THIRD HOMOLOGY OF GENERAL LINEAR GROUPS
OVER RINGS WITH MANY UNITS

BEHROOZ MIRZAI

ABSTRACT. For a commutative ring \( R \) with many units, we describe the kernel of \( H_3(\text{inc}) : H_3(\text{GL}_2(R), \mathbb{Z}) \to H_3(\text{GL}_3(R), \mathbb{Z}) \). Moreover we show that the elements of this kernel are of order at most two. As an application we study the indecomposable part of \( K_3(R) \).

INTRODUCTION

Interest in the study of the homology of general linear groups has arose mostly because of their close connection with the \( K \)-theory of rings. For any ring \( R \) and any positive integer \( n \), there are natural homomorphisms

\[
\begin{array}{cccc}
K_n(R) & \xrightarrow{h_n} & H_n(\text{GL}(R), \mathbb{Z}) \\
h_n' & \downarrow & \uparrow \\
H_n(E(R), \mathbb{Z}) & \xrightarrow{} & \end{array}
\]

where \( E(R) \) is the elementary subgroup of the stable general linear group \( \text{GL}(R) \) and \( h_n \) and \( h_n' \) \(( n \geq 2 \) for \( h_n' \)) are the Hurewicz maps coming from algebraic topology [10, Chap. 2].

It is known that \( K_1(R) \xrightarrow{h_1} H_1(\text{GL}(R), \mathbb{Z}) \), \( K_2(R) \xrightarrow{h_2} H_2(E(R), \mathbb{Z}) \) [10, Chap. 2]. The homomorphism \( h_3' : K_3(R) \to H_3(E(R), \mathbb{Z}) \) is surjective with 2-torsion kernel [12, Corollary 5.2], [9, Proposition 2.5].

Homological stability type theorems, are very powerful tools for the study of \( K \)-theory of rings. Suslin has proved that for an infinite field \( F \), we have the homological stability

\[
H_n(\text{GL}_n(F), \mathbb{Z}) \xrightarrow{\sim} H_n(\text{GL}_{n+1}(F), \mathbb{Z}) \xrightarrow{\sim} H_n(\text{GL}_{n+1}(F), \mathbb{Z}) \xrightarrow{\sim} \cdots ,
\]

and used this to prove many interesting results [11]. For example he showed that we have an exact sequence

\[
H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \xrightarrow{H_n(\text{inc})} H_n(\text{GL}_{n}(F), \mathbb{Z}) \xrightarrow{K} K_n^M(F) \xrightarrow{0} \]

Suslin has conjectured that the kernel of

\[
H_n(\text{GL}_{n-1}(F), \mathbb{Z}) \to H_n(\text{GL}_{n}(F), \mathbb{Z})
\]
is a torsion group [9, Problem 4.13]. These results can be generalized over rings with many units [4], e.g. semilocal rings with infinite residue fields. Also Suslin’s conjecture can be asked in this more general setting [7]. A positive answer to this conjecture only is known for \( n \leq 4 \) [3], [8], [7].

It was known that when \( F \) is an infinite field, the kernel of the homomorphism \( H_3(\text{GL}_2(F), \mathbb{Z}) \to H_3(\text{GL}_3(F), \mathbb{Z}) \) is a 2-power torsion group [8]. In this article we generalize this to all commutative rings with many units. In fact we do more. Here we describe the kernel of \( H_3(\text{inc}) : H_3(\text{GL}_2(R), \mathbb{Z}) \to H_3(\text{GL}_3(R), \mathbb{Z}) \), where \( R \) is a commutative ring with many units. Our main theorem claims that the elements of \( \ker(H_3(\text{inc})) \) are of the form

\[
\sum c(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(c, c^{-1}))
\]

provided that

\[
\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in R^* \otimes_{\mathbb{Z}} K_2^M(R).
\]

Moreover by an easy argument we will show that \( \ker(H_3(\text{inc})) \) is a 2-torsion group. It is highly expected that this kernel should be trivial, at least when \( R \) is a field [5, Section 5].

It is known that, the map \( H_3(\text{inc}) \) is closely related to the indecomposable part of \( K_3(R) \), i.e. \( K_3(R)^{\text{ind}} := K_3(R)/K_3^M(R) \) [8], [5]. As an application of our main theorem we show that

\[
K_3(R)^{\text{ind}} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \cong H_0(R^*, H_3(\text{SL}_2(R), \mathbb{Z}[1/2])).
\]

If \( R^* = R^2 \), then we get the isomorphism

\[
K_3(R)^{\text{ind}} \cong H_3(\text{SL}_2(R), \mathbb{Z}).
\]

Previously these results were only known for infinite fields [8].

