MEAGER COMPOSANTS OF TREE-LIKE CONTINUA

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Abstract. A subset $M$ of a continuum $X$ is called a meager composant if $M$ is maximal with respect to the property that every two of its points are contained in a nowhere dense subcontinuum of $X$. Motivated by questions of Bellamy, Mouron and Ordoñez, we show that no tree-like continuum has a proper open meager composant, and that every tree-like continuum has either 1 or $2^\aleph_0$ meager composants. We also prove a decomposition theorem: If $X$ is tree-like and every indecomposable subcontinuum of $X$ is nowhere dense, then the partition of $X$ into meager composants is upper semi-continuous and the space of meager composants is a dendrite.

1. Introduction

A continuum is a compact connected metric space. Given a continuum $X$ and a point $x \in X$, then

$$M_x = \bigcup \{L \subset X : L \text{ is a nowhere dense subcontinuum of } X \text{ and } x \in L\}$$

is called the meager composant of $x$ in $X$. More generally, $M \subset X$ is a meager composant of $X$ if there exists $x \in X$ such that $M = M_x$. Note that

$$\mathcal{M}_X = \{M_x : x \in X\}$$

partitions $X$ into pairwise disjoint sets. This partition is topological in the sense that homeomorphisms respect its members.

Meager composants of continua were introduced by David Bellamy in [2] as “tendril classes”. They were subsequently investigated by Chris Mouron and Norberto Ordoñez in [6], and by the current author in [4]. Bellamy asked if there exists a continuum with a proper open meager composant (i.e. a meager composant which is open in $X$ but not equal to $X$) [2, Problem 25]. In this article we will show that there is no tree-like example.

Mouron and Ordoñez proved that $\mathcal{M}_X$ is upper semi-continuous if $X$ is locally connected, hereditarily arcwise connected, or irreducible and hereditarily decomposable [6, Corollary 8.2]. They asked to identify other classes of continua for which $\mathcal{M}_X$ is upper semi-continuous [6, Problem 8.1]. Here we will add the class of tree-like continua whose indecomposable subcontinua are nowhere dense. We will also prove that each tree-like continuum has either 1 or $2^\aleph_0$ meager composants. In general it is not evident that the cardinality of $\mathcal{M}_X$ must be 1 or $2^\aleph_0$ [6, Problem 8.5]. There may even be a continuum with exactly 2 meager composants, one of which is a singleton.

Question 1 ([6, Problem 8.8]). Is there a continuum $X = O \cup \{p\}$ where $O$ and $\{p\}$ are meager composants of $X$?

2010 Mathematics Subject Classification. 54F15, 54F50, 54H15.

Key words and phrases. continuum, tree-like, hereditarily unicoherent, upper semi-continuous, meager composant.
Outline of the paper. The only property of tree-like continua that we will use is hereditary unicoherence (defined in Section 2). As such, all theorems will be stated and proved for hereditarily unicoherent continua. The results are organized as follows.

Suppose that $X$ is a hereditarily unicoherent continuum.

- In Section 4 we will prove that $X$ has no proper open meager composant.
- In Section 5 we will show that every meager composant of $X$ is closed if and only if every indecomposable subcontinuum of $X$ is nowhere dense.
- In Section 6 we will show that if every meager composant of $X$ is closed, then $M_X$ is upper semi-continuous and the space $M_X$ is a dendrite.
- In Section 7 we will apply results from Sections 5 and 6 to show $|M_X| \in \{1, 2^\aleph_0\}$.

We suspect that most of these statements are also true in the context of homogeneity. In the Appendix at the end of the paper, we will begin investigating meager composants of homogeneous continua, and pose some questions for further research.

2. Definitions and basic notions

A continuum $X$ is tree-like if for every $\varepsilon > 0$ there is an acyclic graph $T$ (a tree) and a mapping $f : X \to T$ such that $\text{diam}(f^{-1}(t)) < \varepsilon$ for each $t \in T$. A dendrite is a locally connected continuum which contains no simple closed curve. Dendrites are tree-like [7, Theorem 10.32].

A continuum $X$ is hereditarily unicoherent if for every two subcontinua $A$ and $B$ the intersection $A \cap B$ is connected. Tree-like continua are hereditarily unicoherent [7, p. 232].

The composant of a point $x \in X$ is defined to be the union of all proper subcontinua of $X$ that contain $x$. The meager composant of $x \in X$ is the union of all nowhere dense subcontinua of $X$ that contain $x$.

