String Theory and the Size of Hadrons

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Abstract

We begin by outlining the ancient puzzle of off shell currents and infinite size particles in a string theory of hadrons. We then consider the problem from the modern AdS/CFT perspective. We argue that although hadrons should be thought of as ideal thin strings from the 5-dimensional bulk point of view, the 4-dimensional strings are a superposition of “fat” strings of different thickness.

We also find that the warped nature of the target geometry provides a mechanism for taming the infinite zero point fluctuations which apparently produce a divergent result for hadronic radii.

Finally a calculation of the large momentum behavior of the form factor is given in the limit of strong ’t Hooft parameter where the classical gravity limit is appropriate. We find agreement with parton model expectations.
1 The Puzzle of Infinite Size

The historical origins of string theory were an attempt to understand the structure of hadrons [1] [3]. However the theory encountered a number of obstacles. Some obvious difficulties involved the spectrum which invariably included massless vectors, scalars and tensor particles. There were also more subtle problems that seemed insurmountable and which involved the existence of form factors. In other words there was no possibility of constructing matrix elements of local currents needed to describe the interaction of hadrons with electromagnetism and gravitation [4]. The natural candidates, vertex operators like \( \exp ikX \), can not be sensibly continued away from specific discrete “on shell” values of \( k^2 \). Closely connected with this was the divergence encountered in attempting to compute the hadronic electromagnetic or gravitational radius [4] [5]. Thus string theory was abandoned as a theory of hadrons and replaced by QCD. The success of string theory in understanding Regge trajectories and quark confinement was understood in terms of an approximate string-like behavior of chromo-electric flux tubes. According to this view, hadronic strings are not the infinitely thin idealized objects of mathematical string theory but are thick tubes similar to the quantized flux lines in superconductors. The ideal string theory was relegated to the world of quantum gravity.

However more recent developments [6] have strongly suggested that an idealized form of string theory may exactly describe certain gauge theories which are quite similar to QCD [7, 8]. We have returned full circle to the suspicion that hadrons may be precisely described by an idealized string theory, especially in the ’t Hooft limit [9]. The new string theories are certainly more complicated than the original versions and it seems very plausible that the problems with the massless spectrum of particles will be overcome. Less, however, has been studied about the problems connected with local currents. In this paper we will show that the new insights from the AdS/CFT correspondence suggest a solution to the form factor problem.

We begin by reviewing the problem. For definiteness we work in the light cone frame in which string theory has the form of a conventional Galilean-invariant Hamiltonian quantum mechanics. The degrees of freedom of the first-quantized string include \( D - 2 \) transverse coordinates \( X^m(\sigma) \). The Lagrangian for these variables is

\[
L = \frac{1}{4\pi} \int_0^{2\pi P_-} d\sigma \left( \dot{X} \dot{X} - \alpha'^{-2} X' X' \right),
\]

where \( \dot{X} \) and \( X' \) denote derivatives with respect to light-cone time \( \tau \) and string parameter \( \sigma \). The light-cone momentum \( P_- \) is conjugate to the light-like coordinate \( x^- \). All irrelevant
constants have been set to unity.

An important feature of the light-cone theory involves the local distribution of $P_-$ on the string. The rule is that the distribution of $P_-$ is uniform with respect to $\sigma$. In other words the longitudinal momentum $dP_-$ carried on a segment of string $d\sigma$ is exactly $d\sigma/2\pi$.

Let us now consider the transverse density of $P_-$. In a space-time field theory this would be given by

$$\rho(X) = \int dx^- T_{--}(X, x^-), \quad (1.2)$$

where $T$ is the energy momentum tensor of the field theory, and $X$ without a superscript denotes the two noncompact transverse coordinates. Matrix elements of $\rho$ between states of equal $P_-$ define form factors for gravitational interactions of the state and are entirely analogous to electromagnetic form factors.

