Subtleties in QCD theory in Two Dimensions

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ABSTRACT

It is shown that in a formulation of Yang-Mills theory in two dimensions in terms of \( A = i f^{-1}\partial f, \bar{A} = i\bar{f}\partial \bar{f}^{-1} \) with \( f(z, \bar{z}), \bar{f}(z, \bar{z}) \in [SU(N_C)]^c \) the complexification of \( SU(N_C) \), reveals certain subtleties. “Physical” massive color singlet states seem to exist. When coupled to \( N_F \) quarks the coupling constant is renormalized in such a way that it vanishes for the pure Yang-Mills case. This renders the above states massless and unphysical. In the abelian case, on the other hand, the known results of the Schwinger model are reproduced with no need of such a renormalization. The massless \( QCD_2 \) theory is analyzed in similar terms and peculiar massive states appear, with a mass of \( e_c\sqrt{\frac{N_F}{2\pi}} \).

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1. Introduction

Pure Yang-Mills theory on compact Riemann surfaces attracted recently much attention, mainly in relation to a possible underlying string theory.[1]

The structure of the bosonized non-abelian massless $QCD_2$ is that of a gauged WZW model with an additional $F^2$ term of the gauge fields. Apart from the pure gauge term, this is thus a special form of a two dimensional coset model$^{[4,5,6]}$. It was found that translating the gauge fields into scalars $f$ and $\bar{f}$ via $A = if^{-1}\partial f$, $\bar{A} = i\bar{f}\partial\bar{f}^{-1}$ with $f(z, \bar{z}), \bar{f}(z, \bar{z}) \in H^c$, the complexification of $H \equiv SU(N_C)$ leads to a convenient formulation of the model. The main advantage of this approach is that one can then easily decouple the “matter” and the gauge degrees of freedom.

In the present paper we point out that the $F^2$ term requires a special treatment. The formulation of pure YM theory in terms of the $f$ variables seems naively to contain unexpected “physical” massive color singlet states. This result is obviously neither in accordance with our ideas of the degrees of freedom of the model nor with the lattice$^{[7,8]}$ and continuum solution of the theory$^{[9]}$. We show that similar “naive” manipulations in the case of $QED_2$ do reproduce the Schwinger model results. Using a coupling constant renormalization introduced by D. Kutasov,$^{[10]}$ we show that in the limit of no matter degrees of freedom the coupling constant is renormalized to zero. In this case the unexpected states turn into massless “BRST” exact states. In the flavored $QCD_2$ case a similar analysis shows the existence of “physical” flavorless states of mass $m^2 = \frac{N_F e^2}{2\pi}$. It seems that these states that could not be seen in the analysis of ’t Hooft, correspond to the massive states of ref.$^{[11]}$

The paper is organized as follows. In section 2 we introduce the formulation of $QCD_2$ in terms of $A = if^{-1}\partial f$, $\bar{A} = i\bar{f}\partial\bar{f}^{-1}$ with $f(z, \bar{z}), \bar{f}(z, \bar{z}) \in [SU(N_C)]^c$, the complexification of $SU(N_C)$. The gauge invariant and gauge fixed actions are written down. The special case of pure YM theory is introduced in section 3, where the equations of motion, the symmetries and the corresponding currents are discussed. In section 4 we digress to analyze the Schwinger model as the abelian
analog of the formulation that was introduced earlier for $QCD_2$. It is shown that a BRST quantization of the model reproduces the well known spectrum of the model. The pure non-abelian gauge theory on the plane is further discussed in section 5. It is analyzed as a “perturbed” topological $\mathcal{G}/\mathcal{G}$ model. We show that without the introduction of renormalization the “naive” space of physical states include unexpected massive color singlets. An alternative formulation of the Yang Mills theory in terms of an auxiliary field is analyzed in section 6. In section 7 the resolution of the mysterious states is presented. Starting from the $QCD_2$ case we show that in the $N_F \to 0$ limit the unexpected states become massless and indeed decouple. Section 8 is devoted to a brief discussion of the massless $QCD_2$ model. Again this is done using a $\mathcal{G}/\mathcal{G}$ interpretation. A summary and a brief comparison with other methods of treating YM and $QCD_2$ is presented in section 9.

2. The action

The bosonized version of $QCD_2$ was shown\cite{2} to be described by the following action

$$S_{QCD_2} = S_1(u) - \frac{1}{2\pi} \int d^2z Tr(\partial u \bar{\partial} u + i u \bar{\partial} u^{-1} A + \bar{A} u^{-1} A u - A \bar{A})$$

$$+ \frac{m^2}{2\pi} \int d^2 z : Tr_G [u + u^{-1}] : + \frac{1}{e^2} \int d^2 z Tr_H [F^2]$$

(2.1)

where $u \in U(N_F \times N_C)$; $S_k(u)$ is a level $k$ WZW model;

$$S_k(u) = \frac{k}{8\pi} \int d^2 x Tr(\partial \mu u \partial \mu u^{-1}) + \frac{k}{12\pi} \int d^3 y z^{ijk} Tr(u^{-1} \partial_i u)(u^{-1} \partial_j u)(u^{-1} \partial_k u)$$

(2.2)

$A$ and $\bar{A}$ take their values in the algebra of $H \equiv SU(N_C)$; $F = \bar{\partial} A - \partial \bar{A} + i[A, \bar{A}]$; $m^2$ equals $m_q \mu C$, where $\mu$ is the normal ordering mass and $C = \frac{1}{2} e^\gamma$ with $\gamma$ Euler’s constant. Apart from the last two terms which correspond to the quark mass term and the YM term the rest of the action is a level one $\mathcal{G}/\mathcal{H}$ coset model with $G = U(N_F \times N_C)$. 

