A local-global principle for the telescope conjecture

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Abstract

We prove an étale local-global principle for the telescope conjecture and use it to show that the telescope conjecture holds for derived categories of Azumaya algebras on noetherian schemes as well as for many classifying stacks and gerbes. This specializes to give another proof of the fact that the telescope conjecture holds for noetherian schemes.

Key Words  Telescope conjecture, derived categories, Azumaya algebras.

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1 Introduction

Let \( T \) be a compactly generated triangulated category with all coproducts. Recall that a (Bousfield) localization of \( T \) consists of a second triangulated category \( T' \) and a pair of adjoint functors

\[ j : T \rightleftarrows T' : j_p \]

such that \( j_p \) is fully faithful, \( j \) being the left adjoint and \( j_p \) the right. The associated localization functor is the composition \( j_p \circ j \). A localization is called smashing if \( j_p \) preserves coproducts, which is equivalent to saying that \( j_p \circ j \) preserves coproducts.

Conjecture 1.1 (Triangulated telescope conjecture). If \( j : T \rightleftarrows T' : j_p \) is a smashing localization, then \( \ker(j) \) is generated by objects that are compact in \( T \), where \( \ker(j) \) is the full subcategory of \( T \) consisting of objects \( x \) such that \( j(x) \approx 0 \).

As a simple but crucial example, let \( Z \) be a closed subscheme of \( X \) (which we assume to be quasi-compact and quasi-separated) defined by \( n \) equations \( f_1, \ldots, f_n \). Write \( U \) for the complement of \( Z \) in \( X \). Then, the restriction functor \( D_{qc}(X) \to D_{qc}(U) \) is a smashing localization. The conjecture can be verified directly in this case as follows. Let \( K_i \) be the perfect complex \( \mathcal{O}_X \xrightarrow{L} \mathcal{O}_X \), and let \( K = K_1 \otimes \cdots \otimes K_n \) (the Koszul complex). Then, \( K \) is a compact generator of the kernel of the localization functor. This was first observed by Bökstedt and Neeman [8] in the affine case. See [3, Proposition 6.9] for the general case.

The telescope conjecture is not really a conjecture, as it is known to be false in certain circumstances, even for the derived categories \( D(R) \) of commutative rings \( R \). The first example was given...
by Keller [17], and more recent examples, of certain dimension 2 valuation rings, were given by Krause and Šťovíček [19, Example 7.8]. Nevertheless, there is a great deal of interest in cases when it does hold, because it relates the classification of smashing localizations of \( \mathcal{T} \) to the classification of thick subcategories of \( \mathcal{T}^c \), the full subcategory of compact objects, and the latter classification problem is sometimes tractable.

To summarize what is known at present, Hopkins and Neeman [24] gave the first results, establishing the conjecture for \( \mathcal{D}(\mathcal{R}) \), the derived category of a noetherian commutative ring. As a consequence, one finds that there is a bijection between the smashing localizations of \( \mathcal{D}(\mathcal{R}) \) (up to equivalence), thick subcategories of the triangulated category \( \mathcal{D}_{\text{perf}}(\mathcal{R}) \) of perfect complexes on \( \mathcal{R} \), and specialization-closed subsets of \( \text{Spec}(\mathcal{R}) \). In the non-noetherian case, Dwyer and Palmieri [15] showed that the conjecture holds for the derived categories of truncated polynomial algebras in countably many generators, while Stevenson [29] established the conjecture for absolutely flat rings.

In the noncommutative case, Brüning [12] proved the conjecture for the derived categories of finite dimensional hereditary algebras of finite representation type over a field, a result which was then extended to all finite dimensional hereditary algebras over a field by Krause-Šťovíček [19]. In particular, the telescope conjecture holds for \( \mathcal{D}_{\text{per}}(\mathbb{P}_k^1) \), making \( \mathbb{P}_k^1 \) the only non-affine variety for which this form of the telescope conjecture is known to hold.

In another direction, Stevenson proved the conjecture for the singularity categories of noetherian rings with hypersurface singularities in [28] and for quotients of regular local rings by regular sequences.

The triangulated telescope conjecture was originally formulated for the stable homotopy category \( \mathcal{SH} \) by Bousfield [10, Conjecture 3.4]. In the form written here it was given by Ravenel [26, Conjecture 1.33]. The reason for its importance in stable homotopy theory is that if true for the \( p \)-local stable homotopy category \( \mathcal{SH}_{(p)} \), it would give a concrete way of computing the \( K(n) \)-localization of a space via a telescope construction, which is a certain homotopy colimit. Specifically, the thick subcategories of the triangulated category of \( p \)-local finite spectra \( \mathcal{SH}^\text{fin}_{(p)} \) are known: they are precisely the thick subcategories given by the kernels of \( E(n) \)-localization for some \( n \). Writing \( L_n \) for telescopic localization, which for a \( p \)-local finite spectrum of type at least \( n \) can be described as \( v_n \)-localization, the kernel of any localization \( L \) satisfies

\[
\ker(L_n) \subseteq \ker(L) \subseteq \ker(L_{E(n)})
\]

for some uniquely determined non-negative integer \( n \). The telescope conjecture would say that these are equalities. The current state of the telescope conjecture for the stable homotopy category seems unclear. Apparently, it is widely believed to be false, and potential counterexamples have even been produced at various points, but a proof that it is false remains elusive.

There is another version of the telescope conjecture suitable for when a \( \otimes \)-triangulated category \( \mathcal{T} \) acts on a triangulated category \( \mathcal{U} \). Again, we require \( \mathcal{T} \) and \( \mathcal{U} \) to have all coproducts and to be compactly generated. We also require the tensor product map

\[
\otimes : \mathcal{T} \times \mathcal{U} \to \mathcal{U}
\]

to preserve coproducts in each variable. A localizing subcategory of \( \mathcal{U} \) will be called \( \mathcal{T} \)-closed if it is closed under tensor products with \( \mathcal{T} \).

**Conjecture 1.2** (Tensor telescope conjecture). If \( j : \mathcal{U} \rightarrow \mathcal{U}' : j_p \) is a smashing localization where \( \ker(j) \) is \( \mathcal{T} \)-closed, then \( \ker(j) \) is generated by objects that are compact in \( \mathcal{U} \).

We will say that the \( \otimes \)-telescope conjecture holds for \( \mathcal{U} \) under the action of \( \mathcal{T} \) when the conjecture is verified. When \( \mathcal{T} = \mathcal{U} \), we will simply say that the \( \otimes \)-telescope conjecture holds for \( \mathcal{T} \). When the
The tensor telescope conjecture was stated in this form by Stevenson [27], generalizing the situation where $\mathcal{U} = \mathcal{T}$ considered previously. An example of why it is useful to consider the more general situation is that if $\alpha \in \text{Br}(X)$, then $D_{\text{qc}}(X, \alpha)$ is not a $\otimes$-category. But, nevertheless, as one result of our paper, if $X$ is noetherian, then the $\otimes$-telescope conjecture holds for $D_{\text{qc}}(X, \alpha)$ under the action of $\mathcal{T}$.

In [2], the authors show that the $\otimes$-telescope conjecture holds for the derived categories of noetherian formal schemes, extending the Hopkins-Neeman result in particular to the derived categories of quasi-coherent sheaves on noetherian schemes. Balmer and Favi [5] established a local-global principle under which the $\otimes$-telescope holds globally if it holds Zariski locally on Balmer’s spectrum for tensor triangulated categories [4]. Their work gives another proof of the $\otimes$-telescope conjecture for the derived categories of quasi-coherent sheaves $D_{\text{qc}}(X)$ on noetherian schemes. Hovey, Palmieri, and Strickland [16] gave a new proof of Neeman’s result, using the equivalence between the triangulated tensor conjecture and the $\otimes$-telescope conjecture for $D(R)$ when $R$ is noetherian. Their methods also prove the $\otimes$-telescope conjecture for comodules over a finite-dimensional Hopf algebra.

