THE HODGE-LAPLACIAN ON THE ČECH-DE RHAM COMPLEX GOVERNS COUPLED PROBLEMS

WIETSE M. BOON∗, DANIEL F. HOLMEN†, JAN M. NORDBOTTEN†, AND JON E. VATNE‡

Abstract. By endowing the Čech-de Rham complex with a Hilbert space structure, we obtain a Hilbert complex with sufficient properties to allow for well-posed Hodge-Laplace problems. We observe that these Hodge-Laplace equations govern a class of coupled problems arising from physical systems including elastically attached rods, multiple-porosity flow systems and 3D-1D coupled flow models.

Key words. Čech-de Rham complex, Hodge-Laplace equation, mathematical modeling

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1. Introduction. The Čech-de Rham complex, introduced in [25], is a double complex that has primarily been used as a tool in algebraic topology to compute cohomology groups and to prove results in homological algebra. We introduce weighted inner products on the Čech-de Rham complex, granting us a Hilbert complex which admits the compactness property. In turn, the codifferential and the corresponding Hodge-Laplace operator can be derived. Using the theory of evolutionary equations, we can also account for time-dependency in the associated Hodge-Laplace problem. At an abstract level, we obtain well-posedness of these systems, an orthogonal decomposition, the Poincaré inequality, convergent mixed finite element approximations, finite-dimensional cohomology, as well as functional guaranteed a posteriori bounds.

The main motivation of this contribution is to share the observation that several models of physical systems of coupled domains have a Čech-de Rham Hodge-Laplacian structure. In this text, we detail the case of 1D elastic rods [1], multiple porosity models [5, 9, 24], and mixed-dimensional coupling with high dimensionality gap [15, 16, 17]. Additionally, a Čech-de Rham Hodge-Laplacian structure can be identified in several other applications such as in the vibration of elastically connected rods [14] and heat- and fluid transfer in layered materials [26, 6]. The developments presented herein mirror the observation that flow in fractured porous media is governed by a Hodge-Laplacian on a double complex [7].

Subsections 1.1 and 1.2 present the mathematical background by introducing the Čech-de Rham complex and the Hodge-Laplace operator for the de Rham complex, respectively. Section 2 combines these two to form the main focus of this work, namely Hodge Laplace problems on the Čech-de Rham complex, and presents the main results from the perspective of Hilbert complexes. Finally, Section 3 presents three examples

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*MOX Scientific Modeling and Computing, Department of Mathematics, Politecnico di Milano, Piazza Leonardo da Vinci 32, Milano, Italy
†Center for Modeling of Coupled Subsurface Dynamics, Department of Mathematics, University of Bergen, Allégaten 41, Bergen, Norway, daniel.holmen@uib.no
‡Department of Economics, BI Norwegian Business School, Kong Christian Frederiks plass 5, Bergen, Norway,

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from physical applications that are governed by the Hodge-Laplace equations.

1.1. The Čech-de Rham complex. Informally speaking, the Čech-de Rham complex is constructed by combining two cochain complexes; the de Rham complex and the Čech complex. The de Rham complex consists of differential forms on a manifold Ω, with the exterior derivative $d$ acting as a differential operator. The Čech complex, on the other hand, introduces an open cover $U$ of $\Omega$ and employs an operator $\delta$, which takes differences of differential forms on the intersections of sets in $U$.

Herein, we focus on the aspects of this theory that are most relevant for our purposes and refer the reader to [8] for a general exposition. Let $U = \{U_i\}_{i \in I}$ be an open cover, indexed by an ordered set of integers $I$, of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. By convention, we use multi-indices (and sometimes multiple subscripts to emphasize the multi-index) to denote intersection $U_i = U_{i_0} \cap \ldots \cap U_{i_p}$ and we let $I^p$ denote the set of increasing multi-indices $i = (i_0, \ldots, i_p)$ with $U_i \neq \emptyset$. Note that $I = I^0$ corresponds to single indices and $I^p$ are multi-indices of length $p+1$, and we will use the index $i$ for both singular indices $i \in I$ and for multi-indices $i \in I^p$.

We will throughout this text assume to be working with a finite good cover, meaning that $I$ is a finite set and all nonempty intersections $U_i = (i_0, \ldots, i_p)$ are diffeomorphic to $\mathbb{R}^n$. Let then $\Lambda^k(U_i)$ be the space of differential $k$-forms on a given open set $U_i$ for $i \in I^p$. The spaces of differential forms together with the exterior derivative $d$ form a cochain complex, meaning that $d^2 = d \circ d = 0$. The complex $(\Lambda^*(U_i), d)$ is called the de Rham complex:

\[
0 \to \Lambda^0(U_i) \xrightarrow{d} \Lambda^1(U_i) \xrightarrow{d} \ldots \xrightarrow{d} \Lambda^{n-1}(U_i) \xrightarrow{d} \Lambda^n(U_i) \to 0.
\]

For a fixed open cover $U$, we write $\mathcal{A}^{p,q} := \bigoplus_{i \in I^p} \Lambda^q(U_i)$ for the de Rham complex on the open cover and its intersections of degree $p$. We now proceed to define a differential operator for the degree of overlap. Consider first each non-empty overlap $U_i = (i_0, i_1)$ with $i_0 < i_1$ and $i \in I^1$, and let the operator $\delta_i : \Lambda^q(U_{i_0}) \oplus \Lambda^q(U_{i_1}) \to \Lambda^q(U_i)$ compute the difference as $\delta_i(\alpha_{i_0}, \alpha_{i_1}) = (\alpha_{i_1} - \alpha_{i_0})|_{U_i}$. This is formally an abuse of notation for the more cumbersome $\alpha_{i_0} - \alpha_{i_0}|_{U_i}$, but will not lead to any confusion. By considering all overlaps, we obtain the operator $\delta : \mathcal{A}^{0,q} \to \mathcal{A}^{1,q}$, such that $((\delta \alpha))_i = \delta_i(\alpha_{i_0}, \alpha_{i_1})$. This operator generalizes to intersections of higher degree $\delta : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ as follows:

\[
(\delta \alpha)_i = \sum_{j=0}^{p+1} (-1)^j (\alpha_{i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{p+1}})|_{U_i}, \quad \forall i \in I^{p+1}, \alpha \in \mathcal{A}^{p,q}.
\]

As an example, consider an open cover $U = \{U_0, U_1, U_2\}$ and let $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathcal{A}^{0,q}$, i.e. each component $\alpha_i$ is a differential form of degree $q$. Then

\[
\delta \alpha = ((\alpha_1 - \alpha_0)|_{U_{0,1}}, (\alpha_2 - \alpha_0)|_{U_{0,2}}, (\alpha_2 - \alpha_1)|_{U_{1,2}}).
\]

If we again apply the operator $\delta$, we get

\[
\delta^2 \alpha = ((\alpha_2 - \alpha_1) - (\alpha_2 - \alpha_0) + (\alpha_1 - \alpha_0))|_{U_{0,1,2}} = 0.
\]

One can readily show that in likeness to the exterior derivative $d$, the difference operator always satisfies $\delta^2 = 0$ and consequently it defines a cochain complex on $\mathcal{A}^{\bullet\bullet}$, called the Čech complex with values in $\Lambda^q$. We refer to the double complex $(\mathcal{A}^{\bullet\bullet}, (d, \delta))$ as the Čech-de Rham complex. The double-graded complex can be turned
into a single-graded total complex by considering the anti-diagonals \( A^k := \bigoplus_{p+q=k} A^{p,q} \), which we also refer to as the Čech-de Rham complex.

