LIFTING MAPS FROM THE SYMMETRIZED POLYDISK IN SMALL DIMENSIONS

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Abstract. The spectral unit ball $\Omega_n$ is the set of all $n \times n$ matrices with spectral radius less than 1. Let $\pi(M) \in \mathbb{C}^n$ stand for the coefficients of its characteristic polynomial of $M$ (up to signs), i.e. the elementary symmetric functions of its eigenvalues. The symmetrized polydisk is $G_n := \pi(\Omega_n)$.

When investigating Pick-Nevanlinna problems for maps from the disk to the spectral ball, it is often useful to project the map to the symmetrized polydisk (for instance to obtain continuity results for the Lempert function): if $\psi \in \mathcal{O}(\mathbb{D}, \Omega_n)$ and $\psi(\alpha_j) = M_j$, $1 \leq j \leq N$, then $\pi \circ \psi \in \mathcal{O}(\mathbb{D}, G_n)$. Given a map $\varphi \in \mathcal{O}(\mathbb{D}, G_n)$, we are looking for necessary and sufficient conditions for this map to “lift through given matrices”, i.e. find $\psi$ as above so that $\pi \circ \psi = \varphi$.

A natural necessary condition is $\varphi(\alpha_j) = \pi(M_j)$, $1 \leq j \leq N$.

When the matrices $M_j$ are derogatory (i.e. do not admit a cyclic vector) new necessary conditions appear, involving derivatives of $\varphi$ at the points $\alpha_j$. Those conditions are necessary and sufficient for a local lifting. We give a scheme which shows that the necessary conditions are also sufficient for a global lifting in small dimensions ($n \leq 5$), and a counter-example to show that the scheme fails in dimension 6 and above.

1. Motivation and statements

1.1. Definitions. Some problems in Robust Control Theory lead to the study of structured singular values of a matrix (denoted by $\mu$). A special case of this is simply the spectral radius. A very special instance of the “$\mu$-synthesis” problem reduces to a Nevanlinna-Pick problem, i.e. given points $\alpha_j \in \mathbb{D}$, $a_j \in \Omega \subset \mathbb{C}^m$, $1 \leq j \leq N$, determine whether there exists $\Phi$ holomorphic from $\mathbb{D}$ to $\Omega$ such that $\Phi(\alpha_j) = a_j$, $1 \leq j \leq N$. We refer the interested reader to Nicholas Young’s stimulating survey [8].

We study this special case. Let us set some notation.
Let \( M_n \) be the set of all \( n \times n \) complex matrices. For \( A \in M_n \) denote by \( \text{sp}(A) \) and \( r(A) = \max_{\lambda \in \text{sp}(A)} |\lambda| \) the spectrum and the spectral radius of \( A \), respectively.

**Definition 1.** The spectral ball \( \Omega_n \) is given as
\[
\Omega_n := \{ A \in M_n : r(A) < 1 \}.
\]
The symmetrized polydisc \( \mathbb{G}_n \) is defined by
\[
\mathbb{G}_n := \{ \pi(A) : A \in \Omega_n \},
\]
where the mapping \( \pi = (\sigma_1, \ldots, \sigma_n) \) is given, up to alternating signs, by the coefficients of the characteristic polynomial of the matrix:
\[
P_A(t) := \det(tI_n - A) =: \sum_{j=0}^{n} (-1)^j \sigma_j(A) t^{n-j}
\]
\((\sigma_0(A) = 1)\).

We would like to find conditions (necessary and sufficient) on a map \( \phi \in \mathcal{O}(\mathbb{D}, \mathbb{G}_n) \) with \( \phi(\alpha_j) = \pi(A_j) \) for \( j = 1, \ldots, N \), such that there exists a \( \Phi \in \mathcal{O}(\mathbb{D}, \Omega_n) \) satisfying \( \phi = \pi \circ \Phi \) and \( \Phi(\alpha_j) = A_j \) for \( j = 1, \ldots, N \). We say that the map \( \phi \) lifts through the matrices \( A_1, \ldots, A_N \).

We remark right away that whenever there is a solution to the lifting problem for \( \alpha_j, A_j \), then there is one for \( \alpha_j, \tilde{A}_j \), when \( A_j \sim \tilde{A}_j \), i.e. \( A_j \) is similar (conjugate) to \( \tilde{A}_j \), i.e. \( \tilde{A}_j = P^{-1}_j A_j P_j [1, proof of Theorem 2.1] \).

Our main results are a local answer (Proposition 7 in Section 3) and a complete answer when \( n \leq 5 \) (Theorem 16 in Section 5). The method developed along the way in Section 4 may work in a number of other cases (for instances when each of the matrices to interpolate has a single eigenvalue), but fails in general for dimensions greater or equal to 6, as is shown in Section 6.

1.2. **Motivations.** Lifting maps is interesting because it reduces the study of Nevanlinna-Pick interpolation into the spectral ball to a problem on a domain of much smaller dimension, and bounded. The symmetrized polydisk, being taut, has the advantage that any family of maps into it is a normal family, so in particular, when studying the behavior of the problem under perturbations, we can get continuity results as corollaries of solution to the lifting problem if we can characterize the maps \( \phi \) that can be lifted through a given set of matrices by a finite number of conditions on the values and derivatives of \( \phi \) at the points \( \alpha_1, \ldots, \alpha_N \).

We give an instance of this when \( N = 2 \). Recall the definition of the Lempert function.
Definition 2. Let $A, B \in \Omega \subset \mathbb{C}^m$, the Lempert function is

$$\ell_{\Omega}(A, B) := \inf\{ |\alpha| : \exists \varphi \in Hol(\mathbb{D}, \Omega) : \varphi(\alpha) = A, \varphi(0) = B \}.$$ 

The following proposition is implicit in [3, Section 4].

Proposition 3. Let $B \in \mathcal{M}_n(\mathbb{C})$.

Suppose that there exists a continuous affine map

$$\Theta_B : Hol(\mathbb{D}, \mathbb{C}^n) \rightarrow \mathbb{C}^m$$

($\Theta_B(\varphi)$ depends only on $\varphi(k)(0)$, $1 \leq k \leq k_0$),

such that:

$$\exists \Phi \in Hol(\mathbb{D}, \Omega_n) \text{ s.t. } \pi \circ \Phi = \varphi, \Phi(0) = B \text{ and } \Phi(\zeta) \text{ is cyclic, } \zeta \neq 0$$

if and only if $\varphi \in Hol(\mathbb{D}, \mathbb{G}_n)$, and $\Theta_B(\varphi) = 0$.

Then $\ell_{\Omega_n}(\cdot, B)$ is continuous at $A$, for any $A$ cyclic matrix.

A corollary of our results, then, is that $\ell_{\Omega_n}(\cdot, B)$ is continuous at $A$, for any $A$ cyclic matrix and $n \leq 5$.

