The solution of conformable Laguerre differential equation using conformable Laplace transform

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Abstract
In this paper, the conformable Laguerre and associated Laguerre differential equations are solved using the Laplace transform. The solution is found to be in exact agreement with that obtained using the power series. In addition some of properties of the Laguerre polynomial is discussed and the conformable Rodriguez's Formula and generating function are proposed and proved

Keywords: conformable derivative, Rodriguez's Formula, associated Laguerre polynomial.

1 Introduction
Special functions have developed from a wide array of practical challenges that attract not only mathematicians, but also other academics in science who want to learn more about their qualities, characteristics, and applications. The special functions are induced as solutions of well-known differential equations, one of which is the Laguerre differential equation.
A specific sort of integral transform is the Laplace Transform. When a function \( f(t) \) is considered, the appropriate Laplace Transform is \( \mathcal{L}[f(t)] \), where \( \mathcal{L} \) is the operator applied to the time domain function \( f(t) \). A function’s Laplace Transform yields a new function with complex frequencies. The Laplace Transform, like the Fourier Transform, is used to solve differential and integral equations. It’s also widely used in the investigation of transient occurrences in electrical circuits, where frequency domain analysis is employed [1]. The fractional derivative has played an important role in physics, mathematics and engineering sciences [2-11]. The definition of fractional
derivative and fractional integral subject to several approaches\cite{2-13} i.e: Riemann-Liouville fractional derivatives, Caputo, Riesz, Riesz-Caputo, Weyl, Gr" unwald-Letnikov, Hadamard, and Chen derivatives.

Recently, a new definition of derivative of fractional order was presented by Khalil et.al \cite{14} called the conformable derivative. This definition is a natural extension of the usual derivative. The conformable derivative of the constant is zero, and satisfies the standard properties of the traditional derivative i.e the derivative of the product and the derivative of the quotient of two functions and satisfies the chain rule. In addition, one can say the conformable derivative is simple and singular to the standard derivative. For this reason, the last few years, the conformable fractional calculus is applied successfully in various fields (math \cite{15-21}, physics \cite{22-29}, modeling \cite{30-32}, biology \cite{33, 34}). Besides, The conformable fractional Euler-Lagrange equation and Hamiltonian formulation were discussed by Lazo and Torres \cite{35} and the laplace transform in conformable derivative is presented by Abdeljawad \cite{36}.

The power series and the Laplace transformation are two different methods for solving differential equations \cite{37}. The conformable laguerre differential equation is solved using conformable power series in ref \cite{38, 39}. Recently, even though Abdelhakim \cite{40}, pointed that the conformable derivative is not fractional derivative but it is a derivative of fractional order and it is simple and applicable which makes it of interests. Thus, we call it conformable derivative. In this paper, we want to solve the conformable Laguerre and associated Laguerre differential equation using conformable Laplace transform and discuss their properties. Besides, the Conformable Rodriguez’s Formula is proposed and proved.

## 2 Conformable Laplace transform

We start by presenting some definitions related to our work.

**Definition 2.1.** we denote a function \( f \in [0, \infty) \rightarrow \mathbb{R} \). The conformable derivative of \( f \) with order \( \alpha \) is defined by \cite{14}

\[
T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon} \tag{1}
\]

**Definition 2.2.** \( I_\alpha^a(f)(t) = I^a_\alpha(t^{a-1}f) = \int_{t_0}^{t} \frac{f(x)}{x^a} dx \) where the integral is the usual Riemann improper integral and \( \alpha \in (0, 1) \).

**Definition 2.3.** let \( t_0 \in \mathbb{R}, 0 < \alpha \leq 1 \) and \( f : [t_0, \infty) \rightarrow \text{real valued function} \). Then the conformable Laplace transform of order \( \alpha \) is defined by

\[
\mathcal{L}_\alpha f(t_0)(s) = \int_{t_0}^{\infty} e^{-s(t-t_0)^{1-\alpha}} f(t) dt = \int_{t_0}^{\infty} e^{-s(t-t_0)^{1-\alpha}} f(t)(t-t_0)^{\alpha-1} dt. \tag{2}
\]

