Modeling the mobility with memory

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Abstract – We study a random walk model in which the jumping probability to a site is dependent on the number of previous visits to the site, as a model of the mobility with memory. To this end we introduce two parameters called the memory parameter \( \alpha \) and the impulse parameter \( p \). From extensive numerical simulations, we found that various limited mobility patterns such as sub-diffusion, trapping, and logarithmic diffusion could be observed. Through memory, a long-ranged directional anticorrelation kinetically induces sub-diffusive and trapping behaviors, and transition between them. With random jumps by the impulse parameter, a trapped walker can escape from the trap very slowly, resulting in an ultraslow logarithmic diffusive behavior. Our results suggest that the memory of walker’s has-beens can be one mechanism explaining many of the empirical characteristics of the mobility of animated objects.

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Introduction. – Since Einstein’s study on the motions of small particles suspended in liquids [1], diffusion and random walk have been a paradigm in describing and modeling the mobility involving seemingly irregular motions exhibited by both material and animated objects [2]. A multitude of variations of the random walk model with additional details have been widely studied [3,4], covering diverse physical and biological domains ranging from charge transport in disordered conductors to foraging patterns of animals and fishes.

From the random walk perspective, even the movement of a human individual may look irregular and random. In contrast to material objects, however, humans are conscious beings and driven by motivations by their will and purposes modulated by external cause, so their mobility patterns cannot be completely random. Indeed, recent empirical data for human mobility [5,6] have revealed a number of key characteristics of human mobility patterns, three of which we highlight here: First, even though the individual trajectory bears some degree of randomness, most part of it is highly predictable, being embedded in a well-defined region in space with the radius of gyration of the trajectory growing logarithmically in time [6]. Second, the mobile object (a human individual in this case) spends disproportionate times in different locations [7]. Third, at the population level, there is high degree of heterogeneity across individual’s degree of mobility [6].

In this work, our key premise is that at the heart of these human mobility patterns does underlie the role of memory: Conscious beings do have memory of where he or she has been and tend to revisit some of the has-beens, his or her favorite spots. By incorporating such a memory effect in the dynamic rule of a random walk model, we investigate the consequential role of memory in the mobility patterns.

We will demonstrate that the memory induces a long-ranged directional anticorrelation, which kinetically slows down the walker’s mobility, resulting in sub-diffusion and trapping, and transition between them. Logarithmic diffusion can occur by introducing the impulsive random jumps. Therefore, many of the characteristics of limited mobility can be explained by simple rules based on memory.

Model. – Our model is defined on a hypercubic lattice in \( d \)-dimensions. Starting from the origin, the walker takes a random walk with jumping probability depending on the number of previous visitations to the target sites. Specifically, we denote \( n_i(t) \) the number of visitations to the site \( i \) up to \( t \) steps by a random walker starting from the origin at time \( t = 0 \). The jumping probability \( w_{j,i}(t) \) from the site \( i \) to site \( j \) at time \( t \) is given by

\[
W_{j,i}(t) = \begin{cases} 
\frac{[n_j(t) + 1]^{\alpha}}{Z}, & \text{with probability } 1 - p, \\
\frac{1/(2d)}{Z}, & \text{with probability } p,
\end{cases}
\]

with the normalization \( Z = \sum_j [n_j(t) + 1]^{\alpha} \) by the restricted sum over nearest neighbors \( j \) of the site \( i \), and

