FLUCTUATIONS OF THE SYMMETRIC PERCEPTRON

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Abstract. We study the fluctuations in the storage capacity of the symmetric binary perceptron, or equivalently, the fluctuations in the combinatorial discrepancy of a Gaussian matrix. Perkins and Xu [16], and Abbe, Li, and Sly [2] recently established a sharp threshold: for some explicit constant $K_c$, the discrepancy of a Gaussian matrix is $K_c + o(1)$ with probability tending to one, but without a quantitative rate. We sharpen these results significantly. We show the fluctuations around $K_c$ of the discrepancy are at most of order $\log(n)/n$, and provide exponential tail bounds. Up to a logarithmic factor, this yields a tight characterization for the fluctuations of the symmetric perceptron and combinatorial discrepancy.

1. Introduction

The combinatorial discrepancy of a matrix $A \in \mathbb{R}^{m \times n}$ is given by the following optimization problem:

$$\text{disc}(A) := \min_{\sigma \in \{-1, +1\}^n} \| A\sigma \|_\infty.$$ 

Sometimes called the “vector balancing problem,” discrepancy is equivalent to the task of giving $\pm$ signs to $n$ vectors in $\mathbb{R}^m$—namely the columns of $A$—so that their sum is as close to balanced as possible. Discrepancy arises as a fundamental quantity in combinatorics, functional analysis, geometry, and optimization. Much recent work in discrepancy theory focuses on understanding phase transitions for the discrepancy of random matrices [1–4,6,8,9,12,13,16,25].

Recently, Aubin, Perkins, and Zdeborová [4] established a connection between combinatorial discrepancy and the celebrated Binary Perceptron that has already lead to several breakthroughs in both problems. The perceptron is an idealized model of learning that has enjoyed almost six decades of intense study by the statistical physics community [11,14,20,24]. While there is an extensive body of detailed physics predictions for the binary perceptron, these predictions have largely resisted rigorous mathematical treatment. The contribution of [4] was to introduce a “symmetrized” binary perceptron (SBP), which is more amenable to analysis and is conjectured to capture all the interesting behaviours of the original model.

Definition 1.1 (Storage Capacity). Consider a sequence of iid Gaussian vectors $\{a_i\}_{i \in \mathbb{N}}$ with

$$a_i \sim N(0, n^{-1}I_{m \times n})$$

for each $i$. For a fixed parameter $K$, the storage capacity of the SBP associated with $\{a_i\}$ is the largest $\alpha$ so that $\alpha n \in \mathbb{Z}$ and there exists $\sigma \in \{\pm 1\}^n$ with

$$\max_{i \in [\alpha n]} |\langle a_i, \sigma \rangle| \leq K.$$ 

For a random Gaussian matrix, understanding the storage capacity and understanding the discrepancy are the same task: the former corresponds to fixing $K$ and varying $\alpha$, while the latter fixes $\alpha$ and varies $K$. Thus, a pressing question for both communities is to understand the discrepancy of a random matrix as exactly as possible. Consider one of the simplest models:

Definition 1.2 (Gaussian Ensemble). Let $\alpha := \alpha(n)$. We say that an $\alpha n \times n$ matrix $A$ is $(\alpha, n)$-Gaussian if it has iid normal entries with mean 0 and variance $1/n$.

We restrict to $\alpha := \alpha(n)$ with $\lim_{n \to \infty} \alpha \in (0, \infty)$. See Section 2 for discussion of $\alpha \to 0$. The normalization of $A$ is exactly so that $\text{disc}(A)$ is on the constant scale. Indeed, it is easy to check by the first moment method that there exists some critical value $K_c := K_c(\alpha) \in (0, \infty)$ so that $\text{disc}(A) > (K_c - \epsilon)$ with
exponentially high probability, for any \( \epsilon > 0 \). It was also shown by Aubin, Perkins, and Zdeborová [4] via the second moment method,

\[
\forall \epsilon > 0, \quad \liminf_{n \to \infty} \mathbb{P}[\text{disc}(A) < K_c + \epsilon] > 0. \tag{1.1}
\]

They also conjectured that \( K_c \) corresponds to a sharp threshold, namely that this upper bound actually holds with high probability rather than just positive probability. The failure of the second moment method to yield a sharp threshold is a key feature of the original asymmetric perceptron, as well as a variety of other interesting constraint satisfaction problems. Perkins and Xu [16] and Abbe, Li, and Sly [2] simultaneously verified this conjecture. Their results remain the state-of-the-art:

\[
\forall \epsilon > 0, \quad \mathbb{P}[(K_c - \epsilon) < \text{disc}(A) < (K_c + \epsilon)] \to 1. \tag{1.2}
\]

However, no further non-trivial quantification of this result is known. In particular, both the rate at which \( \text{disc}(A) \) is tending to \( K_c \) as well as probabilistic tail bounds are open. Building off of the techniques of Perkins and Xu, our main result is to resolve both of these by giving an almost tight characterization of discrepancy.

**Theorem 1 (Main Result; Discrepancy).** Fix any \( \alpha \in (0, \infty) \) and let \( A \) be from the \((\alpha, n)\)-Gaussian ensemble. There exists positive constants \( c \) and \( C \) depending only on \( \alpha \) so that for any sufficiently large fixed \( x \), the following holds for all sufficiently large \( n \):

\[
\mathbb{P}\left[|\text{disc}(A) - K_c(\alpha)| > \frac{x \log(n)}{n}\right] \leq C \exp\{-cx \log(n)\}. \tag{1.3}
\]

Further, there exists some other positive constants \( c \) and \( C \) depending only on \( \alpha \) so that for all \( n \) sufficiently large,

\[
\frac{c}{n} \leq \text{Var}(\text{disc}(A))^{1/2} \leq \frac{C \log(n)}{n}. \tag{1.4}
\]

Conveniently, one can think interchangeably about fixing \( \alpha \) and increasing \( K \) or fixing \( K \) and decreasing \( \alpha \). The former is adopted in the discrepancy literature, the latter in the binary perceptron literature, and they turn out to be equivalent in our setting (e.g. Lemma 5.1).

**Corollary 1.1 (Main Result; Perceptron).** Fix \( K > 0 \) and consider a sequence of iid Gaussian vectors \( \{a_i\}_{i \in \mathbb{N}} \) with \( a_i \sim N(0, n^{-1}I_{n \times n}) \) for each \( i \). Let \( \alpha^* \) denote the storage capacity of the associated symmetric perceptron. There exists positive constants \( c \), \( C \), and \( \alpha_c \) depending only on \( K \) so that for any sufficiently large fixed \( x \), the following holds for all sufficiently large \( n \):

\[
\mathbb{P}\left[|\alpha^* - \alpha_c(K)| > \frac{x \log(n)}{n}\right] \leq C \exp\{-cx \log(n)\}. \tag{1.5}
\]

Further, there exists some other positive constants \( c \) and \( C \) depending only on \( K \) so that for all \( n \) sufficiently large,

\[
\frac{c}{n} \leq \text{Var}(\alpha^*)^{1/2} \leq \frac{C \log(n)}{n}. \tag{1.6}
\]

Some remarks:

1. General tools for upper bounding the standard deviation of random variables fall quite short of \( \log(n)/n \). The Gaussian Poincaré inequality yields that \( \text{Var}(\text{disc}(A)) \leq 1 \). There is also a result of Xu giving a sharp threshold for the asymmetric perceptron [26], but with a rate of \( (\log(\log(n)))^{-1/10} \).
2. It is natural to compare discrepancy to the smallest singular value since they solve somewhat similar minimization problems. Theorem 1 shows discrepancy is significantly more concentrated, with fluctuations of order \( n^{-1} \), whereas the smallest singular value of a Gaussian matrix follows the Tracy-Widom law with fluctuations of order \( n^{-2/3} \) [21].

We also highlight a significant strengthening of Eq. (1.1) that is needed as an intermediate step in the proof of Theorem 1. We are able to show that the second moment method yields a non-trivial bound even exactly down to the critical threshold \( K_c \). This is new and perhaps surprisingly.

**Theorem 2 (Second Moment Method).** Fix any \( \alpha > 0 \). There exists \( C := C(\alpha) > 0 \) so that

\[
\mathbb{P}[\text{disc}(A) \leq K_c(\alpha)] > C. \tag{1.7}
\]
Since the second moment method can be carried out all the way to criticality via Theorem 2, it is natural to suspect similar improvements are possible in Eq. (1.2), the sharp threshold of [16] and [2]. Indeed, this is the content of Theorem 1. We choose to refine the techniques of [16] because they naturally lead to strong tail bounds, albeit at the cost of an extra logarithm in Eq. (1.4). It seems likely that the techniques of [2] can be used to show \( \text{disc}(A) = K_c + \Theta(1/n) \), resolving this extra log. However, the techniques of [2] are asymptotic and do not easily lead to tail bounds.

Simultaneously providing strong tail bounds and capturing the exact size of fluctuations is left as an open question; we conjecture that our main result can be improved both by removing the logarithmic mismatch in Eq. (1.4), and also by giving subgaussian rather than subexponential tails in Eq. (1.3).

**Conjecture 1.** \( |\text{disc}(A) - K_c| \) is \( 1/n \)-subgaussian.

## 2. Related Works

**Previous Results.** The connection between discrepancy and the perceptron model was introduced by Aubin, Perkins and Zdeborová in 2019 [4]. In their work, they carry out a second moment computation and are able to show—conditional on a numerical conjecture (see Section 3)—that \( \text{disc}(A) \) has positive probability of being less than \( K_c + \epsilon \) for any positive \( \epsilon \). In other words, the fluctuations of \( \text{disc}(A) \) are at most order 1. Their work also establishes the negative result that the second moment method does not suffice to prove a sharp threshold, but they conjectured that a sharp threshold should still occur.

