A semi-variational approach to QCD at finite temperature and baryon density

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Abstract

Recently a new bosonization method has been used to derive, at zero fermion density, an effective action for relativistic field theories whose partition function is dominated by fermionic composites, chiral mesons in the case of QCD. This approach shares two important features with variational methods: the restriction to the subspace of the composites, and the determination of their structure functions by a variational calculation. But unlike standard variational methods it treats excited states on the same footing as the ground state.

I extend this method including states of nonvanishing fermion (baryon) number and derive an effective action for QCD at finite temperature and baryon density. I test the result on a four-fermion interaction model.
1 Introduction

Increasing temperature and baryon density hadronic matter is expected to undergo one or more crossovers and/or phase transitions. Increasing temperature at zero baryon number one might/should meet a phase in which quarks coexist with hadronic (possibly colored) states \[1\]. Increasing baryon density at fixed temperature one should meet a similar phase and possibly a color superconducting phase \[2\] due to a weak attractive channel between quarks of different colors. These new states of matter should be at least partially accessible to experimental investigation in heavy ion collision experiments.

Understanding the behavior of hadronic matter at high temperature and baryon density is relevant for the study of early Universe and neutron stars. But its theoretical properties can be studied only nonperturbatively and the lattice approach, the most powerful tool for first principles, nonperturbative studies, is affected in the case of finite density QCD by the well known sign problem.

Some progress was achieved recently \[3, 4\] by simulations at imaginary chemical potential. Other interesting results were obtained \[5\] within a modified version of the Glasgow re-weighting technique and by an approach which makes use of a Taylor expansion in the chemical potential in the small $\mu/T$ region \[6\].

At last a new approach to simulate QCD at finite temperature and baryon density was developed \[7\], which resembles in some aspects that of the imaginary chemical potential, but seems to have a wider range of applicability \[8\].

The magnitude of quark masses has large effects in numerical simulations. Several arguments lead to the expectation that the evaluation of the fermion determinant is more stable the larger fermion masses are \[9\]. These arguments are relevant to the present work, as explained at the end of Section VI.

To tackle the problem of QCD at finite temperature and baryon density I extend a new method constructed to treat fermionic systems whose partition function is dominated by fermionic composites. This is certainly the case of QCD at low temperature and baryon density, in which the relevant degrees of freedom are mesons and nucleons, and also at high temperature and baryon density according to the expectations reported above. This method was first developed in the framework of many-body nonrelativistic theories \[10\] and then applied to relativistic field theories \[11\] at finite temperature and zero fermion density. In the case of QCD, neglecting nucleons it amounts
to a bosonization. The heuristic motivation is that reformulation of a theory in terms of fields related to physical degrees of freedom should make it simpler. The starting point of this approach is the partition function in operator form, namely the trace of the transfer matrix in the Fock space of the fermions. The physical assumption of composite dominance is then implemented by restricting the trace to fermion composites. This requires an approximation of a projection operator on the subspace of the composites, the approximation being the better, the higher the number of fermion states (called index of nilpotency) in the composites. The approximate projection operator is constructed in terms of coherent states of composites, and evaluation of the trace, which is done exactly, generates a bosonic action in terms of the holomorphic variables appearing in the coherent states.

The structure functions of the composites are determined by a variational procedure. So this approach shares two important features with variational methods: The restriction to a subspace of the Fock space of fermions, the space of chiral mesons in the case of QCD, and the determination of their structure functions by a variational calculation. But unlike standard variational methods in the present one excited states are treated on the same footing as the ground state.

The utility of variational methods and bosonization has been widely appreciated in the theory of many-body systems. But their potentiality has also been considered in the framework of relativistic field theories, in particular gauge theories, for example by R. Feynman [12] who, however, was skeptical about their practical applicability, and recently in connection with QCD at high baryon density [13].

The approach just outlined is compatible with any regularization. But in gauge theories the effective action of the composites will involve vacuum expectation values of invariant functions of gauge fields which cannot be evaluated within the present framework. Therefore a lattice formulation was adopted in order to be able to extract such expectation values from numerical simulations. One is then confronted with the well known difficulty with chiral invariance, which can only in part be overcome by using Kogut-Susskind fermions. However the method can, at least in principle, be used with any other lattice regularization [14] for which a transfer matrix has been explicitly constructed.

The formalism of the transfer matrix does not treat time and space in a symmetric way, and therefore Euclidean invariance of the bosonic action must be checked a posteriori. All other symmetries are instead respected.

Before outlining the extension of this approach I will resume what has been already done. The validity of the method was tested on a model with a
4-fermion interaction in 3+1 dimensions: Euclidean invariance was recovered in the continuum limit and all the known results in the boson sector were exactly reproduced, namely condensation of a composite boson with the right mass, which breaks the discrete chiral invariance of the model. In addition the structure function of the composite was determined, and its radial factor, in a polar representation, turned out to be identical with that of the Cooper pairs of the BCS model of superconductivity.

To study QCD at finite temperature and baryon density as a first step I must introduce quark states in the presence of mesons, namely I must construct a Fock space containing composites and their constituents avoiding double counting. A similar but more difficult problem has been considered since a long time: given a Lagrangian which generates bound states, how to replace it by a physically equivalent Lagrangian in which bound states and constituents are treated on equal footing [15]. I solve my problem defining quasiquark states in such a way that quasiquark-quasiantiquark states are orthogonal to meson states. This constraint corresponds to the condition on the wave function renormalization of composite particles in the Lehman spectral representation of composite operators [15].

The next step, the explicit introduction of baryons and antibaryons constructed in terms of quasiquarks and quasiantiquarks is desirable but not necessary in a variational calculation, because a space of mesons, quasiquarks and quasiantiquarks obviously contains baryons and antibaryons. Therefore in the present paper I will not explicitly include in the partition function baryonic states. For a further simplification, which will be removed in work in progress, I will exclude antiquarks, so that my variational space contains mesons and baryons. This amounts to neglect virtual baryons-antibaryons, and it is justified for not too high temperature and baryon density. In the resulting effective action the expectation value of the chiral sigma field provides a mass to the quasiquarks. On the ground of the arguments concerning the effects of quark masses on numerical simulations quoted above [9], I hope that numerical simulations with such effective action will be more stable. I will also investigate in a separate work the possibility of analytical expansions.

I again test the method on a four-fermion interaction model, reproducing the known results in the fermion sector, namely existence of a free fermion whose mass is half that of the composite boson, and chiral symmetry restoration with increasing fermion density. The mechanism of this restoration is that quasifermions occupy the lowest energy states, from zero energy up to a maximum energy increasing with density, progressively depleting the condensate.
The paper is organized in the following way. In Section 2 I report the general formalism at zero baryon density, in section 3 I define quasiquark states and the approximate projection operator in the subspace of mesons and quasiquarks, in section 4 I derive a first form of their effective action. In Section 5 I apply this action to the study of the four-fermion interaction model, deriving the results described above. In section 6 I derive a second form of the effective action, which has a more transparent interpretation, and allows a crosscheck of the accuracy of the approximation for the projection operator by comparison of the results for the four-fermion interaction model, which coincide with those obtained by the first effective action. In Section 7 I summarize my results with an outlook to possible applications.