For a quasi-compact and quasi-separated scheme, Thomason [30] classified the thick $\otimes$-subcategories of $D_{\text{perf}}(X)$, the triangulated category of perfect complexes on $X$. By the results of [2] and [5], there is a nice description of all smashing $\otimes$-localizations of $D(X)$ for $X$ noetherian. In particular, to any such localization there is a uniquely defined specialization-closed subset of $X$. This subset is precisely the locus where the objects of $\ker(j)$ are supported.

Stevenson [27] considered the theory of supports that arises when $\mathcal{T}$ acts on $\mathcal{U}$ and used this to give yet another proof of the tensor telescope conjecture for the derived categories of noetherian schemes. Stevenson’s proof is conceptually satisfying as it proceeds by actually classifying the tensor closed subcategories, yielding a proof closer in spirit to Neeman’s proof in the affine case. Dell’Ambrogio and Stevenson [13] proved the $\otimes$-telescope conjecture for quasi-projective varieties and for weighted projective spaces by considering a graded version of the telescope hypothesis and then using support theory.

The work of Balmer and Favi shows in a great deal of generality that the $\otimes$-telescope conjecture holds for $\mathcal{T}$ acting on itself when it holds locally on the Balmer spectrum of $\mathcal{T}$. The purpose of this paper is to establish a new local-global principle for the telescope conjecture. Our principle differs in three important ways from theirs. First, it holds for étale covers not just Zariski covers. Second, it works for the action of $\mathcal{T}$ on $\mathcal{U}$, allowing one to establish telescoping in noncommutative situations such as for Azumaya algebras. This perspective is present in Stevenson [27] as well. Third, it requires as input not triangulated categories but enhancements such as stable $\infty$-categories. This restriction is not a barrier for any foreseeable application.

Our methods use in a crucial way the notion of a stack of stable presentable $\infty$-categories over a scheme $X$, as studied in [22] and [3]. The dg category approach to these ideas can be found in [31]. We very briefly describe this theory here, referring the reader more generally to [3, Section 6] and the references there. These stacks provide one method of giving sense to the nonsense notion of a stack of triangulated categories.

We fix a base connective commutative ring spectrum $R$. This might be the Eilenberg-MacLane spectrum of an ordinary commutative ring or of a simplicial commutative ring. For our applica-
tions, we use only ordinary commutative rings, but it seems relevant to note that the theorems hold for quasi-compact and quasi-separated derived schemes, which are schemes with sheaves of local connective commutative ring spectra.

A stable ∞-category C is an ∞-category that has a 0 object, that has fiber and cofiber sequences, and in which fiber and cofiber sequences agree. The homotopy category Ho(C) of C is naturally a triangulated category. By [23, Corollary 1.4.4.2], a stable ∞-category C is presentable if Ho(C) has all coproducts, has hom sets, and has a k-compact generator for some regular cardinal k.

When A is an $A_{\infty}$-algebra spectrum (such as the Eilenberg-MacLane spectrum of an ordinary associative algebra), Mod$_A$ is a stable presentable ∞-category, with homotopy category D(A). For example if $S$ is the sphere spectrum, then Mod$_S$ is an ∞-categorical enhancement of the triangulated stable homotopy category. For a quasi-compact and quasi-separated scheme $X$, there is a stable presentable ∞-category that we will denote by Mod$_X$ with Ho(Mod$_X$) = $D_{qc}(X)$, the triangulated category of complexes of $O_X$-modules with quasi-coherent cohomology sheaves.

If $S$ is a connective commutative $R$-algebra, then an $S$-linear category is a stable presentable ∞-category enriched over Mod$_S$, the stable presentable ∞-category of $S$-module spectra. These objects can be realized as the (left) modules for the commutative ring object Mod$_S$ in the ∞-category $Pr^L_S$.

We denote this category by Cat$_S = Mod_{Mod_S}(Pr^L_S)$.

An $S$-linear category with étale hyperdescent is an $S$-linear category Mod$_S^{\epsilon}$ such that for any connective commutative $S$-algebra $T$ and any étale hypercover Spec $T^\bullet \rightarrow$ Spec $T$, the induced map

$$Mod_T \otimes_{Mod_S} Mod_S^{\epsilon} \rightarrow \lim_{\Delta} Mod_T, \otimes_{Mod_S} Mod_S^{\epsilon}$$

is an equivalence. These define a full subcategory Cat$_S^{\text{des}}$ of Cat$_S$. It is an important fact that these glue together to form a stack Cat$_S^{\text{des}}$ (see [22, Theorem 7.5]).

Let $X$ be an $R$-scheme (which might be derived). An étale hyperstack (henceforth just a stack) of linear categories on $X$ is a map of stacks Mod$_{\leftarrow} : X \rightarrow$ Cat$_S^{\text{des}}$ over Spec $R$. Loosely speaking, Mod$_{\leftarrow}$ assigns to any affine Spec $S \rightarrow X$ a stable presentable $S$-linear category Mod$_S^{\epsilon}$ and to any map $f : Spec T \rightarrow Spec S$ a pull-back map $f^* : Mod_S^{\epsilon} \rightarrow Mod_T^{\epsilon}$ in such a way that if Spec $T^\bullet \rightarrow$ Spec $S$ is an étale hypercover, then the associated map

$$Mod_S^{\epsilon} \rightarrow \lim_{\Delta} Mod_T^{\epsilon},$$

is an equivalence of stable presentable $S$-linear categories. The affines here are Spec $S$ for all connective commutative $R$-algebras, a class that includes the Eilenberg-MacLane spectra of all ordinary $\pi_0R$-algebras. For us, $R$ itself will be such an Eilenberg-MacLane spectra and no truly derived schemes will arise in the paper.

The ∞-category of global sections of Mod$_{\leftarrow}$ is

$$\text{Mod}_X^{\epsilon} = \lim_{\text{Spec } S \rightarrow X} \text{Mod}_S^{\epsilon},$$

where the limit is computed in Cat$_R$.

**Example 1.3.** The stack that assigns to each Spec $S \rightarrow X$ the stable presentable ∞-category Mod$_S$ of $S$-module spectra will be written Mod$_X^{\Omega}$. This can be thought of as the stack of complexes of $O_X$-modules with quasi-coherent cohomology. The homotopy category of Mod$_X = Mod_X^{\Omega}$ is $D_{qc}(X)$.

The stable ∞-category Mod$_X$ is symmetric monoidal, and any other category of global sections Mod$_X^{\alpha}$ comes with a natural action of Mod$_X$. We say that Mod$_X^{\alpha}$ satisfies the Mod$_X$-linear telescope hypothesis if the kernel of any Mod$_X$-linear smashing localization is generated by compact objects of Mod$_X^{\alpha}$. When $\alpha = \emptyset$, this is the ∞-categorical analogue of the $\otimes$-triangulated telescope conjecture.

The local-global principle of the title is encoded in the following result.
Theorem 1.4. Let $X$ be a quasi-compact and quasi-separated scheme, and suppose that $\text{Mod}^\alpha$ is a stack of linear categories on $X$. If there is an étale cover $U \to X$ such that $\text{Mod}^\alpha_U$ satisfies the $\text{Mod}^\alpha_U$-linear telescope hypothesis, then $\text{Mod}^\alpha_X$ satisfies the $\text{Mod}^\alpha_X$-linear telescope hypothesis.

