On the Impact of Information Acquisition and Aftermarkets on Auction Efficiency

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Abstract

A common assumption in auction theory is that the information available to the agents is given exogenously and that the auctioneer has full control over the market. In practice, agents might be able to acquire information about their competitors before the auction (by exerting some costly effort), and might be able to resell acquired items in an aftermarket. The auctioneer has no control over those aspects, yet their existence influences agents’ strategic behavior and the overall equilibrium welfare can strictly decrease as a result.

We show that if an auction is smooth (e.g., first-price auction, all-pay auction), then the corresponding price of anarchy bound due to smoothness continues to hold in any environment with (a) information acquisition on opponents’ valuations, and/or (b) an aftermarket satisfying two mild conditions (voluntary participation and weak budget balance). We also consider the special case with two ex ante symmetric bidders, where the first-price auction is known to be efficient in isolation. We show that information acquisition can lead to efficiency loss in this environment, but aftermarkets do not: any equilibrium of a first-price or all-pay auction combined with an aftermarket is still efficient.

1 Introduction

There is a vast literature in economics and computer science that analyzes welfare and revenue properties of auctions. A common assumption in this literature is that the information of the agents is exogenously given, and no trade occurs after the auction\textsuperscript{1} However, in practice, agents can engage in costly information acquisition about opponents’ values to refine

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\textsuperscript{1}There are two separate strands of literature regarding information acquisition in auctions and analysis with specific resale mechanisms. We briefly discuss these strands below and further extend the discussion in Section 1.2.
their bidding strategies in the auction, and the items acquired in the auction can often be resold in a secondary market (aftermarket) to further increase the agents’ utilities. Such activities are common for many goods sold in auctions, from houses to artwork and more. The endogeneity of information and the potential for resale can change behavior in the primary auction and may encourage socially-wasteful information acquisition and speculation. As a result, the final allocation and welfare might be very different than that of the auction in isolation.

A central issue here is the moral hazard of the agents and the inability for the auctioneer to observe their behavior. Indeed, the auctioneer might not even be aware of the set of feasible information acquisition technologies or the format of the aftermarkets. Taking aftermarkets as an example, there exist known institutions such as StubHub in which event tickets are resold following a primary sale. One might worry that this encourages speculation (i.e., scalping) which could distort the allocation. A seller could attempt to take the StubHub mechanism into account when designing an auction, but even if the StubHub mechanism is publicly known it is not controlled by the seller and may change. And while a centralized aftermarket like StubHub is publicly visible, there are other secondary markets that are not as transparent. For example, agents might enter a private Nash bargaining game to negotiate a trade where the bargaining protocol is not revealed to the auctioneer. Note that even in highly regulated markets like spectrum auctions where resale is prohibited, the auctioneer may be unable to circumvent this issue.

In this paper we allow agents to acquire costly information about others’ types before an auction begins. That is, agent types are drawn from publicly-known prior distributions and each agent observes her own type (as usual), but each agent may be able to purchase additional (possibly type-dependent) signals correlated with the realization of other agent’s types. We also allow them to trade items in a secondary market that follows the auction. To distinguish a “secondary market” (or aftermarket) from a general mechanism, we impose some mild conditions on the form these markets can take. Specifically, we assume that these are trade mechanisms: mechanisms that are budget balanced and do not force participation.

One subtlety is that behavior in the secondary market can depend on the information released after the primary auction, such as whether bids are publicly observed. We want results that are robust to this choice, so we allow an arbitrary revelation of signals correlated with the auction bids and outcomes before the secondary market begins. Finally, we assume that agents are fully aware of the secondary market (and what information they’ll learn about the primary auction outcome) when acquiring information and when playing in the primary auction. We call the resulting mechanism that combines the information acquisition, the

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2 One such example is provided in Hafalir and Krishna (2008): the Canadian firm TIW won a license in the spectrum auction in 2000, then circumvented license resale regulations by selling the entire company to Hutchison.

3 Our results will hold not only for trade mechanisms that are strongly budget balanced (net payment of 0), but also for weakly budget balanced mechanisms (mechanisms that never lose money).

4 If agents naively play in the auction, ignoring the existence of the aftermarket, then clearly any trade in the aftermarket only Pareto improves the utilities of all agents (as the trade mechanism satisfies voluntary participation), and welfare never decreases due to aftermarket trading.
primary auction, and the aftermarket trade mechanism the \textit{combined market}.

How do aftermarkets and information acquisition about opponents’ types affect welfare? The effect of information acquisition before the auction on the equilibrium welfare is ambiguous. On one hand, acquiring more information will help agents to come up with better bidding strategies, which can potentially improve the equilibrium welfare. On the other hand, the information acquisition is costly, and those costs have a negative impact on the equilibrium welfare. In the following example we show that the latter effect may be more prominent and there may exist strict welfare loss under equilibrium when agents can acquire costly information.\footnote{The example illustrates one symmetric equilibrium that is inefficient. In Appendix C we further prove that in this setting all equilibria are inefficient.}

\textbf{Example 1.} Consider selling a single item to two i.i.d. buyers using a first-price auction. With probability $\frac{1}{2}$, the value of the buyer is 0, and with remaining probability the value of the buyer is drawn from the uniform distribution supported on $[0, 1]$. For each buyer $i \in \{1, 2\}$ with value $v_i \in [0, 1]$, she can choose to not acquire any information, or pay cost $\frac{v_i^2}{16}$ to observe whether the opponent has value 0 or not.

By Chawla and Hartline (2013), when there is no information acquisition, in any equilibrium the allocation is efficient (welfare maximizing). In contrast, in the presence of information acquisition there is a symmetric equilibrium of the buyers with positive welfare loss. For each buyer $i \in \{1, 2\}$ with value $v_i \in [0, 1]$, if $v_i = 0$, the buyer will not acquire any information and opt out of the auction (e.g., bid $-1$ which is below the minimum acceptable bid of 0). If $v_i > 0$, the buyer will acquire information paying a cost of $\frac{v_i^2}{16}$. The buyer then bids 0 if she finds out that opponent has value 0, and bids $\frac{v_i}{2}$ otherwise. One can verify that this is an equilibrium strategy for both buyers and the equilibrium welfare is sub-optimal as agents pay positive costs for acquiring information.

Next consider the welfare impact of a secondary market. Clearly, for any secondary market satisfying weak budget balance and voluntary participation, any post-auction trade can only increase welfare relative to the auction allocation. However, the mere existence of the secondary market itself changes the bidding strategies of agents at equilibrium and so can make an auction – efficient in isolation – inefficient in the broader environment. The next example illustrates that the presence of secondary markets can distort the incentives of agents in the auction by enabling speculation, and may result with a significant welfare loss.

\textbf{Example 2.} Consider a single-item setting with two agents. Agent 1 has value 1 for the item for sure. For some $H > 1$, agent 2 has value distribution with CDF $F_2(v) = \frac{v - 1}{v}$ for $v \in [1, H]$ and $F_2(H) = 1$ (i.e., an equal-revenue distribution truncated at $H$). It is efficient to always allocate the item to agent 2, resulting in expected welfare of $\Theta(\log H)$.

Suppose the item is sold in a second-price auction (SPA), followed by an aftermarket trade mechanism in which the item holder can make a take-it-or-leave-it offer to the other agent. Assume each bidder knows her own allocation and payment after the auction, but no additional information about the bid of the other agent.
Consider the following strategies in the combined market: agent 1 bids $H$ in the auction and offers the item at price $H$ in the secondary market. Agent 2 bids 0 in the auction and accepts the offer from agent 1 if her value is $H$. These strategies form an equilibrium. The expected welfare of the final allocation under these strategies is 2, and thus the multiplicative welfare loss is $\Theta(\log H)$, which is unbounded when $H \to \infty$.

One can show that the strategies presented in the example are undominated and form a sequential equilibrium (see Lemma 3 for the proof). While we use a second-price auction in our construction for simplicity, welfare loss as exhibited by this example is not specific to second-price auctions; Hafalir and Krishna (2009) present a first-price auction setting in which the expected equilibrium welfare decreases in the combined market due to speculation, see Section 3 for details.

The examples above show that information acquisition and aftermarkets can decrease welfare at equilibrium. We ask: can this loss of welfare be bounded? We note that in contrast to the unbounded welfare loss in the example with second-price auction (as $H$ grows large), the welfare loss due to speculation in the first-price auction construction of Hafalir and Krishna (2009) is rather small, as is the loss due to information acquisition in our Example 1. So a natural question is whether there exists an instance of a first-price auction such that the welfare loss due to information acquisition or the introduction of an aftermarket is large, under some model of how much information about auction outcomes is revealed to the agents after the auction ends. As it turns out, all such equilibria will be approximately efficient due to the fact that first-price auctions are smooth.

### 1.1 Our Results

In this paper we present general results that upper bound the inefficiency of combined mechanisms while circumventing the need to explicitly characterize equilibrium behavior. We do so by extending our understanding regarding the power of smooth auctions (Roughgarden, 2012; Syrgkanis and Tardos, 2013), showing that smooth auctions are also robust to being combined with information acquisition and aftermarkets, regardless of how much information about auction bids and outcomes are released after the primary auction ends. In particular, as the first-price auction is smooth (Roughgarden, 2012), an application of our results indicates that the welfare loss for first-price auction is small even in the presence of information acquisition and aftermarkets. This welfare loss is formally bounded by upper bounding the price of anarchy (PoA), i.e., the worse ratio between the maximal welfare and the welfare in a Bayesian Nash equilibrium (BNE).

In our main result we show that if an auction is $(\lambda, \mu)$-smooth and it operates in an environment with information acquisition before the auction and a trade mechanism as an

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The strategies used in Example 2 are undominated and sequentially rational, but one might still argue that they feel unnatural due to the underbidding behavior of Agent 2 in the face of indifference. Nevertheless, we present this simple example for the sake of clarity; we note that the (more complex) example of Hafalir and Krishna (2009) (which is presented in Section 3) does not rely on such indifference.
aftermarket, then for independent value distributions the (Bayesian) price of anarchy (PoA) for the combined market is at most $\frac{\mu}{\lambda}$.

**Theorem 1** (informal). *If an auction mechanism is $(\lambda, \mu)$-smooth for some $\lambda > 0$ and $\mu \geq 1$, then for any information acquisition technology for learning about opponents’ values and any aftermarket trade mechanism, the price of anarchy of the combined market within the family of all product distributions is at most $\frac{\mu}{\lambda}$.*

We remark that our result above holds for any Bayesian Nash Equilibria, not only for sequential equilibria (or other refinement of BNE). In Section 3 we prove the theorem and also present a similar result for semi-smooth mechanisms, implying PoA bounds for any priors (not necessarily independent).

This result has many implications, since there are many auctions that are known to be smooth. Our result shows they are all robust to additional information acquisition and aftermarkets. In Table 1 we summarize some known smoothness results and the implication of our result when using these bounds. Note that given the Example 2 we cannot expect to have a positive result for second-price auctions, and indeed this auction is not smooth and does not appear in the table.

| Auction Environment       | Mechanism $\mathcal{M}^1$                  | Smoothness            | PoA in Combined Market $\mathcal{M}^C$ |
|---------------------------|--------------------------------------------|-----------------------|----------------------------------------|
| single-item               | first-price auction                        | $(1 - \frac{1}{e}, 1)^*$ | $\frac{e}{(e - 1)}$                   |
|                           | all-pay auction                            | $(\frac{1}{2}, 1)^*$   | 2                                     |
| combinatorial, submodular | simultaneous first price                    | $(1 - \frac{1}{e}, 1)^\dagger$ | $\frac{e}{(e - 1)}$                   |
|                           | simultaneous all pay                       | $(\frac{1}{2}, 1)^\dagger$ | 2                                     |

* Roughgarden (2012)  † Syrgkanis and Tardos (2013)

Table 1: The first column lists the auction environment and the second the auction mechanism. The third column lists the $(\lambda, \mu)$-smoothness results from the literature, and by Theorem 1 this implies the price of anarchy upper bound for the combined market, when every valuations are independently distributed, as listed in the last column.

7 Theorem 1 is a result about the worse equilibrium (price of anarchy), but what about the best equilibrium (price of stability)? To complete the picture, we observe that the price of stability may strictly increase when agents can acquire costly information (Example 1). In contrast, in the model without information acquisition, in Appendix D we observe that under a mild natural assumption about the secondary market, essentially that there is an action profile of “everyone not participating” in aftermarket trade, the best equilibrium of the auction can be used to create an equilibrium in the combined market with the same welfare (so the price of stability of the combined market is at most the one of the auction).

