1 Introduction

In this paper we generalize the well-known construction of shuffle product algebras by using mixable shuffles, and prove that any free Baxter algebra is isomorphic to a mixable shuffle product algebra. This gives an explicit construction of the free Baxter algebra, extending the work of Rota [15] and Cartier [2].

In an important paper published in 1958, Ree [12] constructed algebras in which the product is expressed in terms of shuffles. He was motivated by Chen’s work on iterated integrals of paths [3], where this shuffle product is derived from the integration by parts formula

\[
\int_0^x f(t)dt \int_0^x g(t)dt = \int_0^x f(t)(\int_0^t g(s)ds)dt + \int_0^x g(t)(\int_0^t f(s)ds)dt. \tag{1}
\]

Subsequently, shuffle product constructions have been studied extensively and have found applications in many areas of pure and applied mathematics.

In another important paper published in 1960, Baxter [11] considered operators \( P \) that satisfies the identity

\[ P(x)P(y) + P(xy) = P(xP(y)) + P(yP(x)) \]

and used this identity to study the theory of fluctuations. Rota studied Baxter’s operators from an algebraic point of view and defined a Baxter algebra to be an algebra \( A \) with an operator \( P \) satisfying the identity

\[ P(x)P(y) + qP(xy) = P(xP(y)) + P(yP(x)) \]

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for some fixed $q$ in the base ring of $A$. Rota [15] and Cartier [2] gave explicit constructions of the free Baxter algebra on a set $X$ in the case when $q = 1$.

It is easy to see that the product in Baxter algebras when $q = 0$ can be described by shuffle products, and the shuffle product algebras considered by Chen and Ree have a canonical Baxter operator. However, there does not appear to be an explicit and systematic study of this connection between Baxter algebras and shuffle products except a remark in a recent paper of Rota [18] implying such a connection.

The motivation for the current paper came from the desire of developing a theory that is “dual” to the beautiful theory of differential algebras obtained by Ritt [13] and Kolchin [10]. Since a differential algebra is an algebra with an operator that satisfies the Leibniz product rule, it seems natural to study “integration algebras”, i.e., algebras with an operator that satisfies an identity similar to the one in equation (1). Of course, these are just Baxter algebras where $q = 0$. Unaware of the above mentioned work on Baxter algebras, we gave a description of the free integration algebra by using shuffle products.

It was Rota who pointed out to us the earlier work on free Baxter algebras and suggested that we extend our shuffle product description of free integration algebras to Baxter algebras. This is carried out in this paper, by making use of a modified shuffle product, called the mixable shuffle product. Thus we not only consider shuffles of two vectors, but also shuffles in which certain components of a shuffle will “merge”. This enables us to construct the mixable shuffle product algebras, generalizing the classical construction of shuffle product algebras, and to give a more intuitive and constructive description of the free Baxter algebra. Also, our description is of free Baxter algebras on any commutative algebra with any value $q$. Furthermore, the free Baxter algebra of Cartier or Rota is in the category of Baxter algebras not necessarily having an identity, while the free Baxter algebra we consider is in the category of Baxter algebras with an identity. When specialized to the case considered by Rota or Cartier, the free Baxter algebra we construct contains their free Baxter algebra as a sub-Baxter algebra in the category of Baxter algebras not necessarily having an identity.

The mixable shuffle product of Baxter algebras allows us to study in more detail the properties of Baxter algebras. This will be the subject of a forthcoming paper. Shuffle products have occurred in many other fields and contexts, such as Hopf algebras, algebraic $K$-theory, algebraic topology and combinatorics, as well as in computational mathematics and applied mathematics; see, for example, [4, 5, 7, 9, 12, 14, 19]. One might hope that the mixable shuffle introduced here will provide interesting and useful generalizations to those theories. One might also hope that the mixable shuffle product will be useful for the applications of Baxter algebras, such as those considered by Baxter and Rota.

In this paper, we first give a brief summary of basic definitions and properties of Baxter algebras in section 2. We then make a careful study of mixable shuffles in section 3. In section 4 we apply properties of mixable shuffles to construct mixable shuffle product algebras and to prove that any free Baxter algebra is isomorphic to such
an algebra. In this section, we also briefly consider variations of the free construction by examining the free Baxter algebra on a set, on a commutative monoid and on a module. The relation between the free Baxter algebras we construct and the free Baxter algebras constructed by Rota and Cartier is described in section 5. We conclude by considering the special case of the free Baxter algebra on the empty set.

2 Definitions and basic properties

In this paper, any ring $R$ is commutative with identity element $1_R$. All notation will be standard unless otherwise noted. In particular, we write $\mathbb{N}$ for the additive monoid of natural numbers $\{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{n \in \mathbb{N} \mid n > 0\}$ for the positive integers. Also, $\binom{n}{k}$ will denote the usual binomial coefficient defined for any $n, k \in \mathbb{N}$ with $k \leq n$ by $\binom{n}{k} = n! / (k!(n-k)!)$.

**Definition 2.1.** Let $C$ be a ring, $q \in C$, and let $R$ be a $C$-algebra. A Baxter operator on $R$ over $C$ is a $C$-module endomorphism $P$ of $R$ satisfying

$$P(x)P(y) + qP(xy) = P(xP(y)) + P(yP(x)), \quad x, y \in R. \quad (2)$$

We will find it convenient for the remainder of this paper to write equation (2) in the form

$$P(x)P(y) = P(xP(y)) + P(yP(x)) + \lambda P(xy), \quad (3)$$

so that our $\lambda$ is $-q$. We will also say that $P$ has weight $\lambda$.

**Definition 2.2.** A Baxter $C$-algebra of weight $\lambda$ is a pair $(R, P)$ where $R$ is a $C$-algebra and $P$ is a Baxter operator of weight $\lambda$ on $R$ over $C$.

If the meaning of $\lambda$ is clear, we will suppress $\lambda$ from the notation.

Note that the mapping $0 : R \to R$ defined by $0(r) = 0$ for all $r \in R$ is trivially a Baxter operator on $R$ over $R$, for any ring $R$. Hence every $C$-algebra can be viewed as a Baxter $C$-algebra. Definitions of basic concepts for $C$-algebras can be similarly defined for Baxter $C$-algebras. In particular, let $(R, P)$ and $(S, Q)$ be two Baxter $C$-algebras of weight $\lambda$. A homomorphism of Baxter $C$-algebras $f : (R, P) \to (S, Q)$ is a homomorphism $f : R \to S$ of $C$-algebras with the property that $f(P(x)) = Q(f(x))$ for all $x \in R$.

Let $\text{Bax}_{C,\lambda}$ denote the category of Baxter $C$-algebras of weight $\lambda$. A Baxter ideal of $(R, P)$ is an ideal $I$ of $R$ such that $P(I) \subseteq I$. For a Baxter ideal $I$ of $(R, P)$, the quotient Baxter $C$-algebra is the quotient algebra $R/I$, together with the $C$-linear endomorphism $\bar{P} : R/I \to R/I$ induced from $P$. If $f : (R, P) \to (S, Q)$ is in $\text{Bax}_{C,\lambda}$, then $\ker f$ is a Baxter ideal of $R$, and $\text{im} f$, with the restriction of $Q$, is a Baxter sub-$C$-algebra of $(S, Q)$.  


3 Mixable shuffle products

This section is a preparation for the next section, where we will give a description of free Baxter algebras in terms of mixable shuffles. We start with a study of mixable shuffles in the context of permutations. We then apply this study to mixable shuffles of vectors. Finally we consider mixable shuffles of tensors, using the mixable shuffle of vectors as a “generic” form. For the purpose of providing a solid foundation for later applications, we give full details of the proofs, even though some of them might be intuitively clear to the expert. So some readers might just want to look at the definitions and results and move on to the next section.

3.1 Permutation shuffles

For \( m, n \in \mathbb{N}_+ \), define the set of \((m, n)\)-shuffles by

\[
S(m, n) = \{ \sigma \in S_{m+n} \mid \sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(m), \\
\quad \sigma^{-1}(m+1) < \sigma^{-1}(m+2) < \ldots < \sigma^{-1}(m+n) \}
\]

Equivalently,

\[
S(m, n) = \{ \sigma \in S_{m+n} \mid \text{if } 1 \leq \sigma(r) < \sigma(s) \leq m \text{ or } m+1 \leq \sigma(r) < \sigma(s) \leq m+n \text{ then } r < s \}.
\]

A pair of indices \((k, k+1), 1 \leq k < m+n\) is called an admissible pair for an \((m, n)\)-shuffle \(\sigma \in S(m, n)\) if \(\sigma(k) \leq m < \sigma(k+1)\). Denote \(T^\sigma\) for the set of admissible pairs for \(\sigma\). For \(\sigma \in S(m, n)\) and \(T \subseteq T^\sigma\), call the pair \((\sigma, T)\) a mixable \((m, n)\)-shuffle. When \(T = \emptyset\), \((\sigma, T)\) is identified with the shuffle \(\sigma\). Denote \(\bar{S}(m, n)\) for the set of mixable \((m, n)\)-shuffles. So

\[
\bar{S}(m, n) = \{ (\sigma, T) \mid \sigma \in S(m, n), \ T \subseteq T^\sigma \}.
\]

Also denote \(s(m, n) = |\bar{S}(m, n)|\).

Intuitively, a \((m, n)\)-shuffle is a permutation \(\sigma\) of \(\{1, \ldots, m, m+1, \ldots, m+n\}\) such that the order of \(\{1, \ldots, m\}\) and \(\{m+1, \ldots, m+n\}\) are preserved in \(\{\sigma(1), \ldots, \sigma(m), \sigma(m+1), \ldots, \sigma(m+n)\}\). Further, a mixable \((m, n)\)-shuffle \((\sigma, T)\) is a \((m, n)\)-shuffle \(\sigma\) in which pairs of indices from \(T\) represent positions where “merging” will occur. The precise meaning of “merging” will be described later.

