Extremes of Censored and Uncensored Lifetimes in Survival Data

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Abstract
The i.i.d. censoring model for survival analysis assumes two independent sequences of i.i.d. positive random variables, \((T^*_i)_{1 \leq i \leq n}\) and \((U_i)_{1 \leq i \leq n}\). The data consists of observations on the random sequence \((T_i) = \text{min}(T^*_i, U_i)\) together with accompanying censor indicators. Values of \(T_i\) with \(T^*_i \leq U_i\) are said to be uncensored, those with \(T^*_i > U_i\) are censored. We assume that the distributions of the \(T^*_i\) and \(U_i\) are in the domain of attraction of the Gumbel distribution and obtain the asymptotic distributions, as sample size \(n \to \infty\), of the maximum values of the censored and uncensored lifetimes in the data, and of statistics related to them. These enable us to examine questions concerning the possible existence of cured individuals in the population.

1 Introduction
In this paper we consider the i.i.d. censoring model in survival analysis, motivated by the fact that, in observed survival data, it is sometimes the case that the lifetimes of some of the longest-lived individuals in the sample are censored at the limit of follow-up time. This can be taken as indicative of the existence in the population of a proportion of “cured” individuals, or individuals “immune” to the event of interest (death of a patient, or recurrence of a disease, etc.) Consequently it is of interest to analyse the maximum values of the censored and uncensored lifetimes in the data, and compare their magnitudes. In

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the present paper we assume a realistic class of distributions for the survival
and censoring distributions – namely, those in the domain of attraction of the
Gumbel distribution – and obtain the joint asymptotic distribution of these
maxima, and of statistics derived from them, and examine questions related
to the existence of cured individuals in the population.

1.1 The Data Model

We assume a general independent censoring model with right censoring. We
have two independent sequences of i.i.d. positive random variables \((T^*_i)_{1\leq i\leq n}\)
and \((U_i)_{1\leq i\leq n}\) having cumulative distribution functions (cdfs) \(\widetilde{F}\) and \(G\) on
\([0, \infty)\). The data in a sample of size \(n\) consists of observations on the random
sequence of (possibly censored) survival times \(T_i = \min(T^*_i, U_i)\), together with
accompanying censor indicators. The censoring distribution \(G\) is assumed
proper (total mass 1), but the distribution \(\widetilde{F}\) of the \(T^*_i\) is in general improper,
with mass at infinity corresponding to cured individuals (who, formally, live
forever). We assume it to be of the form

\[ \widetilde{F}(t) = pF(t), \quad t \geq 0, \tag{1.1} \]

where \(0 < p \leq 1\) and \(F\) is the proper distribution of the “susceptible” indi-
viduals. Only susceptibles can experience the event of interest and have an
uncensored failure time.

An informative way to display the sample data is with the Kaplan-Meier
estimator (KME) of the lifetime distribution; that is, the analogue of the
empirical distribution function after censoring is taken into account. Figure 1
shows the KME constructed from data on 21 leukaemia patients (data from
[7], also in Figure 1.1, p.2, of [9]). The KME jumps at uncensored data times
(full dots in Fig.1) and remains constant at censored points (open circles in
Fig.1).

A significant feature is the levelling of the KME below 1 at the right hand
end (so the empirical distribution is improper) with a number of the largest
observations being censored defining the level stretch. Such long-censored
lifetimes indicate the possibility of cured individuals being present in the pop-
ulation. The useful information for this purpose is in the righthand end of

\(^1\)Figure 1 is a plot of a small, but real, data set. We include it as a schematic to display
the features we are interested in.
the KME and of course these are the largest observations – censored, in this case of interest – suggesting an application of extreme value theory to study the distribution of the largest censored lifetime. Besides this, we also want information on the largest uncensored lifetime, and, furthermore, we need a comparison between the two. The largest uncensored lifetime in Figure 1 is at 23 weeks, the largest censored lifetime is at 35 weeks, and the 12 weeks difference between them is the length of the level stretch at the righthand end of the KME.

Our approach is to assume both distributions $F$ and $G$ are in the domain of attraction of the Gumbel distribution and are comparable in terms of a certain balance condition on their hazard functions. Such distributions include the exponential, normal, lognormal, Weibull, and indeed most of the common distributions in use in survival analysis.

The following theorem encapsulates our main findings.
Theorem 1.1. Suppose $F$ and $G$ are both in the domain of attraction of the Gumbel distribution and their hazard functions satisfy a certain balance condition (Condition (2.12) below) depending on a parameter $\kappa \geq 0$. Then for a sample of size $n$, we have the following results as $n \to \infty$.

1. The largest uncensored lifetime and the largest censored lifetime converge jointly in distribution, after norming and centering, to independent Gumbel random variables.

2. The largest uncensored lifetime and the largest lifetime (overall) converge jointly, after norming and centering, to a bivariate limiting random variable $(L_1, L_2)$.

3. The difference between the largest observation and the largest uncensored lifetime converges in distribution, after norming, to the random variable $L := L_2 - L_1$, having cdf

$$P[L \leq x] = \frac{1}{1 + \kappa e^{-x}}, \quad x \geq 0. \quad (1.2)$$

4. The difference in Part 3, taken as a proportion of the largest observed lifetime, converges in distribution (with no norming or centering needed), to the random variable $R := (L_2 - L_1)/\max(L_1, L_2)$, having the distribution tail in (6.16) below, depending only on the parameter $\kappa$.

The result in Part 3 of Theorem 1.1 is remarkably simple and explicit but its application in practice depends on knowing or estimating the norming sequence $a(n)$ in (6.1) below, as well as the parameter $\kappa$. The result in Part 4 is more easily applicable, requiring only an estimate of $\kappa$. This parameter is related to the ratio of the hazard functions of the lifetime and censoring distributions $F$ and $G$. We give further discussion of this, and examples, in an applications Section 4.

Another measure of the extent of followup in the sample is to count the number of censored lifetimes greater than the largest uncensored observation. In Section 3 we give the asymptotic distribution of this number, under the same assumptions as in Theorem 1.1.

Theorem 1.1 will be proved in Section 6 and Theorem 3.1 in Section 7. Our analysis prior to that, in Sections 2 and 5, proceeds by separating out, notionally, the subsequences of censored and uncensored observations in the sample, applying extreme value techniques to each, then combining the results.
2 Notation and Preliminary Results

Throughout we will assume both $F$ and $G$ are proper cdfs ($F(\infty) = G(\infty) = 1$) with infinite right endpoints (the working can be modified to deal with finite right endpoints if they are the same for each distribution). Let $\overline{F}(t) = 1 - F(t)$ denote the survival function (tail function) of $F$, and similarly for $\overline{G}$. Let $H(t) := P(T_1 \leq t)$ be the distribution of the observed survival times $(T_i)_{i \geq 1}$ with distribution tail $\overline{H}(t) = 1 - H(t) = \overline{F(t)}\overline{G(t)}$.

