Abstract

In contrast with pseudo-gravitational effects that are mathematically analogous but physically quite distinct from gravity, this presentation deals with a kind of quasi-gravitational effect that can act in an asymmetrically moving brane worldsheet in a manner that approximates (and in a crude analysis might be physically indistinguishable from) the effect that would arise from genuine gravitation, of ordinary Newtonian type in non-relativistic applications, and of scalar-tensor (Jordan-Brans-Dicke rather than pure Einstein) type in relativistic applications.

1 Introduction

Most of the talks at this workshop on Analogue Models of Gravity have been concerned with what may be termed pseudo-gravitational effects, meaning effects whose mathematical description is more or less analogous to that of gravity, but whose physical nature is quite distinct. These effects typically involve another Lorentz signature metric that coexists with the ordinary spacetime metric (whose deviations from flatness are interpretable as corresponding to true gravity) but couples to matter in an entirely different way, specifying pseudo light cones that typically govern the propagation not of real light, nor gravity, but of some quite independent excitation such as sound.

The purpose of the present contribution is to draw attention to something rather different, what may be described as quasi-gravitational effects, meaning phenomena that affect matter locally in approximately the same physical manner as true gravity, even though their origin and detailed behaviour may be rather different. In the context of non-relativistic Newtonian gravitation theory, the most familiar example of such a quasi-gravitational effect is the centrifugal field attributable to the rotation of the earth that modifies the locally observable Galilean acceleration field by contributing a term that is to be added to the strictly gravitational contribution due to the Newtonian inverse square law attraction due to the terrestrial matter distribution. Although this centrifugal quasi-gravitational contribution is indistinguishable from the truly gravitational contribution
in a crude laboratory experiment, the difference is of course detectable, via the Coriolis
effect, in more sensitive experiments such as that of the Foucault pendulum.

The kind of quasi gravitational effect I wish to describe here is something that modifies
the induced spacetime metric on the \((q+1)\) dimensional worldsheet of a q-brane in a
higher dimensional background in a manner that approximates the effect of true \((q+1)\)
dimensional gravity, even though its origin and precise nature is essentially different.
It is of particular potential interest in the currently fashionable context of models that
represent our 4 dimensional universe as a 3 brane in a five dimensional background,
though in the kind of scenarios that are most commonly envisaged the effect considered
here would be excluded by the usual assumption of symmetry between the two opposite
sides of the 3-brane, while even if the symmetry assumption were dropped (as has recently
been proposed in cases where the q-brane worldsheet is coupled to a background gauge
\((q+1)\) form \([1, 2, 3]\)) the quasi gravitational effect could still be overwhelmed by much
stronger effects of genuinely gravitational origin (just as the terrestrial centrifugal effect
is overwhelmed by the centrally directed genuinely gravitational attraction).

\section{Equation of motion of brane worldsheet}

The effect to be considered here is derivable directly by perturbing the general purpose
brane worldsheet equation of motion, which is given \([4]\) in terms of the second fundamental
tensor \(K_{\mu^\rho}^\rho\) of the brane worldsheet and of the corresponding worldsheet stress energy
density tensor \(T_{\mu\nu}\) of the brane by

\[
T_{\mu\nu} K_{\mu^\rho}^\rho = \bot_{\mu^\rho}^\rho T_{\mu}^\rho,
\]

where \(\overline{T}_{\mu}\) is the external force density, if any, acting on the brane, and \(\bot_{\mu^\rho}^\rho\) is the orthogonal
projection tensor. The complementary (rank \(p+1\)) tangential projection tensor

\[
\gamma_{\mu^\nu} = g_{\mu^\nu} - \bot_{\mu^\rho}^\rho,
\]

i.e. the first fundamental tensor, defines the tangential covariant differentiation operator

\[
\nabla_{\mu} = \gamma_{\mu^\nu} \nabla_{\nu},
\]
whose action on the first fundamental tensor defines the second fundamental tensor ac-
cording to the specification

\[
K_{\mu^\rho}^\rho = \gamma_{\mu^\nu} \nabla_{\nu} \gamma_{\rho^\sigma}^\sigma,
\]

which is such as to ensure the Weingarten symmetry condition

\[
K_{\mu^\rho}^\rho = K_{\nu^\mu}^\rho,
\]
as a worldsheet integrability condition, as well as having the more obvious tangentiality
and orthogonality properties

\[
K_{\mu^\nu}^\sigma \gamma_{\sigma^\rho}^\rho = 0 = \bot_{\mu^\rho}^\rho K_{\rho^\lambda}^\lambda,
\]
while its trace
\[ K^\rho = K_{\mu}^{\mu \rho} = \nabla_{\nu} \gamma^{\mu \rho}, \]  \hfill (7)
inherits the simple worldsheet orthogonality property
\[ \gamma_{\sigma}^{\rho} K^{\sigma} = 0. \]  \hfill (8)

