Existence and boundary behaviour of radial solutions for weighted elliptic systems with gradient terms

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Abstract
We study the existence and boundary behaviour of positive radial solutions of elliptic systems in \( \Omega \subset \mathbb{R}^N \), where \( \Omega \) is either a ball centered at the origin or the whole space \( \mathbb{R}^N \). Firstly, we determine optimal conditions for the existence of positive radial solutions when the system is posed in a ball corresponding to their behaviour at the boundary. Next, we take \( \Omega = \mathbb{R}^N \) and by the use of dynamical system techniques we are able to describe the behaviour at infinity of such positive radial solutions.

Keywords Radial solutions · Elliptic systems · Nonlinear gradient terms · Dynamical systems

Mathematics Subject Classification 35B40 · 35J05 · 35J61 · 35J91

1 Introduction

In this paper, we study the positive radial solutions of the following weighted semilinear elliptic system:

\[
\begin{align*}
\Delta u &= |x|^a v^p & \text{in } \Omega, \\
\Delta v &= |x|^b v^q f(|\nabla u|) & \text{in } \Omega,
\end{align*}
\]  

(1)

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where $\Omega \subset \mathbb{R}^N$ is either a ball centered at the origin with radius $R > 0$ or the whole space $\mathbb{R}^N$, $a, b, p, q > 0$ and $f \in C^1(0, \infty)$ is an increasing function such that $f(t) > 0$ for all $t > 0$. We start with the assumption that $u$ and $v$ are positive, radially symmetric solutions of (1). There is no prior condition at the boundary for $u$ and $v$, but it will be required as we move forward in our analysis as we are concerned with the classification of all the solutions of (1). In [25], Singh studied system (1) in the case $a = b = q = 0$, that is, the system

$$\begin{cases}
\Delta u = v^p & \text{in } \Omega, \\
\Delta v = f(|\nabla u|) & \text{in } \Omega.
\end{cases}$$

The author considered system (2) both in the case of a ball centered at the origin with positive radius and in the whole space $\mathbb{R}^N$. Optimal conditions for the existence of positive radial solutions in the case $\Omega = \mathbb{R}^N$ were found, as well as the classification of all positive radial solutions of (2) corresponding to their behaviour at the boundary of $\Omega$. In [9], Filippucci and Vinti considered a generalisation of (2),

$$\begin{cases}
\Delta u = v^p & \text{in } \Omega, \\
\Delta v = g(|x|) f(|\nabla u|) & \text{in } \Omega,
\end{cases}$$

where $g$ is a bounded and increasing function, and $h$ is also increasing. The authors found analogous existence conditions to those in [25], as well as a weaker version of the result when considering a quasilinear generalisation of (3). In this paper, we extend the results of [25] in a new direction by considering more general non-linearities in both equations of (2), in particular allowing a weighting function in both equations, as well as dependence on $v$ in the second equation. We now motivate the study of such systems.

In the case where $\Omega$ is a ball, system (2) with $p = 1$ and $f(t) = t^2$, that is,

$$\begin{cases}
\Delta u = v & \text{in } \Omega, \\
\Delta v = |\nabla u|^2 & \text{in } \Omega,
\end{cases}$$

was first studied by Diaz, Lazzo, and Schmidt in [5]. This special choice of $p$ and function $f$ appears in the study of the dynamics of viscous, heat-conducting fluids. In [5], the authors obtained that system (4) has one positive solution which blows up at the boundary and the authors also observed that in case of small dimensions ($N \leq 9$), there exists one sign-changing solution which also blows up at the boundary. In [6, 7], Diaz, Rakotoson, and Schmidt extended these results to time dependent systems.

Problems related to boundary blow-up solutions can be linked back to at least a century ago, for instance, Bieberbach [3] in 1916 studied the boundary blow-up solutions for the equation $\Delta u = e^u$ in a planar domain. From that point in time onwards, there have been many new techniques devised to tackle such problems (see [12, 13, 24]). In the last few decades, semilinear elliptic equations with nonlinear gradient terms have been studied extensively for such boundary blow-up solutions (see [1, 4, 8, 11, 19]).
In [10], Ghergu, Giacomoni and Singh studied the following more general quasilinear elliptic system with nonlinear gradient terms:

\[
\begin{align*}
\Delta_p u &= v^m |\nabla u|^a & \text{in } \Omega, \\
\Delta_p v &= v^\beta |\nabla u|^q & \text{in } \Omega,
\end{align*}
\]

where \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\), \(p > 1\), is the \(p\)-Laplace operator. The authors classified all positive radial solutions in the case that \(\Omega\) is a ball and also obtained the behaviour at infinity of such solutions.

