A new approach to shortest paths on networks based on the quantum bosonic mechanism

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Abstract. This paper presents quantum bosonic shortest path searching (QBSPS), a natural, practical and highly heuristic physical algorithm for reasoning about the recognition of network structure via quantum dynamics. QBSPS is based on an Anderson-like itinerant bosonic system in which a boson’s Green function is used as a navigation pointer for one to accurately approach the terminals. QBSPS is demonstrated by rigorous mathematical and physical proofs and plenty of simulations, showing how it can be used as a greedy routing to seek the shortest path between different locations. In methodology, it is an interesting and new algorithm rooted in the quantum mechanism other than combinatorics. In practice, for the all-pairs shortest-path problem in a random scale-free network with $N$ vertices, QBSPS runs in $O(\mu(N) \ln \ln N)$ time. In application, we suggest that the corresponding experimental realizations are feasible by considering path searching in quantum optical communication networks; in this situation, the method performs a pure local search on networks without requiring the global structure that is necessary for current graph algorithms.

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The problem of finding optimal paths has received considerable attention from the scientific community since it has found extensive applications in the fields of computer science, physics, operations research, transportation engineering, network routing and so on. Among the various interesting issues of these problems, the shortest path problem is, beyond doubt, one of the most well-studied algorithmic graph problem. In recent decades, this problem has been extensively studied in all kinds of classic graphs (networks), directed or undirected, weighted or unweighted. There exist many mathematical algorithms for the shortest path problem. The standard textbook algorithms, such as Dijkstra [1], Bellman–Ford, Floyd–Warshall and others (see [2]), all work in the comparison-addition model with real edge weights. Since then, much progress in shortest path problems has been made by combining or improving these basic methods. There are also combinatorics techniques based on scaling [3], integer matrix multiplication [4] and fast integer sorting [5]. In the most general case, for directed graph with real edge weights, the best-known algorithm [6] solving the all-pairs shortest-path (APSP) problem runs in $O(MN + N^2 \log \log N)$ time, where $M$ and $N$ are the numbers of edges and nodes of the graph, respectively. See [7] for recent developments. The best-known result for the upper bound on the time complexity in this version is $O(N^3 / \log^2 N)$ given by Chan’s algorithm [8], which is marginally subcubic. Trivially, the existing lower bound on the time complexity of APSP is $\Omega(N^2)$ lower bound. For undirected and unweighted graphs, quite a simple and elegant algorithm was given by Seidel [9] running in $\tilde{O}(N^\omega)$ time, where $\omega$ is the exponent of matrix multiplication and the $\tilde{O}$ notation hides polylogarithmic and, in some cases, $n^{\varepsilon}$ factors for an arbitrarily small constant $\varepsilon > 0$. While for undirected graphs with edge weights from 0, 1, . . . , $W$, Shoshan and Zwick [10] proposed an $\tilde{O}(WN^{\omega})$ algorithm.

We must admit that the algorithms for the shortest-path problem based on traditional combinatorics are definitely very important because the graph-theoretic abstraction of network topology is the foundation of path searching. However, there is also the interesting and natural question as to whether it is possible to design shortest-path algorithms with the aid of some physical laws. In fact, most networked systems, natural or man-made, are physical ones; they transport energy, information packets and entities efficiently through the optimal paths in a network maze. There should exist some underlying laws guiding the observed transport process through the shortest paths. This question may become even more significant once people are able to use physical entities, such as optical signals, random diffusing particles, quantum information...
packets and so on, to detect unknown network topologies in the future. An ideal goal is that, utilizing some observable or measurable physical properties, the shortest paths can be found without dealing with global information of the network topology.

In fact, for some large-scale complex systems in many technical fields, e.g. the internet, WWW (World Wide Web), human transportation and communication systems, shortest path searching is also considered as a related problem with routing or navigation on networks. In discussing those problems, computer scientists and physicists have introduced a number of new theories, models and algorithms to illuminate the navigation function of networked systems \[11\]. Recently, Boguñá \textit{et al} \[12\] found that greedy routing (GR) \[13, 14\] that relies on the hidden metric space abstraction of a network \[15\] can find asymptotically the shortest paths efficiently. Although this theory is still a conjecture for some real networks in general, it sparks many interesting discussions on new networking paradigms. This is of practical significance for some large man-made systems, such as the internet, since many experts point out that the existing data network architecture is severely stressed and is reaching its capability limits \[16\].

