Comparison of semi-simplifications of Galois representations

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Introduction

The aim of this paper is to provide a criterion to ensure that the semi-simplifications of $p$-adic finite dimensional Galois representations are isomorphic. Such an isomorphism implies that the Artin $L$-functions of these representations are the same. This can for example be used to compare the Artin $L$-functions obtained from automorphic representations with those issued from algebraic geometry. Another possible use is shown in Section 4 which proves that the two representations considered in [2], one of which is a subrepresentation of the cohomology of a variety while the other is a conjectural automorphic representation, are isomorphic. The main result of this paper, Theorem 3, provides an effective criterion to check whether two semi-simplifications are isomorphic and explains which Frobenius elements suffice to compare the two. In [3] section 4], Ron Livné explained and generalized (by lowering the required number of comparisons) a result of Jean-Pierre Serre giving a sufficient condition for semi-simplifications of $p$-adic Galois representations to be isomorphic. We intend to generalize here the original result of Jean-Pierre Serre. Even though our result is valid for all dimensions and cannot use the fact that a group of exponent 2 is abelian, our result is similar in complexity to the one of Livné (1).

This paper can be considered as an application of the method explained in [4] p27–29].

I would like to thank Bert van Geemen who brought this subject to my attention, gave me some hints and helped remove some errors. This result has been made possible thanks

(1)Livné has a better result because he shows that he does not need to compare the representations for all Frobenius elements but only for a so called “non cubic” family
to the help of Thomas Weigel who helped me understand the complexity of the pro-$p$ groups and suggested the use of the “powerful pro-$p$ groups” which proved to be the right way to tame that complexity. I also would like to thank Karim Belabas who has been of great help for the computational part of the result. I would finally like to thank the referee for his extremely careful reading and his very precise and insightful remarks.

1 The result

1.1 Setup

This section sets up the framework of this work. Let us fix an integer $n \geq 2$, a prime $p$ and define $m$ as the minimum integer such that $p^m \geq n$. We fix a global field $K$ and let $\overline{K}$ be a maximal separable algebraic extension of $K$. All extensions of $K$ considered in this paper are sub-extensions of $\overline{K}$. For any subfield $L$ of $\overline{K}$, we denote $\Gamma_L = \text{Gal}(\overline{K}/L)$.

We denote $\mathcal{M}(n, A)$ the algebra of matrices of size $n \times n$ with coefficients in a ring $A$.

Definition 1 Let $E$ be a finite extension of $\mathbb{Q}_p$ for some prime $p$. Let $M_1$ and $M_2$ be two matrices in $\mathcal{M}(n, E)$. Let $F$ be a finite extension of $E$ containing the eigenvalues of $M_1$ and $M_2$. Denote $\mathcal{O}_F$ the integer ring of $F$, $p_F$ its maximal ideal and $\varpi_F$ a uniformiser. The two matrices $M_1$ and $M_2$ are said to have congruent eigenvalues if there exist $\lambda \in \mathcal{O}_F^\times$ and $v \in \mathbb{Z}$ such that the characteristic polynomials of $\varpi_F^{-v}M_1$ and $\varpi_F^{-v}M_2$ are in $\mathcal{O}_F[\overline{X}]$ and are congruent to $(X - \lambda)^n$ modulo $p_F$.

Remark 2 – The absolute Galois group of a global field is compact, so the eigenvalues of the matrices in the image of a Galois representation necessarily have valuation $v = 0$.

– The condition on the matrices is rather strong: it implies that the $2n$ eigenvalues are congruent to a single one. Therefore the condition is strong even for each matrix separately.

1.2 Construction

For any finite set of places $S$ of $K$, we want to construct an extension $K_S = K_{S,n}$ of $K$ such that the Galois group of $K_S/K$ is sufficient to compare the semi-simplifications of representations of $\Gamma_K$ with values in $\text{GL}(n, E)$, unramified outside $S$, and with all eigenvalues reducing to a single one in the residual field of $E$.

Take $K_0 = K$. Define $K_i$ by induction by taking $K_{i+1}$ to be the maximal abelian extension of $K_i$ unramified outside $S$ and such that $\text{Gal}(K_{i+1}/K_i)$ is a direct product of copies of $\mathbb{F}_p$. Notice that $K_i$ is a Galois extension of $K$ at each step $i$. Let $\epsilon = 0$ if $p \neq 2$ and $\epsilon = 1$ if $p = 2$. Let $r = N^{2(1+\epsilon)}\frac{N(N-1)}{2}$ with $N = n[E : \mathbb{Q}_p]$. Let $\lambda$ be the minimum integer such that $2^\lambda \geq r$. Finally take $K_S = K_{\lambda+\epsilon+m}$. 

