Abstract

Let $G$ be a finite $p$-group acted on faithfully by a group $A$. We prove that if $A$ fixes every element of order dividing $p$ ($4$ if $p = 2$) in a specified subgroup of $G$, then both $A$ and $[G,A]$ behave regularly, that is the elements of order dividing any power $p^i$ in each one of them form a subgroup; moreover $A$ and $[G,A]$ have the same exponent, and they are nilpotent of class bounded in terms of $p$ and the exponent of $A$. This leads in particular to a solution of a problem posed by Y. Berkovich. In another direction we discuss some aspects of the influence of a $p$-group $P$ on the structure of a finite group which contains $P$ as a Sylow subgroup, under assumptions like every element of order $p$ ($4$ if $p = 2$) in a given term of the lower central series of $P$ lies in the center of $P$.

Keywords: automorphisms, finite $p$-groups

1. Introduction

Let $G$ be a finite group acted on by a group $A$. It is convenient to say that $A$ acts $p$-centrally on $G$ if $A$ fixes every element of order dividing $p$ ($4$ if $p = 2$) in $G$.

For a positive integer $k$, the left normed commutator $[x_1, x_2, \ldots, x_k]$ in $k$ elements of an ambient group, can be defined by induction, $[x_1] = x_1$ and $[x_1, x_2, \ldots, x_k] = [[x_1, x_2, \ldots, x_{k-1}], x_k]$.

We define $\gamma_k(G,A)$ to be the subgroup of $G$ generated by all the left normed commutators $[x_1, x_2, \ldots, x_n]$, $n \geq k$, where the $x_i$'s lie in $G \cup A$ in such a way that $x_1 \in G$, and at least $k - 1$ of them lie in $A$. Note that if one takes the natural action of $G$ on itself, $\gamma_k(G,G)$ coincides with $\gamma_k(G)$ the $k$th term of the lower central series of $G$. Moreover we have $\gamma_k(G,A)$ is an $A$-invariant normal subgroup of $G$; this fact will be used freely below.

In [7], M. Isaacs proved that if $A$ is cyclic and acts $p$-centrally on $[G,A]$, for all the primes $p$ dividing $|G|$, then there is a severe restriction on the structure of $[G,A]$ in terms of $n$ the order of $A$; for instance $[G,A]$ is nilpotent of class bounded by $n$, and has exponent dividing $n$. The first purpose of this paper is to show in one hand that an analogue of Isaacs’ result holds under the weaker condition that $A$ is a group of automorphisms of $G$ acting $p$-centrally on $\gamma_p(G,A)$, and on the other hand to show that such a severe restriction applies also on $A$.

**Theorem 1.1.** Let $G$ be a finite $p$-group and $A$ be a group of automorphisms of $G$, such that $A$ acts $p$-centrally on $\gamma_p(G,A)$. Then for all positive integer $i$,

(i) the elements of $[G,A]$ of order dividing $p^i$ form a subgroup;
(ii) the elements of $A$ of order dividing $p'$ form a subgroup;

(iii) $\exp([G,A]) = \exp(A)$;

(iv) the nilpotency class of both $A$ and $[G,A]$ does not exceed $n + p - 2$, where $p^n = \exp(A)$.

This result is the best possible as shows the following example:

Let $E$ be an elementary abelian $p$-group of rank $p + 1$, and let $A$ be the automorphism group of $E$ generated by the matrix

$$
\sigma = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & \cdots & 1 & 1
\end{pmatrix}
$$

Clearly, $\gamma_p(E,A) = [E,p,A]$, and $A$ acts $p$-centrally on it. We have $\exp[E,A] = p$, however it is easy to see that for any positive integer $n$, we have (with the convention $\binom{i}{n} = 0$ for $i \geq n + 1$)

$$
\sigma^n = \begin{pmatrix}
1 & \binom{1}{n} & \binom{2}{n} & \cdots & \binom{p}{n} \\
1 & \binom{1}{n} & \ddots & \cdots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \binom{1}{n} & \binom{2}{n} & 1
\end{pmatrix}
$$

Therefore $\sigma^p \neq 1$, that is $\exp(A) \geq p^2$.

The following is an immediate consequence of Theorem 1.1

**Corollary 1.2.** Let $G$ be a finite $p$-group which acts $p$-centrally on $\gamma_p(G)$. Then $G'$ and $G/Z(G)$ have the same exponent.

