y) \, d\nu_\rho(y)$. Consider the associated nonsingular Bernoulli action $\Gamma \ltimes (X, \mu_\rho)$ given by (1.1), with crossed product factor $M_\rho = L^\infty(X, \mu_\rho) \rtimes_{\alpha_\rho} \Gamma$.

1. $M_\rho$ is a full factor of type III and Connes $\tau$-invariant of $M_\rho$ is the usual topology on $\mathbb{R}$. When $\Lambda$ has infinite conjugacy classes and is not inner amenable, this was proven in [VW17, Proposition 7.1], but the result holds by only assuming that $\Lambda$ is nonamenable, as we show in Lemma 4.7 below.

2. Let $\rho$ and $\rho'$ be nonatomic probability measures on $K$. If $M_\rho \cong M_{\rho'}$, then $\rho \sim \rho'$. If $M_\rho$ admits an embedding with expectation into $M_{\rho'}$, then $\rho \prec \rho'$. Both statements follow from Corollary 4.5: given a nonatomic probability measure $\rho$ on $K$, we have that $(\log d\nu_\rho/d\nu_{\rho'})_\tau(x)$ is a translate of $\nu_\rho$. Since $\rho$ is supported on $K$ and since $\lambda(x + K) = 0$ for every $x \in \mathbb{R}$, it follows from Lemma 4.4 that $\pi_K(\nu_\rho) \vee \pi_K(\nu_{\rho'}) \sim \rho$.

For completeness, we include a proof for the following result, which was proven in [VW17, Proposition 7.1] under the stronger assumption that $\Lambda$ has infinite conjugacy classes and is not inner amenable.

Lemma 4.7. Let $\nu \sim \eta$ be equivalent probability measures on a standard Borel space $Y$. Assume that $\nu$ and $\eta$ are not concentrated on a single atom. Let $\Lambda$ be a countable nonamenable group. Write $\Gamma = \mathbb{Z} \rtimes \Lambda$ and define the nonsingular Bernoulli action $\Gamma \ltimes (X, \mu)$ by (1.1). Then, the factor $M = L^\infty(X, \mu) \rtimes \Gamma$ is full and the $\tau$-invariant is the weakest topology on $\mathbb{R}$ that makes the map

$$\mathbb{R} \to \mathcal{U}(L^\infty(Y, \nu)) : t \mapsto \left(\frac{d\nu}{d\eta}\right)^it$$

continuous, where $\mathcal{U}(L^\infty(Y, \nu))$ is equipped with the strong topology.

Proof. Denote by $\varphi$ the canonical crossed product state on $M$. Write $Q = L^\infty(X, \mu) \rtimes \Lambda$. As in the proof of [VW17, Proposition 7.1], it suffices to show the following: if $x_n \in \mathcal{U}(M)$ is a sequence of unitaries such that $x_n a - ax_n \to 0$ $\ast$-strongly for every $a \in Q$, then $x_n - \varphi(x_n)1 \to 0$ $\ast$-strongly. Note that $Q \subset M^\mathbb{R}$. There thus exists a unique $\varphi$-preserving conditional expectation $E : M \to L(\Lambda)$. In the proof of [VW17, Proposition 7.1], it is shown that $x_n - E(x_n) \to 0$ $\ast$-strongly. Writing $y_n = E(x_n)$, we have found a bounded sequence in $L(\Lambda)$ satisfying $y_n a - ay_n \to 0$ $\ast$-strongly for every $a \in Q$.

For every $h \in \Lambda$, denote by $\pi_h : L^\infty(Y, \eta) \to L^\infty(X, \mu)$ the state preserving embedding as the $h$-th coordinate. Choose an element $a \in L^\infty(Y, \eta)$ such that $\int_Y |a|^2 \, d\eta = 1$ and $\int_Y a \, d\eta = 0$. Denote by $(y_n)_g \in \Lambda$ the canonical unitaries, so that $(y_n)_g := \varphi(y_n u^*_g)$ are the canonical Fourier coefficients. We also write $\|x\|_{\varphi} = \varphi(x^*x)^{1/2}$ for all $x \in M$. Since $\pi_e(a)$ and $\pi_h(a)$ are orthogonal in $L^2(M, \varphi)$ for all $h \neq e$, a direct computation shows that

$$\|y_n \pi_e(a) - \pi_e(a) y_n\|_{\varphi}^2 \geq 2 \sum_{h \in \Lambda \setminus \{e\}} |(y_n)_h|^2 = 2 \|y_n - \varphi(y_n)1\|_2^2.$$

Therefore, $y_n - \varphi(y_n)1 \to 0$ $\ast$-strongly. Then also $x_n - \varphi(x_n)1 \to 0$ $\ast$-strongly. \qed
5 Solidity of nonsingular Bernoulli actions, proof of Theorem C

Recall from [Oza03, VV05] that a von Neumann algebra $M$ is called solid if for every diffuse von Neumann subalgebra with expectation $A \subset M$, the relative commutant $A' \cap M$ is amenable. If $B \subset M$ is a von Neumann subalgebra with expectation, recall from [Mar16] that $M$ is said to be solid relative to $B$ if the following holds: for every nonzero projection $p \in M$ and nonamenable von Neumann subalgebra $Q \subset \bar{p}M\bar{p}$ with expectation and with diffuse center $Z(Q)$, we have that $Q \prec_M B$.

