Eigenvalue asymptotics for a class of multi-variable Hankel matrices

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Abstract

A one-variable Hankel matrix \( H_a \) is an infinite matrix \( H_a = [a(i+j)]_{i,j \geq 0} \). Similarly, for any \( d \geq 2 \), a \( d \)-variable Hankel matrix is defined as \( H_a = [a(i+j)] \), where \( i = (i_1, \ldots, i_d) \) and \( j = (j_1, \ldots, j_d) \), with \( i_1, \ldots, i_d, j_1, \ldots, j_d \geq 0 \). For \( \gamma > 0 \), A. Pushnitski and D. Yafaev proved that the eigenvalues of the compact one-variable Hankel matrices \( H_a \) with \( a(j) = j^{-1}(\log j)^{-\gamma} \), for \( j \geq 2 \), obey the asymptotics \( \lambda_n(H_a) \sim C n^{-\gamma} \), as \( n \to +\infty \), where the constant \( C_\gamma \) is calculated explicitly. This paper presents the following \( d \)-variable analogue. Let \( \gamma > 0 \) and \( a(j) = j^{-d}(\log j)^{-\gamma} \), for \( j \geq 2 \). If \( a(j_1, \ldots, j_d) = a(j_1 + \cdots + j_d) \), then \( H_a \) is compact and its eigenvalues follow the asymptotics \( \lambda_n(H_a) \sim C_{d,\gamma} n^{-\gamma} \), as \( n \to +\infty \), where the constant \( C_{d,\gamma} \) is calculated explicitly.

1 Introduction

1.1 One-variable Hankel operators

Let \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots \} \). Given a complex valued sequence \( a = \{a(j)\}_{j \in \mathbb{N}_0} \), a Hankel operator (matrix) \( H_a \) on \( \ell^2(\mathbb{N}_0) \) is formally defined by

\[
(H_a x)(i) = \sum_{j \in \mathbb{N}_0} a(i+j)x(j), \quad \forall i \in \mathbb{N}_0, \quad \forall x = \{x(j)\}_{j \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0).
\]

The sequence \( a \) is called a \textit{parameter sequence}.

Nehari’s theorem [9] Theorem 1.1] gives a necessary and sufficient condition for \( H_a \) to be bounded on \( \ell^2(\mathbb{N}_0) \). A simple sufficient condition is given by \( a(j) = O(j^{-1}) \), when \( j \to +\infty \). A sufficient condition for compactness is \( a(j) = o(j^{-1}) \), when \( j \to +\infty \). Note that these two conditions are also necessary in the case of positive Hankel operators [14] Theorems 3.1, 3.2].

Let \( \alpha > 0 \) and consider the Hankel operator \( H_a \), where

\[
a(j) = (j+1)^{-\alpha}, \quad \forall j \in \mathbb{N}_0.
\]

For \( \alpha \in (0, 1) \), \( H_a \) is not bounded. When \( \alpha = 1 \), \( H_a \) is bounded but not compact. In this case, \( H_a \) is known as Hilbert’s matrix. Finally, for \( \alpha > 1 \), \( a(j) = o(j^{-1}) \), as \( j \to +\infty \), and so, \( H_a \) is bounded and compact. From this discussion, it is inferred that the exponent \( \alpha = 1 \) is the boundedness-compactness threshold.

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1.2 Multi-variable Hankel operators

In [11] (see the example after Theorem 3.3) it is proved that the eigenvalue asymptotics of the Hankel operators with parameter sequence \(a(j) = (j + 1)^{-\alpha}\), where \(\alpha > 1\), are described by

\[
\lambda_n(H_a) = \exp \left( -\pi \sqrt{2(\alpha - 1)n + o(\sqrt{n})} \right), \quad n \to +\infty.
\]

In [10] A. Pushnitski and D. Yafaev studied a whole class of Hankel operators that lies between the cases \(\alpha = 1\) and \(\alpha > 1\). That was achieved by considering parameter sequences \(a = \{a(j)\}_{j \in \mathbb{N}_0}\) of the following type

\[
a(j) = j^{-1}(\log j)^{-\gamma}, \quad \forall j \geq 2,
\]

where \(\gamma > 0\). They concluded that if \(\{\lambda_n^+(H_a)\}_{n \in \mathbb{N}}\) is the non-increasing sequence of positive eigenvalues of \(H_a\), and \(\lambda_n^-(H_a) = \lambda_n^+(\pm H_a)\), then the eigenvalues of the corresponding Hankel operator \(H_a\) obey the following asymptotic law

\[
\lambda_n^+(H_a) = C_\gamma n^{-\gamma} + o(n^{-\gamma}) \quad \text{and} \quad \lambda_n^-(H_a) = o(n^{-\gamma}), \quad n \to +\infty,
\]

where

\[
C_\gamma = \left[ \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{\pi}{\cosh(\pi x)} \right)^{\frac{1}{\gamma}} \, dx \right]^{\gamma} = 2^{-\gamma} \pi^{1-2\gamma} B \left( \frac{1}{2\gamma}, \frac{1}{2} \right)^{\gamma};
\]

here \(B(\cdot, \cdot)\) is the Beta function.

1.2 Multi-variable Hankel operators

The purpose of this section is to introduce the multi-variable Hankel operators and develop the \(d\)-variable analogue of the asymptotics [11].

From now on, we will denote all the multi-variable functions and their arguments by boldface letters. So, for \(d \geq 2\) consider the set \(\mathbb{N}_0^d = \{ j = (j_1, j_2, \ldots, j_d) : j_i \in \mathbb{N}_0, \text{ for } i = 1, 2, \ldots, d\}\) and the space \(\ell^2(\mathbb{N}_0^d)\) of \(d\)-variable square summable sequences \(x = \{x(j)\}_{j \in \mathbb{N}_0^d}\). Let \(a = \{a(j)\}_{j \in \mathbb{N}_0^d}\) be a complex valued sequence and define, formally, the Hankel operator \(H_a\) on \(\ell^2(\mathbb{N}_0^d)\) by

\[
(H_a x)(i) := \sum_{j \in \mathbb{N}_0^d} a(i + j)x(j), \quad \forall i \in \mathbb{N}_0^d, \quad \forall x = \{x(j)\}_{j \in \mathbb{N}_0^d} \in \ell^2(\mathbb{N}_0^d).
\]

The sequence \(a\) is a parameter sequence.

To the best of the author’s knowledge, no necessary and sufficient conditions for the boundedness or compactness of \(H_a\) are available at present. Heuristically, \(a(j)\) can go to zero at different rates in different directions, which makes the problem more subtle than in the one-variable case. One can make progress by focusing on a subclass of sequences \(a(j)\). In this paper, we consider the following subclass. Let \(a = \{a(j)\}_{j \in \mathbb{N}_0}\) be a one-variable sequence and define

\[
a(j) = a(|j|), \quad \text{where} \quad |j| = \sum_{i=1}^d j_i, \quad \forall j = (j_1, j_2, \ldots, j_d) \in \mathbb{N}_0^d.
\]

In this case, it can be verified that \(H_a\) is bounded if \(a(j) = O(j^{-d})\) and compact if \(a(j) = o(j^{-d})\), when \(j \to +\infty\). Moreover, for \(\alpha > 0\) consider the sequence

\[
a(j) = (j + 1)^{-\alpha}, \quad \forall j \in \mathbb{N}_0.
\]

If \(\alpha \in (0, d)\), then \(H_a\) is unbounded. If \(\alpha = d\), \(H_a\) is bounded but not compact and for \(\alpha > d\), the aforementioned tests imply boundedness and compactness. Therefore, the boundedness-compactness threshold exponent, for this choice of the parameter sequence \(a\), is \(\alpha = d\). The
main result of this paper is the $d$-variable analogue of (1.1). We first give a simple version of our result, Theorem 1.1; a more complete statement is Theorem 1.2 below. In order to formulate Theorem 1.1 let $\mathcal{F}$ be the Fourier transform on the real line; i.e.

$$
(Ff)(x) = \int_{\mathbb{R}} f(y) e^{-2\pi i xy} \, dy, \ \forall x \in \mathbb{R}.
$$

(1.3)

The inverse Fourier transform, $\mathcal{F}^* f$, of $f$ will be often denoted by $\hat{f}$.

**Theorem 1.1.** Let $\gamma > 0$ and consider the parameter sequence $a(j) = a(|j|)$, for all $j \in \mathbb{N}_0^d$, where

$$
a(j) = j^{-d}(\log j)^{-\gamma}, \ \forall j \geq 2.
$$

Moreover, for any $j \in \mathbb{N}$, define the function

$$
\phi_j(x) := \frac{1}{\cosh^d\left(\frac{x}{\pi}\right)}, \ \forall x \in \mathbb{R}.
$$

(1.5)

Then the corresponding Hankel operator $H_a$ is self-adjoint, compact, and it presents power eigenvalue asymptotics of the form below:

$$
\lambda_n^+(H_a) = C_{d,\gamma} n^{-\gamma} + o(n^{-\gamma}) \quad \text{and} \quad \lambda_n^-(H_a) = o(n^{-\gamma}), \ n \to +\infty,
$$

where

$$
C_{d,\gamma} = \frac{1}{2^d(d-1)!} \left( \int_{\mathbb{R}} \phi_d^\frac{1}{2} (x) \, dx \right)^\gamma.
$$

(1.6)

**Remark.** It is worth to notice that relation (1.6) gives (as expected) (1.2) when $d = 1$. For observe that $\phi_1(x) = \frac{2\pi}{\cosh(2\pi x)}$. Then, by applying the change of variables $y = 2\pi x$, we get

$$
C_{1,\gamma} = \frac{1}{2} \left( \int_{\mathbb{R}} \phi_1^\frac{1}{2} (x) \, dx \right)^\gamma = \left[ \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{\pi}{\cosh(\pi y)} \right)^\frac{1}{2} \, dy \right]^\gamma = C_{\gamma},
$$

where $C_{\gamma}$ is the constant that is defined by (1.2).

1.3 Main result

A generalisation of Theorem 1.1 leads to the main result, Theorem 1.2. For any $\gamma > 0$, define

$$
M(\gamma) := \begin{cases} 
0, & \gamma \in \left(0, \frac{1}{2}\right), \\
\lceil \gamma \rceil + 1, & \gamma \geq \frac{1}{2},
\end{cases}
$$

(1.7)

where $\lceil \gamma \rceil = \max\{x \in \mathbb{Z} : x \leq \gamma\}$. In addition, for any sequence $a = \{a(j)\}_{j \in \mathbb{N}_0}$, define the sequence of iterated differences $a^{(m)} = \{a^{(m)}(j)\}_{j \in \mathbb{N}_0}$, where $m \in \mathbb{N}_0$, with

$$
a^{(0)} = a \quad \text{and} \quad a^{(m)}(j) = a^{(m-1)}(j+1) - a^{(m-1)}(j), \ \forall j \in \mathbb{N}_0, \ \forall m \in \mathbb{N}.
$$

**Theorem 1.2.** Let $\gamma > 0$, $b_1, b_{-1} \in \mathbb{R}$, and $a$ be a real valued sequence of $\mathbb{N}_0$, such that

$$
a(j) = (b_1 + (-1)^j b_{-1}) j^{-d}(\log j)^{-\gamma} + g_1(j) + (-1)^j g_{-1}(j), \ \forall j \geq 2,
$$

(1.8)

where both $g_1$ and $g_{-1}$ satisfy the following condition:

$$
g^{(m)}_{\pm 1}(j) = o(j^{-d-m}(\log j)^{-\gamma}), \ j \to +\infty,
$$
for \( m = 0, 1, \ldots, M(\gamma) \). If \( H_a \) is the Hankel operator, where \( a(j) = a(|j|), \forall j \in \mathbb{N}_0^d \), then it is a self-adjoint, compact operator and its eigenvalues satisfy the following asymptotic law

\[
\lambda^\pm(H_a) = C^\pm n^{-\gamma} + o(n^{-\gamma}), \; n \to +\infty. \tag{1.9}
\]

The leading term coefficients are given by

\[
C^\pm = \left( (b_1)^{\frac{1}{\pm}} + (b_{-1})^{\frac{1}{\pm}} \right)^\gamma C_{d,\gamma}, \tag{1.10}
\]

where \( C_{d,\gamma} \) is defined in (1.6) and \( (x)_\pm := \max\{0, \pm x\} \), for any \( x \in \mathbb{R} \).

1.4 Proof outline

In order to derive the spectral asymptotics for the class of operators that were introduced in Theorem 1.2, we follow the steps that are listed below. In the sequel, we give a brief description of each one of them.

- Construction of a model operator (see §3),
- reduction of the model operator to pseudo-differential operators (see §4),
- use of Weyl-type spectral asymptotics of the respective pseudo-differential operators (see §5),
- reduction of the error terms to one-variable weighted Hankel operators (see §6), and
- Schatten class inclusions of the error terms (see §7).

The construction of the model operator aims to give the leading term in the eigenvalue asymptotics. More precisely, the model operator will be a Hankel operator which behaves “similarly” to the initial Hankel operator but whose eigenvalue asymptotics are retrieved much easier and explicitly. By examining for simplicity the case of \( a \) given by (1.4), the model operator will be a Hankel operator of the form \( \tilde{H} := H_{\tilde{a}} \), with parameter sequence \( \tilde{a}(j) = \tilde{a}(|j|) \), for all \( j \in \mathbb{N}_0^d \).

**Remark.** From now on, objects related with the model operator will be declared with the tilde symbol; e.g. \( \tilde{H}, \tilde{a}, \tilde{a}, \) etc.

The sequence \( \tilde{a} \) will be chosen to be the Laplace transform of a suitable function \( w \), i.e.

\[
\tilde{a}(j) = (\mathcal{L}w)(j) = \int_0^{+\infty} w(\lambda) e^{-\lambda j} d\lambda, \; \forall j \in \mathbb{N}_0.
\]

The function \( w \) is chosen in a way such that \( \tilde{a}(j) \sim a(j) \), as \( j \to +\infty \); i.e. \( \frac{\tilde{a}(j)}{a(j)} \to 1 \), as \( j \to +\infty \). The latter is obtained via a lemma for Laplace transform asymptotics.