2 General formalism at zero baryon density

To make the paper reasonably self-contained I report the general formalism developed for relativistic field theories of fermions in the presence of composites dominance at zero fermionic density [11]. I make only a small modification: I do not fix the gauge, because this would prevent numerical simulations and complicate the treatment due to the need to enforce the Gauss constraint in Fock space.

The starting point of this formalism is the standard expression of the partition function of QCD in terms of the transfer matrix

$$Z = \int [dU] \exp \left[ - S_G(U) \right] \text{Tr}^F \left\{ \prod_{t=0}^{L_0-1} \left( \hat{T}_t \hat{V}_t \exp(\mu \hat{n}_B) \hat{T}_{t+1} \right) \right\}$$

(1)

where $L_0$ is the number of links in the temporal direction, $S_G$ is the gluon action and

$$\hat{T}_t = \exp[\hat{u}^\dagger M_t \hat{u} - \hat{v}^\dagger M_t^T \hat{v}] \exp[\hat{v} \hat{N}_t \hat{u}]$$

$$\hat{V}_t = \exp[\hat{u}^\dagger \ln \hat{U}_{0,t} \hat{u} + \hat{v}^\dagger \hat{U}_{0,t}^\dagger \hat{v}]$$

(2)

The $U_{\mu,t}$ are matrices whose matrix elements are the link variables at Euclidean time "t"

$$(U_{\mu,t})_{x_1,x_2} = \delta_{x_1,x_2} U_{\mu,t}(x_1).$$

(3)

Because the formalism treats asymmetrically time and space, I use **boldface** letters, as $x$, to denote spatial coordinates, and *italic* letters to denote space-time coordinates: $x = (t, x)$. $\hat{u}^\dagger$ and $\hat{v}^\dagger$ are, respectively, creation operators of quarks and antiquarks in state $i$, obeying canonical anti-commutation
relations. \( \text{Tr}^F \) is the trace over the Fock space of quarks, \( \mu \) is the chemical potential and \( \hat{n}_B \) the baryon number operator. The matrices \( M_t \) (\( M_t^T \) being the transposed of \( M_t \)) and \( N_t \) are functions of the spatial link variables at time \( t \). They depend on the regularization adopted for the fermions, but what follows is not affected by their explicit expressions, which are reported in Appendix B for Wilson and Kogut-Susskind fermions in the flavor basis.

I include in the gluon action the term

\[
\delta S_G = \sum_t -4 \text{tr}_- M_t \tag{4}
\]

which comes from transformations on the fermion fields going from the functional form to this operator form of the transfer matrix. I introduced the notation, which I will use for any matrix \( \Lambda \)

\[
\text{tr}_\pm \Lambda = \text{tr} \left( P_0^{(\pm)} \Lambda \right) . \tag{5}
\]

The operators \( P_0^{(\pm)} \), which project on the quark antiquark components of the quark field are defined in Appendix B, \( \text{tr}_\pm \) is the trace over quarks or antiquarks intrinsic quantum numbers and spatial coordinates (but not over time).

The expression (11) for the partition function was given by Lüscher [16] in the gauge \( U_0 \sim 1 \), in which \( V_t = 1 \) (but one has to impose the Gauss constraint in the Fock space of fermions).

Under the assumption that at low energy the partition function is dominated by chiral mesons, the trace in the Fock space can be restricted to them. The restricted partition function can be written introducing an operator \( P_m \) which projects in the subspace of mesons

\[
Z_{\text{mesons}} = \int dU \exp \left[ -S_G(U) \right] \text{Tr}^F \left\{ \prod_{t=0}^{N_0-1} \left( P_m \hat{T}_t^\dagger \hat{V}_t \exp(\mu \hat{n}_B) \hat{T}_{t+1} \right) \right\} . \tag{6}
\]

To construct this projector, meson creation operators are introduced

\[
\hat{\Phi}_{x,K}^\dagger = \hat{u}^\dagger \Phi_{xK}^\dagger \hat{v}^\dagger = \sum_{ij} \hat{u}^\dagger_i (\Phi_{xK}^\dagger)_{ij} \hat{v}^\dagger_j , \tag{7}
\]

where \( x \) represents their spatial coordinate, \( K \) their quantum numbers, like radial excitations, spin, flavor, etc, and \( \Phi_{xK} \) their structure functions (wave functions).
Since fermion creation operators are nilpotent, composite creation operators $\hat{\Phi}^\dagger_{x,K}$ can be classified according to their index of nilpotency, which is the highest integer exponent $\Omega$ such that
\[
(\hat{\Phi}^\dagger)^\Omega \neq 0.
\] (8)
$\Omega$ counts the number of fermion states in the composite. By analogy with systems of elementary bosons coherent states of mesons can be constructed
\[
|\phi\rangle = \exp \left( \sum_{x,K} \phi_{xK} \hat{\Phi}^\dagger_{xK} \right) |0\rangle,
\] (9)
where the $\phi_{xK}$'s are holomorphic variables. But since composites operators do not obey canonical commutation relations, the properties of their coherent states can differ from those of canonical bosonic coherent states. For instance the basic property of coherent states cannot be exactly satisfied
\[
\hat{\Phi}_{xK}|\phi\rangle \neq \phi_{xK}|\phi\rangle.
\] (10)
However, if the index of nilpotency of the composites is large enough, the composites system resembles a canonical bosonic system, and the properties of canonical boson coherent states will approximately hold for the composite coherent states, as shown in detail in Refs. [10, 11].

Hence, under the assumption that the composite operators which dominate the partition function have a large index of nilpotency, an approximate projection operator in the Fock space of the fermions can be defined
\[
P_m = \int \left[ \frac{d\phi d\phi^*}{2\pi i} \right] (|\phi\rangle \langle \phi|)^{-1} |\phi\rangle \langle \phi|,
\] (11)
where
\[
\left[ \frac{d\phi d\phi^*}{2\pi i} \right] = \prod_{x,K} \left[ \frac{d\phi_{xK} d\phi^*_{xK}}{2\pi i} \right].
\] (12)
It is important to observe that the space selected by this operator includes 2 physically equivalent states obtained for $\phi = 0, \infty$. They correspond to a completely empty or filled lattice.

The scalar product of coherent states appearing in the definition of the projection operator is
\[
\langle \phi|\phi' \rangle = \det_+ \left[ \mathcal{I} + (\phi \cdot \Phi^\dagger) (\phi' \cdot \Phi) \right],
\] (13)
where
\[ \phi \cdot \Phi^\dagger = \sum_{x,K} \phi_{xK} \Phi^\dagger_{xK} \]  
and for any matrix \( \Lambda \)
\[ \det_\pm \Lambda = \det(P_0^\pm \Lambda) . \]  
\( \mathcal{I} \) is the identity in the space of all the matrices. I remind that the entries of these matrices do not include time. By a little abuse of notation I will often write "1" instead of \( \mathcal{I} \).