In this paper, we first consider the case when \(\Omega = B_R\) is a ball of radius \(R > 0\) and centered at the origin. We obtain that in this case system (1) has positive radially symmetric solutions \((u, v)\) such that \(u\) or \(v\) (or both) blow up around \(\partial \Omega\) if and only if

\[
\int_1^\infty \left( \int_0^s F(t) \, dt \right)^{-p/(2p-q+1)} \, ds < \infty \quad \text{where } F(t) = \int_0^t f(k) \, dk. \tag{5}
\]

We also study the full classification of positive radially symmetric solutions in the case that system (1) is posed in a ball. The condition (5) can be seen as analogous to optimal conditions obtained by Keller [18] and Osserman [20] in the 1950s while studying the existence of a solution to the boundary blow-up problem

\[
\begin{align*}
\Delta u &= f(u) & \text{in } \Omega, \\
u &= \infty & \text{on } \partial \Omega, \tag{6}
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded domain and \(f \in C^1[0, \infty)\) is a nonnegative increasing function. The authors obtained that (6) has \(C^2(\Omega)\) solutions if and only if

\[
\int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty \quad \text{where } F(s) = \int_0^s f(t) \, dt. \tag{7}
\]

Equation (7) has also been seen in various other circumstances as it is related to the maximum principle for nonlinear elliptic inequalities. For example, if \(u \in C^2(\Omega)\) is nonnegative and satisfies \(\Delta u \leq f(u)\) in \(\Omega\), then, if \(u\) vanishes at a point in \(\Omega\), it must vanish everywhere in \(\Omega\). For various extensions of this result, one could refer to Vazquez [26] and to Pucci, Serrin and Zou [21–23]. Next, if \(f(t) = t^s\), \(s \geq 1\), we are able to find the exact rate at which the solution \((u, v)\) grows at infinity. Here, we have used dynamical system techniques for cooperative systems with negative divergence to determine the growth at infinity.
2 Main results

First, we consider system (1) with $\Omega = B_R$, that is, we study the system

$$
\begin{align*}
\Delta u &= |x|^a v^p \quad \text{in } B_R, \\
\Delta v &= |x|^b v^q f(|\nabla u|) \quad \text{in } B_R,
\end{align*}
$$

(8)

where $B_R \subset \mathbb{R}^N$, $N \geq 2$, is an open ball of radius $R > 0$ centred at the origin, $a, b, p, q > 0$ and $f \in C^1[0, \infty)$ is an increasing function satisfying $f(t) > 0$ for all $t > 0$.

One of the following three boundary conditions can be imposed on system (8):

- either $u$ and $v$ are bounded in $B_R$; or
- $u$ is bounded in $B_R$ and $\lim_{|x| \to R} v(x) = \infty$; or
- $\lim_{|x| \to R} u(x) = \lim_{|x| \to R} v(x) = \infty$.

We note that by the first equation in (1), the boundary condition $\lim_{|x| \to R} u(x) = \infty$ and $v$ is bounded in $B_R$ is not possible. Theorem 2.1 below gives optimal conditions for the existence of positive radial solutions to (8) under each of the above boundary conditions.

**Theorem 2.1** Let us suppose that $(u, v)$ is a positive radial solution of (8). Then, we have:

(i) Both $u$ and $v$ are bounded if and only if

$$
\int_1^\infty \left( \int_0^s F(t) \, dt \right)^{-p/(2p-q+1)} \, ds = \infty.
$$

(ii) $u$ is bounded and $\lim_{|x| \to R} v(r) = \infty$ if and only if

$$
\int_1^\infty s \left( \int_0^s F(t) \, dt \right)^{-p/(2p-q+1)} \, ds < \infty.
$$

(iii) $\lim_{|x| \to R} u(x) = \lim_{|x| \to R} v(x) = \infty$ if and only if

$$
\begin{align*}
\int_1^\infty \left( \int_0^s F(t) \, dt \right)^{-p/(2p-q+1)} \, ds &< \infty, \\
\int_1^\infty s \left( \int_0^s F(t) \, dt \right)^{-p/(2p-q+1)} \, ds &= \infty.
\end{align*}
$$

Now, if $f(t) = t^s$, $s \geq 1$, Theorem 2.1 easily yields the following result.

**Corollary 2.2** Consider the system

$$
\begin{align*}
\Delta u &= |x|^a v^p \quad \text{in } B_R, \\
\Delta v &= |x|^b v^q |\nabla u|^s \quad \text{in } B_R.
\end{align*}
$$

(9)
We have:

1. All positive radial solutions to (9) are bounded if and only if
   \[ ps + q - 1 \leq 0. \]

2. There exist positive radial solutions to (9) such that \( u \) is bounded and
   \[ \lim_{|x| \to \infty} v(r) = \infty \] if and only if
   \[ s > 2 \left( 1 + \frac{1 - q}{p} \right). \]

3. There exist positive radial solutions to (9) satisfying
   \[ \lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = \infty \] if and only if
   \[ \frac{1 - q}{p} < s \leq 2 \left( 1 + \frac{1 - q}{p} \right). \]

We now consider system (9) in the whole space \( \mathbb{R}^N \), that is,

\[
\begin{align*}
\Delta u &= |x|^a v^p \quad \text{in } \mathbb{R}^N, \\
\Delta v &= |x|^b v^q |\nabla u|^s \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

where \( a, b, p, q > 0 \) and \( s \geq 1 \). The following results give optimal conditions for the existence of solutions of (10), as well as the asymptotic behaviour of such solutions.