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We now introduce the following parameterization for the gauge fields $A = if^{-1} \partial f$, $\bar{A} = i\bar{f} \bar{\partial} \bar{f}^{-1}$ with $f(z, \bar{z}), \bar{f}(z, \bar{z}) \in SU(N_C)^c$. These type of variables were used frequently in dealing with gauged WZW actions, for instance in computing the effective action of $QCD^2$[12] and in the $\hat{G}$ models.[13] They may be interpreted as Wilson lines along the $z$ and $\bar{z}$ directions.[14] The gauged WZW part of the action, first line of (2.1), then[5,6,12] takes the form

$$S_1(u, A) = S_1(fu\bar{f}) - S_1(f\bar{f})$$

(2.3)

The Jacobian of the change of variables from $A$ to $f$ introduces[5,6] a dimension $(1,0)$ system of anticommuting ghosts $(\rho, \chi)$ in the adjoint representation of $H$. The WZW part of the action thus becomes

$$S_1(u, A) = S_1(fu\bar{f}) - S_1(f\bar{f}) + \frac{i}{2\pi} \int d^2z \text{Tr}_H[\rho\bar{D}\chi + \rho\bar{D}\bar{\chi}]$$

(2.4)

where $D\chi = \partial\chi - i[A, \chi]$. Our integration variables in the functional integral are $if^{-1}df$ and $i\bar{f}d\bar{f}^{-1}$. This action involves an interaction term of the form $Tr_H(\bar{\rho}[f^{-1}\partial f, \bar{\chi}])$ and a similar term for $\rho, \chi$. By performing a chiral rotation $\bar{\rho} \rightarrow f^{-1} \bar{\rho} f$ and $\bar{\chi} \rightarrow f^{-1} \bar{\chi} f$ with $\rho \rightarrow \bar{f} \rho \bar{f}^{-1}$ and $\chi \rightarrow \bar{f} \chi \bar{f}^{-1}$, one achieves a decoupling of the whole ghost system. The price of that is an additional $S^{H(N_C)}(f\bar{f})$ term in the action (here trace over $H$ only) resulting from the corresponding anomaly. This result can be derived by using a non-abelian bosonization of the ghost system.[16]

In this language the ghost action takes the form

$$S_{gh} = S_{N_c}(l_1, A, \bar{A}) + S_{N_c}(l_2, A, \bar{A}) + S_{\text{twist}}(l_1) + S_{\text{twist}}(l_2)$$

(2.5)

where $l_1$ and $l_2$ are in the adjoint representation and $S_{\text{twist}}$ is a twist term given in ref. [16]. Now using the Polyakov-Wiegmann formula[12] we get

$$S^b_{gh} = S_{N_c}(fl_1\bar{f}) - S_{N_c}(f\bar{f}) + S_{\text{twist}}(l_1) + S_{N_c}(fl_2\bar{f}) - S_{N_c}(f\bar{f}) + S_{\text{twist}}(l_2)$$

$$= -S_{2N_c}(f\bar{f}) + \frac{i}{2\pi} \int d^2z \text{Tr}_H[\rho'\partial\chi' + \rho'\bar{\partial}\bar{\chi}]$$

(2.6)

where the last line has been transferred back to the ghost language. Notice that
unlike the ghost fields in (2.4) the new ghost fields $\rho'$ and $\chi'$ are now gauge invariant. It is interesting to note that the action given in (2.4) is non-local in terms of the local degrees of freedom $A$ and $\bar{A}$. Note that had we done the right and left rotations separately, we would have gotten $S_{-2N_c}(f) + S_{-2N_c}(\bar{f})$, which however is not vector gauge invariant, but rather a left-right symmetric scheme.

The full gauge invariant action including the anomaly contribution of the anti-commuting part now reads

$$S_{QCD_2} = S_1(u) + \frac{1}{e_c^2} \int d^2 z \, \text{Tr}_H \left[ F^2 \right] + \frac{m^2}{2\pi} \int d^2 z : \text{Tr}_G \left[ f^{-1} u \bar{f}^{-1} + \bar{f} u^{-1} f \right] : + \left[ S_{-(N_f + 2N_c)}^{(H)}(f \bar{f}) + i \frac{2}{2\pi} \int d^2 z \, \text{Tr}_H \left[ \rho' \partial \bar{\chi}' + \rho' \bar{\partial} \chi' \right] \right]$$

(2.7)

In deriving equation (2.7) we used a redefinition $fu\bar{f} \to u$. This does not require an extra determinant factor $[^5,^6]$. Also, as $S^{(H)}(f \bar{f})$ involves $Tr_H$ rather than the $Tr_G$ in $S_1(f \bar{f})$ of equation (2.4), a factor of $N_f$ appears. Note that had we introduced the special parameterization of the gauge fields in the fermionic formulation of $QCD_2$, we would have arrived at the same action after decoupling the fermionic currents from the gauge fields, by performing chiral rotation and then bosonizing the free fermions. Equation (2.7) was derived without paying attention to possible renormalizations. The latter will be treated in section 7.