For example, \(\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\) is a \((2, 1)\)-shuffle. The pair \((1, 2)\) is an admissible pair for \(\sigma\). Note that \(\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\) is also a \((2, 1)\)-shuffle, but has no admissible pairs.
Denote
\[
\bar{S}_{1,0}(m, n) = \{(\sigma, T) \in \bar{S}(m, n) \mid (1, 2) \notin T, \sigma^{-1}(1) = 1\}, \\
S_{0,1}(m, n) = \{(\sigma, T) \in S(m, n) \mid (1, 2) \notin T, \sigma^{-1}(m + 1) = 1\}, \\
\bar{S}_{1,1}(m, n) = \{(\sigma, T) \in \bar{S}(m, n) \mid (1, 2) \in T\}.
\]
Then we clearly have
\[
\bar{S}(m, n) = \bar{S}_{1,0}(m, n) \cup \bar{S}_{0,1}(m, n) \cup \bar{S}_{1,1}(m, n)
\tag{4}
\]
since, by the definition of \(S(m, n)\), one and only one of \(\sigma^{-1}(1)\) and \(\sigma^{-1}(m + 1)\) equals 1. Similarly we will define the set of \((m, n, \ell)\)-shuffles by
\[
S(m, n, \ell) = \left\{\sigma \in S_{m+n+\ell} \mid \begin{array}{l}
\sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(m), \\
\sigma^{-1}(m + 1) < \sigma^{-1}(m + 2) < \ldots < \sigma^{-1}(m + n), \\
\sigma^{-1}(m + n + 1) < \sigma^{-1}(m + n + 2) \\
< \ldots < \sigma^{-1}(m + n + \ell)
\end{array} \right\}.
\]
For \(\sigma \in S(m, n, \ell)\), define
\[
T^\sigma = T_{1,0}^\sigma \cup T_{0,1,1}^\sigma \cup T_{1,1,1}^\sigma,
\]
where
\[
T_{1,0}^\sigma = \left\{(k, k + 1) \mid 1 \leq k < m + n + \ell - 1, \sigma(k) \leq \sigma(k + 1) \right\},
\]
\[
T_{1,0,1}^\sigma = \left\{(k, k + 1) \mid 1 \leq k < m + n + \ell, \sigma(k) \leq m, m + n < \sigma(k + 1) \right\},
\]
\[
T_{0,1,1}^\sigma = \left\{(k, k + 1) \mid 1 < k < m + n + \ell, \sigma(k) \leq m + n < \sigma(k + 1) \right. \left. \text{ and either } \sigma(k - 1) > m \text{ or } m \geq \sigma(k) \right\},
\]
\[
T_{1,1,1}^\sigma = \left\{(k, k + 1, k + 2) \mid 1 \leq k < m + n + \ell - 1, \sigma(k) \leq m < \sigma(k + 1) \leq m + n < \sigma(k + 2) \right\}.
\]
For \((\sigma, T) \in \bar{S}(m, n, \ell)\), define
\[
\deg T = |T \cap T_{1,0}^\sigma| + |T \cap T_{1,0,1}^\sigma| + |T \cap T_{0,1,1}^\sigma| + 2 |T \cap T_{1,1,1}^\sigma|.
\tag{5}
\]
Denote
\[
\bar{S}(m, n, \ell) = \{(\sigma, T) \mid \sigma \in S(m, n, \ell), T \in T_{\sigma}\},
\]
and
\[
s(m, n, \ell) = |\bar{S}(m, n, \ell)|.
\]
Proposition 3.1. Let $m, n \in \mathbb{N}_+$. 

1. $s(m, n) = s(m - 1, n) + s(m, n - 1) + s(m - 1, n - 1)$.

2. $|\{(\sigma, T) \in \bar{S}(m, n) \mid |T| = i\}| = (m+n-i)\binom{n}{i}.$

3. $s(m, n) = \sum_{i=0}^{n} (m+n-i)\binom{n}{i}.$

4. $s(m, n, \ell) = \sum_{k=0}^{n+\ell} \sum_{i=0}^{n} (m+n+\ell-k)\binom{\ell}{k-i}(m+n-i)\binom{n}{i}.$

Proof. 1. This follows from equation (4).

2. Denote the set on the left hand side of the equation by $\bar{S}(i)(m, n)$. We use induction on $m+n$, with $m, n \geq 1$. When $m+n=2$, this can be verified directly. In general, define $\bar{S}(i)(1,0)(m, n) = \bar{S}(i)(m, n) \cap \bar{S}(1,0)(m, n)$ and similarly for $\bar{S}(i)(0,1)(m, n)$ and $\bar{S}(i)(1,1)(m, n)$. From equation (4), we obtain, 

$$\bar{S}(i)(m, n) = \bar{S}(i)(1,0)(m, n) \cup \bar{S}(i)(0,1)(m, n) \cup \bar{S}(i)(1,1)(m, n).$$

Here $A \cong B$ means that the two sets $A$ and $B$ have the same cardinality. By the inductive assumption, the size of the right hand side is

$$\binom{m+n-i}{m-1}\binom{i}{m} + \binom{m+n-1-i}{m}\binom{i}{m} + \binom{m+n-2-i}{m-1}\binom{i}{m-1}.$$ 

Applying Pascal’s identity, we see that the last sum is also the value of $(m+n-i)\binom{m}{i}.$

3. This follows from part 2 by summing over $i$ for $i = 0, \ldots, n$.

4. The proof is similar to the proof of part 3. For $0 \leq k \leq n+\ell$, denote 

$$\bar{S}(k)(m, n, \ell) = \left\{(\sigma, T) \in \bar{S}(m, n, \ell) \mid \deg T = k\right\}.$$

We will use induction on $j = m+n+\ell$, $m, n, \ell \geq 1$ to prove 

$$|\bar{S}(k)(m, n, \ell)| = \sum_{i=0}^{n} (m+n+\ell-k)\binom{\ell}{k-i}(m+n-i)\binom{n}{i}.$$ (6)
When \( m + n + \ell = 3 \), the equation can be verified directly. Now assume that the equation holds for \( m + n + \ell < j \) and consider the case when \( m + n + \ell = j \). Denote

\[
\bar{S}_{1,0,0}(m, n, \ell) = \{ (\sigma, T) \in \bar{S}(m, n) \mid \sigma^{-1}(1) = 1, (1, 2) \notin T \},
\]
\[
\bar{S}_{0,1,0}(m, n, \ell) = \{ (\sigma, T) \in \bar{S}(m, n) \mid \sigma^{-1}(m + 1) = 1, (1, 2) \notin T \},
\]
\[
\bar{S}_{0,0,1}(m, n, \ell) = \{ (\sigma, T) \in \bar{S}(m, n) \mid \sigma^{-1}(m + n + 1) = 1, (1, 2) \notin T \},
\]
\[
\bar{S}_{1,1,0}(m, n, \ell) = \{ (\sigma, T) \in \bar{S}(m, n) \mid (1, 2) \in T \cap T^\sigma_{1,1,0}, (1, 2, 3) \notin T \},
\]
\[
\bar{S}_{1,0,1}(m, n, \ell) = \{ (\sigma, T) \in \bar{S}(m, n) \mid (1, 2) \in T \cap T^\sigma_{0,1,1}, (1, 2, 3) \notin T \},
\]
\[
\bar{S}_{0,1,1}(m, n, \ell) = \{ (\sigma, T) \in \bar{S}(m, n) \mid (1, 2) \in T \cap T^\sigma_{0,1,1}, (1, 2, 3) \in T \}.
\]

Also denote

\[
\bar{S}_{u,v,w}^{(k)}(m, n, \ell) = \bar{S}_{u,v,w}(m, n, \ell) \cap \bar{S}^{(k)}(m, n, \ell)
\]

for \( u, v, w = 0 \) or 1. It follows from the definition that

\[
\bar{S}(m, n, \ell) = \bar{S}_{1,0,0}(m, n, \ell) \cup \bar{S}_{0,1,0}(m, n, \ell) \cup \bar{S}_{0,0,1}(m, n, \ell)
\]
\[
\cup \bar{S}_{1,1,0}(m, n, \ell) \cup \bar{S}_{1,0,1}(m, n, \ell) \cup \bar{S}_{0,1,1}(m, n, \ell) \cup \bar{S}_{1,1,1}(m, n, \ell).
\]

So

\[
\bar{S}^{(k)}(m, n, \ell) = \bar{S}_{1,0,0}^{(k)}(m, n, \ell) \cup \bar{S}_{0,1,0}^{(k)}(m, n, \ell) \cup \bar{S}_{0,0,1}^{(k)}(m, n, \ell)
\]
\[
\cup \bar{S}_{1,1,0}^{(k)}(m, n, \ell) \cup \bar{S}_{1,0,1}^{(k)}(m, n, \ell) \cup \bar{S}_{0,1,1}^{(k)}(m, n, \ell) \cup \bar{S}_{1,1,1}^{(k)}(m, n, \ell)
\]
\[
\cong \bar{S}_{1,0,0}(m - 1, n, \ell) \cup \bar{S}_{0,1,0}^{(k)}(m - 1, n - 1, \ell) \cup \bar{S}_{0,0,1}^{(k)}(m, n, \ell - 1)
\]
\[
\cup \bar{S}_{1,1,0}^{(k-1)}(m - 1, n - 1, \ell - 1) \cup \bar{S}_{1,0,1}^{(k-1)}(m - 1, n, \ell - 1)
\]
\[
\cup \bar{S}_{0,1,1}^{(k-2)}(m - 1, n - 1, \ell - 1).
\]

Here again \( \cong \) stands for a bijection between sets. Taking cardinalities and applying the induction hypothesis, we see that the left hand side of equation \( \text{(3)} \) is

\[
\sum_{i=0}^{n} \binom{m+n+\ell-k-1}{\ell} \binom{m+n-i-1}{k-i} \binom{n}{i}
\]

7
\[ + \sum_{i=0}^{n-1} \binom{m+n+\ell-k-1}{\ell} \binom{m+n-i}{k-i} \binom{m+n-i-1}{n-i} \]

On the other hand, using Pascal’s identity we see that the right hand side of the equation equals

\[ \sum_{i=0}^{n} \binom{m+n+\ell-k}{\ell} \binom{m+n-i}{k-i} \binom{m+n-i-1}{n-i} = \]

\[ = \sum_{i=0}^{n} \binom{m+n+\ell-k-1}{\ell} \binom{m+n-i}{k-i} \binom{m+n-i-1}{n-i} \]

\[ + \sum_{i=0}^{n} \binom{m+n+\ell-k-1}{\ell-1} \binom{m+n-i}{k-i} \binom{m+n-i-1}{n-i} \]

\[ = \sum_{i=0}^{n} \binom{m+n+\ell-k-1}{\ell} \binom{m+n-i}{k-i} \binom{m+n-i-1}{n-i} \]

\[ + \sum_{i=0}^{n} \binom{m+n+\ell-k-1}{\ell-1} \binom{m+n-i}{k-i} \binom{m+n-i-1}{n-i} \]
Let $\Omega$ be a countable infinite set. Let $F^k$ be the set consisting of finite non-empty subsets of $\Omega$. For $k \leq n$, let

$$\sum_{i=0}^{n} \binom{m+n-1}{\ell} \binom{m-n-1}{k-1} \binom{n}{i} + \sum_{i=0}^{n} \binom{m+n-1}{\ell} \binom{m-n-1}{k-1} \binom{n}{i}$$

