On classification of projective homogeneous varieties up to motivic isomorphism

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Abstract

We give a complete classification of anisotropic projective homogeneous varieties of dimension less than 6 up to motivic isomorphism. We give several criteria for anisotropic flag varieties of type $A_n$ to have isomorphic motives.

1 Introduction

The present paper can be viewed as an application of the methods and results obtained by the authors in [CPSZ].

Let $k$ be a field of characteristic not 2 and $k_s$ denotes its separable closure. For a variety $X$ over $k$ we denote by $X_s$ the base change $X \times_k k_s$. By $\mathcal{M}(X)$ we denote the Chow motive of $X$. Recall (see [MPW, § 1]) that $X$ is a twisted flag variety of inner type over $k$ if $X = \xi(G/P)$ is a twisted form of the projective homogeneous variety $G/P$, where $G$ is an adjoint simple split algebraic group over $k$, $P$ its parabolic subgroup and the twisting is given by a 1-cocycle $\xi \in Z^1(k, G(k_s))$.

The present paper is devoted to the following

**Problem.** Describe all pairs $(X, Y)$ of non-isomorphic twisted flag varieties $X$ and $Y$ of inner type over $k$ which have isomorphic Chow motives.

This problem can be subdivided into two subproblems:

(i) Describe all such pairs $(X, Y)$ with $X_s \simeq Y_s$;
(ii) Describe all such pairs \((X, Y)\) with \(X_s \not\approx Y_s\).

Let us briefly remind what is known so far. The complete solution of the problem (i) is known for quadrics and Severi-Brauer varieties due to Izhboldin, Karpenko, Merkurjev, Rost, Vishik and others (see [Izh98], [Ka96], [Ka00], [Ro98], [Vi03]). Concerning (ii), the example (of dimension 5) was provided by Bonnet in [Bo03]. It deals with twisted flag varieties of type \(G_2\). For exceptional varieties of type \(F_4\) a similar example was provided in [NSZ].

In the present paper we provide a complete solution of the mentioned above problem for projective homogeneous varieties of dimension less than 6. Namely, we prove the following (using the notation of 2.1)

1.1 Theorem. Let \(X\) and \(Y\) be non-isomorphic twisted flag varieties of dimension \(\leq 5\) of inner type over \(k\) which have isomorphic Chow motives.

(i) If \(X_s \simeq Y_s\), then either
\[
X = \text{SB}(A) \quad \text{and} \quad Y = \text{SB}(A^{\text{op}})
\]
are Severi-Brauer varieties corresponding to a central simple algebra \(A\) and its opposite \(A^{\text{op}}\), where \(\deg(A) = 3, 4, 5, 6\) and \(\exp(A) > 2\)

or
\[
X = \text{SB}_{1,2}(A) \quad \text{and} \quad Y = \text{SB}_{1,2}(A^{\text{op}})
\]
are twisted forms of the flag varieties corresponding to central simple algebras \(A\) such that \(\deg(A) = \exp(A) = 4\).

(ii) If \(X_s \not\approx Y_s\), then either
\[
X = \text{SB}_{1,3}(A) \quad \text{and} \quad Y = \text{SB}_{1,2}(A')
\]
are twisted forms of the flag varieties corresponding to central simple algebras \(A\) and \(A'\) such that \(\deg(A) = \deg(A') = 4\) and \(A \simeq A'\) or \(A^{\text{op}}\),

or
\[
X = \xi(G_2/P_1) \quad \text{and} \quad Y = \xi(G_2/P_2)
\]
are twisted forms of the variety \(G/P_i\), \(i = 1, 2\), where \(G\) is a split exceptional group of type \(G_2\) and \(P_i\) is one of its maximal parabolic subgroups,

or
\[
X = \mathbb{P}^n \quad \text{and} \quad Y = Q^n
\]
is the projective space and the split quadric respectively, where \(n = 3, 5\).

or
\[
X = \mathbb{P}^5 \quad \text{and} \quad Y = G_2/P_2
\]
is the projective space and the split Fano variety of type \(G_2\).
1.2 Remark. Observe that the case $X = \xi(G_2/P_1)$ and $Y = \xi(G_2/P_2)$ of the theorem is the example of Bonnet mentioned above and, hence, is the minimal one in the sense of dimension.

1.3 Remark. The case $X = SB_{1,3}(A)$ and $Y = SB_{2,3}(A')$ with $A \simeq A', A^\text{top}$ provides another minimal example of two non-isomorphic varieties that have isomorphic Chow motives.

