ON THE STRUCTURE OF DENSE TRIANGLE-FREE
BINARY MATROIDS

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Abstract. We prove, by means of an exact structural description, that every simple triangle-free binary matroid $M$ with $|M| > \frac{33}{128}2^r(M)$ has critical number at most 2.

1. Introduction

We call a matroid triangle-free if it has no circuit of size at most 3; in particular, triangle-free matroids are simple, which is a slight abuse of terminology. The critical number $\chi(M)$ of a simple rank-$r$ binary matroid $M$, viewed as a restriction of the projective geometry $G \cong \text{PG}(r-1, 2)$, is the smallest integer $c \geq 0$ such that $G$ has a rank-$(r-c)$ flat disjoint from $E(M)$. We prove the following:

**Theorem 1.1.** If $M$ is a rank-$r$ triangle-free binary matroid satisfying $|M| > \frac{33}{128}2^r$, then $\chi(M) \leq 2$.

In fact, we prove a stronger result, giving an exact structure theorem in terms of quotients. Consider a simple rank-$r$ binary matroid $M$ as a restriction of a projective geometry $G \cong \text{PG}(r-1, 2)$. A quotient of $M$ is a matroid of the form $\text{si}((G/W)|E(M))$, where $W$ is a flat of $G$ that is disjoint from $E(M)$. Thus, $M$ has a quotient of rank at most $c$ if and only if $M$ has critical number at most $c$. Moreover, since triangles in $M$ are “projected” onto triangles in the quotient, if $M$ has a triangle-free quotient, then $M$ is triangle-free.

Theorem 1.1 will be derived from our main result, which gives an exact structure theorem for triangle-free binary matroids with density greater than $\frac{33}{128}$.

**Theorem 1.2.** If $M$ is a triangle-free binary matroid with $|M| > \frac{33}{128}2^r(M)$, then $M$ has a triangle-free quotient of rank at most 6.

Date: April 2, 2015.

1991 Mathematics Subject Classification. 05B35.

Key words and phrases. matroids, finite geometry.

This research was partially supported by a grant from the Office of Naval Research [N00014-10-1-0851].
We will refine this a little further in Section 4, showing that $M$ has a quotient isomorphic to one of eight particular triangle-free matroids each with rank at most 6. A binary matroid $M$ is maximally triangle-free if it is triangle-free and is not a proper restriction of a triangle-free binary matroid of the same rank.

**Corollary 1.3.** For each $r \geq 6$, there are, up to isomorphism, exactly eight rank-$r$ maximally triangle-free binary matroids with more than $\frac{33}{128} 2^r$ elements.

Theorem 1.2 contrasts starkly with the main result of [4], which shows that for any $\varepsilon > 0$, the class of triangle-free binary matroids of density at least $\frac{1}{4} - \varepsilon$ is ‘wild’ in a sense that precludes any guaranteed bound on the size of quotients. Neither that result nor Theorem 1.2 applies to matroids with density between $\frac{1}{4}$ and $\frac{33}{128}$. However we expect these to be structurally well-behaved; [4] contains the following conjecture:

**Conjecture 1.4.** For all $\varepsilon > 0$, there is an integer $r_0$ such that, if $M$ is a triangle-free binary matroid with $|M| > (\frac{1}{4} + \varepsilon)2^{r(M)}$, then $\chi(M) \leq 2$ and $M$ has a triangle-free quotient of rank at most $r_0$.

In Section 5 we show that Theorem 1.2 is best-possible in the sense that it fails if we relax the inequality $|M| > \frac{33}{128} 2^r$.

**Graph theory.** Our results are analogous to, and motivated by, results on triangle-free graphs. A quotient $H$ of a simple graph $G$ is a graph obtained by identifying sets of non-adjacent vertices of $G$ and suppressing parallel edges; note that if $H$ is triangle-free then so is $G$. Theorem 1.2 thus resembles the following theorem of Jin [7], which characterises all sufficiently dense triangle-free graphs in terms of a small triangle-free quotient:

**Theorem 1.5.** If $G$ is a simple triangle-free graph with minimum degree greater than $\frac{10}{29} |V(G)|$, then $G$ has a triangle-free quotient with at most 26 vertices.

Our use of the word “quotient” is analogous to its usage in graph theory and is very well-suited to our purposes, but differs from its established definition in matroid theory (see, for example, Oxley [8]) which does not allow for suppression of parallel elements (which ours does) but does allow for the creation of loops (which ours does not).

**Finite geometry.** Our results are related to the following two theorems in finite geometry. The first of these results is a specialization of a theorem of Bose and Burton [1]:
Theorem 1.6. Let $M$ be a rank-$r$ triangle-free binary matroid. Then $|M| \leq \frac{1}{2} 2^r$. Moreover, if $|M| = \frac{1}{2} 2^r$ then $M$ is an affine geometry.

The second, due to Govaerts and Storme [3], is a structure theorem for triangle-free binary matroids with density greater than $\frac{5}{16}$.

Theorem 1.7. Let $M$ be a rank-$r$ triangle-free binary matroid. If $|M| > \frac{5}{16} 2^r(M)$, then $M$ has a quotient isomorphic to $U_{1,1}$. Moreover, if $|M| = \frac{5}{16} 2^r(M)$, then $M$ has a quotient isomorphic to $U_{1,1}$ or $U_{4,5}$.

Govaerts and Storme stated their results in terms of “blocking sets” in binary projective geometries.

