A UNIFORM ESTIMATE FOR AN EQUATION WITH HOLDERIAN CONDITION
AND BOUNDARY SINGULARITY.

SAMY SKANDER BAHOURA

ABSTRACT. We consider the following problem on open set $\Omega$ of $\mathbb{R}^2$:

\[
\begin{cases}
-\Delta u_i = |x - x_0|^{-2\alpha} V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\
u_i = 0 & \text{in } \partial\Omega.
\end{cases}
\]

Here, $x_0 \in \partial\Omega$ and, $\alpha \in (0, 1/2)$.

We assume, for example that:

\[
\int_{\Omega} |x - x_0|^{-2\alpha} V_i e^{u_i} \, dy \leq 16\pi - \epsilon, \quad \epsilon > 0
\]

1) We give, a quantization analysis of the previous problem under the conditions:

\[
\int_{\Omega} |x - x_0|^{-2\alpha} e^{u_i} \, dy \leq C,
\]

and,

\[
0 \leq V_i \leq b < +\infty
\]

2) In addition to the previous hypothesis we assume that $V_i$ is holderian with $1/2 < s \leq 1$, then we have a compactness result, namely:

\[
\sup_{\Omega} u_i \leq c = c(b, C, A, s, \epsilon, x_0, \Omega).
\]

where $A$ is the holderian constant of $V_i$.

1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = \partial_{11} + \partial_{22}$ on open set $\Omega$ of $\mathbb{R}^2$ with a smooth boundary.

We consider the following problem on $\Omega \subset \mathbb{R}^2$:

\[
(P) \begin{cases}
-\Delta u_i = |x - x_0|^{-2\alpha} V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\
u_i = 0 & \text{in } \partial\Omega.
\end{cases}
\]

Here, $x_0 \in \partial\Omega$ and, $\alpha \in (0, 1/2)$.

We assume that,

\[
0 \leq V_i \leq b < +\infty, \quad \int_{\Omega} |x - x_0|^{-2\alpha} e^{u_i} \, dy \leq C, \quad u_i \in W^{1,1}_0(\Omega)
\]

The above equation is called, the Prescribed Scalar Curvature equation in relation with conformal change of metrics. The function $V_i$ is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type (in dimension 2 and higher dimensions) were studied by many authors, see [1-24]. We can see in [8], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on $V_i$, for example we suppose $V_i \geq 0$ and $V_i \in L^p(\Omega)$ or $V_i e^{u_i} \in L^p(\Omega)$ with $p \in [1, +\infty]$.

Among other results, we can see in [8], the following important Theorem,

**Theorem A**(Brezis-Merle [8]). If $(u_i)$, and $(V_i)$, are two sequences of functions relatively to the previous problem $(P)$ with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set $K$ of $\Omega$,
A simple consequence of this theorem is that, if we assume \( u_i = 0 \) on \( \partial \Omega \) then, the sequence \( (u_i) \) is locally uniformly bounded. We can find in [8] an interior estimate if we assume \( a = 0 \), but we need an assumption on the integral of \( e^{u_i} \). We have in [8]:

**Theorem B (Brezis-Merle [8])** If \( (u_i) \) and \( (V_i) \) are two sequences of functions relatively to the previous problem \( (P) \) with, \( 0 \leq V_i \leq b < +\infty \), and,

\[
\int e^{u_i} dy \leq C,
\]

then, for all compact set \( K \) of \( \Omega \),

\[
\sup_K u_i \leq c = c(b, C, K, \Omega).
\]

If, we assume \( V \) with more regularity, we can have another type of estimates, \( \sup + \inf \). It was proved, by Shafrir, see [23], that, if \( (u_i) \) and \( (V_i) \) are two sequences of functions solutions of the previous equation without assumption on the boundary and, \( 0 < a \leq V_i \leq b < +\infty \), then we have the following interior estimate:

\[
C \left( \frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).
\]

We can see in [12], an explicit value of \( C \left( \frac{a}{b} \right) = \sqrt{\frac{a}{b}} \). In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality, see [6]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose \( (V_i) \) uniformly Lipschitzian with \( A \) the Lipschitz constant, then, \( C(a/b) = 1 \) and \( c = c(a, b, A, K, \Omega) \), see Brezis-Li-Shafrir [7]. This result was extended for Hölderian sequences \( (V_i) \) by Chen-Lin, see [12]. Also, we can see in [18], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [19] explicit form, \((8\pi m, m \in \mathbb{N}^*)\), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used.

In [8], Brezis and Merle proposed the following Problem:

**Problem (Brezis-Merle [8]).** If \( (u_i) \) and \( (V_i) \) are two sequences of functions relatively to the previous problem \( (P) \) with,

\[
0 \leq V_i \to V \text{ in } C^0(\Omega),
\]

\[
\int e^{u_i} dy \leq C,
\]

Is it possible to prove that:

\[
\sup_{\Omega} u_i \leq c = c(C, V, \Omega) ?
\]

Here, we assume more regularity on \( V_i \), we suppose that \( V_i \geq 0 \) is \( C^s \) (s-holderian) \( 1/2 < s \leq 1 \) and when we have a boundary singularity. We give the answer where \( bC < 16\pi \) for an equation with boundary singularity.