With periodic boundary conditions for the gauge fields, the partition function is
\[ Z = \text{Tr} F \left\{ \hat{T}_0^\dagger \hat{V}_0 \exp(\mu n_B) \hat{T}_1^\dagger \hat{V}_1 \exp(\mu n_B) \cdots \hat{T}_{L_0-1}^\dagger \hat{V}_{L_0-1} \exp(\mu n_B) \hat{T}_0 \right\} \]  
while its restriction to mesons is
\[ Z_{\text{mesons}} = \text{Tr} F \left\{ \mathcal{P}_m \hat{T}_0^\dagger \hat{V}_0 \mathcal{P}_m \hat{T}_1^\dagger \hat{V}_1 \cdots \mathcal{P}_m \hat{T}_{L_0-1}^\dagger \hat{V}_{L_0-1} \hat{T}_0 \right\} 
= \int \prod_{t=0}^{L_0-1} \left[ \frac{d\phi_t d\phi^*_t}{2\pi i} \right] \frac{1}{\langle \phi_t | \phi_t \rangle} \langle \phi_t | \hat{T}_t^\dagger \hat{V}_t \hat{T}_{t+1} | \phi_{t+1} \rangle \]  
where a copy of the Fock space of the mesons has been introduced at each time slice. The chemical potential has disappeared because it is not active in a space of only mesons. Explicitly
\[ |\phi_t \rangle = \exp \left( \sum_{x,K} \phi_{xK} (t, x) \hat{\Phi}^\dagger_{xK} [U_t] \right) |0\rangle . \]  

I remark that the structure functions \( \Phi_{x,K} \) do not depend explicitly on time, but as they are functions of gauge fields, time will enter as a label of these fields.

In the evaluation of the trace on the fermionic Fock space the only difference with respect to [11] is the presence of the operator \( \hat{V}_t \). But I notice that the product of operators \( \hat{V}_t \hat{T}_{t+1} \) has an expression similar to that of \( \hat{T}_{t+1} \)
\[ \hat{V}_t \hat{T}_{t+1} = \exp \left[ -\hat{u}^\dagger \ln(e^{M_{t+1}^T U_{0,t}^T}) \hat{u} - \hat{v}^\dagger \ln(e^{M_{t+1} T} U_{0,t}) \hat{v} \right] \exp[\hat{v} N_t \hat{u}] . \]  
Then evaluation of the trace over the Fock space proceeds exactly as in [11] with the result
\[ Z_{\text{mesons}} = \int [dU] \exp [S_G(U)] \int \prod_t \left[ \frac{d\phi_t d\phi^*_t}{2\pi i} \right] \exp \left[ -S_{\text{mesons}}(\phi^*, \phi) \right] , \]  

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where
\[ \prod_t \left[ \frac{d\phi_t d\phi_t^*}{2\pi i} \right] = \prod_{x,K} \left[ \frac{d\phi_K(x) d\phi_K^*(x)}{2\pi i} \right] , \tag{21} \]

and
\[ S_{\text{mesons}} = \sum_t \text{tr}_- \left[ -\ln R_t + \ln \mathcal{R}_t + M_t^\dagger \right] . \tag{22} \]

In the last equation
\[ R_t = \left( 1 + \mathcal{F}_t^\dagger \mathcal{F}_t \right)^{-1} , \]
\[ \mathcal{R}_t = \left[ (1 + \mathcal{F}_t^\dagger N) e^{M_{t+1} U_{0,t}^\dagger} e^{M_t^\dagger} (1 + N^\dagger \mathcal{F}_t) \right]^{-1} e^{M_t^\dagger + M_{t+1} U_{0,t}^\dagger} , \tag{23} \]
in which I set
\[ \mathcal{F} = \phi^* \cdot \Phi . \tag{24} \]

Notice that the matrix \( \mathcal{R}_t \) involves gauge fields at time \( t \) and \( t + 1 \). The notation is somewhat different from that of \[11\].

It is remarkable that \( S_{\text{mesons}} \) has been evaluated exactly \[11\], so that the only approximations in the partition function are the physical assumption of boson dominance and the form of the projector over the meson subspace. Since the projector depends on the structure functions \( \Phi_{x,K} \), the effective action is a functional of these functions which are determined by a variational calculation on the quantities of interest. In simple cases, like the four-fermion interaction model, the variational calculation provides the exact form of the structure function. In QCD, unless some analytic progress is made along a way similar to that of the four-fermion interaction model, one has to adopt a trial expression.

In Ref. \[11\] an alternative, equivalent form of the effective action was derived, which has a more transparent interpretation. I will not report it here because I will also derive two forms of the effective action at finite baryon density, but for the second one I will follow a somewhat different procedure.

### 3 Quasiquarks and generalized Bogoliubov transformations

In order to extend the formalism to QCD at finite baryon density, I must introduce in the partition function states with nonvanishing baryon number. In the spirit of composites dominance, I should then construct baryonic
composites and define a projection operator on the subspace of mesons and baryons. But in the present paper I will be satisfied by introducing, in addition to mesons, quasiparticles states with quark quantum numbers, which I will call quasiquarks (and quasiantiquarks). Such a space obviously contains the space of mesons and baryons.

To avoid double counting, mesons and quasiparticles must satisfy a mutual compositeness condition: Mesonic states must be orthogonal to quasiquark-quasiantiquark states. This constraint has the physical meaning of the condition $Z = 0$ for bound states in the Lehmann spectral representation of composite operators [15], namely the the condition required to introduce a bound state on the same footing as the constituents in a Lagrangian.

I will denote by $\hat{\alpha}_i, \hat{\beta}_i$ quasiparticles destruction operators. I will enforce the compositeness condition by requiring that quasiquark-quasiantiquark states annihilate coherent states of mesons

$$\hat{\alpha}_i |\phi\rangle = \hat{\beta}_i |\phi\rangle = 0.$$ (25)

This condition will prove crucial in the application to the 4-fermion model.

