Pivotal Objects in Monoidal Categories and Their Hopf Monads

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Abstract

An object $P$ in a monoidal category $C$ is called pivotal if its left dual and right dual objects are isomorphic. Given such an object and a choice of dual $Q$, we construct the category $C(P, Q)$, of objects which intertwine with $P$ and $Q$ in a compatible manner. We show that this category lifts the monoidal structure of $C$ and the closed structure of $C$, when $C$ is closed. If $C$ has suitable colimits we show that $C(P, Q)$ is monadic and thereby construct a family of Hopf monads on arbitrary closed monoidal categories $C$. We also introduce the pivotal cover of a monoidal category and extend our work to arbitrary pivotal diagrams.

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1 Introduction

Hopf monads were originally introduced as generalisations of braided Hopf algebras in braided monoidal categories, from the setting of braided categories to arbitrary monoidal categories. In [16], Hopf monads were defined as monads which lift the monoidal structure of a monoidal category to their category of modules. These monads are now referred to as bimonads or opmonoidal monads, whereas monads which lift the closed structure of a closed monoidal category as well as its tensor, are called Hopf monads, according to [6]. Hopf monads have proved of particular importance in the study of tensor categories [7] and topological field theories [23]. However, in both settings the categories in consideration are rigid. Although in [6], the theory of Hopf monads was extended to arbitrary monoidal categories, not many examples have been studied when the category is not rigid. Inspired from our work on bimodule connections and Hopf algebroids in [10], under the setting of noncommutative differential geometry [3], in the present work, we study objects in arbitrary monoidal categories, which have isomorphic left and right duals. We call these object pivotal and when provided with such a pivotal pair, $P$ and $Q$ in a monoidal category $C$, we construct the category of $P$ and $Q$ intertwined objects $C(P, Q)$. The constructed category lifts the monoidal structure of $C$ and the closed structure of $C$, when $C$ is closed. Consequently, if $C$ is closed and has suitable colimits, we construct a Hopf monad corresponding to such a pivotal pair in an arbitrary closed monoidal category so that its Eilenberg-Moore
category recovers \( \mathcal{C}(P, Q) \). The merit of this construction is that it does not fall in the realm of Tannaka-Krein type constructions, \([20]\), which are usually utilised and does not require the category to be braided or even to have a non-trivial center. On the other hand, when \( \mathcal{C} \) is rigid, \( \mathcal{C}(P, Q) \) also becomes rigid and although the necessary colimits might not exist for the forgetful functor to be monadic, we show that the forgetful functor is exact. In particular, if \( \mathcal{C} \) is a tensor category in the sense of \([19]\), as in Example 4.9 then \( \mathcal{C}(P, Q) \) becomes an abelian monoidal category.

In \([10]\), we construct a family of Hopf algebroids corresponding to first order differential calculi. The modules of the Hopf algebroid recover a closed monoidal subcategory of bimodule connections, however, the differential calculus must be pivotal for the construction to work. We explain this ingredient in a much more general setting, here. If \( \mathcal{C} \) is a monoidal category and \( P \) an object in \( \mathcal{C} \), one can construct a category of \( P \)-intertwined objects, whose objects are pairs \((A, \sigma)\), where \( A \) is an object of \( \mathcal{C} \) and \( \sigma : A \otimes P \to P \otimes A \) an invertible morphism. This category naturally lifts the monoidal structure of \( \mathcal{C} \), in a similar fashion to the monoidal structure of the center of \( \mathcal{C} \). The key feature in our construction is the following: if \( P \) is pivotal and we chose a dual, \( Q \), of \( P \), any invertible morphism \( \sigma \) induces two \( Q \)-intertwinings on the object \( A \), namely 5 and 6. In order to obtain a closed monoidal category, we must restrict to the subcategory of pairs where these induced \( Q \)-intertwinings are inverse. We denote this category, corresponding to the pivotal pair \( P \) and \( Q \), by \( \mathcal{C}(P, Q) \) and describe its monoidal structure in Theorem 4.1. Our main results are Theorem 4.2 and Corollary 4.3 which show that \( \mathcal{C}(P, Q) \) lifts left and right closed structures on \( \mathcal{C} \), when they exist.

We also discuss the construction in the cases where \( \mathcal{C} \) is rigid, Corollary 4.4, and when \( \mathcal{C} \) has a pivotal structure which is compatible with \( P \) and \( Q \), Theorem 4.5.

Pivotal categories were introduced in \([1]\) and their study is vital for topological field theories, \([23]\). However, a study of individual objects in a monoidal category which have isomorphic left and right duals has not been produced. In \([19]\), the pivotal cover of a rigid monoidal category was introduced, in connection with Frobenius-Schur indicators discussed in \([17]\). We introduce the pivotal cover \( \mathcal{C}^{\text{piv}} \) of an arbitrary monoidal category \( \mathcal{C} \), in Definition 3.5 from a different point of view which arose in \([10]\), namely pivotal morphisms. The pivotal cover of a monoidal category has pivotal pairs as objects, and suitable pivotal morphisms between them, so that a strong monoidal functor from a pivotal category to the original category, must factor through the pivotal cover, Theorem 3.7. The construction in \([19]\) requires all objects to have left duals and a choice of distinguished left dual for each object i.e. for the category to be left rigid, while our construction avoids these issue by taking pivotal pairs as objects of \( \mathcal{C}^{\text{piv}} \).

The applications of our work are spread as examples throughout the article. In Section 3 we observe that dualizable objects in any braided category and ambidextrous adjunctions are simply examples of pivotal objects in monoidal categories. In Section 4.2 of \([10]\), we have presented several other examples, in the format of differential calculi, where the space of 1-forms is a pivotal object in the monoidal category of bimodules over the algebra of noncommutative functions. We explain this setting briefly in Example 3.3. The Hopf monad constructed in this case, becomes a Hopf algebroid, Example 5.8, which is a subalgebra of the Hopf algebroid of differential operators defined in \([10]\). Additionally, to construct the sheaf of differential operators in the setting of \([10]\), we require the wedge product between the space of 1-forms and 2-forms to be a pivotal morphism. A direct consequence of Example 5.6 is that any bicovariant calculus over a Hopf algebra, satisfies this condition.

As alluded to in Example 5.5, the Hopf monads constructed here are in some sense a noncommutative version of the classical Hopf algebra \( \mathcal{O}(GL(n)) \), where instead of
we note that for a finite dimensional vector space, the resulting Hopf algebra becomes precisely a quotient of the free matrix Hopf algebra, $\mathcal{NH}\mathcal{L}(n)$, discussed in [21]. More generally, in Theorem 5.4 we show that the Hopf monad constructed is augmented if and only if the pair $P$ and $Q$ is a pivotal pair in the center of the monoidal category. Consequently, using the theory of augmented Hopf monads from [6], we construct the braided Hopf algebra corresponding to every pivotal pair in the center of the monoidal category.

Organisation: In Section 2, we review the theory of Hopf monads and braided Hopf algebras and the necessary background on duals and closed structures in monoidal categories. In Section 3, we introduce the notion of pivotal objects and morphisms in an arbitrary monoidal categories and introduce the pivotal cover. In Section 4, we construct $\mathcal{C}(P, Q)$ and review some of its properties and in Section 5 we construct its corresponding Hopf monad. In Section 6 we briefly discuss the generalisation of our work to arbitrary pivotal diagrams. The proof of Theorem 4.2 requires several large commutative diagrams which are presented in Section 7 at the end of our work.

2 Preliminaries

We assume basic categorical knowledge and briefly recall the theory of monads and monoidal categories from [12, 23] and the theory of Hopf monad and bimonads from [8, 6].

2.1 Monads

A monad $T$ on a category $\mathcal{C}$, consists of a triple $(T, \mu, \eta)$, where $T : \mathcal{C} \rightarrow \mathcal{C}$ is an endofunctor with natural transformations $\mu : TT \rightarrow T$ and $\eta : id_{\mathcal{C}} \rightarrow T$ satisfying

$$\mu(T \mu) = \mu T \mu \quad \text{and} \quad \mu \eta = \mu T \mu.$$ 

Any monad gives rise to an adjunction $F_T \dashv G_T : \mathcal{C} \rightleftarrows \mathcal{C}$, where $\mathcal{C}_T$ is the Eilenberg-Moore category associated to $T$. The category $\mathcal{C}_T$ consists of pairs $(X, r)$, where $X$ is an object of $\mathcal{C}$ with a $T$-action $r : TX \rightarrow X$ satisfying $r \mu_X = r(T \mu)$ and $r \eta = r \mu_X$ and morphisms which respect $T$-actions. The free functor is defined by $F_T(X) = (TX, \mu_X)$ and the forgetful functor by $U_T(X, r) = X$. Conversely, any adjunction $F \dashv G : \mathcal{D} \rightleftarrows \mathcal{C}$ gives rise to a monad via its unit $\eta : id_{\mathcal{D}} \rightarrow GF$ and counit $\epsilon : FG \rightarrow id_{\mathcal{D}}$. The triple produced is $(GF, Ge_{\mathcal{F}}, \eta)$. Hence, there is a natural functor $K = F_G \circ D \rightarrow C_T$ called the comparison functor. We say functor $G$ is monadic if $K$ is an equivalence of categories. For more detail on monads we refer to Chapter VI of [12], since we will only present Beck’s Theorem and later utilise it.

Theorem 2.1. [Beck’s Theorem] Given an adjunction $F \dashv G : \mathcal{D} \rightleftarrows \mathcal{C}$, $G$ is monadic if and only if the functor $G$ creates coequalizers for parallel pairs $f, g : X \rightarrow Y$ for which $GF, Gf$ has a split coequalizer.

2.2 Monoidal Categories

We call $(\mathcal{C}, \otimes, 1_\otimes, \alpha, \ell, r)$ a monoidal category, where $\mathcal{C}$ is a category, $1_\otimes$ an object of $\mathcal{C}$, $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ a bifunctor and $\alpha : (id_{\mathcal{C}} \otimes id_{\mathcal{C}}) \otimes id_{\mathcal{C}} \rightarrow id_{\mathcal{C}} \otimes (id_{\mathcal{C}} \otimes id_{\mathcal{C}}), \ell : 1_\otimes \otimes id_{\mathcal{C}} \rightarrow id_{\mathcal{C}}$ and $r : id_{\mathcal{C}} \otimes 1_\otimes \rightarrow id_{\mathcal{C}}$ natural isomorphisms satisfying coherence axioms as presented in Section 1.2 of [23]. There exists a corresponding monoidal
structure on \( \mathcal{C} \), the opposite monoidal category, which we denote by \((\mathcal{C}, \otimes^{op})\) and is defined by composing \( \otimes \) with the flip functor \( \text{id} \). \( X \otimes^{op} Y = Y \otimes X \) for pairs of objects \( X, Y \) of \( \mathcal{C} \).

In what follows, we assume that all monoidal categories in question are strict i.e. \( \alpha, \lambda, \text{ and } \tau \) are all identity morphisms.

A functor \( F : \mathcal{C} \to \mathcal{D} \) between monoidal categories is said to be (strong) monoidal if there exists a natural (isomorphism) transformation \( F_2(-, -) : F(-) \otimes_D F(-) \to F(- \otimes_C -) \) and a (isomorphism) morphism \( F_0 : 1 \to F(1) \) satisfying

\[
F_2(X \otimes Y, Z)(F_2(X, Y) \otimes \text{id}_{F(Z)}) = F_2(X, Y \otimes Z)(\text{id}_{F(X)} \otimes F_2(Y, Z))
\]

\[
F_2(X, 1) (\text{id}_{F(X)} \otimes F_0) = \text{id}_{F(X)} = F_2(1, X)(F_0 \otimes \text{id}_{F(X)})
\]

where we have omitted the subscripts denoting the ambient categories, since they are clear from context. A functor is said to be opmonoidal or comonoidal if all morphisms in the above definition are reversed. A strong monoidal functor \( F \) is called strict monoidal if the natural isomorphisms \( F_2 \) and \( F_0 \) are identity morphisms. A monoidal category is said to be braided if there exists a natural isomorphism \( \Psi_{X,Y} : X \otimes Y \to Y \otimes X \) satisfying braiding axioms described in Section 3.1 of [23].

**Notation.** We will abuse notation and write \( X \) instead of the morphism \( \text{id}_X \) whenever it is feasible. We will also omit \( \otimes \) when writing long compositions of morphisms i.e. \( AB \) will denote the morphisms \( \text{id}_A \otimes f \otimes \text{id}_B \) for arbitrary objects \( A \) and \( B \) and morphism \( f \) in \( \mathcal{C} \).

The (lax) center of a monoidal category \((\mathcal{C}, \otimes, 1)\) has pairs \((X, \tau)\) as objects, where \( X \) is an object in \( \mathcal{C} \) and \( \tau : X \otimes - \to - \otimes X \) is a natural (transformation) isomorphism satisfying \( \tau_1 = \text{id}_X \) and \((\text{id}_M \otimes \tau_N)(\tau_M \otimes \text{id}_N) = \tau_{M \otimes N} \), and morphisms \( f : X \to Y \) of \( \mathcal{C} \), satisfying \((\text{id}_{C} \otimes f)\tau = \nu(f \otimes \text{id}_C)\), as morphism \( f : (X, \tau) \to (Y, \nu) \). We denote the lax center and center by \( Z^{\text{lax}}(\mathcal{C}) \) and \( Z(\mathcal{C}) \), respectively. The center is often referred to as the Drinfeld-Majid center and the lax center is sometimes referred to as the prebraided or weak center. The (lax) center has a monoidal structure via

\[
(X, \tau) \otimes (Y, \nu) := (X \otimes Y, (\tau \otimes \text{id}_Y)/(\text{id}_X \otimes \nu))
\]

and \((1, \text{id}_C)\) acting as the monoidal unit, so that the forgetful functor to \( \mathcal{C} \) is strict monoidal. The center \( Z(\mathcal{C}) \) is also braided by \( \Psi_{(X, \tau), (Y, \nu)} = \tau_Y \).