2. Some triangle-free binary matroids

We largely use the standard notation of matroid theory; see Oxley [8]. We will often implicitly use the fact that every binary matroid $M$ is uniquely representable up to projective equivalence, so every binary representation of a restriction of $M$ can be extended to a representation of $M$. The density of a rank-$r$ binary matroid is defined to be the ratio $|M|/2^r$. It may seem unusual to divide by $2^r$ instead of $2^r - 1$, but our results suggest that this definition is more natural.

All matrices throughout have entries in the field $GF(2)$. For positive integers $m$ and $n$, we write $O_{m,n}$ and $J_{m,n}$ for the $m \times n$ all-zeroes and all-ones matrices respectively, abbreviating $O_n$ and $J_n$ when $m = n$.

Affine matroids. A simple binary matroid is affine if it has a quotient isomorphic to $U_{1,1}$. This is the analogue of a bipartite graph, which can be defined as a graph having the single edge $K_2$ as a quotient. Moreover, Brylawski [2] and Heron [6] independently proved that a binary matroid is affine if and only if it does not contain an odd circuit. In particular, affine matroids are triangle-free. Note that the densest affine matroids, the affine geometries, have density $\frac{1}{2}$.

Nearly-affine matroids. We are interested in the structure of triangle-free binary matroids with density greater than $\frac{1}{4}$. There is a rich class of triangle-free binary matroids that just beat this threshold. Let $M'$ be a simple rank-$r$ binary matroid obtained by extending the affine geometry $G \cong AG(r - 1, 2)$ by a single element $e$. The only triangles of $M'$ are those containing $e$, and these triangles partition $E(G)$ into pairs. Let $X$ be a transversal of this partition, then $M'\backslash X$ is triangle-free and $|M'\backslash X| = \frac{1}{2} 2^r + 1$. We call such matroids nearly affine. One example is the unique series extension of $AG(r - 2, 2)$; however, since $X$ is an arbitrary transversal, this class is indeed quite rich.
Doubling. The smallest matroid that is nearly affine, but not affine, is the 5-element circuit $C_5 \cong U_{4,5}$; this matroid has density $\frac{5}{16}$. Starting with $C_5$ we can obtain arbitrarily high-rank triangle-free matroids of this density. Consider a simple binary matroid $M = M[A]$ where $A \in \text{GF}(2)^{r \times n}$. Now consider the binary matroids $M' = M\left[ \begin{array}{c|c} 1 & 1 \\ \hline A & A \end{array} \right]$, and $M'' = M\left[ \begin{array}{c|c|c} 1 & 1 & 1 \\ \hline O_{1,n} & 1 & O_{r,1} \end{array} \right]$.

Let $e$ be the element that we have extended $M'$ by to obtain $M''$. Now $M''/e$ simplifies to $M$. Therefore, $M$ is a quotient of $M'$. Thus

- if $M$ is triangle-free, then $M'$ is triangle-free,
- $M'$ and $M$ have the same density,
- $\chi(M') = \chi(M)$, and
- $r(M') = r(M) + 1$.

We say that $M'$ is obtained from $M$ by doubling. The following result is elementary; we omit the proof.

**Lemma 2.1.** Let $M$ and $N$ be simple binary matroids with $r(M) \geq r(N)$, and let $k = r(M) - r(N)$. Let $N^k$ be the matroid obtained from $N$ by doubling $k$ times. The following are equivalent:

- $M$ has a quotient that is isomorphic to a restriction of $N$; and
- $M$ is isomorphic to a restriction of $N^k$.

Lemma 2.1 has two easy consequences, whose proofs we also omit. Neither result holds for graphs, but this seems particularly notable in the first case.

**Lemma 2.2.** If $M$ is a maximally triangle-free binary matroid and $N$ is a quotient of $M$, then $N$ is at least as dense as $M$.

A triangle-free binary matroid $M$ is irreducibly triangle-free if it has no triangle-free quotient of smaller rank.

**Lemma 2.3.** If $M$ is a maximally triangle-free binary matroid and $|M|$ is odd, then $M$ is irreducibly triangle-free.

A triangle-free binary matroid is critically triangle-free if it is both maximally and irreducibly triangle-free. Theorem 1.2 is equivalent to the claim that all critically triangle-free binary matroids with density greater than $\frac{33}{128}$ have rank at most 6.

3. Finding a Small Quotient

In this section, we consider various restrictions that are forced in sufficiently dense critically triangle-free binary matroids, eventually proving Theorem 1.2. We start with the following result from [5]:
Theorem 3.1. If $M$ is a rank-$r$ triangle-free binary matroid with $|M| > \frac{7}{64}2^r$, then either $M$ is affine, or $M$ has a $C_5$-restriction.

The weight of a vector $v \in \text{GF}(2)^n$ refers to its Hamming weight, which we also denote by $|v|$, and the support of $v$ is the set of indices on which $v$ is one.

Note that $C_5 \cong M[J_5 - I_5]$. This representation is convenient since, despite not having full row-rank, it exhibits all of $C_5$’s symmetries.

Lemma 3.2. $C_5$ is critically triangle-free.

