Eigenvalues of Sturm Liouville problems with discontinuity conditions inside a finite interval

by

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Abstract — In this work, we use the regularized sampling method to compute the eigenvalues of Sturm Liouville problems with discontinuity conditions inside a finite interval. We work out an example by computing a few eigenvalues and their corresponding eigenfunctions.

Keywords: Sturm-Liouville Problems, discontinuity conditions, Shannon’s sampling theory, Regularized Sampling Method, Whittaker-Shannon-Kotel’nikov theorem
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1 Introduction

In [3], this author introduced the regularized sampling method. A method which is based on Shannon’s sampling theory but applied to regularized functions, hence avoiding any (multiple) integration(s) and keeping the number of terms in the cardinal series manageable. It has been demonstrated that the method is capable of delivering higher order estimates of the eigenvalues at a very low cost. The purpose in this paper is to extend the domain of application of this method to the problem at hand.

2 Main results

Consider the following Sturm-Liouville problem with discontinuity inside a finite interval,

\[-y'' + q(x)y = \mu^2 y, \quad x \in (0, \pi)\]
\[y'(0) = 0 = y(\pi)\]
\[y(d + 0) = ay(d - 0)\]
\[y'(d + 0) = a^{-1}y'(d - 0)\]

(2.1)

where \(a > 0, \ a \neq 1, \ 0 < d < \pi\) and \(q \in L^2(0, \pi)\).

It has been shown in [1] that the problem has a point spectrum and each eigenvalue has multiplicity one and accumulate only at \(+\infty\). The purpose in this paper is to compute the eigenvalues of (2.1) with the minimum of effort and a greater precision using the newly introduced regularized sampling method [3], an improvement on the method based on sampling theory introduced in [2].

Let \(y_L(x, \mu)\) and \(y_R(x, \mu)\), be the solutions of the base problems

\[-y''_L + q(x)y_L = \mu^2 y_L, \quad x \in [0, d]\]
\[y_L(0) = 1, \ y'_L(0) = 0\]

(2.2)
and
\[
\begin{cases}
    -y''_R + q(x)y_R = \mu^2 y_R, & x \in [d, \pi] \\
y_R(\pi) = 0, & y'_R(\pi) = 1
\end{cases}
\] (2.3)
respectively.

Using \(y'(\pi, \mu) = \alpha\), an unknown constant to be determined along the eigenvalue parameter \(\mu^2\), the discontinuity conditions give
\[
\begin{align*}
    \alpha y_L(d, \mu) &= a y_R(d, \mu) \\
    \alpha y'_L(d, \mu) &= a^{-1} y'_R(d, \mu)
\end{align*}
\] (2.4)

Note that \(\alpha \neq 0\) since otherwise, \(y_L(d, \mu) = y'_L(d, \mu) = 0\) leading to \(y_L(x, \mu) \equiv 0\) which contradicts \(y_L(0, \mu) = 1\). A necessary and sufficient condition for non trivial solutions is \(\Delta(\mu) = 0\) where the characteristic function \(\Delta\) is defined by,
\[
\Delta(\mu) = \begin{vmatrix}
    a y_L(d, \mu) & y_R(d, \mu) \\
    a^{-1} y'_L(d, \mu) & y'_R(d, \mu)
\end{vmatrix}
\] (2.5)

Thus the eigenvalues of the problem at hand are seen as the square of the zeroes of \(\Delta\). Let \(\mu_k\) be such a zero, \(\alpha\) will take the value
\[
\alpha_k = \frac{a y_L(d, \mu_k)}{y_R(d, \mu_k)}
\] (2.6)
The eigenfunction associated to the simple eigenvalue \(\mu_k\) is defined by
\[
y_{[k]}(x, \mu_k) = \begin{cases}
    y_L(x, \mu_k) & \text{if } 0 \leq x \leq d \\
    \alpha_k y_R(x, \mu_k) & \text{if } d \leq x \leq \pi
\end{cases}
\] (2.7)

In the following we shall present some important properties of the functions \(y_L(x, \mu)\) and \(y_R(x, \mu)\).

We shall need the following well known lemmata,

**Lemma 2.1** \(\sin z/z\) and \(\cos z\) are entire as functions of \(z\) and satisfy the estimates
\[
|\sin z/z| \leq \gamma_0 e^{\gamma_1 |z|/(1 + |z|)} \text{ and } |\cos z| \leq e^{\gamma_2 |z|}
\]
where \(\gamma_0 = 1.72\).

