LOCAL WELL-POSEDNESS FOR NAVIER-STOKES EQUATIONS
WITH A CLASS OF ILL-PREPARED INITIAL DATA

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Abstract. In this paper, we prove that for the ill-prepared initial data of the form

\[ u_\epsilon^0(x) = (v_0^h(x_\epsilon), \epsilon^{-1}v_0^3(x_\epsilon))\mathbf{T}, \quad x_\epsilon = (x_h, \epsilon x_3)\mathbf{T} \]

the Cauchy problem of the incompressible Navier-Stokes equations on \( \mathbb{R}^3 \) is locally well-posed for all \( \epsilon > 0 \), provided that the initial velocity profile \( v_0 \) is analytic in \( x_3 \) but independent of \( \epsilon \).

1. Introduction. In this paper, we consider the Cauchy problem of the 3D incompressible Navier-Stokes equations, which is described by the following system

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla P &= 0, \quad x \in \mathbb{R}^3, \ t > 0, \\
\text{div} \ u &= 0, \quad x \in \mathbb{R}^3, \ t > 0, \\
u(0) &= u_0, \quad x \in \mathbb{R}^3,
\end{aligned}
\]

(1)

where \( u \) represents the velocity field and \( P \) is the scalar pressure. The initial velocity field \( u_0^\epsilon \) is of the form

\[ u_\epsilon^0(x) = (v_0^h(x_\epsilon), \epsilon^{-1}v_0^3(x_\epsilon))\mathbf{T}, \quad x_\epsilon = (x_h, \epsilon x_3)\mathbf{T} \]

which allows slowly varying in the vertical variable \( x_3 \) when \( \epsilon > 0 \) is a small parameter.

This family of initial data are very interesting (as has been pointed out by V. Sverák, see the acknowledgement in [8]) considered by Chemin, Gallagher and Paicu in [8] and Y. Chen, B. Han and Z. Lei in [10]. In [8], the authors proved the global regularity of solutions to the Navier-Stokes equations when \( v_0 \) is analytic in \( x_3 \) and periodic in \( x_h \), and certain norm of \( v_0 \) is sufficiently small but independent of \( \epsilon > 0 \). More precisely, they proved the following Theorem:

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Theorem 1.1 (Chemin-Gallagher-Paicu, Ann. Math. 2011). Let $\alpha$ be a positive number. There are two positive numbers $\epsilon_0$ and $\eta$ such that for any divergence free vector field $v_0$ satisfying
\[
\|e^{\alpha|D_3|}v_0\|_{B^{\frac{7}{2},2}_{2,1}} \leq \eta,
\]
then, for any positive $\epsilon$ smaller than $\epsilon_0$, the initial data (2) generates a global smooth solution to (1) on $\mathbb{T}^2 \times \mathbb{R}$.

As has been pointed out by Chemin, Gallagher and Paicu, the reason why the horizontal variable of the initial data in [8] is restricted to a torus is to be able to deal with very low horizontal frequencies. In the proof of Theorem 1.1 in [8], functions with zero horizontal average are treated differently to the others, and it is important that no small horizontal frequencies appear other than zero. Later on, many efforts are made towards removing the periodic constraint of $v_0$ on the horizontal variables. For instance, see [7, 15, 14, 25, 10] and so on. Particularly, authors in [10] proved that 3D generalized Navier-Stokes equations with initial data (2) admits a global solution on $\mathbb{R}^3$ by using the anisotropic Besov spaces. Further more, authors in [10] proved that there exists a global solution to the 4D Navier-Stokes equations with initial data (2).

In this paper, we want to show that for the initial data of the form (2), the Cauchy problem of 3D incompressible Navier-Stokes equations on $\mathbb{R}^3$ is locally well-posed for all $\epsilon > 0$, provided that $v_0$ is analytic in $x_3$ but independent of $\epsilon$. Our main result states as follows.

Theorem 1.2. Let $\alpha$, $\delta$ and $\epsilon_0$ be three positive constants and $0 < \delta < 1$. For any positive constant $\eta$ such that $0 < \epsilon < \epsilon_0$ and any divergence free vector field $v_0 = (v_0^h, v_0^3)$ satisfying
\[
\|e^{\alpha|D_3|}v_0\|_{B^{\frac{5}{2},1} \cap B^{1-\delta,1-\delta}_{2,1}} + \|e^{\alpha|D_3|}\partial_3 v_0\|_{B^{1-\delta,\frac{1}{2}}_{2,\frac{3}{2}}} \leq \eta,
\]
the Navier-Stokes equations (1) with initial data (2) generates a local smooth solution on $\mathbb{R}^3$.

Before going any further, let us recall some known results on the small-data global regularity of the Navier-Stokes equations. In the seminal paper [23], Leray proved that the 3D incompressible Navier-Stokes equations are globally well-posed if the initial data $u_0$ is such that $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2}$ is small enough. This quantity used by Leray is invariant under the natural scaling of the Navier-Stokes equations. Later on, many authors studied different scaling invariant spaces in which Navier-Stokes equations are well-posed at least for small initial data, which include but are not limited to

\[
\hat{H}^\frac{1}{2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow B^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3) \hookrightarrow BMO^{-1}(\mathbb{R}^3),
\]

where $3 < p < \infty$. The space $BMO^{-1}(\mathbb{R}^3)$ is known to be the largest scaling invariant space so that the Navier-Stokes equations (1) are globally well-posed under small initial data. The readers are referred to [13, 19, 4, 20] as references. We also mention that the work of Lei and Lin [21] was the first to quantify the smallness of the initial data to be 1 by introducing a new space $X^{\epsilon}$.

We remark that the norm in the above scaling invariant spaces are always greater than the norm in the Besov space $B^{-1}_{\infty,\infty}$ defined by
\[
\|u\|_{B^{-1}_{\infty,\infty}} \overset{\text{def}}{=} \sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta} u_0\|_{L^\infty}.
\]
Bourgain and Pavlovic in [3] showed that the cauchy problem of the 3D Navier-Stokes equation is ill-posed in the sense of norm inflation. Partially because of the result of Bourgain and Pavlovic, data with a large $B^{-1}_{\infty,\infty}$ are usually called large data to the Navier-Stokes equations (for instance, see [7, 25]). Towards this line of research, a well-oiled case is the family of initial data which is slowly varying in vertical variable. The well-prepared case was considered by J. Chemin and I. Gallagher in [7]. They proved the global well-posedness of (1) when $u_0^{\epsilon}$ is of the form

$$u_0^{\epsilon}=(v_0^h+\epsilon w_0^h, w_0^3)(x_3, \epsilon x_3).$$

Later, G. Gui, J. Huang, and P. Zhang in [14] generalized this result to the density dependent Navier-Stokes equations with the same initial velocity. Recently, B. Han in [15] consider global regularity of (1) if $u_0^{\epsilon}$ satisfies the form of

$$u_0^{\epsilon}(x)=\epsilon^\delta v_0^h(x_3, \epsilon x_3), \epsilon^{-1}v_0^3(x_3, \epsilon x_3))$$

for any $\delta > 0$, then $u_0^{\epsilon}$ generates a global solution of (1) on $\mathbb{R}^3$. The case $\delta = \frac{1}{2}$ was considered by M. Paicu and Z. Zhang in [25]. The case $\delta = 0$ was proved in [10] for the 3D generalized Navier-Stokes equations on $\mathbb{R}^3$ and 4D Navier-Stokes equations on on $\mathbb{R}^4$. One can note that, all of the initial data is large in $B^{-1}_{\infty,\infty}$, but still generates a global solution. We also mention that for the general 3D incompressible Navier-Stokes equations which possess hyper-dissipation in horizontal direction, D. Fang and B. Han in [12] obtain the global existence result when the initial data belongs to the anisotropic Besov spaces.

In the following, we will introduce the sketch of the proof of Theorem 1.2.

2. Main ideas of the Proof. We will prove the result by construct the bilinear estimate (independent of $\epsilon$) of the rescaled system of (1). The following is the outline of our method.

In preparation, we should rescale the system. As in [8], we define

$$u'(t, x) = (v^h(t, x\epsilon), \epsilon^{-1}v^3(t, x\epsilon))^T, \quad P'(t, x) = q(t, x\epsilon).$$

Denote

$$\Delta_{\epsilon} = \partial_1^2 + \partial_2^2 + \epsilon^2 \partial_3^2.$$

Using the Navier-Stokes equations (1), it is easy to derive the equations governing the rescaled variables $v$ and $q$ (they are still depending on $\epsilon$):

$$\begin{cases}
\partial_t v^h + v \cdot \nabla v^h - \Delta_{\epsilon} v^h + \nabla q = 0, \\
\partial_t v^3 + v \cdot \nabla v^3 - \Delta_{\epsilon} v^3 + \epsilon^2 \partial_3 q = 0, \\
\text{div } v = 0, \quad v(0) = v_0(x).
\end{cases}$$

The rescaled pressure $q$ can be recovered by the divergence free condition as

$$-\Delta_{\epsilon} q = \sum_{i,j} \partial_i \partial_j (v^i v^j).$$

The local existence result of the solutions to system (4) for small initial data $v_0$ will be presented in Section 3 for any positive $\epsilon$. But to best illustrate our ideas, let us here focus on the case of $\epsilon = 0$. Formally, by taking $\epsilon = 0$ in system (4), we
have the following limiting system:
\[
\begin{aligned}
\partial_t v^h + v \cdot \nabla v^h - \Delta_h v^h + \nabla h q &= 0, \\
\partial_t v^3 + v \cdot \nabla v^3 - \Delta_h v^3 &= 0, \\
\text{div } v &= 0, \quad v(0) = v_0(x).
\end{aligned}
\] (5)

The pressure \( q \) in (5) is given by
\[
-\Delta_h q = \sum_{i,j} \partial_i \partial_j (v^i v^j).
\]

Based on the new system, we try to find the best \textit{a priori} estimate.

**An initial estimate.** Observe that in the rescaled system (5), the viscosity is absent in the vertical direction. To make the full use of smoothing effect from horizontal heat kernel \( \partial_t - \Delta_h \), particularly in low frequency part, we will apply the theories in anisotropic homogeneous Besov spaces.

Naturally, we define
\[
\Psi(t) = \|v_k\|_{L^\infty_t(B^{{\frac{1}{2}}}_{2,1})} + \|v_k\|_{L^1_t(B^{{\frac{1}{4}} + \delta}_{2,1})} + \cdots.
\]

The goal is to derive the certain \textit{a priori} estimate of the form:
\[
\Psi(t) \lesssim \Psi(0) + \Psi(t)^2
\]

By Duhamel’s principle, we can write the integral equation of \( v^h \) by
\[
v^h = e^{t\Delta_h} v^h_0 - \int_0^t e^{(t-\tau)\Delta_h} (v^h \cdot \nabla_h v^h + v^3 \partial_3 \partial_3 v^h + \nabla_h q) d\tau.
\]

According to the estimates of heat equation, one can formally has
\[
\Psi(t) \lesssim \Psi(0) + \|v^h \cdot \nabla_h v^h\|_{L^1_t(B^{{\frac{1}{4}} + \delta}_{2,1})} + \|v^3 \partial_3 v^h\|_{L^1_t(B^{{\frac{1}{4}} + \delta}_{2,1})} + \|\nabla_h q\|_{L^1_t(B^{{\frac{1}{4}} + \delta}_{2,1})}.
\]

It is easy to see that
\[
\|v^h \cdot \nabla_h v^h\|_{L^1_t(B^{{\frac{1}{4}} + \delta}_{2,1})} \lesssim \|v^h\|_{L^\infty_t(B^{{\frac{1}{4}} + \delta}_{2,1})} \|\nabla_h v^h\|_{L^1_t(B^{{\frac{1}{4}} + \delta}_{2,1})} \lesssim \Psi(t)^2.
\]

For the pressure term, we split it into three parts
\[
\nabla_h q = -2 \nabla_h (-\Delta_h)^{-1} \partial_3 (v^3 \text{div} h v^h) + 2 \sum_{i=1,2} \nabla_h (-\Delta_h)^{-1} \partial_i \partial_3 (v^i v^3)
\]
\[+ \sum_{i,j=1,2} \nabla_h (-\Delta_h)^{-1} \partial_i \partial_j (v^i v^j).
\]

The \( L^1_t(B^{{\frac{1}{2}} + \delta}_{2,1}) \) norm of the first part in \( \nabla h q \) can be estimated by
\[
\|\nabla_h (-\Delta_h)^{-1} \partial_3 (v^3 \text{div} h v^h)\|_{L^1_t(B^{{\frac{1}{4}} + \delta}_{2,1})} \lesssim \|\partial_3 (v^3 \text{div} h v^h)\|_{L^1_t(B^{{\frac{1}{2}} - 1 + \delta}_{2,1})}.
\]

We have to deal with \( \|fg\|_{B^{{\frac{1}{2}} - 1 + \delta}_{2,1}} \) type estimate as follows. Hence, we should choose that \( \delta \) is a small positive number.