The natural object in string theory to identify with $\rho(X)$ is

$$\frac{1}{2\pi} \int_0^{2\pi P_-} d\sigma \delta^2(X - X(\sigma)). \quad (1.3)$$

(To be precise, we subtract the center-of-mass mode from $X(\sigma)$. ) In other words $\rho(X)$ receives contributions from every element of string localized at $X$. The Fourier transform of $\rho(X)$

$$\tilde{\rho}(k) = \frac{1}{2\pi} \int_0^{2\pi P_-} d\sigma \exp ikX(\sigma) \quad (1.4)$$

defines a system of form factors by its matrix elements between string states.

The mean square radius of the distribution function is given by

$$r^2 = \int d^2X X^2 \langle \rho(X) \rangle \quad (1.5)$$

and can be rewritten in terms of $\tilde{\rho}$,

$$r^2 = -\partial_k \partial_k \langle \tilde{\rho}(k) \rangle |_{k=0}. \quad (1.6)$$

Eq. (1.6) is the standard definition of the mean-square radius in terms of the momentum space form factor. The squared radius is also given by

$$\langle X(\sigma)^2 \rangle \quad (1.7)$$

where the value of $\sigma$ is arbitrary.

For a field theory with a mass gap, such as pure QCD, it is possible to prove that $r^2$ is finite. This follows from the standard analytic properties of form factors. The problem
arises when we attempt to apply the world sheet field theory to compute $\langle X(\sigma)^2 \rangle$. An elementary calculation based on the oscillator representation of $X$ gives a sum over modes

$$\langle X^2 \rangle \sim \alpha' \sum_0^\infty \frac{1}{n} = \alpha' \log \infty .$$

A related disaster occurs when we compute the form factor which is easily seen to have to form

$$\langle \tilde{\rho}(k) \rangle = \exp (-k^2 \langle X^2 \rangle / 2) .$$

Evidently it is only non-zero at $k^2 = 0$.

In a covariant description of string theory the problem has its roots in the fact that the graviton vertex operator is only well defined on the mass shell of the graviton, $k^2 = 0$. Vertex operators to be well defined must correspond to perturbations with vanishing world sheet $\beta$-function. This implies that they should correspond to on shell solutions of the appropriate space-time gravitational theory. For the kinematical situation in which the graviton carries vanishing $k_{\pm}$ the transverse momentum must vanish. Thus, no well defined off shell continuation of the form factor exists.

One might wonder if the divergence of $X^2$ is special to the case of a free world sheet field theory. The answer is that the divergence can only be made worse by interactions. The 2-point function of a unitary quantum field theory is at least as divergent as the corresponding free field theory. This follows from the spectral representation for the two point function and the positivity of the spectral function. Thus it is hard to see how an ideal string theory can ever describe hadrons.

## 2 Light Cone Strings in AdS

There are good reasons to believe that certain confining deformations of maximally supersymmetric Yang Mills theory are string theories albeit in higher dimensions. The strings move in a five-dimensional space that is asymptotically $AdS_5$ In the ’t Hooft limit these theories are believed to be free string theories. Evidently if this is so there must exist a well defined string prescription for form factors in the 4-D theory.

What we will see is that although the theory in bulk of AdS is an ideal thin-string theory, the 4-D boundary field theory is not described by thin strings. That may seem

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1 Strictly speaking the target space is ten-dimensional with the form $AdS_5$ times a compact space such as $S_5$. In this paper the compact factor plays no role.
surprising. Suppose that in the light-cone frame the thin 5-D string has the form

\[(X(\sigma), Y(\sigma))\] (2.1)

where \(X\) are the transverse coordinates of 4-D Minkowski space and \(Y\) is the additional coordinate perpendicular to the boundary of AdS. Then it would seem natural to consider the projection of the string onto the \(X\) plane to define a thin string. According to this view the mean-squared radius would again be \(\langle X^2 \rangle\) and we would be no better off than before. Before discussing the resolution of this problem let us work out the bosonic part of the light-cone string Lagrangian in AdS. We will make no attempt to derive the full supersymmetric form of the theory [10] in this paper. We believe the resolution of the form factor problem does not require this. On the geometric side we will also ignore the 5-sphere component of the geometry implied by the usual R-symmetry of the \(N = 4\) supersymmetry.

