Coded mask imaging of extended sources with Gaussian random fields

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ABSTRACT
A novel method for generating coded mask patterns based on Gaussian random fields (GRF) is proposed. In contrast to traditional algorithms based on cyclic difference sets, it is possible to construct mask patterns that encode a predefined point spread function (PSF). The viability of this approach and the reproducibility of the PSFs is examined, together with studies on the mean transparency, pixel-to-pixel variance and PSF deterioration due to partial shadowing.

Sensitivity considerations suggest the construction of thresholded realizations of Gaussian random fields (TGRF) which were subjected to the same analyses. Special emphasis is given to ray-tracing simulations of the pattern’s performance under finite photon statistics in the observation of point sources as well as of extended sources in comparison to random masks and the pattern employed in the wide field imager onboard BeppoSAX. A key result is that in contrast to traditional mask generation schemes, coded masks based on GRFs are able to identify extended sources at accessible photon statistics. Apart from simulating on-axis observations with varying levels of signal and background photon counts, partial shadowing of the mask pattern in the case of off-axis observations and the corresponding field-of-view is assessed.

Key words: instrumentation: miscellaneous, methods: numerical, techniques: image processing

1 INTRODUCTION
In X-ray astronomy, focusing of radiation is so far feasible only for photon energies up to about 10 keV through grazing incidence reflection. Applied in Wolter-type mirrors, this method can provide a very good angular resolution, i.e. down to 0.5 in the case of Chandra\(^1\) and 4 – 12 for XMM-Newton\(^2\). The collecting area is maximised through the use of nested mirrors. The field-of-view (FOV) is limited by the grazing incidence condition set by the diffractive index of the mirror material to \(\leq 1\). At energies higher than 10 keV, focusing is technologically very hard to achieve. A workaround are coded mask imagers, where a position sensitive detector records the shadow of a mask pattern cast by the sources under investigation. The arrangement of sources can be reconstructed by cross-correlating the recorded shadowgram with the mask pattern.

Coded masks have by now found a widespread use in high energy astrophysics and there is a large number of successful missions such as BeppoSAX\(^3\), currently flying instruments like INTEGRAL\(^4\) and HETE-2\(^5\), and ambitious future projects, for instance SWIFT\(^6\).

In this paper, I propose coded mask patterns based on Gaussian random fields, because they enable the construction of a coded mask device for predefined imaging characteristics, i.e. for a given PSF. The shape of the PSF can be tuned to match the anticipated source profile. A beautiful example of a naturally occurring Gaussian random field is the pattern of fluctuations in the cosmic microwave background (CMB). Analyses of WMAP data carried out among others by Cayón et al.\(^7\) and Komatsu et al.\(^8\) find the CMB consistent with Gaussian primordial fluctuations and have set upper limits on non-Gaussianity.

After a recapitulation of coded mask imaging and existing mask pattern generation schemes in Sect. 2 GRFs are introduced in Sect. 3. The feasibility of GRFs in coded mask imagers is examined in Sect. 4 with special emphasis on the performance of GRFs in realistic scenarios, i.e. under finite photon statistics and in the observation of extended sources (Sect. 5). A summary of the key results in Sect. 6 concludes the paper.

\(^{1}\)http://cxc.harvard.edu/
\(^{2}\)http://xmm.vilspa.esa.es/
\(^{3}\)http://bepposax.gsfc.nasa.gov/bepposax/index.html
\(^{4}\)http://astro.estec.esa.nl/SA-general/Projects/Integral/
\(^{5}\)http://space.mit.edu/HETE/
\(^{6}\)http://swift.gsfc.nasa.gov/
2 CODED MASK IMAGING

Coded mask cameras observe a source by recording the shadow cast by the mask onto the detector. The mask pattern is described by the position dependent transparency $\psi(x)$. A shifted shadowgram $\psi(x - d \tan(\theta))$ is observed if the radiation incides under an angle $\theta$ with respect to the optical axis. The distance between the coded mask and the detector is denoted by $d$. The correlation function $a(x)$, defined as

$$a(x) = \psi(x - d \tan(\theta)) \otimes \psi(x) = \langle \psi(x - d \tan(\theta) + \lambda) \psi(\lambda) \rangle,$$  

(1)

peaks at $x_0 = d \tan(\theta)$, from which the angle of incidence $\theta = \arctan(x_0/d)$ can be inferred. The PSF $c(x)$, defined as the correlation function at normal incidence ($\theta = 0^\circ$), i.e. the auto-correlation function, reads

$$c(x) = \psi(x) \otimes \psi(x) = \langle \psi(x + \lambda) \psi(\lambda) \rangle.$$

(2)

The influence of imperfections of the detector can be modelled by convolution of $\psi(x)$ with suitable kernels describing the positional detector response (see, e.g. Schäfer & Kawai 2003). Techniques for analysing coded mask data have been summarised by Skinner et al. (1987) and Caroli et al. (1987).