Proof. By Lemma 2.3, it suffices to prove that $C_5$ is maximally triangle-free. Note that the columns of $J_5 - I_5$ contain all weight-4 vectors in $\text{GF}(2)^5$, and each weight-2 vector in $\text{GF}(2)^5$ is the sum of two of the columns of $J_5 + I_5$. If $M[J_5-I_5|x]$ is a simple rank-4 binary extension of $C_5$, then $x$ has weight 2 and, hence, $M[J_5-I_5|x]$ has a triangle. □

Define the rank-5 triangle-free binary matroids $C_5^+$ and $M_9$ by

$$C_5^+ = M \begin{bmatrix} O_{1,5} & J_{1,5} \\ J_5 - I_5 & J_5 - I_5 \end{bmatrix}, \quad M_9 = M \begin{bmatrix} O_{1,5} & 1 & 1 & 1 \\ J_5 - I_5 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

Thus, $C_5^+$ is a doubling of $C_5$.

Lemma 3.3. The maximally triangle-free rank-5 binary matroids with a $C_5$-restriction are, up to isomorphism, $C_5^+$ and $M_9$.

Proof. Any such matroid has the form

$$M = M \begin{bmatrix} O_{1,5} & J_{1,t} \\ J_5 - I_5 & P \end{bmatrix}$$

for some $t \geq 1$ and some matrix $P$ whose columns have even weight. Let $S \subseteq \text{GF}(2)^5$ be the set of columns of $P$; since $M$ is triangle-free, no two vectors in $P$ sum to a vector of weight 4. By row operations using the first row, we may assume that $0 \in S$, so every column of $S - \{0\}$ has weight 2. Therefore $S - \{0\}$ is a maximal collection of weight-2 vectors in $\text{GF}(2)^5$, no two of which have disjoint support. This corresponds to a maximal set of edges of $K_5$ that contains no two-edge matching: that is, either a four-edge star or a triangle. These two cases correspond to $C_5^+$ and $M_9$ respectively. □

By Lemma 2.3, the matroid $M_9$ is critically triangle-free. However, $C_5^+$ is a doubling of $C_5$, so it is reducible.
Lemma 3.4. Let $M$ be a rank-$r$ triangle-free binary matroid. If $|M| > \frac{1}{4}2^r + 1$, then $M$ contains a $C_5^+$-restriction.

Proof. Consider $M$ as a restriction of a projective geometry $G \cong \text{PG}(r-1,2)$. By Lemma 3.1, $M$ has a restriction $N \cong C_5$. Since $C_5$ is maximal triangle-free and $|C_5| = \frac{1}{4}2^r(C_5) + 1$, we have $r(M) > 4$. We may assume that $M$ has no $C_5^+$-restriction, and, hence, by Lemma 3.3 each parallel class of $G/E(N)$ contains at most 4 elements of $M$. Therefore $|M| \leq 4(2^r - 4) + |N| \leq \frac{1}{4}2^r + 1$, contrary to the hypotheses of the lemma. \hfill \Box

Let

$$M_{13} = M \left[ \begin{array}{ccc} O_{3,5} & J_{3,5} & I_3 \\ J_5 - I_5 & J_5 - I_5 & O_{5,3} \end{array} \right].$$

For a matrix $A$ and a subset $X$ of its column-indices, we let $A|X$ denote the restriction of $A$ to the columns indexed by elements of $X$.

Lemma 3.5. If $M$ is an irreducible triangle-free binary matroid with a $C_5^+$-restriction, then $M$ has an $M_{13}$-restriction.

Proof. For any such $M$, there is some integer $t \geq 0$ and there are matrices $P, Q, R$ so that each column of $R$ has even weight, and $M = M(A)$, where

$$A = \left[ \begin{array}{ccc} O_{t,5} & O_{t,5} & P \\ O_{1,5} & J_{1,5} & Q \\ J_5 - I_5 & J_5 - I_5 & R \end{array} \right].$$

Let $X$ be the set of column-indices for the first 10 columns of $A$; thus $M|X \cong C_5^+$. Let $w \in \text{GF}(2)^{t+6}$ be the standard basis vector whose $(t+1)$-th entry is 1. Since $M$ is triangle-free, $w$ is not a column of $A$; it follows that $M' = \text{si}(M(A'))$ is a proper quotient of $M$, where $A'$ is obtained from $A$ by removing the $(t+1)$th row. By assumption $M'$ has a triangle, say $T$; thus the columns of $A'|T$ sum to zero. Since the columns of $A|T$ do not sum to zero, they sum to $w$. There is no column $v$ of $A|T$ such that $v + w$ is a column of $A$, since otherwise three columns of $A$ would sum to zero. Therefore $X \cap T = \emptyset$. Since $w$ is in the column-span of both $A|X$ and $A|T$, we have $r(X \cup T) \leq r(X) + 2$. Discarding all other columns of $[P|Q|R]^T$ and discarding redundant rows, we see that $M$ has a restriction $N$ with

$$N = M \left[ \begin{array}{ccc} O_{2,5} & O_{2,5} & P_1 \ p_2 \ p_3 \\ O_{1,5} & J_{1,5} & Q_1 \ q_2 \ q_3 \\ J_5 - I_5 & J_5 - I_5 & R_1 \ r_2 \ r_3 \end{array} \right],$$

where $p_1 + p_2 + p_3 = [0,0]$, $r_1 + r_2 + r_3 = [0,0,0,0,0]$, $q_1 + q_2 + q_3 = [1]$, and $X$ indexes the first 10 columns. Since $N|X \cong C_5^+$, each of $p_1$, $p_2$, $p_3$, $q_1$, $q_2$, $q_3$, $r_1$, $r_2$, $r_3$ has an $O$-restriction on any proper subset of the first 10 columns.
and \( p_3 \) is nonzero and, since they sum to zero, they are distinct. Since \( p_1 \neq p_2 \), we may perform row-operations with the first three rows so that \( p_1 = [1,0] \) and \( p_2 = [0,1] \), and further row-operations can then be used to set \( q_1 = q_2 = [0] \) and \( r_1 = r_2 = [0,0,0,0,0] \). Therefore \( p_3 = [1,1], q_3 = [1] \) and \( r_3 = [0,0,0,0,0] \). Now adding the third row to each of the first and the second gives a representation of \( M_{13} \). □

The next two lemmas are technical, and will be used to simplify the case analysis that follows. The first, whose proof we omit, is equivalent to the easily checked statement that every three-vertex stable set in the cube graph \( Q_8 \) lies entirely on one side of a bipartition.

