UNIQUENESS AND NONDEGENERACY OF POSITIVE SOLUTIONS TO KIRCHHOFF EQUATIONS AND ITS APPLICATIONS IN SINGULAR PERTURBATION PROBLEMS

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ABSTRACT. In the present paper, we establish the uniqueness and nondegeneracy of positive energy solutions to the Kirchhoff equation

\[-(a + b \int_{\mathbb{R}^3} |\nabla u|^2) \Delta u + u = |u|^{p-1}u \quad \text{in } \mathbb{R}^3,\]

where \(a, b > 0, 1 < p < 5\) are constants. Then, as applications, we derive the existence and local uniqueness of solutions to the perturbed Kirchhoff problem

\[-\left(\epsilon^2 a + \epsilon b \int_{\mathbb{R}^3} |\nabla u|^2\right) \Delta u + V(x)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^3\]

for \(\epsilon > 0\) sufficiently small, under some mild assumptions on the potential function \(V : \mathbb{R}^3 \to \mathbb{R}\). The existence result is obtained by applying the Lyapunov-Schmidt reduction method. It seems to be the first time to study singularly perturbed Kirchhoff problems by reduction method, as all the previous results were obtained by various variational methods. Another advantage of this approach is that it gives a unified proof to the perturbation problem for all \(p \in (1, 5)\), which is quite different from using variational methods in the literature. The local uniqueness result is totally new. It is obtained by using a type of local Pohozaev identity, which is developed quite recently by Deng, Lin and Yan in their work “On the prescribed scalar curvature problem in \(\mathbb{R}^N\), local uniqueness and periodicity.” (see J. Math. Pures Appl. (9) 104(2015), 1013-1044).

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1. Introduction and main results

1.1. Introduction. Let $a, b > 0$ and $1 < p < 5$. In this paper, we are concerned with the following equation
\begin{equation}
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + u = u^p, \quad u > 0 \quad \text{in} \quad \mathbb{R}^3
\end{equation}
and the related perturbation problem
\begin{equation}
- \left( \epsilon^2 a + \epsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x) u = u^p, \quad u > 0 \quad \text{in} \quad \mathbb{R}^3,
\end{equation}
where $\epsilon > 0$ is a parameter, $V : \mathbb{R}^3 \to \mathbb{R}$ is a bounded continuous function.

Problems (1.1), (1.2) and their variants have been studied extensively in the literature. It was the physician Kirchhoff [31] that proposed the following time dependent wave equation
\begin{equation}
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \right) \frac{\partial^2 u}{\partial x^2} = 0
\end{equation}
for the first time, in order to extend the classical D’Alembert’s wave equations for free vibration of elastic strings. Bernstein [7] and Pohozaev [40] are examples of early research on the study of Kirchhoff equations. Much attention was received until J.L. Lions [35] introducing an abstract functional framework to this problem. More interesting results can be found in e.g. [3, 12, 14] and the references therein. From a mathematical point of view, the interest of studying Kirchhoff equations comes from the nonlocality of Kirchhoff type equations. For instance, the consideration of the stationary analogue of Kirchhoff’s wave equation leads to the Dirichlet problem
\begin{equation}
\begin{cases}
- \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = f(x,u) & \text{in} \quad \Omega, \\
u = 0 & \text{on} \quad \partial \Omega,
\end{cases}
\end{equation}
where $\Omega \subset \mathbb{R}^3$ is a bounded domain, and to equations of type
\begin{equation}
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u = f(x,u) \quad \text{in} \quad \mathbb{R}^3,
\end{equation}
respectively. In the above two problems, \( f \) denotes some nonlinear functions, a typical example of which is given as in Eq. (1.1). Note that the term \( \left( \int |\nabla u|^2 \, dx \right) \Delta u \) depends not only on the pointwise value of \( \Delta u \), but also on the integral of \( |\nabla u|^2 \) over the whole domain. In this sense, Eqs. (1.1), (1.2), (1.3) and (1.4) are no longer the usual pointwise equalities. This new feature brings new mathematical difficulties that make the study of Kirchhoff type equations particularly interesting. We refer to e.g. [39] and to e.g. [15, 18, 25, 27, 33, 34] for mathematical researches on Kirchhoff type equations on bounded domains and in the whole space, respectively.

Eqs. (1.1) and (1.2) are also closely related to Schrödinger equations. Indeed, notice that when the constant \( b \) vanishes, Eqs. (1.1) and (1.2) reduce to the classical Schrödinger equation

\[-\Delta w + w = w^p, \quad w > 0 \quad \text{in } \mathbb{R}^3\]  

and its perturbation problem

\[-\epsilon^2 \Delta u + V(x)u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^3,\]

respectively. They are special cases of

\[-\epsilon^2 \Delta u + V(x)u = u^q, \quad u > 0 \quad \text{in } \mathbb{R}^n,\]  

where \( 1 < q \) is subcritical and \( n \geq 1 \). It is known that Eq. (1.5) admits a unique positive solution (up to translations) which is also nondegenerate (see e.g. [5, 6, 13, 32]). Based on this uniqueness and nondegeneracy property, Flower and Weinstein [19], Oh [37, 38] and many others proved the existence of solutions to Eq. (1.6) for \( \epsilon > 0 \) sufficiently small (the so called semiclassical solutions), by using the Lyapunov-Schmidt reduction method. Their works motivated us to study the uniqueness and nondegeneracy of positive solutions to problem (1.1) and its application in problem (1.2).

Another motivation of this work is due to the fact that up to now there have no results on local uniqueness of concentrating solutions to singularly perturbed Kirchhoff equations, while quite many works have been devoted to the local uniqueness of concentrating solutions to singularly perturbed Schrödinger equations, see e.g. [8, 9, 22, 23] and the references therein. Here, by local uniqueness, it means that it has only one solution in the given class of solutions. As an example, Cao, Li and the second-named author of the present paper recently considered in [9] the Schrödinger equation (1.6) under the assumptions that \( V \) satisfies

1. \( V \) is a bounded \( C^1 \) function and \( \inf_{\mathbb{R}^n} V > 0 \);
2. There exist \( m > 1 \) and \( \delta > 0 \) such that

\[
\begin{align*}
V(x) &= V(a_j) + \sum_{i=1}^n b_{j,i}|x_i - a_{j,i}|^m + O(|x - a_j|^{m+1}), \quad x \in B_{\delta}(a_j), \\
\frac{\partial V}{\partial x_i} &= mb_{j,i}|x_i - a_{j,i}|^{m-2}(x_i - a_{j,i}) + O(|x - a_j|^m), \quad x \in B_{\delta}(a_j),
\end{align*}
\]

where \( x = (x_1, \cdots, x_n) \in \mathbb{R}^n, a_j = (a_{j,1}, \cdots, a_{j,n}) \in \mathbb{R}^n, b_{j,i} \in \mathbb{R} \) with \( b_{j,i} \neq 0 \) for each \( i = 1, \cdots, n \) and \( j = 1, \cdots, k \). By introducing new ideas such as a type of local Pohozaev identity from Deng, Lin and Yan [16], they showed the local uniqueness of multibump solutions to problem (1.6) concentrating at \( k \) different critical points \( \{a_j\}_{j=1}^k \) of the potential \( V \). Here, by concentrating at \( \{a_j\}_{j=1}^k \), it means that if \( u_\epsilon \) is a solution to Eq.
(1.6), then for any $\delta > 0$, there exist $\epsilon_0 > 0$, $R > 1$, such that $u_\epsilon(x) \leq \delta$ for all $|x - a_j| \geq \epsilon R$ and $\epsilon < \epsilon_0$. Throughout their proof, the nondegeneracy result of Kwong [32] on positive solutions to Eq. (1.5) plays a fundamental role. For more local uniqueness results in this respect, see the references in [9]. Local uniqueness results have important applications, as was found for the first time by Deng, Lin and Yan [16]. Indeed, Deng, Lin and Yan [16] considered solutions of a prescribed scalar curvature problem with infinitely many bubbles. By considering the normalized difference of two such solutions, and establishing various Pohozaev-type identities, they proved that solutions with infinitely many bubbles are unique, which then implied the periodicity of such solutions under the additional assumption that the prescribed scalar curvature function is periodic with respect to one or several variables. Guo, Peng and Yan [24] further extend the results of Deng, Lin and Yan [16] to poly-harmonic problems with critical nonlinearity. Due to the fact that local uniqueness problem for singularly perturbed Kirchhoff equations is unknown, in the present paper, we also aim to establish this type of uniqueness results for Eq. (1.2) under suitable assumptions.

1.2. Uniqueness and nondegeneracy results. It is known that Eq. (1.1) is the Euler-Lagrange equation of the energy functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + u^2) + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - \frac{1}{p + 1} \int_{\mathbb{R}^3} |u|^{p+1}$$

for $u \in H^1(\mathbb{R}^3)$. Thus critical point theories have been devoted to find solutions for Eq. (1.1) and its variants, see e.g. [30, 34, 45] and the references therein. In particular, the existence of positive solutions of Eq. (1.1) was obtained by looking for the so called ground states, which is defined as follows: Consider the set of solutions to Eq. (1.1) and denote

$$m = \inf \{ I(v) : v \in H^1(\mathbb{R}^3) \text{ is a nontrivial solution to Eq. (1.2)} \} . \quad (1.7)$$

A nontrivial solution $u$ to Eq. (1.1) is called a ground state if $I(u) = m$.

The following proposition is summarized from the literature for the readers’ convenience.

**Proposition 1.1.** Let $a, b > 0$ and $1 < p < 5$. Let $m$ be the ground state energy defined as in (1.7). Then, there exists a positive ground state of (1.1), and $m > 0$ holds.

Moreover, for any positive solution $u$, there hold

1. (smoothness) $u \in C^\infty(\mathbb{R}^3)$;
2. (symmetry) there exists a decreasing function $v : [0, \infty) \to (0, \infty)$ such that $u = v(|\cdot - x_0|)$ for a point $x_0 \in \mathbb{R}^3$;
3. (Asymptotics) For any multiindex $\alpha \in \mathbb{N}^n$, there exist constants $\delta_\alpha > 0$ and $C_\alpha > 0$ such that

$$|D^\alpha u(x)| \leq C_\alpha e^{-\delta_\alpha|x|} \quad \text{for all } x \in \mathbb{R}^3.$$

The existence of ground states of equation (1.1) is implied by Proposition 1.1 of Ye [45], where more general existence results on Kirchhoff type equations in $\mathbb{R}^3$ are obtained. In the special cases when $3 < p < 5$ and $2 < p < 3$, the existence has also been proved.

\footnote{This reference was brought to us by Ye.}
by He and Zou [30] and Li and Ye [34], respectively. In particular, in the papers Ye [45] and Li and Ye [34], to apply the Mountain Pass Lemma to find a ground state solution, quite complicated manifolds were constructed in order to find a bounded Palais-Smale sequence. The fact that $m > 0$ follows from Li and Ye [34], Lemma 2.8, see also Ye [45]. Other properties follow easily from the theory of classical Schrödinger equations. For applications of Proposition 1.1, see e.g. He and Zou [30], Li and Ye [34] and Ye [45] and the references therein.

Proposition 1.1 provides a good understanding on ground states of Eq. (1.1). However, we are still left an open problem of uniqueness and nondegeneracy of the ground state. In the literature, there exist several interesting results in this respect. For instance, there hold uniqueness and nondegeneracy of positive solutions to the quasilinear Schrödinger equation

$$- \Delta u - u \Delta |u|^2 + u - |u|^{q-1} u = 0 \quad \text{in } \mathbb{R}^n, \quad (1.8)$$

see e.g. [1, 42, 44], and for ground states of the fractional Schrödinger equations ($0 < s < 1 \leq n$)

$$(-\Delta)^s w + w = w^q, \quad w > 0 \quad \text{in } \mathbb{R}^n,$$

see e.g. [17, 20, 21]. In the above examples, $q$ is an index standing for the nonlinearity of subcritical growth. For a systematical research on applications of nondegeneracy of ground states to perturbation problems, we refer to Ambrosetti and Malchiodi [2] and the references therein. Uniqueness and nondegeneracy results also play an important role in many other problems. It is known that the uniqueness and nondegeneracy of ground states are of fundamental importance when one deals with orbital stability or instability of ground states. It mainly removes the possibility that directions of instability come from the kernel of the corresponding linearized operator. The uniqueness and nondegeneracy of ground states also play an important role in blow-up analysis for the corresponding standing wave solutions in the corresponding time-dependent equations, see e.g. Frank et al. [20, 21] and the references therein. Thus, as our first result in this paper, we establish

**Theorem 1.2.** There exists a unique positive radial solution $U \in H^1(\mathbb{R}^3)$ satisfying

$$- \left( a + b \int_{\mathbb{R}^3} |\nabla U|^2 \right) \Delta U + U = U^p, \quad U > 0 \quad \text{in } \mathbb{R}^3. \quad (1.9)$$

Moreover, $U$ is nondegenerate in $H^1(\mathbb{R}^3)$ in the sense that there holds

$$\text{Ker } \mathcal{L} = \text{span} \{ \partial_{x_1} U, \partial_{x_2} U, \partial_{x_3} U \},$$

where $\mathcal{L} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the linear operator defined as

$$L \varphi = - \left( a + b \int_{\mathbb{R}^3} |\nabla U|^2 \right) \Delta \varphi - 2b \left( \int_{\mathbb{R}^3} \nabla U \cdot \nabla \varphi \right) \Delta U + \varphi - pU^{p-1} \varphi \quad (1.10)$$

for all $\varphi \in L^2(\mathbb{R}^3)$.

We remark that our proof of Theorem 1.2 also shows the existence of positive solutions to Eq. (1.1). Moreover, we obtain an almost explicit expression for the solutions, from which Proposition 1.1 follows easily. Also, our proof is unified for all $p$, $1 < p < 5$, which is quite different from the variational methods mentioned above.
In the end of this subsection, let us sketch the proof of Theorem 1.2. Recall that to deduce the uniqueness and nondegeneracy for positive solutions to the local Schrödinger equations (1.6) and (1.8), corresponding ordinary differential equations are used. That is, to consider the ordinary differential equations

\[- \left( u_{rr} + \frac{n-1}{r} u_r \right) + u(r) - u^p(r) = 0, \quad r > 0, \]

and

\[- \left( u_{rr} + \frac{n-1}{r} u_r \right) - u(r) \left( (u^2)_{rr} + \frac{n-1}{r} (u^2)_r \right) + u(r) - u^p(r) = 0, \quad r > 0, \]

respectively, where \( u_r \) is the derivative of \( u \) with respect to \( r \), see e.g. Kwong [32] and Adachi et al. [1]. Therefore, to prove the uniqueness part of Theorem 1.2, it is quite natural to consider the corresponding ordinary differential equation to Eq. (1.1)

\[- \left( a + b \int_0^\infty 4\pi r^2 u^2(r) \right) \left( u_{rr} + \frac{2}{r} u_r \right) + u(r) - u^p(r) = 0 \]

for \( 0 < r < \infty \). However, it turns out that this idea is not so applicable due to the nonlocality of the term \( \int_0^\infty 4\pi r^2 u^2(r) \). To overcome this difficulty, our key observation is that the quantity \( \int_0^\infty 4\pi r^2 u^2(r) \) is, in fact, independent of the choice of the positive solution \( u \). Hence we conclude that the coefficient \( a + b \int_0^\infty 4\pi r^2 u^2(r) \) is just a positive constant that is independent of the given solution \( u \). At this moment, we are allowed to apply the uniqueness result of Kwong [32] on positive solutions to Eq. (1.5) to prove the uniqueness part of Theorem 1.2.

To prove the nondegeneracy part of Theorem 1.2, we apply the spherical harmonics to turn the problem into a system of ordinary differential equations. It turns out that the key is to show that the problem \( \mathcal{L} \varphi = 0 \) has only a trivial radial solution. In other words, the key step is to show that the positive solution \( u \) of Eq. (1.1) is nondegenerate in the subspace of radial functions of \( H^1(\mathbb{R}^3) \). To this end, again the above observation plays an essential role. To be precise, write \( c = a + b \int_0^\infty 4\pi r^2 u^2(r) \) and keep in mind that \( c \) is a constant that is independent of \( u \). Introduce an auxiliary operator \( A_u \) associated to \( u \) by defining

\[ A_u \varphi = -c \Delta \varphi + \varphi - pu^{p-1} \varphi \]

for \( \varphi \in L^2(\mathbb{R}^3) \). Then solving the problem \( \mathcal{L} \varphi = 0 \), where \( \varphi \) is radial, is equivalent to solving

\[ A_u \varphi = 2b \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \right) \Delta u. \]

Since \( A_u \) is the linearized operator of positive solutions to Eq. (1.5) up to a constant, the theory of the nondegeneracy of positive solutions to Eq. (1.5) are applicable, see Proposition 2.2 and Proposition 2.3 below. Finishing this step, the rest of the proof is standard. We refer the readers to the proof of Theorem 1.2 for details.
1.3. **Existence of semiclassical bounded states.** As applications of Theorem 1.2, we look for solutions of (1.2) in the Sobolev space $H^1(\mathbb{R}^3)$ for sufficiently small $\epsilon$. Following Oh [37], we call the solutions as semiclassical solutions. We also call such derived solutions as concentrating solutions since they will concentrate at certain point of the potential function $V$.

