On improvement of the series convergence in the problem of the vibrations of orthotropic rectangular prism

A D Lyashko
V I Vernadsky Crimean Federal University, 4 Vernadskogo Prospekt, Simferopol, Crimea, Russia
E-mail: knightla@yandex.ru

Abstract. A new analytical presentation of the solution for steady-state oscillations of orthotropic rectangular prism is found. The corresponding infinite system of linear algebraic equations has been deduced by the superposition method. A countable set of precise eigenfrequencies and elementary eigenforms is found. The identities are found which make it possible to improve the convergence of all the infinite series in the solution of the problem. All the infinite series in presentation of solution are analytically summed up. Numerical calculations of stresses in the rectangular orthotropic prism with a uniform along the border and harmonic in time load on two opposite faces have been performed.

1. Introduction
Orthotropic prisms and plates are among the most widespread and modern elements of structures in engineering and construction. Many researches [1-6] applied different variants of the superposition method to problems of vibrations of rectangular prisms and plates. Dynamic behavior of thin orthotropic plates is studied using the Rayleigh-Ritz method most fully in [7]. One of the common numerical methods applied to the study of these problems is finite element method [8].

Grinchenko and Meleshko considered the problem of vibration of isotropic rectangular prism in [1]. They constructed the displacements as sum of two solutions each of which allows satisfying arbitrary conditions on displacement and stress on corresponding pair of parallel sides of the prism. The generalization of this approach for orthotropic plates is considered by Papkov [2]. Even in case of isotropic prisms the use of obtained solutions is difficult because of logarithmic singularities in normal stresses near the corners. The convergence of series in expressions for stresses deteriorates near the sides of the prism. Because of this convergence of series is being improved in one way or another. Additionally a countable set of elementary eigenforms for a countable set of ratios of prism's sides is specified in [1] called Lamé modes.

In this paper all the series in presentation of solution are analytically summed up. The convergence of series is improved with the help of theory of regular infinite system of linear algebraic equations. The generalization of Lamé modes for orthotropic prisms is obtained.

2. Formulation of the problem and analytical presentation of solution
The plane deformation of orthotropic prism, cross-section of which is a rectangle with sides $2a \times 2b$, is considered. Prism's sides are parallel to X and Y axes. The origin of coordinate system is at the
prism's mid-point. The basis vectors are normal to planes of symmetry of orthotropic material. The equations of motion for an elastic solid are written as follows:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = \rho \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2}.
\]  

(1)

The stress-displacement relations are given by the generalized Hooke's law:

\[
\frac{1}{E_i} \sigma_{xx} = \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y}, \quad \frac{1}{E_2} \sigma_{yy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \frac{\sigma_{xy}}{G} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},
\]

(2)

where \( v_1, v_2, E_1, E_2 \) and \( G \) are respectively Poisson's ratios, Young's moduli in direction of \( X \) and \( Y \) axes and shear modulus.

Steady-state vibration of rectangular prism, symmetric about both axes, under load normal to its sides is considered.

\[
\frac{1}{E_i} \sigma_{xx} \bigg|_{x=L,y=0} = f(y)\sin(\omega t), \quad \frac{\sigma_{xx}}{G} \bigg|_{x=L,y=0} = 0,
\]

(3)

\[
\frac{1}{E_2} \sigma_{yy} \bigg|_{y=L,x=0} = g(x)\sin(\omega t), \quad \frac{\sigma_{yy}}{G} \bigg|_{y=L,x=0} = 0.
\]

(4)

Amplitudes of displacements and stresses are introduced as follows:

\[
\bar{u} = u \sin(\omega t), \quad \bar{v} = v \sin(\omega t),
\]

\[
\bar{\sigma}_{xx} = \sigma_{xx} \sin(\omega t), \quad \bar{\sigma}_{yy} = \sigma_{yy} \sin(\omega t), \quad \bar{\sigma}_{xy} = \sigma_{xy} \sin(\omega t).
\]

