INDICATOR FRACTIONAL STABLE MOTIONS

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Abstract
Using the framework of random walks in random scenery, Cohen and Samorodnitsky (2006) introduced a family of symmetric $\alpha$-stable motions called local time fractional stable motions. When $\alpha = 2$, these processes are precisely fractional Brownian motions with $1/2 < H < 1$. Motivated by random walks in alternating scenery, we find a complementary family of symmetric $\alpha$-stable motions which we call indicator fractional stable motions. These processes are complementary to local time fractional stable motions in that when $\alpha = 2$, one gets fractional Brownian motions with $0 < H < 1/2$.

1 Introduction

There are a plethora of integral representations for Fractional Brownian motion (FBM) with Hurst parameter $H \in (0, 1)$, and not surprisingly there are several generalizations of these integral representations to stable processes. These generalizations are often called fractional symmetric $\alpha$-stable (S$\alpha$S) motions, with $0 < \alpha < 2$, and they can be considered analogs of FBM. Two common fractional S$\alpha$S motions include linear fractional stable motion (L-FSM) and real harmonizable fractional stable motion (RH-FSM).

In [CS06], a new generalization of FBM, $H > 1/2$, called local time fractional stable motion (LT-FSM) was introduced. LT-FSM is particularly interesting because it is a subordinated process (this terminology is taken from Section 7.9 of [ST94] and should not be confused with subordination in the sense of time-changes). Subordinated processes are processes constructed from integral representations with random kernels, or said another way, where the stable random measure (of the integral representation) has a control measure related in some way to a probability measure of some other stochastic process (see Section 2 below). We note that subordinated processes are examples of what are known in the literature as doubly stochastic models.

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In this work we introduce another subordinated process which can be considered a natural extension of LT-FSM to $H < 1/2$. The processes we consider have random kernels of a very simple type, namely the indicator function

$$1_{[A_t,A_s]}(x) \quad ([A_t,A_s] := [A_s,A_t] \text{ if } A_t < A_s)$$

with respect to some self-similar stationary increment (SSSI) process $A_t$. As such we call these processes indicator fractional stable motions (I-FSM).

I-FSM’s relation to LT-FSM comes from the idea that the indicator function of a real-valued process $A_t$ can be thought of as an alternating version of the local time of $A_t$ in the following way. Suppose $S_n$, with $S_0 = 0$, is a discrete-time simple random walk on $\mathbb{Z}$. If $e$ is the edge between $k$ and $k+1$, then the discrete local time of $S_n$ at $e$ is the total number of times $S_n$ has gone from either $k$ to $k+1$ or from $k+1$ to $k$, up to time $n$. Now, instead of totaling the number of times $S_n$ crosses over edge $e$, one can consider the parity of the number of times $S_n$ crosses $e$ up to time $n$. The parity of the discrete local time at edge $e$ up to time $n$ is odd if and only if $e$ is between 0 and $S_n$. Thus, heuristically, the edges which contribute to an “alternating local time” are those edges which lie between 0 and $A_t$. This heuristic is discussed more rigorously in [JM11].

We can generalize the motivational discrete model to all random walks on $\mathbb{Z}$. In this case, when $S_n$ goes from $x$ to $y$ on a given step, it “crosses” all edges in between. In terms of the discrete local time, we heuristically think of the random walk as having spent a unit time at all edges between $x$ and $y$ during that time-step.

The first question one must ask is: are these new stable processes a legitimate new class of processes or are they just a different representation of L-FSMs and/or RH-FSMs? Using characterizations of the generating flows for the respective processes (see Section 3 below), [CS06] showed that the class of LT-FSMs is disjoint from the classes of RH-FSMs and L-FSMs. Following their lead, we use the same characterizations to show that when the (discretized) subordinating process $\{A_n\}_{n \in \mathbb{N}}$ is recurrent, the class of I-FSMs is also disjoint from the two classes, RH-FSMs and L-FSMs. Since I-FSMs and LT-FSMs have disjoint self-similarity exponents when $1 < \alpha < 2$, these two classes of processes are also disjoint when $1 < \alpha < 2$. For $\alpha < 1$, the class of I-FSMs has a strictly larger self-similarity range than the class of LT-FSMs.

