Abstract

We continue the study of Specht modules $S_{((n-m),(1^m))}$ labelled by hook bipartitions with quantum characteristic at least three for the cyclotomic Khovanov–Lauda–Rouquier algebra $\mathcal{H}^\Lambda_n$ in level two. We obtain the irreducible labels of the composition factors of $S_{((n-m),(1^m))}$, and hence determine the ungraded decomposition matrix for $\mathcal{H}^\Lambda_n$ comprising rows corresponding to hook bipartitions. Moreover, by finding the graded dimensions of $S_{((n-m),(1^m))}$ and those of its composition factors, we provide a graded analogue.

1 Introduction

We let $F$ be an arbitrary field throughout and recall that, for $q \in F^\times$, the quantum characteristic $e$ is the smallest positive integer such that $1 + q + q^2 + \cdots + q^{e-1} = 0$, setting $e = \infty$ if no such integer exists. Moreover, we set $I := \mathbb{Z}/e\mathbb{Z}$ and identify $I$ with the set $\{0, 1, \ldots, e-1\}$.

In [S], we determined the composition series of Specht modules $S_{((n-m),(1^m))}$ labelled by hook bipartitions in level 2 of the cyclotomic Khovanov–Lauda–Rouquier algebra $\mathcal{H}^\Lambda_n$ (namely, the Iwahori–Hecke algebra of type B), with quantum characteristic $e \geq 3$. We denote the abelian category of all finitely generated graded $\mathcal{H}^\Lambda_n$-modules by $\mathcal{H}^\Lambda_n$-mod. By restriction and induction on $\mathcal{H}^\Lambda_n$-modules, we determine the irreducible labels of the factors of $S_{((n-m),(1^m))}$, and thus find the ungraded multiplicities $[S_{((n-m),(1^m))} : D]$, where $D$ is a composition factor of $S_{((n-m),(1^m))}$. We completely determine the analogous graded multiplicities $[S_{((n-m),(1^m))} : D]_v$, for an arbitrary indeterminate $v$, by exploiting the combinatorial grading on $S_{((n-m),(1^m))}$ as defined in [BKW].

Acknowledgements

This paper was written under the guidance of the author’s supervisor, Dr Matthew Fayers, at Queen Mary University of London, and forms part of her PhD thesis. The author would like to thank Dr Fayers for his many helpful comments and ongoing support.

2 Combinatorics

We introduce necessary combinatorial definitions, most of which dates back to [K3], and adopt notation introduced by Fayers [Fa2].

We first recall several important definitions given in [S]. We write $\mathcal{P}^l_n$ for the set of all $l$-multipartitions of $n$. We draw the Young diagram $[\lambda]$ of an $l$-multipartition $\lambda = \ldots \ldots \ldots}$
(\(\lambda^{(1)}, \ldots, \lambda^{(l)}\)) we draw its \(i\)th component \(\lambda^{(i)}\) above its \((i+1)\)th component \(\lambda^{(i+1)}\), for all \(i \geq 1\). For example, \(((3^2,1),(4,3),(2^2))\) has the Young diagram

```
  0 1 2 0
2 0 1 2
1 2 0 1
```

Each element \((i,j,m) \in [\lambda]\) is called a node of \(\lambda\), and in particular, an \((i,j)\)-node of the \(m\)th component \(\lambda^{(m)}\). For nodes \((i_1,j_1,m),(i_2,j_2,l) \in [\lambda]\), we say that node \((i_1,j_1,m)\) is higher than node \((i_2,j_2,l)\) if \(i_1 < i_2\) and \(m \leq l\).

For \(\lambda \in \mathcal{P}_n^l\), we say that \(A \in [\lambda]\) is a removable node for \(\lambda\) if \([\lambda]\) \{-\{A\}\} is a Young diagram of an \(l\)-multipartition. Similarly, we say that \(A \not\in [\lambda]\) is an addable node for \(\lambda\) if \([\lambda] \cup \{A\}\) is a Young diagram of an \(l\)-multipartition.

We fix an \(e\)-multicharge \(\kappa = (\kappa_1, \ldots, \kappa_l) \in I^l\). The \(e\)-residue of a node \(A = (i,j,m)\) lying in the space \(\mathbb{N} \times \mathbb{N} \times \{1, \ldots, l\}\) is defined by

\[
\text{res}_A := \kappa_m + j - i \pmod{e}.
\]

We call a node of residue \(i\) an \(i\)-node. For \(\lambda \in \mathcal{P}_n^l\), the residue content of \(\lambda\) is defined to be

\[
\text{cont}(\lambda) := \sum_{A \in [\lambda]_{\text{res}_A}} \alpha_{\text{res}_A}.
\]

We call a node \(A \in [\lambda]\) a removable \(i\)-node of \(\lambda\) if \(A\) is a removable node of \(\lambda\) and \(\text{res}_A = i\). Similarly, a node \(A \not\in [\lambda]\) is called an addable \(i\)-node of \(\lambda\) if \(A\) is an addable node of \(\lambda\) and \(\text{res}_A = i\). We denote the total number of removable \(i\)-nodes of \(\lambda\) by \(\text{rem}_i(\lambda)\), and denote the total number of addable \(i\)-nodes of \(\lambda\) by \(\text{add}_i(\lambda)\). For \(\lambda \in \mathcal{P}_n^l\), we write the multipartition obtained by removing all of the removable \(i\)-nodes from \(\lambda\) as \(\lambda^\triangledown_i\), and we write the multipartition obtained by adding all of the addable \(i\)-nodes to \(\lambda\) as \(\lambda^\blacktriangle_i\).

Let \(\lambda \in \mathcal{P}_n^l\). We define the \(i\)-signature of \(\lambda\) by reading the Young diagram \([\lambda]\) from the top of the first component down to the bottom of the last component, writing a \(+\) for each addable node and writing a \(-\) for each removable node, where the leftmost \(+\) corresponds to the highest addable node of \(\lambda\). We obtain the reduced \(i\)-signature of \(\lambda\) by successively deleting all adjacent pairs \(+-\) from the \(i\)-signature of \(\lambda\), always of the form \(-\cdots-++\cdots+\).

**Example 2.1.** Let \(e = 3\), \(\kappa = (0,0)\) and \(\lambda = ((8,4^2),(4))\). The 3-residues of \(\lambda\) are

```
0 1 2 0 1 2 0
2 0 1 2
1 2 0 1
```

and the 0-addable and 0-removable nodes of \(\lambda\) are labelled as follows

```
+  -
  +    +
  
```

The 3-residues of \(\lambda\) are
Thus, by removing all of the removable 0-nodes from \( \lambda \) (corresponding to the outlined nodes), and respectively, adding all of the removable 0-nodes from \( \lambda \) (corresponding to the shaded nodes) we have the Young diagrams of multipartitions

\[
\lambda^{\Box_0} = \begin{array}{ccc}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\end{array} \quad \text{and} \quad \lambda^\Box = \begin{array}{ccc}
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\Box & \Box & \Box \\
\end{array}
\]

Referring to Diagram (2.1), the 0-signature of \( \lambda \) is \(-+\ldots\) (corresponding to the \(-\) and \(+\) labels from top to bottom in the diagram), and the reduced 0-signature is \(-+\) (corresponding to the nodes \((1,7,1)\) and \((2,5,1)\)).

The removable \( i \)-nodes corresponding to the \(-\) signs in the reduced \( i \)-signature of \( \lambda \) are called the normal \( i \)-nodes of \( \lambda \), and similarly, we call the addable \( i \)-nodes corresponding to the \(+\) signs in the reduced \( i \)-signature of \( \lambda \) the conormal \( i \)-nodes of \( \lambda \). We denote the total number of normal \( i \)-nodes of \( \lambda \) by \( \text{nor}_i(\lambda) \), and we denote the total number of conormal \( i \)-nodes of \( \lambda \) by \( \text{conor}_i(\lambda) \). The lowest normal \( i \)-node of \( [\lambda] \), if there is one, is called the good \( i \)-node of \( \lambda \), which corresponds to the last \(-\) sign in the \( i \)-signature of \( \lambda \). Dually, the highest conormal \( i \)-node of \( [\lambda] \), if there is one, is called the cogood \( i \)-node of \( \lambda \), which corresponds to the first \(+\) sign in the \( i \)-signature of \( \lambda \).

For \( 0 \leq r \leq \text{nor}_i(\lambda) \), we denote the multipartition obtained from \( \lambda \) by removing the \( r \) lowest normal \( i \)-nodes of \( \lambda \) by \( \lambda \downarrow_i^r \), and for \( 0 \leq r \leq \text{conor}_i(\lambda) \), we denote the multipartition obtained from \( \lambda \) by adding the \( r \) highest conormal \( i \)-nodes of \( \lambda \) by \( \lambda \uparrow_i^r \). We set \( \uparrow_i^r := \uparrow_i^1 \) for adding the cogood \( i \)-node of \( \lambda \) and \( \downarrow_i^r := \downarrow_i^1 \) for removing the good \( i \)-node of \( \lambda \). For \( \lambda \in \mathcal{P}_n \), it is easy to see that \( A \) is a cogood node for \( \lambda \) if and only if \( A \) is a good node for \( \lambda \cup \{A\} \).

The operators \( \uparrow_i^r \) and \( \downarrow_i^r \) act inversely on a multipartition \( \lambda \in \mathcal{P}_n \) in the following sense

\[
\lambda \downarrow_i^r \uparrow_i^r = \lambda \quad \text{and} \quad \lambda \uparrow_i^r \downarrow_i^r = \lambda,
\]

for \( 0 \leq r \leq \text{nor}_i(\lambda) \) and \( 0 \leq s \leq \text{conor}_i(\lambda) \).

We define the set of all regular \( l \)-multipartitions of \( n \) to be the set

\[
\mathcal{R} \mathcal{P}_n^l = \{ \emptyset \uparrow_i^1 \ldots \uparrow_i^s \mid i_1, \ldots, i_n \in I \}.
\]

If a multipartition \( \lambda \) lies in \( \mathcal{R} \mathcal{P}_n^l \), then \( \lambda \) is called regular. Hence \( \lambda \in \mathcal{R} \mathcal{P}_n^l \) is regular if and only if \( [\lambda] \) is obtained by successively adding cogood nodes to \( \emptyset \). That is, we have a sequence \( \emptyset = \lambda(0), \lambda(1), \ldots, \lambda(n) = \lambda \) such that \( [\lambda(i)] \cup \{A\} = [\lambda(i + 1)] \), where \( A \) is a cogood node of \( \lambda(i) \).

We can alternatively write the set of all regular \( l \)-multipartitions of \( n \) as

\[
\mathcal{R} \mathcal{P}_n^l = \left\{ \lambda \in \mathcal{P}_n^l \mid \lambda \downarrow_i^1 \ldots \downarrow_i^n = \emptyset, \text{for some } i_1, \ldots, i_n \in I \right\}.
\]

**Example 2.2.** Suppose that \( l = 1 \). If \( e \in \{2,3,\ldots\} \) is finite, then \( \mathcal{R} \mathcal{P}_n^e \) coincides with the set of all \( e \)-regular partitions, whereas \( \mathcal{R} \mathcal{P}_n^1 = \mathcal{P}_n^1 \) if \( e = \infty \).

### 3 Graded irreducible \( \mathcal{H}_n^\Lambda \)-modules

We determine a classification of the graded irreducible \( \mathcal{H}_n^\Lambda \)-modules.

We know from [HM1] that Specht modules exhibit a graded cellular basis, so that Specht modules can be analogously studied as graded cell modules. For \( \lambda \in \mathcal{P}_n^l \), we can equip the
graded $H_n^\Lambda$-module $S_\lambda$ with a homogeneous symmetric bilinear form $\langle , \rangle$ of degree zero (see [HM1, §2]). We define the radical of $S_\lambda$ to be

$$\text{rad} S_\lambda = \{ v_T \in S_\lambda | \langle v_T, v_S \rangle = 0, \forall v_S \in S_\lambda \}.$$ 

Since $\langle v_T, v_S \rangle = 0$ whenever $\deg(v_T) + \deg(v_S) \neq 0$ (that is, $\deg(T) + \deg(S) \neq 0$), $\text{rad} S_\lambda$ is a graded $H_n^\Lambda$-submodule of $S_\lambda$. We now define the graded quotient $H_n^\Lambda$-module

$$D_\lambda := S_\lambda / \text{rad} S_\lambda,$$

for each $\lambda \in \mathcal{P}_n$. We know from [HM1, Lemma 2.9] that $D_\lambda$ is absolutely irreducible or zero, and moreover, $D_\lambda$ is a well-defined graded quotient of $S_\lambda$ since $\text{rad} D_\lambda$ is the graded Jacobson radical of $S_\lambda$.

The next result shows that the irreducible heads of $S_\lambda$ only arise from regular multipartitions, first conjectured by Ariki and Mathas in [AM].

**Theorem 3.1.** [HM1, Corollary 5.11] $D_\lambda$ is an absolutely irreducible $H_n^\Lambda$-module if and only if $\lambda \in \mathcal{RP}_n$.

Thus, these non-zero quotient $H_n^\Lambda$-modules labelled by regular multipartitions give a complete classification of the irreducible $H_n^\Lambda$-modules.

**Theorem 3.2.** [BK3, Theorem 4.11] and [HM1, Proposition 2.18]  

1. $\{ D_\lambda(i) | \lambda \in \mathcal{RP}_n, i \in \mathbb{Z} \}$ is a complete set of pairwise non-isomorphic irreducible graded $H_n^\Lambda$-modules. 

2. For all $\lambda \in \mathcal{RP}_n$, $D_\lambda \cong D_\lambda^\circ$ as graded $H_n^\Lambda$-modules.

4 Graded dimensions of Specht modules and their irreducible heads

We now consider the graded dimensions of Specht modules for $H_n^\Lambda$.

For $\lambda \in \mathcal{P}_n$, and $A$ an $i$-node of $\lambda$, recall that

$$d^A(\lambda) := \# \{ \text{addable } i\text{-nodes of } \lambda \text{ strictly above } A \} - \# \{ \text{removable } i\text{-nodes of } \lambda \text{ strictly above } A \}.$$ 

Let $T \in \text{Std}(\lambda)$ where $n$ lies in node $A$ of $\lambda$, and set $\mu = \lambda \backslash \{ A \}$. We set $\deg(\emptyset) := 0$ and recall that the degree of a standard $\lambda$-tableau $T$ is defined recursively via

$$\deg(T) := d^A(\mu) + \deg(T_{e\in n-1}),$$

where $T_{e\in n-1}$ is the standard $\mu$-tableau obtained by removing node $A$ from $T$, which contains entry $n$. By [BKW, §4], we know that the degree of a standard basis vector $v_T$ is defined by the degree of the standard $\lambda$-tableau $T$, that is, $\deg(v_T) = \deg(T)$. Thus we obtain the graded dimension of $S_\lambda$, denoted $\text{grdim}(S_\lambda)$, by summing over each degree of every possible standard $\lambda$-tableau as follows.

**Definition 4.1.** Let $\lambda \in \mathcal{P}_n$. Then the graded dimension of $S_\lambda$ is

$$\text{grdim}(S_\lambda) := \sum_{T \in \text{Std}(\lambda)} v^{\deg(T)},$$

where $v$ is an arbitrary indeterminate.
Naturally, we recover the ungraded dimension of $S_\lambda$ by setting $v = 1$.

**Example 4.2.** Let $e = 3$ and $\kappa = (0,0)$. $S_{((1),(1^4))}$ is spanned by basis vectors labelled by tableaux

\[
T_1 = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array},
T_2 = \begin{array}{c}
2 \\
1 \\
3 \\
4 \\
5 \\
\end{array},
T_3 = \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array},
T_4 = \begin{array}{c}
1 \\
1 \\
2 \\
3 \\
4 \\
\end{array},
T_5 = \begin{array}{c}
1 \\
1 \\
1 \\
2 \\
3 \\
\end{array}.
\]

We find the degree of $T_1$. We note that the degree of any node in the first component is 0, so $d^{(1,1,1)} = 0$. The $e$-residues of $((1),(1))$ are

\[
\begin{array}{c}
0 \\
0 \\
\end{array}.
\]

Thus $(1,1,2)$ has removable 0-node $(1,1,1)$, and hence $d^{(1,1,2)} = -1$. Adding this node, we observe the $e$-residues of $((1),(1^2))$

\[
\begin{array}{c}
0 \\
0 \\
2 \\
\end{array}.
\]

Thus $(2,1,2)$ has addable 2-node $(2,1,1)$, and hence $d^{(2,1,2)} = 1$. Adding this node, we observe the $e$-residues of $((1),(1^3))$

\[
\begin{array}{c}
0 \\
0 \\
2 \\
1 \\
\end{array}.
\]

Thus $(3,1,2)$ has addable 1-nodes $(1,2,1)$ and $(1,2,2)$, and hence $d^{(3,1,2)} = 2$. Finally, adding this node, we observe the $e$-residues of $((1),(1^4))$

\[
\begin{array}{c}
0 \\
0 \\
2 \\
1 \\
0 \\
\end{array}.
\]

Thus $(4,1,2)$ has removable 0-node $(1,1,1)$, and hence $d^{(4,1,2)} = -1$. Thus

\[
\deg(T_1) = d^{(1,1,1)} + d^{(1,1,2)} + d^{(2,1,2)} + d^{(3,1,2)} + d^{(4,1,2)} = 1.
\]

Similarly, one can find that $\deg(T_2) = \deg(T_5) = 3$, $\deg(T_3) = 2$ and $\deg(T_4) = 1$, and hence $\grdim(S_{((1),(1^4))}) = 2v^3 + v^2 + 2v$.

By Theorem 3.2, irreducible $\mathcal{H}^{\Lambda}_n$-modules are self-dual. As $D_\lambda$ and its dual are isomorphic as graded $\mathcal{H}^{\Lambda}_n$-modules, no grading shifts occur which leads us on to the following result.

**Proposition 4.3.** For all $\lambda \in \mathcal{PR}_n$, $\grdim(D_\lambda)$ is symmetric in $v$ and $v^{-1}$. 
Corollary 4.4. Let $\lambda \in \mathcal{R}_n^{\mathcal{P}}$ and $\mathcal{T} = \text{Std}(\lambda)$. Then

$$\text{grdim}(D_\lambda) = v^i \sum_{T \in \mathcal{T}} v^{\text{deg}(T)},$$

where $2i = -\max\text{deg}(\mathcal{T}) - \min\text{deg}(\mathcal{T})$. Moreover, the highest degree in the graded dimension of $D_\lambda$ is $\frac{1}{2}(\max\text{deg}(\mathcal{T}) - \min\text{deg}(\mathcal{T}))$.

Setting $S = \{v_T \mid T \in \mathcal{T}\}$, we have $\text{grdim}(D_\lambda) = v^i \text{grdim}(\text{span} S)$, and in other words $D_\lambda = \text{span} S_{\langle i \rangle}$.

5 Graded decomposition numbers for $\mathcal{H}_n^\Lambda$

Decomposition numbers record information about the structure of Specht modules over $\mathcal{H}_n^\Lambda$. For $\lambda \in \mathcal{P}_n^{\mathcal{P}}$ and $\mu \in \mathcal{R}_n^{\mathcal{P}}$, we denote the ungraded decomposition number by $d_{\lambda,\mu} = [S_\lambda : D_{\mu}]$, which is the multiplicity of $D_{\mu}$ appearing as a composition factor in a composition series of $S_\lambda$. We know that we can afford Specht modules with a grading and study the graded composition factors that arise in their composition series, since a graded version of the Jordan–Hölder theorem exists.

We denote the ungraded decomposition matrix for $\mathcal{H}_n^\Lambda$ by $(d_{\lambda,\mu})$; we write $(d_{\lambda,\mu}^F)$ when we want to emphasise the ground field $F$. We can compute the ungraded decomposition matrices for $\mathcal{H}_n^\Lambda$ over a field of characteristic zero via the generalised Lascoux–Leclerc–Thibon algorithm given by Fayers in [Fa1], whereas, the ungraded decomposition matrices for $\mathcal{H}_n^\Lambda$ over a field of positive characteristic are far more elusive. For $\nu, \mu \in \mathcal{R}_n^{\mathcal{P}}$, we know from [BK3] that there exists an adjustment matrix $(a_{\nu,\mu}^F)$ such that the product $(d_{\lambda,\nu}^F)(a_{\nu,\mu}^F)$ gives us the ungraded decomposition matrices for $\mathcal{H}_n^\Lambda$ over an arbitrary field, and moreover, $(d_{\lambda,\nu}^C)(a_{\nu,\mu}^C) = (d_{\lambda,\nu}^C)$. However, there exists no algorithm for finding the entries in the adjustment matrix over a field of positive characteristic.

For $\lambda \in \mathcal{P}_n^{\mathcal{P}}$ and $\mu \in \mathcal{R}_n^{\mathcal{P}}$, we define the graded decomposition number (or the graded composition multiplicity) to be

$$d_{\lambda,\mu}(v) = [S_\lambda : D_{\mu}]_v := \sum_{i \in \mathbb{Z}} a_i v^i \in \mathbb{N}[v, v^{-1}],$$

where $a_i$ is the composition multiplicity of $D_{\mu}(i)$ appearing as a composition factor of $S_\lambda$. Note that we recover the ungraded decomposition number by setting $v = 1$.

We record these graded multiplicities in a graded decomposition matrix, denoted by $(d_{\lambda,\mu}(v))$, where its rows correspond to Specht modules labelled by multipartitions and its columns correspond to irreducible quotients of Specht modules labelled by regular multipartitions. By [BK1, Corollary 6.3] we know that the graded decomposition matrices for cyclotomic Hecke algebras, and hence for the corresponding cyclotomic Khovanov–Lauda–Rouquier algebras, only depend on the quantum characteristic $e$ and the characteristic of the ground field $F$, and not on $F$ itself, affirming a conjecture by Mathas.

The following result for $\mathcal{H}_n^\Lambda$ is a generalised graded version of [J, Corollary 12.2] for $\mathbb{F}\mathcal{S}_n$.

Theorem 5.1. [BK3, Corollary 5.15] Let $\lambda \in \mathcal{P}_n^{\mathcal{P}}$ and $\mu \in \mathcal{R}_n^{\mathcal{P}}$. Then

1. $d_{\mu,\mu}(v) = 1$;
2. $d_{\lambda,\mu}(v) \neq 0$ only if $\mu \succeq \lambda$. 

Theorem 6.1. [Ma, Corollary 5.8] Let $v \in \mathcal{P}_n$ and $\mu \in \mathcal{R}\mathcal{P}_n$. Then

$$d^F_{\lambda,\mu}(v) = \sum_{\nu \in \mathcal{P}_n} d^C_{\lambda,\nu}(v)a^F_{\nu,\mu}(v),$$

for some $a^F_{\nu,\mu}(v) \in \mathbb{N}[v, v^{-1}]$ with $a^F_{\nu,\mu}(v) = a^F_{\nu,\mu}(v^{-1})$.

We say that $a^F_{\nu,\mu}(v)$ is a graded adjustment number of $\mathcal{H}_n^\Lambda$ over $\mathbb{F}$; the graded adjustment matrix $(a^F_{\nu,\mu}(v))$ is a square unitriangular matrix whose entries are symmetric in $v$ and $v^{-1}$ and whose rows and columns correspond to regular multipartitions, whereby we recover the ungraded adjustment matrix when we set $v = 1$. It follows that we can obtain the graded decomposition matrix for $\mathcal{H}_n^\Lambda$ over a field of positive characteristic by post-multiplying the graded decomposition matrix $\mathcal{H}_n^\Lambda$ over $\mathbb{C}$ by the graded adjustment matrix, that is,

$$(d^F_{\lambda,\mu}(v)) = (d^C_{\lambda,\nu}(v)) (a^F_{\nu,\mu}(v)),$$

for $\lambda \in \mathcal{P}_n$, $\mu, \nu \in \mathcal{R}\mathcal{P}_n$.

6 Induction and restriction of $\mathcal{H}_n^\Lambda$-modules

The decomposition number problem of understanding the multiplicities $[S_1 : D_\mu]$, for all $\lambda, \mu \in \mathcal{P}_n$, is equivalent to the branching problem of understanding the multiplicities

$$[\text{res}_{\mathcal{H}_n^\Lambda}^{\mathcal{H}_{n-1}^\Lambda} D_\lambda : D_\mu],$$

for all $\lambda, \mu \in \mathcal{P}_n$, which provides the motivation for studying the restriction of an irreducible $\mathcal{H}_n^\Lambda$-module to an $\mathcal{H}_{n-1}^\Lambda$-module. The restriction of the ordinary representations of the symmetric group and their composition factors are well understood via the Classical Branching Rule for $\mathfrak{S}_n$ (for example, see [J, Theorem 9.2]). This result was extended to the Ariki–Koike algebras or the cyclotomic Hecke algebras by Ariki–Koike [AK, Corollary 3.12], and hence we introduce an analogous result for the cyclotomic Khovanov–Lauda–Rouquier algebras, recently given by Mathas [Ma]. We simultaneously recall the ‘dual’ results in [HM1] of how Specht modules of the cyclotomic Khovanov–Lauda–Rouquier algebras behave under induction. Induction allows us to understand representations of the cyclotomic Khovanov–Lauda–Rouquier algebra $\mathcal{H}_{n+1}^\Lambda$ from representations of the subalgebra $\mathcal{H}_n^\Lambda$ that are known to us. We write $\text{res}$ to denote the functor restricting a $\mathcal{H}_n^\Lambda$-module to a $\mathcal{H}_{n-1}^\Lambda$-module, and write $\text{ind}$ to denote the functor inducing a $\mathcal{H}_n^\Lambda$-module to a $\mathcal{H}_{n+1}^\Lambda$-module.

We first introduce Brundan and Kleshchev’s $i$-restriction and $i$-induction operators $e_i$ and $f_i$ acting on $\mathfrak{F}\mathfrak{S}_n$-modules, as given in Section 2.2 of [BK1]. These functors originate from Robinson [Rob]; we extend these exact operators to act on $\mathcal{H}_n^\Lambda$-modules.

We let $M$ be a $\mathcal{H}_n^\Lambda$-module. There are $i$-restriction functors $e_i : \mathcal{H}_n^\Lambda$-$\text{mod} \to \mathcal{H}_{n-1}^\Lambda$-$\text{mod}$, for $i \in \mathbb{Z}/e\mathbb{Z}$, such that

$$\text{res}_{\mathcal{H}_n^\Lambda}^{\mathcal{H}_{n-1}^\Lambda} M \cong \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} e_i M,$$

by [BK1, Lemma 2.5], which is an analogous result to [AK, Corollary 3.12].

The graded classical branching rule for Specht modules is given as follows, whereby the ungraded version is recovered by setting $v$ to be 1.

Theorem 6.1. [Ma, Corollary 5.8] Let $\lambda \in \mathcal{P}_n$ and $i \in \mathbb{Z}/e\mathbb{Z}$. Let $A_1, A_2, \ldots, A_m$ be the removable $i$-nodes of $\lambda$ and $\lambda^{(1)} \triangleleft \lambda^{(2)} \triangleleft \cdots \triangleleft \lambda^{(m)}$ be the respective $l$-multipartitions of $n - 1$ such that $\lambda^{(j)} = \lambda \setminus \{A_j\}$ for $1 \leq j \leq m$. Then $e_i S_\lambda$ has a filtration of $\mathcal{H}_{n-1}^\Lambda$-modules

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_m = e_i S_\lambda,$$
where $M_j/M_{j-1} \cong v^{d_{A_j}(\lambda)} S_{\lambda(j)}$, for $1 \leq j \leq m$.

**Example 6.2.** Let $e = 3$, $\kappa = (0, 2)$ and $\lambda = ((6, 5^2, 2), (4, 3, 2))$. We observe that the 3-residues of $\lambda$ and its addable 2-nodes, shading the removable 2-nodes of $\lambda$, are

\[
\begin{array}{cccc}
0 & 1 & 2 & 0 \\
2 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 \\
0 & 1 & 2 & 2 \\
\end{array}
\]

We label the removable 2-nodes as $A_1 = (1, 6, 1)$, $A_2 = (3, 5, 1)$ and $A_3 = (1, 4, 2)$. It follows that $e_2 S_{\lambda}$ has a filtration of $A_{26}^\lambda$-modules

\[0 \subset M_0 \subset M_1 \subset M_2 \subset M_3 = e_2 S_{\lambda},\]

where

\[
\begin{align*}
M_1/M_0 & \cong v^{d_{A_1}(\lambda)} S_{\lambda(1)} \cong S_{((5^3, 2), (4, 3, 2))}, \\
M_2/M_1 & \cong v^{d_{A_2}(\lambda)} S_{\lambda(2)} \cong v^{-1} S_{((6, 5^2, 2), (4, 3, 2))}, \\
M_3/M_2 & \cong v^{d_{A_3}(\lambda)} S_{\lambda(3)} \cong S_{((6, 5^2, 2), (3^2, 2))}.
\end{align*}
\]

Similarly, there are $i$-induction functors $f_i : A_n^\lambda$-mod $\rightarrow A_{n+1}^\lambda$-mod, for $i \in \mathbb{Z}/e\mathbb{Z}$, such that

\[
\text{ind}_{A_{n+1}^\lambda} M \cong \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} f_i M,
\]

by [BK1, Lemma 2.5].