First let us review some known results. He and Zou [30] seems to be the first to study singular perturbed Kirchhoff equations. In their work [30], they considered the problem

$$-\left(\epsilon^2 a + \epsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = f(u), \quad u > 0 \quad \text{in } \mathbb{R}^3,$$

where $V$ is assumed to satisfy the global condition of Rabinowitz [41]

$$\liminf_{|x| \to \infty} V(x) > \inf_{x \in \mathbb{R}^3} V(x) > 0, \quad (1.11)$$

and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function with subcritical growth of type $u^q$ for some $3 < q < 5$. By using variational method, they proved the existence of multiple positive solutions for $\epsilon$ sufficiently small. Among other results, Wang, Tian, Xu and Zhang [43] established similar results for Kirchhoff equations with critical growth

$$-\left(\epsilon^2 a + \epsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = f(u) + u^5, \quad u > 0 \quad \text{in } \mathbb{R}^3,$$

by using variational methods as well, where $V$ and $f$ satisfy similar conditions as that of [30]. Based on “penalization method”, He, Li and Peng [28] improved an existence result of Wang, Tian, Xu and Zhang [43] by allowing that $V$ only satisfies local conditions: there exists a bounded open set $\Omega \subset \mathbb{R}^3$ such that

$$\inf_{\Omega} V < \inf_{\partial \Omega} V, \quad (1.12)$$

Later, by introducing new manifold and applying new approximation method of [18], He and Li [27] proved the existence of solutions for $\epsilon$ sufficiently small to the following problem

$$-\left(\epsilon^2 a + \epsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = u^q + u^5, \quad u > 0 \quad \text{in } \mathbb{R}^3,$$

with $V$ satisfying the local condition (1.12) and $1 < q < 3$. Note that one of their innovations of He and Li [27] is that they assume $1 < q < 3$, which is not considered before their paper. This is due to some drawback of variational methods applied in previous researches. He [26] further improved the results of He and Li [27] by considering Kirchhoff problems with more general nonlinearity.

From the above, we summarize that all existing results on singularly perturbed Kirchhoff problems mentioned above are obtained by variational methods. Moreover, to deal with nonlinearity of type $u^q$ for $q$ in different subintervals of $(1,5]$, different variational methods have to be applied. By Theorem 1.2, it is now possible that we apply Lyapunov-Schmidt reduction to study the perturbed Kirchhoff equation (1.2). Moreover, it is expected that this approach can deal with problem (1.2) for all $p$, $1 < p < 5$, in a unified way, as was shown in Theorem 1.2. Indeed, we will derive semiclassical solutions for problem (1.2) by using Lyapunov-Schmidt reduction for all $p$, $1 < p < 5$, in a unified way.
To state our following results, let introduce some notations that will be used throughout the paper. For \( \epsilon > 0 \) and \( y = (y_1, y_2, y_3) \in \mathbb{R}^3 \), write
\[
 u_{\epsilon,y}(x) = u((x-y)/\epsilon), \quad x \in \mathbb{R}^3.
\]
Assume that \( V: \mathbb{R}^3 \to \mathbb{R} \) satisfies the following conditions:

(V1) \( V \) is a bounded continuous function with \( \inf_{x \in \mathbb{R}^3} V > 0 \);

(V2) There exist \( x_0 \in \mathbb{R}^3 \) and \( r_0 > 0 \) such that
\[
 V(x_0) < V(x) \quad \text{for} \quad 0 < |x-x_0| < r_0,
\]
and \( V \in C^\alpha(B_{r_0}(x_0)) \) for some \( 0 < \alpha < 1 \). That is, \( V \) is of \( \alpha \)th order Hölder continuity around \( x_0 \). Without loss of generality, we assume \( x_0 = 0 \), \( r_0 = 10 \) and \( V(x_0) = 1 \) for simplicity.

The assumption (V1) allows us to introduce the inner product
\[
 \langle u, v \rangle_\epsilon = \int_{\mathbb{R}^3} (\epsilon^2 a \nabla u \cdot \nabla v + V(x) uv)
\]
for \( u, v \in H^1(\mathbb{R}^3) \). We also write
\[
 H_\epsilon = \{ u \in H^1(\mathbb{R}^3) : \|u\|_\epsilon \equiv \langle u, u \rangle_\epsilon^{1/2} < \infty \}.
\]
Denote by \( U \in H^1(\mathbb{R}^3) \) the unique positive radial solution to Eq. (1.9). \( U \) plays the role of a building block in the procedure of finding solutions. Now we state the existence result as follows.

**Theorem 1.3.** Let \( a, b > 0 \) and \( 1 < p < 5 \). Suppose that \( V \) satisfies (V1) and (V2). Then there exists \( \epsilon_0 > 0 \) such that for all \( \epsilon \in (0, \epsilon_0) \), problem (1.2) has a solution \( u_\epsilon \) of the form
\[
 u_\epsilon = U \left( \frac{x - y_\epsilon}{\epsilon} \right) + \varphi_\epsilon
\]
with \( \varphi_\epsilon \in H_\epsilon \), satisfying
\[
 y_\epsilon \to x_0,
\]
\[
 \| \varphi_\epsilon \|_\epsilon = o(\epsilon^{3/2})
\]
as \( \epsilon \to 0 \).

We prove Theorem 1.3 by using Lyapunov-Schmidt reduction based on variational methods. It is known that every solution to Eq. (1.2) is a critical point of the energy functional \( I_\epsilon: H_\epsilon \to \mathbb{R} \), given by
\[
 I_\epsilon(u) = \frac{1}{2}\|u\|^2_\epsilon + \frac{eb}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} u_+^{p+1}
\]
for \( u \in H_\epsilon \), where \( u_+ = \max(u, 0) \). It is standard to verify that \( I_\epsilon \in C^2(H_\epsilon) \). So we are left to find a critical point of \( I_\epsilon \). We will follow the scheme of Cao and Peng [11], and reduce the problem to find a critical point of a finite dimensional function (see more details in the next section). However, due to the presence of the nonlocal term \( (\int_{\mathbb{R}^3} |\nabla u|^2) \Delta u \), it requires more careful estimates on the orders of \( \epsilon \) in the procedure. In particular, the
nonlocal term brings new difficulties in the higher order remainder term, which is more complicated than the case of the Schrödinger equation (1.6).

We remark that to establish Theorem 1.3, we can also assume other types of “critical” points in the assumption (V2). However, for simplicity in the present paper, we will restrict ourselves to the case as assumed in (V2).

1.4. Uniqueness of semiclassical bounded states. Now we state the local uniqueness result. We need the following additional assumption on \( V \):

(V3) \( V \in C^1(\mathbb{R}^3) \) and there exist \( m > 1 \) and \( \delta > 0 \) such that

\[
\begin{align*}
V(x) &= V(x_0) + \sum_{i=1}^3 c_i |x_i - x_{0,i}|^m + O(|x - x_0|^{m+1}), \quad x \in B_\delta(x_0), \\
\frac{\partial V}{\partial x_i} &= mc_i |x_i - x_{0,i}|^{m-2}(x_i - x_{0,i}) + O(|x - x_0|^m), \quad x \in B_\delta(x_0),
\end{align*}
\]

where \( c_i \in \mathbb{R} \) and \( c_i \neq 0 \) for \( i = 1, 2, 3 \).

**Theorem 1.4.** Assume that \( V \) satisfies (V1), (V2) and (V3). If \( u_\epsilon^{(i)} \), \( i = 1, 2 \), are two solutions derived as in Theorem 1.3, then

\[ u_\epsilon^{(1)} \equiv u_\epsilon^{(2)} \]

holds for \( \epsilon \) sufficiently small.

Moreover, let \( u_\epsilon = U_{\epsilon, y_\epsilon} + \varphi_\epsilon \) be the unique solution, then there hold

\[
|y_\epsilon - x_0| = o(\epsilon),
\]

\[
\|\varphi_\epsilon\|_\epsilon = O(\epsilon^{3/2+m(1-\tau)})
\]

for some \( 0 < \tau < 1 \) sufficiently small.

We remark that if \( V \) satisfies (V1), (V2) and (V3), then we must have \( c_i > 0 \) for each \( i = 1, 2, 3 \) in (1.14). In fact, the assumption (V2) in Theorem 1.4 is only for the use of the existence result of Theorem 1.3. The arguments of Theorem 1.4 show that we can replace (V2) by working in the class of solutions that satisfy some properties implied by Theorem 1.3. In this way, the coefficients \( c_i \) in (1.14) are allowed to have different signs.

For the sake of brevity we only present Theorem 1.4 here, but leave the more general local uniqueness result in Section 6 (see Theorem 6.4).

To prove Theorem 1.4, we will follow the idea of Cao, Li and Luo [9]. More precisely, if \( u_\epsilon^{(i)} \), \( i = 1, 2 \), are two distinct solutions derived as in Theorem 1.3, then it is clear that the function

\[ \xi_\epsilon = (u_\epsilon^{(1)} - u_\epsilon^{(2)})/\|u_\epsilon^{(1)} - u_\epsilon^{(2)}\|_{L^\infty(\mathbb{R}^3)} \]

satisfies \( \|\xi_\epsilon\|_{L^\infty(\mathbb{R}^3)} = 1 \). We will show, by using the equations satisfied by \( \xi_\epsilon \), that \( \|\xi_\epsilon\|_{L^\infty(\mathbb{R}^3)} \rightarrow 0 \) as \( \epsilon \rightarrow 0 \). This gives a contradiction, and thus follows the uniqueness. To deduce the contradiction, we will need quite delicate estimates on the asymptotic behaviors of solutions and the concentrating point \( y_\epsilon \). A main tool is a local Pohozaev type identity (see (5.1)). Again, due to the presence of the nonlocal term \( (\int_{\mathbb{R}^3} |\nabla u|^2 \Delta u) \), the local Pohozaev identity is more complicated than the case of the Schrödinger equation (1.6). More careful analysis in the procedure are needed.

Before closing this subsection, let us point out that either in the literature cited as above or in the present work, solutions to Eq. (1.2) are of single peak. That is, solutions
concentrate at only one strict local minima of the potential $V$ with only one peak. It has been known for long time that singularly perturbed Schrödinger equations have multi-peak solutions concentrated at one or more critical points of $V$ (see e.g. Oh [38] and Noussair and Yan [36]). However, it seems that there have no results on multi-peak solutions of singularly perturbed Kirchhoff equations. Thus, a natural question to be considered after this work is to construct multi-peak solutions for problem (1.2) under suitable conditions on the potential $V$, and to show that such constructions are locally unique as well. We will explore this problem in the forthcoming paper.

1.5. Organization of the paper and notations. The paper is organized as follows. In section 2, we prove Theorem 1.2. Then, in Section 3 we give some preliminaries that will be used for the applications of Theorem 1.2 later. In section 4 we first reduce the problem of finding a critical point for $I_\varepsilon$ to that of a finite dimensional function, and then complete the proof of Theorem 1.3. In section 5, we further explore some properties of the solutions derived as Theorem 1.3, and introduce a local Pohozaev type identity for solutions to Eq. (1.2). In section 6, we prove Theorem 1.4. For brevity, some elementary but long calculations are left in Appendix A and Appendix B.

Our notations are standard. Denote $u_+ = \max(u, 0)$ for $u \in \mathbb{R}$. We use $B_R(x)$ (and $\bar{B}_R(x)$) to denote open (and close) balls in $\mathbb{R}^3$ centered at $x$ with radius $R$. For any $1 \leq s \leq \infty$, $L^s(\mathbb{R}^3)$ is the standard Banach space of real-valued Lebesgue measurable functions. A function $u$ belongs to the Sobolev space $H^1(\mathbb{R}^3)$ if $u$ and all of its first order weak partial derivatives belong to $L^2(\mathbb{R}^3)$. We use $H^{-1}(\mathbb{R}^3)$ to denote the dual space of $H^1(\mathbb{R}^3)$. For the properties of the Sobolev functions, we refer to the monograph [46]. By the usual abuse of notations, we write $u(x) = u(r)$ with $r = |x|$ whenever $u$ is a radial function in $\mathbb{R}^3$. We will use $C$ and $C_j$ ($j \in \mathbb{N}$) to denote various positive constants, and $O(t), o(t)$ to mean $|O(t)| \leq C |t|$ and $o(t)/t \to 0$ as $t \to 0$, respectively.

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Throughout this section we denote by $Q \in H^1(\mathbb{R}^3)$ the unique positive radial function that satisfies

$$-\Delta Q + Q = Q^p \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

We refer to e.g. Berestycki and Lions [5] and Kwong [32] for the existence and uniqueness of $Q$, respectively.

2.1. Uniqueness. In this subsection we prove the uniqueness part of Theorem 1.2.

Proof of Uniqueness. Let $u \in H^1(\mathbb{R}^3)$ be an arbitrary positive solution to Eq. (1.1). Write

$$c = a + b \int_{\mathbb{R}^3} |
abla u|^2 \, dx$$

so that $u$ satisfies

$$-c \Delta u + u = u^p \quad \text{in } \mathbb{R}^3.$$

Then, it is direct to verify that $u(\sqrt{c}(\cdot - t))$ solves Eq. (2.1) for any $t \in \mathbb{R}^3$. Thus, the uniqueness of $Q$ implies that

$$u(x) = Q \left( \frac{x - t}{\sqrt{c}} \right), \quad x \in \mathbb{R}^3,$$
for some $t \in \mathbb{R}^3$. In particular, we obtain $\int_{\mathbb{R}^3} |\nabla u|^2 \, dx = \sqrt{c} \int_{\mathbb{R}^3} |\nabla Q|^2 \, dx$. Substituting this equality into the definition of $c$ yields

$$c = a + b\|\nabla Q\|_2^2 \sqrt{c}.$$ 

Since $c > 0$, this equation is uniquely solved by

$$\sqrt{c} = \frac{1}{2} \left( b\|\nabla Q\|_2^2 + \sqrt{b^2\|\nabla Q\|_2^4 + 4a} \right). \quad (2.2)$$

As a consequence, we deduce that

$$u(x) = Q \left( \frac{2(x - t)}{b\|\nabla Q\|_2^2 + \sqrt{b^2\|\nabla Q\|_2^4 + 4a}} \right)$$

for some $t \in \mathbb{R}^3$. At this moment, we can easily conclude that the set

$$\mathcal{M} = \left\{ Q \left( \frac{2(x - t)}{b\|\nabla Q\|_2^2 + \sqrt{b^2\|\nabla Q\|_2^4 + 4a}} \right) : t \in \mathbb{R}^3 \right\}$$

consists of all the positive solutions of Eq. (1.2). This finishes the proof. \(\square\)

Note that (2.2) implies that the value of $c$ is independent of the choice of positive solutions. This fact will be used repeatedly below.

As a consequence, we point out that the following result can be derived naturally.

**Corollary 2.1.** The ground state energy $m$ is an isolated critical value of $I$.

### 2.2. Nondegeneracy

In this subsection we prove the nondegeneracy part of Theorem 1.2. We need the following result.

**Proposition 2.2.** Let $1 < p < 5$ and let $Q \in H^1(\mathbb{R}^3)$ be the unique positive radial ground state of Eq. (2.1). Define the operator $A : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ as

$$A\varphi = -\Delta \varphi + \varphi - pQ^{p-1}\varphi$$

for $\varphi \in L^2(\mathbb{R}^3)$. Then the following hold:

1. $Q$ is nondegenerate in $H^1(\mathbb{R}^3)$, that is,
   $$\text{Ker} A = \text{span} \{ \partial_{x_1} Q, \partial_{x_2} Q, \partial_{x_3} Q \};$$

2. The restriction of $A$ on $L^2_{\text{rad}}(\mathbb{R}^3)$ is one-to-one and thus it has an inverse $A^{-1} : L^2_{\text{rad}}(\mathbb{R}^3) \to L^2_{\text{rad}}(\mathbb{R}^3)$;

3. $AQ = -(p - 1)Q^p$ and $AR = -2Q$,

where $R = \frac{2}{p-1}Q + x \cdot \nabla Q$.

For a brief proof of (1), we refer to Chang et al. [13, Lemma 2.1] (see also the references therein); (2) is an easy consequence of (1) since $Q$ is radial and $\text{Ker} A \cap L^2_{\text{rad}}(\mathbb{R}^3) = \emptyset$; the last result can be obtained by a direct computation, see also Eq. (2.1) of Chang et al. [13].

Next, we introduce an auxiliary operator. Let $u$ be a positive solution of Eq. (1.1). Since Eq. (1.1) is translation invariant, we assume with no loss of generality that $u$ is
radially symmetric with respect to the origin. Write \( c = a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \). Keep in mind that \( c \) is a constant that is independent of the choice of \( u \) by (2.2). Then \( u \) satisfies
\[
-c\Delta u + u - u^p = 0 \quad \text{in } \mathbb{R}^3.
\]
Define the auxiliary operator \( A_u : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3) \) as
\[
A_u \varphi = -c\Delta \varphi + \varphi - pu^{p-1}\varphi
\]
for \( \varphi \in L^2(\mathbb{R}^3) \). The following result on \( A_u \) follows easily from Proposition 2.2.