### 2.3 Rigid and Closed Monoidal Categories

For any object, \( X \), in a monoidal category \( \mathcal{C} \), we say an object \( \nu X \) is a left dual of \( X \), if there exist morphisms \( \text{ev}_X : \nu X \otimes X \to 1 \) and \( \text{coev}_X : 1 \to X \otimes \nu X \) such that

\[(\text{ev}_X \otimes \text{id}_{\nu X})(\text{id}_{\nu X} \otimes \text{coev}_X) = \text{id}_{\nu X}, \quad (\text{id}_X \otimes \text{ev}_X)(\text{coev}_X \otimes \text{id}_X) = \text{id}_X\]

In such a case, we call \( X \) a right dual for \( \nu X \). Furthermore, a right dual of an object \( X \) is denoted by \( X^\ast \), with evaluation and coevaluation maps denoted by \( \text{ev}_X : X \otimes X^\ast \to 1 \) and \( \text{coev}_X : 1 \to X^\ast \otimes X \), respectively. We say an object \( X \) is dualizable if has both a left dual and a right dual. The category \( \mathcal{C} \) is said to be left (right) rigid or autonomous if all objects have left (right) duals. If a category is both left and right rigid, we simply call it rigid. Usually, when a category is said to be left (or right rigid), it is assumed that we have chosen a left dual for all objects and \( \nu X \) denotes this specific choice of left dual for any object \( X \). Given these choices, we have a contravariant functor \( \nu : (\mathcal{C}, \otimes) \to \mathcal{C} \) which sends objects \( X \) to their left duals \( \nu X \) and morphisms
A monad \((T, \mu, \eta)\) on \(\mathcal{C}\) is said to be a bimonad or an opmonoidal monad if it also has a compatible comonoidal structure \((T_2, T_0)\) satisfying

\[
T_2(X, Y)\mu_{X\otimes Y} = (\mu_X \otimes \mu_Y)T_2(TX, TY)T(T_2(X, Y))
\]

\[
T_0\mu_1 = T_0T(\eta_0), \quad T_2(X, Y)\eta_{X\otimes Y} = \eta_X \otimes \eta_Y, \quad T_0\eta_1 = id_1,
\]

where \(X, Y\) are objects of \(\mathcal{C}\). A bimonad is said to be left (right) Hopf if the left (right) fusion operators, denoted by \(H^l\) (resp. \(H^r\)), and defined as

\[
H^l_{X,Y} := (id_{TX} \otimes \mu_Y)T_2(X, T(Y)) : T(X \otimes T(Y)) \rightarrow T(X) \otimes T(Y)
\]

\[
H^r_{X,Y} := (\mu_X \otimes id_{TY})T_2(T(X), Y) : T(T(X) \otimes Y) \rightarrow T(X) \otimes T(Y)
\]

for objects \(X, Y\) of \(\mathcal{C}\), is invertible. A bimonad is called Hopf if it is both left and right Hopf. The above conditions can be reformulated purely in terms of \(F_T\) and \(U_T\): given an adjunction \(F \dashv G : \mathcal{D} \Rightarrow \mathcal{C}\), the induced monad \((GF, G\mu, \eta)\) is a bimonad if and only if \(U\) is strong monoidal. In this case, the adjunction is called comonoidal. The adjunction is called left (right) Hopf if \(U\) is a left (right) closed functor. We briefly recall the main property of these structures and refer the reader to [6] for more detail on bimonads and Hopf monads.

**Theorem 2.2.** If \(T\) is a monad on a monoidal category \(\mathcal{C}\), then

(I) [15] Bimonad structures on \(T\) are in correspondence with liftings of the monoidal structure of \(\mathcal{C}\) onto \(\mathcal{C}_T\) i.e. monoidal structures on \(\mathcal{C}_T\) such that \(U_T\) is strong monoidal.

(II) [6], [8] If \(\mathcal{C}\) is left (right) rigid and \(T\) a bimonad, then \(T\) is left (right) Hopf if and only if \(\mathcal{C}_T\) is left (right) rigid.

(III) [6] If \(\mathcal{C}\) is left (right) closed then, \(T\) being left (right) Hopf is equivalent to \(\mathcal{C}_T\) being left (right) closed and \(U_T\) a left (right) closed functor.
Lemma 3.1. If \( S \) is a Hopf algebra with its invertible antipode defined by \( S \) and \( \xi \), in this section, we define the notion of pivotal objects, pairs and pivotal cover for the proof of Theorem 2.3. (braided Hopf algebras and \[6\] for the more details on augmented Hopf monads and the trivial Hopf monad structure. For an algebra \( M \) in a monoidal category \( C \), \( \xi \), \( \eta \), \( \Delta \) and \( \tau \) are morphisms in \( C \) and \( (m \otimes m)(B \otimes \tau_B \otimes B)(\Delta \otimes \Delta) = \Delta m \) and \( \epsilon m = (\epsilon \otimes \epsilon) \). Hence, the monad \( T = B \otimes - \) has a bimonad structure, with \( T = (B \otimes \tau \otimes \text{id}_C)(\Delta \otimes \text{id}_C \otimes \text{id}_C) \) and \( T_0 = \epsilon \). A bialgebra is called a central Hopf algebra if there exists a morphism \( S : (B, \tau) \to (B, \tau) \) such that \( m(B \otimes S)\Delta = \eta \epsilon = m(S \otimes B)\Delta \) and the mentioned bimonad \( B \otimes - \) is left Hopf in this case and Hopf if \( S \) is also invertible. A braided Hopf algebra in a braided monoidal category \( (C, \Psi) \) is just a central Hopf algebra \( (H, \tau) \) where \( \tau_\epsilon = \Psi H \). We recover the usual notion of Hopf algebras as braided Hopf algebras in the braided monoidal category of vector spaces.

A Hopf monad \( (T, \mu, \eta_1, T_2, T_0) \) on a monoidal category \( C \) is said to be augmented if there exists a bimonad morphism \( \xi : T \to \text{id}_C \), where the identity functor \( \text{id}_C \) has trivial Hopf monad structure. For \( \xi \) to be a bimonad morphism, \( \xi \eta = \text{id}_C \), \( \xi \mu = \xi T(\xi) \), \( (\xi \otimes \xi)T_2 = \xi_0 \) and \( \xi_1 = T_0 \) must hold.

Theorem 2.3. (Theorem 5.7 \[6\]) There is an equivalence of categories between the category of Hopf algebras in the center of \( C \) and augmented Hopf monads on \( C \).

An augmentation on a Hopf monad, provides \( T(1) \) with a central Hopf algebra structure and the Hopf monad \( T \) in this case is shown to be isomorphic to the induced Hopf monad of \( T(1) \). In particular, \( (T(1), T_2(1), 1, T_0) \) forms a comonoid in \( C \), while \( (1_1, H_1, 1) \) forms a monoid in \( C \), where \( u_X = T(T(1))^{-1}(1(X) \otimes \eta X) \), and \( \tau = (\xi \otimes T(1))T_2(T(X), 1) \) defines a braiding, which makes \( T(1, \tau) \) a central Hopf algebra with its invertible antipode defined by \( S = \xi T(1)(H_1^{-1}(1(X) \otimes \eta X)) \) and \( S^{-1} = \xi T(1)(H_1^{-1}(\eta X \otimes T(1))) \). We refer the reader to \[13\] for more details on braided Hopf algebras and \[6\] for the more details on augmented Hopf monads and the proof of Theorem 2.3.

3 Pivotal Objects and Pivotal Cover

In this section, we define the notion of pivotal objects, pairs and pivotal cover for arbitrary monoidal categories. Let \( (C, \otimes, 1) \) be a monoidal category.

Lemma 3.1. If \( P \) is an object of \( C \), then the following statements are equivalent:

(I) The object \( P \) is dualizable and there exists an isomorphism \( {}^\vee P \cong P^\vee \).

(II) There exists an object \( Q \) and morphisms \( \text{coev} : 1 \to P \otimes Q, \text{ev} : Q \otimes P \to 1 \) and \( \text{coev} : 1 \to Q \otimes P, \text{ev} : P \otimes Q \to 1 \), making \( Q \) a left and right dual of \( P \), respectively.

(III) Left duals \( {}^\vee P \) and \( \vee {}^\vee P \) exist and there exists an isomorphism \( P \cong {}^\vee \vee P \).
We say $P$ is a Pivotal object if it satisfies any of the above statements and refer to an ordered pair $(P, Q)$, as in part (II), as a pivotal pair.

Proof. (I)⇒(II) Assume $P$ is dualizable with $\triangledown P$ and $P^\triangledown$, its left and right dual objects, $\text{coev}, ev$ and $\text{coev}, ev$ as the respective coevaluation and evaluation morphisms and let $f : \triangledown P \to P^\triangledown$ be an isomorphism. Hence, $(f^{-1} \otimes P)\text{coev}$ and $ev(P \otimes f)$ make $\triangledown P$ right dual to $P$ and $Q = \triangledown P$ and $P^\triangledown$, $(f^{-1} \otimes P)\text{coev}$, $ev(P \otimes f)$ satisfy the conditions in (II).

(II)⇒(III) By assumption $Q = \triangledown P$ and $P = \triangledown Q$, thereby $P = \triangledown \triangledown P$.

(III)⇒(I) Let $f : P \to \triangledown \triangledown P$ be an isomorphism and $\text{coev} \circ \text{ev} P : 1 \to \triangledown \triangledown P \otimes \triangledown P$, $\text{ev} \circ f : \triangledown P \otimes \triangledown P \to 1$ be the relevant coevaluation and evaluation morphisms. Hence, $(\triangledown P \otimes f)\text{coev} \circ \text{ev} P$ and $(f^{-1} \otimes \triangledown P)\text{ev} \circ f$ make $\triangledown P$ right dual to $P$, and $\triangledown P = P^\triangledown$. □

From this point forward, we assume $P$ is a pivotal object and $Q$ its left and right dual as in statement (II). Notice that $Q$ is also pivotal by definition. Moreover, note that strong monoidal functors preserve pivotal objects, since they preserve duals and isomorphisms.

We now review some examples of pivotal objects in monoidal categories.

**Example 3.2.** Any dualizable object $P$ in a braided monoidal category $(B, \Psi)$ is pivotal. Let $\triangledown P$ and $P^\triangledown$ be the left and right duals of $P$ with the coevaluation and evaluation maps, $\text{coev}, ev$ and $\text{coev}, ev$, respectively. In this case

\[
\text{tr}_P := (P^\triangledown \otimes \text{ev}(P \triangledown))(\text{coev} \otimes \triangledown P) : \triangledown P \to P^\triangledown \tag{1}
\]

\[
\text{tl}_P := (\triangledown P \otimes \text{ev})(\Psi_{P, \triangledown}^{-1} \triangledown \text{coev} \otimes P^\triangledown) : P^\triangledown \to \triangledown P \tag{2}
\]

are inverses and provide an isomorphism between $\triangledown P$ and $P^\triangledown$.

**Example 3.3.** For a $\mathbb{K}$-algebra $A$, the category of $A$-bimodules, denoted by $A\mathcal{M}_A$, has a monoidal structure by tensoring bimodules over $A$. In Section 4.2 of [10], we provided a family of examples of first order differential calculi $(A, \Omega^1, d : A \to \Omega^1)$, where $\Omega^1$ was a pivotal bimodule over $A$. Our examples included Hopf bimodules over Hopf algebras. If $A$ is a Hopf algebra with an invertible antipode, the category of Hopf bimodules has braided monoidal structure and any dualizable object in this category is pivotal. The forgetful functor from the category of Hopf bimodules to the category of bimodules over $A$ is strong monoidal, hence as a bimodule, any dualizable Hopf bimodule is pivotal.

**Example 3.4.** For any category $C$, the category of endofunctors on $C$, denoted by $\mathcal{E}nd(C)$, has a monoidal structure via composition of functors i.e. $F \otimes G = FG$ for $F, G \in \mathcal{E}nd(C)$ and the identity functor $\text{id}_C$ acting as the monoidal unit. In this case, a functor $F$ being left (right) dual to $G$, is exactly equivalent to $F$ being left (right) adjoint to $G$. An endofunctor is thereby called pivotal if it has left and right adjoint functors which are isomorphic. Such adjunctions are referred to as ambidextrous and the monad $GF$ on $C$ is called a Frobenius monad (Theorem 17 [11]).