Since the difference operator is a finite alternating sum of restrictions, it commutes with the exterior derivative. The total differential is given by

\[
D^k : A^k \rightarrow A^{k+1}, \quad D^k = d + (-1)^k \delta.
\]

Since \( D^k \) acts on a direct sum of spaces of different degree (both in terms of forms and intersections), the degree of \( d \) and \( \delta \) is therefore necessarily determined from context based on the element of \( A^k \) they act on. Since the operators \( d \) and \( \delta \) commute, we get that

\[
D^{k+1} \circ D^k = d^2 + (-1)^k d \delta + (-1)^{k+1} \delta d + (-1)^{2k+1} \delta^2 = 0.
\]

Since \( D^k \) acts on each \( A^{p,q} \) with \( p + q = k \), we frequently omit the indexing of the operators \( d \) and \( \delta \). Note that some authors choose to define the two differential operators in such a way that they are anti-commutative and then define the total differential as \( D = d + \delta \).

The diagram below shows the augmented Čech-de Rham complex. The leftmost column is the original de Rham complex on \( \Omega \) and \( r \) is the sum of the restriction map onto the sets \( U_i \in \mathcal{U} \) with \( i \in I \). The remaining columns are the de Rham complexes on the intersections of sets of degree \( p \). The rows are the Čech complexes with values in \( \Lambda^q \), and the total complex consists of the anti-diagonals of the double complex \( A^{p,q} \):

\[
\begin{array}{cccccccc}
\Lambda^n(\Omega) & \xrightarrow{r} & A^{0,n} & \xrightarrow{(-1)^n \delta} & A^{1,n} & \rightarrow & \cdots & \rightarrow & A^{p,n} \\
\uparrow d & & \downarrow d & & \uparrow d & & \uparrow d & & \uparrow d \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\uparrow d & & \downarrow d & & \uparrow d & & \uparrow d & & \uparrow d \\
\Lambda^1(\Omega) & \xrightarrow{r} & A^{0,1} & \xrightarrow{-\delta} & A^{1,1} & \rightarrow & \cdots & \rightarrow & A^{p,1} \\
\uparrow d & & \downarrow d & & \uparrow d & & \uparrow d & & \uparrow d \\
\Lambda^0(\Omega) & \xrightarrow{r} & A^{0,0} & \xrightarrow{\delta} & A^{1,0} & \rightarrow & \cdots & \rightarrow & A^{p,0}
\end{array}
\]

1.2. The Hodge-Laplacian of the exterior derivative. Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain as before, and let \( \mathcal{N} \) denote the index set \( \{1, \ldots, n\} \). Similarly to the previous section, \( \Lambda^k \) denotes the set of increasing multi-indices of length \( k \). For two differential \( k \)-forms \( \alpha, \beta \in L^2 \Lambda^k(\Omega) \), we can write them as

\[
\alpha = \sum_{j \in \mathcal{N}^k} \hat{\alpha}_j \, dx_j, \quad \beta = \sum_{j \in \mathcal{N}^k} \hat{\beta}_j \, dx_j,
\]

with \( \hat{\alpha}_j, \hat{\beta}_j \in L^2(\Omega) \). Using the volume form \( \text{vol}_\Omega = \bigwedge_{j=1}^n dx_j \), the inner product on \( \Lambda^k(\Omega) \) and its induced norm are given by

\[
\langle \alpha, \beta \rangle_{\Lambda^k(\Omega)} := \int_\Omega \sum_{j \in \mathcal{N}^k} \hat{\alpha}_j \hat{\beta}_j \, \text{vol}_\Omega, \quad \| \alpha \|_{\Lambda^k(\Omega)} := \sqrt{\langle \alpha, \alpha \rangle_{\Lambda^k(\Omega)}}.
\]

To obtain a more general class of models, we will introduce spatially varying weights in the inner product. We define the weighted inner product by considering a
collection \( \{w_k\}_{k=0}^{n} \) of bijective, symmetric, bounded linear operators \( w_k : L^2\Lambda^k(\Omega) \rightarrow L^2\Lambda^k(\Omega) \). We can describe the weighted inner product in terms of the unweighted inner product in the following way:

\[
\langle \alpha, \beta \rangle_{\Lambda^k_w(\Omega)} = \langle w_k \alpha, \beta \rangle_{\Lambda^k(\Omega)}.
\]

The weights are identified with material parameters in Section 3. Note that the unweighted inner product corresponds to each \( w_k \) being equal to the identity operator. We will work with weighted inner products, with the understanding that we obtain the standard inner product by considering unit weights.

The inner product gives rise to an adjoint operator \( d^* \) of the exterior derivative, called the codifferential:

\[
\langle \alpha, d^* \beta \rangle_{\Lambda^k_w(\Omega)} = \langle d^k \alpha, \beta \rangle_{\Lambda^{k-1}_w(\Omega)}.
\]

Some authors use \( \delta \) to denote the codifferential. We reserve this letter for the difference operator from (1.2) associated to \( \check{\text{C}} \)ech complex. We sometimes emphasize that the codifferential arises from a weighted inner product by denoting it \( d^*_w \).

**Proposition 1.1.** The weighted adjoint \( d^*_w \) can be expressed in terms of the unweighted adjoint \( d^* \) in the following way:

\[
d^*_w = w_{k-1}^{-1}d^*_{k-1}w_k.
\]

Here the adjoints of the weights are defined in terms of the unweighted inner-products.