The compositeness condition in the above form can be solved exactly, even though composites coherent states do not satisfy exactly the basic property (10) of coherent states of elementary bosons. The operators $\hat{\alpha}_i, \hat{\beta}_i$ are obtained by a generalized Bogoliubov transformation [17]

$$\hat{\alpha}_i = \left[ R^\frac{1}{2} \left( \hat{u} - \mathcal{F}^\dagger \hat{v}^\dagger \right) \right]_i$$
$$\hat{\beta}_i = \left[ \left( \hat{v} + \hat{u}^\dagger \mathcal{F}^\dagger \right) \left( R' \right)^\frac{1}{2} \right]_i,$$ (26)

where

$$R' = (1 + \mathcal{F} \mathcal{F}^\dagger)^{-1}.$$ (27)

Bogoliubov introduced his tranformation to construct a theory of superconductivity in the presence of an electron-phonon interaction, and defines quasiparticles in terms of electron particle-holes states, explicitly breaking the $U(1)$ symmetry related to fermion conservation. The above transformations instead respect all symmetries, which is necessary with gauge interactions, thanks to the presence of the bosonic field $\phi$. The operators $\hat{\alpha}_i, \hat{\beta}_i$ and their Hermitean conjugates

$$\hat{\alpha}_i^\dagger = \left[ \left( \hat{u}^\dagger - \hat{v} \mathcal{F}^\dagger \right) \left( R \right)^\frac{1}{2} \right]_i$$
$$\hat{\beta}_i^\dagger = \left[ \left( R' \right)^\frac{1}{2} \left( \hat{v}^\dagger + \mathcal{F} \hat{u} \right) \right]_i.$$ (28)
satisfy canonical commutation relations. As anticipated in the Introduction I include only mesons and quasiquarks in my variational space, but not quasiantiquarks. My variational space does not contain antibaryons, which are not expected to be important for not too high temperature and baryon density. In any case it will be clear that extension of the formalism to include quasiantiquarks should not present any significant difficulty.

I define "coherent" states of quasiquarks and mesons

\[ |\alpha, \phi \rangle = \exp(-\alpha \cdot \hat{\alpha}^\dagger) \exp(\phi \cdot \hat{\Phi}^\dagger)|0\rangle \]  

(29)

where the \( \alpha_i \) are Grassmann variables and

\[ \alpha \cdot \hat{\alpha} = \sum_i \alpha_i \hat{\alpha}_i. \]  

(30)

These states can be recast in the form

\[ |\alpha, \phi \rangle = \exp(\hat{u}^\dagger R^{-\frac{1}{2}} \alpha + \phi \cdot \hat{\Phi}^\dagger)|0\rangle \]  

(31)

which is more convenient for calculations. The operator which approximately projects on states of mesons and quasiquarks is

\[ P_{m-q} = \int [d\alpha^* d\alpha] \left[ \frac{d\phi^* d\phi}{2\pi i} \right] \langle \alpha, \phi | \alpha, \phi \rangle^{-1} |\alpha, \phi \rangle \langle \alpha, \phi | \]  

(32)

where the measure is

\[ \langle \alpha, \phi | \alpha, \phi \rangle^{-1} = \langle \alpha | \alpha \rangle^{-1} \langle \phi | \phi \rangle^{-1} = \exp\{ \text{tr} - \ln R - \alpha^* \cdot \alpha \}. \]  

(33)

If \( P_{m-q} \) is an approximate projector it must satisfy the equations

\[ \langle 0 | \hat{\Phi}^{m_1} \hat{\alpha}^{n_1} P_{m-q} (\hat{\alpha}^\dagger)^{m_2} (\hat{\Phi}^\dagger)^{m_2} |0\rangle \approx \]  

\[ \langle 0 | \hat{\Phi}^{m_1} \hat{\alpha}^{n_1} (\hat{\alpha}^\dagger)^{m_2} (\hat{\Phi}^\dagger)^{m_2} |0\rangle \propto \delta_{m_1,m_2} \delta_{n_1,n_2}. \]  

(34)

These equations are generated by the following ones

\[ \langle \phi_1 \alpha_1 | P_{m-q} | \phi_2 \alpha_2 \rangle \approx \langle \phi_1 \alpha_1 | \phi_2 \alpha_2 \rangle, \]  

(35)

by taking derivatives with respect to the variables \( \alpha_i, \phi_i \) and setting them equal to zero. The left hand side of (35) is

\[ \langle \alpha_1, \phi_1 | P_{m-q} | \alpha_2, \phi_2 \rangle = \int [d\alpha^* d\alpha] \left[ \frac{d\phi^* d\phi}{2\pi i} \right] \exp\{ \text{tr} - \ln R 

+ \ln(1 + F_1^\dagger F_1) + \ln(1 + F_2^\dagger F_2) 

+ \alpha_1^\dagger R_1^{-\frac{1}{2}} (1 + F_1^\dagger F_1)^{-1}(1 + F_2^\dagger F_2)^{-1} R_2^{-\frac{1}{2}} \alpha_2 \} \].  

(36)
This shows by inspection that the first member of (34) vanishes unless $m_1 = m_2, n_1 = n_2$. Evaluating the integral in the above equation by the saddle point method, as done in [11] for $P_m$, we see that $P_{m-q}$ is approximately a projector if we assume
\[ \text{tr} \left( \Phi^\dagger \Phi \right)^n \simeq \Omega^{-n+1}. \] (37)

I remind that $\Omega$ is the index of nilpotency of $\hat{\Phi}$.

4 First form of the effective action at finite baryon density

I follow the derivation of the effective action outlined in Section 2 for zero baryon density. I skip many intermediate steps because calculations of this kind have been reported in any detail in [11], and can be easily repeated here by the help of the formulæ collected in Appendix A. I start by evaluating the matrix elements of the transfer matrix between coherent states according to
\[
\langle \alpha_t, \phi_t | \hat{T}_t^\dagger \hat{V}_t \exp(\mu \hat{n}_B) \hat{T}_{t+1} | \alpha_{t+1}, \phi_{t+1} \rangle = \int [d\gamma^* d\gamma] [d\delta^* d\delta] \times e^{-\gamma^\dagger \gamma - \delta^\dagger \delta} \langle \alpha_t, \phi_t | \hat{T}_t^\dagger | \gamma \delta \rangle \langle \gamma \delta | \hat{V}_t \exp(\mu \hat{n}_B) \hat{T}_{t+1} | \alpha_{t+1}, \phi_{t+1} \rangle. \] (38)

The last factor is
\[
\langle \gamma | \hat{V}_t \exp(\mu \hat{n}_B) \hat{T}_{t+1} | \alpha_{t+1}, \phi_{t+1} \rangle = \det_{-} (1 + F^\dagger N)_{t+1} \times \exp \left\{ \gamma^\dagger U_{0, t} e^{-M_{t+1}} (1 + F^\dagger N)_{t+1}^{-1} \left[ e^\mu R_{t+1} \frac{1}{2} \alpha_{t+1} + F^\dagger_{t+1} e^{-M_{t+1}} U_{0, t} \delta^\dagger \right] \right\}. \] (39)

A similar result for the other matrix element and integration over $\gamma^*, \gamma, \delta^*, \delta$ leads to the expression
\[
\langle \alpha_t, \phi_t | \hat{T}_t^\dagger \hat{V}_t \exp(\mu \hat{n}_B) \hat{T}_{t+1} | \alpha_{t+1}, \phi_{t+1} \rangle = \det_{-} \left( e^{-M_{t}^\dagger} R_{t}^{-1} \right) \times \exp \left( \alpha_t^* e^\mu R_{t}^\frac{1}{2} \mathcal{R}_{t} U_{0, t} e^{-M_{t+1}} R_{t+1}^\frac{1}{2} \alpha_{t+1} \right). \] (40)