Relative to the sequence \{(T_j^*, U_j), j \geq 1\}, we define the random indices $K_j^{u}$ and $K_j^{c}$ by

\begin{align*}
K_0^{u} &= 0, \quad K_j^{u} = \inf\{m > K_{j-1}^{u} : T_m^* \leq U_m\}, \text{ and} \\
K_0^{c} &= 0, \quad K_j^{c} = \inf\{m > K_{j-1}^{c} : T_m^* > U_m\}. \quad (2.1)
\end{align*}

Then the sequence \{(T_{K_j^{u}}, U_{K_j^{u}}), j \geq 1\} selects out the subsequence of uncensored observations in the sample, and the sequence \{(T_{K_j^{c}}, j \geq 1\} selects out the subsequence of censored observations. Also define

\begin{align*}
N_u(n) &= \sup\{m : K_m^{u} \leq n\} \\
&= \{\text{number of uncensored observations in a sample of size } n\}, \quad (2.2)
\end{align*}

and

\begin{align*}
N_c(n) &= n - N_u(n) = \{\text{number of censored observations in the sample}\}. \quad (2.3)
\end{align*}

With the above notation the largest uncensored lifetime in the first $n$ observations can be written as

\begin{align*}
M_u(n) := \max_{1 \leq i \leq N_u(n)} T_{K_i^{u}}, \quad (2.4)
\end{align*}

and the largest censored lifetime is $M_c(n) := \max_{1 \leq i \leq N_c(n)} U_{K_i^{c}}$. The largest observation in the sample is then

\begin{align*}
M(n) := \max_{1 \leq i \leq n} T_i = \max\{M_u(n), M_c(n)\}.
\end{align*}

The Découpage de Lévy. A remarkable fact due to Lévy (e.g., [12, p.212]) is that, with $K_j^{u}$ and $K_j^{c}$ defined by (2.1), both subsequences \{(T_{K_j^{u}}, U_{K_j^{c}})\} and
\{ (T_{K_j}, U_{K_j}) \}\) are comprised of i.i.d. random vectors. Furthermore, the three sequences
\[ \{ (T_{K_j}, U_{K_j}) \}, j \geq 1 \}, \{ (T_{K_j}, U_{K_j}) \}, j \geq 1 \}, \{ N_u(j), j \geq 1 \} \quad (2.5) \]
are independent of each other, and the sequence \( \{ N_u(j) \} \) is a renewal counting function (a sum of i.i.d. indicator variables rvs). The distribution of the 2-vector \((T_{K_1}, U_{K_1})\) is the conditional distribution of \((T_1, U_1)\) given \(T_1^* \leq U_1\); that is,
\[ (T_{K_1}, U_{K_1}) \overset{d}{=} ((T_1, U_1)|T_1^* \leq U_1) = ((T_1, U_1)|T_1^* \leq U_1). \]
We have for the distribution tail of an uncensored lifetime
\[ P[T_{K_1}^* > x] = P[T_1 > x|T_1^* \leq U_1] = \frac{P[U_1 \geq T_1^* > x]}{P[U_1 \geq T_1^*]} = \frac{\int_{x}^{\infty} \bar{G}(s)F(ds)}{\int_{0}^{\infty} \bar{G}(s)F(ds)}. \quad (2.6) \]
Likewise, for a censored lifetime
\[ (T_{K_1}, U_{K_1}) \overset{d}{=} ((T_1, U_1)|T_1^* > U_1) = ((U_1, U_1)|T_1^* > U_1). \quad (2.7) \]
Interchanging \( F \) and \( G \) in \( (2.6) \), we get the distribution tail of a censored lifetime as
\[ P[T_{K_1}^* > x] = P[T_1 > x|T_1^* > U_1] = P[U_{K_1} > x] = \frac{\int_{x}^{\infty} \bar{F}(s)G(ds)}{\int_{0}^{\infty} \bar{F}(s)G(ds)}. \quad (2.8) \]
*The domain of attraction (DOA) of the Gumbel.* The Gumbel distribution in standard form has cdf
\[ \Lambda(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R}. \quad (2.9) \]
Throughout we will assume both \( F \) and \( G \) are absolutely continuous and both are in the domain of attraction of the Gumbel. (Write this as \( F \in D(\Lambda) \) and refer to [12, Sect. 1.1] or [4, Sect 1.2] for background.) This allows us to calculate asymptotic distributions of maxima of uncensored and censored lifetimes, using extreme value theory applicable to the Gumbel distribution. In fact, consistent with the analysis in [5], we will assume a little more than just the domain of attraction condition, namely, that both \( F \) and \( G \) are Von Mises distributions [12, p. 40] whose tail functions have the form
\[ 1 - F(x) = \bar{F}(x) = k_1 \exp\{-\int_{x_0}^{x} \frac{1}{f(u)}du\}, \quad x > x_0, \quad (2.10) \]
and

\[ 1 - G(x) = \overline{G}(x) = k_2 \exp \left\{ - \int_{x_0}^{x} \frac{1}{g(u)} du \right\}, \quad x > x_0, \quad (2.11) \]

where \( f, g \) are absolutely continuous functions on \([x_0, \infty)\) with densities \( f', g' \) satisfying \( f'(x) \to 0, g'(x) \to 0, x \to \infty \). In (2.10) and (2.11) \( x_0 \) is a lower bound for the interval on which the representations hold and \( k_1, k_2 \) are positive constants.

An important result for us is that, under (2.10) and (2.11), the product \( F \times G \), which is the tail of the distribution of the observed survival time \( T_1 = T_1^* \wedge U_1 \), is also the tail of a Von Mises distribution, as shown in the next theorem. The proof of the theorem is in Section 5.

**Theorem 2.1** (\( \overline{H} \) is the tail of a Von Mises distribution). If \( F \) and \( G \) are Von Mises distribution tails satisfying (2.10) and (2.11), then \( \overline{H} = F \times G \) is a Von Mises distribution tail with auxiliary function \( h := \frac{fg}{f + g} \).

