Complete Integrability of a New Class of Hamiltonian Hydrodynamic Type Systems

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Received September 1, 2021; Revised September 15, 2021; Accepted September 15, 2021

Abstract. In this paper, we consider a new class of Hamiltonian hydrodynamic type systems whose conservation laws are polynomial with respect to one of the field variables.

DOI: 10.1134/S1061920821040099

1. INTRODUCTION

The theory of Hamiltonian hydrodynamic type systems

\[ u_t^k = \eta^{km} \left( \frac{\partial h}{\partial u^m} \right)_x, \quad k, m = 1, 2, \ldots, N, \]

was initiated by the seminal paper [1] written by B. A. Dubrovin and S. P. Novikov in 1983. Here the matrix \( \eta^{ik} \) is constant, while a Hamiltonian density \( h(u) \) is an arbitrary function of \( N \) field variables \( u^k \) only. If \( N = 2 \), this is a “classical” case, because such systems are “linearizable” by a hodograph method. Here “linearization” means a transformation of a quasilinear system to a linear system with variable coefficients. In this paper, we are going to consider the special Hamiltonian case

\[ \begin{align*}
    u_t &= \left( \frac{\partial h}{\partial v} \right)_x, \\
    v_t &= \left( \frac{\partial h}{\partial u} \right)_x,
\end{align*} \tag{1} \]

where the Hamiltonian density \( h \) depends on one of the field variables \( v \) quadratically, i.e.,

\[ h(u, v) = A(u)v^2 + B(u)v + C(u). \tag{2} \]

Here \( A(u), B(u), \) and \( C(u) \) are arbitrary analytic functions. We show that such Hamiltonian systems possess infinitely many particular solutions, which can be found by integrating linear ordinary differential equations of second order only.

The class of Hamiltonian densities (2) under consideration is meaningful, since it contains some physically relevant examples which will be discussed in subsequent sections:

- the shallow water equations with densities
  \[ h_{ESWE} = \frac{1}{2} uv^2 + C(u), \quad h_{LSWE} = \frac{1}{2} v^2 + C(u) \]
  corresponding to Eulerian and Lagrangian coordinates, respectively;

- the two-layer non-Boussinesq shallow water equations [2, 3] with density
  \[ h_{TSWE} = -\frac{1}{4} \frac{(1 - u^2)}{(1 - ru)} v^2 - \frac{1}{4} u^2, \]
  where \( r = \text{const} \) is the Boussinesq parameter;

- the system describing motions of liquid layer on a rotating cylinder [4]:
  \[ h_{LLRC} = \frac{1}{2} v^2 \ln u + \frac{1}{2} v \Omega u \ln u - \frac{1}{8} \Omega^2 u^2 \ln u + \frac{1}{4} \Omega^2 u^2, \]
  where \( \Omega \) stands for constant vorticity.
Since any two-component hydrodynamic type system is linearizable by the hodograph method, it follows that the Hamiltonian system (1) possesses infinitely many conservation laws \((f(u, v))_t = (g(u, v))_x\), where conservation law densities \(f(u, v)\) satisfy a second-order linear equation
\[
 h_{uu}f_{vv} = h_{vu}f_{uv},
\]  
which general solution is parameterized by two arbitrary functions of a single variable. However, this general solution is impossible to find in explicit form for an arbitrary function \(h(u, v)\). For this reason, the possibility to construct infinitely many particular solutions is also very important. The first nontrivial case was found by Ya. Nutku and P. Olver (see [5]). They selected the so-called separable class
\[
p(v)f_{vv} = q(u)f_{uu},
\]  
which is connected with two types of Hamiltonian densities, \(h = A(u)B(v)\) and \(h = A(u) + B(v)\). Later Ya. Nutku and P. Olver formulated the question “how many results presented for the separable class can be generalized to the nonseparable classes?”

