DIFFERENCE OF COMPOSITION OPERATORS BETWEEN WEIGHTED BERGMAN SPACES ON THE UNIT BALL

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ABSTRACT. We obtain some estimates for norm and essential norm of the difference of two composition operators between weighted Bergman spaces $A^p_\alpha$ and $A^q_\beta$ on the unit ball. In particular, we completely characterize the boundedness and compactness of $C_\phi - C_\psi : A^p_\alpha \to A^q_\beta$ for full range $0 < p, q < \infty$, $-1 < \alpha, \beta < \infty$.

Keywords: Bergman space, composition operator, difference, norm, essential norm.

1. INTRODUCTION

Let $B = B_n$ be the open unit ball in $\mathbb{C}^n$. For any two points $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write

$$\langle z, w \rangle = z_1 \overline{w_1} + \ldots + z_n \overline{w_n},$$

and

$$|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \ldots + |z_n|^2}.$$

Let $H(B)$ be the class of holomorphic functions on $B$. Let $\varphi$ be a holomorphic self-map of $B$. The map $\varphi$ induces a composition operator $C_\varphi$ on $H(B)$, which is defined by $C_\varphi f = f \circ \varphi$. We refer to [4, 20] for various aspects on the theory of composition operators acting on holomorphic function spaces.

Let $\rho(z, w)$ be the pseudo-hyperbolic distance between $z, w \in B$. Given two holomorphic self-maps $\varphi, \psi$ of $B$, we put

$$\rho(z) = \rho(\varphi(z), \psi(z))$$

for short. Given $\alpha, \beta > -1$ and $0 < p, q < \infty$, the joint pull-back measure $\omega_{\beta, q, \varphi, \psi}$ is defined by (see [8])

$$\omega_{\beta, q, \varphi, \psi}(E) = \int_{\varphi^{-1}(E)} \rho(z)^q d\nu_\beta(z) + \int_{\psi^{-1}(E)} \rho(z)^q d\nu_\beta(z)$$

for Borel sets $E \subset B$. For the simplicity, we denote $\omega_{\beta, q, \varphi, \psi}$ by $\omega_{\beta, q}$. So, $\omega_{\beta, q}$ is actually the sum of two pull-back measures $\rho^q d\nu_\beta \circ \varphi^{-1}$ and $\rho^q d\nu_\beta \circ \psi^{-1}$. By a

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standard argument one can verify that
\[ \int_B g d\omega_{\beta,q} = \int_B (g \circ \varphi + g \circ \psi) \rho^\beta d\nu_{\beta} \]  
for any positive Borel function \( g \) on \( B \).

Let \( d\nu \) be the normalized volume measure on \( B \). For \( \alpha > -1 \), put
\[ d\nu_{\alpha} = c_\alpha (1 - |z|^2)^\alpha d\nu(z), \]
where the constant \( c_\alpha = \frac{\Gamma(n+1+\alpha)}{n\Gamma(\alpha+1)} \) is chosen so that \( \nu_{\alpha}(B) = 1 \). For \( 0 < p < \infty \) and \( \alpha > -1 \), the weighted Bergman space \( A^p_{\alpha} = A^p_{\alpha}(B) \) is the space of all \( f \in H(B) \) such that
\[ \|f\|_{A^p_{\alpha}} = \int_B |f(z)|^p d\nu_{\alpha}(z) < \infty. \]

We will also let \( L^p_{\alpha} = L^p_{\alpha}(B) \) denote the standard Lebesgue space on \( B \) with respect to the measure \( \nu_{\alpha} \). The space \( A^p_{\alpha} \) equipped with the norm \( \| \cdot \|_{A^p_{\alpha}} \) is a Banach space for \( 1 \leq p \leq \infty \) and a complete metric space for \( 0 < p < 1 \) with respect to the translation-invariant metric \( (f, g) \mapsto \|f - g\|_{A^p_{\alpha}}^p \).

It seems better to clarify the concept of compact operators, since the weighted Bergman space \( A^p_{\alpha} \) are not Banach space when \( 0 < p < 1 \). Suppose \( X \) and \( Y \) are topologies vector spaces whose topologies induced by complete metrics. For a linear operator \( T : X \rightarrow Y \), we denote
\[ \|T\|_{X \rightarrow Y} = \sup_{\|f\|_X \leq 1} \|Tf\|_Y \]
the operator norm of \( T \), where \( \| \cdot \|_X \) and \( \| \cdot \|_Y \) denote the norm or quasi-norm of \( X \) and \( Y \). If \( \|T\|_{X \rightarrow Y} \) is finite, we say that \( T \) is a bounded operator from \( X \) to \( Y \).

A linear operator \( T : X \rightarrow Y \) is said to be compact if the image of every bounded sequence in \( X \) has a subsequence that converges in \( Y \). If \( T : X \rightarrow Y \) is a bounded linear operator, then the essential norm of the operator \( T : X \rightarrow Y \), denote by \( \|T\|_{e,X \rightarrow Y} \), is defined as
\[ \|T\|_{e,X \rightarrow Y} = \inf \{\|T - K\|_{X \rightarrow Y} : K \text{ is compact from } X \text{ to } Y\}. \]

It is obvious that the operator \( T \) is compact if and only if \( \|T\|_{e,X \rightarrow Y} = 0 \).

Efforts to understand the topological structure of the space of composition operators in the operator norm topology have led to the study of the operator \( C_\varphi - C_\psi \) of two composition operators induced by holomorphic self-maps \( \varphi, \psi \) of \( B \). In the setting of the unit disk \( D = B_1 \), by Littlewood’s Subordination Principle, all composition operators, and hence all differences of two composition operators, are bounded on the Hardy space \( H^p(D) \) and the weighted Bergman space \( A^p_{\alpha}(D) \). Thus the question of when the operator \( C_\varphi - C_\psi \) is compact naturally arises. Shapiro and Sundberg [21] raised and studied such a question on the Hardy space, motivated by the isolation phenomenon observed by Berkson [11]. After that, such related problems have been studied between several spaces of analytic functions by many authors. See, for example, [6,14,22] on Hardy spaces and [2,8,9,13,18,19] on weighted Bergman spaces.
In 2005, Moorhouse [13] characterized the compact difference of composition operators on the standard weighted Bergman space $A^2_\alpha(D)$ by angular derivative cancellation property. For $0 < p \leq q < \infty$, Saukko [18] obtained some compactness criterion for difference $C_\varphi - C_\psi$ from $A^p_\alpha(D)$ to $A^q_\alpha(D)$. In [8], Koo and Wang gave some characterizations for the boundedness and compactness of the difference $C_\varphi - C_\psi : A^p_\alpha \to A^q_\alpha$ on the unit ball. It is worth pointing out that the approach in [18, Theorem 4.5(i)] does not work as well for the unit ball. For the essential norm estimate of $C_\varphi - C_\psi : A^p_\alpha \to A^q_\alpha$, Saukko’s approach [18] is only valid for $1 < p \leq q < \infty$. The main idea consists in approximating the identity operator by a sequence of compact operators, which fits well with the study of the difference of composition operators on these spaces. In [18], this sequence was the finite rank operators $S_n$ which map $A^p_\alpha$ to $n$th-partial sum of the Taylor series of $f$. However, this sequence is uniformly bounded only for $p > 1$. The first goal of this paper is to study the norm and the essential norm of the difference of composition operators $C_\varphi - C_\psi : A^p_\alpha \to A^q_\beta$ for $0 < p \leq q < \infty$. Our work requires certain new approach and substantial amount of extra works.

The first two main results of this paper are the following theorems. For the simplicity, we denote
\[
\Gamma(\varphi, \psi) = \left( \frac{(1 - |a|^2)^s}{1 - \langle a, \varphi(z) \rangle^{(\alpha+1+\alpha)s}} + \frac{(1 - |a|^2)^s}{1 - \langle a, \psi(z) \rangle^{(\beta+1+\alpha)s}} \right).
\]

**Theorem 1.1.** Let $0 < p \leq q < \infty$, $-1 < \alpha, \beta < \infty$. Suppose $\varphi$ and $\psi$ are holomorphic self-maps of $\mathbb{B}$, and denote $\lambda = q/p$. Then the operator $C_\varphi - C_\psi$ maps $A^p_\alpha$ into $A^q_\beta$ if and only if the joint pull-back measure $\omega_{\beta,q}$ s a $(\lambda, \alpha)$-Carleson measure. Furthermore,
\[
\|C_\varphi - C_\psi\|_{A^p_\alpha \to A^q_\beta}^q \asymp \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \Gamma(\varphi, \psi) \rho(z)^s d\nu_\beta(z)
\]
for some (or equivalent for all) $s > 0$.

