Dynamic emergence of domino effects in systems of interacting tipping elements in ecology and climate

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Abstract

In ecology, climate and other fields, systems have been identified that can transition into a qualitatively different state when a critical threshold or tipping point in a driving process is crossed. An understanding of those tipping elements is of great interest given the increasing influence of humans on the biophysical Earth system. Tipping elements are not independent from each other as there exist complex interactions, e.g. through physical mechanisms that connect subsystems of the climate system. Based on earlier work on such coupled nonlinear systems, we systematically assessed the qualitative asymptotic behavior of interacting tipping elements. We developed an understanding of the consequences of interactions on the tipping behavior allowing for domino effects and tipping cascades to emerge under certain conditions. The application of these qualitative results to real-world examples of interacting tipping elements shows that domino effects with profound consequences can occur: the interacting Greenland ice sheet and thermohaline ocean circulation might tip before the tipping points of the isolated subsystems are crossed. The eutrophication of the first lake in a lake chain might propagate through the following lakes without a crossing of their individual critical nutrient input levels. The possibility of emerging domino effects calls for the development of a unified theory of interacting tipping elements and the quantitative analysis of interacting real-world tipping elements.

Keywords tipping point, critical threshold, hysteresis, domino effect, Earth system, eutrophication

1 Introduction

Many natural systems exhibit nonlinear dynamics and can undergo a transition into a qualitatively different state when a critical threshold is crossed. Those systems are called tipping elements and the corresponding threshold in terms of a critical parameter is the tipping point of the system. A precise mathematical definition is given in [1]. Examples for tipping elements can be found in ecology as a specific type of regime shifts [2, 3] such as the transition of a shallow lake from a clear to a turbid state [4, 5, 6, 7, 8, 9]. Furthermore, subsystems of the Earth system [1, 10] such as the thermohaline circulation [11, 12, 13, 14, 15, 16, 17] or the Greenland ice sheet [18, 19] have been identified as tipping elements.

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The term tipping point among other roots originated from describing the changing prevalence of ethnically diverse population in an US-community [20, 21, 22, 23, 24] and has been applied to natural systems more recently. However, the underlying concept of systems showing such nonlinear behavior has already been developed within the frameworks of dynamical systems and catastrophe theory [25, 26, 27, 28, 29]. The latter theory received large attention and has been applied to several real-world systems in its beginning [30]. Its extensive use has been criticized [31, 32, 33, 34] so that it became a mathematical theory without much recent influence [35]. Mostly independently of the results given by catastrophe theory, critical transitions, tipping points and regime shifts have been analyzed in ecology [2, 9, 36, 37, 38] using the concepts of multistability and resilience [39, 40]. Some first attempts to define a climatic tipping elements relating to abrupt climate shifts can be found in [41, 42] and [43].

Different types of tipping points are discussed in the literature [9, 22, 36, 37, 38, 44].

1. A qualitative change of the system’s state when a continuously changing control parameter crosses a threshold is called bifurcational tipping [45, 46, 22, 47].

2. Noise can induce a transition into an alternative stable state without a change of the system’s control parameter [22, 37, 48].

3. Rate-induced tipping describes the shift to a qualitatively different state when the rate of change of a control parameter crosses a critical threshold [22, 44, 48, 49].

It needs to be stressed that bifurcational tipping, even though often mentioned, is not the only possible type of tipping [23, 44, 48, 50]. Nevertheless, the response of many natural systems to a control parameter can be described in terms of a double fold bifurcation [47, 51, 52].

Real-world tipping elements are not independent from each other [51] but there may exist complex interactions between them. Potential interactions through various physical mechanisms were revealed for tipping elements in the climate system [53]. Lake chains or rivers can be seen as an ecological example for coupled tipping elements. Each lake or river section in the chain can undergo a transition from a clear to a turbid state in response to nutrient input [5, 6, 8]. The single lakes can in reality be connected through small rivers or streams and can therefore not be considered independently [54, 55, 56, 57].

The tipping probability of a certain tipping element might be influenced by the behavior of other interacting tipping elements [53, 58]. As a consequence, crossing of a critical threshold of a first tipping element could trigger, as a domino effect, a critical transition in a coupled tipping element or even tipping cascades [36, 58, 59, 60, 61, 62, 63]. The heterogeneity of the subsystems as well as the coupling strength were mentioned as important factors that influence the overall system behaviour and should be considered in the analysis of coupled tipping elements [64].