Following to ref \cite{36}, the conformable Laplace transform for some certain functions

\[
(i) \quad \mathcal{L}_\alpha t_0 [1] = \frac{1}{s}, \quad s > 0 \tag{3}
\]
\begin{align}
\text{(ii)} \quad L^0_\alpha[t] &= \frac{t_0}{s} + \alpha s^{\frac{1}{\alpha}} \frac{\Gamma(1 + \frac{1}{\alpha})}{s^{1 + \frac{1}{\alpha}}}, s > 0 \\
\text{(iii)} \quad L^0_\alpha[t^p] &= \alpha s^{\frac{1}{\alpha}} \frac{\Gamma(1 + \frac{p}{\alpha})}{s^{1 + \frac{p}{\alpha}}}, s > 0 \\
\text{(iv)} \quad L^0_\alpha[e^{\frac{t}{\alpha}}] &= \frac{1}{s - 1}, s > 1 \\
\text{(v)} \quad L^0_\alpha[sin \frac{t}{\alpha}] &= \frac{1}{s^2 + \pi^2} \\
\text{(vi)} \quad L^0_\alpha[cos \frac{t}{\alpha}] &= \frac{s}{s^2 + \pi^2}
\end{align}

Some properties: let \( f \) and \( g \) be functions defined on \([0, \infty)\) and let \( \lambda, \mu, a \in \mathbb{R} \) and \( 0 < \alpha \leq 1 \). Then [41, 42]

\begin{align}
\text{(i)} \quad L_\alpha[\lambda f + \mu g](t) &= (\lambda L_\alpha f + \mu L_\alpha g)(t) = \lambda F_\alpha(s) + \mu G_\alpha(s), s > 0 \\
\text{(ii)} \quad L_\alpha[e^{-\frac{at}{\alpha}} f(t)] &= F_\alpha(s + a), s > |a| \\
\text{(iii)} \quad L_\alpha[t^{n\alpha} f(t)] &= (-1)^n \frac{d^n}{ds^n} F_\alpha(s), s > 0 \\
\text{(iv)} \quad L_\alpha[f * g](t) &= F_\alpha(s) * G_\alpha(s) \\
\text{(v)} \quad L_\alpha[T_\alpha f(t)] &= sF_\alpha(s) - f(0) \\
\text{(vi)} \quad L_\alpha[T_\alpha^2 f(t)] &= s^2 F_\alpha(s) - sf(0), \quad 0 < \alpha \leq \frac{1}{2}
\end{align}

In this paper, we adopt \( D^\alpha f \) to denote the conformable derivative of \( f \) with order \( \alpha \).

3 Conformable Laguerre differential equation

The conformable laguerre differential equation can be written by replacing the integer derivative by the conformable derivative of order \( \alpha \) as

\[ x^\alpha D^\alpha D^\alpha y + (\alpha - x^\alpha) \frac{d^n}{ds^n} y + nax y = 0 \quad (15) \]

where \( 0 < \alpha \leq 1 \) and \( n \) is a natural number.

The conformable Laplace transform of this equation is

\[ L_\alpha[x^\alpha D^\alpha D^\alpha y] + \alpha L_\alpha[D^\alpha y] - L_\alpha[x^\alpha D^\alpha y] + n\alpha L_\alpha[y] = 0. \quad (16) \]
Making use of conformable Laplace properties we have

\[ L_{\alpha}[x^{\alpha}D^{\alpha}D^{\alpha}y] = -\alpha \frac{d}{ds} L_{\alpha}[D^{\alpha}D^{\alpha}y] = -\alpha \frac{d}{ds} [s^2 Y_{\alpha}(s) - sy(0)] \]

\[ = -\alpha s^2 \frac{d}{ds} Y_{\alpha}(s) - 2s\alpha Y_{\alpha}(s) + \alpha y(0), \]

\[ = -\alpha s^2 \frac{d}{ds} Y_{\alpha}(s) - 2s\alpha Y_{\alpha}(s) + \alpha \]  \hspace{1cm} (17)

where \( y(0) = 1 \) and

\[ L_{\alpha}[D^{\alpha}y] = sY_{\alpha}(s) - y(0) \]

\[ = sY_{\alpha}(s) - 1. \]  \hspace{1cm} (18)

Also,

\[ L_{\alpha}[x^{\alpha}D^{\alpha}y] = -\alpha \frac{d}{ds} L_{\alpha}[D^{\alpha}y] = -\alpha \frac{d}{ds} [sY_{\alpha}(s) - y(0)] \]

\[ = -\alpha s \frac{d}{ds} Y_{\alpha}(s) + \alpha s \frac{d}{ds} Y_{\alpha}(s) + Y_{\alpha}(s). \]