**Proof.** By the definition of the weighted inner product and adjoint operators, we have that

\[
\langle d^{k-1} \alpha, \beta \rangle_{\Lambda^k_w(\Omega)} = \langle w_k d^{k-1} \alpha, \beta \rangle_{\Lambda^k_w(\Omega)}
\]

\[
= \langle w_k d^{k-1}(w_{k-1}^{-1} w_k^{-1}) \alpha, \beta \rangle_{\Lambda^k_w(\Omega)}
\]

\[
= \langle (w_k d^{k-1}(w_{k-1}^{-1} w_k^{-1}))^{-1} w_k \alpha, \beta \rangle_{\Lambda^k_w(\Omega)}
\]

\[
= \langle w_{k-1} \alpha, (w_k d^{k-1}(w_{k-1}^{-1} w_k^{-1}))^* \beta \rangle_{\Lambda^{k-1}_w(\Omega)}
\]

\[
= \langle \alpha, w_{k-1}^{-1} d^*_{k-1} w_k \beta \rangle_{\Lambda^{k-1}_w(\Omega)}.
\]

In the final equality, we used the symmetry of \( w_k \) and \( w_{k-1} \). The result (1.11) now follows. \( \square \)

The requirement for the weights to be bijective is evident from this proposition, as the expression for \( d^*_w \) involves the inverse \( w_{k-1}^{-1} \), which we require to be bounded. The differential together with the codifferential defines the (weighted) Hodge-Laplacian of the exterior derivative,

\[
\Delta^k_{d,w} = d^{k-1} d^*_{k-1} + d^*_k d^k.
\]

The exterior derivative together with the associated codifferential and Hodge-Laplacian gives an orthogonal decomposition of differential forms which generalizes the Helmholtz decomposition for vector fields, called the Hodge decomposition:

\[
L^2\Lambda^k_w(\Omega) = \text{im} d^{k-1} \oplus \ker \Delta^k_{d,w} \oplus \text{im} d^*_{k+1}.
\]

Furthermore, we can identify the kernel of the Hodge-Laplacian with the cohomology of the manifold: \( \ker \Delta^k_{d,w} \equiv \ker d^k/\text{im} d^{k-1} \). For contractible domains, the de Rham cohomology vanishes for \( k > 0 \), and therefore we can write any differential form as the image of \( d \) and \( d^* \), i.e. we write \( \alpha = d \beta + d^* \eta \).
2. The Hodge-Laplacian on the $L^2$ Čech-de Rham complex. In this section, we construct inner products on the Čech-de Rham complex and show that it forms a Hilbert complex, i.e. a cochain complex where each space is a Hilbert space and the differential operators are closed, densely-defined unbounded linear operators.

Moreover, we define the Hodge-Laplace operator for the Čech-de Rham complex and show that the complex satisfies the compactness property. Several theoretical results that hold for arbitrary closed Hilbert complexes are then summarized. In particular, we obtain the well-posedness of the Hodge-Laplace problem of the Čech-de Rham complex, meaning that mathematical models that fit into this framework will be well-posed.

2.1. The $L^2$ Čech-de Rham complex. In this section, we combine subsection 1.1 and subsection 1.2 to make the Čech-de Rham complex into a Hilbert complex. For a given open cover $\mathcal{U}$, we start by defining the Hilbert spaces containing square-integrable forms $L^2\mathcal{A}^{p,q}$ and the spaces formed by the anti-diagonals of the complex $L^2\mathcal{A}^k$ with $k = p + q$:

\[
L^2\mathcal{A}^{p,q} := \bigoplus_{i \in \mathcal{I}^p} L^2\Lambda^q(U_i), \quad L^2\mathcal{A}^k := \bigoplus_{p+q=k} L^2\mathcal{A}^{p,q}.
\]

In order to define a weighted inner product on the Čech-de Rham complex, we require a collection $\{w_k\}_{k \in \mathbb{Z}}$, where each $w_k$ consists of a set of $k+1$ weights:

\[
w_k = \{w^k_i : L^2\Lambda^q(U_i) \rightarrow L^2\Lambda^q(U_i) : i \in \mathcal{I}^p, p \in \{0, ..., k\}, p + q = k\},
\]

where each $w^k_i$ is a bijective, symmetric, bounded linear operator as before. The weighted inner product is then defined as

\[
\langle \alpha, \beta \rangle_{\mathcal{A}^k_w} = \langle w_k \alpha, \beta \rangle_{\mathcal{A}^k} = \sum_{p=0}^{k} \sum_{i \in \mathcal{I}^p} \langle w^k_i \alpha_i, \beta_i \rangle_{\Lambda^q(U_i)}.
\]

As an example, on an open cover $\mathcal{U} = \{U_0, U_1\}$, the inner products for $f = (f_0, f_1), g = (g_0, g_1) \in \mathcal{A}^0$, are given by

\[
\langle f, g \rangle_{\mathcal{A}^0_w} = \langle w^0_0 f_0, g_0 \rangle_{\Lambda^0(U_0)} + \langle w^0_1 f_1, g_1 \rangle_{\Lambda^0(U_1)}.
\]

Similarly for $\alpha = ((\alpha_0, \alpha_1), (\alpha_{0,0}, \alpha_{0,1}), \beta = ((\beta_0, \beta_1), (\beta_{0,0}, \beta_{0,1})) \in \mathcal{A}^1 = \mathcal{A}^{0,1} \oplus \mathcal{A}^{1,0}$, we have the following expression for the inner product:

\[
\langle \alpha, \beta \rangle_{\mathcal{A}^1_w} = \langle w^1_0 \alpha_0, \beta_0 \rangle_{\Lambda^1(U_0)} + \langle w^1_1 \alpha_1, \beta_1 \rangle_{\Lambda^1(U_1)} + \langle w^1_{0,0} \alpha_{0,0}, \beta_{0,0} \rangle_{\Lambda^0(U_{0,0})}.
\]

We define the $L^2$ Čech-de Rham complex as $L^2\mathcal{A}^k$ equipped with the total differential $D^k := d + (-1)^k \delta$. The total differential $D^k$ is not well-defined on the entire Hilbert space $L^2\mathcal{A}^k$. However, we show that it is a closed densely-defined unbounded linear operator, see e.g. [3, Chap. 3], which is necessary to have a Hilbert complex.

Proposition 2.1. $D^k : \text{dom } D^k \subset L^2\mathcal{A}^k \rightarrow L^2\mathcal{A}^{k+1}$ is a closed densely defined unbounded linear operator (CDUL for short).

Proof. There are three properties that need to be addressed: closed, densely defined and linear. Having an unbounded operator in not necessary, as a bounded operator can also be a CDUL operator. However, $D^k$ will be unbounded because the
exterior derivative is unbounded. Linearity is obvious, since the sum of two linear operators is again linear.