(5)

(6)

Analytical presentation of the solution is built based on the superposition method outlined in [1]. Functions in the solutions and part of coefficients are chosen to satisfy differential equations and boundary conditions for shear stress. To satisfy the boundary conditions for normal stress an infinite system of linear equations is constructed. Under this approach a new analytical presentation of solution for amplitudes of displacements was obtained in the form of the following trigonometric Fourier series:

\[
\begin{align*}
\bar{u} &= \frac{N_1}{b} y_0 \sin\Omega_2 x + \frac{2N_2}{a} \sum_{n=1}^{\infty} X_n (1)^n \left( B_{11}(n, \alpha, p) \frac{\cosh q_{mn}^x}{\sinh p_{mb}} + B_{12}(n, \alpha, p) \frac{\cosh q_{mn}^y}{\sinh p_{mb}} \right) \sin p_{mn}^a - \\
&\quad + \frac{2N_1}{b} \sum_{n=1}^{\infty} Y_n (-1)^n \left( C(\beta_n, q_{mn}^x, E_2) B_{21}(n, m, \beta, q) \frac{\sinh q_{mn}^x}{\sinh q_{mn}^a} \right) \cos \beta_{mn}^y,
\end{align*}
\]

(7)

\[
\begin{align*}
\bar{v} &= \frac{N_2}{a} x_0 \sin\Omega_1 y - \frac{2N_2}{a} \sum_{n=1}^{\infty} X_n (1)^n \left( B_{11}(n, m, \beta, q) \frac{\cosh q_{mn}^x}{\sinh q_{mn}^a} + B_{12}(n, m, \beta, q) \frac{\cosh q_{mn}^y}{\sinh q_{mn}^a} \right) \sin \beta_{mn}^y + \\
&\quad + \frac{2N_1}{b} \sum_{n=1}^{\infty} Y_n (-1)^n \left( C(\alpha_n, p_{mn}^x, E_1) B_{21}(n, \alpha, p) \frac{\sinh p_{mn}^x}{\sinh p_{mb}} \right) \cos \alpha_{mn}^x.
\end{align*}
\]

(8)
Here

\[ \Omega_{11} = \Omega_2 \sqrt{G(1 - \nu_1 v_2)} / E_1, \quad \Omega_{12} = \Omega_2 \sqrt{G(1 - \nu_1 v_2)} / E_2, \quad \Omega_2 = \pi / 2, \quad \Omega = \frac{2 \omega}{\pi \sqrt{G}}, \]

\[ B_{jk}(n, \gamma, r) = \frac{r_{kn} \alpha_n (r_{kn} v_j + r_n^2 - \Omega_{ij}^2)}{(r_{kn}^2 - r_n^2)(\Omega_{ij}^2 - \Omega_{ij}^2)(\Omega_{ij}^2 - \Omega_{ij}^2)}/(1 - v_1 v_2)/(1 + v_1 v_2), \quad j = 3 - j, \]

\[ N_j = \sqrt{1 + v_j}, \quad C(\gamma, r, E) = \frac{(1 - v_1 v_2)(r^2 + \Omega_{ij}^2) - \gamma^2 E / G}{(1 + v_1 E_2 / G - v_1 v_2) r^2}, \quad a_n = \frac{n \pi}{a}, \quad \beta_m = \frac{m \pi}{b}. \]

\[ p_{in}, p_{2m}, q_{lm}, q_{2m} \] are the roots of the following biquadratic equations:

\[ p^2 + \left( \Omega_{11}^2 + \Omega_{12}^2 - \left( \frac{E_1}{G} - 2v_1 \right) \alpha_n^2 \right) p^2 + \left( \frac{2}{v_1} \right) \frac{\alpha_n^2 (\alpha_n^2 - \Omega_{11}^2)(\alpha_n^2 - \Omega_{12}^2)}{0}, \quad (9) \]

\[ q^2 + \left( \Omega_{11}^2 + \Omega_{12}^2 - \left( \frac{E_2}{G} - 2v_2 \right) \beta_m^2 \right) q^2 + \left( \frac{2}{v_1} \right) \frac{\beta_m^2 (\beta_m^2 - \Omega_{11}^2)(\beta_m^2 - \Omega_{12}^2)}{0}. \quad (10) \]

