The Marichev-Saigo-Maeda Fractional Calculus Operators Pertaining to the Generalized $K$-Struve Function

Seema Kabra$^1$, Harish Nagar$^2$, Kottakkaran Sooppy Nisar$^3$, D.L. Suthar$^4$

$^1$Department of Mathematics, Sangam University, Bhilwara, Rajasthan, India, E-mail: kabraseema@rediffmail.com
$^2$Department of Mathematics, Sangam University, Bhilwara, Rajasthan, India, E-mail: drharishngr@gmail.com
$^3$Department of Mathematics, College of Arts & Sciences, Wadi Aldawaser, Prince Sattam bin Abdulaziz University, Saudi Arabia, E-mail: n.sooppy@psau.edu.sa
$^4$Department of Mathematics, Wollo University, Dessie, P.O. Box:1145, Amahara Region, Ethiopia

Abstract

In the present paper, we establish some compositions formulas for Marichev-Saigo-Maeda (MSM) fractional calculus operators with $k$-Struve function $S_{k,c}^\nu$ as of the kernel. The results are presented in terms of generalized $k$-Wright function $p^\nu_{k,q}$. 

Keywords: Generalized $k$-Struve function, Marichev-Saigo-Maeda fractional calculus operators, Generalized $k$-Wright function, $k$-Pochhammer symbol, $k$-Gamma function.

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1 Introduction and Preliminaries

The Wright function play an important role in the partial differential equation of fractional order which is familiar and extensively treated in papers by a number of authors including Gorenflo et al. [6].

For $\zeta, \tau_j \in \mathbb{R}\setminus\{0\}$ and $a_i, b_j \in \mathbb{C}, i = (1, \bar{p}); j = (1, q)$ the generalized form of Wright function is defined by
generalized K-Wright function which is defined as
\[
\psi_k(z) = \psi_q \left[ \left( a_i, \zeta_i \right)_{1, p} \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + n \zeta_i)}{\prod_{j=1}^{q} \Gamma(b_j + n \zeta_j)} \frac{z^n}{n!}, z \in \mathbb{C}, \tag{1.1}
\]
where \( \Gamma(z) \) is the well-known Euler gamma function [4]. The condition for existence of (1.1) with its depiction in terms of Mellin-Barnes integral and the H-function were obtained by Kilbas et al. [10].

The generalized form of the above Wright function (1.1) was given by Gehlot and Prajapati [5], named as generalized K-Wright function which is defined as
\[
\psi_k(z) = \psi_q \left[ \left( a_i, \zeta_i \right)_{1, p} \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma_k(a_i + n \zeta_i)}{\prod_{j=1}^{q} \Gamma_k(b_j + n \zeta_j)} \frac{z^n}{n!}, z \in \mathbb{C}, \tag{1.2}
\]
where \( k \in \mathbb{R}^+ \) and \( (a_i + n \zeta_i), (b_j + n \zeta_j) \in \mathbb{C} \setminus k\mathbb{Z}^- \) for all \( n \in \mathbb{N}_0 \). The generalized \( k \)-gamma function [3] is defined as
\[
\Gamma_k(z) = \int_0^\infty e^{-\frac{z}{t}} t^{-1} dt; \quad (\Re(z) > 0; k \in \mathbb{R}^+) \tag{1.3}
\]
and
\[
\Gamma_k(z) = \lim_{n \to \infty} \frac{n! \kappa(nk)^{z-1}}{(z)_{n,k}}, \quad k \in \mathbb{R}^+, \quad z \in \mathbb{C} \setminus k\mathbb{Z}^- \tag{1.4}
\]
Also
\[
\Gamma_k(z) = k^{z-1} \Gamma\left( \frac{z}{k} \right), \tag{1.5}
\]
where \((z)_{n,k}\) is the \( k \)-Pochammer symbol introduced by Diaz and Pariguan [3] defined for complex \( z \in \mathbb{C} \) and \( k \in \mathbb{R} \) as
\[
(z)_{n,k} = \begin{cases} 1 & \text{if } n = 0, \\ z(z+k)(z+2k)\ldots(z+(n-1)k) & \text{if } n \in \mathbb{N}. \end{cases} \tag{1.6}
\]
On taking \( k = 1 \), then the generalized K-Wright function (1.2) diminishes to the generalized Wright function (1.1).

