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ABSTRACT. We prove that convex viscosity solutions to the quadratic Hessian inequality

\[ \sigma_2(D^2 u) \geq 1 \]

are strictly 2-convex. As a consequence we obtain short proofs of smoothness and interior \( C^2 \) estimates for convex viscosity solutions to \( \sigma_2(D^2 u) = 1 \), which were proven using different methods in recent works of Guan-Qiu [GQ], McGonagle-Song-Yuan [MSY] and Shankar-Yuan [SY2].

1. Introduction

In this note we consider convex viscosity solutions to the quadratic Hessian inequality

\[ \sigma_2(D^2 u) \geq 1. \]

Our main result is their strict two-convexity. That is:

**Theorem 1.1.** Let \( u \) be a convex viscosity solution to (1) in \( \Omega \subset \mathbb{R}^n \), and let \( L \) be a supporting linear function to \( u \) in \( \Omega \). Then

\[ \dim \{ u = L \} \leq n - 2. \]

Theorem 1.1 is sharp in view of the example \( u = x_1^2 + x_2^2 \), with \( L = 0 \).

Local smoothness of convex viscosity solutions to

\[ \sigma_2(D^2 u) = 1 \]

follows from Theorem 1.1, using the classical solvability of the Dirichlet problem [CNS] and the Pogorelov-type interior \( C^2 \) estimate from [CW] (see Section 2). With a compactness argument we can in fact prove a universal modulus of strict 2-convexity (see Proposition 4.1). As a result we obtain:

**Theorem 1.2.** Let \( u \) be a convex viscosity solution of (2) in \( B_1 \subset \mathbb{R}^n \). Then \( u \) is smooth, and

\[ |D^2 u(0)| \leq C \left( n, \| u\|_{L^\infty(B_1)} \right). \]

Inequality (3) was recently proven for smooth convex solutions of (2) in [GQ] and [MSY], and Theorem 1.2 was proven in [SY2]. A subtle issue in passing to the viscosity case is that smooth approximations of convex viscosity solutions may not be convex. An advantage of our approach is that it avoids using a priori estimates for smooth convex solutions, which allows us to bypass this issue. The methods in the above-mentioned works are quite different from ours, based in [GQ] on the

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Bernstein technique, and in [MSY] and [SY2] on the properties of the equation for the Legendre-Lewy transform of $u$.

An interesting question is whether the conclusion of Theorem 1.2 holds without assuming that $u$ is convex. It is true when $n = 2$ (in which case solutions are automatically convex and (2) is the Monge-Ampère equation, [H]) and when $n = 3$ (in which case (2) is equivalent to the special Lagrangian equation, [WY]). It is also known to be true if $u$ is slightly non-convex [SY2]. Finally, an interior $C^2$ estimate of the form (3) was recently obtained in [SY1] for smooth solutions to (2) that satisfy the semi-convexity condition $D^2 u \geq -KI$, with $C$ depending also on $K$. The general case in dimension $n \geq 4$ remains open.

Remark 1.3. Local smoothness and interior $C^2$ estimates are false for convex viscosity solutions to the $k$-Hessian equation

$$\sigma_k(D^2 u) = 1$$

when $k \geq 3$, in view of the well-known Pogorelov example ([P], [U]). The same example shows that convex viscosity solutions to $\sigma_k(D^2 u) \geq 1$ are not always strictly $k$-convex when $k \geq 3$. In particular, Theorems 1.1 and 1.2 are both special to the quadratic Hessian equation.

The paper is organized as follows. In Section 2 we recall a few classical results about the $k$-Hessian equation, and we use them to show that Theorem 1.1 implies that convex viscosity solutions of (2) are smooth. In Section 3 we prove Theorem 1.1. Finally, in Section 4 we prove a quantitative version of Theorem 1.1 using a compactness argument, and we use it to complete the proof of Theorem 1.2.

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2. Preliminaries

In this section we recall a few classical facts about the $k$-Hessian equation. Below $\Omega$ denotes a bounded domain in $\mathbb{R}^n$, and $1 \leq k \leq n$.

