Robust mean-variance hedging in the single period model

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We give an explicit solution of robust mean-variance hedging problem in the single period model for some type of contingent claims. The alternative approach is also considered.

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1 Introduction

The study of mean variance hedging problem was initiated by H. Föllmer and D. Sondermann [7] and the solution of this problem for multiperiod model was given by H. Föllmer and M. Schweizer [6]. In this paper we investigate the single period mean variance hedging problem of contingent claims in incomplete markets, when parameters of asset prices are not known with certainty. Usually such parameters may be appreciate rate (or drift) and volatility coefficients. In such models it is desirable to choose an optimal portfolio for the worst case of parameters. Such type problem one calls the robust hedging problem.

The numerous of publications are concerned to the case when one of these parameters is known exactly. In the case of unknown drift coefficient the existence of saddle point of corresponding minimax problem has been established and characterization of the optimal strategy has been obtained (see [4], [9], [8]). For the case of unknown volatility coefficients the construction of hedging strategy were given in the works [1], [3], [2], [10].
The most difficult case is to characterize the optimal strategy of minimax (or maximin) problem under uncertainty of both drift and volatility terms. Talay and Zheng [13] applied the PDE-based approach to the maximin problem in the continuous time model and characterized the value as a viscosity solution of corresponding Bellman-Isaacs equation. However for robust hedging it is more convenient to consider the minimax problem. Such type of problem was studied for the single period model of financial market by Pinar [12], who consider the computational scheme to find the optimal strategy and optimal initial capital.

The purpose of the present paper is to investigate the robust mean-variance hedging problem in the one-step model, when drift and volatility of the asset are not known exactly. We consider the minimax problem and construct the optimal strategy for some type of contingent claims. Our approach is twofold. The main approach we develop is the randomization of the parameters and change the minimax problem by maximin one. This approach successfully works in the one period model and preliminary results show that it will be productive in multi-period and continuous time models. The other way is to perform maximization and minimization directly as they are given and describe the solution based on result of [5]. This way we call the alternative.

The paper is organized as follows. In section 2 we describe the market model and give a setting of the problem. Using randomization of parameters we argue the existence of saddle point. Further we construct explicit solution of obtained maximin problem. In examples 1-3 are considered the particular cases when the optimal strategy is expressed in a simple form. In section 3 we give an alternative approach to the minimax problem based on the result of [5].

2 The main results

We consider a financial market model with two assets. Let \((S_t, \eta_t), t = 0, 1\) be the price of assets. We suppose that

\[ S_1 = S_0 + \mu + \sigma w, \quad \eta_1 = \beta + \delta \bar{w}, \]

where \(w, \bar{w}\) is random pair with \(Ew = E\bar{w} = 0, \ Dw = D\bar{w} = 1, \ Cov(w, \bar{w}) \neq 0\) and \(\mu, \sigma, \beta, \delta\) are constants. We suppose also that the appreciate rate \(\mu\) and volatility \(\sigma\) of the asset price \(S_t\) are misspecified but stay in rectangle of uncertainty, i.e.

\((\mu, \sigma) \in D = [\mu_-, \mu_+] \times [\sigma_-, \sigma_+]\).

Let \(\beta, \delta\) be known exactly. We denote by \(\pi\) the number of stocks \(S\) bought at time \(t = 0\) and by \(x_0 = \pi S_0\) the initial capital. The wealth at time \(t = 1\) is

\[ X_1 = x_0 + \pi(S_1 - S_0) = x_0 + \pi\mu + \pi\sigma w. \]
The contingent claim $H(\eta)$ we assume depends on the asset $\eta$, which cannot be traded directly. The robust mean-variance hedging problem is

$$\min_{\pi} \max_{\mu, \sigma} E[H - x_0 - \pi \mu - \pi \sigma w]^2.$$  \hfill (2.2)

Let

$$H - x_0 = h_0 + h_1 w + H^\perp$$  \hfill (2.3)

be the decomposition of $H - x_0$ with $h_0 = E(H - x_0)$, $h_1 = EwH$, $EwH^\perp = 0$. Then the problem can be rewritten as

$$\min_{\pi \in R} \max_{(\mu, \sigma) \in D} F(\pi, \mu, \sigma),$$ \hfill (2.4)

where

$$F(\pi, \mu, \sigma) = (h_0 - \pi \mu)^2 + (h_1 - \pi \sigma)^2.$$ \hfill (2.5)

The function $F(\pi, \cdot)$ can be continued on the space of probability measures on $D$ as

$$F(\pi, \nu) = \int_D ((h_0 - \pi \mu)^2 + (h_1 - \pi \sigma)^2) \nu(d\mu d\sigma), \text{ for measure } \nu \text{ on } D$$ \hfill (2.6)