In the sequel, the spectral analysis of the model operator, \( \tilde{H} \), is reduced to that one of a pseudo-differential operator. To see this, consider the inner product

\[
\langle \tilde{H}x, y \rangle = \sum_{i,j \in \mathbb{N}_0^d} \tilde{a}(|i + j|)x(j)y(i).
\]

By using the fact that \( \tilde{a}(j) = (\mathcal{L}w)(j) \), we can swap summation and integration to obtain

\[
\langle \tilde{H}x, y \rangle = \int_0^{+\infty} (Lx)(t)\overline{(Ly)}(t) dt,
\]

for \( m = 0, 1, \ldots, M(\gamma) \). If \( H_a \) is the Hankel operator, where \( a(j) = a(|j|), \forall j \in \mathbb{N}_0^d \), then it is a self-adjoint, compact operator and its eigenvalues satisfy the following asymptotic law

\[
\lambda^\pm(H_a) = C^\pm n^{-\gamma} + o(n^{-\gamma}), \; n \to +\infty. \tag{1.9}
\]

The leading term coefficients are given by

\[
C^\pm = \left( (b_1)^{\frac{1}{\pm}} + (b_{-1})^{\frac{1}{\pm}} \right)^\gamma C_{d,\gamma}, \tag{1.10}
\]

where \( C_{d,\gamma} \) is defined in (1.6) and \( (x)_\pm := \max\{0, \pm x\} \), for any \( x \in \mathbb{R} \).
where \( L : \ell^2(\mathbb{N}_0^d) \to L^2(\mathbb{R}_+) \) is given by
\[
(Lx)(t) = \sqrt{w(t)} \sum_{j \in \mathbb{N}_0^d} e^{-|j|^\beta} x(j), \quad \forall t \in \mathbb{R}_+, \quad \forall x \in \ell^2(\mathbb{N}_0^d);
\]
note that \( \mathbb{R}_+ := (0, +\infty) \).

**Remark.** Notice that in order \( L \) to be well-defined, \( w \) has to be non-negative. For sake of simplicity, in this introduction, we assume that this is true and the general case is addressed properly in the proofs.

Therefore, \( \tilde{H} \) can be expressed as a product of two operators, \( \tilde{H} = L^* L \), and we can apply the following lemma ([3, §8.1, Theorem 4]).

**Lemma 1.3.** Let \( L \) be a linear bounded operator, defined on a Hilbert space \( \mathcal{H} \). Then, the restrictions \( L^* L \upharpoonright (\operatorname{Ker} L^* L)^\perp \) and \( LL^* \upharpoonright (\operatorname{Ker} LL^*)^\perp \) are unitarily equivalent.

**Remark.** We will denote this equivalence by \( \simeq \); e.g. \( L^* L \simeq LL^* \).

Thus, \( \tilde{H} \) is unitarily equivalent (modulo kernels) to \( LL^* : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \). Finally, by an exponential change of variable, \( LL^* \) is proved to be unitarily equivalent (modulo kernels) to a pseudo-differential operator \( \beta(X)\alpha(D)\beta(X) \), where \( D \) is the differentiation operator in \( L^2(\mathbb{R}) \), \( Df = -if' \), and \( X \) is the multiplication operator (in \( L^2(\mathbb{R}) \)) by the function \( \text{id}(x) = x \). Then, by exploiting a Weyl-type spectral asymptotics formula for the operator \( \beta(X)\alpha(D)\beta(X) \), we retrieve its eigenvalue asymptotics and thus, those of \( \tilde{H} \).

**Remark.** The technique of considering the inner product \( (H_a x, y) \) and changing the order of summation and integration was also applied by Widom in [14] for one-variable Hankel operators. In order to derive the eigenvalue asymptotics, Widom also applied Lemma 1.3. This yielded the equivalence to the pseudo-differential operator that we would obtain, if we followed the steps that are described above (for \( d = 1 \)). The same equivalence, but in greater generality, is also obtained by Yafaev in [15, Theorem 7.7].

Finally, the initial Hankel operator, \( H_a \), can be expressed as a sum of operators, \( H_a = \tilde{H} + (H_a - \tilde{H}) \). Having obtained the eigenvalue asymptotics for \( \tilde{H} \), the next step is to prove that the spectral contribution of the operator \( H_a - \tilde{H} \) is negligible, compared to that one of \( \tilde{H} \). This will be achieved by proving certain Schatten-Lorentz class inclusions for \( H_a - \tilde{H} \). These inclusions depend on the range of the exponent \( \gamma \) in (1.4) and are obtained by a combination of interpolation and reduction to one-variable weighted Hankel operators.

### 1.5 List of notation

For the reader’s convenience, we close our introduction by summarising the introduced notation.

**Set notation:** Let \( \mathbb{R} \) be the set of real numbers, \( \mathbb{Z} \) the set of integers, \( \mathbb{N} = \{1, 2, 3, \ldots\} \) be the set of natural numbers, and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). In addition, \( \mathbb{R}_+ = (0, +\infty) \). We denote by \( \mathbb{C} \) the set of complex numbers. Then \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \) and \( \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \). Moreover, \( \mathbb{T} \) can be identified with the interval \([0, 1)\), via the map \( t \mapsto e^{2\pi it} \), for all \( t \in [0, 1) \).

For any \( d \geq 2 \), we can define \( d \)-Cartesian products of the aforementioned sets; e.g. \( \mathbb{R}^d = \{x = (x_1, x_2, \ldots, x_d) : x_i \in \mathbb{R}, \text{ for } i = 1, 2, \ldots, d\} \).

**Dimension notation:** We use the Roman (standard) typeface for one-dimensional/variable objects and boldface letters for \( d \)-dimensional/variable ones. For example, let \( f(x) \) describe a function defined on \( \mathbb{R} \) and \( a = \{a(j)\}_{j \in \mathbb{N}_0^d} \) be a \( d \)-variable sequence.
Sequence notation: We say that two (real valued) sequences \( \{a(j)\}_{j \in \mathbb{N}_0} \) and \( \{b(j)\}_{j \in \mathbb{N}_0} \) present the same asymptotic behaviour at infinity, and denote by \( a(j) \sim b(j) \), as \( j \to +\infty \), when \( \frac{a(j)}{b(j)} \to 1 \), as \( j \to +\infty \). For a (complex valued) sequence \( a = \{a(j)\}_{j \in \mathbb{N}_0} \), define the sequence of iterated differences \( a^{(m)} = \{a^{(m)}(j)\}_{j \in \mathbb{N}_0} \), where \( m \in \mathbb{N}_0 \), with
\[
a^{(0)} = a \quad \text{and} \quad a^{(m)}(j) = a^{(m-1)}(j+1) - a^{(m-1)}(j), \quad \forall j \in \mathbb{N}_0, \quad \forall m \in \mathbb{N}.
\]

Number notation: For any real number \( x \), we define its integer part \( \lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\} \) and its positive (resp. negative) part \( (x)_+ = \max\{0, x\} \) (resp. \( (x)_- = \max\{0, -x\} \)). Furthermore, let \( \langle x \rangle = \sqrt{1 + x^2} \). For any real numbers \( x \) and \( y \), we write \( x \lesssim y \) when there exists a non-zero number \( c \) such that \( x \leq cy \). Finally, for any \( d \geq 2 \),
\[
|j| = \sum_{i=1}^{d} j_i, \quad \forall j = (j_1, j_2, \ldots, j_d) \in \mathbb{N}_0^d.
\]

Fourier transform: For a function \( \phi : \mathbb{T} \to \mathbb{C} \) the sequence of its Fourier coefficients \( \{(\Phi \phi)(n)\}_{n \in \mathbb{Z}} \) is given by
\[
(\Phi \phi)(n) = \int_{0}^{1} \phi(t) e^{-2\pi int} \, dt, \quad \forall n \in \mathbb{Z}.
\]
The Fourier transform \( \mathcal{F}f \) of a function \( f : \mathbb{R} \to \mathbb{C} \) is given by (1.3). We denote by \( \mathcal{F}^* \) its inverse and \( \hat{f} = \mathcal{F}^* f \).

Operator notation: For any operator \( A \), let \( A \upharpoonright S \) be the restriction of \( A \) to a subset \( S \) of its domain. Two operators \( A \) and \( B \), in general defined on different spaces, will be called unitarily equivalent modulo kernels (write \( A \simeq B \)), when they have unitarily equivalent non-zero parts. Namely, when there exists a unitary operator \( U \) such that
\[
A \upharpoonright (\text{Ker} A)^\perp = U^* B \upharpoonright (\text{Ker} B)^\perp U.
\]
We denote by \( H_\mathbf{a} \) (resp. \( H_a \)) all the \( d \)-variable (resp. one-variable) Hankel operators with parameter sequence \( \mathbf{a} \) (resp. \( a \)). Moreover, when \( H_\mathbf{a} \) has been defined, objects related with the model operator \( \mathcal{H} \) that corresponds to \( H_\mathbf{a} \) will be indicated with the tilde symbol; e.g. \( \tilde{a} \) will refer to the parameter sequence of the model operator, so that \( \mathcal{H} = H_{\tilde{a}} \). Finally, for weighted Hankel operators, we use the capital gamma; e.g. \( \Gamma, \Gamma_\mathbf{a}^{\alpha, \beta} \), etc. (see §2.4 for the relevant definitions).

Eigenvalue notation: Let \( A \) be an operator and \( \{\lambda_n^+(A)\}_{n \in \mathbb{N}} \) be the sequence of its positive eigenvalues. Then \( \overline{\lambda_n}(A) = \lambda_n^+(-A), \forall n \in \mathbb{N} \).

## 2 Preliminaries

### 2.1 Besov classes

We define Besov classes of analytic functions on the unit circle \( \mathbb{T} \). If \( C_c^\infty(\mathbb{R}) \) is the set of infinitely many times differentiable functions on \( \mathbb{R} \), with compact support, let \( v \) be a \( C_c^\infty(\mathbb{R}) \) function, such that \( \text{supp}(v) = [2^{-1}, 2] \), \( v(1) = 1 \), and \( v([2^{-1}, 2]) = [0, 1] \); notice that \( v(2^{-1}) = v(2) = 0 \). Then consider a sequence of \( C_c^\infty(\mathbb{R}) \) non-negative valued functions \( \{v_n\}_{n \in \mathbb{N}} \), such that,
\[
v_n(t) = v\left(\frac{t}{2^n}\right), \quad \forall n \in \mathbb{N},
\]
for any $t \in \mathbb{R}$, and
\[ \sum_{n \geq 0} v \left( \frac{t}{2^n} \right) = 1, \forall t \geq 1. \]

Ensuing, define the polynomials
\[ V_0(z) = z + 1 + z, \forall z \in T, \tag{2.1} \]
and, for every $n \in \mathbb{N}$,
\[ V_n(z) = \sum_{j \in \mathbb{N}} v_n(j) z^j = \sum_{j=2^{n-1}}^{2^n+1} v_n(j) z^j, \forall z \in T. \tag{2.2} \]

Then we say that an analytic function $f$ of $T$ belongs to the Besov class $B_{q,r}^p$ if and only if
\[ \|f\|_{B_{q,r}^p} := \left( \sum_{n \in \mathbb{N}_0} 2^{np} \|f \ast V_n\|_q^r \right)^{\frac{1}{r}} < +\infty. \]

The lemma below can be found in [11, Lemma 4.6].

**Lemma 2.1.** Assume that $\gamma \geq \frac{1}{2}$ and let $M(\gamma)$ be as defined in (1.7). Moreover, let $\{a(j)\}_{j \in \mathbb{N}_0}$ be a sequence of complex numbers which satisfies (7.2) and consider the function
\[ \phi(z) = \sum_{j \in \mathbb{N}_0} a(j) z^j, \forall z \in T. \]

If $V_n$ are as defined in (2.2), then, for every $q > \frac{1}{M(\gamma)}$ and every $n \in \mathbb{N}$ such that $2^n-1 \geq M(\gamma)$,
\[ \|\phi \ast V_n\|_{\infty} \leq \sum_{j=2^n-1}^{2^n+1} |a(j)|, \tag{2.3} \]
and
\[ 2^n \|\phi \ast V_n\|_q^q \leq C_q \left( \sum_{m=0}^{M(\gamma)} \sum_{j=2^n-1-M(\gamma)}^{2^n+1} (1+j)^m |a^{(m)}(j)| \right)^q, \tag{2.4} \]
for some positive constant $C_q$, depending only on $q$.

### 2.2 Schatten classes

Consider a compact operator $T$ and the sequence of its singular values $\{s_n\}_{n \in \mathbb{N}}$; i.e. the sequence of (positive) eigenvalues of $\sqrt{T^*T}$. Denote by $S_\infty$ the space of compact operators. For $p \in (0, +\infty)$, we define the Schatten class $S_p$, the Schatten-Lorentz classes $S_{p,q}$ and $S_{p,\infty}$, and the class $S_{0,\infty}^0$ by the following conditions:

\[ T \in S_p \iff \|T\|_{S_p} := \left( \sum_{n \in \mathbb{N}} s_n^p \right)^{\frac{1}{p}} < +\infty \]

\[ T \in S_{p,q} \iff \|T\|_{S_{p,q}} := \left( \sum_{n \in \mathbb{N}} \frac{\left( \frac{n}{\pi s_n} \right)^q}{n^q} \right)^{\frac{1}{q}} < +\infty \]

\[ T \in S_{p,\infty} \iff \|T\|_{S_{p,\infty}} := \sup_{n \in \mathbb{N}} n^{\frac{q}{p}} s_n < +\infty \]

\[ T \in S_{0,\infty}^0 \iff \lim_{n \to +\infty} n^{\frac{1}{p}} s_n = 0. \]
Recall that \( S_p \subset S^0_{p,\infty} \subset S_{p,\infty} \), with \( \| \cdot \|_{S_{p,\infty}} \leq \| \cdot \|_{S_p} \), \( \forall p > 0 \). Finally, Lemma 2.2 suggests that perturbations with \( S^0_{p,\infty} \) operators leave the eigenvalue asymptotics unaffected.

**Lemma 2.2 (K. Fan).** Let \( S \) and \( T \) be two compact, self-adjoint operators on a Hilbert space. If
\[
\lambda^\pm_n(S) = K^\pm n^{-\gamma} + o(n^{-\gamma}), \quad \text{and} \quad s_n(T) = o(n^{-\gamma}), \quad n \to +\infty,
\]
then
\[
\lambda^\pm_n(S + T) = K^\pm n^{-\gamma} + o(n^{-\gamma}),
\]
for some constants \( K^\pm \).

For the next lemma, define \( C^\infty_c(\mathbb{R}^2) \) to be the set of all infinitely differentiable, compactly supported functions \( \kappa : \mathbb{R}^2 \to \mathbb{R} \).