At this point one may choose a gauge. A convenient gauge choice is $\bar{A} = i\bar{f} \bar{\partial} \bar{f}^{-1} = 0$. Notice that since the underlying space-time is a plane this is a legitimate gauge. The gauge fixed action can be written down using the BRST procedure, namely

$$S_{GF} = S_{QCD_2} + S^{(gf)} + S^{(gh)} = S_{QCD_2} + \delta_{BRST}(b\bar{A}) = S_{QCD_2} + Tr_H[B\bar{A}] + Tr_H[b\bar{D}c]$$

(2.8)

where $S_{GF}$, $S^{(gf)}$ and $S^{(gh)}$ are, respectively, the gauge fixed action, the gauge fixing term and the ghost action. The $(b, c)$ fields are yet another $(1,0)$ ghost...
system and $B$ is a dimension one auxiliary field, all in the adjoint representation of $SU(N_C)$. The integration over $B$ introduces a delta function of the gauge choice to the measure of the functional integral. In addition we integrate over the ghosts $b$ and $c$.

It is interesting to note that the $QCD_2$ action can be related to a “perturbed” topological $\frac{H}{\pi}$ coset model. To realize this face of $QCD_2$ we parameterize $u$ as $g h l e^{i \sqrt{\frac{4\pi}{N_C N_F}} \phi}$ and rewrite (2.8) accordingly. The Polyakov-Wiegmann relation implies

$$S[u] = S[ghl] + \frac{1}{2} \int d^2 x \partial_\mu \phi \partial^\mu \phi$$

$$S[ghl] = S[g] + S[l] + S[h] + \frac{1}{2\pi} \int d^2 x \mathrm{Tr} (g^\dagger \partial_+ g h l \partial_- l - h^\dagger \partial_+ h l \partial_- l). \quad (2.9)$$

Since $l$ is a dimension zero field with an associated zero central charge we have $S[l] = 0$ and thus

$$S_{GF} = S_{N_F}(h) + S_{(N_F+2N_C)}^{(H)}(f) + \frac{i}{2\pi} \int d^2 z \mathrm{Tr}_H [\bar{\rho} \partial \bar{\chi} + \rho \partial \chi]$$

$$+ S_{N_C}(g) + \frac{1}{2\pi} \int d^2 z [\partial_\phi \bar{\partial} \phi]$$

$$+ \frac{m^2}{2\pi} \int d^2 z \mathrm{Tr}_G : [f^{-1} g h l e^{i \sqrt{\frac{4\pi}{N_C N_F}} \phi} + e^{-i \sqrt{\frac{4\pi}{N_C N_F}} \phi} l^{-1} h^{-1} g^{-1} f] :$$

$$+ \frac{1}{e^2} \int d^2 z \mathrm{Tr}_H [(\bar{\partial} (f^{-1} \partial f))^2]. \quad (2.10)$$

It is now easy to recognize the first line in the action as the action of $SU(N_C)$ topological theory.

It is interesting to note that a WZW term $S_{-2N_C}(f)$ appears in the action even without the introduction of quarks. We therefore digress to an analysis of the pure YM theory in the formulation introduced above.

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3. Two dimensional Yang-Mills theory

Pure Yang-Mills theory attracted recently much attention along the lines of an underlying string theory\[^1\]. Here we restrict our discussion to the 2D Minkowski or Euclidean space-time, where the rich structure of the model on compact Riemann surface does not show up. In terms of the parameterization introduced in equation (2.7) the gauge invariant action of the pure YM theory is

\[
S_{YM} = S_{\phi(2N_c)}(f \bar{f}) + \frac{i}{2\pi} \int d^2z \text{Tr}_H [\bar{\rho} \partial \bar{\chi} + \rho \bar{\partial} \chi] \\
+ \frac{1}{e_c^2} \int d^2z \text{Tr}_H [F^2]
\]

(3.1)

Here again we remind the reader that the coupling constant undergoes a multiplicative renormalization. This will be discussed in section 7. Let us first discuss the corresponding equations of motion for \( f \) and \( \bar{f} \),

\[
\delta f : \quad \bar{\partial} A - D \bar{A} + \frac{2}{m^2_A} D \bar{D} F = (1 + \frac{2}{m^2_A} D \bar{D}) F = 0 \\
\delta \bar{f} : \quad \partial \bar{A} - \bar{D} A - \frac{2}{m^2_A} \bar{D} D F = -(1 + \frac{2}{m^2_A} \bar{D} D) F = 0
\]

(3.2)

where \( D = \partial - i[A, \cdot] \), \( m_A = e_c \sqrt{\frac{N_c}{\pi}} \) and \( \Box = 2\partial \bar{\partial} \). In fact these two equations are identical, as \([D, \bar{D}] F = 0\). The equation is that of a massive gauge field with self interaction. Note that in this approach, unlike the equations that follow from varying the action with respect to the gauge fields, one gets two derivatives on \( F \).

