Geodesic flows and contact toric manifolds

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Forward

These notes are based on five 1.5 hour lectures on torus actions on contact manifolds delivered at the summer school on Symplectic Geometry of Integrable Hamiltonian Systems at Centre de Recerca Matemàtica in Barcelona in July 2001. Naturally the notes contain more material that could have been delivered in 7.5 hours. I am grateful to Carlos Curràs-Bosch and Eva Miranda, the organizers of the summer school, for their kind invitation to teach a course. Thanks are also due to the staff of the CRM without whom the summer school would not have been a success.

The main theme of these notes is the topological study of contact toric manifolds, a relatively new class of manifolds that I find very interesting. A motivation for studying these manifolds comes from completely integrable systems \( \{ f_1, \ldots, f_n \} \) on punctured cotangent bundles where each function \( f_i \) is homogeneous of degree 1 (one can think of \( f_i \)'s as symbols of first order pseudo-differential operators, but this is not essential). A punctured cotangent bundle is a symplectic cone whose base is naturally a contact manifold (this is explained in detail in Chapter 2). This observation leads to studying completely integrable systems on contact manifolds, whatever those are.

The simplest (symplectic) completely integrable systems are the ones with global action-angle coordinates. The next simplest case is that of Hamiltonian torus actions. If the phase space is compact one ends up with (compact) symplectic toric manifolds. This is the theme of Ana Cannas’s lectures delivered at the summer school. The corresponding case in the contact category is that of compact toric manifolds.

We will use the excuse of studying completely integrable geodesic flows with homogeneous integrals to introduce various ideas essential for the classification of contact and symplectic toric manifolds. More specifically we will discuss in these notes contact moment maps, slices for group actions, sheaves and Čech cohomology, orbifolds and Morse theory on orbifolds.
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Chapter 1

Introduction

We start with an innocuous sounding problem.

**Problem.** Consider the cotangent bundle of the $n$-torus minus the zero section $T^*\mathbb{T}^n \setminus \{0\}$. That is, consider the manifold

$$M = \{(q,p) \in \mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n | p \neq 0\}.$$ 

Suppose further that the torus $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ acts on $M$ effectively and preserves the standard symplectic form $\omega = \sum dp_i \wedge dq_i$ (i.e., the action is symplectic). Suppose further that the action of $G$ commutes with dilations, i.e., the action $\rho$ of $\mathbb{R}$ on $M$ given by

$$\rho_\lambda(q,p) = (q,e^{\lambda}p).$$

Is the action of $G$ necessarily free?

**Remark 1.1.**

1. Recall that an action of the group $G$ on a manifold $M$ is **effective** if the only element of $G$ that fixes all the points of $M$ is the identity.

2. There is an “obvious” action of $\mathbb{T}^n$ on $M$ which has the above properties and is free — it is the lift of left multiplication:

$$a \cdot (q,p) = (aq,p),$$

where $a \in \mathbb{T}^n$, $(q,p) \in M$. The issue is whether an arbitrary action of $\mathbb{T}^n$ which is effective, symplectic and commutes with dilations is necessarily free.

3. We will see later (Proposition 2.6) that an action of a Lie group $G$ on a punctured cotangent bundle which is symplectic and commutes with dilations is necessarily Hamiltonian. That is, there is a moment map $\Phi : M \to g^*$. Recall the definition of the moment map for a symplectic action of a Lie
group $G$ on a symplectic manifold $(M,\omega)$: for any vector $X$ in the Lie algebra $\mathfrak{g}$ of $G$

$$d\langle \Phi, X \rangle = \omega(X_M, \cdot),$$

where $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the canonical pairing and $X_M$ is the vector field induced by the infinitesimal action of $X$: $X_M(x) = \frac{d}{dt}{|}_{t=0} (\exp tX) \cdot x$.

The problem is due to John Toth and Steve Zelditch [TZ]. The context is classical and quantum integrability of geodesic flows. Recall that for a manifold $Q$ with a Riemannian metric $g$ the corresponding geodesic flow is the flow on the cotangent bundle $T^*Q$ of the Hamiltonian vector field $X_h$ of the function $h = h_g \in C^\infty(T^*Q)$ which is the square root of the energy:

$$h_g(q,p) = (g^*(p,p))^{1/2}$$

for all $q \in Q, p \in T^*_q Q$. Here $g^*$ denotes the inner product on the cotangent bundle $T^*Q$ dual to the inner product $g$; $g^*$ is the so called dual metric. For an analyst the function $h$ is the principal symbol of the square root of the Laplace operator $\sqrt{\Delta}$ defined by the metric $g$. A precise definition of $\sqrt{\Delta}$ will play no role in these notes. For a Riemannian geometer it’s important that the integral curves of the vector field $X_h$ project down to geodesics on the manifold $Q$. Toth and Zelditch were interested in the meaning of $L^\infty$ boundedness of the $L^2$-normalized eigenfunctions of $\sqrt{\Delta}$. They observed that the question is easier if $\sqrt{\Delta}$ is quantum completely integrable. Again, it will not be important to us as to what that means precisely. What will matter is that quantum integrability of $\sqrt{\Delta}$ implies (classical) homogeneous complete integrability of the geodesic flow. Namely, it implies that there exist functions $f_1 = h, f_2, \ldots, f_n \in C^\infty(T^*Q \setminus 0), n = \dim Q$, such that

1. the functions $f_1, \ldots, f_n$ are functionally independent on an open dense set $U \subset T^*Q \setminus 0$, i.e., $df_1 \wedge \ldots \wedge df_n \neq 0$ on $U$;
2. the functions Poisson commute with each other: $\{f_i, f_j\} = 0$ for all $1 \leq i, j \leq n$;
3. the functions $f_i$ are homogeneous of degree 1:

$$\rho_\lambda^* f_i = e^{\lambda} f_i$$

for all $\lambda \in \mathbb{R}$, where $\rho_\lambda : T^*Q \setminus 0 \to T^*Q \setminus 0$ again denotes the dilation $\rho_\lambda(q,p) = (q, e^{\lambda} p), q \in Q, p \in T^*_q Q$.

Exercise 1.2. Suppose $f$ is a smooth function on the punctured cotangent bundle $T^*Q \setminus 0$ of a manifold $Q$ which is homogeneous of degree 1, i.e., $\rho_\lambda^* f = e^{\lambda} f$ for all dilations $\rho_\lambda$. Show that its Hamiltonian vector field $X_f$ (relative to the standard symplectic structure on $T^*Q$) satisfies

$$d\rho_\lambda(X_f) = X_f \circ \rho_\lambda.$$

\[ T^*Q \setminus 0 \text{ denotes the punctured cotangent bundle of } Q, \text{ that is, } T^*Q \text{ with the zero section deleted.} \]
Conclude that the flow of $X_h$ commutes with dilations.

Given a completely integrable system, we know that locally around any generic point there exist action-angle variables. Toth and Zelditch observed that things are considerably simpler if the action-angle variables are global. Then there exists an effective action of a torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ on $T^*Q \setminus 0$ preserving the function $h$ and the symplectic form, and commuting with dilations $\rho_\lambda$. Toth and Zelditch proved (op. cit.):

**Theorem 1.3.** Suppose that the Lie group $G = \mathbb{T}^n$ acts effectively on $M = T^*\mathbb{T}^n \setminus 0$, preserving the standard symplectic form and the function $h_g(q,p) = (g^*_q(p,p))^{1/2}$ for some metric $g$ on $\mathbb{T}^n$. Suppose further the action commutes with dilations $\rho_\lambda : M \to M$, $\rho_\lambda(q,p) = (q, e^{\lambda}p)$. Then the action of $G$ is free then the metric $g$ is flat, that is,

$$g = \sum g_{ij} \, dq_i \otimes dq_j$$

for some constants $g_{ij}$ (with $g_{ij} = g_{ji}$).

The eigenfunctions of a flat metric Laplace operator on a torus are well understood.

Let us now go back to the problem. The answer to the question is yes [LS]:

**Theorem 1.4.** Suppose the Lie group $G = \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ acts effectively on the punctured cotangent bundle $M = T^*\mathbb{T}^n \setminus 0$ preserving the standard symplectic form and commuting with dilations $\rho_\lambda : M \to M$, $\rho_\lambda(q,p) = (q, e^{\lambda}p)$. Then the action of $G$ is free.

The main purpose of these notes is to explain why Theorem 1.4 is true. I will now try to motivate the proof and to put it in a broader context. As was remarked previously (Remark 1.1(3)), the action of the torus $G$ on $M = T^*\mathbb{T}^n \setminus 0$ is Hamiltonian. By the dimension count the manifold $M$ together with its natural symplectic structure and the action of $G$ is a symplectic toric manifold. Recall the definition.

**Definition 1.5.** A symplectic toric manifold is a triple $(M, \omega, \Phi : M \to g^*)$ where $M$ is a manifold, $\omega$ is a symplectic form on $M$, and $\Phi$ is a moment map for an effective Hamiltonian action of a torus $G$ on $(M, \omega)$ satisfying $2 \dim G = \dim M$.

Compact symplectic toric manifolds are well understood thanks to a classification theorem of Delzant [D], which says that all such manifolds are classified by the images of the corresponding moment maps. Note that by the Atiyah - Guillemin - Sternberg convexity theorem [A, GS1], the images are convex rational polytopes. Delzant proved that in the toric case the polytopes are simple\(^2\) and additionally satisfy certain integrality conditions. Finally any simple polytope satisfying the integrality conditions occurs as an image of the moment map for a compact symplectic toric manifold.

\(^2\)A polytope in an $n$-dimensional real vector space is simple if there are exactly $n$ edges meeting at each vertex. Equivalently, all the supporting hyperplanes are in general position. Thus a cube and a tetrahedron are simple and an octahedron is not.
Exercise 1.6. Show that a Hamiltonian torus action on a compact symplectic manifold is never free. Hint: any smooth function on a compact manifold has a critical point (in fact it has at least two — a maximum and a minimum).

The manifold $M = T^*\mathbb{T}^n \setminus 0$ we are interested in is not compact. Worse, we will see that the corresponding moment map $\Phi : M \to \mathfrak{g}^*$ is homogeneous (Proposition 2.6):

$$\Phi(\rho\lambda(m)) = e^\lambda \Phi(m)$$

for all $m \in M$, $\lambda \in \mathbb{R}$. This is a bit of bad news — Morse theory is an essential ingredient in the proof of the Atiyah-Guillemin-Sternberg convexity theorem and hence of Delzant’s classification. Morse functions on a noncompact manifold which are not bounded either above or below are in practice impossible to work with.

On the other hand, because of homogeneity, the moment map descends to the quotient of $M$ by the action of $\mathbb{R}$. The quotient is diffeomorphic to the co-sphere bundle $S^*\mathbb{T}^n := \{(q, p) \in T^*\mathbb{T}^n \mid g^*_q(p, p) = 1\}$ for some dual metric $g^*$. Since $S^*\mathbb{T}^n$ is odd-dimensional, it is not a symplectic manifold.\(^3\)

It is also easy to see in an example that the map induced on $S^*\mathbb{T}^n$ by a homogeneous moment map on $T^*\mathbb{T}^n \setminus 0$ behaves rather strangely if one is used to symplectic moment maps:

Example 1.7. Consider the standard action of $G = \mathbb{T}^n$ on the cotangent bundle $T^*\mathbb{T}^n$, the lift of the left multiplication:

$$a \cdot (q, p) = (a \cdot q, p)$$

The corresponding moment map $\Phi : T^*G = G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is given by $\Phi(q, p) = p$. Fix the standard metric on $G$ and identify the dual of the Lie algebra $\mathfrak{g}^*$ with $\mathbb{R}^n$. Then the co-sphere bundle $S^*G$ is $\mathbb{T}^n \times S^{n-1}$. The map $\Phi' = \Phi|_{S^*G} : S^*G \to \mathfrak{g}^*$ is also given by $\Phi'(q, p) = p$. Note that for any nonzero vector $X \in \mathfrak{g}$ the function $\langle \Phi', X \rangle$ has exactly two critical manifolds even though the action of $G$ on $S^*G$ is free!

Compare this with the symplectic situation where critical points of components of moment maps are points with nontrivial isotropy groups. What are we dealing with? We are dealing with moment maps for group actions on contact manifolds.

We finish the section by sketching a strategy for our proof of Theorem 1.4. The effective action of the torus $G$ on $M = T^*\mathbb{T}^n \setminus 0$ descends to an effective action on the quotient $B = S^*\mathbb{T}^n$ of $M$ by dilations. The action of $G$ on $B$ preserves a contact structure $\xi$ (see Definition 2.14 below) making $(B, \xi)$ into a compact connected contact toric $G$-manifold (c.c.c.t.m., see Definition 2.30). We then study all c.c.c.t.m.’s with a non-free torus actions and argue that none of them can have the homotopy type of $B = \mathbb{T}^n \times S^{n-1}$.

\(^3\)The manifold $S^*\mathbb{T}^n$ is contact. See next Chapter and the Appendix.
Chapter 2

Symplectic cones and contact manifolds

In this section we define symplectic cones, contact forms and contact structures. Given a symplectic cone we show how to construct the corresponding contact manifold, and conversely, given a contact manifold we construct the corresponding symplectic cone. Thus symplectic manifolds and contact manifolds are “the same thing.” Next we show that a symplectic action of a Lie group on a symplectic cone induces a contact action on the corresponding contact manifold. This will give us tools to set up a proof of Theorem 1.4 as a study of contact toric manifolds. The material in this section is fairly well known. We now start by defining symplectic cones.

Definition 2.1. A symplectic manifold $(M, \omega)$ is a symplectic cone if

- the manifold $M$ is a principal $\mathbb{R}$ bundle over some manifold $B$, called the base of the cone, and

- the action of the real line $\mathbb{R}$ expands the symplectic form exponentially. That is, $\rho^*_\lambda \omega = e^{\lambda} \omega$, where $\rho_\lambda$ denotes the diffeomorphism define by $\lambda \in \mathbb{R}$.

Definition 2.2. Recall that a map $f : X \to Y$ between two topological spaces is proper if the preimage of a compact set under $f$ is compact. An action of a Lie group $G$ on a manifold $M$ is proper if the map $G \times M \ni (g, m) \mapsto (g \cdot m, m) \in M \times M$ is proper.

By a theorem of Palais [P] the quotient $M/G$ of a manifold $M$ by a free proper action of a Lie group $G$ is a manifold and the orbit map $M \to M/G$ makes $M$ into a principal $G$ bundle (see also Remark 4.5 below). It follows that if a symplectic manifold $(M, \omega)$ has a complete vector field $X$ with the following two properties:
1. the action of \( \mathbb{R} \) induced by the flow of \( X \) is proper, and
2. the Lie derivative of the symplectic form \( \omega \) with respect to the vector field \( X \) is again \( \omega \):
\[
L_X \omega = \omega,
\]
then \( (M, \omega) \) is a symplectic cone relative to the induced action of \( \mathbb{R} \). This gives us an equivalent definition of a symplectic cone.

**Definition 2.3.** A **symplectic cone** is a triple \( (M, \omega, X) \) where
- \( M \) is a manifold,
- \( \omega \) is a symplectic form on \( M \),
- \( X \) is a vector field on \( M \) generating a proper action of the reals \( \mathbb{R} \) such that
\[
L_X \omega = \omega.
\]

**Example 2.4.** Let \( (V, \omega_V) \) be a symplectic vector space. The manifold \( M = V \setminus \{0\} \) is a symplectic cone with the action of \( \mathbb{R} \) given by \( \rho_\lambda(v) = e^{\lambda} v \). Clearly \( \rho_\lambda^* \omega_V = e^{\lambda} \omega_V \). The base is a sphere.

**Example 2.5.** Let \( Q \) be a manifold. Denote the cotangent bundle of \( Q \) with the zero section deleted by \( T^*Q \setminus 0 \). There is a natural free action of the reals \( \mathbb{R} \) on the manifold \( M := T^*Q \setminus 0 \) given by dilations \( \rho_\lambda(q, p) = (q, e^{\lambda} p) \). It expands the standard symplectic form on the cotangent bundle exponentially. Thus \( T^*Q \setminus 0 \) is naturally a symplectic cone. The base is the co-sphere bundle \( S^*Q \).

**Proposition 2.6.** Suppose \( (M, \omega, X) \) is a symplectic cone and suppose a Lie group \( G \) acts on \( M \) preserving the symplectic form \( \omega \) and the expanding vector field \( X \). Then the action of \( G \) on the symplectic manifold \( (M, \omega) \) is Hamiltonian. Moreover we may choose the moment map \( \Phi : M \to \mathfrak{g}^* \) to be homogeneous of degree 1, i.e.,
\[
\Phi(\rho_\lambda(m)) = e^{\lambda} \Phi(m)
\]
for all \( \lambda \in \mathbb{R} \) and \( m \in M \). Here \( \rho_\lambda \) denotes the action of \( \mathbb{R} \) generated by \( X \), that is, the time \( \lambda \) flow of \( X \).