This conjecture was resolved by Abbe, Li, and Sly [2], and Perkins and Xu [16] simultaneously: both groups improved the upper bound on fluctuations to \( o(1) \). Abbe, Li, and Sly used an analogue of the so-called small subgraph conditioning technique of Robinson and Wormald [19], that was recently adapted to dense graphs by Banerjee [5]. Perkins and Xu adapted and improved a technique of Talagrand [23] for the asymmetric perceptron. Sun and Nakajima [15] recently extended the method of Talagrand to give a sharp threshold for wide class of random matrices, yielding comparable results to [2,16] in a broader setting (including the asymmetric problem).

**Algorithms.** The foundational result of combinatorial discrepancy is Spencer’s “Six standard deviations suffice” [22], which asserts for any square matrix \( A \) with entries in \([-n^{-1/2}, n^{1/2}]\), deterministically \( \text{disc}(A) \leq 6 \). Finding an efficient algorithm that achieves Spencer’s bound took almost 25 years. While a number of such algorithms were discovered in the last decade, and while many of these easily generalize beyond bounded entries to our setting of iid random entries (e.g. [7]), they are all far from achieving the optimal constant \( K_c \). There are interesting connections between average-case complexity and the geometry of the “solution space” for optimization problems. We refer the interested reader to [1,2,4,10,16] for further discussion.

**Rectangular matrices.** The discrepancy of \( \alpha \times n \) matrices with \( \alpha \to 0 \) has connections to the long-standing Beck-Fiala conjecture, as well as applications in a surprising range of combinatorial optimization problems. The case of matrices with iid Bernoulli(p) entries is of particular interest in applications. The second moment method readily gives a sharp threshold for Gaussian matrices [25]. However for Bernoulli matrices, there are fundamental obstacles to analyzing the second moment if \( p := \rho(n) \) vanishes quickly. A sharp characterization of discrepancy was first established for large values of \( p \) [6,8,9,12,13,18], and eventually for all \( p \) [3].

## 3. Preliminaries

**Notation:** throughout this work, we use \( c \) and \( C \) to denote positive universal constants that may take different values on different lines. Any important constants will be distinguished with subscripts.

### 3.1. First Moment.** Let us adopt the language of random Constraint Satisfaction Problems (CSP’s). For an \((\alpha,n)\)-Gaussian matrix \( A \), define the sublevel set \( Z \) by:

\[
Z_{K,\alpha} := \{ x \in \{ \pm 1 \}^n : \|Ax\|_\infty \leq K \}, \quad \text{disc}(A) = \inf \{ K : Z_K \neq \emptyset \}.
\]

Then \( Z_{K,\alpha} \) is the set of solutions to the “discrepancy instance” with parameters \( K \) and \( \alpha \) given by the matrix \( A \). The matrix \( A \) encodes a set of constraints: for each row \( a_i \), \( x \) satisfies \( |\langle x, a_i \rangle| \leq K \). It is natural to suspect that for fixed \( \alpha \), the probability of \( Z_K \) being empty undergoes a rapid transition from zero to one as \( K \) increases.
A standard first approach to understanding the value of an optimization problem is the first moment method. If $\mathbb{E} [\|Z_K\|]$ is vanishing, then by Markov’s inequality with high probability there are no solutions. This provides a lower bound on disc($A$). For an $(\alpha, n)$-Gaussian matrix $A$, it is an elementary computation that for $Z$ a standard normal,

$$\frac{1}{n} \log \mathbb{E} [\|Z_{K,\alpha}\|] = \log(2) + \alpha \log p_K,$$

$$p_K := \mathbb{P} [\|Z\| \leq K] = \frac{1}{\sqrt{2\pi}} \int_{-K}^{K} e^{-x^2/2} dx.$$

For fixed $K$, we define $\alpha_c(K)$ the “critical value” of $\alpha$ so that the expected number of solutions is one:

$$\alpha_c(K) := -\frac{\log(2)}{\log p_K} \quad (3.1)$$

Conversely, for fixed $\alpha$, we define $K_c(\alpha)$ the “critical value” of $K$ as the unique solution to

$$\alpha_c(K) = \alpha.$$

A key relation we will repeatedly reference is the expansion:

**Proposition 3.1 (Expansion of $\mathbb{E} [\|Z\|]$).** There exists positive constants $c$ and $C$ so that for any $(K, \alpha)$ with $|K - K_c(\alpha)| < \epsilon$, we have:

$$C^{-1} |K - K_c(\alpha)| \leq (\alpha n)^{-1} \log \mathbb{E} [Z_{K,\alpha}] \leq C |K - K_c(\alpha)|, \quad (3.2)$$

*Proof.* Since $\log \mathbb{E} [\|Z_{K,\alpha}\|] = \alpha n (\log p_K - \log p_K)$, Eq. (3.2) follows by a first-order Taylor expansion of $p_K$ with respect to $(K - K_c)$, which is easily computed by the fundamental theorem of calculus.

By Markov’s inequality, we then immediately have control over the lower tail of disc($A$).

**Corollary 3.1 (Lower tail).** Fix $\alpha \in (0, \infty)$. There exists some positive constants $c$ and $\epsilon$ so that for all $y \in [0, \epsilon]$,

$$\mathbb{P} [(\text{disc}(A) - K_c)_- > y] < e^{-c n y}.$$

We also easily obtain a one-sided variance bound:

**Corollary 3.2.**

$$\mathbb{E} [(\text{disc}(A) - K_c)^2]^{1/2} = \mathcal{O} \left( n^{-1} \right).$$

*Proof of Corollary 3.2.* Integrating the tail bound in Corollary 3.1 for $y \in [0, \epsilon]$, and noting that trivially $(\text{disc}(A) - K_c)_- \leq K_c$ deterministically for $y > \epsilon$,

$$\mathbb{E} [(\text{disc}(A) - K_c)^2] \leq \int_{0}^{\epsilon} 2c e^{-c n y} dy + (K_c - \epsilon) e^{-c n \epsilon} \leq \mathcal{O} \left( n^{-2} \right).$$

In summary, the first moment of $|Z|$ shows that disc($A$) cannot be too much smaller $K_c$. More difficult is the upper tail, i.e. showing that disc($A$) is not too much larger than $K_c$.

### 3.2. Second Moment.

We turn to the second moment method for an upper bound on disc($A$). If the variance of $Z_{K,\alpha}$ is small, then disc($A$) will concentrate around the first value $K$ for which $\mathbb{E} [\|Z_K\|] \geq 1$.

Aubin, Perkins, and Zdeborová [4] established (originally contingent on a numerical hypothesis, later removed by [2])

**Theorem 3 (Positive probability; [4]).** Fix any $\alpha > 0$ and $\epsilon > 0$. Then there exists positive constants $c$ and $C$ with $c > 1$, so that for all $n$ sufficiently large,

$$c \leq \frac{\mathbb{E} [\|Z_{K_c(\alpha) + \epsilon, \alpha}\|^2]}{\mathbb{E} [\|Z_{K_c(\alpha) + \epsilon, \alpha}\|^2]} \leq C.$$
By the Paley-Zygmund inequality, an upper bound on \( \text{disc}(A) \) immediately follows: for any constants \( \alpha > 0 \) and \( \epsilon > 0 \),
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \text{disc}(A) < K_c + \epsilon \right] > 0.
\]
However, since \( c > 1 \), Theorem 3 also establishes that the variance of \( Z_K \) is too large to directly show that \( \text{disc}(A) < K_c + \epsilon \) with high probability. (Of course, \( Z_K \) having large variance certainly does not imply that \( \text{disc}(A) \) has large variance. A sharp threshold for \( \text{disc}(A) \) only requires that the event \( \{|Z_K| > 0\} \) is well-concentrated).

Moving beyond the failure of the second moment method to establish a sharp threshold is often quite difficult, and indeed the results of [2] and [16] required new ideas. Our work builds on [16], which in turn builds on an idea of Talagrand [23]. Let us begin by formally stating the result of [16] that we wish to refine:

**Theorem 4** ([16], Theorem 9). Fix \( K > 0 \) and \( \epsilon > 0 \). Let \( \alpha \) be such that \( \alpha \leq \alpha_c(K) - \epsilon \). Then, for every \( \delta > 0 \), there exists \( M = M(\delta) \) so that
\[
\limsup_{n \to \infty} \mathbb{P} \left[ \log \left( \frac{|Z_{K,\alpha}|}{\mathbb{E}[|Z_{K,\alpha}|]} \right) \geq M \log n \right] \leq \delta.
\] (3.3)

Similarly to Theorem 3, the original statement of Theorem 4 was originally contingent on a numerical conjecture (Eq. (3.6) below) that can be removed by the results of [2]. By the expansion given in Eq. (3.2), for \( K \) close to \( K_c \) we have
\[
\mathbb{E}[|Z_{K,\alpha}|] = \exp \{\Omega(n|K - K_c(\alpha)|)\},
\]
It is easy (see e.g. Lemma 5.1 below) to also write this in terms of \( |\alpha - \alpha_c(K)| \), since
\[
|K - K_c(\alpha)| = \Theta(\alpha - \alpha_c(K))
\]
as \( K - K_c(\alpha) \) tends to 0. Then for any \( \epsilon > 0 \), the expected number of solutions \( |Z| \) is exponential for \( \alpha < \alpha_c - \epsilon \). According to Theorem 4 however, the multiplicative fluctuations of \( |Z| \) are at most polynomial. Thus, our goal is to show the bound of Theorem 4 continues to hold even as \( |\alpha - \alpha_c| \) vanishes.