A rigid monoidal category $\mathcal{C}$ is called pivotal (sometimes called sovereign) if there exists a natural isomorphism $\varrho : \text{id}_\mathcal{C} \to \triangledown (-)$. Equivalently, the condition is sometimes stated as the existence of a monoidal natural isomorphism $\varrho^\triangledown : (-)^\triangledown \to \triangledown (-)$. In [19], the pivotal cover of a rigid monoidal category was introduced by K. Shimizu. Independently, we discovered this notion for general monoidal categories, by encountering the notion of pivotal morphisms between pivotal objects in [10].
Definition 3.5. Let $\mathcal{C}$ be a monoidal category, and $(P_1, Q_1)$ and $(P_2, Q_2)$ pivotal pairs in $\mathcal{C}$ with $\text{coev}_P, ev_P, \text{coev}_Q, ev_Q$ being the relevant coevaluation and evaluation morphisms. We say a morphism $f : P_1 \to P_2$ is pivotal if

$$(ev_2 \otimes Q_1)((Q_2 \otimes f \otimes Q_1)(Q_2 \otimes \text{coev}_1)) = (Q_1 \otimes ev_2)(Q_1 \otimes f \otimes Q_2)(\text{coev}_1 \otimes Q_2)$$

as morphisms from $Q_2$ to $Q_1$. The pivotal cover of $\mathcal{C}$, denoted by $\mathcal{C}^\text{piv}$, has ordered pivotal pairs $(P, Q)$ in $\mathcal{C}$ as objects and pivotal morphisms of $\mathcal{C}$ as morphisms.

There is a subtlety which we must address, a morphisms $f : P_1 \to P_2$ being pivotal does not depend on the choice of $Q_1$ and $Q_2$. In particular, if $Q'_1$ and $Q'_2$ are also left and right duals of $P_1$ and $P_2$ with $\text{coev}'_P, ev'_P, \text{coev}'_Q, ev'_Q$ the relevant coevaluation and evaluation morphisms, then there exist isomorphisms $h_i : Q_i \to Q'_i$ so that $ev'_i = ev_i(h_i^{-1} \otimes P_i), \text{coev}'_i = \text{coev}_i(h_i \otimes P_i), ev'_i = ev_i(P_i \otimes h_i^{-1}), \text{coev}'_i = (h_i \otimes P_i)\text{coev}_i$. Hence, $f$ holds for $f$ in terms of $Q_1$ and $Q_2$ if and only if it holds in terms of $Q'_1$ and $Q'_2$ and $f$ being pivotal is only dependent on $P_1$ and $P_2$.

In the terminology of rigid monoidal categories, if $Q = \vee P = P^\vee$ is the chosen left and right dual of $P$, the left hand morphism in $3$ is exactly $f$ and the right hand morphism, $f^\vee$, and we call a morphism $f$ between objects pivotal, if $f^\vee = f$. It should be clear the $\mathcal{C}^\text{piv}$ is well-defined and pivotal morphisms are closed under composition.

In Section 5.2 of $\cite{10}$, we presented two families of examples of pivotal morphisms. In $\cite{10}$, we were interested in differential graded algebras $\oplus_{i \geq 0} \Omega_i$, where $\Omega_1$ and $\Omega_2$ are pivotal bimodules over $\Omega_0$ and $\wedge : \Omega^1 \to \Omega^2$ is a pivotal morphism. One of the examples presented in $\cite{10}$ was that of Woronowicz’s bicovariant algebras $\cite{24}$ over the group algebra, for any arbitrary group. The Woronowicz construction has been generalised to the language of braided abelian categories by Majid, Section 2.6 $\cite{3}$, in the name of braided exterior algebras.

Example 3.6. [Braided Exterior Algebra] Let $(\mathcal{C}, \otimes, 1)$ be an abelian braided monoidal category and $\Psi$ denote its braiding. Recall from Example 3.2 that any dualizable object in $\mathcal{C}$ is pivotal. Let $P_1$ and $P_2$ be dualizable objects in $\mathcal{C}$ and denote the coequalizer of the parallel pair $\text{id}_{P_1} \otimes P_1 : P_1 \otimes P_1 \Rightarrow P_1 \otimes P_1$ by $\pi : P_1 \otimes P_1 \to P_2$. We claim that $\pi$ is a pivotal morphism. As in Example 3.2, $\vee P_1$ and $\vee P_2$ denote left duals of $P_1$ and $P_2$, respectively, which become right duals by the isomorphism provided, so that $\vee P_1 \otimes \vee P_1$ also becomes a left and right dual of $P_1 \otimes P_1$. Hence, writing the pivotal condition, $3$ for $\wedge$ in terms of $\vee P_1 \otimes \vee P_1$ and $\vee P_2$, reduces to checking if the morphisms

$$(ev_2(\vee P_1)(\vee P_2))((\vee P_2 \wedge (\vee P_1)(\vee P_1))((\vee P_2(P_1, \text{coev}_1(P_1))(\text{coev}_1)),$$

$$(\vee P_1 \vee P_2, ev_2(\vee P_1 \otimes \vee P_2 \otimes \vee P_2))((\vee P_2(\vee P_1, \text{coev}_1(P_1))(\text{coev}_1)) \otimes \text{lt}_{P_2})$$

are equal. By the definition of $\text{lt}$ and $\text{lt}$ and the properties of the braiding, the second morphism simplifies as follows

$$(\vee P_1 \vee P_2, ev_2(\vee P_1 \otimes P_2))(\vee P_1)(\vee P_2)((\vee P_1, \text{coev}_1(P_1))(\text{coev}_1))$$

and

$$(\vee P_1 \vee P_2, ev_2(\vee P_1 \otimes P_2))(\vee P_2)(\vee P_1)(\vee P_1, \text{coev}_1(P_1))(\text{coev}_1))$$

are equal.
By our definition of pivotal morphisms, it should be clear that the trivial identity pivotal functor \(\text{id}_X\) is trivially pivotal.

We can define a natural monoidal structure on \(C^{\text{piv}}\) by \((P_1, Q_1) \otimes (P_2, Q_2) = (P_1 \otimes P_2, Q_1 \otimes Q_2)\), since the tensor of pivotal morphisms is again a pivotal morphism in this way. Thereby, \(C^{\text{piv}}\) lifts the monoidal structure of \(\mathcal{C}\), so that the natural forgetful functor \(H : C^{\text{piv}} \to \mathcal{C}\), which sends a pivotal pair \((P, Q)\) to \(P\), is strict monoidal.

Furthermore, notice that \(C^{\text{piv}}\) is also rigid and admits left and right duality functors \(\overset{\vee}{-}\) and \(\overset{-\vee}{-}\), which are defined by \((P, Q) \overset{\vee}{\to} (Q, P)\) and \(f \overset{\tr}{\to} (ev_2 \circ Q \otimes f \otimes Q_1)(Q_2 \otimes coev_1)\) for \(f : (P_1, Q_1) \to (P_2, Q_2)\), as in Definition 3.5. By our definition of pivotal morphisms, it should be clear that the trivial identity morphism forms an isomorphism between \(\overset{\tr}{-}\) and \(-\overset{\tr}{-}\) and the category \(C^{\text{piv}}\) is trivially pivotal.

In order to discuss the universal property of the pivotal cover, we recall the definition of a pivotal functor from [19]. If \(\mathcal{C}\) and \(\mathcal{D}\) are rigid monoidal categories and \(F : \mathcal{C} \to \mathcal{D}\) is a strong monoidal functor, then we have a natural family of unique isomorphisms \(\zeta : F(\overset{\tr}{-}) \to \overset{\tr}{F}(\overset{-\vee}{-})\) defined by

\[
\zeta_X = (F_0F(ev)F_2(\overset{\tr}{X}, X) \otimes \overset{\tr}{F}(X))(\overset{\tr}{F}(X) \otimes coev_{F(X)})
\]

where \(X\) is an object of \(\mathcal{C}\). If \(\mathcal{C}\) and \(\mathcal{D}\) are pivotal categories with pivotal structures \(\varrho^\mathcal{C} : \text{id}_\mathcal{C} \to \overset{\vee\vee}{-}\) and \(\varrho^\mathcal{D} : \text{id}_\mathcal{D} \to \overset{-\vee\vee}{-}\), we say \(F\) preserves the pivotal structure if

\[
\varrho^\mathcal{D}_{F(X)} = \overset{\tr}{(\zeta_X^{-1})(\zeta_X)}F(\varrho^\mathcal{C}_X)
\]

holds for all objects \(X\) of \(\mathcal{C}\).

**Theorem 3.7.** The pivotal cover \(C^{\text{piv}}\) of \(\mathcal{C}\) is pivotal and satisfies the following universal property: if \(\mathcal{D}\) is a pivotal monoidal category and \(G : \mathcal{D} \to \mathcal{C}\) a strong monoidal functor, then there exists a unique functor \(G' : \mathcal{D} \to C^{\text{piv}}\), so that \(G = HG'\) and \(G'\) respects the pivotal structures.

**Proof.** Let the pivotal structure of \(\mathcal{D}\) be denoted by the natural isomorphism \(\varrho : \text{id}_\mathcal{D} \to \overset{\vee\vee}{-}\). Since strong monoidal functors preserve duals, \(G(X)\) for any object \(X\) in \(\mathcal{D}\) will be a pivotal object in \(\mathcal{C}\). In particular, we define \(G'(X)\) to be the pivotal pair \((G(X), \overset{\vee\vee}{G}(X))\) with coevaluation and evaluation morphisms

\[
\text{coev}_{G'(X)} := G_2^{-1}(X, \overset{\vee\vee}{X})G(\text{coev}_{X})G_0^{-1}, \quad G_2^{-1}(\overset{\tr}{X}, \overset{-\vee\vee}{X})G(\overset{\tr}{X} \otimes g_X)G_0^{-1}
\]

\[
\text{ev}_{G'(X)} := G_0G(\text{ev}_{X})G_2(\overset{-\vee\vee}{X}, X), \quad G_0G(\text{ev}_{X}(\overset{-\vee\vee}{X} \otimes \overset{\tr}{X}))G_2(X, \overset{\tr}{X})
\]

If \(f : X_1 \to X_2\) is a morphism in \(\mathcal{D}\), then \(G(f)\) is a pivotal morphism between \((G(X_1), G(\overset{\tr}{X}_1))\) and \((G(X_2), G(\overset{\tr}{X}_2))\), since \(\varrho_X^{-1}f\overset{\tr}{\varrho}_X = \overset{\tr}{f}\). Hence, by letting \(G'(f) = G(f)\) we have defined a functor \(G' : \mathcal{D} \to C^{\text{piv}}\) such that \(HG' = G\).
Additionally, $G'$ is a pivotal functor: if $\zeta: G'(-) \to ^\vee G(-)$ is the unique natural isomorphism as defined before the Theorem, then $G'(-) = (G(-), G(-))$ and $^\vee G(X) = (G(X), G(X))$ with the appropriate duality morphisms as defined above. Hence,
\[
^\vee (\zeta_X^{-1})(\zeta_X G'(q_X)) = \vee (\zeta_X ^{G'(X)}(q_X)) = G'(q_X) \otimes \text{coev}_{G'(X)} = (G'(q_X) \otimes \text{coev}_{G'(X)}) = (ev_X \otimes \text{coev}_{G'(X)})(G'(q_X) \otimes \text{coev}_{G'(X)}) \]
\[
= (ev_X \otimes \text{coev}_{G'(X)})(G'(q_X) \otimes \text{coev}_{G'(X)})(\text{ev}_{G'} X \otimes \text{coev}_{G'}(X)) = (\text{coev}_{G'} X \otimes \text{ev}_{G'}(X))(G'(q_X) \otimes \text{coev}_{G'(X)})(\text{ev}_{G'} X \otimes \text{coev}_{G'}(X)) \]
\[
= (\text{ev}_{G'} X \otimes \text{coev}_{G'}(X))(G'(q_X) \otimes \text{coev}_{G'}(X)) \]
and by construction $id_{(P,Q)} = P$ for any pair $(P, Q)$ in $C_{\text{prom}}$ and
\[
^\vee (\zeta_X^{-1})(\zeta_X G'(q_X)) = (ev_X \otimes G(X))(G'(q_X) \otimes \text{coev}_{G'(X)}) = (G'(q_X) \otimes \text{coev}_{G'}(X)) \]
\[
= (\text{ev}_{G'} X \otimes \text{coev}_{G'}(X))(G'(q_X) \otimes \text{coev}_{G'}(X)) \]
holds and thereby $G'$ is pivotal.

In [19], the pivotal cover of a left rigid monoidal category $C_{\text{prom}}$ is constructed as the category of “fixed objects” by the endofunctor $^\vee (-) : C \to C$. Constructing the pivotal cover as such has two main drawbacks, namely that we need to assume all objects in $C$ have left duals and $C$ is left rigid so that there is a distinguished choice of left dual for every object. While we will not directly compare the constructions, the universal property above, is also proved in Theorem 4.3 of [19] and thereby the two constructions of $C_{\text{prom}}$ are equivalent when $C$ is left rigid.