In this paper, we present a new shortest-path algorithm from the quantum mechanics point of view and discuss its practicability and validity on several realistic scale-free (SF) networks. We introduce, in particular, an itinerant bosonic model and Green function method to detect the shortest paths in a network. The Green function of the hopping boson is used as a navigation pointer to search out the right sequence of nodes of a shortest-path step by step. For a sufficiently short time interval, the Green function for nodes with shorter distance to the terminal exhibits larger values as shown in the propagation of quantum waves. In this sense, by selecting nodes with the largest Green function values among the neighborhood nodes, we can identify the shortest paths from the mass of nodes. Rigorous mathematical and physical proofs and plenty of numerical simulations are both given to show the correctness and validity of the algorithm.

It is also suggested that a corresponding experiment is feasible on quantum optical networks, where the quantum wave intensity can be measured instead of calculation. In this situation, the shortest paths can be found through local search without knowing the global network structure. We attempt to build a bridge between the computer science and modern physics mechanisms. This will not only bring fresh physical ideas to reconsider the classic graph problem but also flashes in the new calculation mode. Furthermore, we show a heuristic APSP algorithm derived from our formulation runs in \(O(\mu(N) \ln \ln N)\) time for random SF networks, where \(\mu(N)\) is the time required to multiply two \(N \times N\) matrices.

2. Problem description

2.1. Notation and definitions

To begin with, let us first consider a large but finite undirected unweighted network with \(N\) nodes and \(M\) edges connecting them. It is assumed that \(1 \leq N \leq \infty\) and \(0 \leq M \leq \infty\). We use the notation \(G = (V, E)\), where \(V\) denotes the set of nodes and \(E\) the set of edges. Naturally, \(E\) is a subset of the set \(\{(v, w)|v, w \in V\}\) of all pairs of nodes. Define the network adjacency matrix \(A\) by the following rules: if nodes \(i\) and \(j\) are connected by a link, the matrix element \(A_{ij}\) is set to unity; otherwise, \(A_{ij} = 0\). Since the network is undirected, \(A^T = A\) holds true. With these definitions, the degree for a specific node \(i\) is given by \(K_i = \sum_{l \neq i} A_{il}\).

A shortest path \(P\) between node \(v_1\) and \(v_k\) should be the shortest sequence of nodes and edges \((v_1, (v_1, v_2), v_2, \ldots, (v_{k-1}, v_k), v_k)\) such that \((v_i, v_{i+1}) \in E\) for \(i = 1, \ldots, k - 1\) and
\[ v_i \neq v_j \text{ for all } 1 \leq i < j \leq k. \] Nodes \( v_1 \) and \( v_k \) are called the origin and terminal nodes of \( P \), respectively. A path from \( s \) to \( f \) is denoted by \( s-f \) path and the distance from \( s \) to \( f \) is denoted by \( d(s, f) \). \( N_v(v) = \{ w \mid A_{vw} \neq 0 \} \) is the set of neighbor nodes of \( v \).

2.2. Physical and mathematical formulations

We are now ready to present our new QBSPS algorithm. Unlike previous algorithms, here we employ a non-interacting itinerant bosonic model on the network, which is used to study some topological characters of complex networks in our previous works [17]. For the convenience of readers, we briefly recall the model here. In this model, the hopping amplitude between any pair of nodes is determined by the local topology of networks. The corresponding Hamiltonian is given by

\[
\hat{H} = \sum_{\{i\}} \epsilon_i \hat{c}_i^\dagger \hat{c}_i + \sum_{\{i,j\}} H_{ij} \hat{c}_i^\dagger \hat{c}_j,
\]

where \( \hat{c}_i^\dagger (\hat{c}_i) \) denotes the boson creation (annihilation) operator that creates (annihilates) a spinless boson at node \( i \). \( \epsilon_i \) represents the disordered on-site potential and \( H_{ij} \) represents the boson hopping amplitude between nodes \( i \) and \( j \) and the summation is over all the distinct pairs of nodes.

