LENGTHS, AREA AND MODULUS OF CONTINUITY OF SOME
CLASSES OF COMPLEX-VALUED FUNCTIONS

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ABSTRACT. In this paper, we discuss the modulus of continuity of solutions to
Poisson’s equation, and give bounds of length and area distortion for some classes
of $K$-quasiconformal mappings satisfying Poisson’s equations. The obtained
results are the extension of the corresponding classical results.

1. Preliminaries and main results

We use $\mathbb{C}$ to denote the complex plane. For $a \in \mathbb{C}$ and $r > 0$, let $D(a, r) = \{ z : |z - a| < r \}$, $D_r = D(0, r)$ and $D = D_1$, the open unit disk in $\mathbb{C}$. Let $\mathbb{T} = \partial D$ be the
boundary of $D$. Furthermore, we denote by $C^m(\Omega)$ the set of all complex-valued $m$-
times continuously differentiable functions from $\Omega$ into $\mathbb{C}$, where $\Omega$ is a subset of $\mathbb{C}$
and $m \in \{0, 1, 2, \ldots \}$. In particular, $C(\Omega) := C^0(\Omega)$ denotes the set of all continuous
functions in $\Omega$. Let $G$ be a domain of $\mathbb{C}$, and let $\overline{G}$ be the closure of $G$. We use
$d_G(z)$ to denote the Euclidean distance from $z$ to the boundary $\partial G$ of $G$. Especially,
we always use $d(z)$ to denote the Euclidean distance from $z$ to the boundary of $D$.

For a real $2 \times 2$ matrix $A$, we use the matrix norm

$$
\| A \| = \sup \{|Az| : |z| = 1\}
$$

and the matrix function

$$
\lambda(A) = \inf \{|Az| : |z| = 1\}.
$$

For $z = x + iy \in \mathbb{C}$, the formal derivative of a complex-valued function $f = u + iv$
is given by

$$
D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},
$$

so that

$$
\| D_f \| = |f_z| + |f_\bar{z}| \quad \text{and} \quad \lambda(D_f) = |f_z| - |f_\bar{z}|,
$$

where

$$
f_z = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_\bar{z} = \frac{1}{2}(f_x + if_y).
$$

We use

$$
J_f := \det D_f = |f_z|^2 - |f_\bar{z}|^2
$$

to denote the Jacobian of $f$.
For \( z, w \in \mathbb{D} \), let
\[
G(z, w) = \log \left| \frac{1 - z \overline{w}}{z - w} \right|
\]
and
\[
P(z, e^{i\theta}) = \frac{1 - |z|^2}{1 - ze^{-i\theta}}
\]
denote the Green function and (harmonic) Poisson kernel, respectively, where \( \theta \in [0, 2\pi] \).

Let \( \psi : \mathbb{T} \to \mathbb{C} \) be a bounded integrable function and let \( g \in C(\overline{\mathbb{D}}) \). For \( z \in \mathbb{D} \), the solution to the Poisson’s equation
\[
\Delta f(z) = g(z)
\]
satisfying the boundary condition \( f|_{\mathbb{T}} = \psi \in L^1(\mathbb{T}) \) is given by
\[
f(z) = P[\psi](z) - G[g](z),
\]
where
\[
G[g](z) = \frac{1}{2\pi} \int_{\mathbb{D}} G(z, w)g(w)dA(w), \quad P[\psi](z) = \frac{1}{2\pi} \int_{0}^{2\pi} P(z, e^{it})\psi(e^{it})dt,
\]
and \( dA(w) \) denotes the Lebesgue measure on \( \mathbb{D} \). It is well known that if \( \psi \) and \( g \) are continuous in \( \mathbb{T} \) and in \( \overline{\mathbb{D}} \), respectively, then \( f = P[\psi] - G[g] \) has a continuous extension \( \tilde{f} \) to the boundary, and \( \tilde{f} = \psi \) in \( \mathbb{T} \) (see [12, pp. 118-120] and [13, 14]).

A continuous increasing function \( \omega : [0, \infty) \to [0, \infty) \) with \( \omega(0) = 0 \) is called a majorant if \( \omega(t)/t \) is non-increasing for \( t > 0 \). Given a subset \( \Omega \) of \( \mathbb{C} \), a function \( f : \Omega \to \mathbb{C} \) is said to belong to the Lipschitz space \( \mathcal{L}_\omega(\Omega) \) if there is a positive constant \( C \) such that
\[
|f(z) - f(w)| \leq C\omega(|z - w|) \quad \text{for all} \quad z, w \in \Omega.
\]
For \( \delta_0 > 0 \), let
\[
\int_{\delta}^{2\delta} \omega(t)\frac{dt}{t} \leq C \omega(\delta), \quad 0 < \delta < \delta_0,
\]
and
\[
\delta \int_{\delta}^{2\delta} \frac{\omega(t)}{t^2} dt \leq C \omega(\delta), \quad 0 < \delta < \delta_0,
\]
where \( \omega \) is a majorant and \( C \) is a positive constant.

A majorant \( \omega \) is said to be regular if it satisfies the conditions (1.6) and (1.7) (see [8, 9, 20]).