Of course, one might wonder how this statement translates into the original language of triangulated categories. This is spelled out in detail in Section 4: the triangulated versions are equivalent to the $\infty$-categorical versions. As a consequence of the theorem, we prove the telescope hypothesis in the following situations.

1. The $\otimes$-telescope conjecture holds for $D_{qc}(X, \alpha)$ under the action of $D_{qc}(X)$, where $D_{qc}(X, \alpha)$ is the $\alpha$-twisted derived category of a noetherian scheme, for $\alpha \in \text{Br}(X)$. Proving this result was the original motivation for the project. Even for $X$ affine this was unknown. When $X$ is affine and $\alpha = 0$, this was Neeman’s result. For $X$ a general noetherian scheme and $\alpha = 0$, it has been proven by [2], [5], [16], and [27]. Thus, we find a fifth proof, most similar in spirit to that of Balmer and Favi.

2. The $\otimes$-telescope conjecture holds for $D_{qc}(B\mathfrak{G})$ under the action of $D_{qc}(X)$, when $B\mathfrak{G}$ is the classifying stack of a finite tame étale group scheme $\mathfrak{G}$ over a noetherian scheme $X$. Since $D_{qc}(B\mathfrak{G})$ is itself a $\otimes$-triangulated category and there is a $\otimes$-triangulated pullback functor $D_{qc}(X) \to D_{qc}(B\mathfrak{G})$, the $\otimes$-telescope conjecture holds for $D_{qc}(B\mathfrak{G})$ acting on itself as well. A similar comment applies in each of the next cases.

3. The $\otimes$-telescope conjecture holds for $D_{qc}(B\mathcal{A})$ under the action of $D_{qc}(X)$, when $\mathcal{A}$ is a finite abelian group scheme over a noetherian scheme $X$.

4. The $\otimes$-telescope conjecture holds for $D_{qc}(X)$ under the action of $D_{qc}(X)$, where $X \to X$ is a finite abelian gerbe over a noetherian scheme.

Besides these results, we give several examples throughout of new cases of the telescope conjecture. For example, in Example 5.5, we give what we believe to be the first example where telescope holds for the derived category of a dg algebra that is not derived Morita equivalent to an ordinary algebra. We also show that when $X$ is quasi-compact and quasi-separated, the $\otimes$-closed smashing localizing subcategories of $D_{qc}(X, \alpha)$ under the action of $D_{qc}(X)$ correspond bijectively to the specialization closed subsets of $X$ that can be written as the union of closed subsets with quasi-compact complement.

We end the paper by establishing the following classification theorem using recent work of Dubey and Mallick [14].

Theorem 1.5. Let $X$ be a smooth scheme of finite type over a field $k$, and let $\mathfrak{G} \to X$ be a finite étale group scheme of order prime to the characteristic of $k$. Suppose that $\mathfrak{X} \to X$ is a $\mathfrak{G}$-gerbe (a stack over $X$ étale locally equivalent to $B\mathfrak{G}$). Then, there is a bijection between the set of $\otimes$-closed smashing localizations of $D_{qc}(\mathfrak{X})$ and the specialization closed subsets of $X$.

In Section 2 we prove a local-global principle for the property of being compactly generated. Section 3 contains the main definitions, of the telescope hypothesis, the linear telescope hypothesis, and the stacky telescope hypothesis. At the end, we prove a key theorem that says that the stacky telescope hypothesis is equivalent to the linear telescope hypothesis. In Section 4 the $\infty$-categorical telescope hypotheses are compared to the triangulated versions, and are shown to be equivalent where appropriate. Section 5 contains the main theorem, the local-global principle, as
well as the consequences for schemes and Azumaya algebras. Finally, in Section 6, we prove the linear telescope conjecture for classifying stacks of finite étale group schemes in the tame case, for finite abelian group schemes, and for gerbes over these.

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2 The local-global principle for compact generation

We need the following result, which is not quite proved in the union of the papers of Lurie [22], Toën [31], and Antieau-Gepner [3]. The idea is due to Bökstedt and Neeman [8].

**Theorem 2.1.** Let $\alpha : X \to \text{Cat}^{\text{desc}}$ classify a stack of linear categories $\text{Mod}^\alpha$ over $X$, where $X$ is a quasi-compact and quasi-separated scheme. If $\text{Mod}^\alpha$ is étale locally compactly generated, then $\text{Mod}^\alpha_X$ is compactly generated.

A special case of the telescope hypothesis is needed in the proof of the theorem.

**Lemma 2.2.** Let $Z = \text{Spec} \ S$ be an affine scheme and $W \subseteq Z$ a quasi-compact Zariski open inclusion. Let $\alpha : Z \to \text{Cat}^{\text{desc}}$ classify a stack of linear categories $\text{Mod}^\alpha$. If $\text{Mod}^\alpha_Z$ is compactly generated, then the kernel $\text{Mod}^\alpha_{Z-W}$ of $\text{Mod}^\alpha_Z \to \text{Mod}^\alpha_W$ is compactly generated by compact objects of $\text{Mod}^\alpha_Z$.

**Proof.** Because tensor products of stable presentable $\infty$-categories are computed as functors [23, Proposition 6.3.1.16], it follows that the exact sequence

$$\text{Mod}^\alpha_{Z-W} \to \text{Mod}^\alpha_Z \to \text{Mod}^\alpha_W$$

is obtained from

$$\text{Mod}^\alpha_{Z-W} \to \text{Mod}^\alpha_Z \to \text{Mod}^\alpha_W$$

by tensoring with $\text{Mod}^\alpha_Z$ over $\text{Mod}^\alpha_Z$. By [3, Proposition 6.9], $\text{Mod}^\alpha_{Z-W}$ is generated by a single compact object. Since, by hypothesis, $\text{Mod}^\alpha_Z$ is compactly generated, it follows that

$$\text{Mod}^\alpha_{Z-W} \cong \text{Mod}^\alpha_{Z-W} \otimes_{\text{Mod}^\alpha_Z} \text{Mod}^\alpha_Z$$

is compactly generated (see [7, Section 3.1]).

Say that an object of $\text{Mod}^\alpha_X$ is perfect if for every $\text{Spec} \ S \to X$ the pullback $x_S$ is compact in $\text{Mod}^\alpha_S$.

**Proposition 2.3.** In the situation of the theorem, perfect objects of $\text{Mod}^\alpha_X$ are compact.

**Proof.** First, this is true on affine schemes by definition. Second, if $X = U \cup V$ where $U$ and $V$ are open subschemes, and if it is true for $U$ and $V$ and $U \cap V$, then it is true for $X$. Indeed, in this case, $X$ is the finite colimit $U \cap V \Rightarrow U \amalg V \to X$. Thus, $\text{Mod}^\alpha_X$ is the fiber in

$$\text{Mod}^\alpha_U \to \text{Mod}^\alpha_U \times \text{Mod}^\alpha_V \Rightarrow \text{Mod}^\alpha_{U \cup V}.$$

Given objects $x, y \in \text{Mod}^\alpha_X$, this means that we can compute the mapping spectrum $\text{Map}_X(x, y)$ as a limit

$$\text{Map}_X(x, y) \to \text{Map}_U(x_U, y_U) \times \text{Map}_V(x_V, y_V) \Rightarrow \text{Map}_{U \cup V}(x_{U \cup V}, y_{U \cup V}).$$
If \( x \) is perfect in \( \text{Mod}^\alpha_X \), then it is compact on \( U, V, \) and \( U \cap V \) by hypothesis. Since filtered colimits commute with finite limits, it then follows that \( x \) is compact in \( X \), as desired. Finally, the result holds for arbitrary quasi-compact and quasi-separated schemes by the so-called reduction principle [9, Proposition 3.3.1].