**Theorem 1.2.** Let $0 < p \leq q < \infty$, $-1 < \alpha, \beta < \infty$. Suppose $\varphi$ and $\psi$ are holomorphic self-maps of $\mathbb{B}$. Then the operator $C_\varphi - C_\psi$ maps $A^p_\alpha$ into $A^q_\beta$ if and only if the joint pull-back measure $\omega_{\beta,q}$ is a vanishing $(\lambda, \alpha)$-Carleson measure. Furthermore,
\[
\|C_\varphi - C_\psi\|_{A^p_\alpha \to A^q_\beta}^q \asymp \limsup_{|a| \to 1} \int_{\mathbb{B}} \Gamma(\varphi, \psi) \rho(z)^s d\nu_\beta(z),
\]
for some (or equivalent for all) $s > 0$.

Saukko in [19] characterized the bounded difference of composition operators from $A^p_\alpha(D)$ into Lebesgue spaces $L^q(D, \mu)$ when $\alpha > -1$ and $p > q$. In particular, the following result was shown in [19].

**Theorem A.** Let $0 < q < p < \infty$, $\alpha > -1$ and $\mu$ a positive Borel measure on $D$. Denote $s := p/(p - q)$. Let $\varphi$ and $\psi$ be analytic self-maps of $D$, and denote $\sigma(z) = \frac{\varphi(z) - \psi(z)}{1 - \varphi(z) \psi(z)}$ for every $z \in D$. Then the following are equivalent:

(i) the operator $C_\varphi - C_\psi$ maps $A^p_\alpha(D)$ into $L^q(D, \mu)$;
(ii) the operators $\sigma C_\varphi$ and $\sigma C_\psi$ map $A^p_\alpha(D)$ into $L^q(D, \mu)$;
(iii) the function
\[
K_{\varphi,\psi}(z) := \int_D \left| \left( \frac{1 - |z|^2}{1 - \overline{z}\varphi(w)} \right)^{\frac{\alpha+q}{p}} - \left( \frac{1 - |z|^2}{1 - \overline{z}\psi(w)} \right)^{\frac{\alpha+q}{p}} \right|^q d\mu(w)
\]
belongs to $L^s(D, A_\alpha)$.

The last main result of this paper (Theorem 1.3) gives an estimate for the norm of the difference of composition operators $C_\varphi - C_\psi$:

\[A^p_\alpha \rightarrow A^q_\beta, \quad 0 < q < p < \infty.\]

Saukko [19] pointed out that, by Pitt’s theorem, the operator $C_\varphi - C_\psi : A^p_\alpha \rightarrow L^q(\mu)$ is compact, whenever it is bounded, for $1 \leq q < p < \infty$. In fact, we can prove that this phenomenon is true for full range $0 < q < p < \infty$. Our method is new even for the case of the unit disk.

**Theorem 1.3.** Let $0 < q < p < \infty$, $-1 < \alpha, \beta < \infty$. Suppose $\varphi$ and $\psi$ are holomorphic self-maps of $B$. Set $t = \frac{p}{p-q}$. Then the following are equivalent:

(i) the operator $C_\varphi - C_\psi : A^p_\alpha \rightarrow A^q_\beta$ is bounded;
(ii) the operator $C_\varphi - C_\psi : A^p_\alpha \rightarrow A^q_\beta$ is compact;
(iii) the joint pull-back measure $\omega_{\beta,q}$ is a $(\lambda, \alpha)$-Carleson measure. Furthermore,

\[
\|C_\varphi - C_\psi\|_{A^p_\alpha \rightarrow A^q_\beta} \asymp \left\| \int_B \left( \frac{(1 - |a|^2)^s}{1 - \langle a, \varphi(z) \rangle^{n+1+\alpha+s}} + \frac{(1 - |a|^2)^s}{1 - \langle a, \psi(z) \rangle^{n+1+\alpha+s}} \right) \rho(z)^q d\nu_\beta(z) \right\|_{L^t(B, \nu_\alpha)},
\]

for some (or equivalent for all) $s > 0$.

The present paper is organized as follows. In Section 2, we give some notations and preliminary results which will be used later. In Sections 3, we give the proofs for Theorems 1.1, 1.2 and 1.3. For two quantities $A$ and $B$, we use the abbreviation $A \lesssim B$ whenever there is a positive constant $C$ (independent of the associated variables) such that $A \leq CB$. We write $A \asymp B$, if $A \lesssim B \lesssim A$.

### 2. PREREQUISITES

In this section we introduce some notations and recall some well known results that will be used throughout the paper.

#### 2.1. Pseudo-hyperbolic distance.
For any $z \in B$, let $P_z$ be the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[z] = \{\lambda z : \lambda \in \mathbb{C}\}$ generated by $z$, and $Q_z$ be the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [z]$. Thus $P_0(w) = 0$, $Q_0(w) = w$ and

\[
P_z(w) = \frac{\langle w, z \rangle}{|z|^2} z, \quad Q_z(w) = w - \frac{\langle w, z \rangle}{|z|^2} z, \text{ if } z \neq 0.
\]
We denote by $\sigma_z(w)$ the Möbius transformation on $B$ that interchanges the points 0 and $z$. More explicitly,
\[
\sigma_z(w) = \frac{z - P_z(w) - \sqrt{1-|z|^2}Q_z(w)}{1 - \langle w, z \rangle}.
\]
Note that $P_z(w) = w$ when $n = 1$. It is well known that $\sigma_z$ satisfies the following properties:
\[
\sigma_z \circ \sigma_z(w) = w, \quad \text{and}
\]
\[
1 - |\sigma_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle w, z \rangle|^2}, \quad z, w \in B.
\]
For $z, w \in B$, the pseudo-hyperbolic distance between $z$ and $w$ is defined by
\[
\rho(z, w) = |\sigma_z(w)|,
\]
while the Bergman metric is given by
\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.
\]
It is also well known that the pseudo-hyperbolic metric have the following strong form of triangle inequality (see [5]):
\[
\rho(z, w) \leq \frac{\rho(z, a) + \rho(a, w)}{1 + \rho(z, a)\rho(a, w)}
\]
for all $a, z, w \in B$. For $z \in B$ and $r > 0$, the Bergman metric ball at $z$ is denoted by
\[
D(z, r) = \{w \in B : \beta(z, w) < r\},
\]
and the pseudo-hyperbolic ball at $z \in B$ with radius $r \in (0, 1)$ is given by
\[
\triangle(z, r) = \{w \in B : \rho(z, w) < r\}.
\]
Note that $\rho(z, 0) = |z|$ since $\sigma_0(z) = -z$, so $\triangle(0, r)$ is a Euclidean ball $|z| < r$. For any $z \neq 0$, $\triangle(z, r)$ is an ellipsoid consisting of all $w \in B$ such that
\[
\frac{|P_z(w) - c|^2}{r^2t^2} + \frac{|Q_z(w)|^2}{r^2t} < 1,
\]
where
\[
c = \frac{(1 - r^2)z}{1 - r^2|z|^2}, \quad t = \frac{1 - |z|^2}{1 - r^2|z|^2}.
\]
Furthermore, if $0 < r < 1$, then the weighted volume
\[
\nu_{\alpha}(\triangle(z, r)) \asymp (1 - |z|^2)^{n+1+\alpha}.
\]
The following lemma should be known to some experts, but we cannot find a reference. So we give the proof for completeness.

**Lemma 2.1.** The pseudo-hyperbolic metric
\[
\rho(z, w) \leq \left| \frac{z - w}{1 - \langle z, w \rangle} \right|,
\]
for all $z, w \in B$. 

Proof. Since
\[ |z - P_z(w) - \sqrt{1 - |z|^2}Q_z(w)|^2 \]
\[ = |P_z(z - w) + \sqrt{1 - |z|^2}Q_z(z - w)|^2 \]
\[ = |P_z(z - w)|^2 + (1 - |z|^2)|Q_z(z - w)|^2 \]
\[ = (1 - |z|^2)|z - w|^2 + |(z - w, z)|^2 \leq |z - w|^2, \]
we get the desired result. □

2.2. Local estimates and test functions. The following lemmas are crucial in our work and will be repeatedly used throughout the paper.

Lemma 2.2. Let \(0 < p \leq q < \infty, \alpha > -1\) and \(0 < s < r < 1\) be arbitrary. Then there exists a constant \(C = C(p, q, \alpha, s, r)\) such that
\[ |f(z) - f(a)|^q \leq C \rho(z, a)^q |f|_{\alpha}^p \frac{|f(w)|^{p\alpha}}{(1 - |a|^2)^{(n+1+\alpha)q/p}} \]
for all \(a \in \mathbb{B}, z \in \triangle(a, s)\) and \(f \in A_\alpha^p\) with \(\|f\|_{A_\alpha^p} \leq 1\).