To (2.10) and (2.11) we will add a third condition:

**Condition A.**

\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \kappa, \quad 0 \leq \kappa < \infty. \quad (2.12) \]

**Remark 2.1.** The functions \( f, g \) are called auxiliary functions; see [12], p.26. Differentiation of (2.10) and (2.11) shows that they are the reciprocal hazard functions of \( F \) and \( G \) on the interval \([x_0, \infty)\). Condition A specifies a certain kind of balance between the hazard functions corresponding to \( F \) and \( G \), and the magnitude of \( \kappa \) measures the relative heaviness of the tails of \( F \) and \( G \). We discuss the practical implications of these facts in Section 4.

The next step in our development is to compare the distribution tails of the censored and uncensored lifetimes with \( \overline{H} \), using the balance Condition A. Let

\[ p_u = \{ \text{probability an observation is uncensored} \} = P[T_i^* \leq U_i] \quad (2.13) \]

and

\[ p_c = \{ \text{probability an observation is censored} \} = P[T_i^* > U_i] = 1 - p_u. \quad (2.14) \]

A simple calculation gives the formulae

\[ p_u = \int_0^\infty \overline{G}(s)F(ds) \quad \text{and} \quad p_c = \int_0^\infty \overline{F}(s)G(ds). \]
Theorem 2.2 (Tail behaviour of the censored and uncensored lifetimes). Suppose (2.10), (2.11) and (2.12) hold. Then the following are true.

1. There exists a non-decreasing function $U(x)$ such that
   \[ \frac{1}{1 - G(x)} = U\left(\frac{1}{1 - F(x)}\right). \] (2.15)
   The function $U(x)$ is regularly varying with index $\kappa \geq 0$ (slowly varying when $\kappa = 0$).

2. For all $\kappa \geq 0$, the tail of the uncensored lifetime distribution (see (2.6)) satisfies
   \[ P[T_{K_u} > x] \sim \frac{1}{(1 + \kappa)p_u} H(x), \quad x \to \infty. \] (2.16)
   When $\kappa > 0$ the tail of the censored lifetime distribution (see (2.8)) satisfies
   \[ P[U_{K} > x] \sim \frac{\kappa}{(1 + \kappa)p_c} H(x), \quad x \to \infty. \] (2.17)
   When $\kappa = 0$ (2.17) remains true in the sense that
   \[ \lim_{x \to \infty} \frac{P[U_{K} > x]}{H(x)} = \lim_{x \to \infty} \frac{P[T_{K_u} > x]}{H(x)} = 0. \] (2.18)

In view of (2.6), the relation (2.16) can be expressed as
   \[ \lim_{x \to \infty} \frac{1}{H(x)} \int_x^\infty G(s)F(ds) = \frac{1}{1 + \kappa}, \] (2.19)
   valid for $0 \leq \kappa < \infty$. The next theorem provides a partial converse to this.

Theorem 2.3 (partial converse to (2.19)). Suppose $\overline{F}$ and $\overline{G}$ are von Mises functions, thus satisfying (2.10) and (2.11). Assume that
   \[ \lim_{x \to \infty} \frac{1}{H(x)} \int_x^\infty \overline{G}(y)dF(y) = k \in (0, 1), \] (2.20)
   Then
   \[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{1 - k}{k}. \] (2.21)

Theorem 2.3 does not cover the cases $k = 0$ or $k = \infty$, but remains true in these cases under extra assumptions on $f$ or $g$. We omit details of this.

See Section 5 for the proofs of Theorems 2.1, 2.2 and 2.3. With these, we can complete the proof of Theorem 1.1 in Section 6.
3 Numbers of Censored Lifetimes

In this section, rather than the length of the level stretch at the right hand end of the KME, as in Theorem 1.1, we consider the number of censored observations that are bigger than the largest uncensored lifetime. A large number of such observations may be evidence for the presence of immunes in the population. In this section we give the asymptotic distribution of this number.

Recall the definitions of $K_c^j$, $N_c(n)$ and $M_u(n)$ in (2.1), (2.3) and (2.4). We need also the number of censored lifetimes in the sample that exceed a value $t > 0$, defined as

$$N_c(> t) := \sum_{j=1}^{N_c(n)} 1[U_{K_c^j} > t].$$

(3.1)

Theorem 3.1. Assume (2.10), (2.11) and (2.12) and keep $0 < \kappa < \infty$.

(i) Conditional on $M_u(n)$ and $N_c(n)$, the number $N_c(> M_u(n))$ is asymptotically, as $n \to \infty$, Poisson with parameter $\kappa E$ where $E$ is a unit exponential rv. By this we mean

$$\lim_{n \to \infty} P[N_c(> M_u(n)) = j \mid M_u(n), N_c(n)] = P[\text{Pois}ss(\kappa E) = j \mid E],$$

for $j = 0, 1, 2, \ldots$, where Poiss$(\cdot)$ is a Poisson rv with the indicated parameter.

(ii) Unconditionally, $N_c(> M_u(n))$ is asymptotically a geometric rv with success probability $p_\kappa := \kappa/(1 + \kappa)$ and mean $\kappa$; thus,

$$\lim_{n \to \infty} P[N_c(> M_u(n)) = j] = (1 - p_\kappa)p_\kappa^j, j = 0, 1, \ldots.$$  

(3.3)

The proof of Theorem 3.1 is in Section 7. Before moving on to the proofs we give some examples and applications.

4 Examples and Applications

4.1 Parameter $\kappa$ and the heaviness of the censoring

The parameter $\kappa$ measures the relative heaviness of the censoring. When $\kappa = 0$ in (2.12), the hazard for $G$ is strongly dominated by the hazard for $F$, corresponding to relatively very light censoring (large values of $U$ are more likely than for $T^*$, so less censoring tends to occur). Increasing values of $\kappa$ introduce progressively heavier censoring.
When \(0 < \kappa < \infty\) the hazards are comparable, asymptotically, but a finer classification is possible in terms of the tails of \(F\) and \(G\). Under (2.10), (2.11) and (2.12) the function \(U(x)\) in (2.15) is regularly varying with index \(\kappa \geq 0\), so we have ([12, Proposition 0.8.(i), p.22])

\[
\lim_{x \to \infty} \frac{U(x)}{x} = \begin{cases} 
\infty, & \text{if } \kappa > 1, \\
0, & \text{if } 0 \leq \kappa < 1.
\end{cases}
\]

Therefore, from (2.15),

\[
\lim_{t \to \infty} \frac{\overline{F}(t)}{\overline{G}(t)} = \begin{cases} 
\infty, & \text{if } \kappa > 1, \\
0, & \text{if } 0 \leq \kappa < 1.
\end{cases} \tag{4.1}
\]

So we see that the value 1 for \(\kappa\) is critical: for values \(0 < \kappa < 1\), the tail of \(G\) dominates that of \(F\), censoring variables tend to be bigger than lifetimes and thus censoring tends to be lighter; when \(\kappa > 1\), the tail of \(F\) dominates that of \(G\) and censoring tends to be heavier.