This completes the induction, proving equation (3). Then the equation in part 4 is obtained by summing equation (3) for $k = 0, \ldots, n + \ell$. \[\blacksquare\]

### 3.2 Mixable shuffles of vectors

Let $\Omega$ be a countable infinite set. Let $\tilde{\Omega}$ be the set consisting of finite non-empty subsets of $\Omega$. For $F = (F_1, \ldots, F_m) \in \tilde{\Omega}^m$, $\sigma \in S_m$ and $T$ a subset of $\{(\ell, \ell + 1) \mid 1 \leq \ell < m\}$, denote

$$\sigma(F; T) = (F_{\sigma(1)} \triangleright \cdots \triangleright F_{\sigma(m)})$$

where, for each $1 \leq \ell < m$,

$$F_{\sigma(\ell)} \triangleright F_{\sigma(\ell + 1)} = \begin{cases} F_{\sigma(\ell)} \cup F_{\sigma(\ell + 1)} & \text{if } (\ell, \ell + 1) \in T \\ F_{\sigma(\ell)}, F_{\sigma(\ell + 1)} & \text{otherwise} \end{cases}$$

For example, let $x_1, x_2, y, z \in \Omega$, $F_1 = \{x_1, x_2\}$, $F_2 = \{y\}$, $F_3 = \{z\}$, $\sigma = (1\ 2\ 3)$, $\ell = 2$. Let $S_3$, $T = \{(2,3)\}$. Then

$$\sigma(F) = (F_{\sigma(1)}, F_{\sigma(2)}, F_{\sigma(3)}) = (F_2, F_1, F_3)$$

and

$$\sigma(F; T) = (F_2 \triangleright F_1 \triangleright F_3)$$
= (F_2, F_1 \cup F_3) \\
= (\{y\}, \{x_1, x_2, z\}) \in \tilde{\Omega}^2

Let F = (F_1, \ldots, F_m) \in \tilde{\Omega}^m, \; G = (G_1, \ldots, G_n) \in \tilde{\Omega}^n. Denote (F, G) = (F_1, \ldots, F_m, G_1, \ldots, G_n) \in \tilde{\Omega}^{m+n} and denote

\[ U_k = \begin{cases} 
F_k, & 1 \leq k \leq m \\
G_{k-m}, & m + 1 \leq k \leq m + n 
\end{cases} \]

**Definition 3.2.** 1. For \( \sigma \in S(m, n) \), \( F \in \tilde{\Omega}^m \), \( G \in \tilde{\Omega}^n \), \( \sigma(F, G) = (U_{\sigma(1)}, \ldots, U_{\sigma(m+n)}) \in \tilde{\Omega}^{m+n} \)

is called a shuffle of \( F \) and \( G \).

2. Denote

\[ S(F, G) = \{ \sigma(F, G) \mid \sigma \in S(m, n) \} \]

for the set of shuffles of \( F \) and \( G \).

3. Let \( \sigma \in S(m, n) \) and let \( T \) be a subset of \( T_\sigma \). The element \( \sigma((F, G); T) = (U_{\sigma(1)}; U_{\sigma(2)}; \ldots; U_{\sigma(m+n)}) \)

is called a mixable shuffle of \( F \) and \( G \).

4. Denote

\[ \tilde{S}(F, G) = \{ \sigma((F, G); T) \mid \sigma \in S(m, n), \; T \subseteq T_\sigma \} \]

for the set of mixable shuffles of \( F \) and \( G \).

If we further have \( H = (H_1, \ldots, H_\ell) \in \tilde{\Omega}^\ell \), then we denote

\[ \tilde{S}(\tilde{S}(F, G), H) = \bigcup_{U \in \tilde{S}(F, G)} \tilde{S}(U, H), \]

and

\[ S(F, \tilde{S}(G, H)) = \bigcup_{U \in \tilde{S}(G, H)} S(F, U). \]

For \( \sigma \in S(m, n, \ell) \) and \( T \in T_\sigma \), define a mixable shuffle of \( F \), \( G \) and \( H \) by

\[ \sigma((F, G, H); T) = (U_{\sigma(1)}; U_{\sigma(2)}; \ldots; U_{\sigma(m+n+\ell)}), \]

in which

\[ U_k = \begin{cases} 
F_k, & 1 \leq k \leq m, \\
G_{k-m}, & m + 1 \leq k \leq m + n \\
H_{k-m-n}, & m + n + 1 \leq k \leq m + n + \ell 
\end{cases} \]
Then for $1 \leq k < m + n + \ell - 1$, we have

$$U_{\sigma(k)} \cup U_{\sigma(k+1)} \cup U_{\sigma(k+2)} =
\begin{cases}
U_{\sigma(k)} \cup U_{\sigma(k+1)} \cup U_{\sigma(k+2)} & (k, k+1, k+2) \in T \\
U_{\sigma(k)} \cup U_{\sigma(k+1)}, U_{\sigma(k+2)} & (k, k+1, k+2) \notin T, (k, k+1) \in T \\
U_{\sigma(k)}, U_{\sigma(k+1)} \cup U_{\sigma(k+2)} & (k, k+1, k+2) \notin T, (k+1, k+2) \in T \\
U_{\sigma(k)}, U_{\sigma(k+1)}, U_{\sigma(k+2)} & \text{otherwise}
\end{cases}$$

Denote

$$\bar{S}(F, G, H) = \{\sigma((F, G, H); T) \mid \sigma \in S(m, n, \ell), T \subseteq \mathcal{T}_n\}.$$ 

for the set of mixable shuffles of $F$, $G$ and $H$.

**Proposition 3.3.** Let $X_1, \ldots, X_m, Y_1, \ldots, Y_n, Z_1, \ldots, Z_\ell$ be disjoint subsets of $\Omega$. Let $X = (X_i) \in \bar{\Omega}^m$, $Y = (Y_j) \in \bar{\Omega}^n$, $Z = (Z_k) \in \bar{\Omega}^\ell$. Then

1. $|\bar{S}(X, Y)| = s(m, n)$.
2. $\bar{S}(X, Y) = \bar{S}(Y, X)$.
3. $\bar{S}(\bar{S}(X, Y), Z) = \bar{S}(X, \bar{S}(Y, Z))$.

**Proof.** 1. We use induction for $m + n$ with $m, n \geq 1$. When $m = 1$, $S(X, Y)$ contains the vectors

$$(X_1, Y_1, \ldots, Y_n), (Y_1, \ldots, Y_i-1, X_1, Y_i, \ldots, Y_n), i = 2, \ldots, n, (Y_1, \ldots, Y_n, X_1)$$

and

$$(X_1 \cup Y_1, Y_2, \ldots, Y_n), (Y_1, \ldots, Y_i-1, X_1 \cup Y_i, Y_{i+1}, \ldots, Y_n), i = 2, \ldots, n-1,$$

$$(Y_1, \ldots, Y_{n-1}, X_1 \cup Y_n).$$

So there are at least $2n + 1$ elements in $S(X, Y)$. By construction, the set $S(X, Y)$ has no more elements than $S(m, n)$ which is $s(1, n) = 2n + 1$ by Proposition 3.1. So the claim holds in this case. A similar argument verifies the claim in the case when $n = 1$.

Now assume that the claim is true for $m + n < k$ with $m, n > 1$, and consider the case when $m + n = k$. Since $m, n \geq 2$, it makes sense to define

$$\bar{S}_{X_1}(X, Y) = \{(X_1, S) \mid S \in \bar{S}(X_2, \ldots, X_m, Y)\},$$
$$\bar{S}_{Y_1}(X, Y) = \{(Y_1, S) \mid S \in \bar{S}(X, Y_2, \ldots, Y_n)\},$$
$$\bar{S}_{X_1 \cup Y_1}(X, Y) = \{(X_1 \cup Y_1, S) \mid S \in \bar{S}(X_2, \ldots, X_m, Y_2, \ldots, Y_n)\}.$$ 

By assumption, $X_1, Y_1$ and $X_1 \cup Y_1$ are distinct. So we have

$$S(X, Y) \supseteq \bar{S}_{X_1}(X, Y) \cup \bar{S}_{Y_1}(X, Y) \cup \bar{S}_{X_1 \cup Y_1}(X, Y) \quad \text{(7)}$$
By induction hypothesis, the three sets on the right have cardinalities \( s(m-1,n), s(m,n-1) \) and \( s(m-1,n-1) \). So

\[ |\tilde{S}(X,Y)| \geq s(m-1,n) + s(m,n-1) + s(m-1,n-1) \]

which is \( s(m,n) \) by equation (4). So by Proposition 3.1, \( |\tilde{S}(X,Y)| = s(m,n) \). Then from equation (7) we have

\[ \tilde{S}(X,Y) = \tilde{S}_X_1(X,Y) \cup \tilde{S}_Y_1(X,Y) \cup \tilde{S}_{X_1 \cup Y_1}(X,Y). \]  

(8)

2. We prove by induction on \( m + n \), \( m, n \geq 1 \). The statement can be directly verified for \( m + n \leq 2 \). Assume that it is true for \( m + n < k \) and let \( X \in A^m \), \( Y \in A^n \) with \( m + n = k \). Then

\[ \tilde{S}_X_1(X,Y) = \{(X_1,S) \mid S \in \tilde{S}((X_2, \ldots, X_m), Y)\} \]
\[ = \{(X_1,S) \mid S \in \tilde{S}(Y, (X_2, \ldots, X_m))\} \]
\[ = \tilde{S}_X_1(Y,X) \]

and similarly,

\[ \tilde{S}_Y_1(X,Y) = \tilde{S}_Y_1(Y,X) \text{ and } \tilde{S}_{X_1 \cup Y_1}(X,Y) = \tilde{S}_{Y_1 \cup X_1}(Y,X). \]

Then it follows from Equation (8) and its symmetric form for \( \tilde{S}(Y,X) \) that \( \tilde{S}(X,Y) = \tilde{S}(Y,X) \).

