Quarkonium production: velocity-scaling rules
and long-distance matrix elements

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Abstract

The hierarchy of long-distance matrix elements (MEs) for quarkonium production depends on their scaling with the velocity $v$ of the heavy quark in the bound state. Ranges for the velocities in various bound states and uncertainties of colour-singlet MEs are estimated in a quark-potential model. Different possibilities for the scaling with $v$ of the MEs are discussed; they depend on the actual values of $v$ and the QCD scale. As an application, $J/\psi$ polarization in $e^+e^-$ annihilation is discussed. The first non-perturbative estimates of colour-octet MEs are presented and compared with phenomenological determinations. Finally, various predictions of prompt quarkonium production at LEP are compared.

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1 Introduction

Despite the appearance of genuine non-perturbative elements, we can calculate the production of a heavy quarkonium bound state \( H(J^{PC}) \): the cross section is given as a sum of terms

\[
\sigma^H = \sum_{c=1,8} \sum_n f_c(n) \langle O^H_c(n) \rangle ,
\]

(1)
each factorized into two parts, \( f_c(n) \) and \( \langle O^H_c(n) \rangle \), respectively, describing the production of a \( Q\bar{Q} \) pair in a state \( |n_c \rangle \) and the transition of this state into the quarkonium \( H \). Here \( c = 1\) (\( c = 8 \)) denotes a colour-singlet (colour-octet) \( Q\bar{Q} \) pair. Besides spin \( S \), orbital angular momentum \( L \), and total spin \( J \) of the \( Q\bar{Q} \) pair, one more index, the dimension of the operator \( O \), is required in order to specify \( n \). The short-distance part \( f_c(n) \) is calculable in perturbation theory in \( \alpha_s(\mu) \) with \( \mu \) of the order of the heavy-quark mass \( m \). For a non-relativistic system such as the heavy quarkonium, the heavy-quark velocity \( v \) (in the quarkonium rest frame) is the natural scale, which governs the magnitude of the long-distance matrix elements (MEs) \( \langle O^H_c(n) \rangle \). For calculations accurate to a given order in \( v \), only a finite number of terms contribute to (1).

The above-outlined factorization into the short-distance physics of the \( Q\bar{Q} \) pair and the long-distance physics of bound-state formation is a rigorous result of non-relativistic QCD (NRQCD) for inclusive quarkonium decays in the limit of large mass \( m \) of the heavy quark where the multipole expansion is valid[1]. For the case of (inclusive) quarkonium production one needs the additional assumption[1, 2] that all singularities, which neither cancel between real and virtual contributions nor can be absorbed into the MEs, can be factorized into the parton distribution functions (PDF). A study on an extension of the factorization approach to exclusive decays[3] has recently started.

Two facts determine which MEs actually have to be included for an \( O(v^\nu) \) calculation of a given process. First the scaling of the MEs with \( v \), and second the magnitude of the respective short-distance coefficients. The latter is determined by the powers of \( \alpha_s(m) \) and, possibly, by kinematical factors.

2 Short-distance processes

Let us consider a few examples of short-distance reactions. The lowest-order short-distance processes relevant for quarkonium hadroproduction are

\[
\begin{align*}
\text{q\bar{q}} & \rightarrow Q\bar{Q}_{1}(X) & X = 3S_1, 3D_1, \ldots & \alpha^2_{\text{em}} \\
& \rightarrow Q\bar{Q}_{8}(X) & X = 3S_1, 3D_1, \ldots & \alpha^2_s \\
\text{gg} & \rightarrow Q\bar{Q}_{1}(X) & X = 1S_0, 3P_{0,2}, 1D_2, \ldots & \alpha^2_s \\
& \rightarrow Q\bar{Q}_{8}(X) & X = 1S_0, 1P_1, 3P_{0,2}, 2S+1D_J, \ldots & \alpha^2_s 
\end{align*}
\]

(2)
The selection rules in (3) are easily understood. Annihilation of quarks and antiquarks proceeds through one-gluon (one-photon) exchange and is, hence, restricted to $J^P = 1^-$ states. Two gluons in a colour-singlet state are charge-conjugation ($C$)-even, while two gluons in a colour-octet state can be $C$-even or $C$-odd, depending on whether the $d$ or $f$ $SU(3)$ structure constants are involved. Two-gluon production of the $^3P_1$ state is forbidden by virtue of the Landau–Yan theorem, independent of colour (provided the incident gluons can be taken on the mass shell!). The vanishing of the squared matrix element for $gg \to ^3S_1 + g$ (again, true for on-mass-shell gluons) seems to be an accidental cancellation between the two QED-like diagrams and the diagram involving the three-gluon coupling, which appears only in $1^-$ production. It would be interesting to check whether the rates for $^3D_1$ production and the $O(v^2)$ correction to $^3S_1$ are non-zero. Quarkonium production through $^1P_1$ and $D$-wave intermediate states has not yet been included in phenomenological analyses.