8 Second-price auctions satisfy weak smoothness, i.e., smooth under the refinement of undominated strategies, when there is no resales. However, as illustrated in Example 2 the property of weak smoothness does not carry over when there are secondary markets.
In Theorem 1 we provide upper bounds on the welfare loss resulting from the presence of information acquisition and aftermarkets. Although the welfare loss is bounded, there are cases in which inefficiency occurs due to fact that the auction is not running in isolation. A natural question to ask is whether equilibrium welfare in combined markets is maximized in classical settings where the allocation is efficient in all equilibria of the isolated auction. Specifically, we focus on the first-price auction for two agents with i.i.d. valuations. It is well known that any BNE in this setting without information acquisition and aftermarkets is efficient. Example 1 shows that even in this symmetric setting, with information acquisition, there may exist strict welfare loss. In contrast, for secondary markets, under a mild assumption regarding the trade mechanism (that it is ex-post individually rational), all equilibria in the combined market that are monotone, are indeed efficient. This result significantly strengthens a similar result of Hafalir and Krishna (2008) in two ways. First we do not assume a specific aftermarket; hence our results apply to secondary markets with, say, Nash bargaining as well as posted pricings. Second, we allow information to be revealed after the auction, while the prior result does not. The proof for this result is based on the technique of Chawla and Hartline (2013), essentially showing any BNE with bids that are non-decreasing in value must be symmetric, and thus efficient.

Finally, in Appendix E, we briefly consider the impact of aftermarket trade mechanisms on the revenue of a seller in single items settings that is running first-price auction with optimally chosen anonymous reserve. We assume that buyers’ valuations are independently drawn from regular distributions with finite support. We show that if no bidding information is released after the auction, for any aftermarket trade mechanism that satisfies ex-post individual rationality, the price of anarchy for revenue for the combined market is at most 2.62. This illustrates the robustness of first-price auctions to the introduction of aftermarket trade mechanism, not only for welfare but also for revenue.

1.2 Related Work

The smoothness framework is a powerful tool for analyzing the price of anarchy in auctions. This framework is first proposed in Roughgarden (2009) for complete information games and Roughgarden (2012); Syrgkanis (2012) for incomplete information games. The idea of smoothness is further refined and generalized for simultaneous composition and sequential composition of smooth mechanisms in multi-item settings. (e.g., Syrgkanis and Tardos 2013; Feldman et al. 2013). There are various generalizations of the smoothness framework, such as the price of anarchy for correlated distributions (Lucier and Paes Leme 2011), the price of anarchy for revenue maximization (Hartline et al. 2014), the price of anarchy for liquid welfare (Azar et al. 2017), and the price of anarchy in large games (Feldman et al. 2016). The papers listed above is just a glimpse of the huge literature on the smoothness framework, and we refer to the survey of Roughgarden et al. (2017) for a more detailed discussion on the literature.

One closely related line of work is price of anarchy for sequential auctions (Syrgkanis and Tardos 2012; Leme et al. 2012). In Leme et al. (2012), the authors illustrate that although price of anarchy of the sequential composition of first-price auction is small for
unit-demand agents, the result breaks for agents with submodular valuations, and the price of anarchy can be unbounded in the latter case. In contrast, our results indicates that for submodular valuations, any trade mechanism followed by simultaneous first-price auction will have constant price of anarchy for the combined market. The main difference that allows us to handle combinatorial auction in sequential auction format is that all items are sold only in the first market, and the secondary market is only providing the platform for agents to retrade the items, rather than selling items sequentially, with each item sold once in one of the auctions.

Recently Eden et al. (2020) bound the price of anarchy when each agent has externality over the allocation of the other agents. The authors motive the externality by the resale model since the value of the an agent for winning any item depends on the utility gain of potentially reselling the item to other agents, and the latter depends on other agents’ private assessment of the item. However, they assume that those resale behaviors are fixed and hence the externality among agents in the auction are exogenous. In contrast, we assume that the seller faces an unknown secondary market, and the utilities of agents for winning any item in the auction are endogenously determined by the mechanism adopted in the auction and the corresponding equilibrium behavior of all agents in the combined market.

The challenges in analyzing the equilibrium in the combined market was acknowledged in Haile (2003) due to the fact that there exist endogenously induced common value components in the auction. In the simple single-item setting with winner posting prices as secondary markets, Hafalir and Krishna (2008) characterized the equilibrium behavior of the agents in the combined market, and Hafalir and Krishna (2009) adopted the characterization to show that the expected welfare of the first price auction with secondary markets may decrease by a multiplicative factor of \( \frac{2e}{2e - 1} \). In addition to the above discussions, there are many papers discussing various properties of the resale model in the economics literature, including but not limited to the observation of bid shading in the auction (Pagnozzi, 2007), and the revenue ranking of the simple auctions (Lebrun, 2010). See the survey of Susin (2017) for more discussions on the equilibrium properties of the resale model. Finally, there are several recent papers focusing on designing optimal mechanisms when the seller has no control over the secondary market. Carroll and Segal (2019) show that second price auction with reserve prices is the robustly revenue optimal mechanisms with unknown resale opportunities. Dworczak (2020) considers the design of information released to the secondary markets and show that the information structure that induces truthful behaviors are cutoff rules. He also provides sufficient conditions for simple information structure to be optimal.

Our paper also relates to the broad setting of information acquisition in auctions (Crémer and Khalil, 1992; Persico, 2000; Shi, 2012). We focus on a special case of the private-value setting where each agent acquires costly information on the valuation of other agents. This model is first proposed in Bergemann and Valimaki (2005) for analyzing its effect on the first-price auction. Tian and Xiao (2010) establish the revenue ranking between the first-

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This means that the secondary market satisfies voluntary participation and weak budget balance. See Section for a detailed definition of the assumption.

In this model, the authors also assume that no information, especially the bids, are revealed in the secondary market to avoid the ratchet effect.
price auction and the second-price auction in private value setting and show that information acquisition leads to a lower revenue for the first-price auction. [Kim and Koh (2020)] generalize the model to the interdependent value setting, and show that compared to second-price auction, information acquisition increases the revenue while in the mean time leading to an efficiency loss in the first-price auction. Thus the designer faces a novel revenue-efficiency trade-off in this interdependent value setting.

2 Preliminaries

2.1 The Basic Setting

We consider a seller holding a set of $m$ items and $n$ agents (buyers) interested in these items. The seller has no value for the items. The set of feasible allocations of the items is denoted by $\mathcal{X}$. Let $\Theta = \times_i \Theta_i$ be the type space of the agents and $F$ be the prior distribution over the type space. For any agent $i$, her type $\theta_i \in \Theta_i$ is her private information and the marginal distribution of her private type is denoted by $F_i$. Let $v_i(x; \theta_i)$ be the public valuation function of agent $i$ given allocation $x \in \mathcal{X}$ and type $\theta_i$, and her utility given allocation $x$ and payment $p_i$ is $u_i(x, p_i; \theta_i) = v_i(x; \theta_i) - p_i$. Moreover, all agents have von Neumann–Morgenstern expected utility for randomized outcomes, valuing it at its expectation. Note that here we allow agents to have externalities (the valuation of an agent might depend on the entire allocation, not only her own allocation).

The above formulation is very general and captures many auction settings. Here we list some examples. In each example, $x \in \mathcal{X}$ can be represented as $(x_i)_{i \in [n]}$, an allocation for each agent, and the valuation for any agent $i$ is a function of her own allocation $x_i$ only. That is, with slight abuse of notation $v_i(x; \theta_i)$ can be written as $v_i(x_i; \theta_i)$.

Single-item Settings. The seller has a single indivisible item he can sell to at most one of $n$ agents. Here the feasibility constraint is $\mathcal{X} = \{(x_i)_{i \in [n]} : x_i \in \{0, 1\}, \forall i \in [n] \text{ and } \sum_i x_i \leq 1\}$.

Combinatorial Auctions. The seller has a set $M$ of non-identical indivisible items to sell to $n$ agents, and the feasibility constraint is $\mathcal{X} = \{(x_i)_{i \in [n]} : x_i \subseteq M, \forall i \in [n] \text{ and } x_i \cap x_j = \emptyset, \forall i, j \in [n]\}$. We say the combinatorial auction is submodular if for each $\theta_i \in \Theta_i$ it holds that $v_i$ is a submodular function, i.e., $v_i(x_i; \theta_i) + v_i(x'_i; \theta_i) \geq v_i(x_i \cup x'_i; \theta_i) + v_i(x_i \cap x'_i; \theta_i)$ for any $x_i, x'_i \subseteq M$.

A mechanism $\mathcal{M}$ defines a set of actions for each agent, and a (possible random) mapping from profile of actions to a feasible allocation and payment from each agent. Formally, a mechanism $\mathcal{M} = (x^\mathcal{M}, p^\mathcal{M}) : A \rightarrow \Delta(\mathcal{X} \times \mathbb{R}^n)$ is defined by an allocation rule $x^\mathcal{M} : A \rightarrow \Delta(\mathcal{X})$, and a payment rule $p^\mathcal{M} : A \rightarrow \mathbb{R}^n$, where $A = \times_i A_i$, and $A_i$ is the action space of agent $i$ in the mechanism. Thus, for action profile $a = (a_1, a_2, \ldots, a_n) \in (A_1, A_2, \ldots, A_n) = A$ the outcome of the mechanism is the (randomized) allocation $x^\mathcal{M}(a)$, and each agent $i$ is charged (in expectation) a payment of $p_i^\mathcal{M}(a)$. The utility of agent $i$ with type $\theta_i$ when participating in the mechanism $\mathcal{M}$ in which agents take actions $a \in A$ is $u_i(\mathcal{M}(a); \theta_i) = v_i(x^\mathcal{M}(a); \theta_i) - p_i^\mathcal{M}(a)$.

There are several auction formats that are of interests in this paper.
• Single-item auctions: Each agent $i$ simultaneously submit bid $a_i \in A_i = \mathbb{R}_{\geq 0}$ representing her value for the item, and the item is sold to agent $i^* \in \arg\max_i a_i$, with arbitrary tie breaking. Payments are set as follows:

  - first-price auction: The highest bidder $i^*$ pays her bid $a_{i^*}$, and any agent $i \neq i^*$ pays 0.
  - second-price auction: The highest bidder $i^*$ pays the second highest bid $\max_{i \neq i^*} a_i$, and any agent $i \neq i^*$ pays 0.
  - all-pay auction: Every agent $i$ pays her bid $a_i$.

• Combinatorial auctions:

  - simultaneous first-price (all-pay) auction: run first-price (all-pay) auctions simultaneously for all items $j \in [m]$.

A mechanism $\mathcal{M}$ with distribution $F$ defines a game. A strategy $\sigma_i : \theta_i \to \Delta(a_i)$ for agent $i$ is a mapping from her type $\theta_i$ to a distribution over her actions (in private information settings we consider, $\sigma_i$ may not depend on $\theta_{-i}$, but may depend on any other information, as the distribution $F$). With slight abuse of notation denote by $\sigma_{-i}(\theta_{-i})$ the profile of actions taken by agents other than $i$ when each $j \neq i$ of type $\theta_j$ takes action $\sigma_j(\theta_j)$. A strategy $\sigma_i$ is a best response for agent $i$ given strategies of the others $\sigma_{-i}$ if for any strategy $\sigma_i'$ it holds that $E[u_i(\mathcal{M}(\sigma_{i}(\theta_i),\sigma_{-i}(\theta_{-i}));\theta_i)] \geq E[u_i(\mathcal{M}(\sigma'_{i}(\theta_i),\sigma_{-i}(\theta_{-i}));\theta_i)]$ for every type $\theta_i$, where the expectation is over $\theta_{-i}$ as well as any randomness in the mechanism and strategies. Finally, a profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a Bayesian-Nash equilibrium for mechanism $\mathcal{M}$ with distribution $F$, if for every agent $i$, strategy $\sigma_i$ is a best response for agent $i$ given strategies of the others $\sigma_{-i}$.

We use $v$ to denote the vector of valuation functions. The welfare of an allocation $x \in \mathcal{X}$ when the valuations are $v$ and the types are $\theta \in \Theta$ is defined to be $\text{Wel}(\theta; v, x) = \sum x(v_i(x; \theta_i))$. For any type profile $\theta$, let $\text{Wel}(\theta; v, \mathcal{X}) = \sup_{x \in \mathcal{X}} \text{Wel}(\theta; v, x)$ be the optimal (highest) welfare given the valuation functions $v$, types $\theta$, and feasibility constraint $\mathcal{X}$. We say an allocation is efficient if it achieves the optimal welfare. Let $\text{Wel}(F; v, \mathcal{X}) = E_{\theta \sim F}[\text{Wel}(\theta; v, \mathcal{X})]$ be the expected optimal welfare. When functions $v$ and $\mathcal{X}$ are clear from the context, we omit it in the notation and use $\text{Wel}(\theta), \text{Wel}(F)$ to denote the optimal and expected optimal welfare. By slightly overloading the notation, we also denote $\text{Wel}(\mathcal{M}, \sigma, F)$ as the expected welfare obtained in mechanism $\mathcal{M}$ using equilibrium strategy profile $\sigma$. Let the price of anarchy of mechanism $\mathcal{M}$ within the family of distributions $\mathcal{F}$ be

$$\text{PoA}(\mathcal{M}, \mathcal{F}) = \sup_{F \in \mathcal{F}} \frac{\text{Wel}(F)}{\inf_{\sigma \in \text{BNE}(F, \mathcal{M})} \{\text{Wel}(\mathcal{M}, \sigma, F)\}}$$

where BNE($F, \mathcal{M}$) is the set of Bayesian-Nash equilibria given distributions $F$ and mechanism $\mathcal{M}$. 