**Proposition 2.3.** \( A_u \) satisfies the following properties:

1. The kernel of \( A_u \) is given by
   \[
   \text{Ker} A_u = \text{span} \{ \partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u \} ;
   \]
2. The restriction of \( A_u \) on \( L^2_{\text{rad}}(\mathbb{R}^3) \) is one-to-one and thus it has an inverse \( A_u^{-1} : L^2_{\text{rad}}(\mathbb{R}^3) \rightarrow \mathbb{R}^3 \);
3. \( A_u u = -(p-1)u^p \) and
   \[
   A_u S = -2u,
   \]
where \( S = \frac{2}{p-1} u + x \cdot \nabla u \).

**Proof.** Apply Proposition 2.2 to \( \bar{u} \) defined by \( \bar{u}(x) = u(\sqrt{c}x) = Q(x) \). We leave the details to the interested readers. \( \square \)

We will also use the standard spherical harmonics to decompose functions in \( H^j(\mathbb{R}^N) \) for \( j = 0, 1 \), where \( N = 3 \) (see e.g. Ambrosetti and Malchiodi [2, Chapter 4]). So let us introduce some necessary notations for the decomposition. Denote by \( \Delta_{\mathbb{S}^{N-1}} \) the Laplacian-Beltrami operator on the unit \( N-1 \) dimensional sphere \( \mathbb{S}^{N-1} \) in \( \mathbb{R}^N \). Write
\[
M_k = \frac{(N+k-1)!}{(N-1)!k!} \quad \forall k \geq 0, \quad \text{and} \quad M_k = 0 \quad \forall k < 0.
\]
Denote by \( Y_{k,l} \), \( k = 0, 1, \ldots \) and \( 1 \leq l \leq M_k - M_{k-2} \), the spherical harmonics such that
\[
-\Delta_{\mathbb{S}^{N-1}} Y_{k,l} = \lambda_k Y_{k,l}
\]
for all \( k = 0, 1, \ldots \) and \( 1 \leq l \leq M_k - M_{k-2} \), where
\[
\lambda_k = k(N+k-2) \quad \forall k \geq 0
\]
is an eigenvalue of \( -\Delta_{\mathbb{S}^{N-1}} \) with multiplicity \( M_k - M_{k-2} \) for all \( k \in \mathbb{N} \). In particular, \( \lambda_0 = 0 \) is of multiplicity 1 with \( Y_{0,1} = 1 \), and \( \lambda_1 = N-1 \) is of multiplicity \( N \) with \( Y_{1,l} = x_l/|x| \) for \( 1 \leq l \leq N \). Then for any function \( v \in H^j(\mathbb{R}^N) \), we have the decomposition
\[
v(x) = v(r\Omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{M_k - M_{k-2}} v_{kl}(r) Y_{kl}(\Omega)
\]
with \( r = |x| \) and \( \Omega = x/|x| \), where
\[
v_{kl}(r) = \int_{\mathbb{S}^{N-1}} v(r\Omega) Y_{kl}(\Omega) \, d\Omega \quad \forall k, l \geq 0.
\]
Note that \( v_{kl} \in H^j(\mathbb{R}^+, r^{N-1}dr) \) holds for all \( k, l \geq 0 \) since \( v \in H^j(\mathbb{R}^N) \).
Now we start the proof of nondeneracy part of Theorem 1.2. We first prove that $u$ is nondegenerate in $H^1_{\text{rad}}(\mathbb{R}^3)$ (in the sense of the following proposition), which is the key ingredient in the proof.

**Proposition 2.4.** Let $\mathcal{L}$ be defined as in (1.10) and let $\varphi \in H^1_{\text{rad}}(\mathbb{R}^3)$ be such that $\mathcal{L}\varphi = 0$. Then $\varphi \equiv 0$ in $\mathbb{R}^3$.

**Proof.** Let $\varphi \in H^1_{\text{rad}}(\mathbb{R}^3)$ be such that $\mathcal{L}\varphi = 0$. By virtue of the notations introduced above, we can rewrite the equation $\mathcal{L}\varphi = 0$ as below:

$$A_u \varphi = 2b \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \right) \Delta u.$$  

We have to prove that $\varphi \equiv 0$. This is sufficient to show that $\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = 0$, (2.4)

since then $\varphi \in \ker A_u \cap L^2_{\text{rad}}(\mathbb{R}^3)$, which implies that $\varphi \equiv 0$ by Proposition 2.3.

To deduce (2.4), we proceed as follows. Since $u$ is radial and $A_u$ is one-to-one on $L^2_{\text{rad}}(\mathbb{R}^3)$ by Proposition 2.3, $\varphi$ satisfies the equivalent equation

$$\varphi = 2b \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \right) A^{-1}_u(\Delta u),$$

where $A_u^{-1}$ is the inverse of $A_u$ restricted on $L^2_{\text{rad}}(\mathbb{R}^3)$. Next we compute $A_u^{-1}(\Delta u)$. By Eq. (2.3), $\Delta u = (u - u^p)/c$. Hence $A_u^{-1}(\Delta u) = (A_u^{-1}(u) - A_u^{-1}(u^p))/c$. Applying Proposition 2.3 (3), we deduce that

$$A_u^{-1}(\Delta u) = \frac{1}{c} \left( -\frac{S}{2} + \frac{u}{p - 1} \right) = -\frac{1}{2c} x \cdot \nabla u,$$

where $S$ is defined as in Proposition 2.3. Therefore, we obtain

$$\varphi = -\frac{b}{c} \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \right) x \cdot \nabla u = \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \right) \psi,$$

with $\psi = -\frac{b}{c} x \cdot \nabla u$.

Now we can deduce (2.4) from the above formula. Taking gradient on both sides gives

$$\nabla \varphi = \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \right) \nabla \psi.$$  

Multiply $\nabla u$ on both sides and integrate. We achieve

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \right) \int_{\mathbb{R}^3} \nabla u \cdot \nabla \psi.$$  

A direct computation yields that

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \psi = \frac{b}{2c} \int_{\mathbb{R}^3} |\nabla u|^2 = \frac{c - a}{2c} < \frac{1}{2},$$

Hence we easily deduce that $\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = 0$, that is, (2.4) holds. The proof of Proposition 2.4 is complete. □
With the help of Proposition 2.4, we can now finish the proof of Theorem 1.2. The procedure is standard, see e.g. Ambrosetti and Malchiodi [2, Section 4.2]. For the readers’ convenience, we give a detailed proof.

Proof of nondeneracy. Let \( \varphi \in H^1(\mathbb{R}^3) \) be such that \( \mathcal{L}\varphi = 0 \). We have to prove that \( \varphi \) is a linear combination of \( \partial_i u \), \( i = 1, 2, 3 \). The idea is to turn the problem \( \mathcal{L}\varphi = 0 \) into a system of ordinary differential equations by making use of the spherical harmonics to decompose \( \varphi \) into

\[
\varphi = \sum_{k=0}^{\infty} \sum_{l=1}^{M_k-M_k-2} \varphi_{kl}(r)Y_{kl}(\Omega)
\]

with \( r = |x| \) and \( \Omega = x/|x| \), where

\[
\varphi_{kl}(r) = \int_{S^2} \varphi(r\Omega)Y_{kl}(\Omega)d\Omega \quad \forall k \geq 0.
\]  

(2.5)

Note that \( \varphi_{kl} \in H^1(\mathbb{R}^+, r^2dr) \) holds for all \( k, l \geq 0 \) since \( \varphi \in H^1(\mathbb{R}^3) \).

Combining the fact that \( \int_{S^2} Y_{kl}d\sigma = 0 \) hold for all \( k, l \geq 1 \), together with the fact that \( u \) is radial, we deduce

\[
\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^3} (-\Delta u) \varphi = \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi_0,
\]

where \( \varphi_0(x) = \varphi_{0,1}(|x|) \) for \( x \in \mathbb{R}^3 \). Hence, the problem \( \mathcal{L}\varphi = 0 \) is equivalent to the following system of ordinary differential equations: For \( k = 0 \), we have

\[
\mathcal{L}\varphi_0 = 0.
\]  

(2.6)

For \( k = 1 \), we have

\[
A_1(\varphi_{1l}) \equiv \left( -c\Delta_r + \frac{\lambda_1}{r^2} \right) \varphi_{1l} + \varphi_{1l} - pu^{p-1}\varphi_{1l} = 0
\]  

(2.7)

for \( l = 1, 2, 3 \). Here \( \Delta_r = \partial_{rr} + \frac{2}{r} \partial_r \). We also used the fact that \( u \) and \( \Delta u \) are radial functions.

For \( k \geq 2 \), we have that

\[
A_k(\varphi_{kl}) \equiv \left( -c\Delta_r + \frac{\lambda_k}{r^2} \right) \varphi_{kl} + \varphi_{kl} - pu^{p-1}\varphi_{kl} = 0.
\]  

(2.8)

To solve Eq. (2.6), we apply Proposition 2.4 to conclude that \( \varphi_0 \equiv 0 \).

To solve Eq. (2.7), note that \( u' \) is a solution of Eq. (2.7) and \( u' \in H^1(\mathbb{R}^+, r^2) \). Since Eq. (2.7) is a second order linear ordinary differential equation, we assume that it has another solution \( v(r) = h(r)u'(r) \) for some \( h \). It is easy to find that \( h \) satisfies

\[
h''u' + \frac{2}{r}h'u' + 2h(u')' = 0.
\]

If \( h \) is not identically a constant, we derive that

\[
-\frac{h''}{h'} = 2\frac{u''}{u} + \frac{2}{r},
\]

which implies that

\[
h'(r) \sim r^{-2}(u')^2 \quad \text{as } r \to \infty.
\]
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Recall that $Q = Q(|x|), x \in \mathbb{R}^3,$ is the unique positive radial solution of Eq. (2.1). It is well known that $\lim_{r \to \infty} re^r Q'(r) = -C$ holds for some constant $C > 0.$ Hence, by the proof of the uniqueness part of Theorem 1.2, we know that $\lim_{r \to \infty} re^{r/\sqrt{c}} u'(r) = -C_1$ for some $C_1 > 0.$ Combining this fact with the above estimates gives

$$|h(r)u'(r)| \geq Cr^{-1}e^{r/\sqrt{c}}$$

as $r \to \infty.$ Thus $hu'$ does not belong to $H^1(\mathbb{R}^3, r^2 dr)$ unless $h$ is a constant. This shows that the family of solutions of Eq. (2.7) in $H^1(\mathbb{R}^3, r^2 dr)$ is given by $hu'$, for some constant $h$. In particular, we conclude that $\varphi_{kl} = d_l u'$ hold for some constant $d_l$, for all $1 \leq l \leq 3.$

For the last Eq. (2.8), we show that it has only a trivial solution. Indeed, for $k \geq 2,$ we have

$$A_k = A_1 + \frac{\delta_k}{r^2},$$

where $\delta_k = \lambda_k - \lambda_1.$ Since $\lambda_k > \lambda_1,$ we find that $\delta_k > 0.$ Notice that $u'$ is an eigenfunction of $A_1$ corresponding to the eigenvalue 0, and that $u'$ is of constant sign. By virtue of orthogonality, we can easily infer that 0 is the smallest eigenvalue of $A_1.$ That is, $A_1$ is a nonnegative operator. Therefore, $\delta_k > 0$ implies that $A_k$ is a positive operator for all $k \geq 2.$ That is, $\langle A_k \psi, \psi \rangle \geq 0$ for all $\psi \in H^1(\mathbb{R}^3, r^2 dr),$ and the equality attains if and only if $\psi = 0.$ As a result, we easily prove that if $\varphi_{kl}$ is a solution of Eq. (2.8), then $\varphi_{kl} \equiv 0$ holds for all $k \geq 2.$

In summary, we obtain

$$\varphi = \sum_{l=1}^3 d_l u'(r)Y_{1l} = \sum_{l=1}^3 d_l \partial_{x_l} u.$$ 

This finishes the proof. 

\[ \Box \]

3. Some preliminaries

In this section, we explain the strategy of the proof of Theorem 1.3 and present some elementary estimates for later use.

First, denote by $U$ the unique positive radial solution of Eq. (1.9) in $H^1(\mathbb{R}^3)$ in the rest of the paper. A useful fact that will be frequently used is the exponential decay of $U$ of and its derivatives. That is,

$$U(x) + |\nabla U(x)| \leq Ce^{-\sigma_0 |x|}, \quad x \in \mathbb{R}^3$$

for some $\sigma_0 > 0$ and $C > 0$ (see Proposition 1.1).

Next, to find solutions for Eq. (1.2) in the form $U_{\epsilon,y} + \varphi$, we introduce a new functional $J_\epsilon : \mathbb{R}^3 \times H_\epsilon \to \mathbb{R}$ defined by

$$J_\epsilon(y, \varphi) = I_\epsilon(U_{\epsilon,y} + \varphi), \quad \varphi \in H_\epsilon.$$ 

See (1.13) for the definition of $I_\epsilon$. Following the scheme of Cao and Peng [11], we divide the proof of Theorem 1.3 into two steps:

**Step 1:** for each $\epsilon, \delta$ sufficiently small and for each $y \in B_{\delta}(0)$, we will find a critical point $\varphi_{\epsilon,y}$ for $J_\epsilon(y, \cdot)$ (the function $y \mapsto \varphi_{\epsilon,y}$ also belongs to the class $C^1(H_\epsilon)$); then
Step 2: for each $\epsilon, \delta$ sufficiently small, we will find a critical point $y_\epsilon$ for the function $j_\epsilon : B_\delta(0) \to \mathbb{R}$ induced by

$$y \mapsto j_\epsilon(y) \equiv J(y, \varphi_{\epsilon,y}). \quad (3.2)$$

That is, we will find a critical point $y_\epsilon$ in the interior of $B_\delta(0)$.

It is standard to verify that $(y_\epsilon, \varphi_{\epsilon,y})$ is a critical point of $J_\epsilon$ for $\epsilon$ sufficiently small by the chain rule. This gives a solution $u_\epsilon \equiv U_{\epsilon,y_\epsilon} + \varphi_{\epsilon,y_\epsilon}$ to Eq. (1.2) for $\epsilon$ sufficiently small in virtue of the following lemma.

Lemma 3.1. There exist $\epsilon_0 > 0$, $\delta_0 > 0$ satisfying the following property: for any $\epsilon \in (0, \epsilon_0)$ and $\delta \in (0, \delta_0)$, $y \in B_\delta(0)$ is a critical point of the function $j_\epsilon$ define as in (3.2) if and only if

$$u_\epsilon \equiv U_{\epsilon,y} + \varphi_{\epsilon,y}$$

is a critical point of $I_\epsilon$.

Lemma 3.1 can be proved in a standard way by using the arguments as that of Bartsch and Peng [4] (see also Cao and Peng [11]). We leave the details for the interested readers.

To realize Step 1, expand $J_\epsilon(y, \cdot)$ near $\varphi = 0$ for each fixed $y$:

$$J_\epsilon(y, \varphi) = J_\epsilon(y, 0) + l_\epsilon(\varphi) + \frac{1}{2} (\mathcal{L}_\epsilon \varphi, \varphi) + R_\epsilon(\varphi),$$

where $J_\epsilon(y, 0) = I_\epsilon(U_{\epsilon,y})$, and $l_\epsilon$, $\mathcal{L}_\epsilon$ and $R_\epsilon$ are defined as follows: for $\varphi, \psi \in H_\epsilon$, define

$$l_\epsilon(\varphi) = (I'_\epsilon(U_{\epsilon,y}), \varphi) = (U_{\epsilon,y}, \varphi) + \epsilon b \left( \int_{\mathbb{R}^3} |\nabla U_{\epsilon,y}|^2 \right) \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \varphi - \int_{\mathbb{R}^3} U_{\epsilon,y}^p \varphi, \quad (3.3)$$

and $\mathcal{L}_\epsilon : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the bilinear form around $U_{\epsilon,y}$ defined by

$$(\mathcal{L}_\epsilon \varphi, \psi) = (I''_\epsilon(U_{\epsilon,y})(\varphi), \psi) = (\varphi, \psi) + \epsilon b \left( \int_{\mathbb{R}^3} |\nabla U_{\epsilon,y}|^2 \right) \int_{\mathbb{R}^3} \nabla \varphi \cdot \nabla \psi + 2\epsilon b \left( \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \varphi \right) \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \psi - p \int_{\mathbb{R}^3} U_{\epsilon,y}^{p-1} \varphi \psi, \quad (3.4)$$

and $R_\epsilon$ denotes the second order reminder term given by

$$R_\epsilon(\varphi) = J_\epsilon(y, \varphi) - J_\epsilon(y, 0) - l_\epsilon(\varphi) - \frac{1}{2} (\mathcal{L}_\epsilon \varphi, \varphi). \quad (3.5)$$

We remark that $R_\epsilon$ belongs to $C^2(H_\epsilon)$ since so is every term in the right hand side of (3.5).