Proof. The Lempert function is always upper semi continuous, so we only need to show that when $A_p \rightarrow A$,

$$\ell_{\Omega_n}(A, B) \leq \limsup_{p} \ell_{\Omega_n}(A_p, B).$$

Let $\alpha_p \in \mathbb{D}$, $\Phi_p \in Hol(\mathbb{D}, \Omega_n)$ s.t. $\Phi_p(0) = B$, $\Phi_p(\alpha_p) = A_p$.

Let $\varphi_p := \pi \circ \Phi_p$. We may assume $\limsup_p |\alpha_p| < 1$, otherwise there is nothing to prove, so by passing to a subsequence assume $\alpha_p \rightarrow \alpha_\infty$.

By normal families, we may assume that $\varphi_p \rightarrow \varphi_\infty$.

By continuity of $\Theta_B$, $0 = \Theta_B(\varphi_p) \rightarrow \Theta_B(\varphi_\infty)$, so $\exists \Phi_\infty$ such that $\pi \circ \Phi_\infty = \varphi_\infty$, $\Phi_\infty(0) = B$.

Furthermore, $\pi(\Phi_\infty(\alpha_\infty)) = \varphi_\infty(\alpha_\infty) = \lim_p \varphi_p(\alpha_p) = \lim_p \pi(A_p) = \pi(A)$, and since those two matrices have the same spectrum and are cyclic, $\Phi_\infty(\alpha_\infty) \sim A$. \qed

2. First reductions of the problem

We recall a standard definition to fix notations.

Definition 4. Given $a := (a_1, \ldots, a_n) \in \mathbb{C}^n$, the companion matrix is

$$C_{[a]} := \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \ddots & \vdots & \\
\vdots & \ddots & 1 & 0 & \\
0 & 0 & \cdots & 0 & 1 \\
a_n & a_{n-1} & \cdots & a_2 & a_1
\end{pmatrix}.$$ 

The companion matrix of a matrix $M$ is $C_M := C_{[(\pm 1)^k \pi_k(M), 1 \leq k \leq n]}$. 


Then \( \pi(C_{[a]}) = (a_1, -a_2, \ldots, (-1)^{j+1}a_j, \ldots, (-1)^{n+1}a_n) \), thus \( \pi(M) = \pi(C_M) \).

In particular, \( \Phi := C_{[\varphi]} \) always defines a lifting of \( \varphi \).

We need a convenient notation.

**Definition 5.** Given \( a := (a_1, \ldots, a_n) \in \mathbb{C}^{n+1} \), we denote \( P_{[a]}(X) := X^n + \sum_{j=1}^{n}(-1)^j a_j X^{n-j} \).

In particular, the characteristic polynomial of the companion matrix is simply \( P_{C_{[a]}} = P_{[a]} \).

Recall that a matrix \( A \in \mathcal{M}_n \) is said to be cyclic (or non-derogatory) if it admits a cyclic vector, or equivalently if it is conjugate to its companion matrix (for other equivalent properties see [5]).

Therefore, in view of the initial remark, this means that lifting through a set of cyclic matrices can be achieved as soon as the obvious necessary conditions \( \varphi(a_j) = \pi(A_j), 1 \leq j \leq N \), are achieved [2].

We write \( B := A_1 \). The case where \( B \) has only one eigenvalue has been completely studied in [6].

3. Necessary conditions

Let \( B \in \mathcal{M}_n \) such that its Jordan form is

\[
B = \begin{pmatrix}
B_1 & & \\
& \ddots & \\
& & B_s
\end{pmatrix}, \quad B_j \in \mathcal{M}_{n_j}, \quad \sum_{j=1}^{s} n_j = n,
\]

where \( \text{Sp } B_j = \{\lambda_j\} \), and the eigenvalues \( \lambda_j \)'s are distinct.

We need to set up some notation to describe a Jordan block associated to an eigenvalue \( \lambda \) [6]. Let \( B = (b_{i,j})_{1 \leq i,j \leq n} \). Then \( b_{j-1,j} \in \{0, 1\}, 2 \leq j \leq n \). Let \( r \) stand for the rank of \( B - \lambda I \).

Enumerate the (possibly empty) set of column indices where the coefficient \( b_{j-1,j} \) vanishes as

\[
\{ j : b_{j-1,j} = 0 \} = \{ b_2, \ldots, b_{n-r} \}, 2 \leq b_2 < \cdots < b_{n-r} \leq n.
\]

The integer \( b_{l+1} - b_l \) is the smallest \( m \) such that the basis vector \( e_{b_{l+1}} \in \ker(B - \lambda I)^m \). We choose the Jordan form so that \( b_{l+1} - b_l \) is increasing for \( 1 \leq l \leq n - r \), with the convention \( b_{n-r+1} := n + 1 \). It means that the possible zeroes appear for the smallest possible indices \( j \), globally.

**Definition 6.** For \( 1 \leq i \leq n \), let \( d_i := 1 + \#\{ j \geq n-i+2 : b_{j-1,j} = 0 \} \).

One can interpret \( d_j = d_j(B) \) as

\[
\min \{ d : \exists v_1, \ldots, v_d \in \mathbb{C}^n \text{ s.t. } \dim \text{Span} \left( B^{k_1}v_1, \ldots, B^{k_d}v_d, k_l \geq 0 \right) \geq j \}.\]
This is a sort of graded measurement of the failure of cyclicity (in the cyclic case, one vector is enough for any dimension $j$; in the other extreme, the scalar case, one needs $j$ independent vectors to reach dimension $j$), but we shall not need this.

The following proposition gives a set of conditions for lifting which are locally necessary and sufficient. This says in particular that all possible necessary conditions that can be obtained from the behavior of $\Phi$ in a neighborhood of $\alpha \in \mathbb{D}$ are exhausted by (2).

**Proposition 7.** Let $\varphi \in Hol(\omega, G_n)$, where $\omega$ is a neighborhood of $\alpha \in \mathbb{D}$.

Then there exists $\omega' \subset \mathbb{D}$ a neighborhood of $\alpha$ and $\Phi \in Hol(\omega', \Omega_n)$ such that

$$
\pi \circ \Phi = \varphi, \Phi(0) = M \text{ and } \Phi(\zeta) \text{ is cyclic for } \zeta \neq 0
$$

if and only if

$$
d^k P_{[\varphi(\zeta)]}(\lambda_j) = O((\zeta - \alpha)^{d_{n_j-k}(B_j)}), \quad 0 \leq k \leq n_j - 1, 1 \leq j \leq s.
$$

Notice that the condition

$$
d^k P_{[\varphi(\zeta)]}(\lambda_j) = O(\zeta^{d_{n_j-k}}), \quad 0 \leq k \leq n_j - 1, 1 \leq j \leq s,
$$

says exactly that $P_{[\varphi(\alpha)]}(X) = P_B(X)$, in other words, $\varphi(\alpha) = \pi(B)$, which is the obvious necessary condition for the existence of a lifting; and the only one that is needed when $B$ is cyclic, which is also the case when $d_{n_j-k}(B_j) = 1$ for all $j$ and $k$.

