Corrections of order $O(E_e^2/m_N^2)$, caused by weak magnetism and proton recoil, to the neutron lifetime and correlation coefficients of the neutron beta decay

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We calculate the contributions of weak magnetism and proton recoil of order $O(E_e^2/m_N^2) \sim 10^{-5}$, i.e. to next-to-next-to-leading order in the large nucleon mass expansion, to the neutron lifetime and correlation coefficients of the neutron beta decay, where $E_e$ and $m_N$ are the electron energy and the nucleon mass, respectively. We analyze the electron-energy and angular distribution for the neutron beta decay with a polarized neutron, a polarized electron and an unpolarized proton. Together with Wilkinson’s corrections (Nucl. Phys. A 377, 474 (1982) and radiative corrections of order $O(\alpha E_e/m_N) \sim 10^{-5}$ (Phys. Rev. D 99, 093006 (2019)), calculated as next-to-leading order corrections in the large nucleon mass $m_N$ expansion to Sirlin’s corrections of order $O(\alpha/\pi)$ (Phys. Rev. 164, 1767 (1967)), the corrections of order $O(E_e^2/m_N^2) \sim 10^{-5}$ provide an improved level of precision of the theoretical background of the neutron beta decay, calculated in the Standard Model, for experimental searches of contributions of interactions beyond the Standard Model.

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I. INTRODUCTION

Nowadays experimental investigations of the neutron beta decay [1, 2] with a polarized neutron and unpolarized electron and proton are allowed at the level of sensitivity of about $10^{-7}$ and even better [3, 4]. An analogous sensitivity is desirable also for experimental investigations of the neutron beta decay with a polarized neutron, a polarized electron and an unpolarized proton [5]. The model-independent corrections of order of $10^{-3}$, caused by the radiative corrections of order $O(\alpha/\pi)$ (so-called outer radiative corrections [10]), where $\alpha$ is the fine-structure constant [6], calculated to leading order in the large nucleon mass $m_N$ expansion, together with corrections of order $O(E_e/m_N)$, induced by weak magnetism and proton recoil, provide a robust theoretical background for the experimental analysis of the neutron beta decay at the level of $10^{-3}$. The notation “large nucleon mass $m_N$ expansion” means that we make an expansion in powers of $1/m_N$, were the nucleon mass is finite but much larger than the momentum of the proton and energies and momenta of decay leptons. The effect of the finite nucleon mass appears only to next-to-leading and higher orders in the large nucleon mass $m_N$ expansion or in the terms proportional to powers of $1/m_N$. For the neutron beta decay with a polarized neutron and unpolarized electron and proton such a theoretical background, including the radiative corrections of order $O(\alpha/\pi)$ and the corrections $O(E_e/m_N)$, caused by weak magnetism and proton recoil, were calculated in [11, 12] (see also [13, 14]). An universal inner radiative correction $\Delta R_{\alpha/\pi}$ of order $O(\alpha/\pi)$, which does not depend on the electron energy and plays an important role for the correct theoretical description of the neutron lifetime $\tau_n = 879.6(1.1)$ s (see [15] and [16]), was calculated in [19, 20]. In turn, for the neutron beta decay with a polarized neutron, a polarized electron and an unpolarized proton the model-independent outer radiative corrections of order $O(\alpha/\pi)$ and corrections of order $O(E_e/m_N)$, caused by weak magnetism and proton recoil, were calculated in [22, 23]. In order to promote an impetus for an improvement of experimental sensitivities from the level of a few parts of $10^{-4}$ [3, 5, 6] to the level of a few parts of $10^{-5}$ or even better the neutron lifetime and the correlation coefficients of the neutron beta decay should be calculated in the Standard Model (SM) at the level of a few parts of
The step in this direction was done by Wilkinson [15] (see also [18, 23, 28] for so-called Wilkinson’s corrections) and then in [24, 33], where the radiative corrections of order $O(\alpha E_c/m_N) \sim 10^{-5}$ were calculated as next-to-leading order corrections in the large nucleon mass expansion to Sirlin’s outer and inner radiative corrections of order $O(\alpha/\pi)$.

This paper is addressed to the calculation of the model–independent corrections of order $O(E^2_c/m_N^2) \sim 10^{-5}$, caused by weak magnetism and proton recoil. Together with Wilkinson’s corrections [15] (see also [18, 23, 28]) and radiative corrections of order $O(\alpha E_c/m_N) \sim 10^{-5}$ [24, 33], the corrections $O(E^2_c/m_N^2) \sim 10^{-4}$ should provide an improved level of precision of the theoretical background of the neutron beta decay, calculated in the SM. It is initiated to promote experimental searches of contributions of interactions beyond the SM [8, 31–33] with experimental sensitivities of about a few parts of $10^{-5}$.

The paper is organized as follows. In section II, we give the electron–energy and angular distribution of the neutron beta decay for a polarized neutron, a polarized electron, and an unpolarized proton with the account for the corrections of order $O(E^2_c/m_N^2)$. We define the structure of the corrections of order $O(E^2_c/m_N^2)$ to the neutron lifetime and to the correlation coefficients. In section III, we give the analytical expressions for the corrections of order $O(E^2_c/m_N^2)$ to the neutron lifetime and correlation coefficients. We give the corrections $O(E^2_c/m_N^2)$ in the form of polynomials in the variable $E_s/E_0$, where $E_0$ is the end–point energy of the electron–energy spectrum [14, 21]. The coefficients of these polynomials are calculated at the neglect of contributions of order of a few parts of $10^{-6}$ and $10^{-4}$ only. Such a form of corrections under consideration makes them applicable for the analysis of experimental data on asymmetries of the neutron beta decay for the searches of contributions of interactions beyond the SM at the level of $10^{-5}$ [18, 23, 27] or even better at the level of a few parts of $10^{-5}$ [24, 33, 57]. In section IV, we discuss the obtained results. In Appendix A, we calculate the amplitude of the neutron beta decay by taking into account the next-to-next-to-leading order corrections $O(E^2_c/m_N^2)$, caused by weak magnetism and proton recoil. In addition, we take into account the contributions of the isovector and axial–vector form factors of the nucleon defined in the dipole approximation. In Appendix B, we give the contributions of order $O(E_s/m_N)$ and $O(E^2_c/m_N^2)$ to the electron–energy and angular distribution of the neutron beta decay with correlation structures, which cannot be reduced to the correlation structure proposed by Jackson et al. [38]. In Appendix C, we adduce the corrections of order $O(E_c/m_N)$ to the neutron lifetime and correlation coefficients, calculated in [18, 23, 27]. We adduce these corrections for the completeness of the analysis of the corrections of order $O(E^2_c/m_N^2)$ to the neutron lifetime and the correlation coefficients, since they give contributions of order of $10^{-5}$ to the corrections $O(E^2_c/m_N^2)$.

II. ELECTRON–ENERGY AND ANGULAR DISTRIBUTION OF THE NEUTRON BETA DECAY WITH A POLARIZED NEUTRON, A POLARIZED ELECTRON AND AN UNPOLARIZED PROTON

For the calculation of the model–independent corrections of order $O(E^2_c/m_N^2)$, induced by weak magnetism and proton recoil, to the neutron lifetime and the correlation coefficients of the neutron beta decay we use the standard effective Lagrangian of the $V – A$ weak interaction with a real axial coupling constant $g_A$ and the contribution of weak magnetism

$$L_W(x) = -\frac{G_F}{\sqrt{2}} V_{ud} \left\{ (\bar{\psi}_p(x)\gamma_\mu(1 - g_A\gamma_5)^5\psi_n(x)) + \frac{\kappa}{2m_N} \partial^\nu [\bar{\psi}_p(x)\gamma_\mu\gamma_\nu\psi_n(x)] \right\} [\bar{\psi}_e(x)\gamma^\nu(1 - \gamma^5)\psi_\nu(x)]$$

invariant under time reversal, where $G_F$ and $V_{ud}$ are the Fermi weak coupling constant and the Cabibbo–Kobayashi–Maskawa (CKM) matrix element [2], $\psi_p(x), \psi_n(x), \psi_e(x)$ and $\psi_\nu(x)$ are the field operators of the proton, neutron, electron and antineutrino, respectively, $\gamma^\mu$, $\gamma^5$ and $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ are the Dirac matrices [39] and $\kappa = \kappa_p - \kappa_n = 3.7059$ is the isovector anomalous magnetic moment of the nucleon, defined by the anomalous magnetic moments of the proton $\kappa_p = 1.7929$ and the neutron $\kappa_n = -1.9130$ and measured in nuclear magneton [38], and $m_N = (m_n + m_p)/2 = 938.9188$ MeV is the nucleon mass and $m_n = 939.5645$ MeV and $m_p = 938.2721$ MeV are the neutron and proton masses [3], respectively. Then, $G_F$ and $V_{ud}$ are the Fermi weak coupling constant and the Cabibbo–Kobayashi–Maskawa (CKM) matrix element [44].

The electron–energy and angular distribution of the neutron beta decays for a polarized neutron, a polarized electron, and an unpolarized proton was given by Jackson et al. [38]. Following [17, 18, 23, 27] it can be written in the following form (see also [27] with a replacement $\lambda = -g_A$)

$$\frac{d^3\lambda_N(E_c, \vec{E}_c, \vec{E}_\nu, \vec{E}_\nu, \vec{E}_\nu)}{dE_c d\Omega_c d\Omega_\nu} = (1 + 3g_A^2) \frac{G_F^2 |V_{ud}|^2}{32\pi^5} (E_0 - E_c)^2 \sqrt{E_c^2 - m_N^2} E_c F(E_c, Z = 1) \psi_c(E_c) \left\{ 1 + b(E_c) \frac{me}{E_c} \right\}$$

where $E_c$ is the electron energy, $\vec{E}_c$ is the electron momentum, $\vec{E}_\nu$ is the antineutrino momentum, $\psi_c$ is the wave function of the neutron at $E_c$, $b(E_c)$ is a function of the electron energy $E_c$.
where we have followed the notation \( \alpha = 18, 23, 27, 37 \). Then, \( \vec{\xi}_n \) and \( \vec{\xi}_e \) are unit vectors of spin–polarization of the neutron and electron \( 23, 24 \) (see also \( 29 \)), respectively, \( (\vec{e}_e, \vec{k}_e) \) and \( (\vec{e}_\nu, \vec{k}_\nu) \) are the energies and 3–momenta of the electron and antineutrino, respectively, \( d\Omega_e \) and \( d\Omega_\nu \) are infinitesimal solid angles in the directions of electron \( \vec{k}_e \) and antineutrino \( \vec{k}_\nu \) 3–momenta, respectively, \( E_0 = (m_n^2 - m_\pi^2 + m_e^2)/(2m_n) = 1.2926 \text{MeV} \) is the end–point energy of the electron–energy spectrum \( 18 \), \( F(E_e, Z = 1) \) is the relativistic Fermi function equal to (see, for example, \( 12 \) and a discussion in \( 23 \))

\[
F(E_e, Z = 1) = \left(1 + \frac{1}{2}\right) \frac{4(2r_p m_e \beta)^{2\gamma}}{T^2(3 + 2\gamma)} e^{\frac{\alpha}{T(1 - \beta^2)}} \left|1 + \frac{\alpha}{T(1 - \beta^2)}\right|^2,
\]

where \( \beta = k_e/E_e = \sqrt{E_e^2 - m_e^2}/E_e \) is the electron velocity, \( \gamma = \sqrt{1 - \alpha^2} - 1 \), \( r_p \) is the electric radius of the proton. In the numerical calculations we use \( r_p = 0.841 \text{ fm} \) \( 10, 11 \). The correlation function \( \langle \xi(E) \rangle \), responsible for the correct value of the neutron lifetime, and the correlation coefficients \( X(E) \), where \( X = \text{electron} – \text{antineutrino correlation}, \) \( a, \text{electron asymmetry}, \) \( A, \text{antineutrino asymmetry}, \) \( B, \) and so on, have the following structure

\[
\langle \xi(E) \rangle = \langle \xi(E) \rangle_{(LO)} + \langle \xi(E) \rangle_{(NLO)} + \langle \xi(E) \rangle_{(N^2 LO)},
\]

\[
X(E) = X(E)_{(LO)} + X(E)_{(NLO)} + X(E)_{(N^2 LO)},
\]

where the abbreviations \( \text{LO}, \text{NLO} \) and \( \text{N}^2 \text{LO} \) mean “leading-order”, “next-to-leading-order” and “next-to-next-to-leading-order” in the large nucleon mass \( m_N \) expansion, respectively. With respect to an expansion in powers of \( 1/m_N \) these terms are of order \( O(1), O(1/m_N) \) and \( O(1/m_N^2) \), respectively.