**Lemma 3.6.** If \( \alpha_1, \alpha_2, \alpha_3 \in GF(2)^3 \) are distinct and \( |\alpha_i + \alpha_j| \neq 1 \) for all \( i, j \in \{1,2,3\} \), then \( |\alpha_1| \equiv |\alpha_2| \equiv |\alpha_3| \mod 2 \).

**Lemma 3.7.** Let \( \mathcal{A} = GF(2)^3 - \{[0,0,0],[1,1,1]\} \). If \( (S_\alpha : \alpha \in \mathcal{A}) \) is a collection of six subsets of \( GF(2)^3 \) so that

(a) for each \( \alpha \in \mathcal{A} \), each vector in \( S_\alpha \) has weight 2,
(b) for each \( \alpha, \beta \in \mathcal{A} \) with \( \alpha + \beta = 1 \), the sets \( S_\alpha \) and \( S_\beta \) are disjoint,
(c) for each \( \alpha \in \mathcal{A} \), no two vectors in \( S_\alpha \) have disjoint supports, and
(d) for each \( \alpha, \beta \in \mathcal{A} \) with \( \alpha + \beta = [1,1,1] \), there does not exist a vector in \( S_\alpha \) whose support is disjoint from a vector in \( S_\beta \),

then \( \sum_{\alpha \in \mathcal{A}} |S_\alpha| \leq 20 \).

**Proof.** Suppose that \( \sum_{\alpha \in \mathcal{A}} |S_\alpha| \geq 21 \). Note that the edge set of \( K_5 \) is the disjoint union of five two-edge matchings, so there is a partition of the ten weight-2 vectors in \( GF(2)^3 \) into five pairs \( \{x_i, y_i\} : 1 \leq i \leq 5 \) each with disjoint supports. By (a) and a majority argument, there exists \( k \in \{1,\ldots,5\} \) so that \( \sum_{\alpha \in \mathcal{A}} |\{x_k, y_k\} \cap S_\alpha| \geq 5 \). By (c), none of the sets \( (S_\alpha : \alpha \in \mathcal{A}) \) contains both \( x_k \) and \( y_k \), so either \( x_k \) or \( y_k \) (say \( x_k \)) is contained in at least three of the sets \( (S_\alpha : \alpha \in \mathcal{A}) \); let \( \alpha_1, \alpha_2, \alpha_3 \in \mathcal{A} \) be distinct elements with \( x_k \in S_{\alpha_1} \cap S_{\alpha_2} \cap S_{\alpha_3} \). By (b), no two of \( \alpha_1, \alpha_2, \alpha_3 \) be distinct elements with \( x_k \in S_{\alpha_1} \cap S_{\alpha_2} \cap S_{\alpha_3} \). By (b), no two of \( \alpha_1, \alpha_2, \alpha_3 \) add to a weight-1 vector; it follows from Lemma 3.6 that \( |\alpha_1| = |\alpha_2| = |\alpha_3| \) and so \( x_k \) is in exactly three of the sets \( (S_\alpha : \alpha \in \mathcal{A}) \). Therefore \( y_k \in S_\beta \) for some \( \beta \notin \{\alpha_1, \alpha_2, \alpha_3\} \). But \( [1,1,1] - \beta \in \{\alpha_1, \alpha_2, \alpha_3\} \), so \( x_k, y_k \), some \( \alpha_i \) and \( \beta \) give a contradiction to (b). □

**Lemma 3.8.** If \( M \) is a rank-7 triangle-free binary matroid with an \( M_{13} \)-restriction, then \( |M| \leq 33 \).

**Proof.** Suppose for a contradiction that \( |M| \geq 34 \). Therefore

\[
M = M \left[ \begin{array}{ccc}
O_{3,5} & J_{3,5} & I_3 \\
J_5 - I_5 & J_5 - I_5 & O_{5,3} \\
\end{array} \right] P
\]
for some matrices $P$ and $Q$ with at least 21 columns where each column of $Q$ has even weight, such that all columns of the above matrix are distinct.

For each $\alpha \in \text{GF}(2)^3$, let $S_\alpha$ be the set of vectors $x \in \text{GF}(2)^5$ for which $\left[ \begin{array}{c} \alpha \\ x \end{array} \right]$ is a column of $\left[ \begin{array}{c} P \\ Q \end{array} \right]$. Since $C^+_5$ is maximal triangle-free, the sets $S_{[0,0,0]}$ and $S_{[1,1,1]}$ are empty; let $\mathcal{A} = \text{GF}(2)^3 - \{[0,0,0],[1,1,1]\}$, and note that $\sum_{\alpha \in \mathcal{A}} |S_\alpha| \geq 21$. We argue, using the fact that no three columns sum to zero, that the sets $(S_\alpha : \alpha \in \mathcal{A})$ satisfy the conditions $(a),(b),(c)$ and $(d)$ of Lemma 3.7, giving a contradiction.