We will show that \( D^k \) is densely defined in \( L^2 A^k \). For a given \( k \), the exterior derivative \( d^k \) is a CDUL operator on \( L^2 A^{p,q} \) for each pair \( (p, q) \) satisfying \( p + q = k \). By direct summation, it follows that the sum \( \bigoplus_q d^q \) is a CDUL on \( L^2 A^k \). Next, we note that \( \bigoplus_q \text{dom} d^q \subset L^2 A^k \) and \( \bigoplus_p \text{dom} \delta^p = L^2 A^k \) and so \( \text{dom} D^k = \bigoplus_p \text{dom} \delta^p \cap \bigoplus_q \text{dom} d^q = \bigoplus_q \text{dom} d^q \), which is dense in \( L^2 A^k \).

It remains to show that \( D^k \) is closed. Consider a sequence \( \alpha_1, \alpha_2, \ldots \in \text{dom} D^k \) such that \( \alpha_m \to \alpha \) and \( D\alpha_m \to \beta \), as \( m \to \infty \), for some \( \alpha \in L^2 A^k \) and \( \beta \in L^2 A^{k+1} \). By the continuity of \( \delta \), we have that \( d\alpha_m \to (\beta - (-1)^k \delta \alpha) \in L^2 A^{k+1} \). Since \( D^k \) is closed, it follows that \( \alpha \in \text{dom} d^k = \text{dom} D^k \) and \( d\alpha = \beta - (-1)^k \delta \alpha \), which implies \( D\alpha = \beta \).

2.2. The total Hodge-Laplace operator. We continue by defining the total codifferential \( D_k^{*,w} : \text{dom} D_k^{*,w} \subset L^2 A^k \to L^2 A^{k-1} \) through the adjoint relationship:

\[
(D_k^{*,w} \alpha, \beta)_{A^{k-1}} = (\alpha, D^{k-1} \beta)_{A^k}
\]

Remark 2.2. The codifferential \( D_k^{*,w} \) is also a CDUL on \( L^2 A^k \) by [3, Prop. 3.3]. We distinguish the codifferential corresponding to the unweighted inner product and the adjoint corresponding to the weighted inner product, and we write \( D_k^{*,1} \) to emphasize the unweighted adjoint when necessary.

Similarly to how we define the adjoint of the exterior derivative, the adjoint of the difference operator is defined through the equation \( (\delta \alpha, \beta)_{A^{p,q}} = (\alpha, \delta^* \beta)_{A^{p,q}} \). We can describe the adjoint of the difference operator \( \delta \) with the characteristic function.

Proposition 2.3. The adjoint of the total differential takes the following form:

\[
D_k^{*,w} = d^{*,w} - (-1)^k \delta^{*,w}.
\]

Proof. The result follows immediately from the linearity of the inner product and the definition of the adjoint.

\[
(\alpha, D_k^{*,w} \beta)_{A^{k-1}} = (D_k^{k-1} \alpha, \beta)_{A^k} = (\alpha, D_k^{*,1} \beta)_{A^k} + (\delta \alpha, \beta)_{A^k} + (\delta^* \beta, \alpha - (-1)^{k-1} \delta \alpha)_{A^k} = (\alpha, D_k^{*,w} \beta)_{A^{k-1}} + (\alpha, (-1)^{k-1} \delta^{*,w} \beta)_{A^k} = (\alpha, D_k^{*,w} - (-1)^k \delta^{*,w} \beta)_{A^{k-1}}.
\]

Following Subsection 1.2, the unweighted and weighted Hodge-Laplace operators for the Čech-de Rham complex are defined as

\[
\Delta_{D,1}^k := D_k^{k-1} D_k^{*,1} + D_k^{1,*} D_k^k,
\]

\[
\Delta_{D,w}^k := D_k^{k-1} D_k^{*,w} + D_k^{1,*w} D_k^k.
\]

While the weights are important in applications, from the perspective of analysis the distinction is typically not critical, and when no confusion arises, we will in the continuation omit both the weight and the degree \( k \) in the notation of \( D, D^*, \) and \( \Delta_D \). The unweighted and weighted Hodge-Laplace problem are then both defined as follows: Given \( \varphi \perp \ker \Delta_D \), find \( \alpha \in \text{dom} \Delta_D \) such that:

\[
\Delta_D \alpha = \varphi, \quad \alpha \perp \ker \Delta_D.
\]
The operators $\Delta_{D,w}$ and $\Delta_{D,1}$ can be explicitly related using the knowledge of the exterior derivative $d$ and the difference operator $\delta$, according to the following two propositions.

**Proposition 2.4.** The weighted Hodge-Laplacian on the Čech-de Rham complex decomposes to the weighted Hodge-Laplacians of $d$ and $\delta$ plus four coupling terms:

\[
(2.11) \quad \Delta_{D,w} = \Delta_{d,w} + \Delta_{\delta,w} + (-1)^k(d^{*,w}\delta - \delta d^{*,w} + \delta^{*,w}d - d\delta^{*,w}).
\]

**Proof.** We substitute the definition of $D^k$ and the expression for the weighted adjoint $D^*_k$ given in Proposition 2.3 into the definition of the Hodge-Laplacian to get

\[
\Delta^k_{D,w} = D^*_k D^k + D^{k-1}_k D^*_k
\]

\[
= (d^{*,w} + (-1)^k \delta^{*,w}) (d + (-1)^k \delta) + (d + (-1)^k \delta)(d^{*,w} + (-1)^k \delta^{*,w})
\]

\[
= d^{*,w}d + dd^{*,w} + \delta^{*,w}d + d\delta^{*,w} + (-1)^k(d^{*,w}\delta + \delta^{*,w}d - d\delta^{*,w} - \delta d^{*,w}).
\]

Reordering terms and identifying $d^{*,w}d + dd^{*,w}$ with $\Delta_{d,w}$ and $\delta^{*,w}\delta + \delta\delta^{*,w}$ with $\Delta_{\delta,w}$, we arrive at

\[
\Delta_{D,w} = \Delta_{d,w} + \Delta_{\delta,w} + (-1)^k(d^{*,w}\delta - \delta d^{*,w} + \delta^{*,w}d - d\delta^{*,w}).
\]

The unweighted Hodge-Laplacian takes a simpler form, as pointed out in the next corollary.

**Corollary 2.5.** The unweighted Hodge-Laplacian on the Čech-de Rham complex decomposes to the sum of the unweighted Hodge-Laplacians of $d$ and $\delta$:

\[
(2.12) \quad \Delta_{D,1} = \Delta_{d,1} + \Delta_{\delta,1}.
\]

**Proof.** For $\alpha \in \text{dom}(\Delta^k_{D,1})$, we have that $d\delta^{*}\alpha = \delta^{*}d\alpha$ as well as $d^{*}\delta\alpha = \delta d^{*}\alpha$, and hence the coupling terms in (2.11) cancel out, and we are left with the stated corollary.