Proof. For the case \(p = q\) see [8, Lemma 2.2]. The case \(p > q\) can be proved similarly as [18, Lemma 3.1]. □

Let \(e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^n\), where \(1 \leq j \leq n\) and 1 is on the \(j\)-th component. For \(0 < t < 1\), let
\[ t_1 = te_1, \quad t_j = t^2 e_1 + t\sqrt{1 - t^2}e_j \quad (j = 2, \ldots, n). \]
For \(N > 0\) and \(0 < t < 1\), let
\[ t_N = 1 - N(1 - t). \]

We need the following two results from [8].

Lemma 2.3. Suppose \(s > 1\) and \(0 < r_0 < 1\). Then there are \(N = N(r_0) > 1\) and \(C = C(s, r_0)\) such that
\[ \left| \frac{1}{(1 - \langle a, t_1 \rangle^s)} - \frac{1}{(1 - \langle b, t_1 \rangle^s)} \right| + \sum_{j=1}^{n} \left| \frac{1}{(1 - \langle a, t_j t_1 \rangle^s)} - \frac{1}{(1 - \langle b, t_N t_j \rangle^s)} \right| \]
\[ \geq C \rho(a, b) \left| \frac{1}{(1 - \langle a, t_1 \rangle^s)} \right|, \]
for all \(a \in \triangle(te_1, r_0)\) with \(1 - t < \frac{1}{2N}\) and \(b \in \mathbb{B}\).

Lemma 2.4. Suppose \(0 < p < \infty, \alpha > -1\), \(a \in \mathbb{B}\) and \(0 < r_0 < 1\). Let \(\delta > 0\) such that \(t = n + 1 + \alpha + \delta > p\). Let \(N = N(r_0)\) be as in Lemma 2.3, \(|a|_N = 1 - N(1 - |a|)\) and
\[ |a|_1 = |a|e_1, \quad |a|_j = |a|^2 e_1 + |a| \sqrt{1 - |a|^2} e_j \quad (j = 2, \ldots, n). \]

Let
\[ f_{a,0}(z) = \frac{(1 - |a|^2)^{\delta/p}}{(1 - \langle z, \sigma^{-1}(|a|_1) \rangle)^{t/p}}, \]
and
\[
f_{a,j}(z) = \frac{(1 - |a|^2)^{\delta/p}}{(1 - |a|_{N}(z, \sigma^{-1}(|a|_j)))^{\delta/p}} \quad (j = 1, \ldots, n),
\]
where \(\sigma\) is a rotation which maps \(a\) to \(|a| e_1\). Then \(\sum_{j=0}^{n} \|f_{a,j}\|_{A^p_{\alpha}} \lesssim 1\), and
\[
\sum_{j=0}^{n} |f_{a,j}(z) - f_{a,j}(w)| \geq C(\alpha, r_0) \rho(z, w) |f_{a,0}(z)|
\]
for any \(z \in \triangle(a, r_0)\), \(w \in \mathbb{B}\) with \(|a| > 1 - \frac{1}{2N}\).

**Remark.** The above lemma can be found in the proof of Theorem 3.1 of [8].

### 2.3. Carleson measure

Let \(\mu\) be a positive Borel measure on \(\mathbb{B}\). For \(\lambda > 0\) and \(\alpha > -1\), we say that \(\mu\) is a \((\lambda, \alpha)\)-Bergman Carleson measure if for any two positive numbers \(p\) and \(q\) with \(q/p = \lambda\) there is a positive constant \(C > 0\) such that
\[
\int_{\mathbb{B}} |f(z)|^q d\mu(z) \leq C \|f\|_{A^p_{\alpha}}^q
\]
for any \(f \in A^p_{\alpha}\). We also denote by
\[
\|\mu\|_{\lambda, \alpha} = \sup_{f \in A^p_{\alpha}, \|f\|_{A^p_{\alpha}} \leq 1} \int_{\mathbb{B}} |f(z)|^q d\mu(z).
\]

The Bergman Carleson measure was first studied by Hastings [7], and independently by Oleinik and Pavlov [16] and Oleinik [15], and further pursued by Luecking [10, 11], Cima and Wogen [3], and many others. The statement in terms of pseudo-hyperbolic balls, essentially due to Luecking [10], is more convenient to use in this paper. The following result can be found in [17, Theorem A].

**Theorem B.** For a positive Borel measure \(\mu\) on \(\mathbb{B}\), \(0 < p \leq q < \infty\), \(-1 < \alpha < \infty\) and \(0 < r < 1\), the following are equivalent:

(i) There is a constant \(C_1 > 0\) such that for any \(f \in A^p_{\alpha}\)
\[
\int_{\mathbb{B}} |f(z)|^q d\mu(z) \leq C_1 \|f\|_{A^p_{\alpha}}^q.
\]

(ii)
\[
\|\mu\|_{\lambda, \alpha, r} = \sup_{a \in \mathbb{B}} \frac{\mu(\triangle(a, r))}{(1 - |a|^2)^{(\alpha+1+\alpha)q/p}} < \infty.
\]

(iii) There is a constant \(C_2 > 0\) such that, for some (every) \(t > 0\),
\[
\sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \frac{(1 - |a|^2)^t}{|1 - \langle z, a\rangle|^{(n+1+\alpha)q/p+t}} d\mu(z) \leq C_2.
\]

Furthermore, the constants \(C_1, C_2, \text{ and } \|\mu\|_{\lambda, \alpha, r}\) are all comparable to \(\|\mu\|_{\lambda, \alpha}\) with \(\lambda = q/p\).
We say that \( \mu \) is a vanishing \((\lambda, \alpha)\)-Bergman Carleson measure if for any two positive numbers \( p \) and \( q \) satisfying \( q/p = \lambda \) and any sequence \( \{f_k\} \) in \( A^p_\alpha \) with \( \|f\|_{A^p_\alpha} \leq 1 \) and \( f_k(z) \to 0 \) uniformly on any compact subset of \( B \),

\[
\lim_{k \to \infty} \int_B |f_k(z)|^q d\mu(z) = 0.
\]

It is well known that, for \( \lambda \geq 1 \), \( \mu \) is a vanishing \((\lambda, \alpha)\)-Bergman Carleson measure if and only if

\[
\lim_{|a| \to 1} \int_B \frac{(1 - |a|^2)^t}{|1 - \langle z, a \rangle|^{(n+1+\alpha)q/p+t}} d\mu(z) = 0
\]

for some (any) \( t > 0 \). For \( s \in (0, 1) \), denote \( B_s = \{z \in B : |z| < s\} \). Let \( \mu \) be a positive Borel measure on \( B \). We denote by \( \mu|_{B \setminus B_s} \) the restriction of \( \mu \) to \( B \setminus B_s \).

**Lemma 2.5.** Let \( 0 < p \leq q < \infty \), \(-1 < \alpha < \infty\). Suppose the positive Borel measure \( \mu \) on \( B \) is a \((\lambda, \alpha)\)-Bergman Carleson measure with \( \lambda = q/p \). Then

\[
\limsup_{|a| \to 1} \mu(\triangle(a, r)) \frac{\mu(\triangle(a, r))}{(1 - |a|^2)^{(n+1+\alpha)q/p}} = \limsup_{s \to 1} \|\mu|_{B \setminus B_s}\|_{\lambda, \alpha, r}
\]

\[
= \limsup_{|a| \to 1} \int_B \frac{(1 - |a|^2)^t}{|1 - \langle z, a \rangle|^{(n+1+\alpha)q/p+t}} d\mu(z)
\]

for some (any) \( t > 0 \) and some (any) \( r \in (0, 1) \).