In the rest of this section, we consider $l_\epsilon : H_\epsilon \to \mathbb{R}$ and $R_\epsilon : H_\epsilon \to \mathbb{R}$ and give some elementary estimates.

We will repeatedly use the following type of Sobolev inequality: for any $2 \leq q \leq 6$ there exists a constant $C > 0$ depending only on $n, V, a$ and $q$, but independent of $\epsilon$, such that

$$\|\varphi\|_{L^q(\mathbb{R}^3)} \leq C \epsilon^{3/2} \|\varphi\|_\epsilon$$

(3.6)
holds for all \( \varphi \in H_\epsilon \). The proof of (3.6) follows from an elementary scaling argument and the Sobolev embedding theorems. Indeed, by setting \( \tilde{\varphi}(x) = \varphi(\epsilon x) \) and using Sobolev inequality, we deduce
\[
\int_{\mathbb{R}^3} |\varphi|^q = \epsilon^3 \int_{\mathbb{R}^3} |\tilde{\varphi}|^q 
\leq C_1 \epsilon^3 \left( \int_{\mathbb{R}^3} \left( |\nabla \tilde{\varphi}|^2 + |\tilde{\varphi}|^2 \right) \right)^{q/2} 
= C_1 \epsilon^{3-\frac{2q}{q}} \left( \int_{\mathbb{R}^3} \left( \epsilon^2 |\nabla \varphi|^2 + |\varphi|^2 \right) \right)^{q/2} 
\leq C_2 \epsilon^{3-\frac{2q}{q}} \|\varphi\|_q^q,
\]
where \( C_1 \) is the best constant for the Sobolev embedding \( H^1(\mathbb{R}^3) \subset L^q(\mathbb{R}^3) \), and \( C_2 > 0 \) depends only on \( n, a, q \) and \( V \). We also used the assumption \( \inf_{\mathbb{R}^3} V > 0 \) here.

**Lemma 3.2.** Assume that \( V \) satisfies (V1) (V2). Then, there exists a constant \( C > 0 \), independent of \( \epsilon \), such that for any \( y \in B_1(0) \), there holds
\[
|l_\epsilon(\varphi)| \leq C \epsilon^{\frac{3}{2}} (\epsilon^\alpha + (V(y) - V(0))) \|\varphi\|_\epsilon
\]
for \( \varphi \in H_\epsilon \). Here \( \alpha \) denotes the order of the Hölder continuity of \( V \) in \( B_{10}(0) \).

**Proof.** Since \( U \) solves Eq. (1.9), we deduce from the definition (3.3) of \( l_\epsilon \) that
\[
l_\epsilon(\varphi) = \int_{\mathbb{R}^3} (V(x) - V(0)) U_{\epsilon,y} \varphi 
= \int_{\mathbb{R}^3} (V(x) - V(y)) U_{\epsilon,y} \varphi + (V(y) - V(0)) \int_{\mathbb{R}^3} U_{\epsilon,y} \varphi
=: l_1 + l_2.
\]

To estimate \( l_1 \), we split \( l_1 \) into two parts:
\[
l_1 = \int_{B_1(y)} (V(x) - V(y)) U_{\epsilon,y} \varphi + \int_{\mathbb{R}^3 \setminus B_1(y)} (V(x) - V(y)) U_{\epsilon,y} \varphi
=: l_{11} + l_{12}.
\]
Combining the assumption (V2) and \( y \in B_1(0) \), the exponential decay of \( U \) at infinity and (3.6), we easily derive
\[
|l_{11}| \leq C \epsilon^{\frac{3}{2} + \alpha} \|\varphi\|_\epsilon.
\]
By using the boundedness assumption (V1), (3.1), applying Hölder’s inequality and (3.6), it yields
\[
|l_{12}| \leq C \epsilon^{\frac{3}{2} + \alpha} \|\varphi\|_\epsilon.
\]
Therefore,
\[
|l_1| \leq |l_{11}| + |l_{12}| \leq C \epsilon^{\frac{3}{2} + \alpha} \|\varphi\|_\epsilon.
\]
To estimate \( l_2 \), we use a simple scaling argument and (3.6) to get
\[
|l_2| \leq C (V(y) - V(0)) \epsilon^{\frac{3}{2}} \|\varphi\|_\epsilon.
\]
The proof is complete by combining the estimates of \( l_1 \) and \( l_2 \).

Next we give estimates for \( R_\epsilon \) (see (3.5)) and its derivatives \( R^{(i)}_\epsilon \) for \( i = 1, 2 \).
Lemma 3.3. There exists a constant $C > 0$, independent of $\epsilon$ and $b$, such that for $i \in \{0, 1, 2\}$, there hold
\[
\|R^{(i)}_{\epsilon}(\varphi)\| \leq C \epsilon^{-\frac{3(p-1)}{2}} \|\varphi\|_{p-i}^{p+1-i} + C(b+1)\epsilon^{-\frac{3}{2}} \left(1 + \epsilon^{-\frac{3}{2}} \|\varphi\|_{\epsilon}\right) \|\varphi\|_{\epsilon}^{3-i}
\]
for all $\varphi \in H_{\epsilon}$.

The proof of Lemma 3.3 is elementary but long. We leave it in the Appendix B.

4. The existence of semiclassical solutions

In this section we prove Theorem 1.3.

4.1. Finite dimensional reduction. In this subsection we complete Step 1 as mentioned in Section 3. First we consider the operator $L_{\epsilon}$ defined as in (3.4). That is,
\[
\langle L_{\epsilon} \varphi, \psi \rangle = \langle \varphi, \psi \rangle_{\epsilon} + \epsilon b \left(\int_{\mathbb{R}^3} |\nabla U_{\epsilon,y}|^2 \right) \int_{\mathbb{R}^3} \nabla \varphi \cdot \nabla \psi
\]
\[
+ 2 \epsilon b \left(\int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \varphi \right) \left(\int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \psi \right) - p \int_{\mathbb{R}^3} U_{\epsilon,y}^{p-1} \varphi \psi,
\]
for $\varphi, \psi \in H_{\epsilon}$. Define
\[
E_{\epsilon,y} = \{ u \in H_{\epsilon} : \langle u, \partial_{y} U_{\epsilon,y} \rangle_{\epsilon} = 0 \text{ for } i = 1, 2, 3 \}.
\]
When no confuse occurs, we suppress $y$ in the notation $E_{\epsilon,y}$. Note that $E_{\epsilon,y}$ is a closed subspace of $H_{\epsilon}$ for every $\epsilon > 0$ and $y \in \mathbb{R}^3$. The following result shows that $L_{\epsilon}$ is invertible when restricted on $E_{\epsilon,y}$.

Proposition 4.1. There exist $\epsilon_1 > 0$, $\delta_1 > 0$ and $\rho > 0$ sufficiently small, such that for every $\epsilon \in (0, \epsilon_1)$, $\delta \in (0, \delta_1)$, there holds
\[
\|L_{\epsilon} \varphi\|_{\epsilon} \geq \rho \|\varphi\|_{\epsilon}, \quad \forall \varphi \in E_{\epsilon,y}
\]
uniformly with respect to $y \in B_{\delta}(0)$.

Proof. We use a contradiction argument. Suppose that there exist $\epsilon_n, \delta_n \to 0$, $y_n \in B_{\delta_n}(0)$ and $\varphi_n \in E_{\epsilon_n,y_n}$ satisfying
\[
\langle L_{\epsilon_n} \varphi_n, g \rangle_{\epsilon_n} \leq n^{-1} \|\varphi_n\|_{\epsilon_n} \|g\|_{\epsilon_n}, \quad \forall g \in E_{\epsilon_n,y_n}.
\]
Since this inequality is homogeneous with respect to $\varphi_n$, we can assume that
\[
\|\varphi_n\|_{\epsilon_n}^2 = \epsilon_n^3 \quad \text{for all } n.
\]
Denote $\tilde{\varphi}_n(x) = \varphi_n(\epsilon_n x + y_n)$. Then
\[
\int_{\mathbb{R}^3} \left(a |\nabla \tilde{\varphi}_n|^2 + V(\epsilon_n x + y_n) \tilde{\varphi}_n^2 \right) = 1.
\]
As $V$ is bounded and $\inf_{\mathbb{R}^3} V > 0$, we infer that $\{\tilde{\varphi}_n\}$ is a bounded sequence in $H^1(\mathbb{R}^3)$. Hence, up to a subsequence, we may assume that
\[
\tilde{\varphi}_n \rightharpoonup \varphi \quad \text{in } H^1(\mathbb{R}^3),
\]
\[
\tilde{\varphi}_n \to \varphi \quad \text{in } L^{p+1}_{\text{loc}}(\mathbb{R}^3),
\]
\[
\tilde{\varphi}_n \to \varphi \quad \text{a.e. in } \mathbb{R}^3.
\]
for some $\varphi \in H^1(\mathbb{R}^3)$. We will prove that $\varphi \equiv 0$.

First we prove that $\varphi = \sum_{l=1}^{3} c_l \partial_{x_l} U$ for some $c_l \in \mathbb{R}$. To this end, let $\mathcal{E}_n \equiv \{ \tilde{g} \in H_\epsilon : \tilde{g}_\epsilon, y_n \in E_{\epsilon_n, y_n} \}$, that is,

$$\mathcal{E}_n = \{ \tilde{g} \in H_\epsilon : \int_{\mathbb{R}^3} (a \nabla \partial_{x_i} U \cdot \nabla \tilde{g} + V(\epsilon_n x + y_n) \partial_{x_i} U \tilde{g}) = 0 \text{ for } i = 1, 2, 3 \}.$$  

For convenience, denote at the moment

$$\langle u, v \rangle_{*, n} = \int_{\mathbb{R}^3} (a \nabla u \cdot \nabla v + V(\epsilon_n x + y_n) uv) \quad \text{and} \quad ||u||_{*, n}^2 = \langle u, u \rangle_{*, n}.$$  

Then (4.1) can be rewritten in terms of $\tilde{\varphi}_n$ as follows:

$$\langle \tilde{\varphi}_n, \tilde{g} \rangle_{*, n} + b \int_{\mathbb{R}^3} |\nabla U|^2 \int_{\mathbb{R}^3} \nabla \tilde{\varphi}_n \cdot \nabla \tilde{g}$$

$$+ 2b \int_{\mathbb{R}^3} \nabla U \cdot \nabla \tilde{\varphi}_n \int_{\mathbb{R}^3} \nabla U \cdot \nabla \tilde{g} - p \int_{\mathbb{R}^3} U^{p-1} \tilde{\varphi}_n \tilde{g} \leq n^{-1} ||\tilde{g}_n||_{*, n},$$

where $\tilde{g}_n(x) = g(\epsilon_n x + y_n) \in \mathcal{E}_n$.

Now, for any $g \in C_0^\infty(\mathbb{R}^3)$, define $a'_{l,n} \in \mathbb{R}$ ($1 \leq l \leq 3$) by

$$a'_{l,n} = \langle \partial_{x_l} U, g \rangle_{*, n} / \| \partial_{x_l} U \|_{*, n}^2$$

and let $\tilde{g}_n = g - \sum_{l=1}^{3} a'_{l,n} \partial_{x_l} U$. Note that

$$\| \partial_{x_l} U \|_{*, n}^2 \to \int_{\mathbb{R}^3} (a |\nabla \partial_{x_l} U|^2 + (\partial_{x_l} U)^2) > 0,$$

and for $l \neq j$,

$$\langle \partial_{x_l} U, \partial_{x_j} U \rangle_{*, n} = \int_{\mathbb{R}^3} V(\epsilon_n x + y_n) \partial_{x_l} U \partial_{x_j} U \to \int_{\mathbb{R}^3} \partial_{x_l} U \partial_{x_j} U = 0.$$  

Hence the dominated convergence theorem implies that

$$a'_{l,n} \to a' \equiv \int_{\mathbb{R}^3} (a \nabla \partial_{x_l} U \cdot \nabla g + \partial_{x_l} U g) / \int_{\mathbb{R}^3} (a |\nabla \partial_{x_l} U|^2 + (\partial_{x_l} U)^2)$$

and

$$\langle \partial_{x_l} U, \tilde{g}_n \rangle_{*, n} \to 0$$

as $n \to \infty$. Moreover, we infer that

$$||\tilde{g}_n||_{*, n} = O(1).$$  

Now substituting $\tilde{g}_n$ into (4.2) and letting $n \to \infty$, we find that

$$\langle \mathcal{L} \varphi, g \rangle - \sum_{l=1}^{3} a' \langle \mathcal{L} \varphi, \partial_{x_l} U \rangle = 0,$$

where $\mathcal{L}$ is defined as in (1.10). Since $U_{x_i} \in \text{Ker} \mathcal{L}$ by Theorem 1.2, we have $\langle \mathcal{L} \varphi, \partial_{x_i} U \rangle = 0$. Thus

$$\langle \mathcal{L} \varphi, g \rangle = 0, \quad \forall g \in C_0^\infty(\mathbb{R}^n).$$
This implies that $\varphi \in \text{Ker} \mathcal{L}$. Applying Theorem 1.2 again gives $c^l \in \mathbb{R}$ ($1 \leq l \leq 3$) such that

$$\varphi = \sum_{l=1}^{3} c^l \partial_{x_l} U.$$ 

Next we prove $\varphi \equiv 0$. Note that $\tilde{\varphi}_n \in \tilde{E}_n$, that is,

$$\int_{\mathbb{R}^3} \left( a \nabla \tilde{\varphi}_n \cdot \nabla \partial_{x_l} U + V(\epsilon_n x + y_n) \tilde{\varphi}_n \partial_{x_l} U \right) = 0$$

for each $l = 1, 2, 3$. By sending $n \to \infty$, we derive

$$c^l \int_{\mathbb{R}^3} \left( a |\nabla \partial_{x_l} U|^2 + (\partial_{x_l} U) \right) = 0,$$

which implies $c^l = 0$. Hence

$$\varphi \equiv 0 \quad \text{in} \ \mathbb{R}^3.$$ 

Now we can complete the proof of Proposition 4.1. We have proved that $\tilde{\varphi}_n \to 0$ in $H^1(\mathbb{R}^3)$ and $\tilde{\varphi}_n \to 0$ in $L^{p+1}_{\text{loc}}(\mathbb{R}^3)$. As a result we obtain

$$p \int_{\mathbb{R}^3} U^{p-1}_{\epsilon_n,y_n} \varphi_n^2 = p \epsilon_n^3 \int_{\mathbb{R}^3} U^{p-1}_{\epsilon_n,y_n} \tilde{\varphi}_n^2$$

$$= p \epsilon_n^3 \left( \int_{B_R(0)} U^{p-1}_{\epsilon_n,y_n} \tilde{\varphi}_n^2 + \int_{\mathbb{R}^3 \setminus B_R(0)} U^{p-1}_{\epsilon_n,y_n} \tilde{\varphi}_n^2 \right)$$

$$= p \epsilon_n^3 \left( o(1) + o_R(1) \right),$$

where $o(1) \to 0$ as $n \to \infty$ since $\tilde{\varphi}_n \to 0$ in $L^{p+1}_{\text{loc}}(\mathbb{R}^3)$, and $o_R(1) \to 0$ as $R \to \infty$ since $\tilde{\varphi}_n \in H^1(\mathbb{R}^3)$ is uniformly bounded. Take $R$ sufficiently large. We get

$$p \int_{\mathbb{R}^3} U^{p-1}_{\epsilon_n,y_n} \varphi_n^2 \leq \frac{1}{2} \epsilon_n^3$$

for $n$ sufficiently large. However, this implies that

$$\frac{1}{n} \epsilon_n^3 = \frac{1}{n} ||\varphi_n||_{\epsilon_n}^2 \geq \langle \mathcal{L}_{\epsilon_n} \varphi_n, \varphi_n \rangle$$

$$= ||\varphi_n||_{\epsilon_n}^2 + b \epsilon_n \int_{\mathbb{R}^3} |\nabla U_{\epsilon_n,y_n}|^2 \int_{\mathbb{R}^3} |\nabla \varphi_n|^2$$

$$+ 2b \epsilon_n \left( \int_{\mathbb{R}^3} \nabla U_{\epsilon_n,y_n} \cdot \nabla \varphi_n \right)^2 - p \int_{\mathbb{R}^3} U^{p-1}_{\epsilon_n,y_n} \varphi_n^2$$

$$\geq \frac{1}{2} \epsilon_n^3.$$

We reach a contradiction. The proof is complete. \hfill \Box

Proposition 4.1 implies that by restricting on $E_{\epsilon,y}$, the quadratic form $\mathcal{L}_{\epsilon} : E_{\epsilon,y} \to E_{\epsilon,y}$ has a bounded inverse, with $||\mathcal{L}_{\epsilon}^{-1}|| \leq \rho^{-1}$ uniformly with respect to $y \in B_R(0)$. This further implies the following reduction map.
Proposition 4.2. There exist $\epsilon_0 > 0$, $\delta_0 > 0$ sufficiently small such that for all $\epsilon \in (0, \epsilon_0)$, $\delta \in (0, \delta_0)$, there exists a $C^1$ map $\varphi_\epsilon : B_\delta(0) \to H_\epsilon$ with $y \mapsto \varphi_{\epsilon,y} \in E_{\epsilon,y}$ satisfying

$$
\left< \frac{\partial J_\epsilon(y, \varphi_{\epsilon,y})}{\partial \varphi}, \psi \right>_\epsilon = 0, \quad \forall \psi \in E_{\epsilon,y}.
$$

Moreover, we can choose $\tau \in (0, \alpha/2)$ as small as we wish, such that

$$
\|\varphi_{\epsilon,y}\|_\epsilon \leq \epsilon^\frac{2}{\alpha} \left( \epsilon^{\alpha-\tau} + (V(y) - V(0))^{1-\tau} \right). \quad (4.3)
$$

Proof. This existence of the mapping $y \mapsto \varphi_{\epsilon,y}$ follows from the contraction mapping theorem. We construct a contraction map as follows.