**The \( \partial_3 \)-derivative loss.** Now we have the bilinear estimate in the following form:
\[
\Psi(t) \lesssim \Psi(0) + \Psi(t)^2 + \|v^3 \partial_3 v^h\|_{L^1_t(B^{{\frac{1}{4}} + \delta}_{2,1})} + \cdots.
\]
However, it is not easy to estimate $\partial_3 v^h$. By the continuity property of the product in anisotropic Besov spaces (Lemma 3.5), the strategy to bound the last term is

$$\|v^3 \partial_3 v^h\|_{L^1_t(B^{s+\frac{1}{2}}_{2,1})} \lesssim \|v^3\|_{L^1_t(B^{s+\frac{1}{2}}_{2,1})} \|\partial_3 v^h\|_{L^\infty_t(B^{s+\frac{1}{2}}_{2,1})},$$

and then we should add the new norm $\|\partial_3 v^h\|_{L^\infty_t(B^{s+\frac{1}{2}}_{2,1})}$ in the definition of $\Psi(t)$. We find that $\|\partial_3 v^h\|_{L^\infty_t(B^{s+\frac{1}{2}}_{2,1})}$ is the hardest term to estimate. Since we note that by Duhamel’s principle,

$$\partial_3 v^h = e^{t \Delta_h} \partial_3 v_0^h - \int_0^t e^{(t-\tau)\Delta_h} \partial_3 (v^3 \partial_3 v^h) d\tau + \cdots.$$  

There are still $\partial_3$-derivative loss in the a priori estimates.

**Modification.** Aiming at the $\partial_3$-derivative loss, motivated by Chemin-Gallagher-Paicu ([8]), we add an exponential weight $e^{\Phi(t,|D^3|)}$ with

$$\Phi(t,|\xi_3|) = (\alpha - \lambda \theta(t))|\xi_3|.$$  

Here $\theta(t)$ is defined by

$$\theta(t) = \int_0^t \|v_\Phi^3\|_{B^{s+\frac{1}{2}}_{2,1}} d\tau,$$

which will be proved to be small to ensure that $\Phi(t,|\xi_3|)$ satisfies the subadditivity.

Denoting $f_\Phi = e^{\Phi(t,|D^3|)} f$, we have

$$\partial_3 v^h = e^{t \Delta_h} e^{\Phi(t,|D^3|)} \partial_3 v_0^h - \int_0^t e^{(t-\tau)\Delta_h} e^{-\lambda \int_0^\tau \dot{\phi}(t') dt'} |D^3| \partial_3 (v^3 \partial_3 v^h) d\tau + \cdots.$$  

So that we recover the derivative loss by

$$\|\partial_3 v^h\|_{L^\infty_t(B^{s+\frac{1}{2}}_{2,1})} \lesssim \int_0^t \int e^{-\lambda \int_0^\tau \dot{\phi}(t') dt'} |D^3| \partial_3 (v^3 \partial_3 v^h) d\lambda d\tau \|\partial_3 v^h\|_{L^\infty_t(B^{s+\frac{1}{2}}_{2,1})} + \cdots + \sum_{k,j} c_{k,j} \int_0^t \int e^{-\lambda \int_0^\tau \dot{\phi}(t') dt'} |D^3| \partial_3 (v^3 \partial_3 v^h) d\lambda d\tau \|\partial_3 v^h\|_{L^\infty_t(B^{s+\frac{1}{2}}_{2,1})} + \cdots,$$

where the sequence $\{c_{k,j}\}_{(k,j) \in \mathbb{Z}^2}$ satisfies $\|c_{k,j}\|_{L^1(\mathbb{Z}^2)} = 1$.

Now we define the new

$$\Psi(t) = \|v_\Phi\|_{L^\infty_t(B^{s+\frac{1}{2}}_{2,1})} \|\partial_3 v_\Phi^3\|_{L^\infty_t(B^{s+\frac{1}{2}}_{2,1})} + \|v_\Phi\|_{L^1_t(B^{s+\frac{1}{2}}_{2,1})}.$$

**Estimation for $\theta(t)$.** For the introduced term $\theta(t)$, we should prove that for any time $t$, $\theta(t)$ is a small quantity. This can ensure that the phase function $\Phi$ satisfies the subadditivity property. Using the interpolation inequality (Lemma 3.4), we have

$$\theta(t) = \|v_\Phi^3\|_{L^1_t(B^{s+\frac{1}{2}}_{2,1})} \lesssim \left( \|v_\Phi^3\|_{L^2_t(B^{s+\frac{1}{2}}_{2,1})} + \|v_\Phi^3\|_{L^1_t(B^{s+\frac{1}{2}}_{2,1})} \right)^{\frac{1}{2}}.$$  

This means that we should get the a priori estimate when the initial data $v_0^3$ in a low regularity space $B^{s+\frac{1}{2}}_{2,1}$. Then we are going to estimate

$$Y(t) = \|v_\Phi^3\|_{L^2_t(B^{s+\frac{1}{2}}_{2,1})} + \|v_\Phi^3\|_{L^1_t(B^{s+\frac{1}{2}}_{2,1})}.$$
However, when $\epsilon > 0$, we can not get the closed estimate for $Y(T)$. Our observation is to add an extra term $\epsilon v^h$ under the same norm which is hidden in the pressure term $\epsilon \partial_3 q$. Then we define a new quantity:

$$X(t) = \epsilon \|v^h\|_{L_t^2(B^1_2(\cdot, \frac{1}{2}))} + \epsilon \|v^h\|_{L_t^1(B^1_{2,1}(\cdot, \frac{1}{2}))}.$$ 

We will get the closed estimate for $X(t) + Y(t)$.

**Estimation for $v^3$.** To complete the prove, we should estimate $v^3$. In fact, the estimate for $v^3$ is much easier than $v^h$. From the limiting system (5), there is no loss of derivative in vertical direction. Due to divergence free condition, the nonlinear term $v^3 \partial_3 v^3$ can be rewritten as $-v^3 \text{div}_h v^h$.

The remaining part of the paper is organized as follows. In Section 2, we present the basic theories of anisotropic Littlewood-Paley decomposition and anisotropic Besov spaces. Section 3 is devoted to obtaining the a priori estimates of solution. The $\theta(t)$ will be studied in Section 4. Finally, the proof of the main result will be given in Section 5.

3. **Anisotropic Littlewood-Paley theories and preliminary lemmas.** In this section, we first recall the definition of the anisotropic Littlewood-Paley decomposition and some properties about anisotropic Besov spaces. It was introduced by D. Iftimie in [17] for the study of incompressible Navier-Stokes equations in thin domains. Let us briefly explain how this may be built in $\mathbb{R}^3$. Let $(\chi, \varphi)$ be a couple of $C^\infty$ functions satisfying

$$\text{Supp} \chi \subset \left\{ r \leq \frac{4}{3} \right\}, \quad \text{Supp} \varphi \subset \left\{ \frac{3}{4} \leq r \leq \frac{8}{3} \right\},$$

$$\chi(r) + \sum_{k \in \mathbb{N}} \varphi(2^{-k}r) = 1 \quad \text{for} \quad r \in \mathbb{R}$$

and

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}r) = 1 \quad \text{for} \quad r \in \mathbb{R} \setminus \{0\}.$$ 

For $u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3)$, we define the homogeneous dyadic decomposition on the horizontal variables by

$$\Delta^h_k u = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\hat{u}) \quad \text{for} \quad k \in \mathbb{Z}.$$ 

Similarly, on the vertical variable, we define the homogeneous dyadic decomposition by

$$\Delta^v_j u = \mathcal{F}^{-1}(\varphi(2^{-j}|\xi_3|)\hat{u}) \quad \text{for} \quad j \in \mathbb{Z}.$$ 

The anisotropic Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\Delta^h_k \Delta^h_l u \equiv 0 \quad \text{if} \quad |k - l| \geq 2 \quad \text{and} \quad \Delta^h_k (S^h_{l-1} u \Delta^h_l u) \equiv 0 \quad \text{if} \quad |k - l| \geq 5,$$

where $S^h_l$ is defined by

$$S^h_l u = \sum_{|l' - l| \leq 1} \Delta^h_{l'} u.$$ 

Similar properties hold for $\Delta^v_j$. In this paper, we shall use the following anisotropic version of Besov spaces [17]. In what follows, we denote for abbreviation

$$\Delta_{k,j} f = \Delta^h_k \Delta^v_j f.$$
Definition 3.1 (Anisotropic Besov space). Let \((p, r) \in [1, \infty]^2\), \(\sigma, s \in \mathbb{R}\) and \(u \in S'(\mathbb{R}^3)\), we set
\[
\|u\|_{\dot{B}^\sigma_{p,r}} \overset{\text{def}}{=} \|2^{k\sigma} 2^{js} \| \Delta_{k,j} u \|_{L^p_v(L^r)}\|_{r(\mathbb{Z}^2)}.
\]

1. For \(\sigma < \frac{2}{p}, \ s < \frac{1}{2}\) (or \(\sigma = \frac{2}{p}, \ s = \frac{1}{2}\) if \(r = 1\)), we define
\[
\dot{B}^\sigma_{p,r}(\mathbb{R}^3) \overset{\text{def}}{=} \{u \in S'(\mathbb{R}^3) \mid \|u\|_{\dot{B}^\sigma_{p,r}} < \infty\}.
\]

2. If \(k, l \in \mathbb{N}\) and \(\frac{2}{p} + k \leq \sigma < \frac{2}{p} + k + 1, \ \frac{1}{2} + l \leq s < \frac{1}{2} + l + 1\) (or \(\sigma = \frac{2}{p} + k + 1\) and \(s = \frac{1}{2} + l + 1\) if \(r = 1\)), then \(\dot{B}^\sigma_{p,r}(\mathbb{R}^3)\) is defined as the subset of \(u \in S'(\mathbb{R}^3)\) such that \(\partial_k^\alpha \partial_\beta u \in \dot{B}^{-k-\alpha, -l}_{p,r}(\mathbb{R}^3)\) whenever \(|\beta| = k, \alpha = l\).

The study of non-stationary equation requires spaces of the type \(L^p_T(X) = L^p(0, T; X)\) for appropriate Banach spaces \(X\). In our case, we expect \(X\) to be an anisotropic Besov space. So it is natural to localize the equations through anisotropic Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. As in [6], we define the so called Chemin-Lerner type spaces:

Definition 3.2. Let \((p, r) \in [1, \infty]^2\), \(\sigma, s \in \mathbb{R}\) and \(T \in (0, \infty]\), we set
\[
\|u\|_{\dot{L}^p_T(\dot{B}^\sigma_{p,r})} \overset{\text{def}}{=} \|2^{k\sigma} 2^{js} \| \Delta_{k,j} u \|_{L^p_v(L^r)}\|_{r(\mathbb{Z}^2)}
\]
and denote by \(\dot{L}^p_T(\dot{B}^\sigma_{p,r})(\mathbb{R}^3)\) the subset of distributions \(u \in S'(0, T) \times S'(\mathbb{R}^3)\) with finite \(\dot{L}^p_T(\dot{B}^\sigma_{p,r})\) norm.

Before giving some properties of the anisotropic Besov spaces, we recall the Hölder and Young’s inequalities in the framework of anisotropic Lebesgue spaces.