A continuum $X$ is decomposable if $X$ can be written as the union of two of its proper subcontinua; otherwise $X$ is indecomposable. Equivalently, $X$ is indecomposable if every proper subcontinuum of $X$ is nowhere dense [7, Exercise 6.19]. In an indecomposable continuum, composants and meager composants are the same and are not closed [7, Proposition 11.14].

The decomposition $M_X$ is upper semi-continuous if for every closed $A \subset X$ the union $\bigcup \{M_x : x \in A\}$ is closed in $X$ [7, Chapter III]. If $M_X$ is upper semi-continuous, then it is a continuum in the quotient topology [7, Theorem 3.10].

3. Ample propositions

We begin by proving two very useful propositions which involve the notion of an ample subcontinuum. A subcontinuum $A$ of a continuum $X$ is ample if for every neighborhood $U$ of $A$, there is a continuum $K$ such that $A \subset K \subset K \subset U$ [8, Definition 2].

Proposition 1. Let $X$ be a continuum and let $K$ be a subcontinuum $X$. If $K$ is not ample then there is a nowhere dense subcontinuum of $X$ meeting $K$ and $X \setminus K$.

Proof. Suppose that $K$ is not ample. Then there is a compact neighborhood $U$ of $K$ such that if $B$ is the connected component of $K$ in $U$, then $K \not\subset B'$. Fix $x \in K \setminus B'$, and let $x_0 \in U \setminus B$ such that $d(x_0, x) < 1$. By compactness of $U$ there is a relatively clopen subset $A_0$ of $U$ such that $B \subset A_0$ and $x_0 \notin A_0$ (cf. [3, Theorem 6.1.23]). Likewise, assuming $x_0, \ldots, x_{n-1}$ and clopen sets $A_0 \supset \ldots \supset A_{n-1}$ have been defined, there exists $x_n \in A_{n-1} \setminus B$ and a clopen $A_n \subset A_{n-1}$ such that $d(x_n, x) < 1/n$ and $B \subset A_n \subset X \setminus \{x_n\}$. For each $n \geq 0$ let $C_n$ be the connected component of $x_n$ in $U$. Then $\{C_n : n \geq 0\}$ is a discrete collection
of continua in $X$, and every $C_n$ meets $\partial U$ by the boundary bumping principle [7, Theorem 5.4]. We conclude that $H = \bigcup_{n=0}^{\infty} C_n \setminus \bigcup_{n=0}^{\infty} C_n$ is a nowhere dense subcontinuum$^1$ of $X$ which contains $x \in K$ and meets $\partial U \subset X \setminus K$. \hfill $\Box$

**Proposition 2.** Let $M_x$ be a meager composant of a hereditarily unicoherent continuum $X$. If $K \subset X$ is a continuum and $\overline{M_x \cap K} = K$, then every ample subcontinuum of $K$ intersects $M_x$.

**Proof.** Let $K \subset X$ be a continuum such that $M_x \cap K$ is dense in $K$. Let $A$ be an ample subcontinuum of $K$. We may assume $A$ is nowhere dense, otherwise it trivially intersects $M_x$. Let $K_0 \supset K_1 \supset \ldots$ be a decreasing sequence of continua in $K$ such that $A \subset K_0^o$ (the interior in $K$) and $\bigcap_{n=0}^{\infty} K_n = A$. For each $n \geq 0$ choose $x_n \in M_x \cap K_n \setminus A$ and let $L_n$ be a nowhere dense subcontinuum of $X$ containing $x_n$ and $x_{n+1}$. By hereditary unicoherence of $X$, $L_n \cap K_n$ is connected. It follows that $L = \bigcup_{n=0}^{\infty} L_n \cap K_n$ is connected and $L$ is a continuum. Each compact neighborhood in $K \setminus A$ intersects only a finite number of the sets $L_n \cap K_n$. Together with the assumption that $A$ is nowhere dense, this implies that $L$ is nowhere dense. Thus $L \subset M_x$. Further, $L \cap A \neq \emptyset$ because the sequence $(x_n)$ has an accumulation point in $A$. Therefore $A \cap M_x \neq \emptyset$. \hfill $\Box$

4. **Open meager composites**

**Theorem 3.** Let $X$ be a hereditarily unicoherent continuum. If $M_x$ contains a dense open subset of $X$, then $M_x = X$.