To build the topology of the network into this model, we further define \( H_{ij} \) and \( \epsilon_i \) by

\[
H_{ij} = -A_{ij}, \quad \text{if} \ i \neq j, \quad \epsilon_i = \frac{\sum_{l \neq i} A_{il} \sqrt{K_l}}{\sqrt{K_i}}.
\]

While the first part in equation (2) seems natural, the second one demands some explanation. In physics, \( \epsilon_i \) is the local chemical potential of particles. It controls the probability for a particle to stay at node \( i \): a higher value of \( \epsilon_i \) makes such probability lower. On the other hand, for a particle hopping randomly in a network, a large value of \( \sum_{l \neq i} A_{il} \sqrt{K_l} \) implies that it has less chance of returning to node \( i \). Therefore, the visibility of the particle at this node is decreased. It justifies our choice for the numerator of \( \epsilon_i \). In the meantime, we introduce the denominator \( \sqrt{K_i} \) in \( \epsilon_i \) for the purpose of normalization.

Now, by choosing a natural basis of single-particle states \( |i\rangle = \hat{c}_i^\dagger |0\rangle \), we rewrite Hamiltonian (1) into the following matrix,

\[
H = \begin{pmatrix}
\sum_i^J A_{i1} \sqrt{K_1} & -A_{12} & \ldots & \ldots \\
-A_{21} & \sum_i^J A_{i2} \sqrt{K_2} & \ldots & \ldots \\
\vdots & \ddots & \ddots & \ddots \\
-A_{N1} & \ldots & -A_{N2} & \sum_i^J A_{N1} \sqrt{K_N} \nend{pmatrix},
\]

where \( \sum_i^J \) stands for \( \sum_{l \neq i} \).

First, we note that matrix \( H \) is semi-positive definite. This result indicates that the ground state energy \( E_0 \) of this system is 0, which will be useful in the following discussion. On the other hand, a direct calculation reveals that the vector \( \mathbf{u} = (\sqrt{K_1}, \sqrt{K_2}, \ldots, \sqrt{K_N})^T \) is an eigenvector of the matrix \( H \) with eigenvalue \( \lambda = 0 \). Therefore, we find that

\[
\Phi_0 = \frac{1}{\sqrt{\sum_n K_n}} \sum_i \sqrt{K_i} |i\rangle
\]
is one of the normalized ground states of Hamiltonian (1). Apparently, the values of the expansion coefficients of the ground state reveal the degree distribution of the network. The expansion coefficients of the first exciting state also provide useful structure information. For details see [17].

2.3. Quantum bosonic shortest-path searching

In the following part, we reveal the key connection between our quantum mechanism and the shortest-path searching problem on the network. Originally, we introduce the Green function $G(f, \Delta t; s, 0)$ to describe the transition amplitude of a boson that appears in a terminal node $f$ at time $\Delta t$ in the case when it locates on the origin node $s$ at time 0. Subsequently, we show how this Green function can be used as a navigation pointer for the shortest-path search.

To make this clearer, let us consider the key procedures for shortest path searching. Suppose we are to search out the shortest $s-f$ path $P = (s, (s, v_1), v_1, \ldots, (v_{k-1}, f), f)$, the key step is to find out which node $v_i$ should be selected from node $v_{i-1}$’s neighbor set $N_v(v_{i-1})$ as the next member of the path sequence $P$. We propose the following lemma to settle this matter.

Lemma. Suppose $s_v, s_w \in N_v(s)$, for sufficiently short-time interval $\Delta t$, $|G(f, \Delta t; s_v, 0)| < |G(f, \Delta t; s_w, 0)|$ holds true if and only if $d(s, v) > d(s, f)$.

Proof. We start from the general case that

$$G(f, \Delta t; s, 0) = \langle 0|\hat{c}_f(\Delta t)\hat{c}_s^\dagger(0)|0 \rangle = \langle 0|e^{iH\Delta t/\hbar}\hat{c}_f(0)e^{-iH\Delta t/\hbar}\hat{c}_s^\dagger(0)|0 \rangle = \langle 0|\hat{c}_f e^{-iH\Delta t/\hbar}\hat{c}_s^\dagger|0 \rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{-i\Delta t}{\hbar}\right)^n \langle 0|\hat{c}_f H^n\hat{c}_s^\dagger|0 \rangle,$$

