Proper Time Formalism and Gauge Invariance in Open String Interactions

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Abstract

The issue of gauge invariances in the sigma model formalism is discussed at the free and interacting level. The problem of deriving gauge invariant interacting equations can be addressed using the proper time formalism. This formalism is discussed, both for point particles and strings. The covariant Klein Gordon equation arises in a geometric way from the boundary terms. This formalism is similar to the background independent open string formalism introduced by Witten.
1 Introduction

An understanding of the gauge and other symmetries in string theory is of the utmost importance in understanding the physical significance of strings. This is lacking at the moment. From a practical point of view we would like the symmetries to be manifest in the computational scheme also. An approach that looks promising to us in this respect is the loop variable approach [1] which is a generalization of the sigma model renormalization group method [3, 4, 5, 6, 7, 8, 9]. However the work in [1, 2] deals with the free theory. One needs to extend it to include interactions. There are several issues that arise: one is the question of modifying the gauge transformations. The second is the question of massive modes and finally there is the issue of going off shell. There is a well defined answer to these questions in string field theory [13] but we would like to approach it in the loop variable framework because of the computational simplicity. The loop variable approach was developed as an extension of the results of [14, 15] to gauge invariant interactions. In [14] it was shown that the equations of motion of the tachyon in string theory can be written as a proper time equation by analogy with point particles. The connection with the renormalization group follows from the fact that the proper time $\tau$ in string theory is related to the coordinate $z$ of the sigma model by $z = e^{\tau + i\sigma}$ and so $\frac{4}{d\tau}$ is a generator of scale transformations. It was also shown in [14] that if one keeps a finite cutoff one finds that instead of obtaining the low energy non polynomial effective equations of motion where the massive modes are integrated out, one gets an equation in which the massive modes are present and which, for an appropriate choice of the cutoff, is quadratic in the fields. For the special case of a tachyon we showed in [15] that the off shell 3-tachyon vertex of string field theory can be reproduced if we keep a finite cutoff. In the language of vertex operators a finite cutoff is equivalent to a hole of finite radius on the world sheet. If one lets the radius go to zero one recovers the usual punctured world sheet. In this case the vertex operator has to be of dimension (1,1) or equivalently the particle has to be on shell. If we keep a finite radius, on the other hand, the particle can be off shell. In the language of the renormalization group if one is far away from the fixed point and one has all the irrelevant operators then, effectively, you have a cutoff in the theory. When the cutoff goes to zero one is pushed towards the neighbourhood of a fixed point where only the marginal and relevant operators are present. Conversely if one is to keep a finite radius...
(cutoff) then one should keep all the massive modes. All this analysis has been done for the tachyon.

If one keeps track of the reparametrizations of the boundary of this hole in the world sheet, then one needs extra variables in the theory and it turns out that this enables one to write down gauge invariant (free) equations for the massive modes \[1, 2\]. In order to extend the results obtained for the tachyon to higher mass states what needs to be done is to generalize this construction to the interacting case. Fortunately, for the massless vector one does not need all this machinery to maintain gauge invariance. In this paper we concentrate on the massless case and for simplicity we stay close to the mass shell. It will turn out that the proper time formalism can be extended to describe this situation in a straightforward way. We will do it both for the point particle and the string. The results of \[14, 15\] suggest that it should be possible to extend this off the mass shell also. We will also discuss briefly the propagation of a gauge (point) particle.