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1.3 Main result

**Theorem 3** We fix an integer \( n \geq 2 \), a prime \( p \) and define \( m \) to be the minimum integer such that \( p^m \geq n \). Let \( K \) be a global field, \( S \) a finite set of places of \( K \) and \( E \) a finite extension of \( \mathbb{Q}_p \). We assume that if \( k \) is the residual field of \( E \), then \( n \) and \( |k^\times| \) are relatively prime. Let \( K_S \) be the field constructed as in Section 1.2. Fix a set \( T \) of places of \( K \), disjoint from \( S \), such that each maximal cyclic subgroup of \( \text{Gal}(K_S/K) \) has a generator of the form \( \text{Frob}(t^t/t) \) for some \( t \in T \) and some prime \( t \) above \( t \) in \( K_S \). Assume now that

\[
\rho_1, \rho_2 : \Gamma_K \to \text{GL}(n, E)
\]

are continuous representations unramified outside \( S \) and satisfy the following conditions:

1. \( \forall \sigma \in \Gamma_K, \rho_1(\sigma) \) and \( \rho_2(\sigma) \) have congruent eigenvalues (see Definition 7).

2. \( \forall t \in T, \rho_1(\text{Frob} t) \) and \( \rho_2(\text{Frob} t) \) have equal characteristic polynomials (where \( \text{Frob} t \) is any Frobenius element above \( t \)).

Then \( \rho_1 \) and \( \rho_2 \) have isomorphic semi-simplifications.

**Remark 4** – In this theorem, the condition \( (n, |k^\times|) = 1 \) is needed just to ensure that, up to a twist by a character, the residual representations are \( p \)-groups.

– The characteristic polynomial of a matrix \( M \) of size \( n \) has coefficients which are symmetric functions of degree at most \( n \) of the eigenvalues of \( M \). Over a field of characteristic \( 0 \) the sums of the powers of the variables \( (X_i)_{1 \leq i \leq n} \) are a basis of the space of symmetric functions in \( (X_i) \). It follows that there exists a function \( f \) independent of \( M \) such that the characteristic polynomial of \( M \) is \( f(\text{Tr} M, \text{Tr} M^2, \ldots, \text{Tr} M^n) \).

Hence we can modify Condition (2) above as follows:

either

\[
\forall t \in T, \forall 1 \leq k \leq n, \text{Tr} \rho_1((\text{Frob} t)^k) = \text{Tr} \rho_2((\text{Frob} t)^k)
\]

or

\[
\forall t \in T, \left\{ \begin{array}{ll}
1 \leq k \leq n - 1, & \text{Tr} \rho_1((\text{Frob} t)^k) = \text{Tr} \rho_2((\text{Frob} t)^k) \\
\det \rho_1(\text{Frob} t) = \det \rho_2(\text{Frob} t).
\end{array} \right.
\]

– As for the condition \( (n, |k^\times|) = 1 \): observe that, if \( n \) is even, then \( p \) has to be 2. Observe also that if \( n \) is a power of \( p \), or \( k = \mathbb{F}_2 \), then the condition is verified. Finally observe that we can multiply \( n \) by \( [E : \mathbb{Q}_p] \) and assume \( E = \mathbb{Q}_p \). We can thus always apply the theorem if we choose \( p = 2 \) (at the cost of enlarging \( n \), which makes it less interesting because \( K_S \) and \( T \) become larger).

2 Pro-\( p \)-groups

The main result of this section is Proposition 9 which establishes our result for a pro-\( p \) group.
2.1 The result for pro-$p$ groups

**Definition 5** For a $p$-group or pro-$p$ group $G$, we denote by $G^#$ the closure of the intersection of the kernels of all group morphisms from $G$ to finite groups such that all their elements have order dividing $p^m$.

**Remark 6**
1. $G^#$ is also called $\mathcal{O}_m(G)$, at least when $G$ is finite.
2. $G^#$ is normal in $G$.
3. Observe that $G^#$ is also the closure of the subgroup generated by $p^m$-th powers.
4. In case $n = p = 2$, the subgroup $G^#$ is just the Frattini subgroup $G^*$.
5. If $\rho: G \rightarrow H$ is a continuous group morphism, $\rho(G^#) = \rho(G)^# \subseteq H^#$ with equality if $\rho$ is surjective.
6. If $G_1$ and $G_2$ are groups, then $(G_1 \times G_2)^# = G_1^# \times G_2^#$.

The following lemma will be useful later on:

**Lemma 7** Let $G$ be a $p$-group such that any element of $G/G^#$ has a representative in $G$ of order dividing $p^m$. Then $G^# = \{1\}$.

**Proof:** Suppose that $G^# \neq \{1\}$. Observe first that, according to [5, Theorem 1.12, p90], we can find a normal subgroup $N$ of $G$ which is a subgroup of index $p$ of $G^#$. Then $(G/N)^# \simeq G^#/N \simeq \mathbb{F}_p$, so that we can as well assume that $G^# = \mathbb{F}_p$. We have an exact sequence

$$0 \rightarrow \mathbb{F}_p \rightarrow G \rightarrow G/G^# \rightarrow 1$$

such that each element of $G/G^#$ has a representative in $G$ of order dividing $p^m$.

