COMBINATORICS OF FOURIER TRANSFORMS FOR
TYPE A QUIVER REPRESENTATIONS

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Abstract. We describe two new combinatorial algorithms (using the language of “triangular arrays”) for computing the Fourier transforms of simple perverse sheaves on the moduli space of representations of an equioriented quiver of type $A$. (A rather different solution to this problem was previously obtained by Knight–Zelevinsky.) Along the way, we also show that the closure partial order and the dimensions of orbits have especially concise descriptions in the language of triangular arrays.

1. Introduction

Let $Q_n$ be the following quiver, with $n$ vertices and $n-1$ arrows:

(1.1) $\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet$

Given a dimension vector $w \in \mathbb{Z}_{\geq 0}^n$, let $E(w)$ be the moduli space of representations of $Q_n$ of dimension vector $w$. (See Section 2 for additional background, definitions, and notation.)

This paper is the result of the authors’ attempts to do exercises with perverse sheaves on $E(w)$, and specifically to compute Fourier–Sato transforms by hand. These exercises led to combinatorial objects called triangular arrays. Using the language of triangular arrays, we describe:

1. the closure partial order on orbits in $E(w)$ (Theorem 3.3)
2. a dimension formula for orbits in $E(w)$ (Theorem 4.6)
3. two new combinatorial algorithms for computing Fourier–Sato transforms of simple perverse sheaves (Theorem 6.4 and Corollary 6.6)

All of these problems have been previously solved in the language of multisegments, also called Kostant partitions [AD, L, KZ, B] (see also [BG]). Nevertheless, we hope to convince the reader that the language of triangular arrays is worth studying:

- The closure partial order is especially easy in this language (it is the “chute-wise dominance order”), and the dimension formula is also very concise.
- The combinatorics of the Fourier–Sato transform in this paper looks very different from the “multisegment duality” of [KZ]. (Indeed, we were unable to find an elementary relationship between the two.) Perhaps our algorithms will be useful in situations where [KZ] is difficult to apply.

2010 Mathematics Subject Classification. 16G20 (Primary); 05E10 (Secondary).

P.A. received support from NSF Grant No. DMS-1500890. M.K. received support from NSF Grant No. DMS-1601862. J.M. received support from a Department of Education GAANN fellowship (Grant No. P200A120001).
For examples of triangular arrays, see Figures 1 and 2. Figure 1 shows the partial order and dimensions of orbits for the dimension vector $w = (3, 3, 3)$, and Figure 2 shows the involution on this set of orbits induced by the Fourier–Sato transform.

The paper is organized as follows: Section 2 defines triangular arrays, and fixes notation related to quiver representations. In Sections 3 and 4 we determine the closure partial order and the dimensions of orbits in terms of triangular arrays. The main new content of the paper is in Sections 5 and 6. Section 5 contains the definitions of two combinatorial operations on triangular arrays, denoted by $T$ and $T'$. That section also contains the proof that $T$ and $T'$ are both bijections. In Section 6, we prove that both $T$ and $T'$ compute the Fourier–Sato transform for simple perverse sheaves on $E(w)$. (In particular, the geometry shows that the maps $T$ and $T'$ coincide. We do not know a combinatorial proof of this fact.)

Acknowledgments. We are grateful to Pierre Baumann, Tom Braden, Thomas Brüstle, Lutz Hille, Ivan Mirković, Laura Rider, Ralf Schiffler, and Catharina Stroppel for helpful conversations while this work was in progress.

2. Notation and preliminaries

2.1. Triangular arrays. Let $w = (w_1, \ldots, w_n)$ be an $n$-tuple of nonnegative integers. Given such a $w$, we define $P(w)$ to be the following set of collections of nonnegative integers:

$$P(w) = \left\{ (y_{ij})_{1 \leq i \leq n, 1 \leq j \leq n-i+1} \biggm| \sum_{j=1}^{n-i+1} y_{ij} = w_i \text{ for all } i, \text{ and } y_{ij} \geq y_{i-1,j+1} \text{ for all } i \text{ and } j \right\}.$$  

An element $Y \in P(w)$ is called a triangular array of size $n$. It can be drawn as follows:

\[
Y = \begin{array}{c}
\begin{array}{c}
\vdots \\
y_{n-1,2} \\
y_{n,1} \\
y_1 \\
y_{1,2} \\
y_{1,1}
\end{array}
\end{array}
\]

We will refer to portions of this diagram as columns, chutes, and ladders:

\[
\begin{align*}
\text{jth} & \text{ column:} \\
\text{i th} & \text{ chute:} \\
\text{k th} & \text{ ladder:}
\end{align*}
\]

With these notions, we can rephrase the definition of $P(w)$ as follows: it is the set of diagrams of nonnegative integers as in (2.1) that:

- have chute-sums given by $w$, and
- are weakly decreasing (from left to right) along ladders.

For $Y \in P(w)$, we call $w$ the dimension vector of $Y$, and we write $\dim(Y) = w$. 
Figure 1. Partial order and dimensions for $w = (3, 3, 3)$
Figure 2. Fourier–Sato transforms for $w = (3, 3, 3)$
Now let \( Y = (y_{ij}) \) and \( Y' = (y'_{ij}) \) be two elements of \( \mathbf{P}(w) \). We equip \( \mathbf{P}(w) \) with a partial order \( \leq_c \) by declaring that

\[
(2.2) \quad Y \leq_c Y' \quad \text{if for all } i \text{ and } j, \sum_{k=1}^{j} y_{ik} \geq \sum_{k=1}^{j} y'_{ik}.
\]

The condition (2.2) resembles the usual dominance order on partitions, but each inequality involves only entries from a single chute. For this reason, we call \( \leq_c \) the “chutewise dominance order.”

2.2. Moduli spaces of quiver representations. Recall from Section I that \( Q_n \) denotes the quiver \( \begin{array}{c} \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \end{array} \end{array} \) with \( n \) vertices and \( n-1 \) arrows. Let \( \text{Rep}(Q_n) \) denote the category of finite-dimensional complex representations of \( Q_n \). Given an object

\[
M = (M_1 \xrightarrow{x_1} M_2 \xrightarrow{x_2} \cdots \xrightarrow{x_{n-1}} M_n)
\]

in \( \text{Rep}(Q_n) \), we denote by \( \dim M \) its dimension vector:

\[
\dim M = (\dim M_1, \dim M_2, \ldots, \dim M_n) \in \mathbb{Z}_{\geq 0}^n.
\]

Given \( w = (w_1, \ldots, w_n) \in \mathbb{Z}_{\geq 0}^n \), let \( E(w) \) be the moduli space of representations of \( Q_n \) with dimension vector \( w \). Explicitly, we put

\[
E(w) = \text{Hom}(C^{w_1}, C^{w_2}) \times \text{Hom}(C^{w_2}, C^{w_3}) \times \cdots \times \text{Hom}(C^{w_{n-1}}, C^{w_n}).
\]

Given \( x = (x_1, \ldots, x_{n-1}) \in E(w) \), let \( M(x) \) denote the quiver representation

\[
M(x) = (C^{w_1} \xrightarrow{x_1} C^{w_2} \xrightarrow{x_2} \cdots \xrightarrow{x_{n-1}} C^{w_n}).
\]

(The point \( x \) and the object \( M(x) \) consist of the same data, but we think of \( x \) as a point in an algebraic variety, and \( M(x) \) as an object of an abelian category.)

The variety \( E(w) \) is just an affine space of dimension \( w_1 w_2 + w_2 w_3 + \cdots + w_{n-1} w_n \).

It is equipped with an action of the group

\[
G(w) = \text{GL}(w_1) \times \text{GL}(w_2) \times \cdots \times \text{GL}(w_n)
\]

given by the formula

\[
(g_1, \ldots, g_n) \cdot (x_1, \ldots, x_{n-1}) = (g_2 x_1 g_1^{-1}, \ldots, g_n x_{n-1} g_{n-1}^{-1}).
\]

Two points \( x, y \in E(w) \) lie in the same \( G(w) \)-orbit if and only if \( M(x) \) and \( M(y) \) are isomorphic objects of \( \text{Rep}(Q_n) \).

Let us recall the classification of indecomposable objects in \( \text{Rep}(Q_n) \). For \( k = 1, \ldots, n \), let \( e_k \) be the dimension vector

\[
e_k = (\underbrace{0, \ldots, 0}_{k-1 \text{ entries}}, 1, \underbrace{0, \ldots, 0}_{n-k \text{ entries}}).
\]

Then, for \( 1 \leq i \leq j \leq n \), let

\[
\gamma_{ij} = e_i + e_{i+1} + \cdots + e_j.
\]

The \( \gamma_{ij} \) can be identified with the positive roots in a root system of type \( A_n \). (The \( e_k \) are then identified with the simple roots.)

Gabriel’s theorem \( [G] \) says that the indecomposable objects in \( \text{Rep}(Q_n) \) are classified by their dimension vectors, and the vectors that occur as dimension vectors of indecomposable objects are precisely the positive roots. Given integers \( 1 \leq i \leq j \leq n \), let \( R_{ij} \) be the quiver representation given by

\[
R_{ij} = 0 \rightarrow \cdots \rightarrow 0 \rightarrow \mathbb{C} \xrightarrow{id} \cdots \xrightarrow{id} \mathbb{C} \xrightarrow{id} 0 \rightarrow \cdots \rightarrow 0.
\]
Its dimension vector is $\gamma_{ij}$. The $R_{ij}$ exhaust the isomorphism classes of indecomposables.

Consider the set
\[ B(w) = \{(b_{ij})_{1 \leq i \leq j \leq n} \mid \sum b_{ij} \gamma_{ij} = w \}. \]

Gabriel’s theorem implies that there is a canonical bijection
\[ (2.3) \quad \{G(w)\text{-orbits on } E(w)\} \xrightarrow{1-1} B(w). \]

**Lemma 2.1.** There is a bijection $\nu : P(w) \xrightarrow{1-1} B(w)$.

**Proof.** Given $Y \in P(w)$, let $\nu(Y)$ be the element of $B(w)$ given by
\[ \nu(Y)_{ij} = y_{i,j-i+1} - y_{i-1,j-i+2}, \]
where the second term is understood to be 0 if $i = 1$. Conversely, given $b = (b_{ij}) \in B(w)$, let $\bar{\nu}(b)$ be the triangular array in $P(w)$ given by
\[ \bar{\nu}(b)_{ij} = \sum_{1 \leq h \leq i} b_{h,i+j-1}. \]

Straightforward computations show that $\nu$ and $\bar{\nu}$ are both well-defined, and that they are inverse to each other. \qed

For $w = (w_1, \ldots, w_n)$, let $w^* = (w_n, \ldots, w_1)$ be the reverse of $w$.