Note that a particular version of Corollary 1.2 was proved by T. Laffey in [11], where he established it under the condition $\Omega(G) \leq Z(G)$. M. Y. Xu generalized Laffey’s result to the case where $G$ acts $p$-centrally on $\gamma_{p-1}(G)$, with $p$ odd (see [12, Corollary 4]). The result is also known for a special class of $p$-groups of class $\leq p$, more precisely for $p$-groups of maximal class and order $\leq p^{p+1}$. Moreover our proof implies, in fact, that $[\Omega(G/Z(G)), G] = \Omega(G')$, for any positive integer $k$.

As another application of Theorem 1.1 we solve the following problem posed by Y. Berkovich.

**Problem 1891** [3]. Do there exist a prime $p$ and a group $G$ of order $p^n$ and exponent $p$ such that $p^2$ divides $\exp(\text{Aut}(G))$?

Let $G$ be finite group of order $\leq p^n$ and exponent $p$, and let $A$ be a $p$-Sylow of $\text{Aut}(G)$. According to Lemma 2.4 (iii) below, $\gamma_{i+1}(G,A) < \gamma_i(G,A)$, unless $\gamma_i(G,A) = 1$. Therefore $|\gamma_i(G,A)| \leq p^{i+1}$, so that $|\gamma_p(G,A)| \leq p$. It follows that $A$ acts $p$-centrally on $\gamma_p(G,A)$, hence Theorem 1.1 yields
Corollary 1.3. Let $G$ be a non-cyclic group of order $\leq p^n$ and exponent $p$. Then a $p$-Sylow of $\text{Aut}(G)$ has exponent $p$.

A finite group $G$ which acts (by conjugation) $p$-centrally on itself is termed $p$-central (this terminology is due to A. Mann, see [5]). The $p$-central $p$-groups have many nice properties which qualify them to be dual to the powerful $p$-groups, we refer the reader to the introduction of [5] for some basic facts on $p$-central $p$-groups.

J. González-Sánchez and T. Weigel introduced in [5] a class of groups that are more general than the $p$-central ones. They called a group $G$ $p$-central of height $k$, if every element of order $p$ lies in the $k$th term of the upper central series of $G$. The first main result in their paper is

Theorem 1.4 (González-Sánchez and T. Weigel). Let $G$ be a finite $p$-central group of height $k \geq 1$, with $p$ odd. Then $G$ has a normal $p$-complement.

A natural variant of a $p$-central group of height $k$, is a group that acts $p$-centrally on the $k$th term of its lower central series. For $k = 1$ and $p$ odd, the two definitions coincide. A remarkable work in this context, which is not followed up extensively, was done by Ming-Yao Xu in [13]. He proved that a finite $p$-group ($p$ odd) satisfying $\Omega_1(G) \leq Z(G)$ should behave regularly: the exponent of $\Omega_1(G)$ does exceed $p^n$, and moreover $|G : G^p| \leq |\Omega_1(G)|$, for all positive integer $n$.

The next result deals with the analogue of Theorem 1.4 for this dual class. Note that this generalizes Lemma B in [7], with only a slight more effort. The proof follows easily from Frobenius’ normal $p$-complement theorem (see [6, Theorem 4.5, p 253]).

Theorem 1.5. Let $G$ be a finite group which acts $p$-centrally on $\gamma_i(G)$ for some positive integer $i$. Then $G$ has a normal $p$-complement.

Let us note that the original proof of Theorem 1.4 is based on Quillen stratification (see [5, Theorem 3.1]), as well as Quillen’s $p$-nilpotency criterion (see [5, Theorem 3.3]). We will give below a more elementary proof of it, which is based on Theorem 1.5 and hence on the classic Frobenius’ normal $p$-complement theorem. Note also that our proof covers the prime 2, however in that case we have to assume that $G$ is $4$-central of height $k \geq 1$.

Now we turn our attention to $p$-soluble groups. Assume that $G$ is a finite $p$-soluble group, and that a $p$-Sylow of $G$ is $p$-central ($4$-central if $p = 2$) of height $k$. In [5] it is proved that if $k \leq p - 2$ and $p \neq 2$, then the $p$-length of $G$ is $\leq 1$. In a subsequent paper E. Khukhro (see [10]) generalized this result and showed that the $p$-length of $G$ is bounded above by $2m + 1$, where $m$ is the largest integer satisfying $p^m - p^{m-1} \leq k$. By means of a theorem of P. Hall and G. Higman (see [8, Theorem A (ii)]), such a result holds if one can find an appropriate bound on the exponent of a $p$-Sylow of $G/\Omega_{p^r} G$). We have the following analogues of Khukhro’s results.