**Theorem 2.3** System (10) has positive radial solutions if and only if

\[ ps + q - 1 \leq 0. \]

**Theorem 2.4** Assume \( p < 1 \) and \( ps + q < 1 \). Let \( (u, v) \) be a positive radially symmetric solution of (10). If

\[ \frac{p(s - 2)(s + as + b + 2)}{1 - ps - q} \leq 2(N + a - 1), \]

then

\[ \lim_{|x| \to \infty} u(x) |x|^{-\frac{(a+2)(1-ps-q)+ps(a+1)+bp+2q}{1-ps-q}} = \frac{(AB^sK)^{p/(ps+q-1)}}{DK} \]

and

\[ \lim_{|x| \to \infty} v(x) |x|^{-\frac{(a+1)s+b+2}{1-ps-q}} = (AB^sK)^{1/(ps+q-1)}, \]

where

\[ A = 2 + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q}, \]

\[ B = N + a + pA, \quad K = N - 2 + A, \quad D = 2 + a + pA. \]
Remark 2.5 We first note that condition (12) will hold for all \( s \leq 2 \). Now, it is easy to see by direct calculation that (10) has a solution \((u_0, v_0) = (C_1|x|^{k_1}, C_2|x|^{k_2})\), which vanishes at the origin, given explicitly by

\[
\begin{align*}
    u_0(x) &= \frac{(AB^sK)^{p/(ps+q-1)}}{DK}|x|^{(a+2)/(1-ps-q)+ps(a+1)+bp+2p/(1-ps-q)}, \\
    v_0(x) &= (AB^sK)^{1/(ps+q-1)}|x|^{((a+1)s+b+2)/(1-ps-q)}.
\end{align*}
\]

Theorem 2.4 thus tells us that any positive radial solution \((u, v)\) of (10) behaves like \((u_0, v_0)\) as \(|x| \to \infty\).

3 Notes on dynamical systems

Let \( x = (x_1, x_2, x_3) \), \( y = (y_1, y_2, y_3) \) be any two points in \( \mathbb{R}^3 \), then for \( x_i \leq y_i \) we write

\[ x \leq y \]

where \( i = 1, 2, 3 \). For \( x \leq y \) and \( x \neq y \), we write \( x < y \).

Also, the open ordered interval is defined as

\[ \llbracket x, y \rrbracket = \{ z \in \mathbb{R}^3 : x < z < y \} \subset \mathbb{R}^3. \]

Consider the initial value problem

\[
\begin{align*}
    \xi_t &= h(\xi) \quad \text{for} \quad t \in \mathbb{R}, \\
    \xi(0) &= \xi_0,
\end{align*}
\]

where \( h : \mathbb{R}^3 \to \mathbb{R} \) is a \( C^1 \) function. This implies that there exists a unique solution \( \xi \) of (13) defined in a maximal time interval for any \( \xi_0 \in \mathbb{R}^3 \). Let \( \varphi(\cdot, \xi_0) \) denote the flow associated to (13), that is, \( t \mapsto \varphi(t, \xi_0) \) is the unique solution of (13) defined in a maximal time interval. Let us suppose that the vector field \( h \) is cooperative, that is

\[ \frac{\partial h_i}{\partial x_j} \geq 0 \quad \text{for} \quad 1 \leq i, j \leq 3, \quad i \neq j. \]

Next, follow the results due to Hirsch [15, 16].

Theorem 3.1 (see [16, Theorem 1]) Any compact limit set of (13) contains an equilibrium or is a cycle.

Definition 3.2 A finite sequence of equilibria \( \zeta_1, \zeta_2, \ldots, \zeta_n = \zeta_1, \ n \geq 2 \), is known as a circuit such that \( W^u(\zeta_i) \cap W^s(\zeta_{i+1}) \) is non-empty. Here, \( W^u \) and \( W^s \) represent the stable and unstable manifolds respectively.
Note 3.3 There is no circuit in case all the equilibria are hyperbolic and also their stable and unstable manifolds are mutually transverse.

Theorem 3.4 (see [16, Theorem 2]) Let us assume that $L \subset \mathbb{R}^3$ is a compact set such that:

(i) There is no circuit and all the equilibria in $L$ are hyperbolic.
(ii) The number of cycles in $L$ which have period less than or equal to $T$ is finite, where $T > 0$.

Then:
(a) Every limit set in $L$ is an equilibrium or cycle.
(b) $L$ has finite number of cycles.

Theorem 3.5 (see [15, Theorem 7]) Assume that $\xi_1, \xi_2 \in \mathbb{R}^3$ such that $\xi_1 < \xi_2$.

Further, if
\[
\text{div} \, h < 0 \quad \text{in} \quad [\xi_1, \xi_2],
\]

then there are no cycles of (13) in $[\xi_1, \xi_2]$.

4 Proofs of main results

We begin with the proof of Theorem 2.1. In what follows, $C$ denotes some positive constant which may vary with each occurrence, and the notation $r = |x|$ will be used. We divide the proof of Theorem 2.1 into the following two lemmas.

Lemma 4.1 Let $(u, v)$ be a positive radial solution of (8). Then
\[
\lim_{|x| \nearrow R} v(x) = \infty
\]
if and only if
\[
\int_{1}^{\infty} \left( \int_{0}^{s} \sqrt{f(t)} \, dt \right)^{-2p/(2p-q+1)} \, ds < \infty.
\]

Proof First, assume we have a positive radial solution to (8) such that \( \lim_{r \to R^{-}} v(r) = \infty \). We can rewrite our system as
\[
\begin{aligned}
(u'(r))^{N-1} &= r^{N+a-1} v^p(r), \\
(v'(r))^{N-1} &= r^{N+b-1} v^q(r) f(|u'(r)|), \\
u'(0) &= v'(0) = 0.
\end{aligned}
\]