Our main results are:

**Theorem 1.1.** Assume \( \Omega = B_1(0) \), \( x_0 \in \partial \Omega \), \( \alpha \in (0, 1/2) \), and,

\[
\int_{B_1(0)} |x - x_0|^{-2\alpha} V_i e^{u_i} dy \leq 16\pi - \epsilon, \; \epsilon > 0,
\]

\[
u_i(x_i) = \sup_{B_1(0)} u_i \to +\infty.
\]
There is a sequences \((x_i^0), (\delta_i^0)\), such that:

\[
(x_i^0)_i \equiv (x_i)_i, \delta_i^0 = \delta_i = d(x_i, \partial B_1(0)) \rightarrow 0,
\]

and,

\[
u_i(x_i) = \sup_{B_1(0)} u_i \rightarrow +\infty,
\]

\[
u_i(x_i) + 2 \log \delta_i - 2\alpha \log d(x_i, x_0) \rightarrow +\infty,
\]

\[
\forall \epsilon > 0, \limsup_{i \rightarrow +\infty} \int_{B(x_i, \delta_i \epsilon)} |x - x_0|^{-2\alpha} V_i e^{u_i} dy \geq 4\pi > 0.
\]

If we assume:

\[
\forall \epsilon > 0, \sup_{B_1(0) - B(x_i, \delta_i \epsilon)} u_i \leq C_\epsilon
\]

\[
\forall \epsilon > 0, \limsup_{i \rightarrow +\infty} \int_{B(x_i, \delta_i \epsilon)} |x - x_0|^{-2\alpha} V_i e^{u_i} dy = 8\pi.
\]

And, thus, we have the following convergence in the sense of distributions:

\[
\int_{B_1(0)} |x - x_0|^{-2\alpha} V_i e^{u_i} dy \rightarrow \int_{B_1(0)} |x - x_0|^{-2\alpha} V e^{u} dy + 8\pi \delta_{x_0}.
\]

**Theorem 1.2.** Assume that, \(V_i\) is uniformly \(s\)-holderian with \(1/2 < s \leq 1, x_0 \in \partial \Omega, \alpha \in (0, 1/2)\), and,

\[
\int_{B_1(0)} |x - x_0|^{-2\alpha} V_i e^{u_i} dy \leq 16\pi - \epsilon, \epsilon > 0,
\]

then we have:

\[
\sup_{\Omega} u_i \leq c = c(b, C, A, s, \alpha, \epsilon, x_0, \Omega).
\]

where \(A\) is the holderian constant of \(V_i\).

2. **Proofs of the results**

Without loss of generality, we can assume that \(\Omega = B_1(0)\) the unit ball centered on the origin.

Here, \(G\) is the Green function of the Laplacian with Dirichlet condition on \(B_1(0)\). We have (in complex notation):

\[
G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{z}y|}{|x - y|},
\]

Since \(u_i \in W^{1,1}_0(\Omega)\) and \(\alpha \in (0, 1/2)\), we have by the Brezis-Merle result and the elliptic estimates, (see [1]):

\[
u_i \in C^{2}(\Omega) \cap W^{2,p}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})
\]

for all \(2 < p < +\infty\).

Set,

\[
v_i(x) = \int_{B_1(0)} G(x, y) V_i(y)|x - x_0|^{-2\alpha} e^{u_i(y)} dy.
\]

We decompose \(v_i\) in two terms (Newtonian potential):
\[ v_1^i(x) = \int_{B_i(0)} -\frac{1}{2\pi} \log |x-y|V_i(y)|x-x_0|^{-2\alpha}e^{u_i(y)}dy, \]

and,
\[ v_2^i(x) = \int_{B_i(0)} \frac{1}{2\pi} \log |1-x|V_i(y)|x-x_0|^{-2\alpha}e^{u_i(y)}dy, \]

According to the proof in the book of Gilbarg-Trudinger see [15], \( v_1^i, v_2^i \), and thus \( v_i \) are \( C^1(\Omega) \). Indeed, we use the same proof as in [15] (Chapter 4, Newtonian potential), we have for the approximate function \( \partial u_i \), terms of type \( O(\epsilon^{1-2\alpha}) \log \epsilon + O(\epsilon^{1-2\alpha}) \). Since \( \alpha < 1/2 \), this term tends to 0.

We use this fact and the maximum principle to have \( u_i = u \).