The metric of AdS is given by

\[ds^2 = R^2 \frac{2 dx^+ dx^- + dX^2 + dY^2}{Y^2}.\] (2.2)

We have defined the overall scale of the AdS (radius of curvature) to be \(R\).

In order to pass to the light cone frame we must also introduce the world sheet metric \(h_{ij}\). In the usual flat space theory it is possible to fix the world sheet metric to be in both the light cone gauge \(\sigma_0 = \tau = x^+\) and also the conformal gauge \(h_{00} = -h_{11}, h_{01} = 0\). However this is not generally possible since it entails three gauge conditions which is one too many (we do not count the local Weyl symmetry, because none of these conditions fix it; this would require a fourth condition, \(\det h = -1\)). The special feature of flat space which permit the over-fixing of the gauge is not shared by AdS. Thus we must give up the conformal gauge if we wish to work in light-cone gauge.

Let us fix the gauge by choosing two conditions

\[
\begin{align*}
\sigma_0 & = x^+, \\
h_{01} & = 0.
\end{align*}
\] (2.3)

The Lagrangian becomes

\[L = \frac{1}{4\pi} \int d\sigma \left\{ \frac{R^2 E}{Y^2} (2\dot{X}^- \dot{X}^+ + \dot{X} \dot{\bar{X}} + \dot{Y} \dot{\bar{Y}}) - \frac{1}{\alpha'^2 R^2 E Y^2} (X' X' + Y' Y') \right\},\] (2.4)

where we have defined

\[\sqrt{-\frac{h_{11}}{h_{00}}} = E.\] (2.5)
The equation of motion from $X^-$ states that the density $p_-(\sigma) = R^2 E(\sigma)/2\pi Y^2(\sigma)$ is time-independent. We can then use the $\tau$-independent $\sigma$ reparameterization not fixed by eq. (2.3) to set $p^- = 1/2\pi$. The Hamiltonian becomes

$$H = \frac{1}{2} \int_{0}^{2\pi p^-} d\sigma \left\{ P_X P_X + P_Y P_Y + \frac{R^4}{\alpha'^2 Y^4} (X'X' + Y'Y') \right\}.$$ (2.6)

This is a more or less conventional string action with the unusual feature that the effective string tension scales like $1/Y^2$. Thus the tension blows up at the AdS boundary $Y = 0$ and tends to zero at the horizon $Y = \infty$. This of course is a manifestation of the usual UV/IR connection.

The Hamiltonian could be obtained from an action

$$S = \frac{1}{2} \int_{0}^{2\pi p^-} d\sigma d\tau \left\{ \dot{X} \ddot{X} + \dot{Y} \ddot{Y} - \frac{R^4}{\alpha'^2 Y^4} (X'X' + Y'Y') \right\}.$$ (2.7)

This action thought of as a $1 + 1$ dimensional field theory is not Lorentz invariant in the world-sheet sense. However it is classically scale invariant if we assume $X$ and $Y$ are dimension zero. The Hamiltonian has dimension 1 and therefore scales as the inverse length of the $\sigma$ circle which is $\sim P^-$. We recognize this scale symmetry as spacetime longitudinal boost invariance under which $H$ and $P_-$ scale oppositely and $X, Y$ are invariant. No doubt the actual Lagrangian, properly supersymmetrized, retains this symmetry when quantized.