Each vector in $\left( \left[ \begin{array}{c} \alpha \\ x \end{array} \right] : \alpha \in \mathcal{A} \right)$ is the sum of at most two columns of $\left[ \begin{array}{c} I_5 \\ 0_{5,3} \end{array} \right]$, so the vector 0 is not contained in any of the sets $(S_\alpha : \alpha \in \mathcal{A})$. Consider any vector $x \in \text{GF}(2)^5$ with $|x| = 4$ and any $\alpha \in \mathcal{A}$. If $|\alpha| = 1$, then $\left[ \begin{array}{c} \alpha \\ x \end{array} \right]$ is the sum of a column of $\left[ \begin{array}{c} O_{5,5} \\ I_5 \end{array} \right]$ and a column of $\left[ \begin{array}{c} I_5 \\ 0_{5,3} \end{array} \right]$. If $|\alpha| = 2$, then $\left[ \begin{array}{c} \alpha \\ x \end{array} \right]$ is the sum of a column of $\left[ \begin{array}{c} J_{5,5} \\ I_5 \end{array} \right]$ and a column of $\left[ \begin{array}{c} I_5 \\ 0_{5,3} \end{array} \right]$. In either case $x \notin S_\alpha$ and, hence, condition 3.7(a) holds.

If $\alpha,\beta \in \mathcal{A}$ satisfy $|\alpha + \beta| = 1$ and $x \in \text{GF}(2)^5$, then $\left[ \begin{array}{c} \alpha \\ x \end{array} \right]$ and $\left[ \begin{array}{c} \beta \\ x \end{array} \right]$ sum to a column of $\left[ \begin{array}{c} I_5 \\ 0_{5,3} \end{array} \right]$. Therefore $x \notin S_\alpha \cap S_\beta$ and, hence, condition 3.7(b) holds.

Finally, let $x,y \in \text{GF}(2)^5$ with $|x+y| = 4$ and let $\alpha,\beta \in \mathcal{A}$ with $\alpha + \beta \in \{[0,0,0],[1,1,1]\}$. Then $\left[ \begin{array}{c} \alpha \\ x \end{array} \right]$ and $\left[ \begin{array}{c} \beta \\ y \end{array} \right]$ sum to a column of either $\left[ \begin{array}{c} O_{5,5} \\ J_{5,5} \end{array} \right]$ or $\left[ \begin{array}{c} J_{5,5} \\ J_{5,5} \end{array} \right]$. Thus either $x \notin S_\alpha$ or $y \notin S_\beta$. Therefore conditions 3.7(c) and 3.7(d) hold, and we have the required contradiction.

**Lemma 3.9.** If $M$ is a rank-8 triangle-free binary matroid with an $M_{13}$-restriction $M|X$, then $|M \setminus c_M(X)| \geq 33$.

**Proof.** Suppose that $|M \setminus c_M(X)| \geq 34$. Restricting $M$ to the union of $X$ and 34 elements outside $c_M(X)$, we see that $M$ has a restriction $M' = M(A)$, where

$$A = \begin{bmatrix}
O_{1,5} & O_{1,5} & O_{1,3} & J_{1,34} \\
O_{3,5} & J_{3,5} & I_3 & P \\
J_5 - I_5 & J_5 - I_5 & O_{5,3} & Q
\end{bmatrix}$$

for some matrices $P$ and $Q$, where each column of $Q$ has even weight and the 34 columns of $\left[ \begin{array}{c} P \\ Q \end{array} \right]$ are distinct. For each $\alpha \in \text{GF}(2)^3$, let $T_\alpha$ be the set of vectors $x \in \text{GF}(2)^5$ for which $\left[ \begin{array}{c} \alpha \\ x \end{array} \right]$ is a column of $\left[ \begin{array}{c} P \\ Q \end{array} \right]$.

There are 16 distinct vectors $x \in \text{GF}(2)^5$ with even weight, so there is some vector $x_0 \in \text{GF}(2)^5$ that is in at least $\binom{34}{16} = 3$ of the sets $(T_\alpha : \alpha \in \text{GF}(2)^3)$. Let $\alpha_0, \alpha_2, \alpha_3 \in \text{GF}(2)^3$ be distinct elements such that $x_0 \in T_{\alpha_1} \cap T_{\alpha_2} \cap T_{\alpha_3}$. Note that no two of $\left[ \begin{array}{c} \alpha_1 \\ x \end{array} \right], \left[ \begin{array}{c} \alpha_2 \\ x \end{array} \right], \left[ \begin{array}{c} \alpha_3 \\ x \end{array} \right]$ sum to
a column of \([I_3_{O_{5,3}}]\). Then, by Lemma 3.6 we have \(|\alpha_1| \equiv |\alpha_2| \equiv |\alpha_3| \pmod{2}\). By performing row-operations using the first row, we may assume that \([\alpha_1 \alpha_2 \alpha_3] = I_3\) and \(x_0 = 0\), so \([I_3_{O_{5,3}}]\) is a submatrix of \([P]\), giving

\[
A = \begin{bmatrix}
O_{1,5} & O_{1,5} & O_{1,3} & J_1,3 & J_{1,31} \\
O_{3,5} & J_{3,5} & I_3 & I_3 & P' \\
J_5 - I_5 & J_5 - I_5 & O_{5,3} & Q' \\
\end{bmatrix}
\]

for some matrices \(P'\) and \(Q'\), where each column of \(Q'\) has even weight.