4 The Category $C(P, Q)$

Given a pivotal pair $P$ and $Q$, as in Lemma [3.1](II), we define the category of $P$ and $Q$ intertwined objects, denoted by $C(P, Q)$, as the category whose objects are pairs $(X, \sigma)$, where $X$ is an object of $C$ and $\sigma : X \otimes P \to P \otimes X$ an invertible morphism in $C$ such that
\[
(\text{ev} \otimes X \otimes Q)(Q \otimes \sigma \otimes Q)(Q \otimes X \otimes \text{coev}) : Q \otimes X \to X \otimes Q \tag{5}
\]
\[
(Q \otimes X \otimes \text{coev})(Q \otimes \sigma^{-1} \otimes Q)(\text{coev} \otimes X \otimes Q) : X \otimes Q \to Q \otimes X \tag{6}
\]
are inverses. Morphisms between objects $(X, \sigma), (Y, \tau)$ of $C(P, Q)$ are morphisms $f : X \to Y$ in $C$, which satisfy $\tau(f \otimes P) = (P \otimes f)\sigma$. For an object $(X, \sigma)$ in $C(P, Q)$, we call $\sigma$ a $P$-intertwining and denote the induced morphisms $\overline{\sigma}$ and $\overline{\tau}$ by $\overline{\overline{\sigma}}$ and $\overline{\overline{\tau}}$, respectively, and call them induced $Q$-intertwinings.

Observe that the definition of $C(P, Q)$ is dependent on the choice of $Q$: let $P$ and $Q'$ together with $\text{coev}' : 1 \to P \otimes Q', \text{ev}' : Q \otimes P \to 1$ and $\overline{\text{coev}}' : 1 \to Q' \otimes P$, $\overline{\text{ev}}' : P \otimes Q' \to 1$ satisfy the conditions of Lemma [3.1](II). Hence, we have two
induced isomorphisms between $Q$ and $Q'$, $f = (ev \otimes Q')(Q \otimes coev')$ and $f^{-1} = (ev' \otimes Q)(Q' \otimes coev)$, and $g = (Q' \otimes ev)(coev' \otimes Q)$ and $g^{-1} = (Q \otimes ev')(coev \otimes Q')$. Additionally, if for a $P$-intertwinings $(X, \sigma)$, we denote the induced $Q$-intertwinings and induced $Q'$-intertwinings by $\sigma_Q, \sigma_Q^{-1}$ and $\sigma_{Q'}, \sigma_{Q'}^{-1}$, respectively, then $\sigma_{Q'} = (g \otimes X)\sigma_Q(X \otimes g^{-1})$ and $\sigma_{Q'}^{-1} = (X \otimes f)\sigma_Q^{-1}(f^{-1} \otimes X)$. Hence, $\sigma_{Q'}$ and $\sigma_{Q'}^{-1}$ being inverses is not equivalent to $\sigma_{Q'}$ and $\sigma_{Q'}^{-1}$ being inverses unless $f = g$.

On the other hand, the category $C(Q, P)$ is isomorphic to $C(P, Q)$. The isomorphism sends an object $(X, \sigma)$ in $C(P, Q)$ to $(X, Q, \sigma^{-1})$ in $C(Q, P)$. The $Q$-intertwining $\sigma^{-1}$ is invertible and the induced $P$-intertwinings on $X$ in $C(Q, P)$ are precisely $\sigma$ and $\sigma^{-1}$:

\[
\sigma^{-1} = (ev \otimes X \otimes P)(P \otimes \sigma^{-1} \otimes P)(P \otimes X \otimes coev) \\
\sigma = (P \otimes X \otimes ev)(P \otimes \sigma \otimes P)(coev \otimes X \otimes P)
\]

Note that the isomorphism described between $C(Q, P)$ and $C(P, Q)$ commutes with the forgetful functors from each category to $C$.

The monoidal structure of $C$ lifts to $C(P, Q)$ such that the forgetful functor $U : C(P, Q) \rightarrow C$ which sends a pair $(X, \sigma)$ to its underlying object $X$, becomes strict monoidal: for any pair of objects $(X, \sigma)$ and $(Y, \tau)$, the monoidal structure of $C(P, Q)$, denoted by $\otimes$ again, is defined by

\[
(X, \sigma) \otimes (Y, \tau) = (X \otimes Y, (\sigma \otimes Y)(X \otimes \tau))
\]

and $(1, \text{id}_P)$ acts as the monoidal unit. Furthermore, $\otimes$ is defined on pairs of morphisms of $C(P, Q)$, as it is by $tn$ in $C$.

Our construction is very similar to that of the center of a monoidal category, and its generalisation, the dual of a strong monoidal functor [14], but we do not require the $P$-intertwinings $\sigma$ to satisfy the usual braiding conditions.

**Theorem 4.1.** The monoidal structure on $C(P, Q)$, as described above, is well-defined.

**Proof.** The monoidal structure is defined exactly the same as it would be in the case of the center of a monoidal category, and for the proof of why $\otimes$ induces a monoidal structure, we refer the reader to [14]. The only non-trivial fact we need to check in our case is whether $(X, \tau) \otimes (Y, \tau)$ is an object of $C(P, Q)$. In particular, if $\sigma$ and $\tau$ are invertible, it should be clear that $(\sigma \otimes Y)(X \otimes \tau)$ is also invertible, however, we need to prove that the induced $Q$-intertwinings, [5] and [6] for $X \otimes Y$ are inverses. This follows from the fact that the induced $Q$-intertwinings, [5] and [6] for $(X, \tau)$ and $(Y, \tau)$ are inverses:

\[
(\sigma \otimes \tau)(\sigma \otimes \tau^{-1}) = (evXYQ)(Q\sigma \otimes \tau Q)(QXYcoev)(QXYcoev')(Q\sigma \otimes \tau^{-1}Q) \\
(QcoevXYQ) \\
= (evXYQ)(Q\sigma YQ)(QX\tau Q)(QXYcoev)(QXYcoev')(QX\tau^{-1}Q)(Q\sigma^{-1}YQ) \\
(QcoevXYQ) \\
= (evXYQ)(Q\sigma YQ)(QXP_{\text{ev}}YQ)(QXYcoev)(QXP_{\text{ev}}YQ')(QX\tau Q)(QXYcoev) \\
(QXcoevY)(QX\tau^{-1}Q)(QXcoevY)(QXP_{\text{ev}}YQ)(Q\sigma^{-1}YQ) \\
= (evXYQ)(Q\sigma YQ)(QX\tau Q)(QXYcoev)(QXP_{\text{ev}}YQ)(QXcoevY) \\
= (evXYQ)(Q\sigma^{-1}YQ)(QcoevXYQ)(QXYcoev')(Q\sigma YQ)(QX\tau Q)(QXYcoev) \\
= (evXYQ)(Q\sigma^{-1}YQ)(QcoevXYQ)(QXYcoev)(Q\sigma YQ)(QX\tau Q)(QXYcoev) \\
= (evXYQ)(Q\sigma^{-1}YQ)(QcoevXYQ)(QXYcoev)(Q\sigma YQ)(QX\tau Q)(QXYcoev) = \text{id}_{XYQ}
\]
\((\sigma \otimes \tau^{-1})(\sigma \otimes \tau) = (QXY_{\text{ev}})(Q(\sigma \otimes \tau)^{-1}Q)(\text{coev}XY_{Q})(Q\sigma \otimes \tau Q)(QXY_{\text{coev}})\)
\((QX_{\text{ev}})(QX^{-1}Q)(Q\sigma^{-1}Q)(\text{coev}XY_{Q})(Q\sigma Q)(QX_{\text{coev}})\)
\((QX_{\text{ev}})(QX^{-1}Q)(QX_{\text{ev}}PY_{Q})(QX_{\text{coev}}PY_{Q})(Q\sigma^{-1}Q)(\text{coev}XY_{Q})\)
\((QX_{\text{ev}})(QX_{\text{ev}}PY_{Q})(QX_{\text{coev}}PY_{Q})(QX_{\text{coev}})(QX_{\text{coev}})\)
\((QX_{\text{ev}})(QX^{-1}Q)(QX_{\text{ev}}PY_{Q})(QX_{\text{coev}})(QX_{\text{coev}})\)
\((QX_{\text{ev}})(QX^{-1}Q)(QX_{\text{ev}}PY_{Q})(QX_{\text{coev}})(QX_{\text{coev}})\)
\((QX_{\text{ev}})(QX^{-1}Q)(QX_{\text{ev}}PY_{Q})(QX_{\text{coev}})(QX_{\text{coev}})\)

**Theorem 4.2.** If \(\mathcal{C}\) is a left closed monoidal category, then \(\mathcal{C}(P,Q)\) has a left closed monoidal structure which lifts that of \(\mathcal{C}\) and the forgetful functor \(U\) is left closed.

**Proof.** Let \((A, \sigma_A)\) and \((B, \sigma_B)\) be objects in \(\mathcal{C}(P,Q)\). If \(\mathcal{C}\) is left closed, we denote the right adjoint functor to \(\Box \to \to [A, [-]]\), and let \(\eta^A : \to [A, [-]]\) and \(\epsilon^A : [A, [-]] \otimes [A, [-]] \to [-]\) denote the unit and counit of this adjunction. To demonstrate that the left closed structure of \(\mathcal{C}\) lifts to \(\mathcal{C}(P,Q)\), we provide a functorial \(P\)-intertwining on \([A, B]^l\), and demonstrate that the unit and counit morphisms are morphisms in \(\mathcal{C}(P,Q)\).

We claim that \((\sigma_A, \sigma_B)^l\) as defined below is a \(P\)-intertwining with the described inverse:
\[
\sigma_A, \sigma_B)^l := \langle P[A, (\text{ev}B)(Q\sigma_B)](Q[A, B]^l:\sigma_A^{-1})]^l \rangle (P\eta_{Q[A,B]^lP}^{A})
\]
\[
\langle P[A, B]^l:\text{coev} \rangle
\]

Demonstrating that \([A, B]^l : \sigma_A, \sigma_B)^l\) is an object of \(\mathcal{C}(P,Q)\) requires showing that \(\sigma_A, \sigma_B)^l\) and \(\sigma_A, \sigma_B)^l^{-1}\) are inverses and that the induced \(Q\)-intertwining given by
\[
\overline{\sigma_A, \sigma_B)^l} := \langle P[A, (\text{ev}B)(Q\sigma_B)](Q[A, B]^l:\sigma_A^{-1})]^l \rangle (P\eta_{Q[A,B]^lP}^{A})
\]
\[
\overline{\sigma_A, \sigma_B)^l}^{-1} := \langle Q[A, B]^l:\text{coev} \rangle
\]

are inverses. These fact are not hard to show, but reduce to long diagram chases, for which we refer the reader to Section 7. If \(f : (B, \sigma_B) \to (C, \sigma_C)\) is a morphism in \(\mathcal{C}(P,Q)\), then it follows by definition that \([A, f]^l\) is also a morphism in \(\mathcal{C}(P,Q)\):
\[
\langle P[A, f]^l : \sigma_A, \sigma_B)^l\rangle := \langle P[A, f]^l : \sigma_A, \sigma_B)^l\rangle (P\eta_{Q[A,B]^lP}^{A})
\]

\[
\langle \text{coev}[A, B]^l \rangle := \langle \sigma_A, \sigma_C)^l : [A, f]^l \rangle
\]
Hence the assignment
\[
[(A, \sigma_A), [-] : C(P) \to C(P, Q)
\]
\[(B, \sigma_B) \mapsto ([A, B]^I, (\sigma_A, \sigma_B))
\]
is functorial by acting as \([A, -]^I\) on morphisms. In particular, the functor \([(A, \sigma_A), [-]^I\] is a lift of \([A, -]^I\) via the forgetful functor \(U\) so that \(U([(A, \sigma_A), [-]^I] = [A, U(-)]^I\). Hence, it only remains to show that natural transformations \(\eta^A\) and \(\epsilon^A\) lift to \(C(P, Q)\), with respect to the defined \(P\)-intertwinings on \([(A, \sigma_A), [-]^I\]: we must check that
\[
(P\eta^A_B)\sigma_B = (\sigma_A, \sigma_B \otimes \sigma_A)(\eta^A_B P) : B \otimes P \to P \otimes [A, B \otimes A]^I
\]
holds. We must also check that the counit commutes with the \(P\)-intertwinings i.e.
\[
\sigma_B(\epsilon^A_P) = (P\epsilon^A_B)(\langle \sigma_A, \sigma_B \rangle \otimes \sigma_A) : [A, B]^I \otimes A \otimes P \to P \otimes B
\]
holds. Both fact are proved in the form of commutative diagram, which are presented in Section 4. Hence, we have demonstrated that the left closed structure of \(C\) lifts to \(C(P, Q)\) via the forgetful functor \(U\).

**Corollary 4.3.** If \(C\) is a right closed monoidal category, then \(C(P, Q)\) has a right closed monoidal structure which lifts that of \(C\) and the forgetful functor \(U\) is right closed.