9
2.2 Information Acquisition

We allow agents to acquire costly information about other agents’ private types before the auction starts. Information is captured by a signal from the types of the others to some signal space. Specifically, for each agent $i$, let $\Psi_i$ be the set of feasible signal structures for agent $i$. Any signal structure $\psi_i \in \Psi_i$ is a mapping from the opponents’ types $\theta_{-i}$ to a distribution over the signal space $S_i$. Note that the information acquisition is costly, i.e., there is a non-negative cost $c_i(\psi_i; \theta_i)$ for any $\psi_i \in \Psi_i$ and any $\theta_i$. Let $\bar{\psi}$ be the signal that acquires no information with zero cost. We assume that $\bar{\psi} \in \Psi_i$ for any agent $i$. In our model, both the cost function $c_i$ and the set $\Psi_i$ of any agent $i$ are common knowledge among all agents.

Note that the ability to acquire additional information does not affect the optimal welfare we considered in Section 2.1 since the optimal welfare is attained when no agent acquires costly information. Thus the price of anarchy is defined analogously as the ratio between the optimal welfare and the equilibrium welfare when information acquisition is an option.

2.3 Secondary Markets

After information acquisition is complete the mechanism will begin. We consider a setting where an auction mechanism is followed by a secondary market in which agents can trade the goods. We model this scenario as a two-round game $\mathcal{G}$ that consists of two mechanisms, $\mathcal{M}^1$ and $\mathcal{M}^2$, run sequentially. We refer to $\mathcal{M}^1$ as the auction and $\mathcal{M}^2$ as the secondary market. Our game formulation also allows signals to be revealed between the auction and the secondary market.

We describe the game more formally. In the first round, the agents participate in the auction $\mathcal{M}^1$. We denote the action space of $\mathcal{M}^1$ by $A^1 = \times_i A_i^1$. The agents simultaneously choose actions $a^1 \in A^1$, resulting in outcome $x_{\mathcal{M}^1}(a^1)$ and payments $(p_{\mathcal{M}^1}^i(a^1))_i$.

Each agent then observes some information about the outcome of the mechanism, which we formalize as a signaling protocol. Each agent $i$ has a signal space $S_i$ and observes a private signal $s_i \in S_i$. Each $s_i$ is a random variable that can be correlated with $A^1$, $x_{\mathcal{M}^1}(a^1)$, and $(p_{\mathcal{M}^1}^i(a^1))_i$. We will write $\Gamma$ to denote the signaling protocol, which includes the space of signals as well as their correlation structure. That is, $\Gamma(A^1, x_{\mathcal{M}^1}(a^1), (p_{\mathcal{M}^1}^i(a^1)))_i$ is a distribution over signals $(s_i)_i \in \times_i S_i$. We will often suppress the dependence on outcomes and simply write $\Gamma$ for the distribution over signals. This flexibility allows us to capture settings where more or less information about the auction outcome is publicly revealed; we assume only that each agent always observes the auction’s outcome and their own payment. With slight abuse, when $\Gamma$ conveys no information ($|S_i| = 1$ for every $i$), we say that bidders get no signal after the auction and denote $\Gamma$ by $\bot$.

After agents receive signals according to $\Gamma$, the second round starts and the agents participate in the secondary market $\mathcal{M}^2$. Informally, agents can use $\mathcal{M}^2$ to re-trade items after participating in auction $\mathcal{M}^1$, and the starting point of the secondary market is the allocation picked by the auction, which is publicly revealed. More formally, the secondary market $\mathcal{M}^2$ is parameterized by an allocation $x \in \mathcal{X}$, which we think of as an auction
outcome. In a slight abuse of notation we will write \( M^2 : A^2 \times X \to \Delta(\mathcal{X} \times \mathbb{R}^n) \) for the allocation and payment rules of secondary market \( M^2 \) as a function of the initial allocation, which assigns for each profile of actions of the agents in the secondary market \( a^2 = (a^2_1, a^2_2, \ldots, a^2_n) \in (A^2_1, A^2_2, \ldots, A^2_n) = A^2 \) (where \( A^2_i \) is the space of actions of agent \( i \) in \( M^2 \)) and auction allocation \( x \in \mathcal{X} \), a distribution over the final allocation and a second round of transfer payments. Note that the action spaces of the two mechanisms can be different, but they share a common set of feasible allocations.

To summarize, the two-round game \( G(M^1, \Gamma, M^2) \) proceeds as follows:

1. Each agent \( i \) picks an action \( a^1_i \in A^1_i \), all agents choose actions simultaneously. Mechanism \( M^1 \) runs on actions \( a^1 \).

2. Signals are realized according to \( \Gamma \). Each agent \( i \) observes \( x^{M^1}(a^1), p^{M^1}_i(a^1) \), and signal \( s_i \).

3. Each agent \( i \) picks an action \( a^2_i \in A^2_i \), all agents choose actions simultaneously. Mechanism \( M^2 \) runs on actions \( a^2 \), starting from allocation \( x^{M^1}(a^1) \).

4. The total payoff to agent \( i \) in the combined market is \( u_i(M^2(a^2, x^{M^1}(a^1)); \theta_i) - p^{M^1}_i(a^1) \).\(^{11}\)

We emphasize that all of the above occurs after the agents acquire costly information about competitors’ types. To capture our intuitive notion of a secondary market, we will impose two assumptions on the secondary market \( M^2 \). First, we assume voluntary participation, which informally means that each agent can choose not to participate. More formally, we require that each agent has an “opt-out” action that guarantees their utility is not reduced by the secondary market.

**Definition 1.** A secondary market \( M^2 \) satisfies **voluntary participation** if for each agent \( i \) and all types \( \theta_i \), there exists an action \( a^2_i \in A^2_i \) such that \( v_i(x; \theta_i) \leq u_i(M^2(x, (a^2_i, a^2_{-i})); \theta_i) \) for any allocation \( x \) and any action profile \( a^2_{-i} \).

We argue that this condition is quite mild. For example, in the single item setting, if the mechanism is one in which the item holder can suggest a take-it-or-leave-it price to the other, and trade happens if both agree, then the holder might decide not to make an offer (“not participate”), and the other can decide to decline any offer made (again, “not participate”). Each agent can therefore ensure that her utility in the combined market that is the same as the utility in the auction.

We also assume that our secondary market satisfies weak budget balance, which means that it does not run a deficit with respect to payments.

**Definition 2.** A secondary market \( M^2 \) satisfies **weak budget balance** if \( \sum_i p_i^{M^2}(a^2, x) \geq 0 \) for any action \( a^2 \in A^2 \) and feasible allocation \( x \in \mathcal{X} \).

\(^{11}\)Note that this expression includes the payments from both the auction and the secondary market, as the secondary market transfers are included in the utility term.
A mechanism that satisfies both voluntary participation and weak budget balance is called a voluntary-non-subsidized-trade mechanism, or a trade mechanism for short.

Another property that we use in our results in Section 4 and Appendix E is ex-post individual rationality of the secondary market.

**Definition 3.** A secondary market $\mathcal{M}^2$ satisfies ex post individual rationality if for any Bayesian Nash equilibrium strategy $\sigma$ in the secondary market given initial allocation $x$, for any type profile $\theta$ and any agent $i$, we have $v_i(x; \theta_i) \leq u_i(\mathcal{M}^2(x, \sigma(\theta)); \theta_i)$.

Note that the assumption of voluntary participation is orthogonal to the assumption of ex post individual rationality. The former assumes that agents have the option to opt out without imposing further constraints on the ex post utilities in equilibrium, i.e., an agent may participate in the secondary market with positive utility in expectation but negative utility ex post. In contrast, ex-post individual rationality requires that the utility of each agent under equilibrium of the combined market is at least her utility at the end of the auction in ex post sense, and it does not impose constraints on the off-path behaviors. Note that both assumptions are satisfied when secondary markets are price-posting mechanisms or Nash bargaining games.

Note that an instance of the two-round game $G(\mathcal{M}^1, \Gamma, \mathcal{M}^2)$ naturally corresponds to a combined mechanism $\mathcal{M}^C$ which we denote by $\mathcal{M}^C = G(\mathcal{M}^1, \Gamma, \mathcal{M}^2)$, in which an action has two components: (1) an action $a_i^1 \in A_i^1$ for the auction mechanisms, and (2) a mapping for each agent $i$ from the tuple of (allocation, payment, signal) from the auction into an action for the secondary market. The notions of BNE and PoA then extend to such a mechanism $\mathcal{M}^C$ as before.

A remark about the revelation of information between rounds. In the game formulation above, after the auction each agent $i$ observes the auction outcome and their own payment, as well as a signal that can contain any information about the action profile in the auction. Our results regarding smoothness of combined markets presented in Section 3 would continue to hold even if agents receive additional information about the types of the other agents before the secondary market; information that the auction has no direct access to. For clarity of exposition we omit this from the formalization.

### 3 Price of Anarchy via Smooth Framework

When the auction is not isolated, rational agents that are aware of the possibility of information acquisition and the opportunity to trade in the secondary market, will take those into account when playing in the auction. As pointed out in [Bergemann and Valimaki (2005)](#), espionage (acquiring information about others) in the first-price auction may cause the equilibrium to be inefficient in the setting without secondary markets, and [Hafalir and Krishna (2009); Garratt and Tröger (2006)](#) show that for auctions without information acquisition, the expected equilibrium welfare may decrease when there is a secondary market, due to speculation.
Explicitly characterizing the associated welfare loss is a laborious task: it requires one to construct and analyze the equilibrium which can depend on the agents’ distributions in subtle ways. This task is complicated even for relatively simple distributions. Hafalir and Krishna (2009) manage to construct an example in a setting with a secondary market (without information acquisition) in which there is a welfare loss of factor $\frac{2e}{2e-1}$. The setting they study is a single-item 2-agent setting with carefully-chosen type distributions $F_1(v) = (\frac{v}{w})^a$ for $v \in [0, w]$ and $F_2(v) = v^a$ for $v \in [0, 1]$ (for given constants $a > 0, w > 1$). Their example operates in a very simple economic environment – a first price auction followed by a posted price secondary market. Only by transitioning to an i.i.d. setting and imposing additional constraints on the mechanism are they able to obtain an efficiency result.

In this section, we circumvent the challenges of explicitly characterizing the equilibrium strategies by showing that while adding information acquisition or a secondary market might harm welfare, the worst-case welfare guarantees of several classical mechanisms (e.g., single-item first-price auction) will not decrease in the combined market, as long as the auction mechanism satisfies certain smoothness properties. In other words, while the equilibrium welfare may decrease in particular market instances, worst-case guarantees are retained for smooth mechanisms.

Our main result shows that if an auction is smooth, when the auction is followed by a secondary market that satisfies voluntary participation and weak budget balance, then the combined market is smooth as well. In addition, if the combined market is smooth, when it operates in the presence of a pre-auction information-acquisition opportunities, the price of anarchy is bounded for product type distributions, for any information acquisition structure.

Note that Theorem 1 does not depend on the details of the information acquisition structure or mechanism adopted in the secondary market. Moreover, our reduction framework does not require refinements on the equilibrium such as sequential equilibrium in the combined market to show that the price of anarchy is small.

We also note that Lemma 2 extends directly to settings with multiple secondary markets executed sequentially, with information released between each market. This is because combining a smooth auction with a trade mechanism results with a new smooth mechanism (with the same parameters), which we can now view as a smooth auction to be combined with the next trade mechanism.

**Definition 4** (Syrgkanis and Tardos 2013). **Auction** $\mathcal{M}$ **with action space** $A$ **is** $(\lambda, \mu)$-smooth **for** $\lambda > 0$ **and** $\mu \geq 1$, **if for any type profile** $\theta$, **there exists action distributions** $\{D_i(\theta)\}_{i \in [n]}$ **such that for any action profile** $a \in A$,

$$\sum_{i \in [n]} \mathbb{E}_{a_i \sim D_i(\theta)}[u_i(M(a_i', a_{-i}); \theta_i) \geq \lambda \cdot \text{Wel}(\theta) - \mu \cdot \text{Rev}(a; M)$$

**Proposition 1** (Roughgarden 2012, Syrgkanis and Tardos 2013). **Let** $\mathcal{F}^\Pi$ **be the family of all possible product type distributions. If a mechanism** $\mathcal{M}$ **is** $(\lambda, \mu)$-smooth **for** $\lambda > 0$ **and** $\mu \geq 1$, **then the price of anarchy of** $\mathcal{M}$ **within the family of distributions** $\mathcal{F}^\Pi$ **is at most** $\frac{\mu}{\lambda}$, **i.e.,** $\text{PoA}(\mathcal{M}, \mathcal{F}^\Pi) \leq \frac{\mu}{\lambda}$. 13
The smooth mechanisms for the single-item auctions and the combinatorial auctions are summarized in Table 1. In the following theorem, we show that the technique of smoothness generalize when agents can acquire costly information before the auction starts, and there is a secondary market after the auction.