Hence we get

$$F(\pi, \nu) = \int_D (\mu^2 + \sigma^2) \nu(d\mu d\sigma) \left( \pi - \frac{\int_D (h_0 \mu + h_1 \sigma) \nu(d\mu d\sigma)}{\int_D (\mu^2 + \sigma^2) \nu(d\mu d\sigma)} \right)^2$$ \hfill (2.7)

and

$$\min_{\pi \in R} F(\pi, \nu) = h_0^2 + h_1^2 - \frac{\left( \int_D (h_0 \mu + h_1 \sigma) \nu(d\mu d\sigma) \right)^2}{\int_D (\mu^2 + \sigma^2) \nu(d\mu d\sigma)}$$ \hfill (2.8)

and

$$\pi^* = \frac{\int_D (h_0 \mu + h_1 \sigma) \nu(d\mu d\sigma)}{\int_D (\mu^2 + \sigma^2) \nu(d\mu d\sigma)}$$ \hfill (2.9)

Since $F$ is strictly convex in $\pi$ by the Theorem Neumann at al. (see Theorem IX.4.1 of [14]) there exists a saddle point $(\pi^*, \nu^*)$, i.e.

$$F(\pi^*, \nu) \leq F(\pi^*, \nu^*) \leq F(\pi, \nu^*).$$ \hfill (2.10)

Since $\max_{\nu} F(\pi, \nu) = \max_{\mu, \sigma} F(\pi, \mu, \sigma)$ then we obtain

$$\min_{\pi} \max_{\mu, \sigma} F(\pi, \mu, \sigma) = \min_{\pi} \max_{\nu} F(\pi, \nu) = \max_{\nu} \min_{\pi} F(\pi, \nu).$$ \hfill (2.11)

Each pair of random variables $(\mu, \sigma)$ with the distribution $\nu$ may be realized on the probability space $([0,1], \mathcal{B}, P(d\omega) = d\omega)$ where $\mathcal{B}$ is the Borel $\sigma$–algebra on $[0,1]$ and $d\omega$ the Lebesgue measure (see Proposition 26.6 of [11]). Hence the minimization problem

$$\min_{\nu} \left( \frac{\int_D (h_0 \mu + h_1 \sigma) \nu(d\mu d\sigma))^2}{\int_D (\mu^2 + \sigma^2) \nu(d\mu d\sigma)} \right)$$ \hfill (2.12)
can be written as
\[
\min_{(\mu(\omega), \sigma(\omega)) \in D} \frac{\left( \int_0^1 (h_0 \mu(\omega) + h_1 \sigma(\omega)) d\omega \right)^2}{\int_0^1 (\mu^2(\omega) + \sigma^2(\omega)) d\omega}.
\] (2.14)

To solve this problem we consider the deterministic control problem
\[
\max_{(\mu(\omega), \sigma(\omega)) \in D} \int_0^1 (\mu^2(\omega) + \sigma^2(\omega)) d\omega,
\] (2.15)
\[
\frac{dx(\omega)}{d\omega} = \mu(\omega), \quad \frac{dy(\omega)}{d\omega} = \sigma(\omega),
\] (2.16)
\[
x(0) = 0, y(0) = 0, \quad x(1) = x, y(1) = y.
\] (2.17)

Lemma 2.1. The solution of the problem (2.15) is of the form
\[
\mu^*(\omega) = \mu_- X_A(\omega) + \mu_+ X_A'(\omega), \quad \sigma^*(\omega) = \sigma_- X_B(\omega) + \sigma_+ X_B'(\omega),
\] (2.18)
with
\[
P(A) = \frac{x - \mu_-}{\mu_+ - \mu_-}, \quad P(B) = \frac{y - \sigma_-}{\sigma_+ - \sigma_-}
\] (2.19)
and the maximal value is \(2x \mu_M + 2y \sigma_M - \mu_- \mu_+ - \sigma_- \sigma_+\), where \(\mu_M = \frac{\mu_+ + \mu_-}{2}\), \(\sigma_M = \frac{\sigma_+ + \sigma_-}{2}\).

