Decoding and Computing Algorithms for Linear Superposition LDPC Coded Systems

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Abstract

This paper is concerned with linear superposition systems in which all components of the superimposed signal are coded with an identical binary low-density parity-check (LDPC) code. We focus on the design of decoding and computing algorithms. The main contributions of this paper include: 1) we present three types of iterative multistage decoding/computing algorithms, which are referred to as decoding-computing (DC) type, computing-decoding (CD) type and computing-decoding-computing (CDC) type, respectively; 2) we propose a joint decoding/computing algorithm by treating the system as a nonbinary LDPC (NB-LDPC) coded system; 3) we propose a time-varying signaling scheme for multi-user communication channels. The proposed algorithms may find applications in superposition modulation (SM), multiple-access channels (MAC), Gaussian interference channels (GIFC) and two-way relay channels (TWRC). For SM system, numerical results show that 1) the proposed CDC type iterative multistage algorithm performs better than the standard DC type iterative multistage algorithm, and 2) the joint decoding/computing algorithm performs better than the proposed iterative multistage algorithms in high spectral efficiency regime. For GIFC, numerical results show that, from moderate to strong interference, the time-varying signaling scheme significantly outperforms the constant signaling scheme when decoded with the joint decoding/computing algorithm (about 8.5 dB for strong interference). For TWRC, numerical results show that the joint decoding/computing algorithm performs better than the CD type algorithm.

Index Terms

LDPC codes, joint Tanner graph, Gaussian interference channels, multiple-access channels, superposition modulation, two-way relay channels.

This work was supported by 973 Program (No.2012CB316100) and the NSF (No. 61172082) of China.

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I. INTRODUCTION

Low-density parity-check (LDPC) codes, which were originally proposed by Gallager in [1], form a class of capacity-approaching codes. Ever since their rediscovery [2], extensive attention have been paid on their construction [3–6], decoding [6–10] and application [11–15]. In [11], the authors investigated the application of LDPC codes to multi-level coding (MLC) systems. They applied the density evolution to optimize the degree distributions of the component codes. To obtain shaping gain, an MLC scheme based on LDPC codes and trellis shaping was proposed in [12]. LDPC codes have also found applications in multi-user communication systems. In [14], the authors employed LDPC codes to implement the physical-layer network coding [16][17][18] for two-way relay channels (TWRC). In [19], the authors analyzed the distance spectrum of coded Gaussian two-way relay channels with binary input. The application of LDPC codes in relay channels has been considered in [20]. In [21], the authors proposed the multi-edge-type bilayer-expurgated LDPC codes for relay channels. A signal cooperation scheme for LDPC coded relay channels have been proposed in [15].

Superposition is a common phenomenon in communication systems. In superposition modulation (SM) systems [22][23], bipolar signals (possibly with different amplitudes) are superimposed at the transmitter to approach the channel capacity without active shaping. Simulation results in [23] showed that properly designed SM can perform very close to the channel capacity. In non-orthogonal Gaussian multiple-access channels (MAC), signals of different transmitters are superimposed at the receiver. The receiver intends to recover all the messages from these transmitters. Similarly, in Gaussian interference channels (GIFC), signals from different transmitters are superimposed at the receiver. However, the receiver only intends to recover the messages from its corresponding transmitter. In two-way relay channels, the signals of the two transmitters are superimposed at the relay. One protocol based on physical-layer network coding has been proposed in [24][14], in which the relay intends to compute the modulo sum of the messages. Superposition can also be found in relay channels and broadcast channels [25]. In SM and broadcast channels, superposition is designated artificially for bandwidth efficiency. However, in MAC, GIFC and TWRC, superposition is inevitable in non-orthogonal wireless transmission systems.

In this paper, we investigate decoding/computing algorithms for linear superposition LDPC
coded systems, which generalize our precious works on LDPC superposition modulation [26] and LDPC coded Gaussian interference channels [27]. We focus on the case in which all levels are coded with an identical binary LDPC code. The main contributions of this paper include:

1) Three types of iterative multistage decoding/computing algorithms are presented, which are decoding-computing (DC) type, computing-decoding (CD) type and computing-decoding-computing (CDC) type, respectively.

2) We show that the considered superposition systems can be viewed as a system coded with a special class of nonbinary LDPC (NB-LDPC) codes. Based on this, we propose a joint decoding/computing algorithm which works over a compact Tanner graph [28].

3) We propose a time-varying signaling scheme for multi-user communication channels.

The proposed algorithms and the time-varying signaling scheme are applicable to SM, MAC, GIFC and TWRC. In SM, simulation results show that 1) the CDC type iterative multistage algorithm performs better than the DC type iterative multistage algorithm; and 2) the joint decoding/computing algorithm performs better than the iterative multistage algorithms in high spectral efficiency regime and reveals lower error floor. For MAC, simulation results show that the time-varying signaling can be implemented, resulting in a new multiple-access method with a natural multiuser detection/decoding algorithm. In GIFC, simulation results show that 1) for weak interference, the joint decoding/computing algorithm and the iterative multistage algorithms have almost the same performance; 2) for moderate interference, the joint decoding/computing algorithm performs better than the CD and the DC type algorithms but reveals higher error floors; and 3) for strong interference, the joint decoding/computing algorithm performs better than the iterative multistage algorithms. We have also applied the proposed time-varying signaling scheme to GIFCs. Simulation results show that 1) for weak interference, the time-varying signaling scheme incurs a performance degradation, and 2) from moderate to strong interference, the time-varying signaling scheme significantly outperforms the constant signaling scheme. In TWRC, simulation results show that the joint decoding/computing algorithm performs better than the CD type iterative multistage algorithm.

The rest of this paper is organized as follows. In Section II, we describe the system model considered in this paper. The three type of iterative multistage algorithms are given in Section III. The joint normal graphical realization of the linear superposition LDPC coded system and
the corresponding joint decoding/computing algorithms are given in Section IV. Applications and
the simulation results of the proposed algorithms are given in Section V. Also given in Section
V is the proposed time-varying signaling scheme. Section VI concludes this paper.