The ‘dual’ graded branching rule for Specht modules is given as follows.

**Theorem 6.3.** [HM2, Main Theorem] Let $\lambda \in \mathcal{A}_n$ and $i \in \mathbb{Z}/e\mathbb{Z}$. Let $A_1, A_2, \ldots, A_m$ be the addable i-nodes of $\lambda$ and $\lambda^{(1)} \triangleright \lambda^{(2)} \triangleright \cdots \triangleright \lambda^{(m)}$ be the respective $i$-multipartitions of $n + 1$ such that $\lambda^{(j)} = \lambda \cup \{A_j\}$ for $1 \leq j \leq m$. Then $f_i S_{\lambda}$ has a filtration of $A_{n+1}^\lambda$-modules

\[0 \subset M_0 \subset M_1 \subset \cdots \subset M_m = f_i S_{\lambda},\]

with $M_j/M_{j-1} \cong v^{d_{A_j}(\lambda)} S_{\lambda(j)}$ for $1 \leq j \leq m$.

**Example 6.4.** Let $e = 3$, $\kappa = (0, 2)$ and $\lambda = ((6, 5^2, 2), (4, 3, 2))$ as in Example 6.2. We label the addable 2-nodes of $\lambda$ as $B_1 = (4, 3, 1)$, $B_2 = (5, 1, 1)$, $B_3 = (3, 3, 2)$ and $B_4 = (4, 1, 2)$. Then $f_2 S_{\lambda}$ has a filtration of $A_{28}^\lambda$-modules

\[0 \subset M_0 \subset M_1 \subset M_2 \subset M_3 \subset M_4 = f_2 S_{\lambda},\]

where

\[
\begin{align*}
M_1/M_0 & \cong v^{d_{B_1}(\lambda)} S_{\lambda(1)} \cong v^{-2} S_{((6, 5^2, 3), (4, 3, 2))}, \\
M_2/M_1 & \cong v^{d_{B_2}(\lambda)} S_{\lambda(2)} \cong v^{-1} S_{((6, 5^2, 2, 1), (4, 3, 2))}, \\
M_3/M_2 & \cong v^{d_{B_3}(\lambda)} S_{\lambda(3)} \cong v^{-1} S_{((6, 5^2, 2), (4, 3^2))}, \\
M_4/M_3 & \cong v^{d_{B_4}(\lambda)} S_{\lambda(4)} \cong S_{((6, 5^2, 2), (4, 3, 2, 1))}.
\end{align*}
\]
The operators \( e_i \) and \( f_i \) are both left and right adjoint to each other by [K4, Lemma 8.2.2], and so are exact functors.

There are generalisations of the \( i \)-restriction and \( i \)-induction operators to “divided powers” \( e_i^{(r)} \) and \( f_i^{(r)} \). For \( i \in \mathbb{Z}/e\mathbb{Z} \) and \( r \geq 0 \), there are divided power \( i \)-restriction functors \( e_i^{(r)} : \mathcal{H}_n^\Lambda\text{-mod} \to \mathcal{H}_{n-r}^\Lambda\text{-mod} \), which satisfy [BK1, Lemma 2.6]

\[
e_i^{(r)} M \cong \bigoplus_{k=1}^{r!} e_i^{(r)} M.
\]

Similarly, for \( i \in \mathbb{Z}/e\mathbb{Z} \), there are divided power induction \( i \)-functors \( f_i^{(r)} : \mathcal{H}_n^\Lambda\text{-mod} \to \mathcal{H}_{n+r}^\Lambda\text{-mod} \), which satisfy [BK1, Lemma 2.6]

\[
f_i^{(r)} M \cong \bigoplus_{k=1}^{r!} f_i^{(r)} M.
\]

Notice that \( e_i^{(1)} = e_i \) and \( f_i^{(1)} = f_i \). The divided powers \( e_i^{(r)} \) and \( f_i^{(r)} \) are also both left and right adjoint to each other (see [K4, Theorem 8.3.2]), and so are exact functors.

For a non-zero \( \mathcal{H}_n^\Lambda\)-module \( M \), we define

\[
\epsilon_i(M) = \max\{ r \geq 0 \mid e_i^{(r)} M \neq 0 \} \quad \text{and} \quad \varphi_i(M) = \max\{ r \geq 0 \mid f_i^{(r)} M \neq 0 \},
\]

and now define

\[
e_i^{(\text{max})} M = e_i^{(\epsilon_i(M))} M \quad \text{and} \quad f_i^{(\text{max})} M = f_i^{(\varphi_i(M))} M.
\]

Corollary 6.5. Let \( \lambda \in \mathcal{R}_n^l \) and \( i \in \mathbb{Z}/e\mathbb{Z} \).

1. Then \( \epsilon_i(S_\lambda) = \text{rem}_i(\lambda) \) and \( e_i^{(\text{max})} S_\lambda \cong S_{\lambda^{\epsilon_i}} \).
2. Then \( \varphi_i(S_\lambda) = \text{add}_i(\lambda) \) and \( f_i^{(\text{max})} S_\lambda \cong S_{\lambda^{\varphi_i}} \).

7 Modular branching rules for \( \mathcal{H}_n^\Lambda\text{-modules} \)

Kleshchev developed the analogous theory for restricting the modular representations of the symmetric group [K1, K2, K3], which Brundan extended to Hecke algebras of type A [B]. These modular branching rules were generalised for cyclotomic Hecke algebras, proven by Ariki in the proof of [A, Theorem 6.1]. Thus, modular branching rules for the cyclotomic Khovanov–Lauda–Rouquier algebras make sense, which we note here. Recall that \( D_\lambda \) is the irreducible quotient of the Specht module \( S_\lambda \), where \( \lambda \in \mathcal{R}_n^l \).

Theorem 7.1. [BK1, §2.6] Let \( \lambda \in \mathcal{R}_n^l \). Then

\[
\epsilon_i(D_\lambda) = \text{nor}_i(\lambda) \quad \text{and} \quad \varphi_i(D_\lambda) = \text{conor}_i(\lambda).
\]

Moreover,

1. if \( A_1, \ldots, A_{\text{nor}_i(\lambda)} \) are the normal \( i \)-nodes of \( \lambda \), then

\[
e_i^{(\text{max})} D_\lambda \cong D_{\lambda \setminus \{A_1, \ldots, A_{\text{nor}_i(\lambda)}\}}.
\]
2. and if \( A_1, \ldots, A_{\text{conor}_i(\lambda)} \) are the conormal \( i \)-nodes of \( \lambda \), then

\[
f_i^{(\text{max})} D_\lambda \cong D_{\lambda \cup \{A_1, \ldots, A_{\text{conor}_i(\lambda)}\}}.
\]
Example 7.2. Let \( e = 3 \), \( \kappa = (0, 2) \) and \( \lambda = ((9, 6, 2, 1^3), (4, 3, 2)) \). We know that \( S_\lambda \) is irreducible since we can obtain \( \lambda \) from \( (\emptyset, \emptyset) \) by adding certain conormal nodes, that is, \( \lambda = (\emptyset, \emptyset) \uparrow 0 \uparrow 0 \uparrow 2 \uparrow 1 \uparrow 0 \uparrow 2 \uparrow 1 \uparrow 0 \uparrow 2 \uparrow 0 \uparrow 4 \uparrow 2 \uparrow 3 \). We draw the 3-residues of \( \lambda \) and its addable nodes as follows.

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
\end{array}
\]

Observe that \( \lambda \) has 2-signature \(-++-++-\), and hence the reduced 2-signature of \( \lambda \) is \(-++-++\), so that \( \lambda \) has two normal 2-nodes and two conormal 2-nodes. Note that we have also drawn the bipartition obtained by removing the normal 2-nodes from \( \lambda \) (that are outlined), as well as the bipartition obtained by adding the conormal 2-nodes of \( \lambda \) (that are shaded). Then we have

\[
\begin{align*}
\gamma_2^{(2)} \Lambda D_\lambda & \cong D_{((8, 6, 2, 1^3), (3^2, 2))} \\
\delta_2^{(2)} \Lambda D_\lambda & \cong D_{((9, 6, 2, 1^3), (4, 3^2, 1))}.
\end{align*}
\]

For each \( i \in \mathbb{Z}/e\mathbb{Z} \), there is at most one good \( i \)-node of \( \lambda \), and hence at most \( e \) good nodes of \( \lambda \). It follows from [K2, Theorem 0.5] that the socle of the restriction of an irreducible \( \mathcal{H}_n^{\Lambda} \)-module \( D_\lambda \) to an \( \mathcal{H}_n^{\Lambda-1} \)-module is a direct sum of at most \( e \) indecomposable \( \mathcal{H}_n^{\Lambda} \)-summands. Moreover, we also know from [K2] that we can verify that the residue sequence of \( \lambda \setminus \{A\} \) is distinct for each good node of \( \lambda \), so that each summand \( D_{\lambda \setminus \{A\}} \) belongs to a distinct block of \( \mathcal{H}_n^{\Lambda} \). We generalise this result to “divided powers” as follows.

Corollary 7.3. Let \( \lambda \in \mathcal{P}_n \) and \( i \in \mathbb{Z}/e\mathbb{Z} \).

1. If \( r \leq \text{nor}_i(\lambda) \), then

\[
\text{soc} \left( \gamma_i^{(r)} D_\lambda \right) \cong D_{\lambda \setminus \{A\}}.
\]
2. If \( r \leq \text{conor}_i(\lambda) \), then
\[
\text{soc} \left( f_i^{(r)} D_\lambda \right) \cong D_{\lambda_i^r}.
\]

It follows that the modular branching rules for Specht modules of the Khovanov–Lauda–Rouquier algebras \( H_n^\Lambda \), together with the operators \( \uparrow_i^r \) and \( \downarrow_i^r \), provide a combinatorial algorithm for determining labels of \( H_n^\Lambda \)-modules that we know are irreducible.

**Proposition 7.4.** Let \( \lambda \in \mathcal{P}_n^l \). If \( D \) is an irreducible \( H_n^\Lambda \)-module with \( e_i^{(r)} D \cong D_\lambda \), then \( D = D_{\lambda_i^r} \).

**Proof.** Suppose that \( D = D_\mu \) where \( \mu \in \mathcal{R}_n^l \), so that \( e_i^{(r)} D = e_i^{(r)} D_\mu \cong D_\lambda \). We know that \( r \leq \text{nor}_i(\mu) \) since \( e_i^{(r)} D \neq 0 \), then from the first part of Corollary 7.3 we have \( \text{soc} \left( e_i^{(r)} D_\mu \right) \cong D_\nu \) where \( \nu = \mu \downarrow_i^r \). Since \( e_i^{(r)} D_\mu \cong D_\lambda \), we have \( \nu = \lambda \). Then, by eq. (2.2), \( \lambda = \mu \downarrow_i^r \uparrow_i^r = \mu \), as required. \( \square \)

Let \( 0 \leq r \leq \text{nor}_i(\lambda) \) with \( e_i^{(r)} D_\mu \cong D_\lambda \), where \( \lambda, \mu \in \mathcal{R}_n^l \). Then the normal \( i \)-nodes of \( \mu \) and the conormal \( i \)-nodes of \( \lambda \) coincide, and hence
\[
\text{soc} \left( f_i^{(r)} \left( e_i^{(r)} D_\mu \right) \right) \cong D_{\mu_i^r \uparrow_i^r} = D_\mu.
\]

**Example 7.5.** Let \( e = 3 \), \( \kappa = (0, 2) \) and \( \lambda = ((8, 6, 2, 1^3), (3^2, 2)) \). By Example 7.2, we have \( e_2^{(2)} D_\lambda \cong D_\lambda \). The 3-residues of \( ((8, 6, 2, 1^3), (3^2, 2)) \) are

\[
\begin{array}{cccccc}
0 & 1 & 2 & 0 & 1 & 2 \\
2 & 0 & 1 & 2 & 0 & 1 \\
1 & 2 & 0 & 1 & \\
2 & 0 & 1 & \\
2 & 1 & \\
0 & \\
\end{array}

\begin{array}{cccccc}
0 & 1 & 2 & 0 & 1 & 2 \\
2 & 0 & 1 & 2 & 0 & 1 \\
1 & 2 & 0 & 1 & \\
2 & 0 & 1 & \\
2 & 1 & \\
0 & \\
\end{array}

f_2^{(2)}

\begin{array}{cccc}
2 & 0 & 1 & 2 \\
2 & 0 & 1 & 2 \\
0 & 1 & 2 \\
2 & \\
\end{array}

where the removable 2-node of \( \lambda \) is shaded and the addable 2-nodes of \( \lambda \) are outlined. Hence \((8, 6, 2, 1^3), (3^2, 2)\) has 2-signature ++++, and thus reduced 2-signature +++. Adding the highest two conormal 2-nodes, it follows that \( \text{soc} \left( f_2^{(2)} D_{((8, 6, 2, 1^3), (3^2, 2))} \right) \cong D_{((9, 6, 2, 1^3), (4, 3, 2))} \), as expected.

For non-irreducible \( H_n^\Lambda \)-modules, we can determine the labels of their composition factors by applying the same combinatorial algorithm using the following result.

**Proposition 7.6.** For \( r > 0 \) and an \( H_n^\Lambda \)-module \( M \), suppose that \( e_i^{(r)} M \cong D_\mu \), where \( \mu \in \mathcal{R}_n^{l-r} \). Then one of the composition factors of \( M \) is \( D_{\mu_i^r} \), while all the other composition factors of \( M \) are killed by \( e_i^{(r)} \).
Example 7.7. Let $e = 3$, $\kappa = (0, 2)$ and $\lambda = ((7, 5, 1^4), (3^2, 1))$. The 3-residues of $\lambda$

\[
\begin{array}{cccc}
0 & 1 & 2 & 0 \\
2 & 0 & 1 & 2 \\
1 & 2 & 0 \\
0 & 2 \\
1 & 2 & 0 & 2 \\
0 & 2 & 1 \\
1 & 2 & 0 & 2 \\
0 & 2 & 1 \\
\end{array}
\]

where the addable 2-node of $\lambda$ is outlined and the removable 2-nodes of $\lambda$ are shaded. Hence $e^{(2)}_2 S_{((7, 5, 1^4), (3^2, 1))} \cong S_{((8, 6, 2, 1^3), (3^2, 2))}$, which we know is irreducible since $S_{((8, 6, 2, 1^3), (3^2, 2))}$ is irreducible from Example 7.2 and $((8, 6, 2, 1^3), (3^2, 2)) \downarrow_{3, 1, 2} = ((7, 5, 1^4), (3^2, 1))$.

We now observe that the 3-residues of $((7, 5, 1^4), (3^2, 1))$ and its addable 2-nodes are

\[
\begin{array}{cccc}
0 & 1 & 2 & 0 \\
2 & 0 & 1 & 2 \\
1 & 2 & 0 \\
0 & 2 \\
2 & 0 & 1 & 2 \\
1 & 2 & 0 \\
0 & 2 & 1 \\
\end{array}
\]

Observe that $+++$ is the 2-signature of $((7, 5, 1^4), (3^2, 1))$, and thus $\text{soc} \left(f^{(2)}_2 D_{((7, 5, 1^4), (3^2, 1))}\right) \cong D_{((7, 5, 2, 1^3), (4, 3, 1))}$ is a composition factor of $S_\lambda$.

8 Specht module homomorphisms

Let $T \in \text{Std}((n - m), (1^m))$ and recall from [S, §5.1] that we write a homogeneous basis element $v_T$ of $S_{((n - m), (1^m))}$ as

$v_T = \psi_w z_{((n - m), (1^m))},$

where $\psi_w = \Psi_{a_1} \cdots \Psi_{a_m}$ such that $1 \leq a_1 < a_2 < \cdots < a_m \leq n$. We define

$v(a_1, \ldots, a_m) := v_T.$

Informally, $v(a_1, \ldots, a_m)$ corresponds to the standard $((n - m), (1^m))$-tableau $T$ with $a_1, \ldots, a_m$ lying in the leg of $T$.

We recall two results from [S] for the reader’s benefit.

Proposition 8.1. [S, Proposition 5.4]
1. If $n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e}$ and $0 \leq m \leq n - 1$, then there exists the following non-zero homomorphism of Specht modules
\begin{align*}
\gamma_m : S_{(n-m),(1^m)} &\rightarrow S_{(n-m-1),(1^{m+1})} \\
z_{(n-m),(1^m)} &\mapsto v(1, \ldots, m, n).
\end{align*}

2. If $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $1 \leq m \leq n - 1$, then there exists the following non-zero homomorphism of Specht modules
\begin{align*}
\chi_m : S_{((n-m,1^m),\emptyset)} &\rightarrow S_{((n-m),(1^m))} \\
z_{((n-m,1^m),\emptyset)} &\mapsto v(2, 3, \ldots, m + 1).
\end{align*}

3. If $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$, $n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e}$ and $1 \leq m \leq n - 1$, then there exists the following non-zero homomorphism of Specht modules
\begin{align*}
\phi_m : S_{((n-m+1,1^{m-1}),\emptyset)} &\rightarrow S_{((n-m),(1^m))} \\
z_{((n-m+1,1^{m-1}),\emptyset)} &\mapsto v(2, 3, \ldots, m, n).
\end{align*}

The standard basis elements of the kernels and images of the above Specht modules homomorphisms are as follows.

Lemma 8.2. [S, Lemma 5.7]

1. If $n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e}$, then
   
   (a) $\text{im}(\gamma_m) = \text{span} \{ v_T \mid T \in \text{Std} ((n-m-1), (1^{m+1})) , T(m+1,1,2) = n \}$;
   
   (b) $\text{ker}(\gamma_m) = \text{span} \{ v_T \mid T \in \text{Std} ((n-m), (1^m)) , T(m,1,2) = n \}$.

2. If $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$, then
   
   (a) $\text{im}(\chi_m) = \text{span} \{ v_T \mid T \in \text{Std} ((n-m), (1^m)) , T(1,1,1) = 1 \}$;
   
   (b) $\text{ker}(\chi_m) = 0$.

3. If $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e}$, then
   
   (a) If $m < n - 1$, then
   \[
   \text{im}(\phi_m) = \text{span} \{ v_T \mid T \in \text{Std} ((n-m), (1^m)) , T(1,1,1) = 1 , T(m,1,2) = n \}.
   \]
   
   (b) If $m = n - 1$, then
   \[
   \text{im}(\phi_m) = \text{span} \{ v_T \mid T \in \text{Std} ((1), (1^{n-1})) , T(1,1,1) = 1 \}.
   \]

9 Irreducible labels of one-dimensional Specht modules

We determine the irreducible labels of the only one-dimensional Specht modules, namely $S_{((n),\emptyset)}$ and $S_{((\emptyset,1^n))}$, which arise as composition factors of $S_{((n-m),(1^m))}$. We note that we work solely with ungraded cyclotomic Khovanov–Lauda–Rouquier modules up to and including Section 12. Let $l$ be the residue of $\kappa_2 - \kappa_1$ modulo $e$ throughout, so that $l \in \{0, \ldots, e - 1\}$.

We know $S_{((n),\emptyset)} = \{ z_{((n),\emptyset)} \}$ and $S_{((\emptyset,1^n))} = \{ z_{((\emptyset,1^n))} \}$ are both one-dimensional $\mathcal{H}_n^A$-modules, and hence are both irreducible. In fact, $S_{((n),\emptyset)} \cong D_{((n),\emptyset)}$. We now introduce a sgn-functor to determine the bipartition $\mu \in \mathcal{R}_n^2$ where $S_{((\emptyset,1^n))} \cong D_{\mu}$. 
For, $1 \leq r \leq n$, $S_{(\varnothing,(1^r))}$ only has one removable node, namely $(r,1,2)$ where $\text{res}(r,1,2) \equiv \kappa_2 + 1 - r \pmod{e}$. So, $e_{\kappa_2+1-r} : \mathcal{H}_r^\Lambda - \text{mod} \rightarrow \mathcal{H}_{r-1}^\Lambda - \text{mod}$ is the only restriction functor which acts non-trivially on $S_{(\varnothing,(1^r))}$, where $e_{\kappa_2+1-r}S_{(\varnothing,(1^r))} \equiv S_{(\varnothing,(1^{r-1}))}$.

Define the sgn-restriction functor to be $e_{\text{sgn}} := e_{\kappa_2} \circ e_{\kappa_2-1} \circ \cdots \circ e_{\kappa_2+1-n}$, with the property that

$$e_{\text{sgn}} : \mathcal{H}_n^\Lambda - \text{mod} \rightarrow \mathcal{H}_0^\Lambda - \text{mod}; \quad e_{\text{sgn}}S_{(\varnothing,(1^n))} \equiv S_{(\varnothing,\varnothing)}.$$ 

We see that $e_{\text{sgn}}$ is the only composition of restriction functors which acts non-trivially on $S_{(\varnothing,(1^n))}$. Define the sgn-induction functor to be $f_{\text{sgn}} = f_{\kappa_2+1-n} \circ f_{\kappa_2+2-n} \circ \cdots \circ f_{\kappa_2}$, where

$$f_{\text{sgn}} : \mathcal{H}_0^\Lambda - \text{mod} \rightarrow \mathcal{H}_n^\Lambda - \text{mod}.$$ 

The sgn-induction functor acts non-trivially on $S_{(\varnothing,\varnothing)}$; we determine the socle of $f_{\text{sgn}}S_{(\varnothing,\varnothing)}$.

**Definition 9.1.** For $a \in \mathbb{N}_0$, we write

$$\{a\} := \left\{ \frac{a + e - 2}{e - 1}, \frac{a + e - 3}{e - 1}, \ldots, \frac{a}{e - 1} \right\},$$

that is, the weakly decreasing sequence of $e - 1$ integers that sum to $a$ that differ by at most $1$.

We are now ready to give an explicit description of the $e$-regular bipartition $\mu$ where $S_{(\varnothing,(1^n))} \equiv D_\mu$.

**Lemma 9.2.** 1. If $n < l$, then $S_{(\varnothing,(1^n))} \equiv D_{(\varnothing,(1^n))}$.

2. If $n \geq l$, then $S_{(\varnothing,(1^n))} \equiv D_{((n-l),(1^n))}$.

**Proof.** Let $1 \leq r \leq n$ and $S_{(\varnothing,(1^n))} \equiv D_\mu$ for some bipartition $\mu$. By [BK2, Lemma 2.5],

$$\text{res} \mathcal{H}_n^\Lambda S_{(\varnothing,(1^n))} \equiv e_{\text{sgn}}S_{(\varnothing,(1^n))}.$$ 

For any $r > 1$, there is only one removable $(\kappa_2+1-r)$-node of $[(\varnothing,(1^r))]$, so that $e_{\kappa_2+1-r}S_{(\varnothing,(1^r))} = 1$. Thus, by Corollary 6.5,

$$e_{\text{sgn}}S_{(\varnothing,(1^n))} \equiv S_{(\varnothing,(1^n))}^{\text{res}} \equiv S_{(\varnothing,(1^n))}^{\text{res}} \equiv S_{(\varnothing,(1^n))}^{\text{res}} \equiv S_{(\varnothing,\varnothing)}.$$ 

Define $\uparrow^r_{\text{sgn}} (\varnothing,\varnothing) := \uparrow_{\kappa_2+1-n}^1 \uparrow_{\kappa_2+2-n}^1 \cdots \uparrow_{\kappa_2}^1 (\varnothing,\varnothing)$. Since $S_{(\varnothing,(1^n))}$ is irreducible, then $S_{(\varnothing,(1^n))} \equiv D_{\uparrow^r_{\text{sgn}}(\varnothing,\varnothing)}$ by Proposition 7.6. To calculate $\uparrow^r_{\text{sgn}} (\varnothing,\varnothing)$ we add $n$ nodes to $[(\varnothing,\varnothing)]$ by successively adding the highest conormal node of $e$-residues $\kappa_2, \kappa_2 - 1, \ldots, \kappa_2 + 1 - n$, respectively.

Firstly, we successively add the highest $l$ conormal nodes of $e$-residues $\kappa_2, \kappa_2 - 1, \ldots, \kappa_2 - l + 1$, respectively. Since $\kappa_1 \equiv \kappa_2 - l \pmod{e}$, it is easy to see that $(\varnothing,(1^l))$ has $(\kappa_2 - i)$-signature $+$, corresponding to node $(i+1,1,2)$, for each $i \in \{0, \ldots, l-1\}$. Hence

$$\uparrow_{\kappa_2-l+1}^1 \uparrow_{\kappa_2-l+2}^1 \cdots \uparrow_{\kappa_2}^1 (\varnothing,\varnothing) = (\varnothing,(1^l)).$$

In particular, if $n \leq l$, then we are done. So suppose that $n > l$.

We now successively add the highest $e$ conormal nodes to $(\varnothing,(1^l))$ of $e$-residue $\kappa_1, \kappa_1 - 1, \ldots, \kappa_1 + 1$, respectively. Notice that $((1^l),(1^l))$ has $(\kappa_1 - i)$-signature $+$, corresponding to node $(i+1,1,1)$, for $i \in \{0, \ldots, e - 1\}$, except in the following cases.

- $\varnothing$ The $\kappa_1$-signature of $(\varnothing,(1^l))$ is $++$, corresponding to nodes $(1,1,1)$ and $(l+1,1,2)$, respectively. Thus $\uparrow_{\kappa_1} (\varnothing,(1^l)) = ((1),(1^l))$. 


Let $\Lambda$ denote Rouquier modules throughout this section.

It follows that
\[
\uparrow_{\kappa_1+1} \uparrow_{\kappa_1+2} \cdots \uparrow_{\kappa_1} (\emptyset, (1^j)) = (2, 1^{e-2}, (1^j)),
\]
and so the first component of $\uparrow_{\operatorname{sgn}} (\emptyset, \emptyset)$ has $e-1$ rows.

We successively add the remaining nodes to the first component of $(2, 1^{e-2}, (1^j))$, down each column from left to right. There are $n-l-r+1$ nodes in
\[
[\uparrow_{\operatorname{sgn}} (\emptyset, \emptyset) \setminus ((1, 1, 2), \ldots, (l, 1, 2)) \cup ((1, 1, 1), \ldots, (r-1, 1, 1))],
\]
for all $r \in \{1, \ldots, e-1\}$. Since there are $e-1$ rows in the first component of $\mu$, there are $\left\lfloor \frac{n-l-r+1}{e-1} \right\rfloor$ nodes in the $r$th row of the first component of $\uparrow_{\operatorname{sgn}} (\emptyset, \emptyset)$.

\section{Labelling the composition factors of $S_{((n-m),(1^m))}$}

In [S, §6], we found that the composition factors of $S_{((n-m),(1^m))}$ arise as quotients of the images and kernels of the Specht module homomorphisms given in Proposition 8.1. We now determine the irreducible labels of these composition factors. Recall that $l$ is the residue of $\kappa_2 - \kappa_1$ modulo $e$ and note that we work solely with ungraded cyclotomic Khovanov–Lauda–Rouquier modules throughout this section.

\subsection{Labelling the composition factors of $S_{((n-m),(1^m))}$ for $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$}

Let $\kappa_2 \not\equiv \kappa_1 - 1 \pmod{e}$ throughout this subsection.

For $n \not\equiv l+1 \pmod{e}$, we recall from [S, Theorem 6.7] that $S_{((n-m),(1^m))}$ is an irreducible $\mathcal{H}_{\kappa_1}^{\Lambda}$-module, that is, $S_{((n-m),(1^m))} \cong D_{\lambda}$ for some bipartition $\lambda \in \mathcal{R}_n^2$.