**Proof.** First, since this is a local result, it is no loss of generality to assume that $\alpha = 0$.

**Necessary conditions.**

First consider the case where $s = 1$, and $\lambda_1 = 0$. This is settled by [6, Corollary 4.3], which can be restated as follows.

**Lemma 8.** If $\varphi = (\varphi_1, \ldots, \varphi_n) = \pi \circ \Phi$ with $\Phi \in O(\mathbb{D}, \Omega_n)$, $\Phi(0) = B$, $SpB = \{0\}$ then $\varphi_1(\zeta) = O(\zeta^d)$.

Notice that if we write $P_{[\varphi(\zeta)]}^{(k)} := \frac{d^k P_{[\varphi(\zeta)]}}{dX^k}$ (the derivative of the polynomial with respect to the indeterminate $X$, not to be confused with derivatives with respect to the holomorphic variable $\zeta$), the conditions above can be written as $P_{[\varphi(\alpha)]}^{(k)}(0) = O(\zeta^{d_{n-k}}), 0 \leq k \leq n - 1$.

If $SpB = \{\lambda\}$, then $Sp(B - \lambda I_n) = \{0\}$. One sees immediately that $P_{B - \lambda I_n}(X) = P_B(X - \lambda)$, so that the necessary condition in the more general case of a matrix with a single eigenvalue becomes:
Lemma 9. If \( \varphi = (\varphi_1, \ldots, \varphi_n) = \pi \circ \Phi \) with \( \Phi \in \mathcal{O}(D, \Omega_n) \), \( \Phi(0) = B \), \( \text{Sp}B = \{ \lambda \} \) then \( P^{(k)}_{[\varphi(\zeta)]}(\lambda) = O(\zeta^{d_{n-k}}), \, 0 \leq k \leq n-1 \).

Consider now the general case. To prove (2) for each \( j \), without loss of generality, study the Jordan block associated to the eigenvalue \( \lambda_1 \).

In Lemma 3.1 and the remarks following it give a holomorphically varying factorization in a neighborhood \( \omega \) of 0 of the characteristic polynomial of \( \Phi(\zeta) \): \( P_{[\varphi(\zeta)]}(X) = P_{\zeta}(X)P_{\zeta}^2(X) \), and a corresponding splitting of the space \( \mathbb{C}^n \), so that there are maps \( \Phi_1 \in \mathcal{O}(\omega, \Omega_{n_1}) \), \( \Phi_2 \in \mathcal{O}(\omega, \Omega_{n_2 + \ldots + n_s}) \), such that \( P_{\zeta}^1(X) \) is the characteristic polynomial of \( \Phi_1(\zeta) \). Then \( P_{\zeta}^1(X) = P_{B_1}(X) = (X - \lambda_1)^{n_1} \), and \( P_{\zeta}^2(X) = (X - \lambda_2)^{n_2} \cdots (X - \lambda_s)^{n_s} \) accounts for the remaining blocks.

As before, we may consider \( P_{[\varphi(\zeta)]}(X - \lambda_1) \) to reduce ourselves to the case \( \lambda_1 = 0 \). The proof of the necessity of (2) concludes with the following lemma (applied to \( k_j := n_1 - d_j(B_1) \)).

Lemma 10. Let \( P_{\zeta}^0, P_{\zeta}^1, P_{\zeta}^2 \) be polynomials depending holomorphically on \( \zeta \), \( P_{\zeta}^i(X) = \sum_{j=0}^{n_i} a_{ij} X^j \), \( i = 0, 1, 2 \), such that \( P_{\zeta}^0(X) = P_{\zeta}^1(X)P_{\zeta}^2(X) \), \( P_{\zeta}^1(X) = X^{n_1} \) and \( a_{00}^1(0) = P_{\zeta}^0(0) \neq 0 \).

Let \( (k_j, 0 \leq j \leq n_1 - 1) \) be a decreasing sequence of positive integers. Then \( a_j^0 = O(\zeta^{k_j}), 0 \leq j \leq n_1 - 1 \) if and only if \( a_j^0 = O(\zeta^{k_j}), 0 \leq j \leq n_1 - 1 \).

Proof. Since \( P_{\zeta}^0 \) is the product of the other two, \( a_j^0 = \sum_{l=0}^{j} a_{l-j} a_{j-l}^2 \). So if \( a_{l-j} = O(\zeta^{k_l-j}) = O(\zeta^{k_j}) \) because \( (k_j) \) is decreasing, we immediately see \( a_j^0 = O(\zeta^{k_j}) \).

Conversely, proceed by induction on \( j \). For \( j = 0 \), \( a_0^1 = a_{00}^1 a_0^2 = O(\zeta^{k_0}) \) since the denominator does not vanish for \( \zeta = 0 \). Suppose the property is satisfied for \( 0 \leq j' \leq j - 1 \). We have \( a_j^1 = \frac{1}{a_0^1} \left( a_j^0 - \sum_{l=1}^{j} a_{j-l} a_{l-j}^2 \right) \), so by induction hypothesis \( a_j^{1-j} = O(\zeta^{k_j-j}) = O(\zeta^{k_j}) \) because \( (k_j) \) is decreasing, and since \( a_j^0 = O(\zeta^{k_j}) \), we are done.

Sufficient conditions.

Using Lemma 3.1 and the remarks following it, applied repeatedly, we find in \( \omega \), some neighborhood of 0, a holomorphically varying factorization into mutually prime factors \( P_{[\varphi(\zeta)]}(X) = P_{[\varphi_1(\zeta)]}(X) \cdots P_{[\varphi_s(\zeta)]}(X) \), and a splitting of the space \( \mathbb{C}^n \),

\[
\mathbb{C}^n = \bigoplus_{j=1}^s \ker P_{[\varphi_j(\zeta)]}(\Phi(\zeta)).
\]

Reducing \( \omega \) is needed, for each \( i, \varphi_i \in \mathcal{O}(\omega, \mathbb{G}_{n_i}) \). By Lemma 10 it verifies the necessary conditions of vanishing relating to the matrix.
Lemma 11. Suppose that $\text{Sp}(B) = \{\lambda_1, \ldots, \lambda_n\}$. There are some integers $0 = m_0 < m_1 < \cdots < m_s = n$ such that $\lambda_k = \lambda_{k'}$ if and only if there exists $i \in \{1, \ldots, s\}$ such that $m_{i-1} < k, k' \leq m_i$.

The Jordan form $(b_{i,j})_{1 \leq i,j \leq n}$ of $B$ is given by a block decomposition with blocks $B_1, \ldots, B_s$, where $\text{Sp}(B_i) = \{\lambda_{m_i}\}$, $B_j \in \mathcal{M}_{m_i-m_{i-1}}$.