**Proof.** We first prove that

\[
\limsup_{s \to 1} \|\mu|_{B \setminus B_s}\|_{\lambda, \alpha, r} = \limsup_{|a| \to 1} \frac{\mu(\triangle(a, r))}{(1 - |a|^2)^{(n+1+\alpha)q/p}}.
\]

Let \( t_r(s) = \frac{s - r}{1 - sr} \) and \( s > r \). By the ellipsoid description of \( \triangle(a, r) \), we known that the point \( c = \frac{1 - r^2}{1 - r^2|a|^2} a \) is the center of the ellipsoid \( \triangle(a, r) \), \( c \in [a] \), the intersection of \( \triangle(a, r) \) with \([a] \) is a one-dimensional disk of radius \( \frac{1 - |a|^2}{1 - r^2|a|^2} r \) and the intersection of \( \triangle(a, r) \) with \( \mathbb{C}^n \ominus [a] \) is an \((n-1)\)-dimensional Euclidean ball of radius \( r \sqrt{\frac{1 - |a|^2}{1 - r^2|a|^2}} \). Therefore, \( \triangle(a, r) \subseteq B_s \) if and only if

\[
\frac{1 - r^2}{1 - r^2|a|^2}|a| + \frac{1 - |a|^2}{1 - r^2|a|^2} r \leq s.
\]

After a calculation, we get \( \triangle(a, r) \cap (B \setminus B_s) \neq \emptyset \) if and only if \( |a| > t_r(s) \). It is easy to see that \( t_r(s) \) is continuous and increasing on \([r, 1)\), and \( \lim_{s \to 1} t_r(s) = 1 \).
Thus,
\[
\limsup_{|a| \to 1} \frac{\mu(\triangle(a, r))}{(1 - |a|^2)^{(n+1+\alpha)q/p}} = \limsup_{s \to 1} \frac{\mu(\triangle(a, r))}{(1 - |a|^2)^{(n+1+\alpha)q/p}}
\geq \limsup_{s \to 1} \frac{\mu(\triangle(a, r)) \cap (\mathbb{B} \setminus \mathbb{B}_s)}{(1 - |a|^2)^{(n+1+\alpha)q/p}}
= \limsup_{s \to 1} \frac{\mu(\triangle(a, r)) \cap (\mathbb{B} \setminus \mathbb{B}_s)}{(1 - |a|^2)^{(n+1+\alpha)q/p}}
= \limsup_{s \to 1} \|\mu|_{\mathbb{B}\setminus\mathbb{B}_s}\|_{\lambda, \alpha, r}.
\]

On the other hand, denote \( A = \limsup_{s \to 1} \|\mu|_{\mathbb{B}\setminus\mathbb{B}_s}\|_{\lambda, \alpha, r} \). For any \( \epsilon > 0 \), there exists \( 0 < t_* < 1 \), such that if \( t_* \leq s < 1 \), we have
\[
\|\mu|_{\mathbb{B}\setminus\mathbb{B}_s}\|_{\lambda, \alpha, r} < A + \epsilon.
\]

For any fixed \( s \), we know that \( \triangle(a, r) \subset \mathbb{B} \setminus \mathbb{B}_s \), as \( |a| \) close enough to 1. Therefore, there exists a \( 0 < t < 1 \), such that
\[
\|\mu|_{\mathbb{B}\setminus\mathbb{B}_s}\|_{\lambda, \alpha, r} = \sup_{a \in \mathbb{B}} \frac{\mu(\triangle(a, r) \cap (\mathbb{B} \setminus \mathbb{B}_s))}{(1 - |a|^2)^{(n+1+\alpha)q/p}}
\geq \sup_{|a| > t} \frac{\mu(\triangle(a, r))}{(1 - |a|^2)^{(n+1+\alpha)q/p}}.
\]

Hence,
\[
A + \epsilon \geq \limsup_{|a| \to 1} \frac{\mu(\triangle(a, r))}{(1 - |a|^2)^{(n+1+\alpha)q/p}}.
\]

Since \( \epsilon \) is arbitrary, we have
\[
\limsup_{s \to 1} \|\mu|_{\mathbb{B}\setminus\mathbb{B}_s}\|_{\lambda, \alpha, r} \geq \limsup_{|a| \to 1} \frac{\mu(\triangle(a, r))}{(1 - |a|^2)^{(n+1+\alpha)q/p}}.
\]

If \( z \in \triangle(a, r) \), then \( 1 - |a|^2 \leq |1 - \langle z, a \rangle| \). Thus, we have
\[
\limsup_{|a| \to 1} \frac{\mu(\triangle(a, r))}{(1 - |a|^2)^{(n+1+\alpha)q/p}} \leq \limsup_{|a| \to 1} \int_{\triangle(a, r)} \frac{(1 - |a|^2)^t}{1 - \langle z, a \rangle|^{(n+1+\alpha)q/p+t}} d\mu(z)
\leq \limsup_{|a| \to 1} \int_{\mathbb{B}} \frac{(1 - |a|^2)^t}{1 - \langle z, a \rangle|^{(n+1+\alpha)q/p+t}} d\mu(z).
\]

For any fixed \( s \), by Theorem B, we have
\[
\limsup_{|a| \to 1} \int_{\mathbb{B}} \frac{(1 - |a|^2)^t}{1 - \langle z, a \rangle|^{(n+1+\alpha)q/p+t}} d\mu(z)
= \limsup_{|a| \to 1} \left( \int_{\mathbb{B}} + \int_{\mathbb{B}\setminus\mathbb{B}_s} \right) \frac{(1 - |a|^2)^t}{1 - \langle z, a \rangle|^{(n+1+\alpha)q/p+t}} d\mu(z)
\leq \limsup_{|a| \to 1} \frac{(1 - |a|^2)^t \mu(\mathbb{B}_s)}{(1 - s)^{(n+1+\alpha)q/p+t}} + \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} \frac{(1 - |a|^2)^t}{1 - \langle z, a \rangle|^{(n+1+\alpha)q/p+t}} d\mu(\mathbb{B}\setminus\mathbb{B}_s(z))
\leq \|\mu|_{\mathbb{B}\setminus\mathbb{B}_s}\|_{\lambda, \alpha, r}.
\]
Letting \( s \) tend to 1, we get
\[
\limsup_{|a| \to 1} \int_B \frac{(1 - |a|^2)^t}{|1 - \langle z, a \rangle|^{(n+1+\alpha)q/p+t}} d\mu(z) \lesssim \limsup_{s \to 1} \|\mu|_{B_{s}}\|_{\lambda,\alpha,r}.
\]
The proof is complete. \( \square \)

**Definition 2.1.** A sequence \( \{a_k\} \) of distinct points in \( B \) is called a separated sequence in the pseudo-hyperbolic metric if \( \delta_0 := \inf_{i \neq j} \rho(a_i, a_j) > 0 \). The number \( \delta_0 \) is called the separated constant of \( \{a_k\} \). We say that the sequence \( \{a_k\} \) is \( \delta \)-separated, if \( 0 < \delta \leq \delta_0 \).

The following lemma can be found in \([5, \text{Lemma 5}] \) or \([12, \text{Lemma 3}] \).

**Lemma 2.6.** If \( \{a_k\} \) is a separated sequence in \( B \) with separation constant \( \delta_0 \). For \( z \in B \) and \( 0 < r < 1 \), let \( L \) denote the number of points in \( \{a_k\} \) that lie in the pseudohyperbolic ball \( \Delta(z, r) \). Then
\[
L \leq \left( \frac{2}{\delta_0 + 1} \right)^{2n} \frac{1}{(1 - r^2)^{n\alpha}}.
\]

**Definition 2.2.** Suppose \( 0 < r < 1 \). A sequence \( \{a_k\} \) of distinct points in \( B \) is called an \( r \)-lattice in the pseudo-hyperbolic metric if it is \( r \)-separated and \( B = \bigcup_{i=1}^{\infty} \Delta(a_k, r) \).

**Remark.** (1). In \([24]\) the definition of \( r \)-lattice is slightly different to ours but it causes no difficulties as we have only notice that \( D(a_k, r) = \Delta(a_k, \tanh(r)) \).

(2). By Lemma 2.6, similarly as \([12, \text{Lemma 4}] \), it is easy to see that there exists a \( r \)-lattice for any \( 0 < r < 1 \).

The following result is essentially due to Luecking (\([12]\) and can be found in \([17, \text{Theorem B}] \).

**Theorem C.** For a positive Borel measure \( \mu \) on \( B \), \( 0 < q < p < \infty \) and \(-1 < \alpha < \infty \), the following statements are equivalent:

(i) There is a constant \( C_1 > 0 \) such that for any \( f \in A^p_{\alpha} \)
\[
\int_B |f(z)|^q d\mu(z) \leq C_1 \|f\|_{A^p_{\alpha}}^q.
\]

(ii) The function
\[
\hat{\mu}_r(z) := \frac{\Delta(z, r)}{(1 - |z|^2)^{n+1+\alpha}}
\]
is in \( L^{p/(p-q)}(\nu_\alpha) \) for any (some) fixed \( r \in (0, 1) \).

(iii) For any \( r \)-lattice \( \{a_k\} \), the sequence
\[
\{\mu_k\} := \left\{ \frac{\mu(\Delta(a_k, r))}{(1 - |a_k|^2)^{(n+1+\alpha)q/p}} \right\}
\]
belongs to \( L^{p/(p-q)} \) for any (some) fixed \( r \in (0, 1) \).