An integration over $(0, r), 0 < r < R,$ then gives us
\[
\begin{aligned}
u'(r) &= r^{1-N} \int_{0}^{r} t^{N+a-1} v^p(t) \, dt, \\
v'(r) &= r^{1-N} \int_{0}^{r} t^{N+b-1} v^q(t) f(|u'(r)|) \, dt.
\end{aligned}
\]
from which it follows that $u$ and $v$ are increasing in $(0, R)$. We can thus take the first integral equation above and estimate $u'$ as

$$u'(r) \leq r^{1-N} v^p(r) \int_0^r t^{N+a-1} dt = \frac{r^{a+1} v^p(r)}{N+a},$$

so from (8) we have that

$$r^a v^p(r) \leq u''(r) + \frac{N-1}{r} \cdot \frac{r^{a+1} v^p(r)}{N+a},$$

which further implies

$$1 + \frac{a}{N+a} r^a v^p(r) \leq u''(r).$$

Similarly we have

$$v'(r) \leq \frac{r^b v^q(r) f(u'(r))}{N+b},$$

from which it follows

$$1 + \frac{b}{N+b} r^b v^q(r) f(u'(r)) \leq v''(r).$$

We thus have that $u(r)$ and $v(r)$ are convex functions, along with the following two estimates which hold for all $0 < r < R$:

$$1 + \frac{a}{N+a} r^a v^p(r) \leq u''(r) \leq r^a v^p(r),$$

$$1 + \frac{b}{N+b} r^b v^q(r) f(u'(r)) \leq v''(r) \leq r^b v^q(r) f(u'(r)).$$

Now, where $w = u'$, we multiply the right inequality in (15) by $v'$ and integrate over $[0, r]$ to find

$$\frac{v'(r)^2}{2} \leq \int_0^r r^b v^q(t) v'(t) f(w(t)) \, dt \leq r^b f(w(r)) \int_0^r \left( \frac{v^{q+1}(t)}{q+1} \right)' \, dt \leq C f(w(r)) v^{q+1}(r) \quad \text{for all} \quad 0 < r < R,$$

from which it follows

$$v^{-(q+1)/2}(r) v'(r) \leq C \sqrt{f(w(r))}.$$  \hfill (16)

Now, fix $0 < r_0 < R$. From (14), we see that for all $r > r_0$ we have

$$v^p(r) \leq C w'(r)$$
and multiplying this inequality by (16) we obtain

\[ v^{p-(q+1)/2}(r) v'(r) \leq C \sqrt{f(w(r))} w'(r). \]

Integrating both sides of this equation over \((r_0, r)\) gives

\[
\begin{align*}
&v^{p+(1-q)/2}(r) - v^{p+(1-q)/2}(r_0) \leq C \int_{r_0}^{r} w'(t) \sqrt{f(w(t))} \, dt \\
&\quad \leq C \int_{w(r_0)>0}^{w(r)} \sqrt{f(t)} \, dt \\
&\quad \leq C \int_{w(0)=0}^{w(r)} \sqrt{f(t)} \, dt \quad \text{for all } r > r_0.
\end{align*}
\]

Since \(\lim_{r \to R^-} v(r) = \infty\), there exists \(\rho < R\) such that

\[
v^{p+(1-q)/2}(r) = (v^p(r))^{(2p-q+1)/(2p)} \leq C \int_{0}^{w(r)} \sqrt{f(t)} \, dt \quad \text{for all } \rho < r.
\]

Let \(\tilde{r} = \max\{r_0, \rho\}\). Using (14) again we see

\[
(w'(r))^{(2p-q+1)/(2p)} \leq C \int_{0}^{w(r)} \sqrt{f(t)} \, dt \quad \text{for all } \tilde{r} < r,
\]

from which it follows that

\[
w'(r) \left( \int_{0}^{w(r)} \sqrt{f(t)} \, dt \right)^{-2p/(2p-q+1)} \leq C \quad \text{for all } \tilde{r} < r.
\]

We can integrate both sides of this expression over \((\tilde{r}, r)\) to find

\[
\int_{\tilde{r}}^{r} w'(t) \left( \int_{0}^{w(t)} \sqrt{f(s)} \, ds \right)^{-2p/(2p-q+1)} \, dt \leq C \int_{\tilde{r}}^{r} \, dt
\]

or, after a change of variables

\[
\int_{w(\tilde{r})}^{w(r)} \left( \int_{0}^{\sqrt{f(t)}} \, ds \right)^{-2p/(2p-q+1)} \, ds \leq C(r - \tilde{r}) \leq Cr.
\]