**Lemma 2.2** If \(\beta\) is a positive constant and \(\varphi\) is a positive function satisfying,
\[
z(x) \leq \beta + \int_x^\pi \varphi(t) z(t) dt
\]
then
\[
z(x) \leq \beta e^{\int_x^\pi \varphi(t) dt}
\]

**Theorem 2.3** \(y_L(x, \mu)\), \(y'_L(x, \mu)\), \(y_L(x, \mu) - \cos \mu x\), \(y'_L(x, \mu) + \mu \sin \mu x\) \(\in PW_x\) as functions of \(\mu\) for each fixed \(x \in (0, d]\) and satisfy the growth conditions,
\[
\begin{align*}
    |y_L(x, \mu)| &\leq \gamma_1 e^{\gamma_2 |x|\mu} \\
    |y_L(x, \mu) - \cos \mu x| &\leq \frac{\gamma_3}{1 + |\mu|} e^{\gamma_4 |x|\mu} \\
    |y'_L(x, \mu) + \mu \sin \mu x - \int_0^x \cos \mu (x - t) q(t) \cos(\mu t) dt| &\leq \frac{\gamma_5}{1 + |\mu|} e^{\gamma_6 |x|\mu} \\
    |y'_L(x, \mu) + \mu \sin \mu x| &\leq \gamma_7 e^{\gamma_8 |x|\mu}
\end{align*}
\]
\[ y_R(x, \mu), y'_R(x, \mu) + \frac{\sin \mu (x - \pi)}{\mu}, y'_R(x, \mu) - \cos \mu (\pi - x) \in PW_{\pi - x} \text{ as functions of } \mu \text{ for each fixed } x \in [d, \pi) \text{ and satisfy the growth conditions,} \]

\[
|y_R(x, \mu)| \leq \frac{\gamma_5}{1 + |\mu| \pi} e^{(\pi-x)|\text{Im}\mu|} \\
\left| y_R(x, \mu) + \frac{\sin \mu (\pi - x)}{\mu} \right| \leq \frac{\gamma_6}{(1 + |\mu| \pi)^2} e^{(\pi-x)|\text{Im}\mu|} \\
|y'_R(x, \mu) - \cos \mu (\pi - x)| \leq \frac{\gamma_7}{1 + |\mu| \pi} e^{(\pi-x)|\text{Im}\mu|}
\]

where \( \gamma_1, \ldots, \gamma_7 \) are some positive constants.

**Proof:** For \( y_L \) we have,

\[
y_L(x, \mu) = \cos \mu x + \int_0^x \frac{\sin \mu (x - t) q(t) y_L(t, \mu)}{\mu} dt \quad (2.8)
\]

and

\[
y'_L(x, \mu) = -\mu \sin \mu x + \int_0^x \cos \mu (x - t) q(t) y_L(t, \mu) dt. \quad (2.9)
\]

Standard arguments show that \( y_L, y'_L \) are entire functions of \( \mu \) for each \( x \in (0, d] \) and belong to \( PW_x \) for each \( x \in (0, d] \) as function of \( \mu \).

\[
|y_L(x, \mu)| \leq e^{x|\text{Im}\mu|} + \int_0^x \gamma_0 \pi e^{(x-t)|\text{Im}\mu|} |q(t)| \cdot |y_L(t, \mu)| dt \\
\]

\[
e^{-x|\text{Im}\mu|} |y_L(x, \mu)| \leq 1 + \int_0^x \gamma_0 \pi |q(t)| \cdot e^{-t|\text{Im}\mu|} |y_L(t, \mu)| dt
\]

using Gronwall’s lemma gives,

\[
e^{-x|\text{Im}\mu|} |y_L(x, \mu)| \leq \exp \left\{ \gamma_0 \pi \int_0^x |q(t)| \, dt \right\} \leq \exp \left\{ \gamma_0 \pi \int_0^\pi |q(t)| \, dt \right\}
\]

so that,

\[
|y_L(x, \mu)| \leq \exp \left\{ \gamma_0 \pi \int_0^\pi |q(t)| \, dt \right\} e^{x|\text{Im}\mu|} = \gamma_1 e^{x|\text{Im}\mu|} \quad (2.10)
\]

where \( \gamma_1 = \exp \left\{ \gamma_0 \pi \int_0^\pi |q(t)| \, dt \right\} \).

\[
y_L(x, \mu) - \cos \mu x = \int_0^x \frac{\sin \mu (x - t)}{\mu} q(t) y_L(t, \mu) dt \quad (2.11)
\]
where \( \gamma_2 = \pi \gamma_0 \gamma_1 \int_0^\pi |q(t)| \, dt \).