The outline of the rest of the paper is as follows. In the next section we define I-FSMs and show that they are SaS-SSSI processes. In Section 3 we give the necessary background concerning generating flows and characterizations with respect to them. In Section 4, we give the classification of I-FSMs according to their generating flows along with a result on the mixing properties of the stable noise associated with an I-FSM.

## 2 Indicator fractional stable motions

Let $m$ be a $\sigma$-finite measure on a measurable space $(B, \mathcal{B})$, and let

$$\mathcal{B}_0 = \{A \in \mathcal{B} : m(A) < \infty\}.$$

**Definition 2.1.** A SaS random measure $M$ with control measure $m$ is a $\sigma$-additive set function on $\mathcal{B}_0$ such that for all $A_1 \in \mathcal{B}_0$

1. $M(A_1) \sim S_m(m(A_1)^{1/\alpha})$

2. $M(A_1)$ and $M(A_2)$ are independent whenever $A_1 \cap A_2 = \emptyset$
where \( S_\alpha(\sigma) \) is a SaS random variable with scale parameter \( \sigma \) (see Section 3.3 of [ST94] for more details).

Another way to say the second property above is to say that \( M \) is independently scattered.

For context, let us first define LT-FSM. Throughout this paper

\[
\lambda := \text{Lebesgue measure on } \mathbb{R}.
\]

Let \((\Omega', \mathcal{F}', P')\) support a subordinating process \( A_t \). \( A_t \) is either a FBM-\( H' \) or a \( S\beta S\)-Levy motion, \( \beta \in (1,2] \), with jointly continuous local time \( L_\alpha(t,x)(\omega') \). By self-similarity, \( A_0 = 0 \) almost surely.

Suppose a SaS random measure \( M \) with control measure \( P' \times \lambda \) lives on some other probability space \((\Omega, \mathcal{F}, P)\). An LT-FSM is a process

\[
X^H_{\lambda}(t) := \int_{\Omega'} \int_{\mathbb{R}} L_\alpha(t,x)(\omega') M(d\omega', dx), \ t \geq 0,
\]

where \( X^H_{\lambda}(t) \) is a SaS-SSSI process with self-similarity exponent \( H = 1 - H' + H'/\alpha \) and \( H' \) is the self-similarity exponent of \( A_t \) (see Theorem 3.1 in [CS06] and Theorem 1.3 in [DGP08]).

We now define I-FSM which is the main subject of this work. Let \((\Omega', \mathcal{F}', P')\) support \( A_t \), a non-degenerate \( S\beta S\)-SSSI process with \( \beta \in (1,2] \) and self-similarity exponent \( H' \in (0,1) \) (again by self-similarity \( A_0 = 0 \) almost surely). Suppose a SaS random measure \( M \) with control measure \( P' \times \lambda \) lives on some other probability space \((\Omega, \mathcal{F}, P)\).

An indicator fractional stable motion is a process

\[
Y^H_{\lambda}(t) := \int_{\Omega'} \int_{\mathbb{R}} 1_{[0,A_t(\omega')]}(x) M(d\omega', dx), \ t \geq 0.
\]

A nice observation is that the finite dimensional distributions of the process do not change if we replace the kernel \( 1_{[0,A_t(\omega')]}(x) \) with \( \text{sign}(A_t(\omega')) 1_{[0,A_t(\omega')]}(x) \):

\[
\begin{align*}
\sum_{j=1}^{n} \theta_j & \int_{\Omega'} \int_{\mathbb{R}} \text{sign}(A_j(\omega')) 1_{[0,A_j(\omega')]}(x) M(d\omega', dx) \\
= & \sum_{j=1}^{n} \theta_j \int_{\Omega'} \int_{\mathbb{R}^+} 1_{\omega' : A_j(\omega') > 0} 1_{[0,A_j(\omega')]}(x) M(d\omega', dx) \\
& \quad + \sum_{j=1}^{n} \theta_j \int_{\Omega'} \int_{\mathbb{R}^-} -1_{\omega' : A_j(\omega') < 0} 1_{[0,A_j(\omega')]}(x) M(d\omega', dx) \\
= & \sum_{j=1}^{n} \theta_j \int_{\Omega'} \int_{\mathbb{R}} 1_{[0,A_j(\omega')]}(x) M(d\omega', dx).
\end{align*}
\]

where the last line holds since \( M \) is both symmetric and independently scattered.