**Proof.** Note first that since \( L_X \omega = \omega \) and since \( d\omega = 0 \), it follows from Cartan’s formula \( (L_X \omega = \iota(X)d\omega + d(\iota(X)\omega)) \) that \( d(\iota(X)\omega) = \omega \). Since the action of \( G \) preserves \( X \) and \( \omega \), it preserves the contraction \( \iota(X) \omega \). Therefore for any vector \( A \) in the Lie algebra \( \mathfrak{g} \) of \( G \) we have \( L_{A_M}(\iota(X)\omega) = 0 \), where \( A_M \) as before denotes the vector field on \( M \) induced by \( A \). Therefore
\[
0 = d(\iota(A_M)\iota(X)\omega + \iota(A_M)d(\iota(X)\omega) = d(\omega(X, A_M)) + \iota(A_M)\omega,
\]
and consequently
\[
\iota(A_M)\omega = d(\omega(A_M, X)).
\]

We conclude that the map \( \Phi : M \to \mathfrak{g}^* \) defined by
\[
\langle \Phi(m), A \rangle = \omega_m(X(m), A_M(m))
\]
is a moment map for the action of \( G \) on \((M, \omega)\). \( \square \)
Exercise 2.7. Suppose a Lie group $G$ acts on a manifold $M$ preserving a 1-form $\beta$. Define the $\beta$-moment map $\Psi_\beta : M \to \mathfrak{g}^*$ by

$$\langle \Psi_\beta(m), A \rangle = \beta_m(A_M(m))$$

for all $A \in \mathfrak{g}$ and all $m \in M$. Here as usual $A_M$ denotes the vector field induced by $A$ on $M$.

Show that $\Psi_\beta$ is $G$-equivariant, that is, show that for any $a \in G$ and any $m \in M$

$$\Psi_\beta(a \cdot m) = \text{Ad}^\dagger(a) \Psi_\beta(m),$$

where $\text{Ad}^\dagger : G \to \text{GL}(\mathfrak{g}^*)$ denotes the coadjoint representation. Conclude that the map $\Phi$ defined in Proposition 2.6 is equivariant.

**Definition 2.8.** A 1-form $\alpha$ on a manifold $B$ is a contact form if the following two conditions hold:

1. $\alpha_b \neq 0$ for all points $b \in B$. Hence $\xi := \ker \alpha = \{(b, v) \in TB \mid \alpha_b(v) = 0\}$ is a vector subbundle of the tangent bundle $TB$.

2. $d\alpha|_\xi$ is a symplectic structure on the vector bundle $\xi \to B$ (i.e. $d\alpha_b|_{\xi_b}$ is nondegenerate).

**Remark 2.9.**

1. If $\xi \to B$ is a symplectic vector bundle, then the dimension of its fibers is necessarily even. Hence if a manifold $B$ has a contact form, then $B$ is odd-dimensional.

2. A 1-form $\alpha$ on $2n + 1$ dimensional manifold $B$ is contact if and only if the form $\alpha \wedge (d\alpha)^n$ is never zero, i.e., it is a volume form. [Prove this].

**Example 2.10.** The 1-form $\alpha = dz + x dy$ on $\mathbb{R}^3$ is a contact form: $\alpha \wedge d\alpha = dz \wedge dx \wedge dy$.

**Example 2.11.** Let $B = \mathbb{R} \times T^2$. Denote the coordinates by $t, \theta_1$ and $\theta_2$ respectively. The 1-form $\alpha = \cos t d\theta_1 + \sin t d\theta_2$ is contact. [Check this.]

**Lemma 2.12.** Suppose $\alpha$ is a contact form on a manifold $B$. Then for any positive function $f$ on $B$ the 1-form $f \alpha$ is also contact.

*Proof.* Note first that since $f$ is nowhere zero, $\ker f \alpha = \ker \alpha$. Thus to show that $f \alpha$ is contact, it is enough to check that $d(f \alpha)|_\xi$ is nondegenerate, where $\xi = \ker \alpha = \ker f \alpha$. Now $d(f \alpha) = df \wedge \alpha + f d\alpha$ and $\alpha|_\xi = 0$. Therefore $d(f \alpha)|_\xi = f d\alpha|_\xi$. But $f$ is nowhere zero and $d\alpha|_\xi$ is nondegenerate by assumption. Thus $d(f \alpha)|_\xi$ is nondegenerate. □

**Definition 2.13.** We define the conformal class of a 1-form $\alpha$ on a manifold $B$ to be the set $[\alpha] = \{e^h \alpha \mid h \in C^\infty(B)\}$, that is, the set of all 1-forms obtained from $\alpha$ by multiplying it by a positive function.
Thus if a 1-form $\alpha$ on a manifold $B$ is contact, then its conformal class consists of contact forms all defining the same subbundle $\xi$ of the tangent bundle of $B$.

**Definition 2.14.** A (co-orientable) contact structure $\xi$ on a manifold $B$ is a subbundle of the tangent bundle $TB$ of the form $\xi = \ker \alpha$ for some contact form $\alpha$.

A **co-orientation** of a contact structure $\xi$ is a choice of a conformal class of contact forms defining the contact structure.

**Remark 2.15.** More generally a contact structure on a manifold $B$ is a subbundle $\xi$ of the tangent bundle $TB$ such that for every point $x \in B$ there is a contact 1-form $\alpha$ defined in a neighborhood of $x$ with $\ker \alpha = \xi$. There exist contact structures which are not co-orientable. For such structures $\xi$ a one-form $\alpha$ with $\ker \alpha = \xi$ exists only locally. In these notes we will only deal with co-orientable contact structures.

**Exercise 2.16.** Let $\beta$ be a nowhere zero 1-form on a manifold $B$ and let $\eta = \ker \beta$. Let $\eta^o \to B$ denote the annihilator of $\eta$ in $T^*B$: the fiber of $\eta^o$ at $b \in B$ is the vector space 

$$\eta^o_b = \{ p \in T^*_bB \mid p|_b = 0 \}.$$ 

Show that $\beta$ is a nowhere zero section of real line bundle $\eta^o \to B$. Show that any other nowhere zero section $\beta'$ of $\eta^o \to B$ is of the form $\beta' = f\beta$ for some nowhere zero function $f$ on $B$.

Conclude that a contact structure $\xi$ is co-orientable if and only if the punctured real line bundle $\xi^o \setminus 0$ has two components (0 of course denotes the zero section). Show that a choice of co-orientation of $\xi$ is the same as a choice of a component $\xi^o_+$ of the punctured bundle $\xi^o \setminus 0$.

**Definition 2.17.** Let $(B_1, \xi_1 = \ker \alpha_1)$ and $(B_2, \xi_2 = \ker \alpha_2)$ be two co-oriented contact manifolds. A diffeomorphism $\varphi : B_1 \to B_2$ is a contactomorphism if the differential $d\varphi$ maps $\xi_1$ to $\xi_2$ preserving the co-orientations. That is, $\varphi^*\alpha_2 = f\alpha_1$ for some positive function $f$.

**Definition 2.18.** An action of a Lie group $G$ on a manifold $B$ preserves a contact structure $\xi$ and its co-orientation if for every element $a \in G$ the corresponding diffeomorphism $a_B : B \to B$ is a contactomorphism. We will also say that the action of $G$ on $(B, \xi)$ is a contact action.

**Definition 2.19.** Let $\alpha$ be a contact form on a manifold $B$. The **Reeb vector field** $Y_{\alpha}$ of $\alpha$ is the unique vector field satisfying $\iota(Y_{\alpha})d\alpha = 0$ and $\alpha(Y_{\alpha}) = 1$.

**Exercise 2.20.** Why does the definition of the Reeb vector field makes sense?

**Remark 2.21.** The Reeb vector field depends strongly on the contact form. And it is not just its magnitude: if $\alpha$ is a contact form and $Y_{\alpha}$ is its Reeb vector field, then there is no reason for $\iota(Y_{\alpha})d(f\alpha) = 0$ where $f$ is a nowhere zero function.

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1 Since $d\varphi(\xi_1) = \xi_2$, the lift $\tilde{\varphi} : T^*B_1 \to T^*B_2$ of $\varphi$ maps $\xi_1^o$ to $\xi_2^o$. "$d\varphi$ preserves the co-orientation" means that $\tilde{\varphi}$ maps $(\xi_1)^o_+$ to $(\xi_2)^o_+$ (cf. Exercise 2.16).
Exercise 2.22. Compute the Reeb vector field of the contact form $\alpha = dz + x\,dy$ on $\mathbb{R}^3$.

Exercise 2.23. Compute the Reeb vector field of the contact form $\alpha$ of Example 2.11: $\alpha = \cos t\,d\theta_1 + \sin t\,d\theta_2$ on $B = \mathbb{R} \times \mathbb{T}^2$.

Symplectic cones and contact manifolds are intimately related:

**Theorem 2.24.** Suppose a compact connected Lie group $G$ acts effectively on a symplectic cone $(M, \omega, X)$ preserving the symplectic form $\omega$ and the expanding vector field $X$. Then $G$ induces an effective action on the base $B$ of the cone making the projection $\varpi : M \to B$ $G$-equivariant. Moreover, the base $B$ has a natural co-oriented contact structure $\xi$, and the induced action of $G$ on $B$ preserves a contact form $\alpha$ defining $\xi$. In particular the action of $G$ on $(B, \xi)$ is contact.

We start the proof of Theorem 2.24 with an observation that the action of $G$ on $M$ descends to an effective action of $G$ on the base $B$ making the projection $\varpi : M \to B$ $G$-equivariant. Next we prove:

**Proposition 2.25.** Any principal $\mathbb{R}$-bundle $\mathbb{R} \to M \xrightarrow{\varpi} B$ is trivial.

**Proof.** The proposition is true because the real line is contractible. Here is an elementary argument. Note first that if $s : B \to M$ is a (local) section of $M \xrightarrow{\varpi} B$ and $f \in C^\infty(B)$ is a function, then $s - f$ makes sense: it is again a (local) section of $\varpi : M \to B$. To prove that a principal bundle is trivial it is enough to construct a global section. To this end choose an open cover $\{U_\alpha\}$ of $B$ such that for each $U_\alpha$ there is a section $s_\alpha : U_\alpha \to M$. Choose a partition of unity $\tau_\alpha$ subordinate to the cover $\{U_\alpha\}$. Two sections of a principal $\mathbb{R}$-bundle differ by real-valued function. Thus by abuse of notation on an intersection $U_\alpha \cap U_\beta$, $s_\alpha - s_\beta$ is a real-valued function. Now define for each index $\alpha$

$$s'_\beta = s_\beta - \sum_{\alpha \neq \beta} \tau_\alpha(s_\beta - s_\alpha).$$

Then on an intersection $U_\alpha \cap U_\beta$

$$s'_\beta - s'_\gamma = \left(s_\beta - \sum_{\alpha \neq \beta} \tau_\alpha(s_\beta - s_\alpha)\right) - \left(s_\gamma - \sum_{\alpha \neq \gamma} \tau_\alpha(s_\gamma - s_\alpha)\right)$$

$$= s_\beta - s_\gamma - \sum_{\alpha \neq \beta, \gamma} \tau_\alpha(s_\beta - s_\gamma) + \tau_\beta(s_\gamma - s_\beta) - \tau_\gamma(s_\beta - s_\gamma)$$

$$= s_\beta - s_\gamma - \sum_{\alpha} \tau_\alpha(s_\beta - s_\gamma) = 0.$$

Therefore the collection of local sections $\{s'_\alpha\}$ defines a global section of $\varpi : M \to B$. Consequently the bundle is trivial. $\square$
Thus any symplectic cone is of the form $B \times \mathbb{R}$ where $B = M/\mathbb{R}$ is an odd-dimensional manifold.

**Lemma 2.26.** Let $(M, \omega, X)$ be a symplectic cone, let $B$ be its base and let $\varpi : M \to B$ denote the projection. Pick a trivialization $\varphi : B \times \mathbb{R} \to M$. Then $\varphi^* \omega = d(e^t \alpha)$ where $t$ is a coordinate on $\mathbb{R}$ and $\alpha$ is a contact form on $B$. Conversely, if $\alpha$ is contact form on $B$ then $(B \times \mathbb{R}, d(e^t \alpha), \frac{\partial}{\partial t})$ is a symplectic cone.

**Proof.** By Proposition 2.25 the principal $\mathbb{R}$ bundle $\varpi : M \to B$ is trivial. Choose a trivialization $M \simeq B \times \mathbb{R}$. Under this identification the vector field $X$ becomes $\frac{\partial}{\partial t}$.

Since $d\omega = 0$ and $L_X \omega = \omega$, $\delta e(X)\omega = \omega$ (c.f. proof of Proposition 2.6). Let $\beta = \iota(X)\omega$. Then $\iota(X)\beta = 0$ and $L_X \beta = d\iota(X)\beta + \iota(X)d\beta = 0 + \iota(X)\omega = \beta$. Hence for any point $(b, t) \in B \times \mathbb{R}$

$$\beta_{(b,t)} = e^t \beta_{(b,0)}.$$  

Since $\iota(X(b,0))\beta_{(b,0)} = 0$ it follows that $\beta_{(b,0)} = \alpha_b$ for a 1-form $\alpha$ on $B$. It remains to show that $\alpha$ is contact. For this it suffices to show that $\alpha \wedge (da)^d$ is nowhere zero, where $d = \frac{1}{2} \dim M - 1$. Now $\omega = d(e^t \alpha)$ is symplectic. Hence $\omega^{d+1}$ is nowhere vanishing. Now $\omega^{d+1} = (e^t(dt \wedge \alpha + da))^{d+1} = e^{td}dt \wedge \alpha \wedge (da)^d$. Hence $\alpha \wedge (da)^d$ is nowhere vanishing.

Conversely suppose $\alpha$ is a contact 1-from on $B$. Let $\omega = d(e^t \alpha)$ and let $X = \frac{\partial}{\partial t}$. Then $L_X \omega = d(\iota^* \omega) = d(e^t(\iota^* \omega)) = d(e^t \alpha) + e^t \omega).$ It remains to check that $\omega$ is nondegenerate. For any $(b, t) \in B \times \mathbb{R}$, the tangent space $T_{(b,t)}(B \times \mathbb{R})$ decomposes as $T_{(b,t)}(B \times \mathbb{R}) = \ker \alpha_b \oplus \mathbb{R}Y_\alpha(b) \oplus \mathbb{R}$ where $Y_\alpha$ is the Reeb vector field of $\alpha$ (cf. Definition 2.19). Since $\alpha$ is contact $\delta \alpha_b | \ker \alpha_b$ is nondegenerate. The restriction $dt \wedge \alpha_b$ to $\mathbb{R}Y_\alpha(b) \oplus \mathbb{R}$ is nondegenerate as well. Hence $\omega = e^t(dt \wedge \alpha + da)$ is nondegenerate. This proves that $(B \times \mathbb{R}, d(e^t \alpha), \frac{\partial}{\partial t})$ is a symplectic cone. \hfill $\square$

**Exercise 2.27.** Let $(M, \omega, X)$ be a symplectic cone, let $B$ be its base and let $\varpi : M \to B$ denote the $\mathbb{R}$-orbit map. Pick a global section $s : B \to M$ of $\varpi : M \to B$. Show that $\alpha = s^* \iota(X)\omega$. Show that $\alpha$ is a contact form on $B$. Show that it is the same contact form that the proof of Lemma 2.26 would produce from the trivialization $\varphi : B \times \mathbb{R} \to M$, $\varphi(b, t) = \rho_t(s(b))$. Here $\rho_t : M \to M$ denotes the action of $\mathbb{R}$.

**Remark 2.28.** Different choices of trivializations $\varphi$ of $\varpi : M \to B$ give rise to different contact forms on $B$. However, they all define the same contact structure $\xi$ on the base $B$. Intrinsically $\xi$ can be defined as follows: for a point $b \in B$,

$$\xi_b = d\varpi_m(\ker(\iota(X)\omega)_m) \quad \text{for any } m \in \varpi^{-1}(b). \quad (2.1)$$  

It is not hard to check that $\xi$ is well-defined. First note that $\mathbb{R}$ acts transitively on the fiber $\varpi^{-1}(b)$. Second observe that for any $\lambda \in \mathbb{R}$ we have $\rho^*_\lambda(\iota(X)\omega) = e^{\lambda t}(\iota(X)\omega)_m$, and hence $d\rho^*_\lambda(\ker(\iota(X)\omega)_m) = (\ker(\iota(X)\omega)_\rho(m))$. Here again $\rho_t : M \to M$ denotes the action of $\mathbb{R}$.
It follows that the action of $G$ on $B$ induced by an action of $G$ on the symplectic cone $\varpi : M \to B$ preserves the contact structure $\xi$ defined by (2.1). Since $G$ is connected and since the identity map preserves the co-orientation of $\xi$, all the other elements of $G$ also preserve the co-orientation.

It remains to show that there is a $G$-invariant 1-form $\alpha$ with $\ker \alpha = \xi$. By Lemma 2.26 a choice of a trivialization of $\varpi : M \to B$ gives us a 1-form $\alpha$ on $B$ with $\ker \alpha = \xi$, but this form need not be $G$-invariant. It is only $G$-invariant if the trivialization is $G$-equivariant. Therefore we proceed as follows.

**Lemma 2.29.** Suppose a compact Lie group $G$ acts on a manifold $B$ preserving a (co-oriented) contact structure $\xi = \ker \alpha$ for some 1-form $\alpha$. Then there exists a $G$-invariant 1-form $\tilde{\alpha}$ with $\ker \tilde{\alpha} = \xi$.

**Proof.** For every $a \in G$ the corresponding diffeomorphism $a_B : B \to B$ is a contactomorphism. Hence $(a_B)^* \alpha = f_a \alpha$ for some positive function $f_a$ depending smoothly on $a$. Define a new contact form $\tilde{\alpha}$ to be the average of $\alpha$ over the action of $G$:

$$
\tilde{\alpha}_b = \int_G ((a_B)^* \alpha)_b \, da = \int_G (f_a(b) \alpha_b) \, da = \left( \int_G f_a(b) \, da \right) \alpha_b
$$

for all $b \in B$. Here $da$ is a bi-invariant measure on $G$ normalized so that $\int_G da = 1$. Since $f_a > 0$ for all $a \in G$, the integral $(\int_G f_a(b) \, da)$ is positive and $\tilde{\alpha}$ is indeed nowhere zero.

From now on we will always assume that whenever a group action preserves a contact structure it also preserves a contact form defining this structure.

