Global Lorentz gradient estimates for quasilinear equations with measure data for the strongly singular case: $1 < p \leq \frac{3n-2}{2n-1}$

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Abstract

In this paper, we study the global regularity estimates in Lorentz spaces for gradients of solutions to quasilinear elliptic equations with measure data of the form

$$\left\{ \begin{array}{l}
\text{div}(A(x, \nabla u)) = \mu \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{array} \right.$$

where $\mu$ is a finite signed Radon measure in $\Omega$, $\Omega \subset \mathbb{R}^n$ is a bounded domain such that its complement $\mathbb{R}^n \setminus \Omega$ is uniformly $p$-thick and $A$ is a Carathéodory vector valued function satisfying growth and monotonicity conditions for the strongly singular case $1 < p \leq \frac{3n-2}{2n-1}$. Our result extends the earlier results [19, 22] to the strongly singular case $1 < p \leq \frac{3n-2}{2n-1}$ and a recent result [18] by considering rough conditions on the domain $\Omega$ and the nonlinearity $A$.

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1. Introduction and main results

In this paper we study the gradient regularity of solutions to the following quasilinear elliptic equations with measure data

$$\left\{ \begin{array}{l}
\text{div}(A(x, \nabla u)) = \mu \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{array} \right.$$

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where $\Omega$ is a bounded open subset of $\mathbb{R}^n$, $(n \geq 2)$, and $\mu$ is a finite signed Radon measure in $\Omega$. The nonlinearity $A : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Carathéodory vector valued function and satisfies the following growth and monotonicity conditions:

$$|A(x, \xi)| \leq \beta |\xi|^{p-1},$$

(1.2)

$$\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \alpha \left( |\xi|^2 + |\eta|^2 \right)^{(p-2)/2} |\xi - \eta|^2$$

(1.3)

for every $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$ and a.e. $x \in \mathbb{R}^n$. Here $\alpha$ and $\beta$ are positive constants, and $p$ will be considered in the range

$$1 < p \leq \frac{3n - 2}{2n - 1}.$$  

(1.4)

As the regularity of boundary of $\Omega$ is concerned, we assume a capacity density condition on $\Omega$ which is known weaker than the Reifenberg flatness condition. More precisely, by a capacity density condition on $\Omega$ we mean the complement $\mathbb{R}^n \setminus \Omega$ is uniformly $p$-thick, that is, there exist constants $c_0, r_0 > 0$ such that for all $0 < t \leq r_0$ and all $x \in \mathbb{R}^n \setminus \Omega$ there holds

$$\text{cap}_p \left( \overline{B_t(x)} \cap (\mathbb{R}^n \setminus \Omega), B_{2t}(x) \right) \geq c_0 \text{cap}_p \left( \overline{B_t(x)}, B_{2t}(x) \right).$$  

(1.5)

Here for a compact set $K \subset B_{2t}(x)$ we define the $p$-capacity of $K$, $\text{cap}(K, B_{2t}(x))$ by

$$\text{cap}_p (K, B_{2t}(x)) = \inf \left\{ \int_{\Omega} |\nabla \varphi|^p \, dx : \varphi \in C_0^\infty (B_{2t}(x)) \text{ and } \varphi \geq \chi_K \right\},$$

where $\chi_K$ is the characteristic function of $K$. It is noticed that the domain satisfying (1.5) includes Lipschitz domains or domain satisfying a uniform exterior corkscrew condition which means that there exist constants $c_0, r_0 > 0$ such that for all $0 < t \leq r_0$ and all $x \in \mathbb{R}^n \setminus \Omega$, there is $y \in B_t(x)$ such that $B_{t/c_0}(y) \subset \mathbb{R}^n \setminus \Omega$.

Under these conditions, our main goal in this paper is to establish the following global gradient estimate in Lorentz spaces

$$\|\nabla u\|_{L^{s,t}(\Omega)} \leq C \left\| M_1(\mu)^{1/(p-1)} \right\|_{L^{s,t}(\Omega)}$$

(1.6)

where $s$ lies below or near the natural exponent $p$, i.e., $s < p + \varepsilon$ for some small $\varepsilon$ depending on $n, p, \alpha, \beta, \mu$, and $\Omega$, and $t \in (0, \infty]$. Here $M_1$ is the fractional maximal function defined for each nonnegative locally finite measure $\mu$ in $\mathbb{R}^n$ by

$$M_1(\mu)(x) = \sup_{\rho > 0} \frac{\mu(B_\rho(x))}{\rho^{n-1}}, \quad x \in \mathbb{R}^n.$$  

And the Lorentz spaces $L^{s,t}(\Omega)$, with $1 < s < \infty$, and $0 < t \leq \infty$, is the set of measurable functions $f$ on $\Omega$ such that

$$\|f\|_{L^{s,t}(\Omega)} = \left[ s \int_0^\infty (\lambda^s \{ x \in \Omega : |f(x)| > \lambda \})^{t/s} \frac{d\lambda}{\lambda} \right]^{1/t} < \infty.$$
if \( t \neq \infty \). It is also noticed that if \( s = t \) then the Lorentz space \( L^{s,s} (\Omega) \) is the usual Lebesgue space \( L^s (\Omega) \).

If \( t = \infty \) the space \( L^{s,\infty} (\Omega) \) is the weak \( L^s \) or Marcinkiewicz space with quasi-norm

\[
\| f \| = \sup_{\lambda > 0} \lambda \left| \{ x \in \Omega : |f(x)| > \lambda \} \right|^{1/s}.
\]

For \( 1 < r < s < \infty \) then one has

\[
L^s (\Omega) \subset L^{s,\infty} (\Omega) \subset L^r (\Omega).
\]