where $|0 \rangle$ represents the vacuum state of the system. Consider the form of the Hamiltonian $H$; it is not difficult to see that $|\langle 0|\hat{c}_f H^n\hat{c}_s^\dagger|0 \rangle| = 1$ if the index $n \geq d(s, f)$, else 0. To compare $|G(f, \Delta t; s_v, 0)|$ with $|G(f, \Delta t; s_w, 0)|$, one only needs to check the first non-zero item of the expansion series of them. Since $d(s_v, j) > d(s_w, j)$, for sufficiently small $\Delta t$, we have

$$\left| \frac{1}{d(s, f)!} \left(\frac{-i\Delta t}{\hbar}\right)^{d(s, f)} \right| < \left| \frac{1}{d(s_w, f)!} \left(\frac{-i\Delta t}{\hbar}\right)^{d(s_w, f)} \right|. \quad (4)$$

As the following items are high-order infinitesimals, this partial ordering relation cannot be changed. Thus, $|G(f, \Delta t; s_v, 0)| < |G(f, \Delta t; s_w, 0)|$ holds true and vice versa. \hfill $\square$

To illustrate this, we carry out some numerical experiments on two kinds of graph models. To obtain exact numerical results for the Green function, we introduce the following method to calculate the Green function:

$$iG(f, \Delta t; s, 0) = \sum_n \langle 0|\hat{c}_f|\Phi_n \rangle \langle \Phi_n|\hat{c}_s^\dagger|0 \rangle e^{-i(E_n-E_n)\Delta t/\hbar}$$

$$+ \sum_n \langle 0|\hat{c}_s^\dagger|\Phi_n \rangle \langle \Phi_n|\hat{c}_f|0 \rangle e^{-i(E_n-E_n)\Delta t/\hbar}. \quad (5)$$
Here \( \Delta t \) is the time interval and the initial time is set to 0. \( |0\rangle \) represents the vacuum state of the system as we discussed before. Customarily, in the following discussion we take the unit \( \hbar \) as 1 for convenience. Note that the completeness condition is used here:

\[
\sum_n |\Phi_n\rangle \langle \Phi_n| = 1,
\]

(6)

where \( |\Phi_n\rangle \) is the eigenstate of the system Hamiltonian \( H \). Since our model is a free bosonic system, the following conditions are tenable:

\[
\hat{c}_s|0\rangle = 0, \quad \hat{c}_s^\dagger |s\rangle = |s\rangle.
\]

(7)

Therefore, equation (5) can be reduced to

\[
iG(f, \Delta t; s, 0) = \sum_n \langle 0|\hat{c}_f|\Phi_n\rangle \langle \Phi_n|\hat{c}_s^\dagger |0\rangle e^{-i(E_n - E_0)\Delta t}
\]

\[
= \sum_n \langle f|\Phi_n\rangle \langle \Phi_n|s\rangle e^{-i(E_n - E_0)\Delta t}.
\]

(8)

Next, by expanding the wave function of each eigenstate, i.e. \( |\Phi_n\rangle = \sum_k b_k^{(n)} |k\rangle \), where \( b_k^{(n)} \) is the corresponding expansion coefficients, we have

\[
iG(f, \Delta t; s, 0) = \sum_n \sum_k \sum_{k'} b_k^{(n)} b_{k'}^{(n)} \langle f|k\rangle \langle k'|s\rangle e^{-i(E_n - E_0)\Delta t}
\]

\[
= \sum_n b_f^{(n)} b_s^{(n)} \{ \cos(E_n - E_0)\Delta t - i \sin(E_n - E_0)\Delta t \},
\]

(9)

and noting the previous fact that \( E_0 = 0 \), we finally express the Green function as

\[
iG(f, \Delta t; s, 0) = \sum_n b_f^{(n)} b_s^{(n)} \{ \cos(E_n \Delta t) - i \sin(E_n \Delta t) \}.
\]

(10)

Consider the corresponding Hamiltonian (1) on both Erdős–Rényi (ER) random networks and random SF networks [11]; we illustrate \( G(f, \Delta t; s, 0) \) for several node pairs \((s,f)\) with different \(d(s,f)\), as shown in figures 1 and 2, respectively. The Green function \( G(f, \Delta t; s, 0) \) vibrates as the time interval \( \Delta t \) varies. Apparently, one can see that in a certain short time interval \( \Delta t \) from the beginning time, the Green function corresponding to smaller \( d(s,f) \) (blue lines) has strictly larger values in both figures.