Then, Eq. (16) takes the form

\[ -\alpha s(s-1) \frac{d}{ds} Y_{\alpha}(s) + \alpha (n+1-s) Y_{\alpha}(s) = 0, \]

where \( L_{\alpha}[y] = Y_{\alpha}(s) \) and dividing by \(-\alpha s(s-1)\), we have

\[ \frac{d}{ds} Y_{\alpha}(s) - \frac{(n+1-s)}{s(s-1)} Y_{\alpha}(s) = 0. \] \hspace{1cm} (20)

Then, the solution read as

\[ Y_{\alpha}(s) = \frac{(s-1)^n}{s^{n+1}}. \] \hspace{1cm} (21)

Which can be written as

\[ Y_{\alpha}(s) = \frac{1}{s} \left(1 - \frac{1}{s}\right)^n. \] \hspace{1cm} (22)

It can be expanded in series representation as following,

\[ Y_{\alpha}(s) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)k!} \frac{1}{s^{k+1}}. \] \hspace{1cm} (23)

Using inverse conformable Laplace transform \( L_{\alpha}^{-1} \), we have

\[ L_{\alpha}^{-1} Y_{\alpha}(s) = y(x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)k!} L_{\alpha}^{-1} \frac{1}{s^{k+1}}. \] \hspace{1cm} (24)
Setting \( k = \frac{p}{\alpha} \), then

\[
L_{s}^{-1} \frac{1}{s^{\frac{p}{\alpha}+1}} = \frac{\alpha^p \Gamma(1 + \frac{p}{\alpha})}{\alpha^p \Gamma(1 + \frac{p}{\alpha})} \frac{1}{s^{\frac{p}{\alpha}+1}},
\]

\[
= \frac{1}{\alpha^p \Gamma(1 + \frac{p}{\alpha})} \frac{\alpha^p \Gamma(1 + \frac{p}{\alpha})}{s^{\frac{p}{\alpha}+1}},
\]

\[
= \frac{1}{\alpha^p \Gamma(1 + \frac{p}{\alpha})} x^p.
\]

Now, replacing \( p \) by \( k\alpha \), we get

\[
L_{s}^{-1} \frac{1}{s^{k\alpha+1}} = \frac{1}{\alpha^k \Gamma(1 + k)} x^{k\alpha} = \frac{1}{\alpha^k k!} x^{k\alpha}.
\]  

(25)

Thus, the solution of Eq. (15) takes the form

\[
y(x) = \sum_{k=0}^{n} \frac{(-1)^k \alpha^k (n-k)!}{\alpha^k (n-k)! (k!)^2} x^{k\alpha}.
\]  

(26)

Here, we obtained the solution of the laguerre conformable differential equation using conformable Laplace transform and it is found to be in exact agreement with the solution obtained using conformable power series, in Ref [38, 39].

### 3.1 Conformable Rodriguez’s Formula

In this subsection we propose the conformable Rodriguez formula in the following form

\[
L_n \left( \frac{x^{\alpha}}{\alpha} \right) = e^{\frac{x^{\alpha}}{\alpha}} D^{n\alpha} \left[ x^{\alpha} e^{-\frac{x^{\alpha}}{\alpha}} \right].
\]

(27)

Which is equivalent to conformable Laguerre polynomial.

**Proof.** Using conformable Leibniz rule

\[
D^{n\alpha} (fg) = \sum_{k=0}^{n} \binom{n}{k} D^{(n-k)\alpha} f D^{k\alpha} g.
\]

(28)

we get

\[
D^{n\alpha} \left[ x^{\alpha} e^{-\frac{x^{\alpha}}{\alpha}} \right] = \sum_{k=0}^{n} \binom{n}{k} D^{(n-k)\alpha} x^{\alpha} D^{k\alpha} e^{-\frac{x^{\alpha}}{\alpha}},
\]

\[
D^{n\alpha} \left[ x^{\alpha} e^{-\frac{x^{\alpha}}{\alpha}} \right] = \sum_{k=0}^{n} \frac{(-1)^k \alpha^{n-k}(n!)^2}{(k!)^2(n-k)!} x^{k\alpha} e^{-\frac{x^{\alpha}}{\alpha}}.
\]

(29)

