A Brownian Particle and Fields II: 
Radiation Reaction as an Application

Keita Seto∗

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Extreme Light Infrastructure – Nuclear Physics (ELI-NP) / 
Horia Hulubei National Institute for R&D in Physics and Nuclear Engineering (IFIN-HH), 
30 Reactorului St., Bucharest-Magurele, jud. Ilfov, P.O.B. MG-6, RO-077125, Romania.

Abstract

Radiation reaction has been investigated traditionally in classical dynamics and recently in non-linear QED as high-intensity field physics produced by high-intensity lasers. Its quantumness is predicted by the factor in the radiation formula. In this Volume II, the quantization of radiation reaction by using a Brownian scalar electron is discussed for obtaining the above radiation formula. Finally, its stochasticity is found as the origin of the factor of its quantumness in high-intensity field physics regime.

Keyword:
[Physics] Stochastic quantum dynamics, relativistic motion, field generation
[mathematics] Applications of stochastic analysis

∗keita.seto@eli-np.ro
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## Notation and Conventions

| Symbol | Description |
|--------|-------------|
| $c$    | Speed of light |
| $\hbar$ | Planck's constant |
| $m_0$  | Rest mass of an electron |
| $e$    | Charge of an electron |
| $\mathbb{V}_M^4$ | 4-dimensional standard vector space for the metric affine space |
| $g$    | Metric on $\mathbb{V}_M^4$: $g := \text{diag}(+1, -1, -1, -1)$ |
| $A^4(\mathbb{V}_M^4, g)$ | 4-dimensional metric affine space with respect to $\mathbb{V}_M^4$ and $g$ |
| $\mathcal{B}(I)$ | Borel $\sigma$-algebra of a topological space $I$. |
| $\varphi_E := \{ \varphi_A^E | A \in \text{set of indexes} \}$ | Coordinate mapping on $E$; $\varphi_E^A : E \to \mathbb{R}$ |
| $(\Omega, \mathcal{F}, \mathcal{P})$ | Probability space |
| $\mathbb{E}[\hat{X}(\bullet)] := \int_{\Omega} d\mathcal{P}(\omega) \hat{X}(\omega)$ | Expectation of $\hat{X}(\bullet) := \{ \hat{X}(\omega) | \omega \in \Omega \}$ |
| $\mathbb{E}[\hat{X}(\bullet)|\mathcal{C}]$ | Conditional expectation of $\hat{X}(\bullet)$ given $\mathcal{C} \subset \mathcal{F}$ |
| $\mathcal{P}_\tau \subset \mathcal{F}$ | Sub-$\sigma$-algebra in the family of "the past", $\{ \mathcal{P}_\tau \}_{\tau \in \mathbb{R}}$ |
| $\mathcal{F}_\tau \subset \mathcal{F}$ | Sub-$\sigma$-algebra in the family of "the future", $\{ \mathcal{F}_\tau \}_{\tau \in \mathbb{R}}$ |
| $\hat{x}(\circ, \bullet)$ | Dual progressively measurable process (D-progressive, D-process) |
| $\mathcal{V} \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$ | Relativistic kinematics of a scalar (spin-less) electron |
| $\mathcal{F}(+)\mathcal{F}(-) \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ | Retarded field and advanced field |
| $\mathfrak{F} \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ | Effective radiation (reaction) field |
| $\delta \hat{x}(\tau, \omega) \in \mathbb{V}_M^4$ | $\delta \hat{x}(\tau, \omega) := \hat{x}(\tau, \omega) - \mathbb{E}[\hat{x}(\tau, \bullet)]$ |
1 Introduction

This series of the papers proposes quantum dynamics of a stochastic scalar (spin-less) electron interacting with classical light fields. It is equivalent to the system of the Klein-Gordon equation and the Maxwell equation. In this Volume II, we focus the topics on radiation reaction as an application of Volume I [1]. It means the quantization of the Lorentz-Abraham-Dirac (LAD) equation of classical dynamics. Radiation reaction as a model of a radiating electron has become important in the research projects of high-intensity lasers [2], represented by “Extreme Light Infrastructure (ELI)” [3, 4, 5] for the last laser-plasma science [6].

Corresponding to the name of “high-energy particle physics”, let us name such a high-intensity laser science "high-intensity field physics". Before entering into the main topic, we summarize the available theoretical works on radiation reaction.