In a practical situation when cured individuals are present in the population we expect an intermediate value of \(\kappa\) and the observed value of the proportion \(R\) in Part 4 of Theorem 1.1 gives some information on this. A sample value of \(R\) close to 1 means the maximal uncensored lifetime is significantly smaller than the maximal censoring variable, while if an observed value of \(R\) is close to 0, the maximal observation is approximately equal to the maximal uncensored lifetime. Both situations are visible in a KME plot; in particular, when \(R\) is close to 0, the KME plot should be close to 1 at its right extreme. In this case there is little evidence of cured individuals in the population; i.e., \(p \approx 1\) in (1.1). In a practical situation we expect to see an intermediate value of \(R\), with its distribution tail given by the expression in (6.16) below.

We see from (6.16) that \(P(R = 0) = 1\) if and only if \(\kappa = 0\), so a test of the hypothesis \(H_0 : \kappa = 0\) serves as a test for the existence of a cure proportion. Rejection of \(H_0\) implies the possible existence of a cure proportion, and evidence against \(H_0\) is a sample value of \(R\) significantly greater then 0. The distribution tail of \(R\) in (6.16) can be used to calculate critical values for the test if an estimate of \(\kappa\) is available. We follow up on these statistical issues elsewhere, and turn next to some examples of distributions to which the theory applies.
4.2 Distributions in the DOA of the Gumbel.

The Weibull distribution. The Weibull distribution is in the domain of attraction of the Gumbel. We consider it in the form

$$F(x) = 1 - e^{-\lambda x^\alpha}, \ x \geq 0, \ (4.2)$$

in which $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. It has density $F'(x) = \lambda x^{\alpha-1} e^{-\lambda x^\alpha}$, hazard function $\lambda \alpha x^{\alpha-1}$, and the function $f(x)$ for (2.10) is $f(x) = (\lambda \alpha)^{-1} x^{1-\alpha}$, taken for $x > x_0 = 1$, say. Since $\lim_{x \to \infty} f'(x) = 0$, $F$ is in $D(\Lambda)$. Suppose $G$ is also Weibull with corresponding parameters $\beta$ and $\mu$. Then $g(x)$ for (2.11) is $g(x) = (\beta \mu)^{-1} x^{1-\beta}$, $x > 1$, $G \in D(\Lambda)$, and, as $x \to \infty$,

$$\frac{f(x)}{g(x)} = \left( \frac{\beta \mu}{\alpha \lambda} \right) x^{\beta-\alpha} \to \kappa = \begin{cases} 0, & \beta < \alpha; \\
\mu/\lambda, & \beta = \alpha; \\
\infty, & \beta > \alpha. \end{cases}$$

Thus all 3 cases $\kappa = 0$, $0 < \kappa < \infty$, $\kappa = \infty$ in Theorems 2.2 and 2.3 can occur.

The exponential distribution. This is the case $\alpha = \beta = 1$ of the Weibull setup. Then $f(x) = 1/\lambda$, $g(x) = 1/\mu$ and $\kappa = \mu/\lambda$. Thus the censoring is lighter or heavier according as $\mu < \lambda$ or vice-versa (the mean censoring variable is inversely proportional to $\mu$, so smaller values of $\mu$ give higher values of the censoring variable, hence lighter censoring). Similar conclusions hold in the Weibull case.

The lognormal distribution is in $D(\Lambda)$. We write the lognormal cdf in the form

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{y=0}^{x} \exp\left(-\frac{(\log y)^2}{2\sigma_F^2}\right) \frac{dy}{y} = \Phi\left(\frac{\log x}{\sigma_F}\right), \ x > 0,$$

where $\Phi(x)$ is the standard normal cdf with tail $\Phi(x)$ and density $\phi(x)$. Then $F \in D(\Lambda)$ ([12] p.43)). The reciprocal of the hazard function is

$$f(x) = \frac{x \sigma_F \Phi\left(\frac{\log x}{\sigma_F}\right)}{\phi\left(\frac{\log x}{\sigma_F}\right)}, \ x > 0,$$

and since $\Phi(z) \sim z^{-1} \phi(z)$ as $z \to \infty$, we have

$$f(x) \sim \frac{x \sigma_F^2}{\log x}, \text{ as } x \to \infty.$$
If the censoring distribution $G$ is lognormal with parameter $\sigma^2_G$, we have

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\sigma^2_F}{\sigma^2_G},$$

Thus the censoring is heavier or lighter according as $\sigma_F > \sigma_G$ or vice-versa.

The normal distribution. The normal distribution is not usually used as a survival distribution, still we can consider a distribution with a normal-like tail and set $\overline{F}(x) = \Phi(x/\sigma_F), \overline{G}(x) = \Phi(x/\sigma_G), \ x \in \mathbb{R}$. Then ([12 p.43])

$$f(x) \sim \frac{\sigma_F}{x}, \quad g(x) \sim \frac{\sigma_G}{x}, \quad x \to \infty,$$

so

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\sigma_F}{\sigma_G}.$$

Thus just as for the lognormal the censoring is heavier or lighter according as $\sigma_F > \sigma_G$ or vice-versa.

The Weibull with lognormal censoring. Suppose $F$ is Weibull with the cdf in (4.2) and $G$ is lognormal with

$$g(x) \sim \frac{x \sigma^2_G}{\log x}, \quad as \ x \to \infty.$$

Then

$$\frac{f(x)}{g(x)} \sim \frac{1}{\alpha \lambda^2} \log x \ x^\alpha$$

and the only possible case for Theorems 2.2 is $\kappa = 0$.

Similar examples to the above can be constructed from the gamma distribution which is in $D(\Lambda)$ ([4 p.34]).