Denote $H = G/G^#$. Then $H$ is a $p$-group and Aut $\mathbb{F}_p$ has $p-1$ elements, so that the action of $H$ on $\mathbb{F}_p$ is trivial. This means that the extension

$$0 \rightarrow \mathbb{F}_p \rightarrow G \xrightarrow{\pi} H \rightarrow 1$$

is central, i.e. that $G^# \subseteq Z(G)$. Thus every element $g$ of $G$ has order dividing $p^m$ (all elements of $gG^#$ have the order of $g$, except if $g \in G^#$, in which case all elements have order either $p$ or $1$). We deduce that the identity is a morphism from $G$ to a group having elements of order dividing $p^m$. This means that $G^# = \{1\}$, which is impossible. □

**Remark 8** This lemma is a generalization of [3, Lemma 4.5, p257]. The definition of $G^#$ accounts for Remark 4.6.a. below the proof of the lemma in loc. cit.
Proposition 9 Let $G$ be a pro-$p$ group which is topologically finitely generated and let $E$ be a finite extension of $\mathbb{Q}_p$. Recall that the integer $m$ used to define $G^\#$ is the minimum integer such that $p^m \geq n$. Assume

$$\rho_1, \rho_2 : G \rightarrow \text{GL}(n, E)$$

are continuous representations and $\Sigma \subset G$ is a subset satisfying:

1. the image of $\Sigma^* = \{\sigma^k/\sigma \in \Sigma, k \in \mathbb{N}\}$ in $G/G^\#$ is equal to $G/G^#$;

2. $\forall \sigma \in \Sigma$, $\rho_1(\sigma)$ and $\rho_2(\sigma)$ have the same characteristic polynomial.

Then $\rho_1$ and $\rho_2$ have isomorphic semi-simplifications.

Proof: Let $\mathcal{O}$ be the integer ring of $E$. Since $G$ is compact, it preserves a full lattice in $E^n$ when acting via each $\rho_i$, for $i = 1, 2$. Since $\mathcal{O}$ is a discrete valuation ring, such a lattice is free over $\mathcal{O}$. Hence we may assume $\rho_i(\mathcal{O}) \subset \text{GL}(n, \mathcal{O})$ for each $i = 1, 2$.

Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}$ and set $k = \mathcal{O}/\mathfrak{p}$. The reduction modulo $\mathfrak{p}$ of $\rho_i(G)$ is a $p$-group in $\text{GL}(n, k)$. A $p$-Sylow subgroup for $\text{GL}(n, k)$ is the subgroup of upper triangular unipotent matrices. We can thus suppose, up to a base change in the lattices above, that the reduction of $\rho_i(G)$ modulo $\mathfrak{p}$ is included in this subgroup. In particular, for any $g$ in $G$, $(\rho_i(g) - I_n)^n \equiv 0 \pmod{\mathfrak{p}}$. We also have that $\rho_i(g)^{p^m} \equiv I_n \pmod{\mathfrak{p}}$ (in fact we can substitute $p^m$ by any power of $p$ that is at least equal to the nilpotency order of the reduction mod $\mathfrak{p}$ of $\rho_i(g) - I_n$).

Now let $M_n = \mathcal{M}(n, \mathcal{O})$. We define $\rho : G \rightarrow M_n \times M_n$ to be the map $\rho(g) = (\rho_1(\sigma), \rho_2(\sigma))$. Set $M$ to be the linear $\mathcal{O}$-span of $\rho(G)$ in $M_n \times M_n$. Then $M$ is an $\mathcal{O}$-algebra spanned (as an $\mathcal{O}$-module) by $\Gamma = \rho(G)$. Let $R = M/\mathfrak{p}M$ and for $g \in G$, we will denote the image of $\rho(g)$ in $R$ by $\overline{g}$. Set $\overline{\Gamma} = \{\overline{g}/g \in G\}$. Then $R$ is a $k$-algebra with unity $\overline{1} = (I_n, I_n)$ mod $\mathfrak{p}M$ and spanned by $\overline{\Gamma}$ as a $k$-vector space.

We would like to prove that $R$ is spanned over $k$ by $\Sigma^* = \{\overline{\sigma^k}/\sigma \in \Sigma, k \in \mathbb{N}\}$. We claim that, for any $\sigma \in \Sigma$, we have $\overline{(\sigma - \overline{1})}^n = \overline{0}$ and $\overline{\sigma^m} = \overline{\sigma}$. Both these equalities generalize $\overline{\sigma^2} = \overline{\sigma}$ for $p = n = 2$. The point is that equalities in $\text{GL}(n, k)$ can sometimes be translated to equalities in $R$. Let us first observe that the characteristic polynomial of $\rho_i(\sigma)$ mod $\mathfrak{p}$ is $(X - 1)^n$. This polynomial is the reduction modulo $\mathfrak{p}$ of the characteristic polynomial of $\rho_i(\sigma)$. Let $\sum_{r=0}^{n} c_{r,i} X^r$ be the characteristic polynomial of $\rho_i(\sigma)$. Let $a_{r,i} = (-1)^{n-r}(\binom{n}{r}) - c_{r,i}$. Then $(\rho_i(\sigma) - I_n)^n = \sum_{r=0}^{n} a_{r,i} \rho_i(\sigma)^r$ and all $a_{r,i} \in \mathfrak{p}$. From Hypothesis 2 we know that
the characteristic polynomials are equal and thus \( a_{r,1} = a_{r,2} = a_r \). We can deduct that