Also, we can use integration by part (the Green representation formula, see its proof in the first chapter of [15]) to have in \( \Omega \) (and not \( \Omega \)):

\[ u_i(x) = -\int_{B_i(0)} G(x,y) \Delta u_i(y)dy = \int_{B_i(0)} G(x,y) V_i(y)|x-x_0|^{-2\alpha}e^{u_i(y)}dy. \]

We write,

\[ u_i(x_i) = \int_{\Omega} G(x_i,y)|x-x_0|^{-2\alpha}V_i(y)e^{u_i(y)}dx = \int_{\Omega-B(x_i,\delta_i/2)} G(x_i,y)|x-x_0|^{-2\alpha}V_i e^{u_i(y)}dy + \int_{B(x_i,\delta_i/2)} G(x_i,y)|x-x_0|^{-2\alpha}V_i e^{u_i(y)}dy \]

According to the maximum principle, the harmonic function \( G(x_i,.) \) on \( \Omega - B(x_i,\delta_i/2) \) take its maximum on the boundary of \( B(x_i,\delta_i/2) \), we can compute this maximum:

\[ G(x_i,y_i) = \frac{1}{2\pi} \log \left| \frac{1 - \bar{x}_i y_i}{|x_i - y_i|}\right| + \frac{1}{2\pi} \log \left| \frac{1 - \bar{x}_i (x_i + \delta_i \theta_i)}{|\delta_i/2|}\right| = \frac{1}{2\pi} \log(2(1+|x_i|+\theta_i)) < +\infty \]

with \( |\theta_i| = 1/2 \).

Thus,
\[ u_i(x_i) \leq C + \int_{B(x_i,\delta_i/2)} G(x_i,y)|x-x_0|^{-2\alpha}V_i e^{u_i(y)}dy \leq C + e^{u_i(x_i)-2\alpha \log d(x_i,x_0)} \int_{B(x_i,\delta_i/2)} G(x_i,y)dy \]

Now, we compute \( \int_{B(x_i,\delta_i/2)} G(x_i,y)dy \)

we set in polar coordinates,
\[ y = x_i + \delta_i t \theta \]

we find:
\[ \int_{B(x_i,\delta_i/2)} G(x_i,y)dy = \int_{B(x_i,\delta_i/2)} \frac{1}{2\pi} \log \frac{1 - \bar{x}_i y}{|x_i - y|} = \frac{1}{2\pi} \int_0^{1/2} \int_0 2\pi \delta_i^2 \log \frac{1 - \bar{x}_i (x_i + \delta_i t \theta)}{\delta_i t} ttdt \theta = \]

\[ = \frac{1}{2\pi} \int_0^{1/2} \int_0 2\pi \delta_i^2 (\log(|1 + |x_i| + t\theta|) - \log t) ttdt \theta \leq C\delta_i^2. \]

Thus,
\[ u_i(x_i) \leq C + C\delta_i^2 e^{u_i(x_i)-2\alpha \log d(x_i,x_0)} \]

which we can write, because \( u_i(x_i) \to +\infty \),
\[ u_i(x_i) \leq C\delta_i^2 e^{u_i(x_i)-2\alpha \log d(x_i,x_0)} \]

We can conclude that:
\[ u_i(x_i) + 2 \log \delta_i - 2\alpha \log d(x_i,x_0) \to +\infty. \]
Since in $B(x_i, \delta_i \epsilon)$, $d(x, x_0)$ is equivalent to $d(x_i, x_0)$ we can consider the following functions:

$$v_i(y) = u_i(x_i + \delta_i y) + 2 \log \delta_i - 2 \alpha \log d(x_i, x_0), \quad y \in B(0, 1/2)$$

The function satisfies all conditions of the Brezis-Merle hypothesis, we can conclude that, on each compact set:

$$v_i \to -\infty$$

we can assume, without loss of generality that for $1/2 > \epsilon > 0$, we have:

$$v_i \to -\infty, \quad y \in B(0, 2\epsilon) - B(0, \epsilon),$$

**Lemma 2.1.** For all $1/4 > \epsilon > 0$, we have:

$$\sup_{B(x_i, (3/2)\delta_i \epsilon) - B(x_i, \delta_i \epsilon)} u_i \leq C_\epsilon.$$  

**Proof of the lemma**

Let $t_i'$ and $t_i$ the points of $B(x_i, 2\delta_i \epsilon) - B(x_i, (1/2)\delta_i \epsilon)$ and $B(x_i, (3/2)\delta_i \epsilon) - B(x_i, \delta_i \epsilon)$ respectively where $u_i$ takes its maximum.