The reason that this is helpful is because the equality

\[
\text{sign}(A_t) 1_{[0,A_t]}(x) = (A_t - x)_+^0 - (-x)_+^0
\]

makes it intuitively clear that the increments of \( Y^H_{\lambda}(t) \) are stationary.
We note that both LT-FSM and I-FSM can technically be extended to the case where \( A_t \) has self-similarity exponent \( H' = 1 \). In these degenerate cases, the kernels for LT-FSM and I-FSM coincide becoming the non-random family of functions \( \{1_0, t \}_{t \geq 0} \) thereby giving us

\[
\int_{\mathbb{R}} 1_{[0,t]} M(dx), \ t \geq 0.
\]

These are the S\( \alpha \)S Levy motions with \( \alpha \in (0, 2) \).

**Theorem 2.2.** The process \( Y_A^H(t) \) is a well-defined S\( \alpha \)S-SSSI process with self-similarity exponent \( H = H'/\alpha \).

**Proof.** We start by noting that

\[
\int_{\Omega'} \int_{\mathbb{R}} |1_{[0,A_t(\omega')]}(x)|^\alpha dx P'(d\omega') = E' \int_{\mathbb{R}} 1_{[0,A_t(\omega')]}(x) dx = E'|A_t| < \infty
\]

(6)

where the finite expectation follows since \( A_t \) is a S\( \beta \)S process with \( \beta > 1 \). This shows that \( Y_A^H(t) \) is a well-defined S\( \alpha \)S process (see Section 3.2 of [ST94] for details).

Recall that the control measure for \( M \) is \( P' \times \lambda \). Using the alternative kernel given in (4), by Proposition 3.4.1 in [ST94] we have for \( \theta_j \in \mathbb{R} \) and times \( t_j, s_j \in \mathbb{R}^+ \):

\[
E \exp \left( i \sum_{j=1}^k \theta_j (Y_A^H(t_j) - Y_A^H(s_j)) \right)
= \exp \left( - \int_{\mathbb{R}} E' \left[ \sum_{j=1}^k \theta_j \cdot \text{sign}(A_{t_j} - A_{s_j}) 1_{[A_{s_j}, A_{t_j}]}(x) \right]^\alpha dx \right).
\]

(7)

Note that if we had not used the alternative kernel given in (4), then the right-side above would have been more complicated.

Using (7), we have

\[
E \exp \left( i \sum_{j=1}^k \theta_j (Y_A^H(t_j + h) - Y_A^H(h)) \right)
= \exp \left( - \int_{\mathbb{R}} E' \left[ \sum_{j=1}^k \theta_j \cdot \text{sign}(A_{t_j+h} - A_h) 1_{[A_h, A_{t_j+h}]}(x) \right]^\alpha dx \right)
= \exp \left( - \int_{\mathbb{R}} E' \left[ \sum_{j=1}^k \theta_j \cdot \text{sign}(A_{t_j}) 1_{[0,A_{t_j}]}(x) \right]^\alpha dx \right)
= E \exp \left( i \sum_{j=1}^k \theta_j Y_A^H(t_j) \right)
\]

(8)

where the second equality follows since \( A_t \) has stationary increments. The above shows that \( Y_A^H(t) \) has stationary increments.
Using (7) once more, the self-similarity of \( \{A_t\}_{t \geq 0} \), and the change of variables \( y = c^{-H'} x \), we obtain

\[
\begin{align*}
E \exp \left( \int_{\mathbb{R}^+} \sum_{j=1}^k \theta_j Y_A^H(c t_j) \, dx \right) &= \exp \left( - \int_{\mathbb{R}^+} \sum_{j=1}^k \theta_j \cdot \text{sign}(A_{ct_j}) 1_{[0,A_{ct_j}]} \, \left| \frac{a}{y} \right| \, dy \right) \\
&= \exp \left( -c^H \int_{\mathbb{R}^+} \sum_{j=1}^k \theta_j \cdot \text{sign}(A_{t_j}) 1_{[0,A_{t_j}]} \, \left| \frac{a}{y} \right| \, dy \right) \\
&= E \exp \left( \sum_{j=1}^k \theta_j c^{H'/a} Y_A^H(t_j) \right)
\end{align*}
\]

(9)

Remarks.