\[
(p(\sigma) - (I_n, I_n))^n = ((\rho_1(\sigma) - I_n)^n, (\rho_2(\sigma) - I_n)^n)
\]

\[
= \left( \sum_{r=0}^{n} a_r \rho_1(\sigma)^r, \sum_{r=0}^{n} a_r \rho_2(\sigma)^r \right)
\]

\[
= \sum_{r=0}^{n} a_r (\rho_1(\sigma))^r, (\rho_2(\sigma))^r
\]

\[
= \sum_{r=0}^{n} a_r (\rho_1(\sigma), \rho_2(\sigma))^r
\]

\[
= \sum_{r=0}^{n} a_r \rho(\sigma)^r
\]

\[
\in pM
\]

Thus \( N(\sigma) = \sigma - \bar{1} \) is nilpotent of order (at most) \( n \). This means that for any \( r \), \( \sigma^r = (1 + N(\sigma))^r \) is a polynomial in \( N(\sigma) \) of degree at most \( n - 1 \). For \( r = p^m \geq n \), then \( (p^m) \) will be in \( p\mathbb{Z} \subset p \) for all \( i \in [1; p^m - 1] \). Thus \( \sigma^{p^m} = \bar{1} \) (as above we can substitute \( p^m \) by any power of \( p \) that is at least equal to the nilpotency order of each \( \rho_i(\sigma) - I_n \) mod \( p \)). In addition, since we have only used the fact that \( \rho_1(\sigma) \) and \( \rho_2(\sigma) \) have the same characteristic polynomial, this remains true for all powers of all the elements of \( \Sigma \):

\[
\forall \sigma \in \Sigma^*, \begin{cases} 
(\sigma - \bar{1})^n = \bar{0} \\
\sigma^{p^m} = \bar{1},
\end{cases}
\]

which means

\[
\forall \sigma \in \Sigma^*, \begin{cases} 
(\sigma - \bar{1})^n = \bar{0} \\
\sigma^{p^m} = \bar{1}.
\end{cases}
\]

To prove that \( R \) is \( k \)-spanned by \( \Sigma^* \), we first prove that \( \Gamma^\# = \{1\} \). Observe that, since \( \mathcal{O} \) is a principal domain, \( R \) is a finite-dimensional \( k \)-vector space of dimension at most \( 2n^2 \). Hence \( R \) and \( \Gamma \) are finite. We can apply Lemma 7 to show that \( \Gamma^\# = \{1\} \); since \( \rho(G) = \Gamma \), we have \( \Gamma^\# = \rho(G)^\# = G^\# \), which implies \( \Gamma^\# = \Gamma^\# = \bar{1} \) and thus any element of \( \bar{1}/\Gamma^\# \) can be represented by an element of \( \Sigma^\# \) and these elements have order dividing \( p^m \). According to Lemma 7, we have \( \Gamma^\# = \{1\} \) and thus \( \Gamma \sim \bar{1}/\Gamma^\# \sim \bar{1}/\Gamma^\# \sim G/G^\# \subseteq \Sigma^\# \) (the last inclusion is up to the canonical projection from \( G \) to \( G/G^\# \)); since \( \Sigma^\# \subseteq \Gamma \) and both are finite, we conclude that \( \Sigma^* = \Gamma \).

Using the former argument, we can apply Nakayama’s lemma to see that \( \Sigma^* \) generates \( M \) as an \( \mathcal{O} \)-module. Since the characteristic polynomials of \( \rho_1(\sigma) \) and \( \rho_2(\sigma) \) are equal, the traces of \( \rho_1(\sigma^k) \) and \( \rho_2(\sigma^k) \) are equal for all \( \sigma \in \Sigma \) and all \( k \in \mathbb{N} \). Thus the linear form \( \alpha \) on \( M \) defined by \( \alpha(a, b) = \text{Tr } a - \text{Tr } b \) is trivial on a generating set of \( M \) and thus on all of
M. As a consequence, the characteristic polynomials of $\rho_1(g)$ and $\rho_2(g)$ are equal for all $g \in G$.

\[ \square \]

2.2 Structure of pro-$p$ groups

A good reference for the following is [6], and in particular chapter 3.