According to the Brezis-Merle work, we have:

$$u_i(t_i') + 2 \log \delta_i - 2 \alpha \log d(x_i, x_0) \to -\infty$$

We write,

$$u_i(t_i) = \int_{\Omega} G(t_i, y) |x-x_0|^{-2\alpha} V_i(y) e^{u_i(y)} dx = \int_{B(x_i, 2\delta_i \epsilon)} G(t_i, y) |x-x_0|^{-2\alpha} V_i e^{u_i(y)} dy +$$

$$+ \int_{B(x_i, 2\delta_i \epsilon) - B(x_i, (1/2)\delta_i \epsilon)} G(t_i, y) |x-x_0|^{-2\alpha} V_i e^{u_i(y)} dy +$$

$$+ \int_{B(x_i, (1/2)\delta_i \epsilon)} G(t_i, y) |x-x_0|^{-2\alpha} V_i e^{u_i(y)} dy$$

But, in the first and the third integrals, the point $t_i$ is far from the singularity $x_i$ and we know that the Green function is bounded. For the second integral, after a change of variable, we can see that this integral is bounded by (we take the supremum in the annulus and use Brezis-Merle theorem)

$$\delta_i^2 e^{u_i(t_i') - 2\alpha \log d(x_i, x_0) \times I_i}$$

where $I_j$ is a Jensen integral (of the form $\int_0^1 \int_0^{2\pi} (\log(|1 + |x_i| + t\theta| - \log(|\theta_i - t\theta|) t dt d\theta$ which is bounded).

we conclude the lemma.

From the lemma, we see that far from the singularity the sequence is bounded, thus if we take the supremum on the set $B_1(0) - B(x_i, \delta_i \epsilon)$ we can see that this supremum is bounded and thus the sequence of functions is uniformly bounded or tends to infinity and we use the same arguments as for $x_i$ to conclude that around this point and far from the singularity, the sequence is bounded.

The process will be finished, because, according to Brezis-Merle estimate, around each supremum constructed and tending to infinity, we have:

$$\forall \epsilon > 0, \limsup_{i \to +\infty} \int_{B(x_i, \delta_i \epsilon)} |x-x_0|^{-2\alpha} V_i e^{u_i(y)} dy \geq 4\pi > 0.$$  

Finally, with this construction, we have a finite number of “exterior” blow-up points and outside the singularities the sequence is bounded uniformly, for example, in the case of one “exterior” blow-up point, we have:

$$u_i(x_i) \to +\infty.$$
\[ \forall \epsilon > 0, \quad \sup_{B_1(0) - B(x_i, \delta_i \epsilon)} u_i \leq C_\epsilon \]
\[ \forall \epsilon > 0, \quad \limsup_{i \to +\infty} \int_{B(x_i, \delta_i \epsilon)} |x - x_0|^{-2\alpha} V(x_i u_i) dy \geq 4\pi > 0. \]

\( x_i \to x_0 \in \partial B_1(0). \)

**Remark:** For the general case, the process of quantization can be extended to more than one blow-up points.

We have the following lemma:

**Lemma 2.2.** Each \( \delta^k_i \) is of order \( d(x^k_i, \partial B_1(0)) \). Namely: there is a positive constant \( C > 0 \) such that for \( \epsilon > 0 \) small enough:

\[ \delta^k_i \leq d(x^k_i, \partial B_1(0)) \leq (2 + \frac{C}{\epsilon}) \delta^k_i. \]

**Proof of the lemma**

Now, if we suppose that there is another ”exterior” blow-up \((t_i)\), we have, because \((u_i)\) is uniformly bounded in a neighborhood of \( \partial B(x_i, \delta_i \epsilon) \), we have:

\[ d(t_i, \partial B(x_i, \delta_i \epsilon)) \geq \delta_i \epsilon \]

If we set,

\[ \delta'_i = d(t_i, \partial B_1(0) - B(x_i, \delta_i \epsilon)) = \inf \{ d(t_i, \partial B(x_i, \delta_i \epsilon)), d(t_i, \partial B_1(0)) \} \]

then, \( \delta'_i \) is of order \( d(t_i, \partial B_1(0)) \). To see this, we write:

\[ d(t_i, \partial B_1(0)) \leq d(t_i, \partial B(x_i, \delta_i \epsilon)) + d(\partial B(x_i, \delta_i \epsilon), x_i) + d(x_i, \partial B_1(0)), \]

Thus,

\[ \frac{d(t_i, \partial B_1(0))}{d(t_i, \partial B(x_i, \delta_i \epsilon))} \leq 2 + \frac{1}{\epsilon}, \]

Thus,

\[ \delta'_i \leq d(t_i, \partial B_1(0)) \leq \delta'_i (2 + \frac{1}{\epsilon}). \]

Now, the general case follow by induction. We use the same argument for three, four,..., \( n \) blow-up points.