In photoproduction we encounter the following reactions up to and including the order $\alpha_s^2 \alpha_s$ or $\alpha_s \alpha_s^2$:

\[
\begin{align*}
\gamma g & \rightarrow Q\overline{Q}_8(X) & 1^1S_0, 3^1P_0, 2^1D_2 \ldots & \epsilon_Q^2 \alpha_s \alpha_s^2 \\
Q\overline{Q}_1(X) + g & \rightarrow 3^3S_1, 1^1P_1, 3^3D_J \ldots & \epsilon_Q^2 \alpha_s \alpha_s^2 \\
Q\overline{Q}_8(X) + g & \rightarrow 2^{2S+1}L_J \ldots & \epsilon_Q^2 \alpha_s \alpha_s^2 \\
Q\overline{Q}_8(X) + \gamma & \rightarrow 3^3S_1, 1^1P_1, 3^3D_J \ldots & (\epsilon_Q^2 \alpha_s)^2 \alpha_s \\
\gamma q & \rightarrow Q\overline{Q}_8(X) + q & 1^1S_0, 3^1P_1, 1^1D_2 \ldots & \epsilon_Q \alpha_s \alpha_s^2 \\
Q\overline{Q}_8(X) + q & \rightarrow 3^3S_1, 3^3D_1 \ldots & \epsilon_Q^2 \alpha_s \alpha_s^2 
\end{align*}
\]

The selection rules for the reactions involving only one gluon (first and fourth ones) are the same as in QED. The second reaction is again restricted by the definite $C$ property of the two-gluon colour-singlet state. Photon–quark-initiated reactions differ whether the photon couples to the heavy-quark line ($\propto e_Q^2$, where $e_Q$ is the electric charge of the heavy quark in units of $e$) or the light-quark line ($\propto e_Q^2$).

In electron–positron annihilation via the exchange of an $s$-channel photon or $Z^0$-boson, the following final states can be reached:

\[
\begin{align*}
Q\overline{Q}_8(X) + g & \rightarrow 1^1S_0, 3^1P_1, 1^1D_2 \ldots & 3^1S_1, 1^1P_1, 3^1D_J, \ldots & - & \alpha_s \xi \\
Q\overline{Q}_8(X) + gg & \rightarrow 3^3S_1, 1^1P_1, 3^3D_J, \ldots & 1^1S_0, 3^1P_1, 1^1D_2, \ldots & - & \alpha_s^2 \xi \\
Q\overline{Q}_8(X) + Q\overline{Q} & \rightarrow 2^{2S+1}L_J & 2^{2S+1}L_J & - & \alpha_s^2 \\
Q\overline{Q}_8(X) + q\overline{q} & \rightarrow 1^1S_0, 3^1P_1, 1^1D_2 \ldots & 3^1S_1, 1^1P_1, 3^1D_J, \ldots & - & \alpha_s^2 \xi \\
Q\overline{Q}_8(X) + q\overline{q} & \rightarrow - & - & 3^3S_1 & \alpha_s^2 \ln^2 \xi 
\end{align*}
\]
The selection rules for one-photon exchange processes are the same as for the analogous photoproduction processes. The same holds true for the vector-current part of $Z^0$ exchange ($\propto g_Q^2$). The axial-vector current contribution has opposite $C$-transformation property to the vector-current contribution. Therefore any $Q\bar{Q}(X)$ state can be reached through $Z^0$ exchange. I have also indicated the behaviour with \( \xi = (2m/\sqrt{s})^2 \), where $\sqrt{s}$ denotes the $e^+e^-$ c.m. energy.

One immediate consequence of (4) is the value of $A$ in the angular distribution, $d\sigma/d\cos\theta \propto 1 + A \cos^2 \theta$, of (direct) $J/\psi$ production with respect to the beam axis at large $Z = 2E_{J/\psi}/\sqrt{s}$ and for $m_{J/\psi} \ll \sqrt{s} \ll m_{Z^0}$ (CLEO energies, $\sqrt{s} \sim 10$ GeV). If colour-octet production is allowed (as is the case in both the colour-evaporation model (CEM) and in NRQCD, see below) then $^1S_0 + g$ production dominates, resulting\(^4,\, 5\) in
\[
A = 1. 
\]
Explicitly\(^5\), for $r \ll 1$
\[
\frac{d\sigma}{d\cos\theta} = \frac{16\pi^2\alpha_{em}^2\alpha_s}{9ms^2} \left\{ \langle O_{8}^{1/\psi}(^1S_0) \rangle + \frac{3}{m^2} \langle O_{8}^{1/\psi}(^3P_0) \rangle \right\} \left( 1 + \cos^2 \theta \right). \tag{5} 
\]

On the other hand, if colour-singlet production dominates then $(Q\bar{Q})_1(^3S_1)gg$ is the leading process, resulting in\(^5\) $A = -(1 - \xi)/(1 + \xi)$. This is easily understood: the major configuration in the colour-singlet model (CSM) has the two gluons recoiling against the $J/\psi$ with about equal energy. For $\xi \to 0$, i.e. neglecting the $J/\psi$ mass, conservation of parity and angular momentum then predicts $A = -1$. Hence, the measurement of the angular distribution appears as a gold-plated test of the colour-octet mechanism.