**Corollary 2.2.** The sets $P(w)$ and $P(w^*)$ have the same cardinality.

**Proof.** This follows from the fact that there is a bijection $B(w) \to B(w^*)$ given by $(b_{ij}) \mapsto (b_{n-j+1,n-i+1})$. \qed

2.3. **Orbits.** Combining (2.3) and Lemma 2.1, we obtain a bijection between $P(w)$ and the set of $G(w)$-orbits in $E(w)$. For $Y \in P(w)$, let
\[ O_Y \subset E(w) \]
be the corresponding $G(w)$-orbit. Let us write down a concrete representative of this orbit.

**Lemma 2.3.** Let $x \in E(w)$, and let $Y = (y_{ij}) \in P(w)$. The following are equivalent:

1. $x \in O_Y$.
2. Each $\mathbb{C}^{w_i}$ admits a basis
\[ \{u^{(k)}_{ij} \mid 1 \leq j \leq n - i + 1, 1 \leq k \leq y_{ij} \} \]
such that $x_i : \mathbb{C}^{w_i} \to \mathbb{C}^{w_{i+1}}$ is given by
\[ x_i(u^{(k)}_{ij}) = \begin{cases} u^{(k)}_{i+1,j-1} & \text{if } j > 1, \\ 0 & \text{if } j = 1. \end{cases} \]

Note that the set in (2.4) does indeed consist of exactly $w_i$ elements. We will call a basis in which (2.5) holds a *Jordan basis of type $Y$*, by analogy with Jordan normal form for matrices.
Proof: We will first show that part (2) implies part (1). Assume that (2.5) holds. Let \((b_{ij}) = v(Y) \in B(w)\). To show that \(x \in \mathcal{O}_Y\), we must show that the representation \(M(x)\) contains exactly \(b_{ij}\) copies of \(R_{ij}\) as direct summands, for all \(i\) and \(j\) such that \(1 \leq i \leq j \leq n\). Fix such an \(i\) and \(j\). Also fix an integer \(k\) such that
\[
y_{i-1,j-i+2} + 1 \leq k \leq y_{i,j-i+1}.
\]
(If \(i = 1\), then \(y_{-1,j-i+2}\) should be understood to be 0.) Let \(N^h_k \subset \mathbb{C}^{w_h}\) be the subspace given by
\[
N^h_k = \begin{cases} 
0 & \text{if } h < i \text{ or } h > j, \\
\text{span}\{u^{(h)}_{k,j-h+1}\} & \text{if } i \leq h \leq j.
\end{cases}
\]
It can be checked using (2.5) that \(N^h_k = \bigoplus_h N^h_k\) is a subrepresentation of \(x\), and that it is isomorphic to \(R_{ij}\). On the other hand, the span of the basis elements from (2.4) that are not included in \(N_k\) is also a subrepresentation, so \(N_k\) is a direct summand. The number of choices for \(k\) is \(y_{i,j-i+1} - y_{i-1,j-i+2} = b_{ij}\), so we have shown that \(x\) contains at least \(b_{ij}\) copies of \(R_{ij}\) as direct summands. The total dimension vector of the summands we have produced is already equal to \(w\), so in fact \(x\) contains exactly \(b_{ij}\) copies of \(R_{ij}\).

Suppose now that \(x \in \mathcal{O}_Y\). Define a new representation \(z \in E(w)\) by choosing some basis as in (2.4), and then defining the linear maps \(z_i : \mathbb{C}^{w_i} \to \mathbb{C}^{w_{i+1}}\) using the formula (2.5). By the implication we have already proved, we have \(z \in \mathcal{O}_Y\). Since \(M(x)\) and \(M(z)\) are isomorphic, \(M(x)\) also admits a Jordan basis of type \(Y\).

In a Jordan basis, we have \(\ker x_i = \text{span}\{u^{(k)}_{11} \mid 1 \leq k \leq y_{i1}\}\). More generally, we have
\[
\ker x_{i+1} \cdots x_{i+1} x_i = \text{span}\{u^{(k)}_{ih} \mid 1 \leq h \leq j, 1 \leq k \leq y_{ih}\},
\]
(2.6)
\[\dim \ker x_{i+1} \cdots x_{i+1} x_i = y_{i1} + y_{i2} + \cdots + y_{ij}\.
\]

Remark 2.4. A number of basic notions involving quiver representations can be translated into the language of triangular arrays. We list some examples below, using the following notation: for \(Y \in \mathbf{P}(w)\), we let \(M(Y)\) denote the quiver representation corresponding to some point \(x \in \mathcal{O}_Y\).

1. For \(Y \in \mathbf{P}(w)\) and \(Z \in \mathbf{P}(v)\), we have \(M(Y + Z) \cong M(Y) \oplus M(Z)\). (Here \(Y + Z\) is the entrywise sum of \(Y\) and \(Z\).)
2. The module \(M(Y)\) is an injective object in \(\text{Rep}(Q_n)\) if and only if \(Y\) is constant along ladders, i.e., if \(y_{ij} = y_{i-1,j+1}\) for all \(i\) and \(j\).
3. The module \(M(Y)\) is a projective object in \(\text{Rep}(Q_n)\) if and only if \(Y\) has nonzero entries only in the last ladder, i.e., if \(y_{ij} = 0\) whenever \(i + j < n + 1\).

3. The partial order on orbits

Let \(\leq_g\) be the partial order on \(\mathbf{P}(w)\) induced by the closure order on \(G(w)\)-orbits; that is, for \(Y, Y' \in \mathbf{P}(w)\),
\[
Y \leq_g Y' \quad \text{if } \mathcal{O}_Y \subset \overline{\mathcal{O}_{Y'}}.
\]
The goal of this section is to prove that the chutewise dominance order \(\leq_c\) (see (2.2)) and the geometric partial order \(\leq_g\) coincide.

Let \(W\) be the symmetric group on \(n + 1\) letters, i.e., the Weyl group associated to the Dynkin diagram that is the underlying graph of our quiver \(Q_n\). Let \(s_i\) (for \(i = 1, 2, \ldots, n\)) be the transposition that exchanges \(i\) and \(i + 1\). In other words,
these are the simple reflections in $W$. Consider the following reduced expression for the longest element $w_0$ in $W$:

$$w_0 = (s_n)(s_{n-1}s_n)\cdots(s_2s_3\cdots s_n)(s_1s_2\cdots s_n).$$

This reduced expression is “adapted” to our quiver in the sense of \cite{L} §4.7. More precisely, in the notation of \cite{L}, the sequence

$$i = (n, n-1, n, \ldots, 2, 3, \ldots, n, 1, 2, \ldots, n) \in \mathcal{H}$$

is adapted to our quiver. This sequence determines an ordering on the set of positive roots as in \cite{L} §2.8. Denote the positive roots in this order by $\alpha^1, \alpha^2, \ldots, \alpha^{n(n+1)/2}$. They are given by:

$$\gamma_{nn}, \gamma_{n-1,n}, \gamma_{n-1,n-1}, \ldots, \gamma_{ii}, \gamma_{i,n-1}, \ldots, \gamma_{1,n-1}, \ldots, \gamma_{11}.$$  

(Recall that $\gamma_{ij} = e_i + e_{i+1} + \cdots + e_j$.) Note that for $b = (b_{ij}) \in B(w)$, the ordering on the positive roots induces an ordering on the $b_{ij}$. We write $b'$ to denote the number $b_{ij}$ corresponding to the positive root $\alpha^i$.

Next, let $\omega_1^\vee, \ldots, \omega_n^\vee$ be the fundamental coweights, and let $\phi_{ij} = -\omega_{i,j-1}^\vee + \omega_{j,j}^\vee$. Following \cite{B}, \cite{M}, the sequence $i$ determines a sequence of $(n(n+1)/2)$ “chamber coweights” $\lambda^1, \lambda^2, \ldots, \lambda^{n(n+1)/2}$. They are given by:

$$\phi_{nn}, \phi_{n-1,n}, \phi_{n-1,n-1}, \ldots, \phi_{ii}, \phi_{i,n-1}, \ldots, \phi_{1,n-1}, \ldots, \phi_{11}.$$  

We write $\langle -, - \rangle$ for the usual pairing between coweights and weights. We have the following description of $\leq_g$.

**Theorem 3.1** (\cite{B} Proposition 4.1 and Remark 4.2(i)). For $Y, Z \in P(w)$, we have $Y \leq_g Z$ if and only if

$$\sum_{s=1}^t \langle \lambda^s, \alpha^s \rangle \nu(Y)^s \geq \sum_{s=1}^t \langle \lambda^s, \alpha^s \rangle \nu(Z)^s \quad \text{for all } 1 \leq t \leq n(n+1)/2. \quad (3.2)$$

For $Y, Z \in P(w)$, let us write $\nu(Y) = (b_{ij})_{1 \leq i \leq j \leq n}$ and $\nu(Z) = (c_{ij})_{1 \leq i \leq j \leq n}$. Consider the following condition:

$$\sum_{i=\ell}^k \sum_{j=k}^n b_{ij} \geq \sum_{i=\ell}^k \sum_{j=k}^n c_{ij} \quad \text{for all } 1 \leq \ell \leq k \leq n. \quad (3.3)$$

**Lemma 3.2.** Let $Y, Z \in P(w)$. Then (3.2) and (3.3) are equivalent conditions.

**Proof.** Notice that the pairing $\langle \phi_{ik}, \gamma_{ij} \rangle$ appears in the sums from (3.2) if and only if $i > \ell$ or $i = \ell$ and $j \geq k$. Under these conditions, $\langle \phi_{ik}, \gamma_{ij} \rangle = \begin{cases} 1 & \text{if } n \geq j \geq k \text{ and } k \geq i \geq \ell \\ 0 & \text{otherwise.} \end{cases}$  

The claim follows. \hfill $\square$

**Theorem 3.3.** The chutewise dominance order $\leq_e$ on $P(w)$ coincides with the geometric partial order $\leq_g$.

**Proof.** Given $Y = (y_{ij})$ and $Z = (z_{ij})$ in $P(w)$, write $\nu(Y) = (b_{ij})_{1 \leq i \leq j \leq n}$ and $\nu(Z) = (c_{ij})_{1 \leq i \leq j \leq n}$. Recall from the proof of Lemma 2.4 that

$$y_{ij} = \sum_{h=1}^i b_{h, i+j-1} \quad \text{and} \quad z_{ij} = \sum_{h=1}^i c_{h, i+j-1}.$$
We first observe that for any $Y$ and $Z$ (regardless of how they compare under $\leq_g$), the $\ell = 1$ case of (3.3) is actually an equality. Indeed, the two sides simplify to $\sum_{j=k}^{n} y_{k,j-k+1}$ and $\sum_{j=k}^{n} z_{k,j-k+1}$, respectively, and both are equal to $w_k$ by the definition of $P(w)$.