Lemma 3.3. 1) Let \(f \in L^p_h(L^r_v), g \in L^p_h(L^r_v)\) for \(1 \leq r, r', r'', p, p', p'' \leq \infty\). Then \(fg \in L^p_h(L^r_v)\) and satisfies
\[
\|fg\|_{L^p_h(L^r_v)} \lesssim \|f\|_{L^p_h(L^r_v)}\|g\|_{L^p_h(L^r_v)}, \quad (6)
\]
where \(\frac{1}{r} = \frac{1}{r'} + \frac{1}{r''}\) and \(\frac{1}{p} = \frac{1}{p'} + \frac{1}{p''}\).

2) Let \(f \in L^p_h(L^r_v), g \in L^p_h(L^r_v)\) for \(1 \leq r, r', r'', p, p', p'' \leq \infty\). Then \(f * g \in L^p_h(L^r_v)\) and satisfies
\[
\|f * g\|_{L^p_h(L^r_v)} \lesssim \|f\|_{L^p_h(L^r_v)}\|g\|_{L^p_h(L^r_v)}, \quad (7)
\]
where \(1 + \frac{1}{r} = \frac{1}{r'} + \frac{1}{r''}\) and \(1 + \frac{1}{p} = \frac{1}{p'} + \frac{1}{p''}\).

In order to investigate the continuity properties of the products of two temperate distributions \(f\) and \(g\) in anisotropic Besov spaces, we then recall the isotropic product decomposition which is a simple splitting device going back to the pioneering work by J.-M. Bony [2].

Let \(f, g \in S'(\mathbb{R}^3),\)
\[
f \ast g = T(f, g) + \ddot{T}(f, g) + R(f, g),
\]
where the paraproducts \(T(f, g)\) and \(\ddot{T}(f, g)\) are defined by
\[
T(f, g) = \sum_{k \in \mathbb{Z}} S_{k-1} f \Delta_k g, \quad \ddot{T}(f, g) = \sum_{k \in \mathbb{Z}} \Delta_k f S_{k-1} g
\]
and the remainder
\[ R(f,g) = \sum_{k \in \mathbb{Z}} \Delta_k f \Delta_k g \quad \text{with} \quad \Delta_k g = \sum_{k' = k-1}^{k+1} \Delta_{k'} g. \]

Similarly, we can define the decompositions for both horizontal variable \( x_h \) and vertical variable \( x_3 \). Indeed, we have the following split in \( x_h \).
\[ fg = T^h(f,g) + \bar{T}^h(f,g) + R^h(f,g), \]

with
\[ T^h(f,g) = \sum_{k \in \mathbb{Z}} S^h_{k-1} f \Delta^h_k g, \quad \bar{T}^h(f,g) = \sum_{k \in \mathbb{Z}} \Delta^h_k f S^h_{k-1} g \]
and
\[ R^h(f,g) = \sum_{k \in \mathbb{Z}} \Delta^h_k f \bar{\Delta}^h_k g \quad \text{where} \quad \bar{\Delta}^h_k g = \sum_{k' = k-1}^{k+1} \Delta_{k'} g. \]

The decomposition in vertical variable \( x_3 \) can be defined by the same line. Thus, we can write \( fg \) as
\[\begin{align*}
fg &= (T^h + \bar{T}^h + R^h)(T^v + \bar{T}^v + R^v)(f,g) \\
&= T^h T^v (f,g) + T^h \bar{T}^v (f,g) + T^h R^v (f,g) \\
&\quad + \bar{T}^h T^v (f,g) + \bar{T}^h \bar{T}^v (f,g) + \bar{T}^h R^v (f,g) \\
&\quad + R^h T^v (f,g) + R^h \bar{T}^v (f,g) + R^h R^v (f,g).
\end{align*}\] (8)

Each term of (8) has an explicit definition. Here
\[ T^h T^v (f,g) = \sum_{(k,j) \in \mathbb{Z}^2} S^h_{k-1} S^v_{j-1} f \Delta_{k,j} g \]
and
\[ \bar{T}^h T^v (f,g) = \sum_{(k,j) \in \mathbb{Z}^2} \Delta^h_k \Delta^v_{j-1} f S^h_{k-1} \Delta^v_j g. \]

Similarly,
\[ R^h T^v (f,g) = \sum_{(k,j) \in \mathbb{Z}^2} \Delta^h_k \Delta^v_{j-1} f \bar{\Delta}^h_k \Delta^v_j g \]
and
\[ R^h R^v (f,g) = \sum_{(k,j) \in \mathbb{Z}^2} \Delta^h_k \Delta^v_{j-1} f \bar{\Delta}^h_k \bar{\Delta}^v_j g, \]
and so on.

At this moment, we can state some properties concerning the continuity of the product in anisotropic Besov spaces. The first lemma is about the spaces in the framework of \( L^2 \). And similar result was proved in [14], for completeness, we also give the details of the proof here.

**Lemma 3.4.** Let \((\sigma_1, \sigma_2)\) be in \( \mathbb{R}^2 \). If \( \sigma_1 + \sigma_2 \) is positive and \( \sigma_1, \sigma_2 \leq 1 \), then we have for any \( f \in B^{\sigma_1,\frac{1}{2}}_2 \) and \( g \in B^{\sigma_2,\frac{1}{2}}_2 \),
\[ ||fg||_{B^{\sigma_1+\sigma_2-1,\frac{1}{2}}_2} \lesssim ||f||_{B^{\sigma_1,\frac{1}{2}}_2} ||g||_{B^{\sigma_2,\frac{1}{2}}_2}. \] (9)
Proof. According to (8), we first give the bound of $T^h T^v(f, g)$. Indeed, applying Lemma 3.3 and Bernstein inequality, we get that

$$
\| \Delta_{k,j}(T^h T^v(f, g)) \|_{L^2} \lesssim \sum_{|k-k'| \leq 4 \atop |j-j'| \leq 4} \| S^h_{k',-1} S^v_{j',-1} f \|_{L^2_x(L^\infty_t)} \| \Delta^h_{k'} \Delta^v_{j'} g \|_{L^2_x(L^\infty_t)}
$$

$$
\lesssim \sum_{|k-k'| \leq 4 \atop |j-j'| \leq 4} \sum_{k'' \leq k-k' \leq 2} \sum_{j'' \leq j-j' \leq 2} 2^{k''} \sigma_2 2^{j''} \| \Delta^h_{k''} \Delta^v_{j''} f \|_{L^2_x} \cdot 2^{k''} \sigma_2 2^{j''} \| \Delta^h_{k''} \Delta^v_{j''} g \|_{L^2_x}
$$

$$
\lesssim \sum_{|k-k'| \leq 4 \atop |j-j'| \leq 4} \sum_{k'' \leq k-k' \leq 2} \sum_{j'' \leq j-j' \leq 2} 2^{k''} \sigma_2 2^{j''} \| \Delta^h_{k''} \Delta^v_{j''} f \|_{L^2_x} \cdot 2^{k''} \sigma_2 2^{j''} \| \Delta^h_{k''} \Delta^v_{j''} g \|_{L^2_x}
$$

$$
\times g(k''-k')(1-\sigma_1) g(k''-k)(1-\sigma_2) 2^{j''} \| f \|_{B^{\sigma_1}_{2,1}} 2^{j''} \| g \|_{B^{\sigma_2}_{2,1}}.
$$

Since $\sigma_1, \sigma_2 \leq 1$, we obtain by applying Young’s inequality that

$$
\| \Delta_{k,j}(T^h T^v(f, g)) \|_{L^2} \lesssim c_{k,j} \delta^{-(\sigma_1+\sigma_2-1)-\frac{1}{2}j} 2^{-\frac{1}{2}j} \| f \|_{\dot{B}^{\sigma_1}_{2,1}} \| g \|_{\dot{B}^{\sigma_2}_{2,1}}
$$

where the sequence $\{c_{k,j}\}_{(k,j) \in \mathbb{Z}^2}$ satisfies $\|c_{k,j}\|_{\ell^1(\mathbb{Z}^2)} = 1$. This gives the estimate of $T^h T^v(f, g)$. Similarly, for $\bar{T}^h T^v(f, g)$, we have

$$
\| \Delta_{k,j}(\bar{T}^h T^v(f, g)) \|_{L^2} \lesssim c_{k,j} \delta^{-(\sigma_1+\sigma_2-1)-\frac{1}{2}j} 2^{-\frac{1}{2}j} \| f \|_{\dot{B}^{\sigma_1}_{2,1}} \| g \|_{\dot{B}^{\sigma_2}_{2,1}}.
$$

Again, $\sigma_1, \sigma_2 \leq 1$ implies that

$$
\| \Delta_{k,j}(\bar{T}^h T^v(f, g)) \|_{L^2} \lesssim c_{k,j} \delta^{-(\sigma_1+\sigma_2-1)-\frac{1}{2}j} 2^{-\frac{1}{2}j} \| f \|_{\dot{B}^{\sigma_1}_{2,1}} \| g \|_{\dot{B}^{\sigma_2}_{2,1}}.
$$

The remainder operator which concerns with the horizontal variable $R^h T^v(f, g)$ can be bounded as follows:

$$
\| \Delta_{k,j}(R^h T^v(f, g)) \|_{L^2} \lesssim \sum_{k' \geq k-2 \atop |j-j'| \leq 4} \| \Delta^h_{k'} S^v_{j',-1} f \|_{L^2_x(L^\infty_t)} \| \Delta_{k'} \Delta_{j'} g \|_{L^2_x(L^\infty_t)} 2^{k'}
$$

$$
\lesssim \sum_{k' \geq k-2 \atop |j-j'| \leq 4} \sum_{k'' \leq k-k' \leq 2} \sum_{j'' \leq j-j' \leq 2} 2^{k''} \sigma_2 2^{j''} \| \Delta^h_{k''} \Delta^v_{j''} f \|_{L^2_x} \cdot 2^{k''} \sigma_2 2^{j''} \| \Delta^h_{k''} \Delta^v_{j''} g \|_{L^2_x}
$$

$$
\times g(k''-k')(1-\sigma_1) g(k''-k)(1-\sigma_2) 2^{j''} \| f \|_{B^{\sigma_1}_{2,1}} 2^{j''} \| g \|_{B^{\sigma_2}_{2,1}}.
$$

As $\sigma_1 + \sigma_2 > 0$, we have

$$
\| \Delta_{k,j}(R^h T^v(f, g)) \|_{L^2} \lesssim c_{k,j} \delta^{-(\sigma_1+\sigma_2-1)-\frac{1}{2}j} 2^{-\frac{1}{2}j} \| f \|_{\dot{B}^{\sigma_1}_{2,1}} \| g \|_{\dot{B}^{\sigma_2}_{2,1}}.
$$
For the remainder operator on both horizontal and vertical variables $R^h R^v (f, g)$, we get by applying Lemma 3.3 and Bernstein inequality that

$$\|\Delta_{k,j} (R^h R^v (f, g))\|_{L^2} \lesssim \sum_{k' \geq k-2, j' \geq j-2} 2^{k' \cdot 2 \cdot j'} \|\Delta_{k', j'}^h (f, g)\|_{L^1} 2^{k' \cdot 2 \cdot j'} \|\Delta_{k', j'}^v (f, g)\|_{L^2} \lesssim \sum_{k' \geq k-2, j' \geq j-2} 2^{k' \cdot 2 \cdot j'} \|\Delta_{k', j'}^h (f, g)\|_{L^1} 2^{k' \cdot 2 \cdot j'} \|\Delta_{k', j'}^v (f, g)\|_{L^2} \times 2^{(k-k') \cdot (\sigma_2 - \sigma_1) \cdot 2 \cdot j'} \lesssim \sum_{k' \geq k-2, j' \geq j-2} 2^{k' \cdot 2 \cdot j'} \|\Delta_{k', j'}^h (f, g)\|_{L^1} 2^{k' \cdot 2 \cdot j'} \|\Delta_{k', j'}^v (f, g)\|_{L^2} \times 2^{(k-k') \cdot (\sigma_2 - \sigma_1) \cdot 2 \cdot j'} .$$

This implies that

$$\|\Delta_{k,j} (R^h R^v (f, g))\|_{L^2} \lesssim c_{\sigma_2} 2^{-k \cdot (\sigma_2 - \sigma_1) \cdot 2 \cdot j'} \|\tilde{f}\|_{\mathcal{B}^2_{\sigma_1, \frac{2}{2}}} \|g\|_{\mathcal{B}^2_{\sigma_2, \frac{2}{2}}} ,$$

as $\sigma_1 + \sigma_2 > 0$. The other terms can be followed exactly in the same way, here we omit the details. This completes the proof of this lemma.