Now,

\[
y'_L(x, \mu) + \mu \sin \mu x - \int_0^x \cos \mu (x-t)q(t) \cos (\mu t) \, dt = \int_0^x \cos \mu (x-t)q(t) \{ y_L(t, \mu) - \cos \mu t \} \, dt \tag{2.15}
\]

so that,

\[
\left| y'_L(x, \mu) + \mu \sin \mu x - \int_0^x \cos \mu (x-t)q(t) \cos (\mu t) \, dt \right|
\leq \int_0^x e^{(x-t)|\text{Im}\mu|} |q(t)| \left| y_L(t, \mu) - \cos \mu t \right| \, dt
\leq e^{x|\text{Im}\mu|} \frac{\gamma_2}{1 + |\mu| \pi} \int_0^\pi |q(t)| \, dt = \frac{\gamma_3}{1 + |\mu| \pi} e^{x|\text{Im}\mu|} \tag{2.16}
\]

where \( \gamma_3 = \gamma_2 \int_0^\pi |q(t)| \, dt \).

\[
|y'_L(x, \mu) + \mu \sin \mu x| = \left| \int_0^x \cos \mu (x-t)q(t)y_L(t, \mu) \, dt \right|
\leq \gamma_1 e^{x|\text{Im}\mu|} \int_0^\pi |q(t)| \, dt = \gamma_4 e^{x|\text{Im}\mu|} \tag{2.17}
\]

where \( \gamma_4 = \gamma_1 \int_0^\pi |q(t)| \, dt \).

As for \( y_R \) we have,

\[
y_R(x, \mu) = \frac{-\sin \mu (\pi - x)}{\mu} - \int_x^\pi \frac{\sin \mu (x-t)}{\mu} q(t)y_R(t, \mu) \, dt \tag{2.19}
\]

and standard arguments show that \( y_R \) and \( y'_R \) are entire functions of \( \mu \) for each \( x \in [d, \pi) \) and belong to \( PW_{\pi-x} \) for each \( x \in [d, \pi) \) as function of \( \mu \). Also,

\[
y_R(x, \mu) + \frac{\sin \mu (\pi - x)}{\mu} = -\int_x^\pi \frac{\sin \mu (x-t)}{\mu} q(t)y_R(t, \mu) \, dt \tag{2.20}
\]

and

\[
y'_R(x, \mu) - \cos \mu (\pi - x) = \int_x^\pi \cos \mu (x-t)q(t)y_R(t, \mu) \, dt \tag{2.21}
\]