**Lemma 2.3.** Let \( \kappa \in C^\infty_c(\mathbb{R}^2) \). Then the integral operator \( \mathcal{K} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \), with
\[
(\mathcal{K}f)(x) = \int_{\mathbb{R}} f(y) \kappa(x, y) \, dy, \quad \forall x \in \mathbb{R},
\]
belongs to all Schatten classes \( S_p \), for \( p > 0 \).

**Remark.** Lemma 2.3 is a minor modification of [8, Chapter 30.5, Theorem 13]. More precisely, Theorem 13 proves the \( S_1 \) inclusion of \( \mathcal{K} \), but the proof for the rest of \( S_p \) is obtained by the same argument.

### 2.3 Asymptotic orthogonality in \( S_{p,\infty} \)

Let \( A \) and \( B \) be two operators that belong to the class \( S_{p,\infty} \), where \( p > 0 \). Notice that \( B^*A \) and \( BA^* \) belong to \( S^0_{p,\infty} \). We will call \( A \) and \( B \) orthogonal if \( B^*A = BA^* = 0 \), and asymptotically orthogonal if \( B^*A \) and \( BA^* \) belong to \( S^0_{p,\infty} \).

Asymptotic orthogonality plays an important role when we want to obtain the spectral asymptotics of the operator \( A + B \), while we know those of \( A \) and \( B \). More precisely, for compact, self-adjoint operators, there is the following Lemma, which is a special case of [12, Theorem 2.3].

**Lemma 2.4.** Let \( A \) and \( B \) be two self-adjoint operators in \( S_{p,\infty} \), for some \( p > 0 \). Assume that the asymptotics of their positive and negative eigenvalues, \( \lambda^\pm_n(A) \) and \( \lambda^\pm_n(B) \), are given by
\[
\lambda^\pm_n(A) = C^\pm_A n^{-\frac{1}{p}} + o(n^{-\frac{1}{p}}), \quad n \to +\infty;
\]
and
\[
\lambda^\pm_n(B) = C^\pm_B n^{-\frac{1}{p}} + o(n^{-\frac{1}{p}}), \quad n \to +\infty.
\]
If \( A \) and \( B \) are asymptotically orthogonal, then
\[
\lambda^\pm_n(A + B) = \left( (C^\pm_A)^p + (C^\pm_B)^p \right) n^{-\frac{1}{p}} + o(n^{-\frac{1}{p}}), \quad n \to +\infty.
\]
2.4 Weighted Hankel operators

Let \( \{w_1(j)\}_{j \in \mathbb{N}_0} \), \( \{w_2(j)\}_{j \in \mathbb{N}_0} \) and \( a = \{a(j)\}_{j \in \mathbb{N}_0} \) be three complex valued sequences and define, formally, the operator \( \Gamma : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0) \) by \( \Gamma = M_w H_a M_{w_2} \), where \( M_w \) is the multiplication operator by a sequence \( w = \{w(j)\}_{j \in \mathbb{N}_0} \). In addition, for any \( \alpha, \beta > 0 \), define the special class of weighted Hankel operators \( \Gamma_{\alpha,\beta} = M_w H_a M_{w_2} \), where \( w_1(j) = (j+1)^\alpha \) and \( w_2(j) = (j+1)^\beta \), for all \( j \in \mathbb{N}_0 \). A Schatten class criterion for this class of weighted operators is given by the Theorem 2.5 [1, Theorem B].

**Theorem 2.5.** Let \( p \in (0, +\infty) \), \( \alpha, \beta > 0 \), and \( \phi \) be an analytic function on \( \mathbb{T} \) and \( \Phi \phi \) the sequence of its Fourier coefficients. Then

\[
\|\phi\|_{B_{p,q}^{1/2 + \alpha + \beta}} \lesssim \|\Gamma_{\alpha,\beta}\|_{\mathbb{S}_p} \lesssim \|\phi\|_{B_{p,q}^{1/2 + \alpha + \beta}},
\]

where \( \Gamma_{\alpha,\beta} \) is the weighted Hankel operator described by the matrix \([ (i+1)^\alpha (i+j)(j+1)^\beta ]_{i,j \geq 0} \).

The following lemma is a combination of Theorem 2.5 and [9, Theorem 6.4.4]. The reader can find a sketch of proof in the Appendix A.

**Lemma 2.6.** Define the measure space

\[
(\mathcal{M}, \mu) := \bigoplus_{n \in \mathbb{N}_0} (\mathbb{T}, 2^n \mathbf{m}),
\]

where \( \mathbf{m} \) is the Lebesgue measure on \( \mathbb{T} \). Let \( p \in (0, +\infty) \), \( q \in (0, +\infty] \) and \( B_{p,q}^{\frac{1}{d}+d-1} \) be the space of analytic functions \( \phi \) on \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \) such that

\[
\bigoplus_{n \in \mathbb{N}_0} 2^{n(d-1)} \phi \ast V_n \in L^{p,q}(\mathcal{M}, \mu), \tag{2.5}
\]

where the polynomials \( V_n \) are defined in (2.1) and (2.2). Then

\[
\left\| \Gamma_{\alpha,\beta} \frac{1}{\Phi \phi} \right\|_{\mathbb{S}_{p,q}} \lesssim \|\phi\|_{B_{p,q}^{\frac{1}{d}+d-1}},
\]

where \( \Phi \phi \) is the sequence of the Fourier coefficients of \( \phi \).

2.5 Laplace transform estimates

Let \( \mathcal{L} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \) be the Laplace transform, given by

\[
(\mathcal{L}f)(t) = \int_0^{+\infty} f(\lambda)e^{-\lambda t} d\lambda, \quad \forall t > 0, \quad \forall f \in L^2(\mathbb{R}_+).
\]

In this paper, we are interested in the \( t \to +\infty \) asymptotic behaviour of the Laplace transform of functions with logarithmic singularities near zero of the form \( f(\lambda) = \lambda^n |\log \lambda|^{-\gamma} \), for \( \gamma > 0 \). This asymptotic behaviour is obtained by Lemma 2.7 [10, Lemma 3.3].

**Lemma 2.7.** Let

\[
I_n(t) = \int_{\lambda_0}^\lambda \lambda^n |\log \lambda|^{-\gamma} e^{-\lambda t} d\lambda,
\]

where \( \gamma > 0 \), \( n \in \mathbb{N}_0 \) and \( \lambda_0 \in (0, 1) \). Then

\[
I_n(t) = n! \ t^{-1-n} |\log t|^{-\gamma} (1 + O(|\log t|^{-1})) , \quad t \to +\infty.
\]
2.6 Weyl-type spectral asymptotics for pseudo-differential operators

Let $X$ and $D$ be, respectively, the multiplication and the differentiation operator in $L^2(\mathbb{R})$. They are self-adjoint operators, defined on appropriate domains, and given by

$$(Xf)(x) = xf(x), \quad (Df)(x) = -if'(x).$$

The following lemma ([10, Theorem 2.4]) deals with pseudo-differential operators of the form

$$\Psi = \beta(X)\alpha(D)\beta(X).$$

Notice that $\alpha(D) = F^*\alpha(2\pi X)F$, an expression which will prove to be useful in the sequel.

Lemma 2.8. Let $\alpha$ be a real valued function in $C^\infty(\mathbb{R})$, such that

$$\alpha(x) = \begin{cases} 
\alpha(+\infty)x^{-\gamma} + o(x^{-\gamma}), & x \to +\infty \\
\alpha(-\infty)|x|^{-\gamma} + o(x^{-\gamma}), & x \to -\infty,
\end{cases}$$

for some real constants $\alpha(+\infty)$, $\alpha(-\infty)$ and $\gamma > 0$. Now let $\beta$ be a real valued function on $\mathbb{R}$ such that

$$|\beta(x)| \leq M \langle x \rangle^{-s}, \quad \forall x \in \mathbb{R},$$

where $s > \frac{\gamma}{2}$ and $M$ is a non-negative constant. Define the pseudo-differential operator $\Psi = \beta(X)\alpha(D)\beta(X)$ on $L^2(\mathbb{R})$. Then $\Psi$ is compact and obeys the following eigenvalue asymptotic formula:

$$\lambda_n^\pm = C^\pm n^{-\gamma} + o(n^{-\gamma}), \quad n \to +\infty,$$

where

$$C^\pm = \left[ \frac{1}{2\pi} \left( \alpha(+\infty)\frac{\gamma}{2} + \alpha(-\infty)\frac{\gamma}{2} \right) \int_\mathbb{R} \langle x \rangle^{\frac{\gamma}{2}} \, dx \right]^{\gamma}.$$

Above $\langle x \rangle := \sqrt{1 + x^2}, \forall x \in \mathbb{R}$.

3 Construction of the model operator

Consider the cut-off function $\chi_0 \in C^\infty(\mathbb{R})$ such that

$$\chi_0(t) = \begin{cases} 
1, & 0 < t \leq \frac{1}{2} \\
0, & t \geq \frac{3}{4}
\end{cases}, \quad (3.1)$$

and $0 \leq \chi_0 \leq 1$. Let $\gamma > 0$ and define the function

$$w(t) = \frac{1}{(d-1)!} t^{d-1} |\log t|^{-\gamma} \chi_0(t), \quad \forall t > 0. \quad (3.2)$$

If

$$(\mathcal{L}w)(t) = \int_0^{+\infty} w(\lambda)e^{-\lambda t} \, d\lambda, \quad \forall t > 0,$$

let $b_1, b_{-1} \in \mathbb{R}$ and define the sequence $\tilde{a} = \{\tilde{a}(j)\}_{j \in \mathbb{N}}$ by

$$\tilde{a}(j) = b_1 (\mathcal{L}w)(j) + (-1)^j b_{-1} (\mathcal{L}w)(j), \quad \forall j \in \mathbb{N}. \quad (3.3)$$

Then we define the model operator $\tilde{H} := H_{\tilde{a}}$, with parameter sequence $\tilde{a}(j) = \tilde{a}(|j|), \forall j \in \mathbb{N}_0^d$. For the sequence $\tilde{a}$, we have the following lemma.
Lemma 3.1. Let \( w \) be the function described in (3.2) and \( a \) be the sequence defined in (3.3). Then \( a \) satisfies the following formula:

\[
\tilde{a}(j) = (b_1 + (-1)^j b_{-1}) j^{-d} (\log j)^{-\gamma} + \tilde{g}_1(j) + (-1)^j \tilde{g}_{-1}(j), \quad \forall j \geq 2,
\]

(3.4)

where the error sequences \( \tilde{g}_{\pm 1} \) present the following asymptotic behaviour:

\[
\tilde{g}^{(m)}_{\pm 1}(j) = O \left( j^{-d-m} (\log j)^{-\gamma-1} \right), \quad j \to +\infty,
\]

(3.5)

for all \( m \in \mathbb{N}_0 \).

Proof. First assume that \( b_{-1} = 0 \) and \( b_1 \neq 0 \). Then

\[
\tilde{a}(j) = b_1 (\mathcal{L} w)(j), \quad \forall j \in \mathbb{N}_0,
\]

and we aim to prove that

\[
\tilde{a}(j) = b_1 j^{-d} (\log j)^{-\gamma} + \tilde{g}_1(j), \quad \forall j \geq 2,
\]

(3.6)

where the error term \( \tilde{g}_1 \) satisfies (3.5). Moreover, without loss of generality, assume that \( b_1 = 1 \), otherwise work with \( \tilde{a} / b_1 \). Let \( \tilde{g}_1 \) be the function below

\[
\tilde{g}_1(t) = \frac{1}{(d-1)!} \int_0^{+\infty} \lambda^{d-1} |\log \lambda|^{-\gamma} \chi_0(\lambda) e^{-\lambda t} \ d\lambda - t^{-d} |\log t|^{-\gamma}, \quad \forall t > 1,
\]

(3.7)

and notice that \( \tilde{g}_1 \in C^\infty(1, +\infty) \). More precisely, for every \( m \in \mathbb{N} \) and any \( t > 1 \),

\[
\tilde{g}^{(m)}_1(t) = \frac{(-1)^m}{(d-1)!} \int_0^{+\infty} \lambda^{d+m-1} |\log \lambda|^{-\gamma} \chi_0(\lambda) e^{-\lambda t} \ d\lambda
\]

\[
- \sum_{n=0}^{m} \binom{m}{n} \left( \frac{d-n}{dt^n} \left( \frac{d^{m-n} (\log t)^{-\gamma}}{dt^{m-n}} \right) \right).
\]

(3.8)

Moreover, for every \( m \in \mathbb{N}_0 \) and any \( t > 0 \),

\[
\int_0^{+\infty} \lambda^{d+m-1} |\log \lambda|^{-\gamma} \chi_0(\lambda) e^{-\lambda t} \ d\lambda = \int_0^{1/2} \lambda^{d+m-1} |\log \lambda|^{-\gamma} e^{-\lambda t} \ d\lambda
\]

\[
+ \int_{1/2}^{+\infty} \lambda^{d+m-1} |\log \lambda|^{-\gamma} e^{-\lambda t} \chi_0(\lambda) \ d\lambda.
\]

Notice that the second integral converges to zero exponentially fast when \( t \to +\infty \). Thus Lemma 2.7 yields

\[
\int_0^{+\infty} \lambda^{d+m-1} |\log \lambda|^{-\gamma} \chi_0(\lambda) e^{-\lambda t} \ d\lambda = (d+m-1)! t^{-d-m} (\log t)^{-\gamma} \left( 1 + O \left( (\log t)^{-1} \right) \right),
\]

