A spectral bound on independence number and applications to eventown problem

Mikhaylo Antipov* and Danila Cherkashin†

January 6, 2022

Abstract

We provide an eigenbound on the independence number of a graph and obtain several corollaries on eventown problem and its generalizations.

1 Introduction

We start with the following bound.

Theorem 1. Let $G = (V, E)$ be a simple graph. Consider a symmetric real matrix $A$ such that $A_{ij} = 1$ for every pair $\{i, j\} \notin E(G)$. Then

$$\alpha(G) \leq \lambda_{\text{max}}(A),$$

where $\alpha(G)$ is the size of maximal independent set of $G$ and $\lambda_{\text{max}}$ is the maximal eigenvalue of $A$.

Proof. Let $I$ be an independent set, denote by $\chi_I$ its characteristic vector. Then

$$(A\chi_I, \chi_I) = |I|^2.$$

From the other hand, one has

$$(A\chi_I, \chi_I) = \sum c_i^2 \lambda_i \leq \left(\sum c_i^2\right) \lambda_{\text{max}} = |I| \cdot \lambda_{\text{max}},$$

where $\chi_I = \sum c_i v_i$ is the decomposition of $\chi_I$ via orthonormal eigenbasis $\{v_i\}$ of $A$. (We use that symmetric matrix has real spectrum and the length of $\chi_I$ is the same in the standard basis and in $\{v_i\}$, i.e. $\sum c_i^2 = |I|$.)

Also we provide the following corollaries. Suppose that $A$ and $G$ satisfy the conditions of Theorem 1. Let $c$, $\lambda_{\text{max}}$ and $\text{sp}$ stand for the minimal entry, the maximal eigenvalue and the spectral radius of $A$, respectively.

Theorem 2. Let $I$ be a set with at most $\varepsilon |I|^2 / 2$ edges inside. Suppose that $(1 - c) \varepsilon < 1$. Then

$$|I| \leq \frac{\lambda_{\text{max}}}{1 - (1 - c) \varepsilon}.$$ 

Theorem 3. Let $I$ and $J$ be subsets of $V(G)$ with at most $\varepsilon |I| \cdot |J|$ edges between $I$ and $J$ (edges in $I \cap J$ are counted twice here). Suppose that $(1 - c) \varepsilon < 1$. Then

$$|I| \cdot |J| \leq \left(\frac{\text{sp}}{1 - (1 - c) \varepsilon}\right)^2.$$ 

*National Research University Higher School of Economics, Soyuza Pechatnikov str., 16, St. Petersburg, Russian Federation.
†Chebyshev Laboratory, St. Petersburg State University, 14th Line V.O., 29, Saint Petersburg 199178, Russia.
1.1 Relation with Hoffman bound

Celebrated Hoffman bound [4] says that if all-one vector is an eigenvector of a symmetric real nonnegative pseudoadjacency matrix $P$ of $G$ then

$$\alpha(G) \leq |V(G)| \frac{-\lambda_{\text{min}}(P)}{\lambda_{\text{max}}(P) - \lambda_{\text{min}}(P)},$$

where $\lambda_{\text{max}}(P)$ and $\lambda_{\text{min}}(P)$ are the largest and the smallest eigenvalues of $P$.

Since entries of $P$ are non-negative, Perron-Frobenius theorem implies that the all-one vector corresponds $\lambda_{\text{max}}(P)$.

Let us show that bound in Theorem 1 essentially coincides with the Hoffman bound. Put $n = |V(G)|$ and denote the spectrum of $P$ by $\lambda_1 \geq \ldots \geq \lambda_n$. For real $s$ define

$$A(s) = J - sP,$$

where $J$ stands for all-one matrix $n \times n$. Since all-one vector is eigenvector of $P$, matrices $P$, $J$ and $A(s)$ have a common eigenbasis. Then the spectrum of $A(s)$ equals to $n + s\lambda_1, s\lambda_2, \ldots, s\lambda_n$. For negative $s$ the largest eigenvalues of $A(s)$ is $n + s\lambda_1$ or $s\lambda_n$. At $s = \frac{n}{\lambda_n - \lambda_1}$ one has $n + s\lambda_1 = s\lambda_n$. Theorem 1 gives

$$\alpha(G) \leq s\lambda_n = n \frac{-\lambda_n}{\lambda_1 - \lambda_n},$$

which is the Hoffman bound.

In the case of non-regular graph the bound of Theorem 1 seems different from the known generalization of Hoffman bound [2].