Then $B$ is conjugate to $B'$ where $b'_{m_i,1+m_i} = 1 \neq b_{m_i,1+m_i} = 0$, $1 \leq i \leq s-1$, and $b'_{j,k} = b_{j,k}$ for all other values of the indices.

Notice that this means that $b'_{i,j} = 0$ if $j \not\in \{i, i+1\}$, that $b'_{i,i+1} \in \{0, 1\}$ and that if $b'_{i,i+1} = 0$, then $b'_{i,i} = b'_{i+1,i+1} \in \text{Sp}(B)$.

The new basis that we will find will no longer split the space into invariant subspaces, but we still obtain a triangular form. Furthermore $B$ is cyclic if and only if $b'_{i,i+1} = 1$ for all $i$, $1 \leq i \leq n-1$.

Proof. Let $\{e_j, 1 \leq j \leq n\}$ be the ordered basis in which the original Jordan form is given. Write $V_i := \text{Span}\{e_j, m_{i-1} < j \leq m_i\}$ for the generalized eigenspace for the eigenvalue $\lambda_{m_i}$. Finally, let $u$ be the linear mapping associated to $B$.

The result will be a consequence of the following fact, to be proved by induction on $j$, $1 \leq j \leq n$:

$$(P_j) : \exists v_j \in \bigoplus_{i:m_i < j} V_i \text{ s. t. if } \ e'_j := e_j + v_j, \text{ then } u(e'_j) = \lambda_j e'_j + b'_{j-1,j} e'_{j-1},$$

with $b'_{j-1,j}$ as defined in the statement of the Lemma. (Here we understand that $e'_0 = 0$ and an empty sum of subspaces is $\{0\}$).

$(P_1)$ is trivially satisfied with $v_1 = 0$.

Now assume $(P_j')$ holds for $j' < j$. We are looking for $v$ such that

$$u(e_j + v) = \lambda_j (e_j + v) + b'_{j-1,j} (e_{j-1} + v_{j-1}),$$
equivalently
\[(u - \lambda_j \text{Id})(v) = b'_{j-1,j}v_{j-1} + (b'_{j-1,j} - b_{j-1,j})e_{j-1}.\]

**Case 1.** There exists \(i\) such that \(j - 1 = m_i\).
Then (3) becomes \((u - \lambda_{m_{i+1}} \text{Id})(v) = v_{m_i} + e_{m_i}\). This last vector is in \(\bigoplus_{i' \leq i} V_{i'}\), which is stable under the linear map \(u - \lambda_{m_{i+1}} \text{Id}\). Furthermore, the restriction to the subspace of that map does not admit 0 as an eigenvalue, so the equation admits a (unique) solution \(v =: v_{j+1}\) in that subspace, which is the required space since \(m_i < j\), q.e.d.

**Case 2.** For all \(i, j - 1 \neq m_i\).
Then (3) becomes \((u - \lambda_j \text{Id})(v) = b'_{j-1,j}v_{j-1}\), and \(\bigoplus_{i : m_i < j} V_i = \bigoplus_{i : m_i < j} V_i\). Again, that subspace is stable under \(u - \lambda_{m_j} \text{Id}\), the restriction of the map is a bijection, so we get a (unique) solution \(v =: v_{j+1}\in \bigoplus_{i : m_i < j} V_i\).

\[\square\]

From now on we assume that the matrices we want to lift through are in modified Jordan form.

### 4.2. Divided Differences.

**Definition 12.** For a polynomial \(P\), the divided differences are given recursively by:

\[\Delta^0 P = P, \Delta^1 P(x_1, x_2) = \frac{P(x_1) - P(x_2)}{x_1 - x_2}, \ldots,\]
\[\Delta^m P(x_1, \ldots, x_{m+1}) = \frac{\Delta^{m-1} P(x_1, \ldots, x_m) - \Delta^{m-1} P(x_2, \ldots, x_{m+1})}{x_1 - x_{m+1}}.\]

Recall that \(\Delta^m P(x, \ldots, x) = \frac{1}{m!} P^{(m)}(x).\) A good general reference about divided differences is [3].

### 4.3. A meromorphic lift.

The following formula gives a “meromorphic” solution to the lifting problem: some of the matrix coefficients are given by quotients which may have poles. Of course, when the singularities are removable, we extend the functions in the usual way, and the properties claimed below extend by continuity.

**Proposition 13.** Let \(\Phi(\zeta) := \)
\[
\begin{pmatrix}
\phi_{11}(\zeta) & f_2(\zeta) & 0 & \cdots & 0 \\
0 & \phi_{22}(\zeta) & \ddots & \vdots \\
\vdots & \ddots & f_{n-1}(\zeta) & 0 \\
0 & 0 & \ddots & \phi_{n-1,n-1}(\zeta) & f_n(\zeta) \\
\phi_{n,1} & \phi_{n,2} & \cdots & \phi_{n,n-1} & \tilde{\phi}_{n,n}
\end{pmatrix},
\]

From now on we assume that the matrices we want to lift through are in modified Jordan form.
where the \( f_k \) and \( \phi_{kk} \) are chosen arbitrarily, \( 1 \leq k \leq n \), and where
\[
\phi_{n,\ell} := -\frac{\Delta^{\ell-1} P_{\varphi}(\phi_{1,1}, \ldots, \phi_{\ell,\ell})}{\prod_{k=\ell+1}^{n} f_k(\zeta)}, \quad 1 \leq \ell \leq n - 1,
\]
and finally
\[
\phi_{n,n} := \phi_{n,n}(\zeta) - \Delta^{n-1} P_{\varphi}(\phi_{1,1}, \ldots, \phi_{n,n}) = \varphi_1 - (\phi_{1,1} + \cdots + \phi_{n-1,n-1}).
\]
Then \( \pi \circ \Phi = \varphi \), when \( \Phi(\zeta) \) is defined.

If furthermore \( f_k(\alpha) = b_{k-1,k} \), \( 2 \leq k \leq n \), \( \phi_{kk}(\alpha) = \lambda_k \), \( 1 \leq k \leq n \), and \( \phi_{n,k}(\alpha) = 0 \), \( 1 \leq k \leq n - 1 \), then \( \Phi(\alpha) = B' \), with \( B' \) as described in Lemma 11.

Note that we sometimes omit the argument \( \zeta \) in \( \phi_{ij}(\zeta) \) and other functions. This will happen again.