Random mask patterns as used in the HETE-2 satellite in 't Zand et al. (1994) consist of white noise. They are not ideal imagers, because their auto-correlation possess side-lobes and are not perfectly flat. Aiming at $\delta$-like PSFs, mask patterns based on cyclic difference sets have been introduced by Gunson & Polvichronopolus (1976). As pointed out by Fenimore & Cannon (1978), these uniformly redundant arrays (URA) provide even sampling at all spatial scales. URA patterns are less susceptible to noise compared to truly random arrays and their auto-correlation function is a $\delta$-spike with perfectly flat side-lobes in case of complete imaging. In this paper, I propose a method for constructing coded mask pattern encoding arbitrary PSFs. While the traditional masks are optimised for the observation of point sources, the PSFs of masks based on GRFs can be adjusted to the source profile of extended sources and make the observation of extended sources such as extended structures in the Milky Way possible.

3 GAUSSIAN RANDOM FIELDS

3.1 Definitions

The statistical properties of a GRF are homogeneous and isotropic and the phases of different Fourier modes are mutually uncorrelated and random. A consequence of the central limit theorem is then that the amplitudes follow a Gaussian distribution. Due to all correlations above the two-point level being either vanishing in the case of odd moments or being expressible in terms of two-point functions for even moments, the statistics of amplitude fluctuations in a GRF is completely described by its power spectrum $P(k)$ (see eqn. (4)).

Because the imaging characteristics of coded mask imagers are described by the PSF, which is defined to be the auto-correlation function of their mask pattern, i.e. by their power spectrum in case of isotropic PSFs, GRFs provide a tool for generating mask patterns with predefined imaging characteristics. The theory of structure formation in cosmology and the description of the cosmic microwave background makes extensive use of GRFs (c.f. Peacock 1995, Longair 1998). Their application is commonplace in generating initial conditions for simulations of cosmic structure formation and in constructing mock CMB fields for simulating sub-millimetric observations.

3.2 Algorithm

Starting from the PSF $c(x)$, the Fourier transform $C(k)$ is derived:

$$C(k) = \mathcal{F}[c(x)] = \int \frac{d^2x}{(2\pi)^2} c(x) \exp(-ikx).$$

(3)

The power spectrum $P(k)$ is defined as the Fourier-transform $C(k)$ of the auto-correlation function $c(x)$. In more than one dimension, an average of the Fourier transform $C(k)$ of the statistically isotropic random field $c(x)$ over all directions of the wave vector $k$ at fixed length $k = |k|$ needs to be performed:

$$P(k) = \langle |C(k)|^2 \rangle_{k=\text{fixed}}.$$  

(4)

All elementary waves $\exp(ikx)$ with wave vectors in the $k$-space shell $[|k|, |k + \Delta k|]$ contribute to the variance $\sigma^2 = P(k)$ required by the power spectrum on scale $k = |k|$. In discretising, the amplitudes $\Psi(k)$ are set such that their quadratic sum $\sum_{|k|,|k+\Delta k|} |\Psi(k)|^2$ matches $\sigma^2$ with the only exception $\Psi(k = 0)$, which is set to zero in order to ensure a vanishing expectation value of the realisation $\psi(x)$. The normal modes $\exp(ikx)$ are modified by a phase factor $\exp(2\pi i q)$, where $q \in [0, 1)$ is a uniformly distributed random number. By inverse Fourier transform, the normal modes $\Psi(k)$ are brought to interference which finally results in the realisation, the real part of which is denoted by $\psi(x)$:

$$\psi(x) = \Re \left( \mathcal{F}^{-1} \{ \Psi(k) \} \right) = \Re \left[ \int d^2 k \Psi(k) \exp(ikx + 2\pi i q) \right].$$

(5)

Alternatively, one may require the additional symmetry $\Psi(-k) = \Psi^*(k)$ in Fourier space (the complex conjugation is denoted by the asterisk), which forces the realisation to be purely real. The flow chart eqn. (6) summarises all steps:

$$C(k) \quad \xrightarrow{\mathcal{F}} \quad P(k) \quad \xrightarrow{\exp(2\pi i q)} \quad \Psi(k) \quad \xrightarrow{\mathcal{F}^{-1}} \quad c(x).$$  

(6)

Due to the periodic boundary conditions imposed by the Fourier transform, the resulting realisations of the Gaussian random field have cyclic boundaries, which is a desirable feature for coded mask patterns. For reasons of numerical accuracy, it is strongly recommended to use shells in $k$-space with varying thickness $\Delta k \propto |k|^{-1}$, such that approximately the same number of discretised modes contributes to the variance required by the power spectrum $P(k)$.

