Forward-Looking Dynamic Persuasion for Pipeline Stochastic Bayesian Game: A Fixed-Point Alignment Principle

Tao Zhang, Quanyan Zhu

Abstract—This paper studies a general-sum two-player pipeline stochastic game where each period is composed of two stages. The agents have uncertainty about the transition of the state which is characterized by a variable known as shock. The first stage is a cognitive decision-making in which each agent selects one of multiple information sources (ISs) and then acquires information about the shock from his selected IS. In the second stage, each agent takes an action based on the information provided by his selected IS. There is a rational Bayesian persuader who controls one IS and aims to influence the agents’ actions to induce her desired dynamic equilibria by solely crafting the information structure of her IS. We refer to such persuasion as forward-looking dynamic persuasion. We restrict attention to a direct persuasion in which the persuader incentivizes the agents to select her IS at the first stage and consider an equilibrium concept known as pipelined perfect Markov Bayesian equilibrium (PPME). We propose a novel design principle termed fixed-point alignment that captures the observation that the agents’ strategic interactions in the second stage induce an indirect cognitive competition (i.e., IS selection) in the first stage and formulate the direct Bayesian persuasion in PPME as a constrained non-linear optimization problem. By decomposing the optimization problem into a two-stage local process, we then provide a class of verifiable necessary and sufficient condition referred to as local admissibility of the direct Bayesian persuasion in PPME. A case study is provided in which we obtain a myopic version of local admissibility, by solving a series of static problems which is homogeneous with respect to the state variable.

I. INTRODUCTION

Spurred on by technological advancements, enhanced computational intelligence and cost-efficient deployment of sensing systems enable individual participants acting actively in societal-scale cyber-physical systems to be exposed to tons of information and to efficiently process such information to make real-time decisions. However, the information-explosion environment also breeds the generations of uncertainties due to increasingly diversified and complicated information asymmetries.

Tackling uncertainties is a key cognitive components of rationality in decision makings under information-asymmetric environments. In many scenarios, agents acquire additional information from information sources (e.g., experts, service providers) to reduce the negative impact of uncertainties. Consider a transportation network with multiple heterogeneous traffic information providers (TIP) (see, e.g., [1]). Smart vehicles (the agents) subscribe to TIPs to receive information sets about global traffic conditions (i.e., to reduce information asymmetry) and available routes. Within the information sets from the subscribed TIPs, vehicles make route choices.

As rational agents, each smart vehicle chooses TIP subscription by integrating the information precision (e.g., from customer reviewers) and the price of subscription into his expected daily needs of travels (e.g., can be characterized by payoff functions). Supply-chain service providers (SCPs) may prefer more precise yet more expensive TIPs to minimize their expected operation cost due to traffic congestion while individual travelers may be more tolerant of traffic congestion and prefer less precise but cheaper TIPs. On the one hand, the choices of TIPs govern the routing decisions of the smart vehicles which directly influence the traffic conditions of the network. On the other hand, the traffic conditions directly influence the travel cost of every vehicles in the network and also lead to information update of the vehicles by their subscribed TIPs. A self-interested rational SCP could even predict such interactions and strategically chooses TIPs and makes routing choices to manipulate the network as well as his competitors.

Indeed, as rationality promoting, agents’ information acquisition once treated as merely a action of getting informed must now be regarded as an essential component of agents’ rationality. The importance of such expanded rationality is twofold. First, being inundated with useless and irrelevant, noisy and misleading, and even adversarial information, agents should have the ability to identify and collect information that benefits his decision-makings. Second, due to the active interactions with other agents and the environment, the choices of ISs not only influence an agent’s own actions but also others’ decision-makings, as well as the operations of the environments (through the actions) which in turn influence the agent’s own utilities.

A Bayesian agent usually handles uncertainties by relaxing on priors and forming posterior beliefs (see, e.g., [2]) over the unobserved parts of the game or forms belief hierarchies (beliefs about the state as well as others’ beliefs; see, e.g., [3], [4]) in competitive multiagent environments. The priors and the beliefs constitute essential components of the game to shape the agents’ rational decisions of action choices since they characterize the uncertainty of the game for the agents. Bayesian persuasion (e.g., [2], [5], [6], [7], [4]) studies how a principal uses her informational advantage to strategically reveal noisy information about the decision-relevant state to the agents, thereby inducing and manipulating agents’ beliefs, to elicit the agents to behave in her favor.

In this work, we study how a principle can shape the
agents’ decision-makings in a dynamic environment by controlling one information source (IS) while the agents have the freedom to select any IS from a finite set of available options. We assume that the principal is the only strategic information designer while other ISs operate in a non-strategic take-it-or-leave-it manner.

We model each agent’s periodic decision-making of first selecting an IS and then taking an action by a novel general-sum incomplete-information stochastic game defined as pipeline stochastic Bayesian game (PSBG). We restrict attention to a class of dynamic Bayesian persuasion introduced in [4], known as forward-looking dynamic persuasion (FLDP), to PSBG, in which the agents acquire information about the transition law of the (global) state of the game. In particular, the transition law is parameterized by a variable known as shock. We assume that the agents’ uncertainty about the transition law is from their unobservability of the shock in each period.

Each period of PSBG is composed of a two-stage pipeline decision-makings. The first stage is a cognitive-posture formation process where each agent decides how to get informed about the game by selecting one IS from a finite options and then acquiring information about the state. In the second stage, agents play the primitive game by taking an action based on the information from their selected ISs. The state then evolves in a Markovian fashion based on current state, actions, and shocks. Built upon Markov perfect equilibrium and perfect Bayesian equilibrium, we consider a refinement of Nash Equilibrium defined as pipelined perfect Markov Bayesian equilibrium (PPME) as the central concept of stability in PSBG for the consistency of each agent’s individual pipeline decisions as well as interactions with others.

The principal aims to design a FLDP mechanism such that the PSBG admits an obedient PPME in which (i) the agents are incentivized to be obedient by selecting the principal’s IS at the first stage of the pipeline decision in every period and (ii) the agents’ behaviors (i.e., actions) coincide with the principal’s goal of design. Such pipeline setting in dynamic environments distinguishes our work from existing Bayesian persuasion in static environments (e.g., [2], [3], [5], [6]) as well as in dynamic models (e.g., [7], [4], [8]).

We propose a design principle referred to as the fixed-point alignment (FPA) which formulates the dynamic Bayesian persuasion as a non-linear optimization problem by identifying two fixed points in obedient PPME. The FPA selects a persuasion strategy that matches the fixed point from the optimality of the agents’ obedience at the first stage of the pipeline to the fixed point from the optimal action choices at the second stage. We then decompose the optimization problem of FPA into local fixed-point alignment processes and obtain a set of verifiable necessary and sufficient conditions known as local admissibility for implementable dynamic Bayesian persuasion in direct PPME. We show that if there is an algorithm that converges to the local admissibility, then the obtained persuasion strategy induces a PSBG which admits an obedient PPME in which the principal’s goal is achieved.

The remainder of the letter is organized as follows. In Section II, we describe the PSBG model and define the equilibrium concept PPME. Section III formulates the dynamic Bayesian persuasion in PSMG. Then, we propose the fixed-point alignment and characterize the necessary and sufficient conditions in Section IV. A case study is provided in Section V to establish a myopic implementability of FLDP, which considers a special class of PSBG with separable reward and state independent transition. Section VI provides the conclusion remarks.

II. PIPELINE STOCHASTIC BAYESIAN GAME

In this work, we restrict attention to a two-agent game. Our results can be readily extended to games of more than two agents. Formally, a (two-stage) pipeline stochastic Bayesian game (PSBG) is an infinite-horizon game defined by a tuple $PSBG < G, S, \Omega, \{\Theta_i, A_i\}_{i \in \{1,2\}}, T, d_e, d_\omega, \{\tau_i^*\}_{i \in \{1,2\}, \{\delta_i\}_{i \in \{1,2\}}, \delta >}$, where

- $G$ is a finite set of states.
- $S \equiv [m]$ with $1 \leq m < \infty$ is a finite set of information sources (ISs).
- $\Omega$ is a finite set of shocks.
- $\Theta_i$ is a finite set of types for agent $i$.
- $A_i$ is a finite set of actions available to each agent $i$ at each period with $A = \{A_i\}_{i \in \{1,2\}}$.
- $T : g \times A \times \Omega \mapsto \Delta(G)$ is the Markovian transition function of the state, such that $T(\cdot | g_i, A_i, \omega_i) \in \Delta(G)$ specifies the probability distribution of the next-period state when the current state is $g_i$, the current joint action of the agents is $a_i = (a_{1,i}, a_{2,i})$, and the shock is $\omega_i$.
- $d_e \in \Delta(G)$ is the initial distribution of the state.
- $d_\omega \in \Delta(\Omega)$ is the distribution of the shock.
- $\tau_i : G \times \Omega \mapsto \Delta(\Theta_i)$ is IS $i$’s stationary type generation rule (generation rule) for agent $i$. Let $\mathcal{T} = \{\tau_i\}_{i \in \{1,2\}}$ denote the collection of all available generation rules.
- $R_i : G \times A \mapsto \mathbb{R}$ is the reward function of agent $i$, for $i \in \{1,2\}$.
- $\delta \in (0,1)$ is the common discount factor.

