Optimal Fractional Repetition Codes and Fractional Repetition Batch Codes

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Abstract—Fractional repetition (FR) codes is a family of codes for distributed storage systems (DSS) that allow uncoded exact repairs with minimum repair bandwidth. In this work, we consider a bound on the maximum amount of data that can be stored using an FR code. Optimal FR codes which attain this bound are presented. The constructions of these FR codes are based on families of regular graphs, such as Turán graphs and graphs with large girth; and on combinatorial designs, such as transversal designs and generalized polygons. In addition, based on a connection between FR codes and batch codes, we propose a new family of codes for DSS, called fractional repetition batch codes, which allow uncoded efficient exact repairs and load balancing which can be performed by several users in parallel.

I. INTRODUCTION

In distributed storage systems, data is stored across a network of nodes, which can unexpectedly fail. To provide reliability, data redundancy based on coding techniques is introduced in such systems. Moreover, existing erasure codes allow to minimize the storage overhead. In [4], Dimakis et al. introduced a new family of erasure codes, called regenerating codes, which allow efficient single node repairs. In particular, they presented two families of regenerating codes, called minimum storage regenerating (MSR) codes and minimum bandwidth regenerating (MBR) codes, which correspond to the two extreme points on the storage-bandwidth trade-off [4]. An \((n,k,d,\alpha,\beta)\) \(q\)-regenerating code \(C\), where \(k \leq d \leq n-1\), \(\beta \leq \alpha\), is used to store a file in \(n\) nodes; each node stores \(\alpha\) symbols from \(\mathbb{F}_q\), the finite field with \(q\) elements, such that the stored file can be recovered by downloading the data from any set of \(k\) nodes. When a single node fails, a newcomer node which substitutes the failed node contacts with a random set of \(d\) other nodes and downloads \(\beta\) symbols of each node in this set to reconstruct the failed data. This process is called a node repair, and the amount of data downloaded to repair a failed node, \(\beta d\), is called the repair bandwidth.

In [13], [14] Rashmi et al. presented a construction for MBR codes which have the additional property of exact repair by transfer, or exact uncoded repair. In other words, the \((n,k,d,n-1,M) = k\alpha - \binom{k}{2}, \alpha = n-1, \beta = 1\) code proposed in [13], [14] allows efficient exact node repairs where no decoding is needed. Every node participating in a node repair process just passes one symbol which will be directly stored in the newcomer node. This construction is based on a concatenation of an outer MDS code with an inner repetition code based on a complete graph. El Rouayheb and Ramchandran [15] generalized the construction of [13] and defined a new family of codes for DSS which allow exact repairs by transfer for a wide range of parameters. These codes, called DRESS (Distributed Replication based Exact Simple Storage) codes [17], consist of the concatenation of an outer MDS code and the inner repetition code called fractional repetition (FR) code. However, in contrast to MBR codes, where a random set of size \(d\) of available nodes is used for a node repair, the repairs with DRESS codes are table based. This usually allows to store more data compared to MBR codes.

Constructions of FR codes based on some regular graphs and combinatorial designs can be found for example in [7], [9], [10], [15]. However, the optimality of the constructed FR codes regarding the FR capacity, i.e. the maximality of the size of the stored file, was not considered.

In this work, we address the problem of constructing optimal FR codes and hence, optimal DRESS codes. Moreover, based on a connection between FR codes and combinatorial batch codes, we propose a new family of codes for DSS, called fractional repetition batch (FRB) codes, which enable uncoded repairs and load balancing that can be performed by several users in parallel.

The rest of the paper is organized as follows. In Section II we define DRESS codes and FR codes based on regular graphs and combinatorial designs. In Section III we present optimal FR codes based on Turán graphs and on graphs with large girth. In Section IV we consider optimal FR codes based on transversal designs and on generalized polygons. In Section V we define FRB codes and present some examples for their constructions. Conclusion is given in Section VI. We point out that, throughout this paper, proofs are often omitted due to space limitations. Details of all the proofs can be found in [16].

II. PRELIMINARIES

An \((n,\alpha,\rho)\) FR code \(C\) is a collection of \(n\) subsets \(N_1, \ldots, N_n\) of \([\theta] = \{1, 2, \ldots, \theta]\), \(n\alpha = \rho\theta\), such that:

- \(|N_i| = \alpha\) for each \(i, 1 \leq i \leq n\); and
- each symbol of \([\theta]\) belongs to exactly \(\rho\) subsets in \(C\), where \(\rho\) is called the repetition degree of \(C\).

