Abstract – We use the notions of reflexivity and of reflexive dimensions in order to introduce probability measures for lattice polytopes and initiate the investigation of their statistical properties. Examples of applications to discrete geometry include a study of randomness of self-duality of reflexive polytopes and implications for expectation values of the numbers of such polytopes in higher dimensions. We also discuss enumeration problems and related algorithms and point out interesting open problems. In this context we define the notion of IP-confined polytopes. Our new results include the list of IP-simplices in 3 dimensions that are not IP-confined. The main motivation for the study of these issues comes from applications in algebraic geometry and string theory.

1. Introduction

Let us consider $d$-dimensional convex polytopes $\Delta$, which are convex hulls of a finite numbers of points in $\mathbb{R}^d$. Obvious notions of randomness in terms of random selections of coordinates of the vertices prefer simplicial polytopes, whose duals are simple [1]. Probability measures based on such concepts hence are not invariant under the elementary duality exchanging vertices with bounding hyperplanes.

If we focus on lattice polytopes, whose vertices belong to a fixed lattice $\mathbb{Z}^d \subset \mathbb{R}^d$, we are usually interested in affine unimodular equivalence classes, whose representatives differ by lattice automorphisms. We will need some special classes of such polytopes:

**Definition 1.1.** An IP-polytope is a lattice polytope $\Delta$ with exactly one interior lattice point, which we can choose to be the origin $0 \in \mathbb{Z}^d$.

The polar (or dual) polytope $\Delta^*$ of a polytope $\Delta \subset \mathbb{R}^d$ with $0$ in its interior is defined by

$$\Delta^* = \{ y \in \mathbb{Z}^d \mid \bar{y} \cdot \bar{x} \geq -1 \ \forall x \in \Delta \}. \quad (1.1)$$

A reflexive polytope is an IP-polytope $\Delta$ for which $\Delta^*$ is also a lattice polytope.

**Remark 1.2.** The number of reflexive polytopes with given dimension is finite. Because eq. (1.1) defines an involution, the polar polytope $\Delta^*$ is reflexive if and only if $\Delta$ is reflexive. Reflexivity of an IP polytope is equivalent to the property that all facets have lattice distance 1 from the origin, i.e. there are no parallel lattice hyperplanes between the origin and the supporting hyperplanes of the facets.

**Definition 1.3.** Haase and Melnikov [2] have shown that every lattice polytope $\Delta$ is (isomorphic to) a face of a reflexive polytope. The reflexive dimensions $rd(\Delta)$ is the smallest dimension for which such a reflexive polytope exists.

Since the number of reflexive polytopes is finite we can define probability measures on the set of all polytopes that are uniform in fixed dimension and thus respect the polar duality of reflexive polytopes exchanging vertices and facets. Similarly, we can use the fact that the number of polytopes $\Delta$ with fixed reflexive dimension $rd(\Delta)$ is finite to define measures on the set of all lattice polytopes. The numbers of polytopes with $rd(\Delta) \leq 4$ are given in table 2 below.

The content of this paper is organized as follows. In section 2 we summarize results concerning the enumeration of reflexive polytopes and discuss some relevant algorithms. In section 3 we extend the polarity of reflexive polytopes to an involutive duality of IP-confined polytopes in $d > 4$ dimensions and discuss various enumeration problems. Section 4 contains statistical considerations and observations, which are used in an attempt at estimating the number of reflexive polytopes in 5 dimensions.

**Acknowledgements.** The author would like to acknowledge discussions with Benjamin Nill and support by the Austrian Research Fund FWF under grant no. P18679-N16.

2. Reflexivity and weight vectors

The classification of reflexive polytopes in two dimensions can be done by hand on a piece of paper with the result shown in fig. 1. The interest in explicit enumerations in higher dimensions is mainly motivated by the relation between lattice polytopes and algebraic geometry, and in particular by the result of Batyrev [3] that generic hypersurfaces in toric varieties are Calabi–Yau if and only if the toric ambient space is defined in terms of a reflexive polytope. On the algebro-geometric side of this correspondence the polar duality corresponds to an exchange of complex structure and Kähler moduli with important applications to enumerative geometry [4,5]. In terms of string
theory, this amounts to determining instanton corrections from a duality called mirror symmetry, which exchanges matter with anti-matter in particle physics [5].