### 1.1 Saigo fractional calculus operators

Saigo [18] defined the fractional integral and differential operators with the Gauss hypergeometric function as kernel, which are remarkable generalizations of the Riemann-Liouville (R-L) and Erdélyi-Kober fractional calculus operators (see; [11]).

For \( \zeta, \tau, \gamma \in \mathbb{C} \) and \( x \in \mathbb{R}^+ \) with \( \Re(\zeta) > 0 \), the left-hand and the right-hand sided generalized fractional integral operators connected with Gauss hypergeometric function are defined as below:

\[
(I_{0+}^{\zeta, \tau, \gamma} f)(x) = \frac{x^{-\zeta - \tau}}{\Gamma(\zeta)} \int_0^x (x-t)^{\zeta-1} \frac{t^{\gamma-1}}{t^{\gamma + \tau}} \frac{f(t)}{t^\tau} f(t) dt \tag{1.7}
\]
and

\[
(I_{+}^{\zeta, \tau, \gamma} f)(x) = \frac{1}{\Gamma(\zeta)} \int_x^{\infty} \frac{(t-x)^{\zeta-1}}{t^{\gamma + \tau}} \frac{f(t)}{t^\tau} f(t) dt \tag{1.8}
\]
respectively. Here, \(2F_1(\zeta, \tau; \gamma; z)\) is the Gauss hypergeometric function [11] defined for \(z \in \mathbb{C}, |z| < 1\) and \(\zeta, \tau \in \mathbb{C}, \gamma \in \mathbb{C} \setminus \mathbb{Z}_0\) by

\[
2F_1(\zeta, \tau; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\zeta)_n(\tau)_n}{(\gamma)_n} \frac{z^n}{n!},
\]

where \((z)_n = (z)_{n,1}\). The corresponding fractional differential operators are

\[
(D^\varsigma_{0+} \varsigma, \tau; \gamma) f(x) = \left(\frac{d}{dx}\right)^l \left(I_{0+}^{\varsigma+l, -\varsigma, \gamma} f(x)\right)
\]

and

\[
(D^\varsigma_{0-} \varsigma, \tau; \gamma) f(x) = \left(-\frac{d}{dx}\right)^l \left(I_{0+}^{\varsigma+l, -\varsigma, \gamma} f(x)\right)
\]

where \(l = \lfloor \Re(\varsigma) \rfloor + 1\) and \(\lfloor \Re(\varsigma) \rfloor\) is the integer part of \(\Re(\varsigma)\). Substituting \(\tau = -\zeta\) and \(\tau = 0\) in equation (1.7) – (1.10), we get the corresponding R-L and Erdélyi-Kober fractional operators, respectively.

### 1.2 Marichev-Saigo-Maeda fractional operators

Marichev [13] was introduced and studied fractional calculus operators which are the generalization of the Saigo operators, later generalized by Saigo and Maeda [19]. For \(\zeta, \zeta', \tau, \tau', \gamma \in \mathbb{C}\) and \(x \in \mathbb{R}^+\) with \(\Re(\gamma) > 0\), the left-hand and right-hand sided MSM fractional integral and derivative operators associated with third Appell function \(F_3\) are defined as

\[
(I^\varsigma_{0+} \varsigma, \tau; \gamma; f(x)) = \frac{x^{-\varsigma}}{\Gamma(\gamma)} \int_0^x \frac{(t-x)^{\gamma-1}}{t^{\varsigma}} F_3(\zeta, \zeta', \tau, \tau', \gamma, 1 - \frac{t}{x}, 1 - \frac{x}{t}) f(t) dt
\]

and

\[
(I^\varsigma_{0-} \varsigma, \tau; \gamma; f(x)) = \frac{x^{-\varsigma}}{\Gamma(\gamma)} \int_x^m \frac{(t-x)^{\gamma-1}}{t^{\varsigma}} F_3(\zeta, \zeta', \tau, \tau', \gamma, 1 - \frac{t}{x}, 1 - \frac{x}{t}) f(t) dt
\]

\[
(D^\varsigma_{0+} \varsigma, \tau; \gamma; f(x)) = \frac{d}{dx}^m \left(I_{0+}^{\varsigma+l, -\varsigma, \gamma} m f(x)\right)
\]

and

\[
(D^\varsigma_{0-} \varsigma, \tau; \gamma; f(x)) = \frac{d}{dx}^m \left(I_{0+}^{\varsigma+l, -\varsigma, \gamma} m f(x)\right)
\]

respectively, where \(m = \lfloor \Re(\gamma) \rfloor + 1\) and the third Appell function [17], is defined by

\[
F_3(\zeta, \zeta', \tau, \tau', \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\zeta)_m(\zeta')_n(\tau)_m(\tau')_n x^m y^n}{(\gamma)_{m+n} m! n!}, \quad \max\{|x|, |y|\} < 1.
\]

