On minimizing a portfolio’s shortfall probability

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Received: / Accepted:

Abstract We obtain a lower asymptotic bound on the decay rate of the probability of a portfolio’s underperformance against a benchmark over a large time horizon. A diffusion model of the asset prices is assumed, with the mean returns and volatilities being represented by possibly nonlinear functions of an economic factor. The bound is tight, more specifically, we are able to produce \( \epsilon \)-asymptotically optimal portfolios.

Keywords portfolio optimization · shortfall probability · large deviations

Mathematics Subject Classification (2010) 60F10

JEL Classification C6

1 Introduction

As noted by Roll [32, p.13] “Today’s professional money manager is often judged by total return performance relative to a prespecified benchmark, usually a broadly diversified index of assets.” He argues that “This is a sensible approach because the sponsor’s most direct alternative to an active manager is an index fund matching the benchmark.” A typical example, of more than just professional interest to academic readers, is the following statement by the TIAA–CREF Trust Company:

Different accounts have different benchmarks based on the client’s overall objectives... Accounts for clients who have growth objectives with

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an emphasis on equities will be benchmarked heavily toward the appropriate equity index – typically the S&P 500 index – whereas an account for a client whose main objective is income and safety of principal will be measured against a more balanced weighting of the S&P 500 and the Lehman Corporate/Government Bond Index [40, p.3].

How should plan sponsors and the investors they represent evaluate the performance of a fund like this? Nobel Laureate William Sharpe [34, p.32] asserts that

The key information an investor needs to evaluate a mutual fund is (i) the fund’s likely future exposures to movements in major asset classes, (ii) the likely added (or subtracted) return over and above a benchmark with similar exposures, and (iii) the likely risk vis-à-vis the benchmark. This paper will focus on Sharpe’s aforementioned point (iii): how to choose a portfolio that minimizes its shortfall risk vis-à-vis an exogenous benchmark. Sensoy [33, p.26] notes that “the vast majority of actively managed, diversified U.S. equity funds use a S&P or Russell benchmark index that is defined on size or value/growth dimensions.” Evidence that a fund’s investors want to outperform its designated benchmark, i.e. not fall short of doing so, was also found by Sensoy (op.cit., p. 38), who concluded:

Performance relative to the specified benchmark, especially above the benchmark, is a significant determinant of a fund’s subsequent cash inflows, even controlling for performance measures that better capture the fund’s style.

Because “mutual funds generally receive a fixed percentage of assets under management as a fee” [Sensoy (op.cit.,p.33)], the fees received by mutual fund management increase when the fund outperforms its designated benchmark and decrease when it doesn’t. Hence a fund’s managers have strong incentives to minimize the shortfall risk vis-à-vis the fund’s exogenous benchmark. But to study how that objective might be achieved, one first must rigorously define “the shortfall risk vis-à-vis the fund’s exogenous benchmark”. One could, for example, fix a specific investment horizon length $T$, and search for a portfolio that minimizes the probability that its value at $T$ will be less than the value of the designated benchmark at $T$. But how should this horizon length be determined? Does a typical fund investor have a specific horizon length in mind? Even if an investor does have a specific horizon length in mind, won’t a mutual fund wind up with investors with different horizon lengths in mind? In light of these unanswered questions, and with an eye toward endowment, pension, or retirement investors who are interested in relatively “long–run” returns, we will use large deviation asymptotics to characterize portfolios with minimum feasible “long–run” shortfall probabilities. While this will take the form of a continuous time, dynamic optimization problem over an infinite horizon, that does not imply that the optimal portfolio will have bad performance over fixed, shorter horizons that may interest other investors.
The problem of optimizing the probability of underperformance of a financial portfolio over an infinite time horizon by using large deviation asymptotics has been studied by a number of authors. Discrete time setups were considered in Stutzer [36,37,38,39]. Continuous–time models were treated in Hata, Nagai, and Sheu [11] and Nagai [27]. More specifically, the latter authors concern themselves with diffusion models of asset prices. Hata, Nagai, and Sheu [11] assume that the mean returns and volatilities of the security prices are affine functions of the economic factor and that the economic factor is represented by a Gaussian process, that the risk–free interest rate does not depend on the economic factor and that no benchmark is involved. In Nagai [27], a nonlinear model is considered with the risk–free asset as the benchmark. Two kinds of optimal portfolios are obtained in Hata, Nagai, and Sheu [11] and in Nagai [27]. The first one is a time–dependent portfolio. At first, one has to choose investment horizon \( T \) and solve an optimal control problem on \([0, T]\). When horizon \( T \) changes, a different optimal control problem has to be solved. As \( T \to \infty \), the performances of the portfolios approach the optimal value. Understandably, the portfolios are referred to in Hata, Nagai, and Sheu [11] as nearly optimal. The other portfolio is "stationary" in the sense that it is dependent on the value of the economic factor only and is updated "in real time". The underperformance probability delivered by that investment strategy approaches the optimal value as time goes to infinity. In order for this other portfolio to be asymptotically optimal more restrictions have to be placed on the model. The proof of the optimality of the latter portfolio in Hata, Nagai, and Sheu [11] is omitted. The proof in Nagai [27] seems to have a gap, as explained below.

The methods of those papers use duality considerations and rely on connection with risk sensitive control. A Hamilton–Jacobi–Bellman equation on a finite time horizon is analyzed first in order to find an optimal control, and, afterwards, the length of time is allowed to tend to infinity. In this paper, we approach the problem from a different angle. We study the model tackled in Nagai [27] supplemented with a general benchmark. (Interestingly enough, the presence of a volatile benchmark lends regularity.) By a change of variables, the setup is cast as a large deviation problem for coupled diffusions with time scale separation. The economic factor can be assumed to "live in fast time" whereas the portfolio price is associated with a process that "lives in slow time". This insight enables us to take advantage of the methods developed for such diffusions in Liptser [22] and Puhalskii [29]. In particular, the empirical measure of the factor process plays a pivotal role in our study. Another novel technical feature is an extensive use of the saddle–point theory.

In a fairly general situation, we obtain an asymptotic lower bound on the scaled by the length of the time period logarithmic probability of underperformance. Under additional conditions, the bound is shown to be tight in the sense that there exist stationary portfolios that approach the lower bound over time. Those portfolios generalize the stationary portfolios in Hata, Nagai and Sheu [11] and in Nagai [27]. If the assumptions are relaxed, "\( \epsilon \)–optimal" portfolios are available whose performance over time falls short of the optimal
value by an arbitrarily small amount so that another limit needs to be taken. In a standard fashion, one can turn two consecutive limits into one so that an asymptotically optimal portfolio is obtained too. We are able to dispose of a number of assumptions in Nagai [27] some of which are questionable from the modelling perspective, e.g., the requirement that the sum of the squared differences of the risk–free interest rate and the security mean return rates be bounded below by a quadratic function of the economic factor (see the discussion following Remark 2.8 for more detail), which condition is characterized as being "crucial" in Nagai [27]. There is another important distinction with the results of Nagai [27] and Hata, Nagai, and Sheu [11]. Both papers require certain stability conditions which involve the coefficients of both the equations for the economic factor and the equations for the securities. At the same time, the model is set in such a way that the economic factor is not influenced by the security prices, so, the stability conditions are arguably at odds with the model’s logic. We use a different stability condition which is along similar lines as the one in Fleming and Sheu [9] and concerns the properties of the economic factor only. On the technical side, our proofs appear to be less involved than the ones in Nagai [27] which could explain why we are able to tackle a more general model, we also allow a non–deterministic initial condition for the economic factor, whereas it is kept fixed in Hata, Nagai, and Sheu [11] and in Nagai [27].

This is how this paper is organized. In Section 2 the model is defined, the choice of an optimal portfolio is explained intuitively, main results are stated, and relation to earlier contributions is discussed in more detail. Section 3 contains auxiliary results needed for the proofs and the main results are proved in Section 4.

2 A model description and main results

We consider a portfolio consisting of \( n \) risky securities priced \( S^1_t, \ldots, S^n_t \) at time \( t \) and a safe security of price \( S^0_t \). We assume that the security prices follow the equations

\[
\frac{dS^i_t}{S^i_t} = a^i(X_t) \, dt + b^i(X_t)^T \, dW_t, \tag{2.1}
\]

for \( i = 1, 2, \ldots, n \), and

\[
\frac{dS^0_t}{S^0_t} = r(X_t) \, dt ,
\]

where \( S^0_0 > 0 \) and \( X_t \) represents an economic factor. It is governed by the equation

\[
dX_t = \theta(X_t) \, dt + \sigma(X_t) \, dW_t. \tag{2.2}
\]

In these equations, the \( a^i(x) \) and \( r(x) \) are real–valued functions, the \( b^i(x) \) are \( \mathbb{R}^k \)-valued functions, \( \theta(x) \) is an \( \mathbb{R}^l \)-valued function, and \( \sigma(x) \) is an \( l \times k \)-matrix–valued function, all being defined for \( x \in \mathbb{R}^l \) and \( ^T \) being used to denote
Under those hypotheses, the processes \( S \), e.g., Chapter 5 in Karatzas and Shreve [16].

The investor holds \( \pi_l \) shares of risky security \( i \) and \( \pi_0 \) shares of the safe security at time \( t \), so the total wealth is given by \( Z_t = \sum_{i=1}^n \pi_l^i S_t^i + \pi_0^0 S_t^0 \). Portfolio \( \pi_l = (\pi_1^1, \ldots, \pi_n^1)^T \) specifies the proportions of the total wealth invested
in the risky securities so that, for \( i = 1, 2, \ldots, n \), \( \pi_i^t S_t^i = \pi_i^t Z_t \). The processes \( \pi_i = (\pi_i^t, t \in \mathbb{R}_+) \) are assumed to be \( \mathcal{B} \otimes \mathcal{F}_t \)-progressively measurable, where \( \mathcal{B} \) denotes the Borel \( \sigma \)-algebra on \( \mathbb{R}_+ \), and such that \( \int_0^t \pi_i^t S_t^i \, ds < \infty, t \in \mathbb{R}_+ \), a.s. We do not impose any other restrictions on the magnitudes of the \( \pi_i^t \) so that unlimited borrowing and shortselling are allowed.

Let \( L^\pi_t = \frac{1}{t} \ln \left( \frac{Z_t}{Y_t} \right) \).

Given \( q \in \mathbb{R} \), the objective is to minimize \( \liminf_{t \to \infty} (1/t) \ln \mathbb{P}(L^\pi_t \leq q) \) over all portfolios \( \pi = (\pi_t, t \in \mathbb{R}_+) \) and identify portfolios for which the bound is attained.

Since the amount of wealth invested in the safe security is \( (1 - \sum_{i=1}^{n} \pi_i^t) Z_t \), in a standard fashion by using the self financing condition, cf. Nagai [27], we obtain that

\[
\frac{dZ_t}{Z_t} = \sum_{i=1}^{n} \pi_i^t \frac{dS_t^i}{S_t^i} + (1 - \sum_{i=1}^{n} \pi_i^t) \frac{dS_0^i}{S_t^i}.
\]

Assuming that \( Z_0 = Y_0 \) and letting \( c(x) = b(x)b(x)^T \), we have by Itô’s lemma that, cf. Nagai [27] and Pham [28],

\[
L^\pi_t = \frac{1}{t} \int_0^t M(\pi_s^t, X_s^t) \, ds + \frac{1}{\sqrt{t}} \int_0^t N(\pi_s^t, X_s^t)^T \, dW_s^t,
\]

(2.5)

The following piece of notation comes in useful. Given \( z \in \mathbb{R}^d \) and positive definite symmetric \( d \times d \)-matrix \( V \), we denote \( \|z\|^2_{V} = z^T V z \). Let, for \( u \in \mathbb{R}^n \) and \( x \in \mathbb{R}^l \),

\[
M(u, x) = u^T (a(x) - r(x)1) - \frac{1}{2} \|u\|^2_{c(x)} + r(x) - \alpha(x) + \frac{1}{2} |\beta(x)|^2 \quad (2.6a)
\]

and

\[
N(u, x) = b(x)^T u - \beta(x). \quad (2.6b)
\]

A change of variables brings equation (2.5) to the form

\[
L^\pi_t = \int_0^1 M(\pi_{ts}, X_{ts}) \, ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_{ts}, X_{ts})^T \, dW_s^t,
\]

(2.7)

where \( W^t_s = W_{ts}/\sqrt{t} \). We note that \( W^t = (W^t_s, s \in [0, 1]) \) is a Wiener process relative to \( \mathcal{F}^t = (\mathcal{F}_{ts}, s \in [0, 1]) \). The righthand side of (2.7) can be viewed as a diffusion process with a small diffusion coefficient which "lives in slow time"
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represented by the variable $s$, whereas in $X$ and $\pi$ ”time” is accelerated by a factor of $t$. Similar setups have been considered in Liptser [22] and Puhalskii [29]. Those papers show that in order to study the large deviation properties of the ”slow” process it is convenient to work with the pair that comprises the slow process and the empirical measure of the fast process. For equations (2.2) and (2.7), this means working with the pair $(L_t^\pi, \mu_t)$, where $\mu_t = (\mu_t(ds, dx))$ represents the empirical process of $(X_{ts}, s \in [0, 1])$, which is defined by the relation

$$\mu_t([0, s], \Gamma) = \int_0^s \chi_\Gamma(X_{ts}) d\bar{s},$$

with $\Gamma$ representing an arbitrary Borel subset of $\mathbb{R}^l$ and with $\chi_\Gamma(x)$ representing the indicator function of the set $\Gamma$. Letting $\pi_t^s = \pi_{ts}$ and $X_t^s = X_{ts}$ in (2.7) obtains that

$$(2.8)$$

We note that both $X^t = (X^s_t, s \in [0, 1])$ and $\pi^t = (\pi^s_t, s \in [0, 1])$ are $\mathcal{F}^t$-adapted.