### 4.3 Comments and related literature.

A variety of models have been developed to analyse lifetime data of the kind displayed in Figure 1 in which a proportion of the population may be long-term survivors. A systematic formulation and treatment of these issues is in [9] to which we refer for further background information. Since 1996 there has been a steady increase in interest in cure models. For more recent reviews, we mention [10], [11], [13] and [1]. An earlier paper along the lines of the present analysis is [8]. A recent paper also assuming a lifetime distribution in the domain of attraction of the Gumbel is [5]. In [6] a lifetime distribution in the domain of attraction of the Fréchet distribution is assumed.
5 Proofs of Theorems

Proof of Theorem 2.1. Assume (2.10) and (2.11) with the positive constants $k_1, k_2$ and $x_0$. Then with $k = k_1 k_2$ and $h = f g / (f + g)$, we have for $x \geq x_0$

$$F(x) \overline{G}(x) = k \exp \left\{ - \int_{x_0}^x \left( \frac{1}{f(u)} + \frac{1}{g(u)} \right) du \right\} = k \exp \left\{ - \int_{x_0}^x \left( \frac{1}{h(u)} \right) du \right\}.$$

Taking derivatives we get

$$h' = \left( \frac{f g}{f + g} \right)' = \frac{f' g + g' f}{f + g} = \frac{f g}{(f + g)^2} (f' + g') =: A + B.$$

Now $|A(x)| \leq |f'(x)| + |g'(x)| \to 0$ and also

$$|B(x)| \leq \frac{f(x) g(x)}{f^2(x) + g^2(x) + 2 f(x) g(x)}(|f'(x)| + |g'(x)|) \leq \frac{1}{2} (|f'(x)| + |g'(x)|) \to 0, \text{ as } x \to \infty.$$

It follows that $h'(x) \to 0$, $x \to \infty$, and $h$ has the required property for $\overline{H}$ to be a Von Mises distribution tail. □

Proof of Theorem 2.2. When $\kappa > 0$ the assertion in (2.15) follows from [3, Theorem 2.1, page 249], and we only comment on the case $\kappa = 0$ where $U$ must be slowly varying. Define the non-decreasing functions

$$U_F = \frac{1}{1 - F} \quad \text{and} \quad U_G = \frac{1}{1 - G},$$

with inverse functions $U_F^\leftarrow$ and $U_G^\leftarrow$. Then (2.10) and (2.11) imply

$$\lim_{t \to \infty} \frac{U_F(t + x f(t))}{U_F(t)} = e^x.$$

Inverting this we obtain

$$\lim_{t \to \infty} \frac{U_F^\leftarrow(t x) - U_F^\leftarrow(t)}{f(U_F^\leftarrow(t))} = \log x, \quad x > 0.$$

Comparable expressions hold for $U_G$ and $U_G^\leftarrow$. The assumption $\kappa = 0$ implies that there exists $\varepsilon(t) \to 0$ with $f(t) = \varepsilon(t) g(t)$. Define

$$U(x) = U_G \circ U_F^\leftarrow(x).$$
Then for $x > 0$ (but fixed),
\[
\lim_{t \to \infty} \frac{U(tx)}{U(t)} = \lim_{t \to \infty} \frac{1}{U_G(U_F(t))} \times U_G \left( \frac{U_F^+(tx) - U_F^+(t)}{f(U_F^+(t))} \cdot f(U_F^+(t)) + U_F^+(t) \right)
\]
\[
= \lim_{s \to \infty} \frac{U_G(\log x \cdot f(s) + s)}{U_G(s)} = \lim_{s \to \infty} \frac{U_G(\log x \cdot \varepsilon(s)g(s) + s)}{U_G(s)}.
\]
Given $\varepsilon > 0$, for all large $s$, $|\varepsilon(s) \log x| \leq \varepsilon$ so
\[
\limsup_{s \to \infty} \frac{U(tx)}{U(t)} \leq e^\varepsilon,
\]
and similarly the lim inf is bounded below by $e^{-\varepsilon}$. This shows that $U$ is slowly varying when $\kappa = 0$.

To prove (2.16), we have from (2.15)
\[
\int_{x}^{\infty} G(s)F(ds) = \int_{x}^{\infty} \frac{1}{U(1/F(v))} F dv = \int_{1/F(x)}^{\infty} \frac{1}{U(v)v^2} dv,
\]
and applying Karamata’s theorem ([12, p. 17]), this is asymptotic to
\[
\left( \frac{1}{1 + \kappa} \right) \left( \frac{1 - F(x)}{U(1/F(x))} \right) = \left( \frac{1}{1 + \kappa} \right) \overline{F(x)}\overline{G(x)} = \left( \frac{1}{1 + \kappa} \right) \overline{H(x)}, \text{ as } x \to \infty.
\]

When $\kappa > 0$, (2.17) is proved by interchanging the roles of $F, G$ and $f, g$ (so $\kappa$ is replaced by $1/\kappa$) and then applying the proof of (2.16). When $\kappa = 0$, we still have $\int_{x}^{\infty} G(s)F(ds) \sim \overline{F(x)}\overline{G(x)}$, $x \to \infty$, and from Fubini’s theorem
\[
\int_{x}^{\infty} \overline{F(s)}G(ds) = \overline{F(x)}\overline{G(x)} - \int_{x}^{\infty} \overline{G(s)}F(ds)
\]
so
\[
\frac{\int_{x}^{\infty} \overline{F(s)}G(ds)}{\overline{F(x)}\overline{G(x)}} = 1 - \frac{\int_{x}^{\infty} \overline{G(s)}F(ds)}{\overline{F(x)}\overline{G(x)}} \to 1 - 1 = 0.
\]
This completes the proof of Theorem 2.2. □

**Proof of Theorem 2.3** This will be a consequence of the following lemma.
We need the following concept: A function $f(x)$ is *self-neglecting* if it is positive on $[x_0, \infty)$ for some $x_0$ and satisfies
\[
\lim_{x \to \infty} \frac{f(x)}{f(x + yf(x))} = 1, \quad y > 0.
\]
(5.1)
The convergence in (5.1) is locally uniform in $y$ ([12, p.41], [2] p.120, Sect. 2.11)].
Lemma 5.1. Suppose $f(x)$ is self-neglecting and $H$ is any distribution function satisfying
\[ \int_x^\infty \frac{H(y)}{f(y)} \, dy \sim kH(x), \quad x \to \infty, \quad (5.2) \]
where $0 < k < 1$. Then $H$ is in the domain of attraction of the Gumbel with an auxiliary function $h(x)$ satisfying $h(x) \sim kf(x)$ as $x \to \infty$.

