Research Article

A Two-Step Modified Explicit Hybrid Method with Step-Size-Dependent Parameters for Oscillatory Problems

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A new two-step modified explicit hybrid method with parameters depending on the step-size is constructed. This method is derived using the coefficients from a sixth-order explicit hybrid method with extended interval of absolute stability and then imposed each stage of the modified formula to exactly integrate the differential equations with solutions that can be expressed as linear combinations of \( \sin(wx) \) and \( \cos(wx) \), where \( w \) is the known frequency. Numerical results show the advantage of the new method for solving oscillatory problems.

1. Introduction

Second-order ordinary differential equations are important tools for modeling physical phenomena in science and engineering. This paper is concerned with the numerical solution of the second-order ordinary differential equations of the form

\[
y''(x) = f(x, y(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad (1)
\]
having oscillatory solutions. These problems can be numerically solved by general-purpose methods or any other methods specially adapted to the structure of the intended problem. In the case of adapted numerical methods, particular algorithms have been proposed by several authors, see [1–3] to solve these classes of problems.

Franco [4] has established the following class of explicit hybrid methods:

\[
Y_1 = y_{n-1}, \quad Y_2 = y_n, \\
Y_i = (1 + c_i) y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{s-1} a_{ij} f(x_n + c_j h, Y_j), \quad i = 3, \ldots, s, \\
y_{n+1} = 2 y_n - y_{n-1} + h^2 \left[ b_1 f_{n+1} + b_2 f_n + \sum_{j=3}^{s} b_j f(x_n + c_j h, Y_j) \right]. \\
(2)
\]

where \( h \) is the step-size while \( f_{n-1} \) and \( f_n \) represent \( f(x_{n-1}, y_{n-1}) \) and \( f(x_n, y_n) \), respectively. The associated Butcher tableau for this class of methods is given by

\[
\begin{array}{cccccccc}
\cdashline{1-8}
-1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
c_3 & a_{31} & a_{32} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s-1} & 0 \\
\cdashline{1-8}
& b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}
\]

(3)
where
\[
\begin{align*}
A &= [a_{ij}], \\
c &= (-1, 0, c_3 \ldots c_s)^T,
\end{align*}
\]
and \(b^T = (b_1, b_2, \ldots, b_{s-1}, b_s)\).

Kalogiratou et al. [5] have modified each stage of the explicit hybrid methods and the improved version is

\[Y_1 = y_{n-1},
Y_2 = y_n,
Y_i = \sigma_i(1 + c_i)y_n - \mu_i c_i y_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} f(x_n + c_j h, Y_j), \quad i = 3, \ldots, s,
Y_{n+1} = \sigma_{s+1} 2c_n y_n - \mu_{s+1} y_{n-1} + h^2 \left[ b_1 f_{n-1} + b_2 f_n + \sum_{i=3}^{s+1} b_i f(x_n + c_i h, Y_i) \right].
\]

The coefficients \(\sigma_i, \mu_i\) are functions of \(v = wh\), where \(w\) is the known frequency of the second-order problems. The Butcher tableau for the modified hybrid method is given by

\[
\begin{array}{cccccccccc}
-1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
c_3 & \sigma_3 & \mu_3 & a_{31} & a_{32} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
c_s & \sigma_s & \mu_s & a_{s1} & a_{s2} & \ldots & a_{s,s-1} & 0 \\
\sigma_{s+1} & b_1 & b_2 & \ldots & b_{s-1} & b_s
\end{array}
\]

(6)

We develop a sixth-order explicit hybrid method with extended interval of absolute stability based on the class of explicit hybrid methods (2). Using the coefficients from the sixth-order hybrid method, we develop a new modified hybrid method. The construction of hybrid methods is described in Section 2. In Section 3, we give the stability analysis of the class of modified hybrid method (5). Numerical results are presented in Section 4 for several second-order problems.

2. Construction of Hybrid Methods

In this section, we derive the new method with four stages.