(iv) For any \( s > 0 \), the Berezin-type transform \( B_{s, \alpha}(\mu) \) belongs to \( L_{\nu_\alpha}^{p/(p-q)} \).
Furthermore, with \( \lambda = q/p \), one has
\[
\| \hat{\mu} \|_{L^{p/(p-q)}} \lesssim \| \mu_k \|_{L^{p/(p-q)}} \lesssim B_{\lambda,\alpha}(\mu_k) \|_{L^{p/(p-q)}} \lesssim \| \mu \|_{\lambda,\alpha}.
\]

Here, for a positive measure \( \mu \), the Berezin-type transform \( B_{\lambda,\alpha}(\mu) \) is
\[
B_{\lambda,\alpha}(\mu)(z) = \int_{B} \frac{(1-|z|^2)^s}{|1 - \langle z, w \rangle|^{n+1+\alpha+s}} d\mu(w).
\]

It is well known that, when \( 0 < q < p < \infty \), that is \( 0 < \lambda = q/p < 1 \), \( \mu \) is a vanishing \((\lambda, \alpha)\)-Bergman Carleson measure if and only if \( \mu \) is a \((\lambda, \alpha)\)-Bergman Carleson measure (see [23]).

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. We first give the upper estimates for the norm of the operator \( C_{\varphi} - C_{\psi} \). Fix \( 0 < s_0 < r_0 < 1 \), set \( E = \{ z \in B : \rho(z) \geq s_0 \} \) and \( E' = B \setminus E \). Let \( f \in A^p_\alpha \) with \( \| f \|_{A^p_\alpha} \leq 1 \). Then
\[
\|(C_{\varphi} - C_{\psi})f\|_{A^q_\beta}^q = \left( \int_E + \int_{E'} \right) |f \circ \varphi(z) - f \circ \psi(z)|^q d\nu_\beta(z).
\]
Using (1), the first term is uniformly bounded above by
\[
\left( \frac{2}{s_0} \right)^q \int_E |f(z)|^q d\omega_{\beta,q}(z) \lesssim \| \omega_{\beta,q} \|_{\lambda,\alpha}^q.
\]
Applying Lemma 2.2, Fubini’s Theorem and \( 1 - |z|^2 \approx 1 - |w|^2 \) for \( z \in \triangle(w, r_0) \), we see that the second term is bounded by
\[
C \int_B |f(w)|^p \int_{\phi^{-1}(\triangle(w, r_0)) \cap \{ \rho(z) < s_0 \}} \rho(z)^q d\nu_\beta(z) d\nu_\alpha(w)
\lesssim \int_B |f(w)|^p \left( \frac{\omega_{\beta,q}(\triangle(w, r_0))}{(1 - |w|^2)^{(n+1+\alpha)q/p}} d\nu_\alpha(w)
\lesssim \| \omega_{\beta,q} \|_{\lambda,\alpha, r_0}.
\]

Next, we give the lower estimate for the norm of the operator \( C_{\varphi} - C_{\psi} \). Denote \( r_1 = 1 - \frac{1}{2N} \), where \( N \) is defined as in Lemma 2.3. By Lemma 2.4, we have
\[
\| C_{\varphi} - C_{\psi} \|_{A^q_\beta \to A^q_\beta}^q \gtrsim \sup_{|a| > r_1} \sum_{j=0}^n \| (C_{\varphi} - C_{\psi})f_{a,j} \|_{A^q_\beta}^q
\gtrsim \sup_{|a| > r_1} \int_{\phi^{-1}(\triangle(a, r_0))} \rho(z)^q d\nu_\beta(z),
\]
where we used the fact \( |1 - \langle z, a \rangle| \approx 1 - |a| \) for \( z \in \triangle(a, r_0) \). Similarly
\[
\| C_{\varphi} - C_{\psi} \|_{A^q_\beta \to A^q_\beta}^q \gtrsim \sup_{|a| > r_1} \int_{\phi^{-1}(\triangle(a, r_0))} \rho(z)^q d\nu_\beta(z).
\]
Therefore,
\[ \|C_\varphi - C_\psi\|_{A_0^q}^q \geq \sup_{|a| > r_1} \frac{\omega_{\beta,q}(\triangle(a,r_0))}{(1 - |a|^2)^{(n+1+\alpha)q/p}}. \]
For \(|a| \leq r_1\), take \(r_2 = \frac{r_1 + r_1}{1 + r_1}\), then \(\triangle(a,r_0) \subset \triangle(0,r_2)\). Therefore, by Lemma 2.1,
\[
\frac{\omega_{\beta,q}(\triangle(a,r_0))}{(1 - |a|^2)^{(n+1+\alpha)q/p}} \leq \frac{1}{(1 - r_1^2)^{(n+1+\alpha)q/p}} \left( \int_{\varphi^{-1}(\triangle(a,r_0))} \rho(z)^q d\nu_\beta(z) + \int_{\psi^{-1}(\triangle(a,r_0))} \rho(z)^q d\nu_\beta(z) \right)
\leq \frac{1}{(1 - r_2^2)^{(n+1+\alpha)q/p}} \left( \int_{\varphi^{-1}(\triangle(a,r_0))} |\varphi(z) - \psi(z)|^q d\nu_\beta(z) + \int_{\psi^{-1}(\triangle(a,r_0))} |\varphi(z) - \psi(z)|^q d\nu_\beta(z) \right)
\leq \|C_\varphi - C_\psi\|_{A_0^q}^q.
\]
Then the result follows by Theorem B and (I). The proof is complete. \(\Box\)

**Lemma 3.1.** Let \(0 < p \leq q < \infty, -1 < \alpha, \beta < \infty\). There is a constant \(C > 0\) such that
\[ \|C_\varphi - C_\psi\|_{A_0^q}^q, A_0^q \rightarrow A_\beta^q \geq C \lim_{|a| \rightarrow 1} \sup \sum_{j=0}^n \|\(C_\varphi - C_\psi\) f_{a,j}\|_{A_\beta^q}^q. \]
Here \(f_{a,j}\) is defined in Lemma 2.4.

**Proof.** Let \(K\) be a compact operator from \(A_0^p\) into \(A_\beta^q\). Consider the operator on \(H(\mathbb{B})\) defined by
\[ K_m(f)(z) = f\left(\frac{m}{m+1}z\right), \quad m \in \mathbb{N}. \]
Denote \(R_m = I - K_m\). It is easy to see that \(K_m\) is compact on \(A_\beta^q\) and
\[ \|K_m\|_{A_\beta^q \rightarrow A_\beta^q} \leq 1, \quad \|R_m\|_{A_\beta^q \rightarrow A_\beta^q} \leq 2 \]
for any positive integer \(m\). Then we have
\[
2\|C_\varphi - C_\psi\|_{A_0^p}^q - K\|_{A_0^q}^q \geq \|R_m \circ (C_\varphi - C_\psi - K)\|_{A_0^p}^q - K\|_{A_0^q}^q \geq \sup_{a \in \mathbb{B}} \|R_m \circ (C_\varphi - C_\psi - K)(f_{a,j})\|_{A_\beta^q}^q.
\]
Since \(K\) is compact, we can extract a sequence \(\{a_i\} \subset \mathbb{B}\) such that \(|a_i| \rightarrow 1\) and \(K f_{a,i,j}\) converges to some \(f_j \in A_\beta^q\) for \(j = 0, 1, \ldots, n\). So, when \(0 < q < 1\), we get
\[
\|R_m \circ (C_\varphi - C_\psi - K)(f_{a,i,j})\|_{A_\beta^q}^q \geq \|R_m \circ (C_\varphi - C_\psi)(f_{a,i,j})\|_{A_\beta^q}^q - \|R_m \circ K(f_{a,i,j})\|_{A_\beta^q}^q \geq \|C_\varphi - C_\psi(f_{a,i,j})\|_{A_\beta^q}^q - \|K_m \circ (C_\varphi - C_\psi)(f_{a,i,j})\|_{A_\beta^q}^q
\]
\[ - \|R_m(K(f_{a,i,j} - f_j))\|_{A_\beta^q}^q - \|R_m(f_j)\|_{A_\beta^q}^q. \quad (3)\]
Since \( \varphi(\frac{m}{m+1}\mathbb{B}) \) and \( \psi(\frac{m}{m+1}\mathbb{B}) \) are contained in a compact subset of \( \mathbb{B} \), the Cauchy-Schwartz inequality yields, for every \( z \in \mathbb{B} \)

\[
|K_m \circ (C\varphi - C\psi)(f_{a,0})(z)| \leq \frac{(1 - |a|^2)^{\delta/p}}{|1 - \langle \varphi(\frac{m}{m+1}z), \sigma^{-1}(|a|e_1) \rangle |^{1/p}} + \frac{(1 - |a|^2)^{\delta/p}}{|1 - \langle \psi(\frac{m}{m+1}z), \sigma^{-1}(|a|e_1) \rangle |^{1/p}} 
\leq C_m(1 - |a|^2)^{\delta/p}.
\]