It is worth mentioning that the local version of (1.6) was first obtained by G. Mingione in [14] for the regular case \( 2 \leq p \leq n \) and then extended several authors in recent. For example, Nguyen Cong Phuc [19] obtained the global Lorentz estimate for solutions of (1.1) in the ‘possibly singular’ case \( 2 - 1/n < p \leq n \) by using a capacity-density condition on \( \Omega \) and the assumptions (1.2)-(1.3). Therein, the author proved the \( L^{s,t} (\Omega) \) estimates of solution for all \( 0 < s < p + \varepsilon, 0 < t \leq \infty \) for some \( \varepsilon > 0 \). Afterward a similar result is obtained by M. P. Tran [22] to the singular case \( \frac{2n - 2}{n - 2} < p \leq 2 - \frac{1}{n} \) with the rough conditions on the domain \( \Omega \) and the nonlinearity \( A \) in which the author exploits some comparison estimates in [17]. In [17] by using the good-\( \lambda \) type inequality Q-H. Nguyen and Nguyen Cong Phuc proved a global gradient estimates in the weighted Lorentz space for solution to (1.1) in the singular case \( \frac{3n - 2}{3n - 2} < p \leq 2 - \frac{1}{n} \) when the nonlinearity \( A \) satisfies the small BMO condition in the \( x \)-variable and the domain \( \Omega \) satisfies the so-called Reifenberg flatness condition. More precisely the authors showed the \( L^{s,t} (\Omega) \) estimates of solution for all \( 0 < s < \infty, 0 < t \leq \infty \). And in a very recent result [18] under similar conditions on \( A \) and the regularity of \( \Omega \), the authors proved a weighted Calderón-Zygmund type inequality for \( 1 < p \leq \frac{3n - 2}{3n - 1} \).

Our aim in this paper is to extend these results. By considering the remaining ‘strongly singular’ case \( 1 < p \leq \frac{3n - 2}{3n - 1} \) and without the hypothesis of Reifenberg flat domain on \( \Omega \) and small BMO semi-norms of \( A \), we show that the estimate (1.6) holds for all \( 2 - p < s < p + \varepsilon \) and \( 0 < t \leq \infty \) (see Theorem 1.6).

To state our main result, we need some preliminary results on \( p \)-capacity, a decomposition of measure \( \mu \) and the definition of renormalized solution which is can be found in [6].

For \( \mu \in \mathfrak{M}_b (\Omega) \) (the set of finite signed measures in \( \Omega \)), we will tacitly extend it by zero to \( \Omega^c := \mathbb{R}^n \setminus \Omega \). We let \( \mu^+, \mu^- \), and \( |\mu| \) be the positive part, negative part, and the total variation of a measure \( \mu \in \mathfrak{M}_b (\Omega) \) respectively. Let us also recall that a sequence \( \{ \mu_k \} \subset \mathfrak{M}_b (\Omega) \) converges to \( \mu \in \mathfrak{M}_b (\Omega) \) in the narrow topology of measures if

\[
\lim_{k \to \infty} \int_{\Omega} \varphi \, d\mu_k = \int_{\Omega} \varphi \, d\mu,
\]

for every bounded and continuous function \( \varphi \) on \( \Omega \).

We denote by \( \mathfrak{M}_0 (\Omega) \) the set of all measures \( \mu \in \mathfrak{M}_b (\Omega) \) which are absolutely continuous with respect to the \( p \)-capacity, i.e. which satisfy \( \mu (B) = 0 \) for every
Borel set $B \subset \subset \Omega$ such that $\text{cap}_p(B, \Omega) = 0$. We also denote by $\mathcal{M}_s(\Omega)$ the set of all measures $\mu \in \mathcal{M}(\Omega)$ which are singular with respect to the $p$-capacity.

It is known that any $\mu \in \mathcal{M}_b(\Omega)$ can be written uniquely in the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathcal{M}_0(\Omega)$ and $\mu_s \in \mathcal{M}_s(\Omega)$ (see [3]). It is also known that any $\mu_0 \in \mathcal{M}_0(\Omega)$ can be written in the form $\mu_0 = f - \text{div}(F)$ where $f \in L^1(\Omega)$ and $F \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^n)$.

To define renormalized solutions, we need some following tools. For $k > 0$, we define the usual two-sided truncation operator $T_k$ by

$$T_k(s) = \max\{\min\{s, k\}, -k\}, \quad s \in \mathbb{R}.$$

For our purpose, the following notion of gradient is needed. If $u$ is a measurable function defined in $\Omega$, finite a.e., such that $T_k(u) \in W^{1,p}_0(\Omega)$ for any $k > 0$, then there exists a measurable function $v : \Omega \to \mathbb{R}^n$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$ a.e. in $\Omega$ for all $k > 0$ (see [2, Lemma 2.1]). In this case, we define the gradient $\nabla u$ of $u$ by $\nabla u := v$. It is known that $v \in L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$ if and only if $u \in W^{1,1}_{\text{loc}}(\Omega)$ and then $v$ is the usual weak gradient of $u$. On the other hand, for $1 < p \leq 2 - \frac{4}{n}$, by looking at the fundamental solution we see that in general distributional solutions of (1.1) may not even belong to $u \in W^{1,1}_{\text{loc}}(\Omega)$.

We now define the renormalized solutions to (1.1) where the right-hand side is assumed to be in $L^1(\Omega)$ or in $\mathcal{M}_0(\Omega)$ (see [3] for the definition and some other equivalent definitions of renormalized solutions).

**Definition 1.1.** Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(\Omega)$, with $\mu_0 \in \mathcal{M}_0(\Omega)$ and $\mu_s \in \mathcal{M}_s(\Omega)$. A measurable function $u$ defined in $\Omega$ and finite a.e. is called a renormalized solution of (1.1) if $T_k(u) \in W^{1,p}_0(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{n}{n-1}$, and $u$ has the following additional property. For any $k > 0$ there exist nonnegative Radon measures $\lambda^+_k, \lambda^-_k \in \mathcal{M}_0(\Omega)$ concentrated on the sets $\{u = k\}$ and $\{u = -k\}$, respectively, such that $\mu^+_k \to \mu^+_s$, $\mu^-_k \to \mu^-_s$ in the narrow topology of measures and that

$$\int_{\{|u|<k\}} (A(x, \nabla u, \nabla \varphi) dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_\Omega \varphi d\lambda^+_k - \int_\Omega \varphi d\lambda^-_k,$$

for every $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$.

**Remark 1.2.** It is known that if $\mu \in \mathcal{M}_0(\Omega)$ then there is one and only one renormalized solution of (1.1) (see [3, 4]). However, to the best of our knowledge, for a general $\mu \in \mathcal{M}_b(\Omega)$ the uniqueness of renormalized solutions of (1.1) is still an open problem.