This concludes the proof of Theorem 2.24. It follows from the Theorem that if an $n$-torus $G$ acts effectively on the punctured cotangent bundle $M = T^* \mathbb{T}^n \setminus 0$ preserving the symplectic form and commuting with dilations then it acts on the quotient $B = M/\mathbb{R} \cong S^* \mathbb{T}^n$ preserving the corresponding contact structure. Note that $2 \dim G = \dim B + 1$.

**Definition 2.30.** An effective action of a torus $G$ on a manifold $B$ preserving a contact structure $\xi$ is completely integrable if $2 \dim G = \dim B + 1$.

A contact toric $G$-manifold is a co-oriented contact manifold with a completely integrable action of a torus $G$.

We are now in position to reduce Theorem 1.4 to a statement about contact toric manifolds. Consider again the action of $G = \mathbb{T}^n$ on $M = T^* \mathbb{T}^n \setminus 0$ preserving the symplectic form and commuting with dilations. As was remarked previously $M$ is a symplectic cone over $B = S^* \mathbb{T}^n = \mathbb{T}^n \times S^{n-1}$. By Theorem 2.24 the action of $G$ on $M$ induces an effective action on $B$. Moreover $B$ has a $G$-invariant contact structure $\xi$ making $(B, \xi)$ into a compact connected contact toric $G$-manifold. Clearly if the action of $G$ on $B$ is free then the original action of $G$ on $M$ was free as well. Therefore a proof of Theorem 1.4 reduces to
CHAPTER 2. SYMPLECTIC CONES AND CONTACT MANIFOLDS

Theorem 2.31. Let \((B, \xi = \ker \alpha)\) be a compact connected contact toric \(G\)-manifold and suppose the action of \(G\) is not free. Then \(B\) is not homotopy equivalent to \(\mathbb{T}^n \times S^{n-1}\), \(n = \dim G\).

The rest of the notes will be occupied with a proof of Theorem 2.31. The proof uses heavily contact moment maps which are discussed in the next section. We end this section with a partial converse to Theorem 2.24.

We have seen that given a contact form \(\alpha\) on a manifold \(B\) the form \(d(e^t \alpha)\) on \(B \times \mathbb{R}\) is symplectic. The pair \((B \times \mathbb{R}, d(e^t \alpha))\) is called the symplectization\(^2\) of \((B, \alpha)\). It is clearly a symplectic cone.

Different contact forms on \(B\) give rise to different symplectic forms on \(B \times \mathbb{R}\). However there is a symplectic cone that depends only on the contact structure \(\xi = \ker \alpha\) (and its co-orientation) and not on a particular choice of a contact form:

Let \(\xi^0\) denote the component of \(\xi \setminus 0\) giving \(\xi\) its co-orientation (cf. Exercise 2.16). It is not hard to check that \(\xi^0_+\) is a symplectic submanifold of the cotangent bundle \(T^*B\) with its standard symplectic structure. The action of \(\mathbb{R}\) on \(T^*B \setminus 0\) by dilations preserves \(\xi^0_+\) and makes it into a symplectic cone. A section \(\beta\) of \(\xi^0_+ \to B\) is a contact form \(\beta\) on \(B\) with \(\xi = \ker \beta\). Moreover the trivialization \(\varphi_\beta : B \times \mathbb{R} \to \xi^0_+, \varphi_\beta(b, t) = e^t \beta_b\), that \(\beta\) defines pulls back the symplectic form on \(\xi^0_+\) to \(d(e^t \beta)\).

Finally given a symplectic cone \((M, \omega, X)\) with the base \(B\), the orbit map \(\varpi : M \to B\) and the induced contact structure \(\xi\) on \(B\), there is an \(\mathbb{R}\) equivariant symplectomorphism \(\varphi : M \to \xi^0_+\) defined as follows: for a point \(m \in M\) let \(\varphi(m)\) be the covector in \(T^*_{\varpi(m)}B\) such that

\[
\varphi(m)(v) = \langle \iota_X \omega_m, ds_b(v) \rangle
\]

for all \(v \in T_{\varpi(m)}B\) and a section \(s\) of \(\varpi : M \to B\).

The discussion above can be summarized as:

Lemma 2.32. Let \((B, \xi = \ker \alpha)\) be a contact manifold, let \(\xi^0_+\) be a component of \(\xi \setminus 0\), the annihilator of \(\xi\) in \(T^*B\) minus the zero section. The principal \(\mathbb{R}\) bundle \(\xi^0_+ \to B\) is a symplectic cone.

If \((M, \omega, X)\) is a symplectic cone with the base \(B\) and \(\xi\) is the induced contact structure on \(B\), then \(\xi^0_+\) is isomorphic to \((M, \omega, X)\) as a symplectic cone.

\(^2\)Sometimes \((B \times \mathbb{R}, d(e^t \alpha))\) is called the symplectification of \((B, \alpha)\).
Chapter 3

Group actions and moment maps on contact manifolds

Moment maps exist in the category of contact group actions. In fact moment maps exist for all contact actions. This is because a contact form defines a bijection between contact vector fields and smooth functions.

**Definition 3.1.** A vector field $X$ on a contact manifold $(B, \xi = \ker \alpha)$ is **contact** if its flow $\varphi_t$ consists of contactomorphisms. In particular $d\varphi_t(\xi) \subset \xi$. Hence for any section $v$ of the bundle $\xi \to B$, the Lie bracket $[X, v]$ is again a section of $\xi \to B$.

Thus for a contact action of a Lie group $G$ on $(B, \xi)$ the vector fields induced by elements of the Lie algebra $g$ of $G$ are contact.

**Exercise 3.2.** Prove that a Reeb vector field is contact. More generally prove that a vector field $X$ on a contact manifold $(B, \xi = \ker \alpha)$ is contact if and only if $L_X \alpha = h\alpha$ for some function $h$ (h can have zeros).

A choice of a contact form on a contact manifold $(B, \xi)$ identifies contact vector fields with smooth functions.

**Proposition 3.3.** Let $(B, \xi = \ker \alpha)$ be a contact manifold. The linear map from contact vector fields to smooth functions given by $X \mapsto f^X := \alpha(X)$ is one-to-one and onto.

**Proof.** Note that the Reeb vector field $Y_\alpha$ corresponds to the function 1. For any vector field $X$ on $B$ the vector field $X - \alpha(X)Y_\alpha$ is in the kernel of $\alpha$, which is the contact distribution $\xi$. Since $d\alpha|_\xi$ is non-degenerate, $X - \alpha(X)Y_\alpha$ is uniquely determined by $\iota(X - \alpha(X)Y_\alpha)(d\alpha|_\xi)$.

For any section $v$ of $\xi \to B$ and any vector field $X$ we have

$$0 = L_X 0 = L_X(\alpha(v)) = (L_X \alpha)(v) + \alpha([X, v]).$$
Assume now that $X$ is contact. Then $\alpha([X, v]) = 0$. Therefore, by Cartan’s formula
\[
0 = (d\iota(X)\alpha + \iota(X)d\alpha)(v).
\]
Hence for any section $v$ of $\xi$, $d\alpha(X, v) = -d(\alpha(X))(v) = -df^X(v)$. And, of course, $d\alpha(X, v) = d\alpha(X - f^XY, v)$ for all $v$. We conclude that
\[
\iota(X - \alpha(X)Y)(d\alpha|\xi) = -df^X|\xi
\]
for any contact vector field $X$. Thus if $X$ is contact, its component in the direction of the Reeb vector field is $f^XY$, and its component in the direction of the contact distribution is uniquely determined by (3.1).

Conversely, given a function $f$ on $B$ there is a unique section $X'_f$ of $\xi$ such that
\[
\iota(X'_f)d\alpha|\xi = -df|\xi.
\]
The vector field $X_f := X'_f + fY_\alpha$ is a contact vector field with $\alpha(X_f) = f$. \[\square\]

Exercise 3.4. Given a function $f$ on a manifold $B$ with contact form $\alpha$ check that the vector field $X_f := X'_f + fY_\alpha$, where $X'_f$ is defined by (3.2), is contact. That is, show that $L_{X_f}\alpha = h\alpha$ for some function $h$ (cf. Exercise 3.2).

Suppose now that a Lie group $G$ acts on a manifold $B$ preserving a contact 1-form $\alpha$. Then for any vector $A$ in the Lie algebra $\mathfrak{g}$ of $G$, the induced vector field $A_B$\footnote{Recall that the vector field $A_B$ is defined by $A_B(b) = \frac{d}{dt}|_{t=0} \exp(tA) \cdot b$.} satisfies $L_{A_B}\alpha = 0$ and, in particular, is contact. Since the contact form $\alpha$ defines a 1-1 correspondence between contact vector fields and functions, it makes sense to define the $\alpha$-moment map $\Psi_\alpha : B \to \mathfrak{g}^*$ by
\[
\langle \Psi_\alpha(b), A \rangle = \alpha_b(A_B(b))
\]
for all $b \in B$ and all $A \in \mathfrak{g}$. Note that by Exercise 2.7 the $\alpha$-moment map $\Psi_\alpha$ is $G$-equivariant.

The $\alpha$-moment map, as the name suggests, depends rather strongly on $\alpha$: if $f$ is any positive $G$-invariant function on $B$, then $f\alpha$ is another $G$ invariant contact form and clearly
\[
\Psi_{f\alpha} = f\Psi_\alpha.
\]
In particular, unlike in the symplectic case, the image of a moment map is not an invariant of the action and of the contact structure. However the moment cone $C(\Psi_\alpha)$ defined by
\[
C(\Psi_\alpha) = \{t\eta \in \mathfrak{g}^* \mid \eta \in \Psi_\alpha(B), t \in [0, \infty)\}
\]
is an invariant of the action of $G$ on $(B, \xi = \ker \alpha)$.\footnote{Recall that the vector field $A_B$ is defined by $A_B(b) = \frac{d}{dt}|_{t=0} \exp(tA) \cdot b$.}
Remark 3.5. Suppose a Lie group $G$ acts on a manifold $B$ preserving a contact form $\alpha$. The action of $G$ lifts to an action on the cotangent bundle $T^*B$ which preserves the annihilator $\xi^\circ$ of the contact structure $\xi = \ker \alpha$. Moreover, the action preserves the component $\xi^\circ_+ \cup \{0\}$ which contains the image of $\alpha : B \to T^*B$.

The action of $G$ on $T^*B$ is Hamiltonian with a natural moment map $\Phi : T^*B \to g^*$ given by

$$\langle \Phi(b, p), A \rangle = \langle p, A_B(b) \rangle$$

for all $b \in B$, $p \in T_b^*B$, $A \in g$. Since the submanifold $\xi^\circ_+$ is a $G$-invariant symplectic submanifold of $T^*B$ the restriction $\Psi := \Phi|_{\xi^\circ_+}$ is a moment map for the action of $G$ on $\xi^\circ_+$. It is not hard to check that

$$\Psi_{\alpha} = \Psi \circ \alpha.$$

For this and other reasons it makes sense to think of $\Psi : \xi^\circ_+ \to g^*$ as the moment map for the action of $G$ on $(B, \xi)$.\footnote{Note that since $\Phi$ is $G$-equivariant and $\alpha : B \to T^*B$ is $G$-equivariant, $\Psi_{\alpha}$ is $G$-equivariant as well. This gives us an alternative proof that the $\alpha$-moment map is equivariant.} Note also that $C(\Psi_{\alpha}) = \Psi(\xi^\circ_+) \cup \{0\}$. We will thus denote the moment cone $C(\Psi_{\alpha})$ by $C(\Psi)$.

Consequently and by analogy with symplectic toric manifolds we will think of **contact toric $G$-manifolds** as triples $(B, \xi = \ker \alpha, \Psi : \xi^\circ_+ \to g^*)$. 


CHAPTER 3. GROUP ACTIONS AND MOMENT MAPS ON CONTACT MANIFOLDS
Chapter 4

Contact toric manifolds

The rest of the lecture notes will be devoted to a proof of Theorem 2.31. Right from the beginning the proof will bifurcate into two cases: the contact manifold $B$ is 3-dimensional and $\dim B > 3$. If $\dim B = 3$ we will argue directly using slices that the orbit space $B/G$ is homeomorphic to a closed interval $[0, 1]$ and then use this to compute the integral cohomology of $B$. This will show that $B$ cannot be homeomorphic to $S^*T^2 = T^3$.

We will then consider the case where $\dim B > 3$. In this case we have a connectedness and convexity theorem of Banyaga and Molino (see [BM1, BM2]; for a different proof see [L2]):

**Theorem 4.1.** Let $(B, \xi = \ker \alpha, \Psi : \xi^*_\alpha \to g^*)$ be a compact connected contact toric $G$-manifold. Suppose $\dim B > 3$. Then the fibers of the moment map $\Psi$ are $G$-orbits (and in particular are connected) and the moment cone $C(\Psi)$ is a convex polyhedral cone in $g^*$. Moreover $C(\Psi) \neq g^*$ iff the action of $G$ is not free.

Our proof will then bifurcate again. We will consider separately the case where the moment cone contains a linear subspace of dimension $k$, $0 < k < \dim G$ and where no such subspace exists (i.e., the moment cone is proper).

In the first case we will use a uniqueness theorem of Boucetta and Molino [BoM] for symplectic toric manifolds to argue that the symplectization $B \times \mathbb{R}$ of $B$ is diffeomorphic to the manifold $N = T^k \times (\mathbb{R}^k \times \mathbb{C}^l \setminus \{(0, 0)\})$ where $2l + 2k = \dim B + 1$ and $k, l > 0$. It is easy to see that $N$ cannot be homotopy equivalent $T^n \setminus 0, n = k + l$.

In the latter case we will argue following Boyer and Galicki [BG] that there is a locally free $S^1$ action on $B$ such that the quotient $B/S^1$ is a (compact connected) symplectic toric orbifold. Since the action of $S^1$ is locally free there is

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1 The result was rediscovered a few years later by Lerman, Tolman and Woodward (Lemma 7.2 and Proposition 7.3 in [LT]).

2 An action of a Lie group $G$ on a manifold $Z$ is locally free if all the isotropy groups are zero dimensional.
a long exact sequence of rational cohomology groups (the Gysin sequence) tying together the cohomology of $B$ and of $B/S^1$. On the other hand Morse theory on the orbifold $B/S^1$ shows that all odd-dimensional rational cohomology of $B/S^1$ vanishes. Together these two facts will imply that $\dim \mathbb{Q} H^1(B, \mathbb{Q}) \leq 1$. Thus $B$ cannot be $S^1 \times \mathbb{T}^n = \mathbb{T}^n \times S^{n-1}$, $n = \frac{1}{2}(\dim B + 1) > 2$. This completes the preview of our proof of Theorem 2.31.

4.1 Homogeneous vector bundles and slices

Suppose that $G$ is a Lie group, $H \subset G$ a closed subgroup and suppose we have a representation of $H$ on a vector space $W$. Then $H$ acts on the product $G \times W$ by $h \cdot (g, w) = (gh^{-1}, g \cdot w)$ for $h \in H$, $(g, w) \in G \times W$. The quotient $G \times_H W := (G \times W)/H$ is a vector bundle over $G/H$ with typical fiber $W$. We denote the image of $(g, w) \in G \times W$ in $G \times_H W$ by $[g, w]$. Note that the action of $G$ on $G \times W$ given by $a \cdot (g, w) = (ag, w)$ commutes with the action of $H$ and hence descends to an action on $G \times_H W$: $a \cdot [g, w] = [ag, w]$. The projection $G \times_H W \rightarrow G/H$ is $G$-equivariant and the action of $G$ on the base $G/H$ is transitive. This makes $G \times_H W \rightarrow G/H$ into a homogeneous vector bundle.

Conversely if $\pi : E \rightarrow G/H$ is a vector bundle with an action of $G$ by vector bundle maps making $\pi$ equivariant, then $E$ is isomorphic to $G \times_H W$ where $W$ is the fiber of $E$ above the identity coset $1H$. Indeed the map

$$G \times W \rightarrow E, \quad (g, w) \mapsto g \cdot w$$

is onto and is constant along the orbits of $H$. It descends to a vector bundle isomorphism

$$G \times_H W \rightarrow E, \quad [g, w] \mapsto g \cdot w.$$

Next suppose a compact Lie group $G$ acts on a manifold $M$. Consider an orbit $G \cdot x \subset M$. The group $G$ acts on the normal bundle of the orbit $\nu(G \cdot x) = TM|_{G \cdot x}/T(G \cdot x)$ making the projection $\pi : \nu(G \cdot x) \rightarrow (G \cdot x)$ equivariant. Thus $\nu(G \cdot x) = G \times_H W$ where $W = T_x M/T_x(G \cdot x)$.

If we choose a $G$-invariant Riemannian metric on $M$ we can identify $\nu(G \cdot x)$ with the perpendicular of $T(G \cdot x)$ in $TM|_{G \cdot x}$. Furthermore, the Riemannian exponential map $\exp : \nu(G \cdot x) \rightarrow M$ is $G$-equivariant. Hence, by the Tubular Neighborhood theorem we get:

Lemma 4.2. Let $G$ be a compact connected Lie group acting on a manifold $M$ and let $x \in M$ be a point. A neighborhood of $G \cdot x$ in $M$ is $G$-equivariantly diffeomorphic to a neighborhood of the zero section of the homogeneous vector bundle $G \times_{G_x} W$ where $G_x$ denotes the isotropy group of $x$ and $W = T_x M/T_x(G \cdot x)$.