3.3 Choice of the PSF

Although the algorithm outlined in Sect. 3.2 is capable of generating random fields $\psi(x)$ encoding any isotropic PSF $c(x)$, PSFs should be shaped like Lorenzian functions $c_L(x)$ or Gaussian functions $c_G(x)$. The parameter $\sigma_x$ describes the spatial extent:

$$c_L(x) = \frac{\sigma^2}{x^2 + \sigma^2},$$

(7)

$$c_G(x) = \exp \left( -\frac{x^2}{2\sigma^2} \right).$$

(8)

The normalisation has been chosen such that the maximum
correlation strength at \( x = 0 \) is set to one. In the realisation \( \phi(x) \),
the variable \( \sigma_r \), that parameterises the PSF can be interpreted as a
correlation length. [8] have pursued a related idea and have suggested coded masks with two spatial scales.
In contrast, the realisations considered here have an entire spectrum of length scales.

3.4 Scaling applied to the Gaussian random fields

If one aims at employing GRFs in coded mask imagers, the field
has to be scaled such that it assumes values ranging from \( \phi(x) = 0 \)
(opaqueness) to full transparency \( \phi(x) = 1 \). This scaling ensures
that the full dynamical range between is used and the modulation
of the shadowgram as strong as possible. Hence, the sensitivity is
maximised. One could think of two different linear transformations,
the most intuitive being:

\[
\psi(x) \rightarrow \psi^*(x) = \frac{\phi(x) - \min \{\phi(x)\}}{\max \{\phi(x)\} - \min \{\phi(x)\}}.
\]

With the symmetry condition \( \max \{\phi(x)\} = \min \{\phi(x)\} \) being
fulfilled, the mean transparency \( \langle \psi^*(x) \rangle \) is equal to 1/2: The mean
\( \langle \psi(x) \rangle = 0 \) vanishes by construction, because each normal mode
\( \cos(kx) \) has a vanishing expectation value. In general, the realisation
\( \phi(x) \) will not fulfill the above mentioned symmetry condition.

Instead, the scaling

\[
\psi(x) \rightarrow \psi'(x) = \frac{1}{2} \frac{\psi(x)}{\max |\phi(x)|} + 1
\]

ensures \( \langle \psi' \rangle = 1/2 \) and will be used in the remainder of the paper.
It should be noted that none of the above scalings strictly
conserves Gaussianity, because each particular realisation is scaled by
its maximal amplitude and consequently, high amplitudes do not
appear any more in an ensemble of realisations.

Now that the mean transparency \( \langle \psi' \rangle \) is fixed, the absolute flux
from a source can be inferred from the number of measured photons.
The scaling eqn. \( 9 \) may be taken advantage of in designing a mask
that blocks a larger or smaller fraction of photons than the
generic fraction of 1/2: In anticipation of Sect. 3.5 in the case of a realisation of a GRF encoding a Gaussian PSF \( c_{Gm}(x) \) with \( \sigma_r = 8 \)
pixels, the probability density \( p((\psi'^*)d(\psi'^*) \) of the mean transpar-
tency \( t = \langle \psi'^* \rangle \) is described by a Gaussian distribution with mean
\( \mu_t = 0.504\pm0.082 \) and standard deviation \( \sigma_t = 0.028\pm0.006 \) at 95%
certainty. When constructing realisations of Gaussian fields for
coded mask instruments, one obtains patterns with transparencies
\( \langle \psi'^* \rangle \in [\mu_t - \sigma_t, \mu_t + \sigma_t] \) with a probability of erf(1/\( \sqrt{2} \)) = 0.6827.

3.5 Gaussian random fields for circular apertures

For coded-mask experiments with a circular aperture it is possible
to construct GRFs with azimuthal symmetry, in the same way as
hexagonal uniformly redundant arrays (HURA) are an adaptation
of the URA patterns to circular apertures. Instead of constructing a GRF
with plane waves as the solutions of Laplace’s equation \( \Delta \phi(x) = 0 \) in
Cartesian coordinates \( x = (x, y) \) with boundary conditions \( \phi(x = -L) = 0 = \phi(x = L) \) (2L denotes the pattern’s side length) one would
resort to solving \( \Delta \phi(r) = \langle \phi'^* \rangle \) in polar coordinates \( r = (r, \phi) \) with the boundary condition \( \phi(r = R) = 0 \)
\( \forall \phi \), where the radius of the aperture is denoted as \( R \). \( \phi \) is easily
found as the solution to Bessel’s differential equation and reads as:

\[
\psi_{\text{code}}(r, \phi) = J_m[r \cdot Z_m(k \ell)] \cdot \exp(i m \phi).
\]

where the numbers \( \ell \) and \( m \) are only allowed to assume integer values.
\( J_m(\ell) \) is the \( \ell \)-th zero of the Bessel function \( J_m \). In Fig. 1
we may be taken advantage of in designing a mask
that blocks a larger or smaller fraction of photons than the
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coded mask instruments, one obtains patterns with transparencies
\( \langle \psi'^* \rangle \in [\mu_t - \sigma_t, \mu_t + \sigma_t] \) with a probability of erf(1/\( \sqrt{2} \)) = 0.6827.