From the measure appearing in the definition of $P_{m-q}$, Eq.(33), I get the factor
\[
\langle \alpha_t, \phi_t | \alpha_t, \phi_t \rangle^{-1} = \det_{-} R_{t} \exp \left( -\alpha_t^* \cdot \alpha_t \right). \] (41)
Putting these pieces together I get the effective action of mesons interacting with quasiquarks

\[ S_{\text{mesons-quarks}} = S_{\text{mesons}} - \sum_t \alpha_t^* \left[ -\alpha_t + e^\mu R_t^{-\frac{1}{2}} \times \mathcal{R}_t U_{0,t} e^{-M_{t+1}} R_{t+1}^{-\frac{1}{2}} \alpha_{t+1} \right], \tag{42} \]

where \( S_{\text{mesons}} \) is given by Eq. \( \text{(22)} \). I remind that that \( \alpha \) is a 2-spinor with the quark intrinsic quantum numbers. It can be put in a more transparent form

\[ S_{\text{mesons-quarks}} = S_{\text{mesons}} - s \sum_t \alpha_t^*(\nabla_t - \mathcal{H}_t)\alpha_{t+1} \tag{43} \]

by introducing the lattice covariant derivative in the presence of a chemical potential and the lattice Hamiltonian

\[
\nabla_t \alpha_{t+1} = \frac{1}{s} \left( e^\mu U_{0,t} \alpha_{t+1} - \alpha_t \right) \\
\mathcal{H}_t = \frac{1}{s} e^\mu \left[ U_{0,t} - R_t^{-\frac{1}{2}} \mathcal{R}_t U_{0,t} e^{-M_{t+1}} R_{t+1}^{-\frac{1}{2}} \right]. \tag{44} 
\]

The factor \( s \) in the above equations takes the value 2 in the Kogut-Susskind regularization because the quarks live on blocks, and 1 in the Wilson regularization. I notice that the time derivative is not symmetric, so that this action does not give rise to fermion doubling. Integrating over the Grassmann variables I get the purely bosonic effective action

\[ S_{\text{effective}} = - \text{Tr} - \ln \left( R^{-1} e^M - e^\mu U_0 T_0^{(+)} \right) \tag{45} \]

where I adopted the following notations: All matrices (with exception of \( T_{\mu}^{(\pm)} \)) which do not have a time label are diagonal in time with matrix elements

\[
(U_0)_{x_1,t_1,x_2,t_2} = \delta_{t_1,t_2} \delta_{x_1,x_3} U_{0,t_1}(x_1) \\
\mathcal{R}_t_{x_1,t_1,x_2,t_2} = \delta_{t_1,t_2} (\mathcal{R}_t)_{x_1,x_2} \\
\tag{46}
\]

while the matrix elements of space-time translation operators are

\[
(T_{\mu}^{(\pm)})_{x_1,x_2} = \delta_{x_2,x_1\pm s\hat{\mu}}. \tag{47}
\]

"Tr" is the trace on all entries including time, while I remind that "tr" is the trace on intrinsic and spatial quantum numbers only.
5 Application to a four-fermion interaction model with Kogut-Susskind fermions

To get insight in the above result and also to test it I apply it to the four-fermion interaction model adopted as a test at zero fermion density [11]. It is a model in 3+1 dimensions regularized on a lattice with Kogut-Susskind fermions in the flavor basis (I do not know any formulation of the transfer matrix in the spin-diagonal basis which can be used in the present formalism). For each of the four Kogut-Susskind tastes there are $N_f$ degenerated flavors. Hence, the continuum limit will describe a theory with $4N_f$ flavors. In the flavor basis the action reads

$$S = \sum_x \sum_y \bar{\psi}(x) \left[ m \mathbb{I} \otimes \mathbb{I} + Q \right]_{x,y} \psi(y) + \frac{1}{2} \frac{g^2}{4N_f} \sum_x \left( \bar{\psi}(x)\psi(x) \right)^2$$  \hspace{1cm} (48)$$

where $m$ is the mass parameter, $g^2$ the coupling constant, $\psi$ the fermion fields and $Q$ the hopping matrix:

$$Q = \sum_{\mu} \gamma_\mu \otimes \mathbb{I} \left[ P^{(-)}_{\mu}\nabla^{(+)}_{\mu} + P^{(+)}_{\mu}\nabla^{(-)}_{\mu} \right]. \hspace{1cm} (49)$$

The matrices to the left (right) of the symbol $\otimes$ act on Dirac (taste) indices. I denote by $\gamma$ and $t$ the matrices acting on these indices, respectively. The operators

$$P^{(\pm)}_{\mu} = \frac{1}{2} \left[ \mathbb{I} \otimes \mathbb{I} \pm \gamma_\mu \gamma_5 \otimes t_5 t_\mu \right]. \hspace{1cm} (50)$$

are orthogonal projectors. The fermion fields are defined on blocks (see Appendix B for details). The right and left derivatives $\nabla^{(\pm)}_{\mu}$ are given by

$$\nabla^{(\pm)}_{\mu} = \pm \frac{1}{2} \left( T^{(\pm)}_{\mu} - 1 \right). \hspace{1cm} (51)$$

The factor $1/2$ is due to the fact that the operators $T_{\mu}$ translate by one block. The model has a discreet chiral symmetry at $m = 0$:

$$\psi \rightarrow -\gamma_5 \otimes t_5 \psi, \quad \bar{\psi} \rightarrow \bar{\psi} \gamma_5 \otimes t_5. \hspace{1cm} (52)$$

To have an action bilinear in the fermion fields a scalar field $\sigma(x)$ is introduced, whose integration generates the four-fermion coupling:

$$S' = \sum_x \sum_y \bar{\psi}(x) \left[ (m + \sigma) \mathbb{I} \otimes \mathbb{I} + Q \right]_{x,y} \psi(y) + \frac{4N_f}{2g^2} \sum_x \left( \psi^2(x) \right). \hspace{1cm} (53)$$
The partition function now reads
\[
Z = \int [d\sigma][d\bar{\psi}d\psi] \exp [-S'].
\] (54)

Its restriction to fermion composites plus a fermion gas, without antifermions is
\[
Z_{C-F} = \int [d\sigma][d\phi^*d\phi][d\alpha^*d\alpha] \exp \left[ -\frac{4N_f}{2g^2} \sum_x ' \sigma^2(x) - S_{C-F} \right].
\] (55)

\(S_{C-F}\) is given by Eq.\(42\), in which one has to insert the expressions of the matrices \(M, N\) appropriate to Kogut-Susskind fermions in the flavor basis \[18\]: The matrix \(M\) is equal to zero and the matrix \(N\) is reported in Appendix \[B\]. Integration over the fermion fields gives the effective action
\[
S_{\text{effective}} = -\text{Tr}_- \ln \left[ \mathcal{R}^{-1} \mathcal{R} - e^\mu T_0^{(+)} \right].
\] (56)

Now I look for constant values of the fields \(\phi^*,\phi\) and \(\sigma\) which make the action stationary. I put a bar over constant fields and their functions. Then
\[
\overline{S}_{\text{effective}} = -\text{Tr}_- \ln \left[ \overline{\mathcal{R}}^{-1} \mathcal{R} - e^\mu T_0^{(+)} \right]
\] (57)

and I can perform the sum over time getting
\[
\overline{S}_{\text{effective}} = -\frac{1}{2}L_0 \text{tr}_- \left\{ \mu \theta \left[ e^\mu - \overline{\mathcal{R}}^{-1} \mathcal{R} \right] + \ln(\overline{\mathcal{R}} \mathcal{R}^{-1}) \theta \left[ \mathcal{R}^{-1} \mathcal{R} - e^\mu \right] \right\},
\] (58)

where \(\theta\) is the step function. For \(\mu = 0\) I recover the effective action derived \[11\] at zero fermion density.