The correlation function \( \langle \xi(E) \rangle_{(LO)} \) and correlation coefficients \( X(E)_{(LO)} \), calculated to leading order in the large nucleon mass \( m_N \) expansion, contain also the contributions of radiative corrections of order \( O(\alpha/\pi) \). A complete set of corrections of order \( 10^{-3} \), defined by radiative corrections of order \( O(\alpha/\pi) \) and corrections of order \( O(E_e/m_N) \), caused by weak magnetism and proton recoil, which we denote as \( \langle \xi(E) \rangle_{(NLO)} \) and \( X(E)_{(NLO)} \), to the neutron lifetime and to the correlation coefficients with correlation structure independent of the electron spin \( \vec{\xi} \) were calculated in \( 11, 12, 14, 15, 19, 24 \) (see also \( 16, 18 \)). In turn, a complete set of corrections of order of \( 10^{-3} \), including the radiative corrections of order \( O(\alpha/\pi) \) and the weak magnetism and proton recoil corrections of order \( O(E_e/m_N) \), by the correlation coefficients, caused by correlations with the electron spin \( \vec{\xi}_e \), have been calculated in \( 23, 24 \). The radiative corrections of order \( O(E_e/m_N) \), which can be also added to \( \langle \xi(E) \rangle_{(NLO)} \) and \( X(E)_{(NLO)} \), have been recently calculated in \( 29, 30 \). The correlation coefficient \( b(E_e) \) is the Fierz interference term \( 42 \). The structure and the value of the Fierz interference term may depend on interactions beyond the SM. A recent information of the theoretical and experimental status of the Fierz interference term can be found in \( 33, 34, 43, 45 \).

The last two terms in Eq. (2) determine the contributions to the electron–energy and angular distributions of order \( O(E_e/m_N) \) and \( O(E_e^2/m_N^2) \), respectively, induced by weak magnetism and proton recoil, with the correlation structure, which cannot be reduced to the correlation structures by Jackson et al. \( 38 \). We give the analytical expressions of these terms in Appendix B.

Below we propose a detailed analysis of the structure and calculation of the corrections \( \langle \xi(E) \rangle_{(N^2 LO)} \) and \( X(E)_{(N^2 LO)} \) to the neutron lifetime and the correlation coefficients of the neutron beta decay, respectively. For the correlation \( \langle \xi(E) \rangle_{(N^2 LO)} \) we define the following structure

\[
\langle \xi(E) \rangle_{(N^2 LO)} = \langle \xi^{(1)}(E) \rangle_{(N^2 LO)} + \langle \xi^{(2)}(E) \rangle_{(N^2 LO)} + \langle \xi^{(3)}(E) \rangle_{(N^2 LO)} + \langle \xi^{(4)}(E) \rangle_{(N^2 LO)}.
\]

where i) \( \langle \xi^{(1)}(E) \rangle_{(N^2 LO)} \) is induced by \( \delta M_n \) in the amplitude of the neutron beta decay (see Eq. (A-14) and Eq. (A-15)), ii) \( \langle \xi^{(2)}(E) \rangle_{(N^2 LO)} \) is caused by corrections of order \( O(E_e^2/m_N^2) \) to the phase–volume of the neutron beta decay (see
Eq. (A-21); iii) $\zeta^{(3)}(E_c)(N^2\text{LO})$ is the correction, obtained by the quadratic and crossing terms of order $O(E_c/m_N)$ in the amplitude of the neutron beta decay Eq. (A-14) without $\delta M_n$, and iv) $\zeta^{(4)}(E_c)(N^2\text{LO})$ is determined by the corrections of order $O(E_c/m_N)$ to the electron–energy and angular distribution of the neutron beta decay with the account for the contributions of order $O(E_c/m_N)$ from the phase–volume of the neutron beta decay (see Eq. (A-21)).

In order to define the structure of the corrections $X(E_c)(N^2\text{LO})$ to the correlation coefficients $X(E_c) = a(E_c), A(E_c)$ and so on we remind that the correlation coefficients $X(E_c)$ appear in the electron–energy and angular distribution of the neutron beta decay in the form (see, for example, [18])

$$
\zeta(E_c)X(E_c) = \bar{X}(E_c)(\text{LO}) + \tilde{X}(E_c)(\text{NLO}) + \bar{X}(E_c)(N^2\text{LO}),
$$

(6)

where we have added the term $\bar{X}(E_c)(N^2\text{LO})$, which we calculate in this paper, to well–known first two terms $\bar{X}(E_c)(\text{LO})$ and $\tilde{X}(E_c)(\text{NLO})$ (see [1, 12, 14, 17, 18, 25–27]). All terms in the right-hand-side (r.h.s.) of Eq. (6) are calculated from $|M_n|^2$ (see, for example, [18, 25–27]), where $M_n$ is given in Eq. (A-14), for a polarized neutron, a polarized electron and an unpolarized proton by taking into account the contributions of the phase–volume of the neutron beta decay (see Eq. (A-21)) at the neglect of the contributions proportional to $1/m_N^2$ and $1/m_N^4$. In this case the corrections $\bar{X}(E_c)(N^2\text{LO})$ have the structure of the correlation function $\zeta(E_c)(N^2\text{LO})$ (see Eq. (A-1))

$$
\bar{X}(E_c)(N^2\text{LO}) = \bar{X}^{(1)}(E_c)(N^2\text{LO}) + \bar{X}^{(2)}(E_c)(N^2\text{LO}) + \bar{X}^{(3)}(E_c)(N^2\text{LO}) + \bar{X}^{(4)}(E_c)(N^2\text{LO}).
$$

(7)

At the neglect of the contributions of the radiative corrections of order $O(\alpha/\pi)$ and $O(\alpha E_c/m_N)$ the correlation function $\zeta(E_c)$ has the structure

$$
\zeta(E_c) = 1 + (\zeta(E_c)(\text{LO}) + \zeta(E_c)(N^2\text{LO})
$$

(8)

with respect to the large nucleon mass $m_N$ expansion, where we have added the term $\zeta(E_c)(N^2\text{LO})$ to the first two terms (see Eq. (7) of Ref. [18]). Taking into (6) we define the correlation coefficients $X(E_c)$ as follows

$$
X(E_c) = \frac{\bar{X}(E_c)(\text{LO}) + \tilde{X}(E_c)(\text{NLO}) + \bar{X}(E_c)(N^2\text{LO})}{1 + \zeta(E_c)(\text{LO}) + \zeta(E_c)(N^2\text{LO})}
$$

(9)

Expanding the r.h.s. of Eq. (9) in powers of $1/m_N$ and restricting such an expansion by the contributions proportional to $1/m_N^2$ we get

$$
X(E_c) = X(E_c)(\text{LO}) + X(E_c)(\text{NLO}) + X(E_c)(N^2\text{LO}),
$$

(10)

where $X(E_c)(\text{LO}) = \bar{X}(E_c)(\text{LO})$ and $X(E_c)(\text{NLO}) = \tilde{X}(E_c)(\text{NLO}) - \bar{X}(E_c)(\text{LO})\zeta(E_c)(\text{NLO})$. The exact expressions for $X(E_c)(\text{LO})$ and $X(E_c)(\text{NLO})$ have been calculated in [14, 17, 18, 25, 27]. The correction $X(E_c)(N^2\text{LO})$ has the structure

$$
X(E_c)(N^2\text{LO}) = \bar{X}(E_c)(N^2\text{LO}) - \bar{X}(E_c)(\text{LO})\zeta(E_c)(N^2\text{LO}) + \bar{X}(E_c)(\text{LO})^2 \zeta(E_c)(\text{LO}) - \bar{X}(E_c)(\text{NLO})\zeta(E_c)(\text{NLO}).
$$

(11)

For the calculation of the last two terms we use the results for the corrections $\zeta(E_c)(\text{LO})$ and $\bar{X}(E_c)(\text{NLO})$, obtained in [18, 25, 27], which we adduce in Appendix C for completeness of our analysis.

III. ANALYTICAL EXPRESSIONS FOR COMPLETE SET OF CORRECTIONS DEFINING $X(E_c)(N^2\text{LO})$

Following [18, 25, 27, 27] we calculate the complete set of corrections defining the total corrections $\zeta(E_c)(N^2\text{LO})$ and $X(E_c)(N^2\text{LO})$ to the correlation function $\zeta(E_c)$ and correlation coefficients $X(E_c)$ for $X(E_c) = a(E_c), A(E_c)$ and so on. We would like to notice that all corrections of order $O(E_c^2/m_N^2)$ to the correlation coefficients $D(E_c), R(E_c)$ and $L(E_c)$ are either equal to zero or much smaller than $10^{-5}$. The corrections to the Fierz interference term $b(E_c)$ we do not consider, since up to now its experimental status is not well defined [32, 33, 42, 43].