Let \( G \) be a proper subdomain of \( \mathbb{C} \). We say that a function \( f \) belongs to the local Lipschitz space \( \text{loc}\mathcal{L}_\omega(G) \) if (1.5) holds, with a fixed positive constant \( C \), whenever \( z \in G \) and \( |z - w| < \frac{1}{2}d_G(z) \) (cf. [10, 16]). Moreover, \( G \) is said to be a
$L_\omega$-extension domain if $L_\omega(G) = \text{loc}L_\omega(G)$. The geometric characterization of $L_\omega$-extension domains was given by Gehring and Martio [10]. Then Lappalainen [16] generalized their characterization, and proved that $G$ is a $L_\omega$-extension domain if and only if each pair of points $z, w \in G$ can be joined by a rectifiable curve $\gamma \subset G$ satisfying

$$\int_\gamma \frac{\omega(d_G(\zeta))}{d_G(\zeta)} \, ds(\zeta) \leq C\omega(|z - w|)$$

with some fixed positive constant $C = C(G, \omega)$, where $ds$ is the arc length measure on $\gamma$. Furthermore, Lappalainen [16, Theorem 4.12] showed that $L_\omega$-extension domains exist only for majorants $\omega$ satisfying (1.6).

The following result is the classical Hardy-Littlewood type Theorem for analytic functions with respect to the majorant $\omega(t) = \omega_\alpha(t) = t^\alpha$ ($0 < \alpha \leq 1$) for $t \in [0, +\infty)$. In fact, the Hardy-Littlewood type Theorems and the modulus of continuity of analytic functions are closely related.

**Theorem A.** ([7, Theorem 5.1]) Let $f$ be an analytic function in $D$ and continuous in $\overline{D}$. Then

$$|f(e^{i\theta_1}) - f(e^{i\theta_2})| \leq C\omega_\alpha(|\theta_1 - \theta_2|) \text{ for all } 0 \leq \theta_1, \theta_2 < 2\pi$$

if and only if

$$|f'(z)| \leq C\frac{\omega_\alpha(d(z))}{d(z)} \text{ for all } z \in D,$$

where $C$ is a positive constant.

Krantz [15] established the following Hardy-Littlewood type theorem for real harmonic functions.

**Theorem B.** ([15, Theorem 15.8]) Let $u$ be a real harmonic function in $D$, and $\omega(t) = \omega_\alpha(t) = t^\alpha$ be a majorant for $0 < \alpha \leq 1$. Then $u$ satisfies

$$|
abla u(z)| \leq C\frac{\omega_\alpha(d(z))}{d(z)} \text{ for all } z \in D$$

if and only if

$$|u(z) - u(w)| \leq C\omega_\alpha(|z - w|) \text{ for all } z, w \in D,$$

where $C$ is a positive constant.

Moduli of continuity of harmonic quasiregular mappings via Hardy-Littlewood property is considered in [1]. In [17], the authors characterizes the moduli of continuity of a function $f$ by using the square of distance function and module of $\Delta f$ (see the the class $OC^2(G)$ in [17]). In particular, quasiregular versions of the well-known result due to Koebe, [18, Theorem 4.2], is established and, by using this result, an extension of Dyakonov’s theorem for quasiregular mappings in space (without Dyakonov’s hypothesis that it is a quasiregular local homeomorphism), [18, Theorem 4.3], is proved. The characterization of Lipschitz-type spaces for quasiregular mappings by average Jacobian is also established in [18, Theorem 4.3].
For a given \( g \in C(\Omega) \), let
\[
F_g(\Omega) = \{ f \in C(\Omega) \cap C^2(\Omega) : \Delta f(z) = g(z), \ z \in \Omega \},
\]
where \( \Omega \) is a proper subdomain of \( \mathbb{C} \). Obviously, all analytic functions and harmonic mappings defined in \( \Omega \) belong to \( F_0(\Omega) \). We improve Theorems A and B into the following form.

**Theorem 1.1.** Suppose that \( \omega \) is a majorant satisfying (1.6), and \( \Omega \) is a bounded \( L_\omega \)-extension domain. For a given \( g \in C(\Omega) \), let \( f \in F_g(\Omega) \). Then \( f \in L_\omega(\Omega) \) if and only if there exists a constant \( C > 0 \) such that, for all \( z \in \Omega \),
\[
\| Df(z) \| \leq C \frac{\omega(d_\Omega(z))}{d_\Omega(z)}.
\]

A mapping \( f \in C^1(D) \) is called a Bloch type mapping if \( f \) satisfies
\[
\sup_{z \in D} \{ \| Df(z) \| \omega((d(z))^\alpha) \} < +\infty,
\]
where \( \omega \) is a majorant and \( \alpha > 0 \) is a constant. The set of all Bloch type mappings, denoted by the symbol \( B_\alpha^\omega \), forms a complex Banach space with the norm \( \| \cdot \| \) given by
\[
\| f \|_{B_\alpha^\omega} = |f(0)| + \sup_{z \in D} \{ \| Df(z) \| \omega((d(z))^\alpha) \}.
\]

In the following, by using the weighted Lipschitz function, Holland and Walsh [11] gave an equivalent characterization of the analytic Bloch space. For the related investigation of this topic for real functions, we refer to [19, 21].

**Theorem C.** ([11, Theorem 3]) Let \( f \) be analytic in \( D \), and let \( \omega \) be a majorant satisfying \( \omega(t) = t \) for \( t \in [0, +\infty) \). Then \( f \in B_1^\omega \) if and only if
\[
\sup_{z,w \in D, z \neq w} \left\{ \sqrt{(1 - |z|^2)(1 - |w|^2)} |f(z) - f(w)| \right\} < \infty.
\]

In [9], Dyakonov studied the relationship between the modulus of continuity and the bounded mean oscillation on analytic functions in \( D \), and obtained the following result.