This paper is organized as follows: In Section 2 we describe briefly three different schemes for deriving free gauge invariant equations in the sigma model formalism. In Section 3 we describe the proper time formalism for a particle in a background vector field. The mechanism of gauge invariance in the interacting case can be understood from this example. In Section 4 we extend this to strings and discuss the mechanism of gauge invariance there. In Section 5 we give some concluding remarks and point out the similarity with Witten’s formulation of the background independent open string equation.
2 Gauge Invariance in the Sigma Model Formalism

Let us describe three different ways of deriving the equations of motion for a massless vector field, i.e. Maxwell’s equations, in the open string. They each involve imposing some requirements on the vertex operator:

\[
\int dz V(x) \equiv A_\mu(x) \partial_z X^\mu \equiv \int dz \int dk A_\mu(k)e^{ikX^\mu} \partial_z X^\mu \tag{2.1}
\]

**Method I**: We require that \( \frac{\delta}{\delta \sigma} V(x) |_{\sigma=0} = 0 \) where the \( \sigma \)-dependence arises due to ultraviolet divergences that we usually remove by normal ordering. Thus:

\[
V(x) =: V_N(x, \sigma) : \tag{2.2}
\]

To get the \( \sigma \)-dependence of the expression in (2.1) a simple method is to consider the vertex operator \( e^{i(kX + A \partial X)} \) and write it as

\[
\exp(i(kX + A \partial X) + \frac{k^2}{2} < XX > + A.k < X \partial X >) \tag{2.3}
\]

\[
= \exp(i(kX + A \partial X) + k^2 \sigma + A.k \sigma)
\]

We have used \(< XX > = 2\sigma \) and \(< X \partial X > = \partial \sigma \). Expanding the exponent and keeping the term linear in \( A \) we get

\[
A_\mu(k)e^{ikX^\mu} \partial_z X^\mu = A_\mu(k) : e^{ikX^\mu} \partial_z X^\mu : e^{k^2\sigma} \tag{2.4}
\]

\[
-ik.A : e^{ikX} : \partial_\sigma e^{k^2\sigma}
\]

Varying w.r.t. \( \sigma \) gives

\[
(k^2A_\mu(k) - k_\mu k.A) : e^{ikX} \partial X^\mu := 0 \tag{2.5}
\]

which is nothing other than \( \partial_\mu F_{\mu\nu} = 0 \) in momentum space. Note that the crucial point (for gauge invariance) in this derivation is the fact that \( \sigma \) depends on \( z \). This is already a generalization of the usual \( \beta \)-function method where we require \( \frac{dV}{da} = 0 \), where \( a \) is a fixed cutoff. One way to think of this is that the flat world sheet cutoff \( a \) is being replaced by \( ae^\sigma \). To lowest order in \( \sigma \) this is sufficient. To get results accurate to higher orders one can replace the cutoff by the geodesic distance, as has been done for
instance in [16]. There are other ways of obtaining the higher order pieces also. Another crucial feature is that in deriving (2.5) one has to perform an integration by parts. This assumes that there are no surface terms. This will not be true when we include interactions.

**Method II**: We impose

\[ L_0 V = 0 = L_1 V \tag{2.6} \]

where \( L_n \) are the Virasoro generators. \([L_n V = 0 \text{ trivially for } n > 1]\). Naively this imposes two requirements on the vertex operator:

\[ k^2 A^\mu = 0 \text{ and } k.A = 0 \tag{2.7} \]

the so called ’physical state’ conditions. However note that we have the freedom to add to \( V \) vertex operators of the form

\[ B(k)k.\partial X e^{ikX} = B(k)\partial_x e^{ikX} = L_{-1}(B e^{ikX}) \tag{2.8} \]

i.e. a total derivative. Thus (2.7) becomes

\[ k^2 A^\mu + k^\mu k^2 B = 0 \tag{2.9} \]

and

\[ k.A + k^2 B = 0 \]

In the first equation we can replace \( k^2 B \) by \(-k.A\) and obtain eqn(2.5):

\[ k^2 A^\mu - k^\mu k.A = 0 \]

The role played by the Liouville mode is taken over by the auxiliary field \( B \).