Now, substituting in eq. (27)

\[
L_n \left( \frac{x^{\alpha}}{\alpha} \right) = e^{\frac{x^{\alpha}}{\alpha}} \sum_{k=0}^{n} \frac{(-1)^k \alpha^{n-k}(n!)^2}{(k!)^2(n-k)!} x^{k\alpha} e^{-\frac{x^{\alpha}}{\alpha}},
\]

\[
= \sum_{k=0}^{n} \frac{(-1)^k n!}{\alpha^k (k!)^2(n-k)!} x^{k\alpha}.
\]
So, our proposed formula is equivalent to the conformable polynomial eq. \( \text{(26)} \). Figures 1 to 5 depict the behavior of conformable Laguerre functions of several orders for different values of \( \alpha \). One can see that as \( \alpha \) approaches 1, the Laguerre functions approaches to the traditional one.

Figure 1: Plot of \( L_1(\frac{z^\alpha}{\alpha}) = 1 - \frac{z^\alpha}{\alpha} \)

Figure 2: Plot of \( L_2(\frac{z^\alpha}{\alpha}) = 1 + \frac{z^{2\alpha}}{2\alpha} - 2\frac{z^\alpha}{\alpha} \)
Figure 3: Plot of $L_3\left(\frac{x^\alpha}{\alpha}\right) = \frac{-x^{3\alpha} - 3\alpha x^{2\alpha} + 18\alpha^2 x^\alpha - 6\alpha^3}{6\alpha^2}$

Figure 4: Plot of $L_4\left(\frac{x^\alpha}{\alpha}\right) = 1 + \frac{x^{4\alpha}}{24\alpha^4} - 2\frac{x^{3\alpha}}{3\alpha^3} + 3\frac{x^{2\alpha}}{\alpha^2} - 4\frac{x^{\alpha}}{\alpha}$
The conformable Laguerre polynomials developed here allowed us to gradually manage the polynomials we employed by adjusting the alpha value. We can see how conformable Laguerre polynomials behave when alpha values change without affecting their behavior in these figures.

3.2 Some remarks of conformable Laguerre functions

1. The values of the conformable Laguerre polynomial at \( x = 0 \), read as

\[
\begin{align*}
(i) & \quad L_n(0) = 1, \\
(ii) & \quad D^\alpha L_n(0) = -k, \\
(iii) & \quad D^\alpha D^\alpha L_n(0) = \frac{n(n-1)}{2}.
\end{align*}
\]

Proof. (i) Following to Ref. [38], the generating function of the conformable Laguerre functions read as

\[
\frac{1}{1 - t e^{-\frac{x^\alpha}{\alpha}}} = \sum_{n=0}^{\infty} L_n(\frac{x^\alpha}{\alpha}) t^n, \quad |t| < 1
\]

Setting \( x = 0 \), we recovered

\[
\frac{1}{1 - t} = \sum_{n=0}^{\infty} L_n(0) t^n.
\]

Then, we obtain \( L_n(0) = 1 \) where \( \frac{1}{1 - t} = \sum_{n=0}^{\infty} t^n. \)

(ii) The general Laguerre conformable differential equation read as

\[
x^\alpha D^\alpha D^\alpha L_n(x^\alpha) + (\alpha - x^\alpha) D^\alpha L_n(x^\alpha) + k\alpha L_n(x^\alpha) = 0,
\]

taking \( x = 0 \), we have

\[
\alpha D^\alpha L_n(0) + k\alpha L_n(0) = 0.
\]
But \( L_n(0) = 1 \), then \( D^\alpha L_n(0) = -k \).

(iii) Taking the conformable derivative of order \( \alpha \) of both sides with respect to eq. (33) twice, we have the following relation

\[
\frac{1}{1 - t} \frac{t^2}{(1 - t)^3} e^{-\frac{x^\alpha t}{\alpha(1 - t)}} = \sum_{n=0}^{\infty} D^\alpha D^\alpha L_n\left(\frac{x^\alpha}{\alpha}\right)t^n,
\]

substituting \( x = 0 \), we get

\[
\frac{t^2}{(1 - t)^3} = \sum_{n=0}^{\infty} D^\alpha D^\alpha L_n(0)t^n,
\]

the left hand side can be expanded as

\[
t^2[1 + 3t + \frac{3 \cdot 4}{2!} t^2 + \frac{3 \cdot 4 \cdot 5}{3!} t^3 + \cdots + \frac{3 \cdot 4 \cdot 5 \cdots \cdot n}{(n - 2)!} t^{n-2} + \cdots] = \sum_{n=0}^{\infty} D^\alpha D^\alpha L_n(0)t^n.
\]