Recall from [CI08] that a countable nonsingular equivalence relation $R$ on a standard probability space $(X, \mu)$ is called solid if for every Borel subequivalence relation $S \subset R$, there exists a partition of $X$, up to measure zero, into $S$-invariant Borel subsets $(X_n)_{n \geq 0}$ such that $S|_{X_0}$ is amenable and $S|_{X_n}$ is ergodic for all $n \geq 1$. This is equivalent to saying that for every diffuse von Neumann subalgebra $A \subset L^\infty(X)$, the relative commutant $A' \cap L(R)$ is amenable. Finally, an essentially free nonsingular action $\Gamma \curvearrowright (X, \mu)$ is said to be a solid action if the orbit equivalence relation $R(\Gamma \curvearrowright X)$ is solid.

**Definition 5.1.** A faithful normal state $\varphi$ on a von Neumann algebra $M$ is said to be a solid state if for every faithful normal state $\psi$ on $M$ with nonamenable centralizer $M^\psi$, we have that $\psi ≪ \varphi$.

Theorem C is a special case of the following result.

**Theorem 5.2.** Let $G$ be a countable amenable group and let $G \curvearrowright (Z, \zeta)$ be a nonsingular action on a nontrivial standard probability space. Let $\Lambda$ be a countable nonamenable group. Define $\Gamma = G * \Lambda$ and let $\Gamma \curvearrowright (X, \mu)$ be defined by (4.1). Put $M = L^\infty(X, \mu) \rtimes \Gamma$.

1. $\Gamma \curvearrowright (X, \mu)$ is a solid action.
2. The factor $M$ is solid relative to $L(\Lambda)$.
3. If $\Lambda$ is biexact, then $M$ is solid.
4. If $\Lambda$ is biexact and $(\log d(g \cdot \zeta)/d\zeta)_* (\zeta)$ is nonatomic for every $g \in G \setminus \{e\}$, then the crossed product state $\varphi_\mu$ on $M$ is a solid state.

We first prove the following lemma, in which we also introduce some of the notation that will be used in the proof of Theorem 5.2.

**Lemma 5.3.** Make the same assumptions as in Theorem 5.2. Write $I = (G * \Lambda)/G$. Denote by $\varphi$ the crossed product state on $M$ induced by $\mu$. For every $a \in G^{(1)}$, denote by $\mu_a \sim \mu$ the probability measure on $X$ defined by $\mu_a = \prod_{i \in I} a_i \cdot \zeta$. Denote by $\varphi_a$ the corresponding crossed product state on $M$.

Let $p \in c(M)$ be a projection of finite trace. Let $A \subset p c(M)p$ be a von Neumann subalgebra whose relative commutant $A' \cap p c(M)p$ has no amenable direct summand. Then one of the following statements holds.

1. $A \prec_{c(M)} L(\Lambda) \vee L(\varphi(G))$.
2. $A \prec_{c(M)} L(\varphi_a(G))$ for some $a \in G^{(1)}$.

If moreover $\Lambda$ is biexact, then the second statement always holds.

**Proof.** Denote $P = L^\infty(Z, \zeta) \rtimes G$ and let $\omega$ be the crossed product state on $P$ given by $\zeta$. Write $(N, \omega) = (P, \omega)^f$ and $N = N \rtimes \Gamma$. We still denote by $\omega$ the crossed product state on
\( N \). Denote by \( \theta : M \to N \) the embedding given by Lemma 4.2. Note that there is a unique faithful normal conditional expectation \( E : N \to \theta(M) \) such that \( \varphi \circ \theta^{-1} \circ E = \omega \). For the rest of the proof, we view \( M \) as a subalgebra of \( N \) and no longer write the canonical embedding \( \theta \). We then also replace the notation \( \omega \) by \( \varphi \).

The conditional expectation \( E \) induces an embedding \( c(M) \subset c(N) \). Write \( B = L_\varphi(\mathbb{R}) \) and \( Q = N_{p(c(N))}(A)^\vee \). Since \( A' \cap p(c(N))p \supset A' \cap p(c(M))p \) has no amenable direct summand, by Theorem 2.1, we can take a projection \( z \in Z(Q) \) such that \( Az \prec f_{c(N)} B \) and, with \( z' = p - z \), we have \( Qz' \prec_{f,c(N)} B \vee L(\Gamma) \).

Assume that \( z' \neq 0 \). We prove that \( Az' \prec_{c(N)} B \vee L(\Lambda) \). Assume the contrary. In particular, \( Qz' \not\prec_{c(N)} B \vee L(\Lambda) \). We may view \( B \vee L(\Gamma) \) as the tensor product \( L(G \rtimes \Lambda) \otimes B \) and hence, also as the amalgamated free product \( (L(G) \overline{\otimes} B) * B \). Since \( Q \) contains the commuting subalgebras \( A \) and \( A' \cap p(c(N))p \), and since \( A' \cap p(c(N))p \) has no amenable direct summand, our assumption says in particular that \( Az' \not\prec_{c(N)} B \vee L(\Lambda) \) and [CH08, Theorem 4.2] imply that \( Az' \prec_{c(N)} B \vee L(G) \). Our assumption says in particular that \( Az' \not\prec_{c(N)} B \), so that [CH08, Theorem 2.4] implies that \( A' \cap p(c(N))p \not\prec_{c(N)} B \vee L(G) \). This is a contradiction because \( B \vee L(G) \) is amenable, while \( A' \cap p(c(N))p \) has no amenable direct summand.