4 RESULTS

In order to provide a visual impression, two GRFs encoding the
above stated PSF with their auto-correlation functions are presented
(Sect. 4.1). Subsequently, the reproducibility of the chosen PSF
(Sect. 4.2), the pixel-to-pixel variance (Sect. 4.3), the Gaussianity of the distribution of pixel amplitudes (Sect. 4.4) and the shape of the
PSF under partial shadowing (Sect. 4.5) are examined. Finally,
thresholded GRFs are introduced and the deterioration of the PSF
of such thresholded realisations (Sect. 4.6) is addressed.

4.1 Visual impression

Following the above prescription, 100 realisations of GRFs encoding
Gaussian and Lorenzian PSFs of different widths \( \sigma_r \) were generated
on a 2-dimensional square grid with 256\(^2\) mesh cells. Figs. 2
and 3 show a realisation of the GRF and its auto-correlation func-
tion for a Gaussian and a Lorenzian PSF, respectively. In order to
facilitate comparison, the widths of the PSFs have been chosen to
be the same: \( \sigma_r = 8 \) pixels. The random fields are scaled to mean
values of 1/2 (by means of eqn. 10) and the central correlation
strength in the auto-correlation functions is equal to 1. The contours have a linear spacing of 0.1. The auto-correlation functions
have the symmetry property that \( \psi(x) \otimes \psi(x) = \psi(-x) \otimes \psi(-x) \).
In the derivation of auto-correlation and cross-correlation functions,
the balanced correlation scheme was used. The correlation functions
were derived for ideal detectors, i.e. finite position resolution or
similar imperfections were neglected.

In comparing the realisations in Figs. 2 and 3 one notices the
larger abundance of small scale structures in the realisation encoding
the Lorenzian PSF \( c_{Lm}(x) \) in comparison to the realisation derived
for the Gaussian PSF \( c_{Gm}(x) \). This can be explained by the fact that the
power spectrum \( \psi_t(k) \) declines \( \propto k^{-3} \) and thus much slower than the power spectrum \( \psi_c(k) \propto k^{-2} \). Both realisations have been derived with the same random seed, i.e. the relative
phases are identical and one immediately recognises similar structures in \( \psi_c(x) \) and \( \psi_L(x) \).
4.2 Reproducibility of the PSF

An important issue is the reproducibility of a chosen PSF $c(x)$ in realisations generated with differing random seeds. This can be assessed by determining the auto-correlations of the scaled GRFs $\psi(x)$ for all realisations within the data sample. In Fig. 4 the Gaussian target PSF $c_G(x)$ and the auto-correlation functions $\psi_G(x) \otimes \psi_G(x)$ following from two realisations $\psi(x)$ are shown. The error bars denote the sample variance derived from 100 realisations of $\psi_G(x)$ following from different random seeds. The width of the PSF was chosen as $\sigma_x = 8 \sqrt{2}$ pixels for better visibility. Fig. 5 shows the analogous for the Lorenzian target PSF $c_L(x)$ with $\sigma_x = 8 \sqrt{2}$ pixels.

As Figs. 4 and 5 illustrate, the functional shape of the target PSF $c(x)$ can be reproduced with high reliability and the ratio of the peak-height to the correlation noise is $\approx 40$. However, there are minor imaging artefacts, namely very weak sidelobes: This is readily explained by the fact that the Fourier transform of a well localised PSF in real space is extended and affected by the cutoff at the Nyquist frequency $k_{\text{Nyquist}}$, which induces a $\sin(k_{\text{Nyquist}}x)/x$-like modulation. Consequently, the sidelobes are suppressed in PSFs with large $\sigma_x$. The Lorenzian PSF is a bad choice in comparison
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Figure 5. Cross section along the $x$-axis through the central part of the auto-correlation function $\phi'_L(x) \otimes \phi'_L(x)$ for two different realisations $\phi'_L(x)$ (dashed) and the Lorenzian target curve $c_L(x)$ (solid).