In the proof of our main result, we require the similar continuity results in the Chemin-Lerner type spaces $\tilde{L}^p_T (\mathcal{B}^\sigma_{2,1})$ as in Lemma 3.4.

**Lemma 3.5.** Let $\rho \in [1, \infty]$, $(\sigma_1, \sigma_2)$ be in $\mathbb{R}^2$ and $(\rho_1, \rho_2) \in [1, \infty]^2$. Assume that

$$\frac{1}{\rho} \overset{\text{def}}{=} \frac{1}{\rho_1} + \frac{1}{\rho_2} \leq 1.$$

If $\sigma_1 + \sigma_2$ is positive and $\sigma_1, \sigma_2 \leq 1$, then for any $f \in \tilde{L}^{\rho_1}_T (\mathcal{B}^{\sigma_1, \frac{2}{2}}_{2,1})$ and $g \in \tilde{L}^{\rho_2}_T (\mathcal{B}^{\sigma_2, \frac{2}{2}}_{2,1})$, we have

$$\|fg\|_{\tilde{L}^p_T (\mathcal{B}^{\sigma_1 + \sigma_2 - \sigma_1, \frac{2}{2}}_{2,1})} \lesssim \|f\|_{\tilde{L}^{\rho_1}_T (\mathcal{B}^{\sigma_1, \frac{2}{2}}_{2,1})} \|g\|_{\tilde{L}^{\rho_2}_T (\mathcal{B}^{\sigma_2, \frac{2}{2}}_{2,1})} .$$

Throughout this paper, $\Phi$ denotes a locally bounded function on $\mathbb{R}^+ \times \mathbb{R}$ which satisfies the following subadditivity

$$\Phi(t, \xi_3) \leq \Phi(t, \xi_3 - \eta_3) + \Phi(t, \eta_3).$$

For any function $f$ in $\mathcal{S}'(0, T) \times \mathcal{S}'(\mathbb{R}^3)$, we define

$$f_\Phi (t, x_h, x_3) = \mathcal{F}^{-1} (e^{\Phi(t, \xi_3)} \hat{f}(t, x_h, \xi_3)) .$$

Let us keep the following fact in mind that the map $f \mapsto f^+$ preserves the norm of $L^p_\sigma (L^q_\sigma)$, where $f^+ (t, x_h, x_3)$ represents the inverse Fourier transform of $|\hat{f}(t, x_h, \xi_3)|$ on vertical variable, defined as

$$f^+ (t, x_h, x_3) \overset{\text{def}}{=} \mathcal{F}^{-1} |\hat{f}(t, x_h, \xi_3)| .$$

Base on these facts, we have the following weighted inequalities as in Lemma 3.4.

**Lemma 3.6.** Let $(\sigma_1, \sigma_2)$ be in $\mathbb{R}^2$. If $\sigma_1 + \sigma_2$ is positive and $\sigma_1, \sigma_2 \leq 1$, then we have for any $f_\Phi \in \mathcal{B}^{\sigma_1, \frac{2}{2}}_{2,1}$ and $g_\Phi \in \mathcal{B}^{\sigma_2, \frac{2}{2}}_{2,1}$,

$$\|fg\|_{\mathcal{B}^{\sigma_1 + \sigma_2 - \sigma_1, \frac{2}{2}}_{2,1}} \lesssim \|f\|_{\mathcal{B}^{\sigma_1, \frac{2}{2}}_{2,1}} \|g\|_{\mathcal{B}^{\sigma_2, \frac{2}{2}}_{2,1}} .$$
Proof. For fixed $k, j$, we have
\[
\| \Delta_{k,j} (T^h T^v (f, g) \Phi) \|_{L^2} 
\lesssim \sum_{|k-k'| \leq 4, |j-j'| \leq 4} \| \mathcal{F} (S^h_{k'-1} S^v_{j'-1} f) (x_h, \cdot) \ast \mathcal{F} (\Delta^h_{k'} \Delta^v_j g) (x_h, \cdot) \|_{L^2}
\]
\[
\lesssim \sum_{|k-k'| \leq 4, |j-j'| \leq 4} \| \mathcal{F} (S^h_{k'-1} S^v_{j'-1} f \Phi) (x_h, \cdot) \ast \| \mathcal{F} (\Delta^h_{k'} \Delta^v_j g \Phi) (x_h, \cdot) \|_{L^2}
\]
\[
\lesssim \sum_{|k-k'| \leq 4, |j-j'| \leq 4} \sum_{|k''| \leq 2, |j''| \leq 2} \| \mathcal{F} (\Delta^h_{k''} \Delta^v_j f \Phi) (x_h, \cdot) \ast \| \mathcal{F} (\Delta^h_{k'} \Delta^v_j g \Phi) (x_h, \cdot) \|_{L^2} \|_{L^2} 2^{k'' j''}.
\]

Using the fact that $f \mapsto f^+$ preserves the norm of $L^2$, we then get by the similar method as in Lemma 3.4 that
\[
\|(T^h T^v (f, g) \Phi)\|_{\overline{B}^s_{2,1}} \lesssim \| f \Phi \|_{\overline{B}^s_{2,1}} \| g \Phi \|_{\overline{B}^s_{2,1}}.
\]

The other terms in (8) can be estimated by the same method and finally, we have
\[
\|(fg)\Phi\|_{\overline{B}^s_{2,1} + \sigma_2 - 1, \frac{1}{2}} \lesssim \| f \Phi \|_{\overline{B}^s_{2,1} + \frac{1}{2}} \| g \Phi \|_{\overline{B}^s_{2,1} + \frac{1}{2}}.
\]

The following lemma is a direct consequence of Lemma 3.6.

Lemma 3.7. Let $\rho \in [1, \infty]$, $(\sigma_1, \sigma_2)$ be in $\mathbb{R}^2$ and $(\rho_1, \rho_2) \in [1, \infty]^2$. Assume that
\[
\frac{1}{\rho} \overset{\text{def}}{=} \frac{1}{\rho_1} + \frac{1}{\rho_2} \leq 1.
\]
If $\sigma_1 + \sigma_2$ is positive and $\sigma_1, \sigma_2 \leq 1$, then we have for any $f \Phi \in \overline{L}^s_{L^r} (\overline{B}^{s_1}_{2,1} \delta)$ and $g \Phi \in \overline{L}^s_{L^r} (\overline{B}^{s_2}_{2,1} \delta)$,
\[
\|(fg)\Phi\|_{\overline{L}^s_{L^r} (\overline{B}^{s_1 + \sigma_2 - 1, \frac{1}{2})}} \lesssim \| f \Phi \|_{\overline{L}^s_{L^r} (\overline{B}^{s_1}_{2,1} \delta)} \| g \Phi \|_{\overline{L}^s_{L^r} (\overline{B}^{s_2}_{2,1} \delta)}.
\]

4. Estimates for the re-scaled system. This section is devoted to obtaining the a priori estimate for the following system
\[
\begin{cases}
\partial_t v^h + v \cdot \nabla v^h - \Delta v^h + \nabla h = 0, \\
\partial_t v^v + v \cdot \nabla v^v - \Delta v^v + c^2 \partial_3 q = 0, \\
\text{div} v = 0, \quad v(0) = v_0(x).
\end{cases}
\]

The pressure $q$ can be computed by the formula
\[
-\Delta q = \sum_{i,j} \partial_i \partial_j (v^i v^j).
\]

Due to the divergence free condition, the pressure can be split into the following three parts.
Lemma 4.1. There exist two constants which is the key bilinear estimate.

\begin{equation}
\begin{cases}
q^1 = (-\Delta_\varepsilon)^{-1} \sum_{i,j=1,2} \partial_i \partial_j (v^i v^j), \\
q^2 = 2(-\Delta_\varepsilon)^{-1} \sum_{i=1,2} \partial_i \partial_3 (v^i v^3), \\
q^3 = -2(-\Delta_\varepsilon)^{-1} \partial_3 (v^3 \text{div}_h v^h).
\end{cases}
\end{equation}

It is worthwhile to note that there will lose one vertical derivative owing to the term $v^3 \partial_3 v^h$ and pressure terms $q^2, q^3$ which appear in the equation on $v^h$. Thus, we assume that the initial data is analytic in the vertical variable. This method was introduced in [5] to compensate the losing derivative in $x_3$. Therefore, we introduce two key quantities which we want to control in order to obtain the global bound of $v$ in a certain space. We define the function $\theta(t)$ by

$$\theta(t) = \int_0^t \|v^h_\Phi(\tau)\|_{B^{1/4}_{2,1}} d\tau.$$  \hfill (13)

For any $\delta \in (0, 1)$, denote

$$\Psi(t) = \|v^h_\Phi\|_{L^\infty_t(B^{s_1/2}_{2,1})} + \|\partial_3 v^h_\Phi\|_{L^\infty_t(B^{s_1/2}_{2,1})} + \|v^h_\Phi\|_{L^2_t(B^{s_2/2}_{2,1})},$$

$$\Psi(0) = \|v^0_\Phi\|_{B^{s_1/2}_{2,1}} + \|\partial_3 v^0_\Phi\|_{B^{s_2/2}_{2,1}},$$

where $v^h_\Phi$ is defined as in (10). The phase function $\Phi(t, D_3)$ is defined by

$$\Phi(t, \xi_3) = (\alpha - \lambda \theta(t))|\xi_3|$$

for some $\lambda$ that will be chosen later on, $\alpha$ is a positive number. Obviously, we need to ensure that $\theta(t) < \frac{\alpha}{\lambda}$ which implies the subadditivity of $\Phi$.

The following lemma provides the estimate of $v^h_\Phi$ in the anisotropic Besov spaces, which is the key bilinear estimate.

**Lemma 4.1.** There exists two constants $C_1$ and $\lambda_0$ such that for any $\lambda > \lambda_0$ and $t$ satisfying $\theta(t) \leq \frac{\alpha}{2\lambda}$, we have

$$\Psi(t) \leq C_1 \Psi(0) + C_1 \left(t^{\frac{3}{2}} + t^{\frac{3}{2} + \frac{1}{2}}\right) \Psi(t)^2.$$  \hfill (14)

For the horizontal component $v^h$, it is unavoidable that we will lose one derivative in vertical direction. Formally, the terms $v^3 \partial_3 v^h$, $\nabla_h q^2$ and $\nabla_h q^3$ are the main bad elements.

**4.1. Estimates on the horizontal component $v^h$.** According to the definition of $v^h_\Phi$, we find that in each dyadic block, it verifies the following equation

$$\Delta_{k,j} v^h_\Phi(t, x) = e^{(\Delta_\varepsilon + \Phi(t, D_3))} \Delta_{k,j} v^h_0$$

$$- \int_0^t e^{(t-\tau)\Delta_\varepsilon - \lambda |D_3|} f^j_\varepsilon(t') dt' \Delta_{k,j} (v \cdot \nabla v^h_\Phi(\tau)) d\tau$$

$$- \int_0^t e^{(t-\tau)\Delta_\varepsilon - \lambda |D_3|} f^j_\varepsilon(t') dt' \nabla_h \Delta_{k,j} q^h_\Phi(\tau) d\tau.$$  \hfill (15)

Then, taking the $L^2$ norm, we deduce that

\begin{equation}
\|\Delta_{k,j} v^h_\Phi\|_{L^2} \lesssim e^{-c(2^k + \epsilon^2 2^j t)} \|\Delta_{k,j} e^{\alpha |D_3|} v^h_0\|_{L^2}
\end{equation}

$$+ \int_0^t e^{-c(2^k + \epsilon^2 2^j)(t-\tau)} e^{-c\lambda 2^j} f^j_\varepsilon(t') dt' \|\Delta_{k,j} (v \cdot \nabla v^h_\Phi)\|_{L^2} d\tau.$$  \hfill (16)
Thus, we can get by Lemma 3.7 that
\begin{align}
\int e^{-c(2^{kh}+c^{2}2^{2j})}t e^{-\text{c}^{\Lambda^{2j}} f'_x}{\text{d}t'} \mathcal{V}_{h} \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 dt &
\end{align}
Using H"older inequality and interpolation, we have
\begin{align}
I = I_1 + I_2 + I_3.
\end{align}
We first estimate the linear term $I_1$. In fact, we have
\begin{align}
\|e^{-c(2^{kh}+c^{2}2^{2j})}t \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 & \leq c_{k,j} 2^{-k\delta} 2^{-\frac{1}{2}} \mathcal{L}^2 \|e^{\text{c}^{\Lambda^{2j}} f'_x} \mathcal{V}_{h} \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 \|_{B_{2,1}^{\frac{1}{2}}},
\end{align}
and
\begin{align}
\|e^{-c(2^{kh}+c^{2}2^{2j})}t \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 & \leq 2^{-2k} c_{k,j} 2^{-k\delta} 2^{-\frac{1}{2}} \mathcal{L}^2 \|e^{\text{c}^{\Lambda^{2j}} f'_x} \mathcal{V}_{h} \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 \|_{B_{2,1}^{\frac{1}{2}}},
\end{align}
where $\{c_{k,j}\}_{(k,j)\in \mathbb{Z}^2}$ is a two dimensional sequence satisfying $\|c_{k,j}\|_{L^1(\mathbb{Z}^2)} = 1$.