**Notation.** In this article by \( H_i(G) \) we mean the homology of group \( G \) with integral coefficients, namely \( H_i(G, \mathbb{Z}) \). By \( \text{GL}_n \) (resp. \( \text{SL}_n \)) we mean the general (resp. special) linear group \( \text{GL}_n(R) \) (resp. \( \text{SL}_n(R) \)), where \( R \) is a commutative ring with 1. If \( A \to A' \) is a homomorphism of abelian groups, by \( A'/A \) we mean \( \text{coker}(A \to A') \) and we take other liberties of this kind. For a group \( A \), by \( A_{\mathbb{Z}[1/2]} \) we mean \( A \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \).

1. **Third homology of product of two abelian groups**

In this section we will study the homology group \( H_3(A \times B) \), where \( A \) and \( B \) are abelian groups.

First we assume \( A = B = \mathbb{Z}/n \). By applying the Künneth formula [13, Proposition 6.1.13] to \( H_3(\mathbb{Z}/n \times \mathbb{Z}/n) \) and using the calculation
of the homology of finite cyclic groups [13, Theorem 6.2.2, Example 6.2.3], we obtain the exact sequence

\[ 0 \to H_3(\mathbb{Z}/n) \oplus H_3(\mathbb{Z}/n) \to H_3(\mathbb{Z}/n \times \mathbb{Z}/n) \to \text{Tor}_3^\mathbb{Z}(\mathbb{Z}/n, \mathbb{Z}/n) \to 0. \]

If \( p_i : \mathbb{Z}/n \times \mathbb{Z}/n \to \mathbb{Z}/n, i = 1, 2, \) is projection on the \( i \)-th factor, then

\[(p_{1*}, p_{2*}) : H_3(\mathbb{Z}/n \times \mathbb{Z}/n) \to H_3(\mathbb{Z}/n) \oplus H_3(\mathbb{Z}/n)
\]
splits the above exact sequence. Thus we obtain a canonical splitting map

\[ \theta_{n,n} : \text{Tor}_3^\mathbb{Z}(\mathbb{Z}/n, \mathbb{Z}/n) \to H_3(\mathbb{Z}/n \times \mathbb{Z}/n) \]

If \( \langle \bar{1}, n, \bar{1} \rangle \) is the image of \( \bar{1} \in \mathbb{Z}/n \) under the isomorphism

\[ \mathbb{Z}/n \xrightarrow{\sim} \text{Tor}_1^\mathbb{Z}(\mathbb{Z}/n, \mathbb{Z}/n), \]

then one can show that \( \theta_{n,n}(\langle \bar{1}, n, \bar{1} \rangle) = \chi_{n,n}, \) where

\[
\chi_{n,n} := \sum_{i=1}^{n} \left( \left[ (\bar{i}, 0)|\bar{0}, \bar{i}] \right] - \left[ (0, \bar{i})|\bar{0}, \bar{i}] \right] + \left[ (0, \bar{i})|\bar{0}, \bar{i}] \right] \right)
\]

\[
+ \left[ (0, \bar{1})|\bar{0}, \bar{1}] \right] \right) - \left[ (1, 0)|\bar{0}, \bar{1}] \right] \right) \right) \right) \right) \right).
\]

[6, Chap. V, Proposition 10.6], [8, Proposition 4.1]. If \( A = \mathbb{Z}/m \) and \( B = \mathbb{Z}/n \), then the same approach shows that the exact sequence

\[ 0 \to H_3(\mathbb{Z}/m) \oplus H_3(\mathbb{Z}/n) \to H_3(\mathbb{Z}/m \times \mathbb{Z}/n) \to \text{Tor}_3^\mathbb{Z}(\mathbb{Z}/m, \mathbb{Z}/n) \to 0, \]

splits canonically. The splitting map

\[ \theta_{m,n} : \text{Tor}_3^\mathbb{Z}(\mathbb{Z}/m, \mathbb{Z}/n) \to H_3(\mathbb{Z}/m \times \mathbb{Z}/n) \]

can be computed similar to \( \theta_{n,n} \). In fact if \( \langle m/d, d, n/d \rangle \) is the image of \( \bar{1} \in \mathbb{Z}/(m,n) \) under the isomorphism \( \mathbb{Z}/(m,n) \xrightarrow{\sim} \text{Tor}_1^\mathbb{Z}(\mathbb{Z}/m, \mathbb{Z}/n) \), then \( \theta_{m,n}(\langle m/d, d, n/d \rangle) = \chi_{m,n}, \) where

\[
\chi_{m,n} := \sum_{i=1}^{n} \left( \left[ (\bar{i}, 0)|\bar{0}, \bar{i}] \right] - \left[ (0, \bar{i})|\bar{0}, \bar{i}] \right] + \left[ (0, \bar{i})|\bar{0}, \bar{i}] \right] \right)
\]

\[
+ \left[ (0, \bar{1})|\bar{0}, \bar{1}] \right] \right) - \left[ (1, 0)|\bar{0}, \bar{1}] \right] \right) \right) \right) \right) \right) \right).
\]

In the next proposition we extend these results to all abelian groups.