**Theorem D.** ([9, Theorem 1]) Suppose that \( f \) is an analytic function in \( D \) which is continuous up to the boundary of \( D \). If \( \omega \) and \( \omega^2 \) are regular majorants, then
\[
f \in L_\omega(D) \iff P[|f|^2](z) - |f(z)|^2 \leq M\omega^2(d(z)).
\]

Analogy Theorems C and D, we prove the following result.

**Theorem 1.2.** For a given \( g \in C(D) \), let \( f \in F_g(D) \). Then, for \( 1 \leq \alpha < 2 \) and a majorant \( \omega \), the following statements are equivalent:

1. \( f \in B_\alpha^\omega \);
(2) There exists a constant $C > 0$ such that for all $r \in (0, d(z)]$,
\[
\frac{1}{|D(z, r)|} \int_{D(z, r)} |f(\zeta) - f(z)| \, dA(\zeta) \leq \frac{Cr}{\omega(r^\alpha)},
\]
where $|D(z, r)|$ denotes the area of $D(z, r)$.

By [5, Theorem 3] and Theorem 1.2, we obtain the following result which is a generalization of Theorem C.

**Corollary 1.3.** For a given $g \in C(\mathbb{D})$, let $f \in \mathcal{F}_g(\mathbb{D})$. Then, for $0 \leq s < 1$ and $1 \leq \alpha \leq s + 1$, the following are equivalent:

1. $f \in \mathcal{B}^s_\alpha$;
2. There exists a constant $C > 0$ such that for all $r \in (0, d(z)]$,
\[
\frac{1}{|D(z, r)|} \int_{D(z, r)} |f(\zeta) - f(z)| \, dA(\zeta) \leq \frac{Cr}{\omega(r^\alpha)},
\]
where $|D(z, r)|$ denotes the area of $D(z, r)$;
3. There exists a constant $C > 0$ such that for all $z, w \in \mathbb{D}$ with $z \neq w$,
\[
\frac{|f(z) - f(w)|}{|z - w|} \leq \frac{C}{\omega(d^s(z)d^{\alpha-s}(w))}.
\]

For $r \in [0, 1)$, the *perimeter* of the curve $C(r) = \{w = f(re^{i\theta}) : \theta \in [0, 2\pi]\}$, counting multiplicity, is defined by

\[
\ell_f(r) = \int_0^{2\pi} |df(re^{i\theta})| = r \int_0^{2\pi} |f_z(re^{i\theta}) - e^{2i\theta}f_r(re^{i\theta})| \, d\theta,
\]
where $f \in C^1(\mathbb{D})$. In particular, let $\ell_f(1) = \sup_{0 < r < 1} \ell_f(r)$ (cf. [4]).

A sense-preserving homeomorphic mapping $f$ from a domain $\Omega$ onto $\Omega'$, contained in the Sobolev class $W^{1,2}_\text{loc}(\Omega)$, is said to be a $K$-quasiconformal mapping if, for $z \in \Omega$,
\[
||D_f(z)||^2 \leq K|\det D_f(z)|, \quad \text{i.e., } ||D_f(z)|| \leq K\lambda(D_f(z)),
\]
where $K \geq 1$ (cf. [13, 14]). In the following, we will give bounds of length and area distortion for some classes of $K$-quasiconformal mappings satisfying Poisson’s equations.

**Theorem 1.4.** For a given $g \in C(\mathbb{D})$, let $f \in \mathcal{F}_g(\mathbb{D})$. If $f = P[f] - G[g]$ is a $K$-quasiconformal mapping with $\ell_f(1) < +\infty$, then, for $n \geq 1$,

\[
|a_n| + |b_n| \leq \frac{K\ell_f(1)}{2n\pi} + \frac{2}{3n} \|g\|_\infty,
\]

\[
\sup_{z \in \mathbb{D}} \{||D_P[f](z)||(1 - |z|^2)\} \leq \left(\frac{\ell_f(1)K}{4\pi^2} + \frac{4}{9} \|g\|_\infty^2 + \frac{\ell_f(1)K^2}{3\pi} \|g\|_\infty\right)^{\frac{1}{2}}
\]

and $f \in \mathcal{B}^1_\omega$, where $P[f](z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n$ and $\omega(t) = t$. 
In particular, if $K = 1$ and $\|g\|_\infty = 0$, then the estimates (1.10) and (1.11) are sharp, and the extreme function is $f(z) = z$ for $z \in \overline{D}$.

For $\theta \in [0, 2\pi]$, the radial length of the curve $C_{\theta}(r) = \{w = f(\rho e^{i\theta}) : 0 \leq \rho \leq r < 1\}$, counting multiplicity, is defined by

\[
(1.12) \quad \ell'_{f}(r, \theta) = \int_{0}^{r} |df(\rho e^{i\theta})| = \int_{0}^{r} \left| f_{\rho}(\rho e^{i\theta}) + e^{-2i\theta} f_{\theta}(\rho e^{i\theta}) \right| d\rho,
\]

where $f \in C^{1}(\overline{D})$ (cf. [6]). In particular, let

\[
\ell'_{f}(1, \theta) = \sup_{0 \leq r < 1} \ell'_{f}(r, \theta).
\]

**Theorem 1.5.** For a given $g \in C(\overline{D})$, let $f \in \mathcal{F}_{g}(\overline{D})$. If $f = P[f] - G[g]$ is a $K$-quasiconformal mapping with $M = \sup_{\theta \in [0, 2\pi]} \ell'_{f}(\theta, 1) < +\infty$, then

\[
(1.13) \quad |a_{n}| + |b_{n}| \leq KM + \frac{2}{3} \|g\|_\infty \quad \text{for } n \geq 1,
\]

where $P[f](z) = \sum_{n=0}^{\infty} a_{n}z^{n} + \sum_{n=1}^{\infty} b_{n}z^{n}$. In particular, if $K = 1$ and $\|g\|_\infty = 0$, then the estimate (1.13) is sharp and the extreme function is $f(z) = Mz$.