Suppose $1 \leq m \leq k \leq n$. Here are two (somewhat expanded) instances of the $\ell = 1$ case of (3.3):

\begin{align*}
(3.4) & \quad \sum_{i=1}^{m-1} \sum_{j=k}^{n} b_{ij} + \sum_{i=m}^{k} \sum_{j=k}^{n} b_{ij} = \sum_{i=1}^{m-1} \sum_{j=k}^{n} c_{ij} + \sum_{i=m}^{k} \sum_{j=k}^{n} c_{ij}, \\
(3.5) & \quad \sum_{i=1}^{m} \sum_{j=m}^{n} b_{ij} + \sum_{i=1}^{m} \sum_{j=k+1}^{n} b_{ij} = \sum_{i=1}^{m} \sum_{j=m}^{n} c_{ij} + \sum_{i=1}^{m} \sum_{j=k+1}^{n} c_{ij}.
\end{align*}

Combining (3.4) and (3.5), we see that $Y \leq_g Z$ if and only if

$$
\sum_{i=1}^{m-1} \sum_{j=k}^{n} b_{ij} \leq \sum_{i=1}^{m-1} \sum_{j=k}^{n} c_{ij} \quad \text{for all } 1 \leq m \leq k \leq n,
$$

or, equivalently,

$$
\sum_{i=1}^{m} \sum_{j=m}^{n} b_{ij} \leq \sum_{i=1}^{m} \sum_{j=k+1}^{n} c_{ij} \quad \text{for all } 1 \leq m \leq k \leq n.
$$

Next, (3.6) implies that (3.0) holds if and only if

$$
\sum_{i=1}^{m} \sum_{j=m}^{k} b_{ij} \geq \sum_{i=1}^{m} \sum_{j=m}^{k} c_{ij} \quad \text{or} \quad \sum_{j=m}^{m} y_{m,j-m+1} \geq \sum_{j=m}^{m} z_{m,j-m+1}
$$

for all $1 \leq m \leq k \leq n$. This is equivalent to (2.2), so we conclude that $Y \leq_g Z$ if and only if $Y \leq_e Z$.

\[\square\]

4. Dimensions of Orbits

There is an explicit formula for the dimension of any orbit in $E(w)$ going back to [L] [6], in terms of $B(\leq_g w)$. (Like the description of $\leq_g$ given in Section 3, the formula requires enumerating the positive roots based on the choice of an adapted reduced expression for $w_0$.) In this section, we obtain a new dimension formula in terms of $P(w)$. Our formula can probably be deduced combinatorially from Lusztig’s formula [L], but we give a self-contained proof.

**Definition 4.1.** Let $w = (w_1, \ldots, w_n) \in \mathbb{Z}_0^n$ and let $Y \in P(w)$. A kernel flag of type $Y$ is a collection of vector spaces $(V_{ij})_{1 \leq i \leq n, 1 \leq j \leq n-i+1}$ such that

- $0 \subset V_{i1} \subset V_{i2} \subset \cdots \subset V_{in-i+1} = \mathbb{C}^{w_i}$ and $\dim V_{ij} = y_{i1} + y_{i2} + \cdots + y_{ij}$.

A quiver representation $x \in E(w)$ is said to preserve the kernel flag $(V_{ij})$ if

$$
x_i(V_{ij}) \subset \begin{cases} 0 & \text{if } j = 1, \\
V_{i+1,j-1} & \text{if } j > 1.
\end{cases}
$$

This definition implies that if $x$ preserves $(V_{ij})$, then

$$
V_{ij} \subset \ker x_{i+j-1} \cdots x_{i+1}x_i.
$$

(4.1)
This observation is the reason for the name “kernel flag.” The space of all kernel flags of type \( Y \) is denoted by \( \text{FL}_Y \). Note that \( G(w) \) acts transitively on \( \text{FL}_Y \). For any \( V \in \text{FL}_Y \), let \( G(w)V \) be its stabilizer in \( G(w) \). We then have an isomorphism
\[
\text{FL}_Y \cong G(w)/G(w)V.
\]

Next, for any \( V \in \text{FL}_Y \), let
\[
E(w)V = \{ x \in E(w) \mid x \text{ preserves the kernel flag } V \}.
\]
Then let \( \tilde{E}_Y \) be the space of pairs
\[
\tilde{E}_Y = \{ (V, x) \in \text{FL}_Y \times E(w) \mid x \in E(w)V \}.
\]
This space is a vector bundle over \( \text{FL}_Y \), with fibers isomorphic to \( E(w)V \) for any \( V \in \text{FL}_Y \). In particular, \( \tilde{E}_Y \) is a smooth, irreducible variety. We denote by
\[
\pi_Y : \tilde{E}_Y \to E(w)
\]
the projection map onto the second factor. This map is proper. Finally, for another description of \( E_Y \), choose a point \( V \in \text{FL}_Y \). Then there is an isomorphism
\[
G(w) \times G(w)V E(w)V \cong \tilde{E}_Y
\]
given by \( (g, x) \mapsto (gV, g \cdot x) \).

**Lemma 4.2.** Let \( Y \in \mathbf{P}(w) \), and let \( V \in \text{FL}_Y \). Then we have
\[
\dim \text{FL}_Y = \sum_{1 \leq j \leq n-1} \sum_{1 \leq k \leq n-j-1} y_{ij}y_{ik} \quad \text{and} \quad \dim E(w)V = \sum_{1 \leq j \leq n-1} \sum_{1 \leq k \leq n-j-1} y_{ij}y_{ik}.
\]

**Proof.** Let us first compute \( \dim \text{FL}_Y \). We begin by recalling that
\[
\dim GL(w_i) = w_i^2 = (y_{i1} + \cdots + y_{i,n-i+1})^2 = \sum_{1 \leq j \leq n-i+1} y_{ij}y_{ik}.
\]
Consider the point \( V = (V_{ij}) \in \text{FL}_Y \). For each \( i \), let \( GL(w_i)V^* \) denote the stabilizer of the partial flag \( 0 \subset V_{ij} \subset V_{i,j+1} \subset \cdots \subset V_{i,n-1} = \mathbb{C}^{w_i} \). Then \( G(w)V \) is the product of the various \( GL(w_i)V^* \). Let us compute the dimension of the latter. Choose a splitting of the flag, i.e., a vector space isomorphism
\[
\mathbb{C}^{w_i} = V_{i1} \oplus (V_{i2}/V_{i1}) \oplus \cdots \oplus (V_{i,n-i+1}/V_{i,n-i}).
\]
Note that \( \dim V_{ij}/V_{i,j-1} = y_{ij} \). We have
\[
GL(w_i)V^* \cong \prod_{j=1}^{n-i+1} GL(V_{ij}/V_{i,j-1}) \times \prod_{1 \leq k \leq j \leq n-i+1} \text{Hom}(V_{ik}/V_{i,k-1}, V_{ij}/V_{i,j-1}),
\]
where we use the convention that if \( j = 1 \), then \( V_{i,j-1} = 0 \). Therefore,
\[
\dim GL(w_i)V^* = \sum_{j=1}^{n-i+1} y_{ij}^2 + \sum_{1 \leq k \leq j \leq n-i+1} y_{ij}y_{ik} = \sum_{1 \leq k \leq j \leq n-i+1} y_{ij}y_{ik}.
\]
We are now ready to compute the dimension of $F_Y$. We have
\[
\dim F_Y = \dim G(w) - \dim (G(w))^V = \sum_{i=1}^{n} (\dim GL(w_i) - \dim GL(w_i)^{V_w}) = \sum_{i=1}^{n} \sum_{1 \leq j, k \leq n-i+1} y_{ij} y_{ik} - \sum_{1 \leq k \leq j \leq n-i+1} y_{ij} y_{ik},
\]
as desired.

Next, for $x = (x_i) \in E(w)^V$, we must have
\[
ex_i = \prod_{k=2}^{n-i+1} \text{Hom}(V_{ik}/V_{i,k-1}, V_{i+1,k-1}).
\]
The dimension of the space on the right-hand side above is
\[
\sum_{k=2}^{n-i+1} y_{ik}(y_{i+1,1} + y_{i+1,2} + \cdots + y_{i+1,k-1}) = \sum_{1 \leq j < k \leq n-i+1} y_{i+1,j} y_{ik}.
\]
The dimension of $E(w)^V$ is the sum of these quantities over all $i$. \[\square\]

**Lemma 4.3.** There is an open subset $U \subset \tilde{E}_Y$ such that $\pi_Y$ restricts to a bijection $U \to \mathcal{O}_Y$.

**Proof.** Choose a point $V = (V_{ij}) \in F_Y$. Let $U_V \subset E(w)^V$ be the subset consisting of elements $x \in E(w)^V$ such that when $j > 1$, the map of quotient spaces
\[
V_{ij}/V_{i,j-1} \to V_{i+1,j-1}/V_{i+1,j-2}
\]
induced by $x_i$ is injective. Note that $U_V$ is an open subset: with an appropriate choice of bases, the injectivity of these induced maps is equivalent to the nonvanishing of certain minors of the matrix for $x_i$. The quotient map $q : G(w) \times E(w)^V \to G(w) \times E(w)^V, E(w)^V \cong \tilde{E}_Y$ is an open map, so the set $U = q(G(w) \times U_V)$ is open.