Proof of Theorem 2.1. The proof is essentially a transcription of the proof of [3, Theorem 6.11], with a couple of alterations. The base case of the induction step is that if \( X \) is affine then étale local compact generation implies global compact generation. Moreover, in that case any compact objects is perfect. This step is provided by [22, Theorem 6.1]. The compact generation of the kernels is provided by the lemma. The rest of the proof goes through, except that one lifts sets of compact generators up to \( X \) using the gluing methods of [3, Theorem 6.11]. Details are left to the reader. The last step is to note that one has built up perfect objects, which are compact by the proposition. ■

Corollary 2.4. In the situation of the theorem, the compact objects of \( \text{Mod}^\alpha_X \) are precisely the perfect objects.

Proof. Since perfect objects are compact and generate \( \text{Mod}^\alpha_X \), the theorem of Ravenel and Neeman [25, Theorem 2.1] shows that the subcategory of compact objects of \( \text{Mod}^\alpha_X \) is the idempotent completion of the subcategory of perfect objects. But, the subcategory of perfect objects is already idempotent-complete, as can be seen by looking locally. ■

3 Telescopy

A localization of a stable presentable \( \infty \)-category \( \mathcal{C} \) is an adjunction

\[
j : \mathcal{C} \rightleftarrows \mathcal{D} : j_\rho
\]

where the right adjoint \( j_\rho \) is fully faithful.

Recall that if \( \mathcal{M} \) is a symmetric monoidal stable presentable \( \infty \)-category, then we can consider “modules” for \( \mathcal{M} \), which are stable presentable \( \infty \)-categories \( \mathcal{C} \) with a tensor product \( \otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{C} \) satisfying various nice properties, most importantly the preservation of homotopy colimits in each variable. These \( \infty \)-categories together with the \( \otimes \)-structure will be called \( \mathcal{M} \)-linear categories. By working in \( \text{Pr}^\infty \), the symmetric monoidal \( \infty \)-category of presentable \( \infty \)-categories and right adjoint functors, an \( \mathcal{M} \)-linear category \( \mathcal{C} \) is precisely a (left) module for the commutative algebra object \( \mathcal{M} \). See [23, Section 6.3].

If \( \mathcal{C} \) is an \( \mathcal{M} \)-linear category, then a localization \( j : \mathcal{C} \rightleftarrows \mathcal{D} : j_\rho \) is \( \mathcal{M} \)-linear if it is a localization in the \( \infty \)-category of \( \mathcal{M} \)-modules in \( \text{Pr}^\infty \). This can be checked in a more down-to-earth way by showing that \( \text{ker}(j) \) is closed under tensor product with \( \mathcal{M} \).

A localization is smashing if \( j_\rho \) preserves small coproducts. Note that because these stable \( \infty \)-categories are presentable, preserving coproducts is equivalent to preserving all small colimits in the \( \infty \)-categorical sense, by [23, Proposition 1.4.4.1].

A localization of stacks consists of an adjunction

\[
j : \text{Mod}^\alpha \rightleftarrows \text{Mod}^\beta : j_\rho,
\]

where \( \text{Mod}^\alpha \) and \( \text{Mod}^\beta \) are stacks of linear categories, such that \( j_\rho \) is fully faithful. By definition, to give such an adjunction is to give a compatible family of \( S \)-linear adjunctions

\[
j_S : \text{Mod}^\alpha_S \rightleftarrows \text{Mod}^\beta_S : j_{S,\rho}
\]
for every $\text{Spec} \ S \to X$. Then, the functor $j_\rho$ is fully faithful if each $j_{S, \rho}$ is fully faithful. The localization of stacks is smashing if each $j_{S, \rho}$ preserves small coproducts.

Given a localization $j : \mathcal{C} \to \mathcal{D}$, there is a kernel $\ker(j)$, the full subcategory of $\mathcal{C}$ of objects $x$ such that $j(x) \simeq 0$. Given an $\mathcal{M}$-linear localization, the kernel $\ker(j)$ is itself $\mathcal{M}$-linear. For a localization of stacks $j : \text{Mod}^\alpha \to \text{Mod}^\beta$, the family of kernels determines itself a stack of linear categories $\text{Mod}^\beta_j$ by setting $\text{Mod}^\beta_j = \ker(j_S)$. To see this, it suffices to check when $X = \text{Spec} \ S$, in other words in the case of $\mathcal{S}$-linear categories with descent. But, the kernel is a limit in $\text{Cat}^\text{desc}_S$, so it can be computed étale locally, since limits commute.

**Definition 3.1.** Let $\mathcal{C}$ be a compactly generated stable presentable $\infty$-category. Then, $\mathcal{C}$ satisfies the telescope hypothesis (TH) if the kernel of every smashing localization $j : \mathcal{C} \to \mathcal{D}$ is generated by compact objects of $\mathcal{C}$.

Now, suppose that $\mathcal{M}$ is a compactly generated symmetric monoidal stable presentable $\infty$-category and that $\mathcal{C}$ is a compactly generated $\mathcal{M}$-linear category. Say that $\mathcal{C}$ satisfies the $\mathcal{M}$-linear telescope hypothesis (LTH) if the kernel of every $\mathcal{M}$-linear smashing localization $j : \mathcal{C} \to \mathcal{D}$ is generated by compact objects of $\mathcal{C}$.

Finally, suppose that $X$ is an étale sheaf over $R$, and let $\alpha : X \to \text{Cat}^\text{desc}_R$ classify a stack $\text{Mod}^\alpha$ of linear categories. Say that $\text{Mod}^\alpha$ satisfies the stacky telescope hypothesis (STH) if for every smashing localization of stacks $\text{Mod}^{\alpha'} \to \text{Mod}^\beta$ and every map $\text{Spec} \ S \to X$, the kernel of $\text{Mod}^\alpha_S \to \text{Mod}^\beta_S$ is generated by compact objects of $\text{Mod}^\alpha_S$.

In the literature, what is called here the telescope hypothesis is often called the telescope conjecture. Since it is false in general, hypothesis seems more appropriate.

**Lemma 3.2.** If $\mathcal{M}$ is a symmetric monoidal stable presentable $\infty$-category that is compactly generated by its unit, and if $\mathcal{C}$ is a compactly generated $\mathcal{M}$-linear category, then $\mathcal{C}$ satisfies the $\mathcal{M}$-linear telescope hypothesis if and only if $\mathcal{C}$ satisfies the telescope hypothesis.

**Proof.** If $\mathcal{C}$ satisfies the telescope hypothesis, then it satisfies the less restrictive $\mathcal{M}$-linear telescope hypothesis. Conversely, we claim that any localization $j : \mathcal{C} \leftrightarrow \mathcal{D} : j_\rho$ is automatically $\mathcal{M}$-linear. It suffices to show that $\ker(j)$ is closed under tensor product with $\mathcal{M}$. Contemplation of the following four facts completes the proof. The localizing subcategory $\ker(j) \subseteq \mathcal{C}$ is closed under homotopy colimits by definition. The tensor product preserves homotopy colimits in each variable. The symmetric monoidal stable $\infty$-category $\mathcal{M}$ is generated under homotopy colimits by its unit $1_\mathcal{M}$. Obviously, $1_\mathcal{M} \otimes x \in \ker(j)$ for $x \in \ker(j)$.

The conclusion is closely related to an observation of Thomason [30, Corollary 3.11.1(a)]: every thick subcategory of $\text{Perf} (\text{Spec} \ R)$ for a commutative ring $R$ is automatically a $\otimes$-ideal.