Proof. By the maximum principle (see [14]) we have
\[
\mu^* = \arg \max_{\mu_- \leq \mu \leq \mu_+} (\mu^2 + p \mu), \quad \sigma^* = \arg \max_{\sigma_- \leq \sigma \leq \sigma_+} (\sigma^2 + q \sigma),
\]
where \(p, q\) are some constants maintaining the conditions (2.17). Hence the solution of the problem (2.15) is of the form (2.18). The relations
\[
\int_0^1 \mu^*(\omega) d\omega = x, \quad \int_0^1 \sigma^*(\omega) d\omega = y
\]
uniquely determines the probabilities \(P(A), P(B)\) by (2.19) and
\[
\int_0^1 (\mu^*^2(\omega) + \sigma^*^2(\omega)) d\omega = 2x \mu_M + 2y \sigma_M - \mu_- \mu_+ - \sigma_- \sigma_+.
\]

Corollary 2.1.
\[
\min_{(\mu(\omega), \sigma(\omega)) \in D} \frac{\left( \int_0^1 (h_0 \mu(\omega) + h_1 \sigma(\omega)) d\omega \right)^2}{\int_0^1 (\mu^2(\omega) + \sigma^2(\omega)) d\omega} = \min_{(x,y) \in D} \frac{(h_0 x + h_1 y)^2}{2 \mu_M x + 2 \sigma_M y - \mu_- \mu_+ - \sigma_- \sigma_+}.
\] (2.20)

To characterize the minimum point of function
\[
\psi(x, y) = \frac{(h_0 x + h_1 y)^2}{2 \mu_M x + 2 \sigma_M y - \mu_- \mu_+ - \sigma_- \sigma_+}
\] (2.21)
we use the following lemma.
Lemma 2.2. The solution of the system
\[ \frac{\partial \psi}{\partial x}(x, y) = 0, \quad \frac{\partial \psi}{\partial y}(x, y) = 0 \]
(2.22)
satisfies the equation \( h_0 x + h_1 y = 0 \).

Proof. It is easy to see that
\[ \frac{\partial \psi}{\partial x}(x, y) = 2(h_0 x + h_1 y) \times \frac{h_0(2\mu M x + 2\sigma M y - \mu_- \mu_+ - \sigma_- \sigma_+) - h_0 \mu_M x - h_1 \mu_M y}{(2\mu M x + 2\sigma M y - \mu_- \mu_+ - \sigma_- \sigma_+)^2}, \]
\[ \frac{\partial \psi}{\partial y}(x, y) = 2(h_0 x + h_1 y) \times \frac{h_1(2\mu M x + 2\sigma M y - \mu_- \mu_+ - \sigma_- \sigma_+) - h_0 \sigma_M x - h_1 \sigma_M y}{(2\mu M x + 2\sigma M y - \mu_- \mu_+ - \sigma_- \sigma_+)^2}. \]
Solving the system we obtain that either \( h_0 x + h_1 y = 0 \) or
\[ h_0 \mu_M x + (2h_0 \sigma_M - h_1 \mu_M) y = h_0 (\mu_+ \mu_+ + \sigma_- \sigma_+), \]
\[ h_1 \sigma_M y + (2h_1 \mu_M - h_0 \sigma_M) x = h_1 (\mu_+ \mu_+ + \sigma_- \sigma_+). \]
The latter system admits the unique solution
\[ x = \frac{h_1 \mu_- \mu_+ + \sigma_- \sigma_+}{2h_1 \mu_M - h_0 \sigma_M}, \]
\[ y = -\frac{h_0 \mu_- \mu_+ + \sigma_- \sigma_+}{2h_1 \mu_M - h_0 \sigma_M}, \]
which also satisfies the equation \( h_0 x + h_1 y = 0 \).

Corollary 2.2. The minimum of \( \psi(x, y) \in D \) is achieved either on the line \( h_0 x + h_1 y = 0 \) or on the boundary of \( D \).

Lemma 2.3. If there exists the pair \( (\bar{x}, \bar{y}) \) such that \( h_0 \bar{x} + h_1 \bar{y} = 0 \), then
\[ \max_{\nu} \min_{\pi} F(\pi, \nu) = \min_{(x, y) \in D} \psi(x, y) = \psi(\bar{x}, \bar{y}) \]
and \( \pi^* = 0 \).

Proof. It is sufficient to take \( (\mu^*, \sigma^*) = (\bar{x}, \bar{y}) \) and to use (2.10). \qed

From now on we assume that \( h_0 x + h_1 y \neq 0 \) for all \( (x, y) \in D \). For certainty we suppose that \( h_0 x + h_1 y > 0 \). The case \( h_0 x + h_1 y < 0 \) can be considered analogously.