Since, by [22] and Itô’s lemma, for twice continuously differentiable function $f$ on $\mathbb{R}^l$, with $\nabla f$ and $\nabla^2 f$ denoting the gradient and the Hessian of $f$, respectively, and with tr standing for the trace of a square matrix,

$$f(X_t) = f(X_0) + \int_0^t \nabla f(X_s)^T \theta(X_s) ds + \frac{1}{2} \int_0^t \text{tr} (\sigma(X_s)\sigma(X_s)^T \nabla^2 f(X_s)) ds + \int_0^t \nabla f(X_s)^T \sigma(X_s) dW_s \quad (2.9)$$

and since the process

$$(\exp(\int_0^t (-\lambda N(\pi_s, X_s) + \sigma(X_s)^T \nabla f(X_s))^T dW_s$$

$$- \frac{1}{2} \int_0^t (-\lambda N(\pi_s, X_s) + \sigma(X_s)^T \nabla f(X_s))^2 ds), t \in \mathbb{R}_+)$$
is a local martingale relative to $\mathbf{F}$, by (2.8) and (2.9), for $\lambda \geq 0$,

$$
\mathbb{E} \exp \left( -t\lambda L^{\pi}_{t} - t \int_{0}^{1} -\lambda M(\pi^{t}_{s}, X^{t}_{s}) \, ds + f(X_{t}) - f(X_{0}) \right)
- t \int_{0}^{1} \nabla f(X^{t}_{s})^{T} \theta(X^{t}_{s}) \, ds - \frac{t}{2} \int_{0}^{1} \text{tr} (\sigma(X^{t}_{s})\sigma(X^{t}_{s})^{T} \nabla^{2} f(X^{t}_{s})) \, ds
- \frac{t}{2} \int_{0}^{1} \left| -\lambda N(\pi^{t}_{s}, X^{t}_{s}) + \sigma(X^{t}_{s})^{T} \nabla f(X^{t}_{s}) \right|^{2} \, ds \leq 1. \quad (2.10)
$$

Intuitively, if we assume that equality prevails in (2.10), which would be the case under certain integrability conditions on $\pi_{s}$ and $\nabla f(X_{s})$, then $L^{\pi}_{t}$ is "maximized" by minimizing the integrals over $\pi_{t}$, i.e., by choosing $\pi^{t}_{s} = u(X^{t}_{s})$ with $u(x)$ attaining $\inf_{u \in \mathbb{R}^{n}} (-\lambda M(u, x) + | -\lambda N(u, x) + \sigma(x)^{T} \nabla f(x) |^{2}/2)$. For that portfolio,

$$
\mathbb{E} \chi \{ L^{\pi}_{t} \leq q \} \exp \left( -t\lambda q + f(X_{t}) - f(X_{0}) \right)
- t \int_{0}^{1} \left( \inf_{u \in \mathbb{R}^{n}} (-\lambda M(u, x) + \frac{1}{2} | -\lambda N(u, x) + \sigma(x)^{T} \nabla f(x) |^{2}) + \nabla f(x)^{T} \theta(x) + \frac{1}{2} \text{tr} (\sigma(x)\sigma(x)^{T} \nabla^{2} f(x)) \right) \mu_{t}(ds, dx) \leq 1.
$$

Consequently,

$$
\frac{1}{t} \ln \mathbb{E} \chi \{ L^{\pi}_{t} \leq q \} \exp \left( f(X_{t}) - f(X_{0}) \right) \leq \lambda q + \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^{l}} \left( \inf_{u \in \mathbb{R}^{n}} (-\lambda M(u, x) + \frac{1}{2} | -\lambda N(u, x) + \sigma(x)^{T} \nabla f(x) |^{2}) + \nabla f(x)^{T} \theta(x) + \frac{1}{2} \text{tr} (\sigma(x)\sigma(x)^{T} \nabla^{2} f(x)) \right) \nu(dx). \quad (2.11)
$$

The "best" upper bound on the normalized logarithmic shortfall probability $\left(1/t\right) \ln \mathbb{P}(L^{\pi}_{t} \leq q)$ is obtained by minimizing the righthand side over $\lambda \geq 0$ and a suitable collection of functions $f$ so that an optimal portfolio should be apparently associated with $\lambda$ and $f$ that minimize the righthand side of (2.11). The main results of the paper bear out that intuition. Furthermore, we show that the upper bound for that particular portfolio choice furnishes a lower bound for all portfolios.

Before we state the main results, more conditions are in order. We assume the stability condition that there exist bounded Borel–measurable function
\( \Phi(x) \) with values in the set of \( l \times n \)-matrices, where \( x \in \mathbb{R}^l \), and positive definite symmetric \( l \times l \)-matrix \( \Psi \) such that

\[
\limsup_{|x| \to \infty} \left( \theta(x) - \Phi(x)(a(x) - r(x)1) \right)^T \frac{\Psi x}{|x|^2} < 0. \quad (2.12)
\]

If \( \Phi(x) = 0 \) and \( \Psi \) is the \( l \times l \)-identity matrix, then one recovers Has’minkii’s drift condition.

The following nondegeneracy condition is also needed. Let \( I_k \) denote the \( k \times k \)-identity matrix and let \( Q_1(x) = I_k - b(x)^T c(x)^{-1}b(x) \). The matrix \( Q_1(x) \) represents the orthogonal projection operator onto the null space of \( b(x) \) in \( \mathbb{R}^k \). We will assume that

\[ (N) \]

1. The matrix \( \sigma(x)Q_1(x)\sigma(x)^T \) is uniformly positive definite.
2. The quantity \( \beta(x)^T Q_2(x)\beta(x) \) is bounded away from zero, where

\[
Q_2(x) = Q_1(x) \left( I_k - \sigma(x)^T (\sigma(x)Q_1(x)\sigma(x)^T)^{-1}\sigma(x) \right) Q_1(x). \quad (2.13)
\]

This condition admits the following geometric interpretation. The matrix \( \sigma(x)Q_1(x)\sigma(x)^T \) is uniformly positive definite if and only if the ranges of \( \sigma(x)^T \) and \( b(x)^T \) do not have common nontrivial subspaces and, in addition, arbitrary nonzero vectors from those respective ranges are at angles bounded away from zero, if and only if the matrix \( c(x) - b(x)\sigma(x)^T (\sigma(x)\sigma(x)^T)^{-1}\sigma(x)b(x)^T \) is uniformly positive definite. Also, \( \beta(x)^T Q_2(x)\beta(x) \) is bounded away from zero if and only if the projection of \( \beta(x) \) onto the null space of \( b(x) \) is of length bounded away from zero and is at angles bounded away from zero to all nonzero vectors from the projection of the range of \( \sigma(x)^T \) onto that null space. Under part 1 of condition \( (N) \), we have that \( k \geq n + l \) and the rows of the matrices \( \sigma(x) \) and \( b(x) \) are linearly independent. Part 2 of condition \( (N) \) implies that \( \beta(x) \) does not belong to the sum of the ranges of \( \sigma(x)^T \) and \( b(x)^T \) so that \( k > n + l \). Part 1 of condition \( (N) \) is essential for the developments in this paper while part 2 may be disposed of at the expense of certain additional assumptions.

Let \( \mathcal{P} \) represent the set of probability measures \( \nu \) on \( \mathbb{R}^l \) such that \( \int_{\mathbb{R}^l} |x|^2 \nu(dx) < \infty \). Let \( \mathbb{P} \) represent the set of probability densities \( m = (m(x), x \in \mathbb{R}^l) \) such that \( \int_{\mathbb{R}^l} |x|^2 m(x) dx < \infty \). Let \( \mathcal{C}^2, \mathcal{C}^2_0, \mathcal{C}_1, \) and \( \mathcal{C}_1^\infty \) represent the set of real–valued twice continuously differentiable functions on \( \mathbb{R}^l \), the set of real–valued compactly supported twice continuously differentiable functions on \( \mathbb{R}^l \), the set of real–valued continuously differentiable functions on \( \mathbb{R}^l \), and the set of real–valued continuously differentiable functions on \( \mathbb{R}^l \) whose gradients satisfy the linear growth condition, respectively. For \( \mathcal{C}^2_0 \)-function \( f \), density \( m \in \mathbb{P} \) and \( \lambda \geq 0 \), we let

\[
G(\lambda, f, m) = \int_{\mathbb{R}^l} \left( -\lambda \sup_{u \in \mathbb{R}^n} (M(u, x) - \frac{1}{2} \lambda |N(u, x)|^2 + \nabla f(x)^T \sigma(x)N(u, x)) \\
+ \nabla f(x)^T \theta(x) + \frac{1}{2} |\sigma(x)^T \nabla f(x)|^2 + \frac{1}{2} \text{tr} \left( \sigma(x)^T \nabla^2 f(x) \right) \right) m(x) dx. \quad (2.14)
\]
Let us define

$$F(\lambda) = \sup_{m \in \mathbb{P}} \inf_{f \in \mathcal{C}_0^2} G(\lambda, f, m).$$  \hspace{1cm} (2.15)$$

One can see that $F(0) \leq 0$ and, moreover, $F(0) = 0$ provided condition (2.12) holds with $\Phi(x) = 0$. Let

$$J^s_q = \sup_{\lambda \geq 0} (\lambda q - F(\lambda)),$$

the latter quantity being nonnegative by $F(0)$ being nonpositive and superscript "s" standing for "shortfall". We show in Lemma 3.4 and Lemma 3.7 below that the function $F(\lambda)$ is strictly convex, is continuously differentiable for $\lambda > 0$, and converges to $\infty$ superlinearly as $\lambda \to \infty$, so the supremum in (2.16) is attained, at unique $\hat{\lambda}$. Furthermore, by Lemma 3.9 below, if either $\hat{\lambda} > 0$ or condition (2.12) holds with $\Phi(x) = 0$, then the function $\lambda q + G(\lambda, f, m)$, being convex in ($\lambda, f$) and concave in $m$, has saddle point $(\hat{\lambda}, \hat{f})$ in $R^+ \times (\mathcal{C}_1^t \cap \mathcal{C}_2^t) \times \mathbb{P}$, with $\hat{\lambda}$, $\nabla \hat{f}$, and $\hat{m}$ being specified uniquely.

In addition, the following equations are satisfied:

$$-\hat{\lambda}(M(\hat{u}(x), x) - \frac{1}{2} \hat{\lambda} |N(\hat{u}(x), x)|^2 + \nabla \hat{f}(x)^T \sigma(x) N(\hat{u}(x), x)) + \nabla \hat{f}(x)^T \theta(x)$$

$$+ \frac{1}{2} |\sigma(x)|^2 \nabla \hat{f}(x)^2 + \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 \hat{f}(x)) = F(\hat{\lambda}) \hspace{1cm} (2.17a)$$

and

$$\int_{\mathbb{R}^l} \left( \nabla h(x)^T (\hat{\lambda} \sigma(x) N(\hat{u}(x), x) + \theta(x) + \sigma(x) \sigma(x)^T \nabla \hat{f}(x)) \nabla \hat{f}(x) \right)$$

$$+ \frac{1}{2} \text{tr}(\sigma(x) \sigma(x)^T \nabla^2 h(x))) \hat{m}(x) \, dx = 0, \hspace{1cm} (2.17b)$$

with $\hat{u}(x)$ being given by the $u$ that attains supremum in (2.14) for $\lambda = \hat{\lambda}$ and $f = \hat{f}$ so that

$$\hat{u}(x) = \frac{1}{1 + \hat{\lambda}} c(x)^{-1} \left( a(x) - r(x) 1 + \hat{\lambda} b(x) \beta(x) + b(x) \sigma(x)^T \nabla \hat{f}(x) \right) \hspace{1cm} (2.18)$$

and with (2.17a) and (2.17b) holding for all $x \in \mathbb{R}^l$ and for all $h \in \mathcal{C}_2^t$, respectively. Also, $\hat{m}$ can be assumed to be bounded, positive and continuously differentiable. In effect, (2.17a) and (2.17b) represent Euler–Lagrange equations for $G(\hat{\lambda}, f, m)$ at $(\hat{f}, \hat{m})$. Equation (2.17a) is known as an ergodic Bellman equation, see Fleming and Sheu [9], Kaise and Sheu [14], Hata, Nagai, and Sheu [11], and equation (2.17b) signifies that $\hat{m}$ is the invariant density of a certain diffusion. Kaise and Sheu [14], see also Ichihara [12], develop an elegant theory of the ergodic Bellman equation which is essential for our study. One hopes that the portfolio $\hat{\pi} = (\hat{\pi}_t, t \in \mathbb{R^+})$ such that $\hat{\pi}_t = \hat{u}(X_t)$ is asymptotically optimal.
Theorem 2.1 Let us suppose that either \( \lambda > 0 \) or condition \((2.12)\) holds with \( \Phi(x) = 0 \). Then, for arbitrary portfolio \( \pi = (\pi_t, t \in \mathbb{R}_+) \),
\[
\lim_{t \to \infty} \inf \frac{1}{t} \ln \mathbb{P}(L_t^\pi < q) \geq -J_q^\pi.
\]

Remark 2.2 Let \( C_b^2 \) represent the set of twice continuously differentiable functions on \( \mathbb{R}^d \) with bounded second derivatives. It is shown in Lemma 3.11 below that \( F(\lambda) = \inf_{f \in C_b^2} \sup_{x \in \mathbb{R}^d} G(\lambda, f, m) \). Thus, the assertion of Theorem 2.1 is consistent with the intuition provided after \((2.11)\).

Let, given \( \lambda \in \mathbb{R}, f \in C^2 \), and \( v = (v(x), x \in \mathbb{R}^d) \),
\[
\dot{H}(x; \lambda, f, v) = -\lambda M(v(x), x) + \frac{1}{2} |\lambda N(v(x), x) + \sigma(x)^T \nabla f(x)|^2 + \nabla f(x)^T \theta(x) + \frac{1}{2} \text{tr} (\sigma(x) \sigma(x)^T \nabla^2 f(x)). \tag{2.19}
\]

By \((2.17a)\), for all \( x \in \mathbb{R}^d \),
\[
\dot{H}(x; \lambda, f, \hat{u}) = F(\hat{\lambda}). \tag{2.20}
\]
In addition, by Remark 3.8 below, \( F(\lambda) = \inf_{f \in C_b^2} \sup_{x \in \mathbb{R}^d} \dot{H}(x; \lambda, f, \hat{u}) \).

Given \( \tau > 0 \), let \( \hat{u}(x) = \hat{u}(x) \chi_{[0, \tau]}(||x||) \). Let us introduce the following condition
\[
\limsup_{\tau \to \infty} \inf_{f \in C_b^2} \sup_{x \in \mathbb{R}^d} \dot{H}(x; \lambda, f, \hat{u}) \leq F(\lambda). \tag{2.21}
\]
By Lemma 3.11 below, \((2.21)\) holds provided that either \( \lambda = 0 \) or there exist \( g > 0, C_1 > 0 \) and \( C_2 > 0 \) such that, for all \( x \in \mathbb{R}^d \),
\[
(1 + g) \|b(x) \sigma(x)^T \nabla \hat{f}(x)\|_{c(x)-1}^2 - \|a(x) - r(x)1\|_{c(x)-1}^2 \leq C_1 ||x|| + C_2. \tag{2.22}
\]
We also introduce the following stronger version of \((2.22)\):
\[
\lim_{||x|| \to \infty} ((1 + g) \|b(x) \sigma(x)^T \nabla \hat{f}(x)\|_{c(x)-1}^2 - \|a(x) - r(x)1\|_{c(x)-1}^2) = -\infty. \tag{2.23}
\]
Let \( \hat{\pi} = (\hat{u}(X_t), t \in \mathbb{R}_+) \) and \( \hat{\pi}^\tau = (\hat{u}^\tau(X_t), t \in \mathbb{R}_+) \).

Theorem 2.3 Suppose that \((2.12)\) holds with \( \Phi(x) = 0 \).

1. If \((2.21)\) holds, then
\[
\lim_{\tau \to \infty} \liminf_{t \to \infty} \frac{1}{t} \ln \mathbb{P}(L_t^{\hat{\pi}^\tau} < q) = \lim_{\tau \to \infty} \limsup_{t \to \infty} \frac{1}{t} \ln \mathbb{P}(L_t^{\hat{\pi}^\tau} \leq q) = -J_q^{\hat{\pi}^\tau}.
\]

2. If, in addition, \((2.23)\) holds, then
\[
\lim_{t \to \infty} \frac{1}{t} \ln \mathbb{P}(L_t^{\hat{\pi}} < q) = \lim_{t \to \infty} \frac{1}{t} \ln \mathbb{P}(L_t^{\hat{\pi}} \leq q) = -J_q^{\hat{\pi}}.
\]
Remark 2.4 Conditions (2.22) and (2.23) are modeled on respective conditions (2.25) in Nagai [27] and (4.1) in Nagai [26]. As the proof of Lemma 3.11 shows, an upper bound on the right-hand side of (2.22) can be allowed to grow at a subquadratic rate.

Remark 2.5 One can see that
\[ \inf_{f \in C^2} \sup_{x \in \mathbb{R}} \tilde{H}(x; \hat{\lambda}, f, \hat{u}^T) = \inf_{f \in C^2} \sup_{x \in \mathbb{R}} \tilde{H}(x; \hat{\lambda}, f, \hat{u}^T). \]

Remark 2.6 The limit in (2.23) holds provided
\[ \limsup_{|x| \to \infty} \frac{1}{|x|^2} \left( \|b(x)\sigma(x)^T \nabla \hat{f}(x)\|_{\ell_c(x)}^2 - \|a(x) - r(x)1\|_{\ell_c(x)}^2 - \right) < 0. \]

It would be nice to have a condition expressed in terms of the coefficients of the model equations but this seems to be an open problem.

