INVERSE SIGNED DOMINATING FUNCTIONS OF CORONA AND ROOTED PRODUCT GRAPHS

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Abstract: Graph theory is an interesting subject in mathematics. Applications in many fields like Linguistics, Engineering communications, Physical Sciences, Coding theory, Computer networking and Logical Algebra. The theory of domination in graphs has a wide range of applications. Among these applications, the most often discussed is a coding theory and communication networks. Inverse domination theory of graphs which are the important branches of graph theory. In this paper, we study the maximal inverse signed dominating functions of corona product graph of a path with a complete graph and rooted product graph of a path with a cycle.

Keywords: inverse signed dominating functions, inverse signed domination number, corona product graph, rooted product graph.

1. INTRODUCTION

Mostly Product of graphs occurs in discrete mathematics. In 1970, Frucht & Harary [6] introduced a new product on two graphs $G_1$ and $G_2$, called corona product denoted by $G_1\square G_2$. The corona product of a path $P_n$ with a complete graph $K_m$ is a graph obtained by taking one copy of $P_n$ and $n$ copies of $K_m$ and then joining the $i^{th}$ vertex of $P_n$ to every vertex of $i^{th}$ copy of $K_m$ and it is denoted by $P_n\square K_m$, where $n>0$ and $m>0$. In 1978, Godsil and McKay [1] introduced a new product on two graphs $G_1$ and $G_2$, called rooted product denoted by $G_1\circ G_2$. In this paper we consider the rooted product graph like, here $P_n\circ K_m$ be a Path graph with $n$ vertices and $K_m$ be a cycle with a sequence of $n$ rooted graphs $C_{m1}, C_{m2}, C_{m3}, \ldots, C_{mn}$. Then by $P_n(C_m)$ we denote the graph obtained by identifying the root of $C_{mi}$ with the $i^{th}$ vertex of $P_n$. We call $P_n\circ C_m$ the rooted product of $P_n$ by $C_m$ and it is denoted by $P_n\circ C_m$. Every $i^{th}$ vertex of $P_n$ is merging with any one vertex in every $i^{th}$ copy of $C_m$. So in $G=P_n\circ C_m$, $P_n$ contains $n$ vertices and $C_m$ contains $(m-1)$ vertices in each copy of $C_m$.

In 1995, Dunbar, Hedetniemi, Henning and Slater [4] have studied about “Signed Domination in Graphs”. Further we studied about signed domination in [2, 7]. In 1996, Favaron [5] have studied about “Signed domination in regular graphs”. In 2010, Zhong-sheng [3] have studied about “On Inverse Signed Total Domination in Graphs”. By using signed domination related parameters we can find out inverse signed domination parameters on product graphs.

2. RESULTS ON ROOTED PRODUCT GRAPH

**Theorem 2.1:** If $m$ is divisible by 3 then the function $f: V \rightarrow \{-1, 1\}$ is defined by

$$f(v) = \begin{cases} +1, & \text{if } m \equiv 1 (\text{mod } 3) \\ -1, & \text{otherwise.} \end{cases}$$

Then $f$ is a maximal inverse signed dominating function of a graph $G = P_n \circ C_m$ and inverse signed domination number of $G$ is $\gamma_s^0(G) = \frac{-mn}{3}$.

**Proof:** Consider the graph $G = P_n \circ C_m$ with $|V|$ number of vertices and $|E|$ number of edges. Let $f$ be a function defined in the hypothesis. Suppose $m$ is divisible by 3.

Here +1 is assigned to $\left(\frac{m}{3}\right)$ vertices in each copy of $C_m$ in $G$, -1 is assigned to all other vertices in $G$.

Case 1: Suppose $v \in P_n$ be such that

(i) As $d(v) = 4$ in $G$ then

$$\sum_{u \in N[v]} f(u) = [(+1)+(+1)] + [(+1)+(+1)+(+1)+(+1)] = -3.$$  

(ii) As $d(v) = 3$ in $G$ then

$$\sum_{u \in N[v]} f(u) = [(+1)+(+1)] + [(+1)+(+1)+(+1)] = -2.$$  

Case 2: Suppose $v \in C_m$ be such that $d(v) = 2$ in $G$ then $f(v) = -1$, $f(v) = +1$.
(i) Then \(N[v]\) contains 2 vertices of \(C_m\) and one vertex of \(P_n\) in \(G\).