Similarly, when \( j = 1, 2, \ldots, n \),

\[
|K_m \circ (C\varphi - C\psi)(f_{a,j})(z)| \leq C_m(1 - |a|^2)^{\delta/p},
\]

for some finite constant \( C_m \) independent of \( z \). Therefore, letting \( i \to \infty \) and then using Fatou’s Lemma as \( m \to \infty \), by (3), we have

\[
\|C\varphi - C\psi - K\|_{A^p_\alpha \to A^q_\beta} \geq \limsup_{i \to \infty} \|(C\varphi - C\psi)(f_{a,i,j})\|_{A^q_\beta}.
\]

This remains valid for \( 1 \leq q < \infty \) by a similar argument. Therefore

\[
\|C\varphi - C\psi\|_{e,A^p_\alpha \to A^q_\beta} \geq C \limsup_{|a| \to 1} \sum_{j=0}^n \|(C\varphi - C\psi)f_{a,j}\|_{A^q_\beta}.
\]

The proof is complete. \( \square \)

**Proof of Theorem 1.2.** First, we give the upper estimate for the essential norm of the operator \( C\varphi - C\psi \). By the boundedness of the operator \( C\varphi - C\psi : A^p_\alpha \to A^q_\beta \), we have that \( \omega_{\beta,q} \) is a \((\lambda, \alpha)\)-Bergman Carleson measure. Defined by \( K_n = C\varphi_n \) for any \( n \geq 1 \), where \( \varphi_n(z) = \frac{n}{n+z} \). Every \( K_n \) trivially has a norm less than 1 and is compact on every \( A^p_\alpha \) (25). Let \( R_n = I - K_n \). Then,

\[
\|C\varphi - C\psi\|_{e,A^p_\alpha \to A^q_\beta} \leq \limsup_{k \to \infty} \|(C\varphi - C\psi)R_k\|_{A^p_\alpha \to A^q_\beta} 
\leq \limsup_{k \to \infty} \sup_{|f|_{A^p_\alpha} \leq 1} \|(C\varphi - C\psi)R_k f\|_{A^q_\beta}.
\]

Fix \( s_0 \in (0, 1) \), set \( E = \{ z \in \mathbb{B} : \rho(z) \geq s_0 \} \) and \( E' = \mathbb{B} \setminus E \). Let \( f \in A^p_\alpha \) with \( |f|_{A^p_\alpha} \leq 1 \). Then we have

\[
I_k(f) := \int_E |(C\varphi - C\psi) \circ R_k f(z)|^q d\nu_{\beta}(z)
\leq \left( \frac{2}{s_0} \right)^q \int_E |R_k f(z)|^q d\omega_{\beta,q}(z)
\leq \int_{B_s} |R_k f(z)|^q d\omega_{\beta,q}(z) + \int_{\mathbb{B} \setminus B_s} |R_k f(z)|^q d\omega_{\beta,q}(z)
:= I_{1,k}(f) + I_{2,k}(f).
\]
By Cauchy integral formula, we get

\[ |R_k(f)(z)| \leq \frac{1}{k + 1} \sup_{w \in B_s} |\Re f(w)| \lesssim \frac{1}{k + 1} \cdot \frac{2}{1 - s} \sup_{w \in B_s} |f(w)| \]

for any \( z \in B_s \) with fixed \( s \in (0, 1) \). Here \( \Re f \) denote the radial derivative of \( f \).

Therefore, for a fixed \( s \), \( \lim \sup I_{1,k}(f) = 0 \).

Now let us turn to \( I_{2,k} \). We denote by \( \omega_{\beta,q}|B_B s \) the restriction of \( \omega_{\beta,q} \) to \( B_B s \). Since \( C_\varphi - C_\psi : A^p_\alpha \to A^q_\beta \) is bounded by assumption, \( \omega_{\beta,q} \) is a \( (\lambda, \alpha) \)-Bergman Carleson measure, and \( \omega_{\beta,q}|B_B s \) is also a \( (\lambda, \alpha) \)-Bergman Carleson measure. Therefore,

\[ I_{2,k}(f) = \| R_k(f) \|_{L^q(B_B s \cap B_B s)} \lesssim \| \omega_{\beta,q}|B_B s \| \| R_k(f) \|_{A^q_\beta} \]

\[ \lesssim \| \omega_{\beta,q}|B_B s \|_{\lambda, \alpha, r}. \]

Letting \( k \to \infty \) and \( s \to 1 \) in order, we obtain

\[ \lim \sup_{k \to \infty} I_{k}(f) \lesssim \lim \sup_{s \to 1} \| \omega_{\beta,q}|B_B s \|_{\lambda, \alpha, r}. \]

Denote

\[ J_k(f) = \int_{E'} |(C_\varphi - C_\psi) \circ R_k f(z)|^q d\nu_\beta(z) \]

\[ = \left( \int_{E' \cap \varphi^{-1}(B_B s)} + \int_{E' \cap \varphi^{-1}(B \setminus B_B s)} \right) |(C_\varphi - C_\psi) \circ R_k f(z)|^q d\nu_\beta(z) \]

\[ := J_{1,k}(f) + J_{2,k}(f). \]

Let \( r_0 \in (0, 1) \) be arbitrary. It is easy to see that

\[ \lim_{k \to \infty} \sup_{\| f \|_{A^p_\alpha} \leq 1} \sup_{|z| \leq r_0} |R_k(f)(z)| = 0. \]

Thus,

\[ \lim_{k \to \infty} \sup_{\| f \|_{A^p_\alpha} \leq 1} \int_{E' \cap \varphi^{-1}(B_B s)} |C_\varphi \circ R_k(f)(z)|^q d\nu_\beta(z) = 0 \]

and

\[ \lim_{k \to \infty} \sup_{\| f \|_{A^p_\alpha} \leq 1} \int_{E' \cap \varphi^{-1}(B \setminus B_B s)} |C_\psi \circ R_k(f)(z)|^q d\nu_\beta(z) = 0, \]

here we used the fact that \( E' \cap \varphi^{-1}(B_B s) \subset \psi^{-1}(B_{s'}), \) where \( s' = \frac{s + s_{1/3}}{1 + s_{1/3}}. \) Hence

\[ \lim \sup_{k \to \infty} J_k(f) \lesssim \sup_{\| f \|_{A^p_\alpha} \leq 1} \int_F |(C_\varphi - C_\psi)f(z)|^q d\nu_\beta(z), \]

where \( F = E' \cap \varphi^{-1}(B \setminus B_B s). \) In the estimate above we also used the fact that the operators \( R_k \) are uniformly bounded. Using Lemmas 2.1 and 2.2, Fubini’s theorem
and $1 - |ϕ(z)|^2 \geq 1 - |w|^2$ for $ϕ(z) \in Δ(w, r)$ we have

$$
\int_F |(C_ϕ - C_ψ)f(z)|^q dν_β(z)
\leq \int_F ρ(z)^q \int_{Δ(ϕ(z), r)} |f(w)|^p dν_α(w)
\leq \int_F |f(w)|^p \int_{Φ^{-1}(Δ(w, r) \cap F)} ρ(z)^q dν_β(z)
\leq \int_F |f(w)|^p \int_{Φ^{-1}(Δ(w, r) \cap (B\setminus B_s))} ρ(z)^q dν_β(z)
\leq \|f\|_{A_β^q} \|ω_β,q\|_{B\setminus B_s} \|λ, α, r\|_{λ, α, r}.
$$

Letting $k \to ∞$ and $s \to 1$ in order, and using the above estimate, we have

$$
\|C_ϕ - C_ψ\|_{e, A_β^q \to A_β^q} \leq \limsup_{k \to ∞} \|C_ϕ - C_ψ\| \circ R_k \|_{A_β^q \to A_β^q}
\leq \liminf_{s \to 1} \|ω_β,q\|_{B\setminus B_s} \|λ, α, r\|_{λ, α, r}.
$$

Next, we give the lower estimate for the essential norm of the operator $C_ϕ - C_ψ$.

By Lemmas 2.4 and 3.1, we have

$$
\|C_ϕ - C_ψ\|_{e, A_β^q \to A_β^q} \geq \limsup_{|a| \to 1} \int_{Φ^{-1}(Δ(a, r_0))} \frac{ρ(z)^q}{(1 - |a|^2)^{q/2} dν_β(z)},
$$

and

$$
\|C_ϕ - C_ψ\|_{e, A_β^q \to A_β^q} \geq \limsup_{|a| \to 1} \int_{ψ^{-1}(Δ(a, r_0))} \frac{ρ(z)^q}{(1 - |a|^2)^{q/2} dν_β(z)}.
$$

Thus,

$$
\|C_ϕ - C_ψ\|_{e, A_β^q \to A_β^q} \geq \limsup_{|a| \to 1} \frac{ω_β,q(Δ(a, r_0))}{(1 - |a|^2)^{q/2} dν_β(z)}.
$$

The claim now follows by Lemma 2.5, (1) and (2). The proof is complete. □

Let us now turn to the proof of the case $0 < q < p < ∞$. For this we will make use of Khinchine’s inequality. Define the Rademacher function $r_m$ by

$$
r_m(t) = \text{sgn}(\sin(2^m πt)).
$$

The Khinchine’s inequality is the following.

