On the Extremum Control Problem with Pointwise Observation for a Parabolic Equation

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Abstract—In this paper we consider a control problem with pointwise observation for a one-dimensional parabolic equation which arises in a mathematical model of climate control in industrial greenhouses. We study a general equation with variable diffusion coefficient, convection coefficient, and depletion potential. For the extremum problem of minimizing an integral weighted quadratic cost functional, we establish the existence and uniqueness of a minimizing function. We also study exact controllability and dense controllability of the problem. Necessary conditions for an extremum are obtained, and qualitative properties of the minimizing function are studied.

Keywords: parabolic equation, mixed problem, pointwise observation, extremum problem, exact controllability, dense controllability, necessary condition

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We consider the following mixed problem for an equation with a convective term and a depletion potential:

\begin{equation}
\frac{\partial}{\partial t}u(x,t) = \alpha(x,t) \frac{\partial}{\partial x} u(x,t) + h(x,t), \quad (x,t) \in (0,1) \times (0,T), \quad \alpha > 0,
\end{equation}

\begin{equation}
\alpha(t) = \varphi(t), \quad \alpha(1,t) = \psi(t), \quad t \in (0,T),
\end{equation}

\begin{equation}
u(x,0) = \xi(x), \quad x \in (0,1),
\end{equation}

where \( \alpha, h, \) and \( \xi \) are sufficiently smooth functions in \( (0,1) \times (0,T) \), \( \alpha(t) \in L(0,1) \), \( \psi \in W^1(0,1) \), and \( \xi \in L^2(0,1) \).

We study a control problem with pointwise observation. Namely, the task is to make the temperature \( u(x_0,t) \) at some point \( x_0 \) close to a given function \( \xi(t) \in L^1(0,T) \) on the whole interval \( (0,T) \) by controlling the temperature \( \varphi \) at the left endpoint of the interval (the functions \( \xi \) and \( \psi \) are assumed to be fixed). Let \( \Phi \subset W^1(0,T) \) be a set of control functions \( \varphi \), and let \( Z \subset L^2(0,T) \) be a set of target functions \( z \). In what follows, the set \( \Phi \) is assumed to be nonempty, closed, and convex. The quality of the control is estimated by the cost functional

\begin{equation}
J[z,\varphi,\rho] = \int_0^1 (u_\varphi(x_0,t) - z(t))^2 \rho(t) dt,
\end{equation}

where \( u_\varphi \) is the solution of problem (1)–(3) with a given control function \( \varphi \) and \( \rho \in L^2(0,T) \) is a weight function such that \( \inf_{\rho \in L^2(0,T)} \rho(t) = \rho_1 > 0 \).

Assuming that the functions \( z \) and \( \rho \) are fixed, we consider the minimization problem

\begin{equation}
m[z,\rho,\Phi] = \inf_{\varphi \in \Phi} J[z,\varphi,\rho].
\end{equation}

This problem arises in a climate control model for industrial greenhouses (see [1, 2]). Detailed explanations concerning the studied mathematical model can be found in [21]. Note that extremum problems for the heat equation have been considered in numerous works (see, e.g., [3–5, 7]). Problems with final observation have been studied better than others [3–6, 9]. A fairly complete survey of earlier results is given in [6], while more recent results are overviewed in [1, 9, 10, 14, 15]. In contrast to previous works on parabolic control problems, which consider problems with a
final or distributed observation [5, 7, 8, 11], we consider a pointwise observation. The type of the cost functional is also new. In this paper, we develop and generalize the results of [16–21]. Specifically, we study a more general equation in which the diffusion $a$ and convection $b$ coefficients and the depletion potential $h$ are all variable and establish qualitative properties of the corresponding minimizing function. In addition to the study of a more general equation (with variable coefficients $a = a(x, t)$ and $b = b(x, t)$ and an inhomogeneous initial condition), some new results are proved, namely, we establish qualitative properties of the minimizing function and derive necessary optimality conditions. The proof relies on the results and methods presented in [12, 13].