Remark 2.7 Under the hypotheses of part 1 of Theorem 2.3 there exists strictly increasing function \( t(\tau) \) such that \( \lim_{\tau \to \infty} (1/t(\tau)) \ln P(L^s_{\tau(t)} < q) = -J_q^s \). Letting \( \tau(t) \) represent the inverse function to \( t(\tau) \), we have that \( \lim_{\tau \to \infty} (1/t) \ln P(L^s_{\tau(t)} < q) = -J_q^s \), so, \( (\hat{u}^{\tau(t)}(X_1), t \geq 0) \) is an asymptotically optimal portfolio.

Remark 2.8 When \( \beta(x) = 0 \), the proofs of Theorems 2.1 and 2.3 go through and their assertions are maintained provided part 1 of condition (N) is satisfied and \( \inf_{x \in \mathbb{R}} (r(x) - \alpha(x)) < q \), with the \( \epsilon \)-optimal portfolio being defined similarly. If \( \inf_{x \in \mathbb{R}} (r(x) - \alpha(x)) \geq q \), then investing in the safe security only is obviously optimal.

For the case where \( \alpha(x) = r(x) \) and \( \beta(x) = 0 \), the control in (2.18) appears in Theorem 2.5 in Nagai [27], which determines the limit in part 2 of Theorem 2.3. Instead of condition (2.23), it is required in Nagai [27] that \( \|b(x)\sigma(x)^T \nabla \hat{f}(x)\|_{\ell_c(x)}^2 - \|a(x) - r(x)1\|_{\ell_c(x)}^2 < 0 \), for all \( x \) (see (2.25) in Nagai [27]). Since it is assumed in Nagai [27] that \( \|a(x) - r(x)1\|_{\ell_c(x)}^2 \) is bounded below by a quadratic function of \( |x| \) (see (2.21) there) and since \( \nabla \hat{f}(x) \) is, at most, of linear growth, that condition implies (2.22). It does not imply (2.23). As mentioned in the Introduction, we have our doubts as to the proof of Theorem 2.5 in Nagai [27] being sound: the last display of the proof on p.660 doesn’t seem to be substantiated in that it is not clear how the term \( \tilde{\epsilon} \int_0^T e^{-w(X_s)}(-\chi) \) on the preceding line is tackled, \( \chi \) being a positive number, e.g., why should \( \tilde{\epsilon} e^{-w(X_s)} < \infty \), given that \( -w(x) \) grows no slower than quadratically with \( |x| \) ? Similar terms were treated more carefully in Kuroda and Nagai [18] and in Nagai [26], e.g., in Nagai [26] it is required that \( \|b(x)\sigma(x)^T \nabla \hat{f}(x)\|_{\ell_c(x)}^2 - \|a(x) - r(x)1\|_{\ell_c(x)}^2 \to -\infty \), as \( |x| \to \infty \), which condition is equivalent to (2.23) when \( \|a(x) - r(x)1\|_{\ell_c(x)}^2 \) is bounded below by a quadratic function of \( |x| \) and may suffice to conclude the proof in
Nagai [27]. Theorem 2.4 in Nagai [27] does not require the condition but, as we have mentioned in the Introduction, it produces different portfolios for different time horizons $T$. Besides, additional assumptions are introduced both in Theorem 2.4 and in Theorem 2.5 in Nagai [27] (see (2.19) and (2.20) there) along with the requirement that $0 < q < -F'(0^+)$. The right-hand side of the latter inequality ruling out the possibility that $\hat{\lambda} = 0$. (Interestingly enough, the condition that $q > 0$ is consistent with Remark 2.8.) Stability condition (2.12) is assumed to hold in Nagai [27] with $\Phi(x) = \sigma(x)b(x)^Tc(x)^{-1}$ and $\Psi$ being the $l \times l$–identity matrix. As argued in the Introduction, imposing a stability condition on $X_t$ only, which is what the choice $\Phi(x) = 0$ does, is more natural from an application point of view.

In the Gaussian case, finding the portfolio $\hat{\pi}$ reduces to solving an algebraic Riccati equation. Let us assume that $a(x), r(x), \alpha(x),$ and $\theta(x)$ are affine functions of $x$ and that the diffusion coefficients are constant. More specifically, let

$$a(x) = A_1x + a_2,$$
$$r(x) = r_1^Tx + r_2,$$
$$\alpha(x) = \alpha_1^Tx + \alpha_2,$$
$$\theta(x) = \Theta_1x + \theta_2,$$

and

$$b(x) = b, \beta(x) = \beta, \sigma(x) = \sigma,$$

where $A_1 \in \mathbb{R}^{n \times l}, a_2 \in \mathbb{R}^{n}, r_1 \in \mathbb{R}^{l}, r_2 \in \mathbb{R}, \alpha_1 \in \mathbb{R}^{l}, \alpha_2 \in \mathbb{R}, \Theta_1 \in \mathbb{R}^{l \times l}, \theta_1 \in \mathbb{R}^{l}, b$ is an $n \times k$–vector, and $\sigma$ is an $l \times k$–matrix such that the matrix $\sigma \sigma^T$ is positive definite, $\beta$ is a non–zero $k$–vector, and $\beta$ is a non–zero $k$–vector, and $\sigma$ is an $l \times k$–matrix such that the matrix $\sigma \sigma^T$ is positive definite. Condition (2.12) is fulfilled provided the pair $(A_1 - 1r_1^T, \Theta_1)$ is detectable, i.e., there exists $l \times n$–matrix $\Phi$ such that the matrix $\Theta_1 - \Phi(A_1 - 1r_1^T)$ is stable, for, in that case, there exists symmetric positive definite $l \times l$–matrix $\Psi$ such that $(\Theta_1 - \Phi(A_1 - 1r_1^T))^T \Psi + \Psi(\Theta_1 - \Phi(A_1 - 1r_1^T)) = -I_l$, see, e.g., p.252 in Bellman [4] or Theorem 8.7.2 on p.270 in Lancaster [20], so one can take $\Phi(x) = \Phi$, where $I_l$ stands for the $l \times l$–identity matrix. Consequently, (2.12) holds with $\Phi = 0$ when the matrix $\Theta_1$ is stable.

Let

$$A = \Theta_1 - \frac{\hat{\lambda}}{1 + \hat{\lambda}} \sigma b^T \sigma^{-1} (A_1 - 1r_1^T),$$
$$B = \sigma \sigma^T - \frac{\hat{\lambda}}{1 + \hat{\lambda}} \sigma b^T \sigma^{-1} b \sigma^T,$$

and

$$C = (A_1 - 1r_1^T)^T \sigma^{-1} (A_1 - 1r_1^T).$$
Let us suppose that there exists symmetric \( l \times l \)-matrix \( \hat{P}_1 \) that satisfies the algebraic Riccati equation
\[
A^T \hat{P}_1 + \hat{P}_1 A + \hat{P}_1 B \hat{P}_1 - \frac{\hat{\lambda}}{1 + \hat{\lambda}} C = 0.
\]

Conditions for the existence of solutions can be found in Fleming and Sheu [9], Willems [42], and Wonham [43]. For instance, if \( \Theta_1 \) is a stable matrix, then the pairs \((A, \sigma)\) and \((A_1 - 1r_1^T, A)\) are stabilizable and detectable, respectively, so, by Theorem 4.1 in Wonham [43] there exists a negative semidefinite symmetric matrix that satisfies the equation and the matrix \( D = A + B \hat{P}_1 \) is stable. Lemma 3.3 in Fleming and Sheu [9] asserts the uniqueness of \( \hat{P}_1 \), provided that \( \Theta_1 + \Theta_1^T \) is negative definite. With \( D \) being stable, the equation
\[
D^T \hat{p}_2 - \frac{\hat{\lambda}}{1 + \hat{\lambda}} (A_1 - 1r_1^T + b \sigma^T \hat{P}_1)^T c^{-1} (a_2 - r_2 1 + \hat{\lambda} b \beta)
\]
has a unique solution for \( \hat{p}_2 \). The function \( \hat{f}(x) = x^T \hat{P}_1 x/2 + \hat{p}_2^T x \) solves the ergodic Bellman equation (2.17), where
\[
\hat{u}(x) = \frac{1}{1 + \hat{\lambda}} c^{-1} (A_1 - 1r_1^T + b \sigma^T \hat{P}_1) x + \frac{1}{1 + \hat{\lambda}} c^{-1} (a_2 - r_2 1 + \hat{\lambda} b \beta + b \sigma^T \hat{p}_2).
\]

If the matrix \((b \sigma^T \hat{P}_1)^T c^{-1} b \sigma^T \hat{P}_1 - (A_1 - 1r_1^T)^T c^{-1} (A_1 - 1r_1^T)\) is negative definite, then (2.23) holds. By (2.17), \( \hat{m} \) is the invariant density of the linear diffusion
\[
dY_t = D Y_t dt + (\frac{\hat{\lambda}}{1 + \hat{\lambda}} \sigma b^T c^{-1} (a_2 - r_2 1 + \hat{\lambda} b \beta + b \sigma^T \hat{p}_2) + \hat{\lambda} \sigma \beta + \sigma \sigma^T \hat{p}_2 + \hat{p}_2^T \theta_2) dt
\]
and
\[
F(\hat{\lambda}) = -\frac{1}{2} \frac{\hat{\lambda}}{1 + \hat{\lambda}} ||a_2 - r_2 1 + \hat{\lambda} b \beta + b \sigma^T \hat{p}_2||^2_{c^{-1}} - \hat{\lambda}(r_2 - \alpha_2 + \frac{1}{2} ||\beta||^2 - \beta^T \sigma^T \hat{p}_2)
\]
\[
+ \frac{1}{2} \frac{\hat{\lambda}^2 ||\beta||^2}{1 + \hat{\lambda}} + \frac{1}{2} \hat{p}_2^T \sigma \sigma^T \hat{p}_2 + \hat{p}_2^T \theta_2 + \frac{1}{2} \text{tr}(\sigma \sigma^T \hat{P}_1).
\]

For the nonbenchmark case, the portfolio in (2.25) is obtained in Hata, Nagai, and Sheu [11] (see (2.39) and Theorem 2.2 there). For the optimality of \( \hat{\pi} \), those authors, who assume that \( r_1 = 0 \), \( \alpha_1 = 0 \), \( \alpha_2 = 0 \), and \( \beta = 0 \), in addition to requiring that the matrix \( \Theta_1 - \sigma b^T c^{-1} A_1 \) be stable and that the matrix \((b \sigma^T \hat{P}_1)^T c^{-1} b \sigma^T \hat{P}_1 - (A_1 - 1r_1^T)^T c^{-1} (A_1 - 1r_1^T)\) be negative definite, need that \((\Theta_1, \sigma)\) be controllable and that \( q < -F'(0+) \). Our results relax those restrictions as well as incorporate the case of nonzero \( r_1 \), \( \alpha_1 \), \( \alpha_2 \), and \( \beta \). It has to be mentioned that the proof of Theorem 2.2 in Hata, Nagai, and Sheu [11] is omitted and that the authors produce also non time–homogeneous portfolios that are "nearly" optimal under weaker hypotheses but require the same stability condition.
3 Technical preliminaries

In this section, we lay the groundwork for the proofs of the main results. Let, given \( x \in \mathbb{R}^l \), \( \lambda \geq 0 \), and \( p \in \mathbb{R}^l \),

\[
\hat{H}(x; \lambda, p) = -\lambda \sup_{u \in \mathbb{R}^n} (M(u, x) - \frac{1}{2} \lambda |N(u, x)|^2 + p^T \sigma(x)N(u, x)) + p^T \theta(x) + \frac{1}{2} |\sigma(x)^T p|^2 . \tag{3.1}
\]

One can thus write (2.14) more compactly as

\[
G(\lambda, f, m) = \int_{\mathbb{R}^l} (\hat{H}(x; \lambda, \nabla f(x)) + \frac{1}{2} \text{tr}(\sigma(x)\sigma(x)^T \nabla^2 f(x))) m(x) \, dx . \tag{3.2}
\]

Let

\[
T_\lambda(x) = \sigma(x)\sigma(x)^T - \frac{\lambda}{1 + \lambda} \sigma(x)b(x)^T c(x)^{-1}b(x)\sigma(x)^T .
\]

By part 1 of condition (N), \( T_\lambda(x) \) is a uniformly positive definite symmetric \( l \times l \)–matrix. Optimizing on the righthand side of (3.1) yields

\[
\sup_{u \in \mathbb{R}^n} (M(u, x) - \frac{1}{2} \lambda |N(u, x)|^2 + p^T \sigma(x)N(u, x)) = \frac{1}{2} \frac{1}{1 + \lambda} \|a(x) - r(x)1 + \lambda b(x)\beta(x) + b(x)\sigma(x)^T p\|_{c(x)^{-1}}^2,
\]

\[- \frac{1}{2} \lambda |\beta(x)|^2 + r(x) - a(x) + \frac{1}{2} |\beta(x)|^2 - \beta(x)^T \sigma(x)^T p \tag{3.3}
\]

so that

\[
\hat{H}(x; \lambda, p) = \frac{1}{2} p^T T_\lambda(x)p + (\frac{\lambda}{1 + \lambda} (a(x) - r(x))1 + \lambda b(x)\beta(x))c(x)^{-1}b(x)\sigma(x)^T + \lambda \beta(x)^T \sigma(x)^T + \theta(x)^T) p
\]

\[- \frac{\lambda}{2(1 + \lambda)} \|a(x) - r(x)1 + \lambda b(x)\beta(x)\|_{c(x)^{-1}}^2
\]

\[- \lambda (r(x) - a(x) + \frac{1}{2} |\beta(x)|^2) + \frac{1}{2} \lambda^2 |\beta(x)|^2 . \tag{3.4}
\]

Drawing on Bonnans and Shapiro [7], we say that, given topological space \( \mathcal{T} \), function \( h : \mathcal{T} \rightarrow \mathbb{R} \) is inf–compact, respectively, sup–compact, if the sets \( \{ x \in \mathcal{T} : h(x) \leq \delta \} \), respectively, the sets \( \{ x \in \mathcal{T} : h(x) \geq \delta \} \), are compact for all \( \delta \in \mathbb{R} \). (It is worth noting that Aubin and Ekeland [3] only require that the sets above be relatively compact. These two definitions are equivalent if the function in question is, in addition, lower semicontinuous, respectively, upper semicontinuous.) We endow the set \( \mathcal{P} \) of probability measures \( \nu \) on \( \mathbb{R}^l \) such that \( \int_{\mathbb{R}^l} |x|^2 \nu(dx) < \infty \) with the Kantorovich–Rubinstein distance

\[
d_1(\mu, \nu) = \sup \{ |\int_{\mathbb{R}^l} g(x) \mu(dx) - \int_{\mathbb{R}^l} g(x) \nu(dx)| : \frac{|g(x) - g(y)|}{|x - y|} \leq 1 \text{ for all } x \neq y \} .
\]
Convergence with respect to \(d_1\) is equivalent to weak convergence coupled with convergence of the first moments, see, e.g., Villani [41].

We introduce, for \(f \in \mathbb{C}^2\),
\[
H(x; \lambda, f) = \hat{H}(x; \lambda, \nabla f(x)) + \frac{1}{2} \text{tr} \left( \sigma(x)\sigma(x)^T \nabla^2 f(x) \right)
\]  
so that, for \(f \in \mathbb{C}^2_0\),
\[
G(\lambda, f, m) = \int_{\mathbb{R}^d} H(x; \lambda, f)m(x) \, dx.
\]

For \(\kappa > 0\), we define \(f_\kappa(x) = \kappa\|x\|_2^2/2\) and let \(\mathcal{A}_\kappa\) denote the convex hull of \(\mathbb{C}^2_0\) and of the function \(f_\kappa\). The next lemma implies, in particular, that \(F(\lambda)\) is finite–valued.