3. We first prove that

\[ \tilde{S}(\tilde{S}(X,Y), Z) = \bigcup_{U \in \tilde{S}(X,Y)} \tilde{S}(U, Z) \]

is a disjoint union. By assumption, \( X_i, Y_j \) and \( Z_k \) are disjoint subsets of \( \Omega \). Since each component of \( U = (U_r) \in \tilde{S}(X,Y) \) is either a \( X_i \), or \( Y_j \) or a \( X_i \cup Y_j \), it follows that the subset \( U_r \) and \( Z_k \) are also disjoint. Let \( U = (U_1, \ldots, U_r) \) and \( U' = (U'_1, \ldots, U'_s) \) be two distinct mixable shuffles of \( X \) and \( Y \). If \( r \neq s \), then without loss of generality, we could assume that \( r > s \). Then there is a \( U_{r_0} \) that is different from any \( U'_r \) and therefore is disjoint with any \( U'_r \). Thus \( U_{r_0} \) is disjoint with any component of any \( W \in \tilde{S}(U', Z) \). On the other hand, \( U_{r_0} \) has non-trivial intersection with some component of every \( W \in \tilde{S}(U, Z) \). Therefore, \( \tilde{S}(U, Z) \cap \tilde{S}(U', Z) = \phi \).

Now assume that \( r = s \). We use induction on \( r \). For \( r = 1 \), \( U \neq U' \) means \( U_1 \) is different from any components of \( U' \). So \( U_1 \) is disjoint from any components of \( W \in \tilde{S}(U', S) \), and the claim is proved. Assume that the claim is true for \( r \), and let \( U \) and \( U' \) both have length \( r + 1 \). Suppose \( \tilde{S}(U, Z) \cap \tilde{S}(U', Z) \) is not empty. Then there is a \( W \in \tilde{S}(U, Z) \cap \tilde{S}(U', Z) \). Write \( W = (W_1, \ldots, W_k) \), then \( W_1 = U_1 \) or \( Z_1 \) or \( U_1 \cup Z_1 \). If \( W_1 = U_1 \), then since \( U_1 \) is disjoint from any \( Z_\ell \), from \( W \in \tilde{S}(U', Z) \) we get \( W_1 = U'_1 \). This shows that
$$(W_2, \ldots, W_k) \in \bar{S}((U_2, \ldots, U_r), Z) \cap \bar{S}((U'_2, \ldots, U'_r), Z).$$

But since $U_1 = W_1 = U'_1$, from $U \neq U'$ we get $(U_2, \ldots, U_r) \neq (U'_2, \ldots, U'_r)$. Then by induction assumption,

$$\bar{S}((U_2, \ldots, U_r), Z) \cap \bar{S}((U'_2, \ldots, U'_r), Z) = \emptyset.$$

This is a contradiction. For the same reason, $W_1 = Z_1$ or $U_1 \cup Z_1$ also implies contradiction. Therefore, the claim is true for $r + 1$. This proves that

$$\bar{S}(\bar{S}(X,Y), Z) = \bigcup_{U \in \bar{S}(X,Y)} \bar{S}(U, Z).$$

By Proposition 3.1, $\bar{S}(X,Y)$ has $\binom{n}{i} \binom{m+n-i}{m}$ mixable shuffles of length $i$, and for each such mixable shuffle $U$, the cardinality of $\bar{S}(U, Z)$ is

$$\sum_{j=0}^{\ell} \binom{m+n+i-j}{\ell} \binom{n}{j}.$$

Therefore $\bar{S}(\bar{S}(X,Y), Z)$ has

$$\sum_{i=0}^{n} \sum_{j=0}^{\ell} \binom{m+n-i}{n} \binom{m+n-i-j}{m+n+\ell-i} \binom{\ell}{n} = \sum_{k=0}^{n+\ell} \sum_{i=0}^{n} \binom{m+n+i-k}{\ell} \binom{m+n-i}{n} \binom{n}{i}$$

elements. But $\bar{S}(\bar{S}(X,Y), Z)$ is a subset of $\bar{S}(X,Y, Z)$, and by Proposition 3.1 the second set has cardinality bounded by the same number. This proves the first equality in part 3 of the proposition.

To prove the second equality in part 3, define

$$\bar{S}(X,Y)_{X_m} = \{(S, X_m) \mid S \in \bar{S}(X_1, \ldots, X_{m-1}, Y)\}$$

and similarly for $\bar{S}(X,Y)_{Y_n}$, $\bar{S}(X,Y)_{X_m \cup Y_n}$. Then the same argument as above gives

$$\bar{S}(X,Y) = \bar{S}(X,Y)_{X_m} \bigcup \bar{S}(X,Y)_{Y_n} \bigcup \bar{S}(X,Y)_{X_m \cup Y_1} \bigcup \bar{S}(X,Y)_{X_1 \cup Y_1} \bigcup \cdots \bigcup \bar{S}(X,Y)_{X_1 \cup Y_1}$$

(9)

The rest of the proof is similar. ■
3.3 Mixable shuffles of tensors

**Notation:** For the rest of this paper, let $\lambda$ be a fixed element of $C$. For any $C$-modules $M$ and $N$, the tensor product $M \otimes N$ is taken over $C$ unless otherwise indicated. Let $A$ be a $C$-algebra. For $n \in \mathbb{N}$, denote

$$A^{\otimes n} = A \otimes \ldots \otimes A$$

with the convention that $A^{\otimes 0} = C$.

Let $A$ be a $C$-algebra. For $m, n, \ell \in \mathbb{N}_+$, denote $x = x_1 \otimes \ldots \otimes x_m \in A^{\otimes m}$, $y = y_1 \otimes \ldots \otimes y_n \in A^{\otimes n}$ and $z = z_1 \otimes \ldots \otimes z_\ell \in A^{\otimes \ell}$. For $\sigma \in S_n$, denote

$$\sigma(x) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \ldots \otimes x_{\sigma(n)}.$$  

Denote $x \otimes y = x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n \in A^{\otimes (m+n)}$, and, for $\sigma \in S_{m+n}$, denote

$$\sigma(x \otimes y) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \ldots \otimes u_{\sigma(m+n)},$$

where

$$u_k = \begin{cases} 
  x_k, & 1 \leq k \leq m, \\
  y_{k-m}, & m + 1 \leq k \leq m + n. 
\end{cases}$$

Likewise, for $\sigma \in S_{m+n+\ell}$, denote

$$\sigma(x \otimes y \otimes z) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \ldots \otimes u_{\sigma(m+n+\ell)},$$

in which

$$u_k = \begin{cases} 
  x_k, & 1 \leq k \leq m, \\
  y_{k-m}, & m + 1 \leq k \leq m + n \\
  z_{k-m-n}, & m + n + 1 \leq k \leq m + n + \ell. 
\end{cases}$$

**Definition 3.4.** Let $x \in A^{\otimes m}$, $y \in A^{\otimes n}$ and $\sigma \in S(m,n)$.

1. $\sigma(x \otimes y) \in A^{\otimes (m+n)}$ is called a shuffle of $x$ and $y$.

2. Let $T$ be a subset of $T_\sigma$. The element

$$\sigma(x \otimes y, T) = z_{\sigma(1)} \hat{\otimes} z_{\sigma(2)} \hat{\otimes} \ldots \hat{\otimes} z_{\sigma(m+n)},$$

where for each pair $(k, k + 1)$, $1 \leq k < m + n$,

$$z_{\sigma(k)} \hat{\otimes} z_{\sigma(k+1)} = \begin{cases} 
  z_{\sigma(k)} z_{\sigma(k+1)}, & (k, k + 1) \in T \\
  z_{\sigma(k)} \otimes z_{\sigma(k+1)}, & (k, k + 1) \notin T 
\end{cases}$$

is called a mixable shuffle of $x$ and $y$.  

It follows from the universal property of the tensor product $A^\otimes k$ that $\sigma(x \otimes y; T)$ does not depend on the choice of $x_1, \ldots, x_m, y_1, \ldots, y_n$ representing the tensor $x \otimes y$.

Now fix $\lambda \in C$. Define, for $x$ and $y$ as above,

$$x \overset{\lambda}{\odot} y = \sum_{(\sigma, T) \in S(m, n)} \lambda^{|T|} \sigma(x \otimes y; T) \in \bigoplus_{k \leq m+n} A^\otimes k.$$ 

The operation $\overset{\lambda}{\odot}$ extends to a mapping

$$\overset{\lambda}{\odot} : A^\otimes m \times A^\otimes n \to \bigoplus_{k \leq m+n} A^\otimes k, m, n \in \mathbb{N}$$

by $C$-linearity. Let

$$III^+_C(A) = III^+_C(A, \lambda) = \bigoplus_{k \in \mathbb{N}} A^\otimes k = C \oplus A \oplus A^\otimes 2 \oplus \ldots.$$ 

Extending by additivity, the binary operation $\overset{\lambda}{\odot}$ gives a $C$-bilinear map

$$\overset{\lambda}{\odot} : III^+_C(A) \times III^+_C(A) \to III^+_C(A)$$

with the convention that

$$C \times A^\otimes m \to A^\otimes m$$

is the scalar multiplication. This binary operation is called the **mixable shuffle product of weight $\lambda$**.

**Theorem 3.5.** The mixable shuffle product $\overset{\lambda}{\odot}$ defines an associative, commutative binary operation on $III^+_C(A) = \bigoplus_{k \in \mathbb{N}} A^\otimes k$, making it into a $C$-algebra with the identity $1_C \in C = A^\otimes 0$.

$III^+_C(A)$ will be called the **mixable shuffle algebra (of weight $\lambda$)** on $A$. In the special case when $\lambda = 0$, $III^+_C(A)$ is denoted by $\text{Sh}(A)$ in [19].

**Proof.** We only need to verify the commutativity and associativity of the operation $\overset{\lambda}{\odot}$. For this we make use of the mixable shuffle of vectors discussed in the previous section.