Since \( z \) and \( z' \) cannot be both equal to zero, we have proven that \( A \prec_{c(N)} L(\Lambda) \vee L_\varphi(\mathbb{R}) \). To conclude the proof of the lemma, we assume that none of the two statements in the lemma hold and prove that \( A \not\prec_{c(N)} L(\Lambda) \vee L_\varphi(\mathbb{R}) \). Since the two statements in the lemma do not hold, we can choose a sequence of unitaries \( w_n \in \mathcal{U}(A) \) such that

\[ \| E_{L(\Lambda) \vee L_\varphi(\mathbb{R})}(x^* w_n y) \|_{2,Tr} \to 0 \quad \text{and} \quad \| E_{L_\varphi(\mathbb{R})}(x^* w_n y) \|_{2,Tr} \to 0 \quad (5.1) \]

for all \( a \in G(\Gamma) \) and \( x, y \in p(c(M)) \). Here, all conditional expectations are the unique trace preserving ones. It suffices to prove that

\[ \| E_{L(\Lambda) \vee L_\varphi(\mathbb{R})}(x^* w_n y) \|_{2,Tr} \to 0 \quad \text{for all} \quad x, y \in p(c(N)), \quad (5.2) \]

As in the proof of Theorem 4.1, we denote by \( \Gamma \rtimes^\alpha G(\Gamma) \) the natural action by translation, define the generalized wreath product group \( G = G \rtimes^\alpha \Gamma = G(\Gamma) \rtimes \alpha \Gamma \) and view \( N \) as the crossed product \( L^\infty(X) \rtimes G \). Define \( i_0 \in I \) as the coset \( i_0 = G \). Under this identification, \( M = L^\infty(X) \rtimes \theta(\Gamma) \), where \( \theta : \Gamma \to G \) is the injective group homomorphism uniquely determined by \( \theta(g) = \pi_{i_0}(g)g \) and \( \theta(\lambda) = \lambda \) for all \( g \in G \), \( \lambda \in \Lambda \). In particular, every \( a \in G(\Gamma) \) gives rise to a canonical unitary \( u_a \in N \). We have \( \varphi_a \circ E = \varphi \circ Ad u_a \), so that \( L_{\varphi_a}(\mathbb{R}) = u_a L_\varphi(\mathbb{R}) u_a^* \). By density, it suffices to prove (5.2) for \( x = x_0 u_a \) and \( y = y_0 u_b \) with \( x_0, y_0 \in p(c(M)) \) and \( a, b \in G(\Gamma) \).

If \( a = b = e \), then (5.2) follows immediately from (5.1). When \( a \) and \( b \) are not both equal to \( e \), the set \( F = \{ \lambda \in \Lambda \mid \alpha(\Lambda) = a \} \) is finite. Also, for \( g \in \Gamma \), we have that \( a^{-1} \theta(g)b \in \theta(\Lambda) = \Lambda \) if and only if \( g \in F \). So, if \( F \neq \emptyset \), we find that

\[ E_{L(\Lambda) \vee L_\varphi(\mathbb{R})}(u_a x_0^* w_n y_0 u_b) = 0 \]

for all \( n \in \mathbb{N} \). When \( F \neq \emptyset \), a direct computation shows that for every \( x_1 \in c(M) \),

\[ E_{L(\Lambda) \vee L_\varphi(\mathbb{R})}(u_a x_1 u_b) = \sum_{\lambda \in F} u_a^* E_{L_{\varphi_a}(\mathbb{R})}(x_1^* u_a^* \lambda u_a u_b). \]

Since for every \( \lambda \in F \), we have by (5.1) that \( \| E_{L_{\varphi_a}(\mathbb{R})}(x_1^* w_n y_0 u_a^* \lambda) \|_{2,Tr} \to 0 \), again (5.2) follows.

So the first part of the lemma is proven.

Finally assume that \( \Lambda \) is moreover biexact. Above, we have seen that \( Qz' \prec_{c(N)} B \vee L(\Lambda) \). We can view \( B \vee L(\Lambda) \) as the tensor product \( L(\Lambda) \overline{\otimes} B \), where \( B \) is abelian. By [OP03, Propositions 11 and 12], any von Neumann subalgebra \( D \) of a corner of \( L(\Lambda) \overline{\otimes} B \) having a nonamenable
relative commutant, intertwines into $B$. So we find that $A' = \curlyvee_{(\Lambda)} B$. Since $z$ and $z'$ cannot be both equal to zero, we get that $A = \curlyvee_{(\Lambda)} L_{\varphi}(\mathbb{R})$. The argument above then shows that 

$A = \curlyvee_{(\Lambda)} L_{\varphi_s}(\mathbb{R})$ for some $a \in G^{(I)}$. 

\textbf{Proof of Theorem 5.2.} We start by proving that $M$ is solid relative to $L(\Lambda)$. It suffices to prove the following statement: if $e \in M$ is a projection and $A \subseteq eMe$ is a diffuse abelian von Neumann subalgebra with expectation such that the relative commutant $Q = A' \cap eMe$ has no amenable direct summand, then $Q \prec_M L(\Lambda)$. Fix a faithful normal conditional expectation $E : eMe \to A$ and choose a faithful normal state $\psi$ on $A$. We still denote by $\psi$ the state $\psi \circ E$ on $eMe$. We identify $c(eMe) = e c(M)e$. Fix a nonzero projection $p \in L_{\psi}(\mathbb{R})$ of finite trace. Then, $Ap$ and $pc_{\psi}(Q)p$ are commuting von Neumann subalgebras of $pc(M)p$ and $pc_{\psi}(Q)p$ has no amenable direct summand.

By Lemma 5.3 and using the notation introduced in that lemma, one of the following statements holds.