The boundary \( \partial D \) of rectangle \( D \) consists from the sides \( B_{-+}, B_{++}, B_{+-}, B_{++} \), where
\[ B_{-+} = \{(x, y) : x = \mu_-, \sigma_- \leq y \leq \sigma_+\}, \]
\[ B_{++} = \{(x, y) : x = \mu_+ , \sigma_- \leq y \leq \sigma_+\}, \]
\[ B_{+-} = \{(x, y) : y = \sigma_-, \mu_- \leq x \leq \mu_+\}, \]
\[ B_{++} = \{(x, y) : y = \sigma_+, \mu_- \leq x \leq \mu_+\}. \]
Obviously that functions defined on the sides

\[ \varphi_a(t) = \psi(\mu_a, \sigma_a + t(\sigma_a - \sigma_0)), \quad \text{on } B_{-a}, \quad a = -, +, \]

\[ \varphi_- = \psi(\mu_0 + t(\mu_0 - \mu_{-}), \sigma_0), \quad \text{on } B_{+b}, \quad b = -, +, \]

coincide with functions of the Appendix. It is easy to show that the \( t_{ab} = \arg \min \varphi_{ab}(t) \), \( a = -, +, \ b = +, - \) can be computed as (see Appendix)

\[ t_{ab} = \begin{cases} 1, & \text{if } 1 \leq \alpha_{ab} \text{ or } 1 \leq 2\beta_{ab} - \alpha_{ab}, \\ 0, & \text{if } \beta_{ab} \leq \alpha_{ab} \leq 0 \text{ or } \beta_{ab} < 2\beta_{ab} - \alpha_{ab} \leq 0 \\ \alpha_{ab}, & \text{if } 0 < \alpha_{ab} < 1 \\ 2\beta_{ab} - \alpha_{ab}, & \text{if } 0 < 2\beta_{ab} - \alpha_{ab} < 1. \end{cases} \] (2.23)

Hence we have

**Proposition 2.1.** Let \( h_0 x + h_1 y > 0 \) for all \((x, y) \in D\). Then

\[ \min_{(x, y) \in D} \psi(x, y) = \min_{a = \pm, b = \pm} \varphi_{ab}(t_{ab}). \]

Moreover for \((x^*, y^*) = \arg \min_{(x, y) \in D} \psi(x, y)\) we have \((x^*, y^*) \in B_{a^*b^*}, \) where \(a^*b^* = \arg \min_{ab} \varphi_{ab}(t_{ab})\) and \(t^* = t_{a^*b^*}\) is the distance from the end of the side to \((x^*, y^*)\) defined by (2.23).

**Proposition 2.2.** Let \( h_0 x + h_1 y > 0 \) for all \((x, y) \in D\). Then the solution of the optimization problem (2.14) is of the form

\[ (\mu^*, \sigma^*) = \begin{cases} (\mu_0 - \chi_B + \sigma_0 + \chi_{B^*}), & \text{if } (x^*, y^*) \in B_{--}, \\ (\mu_0 + \chi_B + \sigma_0 + \chi_{B^*}), & \text{if } (x^*, y^*) \in B_{++}, \\ (\mu_0 - \chi_A + \sigma_0 + \chi_{A^*}, \sigma_0), & \text{if } (x^*, y^*) \in B_{+-}, \\ (\mu_0 - \chi_A + \sigma_0 + \chi_{A^*}, \sigma_0), & \text{if } (x^*, y^*) \in B_{-+}. \end{cases} \] (2.24)

**Proof.** Let \((x^*, y^*)\) be the minimum point of \(\psi(x, y)\). By Proposition (2.1) \((x^*, y^*)\) belongs on some side. Hence the pair \((\mu^*, \sigma^*)\) such that

\[ P(\mu^* = \mu_0) = \frac{\mu_0 - x^*}{\mu_0 - \mu_-}, \quad P(\sigma^* = \sigma_0) = \frac{\sigma_0 - y^*}{\sigma_0 - \sigma_-} \]

is the optimal pair.

**Example 1.** Let \( H \) be a constant. i.e. \( h_1 = 0 \). It is evident

\[ \min_{(x, y) \in D} \psi(x, y) = \min_{\mu_0 \leq x \leq \mu_0} \frac{h_0 x^2}{2\mu M x - \mu_0 + \mu_+} = \min_{\mu_0 \leq t \leq 1} \frac{h_0^2 (\mu_0 + t(\mu_0 - \mu_-))^2}{\mu_0^2 + \mu_0 + \mu_+} = \min_{\mu_0 \leq t \leq 1} \frac{h_0^2 (\mu_0 + t(\mu_0 - \mu_-))^2}{\mu_0^2 + \mu_0 + \mu_+} = \min_{\mu_0 \leq t \leq 1} \frac{h_0^2 (\mu_0 - x)^2}{\mu_0^2 + \mu_0 + \mu_+}. \]