are entire functions of \( \mu \) for each \( x \in [d, \pi) \) and belong to \( PW_{\pi-x} \) for each \( x \in [d, \pi) \) as function of \( \mu \).
Using the Lemma 2.1, \[2.19\] gives,
\[
|y_R(x, \mu)| \leq \gamma_0 \sqrt{\frac{(\pi - x)}{1 + |\mu|} e^{(\pi - x)|\Im \mu|} + \int_x^\pi \gamma_0 e^{(t-x)|\Im \mu|} |t-x| |q(t)| \cdot |y_R(t, \mu)| dt \tag{2.22}
\]
\[
e^{-(\pi - x)|\Im \mu|} |y_R(x, \mu)| \leq \gamma_0 \frac{\pi}{1 + |\mu|} + \int_x^\pi \gamma_0 |q(t)| \cdot e^{-(\pi - t)|\Im \mu|} |y_R(t, \mu)| dt \tag{2.23}
\]
Gronwall’s inequality (Lemma 2.2) yields,
\[
e^{-(\pi - x)|\Im \mu|} |y_R(x, \mu)| \leq \gamma_0 \frac{\pi}{1 + |\mu|} \exp \left\{ \gamma_0 \int_x^\pi |q(t)| dt \right\} \tag{2.24}
\]
from which we get,
\[
|y_R(x, \mu)| \leq \gamma_0 \frac{\pi}{1 + |\mu|} \exp \left\{ \gamma_0 \int_x^\pi |q(t)| dt \right\} e^{(\pi - x)|\Im \mu|} \leq \frac{\gamma_5}{1 + |\mu|} e^{(\pi - x)|\Im \mu|} \tag{2.25}
\]
where \( \gamma_5 = \gamma_0 \pi \exp \left\{ \gamma_0 \int_0^\pi |q(t)| dt \right\} \cdot \]
\[
y_R(x, \mu) + \frac{\sin \mu(\pi - x)}{\mu} = - \int_x^\pi \frac{\sin \mu(x-t)}{\mu} q(t) y_R(t, \mu) dt
\]
\[
\left| y_R(x, \mu) + \frac{\sin \mu(\pi - x)}{\mu} \right| \leq \left\{ \gamma_0 \frac{\pi}{1 + |\mu|} \frac{\gamma_5}{1 + |\mu|} \int_0^\pi |q(t)| dt \right\} e^{(\pi - x)|\Im \mu|} = \frac{\gamma_6}{(1 + |\mu|)^2} e^{(\pi - x)|\Im \mu|} \tag{2.26}
\]
where \( \gamma_6 = \gamma_0 \gamma_5 \int_0^\pi |q(t)| dt \). Similarly,
\[
|y'_R(x, \mu) - \cos \mu(\pi - x)| \leq \int_x^\pi e^{(t-x)|\Im \mu|} |q(t)| \cdot |y_R(t, \mu)| dt
\]
\[
\leq \left\{ \frac{\gamma_5}{1 + |\mu|} \int_0^\pi |q(t)| dt \right\} e^{(\pi - x)|\Im \mu|}
\]
\[
= \frac{\gamma_7}{1 + |\mu|} e^{(\pi - x)|\Im \mu|} \tag{2.27}
\]
where \( \gamma_7 = \gamma_5 \int_0^\pi |q(t)| dt \).

Thus, \( y_R(x, \mu), y_R(x, \mu) + \frac{\sin \mu(\pi - x)}{\mu}, y'_R(x, \mu) - \cos \mu(\pi - x) \) are entire functions of \( \mu \) for each \( x \in [d, \pi] \) and belong to \( PW_{\pi-x} \) for each \( x \in [d, \pi] \) as function of \( \mu \).

Although we have obtained much higher estimates in \[3\] and \[4\] to the expense of subtracting terms involving multiple integrals, and as in \[3\], we shall stick with the estimates given in Theorem 2.2, hence avoiding any (multiple) integration(s) and show by the same token that we can get a higher order estimate of the eigenvalues of the problem at hand at a very low cost. In fact we do not have even to keep on increasing the number of sampling points.

Let \( PW_\sigma \) denote the Paley-Wiener space
\[
PW_\sigma = \{ f \text{ entire}, |f(\mu)| \leq C e^{\sigma |\Im \mu|}, \int_R |f(\mu)|^2 \, d\mu < \infty \}
\]
Let $h_{kl}$ be defined by

\[
\begin{align*}
    h_{11}(\mu) &= \left(\frac{\sin \theta \mu}{\theta \mu}\right)^m (y_L(d, \mu) - \cos \mu d) \\
    h_{12}(\mu) &= \left(\frac{\sin \theta \mu}{\theta \mu}\right)^m (y'_L(d, \mu) + \mu \sin \mu d) \\
    h_{21}(\mu) &= \left(\frac{\sin \theta \mu}{\theta \mu}\right)^m \left(y_R(d, \mu) + \frac{\sin \mu(\pi - d)}{\mu}\right) \\
    h_{22}(\mu) &= \left(\frac{\sin \theta \mu}{\theta \mu}\right)^m (y'_R(d, \mu) - \cos \mu(\pi - d))
\end{align*}
\]

Then we rewrite $y_L(d, \mu)$, $y'_L(d, \mu)$, $y_R(d, \mu)$ and $y'_R(d, \mu)$ as

\[
\begin{align*}
    y_L(d, \mu) &= h_{11}(\mu) \left(\frac{\sin \theta \mu}{\theta \mu}\right)^{-m} + \cos \mu d \\
    y'_L(d, \mu) &= h_{12}(\mu) \left(\frac{\sin \theta \mu}{\theta \mu}\right)^{-m} - \mu \sin \mu d \\
    y_R(d, \mu) &= h_{21}(\mu) \left(\frac{\sin \theta \mu}{\theta \mu}\right)^{-m} - \frac{\sin \mu(\pi - d)}{\mu} \\
    y'_R(d, \mu) &= h_{22}(\mu) \left(\frac{\sin \theta \mu}{\theta \mu}\right)^{-m} + \cos \mu(\pi - d)
\end{align*}
\]