**Proof.** Suppose that $O \subset X$ is a dense open set and $O \subset M_x$. Let $K$ be a connected component of $X \setminus O$. Then $K$ is a nowhere dense subcontinuum of $X$. If $K$ is not ample, then by Proposition 1 its meager composant must intersect $O$ and we have $K \subset M_x$. In the other case that $K$ is ample, $K \cap M_x \neq \emptyset$ by Proposition 2. Again $K \subset M_x$. This shows that each connected component of $X \setminus O$ is contained in $M_x$. Hence $M_x = X$. \hfill $\Box$

**Theorem 4** (No proper open meager composant). Let $X$ be a hereditarily unicoherent continuum. If $M_x$ is open then $M_x = X$.

**Proof.** Suppose that $M_x$ is open. Then $M_x$ is a dense open meager composant of the hereditarily unicoherent continuum $\overline{M_x}$. By Theorem 3, $M_x = \overline{M_x}$. Thus $M_x$ is a clopen subset of $X$. Since $X$ is connected, this implies $M_x = X$. \hfill $\Box$

**Corollary 5.** No tree-like continuum has a proper open meager composant.

**Example 1.** Theorem 3 cannot be improved along the lines of:

If $U \subset M_x$ is open then $\overline{U} \subset M_x$.

To see that this statement is false, consider the tree-like continuum $X \subset [0,2] \times [0,1]$ that is depicted in Figure 1. The set $I = X \cap [0,1]^2$ is an indecomposable continuum with endpoints $\langle 0,0 \rangle$ and $\langle 1,1 \rangle$ which belong to different composants of $I$. The meager composant of $\langle 0,0 \rangle$ in $X$ contains the open set $U = X \cap (1,2) \times (0,1)$. But $\overline{U}$ contains $[1,2] \times \{1\}$ which belongs to a different meager composant of $X$.

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$^1$To see that the set $H$ is connected, it suffices to let $V$ and $W$ be open subsets of $X$ such that $V \cap H \neq \emptyset$, $W \cap H \neq \emptyset$, and $H \subset V \cup W$, and show $V \cap W \neq \emptyset$. To that end, assume $x \in V$. Then there is an integer $N_1$ such that $x_n \in V$ for all $n \geq N_1$. By compactness of $X$ there exists $N_2$ such that $C_n \subset V \cup W$ for all $n \geq N_2$. Let $n \geq N_1 + N_2$ such that $C_n \cap W \neq \emptyset$. Then $C_n \subset V \cup W$, $C_n \cap V \neq \emptyset$, and $C_n \cap W \neq \emptyset$. Since $C_n$ is connected we have $V \cap W \neq \emptyset$. 

Figure 1. The continuum $X$ for Example 1

5. NON-CLOSED MEAGER COMPOSANTS

In [6, Section 4] it was shown that a hereditarily unicoherent continuum with a non-closed meager composant must contain an indecomposable continuum. In this section we will prove the stronger statement that every such continuum has an indecomposable subcontinuum with interior. The proof will involve the concept of irreducibility.

If $X$ is a continuum and $a, b \in X$ then $X$ is irreducible between $a$ and $b$ if no proper subcontinuum of $X$ contains both $a$ and $b$. In this case we define

$$A = \{ x \in X : X \text{ is irreducible between } x \text{ and } b \}$$

$$B = \{ x \in X : X \text{ is irreducible between } a \text{ and } x \}.$$  

By [7, Theorem 11.4], $A$ and $B$ are connected. Hence $\overline{A}$ and $\overline{B}$ are continua.

**Lemma 6.** Let $X$ be a continuum that is irreducible between $a$ and $b$. If $\overline{A}$ has interior, then $\overline{A}$ is indecomposable (and likewise for $\overline{B}$).

**Proof.** Suppose that $\overline{A}$ has interior in $X$. If $X$ is indecomposable then $\overline{A} = X$ and we are done. Assume now that $X$ is decomposable. Then $C = X \setminus \overline{A}$ is connected and $b \in C$. It follows that $\overline{A}$ is equal to the closure of its interior, otherwise $\overline{C}$ would be a proper subcontinuum of $X$ intersecting $A$ and containing $b$. We are now ready to show that $\overline{A}$ is incomposable. For a contradiction suppose that $\overline{A} = H \cup K$ where $H$ and $K$ are proper subcontinua of $\overline{A}$. Without loss of generality $K \cap \overline{C} \neq \emptyset$. Since $\overline{A}$ is the closure of its interior, $H \setminus K$ has interior in $X$. Therefore $K \cup \overline{C}$ is a proper subcontinuum of $X$. It intersects $A$ and contains $b$, which is a contradiction. Therefore $\overline{A}$ is indecomposable. \qed

A standard consequence of Zorn’s lemma is that for every two points $a$ and $b$ in a continuum $X$ there exists a subcontinuum of $X$ that is irreducible between $a$ and $b$ [7, Exercise 4.35]. This fact will be used in the proof below.