Let $x \in \mathcal{O}_Y$. We will show that $\pi_Y^{-1}(x)$ consists of a single point, and that point lies in $U$. Choose a Jordan basis $\{u_{ij}^{(k)}\}$ for $M(x)$. Comparing (2.6) with (4.1), we see that there is a unique kernel flag of type $Y$ preserved by $x$: namely,
\[
V_{ij} = \ker x_{i+1,j-1} \cdots x_{i+1,x_i}.
\]
In other words, $\pi_Y^{-1}(x)$ consists of a single point. The quotient space $V_{ij}/V_{i,j-1}$ can then be identified with the span of $\{u_{ij}^{(k)} | 1 \leq k \leq y_{ij}\}$, so (2.5) shows us that the induced map $V_{ij}/V_{i,j-1} \to V_{i+1,j-1}/V_{i+1,j-2}$ is injective. Thus, the point $((V_{ij}), x)$ belongs to $U$.

For the opposite direction, we start with a point $((V_{ij}), x) \in U$. We will prove that $x \in \mathcal{O}_Y$. We will construct a certain basis $\{u_{ij}^{(k)}\}_{1 \leq j \leq n-i+1, 1 \leq k \leq y_{ij}}$ for $\mathbb{C}^{w_i}$ with the property that for any $m \leq n - i + 1$,
\[
\{u_{ij}^{(k)} | 1 \leq j \leq m, 1 \leq k \leq y_{ij}\}
\]
is a basis for $V_{im} \subset \mathbb{C}^{w_i}$.

We proceed by induction on $i$. For $i = 1$, choose any basis $\{u_{ij}^{(k)}\}$ satisfying (4.2). For $i > 1$, define
\[
u_{ij}^{(k)} = x(u_{i-1,j+1}^{(k)}) \quad \text{if} \quad 1 \leq k \leq y_{i-1,j+1}.
\]
Assume first that \( q \) is a ladder. Since the image of size \( n \) is irreducible, its image must therefore be contained in the closure of \( \pi \). Similarly, if we instead assume that \( \pi \) is a ladder, its image must therefore be contained in the closure of \( \pi \).

Corollary 4.4. We have \( \dim \overline{E_Y} = \dim \mathcal{O}_Y \).

Corollary 4.5. The image of \( \pi_Y : \overline{E_Y} \to E(w) \) is \( \overline{\mathcal{O}_Y} \).

Proof. Since \( \overline{E_Y} \) is irreducible, it is the closure of the open set \( U \) that was introduced in Lemma 4.3. Its image must therefore be contained in the closure of \( \pi_Y(\overline{E_Y}) = \overline{\mathcal{O}_Y} \). Since \( \pi_Y \) is proper, its image is closed, so its image is precisely \( \overline{\mathcal{O}_Y} \).

Combining the preceding results, we obtain the following dimension formula.

Theorem 4.6. For any \( Y \in \mathcal{P}(w) \), we have

\[
\dim \mathcal{O}_Y = \sum_{1 \leq i \leq n-1, 1 \leq j \leq n-i+1} y_{ij} y_{ik} + \sum_{1 \leq i \leq n-1} y_{i+1,j} y_{ik}.
\]

5. Operations on triangular arrays

This section is the “combinatorial heart” of the paper. We describe a number of constructions one can carry out using triangular arrays, culminating in the definitions of two maps \( T, T' : \mathcal{P}(w) \to \mathcal{P}(w^*) \). The main result of this section states that \( T \) and \( T' \) are both bijections, inverse to one another. (In Section 3 we will learn that \( T \) and \( T' \) are actually the same map, but the proof of this is not combinatorial.)

5.1. Elementary operations on triangular arrays. Consider a triangular array \( Y = (y_{ij})_{1 \leq i \leq n, 1 \leq j \leq n-i+1} \) of size \( n \). We define \( \text{Del}^\prec(Y) \) to be the triangular array of size \( n-1 \) obtained from \( Y \) by deleting the first chute. In other words,

\[
\text{Del}^\prec(Y)_{ij} = y_{i+1,j} \quad \text{for } 1 \leq i \leq n-1, 1 \leq j \leq n-i.
\]

Similarly, \( \text{Del}_\succ(Y) \) is the triangular array of size \( n-1 \) obtained by deleting the last ladder:

\[
\text{Del}_\succ(Y)_{ij} = y_{ij} \quad \text{for } 1 \leq i \leq n-1, 1 \leq j \leq n-i.
\]

On the other hand, let \( Q = (q_1, \ldots, q_{n+1}) \) be a list of \( n+1 \) nonnegative integers. Assume first that \( q_j \geq y_{1,j-1} \) for \( 2 \leq j \leq n+1 \). Let \( Y \cup \succ Q \) be the triangular array of size \( n+1 \) obtained from \( Y \) by making \( Q \) the new topmost chute. In other words,

\[
(Y \cup \succ Q)_{ij} = \begin{cases} q_j & \text{if } i = 1, 1 \leq j \leq n + 1, \\ y_{i-1,j} & \text{if } 2 \leq i \leq n + 1, 1 \leq j \leq n - i + 2. \end{cases}
\]

Similarly, if we instead assume that \( q_1 \geq q_2 \geq \cdots \geq q_{n+1} \), then we can define a new triangular array \( Y \cup \prec Q \) by adjoining \( Q \) as the new bottommost ladder. Explicitly, we put

\[
(Y \cup \prec Q)_{ij} = \begin{cases} y_{ij} & \text{if } 1 \leq i \leq n \text{ and } 1 \leq j \leq n - i + 1, \\ q_{n-i+2} & \text{if } 1 \leq i \leq n + 1 \text{ and } j = n - i + 2. \end{cases}
\]
Let \( \text{Top}(Y) \) denote the topmost chute of \( Y \), regarded as an element of \( \mathbb{Z}^n \):
\[
\text{Top}(Y) = (y_{11}, y_{12}, \ldots, y_{1n}).
\]

Note that \( Y = \text{Del}^\triangledown(Y) \cup \setminus \text{Top}(Y) \).

Next, we define \( \text{Raise}(Y, i, j) \) and \( \text{Lower}(Y, i, j) \) to be the triangular arrays obtained from \( Y \) by replacing the entry in chute \( i \), column \( j \) by \( y_{ij} + 1 \) and by \( y_{ij} - 1 \), respectively. There is a well-definedness issue here: because ladders are required to be weakly decreasing, \( \text{Raise}(Y, i, j) \) only makes sense if \( j = 1 \) or if \( y_{ij} < y_{i+1,j-1} \). Similarly, for \( \text{Lower}(Y, i, j) \) to make sense, we must either have \( i = 1 \) and \( y_{ij} > 0 \), or else \( i > 1 \) and \( y_{ij} > y_{i-1,j+1} \). When they make sense, it is clear from the definitions that
\[
\dim(\text{Raise}(Y, i, j)) = \dim(Y) + \mathbf{e}_i \quad \text{and} \quad \dim(\text{Lower}(Y, i, j)) = \dim(Y) - \mathbf{e}_i.
\]

5.2. **Invariants of triangular arrays.** In this subsection, we define various integer-valued functions on triangular arrays that will be used in the definitions of the algorithms below. As above, let \( Y \) be a triangular array of size \( n \). Let \( k \) be an integer with \( 1 \leq k \leq n \). Let
\[
\mathcal{I}(Y, k) = \begin{cases} 
\text{the smallest integer } j \geq k \text{ such that } y_{1j} > 0, \text{ or} \\
\infty, \text{ if there is no such } j.
\end{cases}
\]

Next, let
\[
\mathcal{J}(Y, k) = \begin{cases} 
\text{the smallest integer } j > \mathcal{I}(Y, k) \text{ such that } y_{1j} < y_{2,j-1}, \\
\infty, \text{ if } 1 < \mathcal{I}(Y, k) < \infty, \text{ or} \\
\infty, \text{ if there is no } j \text{ as in the previous case, or if } \mathcal{I}(Y, k) = \infty.
\end{cases}
\]

In other words, if \( \mathcal{J}(Y, k) < \infty \), then it is the smallest integer \( \geq \mathcal{I}(Y, k) \) such that \( \text{Raise}(Y, 1, \mathcal{J}(Y, k)) \) is defined. In particular, we always have \( \mathcal{J}(Y, k) > 1 \).

Finally, suppose \( 1 \leq i \leq n - k + 1 \). Let
\[
\mathcal{K}_i(Y, k) = \max\left(\{1\} \cup \{j \mid 2 \leq j \leq k \text{ and } y_{ij} < y_{i+1,j-1}\}\right).
\]

In other words, \( \mathcal{K}_i(Y, k) \) is the largest integer \( \leq k \) such that \( \text{Raise}(Y, i, \mathcal{K}_i(Y, k)) \) is defined. It is immediate from the definition that if \( 2 \leq i \leq n - k + 1 \), then we have
\[
\mathcal{K}_i(Y, k) = \mathcal{K}_{i-1}(\text{Del}^\triangledown(Y), k).
\]

5.3. **Advanced operations on triangular arrays.** We will now introduce several more complicated operations on triangular arrays, and we prove a few lemmas about them.

**Procedure a.** This operation takes as input a triple \( (Y, i, k) \) where \( Y \) is a triangular array of size \( n \); \( i \) is an integer such that \( 1 \leq i \leq n \); and \( k \) is an integer such that \( 1 \leq k \leq n - i + 1 \). Its output is also a triple consisting of a triangular array and two integers. It is defined by
\[
a(Y, i, k) = (\text{Raise}(Y, i, \mathcal{K}_i(Y, k)), i - 1, \mathcal{K}_i(Y, k)).
\]

Note that as long as \( i > 1 \), the output of \( a \) satisfies the conditions required of its input, so it makes sense to apply \( a \) repeatedly.

