Dirac Quantization of Gravity-Yang-Mills Systems in 1+1 Dimensions

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Abstract

In two dimensions a large class of gravitational systems including, e.g., $R^2$-gravity can be quantized exactly also when coupled dynamically to a Yang-Mills theory. Some previous considerations on the quantization of pure gravity theories are improved and generalized.

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In recent years the study of two-dimensional exactly solvable field theories has attracted considerable interest. One of the areas of investigations is 2D Yang-Mills (YM) theory (on a cylinder in a Hamiltonian approach \([1, 2]\), or on an arbitrary Riemann surface when evaluating the partition function \([3]\)); other models of interest are gravitational ones such as the one for 2D black hole \([4]\), the Jackiw-Teitelboim model \([5]\), or the Katanaev-Volovich (KV) model \([6]\). The first main purpose of this letter is to show that the exact quantum integrability extends to the combined treatment of the YM theory and a large class of gravitational systems.\(^a\)

The gravitational part of the action considered in this work will be

\[
S_G = \int [\pi_\omega d\omega + \pi_a De^a - V(\pi_\omega, \pi^2) \varepsilon], \quad V = v(\pi_\omega) + \frac{\tau}{2} \pi^2, \tag{1}
\]

in which the basic fields are the zweibein and spin-connection one-forms \(e^a\) and \(\omega\), respectively, as well as the functions \(\pi_a\) and \(\pi_\omega\). \(De^a \equiv de^a + \varepsilon^{ab} \omega \wedge e^b\) is the torsion two-form, \(\pi^2 \equiv \pi^a \pi_a \equiv 2\pi_+ \pi_-\), and \(\varepsilon \equiv e^+ \wedge e^- \equiv e d^2x\) with \(e \equiv \det(e^\mu_a)\) denotes the \(\varepsilon\)-tensor or metric induced volume-form. \(v\) is some potential and \(\tau\) a constant. For the case that \(v\) is chosen as \(-(1/4\gamma)(\pi_\omega)^2 + \lambda\) and \(\tau \neq 0\) our action (1) yields, after elimination of \(\pi_a\) and \(\pi_\omega\) (use \(\star \varepsilon = -1\)), 2D gravity with torsion \([6, 8, 9]\)

\[
S_{KV}^G = \int [\gamma d\omega \wedge \star d\omega - \frac{1}{2\tau} De^a \wedge \star De_a - \lambda \varepsilon], \tag{2}
\]

the most general Lagrangian in two dimensions yielding second order differential equations for \(e^a\) and \(\omega\). The same \(v\) but with \(\tau = 0\) is analogously found to describe torsionless \(R^2\) gravity \([10]\). For \(V \propto \pi_\omega\) the action \(S_G\) describes deSitter gravity (the Jackiw-Teitelboim model \([5, 11]\)), whereas \(V \propto 1/\sqrt{\pi_\omega}\) was claimed to effectively describe 4D spherical symmetric gravity \([12, 13]\). \(V = \text{const}\), furthermore, yields a gravity theory basically equivalent \([14]\) to the string inspired 2D black hole gravity \([4]\) for a redefined metric; this equivalence, however, looses its attractiveness when one couples the action to nonconformal matter using the redefined metric. Most of the specific models have been quantized in a Dirac approach already (cf. citations above); moreover, this is also true for the general action \(S_G\) in the torsionless case \(\tau = 0\) \([13]\). It is the second main purpose of this letter that these quantizations, which came down to the quantization of a one dimensional point particle system, in many cases have to be supplemented by appropriate discrete indices, originating from nontrivial topological properties of the constraint surface.

The Yang-Mills part of our action has the standard form \((1/4\kappa^2) \int tr(F \wedge \star F)\), where \(F = dA + A \wedge A\) and the trace is taken in the adjoint representation. Rewriting this action in first order form, it reads

\[
S_{YM} = \int tr(E F + \kappa^2 E^2 \varepsilon), \tag{3}
\]

the 'electric fields' \(E\) being (Lie algebra valued) functions. The coupling to the gravity sector is seen to be separated to the second term now. For simplicity we will assume the

\(^a\)The classical local integrability of YM coupled to the KV model, defined through \(S_{KV}^G\) below, has been observed already in \([6]\).
gauge group $G$ to be compact and simply connected, which implies also that $G$ is simple. But it would be straightforward to generalize what follows, e.g., to arbitrary compact groups $G$ (gaining a $\Theta$-angle for every $U(1)$-factor, cf., e.g., [2]).