1. Corrections $\zeta^{(1)}(E_c)(N^2\text{LO})$ and $\bar{X}^{(1)}(E_c)(N^2\text{LO})$, induced by the terms of order $O(E_c^2/m_N^2)$ in the amplitude of the neutron beta decay Eq. (A-15) and Eq. (A-11)

Using the standard procedure for the calculation of the electron–energy and angular distribution of the neutron beta decay (see, for example, Appendix A in [18]) we obtain the following analytical expressions for the corrections
\[ \zeta^{(1)}(E_e)(N^{2\text{LO}}) \] and \[ \bar{X}^{(1)}(E_e)(N^{2\text{LO}}) : \]

\[ \zeta^{(1)}(E_e)(N^{2\text{LO}}) = \frac{1}{1 + 3g_A^2 4m_N^2} \left\{ \left( -6g_A^2 + 4\kappa + 2 \right) \left( 1 - \frac{m_e^2}{E_e^2} \right) \frac{E_e}{E_0} \right\} \]

\[ \tilde{a}^{(1)}(E_e)(N^{2\text{LO}}) = \frac{1}{1 + 3g_A^2 4m_N^2} \left\{ \left( 8g_A^2 E_e \frac{E_e}{E_0} \right) \left( 1 - \frac{E_e}{E_0} \right) \right\}, \]

\[ \bar{A}^{(1)}(E_e)(N^{2\text{LO}}) = \frac{1}{1 + 3g_A^2 4m_N^2} \left\{ -4g_A(g_A + \kappa) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) + 2g_A(g_A + \kappa + 1) \frac{m_e^2}{E_0^2} \right\}, \]

\[ \bar{B}^{(1)}(E_e)(N^{2\text{LO}}) = \frac{1}{1 + 3g_A^2 4m_N^2} \left\{ -4g_A(g_A - \kappa) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) - 2g_A \frac{m_e}{E_0} \frac{m_e}{E_e} + 2g_A(g_A + \kappa + 1) \frac{m_e^2}{E_0^2} \right\}, \]

\[ \bar{K}^{(1)}(E_e)(N^{2\text{LO}}) = \frac{1}{1 + 3g_A^2 4m_N^2} \left\{ \left( -4g_A(g_A + \kappa) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) \right\}, \]

\[ \bar{Q}^{(1)}(E_e)(N^{2\text{LO}}) = \frac{1}{1 + 3g_A^2 4m_N^2} \left\{ 4g_A(g_A - \kappa) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) \right\}, \]

\[ \bar{G}^{(1)}(E_e)(N^{2\text{LO}}) = \frac{1}{1 + 3g_A^2 4m_N^2} \left\{ 2g_A^2 (c - 1) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) + \frac{2}{3} g_A (2g_A + 4\kappa) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) \right\}

\[ + \frac{2}{3} \left( g_A^2 + 2\kappa + 1 \right) \frac{m_e}{E_0} \left( \frac{E_e}{E_0} \right), \]

\[ \bar{H}^{(1)}(E_e)(N^{2\text{LO}}) = \frac{1}{1 + 3g_A^2 4m_N^2} \left\{ -2g_A^2 + 2\kappa + 1 \right\} \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) + \frac{2}{3} g_A (2g_A + 4\kappa) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) \right\} \]

\[ \bar{N}^{(1)}(E_e)(N^{2\text{LO}}) = \frac{1}{1 + 3g_A^2 4m_N^2} \left\{ -4g_A(g_A + \kappa) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) + 2g_A(g_A + \kappa + 1) \frac{m_e}{E_0} \right\}

\[ + 2g_A \frac{E_e}{E_0} + 2g_A(g_A - \kappa - 1) \frac{m_e}{E_0} \right\}, \]

\[ \bar{Q}^{(1)}(E_e)(N^{2\text{LO}}) = \frac{1}{1 + 3g_A^2 4m_N^2} \left\{ -2g_A^2 + 2\kappa + 1 \right\} \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) + \frac{2}{3} g_A (2g_A + 4\kappa + 1) \frac{E_e}{E_0} \left( 1 - \frac{E_e}{E_0} \right) \right\}

\times \left\{ 1 + \frac{m_e}{E_e} + \left( -2\kappa - g_A^2 - 2\kappa - 1 \right) \frac{m_e}{E_0} \right\}, \quad (12) \]

These corrections are proportional to the factor \((E_0^2/4m_N^2)/(1 + 3g_A^2) = 8.05 \times 10^{-8} \sim 10^{-7}\), calculated for \(E_0 = 1.2926\text{ MeV}, m_N = (m_a + m_a)/2 = 938.9188\) and \(g_a = 1.2764\). Because of such a factor this sort of corrections are of order \(10^{-6}\) and smaller. They can be neglected in comparison with corrections of order of \(10^{-5}\).

2. Corrections \(\zeta^{(2)}(E_e)(N^{2\text{LO}})\) and \(\bar{X}^{(2)}(E_e)(N^{2\text{LO}})\), induced by next-to-next-to-leading order terms from phase-volume of the neutron beta decay (see Eq. (A-21))

The corrections \(\zeta^{(2)}(E_e)(N^{2\text{LO}})\) and \(\bar{X}^{(2)}(E_e)(N^{2\text{LO}})\), induced by next-to-next-to-leading order terms in the large nucleon mass \(m_N\) expansion of the neutron beta decay (see Eq. (A-21)), are given by

\[ \zeta^{(2)}(E_e)(N^{2\text{LO}}) = \frac{6E_0^2}{m_N^2} \left\{ \left( 1 - \frac{1}{4} \frac{E_0}{E_e} \right) + \frac{1}{3} \left[ 1 - 2 \frac{1 - g_A^2}{1 + 3g_A^2} \frac{1}{8} \frac{E_0}{E_0} \right] \right\} \left( 1 - \frac{m_e^2}{E_e^2} \right) \}

\[ \tilde{a}^{(2)}(E_e)(N^{2\text{LO}}) = \frac{12E_0^2}{m_N^2} \left\{ - \frac{1}{8} \frac{E_0}{E_e} + \frac{1}{3} \frac{g_A^2}{1 + 3g_A^2} \left( 1 - \frac{1}{4} \frac{E_0}{E_e} \right) \right\} \}

\[ \bar{A}^{(2)}(E_e)(N^{2\text{LO}}) = \frac{12E_0^2}{m_N^2} \left\{ \frac{g_A^2 (1 - g_A)}{1 + 3g_A^2} \left( 1 - \frac{1}{4} \frac{E_0}{E_e} \right) + \frac{1}{3} \frac{g_A (1 - g_A)}{1 + 3g_A^2} \left( 1 - \frac{m_e^2}{E_e^2} \right) \right\} \}

\[ \bar{B}^{(2)}(E_e)(N^{2\text{LO}}) = \frac{12E_0^2}{m_N^2} \left\{ \frac{g_A (1 + g_A)}{1 + 3g_A^2} \left( 1 - \frac{1}{4} \frac{E_0}{E_e} \right) \right\} \]
\[ K_n^{(2)}(E_e)(N^2\text{LO}) = 24 \frac{E_e^2}{m_N^2} \left\{ - \frac{g_A(1 - g_A)}{1 + 3g_A^2} \left(1 - \frac{1}{8} \frac{E_0}{E_e}\right) \right\}, \]

\[ \hat{Q}_n^{(2)}(E_e)(N^2\text{LO}) = 24 \frac{E_e^2}{m_N^2} \left\{ - \frac{g_A(1 + g_A)}{1 + 3g_A^2} \left(1 - \frac{1}{8} \frac{E_0}{E_e}\right) \right\}, \]

\[ G^{(2)}(E_e)(N^2\text{LO}) = 6 \frac{E_e^2}{m_N^2} \left\{ - \frac{1}{3} \frac{E_0}{E_e} - \frac{1}{3} \frac{m_e^2}{E_e^2} + \frac{2}{3} \frac{m_e^2}{1 + 3g_A^2} \left(1 - \frac{1}{8} \frac{E_0}{E_e}\right) \right\}, \]

\[ \hat{H}^{(2)}(E_e)(N^2\text{LO}) = 6 \frac{E_e^2}{m_N^2} \left\{ - \frac{1}{1 + 3g_A^2} \frac{m_e}{E_e} \left(1 - \frac{1}{8} \frac{E_0}{E_e}\right) \right\}, \]

\[ N^{(2)}(E_e)(N^2\text{LO}) = 12 \frac{E_e^2}{m_N^2} \frac{m_e}{E_e} \left\{ - \frac{g_A(1 - g_A)}{1 + 3g_A^2} \left(1 - \frac{1}{4} \frac{E_0}{E_e}\right) - \frac{1}{3} \frac{g_A(1 - g_A)}{1 + 3g_A^2} \left(1 - \frac{m_e^2}{E_e^2}\right) \right\}, \]

\[ \hat{Q}_e^{(2)}(E_e)(N^2\text{LO}) = 12 \frac{E_e^2}{m_N^2} \left\{ \frac{g_A(1 - g_A)}{1 + 3g_A^2} \left(1 - \frac{1}{4} \frac{E_0}{E_e}\right) - \frac{1}{3} \frac{g_A(1 - g_A)}{1 + 3g_A^2} \left(1 - \frac{m_e^2}{E_e^2}\right) + \frac{2}{3} \frac{g_A(1 + g_A)}{1 + 3g_A^2} \right\}, \]

\[ K_e^{(2)}(E_e)(N^2\text{LO}) = 12 \frac{E_e^2}{m_N^2} \left\{ \left(1 - \frac{1}{8} \frac{E_0}{E_e}\right) \left(1 + \frac{m_e}{E_e}\right) - \frac{1}{2} \frac{1 - g_A^2}{1 + 3g_A^2} \left(1 - \frac{1}{8} \frac{E_0}{E_e}\right) \right\}. \]
These corrections are proportional to the factor $(E^2_w/4m_N^2)/(1+3g_A^2) = 8.05 \times 10^{-8} \sim 10^{-7}$. In spite of a strong dependence of on the axial coupling constant $g_A = 1.2764$ and the isovector anomalous magnetic moment of the nucleon $\kappa = \kappa_p - \kappa_n = 3.7059$ the value of the corrections in Eq.(15) are at the level of a few parts of $10^{-6}$. We neglect these contributions in our analysis of the corrections of order $\mathcal{O}(E_w^2/m_N^2) \sim 10^{-5}$. 

(15)
4. Corrections $\zeta(E_e)(N^{2\text{LO}})$ and $\bar{X}(E_e)(N^{2\text{LO}})$, obtained by multiplication of the corrections of order $O(E_e/m_N)$ to the electron–energy and angular distribution of the neutron beta decay by the corrections of order $O(E_e/m_N)$ from the phase–volume of the neutron beta decay (see Eq. [A-21]).