**Remark 1.3.** By [4, Lemma 4.1] we have

$$\|\nabla u\|_{L^{\frac{(p-1)n}{n-1}, \infty}(\Omega)} \leq C \left(|\mu|_1(\Omega)\right)^{\frac{1}{p-1}},$$

which implies that

$$\left(\frac{1}{R^n} \int_\Omega |\nabla u|^{\gamma_1} \right)^{1/\gamma_1} \leq C_{\gamma_1} \left[|\mu|_1(\Omega)\right]^{1/(p-1)} \left[\frac{R}{\text{diam}(\Omega)}\right]^{-1/(p-1)}$$

for any $0 < \gamma_1 < \frac{(p-1)n}{n-1}$, where $R = \text{diam}(\Omega)$. 

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Let us also recall the Hardy-Littlewood maximal function $M$ is defined for each locally integrable function $f$ in $\mathbb{R}^n$ by
\[
M(f)(x) = \sup_{\rho>0} \int_{B_\rho(x)} |f(y)| dy, \quad \forall x \in \mathbb{R}^n.
\]

**Remark 1.4.** In [10] the operator $M$ is bounded from $L^s(\mathbb{R})$ to $L^{s,\infty}(\mathbb{R})$ for $s \geq 1$, that is,
\[
|\{x \in \mathbb{R}^n : M(f) > \lambda\}| \leq \frac{C}{\lambda^s} \int_{\mathbb{R}^n} |f|^s dx \quad \text{for all } \lambda > 0.
\]

**Remark 1.5.** In [1, 10] it allows us to present a boundedness property of maximal function $M$ in the Lorentz space $L^{s,t}(\mathbb{R}^n)$ for $s > 1$ as follows:
\[
\|M(f)\|_{L^{s,t}(\mathbb{R}^n)} \leq C \|f\|_{L^{s,t}(\mathbb{R}^n)}.
\]

We are now ready to state the main result of the paper.

**Theorem 1.6.** Let $\mu \in M_b(\Omega)$ and $1 < p \leq \frac{3n-2}{2n-1}$. There exists $\varepsilon > 0$ such that for any $2 - p < s < p + \varepsilon$ and $t \in (0, \infty)$, then there exists a renormalized solution $u$ to (1.1) such that
\[
\|\nabla u\|_{L^{s,t}(\Omega)} \leq C \|\mathcal{M}_1(|\mu|)|^{1/p-1}\|_{L^{s,t}(\Omega)}.
\]

Here the constant $C$ depends only on $n, p, \Lambda, q$, and $\text{diam}(\Omega)/r_0$.

For the proofs of the above theorem: we follow the approach developed by [14, 19] and use some new comparison estimates obtained recently by [18], but technically our present is somewhat different form that of [14, 18]. It is also possible to apply some results developed for quasilinear equations with given measure data, or linear/nonlinear potential and Calderón-Zygmund theories (see [2, 4, 6, 7, 8, 12, 13, 14, 16, 19, 20, 21]), to some new comparison estimates in the singular case $1 < p \leq \frac{3n-2}{2n-1}$.

The paper is organized as follows. In Section 2 we present some important comparison estimates that are needed for the proof of main result and its applications are given in Sections 3. In Section 4 we complete the proof of Theorem 1.6.

**2. Local interior and boundary comparison estimates**

Let $u \in W^{1,p}_1(\Omega)$ be a solution of (1.1) and for each ball $B_{2R} = B_{2R}(x_0) \subset \subset \Omega$, we consider the unique solution $w \in u + W^{1,p}_1(B_{2R})$ to the equation
\[
\begin{aligned}
- \text{div} (A(x, \nabla w)) &= 0 \quad \text{in } B_{2R}, \\
w &= u \quad \text{on } \partial B_{2R}.
\end{aligned}
\]

Then the following well-known version of Gehring’s lemma holds for function $w$ defined above, see [3, Theorem 6.7 and Remark 6.12] and also [19, Lemma 2.1].
Lemma 2.1. Let \( w \) be the solution to (2.1). Then there exists a constant \( \theta_0 = \theta_0(n, p, \alpha, \beta) > 1 \) such that for any \( t \in (0, p] \) and any balls \( B_p(y) \subset B_2R(x_0) \) the following reverse Hölder type inequality holds

\[
\left( \frac{1}{B_{p/2}(y)} \int_{B_{p/2}(y)} |\nabla w|^{p\theta_0} \, dx \right)^{1/p\theta_0} \leq C \left( \frac{1}{B_p(y)} \int_{B_p(y)} |\nabla w|^t \, dx \right)^t
\]

with a constant \( C = C(n, p, \alpha, \beta, t) \).

The next comparison estimate gives an estimate for the difference \( \nabla u - \nabla w \) in terms of the total variation of \( \mu \) in \( B_{2R} \) and the norm of \( \nabla u \) in \( L^{2-p}(B_{2R}) \) in the "strongly singular" case \( 1 < p \leq \frac{3n-2}{2n-1} \). This result was proved by Q.-H. Nguyen [18, Lemma 2.1]. Similar estimates for the other case, i.e., \( p > \frac{3n-2}{2n-1} \) was given in [13, 6, 8, 17].

Lemma 2.2. Let \( u \) and \( w \) be solution to (1.1) and (2.1) respectively and assume that \( 1 < p \leq \frac{3n-2}{2n-1} \). Then

\[
\left( \frac{1}{B_{2R}} \int_{B_{2R}} |\nabla (u - w)|^{\gamma_1} \, dx \right)^{1/\gamma_1} \leq C \left( \frac{1}{B_0(x_0)} \int_{B_0(x_0)} |\mu|^{(B_2R)} \right)^{1/(p-1)} + \frac{1}{B_{2R}} \int_{B_{2R}} |\nabla u|^{2-p} \, dx,
\]

for any \( 0 < \gamma_1 < \frac{n(p-1)}{n-1} \).

Lemmas 2.1 and 2.2 can be extended up to boundary. As \( \mathbb{R}^n \setminus \Omega \) is uniformly \( p \)-thick with constants \( c_0, \rho_0 > 0 \), there exists \( 1 < p_0 = p_0(n, p, c_0) < p \) such that \( \mathbb{R}^n \setminus \Omega \) is uniformly \( p_0 \)-thick with constants \( c_0 = c(n, p, c_0) \) and \( \rho_0 \). This is by now a classical result due to Lewis [11] (see also [15]). Moreover, \( p_0 \) can be chosen near \( p \) so that \( p_0 \in (np/(n+p), p) \). Thus, since \( p_0 < n \), we have

\[
\text{cap}_{p_0} \left( \overline{B_t(x) \cap (\mathbb{R}^n \setminus \Omega)} \cap B_{2t}(x) \right) \geq c \text{cap}_{p_0} \left( B_t(x), B_{2t}(x) \right) \geq C(n, p, c_0) t^{n-p_0},
\]

for all \( 0 < t \leq \rho_0 \) and for all \( x \in \mathbb{R}^n \setminus \Omega \).