We have, by induction and, here we use the fact that \( u_i \) is uniformly bounded outside a small ball centered at \( x^j_i, j = 0, \ldots, k - 1: \)

\[ \delta^j_i \leq d(x^j_i, \partial B_1(0)) \leq C_1 \delta^j_i, \quad j = 0, \ldots, k - 1, \]

\[ d(x^k_i, \partial B(x^j_i, \delta^j_i \epsilon / 2)) \geq \epsilon \delta^j_i, \quad \epsilon > 0, \quad j = 0, \ldots, k - 1, \]

and let’s consider \( x^k_i \) such that:

\[ u_i(x^k_i) = \sup_{B_1(0) - \cup_{j=0}^{k-1} B(x^j_i, \delta^j_i \epsilon)} u_i \to +\infty, \]

take,

\[ \delta^k_i = \inf \{ d(x^k_i, \partial B_1(0)), d(x^k_i, \partial B_1(0) - \cup_{j=0}^{k-1} B(x^j_i, \delta^j_i \epsilon / 2)) \}, \]

if, we have,

\[ \delta^k_i = d(x^k_i, \partial B(x^j_i, \delta^j_i \epsilon / 2)), \quad j \in \{0, \ldots, k - 1\}. \]

Then,
\[ \delta_i^k \leq d(x_i^k, \partial B_1(0)) \leq d(x_i^k, \partial B(x_i^k, \delta_i^k/2)) + d(\partial B(x_i^k, \delta_i^k/2), x_i^k) + d(x_i^k, \partial B_1(0)) \leq (2 + \frac{C_1}{\epsilon})\delta_i^k. \]

To apply lemma 2.1 for \( m \) blow-up points, we use an induction:

We do directly the same approach for \( t_i \) as \( x_i \) by using directly the Green function of the unit ball.

If we look to the blow-up points, we can see, with this work that, after finite steps, the sequence will be bounded outside a finite number of balls, because of Brezis-Merle estimate:

\[ \forall \epsilon > 0, \quad \limsup_{i \to +\infty} \int_{B(x_i^k, \delta_i^k\epsilon)} |x - x_0|^{-2\alpha} V_i e^{u_i} dy \geq 4\pi > 0. \]

Here, we can take the functions:

\[ u_i^k(y) = u_i(x_i^k + \delta_i^k y) + 2 \log \delta_i^k - 2\alpha \log d(x_i, x_0). \]

Indeed, by corollary 4 of the paper of Brezis-Merle, if we have:

\[ \limsup_{i \to +\infty} \int_{B(x_i^k, \delta_i^k\epsilon)} |x - x_0|^{-2\alpha} V_i e^{u_i} dy \leq 4\pi - \epsilon_0 < 4\pi, \]

then, \((u_i^k)^+\) would be bounded and this contradict the fact that \( u_i^k(0) \to +\infty\).

Finally, we can say that, there is a finite number of sequences \((x_i^k)_i, (\delta_i^k), 0 \leq k \leq m\), such that:

\[ (x_i^0)_i \equiv (x_i)_i, \quad \delta_i^0 = \delta_i = d(x_i, \partial B_1(0)), \]

\[ (x_i^1)_i \equiv (t_i)_i, \quad \delta_i^1 = \delta_i = d(t_i, \partial(B(1) - B(x_i, \delta_i\epsilon))), \]

and each \( \delta_i^k \) is of order \( d(x_i, \partial B_1(0)) \).

\[ u_i(x_i^k) = \sup_{B_1(0) - \cup_{j=0}^m B(x_i^j, \delta_i^j\epsilon)} u_i \to +\infty, \]

\[ u_i(x_i^k) + 2 \log \delta_i^k - 2\alpha \log d(x_i^k, x_0) \to +\infty, \]

\[ \forall \epsilon > 0, \quad \sup_{B_1(0) - \cup_{j=0}^m B(x_i^j, \delta_i^j\epsilon)} u_i \leq C_\epsilon \]

\[ \forall \epsilon > 0, \limsup_{i \to +\infty} \int_{B(x_i^k, \delta_i^k\epsilon)} |x - x_0|^{-2\alpha} V_i e^{u_i} dy \geq 4\pi > 0. \]

**The work of YY Li-I.Shafir**

Since in \( B(x_i, \delta_i\epsilon), d(x, x_0) \) is equivalent to \( d(x_i, x_0) \) we can consider the following functions:

\[ v_i(y) = u_i(x_i + \delta_i y) + 2 \log \delta_i - 2\alpha \log d(x_i, x_0). \]

With the previous method, we have a finite number of "exterior" blow-up points (perhaps the same) and the sequences tend to the boundary. With the aid of proposition 1 of the paper of Li-Shafir, we see that around each exterior blow-up, we have a finite number of "interior" blow-ups. Around, each exterior blow-up, we have after rescaling with \( \delta_i^k \), the same situation as around a fixed ball with positive radius. If we assume:

\[ V_i \to V \text{ in } C^0(B_1(0)), \]
we have a function (the usual product of function):
\[ \nu = (1, 0) \] and \( u_i(0, x_2) \equiv 0 \). Also, we set \( x_i \) the blow-up point and \( x_i^0 = (0, x_i^0) \) and \( x_i^1 = (x_i^0, x_i) \) respectively the second and the first part of \( x_i \). Let \( \partial B^+ \) the part of the boundary for which \( u_i \) and its derivatives are uniformly bounded and thus converge to the corresponding function.