In deriving the above, it is convenient to remember that

\[
\delta S_{WZW}(f) = \frac{1}{2\pi} Tr \left\{ (f^{-1} \delta f) \bar{\partial}(f^{-1} \partial f) \right\}.
\]

The YM action equation (3.1) is obviously invariant under the original gauge
transformations
\[ f \rightarrow f v(z, \bar{z}) \quad \bar{f} \rightarrow v^{-1}(z, \bar{z}) \bar{f} \] (3.3)

with \( v \in SU(N_C) \). In addition the action is invariant separately under the holomorphic and anti-holomorphic “color” transformations
\[ f \rightarrow u(z) f \quad \bar{f} \rightarrow \bar{f} w(z) \] (3.4)

where \( u, w \in SU(N_C) \). These are “spurious” transformations since they leave \( A \) and \( \bar{A} \) invariant. The corresponding holomorphic and anti-holomorphic color currents are
\[
\bar{J}^a = -\frac{N_c}{\pi} [i(f \bar{f}) \bar{\partial}(f \bar{f})^{-1} - \frac{2}{m_A^2} f \bar{f} D f^{-1}] \\
J^a = -\frac{N_c}{\pi} [i(f \bar{f})^{-1} \partial(f \bar{f}) + \frac{2}{m_A^2} \bar{f}^{-1} \bar{D} \bar{f}] 
\] (3.5)

The gauge fixed (\( \bar{f} = 1 \)) action takes the form
\[
S_{YM} = S_{-(2N_C)}(f) + \frac{1}{e^2} \int d^2 z \text{Tr}_H [(\bar{\partial}(f^{-1} \partial f))^2] \\
+ i \frac{1}{2\pi} \int d^2 z \text{Tr}_H [\bar{\rho} \partial \bar{\chi} + \rho \bar{\partial} \chi] 
\] (3.6)

As is expected the equation of motion at present is just that of equation (3.2) after setting \( \bar{A} = 0 \). Naturally, the action now lacks gauge invariance, nevertheless, it is invariant under the following residual holomorphic transformations
\[ f \rightarrow u(z) f \quad f \rightarrow f w(z) \] (3.7)

with the corresponding holomorphic and anti-holomorphic currents
\[
J_G = -\frac{N_C}{\pi} [A + \frac{2}{m_A^2} D(\bar{\partial}A)] \\
\bar{J}_G = -\frac{N_C}{\pi} [\bar{A} + \frac{2}{m_A^2} \bar{D}(\partial \bar{A})] 
\] (3.8)

and \( \bar{A} = i f \bar{D} f^{-1} \). Notice that in spite the similar structure, \( \bar{A} \) is not related to \( \bar{A} \) which was set to zero. To better understand the physical picture behind these currents we defer temporarily to the abelian case.
4. Schwinger model revisited

Since in the pure Maxwell theory there is no analog to the \((-2N_C)\) level WZW term of equation (3.1), we study instead the Schwinger model in its bosonized form,

\[
S_{(Sch)} = \frac{1}{2\pi} \int d^2 z [\partial X \bar{\partial} X - \sqrt{2} \partial X \bar{A} + \sqrt{2} \bar{\partial} X A + \frac{\pi}{\epsilon^2} (\partial \bar{A} - \bar{\partial} A)^2] \tag{4.1}
\]

In analogy to the change of variables in the non-abelian case, we now introduce the following parameterization of the gauge fields \(A = \partial \varphi\), \(\bar{A} = \bar{\partial} \bar{\varphi}\). In terms of these fields the action takes the form

\[
S_{(Sch)} = \int d^2 z \left\{ [\partial X \bar{\partial} X - \sqrt{2} \bar{X} \partial \bar{\varphi} - \varphi] + \frac{\pi}{\epsilon^2} [\partial \bar{\partial} (\varphi - \bar{\varphi})] + i [\partial \bar{\partial} \chi + \partial \bar{\partial} \bar{\varphi}] \right\} \tag{4.2}
\]

In the gauge \(\bar{A} = 0\) and after the field redefinition \(\bar{X} = X + \frac{1}{\sqrt{2}} \varphi\), the action is decomposed into decoupled sectors

\[
S_{(Sch)} = S(\bar{X}) + S(\varphi) + S_{(\text{ghost})}
\]

\[
S(\bar{X}) = \frac{1}{4\pi} \int d^2 z [\partial \bar{X} \bar{\partial} \bar{X}], \quad S(\varphi) = \frac{1}{4\pi} \int d^2 z \left\{ \frac{2}{\mu^2} [\partial \bar{\partial} \varphi]^2 - \partial \varphi \bar{\partial} \bar{\varphi} \right\}, \tag{4.3}
\]

where \(\mu^2 = \frac{\epsilon^2}{\pi}\). The corresponding equations of motion are

\[
\partial \bar{\partial} [1 + \frac{2}{\mu^2} \partial \bar{\partial}] \varphi = 0, \quad \partial \bar{\partial} \bar{X} = 0 \tag{4.4}
\]

The invariance under the chiral shifts \(\delta \varphi = \epsilon(\bar{z})\) and \(\delta \varphi = \epsilon(z)\) are generated by the holomorphically conserved currents

\[
J_G = \partial \varphi + \frac{2}{\mu^2} \partial \bar{\partial} \varphi, \quad \bar{J}_G = \bar{\partial} \bar{\varphi} + \frac{2}{\mu^2} \partial \bar{\partial} \bar{\varphi}. \tag{4.5}
\]