Denote by \( B(I) \) the Borel \( \sigma \)-algebra of a topological space \( I \) and a metric affine space \( \mathbb{A}^4(\mathbb{V}_M, g) \) with respect to a 4-dimentional standard vector space \( \mathbb{V}_M^4 \) and its metric \( g := \text{diag}(+1, -1, -1, -1) \), let a measure space \( (\mathbb{A}^4(\mathbb{V}_M, g), B(\mathbb{A}^4(\mathbb{V}_M, g)), \mu) \) be the Minkowski spacetime. The coordinate mappings \( \varphi_E : \{ \varphi_E^A | A \in \text{set of indexes, } \mathbb{V}_M^4 \} \) is introduced such that the index \( A \) becomes \( A = \mu \) if \( E = \mathbb{V}_M^4 \) and \( A = \mu \nu \) when \( E = \mathbb{V}_M^4 \otimes \mathbb{V}_M^4 (\mu, \nu = 0, 1, 2, 3) \), even if we do not declare it explicitly.

1.1 LAD model and QED correction

In the non-relativistic classical dynamics, an electron emits its energy by light by Larmor’s formula such that \( dW_{\text{classical}}/dt = m_0 \tau_0 \dot{v}^2 \) [2]. Hence, the momentum transfer of an electron due to its radiation has to be taken into account. It is associated by the following energy balance relation:

\[
\frac{d}{dt} \left( \frac{1}{2}m_0 \dot{v}^2 \right) = F_{\text{ex}} \cdot \dot{v} - \frac{dW_{\text{classical}}}{dt}
\]

Smearing this equation by its definite integral on the domain of \( (-\infty, \infty) \) with the condition \( [m_0 \tau_0 \dot{v} \cdot \dot{v}]_{-\infty}^{\infty} = 0 \), the Lorentz-Abraham (LA) equation is imposed:

\[
m_0 \dot{v} = F_{\text{ex}} + m_0 \tau_0 \dot{v}
\]

Then, the Lorentz-Abraham-Dirac (LAD) equation is derived as the covariant form of [1] in the Minkowski spacetime. The following is the summary of the LAD model provided by P. A. M. Dirac [3].

**Theorem 1** (Classical dynamics). Consider a single scalar electron system denoted by the equation of motion and the Maxwell equation in \((\mathbb{A}^4(\mathbb{V}_M^4, g), B(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)\):

\[
m_0 \frac{d\nu}{d\tau}(\tau) = -e [F_{\text{ex}}^{\mu\nu}(x(\tau)) + F_{\text{LAD}}^{\mu\nu}(x(\tau))] v_\nu(\tau)
\]

\[
\partial_\mu F^{\mu\nu}(x) = \mu_0 \times \left[ -ec \int_{\mathbb{R}} d\tau \, v^\nu(\tau) \delta^4(x - x(\tau)) \right]
\]

Where \( m_0, \, e, \, \mu_0 \) and \( c \) are the physical constants of the rest mass, the charge of an electron, the vacuum permeability and the speed of light. The vacuum permeability \( \mu_0 \) relates to the vacuum permittivity \( \varepsilon_0 \), i.e., \( \varepsilon_0 \mu_0 c^2 = 1 \) in the SI unit. \( v \in \mathbb{V}_M^4 \) represents the 4-velocity of an electron. The fields \( F_{\text{ex}} \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4 \) such that \( \partial_\mu F_{\text{ex}}^{\mu\nu} = 0 \) denotes an external field and \( F_{\text{LAD}} \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4 \) satisfying \( \partial_\mu F_{\text{LAD}}^{\mu\nu} = 0 \) is, so-called,
"the LAD field (the effective radiation reaction field)" which is the homogeneous solution of (3).

The homogeneous solution \( F_{LAD} \) is expressed by the retarded field and the advanced field which are the solutions of (3) [8, 36].

**Lemma 2** (Retarded, advanced and LAD fields). By solving (3), the retarded field \( F_{\text{ret}} \) and the advanced field \( F_{\text{adv}} \) are obtained. By the substitution \( x = x(\tau) \),

\[
F_e(x(\tau))|_{e=\text{ret,adv}} = \frac{e}{8\pi\varepsilon_0 c^5} \left[ \frac{dv}{d\tau}(\tau) \otimes v(\tau) - v(\tau) \otimes \frac{dv}{d\tau}(\tau) \right] \int_\mathbb{R} d\tau' \frac{\delta(\tau' - \tau)}{|\tau' - \tau|}
\]

\[
-\text{sign}(\epsilon) \times \frac{e}{6\pi\varepsilon_0 c^5} \left[ \frac{d^2v}{d\tau^2}(\tau) \otimes v(\tau) - v(\tau) \otimes \frac{d^2v}{d\tau^2}(\tau) \right].
\]

Where, \( \text{sign}(\text{ret}) = 1 \) and \( \text{sign}(\text{adv}) = -1 \). The LAD field \( F_{LAD} \) is given by

\[
F_{LAD}(x(\tau)) := \frac{F_{\text{ret}}(x(\tau)) - F_{\text{adv}}(x(\tau))}{2}
\]

as the homogeneous solution of the Maxwell equation (3) at \( x = x(\tau) \).