Fix \( x_0 \in \partial \Omega \) and \( 0 < R \leq \rho_0/10 \). With \( u \in W^{1,p}_0(\Omega) \) being a solution to (1.1), we now consider the unique solution \( w \in u + W^{1,p}_0(\Omega_{10R}(x_0)) \) to the following equation

\[
\begin{align*}
- \text{div}(A(x, \nabla w)) &= 0 & \text{in } \Omega_{10R}(x_0), \\
w &= u & \text{on } \partial \Omega_{10R}(x_0),
\end{align*}
\]

where we define \( \Omega_{10R}(x_0) = \Omega \cap B_{10R}(x_0) \) and extend \( u \) by zero to \( \mathbb{R}^n \setminus \Omega \) and \( w \) by \( u \) to \( \mathbb{R}^n \setminus \Omega_{10R}(x_0) \).

Then we have the following version of Lemma 2.1 up to the boundary, see [19, Lemma 2.5] for its proof.

Lemma 2.3. Let \( w \) be solution to (2.2). Then there exists a constant \( \theta_0 = \theta_0(n, p, \alpha, \beta) > 1 \) such that for any \( t \in (0, p] \) the following reverse Hölder type inequality

\[
\left( \frac{1}{B_{p/2}(y)} \int_{B_{p/2}(y)} |\nabla w|^{p\theta_0} \, dx \right)^{1/p\theta_0} \leq C \left( \frac{1}{B_p(y)} \int_{B_p(y)} |\nabla w|^t \, dx \right)^t
\]

holds for any balls \( B_{3p}(y) \subset B_{10R}(x_0) \) with a constant \( C = C(n, p, \alpha, \beta, t) \).
As a consequence, we have another version of reverse Hölder type inequality.

**Lemma 2.4.** Let \( w \) be solution to (2.2). Then there exists a constant \( \theta_0 = \theta_0(n, p, \alpha, \beta) > 1 \) such that for any \( t \in (0, p] \) and for \( 0 < \sigma_1 < \sigma_2 < 1 \) it holds that

\[
\left( \int_{B_{\sigma_1 \rho}(y)} |\nabla w|^{p \theta_0} \, dx \right)^{1/p \theta_0} \leq C \left( \int_{B_{\sigma_2 \rho}(y)} |\nabla w|^t \, dx \right)^t
\]

for any balls \( B_\rho(y) \subset B_{10R}(x_0) \) with a constant \( C = C(n, p, \alpha, \beta, t, \rho_1, \rho_2) \).

**Proof.** Let \( x_1, x_2, \ldots, x_m \in B_{\sigma_1}(0) \) be such that

\[
B_{\sigma_1}(0) \subset \bigcup_{i=1}^m B_{\sigma_2 - \sigma_1}(x_i)
\]

For \( \rho > 0 \), let \( B_{\rho}(y) \subset B_{10R}(x_0) \), then we find that

\[
B_{\sigma_1 \rho}(y) \subset \bigcup_{i=1}^m B_{(\sigma_2 - \sigma_1) \rho}(y + \rho x_i)
\]

(2.3)

Also noticed that since \( B_{(\sigma_2 - \sigma_1) \rho}(y + \rho x_i) \subset B_{\sigma_2 \rho}(y) \) for all \( i = 1, \ldots, m \), by Lemma 2.3 there exist \( \theta_0 = \theta_0(n, p, \alpha, \beta) \) and \( C = C(n, p, \alpha, \beta, t) \) such that for each \( t \in (0, p] \) it holds that, for all \( i = 1, \ldots, m \)

\[
\left( \int_{B_{(\sigma_2 - \sigma_1) \rho}(y + \rho x_i)} |\nabla w|^{p \theta_0} \, dx \right)^{1/p \theta_0} \leq C \left( \int_{B_{(\sigma_2 - \sigma_1) \rho}(y + \rho x_i)} |\nabla w|^t \, dx \right)^t.
\]

(2.4)

From (2.3) and (2.4) for any \( t \in (0, p] \) we have

\[
\left( \int_{B_{\sigma_1 \rho}(y)} |\nabla w|^{p \theta_0} \, dx \right)^{1/p \theta_0} \leq C \sum_{i=1}^m \left( \int_{B_{(\sigma_2 - \sigma_1) \rho}(y + \rho x_i)} |\nabla w|^{p \theta_0} \, dx \right)^{1/p \theta_0}
\]

\[
\leq C \sum_{i=1}^m \left( \int_{B_{(\sigma_2 - \sigma_1) \rho}(y + \rho x_i)} |\nabla w|^t \, dx \right)^t
\]

\[
\leq C \left( \int_{B_{\sigma_2 \rho}(y)} |\nabla w|^t \, dx \right)^t.
\]

Thus the proof is complete. \( \blacksquare \)

We also present here the counterpart of Lemma 2.2 up to the boundary, see [18, Lemma 2.3].
Lemma 2.5. Let $1 < p \leq \frac{3n-2}{2n-1}$, and let $u, w$ be solution to (1.1) and (2.2) respectively. Then we have

$$\left( \int_{B_{10R}(x_0)} |\nabla (u - w)|^{\gamma_1} dx \right)^{1/\gamma_1} \leq C \left[ \frac{\mu(B_{10R}(x_0))}{R^{n-1}} \right]^{1/(p-1)} + C \frac{\mu(B_{10R}(x_0))}{R^{n-1}} \int_{B_{10R}(x_0)} |\nabla u|^{2-p} dx,$$

for any $0 < \gamma_1 < \frac{(p-1)n}{n-1}$.

3. Applications of comparison estimates

Our approach to Theorem 1.6 is based on the following technical lemma which allows one to work with balls instead of cubes. A version of this lemma appeared for the first time in [23]. It can be viewed as a version of the Calderón-Zygmund-Krylov-Safonov decomposition that has been used in [5] and [14]. A proof of this lemma, which uses Lebesgue differentiation theorem and the standard Vitali covering lemma, can be found in [4] with obvious modifications to fit the setting here.