II. SYSTEM DESCRIPTION AND PROBLEM STATEMENT

A. The Linear Superposition LDPC Coded System

Assume that \( \ell \) “users”, which can be levels (in SM) or users (in multi-user networks),
are attempting to transmit \( \ell \) binary sequence \( u^{(i)}(0 \leq i \leq \ell - 1) \) of length \( k \) through the
additive white Gaussian noise (AWGN) channel. The \( i \)-th sequence \( u^{(i)} \) is encoded with a
given binary LDPC code \( C_2[n, k] \) of length \( n \) and dimension \( k \), resulting in a coded sequence
\( c^{(i)}(0 \leq i \leq \ell - 1) \). The coded bit \( c^{(i)}_t \) at time \( t \) is then mapped into a
bipolar signal \( x^{(i)}_t = \alpha^{(i)}_t (1 - 2c^{(i)}_t) \). These coded sequences are then transmitted simultaneously
over the AWGN channel. The received signal at time \( t \) can be written as
\[
y_t = \sum_{0 \leq i \leq \ell - 1} h_ix^{(i)}_t + z_t = \sum_{0 \leq i \leq \ell - 1} h_i\alpha^{(i)}_t (1 - 2c^{(i)}_t) + z_t,
\]
where \( h_i \) is the channel coefficient for the \( i \)-th user and \( z_t \) is a sample from an AWGN ensemble
with variance \( \sigma^2 = N_0/2 \). Note that the channel coefficients \( h_i \)'s are assumed to be time-invariant,
while the amplitudes \( \alpha^{(i)}_t \)'s can be designed to be time-varying. However, it is usually required
that
\[
\frac{1}{n} \sum_{0 \leq t \leq n-1} (\alpha^{(i)}_t)^2 \leq P^{(i)},
\]
where \( P^{(i)} \) is the power of the \( i \)-th user. In this paper, we assume that \( \sigma^2 \equiv 1 \).

Let \( H \) denote the parity-check matrix of \( C_2[n, k] \). Then we have \( c^{(i)} H^T = 0 \), for \( 0 \leq i \leq \ell - 1 \),
where \( H^T \) denotes the transpose of \( H \). Let \( c_t = (c^{(0)}_t, c^{(1)}_t, \ldots, c^{(\ell - 1)}_t)^T \), which is a column vector
collecting all coded bits at time \( t \). Define
\[
\mathcal{C} = (c_0, c_1, \ldots, c_{n-1}) = \begin{pmatrix}
    c^{(0)}_0 & c^{(1)}_0 & \cdots & c^{(0)}_{n-1} \\
    c^{(1)}_0 & c^{(1)}_1 & \cdots & c^{(1)}_{n-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    c^{(\ell - 1)}_0 & c^{(\ell - 1)}_1 & \cdots & c^{(\ell - 1)}_{n-1}
\end{pmatrix}.
\]
It is obvious that $\mathbf{c}^T \mathbf{H}^T = (c_0, c_1, \cdots, c_{n-1})^T \mathbf{H}^T = 0$. Let $\tau$ be a linear mapping from $\mathbb{F}_2^\ell$ to $\mathbb{F}_2^{\ell'}$. Define $\underline{v} = \tau(\underline{c}) \triangleq (\tau(c_0), \tau(c_1), \cdots, \tau(c_{n-1}))$. In this paper, we consider the following general problem.

**How to compute $\underline{v} = \tau(\underline{c})$ from the received sequence $y$?**

- The straightforward solution is to decode $\underline{c}$ first and then compute $\tau(\underline{c})$. Algorithms follow this procedure will be referred to as *decoding-computing type algorithms*. To initialize the decoding algorithms for the $\ell$-level LDPC coded system, we need the following original likelihoods

$$f_o(y_t | c_t) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(y_t - \phi_t(c_t))^2}{2}\right), \quad c_t \in \mathbb{F}_2^\ell, \quad (4)$$

where $\phi_t : \mathbb{F}_2^\ell \mapsto \mathbb{R}$ is determined by

$$\phi_t(c_t) = \sum_{0 \leq i \leq \ell - 1} h_i \alpha_t^{(i)} (1 - 2c_t^{(i)}),$$

for $c_t \in \mathbb{F}_2^\ell$ and $0 \leq t \leq n - 1$. The issue of the decoding-computing solution is that it may be impossible to recover $\underline{c}$ reliably from $y$.

- The second solution is referred to as computing-decoding solution. First, we compute the likelihoods for $v_t = \tau(c_t)$ as

$$f_\tau(y_t | v_t) \propto \sum_{c_t \in \mathbb{F}_2^\ell, \tau(c_t) = v_t} f_o(y_t | c_t), \quad v_t \in \mathbb{F}_2^{\ell'} \quad (5)$$

for $0 \leq t \leq n - 1$. Then, these likelihoods are used to initialize the decoding algorithms for an $\ell'$-level LDPC coded system. Algorithms follow this procedure will be referred to as *computing-decoding type algorithms*.

- The third solution is to recover $\underline{w} = \tilde{\tau}(\underline{c})$ for an invertible linear mapping $\tilde{\tau}$ first and then compute $\tau \tilde{\tau}^{-1}(\underline{w})$. Algorithms follow this procedure will be referred to as *computing-decoding-computing type algorithms*. The decoding algorithms are performed for an $\ell$-level LDPC coded system with initial likelihoods

$$f_\tau(y_t | w_t) = f_o(y_t | \tilde{\tau}^{-1}(w_t)), \quad w_t \in \mathbb{F}_2^\ell \quad (6)$$

for $0 \leq t \leq n - 1$. 
Interestingly, we have found by simulation that, in some applications, the computing-decoding-computing type algorithm becomes more efficient.

To make it more clear, we have illustrated the relationships of all the involved variables in Fig. 1. The decoding-computing type algorithms follow the route $y \rightarrow x \rightarrow c \rightarrow v$, the computing-decoding type algorithms follow the route $y \rightarrow x \rightarrow v$, while the computing-decoding-computing type algorithms follow the route $y \rightarrow x \rightarrow w \rightarrow c \rightarrow v$.

In the following subsections, we consider four well-known applications.

B. Superposition Modulation

Superposition modulation is a low-complexity modulation scheme to approach the capacity of AWGN channels [22][23][29]. In superposition modulation, we can set $h_i \equiv 1$ for $0 \leq i \leq \ell - 1$. The objective of the receiver is to recover all the $\ell$ information sequence $u^{(i)}$ reliably and efficiently. That is, the linear mapping $\tau$ is defined by the $\ell \times \ell$ identity matrix $I$. The amplitudes $\alpha^{(i)}_t$'s can be optimized jointly. The constraint is

$$\sum_{0 \leq i \leq \ell - 1} P^{(i)} \leq P,$$

where $P$ is the average power of the transmitter. The coding rate is defined as $r \overset{\Delta}{=} \frac{\ell k}{n}$ bits/dim.

