The classical \( r \)-matrix in a geometric framework

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Abstract

We use a Riemannian (or pseudo-Riemannian) geometric framework to formulate the theory of the classical \( r \)-matrix for integrable systems. In this picture the \( r \)-matrix is related to a fourth rank tensor, named the \( r \)-tensor, on the configuration space. The \( r \)-matrix itself carries one connection type index and three tensorial indices. Being defined on the configuration space it has no momentum dependence but is dynamical in the sense of depending on the configuration variables. The tensorial nature of the \( r \)-matrix is used to derive its transformation properties. The resulting transformation formula turns out to be valid for a general \( r \)-matrix structure independently of the geometric framework. Moreover, the entire structure of the \( r \)-matrix equation follows directly from a simple covariant expression involving the Lax matrix and its covariant derivative. Therefore it is argued that the geometric formulation proposed here helps to improve the understanding of general \( r \)-matrix structures. It is also shown how the Jacobi identity gives rise to a generalized dynamical classical Yang-Baxter equation involving the Riemannian curvature.

1 Introduction

The classical \( r \)-matrix is a fundamental object in the study of classically integrable systems \([1, 2]\). It appears at an even more fundamental level than the equally important Lax pair structure. However, the nature of the classical \( r \)-matrix has remained somewhat obscure for almost two decades. For example, some \( r \)-matrices are purely numerical while others depend on the configuration variables (see \([3]\) and references therein) and there are also examples where the \( r \)-matrix depends on the momentum variables \([4]\). The purpose of this paper is to contribute towards a better understanding of the classical \( r \)-matrix. In particular we demonstrate how a general \( r \)-matrix can be transformed by a coordinate transformation. The transformation formula is explained in terms of a geometric interpretation of the \( r \)-matrix. In this picture, the \( r \)-matrix appears as a four-index object carrying three tensorial indices and one connection type index. The formulation given here relies on the geometric formulation proposed earlier \([5, 6]\) for the Lax pair equation \( \dot{L} = [L, A] \) (we use boldface symbols for matrices throughout the paper). In that formulation, the Lax matrix itself (\( L \)) is related to a tensor carrying three indices, called the (first) Lax tensor while the second Lax matrix (\( A \)) is related to a connection type object. In \([5]\) a fully covariant formulation was achieved in which the second Lax matrix was replaced by an object (called the second Lax tensor) of the same tensorial type as the first Lax tensor. Similarly, in the geometric version of the \( r \)-matrix structure to be outlined in this letter the \( r \)-matrix is replaced by a fourth rank tensorial object, named the \( r \)-tensor. We now describe this procedure starting with the already known tensorial Lax pair formulation as given in \([3]\). A more detailed account of this work will be published elsewhere \([7]\).
2 The covariant Lax pair equation

The approach taken here is purely differential geometric without any reference to an underlying group theoretic structure. Also, for simplicity, we use a formulation without spectral parameters. However, we expect that the fundamental conclusions will remain valid also in the presence of spectral parameters. It is assumed that the dynamical system under study is described in terms of the phase space variables \((q^\alpha, p_\beta)\) labelled by greek indices, \(\alpha, \beta, \ldots = 1, 2, \ldots, d\). The equations of motion are taken to be the geodesics with respect to a Riemannian or pseudo-Riemannian metric

\[
d s^2 = g_{\alpha\beta}(q) \, dq^\alpha \, dq^\beta .
\]  

(1)

The system can then be represented by the Hamiltonian

\[
H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta ,
\]  

(2)

where \(g^{\alpha\beta}\) is the matrix inverse of \(g_{\alpha\beta}\). The application of this kind of geometric formalism is however not restricted to purely kinetic Hamiltonians. In fact it is well-known that any Hamiltonian system of the form \(H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + U(q)\) can be put in the form \(\ref{eq:hamiltonian}\) by a change of time variable (see e.g. \cite{2, 3} and cf. also \cite{1} for an alternative geometrization scheme using canonical transformations). To arrive at the geometric formulation of the Lax pair equation

\[
\dot{L} := \{L, H\} = [L, A] ,
\]  