Lemma 3.1. Assume that $A \subset \mathbb{R}^n$ is a measurable set for which there exist $c_1, r_1 > 0$ such that

$$|B_t(x) \cap A| \geq c_1 |B_t(x)|$$

holds for all $x \in A$ and $0 < t \leq r_1$. Fix $0 < r < r_1$ and let $C \subset D \subset A$ be measurable sets for which there exists $0 < \varepsilon < 1$ such that

(i) $|C| < \varepsilon r^n |B_r|,$

(ii) for all $x \in A$ and $\rho \in (0, r]$, if $|C \cap B_{\rho}(x)| \geq \varepsilon |B_{\rho}(x)|$, then $B_{\rho}(x) \cap A \subset D.$

Then we have the estimate $|C| \leq \frac{c}{c_1} |D|.$

In order to apply Lemma 3.1 we need the following proposition, whose proof relies essentially on the comparison estimates in the previous section.

Proposition 3.2. There exist constants $A$, $\theta_0 > 1$, depending only on $n, p, \alpha, \beta$, and $c_0$, so that the following holds for any $T > 1$ and $\lambda > 0$. Let $u$ be a solution of (1.1) with $A$ satisfying (1.2) and (1.3). Assume that for some ball $B_{\rho}(y)$ with $10\rho \leq r_0$ we have

$$\left\{ x \in B_{\rho}(y) : \mathcal{M}(\chi_A |\nabla u|^{\gamma_1})(x)^{1/\gamma_1} \leq \lambda, \mathcal{M}_1(\chi_A |\mu|)(x)^{1/(p-1)} \leq T^{-\gamma} \lambda \right\} \neq \emptyset,$$

where $0 < \gamma_1 < \min \left\{ \theta_0 \frac{p \theta_0}{p-1} \frac{(p-1)n}{n-1} \right\}$ and $\gamma = \frac{\rho_0}{\gamma_1 (p-1)} - 1 > 0$. Then

$$\left\{ x \in B_{\rho}(y) : \mathcal{M}(\chi_A |\nabla u|^{\gamma_1})(x)^{1/\gamma_1} > AT \lambda, \mathcal{M}(\chi_A |\nabla u|^{2-p})(x)^{1/(2-p)} \leq T^{\gamma} \lambda \right\} \leq T^{-p \theta_0} |B_{\rho}(y)|$$

(3.3)
Proof. From the assumption (3.2), we imply that there exists \( x_1 \in B_\rho(y) \) such that
\[
[M(\chi_\Omega |\nabla u|^{\gamma_1})(x_1)]^{1/\gamma_1} \leq \lambda \quad \text{and} \quad [M(\chi_\Omega |\mu|^{\gamma_1})(x_1)]^{1/(p-1)} \leq T^{-\gamma}\lambda. \tag{3.4}
\]
On the other hand (3.3) is trivial if the set on the left hand side is empty, so we may assume that there is \( x_2 \in B_\rho(y) \) so that
\[
[M(\chi_\Omega |\nabla u|^{2-p})(x_2)]^{1/(2-p)} \leq TL. \tag{3.5}
\]
It follows from (3.4) that for any \( x \in B_\rho(y) \)
\[
[M(|\chi_\Omega \nabla u|^{\gamma_1})(x)]^{1/\gamma_1} \leq \max\{[M(\chi_{B_{2\rho}(y)} \nabla u|^{\gamma_1})(x)]^{1/\gamma_1}, 3^n\lambda\}. \tag{3.6}
\]
From the last estimate, we observe that (3.3) is also trivial provided \( A \geq 3^n \) and \( B_{4\rho}(y) \subset \mathbb{R}^n \setminus \Omega \). Thus it suffices to consider two cases: the case \( B_{4\rho}(y) \subset \Omega \) and the case \( B_{4\rho}(y) \cap \partial \Omega \neq \emptyset \).