We define the normalized SNR, $\text{SNR}_{\text{norm}}$, as [30]

$$\text{SNR}_{\text{norm}} = 10 \log_{10}(P/(2^{2r} - 1)).$$
C. Multiple-Access Channels

In this application, we consider the symmetric case in which the received powers of different users are equal. This has been shown to be optimal for most modulation and demodulation schemes [31] given that the total received power is fixed. That is, we have $h_i^2 P^{(i)} = h_j^2 P^{(j)}$. Without loss of generality, we can assume $h_i \equiv 1$ and $P^{(i)} = P^{(j)}$ for $0 \leq i, j \leq \ell - 1$. Different from superposition modulation, only predefined (limited) cooperations can be conducted among the users. The linear mapping $\tau$ is determined by the $\ell \times \ell$ identity matrix $I$. The SNR is defined as $\text{SNR}_{\text{norm}} = 10 \log_{10}(P/(2^{2r} - 1))$ where $P = \ell P^{(i)}$ is the total receiving power and $r$ is the sum-rate of the $\ell$ users.

D. Gaussian Interference Channels

Gaussian interference channel (GIFC) [32] is an important model for wireless network communications. In GIFC, the transmitter $i$ attempts to transmit the sequence $c^{(i)}$ to the receiver $i$. In this paper, we restrict our attention to binary-input symmetric GIFC with two pairs of users, which can be represented as

$$
y_i^{(0)} = x_i^{(0)} + h_1 x_i^{(1)} + z_i^{(0)}
$$

$$
y_i^{(1)} = h_1 x_i^{(0)} + x_i^{(1)} + z_i^{(1)}
$$

where $x_i^{(i)}$, $y_i^{(i)}$ and $z_i^{(i)}$ are the channel input, channel output and additive noise of the receiver $i$ at time $t$, respectively. We have $x_i^{(i)} = \alpha_i^{(i)} (1 - 2c_i^{(i)})$, where $\alpha_i^{(i)}$ is the amplitude of the channel input of transmitter $i$ at time $t$. Consider the 0-th receiver, its received signal can be characterized by (1) with $h_0 \equiv 1$. The interference coefficients $h_0$ and $h_1$ are fixed, however, the parameters $\alpha_i^{(i)}$'s can be selected to achieve better performance. We assume that $P^{(0)} = P^{(1)}$, which is reasonable for symmetric GIFCs. Also note that the 0-th receiver only needs to recover $c^{(0)}$ (or $u^{(0)}$), which means that the linear mapping $\tau$ is determined by the matrix $(1 \ 0)$. The SNR is defined as

$$
\text{SNR} = 10 \log_{10}(P^{(0)}) = 10 \log_{10}(P/2).
$$
E. Two-Way Relay Channels

In two-way relay channels (TWRC), two users try to exchange messages with the help of a relay node. The received signal of the relay at time \( t \) is

\[
y_t = \alpha_t(0)(1 - 2c_t(0)) + \alpha_t(1)(1 - 2c_t(1)) + z_t.
\]  

(8)

A protocol for message exchanging based on physical-layer network coding has been proposed [24][14][33], where the relay only needs to recover \( c(0) \oplus c(1) \). This means that the linear mapping \( \tau \) is determined by the matrix \((1, 1)\). We will assume that \( P(0) = P(1) \). The SNR is defined as

\[
\text{SNR} = 10 \log_{10}(P(0)) = 10 \log_{10}(P/2).
\]

It is possible to extend the physical-layer network coding scheme from TWRC to multi-way relay channels. In multi-way relay channels, multiple users want to exchange information with the help of a single relay. A simple protocol for the three-user case is described as follows. The received signal at the relay is

\[
y_t = \alpha_t(0)(1 - 2c_t(0)) + \alpha_t(1)(1 - 2c_t(1)) + \alpha_t(2)(1 - 2c_t(2)) + z_t.
\]  

(9)

Upon receiving the corrupted signal \( y \), the relay attempts to recover \( c(0) \oplus c(1) \) and \( c(0) \oplus c(2) \). That is, the relay attempts to recover \( \tau(c) \), in which the linear mapping \( \tau \) is defined by the matrix

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}.
\]

Assume that the relay has decoded \( \tau(c) \) successfully. It then broadcasts \( \tau(c) \) to the three users. It can be easily verified that the user \( i \) \((0 \leq i \leq 2)\) can recover the information of the other two users successfully if it can recover \( \tau(c) \) successfully.

III. ITERATIVE MULTISTAGE DECODING/COMPUTING ALGORITHMS

Let \( H \) denote the parity-check matrix of the binary LDPC code \( C_2[n, k] \). A high-level normal graph for the superposition system is shown in Fig. 2 (a). In a normal graph, edges represent variables, while vertices represent constraints. In this paper, a message associated with a random variable is defined as its probability mass function (pmf). For example, a message associated
with a random variable $Z$ over the finite field $\mathbb{F}_q$ can be represented by $P_Z(z), z \in \mathbb{F}_q$, a real vector of dimension $q$. For convenience, we use $P_{Z}(z)$ to denote a sequence of messages, where the $t$-th message correspond to the $t$-th random variable $Z_t$. Let $Z$ denote the random variables associated with the edge connecting node $A$ and node $B$ in the normal graph. We will use the notation $P_{A \to B}(z)$ to denote the messages from node $A$ to node $B$.

A. The Decoding-Computing Type Iterative Multistage Algorithm

The decoding-computing type iterative multistage message processing/passing algorithm described below works by exchanging messages over the normal graph shown in Fig. 2 (a).