Let us consider the equal time correlation function $\langle X(0)X(\sigma) \rangle$ in the field theory defined by eq. (2.7), or more precisely in its supersymmetrized version. By inserting a complete set of eigenstates of the energy and (world sheet) momentum we obtain

$$\langle X(0)X(\sigma) \rangle = \sum \int \frac{dE dp}{p^2} e^{ip\sigma} |\langle X(0)|E, p\rangle|^2$$

$$= \int \frac{dE dp}{p^2} e^{ip\sigma} F(E, p)$$ (2.8)$$

with $F \geq 0$. The measure of integration $dE dp/p^2$ follows from the fact that $X$ has “engineering” dimension zero under the longitudinal boost rescaling. Furthermore the assumption that the scale invariance is preserved in the quantum theory requires $F(E, p) = F(E/p)$ for large $p, E$. It follows that as long as $F$ does not go to zero in this limit that the correlation function diverges as $\sigma \to 0$. This would imply $X^2 = \infty$.

3 Dressing the Vertex with $Y$ Dependence

Let us consider the problem from the point of view of the vertex operator $\exp ikX$. One problem that we have emphasized is that it is not a solution of the on-shell condition. We
can try to fix this by replacing it with a solution of the wave equation for a graviton in AdS space \([11]\). The relevant equation is

\[
\left( \partial_\mu \partial^\mu + Y^3 \partial_Y Y^{-3} \partial_Y \right) \Phi = 0 ,
\]  

(3.1)

where \(\mu\) runs over the four dimensions of flat Minkowski space. To be precise, this is the first point at which we have assumed large 't Hooft parameter, so that we can use the low energy field equations.

The particular solutions we are looking for are independent of the \(x^\pm\) and have the form

\[
\Phi = \exp(ikX) f(k, Y) ,
\]  

(3.2)

where \(F\) satisfies

\[
Y^3 \partial_Y Y^{-3} \partial_Y f(k, Y) = k^2 f(k, Y) .
\]  

(3.3)

Thus the on shell vertex operator has the form

\[
\int d\sigma \exp(i k X(\sigma)) f(k, Y(\sigma)) .
\]  

(3.4)

The factor \(f\) is a dressing of the vertex, necessary to make its matrix elements well defined for \(k \neq 0\).

Let us consider the mean square radius of the hadron defined by eq. (1.6),

\[
\overline{r^2} = -\partial_k \partial_k \langle f(k, Y) \exp(ikX) \rangle|_{k=0}
\]

\[
= \langle X^2 f(0, Y) - 2i X \cdot f'(0, Y) - f''(0, Y) \rangle
\]  

(3.5)

where \(f'' \equiv \partial_k \partial_k f\).

For a state of zero angular momentum in the \(X\) plane the term linear in \(X\) vanishes and we have

\[
\overline{r^2} = \langle X^2 f(0, Y) - f''(0, Y) \rangle .
\]  

(3.6)

To compute \(f\) and \(f''\) we Taylor expand \(f(k, Y)\) in powers of \(k\). There are two linearly independent solutions,

\[
f(k, Y) = Y^4 + \frac{1}{12} k^2 Y^6 + \ldots
\]  

(3.7)

and

\[
f(k, Y) = 1 - \frac{1}{4} k^2 Y^2 + \ldots .
\]  

(3.8)

Only the second of these is relevant to the problem of form factors. To see this we need only note that the vertex at \(k = 0\) is just the operator that measures \(P_-\). For states with \(P_- = 1\) this operator in just the identity. This implies that \(F(0, Y) = 1\). Thus we find

\[
\overline{r^2} = \langle (X^2 + Y^2) \rangle .
\]  

(3.9)
This dressed result should be finite, because the on-shell condition in five (actually ten) dimensions arises from the requirement of world-sheet conformal invariance. One might have expected that this would happen as a simple cancellation between a divergence in $X^2$ and a divergence in the dressing. However, this is not the case: $r^2$ as given in eq. (3.9) is the sum of $X^2$ and a positive term. Evidently cancellation is not the answer, and it must be that, in contrast to all experience in field theory, the operator $X^2$ is finite!