**Khinchine’s inequality.** For $0 < p < ∞$, there exist constants $0 < A_p < B_p < ∞$ such that, for all natural numbers $m$ and all complex numbers $c_1, c_2, \cdots, c_m$, we have

$$
A_p \left( \sum_{j=1}^{m} |c_j|^2 \right)^{\frac{q}{2}} \leq \int_0^1 \left( \sum_{i=1}^{m} c_j r_j(t) \right)^p dt \leq B_p \left( \sum_{j=1}^{m} |c_j|^2 \right)^{\frac{q}{2}}.
$$

**Proof of Theorem 1.3.** We first prove that

$$
\|C_ϕ - C_ψ\|_{A_β^q \to A_β^q} \lesssim \|ω_β,q\|_{λ, α}.
$$
Let \( f \in A^p_\alpha \) with \( \|f\|_{A^p_\alpha} \leq 1 \) and \( r \in (0, 1) \) be fixed. We write
\[
\|f \circ \varphi - f \circ \psi\|_{A^q_{\beta}}^q
= \left( \int_{\{z \in \mathcal{B} : \rho(z) \geq r\}} + \int_{\{z \in \mathcal{B} : \rho(z) < r\}} \right) |f \circ \varphi(z) - f \circ \psi(z)|^q d\nu_\beta(z).
\]
By the assumption on the joint pull-back measure \( \omega_{\beta,q} \), the first term is bounded above by \( \|\omega_{\beta,q}\|_{\lambda, \alpha} \). Applying Lemma 2.2, Fubini’s Theorem and \( 1 - |z|^2 \asymp 1 - |w|^2 \) for all \( z \in \Delta(w, r) \) respectively, we see that the second term is bounded above by
\[
C \int_{\mathcal{B}} |f(w)| q \int_{\Delta(w, r')} \rho(z)^q d\nu_\beta(z) \frac{(1 - |w|^2)^{n+1+\alpha}}{(1 - |w'|^2)^{n+1+\alpha}} d\nu_\alpha(w)
\leq C \int_{\mathcal{B}} |f(w)| q \omega_{\beta,q}(\Delta(w, r')) \frac{|a_k|^q d\nu_\alpha(w)}{(1 - |a_k|)^{(n+1+\alpha)q/p'}},
\]
where the constants \( C \) and \( r' \in (0, 1) \) depend only on \( n, \alpha \) and \( r \). Applying the Hölder’s inequality and Theorem C we get the desired result.

Next, we prove that
\[
\|\{\omega_{\beta,q,k}\}\|_{q, \frac{q}{2}} \lesssim \|C_\varphi - C_\psi\|_{A^p_\alpha \to A^q_{\beta}}.
\]
Here
\[
\omega_{\beta,q,k} = \frac{\omega_{\beta,q}(\Delta(a_k, r_0))}{(1 - |a_k|)^{(n+1+\alpha)q/p'}}.
\]
We suppose \( M := \|C_\varphi - C_\psi\|_{A^p_\alpha \to A^q_{\beta}} < \infty \). If \( M = 0 \), it is easy to see that \( \varphi = \psi \) and \( \omega_{\beta,q} = 0 \) and the claim is straightforward. So, we suppose \( M \neq 0 \). Let \( 0 < r_0 < 1 \) and \( N = N(r_0) \) be defined as in Lemma 2.3 and denote \( r_1 = 1 - \frac{1}{2N} \). Let \( \{a_k\} \) be an \( r_0 \)-lattice of \( \mathcal{B} \) in the pseudo-hyperbolic metric with \( |a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots \). By Lemma 2.6, there exist a nonnegative number
\[
L_0 \leq \frac{(\frac{2}{r_0} + 1)^{2n}}{(1 - r_0^2)^n}
\]
such that \( |a_1| \leq |a_2| \leq \cdots \leq |a_{L_0}| \leq r_1 \) and \( |a_k| > r_1 \) for any \( k \geq L_0 + 1 \). For \( \{c_j\} \in l^p \), define
\[
g_j(z) = \sum_{k=1}^{\infty} c_k f_{a_k,j}(z), \quad j = 1, 2, \cdots, n,
\]
where \( f_{a_k,j} \) are defined as Lemma 2.4. Then \( \|g_j\|_{p_\alpha} \lesssim (\sum_{k=1}^{\infty} |c_k|^{p})^{\frac{q}{p}} \). Using the boundedness of the operator \( C_\varphi - C_\psi \), we have
\[
M^q \left( \sum_{k=1}^{\infty} |c_k|^{p} \right)^{\frac{q}{p}} \geq C \|C_\varphi - C_\psi\|_{A^p_\alpha \to A^q_{\beta}}^{q}
\geq C \int_{\mathcal{B}} \left| \sum_{k=1}^{\infty} c_k (C_\varphi - C_\psi) \circ f_{a_k,j}(z) \right|^q d\nu_\beta(z).
\]
Applying Lemma 2.4 and $|z|$. By Lemma 2.6, there exists a $c_k$ such that for any $z \in \triangle(a_k, r_0)$, we get

$$M^q \left( \sum_{k=1}^{\infty} |c_k|^p \right)^{\frac{q}{p}} \geq \int_0^1 \left( \sum_{k=1}^{\infty} c_k r_k(t)(C_\varphi - C_\psi) \circ f_{a_k,j}(z) \right)^q dt \, d\nu_\beta(z)$$

$$\geq A_q \int_\mathbb{B} \left( \sum_{k=1}^{\infty} |c_k|^2 |f_{a_k,j} \circ \varphi(z) - f_{a_k,j} \circ \psi(z)|^2 \right)^{\frac{q}{2}} \, d\nu_\beta(z).$$

Applying Lemma 2.4 and $|1 - \langle z, a_k \rangle| \geq 1 - |a_k|^2$ for $z \in \triangle(a_k, r_0)$, we have

$$\sum_{j=0}^{n} |f_{a_k,j} \circ \varphi(z) - f_{a_k,j} \circ \psi(z)|^2$$

$$\geq |f_{a_k,0} \circ \varphi(z)|^2 \rho(z)^2 \chi_{\varphi^{-1}(\triangle(a_k, r_0))} (z)$$

$$\geq \rho(z)^2 \chi_{\varphi^{-1}(\triangle(a_k, r_0))} (z) / (1 - |a_k|^{(n+1+\alpha)2/p}).$$

By Lemma 2.6, there exists a $K = K(r_0) \leq \left( \frac{r_0}{r_0^2} + 1 \right)^{2n}$ such that for any $z \in \mathbb{B}$, there are at most $K$ elements of $\{a_k\}$ lying in $\triangle(z, r_0)$. Therefore

$$(n + 1) M^q \left( \sum_{k=1}^{\infty} |c_k|^p \right)^{\frac{q}{p}}$$

$$\geq A_q \max \{1, (n + 1)^{\frac{q}{p}} \}$$

$$\cdot \int_\mathbb{B} \left( \sum_{k=1}^{\infty} |c_k|^2 \sum_{j=0}^{n} |f_{a_k,j} \circ \varphi(z) - f_{a_k,j} \circ \psi(z)|^2 \right)^{\frac{q}{2}} d\nu_\beta(z)$$

$$\geq \int_\mathbb{B} \left( \sum_{k=L_0+1}^{\infty} |c_k|^2 \rho(z)^2 \chi_{\varphi^{-1}(\triangle(a_k, r_0))} (z) / (1 - |a_k|^{(n+1+\alpha)2/p}) \right)^{\frac{q}{2}} d\nu_\beta(z)$$

$$\geq \max \{1, K^{\frac{q}{p}} - 1 \} \int_\mathbb{B} \sum_{k=L_0+1}^{\infty} |c_k|^q \rho(z)^q \chi_{\varphi^{-1}(\triangle(a_k, r_0))} (z) / (1 - |a_k|^{(n+1+\alpha)q/p}) d\nu_\beta(z)$$

$$\geq \sum_{k=L_0+1}^{\infty} |c_k|^q \int_{\varphi^{-1}(\triangle(a_k, r_0))} \rho(z)^q d\nu_\beta(z) / (1 - |a_k|^{(n+1+\alpha)q/p}).$$