**Theorem 1.** Let $\mathcal{F}^\Pi$ be the family of all possible product type distributions. For any set of signals $\Psi$, any cost function $c$, any signaling protocol $\Gamma$ and any trade mechanism $M^2$ in the secondary market, if a mechanism $M$ is $(\lambda, \mu)$-smooth for $\lambda \in (0, 1]$ and $\mu \geq 1$, then the price of anarchy of $M$ within the family of distributions $\mathcal{F}^\Pi$ for the combined market with information acquisition and secondary markets is at most $\frac{\mu}{\lambda}$, i.e., $\text{PoA}(M, \mathcal{F}^\Pi) \leq \frac{\mu}{\lambda}$.

Theorem 1 holds by directly combining Lemmas 1 and 2. Lemma 1 establishes a price of anarchy bound for a smooth mechanism in which agents can acquire costly information before the mechanism begins. The proof of Lemma 1 is analogous to the smooth arguments in Roughgarden (2012) and the details are deferred to Appendix B.

**Lemma 1.** Let $\mathcal{F}^\Pi$ be the family of all possible product type distributions. For any set of signals $\Psi$ and any cost function $c$, if the combined mechanism $M$ for auction and secondary market is $(\lambda, \mu)$-smooth for $\lambda \in (0, 1]$ and $\mu \geq 1$, then the price of anarchy of $M$ within the family of distributions $\mathcal{F}^\Pi$ for the combined market with information acquisition is at most $\frac{\mu}{\lambda}$, i.e., $\text{PoA}(M, \mathcal{F}^\Pi) \leq \frac{\mu}{\lambda}$.

Next, Lemma 2 establishes that smoothness is preserved when combining a primary mechanism with a trade mechanism, regardless of what information about bids is revealed after the primary mechanism concludes.

**Lemma 2.** If a mechanism $M^1$ is $(\lambda, \mu)$-smooth for some $\lambda > 0$ and $\mu \geq 1$, then for any signaling protocol $\Gamma$ and any trade mechanism $M^2$, the combined mechanism $M^C = G(M^1, \Gamma, M^2)$ is $(\lambda, \mu)$-smooth.

**Proof.** Let $A^1, A^2$ be the action spaces of mechanisms $M^1, M^2$ respectively, and let $A^C$ be the action space of the combined market. Note that $a^C \in A^C$ is an extensive form action, which is choosing action $a^1 \in A^1$ in the first market, and then choosing action $a^2 \in A^2$ based on the allocation, payment, and the signal realized in the first market. For each agent $i$, by voluntary participation, there exists action $\hat{a}_i^2 \in A_i^2$ such that her payoff is at least her value of the initial allocation in the secondary market. Since mechanism $M^1$ is $(\lambda, \mu)$-smooth, for any type profile $\theta$, there exists action distributions $\{D_i^1(\theta)\}_{i \in [n]}$ such that for any action profile $a^1 \in A^1$,

$$\sum_{i \in [n]} E_{\hat{a}_i^1 \sim D_i^1(\theta)} \left[ u_i(M^1(\hat{a}_i^1, a^-_i^1, \theta_i)) \right] \geq \lambda \text{Wel}(\theta) - \mu \text{Rev}(a^1; M^1).$$

For any type profile $\theta$, let $\{D_i^C(\theta)\}_{i \in [n]}$ be the distributions over actions $\hat{a}_i^C$ for each agent $i$ where $\hat{a}_i^C$ chooses action $\hat{a}_i^1$ according to distribution $D_i^1(\theta)$, and chooses action $\hat{a}_i^2$ regardless
of the signal realization $S_i$, or the allocation and payment in the first market. For any $a^C \in A^C$, we have
\[
\sum_{i \in [n]} \mathbb{E}_{a^C_i \sim D^C_i(\theta)}[u_i(M^C(\hat{a}^C_i, a^C_i); \theta_i)] \geq \sum_{i \in [n]} \mathbb{E}_{\hat{a}^1_i \sim D^1_i(\theta)}[u_i(M^1(\hat{a}^1_i, a^1_i); \theta_i)] \\
\geq \lambda \text{Wel}(\theta) - \mu \text{Rev}(a^1 \mid M^1) \geq \lambda \text{Wel}(\theta) - \mu \text{Rev}(a^C \mid M^C)
\]
where the first inequality holds by the definition of $\hat{a}^C_i$, the second as $M^1$ is $(\lambda, \mu)$-smooth, and the last inequality holds since mechanism $M^2$ is weakly budget balanced.

Thus, for product distributions, we have proven a robust welfare guarantee for the combined market generated by any $(\lambda, \mu)$-smooth mechanism. As Proposition 1 is proven only for independent value distributions, Theorem 1 likewise applies only to product distributions. As it turns out, we can derive similar results for correlated distributions based on semi-smoothness (Lucier and Paes Leme, 2011).

**Definition 5** (Lucier and Paes Leme, 2011; Roughgarden et al., 2017). *Auction $M$ with action space $A$ is $(\lambda, \mu)$-semi-smooth for $\lambda > 0$ and $\mu \geq 1$, if for any type profile $\theta$, there exists action distributions $\{D_i(\theta_i)\}_{i \in [n]}$ such that for any action profile $a \in A$,
\[
\sum_{i \in [n]} \mathbb{E}_{a^1_i \sim D^1_i(\theta_i)}[u_i(M(a^1_i, a_{-i}) \mid \theta_i)] \geq \lambda \text{Wel}(\theta) - \mu \text{Rev}(a \mid M)
\]

The main difference between the definition of semi-smooth and smooth is that for each agent $i$, the deviating action distribution $D_i(\theta_i)$ in semi-smooth only depends on her private type $\theta_i$, not the entire type profile $\theta$.

**Proposition 2** (Lucier and Paes Leme, 2011). *If a mechanism $M$ is $(\lambda, \mu)$-semi-smooth for $\lambda > 0$ and $\mu \geq 1$, then the price of anarchy of $M$ within the family of all distributions $F$ is at most $\frac{\mu}{\lambda}$, i.e., $\text{PoA}(M, F) \leq \frac{\mu}{\lambda}$.

Similarly to Theorem 1, we next show that combining a $(\lambda, \mu)$-semi-smooth auction with any pre-auction information acquisition, and any signaling protocol for the trade mechanism happening aftermarkets, the resulting mechanism in the combined market has small price of anarchy for arbitrary distributions.

**Theorem 2.** *Let $F$ be the family of all possible type distributions. For any set of signals $\Psi$, any cost function $c$, any signaling protocol $\Gamma$ and any trade mechanism $M^2$ in the secondary market, if a mechanism $M$ is $(\lambda, \mu)$-semi-smooth for $\lambda \in (0, 1]$ and $\mu \geq 1$, then the price of anarchy of $M$ within the family of distributions $F$ for the combined market with information acquisition and secondary markets is at most $\frac{\mu}{\lambda}$, i.e., $\text{PoA}(M, F) \leq \frac{\mu}{\lambda}$.

The proof of Theorem 2 is essentially identical to Theorem 1 (up to replacing $D_i(\theta)$ by $D_i(\theta_i)$) and hence omitted here.

We can now use results regarding semi-smooth auction from the literature to prove that the price of anarchy of the corresponding combined markets is bounded.

**Proposition 3** (Roughgarden et al., 2017). *For the single-item setting, the first-price auction is $(1 - 1/e, 1)$-semi-smooth.*
4 Efficiency in Single-Item Symmetric Environments

We have shown that the welfare loss is bounded for smooth auctions (Section 3). In this section, we are interested in understanding conditions under which there will not be any welfare loss. We present such conditions for some single-item auctions (e.g., first-price auctions) in the well-studied case of two agents with i.i.d. valuation distributions. Note that without information acquisition and aftermarkets, Chawla and Hartline (2013) show that for symmetric distributions, the first-price auction has a unique equilibrium, and it is efficient.

4.1 Information Acquisition

When the auction is the first-price auction and the agents can acquire costly information about opponents, even in symmetric environments when there is no aftermarket, there may exist equilibria that are not efficient. This is formally shown in Example 1. Intuitively, this holds because although additional information is not beneficial in the social planner’s perspective, it is profitable for each individual to acquire information for better bidding strategies. In the following proposition we state a stronger result, showing that any equilibrium in Example 1 is inefficient. The proof is deferred in Appendix C.

**Proposition 4.** There exists a single-item setting with two i.i.d. buyers such that with information acquisition, the allocation of any equilibrium is inefficient.

4.2 Aftermarkets

In Hafalir and Krishna (2008), the authors consider the setting with two i.i.d. agents and show that when combining the first-price auction with a secondary market taken from a specific family of secondary markets assuming bids in the auction are not revealed before the secondary market, any BNE of the combined market in which bids in the auction are non-decreasing is welfare-maximizing. We generalize their BNE efficiency result by showing it holds under significantly weaker conditions. First, our result allows the aftermarket to be any trade mechanism that satisfies ex post individual rationality (Definition 3). Second, we allow arbitrary information about bids, outcomes, and payments of the auction to be revealed before the secondary market. Given this information agents update their belief about others’ valuation. As in Hafalir and Krishna (2008), we consider only strategies in which the bids in the auction are non-decreasing[14]. The techniques we use in the proof are similar to the techniques in Chawla and Hartline (2013). It shows that any monotone strategies must be essentially identical.

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12 Second-price auctions have multiple Nash equilibria, so equilibrium selection is an issue there, even within a standalone market.

13 Essentially, in the model of Hafalir and Krishna (2008), the agent winning the item in the auction posts a price to the other agent in the secondary market.

14 We conjecture that this assumption is unnecessary and that any BNE of the combined market is efficient, even without this assumption.
Figure 1: Asymmetric bid functions $b_1$ and $b_2$.

**Theorem 3.** Consider the single-item setting with two i.i.d. buyers with atomless and bounded valuation distribution that has positive density everywhere on the support. Fix the auction $\mathcal{M}^1$ to be either the first-price auction or the all-pay auction. Fix any signaling protocol $\Gamma$ and any trade mechanism satisfying ex post individual rationality. Then in the combined market $\mathcal{M}^C = \mathcal{G}(\mathcal{M}^1, \Gamma, \mathcal{M}^2)$, every Bayesian Nash equilibrium in which the bids of the agents in the auction are non-decreasing in values, is efficient.

We next briefly outline the proof of Theorem 3; the full proof appears in Appendix C. Observe first that if the agents use symmetric strategy profiles in the auction then the allocation of the auction is efficient, and thus by voluntary participation of the secondary market the outcome of the combined market will be efficient as well. So let’s assume the bids are not symmetric.

For the given strategies of the agents in the combined market, we denote by $u_i(v)$ the expected utility of agent $i \in \{1, 2\}$ after the aftermarket, given that the agent’s value is $v$. Suppose there exists a value $v^*$ such that the bid of agent 1 with value $v^*$ in the auction is higher than that for agent 2 with the same value. Consider the maximum-length interval $(\bar{v}, \bar{v}')$ such that the bid of agent 1 is always weakly higher than agent 2 for values within the interval. Since the secondary market satisfies ex post individual rationality, the interim allocation in the combined market for agent 1 between $(\bar{v}, \bar{v}')$ is higher than agent 2, which implies that the expected utility difference satisfies $u_1(\bar{v}') - u_1(\bar{v}) \geq u_2(\bar{v}') - u_2(\bar{v})$ by the characterization in [Myerson (1981)]. Moreover, through the payment format of first-price auction or all-pay auction, since the secondary market is a trade mechanism, we can additionally conclude that $u_1(\bar{v}) - u_2(\bar{v})$ and $u_1(\bar{v}') - u_2(\bar{v}')$. Thus we can conclude that $u_1(\bar{v}') - u_1(\bar{v}) = u_2(\bar{v}') - u_2(\bar{v})$, and the only possible situation for the equality to hold is that the item is resold in the secondary market such that the allocation is efficient.

\[\text{Note that there is at least one efficient equilibrium: agents adopt symmetric strategies in the auction, the resulting auction allocation is efficient, and no trade occurs in the secondary market.}\]
5 Conclusions and Discussions

In this paper, we have shown that although the equilibrium welfare may decrease in the presence of information acquisition or secondary markets, the worst case guarantee will not decrease if the mechanisms adopted in the auction is smooth. With this in mind, one might wonder if the result can be extended to mechanisms that are only smooth under some equilibrium refinement (like undominated strategies). However, as we illustrated in Appendix A, the smoothness result under refinement cannot be generalized to the setting with secondary markets even if there is no information acquisition, since agents can benefit from speculation in the combined market. Moreover, for the first-price auction in the single-item settings with two-agent having i.i.d. distributions, we show that information acquisition can still lead to efficiency loss while all equilibria are efficient under natural assumptions on the secondary markets.

Our paper provides a preliminary understanding of the welfare guarantees when agents can acquire costly information and the auction is followed by secondary markets. Note that our techniques and results are not limited to the exact model we have presented in this paper. For example, the reduction framework we introduced in Section 3 extends easily when the agents discount the value for the items they acquire in the secondary markets. This discounting behavior is natural to consider if we view the secondary markets as a place where agents can trade used items, and people usually value less for the used items.