The dressing of the vertex by the factor $f(k, Y)$ obviously modifies the expression (1.3) for the transverse density $\rho$. If we define the Fourier transform of $f$ with respect to $k$ to be $\tilde{f}(X, Y)$, eq. (1.3) is replaced by

$$\rho(X) \sim \int_0^{2\pi P_\perp} d\sigma \tilde{f}(X - X(\sigma), Y).$$

This means that an ideal thin string in the AdS bulk space is smeared out by the holographic projection onto the boundary. This is of course the familiar UV/IR correspondence at work. Bulk strings near the boundary are projected as very thin strings in the 4-D theory but those far from the boundary are fat. The extra term $\langle Y^2 \rangle$ in eq. (3.9) represents this fattening. Evidently we have only made things worse by including the dressing.

Before discussing the solution to the problem let us make some remarks about confining deformations in the context of AdS/CFT. Bulk descriptions of confining deformations of super Yang Mills theory have an effective infrared “wall” at a value of $Y$ which represents the confinement scale. In these cases the metric (2.2) is modified in the infrared region. A simple model for this is

$$ds^2 = h(Y) \left( 2dx^+ dx^- + dX^2 + dY^2 \right)$$

where, as in the conformal case, $h \sim 1/Y^2$ for $Y \to 0$. Assume that $h$ has a minimum at the confinement scale, $Y = Y^*$. The light-cone hamiltonian is easily worked out,

$$H = \frac{1}{2} \int_0^{2\pi P_\perp} d\sigma \left\{ P_X P_X + P_Y P_Y + h(Y)^2 \alpha'^{-2}(X'X' + Y'Y') \right\}$$

Consider a string stretched along the $X$ direction and choose $\sigma$ so that $\partial_\sigma X = 1$. The potential energy of the string is then given by

$$V(Y) = h(Y)^2 \alpha'^{-2}$$

which has a minimum at $Y = Y^*$. Thus a classical long straight string will be in equilibrium at this value of $Y$. This classical bulk string corresponds to a field theory configuration
which, according to the UV/IR connection, is thickened to a size $\sim Y^*$, that is, the QCD scale.

Quantum fluctuations will cause the wave function of the string to fluctuate away from $Y^*$. The implication is that the QCD string is a superposition of different thickness values extending from infinitely thin to QCD scale. Indeed different parts of the string can fluctuate in thickness over this range. The portions of the string near $Y = 0$ will be very thin and will determine the large momentum behavior of the form factor.

4 Finiteness of $\langle X^2 \rangle$

We believe that despite the argument given at the end of Section 2 the value of $\langle X^2 \rangle$ is finite. This can only be if the function $F = \sum |\langle X(0)|E,p \rangle|^2$ vanishes in the scaling limit of large $E,p$. We will first give an intuitive argument and follow it with a more technical renormalization group analysis.

First suppose the string is “stuck” at some value of $Y$. In that case the action for $X$ in eq. (2.7) is a conventional string action except that the sting tension is replaced by $1/Y^4$. The divergence in $X^2$ would then be given by

$$\langle X^2 \rangle = Y^2 |\log \epsilon|.$$  \hspace{1cm} (4.1)

If we ignore quantum fluctuations of $Y$ we could replace $Y$ by $Y^*$. But $Y$ fluctuates as well as $X$ and can be expected to fluctuate toward the boundary as $\epsilon$ tends to zero. This is just the usual UV/IR connection in AdS. Therefore as we remove the cutoff the fluctuations of $X$ are diminished because the string moves into a region of increasing effective stiffness. If for example the average value of $Y^2$ tends to zero as $|1/\log \epsilon|$ or faster then the fluctuations of $X$ would remain bounded.