The diagram below illustrates how the different components of the weighted Hodge-Laplacian acts on differential forms of degree $(p, q)$. The vertical and horizontal maps are the Hodge-Laplacians $\Delta_{d,w}$ and $\Delta_{\delta,w}$, respectively, which increases or decreases the degree of $q$ or $p$ by one, then maps back to $A^{p,q}$. The coupling terms preserves the total degree $k$, but with codomain $A^{p-1,q+1}$ and $A^{p+1,q-1}$. 

\[
\begin{align*}
A^{p-1,q+1} 
\rightarrow & \quad (\delta^{*,w}d - dd^{*,w}) \\
\downarrow & \quad (-1)^k(d^{*,w}\delta - \delta d^{*,w}) \\
\rightarrow & \quad A^{p,q} \\
\downarrow & \quad \delta^{*,w}d \\
\rightarrow & \quad A^{p+1,q-1}
\end{align*}
\]

(2.13)
2.3. Results from Hilbert complex theory. In the following section we show that the \( L^2 \) Čech-de Rham complex is not just a Hilbert complex, but it also satisfies the compactness property, which in turn implies that the range of \( D^k \) is closed in \( L^2 A^{k+1} \). As a consequence, general results for Hilbert complexes with the compactness property can now be directly applied to all coupled problems that can be identified as a Hodge-Laplace problem on a Čech-de Rham complex.

**Theorem 2.6.** The \( L^2 \) Čech-de Rham complex has the compactness property, i.e. the inclusion \( \text{dom}(D^k) \cap \text{dom}(D^*_k) \subset L^2 A^k \) is compact for all \( k \).

**Proof.** As noted in Proposition 2.1, \( \text{dom} D^{p,q} = \text{dom} d^q \) and by analogous arguments, we have \( \text{dom} D^*_{p,q} = \text{dom} d^*_{p,q} \) and thus, \( \text{dom} D^{p,q} \cap \text{dom} D^*_{p,q} = \text{dom} d^q \cap \text{dom} d^*_{p,q} \), which is compactly embedded in each \( L^2 A^{p,q} \), see [21, 22]. The compact embedding in \( L^2 A^k \) then follows by direct summation. \( \Box \)

We now summarize some general results for Hilbert complexes with the compactness property in the following corollaries, with the understanding that for any of the concrete examples provided in Section 3, sharper results may be obtained by exploiting problem-specific aspects of the examples.

**Corollary 2.7 ([3, Thm. 4.4]).** The \( L^2 \) Čech-de Rham complex is a closed Hilbert complex, meaning that the range of \( D^k \) is closed in \( L^2 A^{k+1} \). Moreover, the \( L^2 \) Čech-de Rham complex is a Fredholm complex, i.e. it has finite-dimensional cohomology.

**Corollary 2.8 ([3, Thm. 4.5]).** For each \( k \), we have the orthogonal Hodge decomposition:

\[
L^2 A^k = \text{im} D^{k-1} \oplus \ker \Delta^k D_k \oplus \text{im} D^*_{k+1}.
\]

**Corollary 2.9 ([3, Thm. 4.6]).** For each \( k \), there exists a constant \( C_k \) such that the Poincaré inequality holds:

\[
\|\alpha\|_{A^k} \leq C_k \|D^k \alpha\|_{A^{k+1}}, \quad \forall \alpha \in \text{dom} D^k \cap (\ker D^k)^\perp.
\]

A major consequence of Theorem 2.6 is that all systems of equations corresponding to a Hodge-Laplace problem are well-posed, as shown in the following corollary.

**Corollary 2.10 ([3, Thm. 4.8]).** The Hodge-Laplace problem on the Čech-de Rham complex (2.10) is well-posed.

2.4. Solution theory of evolutionary equations. In this section, we generalize the Hodge-Laplace problem (2.10) by introducing a time-dependent term. In particular, let us consider the following problem: find \( \alpha \) such that

\[
\partial^\ell \alpha + \Delta_D \alpha = \varphi, \quad \ell \in \{1, 2\}.
\]

For \( \ell = 1 \), we refer to this problem as the parabolic Hodge-heat equation. On the other hand, for \( \ell = 2 \), we obtain the hyperbolic Hodge-wave equation. Since we aim to analyze both problems in a single framework, we turn to the solution theory for evolutionary equations [23].

We start by introducing several preliminary definitions. First, let \( V \) be the Hilbert space given by

\[
V := L^2 A^k \times L^2 A^{k+1} \times L^2 A^{k-1}.
\]
Let the inner product and norm on \( V \) be defined as follows for \( v = (\alpha, \beta, \gamma) \in V \) and \( \hat{v} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma}) \in V: \)

\[
(\hat{v}, v)_V := \langle \alpha, \hat{\alpha} \rangle_{\mathcal{A}_k^0} + \langle \beta, \hat{\beta} \rangle_{\mathcal{A}_k^1} + \langle \gamma, \hat{\gamma} \rangle_{\mathcal{A}_k^{-1}}, \quad \|v\|_V^2 := (v, v)_V.
\]

Next, for given parameter \( \mu \), we define the weighted Bochner space and its associated norm as follows:

\[
L^2_\mu(\mathbb{R}; V) := \left\{ g : \mathbb{R} \to V \mid \|g\|_{L^2_\mu(\mathbb{R}; V)} < \infty \right\}, \quad \|g\|_{L^2_\mu(\mathbb{R}; V)} := \int_\mathbb{R} e^{-\mu t}\|g(t)\|_V^2\,dt.
\]

On this space, the temporal derivative is defined as

\[
\partial_{0, \mu} : \text{dom}(\partial_{0, \mu}) \subseteq L^2_\mu(\mathbb{R}; V) \to L^2_\mu(\mathbb{R}; V), \quad \partial_{0, \mu} := e^{\mu t}(\partial_t + \mu)e^{-\mu t}.
\]

We emphasize that \( \partial_{0, \mu} g = \partial_t g \) for differentiable \( g : \mathbb{R} \to V \), which can be confirmed by direct calculation.

Finally, for an operator \( A : \text{dom}(A) \subseteq V \to V \) let \( \mathcal{G}(A) := \{(v, Av), v \in \text{dom}(A)\} \) be its graph. \( \overline{A} \) then denotes the closure of \( A \) which, if it exists, satisfies \( \mathcal{G}(\overline{A}) = \mathcal{G}(A) \).

With these prerequisites in place, we state the following key result for linear evolutionary equations.