- $Ap \prec_{c(M)} L(\Lambda) \vee L_{\varphi}(\mathbb{R})$.
- $Ap \prec_{c(M)} L_{\varphi_\alpha}(\mathbb{R})$ for some $a \in G^{(I)}$.

We claim that the second statement does not hold. Since $A$ is diffuse, we can choose a sequence $w_n \in \mathcal{U}(A)$ such that $w_n \to 0$ weakly. Whenever $x_0, y_0 \in M$ and $x_1, y_1 \in L_{\varphi_\alpha}(\mathbb{R})$, we get that

$$E_{L_{\varphi_\alpha}(\mathbb{R})}(x_1^* x_0^* w_n y_0 y_1) = x_1^* \varphi_\alpha(x_0^* w_n y_0) y_1.$$ 

Since $w_n \to 0$ weakly, it follows by density that $\|E_{L_{\varphi_\alpha}(\mathbb{R})}(x^* w_n y)\|_{2, Tr} \to 0$ for all $x, y \in c(M)$ with $\text{Tr}(x^* x) < +\infty$ and $\text{Tr}(y^* y) < +\infty$. In particular, $\|E_{L_{\varphi_\alpha}(\mathbb{R})}(x^* w_n y)\|_{2, Tr} \to 0$ for all $x, y \in c(M)$. So, the claim is proven. It follows that $Ap \prec_{c(M)} L(\Lambda) \vee L_{\varphi}(\mathbb{R})$.

We now claim that $A \prec_M L(\Lambda)$. Assume the contrary. Denote by $E_{L(\Lambda)} : M \to L(\Lambda)$ the unique $\varphi$-preserving conditional expectation. Since $A$ is abelian and $A \not\prec_M L(\Lambda)$, we can take a sequence of unitaries $w_n \in \mathcal{U}(A)$ such that $E_{L(\Lambda)}(x^* w_n y) \to 0$ $*$-strongly, for all $x, y \in M$. If now $x_0, y_0 \in M$ and $x_1, y_1 \in L_{\varphi}(\mathbb{R})$, we get that

$$E_{L(\Lambda) \vee L_{\varphi}(\mathbb{R})}(x_1^* x_0^* w_n y_0 y_1) = x_1^* E_{L(\Lambda)}(x_0^* w_n y_0) y_1.$$ 

By density, we get that $\|E_{L(\Lambda) \vee L_{\varphi}(\mathbb{R})}(x^* w_n y)\|_{2, Tr}$ for all $x, y \in c(M)$ with $\text{Tr}(x^* x) < +\infty$ and $\text{Tr}(y^* y) < +\infty$. In particular, $\|E_{L(\Lambda) \vee L_{\varphi}(\mathbb{R})}(x^* w_n y)\|_{2, Tr} \to 0$ for all $x, y \in c(M)$. This contradicts the statement that $Ap \prec_{c(M)} L(\Lambda) \vee L_{\varphi}(\mathbb{R})$. So, the claim that $A \prec_M L(\Lambda)$ is proven.

Choose projections $r \in M$ and $s \in L(\Lambda)$, a nonzero partial isometry $v \in rM$s and a unit norm $*$-homomorphism $\theta : rAr \to sL(\Lambda)s$ such that $av = \theta(a)$ for all $a \in rAr$. Denote $D = \theta(rAr)' \cap sMs$. Let $\Theta : M \to P^I \rtimes \Gamma$ be the embedding given by Lemma 4.2. Then, $\Theta(\theta(rAr))$ is a diffuse von Neumann subalgebra of a corner of $L(\Lambda)$. Since $\Lambda \cap \text{Stab} i = \{e\}$ for every $i \in I$ and $\Theta(\theta(rAr))$ is diffuse, we get for every $i \in I$ that $\Theta(\theta(rAr)) \not\subseteq L(\Gamma) \cap (\text{Stab} i)$. It then follows from Proposition 2.2 that $\Theta(D) \subseteq L(\Gamma)$. Since $\Theta(M) \cap L(\Gamma) = L(\Lambda)$, we conclude that $D \subseteq sL(\Lambda)s$. In particular $s_1 = v^* v$ belongs to $L(\Lambda)$. By construction, $v^* Q v \subseteq D$ and $r_1 = vv^*$ belongs to $Q$. In particular, $Q \prec_M L(\Lambda)$. So we have proven that $M$ is solid relative to $L(\Lambda)$.

If $\Lambda$ is biexact, then $L(\Lambda)$ is solid by [Oza03]. Since $M$ is solid relative to $L(\Lambda)$, it then follows that $M$ is solid, when $\Lambda$ is biexact.

We next prove that $\Gamma \cap (X, \mu)$ is a solid action. Choose a diffuse von Neumann subalgebra $A \subset L^\infty(X)$. We have to prove that $A' \cap M$ is amenable. Assume that $A' \cap M$ is nonamenable.
Since $L^\infty(X) \subset M$ is an inclusion with expectation, also $A \subset M$ and $A' \cap M$ are inclusions with expectation. Since $A \subset \mathcal{Z}(A' \cap M)$ and $M$ is solid relative to $L(\Lambda)$, we find that $A' \cap M \prec_M L(\Lambda)$. A fortiori, $A \prec_M L(\Lambda)$. On the other hand, since $A$ is diffuse, we can take a sequence of unitaries $w_n \in \mathcal{U}(A)$ such that $w_n \to 0$ weakly. For all $g, h \in \Gamma$ and $x, y \in L^\infty(X)$, we have

$$E_{L(\Lambda)}(u^*_g x^* w_n y u_h) = \begin{cases} \varphi(x^* w_n y) \, u^*_g u_h & \text{if } g^{-1}h \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

The conditional expectation $E_{L(\Lambda)}$ is $\varphi$-preserving and the restriction of $\varphi$ to $L(\Lambda)$ is the canonical trace on $L(\Lambda)$. By density, we find that $\|E_{L(\Lambda)}(x^* w_n y)\|_2 \to 0$ for all $x, y \in M$. So, $A \not\prec_M L(\Lambda)$. This contradiction concludes the proof that $\Gamma \cap (X, \mu)$ is a solid action.