Recall that the $\text{Mod}^\alpha_X$-linear category of global sections of $\mathcal{M}$ is

$$\text{Mod}^\alpha_X = \lim_{\text{Spec} \ S \to X} \text{Mod}^\alpha_S.$$

The following theorem allows passage back and forth between the linear and the stacky telescope hypotheses.

**Theorem 3.3.** If $X$ is a quasi-compact and quasi-separated derived scheme over $R$ and $x : X \to \text{Cat}^\text{desc}_R$, then $\text{Mod}^\alpha$ satisfies the stacky telescope hypothesis if and only if $\text{Mod}^\alpha_X$ satisfies the $\text{Mod}^\alpha_X$-linear telescope hypothesis.
Proof. Suppose first that $\text{Mod}^\infty$ satisfies the stacky telescope hypothesis, and let $j : \text{Mod}^\infty_X \to \mathcal{D}$ be a $\text{Mod}_X$-linear smashing localization of $\text{Mod}_X$-linear categories. Because $\text{Mod}^\infty \to 0$ is a smashing localization, the stacky telescope hypothesis for $\text{Mod}^\infty$ says that $\text{Mod}^\infty_X$ is compactly generated for every $\text{Spec} S \to X$. By Theorem 2.1, it follows that $\text{Mod}^\infty_X$ is compactly generated, which in turn implies that $\mathcal{D}$ is compactly generated. Indeed, because $j_\rho$ preserves coproducts, adjunction implies that $j$ preserves compact objects. Since $j_\rho$ is fully faithful, we can take as a set of compact generators of $\mathcal{D}$ the image under $j$ of a set of compact generators of $\text{Mod}^\infty_X$. Define

$$\text{Mod}^\infty_S = \text{Mod}_S \otimes_{\text{Mod}_X} \mathcal{D}.$$ 

Since $\mathcal{D}$ is compactly generated, it follows from [22, Corollary 6.11] that $\text{Mod}^\infty_S$ is indeed a stack of linear categories. We claim that $\text{Mod}^\infty \to \text{Mod}^\infty_S$ is a smashing localization of stacks. But, this is clear because the adjoint

$$j_{\mathcal{S}, \rho} : \text{Mod}^\infty_S \to \text{Mod}^\infty_S$$

can be written as

$$\text{id}_{\text{Mod}_S} \otimes_{\text{Mod}_X} j_\rho : \text{Mod}^\infty_S \cong \text{Mod}_S \otimes_{\text{Mod}_X} \mathcal{D} \to \text{Mod}_S \otimes_{\text{Mod}_X} \text{Mod}^\infty_S \cong \text{Mod}^\infty_S.$$ 

Since $j_\rho : \mathcal{D} \to \text{Mod}^\infty_X$ preserves coproducts, so does $j_{\mathcal{S}, \rho}$. Similarly, $\text{id}_{\text{Mod}_S} \otimes_{\text{Mod}_X} j_\rho$ is fully faithful because $j_\rho$ is. By the stacky telescope hypothesis for $\text{Mod}^\infty$, it follows that each $\ker(j_\rho)$ is compactly generated by objects in $\text{Mod}^\infty_S$. Again, using Theorem 2.1, it follows that the $\infty$-category of sections of the kernel stack is compactly generated. By Corollary 2.4, the compact objects of $\ker(j)$ are perfect. Thus, they are étale locally compact in $\text{Mod}^\infty_S$. So, they are perfect, and hence compact, in $\text{Mod}^\infty_S$. Thus, $\ker(j)$ is compactly generated by compact objects of $\text{Mod}^\infty_S$.

Now, assume that $\text{Mod}^\infty_S$ satisfies the $\text{Mod}_X$-linear telescope hypothesis. Let $\text{Mod}^\infty \to \text{Mod}^\infty_S$ be a smashing localization of stacks. Consider the induced functor $\text{Mod}^\infty_X \to \text{Mod}^\infty_S$, which we claim is a $\text{Mod}_X$-linear smashing localization. Since mapping spaces can be computed locally, the right adjoint $j_{\mathcal{X}, \rho}$ is fully faithful. Similarly, if $\colim y_i \to y$ is a colimit in $\text{Mod}^\infty_S$, then the natural map $\colim_j j_{\mathcal{X}, \rho}(y_i) \to j_{\mathcal{X}, \rho}(y)$ is locally an equivalence. Thus, it is an equivalence in $\text{Mod}^\infty_S$, so that $j_{\mathcal{X}, \rho}$ preserves coproducts. This proves the claim. Now, let $\mathcal{X}_X = \ker(j_\mathcal{X})$, and set $\mathcal{X}_S = \text{Mod}_S \otimes_{\text{Mod}_X} \mathcal{X}_X$ for every $\text{Spec} S \to X$. Since the $\text{Mod}_X$-linear telescope hypothesis applied to $j$ says that $\mathcal{X}_X$ is generated by compact objects of $\text{Mod}^\infty_S$, it follows that $\mathcal{X}_S$ is, in particular, compactly generated, so that $\mathcal{X}$ defines a stack of linear categories by [22, Corollary 6.11]. It follows immediately that $\mathcal{X}_S$ is generated by compact objects of $\text{Mod}^\infty_S \cong \text{Mod}_S \otimes_{\text{Mod}_X} \text{Mod}^\infty_X$. But, $\mathcal{X}_S$ is also the kernel of $j_{\mathcal{S}}$, as tensoring with $\text{Mod}_S$ over $\text{Mod}_X$ preserves the exact sequence

$$\mathcal{X}_S \to \text{Mod}^\infty_X \to \text{Mod}^\infty_S.$$ 

Therefore, the kernel of $j_{\mathcal{S}}$ is generated by compact objects of $\text{Mod}^\infty_S$ for every $\text{Spec} S \to X$. So, $\text{Mod}^\infty$ satisfies the stacky telescope hypothesis.

4 Telecopy for triangulated categories

Just as for stable $\infty$-categories, several notions of telecopy for triangulated categories exist. The first is straightforward. A triangulated category $\mathcal{T}$ satisfies the triangulated telescope hypothesis (TH) if every smashing localization

$$j : \mathcal{T} \leftrightarrow \mathcal{T}' : j_\rho.$$
has a kernel generated by compact objects of $\mathcal{T}$. Note that by definition every smashing localization is a Bousfield localization, so it is determined by the kernel of $j$.

Now, suppose as in the introduction that $\mathcal{T}$ and $\mathcal{U}$ are compactly generated triangulated categories with all coproducts, that $\mathcal{T}$ is a $\otimes$-triangulated category, and that there is a $\otimes$-product $\otimes: \mathcal{T} \times \mathcal{U} \to \mathcal{U}$ that preserves coproducts in each variable separately. Then $\mathcal{U}$ satisfies the $\otimes$-telescope hypothesis under the action of $\mathcal{T}$ ($\otimes$TH) if every smashing localization where the localizing subcategory is closed under the action of $\mathcal{T}$ is generated by compact objects of $\mathcal{U}$.

If $\mathcal{M}$ is a compactly generated symmetric monoidal stable presentable $\infty$-category and $\mathcal{C}$ is $\mathcal{M}$-linear, then $\text{Ho}(\mathcal{M})$ and $\text{Ho}(\mathcal{C})$ satisfy the hypotheses on $\mathcal{T}$ and $\mathcal{U}$ above.

**Lemma 4.1.** The stable presentable $\infty$-category $\mathcal{C}$ satisfies the $\mathcal{M}$-linear telescope hypothesis if and only if the $\otimes$-telescope hypothesis holds for $\text{Ho}(\mathcal{C})$ under the action of $\text{Ho}(\mathcal{M})$.