We first recall some facts about the $\sigma_k$ operator. The function $\sigma_k$ on $\text{Sym}_{n \times n}$ denotes the $k^{th}$ symmetric polynomial of the eigenvalues. It is elliptic on the cone

$$\Gamma_k := \{ M \in \text{Sym}_{n \times n} : \sigma_i(M) > 0 \text{ for each } 1 \leq i \leq k \},$$

and has convex level sets in $\Gamma_k$. Furthermore, the function $\sigma_k$ is uniformly elliptic on compact subsets of $\Gamma_k$.

Next we recall the notion of viscosity solution. We say that a function $u \in C^2(\Omega)$ is $k$-convex if $D^2 u \in \overline{\Gamma_k}$. Given a nonnegative function $f \in C(\Omega)$, we say that a function $u \in C(\Omega)$ is a viscosity solution of

$$\sigma_k(D^2 u) \geq (\leq) f$$

if, whenever a $k$-convex function $\varphi \in C^2(\Omega)$ touches $u$ from above (below) at a point $x_0 \in \Omega$, we have

$$\sigma_k(D^2 \varphi(x_0)) \geq (\leq) f(x_0).$$

We say that $u \in C(\Omega)$ is a viscosity solution of

$$\sigma_k(D^2 u) = f$$
if it is a viscosity solution of both \( \sigma_k(D^2 u) \geq f \) and \( \sigma_k(D^2 u) \leq f \). Viscosity solutions are closed under uniform convergence, and the notions of classical and viscosity solution coincide on \( C^2 \) functions that are \( k \)-convex.

Third we recall the classical solvability of the Dirichlet problem for the \( k \)-Hessian equation, proven in [CNS]:

**Theorem 2.1.** Let \( q \in C^\infty(\partial B_R) \). Then there exists a unique \( k \)-convex solution \( u \in C^\infty(\overline{B_R}) \) to the Dirichlet problem  

\[
\sigma_k(D^2 u) = 1 \text{ in } B_R, \quad u|_{\partial B_R} = g.
\]

The result in fact holds for smooth bounded \( k-1 \)-convex domains.

Finally we recall the Pogorelov-type estimate Theorem 4.1 from [CW]:

**Theorem 2.2.** Assume that \( u \in C^\infty(\overline{\Omega}) \) is a \( k \)-convex solution to  

\[
\sigma_k(D^2 u) = 1 \text{ in } \Omega,
\]

and that there exists a \( k \)-convex function \( w \in C(\overline{\Omega}) \) such that \( u < w \) in \( \Omega \) and \( u = w \) on \( \partial \Omega \). Then

\[
\sup_{\Omega} ((w - u)^4|D^2 u|) \leq C (n, k, \|u\|_{C^1(\Omega)}).
\]

Inequality (4) implies in particular that the equation for \( u \) is uniformly elliptic on compact subdomains of \( \Omega \). By the Evans-Krylov theorem (see [CC]), interior derivative estimates of all higher orders follow.

To conclude the section we show local smoothness of convex viscosity solutions to (2). We assume \( u \) is defined in \( B_1 \subset \mathbb{R}^n \), and it suffices to prove smoothness in a neighborhood of the origin. After subtracting a supporting linear function we may assume that \( u(0) = 0 \) and that \( u \geq 0 \). By Theorem 1.1 we have after a rotation that \( \{u = 0\} \) is contained in the subspace spanned by \( \{e_3, \ldots, e_n\} \). Let  

\[
w_\delta(x) := \delta^2(2(n-2)(x_1^2 + x_2^2) - (x_3^2 + \ldots + x_n^2)),
\]

and notice that \( w_\delta \) is \( 2 \)-convex for all \( \delta > 0 \). Furthermore, we can choose \( \delta, \eta, \mu > 0 \) small (depending on \( u \)) such that  

\[
u > w_\delta + \eta \text{ on } \partial B_{1/2} \quad \text{and} \quad B_R \subset \{u < w_\delta + \eta\}.
\]

Let \( \{v_j\} \) be a sequence of smooth \( 2 \)-convex (but not necessarily convex) solutions to (2) that converge uniformly to \( u \) in \( B_{1/2} \). (One obtains the functions \( v_j \) e.g. by taking smooth approximations to \( u \) on \( \partial B_{1/2} \) and applying Theorem 2.1 with \( R = 1/2 \) and \( k = 2 \).) Applying Theorem 2.2 to \( v_j \) with \( w = w_\delta + \eta \) and \( k = 2 \), we see that the solutions \( v_j \) enjoy uniform derivative estimates of all orders in \( B_\mu \) as \( j \to \infty \). We conclude that \( u \) is smooth in \( B_\mu \).