The algorithm that has been used for the complete enumeration of reflexive polytopes in 3 and 4 dimensions [6, 7] is based on the idea to determine the maximal objects in terms of the polar minimal ones [8] and on an algorithm for the enumeration of all possible linear relations among vertices of the minimal polytopes [9]. For the case of two dimensions the three maximal reflexive polytopes are drawn, together with the dual minimal ones, in the top row of fig. 1. For the other two maximal reflexive polytopes the weight vectors are (1, 1, 2) and (1, 1, 2). For the two simplicial minimal polytopes in fig. 1 the weight vectors are (1, 1, 1) and (1, 1, 2) and they can be obtained as 

\[ w_i = D q_i \]

where \( D \) is the common denominator of the barycentric coordinates \( q_i \) of the origin with respect to the vertices of the minimal simplices, i.e. \( \sum q_i v_j = 0 \) and \( \sum q_i = 1 \). The polar maximal polytopes can be recovered as Newton polytopes (i.e. exponent vectors of the monomials) of generic quasi-homogeneous polynomials of degree \( D \) w.r.t. the weights \( w_i \). The 10 points of the first maximal polytope, for example, correspond to the 10 monomials of degree 3 in 3 variables.

Non-simplicial minimal IP-polytopes are convex hulls of a collection of lower-dimensional minimal IP-simplices and thus define combined weight systems, or weight matrices, whose rows consist of weight vectors augmented by zeros for the vertices that do not belong to the respective simplex. For the third minimal polytope in fig. 1, i.e. the square, the matrix of linear relations reads

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & w_{24} & w_{25}
\end{pmatrix}
\]

Definition 2.4. For \( w_j \in \mathbb{N} \) with \( \gcd(w_j) = 1 \) let \( \Delta(w) = \text{ConvHull}\{\vec{m}\} \) denote the Newton polytope of a generic quasi-homogeneous polynomial \( \sum c_m x_m \) of degree \( D = \sum w_j \), regarded as a lattice polytope on the intersection of the lattice \( \mathbb{Z}^{d+1} \) with the hyperplane \( \vec{m} \cdot \vec{w} = D \). Since \( w_j > 0 \) and \( m_j \geq 0 \) the Newton polytope \( \Delta(w) \) has at most one interior point, namely \( m_j = 1 \). We call \( w \) an IP-weight-vector if \( \Delta(w) \) is an IP-polytope.

Remark 2.5. It follows already from a more general result of Lagarias and Ziegler [10] that the number of IP polytopes, and hence the number of IP weight vectors, is finite in fixed dimension. A constructive proof has been found in terms of an efficient algorithm for the enumeration of all IP-weight-vectors [9, 11].

Remark 2.6. Since all IP-polygons are reflexive, linear relations of vertices of IP-simplices in two dimensions are always IP-weight-vectors. For \( d \geq 3 \), on the other hand, the latter form a proper subset of the set of all linear relations among simplicial IP-polytopes. The lowest degree examples of non-IP-weights of IP simplices in 3, 4 and 5 dimensions are \( P_{1,5,6,8}[20], P_{1,1,5,5,8}[20] \) and \( P_{2,2,2,3,3,1,1}[23] \), respectively. Table 1 contains the complete list for \( d = 3 \) in boldface. Note that \( w_j \leq D/2 \) for all linear relations of IP-simplices because for \( w_0 > w_1 + \ldots + w_d \) the simplex \( \langle v_j \rangle \) with \( \sum w_j v_j = 0 \) has at least two interior lattice points, namely \( 0 \) and \( -v_0 = \sum c_j v_j \) with \( c_0 = \frac{2w_0}{D} - 1 \) and \( c_j = \frac{2w_j}{D} v_j \) for \( j > 0 \), so that \( \sum c_j = 1 \) and \( c_j > 0 \); if \( w_0 = D/2 \) then \( -v_0 \) is in the interior of a facet.

Theorem 2.7 (Skarke [9]). The Newton polytope \( \Delta(w) \) of an IP weight vector \( \vec{w} = (w_0, \ldots, w_d) \) is always reflexive in \( d \leq 4 \) dimensions. More generally, for weight-matrices \( W_{ij} \) with \( 1 \leq i \leq I, 1 \leq j \leq J \) and \( D_i = \sum W_{ij} \) the Newton polytopes \( \Delta(W) \) of generic polynomials of multi-degrees \( D_i \) are always reflexive whenever \( W \) is an IP-weight-matrix (i.e. when \( \Delta(W) \) is an IP polytope) if the dimension \( d = J - I \) of \( \Delta(W) \) obeys \( d \leq 4 \).