(3.9)

when \( t \to +\infty \). Besides, notice that, for every \( k \in \mathbb{N} \),

\[
\frac{d^k}{dt^k} (\log t)^{-\gamma} = O \left( t^{-k} (\log t)^{-\gamma-1} \right), \quad t \to +\infty.
\]
Thus, it is easily verified that,
\[
\sum_{n=0}^{m} \binom{m}{n} \left( \frac{d^n t^{-d}}{dt^n} \right) \left( \frac{d^{m-n}(\log t)^{-\gamma}}{dt^{m-n}} \right) = \frac{(-1)^m}{(d-1)!} (d+m-1)! t^{-d-m}(\log t)^{-\gamma} \\
+ O(t^{-d-m}(\log t)^{-\gamma-1}), \text{ when } t \to +\infty. (3.10)
\]
Then by putting (3.9) and (3.10) back to (3.8), we obtain that for every \( m \in \mathbb{N}_0 \),
\[
\tilde{g}_1^{(m)}(t) = \frac{(-1)^m}{(d-1)!} (d+m-1)! t^{-d-m}(\log t)^{-\gamma} (1 + O((\log t)^{-1})) \\
- \frac{(-1)^m}{(d-1)!} (d+m-1)! t^{-d-m}(\log t)^{-\gamma} + O(t^{-d-m}(\log t)^{-\gamma-1}), \text{ for } t \to +\infty.
\]
Therefore, \( \tilde{g}_1(t) \) satisfies the following smoothness property:
\[
\tilde{g}_1^{(m)}(t) = O(t^{-d-m}(\log t)^{-\gamma-1}), \text{ for } t \to +\infty, \forall m \in \mathbb{N}_0. \quad (3.11)
\]
In addition, by (3.7), the function \( \tilde{a}(t) := b_1(\mathcal{L}w)(t), \forall t > 0 \), satisfies
\[
\tilde{a}(t) = t^{-d} |\log t|^{-\gamma} + \tilde{g}_1(t), \forall t > 1.
\]
Thus, by restricting \( \tilde{a} \) on the set of integers greater than or equal to 2, we get (3.6). The relation (3.5) for \( \tilde{g}_1(j) \) is obtained by noticing that \( \{\tilde{g}_1(j)\}_{j \geq 2} \) is the restriction of the function \( \tilde{g}_1 \) on the set of integers greater than one, so
\[
\tilde{g}_1(j) = O(j^{-d}|\log j|^{-\gamma-1}), \ t \to +\infty,
\]
and also
\[
\tilde{g}_1^{(m)}(j) = \int_{0}^{1} \int_{0}^{1} \cdots \int_{0}^{1} \tilde{g}_1^{(m)}(j + t_1 + t_2 + \cdots + t_m) \, dt_1 \cdots dt_m, \forall j \geq 2, \forall m \in \mathbb{N},
\]
where \( \tilde{g}_1(t) \) satisfies (3.11). As a result,
\[
\tilde{g}_1^{(m)}(j) = O(j^{-d-m}|\log j|^{-\gamma-1}), \ t \to +\infty,
\]
for every \( m \in \mathbb{N} \).
Finally, by repeating the same arguments when \( b_1 = 0 \) and \( b_{-1} \neq 0 \), we obtain that
\[
\tilde{a}(j) = (-1)^j b_{-1} j^{-d}(\log j)^{-\gamma} + (-1)^j \tilde{g}_{-1}(j), \forall j \geq 2, \quad (3.12)
\]
where the error term \( \tilde{g}_{-1} \) satisfies (3.5). By combining (3.6) and (3.12) together we eventually obtain (3.4).

### 4 Reduction to pseudo-differential operators

Let \( \tilde{a} \) (see (3.3)) be the parameter sequence of the model operator \( \tilde{H} \). Then
\[
\tilde{a}(j) = \tilde{a}_1(j) + \tilde{a}_{-1}(j),
\]
where
\[
\tilde{a}_{\pm 1}(j) = (\pm 1)^j b_{\pm 1}(\mathcal{L}w)(j), \forall j \in \mathbb{N}_0, \quad (4.1)
\]
and \( w \) is defined in (3.2). Then \( \tilde{a}_1 \) (resp. \( \tilde{a}_{-1} \)) defines the Hankel operator \( \tilde{H}_1 \) (resp. \( \tilde{H}_{-1} \)), with parameter sequence \( \tilde{a}_1(|j|) \), for all \( j \in \mathbb{N}_0 \) (resp. \( \tilde{a}_{-1}(|j|) \)). Thus, \( \tilde{H} = \tilde{H}_1 + \tilde{H}_{-1} \). We reduce the spectral analysis of \( \tilde{H}_{\pm 1} \) to that of some pseudo-differential operators \( \Psi_{\pm 1} \).
Lemma 4.1. For \( j = 1, 2, \ldots, d - 1 \), let \( R_j : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \) be the integral operator
\[
(R_j f)(t) = \int_0^{+\infty} \sqrt{w(t)} \frac{f(s)}{(s+t)^{p}} \sqrt{w(s)} \, ds , \quad \forall t > 0, \forall f \in L^2(\mathbb{R}_+).
\]
Then \( R_j \in \bigcap_{p>0} S_p \), for all \( j = 1, 2, \ldots, d - 1 \).

Proof. Let \( U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}) \) be the unitary transformation that is given by
\[
(U f)(x) = e^{x}f(e^{x}), \quad \forall x \in \mathbb{R}, \forall f \in L^2(\mathbb{R}_+).
\]
Therefore, by applying the change of variable \( s = e^{y} \) and setting \( x = \log t \),
\[
(R_j f)(e^{x}) = \int_{\mathbb{R}} \sqrt{w(e^{x})} \frac{f(e^{y})e^{y}}{(e^{x} + e^{y})^{p}} \sqrt{w(e^{y})} \, dy , \quad \forall x \in \mathbb{R}, \forall f \in L^2(\mathbb{R}_+).
\]
Moreover, observe that \( e^{x} + e^{y} = 2e^{\frac{x+y}{2}} \cosh(\frac{x-y}{2}) \), so that, for any \( f \in L^2(\mathbb{R}_+) \),
\[
(U R_j f)(x) = \int_{\mathbb{R}} \sqrt{2^{-d}e^{-(j-1)x}w(e^{x})} \frac{(U f)(y)}{\cosh^j(\frac{x-y}{2})} \sqrt{2^{-d}e^{-(j-1)y}w(e^{y})} \, dy , \quad \forall x \in \mathbb{R},
\]
for all \( j = 1, 2, \ldots, d - 1 \). For any \( j = 1, 2, \ldots, d - 1 \), define the functions
\[
\alpha_j(x) = 2^{-d}e^{-(j-1)x}w(e^{x}), \quad \forall x \in \mathbb{R}.
\]
Then
\[
R_j = U^* \alpha_j^{1/2}(X)T_j \alpha_j^{1/2}(X)U,
\]
where the operator \( T_j : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is the convolution operator with the function \( \phi_j \); i.e. for any \( j \in \mathbb{N} \),
\[
(T_j f)(x) = (\phi_j * f)(x) , \quad \forall x \in \mathbb{R}, \forall f \in L^2(\mathbb{R}),
\]
where \( \phi_j \) is given by (1.5). In order \( R_j \) to belong to a Schatten class \( S_p \), it is enough to prove that \( \alpha_j^{1/2}(X)T_j \alpha_j^{1/2}(X) \in S_p \), for \( j = 1, 2, \ldots, d - 1 \). To see this, observe that \( T_j = \mathcal{F} \beta_j(X) \mathcal{F}^* \), where
\[
\beta_j(x) = \sqrt{\hat{\phi}_j(x)}, \quad \forall x \in \mathbb{R}, \forall j \in \mathbb{N}.
\]
Note that \( \hat{\phi}_1(x) = 2\pi(\cosh(2\pi^2 x))^{-1} \), and the latter is positive for any \( x \in \mathbb{R} \). Since the convolution of positive functions is positive, \( \hat{\phi}_j > 0 \), and thus, \( \beta_j \) is well-defined. Then
\[
\alpha_j^{1/2}(X)T_j \alpha_j^{1/2}(X) = \alpha_j^{1/2}(X)\mathcal{F} \beta_j^{2}(X) \mathcal{F}^* \alpha_j^{1/2}(X),
\]
and Lemma 1.3 implies that the latter is unitarily equivalent (modulo kernels) to the pseudodifferential operator \( \beta_j(X)\alpha_j(\frac{1}{2\pi D})\beta_j(X) \). Moreover, (4.3) implies that
\[
\alpha_j(x) = \begin{cases} 0, & \text{when } x \to +\infty, \\ \frac{1}{2^{(d-1)}!}e^{-|x|} |x|^{-\gamma}, & \text{when } x \to -\infty, \end{cases} \quad \forall j = 1, 2, \ldots, d - 1.
\]
Since \( \alpha_j(x) \) decays exponentially fast, when \( x \to -\infty \), Lemma 2.8 indicates that the pseudodifferential operator \( \beta_j(X)\alpha(\frac{1}{2\pi D})\beta_j(X) \) and thus, \( \alpha_j^{1/2}(X)T_j \alpha_j^{1/2}(X) \), belong to \( \bigcap_{p>0} S_p \), for all \( j = 1, 2, \ldots, d - 1 \). \( \square \)

Lemma 4.2. Let \( \tilde{H}_1 \) and \( \tilde{H}_{-1} \) be the Hankel operators that were defined at the start of Section 4, with parameter sequences \( \tilde{a}_1(|j|) \) and \( \tilde{a}_{-1}(|j|) \), for all \( j \in \mathbb{N}_0^d \), respectively; where \( \tilde{a}_{\pm 1} \) have been defined in (4.1). Then there exist two couples of operators \( S_1, S_{-1} \) and \( E_1, E_{-1} \), defined on \( L^2(\mathbb{R}_+) \), such that
(i) $\tilde{H}_{1}$ is unitarily equivalent (modulo kernels) to $S_{1}$, 
(ii) $E_{1} \in \bigcap_{p>0} S_{p}$, and 
(iii) $S_{1} - E_{1}$ is unitarily equivalent (modulo kernels) to a pseudo-differential operator $\Psi_{1} : L^{2}(\mathbb{R}) \rightarrow L^{2}(\mathbb{R})$. More precisely, $\Psi_{1} = \beta(X)\alpha_{1}(\frac{1}{\sqrt{2\pi}}D)\beta(X)$, where

$$\alpha_{1}(x) = 2^{-d}e^{-(d-1)x}b_{1}w(e^{x}), \quad \beta(x) = \sqrt{\phi_{d}(x)}, \quad \forall x \in \mathbb{R}, \quad (4.6)$$

and $\phi_{d}$ is defined in (1.3).

Proof. First of all, notice that in Lemma 4.1, we proved that $\beta$ is well defined, since $\beta = \beta_{d}$, where the latter is given by (1.5). We prove the assertion for $\tilde{H}_{1}$ and the proof for $\tilde{H}_{-1}$ is completely analogous. Moreover, we can assume that $b_{1} = 1$, otherwise work with $\frac{1}{b_{1}}H_{1}$.

(i) Let $x, y \in \ell^{2}(\mathbb{N}_{0}^{d})$. Then

$$(\tilde{H}_{1}x, y) = \sum_{i,j \in \mathbb{N}_{0}^{d}} \hat{a}_{i}(|i| + |j|)x(j)y(i)$$

$$= \sum_{i,j \in \mathbb{N}_{0}^{d}} \int_{0}^{+\infty} w(t) e^{-(|i|+|j|)t} dt x(j)y(i)$$

$$= (L_{1}x, L_{1}y),$$

where $L_{1} : \ell^{2}(\mathbb{N}_{0}^{d}) \rightarrow L^{2}(\mathbb{R}_{+})$ is defined by

$$L_{1}(x)(t) = \sqrt{w(t)} \sum_{j \in \mathbb{N}_{0}^{d}} e^{-|j|t}x(j), \quad \forall t \in \mathbb{R}_{+}, \forall x \in \ell^{2}(\mathbb{N}_{0}^{d}). \quad (4.7)$$

Notice that the interchange of summation and integration is justified by the uniform convergence of $\sum_{j \in \mathbb{N}_{0}^{d}} e^{-|j|t}$, in $\mathbb{R}_{+}$. Therefore, $\tilde{H}_{1} = L_{1}^{*}L_{1}$. Moreover, it is not difficult to verify that the formula for the adjoint operator $L_{1}^{*} : L^{2}(\mathbb{R}_{+}) \rightarrow \ell^{2}(\mathbb{N}_{0}^{d})$ is the following:

$$(L_{1}^{*}f)(j) = \int_{0}^{+\infty} w(t)f(t)e^{-|j|t} dt, \quad \forall j \in \mathbb{N}_{0}^{d}, \forall f \in L^{2}(\mathbb{R}_{+}). \quad (4.8)$$

In addition, Lemma 1.3 implies that the non-zero parts of $\tilde{H}_{1}$ and $S_{1} := L_{1}L_{1}^{*}$ are unitarily equivalent. Now observe that $S_{1} : L^{2}(\mathbb{R}_{+}) \rightarrow L^{2}(\mathbb{R}_{+})$ and

$$(S_{1}f)(t) = \sqrt{w(t)} \sum_{j \in \mathbb{N}_{0}^{d}} \int_{0}^{+\infty} f(s) \sqrt{w(s)} e^{-(t+|j|)s} ds$$

$$= \int_{0}^{+\infty} \sqrt{w(t)} \frac{f(s)}{(1 - e^{-(s+|j|)})^{d}} \sqrt{w(s)} ds, \quad \forall t \in \mathbb{R}_{+}, \forall f \in L^{2}(\mathbb{R}_{+}). \quad (4.9)$$

Remark. Observe that the respective formulae for $L_{-1}$ and $L_{-1}^{*}$, assuming that $b_{-1} = 1$, will be

$$(L_{-1}x)(t) = \sqrt{w(t)} \sum_{j \in \mathbb{N}_{0}^{d}} (-1)^{|j|} e^{-|j|t}x(j), \quad \forall t \in \mathbb{R}_{+}, \forall x \in \ell^{2}(\mathbb{N}_{0}^{d}), \quad (4.10)$$

and

$$(L_{-1}^{*}f)(j) = (-1)^{|j|} \int_{0}^{+\infty} \sqrt{w(t)}f(t)e^{-|j|t} dt, \quad \forall j \in \mathbb{N}_{0}^{d}, \forall f \in L^{2}(\mathbb{R}_{+}), \quad (4.11)$$

so that $S_{-1} = S_{1}$.
(ii) Observe the formula (4.9) for $S_1$ and that
\[
\frac{1}{(1 - e^{-(s+t)})^d} = \frac{1}{(s+t)^d} + \rho(s + t),
\]
where $\rho$ is real analytic with a pole of order $d - 1$ at 0. Now define the operator $E_1 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, with
\[
(E_1f)(t) = \int_0^{+\infty} \sqrt{w(t)} f(s) \rho(s + t) \sqrt{w(s)} \, ds, \quad \forall t \in \mathbb{R}_+, \forall f \in L^2(\mathbb{R}_+).
\]
(4.12)
The function $\rho$ can be written as
\[
\rho(t) = \sum_{j=1}^{d-1} \frac{c_{-j}}{t^j} + \rho_{\text{an}}(t), \quad \forall t \neq 0,
\]
where $\rho_{\text{an}}$ is real analytic and $c_{-j}$ are real constants. Now notice that the function $\sqrt{\chi_0(t)} \rho_{\text{an}}(s + t) \sqrt{\chi_0(s)}$, where $\chi_0$ is defined in (3.1), belongs to $C^\infty(\mathbb{R}^2)$. Then, according to Lemma 2.3, the integral operator with kernel $\sqrt{\chi_0(t)} \rho_{\text{an}}(s + t) \sqrt{\chi_0(s)}$, belongs to any Schatten class $S_p$. Moreover, the function $t^{d-1}|\log t|^{-\gamma}$ is bounded near 0, so that the integral operator with kernel $\sqrt{w(t)} \rho_{\text{an}}(s + t) \sqrt{w(s)}$ belongs to any Schatten class $S_p$. It remains to prove the same for the integral operators $R_j$ with kernel $\sqrt{w(t)}(s + t)^{-j} \sqrt{w(s)}$, where $j = 1, 2, \ldots, d - 1$, which holds true due to Lemma 4.1.