First we show that $A_\epsilon$ exists. Let $y \in B_\delta(0)$ for $\delta < \delta_0$. Recall that

$$
J_\epsilon(y, \varphi) = I_\epsilon(U_{\epsilon,y}) + L_\epsilon(\varphi) + \frac{1}{2}(L_\epsilon \varphi, \varphi) + R_\epsilon(\varphi).
$$

So we have

$$
\frac{\partial J_\epsilon(\varphi)}{\partial \varphi} = l_\epsilon + L_\epsilon \varphi + R_\epsilon'(\varphi).
$$

Since $E_{\epsilon,y}$ is a closed subspace of $H_\epsilon$, Lemma 3.2 and Lemma 3.3 imply that $l_\epsilon$ and $R_\epsilon'(\varphi)$ are bounded linear operators when restricted on $E_{\epsilon,y}$. So we can identify $l_\epsilon$ and $R_\epsilon'(\varphi)$ with their representatives in $E_{\epsilon,y}$. Then, to prove Proposition 4.2, it is equivalent to find $\varphi \in E_{\epsilon,y}$ that satisfies

$$
\varphi = A_\epsilon(\varphi) \equiv -L_\epsilon^{-1} \left( l_\epsilon + R_\epsilon'(\varphi) \right). \quad (4.4)
$$

To solve (4.4), define

$$
N_\epsilon = \left\{ \varphi \in E_{\epsilon,y} : \|\varphi\|_\epsilon \leq \epsilon^\frac{2}{\alpha} \left( \epsilon^{\alpha-\tau} + (V(y) - V(0))^{1-\tau} \right) \right\}
$$
for $\tau \in (0, \alpha/2)$ sufficiently small. We prove that $A_\epsilon : N_\epsilon \to N_\epsilon$ is a contraction map.

First we show that $A_\epsilon(N_\epsilon) \subset N_\epsilon$. Let $\rho > 0$ be the constant defined as in Proposition 4.1 so that $\|L_\epsilon^{-1}\| \leq \rho^{-1}$. Then for $\varphi \in N_\epsilon$ we have

$$
|A_\epsilon(\varphi)| \leq \rho^{-1} \left( \|l_\epsilon\| + \|R_\epsilon'(\varphi)\| \right).
$$

Lemma 3.2 gives

$$
\|l_\epsilon\| \leq C\epsilon^\frac{2}{\alpha} \left( \epsilon^\alpha + (V(y) - V(0)) \right),
$$
where $C$ is independent of $\epsilon$ and $y$. Since $\tau > 0$ and $V(y) \to V(0)$ as $y \to 0$, we may choose $\delta_0$ and $\epsilon_0$ sufficiently small, so that when $\epsilon < \epsilon_0$ and $\delta < \delta_0$ we have

$$
\|l_\epsilon\| \leq \frac{\rho}{2} \left( \epsilon^{\alpha-\tau} + (V(y) - V(0))^{1-\tau} \right). \quad (4.5)
$$

To estimate $\|R_\epsilon'(\varphi)\|$ we use Lemma 3.3. Note that $\varphi \in N_\epsilon$ implies that

$$
\epsilon^{-\frac{2}{\alpha}}\|\varphi\|_\epsilon \leq \epsilon^{\alpha-\tau} + (V(y) - V(0))^{1-\tau} = o(1), \quad (4.6)
$$
where $o(1) \to 0$ as $\epsilon_0, \delta_0 \to 0$. In particular, for $\varphi \in N_\epsilon$ and $\epsilon$ sufficiently small, we have

$$
1 + \epsilon^{-\frac{2}{\alpha}}\|\varphi\|_\epsilon \leq 2. \quad (4.7)
$$

Hence, using Lemma 3.3 and (4.6) gives

$$
\|R_\epsilon'(\varphi)\| \leq C\epsilon^{-\frac{3(p-1)}{2}}\|\varphi\|_e^p + C(b+1)\epsilon^{-\frac{3}{2}}\|\varphi\|_e^2 \leq \frac{\rho}{2}\|\varphi\|_\epsilon. \quad (4.8)
$$
Combining (4.5) and (4.8), we deduce
\[ A \epsilon(\varphi) \leq \epsilon^{\frac{3}{2}} \left( \epsilon^{\alpha-\tau} + (V(y) - V(0))^{1-\tau} \right). \]
This proves that \( A \epsilon(N_\epsilon) \subset N_\epsilon \).

Next we prove that \( A \epsilon : N_\epsilon \rightarrow N_\epsilon \) is a contraction map. For any \( \varphi, \psi \in N_\epsilon \), we have
\[ |A \epsilon(\varphi) - A \epsilon(\psi)| = |L^{-1}_\epsilon(R'_\epsilon(\varphi) - R'_\epsilon(\psi))| \leq \rho^{-1}|(R'_\epsilon(\varphi) - R'_\epsilon(\psi))|. \]
To estimate \( |(R'_\epsilon(\varphi) - R'_\epsilon(\psi))| \), we use the estimate for \( R''_\epsilon \) in Lemma 3.3. Argue similarly as above. We obtain
\[ |(R'_\epsilon(\varphi) - R'_\epsilon(\psi))| \leq C \epsilon^{-3(p-1)} \left( \|\varphi\|_\epsilon^{p-1} + \|\psi\|_\epsilon^{p-1} \right) \|\varphi - \psi\|_\epsilon \leq \frac{p}{2} \|\varphi - \psi\|_\epsilon, \]
where we have used (4.6) and (4.7) again. Hence we deduce that
\[ |A \epsilon(\varphi) - A \epsilon(\psi)| \leq \frac{1}{2} \|\varphi - \psi\|_\epsilon. \]
This shows that \( A \epsilon : N_\epsilon \rightarrow N_\epsilon \) is a contraction map. Thus, there exists a contraction map \( y \rightarrow \varphi_{\epsilon,y} \) such that (4.4) holds.

At last, we claim that the map \( y \rightarrow \varphi_{\epsilon,y} \) belongs to \( C^1 \). Indeed, by similar arguments as that of Cao, Noussair and Yan [10], we can deduce a unique \( C^1 \)-map \( \tilde{\varphi}_{\epsilon,y} : B_{\delta}(0) \rightarrow E_{\epsilon,y} \) which satisfies (4.4). Therefore, by the uniqueness \( \varphi_{\epsilon,y} = \tilde{\varphi}_{\epsilon,y} \), and hence the claim follows. \( \square \)

4.2. **Proof of Theorem 1.3.** First we give the following observation.

**Lemma 4.3.** There holds
\[ \langle L_\epsilon \varphi, \varphi \rangle = O \left( \|\varphi\|^2_\epsilon \right) \]
for \( \varphi \in E_\epsilon \).

**Proof.** The proof is direct. Recall that
\[ \langle L_\epsilon \varphi, \varphi \rangle = \|\varphi\|^2_\epsilon + b \left( \int_{\mathbb{R}^3} |\nabla U_{\epsilon,y}|^2 \int_{\mathbb{R}^3} |\nabla \varphi|^2 + 2 \left( \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \varphi \right)^2 \right) - p \int_{\mathbb{R}^3} U_{\epsilon,y}^{p-1} \varphi^2. \]
Direct computation gives
\[ \epsilon \int_{\mathbb{R}^3} |\nabla U_{\epsilon,y}|^2 \int_{\mathbb{R}^3} |\nabla \varphi|^2 = C_0 \epsilon^2 \int_{\mathbb{R}^3} |\nabla \varphi|^2 \]
with \( C_0 = \int_{\mathbb{R}^3} |\nabla U|^2 \). Thus using this equality and Hölder’s inequality yields
\[ \epsilon \left( \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \varphi \right)^2 = O \left( \|\varphi\|^2_\epsilon \right). \]
The last term \( \int_{\mathbb{R}^3} U_{\epsilon,y}^{p-1} \varphi^2 \) can be estimated by using (3.6). Combing the above estimates together, we complete the proof. \( \square \)
Now we prove Theorem 1.3.

Proof of Theorem 1.3. Let $\epsilon_0$ and $\delta_0$ be defined as in Proposition 4.2 and let $\epsilon < \epsilon_0$. Fix $0 < \delta < \delta_0$. Let $y \mapsto \varphi_{\epsilon,y}$ for $y \in B_\delta(0)$ be the map obtained in Proposition 4.2. As aforementioned in Step 2 in Section 3, it is equivalent to find a critical point for the function $j_\epsilon$ defined as in (3.2) by Lemma 3.1. By the Taylor expansion, we have

$$j_\epsilon(y) = J(y, \varphi_{\epsilon,y}) = I_\epsilon(U_{\epsilon,y}) + l_\epsilon(\varphi_{\epsilon,y}) + \frac{1}{2} \langle L_\epsilon \varphi_{\epsilon,y}, \varphi_{\epsilon,y} \rangle + R_\epsilon(\varphi_{\epsilon,y}).$$

We analyze the asymptotic behavior of $j_\epsilon$ with respect to $\epsilon$ first.

By Proposition A.1, we have

$$I_\epsilon(U_{\epsilon,y}) = A\epsilon^3 + B\epsilon^3 (V(y) - V(0)) + O(\epsilon^{3+\alpha})$$

for some constants $A, B \in \mathbb{R}$. Lemma 3.2 and Proposition 4.2 give

$$l_\epsilon(\varphi_{\epsilon,y}) = O(\epsilon^3) \left( \epsilon^\alpha + (V(y) - V(0)) \left[ \epsilon^{\alpha-\tau} + (V(y) - V(0))^{1-\tau} \right] \right).$$

Lemma 4.3 gives $\langle L_\epsilon \varphi_{\epsilon,y}, \varphi_{\epsilon,y} \rangle = O(\|\varphi_{\epsilon,y}\|^2)$. Lemma 3.3 gives

$$|R_\epsilon(\varphi_{\epsilon,y})| \leq C \left( \epsilon^{-\frac{3(n-1)}{2}} \|\varphi_{\epsilon,y}\|^p + \epsilon^{-\frac{3}{2}} \|\varphi_{\epsilon,y}\|^3 \right) = O(1) \|\varphi_{\epsilon,y}\|^2,$$

where we have used (4.6) since $\varphi_{\epsilon,y} \in N_\epsilon$. Combining the above estimates yields

$$j_\epsilon(y) = A\epsilon^3 + B\epsilon^3 (V(y) - V(0)) + O(\epsilon^{3+\alpha})$$

$$+ O(\epsilon^3) \left( \epsilon^{\alpha-\tau} + (V(y) - V(0))^{1-\tau} \right)^2.$$

Now consider the minimizing problem

$$j_\epsilon(y_\epsilon) \equiv \inf_{y \in B_\delta(0)} j_\epsilon(y).$$

We claim that $y_\epsilon \in B_\delta(0)$. That is, $y_\epsilon$ is an interior point of $B_\delta(0)$.

To prove the claim, we apply a comparison argument. Let $e \in \mathbb{R}^3$ with $|e| = 1$ and $\eta > 1$. We will choose $\eta$ later. Let $z_\epsilon = \epsilon^3 e \in B_\delta(0)$ for a sufficiently large $\eta > 1$. By the above asymptotics formula, we have

$$j_\epsilon(z_\epsilon) = A\epsilon^3 + B\epsilon^3 (V(z_\epsilon) - V(0)) + O(\epsilon^{3+\alpha})$$

$$+ O(\epsilon^3) \left( \epsilon^{\alpha-\tau} + (V(z_\epsilon) - V(0))^{1-\tau} \right)^2.$$

Applying the Hölder continuity of $V$, we derive that

$$j_\epsilon(z_\epsilon) = A\epsilon^3 + O(\epsilon^{3+\alpha})$$

$$+ O(\epsilon^3) \left( \epsilon^{2(\alpha-\tau)} + \epsilon^{2\eta(1-\tau)} \right)$$

$$= A\epsilon^3 + O(\epsilon^{3+\alpha}),$$

where $\eta > 1$ is chosen to be sufficiently large accordingly. Note that we also used the fact that $\tau \ll \alpha/2$. Thus, by using $j(y_\epsilon) \leq j(z_\epsilon)$ we deduce

$$B\epsilon^3 (V(y_\epsilon) - V(0)) + O(\epsilon^3) \left( \epsilon^{\alpha-\tau} + (V(y_\epsilon) - V(0))^{1-\tau} \right)^2 \leq O(\epsilon^{3+\alpha}).$$

That is,

$$B (V(y_\epsilon) - V(0)) + O(1) \left( \epsilon^{\alpha-\tau} + (V(y_\epsilon) - V(0))^{1-\tau} \right)^2 \leq O(\epsilon^\alpha). \quad (4.9)$$
If \( y_\epsilon \in \partial B_\delta(0) \), then by the assumption (V2), we have
\[
V(y_\epsilon) - V(0) \geq c_0 > 0
\]
for some constant \( 0 < c_0 \ll 1 \) since \( V \) is continuous at \( x = 0 \) and \( \delta \) is sufficiently small. Thus, by noting that \( B > 0 \) from Proposition A.1 and sending \( \epsilon \to 0 \), we infer from (4.9) that
\[
c_0 \leq 0.
\]
We reach a contradiction. This proves the claim. Thus \( y_\epsilon \) is a critical point of \( j_\epsilon \) in \( B_\delta(0) \).

**Theorem 1.3** now follows from the claim and Lemma 3.1. \( \square \)

5. **Local Pohozaev identity**

In this section we present some preliminaries for the proof of Theorem 1.4. In particular, we derive a local Pohozaev type identity which plays an important role in the proof of Theorem 1.4.

First we explore some properties of the solutions derived as in Theorem 1.3. Let \( u_\epsilon = U_{\epsilon,y_x} + \varphi_\epsilon \) be a solution of Eq. (1.2). By Theorem 1.3, we know \( \| \varphi_\epsilon \|_\epsilon = O(\epsilon^{3/2}) \).

Thus, a straightforward computation gives
\[
\| u_\epsilon \|_\epsilon = O(\epsilon^{3/2}). \tag{5.1}
\]
Set
\[
\bar{u}_\epsilon(x) = u_\epsilon(\epsilon x + y_\epsilon).
\]
Then \( \bar{u}_\epsilon > 0 \) solves
\[
- \left( a + b \int_{\mathbb{R}^3} |\nabla \bar{u}_\epsilon|^2 \right) \Delta \bar{u}_\epsilon + \bar{V}_\epsilon(x) \bar{u}_\epsilon = \bar{u}_\epsilon^p \quad \text{in} \ \mathbb{R}^3, \tag{5.2}
\]
with \( \bar{V}_\epsilon(x) = V(\epsilon x + y_\epsilon) \). Moreover, there holds
\[
\int_{\mathbb{R}^3} \left( a |\nabla \bar{u}_\epsilon|^2 + \bar{V}_\epsilon \bar{u}_\epsilon^2 \right) = \epsilon^{-3} \| u_\epsilon \|_\epsilon^2 = O(1) \tag{5.3}
\]
by (5.1).

By the assumption (V1), \( \bar{V}_\epsilon \) is bounded uniformly with respect to \( \epsilon \), and
\[
\gamma \equiv \inf_{x \in \mathbb{R}^3} \bar{V}_\epsilon(x) > 0.
\]
Therefore, \( \bar{u}_\epsilon \) satisfies
\[
\begin{cases}
-a \Delta \bar{u}_\epsilon + \gamma \bar{u}_\epsilon \leq \bar{u}_\epsilon^p & \text{in} \ \mathbb{R}^3, \\
\sup_{\epsilon} \| \bar{u}_\epsilon \|_{H^1(\mathbb{R}^3)} \leq C < \infty.
\end{cases}
\]
Using the comparison principle as that of He and Xiang [29], we infer that
\[
\bar{u}_\epsilon(x) \leq C e^{-\eta|x|}, \quad x \in \mathbb{R}^3 \tag{5.4}
\]
holds for some constants \( C, \eta > 0 \) independent of \( \epsilon > 0 \).

We remark that (5.4) is equivalent to
\[
u_\epsilon(x) \leq C e^{-\eta|\epsilon x - y_\epsilon|}, \quad x \in \mathbb{R}^3,
\]
which means that \( u_\epsilon \) concentrates at \( x = 0 \) rapidly as \( \epsilon \to 0 \). In particular, under the additional assumption (V3), we will prove that \( y_\epsilon = o(\epsilon) \), which in turn implies that
$u_{\epsilon}(x) \leq C e^{-\eta |x|/\epsilon}$ for $x \in \mathbb{R}^3$. This shows that the solutions concentrate around the minima of $V$.