Definition 1 [22, p. 15 of the Russian edition]. Let $V^1_2(Q_T)$ denote the Banach space of functions $u \in W^1_2(Q_T)$ that $t \mapsto u(\cdot, t)$ is a continuous mapping from $[0, T]$ to $L_2(0, 1)$, with finite norm

$$
\|u\|_{V^1_2(Q_T)} = \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^2(0, 1)} + \|u_t(\cdot, t)\|_{L^2(0, 1)}.
$$

Let $\tilde{W}^1_2(Q_T)$ denote the set of functions $\eta \in W^1_2(Q_T)$ satisfying the conditions $\eta(x, T) = 0$ and $\eta(0, t) = 0$.

Definition 2. The solution of problem (1)–(3) is a function $u \in V^1_2(Q_T)$ satisfying the condition $u(0, t) = \varphi(t)$ and the integral identity

$$
\int_0^T (a(x, t)u_t(x, t) - b(x, t)u(x, t) - h(x, t)\eta - \eta u_t) dx dt
$$

$$
= \int_0^T \xi(x, t) u(0, t) dx + \int_0^T a(1, t)\psi(t)\eta(1, t) dt
$$

for all $\eta \in \tilde{W}^1_2(Q_T)$.

Theorem 1 ([18, 19]). Problem (1)–(3) has a unique solution $u \in V^1_2(Q_T)$, which satisfies the inequality

$$
\|u\|_{W^1_2(Q_T)} \leq C_1 (\|\varphi\|_{W^1_2(Q_0)} + \|\eta\|_{W^1_2(Q_T)} + \|\xi\|_{L^2(0, T)}),
$$

where $C_1$ is a constant independent of $\varphi$, $\psi$, and $\xi$.

Corollary 1. The mapping $[\xi, \varphi, \psi] \mapsto u$ from the space $L_2(0, 1) \times W^1_2(0, 1) \times W^1_2(0, T)$ to $V^1_2(Q_T)$ is continuous.

To derive the estimate presented below, we need the following positivity principle.

Theorem 2. Let $u$ be a solution of problem (1)–(3) with nonnegative initial and boundary functions: $\text{ess inf}_{t \in (0, T)} \varphi \geq 0$, $\text{ess sup}_{t \in (0, T)} \|\psi\| \leq 0$, $\text{ess inf}_{t \in (0, T)} \xi \geq 0$. Then the solution $u$ is also nonnegative: $\text{ess inf}_{t \in (0, T)} u \geq 0$.

Remark 1. For the functions $\varphi$ and $\psi$, we have $\text{ess inf}_{t \in (0, T)} \varphi = \min \varphi$ and $\text{ess sup}_{t \in (0, T)} \psi = \sup \psi$.

To prove Theorem 2, we change the unknown function, so problem (1)–(3) is reduced to a problem with a Robin boundary condition. Next, the required result is obtained by applying a modified barrier function method.

With the use of Theorem 2, we obtain the following estimate.

**Theorem 3.** Suppose that $a_1 \geq 0$ and $b_1 - h \geq 0$ for $(x, t) \in Q_T$, $b \geq 0$ for $(x, t) \in [0, x_0] \times [0, T]$ with $x_0 \in (0, 1)$, and $b(1, t) \leq 0$ for $t \in [0, T]$. Then the solution of problem (1)–(3) satisfies the inequality

$$
\|u(x_0, \cdot)\|_{L^2(0, T)} \leq \|w\|_{L^2(0, T)} + \frac{x_0}{a_1} (a_2 \|\psi\|_{L^2(0, T)} + \|\xi\|_{L^2(0, 1)}).
$$

Corollary 2. Suppose that the conditions of Theorem 3 for $a, b,$ and $h$ hold. Assume that $\psi = 0$ and $\xi = 0$. Then the solution of problem (1)–(3) satisfies the inequality

$$
\|u(x_0, \cdot)\|_{L^2(0, T)} \leq \|w\|_{L^2(0, T)}.
$$

Theorem 3 implies a lower bound for the norms of control functions in terms of the cost functional value.