Different attempts to analyze the influence of coupling between different tipping elements on their tipping behavior have been followed. The development of critical transitions in lake chains was studied using established models of lake eutrophication [57, 65, 66]. In analogy to wave propagation in discrete media [67, 68, 69, 70], the spread of local disturbances in spatially extended, bistable ecosystems was analyzed for explicit ecological examples [71] and more theoretically [72]. In addition, the possibility of cascades on networks [73, 74] and networks of networks [75, 76, 77, 78, 79] was discussed. [51], [80] and [81] analyzed the system behavior of special cases of coupled cusp catastrophes. The possible appearance of tipping cascades in coupled bifurcational systems and, in particular, in the climate system was supported by results from coupling conceptual models of the Atlantic Meridional Overturning Circulation and El Niño-Southern Oscillation [82].

Consequences of interactions between tipping elements, their nonlinear dynamics as well as the possible development of domino effects and tipping cascades in systems of interacting tipping elements has not been assessed systematically so far. Here, we make an advance in the theory of interacting tipping elements focusing on bifurcational tipping in the form of cusp catastrophes. The special cases presented in [51] and [81] were extended by looking at unidirectional and tipping cascades consisting of two and three elements. We explored the tipping behavior of the interacting system systematically under the influence of different coupling types (positive and negative interactions) and the coupling strength and particularly focused on identifying conditions that favour tipping cascades. In addition, we applied our theoretical results to real-world systems to reveal mathematically possible tipping cascades in ecological systems such as lake chains and in the climate system.
2 Model

We use a conceptual model of tipping elements in order to investigate the qualitative asymptotic behavior of coupled subsystems which each exhibit critical transitions. Therefore, we consider a continuous dynamical system \( x_i(t) = f_i(x_1, \ldots, x_n) \) in \( n \) dimensions. Each component \( x_i(t) \in \mathbb{R} \) corresponds to a generic tipping element \( X_i \).

The dynamics of the tipping elements is modeled with the topological normal form of the cusp bifurcation [51, 81], i.e. the most generic polynomial system exhibiting this type of tipping behavior (Fig. 1). The asymptotic behavior of many real-world systems such as the thermohaline circulation [11, 14, 16, 17], the Greenland ice sheet [18] and shallow lakes [6, 8] is qualitatively represented by a slice of the cusp catastrophe, a double fold bifurcation, showing bistability, hysteresis behavior and transitions to a qualitatively different state when a critical threshold is crossed [83]. In contrast to other possible bifurcations such as the transcritical, pitchfork or Hopf bifurcations, the double fold bifurcation as a ‘dangerous’ bifurcation [45] captures the catastrophic nature of tipping which is of major interest here. A tipping element \( X_i \) is then represented by

\[
\begin{align*}
  f_i^0(x_i) &= a_i x_i(t) - b_i x_i^3(t) + c_i & \text{with} & & a_i, b_i, c_i \in \mathbb{R}. \\
\end{align*}
\]

where \( a_i, b_i > 0 \) and \( f_i^0 \) corresponds to the uncoupled case.

The parameter \( a_i \) corresponds to the height between the two stable equilibria layers of the cusp [81] and the parameter \( b_i \) controls the strength of the nonlinearity in the system. The control parameter \( c_i \) is associated with a tipping parameter the size of which determines whether the system undergoes a critical transition from one stable state to another (Fig. 1).

For a certain range of the control parameter \( -c_{\text{crit}} < c_i < c_{\text{crit}} \) Eq. 1 has one negative stable equilibrium \( x^*_i < 0 \) and a positive stable equilibrium \( x^*_i > 0 \) as alternative stable states. We call \( x^*_i < 0 \) and \( x^*_i > 0 \) the normal and the tipped state, respectively. Increasing the control parameter \( c_i \) such that the threshold \( c_{\text{crit}} \) is crossed, the normal state \( x^*_i < 0 \) disappears and only the tipped state \( x^*_i > 0 \) exists. Given that the system resided in the normal state \( x^*_i < 0 \), it undergoes a critical transition to the tipped state \( x^*_i > 0 \) for \( c_i > c_{\text{crit}} \). Analogously, for \( c_i < -c_{\text{crit}} \), only the normal state \( x^*_i < 0 \) exists and for lowering the control \( c_i \) below \( -c_{\text{crit}} \) the system falls from the disappearing tipped state \( x^*_i > 0 \) to the normal state \( x^*_i < 0 \) (Fig. 1). The transition of the uncoupled tipping elements at the critical manifold [84] given by the roots of the polynomial can be quantified: depending on the sign of the discriminant \( D^0 = (b_i c_i/2)^2 - b_i (a_i/3)^3 \) there are either one \( (D^0 > 0) \) or two \( (D^0 \leq 0) \) stable equilibria determined by \( f_i^0(x_i^*) = 0 \).