**Theorem 2.4** Let $\vartheta$ be a positive constant and $m$ be a positive integer ($m \geq 2$). The functions $h_{kl}$, $(k, l = 1, 2)$ belong to the Paley space $PW_\sigma$ with $\sigma = \sigma_0 + m\vartheta$ and satisfy the estimates

\[
|h_{kl}(\mu)| \leq \frac{\gamma}{(1 + |\mu|)^m} e^{\sigma}|\mu|
\]

$k, l = 1, 2$ for some positive constant $\gamma$, $\sigma_0 = \max \{d, \pi - d\}$.

**Proof:** That $h_{kl}$ are entire and satisfy the given estimates is a direct consequence of Theorem 2.2 and the fact that $\frac{\sin \theta \mu}{\theta \mu}$ is an entire function of $\mu$ and satisfy the estimate in Lemma 2.1.

Since the $h_{kl}(\mu)$ belong to the Paley-Wiener space $PW_\sigma$ for each $k, l = 1, 2$, they can be recovered from their values at the points $\mu_j = j\frac{\pi}{\sigma}$, $j \in \mathbb{Z}$, using the following celebrated theorem,

**Theorem 2.5 (Whitaker-Shannon-Kotel’nikov)** Let $h \in PW_\sigma$, then

\[
h(\mu) = \sum_{j=-\infty}^{\infty} h(\mu_j) \frac{\sin \sigma(\mu - \mu_j)}{\sigma(\mu - \mu_j)}
\]

$\mu_j = j\frac{\pi}{\sigma}$. The series converges absolutely and uniformly on compact subsets of $C$ and in $L^2_{\Omega}(R)$.

For all practical purposes, we consider finite summations, therefore we need to approximate $h_{kl}$ by a truncated series $h_{kl}^{[N]}$. The following lemma gives an estimate for the truncation error.

**Lemma 2.6 (Truncation error)** Let $h_{kl}^{[N]}(\mu) = \sum_{j=-N}^{N} h_{kl}(\mu_j) \frac{\sin \sigma(\mu - \mu_j)}{\sigma(\mu - \mu_j)}$ denote the truncation of $h_{kl}(\mu)$. Then, for $|\mu| < N\pi/\sigma$,

\[
|\hat{h}_{kl}(\mu) - h_{kl}^{[N]}(\mu)| \leq \frac{|\sin \mu|c_2}{\pi(\pi/\sigma)^m - 4^{-m+1}} \left[ 1 + 1 \right] \frac{1}{(N + 1)^{m-1}},
\]

where $c_3 = ||\mu^{m-1}h_{kl}(\mu)||_2$. 


Proof: Since \( \mu^{m-1} h_{kl}(\mu) \in L^2(-\infty, \infty) \), Jagerman’s result (see [11], Theorem 3.21, p.90) is applicable and yields the given estimate for the \( h_{kl}, k, l = 1, 2 \).

An approximation \( B_N \) to the characteristic function \( B \) is provided by replacing the \( h_{kl} \) by its approximation \( h_{kl}^{[N]} \), and we obtain at once,

**Lemma 2.7** The approximate characteristic function \( B_N \) satisfies the estimate,

\[
|B(\mu) - B_N(\mu)| \leq \left| \frac{\sin \theta \mu}{\theta \mu} \right|^{-m} \frac{|\sin \mu| c_4}{\pi (\pi/\sigma)^{m-1} \sqrt{1 - 4^{-m+1}}} \left[ \frac{1}{\sqrt{N\pi/\sigma} - \mu} + \frac{1}{\sqrt{N\pi/\sigma} + \mu} \right] \frac{1}{(N + 1)^{m-1}},
\]

for some positive constant \( c_4 \).