Hence \((x^*, y^*) \in B_{++}\) and we must find \( t_{++} = \arg \min \varphi_{++}(t) \). From (A.6) we have

\[ \alpha_{++} = -\frac{\mu_0 - \mu_0 - \mu_-}{\mu_0 + \mu_0 - \mu_-} < 0, \quad \beta_{++} = -\frac{\mu_0 + \sigma_0 + \mu_0 - \mu_0 - \mu_-}{\mu_0 + \mu_0 - \mu_-}, \quad \gamma_+ = \frac{h_0^2 (\Delta \mu)^2}{\mu_0^2 + \mu_0 + \mu_+} = \frac{\mu_0 + \mu_0 - \mu_-}{\mu_0 + \mu_0 - \mu_-} > 0. \] (2.25)
Moreover
\[ 2\beta_{++} - \alpha_{++} = -\frac{\mu_-^2 - \mu_- \mu_+ + 2\sigma_-^2}{\mu_+^2 - \mu_-^2} < \frac{\mu_-}{\mu_+ + \mu_-} < 1. \] (2.26)

Thus
\[
t_{++} = \begin{cases} 
0, & \text{if } 2\beta_{++} - \alpha_{++} \leq 0 \\
\frac{\mu_- - \mu_+ - 2\beta_{++}}{\mu_+^2 - \mu_-^2}, & \text{if } 0 < 2\beta_{++} - \alpha_{++}.
\end{cases}
\] (2.27)

Simplifying we obtain
\[
t_{++} = \begin{cases} 
0, & \text{if } \mu_+ \mu_- - 2\sigma_-^2 \leq \mu_-^2 \\
\frac{\mu_- - \mu_+ - 2\sigma_-^2}{\mu_+^2 - \mu_-^2}, & \text{if } \mu_-^2 < \mu_+ \mu_- - 2\sigma_-^2.
\end{cases}
\] (2.28)

and
\[
\min_{\nu} \frac{(\int_D h_0 \nu (d\mu d\sigma))^2}{\int_D (\mu^2 + \sigma^2)\nu (d\mu d\sigma)} = \varphi_{++}(t_{++}) = \begin{cases} 
\frac{h_0^2 \mu_-^2}{\mu_+^2 + \sigma_+^2}, & \text{if } \mu_+ \mu_- - 2\sigma_-^2 \leq \mu_-^2 \\
\frac{h_0^2 \mu_- - \sigma_-^2}{\mu_+^2}, & \text{if } \mu_-^2 < \mu_+ \mu_- - 2\sigma_-^2.
\end{cases}
\] (2.29)

By (2.10) the optimal strategy is
\[
\pi^* = h_0 \frac{\mu_- + t_{++}(\mu_+ - \mu_-)}{\mu_-^2 + t_{++}(\mu_+^2 - \mu_-^2) + \sigma_-^2} = \begin{cases} 
\frac{h_0 \mu_-}{\mu_+ + \sigma_-^2}, & \text{if } \mu_+ \mu_- - 2\sigma_-^2 \leq \mu_-^2 \\
\frac{h_0}{\mu_+}, & \text{if } \mu_-^2 < \mu_+ \mu_- - 2\sigma_-^2.
\end{cases}
\] (2.30)

Example 2. Analogously we can consider the case $h_0 = 0$. Then $(x^*, y^*) = (\mu_+, \sigma_- + t_{++}(\sigma_+ - \sigma_-)) \in B_+$,
\[
t_{--} = \begin{cases} 
0, & \text{if } \sigma_+ \sigma_- - 2\mu_+^2 \leq \sigma_-^2 \\
\frac{\sigma_- \sigma_+ - \sigma_-^2 - 2\mu_+^2}{\mu_+^2 - \mu_-^2}, & \text{if } \sigma_-^2 < \sigma_+ \sigma_- - 2\mu_+^2,
\end{cases}
\] (2.31)

\[
\pi^* = \begin{cases} 
\frac{h_1 \sigma_-}{\sigma_-^2 + \mu_+^2}, & \text{if } \sigma_+ \sigma_- - 2\mu_+^2 \leq \sigma_-^2 \\
\frac{h_1}{\sigma_+}, & \text{if } \sigma_-^2 < \sigma_+ \sigma_- - 2\mu_+^2.
\end{cases}
\] (2.32)

Example 3. let $\mu_- = \mu_+ = 0$. Then $\psi(x, y) = \frac{h_0^2 y^2}{2\sigma_M y - \sigma_- \sigma_+}$ and
\[
t^* = \arg \min_{0 \leq t \leq 1} \frac{h_0^2 (\sigma_+ + t(\sigma_- - \sigma_-))^2}{\sigma_-^2 + t(\sigma_+^2 - \sigma_-^2)} = \frac{\sigma_-}{\sigma_+ + \sigma_-}.
\]

Therefore $\pi^* = \frac{h_1}{\sigma_M}$.