**Method III**: We require \( \{Q, cV\} = 0 \) where \( Q \) is the BRST operator and \( c \) is the ghost (fermionic) field. Using

\[ Q = \oint dz c(z)[-1/2\partial X\partial X + \partial cb] \tag{2.10} \]

and \( V \) as before we get

\[ \{Q, cV\} = 1/2(A.k\partial^2 cc(z) - ik^2 A_\mu \partial X^\mu \partial c(z)c(z)) \tag{2.11} \]

Setting the RHS of (2.10) to zero we would get the usual physical state conditions (2.7). However we can add to \( cV \) another operator of the same dimension and ghost number:

\[ W = B(k)\partial_x e^{ikX} \tag{2.12} \]
\[
\{Q, W\} = (Bc\partial^2 c + ik\partial X Bc\partial c)e^{ikX} \tag{2.13}
\]

Thus we should actually require that \(\{Q, cV + W\} = 0\) and this gives two equations:

\[A.k/2 - B = 0 \tag{2.14}\]

and

\[k^2/2A^\mu - k^\mu B = 0 \tag{2.15}\]

which, combined together, give Maxwell’s equation. Note that this method is very similar to method II in that we need an auxiliary field \(B\).

Each of these methods can be generalized to the massive cases as well. Before we describe that let us describe the gauge transformations. In method III it is obvious:

\[\delta(cV) = [Q, \Lambda] \tag{2.16}\]

where \(\Lambda\) has ghost number zero, since \(\{Q, [Q, \Lambda]\} = 0\) identically (in 26 dimensions). That is we can add to the vertex operator \(cV\) the piece \([Q, \Lambda]\) and it does not affect the BRST invariance properties.

Thus letting \(\Lambda = \Lambda_0 e^{ikX}\) we get

\[[Q, \Lambda] = cik^\mu \Lambda \partial X^\mu e^{ikX} + k^2/2\partial c e^{ikX} \tag{2.17}\]

This method is obviously the sigma model version of Witten’s string field theory equation\[13\]:

\[Q\Psi = 0 \tag{2.18}\]

and has the gauge invariance:

\[\delta\Psi = Q\Lambda \tag{2.19}\]

The generalization to higher mass levels is immediate - it is just a matter of writing down the relevant vertex operators. Although we will not need it in this paper we will, for future reference, give very briefly the results for the next mass level. The general vertex operator is

\[W = [S^\mu \partial^2 X^\mu + S^{\mu\nu} \partial X^\mu \partial X^\nu + D\partial^2 c] \tag{2.20}\]
\[ B^\mu \partial c \partial X^\mu + E c \partial c b e^{ikX} \]

The equations are \( \{Q, W\} = 0 \).

\[ - (k^2/2 + 1) S^\mu + B^\mu + i k^\mu D = 0 \quad (2.21) \]
\[ - S^\mu + i k^\mu S^{\mu \nu} + i k^\mu D + B^\mu = 0 \]
\[ i k . S/3 + S^\mu_\mu/6 + D + 2/3 E = 0 \]
\[ (1 + k^2/2) D + i k . B/2 - 3/2 E = 0 \]
\[ (k^2/2 + 1) S^{\mu \nu} - i k^\mu B^\nu + 1/2 \delta^{\mu \nu} E = 0 \]

and the gauge transformations are \([Q, \Lambda]\) with

\[ \Lambda = [\Lambda^\mu \partial X^\mu + \Lambda c b] \quad (2.22) \]

which gives:

\[ \delta S^\mu = \Lambda^\mu - i k^\mu \Lambda \quad (2.23) \]
\[ \delta D = - i k . \Lambda/2 - 3/2 \Lambda \]
\[ \delta E = -(k^2/2 + 1) \Lambda \]
\[ \delta S^{\mu \nu} = i/2 (k^\mu \Lambda^\nu) + 1/2 \delta^{\mu \nu} \Lambda \]
\[ \delta B^\mu = (k^2/2 + 1) \Lambda^\mu \]

(2.19) is invariant under (2.21) only in 26 dimensions.