When \( i > 1 \), we can study how \( a \) interacts with \( \text{Del}^\triangledown \) using (5.1). Suppose
\[
a(Y, i, k) = (X, i - 1, k') \quad \text{and} \quad a(\text{Del}^\triangledown(Y), i - 1, k) = (X', i - 2, k'').
\]
These are related by

\[(5.2) \quad X = X' \cup^\top \text{Top}(Y) \quad \text{and} \quad k' = k''. \]

**Procedure A**. This operation takes as input a triangular array \(Y\) of size \(n\), where \(1 \leq i \leq n\). Its output it also a triangular array of size \(n\). Apply procedure \(a\) \(i\) times to the triple \((Y, i, n)\): the result has the form

\[
\underbrace{a \circ \ldots \circ a}_{\text{i times}}(Y, i, n - i + 1) = (X, 0, k).
\]

We define \(A_i(Y) = X\). Since this sequence of \(a\)'s performs one \(\text{Raise}\) on each of the first \(i\) chutes, we see that

\[(5.3) \quad \text{dim}(A_i(Y)) = \text{dim}(Y) + e_1 + e_2 + \cdots + e_i. \]

**Procedure B**. This operation takes as input a pair \((Y, k)\), where \(Y\) is a triangular array of size \(n\); \(k\) is an integer such that \(1 \leq k \leq n\); and, moreover, we have \(I(Y, k) < \infty\). Its output is again a pair consisting of a triangular array and an integer (not necessarily satisfying any condition with respect to \(I\)). The definition is by induction on \(n\). If \(n = 1\), we necessarily have \(k = 1\). In this case, we put

\[B(Y, 1) = (\text{Lower}(Y, 1, 1), 1).\]

(The assumption that \(I(Y, 1) < \infty\) implies that this use of \(\text{Lower}\) makes sense.)

Suppose now that \(n > 1\), and that \(B\) is already defined for smaller diagrams. If \(J(Y, k) = \infty\), we simply put

\[B(Y, k) = (\text{Lower}(Y, 1, I(Y, k)), 1).\]

On the other hand, if \(J(Y, k) < \infty\), let \(j_0 = J(Y, k)\). Our assumption implies that \(\text{Raise}(Y, 1, j_0)\) makes sense, so \(y_1j_0 < y_2j_0 - 1\). In particular, \(y_2j_0 - 1 \neq 0\), and hence \(I(\text{Del}^<(Y), j_0 - 1) < \infty\). By induction, \(B(\text{Del}^<(Y), j_0 - 1)\) is already defined; let \((Z, r) = B(\text{Del}^<(Y), j_0 - 1)\). Finally, set

\[B(Y, k) = (\text{Lower}(Z \cup^\top \text{Top}(Y), 1, I(Y, k)), r + 1).\]

This completes the definition of \(B\). Note that the definition for \(n = 1\) is a special case of the definition in the case where \(J(Y, k) = \infty\).

**Lemma 5.1.** Suppose that \(I(Y, k) < \infty\), and let \((Y', q) = B(Y, k)\). Then \(\text{dim}(Y') = \text{dim}(Y) - (e_1 + e_2 + \cdots + e_q)\).

**Proof.** We proceed by induction on \(n\). If \(n = 1\), or if \(J(Y, k) = \infty\), we have \(q = 1\), and \(Y'\) is given by a \(\text{Lower}\). The claim is clear in this case.

If \(J(Y, k) < \infty\), suppose \(\text{dim}(Y) = (w_1, \ldots, w_n)\). Then \(\text{dim}(\text{Del}^<(Y)) = (w_2, \ldots, w_n)\). Let \((Z, r)\) be as in the definition of \(B\) above. By induction, \(\text{dim}(Z) = (w_2 - 1, w_3 - 1, \ldots, w_q - 1, w_{q+1}, \ldots, w_n)\). Then \(Y' = \text{Lower}(Z \cup^\top \text{Top}(Y), 1, I(Y, k))\) has the dimension vector claimed in the lemma. \(\square\)

**Lemma 5.2.** Let \((Y', q_1) = B(Y, k)\), and let \((Y'', q_2) = B(Y', k')\) for some \(k' \geq k\). Then \(q_1 \geq q_2\).

**Proof.** If \(J(Y, k) = \infty\), then \(q_1 = q_2 = 1\), and the lemma is verified. Now assume that \(J(Y, k)\) exists. Certainly

\[(5.4) \quad \text{Del}^<(Y'), q_1 - 1 = B(\text{Del}^<(Y'), J(Y, k) - 1).\]
Applying this to \((Y', k')\) yields
\[
(5.5) \quad (\text{Del} \setminus (Y''), q_2 - 1) = B(\text{Del} \setminus (Y'), J(Y', k') - 1).
\]
Note that \(J(Y', k') \geq J(Y, k)\) so that \((5.4)\) and \((5.5)\) match the statement of the lemma for smaller triangles. By induction, \(q_1 - 1 \geq q_2 - 1\), and we are done. \(\square\)

**Lemma 5.3.** Suppose that \(I(Y, k) < \infty\), and let \((Y', q) = B(Y, k)\). Then we have
\[
a^{q}(Y', q, n - q + 1) = (Y, 0, I(Y, k)).
\]
In particular, we have \(A_{q}(Y') = Y\).

**Proof.** We proceed by induction on the size \(n\) of the triangular array. Throughout the proof, we let \(j_0 = I(Y, k)\).

Suppose first that \(J(Y, k) = \infty\). (This includes the special case where \(n = 1\).) Recall that \(j_0\) is the smallest integer \(\geq k\) such that \(y_{1,j_0} \neq 0\). Moreover, if \(j > j_0\), then \(\text{Raise}(Y, 1, j)\) is not defined. From the definition of \(B\), we have \(q = 1\) and \(Y' = \text{Lower}(Y, 1, j_0)\). Since \(Y\) and \(Y'\) differ only at the entries at position \(ij\), we see that \(\text{Raise}(Y', 1, j)\) is also not defined for \(j > j_0\). On the other hand, \(\text{Raise}(Y', 1, j_0)\) clearly is defined: it is equal to \(Y\). We have just shown that \(K_1(Y', n) = j_0\). As a consequence, we have
\[
a(Y', 1, n) = (\text{Raise}(Y', 1, K_1(Y', n)), 0, K_1(Y', n))
\]
\[
= (\text{Raise}(Y', 1, j_0), 0, j_0) = (Y, 0, I(Y, k)),
\]
as desired.

Now suppose that \(J(Y, k) < \infty\), and let \(j_1 = J(Y, k)\). From the definition of \(B\), we see that \(B(\text{Del} \setminus (Y'), j_1 - 1)\) is of the form \((Z, q - 1)\) for some triangular array \(Z\) of size \(n - 1\). By induction, we have
\[
a^{q-1}(Z, q - 1, n - q + 1) = (\text{Del} \setminus (Y'), 0, I(\text{Del} \setminus (Y), j_1 - 1)).
\]
Recall from the definition of \(B\) that
\[
(5.6) \quad Y' = \text{Lower}(Z \cup \setminus \text{Top}(Y'), 1, j_0).
\]
In particular, we have \(\text{Del} \setminus (Y') = Z\). Applying \((5.6)\) \(q - 1\) times, we obtain
\[
a^{q-1}(Y', q, n - q + 1) = (\text{Del} \setminus (Y') \cup \setminus \text{Top}(Y'), 1, I(\text{Del} \setminus (Y), j_1 - 1)).
\]
To finish the proof of the lemma, we must show that if we apply a one more time to this equation, the result is \((Y, 0, j_0)\). Let \(Y'' = \text{Del} \setminus (Y') \cup \setminus \text{Top}(Y')\). It follows from \((5.6)\) that \(Y = \text{Raise}(Y'', 1, j_0)\), so to complete the proof, it is enough to show that \(K_1(Y'', j_1 - 1) = j_0\). Denote the entries of \(Y''\) by \(y''_{ij}\). We have
\[
y''_{1,j_0} = y_{1,j_0} - 1 < y_{2,j_0 - 1} = y''_{2,j_0 - 1},
\]
\[
y''_{1,j} = y_{1,j} = y_{2,j} - 1 = y''_{2,j} \quad \text{for} \quad j_0 < j \leq j_1 - 1,
\]
where the latter holds by the definition of \(J(Y, k)\). These two conditions together tell us that \(K_1(Y'', j_1 - 1) = j_0\), as desired. \(\square\)

**5.4. The combinatorial Fourier transform and its inverse.** We are now ready to define the main combinatorial algorithms in the paper. Let \(Y\) be a triangular array of size \(n\). We will define the *combinatorial Fourier transform of \(Y\)*, denoted by \(T(Y)\), by induction on \(n\). If \(n = 1\), we set \(T(Y) = Y\). Otherwise, we set
\[
T(Y) = A_{y_{n}}^{n,n-1}A_{y_{n-1}}^{n-1,n-2} \ldots A_{y_{1}}^{y_{1},y_{1}-1,2}(T(\text{Del} \setminus (Y)) \cup \setminus (0, \ldots, 0))
\]
Note that \(T(Y)\) is again a triangular array of size \(n\).
We will also define the inverse combinatorial Fourier transform of $Y$, denoted by $T'(Y)$, by induction on $n$. If $n = 1$, we again set $T'(Y) = Y$. Suppose now that $n > 1$, and let $(w_1, \ldots, w_n) = \dim(Y)$. Recall that $B(Y, 1)$ is defined as long as the top chute of $Y$ is nonzero, or equivalently, if $w_1 > 0$. If this is the case, set $(Y', q_1) = B(Y, 1)$. Recall from Lemma 5.3 that the first coordinate of $\dim(Y')$ is $w_1 - 1$. If this is still positive, we can apply $B$ again. In general, we produce a sequence as follows:

$$(Y', q_1) = B(Y, 1),$$

$$(Y'', q_2) = B(Y', 1),$$

\vdots

$$(Y^{(w_1)}, q_{w_1}) = B(Y^{(w_1-1)}, 1).$$

(The top chute of $Y^{(w_1)}$ is zero, so we cannot apply $B$ again.) Define a list of integers $P = (p_1, \ldots, p_n)$ by

$$p_j = \#\{k \mid q_k \geq j\}.$$

(Since $1 \leq q_k \leq n$ for all $k$, we have $p_1 = w_1$.) Finally, we define $T'(Y)$ by

$$T'(Y) = T'(\Del^{\cdot}_{T}(Y^{(w_1)})) \cup_{\cdot} P.$$  

The terminology is justified by Theorem 5.5 below.