Finally assume that $A$ is biexact and that $(\log d(g, \cdot)/d\zeta)_*(\zeta)$ is nonatomic for every $g \in G \setminus \{e\}$. Let $\psi$ be a faithful normal state on $M$ such that $M^{\psi}$ is nonamenable. Take a nonzero central projection $e \in \mathcal{Z}(M^{\psi})$ such that $M^{\psi} e$ has no amenable direct summand. Fix a finite trace projection $q \in L_\psi(\mathbb{R}) \subset c(M)$ such that the projection $p = eq$ is nonzero. Then, $L_\psi(\mathbb{R}) p$ is a von Neumann subalgebra of $pc(M)^p$ whose relative commutant contains $M^{\psi} p$ and hence, has no amenable direct summand. By the second part of Lemma 5.3, we find $a \in G^{(I)}$ such that $L_\psi(\mathbb{R}) p \prec_{c(M)} L^\varphi_a(\mathbb{R})$. By Lemma 3.1, it follows that $\psi \prec \varphi_a$. In particular, $\varphi_a$ has a nonamenable centralizer.

Denote by $\Gamma \cap \alpha G^{(I)}$ the action by translation. Let $i_0 \in I$ be the coset $G$. Denote by $\pi_i : G \to G^{(I)}$ the embedding in the $i$th coordinate. A map $c : \Gamma \to G^{(I)}$ is called an $\alpha$-cocycle if $c(gh) = c(g) \alpha(g)(c(h))$ for all $g, h \in \Gamma$. Let $c : \Gamma \to G^{(I)}$ be the unique $\alpha$-cocycle satisfying $c(g) = \pi_{i_0}(g)$ for all $g \in G$ and $c(\lambda) = e$ for all $\lambda \in \Lambda$. A direct computation gives that $g \cdot \mu_a = \mu_{c(g) \alpha(g)(a)}$ for all $g \in \Gamma$. Define the subgroup $L \subset \Gamma$ by

$$L = \{ g \in \Gamma \mid c(g) = a \alpha_g(a^{-1}) \}.$$ 

Since $(\log d(g, \cdot)/d\zeta)_*(\zeta)$ is nonatomic for every $g \in G \setminus \{e\}$, we get that $(d\mu_{i_0}/d\mu)(x) \neq 1$ for a.e. $x \in X$ and all $b \neq c$ in $G^{(I)}$. It follows that $(d(g, \cdot)/d\mu)(x) \neq 1$ for a.e. $x \in X$ and all $g \in \Gamma \setminus L$. It follows that $M^{\varphi_a}$ is a von Neumann subalgebra with expectation of $L^\infty(X) \times L$. Since $M^{\varphi_a}$ is nonamenable, we conclude that $L$ is a nonamenable group.

For every $b \in G^{(I)}$, we denote $|b| = \# \{ i \in I \mid b_i \neq e \}$. We also define for every $g \in \Gamma = G \ast \Lambda$, the $G$-length $|g|_G$ as the minimal number of elements in $G$ one needs when writing $g$ as a product of elements in $G$ and elements in $\Lambda$. Thus, $|\lambda g|_G = 0$ for all $\lambda \in \Lambda$ and $|g|_G = n$ whenever $g = \lambda_0 g_1 \lambda_1 \cdots \lambda_{n-1} g_n \lambda_n$ with $g_i \in G \setminus \{e\}$ for all $i$, $\lambda_i \in \Lambda \setminus \{e\}$ for all $i \in \{1, \ldots, n-1\}$ and $\lambda_0, \lambda_n \in \Lambda$. Another direct computation shows that $|c(g)| = |g|_G$ for all $g \in \Gamma$. For all $g \in L$, we have that $|g|_G = |c(g)| = |a \alpha_g(a^{-1})| \leq 2 |a|$. Hence, $g \mapsto |g|_G$ is bounded on $L$. Since $L$ is nonamenable, this implies that $L = g_0 \Lambda_0 g_0^{-1}$ for some $g_0 \in \Gamma$, where $\Lambda_0 \subset \Lambda$ is a nonamenable subgroup. Write $b = c(g_0)$. Since $c(\lambda) = e$ for all $\lambda \in \Lambda$, we get that $c(g) = b \alpha_g(b^{-1})$ for all $g \in g_0 \Lambda_0 g_0^{-1}$. Hence, $a \alpha_g(a^{-1}) = b \alpha_g(b^{-1})$ for all $g \in L$. This means that $\alpha_g(b^{-1}a) = b^{-1}a$ for all $g \in L$. Since the nonamenable subgroup $L$ acts with infinite orbits on $I$, we conclude that $a = b$. So, $\mu_a = \mu_{c(g_0)} = g_0 \cdot \mu$. Using the unitary $u_{g_0} \in M$, it follows that $\varphi_a$ and $\varphi$ are unitarily conjugate. We already proved that $\psi \prec \varphi_a$. It follows that $\psi \prec \varphi$. We have thus proven that $\varphi$ is a solid state on $M$.