1.2 A straightforward application to eventown problem

Let $F$ be a family of subsets of $[n]$ is eventown if the intersection of any two members is even (in particular all sets have even size). Berlekamp [1] and Graver [3] independently proved $F$ has at most $2^{\lceil n/2 \rceil}$ members, which is also best possible. The proof is very short up to general linear algebra. Note that every maximal eventown $F$ is a linear subspace of $F_2^n$; otherwise one can replace $F$ with span $F$. Since $F$ lies in the orthogonal complement $F^\perp$ and $\dim F + \dim F^\perp = n$, $|F|$ has the dimension at most $\lceil n/2 \rceil$.

Consider the following Hadamard matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Its spectrum is \{±√2\}. Then the spectrum of $M = A^{\otimes n}$ is \{±2^{n/2}\}. Let us consider $G = (2^{[n]}, E)$ and $(X, Y) \in E$ iff $|X \cap Y|$ is odd. Then we identify elements of $2^{[n]}$ with $\{0, 1\}^n$ by usual way as well as indices for rows and columns of matrix $M$ (where we mean that $A = (a_{ij})_{i,j \in \{0,1\}^n}$). Let $M = (m_{st})_{s,t \in \{0,1\}^n}$. Then for each $X, Y \in 2^{[n]}$ we have

$$m_X(X, Y) = \prod_{r=1}^n a_{(X)r}a_{(Y)r} = (-1)^{|\{r \in \{1, \ldots, n\} : a_{(X)r}a_{(Y)r} = -1\}|} = (-1)^{|X \cap Y|}.$$

Thus we see that graph $G$ and matrix $M$ satisfy the conditions of Theorem 1.

Applying Theorem 1 one has $|F| \leq 2^{n/2}$. For even $n$ we already get another proof of eventown theorem. For even $n$ we should also recall that $F$ is a linear subspace, so $|F| \leq 2^{\lceil n/2 \rceil}$.

Let $op(F)$ denote the number of distinct pairs $f_1, f_2 \in F$ for which $|f_1 \cap f_2|$ is odd. O’Neill [5] showed that for $1 \leq s \leq 2^{\lceil n/2 \rceil} - 2^{\lceil n/4 \rceil}$ there is a family $F$ with $|F| = 2^{\lceil n/2 \rceil} + s$ and $op(F) = s \cdot 2^{\lceil n/2 \rceil} - 1$. Also he conjectured that this example is tight and proved the conjecture for $s = 1, 2$. The application of Theorem 2 gives twice weaker bound for even $n$ (and much weaker bound for odd $n$).

**Theorem 4.** Let $|F| = 2^{\lceil n/2 \rceil} + s$ for some integer $s$. Then

$$op(F) \geq s \cdot 2^{\lfloor \frac{n}{4} \rfloor} - 2.$$

1.3 An application to k-town problem

Let $F$ be a family of vectors from $\{0, \ldots, k - 1\}^n$, such that $(f_1, f_2) = 0$ (mod $k$) for any $f_1, f_2 \in F$ (in particular for $f_1 = f_2$); such $F$ is further called a $k$-town family.

First, for prime $k$ the classical argument gives the tight upper bound $k^{\lfloor n/2 \rfloor}$.

If $k$ is square free we can obtain the same bound from the prime $k$ case in the following way. Let us see that the case $k = pq$ is a formal consequence of the cases $k = p, k = q$ if $p, q$ are coprime. Indeed let $F \subset (\mathbb{Z}/pq\mathbb{Z})^n$ such that $(f_1, f_2) = 0$ for each $f_1, f_2 \in F$. One can suppose that $F$ is a subgroup; otherwise replace $F$ with span $F$. By the finite abelian group classification we have $F \simeq F_p \times F_q$ for some subgroups $F_p, F_q$ with $pF_p = 0, qF_q = 0$, i.e. $F_p \subset (\mathbb{Z}/p\mathbb{Z})^n, F_q \subset (\mathbb{Z}/q\mathbb{Z})^n$. Then we
can consider $F_p, F_q$ as subgroups of $(\mathbb{Z}/p\mathbb{Z})^n, (\mathbb{Z}/q\mathbb{Z})^n$ with standard multiplication (and these subgroups $F_p, F_q$ satisfy orthogonality conditions modulo $p, q$ correspondingly). Using inequalities $|F_p| \leq p^{2\frac{n}{2}}, |F_q| \leq q^{\frac{n}{2}}$ we obtain $|F| \leq (pq)^{\frac{n}{2}}$.

The observations above should be folklore, meanwhile we do not know how to prove the following theorem without spectral graph theory for an arbitrary $k$.