The term $I_2$ can be rewritten as
\begin{align}
I_2 & \leq \int e^{-c(2^{kh}+c^{2}2^{2j})}t e^{-\text{c}^{\Lambda^{2j}} f'_x}{\text{d}t'} \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 dt
\end{align}
For $I_{21}$, by Young’s inequality, we have
\begin{align}
\|I_{21}\|_{L_t^\infty} & \leq \|e^{-c(2^{kh}+c^{2}2^{2j})}t e^{-\text{c}^{\Lambda^{2j}} f'_x}{\text{d}t'} \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 dt\|_{L_t^\infty}
\end{align}
Thus, we can get by Lemma 3.7 that
\begin{align}
\|I_{21}\|_{L_t^\infty} & \leq c_{k,j} 2^{-k\delta} 2^{-\frac{1}{2}} \|I_{21}\|_{L_t^\infty} \|e^{\text{c}^{\Lambda^{2j}} f'_x} \mathcal{V}_{h} \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 \|_{B_{2,1}^{\frac{1}{2}}},
\end{align}
Using H"older inequality and interpolation, we have
\begin{align}
\|v_{0}^h\|_{L_t^1(\mathbb{B}_{2,1}^{\frac{1}{2}})} \leq \|v_{0}^h\|_{L_t^1(\mathbb{B}_{2,1}^{\frac{1}{2}})}
\end{align}
and
\begin{align}
\|v_{0}^h\|_{L_t^1(\mathbb{B}_{2,1}^{\frac{1}{2}})} \leq \|v_{0}^h\|_{L_t^1(\mathbb{B}_{2,1}^{\frac{1}{2}})} + \|v_{0}^h\|_{L_t^1(\mathbb{B}_{2,1}^{\frac{1}{2}})},
\end{align}
where the exponents satisfy the conditions $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{p'} = \frac{1}{2}$.

Then one can deduce that
\begin{align}
\|I_{21}\|_{L_t^\infty} & \leq c_{k,j} 2^{-k\delta} 2^{-\frac{1}{2}} \|\mathcal{V}_{h} \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 \|_{B_{2,1}^{\frac{1}{2}}} \|\mathcal{V}_{h} \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 \|_{B_{2,1}^{\frac{1}{2}}},
\end{align}
and
\begin{align}
\|I_{21}\|_{L_t^\infty} & \leq c_{k,j} 2^{-k\delta} 2^{-\frac{1}{2}} \|\mathcal{V}_{h} \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 \|_{B_{2,1}^{\frac{1}{2}}} \|\mathcal{V}_{h} \mathcal{D}_{b} \mathcal{Q}_{b} \mathcal{L}^2 \|_{B_{2,1}^{\frac{1}{2}}},
\end{align}
(20)
Similarly, we can obtain that
\[
\|I_{21}\|_{L^1_t} \lesssim c_{k,j} 2^{-k(d+2)} 2^{-\frac{1}{2}j} \|v^h \cdot \nabla_h v^h\|_{L^1_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} \\
\lesssim c_{k,j} 2^{-k(d+2)} 2^{-\frac{1}{2}j} \|v^h\|_{L^{\infty}_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} \|v^h\|_{L^1_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} \\
\lesssim c_{k,j} 2^{-k(d+2)} 2^{-\frac{1}{2}j} \|v^h\|_{L^{\infty}_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} \|v^h\|_{L^1_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} (\|v^h\|_{L^{\infty}_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} + \|v^h\|_{L^1_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})}) t^\frac{k}{H}.
\]  
(21)

Now let us estimate \(I_{22}\). By Lemma 3.6, we have
\[
|I_{22}| \leq \int_0^t e^{-c(2^{2k}+2^{2j})(t-\tau)} e^{-c\lambda t^2} f^t_s \vartheta(t')dt' \|\Delta_{k,j}(v^3 \partial_3 v^h)\|_{L^2} dt \tau \\
\lesssim c_{k,j} 2^{-k\delta 2} 2^{-\frac{1}{2}j} \int_0^t e^{-c(2^{2k} + 2^{2j})(t-\tau)} e^{-c\lambda t^2} f^t_s \vartheta(t')dt' \|v^3\|_{L^\infty_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} \|v^3\|_{L^1_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} dt.
\]

Hence, by Young’s inequality, we have
\[
\|I_{22}\|_{L^2} \lesssim c_{k,j} 2^{-k\delta 2} 2^{-\frac{1}{2}j} \|v^3\|_{L^\infty_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} \|v^3\|_{L^1_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})},
\]
\[
\|I_{22}\|_{L^1_t} \leq c_{k,j} 2^{-k(d+2)} 2^{-\frac{1}{2}j} \|v^3\|_{L^\infty_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} \|v^3\|_{L^1_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})}.
\]  
(22)

Now we are left with the study of the pressure term \(I_3\). The pressure can be split into \(q = q^1 + q^2 + q^3\) with \(q^1, q^2, q^3\) defined in (12). For convenience, we denote that
\[
I_{31} = \int_0^t e^{-c(2^{2k}+2^{2j})(t-\tau)} e^{-c\lambda t^2} f^t_s \vartheta(t')dt' \|\nabla_h \Delta_{k,j} q_{\phi}^1\|_{L^2} dt \tau, \\
I_{32} = \int_0^t e^{-c(2^{2k}+2^{2j})(t-\tau)} e^{-c\lambda t^2} f^t_s \vartheta(t')dt' \|\nabla_h \Delta_{k,j} q_{\phi}^2\|_{L^2} dt \tau, \\
I_{33} = \int_0^t e^{-c(2^{2k}+2^{2j})(t-\tau)} e^{-c\lambda t^2} f^t_s \vartheta(t')dt' \|\nabla_h \Delta_{k,j} q_{\phi}^3\|_{L^2} dt \tau.
\]

Hence, using the fact that \((-\Delta_c)^{-1} \partial_i \partial_j\) is a bounded operator applied for frequency localized functions in \(L^2\) when \(i, j = 1, 2\), we get
\[
\|\nabla_h \Delta_{k,j} q_{\phi}^k\|_{L^2} \lesssim \|\Delta_{k,j}(v^h \cdot \nabla_h v^h)\|_{L^2}.
\]

By the same method as in the estimate of \(I_{21}\), we have
\[
\|I_{31}\|_{L^{\infty}_t} \lesssim c_{k,j} 2^{-k\delta 2} 2^{-\frac{1}{2}j} \|v^h\|_{L^{\infty}_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} \left(\|v^h\|_{L^{\infty}_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} + \|v^h\|_{L^1_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})}\right) t^\frac{k}{H}, \\
\|I_{31}\|_{L^1_t} \lesssim c_{k,j} 2^{-k(d+2)} 2^{-\frac{1}{2}j} \|v^h\|_{L^{\infty}_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} \left(\|v^h\|_{L^{\infty}_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})} + \|v^h\|_{L^1_t(\tilde{B}^{\frac{d}{2} + \frac{1}{2}})}\right) t^\frac{k}{H}.
\]  
(23)

Noting that
\[
\nabla_h q^2 = 2(-\Delta_c)^{-1} \nabla_h \partial_i (v^3 \partial_3 v^h - v^h \text{div}_h v^h),
\]
and as the estimate of (23), we have
\[
\|I_{32}\|_{L^\infty_t L^2_x} \lesssim c_{k,j} 2^{-k_3} 2^{-\frac{1}{2}j} \|\partial_3 v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} \|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})} \\
+ c_{k,j} 2^{-k_3} 2^{-\frac{1}{2}j} \left(\|v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} + \|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})}\right) \|v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} t^\frac{3}{2},
\]
\[
\|I_{32}\|_{L^1_t} \lesssim c_{k,j} 2^{-k(\delta+2)} 2^{-\frac{1}{2}j} \|\partial_3 v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} \|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})} \\
+ c_{k,j} 2^{-k(\delta+2)} 2^{-\frac{1}{2}j} \left(\|v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} + \|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})}\right) \|v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} t^\frac{3}{2}.
\]

Similarly, by Hölder and interpolation inequality, we have
\[
\|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})} \lesssim \|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})} t^{\frac{3}{2}},
\]
and
\[
\|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})} \lesssim \|v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} + \|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})},
\]
where the exponents satisfy the conditions $\frac{1}{q} + \frac{1}{\sigma} = 1$ and $\frac{1}{q} = \frac{1+\delta}{2}$. Using
\[
\nabla_h q^3 = 2(-\Delta)^{-1} \nabla_h (\text{div}_h v^h \text{div}_h v^h - v^3 \partial_3 \text{div}_h v^h),
\]
we write $I_{33} = I^1_{33} + I^2_{33}$ and first estimate that
\[
|I^1_{33}| \lesssim \int_0^t e^{-c(2^{k_3} + 2^2)(t-\tau)} e^{-c\lambda_2^j f^j_1 \theta(t')dt'} \\
\cdot \|\Delta_{k,j} (-\Delta)^{-1} \nabla_h (\text{div}_h v^h \text{div}_h v^h)_\Phi\|_{L^2} d\tau \\
\lesssim c_{k,j} 2^{-k_3} 2^{-\frac{1}{2}j} \int_0^t e^{-c(2^{k_3} + 2^2)(t-\tau)} e^{-c\lambda_2^j f^j_1 \theta(t')dt'} \\
\cdot \|\text{div}_h v^h \text{div}_h v^h_\Phi\|_{B^{1,1}} t^{-\frac{1}{2}} d\tau.
\]
Thus, we have
\[
\|I^1_{33}\|_{L^\infty_t L^2_x} \lesssim c_{k,j} 2^{-k_3} 2^{-\frac{1}{2}j} \|v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} \|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})},
\]
\[
\|I^1_{33}\|_{L^1_t} \lesssim c_{k,j} 2^{-k(\delta+2)} 2^{-\frac{1}{2}j} \|v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} \|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})}.
\]

For the term $I^2_{33}$,
\[
|I^2_{33}| \lesssim \int_0^t e^{-c(2^{k_3} + 2^2)(t-\tau)} e^{-c\lambda_2^j f^j_1 \theta(t')dt'} \|\Delta_{k,j} (-\Delta)^{-1} \nabla_h (v^3 \partial_3 \text{div}_h v^h)_\Phi\|_{L^2} d\tau \\
\lesssim c_{k,j} 2^{-k_3} 2^{-\frac{1}{2}j} \int_0^t e^{-c(2^{k_3} + 2^2)(t-\tau)} e^{-c\lambda_2^j f^j_1 \theta(t')dt'} \|v^3 \partial_3 \text{div}_h v^h_\Phi\|_{B^{1,1}} t^{-\frac{1}{2}} d\tau.
\]
Then one can get that
\[
\|I^2_{33}\|_{L^\infty_t L^2_x} \lesssim c_{k,j} 2^{-k_3} 2^{-\frac{1}{2}j} \|\partial_3 v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} \|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})},
\]
\[
\|I^2_{33}\|_{L^1_t} \lesssim c_{k,j} 2^{-k(\delta+2)} 2^{-\frac{1}{2}j} \|\partial_3 v^h_\Phi\|_{L^\infty_t(\dot{B}^{\delta+2}_{2,1})} \|v^h_\Phi\|_{L^1_t(\dot{B}^{\delta+2}_{2,1})}.
\]
Now we are going to estimate the $L_t^\infty(B_{2,1}^{\frac{3}{2}})$ norm of $\partial_3 v_h^h$. According to (16), we find that in each dyadic block $\partial_3 v_h^h$ verifies

$$\Delta_{k,j}\partial_3 v_h^h(t,x) = e^{\tau^2 \alpha^3 \Phi(t,D_3)} \Delta_{k,j}\partial_3 v_0^h$$