Table 1. The mean transparencies $\langle \psi' \rangle$, the variance $\langle \psi'^2 \rangle - \langle \psi' \rangle^2$ and the standard deviation $\sqrt{\langle \psi'^2 \rangle - \langle \psi' \rangle^2}$ together with their respective uncertainties (1σ) for a set of GRFs encoding Gaussian PSFs with differing width $\sigma_x$.

| PSF width | mean transparency $\langle \psi' \rangle$ | variance $\langle \psi'^2 \rangle - \langle \psi' \rangle^2$ | standard deviation $\sqrt{\langle \psi'^2 \rangle - \langle \psi' \rangle^2}$ |
|-----------|------------------------------------------|---------------------------------|---------------------------------|
| $2 \sqrt{2}$ | 1/2 | 0.013 ± 0.002 | 0.116 ± 0.009 |
| $4 \sqrt{2}$ | 1/2 | 0.014 ± 0.002 | 0.121 ± 0.010 |
| $4 \sqrt{2}$ | 1/2 | 0.016 ± 0.003 | 0.128 ± 0.012 |
| $8 \sqrt{2}$ | 1/2 | 0.018 ± 0.003 | 0.135 ± 0.013 |

4.3 Pixel-to-pixel variance

In sensitivity considerations carried out by 't Zand et al. (1994) for purely random masks, i.e. masks consisting of either transparent ($\psi'(x) = 1$) or opaque ($\psi'(x) = 0$) pixels, optimised mean transparency $\langle \psi' \rangle$ and standard deviation $\sqrt{\langle \psi'^2 \rangle - \langle \psi' \rangle^2}$ are derived to be equal to 1. In that way, the variance and therefore the modulation of the signal is maximised. For the GRFs considered here, the variance and hence the modulation of the shadowgram is noticeably smaller. In Table 1 the mean transparencies $\langle \psi' \rangle$, the variance $\langle \psi'^2 \rangle - \langle \psi' \rangle^2$ and the standard deviation $\sqrt{\langle \psi'^2 \rangle - \langle \psi' \rangle^2}$ together with their respective uncertainties for a set of GRFs encoding Gaussian PSFs with differing width $\sigma_x$ are summarised.

One would expect that with increasing PSF width $\sigma_x$ the variance decreases, which would be explained by the fact that the variance is given by a weighted integration over the power spectrum $P(k)$. For increased position resolution, i.e. a narrow PSF $c(x)$, a wide power spectrum $P(k)$ is needed, which in turn would lead to a high variance.

This simple argument however, does not straightforwardly apply to the scaled realisations at hand: As laid down in eqn. (10), the field $\psi(x)$ is modified by a factor depending on the maximal value $|\phi(x)|$ of the particular realisation. The occurrence of a high amplitude is following a Gaussian distribution with variance $\langle \psi'^2 \rangle - \langle \psi' \rangle^2$ and the standard deviation $\sqrt{\langle \psi'^2 \rangle - \langle \psi' \rangle^2}$ together with their respective uncertainties (1σ) for a set of GRFs encoding Gaussian PSFs with differing width $\sigma_x$.

4.4 Distribution of the pixel amplitudes

As Fig. 5 illustrates, the pixel amplitudes $\psi'(x)$ follow a Gaussian distribution, irrespective of the encoded PSF,

$$p(\psi') d\psi' = \frac{1}{\sqrt{2\pi\sigma_\psi}} \exp\left(-\frac{(\psi' - \mu_\psi)^2}{2\sigma_\psi^2}\right) d\psi',$$

as a consequence of the central limit theorem (see Kendall & Stuart (1958)). The mean and variance of that particular realisation have been determined to be $\mu_\psi = 0.5000 \pm 0.001$ and $\sigma_\psi = 0.1277 \pm 0.0007$ at 95% confidence. For illustrative purposes, a Gaussian PSF with $\sigma_x = 2 \sqrt{2}$ pixels has been chosen.

Again, it should be emphasised that the scaling eqn. (10), while being reasonable from the physical point of view, is not conserving Gaussianity. This is for the application at hand not a serious limitation, because the variance of the distribution $p(\psi') d\psi'$ is small compared to 1.

4.5 Partial shadowing

It is interesting to see how partial shadowing affects shape and amplitude of the auto-correlation function. If a source is observed at large off-axis angles, the shadowgram cast by the coded mask onto the detector is incomplete and reconstruction artefacts emerge in the correlation function. In order to examine the extent to which the PSF suffers from partial shadowing, the amplitudes $\psi_c(x)$ in a margin amounting to a fraction of 25%, 50% and 75% of the total area have been set to zero and the cross-correlation function $\psi_c(x) \otimes \psi_{shadow}(x)$ has been determined with the full coded mask.
As Fig. 6 shows for a Gaussian PSF with $\sigma_x = 8$ pixels, the PSF drops in central amplitude according the unshadowed area, but otherwise its shape remains unaltered. A second observation is that the amplitude of the sidelobes is unaffected by the partial shadowing.