In the proof of the sharp threshold Theorem 4, the boundedness of the second moment ratio given by Theorem 3 is an important intermediate step. Correspondingly, as a first step, we strengthen Theorem 3 by showing it holds all the way down to criticality, i.e. that \( \text{disc}(A) \leq K_c \) with positive probability. (C.f. Theorem 3 as written, which only allows for \( K \) constant and strictly separated from \( K_c \), i.e. \( K = K_c + \epsilon \) for \( \epsilon > 0 \). The resolution we consider is a factor of \( n \) smaller). Our goal is:

**Theorem 5** (Second Moment Method). For any positive \( K > 0 \), there exists a strictly positive constant \( C \) so that for any \( \alpha := \alpha(n) \leq \alpha_c(K) \) and \( A \) an \( (\alpha, n) \)-Gaussian matrix,
\[
\frac{\mathbb{E}[|Z_{K,\alpha}|^2]}{\mathbb{E}[|Z_{K,\alpha}|]^2} < C.
\] (3.4)

Consequently,
\[
\mathbb{P} \left[ \text{disc}(A) \leq K_c(\alpha) \right] > C^{-1}.
\] (3.5)

Eq. (3.4) will be used as an important step in our strengthening of Theorem 4, and Eq. (3.5) will be the main ingredient for our lower bound on the variance of \( \text{disc}(A) \). We prove Theorem 5 in the next section.

In order to compute the second moment of \( |Z| \), we study the so-called free energy. The free energy is a measure of how much pairs of corners \((x, y)\) from the discrete hypercube contribute to the second moment \( \mathbb{E}[|Z_i|^2] \), as a function of the normalized Hamming distance \( n^{-1} \sum_{i=1}^n \mathbb{1}(x_i \neq y_i) \).

**Definition 3.1** (Pair Probability). For \( X \) and \( Y \) independent standard normal, let
\[
q_K(\beta) := \mathbb{P} \left[ \sqrt{\beta}X + \sqrt{1 - \beta}Y \leq K, \ |\sqrt{\beta}X - \sqrt{1 - \beta}Y| \leq K \right].
\]
Then \( q_K(\beta)^{\alpha n} \) is the probability that \( x \) and \( y \) with \( \langle x, y \rangle = (2\beta - 1)n \) are both in \( Z_{K,\alpha} \).

**Definition 3.2** (Free Energy). Let \( H \) be binary entropy; define the free energy \( F : [0, 1] \to \mathbb{R} \) by
\[
F(\beta) := F_{K,\alpha,n}(\beta) = H(\beta) + \alpha \log q_K(\beta).
\]
Let us collect some trivial facts about $F$ for intuition. First, Gaussian random variables which are orthogonal are also independent, so $q(1/2) = p^2$ for any $n, K, \alpha$. Second, the functions $q(\beta)$ and $H(\beta)$ are symmetric around $\beta = 1/2$, so $F$ is as well. Finally, by construction of $\alpha_c$, for any $K$ and $n$,

$$F_{K, \alpha_c(K), n}(\frac{1}{2}) = F_{K, \alpha_c(K), n}(0) = -\log(2).$$

Regarding the shape of $F$, the following was conjectured and assumed in [4] and [16] (as well as an earlier version of [2]) in order for second moment method computations to be tractable:

**Conjecture 2** (Shape of the Free Energy; conjecture). For all positive $K$ and $\alpha$ with $F''_{K, \alpha, n}(1/2) < 0$, it further holds that $F := F_{K, \alpha, n}$ has a single critical point in $(0, 1/2)$. In particular, for any $0 \leq a \leq b \leq 1/2$,

$$\max_{a \leq \beta \leq b} F(\beta) \in \max \{F(a), F(b)\}. \quad (3.6)$$

Compelling numerical evidence was supplied in [4]. Later, in [2], a slightly weaker form of this conjecture was recovered, for $K > 0$ and $\alpha < \alpha_c$. The main difference is that for some $K$, it is only easy to rigorously establish that $F$ is decreasing near 0 and increasing near 1/2. In between, it is far easier to simply show that $F$ is much less than $F(1/2)$—rather than controlling the number of critical points—which is more than enough for second moment method applications. We now collect the facts, rigorously established in [2], that we will need:

**Lemma 3.1** (Shape of the Free Energy [2]). Consider the free energy $F := F_{K, \alpha, n}$.

1. For any $\alpha, K, n$,

$$F''(\frac{1}{2}) = 4 \left(-1 + \frac{2\alpha K^2 e^{-K^2}}{\pi p^2}\right). \quad (3.7)$$

2. Fix any $K > 0$. There exists $\epsilon := \epsilon(K) > 0$ sufficiently small so that for any $\alpha \leq \alpha_c(K)$,

$$F''\left(\frac{1}{2}\right) < -\epsilon. \quad (3.8)$$

3. There exists $b := b(K) > 0$ such that, for all $\alpha \leq \alpha_c$, $F_{\alpha}(\beta)$ is decreasing for $\beta \in [0, b]$.

4. Let $K > 0$ and $F_{\alpha_c} := F_{K, \alpha_c(K), n}$. There exists $\epsilon := \epsilon(K) > 0$ such that, for any $a, b \in [0, 1/2]$,

$$\max_{\beta \in [a, b]} F_{\alpha_c}(\beta) \leq \max \{F_{\alpha_c}(a), F_{\alpha_c}(b), F_{\alpha_c}(1/2) - \epsilon\}. \quad (3.9)$$

The proof of Lemma 3.1 is deferred to the appendix, since it is directly reproduced from [2] with only trivial modifications. Upon first read, one could also simply assume Eq. (3.6).

3.3. **Sharp Threshold.** We seek to boost the “positive probability” guarantee of Theorem 5 into a “high probability” guarantee. Our goal is: if $\log(\mathbb{E}[|Z_{K, 0}|]) \gg \log n$, then $|Z_r| \neq 0$. Making this precise and also giving a quantitative tail bound, we will show:

**Theorem 6** (Upper tail). Fix $K > 0$; there exists a constant $c := c(K) > 0$ so that the following holds. Fix any $x > 0$ sufficiently large and define the function $\alpha := \alpha(n)$ with $\alpha = \alpha_c(K) - x n^{-1} \log(n)$. Then for all $n$ sufficiently large,

$$\mathbb{P}[|Z_{K, 0}| = 0] < n^{-cx}. \quad (3.10)$$

Equivalently, for any $\alpha > 0$ there exists some constant $c := c(\alpha) > 0$ so that for any fixed $x$ sufficiently large, for all $n$ sufficiently large

$$\mathbb{P}\left[\text{disc}(A) > K_c + \frac{x \log(n)}{n}\right] < n^{-cx}. \quad (3.11)$$

It is fairly straightforward to control the variance of $\text{disc}(A)$ by integrating this tail bound. The only complication is that our tail bound does not allow for $x$ growing as a function of $n$.

**Corollary 3.3.** Under the conditions of Eq. (3.11), we also have the one-sided variance bound

$$\mathbb{E}\left[(\text{disc}(A) - K_c)^2\right]^{1/2} = O\left(\frac{\log(n)}{n}\right). \quad (3.12)$$
Proof of Corollary 3.3. A crude bound suffices. Consider the collection of (dependent) standard normal random variables generated by taking all possible inner-products between corners of the cube and the rows of the matrix \( A \),

\[
R := \{|z| : z = (A\sigma)_i, \ \sigma \in \{\pm 1\}^n, \ i \in [m]\}.
\]

The cardinality of \( R \) is easily bounded by \( 2^{2n} \), so the maximum of \( R \) is subgaussian with variance proxy at most \( O(n) \). We crudely bound \( \text{disc}(A) \) by the maximum of \( R \). Define

\[
U := (\text{disc}(A) - K_c)_+.
\]

Then for sufficiently large \( x > 0 \), we have by Cauchy-Schwarz:

\[
E\left[ U^2 \right]^{1/2} = E\left[ U^2 \mathbb{1}(U \leq xn^{-1} \log n) \right]^{1/2} + E\left[ U^2 \mathbb{1}(xn^{-1} \log n \leq U) \right]^{1/2}
\]

\[
\leq \frac{x \log(n)}{n} + E\left[ R^4 \right]^{1/4} \mathbb{P}[\text{disc}(A) > K_c + xn^{-1} \log n]
\]

\[
\leq \frac{x \log(n)}{n} + n (n^{-cx})
\]

\[
= \mathcal{O}\left( \frac{\log(n)}{n} \right).
\]

\[ \square \]

4. Second Moment Method

Here we establish that solutions exist with positive probability.

Proof of Theorem 5. Fix \( K > 0 \), and let \( C := C(K) > 0 \) be some sufficiently large constant we choose later. Let \( \alpha := \alpha(n) \leq \alpha_c \). Our goal is to upper bound the ratio

\[
\frac{E\left[ |Z_{K,\alpha}|^2 \right]}{E\left[ |Z_{K,\alpha}| \right]^2} = 2^{-n} \sum_{s=0}^{n} \left( \frac{n}{s} \right)^\alpha \left( \frac{q(s)}{p^2} \right)^{\alpha n}.
\]

(4.1)

We claim that Eq. (4.1) is monotone increasing in \( \alpha \), so that if we establish the claim for \( \alpha_c \), then it follows for all \( \alpha \leq \alpha_c \). Indeed, for any \( \beta \), it easy to check that \( p^2 \leq q(\beta) \) by e.g. the Gaussian Correlation Inequality [17] or direct computation. Thus, it suffices to upper bound Eq. (4.1) for \( \alpha = \alpha_c \). For convenience, we may thus adopt the shorthand \( F := F_{K,\alpha C(K),n} \) and drop the subscript \( \alpha \) wherever possible for the rest of this proof.