Furthermore, by using the Bessel potential, we derive

$$\bar{u}_\epsilon \leq \frac{1}{-a\Delta + \gamma} \bar{u}_\epsilon^p.$$  

Since $p < 5$ is $H^1$-subcritical, the standard potential theory and iteration arguments shows that $\bar{u}_\epsilon \in L^\infty(\mathbb{R}^3)$ and 

$$\|\bar{u}_\epsilon\|_{L^\infty(\mathbb{R}^3)} \leq C < \infty$$

holds for some $C > 0$ uniformly with respect to $\epsilon$. As a consequence of this estimates and the assumption (V1), we further infer from Eq. (5.2) that

$$\|\Delta \bar{u}_\epsilon\|_{L^\infty(\mathbb{R}^3)} \leq C$$  

holds uniformly with respect to sufficiently small $\epsilon > 0$.

Next we derive a local Pohozaev-type identity for solutions of Eq. (1.2).

**Proposition 5.1.** Let $u$ be a positive solution of Eq. (1.2). Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^3$. Then, for each $i = 1, 2, 3$, there hold

$$\int_{\Omega} \frac{\partial V}{\partial x_i} u^2 = \left( \epsilon^2 a + cb \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\partial \Omega} \left( |\nabla u|^2 \nu_i - 2 \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial x_i} \right) 
+ \int_{\partial \Omega} Vu^2 \nu_i - \frac{2}{p+1} \int_{\partial \Omega} u^{p+1} \nu_i.  
$$  

(5.6)

Here $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit outward normal of $\partial \Omega$.

**Proof.** Identity (5.6) is obtained by multiplying $\partial_{x_i} u$ on both sides of Eq. (1.2) and then do integrating by parts. We refer the readers to Proposition 2.3 of Cao, Li and Luo [9] for details. \(\square\)

Now, let $u_{\epsilon} = U_{\epsilon,y_\epsilon} + \varphi_{\epsilon,y_\epsilon}$ be an arbitrary solution of Eq. (1.2) derived as in Theorem 1.3. We know $y_\epsilon = o(1)$ as $\epsilon \to 0$ from Theorem 1.3. We will improve this asymptotics estimate by assuming that $V$ satisfies the additional assumption (V3), and by means of the above Pohozaev type identity. However, before we proceed further, let us give some observations first.

Notice that using polar coordinates, there holds

$$\int_1^2 \int_{\partial B_r(y_\epsilon)} |f| = \int_{\{1 < |x-y_\epsilon| < 2\}} |f| \leq \int_{\mathbb{R}^3} |f|$$

for any $f \in L^1(\mathbb{R}^3)$. So, there exists $d \in (1, 2)$, possibly depending on $f$, such that

$$\int_{\partial B_d(y_\epsilon)} |f| \leq \int_{\mathbb{R}^3} |f|.$$  

Applying this inequality to $f = \epsilon^2 |\nabla \varphi_{\epsilon}|^2 + \varphi_{\epsilon}^2$, we find a constant $d = d_\epsilon \in (1, 2)$ such that

$$\int_{\partial B_d(y_\epsilon)} (\epsilon^2 |\nabla \varphi_{\epsilon}|^2 + \varphi_{\epsilon}^2) \leq \|\varphi_\epsilon\|_{\epsilon}^2.  
$$  

(5.7)
See Lemma 4.5 of [9] for similar applications. Also, for \( d \) defined as above, it follows from (3.1) that
\[
\varepsilon^2 \int_{\partial B_d(y_\varepsilon)} |\nabla U_{\varepsilon,y_\varepsilon}|^2 = O(e^{-\sigma_1/\varepsilon})
\] (5.8)
holds for some \( \sigma_1 > 0 \) independent of \( \varepsilon \). By an elementary inequality, we have
\[
\int_{\partial B_d(y_\varepsilon)} |\nabla u_\varepsilon|^2 \leq 2 \int_{\partial B_d(y_\varepsilon)} |\nabla U_{\varepsilon,y_\varepsilon}|^2 + 2 \int_{\partial B_d(y_\varepsilon)} |\nabla \varphi_\varepsilon|^2.
\]
Hence, for the constant \( d \) chosen as above, we deduce
\[
\varepsilon^2 \int_{\partial B_d(y_\varepsilon)} |\nabla u_\varepsilon|^2 = O(\parallel \varphi_\varepsilon \parallel^2).
\] (5.9)
Now we can improve the estimate for the asymptotic behavior of \( y_\varepsilon \) with respect to \( \varepsilon \).

**Proposition 5.2.** Assume that \( V \) satisfies (V1), (V2) and (V3). Let \( u_\varepsilon = U_{\varepsilon,y_\varepsilon} + \varphi_\varepsilon \) be a solution derived as in Theorem 1.3. Then
\[|y_\varepsilon| = o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0.\]

**Proof.** To analyze the asymptotic behavior of \( y_\varepsilon \) with respect to \( \varepsilon \), we apply the Pohozaev-type identity (5.6) to \( u = u_\varepsilon \) with \( \Omega = B_d(y_\varepsilon) \), where \( d \in (1,2) \) is chosen as in (5.7). Note that \( d \) is possibly dependent on \( \varepsilon \). We get
\[
\int_{B_d(y_\varepsilon)} \frac{\partial V}{\partial x_i} (U_{\varepsilon,y_\varepsilon} + \varphi_\varepsilon)^2 =: \sum_{i=1}^{3} I_i
\] (5.10)
with
\[I_1 = \left( \varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 \right) \int_{\partial B_d(y_\varepsilon)} \left( |\nabla u_\varepsilon|^2 \nu_i - 2 \frac{\partial u_\varepsilon}{\partial \nu} \frac{\partial u_\varepsilon}{\partial x_i} \right),\]
\[I_2 = \int_{\partial B_d(y_\varepsilon)} V u_\varepsilon^2 \nu_i \quad \text{and} \quad I_3 = -\frac{2}{p+1} \int_{\partial B_d(y_\varepsilon)} u_\varepsilon^{p+1} \nu_i.
\]
We estimate each side of (5.10) as follows.

From (5.1) we get
\[
\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 = O(\varepsilon^2).
\]
Thus, from (5.9) we deduce \( I_1 = O(\parallel \varphi_\varepsilon \parallel^2) \). Using similar arguments and choosing a suitable \( d \) if necessary, we also get \( I_2 = O(\parallel \varphi_\varepsilon \parallel^2) \) and \( I_3 = O(\parallel \varphi_\varepsilon \parallel^{p+1}) \). Hence
\[
\sum_{i=1}^{3} I_i = O(\parallel \varphi_\varepsilon \parallel^2).
\] (5.11)
To estimate the left hand side of (5.10), notice that
\[
\int_{B_d(y_\varepsilon)} \frac{\partial V}{\partial x_i} (U_{\varepsilon,y_\varepsilon} + \varphi_\varepsilon)^2 = \int_{B_d(y_\varepsilon)} \frac{\partial V}{\partial x_i} U_{\varepsilon,y_\varepsilon}^2 + O(\varepsilon \parallel \varphi_\varepsilon \parallel) + O(\parallel \varphi_\varepsilon \parallel^2).
\] (5.12)
By the assumption (V3), we deduce, for each $i = 1, 2, 3$,
\[
\int_{B_d(y_i)} \frac{\partial V}{\partial x_i} U_{\epsilon,y_i}^2 = mc_i \int_{B_d(y_i)} |x_i|^{m-2} x_i U_{\epsilon,y_i}^2 + O \left( \int_{B_d(y_i)} |x|^{m} U_{\epsilon,y_i}^2 \right)
= mc_i \epsilon^3 \int_{B_{\frac{\epsilon}{2}}(0)} |\epsilon z_i + y_{\epsilon,i}|^{m-2}(\epsilon z_i + y_{\epsilon,i}) + O \left( \epsilon^3 (\epsilon^m + |y_{\epsilon}|^m) \right) \quad (5.13)
= mc_i \epsilon^3 \int_{\mathbb{R}^3} |\epsilon z_i + y_{\epsilon,i}|^{m-2}(\epsilon z_i + y_{\epsilon,i}) U^2 + O \left( \epsilon^3 (\epsilon^m + |y_{\epsilon}|^m) \right).
\]
We have used (3.1) in the last two lines in the above. (5.12) and (5.13) gives
\[
\int_{B_d(y_i)} \frac{\partial V}{\partial x_i} (U_{\epsilon,y_i} + \varphi_{\epsilon})^2 = mc_i \epsilon^3 \int_{\mathbb{R}^3} |\epsilon z_i + y_{\epsilon,i}|^{m-2}(\epsilon z_i + y_{\epsilon,i}) U^2 \nonumber
+ O \left( \epsilon^2 \|\varphi_{\epsilon}\|_\epsilon + \|\varphi_{\epsilon}\|_\epsilon^2 + \epsilon^3 (\epsilon^m + |y_{\epsilon}|^m) \right). \quad (5.14)
\]
Since $c_i \neq 0$ by assumption (V3), combining (5.10)-(5.14) we deduce
\[
\epsilon^3 \int_{\mathbb{R}^3} |\epsilon z_i + y_{\epsilon,i}|^{m-2}(\epsilon z_i + y_{\epsilon,i}) U^2 = O \left( \epsilon^2 \|\varphi_{\epsilon}\|_\epsilon + \|\varphi_{\epsilon}\|_\epsilon^2 + \epsilon^3 (\epsilon^m + |y_{\epsilon}|^m) \right).
\]
By Proposition 4.2 and (V3),
\[
\|\varphi_{\epsilon}\|_\epsilon = O \left( \epsilon^{3/2}(\epsilon^{m-\tau} + |y_{\epsilon}|^{m(1-\tau)}) \right).
\]
Thus,
\[
\int_{\mathbb{R}^3} |\epsilon z_i + y_{\epsilon,i}|^{m-2}(\epsilon z_i + y_{\epsilon,i}) U^2 = O \left( \epsilon^{m-\tau} + |y_{\epsilon}|^{m(1-\tau)}) \right. \quad (5.15)
\]
On the other hand, let $m^* = \min(m, 2)$. We have
\[
|y_{\epsilon,i}|^m \leq |\epsilon z_i + y_{\epsilon,i}|^m - m|\epsilon z_i + y_{\epsilon,i}|^{m-2}(\epsilon z_i + y_{\epsilon,i})\epsilon z_i \nonumber
+ C \left( |\epsilon z_i + y_{\epsilon,i}|^{m-m^*} |\epsilon z_i|^{m^*} + |\epsilon z_i|^m \right) \quad (5.16)
\]
by the following elementary inequality: for any $e, f \in \mathbb{R}$ and $m > 1$, there holds
\[
|\epsilon + f|^m - |\epsilon|^m - m|\epsilon|^{m-2}ef \leq C \left( |\epsilon|^{m-m^*} |f|^{m^*} + |f|^m \right)
\]
for some $C > 0$ depending only on $m$. So, multiplying (5.16) by $U^2$ on both sides and integrate over $\mathbb{R}^3$. We get
\[
|y_{\epsilon,i}|^m \int_{\mathbb{R}^3} U^2 \leq m \int_{\mathbb{R}^3} |\epsilon z_i + y_{\epsilon,i}|^{m-2}(\epsilon z_i + y_{\epsilon,i})y_{\epsilon,i} U^2 + O \left( \epsilon^m + |y_{\epsilon}|^{m-m^*} \epsilon^{m^*} \right)
\]
for each $i$. Applying (5.15) to the above estimate yields
\[
|y_{\epsilon}|^m = O \left( \left( \epsilon^{m-\tau} + |y_{\epsilon}|^{m(1-\tau)} \right) |y_{\epsilon}| + \epsilon^m + |y_{\epsilon}|^{m-m^*} \epsilon^{m^*} \right)
\]
Recall that $m\tau < 1$. Using $\epsilon$-Young inequality
\[
XY \leq \delta X^m + \delta^{-\frac{m}{m\tau}} Y^{\frac{m}{m\tau}}, \quad \forall \delta, X, Y > 0,
\]
we deduce
\[
|y_{\epsilon}| = O(\epsilon).
\]
We have to prove that \(|y_\epsilon| = o(\epsilon)|. Assume, on the contrary, that there exist \(\epsilon_k \to 0\) and \(y_{\epsilon_k} \to 0\) such that \(y_{\epsilon_k}/\epsilon_k \to A \in \mathbb{R}^3\) with \(A \neq 0\). Then (5.15) gives
\[
\int_{\mathbb{R}^3} \left| z + \frac{y_{\epsilon_k}}{\epsilon_k} \right|^{m-2} \left( z_i + \frac{y_{\epsilon_k}}{\epsilon_k} \right) U^2 = O(\epsilon^{m-\tau}),
\]
Taking limit in the above gives
\[
\int_{\mathbb{R}^3} |z + A|^{m-2} (z + A) U^2(z) = 0.
\]
However, since \(U = U(|z|)\) is strictly decreasing with respect to \(|z|\), we infer that \(A = 0\). We reach a contradiction. The proof is complete. □

As a consequence of Proposition 5.2 and the assumption (V3), we infer that
\[
\|\phi_{\epsilon,y}\|_{\epsilon} = O\left(\epsilon^{3/2} + m(1-\tau)\right).
\]

Here we can take \(\tau\) so small that \(m(1-\tau) > 1\) since \(m > 1\).

6. Proof of Theorem 1.4

This section is devoted to proving Theorem 1.4. We use a contradiction argument as that of Cao, Li and Luo [9]. Assume \(u_{\epsilon}^{(i)} = U_{\epsilon,y_{\epsilon}^{(i)}} + \varphi_{\epsilon}^{(i)}, i = 1, 2\), are two distinct solutions derived as in Theorem 1.3. By (5.4), \(u_{\epsilon}^{(i)}\) are bounded functions in \(\mathbb{R}^3, i = 1, 2\). Set
\[
\xi_{\epsilon} = \frac{u_{\epsilon}^{(1)} - u_{\epsilon}^{(2)}}{\|u_{\epsilon}^{(1)} - u_{\epsilon}^{(2)}\|_{L^\infty(\mathbb{R}^3)}}
\]
and set
\[
\tilde{\xi}_{\epsilon}(x) = \xi_{\epsilon}(\epsilon x + y_{\epsilon}^{(1)}).
\]
It is clear that
\[
\|\tilde{\xi}_{\epsilon}\|_{L^\infty(\mathbb{R}^3)} = 1.
\]
Moreover, by the remark following (5.4), there holds
\[
\tilde{\xi}_{\epsilon}(x) \to 0 \quad \text{as } |x| \to \infty \quad (6.1)
\]
uniformly with respect to sufficiently small \(\epsilon > 0\). We will reach a contradiction by showing that \(\|\tilde{\xi}_{\epsilon}\|_{L^\infty(\mathbb{R}^3)} \to 0\) as \(\epsilon \to 0\). In view of (6.1), it suffices to show that for any fixed \(R > 0\),
\[
\|\tilde{\xi}_{\epsilon}\|_{L^\infty(B_R(0))} \to 0 \quad \text{as } \epsilon \to 0. \quad (6.2)
\]
To this end, we will establish a series of results. First we have

Proposition 6.1. There holds
\[
\|\xi_{\epsilon}\|_\epsilon = O(\epsilon^{3/2}).
\]

Proof. Since both \(u_{\epsilon}^{(i)}, i = 1, 2\), are assumed to be solutions to Eq. (1.2), we obtain that
\[
- \left( \epsilon^2 a + eb \int_{\mathbb{R}^3} |\nabla u_{\epsilon}^{(1)}|^2 \right) \Delta \xi_{\epsilon} + V \xi_{\epsilon}
= eb \left( \int_{\mathbb{R}^3} \nabla(u_{\epsilon}^{(1)} + u_{\epsilon}^{(2)}) \cdot \nabla \xi_{\epsilon} \right) \Delta u_{\epsilon}^{(2)} + C_{\epsilon}(x) \xi_{\epsilon},
\]

and that
\[- \left( \epsilon^2 a + eb \int_{\mathbb{R}^3} \left| \nabla u^{(2)}_\epsilon \right|^2 \right) \Delta \xi_\epsilon + V \xi_\epsilon \]
\[= eb \left( \int_{\mathbb{R}^3} \nabla (u^{(1)}_\epsilon + u^{(2)}_\epsilon) \cdot \nabla \xi_\epsilon \right) \Delta u^{(1)}_\epsilon + C_\epsilon(x) \xi_\epsilon, \]  
(6.4)
where
\[C_\epsilon(x) = p \int_0^1 \left( tu^{(1)}_\epsilon(x) + (1-t)u^{(2)}_\epsilon(x) \right)^{p-1} dt.\]

Adding (6.3) and (6.4) together gives
\[- \left( 2 \epsilon^2 a + eb \int_{\mathbb{R}^3} \left| \nabla u^{(1)}_\epsilon \right|^2 + \left| \nabla u^{(2)}_\epsilon \right|^2 \right) \Delta \xi_\epsilon + 2V \xi_\epsilon \]
\[= eb \left( \int_{\mathbb{R}^3} \nabla (u^{(1)}_\epsilon + u^{(2)}_\epsilon) \cdot \nabla \xi_\epsilon \right) \Delta (u^{(1)}_\epsilon + u^{(2)}_\epsilon) + 2C_\epsilon(x) \xi_\epsilon. \]  
(6.5)

Multiply \( \xi_\epsilon \) on both sides of (6.5) and integrate over \( \mathbb{R}^3 \). By throwing away the terms containing \( b \), we get
\[\| \xi_\epsilon \|^2 \leq \int_{\mathbb{R}^3} C_\epsilon \xi_\epsilon^2 \, dx.\]

On the other hand, note that \( C_\epsilon \leq C \sum_{i=1}^2 (u^{(i)}_\epsilon)^{p-1} \). This implies
\[\int_{\mathbb{R}^3} C_\epsilon \xi_\epsilon^2 \leq C \sum_{i=1}^2 \int_{\mathbb{R}^3} \left( u^{(i)}_\epsilon \right)^{p-1} \xi_\epsilon^2 \]
\[\leq C \sum_{i=1}^2 \left( \int_{\mathbb{R}^3} \left( u^{(i)}_\epsilon \right)^6 \right)^{\frac{p-1}{6}} \left( \int_{\mathbb{R}^3} \left( \xi_\epsilon^2 \right)^{\frac{6}{6-p}} \right)^{\frac{7-p}{6}} \]
\[\leq C \sum_{i=1}^2 \| \nabla u^{(i)}_\epsilon \|_{L^6(\mathbb{R}^3)}^{p-1} \left( \int_{\mathbb{R}^3} \xi_\epsilon^2 \right)^{\frac{7-p}{6}} \]
\[= O(\epsilon^{\frac{p-1}{2}}) \| \xi_\epsilon \|_{L^6(\mathbb{R}^3)}^{(7-p)/3}.\]

In the last inequality we have used the fact that \( \| \xi_\epsilon \|_{L^\infty(\mathbb{R}^3)} = 1 \) and (5.1).