In method II the gauge transformation evidently corresponds to the freedom of adding a piece \( L_{-1} B e^{ikX} \) to the vertex operator \( A_\mu \partial X^\mu e^{ikX} \). The point is that this ambiguity is already allowed for by the addition of (2.8) and hence a fortiori is an invariance of the equations of motion. The generalization to higher mass levels would be to add

\[ L_{-n} \Psi_n \]

(2.24)

to the vertex operator \( V \) and then impose

\[ L_m (V + \Sigma_n L_{-n} \Psi_n) = 0 \quad (2.25) \]

The equations obtained on eliminating the \( \Psi_n \) are guaranteed to have gauge invariance of the form \( V \rightarrow V + L_{-n} \Lambda_n \) [12]. This is the sigma model version.
of the Banks-Peskin string field theory. Of course, as shown there, this naive generalization, while it has all the gauge invariances, does not correspond to string theory. One has to get rid of many redundant fields and gauge invariances associated with those fields. The end result is a fairly involved expression for the equation of motion\cite{12}. Nevertheless one could, if one so desired, transcribe these results to the sigma model framework.

Finally, in method I gauge invariance corresponds to the freedom to add total derivatives of the form $\partial_z \Lambda(X)$ to the action (2.1). The generalization to massive modes is what is described in detail in \cite{1, 2}. It involves introducing an infinite number of new variables $x_n$ and vertex operators are expressed as derivatives in $x_n$ rather than $z$. The freedom to add total derivatives in $z$ is generalized to that of adding total derivatives in $x_n$. This method is closest in spirit to the renormalization group since in the end we still require $\delta \delta \sigma V = 0$. The gauge transformations in this method are fairly simple\cite{1, 2}. We will not describe it here since we are not going to discuss the massive modes.

In this section we have described three approaches to understanding the issue of gauge invariance in the sigma model language, at the free level. We now have to generalize this to the interacting level. The BRST method (III) has been generalized in the string field theory language to the interacting level \cite{13} and in a form more closely related to sigma model and two dimensional field theory \cite{27, 29, 17, 18, 19}. We are looking for an analogous generalization for the first method. At the free level there appear to be certain advantages to this method and the hope is that this may be true at the interacting level also. In this paper we will restrict ourselves to the massless vector (and the tachyon) - so we will not need the extra variables used in the loop variable generalization of the first method.
3 The Proper Time Formalism and Gauge Invariance for Point Particles

The proper time formalism for free particles is well known \[20, 21, 22, 23, 24, 25\]. In \[14\] we modified it to describe a self interacting scalar particle. It was then shown that one could write a very similar equation for strings and this led directly to a proof of the proportionality of the equations of motion and the $\beta$ - function (for the tachyon). Describing gauge theories in the first quantized formalism is a little harder. A lot of work has been done in applying the BRST formalism to this end \[26\]. In this section we want to describe a point particle in a background gauge field using the proper time formalism. We will also discuss briefly the propagation of a gauge particle itself (albeit a free one) which is a little trickier.

The proper time equation for a massless free relativistic particle is

$$\frac{\partial \phi[X, \tau]}{\partial \tau} = \Box \phi[X, \tau] = 0 \quad (3.1)$$

The solution to the first part of the equation is

$$\phi[X, \tau] = \int dX_i \int_{X(0)=X_i}^{X(T)=X_f} D X e^{i/2 \int_0^T d\tau (\frac{\partial X}{\partial \tau})^2} \phi[X, 0] \quad (3.2)$$

The kernel in equation (3.2) is the evolution operator in proper time. Integrating over $T$ from 0 to $\infty$ sets $d\phi/d\tau = 0$ in eqn.(3.1) and gives us the Klein Gordon propagator. We will use (3.2) and require $d\phi/d\tau = 0$ as in \[14\]. We can, if we want, now modify the action to include various backgrounds and then requiring $d\phi/d\tau = 0$ should give the required generalization of (3.1) to the interacting equation. In \[14\] this was done for a self interacting scalar field.