**Theorem 7.** Let $X$ be a hereditarily unicoherent continuum. If $M_x$ is not closed, then $\overline{M_x}$ contains an indecomposable continuum with interior in $X$.

**Proof.** Suppose that $M_x$ is not closed. Let $K$ be a subcontinuum of $\overline{M_x}$ that is irreducible between $a \in M_x$ and $b \in \overline{M_x} \setminus M_x$. Then $K$ has interior in $X$. And by Lemma 6 may assume that $A$ and $B$ (the sets of irreducibility for $K$) are nowhere dense. Let $U$ be a non-empty open subset of $X$ that is contained in $K \setminus (\overline{A} \cup \overline{B})$. Fix $a' \in U \cap M_x$, and let $L$ be a nowhere dense subcontinuum of $X$ with $\{a, a'\} \subset L$. Let $\delta = d(a', \overline{B})$. For each $n \geq 1$ let $C_n$ be the...
connected component of $B$ in the set \( \{ y \in K : d(a', y) \geq \delta/n \} \). Note that \( d(a', C_n) = \delta/n \) by the boundary bumping principle [7, Theorem 5.4]. Let \( K' = \bigcup_{n=1}^{\infty} C_n \). Then \( K' \) is a continuum and \( \{a', b\} \subset K' \). By irreducibility of \( K \) we have \( L \cup K' = K \) and each \( C_n \) misses \( L \). It follows that \( K' \) is irreducible between \( a' \) and \( b \). Let
\[
A' = \{ y \in K' : K' \text{ is irreducible between } y \text{ and } b \}.
\]

We claim that the continuum \( A' \) has interior in \( X \). This will be proved in two cases.

Case 1: \( A' \) is not ample in \( K' \). Then by Proposition 1 there is a nowhere dense continuum \( M \) which intersects \( A' \) and contains a point \( x' \in K' \setminus A' \). Let \( P \) be a proper subcontinuum of \( K' \) that contains \( x' \) and \( b \). The open set \( U \setminus P \) is non-empty as it contains \( a' \), and by irreducibility of \( K \) we have \( K = L \cup A' \cup M \cup P \). Since \( L \) and \( M \) are nowhere dense, this means that \( A' \) has interior in \( X \).

Case 2: \( A' \) is ample in \( K' \). Note that \( C_n \cap M_x = \emptyset \) for each \( n \) by irreducibility of \( K \). So \( C_n \) is nowhere dense in \( X \) and thus \( C_n \subset M_x \). Therefore \( M_x \subset K' \) is dense in \( K' \). By Proposition 2, \( A' \) intersects \( M_{\delta} \). Recall that \( A' \) also contains \( a' \in M_{\delta} \) and \( M_a \neq M_{\delta} \). Therefore \( A' \) has interior in \( X \).

By the preceding claim and Lemma 6, \( A' \) is an indecomposable continuum with interior in \( X \).

We now establish the converse of Theorem 7.

**Lemma 8.** Let \( X \) be a hereditarily unicoherent continuum. If \( K \) is an indecomposable subcontinuum of \( X \) with interior and \( M_x \cap K \neq \emptyset \), then \( M_x \cap K \) is a composant of \( K \). In particular, \( M_x \) is not closed.

**Proof.** Suppose that \( K \) is an indecomposable subcontinuum of \( X \) with interior, and \( y \in M_x \cap K \). Let \( \kappa \) be the composant of \( y \) in \( K \). Then \( \kappa \) is a meager composant of \( K \) and hence \( \kappa \subset M_x \). On the other hand, if \( L \) is any subcontinuum of \( X \) meeting two different composants of \( K \), then since \( L \cap K \) is connected we have \( K \subset L \) and thus \( L \) has interior. This shows that \( M_x \cap K \subset \kappa \). Therefore \( M_x \cap K = \kappa \). Since \( \kappa \) is not closed in \( K \), we conclude that \( M_x \) is not closed.

