Ladder Climbing and Autoresonant Acceleration of Plasma Waves

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(Dated: April 3, 2015)

Classical plasma waves are predicted to exhibit quantumlike ladder climbing, which is achieved by chirped modulations of the background density. An equivalence with the quantum particle is manifestly conserved under mode coupling. This system is mathematically equivalent to a quantum particle in a box is identified and used to calculate the efficiency and the rate of this effect. In the limit of densely spaced spectrum, ladder climbing transforms into continuous autoresonance; plasmons may then be manipulated by chirped background modulations much like electrons are autoresonantly manipulated by chirped fields. Such ladder climbing and autoresonance effects are also predicted for other classical waves by means of a unifying Lagrangian theory.

PACS numbers: 52.35.-g, 52.35.Mw, 42.65.-k, 47.10.Df

Introduction.—Quantum mechanics is well known to be closely related to the mechanics of classical waves\textsuperscript{[1-3].} This permits applying common techniques for manipulating quantum and classical systems and helps bridging seemingly different areas of physics. One important technique to study in this context is ladder climbing (LC), which is the successive transfer of quanta through nonequally-spaced energy levels due to an oscillating driving force with chirped frequency\textsuperscript{[4, 5].} The system energy changes with time in a ladder-like manner during LC, with each transition described by the famous Landau-Zener (LZ) theory\textsuperscript{[6].} In the limit of continuous spectra, the effect has been widely known as classical autoresonance (AR), enjoying numerous applications in physics of plasmas\textsuperscript{[7-8]}, fluids\textsuperscript{[9]}, Josephson junctions\textsuperscript{[10]}, optics\textsuperscript{[11]}, and even planetary dynamics\textsuperscript{[12].} In contrast, the discrete nature of LC is visible only in systems with sufficiently discrete spectra and, so far, has been studied exclusively in quantum contexts\textsuperscript{[13-15].} Whether classical systems can exhibit LC has remained an open question.

Here we report the first theoretical prediction of LC in a classical system, namely, in an ensemble of plasma waves. For simplicity, we consider one-dimensional collisionless plasma, with nondissipative Langmuir waves, whose spectrum is quantized due to the boundary conditions. We derive a Schrödinger-type equation for the “plasmon wave function”, which is a classical measure of the electric field whose norm (the total wave action) is manifestly conserved under mode coupling. This system is mathematically equivalent to a quantum particle in electrostatic potential. Hence, plasmons can be manipulated by resonant modulation of the underlying medium, much like electrons and molecules are manipulated by resonant external fields\textsuperscript{[16, 17].} In particular, we show that plasmons can exhibit both LC and AR and can be controllably transported up and down in momentum space. Finally, we report a unifying Lagrangian formulation of the problem that paves the way for applying these techniques to general classical waves.

Basic equations.—For simplicity, we consider an electron plasma described by a hydrodynamic model.

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x} (n_e u_e) = 0,$$  \hspace{1cm} (1)

$$\frac{\partial}{\partial t} (n_e u_e) + \frac{\partial}{\partial x} (n_e u_e^2) = -\frac{e}{m_e} n_e E - \frac{1}{m_e} \frac{\partial P}{\partial x},$$  \hspace{1cm} (2)

$$\frac{\partial E}{\partial x} = -4\pi e (n_e - Z n_i).$$  \hspace{1cm} (3)

Here, $-e$ and $m_e$ are the electron charge and mass, $P$ is the electron pressure, $Ze$ is the ion charge, and $n_i$ is the ion density. We neglect high-frequency oscillations of $n_i$ and consider $n_i$ to be a slow function, $Z n_i = n_0 + n_d(t, x)$; here $n_0$ is the unperturbed electron density, and $n_d$ is a prescribed driving modulation. Such a modulation can be created by external fields, e.g., by means of ponderomotive forces. Then, $n_e = n_0 + n_d(t, x) + n(t, x)$, where $n < n_0$ determines a small uncompensated charge density due to electron inertia. For simplicity, we adopt an isentropic model, $P = P(n_e) \approx P_0 + 3 m_e v^2_{th} (n + n_d) + R(n + n_d)/2$; here $v_{th}$ is the electron thermal speed, and $R = \partial^2 P/\partial n_e^2|_{n_0}$, which are considered constant. We assume hard-wall boundary conditions, so $u_e|_{x=0} = u_e|_{x=\ell} = 0$, where $\ell$ is the plasma length. We also assume $E|_{x=0} = E|_{x=\ell} = 0$, so any field is representable as a series of $\sin(k_n x)$, where $k_n = \pi n/\ell$. Then, according to Eqs. (1)–(3), the boundary conditions for the density must be $\partial_x n|_{x=0} = \partial_x n|_{x=\ell} = 0$, so $n$ is a series of $\cos(k_n x)$. We consider the external driving modulation to be the $N$th standing-wave mode,

$$n_d(x, t) = n_0 A \cos(k_N x) \cos(\omega_d t).$$  \hspace{1cm} (4)