In this paper, we present a first example belonging to nonseparable classes selected by the choice of Hamiltonian density (2). Then the linear equation (3) reduces to the form
\[
(A''(u)v^2 + B''(u)v + C''(u))f_{vv} = 2A(u)f_{uu}.
\]  
Our claim is that this linear PDE has infinitely many particular polynomial solutions with respect to an independent variable \(v\), selected by infinitely many corresponding second-order ODEs. This relies on the following simple observation. Any linear equation (3) has two particular polynomial solutions of first degree \(f = u\) and \(f = v\). Now consider the ansatz
\[
f = G_{2,2}(u)v^2 + G_{2,1}(u)v + G_{2,0}(u);
\]  
then (4) decomposes into three ODEs for the functions \(G_{2,i}\):
\[
AG''_{2,2} = A''G_{2,2}, \quad AG''_{2,1} = B''G_{2,2}, \quad AG''_{2,0} = C''G_{2,2},
\]  
which can be integrated immediately, giving two particular solutions, and one of them is \(f = h\). Then let us take a cubic polynomial:
\[
f = G_{3,3}(u)v^3 + G_{3,2}(u)v^2 + G_{3,1}(u)v + G_{3,0}(u).
\]  
The corresponding ODE’s are
\[
AG''_{3,3} = 3A''G_{3,3}, \quad AG''_{3,2} = A''G_{3,2} + 3B''G_{3,3}, \quad AG''_{3,1} = B''G_{3,2} + 3C''G_{3,3}, \quad AG''_{3,0} = C''G_{3,2}.
\]  
In the same way, substituting fourth order conservation law densities
\[
f = G_{4,4}(u)v^4 + G_{4,3}(u)v^3 + G_{4,2}(u)v^2 + G_{4,1}(u)v + G_{4,0}(u)
\]  
into (4) leads to the system
\[
AG''_{4,4} = 6A''G_{4,4}, \quad AG''_{4,3} = 3A''G_{4,3} + 6B''G_{4,4}, \quad AG''_{4,2} = A''G_{4,2} + 3B''G_{4,3} + 6C''G_{4,4}, \quad AG''_{4,1} = B''G_{4,2} + 3C''G_{4,3}, \quad AG''_{4,0} = C''G_{4,2}.
\]  
Thus, one can see that the first ODE in (5), (6), and (7) is the same up to the factor of \(A''\). It can also be seen that the second ODE in (6) and (7) has the same homogeneous part as the first one in (5) and (6), respectively. This means that, once the first equation in (5), (6), and (7) is solved, one can integrate all remaining ODE’s in quadratures.

A further investigation of other higher polynomial conservation law densities (with respect to an independent variable \(v\)) shows that, if the first linear ODE (here \(k\) is a positive integer determining the degree of a corresponding polynomial conservation law density with respect to an independent variable \(v\))
\[
AG''_{k,k} = \frac{1}{2}k(k-1)A''G_{k,k}
\]  
can be solved, then the other coefficients \(G_{k,i}\) for \(i < k\) can be found in quadratures. Below we describe our construction in detail.
2. INTEGRABILITY AND COMPLETE INTEGRABILITY

The quasilinear system (1), under the hodograph transform \((u, v) \leftrightarrow (x, t)\), becomes a linear system

\[
x_v = t_u h_{vv} - t_v h_{uv}, \quad x_u = t_u h_{uu} - t_u h_{uv},
\]

(8)

with variable coefficients \(h_{uu}, h_{uv}, h_{vv}\). Eliminating \(x\), one can obtain a linear equation of second order

\[
2t_u h_{uvv} - 2t_v h_{uuv} + t_uh_{vv} - t_vh_{vu} = 0,
\]

(9)

with variable coefficients \(h_{uu}, h_{vv}, h_{uuv}, h_{uvv}\).

The Tsarev generalized hodograph method (see [6]) is based on the concept of commuting flows. This means that the Hamiltonian system (1) possesses infinitely many commuting flows

\[
u_y = \left( \frac{\partial f}{\partial v} \right)_x, \quad v_y = \left( \frac{\partial f}{\partial u} \right)_x,
\]

where \(y\) is the so-called group parameter and the Hamiltonian density \(f(u, v)\) cannot be arbitrary. Indeed, the compatibility conditions \(u_y = (u_t)_y, \ (v_y) = (v_t)_y\) lead to a single second-order linear equation

\[
h_{uu} f_{vv} = h_{vv} f_{uu}.
\]

(10)

In this case, the solutions of the linear equations (9) and (10) are connected by equivalent substitutions

\[
t = \frac{f_{vv}}{h_{uu}} = \frac{f_{uu}}{h_{uv}}.
\]

(11)

Then the dependence \(x(u, v)\) can be found by quadratures from (8):

\[
x = f_{uv} - th_{uv}.
\]

(12)

Thus, in this paper, we consider equation (10) which is simple than (9). As was mentioned in the previous section, we have a second-order linear hyperbolic equation whose general solution depends on two arbitrary functions of a single variable. Now we define our concept of solvability:

Equation (10) is said to be solvable if it possesses infinitely many particular solutions each of which can be found from the corresponding linear ordinary differential equation of second order.

Assume that we are able to construct a general solution parameterized by two arbitrary functions of a single variable. Then we can expand these two functions in a Taylor series. This means that we are able to construct two infinite series of particular solutions. Vice versa, if we are able to construct two infinite series of particular solutions, we expect that they can be combined into a general solution parameterized by two arbitrary functions of a single variable. Thus, here we give our definition of completeness:

Equation (10) is said to be completely solvable if it possesses two infinite series of particular solutions each of which can be found from the corresponding linear ordinary differential equation of second order.