Letting \(r \to R\), we see

\[
\int_{w(\tilde{r})}^{\infty} \left( \int_{0}^{\sqrt{f(t)}} \, ds \right)^{-2p/(2p-q+1)} \, ds \leq CR < \infty.
\]
Hence,  
\[
\int_{1}^{\infty} \left( \int_{0}^{s} \sqrt{f(t)} \, dt \right)^{-2p/(2p-q+1)} \, ds < \infty. \tag{17}
\]
A minor adjustment of [25, Lemma 4.1] gives us that (17) is equivalent to  
\[
\int_{1}^{\infty} \left( \int_{0}^{s} F(t) \, dt \right)^{-p/(2p-q+1)} \, ds < \infty, \tag{18}
\]
giving us the first part of the result. Assume now that (18) holds. We see that (8) can be rewritten as
\[
\begin{cases}
  u(r) = u_0 + \int_{0}^{r} t^{1-N} \left( \int_{0}^{t} s^{a} v^{p}(s) \, ds \right) \, dt, & r > 0, \\
  v(r) = v_0 + \int_{0}^{r} t^{1-N} \left( \int_{0}^{t} s^{b} v^{q}(s) f(|u'(s)|) \, ds \right) \, dt, & r > 0, \\
  u(0) > 0, \quad v(0) > 0, \quad u'(0) = 0, \quad v'(0) = 0.
\end{cases} \tag{19}
\]
Using a contraction mapping argument, we can show that system (19) has a solution \((u, v)\) defined on some maximum interval \([0, R_0)\). Now, fix \(\rho \in (0, R_0)\), and, recalling (14) and (15), we have
\[
\begin{align*}
  f(w(r)) &\leq C v^{-q}(r) v''(r) \quad \text{for all } \rho \leq r < R_0, \\
  w'(r) &\leq C v^{p}(r) \quad \text{for all } \rho \leq r < R_0. \tag{20}
\end{align*}
\]
Multiplying the two inequalities in (20) and integrating over \([\rho, r]\) we find  
\[
F(w(r)) - F(w(\rho)) \leq C v^{p-q}(r) v'(r),
\]
which we can express as  
\[
F(w(r)) \leq C v^{p-q}(r) v'(r).
\]
Using (20) again, we see that the above becomes  
\[
w'(r) F(w(r)) \leq C v^{2p-q}(r) v'(r) \quad \text{for all } \rho \leq r < R_0.
\]
Define  
\[
G(r) := \int_{\rho}^{r} F(t) \, dt \quad \text{for all } \rho \leq r < R_0.
\]
We thus have  
\[
G(w(r)) = \int_{\rho}^{w(r)} F(t) \, dt \leq C \int_{\rho}^{r} v^{2p-q}(t) v'(t) \, dt \\
\leq C [v^{p}(r)]^{(2p-q+1)/p} \\
\leq C [w'(r)]^{(2p-q+1)/p} \quad \text{for all } \rho \leq r < R_0.
\]
Hence
\[ C \leq \frac{w'(r)}{G(w(r))^{p/(2p-q+1)}} \text{ for all } \rho \leq r < R_0. \]

Integrating the above over \([\rho, r]\) gives
\[ C(r - \rho) \leq \int_{\rho}^{r} \frac{w'(t)}{G(w(t))^{p/(2p-q+1)}} \, dt = \int_{w(\rho)}^{w(r)} \frac{dt}{G(t)^{p/(2p-q+1)}}. \]

Now, letting \( r \to R_0 \) we see that
\[ C(R_0 - \rho) \leq C \int_{1}^{\infty} \frac{dt}{G(t)^{p/(2p-q+1)}} \]
from which it follows that
\[ R_0 \leq C \int_{1}^{\infty} \left( \int_{0}^{s} F(t) \, dt \right)^{-p/(2p-q+1)} \, ds < \infty. \]

We have obtained a positive radial solution \((u, v)\) of (8) in \( B_{R_0} \) satisfying \( \lim_{r \to R_0^-} v(r) = \infty \). Now, if \( R > 0 \) is any arbitrary radius, we set
\[ \tilde{f}(t) = \lambda^{b} + a(1-q)/p + 2(1+1-q)/p \cdot f \left( \frac{t}{\lambda} \right) \quad \text{for all } t \geq 0. \]

By the above, we know there exists \((\tilde{u}, \tilde{v})\) satisfying
\[
\begin{cases}
\Delta \tilde{u} = r^{a} \tilde{v}^{p} & \text{in } B_{R}, \\
\Delta \tilde{v} = r^{b} \tilde{v}^{q} \tilde{f}(|\nabla \tilde{u}|) & \text{in } B_{R},
\end{cases}
\]
where \( B_{R} \) is a maximum ball of existence. Let
\[
\begin{cases}
u(r) = \tilde{v} \left( \frac{r}{\lambda} \right) & \text{in } B_{R}, \\
v(r) = \lambda^{-(a+2)/p} \tilde{v} \left( \frac{r}{\lambda} \right) & \text{in } B_{R}.
\end{cases}
\]

Taking \( \lambda = R/R_0 \) we see that \((u, v)\) is a solution to (8) in \( B_{R} \), finishing the proof. □

**Lemma 4.2** System (8) has a positive radial solution satisfying
\[ u \text{ is bounded in } B_{R} \text{ and } \lim_{|x| \nearrow R} v(x) = \infty \quad (21) \]
if and only if
\[ \int_{1}^{\infty} s \left( \int_{0}^{s} F(t) \, dt \right)^{-p/(2p-q+1)} \, ds < \infty. \]
Also, system (8) has a positive radial solution satisfying

\[
\lim_{|x| \to R} u(x) = \lim_{|x| \to R} v(x) = \infty
\]

if and only if

\[
\int_1^\infty \left( \int_0^s F(t) \, dt \right) -p/(2p-q+1) \, ds < \infty, \quad \int_1^\infty s \left( \int_0^s F(t) \, dt \right)^{-p/(2p-q+1)} \, ds = \infty.
\]

**Proof** It is enough to prove this for a solution satisfying (21), so assume that \((u, v)\) is a solution to (8) satisfying this condition. From Lemma 4.1 we know that \(f\) must satisfy (18), and by [25, Lemma 4.1], we also have that

\[
\left( \int_0^{2s} F(t) \, dt \right)^{p/(2p-q+1)} \geq \left( \int_0^s \sqrt{f(t)} \, dt \right)^{2p/(2p-q+1)} \quad \text{for all } s \geq 0.
\]