\footnote{Why is there a representation of $H$ on $W$?}

\footnote{Choose any metric on $M$ and then average it over the group $G$ (cf. proof of Lemma 2.29).}
**Definition 4.3.** Let $G$ be a Lie group acting on a manifold $M$; let $x \in M$ be a point. Denote the isotropy group of $x$ by $G_x$. An embedded submanifold $S \subset M$ is a slice through $x$ for the action of $G$ on $M$ if $x \in S$, $S$ is $G_x$-invariant and if the map $G \times S \to M$ given by $(g,s) \mapsto g \cdot s$ descends to an open embedding $G \times_{G_x} S \to M$, $[g,s] \mapsto g \cdot s$.

Thus by Lemma 4.2 slices exist for actions of compact Lie groups: we may choose as a slice at $x$ the image of a small $G_x$-invariant neighborhood of 0 in $W = T_x M/T_x(G \cdot x)$ under the exponential map.

Here is a typical application of the existence of slices. Locally near $x$ the quotient $M/G$ is homeomorphic to the quotient $(G \times_{G_x} W)/G = W/G_x$. Hence quotients of manifolds by actions of compact Lie groups are modeled on quotients of vector spaces by linear actions of compact Lie groups. The linear action of $G_x$ on $W = T_x M/T_x(G \cdot x)$ is called the slice representation at $x$.

Here is another application of the above construction:

**Lemma 4.4.** Suppose $G$ is a compact abelian group acting effectively on a connected manifold $M$. Then every slice representation is faithful, i.e., no slice representation has a kernel.

**Proof.** Suppose the slice representation of $H = G_x$ on $W = T_x M/T_x(G \cdot x)$ is not faithful at a point $x \in M$. By Lemma 4.2 it is no loss of generality to assume that a neighborhood of $G \cdot x$ in $M$ is the homogeneous vector bundle $G \times_H W \to G/H = G \cdot x$ and that the point $x$ is $[1,0] \in G \times_H W$. If there is an element $a \in H$ such that $a \neq 1$ and yet $a \cdot w = w$ for all $w \in W$, then $a \cdot [g, w] = [ag, w] = [ga, w] = [g, a \cdot w] = [g, w]$ for all $[g, w] \in G \times_H W$ (ag = ga since $G$ is abelian). Thus $a \in H$ fixes an open neighborhood of $x$. Since the set fixed by $a$ is closed and since $M$ is connected it follows that $a$ fixes all of $M$. This contradicts the assumption that the action of $G$ on $M$ is effective.

**Remark 4.5.** The compactness of the Lie group $G$ is not necessary for the existence of slices. According to Palais [P] it is only necessary that its action on a manifold $M$ be proper (see Definition 2.2 above).

Thus if an action of a Lie group $G$ on a manifold $M$ is free and proper, then the existence of slices tell us that a neighborhood of every orbit is equivariantly diffeomorphic to a product of $G$ with some manifold $S$. It is not hard to deduce from this that the orbit space $M/G$ is a manifold and that the orbit map $M \to M/G$ makes $M$ into a principal $G$-bundle.

We now recall a few properties of tori, which for us are compact abelian Lie groups. If $G$ is a torus, then the (Lie group) exponential map $\exp : g \to G$ is a covering map. The kernel $\mathbb{Z}_G$ of $\exp$ is called the integral lattice. Clearly $G = g/\mathbb{Z}_G$. The group $\mathbb{Z}_G$ is isomorphic to the fundamental group of $G$. Also it has the property that for any $X \in \mathbb{Z}_G$ the corresponding 1-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\}$ is a circle. The dual lattice $\mathbb{Z}_G^\ast = \text{Hom}_\mathbb{Z}(\mathbb{Z}_G, \mathbb{Z}) \cong \{\ell \in g^\ast \mid \ell(\mathbb{Z}_G) \subset \mathbb{Z}\}$ is the weight lattice. It parameterizes 1-dimension complex representations of $G$, or,
equivalently, group homomorphisms (characters) \( \chi : G \to S^1 \): Given \( \nu \in \mathbb{Z}/2\mathbb{Z} \), the corresponding character \( \chi_\nu : G = \mathfrak{g}/\mathbb{Z}_G \to S^1 \) is defined by \( \chi_\nu(\exp X) = e^{2\pi i \nu(X)} \) for all \( X \in \mathfrak{g} \). Given a character \( \chi : G \to S^1 \), its differential \( \partial_1 \) at 1 is a weight.

Recall that a complex representation of a compact abelian group is a direct sum of one-dimensional complex representations. Thus a complex representation of a torus is completely characterized by a finite set of weights. The same is true for symplectic representations of tori — one defines weights with respect to some complex structure compatible with the symplectic form. The weights do not depend on the choice of the complex structure; they only depend on the symplectic form.

The Lemma below is a key representation-theoretic fact in the classification of symplectic and contact toric manifolds.

**Lemma 4.6.** Suppose \( \rho : H \to \text{Sp}(V, \omega) \) is a symplectic representation of a compact abelian Lie group \( H \) on a symplectic vector space \( (V, \omega) \). Suppose that \( \rho \) is faithful, i.e., \( \ker \rho = \{1\} \).

Then \( \dim H \leq \frac{1}{2} \dim V \). If \( H = \frac{1}{2} \dim V \), then \( H \) is connected. Moreover, the set of weights for the representation of \( H \) on \( (V, \omega) \) is a basis of the weight lattice \( \mathbb{Z}_H^* \) of \( H \).

**Proof.** Since \( H \) is compact there exists an \( H \)-invariant complex structure \( J \) on \( V \) compatible with \( \omega \) (i.e., \( \omega(JJ^* x, y) = \omega(x, y) \) and for any \( v \neq 0 \) we have \( \omega(J v, v) > 0 \).

The choice of \( J \) identifies \( (V, \omega) \) with \( \mathbb{C}^n \) with the standard symplectic structure \( \sqrt{-1} \sum dz_j \wedge d\bar{z}_j \), \( n = \frac{1}{2} \dim \mathfrak{g} V \). This, in turn, gives us a representation of \( H \) on \( \mathbb{C}^n \) by unitary matrices. Thus we may assume that \( \rho \) is an injective group homomorphism \( \rho : H \to \text{U}(n) \).

The connected component \( H^0 \) of \( H \) is a torus. Therefore \( \rho(H^0) \) is contained in a maximal torus of \( \text{U}(n) \) which is the \( n \)-torus. Since \( \rho \) has no kernel we have \( \dim H = \dim H^0 = \dim \rho(H^0) \leq n = \frac{1}{2} \dim V \).

Now suppose \( \dim H = n \). Since all maximal tori in \( \text{U}(n) \) are conjugate, we may assume that \( \rho(H^0) \) is the standard maximal torus, that is \( \rho(H^0) \) is the set of all diagonal unitary matrices. Since the only unitary matrices which commute with all the diagonal matrices are the diagonal matrices, we see that we must have \( \rho(H) = \rho(H^0) \). Consequently since \( \rho \) is faithful, \( H = H^0 \), i.e., \( H \) is a torus.

Finally the set of weights of the maximal torus \( \mathbb{T}^n \) in \( \text{U}(n) \) for its representation on \( \mathbb{C}^n \) is a lattice basis of weight lattice \( \mathbb{Z}^*_n \). Hence the set of weights for the representation of \( H \) on \( (V, \omega) \) is a basis of the weight lattice \( \mathbb{Z}^*_H \) of \( H \). \( \square \)

**Lemma 4.7.** Let \( (B, \xi = \ker \alpha) \) be a contact toric \( G \)-manifold. Then

1. No \( G \)-orbit is tangent to the contact structure \( \xi \). In particular there are no fixed points. Hence the \( \alpha \)-moment map \( \Psi_\alpha \) does not vanish at any point for any \( G \)-invariant contact form \( \alpha \). Equivalently \( \Psi(\xi^*_+ \alpha) \) does not contain the origin in \( \mathfrak{g}^* \).

2. All isotropy groups are connected.
4.1. HOMOGENEOUS VECTOR BUNDLES AND SLICES

Proof. Let \( b \) be a point in \( B \) and let \( H \) denote its isotropy group. The group \( H \) acts on the tangent space \( T_b B \). Since the contact form \( \alpha \) is \( G \)-invariant, its kernel at \( b \), the hyperplane \( \xi_b \), is an \( H \)-invariant subspace of \( T_b B \). Since the Reeb vector field \( Y_\alpha \) of \( \alpha \) is unique, the vector \( Y_\alpha(b) \) is fixed by \( H \). Thus we have an \( H \)-equivariant splitting

\[ T_b B = \mathbb{R}Y_\alpha(b) \oplus \xi_b. \]

Let \( V = T_b(G \cdot b) \cap \xi_b \). It is an \( H \)-invariant subspace of \( \xi_b \). Note that since \( \dim \xi_b = \dim B - 1 \) we either have \( V = T_b(G \cdot b) \) or \( \dim V = \dim G \cdot b - 1 \). We will argue that the former case cannot occur. But first we argue that \( V \) is an isotropic subspace of the symplectic vector space \( (\xi_b, \omega = d\alpha|_\xi_b) \).

Let \( x, z \in V \) be two vectors. There exist vectors \( X, Z \in \mathfrak{g} \) such that \( X_B(b) = x \) and \( Z_B(b) = z \). Then \( \omega(x, z) = d\alpha(X_B(b), Z_B(b)) \). Since \( G \) is abelian \( [Z_B, X_B] = 0 \). The function \( \alpha(X_B) \) is \( G \)-invariant, hence \( Z_B(\alpha(X_B)) = 0 \). Therefore

\[ \omega(x, y) = d\alpha(X_B, Z_B) = X_B(\alpha(Z_B)) - Z_B(\alpha(X_B)) - d\alpha([X_B, Z_B]) = 0 - 0 + 0. \]

This proves that \( V \) is isotropic.

Since \( H \) is compact, there is an \( H \)-invariant complex structure \( J \) on \( \xi_b \) compatible with \( \omega \). Since \( V \) is isotropic, \( V \cap JV = 0 \) and \( V + JV = V \oplus JV \) is a symplectic subspace of \( (\xi_b, \omega) \). It is \( H \)-invariant. In fact, since \( G \) is abelian, the action of \( H \) on \( T_b(G \cdot b) \) is trivial. Hence the action of \( H \) on \( V \oplus JV \) is trivial as well. Let \( W \) denote the symplectic perpendicular to \( V \oplus JV \). We get a symplectic representation of \( H \) on \( W \).

We now argue that if \( T_b(G \cdot b) \subset \xi_b \) then \( \dim W \) is less than \( 2 \dim H \) and that the representation of \( H \) on \( W \) must be faithful. This by Lemma 4.6 would give us a contradiction. We would then conclude that \( \dim W = 2 \dim H \), which by Lemma 4.6 implies that \( H \) is connected.

Suppose now that \( T_b(G \cdot b) \subset \xi_b \). Then \( \dim V = \dim G \cdot b \). Since \( B \) is toric, \( \dim B = 2 \dim G - 1 \). Since

\[ T_b B = \mathbb{R}Y_\alpha(b) \oplus V \oplus JV \oplus W \tag{4.1} \]

\( \dim W = (2 \dim G - 1) - 1 = 2 \dim G - 2 \dim G - (\dim G - \dim H) = 2 \dim H - 2 \). Since (4.1) is a splitting as \( H \)-representations, the slice representation of \( H \) at \( b \) is \( \mathbb{R}Y_\alpha(b) \oplus JV \oplus W \). As we observed earlier the action of \( H \) on \( \mathbb{R}Y_\alpha(b) \oplus JV \) is trivial. Since the action of \( G \) on \( B \) is effective, the representation of \( H \) on \( W \) must be faithful. Contradiction.

Therefore \( T_b(G \cdot b) \not\subset \xi_b \) and so \( \dim V = \dim G \cdot b - 1 \). In this case the dimension count gives us exactly that \( \dim W = 2 \dim H \).

\[ \text{Lemma 4.8.} \text{ Let } H \subset T^2 \text{ be a closed subgroup isomorphic to } S^1. \text{ Then there is another closed subgroup } K \subset T^2 \text{ isomorphic to } S^1 \text{ such that } T^2 = K \times H. \]

\[ \text{Proof.} \text{ Since } H \text{ is isomorphic to } S^1 \text{ it is of the form } \{ \exp t \nu \mid t \in \mathbb{R} \} \text{ for some vector } \nu = (n_1, m_1) \in \mathbb{Z}^2 = \ker \{ \exp : \mathbb{R} \to T^2 \}. \text{ We may assume that } n_1 \text{ and} \]


$m_1$ are relatively prime. Therefore there exist integers $n_2, m_2$ such that $n_1m_2 - m_1n_2 = 1$. Hence the vectors $(n_1, m_1)$ and $(n_2, m_2)$ form a basis of $\mathbb{Z}^2$. Take $K = \{\exp t(n_2, m_2) \mid t \in \mathbb{R}\}$.

Remark 4.9. More generally if $G$ is a torus and $H \subset G$ is a closed connected subgroup there is another closed connected subgroup $K \subset G$ so that $G = K \times H$. This is a bit harder to prove than the Lemma above.
Chapter 5

Proof of Theorem 2.31 part I: the 3-dimensional case

In this section we prove Theorem 2.31 in the case that \( \dim B = 3 \):

**Lemma 5.1.** Let \( B \) be a compact connected contact toric \( G = \mathbb{T}^2 \) manifold (in particular \( \dim B = 3 \)). Suppose the action of \( G \) is not free. Then there exist two closed subgroups \( K_1, K_2 \subset G \) isomorphic to \( S^1 \) so that \( B \) is homeomorphic to \( ([0, 1] \times G)/\sim \) where \( (0, g) \sim (0, ag) \) for all \( g \in G \), \( a \in K_1 \) and \( (1, g) \sim (1, ag) \) for all \( g \in G \) and \( a \in K_2 \). In other words \( B \) is obtained from the manifold with boundary \([0, 1] \times G\) by collapsing circles in the two components of the boundary by the respective actions of two circle subgroups. (It may happen that \( K_1 = K_2 \)).

**Exercise 5.2.** Consider the standard sphere \( S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \). The torus \( G = \{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid |\lambda_1|^2 = 1, \ |\lambda_2|^2 = 1 \} \) acts on \( S^3 \) by \( (\lambda_1, \lambda_2) \cdot (z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2) \). Show that \( S^3 \) is of the form \( ([0, 1] \times G)/\sim \) for two circle subgroups \( K_1, K_2 \) of \( G \) where \( \sim \) is the equivalence relation in Lemma 5.1. What are the subgroups \( K_1, K_2 \)?

**Proof of Lemma 5.1.** By Lemma 4.7 all isotropy groups for the action of \( G \) on \( B \) are connected and no isotropy group is all of \( G \). Therefore, since \( \dim G = 2 \) the possible isotropy groups are trivial or circles. And points with circle isotropy groups must exist since the action is not free.

If the isotropy group of a point \( b \in B \) is trivial then by Lemma 4.2 and the dimension count a \( G \)-invariant neighborhood \( U \) of \( b \) is equivariantly diffeomorphic to \( G \times I \) where \( I \) is an open interval and \( G \) acts on \( G \times I \) by \( g \cdot (a, t) = (ga, t) \). Hence \( U/G = I \) and there is a map \( s : U/G \to B \) so that \( \pi \circ s = id \) where \( \pi : B \to B/G \) is the orbit map, i.e., \( \pi \) has a local section.

Now consider a point \( b \in B \) with the isotropy group \( G_b \) isomorphic to \( S^1 \).

By Lemma 4.2 a neighborhood of \( b \) in \( B \) is \( G \)-equivariantly diffeomorphic to a

\(^1\)By Lemma 4.7 there are no more possibilities for \( G_b \).
groups is $\pi \varphi G$ diffeomorphism (where $\varphi$ may identify it with $[0,1]$). Some curve $g$ dimensional manifold with boundary. Since $B/G$ is two intervals will be left as an exercise to the reader. Thus we have two sections, open interval or to a half-open interval $[0,1]$ there are only two intervals: $I$ and $I_1 = (1/3, 1]$. The case of more than two intervals will be left as an exercise to the reader. Thus we have two sections $s_0 : [0,2/3) \to B$ and $s_1 : (1/3, 1] \to B$. Since $G$ acts freely on $\pi^{-1}([0,1])$ the map $\varphi : (1/3, 2/3) \times G \to \pi^{-1}((1/3, 2/3))$ given by $\varphi(t,g) = g \cdot s_0(t)$ is a $G$-equivariant diffeomorphism (where $G$ acts on the product $(1/3, 2/3) \times G$ by multiplication on the second factor). We have $\varphi^{-1} \circ s_0(t) = (t,1)$ and $\varphi^{-1} \circ s_1(t) = (t, g(t))$ for some curve $g : (1/3, 2/3) \to G$. Since $(1/3, 2/3)$ is simply connected we may lift $g$.

Note that $S^1 \times D^2 = (\mathbb{T}^2 \times [0,1))/ \sim$ where $(\lambda_1, \lambda_2, 0) \sim (\lambda_1, \mu_2, 0)$ for all $(\lambda_1, \lambda_2) \in \mathbb{T}^2$ and $\mu \in S^1$. We conclude that

1. $U/G \simeq [0,1]$;
2. $U \simeq (G \times [0,1))/ \sim$ where $(g, 0) \sim (ag, 0)$ for all $g \in G$, $a \in G_b$;
3. there is a local section $s : U/G \to B$ of the orbit map $\pi : B \to B/G$;
4. the set of points in $U$ with non-trivial isotropy groups is $G \cdot b = \pi^{-1}(0)$, where again $\pi : U \to [0,1]$ denotes the orbit map.