### 1.3 Generalized k-Struve function

The generalized \(k\)-Struve function was defined by Nisar et al. [14] as

\[
S^k_{\nu,c}(t) = \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + \nu + \frac{3}{2}) \Gamma(n + \frac{3}{2}) n!} \left(\frac{t}{2}\right)^{2n+\nu+1}
\]

\[
(k \in \mathbb{R}^+; c \in \mathbb{R}; \nu > -1)
\]
Fractional Calculus Approach

be used from [9] as below:

Further, to obtain the MSM fractional differentiation of the generalized Struve function, following results will be used from [9] as below:

\[ (\mathcal{D}_0^\zeta,\tau,\gamma)_{t-\rho} f(x) = \frac{\Gamma(\rho)\Gamma(-\zeta + \tau + \gamma + \rho)\Gamma(\zeta + \tau' + \gamma + \rho)}{\Gamma(-\tau + \rho)\Gamma(\zeta + \tau' - \gamma + \rho)\Gamma(\zeta + \tau + \gamma + \rho)} x^{\zeta + \tau' - \gamma - \rho - 1} \]

(ii). If \( \Re(\rho) > \max\{\Re(\tau), \Re(-\zeta' + \gamma), \Re(-\zeta' + \tau + \gamma)\} \), then

\[ (\mathcal{D}_0^\zeta,\tau,\gamma)_{t-\rho} f(x) = \frac{\Gamma(\rho)\Gamma(-\zeta + \tau + \gamma + \rho)\Gamma(\zeta + \tau' + \gamma + \rho)}{\Gamma(-\tau + \rho)\Gamma(\zeta + \tau' - \gamma + \rho)\Gamma(\zeta + \tau + \gamma + \rho)} x^{\zeta + \tau' - \gamma - \rho - 1} \]

(ii). If \( \Re(\rho) > \max\{\Re(-\tau'), \Re(\zeta' + \tau + \gamma), \Re(\zeta' + \tau + \gamma) + [\Re(\gamma) + 1]\} \), then

\[ (\mathcal{D}_0^\zeta,\tau,\gamma)_{t-\rho} f(x) = \frac{\Gamma(\rho)\Gamma(-\zeta - \zeta' + \gamma + \rho)\Gamma(-\zeta' - \tau + \gamma + \rho)}{\Gamma(-\tau + \rho)\Gamma(-\zeta - \zeta' + \tau + \gamma + \rho)\Gamma(-\zeta - \zeta' + \tau + \gamma + \rho)} x^{\zeta + \tau' - \gamma - \rho - 1} \]

2 Fractional Calculus Approach

In this section, the following six theorems for \( k \)-Struve function concerning to MSM fractional integral and differential operators are established here as main results.

**Theorem 1.** Let \( \zeta, \zeta', \tau, \tau', \gamma, \rho \in \mathbb{C} \) and \( k \in \mathbb{R}^+ \) be such that \( \Re(\gamma) > 0, \Re\left(\frac{k}{2}\right) > \max\{0, \Re(\zeta' - \tau'), \Re(\zeta + \zeta' + \tau + \gamma)\} \). Also let \( c \in \mathbb{R} ; \nu > -1 \), then for \( t > 0 \)

\[ (\mathcal{D}_0^{\zeta,\tau,\gamma} f(x)) (t) = \frac{k^{\nu + \frac{1}{2}} x^{-\zeta - \zeta' + \gamma + \frac{k}{2} + \frac{\nu}{2}}}{2^{\nu + 1}} \times \left[ \left( \begin{array}{c} \lambda + \nu + k, 2k, \\ k + 2k, \end{array} \right), \left( \begin{array}{c} -k \zeta' + k \tau + \lambda + \nu + k, 2k, \\ -k \zeta - k \zeta' + k \gamma + \lambda + \nu + k, 2k, \end{array} \right), \left( \begin{array}{c} -k \zeta - k \zeta' - k \tau + k \gamma + \lambda + \nu + k, 2k, \\ -k \zeta' - k \tau + k \gamma + \lambda + \nu + k, 2k, \end{array} \right) \right] \left( \begin{array}{c} c^2 x^k, \\ \left( \begin{array}{c} \frac{3k}{2}, k, \\ \frac{3k}{2}, k, \end{array} \right) \right) \right] . \]