Let us turn to the phase space structure of the theory. Since $S = S_G + S_{YM}$ is already in first order form, we can read off the Poisson brackets and constraints directly. The canonically conjugates are $(e_1^a, \omega_1, A_1; \pi_a, \pi_\omega, E)$, respectively, whereas the zero components of the basic one forms enforce the following first class constraints ($\partial \equiv \partial/\partial x^1$)

$$
\begin{align*}
G_a &= \partial \pi_a + \varepsilon_a^b \pi_b \omega_1 - \varepsilon_{ab} e_1^b [V - \kappa^2 trE^2] \approx 0 \quad (4) \\
G_\omega &= \partial \pi_\omega + \varepsilon^a_b \pi_a e_1^b \approx 0 \quad (5) \\
G &= \nabla_1(A)E \equiv \partial_1 E + [A_1, E] \approx 0 \quad (6)
\end{align*}
$$

and can be regarded as arbitrary Lagrange multipliers within the Hamiltonian

$$
H = - \oint dx^1 e_0^a G_a + \omega_0 G_\omega + tr(A_0 G). \quad (7)
$$

We observe that in two dimensions the addition of a dynamical gravity sector leaves the Yang-Mills’ Gauß law $G \approx 0$ completely unchanged. This is in contrast to four space time dimensions where the covariant derivative $\nabla$ contains also a gravitational connection resulting from the fact that the electric fields are not functions there but one forms on a three manifold (c.f., e.g., [13]). Since, furthermore, $\partial(trE^2) = trGE/2 \approx 0$, on-shell the YM theory modifies the gravitational theory only via dynamically shifting the cosmological constant of the gravity sector by the YM Hamiltonian $H_{YM}^{(0)} \equiv -\kappa^2 \oint trE^2 dx^1$.

2. To quantize the system we choose our wave functionals to depend on $\pi_a, \pi_\omega, A_1$ and replace $e_1^a, \omega_1, E$ by the appropriate derivative operators. Our ordering prescription for the quantum constraints is to put all derivative operators to the right, which yields a consistent quantum algebra. For the solution of the quantum Gauß law $G \Psi = 0$ we can refer the reader to the extensive literature. The basic result is (cf., e.g., [2]) that the functional $\Psi[\pi_b, \pi_\omega, A_1(x^1)]$ can be written as a function $\Psi[\pi_b, \pi_\omega, a]$ of a constant element $a$ of the corresponding Cartan subalgebra (CSA) which is gauge related to $A_1(x^1)$; due to a residual gauge freedom, $\Psi$ is, moreover, invariant under affine Weyl transformations so that the fundamental domain of definition of $\Psi$ as a function of $a$ is the Weyl cell of the CSA. For the simplest case of $SU(2)$, e.g., $\Psi$ is a periodic function of $a \in R$, and similarly for the case of $SU(3)$ a function on a triangle, ‘periodically’ continued to the plane via Weyl reflections and translations. The Hamilton operator of pure Yang-Mills theory $H_{YM}^{(0)}$ projected onto this physical subspace is proportional to the ordinary Laplacian on the CSA. The same projection yields also a natural measure $\mu(a)$ with respect to which $H_{YM}^{(0)}$ is self-adjoint.

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\footnote{This suggests also that there should be some connection to [16] where the authors allowed for a dynamical cosmological constant within the reformulated 2D black hole gravity so as to reinterpret the resulting theory as a connection flat gauge theory for the centrally extended Poincaré group. Indeed, choosing as our potential $V = 1/4\kappa^2$, as YM gauge group the real line, and shifting $E$ by $-1/2\kappa^2$, the limit $\kappa \to 0$ reproduces that theory.}
Due to the finite size of the Weyl cell $H^{(0)}_{YM}$ has a discrete spectrum $\epsilon_k$, $k \in \mathbb{N}$, and we can expand our gravity-Yang-Mills functional $\Psi$ onto an orthogonal set of eigenfunctions $\chi_{k,l}(a)$, $l$ labelling possible degeneracies of $\epsilon_k$:

$$\Psi[\pi_+, \pi_-, \pi_\omega, a] = \sum_{k,l} \psi_{k,l}[\pi_+, \pi_-, \pi_\omega] \chi_{k,l}(a).$$ (8)