(3)

we write the Lax pair matrices, \(L = (L^\alpha_\beta), A = (A^\alpha_\beta)\) using mixed indices with contravariant (up) indices labelling rows and covariant (down) indices labelling columns. This notation is necessary in order to make matrix multiplication a covariant operation. However, it is in fact very convenient in the noncovariant picture as well. For the covariant formulation we also need the matrices defined by \(L^\mu := \partial L/\partial p_\mu = (L^\alpha_\beta^\mu)\) and \(A^\mu := \partial A/\partial p_\mu = (A^\alpha_\beta^\mu)\).

It is assumed in this paper that the Lax matrices in the geometric picture are linear and homogeneous in the momenta. Although this does not give the most general geometrization framework it is the simplest and perhaps most elegant subcase. Also, as shown in \cite{3}, the 3-particle nonperiodic Toda lattice can be given such a linear geometric Lax formulation. Furthermore, it turns out that an important application given in the present letter, namely the transformation properties of the \(r\)-matrix, will be seen to be valid independently of this assumption. The matrices \(L^\mu\) and \(A^\mu\) are then functions on the configuration space. The object with components \(L^\alpha_\beta^\mu\) will be interpreted as a third rank tensor, the first Lax tensor. The second Lax tensor has components \(B^\alpha_\beta^\mu\) and is defined by \(B^\alpha_\beta^\mu := \partial B/\partial p_\mu = (B^\alpha_\beta^\mu)\) where \(B := A - \Gamma\) and \(\Gamma = (\Gamma^\alpha_\beta)\) is a connection matrix with components given by \(\Gamma^\alpha_\beta = g^{\mu\nu}(\Gamma^\alpha_\beta^\mu p_\mu = g^{\mu\nu}\Gamma^\alpha_\beta^\nu p_\nu\) where \(\Gamma^\alpha_\beta^\nu\) are the Christoffel symbols with respect to the metric \(g\). In this picture \(A^\mu\) will therefore represent a connection type object. Using the fact that \(\ref{eq:hamiltonian}\) is homogeneous and quadratic in the momenta we can write the Lax pair equation on the configuration space as

\[
\partial(\mu L_\nu) = \Gamma^\lambda_\mu_\nu L_\lambda + [L(\mu, A_\nu)] .
\]  

(4)

The covariant derivative of a third rank tensor \(T_\mu = (T^\alpha_\beta_\mu)\) can be written in matrix form as \(\nabla_\mu T_\nu = \partial_\mu T_\nu + [\Gamma_\mu, T_\nu] - \Gamma^\lambda_\mu_\nu T_\lambda\). Using this relation to replace the partial derivative in equation \(\ref{eq:hamiltonian}\) by the covariant derivative we obtain the covariant Lax pair equation

\[
\nabla(\mu L_\nu) = [L(\mu, B_\nu)] ,
\]  

(5)

where \(L_\mu = g_{\mu\nu} L^\nu, B_\mu = g_{\mu\nu} B^\nu\) and \(\nabla_\mu\) represents the covariant derivative associated to the connection given by \(\Gamma^\alpha_\beta_\mu\). It is worth pointing out that the entire structure of the Lax pair equation \(\ref{eq:hamiltonian}\) follows from the symmetrized covariant derivative in the left hand side of \(\ref{eq:hamiltonian}\). This is true even if \(B = 0\) so that the right hand side of \(\ref{eq:hamiltonian}\) vanishes since one can still have a nontrivial Lax pair \(L\) and \(A = \Gamma\).
3 The \( r \)-tensor and the covariant \( r \)-matrix equation