3 Perturbed worldsheet configuration

The quasi gravity effect to be considered occurs (in its simplest form) when the total surface stress energy tensor \( \mathcal{T}^{\mu \nu} \) is dominated by an isotropic (Dirac-Nambu-Goto type) contribution specified by a large fixed tension \( T_\infty \) say, together with a small additional contribution \( \tau^{\mu \nu} \) arising from the effect of local fields on the brane (representing the observable matter of the universe in brane - world scenarios) in the form
\[ \mathcal{T}^{\mu \nu} = -T_\infty \gamma^{\mu \nu} + \tau^{\mu \nu}, \]  \hfill (9)
in the presence of an external force of the commonly occurring kind (including a Magnus force on a string and a wind force on a sail) that is automatically orthogonal to the worldsheet, \( \gamma^{\mu \nu} f^{\nu} = 0 \) so that the orthogonal projection on the right in (1) is superfluous. Then if the observable matter contribution \( \tau^{\mu \nu} \) were absent, the dynamical equation of motion (1) would reduce to the simple form
\[ T_\infty K^{\rho} = f^{\rho}. \]  \hfill (10)
Starting from an almost uniform (low curvature) reference configuration of this kind, one can consider an actual configuration that deviates from this due to the presence of a matter distribution \( \tau^{\mu \nu} \) confined within a lengthscale that is relatively small (compared with the reference curvature scale) for which the dominant terms in the dynamical equation obtained by perturbation of (1) can be seen to be given by an expression of the form
\[ T_\infty \delta K^{\rho} = \tau^{\mu \nu} K_{\mu \nu}^{\rho}, \]  \hfill (11)
in which the perturbation \( \delta K^{\rho} \) of the curvature is given in terms of the Dalembertian wave operator \( \Box = \nabla^{\nu} \nabla_{\nu} \) of the \( (q+1) \) dimensional worldsheet metric, and of the surface orthogonal vector field \( \xi^{\mu} \) specifying the displacement of the worldsheet, by an expression of the form
\[ \delta K^{\rho} \simeq \Box \xi^{\rho}, \]  \hfill (12)
that is obtained from the general curvature perturbation formula \( [4] \) by retaining only the gradient terms of highest order, which are the ones that dominate in the localised (short lengthscale) limit.

For a brane worldsheet matter distribution that is approximately specified with respect to the relevant tangent rest frame unit vector \( \bar{u}^{\mu} \) (\( \bar{u}^{\nu} \bar{u}_{\nu} = -1 \)) by a stress-energy density tensor of the non-relativistic form
\[ \tau^{\mu \nu} \simeq \bar{\rho} \bar{u}^{\mu} \bar{u}^{\nu}, \]  \hfill (13)
in terms of a surface mass density \( \bar{\rho} \) whose space section volume integral determines the corresponding total mass \( M \) say, the resulting equation takes the form

\[
T_\infty \Box \xi^\rho \simeq \bar{\rho} \bar{u}^\mu \bar{u}^\nu K_{\mu\nu}^{\rho},
\]

(14)
in which, due to the staticity the (hyperbolic) Dalembertian operator will reduce to a Laplacian operator (of elliptic type), so that for a \((q-2)\) spherically symmetric distribution the solution will be expressible in terms of the radial distance \( r \) from the center by an expression that for \( q \geq 3 \) will have the power law form

\[
T_\infty \xi^\mu = -\frac{M}{(q-2)\Omega[q-1]} r^{q-2} a^\mu,
\]

(15)
in terms of the rest frame orthogonal worldsheet acceleration vector \( a^\mu = \bar{u}^\nu \nabla_\nu \bar{u}^\mu \) given by

\[
a^\rho = \bar{u}^\mu \bar{u}^\nu K_{\mu\nu}^{\rho}.
\]

(16)
For the familiar, experimentally accessible, case of an ordinary membrane, with \( q=2 \), there will be an analogous formula involving radial dependence of logarithmic rather than power law type.