**Theorem 2.11** ([23, Thm. 2.5]). Let \( \mu > 0 \) and \( r > \frac{1}{2\mu} \). Let \( A : \text{dom}(A) \subseteq V \to V \) and let \( M(z) : V \to V \) be a continuous linear operator for all \( z \) in the open disk \( B_\mathbb{C}(r, r) \subset \mathbb{C} \) with center \( r \) and radius \( r \). Assume that:



**A1.** The operator \( A \) is skew-adjoint.

**A2.** There exists \( c > 0 \) such that \( z^{-1}M(z) - cI \) is monotone for all \( z \in B_\mathbb{C}(r, r) \).

Then, for a given \( f \in L^2_\mu(\mathbb{R}; V) \), a unique \( v \in L^2_\mu(\mathbb{R}; V) \) exists that satisfies the evolutionary equation

\[
(\partial_{0, \mu} + M(\partial_{0, \mu}^{-1}) + A) v = f.
\]

Moreover, \( v \) satisfies the bound \( c\|v\|_{L^2_\mu(\mathbb{R}; V)} \leq \|f\|_{L^2_\mu(\mathbb{R}; V)} \).

We are now ready to fit the Hodge-heat and Hodge-wave equations (2.16) in the framework of evolutionary equations and show that these problems are well-posed.

**Corollary 2.12.** For \( \mu > 0 \) and \( \varphi \in L^2_0(\mathbb{R}; L^2_\mathcal{A}^k) \), the Hodge-heat problem (2.16) with \( \ell = 1 \) admits a unique solution \( \alpha \in L^2_0(\mathbb{R}; L^2_\mathcal{A}^k) \) that satisfies

\[
c\|\langle \alpha, D^k \alpha, D^k_\mathcal{A} \rangle\|_{L^2_0, \mu(\mathbb{R}; V)} \leq \|\varphi\|_{L^2_0, \mu(\mathbb{R}; L^2_\mathcal{A}^k)},
\]

for some \( c > 0 \).

**Proof.** We cast the problem in the format of evolutionary equations (2.20) and then invoke the well-posedness result from Theorem 2.11. Let \( \beta = D^k \alpha \) and \( \gamma = D^k_\mathcal{A} \alpha \), so that the problem is rewritten as: find \( (\alpha, \beta, \gamma) \in V \) such that

\[
(\partial_{0, \mu} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -D^k & D^k_\mathcal{A} \\ -D^k_\mathcal{A} & D^k \end{bmatrix}) \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \varphi \\ 0 \\ 0 \end{bmatrix}.
\]

Problem (2.22) has the form (2.20) with \( v := (\alpha, \beta, \gamma) \), \( f := (\varphi, 0, 0) \), and

\[
M(z) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad A := \begin{bmatrix} -D^k & D^k_\mathcal{A} \\ -D^k_\mathcal{A} & D^k \end{bmatrix}.
\]
We now confirm the two assumptions of Theorem 2.11. First, A1 is valid by the definition of the total codifferential (2.6). For the second, we note that $z \in B_C(r, r)$ implies that $\text{Re}(z^{-1}) > \frac{1}{2r}$. In turn, we derive

$$\text{Re} \left( \langle v, z^{-1}M(z)v \rangle_V \right) = \text{Re} \left( \left\langle v, \left[ \begin{array}{c} z^{-1} \\ 1 \end{array} \right] v \right\rangle_V \right) \geq \min \left\{ 1, \frac{1}{2r} \right\} \langle v, v \rangle_V.$$ 

By linearity of $M(z)$, assumption A2 now follows with $c = \min \left\{ 1, \frac{1}{2r} \right\} > 0$. Theorem 2.11 then implies that a unique $v = (\alpha, \beta, \gamma)$ exists, bounded in the $L^2_{0,\mu}(\mathbb{R}; V)$-norm. In turn, the result (2.21) follows by equivalence of the problems (2.16) and (2.22).

**Corollary 2.13.** For $\mu > 0$ and $\varphi \in L^2_{0,\mu}(\mathbb{R}; L^2A^k)$ with $\partial^{-1}_{0,\mu}\varphi \in L^2_{0,\mu}(\mathbb{R}; L^2A^k)$, the Hodge-wave problem (2.16) with $\ell = 2$ admits a unique solution $\alpha \in L^2_{0,\mu}(\mathbb{R}; L^2A^k)$ that satisfies

$$c \| (\alpha, \partial^{-1}_{0,\mu}D^k\alpha, \partial^{-1}_{0,\mu}D^k\gamma) \|_{L^2_{0,\mu}(\mathbb{R}; V)} \leq \| \partial^{-1}_{0,\mu}\varphi \|_{L^2_{0,\mu}(\mathbb{R}; L^2A^k)},$$

for some $c > 0$.

**Proof.** The proof proceeds similar to Corollary 2.12. Let $V$ be defined as in (2.17) and let now $\beta = \partial^{-1}_{0,\mu}D^k\alpha$ and $\gamma = \partial^{-1}_{0,\mu}D^k\gamma$. The problem then becomes: find $(\alpha, \beta, \gamma) \in V$ such that

$$(\partial_{0,\mu}I + A) \left[ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right] = \partial^{-1}_{0,\mu} \left[ \begin{array}{c} \varphi \\ 0 \\ 0 \end{array} \right],$$

with $A$ as in (2.23). Again, the first assumption A1 follows directly from (2.6). Here, A2 is also immediate with $c = \frac{1}{2r}$ since $M(z) := I$. To obtain the result on $\alpha$, we use the equivalence between the problems (2.16) and (2.25).

A more general class of nonlinear problems involving maximal monotone operators can be analyzed by following [23, Thm. 3.2]. We have restricted our exposition to the linear case for conciseness.

**2.5. Further implications for approximation methods methods.** Finally, we mention that the framework of finite element exterior calculus [4] directly provides stable and convergent mixed finite element discretizations for these problems, given a conforming, simplicial grid. Additionally, functional guaranteed a posteriori bounds can be obtained using the techniques summarized in [20].

**3. Examples in mathematical modeling.** We consider three elementary examples in which $U = \{U_0, U_1\}$ forms an open cover of a given domain $\Omega$, as illustrated in Figure 1. In each case, we show that the Hodge-Laplace problem corresponds to the coupled equations that arise in physical applications.

We focus on $\Delta_{D,w}^0$ i.e. the weighted Hodge-Laplace operator defined on $L^2A^0 = L^2A^{0,0}$ in the bottom-left corner of the double complex (1.7). To make the exposition accessible, we will use the canonical proxies of differential forms, through the identification of “scalar-valued” spaces $L^2\Lambda^0(U_i) \cong L^2(U_i)$, as well as “vector-valued” spaces $L^2\Lambda^1(U_i) \cong (L^2(U_i))^n$.