Another example is to generalize our framework to settings where the exact smooth framework does not deliver the tightest analysis, e.g., simultaneous first-price auction for subadditive valuations (Feldman et al., 2013) and sequential first-price auction for unit-demand valuations (Syrgkanis and Tardos, 2012). For those setting, the smooth parameter is large while the price of anarchy is small. The idea for showing small price of anarchy resembles the smooth analysis, where for the given mechanisms, there exist deviating strategies for each agent with sufficiently large utility and hence the total utilities of all agent are sufficiently high under any equilibrium. The latter guarantees small price of anarchy for the given mechanism. One can imitate their proof step by step and show that the price of anarchy results for those settings extend when agents can acquire costly information and there is a secondary market satisfying voluntary participation and weak budget balance.

There are also a lot of interesting open questions remaining. In the symmetric case, we only prove the efficiency result for two agents when all agents adopt monotone bids. A possible direction is to generalize the result to three or more agents, and without any assumption on the strategies. In our work, we only show that the price of anarchy cannot increase when the mechanism is smooth. It would be interesting to see if one can design specific information acquisition structures or specific secondary markets such that the price of anarchy of the combined market strictly decreases compared to the setting with standalone auction.
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A Equilibrium Refinements

Since $\mathcal{M}^C$ proceeds in multiple rounds, it will be helpful to consider notions of equilibrium that ensure that strategies are rational in each round. We will use the notion of sequential equilibrium (SE), a refinement of BNE proposed in [Kreps and Wilson (1982)]. A fully general definition of SE is beyond the scope of this paper; the following description is semi-formal. Roughly speaking, in an SE of our game each agent maintains beliefs about how other agents will act in each round, updates those beliefs using Bayes’ rule in response to information given between the auction and the secondary market, and in each round every agent acts in a utility-maximizing way given their beliefs. An important subtlety is how Bayes’ rule should be applied to events with 0 probability, such as under deviations from the equilibrium. This is handled by thinking of each strategy as a limit of ‘trembling’ strategies in which all possible actions have a positive chance of being observed.

More formally, a strategy profile $\sigma$ is a sequential equilibrium if there exists a belief assessment $\beta$ such that

1. For every agent and given any history of observations, following the strategy $\sigma$ is a best response to the belief specified by $\beta$; and

2. there exists a sequence of totally mixing strategy profiles $\{\sigma^k\}_{k \geq 1}$ converging to $\sigma$ and a sequence of beliefs $\{\beta^k\}_{k \geq 1}$ converging to $\beta$ such that $\beta^k$ is consistent with each agent applying Bayes’ rule to all observations, under the assumption that agents are applying strategy profile $\sigma^k$.

Every SE is a BNE, but the reverse is not always true. All of our positive results will hold for all BNE (even those that are not SE), and all of our negative results will apply even when restricting to SE.

We first show that the strategy profile constructed in Example 2 are sequential equilibrium, and undominated (survives a single round of elimination of weakly dominated strategies).

**Lemma 3.** The strategies presented in Example 2 are undominated. Additionally, they form a sequential equilibrium.

**Proof.** We first show that this is indeed an equilibrium in undominated strategies. The strategy $\sigma_1$ of agent 1 is to bid $H$ in the auction and if she wins, post the utility maximizing price of $H$ to agent 2. We claim there is no other strategy of agent 1 that dominates this one. Consider any other strategy $\sigma'_1$ of agent 1: agent 1 submits a randomized bid $b_1$ sampled from a bid distribution $F_1$ in the auction, and if she wins, she offers the item at price $q_1$ to agent 2. Note that as bidders get no signal after the auction, $q_1$ cannot depend on the bid of

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16 One can view Sequential Equilibrium as a notion of subgame perfection tailored to settings of incomplete information.

17 A belief assessment is a mapping from a history of observations to a belief (distribution) over the actions of others.

18 Note that her utility from price $p \in [1, H]$ is $p \cdot \frac{1}{p} + 1 \cdot (1 - \frac{1}{p}) = 2 - \frac{1}{p}$, which is maximized at $p = H$. 

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agent 2. To prove that $\sigma_1$ is undominated, we claim that $\sigma_1$ is strictly better than $\sigma'_1$ when agent 2 plays the following strategy: (1) places deterministic bid $b' < H$ in the auction such that $b_1 < b'$ with positive probability measure; (2) resells the item with price $H$ conditional on winning; and (3) purchases the item if the price is no more than $H$ conditional on losing. Note that the utility of agent 1 is the same under $\sigma'_1$ and $\sigma_1$ if the realized bid $b_1 \geq b'$, and the utility of agent 1 is strictly smaller under $\sigma'_1$ if the realized bid $b_1 < b'$.

Similarly, the strategy $\sigma_2$ of agent 2 is to bid 0 in the auction and accept the price from agent 1 if and only if the price is at most her value. Consider any other strategy $\sigma'_2$ of agent 2: agent 2 submits a randomized bid $b_2$ sampled from a bid distribution $F_2$ in the auction, and if she wins, she offers the item at price $q_2$ to agent 1. Consider the following strategy of agent 1: (1) places deterministic bid $b' > 0$ in the auction such that $b_2 > b'$ with positive probability measure; (2) resells the item with price 0 conditional on winning; and (3) does not purchase the item in secondary market conditional on losing. Note that the utility of agent 2 is the same under $\sigma'_1$ and $\sigma_1$ if the realized bid $b_2 \leq b'$, and the utility of agent 2 is strictly smaller under $\sigma'_1$ if the realized bid $b_2 > b'$.

We next show that this strategy profile is indeed a sequential equilibrium. First note that for all agents, conditional on losing in the auction, it is dominant strategy to accept the price if and only if the price is below her value. So this price-taking behavior is always sequentially rational regardless of beliefs. Conditional on winning, agent 2 has no uncertainty about the type of agent 1 and would never sell the item. If agent 1 wins, then the rational belief of agent 1 about agent 2’s valuation in the secondary market is identical to the prior. Under this belief, posting price $H$ conditional on winning is the dominant strategy for agent 1. Moreover, this belief is consistent with the equilibrium strategy of the agents, i.e., there exists a sequence of totally mixing strategy $\sigma^k$ where in the auction, agent 1 bids $H$ with probability $1 - \frac{1}{k}$ and bids uniformly in $[0, H]$ with probability $\frac{1}{k}$, and agent 2 bids 0 with probability $1 - \frac{1}{k}$ and bids uniformly in $[0, H]$ with probability $\frac{1}{k}$ regardless her valuation. Note that $\sigma^k \to \sigma$, and the strategy $\sigma^k$ reveals no information of agent’s value for any $k$. Thus $\beta^k$ is identical to the belief $\beta$ that the agent has under equilibrium for all $k$.

Thus, Example 2 shows that the second-price auction is not robust (in the sense of low PoA) to the introduction of a secondary market, even under the refinement of undominated strategies (and of sequential equilibrium). In the next theorem, we show that under the stronger refinement of two round elimination of weakly dominated strategies (TWDS), all equilibria of the combined market are efficient, if we combine the second-price auction with the specific secondary market where the winner in the auction posts a price to other agents, and no information is released from the auction.\footnote{This is the same secondary market we considered in Example 2.}

**Theorem 4.** Consider the single-item setting with buyers with independently drawn valuations. Fix the auction $\mathcal{M}^1$ to be the second-price auction, and $\mathcal{M}^2$ to be a price posting

\footnote{If $b_1 = H$ for sure, then clearly $\sigma_1$ is strictly better than $\sigma'_1 \neq \sigma_1$ since for such $\sigma'_1$ it must hold that $q_2 \neq H$, but $H$ is the unique optimal offer.}
Consider the combined market $\mathcal{M}^C = \mathcal{G}(\mathcal{M}^1, \perp, \mathcal{M}^2)$ in which bidders get no signal after the auction. Then under the refinement of two-round elimination of weakly dominated strategies, all Bayesian Nash equilibria in the combined market are efficient.

Proof. Fix any equilibrium of the combined mechanism that survives iterated elimination of weakly dominated strategies. (We will later show that such an equilibrium is guaranteed to exist.) For any agent $i$, let $p_i$ be the price agent $i$ charges when she resells the item under this equilibrium. Since agent $i$ also maximizes her utility in the secondary market, we have $v_i \leq p_i$. Next we claim that any bid $b_i > p_i$ in the first stage would be weakly dominated by bidding $p_i$ and reselling the item with price $p_i$ in the secondary market. Since no information is revealed in the secondary market, the only difference for agent $i$ is the case that the highest bid of the opponents in the auction is in the range of $(p_i, b_i)$. In that case, she strictly benefits from lowering the bid and not winning the item since her value is $v_i \leq p_i$. We can therefore assume that $b_i \leq p_i$ for each agent $i$ with probability 1.

Conditional on the event that for all agents, the bids in the auction are weakly lower than their price charged for reselling the item in the secondary market, it is a weakly dominant strategy for all agents to bid their value in the auction, since all agents get weakly higher utility for winning in the auction than in the secondary market. Finally, given that all agents are not underbidding in the auction, the allocation is efficient under any Nash equilibrium since if there exists any agent that overbids, in order for this to sustain under equilibrium, anytime the allocation is inefficient in the auction, the winner in auction, denoted by $i$, must have bids $b_i > v_i$, and her utility is strictly less than bidding $v_i$ unless she resales the item to the highest other bidder $i'$ with price $b_{i'} = v_{i'}$. This is efficient since all agents with bid below $b_{i'}$ must also have value below $v_{i'}$ since none of agents are underbidding.

The next example illustrates that the idea of two-rounds of elimination of weakly dominated strategies generates efficiency outcomes does not generalize beyond the single-item auctions. More specifically, we consider the multi-unit auction for agents with decreasing marginal valuations. For uniform price auctions, [de Keijzer et al. (2013)] showed that the price of anarchy is at most 3.15 when agents bid at most their marginal values. Moreover, overbidding is weakly dominated when there is no secondary market. However, those results does not extend in the presence of the secondary market. In the combined market, agents may try to win the item and resell in the secondary market, and thus it is reasonable that under equilibrium, agents will bid in the auction above their marginal valuation for the item. In the following, we first provide a formal definition for the multi-unit setting and the mechanisms we interested in.

Multi-unit Settings. The seller has $m$ identical indivisible items to sell to $n$ agents, and the feasibility constraint is $\mathcal{X} = \{(x_i)_{i \in [n]} : x_i \in \{0, \ldots, m\}, \forall i \in [n] \text{ and } \sum_i x_i \leq m\}$. One might restrict to non-increasing marginal valuations, in which for any type $\theta_i \in \Theta_i$ it holds that $v_i(j; \theta_i) - v_i(j - 1; \theta_i)$ is non-negative and (weakly) decreasing in $j$, for $j \in [m]$. A

\footnote{Here the winner in the auction posts a price $p$ to other agents in the secondary market, and the item is traded at price $p$ if there is at least one other agent willing to accept the price. Ties are broken arbitrarily.}
valuation is sub-additive if \( v_i(j;\theta_i) + v_i(j';\theta_i) \geq v_i(j+j';\theta_i) \) for any \( j, j' \in [m] \) such that \( j + j' \leq m \).

**Mechanisms.** Each agent \( i \) simultaneously submit a bid \((a_{i1},\ldots,a_{im}) \in A_i = \mathbb{R}_{\geq 0}^m\) representing her \( m \) marginal values, with \( a_{i1} \geq \cdots \geq a_{im} \). Then \( m \) units of items are greedily allocated to the bidders of the \( m \) highest marginal bids. Payments are set as follows:

- **discriminatory auction:** Agent \( i \) winning \( x_i \) units of items pays her highest \( x_i \) marginal values \( \sum_{j=1}^{x_i} a_{ij} \).
- **uniform price auction** Agent \( i \) winning \( x_i \) units of items pays for each unit the \((m+1)\)-th highest marginal bid within the \( n \times m \) reported marginals.

Note that the discriminatory auction is a generalization of the first-price auction in the multi-unit setting and the uniform-price auction is a generalization of the second-price auction in the multi-unit setting. In the following example, we consider a uniform-price auction followed by a secondary market where the agents that win items in the auction can sequentially post prices to other agents as a take-or-leave-it offer. The equilibrium constructed in the following example illustrate the failure of extending the approach in Theorem 4 to multi-unit auctions.

**Example 3.** There are two agents and \( m \) units of items in the market.

- **Agent 1** has value 1 for each of the \( m \) units.
- **Agent 2** has value \( m \) for the first unit, and value \( v \) per unit for each of the other \( m-1 \) units, where the cumulative distribution for value \( v \) is \( F(v) = 1 - \frac{1}{v} \) for \( v \in [1,m) \) and \( F(v) = 1 \) for \( v \geq m \).

The following profile of strategies is a sequential equilibrium:

- **Agent 1** bids \( m \) for the first \( m-1 \) units, and 0 for the last unit. **Agent 2** bids \( m \) for the first unit, and bids 0 for the rest of the units.
- **In the secondary market**, agent 1 post price \( m \) to agent 2 for each of the \( m-1 \) units.