**Proof.** On using (1.16) and taking the left-hand sided MSM fractional integral operator inside the summation, the left-hand side of (2.1) becomes

\[ \frac{(-c)^n}{k (nk + \nu + \frac{3k}{2}) n! 2^{2n + \frac{1}{2} + 1}} (\mathcal{D}_0^{\zeta,\tau,\gamma} f(x)) (t) = \frac{(-c)^n}{k (nk + \nu + \frac{3k}{2}) n! 2^{2n + \frac{1}{2} + 1}} (\mathcal{D}_0^{\zeta,\tau,\gamma} f(x)) (t) , \]

\[ \frac{(-c)^n}{k (nk + \nu + \frac{3k}{2}) n! 2^{2n + \frac{1}{2} + 1}} (\mathcal{D}_0^{\zeta,\tau,\gamma} f(x)) (t) . \]
Making use of (1.18), we obtain
\[
= \sum_{n=0}^{\infty} \frac{(-c)^n x^{-\xi - \xi' + \gamma + \frac{2}{k} + \frac{2n + 2}{2n + 2}}}{\Gamma_k(nk + \nu + \frac{3k}{2}) \Gamma(n + \frac{3}{2}) n! 2^{n + 1}} \frac{\Gamma(\frac{2}{k} + \frac{2n + 1}{2} + 1)}{\Gamma(\frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)} \times \frac{\Gamma(\xi - \xi' - \tau + \gamma + \frac{2}{k} + \frac{2n + 1}{2} + 1) \Gamma(-\xi' + \frac{2}{k} + \frac{2n + 1}{2} + 1)}{\Gamma(\xi - \xi' + \gamma + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1) \Gamma(-\xi' + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)}.
\]

Now, using equation (1.5) on above term, then we get
\[
= \frac{x^{-\xi - \xi' + \gamma + \frac{2}{k} + \frac{2n + 2}{2}}}{2^{n + 1} k^{-\frac{2}{k} - \frac{2n + 2}{2}} \sum_{n=0}^{\infty} \frac{\Gamma_k\left(\lambda + v + k + 2nk\right) \Gamma(-\xi' + \frac{2}{k} + \frac{2n + 1}{2} + 1)}{\Gamma_k(k \tau' + \lambda + v + k + 2nk) \Gamma(-\xi' + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)} \Gamma(-\xi' + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)} \times \frac{\Gamma(-\xi + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)}{\Gamma(-\xi + \gamma + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)} \times \frac{\left(-c x^2 k\right)^n}{\left(\frac{3}{2} - \nu\right)^n}.
\]

Using the definition of (1.2) in the above term, we arrive at the result (2.1).

Next theorem gives the right-hand MSM fractional integration of $S_\nu^k(.)$.

**Theorem 2.** Let $\xi, \xi', \tau, \tau', \gamma, \nu, \rho \in \mathbb{C}$ and $k \in \mathbb{R}^+$ be such that $\Re(\gamma) > 0, \Re\left(\frac{2}{k}\right) > \max\{\Re(\tau), \Re(\xi - \xi' + \gamma), \Re(\xi - \tau' + \gamma)\}$. Also let $c \in \mathbb{R}, \nu > 1$, then for $t > 0$
\[
\left(J_{\xi', \tau', \gamma}^k(t) S_\nu^k(t)\right)(x) = \frac{k^\nu x^{-\xi' + \gamma + \frac{2}{k} + \frac{2n + 2}{2}}}{2^{n + 1} k^{-\frac{2}{k} - \frac{2n + 2}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma_k\left(\lambda + v + k + 2nk\right) \Gamma(-\xi' + \frac{2}{k} + \frac{2n + 1}{2} + 1)}{\Gamma_k(k \tau' + \lambda + v + k + 2nk) \Gamma(-\xi' + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)} \Gamma(-\xi' + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)} \times \frac{\Gamma(-\xi + \gamma + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)}{\Gamma(-\xi + \xi' + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)} \times \frac{\left(-c x^2 k\right)^n}{\left(\frac{3}{2} - \nu\right)^n}.
\]