**Proposition 1.1.** Let \( A \) and \( B \) be abelian groups. Then we have the canonical decomposition

\[ H_3(A \times B) = \bigoplus_{i+j=3} H_i(A) \otimes H_j(B) \oplus \text{Tor}_1^\mathbb{Z}(A, B). \]
Proof. By the Künneth formula we have the exact sequence
\[ 0 \rightarrow \bigoplus_{i+j=3} H_i(A) \otimes H_j(B) \rightarrow H_3(A \times B) \rightarrow \text{Tor}_1^Z(A, B) \rightarrow 0. \]
We will construct a canonical splitting map
\[ \text{Tor}_1^Z(A, B) \rightarrow H_3(A \times B). \]
It is known that direct limit with directed set index, is an exact functor and it commutes with the homology group [2, Chap. V, Section 5, Exercise 3] and the functor Tor [13, Corollary 2.6.17]. Since any abelian group can be written as direct limit of its finitely generated subgroups, we may assume that \( A \) and \( B \) are finitely generated abelian groups. On the other hand,
\[ \text{Tor}_1^Z(A, B) \simeq \text{Tor}_1^Z(A_{\text{tor}}, B_{\text{tor}}), \]
where \( A_{\text{tor}} \) is the subgroup of torsion elements of \( A \). So we may even assume that \( A \) and \( B \) are finite abelian groups. Let
\[ A = \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_r, \quad B = \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_s. \]
Now consider the commutative diagram
\[
\begin{array}{c}
0 \rightarrow H_3(\mathbb{Z}/m_i) \oplus H_3(\mathbb{Z}/n_j) \rightarrow H_3(\mathbb{Z}/m_i \times \mathbb{Z}/n_j) \rightarrow \text{Tor}_1^Z(\mathbb{Z}/m_i, \mathbb{Z}/n_j) \rightarrow 0 \\
\downarrow \quad \downarrow \text{inc} \\
0 \rightarrow \bigoplus_{i+j=3} H_i(A) \otimes H_j(B) \rightarrow H_3(A \times B) \rightarrow \text{Tor}_1^Z(A, B) \rightarrow 0.
\end{array}
\]
We have seen that the first row of this diagram splits by the canonical map \( \theta_{m_i, n_j} \). Thus the composition
\[
\text{Tor}_1^Z(\mathbb{Z}/m_i, \mathbb{Z}/n_j) \xrightarrow{\text{inc}_{m_i, n_j} \circ \theta_{m_i, n_j}} H_3(A \times B) \rightarrow \text{Tor}_1^Z(A, B)
\]
is the natural inclusion map. Since
\[
\text{Tor}_1^Z(A, B) = \bigoplus_{1 \leq i \leq r, 1 \leq j \leq s} \text{Tor}_1^Z(\mathbb{Z}/m_i, \mathbb{Z}/n_j),
\]
we obtain a map \( \theta_{A, B} : \text{Tor}_1^Z(A, B) \rightarrow H_3(A \times B) \) that decomposes our exact sequence canonically. In fact \( \theta_{A, B} = \sum_{i,j} \text{inc}_{m_i, n_j} \circ \theta_{m_i, n_j}. \)

2. The third homology of GL_2

A commutative ring \( R \) with 1 is called a ring with many units if for any \( n \geq 2 \) and for any finite number of surjective linear forms \( f_i : R^n \rightarrow R \), there exists a \( v \in R^n \) such that, for all \( i \), \( f_i(v) \in R^* \). Important examples of rings with many units are semilocal rings with infinite residue fields. In particular for an infinite field \( F \), any commutative
finite dimensional $F$-algebra is a semilocal ring and so is a ring with many units. In this article we always assume that $R$ is a commutative ring with many units.

Let

$$R^* \times \text{GL}_n \overset{\text{inc}}{\rightarrow} R^* \times \text{GL}_1 \overset{\text{inc}}{\rightarrow} R^* \times \text{GL}_2 \overset{\text{inc}}{\rightarrow} \text{GL}_3$$

be the natural diagonal inclusions. Here by $R^*_n$ we mean

$$R^* \times \cdots \times R^* \text{ (n-times)}.$$  

Let

$$\sigma_1 := \text{inc} : R^* \times \text{GL}_2 \rightarrow \text{GL}_3,$$

$$\sigma_1^1 : R^* \times \text{GL}_1 \rightarrow R^* \times \text{GL}_2, \quad (a, b, c) \mapsto (b, a, c),$$

$$\sigma_1^2 = \text{inc} : R^* \times \text{GL}_1 \rightarrow R^* \times \text{GL}_2, \quad (a, b, c) \mapsto (a, b, c),$$

$$\sigma_0^1 : R^* \times \text{GL}_0 \rightarrow R^* \times \text{GL}_1, \quad (a, b, c) \mapsto (b, c, a),$$

$$\sigma_0^2 : R^* \times \text{GL}_0 \rightarrow R^* \times \text{GL}_1, \quad (a, b, c) \mapsto (a, c, b),$$

$$\sigma_0^3 = \text{inc} : R^* \times \text{GL}_0 \rightarrow R^* \times \text{GL}_1, \quad (a, b, c) \mapsto (a, b, c).$$