For given \( a_i \) and \( b_i \) and setting \( D^0 = 0 \), the critical value for the control parameter \( c_{\text{crit}}(a_i, b_i) = \pm 2 \sqrt{\frac{1}{b_i} (\frac{a_i}{3})^3} \) can be calculated, where the transition into a regime with only one equilibrium takes place. We call \( c_{\text{crit}}(a_i, b_i) \) the intrinsic tipping points for an uncoupled tipping element as given by Eq. (1).

In the following, we couple the subsystems with each other using a coupling function \( C_i \in \mathbb{R} \) (Fig. 1). Subsystem \( X_i \) then develops as

\[
  f_i(x_i) = a_i x_i(t) - b_i x_i^3(t) + c_i + C_i(x_1(t), x_2(t), \ldots, x_n(t))
\]
with \( a_i, b_i > 0 \).

For simplicity, we choose a linear coupling \([51, 81]\). The linear coupling function for subsystem \( X_i \) then reads

\[
C_i(x_1(t), x_2(t), \ldots, x_n(t)) = \sum_{j=1}^{n} d_{ij} x_j(t) \quad \text{with} \quad i \neq j
\]  

(3)

where a coupling \( d_{ij} \neq 0 \) indicates an influence of another subsystem \( X_j \) to subsystem \( X_i \). Even though Eq. (2) and (3) provide the simplest equations to describe the threshold behavior of \( n \) coupled tipping elements, they can be used for understanding the qualitative features of all systems with the same critical behavior. Using the concept of topological equivalence \([84]\), the critical behavior of more complicated real-world systems can be mapped to the system above. Tab. 1 provides an overview of special cases of coupling between interacting cusp catastrophes investigated in the literature \([51, 80, 81]\).

### Table 1: Overview of linearly coupled tipping elements studied in the literature.

| Reference | Coupling type | Parameter choices |
|-----------|---------------|-------------------|
| \([51]\)  | master-slave system with linear coupling | \(b_1 = b_2 = 1\), \(a_1 = a_2 = 1\), \(C_1(x_1, x_2) = 0\), \(C_2(x_1, x_2) = dx_1\) |
| \([81]\)  | Kadyrov style | \(a_1 = a_2 = 1\), \(b_1 = b_2 = 1\), \(c_1 = c_2 = 0\), \(C_1(x_1, x_2) = d_{21}x_2\), \(C_2(x_1, x_2) = d_{12}x_1\) |

- symmetric coupling: \(d_{21} = d_{12}\)
- asymmetric coupling: \(d_{21} \neq d_{12}\)

| Reference | Coupling type | Parameter choices |
|-----------|---------------|-------------------|
| \([51]\)  | master-slave-slave system with linear coupling | \(b_1 = b_2 = b_3 = 1\), \(a_1 = a_2 = a_3 = 1\), \(C_1(x_1, x_2, x_3) = 0\), \(C_2(x_1, x_2, x_3) = dx_1\), \(C_3(x_1, x_2, x_3) = dx_2\) |

| Reference | Coupling type | Parameter choices |
|-----------|---------------|-------------------|
| \([80]\)  | \(n\) equations coupled in a graph | \(x_i = -x_i + A_{ij}x_j\) |

\(A_{ij}\): matrix of size \(N \times N\)

For \(n = 2\) the corresponding equations read

\[
\dot{x}_1(t) = a_1 x_1(t) - b_1 x_1^2(t) + c_1 + d_{21} x_2(t)
\]

\[
\dot{x}_2(t) = a_2 x_2(t) - b_2 x_2^2(t) + c_2 + d_{12} x_1(t).
\]

(4)

with \(a_i, b_i > 0\). With \(d_{21} = 0 \) and \(d_{12} \neq 0\) we recover a generic master-slave configuration. The stable equilibria can be determined analogously to the uncoupled case with \(f_1(x_1^*) = f_2(x_1^*, x_2^*) = 0\).

The discriminant for the second tipping element \(X_2\) becomes \(D_2 = (b_2(c_2 + d_{12} x_1^*) / 2) - b_2 c_2 - (a_2 / 3)^3\).

Note that \(D_2\) is a function of the control parameter \(c_2\), the coupling strength \(d_2\) and the equilibrium \(x_1^*\). The number of stable equilibria depends on the sign of the discriminant. For \(D_2 \leq 0\) we find two stable equilibria and for \(D_2 > 0\) we find one stable equilibrium. The threshold of the control parameter \(c_2\) at which the number of solutions changes is obtained by solving \(D_2 = 0\) and is given by

\[
c_2 = -d_{12} x_1^* \pm 2 \sqrt{\frac{1}{b_i} \left( \frac{a_i}{3} \right)^3}
\]

(5)
as the effective tipping point of the coupled tipping element $X_2$.