1. The case \( B_{4\rho}(y) \subset \Omega \). Let \( w \in u + W^{1,p}_0(B_{4\rho}(y)) \) be the unique solution to the Dirichlet problem
\[
\begin{align*}
\text{div}A(x, \nabla w) &= 0 \quad \text{in} \quad B_{4\rho}(y), \\
w &= u \quad \text{on} \quad \partial B_{4\rho}(y).
\end{align*}
\]
By Chebyshev inequality and weak type (1,1) estimate for the maximal function we have
\[
\left| \left\{ x \in B_\rho(y) : M(\chi_{B_{2\rho}(y)} |\nabla u|^{\gamma_1})(x) > AT\lambda \right\} \right| \\
\leq \left| \left\{ x \in B_\rho(y) : M(\chi_{B_{2\rho}(y)} |\nabla w|^{\gamma_1})(x) > AT\lambda/2 \right\} \right| \\
+ \left| \left\{ x \in B_\rho(y) : M(\chi_{B_{2\rho}(y)} |\nabla u - \nabla w|^{\gamma_1})(x) > AT\lambda/2 \right\} \right| \\
\leq C(\lambda) -p\theta_0 \int_{B_{2\rho}(y)} |\nabla w|^{p\theta_0} dx + C(\lambda) -\gamma_1 \int_{B_{2\rho}(y)} |\nabla u - \nabla w|^{\gamma_1} dx.
\]
Using Lemma 2.1 we have
\[
\left( \int_{B_{2\rho}(y)} |\nabla w|^{p\theta_0} dx \right)^{\gamma_1/p\theta_0} \leq C \int_{B_{4\rho}(y)} |\nabla u|^{\gamma_1} dx \\
\leq C \int_{B_{4\rho}(y)} |\nabla u|^{\gamma_1} dx + C \int_{B_{4\rho}(y)} |\nabla u - \nabla w|^{\gamma_1} dx
\]
and hence
\[
\left| \left\{ x \in B_\rho(y) : M(\chi_{B_{2\rho}(y)} |\nabla u|^{\gamma_1})(x) > AT\lambda \right\} \right| \\
\leq C(\lambda) -p\theta_0 |B_\rho(y)| \left( \int_{B_{4\rho}(y)} |\nabla u|^{\gamma_1} dx \right)^{p\theta_0/\gamma_1} \\
+ C(\lambda) -\gamma_1 |B_\rho(y)| \int_{B_{4\rho}(y)} |\nabla u - \nabla w|^{\gamma_1} dx \\
+ C(\lambda) -\gamma_1 |B_\rho(y)| \int_{B_{4\rho}(y)} |\nabla u - \nabla w|^{\gamma_1} dx. \tag{3.7}
\]
On the other hand, by Lemma 2.2 we have that
\[
\left( \int_{B_{4p}(y)} |\nabla (u - w)|^{\gamma_1} \, dx \right)^{\frac{1}{\gamma_1}} \leq C \left( \frac{\|B_{4p}(y)\|}{\rho^{n-1}} \right)^{-\frac{1}{p-1}} + \frac{\|B_{4p}(y)\|}{\rho^{n-1}} \int_{B_{4p}(y)} |\nabla u|^{2-p} \, dx
\]
\[
\leq C \left( \frac{\|B_{5p}(x_1)\|}{\rho^{n-1}} \right)^{-\frac{1}{p-1}} + C \frac{\|B_{5p}(x_1)\|}{\rho^{n-1}} \int_{B_{5p}(x_1)} |\nabla u|^{2-p} \, dx,
\]
which implies, due to (3.5) and (3.6)
\[
\int_{B_{4p}(y)} |\nabla (u - w)|^{\gamma_1} \, dx \leq C \left( \frac{\|B_{5p}(x_1)\|}{\rho^{n-1}} \right)^{-\frac{1}{p-1}} + C \frac{\|B_{5p}(x_1)\|}{\rho^{n-1}} \int_{B_{5p}(x_1)} |\nabla u|^{2-p} \, dx.
\]
Combining (3.7) and (3.8) and noting that \( T > 1 \), one has
\[
\left| \left\{ x \in B_{4p}(y) : \mathcal{M} \left( \chi_{B_{2p}(y)} |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > AT\lambda, \quad \mathcal{M} \left( \chi_{\Omega} |\nabla u|^{2-p} \right) (x)^{1/(2-p)} \leq T\lambda \right\} \right| \\
\leq C \left[ (AT)^{-p\theta_0} + A^{-p\theta_0} T^{-p\theta_0(\gamma_1+1)(p-1)} + A^{-\gamma_1} T^{-\gamma_1(\gamma_1+1)(p-1)} \right] |B_{4p}(y)|
\]
\[
\leq \left[ CA^{-p\theta_0} + CA^{-\gamma_1} \right] T^{-p\theta_0} |B_{4p}(y)|.
\]
By choosing \( A \geq \max \left\{ 3^n, (4C)^{1/\gamma_1} \right\} \) we have that
\[
\left| \left\{ x \in B_{4p}(y) : \mathcal{M} \left( \chi_{B_{2p}(y)} |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > AT\lambda, \quad \mathcal{M} \left( \chi_{\Omega} |\nabla u|^{2-p} \right) (x)^{1/(2-p)} \leq T\lambda \right\} \right| \leq \frac{1}{2} T^{-p\theta_0} |B_{4p}(y)|,
\]
which in view of (3.6) yields (3.3).

2. The case \( B_{4p}(y) \cap \partial\Omega \neq \emptyset \). Let \( x_3 \in \partial\Omega \) be a boundary point such that \( |y - x_3| = \text{dist}(y, \partial\Omega) < 4p \). And let us define \( w \in u + W_{10, p}(\Omega_{10p}(x_3)) \) to be the unique solution to the Dirichlet problem
\[
\begin{align*}
\text{div} A(x, \nabla w) &= 0 \quad \text{in} \quad \Omega_{10p}(x_3), \\
w &= u \quad \text{on} \quad \partial\Omega_{10p}(x_3).
\end{align*}
\]
Here we also extend \( u \) by zero to \( \mathbb{R}^n \setminus \Omega \) and then extend \( w \) by \( u \) to \( \mathbb{R} \setminus \Omega_{10p}(x_3) \). Using similar argument as in (3.7) in which Lemma 2.3 is exploited instead of Lemma 2.1.
we have
\[ \left\{ x \in B_\rho(y) : \mathcal{M} \left( \chi_{B_\rho(y)} |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > AT\lambda \right\} \]
\[ \leq C (AT\lambda)^{-\rho_0} |B_\rho(y)| \left( \frac{1}{B_{\rho(y)}(y)} |\nabla u|^{\gamma_1} \, dx \right)^{\rho_0/\gamma_1} \]
\[ + C (AT\lambda)^{-\rho_0} |B_\rho(y)| \left( \frac{1}{B_{\rho(y)}(y)} |\nabla u - \nabla w|^{\gamma_1} \, dx \right)^{\rho_0/\gamma_1} \]
\[ + C (AT\lambda)^{-\gamma_1} |B_\rho(y)| \int_{B_{\rho(y)}(y)} |\nabla u - \nabla w|^{\gamma_1} \, dx. \] (3.9)
On the other hand, since
\[ B_{6\rho}(y) \subset B_{10\rho}(x_3) \subset B_{14\rho}(y) \subset B_{15\rho}(x_1) \subset B_{16\rho}(x_2) \]
it follows from Lemma 2.5 that
\[ \int_{B_{6\rho}(y)} |\nabla (u - w)|^{\gamma_1} \, dx \leq CT^{-\rho_0+\gamma_1} \lambda^{\gamma_1} \] (3.10)
Combining (3.9) and (3.10) and the fact that \( T > 1 \) we have
\[ \left\{ x \in B_\rho(y) : \mathcal{M} \left( \chi_{B_\rho(y)} |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > AT\lambda, \]
\[ \mathcal{M} \left( \chi_{\Omega} |\nabla u|^{2-p} \right) (x)^{1/(2-p)} \leq T\lambda \right\} \leq \left[ CA^{-\rho_0} + CA^{-\gamma_1} \right] T^{-\rho_0} |B_\rho(y)|. \]
By choosing \( A \geq \max \left\{ 3^n, (4C)^{1/\gamma_1} \right\} \) we arrive at
\[ \left\{ x \in B_\rho(y) : \mathcal{M} \left( \chi_{B_\rho(y)} |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > AT\lambda, \]
\[ \mathcal{M} \left( \chi_{\Omega} |\nabla u|^{2-p} \right) (x)^{1/(2-p)} \leq T\lambda \right\} \leq \frac{1}{2} T^{-\rho_0} |B_\rho(y)| \] (3.11)
Thus (3.8) follows from (3.9) and (3.11).