The proofs of Theorems 1.1∼1.5 will be presented in Section 2.

2. The proof of the main results

The following result easily follows from [14, Lemma 2.7].

**Lemma E.** If $g \in C(\overline{D})$, then, for $z \in \overline{D}$,

\[
\max \left\{ \left| \frac{\partial}{\partial z} G[g](z) \right|, \left| \frac{\partial}{\partial \overline{z}} G[g](z) \right| \right\} \leq \frac{1}{3} \|g\|_\infty,
\]

where $G[g]$ is defined in (1.4).

**Proof of Theorem 1.1.** We first prove the necessity. Let $z \in \Omega$ and $r = d_{\Omega}(z)/2$. For $w \in \mathbb{D}(z, r)$, we have

\[
f(w) = J_{1}(w) - J_{2}(w),
\]

where

\[
J_{1}(w) = \frac{1}{2\pi} \int_{0}^{2\pi} P \left( \frac{w - z}{r}, e^{i\theta} \right) f(z + re^{i\theta}) d\theta
\]

and

\[
J_{2}(w) = \frac{r^{2}}{2\pi} \int_{\partial \mathbb{D}} G \left( \frac{w - z}{r}, \zeta \right) g(r\zeta + z) dA(\zeta),
\]

where $G$ and $P$ are defined in (1.1) and (1.2), respectively. By elementary calculations, we have

\[
\frac{\partial}{\partial w} G \left( \frac{w - z}{r}, \zeta \right) = \frac{1}{2} \frac{r(|\zeta|^{2} - 1)}{|r - (w - z)\zeta|(w - z - r\zeta)}
\]
and
\[
\frac{\partial}{\partial \bar{w}} G \left( \frac{w - z}{r}, \zeta \right) = \frac{1}{2} \frac{r(|\zeta|^2 - 1)}{|r - (w - \bar{z})\zeta| (\bar{w} - \bar{z} - r\zeta)},
\]
which give that
\[
\| D_{J_2}(w) \| = \left| \frac{r^2}{4\pi} \int_D \frac{\partial}{\partial \bar{w}} G \left( \frac{w - z}{r}, \zeta \right) g(r\zeta + z) dA(\zeta) \right| \\
+ \left| \frac{r^2}{4\pi} \int_D \frac{\partial}{\partial \bar{w}} G \left( \frac{w - z}{r}, \zeta \right) g(r\zeta + z) dA(\zeta) \right| \\
\leq \frac{r^2\|g\|_{\infty}}{4\pi} \int_D \frac{1 - |\zeta|^2}{|w - z - \zeta| \left| 1 - \frac{(w - z)\zeta}{r} \right|} dA(\zeta) \\
+ \frac{r^2\|g\|_{\infty}}{4\pi} \int_D \frac{1 - |\zeta|^2}{|w - z - \zeta| \left| 1 - \frac{(w - z)\zeta}{r} \right|} dA(\zeta).
\]
By (2.1), Lemma E and by letting \( \xi = \frac{w - z}{r} \), we see that
\[
\| D_{J_2}(w) \| \leq \frac{r^2\|g\|_{\infty}}{4\pi} \int_D \frac{1 - |\zeta|^2}{|w - z - \zeta| \left| 1 - \frac{(w - z)\zeta}{r} \right|} dA(\zeta) \leq \frac{2}{3} \|g\|_{\infty} r^2.
\]
The elementary computations lead to
\[
\frac{\partial}{\partial w} P \left( \frac{w - z}{r}, e^{i\theta} \right) = \frac{-(\bar{w} - \bar{z})|w - z - re^{i\theta}|^2 - (r^2 - |w - z|^2)(\bar{w} - \bar{z} - re^{-i\theta})}{|w - z - re^{i\theta}|^4},
\]
and
\[
\frac{\partial}{\partial \bar{w}} P \left( \frac{w - z}{r}, e^{i\theta} \right) = -\frac{(w - z)|w - z - re^{i\theta}|^2 - (r^2 - |w - z|^2)(w - z - re^{i\theta})}{|w - z - re^{i\theta}|^4}.
\]
Then, for \( w \in \mathbb{D}(z, r/2) \),
\[
\left| \frac{\partial}{\partial w} P \left( \frac{w - z}{r}, e^{i\theta} \right) \right| \leq \frac{|w - z|}{|w - z - re^{i\theta}|^2} + \frac{r^2 - |w - z|^2}{|w - z - re^{i\theta}|^3} \\
\leq \frac{\frac{r}{2}}{r^2} + \frac{r^2}{s} = \frac{10}{r}
\]
and
\[
\left| \frac{\partial}{\partial \bar{w}} P \left( \frac{w - z}{r}, e^{i\theta} \right) \right| \leq \frac{10}{r}.
\]
It follows from (2.2), (2.3) and (2.4) that, for \( w \in \mathbb{D}(z, r/2) \),
(2.5) \[ \|D_f(w)\| \leq \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial w} P(w - z, e^{i\theta}) \left( f(z + re^{i\theta}) - f(z) \right) d\theta \right| + \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial w} P(w - z, e^{i\theta}) \left( f(z + re^{i\theta}) - f(z) \right) d\theta \right| + \|D_{J_2}(w)\| \leq 10 \frac{r}{\pi} \int_0^{2\pi} |f(z + re^{i\theta}) - f(z)| d\theta + \frac{2}{3} \|g\|_{\infty} r^2. \]