Proof of Lemma 5.1: Assume (5.2) with $f$ satisfying (5.1) and define
\[ \chi(x) = \frac{1}{H(y)/f(y)} \int_x^\infty \frac{H(y)}{f(y)} \, dy, \quad x \geq x_0. \quad (5.3) \]
Then
\[ \exp \left( - \int_{x_0}^x \frac{1}{\chi(y)} \, dy \right) = \frac{\int_x^\infty (H(y)/f(y)) \, dy}{\int_{x_0}^x (H(y)/f(y)) \, dy} =: L(x) \]
is the tail of a cdf $L$ defined on $[x_0, \infty)$, and by (5.2)
\[ L(x) \sim \frac{kH(x)}{H(y)/f(y)} , \quad \text{as} \quad x \to \infty. \quad (5.4) \]
From (5.2) and (5.3) we have
\[ \chi(x) = \frac{1}{H(x)/f(x)} \int_x^\infty \frac{H(y)}{f(y)} \, dy \sim \frac{kH(x)}{H(x)/f(x)} = kf(x), \]
and it follows that $\chi$ is self-neglecting from the local uniform convergence in (5.1). Consequently $L$ is in the domain of attraction of the Gumbel with auxiliary function $\chi$, and since by (5.4) $L(x)$ is asymptotically equivalent to a constant times $H(x)$, also $H$ is in the domain of attraction of the Gumbel with auxiliary function $h$, say. Again since $L(x)$ is asymptotically equivalent to a constant times $H(x)$, the auxiliary functions of $L$ and $H$ can be taken the same ([12, p. 67, Proposition 1.19]). Thus $h(x) \sim \chi(x) \sim kf(x)$ as $x \to \infty$.

To complete the proof of Theorem 2.3 assume (2.20) and set $H(x) = F(x)G(x)$ where $F$ and $G$ satisfy (2.10) and (2.11) with auxiliary functions $f$ and $g$. Apply Lemma 5.1 to get
\[ h(x) = \frac{f(x)g(x)}{f(x) + g(x)} \sim kf(x). \]
This implies $f(x) \sim (k^{-1} - 1)g(x)$ as required in (2.21).
6 Proof of Theorem 1.1

Assume (2.10), (2.11) and (2.12). Recall from Theorem 2.1 that $H$ is then the tail of a Von Mises distribution with auxiliary function $h$. Thus $H$ is in the domain of attraction of the Gumbel distribution with the cdf $\Lambda(x)$ in (2.9). Analytically, this means that its auxiliary function $h$ has the property

$$\lim_{t \to \infty} \frac{H(t + xh(t))}{H(t)} = e^{-x}, \quad x > 0,$$

and any positive sequences $a(n)$ and $b(n)$ satisfying

$$\lim_{n \to \infty} nH(b(n)) = 1 \quad \text{and} \quad a(n) = h(b(n))$$

also satisfy

$$\lim_{n \to \infty} nH(a(n)x + b(n)) = e^{-x}, \quad x \in \mathbb{R}.$$  \hspace{1cm} (6.2)

Hence $a(n)$ and $b(n)$ are the appropriate norming and centering sequences such that the maximum observation has the Gumbel limit:

$$\lim_{n \to \infty} P\left[ \frac{M(n) - b(n)}{a(n)} \leq x \right] = \Lambda(x), \quad x \in \mathbb{R},$$

see [12], Prop. 1.1, p.40.

Further to this, there is also functional convergence to an extremal process with standard Gumbel marginals, $(Y(t))_{t>0}$. By this we mean that (6.3) can be extended to

$$\left( \frac{M([nt]) - b(n)}{a(n)} \right)_{t>0} \Rightarrow (Y(t))_{t>0},$$

where the convergence is in $D(0, \infty)$, the space of right continuous $\mathbb{R}$-valued functions with finite left limits on $(0, \infty)$, and $(Y(t))$ satisfies

$$P[Y(t) \leq x] = \Lambda^t(x) = \exp\{-te^{-x}\}, \quad x \in \mathbb{R}.$$  \hspace{1cm} (6.5)

See [12], Prop. 4.20, p.211.

We can prove an analogous result for the joint convergence of the extremes $M_u(n)$ and $M_c(n)$, after recalling the independence property in (2.5) which means we may analyse them separately. The limits will involve two independent extremal processes $\{Y_u(t), t > 0\}$ and $\{Y_c(t), t > 0\}$, each with standard Gumbel marginals, as in (6.5).

\footnote{For background on extremal processes, see [12], Sect. 4.3, p.179.}
First consider the limit distribution for uncensored lifetimes. Recall from (2.4) that \( M_u(n) \) is the maximum of \( N_u(n) \) i.i.d. copies of \( T_{K_1^u} \). To analyse this we look first at the maximum of \( n \) i.i.d. copies of \( T_{K_1^1} \). A standard asymptotic using (6.2) gives

\[
\lim_{n \to \infty} P \left[ \bigvee_{i=1}^n T_{K_1^u}^{u} - b(n) \geq a(n) x + b(n) \right] = \lim_{n \to \infty} \left( 1 - \frac{nP[T_{K_1^u} > a(n) x + b(n) / n]}{n} \right)^n \exp \left\{ - \lim_{n \to \infty} nP[T_{K_1^u} > a(n) x + b(n)] \right\}.
\]

(6.6) Use (2.15) to write the RHS of (6.6) as

\[
\exp \left\{ - \frac{1}{1 + \kappa} \lim_{n \to \infty} n \bar{H}(a(n) x + b(n)) \right\}
\]

and because of (6.3) this equals

\[
(\Lambda(x))^{(1+\kappa)p_u^{-1}} = P \left[ Y_u \left( \frac{1}{p_u(1 + \kappa)} \right) \leq x \right], \quad x > 0. \tag{6.7}
\]

6.1 The case \( \kappa > 0 \).

Keeping \( \kappa > 0 \) now, we have for censored lifetimes, similar to (6.6) and (6.7),

\[
\lim_{n \to \infty} P \left[ \bigvee_{i=1}^{\lfloor nt \rfloor} T_{K_1^c} - b(n) \geq a(n) x \right] = P \left[ Y_c \left( \frac{\kappa}{p_c(1 + \kappa)} \right) \leq x \right], \quad x > 0.
\]

We can combine the separate convergences into a bivariate functional limit theorem using [12, Proposition 4.20, p.211] again, after noting that, by the weak law of large numbers, \( N_u(n)/n \to p_u \) and \( N_c(n)/n \to p_c \). So we get for \( t > 0, \kappa > 0, \)

\[
\left( \left( \frac{\bigvee_{i=1}^{\lfloor nt \rfloor} T_{K_1^u}^{u} - b(n)}{a(n)} \right), \frac{\bigvee_{i=1}^{\lfloor nt \rfloor} T_{K_1^c} - b(n)}{a(n) \sqrt{n}}, \frac{N_u(n)}{n}, \frac{N_c(n)}{n} \right) \Rightarrow \left( Y_u \left( \frac{1}{p_u(1 + \kappa)} t \right), Y_c \left( \frac{\kappa}{p_c(1 + \kappa)} t \right), p_u, p_c \right), \tag{6.8}
\]

where the convergence is in \( D((0, \infty) \mapsto \mathbb{R}^2) \times [0, 1]^2 \mapsto \mathbb{R}^2 \times [0, 1]^2 \).