**Algorithm 1:** The Decoding-Computing Type Iterative Multistage Algorithm (DC-IMSA)

- **Initialization:** The messages $P_{C(i)}^{H \to \Sigma}(c_{(i)}^{(j)})$ are initialized with Bernoulli-1/2 distribution for $0 \leq i \leq \ell - 1$. Select a maximum local iteration number $I_{max}$ and a maximum global iteration number $K_{max}$. Set $K = 0$.
- **Iteration:** While $K < K_{max}$
  1) **Decoding:** for $i = 0, 1, \ldots, \ell - 1$
    a) Compute the extrinsic messages $P_{C(i)}^{\Sigma \to H}(c_{(i)}^{(j)})$ with the following soft-in-soft-out (SISO) demapping algorithm [23].
    - The $t$-th ($0 \leq t \leq n - 1$) component of $P_{C(i)}^{\Sigma \to H}(c_{(i)}^{(j)})$ is computed as
      $$P_{C(i)}^{\Sigma \to H}(m) \propto \sum_{c_{t} \in \mathbb{F}_2, c_{t}^{(i)} = m} f_{o}(y_{t}|c_{t}) \prod_{j \neq i} P_{C(j)}^{H \to \Sigma}(c_{(j)}^{(j)}),$$
      for $m \in \mathbb{F}_2$.
    b) Compute the extrinsic messages $P_{C(i)}^{H \to \Sigma}(c_{(i)}^{(j)})$ with the iterative sum-product algorithm (SPA). The SPA is executed with maximum iteration number $I_{max}$.
    c) Compute the full messages $P_{C(i)}(c_{(i)}^{(l)})$ as
      $$P_{C(i)}(c_{(i)}^{(l)}) \propto P_{C(i)}^{H \to \Sigma}(c_{(i)}^{(l)}) P_{C(i)}^{\Sigma \to H}(c_{(i)}^{(l)})$$
      for $c_{t}^{(i)} \in \mathbb{F}_2$ and $0 \leq t \leq n - 1$. Then find
      $$c_{t}^{(i)} = \arg \max_{c_{t}^{(i)} \in \mathbb{F}_2} P_{C(i)}(c_{(i)}^{(l)}),$$
Fig. 2: High-level normal graphs: (a) DC-IMSA, (b) CD-IMSA and (c) CDC-IMSA.
for $0 \leq t \leq n - 1$.

2) **Computing**: $\hat{v} = \tau(\hat{c})$.

3) If $\hat{v}H_T = 0$, declare the decoding success and exit the iteration; else increment $K$ by one.

- **Failure Report**: If $K = K_{max}$, report a decoding failure.

**Remark**: Note that the DC-IMSA attempts to recover $\tau(c)$ by first recovering $c$ and then applying the linear mapping $\tau$. This algorithm has been shown in [14] to be ineffective for coded TWRC.

### B. The Computing-Decoding Type Iterative Multistage Algorithm

We may rewrite the computing messages $v = \tau(c)$ as

$$v = (v_0, v_1, \ldots, v_{n-1}) = \begin{pmatrix} v^{(0)} \\ v^{(1)} \\ \vdots \\ v^{(l'-1)} \end{pmatrix},$$

where $v_t \in \mathbb{F}_2^l$ ($0 \leq t \leq n - 1$) and $v^{(i)}$ ($0 \leq i \leq l' - 1$) is a binary vector of length $n$. It can be verified that

**Proposition 1**: For each $i$ ($0 \leq i \leq l' - 1$), the vector $v^{(i)}$ is a codeword in $C_2[n, k]$. That is, $v^{(i)}H_T = 0$.

From the above proposition, we have a normal graph as shown in Fig. 2 (b). The computing-decoding type iterative multistage algorithm described below works by exchanging messages over this normal graph.

**Algorithm 2**: The Computing-Decoding Type Iterative Multistage Algorithm (CD-IMSA)

- **Initialization**: The messages $P_{\Sigma^{(i)}}^{H\rightarrow\Sigma}(v^{(i)})$ are initialized with Bernoulli-1/2 distribution for $0 \leq i \leq l' - 1$. Select a maximum local iteration number $I_{max}$ and a maximum global iteration number $K_{max}$. Set $K = 0$.

- **Computing**: Compute the likelihoods $f_\tau(y_t|v_t)$ according to (5).

- **Decoding**: While $K < K_{max}$

  1) For $i = 0, 1, \ldots, l' - 1$,

    a) Compute the extrinsic messages $P_{\Sigma^{(i)}}^{\Sigma\rightarrow H}(v^{(i)})$ as follows.
The \( t \)-th \((0 \leq t \leq n - 1)\) component of \( P_{V(i)}^{\Sigma \rightarrow H}(\underline{y}^{(i)}) \) is computed as

\[
P_{V(i)}^{\Sigma \rightarrow H}(m) \propto \sum_{v_t \in \mathbb{F}_2^n} f_{\tau}(y_t|v_t) \prod_{j \neq i} P_{V(j)}^{H \rightarrow \Sigma}(v_t^{(j)}),
\]

for \( m \in \mathbb{F}_2 \).

b) Compute the extrinsic messages \( P_{V(i)}^{H \rightarrow \Sigma}(\underline{y}^{(i)}) \) with the SPA. The SPA is executed with maximum iteration number \( I_{\text{max}} \).

c) Compute the full messages \( P_{V(i)}(\underline{y}^{(i)}) \) as

\[
P_{V(i)}(v_t^{(i)}) \propto P_{V(i)}^{H \rightarrow \Sigma}(v_t^{(i)}) P_{V(i)}^{\Sigma \rightarrow H}(v_t^{(i)})
\]

for \( v_t^{(i)} \in \mathbb{F}_2 \) and \( 0 \leq t \leq n - 1 \). Then find

\[
\hat{v}_t^{(i)} = \arg \max_{v_t^{(i)} \in \mathbb{F}_2} P_{V(i)}(v_t^{(i)}),
\]

for \( 0 \leq t \leq n - 1 \).

2) If \( \hat{\underline{y}}^{H^T} = 0 \), declare the decoding success and exit the iteration; else increment \( K \) by one.

- **Failure Report:** If \( K = K_{\text{max}} \), report a decoding failure.

**Remarks:**

a) If the linear mapping \( \tau \) is determined by the \( \ell \times \ell \) identity matrix, the CD-IMSA and the DC-IMSA are the same.

b) It has been shown in [14] that the CD-IMSA performs better than the DC-IMSA in TWRC.

**C. The Computing-Decoding-Computing Type Iterative Multistage Algorithm**

An alternative algorithm to recover \( \tau(\underline{c}) \) is based on the computing-decoding-computing procedure. In this algorithm, we first recover \( \hat{\tau}(\underline{c}) \) for an invertible linear mapping \( \hat{\tau} \) and then compute \( \tau(\hat{\tau}^{-1}(\underline{c})) \). The motivation of this algorithm is as follows.