It is easy to see that agent 1 wins \( m-1 \) units and agent 2 wins 1 unit in the auction, and each pays zero. The units won by agent 1 is resold to agent 2 if and only if the value of the additional units for agent 2 is \( m \), which occurs with probability \( \frac{1}{m} \). One can easily verify this strategy profile is indeed a sequential equilibrium.\(^{22}\) The optimal welfare is \( \Theta(m \log m) \) by allocating all items to agent 2, while the expected welfare under the constructed sequential equilibrium is \( \Theta(m) \). Thus the price of anarchy is \( \Omega(\log m) \).

\(^{22}\)This is because in the secondary market, agent 2 are adopting dominant strategy of accepting the prices if and only if the value is at least the price, and agent 1 is posting the unique utility maximization price given her belief about agent 2.
Remark: In contrast to the case of second-price auction for the single-item setting, in uniform-price auction for the multi-unit setting there exist strict incentives for the agents to shade bids in the auction (in Example 3, the utility of agent 2 strictly decreases if she increases her bids in the auction), so such underbidding is not dominated, even if the price posted by the opponent in the secondary market is at least her bid in the auction. However, as illustrated in Theorem 4, with two rounds of elimination of weakly dominated strategies, underbidding in the auction is weakly dominated. Moreover, knowing that opponents will shade their bids in the auction while also willing to accept higher prices in the secondary markets, agents with low values also have incentives to overbid and win the item for resale.

The example above illustrates that for multi-unit uniform-price auction, even though it is smooth under undominated strategies when there is no secondary markets (c.f. de Keijzer et al., 2013), the combined market may have large price of anarchy not only for BNE but also for sequential equilibria as well as for undominated strategies. This is in sharp contrast to the discriminatory auction which is smooth without further assumptions (c.f. de Keijzer et al., 2013), and in Theorem 4 we show that the price of anarchy for smooth auction is small in the combined market for all Bayesian Nash equilibria, even without any refinement.

B Missing Proofs from Section 3

Lemma 1. Let $\mathcal{F}^\Pi$ be the family of all possible product type distributions. For any set of signals $\Psi$ and any cost function $c$, if the combined mechanism $M$ for auction and secondary market is $(\lambda, \mu)$-smooth for $\lambda \in (0, 1]$ and $\mu \geq 1$, then the price of anarchy of $M$ within the family of distributions $\mathcal{F}^\Pi$ for the combined market with information acquisition is at most $\frac{\mu}{\lambda}$, i.e., $\text{PoA}(M, \mathcal{F}^\Pi) \leq \frac{\mu}{\lambda}$.

Proof. Since mechanism $M$ is $(\lambda, \mu)$-smooth, by definition, for any value type $\theta$, there exists action distributions $\{D_i(\theta)\}_{i \in [n]}$ such that for any action profile $a \in A$,

$$\sum_{i \in [n]} \mathbb{E}_{a_i' \sim D_i(\theta)}[u_i(M(a_i', a_{-i}); \theta_i)] \geq \lambda \text{Wel}(\theta) - \mu \text{Rev}(a; M).$$

Suppose in equilibrium, for any agent $i$ the information acquisition strategy is $\bar{\sigma}_i(\theta)$ and the bidding strategy is $\hat{\sigma}_i(\theta, s_i)$. Note that since the information acquisition decisions are not revealed to the opponents, the distribution over bids of any agent $i$ is not affected by the information acquisition decisions taken by agent $j \neq i$. Let $G_i$ be the distribution over actions in the auction for agent $i$ under equilibrium strategies $\bar{\sigma}_i$ and $\hat{\sigma}_i$, when her type is distributed according to $\mathcal{F}_i$.

Now consider the following deviating strategy for agent $i$. Agent $i$ will not acquire any information by adopting signal structure $\tilde{\psi}$. Then in the auction, agent $i$ simulates the behavior of the other agents by first sampling $\hat{\theta}_j$ according distribution $F_j$ for any $j \neq i$, and then follow the action distribution $D_i(\theta_i, \hat{\theta}_{-i})$. The expected utility of all agents given this
deviating strategy is
\[
\sum_{i \in [n]} \mathbb{E}_{\theta_i \sim F} \left[ \mathbb{E}_{a_i' \sim D_i(\theta_i, \hat{\theta}_i)} \left[ u_i(\mathcal{M}(a_i', a_{-i}); \theta_i) \right] \right]
\]
\[
= \sum_{i \in [n]} \mathbb{E}_{\theta \sim F} \left[ \mathbb{E}_{a_i' \sim D_i(\theta); a_{-i} \sim G_{-i}} \left[ u_i(\mathcal{M}(a_i', a_{-i}); \theta_i) \right] \right]
\]
\[
\geq \mathbb{E}_{\theta \sim F} \left[ \lambda \cdot \text{Wel}(\theta) - \mu \cdot \mathbb{E}_{a \sim G}[\text{Rev}(a; \mathcal{M})] \right]
\]
\[
= \lambda \cdot \text{Wel}(F) - \mu \cdot \mathbb{E}_{a \sim G}[\text{Rev}(a; \mathcal{M})]
\]
where the first equality holds by renaming the random variables $\hat{\theta}_i$ as $\theta_i$. The inequality holds by applying the definition of the smoothness and taking expectation over the actions according to distribution $G$. Note that in every equilibrium $\hat{\sigma}(\theta) = (\hat{\sigma}_1(\theta_1), \ldots, \hat{\sigma}_n(\theta_n))$ and $\hat{\sigma}(\theta, s) = (\hat{\sigma}_1(\theta_1, s_1), \ldots, \hat{\sigma}_n(\theta_n, s_n))$, the utility of any agent is at least her utility given the above deviating strategy. Thus,
\[
\sum_{i \in [n]} \mathbb{E}_{\theta \sim F} \left[ \mathbb{E}_{\psi \sim \hat{\sigma}(\theta); s \sim \psi(\theta)} \left[ \mathbb{E}_{a \sim \hat{\sigma}(\theta, s)} [u_i(\mathcal{M}(a); \theta_i)] - c_i(\psi, \theta_i) \right] \right]
\]
\[
\geq \lambda \cdot \text{Wel}(F) - \mu \cdot \mathbb{E}_{a \sim G}[\text{Rev}(a; \mathcal{M})].
\]
By rearranging the terms and noting that the sum of expected utility is the difference between equilibrium welfare and the expected revenue, we also have
\[
\sum_{i \in [n]} \mathbb{E}_{\theta \sim F} \left[ \mathbb{E}_{\psi \sim \hat{\sigma}(\theta); s \sim \psi(\theta)} \left[ \mathbb{E}_{a \sim \hat{\sigma}(\theta, s)} [u_i(\mathcal{M}(a); \theta_i)] - c_i(\psi, \theta_i) \right] \right]
\]
\[
= \text{Wel}(\mathcal{M}, (\hat{\sigma}, \hat{\sigma}), F) - \mathbb{E}_{a \sim G}[\text{Rev}(a; \mathcal{M})].
\]
Multiplying both sides of the equality (2) with factor $\mu$ and combining it with the inequality (1) above, and recalling that $\mu \geq 1$ and the equilibrium utility is non-negative, we have
\[
\mu \cdot \text{Wel}(\mathcal{M}, (\hat{\sigma}, \hat{\sigma}), F) \geq \lambda \cdot \text{Wel}(F),
\]
i.e., $\text{PoA}(\mathcal{M}, \mathcal{F}^\Pi) \leq \frac{\mu}{\lambda}$.

\[\square\]

C Missing Proofs from Section 4

First we observe that Myerson payment identity result for the BNE expected payment of an agent as function of her expected allocation, holds in the combined market when agents do not acquire costly information.

Lemma 4 [Myerson 1981]. In single-item setting, for any combined market, in any BNE and any agent $i$, if $\bar{x}_i(\cdot)$ is the interim allocation of $i$, and $\tilde{p}(\cdot)$ her interim payment function, then
\[
\tilde{p}_i(v_i) = v_i \bar{x}_i(v_i) - \int_0^v \bar{x}_i(z) \, dz + \tilde{p}_i(0)
\]
where $\tilde{p}_i(0)$ is the expected payment of agent $i$ with value 0.
Proposition 4. There exists a single-item setting with two i.i.d. buyers such that with information acquisition, the allocation of any equilibrium is inefficient.

Proof. Consider the distributions in Example 1. Suppose by contradiction the equilibrium is efficient, no agent will acquire costly information and the item is allocated to the highest value agent. By Lemma 4, since the expected payment of agent \( i \) with value 0 must be 0, by solving the following equality,

\[
(v - b)F(v) = u(v) = \int_0^v F(z)dz,
\]

the bid of the agent with value \( v \in [0, 1] \) must be \( b = \frac{v^2}{2(1+v)} \) and the equilibrium utility of the agent is \( \frac{v}{2} + \frac{v^2}{4} \). Note that in this case, a deviating strategy for the agent is to acquire information, and bid 0 to win the item if the opponent has value 0, and bid \( \frac{v^3}{8(2+v)} \) if the opponent has positive value. The expected utility under this strategy is

\[
\frac{v}{2} + \frac{v}{2} \left(v - \frac{v^3}{8(2+v)}\right) - \frac{v^2}{16} \geq \frac{v}{2} + \frac{3v^2}{8} > \frac{v}{2} + \frac{v^2}{4},
\]

which is a contradiction.

Next we prove the efficiency result with the introduction of secondary markets.

Theorem 3. Consider the single-item setting with two i.i.d. buyers with atomless and bounded valuation distribution that has positive density everywhere on the support. Fix the auction \( \mathcal{M}_1 \) to be either the first-price auction or the all-pay auction. Fix any signaling protocol \( \Gamma \) and any trade mechanism satisfying ex post individual rationality. Then in the combined market \( \mathcal{M}_C = \mathcal{G}(\mathcal{M}_1, \Gamma, \mathcal{M}_2) \), every Bayesian Nash equilibrium in which the bids of the agents in the auction are non-decreasing in values, is efficient.

In Theorem 3, we obtain efficiency for both first-price auction and all-pay auction. The proof for those auctions are similar and in the following parts, we only show it for the first-price auction.

Proof of Theorem 3. Let \( v \) be the lowest value in the support, and let \( H \) be the highest value in the support (\( H \) is finite as we assume that the support is bounded). Since the secondary market satisfies ex post individual rational and weak budget balance, for any valuation profile such that the allocation in the auction environment is efficient, no trade occurs in the secondary market, and hence the allocation is efficient in the combined market. If both agents use the same strictly increasing strategy, the allocation is efficient in the auction environment for all valuation profiles, and hence it is efficient in the combined market. Moreover, if both agents use the same monotone strategy while there exists a non-trivial interval \((v', v'')\) with the same bid \( b'' \) for all values in this interval, at least one agent with

\[^{23}\text{Note that there is at least one efficient equilibrium: agents adopt symmetric strategies in the auction, the resulting auction allocation is efficient, and no trade occurs in the secondary market.}\]
value in the interval can strictly increase her bid by an infinitesimal amount to strictly increase her utility, contradicting to the assumption that the strategies are sustained under equilibrium.

It remains to consider the case where the strategies of the two agents are monotone yet asymmetric. For any value \( v^* \) such that \( b_1(v^*) > b_2(v^*) \), let \( \bar{v} \leq v^* \) be the infimum value such that \( b_1(\bar{v}) \geq b_2(\bar{v}) \), and \( \bar{v}' \geq v^* \) be the supremum value such that \( b_1(\bar{v}') \geq b_2(\bar{v}') \). This is illustrated in Figure 1. We first consider the case that \( \bar{v} > v \) and \( \bar{v}' < H \). The boundary case where we have \( b_1(v) \geq b_2(v) \) for any \( v < v^* \), or \( b_1(v) \geq b_2(v) \) for any \( v > v^* \), is considered at the end of the proof. Note that as we assume that \( \bar{v} > v \), the definition of \( \bar{v} \) implies that there exists \( \epsilon' > 0 \) such that \( b_2(v) > b_1(v) \) for any \( v \in (\bar{v} - \epsilon', \bar{v}) \). This further implies that \( \lim_{v \uparrow \bar{v}} b_2(v) \geq \lim_{v \uparrow \bar{v}} b_1(v) \).

For the given strategies of the agents in the combined market, we denote the expected utility of agent \( i \in \{1, 2\} \) with value \( v \) after the aftermarket by \( u_i(v) \). We wish to show that \( u_1(\bar{v}) \geq u_2(\bar{v}) \). The claim that \( u_1(\bar{v}') \leq u_2(\bar{v}') \) holds similarly. The first step to showing that \( u_1(\bar{v}) \geq u_2(\bar{v}) \) is establishing that \( \lim_{v \uparrow \bar{v}} b_1(v) = \lim_{v \uparrow \bar{v}} b_2(v) \).