**Proof.** We compute \( \det(XI_n - \Phi(\zeta)) \) by expanding w.r.t. the last row:
\[
= (X - \tilde{\phi}_{n,n}) \prod_{i=1}^{n-1} (X - \phi_{i,i}) - \sum_{\ell=1}^{n-1} (-1)^{n+\ell} \phi_{n,\ell} \prod_{i=1}^{\ell-1} (X - \phi_{i,i}) \prod_{i=\ell+1}^{n} (-f_i)
\]
\[
= \prod_{i=1}^{n} (X - \phi_{i,i}) + \Delta^{n-1} P_{\varphi}(\phi_{1,1}, \ldots, \phi_{n,n}) \prod_{i=1}^{n-1} (X - \phi_{i,i})
\]
\[
+ \sum_{j=0}^{n-2} \Delta^j P_{\varphi}(\phi_{1,1}, \ldots, \phi_{j+1,j+1}) \prod_{i=1}^{j} (X - \phi_{ii})
\]
\[
= P_{\varphi}(\zeta)(X), \quad \square
\]

4.4. Preliminary Computations.

**Lemma 14.** For natural numbers \( k \leq j \), the divided difference \( \Delta^k X^j \)

is given by the formula
\[
\Delta^k X^j(x_1, x_2, \ldots, x_{k+1}) = \sum_{i_1, \ldots, i_{k+1} \geq 0 \atop i_1 + \cdots + i_{k+1} = j-k} x_1^{i_1} x_2^{i_2} \cdots x_{k+1}^{i_{k+1}}.
\]

**Lemma 15.** Let \( \varphi \in \mathcal{O}(\mathbb{D}, \mathbb{G}_n) \) satisfy the conditions (2) at \( \alpha = 0 \).

(a) Let \( \phi_{1,1}, \phi_{2,2}, \ldots, \phi_{n_1,n_1} \in \mathcal{O}(\mathbb{D}, \mathbb{C}) \) with \( \phi_{i,i}(0) = \lambda_i \) for \( 1 \leq i \leq n_1 \).

Then
\[
\Delta^k P_{\varphi}(\zeta)(\phi_{1,1}, \phi_{2,2}, \ldots, \phi_{k+1,k+1}) = O(\zeta^{d_{n_1-k(B_1)}})
\]

for \( 0 \leq k \leq n_1 - 1 \).
(b) Let $\lambda_1 \neq \lambda_2$. Suppose

$$\Delta^k P_{[\varphi(\zeta)]}(\phi_{1,1}, \phi_{2,2}, \ldots, \phi_{k+1,k+1}) = O(\zeta^d)$$

for $0 \leq k \leq n_1 - 1$ with $d \geq d_{n_2}(B_2)$. Let $\phi_{n_1+1,n_1+1} \in O(\mathbb{D}, \mathbb{C})$ arbitrary function with $\phi_{n_1+1,n_1+1}(0) = \lambda_2$, then

$$\Delta^{n_1} P_{[\varphi(\zeta)]}(\phi_{1,1}, \phi_{2,2}, \ldots, \phi_{n_1+1,n_1+1}) = O(\zeta^{dn_2}(B_2)).$$

Proof. To prove (a), we write $P_{[\varphi(\zeta)]}$ in the form of a Taylor series

$$P_{[\varphi(\zeta)]}(X) = \sum_{j=0}^{n} \frac{P(j)_{[\varphi(\zeta)]}(\lambda_1)}{j!}(X - \lambda_1)^j.$$ 

Put $\lambda_{i,i} = \lambda_1 + \phi_i$ for $1 \leq i \leq n_1$. By linearity of the difference operator $\Delta$,

$$\Delta^k P_{[\varphi(\zeta)]}(\phi_{1,1}, \phi_{2,2}, \ldots, \phi_{k+1,k+1}) =$$

$$= \sum_{j=k}^{n} \frac{P(j)_{[\varphi(\zeta)]}(\lambda_1)}{j!} \Delta^k(X - \lambda_1)^j(\phi_{1,1}, \phi_{2,2}, \ldots, \phi_{k+1,k+1})$$

$$= \sum_{j=k}^{n} \frac{P(j)_{[\varphi(\zeta)]}(\lambda_1)}{j!} \sum_{i_1, \ldots, i_{k+1} \geq 0 \atop i_1 + \ldots + i_{k+1} = j-k} \phi_{i_1}^1 \phi_{i_2}^2 \cdots \phi_{i_{k+1}}^{k+1}$$

$$= \sum_{j=k}^{n} \frac{P(j)_{[\varphi(\zeta)]}(\lambda_1)}{j!} O(\zeta^{j-k}).$$

By the conditions (2) and the facts that $d_j(B_1) \leq d_{j+1}(B_1) \leq d_j(B_1) + 1$ and that $d_j(B_1) \leq j$ for $1 \leq j \leq n_1 - 1$, we get the result.

To prove (b), we use the Newton interpolation formula

$$P_{[\varphi(\zeta)]}(\phi_{n_1+1,n_1+1}) =$$

$$= \sum_{k=0}^{n_1} \Delta^k P_{[\varphi(\zeta)]}(\phi_{1,1}, \ldots, \phi_{k+1,k+1}) \cdot (\phi_{n_1+1,n_1+1}-\phi_{1,1}) \cdots (\phi_{n_1+1,n_1+1}-\phi_{k-1,k-1}).$$

We remark that $(\phi_{n_1+1,n_1+1}-\phi_{1,1}) \cdots (\phi_{n_1+1,n_1+1}-\phi_{k,k}) \neq 0$ at $\zeta = 0$ for $1 \leq k \leq n_1$.

Combining the result in (a) at $\lambda_1$ in the right hand side and at $\lambda_2$ in the left hand side, we deduce part (b) of the Lemma. \qed
5. The case where \( n \leq 5 \)

The case \( n = 2 \) of this theorem is covered by [1] and the case \( n = 3 \) by [6] and [4].

**Theorem 16.** Let \( n \in \mathbb{N}^* \), \( n \leq 5 \), \( A^1, \ldots, A^N \in \mathcal{M}_n \), \( \alpha_1, \ldots, \alpha_N \in \mathbb{D} \) and \( \varphi \in \mathcal{O}(\mathbb{D}, \mathcal{G}_n) \).

Then there exists \( \Phi \in \mathcal{O}(\mathbb{D}, \mathcal{G}_n^* \) satisfying \( \varphi = \pi \circ \Phi \) and \( \Phi(\alpha_j) = A^j \) for \( j = 1, \ldots, N \) if and only if \( \varphi \) satisfies the conditions (2) for \( A^j \) and \( \alpha_j \), \( 1 \leq j \leq N \).

**Proof.** In view of Proposition 13, it will be enough to show that we can choose \( f_k \) and \( \phi_{kk} \) such that for each \( j \), they and the entries \( \phi_{n,k} \) are defined and assume the correct values at \( \alpha_j \).

Write \( A_j^t = (a_{i,j}^t)_{1 \leq i,j \leq n} \) in modified Jordan form. A first requirement is that \( f_k(\alpha_j) = a_{k-1,j}^j, 2 \leq k \leq n, 1 \leq j \leq N \), and \( \phi_{kk}(\alpha_j) = a_{k,k}^j = \lambda_k(A_j) \), \( 1 \leq k \leq n, 1 \leq j \leq N \). We also will require that \( f_k(\zeta) \neq 0 \) unless \( \zeta = \alpha_j \) and \( a_{k-1,k}^j = 0 \), and that all of its zeros are simple.