We assume that $A \ll 1$ and $\omega_d = \varphi \ll \omega_p$, where $\omega^2_p = 4\pi e^2 n_0/m_e$; we also adopt that $n$ is of the order of $A$. By retaining only those terms that are of the first order in $A$ and $\omega_d$\textsuperscript{[18]}, we obtain a dimensionless wave equation
for the electric field,
\[- \frac{\partial^2 E}{\partial t^2} + 3 \frac{\partial^2 E}{\partial x^2} - E = \tilde{n}_d E - R \frac{\partial}{\partial x} \left( \tilde{n}_d \frac{\partial E}{\partial x} \right). \tag{5}\]

Here and further we measure time in units \(\omega_p^{-1}\) and length in units \(v_{th}/\omega_p\); also, \(\tilde{n}_d = n_d/n_0\), and \(R = R\tilde{n}_d/(mv^2_n)\). Next, we decompose the field into unperturbed eigenmodes,
\[E = \text{Re} \sum_{m=1}^{\infty} E_m e^{-i\omega_m t} \sqrt{2/\ell} \sin(k_m x), \tag{6}\]
where \(E_m(t)\) are complex coefficients. To zeroth order in \(A\), Eq. (5) yields the dimensionless dispersion relations \(\omega_m^2 = 1 + \beta m^2\), where \(\beta = 3\pi^2/\ell^2\) is analogous to the anharmonic frequency shift of the quantum oscillator. It is convenient to introduce new variables \(\psi_m\) via \(\rho_m \psi_m(t) = E_m e^{-i\omega_m t}\). Here \(\rho_m\) are constants such that in the unperturbed system, \(|\psi_m|^2\) are the actions of individual modes; specifically one finds \(\rho_m = (8\pi/\omega_m)^{1/2}\). The equations for \(\psi_m\), obtained via Fourier-transforming Eq. (5), are:
\[i \dot{\psi}_m = \omega_m \psi_m + \sum_{m'} h_{m,m'} \psi_{m'}, \tag{7}\]
\[h_{m,m'} = \frac{\rho_m \rho_{m'}}{8\pi \ell} \int_0^t \sin(k_m x) \tilde{F} \sin(k_{m'} x) dx, \tag{8}\]
where \(\tilde{F} = \tilde{n}_d - \partial_x \tilde{R} \tilde{n}_d \partial_x\) is a differential operator. Since \(h_{m,m'}\) is Hermitian, the evolution of \(\psi_m\) is manifestly unitary; i.e., the wave total action \(\sum_m |\psi_m|^2\) is conserved. The vector \(\psi = (\psi_1, \psi_2, \ldots)\) can then be understood as the plasmon wave function in the energy representation. Next, we will assume the resonance condition \(\omega_d \approx \omega_{m,m+N}\), where the level spacing is \(\omega_{m,m+N} = \omega_{m+N} - \omega_m \approx \beta N(2m + N)\). Then, \(\beta \sim \omega_d \ll 1\), and therefore, in the already small coupling term, we must adopt \(\omega_m \approx 1\), i.e., \(\rho_m \approx \sqrt{8\pi}\) and neglect the higher order terms in pressure. This leads to
\[i \dot{\psi}_m = \omega_m \psi_m + A \left( \psi_{m-N} + \psi_{m+N} \right) \cos \varphi_d. \tag{9}\]

Note also that, for \(\beta m^2 \ll 1\), Eq. (9) is equivalent to the energy representation of the Schrödinger equation for a quantum particle in a square potential well [20]. This can be understood as follows: at weak spatial dispersion, all \(\omega_m\) are close to \(\omega_p\), so Eq. (5) permits a quasioptical approximation, turning into the standard quantum Schrödinger equation with \(\tilde{n}_d\) serving as an effective potential. (This analogy has been noted in e.g., Ref. [21].)