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Trajectories up to diffusive, trapped, logarithmic diffusive behaviors, respectively. α-constrained trajectories are favored (note that even the variations in visitation numbers in the three cases. difference in the ranges of view field, as well as the range of coded by the visitation numbers (see the color bar). Notethe occur due to, e.g., occasional purely random jumps, which mayvisited more often. The parameter α is, the walker tends to go back to sites previously visited many times, resulting in a memory-dependent revisitation tendency. We are primarily interested in the cases of positive α, for which the walker tends to revisit the sites previously visited many times, resulting in a constrained trajectory (see fig. 1(a) for example) compared to simple symmetric random walks, similarly to interacting random walks [9,10]. When α = 0, the model becomes memoryless and is reduced to simple symmetric random walk. When α is negative, the walker disfavors revisitation, which would lead to a stretched trajectory, similarly to weakly self-avoiding walks [8]. In the limit α → −∞, the model becomes the self-avoiding walk. In the opposite limit α → ∞, which is of more interest to us, the walk would become so constrained that the walker will be trapped at the origin (see fig. 1(b) for example). The interesting questions are the existence of phase transition to the trapped trajectory at finite αc and how the normal diffusive behavior at α = 0 and trapping at α = αc are interpolated as α increases. Thus in the following we focus on the positive-α regime.

To answer these questions, we perform extensive numerical simulations. To focus on the role of the memory parameter, we fix p = 0 and vary α in the range from 0 to 1 (fig. 2). We measured the average dispersion of the walker in terms of the mean-square displacement, ⟨r²(t)⟩, as a function of the number of steps t. ⟨⋯⟩ denotes the ensemble average. The dispersion increases with the number of steps in a power-law manner at long times,

\[ ⟨r^2(t)⟩ \sim t^{2\nu}. \] (2)

The exponent ν is 1/2 for ordinary diffusion, and ν < 1/2 indicates sub-diffusive behavior. We found that in one dimension, ν decreases continuously from 1/2 at α = 0 to zero near αc ≈ 0.5 (fig. 2(a)). This suggests a

Fig. 1: (Color online) Typical trajectories from the model, corresponding to the parameter combinations (α, p) = (0.4, 0) (a), (0.8, 0) (b), and (1, 0.2) (c), each representing the sub-diffusive, trapped, logarithmic diffusive behaviors, respectively. Trajectories up to \( t = 2^{15} \) in two dimensions are shown, color-coded by the visitation numbers (see the color bar). Note the difference in the ranges of view field, as well as the range of variations in visitation numbers in the three cases.

d the spatial dimension. Thus the jumping probability changes dynamically, depending on the history of the walk.

The two parameters α and p characterize our model. α determines the degree of memory-dependent revisitation tendency. So we call it the memory parameter. The higher α is, the walker tends to go back to sites previously visited more often. The parameter p is introduced to implement occasional purely random jumps, which may occur due to, e.g., impulsive decisions of a mobile walker. So we call p the impulse parameter. With p = 0, the walker always performs a memory-dependent move. With p = 1, the model is reduced to simple symmetric random walk. In fig. 1, typical trajectories from our model in two dimensions with different (α, p)-parameters are shown.

Random walk models in a similar vein had been studied in relation to self-interacting random walks [8–10]. For example, Duxbury and de Queiroz [10] assigned to each (simple) random walk trajectory the “energy” in the form of \( \exp(-g \sum_i n_i^{\alpha}) \), where \( n_i \) is the visitation numbers of the site \( i \) as in our model. When \( g > 0 \) and \( \alpha \geq 0 \), constrained trajectories are favored (note that even the \( \alpha = 0 \) case is not reduced to simple random walk in this model). Depending on \( g \) and \( \alpha \), different behaviors occur, such as sub-diffusion, trapping at origin, or self-avoiding walks [10]. More recently, data-driven human mobility models have been proposed [11,12]. Song et al. [11] derived two rules of human mobility from mobile-phone data, termed the preferential return and the exploration, similar to our memory and impulse parameters, which could explain many characteristics of empirical patterns such as the ultraslow logarithmic dispersion. Szell et al. [12] introduced a mobility model based on the time-ordered memory of individual trajectory extracted from the multiplayer online game data, to demonstrate the sub-diffusive mobility.

These models and ours share key properties in motivations and model rules, but the precise formulation is somewhat different. Most importantly, the parameter set (α, p) in our model covers a broader range, therefore we can address the generic role of these parameters, which might be relevant for novel mobility patterns associated with the memory effect, other than those identified thus far.