Using the next-to-leading order corrections $\zeta(E_e)(N^{\text{LO}})$ and $\bar{X}(E_e)(N^{\text{LO}})$ in the large nucleon mass $m_N$ expansion $O(E_e/m_N)$, calculated in [18–25], and adduced in Appendix C, we may calculate the corrections $\zeta(E_e)(N^{2\text{LO}})$ and $\bar{X}(E_e)(N^{2\text{LO}})$ multiplying the corrections $\zeta(E_e)(N^{\text{LO}})$ and $\bar{X}(E_e)(N^{\text{LO}})$ by the terms of order $O(E_e/m_N)$ from the phase–volume of the neutron beta decay (see Eq. [A-21]). We get:

\[
\begin{align*}
\zeta^{(4)}(E_e)(N^{2\text{LO}}) &= 3 \frac{E_e}{m_N} \zeta(E_e)(N^{\text{LO}}) - \frac{E_e}{m_N} \bar{a}(E_e)(N^{\text{LO}}) \left(1 + \frac{m_e^2}{E_e^2}\right) = -1.97 \times 10^{-5} \frac{E_e}{E_0} + 5.49 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
\bar{a}^{(4)}(E_e)(N^{2\text{LO}}) &= 3 \frac{E_e}{m_N} \bar{a}(E_e)(N^{\text{LO}}) - \frac{E_e}{m_N} \zeta(E_e)(N^{\text{LO}}) = 1.97 \times 10^{-5} \frac{E_e}{E_0} - 5.57 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
\bar{A}^{(4)}(E_e)(N^{2\text{LO}}) &= 3 \frac{E_e}{m_N} \bar{A}(E_e)(N^{\text{LO}}) - \frac{E_e}{m_N} \bar{K}_n(E_e)(N^{\text{LO}}) \left(1 + \frac{m_e^2}{E_e^2}\right) = -1.31 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
B^{(4)}(E_e)(N^{2\text{LO}}) &= 3 \frac{E_e}{m_N} B(E_e)(N^{\text{LO}}) = -1.47 \times 10^{-5} \frac{E_e}{E_0} + 3.91 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
\bar{K}_n^{(4)}(E_e)(N^{2\text{LO}}) &= 3 \frac{E_e}{m_N} \bar{K}_n(E_e)(N^{\text{LO}}) - 3 \frac{E_e}{m_N} \bar{A}(E_e)(N^{\text{LO}}) = 1.51 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
\bar{Q}_n^{(4)}(E_e)(N^{2\text{LO}}) &= 3 \frac{E_e}{m_N} \bar{Q}_n(E_e)(N^{\text{LO}}) - 3 \frac{E_e}{m_N} B(E_e)(N^{\text{LO}}) = 2.79 \times 10^{-5} \frac{E_e}{E_0} - 6.91 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
\bar{G}^{(4)}(E_e)(N^{2\text{LO}}) &= 3 \frac{E_e}{m_N} \bar{G}(E_e)(N^{\text{LO}}) - \frac{E_e}{m_N} \bar{H}(E_e)(N^{\text{LO}}) - \frac{E_e}{m_N} \bar{K}_e(E_e)(N^{\text{LO}}) \left(1 + \frac{m_e}{E_e}\right) = \\
&= 1.97 \times 10^{-5} \frac{E_e}{E_0} - 5.49 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
\bar{H}^{(4)}(E_e)(N^{2\text{LO}}) &= 3 \frac{E_e}{m_N} \bar{H}(E_e)(N^{\text{LO}}), \quad \bar{N}^{(4)}(E_e)(N^{2\text{LO}}) = 3 \frac{E_e}{m_N} \bar{N}(E_e)(N^{\text{LO}}), \\
\bar{Q}_e^{(4)}(E_e)(N^{2\text{LO}}) &= 3 \frac{E_e}{m_N} \bar{Q}_e(E_e)(N^{\text{LO}}) = 1.96 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
\bar{K}_e^{(4)}(E_e)(N^{2\text{LO}}) &= 3 \frac{E_e}{m_N} \bar{K}_e(E_e)(N^{\text{LO}}) - 3 \frac{E_e}{m_N} \bar{G}(E_e)(N^{\text{LO}}) \left(1 + \frac{m_e}{E_e}\right) = 8.30 \times 10^{-5} \frac{E_e^2}{E_0^2}.
\end{align*}
\]

We have given the corrections $\zeta^{(4)}(E_e)(N^{2\text{LO}})$ and $\bar{X}^{(4)}(E_e)(N^{2\text{LO}})$ in the form of polynomials in the variable $E_e/E_0$. The coefficients of the polynomials are calculated at the neglect of the contributions of order of a few parts of $10^{-6}$. The same order of magnitude possess the corrections $\bar{H}^{(4)}(E_e)(N^{2\text{LO}})$ and $\bar{N}^{(4)}(E_e)(N^{2\text{LO}})$, respectively.

5. Corrections $\bar{X}(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}})$

For the corrections $\bar{X}(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}})$ we obtain the following analytical expressions

\[
\begin{align*}
\bar{a}(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}}) &= \frac{1 - g_A^2}{1 + 3g_A^2} \zeta(E_e)(N^{2\text{LO}}), \quad \bar{A}(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}}) = 2 \frac{g_A(1 - g_A)}{1 + 3g_A^2} \zeta(E_e)(N^{2\text{LO}}), \\
B(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}}) &= 2 \frac{g_A(1 + g_A)}{1 + 3g_A^2} \zeta(E_e)(N^{2\text{LO}}), \quad \bar{K}_n(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}}) = \bar{Q}_n(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}}) = 0, \\
\bar{G}(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}}) &= -\zeta(E_e)(N^{2\text{LO}}), \quad \bar{H}(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}}) = -\frac{1 - g_A^2}{1 + 3g_A^2} \frac{m_e}{E_e} \zeta(E_e)(N^{2\text{LO}}), \\
\bar{N}(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}}) &= -2 \frac{g_A(1 - g_A)}{1 + 3g_A^2} \frac{m_e}{E_e} \zeta(E_e)(N^{2\text{LO}}), \quad \bar{Q}_e(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}}) = -2 \frac{g_A(1 - g_A)}{1 + 3g_A^2} \zeta(E_e)(N^{2\text{LO}}), \\
\bar{K}_e(E_e)(L_0)\zeta(E_e)(N^{2\text{LO}}) &= \frac{1 - g_A^2}{1 + 3g_A^2} \zeta(E_e)(N^{2\text{LO}}).
\end{align*}
\]
where we have used the results obtained in \[18, 25, 27\]. According to our analysis, carried out above, the correction \(\zeta(E_c)_{(N^2\text{LO})}\) contains contributions of order of \(10^{-5}\) from \(\zeta^{(2)}(E_c)_{(N^2\text{LO})}\) and \(\zeta^{(4)}(E_c)_{(N^2\text{LO})}\) only. So we may set

\[
\zeta(E_c)_{(N^2\text{LO})} = \zeta^{(2)}(E_c)_{(N^2\text{LO})} + \zeta^{(4)}(E_c)_{(N^2\text{LO})} = -1.97 \times 10^{-5} \frac{E_c}{E_0} + 7.09 \times 10^{-5} \frac{E^2_c}{E_0^2}. \tag{18}
\]

Because of the factors \(g_A(1-g_A)/(1+3g_A^2)\) and \(g_A^2/(1+3g_A^2)\) are of order of a few parts of \(10^{-6}\) or even smaller. As a result, between the corrections in Eq.(17) we may take into account the corrections \(\bar{B}(E_c)_{(LO)}\zeta(E_c)_{(N^2\text{LO})}\) and \(\bar{G}(E_c)_{(LO)}\zeta(E_c)_{(N^2\text{LO})}\), where \(\bar{B}(E_c)_{(LO)}\) and \(\bar{G}(E_c)_{(LO)}\) are of order \(O(1)\). We get

\[
\begin{align*}
\bar{B}(E_c)_{(LO)}\zeta(E_c)_{(N^2\text{LO})} &= -1.94 \times 10^{-5} \frac{E_c}{E_0} + 7.06 \times 10^{-5} \frac{E^2_c}{E_0^2}, \\
\bar{G}(E_c)_{(LO)}\zeta(E_c)_{(N^2\text{LO})} &= 1.97 \times 10^{-5} \frac{E_c}{E_0} - 7.09 \times 10^{-5} \frac{E^2_c}{E_0^2}.
\end{align*} \tag{19}
\]

Thus, the corrections \(\bar{B}(E_c)_{(LO)}\zeta(E_c)_{(N^2\text{LO})}\) and \(\bar{G}(E_c)_{(LO)}\zeta(E_c)_{(N^2\text{LO})}\) possess a required order of a few parts of \(10^{-5}\) only.

### 6. Corrections \(\bar{X}(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)}\)

The corrections \(\bar{X}(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)}\) we calculate by using the results obtained in \[18, 25, 27\]. The correction \(\zeta(E_c)_{(NLO)}\) is adduced also in Appendix C. We get

\[
\begin{align*}
\bar{a}(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)} &= \frac{1}{1 + 3g_A^2} \zeta^2(E_c)_{(NLO)} = -1.05 \times 10^{-5} \frac{E^2_c}{E_0^2}, \\
\bar{A}(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)} &= 2g_A(1-g_A) \frac{E_c}{1 + 3g_A^2} \zeta^2(E_c)_{(NLO)} = -1.17 \times 10^{-5} \frac{E^2_c}{E_0^2}, \\
\bar{B}(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)} &= 2g_A(1+g_A) \frac{E_c}{1 + 3g_A^2} \zeta^2(E_c)_{(NLO)} = -6.98 \times 10^{-5} \frac{E_c}{E_0} + 9.67 \times 10^{-5} \frac{E^2_c}{E_0^2}, \\
\bar{K}_n(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)} &= \bar{Q}_n(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)} = 0, \\
\bar{G}(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)} &= -\zeta^2(E_c)_{(NLO)} = 7.07 \times 10^{-5} \frac{E_c}{E_0} - 9.79 \times 10^{-5} \frac{E^2_c}{E_0^2}, \\
\bar{H}(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)} &= -\frac{1}{1 + 3g_A^2} \frac{m_e}{E_c} \zeta^2(E_c)_{(NLO)}, \\
\bar{N}(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)} &= -2g_A(1-g_A) \frac{m_e}{1 + 3g_A^2} \zeta^2(E_c)_{(NLO)}, \\
\bar{Q}_e(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)} &= -2g_A(1+g_A) \frac{m_e}{1 + 3g_A^2} \zeta^2(E_c)_{(NLO)} = 1.17 \times 10^{-5} \frac{E^2_c}{E_0^2}, \\
\bar{K}_c(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)} &= -\frac{1}{1 + 3g_A^2} \zeta^2(E_c)_{(NLO)} = 1.05 \times 10^{-5} \frac{E^2_c}{E_0^2}, \tag{20}
\end{align*}
\]

where we have neglected the contributions of order of a few parts of \(10^{-6}\). The same order of magnitude possess the corrections \(\bar{H}(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)}\) and \(\bar{N}(E_c)_{(LO)}\zeta^2(E_c)_{(NLO)}\), respectively, the contributions of which we have neglected.

### 7. Corrections \(\bar{X}(E_c)_{(NLO)}\zeta(E_c)_{(NLO)}\)

The calculation of corrections \(\bar{X}(E_c)_{(NLO)}\zeta(E_c)_{(NLO)}\) we may perform by using the results obtained in \[18, 25, 27\], which we have adduced in Appendix C. Since the analytical expressions of these corrections are rather bulky, we give
we give them in the form of polynomials in the variable $E_e/E_0$ only. We get

\begin{align*}
\bar{a}(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} &= 7.17 \times 10^{-5} \frac{E_e}{E_0} - 1.01 \times 10^{-4} \frac{E_e^2}{E_0^2}, \\
\bar{A}(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} &= 1.43 \times 10^{-5} \frac{E_e}{E_0} - 2.90 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
\bar{B}(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} &= -6.92 \times 10^{-5} \frac{E_e}{E_0} + 9.37 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
Q_n(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} &= 5.74 \times 10^{-5} \frac{E_e}{E_0} - 7.19 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
G(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} &= 7.07 \times 10^{-5} \frac{E_e}{E_0} - 9.79 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
H(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} &= -1.94 \times 10^{-5} + 1.51 \times 10^{-5} \frac{E_e}{E_0}, \\
\bar{Q}_n(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} &= -1.44 \times 10^{-5} \frac{E_e}{E_0} + 4.70 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
K_e(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} &= -4.70 \times 10^{-5} \frac{E_e}{E_0} + 1.01 \times 10^{-4} \frac{E_e^2}{E_0^2}. \\
\end{align*}

(21)

The corrections $\bar{K}_n(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}}$ and $\bar{N}(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}}$ are of order of a few parts of $10^{-6}$.