Theorem 1.6. Let $G$ be a finite $p$-soluble group such that $\Omega_p(G) = 1$. If a $p$-Sylow $P$ of $G$ acts $p$-centrally on $\gamma_k(P)$, for some positive integer $k$; then the exponent of a $p$-Sylow of $G/\Omega_p G$ does not exceed $p^m$, where $m$ is the largest integer satisfying $p^m - p^{m-1} \leq k$.

This corollary follows easily from [8, Theorem A (ii)] for $p$ odd, and from [4] for $p = 2$.

Corollary 1.7. Let $G$ be a finite $p$-soluble group such that a $p$-Sylow $P$ of $G$ acts $p$-centrally on $\gamma_k(P)$, for some positive integer $k$. Then the $p$-length of $G$ is bounded above by $2m + 1$, where $m$ is the largest integer satisfying $p^m - p^{m-1} \leq k$. 


The notation in this paper is standard. Note only that $\Omega(G)$ stands for $\Omega_1(G)$ if $p$ is odd, and $\Omega_2(G)$ if $p = 2$; and $\Omega_0(G)$ denotes the set of all elements of $G$ having order dividing $p'$. The basic results on regular $p$-groups can be found in [3, Kap III], as well as in [1]. We shall use them freely in the paper.

The remainder of the paper is divided into two sections. Section 2 is devoted to proving Theorem 1.1, and Section 3 to proving the remaining theorems stated in the introduction.

2. $p$-central action on $p$-groups

First, we collect some basic facts about the series $\gamma_i(G,A)$ defined in the introduction.

Lemma 2.1. Let $G$ be a finite group and $A$ be a group acting on $G$. Then we have

1. $[\gamma_i(G,A),\gamma_j(A)] \leq \gamma_{i+j}(G,A)$, for $i, j \geq 1$;
2. $\gamma_{i+1}(G,A) = (\gamma_i(G,A), A_n G)$, $n \geq 0$;
3. If $A$ and $G$ are finite $p$-groups, then $\gamma_{i+1}(G,A) < \gamma_i(G,A)$, unless $\gamma_i(G,A) = 1$.

Proof. 1. We proceed by induction on $j$. Assume that $j = 1$, then $c$ is a generator of $\gamma_i(G,A)$ and $a \in A$. Since $\gamma_i(G,A)$ is normal in $G$ and $A$-invariant, the claim follows for $j = 1$. Assume now that the result holds for $j$, it follows that $[\gamma_i(G,A), \gamma_j(A), A] \leq [\gamma_{i+j}(G,A), A] \leq \gamma_{i+j+1}(G,A),$ and $[A, \gamma_i(G,A), \gamma_j(A)] \leq [\gamma_{i+1}(G,A), \gamma_j(A)] \leq \gamma_{i+j+1}(G,A)$; the Three Subgroups Lemma yields $[\gamma_{i+j}(G,A), \gamma_j(A)] \leq \gamma_{i+j+1}(G,A)$.

2. It follows from the first property that all the subgroups $[\gamma_i(G,A), A_n G]$ lie in $\gamma_{i+1}(G,A)$. Conversely let $c = [x_1, x_2, \ldots, x_s]$ be a generator of $\gamma_{i+1}(G,A)$, and let be $s = \sup\{i \mid x_i \in A\}$. We have $s \geq i + 1$, so $[x_1, x_2, \ldots, x_{s-1}] \in \gamma_i(G,A)$, hence $c \in [\gamma_i(G,A), A_{s-1} G]$.

3. Assume for a contradiction that $\gamma_{i+1}(G,A) = \gamma_i(G,A) \neq 1$. Let be $N = \gamma_i(G,A) \cap Z(G)$; we have $\gamma_i(G/N, A) = \gamma_i(G,A)/N$. By induction on the order of $G$, if $\gamma_i(G,A) \neq N$ then $\gamma_i(G/N, A) N < \gamma_i(G,A)$ which contradicts our assumption. Thus we have $\gamma_i(G,A) = N$. It follows that $[\gamma_i(G,A), A_n G] = 1$ for all $n \geq 0$, so by the second property we have $\gamma_{i+1}(G,A) = 1$, a contradiction.

In the same spirit, the Three Subgroups Lemma yields

Lemma 2.2. Let $G$ be a group and $A$ be a group acting on $G$. Let $k$ be an integer $\geq 2$, $H = [G, A]$, and assume that $A$ acts $p$-centrally on $\gamma_k(G,A)$. Then $\Omega(\gamma_{k-1}(H)) \leq \Omega(\gamma_k(G,A)) \leq Z(H)$.