Shear stress vanishes on the faces of the prism due to the choice of the presentation of the solution. Substitution of expressions for normal stresses into the boundary conditions after a known procedure of the superposition method leads to an infinite system of linear algebraic equations with respect to indeterminate coefficients \( X_k, Y_k \) \( (k = 0, 1, 2, \mathcal{K}) \). Constant terms in the infinite system are proportional to coefficients of Fourier series expansion of boundary conditions for normal stresses:

\[ f(y) = f_0 + \sum_{m=1}^{\infty} (-1)^m f_m \cos \beta_m y, \quad g(x) = g_0 + \sum_{n=1}^{\infty} (-1)^n g_n \cos \alpha_n x. \quad (11) \]

The infinite system for the solution presentation (7), (8) is as follows:

\[ X_0 \cot(\Omega_{n,b}) N_2 = \frac{v_2}{\Omega_{12}} Y_0 + \frac{2}{b} \sum_{m=1}^{\infty} \frac{\Omega_{12}^2 v_2}{(\beta_m^2 - \Omega_{11}^2)(\Omega_{11}^2 - \Omega_{12}^2)} - a \Omega_1 \frac{v_2}{\Omega_{11}} N_2 g_0, \]

\[ Y_0 \cot(\Omega_{a,b}) N_2 = \frac{v_1}{\Omega_{11}} X_0 + \frac{2}{a} \sum_{n=1}^{\infty} \frac{\Omega_{11}^2 v_1}{(\alpha_n^2 - \Omega_{11}^2)(\Omega_{11}^2 - \Omega_{12}^2)} + b \Omega_1 \frac{v_1}{\Omega_{11}} N_1 f_0, \]

\[ X_0 \Omega_{n,a} N_1 = \frac{1}{b} \frac{v_3}{\alpha_n - \Omega_{11}} Y_0 + \frac{2}{b} \sum_{m=1}^{\infty} \frac{\Omega_{11} v_3}{(\alpha_n^2 + q_{in})(\alpha_n^2 + d_{2m})(\beta_m^2 - \Omega_{11}^2)} + a \frac{2}{2} N_2 g_n \quad (n = 1, 2, \mathcal{K}), \]

\[ Y_0 \Omega_{m,a} N_1 = \frac{1}{a} \frac{v_3}{\beta_m - \Omega_{12}} X_0 + \frac{2}{a} \sum_{m=1}^{\infty} \frac{\Omega_{11} v_3}{(\beta_m^2 + p_{2m})(\beta_m^2 + p_{2m}^2)(\alpha_n^2 - \Omega_{11}^2)} + b \frac{2}{2} N_2 f_m \quad (m = 1, 2, \mathcal{K}), \]

where

\[ A_{n,m} = \frac{\beta_m^2 (\alpha_n^2 (1 - v_1 v_2) - \nu_1 \Omega_{ij}^2) - \nu_1 \Omega_{ij}^2 (\alpha_n^2 - \Omega_{ij}^2)}{\delta_n = B_{11}(n, \alpha, p)(v_1 \alpha_n + C(\alpha_n, p_{in}, E_1) p_{in}) \cot p_{in} b + B_{12}(n, \alpha, p)(v_1 \alpha_n + C(\alpha_n, p_{2m}, E_1) p_{2m}) \cot p_{2m} b}, \]

\[ \delta_m = B_{21}(m, \beta, q)(v_2 \beta_n + C(\beta_n, q_{lm}, E_2) q_{lm}) \cot q_{lm} a + B_{22}(m, \beta, q)(v_2 \beta_n + C(\beta_n, q_{2m}, E_2) q_{2m}) \cot q_{2m} a. \]
3. Lamé modes and infinite system study