2.1. Sixth-Order Explicit Hybrid Method. Consider the explicit hybrid methods (2). The associated Butcher tableau for a class of four-stage explicit hybrid methods is given by

\[
\begin{array}{cccccccccc}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
c_3 & a_{31} & a_{32} & 0 & 0 & 0 \\
c_4 & a_{41} & a_{42} & a_{43} & 0 & 0 \\
c_5 & a_{51} & a_{52} & a_{53} & a_{54} & 0 \\
\sigma_{s+1} & b_1 & b_2 & b_3 & b_4 & b_5
\end{array}
\]

(7)

The sixth-order explicit hybrid method must satisfy the order conditions for a sixth-order hybrid method as stated in [6]. Solving the order conditions, we obtain

\[
b_1 = \frac{(5c_3^2 - 2)}{60(c_3 - 1)(c_3 + 1)},
\]

\[
b_2 = \frac{(25c_3^2 - 3)}{30c_3^2},
\]

\[
b_3 = \frac{-1}{20(c_3 - 1)(c_3 + 1)c_3},
\]

\[
b_4 = \frac{-1}{20(c_3 - 1)(c_3 + 1)c_3^2},
\]

\[
b_5 = \frac{(5c_3^2 - 2)}{60(c_3 - 1)(c_3 + 1)},
\]

\[
a_{31} = \frac{-c_3(c_3 - 1)(c_3 + 1)}{6},
\]

\[
a_{33} = \frac{c_3(c_3^2 + 3c_3 + 2)}{6},
\]

\[
a_{41} = \frac{c_3(c_3^2 + c_3 + 1)(c_3 - 1)}{6(c_3 + 1)},
\]

\[
a_{42} = \frac{-c_3(c_3 - 3)(c_3 - 1)}{6},
\]

\[
a_{43} = \frac{-c_3(c_3 - 1)}{6(c_3 + 1)},
\]

\[
a_{51} = \frac{-3c_3 - 3}{6(c_3 + 1)(5c_3^2 - 2)},
\]

\[
a_{52} = \frac{(30c_3^4 - 11c_3^2 - 3c_3 + 2)}{6c_3^2(5c_3^2 - 2)},
\]

\[
a_{53} = \frac{(c_3^2 - 2c_3^2 + 2c_3 - 1)}{6c_3^2(5c_3^2 - 2)(c_3 + 1)},
\]

\[
a_{54} = \frac{(c_3 - 1)(c_3 + 1)}{6c_3^2(5c_3^2 - 2)},
\]

\[
c_6 = -c_3,
\]

\[
c_5 = 1.
\]
Next, we choose the free parameter $c_3 = (7731/10000)$ to maximize the interval of absolute stability. For detailed explanation on stability properties of hybrid methods, refer [4]. The resulting method has a phase-lag of order 6 and a dissipation error of order 7. The interval of absolute stability of this method is $(0, 4.54)$.

2.2. The New Method with Parameters Depending on Step-Size. This method is derived using the coefficients from the sixth-order explicit hybrid method in Section 2.1. Consider the modified four-stage explicit hybrid method represented by this tableau:

\[
\begin{array}{cccccccc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_3 & c_3 & \mu_3 & a_{31} & a_{32} & 0 & 0 & 0 \\
c_4 & c_4 & \mu_4 & a_{41} & a_{42} & a_{43} & 0 & 0 \\
c_5 & c_5 & \mu_5 & a_{51} & a_{52} & a_{53} & a_{54} & 0 \\
\sigma_6 & \mu_6 & b_1 & b_2 & b_3 & b_4 & b_5 & 0 \\
\end{array}
\]

Associate each formula stage of the modified four-stage explicit hybrid method with the following linear operators:

\[
L[h, a] y(x) = y(x + c_i h) - \sigma_i (1 + c_i h) y(x) + \mu_i c_i y(x - h) - h^2 \sum_{j=1}^{i-1} a_{ij} y''(x + c_i h), \quad i = 3, 4, 5,
\]

\[
L[h, b] y(x) = y(x + h) - 2\sigma_i y(x) + \mu_i y(x - h) - h^2 \sum_{i=1}^{5} b_i y''(x + c_i h).
\]