To determine the magnitude of the condensate I must perform a variation with respect to the boson fields \(\overline{\phi}, \overline{\phi}\), and to determine the form factor of the composite a variation with respect to the matrices \(\overline{\mathcal{F}}^\dagger, \mathcal{F}\). But \(\overline{S}_{\text{effective}}\) does not depend on these variables separately, it is a function of \(\overline{\mathcal{F}}^\dagger\) and \(\mathcal{F}\). The saddle point equations with respect to these matrices for \(e^\mu < \mathcal{R}^{-1}\overline{\mathcal{R}}\), are identical to the ones for zero chemical potential. Using the result of \[11\] I then get
\[
\overline{\mathcal{F}}^\dagger = \frac{N}{2H} \left( \sqrt{1 + H^2} + 1 \right), \quad e^\mu < \mathcal{R}^{-1}\overline{\mathcal{R}},
\] (59)

where
\[
H = \frac{1}{2} \sqrt{N^\dagger N} = \sqrt{(m + \overline{\sigma})^2 - \Delta}
\] (60)
with $\triangle$ given by Eq. (96). I notice that $H$ differs from the lattice Hamiltonian defined above

$$\overline{H} = \frac{1}{2} e^{i\mu} \left( 1 - \overline{R}^{-1} \overline{R} \right) = e^{i\mu} H \left( \sqrt{1 + H^2} - H \right),$$  \hspace{1cm} (61)$$

but they are equal in the formal limit of vanishing lattice spacing. In this limit I can rewrite $S_{C-F}$ in the form

$$S_{C-F} = S_C - 2 \sum_t \alpha_t^* \left( \nabla_t^{(+)} - (H - \mu) \right) \theta(2H - \mu) \alpha_t.$$  \hspace{1cm} (62)$$

For $m = \mu = 0$, I recover the well known result that the fermionic system under consideration in the limit of $N_f \to \infty$ contains free fermions of mass $\overline{\sigma}$ in addition to free bosons of mass $2\overline{\sigma}$. I emphasize that the result recovered in this way is only formal. Indeed after adding one fermion I should determine the new minimum of the action, namely the variation of the structure functions. But it can be justified in a concrete way by evaluating the difference of $\overline{S}_{\text{effective}}$ given by Eq. (64) below at fermion numbers differing by one unit.

Now I impose the condition on the fermion number which determines the chemical potential. From Eq. (58) I get

$$-2L_0 \frac{\partial}{\partial \mu} S_{\text{effective}} = \text{tr} \, \theta \left( \exp \mu - \overline{R}^{-1} \overline{R} \right) = \text{tr} \, \theta \left( \exp \mu - 1 - 2\overline{H} \right) = n_F.$$  \hspace{1cm} (63)$$

For $\mu < 2\overline{\sigma}$, $n_F = 0$. For $\mu > 2\overline{\sigma}$, quasifermions occupy the states from zero energy up to a maximum energy $E_{n_f}$ depending on the fermion number $n_F$.

The effective action at the minimum takes the form

$$\overline{S}_{\text{effective}} = -L_0 \text{tr} \left\{ \ln \left( \sqrt{1 + H^2} + H \right)^2 \theta \left( 2\overline{H} + 1 - \exp \mu \right) \right\}. $$  \hspace{1cm} (64)$$

Stationarity with respect to $\overline{\sigma}$ yields the gap equation which determines the masses and therefore the breaking of chiral invariance

$$\frac{4L_0 N_f}{g^2} \overline{\sigma} = -\frac{\partial}{\partial \overline{\sigma}} \overline{S}_{\text{effective}} = 2L_0 \overline{\sigma} \text{tr} \left\{ \frac{1}{H\sqrt{1 + H^2}} \theta \left( 2\overline{H} + 1 - \exp \mu \right) \right\}. $$  \hspace{1cm} (65)$$

Increasing the fermion density, namely the chemical potential, quasifermions occupy higher and higher energy states depleting the condensate, until only the solution $\overline{\sigma} = 0$ remains and chiral invariance is restored.
6 Second form of the effective action

The expression (59) of the form factors is somewhat surprising, because they are increasing functions of momentum. In [11] a more natural form was deduced by performing a unitary transformation in the fermionic Fock space and deriving the corresponding effective action. This transformation changes the empty lattice into the fully occupied one and particles into holes. In this new Fock space the structure functions are decreasing functions of momentum, and in a polar representation their polar factor is equal to that of the Cooper pairs of the BCS model of superconductivity.

But in addition the second form of the action provided a test of consistency of the approximation for the projection operator \( P_m \). I could follow the same path at nonzero baryon density, but instead I will get a similar result in a different way. First I rearrange the trace in Fock space in the following way

\[
Z = \text{Tr}^F \left\{ \hat{V}_0 \exp(\mu n_B) \hat{T}_1 \hat{T}_1^\dagger \hat{V}_1 \exp(\mu n_B) \cdots \hat{V}_{L_0-1} \exp(\mu n_B) \hat{T}_0 \hat{T}_0^\dagger \right\}. \tag{66}
\]

Then I insert the projection operator \( P_{m-q} \) in the trace according to

\[
Z'_{\text{mesons-quarks}} = \int [dU] \exp [S_G(U)] \text{Tr}^F \left\{ \prod_{t=0}^{L_0-1} \left( P_{m-q} \exp(\mu \hat{n}_B) \hat{V}_t \hat{T}_{t+1} \hat{T}_{t+1}^\dagger \right) \right\}. \tag{67}
\]

I emphasize that \( Z', Z \) need not coincide with each other because \( P_{m-q} \) is not an exact projection operator, but the results obtained by the two forms should agree within the approximation for \( P_{m-q} \). A comparison between these results provides a check of its accuracy.