**Proof.** One could prove this statement directly as done for the left closed structure in Theorem 4.2; however, we take a short-cut in this case. Recall that a right closed structure on \((C, \otimes)\) corresponds to a left closed structure on \((C, \otimes^{op})\). Hence, the forgetful functor \(U : C(P, Q) \to C\) lifting the right closed structure of \(C\) is equivalent to \(U^{op} : C(P, Q)^{op} \to C^{op}\) lifting the left closed structure of \(C^{op}\), where by \(C^{op}\) we mean \((C, \otimes^{op})\). On the other hand, we observe that \(P\) is pivotal in \(C^{op}\) with \(ev\) and \(coev\) making \(Q\) a left dual of \(P\) and \(ev\) and \(coev\) making \(Q\) a right dual of \(P\) in \((C, \otimes^{op})\). Furthermore, we have an isomorphism of categories
\[
L : C^{op}(P, Q) \to C(P, Q)^{op}
\]
\[(A, \sigma : A \otimes^{op} P \to P \otimes^{op} A) \mapsto (A, \sigma^{-1} : A \otimes P \to P \otimes A)
\]
\[
R : C(P, Q)^{op} \to C^{op}(P, Q)
\]
\[(A, \sigma : A \otimes P \to P \otimes A) \mapsto (A, \sigma^{-1} : A \otimes^{op} P \to P \otimes^{op} A)
\]
which is monoidal i.e. for a pair of objects \((A, \sigma_A)\) and \((B, \sigma_B)\) in \(C(P, Q)^{op}\), we have
\[
R([(A, \sigma_A) \otimes^{op} (B, \sigma_B)]) = R((B \otimes A, \sigma_B \otimes \sigma_A)) = (B \otimes A, (\sigma_B \otimes \sigma_A)^{-1})
\]
\[
= (B \otimes A, (B \otimes \sigma_A^{-1})(\sigma_B^{-1} \otimes A))
\]
\[
\cong (A, \sigma_A^{-1}) \otimes^{op} (B, \sigma^{-1}) = R((A, \sigma_A)) \otimes^{op} R((B, \sigma_B))
\]
Moreover, \(U^{op} L\) is precisely the forgetful functor from \(C^{op}(P, Q)\) to \(C^{op}\) which sends a pair \((A, \sigma)\) to \(A\). By Theorem 4.2, we know that \(U^{op} L\) lifts the left closed structure of \(C^{op}\) and since \(L\) is an isomorphism of monoidal categories, we conclude that \(U^{op}\) also lifts the left closed structure of \(C^{op}\).
The proof of Corollary 4.3 allows us to compute the induced $P$-intertwiningso n the right inner homs of $C$ so that the right closed structure of $C$ lifts to $C(P, Q)$. Explicitly, if $(A, \sigma_A)$ and $(B, \sigma_B)$ are objects in $C(P, Q)$, and $\Gamma_A : - \to [A, A \otimes -]^r$ and $\Theta_A : A \otimes [A, -]^r \to -$ denote the unit and counit of $- \otimes A \dashv [A, -]^r$ in $C$, then

$$
\langle \sigma_A, \sigma_B \rangle_r := (P[A, (ev_B)(Q\sigma_B)(Q\Theta_B^A P)([\sigma_A^{-1} [A, B]^r P])^r] \Theta_A^{-1} \otimes X, \sigma_X) (P[A, B]^r \otimes X) (P[T^A, P][A, B]^r P)
$$

$$
\langle \sigma_A, \sigma_B \rangle_r^{-1} := (\sigma_B) (P[A, B]^r \otimes X) (P[A, B]^r P)
$$

define a suitable $P$-intertwining on $[A, B]^r$ so that the endofunctor $[\sigma_A, -]^r$ which sends a pair $(A, B) \in C(P, Q)$ to $([A, B]^r, \langle \sigma_A, \sigma_B \rangle_r)$ is right adjoint to $- \otimes (A, \sigma_A).

**Corollary 4.4.** If $C$ is left (right) rigid, then $C(P, Q)$ is left (right) rigid.

**Proof.** The statement follows directly from Theorem 4.2 and Corollary 4.3, when restricted to the case of a left or right rigid monoidal category. Explicitly, if $(X, \sigma)$ is an object in $C(P, Q)$, then the $P$-intertwining induce on $X^\vee X = [X, 1]^r$ and $X^\vee = [X, 1]^r$ (if they exist), are denote by $\sigma_{X^\vee}$ and $\sigma_{X^\vee}$, respectively, and are given by

$$
\sigma_{X^\vee} = (ev_X P X^\vee) (ev_X P X^\vee) (P^X Q^X P) (P X^\vee X P^X Q X P) (P^X Q X P)
$$

$$
\sigma_{X^\vee}^{-1} = (P X^\vee P X^\vee) (P X^\vee P X^\vee) (P^X Q X P) (P^X Q X P)
$$

providing the left and right duals of $(X, \sigma)$ in $C(P, Q)$. \hfill \Box

**Theorem 4.5.** If $g : \text{id}_C \to \text{id}_C$ is a pivotal structure on $C$ and $P$ is fixed by $\text{id}_C \to \text{id}_C$, $P = Q$ and $g_P = \text{id}_{P}$, then $C(P, Q)$ is pivotal and the forgetful functor $U$ preserves this pivotal structure.

**Proof.** In this case, the pivotal structure of $C$ directly lifts to $C(P, Q)$. Here, we demonstrate that $g_X : (X, \sigma) \to (X, \sigma)$ commutes with the $P$-intertwining for any object $(X, \sigma)$ of $C(P, Q)$. Observe that by Theorem 4.2, $X^\vee \otimes X^\vee = (X^\vee X, (\sigma_{X^\vee} \otimes X^\vee) X^\vee)$ where

$$
(\sigma_{X^\vee} = \text{ev}_X P^X X^\vee X) \text{ev}_X P^X X^\vee P^X X^\vee P
$$

Observe that, in the above statement we are abusing notation since $X^\vee (-)$ is strict monoidal whereas this is not necessarily the case and $\text{id}_C \otimes X$ denotes a morphism from $X^\vee Q \otimes X$ to $X^\vee Q \otimes X$. However, this is not an issue since $g_X$ is a monoidal isomorphism and commutes with the natural isomorphisms $X^\vee Q \otimes X$ since $X^\vee X = g_{X^\vee}^{-1}$. \hfill \Box

Hence $g_X$ is morphism in $C(P, Q)$ and lifts the pivotal structure of $C$ trivially.
Remark 4.6. Notice that in the proof of Theorem 4.3, we only needed \( \sigma \) for an arbitrary object \( (X, \sigma) \) in \( \mathcal{C}(P, Q) \) to commute with \( g_P \). Although this does not hold for arbitrary \( P \)-intertwinings, one could restrict to a subcategory of \( \mathcal{C}(P, Q) \) where this additional condition holds. We will briefly discuss generalisations of this type in Section 6.

Before concluding this section, we show which colimits in \( \mathcal{C} \) lift to \( \mathcal{C}(P, Q) \).

Lemma 4.7. If \( \mathcal{C} \) is closed, the forgetful functor \( U \) creates colimits.

Proof. Consider a diagram \( D : J \to \mathcal{C}(P, Q) \) so that the diagram \( U \circ D : J \to \mathcal{C} \) has a colimit \( A \) in \( \mathcal{C} \) with a family of universal morphisms \( \pi_j : U(j) \to A \) for objects \( j \) in \( \mathcal{C} \). Since \( \mathcal{C}(P, Q) \) is closed, the diagrams \( U \circ D \circ j \to P \) and \( U \circ D \) admit colimits \( A \circ P \) and \( P \circ A \), respectively. By the universal property of \( \mathcal{C}(P, Q) \), there exists a unique morphism \( \sigma_A \) such that \( \sigma_A(\pi \circ \pi) = (P \circ \pi) \pi \). Since \( \pi \) is invertible, it follows by the universal property of \( A \circ P \) that there exists a unique morphism \( \sigma_A^{-1} \) such that \( \sigma_A^{-1}(P \circ \pi) = (P \circ \pi) \pi^{-1} \). It follows that \( \sigma_A \) and \( \sigma_A^{-1} \) are inverses and similarly we conclude that the induced \( Q \)-intertwinings on \( A \) are inverses. Hence, \( (A, \sigma_A) \) is an object of \( \mathcal{C}(P, Q) \) and \( \pi : D \to (A, \sigma_A) \) a cocone of the diagonal. To demonstrate that \( (A, \sigma_A) \) is a colimit, consider another cocone \( \kappa : D \to (B, \sigma_B) \). Since \( A \) is a colimit of \( U \circ D \), there exists a unique morphism \( t : A \to B \) such that \( U \kappa = t(U \pi) \). What remains to be shown is whether \( t \) commutes with the \( P \)-intertwinings of \( A \) and \( B \) which follows from the universality of \( A \circ P \) and the calculation below

\[
(P \circ t) \sigma_A(U \pi \circ P) = (P \circ t)(P \circ U \pi) = (P \circ U \kappa) = \sigma_B(U \kappa \circ P) = \sigma_B(t \circ P)(U \pi \circ P)
\]

Hence \( (P \circ t) \sigma_A = \sigma_B(t \circ P) \) and thereby, \( (A, \sigma_A) \) is a colimit of the original diagram \( D \). \( \square \)

Corollary 4.8. If \( \mathcal{C} \) is a rigid abelian category, then \( \mathcal{C}(P, Q) \) is rigid and abelian and the forgetful functor \( U \) is exact.

Proof. Since in a rigid category \( X \circ - \) and \( - \circ X \) preserve limits as well as colimits, for arbitrary objects \( X \) in \( \mathcal{C} \), a symmetric proof to that of Lemma 4.7 demonstrates that \( U \) creates limits. Furthermore, the additive structure of \( \mathcal{C} \) lifts trivially and since \( U \) creates all finite limits and colimits, \( \mathcal{C}(P, Q) \) becomes abelian and \( U \) exact. \( \square \)

We conclude this section with a small examples of what the category \( \mathcal{C}(P, Q) \) looks like, for a well-known monoidal category.

Example 4.9. Let \( G \) be a finite group and consider the monoidal category of finite dimensional \( G \)-graded vector spaces \( \text{vec}_G \) with the usual monoidal structure, as described in Example 2.3.6 of [9] and denote its simple objects by \( V_g \) where \( g \in G \). Then for any \( g \in G \), \( P = V_g \) is pivotal and \( Q = V_{g^{-1}} \) and the evaluation and co-valuation morphism are trivial identity morphisms of the ground field. Hence, the category \( \mathcal{C}(V_g, V_{g^{-1}}) \) has pairs \( (\bigoplus_{i=1}^n V_{h_i}, \sigma) \) as objects, where \( n \in \mathbb{N}, h_i \in G \) and \( \sigma : \bigoplus_{i=1}^n V_{h_i} \to \bigoplus_{i=1}^n V_{g h_i} \) is a \( G \)-graded isomorphism. Due to the trivial form of the the duality morphisms in \( \text{vec}_G \), for any such \( \sigma, \sigma^{-1} \) will automatically be inverses. Note that for any object \( (\bigoplus_{i=1}^n V_{h_i}, \sigma) \), the set \( \{h_i, 1 \leq i \leq n\} \) is a disjoint union of orbits of the conjugation action of \( g \) on \( G \).
5 Resulting Hopf Monads

In this section, we assume that the category $C$ is closed and has countable colimits. Thereby, $\otimes$ commutes with colimits and the category of endofunctors $End(C)$ also has countable colimits. Utilising this, we construct the Hopf monad whose Eilenberg-Moore category recovers $C(P, Q)$.

Observe that for a pair $(X, \sigma)$ in $C(P, Q)$, we can view $\sigma$ and $\sigma^{-1}$ as certain actions of the functors $Q \otimes - \otimes P$ and $P \otimes - \otimes Q$ on $X$:

$$Q \otimes X \otimes P \xrightarrow{(ev \otimes X)(Q \otimes \sigma)} X$$

Observe that for any pair $(X, \sigma)$ in $C(P, Q)$ we can translate the mentioned actions in terms of the induced $Q$-interwinings since $(X \otimes ev)(\sigma \otimes P) = (ev \otimes X)(Q \otimes \sigma)$ and $(ev \otimes X)(P \otimes \sigma^{-1}) = (X \otimes ev)(\sigma^{-1} \otimes Q)$.

Conversely, when provided with two morphisms $\alpha : Q \otimes X \otimes P \to X$ and $\beta : P \otimes X \otimes Q \to X$, we can recover right and left $P$-interwinings as below:

$$X \otimes P \xrightarrow{(P \otimes \alpha)(\coev \otimes X \otimes P)} P \otimes X$$

If we want the induced $P$-interwinings of $\alpha$ and $\beta$ to be inverses, we need the following equalities to hold:

$$ev \otimes X = \beta(P \otimes \alpha \otimes Q)(P \otimes Q \otimes X \otimes \coev) : P \otimes Q \otimes X \to X$$

(7)

$$X \otimes ev = \alpha(Q \otimes \beta \otimes P)(\coev \otimes X \otimes Q \otimes P) : X \otimes Q \otimes P \to X$$

(8)

Similarly, $\alpha$ and $\beta$ induce $Q$-interwinings,

$$X \otimes X \xrightarrow{(\beta \otimes Q)(P \otimes X \otimes \coev)} X \otimes P$$

If in order for the induced $Q$-interwinings to be inverses, we require the following equalities to hold:

$$ev \otimes X = \alpha(Q \otimes \beta \otimes P)(Q \otimes P \otimes X \otimes \coev) : Q \otimes P \otimes X \to X$$

(9)

$$X \otimes ev = \beta(P \otimes \alpha \otimes Q)(\coev \otimes X \otimes P \otimes Q) : X \otimes P \otimes Q \to X$$

(10)

With this view of $P$-interwinings in mind, we construct the left adjoint functor to $U$.