Apart from Theorem 1.1, we prove the following

1.4 Theorem. Let $X = SB_{n_1,\ldots,n_r}(A)$ and $Y = SB_{m_1,\ldots,m_r}(A')$ be twisted flag varieties of inner type $A_n$, $n \geq 2$, over $k$, where the central simple algebras $A$ and $A'$ have exponents 1, 2, 3, 4, or 6. Assume that

(i) $\mathcal{M}(X_s) \simeq \mathcal{M}(Y_s)$;

(ii) $n_1 = 1$ or $n_r = n$

(iii) $m_1 = 1$ or $m_r = n$.

Then $\mathcal{M}(X) \simeq \mathcal{M}(Y) \iff A \simeq A'$ or $A^\text{top}$.

As an immediate consequence we obtain

1.5 Corollary. Let $X = SB_{1,n}(A)$ and $Y = SB_{n-1,n}(A')$, where $A$ and $A'$ are central simple algebras of degree $n + 1$, $n \geq 3$, and exponent 1, 2, 3, 4 or 6. Then

$\mathcal{M}(X) \simeq \mathcal{M}(Y) \iff A \simeq A'$ or $A^\text{top}$.

1.6 Remark. The varieties $X$ and $Y$ of 1.5 provide examples of twisted flag varieties that satisfy (b). In fact, $X_s \not\simeq Y_s$, since they have different automorphism groups by \cite{De77}.

The paper is organized as follows. In section 2 we consider the case of a split group and state several facts which will be extensively used in the proofs. Section 3 is devoted to the case by case proof of Theorem 1.1. In the section 4 we prove Theorem 1.4 and provide several results that we need for the proof of 1.3.
2 Preliminaries

In the paper we use the following notation.

2.1. Let $G$ be a split simple algebraic group defined over a field $k$. We fix a maximal split torus $T$ of $G$ and a Borel subgroup $B$ of $G$ containing $T$ and defined over $k$. Denote by $\Phi$ the root system of $G$, by $\Pi = \{\alpha_1, \ldots, \alpha_{rkG}\}$ the set of simple roots of $\Phi$ corresponding to $B$, by $W$ the Weyl group, and by $S = \{s_1, \ldots, s_{rkG}\}$ the corresponding set of fundamental reflections. Let $P_\Theta$ be the standard parabolic subgroup corresponding to a subset $\Theta \subset \Pi$, i.e., $P_\Theta = BW_\Theta B$, where $W_\Theta = \{s_\theta, \theta \in \Theta\}$. Denote by $P_i$ the maximal parabolic subgroup $P_{\Pi \setminus \{\alpha_i\}}$. By $\Phi/P_\Theta$ we denote the flag variety $G/P_\Theta$. The root enumeration follows Bourbaki.

The notation $SB_{n_1, \ldots, n_r}(A)$, $1 \leq n_1 < \ldots < n_r \leq n$, is used for the twisted form of the variety $A_n/P_\Theta$, where $\Theta = \Pi \setminus \{\alpha_{n_1}, \ldots, \alpha_{n_r}\}$ and $A$ is a central simple algebra of degree $n + 1$ corresponding to the twisting. Observe that $SB_{n_1, \ldots, n_r}(A) = X(A; n_1, \ldots, n_r)$ in the notation of [MPW Appendix] and $SB(A) = SB_1(A)$ is the usual Severi-Brauer variety defined by $A$. By $\text{ind}(A)$ we denote the index of $A$ and by $\exp(A)$ its exponent. A split projective quadric of dimension $n$ is denoted by $Q^n$.

2.2. According to [Ko91] the Chow motive of the flag variety $X = G/P_\Theta$, when $G$ is a split group, is isomorphic to

$$
\mathcal{M}(X) \simeq \bigoplus_{i=0}^{\dim X} \mathbb{Z}(i)^{\oplus a_i(X)},
$$

where $\mathbb{Z}(i)$ are the twists of the Lefschetz motive and the positive integers $a_i(X)$ are the coefficients of the generating polynomial $p_X(z) = \sum_{i=0}^{\dim X} a_i(X)z^i$. The latter is defined by the following explicit formula:

$$
p_X(z) = \left( \prod_{i=1}^{rkG} \frac{zd_i(W_i) - 1}{z - 1} \right) / \left( \prod_{j=1}^{m} \prod_{i=1}^{d_i(W_j)} \frac{zd_i(W_j) - 1}{z - 1} \right).
$$

Here $W_1 \times \ldots \times W_m$ is the decomposition of $W_\Theta$ into the product of Weyl groups corresponding to the irreducible root systems and $d_i(W_j)$ are the degrees of the respective fundamental polynomial invariants (see [Ca72 9.4 A]).