It is easy to see that the chain of maps

$$H_3(R^* \times \text{GL}_0) \xrightarrow{\sigma_1^0 - \sigma_2^0 + \sigma_3^0} H_3(R^* \times \text{GL}_1) \xrightarrow{\sigma_1^1 - \sigma_2^1} H_3(R^* \times \text{GL}_2) \xrightarrow{\sigma_2^1} H_3(\text{GL}_3) \rightarrow 0$$

is a chain complex. The following result has been proved in [8, Corollary 3.5].

**Theorem 2.1.** The sequence

$$H_3(R^* \times \text{GL}_1) \xrightarrow{\sigma_1^1 - \sigma_2^1} H_3(R^* \times \text{GL}_2) \xrightarrow{\sigma_2^1} H_3(\text{GL}_3) \rightarrow 0$$

is exact.

Using the Künneth formula [13, Proposition 6.1.13], we have the decomposition $H_3(R^* \times \text{GL}_2) = \bigoplus_{i=0}^4 S_i$, where

$$S_0 = H_3(\text{GL}_2),$$

$$S_i = H_i(R^*) \otimes H_{3-i}(\text{GL}_2), \quad 1 \leq i \leq 3,$$

$$S_4 = \text{Tor}_1^Z(R^*, H_1(\text{GL}_2)) \simeq \text{Tor}_1^Z(\mu(R), \mu(R)).$$

Note that by the homological stability, $R^* \simeq H_1(\text{GL}_1) \simeq H_1(\text{GL}_2)$ [4, Theorem 1]. This decomposition is canonical. The splitting map

$$S_4 \simeq \text{Tor}_1^Z(\mu(R), \mu(R)) \rightarrow H_3(R^* \times \text{GL}_2)$$
is given by the composition
\[ S_4 \simeq \text{Tor}^\mathbb{Z}_1(\mu(R), \mu(R)) \xrightarrow{\theta_{R,R}} H_3(R^* \times R^*) \xrightarrow{q_2} H_3(R^* \times \text{GL}_2), \]
where
\[ q : R^* \times R^* \to R^* \times \text{GL}_2, \quad (a, b) \mapsto (a, b, 1), \]
and \( \theta_{R,R} \) is obtained from Proposition 1.1. Using the decomposition
\[ H_2(\text{GL}_2) = H_2(\text{GL}_1) \oplus K^M_2(R) \]
[4, Theorem 2], we have
\[ S_4 = S'_4 \oplus S''_4, \]
where
\[ S'_4 = R^* \otimes H_2(\text{GL}_1), \quad S''_4 = R^* \otimes K^M_2(R). \]

We should remark that the inclusion \( K^M_2(R) \to H_2(\text{GL}_2) \), in the decomposition of \( H_2(\text{GL}_2) \), is given by the formula
\[ \{a, b\} \mapsto c(\text{diag}(a, 1), \text{diag}(b, b^{-1})) \]
[3, Proposition A.11]. For the definition of Milnor’s \( K \)-groups, \( K^M_n(R) \), over commutative rings and their study over rings with many units, we refer the interested readers to subsection 3.2 of [4].

Let us introduce the notation \( c(\cdot, \cdot) \) in a more general setting and state some of its main properties. These will be used frequently in this article. Let \( G \) be a group and set
\[ c(g_1, g_2, \ldots, g_n) := \sum_{\sigma \in \Sigma_n} \text{sign}(\sigma) [g_{\sigma(1)} g_{\sigma(2)} \cdots g_{\sigma(n)}] \in H_n(G), \]
where \( g_1, \ldots, g_n \in G \) pairwise commute and \( \Sigma_n \) is the symmetric group of degree \( n \). Here we use the bar resolution of \( G \) [2, Chapter I, Section 5] to define the homology of \( G \).

**Lemma 2.2.** Let \( G \) and \( G' \) be two groups.
(i) If \( h_1 \in G \) commutes with all the elements \( g_1, \ldots, g_n \in G \), then
\[ c(g_1 h_1, g_2, \ldots, g_n) = c(g_1, g_2, \ldots, g_n) + c(h_1, g_2, \ldots, g_n). \]
(ii) For every \( \sigma \in \Sigma_n, \ c(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = \text{sign}(\sigma) c(g_1, \ldots, g_n). \)
(iii) The cup product of \( c(g_1, \ldots, g_p) \in H_p(G) \) and \( c(g'_1, \ldots, g'_q) \in H_q(G') \) is \( c((g_1, 1), \ldots, (g_p, 1), (1, g'_1), \ldots, (1, g'_q)) \in H_{p+q}(G \times G'). \)