However, also in the CSM differential cross section there is an $A = +1$ component, namely when the energy of one gluon is at its lower limit (and, necessarily, the other gluon’s energy at its maximum value). Integrating the differential cross section over the region $E_i \leq mv^2$, where $E_i$ are the gluon energies in the $Q\bar{Q}$ rest frame, I find in the limit $r \ll 1$ and $v^2 \ll 1$ the NRQCD result\(^6\) with the curly bracket replaced by
\[
\left[ \frac{64}{81} \frac{\alpha_s(mv^2)}{\pi} v^4 \langle O_{1/\psi}^{1/\psi}(^3S_1) \rangle \right]. \tag{6} 
\]
Thus, if physics emphasizes the soft-gluon region the CSM result reproduces the NRQCD one with $\alpha_s(mv^2) \sim 1$.

Alternatively, we can assume that gluons up to energies $k \sim mv$ are important. Integrating the squared matrix element over the region $E_i \leq mv$, I again obtain a NRQCD-type result, but this time with the curly bracket replaced by a term proportional to $\alpha_s(mv)v^2|R(0)|^2$. Identifying $\alpha_s(mv) \sim v$ we now predict $\langle O_{8}^{1/\psi}(^1S_0) \rangle$ to scale like $v^3$ rather than like $v^4$ w.r.t. $\langle O_{1/\psi}^{1/\psi}(^3S_1) \rangle$.

3 Velocity-scaling rules

The velocity scaling of the MEs $\langle O_{c}^{H}(n) \rangle \equiv \langle O_{c}^{H}(d, X) \rangle$ in NRQCD is determined by the number of derivatives in the respective operators and the number of electric or magnetic
dipole transitions between the $Q\overline{Q}$ pair produced at short distances and the $Q\overline{Q}$ pair in the asymptotic quarkonium. The velocity scaling for the leading MEs of the most prominent quarkonium states are listed in Table 1. These leading MEs up to and including the

| $H = \psi(1^{--})$ | Colour-singlet | Colour-octet |
|---------------------|----------------|--------------|
| $X = \psi(1^{--})$ | $v^8$ $1$ $v^8$ $v^8$ $v^8$ $0$ | $v^4$ $v^4$ $v^8$ $v^4$ $v^8$ $0$ |
| NRQCD               | $v^8$ $1$ $v^8$ $v^8$ $v^8$ $v^8$ | $v^4$ $v^4$ $v^8$ $v^4$ $v^8$ $0$ |
| CSM                 | $v^2\lambda^2$ $1$ $v^4\lambda^4$ $v^2\lambda^2$ $v^4\lambda^4$ $v^6\lambda^6$ | $v^4\lambda^4$ $v^2\lambda^2$ $v^4\lambda^4$ $v^6\lambda^6$ |
| VSR1                | $v^7$ $1$ $v^7$ $v^8$ $v^8$ $v^11$ | $v^3$ $v^4$ $v^7$ $v^4$ $v^8$ $v^11$ |
| CEM                 | $1$ $1$ $v^2$ $v^2$ $v^4$ $v^4$ | $1$ $1$ $v^2$ $v^2$ $v^4$ $v^4$ |

| $H = \chi_0(J^{++})$ | Colour-singlet | Colour-octet |
|---------------------|----------------|--------------|
| $X = \chi_0(J^{++})$ | $v^6$ $v^6$ $v^{10}$ $v^2$ $v^{10}$ $v^2$ | $v^6$ $v^6$ $v^6$ $v^6$ $v^6$ $v^10$ |
| NRQCD               | $v^6$ $v^6$ $v^{10}$ $v^2$ $v^{10}$ $v^2$ | $v^6$ $v^6$ $v^6$ $v^6$ $v^6$ $v^10$ |
| CSM                 | $v^2\lambda^2$ $1$ $v^4\lambda^4$ $v^2\lambda^2$ $v^4\lambda^4$ $v^6\lambda^4$ | $v^4\lambda^4$ $v^2\lambda^2$ $v^4\lambda^4$ $v^6\lambda^4$ |
| VSR1                | $v^7$ $1$ $v^7$ $v^9$ $v^{10}$ $v^9$ | $v^5$ $v^2$ $v^5$ $v^6$ $v^6$ $v^9$ |
| CEM                 | $1$ $1$ $v^2$ $v^2$ $v^4$ $v^4$ | $1$ $1$ $v^2$ $v^2$ $v^4$ $v^4$ |

Table 1: Relative scaling of $\langle O^H(X) \rangle$.

order $v^4$ are the ones used in current phenomenological estimates of production rates, in particular those MEs, which are enhanced by the short-distance coefficients. Note, however, that there are non-leading MEs that are less suppressed than some of the MEs given in Table 1. For example, the ME $\langle O^1_1(d = 8, 3S_1) \rangle$ is suppressed by only $v^2$ relative to the leading ME $\langle O^1_0(d = 6, 3S_1) \rangle$ given in Table 1; it therefore gives rise to the first $O(v)$ correction to $1^{--}$ production. Even though it has the same short-distance factor as the leading colour-singlet ME, it may be favoured kinematically and, therefore, result in a sizeable contribution.