The presentation of the solution (7), (8) differs from other known presentations [2] by the fact that a unit solution \( X_k=Y_k=1 \) of infinite system (12) corresponds to the following elementary analytical solution of the steady-state vibration of orthotropic prism:

\[
\begin{align*}
    u &= \frac{1 \pm \nu_1 \nu_2}{\Omega_2} \sin(\nu_2 x \sqrt{G(1 + \nu_2) / E_2}) \sin(\nu_1 x \sqrt{G(1 + \nu_1) / E_1}), \\
    v &= -\frac{1 \pm \nu_1 \nu_2}{\Omega_1} \sin(\nu_2 y \sqrt{G(1 + \nu_2) / E_2}) \cos(\nu_1 y \sqrt{G(1 + \nu_1) / E_1}), \\
    \frac{1 \pm \nu_1 \nu_2}{E_1} \sigma_{xx} &= \frac{\nu_1 \pm \nu_1 \nu_2}{\nu_2} \frac{\nu_2 \pm \nu_1 \nu_2}{E_2} \sigma_{yy}, \quad \frac{\nu_1 \pm \nu_1 \nu_2}{E_1} \sigma_{yy} = 0.
\end{align*}
\]

For the following discrete set of frequencies and prism sides ratios:

\[
\Omega a = (1 + 2k) \sqrt{\frac{E_1}{G(1 + \nu_1)}}, \quad \frac{b}{a} = \frac{1 + 2l}{1 + 2k} \sqrt{\frac{1 + \nu_1}{1 + \nu_2}}, \quad k, l \in \mathbb{Z},
\]

stresses (15), (16) vanish at prism border. Therefore, analytical presentation of solution (7), (8) proposed here leads not only to the exact elementary particular solution of infinite system of linear algebraic equations and the exact elementary solution of the steady-state vibration of orthotropic prism, but also to the infinite discrete set of eigenfrequencies and elementary eigenforms of rectangular orthotropic prism. In case of isotropic prism these eigenfrequencies and elementary eigenforms were found in another way in monograph [1] and called Lamé modes.

Knowing the unit solution of infinite system (12) significantly simplifies study of its regularity. Obtaining all Fourier series coefficients \( f_k, g_k \) for stresses (15), (16) and substituting them alongside with the unit solution \( X_k=Y_k=1 \) into infinite system (12) produces identities from which follow simple exact formulas for sum of coefficients in each equation of infinite system:

\[
\begin{align*}
    S_{m0} &= 1 - \frac{\tan \Omega_{12} b}{\tan(\Omega_2 b \sqrt{\frac{G(1 + \nu_2)}{E_2}})} \frac{1 - \nu_1 \nu_2}{1 + \nu_2}, \quad S_{00} = 1 - \frac{\tan \Omega_{12} a}{\tan(\Omega_2 a \sqrt{\frac{G(1 + \nu_1)}{E_1}})} \frac{1 - \nu_1 \nu_2}{1 + \nu_1}, \\
    S_{m} &= 1 - \frac{1}{\delta_n} \cot \left( \Omega_2 b \sqrt{\frac{G(1 + \nu_2)}{E_2}} \right) \frac{1 - \nu_1 \nu_2}{1 + \nu_2} \Omega_{12} \frac{\nu_1}{\nu_2} \frac{1 - \nu_1 \nu_2}{1 + \nu_1} \Omega_{12}^2, \\
    S_{0m} &= 1 - \frac{1}{\delta_m} \cot \left( \Omega_2 a \sqrt{\frac{G(1 + \nu_2)}{E_1}} \right) \frac{1 - \nu_1 \nu_2}{1 + \nu_1} \frac{\nu_1}{\nu_2} \Omega_{12} \frac{\nu_1}{\nu_2} \frac{1 - \nu_1 \nu_2}{1 + \nu_1} \Omega_{12}^2.
\end{align*}
\]

For sufficiently low vibration frequency all coefficients are positive and these expressions are less than 1. Thus infinite system (12) is regular in this case. Regularity of this infinite system is further studied in article [9].