Using the coefficients from the sixth-order explicit hybrid method and imposing the linear operators to exactly integrate the set \{sin(wx) and cos(wx)\}, we get

\[
\begin{align*}
\sigma_1 & = \frac{120000000000 \cos(\text{v}(\text{v}(7731/10000)v) + 1200000000000 \sin(\text{v}(7731/10000)v) + 126701667265\text{v}(\text{v}(7731/10000)v))}{1200000000000 \cos(\text{v}(\text{v}(7731/10000)v) + 1200000000000 \sin(\text{v}(7731/10000)v) + 126701667265\text{v}(\text{v}(7731/10000)v))} \\
\mu_1 & = \frac{1}{12000000000000 \cos(\text{v}(\text{v}(7731/10000)v))} \\
\sigma_2 & = \frac{5847213000000 \cos(\text{v}(\text{v}(7731/10000)v)) - 3546200000000 \cos(\text{v}(\text{v}(7731/10000)v)) + 3546200000000 \cos(\text{v}(\text{v}(7731/10000)v)) - 230871666612107 \cos(\text{v}(\text{v}(7731/10000)v)) + 5847213000000 \cos(\text{v}(\text{v}(7731/10000)v))}{(12000000000000 \cos(\text{v}(\text{v}(7731/10000)v)))} \\
\mu_2 & = \frac{5847213000000 \cos(\text{v}(\text{v}(7731/10000)v)) - 3546200000000 \cos(\text{v}(\text{v}(7731/10000)v)) + 3546200000000 \cos(\text{v}(\text{v}(7731/10000)v)) - 230871666612107 \cos(\text{v}(\text{v}(7731/10000)v)) + 5847213000000 \cos(\text{v}(\text{v}(7731/10000)v))}{(12000000000000 \cos(\text{v}(\text{v}(7731/10000)v)))} \\
\sigma_3 & = \frac{1580247614567593 \cos(\text{v}(\text{v}(7731/10000)v))}{2741097200000000 \cos(\text{v}(\text{v}(7731/10000)v))} \\
\mu_3 & = \frac{1580247614567593 \cos(\text{v}(\text{v}(7731/10000)v))}{2741097200000000 \cos(\text{v}(\text{v}(7731/10000)v))} \\
\sigma_4 & = \frac{1528491700000000 \cos(\text{v}(\text{v}(7731/10000)v)) + 12586924566812778966 \cos(\text{v}(\text{v}(7731/10000)v)) + 6778474807967392957 \cos(\text{v}(\text{v}(7731/10000)v)) + 90448521221000000 \cos(\text{v}(\text{v}(7731/10000)v))}{(2741097200000000 \cos(\text{v}(\text{v}(7731/10000)v)))} \\
\mu_4 & = \frac{17541639000000000 \cos(\text{v}(\text{v}(7731/10000)v))}{270841452536321 \cos(\text{v}(\text{v}(7731/10000)v)) + 17541639000000000 \cos(\text{v}(\text{v}(7731/10000)v))} \\
\sigma_5 & = \frac{197846341}{23597344} \\
\mu_5 & = \frac{197846341}{23597344}
\end{align*}
\]

and $\mu_6 = 1$, where $v = wh$.

For small $v$, coefficients $\sigma_i$ and $\mu_i$ may cause heavy calculations which lead to inaccuracies; hence, it is often convenient to use Taylor expansions for the coefficients. The resulting method is denoted by MEHM6.

3. Stability Analysis

In this section, we present the stability analysis of the modified hybrid method. Assume that $H = \lambda h$, $e = (1, 1, \ldots, 1)^T$, $\sigma(v) = (0, 0, 0, 0, 0, 0)^T$, and $\mu(v) = (0, 0, \mu_3, \mu_4, \ldots, \mu_5)^T$. Employing the hybrid methods defined by (5) to the standard equation

\[
y''(x) = -\lambda^2 y(x), \quad \lambda > 0,
\]

we get

\[
y_{n+1} - S(H^2, v)y_n + P(H^2, v)y_{n-1} = 0,
\]

where

\[
S(H^2, v) = 2\sigma_{x_1} - H^2 b^T (1 + H^2 A)^{-1} \sigma(v) \times (e + c), P(H^2, v) = \mu_{x_1} - H^2 b^T (1 + H^2 A)^{-1} \mu(v) \times c,
\]

and the symbol “$\times$” denotes component-wise multiplication. The characteristic polynomial which determines the solution (12) is

\[
\pi(\zeta) = \zeta^2 - S(H^2, v)\zeta + P(H^2, v).
\]

Definition 1 (see [5]). For the hybrid methods corresponding to the characteristic equation (13) and $v = wh$, the region in the $H$-$v$ plane, such that

\[
|P(H^2, v)| < 1,
\]

\[
|S(H^2, v)| < 1 + P(H^2, v),
\]
is called the region of absolute stability of the method.