Change the roles of $\varphi$ and $\psi$, we get

$$(n + 1) M^q \left( \sum_{k=1}^{\infty} |c_k|^p \right)^{\frac{q}{p}} \geq \sum_{k=L_0+1}^{\infty} |c_k|^q \int_{\varphi^{-1}(\triangle(a_k, r_0))} \rho(z)^q d\nu_\beta(z) / (1 - |a_k|^{(n+1+\alpha)q/p}).$$
Therefore
\[ M_q\left(\sum_{k=1}^{\infty} |c_k|^p\right)^{\frac{q}{p}} \gtrsim \sum_{k=L_0+1}^{\infty} |c_k|^q \frac{\omega_{\beta,q}(\triangle(a_k, r_0))}{(1 - |a_k|)^{(n+\alpha)q/p}}. \]

Let
\[ b_k = \frac{1}{M_q} \frac{\omega_{\beta,q}(\triangle(a_k, r_0))}{(1 - |a_k|)^{(n+\alpha)q/p}}, k = 1, 2, 3, \ldots. \]

Since \( \{c_k\} \in l^p \) is arbitrary, we deduce
\[ (b_{L_0+1}, b_{L_0+2}, \ldots) \in (l^p)^* = l^{\frac{p}{p-q}} \]
and there exists a constant \( C > 0 \) such that
\[ \sum_{k=L_0+1}^{\infty} \left( \frac{\omega_{\beta,q}(\triangle(a_k, r_0))}{(1 - |a_k|)^{(n+\alpha)q/p}} \right)^{\frac{p}{p-q}} < CM_{\frac{p}{p-q}}. \]

For \( 1 \leq i \leq L_0 \), we have \( |a_i| \leq 1 - \frac{1}{2N} \). Denote \( r_1 = 1 - \frac{1}{2N} \) and \( r_2 = \frac{r_0 + r_1}{1 + r_0 r_1} \). Then \( \triangle(a_i, r_0) \subset \triangle(0, r_2) \). By Lemma 2.1, we have
\[
\begin{align*}
    b_i &\leq \frac{1}{M_q(2N)^{(n+\alpha)q/p}} \left( \int_{\phi^{-1}(\triangle(a_i, r_0))} \rho(z)^q d\nu_{\beta}(z) + \int_{\psi^{-1}(\triangle(a_i, r_0))} \rho(z)^q d\nu_{\beta}(z) \right) \\
    &\leq \frac{1}{M_q(2N)^{(n+\alpha)q/p}} \left( \int_{\phi^{-1}(\triangle(a_i, r_0))} \frac{|\phi(z) - \psi(z)|^q}{(1 - r_2)^q} d\nu_{\beta}(z) + \int_{\psi^{-1}(\triangle(a_i, r_0))} \frac{|\phi(z) - \psi(z)|^q}{(1 - r_2)^q} d\nu_{\beta}(z) \right) \\
    &\leq \frac{1}{M_q(2N)^{(n+\alpha)q/p}} \cdot \frac{2M_q \|z\|^q_{A_\beta^q}}{(1 - r_2)^q} \\
    &\leq C(r, N, n, p, q, \alpha).
\end{align*}
\]

Thus, we get
\[ \{b_1, b_2, \ldots\} \in l^{\frac{p}{p-q}} \]
and there exist a constant \( C > 0 \) such that
\[ \sum_{k=1}^{\infty} \left( \frac{\omega_{\beta,q}(\triangle(a_k, r_0))}{(1 - |a_k|)^{(n+\alpha)q/p}} \right)^{\frac{p}{p-q}} < CM_{\frac{p}{p-q}}. \]

Hence, \( \omega_{\beta,q} \) is a \((\lambda, \alpha)\)-Bergman Carleson measure, and
\[ \|\{\omega_{\beta,q,k}\}\|_{\nu/(\nu-q)} \leq \|C_\varphi - C_\psi\|_{A_\alpha^\lambda \to A_\beta^q}. \]

Therefore, by the above discussion and Theorem C, we get
\[
\begin{align*}
    \|C_\varphi - C_\psi\|_{A_\alpha^\lambda \to A_\beta^q} &\leq \|\omega_{\beta,q}\|_{\lambda, \alpha} \lesssim \|\{\omega_{\beta,q,k}\}\|_{\nu/(\nu-q)} \\
    &\lesssim \|\omega_{\beta,q}\|_{\nu/(\nu-q)} \lesssim \|B_{s,\alpha}(\omega_{\beta,q})\|_{L_\nu/(\nu-q)}. \end{align*}
\]

Finally, we suppose that \( \omega_{\beta,q} \) is a \((\lambda, \alpha)\)-Bergman Carleson measure, and then prove that the operator \( C_\varphi - C_\psi : A_\alpha^p \to A_\beta^q \) is compact. Let \( \{f_k\} \) be any sequence in \( A_\alpha^p \) with \( \|f_k\|_{A_\alpha^p} \leq 1 \) and \( f_k \to 0 \) uniformly on compact subsets of \( \mathbb{B} \). Then by
the Remark after Theorem C, \( \omega_{\beta,q} \) is a vanishing \((\lambda, \alpha)\)-Bergman Carleson measure. Therefore, we have

\[
\lim_{k \to \infty} \int_{\mathbb{B}} |f_k(z)|^q d\omega_{\beta,q}(z) = 0.
\]

For \( r \in (0, 1) \) fixed, we write

\[
\|f_k \circ \varphi - f_k \circ \psi\|_{A_\beta^q}^q = \left( \int_{\{z \in \mathbb{B} : \rho(z) \geq r\}} |f_k \circ \varphi(z) - f_k \circ \psi(z)|^q d\nu_{\beta}(z) \right)^{1/q}.
\]

The first term is uniformly bounded above by

\[
\left( \frac{2}{r'} \right)^q \int_{\{z \in \mathbb{B} : \rho(z) \geq r\}} |f_k(z)|^q d\omega_{\beta,q}(z).
\]

For any fixed \( s \in (0, 1) \), we see that the second term is bounded above by

\[
C \int_{\mathbb{B}} |f_k(w)|^q \frac{\omega_{\beta,q}(\Delta(w, r'))}{(1 - |w|^2)^{n+1+\alpha}} d\nu_{\alpha}(w) = C \left( \int_{\mathbb{B}_s} + \int_{\mathbb{B} \setminus \mathbb{B}_s} \right) |f_k(w)|^q \frac{\omega_{\beta,q}(\Delta(w, r'))}{(1 - |w|^2)^{n+1+\alpha}} d\nu_{\alpha}(w) := I_1 + I_2,
\]

where the constants \( C \) and \( r' \in (0, 1) \) depend only on \( n, \alpha \) and \( r \). Since \( \omega_{\beta,q}(\mathbb{B}) \leq 2 \), we have

\[
I_1 \lesssim \int_{\mathbb{B}_s} |f_k(z)|^q d\nu_{\alpha}(z).
\]

Applying the Hölder’s inequality, we have

\[
I_2 \lesssim \|f_k\|_{A^p_{\beta}}^q \left\| \frac{\omega_{\beta,q}(\Delta(w, r'))}{(1 - |w|^2)^{n+1+\alpha}} \right\|_{L^r(\mathbb{B} \setminus \mathbb{B}_s, \omega_{\beta,q})} \leq \left\| \frac{\omega_{\beta,q}(\Delta(w, r'))}{(1 - |w|^2)^{n+1+\alpha}} \right\|_{L^r(\mathbb{B} \setminus \mathbb{B}_s, \omega_{\beta,q})}.
\]

Letting \( k \to \infty \), we obtain

\[
\limsup_{k \to \infty} \|f_k \circ \varphi - f_k \circ \psi\|_{A_\beta^q}^q \lesssim \left\| \frac{\omega_{\beta,q}(\Delta(w, r'))}{(1 - |w|^2)^{n+1+\alpha}} \right\|_{L^r(\mathbb{B} \setminus \mathbb{B}_s, \omega_{\beta,q})}.
\]

Since \( \omega_{\beta,q} \) is a \((\lambda, \alpha)\)-Bergman Carleson measure, we have

\[
\frac{\omega_{\beta,q}(\Delta(w, r'))}{(1 - |w|^2)^{n+1+\alpha}} \in L^r(\mathbb{B}, \omega_{\beta,q}).
\]

Thus, letting \( s \to 1 \), by the Lebesgue Dominated Convergence Theorem, we get

\[
\lim_{k \to \infty} \|f_k \circ \varphi - f_k \circ \psi\|_{A_\beta^q} = 0.
\]

Therefore, \( C_\varphi - C_\psi : A^p_{\alpha} \to A^q_{\beta} \) is compact. The proof is complete. \( \square \)
Remark. The methods we used to prove Theorem 1.1-1.3 can be generalized to the case when $\nu_\beta$ is replaced by a positive Borel measure $\mu$ by using the same technique.

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