Concretely, the searching process is handled by the following steps:

- **Step 1:** choose an *origin* node \( s \) and a *terminal* node \( f \) in a network.
- **Step 2:** for the chosen \( s \) and \( f \), calculate all the transition amplitudes \( P(f, N_l) = |G(f, \Delta t; N_l, 0)| \) between \( f \) and any neighbor nodes \( N_l \in N_e(s) \) \((l = 1, 2, \ldots, K_s)\) of \( s \); here \( K_s \) is the degree of \( s \).
- **Step 3:** choosing the maximum item of \( P(f, N_l) \), denote as \( P(f, N_m) \), record \( N_m \) and then set \( s \) to \( N_m \).
- **Step 4:** go to step 2 if none of the neighbors of \( s \) is \( f \), else output the sequence of \( N_m \).

Given an appropriate time variable \( \Delta t \), we make an obvious observation that the matrix \( G(\Delta t) = (G(f, \Delta t; s, 0))_{N \times N}, \forall s, f \in V \) stores the necessary information needed for the shortest-path searching. Note that the computation of the whole propagator diagram \( G(\Delta t) \) involves the complete matrix \( H \) and thus the global structure.
Figure 1. $G(f, \Delta t; s, 0)$ versus an index $n$ with $\Delta t = 0.002n$ in an ER network with 300 nodes. The average degree $\langle k \rangle = 18$ and the network is connected. The red line is for $G(f, \Delta t; s_1, 0)$ and the blue line for $G(f, \Delta t; s_2, 0)$. $s_1$ and $s_2$ are neighbors of a randomly selected node $f$ with $d(s_1, f) = 3$ and $d(s_2, f) = 2$, respectively.

Figure 2. $G(f, \Delta t; s, 0)$ versus an index $n$ with $\Delta t = 0.002n$ in a random SF network of 300 nodes generated by the Barabási–Albert (BA) model. The initial network size $m_0 = 10$ and the power exponent $\gamma = 3$. The red line is for $G(f, \Delta t; s_1, 0)$ and the blue line for $G(f, \Delta t; s_2, 0)$. $s_1$ and $s_2$ are neighbors of a randomly selected node $f$ with $d(s_1, f) = 3$ and $d(s_2, f) = 2$, respectively.

3. Reduction and heuristic estimate

The method derived in equation (10) has quite high computational complexity for it needs the whole energy spectrum of Hamiltonian (1) together with all the expansion coefficients of each energy eigenstate. To obtain a more efficient algorithm for $G(\Delta t)$, and consequently for APSP, all that remains is to obtain the appropriate values for each $G(f, \Delta t; s, 0), \forall s, f \in V$. For the sake of simplicity, in the following, we will shortly write $G_{sf}(\Delta t)$ for $G(f, \Delta t; s, 0)$. 

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Theorem. All the transition amplitude $G(\Delta t) = (G_{sf}(\Delta t))_{N \times N}$ can be obtained by computing the product of two $N \times N$ matrices in $O(\mu(N))$ time, where $\mu(N)$ is the time required to multiply two $N \times N$ matrices.

Proof. By using the method of the motion equation to solve the Green function, we express the term $G_{sf}(\Delta t)$ in the differential equation, that is,

$$i \frac{d}{dt} G_{sf}(\Delta t) = \delta(\Delta t) \langle [\hat{c}^+_s, \hat{c}_f] \rangle + \langle [\hat{c}^+_s(\Delta t), H], \hat{c}^+_f \rangle \rangle.$$