The following observation follows from the above isomorphism.
2.3. **The motives of flag varieties** $X$ and $Y$ of dimension $n$ over a separably closed field are isomorphic iff the corresponding sequences of ranks $(a_0(X), \ldots, a_n(X))$ and $(a_0(Y), \ldots, a_n(Y))$ are equal.

We shall need the following two facts:

2.4. (See [Kato00, Criterion 7.1]) Let $A$, $A'$ be central simple algebras over $k$ and $\text{SB}(A)$, $\text{SB}(A')$ be the respective Severi-Brauer varieties. Then

\[ \mathcal{M}(\text{SB}(A)) \simeq \mathcal{M}(\text{SB}(A')) \iff A \simeq A', A'^{\text{op}}. \]

2.5. (see [Izh98, Cor. 2.9 and Prop. 3.1]) Let $q$, $q'$ be regular quadratic forms of rank $n$ and $X_q$, $X_{q'}$ be the respective projective quadrics. If $n$ is odd or $n < 7$, then

\[ \mathcal{M}(X_q) \simeq \mathcal{M}(X_{q'}) \iff X_q \simeq X_{q'}. \]

Finally, we shall need the following observation:

2.6. (See [Kato00, Proof of Lemma 2.3]) Let $X$ and $Y$ be smooth projective varieties over $k$ with isomorphic Chow motives. Then there is an isomorphism of abelian groups

\[ \text{Coker}(\text{CH}_0(X) \xrightarrow{\text{res}} \text{CH}_0(X_s)) \simeq \text{Coker}(\text{CH}_0(Y) \xrightarrow{\text{res}} \text{CH}_0(Y_s)). \]

3 **Small dimensions**

In this section we classify all pairs $(X, Y)$ of non-isomorphic twisted flag varieties of inner type over $k$ of dimension $\leq 5$ which have isomorphic Chow motives and hence prove Theorem 1.1.

**Dimension 1.** Twisted flag varieties of dimension 1 are the twisted forms of the projective line $\mathbb{P}^1$. The twisted forms of $\mathbb{P}^1$ are Severi-Brauer varieties $\text{SB}(H)$, where $H$ is a quaternion algebra. By 2.4

\[ \mathcal{M}(\text{SB}(H)) \simeq \mathcal{M}(\text{SB}(H')) \iff H \simeq H', H'^{\text{op}}. \]

Since $H \simeq H'^{\text{op}}$, we conclude that the motives are isomorphic iff the varieties are isomorphic.
Dimension 2. All twisted flags of dimension 2 are the twisted forms of the projective space $\mathbb{P}^2$ or the split quadric surface $Q^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Observe that $Q^2$ is a projective homogeneous variety for a group of type $D_2$ which is not simple, but semisimple. Nevertheless, we shall consider this case too.

The motives of $\mathbb{P}^2$ and $Q^2$ are not isomorphic, since the respective sequences of ranks $(1, 1, 1)$ and $(1, 2, 1)$ are different.

The twisted forms of $Q^2$ of inner type over $k$ are 2-dimensional quadrics (see [Inv, Cor. (15.12)]). By 2.5 the motives of two quadrics of dimension 2 are isomorphic iff the quadrics are isomorphic.

The twisted forms of $\mathbb{P}^2$ are Severi-Brauer varieties $SB(A)$, where $A$ is a central simple algebra of degree 3. Again by 2.4 we have

$$\mathcal{M}(SB(A)) \simeq \mathcal{M}(SB(A')) \iff A \simeq A', A^{\text{op}}.$$

Since the varieties $SB(A)$ and $SB(A^{\text{op}})$ are isomorphic iff $A$ is split, we conclude that all pairs of non-isomorphic varieties which have isomorphic motives are of the kind $(SB(A), SB(A^{\text{op}}))$, where $A$ is a division algebra of degree 3.

Dimension 3. Computing generating functions (see 2.2) we conclude that there are only three projective homogeneous varieties of dimension 3 over $k_s$. Namely, the projective space $\mathbb{P}^3$, the quadric $Q^3$ and the variety of complete flags $A_2/B$ ($B$ denotes a Borel subgroup). The respective sequences of ranks look as follows:

$$\mathbb{P}^3 \simeq A_3/P_1 : (1, 1, 1, 1)$$
$$Q^3 \simeq B_2/P_1 : (1, 1, 1, 1)$$
$$A_2/B : (1, 2, 2, 1)$$

In particular, we see that the motives of $\mathbb{P}^3$ and $Q^3$ are isomorphic but the motives of $Q^3$ and $A_2/B$ are not.