Since \( f \in \mathcal{L}_\omega(\Omega) \), we know that there is a positive constant \( C_1 \) such that

(2.6) \[ |f(z + re^{i\theta}) - f(z)| \leq C_1 \omega(r). \]

Since \( \Omega \) is a bounded domain, we see that there is a positive constant \( C_2 \) such that

(2.7) \[ \frac{\omega(r)}{r} \geq \frac{\omega(diam(\Omega))}{diam(\Omega)} \geq \frac{2}{3} \|g\|_{\infty} C_2. \]

By (2.5), (2.6) and (2.7), we conclude that there is a positive constant \( C \) such that

\[ \|D_f(w)\| \leq C \omega(r). \]

Next, we show that the sufficiency. Since \( \Omega \) is a \( \mathcal{L}_\omega \)-extension domain, we see that for any \( z_1, z_2 \in \Omega \), by using (1.8), there is a rectifiable curve \( \gamma \subset \Omega \) joining \( z_1 \) to \( z_2 \) such that

\[ |f(z_1) - f(z_2)| \leq \int_{\gamma} \|D_f(\zeta)\| d\zeta \leq C \int_{\gamma} \frac{\omega(d_{\Omega}(\zeta))}{d_{\Omega}(\zeta)} d\zeta \leq C \omega(|z_1 - z_2|) \]

for some constant \( C > 0 \). The proof of this theorem is complete. \( \square \)

Lemma 2.1. For a given \( g \in C(\mathbb{D}) \), let \( f \in F_g(\mathbb{D}) \). Then, for \( a \in \mathbb{D} \), there is a positive constant \( C \) such that

\[ \|D_f(a)\| \leq \frac{1}{\pi r} \int_0^{2\pi} |f(a + re^{i\theta}) - f(a)| d\theta + \frac{2}{3} \|g\|_{\infty} \frac{r}{3}, \]

where \( r \in (0, 1 - |a|) \).

Proof. For \( z \in \mathbb{D}_r \), let

\[ F(z) = f(z + a) - f(a). \]

Then, \( z \in \mathbb{D}_r \),

\[ \Delta F(z) = \Delta f(z + a) = g(z + a). \]

By (1.3), we have

\[ F(z) = \frac{1}{2\pi} \int_0^{2\pi} r^2 - |z|^2 F(re^{i\theta}) d\theta - \frac{r^2}{2\pi} \int_0^{2\pi} \log \left| \frac{r - zw}{z - rw} \right| g(rw + a) dA(w) \]

for \( z \in \mathbb{D}_r \). By calculations, we have
\[ F_z(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{-z - re^{i\theta} + (r^2 - |z|^2)(\overline{z} - re^{-i\theta})}{|z - re^{i\theta}|^4} F(re^{i\theta}) d\theta \\
- \frac{r^3}{4\pi} \int_D \frac{(|w|^2 - 1)}{(r - wz)(z - rw)} g(rw + a) dA(w) \]
and
\[ F_{\overline{z}}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{-z - re^{i\theta} + (r^2 - |z|^2)(\overline{z} - re^{-i\theta})}{|z - re^{i\theta}|^4} F(re^{i\theta}) d\theta \\
- \frac{r^3}{4\pi} \int_D \frac{(|w|^2 - 1)}{(r - wz)(z - rw)} g(rw + a) dA(w), \]
which yields that
\[
\|D_F(0)\| \leq \frac{1}{r\pi} \int_0^{2\pi} |F(re^{i\theta})| d\theta + \frac{r}{2\pi} \int_D \frac{(1 - |w|^2)}{|w|} g(rw + a) dA(w) \\
\leq \frac{1}{r\pi} \int_0^{2\pi} |F(re^{i\theta})| d\theta + \frac{2\|g\|_{\infty} \rho}{3},
\]
The proof of this lemma is complete. \(\square\)