**Theorem 4.** Suppose that the conditions of Theorem 3 are satisfied. Then

$$
\|w\|_{L^2(0, T)} \geq \max \left\{ 0, \|\xi\|_{L^2(0, 1)} - \frac{\sqrt{TJ[z, \rho, \Phi]}}{\rho_1} \right\}.
$$

Corollary 3. Suppose that the conditions of Theorem 3 for $a, b,$ and $h$ hold. Assume that $\psi = 0$ and $\xi = 0$. Then

$$
\|w\|_{L^2(0, T)} \geq \max \left\{ 0, \|\xi\|_{L^2(0, 1)} - \frac{\sqrt{TJ[z, \rho, \Phi]}}{\rho_1} \right\}.
$$

**Theorem 5.** If the set $\Phi$ is bounded, then, for any $z \in L_2(0, T)$, there exists a unique function $\varphi_0 \in \Phi$ such that

$$
m[z, \rho, \Phi] = J[z, \rho, \varphi_0].
$$

We examine properties of the minimizing function $\varphi_0$ as an element of the set $\Phi$.

**Theorem 6.** If $\Phi \subset W^1_2(0, T)$ is a bounded set, the coefficients $a, b,$ and $h$ in Eq. (1) are independent of $t$, and $m[z, \rho, \Phi] > 0$, then $\varphi_0 \in \partial \Phi$. For any convex $\Phi_1$ such that $\Phi_1 \subset \text{Int} \Phi$, it is true that $m[z, \rho, \Phi_1] > m[z, \rho, \Phi]$.

An important issue is the exact controllability of the extremum problem.

**Definition 3.** Problem (1)–(3), (5) is said to be exactly controllable from a set $\Phi$ to a set $Z$ if, for any $z \in Z$, there exists a control function $\varphi_0 \in \Phi$ such that

$$
J[z, \rho, \varphi_0] = 0.
$$

The function $\varphi_0$ is then called an exact control.
The following theorem states that the set \( Z \) of functions \( z \in L_2(0,T) \) admits exact controllability is a sufficiently "small" subset of \( L_2(0,T) \).

**Theorem 7.** The set \( Z \) of all functions \( z \in L_2(0,T) \) admitting exact controllability, i.e., such that \( J[z,p,\varphi] = 0 \) for some \( \varphi \in W^1_2(0,T) \), is a first-category set in \( L_2(0,T) \).

Now we examine the dense controllability of the problem.

**Definition 4.** Problem (1)–(3), (5) is said to be densely controllable from a set \( \Phi \) to a set \( Z \) if \( m[z,p,\varphi] = 0 \) for all \( z \in Z \).

The following theorem establishes the dense controllability of problem (1)–(3), (5) from \( W^1_2(0,T) \) to \( L_2(0,T) \).

**Theorem 8.** Suppose that the coefficients \( a, b, \) and \( h \) in (1) are independent of \( t \). Then, \( m[z,p,W^1_2(0,T)] = 0 \) for any \( z \in L_2(0,T) \).

Note that this result is proved by applying the Titchmarsh convolution theorem ([24, Chap. 11, Theorem 152]).

Another important issue is one of obtaining necessary minimum conditions for \( \varphi_0 \in \Phi \). A necessary condition can be formulated in terms of the conjugate problem. For example, if \( \varphi_0 \in \Phi \) is a minimizing function, then, for any control function \( \varphi \in \Phi \), we have

\[
\int_0^T (u_{\varphi_0}(x_0,t) - z(t)) (u_{\varphi}(x_0,t) - u_{\varphi_0}(x_0,t)) \rho(t) dt \geq 0.
\]

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