**Definition 3** (LAD equation). By substituting (5) for (2),

\[
m_0 \frac{dv^\mu}{d\tau}(\tau) = -eF_{\text{ex}}^{\mu\nu}(x(\tau))v_\nu(\tau) + \frac{m_0\tau_0}{c^2} \left[ \frac{d^2v^\mu}{d\tau^2}(\tau) \cdot v^\nu(\tau) - \frac{d^2v^\nu}{d\tau^2}(\tau) \cdot v^\mu(\tau) \right] v_\nu(\tau)
\]

is named the Lorentz-Abraham-Dirac (LAD) equation [8]. Where, \( \tau_0 := \frac{e^2}{6\pi\varepsilon_0 m_0 c^3} = O(10^{-24}\text{sec}) \).

The non-relativistic limit of (2) tends to (1). The energy loss by radiation is estimated by the invariant form of Larmor’s formula:

\[
\frac{dW_{\text{classical}}}{dt}(\tau) = -m_0\tau_0 \frac{dv^\alpha}{d\tau}(\tau) \cdot \frac{dv^\alpha}{d\tau}(\tau) \quad (\geq 0)
\]

\[
= -\frac{\tau_0}{m_0} g_{\alpha\beta} \left( -eF_{\text{ex}}^{\alpha\mu} v_\mu(\tau) \right) \left( -eF_{\text{ex}}^{\beta\nu} v_\nu(\tau) \right) + O(\tau_0^2)
\]

This LAD equation has been treated as the standard model of radiation reaction in the previous laser-plasma simulations for explaining radiation from classical particles since \( d(m_0c^2\gamma)/dt \approx F_{\text{ex}} \cdot v - dW_{\text{classical}}/dt \) [the 0th component of (6)]. However, its quantum correction has started to be considered in the recent works provided by high-intensity lasers [9, 10]. It is found by the following steps: For an external field of a plane wave, Volkov solutions of a Dirac equation are derived [11, 12]. Then, a family of these Volkov solutions imposes its orthogonality and completeness [13, 14], thus, the family is regarded as a basis of Dirac fields absorbing an external field (a dressed Dirac field) [12, 13]. An S-matrix by a family of Volkov solutions is introduced [16, 17]. Finally, a non-linear Compton scattering as radiation reaction can be calculated [18, 19, 20]. This is so-called the Furry picture. By resulting them, the radiation formula is derived as follows.
Figure 1: Quantumness of radiation, $q(\chi)$. This is the plot of (11) with respect to energies of an electron and laser intensities. When we choose the combination between a laser intensity of $O(10^{22} \text{W/cm}^2)$ and an electron energy of $O(1\text{GeV})$, the factor $q(\chi)$ reaches 0.3 which is the feasible regime produced by the ELI-NP facility [28]. The SLAC E-144 included the experiments of the non-linear Compton scatterings by the combination of $O(10^{18} \text{W/cm}^2 + 46\text{GeV})$ [33].

Many authors have investigated this effect for their future experiments by high-intensity lasers [9, 10, 22, 23, 24, 25, 26, 27, 28, 29, 30]. For example at ELI-NP in Romania [3, 28], they aim the factor $q(\chi) = 0.3$ by a combination between a high-intensity laser of $O(10^{22} \text{W/cm}^2)$ and an electron of $O(1\text{GeV})$ [see Figure 1]. Thus, the behavior of $q(\chi)$ depending on laser intensities is important to understand the transition from classical dynamics to non-linear QED.

However, can we generalize the applicable range of this Furry picture from an external field of a plane wave? Unfortunately, there are the mathematical demonstrations only in the case of a plane wave [13, 14, 31]. It is a major interest for the expression of strong focused or superpositioned lasers [A. Di Piazza]. In addition, quantum dynamics doesn’t provide the method to draw a real trajectory of a particle. It is very helpful for laser-plasma simulations if it is improved. Let us aim the second purpose in this article.
1.2 Proposal: by a Brownian particle and fields

The regime of high-intensity field physics can be found at quantized, relativistic and high-intensity field interactions marked by "the star" in Figure 2. The Furry picture is the way from the relativistic quantum dynamics. The quantization from the LAD equation at classical, relativistic and high-intensity regime shall summarize those ideas. On the other hand, the quantization after reaching high-intensity "classical" dynamics is its second candidate. The Lorentz-Abraham-Dirac (LAD) equation at high-intensity "classical" dynamics (Definition 3) is the standard model of radiation reaction. ELI-NP is the state-of-the-arts laser facility which can perform the green area and proposes the real experiments of radiation reaction [3, 28].