Finally, the presence of a solution of the infinite system that tends to a nonzero constant allows us to make an assumption that in general case solutions of infinite system follow the asymptotic law:

\[
\lim_{k \to \infty} X_k = \lim_{k \to \infty} Y_k = L.
\]
In view of this assumption it is possible to apply method [1] of «improved reduction». In accordance with this method an assumption is made that starting with some number \( P \) all variables are approximately equal to the limit \( L \). The following change of variables:

\[
X_k = \bar{X}_k + X_k L, \: Y_k = \bar{Y}_k + Y_k L \quad (k = 0,1,\ldots P-1), \tag{22}
\]

\[
X_k = L, \: Y_k = L \quad (k = P, P+1,\ldots) \tag{23}
\]

results in two infinite systems of linear equations with the same matrix with respect to \( \bar{X}_k, \bar{Y}_k \) and \( \bar{X}_k, \bar{Y}_k \). To find the limit \( L \) an approach analogous to the one described in [1] for isotropic case is employed. The relation between the variables of solutions included in the infinite system is obtained by equating the difference of normal stresses at the corner point to the value known from boundary conditions (3), (4):

\[
\frac{1 - \nu_1 \nu_2}{E_1} \sigma_{xx}(a,b) - \frac{1 - \nu_1 \nu_2}{E_2} \sigma_{yy}(a,b) = f(b) - g(a). \tag{24}
\]

After substitution of the expressions for normal stresses into equation (24) the expression linear with respect to \( L \) is obtained. The series in this expression converge quickly therefore they are mainly determined by the values of the first variables.

4. Improvement of the series convergence and discussion of results

Direct evaluations show that series in normal stresses diverge in corner points. The asymptotical law (21) and known unit solution allows improving series convergence at the border and within the prism. In order to do so Krylov’s method of convergence improvement [10] is used. The limit \( L \) is added to and subtracted from each indeterminate coefficient. The added terms are separated and summed up with the help of identity for the unit solution. Having done this procedure for normal stress \( \sigma_{xx} \) the following expression is obtained:

\[
\left. \frac{1 - \nu_1 \nu_2}{E_1} \sigma_{xx} \right|_{k=+a} = 1, \quad \left. \frac{1 - \nu_1 \nu_2}{E_2} \sigma_{yy} \right|_{y=+b} = \frac{\tau_{xy}}{G} \left|_{k=+a} = \frac{\tau_{y}}{G} \right|_{y=+b} = 0. \tag{26}
\]
The corresponding infinite system of equations is solved and the stresses along prism’s sides are computed. The results of computation with $P=21$ are shown in table 1 and table 2 for stresses $\sigma_{xx}$ and $\sigma_{yy}$ accordingly.

**Table 1. Stress $\sigma_{xx}$ along the side $x=a$ with $P=21$.**

| y/b  | Without convergence improvement | With convergence improvement |
|------|--------------------------------|-----------------------------|
| 0    | 0.9389                         | 1.0003                      |
| 0.2  | 0.9391                         | 1.0003                      |
| 0.4  | 0.9399                         | 1.0003                      |
| 0.6  | 0.9426                         | 1.0003                      |
| 0.8  | 0.9571                         | 1.0004                      |
| 1.0  | 4.9553                         | 0.9646                      |

**Table 2. Stress $\sigma_{yy}$ along the side $y=b$ with $P=21$.**

| x/a  | Without convergence improvement | With convergence improvement |
|------|--------------------------------|-----------------------------|
| 0    | 0.0942                         | 0.0000                      |
| 0.2  | 0.0940                         | 0.0000                      |
| 0.4  | 0.0928                         | 0.0000                      |
| 0.6  | 0.0891                         | 0.0000                      |
| 0.8  | 0.6899                         | 0.0003                      |
| 1.0  | -7.2397                        | -0.0354                     |

More accurate the boundary condition satisfaction may be achieved by increasing the number of equations in systems for $X_k, \tilde{Y}_k$ and $\tilde{X}_k, \tilde{Y}_k$, i.e. increasing the number of indeterminate coefficients that are not considered approximately equal to $L$. The corresponding stresses computed with $P=101$ are presented in table 3 and table 4. Without convergence improvement the boundary conditions near the corners are not satisfied independently of $P$ due to logarithmic singularity in the series.