Define the endofunctors $F_+, F_- : C \to C$ by

$$F_+(X) = Q \otimes X \otimes P, \quad F_-(X) = P \otimes X \otimes Q$$

Let the endofunctor $F^*$ be defined as the coproduct

$$F^* = \coprod_{n \in \mathbb{N}_0, (i_1, i_2, \ldots, i_n) \in \{-, +\}^n} F_{i_1}F_{i_2} \cdots F_{i_n}$$

where the term $F_{i_1}F_{i_2} \cdots F_{i_n}$ at $n = 0$, is just the identity functor $\text{id}_C$. For arbitrary $n \in \mathbb{N}$ and $(i_1, i_2, \ldots, i_n) \in \{-, +\}^n$, we denote $F_{i_1}F_{i_2} \cdots F_{i_n}$ by $F_{i_1i_2 \cdots i_n}$ and the respective natural transformations $F_{i_1i_2 \cdots i_n} \Rightarrow F^*$ by $\iota_{i_1i_2 \cdots i_n}$. We denote the additional natural transformation $\iota_{i_1i_2 \cdots i_n} \Rightarrow F^*$ by $\iota_0$. Hence, for any $F_{i_1i_2 \cdots i_n}$ we have four parallel pairs:

$$P \otimes Q \otimes F_{i_1i_2 \cdots i_n} \xrightarrow{(P \otimes Q \otimes F_{i_1i_2 \cdots i_n} \otimes \coev)} F^*$$

(11)
As we demonstrated at the beginning of this section, we only need to check that equalities 7, 8, 9 and 10 hold for the defined actions $\Psi_{i_1,i_2,\ldots,i_n}$ with the necessary $\psi$-intermeanings, but we must show that the induced $P$-intermeanings belongs to $C(P,Q)$.

**Lemma 5.1.** For any object $X$ in $C(P,Q)$, the pair 

$$(T(X), (P \otimes \alpha_X)(\text{coev} \otimes T(X) \otimes P))$$

belongs to $C(P,Q)$.

**Proof.** As we demonstrated at the beginning of this section, we only need to check that equalities (7), (8), (9) and (10) hold for the defined actions $\alpha_X$ and $\beta_X$. Consider equation (7). We observe that by construction

$$(\text{coev} \otimes T)(P \otimes Q \otimes \psi_{i_1,i_2,\ldots,i_n}) = \psi_{-,i_1,i_2,\ldots,i_n}(P \otimes Q \otimes F_{i_1,i_2,\ldots,i_n} \otimes \text{coev}) = \beta(P \otimes \psi_{i_1,i_2,\ldots,i_n} \otimes Q)(P \otimes Q \otimes F_{i_1,i_2,\ldots,i_n} \otimes \text{coev}) = \beta(P \otimes \alpha \otimes Q)(P \otimes Q \otimes \psi_{i_1,i_2,\ldots,i_n} \otimes P \otimes Q)(P \otimes Q \otimes F_{i_1,i_2,\ldots,i_n} \otimes \text{coev}) = \beta(P \otimes \alpha \otimes Q)(P \otimes Q \otimes T \otimes \text{coev})(P \otimes Q \otimes \psi_{i_1,i_2,\ldots,i_n})$$

and by the universal property of the functor $T$, we conclude that

$$(\text{coev} \otimes T) = \beta(P \otimes \alpha \otimes Q)(P \otimes Q \otimes T \otimes \text{coev})$$

It should be clear that (7), (8) and (10) follow in a similar manner from the construction of the functor $T$, and we leave the details to the reader.

We denote the natural transformation $(P \otimes \alpha)(\text{coev} \otimes T) : T \otimes P \Rightarrow P \otimes T$ by $\sigma^T$. Hence, we define the functor $F : C \to C(P,Q)$ by $F(X) = (T(X), \sigma^T_X)$ for objects $X$ of $C$ and $F(f) = T(f)$ for morphisms $f$ of $C$. By construction, $F$ is functorial.
Theorem 5.2. The functor $F$ as defined above is left adjoint to $U$, making $F \dashv U$ a Hopf adjunction.

Proof. We provide the unit and counit of the adjunction explicitly and show they satisfy the necessary conditions. The unit of the adjunction was present in our construction as $
u := \psi_0 : \text{id}_C \Rightarrow UF = T$. For the counit, consider a pair $(X, \sigma)$ in $\mathcal{C}(P, Q)$ and denote its induced actions $(ev \otimes X)(Q \otimes \sigma)$ and $(X \otimes ev)(\sigma^{-1} \otimes Q)$ by $\alpha_\sigma : F_1(X) \to X$ and $\beta_\sigma : F^{-1}(X) \to X$, respectively. We can define $\theta_{1,n} : F_{1,n}(X) \to X$, for arbitrary $n \in \mathbb{N}$ and $(i_1, i_2, \ldots, i_n) \in \{-, +\}^n$, by iteratively applying $\alpha_\sigma$ and $\beta_\sigma$ so that $\theta_{1,i_1} = \alpha_\sigma(Q \otimes \theta_{1,i_2} \otimes \ldots \otimes \theta_{1,i_n} \otimes P)$ and $\theta_{1, -i_1, i_2, \ldots, i_n} = \beta_\sigma(P \otimes \theta_{1,i_2, i_3, \ldots, i_n} \otimes Q)$, where $\theta_{1, \ldots, i_n} = \alpha_\sigma$ and $\theta_{-1, \ldots, i_n} = \beta_\sigma$. Together with $\theta_0 = \text{id}_X$, we have a family of morphisms from $F_{1,n}(X)$ to $X$, which must factorise through $F^n(X)$. We denote the unique morphism $F^n(X) \to X$ by $\theta^n$ and observe that the family of morphisms described commute with the parallel pairs $\square$ and $\square$ e.g. for the parallel pair $\square$

$$\theta^*_{1,-, i_1, i_2, \ldots, i_n} : (P \otimes Q \otimes F_{1,i_2, \ldots, i_n} \otimes \text{coev})$$

$$= \theta_{1, -, i_1, i_2, \ldots, i_n} : (P \otimes Q \otimes F_{1,i_2, \ldots, i_n} \otimes \text{coev})$$

$$= \beta(P \otimes Q \otimes \theta_{1,i_2, \ldots, i_n} \otimes Q \otimes P)(P \otimes Q \otimes F_{1,i_2, \ldots, i_n} \otimes \text{coev})$$

$$= \beta(P \otimes (ev \otimes X)(Q \otimes \sigma) \otimes Q)(P \otimes Q \otimes X \otimes \text{coev})(P \otimes Q \otimes \theta_{1,i_2, \ldots, i_n})$$

$$= (X \otimes ev)(\sigma^{-1} \otimes Q)(P \otimes \text{coev})(P \otimes Q \otimes \theta_{1,i_2, \ldots, i_n})$$

$$= (ev \otimes X)(P \otimes Q \otimes \theta_{1,i_2, \ldots, i_n}) = \theta^n(ev \otimes i_{1,i_2, \ldots, i_n})$$

Similar calculations follow for parallel pairs $\square$ and $\square$ from the properties of $\sigma$. Hence, by the universal property of $T(X)$, we conclude that there exists a unique morphism $\theta(X, \sigma) : T(X) \to X$ such that $\theta_X \psi_{1,i_2, \ldots, i_n} = \theta_{1,i_2, \ldots, i_n}$. In fact, $\theta$ is a morphism between $(T(X), \sigma^T_X)$ and $(X, \sigma)$ in $\mathcal{C}(P, Q)$ i.e. $(P \otimes \theta_X)\sigma^T_X = \sigma(\theta_X \otimes P)$ holds: this is equivalent to $\alpha_\sigma(Q \otimes \theta_X \otimes P) = \theta_X \alpha_X$ which holds by definition of $\theta_X$. By universality of $T$, $\theta$ is natural and we have described a natural transformation $\theta : FU \Rightarrow \text{id}_{\mathcal{C}(P,Q)}$. The triangle identities for the unit and counit $\nu$ and $\theta$ follow trivially by the universal property of $\theta$ since $\theta(T(X), \sigma^T_X)\nu_X = \text{id}_X$ by definition. \qed

Corollary 5.3. The adjunction $F \dashv U$ is monadic and the monad $(T, U, \theta_F, \nu)$ is a Hopf monad.

Proof. By Lemma $\square$ and Beck’s Theorem, Theorem $\square$ the adjunction is monadic. By Theorem $\square$ the induced monad $(T, \epsilon, \eta)$ is a Hopf monad. \qed

At this point we would like to take a step back and look at the particular structure of $T$ as a bimonad. At the beginning of the section, we described how a $P$-intertwining $\sigma$ on an object $X$ is equivalent to a pair of suitable actions $\alpha_\sigma$ and $\beta_\sigma$ on $X$. In the proof of Theorem $\square$ we showed that for any object $(X, \sigma)$ in $\mathcal{C}(P, Q)$, there exists a unique morphism $\theta(X, \sigma) : T(X) \to X$ so that $\theta(X, \sigma) \psi_{1,i_2, \ldots, i_n} = \theta_{1,i_2, \ldots, i_n}$, where $\theta_{1,i_2, \ldots, i_n}$ are just the iterative applications of $\alpha_\sigma$ and $\beta_\sigma$. In particular, for any object $X$ of $\mathcal{C}(T(X))$ has naturally suitable actions $\alpha_X$ and $\beta_X$ which satisfy $\alpha_X(Q \otimes (\psi_{1,i_2, \ldots, i_n}X \otimes P) = (\psi_{1,i_2, \ldots, i_n}X \otimes Q \otimes ev)(\psi_{1,i_2, \ldots, i_n}X \otimes Q) = (\psi_{-1,i_2, \ldots, i_n}X \otimes Q)$. Hence, for the pair $F(X) = (T(X), \sigma^T_X)$, $\theta_F(X) : TT(X) \to T(X)$ is the unique morphism such that

$$\theta_F(X)(\psi_{1,i_2, \ldots, i_n})T(X)F_{1,i_2, \ldots, i_n}(\psi_{j_1, \ldots, j_m}X) = (\psi_{1,i_2, \ldots, i_n,j_1, \ldots, j_m})X$$
for arbitrary non-negative integers \( n, m \) and \( i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_m \in \{+, -\}. \) As previously mentioned \( \nu = \psi_0 : \text{id}_C \to T. \) The comonoidal structure of \( T \) arises directly from the monoidal structure of \( \mathcal{C}(P, Q) \). Observe that for pairs \( (X, \sigma) \) and \( (Y, \tau) \), the induced action \( \alpha_{\sigma \otimes \tau} \) on \( A \otimes B \) is the composition \( (\alpha_{\sigma} \otimes \beta_{\tau})(P \otimes X \otimes \text{coev} \otimes Y \otimes Q) \). With this in mind, we observe that the comonoidal structure of \( T \), \( T_2 : T(\cdot \otimes \cdot) \to T(\cdot) \otimes T(\cdot), \) is the unique morphism such that

\[
T_2\psi_{1, \ldots, i_n} = (\psi_{1, \ldots, i_n} \otimes \psi_{1, \ldots, i_n})F_{i_1, \ldots, i_n}(- \otimes \text{coev}_{i_1, \ldots, i_n} \otimes -)
\]

where \( \text{coev}_{i_1, \ldots, i_n} : 1 \to F_{-i_1, \ldots, -i_n}(1) \) are iteratively defined by \( \text{coev}_{+, i_1, i_2, \ldots, i_n} = F_{-i_1, \ldots, -i_n}(\text{coev}) \) and \( \text{coev}_{-, i_1, i_2, \ldots, i_n} = F_{i_1, \ldots, -i_n}(\text{coev}) \) where \( \text{coev}_+ = \text{coev} \) and \( \text{coev}_- = \text{coev} \). Recall that the \( P \)-intertwining making \( 1 \) the unit of the monoidal structure in \( \mathcal{C}(P, Q) \) is simply the identity morphisms \( \text{id}_P \) and its induced actions are \( \alpha_{\text{id}_P} = \text{ev} \) and \( \beta_{\text{id}_P} = \text{ev} \). Hence, morphism \( T_0 : T(1) \to 1 \) is the unique morphism so that \( T_0\psi_{1, i_2, \ldots, i_n} = \text{ev}_{i_1, i_2, \ldots, i_n} \) where \( \text{ev}_{i_1, i_2, \ldots, i_n} : F_{i_1, \ldots, i_n}(1) \to 1 \) is defined iteratively by \( \text{ev}_+ = \text{ev}_- = \text{ev} \).

Theorem 5.4. The Hopf monad \( T \) is augmented if and only if there exists a braiding \( \lambda : P \otimes \text{id}_C \Rightarrow \text{id}_C \otimes P \) and \( \chi : Q \otimes \text{id}_C \Rightarrow \text{id}_C \otimes Q \) so that \( (P, \lambda) \) and \( (Q, \chi) \) are object in \( Z(\mathcal{C}) \) and \( \text{coev}, \text{ev}, \text{coev} \) and \( \text{ev} \) are morphisms in \( Z(\mathcal{C}) \), making \( (P, \lambda) \) and \( (Q, \chi) \) a pivotal pair in \( Z(\mathcal{C}) \).