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1.

**Proof of Theorem 1.1:** Assume by way of contradiction that there exists a supporting linear function \( L \) to \( u \) such that \( \dim\{u = L\} \geq n - 1 \). After subtracting \( L \), translating, rotating, and quadratically rescaling, we may assume that \( u \) is defined in \( B_2 \), that \( u \geq 0 \), and that \( u = 0 \) on \( \{x_n = 0\} \cap B_2 \). After subtracting another supporting linear function of the form \( ax_n \) with \( a \geq 0 \), we may also assume that  

\[
u(te_n) = o(t) \text{ as } t \to 0^+.
\]
Letting $x = (x', x_n)$, it follows that $\{ u < h \}$ contains a cylinder of the form

$$Q_h := \{ |x'| < 1 \} \times (0, H),$$

with $h/H \to 0$ as $h \to 0^+$. For $h$ small, the convex paraboloid

$$P_h := h|x'|^2 + 4 \frac{h}{H^2} (x_n - H/2)^2$$

thus satisfies that $P_h \geq h \geq u|\partial Q_h$, that $P_h(He_n/2) = 0 \leq u$, and that

$$\sigma_2(D^2P_h) = c_1(n)h^2 + c_2(n) \frac{h^2}{H^2} < 1,$$

which contradicts (1). \qed

4. Proof of Theorem 1.2

In this section we prove a quantitative version of Theorem 1.1, and we use it to complete the proof of Theorem 1.2. For a set $S \subset \mathbb{R}^n$ and $r > 0$ we let $S_r$ denote the $r$-neighborhood of $S$.

**Proposition 4.1.** For $K > 0$, $r > 0$ and $n \geq 2$, there exists $\delta(n, K, r) > 0$ such that if $u$ is a convex viscosity solution to (1) in $B_1 \subset \mathbb{R}^n$ with $\| u \|_{L^\infty(B_1)} \leq K$ and $L$ is a supporting linear function to $u$ at 0, then

$$\{ u < L + \delta \} \subset \subset T_r$$

for some $n - 2$-dimensional subspace $T$ of $\mathbb{R}^n$.

**Proof.** Assume not. Then there exist convex viscosity solutions $u_j$ to (1) on $B_1$ with $\| u_j \|_{L^\infty(B_1)} \leq K$ and supporting linear functions $L_j$ at 0 such that the conclusion fails with $\delta = 1/j$. Up to taking a subsequence, the functions $u_j$ converge locally uniformly to a convex viscosity solution $v$ of (1) in $B_1$, and $L_j$ converge to a supporting linear $L$ to $v$ at 0 such that $\{ v = L \}$ is not compactly contained in $T_r$ for any $n - 2$-dimensional subspace $T$. This contradicts Theorem 1.1. \qed

**Proof of Theorem 1.2:** We proved that $u$ is smooth at the end of Section 2. The proof of the estimate (3) follows the same lines. We call a constant universal if it depends only on $n$ and $\| u \|_{L^\infty(B_1)}$. We may assume after subtracting a supporting linear function with universal $C^1$ norm that $u(0) = 0$ and that $u \geq 0$. Write $x = (y, z)$ with $y \in \mathbb{R}^2$ and $z \in \mathbb{R}^{n-2}$. By Proposition 4.1 there exists $\delta > 0$ universal such that, after a rotation, $u > \delta$ on $\{ |y| = 1/(2n) \} \cap B_1$. It follows that

$$u > w := \frac{2(n - 2)|y|^2 - |z|^2 + 1}{8}$$

on the boundary of $B_{3/4} \cap \{ |y| < 1/(2n) \}$. Notice also that $w$ is 2-convex. The estimate (3) follows by applying Theorem 2.2 in the connected component of the set $\{ u < w \}$ that contains the origin. \qed

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