Thus, Equating coefficient of \( t^n \), we get

\[
D^\alpha D^\alpha L_n(0) = \frac{3 \cdot 4 \cdot 5 \cdots \cdot n}{(n - 2)!} = \frac{n!}{(n - 2)!} = \frac{n(n - 1)}{2}.
\]

2. The conformable Laguerre functions are orthogonal

\[
\int_0^{\infty} e^{-\frac{x^\alpha}{\alpha}} L_n\left(\frac{x^\alpha}{\alpha}\right)L_m\left(\frac{x^\alpha}{\alpha}\right) d^\alpha x = \delta_{nm}
\]

**Proof.** Using the generating function

\[
\frac{1}{1 - t} e^{-\frac{x^\alpha}{\alpha}(1 - t)} = \sum_{n=0}^{\infty} L_n\left(\frac{x^\alpha}{\alpha}\right)t^n,
\]

and replacing \( t \) by \( s \)

\[
\frac{1}{1 - s} e^{-\frac{x^\alpha}{\alpha}(1 - s)} = \sum_{m=0}^{\infty} L_m\left(\frac{x^\alpha}{\alpha}\right)s^m.
\]

Thus, we have

\[
\frac{1}{(1 - t)(1 - s)} e^{-\frac{x^\alpha}{\alpha}(1 - s)} e^{-\frac{x^\alpha}{\alpha}(1 - t)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} L_n\left(\frac{x^\alpha}{\alpha}\right)L_m\left(\frac{x^\alpha}{\alpha}\right)s^m t^n.
\]

Multiplying both sides by \( e^{-\frac{x^\alpha}{\alpha}} \) and integrating over \( d^\alpha x \), we have

\[
\frac{1}{(1 - t)(1 - s)} \int_0^{\infty} e^{-\frac{x^\alpha}{\alpha}} e^{-\frac{x^\alpha}{\alpha}(1 - s)} e^{-\frac{x^\alpha}{\alpha}(1 - t)} d^\alpha x = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^{\infty} e^{-\frac{x^\alpha}{\alpha}} L_n\left(\frac{x^\alpha}{\alpha}\right)L_m\left(\frac{x^\alpha}{\alpha}\right)s^m t^n d^\alpha x.
\]
The integral in the left hand side can be calculated as
\[
\int_{0}^{\infty} e^{-\frac{x}{x^\alpha}(1+\frac{1}{1-t})} x^{\alpha-1} dx = \frac{1}{1 + \frac{t}{(1-t)} + \frac{s}{(1-s)}}
\]
Substituting this result, we have
\[
\frac{1}{(1-st)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{0}^{\infty} e^{-\frac{n}{x^\alpha}} L_n\left(\frac{x^\alpha}{\alpha}\right) L_m\left(\frac{x^\alpha}{\alpha}\right) s^m t^n d\alpha x.
\]
Then we have the value of the integral as series representations as
\[
\frac{1}{(1-st)} = \sum_{n=0}^{\infty} s^n t^n.
\]
Then the general relation becomes
\[
\sum_{n=0}^{\infty} s^n t^n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{0}^{\infty} e^{-\frac{n}{x^\alpha}} L_n\left(\frac{x^\alpha}{\alpha}\right) L_m\left(\frac{x^\alpha}{\alpha}\right) s^m t^n d\alpha x.
\]
It is clear that, if \(m = n\), we have
\[
\int_{0}^{\infty} e^{-\frac{n}{x^\alpha}} L_n\left(\frac{x^\alpha}{\alpha}\right) L_m\left(\frac{x^\alpha}{\alpha}\right) d\alpha x = 1.
\]
and if \(m \neq n\), we have
\[
\int_{0}^{\infty} e^{-\frac{n}{x^\alpha}} L_n\left(\frac{x^\alpha}{\alpha}\right) L_m\left(\frac{x^\alpha}{\alpha}\right) d\alpha x = 0.
\]

4 Conformable associated Laguerre differential equation

The conformable associated Laguerre differential equation is defined by
\[
x^\alpha D^\alpha D^\alpha y + (m\alpha + \alpha - x^\alpha) D^\alpha y + n\alpha y = 0 \tag{35}
\]

**Theorem.** If \(S(x)\) is a solution of conformable Laguerre differential equation of order \((m+n)\), then \(D^{\alpha m} S\) will be a solution of conformable associated Laguerre differential equation.