**Theorem 9** (Characterization). If \( X \) is a hereditarily unicoherent continuum, then every meager composant of \( X \) is closed iff every indecomposable subcontinuum of \( X \) is nowhere dense.

**Proof.** (\( \Leftarrow \)): Suppose that \( X \) has a non-closed meager composant. Then by Theorem 7, \( X \) has an indecomposable subcontinuum with interior.

(\( \Rightarrow \)): Suppose that \( X \) has an indecomposable subcontinuum \( K \) with interior. Let \( x \in K \).

By Lemma 8, \( M_x \) is not closed.

**Example 2.** Hereditary unicoherence is critical to each implication in Theorem 9. The continuum featured in [6, Section 5] is hereditarily decomposable but its meager composants are not closed. And it is easy to construct a continuum which has an indecomposable subcontinuum with interior and only 1 (closed) meager composant.

6. **Closed meager composants**

**Proposition 10.** Every closed meager composant of a continuum is ample.

**Proof.** If \( M_x \) is closed then apply Proposition 1 to the continuum \( M_x \subset X \).
Lemma 11. Let $X$ be a hereditarily unicoherent continuum. If every meager composant of $X$ is closed, then the equivalence relation $E = \{ (x, y) \in X \times X : y \in M_x \}$ is closed in $X \times X$.

Proof. Suppose that $(x_0, y_0), (x_1, y_1), \ldots \in E$ and $(x_n, y_n) \to (x, y) \in X \times X$. And suppose that $M_x$ and $M_y$ are closed. We want to show $(x, y) \in E$. To that end, for each $n \geq 0$ let $L_n$ be a nowhere dense subcontinuum of $X$ containing $x_n$ and $y_n$. We may assume that there exists $N$ such that $L_n \cap M_x = \emptyset$ for all $n \geq N$.

For a contradiction, suppose that $(x, y) \notin E$. Then $M_x$ and $M_y$ are disjoint and ample (Proposition 10). Thus there are continuum neighborhoods $C_1 \supset C_2 \supset \ldots$ and $D$ of $M_x$ and $M_y$ respectively such that $C_i \cap D = \emptyset$ and $\bigcap_{i=1}^{\infty} C_i = M_x$. For each $i \geq 1$ choose $n_i \geq N$ such that $x_{n_i} \in C_i$ and $y_{n_i} \in D$. The set

$$(L_{n_i} \cup L_{n_i+1} \cup D) \cap C_i = (L_{n_i} \cup L_{n_i+1}) \cap C_i$$

contains $\{x_{n_i}, x_{n_{i+1}}\}$ and is connected by hereditary unicoherence of $X$. So

$L = \bigcup_{i=1}^{\infty} (L_{n_i} \cup L_{n_{i+1}}) \cap C_i$

is connected and $\overline{L}$ is a continuum. Note that $\overline{L}$ is nowhere dense because it misses the interior of $M_x$ and every compact neighborhood in $X \setminus M_x$ intersects only finitely many constituents of $L$. Thus $\{x_{n_i}, x\} \subset \overline{L}$ implies $x_{n_i} \in M_x$. But $L_{n_i} \cap M_x \neq \emptyset$ contradicts our earlier assumption. Therefore $(x, y) \in E$. □

Theorem 12 (Meager decomposition). Let $X$ be a hereditarily unicoherent continuum. If every indecomposable subcontinuum of $X$ is nowhere dense, then

$${\mathcal{M}}_X = \{ M_x : x \in X \}$$

is an upper semi-continuous decomposition of $X$ and the space $M_X$ is a dendrite.

Proof. Suppose that every indecomposable subcontinuum of $X$ is nowhere dense. By Theorem 9 every meager composant of $X$ is closed. By Lemma 11 and [7, Exercise 7.17], $M_X$ is upper semi-continuous. Further, since each meager composant of $X$ is ample (Proposition 10), if $U$ is any open subset of $M_X$ and $M_x \in U$ then the connected component of $M_x$ in $U \cup {\mathcal{U}}$ is an open union of meager composants. Therefore $M_X$ is locally connected. It does not contain a simple closed curve because $X$ is hereditarily unicoherent and the quotient map $X \to M_X$ is monotone. Therefore $M_X$ is a dendrite. □

Corollary 13. If $X$ is a tree-like continuum and every indecomposable subcontinuum of $X$ is nowhere dense, then $X$ has a monotone upper semi-continuous decomposition into a dendrite whose elements are the meager composants of $X$.