**Table 3. Stress $\sigma_{xx}$ along the side $x=a$ with $P=101$.**

| y/b  | Without convergence improvement | With convergence improvement |
|------|--------------------------------|-----------------------------|
| 0    | 0.9875                         | 1.0000                      |
| 0.2  | 0.9875                         | 1.0000                      |
| 0.4  | 0.9876                         | 1.0000                      |
| 0.6  | 0.9882                         | 1.0000                      |
| 0.8  | 4.9787                         | 1.0004                      |
| 1.0  |                                 |                             |

**Table 4. Stress $\sigma_{yy}$ along the side $y=b$ with $P=101$.**

| x/a  | Without convergence improvement | With convergence improvement |
|------|--------------------------------|-----------------------------|
| 0    | 0.0193                         | 0.0000                      |
| 0.2  | 0.0193                         | 0.0000                      |
| 0.4  | 0.0193                         | 0.0000                      |
| 0.6  | 0.0182                         | 0.0000                      |
| 0.8  | -7.1868                        | 0.0004                      |
| 1.0  |                                 |                             |

The difference of the proposed presentation of solution (7), (8) from other known solutions is that the convergence of series may be easily improved not only along the sides of the prism but also inside it. The results of computation of stresses at the distance of one twentieth of prism length from the sides are presented in table 5 and table 6. The magnitude of stresses may be explained by the proximity to the eigenfrequency since frequency $\Omega = 1,47$ differs from the one specified in [2] by less than 1%. As it can be seen from these tables the relative error of stresses is greater than 0,001 reaching 0,15 for the stress $\sigma_{xx}$ at the prism’s side. Although the values without convergence improvement may be computed more accurately by increasing the number of equations, the closer the stresses are evaluated to the sides the more equations are required.

**Table 5. Stress $\sigma_{xx}$ along $x=0.9a$ with $P=21$.**

| y/b  | Without convergence improvement | With convergence improvement |
|------|--------------------------------|-----------------------------|
| 0    | 2.8901                         | 2.8979                      |
| 0.2  | 2.7831                         | 2.7909                      |
| 0.4  | 2.4642                         | 2.4717                      |
| 0.6  | 1.9409                         | 1.9474                      |
| 0.8  | 1.2402                         | 1.2412                      |
| 1.0  | 0.7544                         | 0.6449                      |


Table 6. Stress $\sigma_{yy}$ along $y=0.9b$ with $P=21$.

| x/a  | 0.0  | 0.2  | 0.4  | 0.6  | 0.8  | 1.0  |
|------|------|------|------|------|------|------|
| Without convergence improvement | 9.87883 | 10.2297 | 11.2212 | 12.6790 | 14.3358 | 15.6916 |
| With convergence improvement   | 9.8522 | 10.2032 | 11.1955 | 12.6556 | 14.3248 | 16.1961 |

5. Conclusion

In this paper the new presentation of a solution for steady-state vibrations of rectangular orthotropic prism is described. Corresponding infinite system of linear algebraic equations was obtained by the superposition method. A countable set of precise eigenfrequencies and elementary eigenforms was found. The identities were created which make it possible to find the intervals where the infinite system is regular and to discover an asymptotic behavior of variables. That asymptotic law allows improving convergence of the infinite series and finding the analytical sums for all the infinite series in the solution of the problem within the prism and on its faces. The numerical comparison of stresses on the faces of the prism with and without convergence improvement is performed.

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