According to our definition, equation (10) is completely solvable when the function \(h\) is of the form

\[
h(u, v) = A(u)v^2 + B(u)v + C(u)
\]

with arbitrary analytic functions \(A, B\) and \(C\). To prove this statement, one must look for solution of the linear equation (10) as a polynomial with arbitrary degree \(k\):

\[
f_k(u, v) = G_{k,k}(u)v^k + \cdots + G_{k,1}(u)v + G_{k,0}(u).
\]

(13)

Substituting this formula into (10), we obtain the following system of inhomogeneous second-order ordinary differential equation for the coefficients \(G_{k,i}(u)\):

\[
2AG_{k,k}'' - k(k - 1)A''G_{k,k} = 0,
\]

\[
2AG_{k,k-1}'' - (k - 1)(k - 2)A''G_{k,k-1} = k(k - 1)B''G_{k,k},
\]

\[
2AG_{k,k-2}'' - (k - 2)(k - 3)A''G_{k,k-2} = (k - 1)(k - 2)B''G_{k,k-1} + k(k - 1)C''G_{k,k},
\]

\[
2AG_{k,k-s}'' - (k - s)(k - s - 1)A''G_{k,k-s} = (k - s)(k - s + 1)B''G_{k,k-s+1}
\]

\[+ (k - s + 1)(k - s + 2)C''G_{k,k-s+2},\]

(14)
where \( s = 3, \ldots, k \). Consider the first equation of system (14):

\[
2AG''_{k,k} - k(k - 1)A''G_{k,k} = 0.
\]

Suppose that the analytic function \( A(u) \) satisfies the equality

\[
\frac{A''(u)}{A(u)} = \frac{a_{-2}}{(u - u_0)^2} + \frac{a_{-1}}{(u - u_0)} + a_0 + a_1(u - u_0) + \cdots
\]

in some neighborhood of the point \( u_0 \). This means that the point \( u_0 \) is weakly singular (or regular when \( a_{-2} = a_{-1} = 0 \)) for equation (15). According to the general analytical theory of ODEs [7], there exists a particular solution which can be represented as a convergent series

\[
G_{k,k}^1(u) = (u - u_0)^\gamma \sum_{m=1}^{\infty} g_m(u-u_0)^m,
\]

where the exponent \( \gamma \) is defined as the bigger root of the characteristic equation

\[
a_{-2} - \gamma(\gamma - 1) = 0.
\]

The second linearly independent solution \( G_{k,k}^2(u) \) is also represented as a convergent series, and its form depends on properties of the parameter \( \gamma \) (see details in [7]). Thus, having obtained the solution of equation (15) in the form

\[
G_{k,k}(u) = c_1 G_{k,k}^1(u) + c_2 G_{k,k}^2(u),
\]

one can easily construct the solution of whole system (14),

\[
AG''_{k,0} = C''G_{k,2}, \quad AG''_{k,1} = B''G_{k,2} + 3C''G_{k,3}.
\]

3. TWO-LAYER NON-BOUSSINESQ SHALLOW WATER SYSTEM

Consider the following system

\[
\xi_t = -\left\{ \frac{(h^2 - \xi^2)\sigma}{(\rho_2 + \rho_1)h - (\rho_2 - \rho_1)\xi} \right\}_x, \tag{16}
\]

\[
\sigma_t = -\frac{1}{2} \left\{ \frac{\rho_2(h - \xi)^2 - \rho_1(h + \xi)^2}{((\rho_2 + \rho_1)h - (\rho_2 - \rho_1)\xi)^2} \sigma^2 + g(\rho_2 - \rho_1)\xi \right\}_x,
\]

arising in describing long small-amplitude nonlinear waves in incompressible two-layered flows in an infinite channel [2, 3]. Here \( \rho_{1,2} \) stands for the density in the upper and lower layer, correspondingly, \( h \) is the total height of the channel, \( \xi \) is the relative thickness (i.e., the difference between layer thicknesses), and \( \sigma \) is the momentum shear. In dimensionless variables (see details in [3]), the system (16) takes the form (1) with the Hamiltonian density

\[
h(u, v; r) = -\frac{1}{4} \frac{(1 - u^2)}{(1 - ru)} v^2 - \frac{1}{4} u^2, \tag{17}
\]

where \( r = (\rho_2 - \rho_1)/(\rho_2 + \rho_1) \) is the Boussinesq parameter.

There are two limit cases:

- **the first case**, as \( r \to 1 \), corresponds to \( \rho_1 \to 0 \), i.e., the zero density in the upper layer. Therefore, the interface can be regarded as a free surface, and our system transforms to the shallow water model up to point transformation. Indeed, as \( r \to 1 \), the Hamiltonian density (17) becomes

\[
h(u, v; 1) = -\frac{1}{4}(1 + u)v^2 - \frac{1}{4} u^2,
\]

which, upon the substitution \( u = 2U - 1, v = V \), gives the shallow water system. The corresponding equation (10) takes the form

\[
f_{VV} - Uf_{UU} = 0,
\]

which has infinitely many polynomial solutions (see [8]).
the second case, as \( r \to 0 \), related to the so-called Boussinesq approximation, means that the densities in both layers nearly coincide. In terms of the parameter \( r \), the relation \( \rho_1 \approx \rho_2 \) corresponds to \( r \to 0 \). Surprisingly, this case is also related to the shallow water system of equations. As noted in [2] (see also [9]), the system (1) with the density

\[
h(u, v; 0) = -\frac{1}{4}(1 - u^2)v^2 - \frac{1}{4}u^2,
\]

coincides, under the mapping

\[
\eta = (1 - u^2)(1 - v^2), \quad \zeta = 2uv,
\]

with the shallow water equations written for the field variables \( \eta \) and \( \zeta \). Thus, the solutions of two-layer non-Boussinesq shallow water system are, in some sense, deformations of those constructed for ordinary shallow water equations [3].