If \(f(v) = -1\) then \[\sum_{u \in N[v]} f(u) = (-1) + [(-1) + (+1)] = -1.\]

If \(f(v) = +1\) then \[\sum_{u \in N[v]} f(u) = (+1) + [(-1) + (+1)] = -1.\]

(ii) Then \(N[v]\) contains 3 vertices of \(C_m\) and zero vertex of \(P_n\) in \(G\).

If \(f(v) = -1\) then \[\sum_{u \in N[v]} f(u) = (-1) + [(-1) + (+1)] = -1.\]

If \(f(v) = +1\) then \[\sum_{u \in N[v]} f(u) = (+1) + [(-1) + (+1)] = -1.\]

From the above cases the function \(f\) is an inverse signed dominating function, because \[\sum_{u \in N[v]} f(u) < 0, \forall v \in V.\]

Now the maximality check for \(f\), define \(g : V \rightarrow \{-1,+1\}\) by
\[g(v) = \begin{cases} +1, & \text{if } m \equiv 0 \pmod{3} \text{ vertices in each copy of } C_m \text{ in } G, \\ -1, & \text{otherwise.} \end{cases}\]

Here two cases are followed.

Case 3: Suppose \(v \in P_n^m\) be such that

(i) As \(d(v) = 4\) in \(G\), then \(N[v]\) contains 2 vertices of \(C_m\) and three vertices of \(P_n\) in \(G\).

Sub case 1: Let \(u_i \in P_n\) in \(i^{th}\) copy of \(G\) then \[\sum_{u \in N[v]} g(u) = [(+1) + (-1)] + [(-1) + (+1)] = -1.\]

Sub case 2: Let \(u_i \notin P_n\) in \(i^{th}\) copy of \(G\) then \[\sum_{u \in N[v]} g(u) = [(+1) + (-1)] + [(-1) + (+1)] = -3.\]

(ii) As \(d(v) = 3\) in \(G\), then \(N[v]\) contains 2 vertices of \(C_m\) and two vertices of \(P_n\) in \(G\).

Sub case 1: Let \(u_i \in P_n\) in \(i^{th}\) copy of \(G\) then \[\sum_{u \in N[v]} g(u) = [(+1) + (-1)] + [(-1) + (+1)] = 0.\]

Sub case 2: Let \(u_i \notin P_n\) in \(i^{th}\) copy of \(G\) then \[\sum_{u \in N[v]} g(u) = [(+1) + (-1)] + [(-1) + (+1)] = -2.\]

Case 4: Suppose \(v \in C_m\) be such that \(d(v) = 2\) in \(G\),

(i) Here \(N[v]\) contains 2 vertices of \(C_m\) and one vertex of \(P_n\) in \(G\) then \(g(v) = -1\) or \(g(v) = +1\).

Sub case 1: Let \(u_i \in P_n\) in \(i^{th}\) copy of \(G\).

If \(g(v) = -1\) then \[\sum_{u \in N[v]} g(u) = (-1) + [(+1) + (+1)] = +1(> 0).\]

If \(g(v) = +1\) then \[\sum_{u \in N[v]} g(u) = (+1) + [(-1) + (+1)] = +1(> 0).\]

Sub case 2: Let \(u_i \notin P_n\) in \(i^{th}\) copy of \(G\).

If \(g(v) = -1\) then \[\sum_{u \in N[v]} g(u) = (-1) + [(+1) + (+1)] = +1(> 0).\]

If \(g(v) = +1\) then \[\sum_{u \in N[v]} g(u) = (+1) + [(-1) + (+1)] = +1(> 0).\]

Sub case 3: Let \(u_i \in P_n\) in \(i^{th}\) copy of \(G\).