Recall that $\Omega$ is an infinite set, and $\tilde{\Omega}$ is the set of finite non-empty subsets of $\Omega$. For $F = (F_1, \ldots, F_m) \in \tilde{\Omega}^m$, $\sigma \in S_m$ and $T$ a subset of $\{(\ell, \ell+1) \mid 1 \leq \ell < m\}$, denote

$$\sigma(F; T) = (F_{\sigma(1)}, \ldots, F_{\sigma(m)})$$

where, for each $1 \leq \ell < m$,

$$F_{\sigma(\ell)} \hat{\odot} F_{\sigma(\ell+1)} = \begin{cases} F_{\sigma(\ell)} \cup F_{\sigma(\ell+1)} & \text{if } (\ell, \ell + 1) \in T \\ F_{\sigma(\ell)}, F_{\sigma(\ell+1)} & \text{otherwise} \end{cases}$$

15
Given a $\varphi : \bigcup_{i=1}^{m} F_i \to A$, denote  
\[
\varphi(\sigma(F;T)) = (a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(m)})
\]
where, for each $1 \leq \ell < m$,  
\[
a_{\sigma(\ell)} = \prod_{f_i \in F_i} \varphi(f_i).
\]

For example, suppose $X_1, X_2, Y, Z \in \Omega$ are distinct. Let $F_1 = \{X_1, X_2\}$, $F_2 = \{Y\}$, $F_3 = \{Z\}$, $\sigma = \left(\begin{array}{ccc}1 & 2 & 3 \\ 2 & 1 & 3\end{array}\right)$ in $S_3$, $T = \{(2, 3)\}$. If $\varphi$ sends each of the capital letters to the corresponding lower case letter in the polynomial algebra $A = C[x_1, x_2, y, z]$, then  
\[
\varphi(\sigma(F;T)) = \varphi(F_{\sigma(1)} \cup F_{\sigma(2)} \cup F_{\sigma(3)}) = y \otimes x_1 x_2 z \in A^{\otimes 2}.
\]

For any fixed $m, n, \ell$ and fixed $x \in A_{\otimes m}$, $y \in A_{\otimes n}$ and $z \in A_{\otimes \ell}$, choose distinct elements $X_1, \ldots, X_m, Y_1, \ldots, Y_n, Z_1, \ldots, Z_\ell$ from the infinite set $\Omega$. Also use the same letters for the singletons $\{X_i\}, \{Y_j\}$ and $\{Z_k\}$ of $\Omega$. Let $\varphi$ be the map sending $X_i$ to $x_i, Y_j$ to $y_j$ and $Z_k$ to $z_k$. Then $\circ^+$ (of weight $\lambda$) could be described as  
\[
x \circ^+ y = \varphi(X) \circ^+ \varphi(Y) = \sum_{U \in S(X,Y)} \lambda^{\deg(U)} \varphi(U),
\]
where $\deg(U) = |T|$ if $U$ is given by $(\sigma, T) \in \bar{S}(m,n)$. The commutativity of $\circ^+$ follows from Proposition \ref{3.3.2}.

Let a mixable shuffle $W \in S(X,Y,Z)$ be given by $(\sigma, T) \in \bar{S}(m,n,\ell)$, i.e., $W = \sigma((X,Y,Z);T)$. Define $\deg(W) = \deg T$, with $\deg T$ defined in equation \ref{5}. By Proposition \ref{3.3.3}, $W$ could also be obtained from a mixable shuffle $V$ in $\bar{S}(U,Z)$, given by $(\sigma_1, T_1)$, where $U$ is from a mixable shuffle in $S(X,Y)$, given by $(\sigma_2, T_2)$. Thus we have $V = \sigma_1((U,Z);T_1)$ and $U = \sigma_2((X,Y);T_2)$. It follows from the definition of $W$ and $\deg W$ that the length of the vector $W$ is $m + n + \ell - \deg(W)$. Since $W$ is also given by $V = \sigma_1((U,Z);T_1)$, its length is also given by  
\[
\text{length of } U + \ell - \deg(V) = m + n - \deg(U) + \ell - \deg(V).
\]
Thus we have $\deg(W) = \deg(U) + \deg(V)$. Then it follows from Proposition \ref{3.3.3} that  
\[
(x \circ^+ y) \circ^+ z = \left(\sum_{U \in S(X,Y)} \lambda^{\deg(U)} \varphi(U)\right) \circ^+ z
\]
\[
= \sum_{U \in S(X,Y)} \lambda^{\deg(U)} \varphi(U) \circ^+ \varphi(Z)
\]
\[
\sum_{U \in \bar{S}(X,Y)} \lambda^{\deg(U)} \sum_{V \in \bar{S}(U,Z)} \lambda^{\deg(V)} \varphi(V)
\]

\[
\sum_{W \in \bar{S}(S(X,Y),Z)} \lambda^{\deg(W)} \varphi(W)
\]

We similarly have

\[
x \vartriangleright^+ (y \vartriangleright^+ z) = \sum_{W \in \bar{S}(X,Y,Z)} \lambda^{\deg(W)} \varphi(W).
\]

This proves the associativity. ■

4 The free Baxter algebra

We now use the mixable shuffle product from last section to describe the free Baxter algebras. We will first construct the free Baxter algebra on a \( C \)-algebra \( A \). We will then give constructions of other types of free Baxter algebras.

4.1 The basic free construction

With the same notations as those in last section, we define \( \Pi_C(A) \) to be the tensor product algebra \( A \otimes_C \Pi_C(A) \). Thus

\[
\Pi_C(A) \cong \bigoplus_{k \in \mathbb{N}} A^{\otimes (k+1)} = A \oplus A^{\otimes 2} \oplus \ldots
\]

as \( A \)-modules and the product on \( \Pi_C(A) \) is defined by the augmented mixable shuffle product (of weight \( \lambda \))

\[
(x_0 \otimes x_1 \otimes \ldots \otimes x_m) \vartriangleright^+ (y_0 \otimes y_1 \otimes \ldots \otimes y_n)
\]

\[
= x_0 y_0 \otimes ((x_1 \otimes \ldots \otimes x_m) \vartriangleright^+ (y_1 \otimes \ldots \otimes y_n))
\]

\[
= x_0 y_0 \otimes \sum_{(\sigma,T) \in \bar{S}(m,n)} \lambda^{|T|} \sigma((x_1 \otimes x_1 \otimes \ldots \otimes x_m) \otimes (y_1 \otimes \ldots \otimes y_n); T)
\]

\[
\in A^{\otimes (m+n+1)}.
\]

Define a \( C \)-linear endomorphism \( P_A \) on \( \Pi_C(A) \) by assigning

\[
P_A(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = 1_A \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_n,
\]

for all \( x_0 \otimes x_1 \otimes \ldots \otimes x_n \in A^{\otimes (n+1)} \) and extending by additivity. Let \( j_A : A \to \Pi_C(A) \) be the canonical inclusion map.
Theorem 4.1. For any $C$-algebra $A$, $(\Pi_C(A), P_A)$, together with the natural embedding $j_A : A \to \Pi_C(A)$, is a free Baxter $C$-algebra on $A$ (of weight $\lambda$) in the sense that the triple $(\Pi_C(A), P_A, j_A)$ satisfies the following universal property: For any Baxter $C$-algebra $(R, P)$ and any $C$-algebra map $\varphi : A \to R$, there exists a unique Baxter $C$-algebra homomorphism $\tilde{\varphi} : (\Pi_C(A), P_A) \to (R, P)$ such that the diagram

\[ A \xrightarrow{j_A} \Pi_C(A) \xrightarrow{\tilde{\varphi}} R \]

commutes.

Remark: By the same argument used to show the uniqueness of other "universal" objects, $\Pi_C(A)$ is the unique free Baxter $C$-algebra on $A$ up to isomorphism. The Baxter $C$-algebra $\Pi_C(A)$ will be called the free Baxter $C$-algebra (of weight $\lambda$) on $A$.

Proof: We first show that $P_A$ is a Baxter operator on $\Pi_C(A)$. For this we only need to verify that for any $x, y \in \Pi_C(A)$,

\[ P_A(x) \circ P_A(y) = P_A(x \circ P_A(y)) + P_A(y \circ P_A(x)) + \lambda P_A(x \circ y). \]

By additivity, we only need to verify this equation for any $x = x_1 \otimes \ldots \otimes x_m \in A^\otimes m$ and $y = y_1 \otimes \ldots \otimes y_n \in A^\otimes n$. By definition,

\[ P_A(x) \circ P_A(y) = (1_A \otimes x_1 \otimes \ldots \otimes x_m) \circ (1_A \otimes y_1 \otimes \ldots \otimes y_n) = 1_A \otimes \sum_{(\sigma, T) \in \bar{S}(m, n)} \lambda^{|T|} \sigma(x \otimes y; T) = P_A(\sum_{(\sigma, T) \in \bar{S}(m, n)} \lambda^{|T|} \sigma(x \otimes y; T)). \]

Recall that equation \[\Pi\] gives us

\[ S(m, n) = S_{1,0}(m, n) \bigcup S_{0,1}(m, n) \bigcup S_{1,1}(m, n) \]

where

\[ S_{1,0}(m, n) = \{ (\sigma, T) \in \bar{S}(m, n) | (1, 2) \notin T, \sigma^{-1}(1) = 1 \}, \]
\[ S_{0,1}(m, n) = \{ (\sigma, T) \in \bar{S}(m, n) | (1, 2) \notin T, \sigma^{-1}(m + 1) = 1 \}, \]
\[ S_{1,1}(m, n) = \{ (\sigma, T) \in \bar{S}(m, n) | (1, 2) \in T \}. \]

Further,

\[ \sum_{(\sigma, T) \in \bar{S}_{1,0}(m, n)} \lambda^{|T|} \sigma(x \otimes y) = x_1 \otimes \sum_{(\sigma, T) \in \bar{S}(m-1, n)} \lambda^{|T|} \sigma((x_2 \otimes \ldots \otimes x_m) \otimes y; T) \]
\[ = x \odot (1_A \otimes y_1 \otimes \ldots \otimes y_n; T) \]
\[ = x \odot P_A(y). \]

where for \( m = 1, \sigma((x_2 \otimes \ldots \otimes x_m) \otimes y; T) = \sigma(y) \) (only \( T = \phi \) is possible). Similarly,

\[
\sum_{(\sigma, T) \in \bar{S}_{0,1}(m, n)} \lambda^{\vert T\vert} \sigma(x \otimes y; T) = P_A(x) \odot y.
\]

and

\[
\sum_{(\sigma, T) \in \bar{S}_{1,1}(m, n)} \lambda^{\vert T\vert} \sigma(x \otimes y; T)
\]
\[ = \sum_{(\sigma, T) \in \bar{S}(m-1, n-1)} \lambda x_1 y_1 \otimes \lambda^{\vert T\vert} \sigma((x_2 \otimes \ldots \otimes x_{m-1}) \otimes (y_2 \otimes \ldots \otimes y_{n-1}); T)
\]
\[ = \lambda P_A(x \odot y). \]

Therefore,

\[ P_A(x) \odot P_A(y) = P_A(x \odot P_A(y)) + P_A(P_A(x) \odot y) + \lambda P_A(x \odot y) \]

This shows that \( P_A \) is a Baxter operator of weight \( \lambda \) on \( III_C(A) \), making it into a Baxter \( C \)-algebra.