Proof of Theorem 1.2. We first prove (1) \(\Rightarrow\) (2). By Lemma 2.1, for \(\rho \in (0, d(z)]\),
\[
\|D_f(z)\| \leq \frac{1}{\pi \rho} \int_0^{2\pi} |f(z + \rho e^{i\theta}) - f(z)| d\theta + \frac{2\|g\|_{\infty}}{3} \rho,
\]
which gives
\[
\int_0^r \rho^2 \|D_f(z)\| d\rho \leq \frac{1}{\pi} \int_0^r \left( \rho \int_0^{2\pi} |f(z + \rho e^{i\theta}) - f(z)| d\theta \right) d\rho + \frac{2\|g\|_{\infty}}{3} \int_0^r \rho^3 d\rho,
\]
where \(r = d(z)\). It follows from (2.8) that
\[
\|D_f(z)\| \leq \frac{3}{\pi r^3} \int_{D(z,r)} |f(z) - f(\zeta)| dA(\zeta) + \frac{\|g\|_{\infty} r}{2} \\
= \frac{3}{r |D(z,r)|} \int_{D(z,r)} |f(z) - f(\zeta)| dA(\zeta) + \frac{\|g\|_{\infty} r}{2} \\
\leq \frac{3C}{\omega(r^\alpha)} + \frac{\|g\|_{\infty} r}{2},
\]
which gives that \(f \in B_\omega^r\).
Now we prove (2) ⇒ (1). Since $f \in \mathcal{B}_\alpha^\omega$, we see that there is a positive constant $C$ such that
\begin{equation}
\|DF(z)\| \leq \frac{C}{\omega(d^\alpha(z))}.
\end{equation}
For $z \in \mathbb{D}$ and $\zeta \in \mathbb{D}(z, r)$, we have
\[\omega(d^\alpha(z + t(\zeta - z))) \geq \omega \left( (d(z) - t|z - \zeta|)^\alpha \right), \quad t \in [0, 1],\]
which, together with (2.9), yields that
\begin{equation}
|f(z) - f(\zeta)| \leq |z - \zeta| \int_0^1 \|DF(z + t(\zeta - z))\| dt
\end{equation}
\begin{align*}
&\leq C|z - \zeta| \int_0^1 \frac{dt}{\omega(d^\alpha(z + t(\zeta - z)))} \\
&\leq C|z - \zeta| \int_0^1 \frac{dt}{\omega \left( (d(z) - t|z - \zeta|)^\alpha \right)} \\
&= C \int_0^{|z - \zeta|} \frac{dt}{\omega \left( (d(z) - t)^\alpha \right)}.
\end{align*}
By (2.10), we conclude that
\begin{equation}
\frac{1}{|D(z, r)|} \int_{D(z, r)} |f(z) - f(\zeta)| dA(\zeta) \leq \frac{C}{|D_r|} \int_{D_r} \left( \int_0^{[\xi]} \frac{dt}{\omega \left( (d(z) - t)^\alpha \right)} \right) dA(\xi)
\end{equation}
\begin{align*}
\begin{split}
&= \frac{2C}{r^2} \int_0^r \rho \left( \int_0^\rho \frac{dt}{\omega \left( (d(z) - t)^\alpha \right)} \right) d\rho.
\end{split}
\end{align*}
By exchanging integral order, we obtain
\begin{align*}
\begin{split}
\int_0^r \rho \left( \int_0^\rho \frac{dt}{\omega \left( (d(z) - t)^\alpha \right)} \right) d\rho &= \int_0^r \left( \int_0^r \rho d\rho \right) \frac{dt}{\omega \left( (r - t)^\alpha \right)} \\
&\leq r \int_0^r \frac{(r - t)^\alpha}{\omega \left( (r - t)^\alpha \right)} (r - t)^{1-\alpha} dt \\
&\leq \frac{r^{\alpha+1}}{\omega(r^\alpha)} \int_0^r (r - t)^{1-\alpha} dt \\
&= \frac{1}{\left( \frac{2 - \alpha}{\omega(r^\alpha)} \right)}.
\end{split}
\end{align*}
It follows from (2.11) and (2.12) that
\[\frac{1}{|D(z, r)|} \int_{D(z, r)} |f(z) - f(\zeta)| dA(\zeta) \leq \frac{2C}{2 - \alpha} \frac{r}{\omega(r^\alpha)}.
\]
The proof of this theorem is complete. □

The following result is well-known (cf. [3]).
Lemma F. Among all rectifiable Jordan curves of a given length, the circle has the maximum interior area.

Proof of Theorem 1.4. We first prove (1.10). Since \( P[f] \) is harmonic in \( D \), we see that \( \partial P[f](z)/\partial z \) and \( \partial P[f](z)/\partial \overline{z} \) are analytic and anti-analytic, respectively. Hence, by Cauchy's integral formula, we have

\[
na_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{\partial P[f](z)}{z^n} \, dz \quad \text{and} \quad nb_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{\overline{\partial P[f](z)}}{z^n} \, dz,
\]

which, together with \( \|DP[f]\| \leq \|Df\| + \|DG[g]\| \), implies that

\[
(2.13) \quad n(|a_n| + |b_n|) = \frac{1}{2\pi} \left( \int_{|z|=r} \frac{\partial P[f](z)}{z^n} \, dz + \int_{|z|=r} \frac{\overline{\partial P[f](z)}}{z^n} \, dz \right) \\
\leq \frac{1}{2\pi r^n} \int_0^{2\pi} r \|DP[f](re^{i\theta})\|d\theta \\
\leq \frac{1}{2\pi r^n} \int_0^{2\pi} r (\|Df(re^{i\theta})\| + \|DG[g](re^{i\theta})\|)d\theta,
\]

where \( r \in (0, 1) \).

