Discrete Translates in Function Spaces

Alexander Olevskii

The talk is based on joint work with Alexander Ulanovskii
Introduction

Given $f \in L^2(\mathbb{R})$, consider the set of the translates

$$\{f(t - \lambda), \lambda \in \mathbb{R}\}.$$

**WIENER:** When the translates span the whole space $L^2(\mathbb{R})$?

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**Theorem (Wiener).** The set of translates $\{f(t - \lambda), \lambda \in \mathbb{R}\}$ spans the whole space $L^1(\mathbb{R})$ if and only if $\mathcal{F}f$ has no zeros on $\mathbb{R}$.
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Z(\hat{f}) := \{ w : \hat{f}(w) = 0 \}.
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Beurling (1951): The set of translates spans $L^p(\mathbb{R})$ if $\text{DIM}_H(Z(\hat{f})) < 2(p - 1)/p$.

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Theorem (N. Lev, A.O., Annals 2011). For every $p, 1 < p < 2$, there are two functions $f_1, f_2 \in (L^1 \cap L^p)(\mathbb{R})$ such that

(i) $Z(\hat{f}_1) = Z(\hat{f}_2)$;

(ii) The set of translates of $f_1$ spans $L^p(\mathbb{R})$, while the set of translates of $f_2$ does not.
Discrete Translates

Let $\Lambda$ be a discrete subset of $\mathbb{R}$. Given $f \in L^2(\mathbb{R})$, consider the set of its $\Lambda$-translates

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**Definition.** $f$ is called a generator for $\Lambda$ if its $\Lambda$-translates span the whole space $L^2(\mathbb{R})$. 

Two examples:

- $\Lambda_1 := \{ \sqrt{n}, n \in \mathbb{Z}^+ \}$
- $\Lambda_2 := \mathbb{Z}$

$\Lambda_1$ admits a generator while $\Lambda_2$ does not.

**DISCUSSION**

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SIZE VERSUS ARITHMETICS!
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**Theorem** (A.O., 1997). *For any almost integer set of translates there is a generator.*

The construction is based on "small denominators" argument.
The case $p > 2$:

**Theorem** (A. Atzmon, A. O., Journal of Approximation Theory, 1996). For every $p > 2$ there is a smooth function $f \in (L^p \cap L^2)(\mathbb{R})$ such that the family \( \{f(t - n), n \in \mathbb{Z}\} \) is complete and minimal in $L^p(\mathbb{R})$.

Hence, $\Lambda = \mathbb{Z}$ admits an $L^p$-generator for every $p > 2$ (and it does not for $p \leq 2$).
$L^1$-generators

No u.d. set $\Lambda$ may admit an $L^1$-generator.

**Theorem** (J. Bruna, A. O., A. Ulanovskii, Rev. Mat. Iberoam., 2006) $\Lambda$ admits an $L^1$-generator iff it has infinite Beurling-Malliavin density.

For $1 < p < 2$ the problem remained open.
Discrete Translates in Function Spaces

Which function spaces can be spanned by a uniformly discrete set of translates of a single function?

All results below are from A.O., A.Ulanovskii:
– Bull. London Math. Soc. (2018) and
– Analysis Mathematica (2018).

Let $X$ be a Banach function space on $\mathbb{R}$, satisfying the condition:
(I) The Schwartz space $S(\mathbb{R})$ is embedded in $X$ continuously and densely;
Then the elements of $X^*$ are tempered distributions.
We also assume
(II) Conditions $g \in X^*$ and $\text{spec} g \subset \mathbb{Z}$ imply $g = 0$. 
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**Theorem 1.** There exist a smooth function $f$ and a uniformly discrete set $\Lambda$ of translates such that the family $\{f(t - \lambda), \lambda \in \Lambda\}$ spans $X$.
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Below we present an explicit construction of $f$ and $\Lambda$ in this result.
Examples

Theorem 1 is applicable to

1. $L^p(\mathbb{R}), p > 1$.
2. Separable symmetric spaces (like Orlitz, Marzienkevich). The only exception is $L^1(\mathbb{R})$.
3. Sobolev spaces $W^{l,p}(\mathbb{R}), p > 1$.
4. Weighted spaces $L^1(w; \mathbb{R})$, where the weight is bounded and vanishes at infinity.
Construction

Definition. $F \in S(R)$ is said to have a deep zero at point $t$ if

$$|F(t + h)| < Ce^{-1/|h|}, \quad |h| < \frac{1}{2}. $$
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GENERATOR: Take an even real function $F$ with deep zeros at all integers (with the same constant) and at infinity, and which has no other zeros. Consider its Fourier transform $f := \hat{F}$.

TRANSLATES: Now define the translates as exponentially small perturbation of integers:

$$\Lambda := \{n + e^{-|n|}, n \in \mathbb{Z}\}.$$  

Theorem 1'. The set of translates $\{f(t - \lambda), \lambda \in \Lambda\}$ is complete in every $X$ satisfying (I) and (II).

Universality!

**Construction**

**Main Lemma.** Let $F$ and $\Lambda$ be as above, $g \in S'$. If the convolution $\hat{F} \ast g$ vanishes on $\Lambda$ then it is zero.
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**Model Example.** If $F$ is as above and $\hat{F}|_{\Lambda} = 0$ then $F = 0$.

**Proof.** $\hat{F}$ is analytic in a strip. Denote

$$H(t) := \sum_{k \in \mathbb{Z}} F(t + k).$$

By the Poisson formula,

$$H(t) = \sum_{n \in \mathbb{Z}} \hat{F}(n) e^{2\pi i nt}.$$

Since $\hat{F}(n)$ is exponentially small, then $H$ is analytic on the circle. And it has a deep zero, so that $H = 0$. Hence, $\hat{F}|_{\mathbb{Z}} = 0$.

Iterate the argument above for $tF$, $t^2F$, ... to get $\hat{F}^{(k)}|_{\mathbb{Z}} = 0$, $k = 1, 2, ...$, so that $\hat{F} = 0$. 
Proof of Theorem 1’

Suppose the translates \( \{ f(t - \lambda), \lambda \in \Lambda \} \) are not complete in \( X \). Then there is a functional \( g \) ”orthogonal” to them, which means

\[ g \ast f|_\Lambda = 0. \]

By the Main Lemma, \( g \ast f = 0 \). That is \( \hat{g}F = 0 \). So, \( \hat{g} = 0 \) on \( \mathbb{R} \setminus \mathbb{Z} \). This means Spec \( g \subset \mathbb{Z} \). Applying Property (II), we get \( g = 0 \).
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Open Problem. Does there exist a set of translates of a single function, which is complete and minimal in \( L^2(\mathbb{R}) \)?
Theorem 2. There are $f_1, f_2 \in S(\mathbb{R})$ such that the $\Lambda$-translates of them span every space $X$, satisfying property (I) only.

This shows an advantage of collective work!
Two generators

**Theorem 2.** There are \( f_1, f_2 \in S(\mathbb{R}) \) such that the \( \Lambda \)-translates of them span every space \( X \), satisfying property (I) only.

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THANKS!