Role of the memory parameter. – The memory parameter α determines the degree of memory-dependent revisitation tendency. We are primarily interested in the cases of positive α, for which the walker tends to revisit the sites previously visited many times, resulting in a constrained trajectory (see fig. 1(a) for example) compared to simple symmetric random walks, similarly to interacting random walks [9,10]. When α = 0, the model becomes memoryless and is reduced to simple symmetric random walk. When α is negative, the walker disfavors revisitation, which would lead to a stretched trajectory, similarly to weakly self-avoiding walks [8]. In the limit α → −∞, the model becomes the self-avoiding walk. In the opposite limit α → ∞, which is of more interest to us, the walk would become so constrained that the walker will be trapped at the origin (see fig. 1(b) for example). The interesting questions are the existence of phase transition to the trapped trajectory at finite αc and how the normal diffusive behavior at α = 0 and trapping at α = αc are interpolated as α increases. Thus in the following we focus on the positive-α regime.

To answer these questions, we perform extensive numerical simulations. To focus on the role of the memory parameter, we fix p = 0 and vary α in the range from 0 to 1 (fig. 2). We measured the average dispersion of the walker in terms of the mean-square displacement, \( ⟨r^2(t)⟩ \), as a function of the number of steps t. \( ⟨⋯⟩ \) denotes the ensemble average. The dispersion increases with the number of steps in a power-law manner at long times,

\[ ⟨r^2(t)⟩ \sim t^{2\nu}. \] (2)

The exponent ν is 1/2 for ordinary diffusion, and ν < 1/2 indicates sub-diffusive behavior. We found that in one dimension, ν decreases continuously from 1/2 at α = 0 to zero near αc ≈ 0.5 (fig. 2(a)). This suggests a
trapping transition at finite $\alpha_c \approx 0.5$ and sub-diffusion with continuously varying exponent $\nu$ for $0 < \alpha < \alpha_c$.

We also measured the average probability that the walker visits to an unvisited site at step $t$, $\langle \Delta(t) \rangle$. For one-dimensional simple symmetric random walk ($\alpha = 0$), it is known [3] that $\langle \Delta(t) \rangle$ decreases with $t$ as a power law,

$$\langle \Delta(t) \rangle \sim t^{-\sigma},$$

with $\sigma = 1/2$. We found that $\sigma$ increases with $\alpha$, becomes $\sigma = 1$ at $\alpha = \alpha_c \approx 0.5$, and further increases for $\alpha > \alpha_c$ (fig. 2(b)). This indicates that the average total number of visited sites $\langle S(t) \rangle$ upto $t$-steps, given by the sum of $\langle \Delta(t) \rangle$ over $t$, exhibits distinct behaviors depending on $\alpha$. $\langle S(t) \rangle$ diverges as $\sim t^{1-\sigma(\alpha)}$ for $\alpha < \alpha_c$ (diffusing), whereas it becomes finite for $\alpha > \alpha_c$ (trapped). At the critical point $\alpha = \alpha_c$, $\langle S(t) \rangle$ increases logarithmically. This result is also in support of the existence of a trapping transition at finite $\alpha_c$.

Thus, with the memory parameter we can obtain slow or limited mobility in the walker’s motion. Depending on its strength, it can kinetically induce sub-diffusion or trapping, and transition between them. The heterogeneity of individual trajectory would follow at the critical point. It can also be thought that the memory parameter $\alpha$ can take different values from person to person, as it characterizes an individual’s degree of revisititation tendency. However, with the memory parameter alone, the walker is either diffusing or gets trapped at some localized region in space. Ultraslow dispersion can only be expected for a single specific value of the memory parameter at the critical point.

**Role of the impulse parameter.** – The impulse parameter $p$ models the degree of impulsive, random decisions in the walker’s motion, introducing occasional unbiased random jumps. Intuitively, introduction of unbiased random jumps ($p > 0$) will give rise to larger dispersion of the walker’s trajectory than the $p = 0$ case. A question of interest is what the effect of random jumps is to a trapped walker. To answer this question, we fix the memory parameter to be $\alpha = 1$, high enough for a strong trapping, as well as in accordance with the empirical data [11], and vary the impulse parameter $p$ in the range from 0 to 1 (fig. 3).