(iii) By recalling the definitions of $S_1$ in (4.9) and $E_1$ in (4.12), $S_1 - E_1$ is also an operator on $L^2(\mathbb{R}_+)$, described by
\[
(S_1 - E_1)f(t) = \int_0^{+\infty} \sqrt{w(t)} \frac{f(s)}{s + t} \sqrt{w(s)} \, ds, \quad \forall t \in \mathbb{R}_+, \forall f \in L^2(\mathbb{R}_+).
\]
Let $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ be the unitary transformation that was defined in (4.2). Then, by applying the change of variable $s = e^y$ and setting $x = \log t$,
\[
(S_1 - E_1)f(e^y) = \int_{\mathbb{R}} \sqrt{w(e^y)} \frac{f(e^y)e^y}{e^x + e^y} e^d \sqrt{w(e^y)} \, dy, \quad \forall x \in \mathbb{R}.
\]
As a result, for any $f \in L^2(\mathbb{R}_+)$,
\[
U(S_1 - E_1)f(x) = \int_{\mathbb{R}} \sqrt{2^{-d}e^{-(d-1)x}w(e^x)} \frac{(Uf)(y)}{\cosh^d\left(\frac{x-y}{2}\right)} \sqrt{2^{-d}e^{-(d-1)y}w(e^y)} \, dy, \quad \forall x \in \mathbb{R}.
\]
Then, $S_1 - E_1 = U^* \alpha_{1/2}^1(X) T_d \alpha_{1/2}^1(X) U$, where
\[
\alpha_1(x) = 2^{-d} e^{-(d-1)x} w(e^x), \quad \forall x \in \mathbb{R},
\]
and $T_d$ is defined in (4.4). Notice that $T_d = F \beta^2(X) F^*$, where $\beta$ is defined in (4.6). Therefore,
\[
S_1 - E_1 = U^* \alpha_{1/2}^1(X) T_d \alpha_{1/2}^1(X) U
= U^* \alpha_{1/2}^1(X) F \beta^2(X) F^* \alpha_{1/2}^1(X) U
\simeq \beta(X) F^* \alpha_1(X) F \beta(X),
\]
where the last equivalence is obtained by Lemma 1.3 and the fact that $U$ is unitary. Therefore, if $\Psi_1 := \beta(X) \alpha_1(\frac{1}{2}D) \beta(X)$, where $\alpha_1$ and $\beta$ are given by (4.6), then $S_1 - E_1$ is unitarily equivalent (modulo kernels) to $\Psi_1$. □
5 Weyl-type spectral asymptotics

In this section we derive Weyl-type spectral asymptotics for the operators $\tilde{H}_\pm$, that were defined in §4 and for the model operator $H$. The latter is obtained by using the asymptotic orthogonality of $H_1$ and $H_{-1}$; see Lemma 5.2.

**Lemma 5.1.** The eigenvalue asymptotics for the operator $\tilde{H}_1$, that was obtained in §4 are given by

$$\lambda_n^\pm(\tilde{H}_1) = C_1^\pm n^{-\gamma} + o(n^{-\gamma}), \ n \to +\infty,$$

(5.1)

where the constants $C_1^\pm$ are given by a formula similar to (1.10):

$$C_1^\pm = \frac{1}{2^d(d-1)!} (b_1)_\pm \left[ \int_{\mathbb{R}} \phi_1^2(x) \, dx \right]^{\gamma},$$

(5.2)

where the function $\phi_1$ is defined in (1.3) and $(b_1)_\pm = \max\{\pm b_1, 0\}$. Similar asymptotics are obtained for $H_{-1}$ by substituting $b_1$ with $b_{-1}$ and thus, obtaining the constants $C_{-1}^\pm$.

**Proof.** In Lemma 4.2 we proved that $\tilde{H}_1$ is unitarily equivalent (modulo kernels) to an operator $S_1$, so that its spectral asymptotics can be retrieved from those of $S_1$. Moreover, $S_1 = (S_1 - E_1) + E_1$, where $E_1$ is also described in Lemma 4.2. In order to obtain the spectral asymptotics of $S_1$, we aim to use Lemma 2.2. In Lemma 4.2, it is proved that $S_1 - E_1$ is unitarily equivalent (modulo kernels) to the pseudo-differential operator $\Psi_1 = \beta(X) \alpha_1(\frac{1}{2\pi}) \beta(X)$, where $\alpha$ and $\beta$ are given by (4.3). Then

$$\alpha_1(\frac{\pi^d}{2\gamma}) = \begin{cases} \frac{b_1(2x)^\gamma}{2^d(d-1)!} |x|^{-\gamma}(1 + o(1)), & \text{when } x \to -\infty \\ 0, & \text{when } x \to +\infty \end{cases}.$$

Moreover, $\beta^2$ belongs to the Schwartz class $S(\mathbb{R})$. Indeed, by differentiating, we can see that $\frac{1}{\cosh^d(\pi)} \in S(\mathbb{R})$ and consequently, $\beta^2 \in S(\mathbb{R})$, too. Therefore,

$$|\beta(x)| = O \left( |x|^{-s} \right), \ x \to +\infty,$$

for every $s > 0$. Thus, all the conditions of Lemma 2.8 are satisfied and therefore, the eigenvalues of $\Psi_1$, $\lambda_n^\pm(\Psi_1)$, follow the asymptotics below:

$$\lambda_n^\pm(\Psi_1) = C_1^\pm n^{-\gamma} + o(n^{-\gamma}), \ n \to +\infty,$$

where the constants $C_1^\pm$ are described by (5.2). Finally, in order to apply Lemma 2.2, it remains to prove that $s_n(E_1) = o(n^{-\gamma})$, for $n \to +\infty$. For notice that, according to Lemma 4.2, $E_1 \in \cap_{p > 0} S_p$. Thus, the singular values of $E_1$ decay faster than any polynomial. As a result, Lemma 2.2 yields that the eigenvalue asymptotics of $H_1$ are given by (5.1). \hfill $\square$

**Lemma 5.2.** Let $\tilde{H}_1$ and $\tilde{H}_{-1}$ be the operators that were defined in §4. Then $\tilde{H}_{-1} \tilde{H}_1$ and $\tilde{H}_1 \tilde{H}_{-1}$ belong to $S_p$, for any $p > 0$. Therefore, $\tilde{H}_1$ and $\tilde{H}_{-1}$ are asymptotically orthogonal.

**Proof.** First assume that both $b_{-1}$ and $b_1$ are equal to 1, otherwise work with $\frac{1}{b_{-1}} \tilde{H}_{-1}$ or $\frac{1}{b_1} \tilde{H}_1$. In the proof of Lemma 4.2 we saw that $\tilde{H}_\pm = L_\pm^* L_\pm$. Recall that $L_1$ and $L_1^*$ are given by (4.7) and (4.8), respectively, while $L_{-1}$ and $L_{-1}^*$ are defined in (4.10) and (4.11). Then

$$\tilde{H}_{-1} \tilde{H}_1 = L_{-1}^* L_{-1} L_1^* L_1.$$
Because $L^*_1$ and $L_1$ are bounded, it is enough to prove that $L_{-1}L_1^* \in S_p$, for all $p > 0$. To this end, we follow the steps that yielded formula (4.9) (for $S_1$). Then, for every $f \in L^2(\mathbb{R}_+)$,

$$(L_{-1}L_1^*)f(t) = \int_0^{+\infty} \sqrt{w(t)} \left( \sum_{j \in \mathbb{N}_0} (-1)^j e^{(t+s)j} \right) f(s) \sqrt{w(s)} \, ds$$

$$= \int_0^{+\infty} \sqrt{w(t)} \frac{f(s)}{(1 + e^{-(t+s)})^d} \sqrt{w(s)} \, ds, \ \forall t \in \mathbb{R}_+.$$ Observe that $(1 + e^{-(t+s)})^{-d} \in C^\infty(\mathbb{R})$. Moreover, by the way that the function $\chi_0$ has been defined (see (5.1)), $\sqrt{\chi_0(t)(1 + e^{-(t+s)})^{-d}} \chi_0(s) \in C^\infty(\mathbb{R})$. Thus, Lemma 2.3 implies that the integral operator with kernel $\sqrt{\chi_0(t)}(1 + e^{-(t+s)})^{-d} \sqrt{\chi_0(s)}$ belongs to any Schatten class $S_p$. Finally, the same holds true for the operator $L_{-1}L_1^*$, since the function $t^{d-1} |\log t|^{-\gamma}$, is bounded near 0. In order to prove that $\hat{H}_1 \hat{H}_{-1} \in \bigcap_{p > 0} S_p$, it is enough to notice that $\hat{H}_1 \hat{H}_{-1} = (\hat{H}_{-1} \hat{H}_1)^*$. Regarding the asymptotic orthogonality, it is enough to notice that, due to Lemma 5.1, both $\hat{H}_{-1}$ and $\hat{H}_1$ belong to $S_{\frac{1}{2}, \infty}$, and that $\hat{H}_{-1} \hat{H}_1$ and $\hat{H}_1 \hat{H}_{-1}$ belong to $\bigcap_{p > 0} S_p \subset S_{\frac{1}{2}, \infty}.$

**Lemma 5.3.** The eigenvalues of the model operator $\hat{H}$ obey the asymptotic formula below:

$$\lambda_n^\pm(\hat{H}) = C^\pm n^{-\gamma} + o(n^{-\gamma}),$$

where the constants $C^\pm$ are defined in (1.11).

**Proof.** According to Lemma 5.1, the eigenvalue asymptotics of $\hat{H}_1$ are described by (5.1) and those of $\hat{H}_{-1}$ by a similar formula (with constants $C^\pm_{\pm}$). But, according to Lemma 5.2, $\hat{H}_{-1}$ and $\hat{H}_1$ are asymptotically orthogonal. Then, since $\hat{H} = \hat{H}_1 + \hat{H}_{-1}$, Lemma 2.4 yields that

$$\lambda_n^\pm(\hat{H}) = ((C_1)^\frac{1}{4} + (C_\gamma)^\frac{1}{4})^\gamma n^{-\gamma} + o(n^{-\gamma}), \ n \to +\infty,$$

which gives (5.3).

## 6 Reduction to one-variable weighted Hankel operators

In this section we demonstrate the reduction of multi-variable Hankel matrices to one-variable weighted Hankel operators. This will prove to be a useful tool for the derivation of spectral estimates for the error terms. Define

$$W_d(j) := \{|k \in \mathbb{N}_0^d : |k| = j\} = \binom{j + d - 1}{d - 1}, \ \forall j \in \mathbb{N}_0;$$

where the last equality can be checked by induction in $d$. In the sequel, consider the linear bounded operator $J : \ell^2(\mathbb{N}_0^d) \to \ell^2(\mathbb{N}_0)$, given by

$$(Jx)(i) = (W_d(i))^{-\frac{1}{2}} \sum_{k \in \mathbb{N}_0^d : |k| = i} \chi(k), \ \forall i \in \mathbb{N}_0, \ \forall x \in \ell^2(\mathbb{N}_0^d).$$

Besides, it is not difficult to check that the adjoint of $J$ is given by

$$(J^*x)(i) = (W_d(|i|))^{-\frac{1}{2}} x(|i|), \ \forall i \in \mathbb{N}_0^d, \ \forall x \in \ell^2(\mathbb{N}_0).$$

In addition, $J^*$ is an isometry. Indeed, for any $x \in \ell^2(\mathbb{N}_0)$,

$$\|J^*x\|^2 = \sum_{i \in \mathbb{N}_0^d} (W_d(|i|))^{-1} |x(|i|)|^2 = \sum_{i \in \mathbb{N}_0} |x(i)|^2 = \|x\|^2.$$
This indicates that \( J \) is a partial isometry. Furthermore, for an arbitrary Hankel operator \( H_a : \ell^2(N_0^d) \to \ell^2(N_0^d) \), with parameter sequence \( a(j) = a(|j|) \), \( \forall j \in N_0^d \) the following relation holds true:

\[
(H_a x, y) = (J^* \Gamma J x, y), \forall x, y \in \ell^2(N_0^d),
\]

(6.1)

where \( \Gamma : \ell^2(N_0) \to \ell^2(N_0) \) is the weighted Hankel operator defined by

\[
(\Gamma x)(i) = \sum_{j \in N_0} \sqrt{W_d(i)a(i+j)} \sqrt{W_d(j)} x(j), \forall i \in N_0, \forall x \in \ell^2(N_0).
\]

(6.2)

Indeed, it is enough to observe that, for any \( x \) and \( y \) in \( \ell^2(N_0^d) \),

\[
(H_a x, y) = \sum_{i,j \in N_0^d} a(i+j)x(j)y(i) \\
= \sum_{i,j \in N_0^d} a(|i+j|)x(j)y(i) \\
= \sum_{i,j \in N_0} a(i+j) \sum_{\{k \in N_0^d : |k| = j\}} x(k) \sum_{\{k \in N_0^d : |k| = i\}} y(k) \\
= (\Gamma J x, J y).
\]

This discussion leads to the following lemma.