The region of stability of the new method is shown in Figure 1.

It is observed that, if \( v < 0 \), then the region of absolute stability collapses into the interval of absolute stability.

4. Numerical Results

The new and existing methods are coded using Microsoft Visual C++ version 6.0 software and applied to some special second-order problems to provide numerical comparisons of the accuracy and execution time of the methods. The accuracy of the methods is measured by maximum global errors, while execution time (in seconds) is measured after the computation of the starting values. For all codes, the starting values are computed using the exact solution formula of each problem. The abbreviations of the codes are as follows:

(i) MEHM6: the modified sixth-order explicit hybrid method with four stages derived in this paper.

(ii) TRIMHLI: trigonometrically fitted multistep hybrid method proposed in [7].

(iii) TRIEFW: two-step trigonometrically fitted explicit hybrid method with four stages derived in [8].

Tables 1–5 show the numerical results of the new and existing methods for solving several second-order problems.

Problem 1 (the two-body problem)

\[
\begin{align*}
\frac{d^2 y_1}{dx^2} &= -\frac{y_1}{(y_1^2 + y_2^2)^{3/2}}, \\
y_1(0) &= 1 - \varepsilon, \\
y_1'(0) &= 0, \\
\frac{d^2 y_2}{dx^2} &= -\frac{y_2}{(y_1^2 + y_2^2)^{3/2}}, \\
y_2(0) &= 0, \\
y_2'(0) &= \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}} \\
\end{align*}
\]

(17)

with \( \varepsilon \) being the eccentricity of the orbit. The theoretical solution of this problem is

\[
\begin{align*}
y_1(x) &= \cos(R) - \varepsilon, \\
y_2(x) &= \sqrt{1 - \varepsilon^2} \sin(R),
\end{align*}
\]

(18)

where \( R \) satisfies the Kepler’s equation \( x = R - \varepsilon \sin(R) \). In this paper, the eccentricity value is chosen to be \( \varepsilon = 0 \). For all codes, \( v = h \) is used.

![Figure 1: Region of absolute stability of method MEHM6.](image)
\[ f_1(x) = \left( \frac{2 \cos(10x) \sin(5x) + (2 \varepsilon \sin(5x) \sin(x) - \cos(10x) \cos(x)) - \varepsilon^2 \sin(2x))}{(\cos^2(10x) + \sin^2(5x) + ((2 \varepsilon \sin(x) \cos(10x)) - \cos(x) \sin(5x)) + \varepsilon^2)} \right) + 99 \varepsilon \sin(x), \tag{20} \]

\[ f_2(x) = \left( \frac{(\cos^2(10x) - \sin^2(5x)) + 2 \varepsilon (\sin(x) \cos(10x) + \cos(x) \sin(5x) - \varepsilon^2 \cos(2x))}{(\cos^2(10x) + \sin^2(5x) + 2 \varepsilon (\sin(x) \cos(10x) - \cos(x) \sin(5x) + \varepsilon^2))} \right) - 24 \varepsilon \cos(x). \]