IV. CORRECTIONS $\zeta(E_e)_{\text{(N2LO)}}$ AND $X(E_e)_{\text{(N2LO)}}$ IN THE FORM OF POLYNOMIALS IN THE VARIABLE $E_e/E_0$

Summing up the contributions of order of $10^{-5}$ we obtain the corrections $\zeta(E_e)_{\text{(N2LO)}}$ and $X(E_e)_{\text{(N2LO)}}$, caused by weak magnetism and proton recoil calculated to next-to-next-to-leading order in the large nucleon mass $m_N$ expansion. We give them in the form of polynomials in the variable $E_e/E_0$:

\begin{align*}
\zeta(E_e)_{\text{(NLO)}} &= \zeta^{(2)}(E_e)_{\text{(N2LO)}} + \zeta^{(4)}(E_e)_{\text{(N2LO)}} + \zeta(E_e)_{\text{FF}} = -3.27 \times 10^{-5} \frac{E_e}{E_0} + 8.39 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
a(E_e)_{\text{(NLO)}} &= \bar{a}^{(2)}(E_e)_{\text{(N2LO)}} + \bar{a}^{(4)}(E_e)_{\text{(N2LO)}} + \bar{a}(E_e)_{\text{LO}} \zeta^{2}(E_e)_{\text{(NLO)}} - \bar{a}(E_e)_{\text{(NLO)}} \zeta^{(E_e)_{\text{(NLO)}}} = -1.43 \times 10^{-5} \frac{E_e}{E_0}, \\
A(E_e)_{\text{(NLO)}} &= \bar{A}^{(4)}(E_e)_{\text{(N2LO)}} + \bar{A}(E_e)_{\text{LO}} \zeta^{2}(E_e)_{\text{(NLO)}} - \bar{A}(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} = -1.43 \times 10^{-5} \frac{E_e}{E_0}, \\
B(E_e)_{\text{(NLO)}} &= \bar{B}^{(2)}(E_e)_{\text{(N2LO)}} + \bar{B}^{(4)}(E_e)_{\text{(N2LO)}} - \bar{B}(E_e)_{\text{LO}} \zeta(E_e)_{\text{(NLO)}} + \bar{B}(E_e)_{\text{LO}} \zeta^{2}(E_e)_{\text{(NLO)}} = -1.62 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
K_n(E_e)_{\text{(NLO)}} &= \bar{K}_n^{(4)}(E_e)_{\text{(N2LO)}} = 1.51 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
Q_n(E_e)_{\text{(NLO)}} &= \bar{Q}_n^{(2)}(E_e)_{\text{(N2LO)}} + \bar{Q}_n^{(4)}(E_e)_{\text{(N2LO)}} - \bar{Q}_n(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} = -2.95 \times 10^{-5} \frac{E_e}{E_0} - 1.97 \times 10^{-5} \frac{E_e^2}{E_0^2}, \\
G(E_e)_{\text{(NLO)}} &= \bar{G}^{(2)}(E_e)_{\text{(N2LO)}} + \bar{G}^{(4)}(E_e)_{\text{(N2LO)}} - \bar{G}(E_e)_{\text{LO}} \zeta(E_e)_{\text{(NLO)}} + \bar{G}(E_e)_{\text{LO}} \zeta^{2}(E_e)_{\text{(NLO)}} = \bar{G}(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} + \bar{G}(E_e)_{\text{FF}} = 0, \\
H(E_e)_{\text{(NLO)}} &= -\bar{H}(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} = 1.94 \times 10^{-5} - 1.51 \times 10^{-5} \frac{E_e}{E_0}, \\
N(E_e)_{\text{(NLO)}} &= -\bar{N}(E_e)_{\text{(NLO)}} \zeta(E_e)_{\text{(NLO)}} = 0,
\end{align*}
\[ Q_e(E_c)_{(N^2 \text{LO})} = \dot{Q}_e(E_c)_{(N^2 \text{LO})} + \dot{Q}_e(E_c)_{(\text{LO})} \xi^2(E_c)_{(\text{NLO})} - \dot{Q}_e(E_c)_{(\text{NLO})} \zeta(E_c)_{(\text{NLO})} = \]
\[ = 1.44 \times 10^{-5} \frac{E_e}{E_0} - 1.57 \times 10^{-5} \frac{E_0^2}{E_0^2} \]
\[ K_e(E_c)_{(N^2 \text{LO})} = \hat{K}_e^{(2)}(E_c)_{(N^2 \text{LO})} + \hat{K}_e^{(4)}(E_c)_{(N^2 \text{LO})} + \hat{K}_e(E_c)_{(\text{LO})} \omega^2(E_c)_{(\text{NLO})} - \hat{K}_e(E_c)_{(\text{NLO})} \zeta(E_c)_{(\text{NLO})} \]
\[ + K_e(E_c)_{\text{FF}} = 3.40 \times 10^{-5} \frac{E_e}{E_0} + 2.95 \times 10^{-5} \frac{E_0^2}{E_0^2}. \] (22)

where we have added the corrections \( X(E_c)_{\text{FF}} \) for \( X = \zeta, \alpha, B, G \) and \( K_e \), induced by the isovector and axial-vector form factors of the neutron taken in the linear approximation for the square four-momentum transfer (see Appendix A). The corrections, given in Eq. (22), illustrate an existence of corrections \( O(E_e^2/m_N^2) \) of order of a few parts of \( 10^{-5} \), caused by weak magnetism and proton recoil and calculated to next-to-next-to-leading order in the large nucleon mass \( m_N \) expansion. These corrections, calculated with a theoretical accuracy of about a few parts of \( 10^{-6} \), are given in the form, which can be, in principle, applied to the description of the SM theoretical background of the neutron beta decay at the level of a few parts of \( 10^{-5} \).

V. CONCLUSION

We have calculated a complete set of the model–independent next-to-next-to-leading order corrections \( O(E_e^2/m_N^2) \) in the large nucleon mass \( m_N \) expansion, caused by weak magnetism and proton recoil, to the neutron lifetime and correlation coefficients of the neutron beta decay for a polarized neutron, a polarized electron and an unpolarized proton. We have given a detailed analysis of the structure of these corrections. In addition we have added corrections, caused by the isovector and axial-vector form factors of the nucleon, calculated to linear approximation in the square of four-momentum transfer with the form factors taken in the dipole approximation. The notation “large nucleon mass \( m_N \) expansion” means that we make an expansion in powers of \( 1/m_N \), were the nucleon mass is finite but much larger than the momentum of the proton and energies and momenta of decay leptons. The effect of the finite nucleon mass appears only to next-to-leading and higher orders in the large nucleon mass \( m_N \) expansion or in the terms proportional to powers of \( 1/m_N \). For the calculation of the corrections \( O(E_e^2/m_N^2) \) we have used a standard technique \( [14, 15, 18, 25, 27] \), which is well expounded in Appendix of Ref. \( 18 \). We have presented the corrections \( O(E_e^2/m_N^2) \) in the form of polynomials in the variable \( E_e/E_0 \) with coefficients of order of \( 10^{-5} \) and even \( 10^{-4} \), calculated at the neglect of the terms of order of a few parts of \( 10^{-6} \). Together with Wilkinson’s corrections of order of \( 10^{-5} \) \( [15] \) (see also \( 18, 25, 27 \)) and radiative corrections of order \( O(\alpha E_e/m_N) \sim 10^{-5} \) \( [29, 30] \) the corrections of order \( O(E_e^2/m_N^2) \sim 10^{-5} \) define the SM theoretical background of the analysis of the neutron beta decay at the level of a few parts of \( 10^{-5} \).

Such a set of corrections should provide an improved level of theoretical investigations of the neutron beta decay within the SM. The representation of the corrections of order \( O(E_e^2/m_N^2) \sim 10^{-5} \) in the form of polynomials in the variable \( E_e/E_0 \) makes them easily applicable for the analysis of experimental data on asymmetries of the neutron beta decay for searches of traces of interactions beyond the SM at the level of a few parts of \( 10^{-5} \). For example, in the experimental electron-energy region \( 0.811 \text{ MeV} \leq E_e \leq 1.211 \text{ MeV} \) \( [6] \) and \( 0.708 \text{ MeV} \leq E_e \leq 1.205 \text{ MeV} \) \( [33] \), the correction \( A^{(W)}(E_c)_{(N^2 \text{LO})} \) to the correlation coefficient \( A^{(W)}(E_c) = A(E_c) + \frac{1}{3} \zeta_n(E_c)_{N^2 \text{LO}} \), defining the electron asymmetry in the neutron beta decay \( [7, 15, 32] \) (see also \( 18 \)), varies in the limits \( -1.77 \times 10^{-5} \geq A^{(W)}(E_c)_{(N^2 \text{LO})} \geq -2.84 \times 10^{-5} \) and \( -1.52 \times 10^{-5} \geq A^{(W)}(E_c)_{(N^2 \text{LO})} \geq -2.82 \times 10^{-5} \), respectively. Recently \( [33] \) the experimental data on the measurements of this asymmetry have been analyzed for a search of the contribution of the Fierz interference term \( b \). The theoretical expression for the correlation coefficient \( A^{(W)}(E_c) \) is calculated in the SM at the level of \( 10^{-3} \) \( [13] \) (see also \( 18 \)), whereas the experimental uncertainties of the electron asymmetry are at the level of a few parts of \( 10^{-4} \), namely, \( A^{(W)} = -0.11985 \pm 0.00021 \) \( [6] \) and \( A^{(W)} = -0.11972 \pm 0.00025 \) \( [33] \), respectively. This implies that the experimental analysis of the neutron beta decay is more precise than the theoretical one. As has been pointed out in \( [33] \), for such uncertainties the experimental data on the electron asymmetry are consistent with the Fierz interference term \( b = 0 \). Formally, non-zero values for the Fierz interference term \( b \), varying in the limits \( -0.018 \geq b \geq 0.052 \), can be extracted from the experimental data on the electron asymmetry for experimental uncertainties of a few parts of \( 10^{-3} \), namely, \( A^{(W)} = -0.1209 \pm 0.0015 \). Such an uncertainty is commensurable with the level of the SM theoretical calculation of the correlation coefficient \( A^{(W)}(E_c) \) \( [13, 18] \). So for an improvement of experimental constraints on the Fierz interference term by measuring the electron asymmetry of the neutron beta decay it is desirable to improve the theoretical level of the SM calculation of the electron asymmetry from \( 10^{-3} \) to \( 10^{-5} \). Such an improvement should make meaningful an improvement of experimental uncertainties by order of magnitude, i.e. from a few parts of \( 10^{-4} \) to a few parts of \( 10^{-5} \). For the SM definition of the electron asymmetry at the level of \( 10^{-5} \) one may use Wilkinson’s corrections \( [15] \) (see also \( 18, 25, 27 \)), the radiative corrections \( O(\alpha E_e/m_N) \sim 10^{-5} \) \( [29, 30] \), calculated as next-to-leading order corrections in the large nucleon mass \( m_N \) expansion to Sirlin’s radiative corrections of order \( O(\alpha/\pi) \).
and the model-independent corrections $O(E_e^2/m_N^2) \sim 10^{-5}$. At the new level of theoretical and experimental analysis of the electron asymmetry of the neutron beta decay one might expect new much better constraints on the Fierz interference term than $-0.018 \leq b \leq 0.052$ reported in \cite{33}. Then, we would like to mention that, according to Paul \cite{3}, the desired accuracy of experimental investigations of the neutron lifetime should be $\delta \tau_n < 0.08$ s, i.e. at the level of sensitivity of a few parts of $10^{-5}$. We are planning to calculate the neutron lifetime and correlation coefficients of the neutron beta decay at the level of a few parts of $10^{-5}$ in our forthcoming publication \cite{46}.