**Claim.** We can determine further conditions (if needed) on a finite number of the successive derivatives of \( \phi_{kk} \) at \( \alpha_j \) to ensure that the entries \( \phi_{n,k}, 1 \leq k \leq n - 1 \), are defined and assume the value 0 at \( \alpha_j \).

If this claim holds, then for instance polynomial interpolation lets us find holomorphic functions \( f_k \) and \( \phi_{kk} \) satisfying the finite union of all those conditions.

The remainder of the argument is devoted to the proof of the above claim. It will be enough to work at one point \( \alpha_j \in \mathbb{D} \), which we may take to be 0 to simplify notations. Likewise the matrix \( A_j^t \) will be denoted \( B \), and will be in modified Jordan form.

### 5.1. The case \( n = 4 \)

We consider different cases according to the values of \( \{b_{12}, b_{23}, b_{34}\} \in \{0,1\}^3 \). If \( b_{k-1,k} = 1 \) for all \( k \), \( B \) is cyclic and the denominators in the formula for \( \phi_{4,k} \) never vanish at \( \zeta = 0 \); since \( \phi_{kk}(0) = \lambda_k \) which is a zero of \( P_A = P_{\phi(0)} \); \( \Delta^\ell P_{\phi(0)} \) always vanishes at \( (\phi_{11}, \ldots, \phi_{1\ell}) \) for \( \ell \leq 3 \), and we are done.

If \( b_{k-1,k} = 0 \) for all \( k \), all the eigenvalues of \( B \) must be equal and the proof is (essentially) done in [6] Proof of Proposition 4.1. In what follows we always assume that \( B \) admits at least two distinct eigenvalues.

If two of the \( b_{k-1,k} \), equal 1 and the last equals 0, we have an eigenspace of dimension 2, say for \( \lambda_1 = \lambda_2 \). The corresponding generalized eigenspace is of dimension 2 or 3. Reverting momentarily to the Jordan form, in the first case the matrix splits into two \( 2 \times 2 \) blocks with distinct eigenvalues and we may assume \( \ker(B - \lambda_1 I_4) = \text{Span}\{e_1, e_2\} \),
\[ \lambda_1 \notin \{\lambda_3, \lambda_4\}. \] In the second case, \( \lambda_3 = \lambda_1 \) and we may assume \( \ker(B - \lambda_1 I_4)^2 = \text{Span}\{e_1, e_2, e_3\} \).

In each case, \( B \) admits the following modified Jordan form:

\[
B = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 \\
0 & 0 & \lambda_3 & 1 \\
0 & 0 & 0 & \lambda_4
\end{pmatrix}, \quad \text{with} \quad \lambda_4 \neq \lambda_1.
\] (4)

If two of the \( b_{k-1,k} \) equal 0 and the last equals 1, there are two possible cases: in the first case, the two 0’s are consecutive and we have an eigenspace of dimension 3 (and since there are at least two distinct eigenvalues, the generalized eigenspace is equal to it), so that \( B \) admits the following modified Jordan form:

\[
B = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_1 & 1 \\
0 & 0 & 0 & \lambda_4
\end{pmatrix}, \quad \text{with} \quad \lambda_4 \neq \lambda_1.
\] (5)

In the second case, we have \((b_{12}, b_{23}, b_{34}) = (0, 1, 0)\), in which case \( B \) admits the following modified Jordan form:

\[
B = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_3
\end{pmatrix}, \quad \text{with} \quad \lambda_3 \neq \lambda_1.
\] (6)

**Case 1:** \( B \) as in (4). Since \( f_3(0) = f_4(0) = 1 \) and \( f_2 \) has a simple zero at 0, we need to prove that \( P_{[\phi(\zeta)]}(\phi_{11}) = O(\zeta^2), \Delta^1 P_{[\phi(\zeta)]}(\phi_{11}, \phi_{22}) = O(\zeta), \Delta^2 P_{[\phi(\zeta)]}(\phi_{11}, \phi_{22}, \phi_{33}) = O(\zeta) \).

In this case, the conditions (2) tell us that

\[
P_{[\phi(0)]}(X) = (X - \lambda_1)^2(X - \lambda_3)(X - \lambda_4),
\]

and

\[
P_{[\phi(\zeta)]}(\lambda_1) = O(\zeta^2).
\] (7)

Apply Lemma 15(a) with \( n_1 = 2 \), we get the first two conditions, and we get the third condition by applying Lemma 15(b) with \( n_2 = 1 \).

**Case 2:** \( B \) as in (5).

In this case, the conditions (2) tell us that

\[
P_{[\phi(0)]}(X) = (X - \lambda_1)^3(X - \lambda_4),
\]
and, since \( d_i(B_1) = i \) for \( i = 1, 2, 3 \),

\[
P_{[\varphi(\zeta)]}(\lambda_1) = O(\zeta^3), \quad P'_{[\varphi(\zeta)]}(\lambda_1) = O(\zeta^2). \tag{8}
\]

Applying Lemma 15(a) with \( n_1 = 3 \), we obtain \( P_{[\varphi(\zeta)]}(\phi_{11}) = O(\zeta^3) \), \( \Delta^1 P_{[\varphi(\zeta)]}(\phi_{11}, \phi_{22}) = O(\zeta^2) \), \( \Delta^2 P_{[\varphi(\zeta)]}(\phi_{11}, \phi_{22}, \phi_{33}) = O(\zeta) \). This is what we need to have a lifting.

**Case 3: \( B \) as in (6).**

This is the most complicated case.

In this case, the conditions (2) tell us that

\[
P_{[\varphi(0)]}(X) = (X - \lambda_1)^2(X - \lambda_3)^2, \tag{9}
\]

and

\[
P_{[\varphi(\zeta)]}(\lambda_1) = O(\zeta^2), \quad P_{[\varphi(\zeta)]}(\lambda_3) = O(\zeta^2). \tag{10}
\]

We need to prove that

\[
P_{[\varphi(\zeta)]}(\phi_{11}) = O(\zeta^3), \tag{11}
\]

\[
\Delta^1 P_{[\varphi(\zeta)]}(\phi_{11}, \phi_{22}) = O(\zeta^2), \tag{12}
\]

\[
\Delta^2 P_{[\varphi(\zeta)]}(\phi_{11}, \phi_{22}, \phi_{33}) = O(\zeta^2). \tag{13}
\]

To obtain this, we will need to choose adequate values for \( \phi'_{11}(0) \) and \( \phi'_{22}(0) \).

**Proof of (11).**

Put \( \phi_{1,1} = \lambda_1 + a, \ \phi_{2,2} = \lambda_1 + b \). Note that \( a'(0) = \phi'_{1,1}(0) \) and \( b'(0) = \phi'_{2,2}(0) \).