Consider the 2-level SM with power allocation \( P^{(0)} = 4 \) and \( P^{(1)} = 1.96 \). The superimposed signal constellation is \( \mathcal{X} \Delta \phi_2(\mathbb{F}_2^\ell) = \{-3.4, -0.6, 0.6, 3.4\} \). If Algorithm 1 is implemented, to compute the initial messages for decoding the second level, the signal set \( \mathcal{X} \) is partitioned into \( \mathcal{X}_0 = \{-0.6, 3.4\} \) and \( \mathcal{X}_1 = \{0.6, -3.4\} \), which correspond to \( c_t^{(1)} = 0 \) and \( c_t^{(1)} = 1 \),
respectively. The distance between $\mathcal{X}_0$ and $\mathcal{X}_1$ is $d(\mathcal{X}_0, \mathcal{X}_1) \overset{\Delta}{=} \min_{a \in \mathcal{X}_0, b \in \mathcal{X}_1} |a - b| = 1.2$. If we apply an invertible mapping $\tilde{\tau}$ defined by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ to $\mathbb{F}_2^2$, we get a new 2-level system. In the new 2-level system, the first level remains unchanged, while the second level becomes the sum (mod 2) of the original two levels. Now the signal set is partitioned into $\mathcal{X}_0 = \{3.4, 3.4\}$ and $\mathcal{X}_1 = \{-0.6, 0.6\}$ for the second level. Since the new partition has a large distance $d(\mathcal{X}_0, \mathcal{X}_1) = 2.8$, it will provide better initial messages for iteratively decoding the second level.

Define

$$w = (w_0, w_1, \cdots, w_{n-1}) \overset{\Delta}{=} \tilde{\tau}(x) = \begin{pmatrix} w_0(0) \\ w_1(1) \\ \vdots \\ w_{\ell-1}(i) \end{pmatrix}.$$ 

Similar to Proposition 1, we have

**Proposition 2:** For each $i$ ($0 \leq i \leq \ell - 1$), $w(i)$ is a codeword of $C_2[n, k]$. That is, $w(i)H^T = 0$.

The computing-decoding-computing type iterative multistage algorithm described below works by exchanging messages over the normal graph shown in Fig. 2 (c).

**Algorithm 3:** The Computing-Decoding-Computing Type Iterative Multistage Algorithm (CDC-IMSA)

- **Initialization:** The messages $P_{W(i)}^{H \rightarrow \Sigma}(w(i))$ are initialized with Bernoulli-1/2 distribution for $0 \leq i \leq \ell - 1$. Select a maximum local iteration number $I_{max}$ and a maximum global iteration number $K_{max}$. Set $K = 0$.
- **Computing:** Compute $f_{\tilde{\tau}}(y_t|w_t)$ according to (6).
- **Iteration:** While $K < K_{max}$
  1) **Decoding:** for $i = 0, 1, \cdots, \ell - 1$
      a) Compute the extrinsic messages $P_{W(i)}^{\Sigma \rightarrow H}(W(i))$ with the following soft-in-soft-out (SISO) demapping algorithm.
      - The $t$-th ($0 \leq t \leq n - 1$) component of $P_{W(i)}^{\Sigma \rightarrow H}(w(i))$ is computed as

      $$P_{W(i)}^{\Sigma \rightarrow H}(m) \propto \sum_{\substack{w_t \in \mathbb{F}_2^n \atop w_t(i) = m}} f_{\tilde{\tau}}(y_t|w_t) \prod_{j \neq i} P_{W(j)}^{H \rightarrow \Sigma}(w_t(j)),$$

      - The $t$-th ($0 \leq t \leq n - 1$) component of $P_{W(i)}^{\Sigma \rightarrow H}(w(i))$ is computed as

      $$P_{W(i)}^{\Sigma \rightarrow H}(m) \propto \sum_{\substack{w_t \in \mathbb{F}_2^n \atop w_t(i) = m}} f_{\tilde{\tau}}(y_t|w_t) \prod_{j \neq i} P_{W(j)}^{H \rightarrow \Sigma}(w_t(j)).$$
for $m \in \mathbb{F}_2$.

b) Compute the extrinsic messages $P_{W_t(i)}^{H \rightarrow \Sigma}(w_t^{(i)})$ with the iterative sum-product algorithm (SPA). The SPA is executed with maximum iteration number $I_{\text{max}}$.

c) Compute the full messages $P_{W_t(i)}(w_t^{(i)})$ as

$$P_{W_t(i)}(w_t^{(i)}) \propto P_{W_t(i)}^{H \rightarrow \Sigma}(w_t^{(i)}) P_{W_t(i)}^{\Sigma \rightarrow H}(w_t^{(i)})$$

for $c_t^{(i)} \in \mathbb{F}_2$ and $0 \leq t \leq n - 1$. Then find

$$\hat{w}_t^{(i)} = \arg \max_{w_t^{(i)} \in \mathbb{F}_2} P_{W_t(i)}(w_t^{(i)})$$

for $0 \leq t \leq n - 1$.

2) **Computing:** $\hat{\tau} = \tau \tau^{-1}(\hat{w})$.

3) If $\hat{\tau}H^T = 0$, declare the decoding success and exit the iteration; else increment $K$ by one.

* **Failure Report:** If $K = K_{\text{max}}$, report a decoding failure.

IV. JOINT DECODING/COMPUTING ALGORITHMS

Recall that $\underline{c} = (c_0, c_1, \ldots, c_{n-1})$. We can treat $c_t \in \mathbb{F}_2^\ell$ as an element in the finite filed $\mathbb{F}_q$ with $q = 2^\ell$. It is obvious that $\underline{c}H^T = (c_0, c_1, \ldots, c_{n-1})H^T = \underline{0}$. Hence, $\underline{c}$ can be treated as a codeword of a special NB-LDPC code $C_q[n, k]$ over $\mathbb{F}_q$. The speciality lies in that all nonzero elements in the parity-check matrix $H$ are equal to the identity of $\mathbb{F}_q$. In particular, this NB-LDPC code is a special class of column-scaled LDPC (CS-LDPC) codes [34]. As a CS-LDPC code, $C_q[n, k]$ has the same minimum Hamming distance, coding rate and graph properties as compared with the original binary LDPC code $C_2[n, k]$. This motivates us to present the following joint decoding/computing algorithm.