**Proof.** Suppose that $\text{Ho}(\mathcal{C})$ satisfies the $\otimes$-telescope hypothesis, and let $j : \mathcal{C} \xleftrightarrow{\sim} \mathcal{D} : j_\rho$ be an $\mathcal{M}$-linear smashing localization. Then,

$$\text{Ho}(j) : \text{Ho}(\mathcal{C}) \xleftrightarrow{\sim} \text{Ho}(\mathcal{D}) : \text{Ho}(j_\rho)$$

is a smashing localization. Moreover, the kernel of $\text{Ho}(j)$ is $\text{Ho}(\mathcal{M})$-closed, since $j$ is $\mathcal{M}$-linear. Therefore, by the $\otimes$-telescope hypothesis for $\text{Ho}(j)$, this kernel is generated by compact objects of $\text{Ho}(\mathcal{C})$. It follows that the kernel of $j$ is generated by compact objects of $\mathcal{C}$.

Now, suppose that $\mathcal{C}$ satisfies the $\mathcal{M}$-linear telescope hypothesis, and let $h : \text{Ho}(\mathcal{C}) \xleftrightarrow{\sim} \mathcal{T} : h_\rho$ be a smashing localization where $\ker(h)$ is $\text{Ho}(\mathcal{M})$-closed. Let $\mathcal{K}$ be the full subcategory of $\mathcal{C}$ consisting of objects $x$ whose homotopy class in $\text{Ho}(\mathcal{C})$ is contained in $\ker(h)$. By hypothesis, $\mathcal{K}$ is closed under tensoring with objects of $\mathcal{M}$. From the existence of the adjoint $h_\rho$, it follows that $\ker(h)$ is well-generated in the sense of triangulated categories (see for instance [18]). Hence, $\mathcal{K}$ is a stable presentable $\infty$-category by [23, Lemma 1.4.4.2]. It follows that $\mathcal{K}$ is also $\mathcal{M}$-linear category (using this closure property and the fact that $\mathcal{M}$ is itself). Since $\mathcal{K}$ is presentable and $\mathcal{K} \to \mathcal{C}$ preserves coproducts, it follows that the inclusion has a right adjoint by the adjoint functor theorem for presentable $\infty$-categories [21, Corollary 5.5.2.9]. Thus, there is a localization $\mathcal{C} \to \mathcal{D}$ with kernel $\mathcal{K}$, which can be identified with the map from $\mathcal{C}$ to the cofiber of $\mathcal{K} \to \mathcal{C}$ in the $\infty$-category of stable presentable $\infty$-categories. By construction, $\text{Ho}(\mathcal{D}) \cong \mathcal{T}$, and the localization is smashing, since this can be checked at the level of homotopy categories. Thus, $\mathcal{K}$ is generated by compact objects of $\mathcal{C}$, and so $\ker(j)$ is generated by compact objects of $\text{Ho}(\mathcal{C})$, as desired.

**Lemma 4.2.** A stable $\infty$-category $\mathcal{C}$ satisfies the telescope hypothesis if and only if its homotopy category $\text{Ho}(\mathcal{C})$ satisfies the triangulated telescope hypothesis.

**Proof.** This is left to the reader. It is straightforward using the techniques of [23, Section 1.4.4] and similar to the proof of the previous lemma.

In the Figure 1, the implications are compiled between the various telescope hypotheses. This paper is essentially about those on the first row. However, the telescope conjecture originally arose in the setting of triangulated categories, so it is useful to be able to go back and forth from that world to this one. The most important conceptual arrow in the figure is the implication proved in the previous section that LTH is equivalent to STH.
Remark 4.3. In general it is difficult to lift constructions at the level of triangulated categories to the level of some model, be it a stable model category, a stable $\infty$-category, or a dg category. However, smashing localizations only make sense in the presence of a Bousfield localization, and these are well enough behaved to be modeled. This is one reason why the classification of smashing localizations is easier than the classification of all localizing subcategories of a triangulated category.

5 The local-global principle

The next theorem is the main result of the paper. In the proof, note that if $i : \mathcal{K} \to \text{Mod}_T^S$ is a fully faithful inclusion of $S$-linear categories with a right adjoint $i_\rho$, and if $\mathcal{K}$ is generated by a set of objects that are compact in $\text{Mod}_T^S$, then every compact object of $\mathcal{K}$ is compact when viewed as an object of $\text{Mod}_T^S$.

Theorem 5.1. Let $X$ be a quasi-compact and quasi-separated derived scheme, and suppose that $\text{Mod}^\alpha$ is a stack of linear categories on $X$. If there is an étale cover $f : U \to X$ such that $\text{Mod}_U^\alpha$ satisfies the $\text{Mod}_U^\alpha$-linear telescope hypothesis, then $\text{Mod}_X^\alpha$ satisfies the $\text{Mod}_X^\alpha$-linear telescope hypothesis.

Proof. By Theorem 3.3, it is enough to show that $\text{Mod}^\alpha$ satisfies the stacky telescope hypothesis. Let

$$j : \text{Mod}^\alpha \rightleftarrows \text{Mod}^\beta : j_\rho$$

be a smashing localization of stacks, and consider the stack of kernels $\mathcal{K}$; that is, $\mathcal{K}_T = \ker(j_T)$ for $\text{Spec } T \to X$. We must show that $\mathcal{K}_T$ is generated by compact objects in $\text{Mod}_T^\alpha$ for every $\text{Spec } T \to X$. Fix a map $g : \text{Spec } T \to X$, and consider the induced étale cover $f_T : U \times_X \text{Spec } T \to \text{Spec } T$ given by pulling back $f : U \to X$. As Theorem 3.3 says that $\text{Mod}^{f_{\ast}\alpha}$ satisfies the stacky telescope hypothesis (over $U$ in this case), it follows immediately that $\text{Mod}^{f_{\ast}\alpha}$ satisfies the stacky telescope hypothesis (over $U \times_X \text{Spec } T$). In particular, by quasi-compactness and quasi-separatedness, there is an affine hypercover $\text{Spec } S^\ast \to \text{Spec } T$ such that each $\text{Mod}^{f_{\ast}\alpha}_S$ satisfies the $S^k$-linear telescope hypothesis. In other words, each $\mathcal{K}_S$ is compactly generated by objects of $\text{Mod}^{f_{\ast}\alpha}_S$. Since $\mathcal{K}_T$ is a $T$-linear category with descent, the vertical arrows of the commutative diagram

$$\begin{array}{ccc}
\mathcal{K}_T & \longrightarrow & \text{Mod}^\alpha_T \\
\downarrow & & \downarrow \\
\lim_{\Delta} \mathcal{K}_S^\ast & \longrightarrow & \lim_{\Delta} \text{Mod}^\alpha_S.
\end{array}$$

are equivalences. In particular, $\mathcal{K}_T$ is étale locally compactly generated, so that it is compactly generated by Theorem 2.1. It suffices now to show that the inclusion functor $i : \mathcal{K}_T \to \text{Mod}^\alpha_T$...
preserves compact objects. Let $x_T$ be a compact object of $\mathcal{X}_T$. Each restriction $x_{S_i}$ is compact in $\mathcal{X}_{S_i}$ by Corollary 2.4, which means that $i(x_{S_i})$ is compact in $\text{Mod}^{\text{qc}}_{S_i}$ by hypothesis. It follows that $i(x)$ is perfect and hence compact, as desired. Therefore, $\mathcal{X}_T$ is generated by compact objects of $\text{Mod}^{\text{qc}}_T$.

**Corollary 5.2.** If $X$ is a noetherian scheme and $\alpha \in \text{Br}'(X)$, then $\text{Mod}^{\text{qc}}_X$ satisfies the $\text{Mod}_X$-linear telescope hypothesis.