In the following section we will elaborate how one can infer the qualitative behavior of the coupled system using this expression.

![Diagram](image)

Figure 2: Phase space portrait of a master-slave system depending on the control parameters $c_1$ and $c_2$ for a low coupling strength $d_{12} > 0$. Stable equilibria are shown in yellow, unstable equilibria are shown in red. The background color indicates the normalized speed $v = \sqrt{x_1^2 + x_2^2}/v_{max}$ going from slow (purple) to fast (yellow-green).

### 3 Results

Different types of tipping behavior of a coupled system can be derived for the governing system of equations (4). For simplicity, let us consider the case $a_i = 1, b_i = 1$ (arbitrary $b_i$ can be achieved by rescaling $x_i$) for $i = 1, 2$ and $d_{21} = 0$, i.e. unidirectional coupling. A critical transition to the tipped state $x_2^* > 0$ takes place for

$$c_2 + d_{12}x_1 \geq c_{2,\text{crit}}(a_2, b_2)$$

(6)
following expression (5) for the effective tipping point of the coupled tipping element $X_2$ in the previous section 2.

Based on this simple system, rules on the spread of tipping processes in a system of coupled tipping elements are formulated in the following. These tipping rules depend on the type of coupling, i.e. whether the subsystems are positively or negatively coupled, as well as on the relation between the control parameter of the given systems and the coupling strength.

Let $d_{12} > 0$, i.e. subsystem $X_2$ is positively coupled to subsystem $X_1$. Then:

- **Amplified tipping**: If subsystem $X_1$ has already tipped and therefore $x_1^* > 0$, subsystem $X_2$ is pushed to its tipping point and undergoes a critical transition to its tipped state $x_2^* > 0$ for $c_2 \geq c_{2,\text{crit}} - d_{12}|x_1^*|$. The effective tipping point of subsystem $X_2$ is lower than its intrinsic tipping point $c_{2,\text{crit}}$. The higher the coupling strength is, the lower is the necessary critical value of the control parameter $c_2$ for which subsystem $X_2$ tips.

- **Delayed tipping**: If subsystem $X_1$ has not tipped yet and therefore $x_1^* < 0$, subsystem $X_2$ is pulled away from its tipping point and undergoes a delayed critical transition for $c_2 \geq c_{2,\text{crit}} + d_{12}|x_1^*|$. The effective tipping point of subsystem $X_2$ is higher than its intrinsic tipping point $c_{2,\text{crit}}$. The higher the coupling strength is, the higher is the necessary critical value of the control parameter $c_2$ for which subsystem $X_2$ tips.

- **Back-tipping**: Let subsystem $X_1$ not be tipped and therefore $x_1^* < 0$. If subsystem $X_2$ has already tipped and therefore $x_2^* > 0$, subsystem $X_2$ tips back to the normal state $x_2^* < 0$ for $c_2 < -c_{2,\text{crit}} - d_{12}|x_1^*|$. This behavior especially occurs for a high coupling strength $d_2$ and small values of the control parameter $c_2$. Subsystem $X_2$ is staying in the tipped state $x_2^* > 0$ if $-c_{2,\text{crit}} + d_{12}|x_1^*| < c_2$. Here subsystem $X_2$ is pushed into the bistable area of the system. This behavior especially occurs for a high coupling strength $d_{12}$ and high values of the control parameter $c_2$.

Let $d_{12} < 0$, i.e. subsystem $X_2$ is negatively coupled to subsystem $X_1$. Then:

- **Delayed tipping**: If subsystem $X_1$ has already tipped and therefore $x_1^* > 0$, subsystem $X_2$ is pulled away from its tipping point and undergoes a delayed critical transition for $c_2 \geq c_{2,\text{crit}} + |d_{12}|x_1^*|$. The effective tipping point of subsystem $X_2$ is lower than its intrinsic tipping point $c_{2,\text{crit}}$. The higher the coupling strength is, the lower is the necessary critical value of the control parameter $c_2$ for which subsystem $X_2$ tips.