Turning back to the Hamiltonian density (17), we have explicit expressions for the functions \( A, B, \) and \( C \):

\[
A(u) = -\frac{1}{4}(1 - u^2), \quad B(u) \equiv 0, \quad C(u) = -\frac{1}{4}u^2.
\]

Hence, the system (14) determining coefficients of the function \( f_k \) in (13) becomes

\[
\begin{align*}
G''_{k,k} - \frac{k(k - 1)(1 - r^2)}{(1 - u^2)(1 - ru)^2} G_{k,k} &= 0, \\
G''_{k,k-1} - \frac{(k - 1)(k - 2)(1 - r^2)}{(1 - u^2)(1 - ru)^2} G_{k,k-1} &= 0, \\
G''_{k,k-s} - \frac{(k - s)(k - s - 1)(1 - r^2)}{(1 - u^2)(1 - ru)^2} G_{k,k-s} &= \frac{(k - s + 1)(k - s + 2)(1 - ru)}{1 - u^2} G_{k,k-s+2},
\end{align*}
\]

with \( s = 2, \ldots, k \). In this case, the general solution to the first equation of the system (18) can be written in a closed form:

\[
G_{k,k} = c_1 G^{1}_{k,k} + c_2 G^{2}_{k,k}, \quad \text{with}
\]

\[
G^{1}_{k,k} = \frac{(1 - u^2)}{(1 - ru)^{k-1}} ( u^{k-2} + R_{k,k-3} u^{k-3} + \cdots + R_{k,1} u + R_{k,0}),
\]

\[
G^{2}_{k,k} = \frac{Q_{k,k-1} u^{k-1} + Q_{k,k-2} u^{k-2} + \cdots + Q_{k,1} u + Q_{k,0}}{(1 - ru)^{k-1}} + \Psi_k G^{1}_{k,k} \ln \left( \frac{1 - u}{1 + u} \right),
\]

where all coefficients \( R_{k,i}, Q_{k,i}, \) and \( \Psi_k \) can be determined uniquely under the substitution of (19) in (18). One can similarly obtain a general solution to the second equation in (18). Then all right-hand sides in the rest of equations are completely determined, and one can construct a general solution of system (18).

Looking at expressions in (19), one can conclude that the complexity of formulas increases as \( k \) grows.

For example, in the case of \( k = 3 \), we have

\[
f_3(u, v) = G_{3,3}(u)v^3 + G_{3,2}(u)v^2 + G_{3,1}(u)v + G_{3,0}(u),
\]

where

\[
G_{3,3} = \frac{c_{31}}{2} \frac{\Gamma^1_{3,3}}{(1 - ru)^2} + \frac{c_{22}}{2} \left\{ \frac{\Gamma^2_{3,3}}{(1 - ru)^2} - 3\frac{\Gamma^3_{3,3}}{(1 - ru)^2} \ln \left( \frac{1 - u}{1 + u} \right) \right\},
\]

\[
G_{3,2} = c_{21} \left\{ \frac{(1 - u^2)}{(1 - ru)} \right\} + \frac{c_{22}}{2} \left\{ \frac{\Gamma^1_{3,2}}{(1 - ru)} + \frac{\Gamma^2_{3,2}}{(1 - ru)} \ln \left( \frac{1 - u}{1 + u} \right) \right\},
\]

\[
G_{3,0} = c_{01} + c_{02} u + c_{21} u^2 + c_{22} \left\{ \frac{u}{2}(1 + 3r^2) + \Gamma^1_{3,0} \ln (1 - u) + \Gamma^2_{3,0} \ln (1 + u) \right\}.
\]
and
\[ G_{3,1} = c_{11} + c_{12}u + \frac{c_{31}}{r^3} \left( \Gamma_{3,1}^1 \ln(1 - ru) + \Gamma_{3,1}^2 \right) - \frac{3c_{32}}{4r^3} \Gamma_{3,1}^3 \left\{ \text{Li}_2 \left( \frac{r(1 - u)}{r - 1} \right) - \text{Li}_2 \left( \frac{r(1 + u)}{r + 1} \right) \right\} \]
\[ - \frac{3c_{32}}{4r^3} \Gamma_{3,1}^4 \ln(1 + u) \ln \left( \frac{1 - ru}{1 + r} \right) - \ln(1 - u) \ln \left( \frac{1 - ru}{1 - r} \right) \]
\[ - \frac{3c_{32}}{4r^3} \left( \Gamma_{3,1}^5 + \Gamma_{3,1}^6 \ln(1 - u) + \Gamma_{3,1}^7 \ln(1 + u) + \Gamma_{3,1}^8 \ln(1 - ru) \right) . \]

Here \( \text{Li}_2(z) \) stands for the dilogarithm or Spence’s function defined by
\[ \text{Li}_2(z) = -\int_0^z \frac{\ln(1 - t)}{t} dt \]
for a complex variable \( z \); the functions \( \Gamma_{j,k}^i \) are polynomials in \( u \) with coefficients polynomial in \( r \). All \( \Gamma_{j,k}^i \) can be found in explicit form by substituting \( G_{3,1} \) into system (18) and are listed in Appendix A.