In the limit of small velocity, the NRQCD MEs for $S$-wave states and, hence, predictions for their production (and decays) reduce to those of the CSM. In the CSM all colour-octet MEs are zero, while the colour-singlet ones are related to the quarkonium wave function at the origin

$$\langle O^1_1(6, 3S_1) \rangle = \frac{3N_C}{2\pi}\left|R_{sS}(0)\right|^2$$
\begin{align}
\langle O_1^{QJ(\alpha P)}(8, 3P_J) \rangle &= (2J + 1) \frac{3N_C}{2\pi} |R'_{nP}(0)|^2 \\
\langle O_1^{nQ(nD)}(10, 1D_2) \rangle &= 5 \frac{15N_C}{8\pi} |R''_{nD}(0)|^2.
\end{align}

The scaling of the wave function and its derivatives at \( r = 0 \) with \( v \), e.g. \( |R_S(0)|^2 \propto (mv)^3 \), \( |R'_P(0)|^2 \propto (mv)^5 \), hence determines the scaling of the MEs in the CSM as shown in Table 1.

Opposite to the scaling rules of the CSM are those of the CEM. Here any \( Q\bar{Q}_c(X) \) state has probability \( v^0 \) to reach any quarkonium state \( H \). The only suppression that occurs is the one in the production of the \( Q\bar{Q}_c(2S+1L_J) \) state \( \propto v^{2L} \), see Table 1.

The velocity scaling rules (VSR) of NRQCD were derived on the basis of consistency requirements of the NRQCD Lagrangian. These are certainly valid for \( \Lambda \) but \( \Lambda \) is still small with respect to \( mv \) for higher-twist corrections, \( \Lambda \) may range from \( \Lambda \) to \( 1 \) GeV. I consider two cases. First, \( mv^2 \) falls below \( \Lambda \), but \( \Lambda \) is still small with respect to \( mv \), i.e. \( \alpha_s(mv) \propto 1 \). Then the ordinary multipole expansion is still valid, but it is \( \alpha \) that cuts of soft gluons rather than the binding energy\(^1\), leading to a double expansion in \( v \) \( (\langle p \rangle /m) \) and \( \lambda = k/(\langle p \rangle) \), where \( k \sim \Lambda \) is the typical momentum of the dynamical (non-Coulombic) gluons. The scaling rules for this alternative scenario are also displayed in Table 1 (labelled VSR1). These can be argued in two ways. Either by considering the energy shift of the \( Q\bar{Q}_c \) Fock state à la BBL\(^2\) or from dimensional arguments. These tell us that electric and magnetic dipole transitions scale as \( \Gamma(E1) \propto g^2k^3r^2 \) and \( \Gamma(M1) \propto g^2k^3/m^2 \), respectively. Hence the probabilities \( \Gamma/k \) scale as \( \lambda^2 \) for \( E1 \) and as \( \lambda^2v^2 \) for \( M1 \) transitions owing to \( r^{-1} \sim mv \). For \( k \sim mv^2 \), the standard NRQCD scaling rules are reproduced.

A second set of velocity-scaling rules (VSR2 in Table 1) is arrived at when \( \alpha_s(mv) \) is not small enough to allow for a perturbative treatment. In non-relativistic QED, the power UV-divergent terms appearing in diagrams containing “soft” photons with energy \( k \sim mv \) can be cancelled perturbatively by corresponding terms in the bare coefficients\(^3\). This leaves only “ultrasoft” photons of energy \( k \sim mv^2 \) for which the usual counting rules of the multipole expansion hold, \( \sim v^2 \) for \( E1 \) and \( \sim v^4 \) for \( M1 \) transitions. However, for the counting rules of soft photons only factors of \( e \) and \( 1/m \) enter, so that \( \psi^\dagger(g/m)\vec{p} \cdot \vec{A} \psi \) and \( \psi^\dagger(g/m)\vec{\sigma} \cdot \vec{B} \psi \) contribute to the same order. Fock states that can be reached by both \( E1 \) and \( M1 \) transitions are suppressed by \( v^3 \). Since nothing can be said about the velocity scaling of the non-perturbative renormalization required in NRQCD for \( \alpha_s(mv) \) not small, \( v^3 \) remains the scaling of spin-flip transitions. Since the \( g\vec{A} \cdot \vec{p} \) term is not renormalized it still scales as \( v^2 \). These general findings are in agreement with the explicit estimate of

\(^1\) This possibility was suggested by Martin Beneke.
\(^2\) After completion of this work I learned about an erratum to Ref.\(^1\), in which spin-flip transitions
the scaling of $\langle O^8_{g/\psi}(1S_0) \rangle$ at the end of section 2, where it was found that this ME scales as $v^4$ ($v^3$) if gluons of energies $mv^2$ ($mv$) are important.

One should bear in mind that the velocity is not a fixed quantity for a given quarkonium system. The velocities of $1^{--}$ and $J^{++}$ states are, in general, different and such is the case for different radial excitations. Universal velocity-scaling rules are only meaningful as long as velocity differences are small compared to the actual velocities. It may even happen that the relation between $\Lambda$ and $mv$ or $mv^2$ differs from one state to the other.

In order to study these problems I calculate the velocity $v$, the momentum $p$ and the kinetic energy $T$ of the heavy quark for the lowest-lying $S$, $P$, and $D$-wave quarkonium states, for a class of potentials of the form $V(r) = \lambda r^\nu + V_0$. A power-like potential successfully describes quarkonium spectroscopy with $\nu \sim 0.1$. Uncertainties of the quark-potential model description can be investigated by studying the power-dependence within a reasonable range, say $\Delta \nu = \pm 0.2$. For every value of $\nu$ I use the $1S$ leptonic width and the $1S$ and $2S$ masses to fix the three remaining parameters $m$, $\lambda$, and $V_0$. Note that I normalize the leptonic width by the quarkonium mass rather than the heavy-quark mass and do not include the known $O(\alpha_s)$ correction.