To see that this happens we consider the renormalization running of the operator $X^2$. We begin with the bare theory defined with a cutoff length $\epsilon$ on the world sheet. We can then ask how a given operator in this bare theory is described in a renormalized version of the theory with a cutoff at some longer distance $l$. A general operator $\phi(X,Y)$ runs to lower momentum scales according to the renormalization group equation

$$(l\partial/\partial l)\phi(X,Y,l) = (\alpha'/2)\nabla^2 \phi(X,Y,l).$$  \hspace{1cm} (4.2)

For example, consider flat space and the operator $X^2$. We look for a solution of (4.2) with $\phi(X,\epsilon) = X^2$.

$$\phi(X,\epsilon) = X^2.$$  \hspace{1cm} (4.3)
The solution is
\[ \phi(X,l) = X^2 + \alpha' \log l/\epsilon. \] (4.4)
Thus if we regulate the theory at some fixed scale, for example \( l \sim 1 \), the matrix elements of \( X^2 \) blow up as we send \( \epsilon \to 0 \).

By contrast, consider the the case of AdS space where
\[ \nabla^2 = R^{-2}(Y^2 \partial_X^2 + Y^5 \partial_Y Y^{-3} \partial_Y). \] (4.5)
For a solution of the form \( X^2 + f(l)Y^2 \) this becomes
\[ (l\partial/\partial l)f = (2\alpha'/R^2)(1 - f). \] (4.6)
With \( f(\epsilon) = 0 \) the solution is
\[ f(l) = 1 - (\epsilon/l)^{2\alpha'/R^2}. \] (4.7)
So if we fix the scale \( l \) and take the cutoff length \( \epsilon \) to zero the matrix elements tend to finite limits and the problem of infinite radii is resolved. If, however, we expand in powers of \( \alpha' \) there are logarithmic divergences. Note that the operator \( X^2 \) runs to a fixed point \( X^2 + Y^2 \) which is just the operator in eq(3.10) which represents the mean squared radius \( \bar{r}^2 \).

Let us make a similar renormalization group calculation of the two-point function. For simplicity we consider the equal-time case; the method extends to unequal times, but the expressions are not as simple. When the renormalization scale is the same as the separation there are no large logarithms and so we obtain the semiclassical result
\[ X^i_\sigma'(\sigma,0)X^j_\sigma_\sigma(0,0) \sim -\delta^{ij} \frac{QY^2_\sigma}{\sigma R^2}. \] (4.8)
Subscripts on operators signify the renormalization scale. The \( \sigma \)-derivative (prime) is taken in order to make this insensitive to IR physics. Now let us write this in terms of operators renormalized at a lower scale \( l \) as in eq. (4.2). The operator \( X \) renormalizes trivially, and so we can drop the subscript. The renormalization of \( Y^2 \) gives
\[ X^i'(\sigma,0)X^j(0,0) \sim -\delta^{ij} (\sigma/l)^{2\alpha'/R^2} \frac{\alpha'Y^2_i}{\sigma R^2}. \] (4.9)

Juan Maldacena raised the issue as to whether the undressed operator \( X^2 \) is physically observable. Once light-cone gauge is fixed, every operator should be observable. A world-sheet scale transformation corresponds to a spacetime boost, so a scale-dependent operator has a nontrivial transformation under this boost.
We can now integrate with respect to $\sigma$,

$$X^i(\sigma,0)X^j(0,0) \sim \text{constant} - \frac{\delta^{ij}}{2}(\sigma/l)^{2\alpha'/R^2}Y_l^2.$$ (4.10)

To complete the evaluation of the two-point function we must run the operators down to the length scale of the string, and then evaluate the expectation value of $Y^2$ in the given string state, which we might take to be a wavepacket centered on some value of $Y$,

$$\langle X^i(\sigma,0)X^j(0,0) \rangle \sim \text{constant} - \frac{\delta^{ij}}{2}\sigma^{2\alpha'/R^2}\langle Y^2 \rangle.$$ (4.11)

The result has the advertised property: a finite limit as $\sigma \to 0$.