A \([(\theta,M), k, (n,\alpha,\rho)]\) DRESS code is a code obtained by the concatenation of an outer \((\theta,M)\) MDS code and an inner \((n,\alpha,\rho)\) FR code \(C\). To store a file \(f \in \mathbb{F}_q^\theta\) in a DSS, \(f\) is first...
encoded by using the MDS code; next, the $\theta$ symbols of the codeword $c_f$ from the MDS code, which encodes the file $f$, are placed in the $n$ nodes defined by $C$, as follows: node $i \in [n]$ of the DSS stores $\alpha$ symbols of $c_f$, indexed by the elements of the subset $N_i$. The encoding scheme for a DRESS code is shown in Fig. 1.

Each symbol of $c_f$ is stored in exactly $\rho$ nodes. It should be possible to reconstruct the stored file $f$ of size $M$ from any set of $k$ nodes, and hence,

$$M \leq \min_{|I|=k} \bigcup_{i \in I} N_i.$$

Since we want to maximize the size of a file that can be stored by using a DRESS code, in the sequel we will always assume that $M = \min_{|I|=k} \bigcup_{i \in I} N_i$. Note, that the same FR code can be used in different DRESS codes, with different $k$’s as reconstruction degrees, and different MDS codes. The file size $M$, which is the dimension of the chosen MDS code, depends on the value of $k$ and hence in the sequel we will use $M(k)$ to denote the size of the file. An $(n, \alpha, \rho)$ FR code is called universally good [15] if for any $k \leq \alpha$ the $[(\theta, M(k)), k, (n, \alpha, \rho)]$ DRESS code satisfies

$$M(k) \geq k\alpha - \binom{k}{2},$$

where the righthand side of equation (2) is the maximum file size (called MBR capacity) that can be stored using an MBR code [4]. Note also that if an FR code $C$ is universally good then $|N_i \cap N_j| \leq 1$, for $N_i, N_j \in C$, $i \neq j \in [n]$ [13]. In the sequel, we will consider only universally good FR codes.

An upper bound on the maximum file size $M(k)$ of a $[(\theta, M(k)), k, (n, \alpha, \rho)]$ DRESS code $(n\alpha = \rho\theta)$, called the FR capacity and denoted in the sequel by $A(n, k, \alpha, \rho)$, was presented in [15]:

$$A(n, k, \alpha, \rho) \leq \varphi(k), \text{ where } \varphi(1) = \alpha,$$

$$\varphi(k + 1) = \varphi(k) + \alpha - \left\lfloor \frac{\rho\varphi(k) - k\alpha}{n - k} \right\rfloor.$$

Note that for any given $k$, the function $A(n, k, \alpha, \rho)$ is determined by the parameters of the inner FR code. We call an FR code $k$-optimal if a file stored by using this code is the maximum possible for the given $k$. We call an FR code optimal if for any $k \leq \alpha$ it is $k$-optimal.

Let $C$ be an $(n, \alpha, \rho)$ FR code. $C$ can be described by an incidence matrix $I(C)$, which is an $n \times \theta$ binary matrix, $\theta = \frac{n\alpha}{\rho}$, whose rows indexed by the nodes and columns indexed by the symbols of the corresponding MDS codeword, such that $(I(C))_{i,j} = 1$ if and only if node $i$ contains symbol $j$. Note that every row of $I(C)$ has $\alpha$ ones and every column of $I(C)$ has $\rho$ ones.

Let $G = (V, E)$ be an $\alpha$-regular graph with $n = |V|$ vertices. We say that an $(n, \alpha, \rho)$ FR code $C$ is based on $G$ if $I(C) = I(G)$, where $I(G)$ is the $|V| \times |E|$ incidence matrix of $G$. Such a code will be denoted by $C_G$.

Let $D = (P, B)$ be a design with $|P| = n$ points such that each block $B \in B$ contains $\rho$ points and each point $p \in P$ is contained in $\alpha$ blocks. We say that an $(n, \alpha, \rho)$ FR code $C$ is based on $D$ if $I(C) = I(D)$, where $I(D)$ is the $|P| \times |B|$ incidence matrix of $D$. Such a code will be denoted by $C_D$.

III. OPTIMAL FR CODES WITH RECEPTION DEGREE $\rho = 2$

In this section we consider optimal FR codes with repetition degree 2. First, we present the following useful lemma which shows a connection between the problem of finding the maximum file size of an FR code based on a graph and the edge isoperimetric problem on graphs [2].