To handle this type of “hybrid” current we suggest the following decomposition of
the massless and massive modes $\varphi = \varphi_1 + \varphi_2$ with

$$\partial \bar{\partial} \varphi_1 = 0 \quad [2\partial \bar{\partial} + \mu^2] \varphi_2 = 0$$

(4.6)

In the holomorphic quantization

$$\Pi = \frac{\delta L}{\delta (\partial \varphi)} = -\frac{1}{\pi \mu^2} \bar{\partial}(\partial \bar{\partial} + \frac{\mu^2}{4}) \varphi = \frac{1}{4\pi} \bar{\partial}(\varphi_2 - \varphi_1).$$

(4.7)

A unique solution to the commutation relations $[\varphi(z, \bar{z}), \Pi(w, \bar{w})]_{z=w} = i\delta(\bar{z} - \bar{w})$, $[\varphi, \varphi] = 0$ and $[\Pi, \Pi] = 0$ is

$$[\varphi_1(z, \bar{z}), \varphi_1(w, \bar{w})]_{z=w} = -\pi \epsilon(\bar{z} - \bar{w})$$

$$[\varphi_2(z, \bar{z}), \varphi_2(w, \bar{w})]_{z=w} = -\pi \epsilon(\bar{z} - \bar{w})$$

$$[\varphi_1(z, \bar{z}), \varphi_2(w, \bar{w})]_{z=w} = 0$$

$$[\bar{X}(z, \bar{z}), \bar{X}(w, \bar{w})]_{z=w} = -\pi \epsilon(\bar{z} - \bar{w})$$

(4.8)

where $\epsilon$ is the standard antisymmetric step function. Notice that the massless degree of freedom has commutation relations which correspond to a negative metric on the phase space. These relations can also be translated to the following OPE’s (choosing the part $\varphi_1(z)$ of $\varphi_1$)

$$\varphi_1(z)\varphi_1(w) = \log(z - w)$$

$$\varphi_2(z, \bar{z})\varphi_2(w, \bar{w}) = -\log|\varphi_1(w)|^2 + O(\mu^2|z - w|^2)$$

$$\varphi_1(z)\varphi_2(w, \bar{w}) = O(z - w)$$

(4.9)

It is thus clear that the model is invariant under a $U(1)$ affine Lie algebra of level $k = -1$ since $J_G(z)J_G(w) = \frac{1}{(z - w)^2}$, as $J_G$ has no contribution from $\varphi_2$ and $J_G = \partial \varphi_1$ (no contribution from $\varphi_2$).

The physical states of the model have to be in the cohomology of the BRST charge. Due to the fact that the current is holomorphically (and the other antiholomorphically) conserved, it follows that the same property holds for the BRST
charge, and thus the space of physical states is an outer product of the cohomology of $Q$ and $\bar{Q}$. The latter are given by \[^{[13]}\]

\[
Q = \chi J = \chi (i\partial \tilde{X} + \partial \varphi_1),
\]

\[
\bar{Q} = \bar{\chi} \bar{J} = \bar{\chi} (-i\partial \tilde{X} + \bar{\partial} \varphi_1).
\]

(4.10)

Expanding the fields $i\partial \tilde{X}$ and $\partial \varphi_1$ in terms of the Laurent modes $\tilde{X}_n$ and $(\tilde{\varphi}_1)_n$ with $[X_n, X_m] = n\delta_{n+m}$ and $[(\varphi_1)_n, (\varphi_1)_m] = n\delta_{n+m}$ we have

\[
Q = \sum_n \tilde{\chi}_n \left[ \tilde{X}_{-n} - i(\varphi_1)_{-n} \right].
\]

Since $J_0 = \{Q, \rho_0\}$, physical states have to have zero eigen-value of $J_0$. The general structure of the states in the $\varphi_1, \tilde{X}, \rho, \chi$ Fock space are as follows

\[
(\tilde{X}_n)^n_x (\varphi_{1m})^{n_f} (\chi_k)^{n_x} (\rho_l)^{n_x} |vac
\]

where obviously $n_\chi$ and $n_\rho$ are either 0 or 1. It is straightforward to realize that only the vacuum state and states of the form $(\tilde{X}_0)^n_x (\varphi_{10})^{n_f}$ are in the BRST cohomology. Recall that being on the plane we exclude zero modes and thus only the vacuum state remains. Since there is no constraint on the modes of $\varphi_2$, the physical states are built solely of $\varphi_2$ which are massive modes. This result is identical to the well known solution of the Schwinger model.
5. Back to the YM theory

Equipped with the lesson from the Schwinger model we return now to the YM case and introduce a decomposition of the group element $f$ so that again the gauge currents obey an affine Lie algebra. Let us write $f = f_2 f_1$ which implies that

$$A = if^{-1}\partial f = if_1^{-1}\partial f_1 + if_1^{-1}(f_2^{-1}\partial f_2) f_1 \equiv J_1 + J_2 \quad (5.1)$$