By 2.4 all non-isomorphic twisted forms of $\mathbb{P}^3$ which have isomorphic motives form pairs $(SB(A), SB(A^{\text{op}}))$, where $A$ is a division algebra of degree 4 and exponent 4. Observe that all non-isomorphic twisted forms of $Q^3$ are quadrics as well and by 2.5 the motive of a quadric determines this quadric uniquely. Therefore it remains to describe all possible motivic isomorphisms between the twisted forms $\zeta(\mathbb{P}^3)$ and $\zeta(Q^3)$ and the twisted forms $\zeta(A_2/B)$ and $\zeta(A_2/B)$ of the variety of complete flags $A_2/B$.

According to Corollary 4.4 there are no non-isomorphic twisted forms of $A_2/B$ which have isomorphic Chow motives. And the next lemma shows that there are no such (non-trivial) twisted forms of $\mathbb{P}^3$ and $Q^3$. 

6
3.1 Lemma. Let $\xi$, $\zeta$ be 1-cocycles. Then $M(\xi\mathbb{P}^3) \simeq M(\zeta Q^3)$ iff $\xi$ and $\zeta$ are trivial.

Proof. This is a particular case of a more general result (see Lemma 4.2) proven using Index Reduction Formula. Here we give an elementary proof. It uses only well-known facts about quadrics and Severi-Brauer varieties.

Observe that any twisted form of $\mathbb{P}^3$ is a Severi-Brauer variety $SB(A)$ for some central simple algebra $A$ of degree 4 and any twisted form of $Q^3$ is a non-singular quadric of dimension 3.

As in 2.6 for a variety $X$ consider the abelian group $\text{Coker}(\text{CH}_0(X) \to \text{CH}_0(X_s))$. If $X = SB(A)$ is a Severi-Brauer variety of a central simple algebra $A$, then this cokernel is equal to $\mathbb{Z}/\text{ind}(A)\mathbb{Z}$ (see [Kn00]), where $\text{ind}(A)$ is the index of $A$. In particular, this cokernel is trivial iff $A$ is split. If $X$ is a quadric then this cokernel is trivial iff $X$ is isotropic (see [Sw89]). In the case $X$ is an anisotropic quadric this cokernel is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

In our case we have two varieties $X = SB(A)$ and $Y = \zeta Q^3$ which have isomorphic motives. Hence, by 2.6 the respective cokernels must be isomorphic.

Hence, if the quadric $Y$ is isotropic, then the algebra $A$ is split. The latter implies that the motive $\mathcal{M}(SB(A))$ splits into the direct sum of Lefschetz motives and so is $\mathcal{M}(Y)$, i.e., $Y$ is split as well by 2.5.

Assume $q$ is anisotropic, then there exists a quadratic field extension $l/k$ such that the Witt index of $Y_l = Y \times_k l$ is one (see [Vd03 §7.2]). Since the motives of $X$ and $Y$ are still isomorphic over $l$, we conclude that $A$ is split over $l$. Then $Y_l$ is split as well. This leads to a contradiction. \(\square\)

3.2 Remark. Observe that the pair of twisted forms $(\xi(B_2/P_1), \xi(B_2/P_2))$ can be viewed as a low-dimensional analog of the pair $(\xi(G_2/P_1), \xi(G_2/P_2))$ considered by Bonnet. The lemma says that contrary to $G_2$-case the motives of $\xi(B_2/P_1)$ and $\xi(B_2/P_2)$ are not isomorphic (if $\xi$ is non-trivial).

Dimension 4. There are three non-isomorphic projective homogeneous varieties of dimension 4 over $k_s$. Namely, the projective space $\mathbb{P}^4$, the 4-dimensional quadric $Q^4 \simeq \text{Gr}(2,4)$ and the variety of complete flags $B_2/B$. The respective sequences of ranks in these cases are all different and look as follows:

\[
\begin{align*}
\mathbb{P}^4 & \simeq A_4/P_1 : (1,1,1,1,1) \\
Q^4 & \simeq A_3/P_2 : (1,1,2,1,1) \\
B_2/B & : (1,2,2,2,1)
\end{align*}
\]
Hence, the motives of $\mathbb{P}^4$, $Q^4$ and $B_2/B$ are non-isomorphic to each other.