**The case of one blow-up point:**

**Theorem 2.3.** If \( V_i \) is \( s \)-Holderian with \( 1/2 < s \leq 1 \) and,
\[
\int_\Omega |x|^{-2\alpha} V_i e^{u_i} dy \leq 16\pi - \epsilon, \quad \epsilon > 0,
\]
we have:
\[
2(1 - 2\alpha) V_i(x_i) \int_{B(x_i, \delta(x_i))} |x|^{-2\alpha} e^{u_i} dy = o(1),
\]
which means that there is no blow-up points.

**Proof of the theorem**

In order to use the Pohozaev identity we need to have a good function \( u_i \), since \( \alpha \in (0, 1/2) \), we have a function \( u_i \) such that:
\[
u = (0, 1, x_1) \geq 0 \]
we have:
\[
\int_{\partial B^+} < (x - x_2^2) \nu \nu_i > d\sigma + \int_{\partial B^+} < (x - x_2^2) \nu \nu_i > \nu |\nabla u_i|^2 d\sigma
\]
We can write it as:
\[
\int_\Omega < (x - x_2^2) \nabla u_i > (V_i - V_i(x_i))|x|^{-2\alpha} e^{u_i} dy = A_i + V_i(x_i) \int_\Omega < (x - x_2^2) \nabla u_i > |x|^{-2\alpha} e^{u_i} dy =
\]
\[
= A_i + V_i(x_i) \int_\Omega < (x - x_2^2) |x|^{-2\alpha} \nabla (e^{u_i}) > dy
\]
And, if we integrate by part the second term, we have (because $x_1 = 0$ on the boundary and $\nu_2 = 0$):

$$
\int_{\Omega} (x - x_2^1) |\nabla u| > (V_i - V_i(x_i))|x|^{-2\alpha}e^{u_i}dy = -2(1 - \alpha)V_i(x_i) \int_{\Omega} |x|^{-2\alpha}e^{u_i}dy +
$$

$$
+2\alpha V_i(x_i) \int_{B(x_i, \delta, e)} x_2 x_1 |x|^{-2\alpha - 2}e^{u_i}dy + 2\alpha V_i(x_i) \int_{\Omega - B(x_i, \delta, e)} x_2 x_1 |x|^{-2\alpha - 2}e^{u_i}dy + B_i
$$

where $B_i$ is,

$$
B_i = V_i(x_i) \int_{\partial B^+} < (x - x_2^1)|\nu| > |x|^{-2\alpha}e^{u_i}dy
$$

and, because of the uniform convergence of $u_i$, we can write:

$$
\int_{\Omega} (x - x_2^1) |\nabla u| > (V - V(0))|x|^{-2\alpha}e^{u}dy = -2(1 - \alpha)V(0) \int_{\Omega} |x|^{-2\alpha}e^{u}dy +
$$

$$
+2\alpha V(0) \int_{B(x_i, \delta, e)} x_2 x_1 |x|^{-2\alpha - 2}e^{u}dy + 2\alpha V(0) \int_{\Omega - B(x_i, \delta, e)} x_2 x_1 |x|^{-2\alpha - 2}e^{u}dy + B,
$$

with,

$$
B = V(0) \int_{\partial B^+} < (x - x_2^1)|\nu| > |x|^{-2\alpha}e^{u}dy
$$

we use the fact that, $u_i$ is bounded outside $B(x_i, \delta, e)$ and the convergence of $u_i$ to $u$ on compact set of $\Omega - \{0\}$, and the fact that $\alpha \in (0, 1/2)$, to write the following:

$$
2(1 - 2\alpha)V_i(x_i) \int_{B(x_i, \delta, e)} |x|^{-2\alpha}e^{u_i}dy + o(1) =
$$

$$
= \int_{\Omega} (x - x_2^1) |\nabla u| > (V_i - V_i(x_i))|x|^{-2\alpha}e^{u_i}dy - \int_{\Omega} < (x - x_2^1)|\nabla u| > (V - V(0))|x|^{-2\alpha}e^{u}dy +
$$

$$
+ (A_i - A) + (B_i - B),
$$

where $A$ and $B$ are,

$$
A = \int_{\partial B^+} < (x - x_2^1)|\nabla u| > < |\nabla u| > |\nabla u|^2 d\sigma + \int_{\partial B^+} < (x - x_2^1)|\nu| > |\nabla u|^2 d\sigma
$$

$$
B = V(0) \int_{\partial B^+} < (x - x_2^1)|\nu| > |x|^{-2\alpha}e^{u}dy
$$

and, because of the uniform convergence of $u_i$, and its derivatives on $\partial B^+$, we have:

$$
A_i - A = o(1) \text{ and } B_i - B = o(1)
$$

which we can write as:

$$
2(1 - 2\alpha)V_i(x_i) \int_{B(x_i, \delta, e)} |x|^{-2\alpha}e^{u_i}dy + o(1) =
$$

$$
= \int_{\Omega} (x - x_2^1) |\nabla u| > (V_i - V_i(x_i))|x|^{-2\alpha}e^{u_i}dy +
$$

$$
+ \int_{\Omega} < (x - x_2^1)|\nabla u| > (V_i - V_i(x_i))|x|^{-2\alpha}e^{u_i}dy +
$$

$$
+ \int_{\Omega} < (x - x_2^1)|\nabla u| > (V_i - V_i(x_i) - (V - V(0))|x|^{-2\alpha}e^{u}dy + o(1)
$$