The results given in Figs. 1 and 2 show that the velocities are rather uncertain: varying $\nu$ between $-0.1$ and $0.3$ changes, e.g. $v_{J/\psi}^2$ from 0.36 to 0.23. An additional uncertainty is caused by the $O(\alpha_s)$ correction $K = 1 - (16/3)\alpha_s/\pi$ to the leptonic width, which enters as $K^{2/3} \sim 0.63$ for charm and $\sim 0.76$ for bottom. It can also be seen that the difference in the $1S$ and $2S$ squared velocities is non-negligible, it can be as large as 20%. Moreover, for charmonia we observe $v^2 \sim v/2 \sim \alpha_s(m)$. The kinetic energy $T$ is about 350 MeV for both the charm and the bottom systems. For bottomonia, however, $v^2$ is small enough for the NRQCD relation $v \sim \alpha_s(mv)$ to be fulfilled. Hence, bottomonia seem to be heavy enough for the NRQCD velocity-scaling rules to hold, while the charm-quark mass is possibly too light for the notion of a single velocity to make sense and a universal scaling to hold for all charmonium states.

4 NRQCD matrix elements

The long-distance MEs can either be calculated using non-perturbative methods or extracted phenomenologically from data. In order to estimate the total $1^{--}$ production rates we need also total production MEs, which include the feed down from higher states. For $\Upsilon(1S)$ the total MEs are defined as follows:

$$\langle O^8_{g/\psi}(3S_1) \rangle_{tot} = \sum_{n=1}^{3} \left\{ \langle O^8_{g/\psi}(3S_1) \rangle \right\}$$

have also been assigned the probability $v^3$. I am indebted to M. Beneke for bringing this work to my attention.
\[
\sum_{J=0}^{2} \langle \mathcal{O}^\chi_{bJ}(nP) (^3S_1) \rangle \text{Br}[\chi_{bJ}(nP) \rightarrow \Upsilon(1S) X] \left( \langle \mathcal{O}_1^\Upsilon(1S) (^3S_1) \rangle \right)_{\text{tot}} = \sum_{n=1}^{3} \langle \mathcal{O}_1^\Upsilon(nS) (^3S_1) \rangle \text{Br}[\Upsilon(nS) \rightarrow \Upsilon(1S) X].
\]

The inclusive branching ratios needed for (8) and the other \( \Upsilon(nS) \) states are compiled in Table 2.

| \( \chi_{bJ}(3P) \) | \( \Upsilon(3S) \) | \( \Upsilon(2S) \) | \( \Upsilon(1S) \) |
|----------------|----------------|----------------|----------------|
| \( \chi_{b2}(3P) \) | 16.2 | 5.4 | 5.4 |
| \( \chi_{b1}(3P) \) | 21 | 7 | 7 |
| \( \chi_{b0}(3P) \) | 4.6 | 1.5 | 1.5 |

| \( \chi_{bJ}(2P) \) | \( \chi_{bJ}(2P) \) | \( \chi_{bJ}(2P) \) | \( \Upsilon(2S) \) | \( \Upsilon(1S) \) |
|----------------|----------------|----------------|----------------|----------------|
| \( \Upsilon(3S) \) | 11.4 \( \pm 0.8 \) | 11.3 \( \pm 0.6 \) | 5.4 \( \pm 0.6 \) | 10.6 \( \pm 0.8 \) | 11.7 \( \pm 0.5 \) |
| \( \chi_{b2}(2P) \) | 16.2 \( \pm 2.4 \) | 12. \( \pm 1.3 \) |
| \( \chi_{b1}(2P) \) | 21 \( \pm 4 \) | 15. \( \pm 1.9 \) |
| \( \chi_{b0}(2P) \) | 4.6 \( \pm 2.1 \) | 2.3 \( \pm 0.9 \) |

| \( \chi_{bJ}(1P) \) | \( \chi_{bJ}(1P) \) | \( \chi_{bJ}(1P) \) | \( \Upsilon(1S) \) |
|----------------|----------------|----------------|----------------|
| \( \Upsilon(2S) \) | 6.6 \( \pm 0.9 \) | 6.7 \( \pm 0.9 \) | 4.3 \( \pm 1.0 \) | 31.1 \( \pm 1.6 \) |
| \( \chi_{b2}(1P) \) | 22. \( \pm 4 \) |
| \( \chi_{b1}(1P) \) | 35 \( \pm 8 \) |
| \( \chi_{b0}(1P) \) | \(< 6 \) |

Table 2: Branching ratios (in per cent) in the \( b\bar{b} \) system; sequential decays have been included for the \( \Upsilon(3S) \), \( \Upsilon(2S) \), and \( \chi_{bJ}(2P) \) branching ratios; based on the PDG data\(^{[12]} \) and, for the \( \chi_{bJ}(3P) \), on Ref.\(^{[12]} \).