By 2.4 all non-isomorphic twisted forms of $\mathbb{P}^4$ which have isomorphic motives form pairs $(SB(A), SB(A^\text{op}))$, where $A$ is a division algebra of degree 5. By Corollary 4.3 there are no non-isomorphic twisted forms of $B_2/B$ which have isomorphic Chow motives. Therefore the only case left is the case of inner twisted forms of $Q^4$.

The inner forms of $Q^4$ are the generalized Severi-Brauer varieties $SB_2(A)$, where $A$ is a central simple algebra of degree 4. The next lemma shows that there are no non-isomorphic forms of $SB_2(A)$ which have isomorphic motives.

3.3 Lemma. Let $A$, $A'$ be central simple algebras of degree 4. Then

$$\mathcal{M}(SB_2(A)) \simeq \mathcal{M}(SB_2(A')) \iff SB_2(A) \simeq SB_2(A')$$

Proof. Let $\mathcal{M}(SB_2(A)) \simeq \mathcal{M}(SB_2(A'))$. It suffices to prove that for all field extensions $l/k$ one has $\text{ind}(A_l) = \text{ind}(A'_l)$. Indeed, by [Ka00, Lemma 7.13] $\langle A \rangle = \langle A' \rangle$ in $\text{Br}(k)$, hence, $A \simeq A'$ or $A^\text{op}$. But $SB_2(A) \simeq SB_2(A^\text{op})$ for any central simple algebra $A$ of degree 4.

Assume that there exists a field extension $l/k$ such that $\text{ind}(A_l) \neq \text{ind}(A'_l)$. Depending on the indices of $A$ and $A'$ we distinguish the following cases:

**Case 1.** $\text{ind}(A) = 4$ and $\text{ind}(A') = 1$ or 2.

In this case $SB_2(A')$ has a rational point. By [Inv, Case A_3 = D_3], the variety $SB_2(A')$ is isotropic, hence, the group

$$\text{Coker}(\text{CH}_0(SB_2(A'))) \to \text{CH}_0(SB_2(A'_{\overline{k}}))$$

is trivial. By 2.6 the cokernel

$$\text{Coker}(\text{CH}_0(SB_2(A)) \to \text{CH}_0(SB_2(A_{\overline{k}})))$$

must be trivial as well. If $\exp(A) = 2$, then $A$ is a biquaternion algebra and by [Inv] Cor. (15.33)] $SB_2(A)$ is an anisotropic quadric. Then the cokernel above must be isomorphic to $\mathbb{Z}/2\mathbb{Z}$, a contradiction. If $\exp(A) = 4$, then by [Inv] Cor. (15.33)] $A \simeq C^\pm(B, \sigma, f)$, where $(B, \sigma, f) \in 1D_3$ and $B$ is a central simple algebra of degree 6 and index 2. By Merkurjev’s theorem (see [Me95]) the cokernel above must be again isomorphic to $\mathbb{Z}/2\mathbb{Z}$, a contradiction.
Case 2. \( \text{ind}(A) = 2 \) and \( \text{ind}(A') = 1 \).

In this case \( A' \) is split, hence, the corresponding variety is a split quadric.

On the other hand, \( \text{SB}_2(A) \simeq X_q \), where \( q \) is some 6-dimensional quadratic form and \( X_q \) is the corresponding projective quadric. Using 2.5, we conclude that \( \text{SB}_2(A) \simeq \text{SB}_2(A') \), a contradiction.

Dimension 5. There are five non-isomorphic projective homogeneous varieties over \( k_s \) of dimension 5. Namely, the projective space \( \mathbb{P}^5 \), the quadric \( Q^5 \), the exceptional Fano variety \( G_2/P_2 \), the flag varieties \( A_3/P_{\{\alpha_1\}} \) and \( A_3/P_{\{\alpha_2\}} \). The respective sequences of ranks look as follows:

\[
\begin{align*}
\mathbb{P}^5 &\simeq A_5/P_1 \quad : \quad (1,1,1,1,1,1) \\
Q^5 &\simeq B_3/P_1 \quad : \quad (1,1,1,1,1,1) \\
G_2/P_2 &\quad : \quad (1,1,1,1,1,1) \\
A_3/P_{\{\alpha_1\}} &\simeq A_3/P_{\{\alpha_3\}} \quad : \quad (1,2,3,2,1) \\
A_3/P_{\{\alpha_2\}} &\quad : \quad (1,2,3,2,1)
\end{align*}
\]

Therefore, the motives of \( \mathbb{P}^5 \), \( Q^5 \) and \( G_2/P_2 \) are isomorphic and the motives of \( A_3/P_{\{\alpha_1\}} \) and \( A_3/P_{\{\alpha_2\}} \) are isomorphic.