**Claim 1.** It holds that \( \lim_{v \uparrow \bar{v}} b_1(v) = \lim_{v \uparrow \bar{v}} b_2(v) \).

**Proof.** Recall that \( \lim_{v \uparrow \bar{v}} b_2(v) \geq \lim_{v \uparrow \bar{v}} b_1(v) \) and assume in contradiction that \( \epsilon = \lim_{v \uparrow \bar{v}} b_2(v) - \lim_{v \uparrow \bar{v}} b_1(v) > 0 \). There exists a value \( \hat{v}_2 \) for agent 2 satisfying \( \hat{v}_2 \in (\bar{v} - \frac{\epsilon}{2}, \bar{v}) \) such that \( b_2(\hat{v}_2) \in [\lim_{v \uparrow \bar{v}} b_2(v) - \frac{\epsilon}{2}, b_2(\bar{v})] \). Thus agent 2 with value \( \hat{v}_2 \) can lower her bid \( b_2(\hat{v}_2) \) by \( \frac{\epsilon}{2} \) without affecting the allocation in the auction, and increase the utility in the auction environment by \( \frac{\epsilon}{2} \) when she wins the item. Thus in any case she wins with bid \( b_2(\hat{v}_2) \), her utility from deviating and bidding \( b_2(\hat{v}_2) - \frac{\epsilon}{2} \) instead, is at least \( \hat{v}_2 - b_2(\hat{v}_2) + \frac{\epsilon}{2} \) in the auction, and this cannot decrease in the combined market as the secondary market satisfies voluntary participation.

We next argue that her utility in the combined market when bidding \( b_2(\hat{v}_2) \) in the auction is at most \( \hat{v}_2 - b_2(\hat{v}_2) + \frac{\epsilon}{4} \), and thus smaller than with the deviation. Indeed, if she wins with bid \( b_2(\hat{v}_2) \) in the auction, her utility from the auction is \( \hat{v}_2 - b_2(\hat{v}_2) \). If she ends up reselling the item in the secondary market, then since it satisfies ex post individual rationality and weak budget balance, her additional utility is bounded by \( v_1 - \hat{v}_2 \), where \( v_1 \) is the value of agent 1. To complete the proof we show that when agent 2 wins in the auction with bid \( b_2(\hat{v}_2) \) it holds that \( v_1 \leq \bar{v} \), and this will prove the claim since \( v_1 - \hat{v}_2 \leq \bar{v} - \hat{v}_2 \leq \frac{\epsilon}{4} \).

Note that if \( b_1(v) > b_2(\hat{v}_2) \) for any \( v > \bar{v} \), the above claim holds. Otherwise, let \( \epsilon'' > 0 \) be the supremum value such that \( b_1(v) \leq b_2(\hat{v}) \) for any \( v \in [\bar{v}, \bar{v} + \epsilon''] \). Moreover, by the assumptions we made on the equilibrium strategy, we have \( b_1(v) \geq b_2(v) \geq b_2(\hat{v}) \) for any \( v \in [\bar{v}, \bar{v} + \epsilon''] \). Thus \( b_1(v) = b_2(v) = b_2(\hat{v}) \) for any \( v \in [\bar{v}, \bar{v} + \epsilon''] \). First we claim that \( b_2(\hat{v}) < \bar{v} + \epsilon'' \). This is because otherwise for agent 2 with value \( \hat{v} \), her utility under equilibrium is negative since conditional on winning, her payment in the auction is at least \( \hat{v} + \epsilon'' \), the value of agent 1 is \( v_1 \leq \bar{v} + \epsilon'' \) (as agent 1 loses the auction), and strictly smaller with positive probability, and the payment from resale is at most \( v_1 \). In the case that \( b_2(\hat{v}) \leq \bar{v} + \epsilon'' \), there exists sufficiently small \( \delta > 0 \) such that agent 2 with value

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\( \lim_{v \uparrow \bar{v}} \) means taking the limit as \( v \) increases to \( \bar{v} \) (limit from below).
\[ \bar{v} + \epsilon'' - \delta \] can simply increase the bid by an infinitesimal amount, which would win the item for all opponent’s value \( v_1 < \bar{v} + \epsilon'' \), which increases her utility. Therefore, we must have \( \lim_{v \uparrow \bar{v}} b_1(v) = \lim_{v \downarrow \bar{v}} b_2(v) \).

Given Claim 1 we can define \( b^* = \lim_{v \uparrow \bar{v}} b_1(v) = \lim_{v \downarrow \bar{v}} b_2(v) \). Fix some \( \delta > 0 \) such that \( b_1(v), b_2(v) \in (b^* - \delta, b^*] \), and let \( \delta' \in (0, \delta) \) be such that \( b_2(v) > b_1(v) \) for any \( v \in (\bar{v} - \delta', \bar{v}) \). The utility of agent 1 with any value \( v \in (\bar{v} - \delta', \bar{v}) \) is

\[
    u_1(v) \geq (v - b^* - \delta) \cdot \Pr_{v \sim F_2} [v \leq \bar{v}] \geq (\bar{\bar{v}} - b_2(v)) \cdot \Pr_{v \sim F_1} [v \leq \bar{v}] - 3\delta \geq u_2(v) - 4\delta.
\]

The first inequality holds because player 1 can always choose to bid \( b^* + \delta \) and wins with probability at least \( \Pr_{v \sim F_2} [v \leq \bar{v}] \) in the auction, and retain at least the same utility in the combined market since the secondary market satisfies voluntary participation. The second inequality holds because \( \bar{v} - v \leq \delta' \leq \delta, b^* - b_2(v) \leq \delta \) and \( \Pr_{v \sim F_1} [v \leq \bar{v}] = \Pr_{v \sim F_2} [v \leq \bar{v}] \) due to the i.i.d. assumption. The final inequality follows because, similar to the argument in the previous paragraph, the payment to agent 2 from reselling the item to agent 1 is at most \( \bar{\bar{v}} \), and the utility increase of agent 2 in the secondary market is at most \( \bar{\bar{v}} - \delta \). Hence \( \lim_{v \uparrow \bar{v}} u_1(v) \geq \lim_{v \downarrow \bar{v}} u_2(v) \). Since the interim utility is a continuous function of the valuation, we conclude that \( u_1(\bar{v}) \geq u_2(\bar{v}) \) as desired.

A symmetric argument establishes that \( u_1(\bar{v}') \geq u_2(\bar{v}') \) as well. The next claim shows that these inequalities imply the desired efficiency.

Claim 2. For any pair of values \( \bar{v}' > \bar{v} \) such that \( b_1(v) \geq b_2(v) \) for any \( v \in (\bar{v}, \bar{v}') \), if \( u_1(\bar{v}) \geq u_2(\bar{v}) \) and \( u_1(\bar{v}') \leq u_2(\bar{v}') \), the allocation is efficient for values between \( \bar{v} \) and \( \bar{v}' \).

Proof. By Lemma 4 the interim allocations of the agents in the combined market under Bayesian Nash equilibrium satisfy

\[
    u_1(\bar{v}') - u_1(\bar{v}) = \int_{\bar{v}}^{\bar{v}'} \bar{x}_1(v)dv \geq \int_{\bar{v}}^{\bar{v}'} \bar{x}_2(v)dv = u_2(\bar{v}') - u_2(\bar{v}).
\]

The inequality holds because (1) the allocation in the auction environment satisfies \( x_1(v) \geq x_2(v) \) since \( b_1(v) > b_2(v) \) for any \( v \in (\bar{v}, \bar{v}') \); and (2) no trade happens in the secondary market if agent \( i \) wins the item and \( v_i > v_{-i} \) since the secondary market satisfies ex post individual rationality and weak budget balance. The two claims implies that the interim allocation satisfies \( x_1(v) \geq x_2(v) \) for any \( v \in (\bar{v}, \bar{v}') \), and hence the inequality holds for the integration of the interim allocation. Moreover, \( u_1(\bar{v}) \geq u_2(\bar{v}) \) and \( u_1(\bar{v}') \leq u_2(\bar{v}') \) implies that \( u_1(\bar{v}') - u_1(\bar{v}) \leq u_2(\bar{v}') - u_2(\bar{v}) \), and hence both inequalities must hold with equality.

In order for the integral of interim allocations to coincide, we have \( \bar{x}_1(v) = \bar{x}_2(v) \) for any \( v \in (\bar{v}, \bar{v}') \), which implies that the item is sold from agent \( -i \) to \( i \) in the secondary market if agent \( i \) loses the item, \( v_i < v_{-i} \), and \( v_i \in (\bar{v}, \bar{v}') \). Therefore, the allocation in the combined market is efficient in range \( (\bar{v}, \bar{v}') \).

Finally, we address the boundary cases where we have either \( b_1(v) \geq b_2(v) \) for any \( v < v^* \), or \( b_1(v) \geq b_2(v) \) for any \( v > v^* \). If \( b_1(v) \geq b_2(v) \) for any \( v < v^* \), then \( u_1(\bar{v}) \geq 0 \) (as the utility
of agent 1 is always non-negative under equilibrium), and \( u_2(v) = 0 \) (since agent 2 with value \( v \) wins with probability zero, as the distribution is atomless, and no trade occurs in the secondary market when agent 2 loses in the auction). This implies that \( u_1(v) \geq u_2(v) \). In the other case that \( b_1(v) \geq b_2(v) \) for any \( v > v^* \), for the highest value \( H \), we have \( u_1(H) = u_2(H) \).

First note that under equilibrium we have \( b_1(H) = b_2(H) \), because otherwise agent 1 with value \( H \) can decrease her bid and retain the same probability of winning in the auction. Since in this case each of the agents when her value is \( H \) wins the item with probability 1 (ties have probability 0 as the distribution has no atom at \( H \)) with payment \( b_1(H) = b_2(H) \), and no trade occurs in the secondary market, we have \( u_1(H) = u_2(H) \). Again by applying Claim 2, the interim allocations for both agents coincide in the combined market, and the allocation is efficient.

We note that we did not make use of the structure of the signal distribution at any point in the argument above. Thus it is apparent that the argument does not depend on the content of the signals released after the auction. In particular, our conclusion that the allocation is efficient holds even if the auction bids are revealed before the secondary market begins.

\[ \square \]

## D Price of Stability

In this section, we analyze the price of stability result in the combined market. First, we made the following assumption on the secondary market \( M^2 \).

**Assumption 6.** For any initial allocation \( x \), there exists an action profile \( \hat{a}^2 = \hat{a}^2_x \in A^2 \) such that for any agent \( i \) and any action \( a^2_i \) we have

\[
u_i(M^2(a^2_i, \hat{a}^2_{-i}, x); \theta_i) \leq u_i(M^2(\hat{a}^2, x); \theta_i) = u_i(x; \theta_i).\]

This assumption implies that for every initial allocation \( x \) in the secondary market, there is a profile of actions (playing \( \hat{a}^2 \)) that corresponds to all agents opting out of the secondary market (so utility is the same as at the end of the auction) which is a Nash equilibrium.

**Observation 1.** For any signaling protocol \( \Gamma \) and any mechanism \( M^2 \) satisfying Assumption 6, consider the combined market \( MC = G(M^1, \Gamma, M^2) \). For any family of distributions \( F \), we have

\[
\text{PoS}(M^C, F) \leq \text{PoS}(M^1, F).
\]

**Proof.** For any allocation \( x \), let \( \hat{a}^2_x \) be the action profile that satisfies Assumption 6 in the secondary market. For any distribution \( F \in F \) and BNE strategy profile \( \sigma^1 \in \text{BNE}(F, M^1) \), let \( \sigma^C \) be the strategy profile in the combined market that follows strategy \( \sigma^1 \) in the first market, and always chooses action profile \( \hat{a}^2_x \) in the secondary market given allocation \( x \), regardless of the payment and the signals realized in the auction. It is easy to verify that since \( \hat{a}^2_x \) is a Nash equilibrium profile in the secondary market for any outcomes realized in the auction, there is no profitable deviation strategy for any player \( i \) in the combined market, i.e.,
$\sigma^C \in \text{BNE}(F, M^C)$. Moreover, the expected welfare given strategy $\sigma^C$ in combined market $M^C$ equals the expected welfare given strategy profile $\sigma^1$ in first market $M^1$. Therefore, the price of stability does not increase.

Note that the observation considers price of stability for BNE. It leaves open the possibility that price of stability of the combined market is higher than the price of stability of the auction for refinements of BNE and other solution concepts.

### E Price of Anarchy for Revenue

In the single-item setting, [Hartline et al. (2014)] showed that the first-price auction (and the all-pay auction) with monopoly reserve guarantees constant approximation to the optimal revenue. However, their result is for stand alone auctions, and their technique does not extend to the setting when there are secondary markets. The main reason is that their analysis requires that agents with value below the reserve price will never get the item, which is not guaranteed in our setting since for some realizations of the valuation profile, agents may buy the item in the secondary market even if their value is below the reserve price in the auction (see Example 4 for an illustration). [Carroll and Segal (2019)] characterized the optimal mechanisms that are robust to the format of the secondary market. They only considered the revenue under the best equilibrium, and the revenue guarantee under worst equilibrium is left as an open problem. We will address this question by considering the price of anarchy for revenue.