Precisely where does the spectral argument (2.8) go wrong? It is in the assumption of scale invariance. Scale invariance is of course broken by the finite coordinate length of the string, but this is an IR effect that should not change the short-distance behavior of correlators. In the present case there is a second source of scale breaking. The operators $Y^q$ have anomalous dimension $(2\alpha'/R^2)q(4-q)$. Since $Y$ is positive, these operators must have expectation values in any state, and this breaks the scale invariance spontaneously. We see this in eq. (4.10): this equation is covariant under scale transformations, but when the operators on the right are replaced with their expectation values the two-point function scales nontrivially. (One might try to interpret the result (4.11) in terms of an anomalous dimension for $X$, but this does not seem sensible). Note the close analogy between $Y$ and a Liouville field, as emphasized in ref. [12].

The reader may wonder how the finiteness of $X^2$ can be explained in covariant gauges such as the conformal gauge in which the world sheet theory has the form of a relativistic field theory. A standard argument insures that the singularity in a two point function can not be less singular than a free field; in this case logarithmic. The argument is based on the positivity of spectral functions which in turn assumes the metric in the space of states is positive. In general this is not the case in covariant gauges.

5 Calculation of the Form Factor

At large 't Hooft parameter, we can use the dual supergravity description to calculate the form factor. To calculate a matrix element

$$\langle A|T_{--}(q)|B \rangle$$ (5.1)

we use the prescription of ref. [12, 13]. That is, the hadronic states $A$ and $B$ correspond to normalizable string states in the bulk, while the local operator insertion corresponds to a
modification of the boundary conditions such that the nonnormalizable modes are excited.
In particular, a perturbation of the bulk metric to
\[
\frac{ds^2}{Y^2} = R^2 \frac{2dx^+dx^- + 2f(X,Y)dx^+dx^+ + dX^2 + dY^2}{Y^2}
\]  \tag{5.2}

corresponds to a perturbation of the boundary metric,
\[
d_{\text{boundary}}^2 = 2dx^+dx^- + 2f(X,Y)dx^+dx^+ + dX^2 dX^2 .
\]  \tag{5.3}

This in turn corresponds to an insertion of the operator
\[
-\frac{i}{4} \int dx^+ dx^- d^2X f(X,0)\mathcal{T}_{--}(x^+, x^-, X) .
\]  \tag{5.4}

The metric perturbation satisfies the same minimal scalar equation (3.3), as a consequence of the field equation \( \mathcal{R}_{++} = 0 \). In AdS\(_5\) the nonnormalizable solution would be
\[
f(X,Y) = \int e^{iqX} q^2 Y^2 K_2(qY) .
\]  \tag{5.5}

In nonconformal theories this is modified at \( Y > Y^* \). Let us first consider hard scattering, where \( qY^* \gg 1 \). The metric perturbation is exponentially suppressed in \( qY^* \) when \( Y \gg Y^* \), and so we can calculate as though in AdS\(_5\) (there is an admixture of the normalizable mode, but again this is exponentially suppressed).

A canonically normalized bulk scalar field \( \Phi \) couples to the metric perturbation as
\[
-\frac{i}{4} \int dx^+ dx^- d^2X dY \frac{R^3}{Y^3} f(X,Y) \partial_- \Phi \partial_- \Phi .
\]  \tag{5.6}

A gauge theory hadron corresponds to a superposition of such states, with the mixing being due to the conformal symmetry breaking. For hard scattering, \( f(X,Y) \) cuts off the integrand at \( Y > q^{-1} \) and so we are interested in the contribution that is least suppressed at small \( Y \). As in ref. [8], if we restrict to scalar fields then the dominant component is that of smallest conformal dimension \( \Delta \). At small \( Y \) the scalar wavefunction is
\[
\Phi \sim R^{-3/2} Y(Y/Y^*)^\Delta \exp(ip_{\mu}x^\mu) .
\]  \tag{5.7}

The normalization is determined as in ref. [8], though it is written differently here because we have reduced to five dimensions from the start.