The first counterexample is \( \vec{w} = (1, 1, 1, 1, 1, 2) \) with

Figure 1. The 16 reflexive polygons in 2 dimensions: 3 are maximal and the last 4 are selfdual.
Remark 2.8. The number of IP weight vectors is $N_{IP}(1) = 1$, $N_{IP}(2) = 3$, $N_{IP}(3) = 95$ and $N_{IP}(4) = 184026$ for $d \leq 4$. By extrapolation we expect that an enumeration might be possible in $d = 5$ but certainly not above because the numbers become too large. For enumeration problems discussed in the next section it would hence be important to find an efficient generalization of Skarke’s algorithm [9] for IP weight vectors with a bounded number of lattice points in their defining minimal simplex.

Definition 2.9. A weight vector $w$ is called transversal if a generic quasi-homogeneous polynomial $f_{\Delta(w)} = \sum_{m\in\Delta} c_m \prod_j x_j^{m_j}$ of degree $D$ is transversal $df(x) = 0 \Rightarrow \vec{x} = 0$, i.e. the gradient only vanishes at the origin $x_j = 0$. Due to the Euler formula $f(x) = \sum q_j x_j \partial f / \partial x_j$ with $q_j = w_j / D$ the equation $f = 0$ thus defines an isolated singularity [12].

Remark 2.10. Transversal weight vectors are a subclass of the IP weight vectors which coincide with the reflexive weights for $d \leq 3$ and imply reflexivity for $d = 4$ [9]. They have been enumerated for $d \leq 5$ [13–16] with $N_T(1) = 1$, $N_T(2) = 3$, $N_T(3) = 7555$ and $N_T(4) = 1100055$. The reflexive weights are much more numerous than the transversal ones already in 4 dimension. Nevertheless, only 252,933 of the transversal weights in 5 dimensions, about 23%, are reflexive. The above example (1, 1, 1, 1, 1, 2) of a non-reflexive IP weight is also the simplest example of a non-reflexive transversal weight.

The complete algorithm for the enumeration of reflexive polytopes is compiled in [11]. A somewhat subtle detail is the possible choice of a sublattice on the side of the maximal polytope, which is constrained by the fact that the surviving lattice points need to span an IP polytope on the sublattice. An implementation of the algorithm and of additional applications to combinatorics and algebraic geometry has been published in the program package PALP [17]. With some further extensions, which are implemented in the current version 1.1, this package has been used for all computations of results that are report in this note. In particular, we enumerated all lattice polytopes with reflexive dimension$^1$ $rd(\Delta) \leq 4$, whose numbers are listed in table 2.

3. IP-confined polytopes

The algorithm for the enumeration of reflexive polytopes is simplified in $d \leq 4$ by the fact that Newton polytopes $\Delta(W)$ of IP weight matrices are automatically reflexive so that all maximal reflexive polytopes are of this form. Since this is no longer true for $d \geq 5$ we are lead to consider a slightly larger class of IP polytopes.

Definition 3.11. Let $\bar{\Delta} := \text{ConvHull}(\Delta \cap \mathbb{Z}^d)$ be the convex hull of the lattice points in $\Delta^*$. An IP polytope is called IP-confined (IPC) if $\bar{\Delta}$ is a lattice polytope.

Table 1. The 104 IP-simplices in $d = 3$ correspond to 95 transversal and 9 (boldface) non-IP-weights.

| $d$ | $w_1 \ldots w_4$ | $d$ | $w_1 \ldots w_4$ | $d$ | $w_1 \ldots w_4$ | $d$ | $w_1 \ldots w_4$ |
|-----|-----------------|-----|-----------------|-----|-----------------|-----|-----------------|
| 4   | 1 1 1 1         | 2   | 2 2 3 5         | 16  | 1 3 4 8         | 20  | 1 4 5 10        |
| 5   | 1 1 1 1         | 2   | 1 2 4 5         | 16  | 1 2 5 8         | 21  | 3 5 6 7         |
| 6   | 1 1 1 1         | 2   | 1 2 3 6         | 17  | 2 3 5 7         | 21  | 1 5 7 8         |
| 7   | 1 1 1 1         | 2   | 1 1 1 6         | 18  | 3 4 5 6         | 21  | 2 3 7 9         |
| 8   | 1 1 1 1         | 2   | 1 3 4 5         | 18  | 1 4 6 7         | 21  | 1 4 7 9         |
| 9   | 1 1 1 1         | 2   | 1 4 6 7         | 19  | 2 3 5 8         | 21  | 1 5 7 10        |
| 10  | 1 1 1 1         | 2   | 1 4 6 7         | 20  | 1 3 7 10        |
| 11  | 1 1 1 1         | 2   | 1 4 6 7         | 20  | 1 5 6 8         |
| 12  | 1 1 1 1         | 2   | 1 4 6 7         | 20  | 1 4 5 9         |
| 13  | 1 1 1 1         | 2   | 1 4 6 7         | 20  | 1 2 3 5         |