**Proof.** The proofs follow from direct computations, so we leave it to the interested readers. \( \square \)

Again using the Künneth formula and Proposition 1.1, we obtain the canonical decomposition
\[ H_3(R^2 \times \text{GL}_1) = \bigoplus_{i=0}^8 T_i, \]
where
$T_0 = H_3(\text{GL}_1),$ \\
$T_1 = \bigoplus_{i=1}^{3} T_{1,i} = \bigoplus_{i=1}^{3} H_i(R^*_1) \otimes H_{3-i}(\text{GL}_1),$ \\
$T_2 = \bigoplus_{i=1}^{3} T_{2,i} = \bigoplus_{i=1}^{3} H_i(R^*_2) \otimes H_{3-i}(\text{GL}_1),$ \\
$T_3 = R^*_1 \otimes R^*_2 \otimes H_1(\text{GL}_1),$ \\
$T_4 = \text{Tor}^Z_1(R^*_1, R^*_2) \simeq \text{Tor}^Z_1(\mu(R), \mu(R)),$ \\
$T_5 = \text{Tor}^Z_1(R^*_1, H_1(\text{GL}_1)) \simeq \text{Tor}^Z_1(\mu(R), \mu(R)),$ \\
$T_6 = \text{Tor}^Z_1(R^*_2, H_1(\text{GL}_1)) \simeq \text{Tor}^Z_1(\mu(R), \mu(R)),$ \\
$T_7 = R^*_1 \otimes H_2(R^*_2),$ \\
$T_8 = H_2(R^*_1) \otimes R^*_2.$

Here by $R^*_i$ we mean the $i$-th component of $R^* \times \cdots \times R^*$. Now we give an explicit description of restriction of the map $\alpha := \sigma^1_{1*} - \sigma^2_{1*}$ on all $T_i$’s. By direct computations one sees that

$\alpha|_{T_0} : T_0 \to S_0, \quad x \mapsto 0,$

$\alpha|_{T_{1,i}} : T_{1,i} \to S_0 \oplus S_i, \quad x_i \otimes x'_i \mapsto (x_i \cup x'_i, -x_i \otimes x'_i), 1 \leq i \leq 3,$

$\alpha|_{T_{2,i}} : T_{2,i} \to S_0 \oplus S_i, \quad y_i \otimes y'_i \mapsto (-y_i \cup y'_i, y_i \otimes y'_i), 1 \leq i \leq 3,$

$\alpha|_{T_3} : T_3 \to S_1, \quad a \otimes b \otimes c \mapsto -b \otimes (a \cup c) - a \otimes (b \cup c),$ 

$\alpha|_{T_4} : T_4 \to S_4, \quad z \mapsto 0,$

$\alpha|_{T_5} : T_5 \to S_0 \oplus S_4, \quad u \mapsto (\sigma^1_{1*}(u), -u),$ 

$\alpha|_{T_6} : T_6 \to S_0 \oplus S_4, \quad v \mapsto (-\sigma^2_{1*}(v), v),$ 

$\alpha|_{T_7} : T_7 \to S_1 \oplus S_2, \quad d \otimes u' \mapsto (-d \otimes u', u' \otimes d),$ 

$\alpha|_{T_8} : T_8 \to S_1 \oplus S_2, \quad v' \otimes e \mapsto (e \otimes v', -v' \otimes e),$ 

where $x \cup y$ is the cup product of $x$ and $y$.

3. The kernel of $H_3(\text{GL}_2) \to H_3(\text{GL}_3)$

Our goal in this article is to study the kernel of the map $\text{inc}_* : H_3(\text{GL}_2) \to H_3(\text{GL}_3)$. So let $x \in \ker(\text{inc}_*)$. Then

$$(x, 0, 0, 0, 0) \in \ker(\sigma^1_{2*}) \subseteq \bigoplus_{i=0}^{4} S_i = H_3(R^* \times \text{GL}_2).$$

By Theorem 2.1 and by the explicit description of $\alpha = \sigma^1_{1*} - \sigma^2_{1*}$ given in the previous section, there exists an element

$l = (0, (x_i \otimes x'_i)_{1 \leq i \leq 3}, (y_i \otimes y'_i)_{1 \leq i \leq 3}, \sum a \otimes b \otimes c, 0, u, v, d \otimes u', v' \otimes e)$
in \( H_3(R^{2} \times GL_1) \) such that \( \alpha(l) = (x, 0, 0, 0) \).

