Application of a method for calculating the sizes of perspective objects

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Abstract: Determining the parameters of perspective images of different objects is a complex problem. Graphic, graphoanalytical and analytical methods are used to solve such problems. Graphic methods are not accurate enough for complex objects compared to analytical methods, which although difficult to apply, are proved to be the most accurate ones. Graphoanalytical methods according to are considered the most appropriate in terms of accuracy and complexity of application. The purpose of this paper is to propose a graphoanalytic method for determining the parameters of perspective images of a parallelepiped. The coordinates of the points which are the vertices of the perspective image and the equations of the lines on which the edges of the perspective image lie are determined both by applying the knowledge of constructing a perspective image and that of analytical geometry.

Keywords: determining, images, Graphic methods, parameters, perspective.

1. Introduction

The current topic includes a number of fundamental questions viewed in the scope of the perspective such as: spatial concepts as part of the evolution of human cognition, the interpretation of space in plastic arts, and others considered to be fundamental in terms of perspective science [1,2,3,4,5]. The broadest perspective topics include many questions deriving from a number of science branches and disciplines that are auxiliary to perspective theory, as well as those that have further arisen based on this theory [6,7,8].

Building perspective images, determining the coordinates of the vertices and the equations of the edges, as well as the equations of the planes provides an aid in solving various practical problems.

The purpose of the present work is to apply a graphoanalytic method for determining the parameters of perspective images of a parallelepiped.

2. The main tasks formulation

In order to achieve this goal, the following main tasks are to be formulated:

- Discovering mathematical models for determining the parameters of the perspective image of a parallelepiped, differently positioned in relation to the projection plane.
- Developing a methodology for modeling perspective images of a parallelepiped and determining its actual dimensions.
The parameters of the perspective image at different locations of the parallelepiped in relation to the perspective plane SS N were determined.

The notations $AG = n$ for the distance from the tip of the parallelepiped $A$ to the perspective plane are employed. Point $G$ is the penetration point of the perpendicular into the perspective plane and $HG = m$. Depending on the length of the segment $n$, the following cases are considered: $n$ to be less than the diagonal of the square base and $n$ be greater than this diagonal.

The methodology also includes another classification regarding the angle at which the parallelepiped is inclined relative to the perspective plane. One particular case is considered where this angle is 45 degrees similar to [4].

This study deals with one of the mentioned cases, where $n$ is less than the diagonal of the base (polygon) and $\angle (SS_1, \overrightarrow{AB}) = 45^0$, at $n = 0$ and $m = 0$, and the formulation of the following problem is considered:

**Figure 1.** Perspective image of a cube.

There is a parallelepiped $\overrightarrow{ABCD}$ of which we know the actual length of the base edge $a$ and the height $q$. The parameters of the perspective image ABCDMNPQ of this cube are to be found, where $h$ is the height of the horizon (formula 1, 2, 3), $d$ is the distance (from the observer to the perspective plane - formula 4). A special case is considered where $h = d$ (Fig. 1).

Construction:

$$S, H = SH = h \ (t. H z S, S) \quad HH_1 \perp SS_1, \quad HH_1 = h \quad \overrightarrow{AH_1}, \overrightarrow{BH_1}, \overrightarrow{CH_1}, \overrightarrow{DH_1}$$

(1)

$$l_i : \begin{cases} zK_1 \\ \perp SS_1 \end{cases} \quad l_5 : \begin{cases} zK_3 \\ \perp SS_1 \end{cases} \quad g : \begin{cases} zG \\ \perp SS_1 \end{cases}$$

is constructed.

(2)

$$l_1 I H_1, S = \{B\}; \quad l_3 I H_1, S = \{D\}; \quad g I t = \{A\} \quad AM = x \ (M z HH_1) \text{ is constructed.}$$

(3)

$$l_1 I SM = \{N\}; \quad l_2 I S_1, M = \{Q\}$$

(4)
The Oxyz coordinate system was then introduced, the beginning of which coincides with the tip of the parallelepiped $\vec{A}$. The axis Ox passes along the edge $\vec{A}\vec{B}$, the axis Oy passes along the edge $\vec{A}\vec{D}$, and the axis Oz – along the edge $\vec{A}\vec{M}$. The following notations were adopted: $\beta$ is the horizontal projection plane (Ox, Oy), $\alpha$ is the projection plane in which the perspective image lies and $\gamma$ is the plane (Oz, O). We assume that the height of the horizon $h$ and the distance $d$ / the distance from the observer $E$ to the perspective plane / are equal. The arrangement of the parallelepiped with respect to the coordinate system thus introduced and the projection planes are shown in Fig. 2 and Fig. 3.