7. Number of meager composants

For a hereditarily unicoherent continuum $X$ the following conditions are easily seen to be equivalent: $|{\mathcal{M}}_X| = 1$; every two points of $X$ are contained in a nowhere dense subcontinuum of $X$; every irreducible subcontinuum of $X$ is nowhere dense. The following theorem states that the only alternative to $|{\mathcal{M}}_X| = 1$ is $|{\mathcal{M}}_X| = 2^\alpha$.

Theorem 14. If $X$ is a hereditarily unicoherent continuum then $|{\mathcal{M}}_X| \in \{ 1, 2^\alpha \}$.

Proof. The proof is by cases.
Case 1: Every indecomposable subcontinuum of $X$ is nowhere dense. Then by Theorem 12 $M_X$ is a continuum. Every non-degenerate continuum has cardinality $2^\aleph_0$, so if $|M_X| > 1$ then $|M_X| = 2^\aleph_0$.

Case 2: There is an indecomposable continuum $K \subset X$ with interior. Then $K$ has $2^\aleph_0$ composants [5], and no two composants of $K$ belong to the same meager composant of $X$ (Lemma 8). Therefore $|M_X| = 2^\aleph_0$.

Corollary 15. Every tree-like continuum has exactly 1 or $2^\aleph_0$ meager composants.

We end with an application to chainable continua. A continuum $X$ is chainable if for every $\epsilon > 0$ there are open sets $U_1, \ldots, U_n$ covering $X$ such that $\text{diam}(U_i) < \epsilon$ and $U_i \cap U_j \neq \emptyset \iff |i - j| \leq 1$ for all $i, j \leq n$.

Corollary 16. Every chainable continuum has $2^\aleph_0$ meager composants.

Proof. If $X$ is a chainable continuum, then $X$ is irreducible [7, Theorem 12.4] and therefore $|M_X| > 1$. Additionally, $X$ is arc-like [7, Theorem 12.11]. By Corollary 15 we have $|M_X| = 2^\aleph_0$.

APPENDIX: HOMOGENEOUS CONTINUA

A continuum $X$ is homogeneous if for every two points $x, y \in X$ there is a homeomorphism $h$ of $X$ onto itself such that $h(x) = y$. Homogeneous continua form a class of spaces with very uniform structures, with fundamental examples such as the circle, Menger universal curve, pseudo-arc, circle of pseudo-arcs, and solenoids. We establish the following classification in terms of meager composants.

Theorem 17. If $X$ is a homogeneous continuum, then precisely one of the following holds.

(a) $X$ has only one meager composant,
(b) $X$ has proper dense meager composants, or
(c) $X$ is a circle of indecomposable continua (i.e. there is continuous decomposition of $X$ into indecomposable subcontinua such that the decomposition space is a simple closed curve).

Proof. By [9, Lemma 4.4] the closures of meager composants $\overline{M_x}$ partition $X$ and are respected by homeomorphisms. From homogeneity of $X$ it follows that if $\overline{M_x}$ has interior then $\overline{M_x}$ is (cl)open and hence $\overline{M_x} = X$. In this event $X$ falls into category (a) or (b). Alternatively, if $\overline{M_x}$ has empty interior then it is a nowhere dense subcontinuum of $X$, and so $M_x = \overline{M_x}$. By [10, Theorem 1] the decomposition $M_X$ is continuous, the space $M_X$ is a (non-degenerate) homogeneous continuum, and each $M_x$ is indecomposable. It remains to show that $M_X$ is a circle. By Proposition 10, $M_x$ is ample and so $M_X$ is locally connected. From monotonicity and lower semi-continuity of the decomposition it can be seen that if $L$ is a nowhere dense subcontinuum of $M_X$ then $\bigcup L$ is a nowhere dense subcontinuum of $X$. So the meager composants of $M_X$ are singletons. Since $M_X$ is path-connected, this implies that it contains an arc with interior. Therefore $M_X$ is a circle [1] and the proof is complete.

A characterization similar to Theorem 9 may exist for homogeneous continua. The following questions would need to be answered.

Question 2. Let $X$ be a homogeneous continuum. If $X$ has an indecomposable subcontinuum with interior, then is $M_x$ not closed?
**Question 3.** Let $X$ be a homogeneous continuum. If $X$ has proper dense meager composants, then does $X$ contain an indecomposable continuum? Is $X$ indecomposable?

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