Set \( \beta := \sigma^1_{0*} - \sigma^2_{0*} + \sigma^3_{0*} \), and consider the following summands of \( H_3(R^{3} \times GL_0) \),

\[
T'_1 := R_1^* \otimes H_2(R_2^*), \quad T'_2 := H_2(R_1^*) \otimes R_2^*.
\]

By easy computations one sees that

\[
\beta|_{T'_1} : T'_1 \rightarrow T_{1,1} \oplus T_{1,2} \oplus T_7, \quad f \otimes w \mapsto (-f \otimes w, w \otimes f, f \otimes w)
\]

\[
\beta|_{T'_2} : T'_2 \rightarrow T_{1,1} \oplus T_{1,2} \oplus T_8, \quad w' \otimes f' \mapsto (f' \otimes w', -w' \otimes f', w' \otimes f').
\]

So we may assume \( d \otimes u' = 0, v' \otimes e = 0 \). Therefore we have

\[
\sum_{i=1}^{3} x_i \cup x'_i - \sum_{i=1}^{3} y_i \cup y'_i + \sigma^1_{1*}(u) - \sigma^2_{1*}(v) = x,
- x_1 \otimes x'_1 + y_1 \otimes y'_1 - \sum [b \otimes (a \cup c) + a \otimes (b \cup c)] = 0,
- x_2 \otimes x'_2 + y_2 \otimes y'_2 = 0,
- x_3 \otimes x'_3 + y_3 \otimes y'_3 = 0,
- u + v = 0.
\]

Therefore we obtain the following relations

\[
x = x_1 \cup x'_1 - y_1 \cup y'_1 \in S_0 = H_3(GL_2),
\]

\[
x_1 \otimes x'_1 - y_1 \otimes y'_1 = - \sum b \otimes (a \cup c) + a \otimes (b \cup c) \in S_1.
\]

Under the decomposition \( H_2(GL_2) = H_2(GL_1) \oplus K_2^M(R) \), we have

\[
a \cup b = c(diag(a, 1), diag(1, b)) = (c(a, b), \{a, b\}).
\]

Thus under the decomposition \( S_1 = S_1' \oplus S_1'' \), we have

\[
(x_1 \otimes x'_1 - y_1 \otimes y'_1 + \sum b \otimes c(a, c) + a \otimes c(b, c), \sum b \otimes \{a, c\} + a \otimes \{b, c\}) = 0,
\]

and hence

\[
x_1 \otimes x'_1 - y_1 \otimes y'_1 = - \sum b \otimes c(a, c) + a \otimes c(b, c),
\]

\[
\sum b \otimes \{a, c\} + a \otimes \{b, c\} = 0.
\]

Therefore

\[
x = - \sum c(diag(a, 1), diag(1, b), diag(1, c)) + c(diag(b, 1), diag(1, a), diag(1, c))
\]

\[
= \sum c(diag(a, 1), diag(1, b), diag(c, c^{-1})),
\]

such that \( \sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \). From now on, we will use the following notation:

\[
l_{a,b,c} = c(diag(a, 1), diag(1, b), diag(c, c^{-1})).
\]

Hence we have proved most parts of the following theorem.
Theorem 3.1. Let $R$ be a commutative ring with many units. Then the kernel of $\text{inc}_*: H_3(\text{GL}_2) \to H_3(\text{GL}_3)$ consists of elements of the form $\sum c(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(c, c^{-1}))$ provided that

$$\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in R^* \otimes K^M_2(R).$$

In particular $\ker(\text{inc}_*) \subseteq R^* \cup H_2(\text{GL}_1) \subseteq H_3(\text{GL}_2)$, where the cup product is induced by the diagonal inclusion $\text{inc}: R^* \times \text{GL}_1 \to \text{GL}_2$. Moreover $\ker(\text{inc}_*)$ is a 2-torsion group.

Proof. The only part that remains to be proved is that $\ker(\text{inc}_*)$ is a 2-torsion group. Let $x \in \ker(\text{inc}_*)$. For simplicity we may assume that $x = l_{a,b,c} = c(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(c, c^{-1}))$, such that $a \otimes \{b, c\} + b \otimes \{a, c\} = 0$. Let $\Phi$ be the following composition

$$R^* \otimes K^M_2(R) \xrightarrow{id_{R^*} \otimes \iota} R^* \otimes H_2(\text{GL}_2) \xrightarrow{\cup} H_3(R^* \times \text{GL}_2) \xrightarrow{\alpha_*} H_3(\text{GL}_2),$$

where $\iota: K^M_2(R) \to H_2(\text{GL}_2)$ is described in the previous section, $\cup$ is the cup product and $\alpha: R^* \times \text{GL}_2 \to \text{GL}_2$ is given by $(a, A) \mapsto aA$. It is easy to see that

$$\Phi(a \otimes \{b, c\}) = c(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1})).$$

Now with easy computations, one sees that

$$0 = \Phi(0) = \Phi(a \otimes \{b, c\} + b \otimes \{a, c\}) = c(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1})) + c(\text{diag}(b, b), \text{diag}(a, 1), \text{diag}(c, c^{-1})) = -2l_{a,b,c}.$$

This completes the proof of the theorem.  