### 3.1 Quasi gravitational metric perturbations

Under the conditions described in the preceding section, the brane worldsheet geometry characterised by the fundamental tensor \( \gamma_{\mu\nu} \) will be subject to a corresponding perturbation, \( \bar{h}_{\mu\nu} = \delta\gamma_{\mu\nu} \) that will be given \([4]\) in terms of the second fundamental tensor of the unperturbed reference state by an expression of the form

\[
\bar{h}_{\mu\nu} = -K_{\mu\nu}^{\rho} \xi_\rho.
\]

(17)
This perturbation will have a time component

\[
\bar{h}_{00} = \bar{u}^\mu \bar{u}^\nu \bar{h}_{\mu\nu},
\]

(18)
given by the formula

\[
\bar{h}_{00} = -2a^\rho \xi_\rho,
\]

(19)
while the trace \( \bar{h}_{\nu} \) will be given by an expression of the analogous form

\[
\bar{h}_{\nu} = -2K^{\rho} \xi_\rho,
\]

(20)
Evaluating \( \bar{h}_{00} \) explicitly using \( \bar{h}_{00} = \bar{u}^\mu \bar{u}^\nu \bar{h}_{\mu\nu} \), one sees that is reducible to an expression of the standard (dimensionally generalised \([3]\)) Newtonian form

\[
\bar{h}_{00} = \frac{2G_{[q+1]}^{[q]} M}{r^{q-2}},
\]

(21)
with the relevant generalised Newton constant given by

$$ G_{[q+1]} = \frac{1}{(q-2)\Omega^{[q-1]}\Omega} a^q a_0 . $$

(22)

Although this mechanism will thus effectively simulate Newtonian type gravitational attraction in so far as its effect on non-relativistic Kepler type orbits is concerned, it leads to a value for the ratio $\bar{h}_{\infty} / \bar{h}_V$ that can be seen from (19) and (20) to be given by (15) as

$$ \frac{\bar{h}_{\infty}}{\bar{h}_V} = a^q a_0 K_0 , $$

(23)

which will not in general agree with the prediction of Einstein’s theory. Although the ensuing prediction for the relativistic behaviour (e.g., of light deflection) will thereby deviate from that of Einstein’s purely tensorial theory of gravity, it gives a result that will be shown to be matchable by a more general theory of the Jordan-Brans-Dicke type to be described in the next section.

### 4 Jordan-Brans-Dicke type theories

The action integral,

$$ I = \int \mathcal{L} \left\| g \right\|^{1/2} dx , \quad \mathcal{L} = \mathcal{L}_D + \mathcal{L}_M $$

(24)

for a Jordan-Bran-Dicke type scalar tensor theory in a (q+1) dimensional space-time, with metric $g_{\mu\nu}$ in a Dicke type conformal gauge (meaning one in which the weak equivalence principle is satisfied) is given by a Lagrangian density consisting of a Dicke type gravitational contribution $\mathcal{L}_D$ involving a dilatonic scalar field $\Phi$ as well as the metric, and an ordinary matter contribution $\mathcal{L}_M$ that is independent of $\Phi$, with the Dicke contribution given in terms of a coupling constant $\omega_D$ by an expression of the form

$$ \mathcal{L}_D = \frac{1}{2(q-1)\Omega^{[q-1]}} \left( \Phi R - \frac{\omega_D}{\Phi} g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu} \right) , $$

(25)

where $R$ is the Ricci scalar for the metric $g_{\mu\nu}$ and $\Omega^{[q-1]}$ is the surface area of the unit (q-1) sphere, which, for an ordinary 4-dimensional spacetime, with space dimension q=3, will be given by $\Omega^{[2]} = 4\pi$.