VI. ACKNOWLEDGEMENTS

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Appendix A: The amplitude of the neutron beta decay with weak magnetism and proton recoil to order $O(E^4/m_N^2)$

Following \[18\] the amplitude of the neutron beta decay we rewrite as follows

$$M(n \rightarrow p e^- \bar{\nu}_e) = \frac{G_F}{\sqrt{2}} V_{ud} \mathcal{M}_n,$$  \hspace{1cm} (A-1)

where $\mathcal{M}_n = [\bar{u}_p O_\mu u_n][\bar{u}_e \gamma^\mu (1 - \gamma^5)v_p]$ and the matrix $O_\mu$ takes the form

$$O_\mu = \gamma_\mu (1 - g_A \gamma^5) + i \frac{\kappa}{2m_N} \sigma_{\mu\nu} (k_p - k_n)^\nu,$$  \hspace{1cm} (A-2)

where $m_N = (m_n + m_p)/2$ is the nucleon mass. In terms of the time and space components of the matrix $O_\mu = (O^0, -\vec{O})$ the amplitude $\mathcal{M}_n$ is defined by \[18\]

$$\mathcal{M}_n = [\bar{u}_p O^0 u_n][\bar{u}_e \gamma_0 (1 - \gamma^5) v_p] - [\bar{u}_p \vec{O} u_n] \cdot [\bar{u}_e \gamma (1 - \gamma^5) v_p].$$  \hspace{1cm} (A-3)

The time $O^0$ and spacial $\vec{O}$ components of the matrix $O_\mu = (O^0, -\vec{O})$ we determine in the large nucleon mass $m_N$ expansion keeping the terms proportional to $1/m_N^2$. We get

$$O^0 = \begin{pmatrix} 1 & -g_A + \frac{\vec{\sigma} \cdot \vec{k}_p}{2m_N} \\ g_A + \frac{\vec{\sigma} \cdot \vec{k}_p}{2m_N} & -1 \end{pmatrix},$$  \hspace{1cm} (A-4)

and

$$\vec{O} = \begin{pmatrix} -g_A \vec{\sigma} + i k \frac{\vec{\sigma} \times \vec{k}_p}{2m_N} \\ -\vec{\sigma} \left(1 + \frac{E_0}{2m_N} + \kappa \frac{E_0^2 - m_N^2 - \vec{k}_p^2}{4m_N^2} \right) \end{pmatrix},$$  \hspace{1cm} (A-5)

where $E_0 = (m_n^2 - m_p^2 + m_e^2)/2m_n = 1.2926$ MeV is the end–point energy of the electron–energy spectrum of the neutron beta decay \[18\]. $\vec{k}_p = -\vec{k}_e - \vec{k}_n$ is 3–momentum transferred momentum in terms of the 3–momenta of the electron $\vec{k}_e$ and antineutrino $\vec{k}_n$, $\vec{k}_p^2 = E_0^2 - m_n^2 + E_0^2 + 2\vec{k}_e \cdot \vec{k}_n$.

For the calculation of the amplitude of the neutron beta decay we use the Dirac wave functions of the neutron and the proton in the momentum representation

$$u_n(\vec{k}, \sigma_n) = \sqrt{2m_n} \begin{pmatrix} \varphi_n \\ 0 \end{pmatrix}, \quad u_p(\vec{k}_p, \sigma_p) = \sqrt{E_p + m_p} \begin{pmatrix} \varphi_p \\ \vec{\sigma} \cdot \vec{k}_p \end{pmatrix},$$  \hspace{1cm} (A-6)

where the Pauli spinorial wave functions $\varphi_n$ and $\varphi_p$ depend on the polarisations $\sigma_n = \pm 1/2$ and $\sigma_p = \pm 1/2$, respectively. For the calculation of the matrix elements $[\bar{u}_p O^0 u_n]$ and $[\bar{u}_p \vec{O} u_n]$ we use the following expansions

$$\sqrt{2m_n(E_p + m_p)} = 2m_n \left(1 - \frac{E_0}{2m_N} - \frac{E_0^2 - 2m_n^2 - \vec{k}_p^2}{8m_N^2} \right), \quad \frac{\vec{\sigma} \cdot \vec{k}_p}{E_p + m_p} = \frac{\vec{\sigma} \cdot \vec{k}_p}{2m_N} + \frac{E_0(\vec{\sigma} \cdot \vec{k}_p)}{4m_N^2}.$$  \hspace{1cm} (A-7)

As a result, we get

$$[\bar{u}_p O^0 u_n] = 2m_n \left\{ \left(1 - \frac{E_0}{2m_N} - \frac{E_0^2 - 2m_n^2 - \vec{k}_p^2}{8m_N^2} \right) [\varphi_p^\dagger \varphi_n] - \frac{g_A}{2m_N} [\varphi_p^\dagger (\vec{\sigma} \cdot \vec{k}_p) \varphi_n] \right\},$$  \hspace{1cm} (A-8)

and

$$[\bar{u}_p \vec{O} u_n] = \frac{1}{2m_N} \left\{ -g_A \left(1 - \frac{E_0}{2m_N} - \frac{E_0^2 - 2m_n^2 - \vec{k}_p^2}{8m_N^2} \right) [\varphi_p^\dagger \vec{\sigma} \varphi_n] + i \frac{\kappa}{2m_N} \left(1 - \frac{E_0}{2m_N} \right) [\varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n] \right\},$$  \hspace{1cm} (A-9)
Using the relation \((\vec{\sigma} \cdot \vec{k}_p)\vec{\sigma} = \vec{k}_p + i (\vec{\sigma} \times \vec{k}_p)\) we rewrite the r.h.s. of Eq. (A-9) as follows

\[
[u_p \hat{O} u_n] = 2m_n \left\{ -g_A \left( 1 - \frac{E_0}{2m_N} - \frac{E_0^2 - 2m_p^2 - \vec{k}_p^2}{8m_N^2} \right) |\varphi_p^\dagger \varphi_n| + \frac{\kappa + 1}{2m_N} \left[ \varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n \right] + \frac{\vec{k}_p}{2m_N} \left( 1 + \frac{E_0}{2m_N} \right) |\varphi_p^\dagger \varphi_n| \right\}.
\]

(A-10)

For the amplitude \(\mathcal{M}_n\) we obtain the following expression

\[
\mathcal{M}_n = 2m_n \left\{ \left( 1 - \frac{E_0}{2m_N} - \frac{E_0^2 - 2m_p^2 + (2\kappa - 1) \vec{k}_p^2}{8m_N^2} \right) |\varphi_p^\dagger \varphi_n| [\vec{u}_c \gamma^0 (1 - \gamma^5) v_0] + g_A \left( 1 - \frac{E_0}{2m_N} - \frac{E_0^2 - 2m_p^2 - \vec{k}_p^2}{8m_N^2} \right) \right.
\]

\[
\times \left[ \varphi_p^\dagger \varphi_n \right] \left[ \vec{u}_c \gamma^0 (1 - \gamma^5) v_0 \right] - \frac{g_A}{2m_N} \left[ \varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n \right] \left[ \vec{u}_c \gamma^0 (1 - \gamma^5) v_0 \right] - \frac{\vec{k}_p}{2m_N} \left( 1 + \frac{E_0}{2m_N} \right) |\varphi_p^\dagger \varphi_n| \left[ \vec{u}_c \gamma^0 (1 - \gamma^5) v_0 \right].
\]

\[
(A-11)
\]

For the transformation of the last term we use the Dirac equations for the electron and antineutrino. We get

\[
-\frac{\vec{k}_p}{2m_N} |\varphi_p^\dagger \varphi_n| \cdot [\vec{u}_c \gamma^0 (1 - \gamma^5) v_0] = \frac{m_n - E_p}{2m_N} |\varphi_p^\dagger \varphi_n| [\vec{u}_c \gamma^0 (1 - \gamma^5) v_0] - \frac{m_e}{2m_N} |\varphi_p^\dagger \varphi_n| [\vec{u}_c (1 - \gamma^5) v_0],
\]

(A-12)

where we have set \(E_e + E_\nu = m_n - E_p\). Making the large nucleon mass expansion and keeping the contributions of order \(1/m_N^2\) we transcribe the right–hand–side (r.h.s.) of Eq. (A-12) into the form

\[
-\frac{\vec{k}_p}{2m_N} |\varphi_p^\dagger \varphi_n| \cdot [\vec{u}_c \gamma^0 (1 - \gamma^5) v_0] = \left( \frac{E_0}{2m_N} + \frac{E_0^2 - 2m_p^2 - \vec{k}_p^2}{4m_N^2} \right) |\varphi_p^\dagger \varphi_n| [\vec{u}_c \gamma^0 (1 - \gamma^5) v_0] - \frac{m_e}{2m_N} |\varphi_p^\dagger \varphi_n| [\vec{u}_c (1 - \gamma^5) v_0].
\]

(A-13)

Substituting Eq. (A-13) into Eq. (A-11) we get

\[
\mathcal{M}_n = 2m_n \left\{ |\varphi_p^\dagger \varphi_n| [\vec{u}_c \gamma^0 (1 - \gamma^5) v_0] + g_A \left( 1 - \frac{E_0}{2m_N} \right) |\varphi_p^\dagger \varphi_n| [\vec{u}_c \gamma^0 (1 - \gamma^5) v_0] - \frac{g_A}{2m_N} |\varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n| [\vec{u}_c \gamma^0 (1 - \gamma^5) v_0] - \frac{\vec{k}_p}{2m_N} |\varphi_p^\dagger \varphi_n| [\vec{u}_c (1 - \gamma^5) v_0] \right\} + \delta \mathcal{M}_n,
\]

(A-14)

where \(\delta \mathcal{M}_n\) defines the contributions of the terms proportional to \(1/m_N^2\). It is equal to

\[
\delta \mathcal{M}_n = 2m_n \left\{ \frac{2\kappa + 1}{E_0^2} \left[ (2\kappa + 1) \frac{q^2}{E_0^2} \right] |\varphi_p^\dagger \varphi_n| [\vec{u}_c \gamma^0 (1 - \gamma^5) v_0] - \frac{g_A}{E_0^2} q^2 - \frac{2m_p^2}{E_0^2} |\varphi_p^\dagger \varphi_n| [\vec{u}_c \gamma^0 (1 - \gamma^5) v_0] \right\} - \frac{2m_e}{E_0} |\varphi_p^\dagger \varphi_n| [\vec{u}_c (1 - \gamma^5) v_0],
\]