The game model PSBG is publicly known at the ante state (i.e., before the game starts). A fictitious player Nature draws a shock $\omega_i \in \Omega$ according to $d_\omega$ and draws a state $g_i$ according to $T(\cdot | g_{i-1}, a_{i-1}, \omega_{i-1})$ (or $d_e(\cdot)$ at the initial period) when the event of the previous period is $(g_{i-1}, a_{i-1}, \omega_{i-1})$. At the beginning of each period, the agents observe the state $g_i$ but not the shock $\omega_i$. With reference to Fig. 1(a), the game play is described as follows. At the first stage of period $t$, each agent $i$ selects an IS $\ell$ from $S$, and then acquires information about the shock from that IS. Let $N_i[\ell] \subseteq \{1,2\}$ denote the set of agents who selects IS $\ell$ in period $t$ with $\prod_{i \in S} N_i[\ell] = \{1,2\}$ for all $t \geq 0$. Upon being selected, IS $\ell$ privately sends a noisy signal about the shock to agent $i$ which determines a type of the agent (coherent hierarchy of beliefs or Harsanyi’s type; see, e.g., [9]), denoted by $\theta^\ell_{i,\ell} \in \Theta_i$, for each $i \in N_i[\ell]$. For simplicity, we omit the the formulations of types due to private signals (see, e.g., [3], [4] for explicit type formation from private signals) and instead
consider that ISs directly generate types for the agents. Note that the shock space $\Omega$ and each type space $\Theta_i$ need not be of the same size for both $i \in \{1, 2\}$. At each period $t$, each IS $\ell$ uses a stationary type generation rule (generation rule) $\tau_\ell^i(\cdot|g_t, \omega_t) \in \Delta(\Theta_i)$ to randomly draw a type for agent $i$ given the shock $g_t$ if agent $i$ selects IS $\ell$.

The solution to the game PSBG is a stationary strategy profile $<\beta, \pi>_\ell$ where each $\beta_i : G \rightarrow S$ and $\pi_i : G \times \Theta \rightarrow \Delta(A_i)$ are each agent $i$'s IS selection policy (selection policy) and action policy, respectively. We refer to each agent $i$'s decision-making of selecting an IS and then taking an action in each period as a pipeline decision (pipeline). We assume that agents' choices of ISs (but not types), denoted by $S^\ell_i$, at the current period are publicly released after each agent receives his signal. Thus, agents do not have to form beliefs over other's selection of ISs at the beginning of the second stage of the pipeline. However, each agent does not observe the realized types of each other and has to form posterior beliefs over other's type. Based on his type $\theta_i$ and the state $g_t$, each agent $i$ forms a belief $\mu_i(\cdot|g_t, \theta_i) \in \Delta(\Omega \times \Theta_{-i})$ over the shock $\omega$ and the other agent's type $\theta_{-i}$, according to the Bayes' rule. Let $\tau_\ell^i$ and $\tau^n_i$ denote the generation rules of the ISs $\ell$ and $n$ which are selected by agent $i$ and agent $-i$, respectively. Then, agent $i$ forms posterior belief $\mu_i(\cdot|\theta_i, \theta_{-i}|g_t) = \tau^n_i(\cdot|g_t, \theta_i)\tau_\ell^i(\cdot|g_t, \omega_t)\mu(\cdot|g_t, \omega_t)d\omega(\cdot)\mu(\cdot|g_t, \omega_t)d\omega(\cdot)^{-1}$.

According to the Ionescu Tulcea theorem [10], $(d\omega, d\theta, T, \tau)$ and the agents’ pipeline strategy profile $<\beta, \pi>$ uniquely define a probability measure, $P_\beta,\pi$, over $(G \times \Omega^2 \times \Theta \times \Theta^2)$. Let $E^\pi_{\beta,\pi}[\cdot|g_t]$ denote the expectation operator with respect to $P_\beta,\pi$. Additionally, given the state $g_t$ and state-type pair $(g_t, \theta_{-i})$, we obtain unique probability measures (as perceived by each agent $i$) $P^s_{\beta,\pi}[g_t|\theta_{-i}]$ and $P^t_{\beta,\pi}[g_t, \theta_{-i}]$ over $G \times \Omega \rightarrow \Theta \times (G \times \Omega \times \Theta)^{\infty}$ and $G \times \Omega \times \Theta \rightarrow \Theta \times (G \times \Omega \times \Theta)^{\infty}$, respectively. In particular, $P^s_{\beta,\pi}[g_t, \theta_{-i}]$ captures the uncertainty at period $t$ for each agent at the beginning of the first stage while $P^t_{\beta,\pi}[g_t|\theta_{-i}]$ captures the uncertainty for each agent $i$ at the beginning of the second stage. Let $E^s_{\beta,\pi}[\cdot|g_t]$ and $E^t_{\beta,\pi}[\cdot|g_t, \theta_{-i}]$, respectively, denote the expectation operators with respect to $P^s_{\beta,\pi}[g_t]$ and $P^t_{\beta,\pi}[g_t, \theta_{-i}]$.

Given $P^s_{\beta,\pi}[g_t]$ agent $i$'s period-$t$ state value function is defined as

$$J^s_i(g_t, \pi, \beta) = E^s_{\beta,\pi}[\sum_{t=1}^{\infty} \delta^t R_i(g_t, \theta_i) | g_t, \beta].$$

With a slight abuse of notation, we write $\mu_i(\omega_t|\theta_{-i})$ and $\mu_i(\omega_t, \theta_{-i}|\theta_i)$ as the marginals of $\mu_i(\omega_t, \theta_{-i}|\theta_i)$ over $\Omega$ and $\Theta_{-i}$, respectively. Given $P^t_{\beta,\pi}[g_t, \theta_{-i}]$, agent $i$'s period-$t$ state-type (ST) value function is defined as

$$V^s_i(g_t, \theta_i, \theta_{-i}) \equiv E^s_{\beta,\pi}[\sum_{t=1}^{\infty} \delta^t R_i(g_t, \theta_i) | g_t, \theta_i].$$

Finally, agent $i$'s period-$t$ state-type-action (STA) value function is defined as

$$Q^s_i(g_t, \theta_i, \theta_{-i}, a_t, \pi, \beta) \equiv R_i(g_t, a_t) + E^s_{\beta,\pi}[\sum_{t=1}^{\infty} \delta^t R_i(g_t, a_t) | g_t, \theta_i].$$

**Definition 1** (Pipeline-Sequential Rationality). We say that an agent's strategy profile $<\beta, \pi>$ is pipelines-sequential rational to $<\beta_i, \pi_{-i}>$ if for any $h_t \in H, \theta_{-i} \in \Theta, \beta_i$ and $\pi_i$,

$$J^s_i(g_t|\beta_i, \beta_{-i}, \pi) \geq J^s_i(g_t|\beta_i', \beta_{-i}, \pi),$$

$$V^s_i(g_t, \theta_i|\beta_i, \pi_{-i}) \geq V^s_i(g_t, \theta_i|\beta_i', \pi_{-i}).$$

**Definition 2** (Pipeline Perfect Markov Bayesian Equilibrium). A strategy profile $<\beta, \pi>$ constitutes a stationary pipelined perfect Markov Bayesian equilibrium (PPME) if each $\beta^*_i, \pi^*_i$ is pipelined-sequential rationality to $<\beta_i^*, \pi_{-i}^*>$, for all $i \in \{1, 2\}$ simultaneously.

A PPME is a refinement of Nash equilibrium in dynamic environments that is built on the concepts of Markov perfect equilibrium (see, e.g., [11]) and perfect Bayesian equilibrium (see, e.g., [12]). Without loss of generality, the subgame perfectness of PPME enables us to restrict attention to one-shot deviation analysis of the game (see, e.g., [13], [14]).
III. FORWARD-LOOKING DYNAMIC PERSUasion

We consider that there is a rational Bayesian persuader (principal, she) who controls one IS in \( S \). Without loss of generality, we index the principal’s IS as \( k \). We assume that all other ISs are non-strategic in the sense that they commit to \( \tau^k = \{ \tau^k_1, \tau^k_2 \} \) in a take-it-or-leave-it manner. The principal has a goal, \( \rho(\tau^k_1|g, \omega) \in \Delta(A) \) for every \( g \in G \) and \( \omega \in \Omega \), which is described by a stationary probability distribution of the agents’ actions conditioning on the state, and she aims to induce the agents to take actions that coincides with her goal \( \rho \) in PPME by strategically generating types according to \( \tau^k \). The principal’s optimal goal is chosen by optimizing her objective, denoted by \( W(\rho) \). The principal may want to maximize her own expected discounted payoff; i.e., \( W(\rho) = \mathbb{E}^p[\sum_{t=0}^{\infty} \beta^t R_t(g, \omega)] \) where \( R^p : G \times A \rightarrow \mathbb{R} \) is the principal’s reward function and \( \mathbb{E}^p[\cdot] \) is the expectation operator corresponding to the probability measure at the ex-ante determined by \( \{ d_g, d_w, T, \tau \} \) and the principal’s goal \( \rho \). She may also want to maximize the social welfare of the agents; i.e., \( W(\rho) = \mathbb{E}^p[\sum_{i=1}^{2} \sum_{t=0}^{\infty} \delta^t R_t(g, \omega)] \). Let \( \text{PSBG}[\tau^k] \) denote the mechanism (model) of the game \( \text{PSBG} \) when the principal uses \( \tau^k \). Since we focus on stationary strategies, we suppress the time indices in the notations in the rest of the letter; unless otherwise stated. With reference to Fig. 1(b), the timing of \( \text{PSBG}[\tau^k] \) is as follows.