Given that the Lax matrices are elements of a vector space \( \mathcal{V} \) the \( r \)-matrix is defined on the Kronecker product space \( \mathcal{V} \otimes \mathcal{V} \). If \( X = (X^\alpha_\beta) \) and \( Y = (Y^\alpha_\beta) \) are elements of \( \mathcal{V} \) then their Kronecker product has components \( (X \otimes Y)^\alpha_\beta = X^\alpha_\beta Y^\mu_\nu \). Any element \( X \) in \( \mathcal{V} \) can be regarded as an element in \( \mathcal{V} \otimes \mathcal{V} \) by taking its left or right Kronecker product with the unit matrix. For this we use a notation with space indicators placed within parentheses on top of the symbols

\[
\begin{align*}
X_1 &:= X \otimes 1, \\
X_2 &:= 1 \otimes X.
\end{align*}
\]

Also, if \( u \) is an element of \( \mathcal{V} \otimes \mathcal{V} \) with components \( u^{ij} \) with respect to a basis \( E_i \) of \( \mathcal{V} \) \( (i, j, \ldots = 1, 2, \ldots, d^2) \) we write

\[
(1) u = u^{ij}E_i \otimes E_j, \\
(2) u = u^{ij}E_i \otimes E_j.
\]

The \( r \)-matrix equation can then be written

\[
\{ (1), (2) \} = \left[ \{ r, (1) \} - \{ r, (2) \} \right].
\]

We wish to write this equation in covariant form in analogy with the covariant formulation of the Lax pair equation. Computing the Poisson bracket we find that the left hand side becomes

\[
\{ (1), (2) \} = \left( - L^\mu \partial_\mu L^\nu + L^\mu \partial_\mu L^{(1)} + L^{(2)} \right) p_\nu = p_\nu L^\mu \wedge \partial_\mu L^\nu,
\]

where we have introduced an exterior Kronecker product, \( X \wedge Y = X \otimes Y - Y \otimes X \). Since the left hand side of (3) is linear in the momenta the same must be true for the right hand side. The Lax matrix itself being linear in the momenta it is therefore natural to demand that the \( r \)-matrix is momentum independent. We can then write the \( r \)-matrix equation on the configuration space as

\[
L^\mu \wedge \partial_\mu L^\nu = \left[ (1), (2) \right] - \left[ (2), (2) \right].
\]

Using the formula \( \nabla_\mu T^\nu = \partial_\mu T^\nu + \left[ \Gamma_\mu, T^\nu \right] + \Gamma^\nu_\lambda T^\lambda \) to rewrite (10) in terms of covariant derivatives we obtain

\[
L^\mu \wedge \nabla_\mu L^\nu = \left[ (1), (12) \right] - \left[ (2), (2) \right],
\]

where \( R := r + s \) and \( s := r \wedge L^A \). This equation is covariant if we interpret \( R \) as a tensor, the \( r \)-covariant tensor. We will therefore refer to (11) as the covariant \( r \)-matrix equation. For a given solution of (11) the \( r \)-matrix itself is given by the relation

\[
r = \mathcal{R} - \Gamma_\lambda \otimes L^A, \\
r^{\alpha_\beta} = \mathcal{R}^{\alpha_\beta} - \Gamma^{\alpha_\beta} L_\mu ^\lambda \cdot
\]

The covariant formulation given here sheds new light on the \( r \)-matrix equation. In particular, the entire structure of the \( r \)-matrix equation can be derived from the left hand side of (11) \( (cf. \) the discussion above of the Lax pair equation). This is because \( R \) being a tensor we may in principle have solutions of (11) with \( R = 0 \) so that the right hand side is zero. We can then still have a nontrivial \( r \)-matrix given by \( r = -\Gamma_\lambda \otimes L^A \). It is also of interest that the \( r \)-matrix in this formulation is independent of the momenta. It should be noted however that momentum dependence can be introduced by a canonical transformation which destroys the geometric form of the Hamiltonian \( (cf. \) (3)).
4 Transforming the $r$-matrix

Another and important consequence of the geometric structure of the $r$-matrix given here is that the relation \( (12) \) can be used to derive the transformation properties of the $r$-matrix under coordinate transformations. Loosely speaking it follows from \( (12) \) that the $r$-matrix is somewhat of a hybrid object; it is neither a tensor nor a connection in this formulation. More precisely, it carries one connection type index and three tensorial indices.