Following \[14\] we write

$$\phi(k', \tau) = \int dk < e^{ik'X(\tau)} e^{ikX(0)}> \phi(k, 0) \quad (3.3)$$

However unlike \[14\] the expectation is calculated using the action

$$\int_0^T d\tau [1/2 (\frac{\partial X}{\partial \tau})^2 + A_\mu \frac{\partial X^\mu}{\partial \tau}] \quad (3.4)$$

The free two point function is given by:

$$<X^\mu(\tau_1)X^\nu(\tau_2)> = \delta^{\mu\nu} | \tau_1 - \tau_2 | \quad (3.5)$$
To lowest order we get using momentum conservation

\[ \phi(k, \tau) = e^{k^2 \tau} \phi(k, 0) \]  

(3.6)

Requiring \( \frac{d\phi}{d\tau} |_{\tau=0} = 0 \) gives \( k^2 \phi = 0 \) - the massless Klein Gordon equation.

To next order we have to calculate

\[ \int_0^T d\tau_1 < e^{ik'X(\tau)} \dot{X}(\tau_1) e^{ipX(\tau_1)} e^{ikX(0)} > \]  

(3.7)

In (3.7) we have written \( A_\mu(x) \frac{\partial X}{\partial \tau} \) as \( \int dp A_\mu(p) e^{ikX(\tau)} \dot{X}^\mu \). The range of integration is restricted from 0 to \( T \). We can simplify the calculation by exponentiating \( \dot{X}(\tau_1) \) into \( e^{i(p.X(\tau_1) + p_1.X(\tau_1))} \) and we will remember in the end to keep the piece linear in \( p_1 \). We get, for (3.7),

\[ \int d\tau_1 \exp(k'.p(\tau - \tau_1) - k'.p_1 + p.k\tau_1 + p_1.k + k'.k\tau) \]  

(3.8)

The linear piece in \( p_1 \) gives

\[ (p_1.k - p_1.k') \int_0^\tau d\tau_1 \exp((k'.p + k'.k)\tau - k'.p\tau_1 + k.p\tau_1) \]  

(3.9)

which in turn gives (using \( k + k' + p = 0 \))

\[ (p_1.k - p_1.k') e^{-k'^2 \tau} \left[ \frac{e^{(k.p-k'.p)\tau} - 1}{p.(k-k')} \right] \]  

(3.10)

Setting \( k'^2 = 0 \) and requiring \( \frac{d}{d\tau} |_{\tau=0} = 0 \) gives the piece (replacing \( p_1 \) with \( A_\mu(p) \))

\[ (A.k - A.k') \phi(k) = (2A(p).k + A(p).p) \phi(k) \]  

(3.11)

To next order we have to calculate

\[ \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 < e^{ik'X(\tau)} e^{i(p.X(\tau_1) + p_1.X(\tau_1))} e^{i(q.X(\tau_2) + q_1.X(\tau_2))} e^{ikX(0)} > \]  

(3.12)

In calculating this expression we need correlators like

\[ < X(\tau_1)X(\tau_2) > \]  

and it is important to keep track of the absolute value prescription in (3.5) (otherwise the correlator vanishes). To lowest order in momentum we have

\[ \lim_{\epsilon \to 0} p_1.q_1 \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 < \left[ \frac{X(\tau_1 + \epsilon) - X(\tau_1 - \epsilon)}{2\epsilon} \right] [\frac{X(\tau_2 + \epsilon) - X(\tau_2 - \epsilon)}{2\epsilon}] > \]  

(3.13)
As long as \( \tau_2 < \tau_1 - 2\epsilon \) the correlator is zero. Otherwise it gives

\[
\int_{\tau_1 - 2\epsilon}^{\tau_1} d\tau_2 (2(\tau_1 - \tau_2) - 4\epsilon) = -4\epsilon^2
\]

(3.14)

Thus (3.13) gives \(-p_1.q_1\tau\) and acting on it with \( \frac{d}{d\tau} \) gives \(-p_1.q_1\) or \(-A^2\).