**Theorem 5.** Let $F$ be a family of vectors from $\{0, \ldots, k^2 - 1\}^n$, such that $(f_1, f_2) \neq 0 \pmod{k}$ for at most $\varepsilon |F|^2$ pairs $f_1, f_2 \in F$. Suppose that $\varepsilon < \frac{k-1}{k^2}$. Then

$$|F| \leq \frac{k^{\frac{n}{2}}}{1 - \frac{k}{k^2}\varepsilon}.$$ 

In particular, if $F$ is a $k$-town family then

$$|F| \leq k^{\frac{n}{2}}.$$ 

Obtaining an example of $k$-town $F$ with $|F| = k^{\frac{n}{2}}$ in the case of prime $k$ and even $n$ is equivalent to finding a set of $\frac{n}{2}$ pairwise (and-self) orthogonal linear independent vectors in $(\mathbb{Z}/k\mathbb{Z})^n$. There are a lot of such sets; for example for prime $k = 4l + 1$ one can consider the vectors of the form $v_j = e_{2j-1} + e_{2j}$ for $1 \leq j \leq \frac{n}{2}$ (here $e_j$ — is the standard basis in $(\mathbb{Z}/k\mathbb{Z})^n$, $\varepsilon$ is a primitive root in $\mathbb{Z}/k\mathbb{Z}$).

For a general prime $k$ one can choose $v_1, v_2, \ldots$ inductively and almost arbitrarily such that $v_j \in (v_1, \ldots, v_{j-1}) \setminus (v_1, \ldots, v_j)$ and $(v_j, v_j) = 0$. This can be done: indeed, for $j < \frac{k}{2} - 2$ we can choose arbitrary $4 < \dim(v_1, \ldots, v_{j-1}) \setminus \dim(v_1, \ldots, v_{j-1})$ linearly independent vectors and find $v_j$ in their span (as any quadratic form with $\geq 3$ variables over finite field has an isotropic vector). When $j = \frac{k}{2} - 2$, we can choose 4-dimensional $V \subset ((v_1, \ldots, v_{j-1}) \setminus (v_1, \ldots, v_{j-1})) \cup 0$ as well, and it is also well-known that there exists $v_{\frac{k}{2} - 1}, v_{\frac{k}{2}} \in V$ such that $(v_{\frac{k}{2} - 1}, v_{\frac{k}{2} - 1}) = (v_{\frac{k}{2} - 1}, v_{\frac{k}{2}}) = (v_{\frac{k}{2}}, v_{\frac{k}{2}}) = 0.$

In some cases were $k$ is non-prime we can obtain examples of different nature. For example, when $k = m^2$ for some natural $m$ one can consider the set of vectors of the form $(mx_1, \ldots, mx_n)$. This example shows that in the case of $k$ being a perfect square the second inequality of Theorem 5 is also tight for an odd $n$.

## 2 Proofs

**Proof of Theorem 2.** Denote by $\chi_I$ the characteristic vector of $I$. Then

$$(A\chi_I, \chi_I) \geq |I|^2 \cdot (1 - (1-c)\varepsilon).$$

From the other hand, one has

$$\langle A\chi_I, \chi_I \rangle = \sum c_i^2 \lambda_i \leq \left( \sum c_i^2 \right) \lambda_{\max} = |I| \cdot \lambda_{\max},$$

where $\chi_I = \sum c_i v_i$ is the decomposition of $\chi_I$ via orthonormal eigenbasis $\{v_i\}$ of $A$.

**Proof of Theorem 3.** Denote by $\chi_I$ and $\chi_J$ the characteristic vectors of $I$ and $J$ respectively. Then

$$(A\chi_I, \chi_J) \geq |I| \cdot |J| \cdot (1 - (1-c)\varepsilon).$$

From the other hand, one has

$$\langle A\chi_I, \chi_I \rangle = \sum c_i d_i \lambda_i \leq \left( \sum |c_i| \cdot |d_i| \right) \cdot |sp| \leq \sqrt{\left( \sum c_i^2 \right) \left( \sum d_i^2 \right)} \cdot |sp| = \sqrt{|I| \cdot |J|} \cdot |sp|,$$

where $\chi_I = \sum c_i v_i$ and $\chi_J = \sum d_i v_i$ are the decompositions of $\chi_I$ and $\chi_J$ via orthonormal eigenbasis $\{v_i\}$ of $A$.

**Proof of Theorem 4.** Consider the following $k \times k$ matrix:

$$a_{jl} = \phi^{il},$$

where $\phi$ is an $k$-th root of unity and $0 \leq j, l \leq k - 1$ and the matrix $M = A^o$. Then we have $A^4 = k^2 E$, therefore $|\lambda| = (k^2)^n = k^2$ for each eigenvalue $\lambda$ of $M$, i.e., $sp(M) = k^2$.