1. For each fixed \( 0 < \alpha < 2 \), I-FSM is a class of \( S_\alpha \)-SSSI processes with self-similarity exponents \( H \) in the feasibility range \( 0 < H < 1/\alpha \). In particular, when \( 1 < \alpha < 2 \), this range of feasible \( H \) complements that of LT-FSM which has the feasibility range \( 1/\alpha < H < 1 \). When \( 0 < \alpha < 1 \), the feasibility range \( 0 < H < 1/\alpha \) of I-FSM is strictly bigger than that of LT-FSM: \( 1 < H < 1/\alpha \).

2. It is not hard to see that I-FSMs are continuous in probability since the subordinating process \( A_t \) is SSSI and continuous in probability. However, it follows from Theorem 10.3.1 in [ST94] that I-FSMs are not sample continuous. This is intuitive since I-FSMs should have continuity properties similar to those of \( S_\alpha \) Levy motions since the latter have the form

\[
\int_{\mathbb{R}^+} 1_{[0,t]}(x) M(dx), \quad t \geq 0
\]

where \( M \) is a \( S_\alpha \) random measure with Lebesgue control measure.

3. By Theorem 11.1.1 in [ST94] an I-FSM has a measurable version if and only if the subordinating process \( A_t \) has a measurable version.

3 Background: Ergodic properties of flows

Throughout this section we suppose that \( 0 < \alpha < 2 \). The general integral representations of \( \alpha \)-stable processes, of the type

\[
X(t) = \int_E f_t(x) M(dx), \quad t \in T
\]

(\( T = \mathbb{Z} \) or \( \mathbb{R} \)) are well-known (see the introduction of [Sam05]). Here \( M \) is a \( S_\alpha \) random measure on \( E \) with a \( \sigma \)-finite control measure \( m \), and \( f_t \in L^\alpha(E,m) \) for each \( t \). We call \( \{f_t(x)\}_{t \in T} \) a spectral representation of \( \{X(t)\} \).
Definition 3.1. A measurable family of functions \( \{ \phi_t \}_{t \in T} \) mapping \( E \) onto itself and such that

1. \( \phi_{t+s}(x) = \phi_t(\phi_s(x)) \) for all \( t, s \in T \) and \( x \in E \),
2. \( \phi_0(x) = x \) for all \( x \in E \)
3. \( m \circ \phi_t^{-1} \sim m \) for all \( t \in T \)

is called a nonsingular flow. A measurable family \( \{ a_t \}_{t \in T} \) is called a cocycle for the flow \( \{ \phi_t \}_{t \in T} \) if for every \( s, t \in T \) we have

\[
a_{t+s}(x) = a_t(x)a_s(\phi_t(x)) \quad m\text{-a.e.}
\]

(12)

In [Ros95] it was shown that in the case of measurable stationary SaS processes one can choose the (spectral) representation in \((11)\) to be of the form

\[
f_t(x) = a_t(x) \left( \frac{dm \circ \phi_t^{-1}(x)}{dm}(x) \right)^{1/\alpha} f_0 \circ \phi_t(x)
\]

(13)

where \( f_0 \in L^\alpha(E, m) \), \( \{ \phi_t \}_{t \in T} \) is a nonsingular flow, and \( \{ a_t \}_{t \in T} \) is a cocycle, for \( \{ \phi_t \}_{t \in T} \), which takes values in \( \{-1,1\} \). Also, note that one may always assume the following full support condition:

\[
supp\{f_t : t \in T\} = E.
\]

(14)

Henceforth we shall assume that \( T = \mathbb{Z} \) and will write \( f_n, \phi_n, \) and \( X(n) \). Note that in the discrete case we may always assume measurability of the process (see Section 1.6 of [Aar97]). Given a representation of the form \((13)\), we say that \( X(n) \) is generated by \( \phi_n \).

In [Ros95] and [Sam05], the ergodic-theoretic properties of a generating flow \( \phi_n \) are related to the probabilistic properties of the SaS process \( X(n) \). In particular, certain ergodic-theoretic properties of the flow are found to be invariant from representation to representation.

In Theorem 4.1 of [Ros95] it was shown that the Hopf decomposition of a flow is a representation-invariant property of stationary SaS processes. Specifically, one has the disjoint union \( E = C \cup D \) where the dissipative portion \( D \) is the union of all wandering sets and the conservative portion \( C \) contains no wandering subset. A wandering set is one such that \( \{ \phi_n(B) \}_{n \in \mathbb{Z}} \) are disjoint modulo sets of measure zero. Since \( C \) and \( D \) are \( \{ \phi_n \} \)-invariant, one can decompose a flow by looking at its restrictions to \( C \) and \( D \), and the decomposition is unique modulo sets of measure zero. A nonsingular flow \( \{ \phi_n \} \) is said to conservative if \( m(D) = 0 \) and dissipative if \( m(C) = 0 \).