Thus, one obtains a solution in implicit form by formulas (11) and (12) using expression (20):
\[ t = 4c_{21} + c_{22} \left\{ \frac{T_{3,1}}{1 - u^2} + T_{3,2} \ln(1 - u) + T_{3,3} \ln(1 + u) \right\} \]
\[ + c_{31} \left\{ \frac{12(u - r)v}{1 - ru} \right\} + c_{32} \left\{ \frac{T_{3,4}}{(1 - ru)(1 - u^2)} + \frac{T_{3,5}}{1 - ru} \ln \left( \frac{1 - u}{1 + u} \right) \right\} , \]
\[ x = 2c_{22} \left\{ \frac{v(1 - ru)}{1 - u^2} \right\} + c_{31} \left\{ \frac{T_{3,1}}{r(1 - u^2)} + \frac{T_{3,2}}{r^2} \ln(1 - ru) \right\} \]
\[ + c_{12} + \frac{3c_{32}}{2r^2} X_{3,3} \left\{ \ln(1 + u) \ln \left( \frac{1 - ru}{1 + r} \right) - \ln(1 - u) \ln \left( \frac{1 - ru}{1 - r} \right) \right\} \]
\[ + \frac{3c_{32}}{2r^2} X_{3,4} \ln(1 - ru) + \frac{X_{3,5}}{(1 - ru)^2} \ln \left( \frac{1 - u}{1 + u} \right) + \frac{X_{3,6}}{(u^2 - 1)(1 - ru)^2} \]
\[ + \frac{3c_{32}}{2r^2} X_{3,7} \left\{ \text{Li}_2 \left( \frac{r(1 - u)}{r - 1} \right) - \text{Li}_2 \left( \frac{r(1 + u)}{r + 1} \right) \right\} + \frac{3c_{32}}{2r^2} X_{3,8} \ln \left( \frac{(1 + u)(1 - u)}{1 + u} \right) , \]
where \( T_{i,j} \) and \( X_{i,j} \) are polynomials in \( u \) and \( v \) with coefficients polynomial in \( r \) (see Appendix A). Here it should also be mentioned that the terms with the coefficients \( c_{21} \) and \( c_{22} \) form the solution for \( k = 2 \). Therefore, the solution for every \( k \) contains already found solutions for lower values of \( k \).

In the theory of stratified fluid dynamics, the Boussinesq approximation is relevant from the physical point of view [9]. Then, for \( r \ll 1 \), the leading order term of (21), (22) has the form
\[ t = 4c_{21} + c_{22} \left\{ \frac{2u}{1 - u^2} + 2 \ln \left( \frac{1 - u}{1 + u} \right) \right\} + 12c_{31}uv + 6c_{32}v \left\{ \frac{2 - 3u^2}{u^2 - 1} + 3u \ln \left( \frac{1 - u}{1 + u} \right) \right\} , \]
\[ x = c_{12} + c_{22} \left\{ \frac{2v}{u^2 - 1} \right\} - 3c_{31} \left\{ u^2(1 + u^2) + u^2 \right\} \]
\[ + \frac{3}{2} c_{32} \left\{ \frac{3u + u^2 + 3u^3(1 + u^2)}{u^2 - 1} - 1 - 4u^2 - 3v^2 + 3u^4(1 + v^2) \ln \left( \frac{1 - u}{1 + u} \right) \right\} . \]

4. LIQUID LAYER ON THE ROTATING CYLINDER SURFACE

Now consider the system
\[ S_t = -\text{\{II ln S\}_x} , \quad \Pi_t = -\text{\{II^2/2S - 1/2Ω^2S - Ω\Pi ln S\}_x} , \quad S = R^2 , \] (23)
where \( R \) is the free surface equation; \( \Pi \) is the velocity circulation, and \( Ω \) is the vorticity constant. System (23) arises in describing a thin liquid layer evolution on the rotating infinite cylinder [4]. This case corresponds to the Hamiltonian density
\[ h(u,v) = -\frac{1}{2} v^2 \ln u + \frac{1}{2} v \Omega u \ln u - \frac{1}{8} Ω^2 u^2 \ln u + \frac{1}{4} Ω^2 u^2 \]
with \( u = S \) and \( v = Π + ΩS/2 \). As for the two-layer shallow water system, there are two limit cases [4]:

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• the first case, \( \Omega \to 0 \), corresponds to nonrotating cylinder. For the reduced system
\[
S_t = -\left\{ \Pi \ln S \right\}_x, \quad \Pi_t = -\left\{ \frac{\Pi^2}{2S} \right\}_x,
\]
one is able to calculate the Riemann invariants in an explicit form:
\[
r_1 = \frac{\ln \Pi + \sqrt{-2\ln S}}{2}, \quad r_2 = \frac{\ln \Pi - \sqrt{-2\ln S}}{2}.
\]
• the second case, when \( \Omega \gg 1 \), is related to the so-called “overturned” shallow water system. Suppose that the free surface variable \( S \) and the parameter \( \Omega \) admit the following asymptotic representation:
\[
S = 1 + \varepsilon \Phi + O(\varepsilon^2), \quad \Omega = \sqrt{2/\varepsilon},
\]
where \( \varepsilon \ll 1 \). The asymptotics corresponds to the situation when the cylinder rotates very rapidly (\( \Omega \gg 1 \) for small \( \varepsilon \)) and free surface disturbances are small as well. Therefore, at the leading order in \( \varepsilon \), one obtains the system
\[
\Phi_t = -(\Phi \Pi)_x, \quad \Pi_t = -(\Pi^2/2 - \Phi)_x,
\]
which coincides with shallow water model up to gravity force direction.

For the Hamiltonian density (24), the explicit expressions for \( A, B, \) and \( C \) are as follows:
\[
A(u) = -\frac{1}{2} \ln u, \quad B(u) = \frac{1}{2} \Omega u \ln u, \quad C(u) = -\frac{1}{8} \Omega^2 u^2 \ln u + \frac{1}{4} \Omega^2 u^2
\]
and system (14) is of the form
\[
G''_{k,k} + \frac{k(k-1)}{2u^2 \ln u} G_{k,k} = 0,
\]
\[
G''_{k,k-1} + \frac{(k-1)(k-2)}{2u^2 \ln u} G_{k,k-1} = -\frac{\Omega(k-1)}{2u \ln u} G_{k,k},
\]
\[
G''_{k,k-s} + \frac{(k-s)(k-s-1)}{2u^2 \ln u} G_{k,k-s} = -\frac{\Omega(k-s)(k-s+1)}{2u \ln u} G_{k,k-s+1} + \frac{(k-s+1)(k-s+2)}{2 \ln u} \left\{ -\frac{1}{4} \Omega^2 + \frac{1}{2} \frac{\Omega^2 \ln u}{2} \right\} G_{k,k-s+2},
\]
with \( s = 2, \ldots, k \). After introducing a new variable \( w = \ln u \) and making the substitution \( G_{k,k}(u) = H_{k,k}(w) \), the above system becomes
\[
H''_{k,k} - H'_{k,k} + \frac{k(k-1)}{2w} H_{k,k} = 0,
\]
\[
H''_{k,k-1} - H'_{k,k-1} + \frac{(k-1)(k-2)}{2w} H_{k,k-1} = -\frac{\Omega(k-1)}{2w} e^w H_{k,k},
\]
\[
H''_{k,k-s} - H'_{k,k-s} + \frac{(k-s)(k-s-1)}{2w} H_{k,k-s} = -\frac{\Omega(k-s)(k-s+1)}{2w} e^w H_{k,k-s+1} - \frac{1}{4} \Omega^2 (k-s+1)(k-s+2) \frac{(1-2w)e^{2w}}{2w} H_{k,k-s+2},
\]
for \( s = 2, \ldots, k \). The first equation in (26) is a particular case of the following ODE:
\[
wH'' + (\alpha + 1 - w)H' + nH = 0,
\]
whose polynomial solutions are called generalized Laguerre polynomials
\[
L_n^{(\alpha)}(w) = \frac{w^{-\alpha}}{n!} \frac{d^n}{dw^n} \left( e^{-w} w^{n+\alpha} \right),
\]
where \( n \) is a nonnegative integer and \( \alpha \) is an arbitrary real number (see [7]). Thus, the general solution to the first equation in (26) has the form
\[
H_{k,k}(w) = c_1 L_{k(k-1)/2}^{(-1)}(w) + c_2 H_{k,k}^{(2)}(w),
\]
where the function \( H_{k,k}^{(2)} \) is defined by Liouville's formula. Thus, all right-hand sides of system (26) are determined, and we can obtain its general solution. In the simplest case \( k = 3 \), for the coefficients \( H_{3,i} \), we have

\[
H_{3,3} = c_{31}P_{3,3}^1 + c_{32}\left\{ e^w P_{3,3}^2 + \text{Ei}(w)P_{3,3}^1 \right\},
\]

\[
H_{3,2} = c_{21} + c_{22}\left\{ w\text{Ei}(w) - e^w \right\} + c_{31}\Omega e^w P_{3,2}^1 + c_{32}\Omega e^{2w} P_{3,2}^1 + 2w\text{Ei}(2w),
\]

\[
H_{3,1} = c_{11}e^w + c_{12} + c_{21}e^{2w}\Omega(1 - w) + c_{22}\Omega e^{2w} (2 - w)\text{Ei}(w) - 2\text{Ei}(2w),
\]

\[
+ c_{31}e^{2w}\Omega^2 P_{3,1}^1 + c_{32}\Omega e^{3w} P_{3,1}^1 e^{2w} P_{3,1}^1 + 2e^{w(2 - w)}\text{Ei}(2w) - 135 \frac{\text{Ei}(3w)}{64},
\]