### Table 2: Numerical results for Problem 2.

| Step-size | Method   | Maximum global error | Execution time |
|-----------|----------|----------------------|----------------|
| 0.2       | TRIMHLI  | 3.54557E-002         | 0.0039083      |
|           | TREFW    | 1.01259E-001         | 0.006824       |
|           | MEHM6    | 6.25211E-006         | 0.0068333      |
| 0.1       | TRIMHLI  | 1.45864E-002         | 0.0084681      |
|           | TREFW    | 2.81845E-006         | 0.011891       |
|           | MEHM6    | 1.26995E-007         | 0.0134972      |
| 0.05      | TRIMHLI  | 3.28175E-004         | 0.0205792      |
|           | TREFW    | 1.56159E-010         | 0.0262821      |
|           | MEHM6    | 1.80522E-009         | 0.0224291      |
| 0.025     | TRIMHLI  | 3.26199E-005         | 0.0366569      |
|           | TREFW    | 2.18646E-012         | 0.0480412      |
|           | MEHM6    | 6.53135E-004         | 0.0488463      |
| 0.0125    | TRIMHLI  | 3.95995E-006         | 0.0650158      |
|           | TREFW    | 8.21140E-013         | 0.112478       |
|           | MEHM6    | 8.42048E-013         | 0.0919229      |

### Table 3: Numerical results for Problem 3.

| Step-size | Method   | Maximum global error | Execution time |
|-----------|----------|----------------------|----------------|
| 0.5       | TRIMHLI  | 5.21409E+000         | 0.0010963      |
|           | TREFW    | 8.67167E-001         | 0.0015327      |
|           | MEHM6    | 8.63619E-002         | 0.0015102      |
| 0.25      | TRIMHLI  | 2.10044E-001         | 0.0021984      |
|           | TREFW    | 4.86877E-003         | 0.0029643      |
|           | MEHM6    | 6.89393E-004         | 0.0028038      |
| 0.125     | TRIMHLI  | 1.11491E-002         | 0.0043691      |
|           | TREFW    | 6.08046E-005         | 0.0058454      |
|           | MEHM6    | 7.65379E-006         | 0.0062404      |
| 0.0625    | TRIMHLI  | 1.06673E-003         | 0.0085956      |
|           | TREFW    | 9.09300E-007         | 0.0137473      |
|           | MEHM6    | 1.08162E-007         | 0.0160981      |
| 0.03125   | TRIMHLI  | 1.25146E-004         | 0.0172246      |
|           | TREFW    | 1.40413E-008         | 0.0231007      |
|           | MEHM6    | 1.61036E-009         | 0.0218682      |

### Table 4: Numerical results for Problem 4.

| Step-size | Method   | Maximum global error | Execution time |
|-----------|----------|----------------------|----------------|
| 0.1       | TRIMHLI  | 2.87798E-002         | 0.0033725      |
|           | TREFW    | 2.72234E-004         | 0.0041873      |
|           | MEHM6    | 3.61139E-004         | 0.0039115      |
| 0.05      | TRIMHLI  | 2.28247E-003         | 0.0057554      |
|           | TREFW    | 3.96310E-006         | 0.0074681      |
|           | MEHM6    | 5.69308E-006         | 0.007701       |
| 0.025     | TRIMHLI  | 2.48929E-004         | 0.0113492      |
|           | TREFW    | 5.84999E-008         | 0.015302      |
|           | MEHM6    | 8.54033E-008         | 0.0132228      |
| 0.0125    | TRIMHLI  | 2.96137E-005         | 0.0227135      |
|           | TREFW    | 9.15825E-010         | 0.0320138      |
|           | MEHM6    | 1.34166E-009         | 0.0254876      |
| 0.00625   | TRIMHLI  | 3.63541E-006         | 0.0449913      |
|           | TREFW    | 1.44656E-011         | 0.0580932      |
|           | MEHM6    | 2.18224E-011         | 0.0544479      |

### Table 5: Numerical results for Problem 5.

| Step-size | Method   | Maximum global error | Execution time |
|-----------|----------|----------------------|----------------|
| 0.5       | TRIMHLI  | 3.20318E-003         | 0.0871526      |
|           | TREFW    | 5.71691E-006         | 0.114812       |
|           | MEHM6    | 4.36318E-007         | 0.112708       |
| 0.25      | TRIMHLI  | 3.54962E-004         | 0.167776       |
|           | TREFW    | 8.77577E-008         | 0.266239       |
|           | MEHM6    | 6.70166E-009         | 0.223963       |
| 0.125     | TRIMHLI  | 4.31670E-005         | 0.356719       |
|           | TREFW    | 1.36480E-009         | 0.475373       |
|           | MEHM6    | 1.04656E-010         | 0.444169       |
| 0.0625    | TRIMHLI  | 5.35121E-006         | 0.66339        |
|           | TREFW    | 2.13232E-011         | 0.918823       |
|           | MEHM6    | 1.03394E-012         | 0.887893       |
| 0.03125   | TRIMHLI  | 6.67693E-007         | 1.3356         |
|           | TREFW    | 4.39348E-013         | 1.8746         |
|           | MEHM6    | 1.93284E-011         | 1.73938        |
Solution 1. \(y_1(x) = \cos(10x) + \varepsilon \sin(x)\) and \(y_2(x) = \sin(5x) - \varepsilon \cos(x)\). For all codes, we use \(v = 10h\) for the first component while \(v = 5h\) for the second component.