![Figure 2. Arrangement of the parallelepiped.](image)

![Figure 3. Horizon $h$ and the distance $d$ is are equal.](image)

3. Determining the vertices coordinates of the parallelepiped images

The coordinates of the vertices of the parallelepiped are determined as follows: $\vec{A}(0,0,0)$, $\vec{B}(a,0,0)$, $\vec{C}(a,a,0)$, $\vec{D}(0,a,0)$, $\vec{M}(0,0,q)$, $\vec{N}(a,0,q)$, $\vec{P}(a,a,q)$, $\vec{Q}(0,a,q)$, the vanishing points being: $S_1(-\frac{h}{\sqrt{2}}; h; h)$ and $S_2(-\frac{h}{\sqrt{2}}; \frac{h}{\sqrt{2}}; h)$, as well as their projections in plane $\beta$ - the points...
$S'_1(-\frac{h}{\sqrt{2}}; \frac{h}{\sqrt{2}}; 0)$ and $S'_2(\frac{h}{\sqrt{2}}; -\frac{h}{\sqrt{2}}; 0)$. Point E – that of the observer has the following coordinates: $E(-\frac{h}{\sqrt{2}}; -\frac{h}{\sqrt{2}}; h)$ (Fig 4).

The vertices of the perspective image of the parallelepiped can be determined analytically. Each vertex of the perspective image is a penetration point of the line passing through the observer point - E and the corresponding vertex of the geometric object $\overline{A \ B \ C \ D \ M \ N \ P \ Q}$ with the projection plane $\alpha$.

Thus points $\{B\} = \alpha \cap (\overline{EB})$, $\{C\} = \alpha \cap (\overline{EC})$, $\{D\} = \alpha \cap (\overline{ED})$, $\{N\} = \alpha \cap (\overline{EN})$, $\{P\} = \alpha \cap (\overline{EP})$, $\{Q\} = \alpha \cap (\overline{EQ})$, and points $\overline{A \ M}$ lie in the plane $\alpha$.

Points $\overline{A}$, $\overline{M}$, $S_1$, $S_2$, $S'_1$ and $S'_2$ are points from plane $\alpha$. The perspective plane equation can be determined in several ways:

- By any three points on the plane;
- By two coplanar to the plane and non-collinear to each other vectors determined by point on the plane.
- By a vector normal to the plane (for example, one of the vectors normal to the plane $\alpha$ vectors $\overrightarrow{AC} \perp \alpha$, $\overrightarrow{EE_1} \perp \alpha$) and a point on the plane.

Here, the perspective plane $\alpha$ equation is defined by the point $\overline{M} (0, 0, q)$ and the normal to the plane vector $\overrightarrow{AC}$ (a, a, 0). The equation of the $\alpha$ plane on which the perspective image lies is as follows: $\alpha : x + y = 0$. 

Figure 4. Determining the vertices coordinates.
The simultaneous solving the equations of the plane $\alpha : x + y = 0$ and the line
\[
(BE) : \frac{\sqrt{2}(a-x)}{h + a\sqrt{2}} = \frac{y\sqrt{2}}{-h} = \frac{z}{h}
\]
determine the coordinates of point
\[
B(-\frac{ah}{2h + a\sqrt{2}}; -\frac{-ah}{2h + a\sqrt{2}}; \frac{ah\sqrt{2}}{2h + a\sqrt{2}}), \text{ since } \{B\} = (BE) \cap \alpha .
\]

Similarly, the coordinates of the other points that are vertices of the perspective image are determined.

The point \{C\} = (CE) \cap \alpha . The plane equation $\alpha : x + y = 0$ and the line equation
\[
(CE) : \frac{a\sqrt{2} - x\sqrt{2}}{h + a\sqrt{2}} = \frac{a\sqrt{2} - y\sqrt{2}}{h + a\sqrt{2}} = z
\]
determine the coordinates of point C(0; 0; \frac{ah\sqrt{2}}{h + a\sqrt{2}}).