**Proof.** On using (1.16) and taking the right-hand side MSM fractional integral operator inside the summation, the left hand side of (2.2) becomes
\[
= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + \nu + \frac{3k}{2}) \Gamma(n + \frac{3}{2}) n! 2^{n + 1}} \left(J_{\xi', \tau', \gamma}^k(t) S_\nu^k(t)\right)(x).
\]

On using (1.19), we get
\[
= \sum_{n=0}^{\infty} \frac{(-c)^n x^{-\xi' + \gamma + \frac{2}{k} + \frac{2n + 2}{2}}}{\Gamma_k(nk + \nu + \frac{3k}{2}) \Gamma(n + \frac{3}{2}) n! 2^{n + 2} + 1} \Gamma(-\tau + \xi' + \gamma + \frac{2}{k} + \frac{2n}{2} - \frac{2}{k} + \frac{2n + 1}{2} + 1) \times \frac{\Gamma(-\xi' + \tau' + \gamma - \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)}{\Gamma(-\xi + \tau' + \gamma - \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)} \times \frac{\Gamma(-\xi' + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)}{\Gamma(\xi + \tau' - \gamma - \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)} \times \frac{\Gamma(-\xi' + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)}{\Gamma(-\xi' + \frac{2}{k} + \frac{2n + 2}{2} + 2n + 1)} \times \frac{\left(-c x^2 k\right)^n}{\left(\frac{3}{2} - \nu\right)^n}.
\]

and the result follows on making use of (1.5) and definition of generalized $k$-Wright function.
Theorem 3. Let \( \zeta, \zeta', \tau, \gamma, \rho \in \mathbb{C} \) and \( k \in \mathbb{R}^+ \) be such that \( \Re(\gamma) > 0, \Re(\frac{\zeta}{\rho}) > \max\{\Re(\tau), \Re(-\zeta - \zeta' + \gamma), \Re(-\zeta - \tau' + \gamma)\} \). Also let \( c \in \mathbb{R}; \nu > -1 \), then for \( t > 0 \)

\[
\left( \tilde{D}_{0^+}^{\zeta', \tau', \gamma} \left( t^{\frac{\nu}{\tau} \Sigma} \right) \right)(x) = \frac{k^{\gamma-rac{1}{2}} x^{\gamma-rac{1}{2}+\frac{1}{2}}}{2^{\frac{1}{2}+1}} \times \left. \Psi_k \left[ \left( -k \zeta + \frac{\lambda - \nu + k, 2k}, (k \zeta + k \zeta' - k \gamma + \lambda + v + k, 2k), \right) \right] \frac{-cx^2k}{4} \right] .
\]

(2.3)

Proof. On using (1.16) and taking the right-hand sided MSM fractional integral operator inside the summation, the left-hand side of (2.3) becomes

\[
= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(nk + v + \frac{3n}{2}) \Gamma(n + \frac{3}{2}) n! 2^{2n+\frac{1}{2}} + 1} \left( \tilde{D}_{0^+}^{\zeta', \tau', \gamma} \left( t^{\frac{\nu}{\tau} \Sigma} \right) \right)
\]

On using (1.19), we obtain

\[
= \frac{(-c)^n t^{\gamma - \zeta' - \gamma + \frac{1}{2} + \frac{1}{2} + 2n + 1}}{\Gamma(nk + v + \frac{3n}{2}) \Gamma(n + \frac{3}{2}) n! 2^{2n+\frac{1}{2}} + 1} \times \frac{\Gamma(-\tau + \frac{\lambda - v}{k} - 2n - 1)}{\Gamma(-\tau + \frac{\lambda - v}{k} - 2n - 1)} \Gamma(\zeta + \zeta' - \gamma + \frac{1}{2} + \frac{1}{2} + 2n - 1)
\]

\[
\quad \times \left( \zeta + \zeta' - \gamma + \frac{1}{2} + \frac{1}{2} + 2n - 1 \right) \Gamma(\zeta + \zeta' - \gamma + \frac{1}{2} + \frac{1}{2} + 2n - 1)
\]