**Proof.** The conformable Laguerre differential equation of order \((m+n)\) can be written as
\[
x^\alpha D^\alpha D^\alpha S + (\alpha - x^\alpha) D^\alpha S + (n + m)\alpha S = 0. \tag{36}
\]
Taking conformable derivative \(m\) times \(D^{\alpha m}\), we get
\[
D^{\alpha m}[x^\alpha D^\alpha D^\alpha S] + D^{\alpha m}[(\alpha - x^\alpha) D^\alpha S] + D^{\alpha m}[(n + m)\alpha S] = 0
\]
Making use of Leibniz rule, we get

\[x^\alpha D^{(m+2)} S + m\alpha D^{(m+1)} S + \alpha D^{(m+1)} S - x^\alpha D^{(m+1)} S - m\alpha D^{\alpha} + (n+m)\alpha D^{\alpha} S = 0,\]

which can be written as

\[x^\alpha D^{(m+2)} S + (m\alpha + \alpha - x^\alpha) D^{(m+1)} S + m\alpha D^{\alpha} S = 0.\]

This equation can be rearranged as

\[x^\alpha D^{\alpha} D^{(m+1)} S + (m\alpha + \alpha - x^\alpha) D^{\alpha} D^{(m+1)} S + m\alpha D^{\alpha} S = 0.\]

Thus, if \(S(x)\) is a solution of conformable Laguerre differential equation, then \(D^{\alpha} S\) is a solution of the conformable associated Laguerre differential equation.

So, we obtain

\[L_n^m(x^\alpha) = D^{\alpha} L_n^{m+1}(x^\alpha),\]

then from the definition, we get

\[L_n^m(x^\alpha) = (-1)^m D^{\alpha} L_n^{m+1}(x^\alpha). \tag{37}\]

This is the conformable associated Laguerre functions.

### 4.1 Conformable Associated Laguerre functions

The conformable associated Laguerre polynomial can be written as

\[L_n^m(x^\alpha) = \sum_{r=0}^{n} \left(-1\right)^r \frac{(n+m)!}{\alpha^r (n-r)!(r+m)!} x^r \alpha^r. \tag{38}\]

**Proof.** According eq. (26), we can write the conformable Laguerre functions of order \((m+n)\) as

\[L_{m+n}(x^\alpha) = \sum_{k=0}^{n} \left(-1\right)^k \frac{(n+m)!}{\alpha^k (n+m-k)! (k)!} x^k \alpha^k. \tag{39}\]

Substituting it in eq. (37), we get

\[L_n^m(x^\alpha) = (-1)^m D^{\alpha} \sum_{k=0}^{n} \left(-1\right)^k \frac{(n+m)!}{\alpha^k (n+m-k)! (k)!} x^k \alpha^k,\]

\[= \sum_{k=m}^{n+m} \left(-1\right)^{k+m} \frac{(n+m)!}{\alpha^k (n+m-k)! (k)!} \alpha k^m x^{(k-m)\alpha},\]

\[= \sum_{k=m}^{n+m} \left(-1\right)^{k+m} \frac{(n+m)!}{\alpha^{k-m} (n+m-k)! (k-m)!} x^{(k-m)\alpha}.\]

Put \(r = k - m\), we have the eq. (38). One may recover the formula (38) by setting \(r = k - m\).
4.2 Conformable Rodriguez’s formula

In this subsection, we show that the conformable Rodriguez formula for the conformable associated Laguerre polynomial takes the form

\[
L_n^m\left(\frac{x^\alpha}{\alpha}\right) = \frac{x^{-m\alpha} e^{\frac{\alpha}{\alpha} n!}}{\alpha^n n!} D^{n\alpha}\left[x^{(n+m)\alpha} e^{-\frac{x^\alpha}{\alpha}}\right].
\] (40)