**Lemma 3.1** Suppose that either \(\lambda > 0\) or that condition \((2.12)\) holds with \(\Phi(x) = 0\). Then, for all \(\kappa > 0\) small enough, the function \(\inf_{f \in \mathbb{C}^2_0} \int_{\mathbb{R}^d} H(x; \lambda, f_\kappa) \nu(dx)\) is sup–compact in \(\nu \in \mathcal{P}\) for the Kantorovich–Rubinstein distance \(d_1\). The function \(\inf_{f \in \mathbb{C}^2_0} \int_{\mathbb{R}^d} H(x; \lambda, f) \nu(dx)\) is sup–compact in \(\nu\). Furthermore, the set \(\bigcup_{\{\lambda : |\lambda - \lambda_0| \leq \pi/2\}} \{\nu \in \mathcal{P} : \inf_{f \in \mathbb{C}^2_0} \int_{\mathbb{R}^d} H(x; \lambda, f) \nu(dx) \geq \delta\}\) is relatively compact, where \(\lambda > 0\) and \(\delta \in \mathbb{R}\).

**Proof** By \((3.4)\),
\[
\begin{align*}
\hat{H}(x; \lambda, \nabla f_\kappa(x)) &= \frac{\kappa^2}{2} \|\Psi x\|_{\hat{H}_2(x)}^2 + \kappa(\theta(x) - \Phi(x)(a(x) - r(x)1))^T \Psi x \\
&+ \kappa\left( -\frac{\lambda}{1 + \lambda} \sigma(x)b(x)^T \epsilon(x)^{-1} + \Phi(x) \right) (a(x) - r(x)1)^T \Psi x \\
&- \frac{\lambda}{2(1 + \lambda)} \|a(x) - r(x)1\|^2_{\epsilon(x)^{-1}} \\
&+ \kappa\left( -\frac{\lambda^2}{1 + \lambda} \beta(x)^T b(x)^T \epsilon(x)^{-1}b(x)\sigma(x)^T + \lambda\beta(x)^T \sigma(x)^T \right) \Psi x \\
&- \frac{\lambda}{2(1 + \lambda)} \left( 2\lambda(a(x) - r(x)1)^T \epsilon(x)^{-1}b(x)\beta(x) + \lambda^2\|b(x)\beta(x)\|^2_{\epsilon(x)^{-1}} \right) \\
&- \lambda(r(x) - a(x)) + \frac{1}{2} |\beta(x)|^2 + \frac{1}{2} \lambda^2 |\beta(x)|^2.
\end{align*}
\]  
Let us suppose that \(\lambda > 0\). Since \(\Phi(x)\) is a bounded function, by the Cauchy inequality, there exists \(K_1 > 0\) such that, for all \(\epsilon > 0\),
\[
\left( -\frac{\lambda}{1 + \lambda} \sigma(x)b(x)^T \epsilon(x)^{-1} + \Phi(x) \right) (a(x) - r(x)1)^T \Psi x
\]  
\[
\leq \frac{1}{2\epsilon^2} \|a(x) - r(x)1\|^2_{\epsilon(x)^{-1}} + K_1 \frac{\epsilon^2}{2} |\Psi x|^2.
\]
By condition (2.12), if $\kappa$ and $\epsilon$ are small enough, then

$$
\lim \sup_{|x| \to \infty} \frac{1}{|x|} \left( \kappa K_{1} \frac{\epsilon^{2}}{2} |\Psi x|^{2} + \kappa^{2} T \Psi T_{\lambda}(x) \Psi x \right.
$$

$$
\left. + \kappa \left( \theta(x) - \Phi(x)(a(x) - r(x)1)^{T} \Psi x \right) < 0 \right).$$

Finally, given $\epsilon$, $\kappa$ can be chosen such that

$$
\frac{\kappa}{2\epsilon^{2}} \|a(x) - r(x)1\|_{\ell}^{2} - \frac{\lambda}{2(1 + \lambda)} \|a(x) - r(x)1\|_{\ell}^{2} \leq 0.
$$

Putting everything together and noting that the terms on the lower two lines of (3.7) grow at most linearly with $|x|$, we conclude that, provided $\kappa$ is small enough, for suitable $K_{2}$ and $K_{3} > 0$,

$$
H(x; \lambda, f_{\kappa}) \leq K_{2} - K_{3}|x|^{2}.
$$

By (2.12) and (3.7), the latter inequality can also be fulfilled if $\lambda = 0$ and $\Phi(x) = 0$.

Therefore, on introducing $\Gamma_{\delta} = \{ \nu \in P : \int_{\mathbb{R}^{l}} H(x; \lambda, f_{\kappa}) \nu(dx) \geq \delta \}$, where $\delta \in \mathbb{R}$, we have that $\sup_{\nu \in \Gamma_{\delta}} \int_{\mathbb{R}^{l}} |x|^{2} \nu(dx) < \infty$. (As a general matter, we assume that $\sup_{\nu} = -\infty$ and $\inf_{\nu} = \infty$.) In addition, by $H(x; \lambda, f_{\kappa})$ being continuous in $x$ and by (3.8), $\int_{\mathbb{R}^{l}} H(x; \lambda, f_{\kappa}) \nu(dx)$ is an upper semicontinuous function of $\nu$, so $\Gamma_{\delta}$ is a closed set. Thus, by Prohorov’s theorem and Lebesgue’s dominated convergence theorem, $\Gamma_{\delta}$ is compact.

By (3.4) and (3.5), the function $H(x; \lambda, f)$ is convex in $f$. Therefore, if $f \in A_{\kappa}$, then, by (3.4) and (3.5), $H(x; \lambda, f)$ is bounded above by an affine function of $x$. Since $H(x; \lambda, f)$ is continuous in $x$, the function $\int_{\mathbb{R}^{l}} H(x; \lambda, f) \nu(dx)$ is upper semicontinuous in $\nu$. Since $f_{\kappa} \in A_{\kappa}$, we obtain that $\inf_{f \in A_{\kappa}} \int_{\mathbb{R}^{l}} H(x; \lambda, f) \nu(dx)$ is sup–compact. Since $\inf_{f \in A_{\kappa}} \int_{\mathbb{R}^{l}} H(x; \lambda, f) \nu(dx) = \inf_{f \in C_{2}} \int_{\mathbb{R}^{l}} H(x; \lambda, f) \nu(dx)$, the latter function is sup–compact.

An examination of the reasoning that led to (3.5) reveals that there exist $K_{2}$ and $K_{3} > 0$ such that $H(x; \lambda, f_{\kappa}) \leq K_{2} - K_{3}|x|^{2}$ if $|\lambda - \bar{x}| \leq \bar{x}/2$. Therefore,

$$
\bigcup_{\{\lambda : |\lambda - \bar{x}| \leq \bar{x}/2\}} \{ \nu \in P : \inf_{f \in C_{2}} \int_{\mathbb{R}^{l}} H(x; \lambda, f) \nu(dx) \geq \delta \}
$$

$$
\subset \bigcup_{\{\lambda : |\lambda - \bar{x}| \leq \bar{x}/2\}} \{ \nu \in P : \int_{\mathbb{R}^{l}} H(x; \lambda, f_{\kappa}) \nu(dx) \geq \delta \}
$$

$$
\subset \{ \nu \in P : K_{3} \int_{\mathbb{R}^{l}} |x|^{2} \nu(dx) \leq K_{2} - \delta \}. \square$$
For $\nu \in \mathcal{P}$, we let $L^2(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ represent the Hilbert space (of the equivalence classes) of $\mathbb{R}^l$-valued functions $h(x)$ on $\mathbb{R}^l$ that are square integrable with respect to $\nu(dx)$ equipped with the norm $(\int_{\mathbb{R}^l} |h(x)|^2 \nu(dx))^{1/2}$ and we let $L^1_{\text{loc}}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ represent the closure in $L^2(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ of the set of gradients of $C^1$-functions, with $C^1_{\text{loc}}$ denoting the set of real-valued compactly supported continuously differentiable functions on $\mathbb{R}^l$. The space $L^1_{\text{loc}}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ is a Hilbert space too. We will use the notation $\nabla f$ for the elements of $L^1_{\text{loc}}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ of those functions might not be proper gradients. Let $\hat{P}$ represent the set of probability densities $m$ such that $m \in \hat{P}$, $m \in \mathbb{W}^{1,2}(\mathbb{R}^l)$, and $\sqrt{m} \in \mathbb{W}^{1,2}(\mathbb{R}^l)$, where $\mathbb{W}$ is used for denoting a Sobolev space, see, e.g., Adams and Fournier [1]. We note that $\hat{P}$ is a convex subset of $\mathbb{P}$. In the next lemma and below, the divergence of a square matrix is defined as the vector whose entries are the divergencies of the rows of the matrix.

**Lemma 3.2** If, for $\nu \in \mathcal{P}$, $\inf_{f \in C^2_0(\mathbb{R}^l)} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) > -\infty$, then $\nu$ admits density which belongs to $\hat{P}$.

**Proof** The reasoning follows that of Puhalskii [29], cf. Lemma 6.1, Lemma 6.4, and Theorem 6.1 there. If there exists $\kappa \in \mathbb{R}$ such that $\int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) \geq \kappa$ for all $f \in C^2_0(\mathbb{R}^l)$, then by (3.5), for arbitrary $\delta > 0$,

$$-\delta \int_{\mathbb{R}^l} \frac{1}{2} \nabla (\sigma(x) \nabla f(x))^2 \nu(dx) \geq \kappa - \int_{\mathbb{R}^l} \hat{H}(x; \lambda, -\delta \nabla f(x)) \nu(dx).$$

Dividing both sides by $-\delta$ and minimizing the right-hand side over $\delta$ obtains with the aid of (3.4) and the linear growth condition (2.4) that there exists constant $K_1 > 0$ such that, for all $f \in C^2_0(\mathbb{R}^l)$,

$$\int_{\mathbb{R}^l} \nabla (\sigma(x) \nabla f(x))^2 \nu(dx) \leq K_1 \left( \int_{\mathbb{R}^l} |\nabla f(x)|^2 \nu(dx) \right)^{1/2}.$$

It follows that the left-hand side extends to a linear functional on $L^1_{\text{loc}}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$, hence, by the Riesz representation theorem, there exists $\nabla g \in L^1_{\text{loc}}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ such that

$$\int_{\mathbb{R}^l} \nabla (\sigma(x) \nabla f(x))^2 \nu(dx) = \int_{\mathbb{R}^l} \nabla g(x)^T \nabla f(x) \nu(dx)$$

and $\int_{\mathbb{R}^l} |\nabla g(x)|^2 \nu(dx) \leq K_1$. Theorem 2.1 in Bogachev, Krylov, and Röckner [6] implies that the measure $\nu(dx)$ has density $m(x)$ with respect to the Lebesgue measure which belongs to $L^p_{\text{loc}}(\mathbb{R}^l)$ for all $\theta < 1/(l - 1)$. It follows that, for arbitrary open ball $S$ in $\mathbb{R}^l$, there exists $K_2 > 0$ such that, for all $C^2_0$-functions $f$ with support in $S$,

$$\int_S |\nabla (\sigma(x) \nabla f(x))^2| m(x) dx \leq K_2 \left( \int_S |\nabla f(x)|^{2\theta/(\theta - 1)} dx \right)^{(\theta - 1)/(2\theta)}.$$
By Theorem 6.1 in Agmon [2], the density $m$ belongs to $W^{1,2}_0(\mathbb{R}^l)$ for all $\zeta < 2l/(2l - 1)$. Furthermore, $\nabla g(x) = -\text{div}(\sigma(x)\sigma(x)^TM(x))/m(x)$ so that $\sqrt{m} \in W^{1,2}(\mathbb{R}^l)$. □

Remark 3.3 Essentially, (3.9) signifies that one can integrate by parts on the lefthand side, so $m(x)$ has to be weakly differentiable.

If $m \in \hat{\mathbb{P}}$, then integration by parts in (2.14) obtains that, for $f \in C^2_0$,

$$G(\lambda, f, m) = \hat{G}(\lambda, \nabla f, m),$$

where

$$\hat{G}(\lambda, \nabla f, m) = \int_{\mathbb{R}^l} \left( \hat{H}(x; \lambda, \nabla f(x)) - \frac{1}{2} \nabla f(x)^T \text{div} (\sigma(x)\sigma(x)^T m(x)) \right) m(x) \, dx.$$  \hspace{1cm} (3.10)

(We assume that $0/0 = 0$.) We will use (3.11) in order to define $\hat{G}(\lambda, \nabla f, m)$ when $\nabla f \in L^{1,2}_0(\mathbb{R}^l, \mathbb{R}^l, m(x) \, dx)$. Furthermore, we will use (3.10) to extend the definition of $G(\lambda, f, m)$ to functions $f \in C^1_1$. It is noteworthy that if $f \in C^1_1 \cap C^2$, then

$$\lim_{R \to \infty} \int_{x \in \mathbb{R}^l: |x| \leq R} \frac{1}{2} \text{tr} \left( \sigma(x)\sigma(x)^T \nabla^2 f(x) \right) m(x) \, dx = \int_{\mathbb{R}^l} -\frac{1}{2} \nabla f(x)^T \text{div} (\sigma(x)\sigma(x)^T m(x)) \, dx.$$  \hspace{1cm} (3.11)

Lemma 3.4 The function $\hat{H}(x; \lambda, p)$ is strictly convex in $(\lambda, p) \in \mathbb{R}_+ \times \mathbb{R}^l$. Given $m \in \hat{\mathbb{P}}$, the function $\hat{G}(\lambda, \nabla f, m)$ is strictly convex in $(\lambda, \nabla f(x)) \in \mathbb{R}_+ \times L^{1,2}_0(\mathbb{R}^l, \mathbb{R}^l, m(x) \, dx)$. The function $G(\lambda, f, m)$ is convex in $(\lambda, f) \in \mathbb{R}_+ \times C^2_0$ and $\inf_{f \in C^2_0} G(\lambda, f, m)$ and $F(\lambda)$ tend to infinity superlinearly, as $\lambda \to \infty$. The function $J_0^f$ is finite and continuous on $\mathbb{R}$.