(A-15)

where \(q^2 = E_0^2 - \vec{k}_p^2 = E_0^2 - (\vec{k}_e + \vec{k}_p)^2\). An additional contribution of order \(O(E_0^2/m_N^2)\) to the electron–energy and angular distribution of the neutron beta decay appears from the phase–volume of the neutron beta decay. Following [18] we denote the contribution of the phase–volume of the neutron beta decay as \(\Phi_{\nu_n}(k_e, \cos \theta_{e\nu})\), where \(\theta_{e\nu}\) is an angle between 3–momenta of the electron and antineutrino such as \(\cos \theta_{e\nu} = \vec{k}_e \cdot \vec{k}_\nu\), and define

\[
\Phi_{\nu_n}(k_e, \cos \theta_{e\nu}) = \int_0^\infty f(E_0) \frac{m_n}{E_p} \frac{E_0^2 dE_p}{(E_0 - E_e)^2},
\]

(A-16)

where the function \(f(E_0)\) is given by the energy conservation in the neutron beta decay \(f(E_0) = m_n - E_p - E_e - E_\nu\) and \(E_p = \sqrt{m_p^2 + (\vec{k}_e + \vec{k}_p)^2} = \sqrt{m_p^2 + k_e^2 + k_p^2 + 2E_eE_\nu \cos \theta_{e\nu}}\) is the proton energy after the integration over the 3–momentum of the proton, giving \(\vec{k}_p = -\vec{k}_e - \vec{k}_\nu\). Using the properties of the \(\delta\)–function the result of the integration over the antineutrino energy \(E_\nu\) is equal to

\[
\Phi_{\nu_n}(k_e, \cos \theta_{e\nu}) = \frac{m_n}{E_p} \frac{E_0^2}{(E_0 - E_e)^2} \left| \frac{df(E_0)}{dE_\nu} \right|_{E_\nu = E_e},
\]

(A-17)
where \( E_e \) is the root of the equation \( f(E_e) = 0 \). It is equal to

\[
E_e = \frac{E_0 - E_e}{1 - \frac{1}{m_n}(E_e - k_e \cos \theta_{ee})}.
\]  

(A-18)

Then, we get [18]

\[
m_n \frac{1}{E_p} \left| \frac{df(E_p)}{dE_p} \right|_{E_p = E_e} = \frac{1}{1 - \frac{1}{m_n}(E_e - k_e \cos \theta_{ee})}.
\]  

(A-19)

The exact expression for the function \( \Phi_n(k_e, \cos \theta_{ee}) \) is

\[
\Phi_n(k_e, \cos \theta_{ee}) = \frac{1}{\left(1 - \frac{1}{m_n}(E_e - k_e \cos \theta_{ee})\right)^3}.
\]  

(A-20)

Replacing \( \Phi_n(k_e, \cos \theta_{ee}) \) by \( \Phi_n(k_e, \bar{k}_e) [18] \), expanding the r.h.s. of Eq. (A-20) in powers of \( 1/m_N \) and keeping the contributions of order \( 1/m_N^2 \) we get

\[
\Phi_n(k_e, \bar{k}_e) = 1 + 3 \frac{E_e}{m_N} \left(1 - \frac{k_e \cdot \bar{k}_e}{E_e E_p}\right) + 6 \frac{E_e^2}{m_N^2} \left(1 - \frac{k_e \cdot \bar{k}_e}{E_e E_p}\right) \left(1 - \frac{k_e \cdot \bar{k}_e}{E_e E_p} - \frac{1}{4} \frac{E_0}{E_e}\right),
\]  

(A-21)

where we have denoted \( |\bar{k}_e| = E_p = E_0 - E_e [18] \).

Thus, the corrections of order \( O(E_e^2/m_N^2) \), caused by weak magnetism and proton recoil, appear in the neutron lifetime and the correlation coefficients of the neutron beta decay from i) the corrections of order \( O(E_e^2/m_N^2) \) to the amplitude of the neutron beta decay, given by Eq. (A-15), ii) the corrections of order \( O(E_e^2/m_N^2) \) to the phase–volume of the neutron beta decay, iii) the corrections of order \( O(E_e^2/m_N^2) \), caused by the quadratic and crossing terms of order \( O(E_e/m_N) \) in Eq. (A-15) without \( \delta M_n \), iv) the corrections of order \( O(E_e^2/m_N^2) \), obtained by the multiplication of the corrections of order \( O(E_e/m_N) \) to the electron–energy and angular distribution, calculated without account for the contributions of the phase-volume of the neutron beta decay, by the second term of order \( O(E_e/m_N) \) in Eq. (A-21) appearing from the phase-volume of the neutron beta decay, and v) the corrections, induced by the expansion of \( 1/\zeta(E_e) \) in powers of \( 1/m_N \) and \( 1/m_N^2 \), respectively.

Of course, having considered corrections of order \( O(E_e^2/m_N^2) \sim 10^{-5} \) we may, in principle, take into account the contributions of the isovector and axial–vector nucleon form factors [42, 47, 50]. Taking the contributions of these form factors in the dipole approximation we rewrite Eq. (A-15) as follows

\[
\delta M_n = 2m_n \left\{ -2 \left( \frac{q^2}{M_V} \frac{E_e}{E_p} \right) \left[ v_e (1 - 3 \gamma) \right] + g_A \left( \frac{q^2}{M_A} \frac{E_e}{E_p} \right) \left[ \bar{v}_e (1 - 3 \gamma) e \right] \right\} + \frac{E^2_e}{m_N^2} \left[ (2k + 1) \frac{q^2}{E_0} \left[ \bar{v}_e (1 - 3 \gamma) e \right] - 8 \frac{m_e}{E_0} \frac{E_e}{E_p} \left[ \bar{v}_e (1 - 3 \gamma) e \right] \right] \left[ \bar{v}_e (1 - 3 \gamma) e \right] \}
\]  

(A-22)

where in front of the brackets in the first line the factor 2 comes from the dipole approximation of the isovector and axial–vector form factors. The slope-parameters \( M_V \) and \( M_A \) we relate to the charge radius of the proton \( r_p = 0.841 \text{ fm} [10, 11] \) and the axial radius \( r_A = 0.635 \text{ fm} \) of the nucleon [12, 51], respectively. This gives \( M_V = \sqrt{12}/r_p = 813 \text{ MeV} \) and \( M_A = \sqrt{12}/r_A = 1077 \text{ MeV} \). As a result, the electron–energy and angular distribution Eq. (2) acquires the following correction

\[
\frac{d^4 \delta \lambda_n(E_e, k_e, \bar{k}_e, \bar{v}_e, \bar{e}_e)}{dE_e d\Omega_e d\Omega_{\bar{v}} d\bar{e}_e} \propto \left\{ \frac{8}{1 + 3g_A} \left( \frac{E^2_e}{M_V^2} + 4g_A^2 \frac{E^2_e}{M_A^2} \right) - \frac{8g_A}{1 + 3g_A} \left( \frac{E^2_e}{M_V^2} + (1 - 3g_A) \frac{E^2_e}{M_A^2} \right) \frac{\bar{v}_e \cdot \bar{k}_e}{E_e} \left[ \bar{v}_e (1 - 3 \gamma) e \right] \left[ \bar{v}_e (1 - 3 \gamma) e \right] \left[ \bar{v}_e (1 - 3 \gamma) e \right] \right\} \times \frac{E_e}{E_0} \left[ 1 - \frac{E_e}{E_0} \right] \left( 1 - \frac{k_e \cdot \bar{k}_e}{E_e E_p} \right).
\]  

(A-23)
Having neglected the contributions of the terms of order of a few parts of $10^{-7}$ we obtain

\[
\begin{align*}
\frac{d^3\delta\lambda(E_c, \vec{k}_e, \vec{k}_p, \xi, \xi_e)}{dE_c d\Omega_d d\Omega_p} &\propto \left\{ -\frac{8}{1+3g_A^2} \left( E_0^2 + 3g_A^2 E_0^2 \right) \frac{\vec{\xi}_e \cdot \vec{\kappa}_e}{E_c} + \frac{8}{1+3g_A^2} \left( E_0^2 + 3g_A^2 E_0^2 \right) \frac{\vec{\xi}_e \cdot \vec{\kappa}_e}{E_c} - \frac{8g_A}{1+3g_A^2} \right\} \\
&\times \left( \frac{E_0^2}{M_
u^2} + (1+2g_A) \frac{E_0^2}{M_A^2} \right) \left( \frac{\vec{\xi}_e \cdot \vec{\kappa}_e}{E_c} \right) + \frac{8g_A}{1+3g_A^2} \left( E_0^2 + 3g_A^2 E_0^2 \right) \frac{\vec{\xi}_e \cdot \vec{\kappa}_e}{E_c} + \frac{8g_A}{1+3g_A^2} \left( E_0^2 + 3g_A^2 E_0^2 \right) \\
&\times \left( \frac{(\xi_e \cdot \vec{\kappa}_e)(\vec{\xi}_e \cdot \vec{\kappa}_e)}{E_c} \right) + \frac{8g_A}{1+3g_A^2} \left( E_0^2 + 3g_A^2 E_0^2 \right) \\
&\times \left( \frac{(\xi_e \cdot \vec{\kappa}_e)(\vec{\xi}_e \cdot \vec{\kappa}_e)}{E_c} \right) + \frac{8g_A}{1+3g_A^2} \left( E_0^2 + 3g_A^2 E_0^2 \right) \\
&\times \frac{E_0}{1-E_0} \left( 1 - \frac{E_0}{E_0} \right) \\
\end{align*}
\]  

(A-24)

The contributions to the correlation function $\zeta(E_c)$ and the correlation coefficients we define as $\zeta(E_c)_F$, $\zeta(E_c)_F$, $B(E_c)_F$, $G(E_c)_F$ and $K_e(E_c)_F$, respectively.

**Appendix B: Electron–energy and angular distributions of the neutron beta decay, beyond the structure introduced by Jackson et al.**