Applying next the remaining quantum constraints $G_\pm, G_\omega$ to this expansion, we see that each of the $\psi_{k,l}$ has to be annihilated by the corresponding operators in which $-\kappa^2 tr E^2(x^1)$ has been replaced by $\epsilon_k$. Let us denote these modified operators by $G_{\pm,(k)}$ and analogously $\psi_k := v + \epsilon_k$, $V(k) := V + \epsilon_k \equiv v(k)(\pi_\omega) + \pi^2/2$. Next one finds the combination

$$\pi^a G_{a,(k)} + V(k) G_\omega = \frac{1}{2} \partial(\pi^2) + V(k) \partial \pi_\omega$$ (9)

to act in a purely multiplicative way on the wave functionals. Multiplying (9) from the left by the integrating factor $\exp(\tau \pi_\omega)$, this yields (no sums)

$$\partial Q(k) \psi_{k,l} = 0$$ (10)

$$Q(k)(\pi^2, \pi_\omega) = \frac{1}{2} \pi^2 \exp(\tau \pi_\omega) + \int \pi_\omega v(k)(u) \exp(\tau u) du.$$ (11)

This is a restriction to the support or the domain of definition of $\psi_{k,l}$. Further inspection of the constraints show that the only 'allowed' functions $\pi_a(x^1)$, $\pi_\omega(x^1)$ containing the 'critical points'

$$\pi_+ = \pi_- = 0, \pi_\omega = \alpha_c, \quad \alpha_c : V(k)(\alpha_c) = 0$$ (12)

are the constant ones. In this restricted domain of definition (!) the wave functional $\exp(\phi[\pi_+, \pi_-, \pi_\omega])$ with

$$\phi = \begin{cases} 
0 & : \pi_+ \equiv \pi_- \equiv 0 \\
-\frac{i}{\hbar} \oint \ln |\pi_+| d\pi_\omega = \frac{i}{\hbar} \oint \ln |\pi_-| d\pi_\omega & : \text{otherwise} 
\end{cases}$$ (13)

can be seen to be a particular (continuous) solution to the quantum constraints. Thus the general solution to the gravity constraints can be written as

$$\tilde{\psi}_{k,l} = \exp(\phi) \tilde{\psi}_{k,l}$$ (14)

with a $\tilde{\psi}_{k,l}$ which is invariant under the Lie derivative part of the $G_{a,(k)}, G_\omega$ constraints, i.e. under infinitesimal classical gauge transformations. At this point it is helpful to interpret $\tilde{\psi}_{k,l}$ (or also $\psi_{k,l}$) as a functional of (parametrized and connected) loops in the three dimensional target space $(\pi_+, \pi_-, \pi_\omega)$, or better on the two-surfaces $\mathcal{M}_q$ generated by setting $Q(k)$ to a (varying) constant $q$ (cf. Eq. (10)). Now, on any of these surfaces the flow of the constraints is transitive, except precisely for the critical points (12), which are fixed points under this flow. This is most easily seen by noting that on any connected part of this surface where $\pi_+ \neq 0$ [resp. $\pi_- \neq 0$] we can use $(G_\omega/\pi_+, G_++/\pi_+)$ $[(G_\omega/\pi_-, G_-/\pi_-)]$ as conjugate variables to the local coordinates $(\pi_+, \pi_\omega)$ $[(\pi_-, \pi_\omega)]$ of $\mathcal{M}_q$; furthermore $\{\pi_a, G_b\} = -\varepsilon_{ab} V \delta$. However, the wavefunctions $\psi_{k,l}$ do not depend
only on \( q \) as one might suppose at first sight. This is so because certainly only loops from the same homotopy class and the same component of \( \mathcal{M}_q \) can be deformed into each other by means of the constraints. Thus \( \tilde{\psi}_{k,l} \) is a function of \( q, \pi_0(\mathcal{M}_q), \) and \( \pi_1(\mathcal{M}_q) \) (as well as the fixed points, if \( q = q_c \equiv Q(k)(0, \alpha_c) \)). Labelling the elements of the latter two discrete groups by \( n_0^q \) and \( n_1^q \), and suppressing the fixed points for a moment, we find

\[
\tilde{\psi}_{k,l} = \tilde{\psi}_{k,l}(q, n_0^q, n_1^q),
\]

which together with (8, 13, 14) describes the general solution of the quantum constraints (1 - 3).