In terms of the trace of the material stress energy density tensor,

$$ T^{\mu\nu}_M = 2 \frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu}} - \mathcal{L}_M g^{\mu\nu} , $$

(26)

the scalar wave equation for such a theory will be given in terms of the Dalembertian operator $\Box = \nabla_\mu \nabla^\mu$ by

$$ \Box \Phi = \alpha_D \Omega^{[q-1]} T^{\mu\nu}_M , $$

(27)
in terms of a dilatonic coupling constant $\alpha_D$ that is given in terms of the original Dicke constant $\omega_D$ by

$$\frac{1}{\alpha_D} = \omega_D + \frac{q}{q - 1}. \quad (28)$$

To deal with the gravitational equations, it is convenient to express the dilatonic amplitude $\Phi$ in terms of some fixed value $\hat{\Phi}$ and of a dimensionless scalar field $\phi$ in the form

$$\Phi = e^{-2\phi} \hat{\Phi}. \quad (29)$$

and to change to what is known as an Einstein gauge by a conformal transformation $g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu}$ that is specified by setting

$$g_{\mu\nu} = e^{2\sigma} \hat{g}_{\mu\nu}, \quad (30)$$

where the field $\sigma$ is given in terms of $\phi$ by by the proportionality relation

$$2\phi = (q - 1)\sigma. \quad (31)$$

In terms of the Einstein type conformal gauge the action (24) will take the form

$$I = \int \hat{\mathcal{L}} \|\hat{\gamma}\|^{1/2} d^{(q+1)}x, \quad \hat{\mathcal{L}} = \hat{\mathcal{L}}_D + \hat{\mathcal{L}}_M \quad (32)$$

with the matter contribution given by

$$\hat{\mathcal{L}}_M = e^{(q+1)\sigma} \mathcal{L}_M, \quad (33)$$

while $\hat{\mathcal{L}}_D$ is given as the sum of an ordinary Einstein-Hilbert type term and a linear scalar field contribution in the form

$$\hat{\mathcal{L}}_D = \frac{\hat{\Phi}}{2(q - 1)\Omega^{[q-1]}} \left( \hat{R} - \frac{4}{\alpha_D} \hat{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right), \quad (34)$$

where $\hat{R}$ is the Ricci scalar for the Einstein metric $\hat{g}_{\mu\nu}$ and $\alpha_D$ is the constant given by (28), while the fixed amplitude $\hat{\Phi}$ now acts as the inverse of the (dimensionally generalised) Newton constant, which can be identified as

$$\hat{G}_{[q+1]} = \frac{1}{\hat{\Phi}}. \quad (35)$$

In this reformulation there will be a matter stress energy density contribution given by

$$\hat{T}_{M,\mu} = e^{(q+1)\sigma} T_{M,\mu}, \quad (36)$$

whose trace will act as the source for the linear wave equation for $\phi$, which will be expressible in the form

$$\Box \phi = - \frac{1}{2} \Omega^{[q-1]} \hat{G}_{[q+1]} \alpha_D \hat{T}_{M,\rho}, \quad (37)$$

where $\Box$ is the Daceblambertian operator for the Einstein metric $\hat{g}_{\mu\nu}$. The corresponding Einstein type gravitational equations will be expressible as

$$\hat{R}_{\mu\nu} - \frac{1}{2} \hat{R} \hat{g}_{\mu\nu} = \frac{2}{\alpha_D} (2\phi_{,\mu} \phi_{,\nu} - \hat{g}_{\mu\nu} \hat{g}^{\rho\sigma} \phi_{,\rho} \phi_{,\sigma}) + (q - 1) \Omega^{[q-1]} \hat{G}_{[q+1]} \hat{T}_{M,\mu\nu}. \quad (38)$$
5 Linearised local scalar tensor field configurations

Let us now consider the weak field, low source density, limit in which the system can be linearised with respect to the dilatonic perturbation field $\phi$ and the Einstein metric perturbation field $\hat{h}_{\mu\nu}$ defined relative to a flat Minkowski background metric $\eta_{\mu\nu}$ by setting

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}.$$  \hspace{1cm} (39)

The equation (37) for $\phi$ is already linear as it stands, while the corresponding linearised Einstein equation for $\hat{h}_{\mu\nu}$ is obtainable from (38) in the standard form

$$\Box \hat{h}_{\mu\nu} = -2\Omega^{(q-1)} \hat{G}_{[q+1]} \left((q-1)\hat{T}_{M\mu\nu} - \hat{T}^\rho_{M\rho} \eta_{\mu\nu}\right).$$ \hspace{1cm} (40)