In this Appendix we adduce the contributions of corrections, caused by weak magnetism and proton recoil to next-to-leading and to next-to-next-to-leading order in the large nucleon mass $m_N$ expansion, which go beyond the correlation structure of the electron–energy and angular distribution of the neutron beta decay, introduced by Jackson et al. We give

\[
\begin{align*}
\frac{d^3\lambda(E_c, \vec{k}_e, \vec{k}_p, \xi, \xi_e)}{dE_c d\Omega_d d\Omega_p} \bigg| \text{(NLO)} & = (1+3g_A^2) \frac{G^2 F^2 |V_{ud}|^2}{32\pi^5} (E_0 - E_c)^2 \sqrt{E_c - m_e^2} E_c F(E_c, Z = 1) \zeta(E_c) \\
&\times \frac{E_0}{m_N} \left\{ -3 \frac{1-g_A^2}{1+3g_A^2} \left( \frac{E_0^2}{E_c} \right)^2 - \frac{1}{3} \right\} + 3 \frac{1-g_A^2}{1+3g_A^2} \left( \frac{\vec{\xi}_e \cdot \vec{\kappa}_e}{E_c} \right) m_e \\
&+ \frac{1}{3} \left( 1 - \frac{m_e}{E_c} \right) \left( \frac{\vec{\xi}_e \cdot \vec{\kappa}_e}{E_c} \right) \right\} \\
(B-1)
\end{align*}
\]
\[
\begin{align*}
&+ \frac{1}{1 + 3g_A^2} \left( \kappa g_A \frac{m_e}{E_0} \left( \xi^a \cdot \vec{k} \right) \left( \xi^b \cdot \vec{k} \right) + \frac{1}{1 + 3g_A^2} \left( 2g_A \frac{E_0 E_\rho}{E_0^2} \right) \left( \xi^a \cdot \vec{k} \right) \left( \xi^b \cdot \vec{k} \right) \right) + \frac{8g_A}{1 + 3g_A^2} \left( \left( \vec{k} \cdot \vec{k} \right) \left( \vec{k} \cdot \vec{k} \right) \right) \left( \vec{k} \cdot \vec{k} \right) \left( \vec{k} \cdot \vec{k} \right) + \frac{1}{1 + 3g_A^2} \left( \kappa g_A \frac{m_e}{E_0} \left( \xi^a \cdot \vec{k} \right) \left( \xi^b \cdot \vec{k} \right) \right) + \frac{8g_A}{1 + 3g_A^2} \left( \left( \vec{k} \cdot \vec{k} \right) \left( \vec{k} \cdot \vec{k} \right) \right) \left( \vec{k} \cdot \vec{k} \right) \left( \vec{k} \cdot \vec{k} \right)
\end{align*}
\]

and

\[
\begin{align*}
\frac{d^2 \lambda^{\text{NLO}}_{10}}{dE_0 \, d\Omega_{\text{d}} \, d\Omega_{\rho}} & = (1 + 3g_A^2) \frac{G_F^2 |V_{ud}|^2}{32\pi^3} \left( E_0 - E_\rho \right)^2 \sqrt{E_0^2 - m_e^2} F(E_0, Z = 1) \left( \xi_\rho \right)
\end{align*}
\]

\[
\begin{align*}
&\times \left( \vec{E}_0^2 \frac{1}{M_{\text{d}}^2} + \frac{1}{1 + 3g_A^2} \left( \frac{1}{1 + 3g_A^2} \left( \xi^a \cdot \vec{k} \right) \left( \xi^b \cdot \vec{k} \right) \right) + \frac{8g_A}{1 + 3g_A^2} \left( \frac{E_0^2}{M_{\text{d}}^2} \right) \left( \xi^a \cdot \vec{k} \right) \left( \xi^b \cdot \vec{k} \right) \right) + \frac{1}{1 + 3g_A^2} \left( \kappa g_A \frac{m_e}{E_0} \left( \xi^a \cdot \vec{k} \right) \left( \xi^b \cdot \vec{k} \right) \right) + \frac{8g_A}{1 + 3g_A^2} \left( \left( \vec{k} \cdot \vec{k} \right) \left( \vec{k} \cdot \vec{k} \right) \right) \left( \vec{k} \cdot \vec{k} \right) \left( \vec{k} \cdot \vec{k} \right)
\end{align*}
\]
respectively, for \( \bar{\nu} \)

They are given by

\[
\bar{\nu}(g_A - (\kappa + 1)) \frac{m_e E_e}{E_0} - g_A (g_A - (\kappa + 1)) \frac{E_e}{E_0} + (\kappa + 1) (g_A - (\kappa + 1)) \frac{E_e}{E_0}
\]

and

\[
+(\kappa + 1) \left( g_A (g_A - (\kappa + 1)) \frac{E_e}{E_0} \right) \left( \frac{\xi_n \cdot \vec{k}_e}{E_e} \right) \cdot \left( \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e m_e} \right) \left( \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e m_e} \right) \left( \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e m_e} \right) \left( \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e m_e} \right)
\]

(B-4)

and

\[
\frac{d^5 \lambda_5^4}{dE_e d\Omega_e d\Omega_{\nu}} \bigg|_{(N^2LO)} = \frac{1}{3 g_\alpha^2} \frac{G_W^2 |V_{ud}|^2}{2 \pi} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} F(E_e, \nu = 1) \zeta(E_e)
\]

\[
\times 3 \frac{E_e}{m_N} \left\{ \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e} - \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e} \right\} - \frac{Q_n(E_e)_{(NLO)}}{E_e E_0} \left( \frac{\vec{\xi}_n \cdot \vec{k}_e}{E_e} \right) \left( \frac{\vec{k}_e \cdot \vec{\xi}_n}{E_e} \right) \left( \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e} \right) - H(E_e)_{(NLO)}
\]

\[
\times \left( \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e E_0} - \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e E_0} \right) - \frac{Q_n(E_e)_{(NLO)}}{E_e E_0} \left( \frac{\vec{\xi}_n \cdot \vec{k}_e}{E_e} \right) \left( \frac{\vec{k}_e \cdot \vec{\xi}_n}{E_e} \right) \left( \frac{\vec{k}_e \cdot \vec{k}_\nu}{E_e} \right)
\]

(B-5)

where the abbreviations NLO and N^2LO mean “next-to-leading” and “next-to-next-to-leading” order in the large nucleon mass \( m_N \) expansion. These terms are proportional to \( 1/m_N \) and \( 1/m_N^2 \), respectively. The contribution to the electron–nucleon and angular distribution, given by Eq. 3[1], was calculated in [14, 17, 18, 25, 26]. The contributions \( \zeta(E_e)_{(NLO)} \) and \( X(E_e)_{(NLO)} \) for \( X = \bar{\alpha}, \bar{A} \) and so on we take from [18, 22, 27]. For completeness of the analysis of the corrections of order \( O(E_e^2/m_N^2) \) we adduce them in Appendix C.

Appendix C: Next-to-leading order corrections of order \( O(E_e/m_N) \), caused by weak magnetism and proton recoil [18, 25, 27]

The next-to-leading order corrections \( O(E_e/m_N) \) in the large nucleon mass expansion, caused by weak magnetism and proton recoil, to the neutron lifetime and correlation coefficients, which we denote as \( \zeta(E_e)_{(NLO)} \) and \( X(E_e)_{(NLO)} \), respectively, for \( X = \bar{\alpha}, \bar{A} \) and so on we take in the form slightly corrected with respect to that calculated in [18, 22, 27]. They are given by

\[
\zeta(E_e)_{(NLO)} = \frac{1}{1 + 3 g_\alpha^2} \frac{E_0}{m_N} \left\{ -2 g_A (g_A + (\kappa + 1)) + (10 g_A^2 + 4 (\kappa + 1) g_A) \right\} \frac{E_e}{E_0}
\]

\[
+ 2 g_A (g_A + (\kappa + 1)) \left( \frac{m_e}{E_0} \right) \left( \frac{m_e}{E_0} \right)
\]

\[
\bar{\alpha}(E_e)_{(NLO)} = \frac{1}{1 + 3 g_\alpha^2} \frac{E_0}{m_N} \left\{ 2 g_A (g_A + (\kappa + 1)) - 4 g_A (3 g_A + (\kappa + 1)) \right\} \frac{E_e}{E_0}
\]

\[
\bar{A}(E_e)_{(NLO)} = \frac{1}{1 + 3 g_\alpha^2} \frac{E_0}{m_N} \left\{ (g_A^2 + g_A (g_A - (\kappa + 1)) - (5 g_A^2 + 3 (\kappa - 4) g_A - (\kappa + 1)) \right\} \frac{E_e}{E_0}
\]

\[
\bar{B}(E_e)_{(NLO)} = \frac{1}{1 + 3 g_\alpha^2} \frac{E_0}{m_N} \left\{ -2 g_A (g_A + (\kappa + 1)) + (7 g_A^2 + 3 (\kappa + 8) g_A + (\kappa + 1)) \right\} \frac{E_e}{E_0}
\]

\[
+ (g_A^2 + (\kappa + 2) g_A + (\kappa + 1)) \left( \frac{m_e}{E_0} \right) \left( \frac{m_e}{E_0} \right)
\]

\[
\bar{K}_n(E_e)_{(NLO)} = \frac{1}{1 + 3 g_\alpha^2} \frac{E_0}{m_N} \left\{ 5 (g_A^2 + (\kappa + 4) g_A - (\kappa + 1)) \right\} \frac{E_e}{E_0}
\]

\[
\bar{Q}_n(E_e)_{(NLO)} = \frac{1}{1 + 3 g_\alpha^2} \frac{E_0}{m_N} \left\{ (g_A^2 + (\kappa + 2) g_A + (\kappa + 1)) - (10 g_A^2 + 4 (\kappa + 1) g_A + 2) \right\} \frac{E_e}{E_0}
\]

\[
\bar{G}(E_e)_{(NLO)} = \frac{1}{1 + 3 g_\alpha^2} \frac{E_0}{m_N} \left\{ (2 g_A^2 + 2 (\kappa + 1) g_A) - (10 g_A^2 + 4 (\kappa + 1) g_A + 2) \right\} \frac{E_e}{E_0}
\]

\[
\bar{H}(E_e)_{(NLO)} = \frac{1}{1 + 3 g_\alpha^2} \frac{E_0}{m_N} \left\{ -2 g_A (g_A + (\kappa + 1)) + (4 g_A^2 + 2 (\kappa + 1) g_A) \right\} \frac{E_e}{E_0}
\]
\[
\begin{align*}
\bar{N}(E_e)_{(NLO)} &= \frac{1}{1 + 3g_A^2/m_N} \frac{E_0}{E_e} \frac{m_e}{m_N} \left[ -\left( \frac{4}{3} g_A^2 + \left( \frac{4}{3} \kappa - \frac{1}{3} \right) g_A - \frac{2}{3} (\kappa + 1) \right) \right. \\
&\quad + \left. \left( \frac{16}{3} g_A^2 + \left( \frac{4}{3} \kappa - \frac{16}{3} \right) g_A - \frac{2}{3} (\kappa + 1) \right) \frac{E_e}{E_0} \right], \\
\bar{Q}_e(E_e)_{(NLO)} &= \frac{1}{1 + 3g_A^2/m_N} \frac{E_0}{E_e} \frac{m_e}{m_N} \left[ -\left( \frac{4}{3} g_A^2 + \left( \frac{4}{3} \kappa - \frac{1}{3} \right) g_A - \frac{2}{3} (\kappa + 1) \right) + \left( 2 g_A^2 + (2\kappa + 1) g_A \right) \frac{m_e}{E_0} \right. \\
&\quad + \left. \left( \frac{22}{3} g_A^2 + \left( \frac{10}{3} \kappa - \frac{10}{3} \right) g_A - \frac{2}{3} (\kappa + 1) \right) \frac{E_e}{E_0} \right], \\
\bar{K}_e(E_e)_{(NLO)} &= \frac{1}{1 + 3g_A^2/m_N} \frac{E_0}{E_e} \frac{m_e}{m_N} \left[ -2g_A (g_A + (\kappa + 1)) + \left( 8 g_A^2 + 2(\kappa + 1)g_A + 2 \right) \frac{m_e}{E_0} \right. \\
&\quad + \left. 4g_A (3g_A + (\kappa + 1)) \frac{E_0}{E_e} \right].
\end{align*}
\] (C-1)

This corrections we give for the completeness of the analysis of corrections of order \(O(E_e^2/m_N^2)\), which we have carried out in this paper. They should be used for the calculation of the contributions \(\bar{X}(E_e)_{(LO)} \bar{\zeta}(E_e)_{(NLO)}\) and \(\bar{X}(E_e)_{(NLO)} \bar{\zeta}(E_e)_{(NLO)}\) to the corrections \(X(E_e)_{(N^2LO)}\) of order \(O(E_e^2/m_N^2)\) (see Eq. (1)).
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