The following result appeared as Corollary 4.2 in [Ros95] and has been adapted to the current context:

Theorem 3.2 (Rosinski). Suppose \( 0 < \alpha < 2 \). A stationary SaS process is generated by a conservative (dissipative, respectively) flow if and only if for some (all) measurable spectral representation \( \{ f_n \}_{n \in \mathbb{Z}^+} \subset L^\alpha(E, m) \) satisfying \((14)\), the sum

\[
\sum_{n \in \mathbb{Z}} |f_n(x)|^\alpha
\]

(15)

is infinite (finite) \( m\text{-a.e.} \) on \( E \).

In [Sam05], another representation-invariant property of flows, the positive-null decomposition of stationary SaS processes, was introduced.

A subset \( B \subset E \) is called weakly wandering if there is a subsequence with \( n_0 = 0 \) such that the sets \( \{ \phi_{n_k}B \}_{k \in \mathbb{N}} \) are disjoint modulo sets of measure zero. The null part \( N \) of \( E \) is the union of all weakly
It is known that stationary $S^\alpha$ has no affect on the distribution of its increments. (A. Gross)

Lemma 4.1 dissipative or conservative null. We first need a result which appeared as Theorem 2.7 of $\text{[Sam05]}$ that the decomposition is representation-invariant modulo sets of measure zero. A null flow is one with $m(P) = 0$ and a positive flow has $m(N) = 0$. Note that dissipative flows are automatically null flows, however in the case of conservative flows, both positive and null flows are possible.

4 Ergodic properties of indicator fractional stable noise

Properties of a $S\alpha$-SSSI process $Y(t)$ are often deduced from its increment process $Z(n) = Y(n) - Y(n - 1), n \in \mathbb{N}$ called a stable noise. In this section, we study the ergodic-theoretic properties (which were introduced in the previous section) of indicator fractional stable noise (I-FSN) which we define as

$$Z_n(n) := \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[0,A_n(N)])}(x) - 1_{[0,A_{n-1}(N)])}(x) M(d\omega, dx), \ n \in \mathbb{N}. \quad (16)$$

We note that in light of the proof of Theorem 2.2, one may deem it natural to instead use the kernel

$$\text{sign}(A_n(N))(x)1_{[0,A_n(N)])}(x) - \text{sign}(A_{n-1}(N))(x)1_{[0,A_{n-1}(N)])}(x).$$

However, as seen in (4), the $\text{sign}(A_n)$ has no affect on the distribution of the process and therefore has no affect on the distribution of its increments.

It is known that stationary $S\alpha$ processes generated by dissipative flows are mixing [SRMC93]. Concerning conservative flows, Theorem 3.1 of [Sam05] states that a stationary $S\alpha$ process is ergodic if and only if it is generated by a null flow, and examples are known of both mixing and non-mixing stationary $S\alpha$ processes generated by conservative null flows (see Section 4 of [GR93]). Our next goal is to show that I-FSN is mixing which implies that its flow is either dissipative or conservative null. We first need a result which appeared as Theorem 2.7 of [Gro94]:

Lemma 4.1 (A. Gross). Suppose $X_n$ is some stationary $S\alpha$ process, and assume $\{f_n\} \subset L^\alpha(E, m)$ is a spectral representation of $X_n$ with respect to the control measure $m$. Then $X_n$ is mixing if and only if for every compact $K \subset \mathbb{R} - \{0\}$ and every $\epsilon > 0$,

$$\lim_{n \to \infty} m\{x : f_0 \in K, |f_n| > \epsilon\} = 0. \quad (17)$$

**Theorem 4.2.** Indicator fractional stable noise is a mixing process.

**Proof.** Using the above lemma, it suffices to show that

$$\lim_{n \to \infty} (P' \times \lambda)\{(\omega', x) : x \in [0, A_1], x \in [A_n, A_{n+1}]\} = 0, \quad (18)$$

recalling that $[A_n, A_{n+1}] := [A_{n+1}, A_n]$ whenever $A_{n+1} < A_n$.