### 4.1 Non-perturbative methods

Since production MEs cannot be determined in lattice gauge theory one has to resort to less rigorous non-perturbative models. The leading-order \( (v) \) colour-singlet MEs can be estimated in the quark-potential model since they are simply related to the wave functions at the origin, see (7). Tables 3 and 4 list the values obtained\(^{[13]} \) from a QCD-motivated potential\(^{[14]} \). Uncertainties in the wave functions arise from several sources:
| $\langle O_1^{J/\psi}(^3S_1) \rangle$ | 1.16 GeV$^3$ | $\langle O_1^{J/\psi}(^3S_1) \rangle_{\text{tot}}$ | 1.59 GeV$^3$ |
| $\langle O_1^{\chi_c}(^3P_1) \rangle$ | 0.32 GeV$^5$ | $\langle O_1^{\psi}(^2S)(^3S_1) \rangle_{\text{tot}}$ | 0.76 GeV$^3$ |

Table 3: Colour-singlet long-distance matrix elements for charmonium production

| $\langle O_1^{Y(1S)}(^3S_1) \rangle$ | 9.28 GeV$^3$ | $\langle O_1^{Y(1S)}(^3S_1) \rangle_{\text{tot}}$ | 11.1 GeV$^3$ |
| $\langle O_1^{\chi_b}(^1P)(^3P_1) \rangle$ | 6.09 GeV$^5$ | $\langle O_1^{Y(2S)}(^3S_1) \rangle_{\text{tot}}$ | 5.01 GeV$^3$ |
| $\langle O_1^{\chi_b}(^2P)(^3P_1) \rangle$ | 7.10 GeV$^5$ | $\langle O_1^{Y(3S)}(^3S_1) \rangle_{\text{tot}}$ | 3.54 GeV$^3$ |
| $\langle O_1^{\chi_b}(^3P)(^3P_1) \rangle$ | 7.71 GeV$^5$ | $\langle O_1^{Y(3S)}(^3S_1) \rangle_{\text{tot}}$ | 3.54 GeV$^3$ |

Table 4: Colour-singlet long-distance matrix elements for bottomonium production

relativistic corrections $\propto v^2$, the form of the potential, and badly determined input parameters. Among the latter the $O(\alpha_s)$ correction to the leptonic width gives by far the biggest uncertainty: the factor $1-(16/3)\alpha_s/\pi$ can change the square of charmonium wave functions by as much as a factor of 2!

In order to assess the size of the other effects I calculate the wave functions for a power-like potential $V(r) \propto \lambda r^\nu$ and study them as functions of $\nu$. In this way we modify not only the form of the potential but also change the velocity $v$ and, hence, the size of relativistic corrections. The $1S$ leptonic width and the $1S$ and $2S$ masses are used as input. The results for $|R_{nl}(0)|^2 = dR_{nl}(r)/dr|_{r=0}$ are shown in Figs. 1 and 2. In the case of charmonia, the $nS$ to $1S$ ratio is rather stable but the ratio $nP$ to $1S$ is quite uncertain, by a factor of almost 2. This uncertainty is not much smaller for the $b$ system, in particular for the higher radial excitations. The $D$-wave wave functions are mostly uncertain. This underlines that the uncertainties of colour-singlet MEs must not be forgotten when estimating the ranges of theoretical predictions for quarkonium production.

Colour-octet production MEs have so far been estimated only phenomenologically. Here I present the first non-perturbative estimates obtained in two different models. One calculation of the colour-octet ME, in collaboration with P. Hoodbhoy, is based on the method of Leutwyler–Voloshin. Here colour-octet MEs are given in terms of the gluon condensate $\langle \alpha_s G^2 \rangle$. We obtain

$$\langle O_8^{J/\psi}(^1S_0) \rangle = 5.6 \times 10^{-3} \text{ GeV}^3 \left[ \frac{\langle \alpha_s G^2 \rangle}{0.1 \text{ GeV}^4} \frac{1.5 \text{ GeV}}{m} \frac{0.26}{\alpha_s} \right]$$
\( n = 1 \)

| Condensate \[13\] | CEM | Tevatron \[17\] | Condensate \[13\] | CEM | Tevatron \[17\] |
|-----------------|-----|---------------|-----------------|-----|---------------|
| \( \langle O^\psi_{s(nS)} (3 S_1) \rangle \) | 14.  | 6.6           | 3.4             | 4.6 |
| \( \langle O^\psi_{s(nS)} (3 S_1) \rangle \) \( \text{tot} \) | 24.  | 14.           | 3.4             | 4.6 |
| \( \langle O^\psi_{s(nS)} (1 S_0) \rangle \) | 5.6  | 2.6           | 33.             | 0.5 | 0.62          | 8.8 |
| \( \langle O^\psi(nS) (3 P_0) \rangle \) | -0.2 | 1.2           | 24.             | 0.29| 6.5           |
| \( \langle O^\chi_{c1}(3 S_1) \rangle \) | 14.  | 9.8           |                 |     |               |

Table 5: Colour-octet long-distance matrix elements \( \langle O^H_{s(2S+1 L_J)} \rangle \) for charmonium production in units of \( 10^{-3} \text{ GeV}^{3+2L} \).

\[
\langle O^J/\psi (3 P_0) \rangle = -\frac{12}{625} m^2 \langle O^J/\psi (1 S_0) \rangle .
\] (9)

Note that \( \langle O^J/\psi (3 P_0) \rangle \) is small and, as a consequence of \( \langle B^2 \rangle = -\langle E^2 \rangle \), negative.