Adding all three contributions gives \((i\partial - A)^2\phi\) the Klein Gordon equation in a background electromagnetic field. The other pieces from (3.12) give zero when we act with \( \frac{d}{d\tau} \bigg|_{\tau=0} \) on them.

From (3.4) one can see that the construction is gauge invariant. The transformation \( A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\Lambda\) does not leave the action invariant but results in a boundary term:

\[
\int_0^T d\tau \dot{X} \frac{d\Lambda}{dX} = \Lambda(T) - \Lambda(0)
\]

(3.15)

This results in a phase, which can be compensated by a gauge transformation

\[
\phi(\tau) \rightarrow e^{i\Lambda(\tau)}\phi(\tau)
\]

(3.16)

As explained in the last section, gauge invariance at the free level is due to the freedom to add total derivatives. However if there are boundary terms then the action is not invariant. This is the situation when one has interactions. We then have to compensate by the transformation (3.16). This is the origin of inhomogeneous terms, i.e. those of the form \( \delta\phi = i\Lambda\phi \), (as against terms of the form \( \delta A_{\mu} = \partial_{\mu}\Lambda \)) - they arise from boundaries of the integration region. It is not obvious in the calculation of the covariant Klein Gordon equation that the interaction terms \( A_{\mu}\partial^\mu\phi, \partial.A\phi, A.A\phi \) also arise in this manner (from surface terms), but this is in fact the case. In the next section we will repeat the calculation in a way that makes this fact manifest.

One can now ask the following question: We understand how gauge invariance is maintained as far as background gauge fields are concerned. What about deriving equations of motion for the gauge particle itself (i.e. Maxwell’s or Yang Mills equations) in this formalism? This is a little tricky since we do not usually treat the electromagnetic field in first quantized form. However motivated by strings we can extend the previous discussion and consider an object of the form

\[
<k_1.\dot{X}(\tau)e^{ik.X(\tau)}A_1.\dot{X}(0)e^{ip.X(0)}>
\]

(3.17)
and require $\frac{d}{d\tau} |_{\tau=0} = 0$ as before. \footnote{\textdoublespace} We immediately run into a problem - that of gauge invariance. In eqn.(3.4) the vertex operator $\dot{X}^\mu(\tau)$ was integrated over $\tau$. So it was a gauge invariant expression (except for surface terms which we took care of by transformong $\phi$). $\dot{X}^\mu(0)$ in the unintegrated form has no such gauge invariance. We will therefore modify (3.17) to

$$\int d\tau_1 \int d\tau_2 \langle k_1 \dot{X}(\tau)e^{ik.X(\tau)} A_1 \dot{X}(0)e^{ip.X(0)} \rangle \quad (3.18)$$

This construction is gauge invariant but now the proper time equation makes no sense - since $\tau_1$ and $\tau_2$ are both integrated over. One must generalize the proper time prescription. We can do as follows: We know that $\langle X(\tau)X(0) \rangle = | \tau |$. Let us treat the entity $\langle X(\tau)X(0) \rangle$ as a field $\Sigma(\tau)$ and require $\frac{\delta}{\delta \Sigma} = 0$. Here $\Sigma$ plays the same role as the Liouville mode $\sigma$ in section 2. As in sec.2 the integrals $\int d\tau_1 \int d\tau_2$ allow us to integrate by parts. In that case (3.18) gives

$$\int d\tau_1 \int d\tau_2 [k_1 A(p) \partial_{\tau_1} \partial_{\tau_2} \langle X(\tau_1)X(\tau_2) \rangle]$$

$$+ k_1 pA.k \partial_{\tau_1} \langle X(\tau_1)X(\tau_2) \rangle \partial_{\tau_2} \langle X(\tau_1)X(\tau_2) \rangle \rangle e^{k.X(\tau_1)X(\tau_2)} = \int d\tau_1 \int d\tau_2 [k_1 A(p) \partial_{\tau_1} \partial_{\tau_2} \Sigma(\tau_1 - \tau_2)]$$

$$+ k_1 pA.k \partial_{\tau_1} \Sigma(\tau_1 - \tau_2) \partial_{\tau_2} \Sigma(\tau_1 - \tau_2) \rangle e^{k.p \Sigma(\tau_1 - \tau_2)}$$