\[
H_{3,0} = c_{01}e^w + c_{02} + c_{21}e^{2w}\Omega P_{3,0}^1 + c_{31}e^{3w}\Omega^3 P_{3,0}^1
\]

\[
+ c_{22}\Omega e^w \left\{ \frac{1}{4} e^{3w} + e^{2w} P_{3,0}^1 e^{2w} \text{Ei}(w) + e^{w} e\text{Ei}(2w) - \frac{1}{2} \text{Ei}(3w) \right\}
\]

\[
+ c_{32}\Omega e^{3w} P_{3,0}^1 e^{2w} P_{3,0}^1 + 135 \frac{\text{Ei}(3w)}{128} e^{w} e\text{Ei}(3w) - \frac{8}{27} \text{Ei}(4w),
\]

where \( P_{i,j}^{i,j} \) are polynomials in \( w \) listed in Appendix B and \( \text{Ei}(z) \) is the exponential integral function defined for complex variable \( z \) by

\[
\text{Ei}(z) = \int_{-\infty}^{z} \frac{e^t}{t} dt.
\]

Thus, formulas (11) and (12) give the solution in implicit form:

\[
t = 2c_{21} + 2c_{22} \left\{ \text{li}(u) - \frac{u}{\ln u} \right\} + c_{32} \left\{ 20u - 4u \ln u - \frac{8u}{\ln u} + \text{li}(u)T_{3,4} \right\},
\]

\[
+ c_{31} \left\{ v T_{3,1} - \Omega u T_{3,2} \right\} + \Omega c_{32} \left\{ 32\text{Ei}(2u) - 9u^2 + u^2 \ln u - \frac{2u^2}{\ln u} - u \text{li}(u)T_{3,3} \right\},
\]

\[
x = \Omega^2 c_{32} \left\{ \frac{25}{16} u^2 + 4\text{Ei}(2u) - \frac{u^2}{8\ln u} + \frac{1}{2} u^2 \ln u - \frac{1}{16} u^2 \ln^2 u + u \text{li}(u)X_{3,6} \right\},
\]

\[
\]

\[
+ c_{32} \left\{ v^2 \left( -\frac{1}{4} + \frac{1}{\ln u} + \frac{1}{\ln u} + \frac{\text{li}(u)}{u} X_{3,3} \right) + \Omega \left\{ \frac{3}{2} u + \frac{u}{\ln u} X_{3,4} + \text{li}(u)X_{3,5} \right\} \right\}
\]

\[
+ c_{21} + c_{22} \left\{ 2\Omega \text{li}(u) - \frac{2v - \Omega u}{\ln u} \right\} + c_{31} \left\{ \frac{3v^2 (\ln u^2 - 6)}{u} + \frac{3}{2} v \Omega X_{3,1} + 3 \frac{u^2}{4} \Omega X_{3,2} \right\},
\]

where the functions \( T_{i,j} \) and \( X_{i,j} \) are polynomials in \( \ln u \) (see Appendix B) and \( \text{li}(u) = \text{Ei}(\ln u) \) is the logarithmic integral function.

5. CONCLUSION

As is well known, two-component two-dimensional hydrodynamic type systems are integrable by the hodograph method. Usually, this means that the corresponding system of two linear equations with variable coefficients can be solved explicitly, i.e., one can construct a general solution (parameterized by two arbitrary functions of a single variable) or at least one can derive two infinite series of particular solutions. One of examples is a polytropic gas (where a pressure depends on the density only). Another example was found by P. Olver and Ya. Nutku [5]. Here we presented a third example, which has applications in fluid dynamics. This case contains three arbitrary functions instead of a single function, as in two above-investigated cases. This advantage enables us to apply our results to a wide range of hydrodynamic type systems.

ACKNOWLEDGMENTS

ZVM and MVP are grateful to A. A. Chesnokov, A. K. Khe, N. I. Makarenko, A. M. Kamchatnov, and M. Yu. Zhukov for very important comments, remarks, and helpful conversations.
FUNDING

ZVM was supported by the project № 2.3.1.2.12 (code FWGG-2021-0011). MVP was supported by the project № 0023-2019-0011.

APPENDIX A

In this section, we list the explicit formulas $\Gamma_{j,k}$ for $G_{3,i}$ in (20) and for $X_{i,j}$ and $T_{i,j}$ in (22) and (21):

$$\Gamma_{3,3}^{1} = (r - u)(u^2 - 1), \quad \Gamma_{3,3}^{2} = u^2(2r^4 - 3r^2 + 3) - u(3r + r^3) - 2 + 6r^2 - 2r^4,$$

$$\Gamma_{3,3}^{3} = (r^2 - 1)^2 \Gamma_{3,3}^{1},$$

$$\Gamma_{3,2}^{1} = (1 + r^2)u - 2r, \quad \Gamma_{3,2}^{2} = (r^2 - 1)(u^2 - 1),$$

$$\Gamma_{3,1}^{1} = 6(r^3 - 1)(ru - 1), \quad \Gamma_{3,1}^{2} = -3r^2u(2 + 2r) + 6ru, \quad \Gamma_{3,1}^{3} = \Gamma_{3,1}^{4} = (r^2 - 1)^2 \Gamma_{3,1}^{1},$$