A compact normal graph is described below and shown in Fig. 3.

1) There are in total $n$ variable nodes, represented by $\oplus$, each of which corresponds to a column of $H$. The $n$ variable nodes are denoted by $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_{n-1}$. The degree of the $j$-th variable node is exactly 1 added by the number of nonzero element in the $j$-th column of $H$.

2) There are in total $m$ check nodes, represented by $\square$, each of which corresponds to a to
Fig. 3: A joint normal graphical realization of linear LDPC coded superposition system.

a row of $H$. The $m$ check nodes are denoted by $C_0, C_1, \cdots, C_{m-1}$. The degree of the $i$-th check node is exactly the number of nonzero element in the $i$-th row of $H$.

3) The check node $C_i$ is connected to the variable node $V_j$ if and only if $h_{i,j}$ is nonzero. The edge (variable) connecting the check node $C_i$ and the variable node $V_j$ is denoted by $X_{i,j}$, which is a random vector over $\mathbb{F}_2^\ell$.

4) All edges (variables) connecting to the $j$-th variable node $V_j$ must take identical values and all edges (variables) connecting to the $i$-th check node $C_i$ must add up to zero.

Remark: The normal graphical realization can be viewed as a realization of the NB-LDPC code $C_q[n, k]$ defined by the parity-check matrix $H$. It should be pointed out that the random variable $X_{i,j}$ in the normal graphical realization can take $q (= 2^\ell)$ possible values, each of which corresponds to a binary vector of length $\ell$. This is different from the normal graphical realization of the binary LDPC code $C_2[n, k]$. This is also different from a general NB-LDPC code since each edge is associated with the identity element. Hence, no message permutation is required during the iterations. For completeness, we include the joint decoding/computing algorithm here.

Upon receiving $y$, the receiver can perform the following iterative message processing/passing algorithm for decoding/computing.

Algorithm 4: The Joint Iterative Decoding/Computing Algorithm

- Initialization: Initialize the messages $P_{V_t \rightarrow V_i}(x) \propto f_{o}(y_t|x)$ for $x \in \mathbb{F}_q$ and $0 \leq t \leq n - 1$. 
All messages from variable nodes to check nodes are initialized by setting $P_{X_{ij}}^{V_j \rightarrow C_i}(x) = P_{X_j}^{C_i \rightarrow V_j}(x)$ for $x \in \mathbb{F}_q$. Select a maximum iteration number $K_{\text{max}}$ and set $K = 0$.

- **Iteration:** While $K < K_{\text{max}}$

  1) **Message processing at check nodes:** for all check nodes, compute the messages $P_{X_{ij}}^{C_i \rightarrow V_j}(x)$ as follows.

$$P_{X_{ij}}^{C_i \rightarrow V_j}(x) = \sum_{x + \sum_{k \neq j} x_{i,k} = 0} \left( \prod_{k \neq j} P_{X_{ij}}^{V_k \rightarrow C_i}(x_{i,k}) \right),$$

for $x \in \mathbb{F}_q$. The equation above is of the form “sum of products”, which can be computed either by the forward-backward trellis algorithms [35] or by a fast Fourier transform (FFT) based procedure [36].

  2) **Message processing at variable nodes:** for all variable nodes, compute the messages $P_{X_{ij}}^{V_j \rightarrow C_i}(x)$ as follows.

$$P_{X_{ij}}^{V_j \rightarrow C_i}(x) \propto P_{X_j}^{V_j \rightarrow V_j}(x) \prod_{k \neq i} P_{X_{kj}}^{C_k \rightarrow V_j}(x),$$

for $x \in \mathbb{F}_q$.

  3) **Making decisions:**

  - For all variable nodes, compute the message

$$P_{X_j}(x_j) \propto P_{X_j}^{V_j \rightarrow V_j}(x_j) \prod_{k \neq i} P_{X_{kj}}^{C_k \rightarrow V_j}(x_j),$$

for $x_j \in \mathbb{F}_q$.

  - Compute the message associated with $V = \tau(C)$

$$P_{V_j}(v_j) = \sum_{x_j \in \mathbb{F}_q^\ell \, : \, \tau(x_j) = v_j} P_{X_j}(x_j)$$

for $v_j \in \mathbb{F}_q^\ell$ and $0 \leq j \leq n - 1$.

  - Find

$$\hat{v}_j = \arg \max_{v_j \in \mathbb{F}_q^\ell} P_{V_j}(v_j),$$

for $0 \leq j \leq n - 1$.

  - If $\hat{H}^T = \mathbf{0}$, declare the decoding/computing success and exit the iteration.
4) Increment $K$ by one.

- **Failure Report:** If $K = K_{\text{max}}$, report a decoding/computing failure.

**Remark:** Note that the complexity of the joint decoding/computing algorithm grows exponentially with the number of levels $\ell$. However, the complexity of the iterative multistage algorithms in Section III grows linearly with the $\ell$. Hence, the joint decoding/computing algorithms are applicable only for systems with small $\ell$.

V. **NUMERICAL RESULTS**

We have presented several decoding/computing algorithms. Apparently, the performances of these algorithms depend on the code as well as amplitudes $\{\alpha^{(i)}_t\}$. If the code is allowed to have large block length $n$ and be optimized, we may first choose $\{\alpha^{(i)}_t\}$ such that the computing rate $\frac{1}{n}I(\tau(C), Y)$ is maximized under certain constraints and then find a code to approach this computing rate. We will not follow this procedure in this paper. Instead, we will fix the code and illustrate by simulation the efficiency of the algorithms and the effect of the amplitudes. We choose a rate 0.5 (3,6)-regular LDPC code with length 10000 and a rate 1/3 Kite code [37] with code length 12288 in our simulations.

A. **Superposition Modulation**

Temporarily, we assume that $\alpha^{(i)}_t = \alpha_i \sqrt{P}$ for $0 \leq i \leq \ell - 1$, which means that the amplitude of the transmitted signals of each level is time-invariant. For simplicity, we use Method 2 in [38] to allocate power. That is,

- the power of the $0$-th level is $P^{(0)} = 10^{\delta/10}$, where $\delta$ is the SNR required by $C_2[n,k]$ to achieve certain bit-error-rate (BER) over AWGN channels;
- the power of the $i$-th level is $P^{(i)} = P^{(0)}(1 + P^{(0)})^i$ for $1 \leq i \leq \ell - 1$.