**Lemma 1.** Let $G = (V, E)$ be an $\alpha$-regular graph and let $C_G$ be the FR code based on $G$. We denote by $G_k$ the family of induced subgraphs of $G$ with $k$ vertices. Then the file size $M(k)$ of $C_G$ is given by

$$M(k) = k\alpha - \max_{G' = (V', E') \in G_k} |E'|.$$

**Proof.** For each induced subgraph $G' = (V', E') \in G_k$ we define $E'_{\text{cut}}$ to be the set of all the edges of $E$ in the cut between $V'$ and $V \setminus V'$, i.e.,

$$E'_{\text{cut}} = \{v, u \in E : v \in V', u \in V \setminus V'\}.$$

Clearly, $k\alpha = 2|E'| + |E'_{\text{cut}}|$ for every $G' \in G_k$. Note that $M(k) = \min_{G' \in G_k} \{|E'| + |E'_{\text{cut}}|\}$ and hence

$$M(k) = \min_{G' \in G_k} \{|E'| + \alpha k - 2|E'|\} = \alpha k - \max_{G' \in G_k} \{|E'|\}.$$
of regular graphs, called Turán graphs, which do not contain a clique of a given size and also have the smallest number of vertices [6]. Let \( r, n \) be two integers such that \( r \) divides \( n \). An \((n, r)\)-Turán graph is defined as a regular complete \( r \)-partite graph, i.e., a graph formed by partitioning the set of \( n \) vertices into \( r \) parts of size \( \frac{n}{r} \) and connecting each two vertices of different parts by an edge. Clearly, an \((n, r)\)-Turán graph does not contain a clique of size \( r + 1 \) and it is an \((r - 1)\frac{n}{r}\)-regular graph.

The following theorem shows that FR codes obtained from Turán graphs attain the upper bound in (3) for all \( k \leq \alpha \) and hence they are optimal FR codes. The proof of this theorem follows from Lemma [1] and by Turán’s theorem [6] p. 58].

Theorem 4. Let \( T = (V, E) \) be an \((n, r)\)-Turán graph, \( r < n \), \( \alpha = (r - 1)\frac{n}{r} \), and let \( k \) be an integer such that \( 1 \leq k \leq \alpha \). Then the \((n, \alpha, 2)\) FR code \( C_T \) based on \( T \) has file size given by

\[
M(k) = k\alpha - \left\lfloor \frac{r - 1}{r} \cdot \frac{k^2}{2} \right\rfloor
\]

which attains the upper bound in (3).

Note that an \((n - 1)\)-regular complete graph \( K_n \) is an \((n, n)\)-Turán graph. Hence, the construction of MBR codes from [13], [14] can be considered as a special case of our construction of the DRESS codes with an inner FR code based on a Turán graph. Note also that an \( \alpha \)-regular complete bipartite graph \( K_{\alpha, n} \) is a \((2\alpha, 2)\)-Turán graph. The following example illustrates Theorem [4] for such a graph.

Example 1. The \((6, 3, 2)\) FR code based on \( K_{3,3} \) and its file size for \( 1 \leq k \leq 3 \) are shown in Fig. 2.

The proof of the following lemma can be easily verified from Lemma [1].

Lemma 5. Let \( C \) be an \((n, \alpha, 2)\) FR code. Then the file size \( M(k) \) of \( C \) for any \( 1 \leq k \leq \alpha \) satisfies

\[
M(k) \leq k\alpha - k + 1.
\]

By Lemma [1] to obtain a large value for \( M(k) \), every induced subgraph with \( k \) vertices should be as sparse as possible. Hence, for the rest of this section we consider graphs where the induced subgraphs with \( k \) vertices, \( 1 \leq k \leq \alpha \), will be cycle-free. These are graphs with girth at least \( k + 1 \), where the girth of a graph is the length of its shortest cycle.

Lemma 6. Let \( G \) be an \( \alpha \)-regular graph with \( n \) vertices and let \( M(k) \) be the file size of the corresponding FR code \( C_G \). The girth of \( G \) is at least \( k + 1 \) if and only if \( M(k) = k\alpha - (k - 1) \).

Corollary 7. For each \( k \leq g - 1 \), an FR code \( C_G \) based on an \( \alpha \)-regular graph \( G \) with girth \( g \) attains the bound in (3), and hence it is \( k \)-optimal. \( C_G \) also attains the bound of Lemma [5].

Corollary 8. An FR code \( C_G \) based on an \( \alpha \)-regular graph \( G \) with girth \( g \geq \alpha + 1 \) is optimal.