**Lemma 5.4.** If $\dim(Y) = w$, then $\dim(T(Y)) = \dim(T'(Y)) = w^*$. 

**Proof.** Let us first prove the statement for $T$. We proceed by induction on the size $n$ of the triangular array involved. If $n = 1$, the statement is obvious. Otherwise, suppose $w = (w_1, \ldots, w_n)$, and let $w' = \dim(\Del_{T}(Y))$. Writing $w' = (w'_1, \ldots, w'_{n-1})$, we clearly have

$$w'_i = w_i - y_{i,n-i+1}.$$ 

By induction, $\dim(T(\Del_{T}(Y))) = (w')^*$, so

$$\dim(T(\Del_{T}(Y)) \cup^{\cdot}_{\cdot} (0, \ldots, 0)) = (0, w'_{n-1} - y_{n-1,2}, \ldots, w_2 - y_{2,n-1}, w_1 - y_{1,n}).$$

Next, (5.3) implies that

$$\dim A^{y_{n,1}}_{n-1} A^{y_{n-1,2}-y_{1,1}}_{n-2} \cdots A^{y_{1,2}-y_{1,1}} \dim(T(\Del_{T}(Y)) \cup^{\cdot}_{\cdot} (0, \ldots, 0))$$

$$= \dim(T(\Del_{T}(Y)) \cup^{\cdot}_{\cdot} (0, \ldots, 0)) + \sum_{i=1}^{n} (y_{n-i+1} - y_{i-1,n-i+2})(e_1 + e_2 + \cdots + e_{n-i+1}).$$

(On the right-hand side, $y_{0,n+1}$ should be understood to be 0.) The coefficient of $e_k$ is $\sum_{i=k+1}^{n} (y_{i,n-i+1} - y_{i-1,n-i+2}) = y_{n-k+1,k}$. Using the fact that $y_{n,1} = w_n$, we conclude that

$$\dim T(Y) = \dim(T(\Del_{T}(Y)) \cup^{\cdot}_{\cdot} (0, \ldots, 0)) + \sum_{k=1}^{n} y_{n-k+1,k} e_k$$

$$= \dim(T(\Del_{T}(Y)) \cup^{\cdot}_{\cdot} (0, \ldots, 0)) + (y_{n,1}, y_{n-1,2}, \ldots, y_{1,n})$$

$$= (w_n, w_{n-1}, \ldots, w_1) = w^*.$$
For $T'$, we again proceed by induction on $n$. Consider the triangular arrays $Y', Y'', \ldots, Y^{(w_1)}$ as in the definition of $T'$. Applying Lemma 5.1 $w_1$ times, we see that
\[
\dim(Y^{(w_1)}) = \dim(Y) - \sum_{k=1}^{w_1} (e_1 + \cdots + e_k).
\]
The coefficient of $e_j$ is $\#\{k \mid q_k \geq j\} = p_j$, so
\[
\dim(Y^{(w_1)}) = w - P = (0, w_2 - p_2, w_3 - p_3, \ldots, w_n - p_n).
\]
By induction, we have $\dim(T'(\text{Del} \wedge (Y^{(w_1)}))) = (w_n - p_n, w_{n-1} - p_{n-1}, \ldots, w_2 - p_2)$, and then
\[
\dim(T'(Y)) = \dim(T'(\text{Del} \wedge (Y^{(w_1)}))) + (p_n, p_{n-1}, \ldots, p_1)
\]
\[
= (w_n, w_{n-1}, \ldots, w_1) = w^*,
\]
as desired. $\Box$

The previous lemma tells us that both $T$ and $T'$ can be regarded as maps from $P(w)$ to $P(w^*)$, or vice versa.

**Theorem 5.5.** Let $w \in \mathbb{Z}^*_\geq 0$. The maps $T : P(w) \to P(w^*)$ and $T' : P(w^*) \to P(w)$ are both bijections, and they are inverse to one another.

**Proof.** We begin by showing that $T \circ T'$ is the identity map on $P(w^*)$. We proceed by induction on $n$. If $n = 1$, the claim is obvious. Otherwise, let $w = (w_1, \ldots, w_n)$. Let $Y \in P(w^*)$, and let $Y', Y'', \ldots, Y^{(w_n)}$ be as in the definition of $T'$. By Lemma 5.3 we have
\[
Y = A_{q_1}A_{q_2} \cdots A_{q_{w_n}} (Y^{(w_n)}).
\]
Next, Lemma 5.2 tells us that $q_1 \geq q_2 \geq \cdots \geq q_{w_n}$. So the preceding equation can be rewritten as
\[
Y = A_n^{p_n}A_{n-1}^{p_{n-1}} \cdots A_2^{p_2-p_0}A_1^{p_1-p_0} (Y^{(w_n)}).
\]
Now, the top chute of $Y^{(w_n)}$ is zero, so $Y^{(w_n)} = \text{Del} \wedge (Y^{(w_n)}) \cup \rho (0, \ldots, 0)$. Since $\text{Del} \wedge (Y^{(w_n)})$ is a triangular array of smaller size, by induction, we have
\[
\text{Del} \wedge (Y^{(w_n)}) = T(T'(\text{Del} \wedge (Y^{(w_n)}))).
\]
Next, from the definition of $T'$, we see that
\[
T'(\text{Del} \wedge (Y^{(w_n)})) = \text{Del} \wedge (T'(Y)).
\]
Combining these observations, we find that
\[
Y^{(w_n)} = \text{Del} \wedge (Y^{(w_n)}) \cup \rho (0, \ldots, 0) = T(T'(\text{Del} \wedge (Y^{(w_n)}))) \cup \rho (0, \ldots, 0)
\]
\[
= T(\text{Del} \wedge (T'(Y))) \cup \rho (0, \ldots, 0).
\]
Finally, we substitute this into (5.7) to obtain
\[
Y = A_n^{p_n}A_{n-1}^{p_{n-1}} \cdots A_2^{p_2-p_0}A_1^{p_1-p_0} (T(\text{Del} \wedge (T'(Y))) \cup \rho (0, \ldots, 0)).
\]
Since the numbers $p_1, p_2, \ldots, p_n$ are precisely those on the bottom ladder of $T'(Y)$, this formula says that $Y = T(T'(Y))$, as desired.

We now know that $T \circ T'$ is the identity map. In particular, $T'$ is injective, and $T$ is surjective. Since the finite sets $P(w)$ and $P(w^*)$ have the same cardinality (Corollary 2.2), we conclude that $T$ and $T'$ are both bijections. $\Box$
6. FOURIER–SATO TRANSFORMS

In this section, we prove the main result of the paper: both \( T \) and \( T' \) compute the Fourier–Sato transforms of simple perverse sheaves on \( E(w) \).

6.1. Fourier–Sato transform. Let \( w \in \mathbb{Z}^n_{\geq 0} \). There is an obvious isomorphism \( G(w) \cong G(w^*) \), given by \((g_1, g_2, \ldots, g_n) \mapsto (g_n, g_{n-1}, \ldots, g_1)\). In this section, we will identify these groups via this isomorphism.

Consider the pairing \( \langle -, - \rangle : E(w) \times E(w^*) \to \mathbb{C} \) defined as follows: for \( x = (x_i)_{1 \leq i \leq n-1} \in E(w) \) and \( y = (y_i)_{1 \leq i \leq n-1} \in E(w^*) \), we put
\[
\langle x, y \rangle = \sum_{i=1}^{n-1} \text{tr}(x_i y_{n-i}).
\]
This pairing is \( G(w) \)-equivariant and nondegenerate, so it identifies \( E(w^*) \) with the dual vector space to \( E(w) \) (as a \( G(w) \)-representation). Following [KS, §3.7], one can define the Fourier–Sato transform, a certain functor \( D^b_{G(w)}(E(w)) \to D^b_{G(w^*)}(E(w^*)) \) denoted in [KS] by \( F \to F^\wedge \). In this paper, it will be more convenient to use the functor
\[
T : D^b_{G(w)}(E(w)) \to D^b_{G(w^*)}(E(w^*))
\]
defined by \( T(F) = (F^\wedge)[\dim E(w)] \). With this additional shift, \( T \) becomes \( t \)-exact for the perverse \( t \)-structure. It is an equivalence of categories (because \( G(w) \)-equivariant sheaves are automatically conic in the sense of [KS, §3.7]), and it is “almost” an involution: its inverse
\[
T' : D^b_{G(w^*)}(E(w^*)) \to D^b_{G(w)}(E(w))
\]
is given by
\[
T'(F) \cong s^* T(F),
\]
where \( s : E(w) \to E(w) \) is the “antipode map” given by \( s(x) = -x \).

6.2. Simple perverse sheaves on \( E(w) \). The following fact is well known. We include the proof since it is so short.

Lemma 6.1. For any point \( x \in E(w) \), the stabilizer of \( x \) in \( G(w) \) is connected.

Proof. Let \( g(w) = \mathfrak{gl}(w_1) \times \cdots \times \mathfrak{gl}(w_n) \), and let this Lie algebra act on \( E(w) \) by
\[
(g_1, \ldots, g_n) \cdot (x_1, \ldots, x_{n-1}) = (g_2 x_1, g_3 x_2, \ldots, g_n x_{n-1}) - (x_1 g_1, x_2 g_2, \ldots, x_{n-1} g_n - 1).
\]
Let \( \mathfrak{z} \) be the stabilizer in \( g(w) \) of a point \( x \in E(w) \). Then \( \mathfrak{z} \) is a vector space. The stabilizer in \( G(w) \) of \( x \) is the Zariski open subset of \( \mathfrak{z} \) consisting of elements with nonzero determinant, so it is connected.