**Proof.** Using Leibniz rule

\[
D^{n\alpha}(fg) = \sum_{r=0}^{n} \binom{n}{r} D^{(n-r)\alpha} f D^{r\alpha} g
\] (41)

we get

\[
D^{n\alpha}\left[x^{(n+m)\alpha} e^{-\frac{x^\alpha}{\alpha}}\right] = \sum_{k=0}^{n} \binom{n}{r} D^{(n-r)\alpha} x^{(n+m)\alpha} D^{r\alpha} e^{-\frac{x^\alpha}{\alpha}}.
\] Which can be written as,

\[
D^{n\alpha}\left[x^{(n+m)\alpha} e^{-\frac{x^\alpha}{\alpha}}\right] = \sum_{r=0}^{n} \frac{(-1)^r \alpha^{n-r} (n+m)!}{(r+m)!(n-r)!r!} x^{(r+m)\alpha} e^{-\frac{x^\alpha}{\alpha}}.
\] (42)

Now, substituting in eq.(40)

\[
L_n^m\left(\frac{x^\alpha}{\alpha}\right) = \frac{x^{-m\alpha} e^{\frac{\alpha}{\alpha} n!}}{\alpha^n n!} \sum_{r=0}^{n} \frac{(-1)^r \alpha^{n-r} (n+m)!}{(r+m)!(n-r)!r!} x^{(r+m)\alpha} e^{-\frac{x^\alpha}{\alpha}},
\]

\[
= \sum_{r=0}^{n} \frac{(-1)^r (n+m)!}{\alpha^r (r+m)!(n-r)!r!} x^{r\alpha}.
\]

It is equal to eq.(38).

Figures 6 to 11 depict the behavior of conformable associated Laguerre functions of different orders for some values of \(\alpha\).
Figure 6: Plot of $L_1\left(\frac{x^n}{\alpha}\right) = 2 - \frac{x^n}{\alpha}$

Figure 7: Plot of $L_2\left(\frac{x^n}{\alpha}\right) = \frac{x^n}{2\alpha^2} - 3\frac{x^n}{\alpha} + 3$
Figure 8: Plot of $L_2^\alpha(x) = \frac{x^{2\alpha}}{2\alpha^2} - 4\frac{x^\alpha}{\alpha} + 6$

Figure 9: Plot of $L_3^\alpha(x) = -\frac{x^{3\alpha}}{6\alpha^3} + 2\frac{x^{2\alpha}}{\alpha^2} - 6\frac{x^\alpha}{\alpha} + 4$
The conformable associated Laguerre functions developed here allowed us to gradually manage the polynomials we employed by adjusting the alpha value. We can see how conformable associated Laguerre polynomials behave when alpha values change without affecting their behavior in these figures.

4.3 Generating function

The generating function for the conformable associated Laguerre polynomial is proposed as

\[
\frac{e^{-\frac{x^\alpha}{\alpha}}}{(1-t)^{m+1}} = \sum_{n=0}^{\infty} L_m^{(\alpha)} \left( \frac{x^\alpha}{\alpha} \right) t^n. \tag{43}
\]

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Using eq. (33) and differentiating it \( m \) times, we have
\[
D^m \frac{1}{1-t} e^{-\frac{x t}{\alpha(1-t)}} = D^m \sum_{n=0}^{\infty} L_n \left( \frac{x}{\alpha} \right) t^n,
\]
\[
\frac{(-1)^m t^m}{(1-t)^{m+1}} e^{-\frac{x t}{\alpha(1-t)}} = D^m \sum_{n=m}^{\infty} L_n \left( \frac{x}{\alpha} \right) t^n.
\]
In the R.H.S put \( n = m + r \), we have
\[
\frac{(-1)^m t^m}{(1-t)^{m+1}} e^{-\frac{x t}{\alpha(1-t)}} = D^m \sum_{r=0}^{\infty} L_{m+r} \left( \frac{x}{\alpha} \right) t^{m+r},
\]
from eq. (37), we get
\[
\frac{(-1)^m t^m}{(1-t)^{m+1}} e^{-\frac{x t}{\alpha(1-t)}} = \sum_{n=0}^{\infty} (-1)^m L_n \left( \frac{x}{\alpha} \right) t^{m+n}.
\]
It is equal the eq. (43).

5 Conclusions

We have solved the conformable Laguerre and associated Laguerre differential equations using the Laplace transform, we have observed the result found to be in exact agreement with the solution obtained using conformable power series, in Ref [38, 39]. In addition, we have given good proposal for Rodriguez’s formula and the generating function and we shown that the conformable Laguerre functions are orthogonal. Moreover from the figures 1 – 11 one may observe that, the obtained results coincide with those of regular Laguerre and associated Laguerre functions as \( \alpha \) approaches 1.

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