Proof. We have $[\Omega(\gamma_k(G,A)), G, A] = [A, \Omega(\gamma_k(G,A)), G] = 1$. The Three Subgroups Lemma yields $[H, \Omega(\gamma_k(G,A))] = 1$. Since $\gamma_k(G,A) \leq \gamma_2(G,A) = H$, it follows that $\Omega(\gamma_k(G,A)) \leq Z(H)$.

Now we claim that $\gamma_{k-1}(H) \leq \gamma_k(G,A)$. For $k = 2$ this is trivial. Assume that this is proved for $k$, and put $K = \gamma_k(G,A)$. We have $[K, A, G] \leq \gamma_{k+1}(G,A)$, and $[G, K, A] \leq [K, A] \leq \gamma_{k+1}(G,A)$. It follows again from the Three Subgroups Lemma that $[H, K] \leq \gamma_{k+1}(G,A)$. By assumption $\gamma_{k-1}(H) \leq K$, therefore $\gamma_k(H) \leq \gamma_{k+1}(G,A)$.

In [2] Ming-Yao Xu showed that if a finite $p$-group $G$, $p$ odd, satisfies $\Omega_1(\gamma_{p-1}(G)) \leq Z(G)$, then $G$ is strongly semi- $p$-abelian, in other words $G$ satisfies the property:

$$(xy^{-1})^{p^n} = 1 \iff x^{p^n} = y^{p^n} \text{ for any positive integer } n.$$
Such a group shares many properties with the regular ones (see [13]). For instance the exponent of $\Omega_2(G)$ does exceed $p^n$, and $|G : G^p| \leq \Omega_2(G)$. The above does not hold for $p = 2$ as shows the quaternion group. However if one requires that $\Omega_2(G) \leq \mathbb{Z}(G)$ we cover the case of 2-central 2-groups which is covered for instance in [7, Corollary 2.2]. Hence we have

**Lemma 2.3** (Ming-Yao Xu). Let $G$ be a finite $p$-group such that $\Omega(\gamma_{p^{-1}}(G)) \leq \mathbb{Z}(G)$. Then for any positive integer $n$, we have

(i) the elements of $G$ of order dividing $p^n$ form a subgroup;

(ii) for $p$ odd, $|G : G^{p^n}| \leq |\Omega_2(G)|$.

The above result combined with Lemma 2.2 yields

**Corollary 2.4.** Let $G$ be a finite $p$-group and $A$ be a group acting on $G$, such that $A$ acts $p$-centrally on $\gamma_p(G, A)$. Then for any positive integer $n$, and for $H = [G, A]$, we have

(i) $\exp(\Omega_2(H)) \leq p^n$;

(ii) for $p$ odd, $|H : H^{p^n}| \leq |\Omega_2(H)|$.

The following lemma generalizes a result of I. M. Isaacs (see [7, Theorem 2.1]).

**Lemma 2.5.** Let $G$ be a finite $p$-group and $A$ be a group of order $p$ acting on $G$, such that $A$ acts $p$-centrally on $\gamma_p(G, A)$. Then $[G, A]$ has exponent at most $p$.

**Proof.** Assume first that $p$ is odd. By induction on $|G|$ we may assume that the result holds for any smaller $p$-group. We have $[G, A] < G$, thus by induction $[G, A, A]$ has at most exponent $p$. This yields $[G, A, A] \leq \Omega_3(H)$; since $\Omega_3(H)$ is normal in $G$, Lemma 2.3(ii) and Corollary 2.4(i) imply that $\gamma_{3}(G, A)$ has exponent $\leq p$. Our condition on the action of $A$ on $G$ implies in particular that $[\gamma_{p}(G, A), A] = 1$, and again Lemma 2.3(ii) yields $\gamma_{p+1}(G, A) = 1$.

Now let $K$ denote the semidirect product $[G, A]A$. We claim that $\gamma_{p}(K) = 1$. This follows at once if one proves that $\gamma_{i}(K) \leq \gamma_{i}(G, A)$ for all $i \geq 3$. Assume that $i = 3$. It follows easily from the Three Subgroup Lemma that $[[G, A], [G, A]] \leq \gamma_{3}(G, A)$, and clearly we have $[[G, A], A] \leq \gamma_{3}(G, A)$. This amounts to saying that $[G, A]/\gamma_{3}(G, A)$ lies in the center of $K/\gamma_{3}(G, A)$, but since $K/\gamma_{3}(G, A) \cong A$ is cyclic, we have $K/\gamma_{3}(G, A)$ is abelian. Hence $\gamma_{3}(K) \leq \gamma_{3}(G, A)$. Now by induction we may assume that $\gamma_{i-1}(K) \leq \gamma_{i}(G, A)$. We have $[\gamma_{i}(G, A), A] \leq \gamma_{i+1}(G, A)$, and the Three Subgroups Lemma implies that $[\gamma_{i}(G, A), [G, A]] \leq \gamma_{i+1}(G, A)$. Thus $[\gamma_{i-1}(K), K] \leq [\gamma_{i}(G, A), K] \leq \gamma_{i+1}(G, A)$.