Now apply the almost surely continuous scaling map \((x(\cdot), y(\cdot), a, b) \mapsto (x(a), y(b))\) from \( D((0, \infty) \mapsto \mathbb{R}^2) \times [0, 1]^2 \mapsto \mathbb{R}^2 \) to (6.8) to deduce

\[
\left( \frac{M_u(n) - b(n)}{a(n)}, \frac{M_c(n) - b(n)}{a(n)} \right) \Rightarrow \left( Y_u \left( \frac{1}{1 + \kappa} \right), Y_c \left( \frac{\kappa}{1 + \kappa} \right) \right). \tag{6.9}
\]
The first component on the left in (6.9) is the centered and normed maximal uncensored observation and the second component is the centered and normed maximal censored lifetime. It follows from (6.9) that
\[
\left( \frac{M_u(n) - b(n)}{a(n)}, \frac{M(n) - b(n)}{a(n)} \right) = \left( \frac{M_u(n) - b(n)}{a(n)}, \frac{M_u(n) \lor M_c(n) - b(n)}{a(n)} \right)
\]
\[\Rightarrow \left( Y_u\left( \frac{1}{1 + \kappa} \right), Y_u\left( \frac{1}{1 + \kappa} \lor Y_c\left( \frac{\kappa}{1 + \kappa} \right) \right) \right) =: (L_1, L_2). \tag{6.10}
\]

The first component on the left in (6.10) is the centered and normed maximal uncensored lifetime, while the second is the centered and normed maximal observation. On the right, the components are in terms of rescaled standard Gumbel extremal processes. Note that the components \((L_1, L_2)\) are not independent. Using (6.5), they satisfy
\[
P[L_1 = L_2] = P\left[ Y_u\left( \frac{1}{1 + \kappa} \right) \lor Y_c\left( \frac{\kappa}{1 + \kappa} \right) > \right]
\]
\[= \int_\mathbb{R} \Lambda^{\kappa/(1+\kappa)}(x)\Lambda^{1/(1+\kappa)}(dx) = \frac{1}{1 + \kappa}. \tag{6.11}
\]

Next we need the asymptotic distribution of the difference between the largest observed lifetime and the largest uncensored lifetime. For this, take differences in (6.10) to get
\[
\frac{M_u(n) \lor M_c(n) - b(n)}{a(n)} - \frac{M_u(n) - b(n)}{a(n)}
\]
\[\Rightarrow Y_u\left( \frac{1}{1 + \kappa} \lor Y_c\left( \frac{\kappa}{1 + \kappa} \right) - Y_u\left( \frac{1}{1 + \kappa} \right) \right)
\]
\[= L_2 - L_1 =: L. \tag{6.12}
\]

Note that it’s important here that the centering sequence \(b(n)\) is the same for both components. Since for \(x > 0\)
\[
P\left[ Y_c\left( \frac{\kappa}{1 + \kappa} \right) \lor y > x \right] = \begin{cases} P\left[ Y_c\left( \frac{\kappa}{1 + \kappa} \right) > x \right], & \text{if } y < x, \\ 1, & \text{if } y > x, \end{cases} \tag{6.12}
\]
we have, using (6.12),
\[
P[L > x] = P\left[ Y_u\left( \frac{1}{1 + \kappa} \lor Y_c\left( \frac{\kappa}{1 + \kappa} \right) - Y_c\left( \frac{1}{1 + \kappa} \right) > x \right] \right]
\]
\[= \int_\mathbb{R} (1 - \Lambda^{\kappa/(1+\kappa)}(x + y)\Lambda^{1/(1+\kappa)}(dy). \tag{6.13}
\]
Evaluating this we obtain
\[ P[L > x] = \frac{\kappa}{e^x + \kappa}, \quad x \geq 0, \quad (6.14) \]

We add to this some mass at 0: from (6.11) we have
\[ P[L = 0] = \frac{1}{1 + \kappa}. \]

Thus we arrive at (1.2).

Finally, for Theorem 1.1, we need the asymptotic distribution of the ratio \( R \) of the difference between the largest observed lifetime and the largest uncensored lifetime taken as a proportion of the largest lifetime. For this, calculate, for \( 0 < x < 1 \),
\[
P[R > x] = \mathbb{P}\left[ \frac{Y_u\left(\frac{1}{1+\kappa}\right) \vee Y_c\left(\frac{\kappa}{1+\kappa}\right) - Y_u\left(\frac{1}{1+\kappa}\right)}{Y_u\left(\frac{1}{1+\kappa}\right) \vee Y_c\left(\frac{\kappa}{1+\kappa}\right)} > x \right]
= \int_{\mathbb{R}} \mathbb{P}\left[ Y_c\left(\frac{\kappa}{1+\kappa}\right) \vee y > \frac{y}{1-x} \right] \mathbb{P}\left[ Y_u\left(\frac{1}{1+\kappa}\right) \in dy \right]
= \int_{\mathbb{R}} (1 - \Phi^{(1+\kappa)}(y/(1-x))) \Phi^{1/(1+\kappa)}(dy) \quad \text{(using (6.12))}
= \int_0^\infty (1 - \exp\left\{ -\kappa e^{-y/(1-x)}/(1+\kappa) \right\} d\left\{ -e^{-y/(1+\kappa)} \right\}).
\quad (6.15)

Some computations reduce the RHS of (6.15) to
\[ P[R > x] = \frac{1 - x}{1 + \kappa} \int_0^\infty (1 - e^{-\kappa u/(1+\kappa)}) e^{-u^{1-x}/(1+\kappa)} u^{-x-1} du, \quad (6.16) \]
for \( 0 < x < 1 \). When \( x = 0 \) we can evaluate the integral and take the complement to get a mass at 0 for \( R \) of \( 1/(1 + \kappa) \). Thus the cdf of \( R \) can be written as the complement of (6.16). With these we complete the proof of Theorem 1.1. □
6.2 The case $\kappa = 0$.