Before verifying that \((III_C(A), P_A)\) satisfies the universal property of a free Baxter \( C \)-algebra over \( A \), we need some preparations. Let \((R, P)\) be a Baxter \( C \)-algebra. For \( x, y \in R \), denote \( P_A(x) = P(xy) \). For \( x_0 \otimes \ldots \otimes x_k \in R^{\otimes (k+1)} \), denote

\[ x_0(\circ_{r=1}^{k} P_{x_r})(y) = x_0(P_{x_1} \circ \ldots \circ P_{x_k})(y), \] (10)

with the convention that \( \circ_{r=1}^{0} P_{x_r} = \text{id}_R \). It follows from the universal property of the tensor product \( R^{\otimes (k+1)} \) and the \( C \)-linearity of the Baxter operator \( P \) that the right hand side of equation (10) is well-defined, and does not depend on the choice of \( x_0, \ldots, x_k \in R \) representing the tensor \( x_0 \otimes \ldots \otimes x_k \). For \( \sigma \in S_n \), denote

\[ \sigma(\circ_{r=1}^{n} P_{x_r}) = \circ_{r=1}^{n} P_{x_{\sigma(r)}}. \]

For \( m, n \in \mathbb{N}_+ \), denote

\[ (\circ_{r=1}^{m} P_{x_r}) \circ (\circ_{s=1}^{n} P_{y_s}) = \circ_{t=1}^{m+n} P_{z_t}, \]

where

\[ z_k = \begin{cases} x_k, & 1 \leq k \leq m, \\ y_{k-m}, & m + 1 \leq k \leq m + n \end{cases} \]

and, for \( (\sigma, T) \in \bar{S}(m, n) \), denote

\[ \sigma((\circ_{r=1}^{m} P_{x_r}) \circ (\circ_{s=1}^{n} P_{y_s}); T) = P_{z_1} \circ \ldots \circ P_{z_{m+n}} \]
where for each \((k, k+1), 1 \leq k < m+n\),
\[
P_{z_k} \circ P_{z_{k+1}} = \begin{cases} P_{z_{2k+1}}, & (k, k+1) \in T \\ P_{z_k} \circ P_{z_{k+1}}, & (k, k+1) \notin T \end{cases}
\]

**Proposition 4.2.** For \(m, n \in \mathbb{N}\) and \(x = x_0 \otimes \ldots \otimes x_m \in R^{\otimes(m+1)}, y = y_0 \otimes \ldots \otimes y_n \in R^{\otimes(n+1)}\),
\[
(x_0 (\sigma_{r=1}^m P_{x_r})(1_R))(y_0 (\sigma_{s=1}^n P_{y_s})(1_R)) = x_0 y_0 \sum_{(\sigma, T) \in S(m, n)} \lambda^{|T|} \sigma((\sigma_{r=1}^m P_{x_r}) \circ (\sigma_{s=1}^n P_{y_s}))(1_R).
\]

**Proof.** It is clear that the equation in general follows from the case when \(x_0 = y_0 = 1\), in which case the equation is
\[
(\sigma_{r=1}^m P_{x_r}(1_R))(\sigma_{s=1}^n P_{y_s}(1_R)) = \sum_{(\sigma, T) \in S(m, n)} \lambda^{|T|} \sigma((\sigma_{r=1}^m P_{x_r}) \circ (\sigma_{s=1}^n P_{y_s}))(1_R)
\]
with \(m, n \geq 1\).

For this we prove by induction on \(k = m+n\). If \(k = 2\), then \(m = n = 1\). The equation to be proved in this case is
\[
P(x_1)P(y_1) = P(x_1P(y_1)) + P(y_1P(x_1)) + \lambda P(x_1y_1)
\]
which is part of the definition of \(P\). Assuming that the equation holds for all \(x \in R^{\otimes(m+1)}, y \in R^{\otimes(n+1)}\) with \(m+n < k\). Let \(x \in R^{\otimes(m+1)}, y \in R^{\otimes(n+1)}\) with \(m+n = k\). Then
\[
(\sigma_{r=1}^m P_{x_r}(1_R)(\sigma_{s=1}^n P_{y_s})(1_R)) = P(x_1(\sigma_{r=2}^m P_{x_r})(1_R))P(y_1(\sigma_{s=2}^n P_{y_s})(1_R))
\]
\[
= P(x_1(\sigma_{r=2}^m P_{x_r})(1_R))P(y_1(\sigma_{s=2}^n P_{y_s})(1_R)) + P(y_1(\sigma_{s=2}^n P_{y_s})(1_R))P(x_1(\sigma_{r=2}^m P_{x_r})(1_R))
\]
\[
+ \lambda P(x_1(\sigma_{r=2}^m P_{x_r})(1_R)y_1(\sigma_{s=2}^n P_{y_s})(1_R))
\]
\[
= P(x_1 \sum_{(\sigma, T) \in S(m-1, n)} \lambda^{|T|} \sigma((\sigma_{r=2}^m P_{x_r}) \circ (\sigma_{s=1}^n P_{y_s}))(1_R))
\]
\[
+ P(y_1 \sum_{(\sigma, T) \in S(m, n-1)} \lambda^{|T|} \sigma((\sigma_{r=1}^m P_{x_r}) \circ (\sigma_{s=2}^n P_{y_s}))(1_R))
\]
\[
+ \lambda P(x_1y_1 \sum_{(\sigma, T) \in S(m-1, n-1)} \lambda^{|T|} \sigma((\sigma_{r=2}^m P_{x_r}) \circ (\sigma_{s=2}^n P_{y_s}))(1_R))
\]

20
This completes the proof of Proposition 4.2.

We now continue with the proof of Theorem 4.1 and verify the universal property for $\Pi_C(A)$. For a given Baxter $C$-algebra $A$ and a $C$-algebra map $\varphi : A \to R$, we extend $\varphi$ to an Baxter $C$-algebra homomorphism $\tilde{\varphi} : \Pi_C(A) \to (R, i)$ as follows. For any $x = x_0 \otimes x_1 \otimes \ldots \otimes x_k \in A^\otimes (k+1)$, define

$$\tilde{\varphi}(x) = \varphi(x_0)(\circ_{j=1}^k P_{\varphi(x_j)})(1_R).$$

This is a well-defined $C$-linear map, hence extends uniquely by additivity to a $C$-module homomorphism

$$\tilde{\varphi} : \Pi_C(A) = \bigoplus_{k \in \mathbb{N}} A^\otimes (k+1) \to R.$$

It follows from the definition of the operation $\circ$ and Proposition 4.2 that $\tilde{\varphi}$ preserves multiplication. Since $\tilde{\varphi}$ is $C$-linear, and

$$\tilde{\varphi}(P_A(x_0 \otimes \ldots \otimes x_k)) = \tilde{\varphi}(1_A \otimes x_0 \otimes \ldots \otimes x_k) = (\circ_{j=0}^k P_{\varphi(x_j)})(1_R) = P(\varphi(x_0)(\circ_{j=1}^k P_{\varphi(x_j)})(1_R)) = P(\tilde{\varphi}(x_0 \otimes \ldots \otimes x_k)),$$

$\tilde{\varphi}$ is a homomorphism of Baxter $C$-algebras. It is clear from its construction that $\tilde{\varphi}$ is the unique homomorphism of Baxter $C$-algebras extending $\varphi$. This proves Theorem 4.1. 

■
Let \( \text{Alg}_C \) be the category of \( C \)-algebras and let \( U_C : \text{Bax}_C \to \text{Alg}_C \) be the forgetful functor.

**Corollary 4.3.**

1. The assignment \( A \mapsto \mathbb{III}_C(A) \) gives a functor

\[
\mathbb{III}_C : \text{Alg}_C \to \text{Bax}_C,
\]

where for a \( C \)-algebra homomorphism \( f : A \to B \), \( \mathbb{III}_C(f) : \mathbb{III}_C(A) \to \mathbb{III}_C(B) \) is defined by \( \oplus_{n \in \mathbb{N}} f^{\otimes n} \) with \( f^{\otimes n} : A^{\otimes n} \to B^{\otimes n} \).

2. \( \mathbb{III}_C \) is the left adjoint functor of the forgetful functor \( U_C \).

3. Any Baxter \( C \)-algebra is isomorphic to a quotient of \( (\mathbb{III}_C(A), P_A) \) for some \( C \)-algebra \( A \).

By Corollary 4.3, the study of Baxter \( C \)-algebras is reduced to studying quotients of free Baxter \( C \)-algebras.

From the proof of the theorem and Proposition 4.2, we also obtain

**Corollary 4.4.**

1. For any sub-\( C \)-algebra \( B \) of a Baxter \( C \)-algebra \( (R, P) \), the Baxter sub-\( C \)-algebra of \( R \) generated by \( B \) is generated by

\[
\{ x_0(\sigma_{i=1}^r P_{x_i})(1_R) \mid x_i \in B, \ 0 \leq i \leq r, \ r \in \mathbb{N} \}
\]

as an additive group.

2. For any \( C \)-algebra homomorphism \( \varphi : A \to R \), the image of \( \tilde{\varphi} : \mathbb{III}_C(A) \to R \) is the Baxter subalgebra of \( (R, P) \) generated by \( \varphi(A) \).

**Proof.** Let \( \hat{B} \) be the Baxter sub-\( C \)-algebra of \( (R, P) \) generated by \( B \). Denote

\[
S = \{ x_0(\sigma_{i=1}^r P_{x_i})(1_R) \mid x_i \in B, \ 0 \leq i \leq r, \ r \in \mathbb{N} \}
\]

and let \( \hat{B}' \) be the additive subgroup of \( R \) generated by \( S \). Since \( S \) is closed under scalar multiplication by \( C \) and the operator \( P \), \( \hat{B}' \) is a \( C \)-module and is closed under \( P \). By Proposition 4.2 \( \hat{B}' \) is closed under multiplication. Therefore, \( \hat{B}' \) is a sub-\( C \)-algebra of \( R \), hence contains \( \hat{B} \). On the other hand, since \( \hat{B} \) contains \( B \) and is closed under multiplication and Baxter operator, it must contain \( S \). Then \( \hat{B} \) must contain \( \hat{B}' \) by closure under addition. This proves the first statement.

By its construction, the Baxter \( C \)-algebra \( \mathbb{III}_C(A) \) is generated by \( j_A(A) \). Since \( \tilde{\varphi} \) is an Baxter \( C \)-algebra homomorphism, \( \tilde{\varphi}(\mathbb{III}_C(A)) \) is also an Baxter \( C \)-subalgebra, and is generated by \( \tilde{\varphi}(j_A(A)) = \varphi(A) \).
4.2 Other free constructions

The construction of the free Baxter $C$-algebra $\mathfrak{III}_C(A)$ on a $C$-algebra $A$ in the last part could be combined with other free constructions to obtain free Baxter algebras on other structures. We now discuss the free Baxter algebra on a set. The free Baxter algebras on a monoid or on a $C$-module will also be considered.