By (1.9), we have

\[
(2.14) \quad \ell_f(1) \geq \ell_f(r) = r \int_0^{2\pi} \left| f_z(re^{i\theta}) - e^{-2i\theta} f_z(re^{i\theta}) \right| d\theta \\
\geq r \int_0^{2\pi} \left( |f_z(re^{i\theta})| - |f_z(re^{i\theta})| \right) d\theta \\
\geq \frac{r}{K} \int_0^{2\pi} \|Df(re^{i\theta})\| d\theta.
\]

It follows from (2.13), (2.14) and Lemma E that

\[
n(|a_n| + |b_n|) \leq \frac{K\ell_f(1)}{2\pi r^n} + \frac{1}{2\pi r^n} \int_0^{2\pi} r \|DG[g](re^{i\theta})\|d\theta \\
\leq \frac{1}{2\pi r^n} \left( K\ell_f(1) + \int_0^{2\pi} \|DG[g](re^{i\theta})\|d\theta \right) \\
\leq \frac{1}{2\pi r^n} \left( K\ell_f(1) + \frac{4\pi}{3} \|g\|_{\infty} \right),
\]

which gives that

\[
|a_n| + |b_n| \leq \inf_{r \in (0, 1)} \left[ \frac{1}{2n\pi r^n} \left( K\ell_f(1) + \frac{4\pi}{3} \|g\|_{\infty} \right) \right] = \frac{K\ell_f(1)}{2n\pi} + \frac{2}{3n} \|g\|_{\infty}.
\]
Next we prove (1.11). Let \( \text{Area}(f(\mathbb{D}_r)) \) denote the area of \( f(\mathbb{D}_r) \), where \( r \in (0, 1) \). Then

\[
\text{Area}(f(\mathbb{D}_r)) = \int_{\mathbb{D}_r} J_f(z) \, dA(z) \geq \frac{1}{K} \int_{\mathbb{D}_r} \| D_f(z) \|^2 \, dA(z).
\]

For \( \theta \in [0, 2\pi] \) and \( z \in \mathbb{D} \), let

\[
H_\theta(z) = \frac{\partial P[f](z)}{\partial z} + e^{i\theta} \frac{\partial P[f](z)}{\partial \bar{z}}.
\]

Then, by the subharmonicity of \( |H_\theta|^2 \), we obtain

\[
|H_\theta(z)|^2 \leq \frac{1}{\pi(1 - |z|^2)^2} \int_{\mathbb{D}_1} |P[f](\bar{z})|^2 \, dA(z) \]

Least 2.16

\[
I = \int_{\mathbb{D}} \left( \| D_G[g](\xi) \| + \| D_f(\xi) \| \right)^2 \, dA(\xi).
\]

By (2.15), Lemma E and Cauchy-Schwarz’s inequality, we get

\[
I = \int_{\mathbb{D}} \left( \| D_G[g](\xi) \| + \| D_f(\xi) \| \right)^2 \, dA(\xi).
\]

Applying Lemma F, we have

\[
\text{Area}(f(\mathbb{D})) \leq \pi \left( \frac{\ell_f(1)}{2\pi} \right)^2 = \frac{\ell_f^2(1)}{4\pi},
\]

which, together with (2.17), yields that
(2.18) \[ I \leq \frac{\ell_f(1)K}{4\pi} + \frac{4\pi}{9}\|g\|_\infty^2 + \frac{\ell_f(1)K^{\frac{1}{2}}}{3}\|g\|_\infty. \]

By (2.16) and (2.18), we conclude that

(2.19) \[ \|D_{f}[f](z)\| = \max_{\theta \in [0, 2\pi]} |H_\theta(z)| \leq \left( \frac{\ell_f(1)K}{4\pi^2} + \frac{4\pi}{9}\|g\|_\infty^2 + \frac{\ell_f(1)K^{\frac{1}{2}}}{3\pi}\|g\|_\infty \right)^{\frac{1}{2}} \]

At last, \( f \in B^1_1 \) follows from (2.19) and Lemma E, where \( \omega(t) = t \). The proof of this theorem is complete. \( \blacksquare \)

The following result is considered to be a Schwarz-type lemma of subharmonic functions.

**Theorem G.** ([2, Theorem 2]) Let \( \phi \) be subharmonic in \( \mathbb{D} \). If, for all \( r \in [0, 1) \),

\[ A(r) = \sup_{\theta \in [0, 2\pi]} \int_0^r \phi(\rho e^{i\theta}) d\rho \leq 1, \]

then \( A(r) \leq r \).

**Proof of Theorem 1.5.** By Cauchy’s integral formula, for \( \rho \in (0, 1) \) and \( n \geq 1 \), we get

\[ na_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{\partial P[f](z)}{z^n} \, dz \quad \text{and} \quad nb_n = \frac{1}{2\pi i} \int_{|z|=\rho} \left( \frac{\partial P[f](z)}{\partial \overline{z}} \right) \frac{1}{z^n} \, dz, \]

which implies that

(2.20) \quad n(|a_n| + |b_n|) = \frac{1}{2\pi} \left| \int_{|z|=\rho} \frac{\partial P[f](z)}{z^n} \, dz \right| + \frac{1}{2\pi} \left| \int_{|z|=\rho} \left( \frac{\partial P[f](z)}{\partial \overline{z}} \right) \frac{1}{z^n} \, dz \right| 

\[ \leq \frac{1}{2\pi \rho^{n-1}} \int_0^{2\pi} \|D_{P[f]}(\rho e^{i\theta})\| \, d\theta. \]