To illustrate the above considerations, let us regard some examples: \( V(k) = \pi_\omega \) (for some fixed \( k \), dropping this index furtheron within the paragraph) implies \( Q = \frac{1}{2}[\pi^2 + (\pi_\omega)^2] \). Putting this to a constant \( q \), we obtain the typical Lorentz orbits in a three dimensional 'Minkowski space' \( \mathbb{R}^3 \) i.e. two-sheet hyperboloids for \( q > 0 \), which implies \( n_0^+ \in \{1, 2\} \) and \( n_1^+ = 0 \), one-sheet hyperboloids for \( q < 0 \), which implies \( n_0^- = 0 \) and \( n_1^- \in \mathbb{Z} \), as well as the future and past lightcone 'separated' by the origin for the critical value \( q = 0 \). Due to the latter parts the resulting orbit space is non-Hausdorff (such as the Lorentz orbit space), and there arises some arbitrariness in determining \( n_0^q, n_1^q \) for this value of \( q \): The origin and the light cones have no disjoint neighborhoods so that, e.g., continuous functions on this space would identify them (\( \Rightarrow n_0^q = 0, n_1^q = 0 \)); on the other hand we know that there are no loops passing through the origin since it is critical (\( \Rightarrow n_0^q \in \{1, 2\}, n_1^q \in \mathbb{Z} \), plus the origin as an own orbit).

As a second example let us consider \( R^2 \)-gravity with potential \( V = 3\pi_\omega^2/2 - 3/4 \) coupled to \( SU(2) \)-YM, yielding \( \epsilon_k = k^2/4 \) (for \( k^2 = 2 \), if we choose \( x^1 = 0 \sim x^1 = 1 \)). We obtain \( 2Q(k) = \pi^2 + \pi_\omega^3 + (2\epsilon_k - 3/2)\pi_\omega \). For \( k = 1 \) there are two critical values of \( q: \alpha_c = \pm 1/\sqrt{3} \to q_c = \mp 1/\sqrt{27} \), whereas for \( k = 2, 3, ... \) there are no critical values of \( q \). In the latter case the resulting two-surfaces are all connected and simply connected. This is also true for \( \epsilon = \epsilon_1 = 1/4 \) and \( q \notin [-1/\sqrt{27}, 1/\sqrt{27}] \), whereas in the case \( k = 0 \) and \( q \in ]-1/\sqrt{27}, 1/\sqrt{27}[, \) the \( Q_0 = q \) surface is connected but has the fundamental group of a pointed torus. At \( q = q_c \) there arises a similar situation as in the first example. Thus in this example the wave functions have the form

\[
\Psi = \exp(\phi)[\tilde{\psi}_1(q, n_1)\chi_1(a) + \sum_{k \geq 2} \tilde{\psi}_k(q)\chi_k(a)]
\]

where \( \phi \) is defined in (13), \( \chi_k \) is a periodic function of one argument, and \( \psi_k \) is a function of one unbounded variable except for \( k = 1, n_1 \neq 0 \), in which case it has support \([-1/\sqrt{27}, 1/\sqrt{27}] \).

More generally the situation can be depicted as follows: For any fixed value \( q \) and \( \epsilon_k \) equation (14) induces a curve in a \( \pi^2 \) over \( \pi_\omega \) diagram. If this curve has no intersections with the \( \pi_\omega \) axis, \( \mathcal{M}_q \) has two simply connected components. Otherwise (and for \( q \neq q_c \)) \( \mathcal{M}_q \) is always connected and the number of basic non-contractible loops is one larger

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\(^*\)This has to do with the fact that the gravitational action for the above potential can be reinterpreted as the one of a \( \pi F \)-theory for gauge group \( SO(2, 1) \sim PSL(2, R) \) (14), or rather its universal covering, as pointed out in [17, 18].
than the number of intersections of the $\pi^2(\pi_\omega)$-curve with the $\pi_\omega$ axis. A change of this number can occur only at critical values of $q$, which correspond to curves having at least one sliding intersection with the $\pi_\omega$ axis. All these surfaces $\mathcal{M}_q$ are noncompact and the spectrum of $q$ ranges over all of $\mathbb{R}$.