**Definition 10.1.** Let $n \not\equiv l+1 \pmod{e}$. For $1 \leq m \leq n-1$, we define
\[
\mu_{n,m} = \begin{cases} 
((n-m),(1^m)) & \text{if } 1 \leq m < l+1 \leq e-1, \\
((n-m,\{m-l-1\}), (1^{l+1})) & \text{if } l+1 \leq m < n - \frac{n}{e}, \\
((\{m-l\}, n-m-1), (1^{l+1})) & \text{if } n - \frac{n}{e} \leq m \leq n-1.
\end{cases}
\]
We claim that $\lambda = \mu_{n,m}$, for $1 \leq m \leq n-1$.

For $n \equiv l+1 \pmod{e}$ and $1 \leq m \leq n-1$, we recall from [S, Theorem 6.10] that $S_{((n-m),(1^m))}$ has two composition factors, namely $\operatorname{im}(\gamma_{m-1})$ and $\operatorname{im}(\gamma_m)$. Thus, $\operatorname{im}(\gamma_{m-1}) \cong D_{\lambda}$ and $\operatorname{im}(\gamma_m) \cong D_{\mu}$ for some bipartitions $\lambda, \mu \in \mathcal{R}_n^2$.

**Definition 10.2.** Let $n \equiv l+1 \pmod{e}$. For $0 \leq m \leq n-1$, we define
\[
\mu_{n,m} = \begin{cases} 
((n-m),(1^m)) & \text{if } 0 \leq m < l+1, \\
((n-m,\{m-l-1\}), (1^{l+1})) & \text{if } l+1 \leq m < n - \frac{n}{e}, \\
((\{m-l+1\}, n-m-2), (1^{l+1})) & \text{if } n - \frac{n}{e} \leq m \leq n-2, \\
((\{n-l\}), (1^l)) & \text{if } m = n-1.
\end{cases}
\]
Notice that $\mu_{n,m-1}$ and $\mu_{n,m}$ are distinct. We claim that the two labels $\lambda, \mu$ of the composition factors of $S_{((n-m),(1^m))}$ are, in fact, $\mu_{n,m-1}$ and $\mu_{n,m}$, respectively, and hence that these factors are non-isomorphic.

We will now give the following combinatorial result in order to confirm our claims.

**Lemma 10.3.**  
1. If $n \equiv l \pmod{e}$, then
   \[
   \mu_{n,m} \uparrow_{\kappa_2-m} = \mu_{n+1,m}. \tag{10.1}
   \]

2. If $n \not\equiv l \pmod{e}$, then
   \[
   \mu_{n,m} \uparrow_{\kappa_2-m} = \mu_{n+1,m+1}. \tag{10.2}
   \]

**Proof.** (i) Let $1 \leq m < l+1 \leq e-1$. Observe that $[(n-m),(1^m)]$ has addable $(\kappa_2-m)$-node $(m+1,1,2)$, as well as $(1,n-m+1,1)$ if $n \equiv l \pmod{e}$, and has removable $(\kappa_2-m)$-node $(1,n-m,1)$ if $n \equiv l+1 \pmod{e}$. We note that addable nodes $(2,1,1)$ and $(1,2,2)$ cannot have residue $\kappa_2-m$ as $m < l+1$. So, if $n \equiv l \pmod{e}$ then $((n-m),(1^m))$ has $(\kappa_2-m)$-signature $++$, corresponding to conormal nodes $(1,n-m+1,1)$ and $(m+1,1,2)$. Adding the higher of these conormal nodes, we have
   \[
   \mu_{n,m} \uparrow_{\kappa_2-m} = ((n-m),(1^m)) \uparrow_{\kappa_2-m} = ((n-m+1),(1^m)) = \mu_{n+1,m}.
   \]
   However, if $n \equiv l+1 \pmod{e}$ then $((n-m),(1^m))$ has $(\kappa_2-m)$-signature $-+$, and if $n - \kappa_2 + \kappa_1 \not\equiv 0,1 \pmod{e}$ then $((n-m),(1^m))$ has $(\kappa_2-m)$-signature $+$. The conormal node in each sequence is $(m+1,1,2)$, whereby adding this node gives
   \[
   \mu_{n,m} \uparrow_{\kappa_2-m} = ((n-m),(1^m)) \uparrow_{\kappa_2-m} = ((n-m),(1^{m+1})) = \mu_{n+1,m+1}.
   \]

(ii) Let $l+1 \leq m < n - \lfloor \frac{n}{e} \rfloor$. Observe that $((n-m),(\{m-l-1\}),(1^{l+1}))$ has the following addable/removable $(\kappa_2-m)$-nodes
   \[
   \begin{align*}
   &\circ \text{addable node } (1,n-m+1,1) \text{ if } n \equiv l \pmod{e}, \\
   &\circ \text{removable node } (1,n-m,1) \text{ if } n \equiv l+1 \pmod{e}, \\
   &\circ \text{addable node at the end of the } [(m+e-l-2)/(e-1)] \text{th column in the first component}, \\
   &\circ \text{addable node } (e+1,1,1) \text{ and removable node } (l+1,1,2) \text{ if } m \equiv l \pmod{e}, \\
   &\circ \text{addable node } (1,2,2) \text{ if } m \equiv -1 \pmod{e}, \\
   &\circ \text{addable node } (l+2,1,2) \text{ if } m \equiv l+1 \pmod{e}.
   \end{align*}
   \]

Suppose that $n \equiv l \pmod{e}$. Then $((n-m),(\{m-l-1\}),(1^{l+1}))$ has $(\kappa_2-m)$-signature
   \[
   \begin{align*}
   &\circ +++-- \text{ if } m \equiv l \pmod{e}, \\
   &\circ ++++ \text{ if } m \equiv -1 \pmod{e} \text{ and } l = e-2, \\
   &\circ +++ \text{ if } m \equiv -1 \pmod{e} \text{ and } l \not\equiv e-2 \text{ or } m \not\equiv -1 \pmod{e} \text{ and } m \equiv l+1 \pmod{e}, \\
   &\circ ++ \text{ for all other cases}.
   \end{align*}
   \]

Adding the highest conormal $(\kappa_2-m)$-node in these sequences, $(1,n-m+1,1)$, we have
   \[
   \mu_{n,m} \uparrow_{\kappa_2-m} = ((n-m),(\{m-l-1\}),(1^{l+1})) \uparrow_{\kappa_2-m} = ((n-m+1),(\{m-l-1\}),(1^{l+1})) = \mu_{n+1,m}.
   \]
Now, suppose that $n \equiv l+1 \pmod{e}$. Then $((n-m),(\{m-l-1\}),(1^{l+1}))$ has $(\kappa_2-m)$-signature
- $-++-$ if $m \equiv l \pmod{e}$,
- $-+++$ if $m \equiv -1 \pmod{e}$ and $l = e - 2$,
- $-++$ if $m \equiv -1 \pmod{e}$ and $l \neq e - 2$ or $m \not\equiv -1 \pmod{e}$ and $m \equiv l + 1 \pmod{e}$,
- $-+$ for all other cases.

And, when $n - l \not\equiv 0, 1 \pmod{e}$, $((n - m, \{m - l - 1\}), (1^{l+1}))$ has $(\kappa_2 - m)$-signature
- $++-\text{ if } m \equiv l \pmod{e}$,
- $+++-\text{ if } m \equiv -1 \pmod{e}$ and $l = e - 2$,
- $++-\text{ if } m \equiv -1 \pmod{e}$ and $l \neq e - 2$ or $m \not\equiv -1 \pmod{e}$ and $m \equiv l + 1 \pmod{e}$,
- $++\text{ for all other cases}.$

So, for $n \not\equiv l \pmod{e}$, we observe that the highest conormal $(\kappa_2 - m)$-node in each signature is the addable node lying at the bottom of the $[(m + e - l - 2)/(e - 1)]$th column in the first component. Adding this node, we have

$$
\mu_{n,m} \uparrow_{\kappa_2 - m} = ((n - m, \{m - l - 1\}), (1^{l+1})) \uparrow_{\kappa_2 - m} = ((n - m, \{m - l\}), (1^{l+1}))
\mu_{n+1,m+1}.
$$

(iii) Let $m \geq n - \left\lceil \frac{n}{e} \right\rceil$. Firstly, suppose that $n \not\equiv l + 1 \pmod{e}$. Observe that $((\{m - l\}, n - m - 1), (1^{l+1}))$ has the following addable/removable $(\kappa_2 - m)$-nodes
- addable node at the bottom of the $[(m + e - l - 2)/(e - 1)]$th column in the first component,
- addable node $(e, n - m, 1)$ if $n \equiv l \pmod{e}$,
- addable node $(e + 1, 1, 1)$ and removable node $(l + 1, 1, 2)$ if $m \equiv l \pmod{e}$,
- addable node $(1, 2, 2)$ if $m \equiv -1 \pmod{e}$,
- addable node $(l + 2, 1, 2)$ if $m \equiv l + 1 \pmod{e}$.

For $n \equiv l \pmod{e}$, it follows that $((\{m - l\}, n - m - 1), (1^{l+1}))$ has $(\kappa_2 - m)$-signature
- $++++-$ if $m \equiv l \pmod{e}$,
- $++-+$ if $m \equiv -1 \pmod{e}$ and $l = e - 2$,
- $++-+$ if $m \equiv -1 \pmod{e}$ and $l \neq e - 2$ or $m \not\equiv -1 \pmod{e}$ and $m \equiv l + 1 \pmod{e}$,
- $++$ for all other cases.

For $n - l \not\equiv 0, 1 \pmod{e}$, $(e, n - m, 1)$ is no longer an addable $(\kappa_2 - m)$-node, so $((\{m - l\}, n - m - 1), (1^{l+1}))$ has $(\kappa_2 - m)$-signatures $++-, +++, ++$ and $++$. So, for $n \not\equiv l + 1 \pmod{e}$, the highest conormal $(\kappa_2 - m)$-node in each $(\kappa_2 - m)$-signature of $((\{m - l\}, n - m - 1), (1^{l+1}))$ is the addable node at the bottom of the $[(m + e - l - 2)/(e - 1)]$th column in the first component. Hence

$$
\mu_{n,m} \uparrow_{\kappa_2 - m} = ((\{m - l\}, n - m - 1), (1^{l+1})) \uparrow_{\kappa_2 - m}
= ((\{m - l + 1\}, n - m - 1), (1^{l+1}))
= \begin{cases} 
\mu_{n+1,m} & \text{if } n \equiv l \pmod{e}, \\
\mu_{n+1,m+1} & \text{if } n - l \not\equiv 0, 1 \pmod{e}.
\end{cases}
$$
Secondly, suppose that \( n \equiv l + 1 \pmod{e} \). Observe that \((\{m-l+1\}, n-m-2), (1^{l+1})\) has the following addable or removable \((\kappa_2 - m)\)-nodes

- addable node \((e, n - m - 1, 1)\),
- addable node \((e + 1, 1, 1)\) and removable node \((l + 1, 1, 2)\) if \( m \equiv l \pmod{e} \),
- addable node \((1, 2, 2)\) if \( m \equiv -1 \pmod{e} \),
- addable node \((l + 2, 1, 2)\) if \( m \equiv l + 1 \pmod{e} \),
- the removable \((\kappa_2 - m)\)-node at the bottom of the \(((m + e - l - 1)/(e - 1)))th column in the first component.

Hence, \((\{m - l + 1\}, n - m - 2), (1^{l+1})\) has \((\kappa_2 - m)\)-signature

- \(-+++\) if \( m \equiv l \pmod{e} \),
- \(-+++\) if \( m \equiv -1 \pmod{e} \) and \( l = e - 2 \),
- \(-++\) if \( m \equiv -1 \pmod{e} \) and \( l \neq e - 2 \) or \( m \neq -1 \pmod{e} \) and \( m \equiv l + 1 \pmod{e} \),
- \(-+\) for all other cases.

The highest conormal \((\kappa_2 - m)\)-node in each sequence is \((e, n - m - 1, 1)\), and adding this node we have

\[
\mu_{n,m} \uparrow_{\kappa_2-m} = \left(\{\{m-l+1\}, n-m-2\}, (1^{l+1})\right) \uparrow_{\kappa_2-m} \\
= \left(\{\{m-l+1\}, n-m-1\}, (1^{l+1})\right) \\
= \mu_{n+1,m+1}.
\]

\[\square\]

**Theorem 10.4.** Let \( n \not\equiv l + 1 \pmod{e} \) and \( 1 \leq m \leq n - 1 \). Then

\[S_{((n-m),(1^m))} \cong D_{\mu_{n,m}}.\]

**Proof.** We proceed by induction on \( n \).

1. Suppose that \( n-l \not\equiv 2 \pmod{e} \). We obtain the irreducible label of \( S_{((n-m),(1^m))} \) by first restricting it to an \( \mathcal{H}_{n-1}^\Lambda \)-module by removing its foot node of residue \( \kappa_2 + 1 - m \) modulo \( e \), and then inducing up to an \( \mathcal{H}_n^\Lambda \)-module by adding its highest conormal \((\kappa_2 + 1 - m)\)-node. We have \( e_{\kappa_2} S_{((n-1),(1))} \cong S_{((n-1),\emptyset)} \). By eq. (10.2), \(((n-1), \emptyset) \uparrow_{\kappa_2} = ((n-1), (1))\), and hence \( S_{((n-1),(1))} \cong D_{((n-1),(1))} \) by Proposition 7.6.

Assuming that \( S_{((n-m),(1^{m-1}))} \cong D_{\mu_{n-1,m-1}} \) for \( m > 1 \), then

\[e_{\kappa_2+1-m} S_{((n-m),(1^m))} \cong S_{((n-m),(1^{m-1}))} \cong D_{\mu_{n-1,m-1}}.\]

By Proposition 7.6, and by \( S_{((n-m),(1^m))} \) being an irreducible \( \mathcal{H}_n^\Lambda \)-module,

\[S_{((n-m),(1^m))} \cong D_{\mu_{n-1,m-1} \uparrow_{\kappa_2-m+1}} = D_{\mu_{n,m}},\]

by eq. (10.2).
2. Suppose that \( n - l \equiv 2 \pmod{e} \). We obtain the irreducible label of \( S_{((n-m),(1^m))} \) by first restricting it to an \( \mathcal{H}_{n-2}^\Lambda \)-module by removing its hand node and its foot node of residue \( \kappa_2 + 1 - m \) modulo \( e \), and then inducing up to an \( \mathcal{H}_n^\Lambda \)-module by adding its two highest conormal \( (\kappa_2 + 1 - m) \)-nodes. We have \( e_{\kappa_2} S_{((n-1),(1))} \cong S_{((n-2),\emptyset)} \). By eq. (10.1) and eq. (10.2), \( ((n-2),\emptyset) \uparrow_{\kappa_2} = ((n-1),\emptyset) \uparrow_{\kappa_2} = ((n-1),(1)) \). Hence \( S_{((n-1),(1))} \cong D_{((n-1),(1))} \) by Proposition 7.6.

Assuming that \( S_{((n-m-1),(1^{m-1}))} \cong D_{\mu_{n-2,m-1}} \) for \( m > 1 \), then

\[
e_{\kappa_2-m+1} S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^{m-1}))} \cong D_{\mu_{n-2,m-1}}.
\]

By Proposition 7.6, and by \( S_{((n-m),(1^m))} \) being an irreducible \( \mathcal{H}_n^\Lambda \)-module,

\[
S_{((n-m),(1^m))} \cong D_{\mu_{n-2,m-1}} \uparrow_{\kappa_2-m+1} = D_{\mu_{n-1,m-1}} \uparrow_{\kappa_2-m+1} = D_{\mu_{n,m}}
\]

eq (eq. (10.1))

\[
= D_{\mu_{n,m}}
\]

eq (eq. (10.2)).

\[\square\]

**Theorem 10.5.** Let \( 1 \leq m \leq n - 1 \). If \( n \equiv l + 1 \pmod{e} \), then the composition factors of \( S_{((n-m),(1^m))} \) are

\[D_{\mu_{n-1,m-1}} \text{ and } D_{\mu_{n,m}}.\]

Moreover, \( D_{\mu_{n,m}} \cong \text{im}(\gamma_m) \).

**Proof.** We obtain the label of each of the two composition factors of a Specht module indexed by a hook bipartition \( ((n-m),(1^m)) \) by first restricting it to an \( \mathcal{H}_{n-1}^\Lambda \)-module by either (1) removing its foot node of residue \( \kappa_2 + 1 - m \), or (2) by removing its hand node of residue \( \kappa_2 - m \), and then inducing up to an \( \mathcal{H}_n^\Lambda \)-module by adding the highest conormal node of residue \( \kappa_2 + 1 - m \) or \( \kappa_2 - m \), respectively.

1. By removing the foot node of \( [((n-m),(1^m))] \), we have

\[
e_{\kappa_2-m+1} S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^{m-1}))},
\]

and by Theorem 10.4, this is isomorphic to \( D_{\mu_{n-1,m-1}} \). By eq. (10.1), \( \mu_{n-1,m-1} \uparrow_{\kappa_2+1-m} = \mu_{n,m-1} \). Then, by Proposition 7.6, \( D_{\mu_{n,m-1}} \) is a composition factor of \( S_{((n-m),(1^m))} \).

2. By removing the hand node of \( [((n-m),(1^m))] \), we have

\[
e_{\kappa_2-m} S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^{m-1}))},
\]

and by Theorem 10.4, this is isomorphic to \( D_{\mu_{n-1,m}} \). By eq. (10.1), \( \mu_{n-1,m} \uparrow_{\kappa_2-m} = \mu_{n,m} \). Then, by Proposition 7.6, \( D_{\mu_{n,m}} \) is a composition factor of \( S_{((n-m),(1^m))} \).

Furthermore, from \([8, \text{Theorem } 6.10]\), we know that \( \text{im}(\gamma_{m-1}) \) and \( \text{im}(\gamma_m) \) must somehow correspond to \( D_{\mu_{n,m-1}} \) and \( D_{\mu_{n,m}} \), for \( 1 \leq m \leq n - 1 \). Moreover, \( \text{im}(\gamma_m) \) is a composition factor of both \( S_{((n-m),(1^m))} \) and \( S_{((n-m-1),(1^{m+1}))} \), and hence must be isomorphic to \( D_{\mu_{n,m}} \), as required.

\[\square\]
10.2 Labelling the composition factors of $S_{(n-m),(1^m)}$ for $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$

Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ throughout this subsection.

For $n \not\equiv 0 \pmod{e}$ and $1 \leq m \leq n - 1$, we recall from [S, Proposition 6.12] that $S_{(n-m),(1^m)}$ has two composition factors, namely $\text{im}(\chi_m)$ and $S_{(n-m),(1^m)}/\text{im}(\chi_m)$. Thus $\text{im}(\chi_m) \cong D_\lambda$ and $S_{(n-m),(1^m)}/\text{im}(\chi_m) \cong D_\mu$ for some bipartitions $\lambda, \mu, \nu, \eta \in \mathcal{P}_n^2$.

**Definition 10.6.** Suppose that $n \not\equiv 0 \pmod{e}$. For $1 \leq m \leq n - 1$, define

$$\mu_{n,2m} = \begin{cases} 
((n-m, \{m\}), \varnothing) & \text{if } 1 \leq m < n - \frac{n}{e}, \\
((\{m+1\}, n - 1 - m), \varnothing) & \text{if } n - \frac{n}{e} \leq m \leq n - 1,
\end{cases}$$

and

$$\mu_{n,2m+1} = \begin{cases} 
((n-m), (1^m)) & \text{if } 1 \leq m < e, \\
((n-m, \{m-e\}), (2, 1^{e-2})) & \text{if } e \leq m < n - \frac{n}{e}, \\
((\{m-e+1\}, n - 1 - m), (2, 1^{e-2})) & \text{if } n - \frac{n}{e} \leq m \leq n - 1.
\end{cases}$$

Notice that $\mu_{n,2m}$ and $\mu_{n,2m+1}$ are distinct. We claim that the labels $\lambda, \mu$ of the two composition factors of $S_{(n-m),(1^m)}$ are $\mu_{n,2m}$ and $\mu_{n,2m+1}$, respectively, and hence that these factors are non-isomorphic.

For $n \equiv 0 \pmod{e}$, we recall from [S, Theorem 6.15] that $S_{(n-m),(1^m)}$ has four composition factors $\text{im}(\phi_m)$, $\text{im}(\phi_{m+1})$, $\text{ker}(\gamma_m)/\text{im}(\phi_m)$ and $\text{ker}(\gamma_{m+1})/\text{im}(\phi_{m+1})$ if $2 \leq m \leq n - 2$, and that $S_{(n-1),(1)}$ and $S_{(1),(n-1)}$ both have three composition factors. Thus $\text{im}(\phi_m) \cong D_\lambda$, $\text{im}(\phi_{m+1}) \cong D_\mu$, $\text{ker}(\gamma_m)/\text{im}(\phi_m) \cong D_\nu$ and $\text{ker}(\gamma_{m+1})/\text{im}(\phi_{m+1}) \cong D_\eta$, for some bipartitions $\lambda, \mu, \nu, \eta \in \mathcal{P}_n^2$.

**Definition 10.7.** Let $n \equiv 0 \pmod{e}$. For $2 \leq m \leq n - 1$, define

$$\mu_{n,2m} = \begin{cases} 
((n-m+1, \{m-1\}), \varnothing) & \text{if } 2 \leq m \leq n - \frac{n}{e}, \\
((\{m+1\}, n - m - 1), \varnothing) & \text{if } n - \frac{n}{e} < m \leq n - 1,
\end{cases}$$

and

$$\mu_{n,2m+1} = \begin{cases} 
((n-m+1), (1^{m-1})) & \text{if } 2 \leq m \leq e, \\
((n-m+1, \{m-e-1\}), (2, 1^{e-2})) & \text{if } e \leq m \leq n - \frac{n}{e}, \\
((\{m-e+1\}, n - m - 1), (2, 1^{e-2})) & \text{if } n - \frac{n}{e} < m \leq n - 1.
\end{cases}$$

We notice that $\mu_{n,2m}$, $\mu_{n,2m+1}$, $\mu_{n,2m+2}$ and $\mu_{n,2m+3}$ are distinct bipartitions. We claim that the labels $\lambda, \mu, \nu, \eta$ of the four composition factors of $S_{(n-m),(1^m)}$ are $\mu_{n,2m}$, $\mu_{n,2m+2}$, $\mu_{n,2m+1}$ and $\mu_{n,2m+3}$ for $2 \leq m \leq n - 2$, and hence these factors are non-isomorphic.

To confirm our claims, we require the following combinatorial result.

**Lemma 10.8.**

1. If $n \not\equiv 0 \pmod{e}$, then

$$\mu_{n,2m} \uparrow_{\kappa_2-m} = \mu_{n+1,2m+2}, \quad (10.3a)$$

$$\mu_{n,2m+1} \uparrow_{\kappa_2-m} = \mu_{n+1,2m+3}. \quad (10.3b)$$

2. If $n \equiv 0 \pmod{e}$, then

$$\mu_{n,2m} \uparrow_{\kappa_2+1-m} = \mu_{n+1,2m}, \quad \text{for } 1 \leq m \leq n - 1, \quad (10.4a)$$

$$\mu_{n,2m+1} \uparrow_{\kappa_2+1-m} = \mu_{n+1,2m+1}, \quad \text{for } 2 \leq m \leq n - 1. \quad (10.4b)$$

**Proof.**

1. Suppose that $n \not\equiv 0 \pmod{e}$ and let $i = \kappa_2 - m$. 


(a) Let $1 \leq m < e$. Then $((n - m, \{m\}), \varnothing)$ has the following addable/removable $i$-nodes
\begin{itemize}
  \item addable node $(1, n - m + 1, 1)$ if $n \equiv l \pmod{e}$,
  \item removable node $(1, n - m, 1)$ if $n \equiv 0 \pmod{e}$,
  \item addable node at the bottom of the $[(m + e - 2)/(e - 1)]$th column in the first component,
  \item addable node $(e + 1, 1, 1)$ if $m \equiv -1 \pmod{e}$,
  \item addable node $(1, 1, 2)$ if $m \equiv 0 \pmod{e}$.
\end{itemize}

So, when $n \equiv l \pmod{e}$, the $i$-signature of $((n - m, \{m\}), \varnothing)$ is either $++$ if $m \not\equiv -1, 0 \pmod{e}$, or is $+++ \mu$ if $m \equiv -1, 0 \pmod{e}$. The highest conormal node in each sequence is $(1, n - m + 1, 1)$, and adding this node gives

$$\mu_{n,2m} \uparrow_i = ((n - m, \{m\}), \varnothing) \uparrow_i = ((n - m + 1, \{m\}), \varnothing) = \mu_{n+1,2m+2}.$$ 

Now, if $n - l \not\equiv 0, 1 \pmod{e}$, then the $i$-signature of $((n - m, \{m\}), \varnothing)$ is either $+$ if $m \not\equiv -1, 0 \pmod{e}$ or is $+++ \mu$ if $m \equiv 0, 1 \pmod{e}$. The highest conormal node in each sequence is the addable node at the end of the $[(m + e - 2)/(e - 1)]$th column in the first component, and hence

$$\mu_{n,2m} \uparrow_i = ((n - m, \{m\}), \varnothing) \uparrow_i = ((n - m, \{m + 1\}), \varnothing) = \mu_{n+1,2m+2}.$$ 

We now observe that $((n - m), \{1^m\})$ has the following addable/removable $i$-nodes
\begin{itemize}
  \item addable node $(1, n - m + 1, 1)$ if $n \equiv l \pmod{e}$,
  \item removable node $(1, n - m, 1)$ if $n \equiv 0 \pmod{e}$,
  \item addable node $(1, 2, 2)$ if $m \equiv -1 \pmod{e}$,
  \item addable node $(m + 1, 1, 2)$.
\end{itemize}

We note that $(2, 1, 1)$ cannot have residue $i$ modulo $e$ as $m < e$.

Suppose that $n \equiv l \pmod{e}$. Then $((n-m), \{1^m\})$ has $i$-signature $++$ if $m < e - 1$, with highest conormal $i$-node $(1, n - m + 1, 1)$. Hence

$$\mu_{n,2m+1} \uparrow_i = ((n - m), \{1^m\}) \uparrow_i = ((n - m + 1), \{1^m\}) = \mu_{n+1,2m+3}.$$ 

However, if $m = e - 1$, then $((n - e + 1), \{1^{e-1}\})$ has $i$-signature $+++$, with highest conormal $(\kappa_2 - m)$-node $(1, n - e + 2, 1)$. Thus $\mu_{n,2e-1} \uparrow_i = \mu_{n+1,2e+1}$. Instead, suppose that $n - l \not\equiv 0, 1 \pmod{e}$. Then $((n-m), \{1^m\})$ has $i$-signature $+$ if $m < e - 1$, corresponding to cornomal node $(m + 1, 1, 2)$. Hence

$$\mu_{n,2m+1} \uparrow_i = ((n - m), \{1^m\}) \uparrow_i = ((n - m), \{1^{m+1}\}) = \mu_{n+1,2m+3}.$$ 

However, if $m = e - 1$, then $((n-m), \{1^m\})$ has $i$-signature $++$, with highest conormal node $(1, 2, 2)$. Hence

$$\mu_{n,2e-1} \uparrow_i = ((n - e + 1), \{1^{e-1}\}) \uparrow_i = ((n - e + 1), (2, 1^{e-2})) = \mu_{n+1,2e+1}.$$

(b) Let $e \leq m < n - \left\lceil \frac{n}{e} \right\rceil$. By the first part, it follows that $\mu_{n,2m} \uparrow_i = \mu_{n+1,2m+2}$ if $n \equiv 0 \pmod{e}$.

We now observe that $((n - m, \{m - e\}), \{2, 1^{e-2}\})$ has the following addable and removable $i$-nodes
\begin{itemize}
  \item addable node $(1, n - m + 1, 1)$ if $n \equiv l \pmod{e}$,
  \item removable node $(1, n - m, 1)$ if $n \equiv 0 \pmod{e}$,
\end{itemize}
addable nodes \((e + 1, 1, 1), (e, 1, 2)\) and removable node \((1, 2, 2)\) if \(m \equiv -1 \pmod{e}\),
addable node \((2, 2, 2)\) if \(m \equiv 0 \pmod{e}\),
addable node \((1, 3, 2)\) and removable node \((e - 1, 1, 2)\) if \(m \equiv -2 \pmod{e}\).