Proof. Assume \( T \) is augmented and there exists a Hopf monad morphism \( \xi : T \Rightarrow \text{id}_C. \) Then \( \xi \) satisfies \( \xi_{\text{id}_C} = \text{id}_C, \xi_\theta = \xi T(\xi), (\xi \otimes \xi)T_2 = \xi_{\text{id}_C} \) and \( \xi_1 = T_0. \)

We consider the natural transformation \( \lambda := (\xi \otimes P)(P \otimes \text{id}_C \otimes \text{coev}). \) We now demonstrate that \( \lambda \) is a braiding as required. First, observe that \( \lambda \) is invertible and \( \lambda^{-1} : (P \otimes \text{id}_C)(\text{coev} \otimes \text{id}_C \otimes P) \) provides its inverse:

\[
\lambda \lambda^{-1} = (\xi \psi_- \otimes P)(P \otimes \text{id}_C \otimes \text{coev})(P \otimes \xi \psi_+)(\text{coev} \otimes \text{id}_C \otimes P)
= (\xi \psi_- \otimes P)(F_-(\xi \psi_+)(\text{coev} \otimes \text{id}_C \otimes P \otimes \text{coev})
= (\xi \psi_- \otimes P)(F_-(\xi \psi_+)(\text{coev} \otimes \text{id}_C \otimes P \otimes \text{coev})
= (\xi \psi_- \otimes P)(\text{coev} \otimes \text{id}_C \otimes P \otimes \text{coev})
= (\xi \psi_- \otimes P)(\text{id}_C \otimes P \otimes \text{coev}) = \text{id}_C
\]

The calculation showing \( \lambda^{-1} \lambda = \text{id}_C \) is completely symmetric and left to the reader. Observe that the braiding conditions follow from the properties of \( \xi \). We can directly deduce that \( \lambda_1 = \text{id}_P \) since

\[
(\xi \psi_-)t_1 \otimes P)(P \otimes 1 \otimes \text{coev}) = (T_0(\psi_-)t_1 \otimes P)(P \otimes 1 \otimes \text{coev})
= (\text{ev} \otimes P)(P \otimes 1 \otimes \text{coev}) = \text{id}_P
\]

and \( \lambda_{X \otimes Y} = (X \otimes \lambda_Y)(\lambda_X \otimes Y) \) hold for any arbitrary pair of objects \( X \) and \( Y \) in \( \mathcal{C} \) since

\[
\lambda_{X \otimes Y} = (\xi_{X \otimes Y}(\psi_-)X \otimes Y) \otimes P)(P \otimes X \otimes Y \otimes \text{coev})
= ((\xi_X \otimes \xi_Y)T_2(X, Y)(\psi_-)X \otimes Y) \otimes P)(P \otimes X \otimes Y \otimes \text{coev})
= (\xi_X(\psi_-)X \otimes \xi_Y(\psi_-)Y \otimes P)(P \otimes X \otimes \text{coev} \otimes Y \otimes \text{coev})
= (X \otimes \xi_Y(\psi_-)Y)(X \otimes P \otimes Y \otimes \text{coev})(\xi_X(\psi_-)X \otimes Y)(P \otimes X \otimes \text{coev} \otimes Y)
\]
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\[(X \otimes \lambda_Y)(\lambda_X \otimes Y)\]

Hence, \((P, \lambda)\) is an object in the center of \(C\). In the same manner, one can deduce that 
\[\chi := (\xi P_+ \otimes Q)(Q \otimes \text{id}_C \otimes \text{coev})\]
a braiding with \(\chi^{-1} := (Q \otimes \xi P_+)(\text{coev} \otimes \text{id}_C \otimes Q)\) as its inverse. What remains to be checked is whether \(\text{coev}, ev, \text{coev}\) and \(ev\) are morphisms in \(Z(C)\) and with the braidings of 1, \(P \otimes Q\) and \(Q \otimes P\). For \(ev\) we must demonstrate that \(ev \otimes \text{id}_C = (\text{id}_C \otimes ev)(\chi \otimes P)(Q \otimes \lambda)\) which follows by considering the parallel pair \([13]\)

\[
(id_C \otimes ev)(\chi \otimes P)(Q \otimes \lambda) = (id_C \otimes ev)(\chi \otimes P)(Q \otimes \xi - \otimes P) \\
(Q \otimes P \otimes id_C \otimes \text{coev})
\]

\[
= (id_C \otimes ev)(\xi P_+ \otimes Q \otimes P)(Q \otimes \xi P_- \otimes \text{coev} \otimes P)(Q \otimes P \otimes id_C \otimes \text{coev})
\]

\[
= \xi T(\xi P_+ \otimes F_+(\psi_+))(Q \otimes P \otimes id_C \otimes \text{coev})
\]

\[
= \xi \theta(\psi_+ \otimes \text{coev})(\chi \otimes P \otimes id_C \otimes \text{coev})
\]

\[
= \xi \psi_+(Q \otimes P \otimes id_C \otimes \text{coev}) = \xi \psi_0(\text{ev} \otimes id_C) = ev \otimes id_C
\]

For \(\text{coev}\) we must show that \((\lambda \otimes Q)(P \otimes \chi)(\text{coev} \otimes id_C) = id_C \otimes \text{coev}\) which follows by considering the parallel pair \([14]\)

\[
(\lambda \otimes Q)(P \otimes \chi)(\text{coev} \otimes id_C) = (\xi \psi_- \otimes P \otimes Q)(P \otimes id_C \otimes \text{coev} \otimes Q)
\]

\[
(P \otimes \xi P_+ \otimes Q)(P \otimes id_C \otimes \text{coev} \otimes Q)
\]

\[
= (\xi \psi_- \otimes P \otimes F_-(\psi_+))(Q \otimes P \otimes F_+ \otimes \text{coev} \otimes Q)(P \otimes id_C \otimes \text{coev})
\]

\[
= (\xi \psi_- \otimes P \otimes \text{coev} \otimes id_C \otimes P \otimes \text{coev} \otimes Q)(id_C \otimes \text{coev})
\]

\[
= (\xi \psi_0 \otimes P \otimes Q)(\text{id}_C \otimes \text{coev} = id_C \otimes \text{coev})
\]

In a symmetric fashion, one can show that \(\text{coev}\) and \(ev\) are also isomorphisms in \(Z(C)\) by looking at \([11]\) and \([12]\).

\((\Leftarrow)\) Assume there exist braidings \(\lambda : P \otimes id_C \Rightarrow id_C \otimes P\) and \(\chi : Q \otimes id_C \Rightarrow id_C \otimes Q\) making \((P, \lambda)\) and \((Q, \chi)\) objects in \(Z(C)\), such that \(\text{coev}, ev, \text{coev}\) and \(ev\) are morphisms in \(Z(C)\). We can iteratively define the natural transformations \(\xi_{i_1 \ldots i_n} : F_{i_1 \ldots i_n} \Rightarrow id_C\) by

\[
\xi_{+, i_1 \ldots i_n} = (id_C \otimes ev)(\chi \otimes P)F_+(\xi_{i_1 \ldots i_n})
\]

\[
\xi_{-, i_1 \ldots i_n} = (id_C \otimes ev)(\lambda \otimes Q)F_-(\xi_{i_1 \ldots i_n})
\]

where \(\xi_+ = (id_C \otimes ev)(\chi \otimes P)\) and \(\xi_- = (id_C \otimes ev)(\lambda \otimes Q)\) and \(\xi_0 = id_C\). Since \(ev\) and \(ev\) are commute with the braidings, then \(\xi_+ = (ev \otimes id_C)(Q \otimes \lambda^{-1})\) and \(\xi_- = (ev \otimes id_C)(P \otimes \chi^{-1})\). It is straightforward to check that \(\xi_{i_1 \ldots i_n}\) commute with the parallel pairs \([11]\) \([12]\) \([13]\) and \([14]\) and therefore induce a unique morphism \(\xi : T \Rightarrow id_C\). Explicitly, for parallel pair \([11]\)

\[
\xi_{-, i_1 \ldots i_n}(P \otimes Q \otimes F_{i_1 \ldots i_n} \otimes \text{coev}) = (id_C \otimes ev)(\lambda \otimes Q)(P \otimes id_C \otimes ev \otimes Q)
\]

\[
(P \otimes \chi \otimes P \otimes Q)F_{-}(\xi_{i_1 \ldots i_n})(P \otimes Q \otimes F_{i_1 \ldots i_n} \otimes \text{coev})
\]

\[
= (ev \otimes id_C)(P \otimes Q \otimes \xi_{i_1 \ldots i_n}) = \xi_{i_1 \ldots i_n}(ev \otimes id_C)
\]

Similar arguments follow for \(\xi_{i_1 \ldots i_n}\) commuting with parallel pairs \([12]\) \([13]\) and \([14]\) Observe that by definition \(\xi \nu = id_C\) and \(\xi_1 = T_0\) since \(\xi_{i_1 \ldots i_n}\) = \(ev_{i_1 \ldots i_n}\).
As we recalled in Theorem 2.3, $S$ commute with the evaluation and coevaluation morphisms. Moreover, we have an isomorphism of bimonads

\[ (\xi \otimes \xi) T_2 (\psi_{i_1, \ldots, i_n}) \circ \sigma = (\xi \psi_{i_1, \ldots, i_n} \otimes \xi \psi_{i_1, \ldots, i_n}) F_{i_1, \ldots, i_n} (- \otimes \coev_{i_1, \ldots, i_n} \otimes -) \]

and from the universal properties of $TT$ and $T(- \otimes -)$, we conclude that $\xi \theta = \xi T(\xi)$ and $(\xi \otimes \xi) T_2 = \xi - \otimes -$ and thereby, $\xi$ is a bimonad morphism.

Recall that an augmentation on a Hopf monad is equivalent to a central Hopf algebra structure on $T(1)$. To be more precise, $T(1)$ together with $T_2(1, 1) : T(1) \to T(1) \otimes T(1)$ and $T_0 : T(1) \to 1$ form a comonoid in $C$. Additionally, if $(P, \lambda)$ and $(Q, \chi)$ is a pivotal pair in the center of $C$, then we have a monoid structure on $T(1)$, $m : T(1) \otimes T(1) \to T(1)$, $\psi_0(1) : 1 \to T(1)$ where $m$ is the unique morphism satisfying

\[ m(\psi_{i_1, \ldots, i_n}(1) \otimes \psi_{j_1, \ldots, j_m}(1)) = \psi_{i_1, \ldots, i_n, j_1, \ldots, j_m}(1) (Q_{i_1} \odot \ldots \odot Q_{i_n} \odot (Q_{j_1} \otimes \ldots \otimes Q_{j_m} \otimes P_{j_1} \otimes \ldots \otimes P_{j_m})) \]

where we denote $P_+ = P$, $P_- = Q_+ = Q$ and $P_- = P$ so that $F_\pm (-) = Q_+ \otimes - \otimes P_+$ and $\lambda_{i_1, \ldots, i_n} : P_{i_1} \otimes \ldots \otimes P_{i_n} \otimes \id_C \to \id_C \otimes P_{i_1} \otimes \ldots \otimes P_{i_n}$ is the induced braiding on $P_{i_1} \otimes \ldots \otimes P_{i_n}$, where $\lambda_+ = \lambda$, $\lambda_- = \chi$ and $\lambda_{i_1, \ldots, i_n} = (\lambda_{i_1} \otimes P_{i_2} \otimes \ldots \otimes P_{i_n})(P_{i_1} \otimes \lambda_{i_2, \ldots, i_n})$. Observe that $T(1)$ has an induced braiding $\varsigma : T(1) \otimes \id_C \Rightarrow \id_C \otimes T(1)$ satisfying

\[ \varsigma(\psi_{i_1, \ldots, i_n}(1) \otimes \id_C) = (\id_C \otimes \psi_{i_1, \ldots, i_n}(1)) \lambda_{-i_1, \ldots, -i_n, i_1, \ldots, i_n} \]

It follows that $(T(1), \varsigma)$ is an object in the center of $C$ and together with $m$, $\psi_0(1)$, $T_2(1, 1)$, $T_0$ forms as central bialgebra. Notice that the bialgebra axioms and the fact that $m$ and $\varsigma$ exists, all follow from the fact that the induced braidings from $\lambda$ and $\chi$ commute with the evaluation and coevaluation morphisms. Moreover, we have an induced isomorphism of bimonads $T : T(-) \Rightarrow T(1) \otimes -$ defined as the unique morphism satisfying

\[ T \psi_{i_1, \ldots, i_n} = (\psi_{i_1, \ldots, i_n}(1) \otimes \id_C)(Q_{i_1} \otimes \ldots \otimes Q_{i_n} \otimes \lambda_{i_1, \ldots, i_n}^{-1}) \]

As we recalled in Theorem 2.3, $T(1)$ becomes a central Hopf algebra and its antipode $S : T(1) \to T(1)$ can be recovered as the unique morphism satisfying

\[ S \psi_{i_1, \ldots, i_n} = (ev_{i_1, \ldots, i_n} \otimes \psi_{i_1, \ldots, i_n}^{-1}(1))(Q_{i_1} \otimes \ldots \otimes Q_{i_n} \otimes \lambda_{i_1, \ldots, i_n}^{-1})(F_{i_1, \ldots, i_n} \otimes -) \]

with its inverse $S^{-1} : T(1) \to T(1)$ defined as the unique morphism satisfying

\[ S^{-1} \psi_{i_1, \ldots, i_n} = (ev_{i_1, \ldots, i_n} \otimes \psi_{i_1, \ldots, i_n}^{-1}(1))(Q_{i_1} \otimes \ldots \otimes Q_{i_n} \otimes \lambda_{i_1, \ldots, i_n}^{-1})(F_{i_1, \ldots, i_n} \otimes -) \]

We now review the structure of the constructed Hopf monad on some familiar categories.
Example 5.5. Let $\mathbb{K}$ be an arbitrary field and consider its symmetric monoidal category of vector spaces $(\text{Vec}, \otimes, \mathbb{K})$, where $\otimes$ denotes the tensor product over the field. Any finite dimensional vector space is dualizable and pivotal. Since the category is symmetric, by Theorem 2.3, any Hopf monad constructed as above is augmented. Hence, the monad should arise from a Hopf algebra. Explicitly, for an $n$-dimensional vector space, the monad $F^*$ is isomorphic to $B \otimes -$, where $B$ is the free algebra $\mathbb{K} \langle f_j, e_j \mid 1 \leq i, j \leq n \rangle$ and generators $f_j$ and $e_j$ correspond to the bases of $Q \otimes P$ and $Q \otimes P$, respectively. Consequently, the monad $T$ is isomorphic to the induced monad, $H \otimes -$, where $H$ is the quotient of the algebra $B$ by relations
\[
\sum_{j=1}^{n} f_j j e_j = \sum_{j=1}^{n} e_j j f_k = \delta_{i,k}
\]
for all $1 \leq i, k \leq n$. The coproduct, counit and antipode, $\Delta, \epsilon, S$ of the Hopf algebra $H$ are defined as
\[
\Delta(\varepsilon f_k) = \sum_{j=1}^{n} f_j \otimes f_k, \quad \epsilon(f_k) = \delta_{i,k}, \quad S(f_k) = \varepsilon_k
\]
\[
\Delta(\varepsilon e_k) = \sum_{j=1}^{n} e_j \otimes e_k, \quad \epsilon(e_k) = \delta_{i,k}, \quad S(e_k) = \varepsilon_k
\]
for all $1 \leq i, k \leq n$.