It follows that the orbit space $B/G$ is locally homeomorphic to either an open interval or to a half-open interval $[0,1)$. Hence $B/G$ is a topological 1-dimensional manifold with boundary. Since $B/G$ is compact and connected we may identify it with $[0,1]$. Note that the set of points with non-trivial isotropy groups is $\pi^{-1}([0,1])$, where by abuse of notation $\pi : B \to [0,1]$ denotes the orbit map. More specifically $\pi^{-1}(0) = G/K_1$, $\pi^{-1}(1) = G/K_2$ for some circle subgroups $K_1, K_2 \subset G$.

We now argue that $\pi : B \to [0,1]$ has a global section $s : [0,1] \to B$, so that $\pi \circ s(t) = t$ for all $t \in [0,1]$. We have seen that sections of $\pi$ exist locally: for every $t \in [0,1]$ there is an interval $I \subset [0,1]$ open in $[0,1]$ and containing $t$ and a map $s : I \to B$ so that $\pi \circ s = \text{id}_t$. We want to patch these local sections into a global section.

Since $[0,1]$ is compact, we can cover it by finitely many intervals $I_j$ so that on each $I_j$ there is a section $s_j : I_j \to B$. Let us now assume for simplicity that there are only two intervals: $I_0 = [0,2/3)$ and $I_1 = (1/3, 1]$. The case of more than two intervals will be left as an exercise to the reader. Thus we have two sections $s_0 : [0,2/3) \to B$ and $s_1 : (1/3, 1] \to B$. Since $G$ acts freely on $\pi^{-1}((0,1))$ the map $\varphi : (1/3, 2/3) \times G \to \pi^{-1}((1/3, 2/3))$ given by $\varphi(t,g) = g \cdot s_0(t)$ is a $G$-equivariant diffeomorphism (where $G$ acts on the product $(1/3, 2/3) \times G$ by multiplication on the second factor). We have $\varphi^{-1} \circ s_0(t) = (t,1)$ and $\varphi^{-1} \circ s_1(t) = (t, g(t))$ for some curve $g : (1/3, 2/3) \to G$. Since $(1/3, 2/3)$ is simply connected we may lift $g$.
to a curve \( \gamma(t) \) the universal cover \( \exp : g \to G \) of \( G \). Choose a smooth function \( \rho : (1/3, 2/3) \to [0, 1] \) with \( \rho(t) = 0 \) for \( t \) near 1/3 and \( \rho(t) = 1 \) for \( t \) near 2/3.

Now consider the curve \( a : (1/3, 2/3) \to G \) given by \( a(t) = \exp(\rho(t)\gamma(t)) \). The map \( s_2(t) = \varphi(t, a(t)) \) is a local section of \( \pi : B \to [0, 1] \) which agrees with \( s_0 \) near 1/3 and with \( s_1 \) near 2/3. Thus we can define a global section \( s : [0, 1] \to B \) by

\[
s(t) = \begin{cases} 
  s_0(t) & t \in [0, 1/3], \\
  s_2(t) & t \in [1/3, 2/3], \\
  s_3(t) & t \in [1/3, 0].
\end{cases}
\]

Now that we have a global section of \( \pi : B \to [0, 1] \) we can define a continuous map \( f : [0, 1] \times G \to B \), \( f(t, g) = g \cdot s(t) \). The map is onto; it descends to a bijective continuous map \( f : ([0, 1] \times G)/ \sim \to B \) where \( \sim \) is the equivalence relation in the statement of the Lemma. Since \( ([0, 1] \times G)/ \sim \) is compact, the map \( f \) is a homeomorphism.

**Exercise 5.3.** Show that if the groups \( K_1 \) and \( K_2 \) in the statement of the Lemma are the same, then \( B \) is \( S^1 \times S^2 \).

**Exercise 5.4.** (This exercise is considerably harder than the one above.) Show that if the groups \( K_1 \) and \( K_2 \) are different, then \( B \) is the quotient of \( S^3 \) by a finite cyclic group.

**Lemma 5.5.** Let \( G = \mathbb{T}^2 \) and let \( K_1, K_2 \subset G \) be two closed subgroups isomorphic to \( S^1 \). Let \( B \) be the topological space \( ([0, 1] \times G)/ \sim \) where \( (0, g) \sim (0, ag) \) for all \( g \in G \), \( a \in K_1 \) and \( (1, g) \sim (1, ag) \) for all \( g \in G \) and \( a \in K_2 \). In other words \( B \) is obtained from the manifold with boundary \([0, 1] \times G \) by collapsing circles in the two components of the boundary by the respective actions of two circle subgroups.

Then either \( H^1(B, \mathbb{Z}) = \mathbb{Z} = H^2(B, \mathbb{Z}) \) or \( H^1(B, \mathbb{Z}) = 0 \) and \( H^2(B, \mathbb{Z}) \) is a finite group. In particular, \( B \) cannot be homeomorphic to the 3-torus \( \mathbb{T}^3 \).

**Proof.** Recall that \( H^1(G, \mathbb{Z}) \) is isomorphic to the weight lattice \( \mathbb{Z}_G^* \) and that the isomorphism is given as follows: A weight \( \nu \in \mathbb{Z}_G^* \) defines a character \( \chi_\nu : G \to S^1 \) by \( \chi_\nu(\exp(X)) = e^{2\pi i \nu(X)} \), the class \( \chi_\nu^* \) of \( d\theta \) is the element in \( H^1(G, \mathbb{Z}) \) corresponding to \( \nu \). Here \( d\theta \) is the obvious 1-form on \( S^1 \).

Consequently if \( G = \mathbb{T}^2 \) and \( K_j \subset G \) is a circle subgroup, then \( \pi_j : G \to G/K_j \simeq S^1 \) is a character and hence the weight \( \nu_j = (d\pi_j)_1 \) defines an element of \( H^1(G, \mathbb{Z}) \). Thus if we identify \( H^1(G/K_j, \mathbb{Z}) \) with \( \mathbb{Z} = H^1(G, \mathbb{Z}) \) with \( \mathbb{Z}_G^* \), then the map \( H^1(G/K_j, \mathbb{Z}) \to H^1(G, \mathbb{Z}) \) becomes the map \( \mathbb{Z} \ni n \to n\nu_j \in \mathbb{Z}_G^* \).

The sets \( U = ([0, 2/3] \times G)/ \sim \) and \( V = ((1/3, 1] \times G)/ \sim \) are two open subsets of \( B \). We have \( B = U \cup V, U \cap V = (1/3, 2/3) \times G \) is homotopy equivalent to \( G, U \) is homotopy equivalent to \( G/K_1, V \) is homotopy equivalent to \( G/K_2 \) and the inclusion maps \( U \cap V \hookrightarrow U, U \cap V \hookrightarrow V \) are homotopy equivalent to projections \( \pi_1 : G \to G/K_1, \pi_2 : G \to G/K_2 \) respectively. Hence under the above identifications of \( H^1(U) \) and \( H^1(V) \) with \( \mathbb{Z} \), the inclusions \( U \cap V \hookrightarrow U, U \cap V \hookrightarrow V \) induce the maps \( \mathbb{Z} \ni n \to n\nu_j \in \mathbb{Z}_G^* \), \( j = 1, 2 \), respectively.
CHAPTER 5. PROOF OF THEOREM?? PART I: THE 3-DIMENSIONAL CASE

We now apply the Mayer-Vietoris sequence to compute the integral cohomology of $B$. We have: $0 \to H^0(B) \to H^0(U) \oplus H^0(V) \to H^0(G) \xrightarrow{\delta} H^1(B) \to H^1(U) \oplus H^1(V) \to H^1(G) \xrightarrow{\delta} H^2(B) \to H^2(U) \oplus H^2(V) \to H^2(G) \xrightarrow{\delta} H^3(B) \to 0$. Clearly the map $H^0(U) \oplus H^0(V) \to H^0(G)$ is onto. Given the identifications above the map $\varphi : H^1(U) \oplus H^1(V) \to H^1(G)$ becomes $\mathbb{Z} \oplus \mathbb{Z} \ni (n, m) \mapsto n\nu_1 + m\nu_2 \in \mathbb{Z}_G^*$. We therefore have $0 \to H^1(B) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\delta} \mathbb{Z}_G^* \xrightarrow{\delta} H^2(B) \to 0 \oplus 0 \to H^2(G) \xrightarrow{\delta} H^3(B) \to 0$. We conclude that

- $H^2(B) = \mathbb{Z}_G^*/(\mathbb{Z}\nu_1 + \mathbb{Z}\nu_2)$
- $H^1(B) = \{(n, m) \in \mathbb{Z} \mid n\nu_1 + m\nu_2 = 0\}$.

Since $\nu_1, \nu_2$ are differentials of projections onto quotients by circle subgroups, either $\nu_1$ and $\nu_2$ are independent over $\mathbb{Z}$ or $\nu_1 = \pm \nu_2$. In the first case $H^1(B) = 0$ and $H^2(B)$ is a finite abelian group. In the second case $H^1(B) = \mathbb{Z}$ and $H^2(B) = \mathbb{Z}_G^*/\mathbb{Z}\nu_1 = \mathbb{Z}$. □

This finishes the proof of Theorem 2.31 in the case that $\dim B = 3$. 
Chapter 6

Proof of Theorem 2.31, part II: uniqueness of symplectic toric manifolds

In this section we sketch a proof of

**Theorem 6.1.** Let \((B, \xi = \ker \alpha, \Psi : \xi^\circ_+ \to g^*)\) be a compact connected contact toric \(G\)-manifold.

Suppose the dimension \(k\) of the maximal linear subspace of the moment cone \(C(\Psi) = \Psi(\xi^\circ_+) \cup \{0\}\) satisfies \(0 < k < \text{dim} \, G\). Then \(B\) is homotopy equivalent to the product of a \(k\)-torus with a sphere. In particular \(B\) is not the co-sphere bundle of the \(n\)-torus, \(n = \frac{1}{2}(\text{dim} \, B + 1) = \text{dim} \, G\).

The main idea of the proof is simple. We will first argue that there is an action of \(G\) on the symplectic manifold

\[
M = T^*\mathbb{T}^k \times \mathbb{C}^l \setminus 0 = \{(q, p, z) \in T^k \times (\mathbb{R}^k)^* \times \mathbb{C}^l = T^*\mathbb{T}^k \times \mathbb{C}^l \mid (p, z) \neq (0, 0)\},
\]

\(l = \frac{1}{2}(\text{dim} \, B + 1 - k) > 0\), with moment map \(\tilde{\Phi} : M \to g^*\) such that

\[
\tilde{\Phi}(M) = \Psi(\xi^\circ_+).
\] (6.1)

We then argue that (6.1) implies that \(M\) is \(G\)-equivariantly symplectomorphic to \(\xi^\circ_+\). Note that \(M\) is homotopy equivalent to \(\mathbb{T}^k \times S^{k+2l-1}\).

We start with the definition of a symplectic slice representation (c.f. proof of Lemma 4.7), which is essential for understanding the local structure of symplectic toric manifolds.

**Definition 6.2.** Let \((M, \omega)\) be a symplectic manifold with a Hamiltonian action of a torus \(G\). Then an orbit \(G \cdot m\) is an isotropic submanifold of \((M, \omega)\).\(^1\) The

\(^1\)For a proof of this easy fact see, for example, [GS].
symplectic slice representation at \( m \) is the representation of the isotropy group \( G_m \) of \( m \) on the symplectic vector space \( V := T_m(G \cdot m)^\omega/T_m(G \cdot m) \). Here, as usual, \( T_m(G \cdot m)^\omega \) denotes the symplectic perpendicular to \( T_m(G \cdot m) \) in the symplectic vector space \( (T_mM, \omega_m) \).

The equivariant isotropic embedding theorem (see for example [GS], Theorem 39.1) asserts that a neighborhood of an orbit \( G \cdot m \) is determined (up to equivariant symplectomorphisms) by the symplectic slice representation at \( m \). In fact, the topological normal bundle of the orbit \( G \cdot m \) in \( M \) is \( G \times_H (\mathfrak{g}/\mathfrak{h} \times V) \) where \( H \) is the isotropy group of \( m \) and \( \mathfrak{h} \) is its Lie algebra (op. cit.).

**Remark 6.3.** In the case of symplectic toric manifolds the dimension of the symplectic slice at \( m \) is twice the dimension of the isotropy group \( G_m \). Hence by Lemma 4.6 the group \( G_m \) is connected and the representation of \( G_m \) on \( V \) is determined by a set of weights \( \{ \nu_i \} \) which forms a basis of the weight lattice of \( G_m \).

Note that the image of \( V \) under the moment map \( \Phi_V : V \to \mathfrak{g}_m^* \) defined by the representation is the cone

\[
\Phi_V(V) = \{ \sum a_i \nu_i \mid a_i \geq 0 \}.
\]

In particular the edges of the cone are spanned by the weights. Alternatively, the isotropy group \( G_m \) is isomorphic to \( \mathbb{T}^l \) for some \( l \) and the slice representation \( \rho : G_m \to \text{Sp}(V) \) is isomorphic to the standard representation of \( \mathbb{T}^l \) on \( \mathbb{C}^l \).

With a little more work one can prove the following two propositions (their proofs can be found, for example, in [D]).

**Proposition 6.4.** Let \((M, \omega, \Phi : M \to \mathfrak{g}^*)\) be a symplectic toric manifold, \( m \in M \) a point and \( U \) a neighborhood of \( G \cdot m \) in \( M \). Then for a sufficiently small ball \( \mathcal{O} \) about \( \eta = \Phi(m) \) in \( \mathfrak{g}^* \) the set \( \Phi(U) \cap \mathcal{O} \) determines the symplectic slice representation at \( m \) and hence a small \( G \)-invariant neighborhood of \( G \cdot m \) in \( M \).

**Proposition 6.5.** Let \((M, \omega, \Phi : M \to \mathfrak{g}^*)\) be a symplectic toric manifold, \( m \in M \) a point, \( \eta = \Phi(m) \) and \( G_m \) the isotropy group of \( m \). Identify \( G_m \) with the standard torus \( \mathbb{T}^d \), \( d = \dim G_m \), and extend it to an identification of \( G \) with \( \mathbb{T}^d \times \mathbb{T}^c \). \(^3 \)

A \( G \)-invariant neighborhood \( U \) of \( G \cdot m \) in \( M \) is equivariantly symplectomorphic to a neighborhood of \( \mathbb{T}^c \times \{(0, 0)\} \) in \( T^* \mathbb{T}^c \times \mathbb{C}^d \cong \mathbb{T}^c \times (\mathbb{R}^c)^* \times \mathbb{C}^d \). Hence

\[
\{ t(\mu - \eta) + \eta \mid t \geq 0, \mu \in \Phi(U) \} = \Phi(T^* \mathbb{T}^c \times \mathbb{C}^d),
\]

where \( \Phi : T^* \mathbb{T}^c \times \mathbb{C}^d \to (\mathbb{R}^c)^* \times (\mathbb{R}^d)^* \cong \mathfrak{g}^* \) is the moment map for the “obvious” action of \( G = \mathbb{T}^c \times \mathbb{T}^d \) on \( T^* \mathbb{T}^c \times \mathbb{C}^d \). \(^4 \)

\(^2\) The standard representation of \( \mathbb{T}^l = \{ (\lambda_1, \ldots, \lambda_l) \in \mathbb{C}^l \mid |\lambda_j| = 1 \} \) on \( \mathbb{C}^l \) is given by \( (\lambda_1, \ldots, \lambda_l) \cdot (z_1, \ldots, z_l) = (\lambda_1 z_1, \ldots, \lambda_l z_l) \).

\(^3\) Here we use Remark 4.9.

\(^4\) The “obvious” action of \( \mathbb{T}^c \) on \( T^* \mathbb{T}^c \) is the lift of left multiplication.
Corollary 6.6. Let \((B, \xi = \ker \alpha, \Psi : \xi^\circ_+)\) be a compact connected contact toric \(G\)-manifold. Suppose the dimension \(k\) of the maximal linear subspace of the moment cone \(C(\Psi) = \Psi(\xi^\circ_+) \cup \{0\}\) satisfies \(0 < k < \dim G\).

Then there is an identification of \(G\) with \(T^k \times T^l\), \(l = \dim G - k\), so that

\[
\Psi(\xi^\circ_+) \cup \{0\} = \tilde{\Phi}(T^k \times C^l)
\]

where \(\tilde{\Phi} : T^*T^k \times C^l \to (\mathbb{R}^k)^* \times (\mathbb{R}^l)^* \simeq \mathfrak{g}^*\) is the moment map for the obvious action of \(T^k \times T^l\) on \(T^*T^k \times C^l\).

Proof. If \(C\) is a cone in \(\mathfrak{g}^*\) whose maximal linear subspace is \(P\) and if \(U\) is a neighborhood in \(\mathfrak{g}^*\) of a point \(\eta \in P\) then \(C\) equals the cone on \(U \cap C\) with the vertex at \(\eta\):

\[
C = \{t(\mu - \eta) + \eta \mid t \geq 0, \mu \in U \cap C\}.
\]

Since the \(B\) is contact toric \(\Psi(\xi^\circ_+)\) does not contain the origin (c.f. Lemma 4.7). Since \(B\) is compact and \(\Psi : \xi^\circ_+ \to \mathfrak{g}^*\) is homogeneous, \(\Psi : \xi^\circ_+ \to \mathfrak{g}^* \setminus \{0\}\) is proper. Hence for any \(\eta \in \Psi(\xi^\circ_+)\) and any neighborhood \(U\) of \(\Psi^{-1}(\eta)\), \(\Psi(U)\) is a neighborhood of \(\eta\) in \(\Psi(\xi^\circ_+)\).

Moreover, since the fibers of \(\Psi\) are \(G\)-orbits (by Theorem 4.1), for any neighborhood \(U\) of an orbit \(G \cdot m\) the set \(\Psi(U)\) is a neighborhood of \(\eta = \Psi(m)\) in \(\Psi(\xi^\circ_+)\). In particular it contains a set of the form \(\mathcal{O} \cap \Psi(\xi^\circ_+)\) where \(\mathcal{O} \subset \mathfrak{g}^*\) is a small ball about \(\eta\).

