A note on local periods for supercuspidal representations

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Abstract

Let $G$ be a $p$-adic reductive group and $H$ a unimodular spherical subgroup of $G$. Let $\pi$ be a unitary supercuspidal representation of $G$. In this note, under a mild assumption, we show that local periods in $\text{Hom}_H(\pi, \mathbb{C})$ can be constructed by integrating the matrix coefficients of $\pi$ over $H$.

1 Main result

Let $F$ be a nonarchimedean local field of characteristic 0. Let $G$ be a reductive group over $F$ and $H$ a unimodular spherical subgroup of $G$, which means that $X = H\backslash G$ is a spherical variety. Write $G = G(F)$ and $H = H(F)$. Denote by $Z$ the center of $G$.

Let $\pi$ be an irreducible admissible representation of $G$ and $V_\pi$ the space of $\pi$. We say that $\pi$ is $H$-distinguished if the space $\text{Hom}_H(\pi, \mathbb{C})$ is nonzero. We call elements of $\text{Hom}_H(\pi, \mathbb{C})$ local periods. If $\pi$ is $H$-distinguished, how to explicitly construct nonzero local periods is an important question. This is part of the local theory of automorphic periods.

In this note, we study the construction of local periods when $\pi$ is unitary supercuspidal. Throughout this note, we make the following assumptions.

Hypothesis 1.1. We assume that

- either $G$ is split and $X$ is wavefront,
- or $X$ is a symmetric space (i.e. $H$ is the subgroup of fix points under an involution on $G$).

Hypothesis 1.2. We assume that $\pi$ is unitary supercuspidal and the central character of $\pi$ is trivial on $H \cap Z$.

Fix a $G$-invariant hermitian inner product $\langle \cdot, \cdot \rangle$ on $V_\pi$. We define a pairing $\mathcal{L}$ on $V_\pi \times V_\pi$ by

$$\mathcal{L}(v, u) = \int_{H/H \cap Z} \langle \pi(h)v, u \rangle \, dh,$$

which is well defined since the matrix coefficient $\phi(g) = \langle \pi(g)v, u \rangle$ is compactly supported modulo $Z$. Note that $\mathcal{L}$ is bi-$H$-invariant. For $u \in V_\pi$, the map $\mathcal{L}_u$ given by

$$\mathcal{L}_u(v) = \mathcal{L}(v, u), \quad v \in V_\pi,$$

belongs to $\text{Hom}_H(\pi, \mathbb{C})$. Denote by $\mathcal{H}$ the subspace of $\text{Hom}_H(\pi, \mathbb{C})$ spanned by $\{\mathcal{L}_u\}_{u \in V_\pi}$.
Theorem 1.3. $\mathcal{H} = \text{Hom}_H(\pi, \mathbb{C})$.

Remark 1.4. When $X$ is a symmetric space and $\pi$ is of the form $\text{ind}_J^G \kappa$ for some open compact subgroup $J$ of $G$ and some irreducible smooth representation $\kappa$ of $J$, results of the above kind have been obtained by J. Hakim, Z. Mao and F. Murnaghan. See [Mu, §8] for a survey. Our method is different from theirs.

Remark 1.5. When $X$ is strongly tempered (cf. [SV, §6.2] for the definition) and $\pi$ is a tempered representation, the pairing $\mathcal{L}$ is well defined. In this case, Y. Sakellaridis and A. Venkatesh [SV, Theorem 6.4.1] showed that $\text{Hom}_H(\pi, \mathbb{C})$ is nonzero if and only if $\mathcal{L}$ is nonzero, by using the Plancherel decomposition of $L^2(X)$ with $X = H \backslash G$. When $(G, H) = (SO_n \times SO_{n+1}, SO_n)$ is in the setting of Gross-Prasad conjecture, A. Ichino and T. Ikeda [II, Proposition 1.1] showed that $X$ is strongly tempered; J.-L. Waldspurger [Wa2, Théorème 1] showed that $\text{Hom}_H(\pi, \mathbb{C}) \leq 1$ (basing on the method of [AGRS]) and [Wa1, Proposition 5.6] also proved that $\text{Hom}_H(\pi, \mathbb{C})$ is nonzero if and only if $\mathcal{L}$ is nonzero by a different method from that of [SV]. Thus, in this case, $\mathcal{H} = \text{Hom}_H(\pi, \mathbb{C})$.

As a consequence, we have the following expression (Corollary 1.6) for the spherical character $\Phi_{\pi, \ell}$ associated to $\ell \in \text{Hom}_H(\pi, \mathbb{C})$. Recall that, for $\ell \in \text{Hom}_H(\pi, \mathbb{C})$, the spherical character $\Phi_{\pi, \ell}$ is defined to be the distribution on $G$ given by

$$\Phi_{\pi, \ell}(f) := \sum_{v \in \text{ob}(\pi)} \ell(\pi(f)v)v, \quad f \in C_c^\infty(G),$$

where $\text{ob}(\pi)$ is an orthonormal basis of $V_\pi$. By Theorem 1.3, there exists $v_0 \in V_\pi$ such that $\ell = \mathcal{L}_{v_0}$. The corollary below is analogous to [Mu, Theorem 6.1] and [IZ, Lemma A.3].

Corollary 1.6. For all $f \in C_c^\infty(G)$, we have

$$\Phi_{\pi, \ell}(f) = \int_{H/H \cap Z} \int_{H/H \cap Z} \left( \int_G f(g)\phi(h_2gh_1) \, dg \right) \, dh_1 \, dh_2,$$

where

$$\phi(g) = \langle \pi(g)v_0, v_0 \rangle, \quad g \in G.$$

Proof. For the proof, see that of [IZ, Lemma A.3].