Varying w.r.t $\Sigma$ gives

$$(k_1 A.k - k_1 pA.k) \partial_{\tau_1} \partial_{\tau_2} \Sigma(\tau_1 - \tau_2) e^{k.p \Sigma(\tau_1 - \tau_2)} \quad (3.20)$$

Set $p_0 = -k_0$ (momentum conservation) and look at the coefficient of $\sigma_1^\mu$: It gives Maxwell’s equation $\partial_\mu F^{\mu\nu} = 0$. The same method obviously works for strings also since we never needed the explicit form of the two point function of $X$.

\footnote{In string theory $\dot{X}$ acts on the ground state and excites it to a vector state. There is no such interpretation for a point particle. Perhaps we can think of $\dot{X} | 0 \rangle$ as a current source for a photon. For our purposes we will not worry about interpreting it but we will formally treat it just as in string theory since that is our real interest in any case.}
To summarize this section, we have derived the gauge invariant equation for a scalar using the proper time method. We have also shown how the proper time formalism can be used for gauge particles at the free level. Both these can be immediately generalized to strings.
4 Proper Time Formalism and Gauge Invariance for Strings

We now apply the proper time formalism to strings: Replace $\tau$ by $\ln z$ to get

$$\left[ \frac{d}{d\ln z} - 2 \right] < e^{ik'X(z)}e^{ikX(0)}> \phi(k) = 0$$

(4.1)

In sec.2 we derived equations of motion by requiring that the vertex operator have dimension one. In eqn.4.1 we have two vertex operators and so it has dimension two and hence should fall off as $1/z^2$ as equation (4.1) indicates.

We will calculate the expectation value using the action

$$\frac{1}{2} \int d^2z \partial_z X \bar{\partial}_z X + \int_0^w A_\mu \partial_z X^\mu$$

(4.2)

The action has the gauge invariance

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \phi \rightarrow e^{i\Lambda} \phi$$

(4.3)

as in the point particle case. The two point function is:

$$< X(z_1)X(z_2) > = \ln(z_1 - z_2), z_1 \neq z_2$$

(4.4)

$$= \ln(ae^\sigma), z_1 = z_2$$

(4.5)

However we will just leave it as $< X(z_1)X(z_2) >$ till the end of the calculation. To lowest order we get from (4.1) $(k^2 - 2)\phi$. At the next order we have

$$< e^{ik'X(z)} \int_w^z dz_1 A_\mu \partial_z X^\mu(z_1)e^{ikX(z_1)}e^{ipX(w)}>$$

(4.6)

which gives

$$\int_w^z dz_1 [iA.k'\partial_{z_1} < X(z)X(z_1)> + iA.p\partial_{z_1} < X(z_1)X(w)>]$$

(4.7)

$$exp(k.k' < X(z)X(z_1)> + k.p < X(z_1)X(w)> + k'.p < X(z)X(w)> )$$

To lowest order we get the surface terms:

$$iA.k'[< X(z)X(z) > - < X(z)X(w) > ] +$$

(4.8)

$$iA.p[< X(z)X(w) > - < X(w)X(w) > ]$$
\[-i(A.k' - A.p)\ln\left(\frac{z - w}{a}\right)\]

This contributes \(-i(A.k' - A.p)\) to the equation of motion.