We claim the following,

**Theorem 2.8** Let \( \mu^2 \) be an exact eigenvalue of \( B \) (multiplicity 1) and denote by \( \mu_N^2 \) the corresponding approximation of a square of a zero of \( B_N \). Then, for \( |\mu_N| < N\pi/\sigma \), we have,

\[
|\mu - \mu_N| \leq \left( \frac{1}{|B'(\tilde{\mu})|} \left| \frac{\sin \theta \mu_N}{\theta \mu_N} \right|^{-m} \frac{|\sin \mu_N| c_4}{\pi (\pi/\sigma)^{m-1} \sqrt{1 - 4^{-m+1}}} \right) \times \left[ \frac{1}{\sqrt{N\pi/\sigma} - \mu_N} + \frac{1}{\sqrt{N\pi/\sigma} + \mu_N} \right] \frac{1}{(N + 1)^{m-1}}
\]

where the inf is taken over a ball centered at \( \mu_N \) with radius \( |\mu_N - \tilde{\mu}| \) and not containing a multiple of \( \pi/\theta \).

Proof Since \( \mu \) is a simple zero of \( B \), we have,

\[
B(\mu) - B(\mu_N) = (\mu - \mu_N) B'(\tilde{\mu})
\]

for some \( \tilde{\mu} \). Thus,

\[
|\mu - \mu_N| = \frac{|B(\mu) - B(\mu_N)|}{|B'(\tilde{\mu})|} \leq \frac{1}{|B'(\tilde{\mu})|} \left| \frac{\sin \theta \mu_N}{\theta \mu_N} \right|^{-m} \frac{|\sin \mu_N| c_4}{\pi (\pi/\sigma)^{m-1} \sqrt{1 - 4^{-m+1}}} \times \left[ \frac{1}{\sqrt{N\pi/\sigma} - \mu_N} + \frac{1}{\sqrt{N\pi/\sigma} + \mu_N} \right] \frac{1}{(N + 1)^{m-1}}
\]

where the inf is taken over a ball centered at \( \mu_N \) with radius \( |\mu_N - \tilde{\mu}| \) and not containing a multiple of \( \pi/\theta \). Thus, the result.
3 Numerical examples

In this section, we shall work out an example to illustrate our method. We shall take $N = 40$, $m = 6$. We have taken $\theta = \sigma_0/(N - m)$ in order to avoid the first singularity of $\left(\sin \theta \mu N \right)^{-1}$. The sampling values were obtained using the Fehlberg 4-5 order Runge-Kutta method.

**Example 3.1**

\[
\begin{cases}
-y'' + q(x)y = \mu^2 y, & x \in (0, \pi) \\
y'(0) = 0 = y(\pi) \\
y(d + 0) = ay(d - 0) \\
y'(d + 0) = a^{-1}y'(d - 0)
\end{cases}
\]  

(3.1)

where $a = 2$, $d = 1$ and $q(x) = x$.

| Index | Exact       | RSM         | Absolute Error | Relative Error     |
|-------|-------------|-------------|----------------|--------------------|
| 1     | 1.22788546912 | 1.227885469249 | $1.31357098 \times 10^{-10}$ | $1.06978298879 \times 10^{-10}$ |
| 2     | 1.83749384727 | 1.837493847255 | $1.6491398678 \times 10^{-11}$ | $8.9749408974 \times 10^{-12}$ |
| 3     | 2.68396812434 | 2.683968124476 | $1.32813906505 \times 10^{-10}$ | $4.94841594058 \times 10^{-11}$ |
| 4     | 3.85661744715 | 3.856617447367 | $2.1294955259 \times 10^{-10}$ | $5.5216647358 \times 10^{-11}$ |

The graphs of the first four exact eigenfunctions and their approximations are displayed in Figure 1. The inner products of any two different approximate eigenfunctions are of the order of $10^{-10}$.

4 Conclusion

In this paper, we have used the *regularized sampling method* introduced recently \[3\] to compute the eigenvalues of Sturm-Liouville problems with discontinuity conditions inside a finite interval. We recall that this method constitutes an improvement upon the method based on Shannon sampling theory introduced in \[2\] since it uses a regularization avoiding any multiple integration. The method allows us to get higher order estimates of the eigenvalues at a very low cost. We have presented an example to illustrate the method and compared the computed eigenvalues with the exact ones. We have also computed the approximate eigenfunctions and compared them with the exact eigenfunctions and checked that the inner products of any two different approximate eigenfunctions are of the order of $10^{-10}$, a point which validates our method. We shall present in a future paper a generalization of the above result together with extensive numerical computations.

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First four eigenfunctions and their approximations

\begin{align*}
\text{#1} & \quad y \quad x \\
\text{#2} & \quad y \quad x \\
\text{#3} & \quad y \quad x \\
\text{#4} & \quad y \quad x
\end{align*}