As in the proof of Lemma 15(a), we have

\[
P_{[\varphi(\zeta)]}(\phi_{1,1}) = \Delta^0 P_{[\varphi(\zeta)]}(\lambda_1 + a)
= P_{[\varphi(\zeta)]}(\lambda_1) + P'_{[\varphi(\zeta)]}(\lambda_1)a + P''_{[\varphi(\zeta)]}(\lambda_1) \frac{a^2}{2!} + O(\zeta^3).
\]

We have \( P_{[\varphi(\zeta)]}(\lambda_1) = O(\zeta^2), \ P'_{[\varphi(\zeta)]}(\lambda_1)a = O(\zeta^2), \) and \( P''_{[\varphi(\zeta)]}(\lambda_1) \neq 0 \)

at \( \zeta = 0 \). So we can choose \( a'(0) \) as a solution of a non-trivial quadratic equation to cancel out the terms of degree 2 in \( \zeta \).

**Proof of (12).**

As in the proof of Lemma 15(a)

\[
\Delta^1 P_{[\varphi(\zeta)]}(\phi_{11}, \phi_{22}) = \Delta^1 P_{[\varphi(\zeta)]}(\lambda_1 + a, \lambda_1 + b)
= P'_{[\varphi(\zeta)]}(\lambda_1) + P''_{[\varphi(\zeta)]}(\lambda_1) \frac{a + b}{2!} + O(\zeta^2).
\]
We have $P'_{\varphi(0)}(\lambda_1) = O(\zeta)$ since $\lambda_1$ is a double root of $P_{\varphi(0)}$, and $P''_{\varphi(0)}(\lambda_1) \neq 0$ at $\zeta = 0$. So we can chose $b'(0)$ to cancel out the terms of degree 2.

**Proof of (13).**

After adequate choices of $\phi_1$ and $\phi_2$, apply Lemma 15(b), we find that

$$\Delta^2 P_{\varphi(0)}(\phi_{11}, \phi_{22}, \phi_{33}) = O(\zeta^2)$$

for every $\phi_{33}$ with $\phi_{33}(0) = \lambda_2$. \(\square\)

5.2. **The case $n = 5$.** As in the case $n = 4$, assume that $B$ admits at least two distinct eigenvalues and $B$ is non-cyclic.

If $B$ admits three distinct eigenvalues, $B$ admits the following modified Jordan form

$$B = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & * & 0 & 0 \\
0 & 0 & \lambda_3 & * & 0 \\
0 & 0 & 0 & \lambda_4 & 1 \\
0 & 0 & 0 & 0 & \lambda_5
\end{pmatrix}, \text{ with } \lambda_5 \neq \lambda_1, \lambda_3, \lambda_4$$

where all the stars stand for 0 or 1.

Suppose that $\Phi(\zeta)$ is as in Proposition 13.

Since we always have $\Delta^2 P_{\varphi(0)}(\phi_{11}, \phi_{22}, \phi_{33}, \phi_{44}) = 0$ at $\zeta = 0$, and $\phi_{5,4} = -\Delta^3 P_{\varphi(0)}(\phi_{11}, \phi_{22}, \phi_{33}, \phi_{44})$ from the formula in Proposition 13, it suffices to consider the divided differences up to order 2, and the proof goes through as in the case $n = 4$.

Therefore, in this case, it suffices to consider the case where $B$ admits exactly two distinct eigenvalues. So $B$ admits the following modified Jordan form

$$B = \begin{pmatrix}
\lambda & b_{12} & 0 & 0 & 0 \\
0 & \lambda & b_{23} & 0 & 0 \\
0 & 0 & \lambda & 1 & 0 \\
0 & 0 & 0 & \mu & b_{45} \\
0 & 0 & 0 & 0 & \mu
\end{pmatrix}, \text{ with } \lambda \neq \mu.$$

If there is a cyclic block for $\lambda$ or $\mu$, i.e $b_{12} = b_{23} = 1$ or $b_{45} = 1$, then we can put that cyclic block in the lower right hand corner of the matrix and reduce the proof as in the case $n = 4$. So we only need to
consider the case where these two blocks are non-cyclic. In this case, $B$ admits one of the following modified Jordan forms:

$$B = \begin{pmatrix}
\lambda & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda & 1 & 0 \\
0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & \mu \\
\end{pmatrix},$$

(16)

or

$$B = \begin{pmatrix}
\mu & 0 & 0 & 0 & 0 \\
0 & \mu & 1 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & \lambda \\
\end{pmatrix},$$

(17)

**Case 1: $B$ as in (16).**

In this case, the conditions \[2\] tell us that

$$P[\varphi(\zeta)](\lambda) = O(\zeta^3), \quad P'[\varphi(\zeta)](\lambda) = O(\zeta^2), \quad P''[\varphi(\zeta)](\lambda) = O(\zeta)$$

and

$$P[\varphi(\zeta)](\mu) = O(\zeta^2), \quad P'[\varphi(\zeta)](\mu) = O(\zeta).$$

We need to prove that

$$P[\varphi(\zeta)](\phi_{11}) = O(\zeta^4),$$

(20)

$$\Delta^1 P[\varphi(\zeta)](\phi_{11}, \phi_{22}) = O(\zeta^3),$$

(21)

$$\Delta^2 P[\varphi(\zeta)](\phi_{11}, \phi_{22}, \phi_{33}) = O(\zeta^2),$$

(22)

$$\Delta^3 P[\varphi(\zeta)](\phi_{11}, \phi_{22}, \phi_{33}, \phi_{44}) = O(\zeta).$$

(23)

To obtain this, we will need to choose adequate values for $\phi'_{11}(0)$, $\phi'_{22}(0)$, and $\phi'_{33}(0)$. Note that as soon as we will have chosen these values, (23) will be automatically satisfied by Lemma 15(b).

**Proof of (20).** Put $\phi_{11} = \lambda + a$, $\phi_{22} = \lambda + b$ and $\phi_{33} = \lambda + c$. As in the proof of Lemma 15(a), we write

$$P[\varphi(\zeta)](\phi_{11}) = D^0 P[\varphi(\zeta)](\lambda + a)$$

$$= P[\varphi(\zeta)](\lambda) + P'[\varphi(\zeta)](\lambda) a + P''[\varphi(\zeta)](\lambda) \frac{a^2}{2!} + P'''[\varphi(\zeta)](\lambda) \frac{a^3}{3!} + O(\zeta^4).$$

From the conditions (18), we find that
and $P''_{p(\zeta)}(\lambda) \neq 0$ at $\zeta = 0$. So we can choose $a'(0)$ as a solution of a non-trivial cubic equation to cancel out the terms of degree 3.

**Proof of (21).** We have

\[ \Delta^1 P_{\varphi(\zeta)}(\phi_{11}, \phi_{22}) = \Delta^1 P_{\varphi(\zeta)}(\lambda + a, \lambda + b) = P'_{\varphi(\zeta)}(\lambda) + P''_{\varphi(\zeta)}(\lambda) \frac{a + b}{2!} + O(\zeta^3) \]

Again, we can choose $b'(0)$ to cancel out the terms of degree 2.