- **Amplified tipping**: If subsystem $X_1$ has not tipped yet and therefore $x_1^* < 0$, subsystem $X_2$ is pushed towards its tipping point and undergoes a critical transition to its tipped state $x_2^* > 0$ for $c_2 \geq c_{2,\text{crit}} - |d_{12}|x_1^*|$. The effective tipping point of subsystem $X_2$ is lower than its intrinsic tipping point $c_{2,\text{crit}}$. The higher the coupling strength is, the lower is the necessary critical value of the control parameter $c_2$ for which subsystem $X_2$ tips.

- **Back-tipping**: Let subsystem $X_1$ be not tipped and therefore $x_1^* < 0$. If subsystem $X_2$ has already tipped and therefore $x_2^* > 0$, subsystem $X_2$ tips back to the normal state $x_2^* < 0$ if $c_2 < -c_{2,\text{crit}} + |d_{12}|x_1^*|$. This behavior especially occurs for a high coupling strength $d_2$ and a low value of the control parameter $c_2$. Subsystem $X_2$ stays in the tipped state if $c_2 \geq -c_{2,\text{crit}} - |d_{12}|x_1^*|$. Here subsystem $X_2$ is pulled to the bistable area of the system. This behavior especially occurs for a high coupling strength $|d_{12}|$ and high values of the control parameter $c_2$.

These tipping rules based on the analytic solution of two unidirectionally coupled tipping elements give an impression of the interplay between the system parameters and their influence on the tipping process. Using numerical calculations, the overall qualitative asymptotic behavior of up to three interacting tipping elements with uni- and bidirectionally coupling of varying sign was assessed. Matrices of stability cards display the number of stable equilibria of the system under consideration depending on the control parameter and the coupling strength (as exemplary given in Fig 4). Switching between the areas of different number of stable equilibria through the variation of one or various parameters is associated with the loss or gain of stable equilibria. Using phase space portraits (see Fig. 2), the different areas in the stability card can be characterized. In particular, assuming that the system resided in a previously stable equilibrium which lost its stability.
and disappears by a changing parameter allows to identify critical transitions in subsystem to a remaining stable state (as given in e.g. Fig. 3).

In the following, results for selected examples of interacting tipping elements are shown. As a first example, a simple master-slave system with a unidirectional coupling is presented in Example I. In Example II, the previous system is extended by an additional negative coupling resulting in a bidirectionally coupled system of two tipping elements. Finally, the propagation in a unidirectionally coupled system consisting of three tipping elements is described in Example III.

**Example I: Master-Slave System for \( d_{12} > 0 \) — e.g. Pair of lakes**

The derived tipping rules are directly realized in the behavior of a master-slave system (\( n = 2 \)) given by Eq. (2) - (3) with \( i = 1, 2 \) and positive coupling \( d_{12} > 0 \) and \( d_{21} = 0 \). This type of coupling can be seen as an example for a pair of interacting lakes. The lakes are connected through an unidirectional water stream and each lake is subject to an external input of nutrients as a control parameter. Critical transitions can be derived using the numerically calculated phase portrait in Fig. 2. The system has four stable equilibria for small values of the control parameters \( c_1 \) and \( c_2 \), which are separated by four saddles and an unstable node in the center of the phase space. Assuming that the system resides in one of the stable equilibria, a critical transition in subsystem \( X_1 \) occurs if \( c_1 > c_{1\text{crit}} \), and therefore its intrinsic tipping point is crossed (see Fig. 3, lower right).

With increasing control parameter \( c_2 \), a critical transition in subsystem \( X_2 \) occurs even if \( c_2 < c_{2\text{crit}} \) given that subsystem \( X_1 \) has already tipped (see Fig. 3, left column). The coupled subsystem \( X_2 \) tips at an effective tipping point lower than its intrinsic tipping point.

For \( c_1 > c_{1\text{crit}} \) and a slight increase of the control parameter \( c_2 \), a domino effect, starting in in the normal state of \( X_1 \) and \( X_2 \), with a critical transition in subsystem \( X_1 \) and a following transition in subsystem \( X_2 \) arises (Fig. 3, upper right).

There is a change in the system behavior for an increasing coupling strength. The previously described area with only one stable fixed point of two tipped subsystems for \( c_1 > c_{1\text{crit}} \) exists for extremely low values of \( c_2 << c_{2\text{crit}} \). Therefore a domino effect can occur for even lower values of the control parameter \( c_2 << c_{2\text{crit}} \) than for a system with lower coupling strength. For \( c_1 < c_{1\text{crit}} \) and low values of the control parameter \( c_2 \), subsystem \( X_2 \) either tips to the tipped state for \( c_2 < c_{2\text{crit}} \) given that subsystem \( X_1 \) has already tipped or subsystem \( X_2 \) tips back from a tipped state to the original state given that subsystem \( X_1 \) has not tipped. For \( c_1 < c_{1\text{crit}} \) and an increased control parameter \( c_2 \), the critical transition of subsystem \( X_2 \) to the tipped state, given that subsystem \( X_1 \) has already tipped, is the only transition that can be observed.