*Proof.* In this case, one can take an étale cover $[\bigcup_i \text{Spec } S_i \rightarrow X$ such that the restriction of $\alpha$ to each $\text{Spec } S_i$ is trivial and such that $S_i$ is noetherian. The result of Hopkins and Neeman [24] says that the telescope conjecture holds for $D(S_i)$ and hence $\text{Mod}_{S_i}$ by Lemma 4.2. In particular, $\text{Mod}_{S_i}$ satisfies the $\alpha$-linear telescope hypothesis. Thus, by the theorem, the $\text{Mod}_X$-linear telescope hypothesis holds for $\text{Mod}^{\text{qc}}_X$.

It follows from the corollary that $D_{\text{qc}}(X, \alpha)$ satisfies the telescope hypothesis for localizations whose kernel is closed under tensor product with complexes in $D_{\text{qc}}(X)$.

**Corollary 5.3.** If $X$ is a noetherian scheme, then $D_{\text{qc}}(X)$ satisfies the $\otimes$-telescope hypothesis.

*Proof.* This follows from the previous corollary, with $\alpha = 0$, and Lemma 4.1.

The second corollary was obtained previously, by [2], [5], [16], and [27]. In flavor, the method used here is most similar to that of Balmer and Favi, although, as the first corollary demonstrates for $\alpha \neq 0$, Theorem 5.1 has much broader consequences. In fact, the first corollary holds even for $\alpha$ in the larger derived Brauer group of $X$ (see [31]). The proof is no different. The power of our method is that we can use étale locality to check for telescoping, rather than just Zariski local methods$^2$.

As a third corollary, we obtain a classification result for the smashing $\otimes$-localizations of $D_{\text{qc}}(X, \alpha)$.

**Corollary 5.4.** Let $X$ be a quasi-compact and quasi-separated scheme, and let $\alpha \in \text{Br}'(X)$. There is a bijection between the smashing $\otimes$-localizing subcategories of $D_{\text{qc}}(X, \alpha)$ under $D_{\text{qc}}(X)$ and the specialization closed subsets of $X$ that can be written as unions of closed subschemes of $X$ with quasi-compact complements.

*Proof.* To any smashing $\otimes$-localizing subcategory $D$ of $D_{\text{qc}}(X, \alpha)$, we can associate the specialization closed subset of $X$ consisting of the union of all supports of all perfect complexes in $D$. Since the support of any $\alpha$-twisted perfect complex is a closed subset with quasi-compact complement (see for instance [30]), we obtain one direction of the correspondence. To get the other direction, we use Thomason’s result [30] that this is true when $\alpha = 0$. It is known, for instance by Toën [31], that $D_{\text{qc}}(X, \alpha)$ is generated by a single $\alpha$-twisted perfect complex, say $E$. Let $V \subseteq X$ be a specialization closed subset, written as $V = \bigcup_i V_i$, where $V_i \subseteq X$ is closed with quasi-compact complement. Then, for each $i$ there is a perfect complex $K_i$ in $D_{\text{qc}}(X)$ with support exactly $V_i$, and any such perfect complex generates $D_{\text{qc}, V_i}(X)$, the smashing subcategory of complexes supported on $V_i$. The collection of objects $K_i \otimes E$ generates a smashing localizing subcategory of $D_{\text{qc}}(X, \alpha)$ whose support is precisely $V$. It thus suffices to show that any two smashing $\otimes$-localizing subcategories of $D_{\text{qc}}(X, \alpha)$ supported on $V$ are equivalent. We can reduce to the case that $V = V_1$ is irreducible with quasi-compact complement. So, assume that $D_1$ and $D_2$ are smashing localizing subcategories of $D_{\text{qc}}(X, \alpha)$ that are closed under tensoring with objects of $D_{\text{qc}}(X)$, and assume moreover that the supports of $D_1$ and $D_2$ are both identically $V$. The dual $E^\vee$ of $E$ is a $(\alpha)$-twisted perfect complex. Note

\footnote{Using the results of Toën [31], these results can be extended to give an fpfp local-global principle for telescoping. However, without any applications in mind, this story is omitted.}
that the (derived) tensor product of an \( \alpha \)-twisted complex and a \( \beta \)-twisted complex is an \( (\alpha + \beta) \)-twisted complex. The \( \otimes \)-localizing subcategories generated by \( D_1 \otimes E \) and \( D_2 \otimes E \) in \( D_{qc}(X) \) have support exactly \( V \), and hence, by Thomason’s result, coincide. It follows that the \( \otimes \)-closed localizing subcategories generated by \( D_1 \otimes E \) and \( D_2 \otimes E \) agree in \( D_{qc}(X, \alpha) \). Hence, \( D_1 = D_2 \).

Now, we consider some examples.

**Example 5.5.** Consider a singular noetherian affine scheme \( X = \text{Spec} \, S \) with a non-zero class \( \alpha \in H^1_{\text{et}}(X, \mathbb{Z}) \) (in which case \( X \) is not normal). For instance, one can take \( S = k[x, y, z]/(y^2 - x^3 + x^5) \). By [31], \( \text{Mod}_A \cong \text{Mod}_A \) for some derived Azumaya \( S \)-algebra \( A \). By construction, \( A \) cannot be derived equivalent to an ordinary associative algebra, for otherwise \( \alpha \in \text{Br}(X) \). Nevertheless, the \( S \)-linear telescope hypothesis holds for \( \text{Mod}_A \) by the theorem. It follows that the telescope hypothesis holds for \( \text{Mod}_A \) and hence that the triangulated telescope hypothesis holds for \( D(A) \). To our knowledge, this is the first example of any version of the telescope hypothesis for a truly derived dg algebra.

**Example 5.6.** In [15], Dwyer and Palmieri give an example of a non-noetherian scheme for which the telescope hypothesis holds, namely the truncated polynomial ring on infinitely many generators

\[
\text{Spec } k[t_1, t_2, \ldots]/(t_i^{n_i})
\]

where \( n_i \geq 2 \) for all \( i \). The theorem says that for an Azumaya algebra over this ring, the telescope hypothesis holds. Any such Azumaya algebra is induced from a central simple algebra over \( k \). But, this fact seems not to lead to an immediate proof of telescope.

## 6 Classifying stacks and gerbes

In this section, a proof is given of telescope for the derived category of gerbes and of classifying spaces of finite group schemes. These cover two of the most important cases of Deligne-Mumford stacks. For instance, the components of the moduli stack of semistable vector bundles on a smooth projective surface are abelian gerbes over noetherian schemes, so the results below apply.

**Theorem 6.1.** Let \( X \) be a noetherian scheme, and let \( \mathcal{G} \to X \) be a finite \( \text{etale} \) group scheme such that the fiber over every point \( x \in X \) is of order prime to the characteristic of \( k(x) \). Then, \( \text{Mod}_{\mathcal{G}} \) satisfies the \( \text{Mod}_X \)-linear telescope hypothesis (and hence the \( \text{Mod}_{\mathcal{B}\mathcal{G}} \)-linear telescope hypothesis), where \( \mathcal{B}\mathcal{G} \) is the classifying stack of \( \mathcal{G} \) over \( X \).