The reconstructed PSF $\psi'_{\text{G}}(x) \otimes \psi_{\text{shadow}}(x)$ for the case of radiation from a source situated at large angles away from the optical axis, where only 1/32 of the mask has been imaged onto the detector is depicted in Fig. 8. Even though a tiny part of the mask amounting to $\approx 3\%$ has been imaged, the correlation peak is clearly recognisable and its peak value is a factor $\approx 4$ above the correlation noise.

4.6 Thresholded realisations

Due to possible technical complications in attempting to build a coded mask pattern based on a GRF with quasi-continuous opaqueness, thresholded realisations are considered. A second argument in favour of thresholded realisations would be their achromatic properties, because the mask has to be constructed from the field $\psi(x)$ for a specific photon distribution in order to assure the maximal modulation of the shadowgram cast onto the detector. Yet another argument in favour of thresholded realisations of GRFs is their better sensitivity, because they imprint a stronger modulation of the shadowgram compared to smoothly varying GRFs.

In thresholded realisations, mask elements are taken to be transparent, if the value $\psi(x)$ of the realisation is greater than zero, conversely, for values $\psi(x) < 0$ the mask element is set to be opaque. An example for a thresholded realisation of a GRF and its PSF is given in Fig. 9.

An important issue is the degradation of the PSF $\psi_{\text{G}}^{(t)}(x) \otimes \psi_{\text{G}}^{(t)}(x)$ imposed by the thresholding. As Fig. 10 illustrates, the resulting auto-correlation function is pointy and its kurtosis is larger than zero (leptokurtic). This results from the fact that small scale power is added by the thresholding: In order to construct a step transition, more small-scale Fourier modes are needed, which leads to an additive power law contribution $\propto k^{-2}$ in the power spectrum $P(k)$, such that the power spectrum acquires Lorenzian wings. The point spread function $\psi_{\text{G}}^{(t)}(x) \otimes \psi_{\text{G}}^{(t)}(x)$, being the inverse Fourier transform of $P(k)$, can then be approximated by two decaying branches of an exponential, which readily explains the pointiness. The target Gaussian PSF $\psi_{\text{G}}(x)$ with $\sigma_x = 8\sqrt{2}$ pixels is shown for comparison. Again, the error of the auto-correlation function is estimated by determining the sample variance in 100 realisations.

The size distribution of the patches as a function of threshold value can be described by means of the Press-Schechter theory well known in cosmology. Press & Schechter (1974), Bond et al. (1991) and Mo & White (2002) provide the mathematical foundation.
5 RAY-TRACING SIMULATIONS INCLUDING FINITE PHOTON STATISTICS

Extensive ray-tracing simulations were performed describing the imaging of point sources with a finite number of photons (Sect. 5.1), and the attainable sensitivity in such an observation was assessed (Sect. 5.2). The analogous was carried out for the Gaussian PSF $c_G(x)$ and the auto-correlation function $\psi_G(x) \otimes \psi_G(x)$ (lower panel).

5.1 Simulation setup

In the following, the performance of the coded mask is examined as a function of photon statistics. The statistical significance $\sigma$ of a simulated observation is defined to be

$$\sigma = \frac{N_{\text{source}}}{\sqrt{N_{\text{bg}}}}$$  \hspace{1cm} (13)

where $N_{\text{source}}$ and $N_{\text{bg}}$ denote the source and background count rates, respectively. Here it should be emphasised, that $\sigma$, $N_{\text{source}}$ and $N_{\text{bg}}$ always refer to the number of actually detected photons which makes a difference when considering the coded mask employed in BeppoSAX’s WFI instrument, in which the average transparency is not equal to 1/2.

Observations were simulated by randomly choosing $2N_{\text{source}}$ homogeneously distributed photon impact positions $x$ across the mask face. In order to emulate the random process of photons penetrating the mask, a homogeneously distributed random number $r$ from the interval $r \in [0...1]$ was drawn for each photon, and compared to the value $\psi(x)$ of the GRF at the same position $x$. In the case $r > \psi(x)$ the photon was assumed to be able to penetrate the mask, whereas in the case $r \leq \psi(x)$ the photon was taken to be absorbed by the mask. For BeppoSAX’s mask pattern, which has an average transparency of 1/3, a total number of $3N_{\text{source}}$ photons was simulated.

For the background, which was assumed to be homogeneous, $N_{\text{bg}}$ photon impact positions were determined and the count rates in the corresponding pixels were increased accordingly. Background count rates were fixed to a value of $N_{\text{bg}} = 10^4$ photons, which are typical for an instrument like WFI in a 100 second exposure.