If \(g(v) = -1\) then \[\sum_{u \in N[v]} g(u) = (-1) + [(+1) + (+1)] = +1(> 0).\]

If \(g(v) = +1\) then \[\sum_{u \in N[v]} g(u) = (+1) + [(-1) + (+1)] = +1(> 0).\]

Now the maximality check for \(g\), define \(h : V \rightarrow \{-1,+1\}\) by
\[h(v) = \begin{cases} +1, & \text{if } m \equiv 0 \pmod{3} \text{ vertices in each copy of } C_m \text{ in } G, \\ -1, & \text{otherwise.} \end{cases}\]

This implies that the function \(g\) is not an inverse signed dominating function. Hence \(f\) is a maximal inverse signed dominating function on \(G\). Now inverse signed total domination number is the sum of the function value of all vertices in \(G\), that is \[\sum_{u \in V(G)} f(u) = \left(\frac{m}{3}\right) + \left(\frac{m}{3}\right) = \frac{mn}{3} \text{ for some } v \in V.\]

Therefore \(\gamma_{ts}^{(0)}(G) = \frac{-mn}{3} \text{ for some } v \in V.\]

**Theorem 2.2:** If \(m = 3k + 1\) or \(3k + 2\) is not divisible by 3 then inverse signed domination number of \(G\) is \(\gamma_{ts}^{(0)}(G) = \begin{cases} n - 1 + \frac{m}{3} & \text{if } m = 3k + 1, \\ n - 1 - \frac{m}{3} & \text{if } m = 3k + 2. \end{cases}\)

**Proof:** Consider the graph \(G = P_n \ast C_m\) with \(|V|\) number of vertices and \(|E|\) number of edges.

**Case 1:** Suppose \(m = 3k + 1\)

Let \(f : V \rightarrow \{-1,+1\}\) be a function defined by \(f(v) = \begin{cases} +1, & \text{if } m \equiv 1 \pmod{3} \text{ vertices in each copy of } C_m \text{ in } G, \\ -1, & \text{otherwise.} \end{cases}\)

Here +1 is assigned to \(\left\lfloor\frac{m}{3}\right\rfloor\) vertices in each copy of \(C_m\) in \(G\), -1 is assigned to all other vertices in \(G\).

**Case 2:** Suppose \(v \in P_n\) be such that

(i) As \(d(v) = 4\) in \(G\) then \[\sum_{u \in N[v]} f(u) = [(+1) + (-1)] + [(-1) + (+1)] + [(-1) + (+1)] = -3.\]

(ii) As \(d(v) = 3\) in \(G\) then \[\sum_{u \in N[v]} f(u) = [(+1) + (-1)] + [(-1) + (+1)] + [(-1) + (+1)] = -2.\]
Case 2: Suppose \( V \in C_m \) be such that \( d(v) = 2 \) in \( G \) then \( f(v) = -1 \), \( f(v) = +1 \).

(i) Here \( N[v] \) contains 2 vertices of \( C_m \) and one vertex of \( P_n \) in \( G \).

If \( f(v) = -1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) = 2 \).

If \( f(v) = +1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) = 2 \).

(ii) Here \( N[v] \) contains 3 vertices of \( C_m \) and zero vertex of \( P_n \) in \( G \).

If \( f(v) = -1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) = 2 \).

If \( f(v) = +1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) = 2 \).

From the above cases the function \( f \) is an inverse signed dominating function, because \( \sum_{u \in N[v]} g(u) < 0, \forall v \in V \).

Now the maximality check for \( f \), define \( g : V \rightarrow \{-1, +1\} \) by

\[
g(v) = \begin{cases} 
+1, & \text{if} \ \sum_{u \in N[v]} g(u) < 0, \\
-1, & \text{otherwise.}
\end{cases}
\]

Here two cases are followed.

Case 3: Suppose \( v \in P_n \) be such that

(i) As \( d(v) = 4 \) in \( G \), then \( N[v] \) contains 2 vertices of \( C_m \) and three vertices of \( P_n \) in \( G \).

Sub case 1: Let \( u_i \in P_n \) in \( i^{th} \) copy of \( G \) then

\[
\sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3.
\]

Sub case 2: Let \( u_i \notin P_n \) in \( i^{th} \) copy of \( G \) then

\[
\sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3.
\]

(ii) As \( d(v) = 3 \) in \( G \), then \( N[v] \) contains 2 vertices of \( C_m \) and two vertices of \( P_n \) in \( G \).

Sub case 1: Let \( u_i \in P_n \) in \( i^{th} \) copy of \( G \) then

\[
\sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3.
\]

Sub case 2: Let \( u_i \notin P_n \) in \( i^{th} \) copy of \( G \) then

\[
\sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3.
\]

Case 4: Suppose \( v \in C_m \) be such that \( d(v) = 2 \) in \( G \),

(i) Here \( N[v] \) contains 2 vertices of \( C_m \) and one vertex of \( P_n \) in \( G \) then \( g(v) = -1 \) or \( +1 \).