The gravitational field that is directly measured by the observation of Kepler type orbits will not be given by this Einstein type metric $\hat{g}_{\mu\nu}$ (to which, due to the involvement of $\phi$ in the relevant stress energy tensor $\hat{T}_{\mu\nu}$, the usual equivalence principle does not apply) but by the original Dicke type metric, $g_{\mu\nu}$, which will be expressible analogously to (39) by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$ \hspace{1cm} (41)

with

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + 2\sigma \eta_{\mu\nu},$$ \hspace{1cm} (42)

to linear order, by (30). Using the relation (11), it can be seen, by combining the wave equations (37) and (41), that the directly observable metric perturbation $h_{\mu\nu}$ will be given, to linear order, by

$$\Box h_{\mu\nu} = -2\Omega^{(q-1)} \hat{G}_{[q+1]} \left((q-1)T_{M\mu\nu} + (1 - \Delta_D) T^\rho_{M\rho} \eta_{\mu\nu}\right).$$ \hspace{1cm} (43)

in which the dilatonic deviation constant is given by

$$\Delta_D = \frac{1}{(q-1)\omega_D + q} = \frac{\alpha_D}{q-1}. \hspace{1cm} (44)$$

Due to the presence of the deviation constant $\Delta_D$, the coefficient $\hat{G}_{[q+1]}$ will not be quite the same as the effective Newtonian coupling constant $G^{(q+1)}$ that will be observed in the static non-relativistic limit for which, in terms of the relevant rest frame unit vector $u^\mu$ (with $u^\mu u_\mu = -1$) the stress-energy density will be approximately of the form

$$T_{M}^{\mu\nu} = \rho u^\mu u^\nu,$$ \hspace{1cm} (45)

in which $\rho$ is the mass density, whose space volume integral will be identifiable in this limit as the total mass $M$. It can be seen that for a spherically symmetric distribution the time component,

$$h_{00} = u^\mu u^\nu h_{\mu\nu},$$ \hspace{1cm} (46)
of the metric perturbation will be given in terms of the radial distance $r$ from the center by an expression of the standard (dimensionally generalised) Newtonian form
\begin{equation}
  h_{00} = \frac{2G_{[q+1]} M}{r^{q-2}},
\end{equation}
but with the effective gravitational coupling constant given by
\begin{equation}
  G_{[q+1]} = \tilde{G}_{[q+1]} \left(1 + \frac{\Delta_D}{q - 2}\right).
\end{equation}
It can be seen that it will be related to the corresponding expression for the trace $h^\rho_\rho$ of the metric perturbation by
\begin{equation}
  h^\rho_\rho = \frac{2 - (q + 1)\Delta_D}{q - 2 + \Delta_D} h_{00},
\end{equation}
which is equivalent to
\begin{equation}
  h_{00} = \frac{(q - 2)\omega_D + q - 1}{2\omega_D + 1} h^\rho_\rho.
\end{equation}

6 Conclusion

It can be seen that the ratio (50) can be matched by the simulation effect leading to the corresponding ratio (23) if the relevant reference frame curvature vector $K^\mu$ and the corresponding acceleration vector $a^\mu$ are related by
\begin{equation}
  a^\rho a_\rho = \frac{(q - 2)\omega_D + q - 1}{2\omega_D + 1} K^\rho a_\rho,
\end{equation}
or equivalently
\begin{equation}
  a^\rho K_\rho = \frac{2 + (q + 1)\Delta_D}{q - 2 + \Delta_D} a^\rho K_\rho.
\end{equation}

This means that, in the linear approximation we have been using, the quasi gravitational effect arising from the extrinsic curvature of the brane simulates what would be predicted by a Jordan-Brans-Dicke theory with $\omega_D$ given by what is obtained by solving (51), namely
\begin{equation}
  \omega_D = \frac{(q - 1)a^\rho K_\rho - a^\rho a_\rho}{2a^\nu a_\nu - (q - 2)a^\nu K_\nu},
\end{equation}
which corresponds to a dilatonic deviation $\Delta_D$ given by
\begin{equation}
  \Delta_D = \frac{2a^\rho a_\rho + (2 - q)a^\rho K_\rho}{(q + 1)a^\nu a_\nu + a^\nu K_\nu}.
\end{equation}

It is to be emphasised that the approximation presented here has been derived only for static configurations in a linearised weak field limit, and cannot be expected to remain accurate when stronger fields or significant deviations from staticity are involved.

The author wishes to express thanks to Mário Novello for initiating the organisation of this workshop, to Matt Visser for ensuring the completion of the edited proceedings, and to Gilles Esposito-Farèse for technical discussions about the simulated worldsheet gravity mechanism in the context of scalar - tensor theories.
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