Now, for each \(\alpha \in \text{GF}(2)^3\), let \(S_\alpha\) be the set of all vectors \(x \subseteq \text{GF}(2)^5\) such that \([\alpha]\) is a column of \([P']_{Q'}\). Let \(A = \text{GF}(2)^3 - \{[0, 0, 0, 0, [1, 1, 1]\}\). Consider \(\alpha \in A\). The vector \([\alpha_{0,1}]\) is either a column of \([I_3_{O_{5,3}}]\) or is the sum of two columns of \([I_3_{O_{5,3}}]\). Then, since \(M(A)\) is triangle-free, \(0 \notin S_\alpha\).

If \(x \in \text{GF}(2)^5\) has weight 4, then \([\alpha_3]\) sums with a column of \([O_{5,3}^3]\) to give a column of \([I_3_{O_{5,3}^3}]\) or \([J_{3,5} - I_5]\). Again, since \(M(A)\) is triangle-free, \(x \notin S_\alpha\). Therefore the sets \((S_\alpha : \alpha \in A)\) satisfy \(3.7(a)\). Moreover, since no two columns of \([P']_{Q'}\) sum to a column of \(P_{13}\), as in the previous lemma, the sets \((S_\alpha : \alpha \in A)\) satisfy conditions \(3.7(c), (d),\) and \(\Box\), so we have \(\sum_{\alpha \in A} |S_\alpha| \leq 20\). Therefore \(|S_{[0,0,0]}| + |S_{[1,1,1]}| \geq 11\), and hence, either \(|S_{[0,0,0]}| \geq 6\) or \(|S_{[1,1,1]}| \geq 6\). In either case, the first five columns of \(A\) lie in a rank-5 subspace with six other columns of \(A\), giving at least 11 elements in a rank-5 triangle-free restriction of \(M'\) that itself has a \(C_5\)-restriction. By Lemma 3.3 this is impossible. \(\Box\)

We now restate and prove Theorem 1.2.

**Theorem 3.10.** If \(M\) is a triangle-free binary matroid with \(|M| > \frac{33}{128}2^{r(M)}\), then \(M\) has a triangle-free quotient of rank at most 6.

**Proof.** Consider a counterexample \(M\). We may assume that \(r(M) \geq 7\) and that \(M\) is maximally triangle-free and irreducible. By Lemmas 3.4 and 3.5 \(M\) has a restriction \(N \cong M_{13}\). Let \(G \cong \text{PG}(r - 1, 2)\) containing \(M\) as a restriction and let \(F = \text{cl}_M(E(N))\).

By Lemma 3.8 we have \(|F| \leq 33\), and, by Lemma 3.9, each parallel class of \(G/F\) contains at most 33 elements of \(M\). Therefore,

\[|M| \leq |F| + 33(2^{r(M/F)} - 1) \leq \frac{33}{128}2^{r(M)},\]

a contradiction. \(\Box\)
4. Low Rank

Henceforth, we denote $U_{1,1}$ by $M_1$. By Lemma 2.3 each of $M_1$, $C_5$, and $M_9$ is irreducibly triangle-free. There are 5 irreducibly triangle-free 17-element binary matroids with rank 6; these are the matroids $M_{17}^1, \ldots , M_{17}^5$ defined in the appendix. We will prove, by case analysis, that $M_1$, $C_5$, $M_9$, or $M_{17}^1, \ldots , M_{17}^5$ are the only irreducible triangle-free binary matroids of rank $r \leq 6$ with $|M| > \frac{1}{4}2^r$; see Lemma 4.3.

Combining this fact with Theorem 1.2 we get the following refinement:

**Theorem 4.1.** If $M$ is a simple binary matroid with $|M| > \frac{33}{128}2^r(M)$, then $M$ is triangle-free if and only if $M$ has a quotient isomorphic to one of $M_1, C_5, M_9$ or $M_{17}^1, \ldots , M_{17}^5$.

Since the density of a binary matroid $M$ cannot exceed that of a quotient of $M$, and the matroids $C_5$, $M_9$ and $M_{17}^i$ have densities $\frac{5}{16}$, $\frac{9}{32}$, and $\frac{17}{64}$ respectively, Theorem 4.1 implies a hierarchy of results for densities greater than $\frac{33}{128}$. The first of these results is implied by the aforementioned result of Govaerts and Storme (Theorem 1.7).

**Corollary 4.2.** Let $M$ be a simple rank-$r$ binary matroid.

- If $|M| > \frac{5}{16}2^r$, then $M$ is triangle-free if and only if $M_1$ is a quotient of $M$ (that is, $M$ is affine).
- If $|M| > \frac{9}{32}2^r$, then $M$ is triangle-free if and only if $M_1$ or $C_5$ is a quotient of $M$.
- If $|M| > \frac{17}{64}2^r$, then $M$ is triangle-free if and only if $M_1$, $C_5$ or $M_9$ is a quotient of $M$.

It remains to prove the following result:

**Lemma 4.3.** If $M$ is a critically triangle-free binary matroid of rank $r \leq 6$ with $|M| > \frac{1}{4}2^r$, then $M$ is isomorphic to one of $M_1, C_5, M_9$, or $M_{17}^1, \ldots , M_{17}^5$.