Table 2. Numbers of lattice polytopes with reflexive dimension $\leq 4$.

| $d$ | rd=1 | rd=2 | rd=3 | rd=4 |
|-----|------|------|------|------|
| d=1 | 1    | 3    | 7    | 54   |
| d=2 | 16   | 328  | 230109 |
| d=3 | 4319 | 45986238 |
| d=4 | 473800776 |
The polytope $\tilde{\Delta} \subset \Delta$ is called the IPC-closure of $\Delta$ and $\Delta$ is called IPC-closed if $\Delta = \tilde{\Delta}$.

**Remark 3.12.** The operation $\Delta \rightarrow \tilde{\Delta}$ extends the polar duality of reflexive polytopes to an involution on the set of IPC-closed polytopes. (Reflexive polytopes are obviously IPC-closed; according to Skarke’s theorem [9] the reverse is true for $d \leq 4$.)

**Corollary 3.13.** Each IP-confined polytope (and hence also each reflexive polytope) is a subpolytope of a maximal IPC-closed polytope $\Delta_M$. These maximal polytopes are of the form $\Delta_M = \text{ConvHull}(\Delta(W) \cap \Gamma)$ where $W$ is an IP-weight-matrix with $\text{rank}(W) \leq d$ and $\Gamma \subset \mathbb{Z}^d$ is a sublattice of the natural lattice of the Newton polytope for which $\Delta(W) \cap \Gamma$ still has an interior point.

**Proof.** The proof is analogous to the one for the reflexive case [11] and uses the generalized duality $\Delta \leftrightarrow \Delta'$ of definition 3.11. The polytopes $\Delta$ that are dual to the maximal IPC-closed polytopes are minimal in the set of IPC-closed polytopes, but not necessarily minimal as IP polytopes. For every minimal IP polytope $\Delta' \subseteq \Delta_M$, however, $\Delta_M$ is the IPC-closure, because otherwise $\Delta_M$ would not be maximal. A weight matrix $W$ with $\Delta_M = \text{ConvHull}(\Delta(W) \cap \Gamma)$ can hence be constructed by regarding $\Delta'$ as the convex hull of lower-dimensional IP-simplices $\Sigma_i$ (which are IP-confined since the duals to the restrictions to the linear subspaces of $\Sigma_i$ are projections of $\Delta_M$, which provide confining IP polytopes for the simplices $\Sigma_i$). Each simplex in the decomposition can be chosen such that it contains at least two vertices of $\Delta'$ that are not contained in a different simplex of the decomposition. The maximal rank $d'$ hence occurs if the maximal polytope is a hypercube. \qed

The numbers of reflexive polytopes in $d$ dimensions are the diagonal entries of table 2. A double-exponential ansatz $N_d \approx 2^{2^{d+1} - 4}$ due to Skarke provides a good fit to these data and would predict $N_5 \approx 1.2 \cdot 10^{18}$ and $N_6 \approx 2.1 \cdot 10^{37}$ in 5 and 6 dimensions, respectively. If this gives the correct order of magnitude an enumeration would already be hopeless in 5 dimensions because the result could not be stored on any existing medium (for the 4d-case we needed, on average, about 20 byte, or five 32-bit integers, to store the data of a reflexive pair). Bounds on the numbers of lattice points have been obtained in [19].

While the case of 4-dimensional reflexive polytopes is of direct relevance to algebraic geometry and string theory because the corresponding toric hypersurfaces are Calabi-Yau 3-folds [3], higher-dimensional cases are important because F-theory applications [20] in physics require (elliptically fibered) Calabi-Yau 4-folds, and, more generally, reflexive polytopes in $n + r$ dimensions are related to complete intersection $n$-folds of codimension $r$ [21–23]. From a pragmatisical point of view, in both contexts the polytopes with a small number of points are the ones that we are most interested in because they encode the combinatorial data for manifolds with (moderately) small Picard numbers. This suggests a different setting for enumeration attempts where we ask for the lists of reflexive polytopes with a certain (maximal) number of points rather than a fixed dimension. Instead of descending from maximal objects, which is not feasible for $d > 4$, one might hence add points to a list of minimal objects.