**Proof.** Let \( \coprod_i \text{Spec } S_i \to X \) be an \( \text{etale} \) cover such that \( \mathcal{G}_{S_i} = \mathcal{G} \times_X \text{Spec } S_i \) is a constant finite group scheme. Then, the restriction of \( \text{Mod}_{\mathcal{B}\mathcal{G}} \) to \( \text{Spec } S_i \) is

\[
\text{Mod}_{\mathcal{B}\mathcal{G}} \cong \text{Mod}_{\mathcal{S}[G]}.
\]

If \( \mathfrak{T} \) is a geometric point of \( \text{Spec } S \), then \( k(\mathfrak{T})[G] \) is a product of matrix algebras (since the order of \( G \) is prime to the characteristic of \( k(\mathfrak{T}) \)). This product does not depend on the geometric point on the connected components of \( X \). Therefore, using for example the arguments of [3, Section 5.3], it follows that \( S[G] \) is \( \text{etale} \) locally a product of matrix algebras over central separable extensions of \( S \).
Since $S$ is noetherian, $\text{Mod}_{S[G]}$ étale locally satisfies the linear telescope hypothesis by Corollary 5.2. But, this implies that $\text{Mod}_{S[G]}$ satisfies the $S$-linear telescope hypothesis by Theorem 5.1, and hence that $\text{Mod}_{S}$ satisfies the $\text{Mod}_X$-linear telescope hypothesis by the same theorem. ■

Recall that a tame Deligne-Mumford stack is one whose stabilizer groups have order prime to the residue characteristics. The classifying stacks appearing in theorem are examples.

**Corollary 6.2.** Let $X$ be a separated noetherian tame Deligne-Mumford stack whose stabilizers groups are locally constant, and assume that the coarse moduli space $X$ of $X$ is a noetherian scheme. Then $\text{Mod}_X$ satisfies the $\text{Mod}_X$-linear telescope hypothesis (and hence the $\text{Mod}_X$-linear telescope hypothesis).

**Proof.** In this case, $X \to X$ is étale locally of the form $[\text{Spec } T/G] \to \text{Spec } S$, where $G$ is a finite group acting on $\text{Spec } T$ with constant stabilizer $H$, by [1, Theorem 3.2]. It follows that $[\text{Spec } T/G]$ is equivalent to the classifying stack of $H$ over $\text{Spec } T^G$. But, $\text{Spec } T^G$ is also noetherian, by hypothesis, so that the corollary follows from the previous theorem. ■

If more was known about the derived categories of algebraic spaces, then the assumption on the coarse moduli space could possibly be dropped in the corollary. In particular, we are led to ask the following question.

**Question 6.3.** Does the $\text{Mod}_X$-linear telescope hypothesis hold for the derived category of a noetherian algebraic space $X$?

In the non-tame case, it is still possible to say something, at least when the stabilizers are abelian. Indeed, in that case, the group algebras $R[G]$ are in fact commutative and noetherian, whence telecopy follows from Neeman’s result. This is summarized in the next proposition, which extends Theorem 6.1. There are analogs of the corollaries as well, although we will leave their formulation to the reader.

**Proposition 6.4.** Let $A$ be a finite étale abelian group scheme over a noetherian scheme $X$. Then, $\text{Mod}_{B,A}$ satisfies the $\text{Mod}_X$-linear telescope hypothesis (and hence the $\text{Mod}_{B,A}$-linear telescope hypothesis).

**Proof.** Indeed, étale locally on $X$, $A$ is a constant abelian group. If $\text{Spec } S \to X$ is a map where $A_S$ is the constant abelian group scheme $A$, then $\text{Mod}_{B,A} \cong \text{Mod}_{S[A]}$. But, $S[A]$ is a commutative noetherian ring, so that the $S$-linear telescope hypothesis holds for $\text{Mod}_{S[A]}$. The rest of the proof follows now familiar lines. ■

**Corollary 6.5.** If $X$ is a noetherian scheme and $X \to X$ is a finite abelian gerbe, then $\text{Mod}_X$ satisfies the $\text{Mod}_X$-linear telescope hypothesis (and hence the $\text{Mod}_X$-linear telescope hypothesis).

**Proof.** In this case, $X \to X$ is étale locally on $X$ the classifying stack of a finite étale abelian group scheme. The corollary follows from the application of Proposition 6.4 followed by Theorem 5.1. ■

**Example 6.6.** Suppose that $X$ is a smooth projective surface over a field, and that $M$ is the moduli stack of geometrically stable vector bundles on $X$ of rank $r$, determinant $L$, and second Chern class $c \in \mathbb{Z}$. Then, the $\text{Mod}_M$-linear telescope hypothesis holds for $\text{Mod}_M$, where $M$ is the coarse moduli space of $M$. In fact, when $X \to X$ is a $\mathbb{G}_m$-gerbe, this is true for $N_X$ as well, where $N_X$ is the moduli stack of geometrically stable $X$-twisted vector bundles of rank $r$, determinant $L$, and second Chern class $c$. For details on these stacks, see [20].
Example 6.7. For a final example, let \( \mathcal{X} \) be a smooth Deligne-Mumford stack over \( \mathbb{C} \) of dimension at most 3, with coarse moduli space a noetherian scheme \( X \). Assume also that the canonical bundle of \( \mathcal{X} \) is trivial. This is precisely the situation in which the Bridgeland-King-Reid theorem \([11]\) holds. Thus, the coarse moduli space \( X \) has a crepant resolution, say \( V \to X \), and there is a derived equivalence \( \text{Mod}_V \cong \text{Mod}_{\mathcal{X}} \). The equivalence turns \( \text{Mod}_{\mathcal{X}} \) into a \( \text{Mod}_V \)-linear category, and since \( V \) is a noetherian scheme, it follows from the previous section that \( \text{Mod}_{\mathcal{X}} \) satisfies the \( \text{Mod}_V \)-linear telescope hypothesis.

Given the numerous positive results in this section, the next question is rather natural.

Question 6.8. Does the \( \text{Mod}_{\mathcal{X}} \)-linear telescope hypothesis hold for \( \text{Mod}_X \) when \( \mathcal{X} \) is a noetherian Deligne-Mumford stack?

Another positive answer is provided by Dell’Ambrogio and Stevenson \([13]\), who establish the linear telescope hypothesis for the derived categories of weighted projective stacks.

The question is especially important when \( \mathcal{X} \) has a coarse moduli scheme \( X \). If moreover \( \mathcal{X} \) is smooth, a recent paper of Dubey and Mallick \([14]\) together with a positive answer to the question would produce a classification of all \( \otimes \)-closed smashing localizations of \( D_{\text{qc}}(X) \): they would be in bijection with specialization closed subsets of \( X \). In particular, the theorems and statements of this section all lead to classification theorems. We end with one example of such a classification theorem.

Theorem 6.9. Let \( X \) be a smooth scheme of finite type over a field \( k \), and let \( \mathfrak{g} \to X \) be as in Theorem 6.1. Suppose that \( \mathfrak{g} \to X \) is a \( \mathfrak{g} \)-gerbe (a stack over \( X \) etale locally equivalent to \( B\mathfrak{g} \)). Then, there is a bijection between the set of \( \otimes \)-closed smashing localizations of \( D_{\text{qc}}(X) \) and the specialization closed subsets of \( X \).

Proof. The coarse moduli space of \( \mathcal{X} \) is \( X \), so by \([14]\) there is an isomorphism \( \text{Spc } D_{\text{perf}}(X) \cong \text{Spc } D_{\text{perf}}(\mathcal{X}) \), where \( \text{Spc} \) denotes the spectrum of Balmer \([4]\). This means that there is a bijection between the thick \( \otimes \)-ideals in these two \( \otimes \)-triangulated categories. The result follows since, by Theorem 6.1, any \( \otimes \)-smashing localization is generated by its intersection with \( D_{\text{perf}}(X) \) and from Thomason’s classification of the thick \( \otimes \)-ideals of \( D_{\text{perf}}(X) \) \([30]\). \( \blacksquare \)

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