As a consequence, the only \( G(w) \)-equivariant irreducible local system on any \( G(w) \)-orbit is the trivial local system. Every simple \( G(w) \)-equivariant perverse sheaf is therefore of the form \( IC(O_Y) \) for some \( Y \in P(w) \). Given such a perverse sheaf, its Fourier–Sato transform \( T(IC(O_Y)) \) is a \( G(w) \)-equivariant simple perverse sheaf on \( E(w^*) \), so it must be isomorphic to \( IC(O_{Y'}) \) for some \( Y' \in P(w) \). We thus obtain a map
\[
T : P(w) \to P(w^*)
\]
characterized by the property that $T(\text{IC}(O_Y)) \cong \text{IC}(O_{\text{IC}(Y)})$. This is a bijection. Note that the antipode map $s : E(w) \rightarrow E(w)$ preserves every $G(w)$-orbit. It follows that at the combinatorial level, $T$ is an involution:

\begin{equation}
T(T(Y)) = Y \quad \text{for all } Y \in P(w).
\end{equation}

**Lemma 6.2.** Suppose we have a short exact sequence of representations $0 \rightarrow M(x) \rightarrow M(x') \rightarrow M(e_i) \rightarrow 0$. Let $v$ be a vector in $M(x')$ whose image in $M(e_i)$ is nonzero, and let $k$ be the smallest integer such that $(x')^k(v) = 0$. If $x \in O_Y$ and $x' \in O_{Y'}$, then

$Y' = \text{Raise}(Y, i, j)$ \quad \text{for some } j \text{ such that } 1 \leq j \leq K_i(Y, k).

**Proof.** Choose a Jordan basis $\{u^{(r)}_{pq}\}$ for $M(x)$. Write $x'(v)$ in this basis:

$$x'(v) = \sum_{1 \leq q \leq k-1} c_q u^{(r)}_{i+1, q}.$$ 

Here, we can take the sum just over $1 \leq q \leq k-1$ instead of $1 \leq q \leq n-i$ because if some basis element $u^{(r)}_{i+1, q}$ with $q \geq k$ occurred with nonzero coefficient in the expansion of $x'(v)$, it would follow that $(x')^q(v) = x^{q-1}(x(v)) \neq 0$, a contradiction.

Next, we break up this sum according to whether $r \leq y_{i+1, q} + 1$ or $r > y_{i+1, q} + 1$:

$$x'(v) = \sum_{1 \leq q \leq k-1} c_q u^{(r)}_{i+1, q} + \sum_{1 \leq q \leq K_i(Y, k)-1} \sum_{y_{i+1, q} + 1 < r \leq y_{i+1, q}} c_q u^{(r)}_{i+1, q}.$$ 

Here, the second sum is just over $1 \leq q \leq K_i(Y, k) - 1$ rather than $1 \leq q \leq k - 1$ because if $K_i(Y, k) < q + 1 \leq k$, then, by the definition of $K_i(Y, k)$, we must have $y_{i+1, q + 1} = y_{i+1, q}$.

Consider the vector

$$v' = v - \sum_{1 \leq q \leq k-1} c_q u^{(r)}_{i+1, q+1}.$$ 

This vector, like $v$, has nonzero image in $M(e_i)$. Moreover, we have

$$x'(v') = \sum_{1 \leq q \leq K_i(Y, k) - 1} \sum_{y_{i+1, q} + 1 < r \leq y_{i+1, q}} c_q u^{(r)}_{i+1, q}.$$ 

Let $j$ be the largest integer such that some coefficient $c^{(r)}_{j-1}$ with $y_{ij} < r \leq y_{i+1, j-1}$ is nonzero. By relabeling the elements of our Jordan basis, we may assume, in particular, that $c^{(r)}_{j-1} \neq 0$ for $r = y_{ij} + 1$. Then the vectors

\begin{equation}
(x'(v')), (x'(v'))^2(v'), \ldots, (x'(v'))^{j-1}(v')
\end{equation}

are all nonzero, but $(x')^j(v') = 0$. We can now equip $M(x')$ with a basis as follows: starting from the original Jordan basis $\{u^{(r)}_{pq}\}$, delete the vectors

$$u^{(y_{i+1, j-1})}_{i+1, j-1}, u^{(y_{i+1, j-2})}_{i+1, j-2}, \ldots, u^{(y_{i+1, j-1})}_{i+1, j-1},$$

and then add the vectors in $\{0.2\}$. More concisely, we are considering the basis

$$\{u^{(r)}_{pq} \mid p + q = i + j, \text{ then } r \neq y_{ij} + 1 \} \cup \{x'(v'), (x'(v'))^2(v'), \ldots, (x'(v'))^{j-1}(v')\}$$

It is straightforward to see that this is a Jordan basis of type $\text{Raise}(Y, i, j)$, as desired. □
Remark 6.3. In the proof of Lemma 6.2, we constructed a Jordan basis for \( M(x') \), a subset of which constitutes a Jordan basis for \( M(x) \). In other words, \( M(x) \) admits a Jordan basis that extends to a Jordan basis for \( M(x') \).

Theorem 6.4. For any \( Y \in \mathbf{P}(w) \), we have \( T(Y) \cong T(Y) \).

To prove this theorem, we need to recall some general results about algebraic group actions on vector spaces. Let \( H \) be a complex algebraic group acting on a vector space \( V \) with finitely many orbits, and let \( O \subset V \) be an \( H \)-orbit. Then one can consider its conormal bundle \( N^*O \subset V \times V^* \). According to [Z], there is a natural bijection

\[
Z : \{ H\text{-orbits in } V \} \to \{ H\text{-orbits in } V^* \}
\]
determined by the condition that \( N^*(Z(O)) = N^*O \subset V \times V^* \). Next, according to [EM] Proposition 7.2, this bijection coincides with the one induced by Fourier transform:

\[
(6.3) \quad T(IC(O)) \cong IC(Z(O)).
\]

On the other hand, for quiver representations, there is another description of the bijection \( Z \), due to Zelevinsky. Consider a pair of quiver representations \( (x,y) \in E(w) \times E(w^*) \). We can draw this pair as

\[
\begin{array}{ccc}
\mathbb{C}^{w_1} & \xrightarrow{x_1} & \mathbb{C}^{w_2} & \xrightarrow{x_2} & \cdots & \xrightarrow{x_{n-1}} & \mathbb{C}^{w_n}
\end{array}
\]

We say that \( x \) and \( y \) commute if

\[
x_iy_{n-i} + y_{n-i}x_{i+1} = 0 \quad \text{for } i = 0,1,\ldots,n-1.
\]

(To make sense of this equation for \( i = 0 \) or \( i = n-1 \), we adopt the convention that \( x_0y_n = 0 \) and \( y_0x_n = 0 \).) For any \( x \in E(w) \), let

\[
C(x) = \{ y \in E(w^*) \mid x \text{ and } y \text{ commute} \}.
\]

Of course, \( C(x) \) is a linear subspace of \( E(w^*) \). There is a unique orbit \( O \subset E(w^*) \) such that \( O \cap C(x) \) is dense in \( C(x) \).

Proposition 6.5 (Zelevinsky). Let \( Y \in \mathbf{P}(w) \) and \( Y' \in \mathbf{P}(w^*) \). Then \( Z(O_Y) = O_{Y'} \) if and only if for any point \( x \in O_Y \), the set \( O_{Y'} \cap C(x) \) is an open dense subset of \( C(x) \) in the Zariski topology.

For a proof, see [Z] Proposition 4.4.

Proof of Theorem 6.4. Suppose \( w = (w_1,\ldots,w_n) \). We will prove this latter statement by a double induction on \( n \) and on \( w_n \).

If \( n = 1 \), then \( \mathbf{P}(w) \) and \( \mathbf{P}(w^*) \) each consist of a single element, and the claim is obvious. Suppose from now on that \( n > 1 \). If \( w_n = 0 \), then the space \( E(w) \) can be identified with \( E(w_1,\ldots,w_{n-1}) \). In this case, the theorem holds because it reduces to the claim for triangular arrays of size \( n-1 \).

Suppose now that \( n > 1 \) and \( w_n > 0 \). Given \( Y \in \mathbf{P}(w) \), let \( i_0 \) be the smallest integer such that \( y_{i_0,n+1-i_0} \neq 0 \). (Since \( w_n > 0 \), some such \( i_0 \) exists.) Define a triangular array \( Y' \) by

\[
Y'_{ij} = \begin{cases} y_{ij} & \text{if } j < n+1-i, \text{ or if } i < i_0, \\ y_{ij} - 1 & \text{if } j = n+1-i \text{ and } i \geq i_0. \end{cases}
\]
Thus, $Y'$ differs from $Y$ only in the last ladder, and we have
\[
y_{i,n+1-i} - y'_{i-1,n+2-i} = \begin{cases} 
0 = y_{i,n+1-i} - y_{i-1,n+2-i} & \text{if } i < i_0, \\
y_{i,n+1-i} - y_{i-1,n+2-i} - 1 & \text{if } i = i_0, \\
y_{i,n+1-i} - y_{i-1,n+2-i} & \text{if } i > i_0.
\end{cases}
\]
From the formula in Section 5.4 we see that
\[
T(Y) = A_{n+1-i_0} T(Y').
\]
Choose a point $x \in O_Y$, and choose a Jordan basis $\{u_{ij}^{(k)}\}$ for $M(x)$. Let $\mathbf{w}' = \dim(Y')$, and identify $\mathbb{C}^w$ with the span of
\[
\{u_{ij}^{(k)} \mid j = n+1-i, \text{then } j \geq 2 \}.
\]
This subspace is clearly preserved by $x$. Let $x' = x|_{\mathbb{C}^w}$. Then $x'$ is of type $Y'$, and the basis above is a Jordan basis for it. By induction and (6.3), we have $T(Y') = Z(Y') = T(Y')$. By Proposition 5.5 we have (6.4)
\[
\mathcal{O}_{T(Y')} \cap C(x') \neq \emptyset.
\]
The remainder of the proof is broken up into several steps.

**Step 1.** The map $p : C(x) \to C(x')$ given by restricting to $\mathbb{C}^w'$ is surjective. Given $\bar{x}' \in C(x')$, we must show how to extend it to a representation $\bar{x}$ on $\mathbb{C}^w$ that commutes with $x$. To define $\bar{x}$, we must specify its values on basis vectors of the form $u_{i,n+1-i}^{(1)}$ with $i \geq i_0$. Choose any vector $v$ in the span of $\{u_{i_0-1,j'}^{(k)} \mid j' \leq n+1-i_0\}$, and then set
\[
\bar{x}(u_{i,n+1-i}^{(1)}) = x^{i-i_0}(v).
\]
We must show that $x$ and $\bar{x}$ commute, or in other words, that
\[
x \bar{x}(u_{ij}^{(k)}) = \bar{x} x(u_{ij}^{(k)})
\]
for any basis vector $u_{ij}^{(k)}$. If $j \leq n-i$, or if $j = n+1-i$ and $k \geq 2$, this holds because $x'$ and $\bar{x}'$ commute. On the other hand, if $j = n+1-i$ and $k = 1$, then (6.6) follows easily from (6.5). This completes the proof of Step 1.