We can write the second term as:

$$
\int_{\Omega} < (x - x_2^1)|\nabla u| > (V_i - V_i(x_i))|x|^{-2\alpha}e^{u_i}dy = \int_{\Omega - B(0, \delta)} < (x - x_2^1)|\nabla u| > (V_i - V_i(x_i))(e^{u_i} - e^{u})|x|^{-2\alpha}dy +
$$
because of the uniform convergence of \( u_i \) to \( u \) outside a region which contain the blow-up and the uniform convergence of \( V_i \). For the third integral we have the same result:

\[
\int_\Omega < x - x_2^1, \nabla > (V_i - V_i(x_i)) (V - V(0)) |x|^{-2\alpha} e^{u_i} dy = o(1),
\]

because of the uniform convergence of \( V_i \) to \( V \).

Now, we look to the first integral:

\[
\int_\Omega < x - x_2^1, \nabla > (V_i - V_i(x_i)) |x|^{-2\alpha} e^{u_i} dy = \int_\Omega < x - x_i, \nabla > (V_i - V_i(x_i)) |x|^{-2\alpha} e^{u_i} dy + \int_\Omega < x_1^1, \nabla > (V_i - V_i(x_i)) |x|^{-2\alpha} e^{u_i} dy,
\]

Thus, we have proved by using the Pohozaev identity the following equality:

\[
\int_\Omega < x - x_i, \nabla > (V_i - V_i(x_i)) |x|^{-2\alpha} e^{u_i} dy + \int_\Omega < x_1^1, \nabla > (V_i - V_i(x_i)) |x|^{-2\alpha} e^{u_i} dy = 2(1 - 2\alpha)V_i(x_i) \int_{\Omega} |x|^{-2\alpha} e^{u_i} dy + o(1)
\]

We can see, because of the uniform boundedness of \( u_i \) outside \( B(x_i, \delta_i) \) and the fact that :

\[
||\nabla (u_i - u)||_1 = o(1),
\]

it is sufficient to look at the integral on \( B(x_i, \delta_i) \).

Assume that we are in the case of one blow-up, it must be \( (x_i) \) and isolated, we can write the following inequality as a consequence of YY.Li-I-Shafrir result:

\[
u_i(x) + 2 \log |x - x_i| - 2\alpha \log d(x, 0) \leq C,
\]

We use this fact and the fact that \( V_i \) is s-hölderian to have that, on \( B(x_i, \delta_i) \),

\[
|(x - x_i)(V_i - V_i(x_i))| x|^{-2\alpha} e^{u_i} | \leq \frac{C}{|x - x_i|^{(1-s)}} \in L^{(2-s')/(1-s)}, \forall \epsilon' > 0,
\]

and, we use the fact that:

\[
||\nabla (u_i - u)||_q = o(1), \forall 1 \leq q < 2
\]

to conclude by the Holder inequality that:

\[
\int_{B(x_i, \delta_i)} < x - x_2^1, \nabla > (V_i - V_i(x_i)) |x|^{-2\alpha} e^{u_i} dy = o(1),
\]

For the other integral, namely:

\[
\int_{B(x_i, \delta_i)} < x_1^1, \nabla > (V_i - V_i(x_i)) |x|^{-2\alpha} e^{u_i} dy,
\]

We use the fact that, because our domain is a half ball, and the sup + inf inequality to have:

\[
x_1^1 = \delta_i,
\]
\[ u_t(x) + 4 \log \delta_i - 4\alpha \log d(x,0) \leq C \]

and,
\[ |x|^{-\alpha s} e^{(s/2)u_i(x)} \leq |x - x_i|^{-s}, \]

\[ |V_i - V_i(x_i)| \leq |x - x_i|^s, \]

Finally, we have:
\[ |\int_{B(x_i, \delta_i)} < x_i^t |\nabla (u_i - u) > (V_i - V_i(x_i)) |x|^{-2\alpha} e^{u_i} dy | \leq \]
\[ \leq C \int_{B(x_i, \delta_i)} |\nabla (u_i - u)| (|x|^{-2\alpha} e^{u_i})^{(3/4-s/2)}, \]

But in the second member, for \( 1/2 < s \leq 1 \), we have \( q_s = 1/(3/4 - s/2) > 2 \) and thus \( q_s' < 2 \) and,
\[ (|x|^{-2\alpha} e^{u_i})^{3/4-s/2} \in L^{q_s} \]
\[ |x|^{-2\alpha(3/4-s/2)} e^{((3/4)-(s/2))u_i} \in L^{q_s'} \]

one conclude that:
\[ \int_{B(x_i, \delta_i)} < x_i^t |\nabla (u_i - u) > (V_i - V_i(x_i)) |x|^{-2\alpha} e^{u_i} dy = o(1) \]

Finally, with this method, we conclude that, in the case of one blow-up point and \( V_i \) is \( s \)-Holderian with \( 1/2 < s \leq 1 \):
\[ 2(1 - 2\alpha) V_i(x_i) \int_{B(x_i, \delta_i)} |x|^{-2\alpha} e^{u_i} dy = o(1) \]

which means that there is no blow-up, which is a contradiction.