The case $\kappa = 0$ of very light censoring is of interest. When $\kappa = 0$, (2.15) holds with $U$ slowly varying and from (2.18), with $a(\cdot), b(\cdot)$ chosen as in (6.1),

$$P\left[ \max_{1 \leq i \leq n} U_{K_i} \leq a(n)x + b(n) \right] = \left( 1 - \frac{nP[U_{K_i} > a(n)x + b(n)]}{n} \right)^n \rightarrow e^{-x} = 1, \quad x \in \mathbb{R}.$$ 

Since this is true for any $x \in \mathbb{R}$, we have

$$\max_{1 \leq i \leq n} U_{K_i} - b(n) \overset{a(n)}{\Rightarrow} -\infty.$$ 

So the analogue of (6.9) has limit $(Y_u(1), -\infty)$ and (6.10) has the degenerate limit

$$(L_1, L_1) = (Y_u(1), Y_u(1)).$$

Thus, for the case $\kappa = 0$,

$$L = L_2 - L_1 = 0$$

and as $n \rightarrow \infty$, the difference between the maximal observation and the maximal uncensored lifetime vanishes asymptotically.

Consequently, in this situation we observe a vanishingly small level stretch at the righthand end of the KME, asymptotically. Likewise, $R = 0 \text{ w.p.}1$ in this case.

7 Proof of Theorem 3.1

We employ the decoupage again, this time applying it twice. Now the pairs $\{(T_i^*, U_i), i \geq 1\}$ are split into independent sets according to whether they are above the diagonal in the $(t, u)$-plane or below. Uncensored lifetimes $\{T_{K_j}, 1 \leq j \leq N_u(n)\}$ depend on observations above the diagonal and the independent random variable $N_u(n)$ and are independent of censored observations which are below the diagonal. The maximal uncensored lifetime given in (2.4), $M_u(n) = \vee_{i=1}^{N_u(n)} T_{K_i}$, is independent of censored observations.

Take the i.i.d. collection $\{C_j := U_{K_j^*}, j \geq 1\}$ of censored lifetimes. Some of these are (typically) less than $M_u(n)$ and some are greater. The variable
\(M_u(n)\) is independent of censored observations. We condition on \(M_u(n) = t\) and split \(\{C_j\}\) into two independent subsets via the decoupage, into the i.i.d. collection of those observations less than \(t\) and those greater than \(t\). The number of censored lifetimes in the sample of size \(n\) of lifetimes is \(N_c(n) = n - N_u(n) \sim np_c\). Recall the definition of \(N_c(> t)\) in \((3.1)\). (Jointly) conditional on \(M_u(n) = t\) and \(N_c(n)\), \(N_c(> t)\) is binomial with the number of trials being \(N_c(n)\) and success probability according to \((2.8)\) being

\[
p(t) := \frac{\int_t^\infty \bar{F}(s)G(ds)}{\int_0^\infty F(s)G(ds)}.
\]

(7.1)

Throughout this section we assume \((2.10)\), \((2.11)\) and \((2.12)\) and keep \(0 < \kappa < \infty\).

### 7.1 Asymptotics of \(p(t)\).

Since the argument requires that we condition on \(M_u(n) = t\), we consider the large sample distribution of \(p(M_u(n))\).

**Proposition 7.1.** We have with \(p_c\) given in \((2.14)\),

\[
np(M_u(n)) \Rightarrow \frac{\kappa}{p_c}E,
\]

where \(E\) is a standard exponential random variable.

**Proof of Proposition 7.1** Set \(Y_n := (M_u(n) - b(n))/a(n)\) so \(M_u(n) = a(n)Y_n + b(n)\). Recall from \((2.4)\) and \((6.10)\) that

\[
Y_n \Rightarrow Y_u(\frac{1}{1 + \kappa}) \overset{d}{=} Y_u(1) + \log \frac{1}{1 + \kappa}.
\]

By the Skorohod embedding theorem ([12, p.6]) convergence in distribution may be replaced by almost sure convergence. Doing this, we get

\[
np(M_u(n)) = n \frac{\int_{M_u(n)}^\infty \bar{F}(s)G(ds)}{\int_0^\infty F(s)G(ds)} \\
\sim \frac{n\kappa}{(1 + \kappa)p_c} \bar{H}(M_u(n)) \quad \text{(from \((2.17)\))}
\]

\[
= n \bar{H}(b(n)) \frac{\kappa}{(1 + \kappa)p_c} \frac{\bar{H}(M_u(n))}{\bar{H}(b(n))} \\
\sim \frac{\kappa}{(1 + \kappa)p_c} \frac{\bar{H}(a(n)Y_n + b(n))}{\bar{H}(b(n))} \\
\sim \frac{\kappa}{(1 + \kappa)p_c} e^{-Y_u(1)} \overset{d}{=} \frac{\kappa}{p_c}E \quad \text{(from \((6.10)\))},
\]

completing the proof of Proposition 7.1. \(\square\)
7.2 Asymptotics of \( N_c(>t) \).

We first prove the conditioned limit result in Theorem 3.1 and then remove the conditioning. Let

\[
E^{(n)}(\cdot) = E(\cdot | M_u(n), N_c(n))
\]

be the conditional expectation given \( M_u(n) \) and \( N_c(n) \).

For \( 0 < s < 1 \) the conditional generating function of the binomial rv \( N_c(> M_u(n)) \) is

\[
E^{(n)}\left(s^{N_c(>M_u(n))}\right) = \left(1 - p(M_u(n))(1 - s)\right)^{N_c(n)}
\]

\[
= \left(1 - \frac{np(M_u(n))(1 - s)}{n}\right)^{n(N_c(n)/n)}.
\]

Applying (7.2) this converges to

\[
\exp\left\{-\frac{\kappa}{p_c}E(1 - s)p_c\right\} = \exp\{-\kappa E(1 - s)\},
\]

which is the generating function of a Poisson random variable with parameter \( \kappa E \).

For the unconditional generating function, by dominated convergence

\[
E_s^{N_c(>M_u(n))} = E\left(E^{(n)}s^{N_c(>M_u(n))}\right)
\]

\[
\to E\left(\exp\{-\kappa E(1 - s)\}\right)
\]

\[
= \frac{1}{1 + \kappa(1 - s)}
\]

\[
= \frac{1 - \frac{\kappa}{1+\kappa}}{1 - s\frac{\kappa}{1+\kappa}},
\]

which is the generating function of the geometric distribution. \(\square\)

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