With no loss of generality we take $\bar{\partial} f_1 = 0$ implying also $\bar{\partial} J_1 = 0$. Inserting these expressions into $J_G$ of equation (3.8) one finds

$$J_G = -\frac{N_C}{\pi} [J_1 + J_2 + \frac{2}{m_A^2} (\bar{\partial}\bar{\partial} J_2 + i[\bar{\partial} J_2, J_1 + J_2])]. \quad (5.2)$$

If one can consistently require that

$$J_2 + \frac{2}{m_A^2} (\bar{\partial}\bar{\partial} J_2 + i[\bar{\partial} J_2, J_1 + J_2]) = 0 \quad (5.3)$$

then, in a complete analogy with the abelian case, $J_G = -\frac{N_C}{\pi} J_1$. The latter is an affine current of level $k = -2N_C$. One can in fact show that (5.3) can be assumed without a loss of generality. $\bar{\partial}J_G = 0$ implies that $J_2 + \frac{2}{m_A^2} (\bar{\partial}\bar{\partial} J_2 + i[\bar{\partial} J_2, J_1 + J_2]) = u(z)$, where $u(z)$ is some holomorphic function. We then introduce the shifted currents $\tilde{J}_2 = J_2 - u(z)$, $\tilde{J}_1 = J_1 + u(z)$. Now $\bar{\partial}\tilde{J}_1 = 0$ as does $J_1$, and $\tilde{J}_2$ obeys equation (5.3) with $\tilde{J}_1$ replacing $J_1$. It is easy to check that the shifts in the currents correspond to $f_1 \rightarrow v(z) f_1$, $f_2 \rightarrow f_1 v(z)^{-1}$ with $u(z) = if_1^{-1}(v(z)^{-1}\partial v(z)) f_1$.

Note that the equation for $J_2$ involves a coupling to $J_1$. This is related to the fact that, unlike the abelian case, one cannot write the action as a sum of decoupled terms which are functions of $J_1$ and $J_2$ separately.

Once the color current $J_G$ is expressed in terms of the holomorphic current $J_1$, the analysis of the space of physical states is directly related to that of the topological $G_G$ model at $k = 0$. The physical states have to be in the cohomology...
of the BRST charge, which corresponds to the following holomorphically conserved BRST current

\[ Q(z) = \chi^a(J_G^a + \frac{1}{2} J_{gh}^a) = -\frac{N_C}{\pi} \chi^a([A + \frac{2}{m^2} D(\bar{\partial}A)]^a + i \frac{1}{2} f_{bc}^a \rho^b \chi^c). \]  

(5.4)

An anti-holomorphic BRST current \( \bar{Q}(z) \) determines the condition for physical states in the analogous manner to \( Q \). From here on we restrict our description to the latter. We define now the zero level affine Lie algebra current

\[ J_{(tot)}^a = J_G^a + J_{(gh)}^a = J_G^a + i f_{bc}^a \chi_b \rho_c \]  

(5.5)

and the \( c = 0 \) Virasoro generator \( T \)

\[ T(z) = -\frac{1}{N_C} : J_G^a J_G^a : + \rho^a \partial \chi^a. \]  

(5.6)

as well as dimension (2,0) fermionic current

\[ G = -\frac{1}{2N_C} \rho_J G^a, \]

and realize the existence of the “topological coset algebra”\[^{[18,19]}\]

\[ T(z) = \{ Q, G(z) \}, \quad Q(z) = \{ Q, j^\#(z) \}, \quad J_{(tot)}^a = \{ Q, \rho^a(z) \}, \]

\[ \{ Q, Q(z) \} = 0, \quad \{ G, G(z) \} \equiv W(z), \]

\[ W(z) = \{ Q, U(z) \}, \quad [W, W(z)] = 0, \]  

(5.7)

where \( j^\# = \chi_a \rho^a \) is “ghost number current”,

\[ W(z) = \frac{1}{4N_c} f_{abc} J_G^a \rho^b \rho^c + \partial \rho^a \rho_a \]

and

\[ U = \frac{1}{12N_c} f_{abc} J_G^a \rho^b \rho^c. \]

A direct consequence is that any physical state has to obey

\[ J_{(tot)}^0 \vert_{phys} = 0, \quad L_0 \vert_{phys} = 0, \quad W_0 \vert_{phys} = 0, \]  

(5.8)

where \( J_{(tot)}^i, \bar{L}_n \) and \( W_n \) are the Laurent modes of \( J_{(tot)}^i \) the Cartan sub-algebra
currents, $T$ and $W$ respectively. In fact the BRST cohomology of the present model is a special case of the set of models of refs. [13,15]. We therefore refer the reader to those works and present here only the result. On the plane where no ghost zero modes are allowed, the only state in the cohomology is the zero ghost number vacuum state of $J_1$.

This state can be tensor-product with oscillators of the massive modes of $J_2$. Unlike the abelian case, $J_G$ does not commute with $J_2$ so that in general the $J_2$ modes are not obviously in the BRST cohomology. However, there is no reason to believe that all the $J_2$ modes will be excluded by the BRST condition. Those $J_2$ modes that remain are by definition color singlets. This result contradicts previous results on $YM_2$. Usually one believes that pure gluodynamics on the plane is an empty theory since all local degrees of freedom can be gauged away.