**Proof.** Since $|M| > \frac{1}{4}2^r > \frac{7}{64}2^r$, Theorem 3.1 implies that either $M$ is affine (in which case $M_1$ is a quotient of $M$, so $M \cong M_1$ by irreducibility), or $M$ contains $C_5$, in which case $r \geq 4$. Let $M|C \cong C_5$. If $r = 4$, then $M \cong C_5$, since $C_5$ is maximally triangle-free. If $r = 5$ then $|M| \geq 9$, so by Lemma 3.3 there is a rank-5 matroid $M'$, having $M$ as a restriction, that is isomorphic either to $C_5^+$ (in which case $C_5$ is a quotient of $M$, contradicting irreducibility), or to $M_9$, in which case $M = M' \cong M_9$.

It remains to consider $r = 6$; note that $|M| \geq 17$ in this case. Since $M|C$ is maximally triangle-free, and each rank-4 flat of PG(4,2) is contained in exactly three rank-5 flats, there are restrictions $N_1, N_2, N_3$.
of $M$ of rank at most 5 so that $E(M)$ is the disjoint union of $E(N_1) - C$, $E(N_2) - C, E(N_3) - C$ and $C$. Now $17 \leq |M| = 5 + \sum_{i=1}^{3} |E(N_i) - C|$, so $\sum_{i=1}^{3} |E(N_i) - C| \geq 12$.

If $|E(N_i) - C| \geq 5$ for some $i$ (say $i = 1$) then $N_1$ is a triangle-free matroid with a $C_5$-restriction and with at least 10 elements, so $N_1 \cong C_5^+$ by Lemma 3.3 and $|E(N_1) - C| = 5$. By fixing a representation of $N_1$ in the first ten columns, we have $M = M[A]$, where

$$A = M \begin{bmatrix} O_{2,5} & J_{2,5} & P \end{bmatrix};$$

here, $[P]_Q$ has at least seven columns, and the columns of $Q$ have even weight. Since $N_1$ is maximally triangle-free, each column of $P$ is either $[0, 1]$ or $[1, 0]$; for $\alpha \in \{[0, 1], [1, 0]\}$ let $S_\alpha$ be the set of all $v \in \text{GF}(2)^5$ for which $[v]_Q$ is a column of $[P]_Q$. By row-operations we may assume that $|S_{[1, 0]}| \geq |S_{[0, 1]}|$ and so $|S_{[1, 0]}| \geq 4$. If $S_{[0, 1]} \subseteq S_{[1, 0]}$ then note that the matroid

$$N' = M \begin{bmatrix} O_{1,5} & J_{1,5} \end{bmatrix},$$

is a triangle-free proper quotient of $M$, where $Q'$ is the matrix with column set $S_{[1, 0]}$, and $t = |S_{[1, 0]}|$. ($N'$ is triangle-free because no vector in $S_{[1, 0]} + S_{[1, 0]}$ has weight 4, and $N'$ is a quotient of $M$ because the vector $v = [1, 1, 0, 0, 0, 0, 0]$ is not a column of $[P]_Q$: $N'$ is obtained from $M$ by appending $v$ to the above matrix, contracting the extension element, and simplifying). This contradicts irreducibility, so we may assume that $S_{[0, 1]} \setminus S_{[1, 0]} \neq \emptyset$. By row-operations we can further assume that $S_{[0, 1]} \setminus S_{[1, 0]}$ contains the zero vector. Since no vector in $S_{[1, 0]} + S_{[0, 1]}$ has weight 4, it follows that each vector in $S_{[1, 0]}$ has weight 2. Since there is no vector of weight 4 in $S_{[1, 0]} + S_{[1, 0]}$ and any set of four edges in $K_5$ containing no two-edge matching is a star, we have $S_{[1, 0]} = \{[1, 1, 0, 0, 0], [1, 0, 1, 0, 0], [1, 0, 0, 1, 0], [1, 0, 0, 0, 1]\}$ after a permutation of co-ordinates. Since $[0, 0, 0, 0, 0] \not\in S_{[1, 0]}$ and $M$ is maximally triangle-free, there must be two columns of $A$ that sum to $[1, 0, 0, 0, 0, 0, 0]$ and therefore $S_{[0, 1]}$ contains a vector $w$ of weight 4. For any such $w$, there is some $x \in S_{[0, 1]}$ so that $|w + x| = 4$, so $w + x$ is a column of $[J_{2,5}]_{I_5-I_5}$, and thus $M$ contains a triangle, a contradiction.

We therefore have $|E(N_i) - C| \leq 4$ for each $i \in \{1, 2, 3\}$, so $|M| = 17$ and $|H_i - C| = 4$ for each $i$. Each $N_i$ is thus a 9-element triangle-free matroid containing $C$ and with rank at most 5; by Lemma 3.3 each $N_i$ is either isomorphic to $G_9$ or is a restriction of $C_5^+$. The matroid $C_5^+$ only contains one nine-element restriction up to isomorphism. We
consider two cases, in each of which we show that $M$ is represented by one of six possible matrices.