**Question 3.14.** Is there an efficient algorithm for the enumeration of weight vectors $\vec{w}$ that define IP-confined simplices $\sum_j w_j v_j = 0$ with a bounded number of lattice points?

While we expect the number of IP-weight-vectors to be too large for $d > 5$ for a complete enumeration, we also expect that the enumeration of all reflexive polytopes with less than, say, 12 or 15 lattice points should be feasible if the questions 3.14 has a positive answer. Here the natural objects of interest are IP-confined lattice polytopes because every IPC-closed polytope, and hence every reflexive polytope, can be constructed by successively adding lattice points to the minimal objects in a finite number of ways as determined by the relevant weight vectors.

### 4. Statistics and numerical experiments

We now turn to statistical considerations that may provide additional information in cases where a complete enumeration is not possible. The first question we want to address is whether the property of self-duality of reflexive lattice polytopes is random in the sense that the number of selfdual objects can be estimated by a probabilistic calculation.

Let $N$ be the number of reflexive polytopes in fixed dimension $d$. Then the number of (formal) duality assignments with $S$ self-mirrors, i.e. the number of involutions with $S$ fixed points, is $\binom{N}{S}$ for the choice of $S$ self-mirrors times $(N - S - 1) (N - S - 3) \ldots 3 \cdot 1$ for the possible selections of the remaining dual pairs, hence

$$n_S = \frac{N!}{S! \cdot 2^{\frac{N - S}{2}} (\frac{N - S}{2})!} \quad \text{with} \quad S - N \in 2\mathbb{Z}. \quad (4.2)$$

Let $Z_N = \sum_{S \leq N} n_S$ be the total number of involutions. The asymptotic expansion for large $N$ can be derived from the generating function [24, section 3.8]

$$e^{x + \frac{1}{4} x^2} = \sum_{N \geq 0} \frac{Z_N}{N!} x^N. \quad (4.3)$$

Since it happens $Z_{N-1}$ times for the $Z_N$ involutive permutations of $N$ objects that a given object is a fixed point, we obtain the following formula for the
expectation value \[ \langle S \rangle = \frac{1}{Z_N} \sum S n_S S = N \frac{Z_{N-1}}{Z_N}. \]  

Assuming a uniform probability distribution of involutions we thus expect
\[ \langle S \rangle = \sqrt{N} - \frac{1}{2} + \frac{1}{2\sqrt{N}} + O\left(\frac{1}{N}\right) \]

self-dual polytopes for large \( N \). As shown in table 3 this roughly explains the size but underestimates the correct numbers for \( d \leq 4 \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
 d & 1 & 2 & 3 & 4 \\
\hline
 N_P(d) & 1 & 16 & 4319 & 473800776 \\
 N_{selfdual} & 1 & 4 & 79 & 41710 \\
 \langle S \rangle & 1 & 3.6 & 65.2 & 21766.5 \\
\hline
\end{array}
\]

Table 3. Numbers of self-dual polytopes and probabilistic expectations.

As this result seems to indicate that a statistical approach makes sense we now want to use similar considerations for predicting the number of reflexive polytopes in 5 dimensions on the basis of incomplete lists. In a random set of \( p > \sqrt{N} \) reflexive polytopes in fixed dimension the formula \( \langle S \rangle \approx \sqrt{N} \) implies that we expect \( s \approx \sqrt{N} \cdot p/N \) self-mirror polytopes. This leads to the prediction \( N \approx (p/s)^2 \) if we find \( s \) self-dual polytopes in the sample. Similarly, if we ignore the relatively small number of self-mirrors for large \( N \) and count the number \( m \) of mirrors pairs in a sample of \( p \) polytopes we obtain the prediction \( N \approx p^2/(2m) \). If we increase the size of a random sample we expect \( s \) to grow linearly and \( m \) quadratically with \( p \), and more precisely \( s \approx \sqrt{2m} \).