Suppose that \(n \equiv l \pmod{e}\). Then \(((n - m, \{m - e\}), (2, 1^{e-2}))\) has \(i\)-signature \(+++\) if \(m \equiv 0 \pmod{e}\), \(+++\) if \(m \equiv -1 \pmod{e}\), \(++-\) if \(m \equiv -2 \pmod{e}\), and \(++\) otherwise. The highest conormal node in each of these sequences is \((1, n - m + 1, 1)\), and hence

\[
\mu_{n,2m+1} \uparrow_i = ((n - m, \{m - e\}), (2, 1^{e-2})) \uparrow_i = ((n - m + 1, \{m - e\}), (2, 1^{e-2})) = \mu_{n+1,2m+3}.
\]

Now, suppose that \(n - l \not\equiv 0,1 \pmod{e}\). Then \(((n - m, \{m - e\}), (2, 1^{e-2}))\) has \(i\)-signature \(++\) if \(m \equiv 0 \pmod{e}\), \(+++\) if \(m \equiv -1 \pmod{e}\), \(++-\) if \(m \equiv -2 \pmod{e}\), and \(+\) otherwise. The highest conormal node in each of these sequences is the addable node at the bottom of the \([(m - e)/(e - 1)]\)th column in the first component. Hence

\[
\mu_{n,2m+1} \uparrow_i = ((n - m, \{m - e\}), (2, 1^{e-2})) \uparrow_i = ((n - m, \{m - e + 1\}), (2, 1^{e-2})) = \mu_{n+1,2m+3}.
\]

(c) Let \(\lfloor \frac{m}{e} \rfloor \leq m \leq n - 1\). We observe that \(((\{m + 1\}, n - m - 1), \emptyset)\) has the following addable/removable \(i\)-nodes

\begin{itemize}
  \item addable node at the bottom of the \([(m + e - 1)/(e - 1)]\)th column in the first component,
  \item addable node \((e, n - m, 1)\) if \(n \equiv l \pmod{e}\),
  \item removable node \((e, n - m - 1, 1)\) if \(n \equiv 0 \pmod{e}\),
  \item addable node \((e + 1, 1, 1)\) if \(m \equiv -1 \pmod{e}\),
  \item addable node \((1, 1, 2)\) if \(m \equiv 0 \pmod{e}\).
\end{itemize}

So, suppose that \(n \equiv l \pmod{e}\). Then \(((\{m + 1\}, n - m - 1), \emptyset)\) has \(i\)-signature \(++\) if \(m \not\equiv -1,0 \pmod{e}\), and \(+++\) if \(m \equiv -1,0 \pmod{e}\). Whereas, when \(n - l \not\equiv 0,1 \pmod{e}\), \(((\{m + 1\}, n - m - 1), \emptyset)\) has \(i\)-signature \(+\) if \(m \not\equiv -1,0 \pmod{e}\), and \(++\) if \(m \equiv -1,0 \pmod{e}\). The highest conormal node in each of these sequences is the addable node at the bottom of the \([(m + e - 1)/(e - 1)]\)th column in the first component. Hence

\[
\mu_{n,2m} \uparrow_i = ((\{m + 1\}, n - m - 1), \emptyset) \uparrow_i = ((\{m + 2\}, n - m - 1), \emptyset) = \mu_{n+1,2m+2}.
\]

We now observe that \(((\{m - e + 1\}, n - m - 1), (2, 1^{e-2}))\) has the following addable/removable \(i\)-nodes

\begin{itemize}
  \item addable node in the \([(m - 1)/(e - 1)]\)th column of the first component,
  \item addable node \((e, n - m, 1)\) if \(n \equiv l \pmod{e}\),
  \item removable node \((e, n - m - 1, 1)\) if \(n \equiv 0 \pmod{e}\),
  \item addable node \((2, 2, 2)\) if \(m \equiv 0 \pmod{e}\),
\end{itemize}
Suppose that $n \equiv l \pmod{e}$. Then $((m - e + 1), n - m - 1, (2, 1^{e-2}))$ has i-signature $+++$ if $m \equiv 0 \pmod{e}$, $+++-$ if $m \equiv -1 \pmod{e}$, $+ + +$ if $m \equiv -2 \pmod{e}$, $++$ otherwise. Instead, suppose that $n - l \not\equiv 0, 1 \pmod{e}$. Then $((m - e + 1), n - m - 1, (2, 1^{e-2}))$ has i-signature $+++$ if $m \equiv 0 \pmod{e}$, $+++-$ if $m \equiv -1 \pmod{e}$, $+ + +$ if $m \equiv -2 \pmod{e}$, $++$ otherwise. In each of these sequences, the highest conormal $i$-node is the addable node in the $[(m - 1)/(e - 1)]$th column of the first component, and hence

$$\mu_{n,2m+1} \uparrow_i = (\{(m - e + 1), n - m - 1, (2, 1^{e-2})\} \uparrow_i$$

$$= (\{(m - e + 2), n - m - 1, (2, 1^{e-2})\})$$

$$= \mu_{n+1,2m+3}.$$

2. Suppose that $n \equiv 0 \pmod{e}$ and let $i = \kappa_2 + 1 - m$.

(a) Let $1 \leq m \leq e$. We observe that $((n - m + 1, \{m - 1\}), \emptyset)$ has the following removable/addable $i$-nodes

- removable node $(1, n - m + 1, 1)$,
- addable node at the bottom of the $[(m + e - 3)/(e - 1)]$th column in the first component,
- addable node $(e + 1, 1, 1)$ if $m \equiv 0 \pmod{e}$,
- addable node $(1, 1, 2)$ if $m \equiv 1 \pmod{e}$.

The i-signature of $((n - m + 1, \{m - 1\}), \emptyset)$ is $--$ if $m \not\equiv 0, 1 \pmod{e}$, or is $--++$ if $m \equiv 0, 1 \pmod{e}$. The highest conormal $i$-node in each sequence is the addable node at the bottom of the $[(m + e - 3)/(e - 1)]$th column in the first component, and adding this node we have

$$\mu_{n,2m} \uparrow_{\kappa_2+1-m} = (\{(n - m + 1, \{m - 1\}), \emptyset\} \uparrow_{\kappa_2+1-m}$$

$$= (\{(n - m + 1, \{m\}), \emptyset\})$$

$$= \mu_{n+1,2m}.$$

For $m > 1$, we now observe that $((n - m + 1), (1^{m-1}))$ has the following removable/addable $i$-nodes

- removable node $(1, n - m + 1, 1)$,
- addable node $(1, 2, 2)$ if $m = e$,
- addable node $(m, 1, 2)$.

The i-signature of $((n - m + 1), (1^{m-1}))$ is $--++$ if $m \not\equiv e$, and is $++$ if $m = e$. The highest conormal $i$-node in each sequence is $(m, 1, 2)$, and adding this node we have

$$\mu_{n,2m+1} \uparrow_{\kappa_2+1-m} = (\{(n - m + 1), (1^{m-1})\} \uparrow_{\kappa_2+1-m}$$

$$= (\{(n - m + 1), (1^m)\})$$

$$= \mu_{n+1,2m+1}.$$

(b) Let $e < m \leq n - \frac{2}{e}$. By the previous part, $\mu_{n,2m} \uparrow_{\kappa_2+1-m} = \mu_{n+1,2m}$.

We observe that $((n - m + 1, \{m - e - 1\}), (2, 1^{e-2}))$ has the following addable and removable $i$-nodes
removable node \((1, n - m + 1, 1)\),
\(\diamond\) addable node at the bottom of the \([m - e]/(e - 1)\)th column in the first component,
\(\diamond\) addable node \((1, 3, 2)\) and removable node \((e - 1, 1, 2)\) if \(m \equiv -1 \pmod{e}\),
\(\diamond\) addable nodes \((e + 1, 1, 1)\) and \((e, 1, 2)\), and removable node \((1, 2, 2)\) if \(m \equiv 0 \pmod{e}\),
\(\diamond\) addable node \((2, 2, 2)\) if \(m \equiv 1 \pmod{e}\).

So \(((n - m + 1, \{m - e - 1\}, (2, 1^{e-2}))\) has \(i\)-signature \(-++\) if \(m \not\equiv -1, 0, 1 \pmod{e}\),
\(-++\) if \(m \equiv -1 \pmod{e}\), \(-+++\) if \(m \equiv 0 \pmod{e}\), and \(-++\) if \(m \equiv 1 \pmod{e}\). The highest conormal \(i\)-node in each of these sequences corresponds to the addable node at the bottom of the \([m - e]/(e - 1)\)th column in the first component, and adding this node we have

\[
\mu_n,2m+1 \uparrow_{\kappa_2+1-m} = ((n - m + 1, \{m - e - 1\}), (2, 1^{e-2})) \uparrow_{\kappa_2+1-m} \\
= ((n - m + 1, \{m - e\}), (2, 1^{e-2})) \\
= \mu_{n+1,2m+1}.
\]

\(c\) Let \(m - \frac{e}{2} < m \leq n - 1\). We observe that \(((m + 1), n - m - 1), \emptyset)\) has the following addable/removable \(i\)-nodes

\(\diamond\) the removable node at the bottom of the \([m + e - 1]/(e - 1)\)th column in the first component,
\(\diamond\) addable node \((e, n - m, 1)\),
\(\diamond\) addable node \((e + 1, 1, 1)\) if \(m \equiv 0 \pmod{e}\),
\(\diamond\) addable node \((1, 1, 2)\) if \(m \equiv 1 \pmod{e}\).

The \(i\)-signature of \(((m + 1), n - m - 1), \emptyset)\) is \(-++\) if \(m \not\equiv 0, 1 \pmod{e}\), and is \(-+++\) if \(m \equiv 0, 1 \pmod{e}\). The highest conormal \(i\)-node in each sequence corresponds to \((e, n - m, 1)\), and adding this node we have

\[
\mu_n,2m \uparrow_{\kappa_2+1-m} = ((m + 1, n - m - 1), \emptyset) \uparrow_{\kappa_2+1-m} \\
= ((m + 1), n - m), \emptyset) \\
= \mu_{n+1,2m}.
\]

We now observe that \(((m - e + 1), n - m - 1), (2, 1^{e-2}))\) has removable/addable \(i\)-nodes

\(\diamond\) the removable node at the bottom of the \([m - 1]/(e - 1)\)th column in the first component,
\(\diamond\) addable node \((e, n - m, 1)\),
\(\diamond\) addable nodes \((e + 1, 1, 1)\) and \((e, 1, 2)\), and removable node \((1, 2, 2)\) if \(m \equiv 0 \pmod{e}\),
\(\diamond\) addable node \((1, 3, 2)\) and removable node \((e - 1, 1, 2)\) if \(m \equiv -1 \pmod{e}\),
\(\diamond\) addable node \((2, 2, 2)\) if \(m \equiv 1 \pmod{e}\).

The \(i\)-signature of \(((m - e + 1), n - m - 1), (2, 1^{e-2}))\) is \(-++\) if \(m \not\equiv -1, 0, 1 \pmod{e}\), \(-+++\) if \(m \equiv 0 \pmod{e}\), \(-+++\) if \(m \equiv -1 \pmod{e}\), and \(-++\) if \(m \equiv 1 \pmod{e}\). The highest conormal \(i\)-node in each of these sequences is \((e, n - m, 1)\), and adding this node we have

\[
\mu_n,2m+1 \uparrow_{\kappa_2+1-m} = ((m - e + 1), n - m - 1), (2, 1^{e-2})) \\
\mu_n,2m+1 \uparrow_{\kappa_2+1-m} = ((m - e + 1), n - m), (2, 1^{e-2})) \\
= \mu_{n+1,2m+1}.
\]
**Theorem 10.9.** Suppose that $n \not\equiv 0 \pmod{e}$ and $1 \leq m \leq n - 1$. Then $S_{((n-m),(1^m))}$ has composition factors $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$.

Moreover, $D_{\mu_{n,2m}} \cong \text{im}(\chi_m)$ and $D_{\mu_{n,2m+1}} \cong S_{((n-m),(1^m))}/\text{im}(\chi_m)$.

**Proof.** We first show that $D_{\mu_{n,3}}$ is a composition factor of $S_{((n-1),(1))}$. We have

$$f^{(2)}_{\kappa_2-1}S_{((n-1),(1))} \cong S_{((n),(1^2))} \quad \text{if } n \equiv -1 \pmod{e},$$

and

$$f^{(2)}_{\kappa_2-1}S_{((n-1),(1))} \cong S_{((n-1),(1^2))} \quad \text{if } n \not\equiv -1 \pmod{e}.$$

For $n \equiv -1 \pmod{e}$, $S_{((n),(1^2))}$ has composition factor $D_{\mu_{n+2,5}}$, by downwards induction on $n$. Hence, by Proposition 7.6, $S_{((n-1),(1))}$ has composition factor $D_{\mu_{n+2,5}} \downarrow_{\kappa_2-1}^2$. We have

$$(n-1),(1) \uparrow_{\kappa_2-1}^2 \mu_{n,3} \uparrow_{\kappa_2-1}^2 = \mu_{n+1,5} \uparrow_{\kappa_2-1}^2 = \mu_{n+2,5} \quad (\text{eq. (10.3b)})$$

and

$$\mu_{n,3} \uparrow_{\kappa_2-1}^2 \mu_{n+1,5} = \mu_{n+1,5} = ((n-1),(1^2)), \quad (\text{eq. (10.4b)})$$

by eq. (10.3b). Its inverse gives us $\mu_{n,3} = \mu_{n+1,5} \downarrow_{\kappa_2-1}$, and hence $D_{\mu_{n,3}}$ is a composition factor of $S_{((n-1),(1))}$.

Similarly, for $n \not\equiv -1 \pmod{e}$, $S_{((n-1),(1^2))}$ has composition factor $D_{\mu_{n+1,5}}$. So by Proposition 7.6, $S_{((n-1),(1))}$ has composition factor $D_{\mu_{n+1,5}} \downarrow_{\kappa_2-1}^2$. Observe that

$$(n-1),(1) \uparrow_{\kappa_2-1} = \mu_{n,3} \uparrow_{\kappa_2-1} = \mu_{n+1,5} = ((n-1),(1^2)),$$

and

$$(n-1),(1) \downarrow_{\kappa_2-1} \mu_{n,3} \downarrow_{\kappa_2-1} = \mu_{n+1,5} = ((n-1),(1^2)),$$

by eq. (10.3b). Its inverse gives us $\mu_{n,3} = \mu_{n+1,5} \downarrow_{\kappa_2-1}$, and hence $D_{\mu_{n,3}}$ is a composition factor of $S_{((n-1),(1))}$.

1. Suppose that $n-l \not\equiv 2 \pmod{e}$. We have $e_{\kappa_2+1-m}S_{((n-m),(1^m))} \cong S_{((n-m),(1^{m-1}))}$, and by induction, $S_{((n-m),(1^{m-1}))}$ has composition factors $D_{\mu_{n-1,2m-2}}$ and $D_{\mu_{n-1,2m-1}}$. It follows from Proposition 7.6 that $S_{((n-m),(1^m))}$ has composition factors $D_{\mu_{n-1,2m-1} \downarrow_{\kappa_2+1-m}}$ and $D_{\mu_{n-1,2m-1} \uparrow_{\kappa_2+1-m}}$. We observe that $\mu_{n-1,2m-1} \uparrow_{\kappa_2+1-m} = \mu_{n,2m}$ by eq. (10.3a), and $\mu_{n-1,2m-1} \downarrow_{\kappa_2+1-m} = \mu_{n+1,2m+1}$ by eq. (10.3b). Thus $S_{((n-m),(1^m))}$ has composition factors $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$.

2. Suppose that $n-l \equiv 2 \pmod{e}$. We have $e_{\kappa_2+1-m}S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^{m-1}))}$, and by induction, $S_{((n-m-1),(1^{m-1}))}$ has composition factors $D_{\mu_{n-2,2m-2}}$ and $D_{\mu_{n-2,2m-1}}$. We have

$$\mu_{n-2,2m-2} \uparrow_{\kappa_2+1-m} = \mu_{n-2,2m-2} \uparrow_{\kappa_2+1-m} = \mu_{n,2m} \quad (\text{eq. (10.4a)})$$

and

$$\mu_{n-2,2m-1} \uparrow_{\kappa_2+1-m} = \mu_{n-2,2m-1} \uparrow_{\kappa_2+1-m} = \mu_{n,2m+1} \quad (\text{eq. (10.3b)})$$

So, by Proposition 7.6, $D_{\mu_{n,2m}}$ is a composition factor of $S_{((n-m),(1^m))}$. We also have that

$$\mu_{n-2,2m-1} \uparrow_{\kappa_2+1-m} = \mu_{n-2,2m-1} \uparrow_{\kappa_2+1-m} = \mu_{n,2m+1} \quad (\text{eq. (10.3b)})$$

Thus, by Proposition 7.6, $D_{\mu_{n,2m+1}}$ is another composition factor of $S_{((n-m),(1^m))}$.\[\square\]
Furthermore, from [S, Proposition 6.12], the composition factors $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$ of $S_{((n-m),(1^m))}$ must correspond to $\text{im}(\chi_m)$ and $S_{((n-m),(1^m))}/\text{im}(\chi_m)$. By Lemma 8.2,

$$\text{im}(\chi_m) = \text{span} \{ v_T \mid T \in \text{Std}((n-m),(1^m)), T(1,1,1) = 1 \},$$

and hence

$$S_{((n-m),(1^m))}/\text{im}(\chi_m) = \text{span} \{ v_T \mid T \in \text{Std}((n-m),(1^m)), T(1,1,2) = 1 \}.$$

Now let $S,T \in \text{Std}((n-m),(1^m))$ such that 1 lies in the arm of $T$ and 1 lies in the leg of $S$. Clearly, every tableau $T$ has residue sequence $(\kappa_1, i_2, \ldots, i_n)$, and every tableau $S$ has residue sequence $(\kappa_2, i_2, \ldots, i_n)$. The first component of $\mu_{n,2m}$ is its only non-zero component, whereas both of the components of $\mu_{n,2m+1}$ are non-zero. Thus, only the residue sequence of $\mu_{n,2m+1}$ can begin with residue $\kappa_2$, and hence $D_{\mu_{n,2m}} \cong \text{im}(\chi_m)$ and $D_{\mu_{n,2m+1}} \cong S_{((n-m),(1^m))}/\text{im}(\chi_m)$, as required.

Theorem 10.10. If $n \equiv 0 \pmod{e}$, then the following statements hold.

1. $S_{((n-1),(1))}$ has composition factors $S_{((n),\emptyset)}$, $D_{\mu_{n,4}}$ and $D_{\mu_{n,5}}$.

2. For $2 \leq m \leq n-2$, $S_{((n-m),(1^m))}$ has composition factors $D_{\mu_{n,2m}}$, $D_{\mu_{n,2m+1}}$, $D_{\mu_{n,2m+2}}$ and $D_{\mu_{n,2m+3}}$.

3. $S_{((1),(1^n))}$ has composition factors $S_{((2),(1^n))}$, $D_{\mu_{n,2n-2}}$ and $D_{\mu_{n,2n-1}}$.

Moreover, $D_{\mu_{n,2m}} \cong \text{im}(\phi_m)$ and $D_{\mu_{n,2m+1}} \cong \ker(\gamma_m)/\text{im}(\phi_m)$.

Proof. (i) Firstly, by removing the foot node of $S_{((n-1),(1))}$, we have

$$e_{\kappa_2}S_{((n-1),(1))} \cong S_{((n-1),\emptyset)} \cong D_{((n-1),\emptyset)}.$$

The $\kappa_2$-signature of $((n-1),\emptyset)$ is $++$, corresponding to conormal nodes $(1,n,1)$ and $(1,1,2)$. Adding the higher of these nodes, $((n-1),\emptyset) \uparrow \kappa_2 = ((n),\emptyset)$, and by Proposition 7.6, $D_{((n),\emptyset)}$ is a composition factor of $S_{((n-1),(1))}$.

Now suppose that $2 \leq m \leq n-1$. By removing the foot node of $((n-m),(1^m))$, we have

$$e_{\kappa_2+1-m}S_{((n-m),(1^m))} \cong S_{((n-m),(1^{m-1}))}.$$

We know that $S_{((n-m),(1^{m-1}))}$ has composition factors $D_{\mu_{n-1,2m-2}}$ and $D_{\mu_{n-1,2m-1}}$, by Theorem 10.9. Observe that $\mu_{n-1,2m-2} \uparrow \kappa_2 = \mu_{n,2m}$ by eq. (10.3a), and that $\mu_{n-1,2m-1} \uparrow \kappa_2 = \mu_{n,2m+1}$ by eq. (10.3b). Thus, by Proposition 7.6, $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+1}}$ are composition factors of $S_{((n-m),(1^m))}$.

(ii) First suppose that $1 \leq m \leq n-2$. By removing the hand node of $((n-m),(1^m))$, we have

$$e_{\kappa_2-m}S_{((n-m),(1^m))} \cong S_{((n-m-1),(1^{m-1}))}.$$

By Theorem 10.9, $S_{((n-m-1),(1^{m-1}))}$ has composition factors $D_{\mu_{n-1,2m}}$ and $D_{\mu_{n-1,2m+1}}$. Observe that $\mu_{n-1,2m} \uparrow \kappa_2 = \mu_{n,2m}$ by eq. (10.3a), and that $\mu_{n-1,2m+1} \uparrow \kappa_2 = \mu_{n,2m+3}$ by eq. (10.3b). Thus, $S_{((n-m),(1^m))}$ also has composition factors $D_{\mu_{n,2m+2}}$ and $D_{\mu_{n,2m+3}}$ by Proposition 7.6.

Secondly, by removing the hand node of $S_{((1),(1^{n-1}))}$, we have

$$e_{\kappa_1}S_{((1),(1^{n-1}))} \cong S_{((2),(1^{n-1}))} \cong D_{((n-e),(1^{e-1}))},$$
by Lemma 9.2. The \( \kappa_1 \)-signature of \( \{(n-\ell),(1^{\ell-1})\} \) is \(+ + +\), corresponding to conormal nodes \((1, \lfloor (n-2)/(\ell-1) \rfloor + 1, 1), (1, 2, 2)\) and \((\ell, 1, 2)\). Adding the highest of these nodes, we have \( \{(n-\ell),(1^{\ell-1})\} \uparrow \kappa_1 = \{(n-\ell+1),(1^{\ell-1})\}\). By Lemma 9.2, \( D_{(\ell-\ell+1),(1^{\ell-1})} \cong S_{(\ell,1^n)} \), and by Proposition 7.6, \( S_{(\ell,1^n)} \) is a composition factor of \( S_{(\ell,1^n)} \).

Furthermore, from [S, Theorem 6.15], we know that the composition factors \( D_{\mu_{n,2m+1}}, D_{\mu_{n,2m+2}} \) and \( D_{\mu_{n,2m+3}} \) of \( S_{(n-m),(1^m)} \) must somehow correspond to \( \text{im}(\phi_m) \), \( \text{im}(\phi_{m+1}) \), \( \ker(\gamma_m)/\text{im}(\phi_m) \) and \( \ker(\gamma_{m+1})/\text{im}(\phi_{m+1}) \), for \( 2 \leq m \leq n-2 \). Moreover, \( \ker(\gamma_m) \) and \( \ker(\gamma_{m+1})/\text{im}(\phi_{m+1}) \) are composition factors of both \( S_{(n-m),(1^m)} \) and \( S_{(n-m-1),(1^{m+1})} \), and hence must somehow correspond to \( D_{\mu_{n,2m+2}} \) and \( D_{\mu_{n,2m+3}} \).

We now let \( v_T \in S_{((n-m),(1^m))} \). By [S, Lemma 5.6] and Lemma 8.2, we have that

\[
\text{im}(\phi_{m+1}) \cong \text{span}\{v_T \mid T \in \text{Std}((n-m),(1^m)), T(1,1,1) = 1, T(1, n-m, 1) = n\}
\]

and

\[
\ker(\gamma_{m+1}) \cong \text{span}\{v_T \mid T \in \text{Std}((n-m),(1^m)), T(1,1,2) = 1, T(1, n-m, 1) = n\}.
\]

Hence

\[
\ker(\gamma_{m+1})/\text{im}(\phi_{m+1}) \cong \text{span}\{v_T \mid T \in \text{Std}((n-m),(1^m)), T(1,1,2) = 1, T(1, n-m, 1) = n\}.
\]

It follows, together with Lemma 8.2, that

\( \diamond T(1,1,1) = 1 \) if \( v_T \) lies in \( \text{im}(\phi_m) \) or \( \text{im}(\phi_{m+1}) \);

\( \diamond T(1,1,2) = 1 \) if \( v_T \) lies in \( \ker(\gamma_m)/\text{im}(\phi_m) \) or \( \ker(\gamma_{m+1})/\text{im}(\phi_{m+1}) \).

Now observe that only the first component of \( \mu_{n,2m} \) is non-empty, whereas both components of \( \mu_{n,2m+1} \) are non-empty. It follows that 1 can only lie in the leg of \( T \) if \( v_T \) lies in \( D_{\mu_{n,2m+1}} \) or \( D_{\mu_{n,2m+3}} \), and hence \( D_{\mu_{n,2m}} \cong \text{im}(\phi_m) \) and \( D_{\mu_{n,2m+1}} \cong \ker(\gamma_m)/\text{im}(\phi_m) \), as required. \( \square \)

11 Ungraded decomposition numbers

Recall that we found the composition factors of Specht modules labelled by hook bipartitions in [S, §6], and further, in Section 10.1 and Section 10.2 we established the irreducible label of each of these composition factors. We can thus find the multiplicities \( |S_{(n-m),(1^m)} : D_{\lambda}| \), for all \( \lambda \in \mathcal{R} \mathcal{P}^2_{n} \), without taking into consideration the grading on \( S_{(n-m),(1^m)} \) and its composition factors.

For all \( n \) and \( \kappa \), recall that the trivial representation for \( \mathcal{R}^\Lambda_n \) is \( S_{(n,\varnothing)} \cong D_{(n,\varnothing)} \), and by Lemma 9.2, the sign representation is

\[
S_{(\varnothing,1^n)}R \cong D_{(\varnothing,1^n)}R = \begin{cases} 
D_{(\varnothing,1^n)} & \text{if } n < l; \\
D_{((n-l),(1^l))} & \text{if } n \geq l,
\end{cases}
\]

where \( l \) is the residue of \( \kappa_2 - \kappa_1 \) modulo \( e \).

11.1 Case I: when \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv l + 1 \pmod{e} \)

We remind the reader that we found that \( S_{(n-m),(1^m)} \) is irreducible in [S, Theorem 6.7], for all \( 0 \leq m \leq n \), and by Theorem 10.4, \( S_{(n-m),(1^m)} \cong D_{\mu_{n,m}} \), leading us to the following result.
**Theorem 11.1.** Let \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv l + 1 \pmod{e} \). Then, by a specific ordering on the columns, part of the (ungraded) decomposition matrix for \( \mathcal{H}_n^\Lambda \) comprising of rows corresponding to hook bipartitions is

\[
\begin{pmatrix}
S_{(n)} & 1 & 0 & 0 \\
S_{(n-1,1)} & 1 & 1 & 0 \\
S_{(n-2,1^2)} & 0 & 1 & 1 \\
S_{(1^2,1)} & 0 & \cdots & 1 \\
\end{pmatrix}
\]

**11.2 Case II: when \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv l + 1 \pmod{e} \)**

From Theorem 10.5, the composition factors of \( S_{((n-m),(1^m))} \) are \( D_{\mu_n,m-1} \) and \( D_{\mu_n,m} \), for \( 1 \leq m \leq n - 1 \), and hence \( D_{\mu_n,m} \) is a composition factor of both \( S_{((n-m),(1^m))} \) and \( S_{((n-m-1),(1^{m+1}))} \), for \( 1 \leq m \leq n - 2 \). Also, note that \( D_{\mu_n,0} = D_{((n),(\emptyset))} \) and \( D_{\mu_n,1} = D_{((n),(1^1))} \). Furthermore, since the bipartitions \( \mu_{n,0}, \mu_{n,1}, \ldots, \mu_{n,n-1} \) are distinct, the irreducible modules \( D_{\mu_n,0}, D_{\mu_n,1}, \ldots, D_{\mu_n,n-1} \) are non-isomorphic, which leads us to the following result.