Proof The Hessian matrix of $\hat{H}(x; \lambda, p)$ with respect to $(\lambda, p)$ is given by

$$\hat{H}_{\lambda p}(x; \lambda, p) = \frac{1}{1 + \lambda} \sigma(x)b(x)^Tc(x)^{-1}b(x)\sigma(x)^T + \sigma(x)Q_1(x)\sigma(x)^T,$$

$$\hat{H}_{\lambda \lambda}(x; \lambda, p) = \frac{1}{(1 + \lambda)^3} \|a(x) - r(x)1 + b(x)\sigma(x)^Tp - b(x)\beta(x)\|_{c(x)}^2 + \beta(x)^T Q_1(x) \beta(x),$$

$$\hat{H}_{\lambda p}(x; \lambda, p) = \frac{1}{(1 + \lambda)^2} \left( a(x) - r(x)1 + b(x)\sigma(x)^Tp - b(x)\beta(x) \right)^T c(x)^{-1}b(x)\sigma(x)^T + \beta(x)^T Q_1(x)\sigma(x)^T.$$
We show that it is positive definite. More specifically, we prove that for all \( z \in \mathbb{R} \) and \( y \in \mathbb{R}^l \) such that \( z^2 + |y|^2 \neq 0 \),
\[
z^2 \tilde{H}_{\lambda \lambda}(x; \lambda, p) + y^T \tilde{H}_{pp}(x; \lambda, p)y + 2z \tilde{H}_{xp}(x; \lambda, p)y > 0.
\]
Since \( \tilde{H}_{pp}(x; \lambda, p) \) is a positive definite matrix by condition (N), the latter inequality holds when \( z = 0 \). Assuming \( z \neq 0 \), we need to show that
\[
\tilde{H}_{\lambda \lambda}(x; \lambda, p) + y^T \tilde{H}_{pp}(x; \lambda, p)y + 2\tilde{H}_{xp}(x; \lambda, p)y > 0. \tag{3.12}
\]

Let, for \( e_1 = (v_1(x), w_1(x)) \) and \( e_2 = (v_2(x), w_2(x)) \), where \( v_1(x) \in \mathbb{R}^n, w_1(x) \in \mathbb{R}^k, v_2(x) \in \mathbb{R}^n, w_2(x) \in \mathbb{R}^k \), and \( x \in \mathbb{R}^l \), the inner product be defined by \( e_1 \cdot e_2 = v_1(x)^T c(x)^{-1} v_2(x) + w_1(x)^T w_2(x) \). By the Cauchy-Schwarz inequality, applied to \( e_1 = ((1 + \lambda)^{-3/2}(a(x) - r(x))1 + b(x)\sigma(x)^T p - b(x)\beta(x)), Q_1(x)\beta(x)) \) and \( e_2 = ((1 + \lambda)^{-1/2}b(x)\sigma(x)^T y, Q_1(x)\sigma(x)^T y) \), we have that \( (\tilde{H}_{xp}(x; \lambda, p)y)^2 < y^T \tilde{H}_{pp}(x; \lambda, p)y\tilde{H}_{\lambda \lambda}(x; \lambda, p) \), with the inequality being strict because, by condition (N), \( Q_1(x)\beta(x) \) is not a scalar multiple of \( Q_1(x)\sigma(x)^T y \). Thus, \( (5.12) \) holds, so the function \( H(x; \lambda, p) \) is strictly convex in \( (\lambda, p) \) on \( \mathbb{R}_+ \times \mathbb{R}^l \) for all \( x \in \mathbb{R}^l \). By \( (3.11) \), \( G(\lambda, \nabla f, m) \) is strictly convex in \( (\lambda, f) \), provided \( m \in \mathbb{P} \), and by \( (3.2) \), \( G(\lambda, f, m) \) is convex in \( (\lambda, f) \). Thus, \( \inf_{f \in C_2^0} G(\lambda, f, m) \) is convex in \( \lambda \), so \( F(\lambda) \) is convex and, hence, continuous.

By \( (3.14) \) and \( (2.13) \), as \( \lambda \to \infty \),
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda^2} \inf_{p \in \mathbb{R}^l} (\tilde{H}(x; \lambda, p) - \frac{1}{2} p^T \text{div} (\sigma(x)\sigma(x)^T m(x)) = \frac{1}{2} \|\beta(x)\|^2_{Q_2(x)}.
\]
The latter quantity being positive by condition (N) implies, by \( (3.10) \) and Fatou’s lemma, that
\[
\liminf_{\lambda \to \infty} \frac{1}{\lambda^2} \inf_{f \in C_2^0} G(\lambda, f, m) > 0.
\]
Hence, by \( (2.13) \), \( \liminf_{\lambda \to \infty} F(\lambda)/\lambda^2 > 0 \). Since \( F(\lambda) \to \infty \) superlinearly, as \( \lambda \to \infty \), the supremum on the right of \( (2.10) \) can be taken over the same compact set of \( \lambda \) when the values of \( q \) come from a bounded set, implying that \( J_q^* \) is finite and continuous. \( \square \)

**Remark 3.5** If \( \beta(x) = 0 \), then part 2 of Condition (N) does not hold but the proof of Lemma \( (3.14) \) still goes through except for the last property in that \( F(\lambda)/\lambda^2 \) tends to zero as \( \lambda \to \infty \). Still,
\[
\liminf_{\lambda \to \infty} \frac{1}{\lambda} \inf_{f \in C_2^0} G(\lambda, f, m) \geq -\int_{\mathbb{R}^l} (r(x) - \alpha(x))m(x) \, dx,
\]
so that
\[
\liminf_{\lambda \to \infty} \frac{F(\lambda)}{\lambda} \geq -\inf_{x \in \mathbb{R}^l} (r(x) - \alpha(x))\).

Consequently, if \( \inf_{x \in \mathbb{R}^l} (r(x) - \alpha(x)) < q \), then \( -\lambda q - F(\lambda) \) tends to \( -\infty \) as \( \lambda \to \infty \), so \( \sup_{\lambda \geq 0} (-\lambda q - F(\lambda)) \) is attained and \( J_q^* \) is finite and continuous.
That might not be the case if \( \inf_{x \in \mathbb{R}} (r(x) - \alpha(x)) \geq q \). For instance, if the functions \( a(x) \), \( r(x) \), \( b(x) \), and \( \sigma(x) \) are constant, \( \alpha(x) = 0 \), and \( q \) is small enough, then the derivative of \(-\lambda q - F(\lambda)\) is positive for all \( \lambda \) and \( J^*_q = \infty \). As a result, \( J^*_q \) might fail to be continuous at \( \inf_{x \in \mathbb{R}} (r(x) - \alpha(x)) \), although it is rightcontinuous regardless.

**Remark 3.6** The convexity property of \( \hat{H}(x; \lambda, p) \) could be expected because, by (3.1),

\[
\hat{H}(x; \lambda, p) = - \sup_{u \in \mathbb{R}^k} \left( \lambda M(u, x) - \frac{1}{2} |\lambda N(u, x) + \sigma(x)^T p|^2 \right) + p^T \theta(x).
\]

By (3.1), (2.15), and by the set of the gradients of functions from \( \mathcal{C}_0^2 \),

\[
F(\lambda) = \sup_{m \in \mathbb{P}} \inf_{\nabla f \in L^1_0(\mathbb{R}^l, \mathbb{R}^l, m(x) \, dx)} \hat{G}(\lambda, \nabla f, m).
\]

Since the matrix \( T_{\lambda}(x) \) is uniformly positive definite, by (3.1), (3.4), and (3.13), \( \hat{G}(\lambda, \nabla f, m) \) tends to infinity as the \( L^2(\mathbb{R}^l, \mathbb{R}^l, m(x) \, dx) \)-norm of \( \nabla f \) tends to infinity. Since \( \hat{G}(\lambda, \nabla f, m) \) is strictly convex in \( \nabla f \), the infimum in (3.13) is attained at a unique point, see, e.g., Proposition 1.2 on p.35 in Ekeland and Temam [8]. Furthermore, since

\[
\sup_{m \in \mathbb{P}} \inf_{\nabla f \in L^1_0(\mathbb{R}^l, \mathbb{R}^l, m(x) \, dx)} \hat{G}(\lambda, \nabla f, m) = \sup_{\nu \in \mathbb{P}} \inf_{f \in \mathcal{C}_0^2} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx)
\]

\[
= \sup_{\nu \in \mathbb{P}} \inf_{f \in \mathcal{A}_0} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) \tag{3.14}
\]

and, for \( \lambda > 0 \), by Lemma 3.1, Lemma 3.2, and (3.10), the function \( \inf_{f \in \mathcal{A}_0} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) \) is sup–compact in \( \nu \), we have that the supremum in (3.13) is attained too, provided \( \lambda > 0 \).

**Lemma 3.7** Suppose that either \( \lambda > 0 \) or that condition (2.12) holds with \( \Phi(x) = 0 \). Then there exists \( (f^\lambda, m^\lambda) \in (\mathcal{C}_0^1 \cap \mathcal{C}^2) \times \mathbb{P} \) that is a saddle point of \( G(\lambda, f, m) \) as a function of \( (f, m) \) so that

\[
\inf_{f \in \mathcal{C}_0^1 \cap \mathcal{C}^2} \sup_{m \in \mathbb{P}} G(\lambda, f, m) = \inf_{f \in \mathcal{C}_0^1 \cap \mathcal{C}^2} \sup_{m \in \mathbb{P}} G(\lambda, f, m) = \sup_{m \in \mathbb{P}} \inf_{f \in \mathcal{C}_0^1 \cap \mathcal{C}^2} G(\lambda, f, m) = \sup_{m \in \mathbb{P}} \inf_{f \in \mathcal{C}_0^1 \cap \mathcal{C}^2} G(\lambda, f, m) = F(\lambda), \tag{3.15}
\]

with the infimum on the leftmost side being attained at \( f^\lambda \) and the supremum on the rightmost side being attained at \( m^\lambda \). The function \( f^\lambda \) satisfies the ergodic Bellman equation

\[
H(x; \lambda, f) = F(\lambda), \tag{3.16}
\]
for all \( x \in \mathbb{R}^l \), and \( m^\lambda(x) \) is the invariant density of a diffusion:

\[
\int_{\mathbb{R}^l} \left( \nabla h(x)^T (-\lambda \sigma(x)N(u^\lambda(x), x) + \theta(x) + \sigma(x)^T \nabla f^\lambda(x)) + \frac{1}{2} \text{tr}(\sigma(x)^T \nabla^2 h(x)) \right) m^\lambda(x) \, dx = 0, \tag{3.17}
\]

for all \( h \in \mathbb{C}^2_+ \), where

\[
u^\lambda(x) = \frac{1}{1 + \lambda} \sigma(x)^{-1} (\sigma(x)^{-1} - r(x)1 + \lambda \beta(x) + b(x)^T \nabla f^\lambda(x)). \tag{3.18}
\]

The density \( m^\lambda(x) \) may be chosen positive, bounded and of class \( \mathbb{C}^1 \). The functions \( \nabla f^\lambda(x) \) and \( m^\lambda(x) \) are specified uniquely.

In addition, the function \( F(\lambda) \) is strictly convex and is continuously differentiable, provided \( \lambda > 0 \), and the righthand derivative at \( \lambda \geq 0 \) is given by

\[
F'_+ (\lambda) = \int_{\mathbb{R}^l} \left( -M(u^\lambda(x), x) + \lambda|N(u^\lambda(x), x)|^2 - \nabla f^\lambda(x)^T \sigma(x)N(u^\lambda(x), x) \right) m^\lambda(x) \, dx. \tag{3.19}
\]

**Proof** Since \( \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) \) is an upper semicontinuous and concave function of \( \nu \in \mathcal{P} \), for all \( f \in \mathcal{A}_\kappa \), is convex in \( f \in \mathcal{A}_\kappa \), and \( \int_{\mathbb{R}^l} H(x; \lambda, f_\kappa) \nu(dx) \) is sup–compact in \( \nu \) by Lemma 3.1, an application of Theorem 7 on p.319 in Aubin and Ekeland yields

\[
\sup_{\nu \in \mathcal{P}} \inf_{f \in \mathcal{C}^2_+} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) = \sup_{\nu \in \mathcal{P}} \inf_{f \in \mathcal{A}_\kappa} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx)
\]

\[
= \inf_{f \in \mathcal{A}_\kappa} \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) \geq \inf_{f \in \mathcal{C}^2_+} \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx),
\]

the supremum on the leftmost side being attained at some \( \nu^\lambda \). It follows that

\[
\sup_{\nu \in \mathcal{P}} \inf_{f \in \mathcal{C}^2_+} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) = \inf_{f \in \mathcal{C}^2_+} \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx).
\]

By Lemma 3.2

\[
\sup_{\nu \in \mathcal{P}} \inf_{f \in \mathcal{C}^2_+} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) = \sup_{m \in \mathcal{P}} \inf_{f \in \mathcal{C}^2_+} \int_{\mathbb{R}^l} H(x; \lambda, f) m(x) \, dx
\]

and \( \nu^\lambda(dx) = m^\lambda(x) \, dx \), where \( m^\lambda \in \mathcal{P} \), and, by an approximation argument,

\[
\sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) = \sup_{m \in \mathcal{P}} \int_{\mathbb{R}^l} H(x; \lambda, f) m(x) \, dx.
\]
We obtain that
\[
\inf_{f_1 \in C^2_0} \sup_{m \in \mathcal{P}} G(\lambda, f, m) = \sup_{f \in \mathcal{E}} \inf_{m \in \mathcal{P}} G(\lambda, f, m) = \inf_{f \in \mathcal{E}} G(\lambda, f, m^\lambda).
\]

Therefore, on applying Lemma 3.2
\[
\inf_{f \in \mathcal{E}_1 \cap \mathcal{E}_2} \sup_{m \in \mathcal{P}} G(\lambda, f, m) \leq \sup_{m \in \mathcal{P}} \inf_{f \in \mathcal{E}_1} G(\lambda, f, m) = \sup_{m \in \mathcal{P}} \inf_{f \in \mathcal{E}_1 \cap \mathcal{E}_2} G(\lambda, f, m).
\]

The leftmost side not being less than the rightmost side obtains (3.15).

By (3.15),
\[
F(\lambda) = \inf_{f \in \mathcal{E}_1} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f).
\]

(The right-hand side is finite: take \( f = f_\kappa \).) Applying the reasoning on pp.289–294 in Kaise and Sheu [14], one can see that, for arbitrary \( \epsilon > 0 \), there exists \( C^2 \)-function \( f^{(\epsilon)} \) such that, for all \( x \in \mathbb{R}^l \), \( H(x; \lambda, f^{(\epsilon)}) = F(\lambda) + \epsilon \). Considering that some details are omitted in Kaise and Sheu [14], we give an outline of the proof, following the lead of Ichihara [12]. By the definition of the infimum, there exists function \( f_1 \in C^2_0 \) such that \( H(x; \lambda, f_1) \leq F(\lambda) + \epsilon \) for all \( x \). Given open ball \( S \), centered at the origin, by Theorem 6.14 on p.107 in Gilbarg and Trudinger [10], there exists \( C^2 \)-solution \( f_2 \) of the linear elliptic boundary value problem \( H(x; \lambda, f) - (1/2) \nabla f(x)^T T_\lambda(x) \nabla f(x) = F(\lambda) + \epsilon \) when \( x \in S \) and \( f(x) = f_\kappa(x) \) when \( x \in \partial S \), with \( \partial S \) standing for the boundary of \( S \). Therefore, \( H(x; \lambda, f_2) \geq F(\lambda) + \epsilon \) in \( S \). By Theorem 8.4 on p.302 of Chapter 4 in Ladyzhenskaya and Ural’tseva [19], for any ball \( S' \) contained in \( S \) and centered at the origin, there exists \( C^2 \)-solution \( f_{S'} \) to the boundary value problem \( H(x; \lambda, f) = F(\lambda) + \epsilon \) in \( S' \) and \( f(x) = f_\kappa(x) \) on \( \partial S' \). Since \( f_{S'} \) is a solution of the boundary value problem \( (1/2) \text{tr} (\sigma(x)\sigma(x)^T \nabla^2 f(x)) = -H(x; \lambda, \nabla f_{S'}(x)) + F(\lambda) + \epsilon \) when \( x \in S' \) and \( f(x) = f_{S'}(x) \) when \( x \in \partial S' \), we have by Theorem 6.17 on p.109 of Gilbarg and Trudinger [10] that \( f_{S'}(x) \) is thrice continuously differentiable. Letting the radius of \( S' \) (and that of \( S \)) go to infinity, we have, by p.294 in Kaise and Sheu [14], see also Proposition 3.2 in Ichihara [12], that the \( f_{S'} \) converge locally uniformly and in \( W_{loc}^{1,2}(\mathbb{R}^l) \) to \( f^{(\epsilon)} \) which is a weak solution to \( H(x; \lambda, f) = F(\lambda) + \epsilon \). Furthermore, by Lemma 2.4 in Kaise and Sheu [14], the \( W^{1,\infty}(S'') \)-norms of the \( f_{S'} \) are uniformly bounded over balls \( S' \) for any fixed ball \( S'' \) contained in \( S' \). Therefore, \( f^{(\epsilon)} \) belongs to \( W_{loc}^{1,\infty}(\mathbb{R}^l) \). By Theorem 6.4 on p.284 in Ladyzhenskaya and Ural’tseva [19], \( f^{(\epsilon)} \) is thrice continuously differentiable.