The proof of the following theorem follows from Lemma [6] and the fact that any two cycles in a graph with girth \( g \) have at most \( \lfloor g/2 \rfloor + 1 \) common vertices.

Theorem 9. If \( G \) is a graph with girth \( g \), then the file size \( M(k) \) of an FR code \( C_G \) based on \( G \) satisfies

\[
M(k) = \begin{cases} 
  k\alpha - k + 1 & \text{if } k \leq g - 1 \leq \lfloor g/2 \rfloor, \\
  k\alpha - k & \text{if } g - 1 < k \leq g + \lfloor g/2 \rfloor - 2.
\end{cases}
\]

A \((d, g)\)-cage is a \( d \)-regular graph with girth \( g \) and minimum number of vertices. Let \( N(d, g) \) be the minimum number of vertices in a \((d, g)\)-cage. A lower bound on \( N(d, g) \), known as Moore bound [3] p. 180], is given by

\[
n_0(d, g) = \begin{cases} 
  1 + d + \sum_{i=0}^{g-2} (d - 1)^i & \text{if } g \text{ is odd}, \\
  2 + \sum_{i=0}^{g-2} (d - 1)^i & \text{if } g \text{ is even}.
\end{cases}
\]

Lemma 10. The bound in (3) is not tight for \( \rho = 2 \) if

\[
\alpha k - \alpha - k + 3 \leq n < N(\alpha, k + 1).
\]

As a consequence of Lemma [10] we have that the bound in (3) is not always tight and hence we have a similar better bound on \( A(n, k, \alpha, \rho) \):

\[
A(n, k, \alpha, \rho) \leq \varphi'(k), \text{ where } \varphi'(1) = \alpha, \\
\varphi'(k + 1) = A(n, k, \alpha, \rho) + \alpha - \left[ \frac{\rho A(n, k, \alpha, \rho) - k\alpha}{n - k} \right].
\]

IV. OPTIMAL FR CODES WITH REPETITION DEGREE \( \rho > 2 \)

In this section, we consider FR codes with repetition degree \( \rho > 2 \). Note, that while codes with \( \rho = 2 \) have the maximum data/storage ratio, codes with \( \rho > 2 \) provide multiple choices for node repairs. In other words, when a node fails, it can be repaired from different \( d \)-subsets of available nodes.

We present generalizations of the constructions from the previous section which were based on Turán graphs and graphs with a given girth. These generalizations employ transversal designs and generalized polygons, respectively.

A transversal design of group size \( h \) and block size \( \ell \), denoted by \( \text{TD}(\ell, h) \), is a triple \((\mathcal{P}, \mathcal{G}, \mathcal{B})\), where

1) \( \mathcal{P} \) is a set of \( \ell h \) points;
2) \( \mathcal{G} \) is a partition of \( \mathcal{P} \) into \( \ell \) sets (groups), each one of size \( h \);
3) \( \mathcal{B} \) is a collection of \( \ell \)-subsets of \( \mathcal{P} \) (blocks);
4) each block meets each group in exactly one point;
5) any pair of points from different groups is contained in exactly one block.

The properties of a transversal design \( \text{TD}(\ell, h) \) which will be useful for our constructions are summarized in the following lemma [J].

Lemma 11. Let \( (\mathcal{P}, \mathcal{G}, \mathcal{B}) \) be a transversal design \( \text{TD}(\ell, h) \). The number of points is given by \( |\mathcal{P}| = \ell h \), the number of groups is given by \( |\mathcal{G}| = \ell \), the number of blocks is given by \( |\mathcal{B}| = h^2 \), and the number of blocks that contain a given point is equal to \( h \).

Let TD be a transversal design \( \text{TD}(\rho, \alpha), \rho \leq \alpha + 1 \), with block size \( \rho \) and group size \( \alpha \). Let \( \text{CTD} \) be an \((n, \alpha, \rho)\) FR code based on a transversal design \( \text{TD}(\rho, \alpha) \) we have

\[
M(k) \geq k\alpha - \binom{k}{2} + \rho\binom{b}{2} + bt.
\]

Remark 1. Note, that for all \( k \geq \rho + 1 \), the file size of the FR code \( \text{CTD} \) is strictly larger than the MBR capacity.

Note that the incidence matrix of the transversal design \( \text{TD}(2, \alpha) \) is equal to the incidence matrix of the \((2\alpha, 2)\)-Turán graph, and hence in this case \( \text{CTD} = \text{CT} \).