Then, we have $\alpha_i = \sqrt{P^{(i)} / P}$. For convenience, The vector $\left(\alpha_0, \alpha_1, \ldots, \alpha_{\ell-1}\right)$ is loosely referred to as the power allocation ratios of the $\ell$ levels.

**Example I:** Consider the transmission at rate 1.0 bits/dim with the (3,6)-regular LDPC code. That is $\ell = 2$. We set $\delta = 1.2$ dB. Based on the power allocation method, we have $\alpha_0^2 = 0.301$ and $\alpha_1^2 = 0.699$. The DC-IMSA (CD-IMSA), CDC-IMSA and the joint decoding algorithm are implemented for decoding. The invertible mapping $\tilde{\tau}$ in the CDC-IMSA is determined
by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The DC-IMSA and the CDC-IMSA are implemented with maximum global iteration number $K_{max} = 30$ and maximum local iteration number $I_{max} = 50$. The joint decoding/computing algorithm is implemented with maximum iteration number $K_{max} = 200$. The error performances of these decoding algorithms in terms of BER are shown in Fig. 4. It can be seen that at BER $= 10^{-5}$

- the CDC-IMSA performs slightly better than the joint decoding algorithm;
- both the CDC-IMSA and the joint decoding algorithm perform about 0.3 dB better than DC-IMSA.

**Example I (Continued):** We have also simulated the performances of the considered system when coded with an optimized irregular LDPC [5] with rate 1/2. The block length of the code is 10000. The simulation results are shown in Fig. 5. It can be seen from these simulation results that better performances can be obtained if optimized irregular LDPC codes are adopted. For example, at BER $= 10^{-5}$, the optimized irregular LDPC code performs about 1.5 dB away from the Shannon limit and performs about 0.4 dB better than (3,6)-LDPC code.
Fig. 5: Error performances of different decoding algorithms: irregular LDPC code.

**Example II:** Consider the transmission at rate 1.5 bits/dim with the (3,6)-regular LDPC code. That is $\ell = 3$. We set $\delta = 1.2$ dB. Based on the power allocation method, we have $\alpha_0^2 = 0.115$, $\alpha_1^2 = 0.267$ and $\alpha_2^2 = 0.618$. The DC-IMSA, CDC-IMSA and the joint decoding algorithm are implemented for decoding. The invertible mapping $\tilde{\tau}$ in the CDC-IMSA is determined by the matrix

$$
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
$$

The DC-IMSA and the CDC-IMSA are implemented with maximum global iteration number $K_{max} = 30$ and maximum local iteration number $I_{max} = 50$. The joint decoding/computing algorithm is implemented with maximum iteration number $K_{max} = 200$. The error performances of these decoding algorithms in terms of BER are shown in Fig. 6. It can be seen that at BER = $10^{-5}$

- the joint decoding algorithm performs slightly better than the CDC-IMSA;
- the CDC-IMSA reveals higher error floor than the joint decoding algorithm;
Fig. 6: Error performances of different decoding algorithms in 3-level SM with $\alpha_0^2 = 0.115$, $\alpha_1^2 = 0.267$ and $\alpha_2^2 = 0.618$.

- both the joint decoding algorithm and the CDC-IMSA perform about 0.4 dB better than DC-IMSA.

B. Multiple-Access Channels

Similar to SM, the receiver attempts to recover the messages of the $\ell$ transmitters in MAC. The parameters $\{\alpha_i, 0 \leq i \leq \ell - 1\}$ selected in SM can be chosen for MAC. However, this will result in unequal receiving power among transmitters. To solve this problem, we propose the following time-varying signaling scheme. At time slot $t$, we set

$$\alpha_t^{(i)} = \alpha_t^{(i+t)\mod\ell} \sqrt{P}.$$  

The above time-varying signalling scheme can be described in words as follows. At the initial time $t = 0$, let $(\alpha_0, \alpha_1, \cdots, \alpha_{\ell-1})$ be the power allocation ratios for the $\ell$ users, respectively. Then, at time $t$, the power allocation ratios are cyclicly shifted to left $t$ positions. It can be easily verified that, for large $n$, we have $P^{(i)} = P/\ell$ for $0 \leq i \leq \ell - 1$. Take $\ell = 2$ as an example. From SM, we have $\alpha_0 = \sqrt{0.301}$ and $\alpha_1 = \sqrt{0.699}$. Then at time slot $t$,
• if \( t \equiv 0 \pmod{2} \), we set \( \alpha_t^{(0)} = \alpha_0 \sqrt{P} \) and \( \alpha_t^{(1)} = \alpha_1 \sqrt{P} \);
• if \( t \equiv 1 \pmod{2} \), we set \( \alpha_t^{(0)} = \alpha_1 \sqrt{P} \) and \( \alpha_t^{(1)} = \alpha_0 \sqrt{P} \).

**Example III:** Consider the two-user MAC coded with the (3,6)-regular LDPC code. We set \( \alpha_0 = \sqrt{0.301} \) and \( \alpha_1 = \sqrt{0.699} \). The error performance is shown in Fig. 7. The joint decoding/computing algorithm is implemented with maximum number of iteration \( K_{\text{max}} = 200 \). Also shown in Fig. 7 is the error performance of SM with the same parameters when decoded with the joint decoding/computing algorithm. It can be seen that, at BER = \( 10^{-5} \), the time-varying signaling scheme performs 0.25 dB better than the constant signaling scheme.

**Example III (Continued):** Simulation results have shown that, for the time-varying signalling scheme, only the joint decoding/computing algorithm works effectively. Since the power allocation ratios are obtained based on iterative successive cancellation algorithms, we wonder if the performance can be improved by adjusting the power allocation ratios under the joint decoding/computing algorithm. For this purpose, we have simulated **Example III** with two different power allocation ratios \( \{ \alpha_0 = \sqrt{0.293}, \alpha_1 = \sqrt{0.707} \} \) and \( \{ \alpha_0 = \sqrt{0.290}, \alpha_1 = \sqrt{0.710} \} \). The error performances are also shown in Fig. 7. The joint decoding/computing algorithm is implemented with maximum number of iteration \( K_{\text{max}} = 200 \). It can be seen that the system with \( \{ \alpha_0 = \sqrt{0.293}, \alpha_1 = \sqrt{0.707} \} \) has a lower error floor.