6. An Alternative Formulation

To get a better understanding of the subtleties of the Yang Mills theory when expressed in terms of $A = i f^{-1} \partial f$, $\bar{A} = i \bar{f} \bar{\partial} \bar{f}^{-1}$, and for future application, we compare now with another formulation of the theory.

Consider the following functional integral

$$Z = \int DAD\bar{AD}Be^{iS(A,\bar{A},B)}$$
$$S = -\int d^2z Tr_H \left[ \frac{1}{\epsilon_c} F B + \frac{1}{4} B^2 \right]$$

where $B$ is a pseudoscalar field in the adjoint representation. Obviously the integration over $B$ produces the usual $Tr[F^2]$ action. It is also easy to realize that the action is invariant under the ordinary gauge symmetry provided that $\delta B = i [\epsilon, B]$. In terms of the $f$ variables after imposing the gauge $\bar{f} = 1$ one finds

$$S_{YM_2} = S_{-(2N_c)}(f) + \int d^2z Tr_H \left[ (\frac{i}{\epsilon_c} (f^{-1} \partial f) \bar{\partial} B) - \frac{1}{4} B^2 \right] + S^{(gh)}$$
where $S^{(gh)} = \frac{i}{2\pi} \int d^2 z \text{Tr}_H[\bar{\rho} \partial \bar{\chi} + \rho \partial \chi]$. One should again bear in mind that the coupling constant undergoes a multiplicative renormalization. This will be discussed in the next section. Using the results of ref. [12] we get

$$S_{YM_2}(B) = \Gamma_{2N_C}(B) - \frac{1}{4} \text{Tr}_H[B^2] - S^{(2N_C)}(vf) + S^{(gh)}$$

(6.3)

where $\Gamma_k(B) = S_k(v)$ with $\frac{1}{e_c} \partial B = \frac{2N_C}{2\pi}(iv\partial v^{-1})$. The second line in (6.3) is a “topological system”. Since the underlying Minkowski space-time does not admit zero modes we can safely integrate over the corresponding fields. We can further pass from functional integrating over $B$ to $iv\partial v^{-1}$. This involves the insertion of $\text{det} \frac{D}{\partial}$ which will introduce a $\Gamma_{-2N_C}(B)$ term with no additional ghost terms. The functional integral (6.1) thus takes the final form

$$Z = \int D[v] e^{-i\left(\frac{1}{4} \int d^2 z \text{Tr}_H[|B^2|]\right)}.$$ 

(6.4)

It is thus clear that the in the present formulation there is no trace of the massive “physical modes” discussed in the previous section.

7. The resolution of the puzzle

Encouraged by the result of the last section, we proceed now to reexamine the steps that led to the unexpected massive modes in the pure $YM_2$ theory. In particular, we would like to check whether in addition to the implementation of proper determinants there is no coupling constant renormalization that has to be invoked when passing to the quantum theory expressed in the $f$ variables. For this purpose we turn on again the matter degrees of freedom. We introduce $N_F$ quarks in the fundamental color representation and explore the behavior of the system in the limit $N_F \rightarrow 0$. Recall that the action of this model is given in equation (2.10). Starting actually from equation (2.1), taking the massless limit, writing $A$


in terms of the $f$ in the action but still with $A$ as an integration variable, and using the formulation presented in the previous section, the path integral of the colored degrees of freedom now reads

$$Z^{(col)} = \int [DA][DB][Dh] e^{iS^{(col)}}$$

$$S^{(col)} = S_{NF}(h) + \frac{N_F}{2\pi} \int d^2z \text{Tr}_H[h\bar{\partial}h^{-1}f^{-1}\partial f]$$

$$+ \int d^2z \text{Tr}_H \left[ \left( \frac{i}{e_c} (f^{-1}\partial f)\bar{\partial}B \right) - \frac{1}{4} B^2 \right]$$

(7.1)

where we have also gone from $u$ to $h$ as in section 2.

It was found out\[10\] that quantum consistency imposes finite renormalization on the coupling constant of the current-gauge field interaction. This renormalization is expressed in the following equality,

$$Z(\bar{J}) \equiv \int DA e^{i[S_k(f) + \frac{i}{4\pi} \int d^2z \text{Tr}_H[i(f^{-1}\partial f)\bar{J}]}$$

$$= \int Df e^{i[S_{k-2N_C}(f) + \frac{i}{4\pi} \int d^2z \text{Tr}_H[(f^{-1}\partial f)\bar{J}]} \int D(gh) e^{iS^{(gh)}}$$

$$= e^{i\Gamma_{k-2N_C}} \left[ \left( \frac{e^{(k-1)} e^{(k)}}{k+2N_C} \right) \bar{J} \right] \int D(gh) e^{iS^{(gh)}}.$$  