Let \(P \subset C(\Psi)\) be the maximal linear subspace and let \(0 \neq \eta \in P\). Then, as noted at the beginning of the proof, for any ball \(\mathcal{O} \subset \mathfrak{g}^*\) about \(\eta\) we have

\[
C(\Psi) = \{t(\mu - \eta) + \eta \mid t \geq 0, \mu \in \mathcal{O} \cap \Psi(\xi^\circ_+)\}
\]

= \(\{t(\mu - \eta) + \eta \mid t \geq 0, \mu \in \Psi(U)\}\).

On the other hand, it follows from Proposition 6.5 that

\[
\{t(\mu - \eta) + \eta \mid t \geq 0, \mu \in \Psi(U)\} = \tilde{\Phi}(T^*T^c \times C^d) \simeq \mathbb{R}^c \times (\mathbb{R}_{\geq 0})^d
\]

for some integers \(c\) and \(d\). It follows that \(c = k, d = \dim G - k = l\) and that

\[
C(\Psi) = \tilde{\Phi}(T^*T^k \times C^l).
\]

\(\square\)

Note that since \(\tilde{\Phi}^{-1}(0) = T^k \times \{(0, 0)\}\), we get

\[
\Psi(\xi^\circ_+) = \tilde{\Phi}(M),
\]

where \(M = (T^*T^k \times C^l) \setminus (T^k \times \{(0, 0)\})\).

Definition 6.7. A symplectic toric \(G\)-manifold \((M, \omega, \Phi : M \to \mathfrak{g}^*)\) is \textit{good} (for the purposes of these lectures) if
1. the fibers of \( \Phi : M \to g^* \) are connected,
2. the set \( \Phi(M) \) is contractible and
3. there is an open set \( U \subset g^* \) such that \( \Phi(M) \subset U \) and the map \( \Phi : M \to U \) is proper.

The definition is designed in such a way that it includes compact symplectic toric manifolds, symplectizations of compact contact toric manifolds of dimension bigger than 3 and the manifolds of the form \( M = (T^*T^k \times C^l) \setminus (T^k \times \{0,0\}) \), \( 0 < k, l \). As a consequence of the definition and of Proposition 6.4 we have

**Lemma 6.8.** Let \((M,\omega,\Phi : M \to g^*)\) be a good symplectic toric manifold. For any point \( \eta \in \Phi(M) \) and any sufficiently small ball \( O \) centered at \( \eta \), the set \( O \cap \Phi(M) \) determines the symplectic toric manifold \((\Phi^{-1}(O),\omega|_{\Phi^{-1}(O)},\Phi|_{\Phi^{-1}(O)})\) (up to a \( G \)-equivariant symplectomorphism).

**Definition 6.9.** Two symplectic toric \( G \)-manifolds \((M,\omega,\Phi : M \to g^*)\) and \((M',\omega',\Phi' : M' \to g^*)\) are **isomorphic** if there is a \( G \)-equivariant diffeomorphism \( \sigma : M \to M' \) such that \( \sigma^*\omega' = \omega \) and \( \sigma^*\Phi' = \Phi \).

We denote by \( \text{Iso}(M,\omega,\Phi) = \text{Iso}(M) \) the group of isomorphisms of a symplectic toric manifold \((M,\omega,\Phi)\).

Note that the last condition on \( \sigma \) is almost redundant — if \( \sigma \) is symplectic and \( G \)-equivariant then \( \sigma^*\Phi' = \Phi + c \) for some constant vector \( c \in g^* \). We impose the last condition for technical convenience.

**Definition 6.10.** Two symplectic toric \( G \)-manifolds \((M,\omega,\Phi : M \to g^*)\) and \((M',\omega',\Phi' : M' \to g^*)\) are **locally isomorphic** over a set \( \Delta \subset g^* \) if

1. \( \Phi(M) = \Delta = \Phi'(M') \) and
2. for any \( \eta \in \Delta \) and any sufficiently small ball \( O \subset g^* \) centered at \( \eta \) the symplectic toric manifolds \((\Phi^{-1}(O),\omega|_{\Phi^{-1}(O)},\Phi|_{\Phi^{-1}(O)})\) and \((\Phi'^{-1}(O),\omega'|_{(\Phi')^{-1}(O)},\Phi'|_{(\Phi')^{-1}(O)})\) are isomorphic.

Given the above definitions we see that Lemma 6.8 implies

**Lemma 6.11.** Suppose \((M,\omega,\Phi)\) and \((M',\omega',\Phi')\) are two good symplectic toric \( G \)-manifolds with \( \Phi(M) = \Phi'(M') \). Then \((M,\omega,\Phi)\) and \((M',\omega',\Phi')\) are locally isomorphic over \( \Delta = \Phi(M) \).

Therefore the proof of Theorem 6.1 reduces to

**Proposition 6.12.** Any good symplectic toric \( G \)-manifold locally isomorphic to a given symplectic toric \( G \)-manifold \((M,\omega,\Phi)\) is actually isomorphic to \((M,\omega,\Phi)\).
The proposition could be stated for a larger class of symplectic toric manifolds. We leave it to the reader to find the most general form of the statement (and prove it). The rest of the section is occupied with a proof of the proposition.

Suppose a symplectic toric $G$-manifold $(M', \omega', \Phi')$ is locally isomorphic to $(M, \omega, \Phi)$. Then for any $\eta \in \Phi(M)$ and for any sufficiently small ball $O \subset g'$ about $\eta$ the symplectic toric manifold $\Phi^{-1}(O)$ is isomorphic to $(\Phi')^{-1}(O)$. Choose a locally finite cover $\{O_i\}_{i \in I}$ of $\Phi(M)$ by such balls. Then for each index $i$ we have an isomorphism

$$f_i : \Phi^{-1}(O_i) \to (\Phi')^{-1}(O_i).$$

Let $O_{ij} = O_i \cap O_j$. Define

$$g_{ij} : \Phi^{-1}(O_{ij}) \to \Phi^{-1}(O_{ij}), \quad g_{ij} = f_i^{-1} \circ f_j$$

(to keep the notation manageable we wrote $f_i$ for the restriction $f_i|_{O_{ij}}$ etc.; we will continue to omit restrictions in the rest of the section). It is easy to see that

$$g_{ii} = id, \ g_{ij} \circ g_{ji} = id, \text{ and } g_{ij} \circ g_{jk} \circ g_{ki} = id \quad (6.2)$$

wherever these equations make sense.

The data $\{O_i\}, \{g_{ij}\}$ and $(M, \omega, \Phi)$ allow us to reconstruct $(M', \omega', \Phi')$. Indeed, let $\tilde{M} = \bigsqcup \Phi^{-1}(O_i)$. Define a relation $\sim$ on $\tilde{M}$ by $\Phi^{-1}(O_i) \ni x_i \sim x_j \in \Phi^{-1}(O_j)$ iff $x_j = g_{ij}(x_i)$. Equations (6.2) imply that $\sim$ is an equivalence relation. It follows that $\tilde{M}/\sim$ is a symplectic toric manifold. Moreover the map $\tilde{F} : \tilde{M} \to M'$ defined by $\tilde{F}|_{\Phi^{-1}(O_i)} = f_i$ descends to a well defined map $F : \tilde{M}/\sim \to M'$. The map $F$ is an isomorphism.

Suppose $\tilde{f}_i : \Phi^{-1}(O_i) \to (\Phi')^{-1}(O_i)$ is another collection of isomorphisms. Let $\tilde{g}_{ij} = (\tilde{f}_i)^{-1} \circ \tilde{f}_j$. Clearly

$$g_{ij} = h_i \circ \tilde{g}_{ij} \circ h_j^{-1} \quad (6.3)$$

where $h_i = f_i^{-1} \circ f_i \in \text{Iso}(\Phi^{-1}(O_i))$.

We also get a different set of data if we choose a different cover. However, all these sets of data define one object — a class in the first Čech cohomology with coefficients in a certain sheaf. Let us now review the notions of sheaves and Čech cohomology. There are many good references for this material. I will be following [WW].

**Definition 6.13.** A **sheaf** of groups $S$ on a topological space $X$ is an assignment

$$S : \{\text{open sets in } X\} \to \text{groups}, \ U \mapsto S(U)$$

satisfying two conditions:

1. For a pair of open sets $U \subset W$ in $X$ there is a restriction map $\rho^W_U : S(W) \to S(U)$ such that for any three sets $U \subset W \subset V$ of $X$

$$\rho^W_U \circ \rho^V_W = \rho^V_U.$$
Elements of $S(U)$ are called sections. Given a section $\varphi \in S(W)$ we write $\varphi|_U$ for $\rho^W_U(\varphi)$.

2. Given an open cover $\{U_i\}$ of an open set $U$ (so that $U = \bigcup U_i$) and a collection of sections $\varphi_i \in S(U_i)$ such that

$$
\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}
$$

for all indices $i, j$ there is a unique section $\varphi \in S(U)$ such that

$$
\varphi|_{U_i} = \varphi_i
$$

for all $i$.

The sheaf $S$ is abelian if $S(U)$ is an abelian group for all open sets $U$.

Three examples of sheaves will be important to us. Check that they are indeed sheaves.

**Example 6.14.** Let $(M, \omega, \Phi)$ be a good symplectic toric $G$-manifold. The assignment $\text{Iso} : U \mapsto \text{Iso}(U) (U \subset \Phi(M) \text{ open})$ is a sheaf on $\Phi(M)$. The group operation is composition.

**Example 6.15 (Locally constant sheaf).** Let $H$ be a group and $X$ a topological space. The assignment that associates the group $H$ to every open connected subset of $X$ is a sheaf, called a locally constant sheaf. It is denoted by $H$. Thus $H(U) = H$ for every connected open set $U \subset X$. The group operation is the multiplication in $H$.

**Example 6.16.** Let $(M, \omega, \Phi)$ be a good symplectic toric $G$-manifold. Define a sheaf $\mathcal{C}$ on $\Phi(M)$ by

$$
\mathcal{C}(U) = C^\infty(\Phi^{-1}(U))^G, \quad G\text{-invariant smooth functions on } \Phi^{-1}(U).
$$

The group operation is addition of functions.

**Definition 6.17.** Let $S_1, S_2$ be sheaves on a topological space $X$. A map of sheaves $\tau : S_1 \to S_2$ is a family of group homomorphisms

$$
\tau_U : S_1(U) \to S_2(U), \quad U \subset X \text{ open}
$$

compatible with the restrictions:

$$(\rho_2)_U^W \circ f_W = f_U \circ (\rho_1)_U^W$$

for all pairs $U \subset W$ of open sets in $X$.

**Example 6.18.** Consider the sheaves Iso and $\mathcal{C}$ defined in Examples 6.14 and 6.16 above. A section $f \in \mathcal{C}(U)$ is a $G$-invariant function on $\Phi^{-1}(U)$. Its time $t$ flow $\varphi^t$ preserves the fibers of $\Phi$ and is a $G$-equivariant symplectomorphism of $\Phi^{-1}(U)$. Hence $\varphi^t_U$ is a section of Iso($U$). This gives us for each open set $U \subset \Phi(M)$ a map $\tau_U : \mathcal{C}(U) \to \text{Iso}(U)$, $\tau_U(f) = \varphi^t_U$. Moreover, any two functions $f_1, f_2 \in \mathcal{C}(U)$ Poisson commute [prove this]. Hence $\varphi^{t_1}_U \circ \varphi^{t_2}_U = \varphi^{t_1+t_2}_U$ and therefore $\tau_U$ is a group homomorphism. Thus we get a map of sheaves $\tau : \mathcal{C} \to \text{Iso}$. 
Given a map of sheaves $\tau: S_1 \to S_2$ one can define the sheaves the kernel and image sheaves $\ker \tau$ and $\text{im} \tau$: for an open set $U$ \((\ker \tau)(U) = \ker \tau_U, (\text{im} \tau)(U) = \text{im} \tau_U\). Hence it makes sense to say that a map of sheaves is onto and more generally talk about exact sequences of sheaves.\footnote{Warning: the map $\tau: S_1 \to S_2$ being onto does not mean that $\tau_U$ is onto for every open set $U$. See [WW] or any other good book on sheaves for more details.}

**Proposition 6.19.** Let $(M, \omega, \Phi)$ be a good symplectic toric manifold. The map of sheaves $\tau: \mathcal{C} \to \text{Iso}$ defined above is onto. Hence $\text{Iso}$ is an abelian sheaf.

The kernel of $\tau$ is the locally constant sheaf $\mathbb{R} \times \mathbb{Z}_{\mathbb{G}}$. We thus have a short exact sequence of abelian sheaves:

$$0 \to \mathbb{R} \times \mathbb{Z}_{\mathbb{G}} \to \mathcal{C} \to \text{Iso} \to 0.$$ 

The proposition is due to Boucetta and Molino [BoM]. See also [LT].

### 6.1 Čech cohomology

In this subsection we “review” the notion of Čech cohomology with coefficients in an abelian sheaf. There are many good references, such as [WW], for the nontrivial facts that we list below without proofs.

Let $X$ be a topological space $\{U_i\}$ an open locally finite cover of $X$ and $\mathcal{S}$ an abelian sheaf on $X$. A 0 Čech cochain is a function that assigns to each index $i$ an element $f_i$ of $\mathcal{S}(U_i)$, i.e., the group of 0-cochains $C^0(\{U_i\}, \mathcal{S})$ is the product $\prod \mathcal{S}(U_i)$. A 1 Čech cochain assigns to an ordered pair of indices $ij$ an element $g_{ij}$ of $\mathcal{S}(U_{ij})$ where $U_{ij} = U_i \cap U_j$. Moreover we require that $g_{ij} = -g_{ji}$ (we now think of the groups $\mathcal{S}(U)$ additively). More generally a $p$-cochain assigns to an ordered $p+1$ tuple of indices $i_0 \ldots i_p$ an element $s_{i_0 \ldots i_p} \in \mathcal{S}(U_{i_0 \ldots i_p})$ where $U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ and $s_{i_0 \ldots i_p}$ is skew-symmetric in the indices. The coboundary operator $\delta: C^p(\{U_i\}, \mathcal{S}) \to C^{p+1}(\{U_i\}, \mathcal{S})$ is defined by

$$\delta s_{i_0 \ldots i_p+1} = \sum (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_p+1}$$

where $\hat{i}_j$ means that the index is omitted, and where we omitted writing the restrictions of the terms on the right hand side to $U_{i_0 \ldots i_{p+1}}$. One proves that $\delta^2 = 0$. The cohomology of the complex $(C^p(\{U_i\}, \mathcal{S}), \delta)$ denoted by $\check{H}^*(\{U_i\}, \mathcal{S})$ is called the Čech cohomology of the cover $\{U_i\}$ with coefficients in the sheaf $\mathcal{S}$.

Given a refinement $\{V_j\}$ of the cover $\{U_i\}$ the restrictions give rise to a chain map $C^p(\{U_i\}, \mathcal{S}) \to C^p(\{V_j\}, \mathcal{S})$, which in turn gives rise to a map in cohomology $\check{H}^*(\{U_i\}, \mathcal{S}) \to \check{H}^*(\{V_j\}, \mathcal{S})$. Taking the direct limit over all locally finite covers we get a well-defined cohomology group

$$\check{H}^*(X, \mathcal{S}) = \lim_{\to} \check{H}^*(\{U_i\}, \mathcal{S}),$$

$$\check{H}^0(X, \mathcal{S}) = \hat{\mathcal{H}}^0(X, \mathcal{S}),$$

$$\hat{\mathcal{H}}^0(X, \mathcal{S}) = \hat{\mathcal{H}}^0(\mathcal{S}) := \mathcal{H}^0(X, \mathcal{S}).$$
the Čech cohomology of $X$ with coefficients in the sheaf $S$.

Now let $(M, \omega, \Phi : M \to g^*)$ be a good symplectic toric $G$-manifold and $\{O_i\}$ a locally finite cover of $\Phi(M)$ by sufficiently small balls. If $\{g_{ij}\} \in C^1(\{O_i\}, \text{Iso})$ is a 1-cochain, then $\delta(\{g_{**}\}) = 0$ means that for all triples of indices $ijk$ we have

$$0 = \delta(\{g_{**}\})_{ijk} = -g_{jk} + g_{ik} + g_{ij},$$

which is (6.2) in additive notation (where on the right hand side we omitted the restrictions to $U_{ijk}$). Similarly if $\{g_{ij}\}, \{\tilde{g}_{ij}\} \in C^1(\{O_i\}, \text{Iso})$ are two 1-cochains that differ by $\delta(\{h_{**}\})$ for some 0-cochain $\{h_{i}\}$ then

$$\tilde{g}_{ij} - g_{ij} = -h_{i} + h_{j}$$

hence

$$g_{ij} = h_{i} + \tilde{g}_{ij} - h_{j},$$

which is (6.3) in additive notation. Thus the discussion above shows that to every element of $H^1(\{O_i\}, \text{Iso})$ there corresponds a good symplectic toric manifold $(M', \omega', \Phi' : M \to g^*)$ locally isomorphic to $(M, \omega, \Phi : M \to g^*)$. More generally one can check that there is a one-to-one correspondence between cohomology classes in $H^1(\Phi(M), \text{Iso})$ and isomorphism classes of good symplectic toric manifolds locally isomorphic to $(M, \omega, \Phi : M \to g^*)$. Thus to complete the proof of Proposition 6.12 (and thereby Theorem 6.1) it remains to show that the group $H^1(\Phi(M), \text{Iso})$ is trivial for any good symplectic toric manifold $M$. For this we use Proposition 6.19, two properties of Čech cohomology and a property of the sheaf $C$ defined in Example 6.16. The first property of Čech cohomology that we need is

**Theorem 6.20.** A short exact sequence of abelian sheaves $0 \to S_1 \to S_2 \to S_3 \to 0$ on a space $X$ induces a long exact sequence in Čech cohomology

$$\cdots \to H^p(X, S_1) \to H^p(X, S_2) \to H^p(X, S_3) \xrightarrow{\delta} H^{p+1}(X, S_1) \to \cdots$$

The second property that we will use is

**Theorem 6.21.** Let $X$ be a simply connected topological space and $H$ an abelian group. The Čech cohomology $\check{H}^*(X, H)$ of $X$ with coefficients in the locally constant sheaf $H$ is isomorphic to the singular cohomology $H^*(X, H)$ of $X$ with coefficients in the abelian group $H$.