**Example IV:** Consider the three-user MAC coded with the (3,6)-regular LDPC code. We set \( \alpha_0 = \sqrt{0.115} \), \( \alpha_1 = \sqrt{0.267} \) and \( \alpha_2 = \sqrt{0.618} \). The error performance is shown in Fig. 8. The joint decoding/computing algorithm is implemented with maximum number of iteration \( K_{\text{max}} = 200 \). Also shown in Fig. 8 is the error performance of SM with same parameters when decoded with the joint decoding/computing algorithm. It can be seen that the time-varying signaling scheme shows no gain over the constant signaling scheme.

**Example IV (Continued):** We have also simulated **Example IV** for other two sets of parameters \( \{ \alpha_0 = \sqrt{0.110}, \alpha_1 = \sqrt{0.262}, \alpha_2 = \sqrt{0.628} \} \) and \( \{ \alpha_0 = \sqrt{0.150}, \alpha_1 = \sqrt{0.250}, \alpha_2 = \sqrt{0.600} \} \). The error performances are also shown in Fig. 8. The joint decoding/computing algorithm are implemented with maximum number of iteration \( K_{\text{max}} = 200 \). It can be seen that the system with \( \{ \alpha_0 = \sqrt{0.150}, \alpha_1 = \sqrt{0.250}, \alpha_2 = \sqrt{0.600} \} \) has an extra coding gain of 0.3 dB at BER = \( 10^{-6} \).
Fig. 7: Error performances of the joint decoding algorithms in two-user MAC with different time-varying signaling schemes.

Fig. 8: Error performances of the joint decoding algorithms in three-user MAC with different time-varying signaling schemes.
C. Gaussian Interference Channels

Example V: Consider the two-user symmetric Gaussian interference channels coded with the (3,6)-regular LDPC code. We will assume that $P^{(0)} = P^{(1)}$, which is reasonable for symmetric Gaussian interference channels. The total power of the two transmitters is $P = P^{(0)} + P^{(1)}$. The following four algorithms can be employed for decoding: the DC-IMSDA, the CD-IMSDA, the CDC-IMSDA and the joint decoding/computing algorithm. The invertible mapping $\tilde{\tau}$ in the CDC-IMSDA is determined by the matrix

$$
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}.
$$

The DC-IMSDA and the CDC-IMSDA are implemented with maximum global iteration number $K_{max} = 30$ and maximum local iteration number $I_{max} = 50$. The CD-IMSDA is implemented with maximum iteration number 200. The joint decoding/computing algorithm is implemented with maximum iteration number $K_{max} = 200$. We have simulated three different interference channels whose interference coefficients are $h_2^1 = 0.3$, $h_2^1 = 0.5$ and $h_2^1 = 0.75$, respectively. The simulation results are shown in Fig. 9, 10 and 11, respectively. We have the following observations.

1) If the interference is weak (i.e. $h_2^1 = 0.3$), the four decoding/computing algorithms have almost the same error performances.

2) If the interference is moderate (i.e. $h_2^1 = 0.5$), at BER $= 10^{-4}$, the CDC-IMSDA and the joint decoding/computing algorithm perform better than DC-IMSDA and CD-IMSDA, but reveal higher error floors.

3) If the interference is strong (i.e. $h_2^1 = 0.75$), the joint decoding algorithm performs better than the other three algorithms. At BER $= 10^{-4}$, it performs about 0.4 dB better than the CDC-IMSDA, 0.8 dB better than the DC-IMSDA and 2.3 dB better than the CD-IMSDA.

Example V (continued): We have shown by simulations in MAC that time-varying signaling scheme may be better than constant signaling scheme in terms of BER. This motivates us to apply the time-varying signaling scheme to GIFCs. We have simulated the time-varying signaling scheme with different power allocation ratios for the considered GIFCs. The error performances are shown in Fig. 9, Fig. 10 and Fig. 12. The joint decoding/computing algorithm is implemented with maximum iteration number $K_{max} = 200$. It can be seen that at BER $= 10^{-4}$,
Fig. 9: Error performances of different decoding algorithms and different signaling schemes in GIFC: $h_1^2 = 0.3$.

Fig. 10: Error performances of different decoding algorithms and different signaling schemes in GIFC: $h_1^2 = 0.5$. 
Fig. 11: Error performances of different decoding algorithms in GIFC: $h_1^2 = 0.75$.

1) for weak interference (i.e. $h_1^2 = 0.3$), the time-varying signaling schemes incur performance degradations.

2) for moderate interference (i.e. $h_1^2 = 0.5$), the time-varying signaling scheme with $\{\alpha_0 = \sqrt{0.25}, \alpha_1 = \sqrt{0.75}\}$ is about 1.2 dB better than constant signaling scheme.

3) for strong interference (i.e. $h_1^2 = 0.7$), the time-varying signaling scheme with $\{\alpha_0 = \sqrt{0.3}, \alpha_1 = \sqrt{0.7}\}$ is about 8.5 dB better than constant signaling scheme.

D. Two-way Relay Channels

Example VI: Consider the two-way relay channels coded with the rate 1/3 Kite code of length 12288. We assume that $P^{(0)} = P^{(1)}$. The CD-IMSDA and the joint decoding/computing algorithm are implemented for decoding. The simulation results are shown in Fig. 13. Both the CD-IMSDA and the joint decoding/computing algorithm are implemented with maximum iteration number $K_{\text{max}} = 200$. It can be seen that the joint decoding/computing algorithm performs about 0.2 dB than the CD-IMSDA at BER $= 10^{-5}$.

Remark: We have also simulated the (3,6)-regular LDPC (rate 0.5) coded two-way relay channels. Simulation results (which are not given here) show that the two algorithms have
Fig. 12: Error performances of different signaling schemes for GIFC; $h_1^2 = 0.75$.

almost the same error performances.

VI. Conclusion

In this paper, we investigated the decoding/computing algorithms for linear superposition LDPC coded systems. We presented three type of iterative multistage decoding algorithms, which are DC-IMSDA, CD-IMSDA and the CDC-IMSDA. We show that the considered system can be treated as a special NB-LDPC coded system, based on which a joint decoding/computing algorithm is proposed. In addition, we proposed a time-varying signaling for multi-user communication channels, which may find applications in multiple-access channels and Gaussian interference channels.

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