Sub case 1: Let \( u_i \in P_n \) in \( i^{th} \) copy of \( G \).

If \( g(v) = -1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3 \).

If \( g(v) = +1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3 \).

Sub case 2: Let \( u_i \notin P_n \) in \( i^{th} \) copy of \( G \).

If \( g(v) = -1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3 \).

If \( g(v) = +1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3 \).

Sub case 2: Let \( u_i \notin P_n \) in \( i^{th} \) copy of \( G \).

If \( g(v) = -1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3 \).

If \( g(v) = +1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3 \).

Sub case 2: Let \( u_i \notin P_n \) in \( i^{th} \) copy of \( G \).

If \( g(v) = -1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3 \).

If \( g(v) = +1 \) then \( \sum_{u \in N[v]} g(u) = (1) + (1) + (1) = 3 \).
If \( f(v) = +1 \Rightarrow \sum_{u \in N[v]} f(u) = (+1) + (-1) = -1 \).

From the above cases the function \( f \) is an inverse signed dominating function, because \( \sum_{u \in N[v]} f(u) \leq 0, \forall v \in V \).

Now maximality check for \( f \), define \( g: V \rightarrow \{-1, +1\} \) by
\[
g(v) = \begin{cases} +1, & \text{if one vertex } v = u_i \in P_n \text{ in } G, \\ -1, & \text{otherwise.} 
\end{cases}
\]

Here two cases are followed.

Case 2: Let \( v \in P_n \) be such that \( g(v) = +1 \).

Case 3: Suppose \( v \in P_n \) be such that \( g(v) = -1, \forall v \in V \).

Sub case 1: Let \( u_i \in P_n \) in \( i \)th copy of \( G \) then
\[
\sum_{u \in N[v]} g(u) = [(+1) + (+1)] + [(-1) + (-1) + (+1)] = +1(> 0).
\]

Sub case 2: Let \( u_i \notin P_n \) in \( i \)th copy of \( G \) then
\[
\sum_{u \in N[v]} g(u) = [(+1) + (+1)] + [(-1) + (-1) + (-1)] = -1.
\]

Case 4: Suppose \( v \in C_m \) be such that \( d(v) = 2 \) in \( G \).

(i) Here \( N[v] \) contains 2 vertices of \( C_m \) and one vertex of \( P_n \) in \( G \) then \( g(v) = +1 \).

Sub case (1): Let \( u_i \in P_n \) in \( i \)th copy of \( G \) then
\[
\sum_{u \in N[v]} g(u) = [(+1)] + [(-1) + (-1)] = -1.
\]

Sub case (2): Let \( u_i \notin P_n \) in \( i \)th copy of \( G \) then
\[
\sum_{u \in N[v]} g(u) = [(+1)] + [(-1) + (-1)] = -1.
\]

(ii) Then \( N[v] \) contains 3 vertices of \( C_m \) and zero vertex of \( P_n \) in \( G \) then \( g(v) = -1 \) and \( g(v) = +1 \).

If \( g(v) = -1 \Rightarrow \sum_{u \in N[v]} g(u) = (-1) + [(+1) + (+1)] = -1 \).

If \( g(v) = +1 \Rightarrow \sum_{u \in N[v]} g(u) = (+1) + [(+1) + (-1)] = -1 \).

From the above cases, we get \( \sum_{u \in N[v]} g(u) > 0, \forall v \in V \).

This implies that the function \( g \) is not an inverse signed dominating function. Hence \( f \) is a maximal inverse signed dominating function on \( G \). Now inverse signed domination number is the sum of the function value of all vertices in \( G \), that is
\[
\sum_{u \in V(G)} f(u) = \left(\frac{m}{3}\right)(+1) + \left(\frac{m}{3}\right)(-1) = -m + \frac{2m}{3}.
\]

Therefore \( \gamma^0_s(G) = n \left[\left\lfloor\frac{m}{3}\right\rfloor - m\right] \).