Long-distance MEs can also be estimated by comparing NRQCD calculations with CEM ones. Leaving the details to a further publication I simply quote the results

\[
\langle O^H(3 S_1) \rangle \big|_{\text{dir}} = F[H] \big|_{\text{dir}} \frac{4}{\pi^2} \left( 2 m_Q \bar{\Lambda} + \bar{\Lambda}^2 \right)^{3/2}
\]

\[
\langle O^H(1 S_0) \rangle = \frac{1}{3 + 9 v^2} \langle O^H(3 S_1) \rangle \big|_{\text{dir}}
\]

\[
\langle O^H(3 P_0) \rangle = m^2 v^2 \langle O^H(1 S_0) \rangle .
\] (10)

Here \( \bar{\Lambda} = m(H_Q) - m_Q \) is the difference between the mass of the heavy quark and that of the lightest meson containing it. The long-distance factors \( F[H] \) can be extracted from the ratios of the production rates for the various \( \Upsilon(nS) (\psi(2S), J/\psi) \) states known from fixed-target experiments together with spin symmetry and an ansatz for direct \( \Upsilon(nS) \) production, which relates these to the respective leptonic widths. The results are given in Tables 3 and 4. In the case of bottomonia, I present my estimates as a function of the size of the cross section of the as yet unobserved \( 3 P \chi_{b1} \) states. The models CEM0, CEM1, and CEM100 are defined by \( R_{3P} = 0, 1, \) and \( 100, \) respectively, where

\[
R_{3P} = \frac{\sigma[\chi_{b1}(3 P)]}{\sigma_{\text{dir}}[\Upsilon(3 S)]} .
\] (11)

We consider CEM1 as the most sensible choice.
Table 6: Colour-octet long-distance matrix elements $\langle O^{H(2S+1L,J)}_8 \rangle$ for bottomonium production in units of $10^{-3}$ GeV$^{3+2L}$.

## 4.2 Phenomenological determination of MEs

Tables 5 and 6 quote also the values of colour-octet MEs determined in a study of quarkonium production at the Tevatron [17]. The usual assumption $m^2\langle \mathcal{O}^{H(1S_0)}_8 \rangle = 3\langle \mathcal{O}^{H(3P_0)}_8 \rangle$ is made and, for bottomonium, I have identified the 3$L$ MEs with the corresponding 2$L$ ones ($L = S, P$).

The values of phenomenological determinations of (in particular colour-octet) MEs should be considered with great care. Numbers found in different processes may deviate by large factors and, yet, be consistent if proper error estimates were indicated. Uncertainties arise from several sources. First, a substantial uncertainty is caused by the large value of $\alpha_s(\mu)$, which enters the various processes with different powers. Realistic error estimates should take into account the variations of $\Lambda_{\text{QCD}}$ and the scale $\mu \sim \sqrt{m^2 + p_T^2}$. 

| Condensate $\Sigma^T_{3S_1}$ | CEM0 | CEM1 | CEM100 | Tevatron |
|-----------------------------|------|------|--------|----------|
| $\langle O^{T(1S)}_8 (3S_1) \rangle$ | 144. | 94.  | 2.8    | 5.9      |
| $\langle O^{T(1S)}_8 (3S_1) \rangle_{\text{tot}}$ | 228. | 228. | 228.   | 480.     |
| $\langle O^{T(1S)}_8 (1S_0) \rangle$ | 2.3  | 41.  | 0.80   | 20.      |
| $\langle O^{T(1S)}_8 (3P_0) \rangle$ | -1.1 | 55.  | 1.1    | 94.      |
| $\langle O^{\chi_b(1P)}_8 (3S_1) \rangle$ | 24.  | 79.  | 189.   | 420.     |
| $\langle O^{T(2S)}_8 (3S_1) \rangle$ | 55.  | 36.  | 1.0    | 4.1      |
| $\langle O^{T(2S)}_8 (3S_1) \rangle_{\text{tot}}$ | 119. | 119. | 119.   | 220.     |
| $\langle O^{T(2S)}_8 (1S_0) \rangle$ | 15.  | 10.  | 0.29   | 23.      |
| $\langle O^{T(2S)}_8 (3P_0) \rangle$ | 21.  | 14.  | 0.40   | 110.     |
| $\langle O^{\chi_b(2P)}_8 (3S_1) \rangle$ | 129. | 159. | 219.   | 320.     |
| $\langle O^{T(3S)}_8 (3S_1) \rangle$ | 39.  | 26.  | 0.75   | 4.1      |
| $\langle O^{T(3S)}_8 (3S_1) \rangle_{\text{tot}}$ | 39.  | 39.  | 39.    | 160.     |
| $\langle O^{T(3S)}_8 (1S_0) \rangle$ | 11.  | 7.3  | 0.21   | 23.      |
| $\langle O^{T(3S)}_8 (3P_0) \rangle$ | 15.  | 9.9  | 0.28   | 110.     |
| $\langle O^{\chi_b(3P)}_8 (3S_1) \rangle$ | 0.   | 25.  | 75.    | 320.     |
| $\langle \sum O^{T(3S_1)}_8 \rangle$ | 387. | 387. | 387.   | 850.     |
A second source of uncertainty is introduced by the parton distribution functions (PDFs). Quark and gluon distribution functions enter in different combinations, at different scales, and in different \( x \) regions. Hence, comparisons of \( B \) decays into charmonium (independent of PDFs), photoproduction of \( J/\psi \) at low energies (dominated by single parton-distributions at large \( x \)), and \( J/\psi \) production at the Tevatron (governed by the product of two gluon-distributions at small \( x \)) are highly non-trivial.