For Steps 2–4 of the proof, we let $\bar{x}'$ be any element of $\mathcal{O}_{T(Y')} \cap C(x')$ (such an element exists by (6.4)), and let $\bar{x}$ be any element of $p^{-1}(\bar{x})$ (such an element exists by Step 1).

**Step 2.** Notation related to procedure $A_{i_0}$. Let $\mathbf{w}' = \dim(Y')$. From the definition, we see that $\mathbf{w} = \mathbf{w}' + (e_{i_0} + e_{i_0+1} + \cdots + e_n)$. We now define a sequence of intermediate dimension vectors
\[
\mathbf{w}' = w_{i_0-1}, w_{i_0}, w_{i_0+1}, \ldots, w_{n-1}, w_n = \mathbf{w}
\]
by
\[
\mathbf{w}_m = \mathbf{w}' + (e_{i_0} + e_{i_0+1} + \cdots + e_m) = \mathbf{w}_{m-1} + e_m
\]
Next, define a sequence of integers $q_{i_0-1}, q_{i_0}, \ldots, q_n$ and a sequence of triangular arrays $Z_{i_0-1}, Z_{i_0}, \ldots, Z_n$ with $Z_m \in P(\mathbf{w}_m^*)$ as follows: we first set $q_{i_0-1} = i_0$, and $Z_{i_0-1} = T(Y') \in P(\mathbf{w}_0^*)$. If $Z_{m-1}$ and $q_{m-1}$ are already defined, we set
\[
q_m = K_{n+1-m}(Z_{m-1}, q_{m-1}),
\]
\[
Z_m = \text{Raise}(Z_{m-1}, n+1-m, q_m) \in P(\mathbf{w}_m^*).
\]
This is just an unpacking of the definition of procedure $A_{n+1-i_0}$: it is easy to see from the definitions that

$$a \circ \cdots \circ a \quad (T(Y'), n + 1 - i_0, i_0) = (Z_m, n - m, q_m).$$

In particular, $Z_n = A_{n+1-i_0}(T(Y')) = T(Y)$.

**Step 3. Notation related to the orbit of $\bar{x}$**. Identify $\mathbb{C}^w_m$ with the span of

$$\{u^{(k)}_{ij} \mid i > m \text{ and } j = n + 1 - i, \text{ then } k \geq 2\}.$$  

Each of these spaces is preserved by $\bar{x}$. They are not preserved by $x$ (except in the extreme cases $m = i_0 - 1$ or $m = n$). Instead, in general, $x$ restricts to a map $\mathbb{C}^w_m \to \mathbb{C}^{w_{m+1}}$.

Let $\bar{x}'_m = \bar{x}|_{\mathbb{C}^w_m}$. Then $\bar{x}'_m \in E(w^*_m)$, and the sequence

$$\bar{x}' = \bar{x}'_0 - 1, \bar{x}'_0, \bar{x}'_1, \ldots, \bar{x}'_{n-1}, \bar{x}'_m = \bar{x}$$

can be thought of as “interpolating” between $\bar{x}'$ and $\bar{x}$. Note that for each $m \in \{i_0, \ldots, n\}$, there is a short exact sequence

$$0 \to M(\bar{x}'_{m-1}) \to M(\bar{x}'_m) \to M(e_{n+1-m}) \to 0.$$

If $m \in \{i_0 + 1, \ldots, n\}$, then this can be enlarged to a commutative diagram

$$
\begin{array}{c}
0 \quad 0 \\
\downarrow \quad \downarrow \\
M(\bar{x}'_{m-2}) \quad M(\bar{x}'_{m-1}) \\
\downarrow \quad \downarrow \\
M(\bar{x}'_{m-1}) \quad M(\bar{x}'_m) \\
\downarrow \quad \downarrow \\
M(\bar{x}'_m) \quad M(e_{n+1-m}) \\
\downarrow \\
0 \quad 0
\end{array}
$$

(6.9)

Let $Z'_m \in P(w^*_m)$ be the label of the orbit of $\bar{x}'_m$. By Lemma 6.2 there is an integer $q'_{m}$ such that

$$Z'_m = \text{Raise}(Z'_{m-1}, n + 1 - m, q'_{m}).$$

Note that $Z'_m$ is the label of the orbit of $\bar{x}$.

**Step 4.** For $m \in \{i_0, i_0 + 1, \ldots, n\}$, we have $q'_m \leq q_m$ and $Z'_m \leq Z_m$. We proceed by induction on $m$. For $m = i_0$, the integer $q_{i_0} = K_{n+1-i_0}(Z_{i_0-1}, i_0)$ is the largest integer $q$ such that $\text{Raise}(Z_{i_0-1}, n + 1 - i_0, q)$ is defined. Since $Z'_{i_0-1} = Z_{i_0-1}$, and since $\text{Raise}(Z_{i_0-1}, n + 1 - i_0, q'_{i_0})$ is also defined, we have

$$q'_{i_0} \leq q_{i_0}.$$

The triangular arrays $Z'_m$ and $Z_{i_0}$ differ only in the $(n + 1 - i_0)$th chute. It is clear from (6.8), (6.10), and the definition of the partial order that $Z'_m \leq Z_{i_0}$.

Now suppose that $m > i_0$, and that $q'_{m-1} \leq q_{m-1}$. By Remark 6.9, we may choose a Jordan basis for $M(\bar{x}'_{m-2})$ that extends to a Jordan basis for $M(\bar{x}'_{m-1})$.

The latter adds one extra basis element $u$ with the property that $(\bar{x}'_{m-1})^{q_{m-1}}(u) = 0$. The commutative diagram (6.9) shows that $M(\bar{x}'_m)$ is spanned by $M(\bar{x}'_{m-1})$ and $x(u)$. Let $k$ be the smallest integer such that $(\bar{x}'_m)^k(x(u)) = 0$. As $(\bar{x}'_m)^{q_{m-1}}(x(u)) = x(\bar{x}'_{m-1})^{q_{m-1}}(u) = 0$, we clearly have $k \leq q'_{m-1} \leq q_{m-1}$. By Lemma 6.2 and the definition of $q_m$, we have

$$q'_m \leq K_{n+1-m}(Z'_{m-1}, k) \leq k \leq q'_{m-1}.$$
We will now show that \( q'_m \leq q_m \). If \( q_m > k \), the claim is obvious. Suppose instead that \( q_m \leq k \). Then we can replace (6.7) by

\[
q_m = \mathcal{K}_{n+1-m}(Z_{m-1}, k).
\]

Since \( k \leq q'_m \leq q_{m-1} \) as well, the first \( k-1 \) entries of the \((n+2-m)\)th chute of \( Z'_{m-1} \) and \( Z_{m-1} \) agree (and coincide with the corresponding entries of \( \mathcal{T}(Y') \)). Of course, the \((n+1-m)\)th chute of \( Z'_{m-1} \) and \( Z_{m-1} \) also agree with the \((n+1-m)\)th chute of \( \mathcal{T}(Y') \). Since \( Z'_{m-1} \) and \( Z_{m-1} \) agree on all entries relevant to the computation of \( \mathcal{K}_{n+1-m}(-, k) \), we conclude that \( \mathcal{K}_{n+1-m}(Z'_{m-1}, k) = \mathcal{K}_{n+1-m}(Z_{m-1}, k) \), and hence that \( q'_m \leq q_m \), as desired.

It remains to show that \( Z'_m \leq Z_m \). The triangular arrays \( Z'_m \) and \( Z_m \) differ from \( Z'_{m-1} \) and \( Z_{m-1} \), respectively, only in the \((n+1-m)\)th chute. Since \( Z'_{m-1} \leq Z_{m-1} \), in order to compare \( Z'_m \) and \( Z_m \), we need only compare their \((n+1-m)\)th chutes. It is clear from (6.8) and (6.10) that

\[
\sum_{p=1}^{j} (Z'_m)_{n+1-m,p} \geq \sum_{p=1}^{j} (Z_m)_{n+1-m,p}
\]

for all \( j \) (indeed, they are equal unless \( q'_m < j < q_m \), in which case the left-hand side is larger by 1). We conclude that \( Z'_m \leq Z_m \), as desired.

**Step 5. Conclusion of the proof.** By (6.3) and Proposition 6.5, \( \mathcal{O}_{\mathcal{T}(Y)} \cap C(x) \) is a Zariski-open dense subset of \( C(x) \). The surjective linear map \( p : C(x) \rightarrow C(x') \) is an open map, so \( p(\mathcal{O}_{\mathcal{T}(Y)} \cap C(x)) \) is a Zariski-open dense subset of \( C(x') \). The same holds for \( \mathcal{O}_{\mathcal{T}(Y')} \cap C(x') \) (see the remarks preceding (6.4), so

\[
p(\mathcal{O}_{\mathcal{T}(Y)} \cap C(x)) \cap \mathcal{O}_{\mathcal{T}(Y')} \cap C(x') \neq \emptyset.
\]

Choose \( x' \) in this set, and then choose \( x \in \mathcal{O}_{\mathcal{T}(Y)} \cap C(x) \). Apply Steps 2–4 to these elements. From Steps 2 and 3, we have \( Z_n = \mathcal{T}(Y) \) and \( Z'_n = \mathcal{T}(Y') \). Step 4 then tells us that

\[
\mathcal{T}(Y) \leq \mathcal{T}(Y').
\]

This inequality holds for all \( Y \in \mathcal{P}(w) \). But since \( \mathcal{T} \) and \( \mathcal{T}' \) are both bijections, this inequality actually implies that \( \mathcal{T}(Y) = \mathcal{T}(Y') \) for all \( Y \).

**Corollary 6.6.** For all \( Y \in \mathcal{P}(w) \), we have \( \mathcal{T}(Y) = \mathcal{T}'(Y) \).

**Proof.** The map \( \mathcal{T}' \) is the inverse of \( \mathcal{T} \), but by Theorem 6.3 and 6.4, \( \mathcal{T} \) is an involution. \( \square \)

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