**Example II: Bidirectional interaction of two tipping elements — e.g. Greenland ice sheet and Atlantic Meridional Overturning Circulation**

Consider a system consisting of two (\( n = 2 \)) tipping elements given by Eq. (2) - (3) with \( i = 1, 2 \) and positive coupling \( d_{12} > 0 \) and \( d_{21} = 0 \). This type of coupling can for instance be found in the interaction of the Greenland ice sheet (GIS) and the Atlantic meridional overturning circulation (AMOC): increased meltwater influx into the North Atlantic due to tipping of the GIS could lead to a weakening or even shutdown (tipping) of the AMOC [85], i.e., introducing a positive coupling. At the same time, a slowdown of the AMOC leads to a relative cooling around Greenland and hence corresponds to a negative coupling [53]. The system is analyzed for a low and a high coupling strength (where \( d_{21} \) and \( d_{12} \) have opposite signs but the same magnitude). The number of stable equilibria and possible critical transitions for different parameter settings are shown in Fig. 4 and Fig. 5, respectively. Note that Fig. 5 is a zoom into Fig. 4 for low coupling strengths.

With increasing control parameter \( c_1 \), a critical transition of the GIS as subsystem \( X_1 \) is possible for \( c_1 < c_{1\text{crit}} \), given that the AMOC as subsystem \( X_2 \) has not tipped so far. The GIS might tip at an effective tipping point which is lower than the intrinsic tipping point of the isolated subsystem. A delayed critical transition of the GIS for a tipped AMOC is possible with a further increase of control parameter \( c_1 \) for \( c_1 \gg c_{1\text{crit}} \).

With increasing control parameter \( c_2 \), a critical transition of the AMOC as subsystem \( X_2 \) to a state with weakened strength is possible for \( c_2 < c_{2\text{crit}} \) given that the GIS as subsystem...
Figure 3: Overview of possible critical transitions in a master-slave system with a low positive coupling $d_{12} > 0$ depending on control parameter $c_1$ and $c_2$. Shown are the states of the system which lose stability and disappear for a transition to the corresponding coloured area as a result of the variation of control parameter $c_1$ (horizontal axis) and $c_2$ (vertical axis). Furthermore, the stable state of the system is shown which is achieved after the critical transition. The critical transition is derived from the phase space portrait. A blue dot represents a not-tipped subsystem while a grey dot represents a tipped subsystem. The transitions between the coloured areas are idealized and correspond only approximately to the analytic bifurcation line. For the marked critical value of the control parameter $c_{i_{\text{crit}}}$ the isolated subsystem $X_i$ undergoes a critical transition.
$X_1$ has already tipped. The AMOC might tip at an effective tipping point which is lower than the intrinsic tipping point of the isolated subsystem. Given that the GIS is in its normal state, a delayed critical transition of the AMOC is possible with a further increase of the control parameter $c_2$ for $c_2 \gg c_{2,\text{crit}}$ at an effective tipping point higher than its intrinsic tipping point. As a result of the model formulation, the GIS would still pull the THC away from its tipping point even though ice has already started to melt but has not tipped to $x_1^* > 0$ (i.e., $0 > x_1^* > -1$). It would be possible to adjust the coupling function or the dynamics of each tipping element (e.g., [74]) so that already a slight change of the GIS state towards the tipped state without a complete critical transition to full loss of the ice sheet would push the AMOC towards its own tipping point.

For a slight increase of both control parameters $c_1$ and $c_2$, a critical transition of the GIS as well as the AMOC to the tipped state is possible for $c_1 < c_{1,\text{crit}}$ and $c_2 < c_{2,\text{crit}}$ before their respective intrinsic tipping points are crossed. In contrast to the previous Example I, the additional negative coupling results in the tipping of both interacting subsystems at an effective tipping point below their intrinsic tipping points. In a master-slave system (Example I) the master system $X_1$ needs to tip through an increase of its control parameter above its intrinsic tipping point $c_1 > c_{1,\text{crit}}$ to trigger a critical transition in the slave system $X_2$ at an effective tipping point $c_3 < c_{3,\text{crit}}$.

With increasing coupling strengths $d_{21}$ and $d_{12}$ the system behavior changes. The system has only one unstable fixed point and Kadyrov oscillations [81] occur for a wide range of the control parameters in the considered part of the $(c_1, c_2)$-parameter space (upper left of Fig. 4).