Actually, the formula \( N \approx p^2/(2m) \) has been used already several years ago when we enumerated the reflexive polytopes in 4 dimensions [7], which required two years of program improvements and computation time after the 3-dimensional case [6]. We thus could check the sufficiency of the implemented data structures for the storage of the result at an early stage of the project. The starting point of the calculation was the list of 308 weight matrices (206 weight vectors and 102 matrices with \( 2 \leq \text{rank} \leq 4 \)) of maximal reflexive polytopes \( \Delta_M \) for which the reflexive subpolytopes were computed in the order of an increasing number of lattice points. The first Newton polytope in this list is defined by the weight vector \((3, 3, 4, 4, 10)\) with degree 24. It has 47 lattice points and 6 vertices. After fetching and compiling PALP its subpolytopes can be computed with the following commands,

```bash
$ gunzip palp-*.tar.gz; tar -xvf palp-*.tar; cd palp; # unpack
$ wget hep.itp.tuwien.ac.at/~kreuzer/CY/palp/palp-1.1.tar.gz # fetch
$ Wdir=$PWD # working directory
$ Bdir=$HOME/bin # directory for binary files (check $PATH)
$ cd /tmp # temporary directory
$ echo '24 3 3 4 4 10' | class.x -f -po /tmp/zbin.47
$ make; mv *.x $Bdir; cd $Wdir # and compile PALP
```

Within two minutes we thus obtain the values 269 million for \( p^2/2m \) with \( p=798878 \) and \( m=1181 \) and 3.25 billion for \( p^2/s^2 \) with \( s=14 \) on a standard 3GHz PC (cf. the last output line) and hence a good approximation of the correct value 473 800 776 with a production rate of about 7000 polytopes per second (back in 1998 the CPU time was almost 1 hour).

For 5 dimensions it is, of course, much harder to get a reliable statistics. With an expectation of \( N \approx 10^{18} \) according to Skarke’s guess, the storage of \( \sqrt{N} \) polytopes would already required some 30GB of disk space so that we can only go above that value by 1-2 orders of magnitude with currently available hardware. Nevertheless, it should be possible to either verify that the number is not much smaller or to get a reasonable prediction if it is. For a first attempt we defined data samples in terms of the Newton polytopes of transversal reflexive weight vectors ordered according to increasing numbers of lattice points. For the increasing series data samples consisting of all reflexive subpolytopes of transversal Newton polytopes with \( \leq 36 \ldots \leq 65 \) points the predictions are plotted in fig. 2.

The result is obviously inconclusive, but certainly compatible with the guess \( 10^{18} \). Unfortunately the

\[ \begin{align*}
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\end{align*} \]
bias of the a priori independent predictions $p^2/2m$ and $p^2/s^2$ is strongly correlated in our data samples. For the $9.025 \cdot 10^9$ polytopes of the largest sample with $m = 86323$ and $s = 354$, which occupies 239 GB of disk space, we find $p^2/2m \approx 4.7 \cdot 10^{14}$ and $p^2/s^2 \approx 6.5 \cdot 10^{14}$.

5. Conclusions

In this note we determined all lattice polytopes with reflexive dimension $rd \leq 4$ and discussed enumeration problems and algorithmic aspects with applications to algebraic geometry and string theory. We pointed out the need for an efficient algorithm for the enumeration of IP weight vectors $w$ with a bounded number of lattice points in the convex hull of the simplex defined by the linear relations $\sum w_i v_j = 0$. Such an algorithm could be used for the enumeration of reflexive polytopes with fixed number of points rather than fixed dimension.

We introduced the concept of IP-confined polytopes, which are a subclass of IP polytopes, and extended the polar duality of reflexive polytopes to IPC-closed polytopes. Maximal IPC-closed polytopes contain all reflexive polytopes in arbitrary dimensions and hence lead to a simplification of the classification program. In turn, we pointed out the existence of IP-simplices that are not IP-confined and enumerated them for the case of 3 dimensions. A constructive classification of such simplices in higher dimensions is another interesting open problem.

We suggested a statistical approach to the enumeration of reflexive polytopes which should at least allow us to obtain probabilistic lower bounds, depending essentially on the size of the available hard-disks for storage of the data. As a first attempt we constructed a data-base containing about $9 \cdot 10^9$ pairs of reflexive 5-dimensional polytopes, which can also be used to produce incomplete lists of polytopes with reflexive dimension 5. Studies of correlations of polytope data like $f$-vectors (numbers of faces) can thus be initiated and may be useful for selecting appropriate data samples for statistical applications.

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