As $\gamma_{p}(K) = 1$, it follows that $K$ is regular; moreover as $K' \leq \gamma_{3}(G, A)$, we have $\exp(K') \leq p$. Thus $(xy)^p = x^p y^p$ for all $x, y \in K$; in other words $K$ is $p$-abelian.

Let be $g \in G, \sigma \in A$ and set $x = [g, \sigma]$. We have

$$g^{-1}g^\sigma = x^{1 + \sigma + \ldots + \sigma^{p-1}} = 1$$

Since $K$ is $p$-abelian, we have

$$x^{1 + \sigma + \ldots + \sigma^{p-1}} = (\sigma^{p-1})^{\sigma} = x^p \sigma^{p} \sigma^p = x^p$$

This shows that $[G, A]$ is generated by elements of order not exceeding $p$; as $K$ is regular it follows that the exponent of $[G, A]$ is at most $p$. 

5
For $p = 2$, we may assume that $[G, A, A]$ has exponent $\leq 2$. Under the above notation let be $y = [x, \sigma]$. We have $1 = g^{-1} g^2 = x^2$ and $y^2 = 1$, hence $x^4 = 1$. As $A$ fixes every element of order 4 in $[G, A]$, we have $y = [x, \sigma] = 1$. Thus $x^2 = 1$ and on the other hand $\gamma_2(G, A) = 1$.

Now as $[[G, A], [G, A]] \leq \gamma_3(G, A) = 1$, it follows that $[G, A]$ is abelian, and it is generated by elements of order $\leq 2$, the result follows.

The following lemma is useful for induction.

**Lemma 2.6.** Let $G$ be a finite $p$-group and $A$ be a $p$-group acting on $G$. If $A$ acts $p$-centrally on $\gamma_k(G, A)$, for some $1 \leq k \leq p$, then the same holds if we replace $G$ by $G/\Omega_2(H)$ for any positive integer $i$, where $H$ denotes $[G, A]$.

**Proof.** Assume first that $i = 1$ and let us denote $G/\Omega_2(H)$ by $\overline{G}$. We have $\gamma_k(\overline{G}, A) = \gamma_k(G, A)$. Let $\overline{x} \in \gamma_k(\overline{G}, A)$ be an element of order $\leq p$ ($\leq 4$ if $p = 2$), so we can assume that $x \in \gamma_k(G, A)$. As $\gamma_p(G, A) \leq \gamma_k(G, A)$, it follows from Corollary 2.4(i) that $x^p \in \Omega(\gamma_k(G, A))$. Therefore $[x^p, A] = 1$, and by Lemma 2.2 $x^p$ lies in the center of $H$. Thus $x$ induces on $K = HA$ (by conjugation) an automorphism of order $\leq p$.

It follows easily that $\gamma_k(K, (x)) \leq \gamma_{k-1}(H, (x)) \leq \gamma_{k-1}(H)$. By Lemma 2.2 $\gamma_{k-1}(H) \leq Z(H)$, thus the inner automorphism induced by $x$ on $K$ acts $p$-centrally on $\gamma_k(K, (x))$, so it acts $p$-centrally on $\gamma_k(K, (x))$. It follows at once from Lemma 2.3 that $[x, A]^p = 1$, that is $[\overline{x}, A] = 1$.

Now by induction we may assume the result is true for $G/\Omega_2(H)$. Since $[G/\Omega_2(H), A] = H/\Omega_2(H)$, Corollary 2.4(i) implies that $\Omega_1([G/\Omega_2(H), A]) = \Omega_{p+1}(H)/\Omega_2(H)$, so by the first step the result holds for $G/\Omega_{2+1}(H)$.

It follows from the lemma above that

**Lemma 2.7.** Let $L$ denote $\gamma_k(G, A)$. Under the assumptions of Lemma 2.6 we have $A$ acts $p$-centrally on $L/\Omega_3(L)$, and $[A, \Omega_3(L)] \leq \Omega_{2+1}(L)$, for all positive integer $i$.