By measuring the mean-square displacement $\langle r^2(t) \rangle$, we find that there exists a critical impulse parameter $p_c \approx 0.5$, above which the walker is no longer confined in the trap and diffuses (fig. 3(a)). For $p > p_c$, $\nu$ increases continuously from zero and becomes 1/2 for $p \to 1$, crossing over to normal diffusion. This result is corroborated by the average probability of visiting an unvisited site $\langle \Delta(t) \rangle$ decaying as eq. (3) with the exponent $\sigma$ varying from 1 to 1/2 as $p$ increases from $p_c \approx 0.5$ to 1 (fig. 3(b)).

The role of impulse parameter for $p < p_c$ is more intriguing. Measuring $\nu$, we have $\nu \approx 0$ for $p < p_c$, suggesting that the walker is still trapped (fig. 3(a)). However, $\sigma$ is stuck at $\sigma = 1$, that is, $\langle \Delta(t) \rangle \sim t^{-1}$, regardless of $p < p_c$ (fig. 3(b)). This means that the average total number of visited sites $\langle S(t) \rangle$ still increases logarithmically, $\langle S(t) \rangle \sim \log t$, suggesting that the walker is in fact marginally escaping from the trap. Such behavior can be seen from the example trajectory shown in fig. 1(c), where one can identify more than one traps.

To illustrate mobility patterns exhibited by different phases, we calculated the probability distribution $P_i(t)$, defined as the average probability to find the walker starting from origin at site $i$ after $t$-steps (fig. 4). For the memory-only model ($p = 0$), we have diffusing ($\alpha < \alpha_c$), critical ($\alpha = \alpha_c$), and trapped phases ($\alpha > \alpha_c$) (figs. 4(a)–(c), respectively). In particular, in the trapped phase, $P_i(t)$ becomes time independent at long times, suggesting the walker is completely trapped, visiting a set of trapping sites repeatedly ad infinitum. With the impulse parameter, the behavior is similar for the diffusing phase (fig. 4(d)). For $p \leq p_c$, the random jumps
Kinetic origins of the anomalous diffusion. –

Our numerical-simulation results show that the memory parameter \( \alpha \) can modulate the diffusivity of the mobility with memory, from sub-diffusion to trapping. In order to gain more microscopic understanding on how such limited mobility is generated by memory, we measured the directional correlation function, \( C(s) \), defined as the average correlation between the directions of two steps separated by a lag of \( s \) steps, that is, \( C(s) = \langle \eta(t)\eta(t+s) \rangle \), where \( \eta(t) = r(t) - r(t - 1) \) is the step direction in one dimension and the overbar denotes the time average. In simple symmetric random walk \( (\alpha = 0) \), \( C(s) = 0 \). We found that for \( \alpha > 0 \), this directional correlation is non-zero.

In the sub-diffusive regime \( (0 < \alpha < 0.5) \), two notable features emerge for \( C(s) \) (fig. 5(a)). First, \( C(s) \) is on average negative in the entire range of \( s \), and its absolute value for small \( s \) increases with \( \alpha \). This means that the walk becomes more and more strongly antibiased towards the current position anytime. Second, \( |C(s)| \) decays as a power law, \( |C(s)| \sim s^{-\xi} \), with a crossover between two regimes. The power-law exponent is \( \xi \approx 1/2 \) for small \( s \) and \( \xi \approx 2 \) for large \( s \), and the crossover timescale decreases with \( \alpha \). This long-ranged directional anticorrelations might be key kinetic origin for the sub-diffusive behavior, with a similar mechanism as in the continuous-time random walks with anticorrelated fractional Gaussian noise [14]. Meanwhile, in the trapped regime \( (\alpha \gtrsim 0.5) \), the directional correlation develops an alternating behavior around zero, reflecting the back-and-forth motion within the trap (fig. 5(a), inset). The amplitude of alternation increases with \( \alpha \), as the trapping becomes stronger. These kinetically induced long-ranged anticorrelated motions can be considered another class of general correlated random walk process for anomalous diffusion [15].