**Lemma 6.1.** Let \( a = \{a(n)\}_{n \in N_0} \) and \( H_a : \ell^2(N_0^d) \to \ell^2(N_0^d) \) be a Hankel operator with parameter sequence \( a(j) = a(|j|) \), for all \( j \in N_0^d \). Then \( H_a \) belongs to any of the ideals \( S_p, S_{p, q}, S_{p, \infty} \), where \( p > 0 \) and \( q \in (0, +\infty) \), if and only if \( \Phi(\phi)(n) = a(n) \), for all \( n \in N_0 \), and \( \Gamma_a = \Gamma^{d-1} \Phi(\phi)^{d-1} \) does; where \( \Phi(\phi)(n) = a(n) \), for any \( n \in N_0 \), and \( \Gamma_{a, b} \) the weighted Hankel operator that is described by the matrix \([ (i+1)\frac{d-1}{2} \Phi(\phi)(i+j)(j+1)\frac{d-1}{2} ]_{i,j \geq 0} \).

**Proof.** In (6.1), we showed that \( H_a \) and \( \Gamma \) have unitarily equivalent non-zero parts. Thus, the operators \( H_a \) and \( \Gamma \) have identical non-zero spectra. Besides, it is easily verified that \( \Gamma = \mathcal{D} \Gamma^{d-1} \Phi(\phi)^{d-1} \mathcal{D} \), where \( \mathcal{D} = [\mathcal{D}(j)]_{j \in N_0} \) is a diagonal matrix defined by \( \mathcal{D}(j) = \left( \frac{W_d(j)}{\gamma(j+1)^{d-1}} \right)^{\frac{1}{2}} \), and \( \Phi(\phi)(j) = a(j) \), \( \forall j \in N_0 \). Notice that \( \mathcal{D} \) is an invertible bounded operator on \( \ell^2(N_0^d) \). The boundedness of \( \mathcal{D} \) can be checked by noticing that \( W_d(j) \sim j^{d-1}/(d-1)! \), when \( j \to +\infty \). Therefore, since the classes \( S_p \), \( S_{p, q} \) and \( S_{p, \infty} \) are ideals of compact operators, and \( \mathcal{D} \) is an invertible bounded operator, \( \Gamma \) belongs to any of these ideals if and only if \( \Gamma^{d-1} \Phi(\phi)^{d-1} \) does. Thus, the observation that the non-zero spectra of \( H_a \) and \( \Gamma \) are identical gives the result. \( \square \)

### 7 Schatten class inclusions of the error terms

In this section we present the spectral estimates of Hankel matrices \( H_a \) with parameter sequence \( a(j) = a(|j|) \), where \( a(j) \) decays faster than \( j^{-d}|\log j|^{-\gamma} \) at infinity, for some positive \( \gamma \). These estimates will eventually yield the spectral estimates of the error terms \( g_1(j) \) and \((-1)^j g_{-1}(j) \), that are defined in Theorem 1.2.

Let \( v = \{v(j)\}_{j \in N_0^d} \) be a sequence that attains positive values. For any \( p \in (0, +\infty) \), we
define the spaces $\ell^p_\gamma(N_0^d)$ and $\ell^{p,\infty}_\gamma(N_0^d)$ as follows:

$$x \in \ell^p_\gamma(N_0^d) \iff \|x\|_{\ell^p_\gamma} := \left(\sum_{j \in N_0^d} |x(j)|^p \nu(j)\right)^{\frac{1}{p}} < +\infty, \ p \in (0, +\infty),$$

$$x \in \ell^{p,\infty}_\gamma(N_0^d) \iff \|x\|_{\ell^{p,\infty}_\gamma} := \sup_{\lambda > 0} \left(\sum_{\{j \in N_0^d : |x(j)| > \lambda\}} \nu(j)\right)^{\frac{1}{p}}.$$

For $p = +\infty$, the space $\ell^{\infty}_\gamma(N_0^d)$ is identified with the usual $\ell^\infty(N_0^d)$. The case of $\gamma \in (0, \frac{1}{2})$ will be addressed by using the following interpolation lemma.

**Lemma 7.1.** Let $H_\alpha$ be a Hankel matrix with parameter sequence $\alpha$ and, $\nu = \{\nu(j)\}_{j \in N_0^d}$, with $\nu(j) = (|j| + 1)^{-d}$, $\forall j \in N_0^d$. Then, for any $p \in [2, +\infty)$, there exists a positive constant $M_p$ such that

$$\|H_\alpha\|_{s_p,\infty} \leq M_p \|\frac{\alpha}{\nu}\|_{\ell^p,\infty}. \quad (7.1)$$

**Proof.** The proof is based on the real interpolation method (cf. [2] Chapter 3). For observe that

$$\|H_\alpha\|_{s_2}^2 = \sum_{i,j \in N_0^d} |\alpha(i + j)|^2$$

$$= \sum_{i_1,i_2,j \geq 0} \cdots \sum_{i_d,j_d \geq 0} |\alpha(i_1 + j_1,i_2 + j_2, \ldots, i_d + j_d)|^2$$

$$= \sum_{j_1,j_2,\ldots,j_d \geq 0} (j_1 + 1)(j_2 + 1)\cdots(j_d + 1)|\alpha(j_1,j_2,\ldots,j_d)|^2$$

$$\leq \sum_{j \in N_0^d} (|j| + 1)^d|\alpha(j)|^2$$

$$= \sum_{j \in N_0^d} (|j| + 1)^{2d}|\alpha(j)|^2(|j| + 1)^{-d},$$

so that $\|H_\alpha\|_{s_2} \leq \frac{\|\alpha\|_{\ell^\infty}}{(|j| + 1)^d}$. In addition, if $\frac{\alpha}{\nu} \in \ell^\infty$, then

$$|\alpha(j)| \leq \frac{\|\frac{\alpha}{\nu}\|_{\ell^\infty}}{(j_1 + 1)(j_2 + 1)\cdots(j_d + 1)}, \ \forall j \in N_0^d.$$ 

Thus,

$$|(H_\alpha x, y)| \leq \sum_{i,j \in N_0^d} |\alpha(i + j)||x(j)||y(i)|$$

$$\leq \left\|\frac{\alpha}{\nu}\right\|_{\ell^\infty} \sum_{i_1,\ldots,i_d,j_1,\ldots,j_d \geq 0} \frac{|x(j)| |y(i)|}{(i_1 + j_1 + 1)\cdots(i_d + j_d + 1)}$$

$$\leq \pi^d \left\|\frac{\alpha}{\nu}\right\|_{\ell^\infty} \|x\| \|y\|, \ \forall x, y \in \ell^2(N_0^d),$$

where the last line is derived from the boundedness of the tensor product of $d$ Hilbert matrices. Therefore, we have shown that there are constants $M_2 = 1$ and $M_\infty = \pi^d$ such that

$$\|H_\alpha\|_{s_2} \leq M_2 \left\|\frac{\alpha}{\nu}\right\|_{\ell^2} \quad \text{and} \quad \|H_\alpha\| \leq M_\infty \left\|\frac{\alpha}{\nu}\right\|_{\ell^\infty},$$

and the real interpolation implies that, for any $p \in (2, +\infty)$, there exists a positive constant $M_p$ such that (7.1) holds true. \qed
Lemma 7.2. Let $\gamma > 0$, $M(\gamma)$ be defined in (7.1), and $\{a(j)\}_{j \in \mathbb{N}_0}$ be a real valued sequence that satisfies
\[
a^{(m)}(j) = O\left(j^{-d-m}(\log j)^{-\gamma}\right), \quad j \to +\infty, \tag{7.2}
\]
for every $m = 0, 1, \ldots, M(\gamma)$. Then the Hankel operator $H_a$, with parameter sequence $a(j) = a(|j|)$, for all $j \in \mathbb{N}_0$, is compact and its singular values satisfy the following estimate
\[
s_n(H_a) = O(n^{-\gamma}), \quad n \to +\infty. \tag{7.3}
\]
In addition, there exists a positive constant $C_\gamma = C(\gamma)$ such that
\[
\|H_a\|_{s_p, \infty} \leq C_\gamma \sum_{m=0}^{M(\gamma)} \frac{1}{(j+1)^d} \sup_{j \geq 0} \left(\log(j+2)\right)^\gamma |a^{(m)}(j)|, \tag{7.4}
\]
where $p = \frac{1}{\gamma}$.

Proof. We split the proof in steps. In the first step, we prove the result when $\gamma \in (0, \frac{1}{2})$. In the second step, we treat the case of $\gamma \geq \frac{1}{2}$.

Step 1: Let $\gamma \in (0, \frac{1}{2})$. Then $p = \frac{1}{\gamma} \in (2, +\infty)$ and thus, in order to prove that $H_a \in S_{p, \infty}$, it is enough to apply Lemma 7.1. To this end, it only needs to show that if $a$ satisfies (7.2), then $\frac{a}{\nu} \in L_p^{\infty}$, for every $p \in (2, +\infty)$, where $\nu$ is defined in Lemma 7.1. For $\lambda > 0$,
\[
\left\{ j \in \mathbb{N}_0^d : \frac{|a(j)|}{\nu(v(j))} > \lambda \right\} = \left\{ j \in \mathbb{N}_0^d : (|j|+1)^d |a(|j|)| > \lambda \right\} \subseteq \left\{ j \in \mathbb{N}_0^d : \log(|j|+2) < \left(\frac{A_0}{\lambda}\right)^p \right\},
\]
where $A_0 := \sup_{j \geq 0} (j+1)^d (\log(j+2))^{\gamma} |a(j)|$. Therefore,
\[
\sum_{\{j \in \mathbb{N}_0^d : \frac{|a(j)|}{\nu(v(j))} > \lambda \}} \frac{1}{(|j|+1)^d} \lesssim \sum_{\{j \in \mathbb{N}_0^d : \log(|j|+2) < \left(\frac{A_0}{\lambda}\right)^p \}} \frac{1}{(|j|+2)^d} = \sum_{\{j \in \mathbb{N}_0 : \log(|j|+2) < \left(\frac{A_0}{\lambda}\right)^p \}} \frac{W_d(j)}{(j+2)^d} \lesssim \sum_{\{j \in \mathbb{N}_0 : \log(|j|+2) < \left(\frac{A_0}{\lambda}\right)^p \}} \frac{(j+2)^{d-1}}{(j+2)^d} \leq \int_{\left\{ \log(x+2) < \left(\frac{A_0}{\lambda}\right)^p \right\}} \frac{1}{x+2} \, dx \lesssim \left(\frac{A_0}{\lambda}\right)^p.
\]
Thus, there exists a positive constant $C$ such that
\[
\lambda^p \sum_{\{j \in \mathbb{N}_0^d : \frac{|a(j)|}{\nu(v(j))} > \lambda \}} \frac{1}{(|j|+1)^d} \leq (CA_0)^p,
\]
which implies, by taking supremum over positive $\lambda$’s, that $\frac{a}{\nu} \in L_p^{\infty} \leq CA_0$. From the last relation and Lemma 7.1 we conclude that
\[
\|H_a\|_{s_p, \infty} \leq M_p \left\| \frac{a}{\nu} \right\|_{L_p^{\infty}} \leq M_p CA_0,
\]
so that relation (7.4) comes true, by setting $C_\gamma = M_p C$.

**Step 2:** Assume that $\gamma \geq \frac{1}{\beta}$ and let $\phi$ be given by

$$
\phi(z) = \sum_{j \in \mathbb{N}_0} a(j) z^j, \ \forall z \in \mathbb{D}.
$$

According to Lemma 6.1 $H_\phi$ and $F_{\phi \phi}^{d+1, d+1}$ satisfy the same Schatten class inclusions, where $(\Phi \phi)(j) = a(j)$. Therefore, in order to derive (7.3), Lemma 2.6 suggests that it is enough to show that $\bigoplus_{n \geq 0} 2^{n(d-1)} \phi * V_n \in L^{p, \infty}(\mathcal{M}, \mu)$, or, in other words, that

$$
\sup_{s > 0} s^p \sum_{n \in \mathbb{N}_0} 2^n \{t \in [-\pi, \pi) : |2^{n(d-1)} (\phi * V_n)(e^{it})| > s\} < +\infty. \quad (7.5)
$$

For every non-negative integer $n$ and any positive number $s$, set

$$
E_n(s) := \{t \in [-\pi, \pi) : |2^{n(d-1)} (\phi * V_n)(e^{it})| > s\}.
$$

The goal is to find an estimate for $|E_n(s)|$ which proves the finiteness of (7.3). First of all, notice that $E_n(s) = \emptyset$, for every $s \geq \|2^{n(d-1)} \phi * V_n\|_{\infty}$. An application of (2.3) gives that $E_n(s) = \emptyset$, for every $s \geq 2^{n(d-1)} \sum_{j=2^n-1}^{2^{n+1}-1} |a(j)|$. Let

$$
A_n := \sup_{j \geq 0} |a^{(m)}(j + 1)| (j + 1)^{d+m} (\log(j + 2))^\gamma, \ \forall m = 0, 1, \ldots, M(\gamma).
$$

Therefore, condition (7.2) implies that $E_n(s) = \emptyset$ when

$$
s \geq 2^{n(d-1)} A_0 \sum_{j=2^n-1}^{2^{n+1}-1} (j + 1)^{-d} (\log(j + 2))^{-\gamma}.
$$

Besides, for every $n \geq 3$,

$$
\sum_{j=2^n-1}^{2^{n+1}-1} (j + 1)^{-d} (\log(j + 2))^{-\gamma} \leq \int_{2^n-1}^{2^{n+1}-1} (t + 1)^{-d} (\log(t + 2))^{-\gamma} dt
\lesssim \int_{n-1}^{n+1} 2^{-s(d-1)} s^{-\gamma} ds \quad \text{(change of variable } s = \log_2 t) 
\lesssim 2^{-n(d-1)} n^{-\gamma},
$$

so that in general,

$$
\sum_{j=2^n-1}^{2^{n+1}-1} (j + 1)^{-d} (\log(j + 2))^{-\gamma} \leq C 2^{-n(d-1)} \langle n \rangle^{-\gamma}, \ \forall n \geq 0,
$$

for some positive constant $C$; without loss of generality, we may assume that $C = C_q$, where $C_q$ appears in (2.4). Therefore, $E_n(s) = \emptyset$, for every $n \geq 0$ such that $\langle n \rangle \geq N(s)$, where $N(s) := (C_q A_0)^p$, $\forall s > 0$. Besides, by following exactly the same steps, it can be shown that

$$
\sum_{j=2^n-1-M(\gamma)}^{2^{n+1}-1} (j + 1)^m |a^{(m)}(j)| \lesssim A_m 2^{-n(d-1)} \langle n \rangle^{-\gamma}, \ \forall m = 1, 2, \ldots, M(\gamma).
$$