The Proposition 3.2 can be rewritten as follows.

**Proposition 3.3.** There exist constants \( A, \theta_0 > 1 \), depending only on \( n, p, \alpha, \beta, \) and \( c_0 \), so that the following holds for any \( T > 1 \) and \( \lambda > 0 \). Let \( u \) be a solution of (1.1) with \( A \) satisfying (1.2) and (13). Suppose that for some ball \( B_\rho(y) \) with \( 10\rho \leq r_0 \) we have
\[ \left\{ x \in B_\rho(y) : \mathcal{M} \left( \chi_{\Omega} |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > AT\lambda, \]
\[ \mathcal{M} \left( \chi_{\Omega} |\nabla u|^{2-p} \right) (x)^{1/(2-p)} \leq T\lambda \right\} \geq T^{-\rho_0} |B_\rho(y)|. \]
Then
\[ B_\rho(y) \subset \left\{ x \in \mathbb{R}^n : \mathcal{M} \left( \chi_{\Omega} |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > \lambda \text{ or } \mathcal{M}_1 \left( \chi_{\Omega} |\nabla u| \right) (x)^{1/(p-1)} > T^{-\gamma_1} \lambda \right\}, \]
where \( \gamma \) and \( \gamma_1 \) as in Proposition 3.3.
Applying Lemma 3.1 with $t < A T > x$

Proof. Let $\Lambda$, $\rho$, $\alpha$, and $\beta$ be constants satisfying (1.2), so that the following holds for any $A T > 0$.

With the aid of Lemma 3.1 and Proposition 3.3 we get the main result of this section which is used later.

Lemma 3.4. There exist constants $A$, $\theta_0 > 1$, depending only on $n, p, \alpha, \beta$, and $c_0$, so that the following holds for any $T > 1$. Let $u$ be a solution of (1.1) with $A$ satisfying (1.2) and (1.3). Let $B_0$ be a ball of radius $R_0$. Fix a real number $0 < r \leq \min\{r_0, 2R_0\}/10$ and suppose that there exists $\lambda > 0$ such that

$$\left| \left\{ x \in \mathbb{R}^n : \mathcal{M} \left( \chi_\Omega |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > \Lambda \right\} \right| < T^{-\rho_0 r^n} |B_1|. \quad (3.12)$$

Then for any integer $i \geq 0$ it holds that

$$\left| \left\{ x \in B_0 : \mathcal{M} \left( \chi_\Omega |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > \Lambda (A T)^{i+1} \right\} \right|$$

$$\leq c(n) T^{-\rho_0} \left| \left\{ x \in B_0 : \mathcal{M} \left( \chi_\Omega |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > \Lambda (A T)^i \right\} \right|$$

$$+ c(n) \left| \left\{ x \in B_0 : \mathcal{M}_1 \left( \chi_\Omega |\mu| \right) (x)^{1/(p-1)} > \Lambda T^{-\gamma} (A T)^i \right\} \right|$$

$$+ \left| \left\{ x \in B_0 : \mathcal{M} \left( \chi_\Omega |\nabla u|^{2-p} \right) (x)^{1/(2-p)} > \Lambda (A T)^{i+1} \right\} \right| .$$

Proof. Let $A$ and $\theta_0 > 1$ be as in Proposition 3.3 and set

$$C = \left\{ x \in B_0 : \mathcal{M} \left( \chi_\Omega |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > \Lambda (A T)^{i+1}, \right. \right.

$$\text{and } \mathcal{M} \left( \chi_\Omega |\nabla u|^{2-p} \right) (x)^{1/(2-p)} \leq \Lambda (A T)^{i+1} \left\}, \right. \right.

and

$$D = \left\{ x \in B_0 : \mathcal{M} \left( \chi_\Omega |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > \Lambda (A T)^i \right. \right.

$$\text{or } \mathcal{M}_1 \left( \chi_\Omega |\mu| \right) (x)^{1/(p-1)} > \Lambda T^{-\gamma} (A T)^i \left\} \right. \right.$$

It is first noticed that since $A T > 1$ we deduce from (3.12) that

$$|C| \leq T^{-\rho_0 r^n} |B_1| .$$

On the other hand, if $x \in B_0$ and $\rho \in (0, r]$ hold $|C \cap B_\rho(x)| \geq T^{-\rho_0} |B_\rho(x)|$, then $10 \rho \leq r_0$ and therefore by applying Proposition 3.3 with $\lambda = \Lambda (A T)^i$ we have

$$B_\rho(x) \cap B_0 \subset D.$$ 

Applying Lemma 3.1 with $A = B_0$ and $\varepsilon = T^{-\rho_0}$ with noting that the condition (3.1) holds for all $0 < t < 2R_0$ we obtain

$$|C| \leq c(n) T^{-\rho_0} |D|$$

$$\leq c(n) T^{-\rho_0} \left| \left\{ x \in B_0 : \mathcal{M} \left( \chi_\Omega |\nabla u|^{\gamma_1} \right) (x)^{1/\gamma_1} > \Lambda (A T)^i \right\} \right|$$

$$+ c(n) T^{-\rho_0} \left| \left\{ x \in B_0 : \mathcal{M}_1 \left( \chi_\Omega |\mu| \right) (x)^{1/(p-1)} > \Lambda T^{-\gamma} (A T)^i \right\} \right| .$$
Thus the proof is complete.

4. Proof of Theorem 1.6

Let $B_0$ be a ball of radius $R_0 \leq 2 \text{diam} (\Omega)$ that contains $\Omega$. Then it is noticed that $\text{diam} (\Omega) \leq 2R_0$. We also extend $u$ and $\mu$ to be zero in $\mathbb{R}^n \setminus \Omega$. We will show that

$$\| \nabla u \|_{L^{s,t} (\Omega)} \leq C \left\| \mathcal{M}_1 (|\mu|)^{1/p-1} \right\|_{L^{s,t} (B_0)}$$

(4.1)

where $2 - p < s < p + \varepsilon$ and $0 < t \leq \infty$. Here $\varepsilon > 0$ is a small number depending only on $n, p, \alpha, \beta,$ and $c_0$. In what follows we consider only the case $t \neq \infty$ as for $t = \infty$ the proof is similar. Moreover, to prove (4.1) we may assume that

$$\| \nabla u \|_{L^{s,t} (\Omega)} \neq 0.$$

Let $r = \min \{ r_0, \text{diam} (\Omega) \} / 10$. For $T > 1$ we first claim that there is $\Lambda > 0$ such that

$$\left\| \chi_{\Omega} \right\|_{L^{\gamma_1} (\Omega)} (x)^{1/\gamma_1} > \Lambda \right\| < T^{-p\theta_0} r^n |B_1|.$$