Finally, for one blow-up point and \( V_i \) is is \( s \)-Holderian with \( 1/2 < s \leq 1 \), the sequence \( (u_i) \) is uniformly bounded on \( \Omega \).

REFERENCES

[1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag 1998
[2] S.S Bahoura. Majorations du type sup u × inf u ≤ c pour l’équation de la courbure scalaire sur un ouvert de \( \mathbb{R}^n \), n ≥ 3. J. Math. Pures. Appl.(9) 83 2004 no, 9, 1109-1150.
[3] S.S Bahoura. Harnack inequalities for Yamabe type equations. Bull. Sci. Math. 133 (2009), no. 8, 875-892
[4] S.S Bahoura. Lower bounds for sup+inf and sup × inf and an extension of Chen-Lin result in dimension 3. Acta Math. Sci. Ser. B Engl. Ed. 28 (2008), no. 4, 749-758
[5] S.S Bahoura. Estimations uniformes pour l’équation de Yamabe en dimensions 5 et 6. J. Funct. Anal. 242 (2007), no. 2, 550-562.
[6] C. Bandle. Isoperimetric inequalities and Applications. Pitman. 1980.
[7] H. Brezis, Y.Y. Li , I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.
[8] H.Brezis and F.Merle, Uniform estimates and blow-up behavior for solutions of \(-\Delta u = Ve^u\) in two dimensions, Commun Partial Differential Equations 16 (1991), 1223-1253.
[9] L. Caffarelli, B. Gidas, J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 37 (1984) 369-402.
[10] W. Chen, C. Li. A priori Estimates for solutions to Nonlinear Elliptic Equations. Arch. Rational. Mech. Anal. 122 (1993) 145-157.
[11] C.-C.Chen, C.-S. Lin. Estimates of the conformal scalar curvature equation via the method of moving planes. Comm. Pure Appl. Math. L(1997) 0971-1017.
[12] C.-C.Chen, C.-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in \( \mathbb{R}^2 \). Commun. Anal. Geom. 6, No.1, 1-19 (1998).
[13] C-C. Chen, C-S. Lin. Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. Comm. Pure Appl. Math. 55 (2002), no. 6, 728-771.

[14] B. Gidas, W-Y. Ni, L. Nirenberg. Symmetry and related properties via the maximum principle. Comm. Math. Phys. 68 (1979), no. 3, 209-243.

[15] D. Gilbarg, N.S. Trudinger. Second elliptic equations. Springer.

[16] J.M. Lee, T.H. Parker. The Yamabe problem. Bull. Amer. Math. Soc. (N.S) 17 (1987), no. 1, 37-91.

[17] YY. Li. Prescribing scalar curvature on $S^n$ and related Problems. C.R. Acad. Sci. Paris 317 (1993) 159-164. Part I: J. Differ. Equations 120 (1995) 319-410. Part II: Existence and compactness. Comm. Pure Appl. Math. 49 (1996) 541-597.

[18] YY. Li. Harnack Type Inequality: the Method of Moving Planes. Commun. Math. Phys. 200; 421-444 (1999).

[19] YY. Li, I. Shafrir. Blow-up Analysis for Solutions of $-\Delta u = Ve^u$ in Dimension Two. Indiana. Math. J. Vol 3, no 4. (1994), 1255-1270.

[20] YY. Li, L. Zhang. A Harnack type inequality for the Yamabe equation in low dimensions. Calc. Var. Partial Differential Equations 20 (2004), no. 2, 133-151.

[21] YY. Li, M. Zhu. Yamabe Type Equations On Three Dimensional Riemannian Manifolds. Comm. Contem.Mathematics, vol 1. No.1 (1999) 1-50.

[22] L. Ma, J-C. Wei. Convergence for a Liouville equation. Comment. Math. Helv. 76 (2001) 506-514.

[23] I. Shafrir. A sup+inf inequality for the equation $-\Delta u = Ve^u$. C. R. Acad. Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.

[24] L. Zhang. Blowup solutions of some nonlinear elliptic equations involving exponential nonlinearities. Comm. Math. Phys. 268 (2006), no. 1, 105-133.

DEPARTEMENT DE MATHEMATIQUES, UNIVERSITE PIERRE ET MARIE CURIE, 2 PLACE JUSSIEU, 75005, PARIS, FRANCE.

E-mail address: samybahours@yahoo.fr, samybahours@gmail.com