- At the ex-ante stage, the principal chooses \( \tau^k \) and publicly releases it, given other ISs’ \( \tau^k \).
- At each period \( t \), each agent \( i \) selects an IS from \( S \) according to \( \beta_{i} \) and requests a type \( \theta_i \) from that IS.
- By observing the current state \( g \), each IS \( k' \) privately sends a type \( \theta_{i} \) according to \( \tau^k_{1t} \) to each agent \( i \in N[k'] \).
- After receiving \( \theta_{i} \), each agent \( i \) takes an action \( a_{i,t} \) based on \( g \) and \( \theta_{i} \), according to \( \pi_{i} \).
- At the end of period \( t \), each agent \( i \) receives a reward \( R_i(g, a_{i,t}) \). The state \( g \) and the joint actions \( a \) are publicly disclosed. Then, the state \( g \) is transitioned to the next state \( g' \) according to \( T \).

**Definition 3** (Direct Information Design). We say that the principal’s information design is direct if each agent \( i \) chooses the principal’s IS \( k \) at every period. That is, each \( \beta_{i}(g) = k \), for all \( g \in G \). We call such selection policy as obedient selection policy.

When the principal performs direct information design, we refer to the game \( \text{PSBG}[\tau^k] \) as direct mechanism.

**Definition 4** (Attainable Goal). In a direct mechanism \( \text{PSBG}[\tau^k] \), the principal’s goal \( \rho \) is attainable by \( \pi \) if

\[
\rho(a|g, \omega) = \sum_{\theta \in \Theta} \pi(a|\theta, g) \tau^k(\theta|g, \omega).
\]

Since other ISs are non-strategic, we rewrite the expectation operators and value functions by only showing the principal’s generation rule \( \tau^k \), e.g., \( \mathbb{E}^{\pi}_{\theta, r}[\cdot|\cdot] \) and \( \mathcal{J}^{\tau^k} \).

**Definition 5** (Implementability). We say that a direct mechanism \( \text{PSBG}[\tau^k] \) is implementable in PPME with \( \rho \) if \( \text{PSBG}[\tau^k] \) induces a profile \( \beta^0, \pi^0 > \) that constitutes a PPME where \( \beta^0 \) is obedient and the principal’s goal \( \rho \) is attainable by \( \pi^0 \). We refer to such PPME as obedient PPME.

The notion of implementability requires that when the principal uses the generation rule profile \( \tau^k \), each agent's obedient goal decision in each period while believing with probability 1 that others make obedient pipeline decisions is a PPME. To achieve an implementable (direct) mechanism \( \text{PSBG}[\tau^k] \), the principal needs to (i) incentivize agents to choose her as their IS at every period and (ii) induce actions that coincide with her goal (i.e., induce action policy profile that attains the goal). We assume that, as is standard in the literature of mechanism design, when there are multiple equilibria action policy profiles, tie-breaking is in the principal’s favor or there is certain choice rule in place.

Let

\[
\mathcal{J}^{\tau^k}(g|\theta, \pi; V_t^{\tau^k}) = \mathbb{E}^{\beta_{i}}{\left[ V_{i}^{\tau^k}(h, \theta_{i}, \tilde{\theta}_{-i}|g^{0}, \pi) \right| g, \theta_{i}}\right].
\]

The obedience is incentivized by the obedience compatibility (OC) constraints which enforce the obedient pipeline decision as a PPME. That is, for all \( i \in \{1,2\}, \beta^i_1, \pi^i_1 \).

\[
\mathcal{J}^{\tau^k}(g|\theta_{i}, \pi; O_t^{\tau^k}, V_t^{\tau^k}) \geq \mathcal{J}^{\tau^k}(g|\theta_{i}, \beta^i_1, \pi^i_1), \quad (\text{OC1})
\]

\[
\mathcal{J}^{\tau^k}(g|\theta_{i}, \beta^i_1, \pi; V_t^{\tau^k}) \geq \mathcal{J}^{\tau^k}(g|\theta_{i}|\beta^i_1, \pi^i_1, \pi_{-i}; V_t^{\tau^k}). \quad (\text{OC2})
\]

We refer to \( \text{PSBG}[\tau^k] \) as obedience compatible (OC) mechanism if (OC1) and (OC2) are satisfied. Note that the OC does not imply implementability since the principal’s goal is not necessarily attainable.

Following standard dynamic programming argument (see, e.g., [15]), we represent (1), (2), and (3) recursively as follows:

\[
\mathcal{J}^{\tau^k}(g|\theta_{i}, \pi, V_t^{\tau^k}) = \sum_{\theta_{oo}} V_{i}^{\tau^k}(g, \theta_{i}|\pi) \tau^k(\theta_{i}|g, \omega)d_{\omega}(\omega), \quad (8)
\]

\[
\mathcal{V}^{\tau^k}(g, \theta_{i}|\pi) = \sum_{\omega} \pi(a|g, \theta_{i}, \omega) Q^{\tau^k}(g, \theta_{i}, a), \quad (9)
\]

\[
Q^{\tau^k}(g, \theta_{i}, a|V_t^{\tau^k}) = R_{i}(g, a) + \delta \sum_{g', \omega} J^{\tau^k}(g'|\pi; V_t^{\tau^k}) T(g'|g, a) \mu_{i}(\omega|\theta_{i}). \quad (10)
\]

Here, we denote \( J^{\tau^k}(\cdot; V_t^{\tau^k}) \) and \( Q^{\tau^k}(\cdot; V_t^{\tau^k}) \) with \( V_t^{\tau^k} \) to highlight their dependence on \( V_t^{\tau^k} \) from the Bellman recursions (8)-(10).

Consider the following constrained optimization problem \((\text{OPT}[\tau^k])\):

\[
\min_{\pi, V} Z^{\tau^k}(\pi, V) = \sum_{i=1}^{2} \sum_{\theta_{i}} \left( \begin{array}{c}
\mathcal{J}^{\tau^k}(g, \theta_{i}, a|V_t^{\tau^k}) \\
\mathcal{V}^{\tau^k}(g, \theta_{i}) - \sum_{a} Q^{\tau^k}(g, \theta_{i}, a|V_t^{\tau^k}) \pi(a|g, \theta_{i})
\end{array} \right) \quad (\text{OPT}[\tau^k])
\]

subject to, for all \( i \in \{1,2\}, a_{i} \in A_{i}, g \in G, \theta_{i} \in \Theta_{i} \) with \( \sum_{g} \tau^k(\theta_{i}|g) T(g|h) > 0 \),

\[
\pi_{i}(a_{i}|\theta_{i}) \geq 0, \quad (\text{RG})
\]
\[
\sum_{a_i \in A_i} \pi_i(a_i|\theta_i) = 1, \quad \text{(FE)}
\]
\[
EV^t_i(g, \theta_i|\pi, V_i) \geq \mathbb{E}^t_{\pi_{-i}} \left[ Q^t_i(g, \theta_i, a_i, \tilde{a}_{-i} | V_i) | g, \theta_i \right], \quad \text{(EQ)}
\]
\[
J^t_i(g | \pi, V_i) \geq \sum_{g, \theta_i} V_i(g, \theta_i | \pi) t^t_i(\theta_i, g| T^t_i(g| h) \right). \quad \text{(OB)}
\]

**Proposition 1.** A strategy profile \( \beta^0, \pi^0 \) constitutes an obedient PPME of PSBG\( [\tau^0] \) if and only if it is a global minimum of \( \langle \text{OPT}[\tau^0] \rangle \) with \( Z^t(\pi^0, V^0) = 0 \) where \( V^0 \) is the corresponding optimal ST value function.

**Proof.** See Appendix A of [16]. \( \square \)

Proposition 1 extends the fundamental formulation of finding a Nash equilibrium of a stochastic game as a nonlinear programming (Theorem 3.8.2 of [17]; see also, [18], [19]). Here, the constraints (RG) and (FE) ensure that each candidate \( \pi \) is a valid conditional probability distribution and rules out the possible trivial solution \( \pi_i = 0 \) for all \( i \in \{1, 2\} \). The conditions (EQ) and (OB) are two necessary conditions for an obedient PPME of the game PSBG\( [\tau^0] \) deriving from the optimality of PPME and the Bellman recursions (8) and (9). Here, the obedient selection policy profile \( \beta^0 \) is not a solution of the problem \( \langle \text{OPT}[\tau^0] \rangle \); instead, the optimality of being obedient is constrained by (OB) in terms of \( \pi \) and \( V \) given \( \tau^0 \).

**IV. DESIGN PRINCIPLE: FIXED-POINT ALIGNMENT**

In this section, we first propose a design principle referred to as fixed-point alignment (FPA) for the principal to design an implementable mechanism PSBG\( [\tau^0] \). Then, we provide a set of verifiable necessary and sufficient conditions for implementable mechanisms.

Suppose that the principal’s IS \( k \) is the only IS available to the agents. Then, the principal’s feasible information design should guarantee that her generation rule profile \( \tau^0 \) induces an equilibrium \( \pi, V > 0 \). Since now the agents are obedient by default, Proposition 1 implies that the point \( \pi, V > 0 \) needs to be a global minimum of \( \langle \text{OPT}[\tau^0] \rangle \) that satisfies (RG) (FE), and (EQ) with \( Z^t(\pi, V) = 0 \). In other words, \( V \) has to be a fixed point of the following equation, given \( \pi_{-i} \), for all \( i \in \{1, 2\} \), \( g \in G, \theta_i \in \Theta_i \):

\[
EV^t_i(g, \theta_i|\pi) = \max_{a_i \in A_i} \mathbb{E}^t_{\mu^t_{-i}} \left[ Q^t_i(g, \theta_i, a_i, \tilde{a}_{-i} | V_i) | g, \theta_i \right], \quad \text{(EQ1)}
\]

where the operator \( \mathbb{E}^t_{\mu_{-i}} [\cdot | g, \theta_i] \) takes expectation of the other agent’s action given his policy \( \pi_{-i} [\cdot | \theta_{-i}] \) in which the knowledge of \( \theta_{-i} \) is estimated by agent \( i \) through the belief \( \mu_i [\cdot | g, \theta_i] \).