We will use the following property of the sheaf $C$ (cf. [LT], Proposition 7.3)

**Lemma 6.22.** The sheaf $C$ defined in Example 6.16 is acyclic, that is, $\check{H}^q(\Phi(M), C) = 0$ for all $q > 0$.

Now putting Theorem 6.20, Lemma 6.22 and Proposition 6.19 together we see that if $(M, \omega, \Phi)$ is a good symplectic toric $G$-manifold and $\text{Iso}$ the sheaf defined in Example 6.14 then the cohomology group $H^1(\Phi(M), \text{Iso})$ is isomorphic
to $\check{H}^2(\Phi(M), \mathbb{R} \times \mathbb{Z}_G)$. The latter group is isomorphic to the singular cohomology group $H^2(\Phi(M), \mathbb{Z}_G \times \mathbb{R})$ by Theorem 6.21. But $\Phi(M)$ is contractible, so $\check{H}^2(\Phi(M), \mathbb{Z}_G \times \mathbb{R}) = 0$. Therefore $\check{H}^1(\Phi(M), \text{Iso}) = 0$, which proves Proposition 6.12 and thereby Theorem 6.1.
Chapter 7

Proof of Theorem 2.31, part III: Morse theory on orbifolds

The goal of this section is to prove

**Theorem 7.1.** Let \((B, \xi = \ker \alpha, \Psi : \xi^*_+ \to g^*)\) be a compact connected contact toric \(G\)-manifold with \(\dim B \geq 3\). Suppose there is a vector \(X\) in the Lie algebra \(g\) of \(G\) such that the function \(\langle \Psi, X \rangle\) is strictly positive on \(B\). Then \(\dim H^1(B, \mathbb{R}) \leq 1\). In particular \(B\) is not the co-sphere bundle \(S^*G = T^n \times S^{n-1}\), \(n = \dim G = \frac{1}{2}(\dim B + 1) \geq 2\).

The proof of Theorem 7.1 above will complete our proof of Theorem 2.31. Since Theorem 2.31 implies the main result of the notes, Theorem 1.4, this, in turn, will finish the proof of the main result. As was sketched out at the beginning of Chapter 4 our proof of Theorem 7.1 has several steps. The first one is a theorem implicit in a paper of Boyer and Galicki [BG]:

**Theorem 7.2.** Let \((B, \xi = \ker \alpha, \Psi : \xi^*_+ \to g^*)\) be a compact connected contact toric \(G\)-manifold with \(\dim B \geq 3\). Suppose there is a vector \(X\) in the Lie algebra \(g\) of \(G\) such that the function \(\langle \Psi, X \rangle\) is strictly positive on \(B\). Then there exists on \(B\) a locally free circle action so that the quotient \(M = B/\mathbb{S}^1\) is a (compact) symplectic toric orbifold.

The second step is the argument that if \(M\) is a compact connected symplectic toric orbifold then \(H^q(M, \mathbb{R}) = 0\) for all odd degrees \(q\). This step uses Morse theory on orbifolds.

Let us now see why these two steps give us a proof of Theorem 7.1. Consider the circle action produced by Theorem 7.2 and the corresponding \(S^1\) orbit map.
π : B → M. If the circle action is actually free, then π is a circle fibration and we have the Gysin sequence

\[ 0 = H^{-1}(M, \mathbb{R}) \to H^1(M, \mathbb{R}) \to H^1(B, \mathbb{R}) \to H^0(M, \mathbb{R}) \to H^2(M, \mathbb{R}) \to \cdots. \]  

(7.1)

If the action of S^1 is locally free, the long exact sequence (7.1) still exists. The reason is that the Gysin sequence arises from a collapse of the Leray-Serre spectral sequence for a sphere bundle. For locally free S^1 actions the orbit map π : B → M is not a fibration, but the corresponding spectral sequences still collapses if we use real coefficients.\(^1\) Now, since H^1(M, \mathbb{R}) = 0 by the second step, it follows from (7.1) that dim H^1(B, \mathbb{R}) ≤ dim H^0(M, \mathbb{R}) = 1, which proves Theorem 7.1.

Here is an outline of the rest of the section. We start with a review of the notion of orbifold. We recall how orbifolds arise as quotients of manifolds by locally free actions of compact Lie groups. We then review the symplectic reduction theorem, in particular the fact that generically symplectic reduced spaces are orbifolds and use it to prove Theorem 7.2. Next we define Morse functions on orbifolds, and discuss the two fundamental results of Morse theory on orbifolds. Finally we argue that a compact symplectic toric orbifold M has a Morse function with all indices even and use this to conclude that the real cohomology of M vanishes in odd degrees.

We now define orbifolds and related differential geometric notions. For more details, see Satake [Sa1, Sa2] and, for a more modern point of view, Ruan [R]. The notion of orbifold was introduced by Satake in 1956 under the name of V-manifold. Orbifolds are designed to generalize manifolds in the following sense: an n-dimensional manifold is locally modeled on an open subset of \( \mathbb{R}^n \). An n-dimensional orbifold is locally modeled on a quotient \( \tilde{U}/\Gamma \) where \( \tilde{U} \) is an open subset of \( \mathbb{R}^n \) and \( \Gamma \) is a finite group acting smoothly on \( \tilde{U} \). To give a precise definition we need a few preliminary notions — we need to define charts, atlases and compatibility of atlases.

Let \( U \) be a connected topological space, \( \tilde{U} \) a connected n-dimensional manifold and \( \Gamma \) a finite group acting smoothly on \( \tilde{U} \). An n-dimensional uniformizing chart\(^2\) on \( U \) is a triple \( (\tilde{U}, \Gamma, \varphi) \) where \( \varphi : \tilde{U} \to U \) is a continuous map inducing a homeomorphism between \( \tilde{U}/\Gamma \) and \( U \) (thus, in particular, \( \varphi \) is constant on the orbits of \( \Gamma \)). We will only consider charts where the set of points in \( \tilde{U} \) fixed by \( \Gamma \) is either all of \( \tilde{U} \) or is of codimension 2 or greater. Note that we do not require that \( \Gamma \) acts effectively.

Two uniformizing charts \( (\tilde{U}_1, \Gamma_1, \varphi_1) \) and \( (\tilde{U}_2, \Gamma_2, \varphi_2) \) of \( U \) are isomorphic if there is a diffeomorphism \( \psi : \tilde{U}_1 \to \tilde{U}_2 \) and an isomorphism \( \lambda : \Gamma_1 \to \Gamma_2 \) such that \( \psi \) is \( \lambda \)-equivariant (i.e., \( \psi(g \cdot x) = \lambda(g) \cdot \psi(x) \) for all \( g \in \Gamma_1, x \in \tilde{U}_1 \)) and \( \varphi_2 \circ \psi = \varphi_1 \). For example, fix \( a \in \Gamma \). Define \( \psi : \tilde{U} \to \tilde{U} \) by \( \psi(x) = a \cdot x \). Let \( \lambda(g) = aga^{-1} \). Then \( (\psi, \lambda) : (\tilde{U}, \Gamma, \varphi) \to (\tilde{U}, \Gamma, \varphi) \) is an isomorphism.

\(^1\) The reader not familiar with spectral sequences may wish to take this claim on faith.

\(^2\)Ruan calls it a uniformizing system.
Let \( \iota : U' \hookrightarrow U \) be a connected open subset of \( U \). We say that a uniformizing chart \((\tilde{U}', \Gamma', \varphi')\) is \textit{induced from} a uniformizing chart \((\tilde{U}, \Gamma, \varphi)\) on \( U \) if there is a monomorphism \( \lambda : \Gamma' \to \Gamma \) and a \( \lambda \)-equivariant embedding \( \psi : \tilde{U}' \to \tilde{U} \) such that \( \iota \circ \varphi' = \varphi \circ \psi \). In this case \((\psi, \lambda) : (\tilde{U}', \Gamma', \varphi') \to (\tilde{U}, \Gamma, \varphi)\) is called an \textit{injection}.

An \textit{orbifold atlas} on a Hausdorff topological space \( M \) is an open cover \( \mathcal{U} \) of \( M \) satisfying the following conditions:

1. Each element \( U \) of \( \mathcal{U} \) is uniformized, say by \((\tilde{U}, \Gamma, \varphi)\).
2. If \( U, U' \in \mathcal{U} \) and \( U' \subset U \), then there is an injection \((\tilde{U}', \Gamma', \varphi') \to (\tilde{U}, \Gamma, \varphi)\).
3. For any point \( p \in U_1 \cap U_2, U_1, U_2 \in \mathcal{U} \), there is a connected open set \( U_3 \in \mathcal{U} \) with \( p \in U_3 \subset U_1 \cap U_2 \) (and hence there are injections \((\tilde{U}_3, \Gamma_3, \varphi_3) \to (\tilde{U}_1, \Gamma_1, \varphi_1) \) and \((\tilde{U}_3, \Gamma_3, \varphi_3) \to (\tilde{U}_2, \Gamma_2, \varphi_2)\)).

Suppose that \( \mathcal{V} \) and \( \mathcal{U} \) are two orbifold atlases on a space \( M \), that \( \mathcal{V} \) is a refinement of \( \mathcal{U} \) and that for every \( V \in \mathcal{V} \) and \( U \in \mathcal{U} \) with \( V \subset U \), we have an injection \((V, \Delta, \phi) \hookrightarrow (U, \Gamma, \varphi)\), where \((V, \Delta, \phi)\) is the respective uniformizing charts. Then we say that \( \mathcal{V} \) and \( \mathcal{U} \) are directly equivalent orbifold atlases. Now take the smallest equivalence relation on the orbifold atlases on \( M \) so that any two directly equivalent atlases on \( M \) are equivalent. We now define an \textit{orbifold} to be a Hausdorff topological space together with an equivalence class of orbifold atlases.

Let \( x \) be a point in an orbifold \( M \), and let \((\tilde{U}, \Gamma, \varphi)\) be a uniformizing chart with \( x \in \varphi(\tilde{U}) \). The \textit{(orbifold) structure group} of \( x \) is the isotropy group of a point in the fiber \( \varphi^{-1}(x) \). It is well-defined as an abstract group: if \( x_1, x_2 \in \varphi^{-1}(x) \), then the corresponding isotropy groups are conjugate in \( \Gamma \).

**Remark 7.3.** It is not hard to show that if \((\tilde{U}, \Gamma, \varphi)\) is a uniformizing chart of \( U \), then for any point \( x \in U \) there is a neighborhood \( U' \) and a uniformizing chart \((\tilde{U}', \Gamma', \varphi')\) induced from \((\tilde{U}, \Gamma, \varphi)\) such that \((\varphi')^{-1}(x)\) is a single point \( \tilde{x} \) (and hence \( \Gamma' \) fixes \( \tilde{x} \)). We will refer to \((\tilde{U}', \Gamma', \varphi')\) as a chart centered at \( x \).

Let \( M \) be an orbifold with an atlas \( \{U_i, \Gamma_i, \varphi_i\} \). A \textit{smooth function} \( f \) on \( M \) is a collection of smooth \( \Gamma_i \)-invariant functions \( \tilde{f}_i \) on \( \tilde{U}_i \) such that for any injection \((\psi_{ij}, \lambda_{ij}) : (\tilde{U}_j, \Gamma_j, \varphi_j) \hookrightarrow (\tilde{U}_i, \Gamma_i, \varphi_i)\), we have \( \psi_{ij}^* \tilde{f}_i = \tilde{f}_j \). Naturally each \( \tilde{f}_i \) defines a continuous map \( f_i : \tilde{U}_i / \Gamma_i = U_i \to \mathbb{R} \). Thanks to the compatibility conditions above, these maps glue together to define a continuous map from the topological space underlying \( M \) to \( \mathbb{R} \). By abuse of notation we may write \( f : M \to \mathbb{R} \). Similarly, a \textit{differential k-form} \( \sigma \) on the orbifold \( M \) is a collection of \( \Gamma_i \)-invariant \( k \)-forms \( \tilde{\sigma}_i \) on \( \tilde{U}_i \) such that for any injection \((\psi_{ij}, \lambda_{ij}) : (\tilde{U}_j, \Gamma_j, \varphi_j) \hookrightarrow (\tilde{U}_i, \Gamma_i, \varphi_i)\), we have \( \psi_{ij}^* \tilde{\sigma}_i = \tilde{\sigma}_j \). We denote the collection of all differential forms on \( M \) by \( \Omega^*(M) \). Note that \( \Omega^*(M) \) has a well-defined exterior multiplication (since exterior multiplication behaves well under pull-backs) and exterior differentiation \( d : \Omega^r(M) \to \Omega^{r+1}(M) \) (for the same reason). In the same manner one defines vector fields, Riemannian metrics, quadratic forms and other differential geometric objects on orbifolds.
Finally we define maps of orbifolds. The reader should be aware that there are several notions of maps orbifolds. For example Satake in his two papers gave two inequivalent definitions. The one we give below is the simplest; it is not the best. It will, however, suffice for our purposes. See Ruan’s survey [R] for the modern point of view. Let \((M, \{\tilde{U}_i, \Gamma_i, \varphi_i\})\) and \((N, \{\tilde{V}_j, \Delta_j, \phi_j\})\) be two orbifolds and let \(F : M \to N\) be a continuous map of underlying topological spaces. The map \(F\) is a (smooth) map of orbifolds if for every point \(x \in M\) there are charts \((\tilde{U}_i, \Gamma_i, \varphi_i)\), \((\tilde{V}_j, \Delta_j, \phi_j)\) with \(x \in \varphi(\tilde{U}_i)\), \(F(\varphi(\tilde{U}_i)) \subset \phi_j(\tilde{V}_j)\) and a \(C^\infty\) map \(\tilde{F}_{ji} : \tilde{U}_i \to \tilde{V}_j\) such that
\[
\phi_j \circ \tilde{F}_{ji} = F \circ \varphi_i.
\]
The reason for the appearance of orbifolds in these notes is the symplectic reduction theorem of Marsden, Weinstein and Meyer [MW, Me]:

**Theorem 7.4.** Let \((M, \omega)\) be a symplectic manifold with a Hamiltonian action of a Lie group \(G\). Let \(\Phi : M \to \mathfrak{g}^*\) denote a corresponding moment map. Suppose \(\eta \in \mathfrak{g}^*\) is a regular value of \(\Phi\) and suppose that the action of the isotropy group \(G_\eta\) of \(\eta\) on \(\Phi^{-1}(\eta)\) is proper. Then the action of \(G_\eta\) on \(\Phi^{-1}(\eta)\) is locally free and the quotient \(M_\eta := \Phi^{-1}(\eta)/G_\eta\) is naturally a symplectic orbifold.

We will not discuss the proof of this well known theorem. Instead let me briefly explain why quotients by locally free actions are orbifolds. The reason is the slice theorem: if the action of a Lie group \(G\) is locally free and proper on a manifold \(Z\), then for any point \(z \in Z\) there is a slice \(S_z\) for the action of \(G\) and the isotropy group \(G_z\) is finite. The quotient \(S_z/G_z\) is homeomorphic to a neighborhood of the orbit \(G \cdot z\) in the orbit space \(Z/G\). The triple \((S_z, G_z, \varphi : S_z \to S_z/G_z \hookrightarrow Z/G)\) is a uniformizing chart of the orbifold \(Z/G\).

Let us now prove Theorem 7.2.

**Proof of Theorem 7.2.** Since \(B\) is compact, the image \(\Psi_\alpha(B)\) is compact. Therefore the set of vectors \(X' \in \mathfrak{g}\) such that the function \(\langle \Psi_\alpha, X' \rangle\) is strictly positive on \(B\), is open. Hence we may assume that \(X\) lies in the integral lattice \(\mathbb{Z}_G := \ker(\exp : \mathfrak{g} \to G)\) of the torus \(G\). Let \(H = \{\exp tx \mid t \in \mathbb{R}\}\) be the corresponding circle subgroup of \(G\).

Let \(f(x) = 1/(\langle \Psi_\alpha(x), X' \rangle)\) and let \(\alpha' = f\alpha\). The form \(\alpha'\) is another \(G\)-invariant contact form with \(\ker \alpha' = \zeta\). The moment map \(\Psi_{\alpha'}\) defined by \(\alpha'\) satisfies \(\Psi_{\alpha'} = f\Psi_\alpha\). Therefore \(\langle \Psi_{\alpha'}, X' \rangle = 1\) for all \(x \in B\).