Thus, we can get, by Lemma 3.7, that

$$\|\Delta_{k,j}\partial_3 v_h^h\|_{L_2} \leq e^{-c(2k^2+c^2)}\|\Delta_{k,j}e^{\alpha|D_3|}\partial_3 v_0^h\|_{L_2} + \int_0^t e^{-(2k^2+c^2)} \|\Delta_{k,j}e^{\alpha|D_3|}\partial_3 v_0^h\|_{L_2} dt$$

$$+ \int_0^t e^{-c(2k^2+c^2)} f(t') \|\Delta_{k,j}\partial_3(v \cdot \nabla v^h)\|_{L_2} dt$$

Then, taking the $L^2$ norm on both sides of (27), we have

$$\|\Delta_{k,j}\partial_3 v_h^h\|_{L_2} \leq e^{-c(2k^2+c^2)}\|\Delta_{k,j}e^{\alpha|D_3|}\partial_3 v_0^h\|_{L_2}$$

$$+ \int_0^t e^{-c(2k^2+c^2)} f(t') \|\Delta_{k,j}\partial_3(v \cdot \nabla v^h)\|_{L_2} dt$$

$$\def \defeq I_4 + I_5 + I_6.$$  

For fixed $k,j$, the $L_t^\infty$ norm of $I_4$ can be bounded by

$$\|e^{-c(2k^2+c^2)}\|_{L_2} \|\Delta_{k,j}e^{\alpha|D_3|}\partial_3 v_0^h\|_{L_2} \lesssim \|\Delta_{k,j}e^{\alpha|D_3|}\partial_3 v_0^h\|_{L_2}$$

$$\def \defeq c_{k,j}2^{-k^2}2^{-\frac{1}{2}}\|e^{\alpha|D_3|}\partial_3 v_0^h\|_{L_2^{\alpha,\frac{1}{2}}}.$$  

The term $I_5$ can be rewritten as

$$I_5 \lesssim \int_0^t e^{-c(2k^2+c^2)}((t-\tau)) \|\Delta_{k,j}\partial_3(v^h \cdot \nabla v^h)\|_{L_2} dt$$

$$\def \defeq I_{51} + I_{52}.$$  

By Young’s inequality, we can deduce that

$$I_{51} L_t^\infty \lesssim \int_0^t e^{-c(2k^2+c^2)}((t-\tau)) \|\Delta_{k,j}(\partial_3 v^h \cdot \nabla v^h)\|_{L_2} dt$$

$$\|\Delta_{k,j}(\partial_3 v^h \cdot \nabla v^h)\|_{L_2} \lesssim c_{k,j}2^{-k^2}2^{-\frac{1}{2}}\|\partial_3 v^h\|_{L_2^{\alpha,\frac{1}{2}}}$$

Thus, we can get by Lemma 3.7 that

$$\|I_{51}\|_{L_t^\infty} \lesssim c_{k,j}2^{-k^2}2^{-\frac{1}{2}}\|\partial_3 v_0^h\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})} \|v_h^h\|_{L_t^\infty(B_{2,1}^{\frac{3}{2}})}.$$
By Hölder and interpolation inequality, we have
\[ \|v_0^h\|_{L^2_t(B^{s+1}_{r, r})} \lesssim \|v_0^h\|_{L^1_t(B^{s+1}_{r, r})} t^{\frac{s}{r} - \frac{1}{p}} \]
and
\[ \|v_0^h\|_{L^1_t(B^{s+1}_{r, r})} \lesssim \|v_0^h\|_{L^\infty_t(B^{s+1}_{r, r})} + \|v_0^h\|_{L^1_t(B^{s+2+\frac{1}{2}}_{r, r})} \]
where the exponents satisfy the conditions \( \frac{1}{p} + \frac{1}{r} = \frac{1}{2} \) and \( \frac{1}{p} = \frac{\delta}{2} \).

Then one can deduce that
\[ \|I_{51}\|_{L^\infty_t} \lesssim c_{k,j} 2^{-k\delta - 2 - \frac{1}{2}} \|\partial_3 v^h_\Phi\|_{L^\infty_t(B^{s}_{2, r})} \left( \|v_0^h\|_{L^1_t(B^{s+1}_{r, r})} + \|v_0^h\|_{L^1_t(B^{s+2+1}_{r, r})} \right) t^{\frac{s}{2}}. \]  
(30)

Now let us estimate \( I_{52} \). By Lemma 3.6, we have

\[ |I_{52}| \leq \int_0^t e^{-c(2^{2k} + c^2q^2)(t - \tau)} e^{-c\lambda 2^j \int_0^\tau \dot{\theta}(t') dt'} \|\Delta_{k,j} \partial_3 (v^3 \partial_3 v^h)\|_{L^2_t} d\tau \]
\[ \lesssim c_{k,j} 2^{-k\delta - 2 - \frac{1}{2}} \int_0^t e^{-c(2^{2k} + c^2q^2)(t - \tau)} e^{-c\lambda 2^j \int_0^\tau \dot{\theta}(t') dt'} 2^j \|\Delta_{k,j} \partial_3 (v^3 \partial_3 v^h)\|_{L^{2}_{2, r}} d\tau \]
\[ \lesssim c_{k,j} 2^{-k\delta - 2 - \frac{1}{2}} \int_0^t e^{-c\lambda 2^j \int_0^\tau \dot{\theta}(t') dt'} 2^j \|\Delta_{k,j} \partial_3 (v^3 \partial_3 v^h)\|_{L^{2}_{2, r}} d\tau \]
\[ \lesssim \frac{1}{\lambda} c_{k,j} 2^{-k\delta - 2 - \frac{1}{2}} \|\partial_3 v^h_\Phi\|_{L^{2}_{2, r}} \|\nabla_h \Delta_{k,j} \partial_3 (\partial_3 (v^3 \partial_3 v^h))\|_{L^2_t} d\tau. \]

As for \( I_6 \), for convenience, we denote that

\[ I_{61} = \int_0^t e^{-c(2^{2k} + c^2q^2)(t - \tau)} e^{-c\lambda 2^j \int_0^\tau \dot{\theta}(t') dt'} \|\nabla_h \Delta_{k,j} \partial_3 q^1_\Phi\|_{L^2_t} d\tau, \]
\[ I_{62} = \int_0^t e^{-c(2^{2k} + c^2q^2)(t - \tau)} e^{-c\lambda 2^j \int_0^\tau \dot{\theta}(t') dt'} \|\nabla_h \Delta_{k,j} \partial_3 q^2_\Phi\|_{L^2_t} d\tau, \]
\[ I_{63} = \int_0^t e^{-c(2^{2k} + c^2q^2)(t - \tau)} e^{-c\lambda 2^j \int_0^\tau \dot{\theta}(t') dt'} \|\nabla_h \Delta_{k,j} \partial_3 q^3_\Phi\|_{L^2_t} d\tau. \]

By the same method as in the estimate of \( I_5 \), we have

\[ \|I_{61}\|_{L^\infty_t} + \|I_{62}\|_{L^\infty_t} \lesssim c_{k,j} 2^{-k\delta - 2 - \frac{1}{2}} \|\partial_3 v^h_\Phi\|_{L^\infty_t(B^{s+1}_{2, r})} \left( \|v_0^h\|_{L^1_t(B^{s+1}_{r, r})} + \frac{1}{\lambda} \right). \]  
(31)

Finally, \( I_{63} \) can be estimate as follows

\[ |I_{63}| \leq \int_0^t \|\Delta_{k,j} (-\Delta_e)^{-1} \nabla_h (\text{div}_h v^h \partial_3 \text{div}_h v^h)\Phi\|_{L^2_t} d\tau \]
\[ + \int_0^t e^{-c\lambda 2^j \int_0^\tau \dot{\theta}(t') dt'} \|\Delta_{k,j} (-\Delta_e)^{-1} \nabla_h \partial_3 (v^3 \partial_3 \text{div}_h v^h)\Phi\|_{L^2_t} d\tau \]
\[ \lesssim c_{k,j} 2^{-k\delta - 2 - \frac{1}{2}} \int_0^t \|\text{div}_h v^h \partial_3 \text{div}_h v^h\Phi\|_{B^{s-1+\frac{1}{2}}_{2, r}} d\tau \]
\[ + c_{k,j} 2^{-k\delta - 2 - \frac{1}{2}} \int_0^t e^{-c\lambda 2^j \int_0^\tau \dot{\theta}(t') dt'} 2^j \|\partial_3 (v^3 \partial_3 \text{div}_h v^h)\Phi\|_{B^{s-1+\frac{1}{2}}_{2, r}} d\tau. \]
Thus, we can get that
\[
\| I_{\delta k,j} \|_{L^\infty_t} \lesssim c_{k,j} 2^{-k \delta} 2^{-\frac{3j}{2}} \| \partial_3 v^h_k \|_{L^\infty_t (B_{2,1}^\delta)} \| v^h_k \|_{L^2_t (B_{2,1}^{\delta + \frac{1}{2}})} + \frac{1}{\lambda} c_{k,j} 2^{-k \delta} 2^{-\frac{3j}{2}} \| \partial_3 v^h_k \|_{L^\infty_t (B_{2,1}^\delta)} .
\]  
(32)

Together with the above estimates (18)-(32), we obtain that
\[
\| v^h_k \|_{L^\infty_t (B_{2,1}^\delta)} + \| \partial_3 v^h_k \|_{L^\infty_t (B_{2,1}^{\delta + \frac{1}{2}})} + \| v^h_k \|_{L^2_t (B_{2,1}^{\delta + 2, \frac{1}{2}})} \lesssim \Psi(0) + \frac{1}{\lambda} \Psi(t) + \left( t^{\frac{3}{2}} + t^{\frac{3j}{4}} \right) \Psi(t)^2 .
\]  
(33)

4.2. Estimates on the vertical component $v^3$. We begin this part by studying the equation of $v^3$, which is stated as follows
\[
\partial_t v^3 - \Delta v^3 + v \cdot \nabla v^3 + \epsilon^2 \partial_3 q = 0.
\]

Observing that in the above equation, one can expect that there is no loss of derivative in vertical direction. More precisely, due to divergence free condition, the nonlinear term $v^3 \partial_3 v^3$ can be rewritten as $-v^3 \text{div}_h v^h$. Thus the estimate on $v^3$ is different from $v^h$.