Example III: Master-slave-slave system – e.g. Propagation of critical transitions in lake chains

Consider a system consisting of three ($n = 3$) unidirectionally coupled tipping elements given by Eq. (2) - (3) with $i = 1, 2, 3$ and $d_{12}, d_{23} > 0$ and $d_{21}, d_{32}, d_{31}, d_{13} = 0$. This type of coupling corresponds to the behavior of a lake chain subject to an external input of nutrients as a control parameter (as, e.g., in [55]). The lakes are connected by an unidirectional water stream. The behavior of subsystem $X_3$ corresponds to the behavior of an uncoupled tipping element. Therefore the eutrophication of the first lake in the lake chain, i.e., the tipping of subsystem $X_1$, is possible with an increase of its control parameter $c_1 > c_{1,\text{crit}}$.

With increasing control parameter $c_2$ or $c_3$, a critical transition in the corresponding subsystems is possible for $c_2 < c_{2,\text{crit}}$ or $c_3 < c_{3,\text{crit}}$ given that the preceding subsystem has already tipped (see Fig. 6) or undergoes a transition into the tipped state through a continuously changing control parameter (as for example shown for $X_2$ in Fig. 7, lower right). Lake $X_2$ and $X_3$ in the lake chain can therefore become eutrophic before the intrinsic critical level of nutrient input of an isolated lake is crossed, given that the preceding lake has already become eutrophic.

For a simultaneous, slight increase of the control parameters $c_2$ and $c_3$ of both subsystems $X_2$ and $X_3$, a critical transition in both subsystems $X_2$ and $X_3$ is possible for $c_2 < c_{2,\text{crit}}$ and $c_3 < c_{3,\text{crit}}$ and, as a result, a domino effect can be observed given subsystem $X_1$ has already tipped (Fig. 6, center) or tips for $c_1 > c_{1,\text{crit}}$ (Fig. 7, upper right). Consequently, after the eutrophication of the first lake, a critical transition to the turbid state of a lake can spread in the lake chain even if the intrinsic critical value of nutrient input known from an isolated lake is not crossed.

4 Discussions and Conclusions

The qualitative asymptotic behavior of interacting, cusp-like tipping elements has been assessed in a simple analytic and a systematic numerical analysis of a conceptual model. Depending on the type of coupling and the coupling strength, qualitatively different behaviors in terms of possible domino effects of the systems of interacting tipping elements were observed.

Simple analytic calculations resulted in the formulation of tipping rules for the spread of tipping processes in systems of interacting tipping elements: a shift in the threshold value of the control parameter, at which an interacting tipping element undergoes a transition into a qualitatively different state, can occur. We call the threshold value of the isolated subsystem the intrinsic tipping point of the tipping element. If an interaction with another tipping element exists, the tipping process takes place when the so called effective tipping point is crossed. Depending on the coupling direction, the effective tipping point can occur at either lower (amplified tipping) or higher (delayed tipping) values of the control parameter than the intrinsic tipping point.
Figure 4: Number of stable equilibria of the system consisting of two bidirectionally coupled tipping elements depending on the coupling strengths $d_{21} < 0$ and $d_{12} > 0$ as well as the control parameters $c_1$ and $c_2$. Coloured areas in the small squares represent the number of fixed points depending on the control parameters $c_1$ and $c_2$ for a specific coupling strength. The position of a small square in the matrix is determined by the coupling strength. In the yellow region for high coupling strengths with opposite sign but same magnitude indicating the absence of stable equilibria a stable limit cycle (Kadyrov oscillations in [81]) can be observed.
We have generalized and extended existing studies of special cases of coupled cusp-like tipping elements [51, 81] through a systematic numerical analysis of two and three interacting tipping elements with one- or bidirectional coupling of varying direction. The behavior of the special cases including the window of synchronization in a simple master-slave system [51], a tipping cascade in a positively coupled master-slave-slave system [51] and the Kadyrov-oscillator [81] for two bidirectionally coupled tipping elements for a high coupling strength of same magnitude but with opposite signs is consistent with the system behavior observed in our analysis.