Problem 3 (linear oscillatory problem)

\[
y''_1 = -13y_1 + 12y_2 + 9\cos(2x) - 12\sin(2x),
\]
\[
y_1(0) = 1,
\]
\[
y'_1(0) = -4,
\]
\[
y''_2 = 12y_1 - 13y_2 - 12\cos(2x) + 9\sin(2x),
\]
\[
y_2(0) = 0,
\]
\[
y'_2(0) = 8, \quad 0 \leq x \leq 10.
\]

Solution 2
\[
y_1(x) = \sin(x) - \sin(5x) + \cos(2x) \quad \text{and} \quad y_2(x) = \sin(x) + \sin(5x) + \sin(2x).
\]

For MEHM6 and TRIEFW, we choose \(v = h\) while for TRIMHLI, \(v = 5h\) as given in [7].

Problem 4 (nonlinear oscillatory problem)

\[
y''_1 = -4x^2y_1 - \frac{2y_2}{\sqrt{y_1^2 + y_2^2}},
\]
\[
y_1(0) = 1,
\]
\[
y'_1(0) = 0,
\]
\[
y''_2 = -4x^2y_2 + \frac{2y_1}{\sqrt{y_1^2 + y_2^2}},
\]
\[
y_2(0) = 0,
\]
\[
y'_2(0) = 0,
\]
\[
0 \leq x \leq 5.
\]

Solution 3. \(y_1(x) = \cos(x^2)\), \(y_2(x) = \sin(x^2)\).

For all codes, \(v = h\).

Problem 5 (the almost periodic problem)

\[
y''_1 = -y_1 + 0.001\cos(x),
\]
\[
y_1(0) = 1,
\]
\[
y'_1(0) = 0,
\]
\[
y''_2 = -y_2 + 0.001\sin(x),
\]
\[
y_2(0) = 0,
\]
\[
y'_2(0) = 0.9995,
\]
\[
0 \leq x \leq 1000.
\]

Solution 4. \(y_1(x) = \cos(x) + 0.0005x \sin(x)\) and \(y_2(x) = \sin(x) - 0.0005x \cos(x)\). For all codes, we use \(v = h\).

It is observed from Table 1 that MEHM6 solves Problem 1 with very close accuracy to TRIMHL1. From the results in Tables 3 and 5, MEHM6 gives the best accuracy as compared to the other codes for most of the step-sizes, while in Table 2, MEHM6 is the most accurate for bigger step-sizes. For smaller step-sizes, the accuracy of MEHM6 is close to TRIEFW as shown in Table 2. Table 4 shows that both MEHM6 and TRIEFW codes have the same order of accuracy for all step-sizes.

On the other hand, TRIMHLI has the shortest execution time for all problems considered. This is mainly due to the fact that TRIMHLI has more starting values than that for MEHM6. Hence, less number of integration steps is needed by TRIMHLI to advance the computation as compared to MEHM6.

5. Conclusions

In this paper, a new two-step modified explicit hybrid method is developed where each stage of the modified method exactly integrates differential equations with solutions that are linear combinations of \(\sin(wx)\) and \(\cos(wx)\). From the numerical results, the new method gives the best accuracy when compared with the multistep methods in [7, 8], particularly for linear oscillatory and almost periodic problems. The new method has two starting values, but the execution time is nevertheless acceptable. Hence, the new method is as competitive as the existing methods for solving oscillatory problems.

Data Availability

The maximum global error and execution time data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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