To write down the transformation formula for the $r$-matrix we use relation \( (12) \) together with the known transformation properties of $R$, $\Gamma_\mu$, and $L^\mu$. A general coordinate transformation is defined by a matrix $U = (U^\mu_\nu)$ and its inverse $U^{-1} = (U^\mu_\nu')$ with components given by

$$U^\mu_\nu = \frac{\partial q^\mu}{\partial q^\nu} \quad U^\mu_\nu' = \frac{\partial q^\mu}{\partial q'^\nu}.$$ \hspace{1cm} (13)

Then since $R$ and $L^\mu$ are tensors they transform according to the formulas

$$L^\mu_\nu' = U^\mu_\nu L^\mu U^{-1}, \quad R' = w R w^{-1},$$ \hspace{1cm} (14)

where $w = U \otimes U$ while $\Gamma_\mu$ being a connection transforms as

$$\Gamma'_{\nu'} = U^\mu_\nu \Gamma^\mu U^{-1} + U_{\partial \nu} U^{-1}.$$ \hspace{1cm} (15)

Using \( (12) \) we then obtain the transformation for the $r$-matrix as

$$r' = wrw^{-1} + (\partial_{\nu} U \otimes U L^\mu) w^{-1} = w (r + U^{-1} \partial_{\nu} U \otimes L^\mu) w^{-1},$$ \hspace{1cm} (16)

Although this formula has been derived under the assumptions of the geometric framework used in this paper, it is nevertheless valid regardless of those assumptions. In fact, it is straightforward to derive it directly from the general form \( (8) \) of the $r$-matrix equation. Essentially the same equation was recently found in that way in \( [10] \). However, the formulation given here has the advantage of yielding a geometric explanation of the transformation formula.

5 The generalized dynamical classical Yang-Baxter equation

In this paragraph we discuss the classical Yang-Baxter equation (CYBE) in the generalized dynamical form which arises in the geometric framework. Since the left hand side of the $r$-matrix equation \( (8) \) involves a Poisson bracket, the right hand side is subject to certain restrictions. It is obvious that the antisymmetry property of the Poisson bracket is automatically incorporated in the structure of the right hand side. However, it turns out that the Jacobi identity leads to nontrivial restrictions. To write down the Jacobi identity in the matrix formalism we need to extend the notation \( (6) \) to the Kronecker product space $V \otimes V \otimes V$

$$X := X \otimes 1 \otimes 1, \quad Y := 1 \otimes X \otimes 1, \quad Z := 1 \otimes 1 \otimes X.$$ \hspace{1cm} (17)

Although the notation then becomes ambiguous (for example $X$ can be an element in both $V \otimes V$ and $V \otimes V \otimes V$) it is commonly used in the literature and does not lead to confusion as long as we keep in mind in which space we are working. We also extend the notation \( (8) \) in a similar way by forming appropriate Kronecker products with the identity matrix in $V$, e.g. \( (13) \) = $u^i_\mu E_i \otimes 1 \otimes E_j$.

A straightforward but somewhat tedious calculation using \( (8) \) then gives

$$\{ \{ L, L \}, L \} + \{ \{ L, L \}, L \} + \{ \{ L, L \}, L \} = [ f, L ] + [ f, L ] + [ f, L ].$$ \hspace{1cm} (18)
where $f := r + q$ and we are using the notation

\[
\begin{align*}
& r := \left[ r, r \right] + \left[ r, h \right] + \left[ r, r \right], \\
& q := \{ L, r \} - \{ r, r \},
\end{align*}
\]

(19a)

(19b)

where the triplet $(a,b,c)$ is a permutation of $(1,2,3)$. From (18) it follows that the Jacobi identity implies the relation

\[
\left[ f, L \right] + \left[ f, L \right] + \left[ f, L \right] = 0.
\]