When there are multiple ISs, the notion of obedient PPME requires that agents’ obedient IS selection is optimal at the first stage of the equilibrium pipeline decisions. Given any \( J_i \), define, for \( i \in \{1, 2\} \), \( \theta_i \in \Theta_i \), \( g \in G \):

\[
\lambda_i^t = \sum_{a_i \in A_i, \theta_i, g} R_i(g, a) + \delta \sum_g J_i(g') T(g'| a, \omega) \times \pi(a| g, \theta) t^t_i(\theta | g, \omega) d_\omega(\omega), \quad \text{(11)}
\]

The optimality of obedience (i.e., constraint (OB)) implies that the optimal history value function \( J_i \) for each agent \( i \) needs to be a fixed point (while fixing others’ obedient selections and action policies) for both \( i \in \{1, 2\} \), \( g \in G \),

\[
J_i(g) = \max_{\theta_i \in \Theta_i} J^t_i(g, \theta_i | J_i, \pi), \quad \text{(OB1)}
\]

which is independent of \( V \).

The underlying idea of FPA is to choose a generation rule profile \( \tau^0 \) such that \( V_i \) is a fixed point of (EQ1) if and only if \( J_i \) is a fixed point of (OB1), for \( i \in \{1, 2\} \), when the principal’s goal is attainable by the agents’ action policy profile. Given a goal \( \rho \), one possible objective function is

\[
Z_{\text{FPA}}^t(\tau^0, J, V; \rho) = \sum_{g, \theta_i} \left( J_i(g) - \sum \left( V_i(g, \theta_i) t^t_i(\theta_i | g, \omega) d_\omega(\omega) \right) \right), \quad \text{(FPA[\rho])}
\]

subject to (EQ1), (OB1), and \( \forall i \in \{1, 2\}, g \in G, \omega \in \Omega, \theta_i \in \Theta_i \),

\[
t^t_i(\theta_i | g, \omega) \geq 0, \quad \text{(RGt)}
\]

\[
\sum_{\theta_i} t^t_i(\theta_i | g, \omega) = 1, \quad \text{(FT)}
\]

\[
\rho(a| g, \omega) = \sum_{\theta_i} \pi(a| g, \theta_i) t^t_i(\theta_i | g, \omega). \quad \text{(AT)}
\]

**Proposition 2.** Fix a goal \( \rho \). Let \( \langle \hat{\tau}^t, \hat{\pi} \rangle \) satisfy (AT). Suppose that \( \langle \hat{\tau}^t, \hat{\pi} \rangle \) satisfy the Bellman recursions (8)-(10). Then, \( \langle \hat{\tau}^t, \hat{\pi} \rangle \) is a global minimum of (FPA[\rho]) with \( Z_{\text{FPA}}^t(\hat{\tau}^t, \hat{\pi}, \hat{\nu}; \rho) = 0 \) if and only if \( \langle \hat{\tau}^t, \hat{\pi} \rangle \) is a global minimum of \( \langle \text{OPT}[\tau^0] \rangle \) with \( Z^t(\hat{\tau}^t, \hat{\pi}, \hat{\nu}) = 0 \).

**Proof.** See Appendix B of [16]. \( \square \)

Proposition 2 shows that if (FPA[\rho]) has a global minimum \( \langle \hat{\tau}^t, \hat{\pi} \rangle \) with \( Z_{\text{FPA}}^t(\hat{\tau}^t, \hat{\pi}, \hat{\nu}; \rho) = 0 \) then the mechanism PSBG\( [\tau^0] \) is implementable in PPME and the principal’s goal \( \rho \) is attainable.

**A. Local FPA**

For each type space \( \Theta_i \) and any generation rule profile \( \tau^0 \), for \( i \in \{1, 2\} \), we decompose it into \( \Theta_i \equiv \Theta_i^t \cup \{ \theta_i \} \), such that the constraint (FT) can be decomposed into the following three: for \( i \in \{1, 2\} \), \( \theta_i \in \Theta_i \), \( g \in G \), \( \omega \in \Omega \), \( t^t_i(\theta_i | g, \omega) \geq 0 \),

\[
\sum_{\theta_i \in \Theta_i^t} t^t_i(\theta_i | g) \leq 1, \quad t^t_i(\theta_i | g) + \sum_{\theta_i \in \Theta_i^t} t^t_i(\theta_i | g) = 1. \quad \text{Let, for both} \ i \in \{1, 2\}
\]

\[
IV^t_i(g, \theta_i | V_i) = \sum_{\theta_i, \omega} V_i(g, \theta_i, \theta_i, g) t^t_i(\theta_i | g, \omega) d_\omega(\omega), \quad \text{(12)}
\]

which is independent of action policies. Let \( X^t_{\rho, \omega} \equiv (J_i, V_i, t^t_i(\cdot | g, \omega)) \). Define, for any \( g \in G \), \( \omega \in \Omega \), \( \theta_i \in \Theta_i \),

\[
\lambda_i(X^t_{\rho, \omega}; g, \theta_i) = J_i(g) - IV^t_i(g, \theta_i | V_i).
\]
Here, \( \lambda_i \) is a function of \((J_i, V_i, \pi_i, \varphi_i(\cdot | g, \omega))\) and is independent of agents’ policy profile \( \pi \). For any \( g \in G, \omega \in \Omega, \theta_i \in \Theta_i \), define the local fixed-point misalignment (misalignment) as follows:
\[
M_i(X_i^{g, \omega}, \tau_i^g, g, \omega) \equiv \sum_{\theta_i' \in \Theta_i^g} \lambda_i(X_i^{g, \omega}, g, \theta_i') \tau_i^g(\theta_i' | g, \omega) + \lambda_i(X_i^{g, \omega}, g, \tilde{\theta}_i) \tau_i^g(\tilde{\theta}_i | g, \omega).
\]
\[\text{(13)}\]
Given \( i \in \{1, 2\} \), \( g \in G \), and \( \omega \in \Omega \), we define the local (fixed-point) alignment process as the process to minimize the misalignment:
\[
\min_{X_i^{g, \omega}, \tau_i^g} M_i(X_i^{g, \omega}, \tau_i^g, g, \omega)
\]
subject to:
\[
\tau_i^g(\theta_i' | g, \omega) \geq 0, \forall \theta_i' \in \Theta_i^g, \tau_i^g(\tilde{\theta}_i | g, \omega) \geq 0, \lambda_i(X_i^{g, \omega}, g, \theta_i'') \geq 0, \lambda_i(X_i^{g, \omega}, g, \tilde{\theta}_i) \geq 0, \forall i, \tau_i^g(\cdot | g, \omega), \lambda_i(X_i^{g, \omega}, g, \cdot), \Theta_i.
\]
Let \( b \equiv \{ b[\theta_i'] \}_{\theta_i' \in \Theta_i^g} \), \( f \equiv \{ f[\theta_i'] \}_{\theta_i' \in \Theta_i}, \) respectively, denote the Lagrange multipliers with respect to the constraints \( \{ \tau_i^g(\theta_i' | g, \omega) \geq 0, \theta_i' \in \Theta_i^g, \} \), \( \{ \tau_i^g(\tilde{\theta}_i | g, \omega) \geq 0, \lambda_i(X_i^{g, \omega}, g, \theta_i'') \geq 0, \theta_i' \in \Theta_i \} \) of \( \text{LMM}_i(g, \omega) \); and the corresponding slack variables are denoted by \( q \equiv \{ q[\theta_i'] \}_{\theta_i' \in \Theta_i^g} \), \( w \equiv \{ w[\theta_i] \}_{\theta_i \in \Theta_i}, \) respectively. We construct the Lagrange function of \( \text{LMM}_i(g, \omega) \) as:
\[
L_i(X_i^{g, \omega}, \tau_i^g, b, c, f, q, w, v, z; g, \omega) \equiv M_i(X_i^{g, \omega}, \tau_i^g, g, \omega) + \sum_{\theta_i \in \Theta_i} c_i[\theta_i] | f[\theta_i] - \lambda_i(X_i^{g, \omega}, g, \theta_i) | + c(w - \tau_i^g(\tilde{\theta}_i | g, \omega)).
\]
\[\text{(14)}\]
By taking derivative of \( L_i \) with respect to \( X_i^{g, \omega} \) and \( \tau_i^g(\cdot | g, \omega) \), respectively, we obtain:
\[
\Delta_i(X_i^{g, \omega}, \tau_i^g, f, g, \omega) \equiv \nabla_{X_i^{g, \omega}} M_i(X_i^{g, \omega}, \tau_i^g, g, \omega) - \sum_{\theta_i \in \Theta_i} f[\theta_i] | \nabla_{X_i^{g, \omega}} \lambda_i(X_i^{g, \omega}, g, \theta_i) |, \text{and, for } \forall \theta_i \in \Theta_i,
\]
\[
d(X_i^{g, \omega}, \tau_i^g(\cdot | g, \omega), b[\theta_i], c, g, \omega) \equiv b[\theta_i] + c \frac{\partial}{\partial \tau_i^g(\theta_i' | g, \omega)} M_i(X_i^{g, \omega}, \tau_i^g(\theta_i' | g, \omega), g, \omega).
\]
For any \( \theta_i \in \Theta_i \), define
\[
\hat{\mathcal{Y}}_i^g(J_i, V_i, \pi_i; \theta_i, a_i) = \text{EV}_i^g(g, \theta_i, a_i, \tilde{\theta}_i | g, \omega).
\]
\[\text{(15)}\]
Definition 6 (Local Admissibility). We say a generation rule profile \( \tau^g \) is locally admissible if there exists a goal \( p \) and a path \( J \prec V \), such that, for every local alignment process \( \text{LMM}_i(g, \omega) \)
\[\text{(i)}\]
there exist \( b \) and \( f \), such that, for each \( i \in \{1, 2\} \),
\[
\Delta_i(X_i^{g, \omega}, \tau_i^g, g, \omega) = 0
\]
\[\text{(16)}\]
\[\text{(ii)}\]
and there exists \( \pi \) such that, for each \( i \in \{1, 2\} \), \( \theta_i \in \Theta_i \), with \( \tau_i^g(\theta_i' | g, \omega) > 0, a_i \in A_i, a \in A \),
\[
\rho(a_i, g, \omega) = \sum_{\theta_i} \pi(a_i, \theta_i) \rho(\theta_i | g, \omega)
\]
\[\text{(17)}\]
Here, \( \rho \) is the reward function, and \( \rho(\theta_i | g, \omega) \) is the reward function associated with \( \theta_i \) and \( g \).
Note that the conditions in \( (16) \) are in general not the Karush–Kuhn–Tucker (KKT) conditions for the local alignment process \( \text{LMM}_i(g, \omega) \) because the local admissibility does not require the feasibility of \( f \) and \( b \) as Lagrange multipliers as in KKT conditions. The conditions in \( (17) \) perform local fixed-point alignment. In particular, \( (17) \) requires the generation rule profile \( \tau^g \) to align the fixed-points \( J_i \) and \( V_i \) through the policy profile \( \pi \) (the second line of \( (17) \)) that attains the goal \( \rho \) (the first line of \( (17) \)). The third line on \( (17) \) constrains the feasibility of agents’ action policies.
Let \( \gamma_i^{g, \omega} \equiv \{ \nabla_{X_i^{g, \omega}} \lambda_i(X_i^{g, \omega}, g, \theta_i) : \theta_i \in \Theta_i \} \) denote a vector of gradients of \( \lambda_i \) with respect to \( X_i^{g, \omega} \).