(7.2)

where $k$ is an arbitrary level and $\Gamma_k(L) = S_k(w)$ for $L = iw\bar{\partial}w^{-1}$. The renormalization factor $e(k)$ has to satisfy $\frac{e^{(k-2N_C)}}{e^{(k)}} = \frac{k}{k+2N_C}$. In addition it is clear from equation (7.2) that it has to be singular at the origin. In ref. [10], $e(k)$ was taken to be $e(k) = \sqrt{\frac{k+2N_C}{k}}$. Implementing this renormalization in our case equation (7.1) takes the form

$$S^{(col)} = S_{NF}(\tilde{h}) + S_{-(N_F+2N_C)}(f)$$

$$+ \int d^2z \text{Tr}_H \left[ \left( i\sqrt{\frac{N_F + 2N_C}{N_F}} \frac{1}{e_c} (f^{-1}\partial f)\bar{\partial}B \right) - \frac{1}{4} B^2 \right]$$

$$+ S^{(gh)}$$

(7.3)
where $\tilde{h} = fh$. After integrating the auxiliary field $B$ the action becomes

$$S^{(col)} = S_{N_F}(\tilde{h}) + S_{-(N_F+2N_C)}(f) + \int d^2z \, \text{Tr} H \left[ \left( \frac{N_F + 2N_C}{e^{-2}N_F} \right) [\bar{\partial}(f^{-1}\partial f)]^2 \right] + S^{(gh)}$$

(7.4)

It is now straightforward to realize that the equation of motion which follows from the variation with respect to $f$ is that of equation (3.2) where now $m_A = e_c\sqrt{\frac{N_F}{2\pi}}$. Thus, the coupling constant renormalization turned the massive modes into massless ones in the case of pure YM theory ($N_F = 0$). Notice that to reach this conclusion it is enough to use the fact that $e(k)$ has to be singular at $k = 0$ and the explicit expression of $e(k)$ is really not needed. Following the arguments presented in section 5, it is clear that these states that became massless are not in the BRST cohomology and thus not in the physical spectrum.

A somewhat similar derivation of the triviality of the model in the $N_F = 0$ limit is the following. We integrate in equation (7.3) over the ghost fields and over $f$ using again the coupling constant renormalization and find

$$Z = \int D[v]e^{-\left\{ iS_{N_F}(v) + \frac{e^2N_F^2}{4} \int d^2z \text{Tr} H [B^2(v)] \right\}}.$$  

(7.5)

It is now clear that the action vanishes at $N_F = 0$ and hence again, on trivial topology, the theory is empty. Notice, however, that the implementation of renormalization modifies also the result of the previous section.

The final conclusion is that in both methods one finds that indeed the pure YM theory has an empty space of physical states as of course is implied by the original formulation in terms of $A$. We have demonstrated that in this formulation it follows only after taking subtleties of renormalization into account.
8. On bosonized $QCD_2$

To resolve the puzzle of the YM theory we were led to analyze the color and flavor sectors of $QCD_2$. The full bosonized $QCD_2$ includes in addition the baryon number degrees of freedom. The corresponding action is given by equations (2.7), (2.8) or by equation (2.10). In the past the low lying baryonic spectrum in the strong coupling limit $m_q \to 0$ was extracted using a semiclassical quantization.$^{[2,3]}$

In addition the “quark” structure of the model was analyzed in ref. [17]. In the present work our analysis was based on switching off the mass term, $m_q = 0$. This limit cannot be treated by the semi-classical approach, as the soliton solution is not there for $m_q = 0$. In our case here one finds a decoupled $WZW$ action for the flavor degrees of freedom $S_{NC}(g)$ and a decoupled free field action for the baryon degree of freedom, in addition to the action of the colored degrees of freedom which is given in equation (7.4) or equation (7.5). The general structure of a physical state in this case is that of a tensor product of $g$ and $\phi$ with the colored degrees of freedom $f$, $h$ and the ghosts. The structure of $QCD_2$ which emerged from the semiclassical quantization$^{[2,3]}$ for $m_q \neq 0$ involves $g$ and $\phi$ only. In our case here the $f$ colored degrees of freedom acquire mass $m_A = e_c \sqrt{\frac{N_F}{2\pi}}$ while the $h$ degrees of freedom remain massless. In the limit $e_c \to \infty$ the $f$ degrees of freedom decouple. It is thus clear that one has to introduce the mass term which couples the three sectors. The massless limit of $QCD_2$ can then be derived by taking the limit $m_q \to 0$ after solving for the physical states. Indeed, it was shown$^{[2]}$ in the limit of $e_c \to \infty$ that turning on $m_q \neq 0$ results in a hadronic spectrum where the flavor representation and the baryon number were correlated. The analysis of the spectrum of the massive multi-flavor $QCD_2$ in the approach of this work remains to be worked out.
9. Summary and Discussion

In this work we have analyzed 2D YM and QCD theories using a special parametrization of the gauge fields in terms of group elements. In the $m_q = 0$ case it enabled us to decouple the matter and gauge degrees of freedom. However, this formulation led, in a naive treatment, to unexpected massive modes. Even though we did not present a full solution of the theory we had reasonable arguments to believe that the BRST projection would not exclude these modes. The fact that a similar approach to $QED_2$ reproduced the known results of the Schwinger model, enhanced the puzzling phenomenon. Eventually, we showed that a coupling constant renormalization, which was originally introduced in ref. [10], renders the unexpected massive modes into massless unphysical states. The benefit of this detective work is the appearing of “physical” massive states in massless $QCD_2$. These states of mass $m_A = e_c \sqrt{ \frac{N_c}{2\pi} }$ were actually discovered in ref. [11] using another approach. The general task of a full solution of $QCD_2$ still remains to be worked out.

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