Therefore,
\[\| \xi_\epsilon \|^2 = O(\epsilon^{\frac{p-1}{2}}) \| \xi_\epsilon \|_{L^6(\mathbb{R}^3)}^{(7-p)/3},\]
which implies the desired estimate. The proof is complete.

Next we study the asymptotic behavior of \( \bar{\xi}_\epsilon \).

**Proposition 6.2.** Let \( \bar{\xi}_\epsilon = \xi_\epsilon(\epsilon x + y^{(1)}_\epsilon) \). There exist \( d_i \in \mathbb{R} \), \( i = 1, 2, 3 \), such that (up to a subsequence)
\[\bar{\xi}_\epsilon \rightarrow \sum_{i=1}^3 d_i \partial x_i U \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}^3)\]
as \( \epsilon \rightarrow 0 \).
Proof. It is straightforward to deduce from (6.3) that $\bar{\xi}_\epsilon$ solves
\[
- \left( a + \epsilon^{-1} b \int_{\mathbb{R}^3} |\nabla u^{(1)}_\epsilon|^2 \right) \Delta \bar{\xi}_\epsilon + V(\epsilon x + y^{(1)}_\epsilon) \bar{\xi}_\epsilon \\
= \epsilon^{-1} b \left( \int_{\mathbb{R}^3} \nabla (u^{(1)}_\epsilon + u^{(2)}_\epsilon) \cdot \nabla \xi \right) \Delta \left( u^{(2)}_\epsilon (\epsilon x + y^{(1)}_\epsilon) \right) + C_\epsilon (\epsilon x + y^{(1)}_\epsilon) \bar{\xi}_\epsilon.
\]
(6.6)

For convenience, we introduce $\bar{u}^{(i)}_\epsilon(x) = u^{(i)}_\epsilon(\epsilon x + y^{(1)}_\epsilon)$ and $\bar{\varphi}^{(i)}_\epsilon = \varphi^{(i)}_\epsilon (\epsilon x + y^{(1)}_\epsilon)$ for $i = 1, 2$. Then, we have
\[
\epsilon^{-1} \int_{\mathbb{R}^3} |\nabla u^{(1)}_\epsilon|^2 = \int_{\mathbb{R}^3} |\nabla \bar{u}^{(1)}_\epsilon|^2
\]
and
\[
\epsilon^{-1} b \left( \int_{\mathbb{R}^3} \nabla (u^{(1)}_\epsilon + u^{(2)}_\epsilon) \cdot \nabla \xi \right) = b \int_{\mathbb{R}^3} \nabla \left( \bar{u}^{(1)}_\epsilon + \bar{u}^{(2)}_\epsilon \right) \cdot \nabla \bar{\xi}_\epsilon,
\]
which are uniformly bounded for $\epsilon$ by (5.3) and by (6.10) below. Moreover, we have
\[
\int_{\mathbb{R}^3} |\nabla \bar{\varphi}^{(i)}_\epsilon|^2 = \epsilon^{-3} O \left( \|\bar{\varphi}^{(i)}_\epsilon\|_2^2 \right) = O(\epsilon^{2m(1-\tau)})
\]
by (5.17), and
\[
\int_{\mathbb{R}^3} |\nabla \bar{\xi}_\epsilon|^2 = \epsilon^{-1} \int_{\mathbb{R}^3} |\nabla \xi|^2 = O(1)
\]
by (6.10).

Thus, in view of $\|\bar{\xi}_\epsilon\|_{L^\infty(\mathbb{R}^3)} = 1$ and (5.5) and estimates in the above, the elliptic regularity theory implies that $\bar{\xi}_\epsilon$ is locally uniformly bounded with respect to $\epsilon$ in $C^{1,\beta}_{\text{loc}}(\mathbb{R}^3)$ for some $\beta \in (0, 1)$. As a consequence, we assume (up to a subsequence) that $\bar{\xi}_\epsilon \to \bar{\xi}$ in $C^{1}_{\text{loc}}(\mathbb{R}^3)$.

We claim that $\bar{\xi} \in \text{Ker} \mathcal{L}$, that is,
\[
- \left( a + b \int_{\mathbb{R}^3} |\nabla U|^2 \right) \Delta \bar{\xi} - 2b \left( \int_{\mathbb{R}^3} \nabla U \cdot \nabla \bar{\xi} \right) \Delta U + \bar{\xi} = pU^{p-1} \bar{\xi}.
\]
(6.11)

Then $\bar{\xi} = \sum_{i=1}^3 d_i \partial_{x_i} U$ follows from Theorem 1.2 for some $d_i \in \mathbb{R}$ ($i = 1, 2, 3$), and thus Proposition 6.2 is proved.

To deduce (6.11), we only need to show that (6.11) is the limiting equation of Eq. (6.6). It follows from (6.7) and (6.9) that
\[
\epsilon^{-1} b \int_{\mathbb{R}^3} |\nabla u^{(1)}_\epsilon|^2 - b \int_{\mathbb{R}^3} |\nabla U|^2 = b \int_{\mathbb{R}^3} \left( |\nabla \bar{u}^{(1)}_\epsilon|^2 - |\nabla U|^2 \right) \\
= b \int_{\mathbb{R}^3} \left( |\nabla U + \nabla \bar{\varphi}^{(1)}_\epsilon|^2 - |\nabla U|^2 \right) \\
= O(\epsilon^{m(1-\tau)}) \to 0.
\]
(6.12)
Similarly, we deduce from (6.8) (6.9) and (6.10) that
\[
\int_{\mathbb{R}^3} \nabla \left( \bar{u}_\epsilon^{(1)} + \bar{u}_\epsilon^{(2)} - 2U \right) \cdot \nabla \tilde{\xi}_\epsilon = \int_{\mathbb{R}^3} \nabla \left( U \left( x + (y^{(1)}_\epsilon - y^{(2)}_\epsilon)/\epsilon \right) - U \right) \cdot \nabla \tilde{\xi}_\epsilon \\
+ \int_{\mathbb{R}^3} \nabla \left( \tilde{\varphi}_\epsilon^{(1)} + \tilde{\varphi}_\epsilon^{(2)} \right) \cdot \nabla \tilde{\xi}_\epsilon \\
= o(1),
\]
and that, for any \( \Phi \in C_0^\infty(\mathbb{R}^3) \),
\[
\int_{\mathbb{R}^3} \nabla \left( \bar{u}_\epsilon^{(2)} - U \right) \cdot \nabla \Phi = \int_{\mathbb{R}^3} \nabla \left( U \left( x + (y^{(1)}_\epsilon - y^{(2)}_\epsilon)/\epsilon \right) - U \right) \cdot \nabla \Phi \\
+ \int_{\mathbb{R}^3} \nabla \tilde{\varphi}_\epsilon^{(2)} \cdot \nabla \Phi \to 0.
\]
Here, we have used Proposition 5.2, which implies \((y^{(1)}_\epsilon - y^{(2)}_\epsilon)/\epsilon \to 0\) as \( \epsilon \to 0 \). Combining the above two formulas and (6.8) and \( \tilde{\xi}_\epsilon \to \tilde{\xi} \) in \( C_0^\infty(\mathbb{R}^3) \), we conclude that
\[
\frac{b}{\epsilon} \left( \int_{\mathbb{R}^3} \nabla (u^{(1)}_\epsilon + u^{(2)}_\epsilon) \cdot \nabla \xi \right) \Delta \left( u^{(2)}_\epsilon (\epsilon x + y^{(1)}_\epsilon) \right) \to 2b \left( \int_{\mathbb{R}^3} \nabla U \cdot \nabla \xi \right) \Delta U \quad (6.13)
\]
in \( H^{-1}(\mathbb{R}^3) \).

Also, similar to Lemma 3.2 of Cao, Li and Luo [9], we have for any \( \Phi \in C_0^\infty(\mathbb{R}^3) \),
\[
\int_{\mathbb{R}^3} C_\epsilon (\epsilon x + y^{(1)}_\epsilon) \xi \Phi - p \int_{\mathbb{R}^3} U^{p-1} \xi \Phi = o(1). \quad (6.14)
\]

Finally, combining (6.12) (6.13) (6.14), we obtain (6.11). The proof is complete. □

Now we prove (6.2) by showing the following lemma.

**Lemma 6.3.** Let \( d_i \) be defined as in Proposition 6.2. Then \( d_i = 0 \) for \( i = 1, 2, 3 \).

**Proof.** We use the Pohozaev-type identity (5.6) to prove this lemma. Apply (5.6) to \( u^{(1)}_\epsilon \) and \( u^{(2)}_\epsilon \) with \( \Omega = B_d(y^{(1)}_\epsilon) \), where \( d \) is chosen in the same way as that of Proposition 5.2. We obtain
\[
\int_{B_d(y^{(1)}_\epsilon)} \frac{\partial V}{\partial x_i} \left( \left( u^{(1)}_\epsilon \right)^2 - \left( u^{(2)}_\epsilon \right)^2 \right) \\
= \left( \epsilon^2 a + eb \int_{\mathbb{R}^3} |\nabla u^{(1)}_\epsilon|^2 \right) \int_{\partial B_d(y^{(1)}_\epsilon)} \left( |\nabla u^{(1)}_\epsilon|^2 \nu - 2 \frac{\partial u^{(1)}_\epsilon}{\partial \nu} \frac{\partial u^{(1)}_\epsilon}{\partial x_i} \right) \\
- \left( \epsilon^2 a + eb \int_{\mathbb{R}^3} |\nabla u^{(2)}_\epsilon|^2 \right) \int_{\partial B_d(y^{(1)}_\epsilon)} \left( |\nabla u^{(2)}_\epsilon|^2 \nu - 2 \frac{\partial u^{(2)}_\epsilon}{\partial \nu} \frac{\partial u^{(2)}_\epsilon}{\partial x_i} \right) \\
+ \int_{\partial B_d(y^{(1)}_\epsilon)} V(x) \left( \left( u^{(1)}_\epsilon \right)^2 - \left( u^{(2)}_\epsilon \right)^2 \right) \nu_i \\
- \frac{2}{p+1} \int_{\partial B_d(y^{(1)}_\epsilon)} \left( \left( u^{(1)}_\epsilon \right)^{p+1} - \left( u^{(2)}_\epsilon \right)^{p+1} \right) \nu_i.
\]
In terms of $\xi_i$, we get
\[
\int_{B_d(y^{(1)}_i)} \frac{\partial V}{\partial x_i} \left( u^{(1)}_\varepsilon + u^{(2)}_\varepsilon \right) \xi_i \\
\quad = \left( \varepsilon^2 a + e b \int_{\mathbb{R}^3} \left| \nabla u^{(1)}_\varepsilon \right|^2 \right) \int_{\partial B_d(y^{(1)}_i)} \left| \nabla u^{(1)}_\varepsilon \right|^2 \nu_i - 2 \frac{\partial u^{(1)}_\varepsilon}{\partial \nu} \frac{\partial u^{(1)}_\varepsilon}{\partial x_i} \\
\quad - \left( \varepsilon^2 a + e b \int_{\mathbb{R}^3} \left| \nabla u^{(2)}_\varepsilon \right|^2 \right) \int_{\partial B_d(y^{(1)}_i)} \left| \nabla u^{(2)}_\varepsilon \right|^2 \nu_i - 2 \frac{\partial u^{(2)}_\varepsilon}{\partial \nu} \frac{\partial u^{(2)}_\varepsilon}{\partial x_i} \\
\quad + \int_{\partial B_d(y^{(1)}_i)} V \left( u^{(1)}_\varepsilon + u^{(2)}_\varepsilon \right) \xi_i \nu_i - 2 \int_{\partial B_d(y^{(1)}_i)} A_i \xi_i \nu_i,
\]
where $A_i(x) = \int_0^1 (tu^{(1)}_\varepsilon(x) + (1-t)u^{(2)}_\varepsilon(x))^p$.

We estimate (6.15) term by term. Note that
\[
\varepsilon^2 a + e b \int_{\mathbb{R}^3} \left| \nabla u^{(i)}_\varepsilon \right|^2 = O(\varepsilon^2)
\]
holds by (5.1) for each $i = 1, 2$. Moreover, by similar arguments as that of Proposition 5.2, we have
\[
\int_{\partial B_d(y^{(1)}_i)} \left| \nabla u^{(i)}_\varepsilon \right|^2 = O(\|\nabla \varphi^{(i)}_\varepsilon\|_{L^2(\mathbb{R}^3)}^2).
\]
Thus, by (6.17),
\[
2 \left( \varepsilon^2 a + e b \int_{\mathbb{R}^3} \left| \nabla u^{(i)}_\varepsilon \right|^2 \right) \int_{\partial B_d(y^{(1)}_i)} \left| \nabla u^{(i)}_\varepsilon \right|^2 \nu_i - 2 \frac{\partial u^{(i)}_\varepsilon}{\partial \nu} \frac{\partial u^{(i)}_\varepsilon}{\partial x_i} \\
\quad = \sum_{i=1}^2 O \left( \varepsilon^2 \|\nabla \varphi^{(i)}_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 \right) \\
\quad = O(\varepsilon^{3+2m(1-\tau)}).
\]
Also, similar to that of Cao, Li and Luo [9], we have
\[
\int_{\partial B_d(y^{(1)}_i)} V(x) \left( u^{(1)}_\varepsilon + u^{(2)}_\varepsilon \right) \xi_i \nu_i = O(\varepsilon^{3+m(1-\tau)})
\]
and
\[
\int_{\partial B_d(y^{(1)}_i)} A_i \xi_i \nu_i = O(\varepsilon^{3+m(1-\tau)})^p.
\]
Hence we conclude that
\[
\text{the RHS of (6.15)} = O(\varepsilon^{3+m(1-\tau)}).
\]