**Theorem 11.2.** Let \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e} \). Then, by a specific ordering on the columns, part of the (ungraded) decomposition matrix for \( \mathcal{H}_n^\Lambda \) comprising of rows corresponding to hook bipartitions is

\[
\begin{pmatrix}
S_{((n),\emptyset)} & 1 & 1 & 0 \\
S_{(n-1),(1^1)} & 1 & 1 & 0 \\
S_{(n-2),(1^2)} & 1 & 1 & 0 \\
S_{(n-3),(1^3)} & 1 & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots \\
S_{((1),(1^{n-1}))} & 0 & 1 & 1 \\
S_{((\emptyset),(1^n))} & 0 & 1 & 1 \\
\end{pmatrix}
\]

**11.3 Case III: when \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv 0 \pmod{e} \)**

From Theorem 10.9, the composition factors of \( S_{((n-m),(1^m))} \) are \( D_{\mu_{n,2m}} \) and \( D_{\mu_{n,2m+1}} \), for \( 1 \leq m \leq n - 1 \). Furthermore, since the bipartitions \( ((n),(\emptyset)), \mu_{n,2}, \mu_{n,3}, \ldots, \mu_{2n-1}, (\emptyset,(1^n)) \) are distinct, the irreducible modules \( D_{((n),(\emptyset))}, D_{\mu_{n,2}}, D_{\mu_{n,3}}, \ldots, D_{\mu_{2n-1}}, D_{((\emptyset),(1^n))} \) are non-isomorphic, leading us to the following result.

**Theorem 11.3.** Let \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv 0 \pmod{e} \). Then, by a specific ordering on the columns, part of the (ungraded) decomposition matrix for \( \mathcal{H}_n^\Lambda \) comprising of rows corresponding to hook bipartitions is

\[
\begin{pmatrix}
S_{((n),(\emptyset))} & 1 & 1 & 0 \\
S_{((n-1),(1^1))} & 1 & 1 & 0 \\
S_{((n-2),(1^2))} & 1 & 1 & 0 \\
S_{((n-3),(1^3))} & 1 & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots \\
S_{((1),(1^{n-1}))} & 0 & 1 & 1 \\
S_{((\emptyset),(1^n))} & 0 & 1 & 1 \\
\end{pmatrix}
\]
11.4 Case IV: when \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv 0 \pmod{e} \)

From Theorem 10.10, the composition factors of \( S_{((n-m),(1^m))} \) are

- \( D_{((n),(\emptyset))}, D_{\mu_n,4} \) and \( D_{\mu_n,5} \), for \( m = 1 \);
- \( D_{\mu_{n,2m}}, D_{\mu_{n,2m+1}}, D_{\mu_{n,2m+2}} \) and \( D_{\mu_{n,2m+3}} \), for \( 2 \leq m \leq n - 2 \);
- \( D_{\mu_{n,2n-2}}, D_{\mu_{n,2n-1}} \) and \( D_{(\emptyset,(1^n))}^R \), for \( m = n - 1 \),

and hence \( D_{\mu_{n,2m+2}} \) and \( D_{\mu_{n,2m+3}} \) are composition factors of both \( S_{((n-m),(1^m))} \) and \( S_{((n-m-1),(1^{m+1}))} \), for \( 1 \leq m \leq n - 2 \). Furthermore, since the bipartitions \( ((n),\emptyset), \mu_{n,4}, \ldots, \mu_{n,2n-1}, (\emptyset,(1^n))^R \) are distinct, the irreducible modules \( D_{((n),(\emptyset))}, D_{\mu_{n,4}}, \ldots, D_{\mu_{n,2n-1}}, D_{(\emptyset,(1^n))}^R \) are non-isomorphic, which leads us to the following result.

**Theorem 11.4.** Let \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv 1 + \kappa_2 - \kappa_1 \pmod{e} \). Then, by a specific ordering on the columns, part of the (ungraded) decomposition matrix for \( \mathcal{H}_n^\lambda \) comprising of rows corresponding to hook bipartitions is

\[
\begin{pmatrix}
1 & 0 \\
1 1 1 0 & 1 1 1 1 \\
1 1 1 1 & 1 1 1 1 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 1 1 1 1 & 1 1 1 1 \\
& & & 1 1 1 1 \\
\end{pmatrix}
\]

12 Graded dimensions of Specht modules and their irreducible heads

We now determine the graded dimensions of graded Specht modules labelled by hook bipartitions and their irreducible quotients, considering the combinatorial \( \mathbb{Z} \)-grading defined on these \( \mathcal{H}_n^\lambda \)-modules. Recall from Definition 4.1 that the graded dimension of \( S_\lambda \), denoted \( \text{grdim}(S_\lambda) \), is the Laurent polynomial

\[
\sum_{\tau \in \text{Std}(\lambda)} v^{\deg(\tau)} \quad (\lambda \in \mathcal{P}_n^1).
\]

We first understand how the entries in the leg of a standard \( ((n-m),(1^m)) \)-tableau affect its degree.

**Lemma 12.1.** Suppose \( 1 \leq i \leq k \). Then \( ((k-i),(1^i)) \) has neither an addable nor a removable \((\kappa_2+1-i)\)-node in the first row of the first component, except in the following cases.

(i) If \( k \equiv l + 1 \pmod{e} \), then \( (1,k-i+1,1) \) is an addable \((\kappa_2+1-i)\)-node for \( ((k-i),(1^i)) \).

(ii) If \( k \equiv l + 2 \pmod{e} \) and \( k > i \), then \( (1,k-i,1) \) is a removable \((\kappa_2+1-i)\)-node for \( ((k-i),(1^i)) \).
\begin{proof}
Let $T \in \text{Std}(\binom{n-m}{m})$. Since we want to determine if $((k-i),(1^i))$ has an addable or a removable $(\kappa_2 + 1 - i)$-node, which will lie in the first row of the first component if one exists, we have $T(i,1,2) = k$.

1. Suppose that $T(i,1,2) = l+1+\alpha e$ for some $\alpha \geq 0$. Then the entries $1, \ldots, l+\alpha e$ lie in the set of nodes $\{(1,1,2), \ldots, (i-1,1,2)\} \cup \{(1,1,1), \ldots, (i,j,1)\}$, where $j = l+\alpha e-i+1$. There are $j = l+\alpha e-i+1$ entries strictly smaller than $l+1+\alpha e$ in the arm of $T$ and there are $i-1 = l+\alpha e-j$ entries strictly smaller than $l+1+\alpha e$ in the leg of $T$. We now observe that

$$\text{res}(1,j+1,1) = \kappa_1 + j = \kappa_1 + l + \alpha e - i + 1 = \kappa_2 - i + 1 = \text{res}(i,1,2),$$

and since $T(i,1,2) > T(1,j,1)$, it follows that $(1,j+1,1)$, which equals $(1,k-i+1,1)$, is an addable $(\kappa_2 + 1 - i)$-node for $((k-i),(1^i))$.

2. Suppose that $T(i,1,2) = l + 2 + \alpha e$ for some $\alpha \geq 0$. Then the entries $1, \ldots, l + 1 + \alpha e$ lie in the set of nodes $\{(1,1,2), \ldots, (i-1,1,2)\} \cup \{(1,1,1), \ldots, (i,j,1)\}$, where $j = l + 2 + \alpha e - i$. There are $j = l + 2 + \alpha e - i$ entries in the arm of $T$ strictly smaller than $l + 2 + \alpha e$ and there are $i - 1 = l + 1 + \alpha e - j$ entries in the leg of $T$ strictly smaller than $l + 2 + \alpha e$. We now observe that

$$\text{res}(1,j,1) = \kappa_1 + j - 1 = \kappa_1 + l + 1 + \alpha e - i = \kappa_2 - i - 1 = \text{res}(i,1,2),$$

and since $T(1,j,1) < T(i,1,2)$, it follows that $(1,j,1)$, which equals $(1,k-i)$, is a removable $(\kappa_2 + 1 - i)$-node for $((k-i),(1^i))$. Moreover, if $T(i,1,2) > T(1,1,1)$, then it is clear that $((k-i),(1^i))$ does not have a removable $(\kappa_2 + 1 - i)$-node.

3. Suppose that $T(i,1,2) = l + k + \alpha e$ for some $\alpha \geq 0$ such that $k \in \{3, \ldots, e\}$. Then the entries $1, \ldots, l + k + \alpha e - 1$ lie in the set of nodes $\{(1,1,2), \ldots, (i-1,1,2)\} \cup \{(1,1,1), \ldots, (1,j,1)\}$, where $j = l + k + \alpha e - i$. There are $j = l + k + \alpha e - i$ entries strictly smaller than $l + k + \alpha e$ in the arm of $T$ and there are $i = 1 = l + k + \alpha e - i - j$ entries strictly smaller than $l + k + \alpha e$ in the leg of $T$. We observe that $\text{res}(i,1,2) = \kappa_2 + 1 - i$, whereas

$$\text{res}(1,j,1) = \kappa_1 + j - 1 = \kappa_1 + l + k + \alpha e - i - 1 = \kappa_2 + k - i - 1 = \text{res}(1,j+1,1) - 1.$$

Hence $\text{res}(i,1,2) \neq \text{res}(1,j,1) \text{res}(1,j+1,1)$ since $k \neq 1, 2$, and it thus follows that $((k-i),(1^i))$ does not have a removable or an addable $(\kappa_2 + 1 - i)$-node in the first row of the first component.

$\square$

Now we are able to obtain the degree of an arbitrary standard $((n-m),(1^m))$-tableau.

\begin{lemma}
Let $T \in \text{Std}(\binom{n-m}{m}) \text{ and } 1 \leq i \leq m$. Then the degree of $T$ is

$$\left\lceil \frac{m+i-1}{e} \right\rceil + \left\lceil \frac{m+1}{e} \right\rceil + \#\{i \mid T(i,1,2) \equiv l + 1 \pmod{e}\} - \#\{i \mid T(i,1,2) \equiv l + 2 \pmod{e}\}.$$\end{lemma}

\begin{proof}
Suppose that $T(i,1,2) = k$ for $i \leq k \leq n$, so that $T_{\leq k}$ has shape $((k-i),(1^i))$. Let $\overline{\text{deg}}(T_{\leq k})$ be the summand in $\text{deg}(T_{\leq k})$, defined similarly to $\text{deg}(T_{\leq k})$, except that we only attach a non-zero degree to a $(\kappa_2 + 1 - i)$-node $(i,1,2)$ in $T_{\leq k}$ if $((k-i),(1^i))$ has either an
we only attach a non-zero degree to a \((\kappa + 1 - i)\)-node in the first row of the first component. Thus, it follows from Lemma 12.1 that
\[
\overline{\deg(T_{\kappa k})} = \begin{cases} 
\deg(T_{\kappa k}) + 1 & \text{if } k \equiv l + 1 \pmod{e}; \\
\deg(T_{\kappa k}) - 1 & \text{if } k \equiv l + 2 \pmod{e} \text{ and } k > i.
\end{cases}
\]

The two remaining summands in \(\deg(T_{\kappa k})\) are defined similarly to \(\deg(T_{\kappa k})\), except that we only attach a non-zero degree to a \((\kappa + 1 - i)\)-node in the first row of the first component. Thus, it follows from Lemma 12.1 that
\[
\overline{\deg(T_{\kappa k})} = \begin{cases} 
\deg(T_{\kappa k}) + 1 & \text{if } k \equiv l + 1 \pmod{e}; \\
\deg(T_{\kappa k}) - 1 & \text{if } k \equiv l + 2 \pmod{e} \text{ and } k > i.
\end{cases}
\]

The following result is a trivial consequence of Lemma 12.2.

The number of nodes in the leg of \(T\) with residue \(\kappa_1 - 1\) is
\[
\begin{align*}
\left\lceil \frac{m+l-2}{e} \right\rceil + \left\lfloor \frac{i+1}{e} \right\rfloor &= \begin{cases} 
\left\lceil \frac{m+l-2}{e} \right\rceil & \text{if } l \neq e - 1 \\
\left\lfloor \frac{m-1}{e} \right\rfloor + 1 & \text{if } l = e - 1,
\end{cases}
\end{align*}
\]
and there are \(\left\lceil \frac{m}{e} \right\rceil\) nodes in the leg of \(T\) with residue \(\kappa_2 + 1\). Hence

\[
\deg(T) = \#\{i \mid (i, 1, 2) \text{ has addable } (\kappa_1 - 1)\text{-node } (2, 1, 1)\} + \#\{i \mid (i, 1, 2) \text{ has addable } (\kappa_2 + 1)\text{-node } (1, 2, 2)\} + \#\{i \mid (i, 1, 2) \text{ has addable } (\kappa_2 + 1 - i)\text{-node in the first row of } T\} - \#\{i \mid (i, 1, 2) \text{ has removable } (\kappa_2 + 1 - i)\text{-node in the first row of } T\}
\]

\[
= \#\{i \mid i \equiv l + 2 \pmod{e}, k > i\} + \#\{i \mid i \equiv 0 \pmod{e}\} + \#\{i \mid k \equiv l + 1 \pmod{e}\} - \#\{i \mid k \equiv l + 2 \pmod{e}, k > i\}
\]

\[
= \#\{i \mid i \equiv l + 2 \pmod{e}\} - \#\{i \mid i \equiv l + 2 \pmod{e}, k = i\} + \#\{i \mid i \equiv 0 \pmod{e}\} + \#\{i \mid k \equiv l + 1 \pmod{e}\} - \#\{i \mid k \equiv l + 2 \pmod{e}\} + \#\{i \mid k \equiv l + 2 \pmod{e}, k = i\}
\]

\[
= \#\{i \mid i \equiv l + 2 \pmod{e}\} + \#\{i \mid i \equiv 0 \pmod{e}\} + \#\{i \mid k \equiv l + 1 \pmod{e}\} - \#\{i \mid k \equiv l + 2 \pmod{e}\},
\]
and thus we obtain our desired result. \(\Box\)

The following result is a trivial consequence of Lemma 12.2.

**Lemma 12.3.** We have
\[
\grdim(S_{(n), \sigma}) = 1; \quad \grdim(S_{(\sigma, (1^n))}) = \left\lfloor \frac{n}{e} \right\rfloor + \left\lceil \frac{n+e-2}{e} \right\rceil + \left\lfloor \frac{i+1}{e} \right\rfloor.
\]

To our end, it is sufficient to only obtain the leading and trailing terms, and in some cases the second leading and second trailing terms too, of the graded dimensions of Specht modules labelled by hook bipartitions. For any \(T \in \text{Std}(n - m, (1^m))\), we define
\[
a_T := \#\{i \mid T(i, 1, 2) \equiv l + 1 \pmod{e}\} - \#\{i \mid T(i, 1, 2) \equiv l + 2 \pmod{e}\}.
\]
Then, for any non-empty subset \( \mathcal{T} \) of \( \text{Std}((n-m),(1^m)) \), we define the set

\[
A_{\mathcal{T}} := \{ a_T \mid T \in \mathcal{T} \}.
\]

We define the maximum degree of \( \mathcal{T} \) to be the largest degree of all tableaux in \( \mathcal{T} \), that is,

\[
\text{maxdeg}(\mathcal{T}) := \max\{ \deg(T) \mid T \in \mathcal{T} \}.
\]

Similarly, the minimum degree of \( \mathcal{T} \) is defined to be the smallest degree of all tableaux in \( \mathcal{T} \), that is,

\[
\text{mindeg}(\mathcal{T}) := \min\{ \deg(T) \mid T \in \mathcal{T} \}.
\]

By Lemma 12.2, it follows that

\[
\text{maxdeg}(\mathcal{T}) = \left\lfloor \frac{m+e-l-2}{e} \right\rfloor + \left\lfloor \frac{l+1}{e} \right\rfloor + \left\lfloor \frac{m}{e} \right\rfloor + \text{max}(A_{\mathcal{T}})
\]

and

\[
\text{mindeg}(\mathcal{T}) = \left\lfloor \frac{m+e-l-2}{e} \right\rfloor + \left\lfloor \frac{l+1}{e} \right\rfloor + \left\lfloor \frac{m}{e} \right\rfloor + \text{min}(A_{\mathcal{T}}).
\]

We now set

\[
a := \# \{ i \mid 1 \leq i \leq n, i \equiv l+1 \pmod{e} \},
\]

\[
b := \# \{ i \mid 1 \leq i \leq n, i \equiv l+2 \pmod{e} \},
\]

\[
c := \# \{ i \mid 1 \leq i \leq n, i-l \not\equiv 1,2 \pmod{e} \}.
\]

**Lemma 12.4.** Let \( \mathcal{T} = \text{Std}((n-m),(1^m)) \).

1. If \( 1 \leq m \leq \frac{n}{e} \), then \( \text{max}(A_{\mathcal{T}}) = m \) and \( \text{min}(A_{\mathcal{T}}) = -m \).
2. If \( \frac{n}{e} < m < n - \frac{n}{e} \), then \( \text{max}(A_{\mathcal{T}}) = a \) and \( \text{min}(A_{\mathcal{T}}) = -b \).
3. If \( n - \frac{n}{e} \leq m \leq n - 1 \), then \( \text{max}(A_{\mathcal{T}}) = n - m + a - b \) and \( \text{min}(A_{\mathcal{T}}) = m - n + a - b \).

**Proof.** Let \( S, T \in \mathcal{T} \) where \( \deg(T) = \text{maxdeg}(\mathcal{T}) \) and \( \deg(S) = \text{mindeg}(\mathcal{T}) \). We have \( a, b \in \{ \left\lfloor \frac{n}{e} \right\rfloor, \left\lfloor \frac{n}{e} \right\rfloor + 1 \} \), depending on \( \kappa \) and \( n \).

1. We can place an entry congruent to \( l+1 \) modulo \( e \) in each node in the leg of \( T \), and similarly, we can place an entry congruent to \( l+2 \) modulo \( e \) in each node in the leg of \( S \).
2. The legs of \( S \) and \( T \) contain at least \( \left\lfloor \frac{n}{e} \right\rfloor + 1 \) nodes, and so we can place every entry congruent to \( l+2 \) modulo \( e \) in the arm of \( T \), and similarly, we can place every entry congruent to \( l+1 \) modulo \( e \) in the arm of \( S \).
3. The arms of \( S \) and \( T \) contain at most \( \left\lfloor \frac{n}{e} \right\rfloor + 1 \) nodes. Thus we place an entry congruent to \( l+2 \) modulo \( e \) in every node in the arm of \( T \). Then the leg of \( T \) contains \( a \) entries congruent to \( l+1 \) modulo \( e \), \( b-n+m \) entries congruent to \( l+2 \) modulo \( e \), and \( n-a-b \) entries congruent to neither \( l+1 \) nor congruent to \( l+2 \) modulo \( e \). Similarly, we place an entry congruent to \( l+1 \) modulo \( e \) in every node in the arm of \( S \). Then the leg of \( S \) contains \( a-n+m \) entries congruent to \( l+1 \) modulo \( e \), \( b \) entries congruent to \( l+2 \) modulo \( e \), and \( n-a-b \) entries congruent to neither \( l+1 \) nor congruent to \( l+2 \) modulo \( e \).
Proposition 12.5. Let $\mathcal{T} = \text{Std}((n-m),(1^m))$. Then the graded dimension of $S_{(n-m),(1^m)}$ is
\[
\sum_{i=0}^{\text{max}(A_T) - \text{min}(A_T)} \sum_{j=0}^{\text{max}(A_T)} \left( \binom{a}{m-i+j} \binom{b}{j} \binom{c}{i-2j} \right) v^{(m-i+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)}.
\]

Proof. Let $T \in \mathcal{T}$. By Lemma 12.2, there are at most $\text{max}(A_T)$ entries in the leg of $T$ congruent to $l+1$ modulo $e$, and at most $\text{min}(A_T)$ entries congruent to $l+2$ modulo $e$. Thus, there exists a tableau with degree $\text{max}(A_T) - i + \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor$ for all $i \in \{0, \ldots, \text{max}(A_T) - \text{min}(A_T)\}$, so $\text{grdim}(S_{(n-m),(1^m)})$ has $\text{max}(A_T) - \text{min}(A_T)$ terms.

Suppose that $T$ has degree $\text{max}(A_T) - i + \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor$ for some $i$, and suppose that there are $j$ entries congruent to $l+2$ modulo $e$ in the leg of $T$. These $j$ entries contribute $-j$ to the degree of $T$. Hence, there must be $m - i + j$ entries congruent to $l+1$ modulo $e$ in the leg of $T$, and the remaining $i - 2j$ nodes in the leg of $T$ must contain entries congruent to neither $l+1$ modulo $e$ nor congruent to $l+2$ modulo $e$. Thus, there are $\binom{a}{m-i+j} \binom{b}{j} \binom{c}{i-2j}$ standard $((n-m),(1^m))$-tableaux with this combination of entries in its leg for some $j \in \{0, \ldots, \left\lfloor \frac{i}{2} \right\rfloor\}$, and summing over $j$ gives the number of standard $((n-m),(1^m))$-tableaux with degree $\text{max}(A_T) - i + \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor$. \qed

Corollary 12.6. The first and last two terms in the graded dimension of $S_{(n-m),(1^m)}$, respectively, are given in the following cases.

1. For $1 \leq m \leq \frac{n}{e}$,
\[
\left( \binom{a}{m} v^{(m+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} \right) + c \left( \binom{a}{m-1} v^{(m-1+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} + \ldots + c \left( \binom{b}{m-1} v^{(1-m+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} + \left( \binom{b}{m} v^{(-m+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} \right).
\]

2. For $\frac{n}{e} < m < n - \frac{n}{e}$,
\[
\left( \binom{c}{m-a} v^{(a+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} \right) + \left( \binom{a}{m-a+1} v^{(a+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} + \ldots + \left( \binom{b}{m-b+1} v^{(1-b+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} + \left( \binom{c}{m-b} v^{(-b+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} \right).
\]

3. For $n - \frac{n}{e} \leq m \leq n - 1$,
\[
\left( \binom{b}{n-m} v^{(n-m+a-b+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} \right) + \left( \binom{a}{n-m-1} v^{(1+m-n+a-b+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} + \left( \binom{a}{n-m} v^{(n-m+a-b+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m^2+e-l-2}{e} \right\rfloor + \left\lfloor \frac{i+1}{e} \right\rfloor)} \right).
\]
Proof. 1. We obtain the leading term in the graded dimension of \( S_{((n-m),(1^m))} \) by setting \( i = j = 0 \) in Proposition 12.5, the second term by setting \( i = 1 \) and \( j = 0 \), the trailing term by setting \( i = 2m \) and \( j = m \), and the second trailing term by setting \( i = 2m - 1 \) and \( j = m - 1 \).

2. We obtain the leading term in the graded dimension of \( S_{((n-m),(1^m))} \) by setting \( i = m - a \) and \( j = 0 \) in Proposition 12.5, the second leading term by setting \( i = m - a + 1 \) and \( j \in \{0,1\} \), the trailing term by setting \( i = m + b \) and \( j = b \), and the second leading term by setting \( i = m + b - 1 \) and \( j \in \{b - 1,b\} \).

3. We obtain the leading term in the graded dimension of \( S_{((n-m),(1^m))} \) by setting \( i = 2m - 2a - c \) and \( j = m - a - c \) in Proposition 12.5, the second leading term by setting \( i = 2m - 2a - c + 1 \) and \( j = m - a - c + 1 \), the trailing term by setting \( i = 2b + c \) and \( j = b \), the second trailing term by setting \( i = 2b + c - 1 \) and \( j = b \).

We now apply this result to Specht modules labelled by hook bipartitions dependent on whether \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) or not and whether \( n \equiv \kappa_2 - \kappa_1 + 1 \pmod{e} \) or not, explicitly giving the corresponding result in each case, which will be useful to refer to in section 14. We let \( T \in \mathrm{Std}((n-m),(1^m)) \).

12.1 Case I: \( \mathrm{grdim} \left( S_{((n-m),(1^m))} \right) \) for \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv l + 1 \pmod{e} \)

For \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv l + 1 \pmod{e} \), there are \( \lfloor \frac{n}{e} \rfloor \) entries in \( T \) that are congruent to \( l + 1 \) modulo \( e \), and \( \lceil \frac{n}{e} \rceil \) entries in \( T \) that are congruent to \( l + 2 \) modulo \( e \), leading us to the following result by Corollary 12.6.

**Proposition 12.7.** Let \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv l + 1 \pmod{e} \). Then the leading and trailing terms of \( \mathrm{grdim}(S_{((n-m),(1^m))}) \), respectively, are as follows.

1. \( \left( \frac{n}{e} \right) + 1 \) \( v \left( m + \left\lfloor \frac{n}{e} \right\rfloor + \left\lceil \frac{m + e - l - 2}{e} \right\rceil \right) \) and \( \left( \frac{n}{e} \right) + 1 \) \( v \left( -m + \left\lfloor \frac{n}{e} \right\rfloor + \left\lceil \frac{m + e - l - 2}{e} \right\rceil \right) \) if \( 1 \leq m \leq \frac{n}{e} + 1 \),

2. \( \frac{n - 2 \left( \frac{n}{e} \right) - 1}{m - \left\lfloor \frac{n}{e} \right\rfloor} \) \( v \left( \frac{n}{e} + \left\lfloor \frac{n}{e} \right\rfloor + \left\lceil \frac{m + e - l - 2}{e} \right\rceil \right) \) and \( \frac{n - 2 \left( \frac{n}{e} \right) - 1}{m - \left\lfloor \frac{n}{e} \right\rfloor} \) \( v \left( -\frac{n}{e} + \left\lfloor \frac{n}{e} \right\rfloor + \left\lceil \frac{m + e - l - 2}{e} \right\rceil \right) \) if \( \frac{n}{e} + 1 < m < n - \frac{n}{e} - 1 \),

3. \( \frac{n}{e} + 1 \) \( v \left( n - m + \left\lfloor \frac{n}{e} \right\rfloor + \left\lceil \frac{m + e - l - 2}{e} \right\rceil \right) \) and \( \frac{n}{e} + 1 \) \( v \left( m - n + \left\lfloor \frac{n}{e} \right\rfloor + \left\lceil \frac{m + e - l - 2}{e} \right\rceil \right) \) if \( n - \frac{n}{e} - 1 \leq m \leq n - 1 \).

**Example 12.8.** Let \( e = 3, \kappa = (0,1) \). There are six tableaux that index the basis vectors of \( S_{((2),(1^2))} \), namely

\[
\begin{align*}
T_1 &= \begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}, \quad T_2 = \begin{array}{cc}
2 & 4 \\
1 & 3
\end{array}, \quad T_3 = \begin{array}{cc}
1 & 4 \\
2 & 3
\end{array}, \quad T_4 = \begin{array}{cc}
1 & 4 \\
2 & 3
\end{array}, \quad T_5 = \begin{array}{cc}
1 & 3 \\
2 & 4
\end{array}, \quad T_6 = \begin{array}{cc}
1 & 2 \\
3 & 4
\end{array}
\end{align*}
\]

It is easy to check that \( \deg(T_1) = \deg(T_3) = 1 \), \( \deg(T_2) = \deg(T_6) = -1 \) and \( \deg(T_4) = \deg(T_5) = 0 \), so that \( \mathrm{grdim} \left( S_{((2),(1^2))} \right) = 2v + 2 + 2v^{-1} \).
12.2 Case II: grdim \( (S_{(m,1)}) \) for \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv l + 1 \pmod{e} \)

For \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv l + 1 \pmod{e} \), there are \( \left\lfloor \frac{n}{e} \right\rfloor + 1 \) entries in \( T \) congruent to \( l + 1 \) modulo \( e \), and \( \left\lfloor \frac{n}{e} \right\rfloor \) entries in \( T \) congruent to \( l + 2 \) modulo \( e \), which leads us to the following result by Corollary 12.6.

**Proposition 12.9.** Let \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv l + 1 \pmod{e} \). Then the leading and trailing terms of grdim \( (S_{(m,1)}) \), respectively, are as follows.

1. \( \left( \frac{n}{e} \right) + 1 \) if \( 1 \leq m \leq \frac{n}{e} \).
2. \( \left( \frac{n}{e} \right) + 1 \) if \( \frac{n}{e} < m < n - \frac{n}{e} \).
3. \( \left( \frac{n}{e} \right) + 1 \) if \( n - \frac{n}{e} \leq m \leq n - 1 \).

12.3 Case III: grdim \( (S_{(m,1)}) \) for \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv 0 \pmod{e} \)

For \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv 0 \pmod{e} \), there are \( \left\lfloor \frac{n}{e} \right\rfloor \) entries in \( T \) congruent to \( l + 1 \) modulo \( e \), and \( \left\lfloor \frac{n}{e} \right\rfloor + 1 \) entries in \( T \) congruent to \( l + 2 \) modulo \( e \), which leads us to the following result by Corollary 12.6.

**Proposition 12.10.** Let \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv 0 \pmod{e} \). Then the leading and trailing terms of grdim \( (S_{(m,1)}) \), respectively, are as follows.