Making use of (1.5), we get

\[
= \frac{x^{\gamma - \zeta' - \gamma + \frac{1}{2} + \frac{1}{2} + 1}}{k^{\gamma - \zeta' - \gamma + \frac{1}{2} + \frac{1}{2} + 1}} \sum_{n=0}^{\infty} \frac{(-c x^2)^n}{4^n \Gamma(nk + v + \frac{3n}{2}) \Gamma(n + \frac{3}{2}) n! 2^{2n+\frac{1}{2}} + 1} \Gamma(-k \zeta + \lambda - v - k - 2n - 1) \Gamma(-k \zeta - k \tau + \lambda - v - k - 2n - 1)
\]

\[
\quad \times \Gamma(k \zeta + k \zeta' - k \gamma + \lambda - v - k - 2n - 1) \Gamma(k \zeta + k \zeta' - k \gamma + \lambda - v - k - 2n - 1) \Gamma(nk + v + \frac{3n}{2}) \Gamma(nk + \frac{3n}{2})
\]

This on expressing in terms of \( k \)-Wright function \( \zeta \Psi_k \) using (1.2) leads to the right-hand side of (2.3). This completes the proof of theorem.

The next theorem obtains the left-hand sided MSM fractional differentiation of \( k \)-Struve function.

Theorem 4. Let \( \zeta, \zeta', \tau, \gamma, \rho \in \mathbb{C} \) and \( k \in \mathbb{R}^+ \) be such that \( \Re(\frac{\zeta}{\rho}) > \max\{0, \Re(-\zeta + \tau), \Re(-\zeta - \zeta' + \gamma), \Re(-\zeta - \tau' + \gamma)\} \).

Also let \( c \in \mathbb{R}; \nu > -1 \), then for \( t > 0 \)

\[
\left( \tilde{D}_{0^+}^{\zeta', \tau', \gamma} \left( t^{\frac{\nu}{\tau} \Sigma} \right) \right)(x) = \frac{k^{\gamma-rac{1}{2}} x^{\gamma-rac{1}{2}+\frac{1}{2}}}{2^{\frac{1}{2}+1}} \times \left. \Psi_k \left[ \left( -k \zeta + \frac{\lambda + v + k, 2k}, (k \zeta + k \zeta' - k \gamma + \lambda + v + k, 2k), \right) \right] \frac{-cx^2k}{4} \right] .
\]

(2.4)

Proof. On using (1.16) and taking the left-hand sided MSM fractional derivative inside the summation, the left-hand side of (2.4) becomes

\[
= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(nk + v + \frac{3n}{2}) \Gamma(n + \frac{3}{2}) n! 2^{2n+\frac{1}{2}} + 1} \left( \tilde{D}_{0^+}^{\zeta', \tau', \gamma} \left( t^{\frac{\nu}{\tau} \Sigma} \right) \right)
\]
Using (1.20) in above term, we obtain

\[
\begin{align*}
&= \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(\frac{2}{3} + \frac{2}{3} + 2n + 1)}{\Gamma_k(nk + v + \frac{3k}{2}) \Gamma(n + \frac{3}{2}) n! 2^{\frac{3}{2} + 2n + 1}} \Gamma(-\tau + \frac{2}{3} + \frac{2}{3} + 2n + 1) \\
&\quad \times \frac{\Gamma(\xi + \frac{2}{3} + \frac{2}{3} + 2n + 1)}{\Gamma(\xi + \frac{2}{3} + \frac{2}{3} + 2n + 1)} \Gamma(\xi + \frac{2}{3} + \frac{2}{3} + 2n + 1)
\end{align*}
\]

\[
\begin{align*}
\gamma
\end{align*}
\]

In above term, we use equation (1.5), and the result follows by using (1.2), then we arrive at (2.4).

The next theorem gives the right-hand sided MSM fractional derivative of \(k\)-Struve function.

**Theorem 5.** Let \(\xi, \xi', \tau, \gamma, \rho \in \mathbb{C}\) and \(k \in \mathbb{R}^+\) be such that \(\Re(\frac{k}{3}) > \max\{\Re(-\tau'), \Re(\xi' + \tau - \gamma), \Re(\xi + \xi' - \gamma) + [\Re(\gamma)] + 1\}\). Also let \(c \in \mathbb{R}; \nu > -1\), then for \(t > 0\)

\[
\left( D_{-}^{\xi, \xi', \tau', \gamma} \left( t^{\frac{1}{4}-1} S_{\nu, c}(t) \right) \right) (x) = \frac{k^{-\gamma + \frac{1}{2} + \frac{\xi + \xi' + \gamma}{2}}}{2^{\frac{3}{2} + 1}}
\]

\[
\begin{align*}
&\times \Psi_{\frac{1}{3}} \left[ \left( \begin{array}{c}
(\xi'-\lambda-\nu, -2k), (\xi' - k\xi' + k\gamma - \lambda - \nu, -2k), \\
(-\lambda - \nu, -2k), (\xi' - k\xi' + k\tau' - \lambda - \nu, -2k), \\
(-\lambda - \nu, -2k), (\lambda - \nu, -2k), (v + \frac{3k}{2}, k), (\frac{3k}{2}, k) \end{array} \right) \right] \\
&\left( -c^{\nu+2k} \right) \end{align*}
\]