Applying the anisotropic dyadic decomposition operator $\Delta_{k,j}$ to the equation of $v^3$, then in each dyadic block, $v^3$ satisfies
\[
\partial_t \Delta_{k,j} v^3 - \Delta_k \Delta_{k,j} v^3 = -\Delta_k (v^h \cdot \nabla_h v^3) + \Delta_{k,j} (v^3 \text{div}_h v^h) - \epsilon^2 \Delta_{k,j} \partial_3 q.
\]  
(34)

Let us define $G \overset{\text{def}}{=} v^h \cdot \nabla_h v^3 - v^3 \text{div}_h v^h$. Writing the solution of (36) in terms of the Fourier transform in vertical variable, we get that
\[
\mathcal{F}(\Delta^3 v^3_k (t, x_h, \xi_3)) = \mathcal{F}(e^{t \Delta_k \Delta_{k,j} v^3_0_k} (x_h, \xi_3) e^{\alpha (|\xi|)}
+ \int_0^t e^{(t-\tau) \Delta_k \Delta_{k,j} e^{\alpha (|\xi|)}} e^{-\lambda (|\xi|) } \int_0^\tau \dot{\theta}(\tau') d\tau' \mathcal{F}(\Delta_{k,j} G (\tau, x_h, \xi_3)) d\tau
+ \epsilon^2 \int_0^t e^{(t-\tau) \Delta_k \Delta_{k,j} e^{\alpha (|\xi|)}} e^{-\lambda (|\xi|) } \int_0^\tau \dot{\theta}(\tau') d\tau' \mathcal{F}(\Delta_{k,j} \partial_3 q (\tau, x_h, \xi_3)) d\tau.
\]  
(35)

Taking the $L^2(\mathbb{R}^3)$ norm on both sides of (35), we can obtain that
\[
\| \Delta_{k,j} v^3_0 \|_{L^2} \lesssim e^{-c \epsilon 2^{k \lambda} (t-\tau)} \| \Delta_{k,j} e^{\alpha |\xi|} v^3_0 \|_{L^2} + \int_0^t e^{-c \epsilon 2^{k \lambda} (t-\tau)} \| \Delta_{k,j} G (\tau) \|_{L^2} d\tau
+ \epsilon^2 \int_0^t e^{-c \epsilon 2^{k \lambda} (t-\tau)} \| \Delta_{k,j} \partial_3 q \|_{L^2} d\tau.
\]  
(36)

By Young’s inequality, one can infer that
\[
\| \Delta_{k,j} v^3_0 \|_{L^\infty_t (L^2)} + 2^{2k} \| \Delta_{k,j} v^3_0 \|_{L^2_t (L^2)} \lesssim \| \Delta_{k,j} e^{\alpha |\xi|} v^3_0 \|_{L^2} + \| \Delta_{k,j} G \|_{L^2_t (L^2)} + \epsilon^2 \| \Delta_{k,j} \partial_3 q \|_{L^2_t (L^2)} .
\]  
(37)

Multiplying both sides of (37) by $2^{k \lambda} 2^{\frac{3j}{2}}$ and taking the sum over $k, j$, we finally get
\[
\| v^3_k \|_{L^\infty_t (B_{2,1}^{\delta + \frac{1}{2}})} + \| v^3_k \|_{L^2_t (B_{2,1}^{\delta + 2, \frac{1}{2}})} \lesssim \| e^{\alpha |\xi|} v^3_0 \|_{B_{2,1}^{\delta + \frac{1}{2}}} + \| G \|_{L^2_t (B_{2,1}^{\delta + \frac{1}{2}})} + \epsilon^2 \| \partial_3 q \|_{L^2_t (B_{2,1}^{\delta + \frac{1}{2}})} .
\]  
(38)
Then Lemma 3.7 implies that
\[ \|G_\Phi\|_{L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})} \lesssim \|v_\Phi\|_{L^\infty_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})} \|v_\Phi\|_{L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})}. \]

For the pressure term, due to the fact that \( \epsilon (-\Delta)\cdot \partial \partial_3 \) and \( \epsilon^2 (-\Delta)\cdot \partial_3^2 \) are bounded operators in \( L^2 \) for \( i = 1, 2 \), we get by applying Lemma 3.7 that
\[ \epsilon^2 \|\partial_3 q_\Phi\|_{L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})} \lesssim \|v_\Phi\|_{L^\infty_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})} \|v_\Phi\|_{L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})}. \]

By the result of interpolation inequalities above, we infer that
\[ \|v_3^k\|_{L^\infty_t(B^{\frac{d}{2}}_{2,1})} + \|v_3^j\|_{L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})} \lesssim \Psi(0) + \Psi(t)^2 t^3. \]

Together with (33), we finally get that
\[ \Psi(t) \lesssim \Psi(0) + \frac{1}{\lambda} \Psi(t) + \left( t^\frac{3}{4} + t^\frac{7}{4} \right) \Psi(t)^2. \]

This completes the proof of Lemma 4.1 by choosing \( \lambda \) large enough.

5. Estimates for \( \theta(t) \). In the above section, we have used the fact that \( \Phi(t) \) is a subadditivity function. This means we should ensure that \( \theta(t) < \frac{\lambda}{4} \). Thus, it is sufficient to prove that for any time \( t \), \( \theta(t) \) is a small quantity. By the definition of \( \theta(t) \), naturally, we assume that \( e^{\alpha[D_3]}v_0^3 \) belongs to \( B^{\frac{d}{2}+\frac{1}{2}}_{2,1} \). According to the property of heat kernel, then we can get the bounds for \( v_3^k \) in \( L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1}) \). However, we can not get the closed estimate for \( v_3^j \) in \( L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1}) \). Our observation is to add an extra term \( \epsilon v_h \) under the same norm which is hidden in the pressure term \( \epsilon^2 \partial_3 q^1 \).

Hence, we first denote that
\[ X(t) = \epsilon \|v_3^k\|_{L^2_t(B^{\frac{d}{2}}_{2,1})} + \epsilon \|v_3^j\|_{L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})}, \]
\[ Y(t) = \|v_3^k\|_{L^2_t(B^{\frac{d}{2}}_{2,1})} + \|v_3^j\|_{L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})}, \]
\[ X_0 = \epsilon \|e^{\alpha[D_3]}v_0^3\|_{B^{\frac{d}{2}+\frac{1}{2}}_{2,1}}, \quad Y_0 = \|e^{\alpha[D_3]}v_0^3\|_{B^{\frac{d}{2}+\frac{1}{2}}_{2,1}}. \]

In order to get the desired estimates, it suffices to prove the following lemma.

**Lemma 5.1.** There exists a constant \( C_2 \) such that for any \( \lambda \) and \( t \) satisfying \( \theta(t) \leq \frac{\lambda}{2\pi} \), for any \( M_1 > 0 \), there exists \( N_1 \in \mathbb{Z}^+ \), we have
\[ X(t) + Y(t) \leq C_2 M + C_2 (2^{N_1} t^\frac{3}{4} + 2^{2N_1} t) \left( X_0 + Y_0 \right) + C_2 t^\frac{3}{4} \left( X(t) + Y(t) \right). \]

**Proof.** We apply the same method as in the above section to prove \( Y(t) \). For (36), we use Young’s inequality to obtain
\[ \|\Delta_{k,j} v_3^k\|_{L^2_t(L^2)} \lesssim \|e^{-c^22^{2k}t}\|_{L^2_t} \|\Delta_{k,j} e^{\alpha[D_3]} v_3^k\|_{L^2} + 2^{-k} \|\Delta_{k,j} G_\Phi\|_{L^1_t(L^2)} \]
\[ + \epsilon^2 2^{-k} \|\Delta_{k,j} \partial_3 q_\Phi\|_{L^1_t(L^2)}. \]

Multiplying both sides of (42) by \( 2^{k\delta}2^{-\frac{d}{2}} \) and taking the sum over \( k, j \), we can get that
\[ \|v_3^k\|_{L^2_t(B^{\frac{d}{2}}_{2,1})} \lesssim \sum_{k,j} 2^{k\delta}2^{-\frac{d}{2}} \|e^{-c^22^{2k}t}\|_{L^2_t} \|\Delta_{k,j} e^{\alpha[D_3]} v_3^k\|_{L^2} \]
\[ + \|G_\Phi\|_{L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})} + \epsilon^2 \|\partial_3 q_\Phi\|_{L^1_t(B^{\frac{d}{2}+\frac{1}{2}}_{2,1})}. \]
Similarly, we can get that
\[
\|v_0^3\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} \lesssim \sum_{k,j} 2^{(\delta+1)\frac{j}{2}} 2^{\frac{j}{2}} \|e^{-c2^{2k}t} \|_{L_t^1} \|\Delta_{k,j} e^{\alpha|D_3|} v_0^3\|_{L^2} \\
+ \|G\Phi\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} + \epsilon^2 \|\partial_3q\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})}.
\]  
(44)

Concerning the first term in (43) and (44), we shall split \(v_0\) into a small part in \(\dot{B}_{2,1}^{s-1,\frac{1}{2}}\) and a large part which could be controlled by \(Y_0\). For some fixed \(M_1 > 0\), there exits some positive real number \(N_1\), such that
\[
\sum_{k>N_1} 2^{(\delta-1)\frac{j}{2}} \|\Delta_{k,j} e^{\alpha|D_3|} v_0\|_{L^2} \leq M_1.
\]  
(45)

Thus, we have
\[
\sum_{k,j} 2^{\delta\frac{j}{2}} 2^{\frac{j}{2}} \|e^{-c2^{2k}t} \|_{L_t^1} \|\Delta_{k,j} e^{\alpha|D_3|} v_0^3\|_{L^2} \\
\leq \sum_{k \leq N_1, j} 2^{\delta\frac{j}{2}} 2^{\frac{j}{2}} \|\Delta_{k,j} e^{\alpha|D_3|} v_0^3\|_{L^2} + \sum_{k>N_1, j} 2^{\delta\frac{j}{2}} 2^{\frac{j}{2}} \|\Delta_{k,j} e^{\alpha|D_3|} v_0^3\|_{L^2} \\
\leq 2^{N_1} t Y_0 + M_1.
\]  
(46)

and
\[
\sum_{k,j} 2^{(\delta+1)\frac{j}{2}} 2^{\frac{j}{2}} \|e^{-c2^{2k}t} \|_{L_t^1} \|\Delta_{k,j} e^{\alpha|D_3|} v_0^3\|_{L^2} \\
\leq \sum_{k \leq N_1, j} 2^{(\delta+1)\frac{j}{2}} 2^{\frac{j}{2}} \|\Delta_{k,j} e^{\alpha|D_3|} v_0^3\|_{L^2} + \sum_{k>N_1, j} 2^{(\delta+1)\frac{j}{2}} 2^{\frac{j}{2}} \|\Delta_{k,j} e^{\alpha|D_3|} v_0^3\|_{L^2} \\
\leq 2^{2N_1} t Y_0 + M_1.
\]  
(47)

According to Lemma 3.7, we can obtain the estimates of nonlinear terms that
\[
\|\text{div}_h(v^3v^h)\Phi\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} \lesssim \|v_0^3\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} \|\Phi\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})},
\]
\[
\|(v^3\text{div}_h v^h)\Phi\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} \lesssim \|v_0^3\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} \|\Phi\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})}.
\]

Using interpolation inequality, this implies that
\[
\|G\Phi\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} \lesssim t^\frac{1}{2} Y(t) \Psi(t).
\]

While for the pressure term, we use the decomposition \(q = q^1 + q^2 + q^3\) in (12). For \(q_1\), since \(\epsilon(-\Delta_x)^{-1}\partial_1\partial_3\) is a bounded operator applied for frequency localized functions in \(L^2\) if \(i = 1, 2\), we have
\[
\epsilon^2 \|\partial_3q_1^h\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} = \epsilon^2 \|(-\Delta_x)^{-1}\partial_1\partial_3(v^i v^i)\Phi\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} \\
\lesssim \epsilon \|\nabla_h (v^h v^h)\Phi\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})}.
\]

Therefore, we get by using Lemma 3.7 that
\[
\epsilon^2 \|\partial_3q_1^h\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} \lesssim \epsilon \|v_0^h\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} \|\Phi\|_{L_t^1(\dot{B}_{2,1}^{s-1,1})} \lesssim t^\frac{1}{2} X(t) \Psi(t).
\]
Similarly, the fact that \( e^2(-\Delta)^{-1} \partial_2^3 \) is a bounded operator applied for frequency localized functions in \( L^2 \) implies
\[
e^2 \left\| \partial_3 q^k_n \right\|_{L_k^{1}(\mathcal{B}_{2^{-1}}^{e^{-1},\frac{1}{2}})} \lesssim \left\| \nabla_h(v^3 v^h) \phi \right\|_{L_k^{1}(\mathcal{B}_{2^{-1}}^{e^{-1},\frac{1}{2}})}.
\]

Thus, applying Lemma 3.7 again, we have
\[
e^2 \left\| \partial_3 q^k_n \right\|_{L_k^{1}(\mathcal{B}_{2^{-1}}^{e^{-1},\frac{1}{2}})} \lesssim \left\| (v^3 \text{div}_h v^h) \phi \right\|_{L_k^{1}(\mathcal{B}_{2^{-1}}^{e^{-1},\frac{1}{2}})}.
\]

Then we obtain that
\[
e^2 \left\| \partial_3 q^k_n \right\|_{L_k^{1}(\mathcal{B}_{2^{-1}}^{e^{-1},\frac{1}{2}})} \lesssim \left( X(t) + Y(t) \right) \Psi(t).
\]

Combining all the above estimates, we can get the bound of \( Y(t) \) by the following:
\[
Y(t) = \left\| v^3_0 \right\|_{L_k^{1}(\mathcal{B}_{2^{-1}}^{e^{-1},\frac{1}{2}})} + \left\| v^3_0 \right\|_{L_k^{1}(\mathcal{B}_{2^{-1}}^{e^{-1},\frac{1}{2}})} \lesssim 2 M_1 + (2^{N_1} t^{\frac{1}{2}} + 2^{2N_1} t)Y_0 + t^{\frac{1}{2}} \left( X(t) + Y(t) \right) \Psi(t).
\]

This completes the proof of \( Y(t) \) in Lemma 5.1.