Remark 3.2. One can show directly that if $a \otimes \{b, c\} + b \otimes \{a, c\} = 0$, then $l_{a,b,c} \in \ker(\text{inc}_*: H_3(\text{GL}_2) \to H_3(\text{GL}_3))$. To see this, let $\Psi$ be the following composition

$$R^* \otimes K^M_2(R) \xrightarrow{id_{R^*} \otimes \iota} R^* \otimes H_2(\text{GL}_2) \xrightarrow{\cup} H_3(R^* \times \text{GL}_2) \xrightarrow{\alpha_*} H_3(\text{GL}_3).$$

Then it is easy to see that

$$\Psi(a \otimes \{b, c\}) = c(\text{diag}(a, 1, 1), \text{diag}(b, 1, 1), \text{diag}(1, c, c^{-1})).$$
Now we have
\[ \text{inc}_*(l_{a,b,c}) = +c(\text{diag}(1, a, 1), \text{diag}(1, 1, b), \text{diag}(1, c, c^{-1})) \]
\[ = +c(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(c, c^{-1}, 1)) \]
\[ = -c(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(c, c^{-1}, 1)) \]
\[ = -c(\text{diag}(b, 1, 1), \text{diag}(1, a, 1), \text{diag}(1, c, 1)) \]
\[ = -c(\text{diag}(b, 1, 1), \text{diag}(1, a, 1), \text{diag}(1, c, c^{-1})) \]
\[ = -c(\text{diag}(b, 1, 1), \text{diag}(1, a, 1), \text{diag}(1, c, c^{-1})) \]
\[ = -\Psi(a \otimes \{b, c\} + b \otimes \{a, c\}) \]
\[ = 0. \]

**Corollary 3.3.** Let \( R \) be a ring with many units.

(i) The natural map \( \text{inc}_* : H_3(\text{GL}_2, \mathbb{Z}[1/2]) \to H_3(\text{GL}_3, \mathbb{Z}[1/2]) \) is injective.

(ii) If \( R^* = R^{*2} = \{a^2 | a \in R^*\} \), then \( \text{inc}_* : H_3(\text{GL}_2) \to H_3(\text{GL}_3) \) is injective.

**Proof.** The part (i) immediately follows from Theorem 3.1. Let \( R^* = R^{*2} \). By Theorem 3.1, we may assume that \( x \in \ker(\text{inc}_*) \) is of the form \( l_{a,b,c} \in H_3(\text{GL}_2) \) such that \( a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \). Let \( c = c'^2 \) for some \( c' \in R^* \). Then \( l_{a,b,c} = 2l_{a,b,c'} \) and \( 2(\{b, c'\} + b \otimes \{a, c'\}) = 0 \). Since \( K_2^M(R) \) is uniquely 2-divisible \([1, \text{Proposition 1.2}]\), \( R^* \otimes K_2^M(R) \) is uniquely 2-divisible too. Hence \( a \otimes \{b, c'\} + b \otimes \{a, c'\} = 0 \). Now from Theorem 3.1, it follows that \( 2l_{a,b,c'} = 0 \). Therefore \( l_{a,b,c} = 0 \) and hence \( \text{inc}_* : H_3(\text{GL}_2) \to H_3(\text{GL}_3) \) is injective. \( \square \)

**Example 3.4.** Let \( R = \mathbb{R} \). It is well-know that \( K_2^M(\mathbb{R}) \cong \{-1, -1\} \oplus V \), where \( V \) is uniquely divisible and is generated by elements \( \{a, b\} \) with \( a, b > 0 \). Let \( l_{a,b,c} \in H_3(\text{GL}_2(\mathbb{R})) \) such that \( a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \). If \( a > 0 \), then \( a \otimes \{b, c\} = a \otimes \{-b, c\} = a \otimes \{b, -c\} = a \otimes \{-b, -c\} \), so we may assume that \( b, c > 0 \). Now with an argument as in the proof of the previous corollary, one sees that \( l_{a,b,c} = 0 \). A similar argument works if \( b > 0 \) or if \( c > 0 \). If \( a, b, c < 0 \), then one can easily reduce the problem to the case that \( a = b = c = -1 \), and it is trivial to see that \( l_{-1,-1,-1} = 0 \). Therefore \( \text{inc}_* : H_3(\text{GL}_2(\mathbb{R})) \to H_3(\text{GL}_3(\mathbb{R})) \) is injective.