A well-known mechanism for sub-diffusion is the power-law distribution of waiting times between consecutive
steps, or pausing times [16]. When the unbiased random walk with finite step length has the waiting time distribution $\psi(\tau)$ of a power-law form, $\psi(\tau) \sim \tau^{-(1+\mu)}$, with $0 < \mu < 1$, its dispersion becomes sub-diffusive as $\langle r^2 \rangle \sim t^\mu$ at long times. To consider how such an effect manifests in our model, we divide the lattice into non-overlapping linear blocks of size $\ell$ (typically $\ell = 7$, to cover the typical size of a trap), and calculated the histogram of the time interval $\tau$ between two consecutive crossings of the block boundaries by the walker, which we call the block waiting time distribution $\psi_B(\tau)$.

We found that in the trapped regime ($\alpha \gtrsim 0.5$), the block waiting time distribution $\psi_B(\tau)$ exhibits a peak at $\tau = 1$ (corresponding to an immediate returning), followed by a power-law regime $\psi_B(\tau) \sim \tau^{-\beta}$ with $\beta \approx 1.0$ and exponential cutoff which increases with $\alpha$ and $\ell$ (fig. 5(b)). The power-law behavior with $\mu \approx 0$ in the trapped regime is apparently consistent with the lack of dispersion. Meanwhile, in the sub-diffusive regime ($0 < \alpha \lesssim 0.5$), the power-law regime is absent and $\psi_B(\tau)$ decays exponentially (fig. 5(b)). We also observed different ensemble average $\langle r^2(t) \rangle_{\text{ens}}$ and temporal moving average $\langle r^2(t) \rangle_T$ of the mean-square displacement, suggestive of ergodicity breaking in the trapped regime [17].

![Fig. 5](image)

Fig. 5: (Color online) Detailed kinetics of the model. (a) Directional correlation function $C(s)$ as a function of the step lag $s$ in double-logarithmic plot. Straight lines are guidelines with slopes $-0.5$ (dotted) and $-2$ (dashed). (b) Block waiting time distribution $\psi_B(\tau)$ with the blocksize $\ell = 7$ in double-logarithmic scale. The straight line is a guideline with slope $-1$. Calculations are based on trajectories with $10^8$ steps, and same set of symbols are used in (a) and (b).

![Fig. 6](image)

Fig. 6: (Color online) Color-coded plots of the exponent $\nu$ in the $(\alpha, p)$-parameter space in one (a) and two (b) dimensions. Dashed lines are guidelines of $\alpha(1 - p) = 1/2$.

**In two dimensions.** We performed numerical simulations of our model in one dimension for other combinations of $(\alpha, p)$-parameters. We found that the critical value of the memory parameter $\alpha_c$ depends on the impulse parameter $p$; starting from $\alpha_c \approx 0.5$ at $p = 0$, $\alpha_c$ increases with $p$ (fig. 6(a)). We also performed numerical simulations in a two-dimensional square lattice, finding overall similar behaviors. A notable difference is that the logarithmic dispersion for $\alpha > \alpha_c$ and $p > 0$ is seen more prominent in two dimensions (fig. 6(b)). Our numerical data suggest the dispersion in the logarithmic regimes as $\sim (\log t)^\gamma$ with $\gamma \approx 0.3$ in one dimension and $\gamma \approx 1.5$ in two dimensions. This result shows that the effect of random jumps by the impulse parameter becomes manifestly more significant in higher dimensions.