Arguing in a similar way to Lemma 4.1, we find that there exists \(\rho \in (0, R)\) such that

\[
\int_1^\infty \left( \int_0^s F(t) \, dt \right) -p/(2p-q+1) \, ds \leq \int_1^{w(r)} \left( \int_0^s \sqrt{f(t)} \, dt \right)^{2p/(2p-q+1)} \, ds \leq C_1 (R-r)
\]

and

\[
\int_1^\infty \left( \int_0^s F(t) \, dt \right) -p/(2p-q+1) \, ds \geq C_2 (R-r)
\]

for all \(\rho < r < R\). Now, let \(\Phi : (0, \infty) \to (0, \infty)\) be defined as

\[
\Phi(t) = \int_t^\infty \left( \int_0^s F(\sigma) \, d\sigma \right)^{-p/(2p-q+1)} \, ds.
\]

We note that \(\Phi\) is decreasing and by (18) we see \(\lim_{t\to\infty} \Phi(t) = 0\). From (23) and (24) we find

\[
\Phi(2w(r)) \leq C_1 (R-r) \quad \text{and} \quad \Phi(w(r)) \geq C_2 (R-r) \quad \text{for all } \rho \leq r < R.
\]

Since \(\Phi\) is decreasing, this then implies

\[
2w(r) \geq \Phi^{-1}(C_1 (R-r)) \quad \text{for all } \rho \leq r < R,
\]

\[
w(r) \leq \Phi^{-1}(C_2 (R-r)) \quad \text{for all } \rho \leq r < R.
\]

Now, recalling that

\[
u(r) = u(\rho) + \int_\rho^r w(t) \, dt \quad \text{for all } \rho \leq r < R,
\]
we see that \( \lim_{r \to R^-} u(r) = \infty \) if and only if

\[
\int_0^R w(t) \, dt = \infty
\]

if and only if

\[
\int_0^R \Phi^{-1}(C(R - r)) \, dt = \infty,
\]

for some \( C > 0 \). Hence \( \lim_{r \to R^-} u(r) = \infty \) if and only if

\[
\int_0^{C(R - \rho)} \Phi^{-1}(u) \, du = \infty
\]

if and only if

\[
\int_0^1 \Phi^{-1}(u) \, du = \infty.
\]

The change of variables \( t = \Phi^{-1}(u) \) then gives us that \( \lim_{r \to R^-} u(r) = \infty \) if and only if

\[
\int_1^\infty s \left( \int_0^s F(t) \, dt \right)^{-p/(2p-q+1)} \, ds = \infty.
\]

To show that the opposite implication holds, one can proceed as in Lemma 4.1 to obtain the local existence of a solution, and then use a scaling argument. \( \square \)

**Proof of Theorem 2.3** First, we assume that (11) holds. We have seen that this implies the existence of a positive, radial solution \((u, v)\) in a maximum ball. Both \(u\) and \(v\) are increasing, and by Corollary 2.2 we know that \(u\) and \(v\) are both bounded. Hence the domain of existence must be \(\mathbb{R}^N\).

Conversely, assume that (11) does not hold, and let \((U, V)\) be a solution to (10). We know from Lemma 4.2 that there exists a solution \((\tilde{u}, \tilde{v})\) satisfying

\[
\lim_{r \to 1^-} \tilde{v}(r) = \infty.
\]

Now, if \((U, V)\) is a solution to (10) in \(\mathbb{R}^N\) then so is \((U_\lambda, V_\lambda)\), where

\[
U_\lambda(r) = \lambda^{p(b+2(s-a+2)(q-1))} U(\lambda r),
\]

\[
V_\lambda(r) = \lambda^{b+2(s-a+1)} V(\lambda r).
\]

Therefore, by considering small enough \(\lambda > 0\), we are justified in assuming that \(V(0) > \tilde{v}(0) > 0\). Define

\[
R := \sup \{ r \in (0, 1) \mid V(t) > \tilde{v}(t) \text{ in } (0, r) \},
\]
and assume \( R \neq 1 \). Now, for all \( 0 < r < R \) we have, where \( W = U' \) and \( \tilde{w} = \tilde{u}' \),
\[
(W r^{N-1})' = r^{N+a-1} V^p(r) > r^{N+a-1} \tilde{v}^p(r) = (\tilde{w} r^{N-1})'.
\]
Integration over \([0, r]\), where \( 0 < r \leq R \), yields \( W > \tilde{w} \) on \((0, R]\). Using a similar strategy we see
\[
(V' r^{N-1})' = r^{N+b-1} V^q(r) W^s(r) > r^{N+b-1} \tilde{v}^q(r) \tilde{w}^s(r) = \lambda^B (v'(r) r^{N-1})'.
\]
This implies that \( V' > \tilde{v}' \) on \((0, R]\) and hence \( V > \tilde{v}' \) on \([0, R]\), contradicting the fact that \( R \neq 1 \). Hence \( R = 1 \), and so \( V > \tilde{v} \) on \((0, 1)\). But this then implies that \( \lim_{r \to 1^-} V(r) = \infty \), which contradicts the fact that \( V(r) \) is a solution of (10). \(\square\)