Here $\delta(\Delta t)$ is the characteristic function that equals 0 if $\Delta t \neq 0$, else equals 1. For convenience we set the initial condition as $G_{sf}(0) = -i\delta_{sf}$ and denote $\epsilon_i$ as $H_{ii}$ in the derivation. Given a small time interval $\Delta t$, we calculated $G_{sf}(\Delta t)$ analytically in the following way:

$$i \frac{d}{dt} G_{sf}(\Delta t) = \langle [\hat{c}^+_s(\Delta t), H], \hat{c}^+_f \rangle \rangle = \langle [e^{iH_{sf}} \hat{c}^+_s e^{-iH_{sf}}], \hat{c}^+_f \rangle \rangle$$

$$= \langle [e^{iH_{sf}} \sum_{lk} H_{lk} \hat{c}^+_l \hat{c}^+_k e^{-iH_{sf}}], \hat{c}^+_f \rangle \rangle$$

$$= \langle [e^{iH_{sf}} \sum_{lk} H_{lk} \hat{c}^+_l \delta_{sk} e^{-iH_{sf}}], \hat{c}^+_f \rangle \rangle$$

$$= \sum_l H_{ls} \langle \langle \hat{c}^+_l(\Delta t), \hat{c}^+_f \rangle \rangle$$

$$= \sum_l H_{ls} G_{lf}(\Delta t).$$

The above differential equation can be transformed into a Gauss integral. That is,

$$i \int_0^{\Delta t} \frac{d}{dt} G_{sf}(t) \ dt = \int_0^{\Delta t} \left[ \sum_{l \neq s} H_{ls} G_{lf}(\Delta t) + \epsilon_s G_{sf}(\Delta t) \right] \ dt.$$  

\[
\square
\]

For a sufficiently small integral interval $\Delta t$, we obtain the approximate form

$$i[G_{sf}(\Delta t) - G_{sf}(0)] = \left[ \sum_{l \neq s} H_{ls} G_{ls}(0) + \epsilon_s G_{sf}(0) \right] \Delta t, \tag{11}$$
the corresponding matrix form is given by
\[ G(\Delta t) = (-i \Delta t \times H + I) \times G(0), \]
where \( G(\Delta t) = (G_{sf}(\Delta t))_{N \times N}, G(0) = (G_{sf}(0))_{N \times N} \) and \( I \) is the identity matrix. Once we set an initial condition for \( G(0) \), for a sufficiently short time interval \( \Delta t \), we need to repeat the operation in equation (12) \( L_{\text{average}} \) times by replacing \( G(0) \) with \( G(\Delta t) \), which is obtained at the previous step in order to get the whole diagram of \( G(\Delta t) \). Generally, \( L_{\text{average}} \) is expected to be the average length of the shortest paths in a network. Then for all node pairs \((s, f)\), execute steps 2, 3 and 4, all-pairs shortest paths can be obtained. Searching the matrix and updating the sequence of \( N_m \) takes \( O(\mu(N)) \) time, where \( \mu(N) \) is the time required for multiplying two \( N \times N \) matrices. Here, we note that recent studies have shown that many random SF networks are ultrasmall worlds and sparse [18, 19] and \( L_{\text{average}} \) in these networks of size \( N \) scales as \( \ln \ln N \). In this situation, the total operands of our algorithm take the order of \( O(\mu(N)\ln \ln N) \).

When we use breadth-first search (BFS), the running time is \( O(N^3) \).

To check the accuracy of our algorithm, extensive numerical simulations are performed on several realistic networks. Concretely, we simulate the bosonic shortest-path searching for all possible source–target pairs on various networks and calculate the accuracy ratio \( p_s \), which is defined as the ratio of the length of paths selected by the quantum bosonic method to the actual shortest-path length. We note that an important independent parameter in the algorithm is the time interval \( \Delta t \). Setting \( \Delta t \) with sufficiently small values helps us to keep the algorithm accurate in various kinds of real networks.

Figure 3 shows the impact of the time interval \( \Delta t \) on the accuracy ratio \( p_s \) of QBSPS for four kinds of realistic networks. It is observed that all the actual shortest paths can be exactly found by our algorithm. The accuracy is very good for values of \( \Delta t \) close to 0 and deteriorates as \( \Delta t \) grows larger. This deterioration phenomenon is an indirect consequence of the vibration of the time-dependent Green function in a long time interval. In a sufficiently short time interval, the Green function or the so-called navigator is able to choose the closest neighbor to the target node exactly by identifying the largest corresponding values due to the local monotonicity. As time elapses, the vibration breaks down the monotonicity and the navigator becomes imprecise in finding the appropriate nodes, which reduces the accuracy. In fact, for \( \Delta t \) approaching 0.05, there is a distinct drop in accuracy for most kinds of networks. These observations mean that for a certain network, there exists a critical value of the parameter \( \Delta t \) below which the algorithm remains quite accurate, but above which its accuracy deteriorates.