**Definition 10** A powerful pro-$p$ group is a pro-$p$ group $G$ such that $G/G^p$ (resp. $G/G^4$ if $p = 2$) is abelian, where $G^p$ (resp. $G^4$) is the subgroup generated by $p$-th (resp. fourth) powers of elements of $G$.

**Proposition 11** For each finitely generated pro-$p$ group $G$ with a powerful open subgroup, there is a number $r$ such that any subgroup of $G$ has at most $r$ generators.

**Definition 12** The minimal number $r$ above is called the rank of the pro-$p$ group $G$.

For any integer $r \geq 1$ we define the integer $\lambda(r)$ as the minimum $\ell$ such that $2^\ell \geq r$.

A proof of the following result is included in the proof of [6, Theorem 3.10].

**Theorem 13** For any pro-$p$ group $G$ of rank $r$, there exists a $t \leq \lambda(r) + \epsilon$ and a filtration

\[ G_t \subseteq G_{t-1} \subseteq \cdots \subseteq G_0 = G \]

with abelian quotients of exponent $p$ such that $G_t$ is powerful. Recall that $\epsilon = 1$ if $p = 2$ and $\epsilon = 0$ otherwise.

3 Reinterpretation of $G/G^\#$ in the Galois group

**Proof of Theorem 3** We take $k$ to be the residual field of $E$ and $q = |k|$. Since $(n, q-1) = 1$, the map $x \mapsto x^n$ is injective and thus surjective and bijective in $k$. Therefore there exists a unique character

\[ \chi : \Gamma_K \rightarrow k^\times \]

satisfying $\chi^i = \det \rho_i \pmod{p}$ for $i = 1, 2$. Let $\chi$ be the Teichmüller lift of $\chi$. Then all the eigenvalues of $\chi^{-1}(g)\rho_i(g)$ will be in some finite extension $F$ of $E$ and they will reduce to the same $\lambda$ in some finite extension $k'$ of $k$. The characteristic polynomial of the reduction mod $p$ of each $\chi^{-1}(g)\rho_i(g)$ will be of the form $P_i(X) = (X - \lambda)^n$. We write $n = p^\nu m$ with $(m, p) = 1$. We then have

\[ P(X) = (X^{p^\nu} - \lambda^{p^\nu})^m = X^n - m\lambda^{p^\nu}X^{n-p^\nu} + \cdots + \lambda^n \]

so that, since $m \neq 0$ in $k$, $\lambda^{p^\nu} \in k$. This shows that $\lambda \in k$. Since $\lambda^n = \det \rho_i(g)\chi^{-1}(g) = 1$, we obtain $\lambda = 1$. Thus the image of $\Gamma_K$ under the map $\rho(g) = \chi^{-1}(g)(\rho_1(g), \rho_2(g))$ is a pro-$p$ group $G \subset \text{GL}(n, E)^2$. This can easily be seen from [6, Proposition 1.11, p22]: change basis so that both reductions mod $p$ of $\rho_i(\Gamma_K)$, for $i = 1, 2$, have image in the
subgroup $U_k$ of unipotent upper triangular matrices. Let $U_p$ be the inverse image of $U_k$ in $GL(n, \mathcal{O})$. Then $U_k$ is a $p$-group and the kernel of the reduction mod $p$ is the normal subgroup $V = I_n + \mathfrak{m}M(n, \mathcal{O})$ which is a pro-$p$ group. Hence $U_p$ is a pro-$p$ group and $\rho_1(\Gamma_K)$ and $\rho_2(\Gamma_K)$ are closed in $U_p$, because they are compact, thus they also are pro-$p$ groups.

We want to compute the ranks of $G_1 = \rho_1(\Gamma_K)$ and $G_2 = \rho_2(\Gamma_K)$. We begin by embedding $GL(n, \mathcal{O})$ in $GL(n, \mathbb{Z}_p)$ by using a basis of $\mathcal{O}$ over $\mathbb{Z}_p$ to identify $\mathcal{O}^n$ and $\mathbb{Z}_p^N$. Let $M$ be a matrix of $GL(n, \mathcal{O})$ with characteristic polynomial $P(X)$. Its embedding $M_r$ in $GL(N, \mathbb{Z}_p)$ has characteristic polynomial equal to $\prod P^r(X)$, where $\sigma$ runs over the embeddings of $E$ in a fixed algebraic closure of $E$ and $P^r$ is the polynomial obtained from $P$ by applying $\sigma$ to its coefficients. In particular, if $M$ reduces to an unipotent matrix in $GL(n, k)$, its characteristic polynomial is congruent to $(X-1)^n$ modulo $p$ so that the characteristic polynomial of $M_r$ is congruent to $(X-1)^N$ modulo $p$. This means that $M_r$ reduces to an unipotent matrix in $GL(N, \mathbb{F}_p)$. The group of unipotent matrices of $GL(N, \mathbb{F}_p)$ has rank at most $\frac{N(N-1)}{2}$. The kernel of the reduction mod $p$ in $GL(N, \mathbb{Z}_p)$ is $V = I_n + p\mathcal{O}M(N, \mathbb{Z}_p)$.