Example 2. Let \( \text{TD} \) be a transversal design \( \text{TD}(3,4) \) defined as follows: \( \mathcal{P} = \{1, 2, \ldots, 12\} \); \( \mathcal{G} = \{G_1, G_2, G_3\} \), where \( G_1 = \{1, 2, 3, 4\} \); \( G_2 = \{5, 6, 7, 8\} \); and \( G_3 = \{9, 10, 11, 12\} \); \( \mathcal{B} = \{B_1, B_2, \ldots, B_{16}\} \), where \( B_1 = \{1, 5, 9\} \); \( B_2 = \{1, 6, 10\} \); \( B_3 = \{1, 7, 11\} \); \( B_4 = \{1, 8, 12\} \); \( B_5 = \{2, 5, 10\} \); \( B_6 = \{2, 6, 9\} \); \( B_7 = \{2, 7, 12\} \); \( B_8 = \{2, 8, 11\} \); \( B_9 = \{3, 5, 12\} \); \( B_{10} = \{3, 6, 11\} \); \( B_{11} = \{3, 7, 10\} \); \( B_{12} = \{3, 8, 9\} \); \( B_{13} = \{4, 5, 11\} \); \( B_{14} = \{4, 6, 12\} \); \( B_{15} = \{4, 7, 9\} \); and \( B_{16} = \{4, 8, 10\} \).

The placement of symbols from a codeword of the corresponding MDS code of length 16 is shown in Fig. 3. The values of a file size \( M(k) \) for \( 1 \leq k \leq 4 \) are given in the following table.

| \( k \) | \( M(k) \) |
|---|---|
| 1 | 4 |
| 2 | 7 |
| 3 | 9 |
| 4 | 11 |

Remark 2. The conditions on the parameters of TD such that the bound on the file size of an FR code \( \text{CTD} \) from Theorem [12] attains the recursive bound in [13] can be found in [16].

Similarly to an FR code \( C_G \) with \( \rho = 2 \) based on a graph \( G \) with girth \( g \), one can consider an FR code \( C_{gp} \) based on a generalized \( g \)-gon (generalized polygon \( GP \) [3]) for \( \rho > 2 \). One can prove that the size of \( C_G \) is identical to the file size of \( C_{gp} \) for \( k \leq g + \lceil \frac{g}{2} \rceil - 2 \) given in Theorem [9]. However, a generalized \( g \)-gon is known to exist only for \( g \in \{3, 4, 6, 8\} \). This observation also holds for a general biregular bipartite graph of girth \( 2g \), not only the incidence graph of a generalized polygon.

Remark 3. Note that the problem of constructing FR codes with \( \rho > 2 \) also can be considered in terms of bipartite expander graphs (see e.g. [3]). Let \( G_{Ex} = (L \cup R, E) \) be a bipartite expander and let \( C_{Ex} \) be the FR code such that the subset \( N_i, 1 \leq i \leq n \), corresponds to the \( i \)-th vertex in \( L \) and the symbol \( j, 1 \leq j \leq \theta \), corresponds to the \( j \)-th vertex in \( R \). \( |L| = n \) and \( |R| = \theta \). Then calculating \( M(k) \) can be described by calculating the number of neighbours of any subset of \( L \) of size \( k \). In other words, for an FR code with file size \( M(k) \) it should hold that \( |\Gamma(A)| \geq M(k) \) for every \( A \subseteq L \) of size \( k \), where \( \Gamma(A) \) denotes the set of neighbours of \( A \). Hence, to have an FR code with file size \( M(k) \), one need to construct a \((k, \frac{M(k)}{k})\) expander graph, where \( \frac{M(k)}{k} \) is its expansion factor [5].

V. FRACTIONAL REPETITION BATCH CODES

In this section we propose a new type of codes for DSS, called fractional repetition batch (FRB) codes, which enable uncoded efficient exact node repairs and load balancing which can be performed by several users in parallel. An FRB code is a combination of an FR code and an uniform combinatorial batch code.

The family of codes called batch codes was proposed in [8] for load balancing in distributed storage. A batch code stores \( \theta \) (encoded) data symbols in \( n \) system nodes in such a way that any batch of \( t \) data symbols can be decoded by reading at most one symbol from each node. In a \( \rho \)-uniform combinatorial batch code, proposed in [12], each node stores a subset of data symbols and no decoding is required during retrieval of any batch of \( t \) symbols. Each symbol is stored in exactly \( \rho \) nodes and hence it is also called a replication based batch code. A \( \rho \)-uniform combinatorial batch code is denoted by \( p-(\theta, N, t, n)\)-
CBC, where $N = \rho \theta$ is the total storage over all the $n$ nodes. These codes were studied in [8], [12], [17].