Let \( m := \alpha_c n \) be the number of rows in the matrix \( A \). Consider \( \beta := \frac{1}{2} + \frac{m}{2n} \). A direct computation via Taylor expansion yields

\[
q(\beta) = p^2 \left( 1 + \frac{(1-\mu_2)^2 (s^2/n - 1)}{2n} + \mathcal{O}\left( \frac{s^4}{n^4} \right) \right),
\]

(4.2)

where we adopt the following notation of [2]:

\[
\mu_2 := p^{-1} \int_{-k}^{k} x^2 e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = -p^{-1} \sqrt{\frac{2}{\pi}} k e^{-k^2/2} + 1.
\]

(4.3)

We are ready to bound Eq. (4.1). Split the sum into three annular regions and treat them separately. (For the first region, highly similar computations appear in e.g. Lemma 3.1 in [2]; Lemma 7 of [4]; or Lemma 3 of [3].)
(1) \((1/2 - \delta \leq \beta \leq 1/2)\) Let \(\beta_s := 1/2 + s/(2n)\) and let \(\delta > 0\) be some arbitrarily small constant that we fix later. For \(\delta\) sufficiently small, we apply Stirling’s formula and Eq. (4.2).

\[
I_1 := \sum_{|s| \leq 2\delta n} 2^{-n} \left( \frac{n}{2} \right)^{\frac{q(\beta_s)}{p^2}}^m
\]

\[
= \sum_{|s| \leq 2\delta n} \frac{2}{n\pi} \exp \left\{ -\frac{s^2}{8n} (1 + \mathcal{O}(\delta)) \right\} \exp \left\{ \frac{\alpha}{2} (1 - \mu_2)^2 \left( \frac{s^2}{n} - 1 \right) (1 + \mathcal{O}(\delta)) \right\}
\]

\[
\leq \mathcal{O} \left( \frac{1}{1 - 4\alpha(1 - \mu_2)^2(1 + \mathcal{O}(\delta))} \right).
\]

Denote for shorthand \(B := \sqrt{\alpha(1 - \mu_2)/2}\). Since \(\delta\) can be taken arbitrarily small, it suffices to show \(|B|\) is strictly less than \(1/2\), uniformly for sufficiently large \(n\). This follows immediately by combining Eq. (3.7) with Eq. (4.3). Indeed, by Eq. (3.8), there is some \(\epsilon := \epsilon(K) > 0\) so that

\[
F_{\alpha}'' \left( \frac{1}{2} \right) = 4(-1 + 4B^2) < -\epsilon < 0.
\]

Thus \(|B|\) is strictly bounded from \(1/2\), so taking \(\delta\) sufficiently small, we obtain for some implicit constant depending only on \(K\) that \(I_1 = \mathcal{O}(1)\).

(2) \((\delta \leq \beta \leq 1/2 - \delta)\) Let \(I_2\) be

\[
I_2 := \sum_{s : 2\delta n \leq |s| < (1 - 2\delta)n} 2^{-n} \left( \frac{n}{2} \right)^{\frac{q(\beta_s)}{p^2}}^m.
\]

Applying Stirling’s formula,

\[
I_2 < n(2^{-n}p^{-2m}) \max_{\beta \in [\delta, \frac{1}{2} - \delta]} \exp \{ nF(\beta) + \mathcal{O}(\log n) \} = \max_{\beta \in [\delta, \frac{1}{2} - \delta]} \exp \{ n(\Theta(\delta)) + \mathcal{O}(\log n) \}.
\]

It suffices to show the exponent is strictly negative and of order \(n\) on this interval. Applying Eq. (3.9), we have for some \(\epsilon_0 := \epsilon_0(K, \delta) > 0\),

\[
\max_{\beta \in [\delta, 1/2 - \delta]} F(\beta) - F(1/2) \leq \max \{ F(\delta) - F(1/2), F(1/2 - \delta) - F(1/2), -\epsilon \} \leq -\epsilon_0.
\]

Thus for some implicit constant depending only on \(K\) and \(\delta\), \(I_2 = \exp \{-|\Theta(\delta)|\}\).

(3) \((0 \leq \beta < \delta)\) Define \(I_3\), the contribution of the remaining terms, by

\[
I_3 := \sum_{|s| > (1 - 2\delta)n} 2^{-n} \left( \frac{n}{2} \right)^{\frac{q(\beta_s)}{p^2}}^m.
\]

By Stirling’s formula, as well as the definition of \(\alpha\) as \(\alpha_s(K)\), we also have the identity

\[
I_3 = \sum_{|s| > (1 - 2\delta)n} p^{-m} \exp \{ nF(\beta_s) + \mathcal{O}(\log n) \}.
\]

By Stirling’s formula and as \(\alpha_s(K)\), we also have the identity

\[
I_3 = \sum_{|s| > (1 - 2\delta)n} p^{-m} \exp \{ nF(\beta_s) + \mathcal{O}(\log n) \}.
\]

Since we are able to take \(\delta\) an arbitrarily small constant in all the previous parts of this proof, let \(\delta\) be sufficiently small so that by Lemma 3.1, \(F\) is decreasing on \([0, \delta]\). The idea is to say \(F(\beta)\) is already extremely negative even for \(\beta = 1/n\). Intuitively, such a dramatic decay is possible due to the “frozen” nature of typical solutions or, equivalently, the fact that \(F''(\beta) \to -\infty\) as \(n \to \infty\). Note the minor subtlety that \(F\) being decreasing does not directly imply that the summand of \(I_3\) is decreasing, due to the logarithmic error terms in Stirling’s formula. Conveniently, \(F(1/n)\) is negative enough to outweigh these lower-order corrections.

First, computing the summand for \(\beta = 0\), we have:

\[
2^{-n} \left( \frac{n}{2} \right)^m = 2^{-n}p^{-m} = 1.
\]
Next, in order to compute the summand for \( \beta = 1/n \), we claim: (c.f. Lemma 6 of [16] for a related result on the “planted” model)

\[
q(1/n) := \frac{q(1/n)}{p} \leq 1 - \Omega \left( \frac{1}{\sqrt{n}} \right).
\]

(4.5)

Indeed, if \( \beta := 1/n \) and \( \delta := cn^{-1/2} \) for \( c := c(K) > 0 \) sufficiently small that we fix later, we have by definition of \( q \):

\[
q(\beta) = \frac{1}{2\pi} \int_{-K}^{K} e^{-y^2/2} \left[ \int_{-K+(1-2\beta)y/(2\sqrt{\beta(1-\beta)})}^{K+(1-2\beta)y/(2\sqrt{\beta(1-\beta)})} \exp \left\{ -x^2/2 \right\} dx \right] dy
\]

\[
\leq \frac{1}{\sqrt{2\pi}} \int_{-K}^{-K-\delta} e^{-y^2/2} dy + \frac{1}{2\pi} \int_{K-\delta}^{K} e^{-y^2/2} \left[ \int_{-K+(1-2\beta)y/(2\sqrt{\beta(1-\beta)})}^{K+(1-2\beta)y/(2\sqrt{\beta(1-\beta)})} \exp \left\{ -x^2/2 \right\} dx \right] dy
\]

\[
\leq p_K - \delta \left( \frac{e^{-K^2/2}}{\sqrt{2\pi}} \right) + \frac{1}{2\pi} \int_{K-\delta}^{K} e^{-y^2/2} \left[ \int_{-K+(1-2\beta)y/(2\sqrt{\beta(1-\beta)})}^{K+(1-2\beta)y/(2\sqrt{\beta(1-\beta)})} \exp \left\{ -x^2/2 \right\} dx \right] dy.
\]

Examining the bounds of integration,

\[
-K + (1-2\beta)(K-\delta) = \frac{-2K\beta - \delta}{2\sqrt{\beta(1-\beta)}} = -\delta \sqrt{n} \left( 1 + \mathcal{O} \left( \frac{1}{n} \right) \right) = -\frac{\delta \sqrt{n}}{2} \left( 1 + \mathcal{O} \left( \frac{1}{n} \right) \right).
\]

Thus, we obtain in total:

\[
q \left( \frac{1}{n} \right) \leq p_K - \delta \left( \frac{e^{-K^2/2}}{\sqrt{2\pi}} \right) + \frac{1 + \mathcal{O} \left( \frac{1}{n} \right)}{\sqrt{2\pi}} \int_{K-\delta}^{K} e^{-y^2/2} \left( \int_{-\infty}^{0} \exp \left\{ -x^2/2 \right\} dx \right) dy
\]

\[
= p_K + \delta \left( \frac{e^{-K^2/2}}{\sqrt{2\pi}} + \frac{e^{-K^2/2}}{2\sqrt{2\pi}} + \Theta (c) \right) + \mathcal{O} \left( \frac{1}{n} \right)
\]

\[
\leq p_K - \Theta \left( \frac{1}{\sqrt{n}} \right).
\]

The last line follows by taking e.g. \( c < e^{-K^2/2}/100 \). We have thus established the desired bound Eq. (4.5). Raising this inequality to the \( m \)’th power yields

\[
q(1/n)^m \leq p^m \exp \left\{ -\Omega \left( \sqrt{n} \right) \right\},
\]

which is far smaller than even what is needed. Indeed, returning to Eq. (4.4) and recalling that \( F \) is strictly decreasing, as well as noting crudely that \( H(1/n) < n^{-9/4} \), we obtain:

\[
I_3 = \sum_{s=0}^{\delta n} p^{-m} \exp \{ nF(\beta_s) + \mathcal{O} (\log n) \}
\]

\[
\leq 1 + \delta n \left( \frac{q(1/n)}{p} \right)^m \exp \left\{ n^{1/4} + \mathcal{O} (\log n) \right\}
\]

\[
\leq 1 + \exp \left\{ -\Omega \left( \sqrt{n} \right) \right\}.
\]

Thus, \( I_3 = 1 + o(1) \).

This completes our casework. Summing \( I_1, I_2, \) and \( I_3 \), we have shown that the second moment ratio (4.1) is \( \mathcal{O} \left( 1 \right) \). By our previous considerations on the monotonicity of Eq. (4.1) with respect to \( \alpha \), Eq. (3.4) is then established for all \( \alpha := \alpha(n) \leq \alpha_c \). By the Paley-Zygmund inequality, Eq. (3.5) follows immediately and the lemma is complete.

\[ \Box \]

5. Sharp Threshold

5.1. Sketch of Theorem 6. Let us begin by observing that a perturbation of \( \alpha \) is equivalent to a perturbation of \( K \) of the same multiplicative order, so that Eq. (3.10) implies Eq. (3.11).
Lemma 5.1. Fix some positive $\alpha$ and $C$, and fix some function $\epsilon(n)$ such that $\epsilon(n) \to 0$ as $n \to \infty$. If $K$ is such that $K \geq K_{c}(\alpha)(1 + C\epsilon(n))$, then $\alpha \leq \alpha_{c}(K)(1 - C'\epsilon(n))$ for some other constant $C' := C'(\alpha, C) > 0$ and $n$ sufficiently large.