According to [6] Theorem 5.2, if $p$ is odd then $V$ is powerful of rank $N^2$ while if $p = 2$ then the subgroup $V' = I_n + 4\mathcal{O}M(N, \mathbb{Z}_2)$ is powerful of rank $N^2$ and $V/V'$ is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^{N^2}$, which means that it is a 2-group of rank at most $N^2$. Putting all three terms together, we see that the group of matrices that reduce to the subgroup of unipotent matrices in $GL(n, k)$ has rank at most $r = N^2 \cdot (N^2)^r \cdot \frac{N(N-1)}{2}$. This means that the ranks of $G_1$ and $G_2$ are at most $r$. We can apply Theorem [13] to $G_i$, for $i = 1, 2$: for some $t \leq \lambda(r) + \epsilon$, we get a filtration

$$V_i = G_{i,t} \subseteq G_{i,t-1} \subseteq \cdots \subseteq G_{i,1} \subseteq G_{i,0} = G_i$$

with all quotients $G_{i,s}/G_{i,s+1}$ abelian of exponent $p$ and $V_i$ a powerful pro-$p$ group. Since $V_i$ is powerful, with $m$ more filtration steps we get $V_i^\#$. It is clear that since $V_i \subseteq G_i$, we have $V_i^\# \subseteq G_i^\#$. Since $G^\#$ is the closure of the subgroup generated by the $p^n$-th powers, we see that $G^\# \subseteq G_1^\# \times G_2^\#$. This means that a filtration with at most $\lambda(r) + \epsilon + m$ steps is sufficient to get a subgroup $V^\#$ of $G^\#[2]$. 

On the field side, the $i$-th step of the filtration corresponds to an extension of $K_i$ by an abelian extension of exponent $p$, i.e. the compositum of cyclic extensions of order $p$. This means that $\rho(\Gamma_{K_i}) \subseteq V^\# \subseteq G^\#$.

Then Proposition [9] gives the result. 

\[ \Box \]

**Proposition 14** Let $K$, $n$, $p$, $E$, $\mathcal{O}$, $p$, $k$, $q$ and $S$ be as in Theorem [3] and its proof. Let $\rho_1, \rho_2 : \Gamma_K \rightarrow GL(n, E)$ be two representations unramified outside $S$. Let $K'$ be the compositum of all extensions of $K$ unramified outside $S$ with degree $d$ such that:

$$-d \mid \#GL(n, k)$$

\[ (2) \text{Observe that } r \text{ is not an upper bound for the rank of } G: \text{ the rank of } G \text{ is at most } 2r \text{ but can be greater than } r. \]

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\[-(d, p) = 1\]

\[-d \leq \frac{q^n - 1}{q - 1}.

Denote \(\rho_1\) and \(\rho_2\) the respective restrictions of \(\rho_1\) and \(\rho_2\) to \(\Gamma_{K'}\). Then \(\rho_1\) and \(\rho_2\) satisfy Condition 3 of Theorem 3.

**Proof:** Let \(G\) be a subgroup of \(GL(n, k)\). We consider a flag \(V_0 = \{0\} \subset V_1 \subset \cdots \subset V_{t} = k^n\) such that each \(V_i\) is stable under the action of \(G\) and the action of \(G\) on each quotient \(V_i = V_i / V_{i-1}\) is irreducible. We will denote \(G_i\) the image of \(G\) in Hom\(\left(\bigoplus V_i\right)\) and \(d_i = \dim_k V_i\). In a basis adapted to the flag \((V_i)\), the matrices representing the action of \(G\) on \(k^n\) are blockwise upper-triangular and the \(i\)-th diagonal block of an element \(g \in G\) is equal to the projection of \(g\) in \(G_i\).

Then for any \(i \in \{1, \ldots, \ell\}\), \(G_i\) is a finite group and a subgroup of a general linear group. If \(P\) is a \(p\)-Sylow subgroup of \(G_i\), then the elements of \(P\) are the elements \(g \in G_i\) such that, for certain basis \((e_j)\) of \(V_i\), \(g(e_j) = e_j + \sum_{k < j} \lambda_{k,j} e_k\). In particular a \(p\)-Sylow of \(G_i\) fixes at least one vector in \(V_i\). Let \(e_i\) be such a vector, \(\{e_{i,j}\}_{1 \leq i \leq n}\) its images under the action of \(G_i\) (with \(e_{i,1} = e_i\) and \(H_{i,j} = \text{Fix}_{G_i}(e_{i,j}) = \{g \in G_i : g(e_{i,j}) = e_{i,j}\}\)). We see that \(H_{i,j}\) is a conjugate of \(H_{i,1}\) and thus contains a \(p\)-Sylow of \(G_i\), in particular its index in \(G_i\) is prime to \(p\). Let \(H_i = \bigcap_j H_{i,j}\). Since \(V_i\) is irreducible under the action of \(G_i\), for any \(v \in V_i\) there exists \((\lambda_j) \in k^{d_i}\) such that \(v = \sum_j \lambda_j e_{i,j}\). In particular \(\forall g \in H_i\), \(g(v) = v\) which means that \(H_i = \{1\}\).