At the next order we have

\[< e^{ik'X(z)} \int_{w}^{z} du \int_{w}^{u} dv A(k) \partial X(u) e^{iqX(u)} A(q) \partial X(v) e^{ipX(w)}> (4.9)\]

Again to lowest order in momenta we get

\[\int_{w}^{z} du \int_{w}^{u} dv A(k) A(q) < \partial_{u} X(u) \partial_{v} X(v)> (4.10)\]

\[= \int_{w}^{z} du A(k) A(q) [\partial_{u} X(u) X(u) - < \partial_{u} X(u) X(w)>] \]

\[= \int_{w}^{z} du [1/2 \partial_{u} X(u) X(u) - \partial_{u} X(u) X(w)] \]

\[= A(k) A(q) [1/2 < X(z) X(z) > - < X(w) X(w) >] \]

\[- < X(z) X(w) > + < X(w) X(w) >] \]

\[= A.A \ln\left(\frac{z - w}{a}\right) (4.11)\]

Adding up all the pieces we get \((\partial - A)^2 \phi = 0\). In following the steps from (4.6) to (4.10) one can see how each contribution is the surface term in an integral and how they conspire to reproduce the gauge invariance as described in eqn.(4.3). All this works exactly the same way as for the point particle since we never really needed to know the functional form of the two point function. In fact as indicated at the end of the last section we could have just required \(\delta \frac{\delta}{\delta <X(z)X(w)>} = 2\) instead of \(\frac{d}{d\ln(z-w)} = 2\).

In this section we have concentrated on understanding the features that are common to particles and strings, in particular, those that deal with the massless gauge invariance. We have shown that the proper time formalism can be made gauge invariant. In this section we kept only the lowest order (in momentum) terms. For point particles if we had similarly kept only the lowest order terms the result (i.e. the Klein-Gordon equation) would still be exact, as the calculation in Section 3 shows. Thus the higher order terms must

\[\delta <X(z)X(w)>, \int dz \int dw < \partial_z X e^{ikX} \partial_w X e^{ipX} > = 0.\]
vanish. This is not so for strings, however. There are higher order corrections to the Klein Gordon equation that ought to be evaluated. Some of these have been calculated in various approximation schemes [28, 27, 29]. It should be possible, however, to do it in a systematic way where the degree to which the massive modes are integrated out can be controlled. The parameter that controls this would be the cutoff of the two dimensional field theory. The proper time formalism [14, 15] appears to be a way of implementing this idea.
5 Conclusion

In this paper we have attempted to understand gauge invariance in the framework of the renormalization group both at the free level and interacting case. Our aim is to have an understanding at the computational level rather than a formal proof of gauge invariance. To this end we have made some progress in understanding gauge invariance of the massless particle at the interacting level provided we stay close to the mass shell. One can also address these questions in the BRST framework. We saw in the second section the similarities between the two approaches at the free level. In fact proceeding to the interacting theory we can see that eqn.(4.1) is very similar to the equation based on the Batalin-Vilkovisky formalism used in [17, 18, 19]. Instead of \( d/d\ln z \) acting on the two point function one can have \( Q_{BRST} \) act on it. Witten’s anti bracket is essentially the Zamolodchikov metric-the two point function. If we were to include ghosts and use \( cV \) instead of \( V \) in (4.1) (\( c \) being the reparametrization ghost) we would have Witten’s anti bracket. In fact we have already seen in Sect3 that when dealing with gauge particles the vertex operator should be integrated over. Thus we should have \( \int dzV \) (which has the same dimension as \( cV \)). Thus this formalism seems very similar to that of [17, 18, 19].

We would like to extend the results of this paper by going off shell and including the massive modes. This issue can be hopefully addressed in this formalism, just as was done for the case of the tachyon, by keeping a finite cutoff. As we change the value of the cutoff one should be able to interpolate continuously from a string field theory where all the modes are present to a low energy effective action obtained via the sigma model formalism. Presumably the extra coordinates of [1] will need to be introduced to maintain reparametrization invariance. We hope to return to these questions.

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