Condition 1. \( \gamma_i^{g, \omega} \) is a set of linearly independent vectors for all \( i \in \{1, 2\} \), \( g \in G \), \( \omega \in \Omega \),

Condition 1 invokes a regularity condition similar to the linear independence constraint qualification (LICQ) that is usually used in KKT conditions. However, Condition 1 imposes linear independence only for the constraints \( \{ \lambda_i(X_i^{g, \omega}, g, \cdot) \geq 0, \forall \theta_i' \in \Theta_i \} \) of \( \text{LMM}_i(g, \omega) \).

Theorem 1. Fix a goal \( p \). Under Condition 1, a direct mechanism \( \text{PSBG}[\tau^g] \) is implementable in obedient PPME with \( \rho \) if and only if the generation rule profile \( \tau^g \) is locally admissible.

Proof. See Appendix C of [16].

Theorem 1 provides necessary and sufficient conditions for characterizing the implementability of the principal’s information design with a goal \( p \). In particular, if there is an algorithm converges to a local admissible \( \tau^g \) that attains \( p \), then, under Condition 1, the direct mechanism \( \text{PSBG}[\tau^g] \) admits an obedient PPME in which the principal’s goal \( p \) is attained. In other words, locally admissible \( \tau^g \) with the corresponding \( J \) and \( V \) achieves \( Z_{\text{PSBG}}(\tau^g, J, V, p) = 0 \), the global minimum of \( \text{FFP}[\rho] \). Since usually \[X_i^g \supseteq \Theta_i \], for all \( i \in \{1, 2\} \), it is straightforward to see that satisfying Condition 1 is much less restrictive than satisfying LICQ of all the constraints of \( \text{FFP}[\rho] \).

V. MYOPIC IMPLEMENTABILITY: A SER-SIT CASE STUDY

We say that the implementability (or local admissibility) of the forward-looking dynamic persuasion is myopic if it can be obtained by solving a series of static problems. In this section, we restrict attention to a class of PSBG built upon the separable reward state independent transition (SER-SIT) model ([20]; see also, e.g., [20], [17], [21]). In such game model, we obtain a design principle with a class of verifiable necessary and sufficient conditions which is homogeneous with respect to the state variable.

A. Myopic Pipeline Decision Model

We first transform the agents’ PSBG into a myopic pipeline decision model.

Definition 7 (SER-SIT PSBG). A PSBG is SET-SIT if it satisfies the following two properties: for \( i \in \{1, 2\} \), \( g \in G \), \( \omega \in \Omega \), \( a_i \in A_i, a \in A \),
\[
R_i(g, a_1, a_2) = c_i(g) + r_i(a_1, a_2).
\]
\[\text{(SER)}\]
\[
T_i(g, a_1, a_2, o) = T(g) | a_1, a_2, o \rangle.
\]
In SET-SIT PSBG, each agent’s reward function is decomposed into a state-dependent (but action-independent) component \( c_i : G \rightarrow \mathbb{R} \) and an action-dependent (but state-independent) component \( r_i : A \rightarrow \mathbb{R} \), and the transition probability \( T \) is independent of current state. In other words, a fixed action-shock tuple \( (a_1, a_2, o) \)
determines the same transition law of the states regardless of the current state.

Given any generation rule profile $\tau^i$ and any type pairs $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$, we define auxiliary functions as follows: for $i \in \{1, 2\}$, $a_i \in A_i$, $\theta_i \in \Theta_i$,

\[
U_i(a, a_2, \theta, \tau^i) = r_i(a_1, a_2) + \delta \sum_{a_1} c_i(g') T(g'|a_1, a_2, \omega) \mu_i(\omega|\theta_i). \tag{18}
\]

Here, the dependence of $U_i(\cdot; \tau^i)$ on $\tau^i$ is through the (marginal) belief $\mu_i(\omega|\theta_i)$. Additionally, $U_i(\cdot; \tau^i)$ is independent of the states.

Given any action profile profile $\pi$, we define the supplemental reward and transition functions, for $i \in \{1, 2\}$

\[
R_i(\theta_1, \theta_2, \omega, \pi) = \sum_{a_i, a_2} U_i(a, a_2, \theta, \tau^i) \sum_{a_1} c_i(g') T(g'|a_1, a_2, \omega). \tag{19}
\]

Similar to the auxiliary functions, we define the following supplemental functions, given any action profile profile $\pi$ and any shock $\omega \in \Omega$: for $i \in \{1, 2\}$

\[
F_i(\theta_1, \theta_2, \omega, \pi) = R_i(\theta_1, \theta_2, \omega, \pi) + \delta \sum_{a_1} c_i(g') T(g'|\theta_1, \theta_2, \omega). \tag{20}
\]

Given any $\tau^i$ and $\pi$, we construct auxiliary and supplemental matrices corresponding to $U_i$ and $F_i$, respectively, for all $i \in \{1, 2\}$, $\theta_i \in \Theta_i$, $\omega \in \Omega$,

\[
\mathcal{U}[\theta_i, \tau^i] = \left[ U_i(a, a_2, \theta, \tau^i) \right]_{a_i \in A_i, a_2 \in A_2}, \tag{21}
\]

\[
\mathcal{F}[\omega, \pi] = \left[ F_i(\theta_1, \theta_2, \omega, \pi) \right]_{\theta_1, \theta_2 \in \Theta_i}. \tag{22}
\]

Definition 8 (MDPM). A myopic pipeline decision model (MDPM) is defined by a tuple

\[
\text{MDPM}^{[i]}(\pi, \omega) = (\{\text{SM}[\pi, \omega]\}_{\omega \in \Omega}, \{\text{AG}^i(\pi, \omega)\}_{\omega \in \Omega}),
\]

where $\text{SM}[\pi, \omega]$ and $\text{AG}(\pi, \omega)$, respectively, are the first and the section stages of MDPD at $\omega$ and $(\theta_1, \theta_2)$.

\[
\text{SM}[\pi, \omega] \equiv (\Theta, \mathcal{F}[\pi, \omega], \mathcal{F}[\omega, \pi]). \tag{23}
\]

\[
\text{AG}(\pi, \omega) \equiv (\Theta, \mathcal{U}(\pi, \omega), \mathcal{U}(\omega, \pi), \mathcal{U}(\omega, \pi)). \tag{24}
\]

For ease of exposition, we refer to the first-stage MDPM $\text{SM}[\pi, \omega]$ and the second-stage MDPM $\text{AG}^i(\pi, \omega)$ as the supplemental (bi-matrix) game and the auxiliary (bi-matrix) game, respectively, played by two fictitious players $i\{1, 2\}$. In particular, each player $i$ at the supplemental game determines a mixed-strategy policy $\sigma_i(\omega|\cdot) \subseteq A_i$ to choose a type $\theta_i \in \Theta_i$ while at the auxiliary game he determines a mixed-strategy policy $\phi_i(\omega|\cdot) \subseteq A_i$ to take an action $a_i \in A_i$. We consider the following two equilibrium concepts.

Definition 9 (Nash Equilibrium). Fix $\pi$. A policy profile $\sigma^*(\omega) = (\sigma_1^*(\omega), \sigma_2^*(\omega))$ constitutes a Nash equilibrium (NE) of $\text{SM}[\pi, \omega]$ if, for all $i \in \{1, 2\}$, $\theta_i \in \Theta_i$ with $\sigma_i^*(\omega)(\theta_i) > 0$, $\theta'_i \in \Theta_i$,

\[
\sum_{\theta_i} \sigma_i^*(\omega)(\theta_i) F_i(\theta_i, \theta'_i, \omega, \pi) \geq \sum_{\theta_i} \sigma_i^*(\omega)(\theta_i) F_i(\theta'_i, \theta_i, \omega, \pi). \tag{25}
\]

Let $N(\pi, \omega)$ denotes the set of NE policy profiles given $\pi$ and $\omega$ with $N(\pi, \omega) \equiv \{\sigma_i(\omega|\cdot) \subseteq A_i \mid \text{SM}[\pi, \omega] \}$. Notice that $N(\pi, \omega)$ is the same for all $\pi$ and $\omega$.