The following is devoted to getting the estimate of \( X(t) \). The horizontal component \( v^h \) in each dyadic block satisfies
\[
\partial_k \Delta_{k,j} v^h - \Delta_{k,j} v^h = -\Delta_{k,j} \text{div}_h (v^h \otimes v^h) - \Delta_{k,j} \partial_3 (v^3 v^h) - \nabla_h \Delta_{k,j} q.
\]

Denote \( F \overset{\text{def}}{=} -\text{div}_h (v^h \otimes v^h) - \partial_3 (v^3 v^h) \), then we infer that
\[
\left\| \Delta_{k,j} v^h \right\|_{L^2} \lesssim e^{-c(2^{2k} + e^{2^{2j}})(t-\tau)} \left\| \Delta_{k,j} e^{\alpha [D_2]} g^0 \right\|_{L^2} + \int_{0}^{t} e^{-c(2^{2k} + e^{2^{2j}})(t-\tau)} \left\| \Delta_{k,j} F \right\|_{L^2} d\tau.
\]

We note that
\[
\int_{0}^{t} e^{-c(2^{2k} + e^{2^{2j}})(t-\tau)} \left\| \Delta_{k,j} \partial_3 (v^3 v^h) \right\|_{L^2} d\tau \lesssim \frac{1}{\epsilon} \int_{0}^{t} e^{-c e^{2^{2j}}(t-\tau)} 2^j \left\| \Delta_{k,j} (v^3 v^h) \right\|_{L^2} d\tau.
\]

Then we get by Young’s inequality that
\[
\int_{0}^{t} e^{-c(2^{2k} + e^{2^{2j}})(t-\tau)} \left\| \Delta_{k,j} \partial_3 (v^3 v^h) \right\|_{L^2} d\tau \lesssim \frac{1}{\epsilon} \left\| \Delta_{k,j} (v^3 v^h) \right\|_{L_k^{1}(L^2)}
\]

and
\[
\int_{0}^{t} e^{-c(2^{2k} + e^{2^{2j}})(t-\tau)} \left\| \Delta_{k,j} \partial_3 (v^3 v^h) \right\|_{L^2} d\tau \lesssim \frac{1}{\epsilon} \left\| \Delta_{k,j} (v^3 v^h) \right\|_{L_k^{1}(L^2)} - 2^{-k} \left\| \Delta_{k,j} (v^3 v^h) \right\|_{L_k^{1}(L^2)}.
\]
Therefore, by taking $L^2$ norm on $[0, t]$ for (50), we deduce that
\[
\begin{aligned}
\| \Delta_{k,j} v^h_t \|_{L^2_t \cap L^2} \lesssim & e^{-c(2^{2k+e^2}2^j)t} \| L^2_t \| e^{|D_3|} \| \Delta_{k,j} v^h_t \|_{L^2} \\
+ & 2^{-k} \| \Delta_{k,j} \mathcal{D}_h (v^h \otimes v^h) \|_{L^2_t} + \frac{1}{e} \| \Delta_{k,j} (v^3 v^h) \|_{L^2_t} \\
+ & 2^{-k} \| \Delta_{k,j} \nabla_h q \|_{L^2_t}.
\end{aligned}
\] (51)

Multiplying both sides of (51) by $2^{k\delta} 2^{j/2}$ and taking the sum over $k, j$, we finally get
\[
\begin{aligned}
\| v^h \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \lesssim & \sum_{k,j} 2^{k\delta} 2^{j/2} \| e^{-c(2^{2k+e^2}2^j)t} \|_{L^2_t} \| \Delta_{k,j} e^{|D_3|} v^h \|_{L^2} \\
+ & \| (v^h \otimes v^h) \|_{L^2_t (B^{s-\frac{1}{2}}_2)} + \frac{1}{e} \| (v^3 v^h) \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \\
+ & \| \nabla_h q \|_{L^2_t (B^{s-\frac{1}{2}}_2)}.
\end{aligned}
\] (52)

For the first term about $v^h_t$, by (46), for some fixed $M_1 > 0$, we have
\[
\sum_{k,j} 2^{k\delta} 2^{j/2} \| e^{-c(2^{2k+e^2}2^j)t} \|_{L^2_t} \| \Delta_{k,j} e^{|D_3|} v^h \|_{L^2_t} \lesssim \frac{1}{e} 2^{N_1 \frac{t}{2}} X_0 + M_1.
\] (53)

For the pressure term $q = q^1 + q^2 + q^3$, we find that
\[
\| \nabla_h q \|_{L^2_t (B^{s-\frac{1}{2}}_2)} = \| (-\Delta_x)^{-1} \nabla_h \partial_i \partial_j (v^i v^j) \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \lesssim \| (v^h v^h) \|_{L^2_t (B^{s-\frac{1}{2}}_2)},
\]
where we have used that $(-\Delta_x)^{-1} \partial_i \partial_j$ is a bounded operator for frequency localized functions in $L^2$.

Similarly, we have
\[
\| \nabla_h q \|_{L^2_t (B^{s-\frac{1}{2}}_2)} = \| (v^h v^h) \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \lesssim \frac{1}{e} \| v^3 v^h \|_{L^2_t (B^{s-\frac{1}{2}}_2)}
\]
and
\[
\| \nabla_h q \|_{L^2_t (B^{s-\frac{1}{2}}_2)} = \| (v^h v^h) \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \lesssim \frac{1}{e} \| v^3 v^h \|_{L^2_t (B^{s-\frac{1}{2}}_2)}.
\]

According to Lemma 3.7, the right hand side of (52) can be bounded by following:
\[
e \| (v^h \otimes v^h) \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \leq e \| v^h \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \| v^h \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \| v^h \|_{L^2_t (B^{s-\frac{1}{2}}_2)}
\]
and
\[
e \| \nabla_h q \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \leq e \| v^h \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \| v^h \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \| v^h \|_{L^2_t (B^{s-\frac{1}{2}}_2)}
\]
These imply that
\[
e \| v^h \|_{L^2_t (B^{s-\frac{1}{2}}_2)} \lesssim e M_1 + 2^{N_1 \frac{t}{2}} X_0 + t^\frac{1}{2} (X(t) + Y(t)) \Psi(t).
\] (54)
Then, taking $L^1$ norm on $[0, t]$ for (50), by the same method, we deduce that

\[
\| \Delta_{k,j}^h \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1(B_{2,1}^\delta)} \lesssim e^{-c(2^{k+3}+e^2t^2)} \| \Delta_{k,j}^h \|_{L_t^1} \| \Delta_{k,j}^h \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1}^2 \\
+ 2^{-2k} \| \Delta_{k,j}^h \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1} \| \Delta_{k,j}^h \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1} + \frac{1}{\epsilon} 2^{-k} \| \Delta_{k,j}^h \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1} + 2^{-2k} \| \Delta_{k,j}^h \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1}.
\]

(55)

Multiplying both sides of (55) by $2^{(\delta+1)k}2^j$ and taking the sum over $k, j$, we finally get

\[
\| v_h^\delta \|_{L_t^1(B_{2,1}^\delta)} \lesssim \sum_{k,j} 2^{(\delta+1)k}2^j e^{-c(2^{k+3}+e^2t^2)} \| \Delta_{k,j}^h \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1} + \frac{1}{\epsilon} \| \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1} + \| \nabla \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1}.
\]

(56)

For the first term about $v_h^\delta$, by (47), for some fixed $M_1 > 0$, we have

\[
\sum_{k,j} 2^{(\delta+1)k}2^j e^{-c(2^{k+3}+e^2t^2)} \| \Delta_{k,j}^h \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1} \| \Delta_{k,j}^h \Phi^{\delta+1, \frac{1}{2}} \|_{L_t^1} \leq \frac{1}{\epsilon} 2^{2N_1} t X_0 + M_1.
\]

(57)

By the same methods as the above estimates, we can also get the bound of $\| v_h^\delta \|_{L_t^1(B_{2,1}^\delta)}$ by the following:

\[
\| v_h^\delta \|_{L_t^1(B_{2,1}^\delta)} \leq \epsilon M_1 + 2^{2N_1} t X_0 + \frac{1}{\epsilon} \left( X(t) + Y(t) \right) \Psi(t).
\]

(58)

Combining (48) with (58) and (54), we finally obtain that there exists a constant $C_2$ such that

\[
X(t) + Y(t) \leq C_2 M_1 + C_2 (2^{N_1} t^2 + 2^{2N_1} t) (X_0 + Y_0) + C_2 t^2 \left( X(t) + Y(t) \right).
\]

(59)

This completes the proof of Lemma 5.1.

\[\square\]

Proof of Theorem 1.2. In this section, we will prove the Theorem 1.2. It relies on a continuation argument. For any $\lambda > \lambda_0$ and $\eta$, we assume that:

\[
X(t) + Y(t) \leq 2M, \quad \Psi(t) \leq 2M.
\]

(60)

We can choose $M$ large enough such that

\[
C_1 \Psi(0) + C_2 \left( X_0 + Y_0 \right) \leq C\eta \leq \frac{M}{2}.
\]

From

\[
\theta(t) \leq CY(t) t^2 < 2CM t^2 < \frac{\alpha}{2\lambda},
\]

we have

\[
t \leq \left( \frac{\alpha}{4CM\lambda} \right)^{\frac{1}{2}}.
\]

(61)

Here we can assume that $t < 1$ since $M$ is large enough. For such fixed $M$, we choose $M_1$ small enough such that

\[
C_2 M_1 \leq \frac{M}{2}.
\]
According to the estimates in Lemma 4.1 and 5.1.

\[
X(t) + Y(t) \leq \frac{M}{2} + \frac{M}{2} (2^{N_1} t^\frac{1}{2} + 2^{2N_1} t) + C_2 t^\frac{\delta}{4} 4M^2 \\
\leq \frac{M}{2} + \frac{M}{2} (2^{N_1} + 2^{2N_1}) t^\frac{1}{2} + C_2 t^\frac{\delta}{4} 4M^2 \\
\leq \frac{M}{2} + \frac{M}{2} (2^{N_1} + 2^{2N_1} + 8C_2 M) t^\frac{\delta}{2}, \\
\Psi(t) \leq \frac{M}{2} + C_1 (t^\frac{\delta}{2} + t^\frac{\alpha}{4}) 4M^2 \\
\leq \frac{M}{2} + 8C_1 M^2 t^\frac{\delta}{2}.
\]

(62)

Combining (62) with (61), for some positive constants \(N_1\) and \(M\), we can find \(T^*\) in the form

\[
T^* = \min \left\{ \left( \frac{1}{2^{N_1} + 2^{2N_1} + 8MC_2} \right)^\frac{1}{\gamma}, \left( \frac{1}{16MC_1} \right)^\frac{1}{\gamma}, \left( \frac{\alpha}{4MC_1} \right)^\frac{1}{\gamma} \right\}.
\]

Hence, when \(t \leq T^*\), we have

\[
X(t) + Y(t) \leq M, \quad \Psi(t) \leq M.
\]

The conclusion of Theorem 1.2 follows immediately. \(\square\)

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