**Proof of (22).**

\[ \Delta^2 P_{\varphi(\zeta)}(\phi_{11}, \phi_{22}, \phi_{33}) = \Delta^2 P_{\varphi(\zeta)}(\lambda + a, \lambda + b, \lambda + c) = P''_{\varphi(\zeta)}(\lambda) \frac{1}{2!} + P'''_{\varphi(\zeta)}(\lambda) \frac{a + b + c}{3!} + O(\zeta^2) \]

Again, we can choose $c'(0)$ to cancel out the terms of degree 1.

**Case 2: $B$ as in (17).**

In this case, the conditions (24) tell us that

(24) $P_{\varphi(\zeta)}(\mu) = O(\zeta^2)$, $P'_{\varphi(\zeta)}(\mu) = O(\zeta)$,

and

(25) $P_{\varphi(\zeta)}(\lambda) = O(\zeta^2)$, $P'_{\varphi(\zeta)}(\lambda) = O(\zeta)$, $P''_{\varphi(\zeta)}(\lambda) = O(\zeta)$

We need to prove that

(26) $P_{\varphi(\zeta)}(\phi_{11}) = O(\zeta^3)$,

(27) $\Delta^1 P_{\varphi(\zeta)}(\phi_{11}, \phi_{22}) = O(\zeta^2)$,

(28) $\Delta^2 P_{\varphi(\zeta)}(\phi_{11}, \phi_{22}, \phi_{33}) = O(\zeta^2)$,

(29) $\Delta^3 P_{\varphi(\zeta)}(\phi_{11}, \phi_{22}, \phi_{33}, \phi_{44}) = O(\zeta)$.

**Proof of (26).** We have

\[ P_{\varphi(\zeta)}(\phi_{11}) = \Delta^0 P_{\varphi(\zeta)}(\mu + a) \]

\[ = P_{\varphi(\zeta)}(\mu) + P'_{\varphi(\zeta)}(\mu)a + P''_{\varphi(\zeta)}(\mu) \frac{a^2}{2!} + O(\zeta^3). \]
From the conditions (24), we find that

$$\text{ord}_{\zeta=0} \left\{ P_{[\phi(\zeta)]}(\mu) + P'_{[\phi(\zeta)]}(\mu) a \right\} \geq 2$$

and $P''_{[\phi(\zeta)]}(\mu) \neq 0$ at $\zeta = 0$, so we can choose $a'(0)$ as a solution of a non-trivial quadratic equation to cancel out the terms of degree 2.

**Proof of (27).**

$$\Delta^1 P_{[\phi(\zeta)]}(\phi_{11}, \phi_{22}) = \Delta^1 P_{[\phi(\zeta)]}(\mu + a, \mu + b)$$

$$= P'_{[\phi(\zeta)]}(\mu) + P''_{[\phi(\zeta)]}(\mu) \frac{a + b}{2!} + O(\zeta^2).$$

Again, we can choose $b'(0)$ to cancel out the terms of degree 1.

**Proof of (28).** This follows automatically from Lemma 15(b).

6. **Counter-examples for the candidate function when $n \geq 6$**

Suppose that $\Phi(\zeta)$ is given by the formula stated in the proposition 13 and that $\Phi(0) = B'$, with $B'$ as described in Lemma 11. In particular we should have $\phi_{n,1}(\zeta) = O(\zeta)$, and we would then have

$$\text{ord}_{\zeta=0}(P_{[\phi(\zeta)]}(\phi_{1,1})) \geq \text{ord}_{\zeta=0} \phi_{n,1} + \text{ord}_{\zeta=0}(f_2 f_3 \ldots f_n)$$

$$\geq 1 + d_{n_1}(B_1) - 1 + \text{ord}_{\zeta=0}(f_{n_1+1} f_{n_1+2} \ldots f_n)$$

$$\geq d_{n_1}(B_1) + \text{ord}_{\zeta=0}(f_{n_1+1} f_{n_1+2} \ldots f_n).$$

To look for a counter-example, we look for a situation where the above inequality does not occur.

Consider the following matrix of size $k$

$$B_k^\lambda = \begin{pmatrix}
\lambda & 0 & 0 & \cdots & 0 \\
\lambda & \lambda & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda & & & \lambda & 0 \\
& & & \lambda & 1 \\
\end{pmatrix}$$
with \( \lambda \in \mathbb{D} \). As usual, we understand that there are 0’s in the places where no entry is indicated. Put

\[
B^\lambda_k(\zeta) = \begin{pmatrix}
\lambda & \zeta & \cdots & \zeta \\
\lambda & \cdots & \cdots & \cdots \\
\lambda & \zeta & \cdots & \zeta \\
\zeta & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

with \( \zeta \in \mathbb{D} \). Its characteristic polynomial is given by the formula

\[
P_{B^\lambda_k}(X) = \det(XI - B^\lambda(\zeta)) = (X - \lambda)^k - \zeta^{k-1}.
\]

Decompose \( n = k + l \) with \( k, l \geq 3 \).

Consider the matrix

\[
B = \begin{pmatrix}
B_k^\lambda_1 \\
B_l^\lambda_2
\end{pmatrix}
\]

with \( \lambda_1 \neq \lambda_2 \in \mathbb{D} \). Put

\[
B(\zeta) = \begin{pmatrix}
B_k^\lambda_1(\zeta) \\
B_l^\lambda_2(\zeta)
\end{pmatrix}
\]

Setting \( \varphi(\zeta) = (\pi \circ B)(\zeta) \), we have

\[
P_{[\varphi(\zeta)]}(X) = P_{B(\zeta)}(X) = P_{B_k^\lambda_1(\zeta)}(X)P_{B_l^\lambda_2(\zeta)}(X).
\]

Suppose now that we can find a lifting \( \Phi(\zeta) \) in the form stated in Proposition 13. Inequality (30) gives the estimate

\[
\text{ord}_{\zeta=0} P_{[\varphi(\zeta)]}(\phi_{1,1}) \geq 1 + (k - 2) + (l - 2) = k + l - 3 \geq k.
\]

Since \( P_{B_{l}^{\lambda_{2}}(\zeta)}(\phi_{1,1}) \neq 0 \) at \( \zeta = 0 \),

\[
\text{ord}_{\zeta=0} P_{[\varphi(\zeta)]}(\phi_{1,1}) = \text{ord}_{\zeta=0} P_{B_{k}^{\lambda_{1}}(\zeta)}(\phi_{1,1})
\]

\[
= \text{ord}_{\zeta=0} \{(\phi_{1,1} - \lambda_1)^k - \zeta^{k-1}\}
\]

\[
= k - 1.
\]

So this gives a contradiction. One will object that the range of \( \varphi \) may not be in \( G_n \). However, \( \varphi(0) \in G_n \), so we can always reparametrize \( \varphi \) by replacing \( \zeta \) by \( \varepsilon \zeta \) with \( \varepsilon \) small enough.

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