The resulting field $\psi^{\text{(sim)}}$ containing the number of photons that struck a certain pixel was then correlated with the original mask pattern $\psi$, again using balanced correlation. In the next step, the highest peak was localised in the correlated data field $\psi^{\text{(sim)}} \otimes \psi$ and its significance $\Sigma$ was determined by comparing the peak height $a_{\text{max}} = \max(\psi^{\text{(sim)}} \otimes \psi)$ to the level of fluctuations

$$\sigma^2 = \frac{\langle (\psi^{\text{(sim)}} \otimes \psi)^2 \rangle}{\langle (\psi^{\text{(sim)}} \otimes \psi) \rangle^2}$$

in the field. If the peak had a significance $\Sigma \geq a_{\text{max}}/(\sigma_x)$ exceeding 3 and was located at a position which deviated less than half a PSF width from the nominal position, the simulated detection was taken to be successful. A particular realisation of a Gaussian random field was exposed to 100 simulated photon distributions from which the detection probability $p$ (i.e. the occurrence of a $\geq 3\sigma_x$-peak located at the correct position) and the false detection probability $q$ (i.e. the occurrence of a $\geq 3\sigma_q$-peak at a wrong position) was derived. The sample variance in comparing 100 realisations of Gaussian random fields was used to derive er-

![Figure 9](image1.png)

Figure 9. A thresholded realisation $\psi_G^{\text{(th)}}(x)$ of a GRF $\psi_G(x)$ (upper panel) for the Gaussian PSF $c_G(x)$ and the auto-correlation function $\psi_G^{\text{(th)}}(x) \otimes \psi_G^{\text{(th)}}(x)$ (lower panel).

![Figure 10](image2.png)

Figure 10. Deterioration of the auto-correlation function $\psi_G^{\text{(th)}}(x) \otimes \psi_G^{\text{(th)}}(x)$ from two thresholded realisations $\psi_G^{\text{(th)}}(x)$ (dashed) in comparison to the initial Gaussian PSF $c_G(x)$ (solid).
5.2 Point source sensitivity of a set of Gaussian random fields

Fig. 11 shows the detection probability \( p(\sigma) \) and the false detection probability \( q(\sigma) \) as a function of photon statistics, expressed in terms of statistical significance \( \sigma \) for GRFs, a purely random mask, and BeppoSAX’s URA pattern. The source was assumed to lie on the optical axis, i.e., the mask pattern is imaged completely onto the detector. Common to all mask patterns is the fact that \( p \) rises with statistical significance, and that \( q \) drops accordingly. But while reliable observations can be done using the BeppoSAX-pattern or random patterns even at low photon statistics of \( 2 - 3\sigma \), the patterns based on GRFs require high photon fluxes. For them, observations are feasible starting from \( \approx 9\sigma \). The reason why GRFs are less sensitive to the traditional mask pattern is the fact that they imprint a weaker modulation of the shadowgram. Furthermore, one immediately notices the trend that the patterns are more sensitive for wider PSF widths due to the increase in variance of the mask pattern with the source profile. Again, one attains higher sensitivities for wider PSFs in the case of patterns based on GRFs.

Common to all figures is the fact, that the curves \( p(\sigma) \) and \( q(\sigma) \) are not adding up to one, which is caused by the combined criterion where apart from the correct peak position a minimal peak height above the correlation background is required, which is often not fulfilled in the cases of low photon statistics.

5.3 Sensitivity in observations of extended sources

In addition, suitable simulations were carried out in order to assess the performance of GRFs in the observation of extended sources, such as supernova remnants, structures in the Milky Way and clusters of galaxies. Typical sizes of those sources range between arcminutes and a degree. For simplicity, the source was assumed to be described by a Gaussian profile with extension \( \sigma_{\text{profile}} = 2 \) pixels. The shadowgram recorded in observations of extended sources are superpositions of slightly displaced point source shadowgrams, where the relative intensities follow from the source profile. Consequently, the imaging of extended sources is simulated by convolving the mask pattern with the source profile prior to the ray-tracing. Despite that, the image reconstruction has been carried out with the unconvolved mask pattern.

Fig. 13 gives the dependence of the detection probability \( p \) and the corresponding \( q \) on the photon counting statistic \( \sigma \). In the observation of extended sources, the patterns based on GRFs are superior to the traditional approaches: While reliable detections can be achieved starting from \( \sigma \gtrsim 10 \) (for \( \sigma_{\text{profile}} = 4 \sqrt{2} \)) up to \( \sigma \gtrsim 20 \) (for \( \sigma_{\text{profile}} = 2 \)), the performance of the traditional masks is notably worse. At the examined levels of photons statistics, the detection probability \( p \) stays close to zero and shows but a shallow increase with \( \sigma \) in the case of BeppoSAX’s URA pattern.