Having proven Theorem C, it is tempting to believe that for any nonsingular Bernoulli action $\Gamma \cap (X, \mu)$ of a biexact group, the crossed product state $\varphi_\mu$ on $M = L^\infty(X, \mu) \rtimes \Gamma$ is a solid state. Having proven Theorem A, it is equally tempting to believe that one may recover the measure class class$(\varphi_\mu)$ as an isomorphism invariant for any nonsingular Bernoulli crossed product $L^\infty(X, \mu) \rtimes \Gamma$ whenever $\mu$ is $\Lambda$-invariant for a nonamenable subgroup $\Lambda \subset \Gamma$. The
following example shows that both statements are wrong. The example is very similar to the construction in (1.1), except that we consider the free product of two nonamenable groups.

**Example 5.4.** Let $\Gamma = \Gamma_1 * \Gamma_2$ be an arbitrary free product of two countable nonamenable groups. The following construction provides a nonsingular Bernoulli action $\Gamma \curvearrowright (X, \mu)$ with the following properties.

1. The crossed product state $\varphi_\mu$ on $M = L^\infty(X, \mu) \rtimes \Gamma$ is not a solid state.
2. The measure $\mu$ is $\Gamma_1$-invariant. There exists an equivalent product measure $\mu' \sim \mu$ that is $\Gamma_2$-invariant. The measure classes $\text{class}(\varphi_\mu)$ and $\text{class}(\varphi_{\mu'})$ are not equivalent.

Since $\Gamma_2$ is nonamenable, not every element of $\Gamma_2$ has order 2. Fix an element $a \in \Gamma_2$ of order at least 3 (and possibly of infinite order). Define the map $\pi : \Gamma \to \Gamma_2$ by $\pi(h) = h$ for all $h \in \Gamma_2$ and $\pi(wh) = h$ whenever $h \in \Gamma_2$ and $w \in \Gamma$ is a reduced word in the free product $\Gamma_1 * \Gamma_2$ ending with a letter from $\Gamma_1 \setminus \{e\}$. Let $Y$ be a standard Borel space with equivalent probability measures $\nu \sim \eta$ on $Y$. Assume that $\nu$ and $\eta$ are not concentrated on a single atom. We will specify these measures later.

Define the subset $W \subset \Gamma$ by $W = \pi^{-1}(\{e, a\})$. For every $g \in \Gamma$, define $\mu_g = \nu$ if $g \in W$ and $\mu_g = \eta$ if $g \in \Gamma \setminus W$. Since $\pi(gv) = \pi(v)$ for all $g \in \Gamma_1$ and $v \in \Gamma$, we have that $gW = W$ for all $g \in \Gamma$. When $h \in \Gamma_2$, one has $hW \setminus W = \{h, ha\} \setminus \{e, a\}$ and $W \setminus hW = \{e, a\} \setminus \{h, ha\}$. We conclude that $gW \triangle W$ is a finite set for every $g \in \Gamma$. So, $\Gamma \curvearrowright (X, \mu) = \prod_{g \in \Gamma} (Y, \mu_g)$ is a nonsingular Bernoulli action. The action is essentially free. By construction, the measure $\mu$ is $\Gamma_1$-invariant. So, $\Gamma_1 \curvearrowright (X, \mu)$ is a pmp Bernoulli action, which is thus ergodic. A fortiori, $\Gamma \curvearrowright (X, \mu)$ is ergodic.

Next define $W' = W \setminus \Gamma_2 = W \setminus \{e, a\}$. Define $\mu'_g = \nu$ if $g \in W'$ and $\mu'_g = \eta$ if $g \in \Gamma \setminus W'$. Define the product measure $\mu' = \prod_{g \in \Gamma} \mu'_g$. Since $W' \triangle W$ is a finite set, we have that $\mu' \sim \mu$. We now have by construction that $hW' = W'$ for all $h \in \Gamma_2$. So, the measure $\mu'$ is $\Gamma_2$-invariant.

Also note that for every $g \in \Gamma_1 \setminus \{e\}$, we have that $gW' \setminus W' = \{e, a\}$ and $W' \setminus gW' = \{g, ga\}$.

Since $\Gamma_1 \curvearrowright (X, \mu)$ and $\Gamma_2 \curvearrowright (X, \mu')$ are ergodic, the centralizer of both states $\varphi_\mu$ and $\varphi_{\mu'}$ is a nonamenable factor. We prove that for the appropriate choice of $\nu$ and $\eta$, we have $\text{class}(\varphi_\mu) \neq \text{class}(\varphi_{\mu'})$. It then follows from point 2 of Proposition 3.2 that $\varphi_\mu \not\sim \varphi_{\mu'}$. Since the relation $\sim$ between states is symmetric, also $\varphi_{\mu'} \not\sim \varphi_\mu$, so that $\varphi_\mu$ is not a solid state.

The measure classes $\text{class}(\varphi_\mu)$ and $\text{class}(\varphi_{\mu'})$ can be easily computed as follows. Define $\gamma = (\log d\nu/d\eta)_* (\eta)$. For every $g \in \Gamma$, we have that

\[
\begin{align*}
(\log d(g \cdot \mu)/d\mu)_* (\mu) &\sim \gamma^{sk} \cdot \tilde{\gamma}^{sl} \quad \text{with } k = |gW \setminus W| \text{ and } l = |W \setminus gW|, \\
(\log d(g \cdot \mu')/d\mu')_* (\mu') &\sim \gamma^{sk} \cdot \tilde{\gamma}^{sl} \quad \text{with } k = |gW' \setminus W'| \text{ and } l = |W' \setminus gW'|.
\end{align*}
\]