In particular, we will denote proxies by Latin letters: each 0-form $\alpha_i \in L^2\Lambda^0(U_i)$ is identified by a scalar proxy $a_i \in L^2(U_i)$ and each 1-form $\beta_i \in L^2\Lambda^1(U_i)$ can be identified by a vector proxy $b_i \in (L^2(U_i))^n$. 
On $L^2 \mathcal{A}^0$, the differential and difference operators then correspond to $d \alpha_i = \nabla a_i$ and $\delta \alpha = (a_1 - a_0)|_{\overline{U}_{0,1}}$, respectively. These definitions lead to the following diagram:

\[
\begin{array}{c}
(b_0, b_1) \\ \nabla \\
\end{array}
\begin{array}{c}
\delta \\
\nabla \\
\end{array}
\begin{array}{c}
(a_0, a_1) \\
\delta \\
\end{array}
\quad \begin{array}{c}
g_{0,1} \\
\vspace{1cm} \\
\delta \\
\vspace{1cm} \\
\end{array}
\]

\[(3.1)\]

When considering weighted inner products as defined in Section 2.1, we will consistently use the convention that we scale the problem to accept unit weights for the inner product on $\mathcal{A}^0$, while we apply weighted inner products on $\mathcal{A}^1$, denoted $w_0$, $w_1$ and $w_{0,1}$. The adjoint operator for $D_1^*$ is then calculated as

\[
\langle D_1^* \beta, \alpha \rangle_{\mathcal{A}^0} = \langle \beta, D_0^* \alpha \rangle_{\mathcal{A}^1} = \langle -\nabla \cdot (w_i \nabla a_i) - (a_1 - a_0)|_{\overline{U}_{0,1}}, a_i \rangle_{\mathcal{A}^0} + \sum_i \int_{\partial U_i} a_0 w_{0,1} b_{0,1} \cdot dA
\]

\[(3.2)\]

We limiting the domain of the codifferential so that the surface integrals vanish. Thus the codifferential $D_1^*$, expressed on the proxies, is composed of the divergence operator restricted to functions with homogeneous boundary conditions $\nabla \cdot$, and the indicator function of the overlap $U_{0,1} \subset U_i$ denoted $\mathbb{1}_{0,1}^i$:

\[
D_1^* \beta = -\nabla \cdot (w_i b_i) + (-1)^{i+1} \mathbb{1}_{0,1}^i (w_{0,1} b_{0,1}), \quad \text{on } U_i.
\]

Composing the differential and codifferential operators, the Hodge-Laplace problem becomes as follows for given $\varphi = \{f_0, f_1\} \in L^2 \mathcal{A}^0$: Find $\alpha \in \text{dom } \Delta_D \subset L^2 \mathcal{A}^0$ such that $\Delta_D \alpha = \varphi$. Written using proxies, the Hodge-Laplace problem becomes

\[
\begin{align*}
\text{(3.3a)} & \quad -\nabla \cdot (w_0 \nabla a_0) - \mathbb{1}_{0,1}^0 (w_{0,1} a_0 - a_0) = f_0, \quad \text{on } U_0, \\
\text{(3.3b)} & \quad -\nabla \cdot (w_1 \nabla a_1) + \mathbb{1}_{0,1}^1 (w_{0,1} a_1 - a_0) = f_1, \quad \text{on } U_1.
\end{align*}
\]

The domain of the adjoint differential imposes the following boundary conditions:

\[
\text{(3.3c)} \quad \nu \cdot (w_i \nabla a_i) = 0, \quad \text{on } \partial U_i,
\]
in which \( \nu \) is the outward unit normal of \( \partial U_i \). Finally, the kernel of the Hodge-Laplacian, \( \ker \Delta_D \) is given by the constants, thus orthogonality to the kernel imposes the constraint

\[
(3.3d) \quad \langle a_0, 1 \rangle_{U_0} + \langle a_1, 1 \rangle_{U_1} = 0.
\]

We can similarly state the mixed formulation of the Hodge-Laplace problem (3.3) in terms of the canonical proxies: Find \( (\alpha, \beta) \in \text{dom} D^0 \oplus \text{dom} D^1 \subset L^2 A^0 \oplus L^2 A^1 \) such that

\[
(3.4a) \quad b_0 = \nabla a_0, \quad b_1 = \nabla a_1, \quad b_{0,1} = a_1 - a_0,
\]

\[
(3.4b) \quad -\nabla \cdot (w_0 b_0) - \mathbb{1}_{0,1}^0 (w_{0,1} b_{0,1}) = f_0, \quad -\nabla \cdot (w_1 b_1) + \mathbb{1}_{0,1}^1 (w_{0,1} b_{0,1}) = f_1,
\]

subject to the boundary conditions

\[
(3.4c) \quad \nu \cdot (w_i b_i) = 0, \quad \text{on } \partial U_i.
\]

The same constraint, equation (3.3d), as in the primal formulation still holds.

**Remark 3.1.** Problem (3.4) is one of \( (n + 2) \) Hodge-Laplace equations arising from the cover \( \mathcal{U} = \{ U_0, U_1 \} \), since the order \( k \) ranges from zero to \( (n + 1) \). Using Corollary 2.5, the (unweighted) Hodge-Laplace operator for \( \gamma \in L^2 A^k \) with \( 1 \leq k \leq n \) can be written in terms of proxies as

\[
\Delta_D^0 \gamma = -\Delta g_i + \mathbb{1}_{0,1}^0 (g_i - g_j) \quad \text{on } U_i, \quad i + j = 1,
\]

\[
\Delta_D^1 \gamma = -\Delta g_{0,1} + 2 g_{0,1} \quad \text{on } U_{0,1}.
\]

Here, \(-\Delta g_i\) denotes the vector-Laplacian \( \nabla \times \nabla \times g_i - \nabla (\nabla \cdot g_i) \) if the proxy \( g_i \) is vector-valued \( (n = 3 \text{ and } k \in \{1, 2\}) \). For \( \gamma \in L^2 A^{n+1} \) at the end of the complex, we derive:

\[
\Delta_D^{n+1} \gamma = -\Delta g_{0,1} + 2 g_{0,1} \quad \text{on } U_{0,1}.
\]

All \( (n + 2) \) distinct Hodge-Laplace problems are well-posed (subject to appropriate boundary conditions and orthogonality constraints as given above) due to Corollary 2.10.

**Remark 3.2.** If the open cover contains more sets, i.e. \( \mathcal{U} = \{ U_0, ..., U_N \} \), then the weighted Hodge-Laplace operator on \( L^2 A^0 \) is given by:

\[
\Delta_D^0 \alpha = -\nabla \cdot (w_i \nabla a_i) + \sum_{j \neq i} \mathbb{1}_{i,j}^i w_{j,i} (a_i - a_j) \quad \text{on } U_i, \quad i \in \mathcal{I},
\]

using the convention \( \mathbb{1}_{i,j}^i = -\mathbb{1}_{j,i}^i \).