Proof. Since $\alpha_{c}(K)$ is strictly increasing in $K$, it further suffices to just check the case $K = K_{c}(\alpha)\sqrt{n}(1 + C\epsilon(n))$. Recalling Eq. (3.1), we have by definition of $K_{c}$ and $\alpha_{c}$, and then applying a first-order Taylor expansion,

$$\alpha_{c}(K) - \alpha = -\log(2) \left( \frac{1}{\log p_{K}} - \frac{1}{\log p_{K_{c}(\alpha)}} \right)$$

$$= -\log(2) \left( \frac{1}{\log \left( p_{K_{c}(\alpha)}(1 + O(\epsilon(n))) \right)} - \frac{1}{\log p_{K_{c}(\alpha)}} \right)$$

$$= |O(\epsilon(n))|. \qed$$

In order to establish Eq. (3.10), we need to refine Theorem 4 of Perkins and Xu to hold even for $\alpha$ allowed to vary with $n$. In particular, it should hold for $\alpha \leq \alpha_{0} := \alpha_{c} - \epsilon n^{-1} \log(n)$. We also would like quantitative tail-bounds. Let us briefly survey the proof.

We follow an approach that Talagrand developed for the asymmetric binary perceptron [23]. Talagrand’s approach was carefully adapted to the setting of the symmetric perceptron by Perkins and Xu [16]. The outline is to build a constraint satisfaction problem by adding one clause at a time, and then show the expected number of solutions concentrates on the log-scale via martingale arguments. For us, this corresponds to adding one row of the matrix $A$ at a time, and then counting how many $\{\pm 1\}^{n}$ vectors have small inner product with all rows of the matrix so far.

Reusing the notation of [16] for clarity of comparison, define a time-indexed process $\{S_{t}\}$ of the solutions to the first $t$ rows:

$$S_{t} := \{ \sigma \in \{\pm 1\}^{n} : |(A\sigma)_{j}| \leq K, \forall 1 \leq j \leq t \}, \quad E[|S_{t}|] = 2^{n}(p_{K})^{t} \tag{5.1}$$

The correspondence with our previous notation of $Z$ is simply that $Z_{K,\alpha}$ can be coupled with $S_{t}$ for $t = \alpha n$. Denote the deviation of $\log |S_{t}|$ from its mean by $Q_{t}$, which we write as a telescoping sum:

$$Q_{t} := \log \left( \frac{|S_{t}|}{E[|S_{t}|]} \right) = \sum_{i=1}^{t} \left[ \log \left( \frac{|S_{i}|}{E[|S_{i}|]} \right) - \log \left( \frac{|S_{i-1}|}{E[|S_{i-1}|]} \right) \right].$$

Rewriting $Q_{t}$ in terms of a martingale difference $Y_{t}$,

$$Q_{t} = \sum_{i=1}^{t} \log(1 + Y_{i}), \quad Y_{i} := \frac{1}{p} \left( \frac{|S_{i}|}{|S_{i-1}|} - p \right).$$

Then, as long as $Y_{i}$ is small for each $i \leq \alpha_{0} n$, we can use the Taylor Expansion

$$Q_{t} = \sum_{i=1}^{t} Y_{i} - \frac{Y_{i}^{2}}{2} + O(Y_{i}^{3}).$$

The $Y_{i}$ are centered and should be roughly $Y_{i} \approx n^{-1/2}$, so that both the sum of the $Y_{i}$ and the sum of the $Y_{i}^{2}$ are constant order, and the sum of the $Y_{i}^{3}$ is vanishing. We then expect $|Q_{t}| = O(\log(n))$ with exponential tails, which would yield the theorem. Indeed, $\log E[|S_{\alpha n}|]$ is at least $\Omega(x \log(n))$. If we take $x$ sufficiently large then $|Q_{t}| < \log E[|S_{i}|]$ for all $t \leq \alpha_{0} n$ with exponentially large probability. Thus the fluctuations of $|S_{t}|$ are much less than the mean, which immediately yields the theorem.

The analysis is by induction. The key idea for this induction is a geometric notion of regularity for $S_{t}$ that asserts solutions are sufficiently “well-spread”. We begin by assuming for induction that $S_{t'}$ is regular for all $t' \leq t$ (precise definition given shortly). Then we follow with three estimates (c.f. Lemmas 11, 12, and 13 of [16] respectively). First, if $S_{t}$ is regular, then $Y_{t+1}$ enjoys good tail bounds. Second, if $Y_{t}$ has good tails for all $t' \leq t + 1$, then martingale concentration yields exponential tail bounds for $Q_{t+1}$. Third, if $Q_{t+1}$ is small, then $S_{t+1}$ is also regular with good probability.
We emphasize that this structure was innovated in [23] and refined in [16]. We will directly use the first estimate (tail bound on $Y_t$) from [16]. For the second and third estimates, we also significantly borrow from the structure of the proofs in [16], albeit with exponential improvements. This concludes our sketch.

5.2. Proof of Theorem 6 (c.f. Theorem 9 of [16]). Let $S_t$, $Q_t$, and $Y_t$ be as defined in the sketch. Fix $K > 0$. All other constants in what follows will depend on $K$; we treat $K$ as a universal constant and suppress this dependence in our notation. Fix $x$ sufficiently large. Define $\alpha_0 := \alpha_0(n)$ by $\alpha_0 := \alpha_0 n - x \log(n)$.

We emphasize that no constants in what follows will depend implicitly on $x$. Recall our convention that $c$ and $C$ denote generic constants that may vary between lines; important constants will have a subscript.

In order to use the martingale structure of $Y_t$, define the filtration $\mathcal{F}_t$ generated by revealing rows $1, \ldots, t$ of the matrix $A$. Define the conditional measure $P_t[\cdot]$ by

$$P_t[\cdot] \triangleq P[\cdot \mid \mathcal{F}_t] \quad (5.2)$$

With this notation, we are ready to formally state our main definitions and lemmas.

**Definition 5.1 (Regular).** We say $S_t \subset \{\pm 1\}^n$ is $t$-regular if $S_t \neq \emptyset$ and, for $\sigma_t^{(1)}$, $\sigma_t^{(2)}$ drawn uniformly and independently (with replacement) from $S_t$,

$$P_t\left[\left|\langle \sigma_t^{(1)}, \sigma_t^{(2)} \rangle \right| > \sqrt{C_r n (x \log(n) + |Q_t|)} \right] \leq n^{-c_r x},$$

where $C_r$ and $c_r$ are some positive universal constants.

The parameters $C_r$ and $c_r$ are explicitly chosen below in Eq. (5.16) and Eq. (5.21) respectively. Define two stopping times: $\tau_S$ the first time $t$ that $S_t$ is not $t$-regular and $\tau_Q$ the first time that $|Q_t|$ is large. Also define the first time $\tau$ that either of these “bad” events occurs. More precisely, for $C_q \in (0, 1)$ some small universal constant (explicitly chosen in Eq. (5.14) below),

$$\tau_S \triangleq \inf \left\{t : S_t \text{ is not } t\text{-regular} \right\},$$

$$\tau_Q \triangleq \inf \left\{t : |Q_t| > C_q n \log(n) \right\},$$

$$\tau \triangleq \tau_S \wedge \tau_Q.$$

The first estimate of the induction is that if $S_t$ is regular then $Y_{t+1}$ is sub-exponential. Up to some trivial rewriting (we use $\tau_Q$ to suppress $|Q|$), the following is exactly Lemma 11 of [16].

**Lemma 5.2 ($Y_{t+1}$ is small; [16], Lemma 11).** There exists positive constants $c$ and $C$ so that for any $t$, for all sufficiently large $n$ and $y$,

$$\mathbb{I}(\tau > t) P_t \left[|Y_{t+1}| > C y \sqrt{\frac{(1 + C_q x) \log n}{n}} \right] \leq \exp \left\{-cy \right\}. \quad (5.3)$$

In particular, $y$ may be taken as a function of $n$. Next, we would like to say that $Q_t$ is small for $t \leq \tau_S$. Only a first-moment bound on $|Q_t|$ is given in [16]; the refinement we require is a bound on the moment-generating function. This will both allow us take $t$ much closer to $\alpha_c$ as well as yield strong tail bounds on $|S_t|$. We prove these lemmas in the next subsection. Let us first check that they imply Theorem 6, our desired bound on the upper tail of $\text{disc}(A)$.
Proof of Theorem 6. Combining Lemma 5.1 and Eq. (3.2) to estimate \( E \| Z_{K,\alpha_0} \| \) yields for some new universal constant \( C_z > 0 \),
\[
\log E [S_t] \geq C_z x \log(n), \quad \forall t \leq \alpha_0 n. \quad (5.5)
\]
Then, on the event \( \{ \tau_Q > t \} \), we have \( |S_t| \neq 0 \) if \( C_q < C_z \), which will indeed follow from our choice of \( C_q \). By Lemma 5.4, with probability at least \( 1 - n^{-c_\varepsilon} \), we have \( \{ \tau_S > \alpha_0 n \} \). Observe that by definition of \( \tau_Q \),
\[
\bigcup_{t=1}^{\alpha_0 n} \{ |Q_{i:t} | < C_q x \log(n) \} \subset \{ \tau_Q > \tau_S \} \cup \{ \tau_Q > \alpha_0 n \},
\]
hence Lemma 5.3 implies that with probability at least \( 1 - n^{-c_\varepsilon} \), we have that \( \{ \tau_Q > \alpha_0 n \} \) and thus
\[
|Z_{K,\alpha_0,n} - x \log(n)| > 0.
\]
\[\square\]