$$\Gamma_{3,1}^{5} = 2r^2u(3r^2 + 4r^3 - 3) + 6(r^2 - 1)^2 \ln \left(\frac{1 + r}{1 - r}\right),$$

$$\Gamma_{3,1}^{6} = r(r - 1)(u - 1)(8r^3 + r^2 - 9r - 6 + 3ru + 3r^2u),$$

$$\Gamma_{3,3}^{7} = r(r + 1)(u + 1)(8r^3 - r^2 - 9r + 6 - 3ru + 3r^2u),$$

$$\Gamma_{3,3}^{8} = -4r(1 - r^2)(3 - 5r^2 + 2r^4),$$

$$T_{3,1} = 2u(r + 1)(u - 1), \quad T_{3,2} = (r^2 - 1), \quad T_{3,3} = r^2 + 1, \quad T_{3,4} = 6v \Gamma_{3,3}^{2},$$

$$T_{3,5} = -9v(-1 - r^2)(u - 1),$$

$$X_{3,1} = 3r^2u(2 - ru + 1), \quad X_{3,2} = -6(1 - r^2), \quad X_{3,3} = -3(r^2 - 1)^2,$$

$$X_{3,4} = 2r(3 - 5r^2 + 2r^4),$$

$$X_{3,5} = 12v^2r(6r^4 - 6r^2 + u^2r(12r - 10r^3 + 2r^5 + (3r - 3r^3)v^2)$$

$$+ ur(-6 + 2r^2 - 4r^4 + (6r^2 - 6r^2)v^2) + u^2v^2(3r^2 + 3r^3) + r(2r + 2r^3),$$

$$X_{3,6} = u^4(6r^3 - 6r^5) + u^3(-12r^2 + 12r^4 + (-3r^2 + 3r^4 - 4r^6)v^2)$$

$$+ u^2(6r - 12r^3 + 6r^5 + (6r^2 + 3r^3)v^2) + u(12r^2 - 12r^4 + (-r^2 - 15r^4 + 4r^6)v^2)$$

$$+ v^2(6r^3 - 2r^5 + 6r^3 - 6r),$$

$$X_{3,7} = 3(r^2 - 1), \quad X_{3,8} = -3r + 5r^3.$$

APPENDIX B

In this section, we list the explicit formulas $P_{j,k}^i$ for $H_{3,i}$ and $X_{i,j}$ and $T_{i,j}$ in (28) and (27):

$$P_{3,3}^{1} = 6w - 6w^2 + w^3, \quad P_{3,3}^{2} = -\frac{1}{6} + \frac{5}{12}w - \frac{1}{12}w^2,$$

$$P_{3,2}^{1} = -\frac{33}{2}w + \frac{15}{2}w^2 - \frac{3}{4}w^3, \quad P_{3,2}^{2} = -\frac{1}{8} + \frac{9}{16}w + \frac{1}{16}w^2, \quad P_{3,2}^{3} = -\frac{11}{8}w + \frac{5}{8}w^2 - \frac{1}{16}w^3,$$

$$P_{3,1}^{1} = \frac{3}{16}(-121 + 130w - 42w^2 + 4w^3), \quad P_{3,1}^{2} = -\frac{1}{32}(16 - 19w + 2w^2), \quad P_{3,1}^{3} = \frac{1}{12}P_{3,1}^{1},$$

$$P_{3,0}^{1} = \frac{1}{4}(w - 2), \quad P_{3,0}^{2} = P_{3,0}^{3} = \frac{1}{288}(-835 + 858w - 234w^2 + 18w^3),$$

$$P_{3,0}^{4} = \frac{1}{576}(-34 - 36w + 3w^2), \quad P_{3,0}^{5} = \frac{1}{12}P_{3,0}^{3}, \quad P_{3,0}^{6} = 2P_{3,0}^{4},$$

$$T_{3,1} = 36 - 36\ln u + 6\ln^2 u, \quad T_{3,2} = 33 - 15\ln u + \frac{3}{2}\ln^2 u, \quad T_{3,3} = 22 - 10\ln u + \ln^2 u,$$

$$T_{3,4} = 24 - 24\ln u + 4\ln^2 u,$$

$$X_{3,1} = \ln^3 u - 2\ln^2 u - 12\ln u + 12, \quad X_{3,2} = -50 + \ln^3 u - 9\ln^2 u + 32\ln u,$$

$$X_{3,3} = -\frac{3}{2} + \frac{1}{4}\ln^2 u, \quad X_{3,4} = \frac{1}{4} + \frac{1}{8}\ln u - \frac{1}{8}\ln^3 u,$$

$$X_{3,5} = \frac{3}{2} - \frac{3}{4}\ln u - \frac{1}{8}\ln^2 u + \frac{1}{8}\ln^3 u, \quad X_{3,6} = \frac{-25}{8} + 2\ln u - \frac{9}{16}\ln^2 u + \frac{1}{16}\ln^3 u.$$
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