A direct computation shows that $|gW \setminus W| = |W \setminus gW|$ for all $g \in \Gamma$ and that all elements of $\{0, 1, 2, \ldots\}$ appear as values. On the other hand, $|gW' \setminus W'| = |W' \setminus gW'|$ for all $g \in \Gamma$ but only the elements of $\{0, 2, 3, \ldots\}$ appear as values. We conclude that

\[
\text{class}(\varphi_\mu) = \delta_0 \vee \bigcup_{k=1}^{\infty} (\gamma \cdot \tilde{\gamma})^sk \quad \text{and} \quad \text{class}(\varphi_{\mu'}) = \delta_0 \vee \bigcup_{k=2}^{\infty} (\gamma \cdot \tilde{\gamma})^sk.
\]

Assume that $\gamma$ is nonatomic and that $K \subset \mathbb{R}$ is an independent Borel set such that $\gamma(K) > 0$. Denote by $\gamma_0$ the restriction of $\gamma$ to $K$. Clearly, $\gamma_0 \ast \tilde{\gamma}_0 \sim \text{class}(\varphi_\mu)$. We claim that $\gamma_0 \ast \tilde{\gamma}_0$ is orthogonal to $\text{class}(\varphi_{\mu'})$. To prove this claim, it suffices to observe that for every $x \in \mathbb{R} \setminus \{0\}$, the set $(x + (K - K)) \cap (K - K)$ is contained in finitely many translates of $K \cup (-K)$. Arguing as in the proof of Lemma 4.4, it follows that $(\eta \ast \gamma \ast \tilde{\gamma})(K - K) = 0$ for every nonatomic
probability measure $\eta$. So, the restriction of class($\varphi_{\nu}$) to $K - K$ equals $\delta_0$. On the other hand, $\gamma_0 * \tilde{\gamma}_0$ is a nonatomic probability measure that is concentrated on $K - K$, hence proving the claim.

Using the construction around (1.3), we can give concrete examples where $\gamma$ is a nonatomic probability measure that is supported on $K$.

6 Conjugacy results and proof of Proposition D

In this section, we prove Proposition D. We use the following well known lemma and provide a proof for completeness.

**Lemma 6.1.** Let $\eta \sim \nu$ be equivalent, but distinct probability measures on the standard Borel space $Y$. Define $\mu_n = \nu$ when $n \in \mathbb{N}$ and $\mu_n = \eta$ when $n \in \mathbb{Z} \setminus \mathbb{N}$. Then, the nonsingular Bernoulli action

$$Z \curvearrowright (Z, \zeta) = \prod_{n \in \mathbb{Z}} (Y, \mu_n)$$

is totally dissipative.

**Proof.** Since $\eta \neq \nu$, we can choose a Borel set $U \subset Y$ such that $\eta(U) \neq \nu(U)$. Since $\eta \sim \nu$, we have that $\eta(U)$ and $\nu(U)$ are different from 0 and 1. Define the probability measures $\gamma_n$ on $\{0, 1\}$ by $\gamma_n(0) = \nu(U)$ for $n \in \mathbb{N}$ and $\gamma_n(0) = \eta(U)$ for $n \in \mathbb{Z} \setminus \mathbb{N}$. Define the factor map $\pi : Y \to \{0, 1\}$ by $\pi(y) = 0$ iff $y \in U$. By construction, $\pi_* (\mu_n) = \gamma_n$ for all $n \in \mathbb{Z}$. So, the nonsingular Bernoulli action $Z \curvearrowright \prod_{n \in \mathbb{Z}} \{0, 1\}, \gamma_n$ is a factor of $Z \curvearrowright (Z, \zeta)$. By [Ham81, Theorem 1], the former is totally dissipative, so that also $Z \curvearrowright (Z, \zeta)$ is totally dissipative. $\square$

**Proof of Proposition D.** Write $\mu_{i,n} = \nu_i$ when $n \in \mathbb{N}$ and $\mu_{i,n} = \eta_i$ when $n \in \mathbb{Z} \setminus \mathbb{N}$. Consider the nonsingular Bernoulli actions $Z \curvearrowright (Z, \zeta_i) = \prod_{n \in \mathbb{Z}} (Y_i, \mu_{i,n})$. Since $\Gamma \curvearrowright (X_i, \mu_i)$ is isomorphic with the action associated in (4.1) to $Z \curvearrowright (Z, \zeta_i)$, it suffices to prove that there exists a measure preserving conjugacy between $\beta_1$ and $\beta_2$.

Denote by $\nu$ the probability measure on $\mathbb{R}$ given by $(\log d\nu_1/d\eta_1)_*(\nu_1) = (\log d\nu_2/d\eta_2)_*(\nu_2)$. Since

$$\int_{\mathbb{R}} \exp(-t) d\nu(t) = \int_{Y_1} \frac{d\eta_1}{d\nu_1} d\nu_1 = 1,$$

we can define the equivalent probability measure $\eta \sim \nu$ on $\mathbb{R}$ such that $(d\eta/d\nu)(t) = \exp(-t)$. Then $\pi_i = \log d\nu_i/d\eta_i$ is a factor map $\pi_i : Y_i \to \mathbb{R}$ satisfying $\pi_i_* (\nu_i) = \nu$ and $\pi_i_* (\eta_i) = \eta$ for all $i \in \{1, 2\}$.