The overlap integral (5.6) is peaked at \( Y \sim q^{-1} \), giving
\[
\langle A|T_{--}(q)|B \rangle \sim p_{--}(q^2 Y^*)^{1-\Delta} .
\]  \tag{5.8}
The numerical normalization depends on the coefficient of the leading piece \( \Sigma \) of the wavefunction, which is determined by the details of the geometry at \( Y \sim Y^* \). As with exclusive scattering \([8]\), the behavior agrees with QCD, with the dimension \( \Delta \) interpreted as the number of partons. Thus, for \( \Delta = 1 \), which cannot actually be attained in AdS/CFT, the hadron has a pointlike form factor. For \( \Delta > 1 \) there is a suppression from the need to transfer momentum to all partons \([14]\). For bulk fields with spin, the matrix element contains an additional factor of \( q^{sA+sB} \) and the effective number of partons is given by the twist \( \tau = \Delta - s \) \([8]\).

The crossover from the hard behavior occurs when \( qY^* \sim 1 \), so the effective size of hadrons is

\[
\langle X^2 \rangle \sim Y^{*2}.
\] (5.9)

This corresponds to the confinement scale as measured by the masses of the lightest hadrons, coming from supergravity modes. As usual in AdS/CFT duality, the length scale set by the flux tube tension is smaller by a factor of \( (gN)^{1/4} \), being given by the redshift of the fundamental string scale,

\[
\alpha_{\text{gauge theory}}^{1/2} \sim \frac{Y^*}{R} \alpha^{1/2} \sim (gN)^{-1/4} Y^*.
\] (5.10)

Thus the size is determined by the holographic spreading and not by the internal wavefunction of the string, except for very highly excited string states. The detailed form of the form factor at small \( q \) can be obtained from the supergravity dual, but it depends on the details of the conformal symmetry breaking.

These same methods can be applied to obtain other amplitudes such as that for deep inelastic scattering, which is expressed in terms of two-current matrix elements \([15]\).

6 Discussion

The original attempt to describe hadrons as idealized strings was frustrated by the infinite zero point oscillations in the size of strings. Early ideas for modifying string theory such as replacing the idealized strings by fat flux tubes or as collections of partons which approximate strings fit well with QCD but seemed to preclude an idealized mathematical string description.

More recent evidence from AdS/CFT type dualities suggests that idealized string theory in higher dimensions may provide an exact description of the ’t Hooft limit of QCD-like theories. In this paper we have argued that an ideal bulk string theory in five dimensions is fully compatible with a fat non-ideal string in four dimensions.
The fifth dimension can be divided into two regions. The “wall” region near $Y = Y^*$ corresponds to the confinement scale $\Lambda$. If we ignore high frequency fluctuations, the string spends most of its time in this region. The usual UV/IR spreading gives the string a thickness of order $\Lambda$. High frequency fluctuations of small sections of string can occur which cause it to fluctuate toward $Y = 0$, the region corresponding to short distance behavior in space-time. These fluctuations will control the large momentum behavior of form factors as well as deep inelastic matrix elements. Such fluctuations should give the string a parton-like makeup. We have also seen that these fluctuations stiffen the effective string tension so much that the infinite zero point size is eliminated.

Vertex operators of the form (3.2) define matrix elements of local currents such as the energy momentum tensor. Similar vertices can be constructed for vector currents of $R$-charge. Products of such vertices can be studied to uncover the behavior of deep inelastic structure functions in the confining deformations of conformal theories.

Our work and ref. [8] show that even at large ’t Hooft parameter the short-distance behavior of gauge theories has much in common with QCD. However, to make contact with real QCD we need a better understanding of the strongly coupled world-sheet theory at small $R$. Even if we truncate to the single field $Y$ the theory (2.7) is nontrivial, and it would be interesting to understand its behavior at small $R$.

7 Acknowledgements

We would like to thank Juan Maldacena, Mimi Schwarz, and Matt Strassler for helpful discussions. This work was supported by NSF grants PHY98-70115, PHY99-07949 and PHY00-98395.

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