Third, essentially all analyses are leading order in both \( v \) and \( \alpha_s(m^2) \) but, the size of higher-order corrections in \( v \) and/or \( \alpha_s \) may well be process-dependent. Moreover, it should be stressed that existing calculations are separately of leading order in \( v \) for the colour-singlet and colour-octet contributions, i.e. do not consistently include all terms of a given order in \( v \).

Last but not least, there exist higher-twist corrections, which enter at different levels. Examples are diffractive contributions to the \( z \to 1 \) behaviour of \( J/\psi \) photoproduction \( \propto \Lambda^2/(1-z) m_c^2 \), \( \Lambda/(1-x_F) m_c^2 \) effects at large-\( x_F \) charmonium production, and effects of intrinsic \( k_T \) at low-\( p_T \) quarkonium production at the Tevatron.

5 Discussion

We start by comparing the values of the charmonium colour-octet MEs extracted from the Tevatron with the non-perturbative results. The MEs \( \langle \mathcal{O}_8^H(\bar{3}S_1) \rangle \) come out quite similar for \( H = J/\psi, \psi(2S) \), and \( \chi_{cJ} \). This is not surprising since ratios of production cross sections for these states \( H \) are rather universal\cite{18} and, hence, the MEs are well constrained. On the other hand, where there are less data, differences show up: the Tevatron values for \( \langle \mathcal{O}_8^H(1S_0, \bar{3}P_0) \rangle \) are considerably larger than the non-perturbative ones.

Turning now to bottomonia, we find similar values for \( \langle \mathcal{O}_8^X(1S_0) \rangle \) if \( R_{3P} \) (see (11)) is close to our preferred value \( \sim 1 \). This also holds for the actual combination extracted from the Tevatron, \( \langle \mathcal{O}_8^X(1S_0) \rangle/3 + \langle \mathcal{O}_8^X(\bar{3}P_0) \rangle/m_b^2 \), which was found\cite{17} to be 7.9, to be compared with 16, 10, 0.31 (\( \times 10^{-3} \) GeV\(^3\)) for \( R_{3P} = 0, 1, 100 \), respectively.

For the choice \( R_{3P} = 1 \), however, the Tevatron value for \( \langle \mathcal{O}_8(\bar{3}S_1) \rangle \) is much larger than the CEM1 one. On the other hand, the situation is reversed for \( \langle \mathcal{O}_8^\top(\bar{3}S_1) \rangle \)\_dir so that, in fact, the values for \( \langle \mathcal{O}_8^\top(\bar{3}S_1) \rangle \)\_tot are rather similar. Again not a surprise, given that so far only total (direct plus indirect) \( \Upsilon(nS) \) cross sections have been measured. Since the Tevatron values correspond to very large production cross sections for the as yet unobserved \( \chi_{bJ}(3P) \) states, it is likely that the \( \bar{3}S_1 \) octet MEs for \( \chi_{bJ}(nP) \) are overestimated while the ones for \( \Upsilon(nS) \) are underestimated.

Finally, I consider prompt quarkonium production from \( Z^0 \) decays at LEP and calculate branching ratios in both NRQCD and the CEM. I improve previous calculations in the CEM\cite{19, 20, 21} and the CSM\cite{22, 23, 24, 6}/NRQCD\cite{25} by carefully including in the \( J/\psi \) and \( \sum \Upsilon \) cross sections the feed down from higher states. Moreover, branching ratios for various other states are also presented. Also the CEM prediction for the \( (Q\overline{Q})_1 + Q\overline{Q} \)
The final state is calculated here for the first time. The results (Table 7) are obtained with the help of the MEs given in Tables 3–6. For direct $\Upsilon(1S)$ production, estimates corresponding to $R_{3P} = 0, 1, \text{ and } 100$ are given.

It can be seen that NRQCD and CEM predictions are rather similar for the colour-octet processes, but differ substantially for the colour-singlet one. Both predictions are compatible with the data. Concerning the NRQCD predictions: one can see that the ratio of colour-singlet to colour-octet production decreases when going from $J/\psi$ to the sum of the $\Upsilon$ states. This is in contrast with the expectation, since the octet contribution scales as $v^4 \ln^2 \xi$ ($\sqrt{\xi} = 2m_Q/\sqrt{s}$) and both factors are larger for charmonium. This supports the observation made above that the colour-octet MEs of the $P$-wave bottomonium states as obtained[17] from the Tevatron are overestimated.

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Figure 1: Average squared velocity, momentum, and kinetic energy of the charm quark as functions of $\nu$ for charmonium states: $S$-waves (solid lines), $P$-waves (dashed lines), $D$-waves (dotted lines). Also shown is $|R_{nl}^{(l)}(0)|^2 = d^3R_{nl}(r)/dr^3|_{r=0}$. 
Figure 2: Same as Fig. 1 but for bottomonia.