**Proof of Theorem 2.4** We obtained that \( u', v', u, v \) are increasing in the proof of Theorem 2.1 and
\[
\begin{align*}
\left\{ \begin{array}{l}
u'(r) = r^{1-N} \int_0^r t^{N-1+a} v^p(t) \, dt & \text{for all } r > 0, \\v'(r) = r^{1-N} \int_0^r t^{N-1+b} v^q(t)(u')^s(t) \, dt & \text{for all } r > 0,
\end{array} \right.
\end{align*}
\]
which gives us
\[
\frac{r^{a+1} v^p(0)}{N + a} \leq u'(r) \leq \frac{r^{a+1} v^p(r)}{N + a} \quad \text{for all } r > 0 \tag{25}
\]
and
\[
\frac{v^{ps+q}(0)r^{(a+1)s+b+1}}{(N + b)(N + a)^s} \leq v'(r) \leq \frac{r^{b+1}v^q(r)u'^s(r)}{N + b} \quad \text{for all } r > 0. \tag{26}
\]
Using (25) and (26) we obtain that \( u'(r), v'(r), u(r), v(r) \) tend to infinity as \( r \to \infty \). Next, we make the following change of variables (see [2, 14, 17]):
\[
\begin{align*}
X(t) &= \frac{ru'(r)}{u(r)}, & Y(t) &= \frac{rv'(r)}{v(r)}, \\
Z(t) &= \frac{r^{a+1} v^p(r)}{u'(r)}, & W(t) &= \frac{r^{b+1} v^q(r) u'^s(r)}{v'(r)},
\end{align*}
\]
where \( t = \ln(r) \) for \( r \in (0, \infty) \). Direct computation shows that \((X(t), Y(t), Z(t), W(t))\) satisfies
\[
\begin{align*}
X_t &= X(Z - (N - 2) - X) & \text{for all } t \in \mathbb{R}, \\
Y_t &= Y(W - (N - 2) - Y) & \text{for all } t \in \mathbb{R}, \\
Z_t &= Z(N + a + pY - Z) & \text{for all } t \in \mathbb{R}, \\
W_t &= W(sZ + N - sN + s + b + qY - W) & \text{for all } t \in \mathbb{R}.
\end{align*}
\]
Also, by L’Hôpital’s rule we deduce that \( \lim_{t \to \infty} X(t) = 2 - N + \lim_{t \to \infty} Z(t) \). Hence, it is enough to study the last three equations of (27), that is,

\[
\begin{align*}
Y_t &= Y(W - (N - 2) - Y) \quad \text{for all } t \in \mathbb{R}, \\
Z_t &= Z(N + a + pY - Z) \quad \text{for all } t \in \mathbb{R}, \\
W_t &= W(sZ + N - sN + s + b + qY - W) \quad \text{for all } t \in \mathbb{R}.
\end{align*}
\]

(28)

Our system can be rewritten as

\[
\dot{\xi} = h(\xi)
\]

where

\[
\xi = \begin{pmatrix} Y(t) \\ Z(t) \\ W(t) \end{pmatrix} \quad \text{and} \quad h(\xi) = \begin{pmatrix} Y(W - (N - 2) - Y) \\ Z(N + a + pY - Z) \\ W(sZ + N - sN + s + b + qY - W) \end{pmatrix}.
\]

One notices that system (29) is cooperative. Therefore, the following comparison principle holds.

**Lemma 4.3** Let us assume that

\[
\xi(t) = \begin{pmatrix} Y(t) \\ Z(t) \\ W(t) \end{pmatrix} \quad \text{and} \quad \tilde{\xi}(t) = \begin{pmatrix} \tilde{Y}(t) \\ \tilde{Z}(t) \\ \tilde{W}(t) \end{pmatrix}
\]

are the two nonnegative solutions of (29) such that

\[
Y(t_0) \geq \tilde{Y}(t_0), \quad Z(t_0) \geq \tilde{Z}(t_0), \quad W(t_0) \geq \tilde{W}(t_0)
\]

for some \( t_0 \in \mathbb{R} \). Then, we have

\[
Y(t) \geq \tilde{Y}(t), \quad Z(t) \geq \tilde{Z}(t), \quad W(t) \geq \tilde{W}(t) \quad \text{for all } t \geq t_0.
\]

Using (25) and (26) we deduce that \( Z \geq N + a \) and \( W \geq N + b \). Hence, we only have two equilibria of (28) which satisfy \( Z \geq N + a \) and \( W \geq N + b \), that is,

\[
\xi_1 = \begin{pmatrix} 0 \\ N + a \\ N + s(a + 1) + b \end{pmatrix} \quad \text{and} \quad \xi_2 = \begin{pmatrix} 2 + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q} \\ N + a + p \left( 2 + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q} \right) \\ N + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q} \end{pmatrix}.
\]

**Lemma 4.4** \( \xi_2 \) is asymptotically stable.

**Proof** At \( \xi_2 \), we have the following linearized matrix:

\[
M = \begin{bmatrix} -Y_2 & 0 & Y_2 \\ pZ_2 & -Z_2 & 0 \\ qW_2 & sW_2 & -W_2 \end{bmatrix}.
\]
and the associated characteristic polynomial of $M$ is

$$P(\lambda) = \det(\lambda I - M) = \lambda^3 + \alpha \lambda^2 + \beta \lambda + (1 - ps - q)\gamma,$$

where

$$\alpha = Y_2 + Z_2 + W_2,$$

$$\beta = Y_2 Z_2 + Z_2 W_2 + (1 - q)Y_2 W_2,$$

$$\gamma = Y_2 Z_2 W_2.$$

As $ps + q < 1$ and $\alpha, \beta, \gamma > 0$, we have that $P(\lambda) > 0$ for all $\lambda \geq 0$. In case $P$ has three real roots, then all of them are negative, which makes $\xi_2$ asymptotically stable.