2 Proof of Theorem 1.3

We say that $\pi$ is relatively supercuspidal (with respect to $H$) if for all $\ell \in \text{Hom}_H(\pi, \mathbb{C})$ and $v \in V_\pi$ the generalized matrix coefficients $\varphi_{\ell, v}$, defined by

$$\varphi_{\ell, v}(g) = \ell(\pi(g)v), \quad g \in G,$$

are compactly supported modulo $ZH$.

Lemma 2.1. If $\pi$ is supercuspidal, it is relatively supercuspidal.

Proof. When $X$ is a symmetric space, it was proved by S. Kato and K. Takano [KT, Proposition 8.1].

When $G$ is split and $X$ is wavefront, the lemma follows from [SV, Proposition 5.3.4] on asymptotics. The basic idea is that the asymptotics behavior of the generalized matrix coefficients can be interpreted by that of the matrix coefficients of $\pi$.

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Now we define an hermitian inner product \((\cdot, \cdot)\) on \(\text{Hom}_H(\pi, \mathbb{C})\). Denote by \(\bar{\pi}\) the complex conjugate of \(\pi\). For \(\ell_1, \ell_2 \in \text{Hom}_H(\pi, \mathbb{C})\) and \(v, v' \in V_\pi\), by the above lemma, the following integral is well defined:

\[
\langle v, v' \rangle_{\ell_1, \ell_2} := \int_{ZH \backslash G} \varphi_{\ell_1, v}(g)\overline{\varphi_{\ell_2, v'}(g)} \, dg.
\]

Then

\[
v \otimes v' \mapsto \langle v, v' \rangle_{\ell_1, \ell_2}
\]

defines a morphism in \(\text{Hom}_G(\pi \otimes \bar{\pi}, \mathbb{C})\). Since \(\dim \text{Hom}_G(\pi \otimes \bar{\pi}, \mathbb{C}) = 1\), there exists \(d_{\ell_1, \ell_2} \in \mathbb{C}\) such that

\[
\langle v, v' \rangle_{\ell_1, \ell_2} = d_{\ell_1, \ell_2} \langle v, v' \rangle
\]

for any \(v, v' \in V_\pi\). Define

\[
(\ell_1, \ell_2) = d_{\ell_1, \ell_2}.
\]

(1)

It is obvious that \((\cdot, \cdot)\) is an hermitian inner product on \(\text{Hom}_H(\pi, \mathbb{C})\).

**Proof of Theorem 1.3.** Let \(\mathcal{H}^\perp\) be the orthogonal complement of \(\mathcal{H}\) in \(\text{Hom}_H(\pi, \mathbb{C})\) with respect to the inner product \((\cdot, \cdot)\) defined as (1). When \(X\) is a symmetric space, P. Delorme \([De,\text{Theorem 4.5}]\) showed that \(\dim \text{Hom}_H(\pi, \mathbb{C}) < \infty\) for any irreducible admissible representation \(\pi\). When \(G\) is split and \(X\) is wavefront, Sakellaridis and Venkatesh \([SV,\text{Theorem 5.1.5}]\) also showed that \(\dim \text{Hom}_H(\pi, \mathbb{C}) < \infty\) for any irreducible admissible representation \(\pi\). Hence, to show \(\mathcal{H} = \text{Hom}_H(\pi, \mathbb{C})\), it suffices to show that \(\mathcal{H}^\perp\) is zero.

From now on, for simplicity and without loss of generality, we assume that the center \(Z\) is anisotropic. Suppose that \(\mathcal{H}^\perp\) is nonzero and choose a nonzero element \(\ell\) of \(\mathcal{H}^\perp\). We will show that there exists a vector \(v_0 \in V_\pi\) so that

\[
(\ell, \mathcal{L}_{v_0}) \neq 0,
\]

which is a contradiction. The arguments are analogous to those in the proof of \([Wa1,\text{Proposition 5.6}]\). For \(u, v \in V_\pi\), set \(\phi(g) = \langle \pi(g)v, u \rangle\) for the matrix coefficient associated to \(u, v\). For \(v_0 \in V_\pi\), we have

\[
\langle v, v_0 \rangle_{\mathcal{L}_{\ell}, \mathcal{L}_{v_0}} = \int_{H \backslash G} \varphi_{\mathcal{L}_{v_0}, v}(g)\overline{\varphi_{\ell, v_0}(g)} \, dg
\]

\[
= \int_{H \backslash G} \left( \int_H \phi(hg) \, dh \right) \overline{\varphi_{\ell, v_0}(g)} \, dg
\]

\[
= \int_G \phi(g)\overline{\ell(\pi(g)v_0)} \, dg
\]

\[
= \ell \left( \int_G \phi(g)\overline{\pi(g)v_0} \, dg \right)
\]

\[
= \ell(\pi(\phi)v_0).
\]
Every step in the above equalities makes sense since $\phi \in C_c^\infty(G)$ and thus the integral over $G$ is actually a finite sum. Now we choose some specific $v_0 \in V$, so that $\ell(v_0) \neq 0$ and set $\phi_0(g) = \langle v_0, \pi(g)v_0 \rangle$. Then, by Schur orthogonality relations, $\pi(\phi_0)v_0 = \lambda v_0$ for some nonzero $\lambda \in \mathbb{C}$. Thus
\[
\langle v_0, v_0 \rangle_{\mathcal{L}, \ell} = \ell(\pi(\phi_0)v_0) = \lambda \ell(v_0) \neq 0.
\]
Therefore, $(\mathcal{L}, \ell) \neq 0$, which completes the proof. 

**Acknowledgements.** The author thanks Wen-Wei Li and Yiannis Sakellaridis for useful conversations.

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