**LC regime.**— Classical AR was previously studied in the infinite square potential well as a limiting case \((j \to \infty)\) of the potential \(V_0 = x^{2j}\) [22] but quantum LC was not [23]. For studying LC in our system, we consider the modulation [4] with \(N = 1\) and a monotonically increasing driving frequency, \(\omega_d = \omega_{1,2} + \alpha t\), where \(\alpha > 0\). At \(t = 0\), this will initiate resonant transitions between levels \(m = 1\) and \(m = 2\), which we denote as \(1 \to 2\). Later, transitions between higher levels, \(m \to m + 1\), occur when the resonance condition, \(\omega_d = \omega_{m,m+1}\), is satisfied. Following the quantum LC theory [4,5], we define slow time \(\tau = \sqrt{\alpha} t\), driving parameter \(P_1 = A/(4\sqrt{\alpha})\), and nonlinearity parameter, \(P_2 = \beta/\sqrt{\alpha}\), which plays the role of an effective Planck constant in the classical system. If \(P_2 \gg 1 + P_1\), the system remains in the quantum LC regime, when only two levels are resonantly coupled at any given time. Then, transitions \(m \to m + 1\) occur at times \(\tau = mP_2\), where each such transition can be described by the commonly known LZ theory [2,3] and thus has a probability
\[P_{m \to m+1} = 1 - \exp \left(-\pi P_2^2/2\right). \tag{10}\]

At large enough \(P_1\), \(P_{m \to m+1} \approx 1\), i.e., all quanta are being transferred, and the dynamics in this limit is characterized by successive two-level LZ transitions.

In Fig. 1 we illustrate this LC dynamics of Langmuir modes by numerically simulating Eq. (9) with \(N = 1\) and initial conditions \(\psi_m(t = t_0) = \delta_{m,1}\). We choose \(\{\alpha, \beta, A\} = \{10^{-8}, 10^{-3}, 6 \times 10^{-5}\}\), so \(P_2 = 10\). In the figure, snapshots of the levels population (a,c,e,g) and the electric field (b,d,f,h) are shown at times \(\tau = [-10, 15, 35, 45]\), respectively. The inset also shows the total energy, \(\sum_m \omega_m |\psi_m|^2\), which is seen to increase with time in a ladder-like manner. The moments of time at
which $m \to m+1$ transitions occur agree with the theory, which predicts $r_m = mP_2$ \cite{2}. The $m$th-level occupation numbers, $\lvert \psi_m \rvert^2$, also agree with the theoretical predic-
tions, namely, $\lvert \psi_m \rvert^2 = \sum_{j=m}^{m-1} P_{j-j+1}$, where $P_{j-j+1}$ are given by Eq. (10). For example, at the adopted parameters, one has $\lvert \psi_5 (\tau = 55) \rvert^2 = 0.83$, which deviates by only about 1% from the value obtained in the simulation.

**AR regime.** — In contrast to the LC dynamics, in the
limit, $P_2 \ll 1$, many levels are coupled simultaneously.
It can be shown then (e.g., using the Wigner phase space
approach, as in Ref. \cite{2}) that quantum LC continuously
transforms into the classical AR by decreasing the ef-
fective Planck constant, $P_2$. The electric field dynamics is then understood as AR acceleration of plasmons that satisfy $\omega_d = k_d v_g$, where

$$v_g = \frac{\omega_{m+1} - \omega_m}{k_{m+1} - k_m} \approx \frac{\partial \omega}{\partial k}. \quad \text{(11)}$$

Notably, this is the same “group-resonance” condition that was recently discussed in Ref. \cite{24} (see Ref. \cite{25} too). Also notably, the AR acceleration of plasmons that we report here is akin to AR acceleration of resonant electrons in phase-mixed nonlinear waves such as Bernstein-Green-Kruskal waves \cite{8,11,20}. The AR dynamics of Langmuir waves is illustrated in Fig. 2 for which we adopted $\{\alpha, \beta, A\} = \{10^{-8}, 10^{-5}, 4.8 \times 10^{-3}\}$, so $P_2 = 0.1$. The simultaneous coupling of many levels is clearly seen in the left subplots, while the right sub-
plots present the electric field of the autoresonant plas-
mon (solid blue lines) driven by the chirped density mod-
ulations, $\bar{n}_d$ (red dashed lines).

Interestingly, the dynamics is reversible, at least ap-
proximately. In fact, one can just as well capture a
wave envelope at large $k$ and then transport it down
the spectrum, much like a trapped charged particle
can be decelerated by a resonant field. We demon-
strate the effect in Fig. 3 where the initial condi-
tions are $E_0 (x) \sim \exp(- (x - x_0)^2 / 2 \sigma^2) \sin (k_m x)$, where $m_r$ is the resonant wave number. In this
example, we apply down-chirped driving phase $\varphi_d = \omega_{m_r, m_r+1} t - \alpha t^2 / 2 + \varphi_0$ with $\{\beta, x_0, \sigma, m_r, \alpha, \varphi_0\} = \{10^{-5}, 0.05, 100, 10^{-5}, 0.004, \pi\}$, where $\varphi_0$ was chosen such that the plasmon is initially phase-locked with $n_d$.