3. Let us conclude with some remarks. Firstly, already the example of Lorentz transformations in a Minkowski space shows that in general orbits cannot be (uniquely) characterized by means of continuous invariants only: Beside the invariant length of a vector of this space, one needs also some (discontinuous) sign functions and, to distinguish the origin from the light cones, even a distribution. This is the reason for the quantum numbers $n_0$ and $n_1$ within the wave functions $\tilde{\psi}_{k,l}$: $Q(k)$ is the continuous invariant on the underlying function space, $n_0$ and $n_1$ correspond to discontinuos (invariant) functions on the latter, and (12) is the counterpart to the origin of the Minkowski space example above. From this perspective it comes at no surprise that beside (15) also

$$ \sum_c C_c \delta[\pi_+] \delta[\pi_-] \delta[\pi_\omega(x^1) - \alpha_c], \quad C_c \in \mathbb{C} \tag{17} $$

solve the quantum constraints.

Secondly, we still have to define an inner product for the $\tilde{\psi}_{k,l}$ (for fixed $k$ and $l$ only, since the $\chi_{k,l}$ in (8) are orthogonal by construction). On parts of the phase space which do not contain critical points (12) the Dirac observable conjugate to $q = \oint Q(k) dx^1$ can be put into the form

$$ p = \oint \exp(-\tau \pi_\omega) \frac{e_1^-}{\pi_+} dx^1 \sim \oint \exp(-\tau \pi_\omega) \frac{e_1^+}{\pi_-} dx^1. \tag{18} $$

Replacing $e_1^\pm$ by the corresponding functional derivative operator, it acts as $(\hbar/i)(d/dq)$ on $\tilde{\psi}_{k,l}$. Requiring that this fundamental Dirac observable shall be represented by a hermitean operator, restricts the measure to be proportional to $dq$ within any interval of $q$ not containing a critical value $q_c$. The implementation of the quantum numbers $n_0$ and $n_1$, however, seems not determined by this procedure and a further investigation of this point would be interesting.

Let me further remark that it is probably incorrect to just neglect the solutions (17). This becomes most apparent in the extreme case $V(k) \equiv 0$, where any constant loop on the $\pi_\omega$ axis becomes critical; on the classical level these solutions are pared with the compactifications of Minkowski space along the boost orbits, yielding Misners two dimensional analogue of a Taub-Nut space [21]. So, neglecting the solutions (17) in this case comes down to throwing away about a third of the reduced phase space. (The remaining 'two thirds', represented by (13) on the quantum level, correspond to a Minkowski space factored along the translational isometries of the flat metric, thus, in part, also incorporate classical solutions with closed timelike curves). In the more generic

\textsuperscript{d}This picture is changed when regarding the gravitational theories corresponding to a Euclidean signature. Some values of $q$ generate compact target-space surfaces $\mathcal{M}_q$ then; on the latter the Wick rotated phase factor (13) is globally defined only for some values of $q$, which leads to a discretization of the corresponding part of the spectrum, [18], [19], [20].
case $V(k) \neq 0$ the neglection of (17) would still change the degeneracy of the spectrum of the Dirac observable $q$. (In the Euclidean formulation of the theory the omission of (17) may even lead to a change of the spectrum of $q$ [15].)

In this letter we carefully constructed the general solutions to the quantum constraints of many gravity theories coupled to YM. Open technical questions concern the construction of an inner product and a possible inclusion of the solutions (17). Furthermore, the treatment of conceptual questions of quantum gravity seems rewarding at this point: firstly, since the classical solutions include black hole type solutions for some choices of $V$, and, secondly, since $S$ reduces to a reparametrization invariant formulation of a pure 2D YM theory for $V \equiv 0$ so that the models comprised in the action (1, 3) may well be used for testing and developing concepts to solve the ‘problem of time’ [22]. From the mathematical point of view the evaluation of the partition function for $S_G$ would be an interesting open task (cf. also [23]). Also, $S_G$ can be regarded as a generalization of an $SO(2, 1)$ gauge theory; proceeding similarly for other groups or target space dimensions one obtains further new topological 2D field theories [19].

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