As in Theorem 4.2 in Kaise and Sheu [14], by using the gradient bound in Lemma 2.4 there, we have that the \( f^{(\epsilon)} \) converge along a subsequence uniformly on compact sets as \( \epsilon \to 0 \) to a \( C^2 \)-solution of \( H(x; \lambda, f) = F(\lambda) \). That solution, which we denote by \( f^\lambda \), delivers infimum on the leftmost side of (3.15) and satisfies the Bellman equation, with the gradient \( \nabla f^\lambda(x) \) obeying the linear growth condition, see Remark 2.5 in Kaise and Sheu [14].
Since $f^\lambda$ delivers infimum on the leftmost side of (3.15) and $m^\lambda$ delivers supremum on the rightmost side, by Proposition 2.156 on p.104 in Bonnans and Shapiro \[7\], or by Proposition 1.2 on p.167 in Ekeland and Temam \[8\], the pair $(f^\lambda, m^\lambda)$ is a saddle point of $G(\lambda, f, m)$ as a function of $(f, m)$. Equation \[8.17\] expresses the requirement of the directional derivative of $G(\lambda, f, m)$ with respect to $f$ in the direction $h$ being equal to zero at $(f^\lambda, m^\lambda)$, cf. Proposition 1.6 on p.169 in Ekeland and Temam \[8\]. In some more detail, either by Theorem 4.13 on p.273 or by Theorem 4.17 on p.276 in Bonnans and Shapiro \[7\], or by a direct calculation, the function \[L^\lambda = (\sigma(x) N(u(x), x) + \lambda N(u(x), x)) \text{ having at } x \text{ positive, bounded and is of class } C^1 \text{ by Corollaries 2.10 and 2.11 in Bogachev, Krylov, and Röckner \[5\], and by Agmon \[2\], see also Theorem 4.1(ii) and p.413 in Metafune, Pallardi, and Rhandi \[25\].}

Since $(f^\lambda, m^\lambda)$ is a saddle point of $G(\lambda, f, m)$, with $m^\lambda$ being specified uniquely, the supremum in \[3.13\] and the rightmost side of \[3.14\] are attained at unique $\nu$ which is $m^\lambda$. Both the infimum and supremum in \[3.13\] being attained when $\lambda > 0$ and the function $G(\lambda, \nabla f, m)$ being strictly convex in $(\lambda, \nabla f)$, the function $F(\lambda)$ is strictly convex.

We address the differentiability of $F(\lambda)$. By Theorem 4.13 on p.273 in Bonnans and Shapiro \[7\] and dominated convergence, we have on recalling \[3.11\] and \[3.13\], that if $m \in \tilde{P}$ and $\nabla f \in L^{1,2}_0(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)$, then $G(\lambda, \nabla f, m)$ is differentiable in $\lambda$ at $\lambda > 0$ with the derivative being equal to

$$\int_{\mathbb{R}^l} \left(-M(\tilde{u}(x), x) + \lambda |N(\tilde{u}(x), x)|^2 - \nabla f(x)^T \sigma(x) N(\tilde{u}(x), x)\right) m(x) dx$$

and with $\tilde{u}(x)$ being defined earlier in this proof. Furthermore, $\inf_{\nabla f \in L^{1,2}_0(\mathbb{R}^l, \mathbb{R}^l, m(x) dx)} G(\lambda, \nabla f, m)$ is attained at $\nabla \tilde{f}$ such that the Fréchet
derivative of \( \hat{G}(\lambda, \nabla f, m) \) with respect to \( \nabla f \) equals zero, so, by (3.1), (3.4), and (3.21),

\[
\int_{\mathbb{R}^l} \nabla h(x)^T \left( -\lambda \sigma(x) \left( b(x)^T \frac{1}{1 + \lambda} c(x)^{-1} (a(x) - r(x)1 + \lambda b(x)) - \beta(x) \right) + \theta(x) \frac{1}{2} \frac{\text{div} (\sigma(x) \sigma(x)^T m(x))}{m(x)} + T_\lambda(x) \nabla \hat{f}(x) \right) m(x) \, dx = 0 , \tag{3.22}
\]

for all \( \nabla h \in \mathbb{L}^{1,2}([0, \infty), \mathbb{R}^l, m(x) \, dx) \). The mapping that associates with an element \((\lambda, \nabla f)\) of \( [0, \infty) \times \mathbb{L}^{1,2}([0, \infty), \mathbb{R}^l, m(x) \, dx) \) the linear functional on \( \mathbb{L}^{1,2}([0, \infty), \mathbb{R}^l, m(x) \, dx) \) that is defined, for \( \nabla h \in \mathbb{L}^{1,2}([0, \infty), \mathbb{R}^l, m(x) \, dx) \), by the lefthand side of (3.22) with \( \nabla f \) as \( \nabla \hat{f} \), is continuously differentiable. Since the matrix \( T_\lambda(x) \) is uniformly positive definite, by the Riesz representation theorem, the partial derivative with respect to \( \nabla f \) is a linear homeomorphism. By the implicit mapping theorem, \( \nabla \hat{f} \) is continuously differentiable in \( \lambda \), see, e.g., Theorem 2.1 on p. 364 in Lang [21]. Since the Fréchet derivative of \( \hat{G}(\lambda, \nabla f, m) \) with respect to \( \nabla f \) equals zero at \( \nabla \hat{f} \), we obtain by the chain rule that \( \hat{G}(\lambda, \nabla \hat{f}, m) \) is continuously differentiable in \( \lambda \) with the full \( \lambda \)-derivative being equal to its partial \( \lambda \)-derivative evaluated at \((\lambda, \nabla \hat{f}, m)\), i.e.,

\[
\int_{\mathbb{R}^l} \left( -M(\tilde{u}(x), x) + \lambda |N(\tilde{u}(x), x)|^2 - \nabla \hat{f}(x)^T \sigma(x) N(\tilde{u}(x), x) \right) m(x) \, dx .
\]

By (4.10),

\[
\inf_{\nu \in P}(\lambda, f, m) \hat{G}(\lambda, \nabla f, m) = \inf_{f \in C_0^2} \int_{\mathbb{R}^l} H(x; \lambda, f) m(x) \, dx ,
\]

so the latter function is differentiable in \( \lambda \) too, with the same derivative. This proves (4.19) when \( \lambda > 0 \). The case where \( \lambda = 0 \) is obtained by an application of Theorem 24.1 on p. 227 in Rockafellar [31].

Given \( \lambda > 0 \), if \( \lambda \) is close enough to \( \lambda \), then

\[
\sup_{\nu \in \overline{P}} \inf_{f \in C_0^2} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) = \sup_{\nu \in \overline{P}} \inf_{f \in C_0^2} \int_{\mathbb{R}^l} H(x; \lambda, f) \nu(dx) , \tag{3.23}
\]

where \( \overline{P} = \cup_{|\lambda - \lambda'| \leq \lambda/2} \{ \nu \in P : \inf_{f \in C_0^2} \int_{\mathbb{R}^l} H(x; \lambda', f) \nu(dx) \geq F(\lambda') - 1 \} \). By Lemma 5.2, the measures from \( \overline{P} \) possess densities which belong to \( \overline{P} \). Hence, the function in the supremum on the right of (3.23) is differentiable in \( \lambda \) for \( \nu \in \overline{P} \). It is also convex in \( \lambda \) and upper semicontinuous in \( \nu \). By Lemma 5.1 the set \( \overline{P} \) is relatively compact. In addition, \( \nu(dx) = m(x) \, dx \) is the only point at which the supremum on the lefthand side of (3.23) is attained for \( \lambda = \lambda' \). Theorem 3 on p. 201 in Ioffe and Tihomirov [13] enables us to conclude that the righthand side of (3.23) is differentiable in \( \lambda \) at \( \lambda \), with the derivative being equal to

\[
\int_{\mathbb{R}^l} \left( -M(u^\lambda(x), x) + \lambda |N(u^\lambda(x), x)|^2 - \nabla f^\lambda(x)^T \sigma(x) N(u^\lambda(x), x) \right) \tilde{m}(x) \, dx .
\]

By (3.13) and (3.14), this is true of \( F(\lambda) \). \( \square \)
Remark 3.8 By Theorem 6.4 on p. 284 in Ladyzhenskaya and Ural’tseva [19], \( f^\lambda \) is thrice continuously differentiable. Furthermore, by (3.20), \( F(\lambda) \) is the smallest \( A \) such that there exists \( C^2 \)-function \( f \) that satisfies the equation \( H(x; \lambda, f) = A \), for all \( x \in \mathbb{R}^l \). One can thus infer the existence of \( m^\lambda(x) \) satisfying (3.17) from the results of Kaise and Sheu [14] and Ichihara [12]. Besides, we have that

\[
F(\lambda) = \inf_{f \in C^2} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f) = \inf_{f \in C^2} \sup_{x \in \mathbb{R}^l} H(x; \lambda, f).
\]

According to Lemma 3.4 and Lemma 3.7, \( \sup_{\lambda \geq 0} (-\lambda q - F(\lambda)) \) is attained at unique \( \lambda \) which is denoted by \( \hat{\lambda} \).

Lemma 3.9 Suppose that either \( \hat{\lambda} > 0 \) or condition (2.12) holds with \( \Phi(x) = 0 \). Then there exists \( (\hat{f}, \hat{m}) \in (C^1_l \cap C^2) \times \hat{P} \) such that \( ((\hat{\lambda}, \hat{f}), \hat{m}) \) is a saddle point of the function \( \lambda q + G(\lambda, f, m) \) in \( \mathbb{R}_+ \times (C^1_l \cap C^2) \times \hat{P} \). Furthermore, \( \nabla \hat{f} \) and \( \hat{m} \) are specified uniquely. The density \( \hat{m} \) is positive, bounded, and is of class \( C^1 \). Equations (2.17a), (2.17b), and (2.18) hold, and

\[
\int_{\mathbb{R}^l} (M(\hat{u}(x), x) - \hat{\lambda}|\nabla \hat{u}(x), x|\sigma(x) N(\hat{u}(x), x)) \hat{m}(x) dx \leq q,
\]

with \( \hat{u}(x) \) being defined by (2.18). If \( \hat{\lambda} > 0 \), then equality prevails in (3.24).

Proof We let \( \hat{f} = f^\hat{\lambda} \) and \( \hat{m} = m^\hat{\lambda} \). Equations (2.17a), (2.17b), and (2.18) hold by Lemma 3.7. Since \( G(\lambda, f, m) \) is convex in \( (\lambda, f) \) and is concave in \( m \), those equations imply that \( ((\hat{\lambda}, \hat{f}), \hat{m}) \) is a saddle point of \( \lambda q + G(\lambda, f, m) \), cf. Proposition 1.7 on p. 170 in Ekeland and Temam [8]. The pair \( (\nabla \hat{f}, \hat{m}) \) is specified uniquely by Lemma 3.7. The inequality in (3.24) follows from (3.19) and the fact that \( q + F^*_*(\hat{\lambda}) \geq 0 \). If \( \hat{\lambda} > 0 \), then the latter inequality is equality.

Remark 3.10 By (2.15) and (2.16), under the hypotheses of the theorem, \( J^*_q = -\lambda q - G(\hat{\lambda}, \hat{f}, \hat{m}) \).

Lemma 3.11 Suppose that (2.12) holds with \( \Phi(x) = 0 \). Suppose that either \( \hat{\lambda} = 0 \) or there exist \( q > 0 \), \( C_1 > 0 \) and \( C_2 > 0 \) such that (2.22) holds for all \( x \in \mathbb{R}^l \). Then (2.21) holds.

Proof By (4.18),

\[
\sup_{\nu \in \mathcal{P}} \inf_{f \in C^2} \int_{\mathbb{R}^l} \hat{H}(x; \hat{\lambda}, f, u^\nu) \nu(dx) = \inf_{f \in C^2} \sup_{x \in \mathbb{R}^l} \hat{H}(x; \hat{\lambda}, f, u^\nu).
\]
For function $f$ and $\tau > 0$, we denote $f(x)^\tau = f(x)\chi_{[0,\tau]}(|x|)$. By (2.6m), (2.6n), (2.18), and (2.19),

\[
\hat{H}(x; \hat{\lambda}, f, \hat{u}^\tau) = \frac{\hat{\lambda}}{2(1 + \hat{\lambda})} \left( \|b(x)\sigma(x)^T \nabla \hat{f}(x)^\tau\|_{c(x)}^2 - \|(a(x) - r(x)) \mathbf{1}^T\|_{c(x)}^2 \right)
\]

\[
- \hat{\lambda}(r(x) - \alpha(x) + \frac{1}{2} \|\beta(x)\|^2) - \frac{\hat{\lambda}}{2(1 + \hat{\lambda})} \|\hat{\lambda}b(x)\beta(x)^T\|_{c(x)}^2
\]

\[
- \frac{\hat{\lambda}}{1 + \hat{\lambda}} \left( \left( (a(x) - r(x)) \mathbf{1}^T \right) c(x)^{-1} b(x) \hat{\lambda} \beta(x) + \left( (a(x) - r(x)) \mathbf{1} + \hat{\lambda} b(x) \beta(x) + b(x) \sigma(x)^T \nabla \hat{f}(x)^\tau \right) c(x)^{-1} b(x) \sigma(x)^T \nabla f(x) \right)
\]

\[
+ \frac{1}{2} \hat{\lambda} \beta(x) + \sigma(x)^T \nabla f(x) |^2 + \nabla f(x)^T \theta(x) + \frac{1}{2} \text{tr} \left( \sigma(x) \sigma(x)^T \nabla^2 f(x) \right).
\]

(3.26)

As in the proof of Lemma 3.1 it follows that, under the hypotheses, there exist $\hat{\kappa} > 0$, $\hat{K}_1 > 0$ and $\hat{K}_2 > 0$ such that $\hat{H}(x; \hat{\lambda}, f, \hat{u}^\tau) \leq \hat{K}_1 - \hat{K}_2 |x|^2$, for all $x \in \mathbb{R}^l$ and all $\tau > 0$. Consequently, $\inf_{f \in C^2_0} \int_{\mathbb{R}^l} \hat{H}(x; \hat{\lambda}, f, \hat{u}^\tau) \nu(dx)$ is a $\nu$-compact function of $\nu \in \mathcal{P}$, so, the supremum over $\nu$ on the lefthand side of (3.25) is attained at some $\nu_\tau$. Moreover, if the lim sup on the lefthand side of (2.21) is greater than $-\infty$, then

\[
\limsup_{\tau \to \infty} \int_{\mathbb{R}^l} |x|^2 \nu_\tau(dx) < \infty,
\]

(3.27)

so, the $\nu_\tau$ make up a relatively compact subset of $\mathcal{P}$.