**Proof.** We have $\gamma_k(L/\Omega_3(L), A) = \gamma_k(G, A)/\Omega_3(H)/\Omega_2(H) = L/\Omega_3(H)/\Omega_2(H)$. Also $L/\Omega_3(H)/\Omega_2(H)$ and $L/L \cap \Omega_3(H)$ are canonically isomorphic as $A$-groups. It follows from Corollary 2.4(i) that $L/\Omega_3(H) = \Omega(L)$. Now Lemma 2.6 implies that $A$ acts $p$-centrally on $L/\Omega_3(L)$. Also Corollary 2.4(i) implies that $\Omega_1(L/\Omega_{2+1}(L)) = \Omega_1(L)/\Omega_{2+1}(L)$, thus $[A, \Omega_3(L)] \leq \Omega_{2+1}(L)$.

In Lemma 2.6 as well as Lemma 2.7 we assumed that $A$ is a $p$-group. The following result shows that this assumption can be dropped if one assumes that $A$ acts faithfully on $G$. The result in a seemingly weaker form is classic (see [9, Satz IV.5.12]).

**Proposition 2.8.** Let $G$ be a finite $p$-group and $A \leq \text{Aut}(G)$. If $A$ acts $p$-centrally on $\gamma_i(G, A)$ for some $i \geq 1$, then $A$ is a $p$-group.

**Proof.** Let $Q$ be a $q$-Sylow of $A$, $q \neq p$. By [10, Theorem 3.6, p 181], $[G, Q, Q] = [G, Q]$, so $[G, Q] = [G, Q]$. As $[G, Q] \leq \gamma_{i+1}(G, Q)$, it follows that $Q$ acts $p$-centrally on $[G, Q]$, and from [7, Lemma 4.1] it follows that $Q$ is a $p$-group. Thus $Q = 1$.

The collection of the lemmas above yields the following key result.
Proposition 2.9. Let $G$ be a finite $p$-group and $A$ be a group of automorphisms of $G$, such that $A$ acts $p$-centrally on $\gamma_p(G,A)$. Then an element $\sigma \in A$ satisfies $\sigma^p = 1$ if and only if $\sigma$ acts trivially on $G/\Omega_p(H)$, where $H = [G,A]$.

Proof. For $n = 1$, let be $\sigma \in A$ such that $\sigma^p = 1$. It follows immediately from Lemma 2.8 that $[G, \sigma]$ has exponent $\leq p$, that is $\sigma$ acts trivially on $G/\Omega_p(H)$.

Conversely, assume that $[g, \sigma] \in \Omega_1(H)$, for all $g \in G$. Let $K = [G, \langle \sigma \rangle]/\langle \sigma \rangle$; as we see in the proof of Lemma 2.8, $\gamma_p(K) \leq \gamma_{p+1}(G, \langle \sigma \rangle)$, and as $\gamma_p(G, \langle \sigma \rangle)$ has exponent $p$, it follows that $\gamma_{p+1}(G, \langle \sigma \rangle) = 1$. Therefore $K$ has class $\leq p - 1$, so it is regular, and since $K' \leq [G, \langle \sigma \rangle]$ has exponent $p$, $K$ is $p$-abelian. Now let be $g \in G$ and $x = [g, \sigma]$. We have

$$g^{-1} s^\sigma = x^{1 + \sigma + \ldots + \sigma^{p-1}} = (x\sigma^{p-1})^p = x^p \sigma^p = 1$$

thus $\sigma$ has order $\leq p$.

Now we proceed by induction on $n$. If $\sigma^{p^{n-1}} = 1$, then $\sigma^p$ has order at most $p^n$, so by induction $g^{-1} \sigma^p(g) \in \Omega_p(H)$ for all $g \in G$. Therefore $\sigma$ acts on $G/\Omega_p(H)$ as an automorphism of order $\leq p$.

By Lemma 2.6, $\sigma$ acts $p$-centrally on $\gamma_p(G/\Omega_p(H),A)$, therefore $[G/\Omega_p(H), \langle \sigma \rangle]$ has exponent $\leq p$ by Lemma 2.8. Thus we have $[g, \sigma]^p \in \Omega_p(H)$ for all $g \in G$, and by Corollary 2.4 (i), $[g, \sigma]^{p^{n-1}} = 1$ for all $g \in G$.

Similarly, assume that $[G, \sigma^p] \leq \Omega_{n+1}(H)$. By Corollary 2.8(i), $\exp(\Omega_{n+1}(H)) \leq p^{n+1}$, hence we have $[g, \sigma]^{p^{n+1}} = 1$, for all $g \in G$. Therefore the automorphism $\sigma^p$ induced by $\sigma$ on $G/\Omega_p(H)$ satisfies $[g\Omega_p(H), \sigma^p] = 1$ for all $g \in G$. We deduce from the first step that $\sigma^p$ has order $\leq p$. Thus $g^{-1} \sigma^p(g) \in \Omega_p(H)$ for all $g \in G$. By induction the order of $\sigma^p$ is at most $p^n$, the result follows.