(20)

This relation may be interpreted as a constraint on the $r$-matrix imposed by the Jacobi identity. Trying to write (20) in covariant form we first note that if the $r$-matrix has no momentum dependence then (19a) becomes

\[
\begin{align*}
& q = L^a \partial_a r - L^a \partial_a r .
\end{align*}
\]

(21)

The next step is to use (12) to replace the $r$-matrix by the $r$-tensor, $\mathcal{R}$. We must also replace the partial derivative by the covariant derivative. For a fourth rank tensor $u = (u^{\alpha\beta\mu\nu})$ (such as the $r$-tensor) the covariant derivative can be written in matrix form as

\[
\nabla_\lambda (a^b) u = \partial_\lambda (a^b) u + \{ \Gamma_\lambda + \Gamma_{\lambda, a^b} u \}.
\]

(22)

After another rather long calculation the result is

\[
\begin{align*}
& f = \mathcal{R} + L^\mu \nabla_\mu (a^b) - L^\mu \nabla_\mu (a^b) \mathcal{R} + L^\mu L^\nu (a^b) F_{\mu\nu} ,
\end{align*}
\]

(23)

where the $F_{\mu\nu}$ are curvature matrices representing the components of the Riemann tensor according to

\[
F_{\mu\nu} = (R^{\alpha\beta\mu\nu}) = 2\partial_{[\mu} \Gamma_{\nu]} + [\Gamma_{\mu}, \Gamma_{\nu}] .
\]

(24)

The relation (23) together with (24) represents the covariant form of the restriction on the $r$-matrix implied by the Jacobi identity. It can therefore be viewed as a covariant generalized dynamical CYBE. One often imposes additional restrictions which serve as sufficient conditions to satisfy the Jacobi identity constraint (20). The most general sufficient conditions so far were given in (3). A commonly used restriction is to assume that the $r$-matrix is purely numerical and satisfies the unitarity condition $(12) r = -(21) r$. Then any solution of the original CYBE (note the difference in the space indicators in the last term compared to (19a))

\[
\left[ r, r \right] + \left[ r, h \right] + \left[ r, r \right] = 0 ,
\]

(25)

satisfies the Jacobi identity constraint (20). In the geometric context such restrictions can be regarded as gauge conditions leading to noncovariant Yang-Baxter equations. Judging from the form of (23) there seems to be no simple covariant restriction on $\mathcal{R}$ which solves (24) for general metrics $g_{\mu\nu}$. If the geometry is flat so that $F_{\mu\nu} = 0$, then the covariant condition $\mathcal{R} = 0$ solves (24), leading to the expression $r = -\Gamma_\lambda \otimes L^\lambda$ for the $r$-matrix. It should be noted that the flat case is far from trivial. For example, suppose $L = L^\mu p_\mu$ is a Lax matrix for the flat geometry and $\mathcal{R}$ is an accompanying $r$-tensor (possibly zero). Define a new geometry by $h^{\mu\nu} := \text{Tr}(L^\mu L^\nu)$. In general, $h^{\mu\nu}$ is a non-flat metric and the corresponding Hamiltonian system is therefore integrable but physically different from the original flat system.
6 Outlook

The formalism introduced in this paper by itself provides a new view of the nature of the classical $r$-matrix. However, to obtain additional insights into the properties of the classical $r$-matrix it would be useful to have explicit examples of $r$-tensors. In fact it seems natural to try to find such an example by looking for an $r$-tensor for the geometric version of the non-periodic Toda lattice given in [6]. Given that such an $r$-tensor exists, the corresponding $r$-matrix would depend on the configuration variables but not on the momenta. However, going back to the original non-geometric formulation using the inverse of the canonical transformation given in [6], the $r$-matrix would seem to pick up a momentum dependence. Presumably that momentum dependence could then be removed by using the freedom in the construction of the $r$-matrix recently investigated by Braden [11].

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