Since the function \(\langle \Psi_\alpha, X \rangle\) is nowhere zero, the action of \(H\) on \(B\) is locally free. Consequently the induced action of \(H\) on the symplectization \((N, \omega) = (B \times \mathbb{R}, d(e^{t\alpha'})\) is locally free as well. Hence any \(a \in \mathbb{R}\) is a regular value of the \(X\)-component \(\Phi, X\) of the moment map \(\Phi\) for the action of \(G\) on the symplectization \((N, \omega)\). Note that \(\Phi(x, t) = -e^t\Psi_{\alpha'}(x)\). The manifold \(B \times \{0\}\) is the \(-1\) level set of \(\langle \Phi, X \rangle\). Therefore by the reduction theorem \(M := (\langle \Phi, X \rangle)^{-1}(-1)/H \simeq B/H\) is a (compact connected) symplectic orbifold. The action of \(G\) on \((\langle \Phi, X \rangle)^{-1}\) descends to an effective Hamiltonian action of \(G/H\) on \(M\). A dimension count shows that
7.1 Morse theory on orbifolds

Let us start by briefly reviewing the fundamental results of Morse theory on manifolds. Let $f: M \to \mathbb{R}$ be a smooth function. A critical point of $f$ is a point $p$ where the differential $df$ is 0. The image $f(p)$ of a critical point $p$ is a critical value of $f$. A critical point $p$ of $f$ is nondegenerate if the Hessian $d^2 f_p$ is a non-degenerate quadratic form (in local coordinates $d^2 f_p$ is the matrix of second order partials $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$; thus $d^2 f_p(x) = \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} x_i x_j$). The index of a nondegenerate critical point $p$ is the number of negative eigenvalues of the Hessian $d^2 f_p$ (counted with multiplicities.) The two and a half fundamental results of Morse theory are the two theorems and the lemma below.

**Theorem 7.5.** Let $f$ be a smooth function on a manifold $M$, and $M_a$ the set $f^{-1}((-\infty, a])$. If $f^{-1}([a, b])$ is compact and contains no critical points then $M_a$ is homotopy equivalent to $M_b$.

*Sketch of proof.* Fix a Riemannian metric on $M$. Since $f$ has no critical points in $f^{-1}([a, b])$, the unit vector field $X = -\nabla f/||\nabla f||$ is well-defined. Extend $X$ to all of $M$. The flow of $X$ gives a retraction of $M_b$ onto $M_a$.

**Theorem 7.6.** Let $f$ be a smooth function on a manifold $M$. Suppose $f^{-1}([a, b])$ is compact and contains exactly one nondegenerate critical point $p$ in its interior. Then $M_b$ has the homotopy type of $M_a$ with a $\lambda$-dimensional disk $D^\lambda$ attached along the boundary $\partial D^\lambda = S^{\lambda-1}$ where $\lambda$ is the index of $p$.

We omit the proof of the theorem which is well known noting only that the key ingredient of the proof is

**Lemma 7.7 (Morse Lemma).** Let $p$ be a nondegenerate critical point of index $\lambda$ of a function $f: M \to \mathbb{R}$. There is a neighborhood $U$ of $p$ in $M$ and an open embedding $\varphi: U \to M$ such that $f \circ \varphi(x) = f(p) + d^2 f_p(x)$ for all $x \in U$. There is, there is a change of coordinates near $p$ so that in new coordinates $f$ is a quadratic form (up to a constant).

Moreover, if a compact Lie group $G$ acts on $M$ fixing $p$ and preserving $f$, we may arrange for $U$ to be $G$-invariant and for $\varphi$ to be $G$-equivariant. Note that in this case the Hessian $d^2 f_p(x)$ is a $G$-invariant quadratic form.

Suppose now that $f$ is a smooth function on an orbifold $M$. Then the 1-form $df$ still makes sense. We define $p$ to be a critical point of $f$ if $df_p = 0$. The effective action of $G/H$ on $M$ is completely integrable, i.e., $M$ is a symplectic toric orbifold. 

The rest of the section is a proof that real odd dimensional cohomology vanishes for symplectic toric orbifolds. It uses Morse theory.

A critical point \( p \) is **nondegenerate** if for any uniformizing chart \((U, \Gamma, \varphi)\) with \( p \in U = \varphi(U) \), a point \( \tilde{p} \in \varphi^{-1}(p) \) is a nondegenerate critical point of \( \tilde{f} = f \circ \varphi \). The **index** of \( p \) is the index of the \( \Gamma \)-invariant quadratic form \( d^2 f_{\tilde{p}} \).

It is not hard to believe that Theorem 7.5 holds for orbifolds with no changes and that essentially the same proof still works. Theorem 7.6 requires a small modification, see [LT].

**Theorem 7.8.** Let \( f \) be a smooth function on an orbifold \( M \). Suppose \( f^{-1}([a,b]) \) is compact and contains exactly one nondegenerate critical point \( p \) in its interior. Then \( M_p \) has the homotopy type of \( M_a \) with the quotient \( D^\lambda/\Gamma \) attached along the boundary \((\partial D^\lambda)/\Gamma = S^{\lambda - 1}/\Gamma\) where \( \lambda \) is the index of \( p \) and \( \Gamma \) is structure group of \( p \). Here again \( D^\lambda \) and \( S^{\lambda - 1} \) denote the disk of dimension \( \lambda \) and the sphere of dimension \( \lambda - 1 \) respectively.

**Corollary 7.9.** Let \( f : M \to \mathbb{R} \), \( p \), \( \lambda \) and \( \Gamma \) be as above. Then \( H^*(M_b, M_a; \mathbb{R}) = H^*(D^\lambda/\Gamma, S^{\lambda - 1}/\Gamma; \mathbb{R}) \). Hence, if \( \Gamma \) is orientation preserving, \( H^q(M_b, M_a; \mathbb{R}) = \mathbb{R} \) for \( q = \lambda \) and 0 otherwise.

**Proof.** By excision \( H^*(M_b, M_a; \mathbb{R}) = H^*(D^\lambda/\Gamma, S^{\lambda - 1}/\Gamma; \mathbb{R}) \). If \( \Gamma \) is orientation preserving, then \( H^q(D^\lambda/\Gamma, S^{\lambda - 1}/\Gamma; \mathbb{R}) = \mathbb{R} \) for \( q = \lambda \) and 0 otherwise. Here \( \mathbb{R} \) denotes the reduced cohomology.

As a consequence of the corollary above we get

**Corollary 7.10.** Let \( f \) be a smooth function on a compact orbifold \( M \). Suppose all indices of \( f \) are even. Then \( H^q(M, \mathbb{R}) = 0 \) for all odd indices \( q \).

**Proof.** The proof is inductive. Suppose that \( c \) is a critical value of \( f \) and suppose we know that for all \( a < c \) we have \( H^q(M_a, \mathbb{R}) = 0 \) for \( q \) odd. Assume for simplicity that \( f^{-1}(c) \) contains only one critical point (this keeps the notation more manageable) and that its index is \( 2k \) for some integer \( k \). Then using the long exact sequence for the pair \((M_{c+\epsilon}, M_{c-\epsilon})\) (for some sufficiently small \( \epsilon > 0 \)) and Corollary 7.9 we see that \( H^q(M_{c+\epsilon}, \mathbb{R}) = H^q(M_{c-\epsilon}, \mathbb{R}) \) for \( q \neq 2k - 1, 2k \), that \( H^{2k-1}(M_{c+\epsilon}, \mathbb{R}) \) embeds in \( H^{2k-1}(M_{c-\epsilon}, \mathbb{R}) = 0 \) and that \( H^{2k}(M_{c+\epsilon}, \mathbb{R}) = H^{2k}(M_{c-\epsilon}, \mathbb{R}) \oplus \mathbb{R} \). The result follows.

Now suppose \((M, \omega, \Phi : M \to g^*)\) is a compact symplectic toric orbifold. Just as for symplectic toric manifolds the image \( \Phi(M) \) is a simple polytope and the moment map sends the fixed points \( M^G \) in one-to-one fashion to vertices of \( \Phi(M) \) (see [LT]). Therefore, for a generic vector \( X \in g \) the function \( f = \langle \Phi, X \rangle \) takes distinct values at fixed points. We will now argue that the critical points of \( f \) are exactly the fixed points. We will then argue that \( f \) is Morse and that all indices of \( f \) are even.

For an action of a torus \( G \) on a compact orbifold \( M \) only finitely many isotropy groups can occur. This is a consequence of compactness and existence
of slices. Therefore the set of subalgebras \( g_x \), which are Lie algebras of isotropy groups for the action of \( G \) on \( M \), is finite. Hence the set

\[
\mathcal{U} = g \setminus \bigcup_{x \in M} \{ g_x \mid g_x \neq g \}
\]

is open and dense. Now take \( X \) to be a vector in \( \mathcal{U} \). Then

\[
0 = df_x = d(\Phi, X)_x = (\iota(X_M)\omega)_x \Leftrightarrow X_M(x) = 0
\]

\[
\Leftrightarrow \exp tX \cdot x = x \text{ for all } t
\]

\[
\Leftrightarrow \exp tX \in G_x \text{ for all } t
\]

\[
\Leftrightarrow X \in g_x.
\]

Since \( X \in \mathcal{U} \) we see that \( df_x = 0 \Leftrightarrow g_x = g \Leftrightarrow x \) is fixed by \( G \). It remains to check that \( f \) is Morse and that all the indices of \( f \) are even.

Let \( x \in M^G \) be a fixed point and let \((\hat{U}, \Gamma, \varphi)\) be a uniformizing chart centered at \( x \) (cf. Remark 7.3). We may assume that \( U = \varphi(\hat{U}) \) is \( G \)-invariant. Denote the symplectic form on \( \hat{U} \) by \( \hat{\omega} \). The map \( \hat{\Phi} = \Phi \circ \varphi : \hat{U} \to g^* \) is a moment map for an action of a torus \( \hat{G} \) on \( \hat{U} \). One can show arguing as in the proof of Lemma 4.6 that \( \Gamma \subset \hat{G} \) and that \( G = \hat{G}\Gamma \). In particular \( \hat{G} \) has \( g \) as its Lie algebra. We now apply the equivariant Darboux theorem to \( \hat{U}, \hat{\omega} \) and \( \hat{G} \). We get a \( \hat{G} \)-invariant neighborhood \( \hat{V} \) of \( \hat{x} = \varphi^{-1}(x) \), an open neighborhood \( \hat{V}_0 \) of 0 in \( T_{\hat{x}}\hat{U} \) and a \( \hat{G} \)-equivariant diffeomorphism \( \tau : \hat{V}_0 \to \hat{V} \) such that \( \tau^*\hat{\omega} = \hat{\omega}_0 \) where \( \hat{\omega}_0 \) is the constant coefficient form \( \hat{\omega}_0 \) on the vector space \( T_{\hat{x}}\hat{U} \).

The action of \( \hat{G} \) on \( T_{\hat{x}}\hat{U} \) is linear. Hence the corresponding moment map \( \hat{\Phi}_0 : T_{\hat{x}}\hat{U} \to g^* \) is quadratic. In fact by choosing a \( \hat{G} \)-invariant complex structure on \( T_{\hat{x}}\hat{U} \) compatible with the symplectic form we can identify \( (T_{\hat{x}}\hat{U}, \hat{\omega}_0) \) with \( (\mathbb{C}^n, \sqrt{-1} \sum dz_j \wedge d\bar{z}_j) \) so that the action of \( \hat{G} \) is given by

\[
a \cdot (z_1, \ldots, z_n) = (\chi_1(a)z_1, \ldots, \chi_n(a)z_n)
\]

for some characters \( \chi_j : \hat{G} \to S^1 \). Then

\[
\hat{\Phi}_0(z_1, \ldots, z_n) = \sum |z_j|^2 \nu_j
\]

where \( \nu_j \) are the corresponding weights. Since \( \tau \) is a \( \hat{G} \)-equivariant symplectomorphism, \( \hat{\Phi} \circ \tau = \hat{\Phi}_0 + c \) for some constant \( c \in g^* \). Hence

\[
f \circ \varphi \circ \tau = \langle \Phi, X \rangle \circ \varphi \circ \tau = \langle \hat{\Phi}_0, X \rangle \circ \tau = \langle \hat{\Phi}_0, X \rangle + c, \text{ i.e.,}
\]

\[
f \circ \varphi \circ \tau(z_1, \ldots, z_n) = \sum \nu_j(X)|z_j|^2 + c.
\]

By the choice of \( X \), \( \nu_j(X) \neq 0 \) for any \( j \). Thus \( f \circ \varphi \) is Morse and its index at \( \hat{x} \) is twice the number of weights \( \nu_j \) with \( \nu_j(X) < 0 \). Therefore, by definition, \( f \) is a Morse function on \( M \) and all indices of \( f \) are even.
Appendix A

Hypersurfaces of contact type

Contact manifolds often arise as codimension 1 submanifolds of symplectic manifolds, i.e., as hypersurfaces.

**Definition A.1.** Let \((M, \omega)\) be a symplectic manifold. A hypersurface \(\Sigma\) of \(M\) is of contact type if there is a neighborhood \(U\) of \(\Sigma\) in \(M\) and a vector field \(X\) on \(U\) such that

1. \(T_m M = T_m \Sigma \oplus \mathbb{R}X(m)\) for any point \(m \in \Sigma\), i.e., the vector field \(X\) is nowhere tangent to \(\Sigma\);

2. the flow of \(X\) expands the symplectic form \(\omega\) exponentially, i.e., \(L_X \omega = \omega\), where as usual \(L_X\) denotes the Lie derivative with respect to \(X\).

The vector field \(X\) with above properties is often called a **Liouville vector field**.

We now prove that hypersurfaces of contact type are indeed contact manifolds.

**Proposition A.2.** Let \(\Sigma\) be a hypersurface of contact type in a symplectic manifold \((M, \omega)\) and let \(X\) be a Liouville vector field defined on a neighborhood \(U\) of \(\Sigma\). The 1-form \(\alpha = (\iota(X)\omega)|_{\Sigma}\) is contact.

**Proof.** Note first that \(d(\iota(X)\omega) = d(\iota(X)\omega) + \iota(X)d\omega = L_X \omega = \omega\). Hence \(d\alpha = d(\iota(X)\omega)|_{\Sigma} = \omega|_{\Sigma}\). Since \(\omega\) is symplectic and \(\Sigma\) is of codimension 1 in \(M\), the form \(\omega_m|_{T_m\Sigma}\) has a 1-dimensional kernel (for any point \(m \in \Sigma\)). Thus there is a vector \(Y_m \in T_m\Sigma\) such that \(\omega_m(Y_m, v) = 0\) for any \(v \in T_m\Sigma\). Since \(\omega\) is symplectic and since by assumption \(T_m M = T_m \Sigma \oplus \mathbb{R}X(m)\) for all \(m \in \Sigma\) we have

\[
0 \neq \omega_m(Y_m, X(m)) = -\iota(X)\omega_m(Y_m) = -\alpha_m(Y_m).
\]
Hence $\alpha_m \neq 0$ for all $m \in \Sigma$ and consequently $\xi = \ker \alpha$ is a codimension 1 distribution on $\Sigma$. It remains to show that for any $m \in \Sigma$

$$d\alpha_m|_{\xi_m} = \omega_m|_{\xi_m}$$

is non-degenerate. On the other hand $d\alpha_m = \omega_m|_{T_m\Sigma}$ and $\xi_m = \{v \in T_m\Sigma \mid \omega_m(v, X(m)) = 0\}$. Since $\omega_m(v, Y_m) = 0$ for any $v \in T_m\Sigma$, the subspace $\xi_m$ of $T_mM$ lies in the symplectic perpendicular to the 2-plane $\text{span}_\mathbb{R}\{Y_m, X(m)\}$ in $(T_mM, \omega_m)$:

$$\xi_m \subset (\text{span}_\mathbb{R}\{Y_m, X(m)\})^\omega.$$  

Since $\dim \xi_m = \dim M - 2 = \dim(\text{span}_\mathbb{R}\{Y_m, X(m)\})^\omega$, we have equality: $\xi_m = (\text{span}_\mathbb{R}\{Y_m, X(m)\})^\omega$. Since $(\text{span}_\mathbb{R}\{Y_m, X(m)\})$ is a symplectic subspace, its symplectic perpendicular $\xi_m$ is symplectic as well.  

**Example A.3.** Let $(Q, g)$ be a Riemannian manifold. Let $g^*$ denote the dual metric on $T^*Q$. The co-sphere bundle $S^*Q$ is the set of covectors in $T^*Q$ of length 1: $S^*Q = \{(q, p) \in T^*Q \mid q \in Q, p \in T_q^*Q, g^*_q(p, p) = 1\}$. It is a hypersurface of contact type in $T^*Q$ relative to the standard symplectic structure. The Liouville vector field is the generator of dilations. In local coordinates $X(q, p) = \sum p_j \frac{\partial}{\partial q_j}$.

**Exercise A.4.** A codimension 1 hypersurface $\Sigma \subset \mathbb{C}^n$ is **star-shaped about the origin** if for any nonzero vector $v \in \mathbb{C}^n$ the ray $\{tv \mid t \in (0, \infty)\}$ intersects $\Sigma$ transversely in exactly one point. In particular $\Sigma$ is the image of an embedding $\iota: S^{2n-1} \rightarrow \mathbb{C}^n$ of the $(2n-1)$-dimensional sphere.

Show that any star-shaped hypersurface is a hypersurface of contact type in $(\mathbb{C}^n, \omega = \sqrt{-1} \sum dz_j \wedge d\bar{z}_j)$. Show that any two star-shaped hypersurfaces are isomorphic as contact manifolds. The contact structure in question is called the **standard contact structure on $S^{2n-1}$**.

Prove a converse: given a contact form $\alpha$ on $S^{2n-1}$ defining the standard contact structure there is an embedding $\iota : S^{2n-1} \rightarrow \mathbb{C}^n$ such that $\iota^*(\iota(X)\omega) = \alpha$ where $X$ is the radial vector field on $\mathbb{C}^n$: $X(z) = \frac{1}{2} \sum (z_j \frac{\partial}{\partial \bar{z}_j} + \bar{z}_j \frac{\partial}{\partial z_j})$. 


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