5.3. Proofs of Lemmas.

Proof of Lemma 5.3. Fix \( t \leq \alpha_0 n \) and define the event \( E \) that all the \( Y_i \) are bounded for \( i \leq t \wedge \tau \):
\[
E \overset{\Delta}{=} \bigcap_{i \leq t-1} \{ 1_{\tau > i} |Y_{i+1}| < 1/\log(n) \}
\]
By Lemma 5.2 and union bound, the event \( E \) holds with probability at least \( 1 - e^{-n^{1}} \). Let \( \lambda \in (0,1) \), where \( \lambda \) does not vary with \( n \) and will be taken sufficiently small later; our goal is to show
\[
E \left[ \exp \{ \lambda |Q_{i:t} | \} I_E \right] \leq n^{\lambda^2 C z}.
\]
Let us simplify Eq. (5.6). Since \( e^{Q_t} \) is a martingale for all \( t \) by construction and \( \lambda \in (0,1) \), the optional stopping theorem and Jensen’s inequality yield
\[
E \left[ \exp \{ \lambda Q_{i:t} \} I_E \right] \leq 1.
\]
Thus, to establish Eq. (5.6), it suffices to show:
\[
E \left[ \exp \{ -\lambda Q_{i:t} \} I_E \right] \leq n^{\lambda^2 C z}.
\]
By Taylor expansion, we have on the event \( E \) that
\[
\exp \{ -\lambda \log(1 + Y_i) \} \leq 1 - C \lambda Y_i + C \lambda^2 Y_i^2, \quad \forall i \leq t \wedge \tau.
\]
Additionally, by integrating the tail bound of Lemma 5.2, we obtain for some new positive constant \( C > 0 \):
\[
1_{\tau > i} E_i [Y_{i+1}^2] \leq C n^{-1} (1 + C_x n) \log n, \quad \forall i.
\]
Recall that \( \{ Y \} \) is a martingale difference sequence under the filtration generated by the rows of \( A \). Combining Eq. (5.8) and Eq. (5.9) yields
\[
E \left[ \exp \{ -\lambda Q_{i:t} \} I_E \right] = E \left[ \exp \{ -\lambda Q_{(t-1):\tau} - 1_{\tau > i-1} \lambda \log(1 + Y_i) \} I_E \right]
= E \left[ \exp \{ -\lambda Q_{(t-1):\tau} \} \exp \{ -1_{\tau > i-1} \lambda \log(1 + Y_i) \} \right] I_E
\leq E \left[ \exp \{ -\lambda Q_{(t-1):\tau} \} \left( 1 - C \|1_{\tau > i-1} \lambda |E_{i-1}[Y_i] + C \|1_{\tau > i-1} \lambda^2 E_{i-1}[Y_i^2] \right) I_E \right]
\leq E \left[ \exp \{ -\lambda Q_{(t-1):\tau} \} I_E \left( 1 + \lambda^2 C(1 + C_x n) \log n \right) \right] .
\]
We have used that the linear term of \( Y \) in Eq. (5.8) vanishes in expectation. Since \( Q_0 = 0 \) by definition, we obtain the desired result (5.7) by an easy induction:
\[
E \left[ \exp \{ -\lambda Q_{i:t} \} I_E \right] \leq \left( 1 + \lambda^2 C(1 + C_x n) \log n \right)^t \leq n^{\lambda^2 C(1 + C_x n) / n} \leq n^{\lambda^2 C z} ,
\]
where the last inequality follows for some new constant \( C \), since \( C_q \) is an explicit universal constant independent of \( x \), and \( x \) is assumed to be sufficiently large. Finally, fix \( \lambda = \min \{ C_q/(2C), \ 1/2 \} \), so that \( \lambda \in (0,1) \) as promised. Then
\[
P \left[ |Q_{i:t} | > C_q x \log(n) \right] \leq n^{\lambda^2 C z} n^{-\lambda C_q x} + P \left[ E^c \right] \leq \min \left\{ n^{-C_q x/(2C)}, n^{-C_q x/2} \right\} + e^{-n^{1/2}} =: n^{-c_\varepsilon} .
\]
\[\square\]
Proof of Lemma 5.4. By union bound and the tower property,
\[ P[\tau \leq \alpha_0 n] = \sum_{i=1}^{\alpha_0 n} P[\tau = t] \leq n^{-cx} + \sum_{i=1}^{\alpha_0 n} P[\tau_S = t, \tau_Q > t] \]
\[ \leq n^{-cx} + \sum_{i=1}^{\alpha_0 n} P[\tau_S = t, \tau_Q > t] \]

The last line follows by Lemma 5.3 and our assumption that \( x \) is sufficiently large. It then suffices to show each summand is bounded above by \( n^{-cx} \). Formally, our goal is:
\[ P[\tau_S = t, \tau_Q > t] \leq n^{-cx}, \quad \forall t \leq \alpha_0 n. \]  
(5.10)

By Lemma 3.1, there exists some constant \( c_f > 0 \) so that for all \( n \) sufficiently large,
\[ \max_{\beta \in [0,1/2-\lambda_n^{-1/2}]} F_{K,t/n,n}(\beta) \leq -c_f \min \{ \lambda^2, E[Z_{K,t/n}] \}. \]  
(5.11)

As a convenient short-hand, we write
\[ R_t(\lambda) := c_f(\lambda^2 \wedge L_t), \quad L_t := \log (E[Z_{K,t/n}]). \]
Then, for two vectors \( \sigma_{t}^{(1)} \) and \( \sigma_{t}^{(2)} \) drawn uniformly and independently from \( S_t \), we have:
\[ E \left[ \left\{ (\sigma_{t}^{(1)}, \sigma_{t}^{(2)}) \in S_t : \left| \langle \sigma_{t}^{(1)}, \sigma_{t}^{(2)} \rangle \right| \geq \lambda \sqrt{n} \right\} \right] \leq \exp \left\{ -R_t(\lambda) \right\} E[|S_t|^2], \]
and in particular,
\[ E \left[ \left| \langle \sigma_{t}^{(1)}, \sigma_{t}^{(2)} \rangle \right| \geq \lambda \sqrt{n} \frac{|S_t|^2}{E[|S_t|^2]} \right] \leq \exp \left\{ -R_t(\lambda) \right\}. \]  
(5.12)

We would now like to show an exponential tail bound on the overlap of a pair of solutions. Let us give this a short-hand name for convenience:
\[ O^t := \frac{1}{\sqrt{n}} \left| \langle \sigma_{t}^{(1)}, \sigma_{t}^{(2)} \rangle \right|. \]

Then
\[ E \left[ \exp \left\{ \frac{R_t(O^t)}{2} + 2Q_t \right\} \right] = E \left[ \exp \left\{ \frac{1}{2} \left( \frac{c_f \left| \langle \sigma_{t}^{(1)}, \sigma_{t}^{(2)} \rangle \right|^2}{n} \wedge L_t \right) + 2Q_t \right\} \right] \]
\[ = E \left[ \int_0^{\infty} \mathbb{P}_t \left[ \exp \left\{ \frac{1}{2} \left( \frac{c_f \left| \langle \sigma_{t}^{(1)}, \sigma_{t}^{(2)} \rangle \right|^2}{n} \wedge L_t \right) \right] \geq y \right] dy \frac{|S_t|^2}{E[|S_t|^2]} \right] \]
\[ = E \left[ \int_{0}^{\exp(L_t/2)} \mathbb{P}_t \left[ \exp \left\{ \frac{c_f \left| \langle \sigma_{t}^{(1)}, \sigma_{t}^{(2)} \rangle \right|^2}{2n} \geq y \right\} \right] dy \frac{|S_t|^2}{E[|S_t|^2]} \right]. \]

Applying the change of variables \( z := \sqrt{2 \log(y)/c_f} \), we obtain by Markov’s inequality
\[ \int_0^{\exp(L_t/2)} \mathbb{P}_t \left[ \exp \left\{ \frac{c_f \left| \langle \sigma_{t}^{(1)}, \sigma_{t}^{(2)} \rangle \right|^2}{2n} \geq y \right\} \right] dy = \int_0^{\sqrt{L_t/c_f}} c_f z e^{c_f z^2/2} E \left[ \mathbb{P}_t \left[ \left| \langle \sigma_{t}^{(1)}, \sigma_{t}^{(2)} \rangle \right| \geq z \sqrt{n} \right] \right] dz \]
\[ \leq \int_0^{\sqrt{L_t/c_f}} c_f z e^{-c_f z^2/2} dz \leq 1. \]

Let \( C > 0 \) be the constant in Eq. (3.4). Since \( t < \alpha_0 n \), we have in total:
\[ E \left[ \exp \left\{ \frac{R_t(O^t)}{2} + 2Q_t \right\} \right] \leq E \left[ \frac{|S_t|^2}{E[|S_t|^2]} \right] \leq C. \]  
(5.13)
We would like to apply Markov’s inequality using Eq. (5.13). Recall Eq. (5.5), namely that there exists some universal constant \( C_z > 0 \) with

\[
L_t \geq C_z x \log(n), \quad \forall t \leq \alpha_0 n.
\]

We have not yet fixed \( C_q \) up to this point; here we will need \( C_q < C \), e.g. \( C_q \leq C/10 \). For the proof of Theorem 6, we also would like \( C_q < C_z \). Thus, set:

\[
C_q := \min \{ C/10, \ C_z/2 \}.
\]  

(5.14)

In order to establish Eq. (5.10), note that if \( \tau_Q > t \), we have

\[
|Q_t| < .1L_t.
\]  

(5.15)

We also have yet to fix \( C_r \), one of the two parameters in the definition of \( t \)-regular. Set

\[
C_r := 2C/c_f \vee 1.
\]  

(5.16)

(We only take the maximum with one for notation convenience later). On the event \( \{ \tau_Q > t \} \), this choice yields

\[
R_t \left( \sqrt{C_r x \log(n) + |Q_t|} \right) + 2Q_t > \min \{ (c_f C_r + 2C_q)x \log(n), .9c_f L_t \} \\
\geq .9R_t \left( \sqrt{C_r x \log(n)} \right).
\]  

(5.17)