**Continuum limit.** The discrete-time master equation for $P_i(t)$ can be written as

$$P_i(t+1) - P_i(t) = \sum_j [w_{i,j}(t)P_j(t) - w_{j,i}(t)P_i(t)],$$  \hspace{1cm} (4)

where the transition probability $w_{j,i}(t)$ is given by eq. (1) and the restricted summation runs over nearest neighbors $j$ of $i$. Decomposing $w_{j,i}$ into memory-dependent and random jumps, we can rewrite the above equation as

$$P_i(t+1) - P_i(t) = (1 - p)\sum_j [\mu_{i,j}(t)P_j(t) - \mu_{j,i}(t)P_i(t)] + p\sum_j \frac{1}{2d} [P_j(t) - P_i(t)].$$  \hspace{1cm} (5)

Here $\mu_{j,i}(t) = \phi_j(t)/\sum_k \phi_k(t)$ with $\phi_j(t) = |n_j(t) + 1|^{\alpha}$, denotes the memory-dependent jumping probability. Let us consider a continuous time and space limit of eq. (5) in one dimension. One can rewrite $\mu_{j,i}(t)$ as $\mu_{i+1,i}(t) = \frac{1}{2} - \frac{1}{2} \frac{\phi_{i+1}(t) - \phi_{i-1}(t)}{\phi_{i+1}(t) + \phi_{i-1}(t)} \approx \frac{1}{2} - \frac{1}{2} \frac{\partial \phi(x,t)}{\partial x} / \phi(x,t)$, etc., assuming that $\phi(x,t)$, the continuum analog of $\phi_i(t)$, is analytic at $x$. 

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This leads to the continuum limit of eq. (5) as
\[
\frac{\partial P(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2 P(x,t)}{\partial x^2} - (1-p) \frac{\partial}{\partial x} \left[ P(x,t) \frac{\partial (\phi(x,t)/\phi(x,t))}{\partial x} \right].
\]
(6)

Taking advantage of the relation between \(\phi(x,t)\) and \(P(x,t)\), one finally obtains a highly nonlinear integro-differential equation for \(P(x,t)\),
\[
\frac{\partial P(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2 P(x,t)}{\partial x^2} - \alpha(1-p) \frac{\partial}{\partial x} \left\{ P(x,t) \frac{\partial}{\partial x} \log \left[ 1 + \int_0^t P(x,t')dt' \right] \right\}.
\]
(7)

Equation (7) is too nonlinear to be solved exactly, yet it can be used to locate the transition point between diffusion and trapping as follows. Approaching to the transition point, \(P(x,t)\) starts to become time independent. Looking for the condition of time-independent \(P(x,t)\) for large \(t\), equivalently \(n(x,t) \approx tP(x)\), we find the transition point to trapping as \(\alpha(1-p) = 1/2\), which is in good agreement with the numerical-simulation results (fig. 6).

**Summary and discussion.** We have studied a random walk model in which the jumping probability to a site is dependent on the number of previous visits to the site, as a model of the mobility with memory. Through extensive numerical simulations, we found that by varying the memory parameter \(\alpha\) and the impulse parameter \(p\), various limited mobility patterns such as sub-diffusion, trapping, and logarithmic diffusion could be observed. With the memory parameter \(\alpha\), a long-ranged directional anticorrelation is developed in the walker’s motion, resulting in sub-diffusive behaviors. At the critical value of \(\alpha_c\), the transition from sub-diffusive to trapped motion occurs, above which the walker’s trajectory becomes confined in a localized trapping region. By allowing random jumps with the impulse parameter \(p\), the walker can escape from the trap, albeit very slowly, but gets trapped to another trap kinetically generated by the walker’s own motion, resulting in an ultraslow logarithmic diffusive behavior.

In our model, heterogeneity in both space and time is induced kinetically as the walker moves. In this sense our work enlarges the concept of the sub-diffusion of mixed origins [18], to dynamically generated spatiotemporal heterogeneity. Our results also demonstrate that the memory of walker’s has-beens can be a mechanism explaining many of empirical human mobility characteristics [5–7]. The revisitation tendency driven by the memory has also been proposed and supported by empirical data in the more detailed, data-driven modeling of human mobility [11,12]. Our work extends this approach to encompass broader spectrum of limited mobility patterns, including sub-diffusion and trapping, and transition between them. It would be interesting to see if our modeling framework with the memory and impulse parameters could be applied to various empirical data for human and animal’s mobility, which might uncover further universality and specificity in the physics of mobility of animated objects [19].

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