If either $\hat{\lambda} = 0$ or (2.22) holds, then, given $\hat{f} \in C^2_0$, by (3.26), there exist $\hat{C}_1$ and $\hat{C}_2$, such that, for all $x \in \mathbb{R}^l$ and all $\tau > 0$,

\[
\hat{H}(x; \hat{\lambda}, \hat{f}, \hat{u}^\tau) \leq \hat{C}_1 |x| + \hat{C}_2.
\]

(3.28)

Assuming that $\nu_\tau \to \nu$, we have, by the convergence $\hat{H}(x_\tau; \hat{\lambda}, \hat{f}, \hat{u}^\tau) \to \hat{H}(\hat{x}; \hat{\lambda}, \hat{f}, \hat{u})$ when $x_\tau \to \hat{x}$, by (3.27), (3.28), the definition of the topology on $\mathcal{P}$, Fatou’s lemma, and the dominated convergence theorem, that

\[
\limsup_{\tau \to \infty} \int_{\mathbb{R}^l} \hat{H}(x; \hat{\lambda}, \hat{f}, \hat{u}^\tau) \nu_\tau(dx) \leq \int_{\mathbb{R}^l} \hat{H}(x; \hat{\lambda}, \hat{f}, \hat{u}) \nu(dx),
\]

so, on recalling (2.20),

\[
\limsup_{\tau \to \infty} \inf_{f \in C^2_0} \int_{\mathbb{R}^l} \hat{H}(x; \hat{\lambda}, f, \hat{u}^\tau) \nu_\tau(dx) \leq \inf_{f \in C^2_0} \int_{\mathbb{R}^l} \hat{H}(x; \hat{\lambda}, f, \hat{u}) \nu(dx) \leq F(\hat{\lambda}).
\]

□
4 Proofs of the main results

Proof of Theorem 2.1. Let $\tilde{q} < q$. By the continuity of $J_s^q$, it suffices to prove that if $\tilde{q}$ is close enough to $q$, then

$$\liminf_{t \to \infty} \frac{1}{t} \ln \mathbb{P}(L_t^n < q) \geq -J_s^q.$$ 

If $\hat{\lambda} > 0$, then $F'(\hat{\lambda}) = q$. Since the function $F(\lambda)$ is strictly convex, $F'(\lambda)$ is a strictly increasing function. By it being continuous according to Lemma 3.7 and the intermediate value theorem, the equation $F'(\lambda) = \tilde{q}$ has positive solution $\tilde{\lambda}$ provided $\tilde{q}$ is close enough to $q$. If $\hat{\lambda} = 0$, then, by hypotheses, (2.12) holds with $\Phi(x) = 0$ and we let $\tilde{\lambda} = 0$. In either case, Lemma 3.9 yields the existence of saddle point $(f(\tilde{\lambda}), m(\tilde{\lambda}))$ of the function $G(\tilde{\lambda}, f, m)$ such that $(f(\tilde{\lambda}), m(\tilde{\lambda})) \in (\mathbb{C}_1 \cap \mathbb{C}_2) \times \mathbb{P}$. In addition, the density $m(\tilde{\lambda})$ is continuously differentiable, positive and bounded. By Remark 3.10, $J_s^\tilde{q} = -\tilde{\lambda} \tilde{q} - G(\tilde{\lambda}, f, m(\tilde{\lambda}))$. Therefore, one needs to prove that

$$\liminf_{t \to \infty} \frac{1}{t} \ln \mathbb{P}(L_t^n < q) \geq -\tilde{\lambda} \tilde{q} + G(\tilde{\lambda}, f, m(\tilde{\lambda})).$$  (4.1)

Let $u(\tilde{\lambda})$ represent the point at which the supremum is attained in (2.14) for $\lambda = \tilde{\lambda}$, $m = m(\tilde{\lambda})$, $f = f(\tilde{\lambda})$ so that, as in (2.18) and (3.18),

$$u(\tilde{\lambda}) = \frac{1}{1 + \tilde{\lambda}} c(x)^{-1} (a(x) - r(x)1 + \tilde{\lambda}b(x)\beta(x) + b(x)\sigma(x)^T \nabla f(\tilde{\lambda})(x)).$$  (4.2)

We prove (4.1) by showing that

$$-\tilde{\lambda} \tilde{q} - G(\tilde{\lambda}, f, m(\tilde{\lambda})) = \frac{1}{2} \int_{\mathbb{R}^i} | -\tilde{\lambda} N(u(\tilde{\lambda})(x), x) + \sigma(x)^T \nabla f(\tilde{\lambda})(x)|^2 m(\tilde{\lambda})(x) dx$$  (4.3)

and that, for $\delta > 0$,

$$\liminf_{t \to \infty} \frac{1}{t} \ln \mathbb{P}(L_t^n < \tilde{q} + 3\delta) \geq \frac{1}{2} \int_{\mathbb{R}^i} | -\tilde{\lambda} N(u(\tilde{\lambda})(x), x) + \sigma(x)^T \nabla f(\tilde{\lambda})(x)|^2 m(\tilde{\lambda})(x) dx - 2\delta.$$  (4.4)

- Proof of (4.3). By Lemma 3.9

$$\int_{\mathbb{R}^i} (M(u(\tilde{\lambda})(x), x) - \tilde{\lambda} |N(u(\tilde{\lambda})(x), x)|^2 + \nabla f(\tilde{\lambda})(x)^T \sigma(x)N(u(\tilde{\lambda})(x), x)) m(\tilde{\lambda})(x) dx \leq \tilde{q},$$  (4.5)
with equality prevailing, provided $\tilde{\lambda} > 0$. Therefore, no matter $\tilde{\lambda}$, by (3.11) and (3.13),

$$-\tilde{\lambda} q - G(\tilde{\lambda}, f^{\tilde{\lambda}}, m) = -\tilde{\lambda} \int_{\mathbb{R}^l} (M(u^{\tilde{\lambda}}(x), x) - \tilde{\lambda} |N(u^{\tilde{\lambda}}(x), x)|^2$$
$$+ \nabla f^{\tilde{\lambda}}(x)^T \sigma(x) N(u^{\tilde{\lambda}}(x), x)) m^{\tilde{\lambda}}(x) \, dx$$
$$- \int_{\mathbb{R}^l} (-\tilde{\lambda} M(u^{\tilde{\lambda}}(x), x) + \frac{1}{2} \tilde{\lambda}^2 |N(u^{\tilde{\lambda}}(x), x)|^2 - \tilde{\lambda} \nabla f^{\tilde{\lambda}}(x)^T \sigma(x) N(u^{\tilde{\lambda}}(x), x)$$
$$+ \nabla f^{\tilde{\lambda}}(x)^T \theta(x) + \frac{1}{2} |\sigma(x)|^2 \nabla f^{\tilde{\lambda}}(x)|^2$$
$$- \nabla f^{\tilde{\lambda}}(x)^T \frac{\text{div}(\nabla f^{\tilde{\lambda}}(x)^T \sigma(x)^T m^{\tilde{\lambda}}(x))}{2 m^{\tilde{\lambda}}(x)} ) m^{\tilde{\lambda}}(x) \, dx$$
$$= \int_{\mathbb{R}^l} \frac{1}{2} \tilde{\lambda}^2 |N(u^{\tilde{\lambda}}(x), x)|^2 m^{\tilde{\lambda}}(x) \, dx$$
$$- \int_{\mathbb{R}^l} (\nabla f^{\tilde{\lambda}}(x)^T \theta(x) + \frac{1}{2} |\sigma(x)|^2 \nabla f^{\tilde{\lambda}}(x)|^2$$
$$- \nabla f^{\tilde{\lambda}}(x)^T \frac{\text{div}(\sigma(x)^T m^{\tilde{\lambda}}(x))}{2 m^{\tilde{\lambda}}(x)} ) m^{\tilde{\lambda}}(x) \, dx. \quad (4.6)$$

By (3.17) in Lemma 3.9, the inclusion $m^{\tilde{\lambda}} \in \mathbb{P}$, and integration by parts, for $h \in C^2_0$, 

$$\int_{\mathbb{R}^l} \nabla h(x)^T (-\tilde{\lambda} \sigma(x) N(u^{\tilde{\lambda}}(x), x) + \theta(x) + \sigma(x)^T \nabla f^{\tilde{\lambda}}(x)$$
$$- \frac{\text{div}(\sigma(x)^T m^{\tilde{\lambda}}(x))}{2 m^{\tilde{\lambda}}(x)} ) m^{\tilde{\lambda}}(x) \, dx = 0. \quad (4.7)$$

The facts that $|\nabla f^{\tilde{\lambda}}(x)|$ grows at most linearly with $|x|$, that $u^{\tilde{\lambda}}(x)$ is a linear function of $\nabla f^{\tilde{\lambda}}(x)$ by (4.2), that $\int_{\mathbb{R}^l} |x|^2 m^{\tilde{\lambda}}(x) \, dx < \infty$, and that $\int_{\mathbb{R}^l} |\nabla m^{\tilde{\lambda}}(x)|^2 / m^{\tilde{\lambda}}(x) \, dx < \infty$, imply that the expression in parentheses under the integral in (4.7) represents a function which is an element of $L^2(\mathbb{R}^l, \mathbb{R}^l, m^{\tilde{\lambda}}(x) \, dx)$. Therefore, the integral on the lefthand side extends to a linear functional on $L^1_{\nu}^{\frac{1}{2}}(\mathbb{R}^l, \mathbb{R}^l, m^{\tilde{\lambda}}(x) \, dx)$. Consequently, one can sub-
stitute $\nabla f \hat{\lambda}(x)$ for $\nabla h(x)$ to obtain that

$$
\int_{\mathbb{R}^d} \nabla f \hat{\lambda}(x)^T (-\hat{\lambda} \sigma(x) N(u \hat{\lambda}(x), x) + \theta(x) + \sigma(x) x^T \nabla f \hat{\lambda}(x)
$$

$$
- \frac{\text{div}(\sigma(x) x^T m \hat{\lambda}(x))}{2 m \hat{\lambda}(x)} ) m \hat{\lambda}(x) \, dx = 0. \quad (4.8)
$$

Substitution on the rightmost side of (4.6) yields (4.3).

**Proof of (4.4).** We apply a Girsanov change of a probability measure. Let

$$
\tilde{W}_t^s = W_t^s - \sqrt{t} \int_0^s (-\tilde{\lambda} N(u \hat{\lambda}(X^s_t), X^s_t) + \sigma(X^s_t)^T \nabla f \hat{\lambda}(X^s_t)) \, d\tilde{s}
$$

and

$$
\frac{d\tilde{P}_t}{dP} = \exp\left(\sqrt{t} \int_0^1 (-\tilde{\lambda} N(u \hat{\lambda}(X^s_t), X^s_t) + \sigma(X^s_t)^T \nabla f \hat{\lambda}(X^s_t)) dW_t^s
$$

$$
- \frac{t}{2} \int_0^1 | -\tilde{\lambda} N(u \hat{\lambda}(X^s_t), X^s_t) + \sigma(X^s_t)^T \nabla f \hat{\lambda}(X^s_t)|^2 \, ds \right). \quad (4.10)
$$

A multidimensional extension of Theorem 4.7 on p.137 in Liptser and Shiryayev [23], which is proved similarly, obtains that, given $t > 0$, there exists $\gamma' > 0$ such that $\sup_{s \leq t} E e^{\gamma' |X_s|} < \infty$. By Example 3 on pp.220,221 in Liptser and Shiryayev [23] and $\nabla f \hat{\lambda}(x)$ obeying the linear growth condition, the expectation of the righthand side of (4.10) with respect to $P$ equals unity. Therefore, $\tilde{P}_t$ is a valid probability measure and the process $(\tilde{W}_t^s, s \in [0,1])$ is a standard Wiener process under $\tilde{P}_t$, see Lemma 6.4 on p.216 in Liptser and Shiryayev [23] and Theorem 5.1 on p.191 in Karatzas and Shreve [16].

As a stepping-stone to the proof of (4.4), we establish the ergodic property that if $|g(x)| \leq K (1 + |x|^2)$, for some $K > 0$, then, for arbitrary $\epsilon > 0$,

$$
\lim_{t \to \infty} \tilde{P}_t \left( \int_{\mathbb{R}^d} g(x) \nu_t(dx) - \int_{\mathbb{R}^d} g(x) m \hat{\lambda}(x) \, dx \right) > \epsilon = 0, \quad (4.11)
$$

where we let

$$
\nu_t(dx) = \mu_t([0,1], dx). \quad (4.12)
$$

By (2.2) and (4.9),

$$
\dot{X}_t^s = t \theta(X^s_t) \, ds + t \sigma(X^s_t) \left( -\tilde{\lambda} N(u \hat{\lambda}(X^s_t), X^s_t) + \sigma(X^s_t)^T \nabla f \hat{\lambda}(X^s_t) \right) ds
$$

$$
+ \sqrt{t} \sigma(X^s_t) d\tilde{W}_s^t. \quad (4.13)
$$
Hence, the process \( X = (X_s, s \geq 0) = (X^i_{s/t}, s \geq 0) \) satisfies the equation
\[
dX_s = \theta(X_s) \, ds + \sigma(X_s) \left( -\lambda \tilde{N}(u \tilde{\Lambda}(X_s), X_s) + \sigma(X_s)^T \nabla f \Lambdabar(X_s) \right) \, ds
+ \sigma(X_s) d\tilde{W}^t_s,
\]
(\( \tilde{W}^t_s \)) being a standard Wiener process under \( \tilde{P}^t \). By Theorem 10.1.3 on p.251 in Stroock and Varadhan [55] the distribution of \( X \) under \( \tilde{P}^t \) is specified uniquely. In addition, by Theorem 9.1.9 on p.220 and Lemma 9.2.2 on p.234 in Stroock and Varadhan [33], \( X \) is a regular Feller process. (See p.399 in Kallenberg [15] for the definition.) By (4.8), \( X \) has the invariant distribution \( m^\lambda(x) \, dx \), see, e.g., Theorem 1.5.13 in Bogachev, Krylov, and Röckner [5]. The process \( X \) is therefore positive Harris recurrent, see Theorem 20.17 on p.405 and Theorem 20.20 on p.408 in Kallenberg [15]. Since \( m^\lambda \in \tilde{\mathbb{P}} \), we have that \( \int_{\mathbb{R}} |x|^2 m^\lambda(x) \, dx < \infty \), so \( \int_0^1 g(X_s) \, ds \) is an integrable additive functional. The limit in (4.11) now follows by Theorem 3.12 on p.397 and the discussion on p.398 in Revuz and Yor [30].