Next we estimate the left hand side of (6.15). We have
\[
\int_{B_d(y^{(1)}_i)} \frac{\partial V}{\partial x_i} \left( u^{(1)}_\varepsilon + u^{(2)}_\varepsilon \right) \xi_i \\
\quad = mc_i \int_{B_d(y^{(1)}_i)} |x_i|^{m-2} x_i \left( u^{(1)}_\varepsilon + u^{(2)}_\varepsilon \right) \xi_i + O \left( \int_{B_d(y^{(1)}_i)} |x_i|^m \left( u^{(1)}_\varepsilon + u^{(2)}_\varepsilon \right) \xi_i \right).
\]
Observe that
\[
mc_i \int_{B_d(y_i^{(1)})} |x_i|^{m-2} x_i \left( u^{(1)}_\epsilon + u^{(2)}_\epsilon \right) \xi_\epsilon \\
= mc_i \epsilon^3 \int_{B_{\epsilon \xi} (y_i^{(0)})} |\epsilon y_i + y_i^{(1)}|^{m-2} \left( \epsilon y_i + y_i^{(1)} \right) \left( U(y) + U \left( y + \frac{y^{(1)}_\epsilon - y^{(2)}_\epsilon}{\epsilon} \right) \right) \bar{\xi}_\epsilon \\
+ mc_i \int_{B_d(y_i^{(1)})} |x_i|^{m-2} x_i \left( \phi^{(1)}_\epsilon + \phi^{(2)}_\epsilon \right) \xi_\epsilon.
\]
Since \( U \) decays exponentially at infinity and \( y_i^{(i)} = o(\epsilon) \), using Proposition 6.2 we deduce
\[
mc_i \epsilon^3 \int_{B_{\epsilon \xi} (y_i^{(0)})} |\epsilon y_i + y_i^{(1)}|^{m-2} \left( \epsilon y_i + y_i^{(1)} \right) \left( U(y) + U \left( y + \frac{y^{(1)}_\epsilon - y^{(2)}_\epsilon}{\epsilon} \right) \right) \bar{\xi}_\epsilon \\
= 2mc_i \epsilon^{m+2} \sum_{j=1}^{3} d_j \int_{\mathbb{R}^3} |y_j|^{m-2} y_j U(y) \partial_{x_j} U + o(\epsilon^{m+2}) \\
= D_i d_i \epsilon^{m+2} + o(\epsilon^{m+2}),
\]
where
\[
D_i = 2mc_i \int_{\mathbb{R}^3} |y|^{m-2} y U(y) \partial_{x_i} U \neq 0.
\]
In the last equality of (6.18), we used the fact that \( U \) is a radially symmetric function. On the other hand, by Hölder’s inequality, (5.17) and Proposition 6.1, we have
\[
mc_i \int_{B_d(y_i^{(1)})} |x_i|^{m-2} x_i \left( \phi^{(1)}_\epsilon + \phi^{(2)}_\epsilon \right) \xi_\epsilon = \sum_{i=1}^{2} O \left( \int_{\mathbb{R}^3} |\phi^{(i)}_\epsilon| \|\xi_\epsilon\| \right) \\
= \sum_{i=1}^{2} O(\|\phi^{(i)}_\epsilon\| \|\xi_\epsilon\|) = O(\epsilon^{3m(1-\tau)}).
\]
Therefore, from (6.18) and (6.20), we deduce
\[
mc_i \int_{B_d(y_i^{(1)})} |x_i|^{m-2} x_i \left( u^{(1)}_\epsilon + u^{(2)}_\epsilon \right) \xi_\epsilon dx = D_i d_i \epsilon^{m+2} + o(\epsilon^{m+2}),
\]
with \( D_i \neq 0 \) given by (6.19). Similar arguments gives
\[
O \left( \int_{B_d(y_i^{(1)})} |x_i|^{m} \left( u^{(1)}_\epsilon + u^{(2)}_\epsilon \right) \xi_\epsilon dx \right) = O(\epsilon^{m+3}).
\]
Hence, combining (6.21) and (6.22), we obtain
\[
\text{the RHS of (6.15) = } D_i d_i \epsilon^{m+2} + o(\epsilon^{m+2}).
\]
At last, this lemma follows from (6.16) and (6.23). The proof is complete. □

Now we can prove Theorem 1.4.
Proof of Theorem 1.4. If there exist two distinct solutions \( u^{(i)}_\epsilon, i = 1, 2 \), then by setting \( \xi_\epsilon \) and \( \bar{\xi}_\epsilon \) as above, we find that

\[
\|\bar{\xi}_\epsilon\|_{L^\infty(\mathbb{R}^3)} = 1
\]

by assumption, and that

\[
\|\bar{\xi}_\epsilon\|_{L^\infty(\mathbb{R}^3)} = o(1) \quad \text{as} \quad \epsilon \to 0
\]

by (6.1) and (6.2). We reach a contradiction. The proof is complete. \( \square \)

We close this paper by remarking that the proof of Theorem 1.4 implies the following slightly more general uniqueness result, which allows \( c_i \) has different signs for \( i = 1, 2, 3 \) in the assumption (V3).

**Theorem 6.4.** Let \( a, b > 0 \) and \( 1 < p < 5 \). Assume that \( V \) satisfies \((V1)\) and \((V3)\). If \( u^{(i)}_\epsilon = U_{\epsilon, y^{(i)}_\epsilon} + \varphi^{(i)}_{\epsilon, y^{(i)}_\epsilon} \), \( i = 1, 2 \), are two solutions to Eq. \((1.2)\) satisfying \( y_\epsilon \to 0 \) and

\[
\left\| \varphi^{(i)}_{\epsilon, y^{(i)}_\epsilon} \right\|_\epsilon \leq \frac{3}{2} \left( \epsilon^{\alpha - \tau} + (V(y^{(i)}_\epsilon) - V(0))^{1-\tau} \right)
\]

for some \( \tau > 0 \) sufficiently small. Then

\[
u^{(1)}_\epsilon \equiv u^{(2)}_\epsilon.
\]

Moreover, writing \( u_\epsilon = U_{\epsilon, y_\epsilon} + \varphi_\epsilon \) as the unique solution, there holds

\[
y_\epsilon = o(\epsilon),
\]

\[
\|\varphi_\epsilon\|_\epsilon = O \left( \epsilon^{\frac{3}{2} + m(1-\tau)} \right).
\]

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**Appendix A. The unperturbed problem**

Let \( U \) be the unique positive radial solution of Eq. \((1.9)\) (see Proposition 1.2). Then, for any \( \epsilon > 0 \) and \( y \in \mathbb{R}^3 \), the function

\[
U_{\epsilon, y}(x) \equiv U \left( \frac{x - y}{\epsilon} \right), \quad x \in \mathbb{R}^3
\]

are the unique positive solutions to equation

\[
- \left( \epsilon^2 a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + u = u^p \quad \text{in} \ \mathbb{R}^3.
\]

We also recall that in the assumption \((V2)\) we assumed that \( x_0 = 0, r_0 = 10 \) and \( V(0) = 1 \).

**Proposition A.1.** Assume that \( V \) satisfies \((V1)\) and \((V2)\). Let \( y \in B_1(0) \). Then , for \( \epsilon > 0 \) sufficiently small, we have

\[
L(U_{\epsilon, y}) = A \epsilon^3 + B \epsilon^3 (V(y) - V(0)) + O(\epsilon^{3+\alpha}),
\]

where

\[
A = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla U|^2 + U^2) + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla U|^2 \right)^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} U^{p+1}
\]
and

\[ B = \frac{1}{2} \int_{\mathbb{R}^3} U^2. \]

**Proof.** By direct computation, we obtain

\[
I_\epsilon(U_{\epsilon,y}) = \frac{1}{2} \int_{\mathbb{R}^3} \left( \epsilon^2 a |\nabla U_{\epsilon,y}|^2 + V(x) U_{\epsilon,y}^2 \right) + \frac{\epsilon b}{4} \left( \int_{\mathbb{R}^3} |\nabla U_{\epsilon,y}|^2 \right)^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} U_{\epsilon,y}^{p+1}
\]

\[
= A \epsilon^3 + \frac{1}{2} \int_{\mathbb{R}^3} (V(x) - V(0)) U_{\epsilon,y}^2
\]

\[
= A \epsilon^3 + B \epsilon^3 (V(y) - V(0)) + \frac{1}{2} \int_{\mathbb{R}^3} (V(x) - V(y)) U_{\epsilon,y}^2,
\]

where \( A, B \) are given as in the result.

Split the last term into two terms:

\[
\int_{\mathbb{R}^3} (V(x) - V(y)) U_{\epsilon,y}^2 = \int_{B_1(y)} (V(x) - V(y)) U_{\epsilon,y}^2 + \int_{\mathbb{R}^3 \setminus B_1(y)} (V(x) - V(y)) U_{\epsilon,y}^2.
\]

Using the assumption (V2) and \( y \in B_1(0) \), we deduce

\[
\int_{B_1(y)} |V(x) - V(y)| U_{\epsilon,y}^2 = O(\epsilon^{3+\alpha}).
\]

Using the boundedness of \( V \) and the exponential decay of \( U \), we deduce

\[
\int_{\mathbb{R}^3 \setminus B_1(y)} |V(x) - V(y)| U_{\epsilon,y}^2 = O(\epsilon^{3+\alpha}).
\]

Combining above estimates gives the desired estimates. The proof is complete. \( \square \)

**Appendix B. Proof of Lemma 3.3**

This section is devoted to the proof of Lemma 3.3. Recall that \( R_\epsilon \) is defined as in (3.5), which gives

\[
R_\epsilon(\varphi) = A_1(\varphi) - A_2(\varphi),
\]

where

\[
A_1(\varphi) = \frac{b \epsilon}{4} \left( \left( \int_{\mathbb{R}^3} |\nabla \varphi|^2 \right)^2 + 4 \int_{\mathbb{R}^3} |\nabla \varphi| \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \varphi \right)
\]

and

\[
A_2(\varphi) = \frac{1}{p+1} \int_{\mathbb{R}^3} \left( (U_{\epsilon,y} + \varphi)^{p+1} - U_{\epsilon,y}^{p+1} - (p+1) U_{\epsilon,y}^p \varphi - \frac{p(p+1)}{2} U_{\epsilon,y}^{p-1} \varphi^2 \right).
\]

Use \( R_\epsilon^{(i)} \) to denote the \( i \)th derivative of \( R_\epsilon \), and also use similar notations for \( A_1 \) and \( A_2 \). By direct computations, we deduce that, for any \( \varphi, \psi \in H_\epsilon \),

\[
\langle R_\epsilon^{(1)}(\varphi), \psi \rangle = \langle A_1^{(1)}(\varphi), \psi \rangle - \langle A_2^{(1)}(\varphi), \psi \rangle
\]

where

\[
\langle A_1^{(1)}(\varphi), \psi \rangle = b \epsilon \int_{\mathbb{R}^3} |\nabla \varphi|^2 \int_{\mathbb{R}^3} \nabla \varphi \cdot \nabla \psi + \int_{\mathbb{R}^3} |\nabla \varphi|^2 \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \psi
\]

\[+ 2b \epsilon \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \varphi \int_{\mathbb{R}^3} \nabla \varphi \cdot \nabla \psi. \]
and
\[
\langle A_2^{(1)}(\varphi), \psi \rangle = \int_{\mathbb{R}^3} ((U_{\epsilon,y} + \varphi)^p \psi - U_{\epsilon,y}^p \psi - pU_{\epsilon,y}^{p-1} \varphi \psi).
\]
We also deduce, for any \( \varphi, \psi, \xi \in H_\epsilon \), that
\[
\langle R_\epsilon^{(2)}(\varphi)[\psi], \xi \rangle = \langle A_1^{(2)}(\varphi)[\psi], \xi \rangle - \langle A_2^{(2)}(\varphi)[\psi], \xi \rangle,
\]
where
\[
\langle A_1^{(2)}(\varphi)[\psi], \xi \rangle = be \left( 2 \int_{\mathbb{R}^3} \nabla \varphi \cdot \nabla \psi \int_{\mathbb{R}^3} \nabla \varphi \cdot \nabla \xi + \int_{\mathbb{R}^3} |\nabla \varphi|^2 \int_{\mathbb{R}^3} \nabla \xi \cdot \nabla \psi \right)
\]
\[
+ 2be \left( \int_{\mathbb{R}^3} \nabla \varphi \cdot \nabla \psi \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \xi + \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \psi \int_{\mathbb{R}^3} \nabla \varphi \cdot \nabla \xi \right)
\]
\[
+ 2be \int_{\mathbb{R}^3} \nabla U_{\epsilon,y} \cdot \nabla \varphi \int_{\mathbb{R}^3} \nabla \xi \cdot \nabla \psi
\]
and
\[
\langle A_2^{(2)}(\varphi)[\psi], \xi \rangle = \int_{\mathbb{R}^3} \left( p(U_{\epsilon,y} + \varphi)^{p-1} \psi \xi - pU_{\epsilon,y}^{p-1} \psi \xi \right).
\]
Now we prove Lemma 3.3.

**Proof of Lemma 3.3.** First, we estimate \( A_1(\varphi) \), \( A_1'(\varphi) \) and \( A_1''(\varphi) \). Notice that
\[
\|\nabla U_{\epsilon,y}\|_{L^2(\mathbb{R}^3)} = C_0 \epsilon^{1/2}
\]
with \( C_0 = \|\nabla U\|_{L^2(\mathbb{R}^3)} \), and that
\[
\|\nabla \varphi\|_{L^2(\mathbb{R}^3)} \leq C_1 \epsilon^{-1}\|\varphi\|_\epsilon, \quad \varphi \in H_\epsilon
\]
holds for some \( C_1 > 0 \) independent of \( \epsilon \). Combining above two estimates together with Hölder’s inequality yields
\[
\int_{\mathbb{R}^3} |\nabla \varphi \cdot \nabla \psi| \int_{\mathbb{R}^3} |\nabla U_{\epsilon,y} \cdot \nabla \xi| \leq C \epsilon^{-5/2}
\]
and that
\[
\int_{\mathbb{R}^3} |\nabla \varphi \cdot \nabla \psi| \int_{\mathbb{R}^3} |\nabla \eta \cdot \nabla \xi| \leq C \epsilon^{-4}
\]
for all \( \varphi, \psi, \eta, \xi \in H_\epsilon \). These estimates imply that
\[
|A_1^{(i)}(\varphi)| \leq C \epsilon^{-\frac{3}{2}} \left( 1 + \epsilon^{-\frac{3}{2}}\|\varphi\|_\epsilon \right) \|\varphi\|_\epsilon^{-i}
\]
for some constant \( C > 0 \) independent of \( \epsilon \).

Next we estimate \( A_2^{(i)}(\varphi) \) (the \( i \)th derivative of \( A_2(\varphi) \)) for \( i = 0, 1, 2 \). We consider the case \( 1 < p \leq 2 \) first.

To estimate \( A_2(\varphi) \), we apply the following elementary inequality: for any \( e, f \in \mathbb{R} \), there exists \( C_1(p) > 0 \) depending only on \( p \), so that
\[
\left| (e + f)^{p+1} - e^{p+1} - (p+1)e^pf - \frac{p(p+1)}{2}e^{p-1}f^2 \right| \leq C_1(p)|f|^{p+1}.
\]
Then there holds
\[
|A_2(\varphi)| \leq C \int_{\mathbb{R}^3} |\varphi|^{p+1} \leq C \epsilon^{-\frac{3(p+1)}{2}}\|\varphi\|_\epsilon^{p+1},
\]
where we have used (3.6) to derive the second term.
To estimate $A_2^{(1)}(\varphi)$, we apply the following elementary inequality: for any $e, f \in \mathbb{R}$, there exists $C_2(p) > 0$ depending only on $p$, so that
\[
\left| (e + f)^p_+ - e^p_+ - pe_+^{p-1}f \right| \leq C_2(p)|f|^p.
\]
Then there holds
\[
|\langle A_2^{(1)}(\varphi), \psi \rangle| \leq C_p \int_{\mathbb{R}^3} |\varphi|^p|\psi| \leq C e^{\frac{3(p-1)}{2}} \|\varphi\|_p^p \|\psi\|_p^p,
\]
where we have used (3.6) to derive the second term. This gives
\[
\|A_2^{(1)}(\varphi)\| \leq C e^{-\frac{3(p-1)}{2}} \|\varphi\|_p^p.
\]
To estimate $A_2^{(2)}(\varphi)$, we apply the following elementary inequality: for any $e, f \in \mathbb{R}$, there exists $C_3(p) > 0$ depending only on $p$, so that
\[
\left| (e + f)^{p-1}_+ - e^{p-1}_+ \right| \leq C_3(p)|f|^{p-1}.
\]
Then there holds
\[
|\langle A_2^{(2)}(\varphi)[\psi], \xi \rangle| \leq C_3(p) \int_{\mathbb{R}^3} |\varphi|^{p-1}|\psi||\xi| \leq C e^{\frac{3(p-1)}{2}} \|\varphi\|_p^{p-1} \|\psi\|_p \|\xi\|_p,
\]
where we have used Hölder’s inequality and (3.6) to derive the second term. This gives
\[
\|A_2^{(2)}(\varphi)\| \leq C e^{-\frac{3(p-1)}{2}} \|\varphi\|_p^{p-1}.
\]
Combining the above estimates yields the result in Lemma 3.3 in the case $1 < p \leq 2$.

In the case $2 < p < 5$, we can estimate $A_2^{(1)}(\varphi)$ similarly as above. So we only point out the following elementary inequalities that are needed. For any $e, f \in \mathbb{R}$, there exist $\tilde{C}_i(p) > 0$ ($i = 1, 2, 3$) such that
\[
\left| (e + f)^{p+1}_+ - e^{p+1}_+ - (p + 1)e^p_+f - \frac{p(p + 1)}{2}e^{p-1}_+f^2 \right| \leq \tilde{C}_1(p)(|e|^{p-2} + |f|^{p-2})|f|^3,
\]
\[
\left| (e + f)^p_+ - e^p_+ - pe^{p-1}_+f \right| \leq \tilde{C}_2(p)(|e|^{p-2} + |f|^{p-2})|f|^2
\]
and
\[
\left| (e + f)^{p-1}_+ - e^{p-1}_+ \right| \leq \tilde{C}_3(p)(|e|^{p-2} + |f|^{p-2})|f|.
\]
The proof of Lemma 3.3 is complete. \qed

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