A PTAS for Scheduling with Tree Assignment Restrictions

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September 24, 2010

Scheduling with assignment restrictions is an important special case of scheduling unrelated machines which has attracted much attention in the recent past. While a lower bound on approximability of 3/2 is known for its most general setting, subclasses of the problem admit polynomial-time approximation schemes. This note provides a PTAS for tree-like hierarchical structures, improving on a recent 4/3-approximation by Huo and Leung [HL10].

1 Introduction

Scheduling on unrelated machines to minimize the makespan is one of the classical problem in optimization; here, we are given a set of $n$ jobs and $m$ machines, such that execution of a job $j$ on machine $i$ takes time $p_{ij} \in \mathbb{N}$. The objective is to find a schedule, i.e. an assignment $\sigma: \{1, \ldots, n\} \to \{1, \ldots, m\}$ of the jobs to the machines that minimizes the makespan $C_{\text{max}} = \max\{\sum_{\sigma(j) = i} p_{ij} : i \in \{1, \ldots, m\}\}$.

Despite its formal simplicity, it is still not understood completely: no approximation result is known that is asymptotically better than the seminal 2-approximation of Lenstra, Shmoys and Tardos [LST90], with asymptotical improvements made by Vakhania and Shechpin [SV05]; however, the known lower bound on approximability is only 3/2, also due to Lenstra, Shmoys and Tardos.

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A seemingly simpler problem is that of assignment restrictions: here, for every job \( j \) we have a length \( p_j \in \mathbb{N} \) and a set \( M_j \subset \{1, \ldots, m\} \) of feasible machines, i.e. we have \( p_{ij} = p_j \) for all \( i \in M_j \) and \( p_{ij} = \infty \) for all \( i \notin M_j \).

Related results As shown already by Lenstra et al. [LST90], scheduling with arbitrary assignment restrictions is also impossible to approximate better than within a factor of \( 3/2 \), unless \( P = NP \), and for the general case, no algorithm better than the 2-approximation for the unrelated machine case is known. However, better results are known for special structures of the sets \( M_j \). If we have \( |M_j| \leq 2 \), we can think of jobs as edges in a graph whose vertices are the machines, and orienting the edge in one direction will increase the load of one of its endpoints. In this graph balancing setting, Ebenlendr et al. [EKS08] give a \( 7/4 \)-approximation. If the graph additionally a tree, Lee et al. [LLP09] give an FPTAS.

Another type of restriction studied is that of the relation between the \( M_j \) sets: the most recent results being a PTAS by Muratore et al. [MSW10] for the case of nested restrictions, i.e. for each two \( M_j, M_j' \), one of \( M_j \subseteq M_j' \), \( M_j \supseteq M_j' \) or \( M_j \cap M_j' = \emptyset \) holds, and a \( 4/3 \)-approximation for tree-hierarchical assignment restrictions by Huo and Leung [HL10]. In this setting, again machines are considered vertices of a graph, a rooted tree in particular, and we impose that the sets \( M_j \) must correspond to the machines on a path from a node to the root.

For older results, we refer the reader to the survey [LL08] by Leung and Li.

**Contribution of this note.** We consider the tree-hierarchical assignment case by Huo and Leung and prove the following result:

**Theorem 1** (label=thm:tree-ptas). *Scheduling with tree-hierarchical assignment restrictions admits a PTAS, i.e. for every \( \varepsilon > 0 \) there is an \( (1 + O(\varepsilon)) \)-approximation with running time polynomial in the input size (but exponential in \( 1/\varepsilon \)).*

## 2 Rounding and simplifying the instance

Our algorithm combines some of the usual techniques for PTAS design such as partition into job sizes and geometric rounding with a hierarchical dynamic programming approach bottom-up through the tree. In this section, we describe the rounding and simplification steps we take to make the problem treatable by dynamic programming.

Throughout the following, let \( \varepsilon > 0 \). To simplify the analysis, our algorithm will create a solution of length at most \( (1 + 4\varepsilon) \) times the optimal value \( OPT \). (For simplicity, we use \( OPT \) to refer to both an optimal schedule and its makespan, since the distinction is clear from context.) Note \( OPT \) must be integral since all jobs lengths are,
and it is bounded pseudopolynomially in the instance size, for example by $\sum_{j=1}^n p_j$. Hence we may, in polynomial time, perform binary search over the range of feasible makespans and it is sufficient to give a relaxed decision procedure that for a guessed target makespan $C$ yields a schedule of length at most $(1 + 4\varepsilon)C$ whenever a schedule of length at most $C$ exists.

In the following, we call a job small if $p_j \leq \varepsilon C$, otherwise, we call it large. We will round up every large job to be of the form $\varepsilon C \cdot (1 + \varepsilon)^k$ for integral $k$. The number $K = O(\log_{1 + \varepsilon} 1/\varepsilon)$ of values $k$ that can occur only depends on $\varepsilon$, i.e. it is a constant for purposes of running time. The following classical result holds for this rounding:

**Lemma 2.** If there is a schedule of length $C$ of the original instance, there exists a schedule of the rounded instance with a constant number $K$ of large job sizes which has length at most $(1 + \varepsilon)C$.

It is also clear that a feasible schedule of the rounded instance is feasible for the original instance by replacing rounded large jobs with their (possibly slightly smaller) unrounded counterparts.