With the proof of (4.11) being out of the way, we mount a final assault on (4.12). By (4.10),
\[
\int_0^1 M(\pi^i_s, X^i_s) \, ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi^i_s, X^i_s)^T \, d\tilde{W}^t_s = \int_0^1 M(\pi^i_s, X^i_s) \, ds
+ \int_0^1 N(\pi^i_s, X^i_s)^T (-\tilde{\lambda} \tilde{N}(u \tilde{\Lambda}(X^i_s), X^i_s) + \sigma(X^i_s)^T \nabla f \Lambdabar(X^i_s)) \, ds
+ \frac{1}{\sqrt{t}} \int_0^1 N(\pi^i_s, X^i_s)^T \, d\tilde{W}^t_s = \frac{1}{t} \ln \mathcal{E}^i_s + \int_0^1 M(u \tilde{\Lambda}(X^i_s), X^i_s) \, ds
+ \int_0^1 N(u \tilde{\Lambda}(X^i_s), X^i_s)^T (-\tilde{\lambda} \tilde{N}(u \tilde{\Lambda}(X^i_s), X^i_s) + \sigma(X^i_s)^T \nabla f \Lambdabar(X^i_s)) \, ds
+ \frac{1}{\sqrt{t}} \int_0^1 N(u \tilde{\Lambda}(X^i_s), X^i_s)^T \, d\tilde{W}^t_s, \tag{4.13}
\]
where \( \mathcal{E}^i_s \) represents the stochastic exponential defined by
\[
\mathcal{E}^i_s = \exp\left( \sqrt{t} \int_0^s (\pi^i_t - u \tilde{\Lambda}(X^i_t))^T b(X^i_t) \, d\tilde{W}^t_t - \frac{t}{2} \int_0^s \|\pi^i_t - u \tilde{\Lambda}(X^i_t)\|^2_{c(X^i_t)} \, ds \right).
\]
Since \( \mathbb{E} \mathcal{E}^i_t \leq 1 \), Markov’s inequality yields the convergence
\[
\lim_{t \to \infty} \mathbb{P}^t\left( \frac{1}{t} \ln \mathcal{E}^i_t < \delta \right) = 1. \tag{4.14}
\]
By (4.10) and (4.13),

\[
P(L_t^* \lt \tilde{q} + 3\delta) = \tilde{E}_t^X \frac{1}{t} \int_0^1 M(\pi^*_s, X^*_s) \, ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi^*_s, X^*_s)^T \, dW^*_s \]

\[
\exp\left(-\sqrt{t} \int_0^1 (-\tilde{\lambda}N(u^{\tilde{\lambda}}(X^*_t^s), X^*_t) + \sigma(X^*_s)^T \nabla f^{\tilde{\lambda}}(X^*_s)) \, d\tilde{W}_s^t\right)
\]

\[
- \frac{t}{2} \int_0^1 |-\tilde{\lambda}N(u^{\tilde{\lambda}}(X^*_t^s), X^*_t) + \sigma(X^*_s)^T \nabla f^{\tilde{\lambda}}(X^*_s)|^2 \, ds
\]

\[
\geq \tilde{E}_t^X \left\{ \frac{1}{t} \ln \epsilon^*_t < \delta \right\} \left\{ \frac{1}{\sqrt{t}} \int_0^1 |N(u^{\tilde{\lambda}}(X^*_s), X^*_s)^T \, d\tilde{W}_s^t| < \delta \right\}
\]

\[
\chi \left\{ \int_0^1 \left( M(u^{\tilde{\lambda}}(X^*_s), X^*_s) + N(u^{\tilde{\lambda}}(X^*_s), X^*_s)^T (-\tilde{\lambda}N(u^{\tilde{\lambda}}(X^*_s), X^*_s)) \right. \right.
\]

\[
\left. + \sigma(X^*_s)^T \nabla f^{\tilde{\lambda}}(X^*_s)) \right\} ds < \tilde{q} + \delta \}
\]

\[
\chi \left\{ \frac{1}{\sqrt{t}} \int_0^1 (-\tilde{\lambda}N(u^{\tilde{\lambda}}(X^*_s), X^*_s) + \sigma(X^*_s)^T \nabla f^{\tilde{\lambda}}(X^*_s)) \, d\tilde{W}_s^t \right\} < \delta\}
\]

\[
\chi \left\{ \int_0^1 |\tilde{\lambda}N(u^{\tilde{\lambda}}(X^*_s), X^*_s) + \sigma(X^*_s)^T \nabla f^{\tilde{\lambda}}(X^*_s)|^2 \, ds \right.
\]

\[
- \int_{\mathbb{R}^t} |\tilde{\lambda}N(u^{\tilde{\lambda}}(x), x) + \sigma(x)^T \nabla f^{\tilde{\lambda}}(x)|^2 m^{\tilde{\lambda}}(x) \, dx < 2\delta \}
\]

\[
\exp\left(-2\delta t - \frac{t}{2} \int_{\mathbb{R}^t} |\tilde{\lambda}N(u^{\tilde{\lambda}}(x), x) + \sigma(x)^T \nabla f^{\tilde{\lambda}}(x)|^2 m^{\tilde{\lambda}}(x) \, dx \right). \quad (4.15)
\]

We work with the terms on the righthand side of (4.15) in order. By (2.4), (2.6a), (2.6b), (4.2), and by \( \nabla f^{\tilde{\lambda}}(x) \) satisfying the linear growth
on minimizing a portfolio’s shortfall probability

\[
\lim_{t \to \infty} \mathcal{P}^t \left( \left| \int_0^1 -\tilde{\lambda} N(u^{\hat{\lambda}}(X^s_t), X^s_t) + \sigma(X^s_t)^T \nabla f^{\hat{\lambda}}(X^s_t) \right|^2 ds \right.
\]

\[
- \int_{\mathbb{R}^d} \left| -\tilde{\lambda} N(u^{\hat{\lambda}}(x), x) + \sigma(x)^T \nabla f^{\hat{\lambda}}(x) \right|^2 m^{\hat{\lambda}}(x) dx \left| < 2\delta \right) = 1.
\]

Similarly, by (4.11) and (4.5),

\[
\lim_{t \to \infty} \mathcal{P}^t \left( \left| \int_0^1 (M(u^{\hat{\lambda}}(X^s_t), X^s_t) + N(u^{\hat{\lambda}}(X^s_t), X^s_t)^T \left( -\tilde{\lambda} N(u^{\hat{\lambda}}(X^s_t), X^s_t) \right. \right. 
\]

\[
+ \sigma(X^s_t)^T \nabla f^{\hat{\lambda}}(X^s_t)) ds < \tilde{q} + \delta \right) = 1.
\]

Since, for \( \epsilon > 0 \), by the Lénglart–Rebolledo inequality, see Theorem 3 on p.66 in Liptser and Shiryaev [24],

\[
\mathcal{P}^t \left( \left| \frac{1}{\sqrt{t}} \int_0^1 -\tilde{\lambda} N(u^{\hat{\lambda}}(X^s_t), X^s_t) + \sigma(X^s_t)^T \nabla f^{\hat{\lambda}}(X^s_t) \right|^2 ds \geq \delta \right)
\]

\[
\leq \frac{\epsilon}{\delta^2} + \mathcal{P}^t \left( \left| \frac{1}{\sqrt{t}} \int_0^1 -\tilde{\lambda} N(u^{\hat{\lambda}}(X^s_t), X^s_t) + \sigma(X^s_t)^T \nabla f^{\hat{\lambda}}(X^s_t) \right|^2 ds \geq \epsilon t \right),
\]

imply that

\[
\lim_{t \to \infty} \mathcal{P}^t \left( \left| \frac{1}{\sqrt{t}} \int_0^1 -\tilde{\lambda} N(u^{\hat{\lambda}}(X^s_t), X^s_t) + \sigma(X^s_t)^T \nabla f^{\hat{\lambda}}(X^s_t) \right|^2 ds < \delta \right) = 1.
\]

Similarly,

\[
\lim_{t \to \infty} \mathcal{P}^t \left( \left| \frac{1}{\sqrt{t}} \int_0^1 N(u^{\hat{\lambda}}(X^s_t), X^s_t)^T d\tilde{W}^s_t \right| < \delta \right) = 1.
\]

Letting \( t \to \infty \) in (4.15) and recalling (4.14) obtains (4.4).

\( \square \)

Remark 4.1 The change of measure in (4.10) is implicit in Puhalskii [29]. The idea of using a stochastic exponential in order to "absorb" control, as in (4.13), is borrowed from Hata, Nagai, and Sheu [11].

Proof of Theorem 2.3 We start with proving part 1. By Theorem 2.1, it suffices to prove that

\[
\limsup_{\tau \to \infty} \limsup_{t \to \infty} \frac{1}{t} \ln \mathcal{P}(L^{\pi \tau}_t \leq q) \leq -J^\pi_q.
\]

(4.17)
Let $f \in A_\kappa$. In analogy with (2.10),
\[ E \exp\left(-t\hat{\lambda}L_{t}^{\hat{\pi},\tau} + f(X_{t}) - f(X_{0}) - t \int_{\mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}) \nu(dx)\right) \leq 1. \]

Thanks to Jensen’s inequality,
\[ E \chi\{L_{t}^{\hat{\pi},\tau} \leq q\} \exp(f(X_{t}) - f(X_{0})) \leq e^{t\hat{\lambda}q} \exp\left(t \sup_{x \in \mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau})\right). \]

By reverse Hölder’s inequality, for $\epsilon \in (0, 1)$,
\[ E \chi\{L_{t}^{\hat{\pi},\tau} \leq q\} \exp(f(X_{t}) - f(X_{0})) \geq P(L_{t}^{\hat{\pi},\tau} \leq q) \frac{1}{1 + \epsilon} \left( E \exp\left(-\frac{1}{\epsilon} (f(X_{t}) - f(X_{0}))\right)\right)^{1/2}. \]

Since $-k \leq f(x) \leq \kappa |x|^2 + k$, for some $k > 0$, if $\kappa < \epsilon \gamma$, then by (2.3)
\[ \lim_{t \to \infty} \left( E \exp(-f(X_{t}) - f(X_{0})) / \epsilon\right)^{1/2} \leq 1, \]
which implies that
\[ \limsup_{t \to \infty} \frac{1}{t} \ln P(L_{t}^{\hat{\pi},\tau} \leq q) \leq \hat{\lambda}q + \inf_{f \in A_\kappa} \sup_{x \in \mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}). \]

In analogy with the proof of Lemma 5.3
\[ \inf_{f \in A_\kappa} \sup_{x \in \mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}) = \inf_{f \in A_\kappa} \sup_{x \in \mathbb{R}^{l}} \int_{\mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}) \nu(dx) \]
\[ = \sup_{\nu \in \mathcal{P}} \inf_{f \in A_\kappa} \int_{\mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}) \nu(dx) = \sup_{\nu \in \mathcal{P}} \inf_{f \in C^{2}_{b}} \int_{\mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}) \nu(dx) \]
\[ \leq \inf_{f \in C^{2}_{b}} \sup_{\nu \in \mathcal{P}} \int_{\mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}) \nu(dx) = \inf_{f \in C^{2}_{b}} \sup_{x \in \mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}). \] (4.18)

Hence,
\[ \inf_{f \in A_\kappa} \sup_{x \in \mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}) = \inf_{f \in C^{2}_{b}} \sup_{x \in \mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}), \]
so, by $\epsilon$ being arbitrarily small,
\[ \limsup_{t \to \infty} \frac{1}{t} \ln P(L_{t}^{\hat{\pi},\tau} \leq q) \leq \hat{\lambda}q + \inf_{f \in C^{2}_{b}} \sup_{x \in \mathbb{R}^{l}} \hat{H}(x; \hat{\lambda}, f, u^{\tau}) \]
and the required property follows by (2.21).

We prove now part 2. By Theorem 2.1 it suffices to prove that
\[ \limsup_{t \to \infty} \frac{1}{t} \ln P(L_{t}^{\pi} \leq q) \leq -J_{q}. \] (4.19)
We borrow from Koncz [17] and Nagai [26]. Similarly to the proof of Theorem 2.1, we introduce the change of measure

\[
\frac{d\hat{\mathbf{P}}}{d\mathbf{P}}|_{\mathcal{F}_t} = \exp\left(\int_0^t \left( -\hat{\lambda} \mathbf{N}(\hat{\mathbf{u}}(X_s), X_s) + \mathbf{\sigma}(X_s)^T \nabla \hat{f}(X_s) \right) dW_s - \frac{1}{2} \int_0^t |\mathbf{\sigma}(X_s) + \mathbf{\sigma}(X_s)^T \nabla \hat{f}(X_s)|^2 ds \right).
\]

Then \((\hat{W}_t, t \geq 0)\) is a standard Wiener process with respect to \(\hat{\mathbf{P}}\), where

\[
\hat{W}_t = W_t - \int_0^t \left( -\hat{\lambda} \mathbf{N}(\hat{\mathbf{u}}(X_s), X_s) + \mathbf{\sigma}(X_s)^T \nabla \hat{f}(X_s) \right) ds.
\]

By (2.2) and Itô’s lemma,

\[
dX_t = (\theta(X_t) + \mathbf{\sigma}(X_t)(-\hat{\lambda} \mathbf{N}(\hat{\mathbf{u}}(X_s), X_s) + \mathbf{\sigma}(X_s)^T \nabla \hat{f}(X_s))) dt + \mathbf{\sigma}(X_t) d\hat{W}_t
\]

and

\[
d\hat{f}(X_t) = \left( \nabla \hat{f}(X_t)^T \theta(X_t) + \mathbf{\sigma}(X_t)(\hat{\lambda} \mathbf{N}(\hat{\mathbf{u}}(X_s), X_s) + \mathbf{\sigma}(X_s)^T \nabla \hat{f}(X_s)) \right)
+ \frac{1}{2} \text{tr}(\mathbf{\sigma}(X_t) \mathbf{\sigma}(X_t)^T \nabla^2 \hat{f}(X_t)) dt + \nabla \hat{f}(X_t)^T \mathbf{\sigma}(X_t) d\hat{W}_t.
\]

By (2.5), (2.6a), (2.6b), and (2.17a),

\[
\mathbf{E} e^{-\hat{\lambda} L_s^\pi} = e^{t F(\hat{\lambda})} \hat{\mathbf{E}} e^{f(X_0) - f(X_t)}.
\]  

By Itô’s lemma, (2.17a), and (2.19),

\[
e^{f(X_0) - f(X_t)} = 1 + \int_0^t e^{f(X_0) - f(X_s)} \left( -\nabla \hat{f}(X_s)^T \theta(X_s) \right)
+ \hat{\lambda} \nabla \hat{f}(X_s)^T \mathbf{\sigma}(X_s) \mathbf{\sigma}(\hat{\mathbf{u}}(X_s), X_s) + \frac{1}{2} |\mathbf{\sigma}(X_s) + \mathbf{\sigma}(X_s)^T \nabla \hat{f}(X_s)|^2
- \frac{1}{2} \text{tr}(\mathbf{\sigma}(X_s) \mathbf{\sigma}(X_s)^T \nabla^2 \hat{f}(X_s)) ds
- \int_0^t e^{f(X_0) - f(X_s)} \nabla \hat{f}(X_s)^T \mathbf{\sigma}(X_s) d\hat{W}_s
= 1 + \int_0^t e^{f(X_0) - f(X_s)} \left( \mathbf{H}(X_s; \lambda, \hat{\mathbf{u}}) - F(\hat{\lambda}) \right) ds
- \int_0^t e^{f(X_0) - f(X_s)} \nabla \hat{f}(X_s)^T \mathbf{\sigma}(X_s) d\hat{W}_s.
\]
where $0$ stands for the zero function. Let
\[
\hat{\tau}_R = \inf\{t \geq 0 : |X_t| > R\},
\]
where $R > 0$. Since $(\int_0^{\hat{\tau}_R} e^{\hat{f}(X_s) - f(X_s)}\nabla \hat{f}(X_s)^T \sigma(X_s) d\hat{W}_s, t \geq 0)$ is a martingale with respect to $\hat{P}$,
\[
\hat{E}e^{\hat{f}(X_0) - f(X_{\hat{\tau}_R})} = 1 + \hat{E} \int_0^{\hat{\tau}_R} e^{\hat{f}(X_s) - f(X_s)} (\hat{H}(X_s; \lambda, \hat{u}) - F(\hat{\lambda})) \, ds.
\]
By (3.26) (with $\tau = \infty$) and (2.23), there exists $K > 0$ such that $\hat{H}(x; \lambda, \hat{u}) - F(\hat{\lambda}) < 0$ if $|x| > K$. Therefore,
\[
\hat{E}e^{\hat{f}(X_0) - f(X_{\hat{\tau}_R})} \leq 1 + \sup_{|x| \leq K} e^{2\hat{f}(x)} \sup_{|x| \leq K} (|\hat{H}(x; \lambda, \hat{u}) - F(\hat{\lambda})|) t,
\]
so, by Fatou's lemma,
\[
\hat{E}e^{\hat{f}(X_0) - f(X_{\hat{\tau}_R})} \leq 1 + \sup_{|x| \leq K} e^{2\hat{f}(x)} \sup_{|x| \leq K} (|\hat{H}(x; \lambda, \hat{u}) - F(\hat{\lambda})|) t,
\]
which implies, by (4.20), that
\[
\limsup_{t \to \infty} \frac{1}{t} \ln \hat{E}e^{-1\lambda L^T_t} \leq F(\hat{\lambda}).
\]
Hence,
\[
\limsup_{t \to \infty} \frac{1}{t} \ln P(L^T_t \leq q) \leq \hat{\lambda} q + \limsup_{t \to \infty} \frac{1}{t} \ln \hat{E}e^{-1\lambda L^T_t} \leq \hat{\lambda} q + F(\hat{\lambda}).
\]
\[\Box\]

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