Since \( R_t(\lambda) \) is weakly increasing in \( \lambda \), we have:

\[
P \left[ \left| \left\langle \sigma^{(1)}_t, \sigma^{(2)}_t \right\rangle \right| > \sqrt{n(C_r x \log(n) + |Q_t|)}, \ \tau_Q > t \right] \\
\leq P \left[ \exp \left\{ R_t(O^r) + 2Q_t \right\} > \exp \left\{ R_t \left( \sqrt{C_r x \log(n) + |Q_t|} \right) + 2Q_t \right\}, \ \tau_Q > t \right].
\]  

(5.18)

By tower property, and then applying Markov’s inequality to Eq. (5.18) via Eq. (5.13) and Eq. (5.17),

\[
E \left[ 1_{\tau_Q > t} P_t \left[ \left| \left\langle \sigma^{(1)}_t, \sigma^{(2)}_t \right\rangle \right| > \sqrt{n(C_r x \log(n) + |Q_t|)} \right] \right] = P \left[ \left| \left\langle \sigma^{(1)}_t, \sigma^{(2)}_t \right\rangle \right| > \sqrt{n(C_r x \log(n) + |Q_t|)}, \ \tau_Q > t \right] \\
\leq C e^{-R_t \left( \sqrt{C_q x \log(n) + |Q_t|} \right)}.
\]

By a final application of Markov’s inequality,

\[
P_t \left[ \left| \left\langle \sigma^{(1)}_t, \sigma^{(2)}_t \right\rangle \right| > \sqrt{n(C_r x \log(n) + |Q_t|)} \right] > e^{-R_t \left( \sqrt{C_q x \log(n)} \right) / 2}, \ \tau_Q > t \]  

(5.19)

\[
\leq O \left( e^{-c_r R_t \left( \sqrt{C_q x \log(n)} \right)} \right) \\
\leq C \left( n^{-c_r} + e^{-c_L L} \right).
\]  

(5.20)

In conclusion, since \( E \|Z_{k,t}\| \geq C x \log(n) \) for all \( t \leq \alpha_0 n \), we may simply upper bound \( e^{-c_L L} \) by \( n^{-c_L} \) and also set \( c_r \) from the definition of \( t \)-regular to

\[
e_c := c_f C_q x / 2.
\]  

(5.21)

With this notation, Eq. (5.20) is equivalent to our goal Eq. (5.10), completing the lemma. Since \( C_r \geq 1 \),

\[
P \left[ \tau_S = t, \ \tau_Q > t \right] = P \left[ P_t \left[ \left| \left\langle \sigma^{(1)}_t, \sigma^{(2)}_t \right\rangle \right| > \sqrt{C_r n(x \log(n) + |Q_t|)} \right] > n^{-c_r x}, \ \tau_Q > t \right] \\
\leq P \left[ P_t \left[ \left| \left\langle \sigma^{(1)}_t, \sigma^{(2)}_t \right\rangle \right| > \sqrt{n(C_r x \log(n) + |Q_t|)} \right] > e^{-R_t \left( \sqrt{C_q x \log(n)} \right) / 2}, \ \tau_Q > t \right] \\
\leq n^{-c_r x}.
\]  

\[\square\]
6. Lower Bound on Fluctuations

Lemma 6.1. Fix positive \( \alpha \), and \( K \) with \( K \leq 2K_c(\alpha) \). For all \( n \) sufficiently large and positive \( \epsilon > 0 \) sufficiently small, there exists some \( C := C(\alpha) > 0 \) such that

\[
\mathbb{P} \left[ \left( K - \frac{\epsilon}{n} \right) \leq \text{disc}(A) \leq K \right] < C\epsilon \mathbb{E} \left[ |Z_K| \right].
\]

(6.1)

Note in particular that \( C \) does not depend on \( n \) or \( \epsilon \). The constant 2 is arbitrary; any constant greater than one will suffice. As an easily corollary, we obtain a lower bound on the variance of \( \text{disc}(A) \), completing our main result Theorem 1. In fact, we obtain a lower bound on fluctuations, which is strictly stronger than a bound on the variance.

Definition 6.1 (Fluctuations). We say that a sequence of random variables \( X_n \) has fluctuations at least of order \( \sigma_n \) if there exists \( \delta > 0 \) and \( C > 0 \) such that, for \( n \) sufficiently large and any \( |a - b| > C\sigma_n \),

\[
\mathbb{P} [a < X_n < b] < 1 - \delta.
\]

Corollary 6.1. The fluctuations of \( \text{disc}(A) \) are at least order \( 1/n \).

Proof of Corollary 6.1. By Theorem 5, we know that for some \( \delta > 0 \)

\[
\mathbb{P} [\text{disc}(A) \leq K_c(\alpha)] > \delta > 0.
\]

So, for any \( a \) and \( b \) possibly functions of \( n \), with \( K_c \leq a \leq b \), we have uniformly

\[
\mathbb{P} [\text{disc}(A) \in [a, b]] < 1 - \epsilon.
\]

Next consider any \( a := a(n) \) and \( b := b(n) \) with \( a < b \) and \( a \leq K_c \). Assume \( |a - b| \leq \epsilon/n \) for some sufficiently small constant \( \epsilon > 0 \) we choose shortly. By definition of \( K_c(\alpha) \) as well as the expansion Eq. (3.2), we have for some \( C > 0 \),

\[
\mathbb{E} \left[ |Z_0| \right] \leq 1 + Ce^{c(n-b-K_c)\epsilon} \leq C \exp \{C\epsilon\}.
\]

Then by Eq. (6.1), for \( \epsilon \) sufficiently small,

\[
\mathbb{P} [\text{disc}(A) \in [a, b]] < C\epsilon \exp \{C\epsilon\} \leq \frac{1}{2}.
\]

This completes the proof. Note that this lower bound on fluctuations also trivially implies the same lower bound on the variance.

Proof of Lemma 6.1. Let \( \Xi_{K,\epsilon} \) be the set of solutions with discrepancy inside an \( \epsilon/n \) window around \( K \), namely

\[
\Xi := \Xi_{K,\epsilon} := (Z_{K-\epsilon/n})^c \cap Z_K.
\]

We will show \( \Xi \) is empty with non-zero probability by bounding its expectation. Let \( T_i(x) \) denote the event that row \( i \) is tight, namely

\[
T_i := \{(Ax)_i \in I_{K,\epsilon}\}, \quad I_{K,\epsilon} := [K - \epsilon/n, K].
\]

Correspondingly, define the number of tight rows \( T \) by

\[
T(x) := \sum_i \mathbb{1}(T_i(x)).
\]

Let \( x \) be an arbitrary vector in \( \{\pm 1\}^n \). Conditioning over \( T \), for some sufficiently large \( C_0 := C_0(\alpha) > 0 \),

\[
\mathbb{E} [\Xi] = 2^n \sum_{t=0}^m \mathbb{P} [x \in \Xi | T(x) = t] \mathbb{P} [T(x) = t]
\]

\[
\leq 2^n \sum_{t=1}^m \left( \frac{m}{t} \right) \mathbb{P} [(Ax)_1 \in I_{K,\epsilon}]^t \mathbb{P} [(Ax)_1 < K]^t
\]

\[
\leq 2^n \sum_{t=1}^m \left( \frac{m}{t} \right) \left( \frac{C_0\epsilon}{n-p_K} \right)^t p_K^{m-t}.
\]
where, as usual, $p^K_\alpha = \mathbb{P}(|Z| \leq K)$ for $Z$ a standard Gaussian. Note that we are able to let $C_0$ only depend on $\alpha$ because $K$ is bounded above, e.g. by $2K_\alpha(\alpha)$. Thus, by the binomial theorem, for some other sufficiently large $C := C(\alpha) > 0$,

$$
\mathbb{E}(|\Xi|) \leq (2^n p^K_\alpha) \sum_{t=1}^{m} \binom{m}{t} \left( \frac{C_0 \alpha}{n} \right)^t 1^{m-t} = \mathbb{E}[Z_K] \left( \left( 1 + \frac{C_0 \alpha}{n} \right)^m - 1 \right) \leq \mathbb{E}[Z_K] C\alpha.
$$

By Markov’s inequality, as well as the fact that $|Z| < 1$ is equivalent to $|Z| = 0$, we are done. \hfill \Box

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8. Appendix

Proof of Lemma 3.1. All claims follow directly from a combination of lemmas in [2] and [4]. The goal of this proof is simply to organize their contributions. We refer the reader to these two papers for details.

Claims 1 and 2. The first claim is an easy computation. The second claim is proven in [4] (page 7). Indeed, they show that $F''(1/2) < 0$ for any $K > 0$ if $\alpha \leq \alpha_c$. Fixing $K$, we thus have by monotonicity in $\alpha$ of Eq. (3.7), a uniform upper bound on $F''(1/2)$.

Claim 3 Simply compute $F'(0)$ (see e.g. Eq. 12 of [4]) and note
\[
\lim_{\beta \to 0^+} F'(\beta) = -\infty.
\]
Since $F$ is differentiable on $(0, 1/2]$ and continuous at 0, there is then some positive radius around 0 for which $F$ is decreasing.

Claim 4. It will suffice to organize the results developed in [2] for the proof of their Lemma 3.4. They treat separately the three cases of: $K > 4$, $K \in [1, 4]$, and $K \in [0, 1]$ respectively in their Lemmas 4.6; 4.10 and 4.11; and 4.7. We summarize these results. For $K > 4$, one can show $F'$ is negative and then positive as $\beta$ from 0 to 1/2, with no other sign changes. It then follows for $K > 4$ that,
\[
\max_{\beta \in [a, b]} F_{\alpha_c}(\beta) \leq \max\{F_{\alpha_c}(0), F_{\alpha_c}(b)\}
\]
In the remaining two cases, while conjecturally the same picture holds, the actual picture that is manageable to prove is slightly different. It is established in [2] only that there exists some constants $0 < b_1 \leq b_2 < 1/2$ (depending only on $K$) so that $F$ is decreasing on $(0, b_1)$, increasing on $(b_2, 1/2)$, and $F_{\alpha_c}(\beta) < F_{\alpha_c}(1/2) - \epsilon$ for some $\epsilon := \epsilon(K) > 0$ in between. This will yield Eq. (3.9), which is enough for our purposes. We will reproduce most of the proof for $K > 4$ to make this more concrete, since this case has the cleanest analysis. Then, we will remark on how $b_1$ and $b_2$ are chosen for the remaining cases and try to highlight which parts are easy and difficult for the three cases. We leave many details of this summary—especially the actual implementation of the grid search—for the reader to find in [2]. We also emphasize again that the following arguments and ideas we attempt to outline are not our own, but rather completely from [2].