Denote the expected revenue of $\mathcal{M}$ given type distribution $F$ and equilibrium strategy $\sigma$ by $\text{Rev}(\mathcal{M}, \sigma, F) = \mathbb{E}_{\theta \sim F} [\sum_i p^M_i(\sigma(\theta))]$. The optimal revenue given distribution $F$ is defined as $\text{Rev}(F) = \sup_{\mathcal{M} \in \mathbb{M}, \sigma \in \text{BNE}(F, \mathcal{M})} \{\text{Rev}(\mathcal{M}, \sigma, F)\}$ where $\mathbb{M}$ is the set of auction mechanisms satisfying voluntary participation. Then the price of anarchy for revenue of mechanism $\mathcal{M}$ within the family of distributions $\mathcal{F}$ is

$$\text{PoAR}(\mathcal{M}, \mathcal{F}) = \sup_{F \in \mathcal{F}} \frac{\text{Rev}(F)}{\inf_{\sigma \in \text{BNE}(F, \mathcal{M})} \{\text{Rev}(\mathcal{M}, \sigma, F)\}}.$$  

In this section, we consider combined markets created by any ex-post IR trade mechanism, running after first-price auction with an optimal anonymous reserve (but with no information revealed after the auction). We assume that the valuation distributions are regular and bounded. We show that even under the worst equilibrium of the combined market, the revenue of the auctioneer is at least a constant fraction of the optimal revenue. Our result holds under the standard Myerson regularity assumption:

**Definition 7.** A distribution $F$ is regular if the virtual value $\phi(v) = v - \frac{1-F(v)}{f(v)}$ is non-decreasing in $v$.  

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25Note that, by the revelation principle, this is equivalent to maximizing revenue over Bayesian incentive compatible and individual rational mechanisms.
Theorem 5. Consider the single-item setting with buyers with independently drawn valuations, each from a regular distribution with bounded support. Fix the auction $\mathcal{M}^1$ to first-price auction with optimally chosen anonymous reserve $r$, and fix any trade mechanism $\mathcal{M}^2$ satisfying ex post individual rationality. Consider the combined market $\mathcal{M}^C = \mathcal{G}(\mathcal{M}^1, \bot, \mathcal{M}^2)$ in which bidders get no signal after the auction. Then the price of anarchy for revenue for the combined market is at most 2.62.

To prove Theorem 5 we make use of a result by Jin et al. (2019), which shows that (in a setting without secondary markets) the best revenue achievable with an anonymous posted price is a 2.62-approximation to the optimal revenue, for regular distributions. As we will show in Lemma 5, a first-price auction with reserve price $r$ followed by a secondary market will generate at least as much revenue as posting an anonymous price $r$. Combining the results, we obtain that price of anarchy on revenue is at most 2.62 and Theorem 5 holds.

Lemma 5. Consider the single-item setting with buyers with independently drawn valuations with bounded support. Fix the auction $\mathcal{M}^1$ to be first-price auction with anonymous reserve $r$, and fix any trade mechanism $\mathcal{M}^2$ satisfying ex post individual rationality. Consider the combined market $\mathcal{M}^C = \mathcal{G}(\mathcal{M}^1, \bot, \mathcal{M}^2)$ in which bidders get no signal after the auction. Then in any BNE of $\mathcal{M}^C$, in any realization of the valuations except a set with measure 0, if there exists an agent with value strictly above the reserve price $r$, the item is sold to some agent in the auction with probability 1.

Intuitively, for any agent with value strictly above the reserve price $r$, if her bid in the auction is below $r$, then consider what would happen if she increases her bid to $r$. There are two cases:

- if the item was previously not being sold at all, then the agent will win the item by raising her bid to the reserve price. In this case, her utility in the combined market strictly increases by this deviation (since originally her utility in the combined mechanism was 0, as the item was going unallocated).

- if the item was previously being sold to some other agent, then the agent’s utility is not impacted by this bid increase in the auction, since information on bids is not revealed in the secondary market so the auction winner cannot observe this deviation.

We conclude that in order for an agent to bid below $r$ at equilibrium, it must be that the first case occurs with probability 0. This means that the item is always sold, as claimed by Lemma 5. This intuitive argument does not account carefully for tie-breaking between bids, which is important since the proposed deviation involves bids that are equal to the reserve price. The formal proof is provided at the end of the section with additional careful analysis of the tie-breakings in the auction.

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26 We assume that the item is sold if there is at least one bidder that bids the reserve or higher.
27 That is, no information about the bids is being revealed after the auction. All each agent knows is the allocation and her own payment.
Note that in Lemma 5, it is crucial to assume that the bids are not revealed in the secondary market. This is a natural assumption if the mechanism designer has control over the bid information and chooses not to reveal such information to the secondary market. The following example illustrate the failure of Lemma 5 if bids are revealed.

**Example 4.** Consider there are 3 agents.

- Agent 1 has value 1.1.
- Agent 2 has value 0 with probability 3/4 and value 2 with probability 1/4.
- Agent 3 has value 0 with probability 1/4 and value 10 with probability 3/4.

Suppose the auctioneer chooses first-price auction with reserve price 1. Now consider the specific secondary market where if the bid profile in auction is (0, 2, 1), then agent 2 has the option to sell the item to agent 3, and if there is a sale then agent 2 receives a payment of 3 and agent 3 pays 4 for the item, and additionally, if the trade happens then agent 1 also receives a payment of 1 (making the mechanism strongly budget balanced.) If the bid profile in auction is not (0, 2, 1), no trade can occur in the secondary market.

It is easy to verify that in this setting, the following is a Bayesian Nash equilibrium strategy for all agents. Agent 1 bids 0, agent 2 bids 2 if her value is 2 and 0 otherwise, and agent 3 bids 1 if her value is 10 and 0 otherwise. Moreover, under this equilibrium, the item is not sold if both agent 2 and agent 3 has value 0, which occurs with probability 3/16, even though agent 1 has value strictly above reserve price 1 with probability 1.

**Proof of Lemma 5.** If the measure of profiles of agents types such that for some agent $i$ it holds that $v_i > r$ and $b_i(v_i) < r$ is 0, Lemma 5 clearly holds. Assume that this measure is positive. We show that in this case, the item is sold with probability 1.

Let $H < \infty$ be larger than the highest value of the support of any agent. Let $\mathcal{E}$ be the event such that the highest bid among all agents is $r$. First we consider the case that the probability of event $\mathcal{E}$ is strictly positive, and denote the probability by $p$.

**Claim 3.** Conditional on event $\mathcal{E}$, for any agent $i$ with value $v_i > r$ and equilibrium bid $b_i(v_i) = r$, if agent $i$ loses in the auction with positive probability $p' > 0$ then the expected per-unit price (expected payment divided by expected allocation) for purchasing the item in the secondary market is at most $r$.

**Proof.** Assume by contradiction that the per-unit price paid by agent $i$ is $r + \epsilon$ for some constant $\epsilon > 0$. Let $\epsilon' < \frac{p p'}{4}$ be the number such that $r + \epsilon' < v_i$ and the probability the highest bid between $(r, r + \epsilon')$ is at most $\frac{p p'}{4}$ Note that for agent $i$, one possible deviation strategy is to bid $r + \epsilon'$ in the auction, and follow the equilibrium strategy in the secondary market. To analyze the utility obtained under such a deviation, consider cases for the highest competing bid in the auction.

- Case 1: $\max_{j \neq i} b_j > r + \epsilon'$. In this case agent $i$ does not win the item, and since the bid of agent $i$ is not revealed, the utility of agent $i$ remains the same.
Case 2: \( \max_{j \neq i} b_j \in (r, r + \epsilon'] \). The probability of this case is at most \( \frac{\epsilon p'}{4H} \), and hence the maximum possible expected utility loss that agent \( i \) can experience due to this case is at most \( \frac{\epsilon p'}{4} \).

Case 3: \( \max_{j \neq i} b_j < r \). In this case agent \( i \) will win the item with a bid of \( r + \epsilon' \). This can lead to a utility loss only if, at equilibrium, agent \( i \) is bidding exactly \( r \) and winning the item. In this case the utility of agent \( i \) decreases by \( \epsilon' \), since his bid increases by \( \epsilon' \). The expected utility loss due to this case is therefore at most \( \epsilon' \leq \frac{\epsilon p'}{4} \).

Case 4: \( \max_{j \neq i} b_j = r \). In this case agent \( i \) will win the item. If at equilibrium agent \( i \) bids \( r \) and wins, then the utility of agent \( i \) decreases by \( \epsilon' \leq \frac{\epsilon p'}{4} \). If under equilibrium agent \( i \) loses (either with a bid of \( r \) or strictly less than \( r \)), which happens with probability at least \( p \cdot p' \), then the utility of agent \( i \) is increased by \( \epsilon - \epsilon' > \frac{3\epsilon}{4H} \) since originally agent \( i \) pays \( r + \epsilon \) per-unit price (in the secondary market), while currently the agent \( i \) wins with payment \( r + \epsilon' \) and the resale will not decrease the utility of agent \( i \). Thus the expected utility increase due to this case is strictly larger than \( \frac{3\epsilon p' p}{4} \).

Combining all these cases, we note that total expected utility increases under this deviation for agent \( i \), which contradicts to the assumption that bidding at most \( r \) is an equilibrium strategy. We conclude that for any agent \( i \) with value \( v_i > r \) and bid \( b_i(v_i) \leq r \), who loses in the auction with positive probability, the expected per-unit price in the secondary market is at most \( r \). □

On the other hand, for any agent with \( v_i \leq r \), the expected per-unit price paid by agent \( i \) in the secondary market conditional on losing in the auction is at most \( r \) given any valuation profile of other agents, since the secondary market satisfies ex post individual rationality. This implies the following claim.

**Claim 4.** For any agent \( i \) with value \( v_i > r \) and equilibrium bid \( b_i(v_i) < r \), except for a set of measure 0, the per-unit price paid by agent \( i \) in the secondary market is at least \( r \) conditional on event \( \mathcal{E} \).

**Proof.** To see why this is true, suppose otherwise. Then conditional on event \( \mathcal{E} \) happening, for any agent \( j \) not winning the auction, the bid of agent \( j \) is at most \( r \), and the per-unit price paid by agent \( j \) is at most \( r \). This implies that the average per-unit price paid by agents losing in the auction is strictly less than \( r \). Thus there exists an agent \( j^* \) with value \( v_{j^*} \) and bid \( b_{j^*}(v_{j^*}) = r \) winning the auction such that the expected per-unit price for reselling the item in the secondary market is strictly below \( r \) conditional on agent \( j \) winning the item. If \( v_{j^*} \geq r \), this implies that the secondary market does not satisfy voluntary participation for agent \( j^* \) and if \( v < r \), the utility of agent \( j \) is negative in the combined market, and agent \( j \) can deviate to bid 0 in the auction to retain utility 0, which violates the assumption of the equilibrium. So in either case we arrive at a contradiction, and hence the claim follows. □

Note that since we assumed the measure of profiles of agents types such that for some agent \( i \) it holds that \( v_i > r \) and \( b_i(v_i) < r \) is strictly positive, Claim 4 implies that there exists
an agent $i^*$ such that $v_i > r$, $b_i(v_i) < r$, and the per-unit price paid by agent $i^*$ conditional on event $\mathcal{E}$ is at least $r$.

**Claim 5.** If event $\mathcal{E}$ happens with positive probability, if there exists an agent $i^*$ such that $v_i > r$, $b_i(v_i) < r$, and the per-unit price paid by agent $i^*$ conditional on event $\mathcal{E}$ is at least $r$, then the item is sold with probability 1.

**Proof.** One possible deviation strategy for agent $i^*$ is to bid $r$ in the auction, and follow the equilibrium strategy in the secondary market. We consider three different cases for the bid of other agents.

1. Suppose $\max_{j \neq i^*} b_j < r$. In this case, the item is not sold to any agent, which implies that agent $i^*$ strictly benefits from deviating to bid $r$ and win the item with price $r$ since $b_{i^*} - r > 0$.

2. Consider the case where $\max_{j \neq i^*} b_j \geq r$ and the item is sold to agent $j^* \neq i^*$ even if agent $i^*$ has bid $r$. For agent $i^*$, by deviating the bid to $r$, the allocations and payments of the auction are not affected, and since the bids are not revealed in the secondary market, the utility of agent $i^*$ remains the same in the combined market.

3. Consider the case where $\max_{j \neq i^*} b_j = r$, i.e., conditional on event $\mathcal{E}$, and when agent $i^*$ deviate the bid to $r$, she wins the item. By the definition of $i^*$, the per-unit price paid to other agents for reselling the item in the secondary market is at least $r$, which implies that the utility of agent $i^*$ in the combined market is at most $v_{i^*} - r$. Moreover, by bidding $r$ and winning the item, the utility of agent $i^*$ is at least $v_{i^*} - r$.

Combining these cases, we conclude that for agent $i^*$ with value $v_{i^*} > r$, the utility for bidding $r$ is weakly higher than the utility for bidding strictly below $r$. Moreover, the equality holds only when the probability of case 1 is 0, which implies that the item is sold with probability 1.

Finally consider the case that event $\mathcal{E}$ happens with probability 0. In this case for any agent $i$ with value $v_i > r$, the utility for bidding $r$ is weakly higher than the utility for bidding strictly below $r$ since the third case discussed in Claim 5 happens with probability 0. Moreover, the equality holds only when the probability of case 1 is 0, which implies that the item is sold with probability 1.