Remark 5.6. The Hopf algebra constructed in the above Example can be viewed as a suitable quotient of the free Matrix Hopf algebra $\mathcal{NGL}(n)$ discussed in [21]. The free matrix bialgebra of rank $n^2$ is exactly the free algebra $\mathbb{K} \langle f_{ij} \mid 1 \leq i, j \leq n \rangle$ with the coproduct and counit defined as in Example 5.5. The Hopf envelope of this bialgebra as defined by Manin [15], would be the quotient of the $\prod_{l \in \mathbb{N}} B_l$ where $B_l = \mathbb{K} \langle f_{ij} \mid 1 \leq i, j \leq n \rangle$ by the relations $\sum_{j=1}^{n} f_{lj} j e_{l+1} = \sum_{j=1}^{n} e_{lj} j f_{l+1} = \delta_{i,k}$ for all $l \in \mathbb{N}$ and the antipode is defined as the shift $S(f_{ij}) = j f_{l+1}$. The Hopf algebra of Example 5.5 is just the quotient of this Hopf algebra by requiring the $B_l$ components to be equal for odd $l$ and similarly for even $l$, so that $S^2 = \text{id}$.

Example 5.7. If $C$ is braided with braiding $\Psi$, then $(P, \Psi_{M,-})$ is an object of $Z(C)$ and by Theorem 2.3 $T$ is augmented. In particular, $T \cong T(1) \otimes -$ where $T(1)$ is in fact a braided Hopf algebra in $C$ since the induce braiding on $T(1)$ will naturally coincide with the braiding $\Psi_{T(1),-}$.

Example 5.8. [Theorem 4.11 [10]] If $A$ is a $\mathbb{K}$-algebra and $P$ a pivotal object in the category of $A$-bimodules, $A^\text{M}_A$, the arising Hopf monad was constructed in [10]: as proven in [22], additive Hopf monads on $A^\text{M}_A$ which admit a right adjoint correspond to left Hopf algebroids over $A$, in the sense of Schauenburg [15]. We adapt the notion of [12] to describe the Hopf algebroid in question and refer the reader to [10] where the construction is described in full detail. Consider the $A \otimes \mathbb{K} A^{op}$-bimodule structure induced on $Q \otimes \mathbb{K} P$ (resp. $P \otimes \mathbb{K} Q$) where we regard $Q$ (resp. $P$) as an $A$-bimodule and $P$ (resp. $Q$) as an $A^{op}$-bimodule, where $A^{op}$ denotes the opposite algebra to $A$. We denote arbitrary elements of $Q \otimes \mathbb{K} P$ and $P \otimes \mathbb{K} Q$ by $(q, p)$ and $(p, q)$, respectively. We define the Hopf algebroid $H$ as the quotient of the free $A \otimes \mathbb{K} A^{op}$-algebra $T_{A \otimes \mathbb{K} A^{op}}(Q \otimes \mathbb{K} P \oplus P \otimes \mathbb{K} Q)$ by relations
\[
\sum_{i=1}^{n} (\omega_i, q)(x_i, p) = \text{ev}(p \otimes q) \quad \sum_{j=1}^{m} (y_j, p)(\rho_j, q) = \text{ev}(q \otimes p)
\]
for all $p \in P$ and $q \in Q$, where $\text{coev}(1) = \sum_{i=1}^{n} \omega_{i} \otimes x_{i}$ and $\text{coev}(1)$ = $\sum_{j=1}^{m} y_{j} \otimes \rho_{j}$ for positive integers $n, m$ and we denote elements of $A^{op}$ in $H$ by a line over head i.e. $a \in A \subset H$ and $\pi \in A^{op} \subset H$. The coproduct and counit of $H$, $\Delta$ and $\epsilon$, are defined by

$$\Delta(a b) = a \otimes b, \quad \epsilon(a b) = b a \quad \epsilon((q, p)) = \text{ev}(p \otimes q) \quad \epsilon((p, q)) = \text{ev}(q \otimes p)$$

$$\Delta((q, p)) = \sum_{i=1}^{n} (q, \tau_{i}) \otimes (x_{i}, p) \quad \Delta((p, q)) = \sum_{j=1}^{m} (p, y_{j}) \otimes (\rho_{j}, q)$$

In [10], we show that $H$ is not only a left Hopf algebroid in the sense of Schauenburg, but furthermore a Hopf algebroid in the sense of Böhm, and Szlachányi [5] and admits an invertible antipode $S$ acting as $S((p, q)) = (q, p)$ with $S = S^{-1}$.

6 On Generalisation to Pivotal Diagrams

By a pivotal diagram, we mean a functor $\mathbb{D} : \mathcal{J} \to C_{0}^{\text{piv}}$ from a small category $\mathcal{J}$ to the category $C_{0}^{\text{piv}}$ which has pivotal pairs $(P, Q)$ as objects and morphism all $f : P_{1} \to P_{2}$ as morphism between $(P_{1}, Q_{1})$ and $(P_{2}, P_{2})$. Hence the datum for a pivotal diagram consists of sets of pivotal pairs $(P_{i}, Q_{i})$ and pivotal morphisms $f_{j} : P_{j_{i}} \to P_{j}$ between them, where $i, j_{i}, j \in I$ and $j \in J$ for index sets $I, J$. We define the category of $\mathbb{D}$-intertwinings objects in $C$, denoted by $C(\mathbb{D})$, as follows: the objects of $C(\mathbb{D})$ are pairs $(X, \{\sigma_{i}\}_{i \in I})$, where $X$ is an object of $C$ and $\{\sigma_{i}\}_{i \in I}$ is a family of morphisms $\sigma_{i} : X \otimes P_{i} \to P_{i} \otimes X$ for $i \in I$, so that for all $i \in I$, the pair $(X, \sigma_{i})$ belongs to $C(P_{i}, Q_{i})$, and for any $j \in J$,

$$(f_{j} \otimes X)\sigma_{j_{i}} = \sigma_{j_{i}}(X \otimes f_{j}) : X \otimes P_{j_{i}} \to P_{j_{i}} \otimes X$$

holds. Morphisms between objects $(X, \{\sigma_{i}\}_{i \in I})$ and $(Y, \{\tau_{i}\}_{i \in I})$ are morphisms $f : X \to Y$ in $C$, which satisfy $\tau_{i}(f \otimes P_{1}) = (P_{1} \otimes f)\sigma_{i}$ for all $i \in I$.

Observe that $C(\mathbb{D})$ lifts the monoidal structure of $C$: we can define the tensor of two objects $(X, \{\sigma_{i}\}_{i \in I})$ and $(Y, \{\tau_{i}\}_{i \in I})$ of $C(\mathbb{D})$ as

$$(X, \{\sigma_{i}\}_{i \in I}) \otimes (Y, \{\tau_{i}\}_{i \in I}) = (X \otimes Y, \{(\sigma_{i} \otimes Y)(X \otimes \tau_{i})\}_{i \in I})$$

By this definition, for any $i \in I$, the forgetful functor $U_{i} : C(\mathbb{D}) \to C(P_{i}, Q_{i})$ which sends a pair $(X, \{\sigma_{i}\}_{i \in I})$ to $(X, \sigma_{i})$, is strict monoidal. Consequently, the forgetful functor $U_{\mathbb{D}} : C(\mathbb{D}) \to C$ sending pairs $(X, \{\sigma_{i}\}_{i \in I})$ to their underlying objects, $X$, also becomes strict monoidal. We must emphasize that the monoidal structure well-defined because for any $j \in J$, $f_{j}$ commutes with the relevant $P_{i}$-intertwinings, and thereby

$$(f_{j} \otimes X \otimes Y)(\sigma_{j_{i}} \otimes Y)(X \otimes \tau_{j_{i}}) = (\sigma_{j_{i}} \otimes Y)(X \otimes f_{j} \otimes Y)(X \otimes \tau_{j_{i}})$$

$$= (\sigma_{j_{i}} \otimes Y)(X \otimes \tau_{j_{i}})(X \otimes f_{j})$$

holds.
Theorem 6.1. If \( C \) is a left (right) closed monoidal category and \( \mathbb{D} \) a pivotal diagram as described above, then \( C(\mathbb{D}) \) has a left (right) closed monoidal structure which lifts that of \( C \) and the forgetful functor \( U_{\mathbb{D}} \) is left (right) closed.

Proof. In Theorem 4.2 we have provided suitable \( P_i \)-intertwinings for inner homs of two objects in \( C(P_i, Q_i) \) and demonstrated that the unit and counits of the adjunctions commute with these intertwinings. Hence, if \( (X, \{ \sigma_i \}_{i \in I}) \) and \( (Y, \{ \tau_i \}_{i \in I}) \) are objects of \( C(\mathbb{D}) \), we only need to check whether the induced \( \mathbb{D} \)-intertwinings \( \{ \langle \sigma_i, \tau_i \rangle \}_{i \in I} \) and \( \{ \langle \sigma_i, \tau_i \rangle \}_{i \in I} \) commute with morphisms \( f_j \) so that \( \{ [X, Y]^i, \{ \langle \sigma_i, \tau_i \rangle \}_{i \in I} \} \) and \( \{ [X, Y]^r, \{ \langle \sigma_i, \tau_i \rangle \}_{i \in I} \} \) provide inner homs in \( C(\mathbb{D}) \), which lift the closed structure of \( C \). Let \( j \in J \), then

\[
\langle \sigma_j, \tau_j \rangle ([X, Y]^i f_j) = (P_j, [A, (\text{ev}_j, B)](Q_j, \tau_j)(Q_j, \sigma_j^A P_j))([X, Y]^i [A, B]^i \sigma_j^{-1}])
\]

holds, where \( \langle X, Y \rangle^r f_j := (\text{ev}_j, Q_j, f_j, Q_j, \text{coev}_j) \). A similar computation follows for \( \{ \langle \sigma_i, \tau_i \rangle \}_{i \in I} \) and

\[
\langle \sigma_j, \tau_j \rangle ([X, Y]^r f_j) = (P_j, [A, (\text{ev}_j, B)](Q_j, \tau_j)(Q_j, \sigma_j^A P_j)(\sigma_j^{-1} [A, B]^r P_j))
\]

holds. We are using the fact that \( \sigma_j^{-1} (\text{ev}_j A) = (\text{ev}_j A) \sigma_j^{-1} \) which follows from \( (\text{ev}_j A) \sigma_j^{-1} = \sigma_j^{-1} (\text{ev}_j A) \). 

As in Section 5, one can construct the relevant Hopf monad for a pivotal diagram, when suitable colimits exist. The monad will be a quotient of the coproduct of \( T_i \), where \( T_i \) are the respective Hopf monads of each pair \( (P_i, Q_i) \). We would also like
to point out that as mentioned in Remark 4.6, if one considers the pivotal diagram $D$, consisting of the object $(P, Q)$ and the morphism $g_P : (P, Q) \to (P, Q)$ in a pivotal category $C$, the obtained category $C(D)$ will lift the pivotal structure of $C$.

7 Diagrams and Proof of Theorem 4.2

In this section, we provide the necessary diagrams for the proof of Theorem 4.2.

Figure 1: Proof of unit commuting with $P$-intertwinings

Figure 2: Proof of counit commuting with $P$-intertwinings
Figure 3: Proof of \( (\sigma_A, \sigma_B)^{-1} \cdot (\sigma_A, \sigma_B) = id_{A,B} \)
Figure 4: Proof of $\langle \sigma_A, \sigma_B \rangle |_{\langle \sigma_A, \sigma_B \rangle^{-1}} = \text{id}_{P[A,B]}$
Figure 5: Proof of $\langle \sigma_A, \sigma_B \rangle^{-1} \langle \sigma_A, \sigma_B \rangle = \text{id}_{Q[A,B]}$
Figure 6: Proof of \( (\sigma_A, \sigma_B) \circ (\sigma_A, \sigma_B) \circ^{-1} = \text{id}_{[A,B]Q} \)
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