Let us see that $A$ (and, as a consequence, $M$) has an eigenbasis in $\mathbb{R}^n$. Let $\{e_i\}$ be a standard basis. Then moving to the basis $\{e_1, 1, \sqrt{k} (e_1 \pm e_{k+2-j})\}$ (note that it is a unitary transformation) we obtain a block matrix with two blocks of the form $A_1, iA_2$, where $A_1, A_2$ are the real symmetric matrices (they have the sizes $\frac{k+1}{2} \times \frac{k-1}{2}$ respectively for odd $k$ and $\frac{k}{2} + 1, \frac{k}{2} + 1$ for even $k$). As a matter of fact, for each $2 \leq j \leq k$ we have

$$A(e_j + e_{k+2-j}) = \sum_{l=1}^{k} \frac{k^2 |(j-l-1)| e_l + \sum_{l=1}^{k} \frac{k^2 |(k+2-j-l)| e_l}{k^2} = 2 \sum_{l=2}^{k} \Re(\phi^{(j-l-1)}(e_l + e_{k+2-j}))$$
and for each $2 \leq j \leq k, j \neq \frac{k}{2}$

$$A(e_j - e_{k+2-j}) = \sum_{l=1}^{k} \phi^{(j-1)(l-1)} e_l - \sum_{l=1}^{k} \phi^{(k+1-j)(l-1)} e_l = \sum_{l=2}^{k} (\phi^{(j-1)(l-1)} - \phi^{(j-1)(l-1)}) e_l = i \sum_{l=2}^{k} 2 \cdot \Im(\phi^{(j-1)(l-1)})(e_l - e_{k+2-l}).$$

Thus we see that this change of the base (over $\mathbb{R}$) leads to real symmetric matrix and to pure imaginary symmetric matrix, which have common real eigenbasis, therefore $A$ has real eigenbasis as well. Hence $N := \Re M$ has the same real eigenbasis with $M$. Obviously, all eigenvalues of $N$ lie in $\{\pm k^{n/2}, 0\}$.

Let us consider $G = ((\mathbb{Z}/k\mathbb{Z})^n, E)$, where $(X, Y) = ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \in E$ iff $\sum_{r=1}^{n} x_r y_r \neq 0$. Then we also identify $(\mathbb{Z}/k\mathbb{Z})^n$ as indices for rows and columns of matrix $M$ (taking all indices in $A$ modulo $k$). Let $M = (m_{st})_{s,t \in \{0,\ldots,k-1\}^n}$. Now for each $X, Y \in ((\mathbb{Z}/k\mathbb{Z})^n)$, $(X, Y) \notin E$ we have

$$m_{X,Y} = \prod_{r=1}^{n} a_{x_r,y_r} = \phi^{\sum_{r=1}^{n} x_r y_r} = \phi^{0} = 1.$$ 

Thus graph $G$ and matrix $M$ (and also $N$) satisfy the conditions of Theorem 2.

An immediate application of Theorem 2 gives

$$|F| \leq \frac{k^{\frac{n}{2}}}{1 - t \epsilon},$$

where $t = 1 - \cos \left(\frac{k}{2} \frac{n}{k} \pi\right)$, which tends to 2 with $k \to \infty$. So we modify the proof of Theorem 2 in the following way.

For every root $\phi$ the matrix $N = N(\phi)$ satisfy

$$(N(\phi)\chi_F, \chi_F) = \sum_{i,j \in F} N_{ij}(\phi)$$

and

$$(N(\phi)\chi_F, \chi_F) \leq |F| \cdot k^{\frac{n}{2}}.$$ 

Summing up these inequalities for all $k$-th roots except 1 one has

$$(k - 1) \cdot |F|^2 \cdot (1 - \epsilon) - \epsilon \cdot |F|^2 \leq (k - 1) \cdot |F| \cdot k^{\frac{n}{2}}$$

(1) since for every $i, j$ that corresponds to sets with nonzero scalar product

$$\sum_{\phi \neq 1} N_{ij}(\phi) = -1,$$

and for $i, j$ that corresponds to sets with zero scalar product

$$\sum_{\phi \neq 1} N_{ij}(\phi) = k - 1.$$

Rewriting (1) finishes the proof.

\[\square\]

**Acknowledgments.** The authors are grateful to Fedor Petrov and Andrey Kupavskii for useful discussions. The work of Danila Cherkaoshin was supported by the Russian Science Foundation grant 21-11-00040.

**References**

[1] E. R. Berlekamp. On subsets with intersections of even cardinality. *Canadian Mathematical Bulletin*, 12(4):471–474, 1969.

[2] Chris D. Godsil and Mike W. Newman. Eigenvalue bounds for independent sets. *Journal of Combinatorial Theory, Series B*, 98(4):721–734, 2008.

[3] Jack E. Graver. Boolean designs and self-dual matroids. *Linear Algebra and its Applications*, 10(2):111–128, 1975.

[4] Willem H. Haemers. Hoffman’s ratio bound. *Linear Algebra and its Applications*, 617:215–219, 2021.

[5] Jason O’Neill. Towards supersaturation for oddtown and eventown. *arXiv preprint arXiv:2109.09925*, 2021.