**Case 1:** Some $N_i$ is contained in $C^+_5$ as a restriction.

Suppose that $N_1$, say, is contained in $C^+_5$. By fixing a representation of $N_1$ in the first nine columns with $C$ in the first five, we have

$$M = M \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
J_5 - I_5 & J_{1,4} & I_4 & Q_{01} & Q_{10}
\end{bmatrix},$$

for some $5 \times 4$ binary matrices $Q_{01}$ and $Q_{10}$ whose columns have even weight, where $H_2 - C$ and $H_3 - C$ are represented by the last two blocks of columns. Since $M$ is not a restriction of a larger triangle-free rank-6 matroid, the vector $[1, 1, 0, 0, 0, 0]$ is the sum of two columns of the above matrix; so $Q_{01}$ and $Q_{10}$ have a column in common. By row-operations, we may assume that this is the zero column. Setting $S_{01}$ and $S_{10}$ to be the sets containing the three nonzero columns of $Q_{01}$ and $Q_{10}$ respectively, the fact that $M$ is triangle-free gives $|x| = 2$ for all $x \in S_{01} \cup S_{10}$, as well as $|x| \in \{0, 2\}$ for all $x \in S_{01} + S_{01}$ and all $x \in S_{10} + S_{10}$. Moreover, the set $S_{01} + S_{10}$ contains no column of $[J_{1,4}]$. A routine case analysis (for which it is convenient to think of the elements of $S_{01}$ and $S_{10}$ as edges of $K_5$) gives six possibilities for $(Q_{01}, Q_{10})$ up to permutation of the last four rows of the matrix above.

**Case 2:** $N_i \cong M_9$ for all $i \in \{1, 2, 3\}$.

Fixing a representation of $N_1 \cong M_9$ in the first nine columns with $C$ in the first five, we have

$$M = M \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
J_5 - I_5 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},$$

for some $5 \times 4$ binary matrices $Q_{01}$ and $Q_{10}$ whose columns have even weight. Since $H_2 \cong H_3 \cong M_9$, the columns of each of $Q_{01}$ and $Q_{10}$ sum to zero. Let $S_{01}$ and $S_{10}$ be their respective column sets, and define $S_{11} \subseteq \text{GF}(2)^5$ analogously. By row-operations, we may assume that $0 \in S_{01}$; it follows that $|x| \in \{0, 2\}$ for all $x \in Q_{01}$. Given that columns of $Q_{01}$ sum to zero, this gives three possibilities for $S_{01}$ up to permutation of the last five rows of the matrix (thinking of the three
nonzero elements of $S_{11}$ as the three edges of a fixed triangle $T$ of the graph $K_5$, the nonzero vectors in $S_{01}$ form another triangle of $K_5$ which shares either one, two or three vertices with $T$). Using the fact that the columns of $Q_{10}$ sum to zero and that $(S_{10} + S_{01}) \cap S_{11} = \emptyset$ and $|x| \neq 4$ for all $x \in S_{10} + S_{10}$, these three possibilities together give rise to six candidates in total for $(Q_{01}, Q_{10})$.

Combining the six possibilities in the first case with the six in the second, it follows that $M$ is isomorphic to one of at most twelve distinct binary matroids. A computer isomorphism test, performed using the SageMath software, shows that there are only five nonisomorphic matroids among these twelve; they are the matroids $M_{i17}$ defined in the appendix. (The representations in the appendix have a somewhat different form from those considered in this proof; this is to highlight other structural aspects of the $M_{i17}$.)

\[ \Box \]

5. Rank-seven examples

The inequality $|M| > \frac{33}{128} 2^r(M)$ in Theorem 1.2 cannot be relaxed, since there exist critically triangle-free binary matroids of rank 7 with 33 elements. Nearly affine matroids of rank 7 are triangle-free and have the appropriate number of elements, but they are not necessarily critically triangle-free. By Lemma 2.3 and Theorem 1.2 these examples are critically triangle-free if and only if they are maximally triangle-free. We performed a computer search of all rank-7 nearly affine binary matroids with rank 7 and found 44 pairwise non-isomorphic examples, 36 of which are critical. (The numbers may in fact be larger than this since we did not check isomorphism exhaustively; we used a matroid invariant.)

The matroids $M_1$, $C_5$, $M_9$, and $M_{17}^1, \ldots, M_{17}^5$ are all nearly affine. However, we know of two critically triangle-free binary matroids of rank-7 with 33 elements that are not nearly affine; these are the matroids $M_{33}^1$ and $M_{33}^2$ defined in the appendix. These matroids both contain a $C_5^+$-restriction which precludes them from being nearly affine.

We conclude with the following conjecture which propounds the best outcome we could hope for in our current state of ignorance.

**Conjecture 5.1.** If $M$ is a rank-$r$ critically triangle-free binary matroid, then $|M| \leq \frac{1}{7} 2^r + 1$. 
6. Appendix

For $i \in \{1, 2, 3, 4, 5\}$, define matrices $A^i_{17}$ by

\[
A^1_{17} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix},
\]

\[
A^2_{17} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix},
\]

\[
A^3_{17} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
A^4_{17} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
A^5_{17} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix},
\]

and let $M^i_{17} = M(A^i_{17})$. Note for each $i$ that $M^i_{17}$ contains $C_5$ represented by its first five columns (so $\chi(M^i_{17}) \geq 2$), and is nearly affine, so $\chi(M^i_{17}) = 2$. 
Let $M_{33}^1$ and $M_{33}^2$ be the binary matroids represented by the following two matrices:

$$
\begin{array}{cccccccccc}
O_{3,5} & J_{3,5} & I_3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
& & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
& & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
J_5 - I_5 & J_5 - I_5 & O_{5,3} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
& & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
& & & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
& & & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
O_{3,5} & J_{3,5} & I_3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
& & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
& & & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
J_5 - I_5 & J_5 - I_5 & O_{5,3} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
& & & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
& & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
& & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
$$

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