Fix $K$ and $\alpha = \alpha_c(K)$, and denote for short-hand $F := F_{K, \alpha_c, n}$. By an easy computation,
\[
F'(\beta) = -\log(\beta) + \log(1 - \beta) + \alpha_c - \exp\left\{\frac{-K^2}{2(1-\beta)}\right\} + \exp\left\{\frac{-K^2}{2\beta}\right\} \frac{\pi q_k(\beta)}{\sqrt{\beta(1-\beta)}}
\]
Rearranging this equation, [2] define another quantity $\mathcal{L}$ that they bound:
\[
\mathcal{L}(\beta) := \frac{\alpha_c}{\pi} \frac{\exp\left\{-\frac{K^2}{2(1-\beta)}\right\}}{\sqrt{\beta(1-\beta)}} - \frac{\exp\left\{-\frac{K^2}{2\beta}\right\}}{\sqrt{\beta(1-\beta)}} (8.2)
\]
Note that $F'(\beta) > 0$ is equivalent to $\mathcal{L}(\beta) < q(\beta)$.

Case 1: $4 < K$. Consider $\beta \in [2, 5]$. It is easy to check that
\[
q_K(\beta) \geq q_4(2) > .9
\]
so our goal is to show $\mathcal{L}(\beta) < .9$ for all $\beta \in [2, .5]$. We have
\[
\frac{\log(1 - \beta) - \log(\beta)}{1/2 - \beta} \geq 4
\]
and
\[
\frac{d^2}{d\beta^2} \exp\left\{-\frac{K^2}{2\beta}\right\} > 0,
\]
Combined with Taylor expansion, Eq. (8.4) yields
\[
\exp\left\{-\frac{K^2}{2(1-\beta)}\right\} - \exp\left\{-\frac{K^2}{2\beta}\right\} \leq \frac{K^2 \exp\left\{-\frac{K^2}{2(1-\beta)}\right\}}{(1-\beta)^2} (8.5)
\]
By Eq. (8.3) and Eq. (8.5), we simplify the desired inequality:

\[
\mathcal{L}(\beta) \leq \frac{\alpha_c}{\pi} \exp\left\{-\frac{K^2}{2(1-\beta)}\right\} - \exp\left\{-\frac{K^2}{2}\right\}
\]

\[
\leq \frac{\alpha_c}{\pi} K^2 \exp\left\{-\frac{K^2}{2(1-\beta)}\right\}
\]

Next, using standard asymptotics for the error function,

\[
1 - p_K \geq \exp\left\{-K^2/2\right\} \sqrt{\frac{2}{\pi}} \left(\frac{1}{K} - \frac{1}{K^2}\right),
\]

so we have

\[
\alpha_c := -\frac{\log(2)}{\log(p)} \leq \frac{\log 2}{1 - p_K} \leq \frac{16 K^2}{15} \frac{\pi}{2} \exp\left\{\frac{K^2}{2}\right\} \leq \exp\left\{K^2\left(\frac{1}{2} - \frac{1}{2(1-\beta)}\right)\right\} \frac{K^3}{(1-x)^2}.19
\]

In the sub-case that \(x \in [0.2, 0.35]\), replace Eq. (8.5) with the trivial inequality

\[
\exp\left\{-\frac{K^2}{2(1-\beta)}\right\} - \exp\left\{-\frac{K^2}{2}\right\} \leq \exp\left\{-\frac{K^2}{2(1-\beta)}\right\}
\]

Then an identical computation easily yields \(\mathcal{L}(\beta) < .9\). So, for all \(\beta \in [0.2, 0.5]\), we have \(F'(\beta) > 0\).

Suppose that \(\mathcal{L}'(\beta) < q'(\beta)\) for \(\beta \leq .2\). Then \(L\) and \(q\) can cross at most once. Note \(F' < 0\) for some ball of positive radius around \(\beta = 0\) by claim three, so \(L > q\) for all \(\beta\) sufficiently small. We have also already shown that \(L < q\) for \(\beta > .2\). Thus, \(L\) and \(q\) cross exactly once, and we have the desired picture: \(F\) is decreasing and then increasing, with no other changes. In order to establish \(\mathcal{L}'(\beta) < q'(\beta)\) for \(\beta \leq .2\), one can differentiate the definition of \(L\), and then apply standard tail-bounds on the error function.

**Case 2:** \(.1 \leq K \leq 4\). The strategy is almost the same. We know that \(F\) is decreasing in some neighborhood of 0, so a natural first step is to, again, show that \(F\) is increasing in some neighborhood of 1/2. In the previous case, we were able to establish \(F\) is increasing in \([.2, .5]\). Here, [2] were only able to show this for \(\beta in [.3, .5]\). The proof is quite similar, except that, roughly speaking, while bounds derived from first-order Taylor expansion sufficed for \(K > 4\), second-order expansions are needed for \(K \in [.1, .4]\). Next, we again try and make Claim 3 (namely that \(F\) is decreasing near 0) quantitative. By looking very close to 0, namely \(\beta \in [0, .005]\), we can actually prove a stronger result with easier computations. Since trivially \(p \geq q\), it of course suffices to show \(L > p_K\) for \(\beta\) near 0. The strategy is to then show that \(\mathcal{L}(.005) > p_K\) and \(\mathcal{L}' < 0\) for \(\beta \in [0, .005]\). These are simple computations. Finally, in the remaining region \(\beta \in [.005, .3]\), we would like to check that \(F(\beta - F(1/2) < -\epsilon\) for some \(\epsilon := c(K) > 0\). This is done in two steps: first, it is relatively straight-forward to check that the derivative of \(F_K(\beta) - F_K(1/2)\) with respect to \(\beta\) and the derivative with respect to \(K\) are both bounded in absolute value by e.g. 6. Second, do a computerized grid search over the region

\[
\{ (\beta, K) : \beta \in [.005, .3], K \in [.14] \}
\]

Since we have a bound on how fast \(F_K(\beta) - F_K(1/2)\) can change, a step-size can be picked appropriately to obtain a provably tolerable error and conclude. In summary, taking \(b_1\) and \(b_2\) as .005 and .3 respectively, the picture we have established is that \(F\) is decreasing on \([0, b_1]\); increasing on \([b_2, 1/2]\); and strictly negative between.
Case 3: $0 \leq K \leq .1$. Here, $K$ being small will actually make the analysis quite tractable. Indeed, we can for example get decent control over $p$ and $q$, which will be used repeatedly. Using a zero-order and first-order Taylor expansion of $\exp \{-x^2/2\}$ for $x \in [-K, K]$ for the upper and lower bound respectively, we have trivially
\[ 0.99K \sqrt{2\pi} \leq p_K \leq 2K \sqrt{2\pi} \]
One can also bound $q_K(\beta)$ in a similarly crude way:
\[ \frac{K^2}{2\pi \sqrt{1-\beta}} \exp \left\{ -\frac{K^2}{2\beta} \right\} \leq q_K(\beta) \leq \frac{K^2}{\pi \sqrt{1-\beta}} \]
Note that since $K$ is small, the gap between the upper and lower bounds here is very small if $\beta$ is not too small as well. With these bounds, we are now ready to prove the desired picture. First, we claim for $\beta \in [0.27, 0.5]$ that $L(\beta) \leq q(\beta)$. The main difference from the first case ($K > 4$) is that now $d^2/d\beta^2 \exp \left\{ -\frac{K^2}{2\beta} \right\} < 0$, so we replace Eq. (8.5) by
\[ \frac{1}{\beta-1/2} \leq \frac{K^2 \exp \left\{ -\frac{K^2}{2\beta} \right\}}{\beta^2} \]
Next, for $\beta \in (0, K^2/2)$, we will easily verify that $L(\beta) > q(\beta)$. Indeed, since $\sqrt{\beta} \log(\beta)$ is decreasing for $\beta \leq .005$ (and here we are only considering $\beta \leq K^2/2 \leq .005$),
\[ -\log(p)q(\beta) \geq \frac{\log 2}{\pi} \frac{1/2}{\sqrt{\beta}(-\log(\beta))} \geq .29 \]
It is trivial to check that $-\log(p_K)q(\beta) > .29$, so indeed $F$ is decreasing on $\beta \in [0, K^2/2]$.

Carrying out Taylor expansions to another term, it turns out both of these bounds can be extended to see that actually $F$ is decreasing on $[0, K/12]$ and then increasing on $[.04, 1/2]$. Finally, for the remaining region of $[K/12, .04]$, we can show directly that $F(\beta) < F(1/2) - \epsilon$ for some $\epsilon := \epsilon(K) > 0$. Indeed, recalling that $F_{\alpha_c(K)}(1/2) = \log(2) + 2\alpha_c(K) \log(p)$, the desired claim follows quickly from the inequality
\[ \log \left( \frac{q(\beta)}{p^2} \right) < \log \left( \frac{1}{.99^2 \sqrt{\beta}} \right) < \log \left( \frac{2}{\sqrt{K}} \right) \]
In summary, for $b_0$ and $b_1$ equal to $K/12$ and .04 respectively, we have again established the picture that $F$ is decreasing on $[0, b_1]$; increasing on $[b_2, 1/2]$; and strictly negative between. This concludes the analysis of the final case $K \leq .01$, and thus the outline of claim four. \[ \square \]