Let \(G_{i,j}\) be the inverse image of each \(H_{i,j}\) in \(G\). Since \(G_{i,j}\) is a subgroup of \(G\), we have \([G : G_{i,j}]|\#G\). Since \(G_i\) is a projection of \(G\), we also have \([G : G_{i,j}] = [G_i : H_{i,j}] \leq q^{d_i} - 1\) and \([G : G_{i,j}], p) = ([G_i : H_{i,j}], p) = 1\). It is clear that \(G_{i,j} \cap Z(GL(n, k)) = \{1\}\) because all the elements of \(H_{i,j}\) have eigenvalues equal to 1. Consider \(G' = k^n G\) then we also have \([G' : G_{i,j}] \leq q^{d_i} - 1\) (because \(G_{i,j}\) is the fixator of \(e_{i,j}\) also in \(G'\)). This means that \([G' : k^n G_{i,j}] \leq \frac{q^{d_i} - 1}{q - 1}\). This in turns implies that if \(Z_0 = Z(GL(n, k)) \cap G\) and \(G'_{i,j} = Z_0 G_{i,j}\), then \([G' : k^n G_{i,j}] = [G : G'_{i,j}]\) so that \([G : G'_{i,j}] \leq \frac{q^{d_i} - 1}{q - 1}\). The intersection of all the \(G_{i,j}\) project trivially in each \(G_i\), which means that its elements have eigenvalues equal to 1; we thus see that any \(g \in \bigcap_{i,j} G'_{i,j}\) have all its eigenvalues equal.

To finish the proof, take \(G = (\rho_1 \times \rho_2)(\Gamma_K)\) acting on \(k^n \times k^n\). Observe that \(k^n \times \{0\}\) and \(\{0\} \times k^n\) are both stabilized by \(G\) so that all \(d_i \leq n\). The inverse image in \(\Gamma_K\) of \(G'_{i,j}\) defines an extension that have the properties listed in the hypothesis of the proposition. This means that if \(K_1\) is their compositum, then the elements of \((\rho_1 \times \rho_2)(\Gamma_{K_1})\) have all their eigenvalues equal thus, since \(K_1 \subseteq K'\), \(\rho_1\) and \(\rho_2\) verify Condition 1 of Theorem 3. \(\square\)
4 Numerical application

4.1 Short version

In [2], the authors give an example of two non self-dual representations of Gal($\mathbb{Q}/\mathbb{Q}$) (one should note that the representation coming from the automorphic side is only conjectural) and show that they have equal trace for all primes from 3 to 67. We can apply our result to their example. In our terms, we have $n = 3$, $p = 2$ (so that $m = 2$), $K = \mathbb{Q}$, $E = \mathbb{Q}_2[i]$ and $S = \{2\mathbb{Z}, \infty\}$. We denote by $\rho_1$ and $\rho_2$ the representations they compare. There are no degree 3 and 7 extensions of $\mathbb{Q}$ that ramify only in $S$ so that, according to Proposition 14, Condition I of the theorem is verified. We made a script in gp/pari to search for the extensions described in the construction of the field $\mathbb{Q}_S$. We found that the final compositum is a degree 64 field, which we denote $\mathbb{Q}_{(2)}$. In the paper [2], it is shown that the characteristic polynomial of the image of a Frobenius element $\text{Frob}_p$ depends only on its trace. As a consequence, all the eigenvalues of $\rho_i(\text{Frob}_p)$ are determined by $\text{Tr}\rho_i(\text{Frob}_p)$. The eigenvalues of $\rho_i(\text{Frob}_p^k) = \rho_i(\text{Frob}_p)^k$ are powers of the eigenvalues of $\rho_i(\text{Frob}_p)$, so that the characteristic polynomial of all the $\rho_i(\text{Frob}_p^k)$ are determined by $\text{Tr}\rho_i(\text{Frob}_p)$. This means that we can restrict the comparison to the traces of the images of the elements generating maximal cyclic subgroups. Thanks to gp/pari, we found a list of primes $p$ such that any element of the Galois group of $\mathbb{Q}_{(2)}$ over $\mathbb{Q}$ is (conjugate to) the power of a Frobenius element above $p$. This list is $\{5, 7, 11, 17, 23, 31\}$. The prime 3 is not included just because of the method (and the particular polynomial defining $\mathbb{Q}_{(2)}$) used. Observe that all of the primes have already been checked in the paper [2].

Professor Luis Dieulefait, from Universitat de Barcelona, made me observe that on page 400 of the aforementioned article, the authors note that the geometric representation is absolutely irreducible, which means in particular that it is equal to its semi-simplification. The remark applies obviously also to the conjectural automorphic representation.