**Remark 3.5.** Consider the following chain of maps
\[ R^* \otimes^3 K_0^M(R) \xrightarrow{\delta_3^0} R^* \otimes^2 K_1^M(R) \xrightarrow{\delta_1^0} R^* \otimes K_2^M(R) \xrightarrow{\delta_2^0} K_3^M(R) \to 0, \]
where
\[
\delta_2^{(3)}: a \otimes \{b, c\} \mapsto \{a, b, c\}
\]
\[
\delta_1^{(3)}: a \otimes b \otimes \{c\} \mapsto a \otimes \{b, c\} + b \otimes \{a, c\}
\]
\[
\delta_0^{(3)}: a \otimes b \otimes c \mapsto b \otimes c \otimes \{a\} + a \otimes c \otimes \{b\} + a \otimes b \otimes \{c\}.
\]
It is easy to see that this is, in fact, a chain complex. It is not difficult to see that ker(\(\delta_2^{(3)}\)) = im(\(\delta_1^{(3)}\)) (see the proof of Theorem 3.2 in [5]).

Under the composition
\[
R^* \otimes^3 \rightarrow R^* \otimes H_2(R^*) \rightarrow H_3(\text{GL}_2),
\]
defined by
\[
a \otimes b \otimes c \mapsto a \otimes c (b, c) \mapsto c(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(1, c)),
\]
one can see that im(\(\delta_0^{(3)}\)) maps to zero. Thus we obtain a surjective map
\[
\ker(\delta_1^{(3)})/\text{im}(\delta_0^{(3)}) \rightarrow \ker(H_3(\text{GL}_2) \rightarrow H_3(\text{GL}_3)),
\]
\[
\sum a \otimes b \otimes c + \text{im}(\delta_0^{(3)}) \mapsto \sum l_{a,b,c}.
\]

**Lemma 3.6.** Let \(R\) be a ring with many units.

(i) We have the exact sequence
\[
0 \rightarrow H_3(\text{SL}_2, \mathbb{Z}[1/2])_{R^*} \rightarrow H_3(\text{SL}, \mathbb{Z}[1/2]) \rightarrow K_3^M(R)_{\mathbb{Z}[1/2]} \rightarrow 0.
\]

(ii) If \(R^* = R^2 = \{a^2|a \in R^*\}\), then we have the exact sequence
\[
0 \rightarrow H_3(\text{SL}_2) \rightarrow H_3(\text{SL}) \rightarrow K_3^M(R) \rightarrow 0.
\]

**Proof.** The proof is similar to the proof of Theorem 6.1 and Corollary 6.2 in [8].

**Theorem 3.7.** Let \(R\) be a ring with many units.

(i) We have the isomorphism
\[
K_3(R)^{\text{ind}} \otimes \mathbb{Z}[1/2] \simeq H_3(\text{SL}_2, \mathbb{Z}[1/2])_{R^*}.
\]

(ii) If \(R^* = R^2 = \{a^2|a \in R^*\}\), then
\[
K_3(R)^{\text{ind}} \simeq H_3(\text{SL}_2).
\]

**Proof.** The proof is similar to the proof of Theorem 6.4 in [8].

**Remark 3.8.** Previously Lemma 3.6 and Theorem 3.7 were only known for infinite fields [8, Corollary 6.2, Proposition 6.4].

**Acknowledgments.** Part of this work has done during my visit to ICTP on August 2011. I would like to thank them for their support and hospitality.
REFERENCES

[1] Bass, H., Tate, J. The Milnor ring of a global field. Lecture Notes in Math., Vol. 342, 1973, 349–446.

[2] Brown, K. S. Cohomology of Groups. Graduate Texts in Mathematics, 87. Springer-Verlag, New York, 1994.

[3] Elbaz-Vincent, P. The indecomposable $K_3$ of rings and homology of $SL_2$. J. Pure Appl. Algebra 132 (1998), no. 1, 27–71.

[4] Guin, D. Homologie du groupe linéaire et $K$-théorie de Milnor des anneaux. J. Algebra 123 (1989), no. 1, 27–59.

[5] Hutchinson, K., Tao, L. The third homology of the special linear group of a field. J. Pure Appl. Algebra 213 (2009), no. 9, 1665–1680.

[6] Mirzaii, B. Homology of $GL_n$: injectivity conjecture for $GL_4$. Math. Ann. 304 (2008), no. 1, 159–184.

[7] Mirzaii, B. Third homology of general linear groups. J. Algebra 320 (2008), no. 5, 1851–1877.

[8] Sah, C. Homology of classical Lie groups made discrete. III. J. Pure Appl. Algebra 56 (1989), no. 3, 269–312.

[9] Srinivas, V. Algebraic $K$-Theory. Second edition. Progress in Mathematics, 90. Birkhäuser Boston, 1996.

[10] Suslin, A. A. Homology of $GL_n$, characteristic classes and Milnor $K$-theory. Proc. Steklov Inst. Math. 3 (1985), 207–225.

[11] Suslin, A. A. $K_3$ of a field and the Bloch group. Proc. Steklov Inst. Math. 183 (1991), no. 4, 217–239.

[12] Weibel, C. A. An Introduction to Homological Algebra. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

Department of Mathematics,
Institute for Advanced Studies in Basic Sciences,
P. O. Box. 45195-1159, Zanjan, Iran.
email: bmirzaii@iasbs.ac.ir