Next, we provide a formal definition of FRB codes. This definition is based on the definitions of a DRESS code and a uniform combinatorial batch code. Let $f \in \mathbb{F}_q^n$ be a file of size $M$ and let $c_f \in \mathbb{F}_q^n$ be a codeword of an $(\theta, M)$ MDS code which encodes the data $f$. Let $\{N_1, \ldots, N_m\}$ be a collection of $\alpha$-subsets of the set $[\theta]$. A $\rho - (n, M, k, \alpha, t)$ FRB code $C$, $k \leq \alpha$, $t \leq M$, represents a system of $n$ nodes with the following properties:

1. Every node $i$, $1 \leq i \leq n$, stores $\alpha$ symbols of $c_f$ indexed by $N_i$;
2. Every symbol of $c_f$ is stored in $\rho$ nodes;
3. From any set of $k$ nodes it is possible to reconstruct the stored file $f$, in other words, $M = \min_{i=1}^n |\cup_i N_i|$;
4. Any batch of $t$ symbols from $c_f$ can be retrieved by downloading at most one symbol from each node.

Note that the retrieval of any batch of $t$ symbols can be performed by $t$ different users in parallel, where each user gets a different symbol.

In the following, we present our constructions of FRB codes which are based on the uniform batch codes from [12] and [17] and on FR codes considered in Sections III and IV.

**Theorem 13.**

1. If $K_{a,\alpha}$ is a complete bipartite graph with $\alpha > 2$, then $C_{K_{a,\alpha}}$ is a $2 - (2\alpha, M, k, \alpha, 5)$ FRB code with $M = k\alpha - \left[\frac{k\alpha}{2}\right]$.
2. If $G$ is an $\alpha$-regular graph on $n$ vertices with girth $g$, then $C_G$ is a $2 - (n, M, k, \alpha, 2g - \left[\frac{g}{2}\right] - 1)$ FRB code with $M = \left\{ \begin{array}{ll} k\alpha - k + 1 & \text{if } k \leq g - 1 \\ k\alpha - k & \text{if } g \leq k \leq g + \left[\frac{g}{2}\right] - 2 \end{array} \right.$.
3. Let TD be a resolvable transversal design $TD(\alpha - 1, \alpha)$, for a prime power $\alpha$. $C_{TD}$ is an $(\alpha - 1 - (\alpha^2 - \alpha, M, k, \alpha, 2\alpha - 2 - 1)$ FRB code with $M \geq k\alpha - \left(\frac{k\alpha}{k}\right) + xy$, where $x, y$ are nonnegative integers which satisfy $k = x(\alpha - 1) + y$, $y \leq \alpha - 2$.

**Example 3.**

- Consider the code $C_{K_{3,3}}$ based on $K_{3,3}$ (see also Example for an FR code based on $K_{3,3}$). By Theorem for $k = 3$, $C_{K_{3,3}}$ is a $2 - (6, 7, 3, 3, 5)$ FRB code.
- Consider the code $C_{TD}$ based on the resolvable transversal design $TD = TD(3, 4)$ (see also Example for an FR code based on $TD(3, 4)$). By Theorem for $k = 4$, $C_{TD}$ is a $3 - (12, 11, 4, 4, 11)$ FRB code, which stores a file of size $11$ and allows for retrieval of any (coded) $11$ symbols, by reading at most one symbol from a node. In particular, when using a systematic MDS code, $C_{TD}$ provides load balancing in data reconstruction.

**Remark 4.** Similarly to FR codes, the problem of constructing for FRB codes can be considered in terms of bipartite expanders (see Remark). The construction of batch codes based on (unbalanced) expander graphs was proposed in [8].

To construct an FRB code, one need a bipartite expander with two different expansion factors, $M(k)/k$ and $I$, for two sides $L$ and $R$ of a graph, respectively.

**VI. Conclusion**

We considered the problem of constructing optimal FR codes and as a consequence, optimal DRESS codes. We presented constructions of FR codes based on Turán graphs, graphs with a given girth, transversal designs, and generalized polygons. Based on a connection between FR codes and batch codes, we proposed a new family of codes for DSS, FRB codes, which have the properties of batch codes and FR codes simultaneously. These are the first codes for DSS which allow uncoded efficient exact repairs and load balancing.

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