Let $c_i$ be constants such that for all $M > 0$, $P'(A_1 > M) < c_1 M^{-\beta}$ and

$$\int_{M}^{\infty} P'(A_1 > x) dx < c_2 M^{-\beta+1} \quad (19)$$
where $\beta > 1$. Also, recall that $0 < H' < 1$ is the self-similarity exponent of $A_t$. We have that

\[ (P' \times \lambda)\{(\omega', x) : x \in [0, A_1], x \in [A_n, A_{n+1}]\} \]

\[ = (P' \times \lambda)\{(\omega', x) : |x| > M, x \in [0, A_1], x \in [A_n, A_{n+1}]\} \]

\[ + (P' \times \lambda)\{(\omega', x) : |x| \leq M, x \in [0, A_1], x \in [A_n, A_{n+1}]\} \]

\[ \leq 2 \int_M^\infty P'(A_1 > x) \, dx + (P' \times \lambda)\{(\omega', x) : |x| \leq M, x \in [A_n, A_{n+1}]\} \]

\[ \leq 2c_2M^{-\beta+1} + 2M \sup_{x \in [-M, M]} P'\{\omega' : x \in [A_n(\omega'), A_{n+1}(\omega')]\} \]

\[ \leq 2c_2M^{-\beta+1} + 2MP'\{|A_n| \leq M\} \cup \{|A_{n+1}| \leq M\} \]

\[ + 2MP'\{|A_n < -M, A_{n+1} > M\} \cup \{|A_n > M, A_{n+1} < -M\}\} \]

\[ \leq 2c_2M^{-\beta+1} + 4MP'\{|A_1| \leq M/n^{H'}\} + 2M \cdot 2c_1M^{-\beta}. \tag{20} \]

where the first inequality uses the symmetry of $A_1$. The second inequality uses (19), and the third inequality uses the fact that for $x \in [-M, M]$, the event $\{\omega' : x \in [A_n(\omega'), A_{n+1}(\omega')]\}$ is contained by the event that either $A_n$ or $A_{n+1}$ is in $[-M, M]$ or that $[A_n, A_{n+1}]$ (which we defined as equivalent to $[A_{n+1}, A_n]$) contains $[-M, M]$. The final inequality uses both self-similarity and stationarity of increments.

Since the right side of (20) can be made arbitrarily small by choosing $M$ and then $n$ appropriately, the result is proved. \qed

Since I-FSN is mixing, it is generated by a flow which is either dissipative or conservative null. Our next result classifies the flow of I-FSN as conservative if almost surely

\[ \limsup_{n \to \infty} A_n = +\infty \quad \text{and} \quad \liminf_{n \to \infty} A_n = -\infty \quad \text{where} \ n \in \mathbb{N}. \tag{21} \]

This holds, for example, when $A_t$ is a FBM or a $S\beta$S Levy motion with $\beta > 1$.

**Theorem 4.3.** If the subordinating process $A_t$ satisfies (21), then the indicator fractional stable noise, $(Z_A(n))_{n \in \mathbb{Z}}$, is generated by a conservative null flow.

**Proof.** By (21), we have that $P'$-almost surely

\[ \sum_{n=0}^\infty |1_{[0,A_n(\omega')]}(x) - 1_{[0,A_{n+1}(\omega')]}(x)|^\alpha \]

\[ = \sum_{n=0}^\infty 1_{[A_n(\omega'),A_{n+1}(\omega')]}(x) = \infty \quad \text{for every} \ x. \tag{22} \]

Hence by Theorem 3.2 we have that $Z_A(n)$ is generated by a conservative flow. By Theorem 4.2 the flow is also null. \qed

**Remarks.**
1. When $A_n$ satisfies (21), the fact that I-FSMs are generated by conservative null flows implies they form a class of processes which are disjoint from the class of RH-FSMs (positive flows) and disjoint from the class of L-FSMs (dissipative flows). We have already seen that the classes of I-FSMs and LT-FSMs are disjoint when $1 < \alpha < 2$ due to their self-similarity exponents.

2. Another useful property of conservative flows comes from Theorem 4.1 of [Sam04]: If $Z_n(n)$ is generated by a conservative flow, then it satisfies the following extreme value property:

$$n^{-1/\alpha} \max_{j=1,...,n} Z_n(n) \overset{p}{\to} 0.$$  \hspace{1cm} (23)

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