Definition 10 (Bayesian Nash Equilibrium). Fix $\tau^i$. A policy profile $\phi^*(\theta) = (\phi_1^*(\theta), \phi_2^*(\theta))$ constitutes a Bayesian Nash equilibrium (BNE) of $\text{AG}^i(\tau^i, \theta)$ if, for all $i \in \{1, 2\}$, $a_i \in A_i$ with $\phi_i^*(\theta)(a_i) > 0$, $a'_i \in A_i$,

\[
\sum_{a_{-i}, \theta_{-i}, \omega} U_i(a_i, a_{-i}, \theta_i, \phi^*_{-i}(\theta_{-i}) \tau^i_{-i}(\theta_{-i}) A_{-i}(\omega)) \geq \sum_{a_{-i}, \theta_{-i}, \omega} U_i(a'_i, a_{-i}, \theta_i, \phi^*_{-i}(\theta_{-i}) \tau^i_{-i}(\theta_{-i}) A_{-i}(\omega)). \tag{26}
\]

Let $\mathcal{B}(\pi, \omega)$ denote the set of BNE policy profiles given $\pi$ and $\omega$ with $\mathcal{B}(\pi, \omega) \equiv \{\phi_{-i}(\theta_{-i}) \subseteq A_{-i} \mid \text{AG}(\tau^i, \theta) \}$. The (action-independent) state-dependent reward functions as defined as:

\[
\begin{align*}
\mathcal{C}_1(g^i) &= \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \\
\mathcal{C}_2(g^i) &= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \\
\mathcal{C}_3(g^i) &= \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix}.
\end{align*}
\]
The state-independent transition functions specified given as follows:
\[
T(g'|a_1,a_2,\omega^2) : T(g'|a_1,a_2,\omega^2) = \begin{bmatrix} 1 & 0.9 & 0 & 0 \\ 0.5 & 0.2 & 0.5 & 0.7 \end{bmatrix},
\]
\[
T(g'|a_1,a_2,\omega^1) : T(g'|a_1,a_2,\omega^1) = \begin{bmatrix} 0 & 0.1 & 1 & 1 \\ 0.5 & 0.8 & 0.5 & 0.3 \end{bmatrix}.
\]
We consider that principal aims to maximize the social welfare of the agents. Let \(\Gamma[\pi]\) denote the set of all goals attained by a locally admissible \(\pi\). Then, the principal’s optimization problem is
\[
\max_{\rho} W(\rho) = \mathbb{E}[\sum_{i=1}^{\infty} \delta^i r_i(\bar{a}_i) + \delta c_i(\bar{g}_i)],
\]
s.t. there exists a locally admissible \(\pi^*\) such that \(\rho \in \Gamma[\pi^*]\).

Let \(\Pi[\pi]\) denote the set of action policy profiles associated with a locally admissible \(\pi\). Due to the definition of locally admissible in Definition 6, the principal’s problem of finding an optimal \(\rho\) can be fully characterized by finding an locally admissible \(\pi^*\) and a corresponding \(\pi\). That is, (23) is equivalent to
\[
\max_{\pi^*} \mathbb{E}[\sum_{i=1}^{\infty} \delta^i r_i(\bar{a}_i) + \delta c_i(\bar{g}_i)],
\]
s.t. \(\pi^*\) is locally admissible \(\pi \in \Pi[\pi]\).

Due to Theorem 2, we can solve (24) by solving the corresponding myopic pipeline decision model. The following corollary summarizes the result of this example.

**Corollary 2.** There exists a unique locally admissible generation rule profile \(\pi^*\) which is truth-revealing: i.e., each \(\mu_i(\omega^j|\theta_i^j)| = 1\), for \(i \in \{1,2\}\). In particular,
\[
\pi^*_{\theta_i^j}(g,\omega^1) = \begin{cases} 1 & \pi^*_{\theta_i^j}(g,\omega^1) = 0 \end{cases},
\]

The equilibrium action profiles are \(\pi^*(g,\omega^1) = (0.5,0.5),\)
\(\pi^*(g,\omega^2) = (0.75,0.25),\)
\(\pi^*(g,\omega^3) = (0.6520,0.3483),\)
and \(\pi^*(g,\omega^4) = (0.6562,0.3438),\) for all \(g \in G\). In addition, the equilibrium values of myopic pipeline decision models are
\[
u(\omega^j, \pi^j) = (1.5,0.75), v(\omega^j, \pi^j) = (0.9281,0.6125).
\]

Hence, according to Theorem 2, it is straightforward to verify that the truth-reveal profile \(\pi^*\) is locally admissible.

**VI. CONCLUSION**

In this paper, we have studied a dynamic Bayesian persuasion defined as forward-looking dynamic persuasion (FLDP) in a novel pipelined stochastic Bayesian game (PSBG) in which each agent periodically makes a pipeline decision of selecting an information source (IS) and then taking an action based on the information acquired from the selected IS. A principal who controls one of the ISs aims to conduct FLDP by first incentivizing the agents to select her IS and then take actions that coincide with her goal. We have proposed a design principle termed fixed-point alignment that formulates the principal’s design as a non-linear optimization problem and characterized a set of verifiable necessary and sufficient conditions defined as local admissibility for the implementability of the design in a refinement of Nash equilibrium referred to as the pipelined perfect Markov Bayesian equilibrium. A case study has been shown to establish local admissibility by solving a series of static problem which is homogeneous with respect to the states. Designing algorithms that converges to the local admissibility will be our future work.

**REFERENCES**

[1] M. Wu, S. Amin, and A. E. Ozdaglar, “Value of information in bayesian routing games,” *Operations Research*, vol. 69, no. 1, pp. 148–163, 2021.
[2] E. Kamenica and M. Gentzkow, “Bayesian persuasion,” *American Economic Review*, vol. 101, no. 6, pp. 2590–2615, 2011.
[3] L. Mathevet, J. Perego, and I. Taneva, “On information design in games,” *Journal of Political Economy*, vol. 128, no. 4, pp. 1370–1404, 2020.
[4] T. Zhang and Q. Zhu, “Bayesian promised persuasion: Dynamic forward-looking multiagent delegation with informational burning,” *arXiv preprint arXiv:2201.06081*, 2022.
[5] A. Celli, S. Coniglio, and N. Gatti, “Private bayesian persuasion with sequential games,” in *Proceedings of the AAAI Conference on Artificial Intelligence*, vol. 34, no. 02, 2020, pp. 1886–1893.
[6] Y. Babichenko, I. Talgam-Cohen, and K. Zabarnyi, “Bayesian persuasion under ex ante and ex post constraints,” in *Proceedings of the AAAI Conference on Artificial Intelligence*, vol. 35, no. 6, 2021, pp. 5127–5134.
[7] J. Gan, R. Majumdar, G. Radanovic, and A. Singla, “Bayesian persuasion in sequential decision-making,” *arXiv preprint arXiv:2106.03537*, 2021.
[8] J. Wu, Z. Zhang, Z. Feng, Z. Wang, Z. Yang, M. I. Jordan, and H. Xu, “Sequential information design: Markov persuasion process and its efficient reinforcement learning,” *arXiv preprint arXiv:2202.10678*, 2022.
[9] S. Zamir, “Bayesian games: Games with incomplete information,” *Complex Social and Behavioral Systems: Game Theory and Agent-Based Models*, pp. 119–137, 2020.
[10] O. Hernández-Lerma and J. B. Lasserre, *Discrete-time Markov control processes: basic optimality criteria*. Springer Science & Business Media, 2012, vol. 30.
[11] E. Maskin and J. Tirole, “Markov perfect equilibrium: I. observable actions,” *Journal of Economic Theory*, vol. 100, no. 2, pp. 191–219, 2001.
[12] D. Fudenberg and J. Tirole, *Game Theory*. Ane Books, 2005, [Online]. Available: https://books.google.com/hk/books?id=j7WQwAAQAAJ
[13] D. Blackwell, “Discounted dynamic programming,” *The Annals of Mathematical Statistics*, vol. 36, no. 1, pp. 226–235, 1965.
[14] T. Zhang and Q. Zhu, “On incentive compatibility in dynamic mechanism design with exit option in a markovian environment,” *Dynamic Games and Applications*, pp. 1–45, 2021.
[15] R. Bellman, “Dynamic programming,” *Science*, vol. 153, no. 3731, pp. 34–37, 1966.
[16] T. Zhang and Q. Zhu, “Fixed-point alignment: Incentive bayesian persuasion for pipeline stochastic bayesian game,” *arXiv preprint arXiv:2203.09725*, 2022.
[17] J. Filar and K. Vrieze, “Competitive markov decision processes-theory, algorithms, and applications,” 1997.
[18] H. Prasad, P. LA, and S. Bhatnagar, “Two-timescale algorithms for learning nash equilibria in general-sum stochastic games,” in *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*, 2015, pp. 1371–1379.
[19] J. Song, H. Ren, D. Sadigh, and S. Ermon, “Multi-agent generative adversarial imitation learning,” *Advances in neural information processing systems*, vol. 31, 2018.
[20] T. Parthasarathy, S. Tij, and O. Vrieze, “Stochastic games with state independent transitions and separable rewards,” in *Selected topics in operations research and mathematical economics*. Springer, 1984, pp. 262–271.
[21] Y. Shoham and K. Leyton-Brown, *Multiagent systems: Algorithmic, game-theoretic, and logical foundations*. Cambridge University Press, 2008.
**APPENDIX**

A. **Proof of Proposition 1**

Suppose that $<\beta^0, \pi^O>$ is an obedient PPME. Then, we can construct $V^O$ given $\pi^O$. By the definition of PPME, it is straightforward to show that the constraints (EQ), (EQ), and (GB) are satisfied. Therefore, $<\pi^O, V^O>$ is a feasible solution of (OPT[$\pi^O$]).