In addition, our systematic analysis allowed to identify types of coupling that favor critical tipping scenarios. Conditions in terms of coupling strength and control parameters of the subsystem under which the tipping scenarios occur were determined. Domino effects of tipping processes that occur before the crossing of intrinsic tipping points, i.e. where the effective tipping point lies at lower values than the intrinsic tipping point of the uncoupled tipping element, are of special interest. In a simple master-slave system with positive coupling, a critical transition in the master system due to a crossing of its intrinsic tipping point triggers a critical transition of the slave system at an effective tipping point lower than its intrinsic tipping point. In contrast, a negative coupling would prevent an amplified tipping of the slave system. In a system of two tipping elements with bidirectional coupling, a tipping cascade is favored if one of the coupling terms is negative. In a master-slave-slave system with $d_{12} > 0$ and $d_{23} > 0$, the initial tipping of the master system can trigger cascading tipping processes in the following subsystems before the intrinsic threshold of the corresponding control parameter is crossed. Such a domino effect before the crossing of the corresponding intrinsic thresholds cannot be observed after the introduction of a negative coupling (results not shown here). Tipping processes are suppressed instead in this case and do not spread into all subsystems.

Applying the qualitative system behavior to selected interacting real-world tipping elements revealed possible tipping scenarios, which are relevant for the future development of the Earth system and in addition, due to the consequences of tipping such as sea level rise [86, 87], for economy, infrastructure and society more broadly. In particular, the analysis of the qualitative asymptotic system behavior of two bidirectionally coupled tipping elements with opposite sign but same magnitude suggests that the Greenland ice sheet and the AMOC might tip before their
Figure 6: Overview of possible critical transitions of a master-slave-slave system depending on control parameter $c_2$ and $c_3$ for $c_1 < c_{1\text{crit}}$, with $d_{12} > 0$ and $d_{23} > 0$ and low coupling strength where $|d_{12}| \approx |d_{23}|$. Same quantities as in Fig. 3 with the following differences: Shown are the initial states of critical transitions and backtipping processes which lose stability and disappear for variation of the control parameters $c_2$ (horizontal axis) and $c_3$ (vertical axis) and the corresponding transition to the colored area of stable fixed points. In addition, the subsystem $X_i$, $i = 1, 2, 3$ is given, which undergoes the transition. The order of the subsystems given does not represent the order order of tipping. The order can vary within a colored area depending on the relation between coupling parameter and coupling strength.
intrinsic tipping points are reached. In other words, the meltdown of the Greenland ice sheet and the slowdown of the AMOC might begin before their known intrinsic threshold value is crossed. The possible existence of such domino effects increases the risks that anthropogenic climate change poses to human societies, since the intrinsic threshold ranges of some climatic tipping elements including the Greenland ice sheet are assumed to lie even within the 1.5-2°C target range of the Paris agreement [88].

When it comes to the application of tipping behavior to real-world systems, it should be noted that tipping elements were described in an idealized way using the normal form of the cusp catastrophe onto which, by the concept of topological equivalence [84], the critical behavior of real world systems can be mapped. The proposed model of interacting tipping elements therefore shows a hypothetical, but mathematically possible system behavior. It was motivated by its catastrophic features [45, 83] in contrast to other bifurcational systems and the appearance of the double fold bifurcation in many real world systems [6, 8, 11, 14, 16, 17, 18, 89]. However, processes which are not taken into account in the conceptual representation of tipping elements, but are present in the real world, might influence the system and its tipping behavior. In addition to a direct coupling of tipping elements in the climate system [53], an indirect, "diffusive" interaction through, e.g. the global mean temperature [1] could be considered as a potential coupling mechanism. Furthermore, chains and pairs of tipping elements were analyzed isolated from the larger network of interacting climatic tipping elements [53], i.e. possible interactions with other climatic tipping elements were neglected. Here, we focused on bifurcational tipping assuming that the control parameter varies sufficiently slowly for the system to keep track with the stable states. It should be noted that a change of the control parameter with a high rate is likely given the increasing influence of humans on the Earth system, possibly giving rise to rate-induced tipping [44, 49].

The qualitative and theoretically possible system behaviors studied here and their application to real-world systems therefore introduces further research questions regarding tipping elements and their interactions in ecology, climate science and other fields. The conceptual approach should be extended to networks of tipping elements as already suggested in [80, 74] and motivated by [53]. Networks of interacting tipping elements can be analysed using methods of statistical mechanics.
Taking the important interactions of climatic tipping elements into account in a network approach, realistic complex models must be used for quantitatively approximating the effective tipping point. In addition, interacting tipping elements with heterogeneous intrinsic threshold and varying internal time scales should be considered as, for example, the critical nutrient input of lakes varies with their depth. Finally, generic early warning signals for tipping cascades comparable to already existing indicators for critical transitions of isolated tipping elements are needed to forecast domino effects and counteract undesired consequences of tipping (cascades). A first step towards early warning indicators of tipping cascades has been presented only recently.

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