As was mentioned before, the twisted forms of \( \mathbb{P}^5 \) and \( Q^5 \) were completely classified up to motivic isomorphisms by Karpenko and Izhboldin (see 2.4 and 2.5). Moreover, by Lemma 4.2 there there is only one pair \( (\xi_{\mathbb{P}^5}, \zeta_{Q^5}) \) of twisted forms that have isomorphic motives.

By the result of Bonnet [Bo03] the motive of the twisted form \( \xi(G_2/P_2) \) is isomorphic to the motive of \( \xi(G_2/P_1) \) which is a 5-dimensional quadric.

By Corollary 1.5 the motives of the twisted forms of \( A_3/P_{\{\alpha_1\}} \) and \( A_3/P_{\{\alpha_2\}} \) are isomorphic if and only if the respective central simple algebras of degree 4 are isomorphic or opposite. This provides the last example (see 1.1) of a pair of non-isomorphic varieties of dimension 5 that have isomorphic motives.

4 Arbitrary dimensions

In the present section we prove several classification results. We start with the following

4.1 Lemma. Let \( X \) and \( Y \) be twisted flag varieties of inner type over \( k \) which have isomorphic Chow motives. Assume \( X \) is not of type \( E_8 \) and splits over its function field \( k(X) \), i.e., the group corresponding to \( X \) splits over \( k(X) \). Then \( X \) splits over the function field of \( Y \).
Proof. Since the motives are isomorphic, there is an isomorphism of cokernels (see 2.6) and, hence, an isomorphism of cokernels over $k(Y)$

$$\text{Coker}(\text{CH}_0(X_{k(Y)}) \to \text{CH}_0(X_{k(Y)})) \cong \text{Coker}(\text{CH}_0(Y_{k(Y)}) \to \text{CH}_0(Y_{k(Y)}))$$

Since $Y_{k(Y)}$ is isotropic, the right cokernel is trivial and so is the left one. The fact that the map $\text{res} : \text{CH}_0(X_{k(Y)}) \to \text{CH}_0(X_{k(Y)}))$ is surjective and the group $\text{CH}_0(X_{k(Y)})$ is a free abelian group of rank one generated by the class of a rational point $[pt]$ implies that the preimage $\text{res}^{-1}([pt])$ is a 0-cycle of degree 1 in $\text{CH}_0(X_{k(Y)})$. Then, the variety $X_{k(Y)}$ is isotropic (see [1604, Q. 0.2]).

By [KR94, 3.16.(iii)] the function field $k(X)$ is a generic splitting field for the respective parabolic subgroup $P$. Since $X_{k(Y)}$ is isotropic, the field $k(Y)$ is a $k$-specialization of $k(X)$ (see [KR94, Def. 1.2]). Since $X$ splits over $k(X)$, $k(X)$ is a splitting field for the respective group $G$. Then, by [KR94, 3.9.(iii)], $k(Y)$ is a splitting field of $G$ as well, i.e., $X_{k(Y)}$ splits.

4.2 Proposition. Let $\gamma, \delta$ be 1-cocycles and $X = \gamma \mathbb{P}^n$, $Y = \delta \mathbb{Q}^n$ be the respective twisted forms for $n > 1$ odd. Then

$$\mathcal{M}(X) \cong \mathcal{M}(Y) \iff \gamma \text{ and } \delta \text{ are trivial.}$$

Proof. Observe that $X$ is a Severi-Brauer variety corresponding to a central simple algebra $A$ and $Y$ is an $n$-dimensional quadric.

Assume that $\mathcal{M}(X) \cong \mathcal{M}(Y)$ and $\gamma$ is not trivial. By Lemma 4.1 applied to $X$ and $Y$, the algebra $A_{k(Y)}$ splits, i.e., $\text{ind}(A_{k(Y)}) = 1$. From the other hand by Index Reduction Formula (see [MPW]) we obtain

$$\text{ind}(A_{k(Y)}) = \min\{\text{ind}(A), 2^{(n-1)/2}\text{ind}(A \otimes_k C_0(q))\} > 1,$$

where $C_0(q)$ is the even part of the Clifford algebra of the quadric corresponding to $Y$. This leads to a contradiction.