(2.5)

**Proof.** On using (1.16) and taking the left-hand sided MSM fractional derivative inside the summation, the left-hand side of (2.5) becomes

\[
\begin{align*}
&= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma_k(nk + v + \frac{3k}{2}) \Gamma(n + \frac{3}{2}) n! 2^{\frac{3}{2} + 2n + 1}} \left( D_{-}^{\xi, \xi', \tau', \gamma} \left( t^{\frac{1}{4}-1} S_{\nu, c}(t) \right) \right)
\end{align*}
\]
Using (1.21) in above term, we obtain

\[
\begin{align*}
&= \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(\tau' - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)}{\Gamma(n + \frac{3}{2}) \Gamma(n + \frac{3}{2}) n! 2^{\frac{1}{2} + 2n} + 1} \\
& \times \frac{\Gamma(-\zeta - \zeta' + \gamma - \frac{\lambda}{k} - \frac{\nu}{k} - 2n) \Gamma(\zeta' - \tau + \gamma - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)}{\Gamma(-\zeta - \zeta' + \tau - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)} x^{\zeta + \zeta' - \gamma + \frac{1}{2} + \frac{3}{2} + 2n} \\
&= \frac{x^{\zeta + \zeta' - \gamma + \frac{1}{2} + \frac{3}{2}}}{2^{\frac{1}{2} + 1}} \sum_{n=0}^{\infty} \frac{(-cx^2)^n}{n! 2^{\frac{1}{2} + 1}} \frac{\Gamma(\tau' - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)}{\Gamma(-\zeta - \zeta' + \gamma - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)} \\
& \times \frac{\Gamma(-\zeta - \zeta' + \tau - \frac{\lambda}{k} - \frac{\nu}{k} - 2n) \Gamma(\zeta' - \tau + \gamma - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)}{\Gamma(-\zeta - \zeta' + \tau - \frac{\lambda}{k} - \frac{\nu}{k} - 2n) \Gamma(\zeta' - \tau + \gamma - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)} \\
&= \sum_{n=0}^{\infty} \frac{(-c)^n \Gamma(\tau' - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)}{\Gamma(n + \frac{3}{2}) \Gamma(n + \frac{3}{2}) n! 2^{\frac{1}{2} + 2n} + 1} \\
& \times \frac{\Gamma(-\zeta - \zeta' + \gamma - \frac{\lambda}{k} - \frac{\nu}{k} - 2n) \Gamma(\zeta' - \tau + \gamma - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)}{\Gamma(-\zeta - \zeta' + \tau - \frac{\lambda}{k} - \frac{\nu}{k} - 2n) \Gamma(\zeta' - \tau + \gamma - \frac{\lambda}{k} - \frac{\nu}{k} - 2n)} x^{\zeta + \zeta' - \gamma + \frac{1}{2} + \frac{3}{2} + 2n}.
\end{align*}
\]

Thus, in accordance with (1.2), we get the required result (2.5).

**Theorem 6.** Let \(\zeta, \zeta', \tau, \tau', \gamma, \rho \in \mathbb{C}\) and \(k \in \mathbb{R}^+\) be such that \(\Re(\frac{1}{k}) > \max\{\Re(-\tau'), \Re(\zeta' + \tau - \gamma), \Re(\zeta + \zeta' - \gamma) + [\Re(\gamma) + 1]\}\). Also let \(c \in \mathbb{R}; \nu > -1, \) then for \(t > 0\)