For a given set $X$, let $C[X]$ be the polynomial $C$-algebra on $X$ with the natural embedding $X \hookrightarrow C[X]$. Let $(\mathfrak{III}_C(X), P_X)$ be the Baxter $C$-algebra $(\mathfrak{III}_C(C[X]), P_{C[X]})$.

**Proposition 4.5.** $(\mathfrak{III}_C(X), P_X)$, together with the set embedding $j_X : X \hookrightarrow C[X] \xrightarrow{j_{C[X]}} \mathfrak{III}_C(C[X])$, is a free Baxter $C$-algebra on the set $X$, described by the following universal property: For any Baxter $C$-algebra $(R, P)$ over $C$ and any set map $\varphi : X \rightarrow R$, there exists a unique Baxter $C$-algebra homomorphism $\tilde{\varphi} : (\mathfrak{III}_C(X), P_X) \rightarrow (R, i)$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{j_X} & \mathfrak{III}_C(X) \\
\varphi \downarrow & & \tilde{\varphi} \downarrow \\
R & & 
\end{array}
$$

commutes.

**Remark:** When $\lambda = -1$, it is easy to show that $\mathfrak{III}_C(X)$ is closely related to the free Baxter algebra constructed by Cartier[2]. See section 5 for detail.

**Proof.** We only need to verify that $(F_C(X), P_X)$ satisfies the universal property of a free Baxter $C$-algebra on $X$. Fix a given Baxter algebra $(R, P)$ over $C$, and a set map $\varphi : X \rightarrow R$. By the universal property of the $C$-algebra $C[X]$, $\varphi$ extends uniquely by multiplicity and $C$-linearity to a $C$-algebra homomorphism $\bar{\varphi} : C[X] \rightarrow R$. Then by Theorem 4.1, $\bar{\varphi}$ extends uniquely to a homomorphism of Baxter $C$-algebras

$$
\tilde{\varphi} : (\mathfrak{III}_C(X), P_X) \cong (\mathfrak{III}_C(C[X]), P_{C[X]}) \rightarrow (R, i).
$$

This proves the proposition. $\blacksquare$

Note that the free Baxter $C$-algebra on a set $X$ can be described as the composite of two free constructions. First we construct the free $C$-algebra $C[X]$ on $X$, and then we construct the free Baxter $C$-algebra on $C[X]$. In a similar manner, we can construct the free Baxter $C$-algebra on a monoid $M$ by first constructing the free $C$-algebra $C <M>$ on $M$ and then the free Baxter $C$-algebra on $C <M>$. Here, for a given commutative monoid $M$, $C <M>$ is the free $C$-module $\oplus_{x \in M} Cx$, where the multiplication on $C <M>$ is induced by the multiplication on $M$.

Likewise, we can construct the free Baxter $C$-algebra on a $C$-module $N$ by first constructing the tensor $C$-algebra $T_C(N)$ on $N$ and then the free Baxter $C$-algebra on $T_C(N)$.
5 Relation to constructions of Cartier and Rota

We now give an explicit description of the relation between our construction of free Baxter algebras and the constructions of Cartier [2] and Rota [15].

As before, let $C$ be a commutative ring with identity. Let $\text{Alg}_C^0$ be the category in which the objects are $C$-algebras not necessarily having an identity and the morphisms preserve the addition and the multiplication, but do not necessarily preserve the identity. Rota and Cartier considered the category $\text{Bax}_C^0$ whose objects are pairs $(R, P)$ where $R$ is an object in $\text{Alg}_C^0$ and $P$ is a Baxter operator (of weight $-1$), and whose morphisms are morphisms in $\text{Alg}_C^0$ that commute with the Baxter operators.

For any non-empty set $X$, the existence of a free Baxter algebra on $X$ in either of the two categories $\text{Bax}_C$ or $\text{Bax}_C^0$ can be proved by general results from universal algebra [6, 11]. Rota and Cartier have given explicit descriptions of the free Baxter algebras in $\text{Bax}_C^0$. Below we will focus on the relation of our construction of free Baxter algebras in $\text{Bax}_C$ with Cartier’s construction of free Baxter algebras in $\text{Bax}_C^0$. This will also explain the relation with Rota’s construction of free Baxter algebras in $\text{Bax}_C^0$, since, by the uniqueness of universal objects in $\text{Bax}_C^0$, the free Baxter algebras of Cartier are isomorphic to the free Baxter algebras of Rota.

We first recall the free Baxter algebra in $\text{Bax}_C^0$ constructed by Cartier. Let $M$ be the free commutative semigroup with identity on $X$. Let $\tilde{X}$ denote the set of symbols of the form

$$u_0 \cdot [ ] \quad u_0 \in M, u_0 \neq 1,$$

$$u_0 \cdot [u_1, \ldots, u_m] \quad m \geq 1, u_0, u_1, \ldots, u_m \in M, u_m \neq 1.$$

Let $\mathfrak{B}(X)$ be the free $C$-module on $\tilde{X}$. Cartier gave a $C$-bilinear multiplication $\diamond_c$ on $\mathfrak{B}(X)$ by defining

$$(u_0 \cdot [ ]) \diamond_c (v_0 \cdot [ ]) = u_0 v_0 \cdot [ ],$$

$$(u_0 \cdot [ ]) \diamond_c (v_0 \cdot [v_1, \ldots, v_n]) = (v_0 \cdot [v_1, \ldots, v_n]) \diamond_c (u_0 \cdot [ ]) = u_0 v_0 \cdot [v_1, \ldots, v_n],$$

and

$$(u_0 \cdot [u_1, \ldots, u_m]) \diamond_c (v_0 \cdot [v_1, \ldots, v_n])$$

$$= \sum_{(k,P,Q) \in \tilde{S}_c(m,n)} (-1)^{k+p+q} u_0 v_0 \cdot \Phi_{k,P,Q}([u_1, \ldots, u_m],[v_1, \ldots, v_n]).$$

Here $\tilde{S}_c(m,n)$ is the set of triples $(k, P, Q)$ in which $k$ is an integer between 1 and $m + n$, $P$ and $Q$ are ordered subsets of $\{1, \ldots, k\}$ with the natural ordering such that $P \cup Q = \{1, \ldots, k\}$, $|P| = m$ and $|Q| = n$. For each $(k, P, Q) \in \tilde{S}_c(m,n)$, $\Phi_{k,P,Q}([u_1, \ldots, u_m],[v_1, \ldots, v_n])$ is the element $[c_1, \ldots, c_k]$ in $\tilde{X}$ defined by
Define a $C$-linear operator $P_X^c$ on $\mathfrak{B}(X)$ by

$$P_X^c(u_0 \cdot [\ ]) = 1 \cdot [u_0],$$
$$P_X^c(u_0 \cdot [u_1, \ldots, u_m]) = 1 \cdot [u_0, u_1, \ldots, u_m].$$

Cartier proved that the pair $(\mathfrak{B}(X), P_X^c)$ is a free Baxter algebra on $X$ in the category $\text{Bax}^0_C$.

On the other hand, since $C[X]$ is a free $C$-module on $M$, it follows from our construction of the mixable shuffle product algebra $\mathbb{III}_C(X) = \mathbb{III}_C(C[X])$ that $\mathbb{III}_C(X)$ is a free $C$-module on the set $\hat{X}$ of tensors

$$u_0 \otimes \ldots \otimes u_k, \ u_k \in M, \ k \geq 0.$$

The Baxter operator $P_X$ on $\mathbb{III}_C(X)$ is defined by

$$P_X(u_0 \otimes \ldots \otimes u_m) = 1 \otimes u_0 \otimes \ldots \otimes u_m.$$

We define a map $f : \hat{X} \to \hat{X}$ by

$$f(u_0 \cdot [\ ]) = u_0;$$
$$f(u_0 \cdot [u_1, \ldots, u_m]) = u_0 \otimes u_1 \otimes \ldots \otimes u_m,$$

and extend it by $C$-linearity to a $C$-linear map

$$f : \mathfrak{B}(X) \to \mathbb{III}_C(X).$$

**Proposition 5.1.** $f$ is an injective morphism in $\text{Bax}^0_C$, identifying $\mathfrak{B}(X)$ with the sub-Baxter algebra of $\mathbb{III}_C(X)$ with the $C$-basis

$$\{u_0 \otimes \ldots \otimes u_m \mid m \geq 0, \ u_i \in M, \ i = 0, \ldots, m, \ u_m \neq 1\}.$$

**Proof.** Define $h : X \to \mathfrak{B}(X)$ by $h(x) = x \cdot [\ ]$. Cartier showed that the pair $(\mathfrak{B}(X), h)$ is a free Baxter algebra on $X$ in the category $\text{Bax}^0_C$. Define $g : X \to \mathbb{III}_C(X)$ by $g(x) = x \in \mathbb{III}_C(X), \ x \in X$, and regard $\mathbb{III}_C(X)$ as an element in $\text{Bax}^0_C(X)$. By the universal property of $\mathfrak{B}(X)$, there is a unique morphism

$$\tilde{g} : \mathfrak{B}(X) \to \mathbb{III}_C(X)$$

in $\text{Bax}^0_C$ such that $\tilde{g}(x \cdot [\ ]) = x$. It follows from the fact that $\tilde{g}$ preserves multiplication that $\tilde{g}(u_0 \cdot [\ ]) = u_0$ for $u_0 \in M, \ u_0 \neq 1$.
It remains to prove, for \( m \geq 1 \), \( u_0, \ldots, u_m \in M \), \( u_m \neq 1 \), that
\[
\tilde{g}(u_0 \cdot [u_1, \ldots, u_m]) = u_0 \otimes u_1 \otimes \ldots \otimes u_m.
\] (11)

We will prove this by induction. Let \( m = 1 \). If \( u_0 = 1 \), we have
\[
\tilde{g}(u_0 \cdot [u_1]) = \tilde{g}(P_X^c(u_1)) = P_X(\tilde{g}(u_1)) = 1 \otimes u_1.
\]
If \( u_0 \neq 1 \), we have
\[
\tilde{g}(u_0 \cdot [u_1]) = \tilde{g}(u_0 \cdot [\cdot]) \circ_c (1 \cdot [u_1])
= \tilde{g}(u_0 \cdot [\cdot]) \circ \tilde{g}(1 \cdot [u_1])
= \tilde{g}(u_0 \cdot [\cdot]) \circ \tilde{g}(P_X^c(u_1 \cdot [\cdot]))
= u_0 \circ P_X(\tilde{g}(u_1 \cdot [\cdot]))
= u_0 \circ P_X(u_1)
= u_0 \circ (1 \otimes u_1)
= u_0 \otimes u_1.
\]