Note that the same proof works for twisted forms of types $B_n$ and $C_n$. Namely,

4.3 Proposition. Let $\gamma, \delta$ be 1-cocycles and $X = \gamma (C_n/P_l)$, $Y = \delta (B_n/P_l)$ be the respective twisted forms for an odd $1 \leq l < n$. Then

$$\mathcal{M}(X) \cong \mathcal{M}(Y) \iff \gamma \text{ and } \delta \text{ are trivial.}$$
Proof. For any simple algebraic group $G$ as above consider a twisted flag variety $W = \xi(G/P_{\Theta})$ over $k$. On the Tits diagram (see [Ti66]) of $G$ over $k(W)$ all vertices corresponding to simple roots from $\Pi \setminus \Theta$ are circled.

In our case since $l$ is odd, this implies that $X_k(X)$ is split (see [Ti66] for a complete list of Tits diagrams). The rest of the proof repeats the proof of 4.2.

The rest of this section is devoted to the twisted forms of flag varieties. In particular, we obtain the description of motivic isomorphisms for twisted forms of the flag varieties $A_2/B$, $B_2/B$ and $A_3/P_{\{\alpha_i\}}$, $i = 1, 2, 3$. We start with the proof of Theorem 1.4.

Proof of Theorem 1.4. Assume $\mathcal{M}(X) \simeq \mathcal{M}(Y)$. Since $X$ and $Y$ are twisted forms of flags containing the subspace of dimension 1 (we may assume $n_1 = m_1 = 1$), the motives of $X$ and $Y$ can be decomposed into a direct sum of twisted motives of Severi-Brauer varieties (see [CPSZ, Thm. 2.1]). Namely,

$$\mathcal{M}(X) \simeq \bigoplus_i \mathcal{M}(\text{SB}(A))(i), \quad \mathcal{M}(Y) \simeq \bigoplus_j \mathcal{M}(\text{SB}(A'))(j).$$

(*)

This together with 2.6 implies the isomorphism of abelian groups

$$\text{Coker}(\text{CH}_0(\text{SB}(A)) \to \text{CH}_0(\mathbb{P}^n)) \simeq \text{Coker}(\text{CH}_0(\text{SB}(A') \to \text{CH}_0(\mathbb{P}^n))$$

and, hence, the isomorphism $\mathbb{Z}/\text{ind}(A)\mathbb{Z} \simeq \mathbb{Z}/\text{ind}(A')\mathbb{Z}$, i.e., $\text{ind}(A) = \text{ind}(A')$. Since the motivic isomorphism is preserved under the base extensions, we obtain that $\text{ind}(A_l) = \text{ind}(A'_l)$ for any finite field extension $l/k$. In fact, by [Ka00, Lemma 7.13] the latter is equivalent to $\langle A \rangle = \langle A' \rangle$ in $\text{Br}(k)$. In particular, if $\exp(A) = \exp(A')$ is 2, 3, 4, 6, we obtain $A \simeq A'$ or $A^{op}$.

In the opposite direction, let $A \simeq A'$ or $A^{op}$. By conditions (i) and (iii) one has two motivic decompositions (*) with the same sets of indices $\{i\}$ and $\{j\}$. Now according to 2.4 the motives of $\text{SB}(A)$ and $\text{SB}(A')$ are isomorphic and, hence, so are $\mathcal{M}(X)$ and $\mathcal{M}(Y)$.

The following obvious consequences of Theorem 1.4 are used in the proof of Theorem 1.1.

4.4 Corollary. Let $X = \text{SB}_{1,\ldots,n}(A)$ and $Y = \text{SB}_{1,\ldots,n}(A')$ be twisted forms of the variety of complete flags of type $A_n$. Assume the respective central simple algebras $A$ and $A'$ have exponents 1, 2, 3, 4 or 6. Then

$$\mathcal{M}(X) \simeq \mathcal{M}(Y) \iff X \simeq Y.$$
4.5 Corollary. Let $X$ and $Y$ be twisted forms of the variety of complete flags $B_2/B$. Then

$$M(X) \simeq M(Y) \iff X \simeq Y.$$ 

Proof. The proof repeats the proof of 1.4 observing that the motivic decompositions (*) is provided by [CPSZ, Cor. 2.9].