The point \{D\} = (DE) \cap \alpha . The plane equation $\alpha : x + y = 0$ and the line equation
\[
(DE) : \frac{x\sqrt{2}}{-h} = \frac{\sqrt{2}(a-y)}{h + a\sqrt{2}} = \frac{z}{h}
\]
determine the coordinates of point
\[
D(-\frac{-ah}{2h + a\sqrt{2}}; \frac{ah}{2h + a\sqrt{2}}; \frac{ah\sqrt{2}}{2h + a\sqrt{2}}).
\]
The point A (0; 0; 0) coincides with its perspective image point A (0; 0; 0).

Obviously (from the drawing, relative to the arrangement of the parallelepiped), points B and D have coordinates in x and y that differ only in sign, and their third coordinates in z are equal. Similarly, with respect to the Ox, Oy and Oz axes, the points A and C have equal first and second coordinates and differ only in the third coordinate z.

Knowing the coordinates of points A, B, C, D, you can easily find the coordinates of points M, N, P, and Q. Since the lines AM, BN, CP, DQ are parallel to each other, the first and second coordinates of points A and B, M and N, C and P, D and Q will be equal and will differ only in z.

As a crossing point of plane $\alpha : x + y = 0$ and line (NE)
\[
\frac{\sqrt{2}(a-x)}{h + a\sqrt{2}} = \frac{y\sqrt{2}}{-h} = \frac{z}{h} - a
\]
the N point coordinates are as follows: \[N(-\frac{ah}{2h + a\sqrt{2}}; -\frac{-ah}{2h + a\sqrt{2}}; \frac{ah(2 + \sqrt{2})}{2h + a\sqrt{2}}).
\]

As a crossing point of plane $\alpha : x + y = 0$ and line (PE)
\[
\frac{\sqrt{2}(a-x)}{h + a\sqrt{2}} = \frac{\sqrt{2}(a-y)}{h + a\sqrt{2}} = z\frac{a}{h} - a
\]
the P point coordinates are as follows: \[P(0; 0; \frac{ah(1 + \sqrt{2})}{h + a\sqrt{2}}).
\]

As a crossing point of plane $\alpha : x + y = 0$ and line (QE)
\[
\frac{\sqrt{2}(a-x)}{h + a\sqrt{2}} = \frac{\sqrt{2}(a-y)}{h + a\sqrt{2}} = z\frac{a}{h} - a
\]
the Q point coordinates are as follows: \[Q(-\frac{-ah}{2h + a\sqrt{2}}; \frac{ah}{2h + a\sqrt{2}}; \frac{ah(2 + \sqrt{2})}{2h + a\sqrt{2}}).
\]

4. From the mathematical models derived to find the coordinates of the points

Vertices of the perspective image of a parallelepiped, the following analysis can be made:

- The coordinates x and y of points A, M, C, P are the same, and for points B, N, D, Q are equal in modulus. The points B and N and D and Q have equal first and second coordinates, and the x and y
coordinates of the points B, N and D, Q differ only in sign.

- The third coordinate of point M is as much greater than the third coordinate of point A as the actual edge length of the parallelepiped.

- The third coordinates of the points C and P differ only in the numerical coefficient by which they are multiplied. At point C it is $\sqrt{2}$, and at point P - $(1 + \sqrt{2})$.

- The third coordinates of the points B, N, D, Q also differ only in numerical coefficient. For the points B and D this coefficient is the same and it is $\sqrt{2}$, and for the points N and Q the numerical coefficient is $(2 + \sqrt{2})$.

5. Conclusion

From these dependencies, the coordinates of the points - vertices of the perspective image can be obtained immediately, given the actual length of the edge of the cube and the distance (the distance from the observer to the perspective plane).

Based on the analysis and calculations, the coordinates of the perspective image of the parallelepiped were found. From the coordinates of the vertices of the perspective image, the equations of all perspective rays can be determined.

The following conclusions can be drawn from the statement made:

- Mathematical models can be constructed to determine the vertices of the perspective image of the parallelepiped and the equations of all perspective rays, using a given parallelepiped and a properly oriented coordinate system.

- The mathematical models constructed depend on four parameters $a$ - the edge of the cube, $q$ - the height of the parallelepiped, $h$ - the height of the horizon and the distance (d = h).

- Mathematical models constructed can be used to find a perspective image of any other edge object.

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