In the same way as for the first form of the effective action I evaluate the matrix elements

\[
\langle \alpha_t, \phi_t | \exp(\mu \hat{n}_B) \hat{V}_t \hat{T}_{t+1} \hat{T}_{t+1}^\dagger | \alpha_{t+1}, \phi_{t+1} \rangle = \exp \left\{ -\text{tr}_+ \mathcal{R}'_t \right\}
\]

\[
+ \alpha_t^* e^\mu R_t^{-\frac{1}{2}} U_{0,t} e^{-M_{t+1}} R_{t+1}^{-\frac{1}{2}} \mathcal{R}'_{t+1} R_{t+1}^{-\frac{1}{2}} \alpha_{t+1} \right\}, \tag{68}
\]

where

\[
\mathcal{R}'_t = \left[ 1 + \left( N_t + e^{-M_t} U_{0,t-1}^\dagger F_{t-1} U_{0,t-1} e^{-M_t} \right)^\dagger \right]^{-1} e^{-M_t}. \tag{69}
\]
Including the contribution \([53]\) from the measure I get

\[
S'_{\text{mesons-quarks}} = S'_{\text{mesons}} - s \sum_t \alpha_t^* \left( \nabla_t - \mathcal{H}_t' \right) \alpha_{t+1} \tag{70}
\]

where

\[
S'_{\text{mesons}} = \sum_t \text{tr}_- \left[ - \ln R_t + \ln \mathcal{R}_t' + M_t^\dagger \right]
\]

\[
\mathcal{H}' = \frac{1}{s} e^\mu \left[ U_{0,t} - R_t^{-\frac{1}{2}} U_{0,t} e^{-M_{t+1} \mathcal{R}_{t+1}} R_{t+1}^{-\frac{1}{2}} \right]. \tag{71}
\]

Integrating over \(\alpha^*, \alpha\) I get the purely bosonic action

\[
S'_{\text{effective}} = - \text{Tr}_- \ln \left[ - R \left( \mathcal{R}' \right)^{-1} e^M + e^\mu U_0 T^{(+)}_0 \right]. \tag{72}
\]

By exploiting the cyclic property of the trace it can be rewritten

\[
S'_{\text{effective}} = - \text{Tr}_- \ln \left[ \left( \mathcal{R}' \right)^{-1} R e^M - e^\mu U_0 T^{(+)}_0 \right]. \tag{73}
\]

This expression differs from \(S_{\text{effective}}, \text{Eq.(45)}\), by the replacement of \(\mathcal{R}\) by \(\mathcal{R}'\).

I use this second form of the effective action for the four-fermion interaction model with Kogut-Susskind fermions in the saddle point approximation. By means of the results of ref.\([11]\) I find

\[
F_t^\dagger = N \frac{1}{2H} \left( \sqrt{1 + H^2} - H \right), \quad 2\mathcal{H}' > e^\mu - 1. \tag{74}
\]

Now the structure function is a decreasing function of the constituent fermions energy. Using the above expression I find that

\[
\mathcal{H}' = \mathcal{H}, \quad \tag{75}
\]

so that the results concerning mass of the uncorrelated fermions and restoration of chiral symmetry derived by the first form of the action are recovered.

I conclude this Section by an observation about the way the arguments of Ref.\([9]\) concerning the stability of numerical simulations might apply to the present effective action, comparing QCD with the four-fermion model. In this model after linearization there is a field, the sigma field, which appears in the matrix \(N\) in the same position as the fermion mass, so that its expectation value provides in a natural way a mass to the fermion in the broken
phase. In QCD there is no such field, but the chiral sigma field can play the same role. Indeed its expectation value appears in Eq. (69) as an addendum to the matrix $N$ and therefore as an addendum to the bare quark mass. To the extent that high quark masses can stabilize numerical simulations, use of the present effective action should make these calculations easier.

7 Summary and outlook

I extended the formalism of composite boson dominance to the case of non-vanishing fermion number. This required a definition of fermion and antifermion states in the presence of bosonic composites satisfying a compositeness condition to avoid double counting. These fermion (antifermion) states are called quasifermions (quasiantifermions). Their definition is achieved by a generalized Bogoliubov transformation.

In the application to QCD I restricted myself to a space of mesons and quasiquarks, excluding quasiantiquarks, which contains the space of mesons and baryons. Neglecting quasiantiquarks amounts to neglect virtual baryons-antibaryons, which is justified for not too high temperature and baryon density. Obviously if one wants to investigate any high temperature and/or baryon density phase transition quasiantiquarks must be included, but I cannot foresee any obstruction in this extension.

I derived two forms of the effective action and I applied both of them to a four-fermi interaction model at zero temperature but finite fermion density. I recovered in both ways the known results, which provides a crosscheck of the approximation of the projection operator introduced to restrict the fermionic Fock space in the partition function. The discrete chiral invariance of the model (at zero fermion mass) is broken by composite boson condensation and the spectrum of the broken phase contains, in addition to a composite boson, a free fermion whose mass is half that of the boson. Increasing the fermion density, quasifermions occupy the lowest energy states up to an energy which increases with increasing density depleting the condensate, until chiral symmetry is restored. The compositeness condition is crucial to get these results. Since the action of the composite boson is known [11], one could study the system also at finite temperature and density.

Possible applications of the present formalism include numerical studies of the evolution of the state of baryon matter with temperature and density and the associated phase transitions. In this connection I remind the way the arguments of Ref. [9] concerning the stability of numerical simulations might apply: the expectation value of the chiral sigma field appears in Eq. (69) as...
an addendum to the matrix \( N \) and therefore as an addendum to the bare quark mass. To the extent that high quark masses can stabilize numerical simulations, use of the present effective action should make these calculations easier.

Also exotic states of baryon matter can be explored. For instance it is not difficult, as it will be shown in a separate paper, to introduce in the present formalism diquark states. Among abnormal states of hadronic matter I would like to mention the layered spin-isospin phase \[21\). This is a state with one-dimensional crystallization (which distinguishes it from usual pion condensation) in which layers of spin-up protons and spin-down neutrons alternate with layers of spin-up neutrons and spin-down protons. An investigation of a dynamical realization of such a phase in light deformed nuclei showed that the spin-isospin nucleon-nucleon interaction is not sufficiently strong to produce it \[22\), while the critical density for a static phase in neutron stars has been estimated \[23\) to be \( 3 \sim 4 \) times normal nuclear density. A first principles calculation might nevertheless be worth while to make an assessment of some simplifications done in the quoted works.

To perform numerical simulations it is necessary to adopt a trial expression of the mesons structure functions, which should be a function of gauge fields depending on temperature and baryon density, as suggested by the example of the four-fermion model. To this end any analytical investigation of the effective action, for instance according to an expansion in inverse powers of the index of nilpotency, might be of great help. In its absence, the form of trial structure functions can be suggested by existing results about the spatial structure of hadrons, of which a few examples can be found in \[24\).