By construction, $Z^\pi(<\pi^O, V^O>) = 0$. From the feasibility, $<\beta^0, \pi^O>$ is a global minimum of (OPT[$\pi^O$]).

Conversely, suppose that $<\pi^O, V^O>$ is feasible for (OPT[$\pi^O$]) with $Z^\pi(<\pi^O, V^O>) = 0$. Then, the constraints (EQ) and (GB) imply that, for every $i \in \{1, 2\}$, $\mathcal{g}_i \in \mathcal{G}$, $\tau_{i,t} \in \Theta$, with $\tau_i^\pi(\mathcal{g}_i, \theta, t, \omega) > 0$ where $\omega_i \in \Omega$ with $d(\omega_i, \omega) > 0$,

$$V_i(g_i, \theta, t, \omega) \geq \sum_{a_i, \omega_{i,t}} Q_i^\pi(g_i, \theta, t, \omega) \pi^\pi(a_i|\theta, t) \mu_i(\theta_{-i}|\omega^{\Theta_i}).$$

(25)

From $Z^\pi(<\pi^O, V^O>) = 0$, we have for every $i \in \{1, 2\}$, $\mathcal{g}_i \in \mathcal{G}$, $\tau_{i,t} \in \Theta$ with $\tau_i^\pi(\mathcal{g}_i, \theta, t, \omega) > 0$ where $\omega_i \in \Omega$ with $d(\omega_i, \omega) > 0$,

$$\sum_{\mathcal{g}_i, \theta_i, \omega_i} \left( V_i(g_i, \theta, t, \omega) - \sum_{a_i, \omega_{i,t}} Q_i^\pi(g_i, \theta, t, \omega) \pi^\pi(a_i|\theta, t) \mu_i(\theta_{-i}|\omega^{\Theta_i}) \right) = 0.$$

Since (25) holds for $<\pi^O, V^O>$, we have

$$V_i(g_i, \theta, t, \omega) = \sum_{a_i, \omega_{i,t}} Q_i^\pi(g_i, \theta, t, \omega) \pi^\pi(a_i|\theta, t) \mu_i(\theta_{-i}|\omega^{\Theta_i}).$$

From iteration, we have that $V^O$ is the unique optimal HT value function associated with $\pi^O$. In addition, (GB) implies that given $V^O$, obedient selection $\beta^O$ is a PPME selection policy. Therefore, $<\beta^0, \pi^O>$ is an obedient PPME.

B. **Proof of Proposition 2**

For given $\rho \in \Gamma[\beta^O]$, let $<\tau^\rho, \pi>$ satisfy (AT). The constraint (AT) is satisfied automatically by $<\tau^\rho, \pi^O>$. Suppose that $<\beta^O, \pi^O, V^O>$ is a global minimum of (OPT[$\tau^\rho$]) with $Z^\pi(\pi, V) = 0$. Then, the constraints (GRT) and (FT) are trivially satisfied. Proposition 1 implies that $<\beta^O, \pi>$ is an obedient PPME that attains the goal $\rho$. From the construction of $Z^\pi(\cdot)$ and the constraint (EQ), we obtain that $<\tau^\rho, \pi>$ satisfies the constraint (EQ1). According to (8), we construct $J$ as

$$J(g) = \sum_{\theta, \omega} V_i(g, \theta, \pi) \tau_i^\rho(\theta, \omega)d(\omega).$$

(26)

Then, $Z^\tau_{PK}(\tau^\rho, J, V, \rho) = 0$. Since $V, \pi$ satisfies constraint (GB) given $\tau^\rho$,

$$J(g) \geq \sum_{\theta, \omega} V_i(g, \theta, \tau_{i,t}|g) \tau_i^\rho(\theta|g, \omega)d(\omega),$$

for all $\theta$ and $g$, which implies constraint (GB1).

From the constraints (EQ1) and (GB1), we know that for any feasible $<\tau^\rho, J, V, \rho>$, $Z^\tau_{PK}(\tau^\rho, J, V|\rho) > 0$. Therefore, $<\tau^\rho, J, V>$ is a global minimum of (FPA[\rho]) with $Z^\tau_{PK}(\tau^\rho, J, V|\rho) = 0$.

Conversely, let $<\tau^\rho, J, V>$ be a global minimum of (FPA[\rho]) with $Z^\tau_{PK}(\tau^\rho, J, V, \rho) = 0$. Then,

$$J_i(g) = \sum_{\theta, \omega} V_i(g, \theta, \tau_{i,t}|g) \tau_i^\rho(\theta|g, \omega)d(\omega).$$

(27)

The constraint (EQ1) directly implies the constraint (EQ). The constraint (GB1) implies

$$J_i(g) \geq \sum_{\theta, \omega} R_i(g, \omega) \tau_i^\rho(\theta|g, \omega)d(\omega).$$

(28)

The right-hand side (RHS) of (28) can be written as:

$$\text{RHS of (28)} = \sum_{\theta, \omega} \left( R_i(g, \omega) + \delta \sum_g f_i(g')T(g'g \omega, \mu_i(\theta|g, \omega)) \times \pi(\theta|g, \omega) \tau_i^\rho(\theta|g, \omega)d(\omega). \right)$$

(29)

Construct

$$\hat{Q}_i(g, \theta, a|\hat{V}_i) = R_i(g, a) + \delta \sum_{\theta', \omega'}\left( \sum_{\mathcal{g}', a'} \hat{V}_i(g', \theta') \tau_i^\rho(\theta'|g', a')T(g'|g, a, \omega) \mu_i(\theta|\theta_i) \right).$$

Then,

$$\text{RHS of (28)} = \sum_{\theta, \omega} \hat{Q}_i(g, \theta, a|\hat{V}_i) \pi(\theta|a, g) \tau_i^\rho(\theta|g, \omega)d(\omega).$$

The constraint (EQ1) implies

$$\hat{V}_i(g, \theta, \tau_{i,t}|g, \omega) \pi(\theta|a, g) \tau_i^\rho(\theta|g, \omega)d(\omega),$$

and thus $Z^\tau_{PK}(\pi, \hat{V}) = 0$. Hence, from (27), we have

$$\sum_{\theta, \omega} \hat{V}_i(g, \theta, \tau_{i,t}|g, \omega)d(\omega) \geq \sum_{\theta, \omega} \hat{V}_i(g, \theta, \tau_{i,t}|g, \omega)d(\omega),$$

which implies the constraint (GB). Therefore, $<\pi, \hat{V}>$ is a global minimum of (OPT[$\tau^\rho$]) with $Z^\pi(\pi, \hat{V}) = 0$.

□

C. **Proof of Theorem 1**

To prove Theorem 1, we need to show that $<\tau^\rho, J, V>$ is a global minimum of (FPA[\rho]) (implementability) with $Z^\tau_{PK}(\tau^\rho, J, V|\rho) = 0$ for any goal $\rho$ and if only if $\tau^\rho$ is locally admissible with the goal $\rho$ (local admissibility). Our proof is based on any fixed goal $\rho$.

C2. **Local Admissibility ⇒ Implementability**

Fix any $\alpha \in \Omega$ and $\mathcal{g} \in \mathcal{G}$. Suppose that $\tau^\rho$ is locally admissible with goal $\rho$. Because $\Delta_i(X^{\rho, \theta_i}, \tau_i^\rho, f; g, \omega) = 0$, we have

$$\sum_{\theta, \omega} \nabla V\cdot=\lambda_i(X^{\rho, \theta_i}; g, \theta') \tau_i^\rho(\theta'_|g, \omega) + \nabla V\cdot=\lambda_i(X^{\rho, \theta_i}; g, \hat{\theta}) \tau_i^\rho(\hat{\theta}|g, \omega)$$

$$= \sum_{\theta, \omega} f(\theta)\nabla V\cdot=\lambda_i(X^{\rho, \theta_i}; g, \theta) = 0.$$

Then, Condition 1 implies

$$f(\theta) = \tau_i^\rho(\hat{\theta}|g, \omega),$$

for all $\theta_i \in \Theta_i$.