3. RESULTS ON CORONA PRODUCT GRAPH

**Theorem 3.1:** A function \( f : V \rightarrow \{-1, +1\} \) is defined by
\[
f(v_i) = \begin{cases} +1, & \text{if } 1 \leq i \leq \left\lfloor\frac{m}{2}\right\rfloor, \\ -1, & \text{otherwise.} 
\end{cases}
\]

is a maximal inverse signed dominating function of a graph \( G = P_n \square K_m \) and inverse signed domination number is \( \gamma^0_s(G) = n, \forall m \in \mathbb{N} \).

**Proof:** Consider the graph \( G = P_n \square K_m \) with \(|V|\) number of vertices and \(|E|\) number of edges. Let \( f \) be a function defined in the hypothesis.

Case 1: Let \( v_i \in P_n \) be such that \( d(v_i) = m+2 \) in \( G \) then
\[
\sum_{u \in N[v_i]} f(u) = (-1) + (-1) + (-1) + [(\frac{m}{2}) + (\frac{m}{2})] = -3.
\]

Case 2: Let \( v_i \in P_n \) be such that \( d(v_i) = m+1 \) in \( G \) then
\[
\sum_{u \in N[v_i]} f(u) = (-1) + (-1) + [(\frac{m}{2}) + (\frac{m}{2})] = -2.
\]

Case 3: Let \( v_i \in K_m \) be such that \( d(v_i) = m \) in \( G \) and \( f(v_i) = -1 \) or +1.

If
\[
f(v_i) = \pm 1 \Rightarrow \sum_{u \in N[v_i]} f(u) = (-1) + [(\frac{m}{2}) + (\frac{m}{2})] = -1.
\]

Hence for all the above possibilities, we get \( \sum_{u \in N[v_i]} f(u) < 0, \forall v_i \in V \). This implies that the function \( f \) is an inverse signed dominating function. Now the maximality check for \( f \), define \( g: V \rightarrow \{-1, +1\} \) by
\[
g(v_i) = \begin{cases} +1, & \text{if } 1 \leq i \leq \left\lfloor\frac{m}{2}\right\rfloor, \\ -1, & \text{otherwise.} 
\end{cases}
\]

Case 1: Let \( v_i \in P_n \) be such that \( d(v_i) = m+2 \) in \( G \).
Theorem 3.2: A function \(f : V \to \{-1, +1\}\) is defined by

\[ f(v_i) = \begin{cases} +1, & \text{if } 1 \leq i \leq \frac{m+1}{2} \text{ of each copy of } K_m \text{ in } G, \\ -1, & \text{otherwise}. \end{cases} \]

is a maximal inverse signed dominating function of a graph \(G = P_n \cup K_m\) and inverse signed domination number is \(\gamma_s^0(G) = 0\), if \(m\) is odd.

**Proof:** Let \(f\) be a function defined in the hypothesis.

Case 1: Let \(v_i \in P_n\) be such that \(d(v_i) = m + 2\) in \(G\) then

\[\sum_{v_i \in N[v_i]} f(u) = (-1) + (-1) + (-1) + \left(\frac{m+1}{2}\right)(+1) + \left(\frac{m-1}{2}\right)(-1) = -2.\]

Case 2: Let \(v_i \in P_n\) be such that \(d(v_i) = m + 1\) in \(G\) then

\[\sum_{v_i \in N[v_i]} f(u) = (-1) + (-1) + \left(\frac{m+1}{2}\right)(+1) + \left(\frac{m-1}{2}\right)(-1) = -1.\]

Case 3: Let \(v_i \in K_m\) be such that \(d(v_i) = m\) in \(G\) and

\[f(v_i) = \pm1 \Rightarrow \sum_{v_i \in N[v_i]} f(u) = (-1) + \left(\frac{m+1}{2}\right)(+1) + \left(\frac{m-1}{2}\right)(-1) = 0.\]

Hence for all the above possibilities, we get

\[\sum_{u \in V(G)} f(u) = (-1) + \ldots + (-1) + \left(\frac{m+1}{2}\right)(+1) + \left(\frac{m-1}{2}\right)(-1) = -n.\]

Finally \(\gamma_s^0(G) = -n\), if \(m\) is even.

This implies that \(G\) is not an inverse signed dominating function because \(\sum_{u \in V(G)} f(u) > 0\), for some \(v_i \in V\).

4. **Conclusion**

It is interesting to study the inverse signed dominating functions of corona product graph of complete graph with a path and rooted product graph of a path with cycle. This work gives the scope for an extensive study of various inverse dominating functions of these graphs.
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