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### A Grassmann integrals and coherent states

If \( |\alpha\rangle \) is a fermionic coherent state

\[
|\alpha\rangle = \exp(-\alpha \hat{u}^\dagger)|0\rangle
\]  
(76)

then

\[
\langle \alpha|\alpha\rangle = \exp(\alpha^* \alpha)
\]  
(77)
and the identity can be written
\[
\int [d\alpha d\alpha^*] \langle \alpha | \alpha \rangle^{-1} |\alpha\rangle \langle \alpha | = 1. \quad (78)
\]

I remind the fundamental property of coherent states
\[
\hat{u} |\alpha\rangle = \alpha |\alpha\rangle \quad (79)
\]
which implies the relations
\[
\langle \alpha \beta | \exp(\hat{v} \hat{u}) |\gamma \delta\rangle = \exp(\delta N \gamma + \alpha^* \gamma + \beta^* \delta) \quad (80)
\]
\[
\langle \gamma \delta | \exp(\hat{u}^\dagger \hat{F}^\dagger \hat{v}^\dagger) |0\rangle = \langle 0 | \exp(\hat{v} \hat{F} \hat{u}) |\gamma \delta\rangle^* = \exp(\gamma^* \hat{F}^\dagger \delta^*) \quad (81)
\]

With the help of these formulae one can compute matrix elements of the type
\[
\langle \alpha \beta | e^{\delta N \hat{u}} e^{\hat{F}^\dagger \hat{v}^\dagger} |0\rangle \quad (82)
\]
\[
= \int \left[ \frac{d\gamma^* d\gamma d\delta^* d\delta}{\langle \gamma \delta | \gamma \delta \rangle} \right] \langle \alpha \beta | e^{\delta N \hat{u}} |\gamma \delta\rangle \langle \gamma \delta | e^{\hat{F}^\dagger \hat{v}^\dagger} |0\rangle \quad (83)
\]
\[
= \int [d\gamma^* d\gamma d\delta^* d\delta] e^{-\gamma^* \gamma - \delta^* \delta + \delta N \gamma + \alpha^* \gamma + \beta^* \delta + \gamma^* \hat{F}^\dagger \delta^*} \quad (84)
\]
\[
= \int [d\delta^* d\delta] e^{-\delta^* (1 + \hat{F}^* \hat{N}^T) \delta + \beta^* \delta - \delta^* \beta^*} \quad (85)
\]
\[
= \exp \left\{ \text{tr} - \ln(1 + \hat{F}^* \hat{N}^T) - \beta^* (1 + \hat{F}^* \hat{N}^T)^{-1} \hat{F}^* \alpha^* \right\} , \quad (86)
\]
by use of the identity
\[
\int [d\alpha^* d\alpha] \exp(-\alpha^* A \alpha + J^* \alpha + \alpha^* J) = \det A \exp(J^* A^{-1} J) . \quad (87)
\]

B  The matrices $M, N$ of the transfer matrix

In this Appendix I report the expressions of the matrices $M, N$ appearing in the definition of the transfer matrix for the Kogut-Susskind and the Wilson regularization. Their common feature is that they depend only on the spatial link variables.

B.1  Kogut-Susskind’s regularization

Kogut-Susskind fermions in the flavor basis are defined on hypercubes whose sides are twice the basic lattice spacing. While in the text intrinsic quantum
numbers and spatial coordinates were comprehensively represented by one index \( i \), here I distinguish the spinorial index \( \alpha = \{1, \ldots, 4\} \), the taste index \( a = \{1, \ldots, 4\} \) and the flavour index \( i = \{1, \ldots, N_f\} \), while \( x = \{t, x_1, \ldots, x_3\} \) is a 4-vector of \textit{even} integer coordinates ranging in the intervals \([0, L_t - 1]\) for the time component and \([0, L_s - 1]\) for each of the spatial components. I distinguish summations over basic lattice and hypercubes according to

\[ \sum' := 2^d \sum_x . \]  

The projection operators over fermions-antifermion states are

\[ P_0^{(\pm)} = \frac{1}{2} (\mathbb{I} \otimes \mathbb{I} \pm \gamma_0 \gamma_5 \otimes t_5 t_0) . \]  

The relation between the variables \( u, v \) and the quark \( q \) field is

\[ P_0^{(+)} q = \frac{1}{4} u, \quad P_0^{(-)} q = \frac{1}{4} v^\dagger . \]  

In the presence of the scalar field \( \sigma \) and of gauge fields, neglecting an irrelevant constant, \( M = 0 \), while \( N \) is

\[ N = -2 \left\{ (m + \sigma) \gamma_0 \otimes \mathbb{I} + \sum_{j=1}^{3} \gamma_0 \gamma_j \otimes \mathbb{I} \left[ P_j^{(-)} \nabla_j^{(+)} + P_j^{(+)} \nabla_j^{(-)} \right] \right\} . \]  

where

\[ \nabla_j^{(+)} = \frac{1}{2} \left( U_j T_j^{(+)} - 1 \right) \]  

\[ \nabla_j^{(-)} = \frac{1}{2} \left( 1 - T_j^{(-)} U_j^\dagger \right) \]  

are the lattice covariant derivative as the \( T_j^{(\pm)} \) are the forward and backward translation operators of one block, that is of two lattice spacing in the original lattice, in the \( \mu \) direction (with unit versor \( \hat{\mu} \))

\[ [T^{(\pm)}_{\mu}]_{x_1, x_2} = \delta_{x_2, x_1 \pm 2 \hat{\mu}} . \]  

I set

\[ N^\dagger N = 4H^2 . \]  

In the absence of gauge fields

\[ H^2 = (m + \sigma)^2 - \Delta . \]
with
\[ \Delta = \frac{1}{4} \sum_{i=1,3} \left( T_i^{(+)} + T_i^{(-)} - 2 \right) \] (96)

The eigenvalues of \( H^2 \) are therefore the fermion energies
\[ E_q^2 = m^2 + \tilde{p}^2, \] (97)
where momentum component \( \tilde{p}_i^2 \) is
\[ \tilde{p}_i^2 = \frac{1}{2} (1 - \cos 2 p_i). \] (98)
and
\[ \tilde{p}^2 = \sum_{i=1}^{3} \tilde{p}_i^2 \] (99)

**B.2 Wilson’s regularization**

The projection operators over fermions-antifermions are
\[ P_0^{(\pm)} = \frac{1}{2} (1 \pm \gamma_0). \] (100)
in a basis in which \( \gamma_0 = \text{diag}(1, 1, -1, -1) \).

The relations between the quark field \( q \) and its upper and lower components \( u, v \) are
\[ P_0^{(+)} q = B^{-\frac{i}{2}} u, \quad P_0^{(-)} q = B^{-\frac{i}{2}} v^\dagger, \] (101)
where
\[ B = 1 - K \sum_{j=1}^{3} \left( U_j T_j^{(+)} + T_j^{(-)} U_j^\dagger \right) \gamma_j \] (102)
and \( K \) is the hopping parameter. The matrices \( M, N \) are
\[ M = -\frac{1}{2} \ln \left( \frac{B}{2K} \right), \]
\[ N = 2K B^{-\frac{i}{2}} c B^{-\frac{i}{2}}, \] (103)
where
\[ c = \frac{1}{2} \sum_{j=1}^{3} i \left( U_j T_j^{(+)} - T_j^{(-)} U_j^\dagger \right) \sigma_j. \] (104)
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