Now we can prove our first main theorem.

Proof of Theorem 1.1. (i) this is Corollary 2.8(i).

(ii) Proposition 2.9 implies that $\Omega_0(A) = C_A(G/\Omega_p(H))$, thus $\Omega_0(A)$ is a subgroup of $A$.

(iii) Let be $\exp(H) = p^n$ and $\exp(A) = p^m$. We have $\Omega_0(A) = C_A(G/\Omega_p(H)) = C_A(G/H) = A$, it follows from (ii) that $p^m \leq p^n$. Again we have $A = \Omega_m(A) = C_A(G/\Omega_m(H))$. Therefore $[G, A] = H = \Omega_m(H)$. By Corollary 2.8(i), $p^n \leq p^m$.

(iv) By Lemma 2.9 $H$ acts $p$-centrally on $\gamma_p(H)$. Lemma 2.7 yields $[\Omega_0(\gamma_p(H)), H] \leq \Omega_{i-1}(\gamma_{p-1}(H))$, for all $i \geq 1$. Therefore

$$1 \leq \Omega_1(\gamma_{p-1}(H)) \leq \Omega_2(\gamma_{p-1}(H)) \leq \ldots \leq \Omega_n(\gamma_{p-1}(H)) = \gamma_{p-1}(H) \leq \gamma_p(\gamma_{p-1}(H)) \leq H$$

is a central series of $H$. This proves that $H$ is nilpotent of class $\leq n + p - 2$.

Similarly, let be $L = \gamma_p(G, A)$; Lemma 2.7 yields $[A, \Omega_i(L)] \leq \Omega_{i-1}(L)$, for all positive integer $i$. Therefore $A$ stabilizes the normal series

$$1 \leq \Omega_1(L) \leq \Omega_2(L) \leq \ldots \leq \Omega_n(L) = \gamma_p(G, A) \leq \gamma_{p-1}(G, A) \leq \ldots \leq \gamma_2(G, A) \leq G$$

It follows at once from the well known result of Kaloujnine (see Satz III.2.9) that $A$ is nilpotent of class $\leq n + p - 2$. 
\]
3. \textit{p-central action, p-nilpotency and p-solubility length}

In the following proofs we need only Proposition 2.8 which is by the way independent from the material developed in the previous section.

\textbf{Proof of Theorem 1.6.} Let $P$ be a non-trivial $p$-subgroup of $G$. We have $A = N_{G}(P)/C_{G}(P)$ acts faithfully on $P$, and $p$-centrally on $\gamma_{r}(P, A)$. By Proposition 2.8, $A$ is a $p$-groups. The result follows now from Frobenius’ criterion of $p$-nilpotency (see [6, Theorem 4.5, p 253]).

\textbf{Elementary proof of Theorem 1.4.} Let $P$ be a finite $p$-group. Let $C = C_{G}(\Omega(G))$. As $\Omega(C) \leq \Omega(G)$, we have $\Omega(C) \leq Z(C)$. Therefore $C$ acts $p$-centrally on itself. It follows from Theorem 1.5 that $C$ has a normal $p$-complement $N$, and since $N$ is characteristic in $C$, $N$ is also normal in $G$. Now we have only to show that $N$ has a $p$-power index, indeed, $\Omega(G) \leq Z_{k}(G)$, it follows that $\Omega(G)$ is nilpotent, so it is a $p$-group. On the other hand it follows by induction that $[\Omega(G)]_{g} \leq Z_{k-1}(G)$, for $i \leq k$. In particular $[\Omega(G)]_{k} \leq G$. Therefore $A = G/C$ acts faithfully on $\Omega(G)$ and $p$-centrally on $\gamma_{r}(G, A)$. From Proposition 2.8 one deduces that $|G : C|$ is a $p$-power, and clearly $|C : N|$ is a $p$-power.

The remainder part is devoted to proving Theorem 1.6. The proof is inspired by that of Khukhro in [10]. Let us quote some celebrated results that we need in the sequel.

Let $P$ be a finite $p$-group. Recall that a Thompson critical subgroup of $P$ is a characteristic subgroup $C$ having the following properties:

1. $[C, P] \leq Z(C)$;
2. $C/Z(C)$ is elementary abelian;
3. $C$ is self centralizing, that is $C_{G}(C) = Z(C)$;
4. any non trivial $p'$-automorphism of $P$ acts non trivially on $C$.