Assume that equation (11) holds for \( m \geq 1 \) and consider the element
\[ u_0 \cdot [u_1, \ldots, u_{m+1}], \ u_i \in M, u_{m+1} \neq 1. \]

If \( u_0 = 1 \), then
\[
\tilde{g}(u_0 \cdot [u_1, \ldots, u_{m+1}]) = \tilde{g}(P_X^c(u_1 \cdot [u_2, \ldots, u_{m+1}]))
= P_X(\tilde{g}(u_1 \cdot [u_2, \ldots, u_{m+1}]))
= P_X(u_1 \otimes \ldots \otimes u_{m+1})
= 1 \otimes u_1 \otimes \ldots \otimes u_{m+1}
= u_0 \otimes u_1 \otimes \ldots \otimes u_{m+1}
\]

If \( u_0 \neq 1 \), we have
\[
\tilde{g}(u_0 \cdot [u_1, \ldots, u_{m+1}]) = \tilde{g}((u_0 \cdot [\cdot]) \circ_c (1 \cdot [u_1, \ldots, u_{m+1}]))
= \tilde{g}(u_0 \cdot [\cdot]) \circ \tilde{g}(P_X^c(u_1 \cdot [u_2, \ldots, u_{m+1}]))
= u_0 \circ P_X(\tilde{g}(u_1 \cdot [u_2, \ldots, u_{m+1}]))
= u_0 \circ P_X(u_1 \otimes u_2 \otimes \ldots \otimes u_{m+1})
= u_0 \circ (1 \otimes u_1 \otimes \ldots \otimes u_{m+1})
= u_0 \otimes u_1 \otimes \ldots \otimes u_{m+1}.
\]

Thus \( \tilde{g} \) is the \( C \)-linear map \( f \) defined above, proving that \( f \) is a morphism in \( \mathbf{Bax}_C^0 \).

Since \( f \) is \( C \)-linear and sends the \( C \)-basis \( X \) of \( \mathfrak{B}(X) \) injectively to the \( C \)-linearly independent set
we see that $f$ is injective. ■

Proposition 5.1 enables us to identify $\mathcal{B}(X)$ as a sub-Baxter algebra of $\mathfrak{III}_C(X)$ in the category $\text{Bax}_C^0$. We further have

**Proposition 5.2.** The injective morphism $f : \mathcal{B}(X) \to \mathfrak{III}_C(X)$ in $\text{Bax}_C^0$, satisfies the following universal property. For any element $A$ in $\text{Bax}_C$, also regarded as an element in $\text{Bax}_C^0$, and any morphism $\phi : \mathcal{B}(X) \to A$ in $\text{Bax}_C^0$, there is a unique morphism $\bar{\phi} : \mathfrak{III}_C(X) \to A$ in $\text{Bax}_C$ such that $\bar{\phi} \circ f = \phi$.

**Proof.** With the notations introduced in the proof of Proposition 5.1, we have the following diagram

$$
\begin{array}{c}
X \\
\downarrow g
\end{array} \quad \begin{array}{c}
\mathfrak{III}_C(X) \\
\bar{\phi} \downarrow \phi
\end{array} \quad \begin{array}{c}
\mathcal{B}(X) \\
\downarrow f
\end{array}
$$

We only need to find $\bar{\phi} : \mathfrak{III}_C(X) \to A$ in $\text{Bax}_C$ such that the lower right triangle commutes. By the universal property of $\mathfrak{III}_C(X)$ in $\text{Bax}_C$, the map $\phi \circ h : X \to A$ induces a morphism $\psi : \mathfrak{III}_C(X) \to A$ in $\text{Bax}_C$ such that $\psi \circ g = \phi \circ h$. We then have

$$
\psi \circ f \circ h = \psi \circ g = \phi \circ h.
$$

From the universal property of $\mathcal{B}(X)$ in $\text{Bax}_C^0$, we obtain

$$
\psi \circ f = \phi.
$$

So we can take $\bar{\phi} = \psi$. ■

### 6 The free Baxter $C$-algebra on $C$

As a particular example, we consider $\mathfrak{III}_C(C)$, the free Baxter $C$-algebra on $C$. $\mathfrak{III}_C(C)$ is also $\mathfrak{III}_C(\phi)$, the free Baxter $C$-algebra on the empty set $\phi$, defined in Corollary 4.5. This free Baxter algebra not only provides the simplest example of a free Baxter algebra, it also helps to explain an important difference between our construction of the free Baxter algebra and the construction of the free Baxter algebra of Rota or Cartier. As we see from the previous section, the free Baxter algebras of Rota and Cartier are in $\text{Bax}_C^0$ and have no identity. In fact, the free Baxter algebra on $\phi$ in $\text{Bax}_C^0$ is the zero algebra with the zero Baxter operator.
If we choose $A = C$ in the construction of free Baxter $C$-algebras, then we get

$$\mathfrak{III}_C(C) = \bigoplus_{n=0}^{\infty} C^{\otimes(n+1)}.$$  

Since the tensor product is over $C$, we have $C^{\otimes(n+1)} = C 1^{\otimes(n+1)}$, where $1^{\otimes(n+1)} = 1_C \otimes \ldots \otimes 1_C$. In particular, $1^{\otimes 1} = 1_C$. Thus $\mathfrak{III}_C(C)$ is a free $C$-module on the basis $1^{\otimes n}$, $n \geq 1$.

**Proposition 6.1.** For any $m, n \in \mathbb{N}$,

$$1^{\otimes(m+1)} \odot 1^{\otimes(n+1)} = \sum_{k=0}^{m} \binom{m+n-k}{n} \lambda^k 1^{\otimes(m+n+1-k)}.$$  

*Proof.* This is immediate from part 2 of Proposition 3.1. \[\Box\]

We also have the following consequence of Theorem 4.1.

**Corollary 6.2.** $\mathfrak{III}_C(C)$ is the initial object in the category $\text{Bax}_C$ of Baxter $C$-algebras. In other words, for any Baxter $C$-algebra $(R, P)$, there is a unique Baxter $C$-algebra homomorphism $(\mathfrak{III}_C(C), P_C) \rightarrow (R, P)$.

Now we consider the special case when $\lambda = 0$. In this case this $C$-algebra has been studied earlier [19, §12.3], as the shuffle algebra $\text{Sh}(C)$ over $C$. The following facts from there can easily be verified.

**Proposition 6.3.** Let $\lambda = 0$.

1. The free Baxter $C$-algebra $\mathfrak{III}_C(C)$ is a bialgebra.

2. For any $m, n \in \mathbb{N}$, $1^{\otimes(m+1)} \odot 1^{\otimes(n+1)} = \binom{m+n}{n} 1^{\otimes(m+n+1)}$.

Now let $HC$ be the ring of Hurwitz series over $C$ [8], defined to be the set of sequences

$$\{(a_n) \mid a_n \in C, n \in \mathbb{N}\}$$

in which the addition is defined componentwise and the multiplication is defined by

$$(a_n)(b_n) = (c_n)$$

with

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$  

Denote $e_n$ for the sequence $(a_k)$ in which $a_n = 1_C$ and $a_k = 0$ for $k \neq n$. Since $e_n e_m = \binom{m+n}{n} e_{m+n}$, from Proposition 6.3 we obtain
Proposition 6.4. Let $\lambda = 0$. The assignment 

$$1^{\otimes (n+1)} \mapsto e_n, \ n \geq 0$$

defines an injective homomorphism of Baxter $C$-algebras from $\mathcal{III}_C(C)$ to $HC$, identifying $\mathcal{III}_C(C)$ with the subalgebra of “Hurwitz polynomials”

$$\{(a_n) \in HC \mid a_n = 0, n \gg 0\} = \{ \sum_{n \geq 0} a_n e_n \mid a_n = 0, n \gg 0\}.$$ 

References

[1] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.* 10 (1960), 731-742.

[2] P. Cartier, On the structure of free Baxter algebras, *Adv. in Math.* 9 (1972), 253-265.

[3] K.T. Chen, Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula, *Ann. of Math.* 65 (1957), 163-178.

[4] K.T. Chen, Algebraic paths, *J. Algebra*, 10 (1968), 8-36.

[5] K.T. Chen, Iterated path integrals, *Bull. AMS*, 83 (1977), 831-879.

[6] P. M. Cohn, “Universal Algebra,” Harper and Row, New York, 1965.

[7] P.E. Crouch and F. Mammabhi-Lagarrigue, Algebraic and multiple integral identities, *Acta Applicandae Mathematicae* 15 (1989), 235-274.

[8] W. Keigher, On the ring of Hurwitz series, *Comm. Algebra*, 25 (1997), 1845-1859.

[9] B. Köck, Shuffle products in higher $K$-theory, *Comm. Algebra*, 92 (1994), 269-307.

[10] E. Kolchin, “Differential Algebras and Algebraic Groups,” Academic Press, New York, 1973.

[11] S. MacLane, “Categories for the Working Mathematician,” Springer-Verlag, New York, 1971.

[12] R. Ree, Lie elements and an algebra associated with shuffles, *Ann. Math.*, 68(1958), 210-220.
[13] J. F. Ritt, “Differential Equations from the Algebraic Standpoint,” Amer. Math. Soc. Colloq. Publ., 14, Amer. Math. Soc., New York, 1932.

[14] M. Ronco, On the Hochschild homology decompositions, Comm. Algebra, 21(1993), 4699-4712.

[15] G. Rota, Baxter algebras and combinatorial identities I, Bull. AMS, 5 (1975), 325-329.

[16] G. Rota, Baxter algebras and combinatorial identities II, Bull. AMS, 5 (1975), 330-334.

[17] G. Rota, Baxter operators, an introduction, In: “Gian-Carlo Rota on Combinatorics, Introductory Papers and Commentaries”, Joseph P.S. Kung, Editor, Birkhäuser, Boston, 1995.

[18] G. Rota, Ten mathematics problems I will never solve, Invited address at the joint meeting of the American Mathematical Society and the Mexican Mathematical Society, Oaxaca, Mexico, December 6, 1997. DMV Mittel lungen Heft 2, 1998, 45-52.

[19] M. Sweedler, “Hopf Algebras,” Benjamin, New York, 1969.