1. \( \left( \frac{n}{e} \right) + 1 \) if \( 1 \leq m \leq \frac{n}{e} \).
2. \( \left( \frac{n}{e} \right) + 1 \) if \( \frac{n}{e} < m < n - \frac{n}{e} \).
3. \( \left( \frac{n}{e} \right) + 1 \) if \( n - \frac{n}{e} \leq m \leq n - 1 \).

12.4 Case IV: grdim \( (S_{(m,1)}) \) for \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv 0 \pmod{e} \)

For \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv 0 \pmod{e} \), there are \( \left\lfloor \frac{n}{e} \right\rfloor \) entries in \( T \) congruent to \( l + 1 \) modulo \( e \), and \( \left\lfloor \frac{n}{e} \right\rfloor + 1 \) entries in \( T \) congruent to \( l + 2 \) modulo \( e \), leading us to the following result by Corollary 12.6.

**Proposition 12.11.** Let \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv 0 \pmod{e} \). Then the first and last two terms of grdim \( (S_{(m,1)}) \) are given in the following cases.

1. For \( 1 \leq m \leq \frac{n}{e} \),

\[
\left( \frac{n}{e} \right) v(m-1+\left\lfloor \frac{n}{e} \right\rfloor + \left\lfloor \frac{n}{e} \right\rfloor + 1) + \ldots + \left( \frac{n}{m-1} \right) v(m-1+\left\lfloor \frac{n}{e} \right\rfloor + \left\lfloor \frac{n}{e} \right\rfloor + 1) + \ldots + \left( \frac{n}{e} \right) v(-m+\left\lfloor \frac{n}{e} \right\rfloor + \left\lfloor \frac{n}{e} \right\rfloor + 1).
\]
2. For \( \frac{n}{e} < m < \frac{n(e-1)+e}{e} \),

\[
\left(\frac{(e-2)n}{e}\right) \left(\frac{n}{em-n}\right) v\left(\frac{m}{e} + \left\lfloor \frac{m}{e} \right\rfloor + 1\right) + \frac{n}{e} \left(\frac{(e-2)n}{e}\right) \left(\frac{m}{e} - \frac{n}{e} + 1\right) + \frac{(e-2)n}{e} \left(\frac{n}{e} - m - 1\right) + \cdots + \left(\frac{n}{e}\right) v\left(\frac{m}{e} + \left\lfloor \frac{m}{e} \right\rfloor + 1\right)
\]

3. For \( \frac{n(e-1)+e}{e} \leq m \leq n - 1 \),

\[
\left(\frac{n}{e}\right) v\left(n - m + \left\lfloor \frac{m}{e} \right\rfloor + 1\right) + \left(\frac{n}{e}\right) v\left(n - m + \left\lfloor \frac{m}{e} \right\rfloor + 1\right) + \cdots + \left(\frac{n}{e}\right) v\left(m - n + \left\lfloor \frac{m}{e} \right\rfloor + 1\right)
\]

### 13 Graded dimensions of composition factors of \( S_{((n-m),(1^m))} \)

We now study the graded composition factors of Specht modules labelled by hook bipartitions, which arise as ungraded composition factors together with a grading shift, and determine results concerning their graded dimensions. Our results rely on the basis elements of these irreducible \( \mathcal{H}_n^\Lambda \)-modules, which we deduce from the bases of the images and kernels of certain Specht module homomorphisms given in Lemma 8.2.

For \( \lambda \in \mathcal{R}_n^\Lambda \), recall from Proposition 4.3 that the graded dimension of the irreducible \( \mathcal{H}_n^\Lambda \)-module \( D_\lambda \) is symmetric in \( v \) and \( v^{-1} \), and by Corollary 4.4, the graded dimension of \( D_\lambda \) that is spanned by \( \{v_T \mid T \in \mathcal{T}\} \) is

\[
grdim(D_\lambda) = v^i \sum_{T \in \mathcal{T}} v^{\text{deg}(T)},
\]

where \( 2i = \max\text{deg}(\mathcal{T}) - \min\text{deg}(\mathcal{T}) \).

We now determine the leading terms in the graded dimensions of composition factors of \( S_{((n-m),(1^m))} \), dependent on whether \( \kappa_2 \equiv \kappa_1 - 1 \pmod{e} \) or not and whether \( n \equiv l + 1 \pmod{e} \) or not. By the symmetry of the graded dimensions of irreducible \( \mathcal{H}_n^\Lambda \)-modules, we automatically recover their trailing terms.

#### 13.1 Case I: \( \kappa_2 \not\equiv \kappa_1 - 1 \) and \( n \not\equiv l + 1 \) modulo \( e \)

Let \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv l + 1 \pmod{e} \). Recall from [S, Theorem 6.7] that \( S_{((n-m),(1^m))} \) is an irreducible \( \mathcal{H}_n^\Lambda \)-module, and by Theorem 10.4, \( S_{((n-m),(1^m))} \cong D_{\mu_{n,m}}(i) \) as graded \( \mathcal{H}_n^\Lambda \)-modules for some \( i \in \mathbb{Z} \).
Proposition 13.1. Let \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \not\equiv l + 1 \pmod{e} \). Then the leading term of \( \text{grdim}(D_{\mu,n,m}) \) is

1. \( \left( \left\lceil \frac{n}{e} \right\rceil + 1 \right) v^m \) if \( 1 \leq m \leq \frac{n}{e} + 1 \),
2. \( \frac{n - 2 \left( \left\lceil \frac{n}{e} \right\rceil - 1 \right)}{m - \left\lceil \frac{n}{e} \right\rceil} \) if \( \frac{n}{e} + 1 < m < n - \frac{n}{e} - 1 \),
3. \( \frac{\left\lceil \frac{n}{e} \right\rceil + 1}{n - m} v^{n-m} \) if \( n - \frac{n}{e} - 1 \leq m \leq n - 1 \).

Moreover, \( D_{\mu,n,m} \cong S_{(n-m),(1^l)} \langle -\left\lfloor \frac{m}{e} \right\rfloor - \left\lfloor \frac{m+l-2}{e} \right\rfloor \rangle \).

Proof. Since \( S_{(n-m),(1^l)} \) is irreducible, the coefficient of the leading degree in the graded dimension of \( D_{\mu,n,m} \) equals the coefficient of the leading degree in the graded dimension of \( S_{(n-m),(1^l)} \), established in Proposition 12.7.

Let \( T \) be the set of all standard \((n-m),(1^l)\)-tableaux. If \( 1 \leq m \leq \frac{n}{e} + 1 \), then \( \text{maxdeg}(T) = m + \left\lceil \frac{n}{e} \right\rceil + \left\lfloor \frac{m+l-2}{e} \right\rfloor \) and \( \text{mindeg}(T) = -m + \left\lfloor \frac{m}{e} \right\rceil + \left\lfloor \frac{m+l-2}{e} \right\rfloor \), by Proposition 12.7. Hence the highest degree in the graded dimension of \( D_{(n-m),(1^l)} \) is \( \frac{1}{2} \left( \text{maxdeg}(T) - \text{mindeg}(T) \right) = m \). One can similarly deduce the leading degree in the other two cases.

Moreover, we determine \( i \in \mathbb{Z} \) where \( D_{(n-m),(1^l)} \cong S_{(n-m),(1^l)} \langle i \rangle \). If \( 1 \leq m \leq \frac{n}{e} + 1 \), then \( \text{maxdeg}(T) = m + \left\lceil \frac{n}{e} \right\rceil + \left\lfloor \frac{m+l-2}{e} \right\rfloor \) and \( \text{mindeg}(T) = -m + \left\lfloor \frac{m}{e} \right\rceil + \left\lfloor \frac{m+l-2}{e} \right\rfloor \), by Proposition 12.7. By the definition of the graded dimension of \( D_{(n-m),(1^l)} \), we know that \( i = -\frac{1}{2} \text{maxdeg}(T) - \frac{1}{2} \text{mindeg}(T) = -\left\lfloor \frac{m}{e} \right\rceil - \left\lfloor \frac{m+l-2}{e} \right\rfloor \), as required. We similarly determine the same grading shift for \( \frac{n}{e} + 1 < m \leq n - 1 \).

\( \square \)

13.2 Case II: \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv l + 1 \pmod{e} \)

Let \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv l + 1 \pmod{e} \). For \( 1 \leq m \leq n - 1 \), recall from Theorem 10.5 that \( S_{(n-m),(1^l)} \) has graded composition factors \( D_{\mu,n,m-1} \cong \text{im}(\gamma_{m-1}) \langle i \rangle \) and \( D_{\mu,n,m} \cong \text{im}(\gamma_m) \langle j \rangle \), for some \( i, j \in \mathbb{Z} \).

Proposition 13.2. Let \( \kappa_2 \not\equiv \kappa_1 - 1 \pmod{e} \) and \( n \equiv l + 1 \pmod{e} \). Then the leading term of \( \text{grdim}(D_{\mu,n,m}) \) is

1. \( \left( \left\lceil \frac{n}{e} \right\rceil + 1 \right) v^m \) if \( 0 \leq m \leq \frac{n}{e} \),
2. \( \frac{n - 2 \left( \left\lceil \frac{n}{e} \right\rceil - 1 \right)}{m - \left\lceil \frac{n}{e} \right\rceil} \) if \( \frac{n}{e} < m < n - \frac{n}{e} \),
3. \( \frac{\left\lceil \frac{n}{e} \right\rceil + 1}{n - m} v^{n-m-1} \) if \( \frac{n}{e} \leq m \leq n - 1 \).

Moreover, \( D_{\mu,n,m} \cong \text{im}(\gamma_m) \langle -\left\lfloor \frac{m+l-2}{e} \right\rfloor - \left\lfloor \frac{m}{e} \right\rceil \rangle \).

Proof. Let \( T = \{ T \in \text{Std}((n-m),(1^l)) \mid T(1,n-m,1) = n \} \). By Lemma 8.2, we know \( \{ v_T \mid T \in \mathcal{F} \} \) is a basis for \( \text{im}(\gamma_m) \), where \( T \in \mathcal{F} \). By Corollary 4.4, we have

\[
\text{grdim} \left( D_{\mu,n,m} \right) = v^i \text{grdim} \left( \text{im}(\gamma_m) \right) = v^i \sum_{T \in \mathcal{F}} v^{\deg(T)},
\]
where $2i = -\maxdeg(\mathcal{T}) - \mindeg(\mathcal{T})$. Now suppose that $T \in \mathcal{T}$. We note that there are $\lfloor \frac{n}{e} \rfloor + 1$ entries in $T$ congruent to $l + 1$ modulo $e$, $n$ being one such entry, and $\lceil \frac{n}{e} \rceil$ entries in $T$ congruent to $l + 1$ modulo $e$, and hence there are $n - 2 \lfloor \frac{n}{e} \rfloor - 1$ entries in $T$ congruent to neither $l + 1$ nor $l + 2$ modulo $e$. We now consider the three cases in the proposition.

1. The tableaux in $\mathcal{T}$ with the maximum degree are those tableaux formed by placing $m$ of the remaining $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l + 1$ modulo $e$ in the $m$ nodes in their legs. Hence, $\max(A_T) = m$. Whereas, the tableaux in $\mathcal{T}$ with the minimum degree are those tableaux formed by placing $m$ of the $\lceil \frac{n}{e} \rceil$ entries congruent to $l + 2$ modulo $e$ in the $m$ nodes in their legs. Hence $\min(A_T) = -m$. Thus, $\frac{1}{2} (\maxdeg(\mathcal{T}) - \mindeg(\mathcal{T})) = \frac{1}{2} (\max(A_T) - \min(A_T)) = m$.

2. The tableaux in $\mathcal{T}$ with the maximum degree are those tableaux formed by placing the remaining $\lceil \frac{n}{e} \rceil$ entries congruent to $l + 1$ modulo $e$ in their legs, together with $m - \lfloor \frac{n}{e} \rfloor$ entries congruent to neither $l + 1$ nor $l + 2$ modulo $e$. Hence, $\max(A_T) = \lfloor \frac{n}{e} \rfloor$. Whereas, the tableaux in $\mathcal{T}$ with the minimum degree are those tableaux formed by placing the $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l + 2$ modulo $e$ in their legs, together with $m - \lceil \frac{n}{e} \rceil$ entries congruent to neither $l + 1$ nor $l + 2$ modulo $e$. Hence, $\min(A_T) = -\lceil \frac{n}{e} \rceil$. Thus, $\frac{1}{2} (\maxdeg(\mathcal{T}) - \mindeg(\mathcal{T})) = \frac{1}{2} (\max(A_T) - \min(A_T)) = \lfloor \frac{n}{e} \rfloor$.

3. The tableaux in $\mathcal{T}$ with the maximum degree are those tableaux with the least amount of entries in their legs congruent to $l + 2$ modulo $e$, that is, we place $n - m - 1$ of the $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l + 2$ modulo $e$ in the remaining $n - m - 1$ nodes in their arms. Thus in the legs of these tableaux there are $\lfloor \frac{n}{e} \rfloor$ nodes congruent to $l + 1$ modulo $e$, $\lfloor \frac{n}{e} \rfloor - n + m + 1$ nodes congruent to $l + 2$ modulo $e$, and $n - 2 \lfloor \frac{n}{e} \rfloor - 1$ nodes congruent to neither $l + 1$ nor $l + 2$ modulo $e$. Hence, $\max(A_T) = n - m - 1$. Whereas, the minimum degree are those tableaux with the least amount of entries in their legs congruent to $l + 1$ modulo $e$, that is, we place $n - m - 1$ of the remaining $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l + 2$ modulo $e$ in the remaining $n - m - 1$ nodes in their arms. So in the legs of these tableaux there are $\lfloor \frac{n}{e} \rfloor$ entries congruent to $l + 2$ modulo $e$, $\lfloor \frac{n}{e} \rfloor - n + m + 1$ nodes congruent to $l + 1$ modulo $e$, and $n - 2 \lfloor \frac{n}{e} \rfloor - 1$ nodes congruent to neither $l + 1$ nor $l + 2$ modulo $e$. Hence, $\min(A_T) = m - n + 1$. Thus, $\frac{1}{2} (\maxdeg(\mathcal{T}) - \mindeg(\mathcal{T})) = \frac{1}{2} (\max(A_T) - \min(A_T)) = n - m - 1$.

Moreover, notice that $\min(A_T) = -\max(A_T)$, for all $m$. Thus, $2i = -\maxdeg(\mathcal{T}) - \mindeg(\mathcal{T}) = -2\left[\frac{n + e - l - 2}{e}\right] - 2\lfloor \frac{n}{e} \rfloor$, as required.

**Example 13.3.** Let $e = 3$, $\kappa = (0, 0)$, $n = 7$ and $\mathcal{T} = \{T \in \text{Std}((5), (1^2)) \mid T(2, 1, 2) = 7\}$. By Lemma 8.2, it follows that $\text{im}(\gamma_1)$ is spanned by $\{v_T \mid T \in \mathcal{T}\}$. There are six possible such tableaux, namely

$$S = \begin{array}{cccccc}
2 & 3 & 4 & 5 & 6 \\
1 & 7
\end{array},$$

$$s_1S = \begin{array}{cccccc}
1 & 3 & 4 & 5 & 6 \\
2 & 7
\end{array},$$

$$s_2s_1S = \begin{array}{cccccc}
1 & 2 & 3 & 5 & 6 \\
3 & 7
\end{array}.$$

$$s_3s_2s_1S = \begin{array}{cccccc}
1 & 2 & 3 & 5 & 6 \\
4 & 7
\end{array},$$

$$s_4s_3s_2s_1S = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 6 \\
5 & 7
\end{array},$$

$$s_5s_4s_3s_2s_1S = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
6 & 7
\end{array}.$$

By Lemma 12.1, if $T \in \mathcal{T}$ where $\text{deg}(T) = \maxdeg(\mathcal{T})$, then $T(1, 1, 2) \in \{1, 4\}$, whereas, if $T \in \mathcal{T}$ where $\text{deg}(T) = \mindeg(\mathcal{T})$, then $T(1, 1, 2) \in \{2, 5\}$. Hence, $\text{deg}(S) = \text{deg}(s_3s_2s_1S) > \text{deg}(S)$.
deg (s_2s_1S) = deg (s_5s_4s_3s_2s_1S) > deg (s_1S) = deg (s_4s_3s_2s_1S). One can check that deg(S) = 3, deg (s_1S) 1 and deg (s_2s_1S) = 2, so that
\[ \text{grdim}(\text{im}(\gamma_1)) = 2v^3 + 2v^2 + 2v. \]

By Lemma 10.8, im(\gamma_1) \cong D_{\nu_7,1} = D_{((6),(1))} as ungraded $\mathcal{H}_7^A$-modules, and so by shifting the grading for im(\gamma_1) we see that
\[ \text{grdim}(D_{((6),(1))}) = \text{grdim}(\text{im}(\gamma_1)(-2)) = 2v + 2 + 2v^{-1}. \]

13.3 Case III: $\kappa_2 \equiv \kappa_1 - 1$ and $n \not\equiv 0$ modulo $e$

Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$. For $1 \leq m \leq n - 1$, we recall from Theorem 10.9 that $S_{((n-m),(1^m))}$ has graded composition factors $D_{\mu_{n,2m}} \cong \text{im}(\chi_m)(i)$ and $D_{\mu_{n,2m+1}} \cong (S_{((n-m),(1^m))}/\text{im}(\chi_m))(j)$, for some $i, j \in \mathbb{Z}$.

Proposition 13.4. Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$.

1. The leading term of \text{grdim}(D_{\mu_{n,2m}}) is
   \[
   (a) \left( \left[ \frac{n}{e} \right] \right) v^m \text{ if } 1 \leq m \leq \frac{n}{e},
   
   (b) \left( \frac{n - 2\left[ \frac{n}{e} \right] - 1}{m - \left[ \frac{n}{e} \right]} \right) v^{\left[ \frac{n}{e} \right]} \text{ if } \frac{n}{e} < m < n - \frac{n}{e},
   
   (c) \left( \frac{\left[ \frac{n}{e} \right]}{n - m - 1} \right) v^{(n-m-1)} \text{ if } n - \frac{n}{e} \leq m \leq n - 1.
   
   Moreover, $D_{\mu_{n,2m}} \cong \text{im}(\chi_m)(-\left[ \frac{n}{e} \right] - \left[ \frac{n-1}{e} \right] - 1)$.

2. The leading term of \text{grdim}(D_{\mu_{n,2m+1}}) is
   \[
   (a) \left( \left[ \frac{n}{e} \right] \right) v^{m-1} \text{ if } 1 \leq m \leq \frac{n}{e},
   
   (b) \left( \frac{n - 2\left[ \frac{n}{e} \right] - 1}{m - \left[ \frac{n}{e} \right]} \right) v^{\left[ \frac{n}{e} \right]} \text{ if } \frac{n}{e} < m < n - \frac{n}{e},
   
   (c) \left( \frac{\left[ \frac{n}{e} \right]}{n - m} \right) v^{n-m} \text{ if } n - \frac{n}{e} \leq m \leq n - 1
   
   Moreover, $D_{\mu_{n,2m+1}} \cong S_{((n-m),(1^m))}/\text{im}(\chi_m)(-\left[ \frac{m-1}{e} \right] - \left[ \frac{m}{e} \right])$.

Proof. We note that in the set $\{1, \ldots, n\}$ there are $\left[ \frac{n}{e} \right]$ entries congruent to $l + 1$ modulo $e$, $\left[ \frac{n}{e} \right] + 1$ entries congruent to $l + 2$ modulo $e$ (including 1), and hence $n - 2\left[ \frac{n}{e} \right] - 1$ congruent to neither $l + 1$ nor $l + 2$ modulo $e$.

1. Let $\mathcal{T} = \{ T \in \text{Std}((n-m),(1^m)) \mid \text{Tr}(1,1,1) = 1 \}$. By Lemma 8.2, we know that \text{im}(\chi_m) is spanned by $\{v_T \mid T \in \mathcal{T}\}$. By Corollary 4.4, we have
   \[ \text{grdim}(D_{\mu_{n,2m}}) = v^i \text{grdim}(\text{im}(\chi_m)) = v^i \sum_{T \in \mathcal{T}} v^{\deg(T)}, \]
   where $2i = -\text{maxdeg}(\mathcal{T}) - \text{mindeg}(\mathcal{T})$. 

(a) The tableaux in $\mathcal{F}$ with the maximum degree are those tableaux constructed by placing $m$ of the $\left\lfloor \frac{m}{e} \right\rfloor$ entries congruent to $l + 1$ modulo $e$ in the $m$ nodes in their legs. Hence, $\max(\mathcal{A}_\mathcal{F}) = m$. Whereas, the tableaux in $\mathcal{F}$ with the minimum degree are those tableaux constructed by placing $m$ of the remaining $\left\lfloor \frac{m}{e} \right\rfloor$ entries congruent to $l + 2$ modulo $e$ in the $m$ nodes in their legs. Hence, $\min(\mathcal{A}_\mathcal{F}) = -m$.

(b) The tableaux in $\mathcal{F}$ with the maximum degree are those tableaux constructed by placing the $\left\lfloor \frac{m}{e} \right\rfloor$ entries congruent to $l + 1$ in their legs, together with $m - \left\lfloor \frac{m}{e} \right\rfloor$ of the $n - 2\left\lfloor \frac{n}{e} \right\rfloor - 1$ entries congruent to neither $l + 1$ nor $l + 2$ modulo $e$. Hence, $\max(\mathcal{A}_\mathcal{F}) = \left\lfloor \frac{m}{e} \right\rfloor$. Whereas, the tableaux in $\mathcal{F}$ with the minimum degree are those tableaux constructed by placing the remaining $\left\lfloor \frac{m}{e} \right\rfloor$ entries congruent to $l + 2$ in their legs, together with $m - \left\lfloor \frac{m}{e} \right\rfloor$ of the $n - 2\left\lfloor \frac{n}{e} \right\rfloor - 1$ entries congruent to neither $l + 1$ nor $l + 2$ modulo $e$. Hence, $\min(\mathcal{A}_\mathcal{F}) = -\left\lfloor \frac{m}{e} \right\rfloor$.

(c) The tableaux in $\mathcal{F}$ with the maximum degree are those tableaux with the least amount of entries congruent to $l + 2$ modulo $e$ in their legs, that is, we place $n - m - 1$ of the remaining $\left\lfloor \frac{n}{e} \right\rfloor$ nodes congruent to $l + 1$ modulo $e$ in the remaining $n - m - 1$ nodes in their arms. Thus, in the legs of these tableaux there are $\left\lfloor \frac{m}{e} \right\rfloor$ entries congruent to $l + 1$ modulo $e$, $\left\lfloor \frac{m}{e} \right\rfloor - n + m + 1$ entries congruent to $l + 2$ modulo $e$, and $n - 2\left\lfloor \frac{n}{e} \right\rfloor + 1$ entries congruent to neither $l + 1$ nor $l + 2$ modulo $e$. Hence, $\max(\mathcal{A}_\mathcal{F}) = n - m - 1$. Whereas, the tableaux in $\mathcal{F}$ with the minimum degree are those tableaux with the least amount of entries congruent to $l + 1$ modulo $e$ in their legs, that is, we place $n - m - 1$ of the $\left\lfloor \frac{n}{e} \right\rfloor$ entries congruent to $l + 1$ modulo $e$ in the remaining $n - m - 1$ nodes in their arms. Thus, in the legs of these tableaux there are $\left\lfloor \frac{m}{e} \right\rfloor$ entries congruent to $l + 2$ modulo $e$, $\left\lfloor \frac{m}{e} \right\rfloor - n + m + 1$ entries congruent to $l + 1$ modulo $e$, and $n - 2\left\lfloor \frac{n}{e} \right\rfloor + 1$ entries congruent to neither $l + 1$ nor $l + 2$ modulo $e$. Hence, $\max(\mathcal{A}_\mathcal{F}) = -n + m + 1$.

Moreover, notice that $\min(\mathcal{A}_\mathcal{F}) = -\max(\mathcal{A}_\mathcal{F})$ for all $m$. Thus, $2i = -\max\deg(\mathcal{F}) - \min\deg(\mathcal{F}) = -2\left\lfloor \frac{m}{e} \right\rfloor - 2\left\lfloor \frac{m-1}{e} \right\rfloor - 2$, as required.

2. Let $\mathcal{F} = \{T \in \text{Std}((n-m),(1^m)) \mid T(1,1,2) = 1\}$. By Lemma 8.2, we know that $S_{(n-m),(1^m)}/\im(\chi_m)$ is spanned by $\{v_T \mid T \in \mathcal{F}\}$. By Corollary 4.4, we have

$$\text{grdim} \left(D_{m_n,2m+1} \right) = v^i \text{grdim} \left(S_{(n-m),(1^m)}/\im(\chi_m) \right) = v^i \sum_{T \in \mathcal{F}} v^{\deg(T)},$$

where $2i = -\max\deg(\mathcal{F}) - \min\deg(\mathcal{F})$.

(a) The tableaux in $\mathcal{F}$ with the maximum degree are those tableaux constructed by placing $m - 1$ of the $\left\lfloor \frac{m}{e} \right\rfloor$ entries congruent to $l + 1$ modulo $e$ in the remaining $m - 1$ nodes in their legs. Hence, $\max(\mathcal{A}_\mathcal{F}) = m - 2$. Whereas, the tableaux in $\mathcal{F}$ with the minimum degree are those tableaux constructed by placing $m - 1$ of the $\left\lfloor \frac{m}{e} \right\rfloor - 1$ entries congruent to neither $l + 1$ nor $l + 2$ modulo $e$ in the remaining $m - 1$ nodes in their legs. Hence, $\min(\mathcal{A}_\mathcal{F}) = -m$.

(b) The tableaux in $\mathcal{F}$ with the maximum degree are those tableaux constructed by placing the $\left\lfloor \frac{m}{e} \right\rfloor$ entries congruent to $l + 1$ modulo $e$, together with $m - \left\lfloor \frac{m}{e} \right\rfloor - 1$ of the $n - 2\left\lfloor \frac{n}{e} \right\rfloor - 1$ entries congruent to neither $l + 1$ nor $l + 2$ modulo $e$, in the remaining $m - 1$ nodes in their legs. Hence, $\max(\mathcal{A}_\mathcal{F}) = \left\lfloor \frac{m}{e} \right\rfloor - 1$. Whereas, the tableaux in $\mathcal{F}$ with the minimum degree are those tableaux constructed by placing the remaining $\left\lfloor \frac{m}{e} \right\rfloor$ entries congruent to $l + 2$ modulo $e$, together with $m - \left\lfloor \frac{m}{e} \right\rfloor - 1$ of the $n - 2\left\lfloor \frac{n}{e} \right\rfloor - 1$ entries congruent to neither $l + 1$ nor $l + 2$ modulo $e$, in the remaining $m - 1$ nodes in their legs. Hence, $\min(\mathcal{A}_\mathcal{F}) = -\left\lfloor \frac{m}{e} \right\rfloor - 1$. 


(c) The tableaux in $\mathcal{T}$ with the maximum degree are those tableaux with the least amount of entries congruent to $l + 2$ modulo $e$ in their legs, that is, we place $n - m - 1$ entries of the remaining $\left\lfloor \frac{n}{e} \right\rfloor$ congruent to $l + 2$ modulo $e$ in their arms. So, there are $\left\lfloor \frac{n}{e} \right\rfloor$ entries congruent to $l + 1$ modulo $e$, $\left\lceil \frac{n}{e} \right\rceil - n + m + 1$ entries congruent to $l + 2$ modulo $e$, and $n - 2\left\lceil \frac{n}{e} \right\rceil - 1$ entries congruent to neither $l + 1$ nor $l + 2$ modulo $e$. Hence, $\max(A_\varphi) = n - m - 1$. Whereas, the tableaux in $\mathcal{T}$ with the maximum degree are those tableaux with the least amount of entries congruent to $l + 1$ modulo $e$ in their legs, that is, we place $n - m - 1$ entries of the $\left\lceil \frac{n}{e} \right\rceil$ entries congruent to $l + 1$ modulo $e$ in their arms. Hence, $\min(A_\varphi) = m - n - 1$.