**3.1. Two joined, elastic rods in 1D.** We start with the geometry illustrated in the left of Figure 1, in which the dimension \( n = 1 \), and the sets \( U_0 := (-1, \varepsilon) \) and \( U_1 := (-\varepsilon, 1) \) for some \( 0 < \varepsilon < 1 \) form an open cover of \( \Omega = (-1, 1) \). This case can be physically realized as illustrated in Figure 2, where the model for displacement and strain correspond to the \( k = 0 \) Hodge-Laplace problem.

For this case, the system (3.4) describes two linearly elastic rods that are elastically connected. As is clear from equation (3.4a), the variables \( a_i \) and \( b_i \) model the displacement and strain, respectively, and \( U_i \) is the initial domain of rod \( i \), with
i ∈ \{0, 1\}. A third strain variable \( b_{0,1} \) on the overlap \( U_{0,1} \) captures the elongation of the connecting welding (for illustrative purposes represented by a lashing using elastic strings in tension).

Equation (3.4b) includes Hooke’s law in the sense that the linearly weighted strain variables correspond to elastic stress. Similarly, the strain \( b_{0,1} \) may in general be considered monotonically related to elastic stress associated with the stretching the connecting strings. In a linearized regime, we also here obtain proportionality between displacement difference (discrete strain) and stress. Finally, (3.4b) also includes the momentum balance due to the presence of the divergence operator. The boundary conditions (3.4c) imply tension-free conditions on both ends of both rods, while the constraint (3.3d) defines the mean position of two rods to be at the origin on the real line. Forces acting on the two rods are incorporated in the right-hand side terms \( f_i \).

This model can be extended by including the term \( \partial^2_t \alpha \) on the left-hand side to model longitudinal acceleration of the rods in time. In particular, the equation \( \partial^2_t \alpha + \Delta_D \alpha = \varphi \) then allows for wave propagation along the joined rods.

### 3.2. Multiple continuum models of porous materials

Let us consider the example illustrated in the middle of Figure 1, in which \( \mathcal{U} \) is an open cover of \( \Omega \subset \mathbb{R}^2 \) with \( U_0 = U_1 = \Omega \).

---

**Fig. 3.** Left and middle: Example of a fractured porous medium represented as two fully overlapping domains. Right and middle: Example of a system with two aquifers, each modeled as two-dimensional, connected by a low-permeable aquitard, again represented as two fully overlapping domains.

We provide two physical interpretation of this model. In the left part of Figure 3, the model corresponds to one where two coupled elliptic variables are used to model a process on the whole domain, such as is common for so-called ”double-continuum” models of fractured porous media [5, 9, 24]. A characteristic illustration of such a material is given in Figure 3. The physical interpretation of this modeling concept is that for percolating fracture networks, the timescales of flow in the fracture network and porous rock separate, such that it is appropriate to consider two fluid pressures: One representing the pressure in the fractures and one representing the pressure in
the rock. In a homogenized model, each of these pressures are defined on the full domain, leading to the double-continuum concept [2].

Concretely, the Hodge-Laplace problem (3.4) describes exactly the double porosity model for this problem as formulated in the above references. We identify that as the domains fully overlap, then $U_0 = U_1 = U_{0,1}$ and thus $I_{0,1} = 1$. Furthermore, $a_i$ describes the fluid pressure inside porous medium compartment $i$, the variable $b_i$ corresponds to the driving force for fluid motion (the gradient of pressure) within compartment $i$ and $b_{0,1}$ driving force for exchange between $U_0$ and $U_1$ (the pressure difference). As is clear from the material properties included in equation (3.4b), the governing equations are given proportionality between driving force and fluid flow (known as Darcy’s law) and the mass (or volume) balance equations. The boundary conditions (3.4c) describe zero fluid flux across the boundary, while the constraint (3.3d) fixes the mean value of the pressure across the system. Fluid compressibility can be incorporated in the system by including the time-dependent term $\partial_\alpha$.

Returning to Figure 3, the double-porosity model is mathematically identical to the models of two interconnected aquifers separated by an aquitard, as studied in the hydrology literature [19, 13, 6] For the case of two aquifers, this system is directly included in the framework as described herein. For the case of more than two aquifers, where the aquifers and aquitards form an alternating stack, this system is a degenerate limit of our exposition, which is obtained by letting the weights $w_{i,j}$ in Remark 3.2 equal zero whenever $i - j > 1$.

Finally, we emphasize that the so-called multiple-network models discussed in e.g. [18], used to model flow of extravascular fluids in the brain, are also structurally identical to the equations given in Remark 3.2.

### 3.3. Mixed-dimensional coupling with high dimensionality gap.

Our third example is illustrated by the sketch to the right in Figure 1. Here the cover $\mathcal{U}$ corresponds to a three-dimensional domain $\Omega$ such that $U_0 := \Omega$, while $U_1 \subset U_0$ is an embedded, vertical cylinder with small radius $\varepsilon > 0$. Such models commonly arise for thin inclusions (such as fiber reinforced materials), and a particular topical example is that of blood vessels within a tissue, as illustrated in Figure 4.

As in the previous example, we interpret $a_0$ and $a_1$ as fluid pressures in the surroundings and the cylinder, respectively, $b_0, b_1$ are the corresponding fluxes, and $b_{0,1}$ describes the mass exchange between the cylinder and the bulk. We recognize the material parameters as the permeability in the bulk ($w_0$), the resistance appearing in the Hagen-Poiseuille law for flow in the cylinder ($w_1 \sim \varepsilon^4$), and an exchange coefficient, associated with filtration between the cylinder and the bulk proportional to a pressure difference. The system (3.4) therefore governs porous medium flow in subsurface systems with an injection or production well. The same equations are encountered in blood perfusion models of vascularized, biological tissue, see e.g. [11].

When the ratio between the length and radius of the cylinder is large, it becomes attractive to approximate the pressure and flux inside the cylinder by using subspaces of functions that are constant along the cross-section. Such techniques lead to the simplified systems that are referred to as mixed-dimensional, which are analyzed in [15, 16, 17, 10, 12].

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FIG. 4. Blood flow in living tissue, such as this figure of the human brain, is frequently modeled as mixed-dimensional. In such a setting, the veins and arteries that are big enough to be resolved (visualized as blue and red, respectively) are modeled as graphs of one-dimensional segments, while the remaining tissue (is modeled as a three-dimensional domain). Figure from [12].

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