**Variational formulation.** — Let us now recast our the-
ory in a form that is not restricted to Langmuir waves but
allows extending the above results to general nondissipative
linear waves. Any such wave, described by some field $E(t, x)$ (electric field being an example), can be assigned a Lagrangian bilinear in $E$, namely, of the form \cite{3}

$$L = \int \mathbf{E} \cdot \mathbf{D}(t, \mathbf{x}, \mathbf{i} \partial_t, -i \nabla) \cdot \mathbf{E} \, d^3 x. \quad \text{(12)}$$

The differential operator $\mathbf{D}$ can be considered Hermitian without loss of generality. Suppose now that $\mathbf{D} = \mathbf{D}_0 + \mathbf{D}_d$, where $\mathbf{D}_0$ is some Hermitian operator that determines the Lagrangian in stationary homogeneous medium, and $\mathbf{D}_d$ is a Hermitian operator that governs the mode interaction driven by some weak modulation. Let us represent the field as $E(t, x) = \Re \sum_m \mathcal{E}_m (t) e_m (x)$. Here $\mathcal{E}_m$ are complex amplitudes and $e_m$ are orthonormal eigenmodes corresponding to the eigenfrequencies $\omega_m$ of $\mathbf{D}_0$. Then, $L = \sum_m L_m + L_d$, where the two terms are,
respectively, due to $\hat{D}_0$ and $\hat{D}_d$. Specifically 3,

$$L_m = \frac{i}{2} (\hat{\psi}_m^* \hat{\psi}_m - \hat{\psi}_m^2 \hat{\psi}_m) - \omega_m |\psi_m|^2,$$  

(13)

where the complex amplitudes $\psi_m$ are defined such that $|\psi_m|^2$ is the action of the $m$th unperturbed mode; i.e., $\rho_m \psi_m = \mathcal{E}_m$, where $\rho_m^2 = dL_m/d\omega_m$, $L_m = (1/2) \int \mathbf{e}_m^* \cdot \mathbf{D}_0 \cdot \mathbf{e}_m \, d^3x$, and $\omega_m$ are found by solving $L_m(\omega_m) = 0$. Also, it is easy to see that

$$L_d \approx \sum_{m, m'} \hat{\psi}_m^* \hat{h}_{m,m'} \hat{\psi}_{m'},$$  

(14)

$$h_{m,m'} = \frac{\rho_m \rho_{m'}}{2} \int \mathbf{e}_m^* \cdot \mathbf{D}_d \cdot \mathbf{e}_m \, d^3x.$$  

(15)

Then the equation for $\psi_m$ has the form (7), and, since $h_{m,m'}$ is Hermitian, it manifestly conserves the total action, $\sum_m |\psi_m|^2$. For electromagnetic waves in particular, one has $\mathbf{D}_d = \chi_d/(8\pi)$, where $\chi_d$ is the modulation-driven perturbation to the medium susceptibility. This can be seen, for instance, by comparing Eq. (12) with its geometrical-optics limit 27. In the case of Langmuir waves, $\chi_d = \hat{F}$, and $\mathbf{D}_0 = \hat{\epsilon}/(8\pi)$, where $\hat{\epsilon}$ is the dielectric permittivity operator; then one recovers Eq. (9). Finally, we note that it is straightforward to apply this approach to other classical waves 28; hence, our further observations of LC and AR apply too.

**Summary.**—We reported the first theoretical prediction of LC in a classical system. Specifically, we show that quasiperiodic chirped modulations of the background density can couple discrete eigenmodes of bounded plasma to produce controllable shift of the wave spectral energy distribution. Apart from academic interest, the new method of continuously controlling the wavelength of Langmuir oscillations might find practical applications, such as regulating coherent Raman scattering of laser radiation in plasma for generating short ultraintense pulses 29. Our results indicate how similar techniques might be practiced with other plasma modes too, or in other settings such as waveguides or photonic crystals. Our work also bridges a number of effects that were previously considered unrelated. In particular, plasma acceleration reported here can be seen as the resonant counterpart of the adiabatic ponderomotive effects on waves discussed recently in Ref. 21. It is also akin to the resonant acceleration of charged particles trapped by chirped nonlinear plasma waves 16,24. Finally, our results further advance the general idea 1,3 that posing classical waves in quantumlike terms can be quite fruitful.

The work was supported by NNSA grant DE274-FG52-08NA28553, DOE contract DE-AC02-09CH11466, and DTRA grant HDTRA1-11-1-0037.

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