The good performance of the GRFs, and their decreasing performance with correlation length, i.e. PSF width \( \sigma_x \), is of course to be traced back to the fact, that mask patterns with large structures are less affected by the convolution with the source profile than mask patterns exhibiting small structures; in the extreme case of random masks or BeppoSAX’s pattern, the structures are washed out and consequently, the modulation of the shadowgram is very weak. This can be circumvent, however, by tuning the angular size of a mask pixel to match the angular size of the source to be observed.

5.4 Field-of-view in the observation of point sources

Now, the size of the field-of-view, i.e. the minimal fraction of the mask pattern required to be imaged onto the detector in order to
yield a significant detection peak is investigated. For that purpose, the point source detection probability \( p \) and the false detection probability \( q \) are considered to be functions of the obscuration \( Q \), which is defined as the fraction of the mask area imaged onto the detector. The number of background photons was kept fixed to be \( N_{bg} = 10^7 \), while the number of source photons \( N_{source} \) was diminished by this factor of \( Q \) prior to the ray-tracing. Their number was fixed to yield a significance of \( \sigma = 20 \) for \( Q = 1 \), i.e. for the case of complete imaging. The background photons were assumed to be homogeneously distributed. The simulation and the derivation of the values for \( p(Q) \) and \( q(Q) \) were carried out in complete analogy to Sect. 5.2.

The results are depicted in Fig. 13. While the traditional patterns show a good performance and have a high detection probability \( p(Q) \) for values of \( Q \geq 0.1 \) (BeppoSAX’s pattern) and \( Q \geq 0.2 \) (random mask), the GRFs fall behind significantly in performance. Imaging is only possible in the cases where a fraction of at least \( Q = 0.5 \ldots 0.6 \) of the mask has been imaged onto the detector, resulting in a decrease of the field-of-view of about a factor of \( 3 \ldots 5 \), which renders the usage of GRFs very unlikely in survey missions. Again, the GRF patterns encoding wide PSFs are more sensitive and yield larger fields-of-view than GRFs with narrow PSFs.

6 SUMMARY AND OUTLOOK

In this article, a new algorithm for generating coded masks is presented that allows the construction of a mask with defined imaging properties, i.e. point spread functions.

- The viability of constructing a coded mask for a predefined PSF as a realisation of a GRF has been shown. For realisations generated with differing random seeds, the shape of the PSF is reproducible with high accuracy. Due to the reproducibility of the PSF, the parameter space is greatly reduced and the necessity of running extensive Monte-Carlo simulations is alleviated.

- The generation of 2-dimensional URA patterns requires the number of pixels in each direction to be nonmatches of the maximal values of the realisation, however, is smaller in the case of GRFs compared to purely random fields, which results in a weaker modulation of the shadogram and hence the sensitivity is expected to be smaller. The variance shows the trend of decreasing with increasing PSF width, which is caused by the scaling with the maximal values of the realisation.

- Coded masks based on GRFs are chromatic in contrast to purely random fields: The mask pattern has to be designed for a specific spectral distribution of photons due to semi-transparent mask elements. Any mismatch in the photon spectrum of a source will be observed and hence the sensitivity is expected to be smaller. The variance shows the trend of decreasing with increasing PSF width, which is caused by the scaling with the maximal values of the realisation.

- Ray-tracing simulations including finite photons statistics and background noise show, that the sensitivity of GRFs falls behind that of purely random masks and URA patterns like the one employed in BeppoSAX by a factor of \( 2 \ldots 3 \) in the observation of point sources, depending on PSF width. For GRFs, the sensitivity was found to depend exponentially on PSF width, one is trading sensitivity for position resolution.
• The sensitivity of patterns based on GRFs is significantly better in the observation of extended sources because their comparably large structures are less affected by the convolution with the source profile than traditional masks that possess pronounced structures on small scales.

• Finally, the size of the field-of-view of GRFs in comparison to traditional masks is examined. It is found that reliable imaging can only performed with GRFs, if a large fraction of the mask is imaged onto the detector. In contrast, purely random masks and especially BeppoSAX’s URA pattern enable imaging at large off-axis angles. Comparing the resulting fields-of-view for the preset number of photons shows, that the field-of-view of patterns based on GRFs are smaller by a factor of 3 . . . 5 (depending on PSF width).

Although the shortcomings of Gaussian random fields with respect to point source sensitivity, chromaticity and localisation accuracy make their usage in observing point sources doubtful, they may find application in observations of extended sources, while simultaneously providing a moderate performance in the observation of point sources. Coded mask patterns on the basis of GRFs may be aesthetically pleasing because they utilise an abstract cosmological concept for a technological application.

ACKNOWLEDGMENTS

I would like to thank Matthias Bartelmann and Christoph Pfrommer for careful reading of the manuscript, Eugene Churazov for constructive and valuable comments and Saleem Zaroubi for many clarifying discussions. Furthermore, I am grateful to Kazuo Makishima for giving me details of the WFI experiment on BeppoSAX.

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