4.6. Consider the pseudo-abelian completion $M(G, R)$ of the category of motives of projective $G$-homogeneous varieties with $R$-coefficients, where $G$ is a group of inner type $A_n$ and $R$ is a ring of coefficients. Such categories were defined and extensively studied in [CM04]. In particular, it was proven that any object of $M(G, R)$, where $R$ is a discrete valuation ring, has a unique direct sum decomposition into indecomposable objects. Modulo this result the proof of 1.4 immediately implies the following

4.7 Corollary. Let $G$ be an adjoint semi-simple group of inner type $A_n$. Let $R$ be a ring such that any object of $M(G, R)$ has a unique direct sum decomposition into indecomposable objects. Let $X = SB_{n_1, \ldots, n_r}(A)$ and $Y = SB_{m_1, \ldots, m_r}(A')$ be two twisted flag varieties of type $A_n$ given by central simple algebras $A$ and $A'$ of prime degree. Assume that $M(X_s) \simeq M(Y_s)$. Then

$$M(X) \simeq M(Y) \text{ in } M(G, R) \iff \langle A \rangle = \langle A' \rangle \text{ in } Br(k).$$

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References

[Ar82] M. Artin. Brauer-Severi varieties. In A. Dold, B. Eckmann (eds), *Lecture Notes in Mathematics*, 917, Springer-Verlag, Berlin-Heidelberg-New York, 1982, 194–210.

[Bo03] J.-P. Bonnet. Un isomorphisme motivique entre deux variétés homogènes projectives sous l’action d’un groupe de type $G_2$. Doc. Math. 8, 2003, 247–277.
[Ca72] R.W. Carter. Simple groups of Lie type. Pure and Applied Math. Eds. R. Courant, L. Bers, J.J. Stoker, vol. XXVIII, John Wiley and Sons, 1972.

[CM04] V. Chernousov, A. Merkurjev. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. Preprint 2004.

[CPSZ] B. Calmès, V. Petrov, N. Semenov, K. Zainoulline. Chow motives of twisted flag varieties. Preprint 2005, 28pp. Available from \url{http://www.math.uni-bielefeld.de/LAG/man/186.html}

[De77] M. Demazure. Automorphismes et déformations des variétés de Borel. Inv. Math., 39 (1977), 179–186.

[Fu98] W. Fulton. Intersection theory. Second edition, Springer-Verlag, Berlin-Heidelberg, 1998.

[Izh98] O. Izhboldin. Motivic equivalence of quadratic forms. Doc. Math. 3 (1998), 341–351.

[Inv] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol. The book of involutions. AMS Colloquium Publications, vol. 44, 1998.

[Ka96] N. Karpenko. The Grothendieck Chow-motifs of Severi-Brauer varieties. St. Petersburg Math. J. 7 (1996), no. 4, 649-661.

[Ka00] N. Karpenko. Criteria of motivic equivalence for quadratic forms and central simple algebras. Math. Ann. 317 (2000), no. 3, 585-611.

[Ko91] B. Köck. Chow motif and higher Chow theory of \(G/P\). Manuscripta Math. 70, 1991, 363–372.

[Ma68] Y. Manin, Correspondences, motives and monoidal transformations. Matematicheskij Sbornik 77 (119) (1968), no. 4, 475–507 (in Russian). Engl. transl.: Math. USSR Sb. 6 (1968), 439–470.

[Me95] A.S. Merkurjev. Zero cycles on some involution varieties. Zap. Nauchn. Sem. POMI, 227, 1995, 93–105. Transl.: J. Math. Sci., 89, no. 2 (1998), 1141–1148.

[MPW] A.S. Merkurjev, I.A. Panin, A.R. Wadsworth. Index reduction formulas for twisted flag varieties. I. K-Theory, 10 (1996), no. 6, 517–596.
[NSZ] S. Nikolenko, N. Semenov, K. Zainoulline. Motivic decomposition of anisotropic varieties of type $F_4$ and generalized Rost motives. Preprint, 2005. Available from Arxiv Preprint Server, math.AG/0502382

[KR94] I. Kersten, U. Rehmann. General splitting of reductive groups. Tohoku Math. J. 46 (1994), 35–70.

[Ro98] M. Rost. The motive of a Pfister form. Preprint, 1998 Available from http://www.math.uni-bielefeld.de/~rost/

[Sw89] R.G. Swan. Zero cycles on quadric hypersurfaces. Proc. AMS, 107, no. 1, (1989), 43–46.

[Ti66] J. Tits. Classification of algebraic semisimple groups. In Algebraic Groups and Discontinuous Subgroups (Proc. Symp. Pure Math.), Amer. Math. Soc., Providence, R.I., 1966.

[To04] B. Totaro. Splitting fields for $E_8$-torsors. Duke Math. J. 121 (2004), no. 3, 425–455.

[Vi03] A. Vishik. Motives of quadrics with applications to the theory of quadratic forms. Lect. Notes Math., 1835, 2003, 25–101.