By calculations, for \( \theta \in [0, 2\pi] \), we obtain

\[ \ell'_f(\theta, r) = \int_0^r |f_z(\rho e^{i\theta}) + e^{-2i\theta} f_\theta(\rho e^{i\theta})| \, d\rho \]

\[ \geq \int_0^r \lambda(D_f)(\rho e^{i\theta}) \, d\rho \]

\[ \geq \frac{1}{K} \int_0^r \|D_f(\rho e^{i\theta})\| \, d\rho, \]

which gives

(2.21) \quad \int_0^r \|D_f(\rho e^{i\theta})\| \, d\rho \leq K \ell'_f(\theta, r) \leq KM.
It follows from (2.21) and Lemma E that

\begin{equation}
\int_0^r \| D_{P[f]}(pe^{i\theta}) \| \, d\rho \leq \int_0^r \| D_f(pe^{i\theta}) \| \, d\rho + \int_0^r \| D_G[g](pe^{i\theta}) \| \, d\rho \\
\leq K M + \frac{2}{3} \| g \|_\infty \r.
\end{equation}

By (2.21), the subharmonicity of $D_{P[f]}(pe^{i\theta})$ and Theorem G, we have

\begin{equation}
\int_0^r \| D_{P[f]}(pe^{i\theta}) \| \, d\rho \leq \left( K M + \frac{2}{3} \| g \|_\infty \right) \r.
\end{equation}

By (2.20) and (2.23), we get

\[
2\pi n (|a_n| + |b_n|) \int_0^r \rho^{n-1} \, d\rho = \int_0^r \left( \int_0^{2\pi} \| D_{P[f]}(pe^{i\theta}) \| \, d\theta \right) \, d\rho \\
= \int_0^{2\pi} \left( \int_0^r \| D_{P[f]}(pe^{i\theta}) \| \, d\rho \right) \, d\theta \\
\leq 2\pi \left( K M + \frac{2}{3} \| g \|_\infty \right) \r,
\]

which yields that

\[
|a_n| + |b_n| \leq \inf_{r \in (0,1)} \left( \frac{K M + \frac{2}{3} \| g \|_\infty}{r^{n-1}} \right) = K M + \frac{2}{3} \| g \|_\infty \text{ for } n \geq 1.
\]

The proof of this theorem is complete. \qed

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**References**

1. A. Abaob, M. Arsenovic and M. Mateljević, Moduli of continuity of harmonic quasiregular mappings on bounded domains, *Ann. Acad. Sci. Fenn. Math.*, 38 (2013), 839–847.
2. E. F. Beckenbach, A relative of the lemma of Schwarz, *Bull. Amer. Math. Soc.*, 44 (1938), 698–707.
3. T. Carleman, Zur Theorie der Minimalflächen, *Math. Z.*, 9 (1921), 154–160.
4. Sh. Chen, G. Liu and S. Ponnusamy, Linear measure and K-quasiconformal harmonic mappings (in Chinese), *Sci. Sin. Math.*, 47(2017), 565–574.
5. Sh. Chen, S. Ponnusamy and A. Rasila, On characterizations of Bloch-type, Hardy-type and Lipschitz-type spaces, *Math. Z.*, 279(2015), 163–183.
6. Sh. Chen, S. Ponnusamy and A. Rasila, Lengths, area and Lipschitz-type spaces of planar harmonic mappings, *Nonlinear Anal.*, 115 (2015), 62–80.
7. P. Duren, Theory of $H^p$ spaces, 2nd ed., Dover, Mineola, N. Y., 2000.
8. K. M. Dyakonov, Holomorphic functions and quasiconformal mappings with smooth moduli, *Adv. Math.*, 187 (2004), 146–172.
9. K. M. Dyakonov, Equivalent norms on Lipschitz-type spaces of holomorphic functions, *Acta Math.*, 178 (1997), 143–167.
10. F. W. Gehring and O. Martio, Lipschitz-classes and quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 10 (1985), 203–219.
11. F. Holland and D. Walsh, Criteria for membership of Bloch space and its subspace, *BMOA, Math. Ann.*, 273 (1986), 317–335.
12. L. Hörmander, *Notions of convexity*, Progress in Mathematics, Vol. 127, Birkhäuser Boston Inc, Boston 1994.
13. D. Kalaj and M. Pavlović, On quasiconformal self-mappings of the unit disk satisfying Poisson’s equation, *Trans. Amer. Math. Soc.*, 363 (2011), 4043–4061.
14. D. Kalaj, Cauchy transform and Poisson’s equation, *Adv. Math.*, 231 (2012), 213–242.
15. S. G. Krantz, Lipschitz spaces, smoothness of functions, and approximation theory, *Expo. Math.*, 3 (1983), 193–260.
16. V. Lappalainen, LipΔ-extension domains, *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, 56 (1985).
17. M. Mateljević and M. Vuorinen, On harmonic quasiconformal quasi-isometries, *J. Inequa. Appl.*, Volume 2010, Article ID 178732, 19 pages, doi:10.1155/2010/178732.
18. M. Mateljević, Distortion of quasiregular mappings and equivalent norms on Lipschitz-type spaces, *Abstr. Appl. Anal.*, Volume 2014, Article ID 895074, 20 pages, http://dx.doi.org/10.1155/2014/895074.
19. M. Pavlović, On the Holland-Walsh characterization of Bloch functions, *Proc. Edinb. Math. Soc.*, 51 (2008), 439–441.
20. M. Pavlović, On Dyakonov’s paper Equivalent norms on Lipschitz-type spaces of holomorphic functions, *Acta Math.*, 183 (1999), 141–143.
21. G. Ren and U. Kähler, Weighted Lipschitz continuity and harmonic Bloch and Besov spaces in the real unit ball, *Proc. Edinb. Math. Soc.*, 48 (2005), 743–755.

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