(4.2)

Indeed, by weak type $(1,1)$ estimate for the maximal function we have

$$\left\| \chi_{\Omega} \right\|_{L^{\gamma_1} (\Omega)} (x)^{1/\gamma_1} > \Lambda \right\| < \frac{c(n)}{\Lambda^{\gamma_1}} \int_{\Omega} |\nabla u|^{\gamma_1} dx.$$

By choosing $\Lambda$ such that

$$\frac{c(n)}{\Lambda^{\gamma_1}} \int_{\Omega} |\nabla u|^{\gamma_1} dx = T^{-p\theta_0} r^n |B_1|.$$

(4.3)

Let $A, \theta_0 > 1$ be as in Lemma 3.4 and let $0 < \gamma_1 < \min \left\{ \frac{\theta_0}{p-1}, \frac{(p-1)n}{n-1} \right\}$ and $\gamma = p\theta_0/\gamma_1 (p-1) - 1$. For $0 < t < \infty$ we consider the sum

$$S = \sum_{i=1}^{\infty} \left( \left( AT \right)^{s_i} \left\| \mathcal{M}_i (|\nabla u|^{\gamma_1})^{1/\gamma_1} > \Lambda (AT)^i \right\| \right)^{t/s}$$
It is noticed that
\[ C^{-1} S \leq \left\| M (|\nabla u| / \Lambda)^{1/\gamma_1} \right\|_{L^s,t(B_0)}^{f} \leq C \left[ S + |B_0|^{1/\gamma_1} \right] \]  \tag{4.4}

By (4.5) and Lemma 3.4, we have
\[ S \leq c(n) \sum_{i=1}^{\infty} \left[ (AT)^{s_i} T^{-p_{i,0}} \right] \left\{ x \in B_0 : M \left| \nabla u \right|^{1/\gamma_1} > \Lambda (AT)^{-i-1} \right\} \right]^{f/\gamma_1}
+ c(n) \sum_{i=1}^{\infty} \left[ (AT)^{s_i} \left\{ x \in B_0 : M_1 \left( |\mu| / \Lambda^{p-1} \right)^{1/(p-1)} \right\} \right]^{f/\gamma_1}
+ \sum_{i=1}^{\infty} \left[ (AT)^{s_i} \left\{ x \in B_0 : M_1 \left( |\mu| / \Lambda^{p-1} \right)^{1/(p-1)} \right\} \right]^{f/\gamma_1}
\leq C \left[ (AT)^{s_i} T^{-p_{i,0}} \right]^{f/\gamma_1} \left( S + |B_0|^{f/\gamma_1} \right) + C \left\| M_1 \left( |\mu| / \Lambda^{p-1} \right)^{1/(p-1)} \right\|_{L^{s,t}(B_0)}^{f/\gamma_1}
+ \left\| \left[ M \left( \left| \nabla u \right| / \Lambda T \right)^{2-p} \right]^{1/(2-p)} \right\|_{L^{s,t}(B_0)}^{f/\gamma_1} \tag{4.5}

Since \( A \) and \( C \) are fixed, then \( T^{1-\frac{s}{2p}} \ll 1 \) for \( T \) large enough if \( s < p \), that is, \( s < p + \varepsilon \) with \( \varepsilon = p (\theta_0 - 1) \). In this case, the estimate (4.5) gives us
\[ S \leq C \left| B_0 \right|^{f/\gamma_1} + C \left\| M_1 \left( |\mu| / \Lambda^{p-1} \right)^{1/(p-1)} \right\|_{L^{s,t}(B_0)}^{f/\gamma_1}
+ \left\| \left[ M \left( \left| \nabla u \right| / \Lambda T \right)^{2-p} \right]^{1/(2-p)} \right\|_{L^{s,t}(B_0)}^{f/\gamma_1}. \tag{4.6} \]

Combining (4.4) and (4.6) and using the boundedness of \( M \) on \( L^{s/(2-p),t}(\mathbb{R}^n) \) where \( s/(2-p) > 1 \) and \( t \geq 0 \), we have
\[ \left\| \nabla u \right\|_{L^{s,t}(\Omega)} \leq C \left( B_0 \right)^{1/\gamma_1} \Lambda + \left\| M_1 \left( |\mu| / \Lambda^{p-1} \right)^{1/(p-1)} \right\|_{L^{s,t}(B_0)}^{f/\gamma_1} + T^{-1} \left\| \nabla u \right\|_{L^{s,t}(B_0)}^{f/\gamma_1}, \]
and hence for \( T \) sufficiently large one has
\[ \left\| \nabla u \right\|_{L^{s,t}(\Omega)} \leq C \left( B_0 \right)^{1/\gamma_1} \Lambda + \left\| M_1 \left( |\mu| / \Lambda^{p-1} \right)^{1/(p-1)} \right\|_{L^{s,t}(B_0)}^{f/\gamma_1}. \tag{4.7} \]

We now estimate \( \Lambda \). It follows from (4.8) and \( \gamma_1 < \frac{n-1}{n+q} \) and the standard estimate for equations with measure data (see [2, Theorem 4.1]) that
\[ \Lambda \leq C r^{-n/\gamma_1} \left\| \nabla u \right\|_{L^{\gamma_1}(\Omega)} \]
\[ \leq C \min \left\{ r_0, \text{diam} \left( \Omega \right) \right\}^{-n/\gamma_1} \text{diam} \left( \Omega \right)^{n/\gamma_1} \left( \frac{|\mu| (\Omega) \left( \text{diam} \left( \Omega \right)^{n-1} \right)^{1/(p-1)} \right), \]

On the other hand, since \( R_0 \leq 2 \text{diam} \left( \Omega \right) \), we have
\[ \Lambda \leq C \left( n, p, \text{diam} \left( \Omega \right) / r_0 \right) M_1 \left( |\mu| \right) \left( x \right)^{1/(p-1)} \tag{4.8} \]
for any \( x \in B_0 \). Finally the proof follows from (4.7) and (4.8).
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