Another important parameter for the algorithm is the success ratio \( p_s \), defined as the percentage of successful paths. It can also be considered as the fraction of particles that reach the destination. Generally, as in the cases we discussed above, the accuracy of QBSPS can be 100%, which indicates that \( p_s \) should also be 100% naturally. However, is there any possibility that the Green function leads to incorrect directions or trapped explorations and that the success ratio is not one? This situation happens if there exists localization of bosons in the network. According to Anderson localization theory, the Green function of localized bosons could decay exponentially as time elapses. Even if we set \( \Delta t \) with a sufficiently small value, the accuracy of the numerical value might decrease, restricting the computer's numerical accuracy. Thus, if the Green function navigator cannot distinguish the tiny differences among pairs of selected nodes due to the limited numerical accuracy, it would be confused in deciding which node should be the next hop and the success ratio could be decreased. In fact, the localization phenomenon could be induced by strong disorder in quantum complex networks.
Figure 3. Accuracy ratio $p_a$ as a function of $\Delta t$ for several real complex networks. The horizontal axis is an index $n$ with $\Delta t = 0.001n$. In all cases, numerical experiments confirm that the accuracy ratio of our algorithm can reach 1 with a sufficiently small value of $\Delta t$. US Airline (a) is the network of the public air transportation system with 332 nodes and 2126 edges [20]; Metabolic (b) is the metabolic network of *C. elegans* with 453 nodes and 4596 edges [21]; Java (c) is the network of Java (compile-time) dependence with 1538 nodes and 8032 edges [20]; and Yeast (d) is a protein–protein interaction network in budding yeast with 2361 nodes and 7182 edges [22]. All the networks are undirected.

show that strong clustering of links in an SF network with high power law exponent can induce a localization–delocalization quantum phase transition. However, such kinds of networks rarely exist in the real world and should be avoided in designing quantum wave packet transportation networks for particles might be trapped.

A fresh view of the application of our method emerges on considering quantum optical communication networks. Such an optical network may be considered in the form of a graph where edges represent optical fibers (or waveguides) and nodes represent optical units (essentially beam splitters) [23] which can redistribute incoming waves into outgoing fibers. The physical entity we choose here is photons or light wave packets. It is known that photons do not interact with one another and can be regarded as free bosons. We suggest that our shortest-path searching method should be realized on measuring the intensity of light waves on relevant nodes. At step 2 of the algorithm, the mathematical calculation for the transition amplitude can be substituted by measurement, which is realizable by setting measurement devices on nodes. Thus, without a global view of the network structure, by spreading photons or light wave packets in this quantum optical communication network, one can easily find the shortest paths between nodes by measuring the intensity of photons. In this situation, QBSPS is a pure local greedy

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searching strategy with considerable efficiency. We argue that the quantum optical network will bring us bright prospects for future communication networks and QBSPS may act as a nice path searching strategy.

4. Discussion and conclusion

We propose a special physical algorithm QBPS for the shortest-path problem in networks. Our motivation is based on the following observation: it is the structure of a network (represented by a graph) that determines the flow of particles in it. Therefore, our knowledge of particle dynamics should be conversely useful for uncovering the topology of the network. In fact, we show that the Green function of free bosons in the quantum model introduced in this paper may be used as an indicator for seeking out the shortest path from the origin node to the terminal node. This method not only provides a novel computational algorithm for the shortest-path problem from the quantum mechanism point of view but also a shortcut to solve the problem in special quantum optical networks. The physical mechanism of this algorithm is discussed in detail both for correctness and validity.

Practically, we apply this algorithm to undirected, unweighted networks. In particular for random SF networks, which is a hot topic for scientists to describe real-world networks, our algorithm is supposed to run in $O(\mu(N) \ln \ln N)$ time. We expect that this method will indeed be applicable in quantum optical communication networks and enable experimental physicists to make precise measurement.

In future works, the deeper relationship between the complex network substrate and the quantum bosonic process may be explored. We expect that this network dynamic-based mechanism may be extended to other network problems, such as network flow, network design, evaluation of the inner connection and so on.

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