Define the probability measures $\mu_n$ on $\mathbb{R}$ by $\mu_n = \nu$ when $n \in \mathbb{N}$ and $\mu_n = \eta$ when $n \in \mathbb{Z} \setminus \mathbb{N}$. Consider the nonsingular Bernoulli action $Z \curvearrowright (Z, \zeta) = \prod_{n \in \mathbb{Z}} (\mathbb{R}, \mu_n)$. Then $\psi_i : (Z_i, \zeta_i) \to (Z, \zeta) : (\psi_i(z))_n = \pi_i(z_n)$ is a measure preserving, $\mathbb{Z}$-equivariant factor map.

Denote by $(\nu_i, t) \in \mathbb{R}$ the disintegration of $\nu_i$ along the measure preserving factor map $\pi_i : (Y_i, \nu_i) \to (\mathbb{R}, \nu_i)$. Similarly, denote by $(\eta_i, t) \in \mathbb{R}$ the disintegration of $\eta_i$. Then, the disintegration $(\zeta_i, z) \in \mathbb{Z}$ of $\zeta_i$ along the factor map $\psi_i$ is given by

$$\zeta_{i,z} = \prod_{n \in \mathbb{N}} \nu_{i,z_n} \times \prod_{n \in \mathbb{Z} \setminus \mathbb{N}} \eta_{i,z_n}.$$

We assumed that the function $\pi_i$ is not essentially one-to-one. We thus find $\varepsilon > 0$ such that the set

$$U_i = \{ z \in \mathbb{R} \mid \text{the largest atom of } \nu_{i,z} \text{ has weight less than } 1 - \varepsilon \}$$
has positive measure, \( \nu(U_i) > 0 \). For \( \zeta \)-a.e. \( z \in Z \), there are infinitely many \( n \in \mathbb{N} \) with \( z_n \in U_i \). It follows that for \( \zeta \)-a.e. \( z \in Z \), the product measure \( \zeta_{i,z} \) is nonatomic.

Denote by \( \lambda \) the Lebesgue measure on \([0,1]\). By the classification of factor maps, at least going back to [Mah83] (see also [GM87, Theorem 2.2]), we can choose a measure preserving isomorphism \( \theta_1 : (Z_i, \zeta_i) \to (Z \times [0,1], \zeta \times \lambda) \) such that \( p_Z(\theta_1(z)) = \psi_1(z) \) for \( \zeta_i \)-a.e. \( z \in Z_i \), where \( p_Z(z,t) = z \) for all \( (z,t) \in Z \times [0,1] \). We denote \( \theta = \theta_1^{-1} \circ \theta \) and have found a measure preserving isomorphism \( \theta : (Z_1, \zeta_1) \to (Z_2, \zeta_2) \) satisfying \( \psi_2(\theta(z)) = \psi_1(z) \) for \( \zeta_1 \)-a.e. \( z \in Z_1 \).

By Lemma 6.1, the Bernoulli action \( Z \curvearrowright (Z, \zeta) \) is totally dissipative. We can thus choose a Borel set \( U \subset Z \) such that the sets \((n \cdot U)_{n \in \mathbb{Z}}\) form a partition of \( Z \), up to measure zero. Write \( U_i = \psi_i^{-1}(U) \). Then, \( U_i \subset Z_i \) is a fundamental domain for the action \( Z \curvearrowright^\lambda_i (Z_i, \zeta_i) \). By construction, \( \theta(U_1) = U_2 \), up to measure zero. We can thus, essentially uniquely, define the nonsingular isomorphism

\[
\Theta : Z_1 \to Z_2 : \Theta(n \cdot z) = n \cdot \theta(z) \quad \text{if } z \in U_1 \text{ and } n \in \mathbb{Z}.
\]

By construction, \( \Theta \) is \( \mathbb{Z} \)-equivariant. We claim that \( \Theta \) is measure preserving.

By construction,

\[
\frac{d(n \cdot \zeta_i)}{d\zeta} = \frac{d(n \cdot \zeta)}{d\zeta} \circ \psi_1.
\]

Therefore,

\[
\frac{d(n \cdot \zeta_2)}{d\zeta_2} \circ \theta = \frac{d(n \cdot \zeta)}{d\zeta} \circ \psi_2 \circ \theta = \frac{d(n \cdot \zeta)}{d\zeta} \circ \psi_1 = \frac{d(n \cdot \zeta_1)}{d\zeta_1}.
\]

Since \( \theta \) is measure preserving, it then follows that the maps \( z \mapsto n \cdot \theta((-n) \cdot z) \) are measure preserving for all \( n \in \mathbb{Z} \). Hence, \( \Theta \) is measure preserving and the claim is proven. This concludes the proof of the proposition.

**Example 6.2.** Let \( \gamma \in (0,1) \). On the finite set \( Y_1 = \{1, 2, 3\} \), we consider the probability measures \( \nu_1(1) = 1/2, \nu_1(2) = \nu_1(3) = 1/4 \) and \( \eta_1(1) = \gamma, \eta_1(2) = \eta_1(3) = (1 - \gamma)/2 \).

On the interval \( Y_2 = [0,1] \), we consider the probability measures \( \nu_2 \sim \eta_2 \) where \( \nu_2 \) is the Lebesgue measure and

\[
\frac{d\nu_2}{dv_2}(t) = \begin{cases} 2\gamma & \text{if } 0 \leq t \leq 1/2, \\ 2(1-\gamma) & \text{if } 1/2 < t \leq 1. \end{cases}
\]

For every countable group \( \Lambda \), the associated nonsingular Bernoulli actions \( \Gamma \curvearrowright (Y_1^\Gamma, \mu_i) \) of \( \Gamma = \mathbb{Z} \ast \Lambda \) admit a measure preserving conjugacy, even though one base space is finite and the other base space is diffuse.

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