We now want to approximately describe every subset of the rounded instance by a $(K + 1)$-element configuration tuple. For large jobs, we simply count the number of jobs of each job size, which must be in $\{0, \ldots, n\}$. For small jobs, we count the total space taken up by them, in integral multiples of $\varepsilon C$, rounding up. Since every small job has size $\leq \varepsilon C$, the total size of all small jobs is at most $n \cdot \varepsilon C$, so this size indicator for small jobs is also from the set $\{0, \ldots, n\}$. In total, the number of configuration tuples is at most $(n + 1)^{K+1}$, in particular, it is polynomial in the input size.

We can in this way associate with each node $v$ in the tree the configuration tuple $c_v$ of jobs $j$ whose set $M_j$ is the path starting in $v$. If $s_v$ is the size multiplicity of the small jobs among them, i.e. their total size is in the interval $[s_v - 1 \cdot \varepsilon C, s_v \cdot \varepsilon C]$, we add up to one dummy job of size up to $\varepsilon C$ to make the total size exactly $s_v \cdot \varepsilon C$. By leaving that job on machine $v$ in the schedule, we obtain

**Lemma 3.** If there is a schedule of length at most $(1 + \varepsilon)C$ in the rounded instance, there is a schedule of length at most $(1 + 2\varepsilon)C$ in the rounded and modified instance.

Let us now consider such a schedule $\sigma$ of length at most $(1 + 2\varepsilon)C$. On every machine (node) $v$, a certain subset $\sigma^{-1}(v)$ of jobs is scheduled. Hence, it has a corresponding configuration tuple associated with it, the total size of which is at most $(1 + 3\varepsilon)C$. The additional loss is again incurred because the small jobs in $\sigma^{-1}(v)$ might not be an integral multiple of $\varepsilon C$. It is these configurations that we will find by dynamic programming.
3 The algorithm

In this section, we describe how to find a feasible assignment of configuration tuples to machines, if it exists, and how to convert this back into a schedule with a small increase in makespan.

The core of our algorithm is a local procedure which works as follows for a node \(v\):

1. In the first step, we accumulate the possible subsets of not-yet-scheduled jobs that \(v\) may need to accept from its children. We maintain a set of possible subset configuration tuples \(S\), which initially contains only the all-zero tuple. Then, for each child of \(v\) in turn, we consider the set \(S'\) of tuples it pushes towards the root and set \(S := S + S' = \{c + c' : c \in S, c' \in S'\}\). Since the size of \(S\) and \(S'\) is always polynomial, this can be done in polynomial time for every child, and since there are at most \(n\) children, finding the ultimate \(S\) with all children taken into account also takes polynomial time.

2. Then, we augment \(S\) by adding to each tuple the tuple \(c_v\) of jobs that are only available for scheduling on \(v\) and its ancestors. The resulting set, which we still denote \(S\), still has polynomial size.

3. For each \(c \in S\), we consider every possible subconfiguration \(\hat{s}\) that can be scheduled on \(v\), i.e. of total size at most \((1 + 3\varepsilon)C\). Then, the relative complement \(c - \hat{c}\) corresponds to jobs that would need to be pushed towards \(v\)’s parent node if we schedule according to \(\hat{c}\) on \(v\). Again, since \(S\) is polynomially bounded and the number of possible \(\hat{c}\) is as well, this can be done in polynomial time and yields a polynomially-sized set of configurations that are possibly pushed upwards.

Our algorithm, for a given target makespan \(C\), will execute this procedure in any leaf-to-root order, i.e. it is always run on the children of a node before it is run on the node itself. We return that a feasible schedule exists if it is possible to push up the all-zero configuration tuple from the root. The configuration tuples themselves can be obtained by standard bookkeeping techniques, i.e. storing, for each sum-of-configurations configuration that occurs one (and only one) set of witness summands.

Clearly, if there is a feasible assignment of configurations to machines of length at most \((1 + 3\varepsilon)C\), the algorithm will find one, too, since all configuration tuples that can be pushed into a node are considered.

To complete the proof of ??, it remains to show how to assign the jobs. This is trivial for large jobs: we select feasible jobs of that size in an arbitrary fashion bottom-up, pushing the remainder upwards. Since nothing is pushed beyond the root, all large jobs are assigned. The situation for small jobs is slightly more complicated, since
we do not know the exact total size of the small jobs. However, we can simply fill the available space in a greedy manner until it is fully used (or we run out of small jobs), i.e. the last small job may protrude beyond the allotted size. Since the last job’s size is at most \( \varepsilon \mathcal{C} \) by definition, this will increase the makespan of the schedule we generate by another \( +\varepsilon \mathcal{C} \) to at most \( (1 + 4\varepsilon)\mathcal{C} \), and it will at most decrease the total size of small jobs pushed towards the root, which clearly maintains feasibility of the remaining configurations.

4 Conclusion

This note shows another case, tree-hierarchical structures, in which scheduling with assignment restrictions can be approximated within arbitrary accuracy. This mostly settles the complexity: an FPTAS cannot exist since the setting generalizes the strongly NP-hard problem \( P||C_{\text{max}} \), the existence of an EPTAS is still open.

For other important settings, the question of inapproximability vs. PTAS is still open: in particular, two natural cases would be cross-free families, where for two sets \( M_j, M'_j \), \( M_j \cup M'_j = \{1, \ldots, m\} \) may also occur in addition to the three cases defining nested families as given in the introduction, and interval restrictions, where every \( M_j \) is of the form \( \{\alpha_j, \ldots, \omega_j\} \) for a fixed permutation of the machines.

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