Corollary 15 The representation and the tentative representation compared in [2] are isomorphic.

Professor Dieulefait also observed that what is said about $P_7$ in the aforementioned article is also true for $P_7 = X^3 - (1 + 4i)X^2 + 7(1 + 4i)X - 7^3$ (the field generated by one root of $P_7$ is of degree 6 over $\mathbb{Q}$, contains only fourth roots of unity and it is immediate to see that no rational multiple of $i$ is a root of $P_7$). This means that all the members of the family of $\ell$-adic representations are absolutely irreducible, hence semi-simple.

4.2 Longer version

The script used above to look for $K_S$ computes the sequence of fields $(K_i)$. At each step, it computes linearly independent Kummer extensions of $K_i$ and takes their compositum. Since the ramification is rather limited, we tried to detect early (i.e. before computing the compositum) whether an extension is not a sub-extension of $K_{i+1}$. For that purpose, we used the fact that the residual extensions are cyclic, therefore we could not have residual extensions larger than $p^m = 4$. At each step the residual extension is easily computed using
Its Galois group is identified in GAP’s small group library as $[64\times34]$, number such that there is a Frobenius element above if a cyclic subgroup is $\{g^i \mid g \text{ is a generator of the group} \}$. We list them below using the following convention:

- $K_1$ has 6 maximal cyclic subgroups. We list them below using the following convention:
  
  - The center of the group is a two element subgroup generated by $\text{Fr}(337)$.
  
  - $S_3$ ramifying only in $K$ extensions of $Q$ of dimension 3 or 4 over any finite extension of $Q$. Hence, to test for the isomorphism of semi-simplification of representations of $\text{Gal}(\overline{Q}/Q)$ of dimension 3 or 4 over any finite extension of $Q_2$ having $F_2$ as residual field, ramifying only at 2 and $\infty$, it is sufficient to either test
    
    - the traces at primes $\{5, 7, 11, 13, 17, 19, 23, 31, 73, 137, 257, 337\}$;
    
    or
    
    - the characteristic polynomials at primes $\{5, 7, 11, 17, 23, 31\}$.

\[x^{64} - 16x^{61} - 96x^{60} + 144x^{59} + 640x^{58} + 1424x^{57} + 1184x^{56} - 18960x^{55} - 41760x^{54} + 1376x^{53} + 197184x^{52} + 686112x^{51} + 503136x^{50} - 361488x^{49} - 32684x^{48} - 422688x^{47} + 3328944x^{46} + 1914144x^{45} - 9106992x^{44} + 12742688x^{43} - 13880240x^{42} - 2172064x^{41} + 42205032x^{40} - 81439424x^{39} + 70223264x^{38} + 5170976x^{37} - 112924176x^{36} + 181443744x^{35} - 120283616x^{34} - 73923872x^{33} + 28855952x^{32} - 363513856x^{31} + 215744096x^{30} + 79679200x^{29} - 318677792x^{28} + 319483168x^{27} - 79843680x^{26} - 217273248x^{25} + 333944272x^{24} - 161711328x^{23} + 190908864x^{22} + 496539520x^{21} - 579760224x^{20} + 422942592x^{19} - 146636736x^{18} - 98472864x^{17} + 232483000x^{16} - 266632896x^{15} + 254039136x^{14} - 234357888x^{13} + 215933024x^{12} - 190302336x^{11} + 152557600x^{10} - 108211328x^{9} + 67231888x^{8} - 36439104x^{7} + 17140160x^{6} - 6942400x^{5} + 2395872x^{4} - 691136x^{3} + 159168x^{2} - 26240x + 2308.\]

Its Galois group is identified in GAP’s small group library as $[64\times34]$. Up to conjugacy, this group has 6 maximal cyclic subgroups. We list them below using the following convention:

- if a cyclic subgroup is $\{1, g, g^2, \ldots, g^k\}$, we write it as $(1, p_1, p_2, \ldots, p_k)$ where $p_i$ is a prime number such that there is a Frobenius element above $p_i$ that is equal to $g^i$.

The list is:

\[
(1, 5, 137, 13); (1, 7, 257, 7); (1, 11, 73, 19); (1, 17, 337, 17); (1, 23, 257, 23); (1, 31)
\]

The center of the group is a two element subgroup generated by $\text{Fr}(337)$.

Since we have $K_3 = K_4$ and there are no extensions degree 3, 5, 7, 9 or 15 of $Q$ ramifying only in $S$, the discussion above applies also to $n = 4$.

\[\text{Gal}(K_3/K_2) \text{ is the Frattini subgroup of } \text{Gal}(K_3/K_2) \text{ and can be of index 2 if and only if } \text{Gal}(K_3/K_2) \text{ is cyclic of order some power of 2, here at most 4.} \]

\[\text{As an additional proof, we checked with GAP that there is no order 128 group admitting such a chain of Frattini quotients.} \]
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