Remark 2.1. The quantity
\[
\max_{\nu} \min_{\pi} F(\pi, \nu).
\] (2.33)

is a function of initial capital $x_0$. Minimizing this expression by $x_0$ we find $x_0^*$ and further construct the optimal $(\pi^*, \mu^*, \sigma^*)$ assuming $h_0 = EH - x_0^*$. Therefore we find the solution of the problem
\[
\min_{x_0, \pi, (\mu, \sigma)} \max F(\pi, \mu, \sigma).
\] (2.34)
3 The alternative approach

For two distinct pairs \((a, b), (c, d)\) \(\in \{+,-\} \times \{+,-\}\) such that \(\mu_a^2 + \sigma_b^2 \leq \mu_c^2 + \sigma_d^2\) we define the functions \(f_{abcd}(\pi) = \max(f_{ab}(\pi), f_{cd}(\pi))\), where \(f_{ab}(\pi) = F(\pi, \mu_a, \sigma_b)\). Obviously that

\[
f_{abcd}(\pi) \leq \max_{(a,b)\in\{+,-\}^2} (f_{ab}(\pi))\quad \text{and}\quad \min_{\pi} f_{abcd}(\pi) \leq \min_{\pi} \max_{(a,b)\in\{+,-\}^2} (f_{ab}(\pi))
\]

Hence by Theorem 3.3 of [3] (Chapter VI p.197)

\[
\min_{\pi} \max_{(\mu,\sigma)\in D} F(\pi, \mu, \sigma) = \min_{\pi} \max_{(a,b)\in\{+,-\}^2} (f_{ab}(\pi)) = \max_{(abcd)} \min_{\pi} f_{abcd}(\pi).
\]

**Lemma 3.1.** For \(\pi_{abcd} = \arg \min_{\pi} f_{abcd}(\pi)\) we have

\[
\pi_{abcd} = \begin{cases}
0, & \text{if } (h_0\mu_a + h_1\sigma_b)(h_0\mu_c + h_1\sigma_d) \leq 0, \\
\frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, & \text{if } \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} > \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \\
\frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2}, & \text{if } \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \leq \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \\
\frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2}, & \text{if } \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \leq \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \leq \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \\
or \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \leq \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} < \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, & \text{if } \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \leq \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \leq \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2}.
\end{cases}
\]

Moreover

\[
f_{abcd}(\pi_{abcd}) = \begin{cases}
\frac{h_0^2 + h_1^2}{\mu_a^2 + \sigma_b^2}, & \text{if } (h_0\mu_a + h_1\sigma_b)(h_0\mu_c + h_1\sigma_d) \leq 0, \\
\frac{h_0^2 + h_1^2}{\mu_a^2 + \sigma_b^2} - \frac{(h_0\mu_a + h_1\sigma_b)^2}{\mu_a^2 + \sigma_b^2}, & \text{if } \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} > \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \\
\frac{h_0^2 + h_1^2}{\mu_a^2 + \sigma_b^2} - \frac{(h_0\mu_c + h_1\sigma_d)^2}{\mu_c^2 + \sigma_d^2}, & \text{if } \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \leq \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \\
\frac{h_0^2 + h_1^2}{\mu_a^2 + \sigma_b^2} - \frac{(h_0\mu_a + h_1\sigma_b)^2}{\mu_a^2 + \sigma_b^2} + \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} - \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}^2, & \text{if } \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \leq \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \leq \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \\
or \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2} \leq \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \leq \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} < \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, & \text{if } \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \leq \frac{2h_0(\mu_c - \mu_a) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \leq \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2}.
\end{cases}
\]
Proof. The minimal value of \( f_{ab}(\pi) \), \( f_{cd}(\pi) \) are achieved at \( \frac{h_0\mu_a + h_1\sigma_a}{\mu_c^2 + \sigma_d^2} \) and \( \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \) respectively. If \( (h_0\mu_a + h_1\sigma_b)(h_0\mu_c + h_1\sigma_d) \leq 0 \) then by continuity of \( h_0x + h_1y \) there exists \( (x, y) \in D \) such that \( h_0x + h_1y = 0 \) and \( \pi^* = 0 \). If \( (h_0\mu_a + h_1\sigma_b)(h_0\mu_c + h_1\sigma_d) > 0 \) then we assume \( h_0\mu_a + h_1\sigma_b > 0 \), \( h_0\mu_c + h_1\sigma_d > 0 \). The roots of the equation \( f_{ab}(\pi) = f_{cd}(\pi) \) are \( \pi = 0 \) and \( \pi = 2\frac{h_0(\mu_a - \mu_c) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \). There exists three possibilities:

1) \( \frac{h_0\mu_a + h_1\sigma_a}{\mu_c^2 + \sigma_d^2} \), \( \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} > 2\frac{h_0(\mu_a - \mu_c) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \),

2) \( \frac{h_0\mu_a + h_1\sigma_a}{\mu_c^2 + \sigma_d^2} \), \( \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \leq 2\frac{h_0(\mu_a - \mu_c) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \),

3) \( h_0\mu_a + h_1\sigma_a \leq 2\frac{h_0(\mu_a - \mu_c) + h_1(\sigma_d - \sigma_b)}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \), \( \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2} \leq \frac{h_0\mu_a + h_1\sigma_a}{\mu_c^2 + \sigma_d^2} \).

In each cases the corresponding minimal value calculated by the equation (3.2).

Corollary 3.1. The solution of minmax problem (2.4) can be given as

\[
\pi^* = \pi_{a^*b^*c^*d^*},
\]

where \( a^*b^*c^*d^* = \text{arg max}_{(a,b,c,d)} f_{abcd}(\pi_{abcd}) \)

### A Appendix

We need to find the measure \( \nu_t = t\delta_{(\mu_a, \sigma_b)} + (1 - t)\delta_{(\mu_c, \sigma_d)} \) minimizing the expression

\[
\min_{\nu} \left( \int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma) \right)^2 / \int_D (\mu^2 + \sigma^2)^2 \nu(d\mu d\sigma)
\]

(A.1)

for \((a, b), (c, d) \in \{-, +\} \times \{-, +\}\). We consider only the case \(0 < \mu_- < \mu_+, \; 0 < \sigma_- < \sigma_+\).

Let

\[
\varphi(t) = \frac{\left( \int_D (h_0\mu + h_1\sigma)\nu_t(d\mu d\sigma) \right)^2}{\int_D (\mu^2 + \sigma^2)^2 \nu_t(d\mu d\sigma)}
\]

(A.2)

\[
= \frac{(h_0\mu_a + h_1\sigma_b + t(h_0\Delta\mu + h_1\Delta\sigma))^2}{(\mu_a^2 + \sigma_b^2 + t(\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2))}
\]

(A.3)

When \( \mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2 = 0 \) and \( h_0\Delta\mu + h_1\Delta\sigma = 0 \), then \( \varphi(t) = \text{const} \). If \( \mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2 = 0 \) and \( h_0\Delta\mu + h_1\Delta\sigma \neq 0 \) then

\[
\varphi(t) = \frac{1}{\mu_a^2 + \sigma_b^2}(h_0\mu_a + h_1\sigma_b + t(h_0\Delta\mu + h_1\Delta\sigma))^2.
\]

(A.4)

If \( \mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2 \neq 0 \) and \( h_0\Delta\mu - h_1\Delta\sigma \neq 0 \) then

\[
\varphi(t) = \gamma \frac{(t - \alpha)^2}{t - \beta}
\]

(A.5)
where $\Delta \mu = \mu_c - \mu_a$, $\Delta \sigma = \sigma_d - \sigma_b$

$$\alpha = -\frac{h_0 \mu_a + h_1 \sigma_b}{h_0 \Delta \mu + h_1 \Delta \sigma}, \quad \beta = -\frac{\mu_a^2 + \sigma_b^2}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2}, \quad \gamma = \frac{(h_0 \Delta \mu + h_1 \Delta \sigma)^2}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2}$$  \hspace{1cm} (A.6)

**Proposition A.1.** Let $t^* = \arg \min_{t \in [0,1]} \varphi(t)$ and $\mu_a^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2 \neq 0$ is satisfied. Then for the case $\gamma < 0$

$$t^* = \begin{cases} 1, & \text{if } 1 \leq \alpha \leq \beta \text{ or } 1 \leq 2\beta - \alpha < \beta, \\ 0, & \text{if } \alpha \leq 0 \text{ or } 2\beta \leq \alpha \\ \alpha, & \text{if } 0 < \alpha < 1 \\ 2\beta - \alpha, & \text{if } 0 < 2\beta - \alpha < 1 \end{cases}$$  \hspace{1cm} (A.7)

and for the case $\gamma > 0$

$$t^* = \begin{cases} 1, & \text{if } 1 \leq \alpha \text{ or } 1 \leq 2\beta - \alpha, \\ 0, & \text{if } \beta \leq \alpha \leq 0 \text{ or } \beta < 2\beta - \alpha \leq 0 \\ \alpha, & \text{if } 0 < \alpha < 1 \\ 2\beta - \alpha, & \text{if } 0 < 2\beta - \alpha < 1 \end{cases}$$  \hspace{1cm} (A.8)

**Proof.** Obviously that

$$\varphi(t) = \gamma \left( t - \alpha + 2(\alpha - \beta) + \frac{(\alpha - \beta)^2}{t - \beta} \right)$$  \hspace{1cm} (A.9)

and

$$\varphi'(t) = \gamma \frac{(t - 2\beta + \alpha)(t - \alpha)}{(t - \beta)^2}. \hspace{1cm} (A.10)$$

The case $\alpha = \beta$ is trivial.