Moreover, notice that $\min(A_\varphi) = -\max(A_\varphi) - 2$. Thus, $2i = -\max\deg(\mathcal{T}) - \min\deg(\mathcal{T}) = -2\left\lceil \frac{m-1}{e} \right\rceil - 2\left\lfloor \frac{m}{e} \right\rfloor$, as required.

\[\square\]

**Example 13.5.** Let $e = 3$, $\kappa = (0, 2)$, $n = 5$ and $\mathcal{T} = \{T \in \text{Std}((3), (1^2)) | T(1, 1, 1) = 1\}$. By Lemma 8.2, $\text{im}(\chi_2)$ is spanned by $\{v_T | T \in \mathcal{T}\}$. There are six tableaux in $\mathcal{T}$, namely

\[T = \begin{array}{cccc}
1 & 4 & 5 \\
2 & 3 & \\
\end{array},
\quad
s_3T = \begin{array}{cccc}
1 & 3 & 5 \\
2 & 4 & \\
\end{array},
\quad
s_4s_3T = \begin{array}{cccc}
1 & 3 & 4 \\
2 & 5 & \\
\end{array},
\quad
s_2s_3T = \begin{array}{cccc}
1 & 2 & 5 \\
2 & 4 & \\
\end{array},
\quad
s_2s_4s_3T = \begin{array}{cccc}
1 & 2 & 4 \\
3 & 5 & \\
\end{array},
\quad
s_3s_2s_3T = \begin{array}{cccc}
1 & 2 & 3 \\
4 & 5 & \\
\end{array}.
\]

By Lemma 12.1, one can check that $\deg(T) = \deg(s_4s_3T) = 2$, $\deg(s_3T) = \deg(s_3s_2s_3T) = 0$ and $\deg(s_4s_3T) = \deg(s_2s_3T) = 1$, and hence

\[\text{grdim}(\text{im}(\chi_2)) = 2v^2 + 2v + 2.\]

By Theorem 10.9, $\text{im}(\chi_2) \cong D_{\mu_{5,4}} = D_{((3, 1^2), \varnothing)}$ as ungraded $H^\Lambda_5$-modules, and one sees that we obtain $D_{((3, 1^2), \varnothing)}$ by shifting the degree of $\text{im}(\chi_2)$ by $-1$. In other words,

\[\text{grdim}(S_{((3, 1^2), \varnothing)}) = \text{grdim}(\text{im}(\chi_2)(-1)) = 2v + 2 + 2v^{-1}.\]

Let $\mathcal{S} = \{S \in \text{Std}((3), (1^2)) | S(1, 1, 2) = 1\}$. It follows from Lemma 8.2 that, $S_{((3), (1^2))}/\text{im}(\chi_2)$ is spanned by $\{v_S | S \in \mathcal{S}\}$. There are four tableaux in $\mathcal{S}$, namely

\[S = \begin{array}{cccc}
3 & 4 & 5 \\
1 & 2 & \\
\end{array},
\quad
s_2S = \begin{array}{cccc}
2 & 4 & 5 \\
1 & 3 & \\
\end{array},
\quad
s_3s_2S = \begin{array}{cccc}
2 & 3 & 5 \\
1 & 4 & \\
\end{array},
\quad
s_4s_3s_2S = \begin{array}{cccc}
2 & 3 & 4 \\
1 & 5 & \\
\end{array}.
\]

It can be easily checked that $\deg(S) = \deg(s_4s_3s_2S) = 0$, $\deg(s_2S) = 1$ and $\deg(s_3s_2S) = -1$, and hence

\[\text{grdim}(S_{((3), (1^2))}/\text{im}(\chi_2)) = v + 2 + v^{-1}.\]

We know $S_{((3), (1^2))}/\text{im}(\chi_2) \cong D_{\mu_{5,5}} = D_{((3), (1^2))}$ as ungraded $H^\Lambda_5$-modules by Theorem 10.9. Since $\text{grdim}(S_{((3), (1^2))}/\text{im}(\chi_2))$ is symmetric in $v$ and $v^{-1}$, $S_{((3), (1^2))}/\text{im}(\chi_2) \cong D_{((3), (1^2))}$ as graded $H^\Lambda_5$-modules.
13.4 Case IV: $\kappa_2 \equiv \kappa_1 - 1$ and $n \equiv 0$ modulo $e$

Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$. We know that $S_{(n-m),(1^m)}$ has ungraded composition factors $D_{\mu_{n,2m}} \cong \text{im}(\phi_m)$ and $D_{\mu_{n,2m+1}} \cong \ker(\gamma_m)/\text{im}(\phi_m)$ by Theorem 10.10. Under grading shifts, $D_{\mu_{n,2m}}(i)$ and $D_{\mu_{n,2m+1}}(j)$ are graded composition factors of $S_{(n-m),(1^m)}$, for some $i, j \in \mathbb{Z}$, which we determine. In this section, we not only find the leading terms in the graded dimensions of the graded composition factors $D_{\mu_{n,2m}}(i)$ and $D_{\mu_{n,2m+1}}(j)$ of $S_{(n-m),(1^m)}$, and hence the trailing terms, but the second leading terms, and hence the second trailing terms too. We will see in section 14 that these extra terms are necessary to determine the graded decomposition numbers in this case, since from Theorem 10.10 we know that $S_{(n-m),(1^m)}$ has up to four composition factors.

Proposition 13.6. Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$.

1. The first two leading terms of $\text{grdim}(D_{\mu_{n,2m}})$ are

   \[
   \begin{align*}
   (a) & \left(\frac{n-e}{m-1}\right) v^{m-1}, \left(\frac{(e-2)n}{m-2}\right) v^{m-2} \text{ if } 1 \leq m \leq \frac{n}{e}, \\
   (b) & \left(\frac{(e-2)n}{em-n}\right) v^{\frac{n-e}{e}}, \left(\frac{(e-2)n}{m-n-1}\right) v^{\frac{n-e}{e}} \text{ if } \frac{n}{e} < m \leq \frac{n(e-1)}{e}, \\
   (c) & \left(\frac{n-e}{n-m}\right) v^{n-m-1}, \left(\frac{(e-2)n}{n-m-2}\right) v^{n-m-1} \text{ if } \frac{n(e-1)}{e} + \frac{1}{e} \leq m \leq n-1.
   \end{align*}
   \]

   Moreover, $D_{\mu_{n,2m}} \cong \text{im}(\phi_m) \left\langle -\left[\frac{m-1}{e}\right] - \left[\frac{m}{e}\right] - 2 \right\rangle$.

2. The first two leading terms of $\text{grdim}(D_{\mu_{n,2m+1}})$ are

   \[
   \begin{align*}
   (a) & \left(\frac{n-e}{m-2}\right) v^{m-2}, \left(\frac{(e-2)n}{m-3}\right) v^{m-3} \text{ if } 2 \leq m \leq \frac{n}{e}, \\
   (b) & \left(\frac{(e-2)n}{e(m-1)-n}\right) v^{\frac{n-e}{e}}, \left(\frac{(e-2)n}{e(m-2)-n}\right) v^{\frac{n-e}{e}} \text{ if } \frac{n}{e} < m \leq \frac{n(e-1)}{e}, \\
   (c) & \left(\frac{n-e}{n-m}\right) v^{n-m}, \left(\frac{(e-2)n}{n-m-1}\right) v^{n-m} \text{ if } \frac{n(e-1)+1}{e} \leq m \leq n-1.
   \end{align*}
   \]

   Moreover, $D_{\mu_{n,2m+1}} \cong \ker(\gamma_m)/\text{im}(\phi_m) \left\langle -\left[\frac{m-1}{e}\right] - \left[\frac{m}{e}\right] - 1 \right\rangle$.

Proof. In the set $\{1, \ldots, n\}$, there are $\frac{n}{e}$ entries congruent to $l + 1$ modulo $e$ (including $n$), and there are $\frac{n}{e}$ entries congruent to $l + 2$ modulo $e$ (including 1).

1. Let $\mathcal{T} = \{T \in \text{Std}((n-m),(1^m)) \mid T(1,1,1) = 1, T(m,1,2) = n\}$. By Lemma 8.2, we know that $\{v_T \mid T \in \mathcal{T}\}$ is a basis for $\text{im}(\phi_m)$. By Corollary 4.4, we have

   \[
   \text{grdim}(D_{\mu_{n,2m}}) = v^i \text{grdim}(\text{im}(\phi_m)) = v^i \sum_{T \in \mathcal{T}} v^{\text{deg}(T)},
   \]

   where $2i = -\text{maxdeg}(\mathcal{T}) - \text{mindeg}(\mathcal{T})$. We let $S, T \in \mathcal{T}$ such that $\text{max}(A_T) = a_T$ and $\text{min}(A_T) = a_3$.

   (a) The degree of $T$ is obtained by placing the remaining $\frac{n-e}{e}$ entries congruent to $l + 1$ modulo $e$ in the remaining $m - 1$ nodes in the leg of $T$. Thus every entry in the leg of $T$ is congruent to $l + 1$ modulo $e$. Hence $\text{max}(A_T) = m$.

   Whereas, we obtain the degree of $S$ by placing the remaining $\frac{n-e}{e}$ entries congruent to $l + 2$ modulo $e$ in the remaining $m - 1$ nodes in the leg of $T$. Thus there is one
Specht modules labelled by hook bipartitions II

Let $\gamma$ be a bipartition, and let $\gamma_m$ be the hook of $\gamma$. Thus, $\min(\gamma_m) = \frac{m}{e}$, and $\max(\gamma_m) = \frac{n}{e}$, with $n > m$. Moreover, notice that $\min(\gamma_m) \geq \min(\gamma)$. Hence, $\min(\gamma_m) = \frac{m}{e}$.

Thus, $\frac{1}{2}(\maxdeg(\gamma) - \mindeg(\gamma)) = \frac{1}{2}(\max(\gamma_m) - \min(\gamma_m)) = m - 1$, as required.

(b) We obtain the degree of $T$ by placing the remaining $\frac{n-e}{e}$ entries congruent to $l + 1$ modulo $e$, together with $m - \frac{n}{e}$ entries congruent to neither $l + 1$ modulo $e$ nor congruent to $l + 2$ modulo $e$, in the remaining $m - 1$ nodes in the leg of $S$. Hence, $\max(\gamma) = \frac{n}{e}$.

Whereas, we obtain the degree of $S$ by placing the remaining $\frac{n-e}{e}$ entries congruent to $l + 2$ modulo $e$, together with $m - \frac{n}{e}$ entries congruent to neither $l + 1$ modulo $e$ nor congruent to $l + 2$ modulo $e$, in the remaining $m - 1$ nodes in the leg of $S$. Thus there is one entry in the leg of $T$ congruent to $l + 1$ modulo $e$. Hence, $\max(\gamma) = \frac{n}{e}$.

Thus, $\frac{1}{2}(\maxdeg(\gamma) - \mindeg(\gamma)) = \frac{1}{2}(\max(\gamma_m) - \min(\gamma_m)) = \frac{n-e}{e}$, as required.

(c) We obtain the degree of $T$ by placing the remaining $\frac{n-e}{e}$ entries congruent to $l + 2$ modulo $e$, together with $m - \frac{n}{e}$ entries congruent to neither $l + 1$ modulo $e$ nor congruent to $l + 2$ modulo $e$, in the remaining $m - 1$ nodes in the leg of $S$. Thus, $\max(\gamma) = n - m - 1$.

Whereas, we obtain the degree of $S$ by placing the remaining $\frac{n-e}{e}$ entries congruent to $l + 1$ modulo $e$ in the remaining $n - m - 1$ nodes in the arm of $T$. Thus there are $\frac{n}{e}$ entries congruent to $l + 1$ modulo $e$ in the leg of $T$, and there are $\frac{n}{e} - n + m$ entries congruent to $l + 2$ modulo $e$ in the leg of $T$. Hence, $\max(\gamma) = n - m - 1$.

Thus, $\frac{1}{2}(\deg(\gamma) - \deg(\gamma)) = \frac{1}{2}(\max(\gamma) - \min(\gamma)) = n - m - 1$, as required.

Moreover, notice that $\min(\gamma_m) = -\max(\gamma_m) + 2$, and thus $2i = -\max(\gamma_m) - \min(\gamma_m) = -2\lceil \frac{m}{e} \rceil - 2\lfloor \frac{m-1}{e} \rfloor - 4$, as required.

2. Let $\mathcal{S} = \{T \in \text{Std}((n-m),(1^m)) | T(1,1,2) = 1, T(m,1,2) = n \}$. By Lemma 8.2, we know that $\ker(\gamma_m)/\text{im}(\phi_m)$ is spanned by $\{v_T | T \in \mathcal{S}\}$. By Corollary 4.4, we have

$$\text{grdim}(D_{m,n,2m+1}) = v^\gamma \text{grdim}(\ker(\gamma_m)/\text{im}(\phi_m)) = v^\gamma \sum_{T \in \mathcal{S}} v^{\deg(T)},$$

where $2i = -\maxdeg(\mathcal{S}) - \mindeg(\mathcal{S})$. We let $S, T \in \mathcal{S}$ such that $\max(\mathcal{S}) = a_\gamma$ and $\min(\mathcal{S}) = a_\gamma$.

(a) We obtain the degree of $T$ by placing the remaining $\frac{n-e}{e}$ nodes congruent to $l + 1$ modulo $e$ in the remaining $m - 2$ nodes in the leg of $T$. Thus there are $m - 1$ entries in the leg of $T$ congruent to $l + 1$ modulo $e$, and there is one entry in the leg of $T$ congruent to $l + 2$ modulo $e$. Hence, $\max(A) = m - 2$.

Whereas, we obtain the degree of $S$ by placing the remaining $\frac{n-e}{e}$ nodes congruent to $l + 2$ modulo $e$ in the remaining $m - 2$ nodes in the leg of $S$. Thus there is one entry in the leg of $S$ congruent to $l + 1$ modulo $e$, and there are $m - 1$ entries in the leg of $S$ congruent to $l + 2$ modulo $e$. Hence, $\min(A) = 2 - m$.

Thus, $\frac{1}{2}(\maxdeg(\mathcal{S}) - \mindeg(\mathcal{S})) = \frac{1}{2}(\max(\mathcal{S}) - \min(\mathcal{S})) = m - 2$, as required.

(b) We obtain the degree of $T$ by placing the remaining $\frac{n-e}{e}$ entries congruent to $l + 1$ modulo $e$ in the leg of $T$, and then place $\frac{n(e-2)}{e}$ entries congruent to neither $l + 1$
modulo $e$ nor congruent to $l + 2$ modulo $e$ in the remaining $m - \frac{n}{e} - 1$ nodes in the leg of $T$. Hence, max$(A_\gamma) = \frac{n-e}{e}$.

Whereas, we obtain the degree of $S$ by placing all of the remaining $\frac{n-e}{e}$ entries congruent to $l + 2$ modulo $e$ in the leg of $S$, together with $\frac{n}{e} - 2$ entries congruent to neither $l + 1$ modulo $e$ nor congruent to $l + 2$ modulo $e$ in the remaining $m - \frac{n}{e} - 1$ nodes in the leg of $S$. Hence, min$(A_\gamma) = \frac{e-n}{e}$.

Thus, $\frac{1}{2}$ \text{max}\deg(\gamma) = \frac{1}{2}(\text{max}(A_\gamma) - \text{min}(A_\gamma)) = \frac{n-e}{e}$, as required.

(c) We obtain the degree of $T$ by placing the remaining $\frac{n-e}{e}$ entries congruent to $l + 2$ modulo $e$ in the arm of $T$. Thus $T$ has $\frac{n}{e}$ entries congruent to $l + 1$ modulo $e$ in the leg of $T$, together with $\frac{n}{e} - n + m$ entries congruent to $l + 2$ modulo $e$ in the leg of $T$. Hence, max$(A_\gamma) = n - m$.

Whereas, we obtain the degree of $S$ by placing the remaining $\frac{n-e}{e}$ entries congruent to $l + 1$ modulo $e$ in the arm of $S$. Thus $S$ has $\frac{n}{e}$ entries congruent to $l + 2$ modulo $e$ in the leg of $S$, together with $\frac{n}{e} - n + m$ entries congruent to $l + 1$ modulo $e$ in the leg of $S$. Hence, min$(A_\gamma) = m - n$.

Thus, $\frac{1}{2}$ \text{max}\deg(\gamma) = \frac{1}{2}(\text{max}(A_\gamma) - \text{min}(A_\gamma)) = n - m$, as required.

Moreover, notice that min$(A_\gamma) = -\text{max}(A_\gamma)$, and thus $2i = -\text{max}(A_\gamma) - \text{min}(A_\gamma) = -2\lfloor \frac{m}{e} \rfloor - 2\lfloor \frac{m-1}{e} \rfloor - 2$, as required.

\[\square\]

\textbf{Example 13.7.} Let $e = 3$, $\kappa = (0, 2)$, $n = 6$ and $\mathcal{T} = \{T \in \text{Std}((3), (1^3)) \mid T(1, 1, 1) = 1, T(3, 1, 2) = 6\}$. From Lemma 8.2, im($\phi_3$) is spanned by $\{v_T \mid T \in \mathcal{T}\}$. There are six tableaux in $\mathcal{T}$, namely

\[
\begin{align*}
T &= \begin{array}{ccc}
1 & 4 & 5 \\
2 & 3 & 6
\end{array}, & s_3T &= \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 & 6
\end{array}, & s_4s_3T &= \begin{array}{ccc}
1 & 3 & 4 \\
2 & 4 & 6
\end{array}, \\
2s_3T &= \begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & 6
\end{array}, & s_2s_4s_3T &= \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 6
\end{array}, & s_3s_2s_4s_3T &= \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array},
\end{align*}
\]

One can check that \text{deg}(T) = \text{deg}(s_2s_4s_3T) = 4, \text{deg}(s_3T) = \text{deg}(s_3s_2s_4s_3T) = 2 and \text{deg}(s_4s_3T) = \text{deg}(s_2s_3T) = 3, and hence

\[\text{grdim } (\text{im}(\phi_3)) = 2v^4 + 2v^3 + 2v^2.\]

We know im($\phi_3$) $\cong D_{6,6} = D_{((4,1^2),2)}$ as ungraded $\mathcal{H}_6^\Lambda$-modules. Now, by shifting the grading on im($\phi_3$) so that its graded dimension is symmetric in $v$ and $v^{-1}$, we have $D_{((4,1^2),2)} \cong \text{im}(\phi_3)(-3)$ as graded $\mathcal{H}_6^\Lambda$-modules.

Let $\mathcal{S} = \{S \in \text{Std}((3), (1^3)) \mid S(1, 1, 2) = 1, S(3, 1, 2) = 6\}$. Also from Lemma 8.2, ker($\gamma_3$)/im($\phi_3$) is spanned by $\{v_S \mid S \in \mathcal{S}\}$. There are four tableaux in $\mathcal{S}$, namely

\[
\begin{align*}
S &= \begin{array}{ccc}
3 & 4 & 5 \\
1 & 3 & 6
\end{array}, & s_2S &= \begin{array}{ccc}
2 & 4 & 5 \\
1 & 3 & 6
\end{array}, & s_3s_2S &= \begin{array}{ccc}
2 & 3 & 5 \\
1 & 4 & 6
\end{array}, & s_4s_3s_2S &= \begin{array}{ccc}
2 & 3 & 4 \\
1 & 5 & 6
\end{array},
\end{align*}
\]

\[\square\]
One can check that \( \deg(S) = \deg(s_4s_3s_2S) = 2 \), \( \deg(s_2S) = 3 \) and \( \deg(s_3s_2S) = 1 \), and hence \( \text{grdim}(\ker(\gamma_3)/\text{im}(\phi_3)) = v^3 + 2v^2 + v \).

We have \( \ker(\gamma_3)/\text{im}(\phi_3) \cong D_{p_6,7} = D_{((4),(1^2))} \) as ungraded \( \mathcal{H}_6^A \)-modules. By shifting the grading on \( \ker(\gamma_3)/\text{im}(\phi_3) \) so that its graded dimension is symmetric in \( v \) and \( v^{-1} \), we have \( D_{((4),(1^2))} = \ker(\gamma_3)/\text{im}(\phi_3)(-2) \) as graded \( \mathcal{H}_6^A \)-modules.

14 Graded decomposition numbers

Recall that we discovered the ungraded decomposition numbers for \( \mathcal{H}_n^A \) corresponding to Specht modules labelled by hook bipartitions in Section 11, and then in Section 12 and Section 13, we determined the graded dimensions of Specht modules labelled by hook bipartitions and of their composition factors, respectively. These findings are equivalent to solving part of the Graded Decomposition Problem, corresponding to hook bipartitions, which we now provide an answer to.

Recall from Section 5 that the graded decomposition numbers are defined to be the Laurent polynomials \( [S_\lambda : D_\mu]_v = \sum_{\mu \in \mathcal{P}_d} [S_\lambda : D_\mu(i)]v^i \), for \( \lambda \in \mathcal{P}_d^l \) and \( \mu \in \mathcal{Q}^l \).

We now find \( i \in \mathbb{Z} \) where \( S_{(\varnothing, (1^n))} \cong D_\lambda(i) \).

**Lemma 14.1.**

1. If \( \kappa_2 \equiv \kappa_1 - 1 \) (mod \( e \)), then \( [S_{(\varnothing, (1^n))} : D_\lambda]_v = v^{2\left\lceil \frac{m}{e} \right\rceil} \).

2. If \( \kappa_2 \not\equiv \kappa_1 - 1 \) (mod \( e \)), then \( [S_{(\varnothing, (1^n))} : D_\lambda]_v = v^{l \left( 1 + \left\lceil \frac{m-l-1}{e} \right\rceil \right) + 1} \).

**Proof.** We have \( \text{grdim}(D_\lambda) = 1 \), whereas \( \text{grdim}(S_{(\varnothing, (1^n))}) = \deg(T_{(\varnothing, (1^n))} \rangle) \), so that \( [S_{(\varnothing, (1^n))} : D_{(\varnothing, (1^n))}]_v = v^{\deg(T_{(\varnothing, (1^n))} \rangle)} \).

If \( e \mid i \), then \( (\varnothing, (1^i)) \) has addable \( (\kappa_2 + 1) \)-node \((1, 2, 2)\) strictly above \((i, 1, 2)\), for \( 1 \leq i \leq n \). Hence, \( d^{(i,1,2)}(\varnothing, (1^i)) = 1 \) when \( e \mid i \). Further, there are \( \left\lfloor \frac{m}{e} \right\rfloor \) nodes \((i, 1, 2)\) where \( e \not\mid i \).

1. If \( e \mid i \), then \( (\varnothing, (1^i)) \) also has addable \( \kappa_1 \)-node \((1, 1, 1)\) strictly above \((i, 1, 2)\), for \( 1 \leq i \leq n \). So \( d^{(i,1,2)}(\varnothing, (1^i)) = 2 \) when \( e \mid i \), and hence, \( \deg(T_{(\varnothing, (1^n))}) = 2 \left[ \frac{m}{e} \right] \).

2. For \( \alpha \geq 0 \), \((\varnothing, (1^{(1+\ell+\alpha e)}))\) has addable \( \kappa_2 \)-node \((1, 1, 1)\) strictly above \((1+\ell+\alpha e, 1, 2)\), so \( d^{(i,1+\alpha e,1,2)}(\varnothing, (1^{(1+\ell+\alpha e)})) = 1 \). There are \( \left\lfloor \frac{m-l-1}{e} \right\rfloor + 1 \) nodes \((i, 1, 2)\) with \( i \equiv l + 1 \) (mod \( e \)). Thus, \( \deg(T_{(\varnothing, (1^n))}) = \left[ \frac{m}{e} \right] + \left[ \frac{m-l-1}{e} \right] + 1 \).

In other words, if \( \kappa_2 \equiv \kappa_1 - 1 \) (mod \( e \)), then \( S_{(\varnothing, (1^n))} \cong D_\lambda \langle \left[ \frac{m}{e} \right] \rangle \), and if \( \kappa_2 \not\equiv \kappa_1 - 1 \) (mod \( e \)), then \( S_{(\varnothing, (1^n))} \cong D_\lambda \langle \left[ \frac{m}{e} \right] + \left[ \frac{m-l-1}{e} \right] + 1 \rangle \), as graded \( \mathcal{H}_n^A \)-modules.

For \( \lambda \in \mathcal{Q}_n^2 \), we now establish the graded composition multiplicities \( [S_{((n-m),(1^m))} : D_\lambda]_v \) of irreducible \( \mathcal{H}_n^A \)-modules \( D_\lambda \) arising as composition factors of \( S_{((n-m),(1^m))} \), for \( 1 \leq m \leq n-1 \), depending on whether \( \kappa_2 \equiv \kappa_1 - 1 \) (mod \( e \)) or not and whether \( n \equiv l + 1 \) (mod \( e \)) or not.
15 Case I: $\kappa_2 \not\equiv \kappa_1 - 1 \ (\text{mod } e)$ and $n \not\equiv l + 1 \ (\text{mod } e)$

Let $\kappa_2 \not\equiv \kappa_1 - 1 \ (\text{mod } e)$ and $n \not\equiv l + 1 \ (\text{mod } e)$. We recall from Theorem 10.4 that $S_{(n-m),(1^m)}$ is irreducible and isomorphic to $D_{\mu_n,m}$ as an ungraded $\mathcal{H}_n^\Lambda$-module, for all $m \in \{1, \ldots, n\}$. To find the graded multiplicity of $D_{(n-m),(1^m)}$ arising as a composition factor of $S_{(n-m),(1^m)}$, it suffices to find the grading shift on $D_{(n-m),(1^m)}$ so that it is isomorphic to $S_{(n-m),(1^m)}$ as a graded $\mathcal{H}_n^\Lambda$-module.

**Theorem 15.1.** Let $\kappa_2 \not\equiv \kappa_1 - 1 \ (\text{mod } e)$ and $n \not\equiv l + 1 \ (\text{mod } e)$. Then

$$[S_{(n-m),(1^m)} : D_{((n-m),(1^m))}]_v = v\left(\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-2-1}{e} \right\rfloor + 1\right).$$

**Proof.** We determine $i \in \mathbb{Z}$ where $[S_{(n-m),(1^m)} : D_{((n-m),(1^m))}]_v = v^i$, which is equivalent to finding $i \in \mathbb{Z}$ where $S_{(n-m),(1^m)} \cong D_{((n-m),(1^m))}^i$ as graded $\mathcal{H}_n^\Lambda$-modules. Thus, the result follows from Proposition 13.1. 

**Example 15.2.** Let $e = 3$, $\kappa = (0, 0)$. Then the submatrix with rows corresponding to Specht modules labelled by hook bipartitions is

$$
\begin{array}{c|ccc}
S_{(8), (0)} & 1 & v^2 & 0 \\
S_{(7), (1)} & v & 0 & \\
S_{(6), (2)} & v^2 & 0 & \\
S_{(5), (3)} & v^3 & 0 & \\
S_{(4), (4)} & v^4 & 0 & \\
S_{(3), (5)} & v^4 & 0 & \\
S_{(2), (6)} & v^4 & 0 & \\
S_{(1), (7)} & v^4 & 0 & \\
S_{(0), (8)} & v^4 & 0 & \\
\end{array}
$$

16 Case II: $\kappa_2 \not\equiv \kappa_1 - 1 \ (\text{mod } e)$ and $n \equiv l + 1 \ (\text{mod } e)$

Let $\kappa_2 \not\equiv \kappa_1 - 1 \ (\text{mod } e)$ and $n \equiv l + 1 \ (\text{mod } e)$. Recall from Theorem 10.5 that $S_{(n-m),(1^m)}$ has ungraded composition factors $D_{\mu_n,m-1}$ and $D_{\mu_n,m}$, for $1 \leq m \leq n - 1$. We now determine the grading shifts $i, j \in \mathbb{Z}$ so that $D_{\mu_n,m-1}^i$ and $D_{\mu_n,m}^j$ are graded composition factors of $S_{(n-m),(1^m)}$.

**Theorem 16.1.** Let $\kappa_2 \not\equiv \kappa_1 - 1 \ (\text{mod } e)$ and $n \equiv l + 1 \ (\text{mod } e)$. Then, for all $m \in \{1, \ldots, n - 1\}$,

1. $[S_{(n-m),(1^m)} : D_{\mu_n,m-1}]_v = v\left(\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-2-1}{e} \right\rfloor + 1\right)$,

2. $[S_{(n-m),(1^m)} : D_{\mu_n,m}]_v = v\left(\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-2-1}{e} \right\rfloor \right)$.

**Proof.** We determine $x, y \in \mathbb{Z}$ where $\text{grdim}(S_{(n-m),(1^m)}) = v^x \text{grdim}(D_{\mu_n,m-1}) + v^y \text{grdim}(D_{\mu_n,m})$.