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Thus, Lemma 2.1 gives that, for every \( q > \frac{1}{M(\gamma)} \) and \( n \in \mathbb{N}_0 \) such that \( M(\gamma) \leq 2^{n-1} \),
\[
2^n \| \phi * V_n \|_q^q \lesssim C_q A^q 2^{-n(d-1)q} \langle n \rangle^{-\gamma q},
\]
where \( A := \sum_{m=0}^{M(\gamma)} A_m \). Now notice that, for any positive \( q \),
\[
s^q |E_n(s)| = \int_{\{t \in [-\pi, \pi]: 2^n(\pi^2 - (\phi * V_n)(et)) > s\}} s^q dt 
\leq 2^n(\pi^2)\| \phi * V_n \|_q^q, \forall n \in \mathbb{N}_0.
\]
Putting all these together results that, for every \( q \in (\frac{1}{M(\gamma)}, p) \) and \( s > 0 \),
\[
s^p \sum_{n \in \mathbb{N}_0} 2^n |E_n(s)| = s^{p-q} \left( s^q \sum_{\langle n \rangle \leq N(s)} 2^n |E_n(s)| \right) 
\leq s^{p-q} \sum_{\langle n \rangle \leq N(s)} 2^n 2^n(\pi^2)\| \phi * V_n \|_q^q 
\lesssim s^{p-q} C_q A^q \sum_{\langle n \rangle \leq N(s)} 2^n(\pi^2) 2^{-n(d-1)q} \langle n \rangle^{-\gamma q} 
\lesssim C_q A^q s^{p-q} N^{1-\gamma q}(s), \text{ (since } \gamma q = \frac{q}{p} \text{ and } q < p).\]
Finally, notice that \( s^{p-q} N^{1-\gamma q}(s) = s^{p-q}(C_q A_0)^{p-q} s^{-(p-q)} = (C_q A_0)^{p-q}, \) so there is a positive constant \( K \), independent of \( s \), such that
\[
s^p \sum_{n \in \mathbb{N}_0} 2^n |E_n(s)| \leq K^p A^p, \forall s > 0,
\]
and this proves the desired result. Finally, Lemma 2.6 suggests that there is a positive constant \( K_\gamma \) such that
\[
\| H_\mathbf{a} \|_{\mathbf{s}_{p,\infty}} = \| \Gamma \|_{\mathbf{s}_{p,\infty}} \leq K_\gamma \| \phi \|_{B_{p,\infty}^{\frac{1}{p}+d-1}} \leq K_\gamma KA,
\]
where \( \Gamma \) is given by (6.2). This gives relation (7.3), with \( C_\gamma = K_\gamma K \).

**Lemma 7.3.** Let \( \gamma > 0 \) and \( \{a(j)\}_{j \in \mathbb{N}_0} \) be a real valued sequence such that
\[
a^{(m)}(j) = o(j^{-d-m}(\log j)^{-\gamma}), \ j \to +\infty, \quad (7.6)
\]
for every \( m = 0, 1, \ldots, M(\gamma) \). Then the Hankel operator \( H_\mathbf{a} \), with parameter sequence \( a(j) = a(|j|) \), for all \( j \in \mathbb{N}_0 \), is compact and its singular values satisfy the following estimate
\[
s_n(H_\mathbf{a}) = o(n^{-\gamma}), \ n \to +\infty.
\]

**Proof.** The goal is to show that \( H_\mathbf{a} \in \mathbf{S}_{p,\infty}^0 \), for \( p = \frac{1}{\gamma} \). The ideal \( \mathbf{S}_{p,\infty}^0 \) is the \( \| \cdot \|_{\mathbf{s}_{p,\infty}} \)-closure of finite rank operators. So, it is enough to approximate \( H_\mathbf{a} \) by finite rank operators in the \( \| \cdot \|_{\mathbf{s}_{p,\infty}} \) quasi-norm. For consider the cut-off function
\[
\chi_0(t) = \begin{cases} 
1, & t \in [0, 1] \\
0, & t \geq 2,
\end{cases}
\]
such that \( \chi_0 \in C^\infty(\mathbb{R}_+) \) and \( 0 \leq \chi_0 \leq 1 \). In addition, for every \( N \in \mathbb{N} \), define the sequences
\[
q_N(j) = a(j)\chi_0(\frac{j}{N}) \quad \text{and} \quad h_N(j) = a(j) - q_N(j), \ \forall j \in \mathbb{N}_0.
\]
Let \( H_{q_N} \) and \( H_{h_N} \) be the Hankel operators, with parameter sequences \( q_N(j) = q_N(\lfloor j \rfloor) \) and \( h_N(j) = h_N(\lfloor j \rfloor), \forall j \in \mathbb{N}_0 \), respectively. In other words, \( H_{h_N} = H_a - H_{q_N} \). Then, by using the Leibniz rule,

\[
(h_N)^{(m)}(j) = \sum_{n=0}^{m} \binom{m}{n} a^{(m-n)}(j+n)(1-\chi_0)^{(n)}(\frac{t}{N}), \forall j \in \mathbb{N}_0.
\]

Therefore, for every \( j \geq 2 \),

\[
\left| (h_N)^{(m)}(j) j^{d+m} (\log j)^\gamma \right| \leq \sum_{n=0}^{m} \binom{m}{n} |a^{(m-n)}(j+n)| j^{d+m-n} (\log j)^\gamma j^n (1-\chi_0)^{(n)}(\frac{t}{N})
\]

\[
\leq \sum_{n=0}^{m} \binom{m}{n} |a^{(m-n)}(j+n)| (j+n)^{d+m-n} (\log(j+n))^\gamma j^n (1-\chi_0)^{(n)}(\frac{t}{N}).
\]

Moreover, observe that, for any \( n \in \mathbb{N} \),

\[
t^n (1-\chi_0)^{(n)}(\frac{t}{N}) = - (\frac{t}{N})^n \chi_0^{(n)}(\frac{t}{N}), \quad \forall t \in \mathbb{R}_+.
\]

As a result, by recalling that \( \chi_0 \) is compactly supported, there exist positive constants \( K_n, n = 1, 2, \ldots, M(\gamma), \) independent of \( N \), such that

\[
\sup_{t>0} t^n (1-\chi_0)^{(n)}(\frac{t}{N}) \leq K_n.
\]

By considering \( K := \max_{1 \leq n \leq M(\gamma)} \{K_n\}, \) (7.7) gives

\[
\left| (h_N)^{(m)}(j) j^{d+m} (\log j)^\gamma \right| \leq K \sum_{n=0}^{m} \binom{m}{n} |a^{(m-n)}(j+n)| (j+n)^{d+m-n} (\log(j+n))^\gamma, \forall j \geq 2.
\]

By taking supremum,

\[
\sup_{j>N} \left| (h_N)^{(m)}(j) j^{d+m} (\log j)^\gamma \right| \leq \sum_{n=0}^{m} \binom{m}{n} \sup_{j>N} |a^{(m-n)}(j+n)| (j+n)^{d+m-n} (\log(j+n))^\gamma. \tag{7.8}
\]

Under the assumption (7.6) for \( a \), we see that, for any \( N \in \mathbb{N} \), \( h_N \) satisfies assumption (7.2) of Lemma 7.2 and consequently, \( H_{h_N} \) satisfies relation (7.4). Thus, there exists a constant \( C_\gamma \) such that

\[
\|H_a - H_{q_N}\|_{\mathcal{S}_{p,\infty}} \leq C_\gamma \sum_{m=0}^{M(\gamma)} \sup_{j \in \mathbb{N}_0} \left| (h_N)^{(m)}(j) (j+1)^{d+m} (\log(j+2))^\gamma \right|
\]

\[
= C_\gamma \sum_{m=0}^{M(\gamma)} \sup_{j>N} \left| (h_N)^{(m)}(j) (j+1)^{d+m} (\log(j+2))^\gamma \right|.
\]

Then (7.5) implies that

\[
\|H_a - H_{q_N}\|_{\mathcal{S}_{p,\infty}} \lesssim \sum_{m=0}^{M(\gamma)} \sum_{n=0}^{m} \binom{m}{n} \sup_{j>N} |a^{(m-n)}(j+n)| (j+n)^{d+m-n} (\log(j+n))^\gamma. \tag{7.9}
\]

Notice that assumption (7.5) implies that, for any \( n \in \mathbb{N}, \)

\[
\limsup_{j \to +\infty} |a^{(m)}(j+n)| (j+n)^{d+m} (\log(j+n))^\gamma = 0, \forall m = 0, 1, \ldots, M(\gamma).
\]

Thus, letting \( N \to +\infty \) in (7.9) results that the right hand side converges to zero and the result is obtained.
8 Proof of Theorem 1.2

Proof. Let \( \tilde{a} \) be the sequence that is defined in (3.3) and generates the model operator \( \tilde{H} \). Then \( H_a = \tilde{H} + (H_a - \tilde{H}) \), where \( H_a - \tilde{H} = H_{a-\tilde{a}} \) is a Hankel operator with parameter sequence \( (a - \tilde{a}) \) \( j \) = \( (a - \tilde{a})(j) \), for any \( j \in \mathbb{N}_0 \). By observing (1.8) and Lemma 3.1, we see that the sequences \( a \) and \( \tilde{a} \) present the same asymptotic behaviour, modulo some error terms. Thus,

\[
(a - \tilde{a})(j) = h_1(j) + (-1)^j h_{-1}(j), \quad \forall j \geq 2,
\]

where \( h_{\pm 1}(j) := (g_{\pm 1} - \tilde{g}_{\pm 1})(j) \), \( \forall j \in \mathbb{N}_0 \). Furthermore, notice that relation (3.5) in Lemma 3.1 implies that

\[
\tilde{g}^{(m)}(j) = o(j^{-d-m}(\log j)\gamma), \quad j \to +\infty.
\]

The same relation is satisfied by \( g_{\pm 1} \), as well, by assumption. Therefore,

\[
h_{\pm 1}^{(m)} = o(j^{-d-m}(\log j)\gamma), \quad j \to +\infty.
\]

Consider the Hankel operators \( H_{h_{\pm 1}} : \ell^2(N_0^d) \to \ell^2(N_0^d) \), with parameter sequence \( h_{\pm 1}(j) = h_{\pm 1}^{(m)}(j) \), for all \( j \in \mathbb{N}_0^d \). Then Lemma 7.3 yields that their singular values satisfy the following asymptotic law:

\[
s_n(H_{h_{\pm 1}}) = o(n^{-\gamma}), \quad n \to +\infty.
\]

(8.1)

Let \( h_{-1}(j) := (-1)^j h_{-1}(j) \), for all \( j \in \mathbb{N}_0 \), and consider the Hankel operator \( H_{h_{-1}} : \ell^2(N_0^d) \to \ell^2(N_0^d) \), with parameter sequence \( h_{-1}(j) = h_{-1}^{(m)}(j) \), for all \( j \in \mathbb{N}_0^d \). Then \( H_{h_{-1}} \) and \( H_{h_{+1}} \) are unitarily equivalent. Indeed, by defining the unitary operator \( Q : \ell^2(N_0^d) \to \ell^2(N_0^d) \), with

\[
(Qx)(j) = (-1)^j x(j), \quad \forall j \in \mathbb{N}_0^d, \quad \forall x \in \ell^2(N_0^d),
\]

it is checked easily that \( H_{h_{-1}} = Q^* H_{h_{+1}} Q \). Thus, the singular values of \( H_{h_{-1}} \) satisfy (8.1).

Notice that \( H_{a} - \tilde{H} = H_{h_{+1}} + H_{h_{-1}} \). Therefore, since the space \( S_{p,q}^0 \) is linear, the singular values of \( H_{a} - \tilde{H} \) satisfy (8.1). Finally, recall that the eigenvalue asymptotics of \( \tilde{H} \) are given in Lemma 5.3. Thus by combining with the fact that

\[
s_n(H_{a} - \tilde{H}) = o(n^{-\gamma}), \quad n \to +\infty,
\]

Lemma 2.2 yields the asymptotic law (1.9). \( \square \)

A Discussion of Lemma 2.6

The discussion is provided in order to clarify some subtle points that exist in the proof of [9, Theorem 6.4.4]. The proof is based on the retract argument, which requires the construction of two specific bounded operators \( \mathcal{I} \) and \( \mathcal{K} \); for details, see below. This discussion mainly aims to clarify the boundedness of \( \mathcal{K} \), which is defined in (A.3).

The claim is equivalent to the fact that the mapping \( \phi \mapsto \Phi_{\phi} \frac{d^{-1}}{d^{-1}} \frac{d^{-1}}{2} \), where \( \Phi_{\phi} \) is the sequence of the Fourier coefficients of \( \phi \), is a bounded linear operator from \( B_{p,q}^{\frac{1}{2}+d-1} \) to \( S_{p,q} \). The boundedness is proved via interpolation arguments.

For the reader’s convenience, we recall a real interpolation method (the K-method) (cf. [2 §3.1]) and the reiteration theorem (cf. [2 §3.5]). If \( X_0 \) and \( X_1 \) are two quasi-Banach spaces which are continuously embedded into the same Hausdorff topological space, then \( X_0 \) and \( X_1 \) are called compatible. If \( (X_0, X_1) \) is a compatible pair of quasi-Banach spaces, then real interpolation with parameters \( \theta \in (0,1) \) and \( q \in [1, +\infty] \), or \( \theta \in [0,1] \) and \( q = +\infty \), results an intermediate quasi-Banach space \( X_{\theta,q} := (X_0, X_1)_{\theta,q} \).
Theorem A.1 (Reiteration Theorem). Let \( X_{\theta_0, q_0} \) and \( X_{\theta_1, q_1} \) be two interpolation spaces created by the K-method from the compatible couple of quasi-Banach spaces \((X_0, X_1)\). Then, for any \( q \in [1, +\infty) \) and \( t \in (0, 1) \),

\[
(X_{\theta_0, q_0}, X_{\theta_1, q_1})_{t,q} = X_{\theta,q} \quad \text{where} \quad \theta = (1-t)\theta_0 + t\theta_1.
\]

An application of real interpolation and the reiteration theorem yields that, for every \( p_0 \in (0, +\infty) \), \( \theta \in (0, 1) \) and \( q \in (0, +\infty) \),

\[
(S_{p_0}, S_{\infty})_{\theta,q} = S_{p,q}, \quad \text{where} \quad p = \frac{p_0}{1 - \theta}.
\] (A.1)

For the reader’s convenience we give a sketch of proof for (A.1) and we refer to \cite{7} for further details. Relation (A.1) holds true for \( p_0 < q \) (cf. \cite{7} (11)). Then an application of the reiteration theorem yields

\[
(S_{p_0, q_0}, S_{p_1, q_1})_{\theta,q} = S_{p,q}, \quad \text{where} \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.
\] (A.2)

for any \( \theta \in (0, 1) \) and \( q \in (0, +\infty) \) (cf. \cite{7} (14)). Finally, by using (A.2) together with the reiteration theorem, one obtains (A.1) for any \( q \in (0, +\infty] \).