\[
\begin{align*}
&\left(D^{\zeta, \zeta', \tau, \tau', \gamma}_{t^{-\frac{1}{2}}, S_{\tau', c}(t)} \frac{(t^{-\frac{1}{2}} S_{\tau', c}(t))}{2^{\frac{1}{2} + 1}} x^{\zeta + \zeta' - \gamma + \frac{1}{2} + \frac{3}{2} + 2n} \right) \\
&\times \psi^k \left[ \left( k \tau' + \lambda - \nu - k, -2k \right), \left( -k \zeta - k \zeta' + k \gamma - \lambda - \nu - k, -2k \right), \left( -k \zeta' - k \tau + k \gamma + \lambda - \nu - k, -2k \right), \left( -k \zeta - k \zeta' - k \tau + k \gamma - \lambda - \nu - k, -2k \right) \right],
\end{align*}
\]

\[
(\nu + \frac{3}{2}, k), (\frac{3}{2}, k), \left( \frac{3}{4}, k \right) \mid \frac{-cx^2 k}{4}.
\]

**Proof.** On using (1.16) and taking the right-hand sided MSM fractional derivative inside the summation, the left-hand side of (2.6) becomes

\[
\begin{align*}
&= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(n + \frac{3}{2}) \Gamma(n + \frac{3}{2}) n! 2^{\frac{1}{2} + 2n} + 1} \left(D^{\zeta, \zeta', \tau, \tau', \gamma}_{t^{-\frac{1}{2} + 2n + 1}} \frac{(t^{-\frac{1}{2}} + 2n + 1)}{2^{\frac{1}{2} + 1}} \right) \\
&= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(n + \frac{3}{2}) \Gamma(n + \frac{3}{2}) n! 2^{\frac{1}{2} + 2n} + 1} \left(D^{\zeta, \zeta', \tau, \tau', \gamma}_{t^{-\frac{1}{2} + 2n + 1}} \frac{(t^{-\frac{1}{2}} + 2n + 1)}{2^{\frac{1}{2} + 1}} \right).
\end{align*}
\]

Using (1.21), we have

\[
\begin{align*}
&= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(n + \frac{3}{2}) \Gamma(n + \frac{3}{2}) n! 2^{\frac{1}{2} + 2n} + 1} \left(D^{\zeta, \zeta', \tau, \tau', \gamma}_{t^{-\frac{1}{2} + 2n + 1}} \frac{(t^{-\frac{1}{2}} + 2n + 1)}{2^{\frac{1}{2} + 1}} \right) \\
&= \sum_{n=0}^{\infty} \frac{(-c)^n}{\Gamma(n + \frac{3}{2}) \Gamma(n + \frac{3}{2}) n! 2^{\frac{1}{2} + 2n} + 1} \left(D^{\zeta, \zeta', \tau, \tau', \gamma}_{t^{-\frac{1}{2} + 2n + 1}} \frac{(t^{-\frac{1}{2}} + 2n + 1)}{2^{\frac{1}{2} + 1}} \right).
\end{align*}
\]
Making use of (1.5), we obtain
\[
= \sum_{n=0}^{\infty} \frac{(-ckt)^n}{n!4^n k^{n+1}} \left( \Gamma_k(k\tau' + \lambda - \nu - k - 2nk) \right.
\times \Gamma_k(-k\xi - k\zeta' - k\tau + k\gamma + \lambda - \nu - k - 2nk) \\
\left. \times \Gamma_k(-k\xi' - k\zeta + k\gamma + \lambda - \nu - k - 2nk) \right)
\]
\[
\times \Gamma_k(k\tau' + \lambda - \nu - k - 2nk) \Gamma_k(k\tau' + \lambda - \nu - k - 2nk)
\]
\[
\times \Gamma_k(k\tau' + \lambda - \nu - k - 2nk) \Gamma_k(nk + \nu + \frac{3k}{2}) \Gamma_k(nk + \frac{3k}{2})
\]
This on expressing in terms of \( k \)-Wright function \( _p \Psi^k_q \) using (1.2) leads to the right-hand side of (2.6). This completes the proof.

3 Concluding Remark

MSM fractional calculus operators have more advantage due to the generalize of Riemann-Liouville, Weyl, Erdélyi-Kober, and Saigo’s fractional calculus operators; there- fore, many authors are called as general operator. Now we are going to conclude of this paper by emphasizing that our leading results (Theorems 1 – 6) can be derived as the specific cases involving familiar fractional calculus operators as above said. On other hand, the \( k \) Struve function defined in (1.16) possesses the lead that a number of special functions occur to be the particular cases. Some of special cases respect to the integrals relating with \( k \)Struve function have been discovered in the earlier research works by various authors with not the same arguments.

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