1. Let $0 \leq m \leq \left\lfloor \frac{n}{e} \right\rfloor$. By Proposition 12.9, the leading and trailing terms, respectively, in the graded dimension of $S_{(n-m),(1^m)}$ are

$$v^{\left\lfloor \frac{m}{e} \right\rfloor + 1} v^{m + \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-2-1}{e} \right\rfloor} \quad \text{and} \quad v^{-m + \left\lfloor \frac{m}{e} \right\rfloor + 1} v^{m + \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-2-1}{e} \right\rfloor}.$$
and by Proposition 13.2, the leading terms in the graded dimensions of $\text{im}(\gamma_{m-1})$ and $\text{im}(\gamma_m)$, respectively are

$$\left(\frac{n}{e} \right) \nu^{m-1} \text{ and } \left(\frac{n}{m} \right) \nu^m.$$ 

Firstly, the graded dimensions of $D_{\mu_{n,m}}$ and $S_{((n-m),(1^m))}$ both have $2m + 1$ terms, and hence $y = \left[\frac{m}{e}\right] + \left[\frac{m+e-2-l}{e}\right]$. Thus, $x - \left[\frac{m}{e}\right] - \left[\frac{m+e-2-l}{e}\right] = 0, 1$ since the trailing coefficients in the graded dimensions of $D_{\mu_{n,m}}$ and $S_{((n-m),(1^m))}$ are equal. Now observe that the sum of the leading coefficients in the graded dimensions of $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$ give the leading coefficient in the graded dimension of $S_{((n-m),(1^m))}$. Hence, $x = \left[\frac{m}{e}\right] + \left[\frac{m+e-2-l}{e}\right] + 1$.

2. Let $\left[\frac{n}{e}\right] < m < n - \left[\frac{n}{e}\right]$. By Proposition 12.9, the leading and trailing terms in the graded dimension of $S_{((n-m),(1^m))}$, respectively are

$$\left(n - 2\left[\frac{n}{e}\right] - 1 \right) \nu\left(\left[\frac{n}{e}\right] + 1 + \left[\frac{m+e-2-l}{e}\right]\right) \text{ and } \left(n - 2\left[\frac{n}{e}\right] - 1 \right) \nu\left(-\left[\frac{n}{e}\right] + \left[\frac{m}{e}\right] + \left[\frac{m+e-2-l}{e}\right]\right).$$

By Proposition 13.2, the leading terms in the graded dimensions of $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$, respectively, are

$$\left(n - 2\left[\frac{n}{e}\right] - 1 \right) \nu\left[\frac{n}{e}\right] \text{ and } \left(n - 2\left[\frac{n}{e}\right] - 1 \right) \nu\left[\frac{n}{e}\right].$$

Observing that the leading coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,m-1}}$ are equal, we deduce that $x = \left[\frac{m}{e}\right] + \left[\frac{m+e-2-l}{e}\right] + 1$. Similarly, observing that the trailing coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,m}}$ are equal, we deduce that $y = \left[\frac{m}{e}\right] + \left[\frac{m+e-2-l}{e}\right]$.

3. Let $\left[\frac{n}{e}\right] \leq m \leq n - 1$. By Proposition 12.9, the leading and trailing terms in the graded dimension of $S_{((n-m),(1^m))}$ are

$$\left(\frac{n}{n-m} \right) \nu\left(n-m+1 + \left[\frac{m}{e}\right] + \left[\frac{m+e-2-l}{e}\right]\right) \text{ and } \left(\frac{n}{n-m} \right) \nu\left(n-m+1 + \left[\frac{m}{e}\right] + \left[\frac{m+e-2-l}{e}\right]\right),$$

respectively, and by Proposition 13.2, the leading terms in the graded dimension of $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$, respectively, are

$$\left(\frac{n}{n-m} \right) \nu^{n-m} \text{ and } \left(\frac{n}{n-m} \right) \nu^{n-m-1}.$$ 

Firstly, the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,m-1}}$ both have $2n - 2m + 1$ terms, and hence $x = \left[\frac{m}{e}\right] + \left[\frac{m+e-2-l}{e}\right] + 1$. Thus, $y - \left[\frac{m}{e}\right] - \left[\frac{m+e-2-l}{e}\right] = 0, 1$, since the leading coefficients in the graded dimensions of $S_{((n-m),(1^m))}$ and $D_{\mu_{n,m-1}}$ are equal. Now observe that the sum of the trailing coefficients in the graded dimensions of $D_{\mu_{n,m-1}}$ and $D_{\mu_{n,m}}$ gives the trailing coefficient in the graded dimension of $S_{((n-m),(1^m))}$. Hence, $y = \left[\frac{m}{e}\right] + \left[\frac{m+e-2-l}{e}\right]$.
Example 16.2. Let $e = 3$, $\kappa = (0,0)$. Then the submatrix with rows corresponding to Specht modules labelled by hook bipartitions is

\[
\begin{pmatrix}
1 & v & 0 \\
v & v^2 & v \\
v^3 & v^2 & v^3 \\
v^4 & v^3 & v^4 \\
v^5 & v^4 & v^5
\end{pmatrix}
\]

17 Case III: $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$

Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$. Recall from Theorem 10.9 that the ungraded composition factors of $S_{(n-m),(1^m)}$ are $D_{\mu_n,2m}$ and $D_{\mu_n,2m+1}$, for all $m \in \{1, \ldots, n-1\}$. Hence as graded $\mathcal{A}_n^\Lambda$-modules, the composition factors of $S_{(n-m),(1^m)}$ are $D_{\mu_n,2m} \langle i \rangle$ and $D_{\mu_n,2m+1} \langle j \rangle$ for some integers $i$ and $j$, which we now determine.

Theorem 17.1. Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \not\equiv 0 \pmod{e}$. Then, for all $m \in \{1, \ldots, n-1\}$,

1. $[S_{(n-m),(1^m)}] : D_{\mu_n,2m} v = v^{(\lfloor \frac{m}{e} \rfloor + \lceil \frac{m+e-1}{e} \rceil)}$.

2. $[S_{(n-m),(1^m)}] : D_{\mu_n,2m+1} v = v^{(\lfloor \frac{m}{e} \rfloor + \lceil \frac{m+e-1}{e} \rceil - 1)}$.

Proof. We determine $x, y \in \mathbb{Z}$ where

\[
\text{grdim}(S_{(n-m),(1^m)}) = v^x \text{grdim}(D_{\mu_n,2m}) + v^y \text{grdim}(D_{\mu_n,2m+1})
\]

1. Let $1 \leq m \leq \lfloor \frac{n}{e} \rfloor$. By Proposition 12.10, the leading and trailing terms in the graded dimension of $S_{(n-m),(1^m)}$, respectively, are

\[
\left( \lfloor \frac{m}{e} \rfloor \right) v^{(m+1) + \lceil \frac{m+e-1}{e} \rceil} \quad \text{and} \quad \left( \frac{m}{e} \right) v^{(-m+1) + \lfloor \frac{m+e-1}{e} \rfloor}.
\]

By Proposition 13.4, the leading terms in the graded dimensions of $D_{\mu_n,2m}$ and $D_{\mu_n,2m+1}$, respectively, are

\[
\left( \frac{m}{e} \right) v^m \quad \text{and} \quad \left( \frac{m}{e} \right) v^{(m-1)}.
\]

Firstly, the graded dimensions of $S_{(n-m),(1^m)}$ and $D_{\mu_n,2m}$ both have $2m + 1$ terms, and hence $x = \lfloor \frac{m}{e} \rfloor + \lceil \frac{m+e-1}{e} \rceil$. Thus, we have $y - \lfloor \frac{m}{e} \rfloor - \lfloor \frac{m+e-1}{e} \rfloor = -1$, so the leading coefficients in the graded dimensions of $S_{(n-m),(1^m)}$ and $D_{\mu_n,2m}$ are equal. Observing that the sum of the trailing coefficients in the graded dimensions of $D_{\mu_n,2m}$ and $D_{\mu_n,2m+1}$ equals the trailing coefficient in the graded dimension of $S_{(n-m),(1^m)}$, and hence $y = \lfloor \frac{m}{e} \rfloor + \lceil \frac{m+e-1}{e} \rceil - 1$.

2. Let $\lfloor \frac{n}{e} \rfloor < m < n - \lfloor \frac{n}{e} \rfloor$. By Proposition 12.10, the leading and trailing terms in the graded dimension of $S_{(n-m),(1^m)}$ are

\[
\left( n - 2 \frac{m}{e} - 1 \right) v^{(\frac{m}{e} + \lfloor \frac{m+e-1}{e} \rfloor)} \quad \text{and} \quad \left( n - 2 \frac{m}{e} - 1 \right) v^{(-\lfloor \frac{m}{e} \rfloor + \lfloor \frac{m+e-1}{e} \rfloor)}.
\]
Let \( \kappa \) be as before.

By Proposition 13.4, the leading terms in the graded dimensions of \( D_{\mu,n,2m} \) and \( D_{\mu,n,2m+1} \) are
\[
\left( n - 2\left\lfloor \frac{n}{m} \right\rfloor - 1 \right) v^\left\lfloor \frac{n}{m} \right\rfloor \text{ and } \left( n - 2\left\lfloor \frac{n}{m} \right\rfloor - 1 \right) v^\left\lfloor \frac{n}{m} \right\rfloor,
\]
respectively. Observe that the graded dimension of \( S_{((n-m),(1^m))} \) has \( 2\left\lfloor \frac{n}{m} \right\rfloor + 2 \) terms, whereas the graded dimension of \( D_{\mu,n,2m} \) and \( D_{\mu,n,2m+1} \) both have \( 2\left\lfloor \frac{n}{m} \right\rfloor + 1 \) terms. Hence, \( x = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n-1}{e} \right\rfloor = y + 1 \) or \( y = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n-1}{e} \right\rfloor = x + 1 \). Observing that the leading coefficients in the graded dimensions of \( S_{((n-m),(1^m))} \) and \( D_{\mu,n,2m} \) are equal, and the trailing coefficients in the graded dimensions of \( S_{((n-m),(1^m))} \) and \( D_{\mu,n,2m+1} \) are equal, the former case holds.

3. Let \( n - \left\lfloor \frac{n}{m} \right\rfloor \leq m \leq n - 1 \). By Proposition 12.10, the leading and trailing terms in the graded dimension of \( S_{((n-m),(1^m))} \) are
\[
\left( \frac{n}{m} + 1 \right) v^{(n-m-1)+\left\lfloor \frac{m}{e} \right\rfloor} \text{ and } \left( \frac{n}{m} \right) v^{(n-m-1)+\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+1}{e} \right\rfloor},
\]
respectively. By Proposition 13.4, the leading terms in the graded dimensions of \( D_{\mu,n,2m} \) and \( D_{\mu,n,2m+1} \) are
\[
\left( \frac{n}{m} \right) v^{(n-m-1)} \text{ and } \left( \frac{n}{m} \right) v^{(n-m)},
\]
respectively. The graded dimensions of \( S_{((n-m),(1^m))} \) and \( D_{\mu,n,2m+1} \) both have \( 2m + 1 \) terms, and hence \( y = \left\lfloor \frac{n}{e} \right\rfloor + \left\lfloor \frac{m+1}{e} \right\rfloor - 1 \). Observing that the sum of the leading coefficients in the graded dimensions of \( D_{\mu,n,2m} \) and \( D_{\mu,n,2m+1} \) equals the leading coefficient in the graded dimension of \( S_{((n-m),(1^m))} \), we have \( x = \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+1}{e} \right\rfloor \).

\section*{Example 17.2.}
Let \( e = 3, \kappa = (0,2) \). Then the submatrix with rows corresponding to Specht modules labelled by hook bipartitions is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\vdots & v & v & 0 \\
\vdots & v^2 & v^2 & 0 \\
\vdots & v^3 & v^3 & 0 \\
0 & v^4 & v^4 & 0 \\
\end{pmatrix}
\]

\section*{18 Case IV: \( \kappa_2 \equiv \kappa_1 - 1 \mod e \) and \( n \equiv 0 \mod e \)}

Let \( \kappa_2 \equiv \kappa_1 - 1 \mod e \) and \( n \equiv 0 \mod e \). Recall from Theorem 10.10 that \( D_{\mu,n,2m} \), \( D_{\mu,n,2m+2} \), \( D_{\mu,n,2m+1} \) and \( D_{\mu,n,2m+3} \) are the ungraded composition factors of \( S_{((n-m),(1^m))} \), for \( 2 \leq m \leq n - 2 \). \( S_{((n-1),(1))} \) and \( S_{((1),(1^n))} \) both have three composition factors. Hence as graded \( \mathcal{H} \) modules, \( S_{((n-m),(1^m))} \) has composition factors \( D_{\mu,n,2m}(i_1), D_{\mu,n,2m+2}(i_2), D_{\mu,n,2m+1}(i_3) \) and \( D_{\mu,n,2m+3}(i_4) \), for some \( i_1, i_2, i_3, i_4 \in \mathbb{Z} \), which we now determine.

Firstly, one observes that the graded dimension of \( D_{\mu,n,2m} \) equals the graded dimension of \( D_{\mu,n,2m+3} \) under a grading shift.
Lemma 18.1. Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$. For all $1 \leq m \leq n - 2$, 

$$v^2[S_{((n-m),(1^m))} : D_{\mu_{n,2m+3}}]v = [S_{((n-m),(1^m))} : D_{\mu_{n,2m}}]v.$$ 

Proof. Follows immediately from Proposition 13.6. \qed

Theorem 18.2. Let $\kappa_2 \equiv \kappa_1 - 1 \pmod{e}$ and $n \equiv 0 \pmod{e}$. Then 

1. $[S_{((n-m),(1^m))} : D_{\mu_{n,2m}}]v = v^{\left(\frac{n}{e} + \left\lfloor \frac{m+1}{e} - 1 \right\rfloor \right)}$, for $1 \leq m \leq n - 1$;
2. $[S_{((n-m),(1^m))} : D_{\mu_{n,2m+2}}]v = v^{\left(\frac{n}{e} + \left\lfloor \frac{m}{e} + \frac{1}{e} - 1 \right\rfloor \right)}$, for $1 \leq m \leq n - 2$;
3. $[S_{((n-m),(1^m))} : D_{\mu_{n,2m+1}}]v = v^{\left(\frac{n}{e} + \left\lfloor \frac{m+1}{e} - 1 \right\rfloor \right)}$, for $2 \leq m \leq n - 2$;
4. $[S_{((n-m),(1^m))} : D_{\mu_{n,2m+3}}]v = v^{\left(\frac{n}{e} + \left\lfloor \frac{m}{e} + \frac{1}{e} - 1 \right\rfloor - 1 \right)}$, for $1 \leq m \leq n - 2$.

Proof. 1. We have $D_{\mu_{n,5}} \cong \ker(\gamma_2) / \im(\phi_2)$, $D_{\mu_{n,4}} \cong \im(\phi_2)$ and $D_{\mu_{n,2}} \cong \im(\phi_1) \cong S_{((n),\emptyset)}$ as ungraded $\mathbb{F}_e$-modules.

By Lemma 18.1, $v^2[S_{((n-1),(1))} : D_{\mu_{n,3}}]v = [S_{((n-1),(1))} : D_{\mu_{n,2}}]v$. So we determine $x, y \in \mathbb{Z}$ where 

$$\text{grdim}(S_{((n-1),(1))}) = v^x \text{grdim}(D_{\mu_{n,2}}) + v^y \text{grdim}(D_{\mu_{n,4}}) + v^{x-2} \text{grdim}(D_{\mu_{n,5}}).$$

By Proposition 12.11 and Proposition 13.6, we have 

$$\text{grdim}(S_{((n-1),(1))}) = \frac{n}{e} v^2 + \frac{(e-2)n}{e} v + \frac{n}{e} = v^x + v^y \left( \frac{n-e}{e} v + \frac{(e-2)n}{e} + \frac{n-e}{e} v^{-1} \right) + v^{x-2}.$$

Thus, by equating terms, $y = 1 = x - 1$.

2. By Lemma 18.1, $v^2[S_{((n-m),(1^m))} : D_{\mu_{n,2m+3}}]v = [S_{((n-m),(1^m))} : D_{\mu_{n,2m}}]v$. holds for $2 \leq m \leq n - 2$. So we determine $x, y, z \in \mathbb{Z}$ where 

$$\text{grdim}(S_{((n-m),(1^m))}) = v^x \text{grdim}(D_{\mu_{n,2m}}) + v^y \text{grdim}(D_{\mu_{n,2m+1}}) + v^z \text{grdim}(D_{\mu_{n,2m+2}}) + v^{x-2} \text{grdim}(D_{\mu_{n,2m+3}}).$$

(a) Firstly, let $2 \leq m \leq \frac{n}{e}$. By the first part of Proposition 12.11, the first two leading terms of $\text{grdim}(S_{((n-m),(1^m))})$ are 

$$\left(\frac{n}{e} \right) v^{(m+\left\lfloor \frac{m+1}{e} \right\rfloor - 1)} \text{ and } \frac{(e-2)n}{e} \left( \frac{n}{m-1} \right) v^{(m-1+\left\lfloor \frac{m}{e} + \frac{1}{e} - 1 \right\rfloor),$$

respectively, and the last two trailing terms of $\text{grdim}(S_{((n-m),(1^m))})$ are 

$$\frac{(e-2)n}{e} \left( \frac{n}{m-1} \right) v^{(1-m+\left\lfloor \frac{m}{e} \right\rfloor - 1)} \text{ and } \frac{n}{e} \left( \frac{n}{m} \right) v^{(-m+\left\lfloor \frac{m}{e} \right\rfloor - 1)},$$

respectively. By Proposition 13.6, the first two leading terms of the graded dimensions of $D_{\mu_{n,2m}}$ and $D_{\mu_{n,2m+2}}$ are 

$$\left( \frac{n-e}{m-1} \right) v^{m-1}, \frac{(e-2)n}{e} \left( \frac{n-e}{m-2} \right) v^{m-2}.$$
and
\[
\left(\frac{n-e}{m}\right)v^m, \left(\frac{e-2}{e}\right)\left(\frac{n-e}{m-1}\right)v^{m-1},
\]
respectively, and the first two leading terms of \(D_{\mu_{n,2m+1}}\) and \(D_{\mu_{n,2m+3}}\) are
\[
\left(\frac{n-e}{m-2}\right)v^{m-2}, \left(\frac{e-2}{e}\right)\left(\frac{n-e}{m-3}\right)v^{m-3}
\]
and
\[
\left(\frac{n-e}{m-1}\right)v^{m-1}, \left(\frac{e-2}{e}\right)\left(\frac{n-e}{m-2}\right)v^{m-2},
\]
respectively.

The graded dimensions of \(S_{((n-m),(1^m))}\) and \(D_{\mu_{n,2m+2}}\) both have \(2m+1\) terms, and hence \(z = \left[\frac{m}{e}\right] + \left[\frac{m+e-1}{e}\right]\).

Observe that the graded dimensions of \(D_{\mu_{n,2m}}\) and \(D_{\mu_{n,2m+3}}\) both have \(2m-1\) terms, so together with Lemma 18.1, \(x = \left[\frac{m}{e}\right] + \left[\frac{m+e-1}{e}\right] + 1\).

Clearly, \(-2 \leq y - \left[\frac{m}{e}\right] - \left[\frac{m+e-1}{e}\right] \leq 2\). Now observe that the sum of the second leading coefficients in the graded dimensions of \(D_{\mu_{n,2m}}\) and \(D_{\mu_{n,2m+2}}\) form the second leading coefficient in the graded dimension of \(S_{((n-m),(1^m))}\). Also, the sum of the second trailing coefficients in the graded dimensions of \(D_{\mu_{n,2m+2}}\) and \(D_{\mu_{n,2m+3}}\) form the second trailing coefficient in the graded dimension of \(S_{((n-m),(1^m))}\). Hence, \(y = \left[\frac{m}{e}\right] + \left[\frac{m+e-1}{e}\right]\).

(b) Let \(\frac{n}{e} < m < \frac{n(e-1)}{e}\). By the second part of Proposition 12.11, the leading and trailing terms of grdim \(S_{((n-m),(1^m))}\) are
\[
\left(\frac{e-2}{e}\right)\left(\frac{n}{e}\right)v^{\left(\left[\frac{n}{e}\right] + \left[\frac{m+e-1}{e}\right]\right)} \quad \text{and} \quad \left(\frac{e-2}{e}\right)\left(\frac{n}{e}\right)v^{\left(\left[\frac{n}{e}\right] + \left[\frac{m+e-1}{e}\right]\right)},
\]
respectively. By Proposition 13.6, the leading terms of grdim \(D_{\mu_{n,2m}}\) and grdim \(D_{\mu_{n,2m+1}}\) are, respectively,
\[
\left(\frac{e-2}{e}\right)\left(\frac{n}{e}\right)v^{\left(\left[\frac{n}{e}\right] + \left[\frac{m+e-1}{e}\right]\right)} \quad \text{and} \quad \left(\frac{e-2}{e}\right)\left(\frac{n}{e}\right)v^{\left(\left[\frac{n}{e}\right] + \left[\frac{m+e-1}{e}\right]\right)},
\]
respectively.

The graded dimension of \(S_{((n-m),(1^m))}\) has \(\frac{2n}{e} + 1\) terms, whereas \(D_{\mu_{n,2m}}\) and \(D_{\mu_{n,2m+3}}\) both have \(\frac{2n}{e} - 1\) terms. Hence \(x = 1 + \left[\frac{m}{e}\right] + \left[\frac{m+e-1}{e}\right]\).

Clearly, \(\left[\frac{m}{e}\right] + \left[\frac{m+e-1}{e}\right] - 1 \leq y, z \leq \left[\frac{m}{e}\right] + \left[\frac{m+e-1}{e}\right] + 1\). Now, observing that the leading coefficients in the graded dimensions of \(S_{((n-m),(1^m))}\) and \(D_{\mu_{n,2m}}\) are equal, and that the trailing coefficients in the graded dimensions of \(S_{((n-m),(1^m))}\) and \(D_{\mu_{n,2m+3}}\) are equal, we deduce that \(y = z = \left[\frac{m}{e}\right] + \left[\frac{m+e-1}{e}\right]\), as required.

(c) Let \(\frac{n(e-1)}{e} \leq m \leq n - 2\). By the third part of Proposition 12.11, the first two leading terms in grdim \(S_{((n-m),(1^m))}\) are
\[
\left(\frac{n}{e}\right)v^{\left(n-m+\left[\frac{m}{e}\right]+\left[\frac{m+e-1}{e}\right]\right)} + \left(\frac{e-2}{e}\right)\left(\frac{n}{e}\right)v^{\left(n-m+\left[\frac{m}{e}\right]+\left[\frac{m+e-1}{e}\right]\right)},
\]
respectively, and the last two trailing terms in grdim \(S_{((n-m),(1^m))}\) are
\[
\left(\frac{e-2}{e}\right)\left(\frac{n}{e}\right)v^{\left(1-n+m+\left[\frac{m}{e}\right]+\left[\frac{m+e-1}{e}\right]\right)}, \left(\frac{n}{e}\right)v^{\left(m-n+\left[\frac{m}{e}\right]+\left[\frac{m+e-1}{e}\right]\right)},
\]
By Proposition 13.6, the first two terms in the graded dimension of \( D_{\mu_n,2m} \) are
\[
\left( \frac{n-e}{n-m-1} \right) v^{n-m-1} \quad \text{and} \quad \left( \frac{n-2}{(n-m-2)} \right) v^{n-m-2},
\]
respectively, and the first two terms in the graded dimension of \( D_{\mu_n,2m+1} \) are
\[
\left( \frac{n-e}{n-m} \right) v^{n-m} \quad \text{and} \quad \left( \frac{n-2}{(n-m-1)} \right) v^{n-m-1},
\]
respectively.

Since the graded dimensions of \( S_{((n-m),(1^m))} \) and \( D_{\mu_n,2m+1} \) both have \( 2n-2m+1 \) terms, \( y = \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-1}{e} \right\rfloor \).

Observe that the graded dimensions of \( D_{\mu_n,2m} \) and \( D_{\mu_n,2m+3} \) both have \( 2n-2m-1 \) terms, so together with \( v^2 [S_{((n-m),(1^m))}] : D_{\mu_n,2m+3} ] \) \( v = [S_{((n-m),(1^m))}] : D_{\mu_n,2m} ] \),
\[
x = \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-1}{e} \right\rfloor + 1.
\]

Clearly, \( -2 \leq z = \left\lfloor \frac{m}{e} \right\rfloor - \left\lfloor \frac{m+e-1}{e} \right\rfloor \leq 2 \). One observes that the sum of the second leading coefficients in the graded dimensions of \( D_{\mu_n,2m} \) and \( D_{\mu_n,2m+1} \) form the second leading coefficient in the graded dimension of \( S_{((n-m),(1^m))} \). Similarly, the sum of the second trailing coefficients in the graded dimensions of \( D_{\mu_n,2m+1} \) and \( D_{\mu_n,2m+3} \) equal the second trailing coefficient in the graded dimension of \( S_{((n-m),(1^m))} \). Hence \( z = \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-1}{e} \right\rfloor \).

3. By Lemma 9.2, we know \( S_{(\varnothing,(1^n))} \cong D_{\lambda} \) as ungraded \( \mathcal{H}_{n}^{\lambda} \)-modules, where
\[
\lambda = \begin{cases} (\{(n-l),(1^l)\}) & \text{if } n \geq l \\ (\varnothing, (1^n)) & \text{if } n < l. \end{cases}
\]

We observe that \( \text{grdim}(D_{\mu_n,2n-2}) = \text{grdim}(S_{(\varnothing,(1^n))}) = 1 \), where \( D_{\mu_n,2n-2} \cong \text{im}(\phi_{n-1}) \) as ungraded \( \mathcal{H}_{n}^{\lambda} \)-modules. We know that \( \text{im}(\phi_{n-1}) \) is spanned by \( \psi_1 \psi_2 \ldots \psi_m z_{((1),(1^{n-1}))} \), and \( S_{(\varnothing,(1^n))} \) is spanned by \( z_{((1),(1^{n-1}))} \). One finds that
\[
\text{deg}(\psi_1 \psi_2 \ldots \psi_m z_{((1),(1^{n-1}))}) = 2 \left\lfloor \frac{m}{e} \right\rfloor + 2 = \text{deg}(z_{((1),(1^{n-1}))}) + 2,
\]
so
\[
v^2 [S_{((1),(1^{n-1}))}] : S_{(\varnothing,(1^n))}] \] \( v = [S_{((1),(1^{n-1}))}] : \text{im}(\phi_{n-1})] \) \( v,
\]
and hence \( v^2 [S_{((1),(1^{n-1}))}] : D_{\lambda} v = [S_{((1),(1^{n-1}))}] : D_{\mu_n,2n-2}] \) \( v. \) Thus we determine \( x, y \in \mathbb{Z} \) where
\[
\text{grdim}(S_{((1),(1^{n-1}))}) \]
\[
= v^x \text{grdim}(D_{\mu_n,2n-2}) + v^y \text{grdim}(\ker(D_{\mu_n,2n-2})) + v^{x-2} \text{grdim}(D_{\lambda}).
\]

By Proposition 12.11 and Proposition 13.6, we have
\[
\text{grdim}(S_{((1),(1^{n-1}))}) = \left( \frac{n}{e} \right) v^{(1 + \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-1}{e} \right\rfloor + 1)} + \left( \frac{e-2}{e} \right) v^{(\left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-1}{e} \right\rfloor - 1)} \]
\[
= v^x + v^y \left( \frac{n-e}{e}v + \frac{e-2}{e} + \frac{n-e}{e}v^{-1} \right) + v^{x-2}.
\]
Equating terms, we deduce that \( y = \left\lfloor \frac{m}{e} \right\rfloor + \left\lfloor \frac{m+e-1}{e} \right\rfloor = x - 1. \)
Example 18.3. Let $e = 3$, $\kappa = (0, 2)$. Then the submatrix with rows corresponding to Specht modules labelled by hook bipartitions is

$$
\begin{bmatrix}
S_{(6),\emptyset} & 1 & v & v^2 & 0 \\
S_{(5),(1)} & 0 & v & v^2 & v^3 \\
S_{(4),(1^2)} & 0 & v^3 & v^4 & v^4 \\
S_{(3),(1^3)} & 0 & v^2 & v^4 & v^3 \\
S_{(2),(1^4)} & 0 & v^2 & v^3 & v^2 \\
S_{(1),(1^5)} & 0 & v & 0 & v \\
S_{(\emptyset),(1^6)} & 0 & 0 & 0 & 0
\end{bmatrix}
$$

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