A celebrated result of Thompson asserts that every finite $p'$-group $P$ has a Thompson critical subgroup $C$ (see [6, Theorem 5.3.11]). By Proposition 2.8 any non trivial $p'$-automorphism of $P$ acts non trivially on $K = \Omega(C)$, and the first two properties of $C$ imply that $K$ has exponent $p$ ($\leq 4$ if $p = 2$). Hence we have the following well-known result.

\textbf{Theorem 3.1} (Thompson). Let $P$ be a finite $p$-group. Then $P$ has a subgroup $K$ of exponent $p$ ($\leq 4$ if $p = 2$) and class at most 2, such that any non trivial $p'$-automorphism of $P$ acts non trivially on $K$.

We need also the following weaker form of the Hall-Higman theorem (see [6, Theorem B]).

\textbf{Theorem 3.2} (Hall-Higman). Let $V$ be a vector space over a field of characteristic $p$, and $G$ be a $p$-soluble group of automorphisms of $V$ such that $O_{p}(G) = 1$. If $g$ is an element of $G$ of order $p^{n}$, then the minimal polynomial of $g$ is $(X - 1)^{r}$, where $r$ satisfies $p^{n} - p^{n-1} \leq r \leq p^{n}$.

\textbf{Proof of Theorem 1.6.} Let $p^{n}$ be the exponent of a $p$-Sylow of $G/O_{p}(G) = \overline{G}$, and $Q$ be a $p$-complement in $O_{p^{n}}(G)$.

\textbf{Claim 1.} There is $g \in G$ which normalizes $Q$ such that $g = O_{p}(G)$ has order $p^{n}$.
Indeed, let be \( y \in G \) such that \( \varphi \) has order \( p^r \). As any two \( p \)-complements in \( O_{pp'}(G) \) are conjugate, \( Q \) and \( Q' \) are conjugate in \( O_{pp'}(G) \). Moreover since \( O_{pp'}(G) = O_p(G)Q \), there is \( x \in O_p(G) \) such that \( Q' = Q^x \). Now it suffices to take \( g = xy^{-1} \).

**Claim 2.** \( \varphi \) acts faithfully on \( \overline{Q} = QO_p(G)/O_p(G) \), so that \( [Q, \varphi^{p-1}] \neq 1 \).

Indeed, as \( \overline{Q} = O_p(G) \), it follows from [6, Theorem 6.3.2] that \( \overline{Q} \) is self centralizing. Thus \( (\varphi)^p \cap \overline{Q} = 1 \).

Now let \( K \) be a subgroup of \( O_p(G) \) as in Theorem 3.1. Consider a series of \( O_p(G) \)-invariant subgroups
\[
1 \leq K_1 \leq \ldots \leq K_n = K
\]
with elementary abelian sections, on which \( O_p(G) \) acts trivially. Whence there is a well defined action (induced by conjugation) of the semi-direct product \( A = Q(\varphi) \) on each section \( K_{i+1}/K_i \).

Since \( O_p(G) = 1 \), \( O_p(G) \) is self centralizing by [6, Theorem 6.3.2]; therefore \( [Q, \varphi^{p-1}] \) is a non trivial \( p' \)-group of automorphisms of \( O_p(G) \), and it acts non trivially on \( K \) by Theorem 3.1.

Therefore \( [Q, \varphi^{p-1}] \) acts non trivially on some section \( V = K_{i+1}/K_i \).

**Claim 3.** \( O_p(A) = 1 \), and \( \overline{g} \) acts faithfully on \( V \).

Assume first that Claim 3 is true. It follows from Theorem 3.2 that \( (\overline{g} - 1)^r \neq 0 \), where \( s = p^r - p^{r-1} - 1 \). Thus for some \( x \in K \), \( [x, g] \neq 1 \). On the other hand any \( p \)-Sylow \( P \) of \( G \) acts \( p \)-centrally on \( x(P) \), and since \( K \) has exponent \( p \) if \( p = 2 \) we have \( [x, g] = 1 \). This implies that \( k \geq s + 1 = p^r - p^{r-1} \).

It remains to prove Claim 3. As \( (\overline{g})^p \) is a \( p \)-Sylow in \( A \), if \( O_p(A) \neq 1 \), then \( \overline{g}^{p-1} \in O_p(A) \). Hence \( 1 \neq [Q, \varphi^{p-1}] \leq O_p(A) \cap O_p(A) = 1 \), a contradiction. Also by definition of \( V \), \( [Q, \varphi^{p-1}] \) acts non trivially on it, so \( \overline{g}^{p-1} \) acts non trivially on \( V \). Thus \( \overline{g} \) acts faithfully on \( V \).

\[
\square
\]

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