Now we go back to our main claim; namely, the mapping \( \phi \mapsto \Gamma^{d-1}_{\phi \theta} \) is a bounded linear operator from \( B_{\theta,q}^{\frac{1}{d}+1} \) to \( S_{p,q} \). Theorem 2.5 implies that this mapping represents a bounded linear operator from \( B_{p}^{\frac{1}{d}+1} \) to \( S_{p} \subset S_{\infty} \), \( \forall p \in (0, +\infty) \). Therefore, with due regard to (A.1), it is enough to prove that, for every \( p_0, p_1 \in (0, +\infty) \), \( \theta \in (0, 1) \) and \( q \in (0, +\infty] \),

\[
\left( B_{p_0}^{\frac{1}{d}+1}, B_{p_1}^{\frac{1}{d}+1} \right)_{\theta,q} = B_{p}^{\frac{1}{d}+1}, \quad \text{where} \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.
\] (A.3)

The relation (A.3) will be proved by using the retract argument \cite{2} Theorem 6.4.2]. Briefly, if \( X \) and \( Y \) are two quasi-Banach spaces, then \( X \) is a retract of \( Y \) if there are bounded linear mappings \( J: X \to Y \) and \( K: Y \to X \) such that \( KJ \) is the identity map on \( X \).

Notice that for every \( p_0, p_1 \in (0, +\infty) \), \( \theta \in (0, 1) \) and \( q \in (0, +\infty] \),

\[
\left( L^{p_0}(M, \mu), L^{p_1}(M, \mu) \right)_{\theta,q} = L^{p,q}(M, \mu), \quad \text{where} \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.
\] (A.4)

The relation above can be found in \cite{2} Theorem 5.3.1]. The goal is to construct two mappings \( J: B_{p_0}^{\frac{1}{d}+1} + B_{p_1}^{\frac{1}{d}+1} \to L^{p_0}(M, \mu) + L^{p_1}(M, \mu) \) and \( K: L^{p_0}(M, \mu) + L^{p_1}(M, \mu) \to B_{p_0}^{\frac{1}{d}+1} + B_{p_1}^{\frac{1}{d}+1} \) such that:

(i) \( B_{p_0}^{\frac{1}{d}+1} \) is retract of \( L^{p_0}(M, \mu) \), under the mappings \( J: B_{p_0}^{\frac{1}{d}+1} \to L^{p_0}(M, \mu) \) and \( K: L^{p_0}(M, \mu) \to B_{p_0}^{\frac{1}{d}+1} \); and

(ii) \( B_{p_1}^{\frac{1}{d}+1} \) is a retract of \( L^{p_1}(M, \mu) \) under the mappings \( J: B_{p_1}^{\frac{1}{d}+1} \to L^{p_1}(M, \mu) \) and \( K: L^{p_1}(M, \mu) \to B_{p_1}^{\frac{1}{d}+1} \).

For let \( Hol(\mathbb{D}) \) be the space of the holomorphic functions on \( \mathbb{D} \) and define the linear operator

\[
J\phi = \bigoplus_{n \in \mathbb{N}_0} 2^{n(d-1)} \phi \ast V_n, \quad \forall \phi \in Hol(\mathbb{D}),
\]

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where the polynomials $V_n$ are defined in (2.1) and (2.2). Then, by the definition of the Besov space $B^{\frac{1}{2}+d-1}_p$, $J$ is an isometry from $B^{\frac{1}{2}+d-1}_p$ to $L^p(M, \mu)$. In addition, consider the polynomials

$$
V_0(z) = V_0(z) + V_1(z), \quad \forall z \in \mathbb{T},
$$

and, for every $n \in \mathbb{N}$,

$$
\tilde{V}_n(z) = V_{n-1}(z) + V_n(z) + V_{n+1}(z), \quad \forall z \in \mathbb{T}.
$$

Notice that $V_n \ast \tilde{V}_n = V_n$, for every $n \in \mathbb{N}_0$. Now define the linear operator

$$
K \bigoplus \phi_n = \sum_{n \in \mathbb{N}_0} 2^{-n(d-1)} \phi_n \ast \tilde{V}_n, \quad \forall \bigoplus \phi_n \in L^p(M, \mu),
$$

which is bounded from $L^p(M, \mu)$ to $B^{\frac{1}{2}+d-1}_p$. To see this, it is enough to check that

$$
\sum_{n \in \mathbb{N}_0} 2^{n[1+p(d-1)]} \| \sum_{m \in \mathbb{N}_0} 2^{-m(d-1)} \phi_m \ast \tilde{V}_m \ast V_n \|_p^p < +\infty, \quad \forall \bigoplus \phi_n \in L^p(M, \mu). \quad (A.6)
$$

Moreover, notice that $KJ \phi \phi = \phi$, for all $\phi$.

It has been proved that $J$ is an isometry from $B^{\frac{1}{2}+d-1}_p$ to $L^p(M, \mu)$. Therefore, due to (A.4), if $\phi$ belongs to $\left( B_{p_0}^{\frac{1}{2}+d-1}, B_{p_1}^{\frac{1}{2}+d-1} \right)$, then $J \phi \in L^{p,q}(M, \mu)$, where $p$ is described in (A.3). According to the definition of $B_{p,q}^{\frac{1}{2}+d-1}$ (see (2.3)), $\phi \in B_{p,q}^{\frac{1}{2}+d-1}$ and thus,

$$
\left( B_{p_0}^{\frac{1}{2}+d-1}, B_{p_1}^{\frac{1}{2}+d-1} \right)_{\theta,q} \subset B_{p,q}^{\frac{1}{2}+d-1}. \quad (A.7)
$$

On the other hand, let $\phi \in B_{p,q}^{\frac{1}{2}+d-1}$ or equivalently, $J \phi \in L^{p,q}(M, \mu)$. Moreover, $\phi = KJ \phi$ and $K$ is bounded from $L^p(M, \mu)$ to $B^{\frac{1}{2}+d-1}_p$. Therefore, due to (A.4), $\phi \in \left( B_{p_0}^{\frac{1}{2}+d-1}, B_{p_1}^{\frac{1}{2}+d-1} \right)_{\theta,q}$.

This results

$$
B_{p,q}^{\frac{1}{2}+d-1} \subset \left( B_{p_0}^{\frac{1}{2}+d-1}, B_{p_1}^{\frac{1}{2}+d-1} \right)_{\theta,q}. \quad (A.8)
$$

Therefore, (A.7) and (A.8) yield (A.3).

In order to complete the proof, it only remains to verify the validity of (A.6). For notice that

$$
\sum_{m \in \mathbb{N}_0} \phi_m \ast \tilde{V}_m \ast V_n = \phi_{n-1} \ast \tilde{V}_{n-1} \ast V_n + \phi_n \ast V_n + \phi_{n+1} \ast \tilde{V}_{n+1} \ast V_n, \quad \forall n \in \mathbb{N}.
$$

Thus, for any $p > 0$ and every $n \in \mathbb{N}$,

$$
\| \sum_{m \in \mathbb{N}_0} 2^{-m(d-1)} \phi_m \ast \tilde{V}_m \ast V_n \|_p^p \lesssim \| 2^{-(n-1)(d-1)} \phi_{n-1} \ast \tilde{V}_{n-1} \ast V_n \|_p^p + \| 2^{-n(d-1)} \phi_n \ast V_n \|_p^p 
$$

$$
+ \| 2^{-(n+1)(d-1)} \phi_{n+1} \ast \tilde{V}_{n+1} \ast V_n \|_p^p. \quad (A.9)
$$

Observe that if the convolution with $V_n$ is a bounded operator whose norm does not depend on $n$, then (A.9) becomes

$$
\| \sum_{m \in \mathbb{N}_0} 2^{-m(d-1)} \phi_m \ast \tilde{V}_m \ast V_n \|_p^p \lesssim \| 2^{-(n-1)(d-1)} \phi_{n-1} \|_p^p + \| 2^{-n(d-1)} \phi_n \|_p^p + \| 2^{-(n+1)(d-1)} \phi_{n+1} \|_p^p.
$$
Then it will not be difficult to see that
\[
\sum_{n \in \mathbb{N}_0} 2^{n[1+p(d-1)]} \left\| \sum_{m \in \mathbb{N}_0} 2^{-m(d-1)} \phi_m * \hat{V}_m * V_n \right\|_p^p \lesssim \bigoplus_{n \in \mathbb{N}_0} \left\| \phi_n \right\|_p^p, \quad \forall \bigoplus_{n \in \mathbb{N}_0} \phi_n \in L^p(\mathcal{M}, \mu),
\]
which actually proves \((A.6)\).

So it remains to prove that convolution with \(V_n\) is a bounded operator whose norm does not depend on \(n\). With a closer look, it only needs to prove that \(\{V_n\}_{n \in \mathbb{N}_0}\) forms a sequence of uniformly bounded multipliers. This requires to split the range of \(p\) in two intervals, \((0,1]\) and \((1, +\infty)\). The first case requires the extra condition of analyticity. Namely, \(\{V_n\}_{n \in \mathbb{N}_0}\) should be a uniformly bounded sequence of \(H^p\) multipliers. Respectively, when \(p > 1\), \(\{V_n\}_{n \in \mathbb{N}_0}\) should be a uniformly bounded sequence of \(L^p\) multipliers.

Regarding the case where \(p \in (0,1]\), notice that the analyticity condition does not cause any harm, since, for every \(n \in \mathbb{N}\), \(V_n\) acts (as a multiplier) on the analytic projection of \(L^p\). So the approach with Hardy multipliers is allowed. For we use the following theorems. Theorem \((A.2)\) can be found in \([13, Théorème 1]\) and Theorem \((A.3)\) in \([4, Théorème 5.1]\).

**Theorem A.2.** Let \(p \in (0,1]\) and consider the Hardy space \(H^p(\mathbb{R})\). Let \(k \in \mathbb{N}\) such that \(k^{-1} < p\) and \(\rho : \mathbb{R}_+ \to \mathbb{C}\) which satisfies the following conditions:

(i) \(|\rho(t)| \leq A, \, \forall t \in \mathbb{R}_+;\)

(ii) \(\rho \in C^k(\mathbb{R}_+)\) and
\[
\int_{\mathbb{R}} |\rho^{(l)}(t)|^2 \, dt \leq AR^{-2l+1}, \quad \forall R > 0, \forall l = 1, \ldots, k;
\]
where \(A\) is a positive constant. Then \(\rho\) is a multiplier on \(H^p(\mathbb{R})\).

**Theorem A.3.** Let \(\rho : \mathbb{R} \to \mathbb{C}\) be a continuous function that gives rise to a multiplier on the Hardy space \(H^p(\mathbb{R})\), for some \(p \in (0,1]\). Then, for any \(t > 0\), the sequence \(\rho_t = \{\rho(tn)\}_{n \in \mathbb{Z}}\) is a multiplier on the Hardy space \(H^p(\mathbb{T})\) and furthermore, the multiplier norm \(\|\rho_t\|_{\mathcal{M}(H^p(\mathbb{T}))}\) is uniformly bounded with respect to \(t\).

Notice that the polynomials \(V_n\) are constructed by scaling the function \(v\) (see \(\S 2.1\) for definitions). Therefore, according to Theorem \((A.3)\) it is enough to prove that \(v\) defines an \(H^p(\mathbb{R})\) multiplier. The latter is achieved by applying Theorem \((A.2)\).

Finally, if \(p > 1\), then the following theorems are needed. Theorem \((A.4)\) can be found in \([2, Théorème 6.1.6]\), Theorem \((A.5)\) in \([2, Théorème 6.1.3]\) and Theorem \((A.6)\) in \([6, Théorème 4.3.7]\).

**Theorem A.4** (Mikhlin’s Multiplier Theorem). Let \(\rho : \mathbb{R} \to \mathbb{C}\) be a function which satisfies
\[
|\rho^{(n)}(x)| \leq A \langle x \rangle^{-n}, \quad \forall x \in \mathbb{R}, \, n = 0, 1.
\]
Then \(\rho \in \mathcal{M}_p(\mathbb{R})\), for every \(p \in (1, +\infty)\), and, more precisely, there exists a positive constant \(C_p\) which depends only on \(p\) such that
\[
\|\rho\|_{\mathcal{M}_p} \leq C_p A.
\]

**Theorem A.5.** Let \(\rho : \mathbb{R} \to \mathbb{C}\) belong to \(\mathcal{M}_p(\mathbb{R})\). Then, for any \(t \in \mathbb{R} \setminus \{0\}\), the function \(\rho_t : \mathbb{R} \to \mathbb{C}\) which maps \(x\) to \(\rho(tx)\) belongs to \(\mathcal{M}_p(\mathbb{R})\) with \(\|\rho_t\|_{\mathcal{M}_p} \leq \|\rho\|_{\mathcal{M}_p}\).

**Theorem A.6.** Let \(\rho : \mathbb{R} \to \mathbb{C}\) be a continuous function such that \(\rho \in \mathcal{M}_p(\mathbb{R})\), for some \(p \in (1, +\infty)\). Then, for any \(t > 0\), the sequence \(\rho_t = \{\rho(tn)\}_{n \in \mathbb{Z}}\) belongs to \(\mathcal{M}_p(\mathbb{T})\) and moreover,
\[
\sup_{t > 0} \|\rho_t\|_{\mathcal{M}_p(\mathbb{T})} \leq \|\rho\|_{\mathcal{M}_p(\mathbb{R})}.
\]
According to Theorem \((A.5)\) it is enough to prove that \(v\) gives rise to an \(L^p(\mathbb{R})\) multiplier. Then Theorem \((A.6)\) will give the desired result. The fact that \(v\) defines an \(L^p(\mathbb{R})\) multiplier can be checked by Theorem \((A.4)\).

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