(31)

The decomposition $\Theta_i = \Theta_i^\rho \cup \{\hat{\theta}_i\}$ implies that $\hat{\theta}_i$ can be fully characterized by $\Theta_i^\rho$. That is,

$$\tau_i^\rho(\hat{\theta}_i|g, \omega) = 1 - \sum_{\theta_i \in \Theta_i^\rho} \tau_i^\rho(\theta_i|g, \omega).$$

From $d(X^{\rho, \theta_i}, \tau_i^\rho(\theta_i|g, \omega), b[\theta_i]; c, g, \omega) = 0$, we have

$$b[\theta_i] = c + \lambda_i(X^{\rho, \theta_i}; g, \hat{\theta}_i) - \lambda_i(X^{\rho, \theta_i}; g, \theta_i) + c.$$
which implies

\[ b[\theta] = -\lambda_i(X_{i,\theta}^{g,0};g,\theta), \forall \theta_i \in \Theta_f, \]

\[ c = -\lambda_i(X_{i,\theta}^{g,0};g, \hat{\theta}). \]

Therefore, the conditions in (16) leads to \( M_i(X_{i,\theta}^{g,0};\tau_i;g) = 0 \) which implies \( Z^{\text{FPA}}(\tau_i^i; J, V; \rho) = 0 \). Moreover, the condition \( \pi_0(a_i|g, \theta_i) \gamma_i g_i \left(J_i(V_i, \pi_i|.; g, \theta_i, a_i); g, \theta_i, a_i = 0, \forall a_i \in A_i \right) \) of (17) implies that \( Z^{\text{F}}(\pi, V) = 0 \). From Proposition 1, \( \pi, V \) constitutes an obedient PMPE of \( \text{PSBG}[^{\text{t}}] \). Then, based on Proposition 2 and \( \rho(a|g, \omega) = \sum_{\theta_i \in \Theta_f} \pi(a_i|g, \theta_i) \gamma_i g_i \theta_i \) of (17), we have that \( \tau_i^i, J, V \) is a global minimum of \( \text{FPA}[\rho] \) with \( Z^{\text{FPA}}(\tau_i^i, J, V; \rho) = 0 \).

\[ C.1. \text{Implementability} \Rightarrow \text{Local Admissibility} \]

Suppose that a direct mechanism \( \text{PSBG}[^{\text{t}}] \) is implementable in obedient PMPE that achieves the goal \( \rho \). Hence, there exists a \( \tau_i^i, J, V \) is a global minimum of \( \text{FPA}[\rho] \) with \( Z^{\text{FPA}}(\tau_i^i, J, V; \rho) = 0 \). According to the constraints \( \text{FPA}[\rho] \), every \( J_i(g) - \sum_{\theta_i, \omega} V_i(g, \theta) \gamma_i g_i (g, \theta, \omega)d_\omega(\omega) \geq 0 \),

which implies that every \( \lambda_i(X_{i,\theta}^{g,0};g, \theta) \geq 0 \). Because \( Z^{\text{FPA}}(\tau_i^i, J, V; \rho) = 0 \), we have

\[ J_i(g) - \sum_{\theta_i, \omega} V_i(g, \theta) \gamma_i g_i (g, \theta, \omega)d_\omega(\omega) = 0. \]

Then, from the definition of \( M_i \) in (13), \( M_i = 0 \). Since \( \lambda_i(X_{i,\theta}^{g,0};g, \theta) \geq 0 \) for \( \theta_i, \omega_i \), we have \( \gamma_i g_i (0) \gamma_i g_i (g, \theta, \omega)d_\omega(\omega) = 0 \).

By constructing \( J_i, b[\theta], \) and \( c \) according to (31) and (32), respectively, we can show that there exist Lagrange multipliers such that the conditions in (16) are satisfied. From Proposition 2, \( \pi, V \) constitutes an obedient PMPE and is a global minimum of \( \text{OPT}[^{\text{t}}] \) with \( Z^{\text{F}}(\pi, V) = 0 \). From the constraints \( \text{OPT}[^{\text{t}}] \), we have both \( \gamma_i g_i (g, \theta, \omega)d_\omega(\omega) \geq 0 \),

Thus, \( \tau_i^i \) is locally admissible.

\[ \square \]

\[ D. \text{Proof of Theorem 2} \]

We prove Theorem 2 in two steps.

1) Step 1: Suppose \( \tau_i^i \) is locally admissible. Let \( V = (V_1, V_2) \) and \( \pi = (\pi_1, \pi_2) \) denote the corresponding state-type value function and action policy profile, respectively.

In this step, we want to show that the followings hold:

(D1) For any \( \tau_i^i \), a policy profile \( \phi[\theta] \equiv (\phi_1[\theta])_{\tau_i^i \in \Theta_f} \) constitutes a BNE of the game \( G[\tau_i^i, \theta] \) if and only if \( \phi_i[\theta](.) = \pi_i[\theta], \forall \theta_i \in \Theta_f, g \in G \).

(D2) For the policy profile \( \pi \), let \( \tau_i^i \) be the state value function profile corresponding to the locally admissible \( \tau_i^i \), then, for all \( i \in \{1, 2\}, g \in G \).

\[ J_i(g) = \gamma_i g_i (g, \theta_i, a_i) = 0, \forall a_i \in A_i. \]

Thus, \( \tau_i^i \) is locally admissible.

For notational compactness, we denote \( IV_i^{\phi_i}(g, \theta) \) defined in (12) by suppressing \( V_i \). From the Bellman recursions in (8)-(9), we have the following recursive representation:

\[ IV_i^{\phi_i}(g, \theta) = \sum_{\theta_i} \left( c_i(g) + r_i(a) \right) + \delta \sum_{\theta_i, \theta_i} \pi_i(g, \theta, \theta_i) \gamma_i g_i (g, \theta, \theta_i) \mu_i(\theta_i|\theta_i). \]

\[ \hat{T}(g', \theta_i) = \sum_{\omega} T_i(g'|\omega) \mu_i(\theta_i|\theta_i). \]

where

\[ \hat{T}(g', \theta_i) = \sum_{\omega} T_i(g'|\omega) \mu_i(\theta_i|\theta_i). \]

Due to the properties of SER and SIT, (33) can be rewritten as

\[ IV_i^{\phi_i}(g, \theta) = c_i(g) + \sum_{\theta_i} U_i(a_i, a_2, \theta_i|\tau_i^i \hat{T}(g', \theta_i) \gamma_i g_i (g, \theta, \theta_i) \mu_i(\theta_i|\theta_i). \]

Let \( IV_i[\theta_i; \tau_i^i] \equiv IV_i^{\phi_i}(g, \theta) \in [G \neq G \neq G \hat{T}(g', \theta_i) \gamma_i g_i (g, \theta, \theta_i) \mu_i(\theta_i|\theta_i). \]

\[ IV_i[\theta_i; \tau_i^i] = \gamma_i g_i (0) \gamma_i g_i (g, \theta, \omega)d_\omega(\omega) = 0. \]

Thus, we obtain (D1) and (D2) (hence, part (ii) of Theorem 2).

2) Step 2: In this step, we want to show that the followings hold:

(D3) For any \( \pi_0 \), a generation profile \( \tau_i^i \) is locally admissible if and only if \( \tau_i^i (\theta, \omega) = \sigma_i(\theta, \omega), \forall \omega \in G \), where \( \sigma_i(\theta, \omega) \equiv (\sigma_1(\theta, \omega), \sigma_2(\theta, \omega)) \) constitute a Nash equilibrium of \( \text{SM}[\pi, \omega] \), for all \( \omega \in \Omega, \theta_i \in \Theta_f, g \in G \).

(D4) Part (ii) of Theorem 2 holds. That is, if \( J_i = (J_1, J_2) \) is the state value function profile corresponding to the locally admissible \( \tau_i^i \), then, for all \( i \in \{1, 2\}, g \in G \).

\[ J_i(g) = \gamma_i g_i (g, \theta_i, a_i) = 0, \forall a_i \in A_i. \]

Define

\[ Y_i(g, \omega) = c_i(g) + \sum_{\theta_i} \gamma_i g_i (g, \theta_i, a_i) + \delta \sum_{\theta_i, \theta_i} \gamma_i g_i (g', \theta_i) \gamma_i g_i (g, \theta, \theta_i) \mu_i(\theta_i|\theta_i). \]
Due to the properties of SER and SIT, $Y_i$ can be written as

$$Y_i(g, \omega) = c_i(g) + \sum_{\theta} F_i(\theta, \omega; \pi) \pi^k(\theta|g, \omega)$$

$$+ \delta \sum_{g', \omega'} d_{\omega}(\omega'') T(g'|\theta, \omega) \left( Y_i(g', \omega') - c_i(g') \right) \pi^k(\theta|g, \omega).$$

Similar to the Step 1, let $Y_i[\omega] \equiv [Y_i(g, \omega)]_{g \in G}$ denote the vectors of $Y_i(\cdot, \omega)$. In addition, let $T^k(\tau^k, \omega)$ denote the transition matrix given $\tau^k$ and $\omega$. When $\sigma[\omega]$ constitute a Nash equilibrium of $SM[\pi, \omega]$, $\sum_\theta F_i(\theta, \omega; \pi) \pi^k(\theta|g, \omega) = v_i[\omega, \pi]$. Then, we have

$$Y_i[\omega] = \bar{c}_i + v_i[\omega, \pi] 1 + \delta T^k(\tau^k, \omega) (Y_i[\omega] - \bar{c}_i),$$

which yields

$$Y_i[\omega] = \bar{c}_i + (1 - \delta)^{-1} v_i[\omega, \pi] 1,$$

(37)

where the second term only depends on the state $g$ through the generation rule. The only if part of (D3) is straightforward and we suppress it here and only prove the if part of (D3). Suppose that $\tau^k(\theta_i|g, \omega) = \sigma_i[\omega](\theta_i)$, for all $i \in \{1, 2\}$, $g \in G$. Then, the second term of (37) becomes homogeneous with respect to the state $g$. Then, it holds that

$$(1 - \delta) T^k(\tau^k, \omega) = (1 - \delta)^{-1} T^k[\tau^k, \omega],$$

which leads to

$$Y_i[\omega] = \bar{c}_i + (1 - \delta)^{-1} v_i[\omega, \pi] T^k[\tau^k, \omega] 1$$

$$= \bar{c}_1 + (1 - \delta)^{-1} v_i[\omega, \pi] 1.$$

Thus, for each $g$, we have

$$Y_i(g, \omega) = c_i(g) + (1 - \delta)^{-1} v_i[\omega, \pi].$$

Since $J_i(g) = \sum_{\omega} Y_i(g, \omega) d_{\omega}(\omega)$, we establish (D4) (hence, part (iii) of Theorem 2).