I) Let $\gamma < 0$ and $\alpha \neq \beta$ then $\beta > 1$ and

$$\lim_{t \downarrow \beta} \varphi(t) = \infty, \quad \lim_{t \to -\infty} \varphi(t) = \infty$$  \hspace{1cm} (A.11)

$$\lim_{t \downarrow \beta} \varphi(t) = -\infty, \quad \lim_{t \to -\infty} \varphi(t) = -\infty$$  \hspace{1cm} (A.12)

Hence $\varphi(t)$ has a minimum on $(-\infty, \beta)$ and has a maximum on $(\beta, \infty)$. Thus if $\alpha < \beta$ then as follows from (A.10) the local minimum is attained at $t = \alpha$, and if $\alpha > \beta$ then $2\beta - \alpha < \beta$ and the local minimum is attained at $t = 2\beta - \alpha$.

II) Let $\gamma > 0$ and $\alpha \neq \beta$ then $\beta < -1$ and

$$\lim_{t \downarrow \beta} \varphi(t) = -\infty, \quad \lim_{t \to -\infty} \varphi(t) = -\infty$$  \hspace{1cm} (A.13)

$$\lim_{t \downarrow \beta} \varphi(t) = \infty, \quad \lim_{t \to -\infty} \varphi(t) = \infty$$  \hspace{1cm} (A.14)

Hence $\varphi(t)$ has a maximum on $(-\infty, \beta)$ and has a minimum on $(\beta, \infty)$. Thus if $\alpha > \beta$ then as follows from (A.10) $t = \alpha$ is the point of local minimum, and if $\alpha < \beta$ then $2\beta - \alpha > \beta$ and $t = 2\beta - \alpha$ is the point of local minimum.
Denote by $\varphi_-(t), \varphi_+(t), \varphi_++(t), \varphi_-^+(t)$ the function $\varphi(t)$ for the cases $(a, b, c, d) = (-, -, +, +), (+, -, +, +), (-, +, -, +), (-, +, +, +)$ respectively. We may say that they are functions defined on sides of the rectangle $D$. Then (A.6) takes the form

$$\alpha_{-\pm} = \frac{-h_0\mu_\pm + h_1\sigma_-}{h_1\Delta\sigma}, \quad \beta_{-\pm} = -\frac{\mu_\pm^2 + \sigma_-^2}{\sigma_\pm^2 - \sigma_-^2}, \quad \gamma_- = \frac{(h_1\Delta\sigma)^2}{\sigma_\pm^2 - \sigma_-^2} \quad \text{(A.15)}$$

$$\alpha_{+\pm} = \frac{-h_0\mu_\pm + h_1\sigma_\pm}{h_0\Delta\mu}, \quad \beta_{+\pm} = -\frac{\mu_\pm^2 + \sigma_\pm^2}{\mu_\pm^2 - \mu_-^2}, \quad \gamma_+ = \frac{(h_0\Delta\mu)^2}{\mu_\pm^2 - \mu_-^2} \quad \text{(A.16)}$$

Obviously that $\mu_\pm^2 - \mu_\pm^2 + \sigma_\pm^2 - \sigma_-^2 \neq 0$ and $\gamma > 0$ for this cases. Hence from the previous Proposition we obtain

**Proposition A.2.** Let $t_{ab} = \arg \min_{t \in [0, 1]} \varphi_{ab}(t)$, for $(a, b) \in \{-, +\}^2$. Then

$$t_{ab} = \begin{cases} 1, & \text{if } 1 \leq \alpha_{ab} \text{ or } 1 \leq 2\beta_{ab} - \alpha_{ab}, \\ 0, & \text{if } \beta_{ab} \leq \alpha_{ab} \leq 0 \text{ or } \beta_{ab} < 2\beta_{ab} - \alpha_{ab} \leq 0 \\ \alpha_{ab}, & \text{if } 0 < \alpha_{ab} < 1 \\ 2\beta_{ab} - \alpha_{ab}, & \text{if } 0 < 2\beta_{ab} - \alpha_{ab} < 1. \end{cases} \quad \text{(A.17)}$$

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