Next, we need to consider the case where $P$ has exactly one real root. So, let us assume that $\lambda_1 \in \mathbb{R}$ and $\lambda_2, \lambda_3 \in \mathbb{C} \setminus \mathbb{R}$ be the roots of characteristic polynomial $P$. We need to show that $\text{Re}(\lambda_2) = \text{Re}(\lambda_3) < 0$, that is,

$$P(-\alpha) = -\beta \alpha + (1 - ps - q)\gamma < 0,$$

which is same as claiming, $\beta \alpha > (1 - ps - q)\gamma$. By the use of AM-GM inequality we get that

$$\alpha \geq 3\sqrt[3]{Y_2 Z_2 W_2} \quad \text{and} \quad \beta \geq 3(1 - q)^{1/3}\sqrt[3]{(Y_2 Z_2 W_2)^2}$$

which further gives us the desired result, that is, $\alpha \beta > (1 - ps - q)\gamma$. Hence, $\xi_2$ is asymptotically stable. 

**Lemma 4.5** For all $t \in \mathbb{R}$, we have

$$0 \leq Y(t) \leq 2 + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q},$$

$$N + a \leq Z(t) \leq N + a + p\left(2 + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q}\right),$$

$$N + s(a + 1) + b \leq W(t) \leq N + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q}.$$ 

**Proof** As $v'(0) = 0$ and $v(0) > 0$, we deduce that $\lim_{t \to -\infty} Y(t) = \lim_{r \to 0} \frac{rv'(r)}{v(r)} = 0$. Next, we show that there exists $t_j \to -\infty$ such that

$$\begin{cases}
  Y(t_j) \leq Y_2, \\
  Z(t_j) \leq Z_2, \\
  W(t_j) \leq W_2.
\end{cases} \quad (30)$$

Since $\lim_{t \to -\infty} Y(t) = 0$ and $\lim_{t \to -\infty} Z(t) = N + a$, we only need to prove the last part of (30). So, let us assume by contradiction that this is not true. We thus have
$W > W_2$ in $(-\infty, t_0)$ for some $t_0 \in \mathbb{R}$. This implies

$$W_t = W(sZ + N - sN + s + b + qY - W) < 0 \text{ in } (-\infty, t_0)$$

for small enough $t_0$. Therefore, $W$ is decreasing in the neighbourhood of $-\infty$, that is, there exists $\ell = \lim_{t \to -\infty} W(t)$. Again, by the use of L’Hôpital’s rule we obtain

$$\ell = \lim_{t \to -\infty} W(t) = \lim_{r \to 0} \frac{r^{b+1}v^q(r)u^b(r)}{v'(r)} = \frac{s(N + a) - s(N - 1) + b + 1}{1 - (N - 1)/\ell},$$

which yields $\ell = N + s(a + 1) + b < W_2$, and this contradicts the assumption that $W > W_2$ in a neighbourhood of $-\infty$. Hence, with this we have proven the last part of (30).

Next, we apply the Comparison Lemma 4.3 on all the intervals $[t_j, \infty)$ for $j \geq 1$ in order to obtain the upper bound inequalities in Lemma 4.5. Similarly, the lower bound inequalities can be obtained.

Now, assume $L = \{\xi_1, \xi_2\} \subset \mathbb{R}^3$. We have that $\omega(\xi) \subseteq L$ by Lemma 4.5. As $\xi_2$ is asymptotically stable, we get that $L$ has no circuits. Also, by (12), one could obtain that

$$\div h(\xi) = -W + (s - 2)Z + (-2 + p + q)Y + 2 + a + N(1 - s) + s + b < 0 \text{ in } L.$$

By the use of Theorems 3.4 and 3.5 we get that $\omega(\xi)$ reduces to one of the equilibria $\xi_1$ or $\xi_2$. In case $\xi(t) \to \xi_1$ as $t \to \infty$, then we have that $Y(t) \to 0$ as $t \to \infty$. Also, we obtain that $Y_t > 0$ in a neighbourhood of infinity by using the second equation of (27) which is impossible given that $Y(t) > 0$ in $\mathbb{R}$. Therefore, $\xi(t) \to \xi_2$ as $t \to \infty$, that is

$$\lim_{t \to \infty} X(t) = 2 + a + p \left(2 + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q}\right),$$

$$\lim_{t \to \infty} Y(t) = 2 + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q},$$

$$\lim_{t \to \infty} Z(t) = N + a + p \left(2 + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q}\right),$$

$$\lim_{t \to \infty} W(t) = N + \frac{b + 2q + s(1 + a + 2p)}{1 - ps - q}.$$

Finally, using the definition of $(X(t), Y(t), Z(t), W(t))$, we get

$$\lim_{|x| \to \infty} u(x)|x|^{-(a+2)ps+q+ps(a+1)+b+2} = \frac{(AB^sK)^{p/(ps+q-1)}}{DK},$$

and

$$\lim_{|x| \to \infty} v(x)|x|^{-(a+1)s+b+2}/(1-ps-q) = (AB^sK)^{1/(ps+q-1)},$$
where,
\[
A = \lim_{t \to \infty} Y(t), \quad B = \lim_{t \to \infty} Z(t), \\
K = \lim_{t \to \infty} W(t), \quad D = \lim_{t \to \infty} X(t).
\]

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