Consider two measurable spaces \((X, \mathcal{X})\) and \((Y, \mathcal{Y})\), a \(\mathcal{X}/\mathcal{Y}\)-measurable mapping \(f : X \to Y\) satisfying \(f^{-1}(A) := \{x \in X | f(x) \in A\} \subset \mathcal{X}\) for all \(A \in \mathcal{Y}\). By employing \(L^p(E)\) as a family of \(\mathcal{B}(\mathbb{R}) \times \mathcal{F}/\mathcal{B}(E)\)-measurable mappings for a topological space \(E\), let us introduce \(L^p_{\text{loc}}(\{\mathscr{P}_\tau\}; E)\) and \(\mathcal{L}^p_{\text{loc}}(\{\mathscr{F}_\tau\}; E)\) as the families of stochastic processes as follows:

\[
\mathcal{L}^p_{\text{loc}}(\{\mathscr{P}_\tau\}; E) := \left\{ \mathcal{L}^p_{\text{loc}}(\mathcal{P}_\tau) \bigg| \mathcal{L}^p_{\text{loc}}(\mathcal{P}_\tau) \right\}
\]

where \(\mathcal{L}^p_{\text{loc}}(\mathcal{P}_\tau) := \left\{ X(\omega, \bullet) = \mathbb{E}\left[ X(\tau, \omega) | \mathcal{F}_\tau \right] : \mathbb{P}\left[ X(\tau, \omega) \in \mathcal{X} \right] < \infty \text{ a.s.} \right\}\).
\[ L_{\text{loc}}^p(\mathcal{F}_\tau; E) := \left\{ \hat{X}(\cdot, \bullet) \in L^p_{[\tau, \infty)}(E) \right\} \forall \tau \in \mathbb{R}, \hat{X}\mu(\cdot, \bullet) \text{ is } \{\mathcal{F}_\tau\}-\text{adapted} \]

**Definition 5** 
Let \( W_+ (\cdot, \bullet) \) and \( W_- (\cdot, \bullet) \) be \( \{\mathcal{F}_\tau\}\)-Wiener processes satisfying the following for \( \mu, \nu = 0, 1, 2, 3 \):

\[ \mathbb{E} \left[ \int_{\tau}^{\tau+\delta\tau} dW_{\pm}^\mu (\tau', \bullet) \right] = 0, \quad \mathbb{E} \left[ \int_{\tau}^{\tau+\delta\tau} dW_{\pm}^\mu (\tau', \bullet) \times \int_{\tau}^{\tau+\delta\tau} dW_{\pm}^\nu (\tau'', \bullet) \right] = \delta^{\mu\nu} \times \delta\tau, \]

The Itô formula of a \( C^2 \)-function \( f : \mathbb{A}^4(\mathcal{V}_M, g) \to \mathbb{C} \) on \( W_{\pm}(\cdot, \cdot) \) is

\[ f(W_+(\tau_b, \omega)) - f(W_+(\tau_a, \omega)) = \int_{\tau_a}^{\tau_b} \partial_\mu f(W_+(\tau, \omega)) dW_+^\mu (\tau, \omega) \pm \frac{\lambda^2}{2} \int_{\tau_a}^{\tau_b} \partial_\mu \partial^\mu f(W_+(\tau, \omega)) d\tau \text{ a.s.} \]

**Theorem 6** (*Kinematics [1]*)  
For \((\Omega, \mathcal{F}, \mathcal{P})\), “A dual-progressively measurable process”, or by shortening “A D-process” and “A D-progressive” \( \hat{x}(\cdot, \bullet) := \{ \hat{x}(\tau, \omega) \in \mathbb{A}^4(\mathcal{V}_M, g) | \tau \in \mathbb{R}, \omega \in \Omega \} \) is a 4-dimensional \( \{\mathcal{P}_\tau \cap \mathcal{F}_\tau\}\)-adapted process, such that

\[ d_+ \hat{x}(\tau, \omega) = V_+ (\hat{x}(\tau, \omega)) d\tau + \lambda \times dW_+(\tau, \omega), \]

where \( \lambda = \sqrt{\hbar/m_0} \), \( \mathcal{V}_+(\hat{x}(\cdot, \bullet)) \in L^1_{\text{loc}}(\{\mathcal{P}_\tau\}; \mathcal{V}_M^4) \) and \( \mathcal{V}_-(\hat{x}(\cdot, \bullet)) \in L^1_{\text{loc}}(\{\mathcal{F}_\tau\}; \mathcal{V}_M^4). \) For a \( C^2 \)-function \( f : \mathbb{A}^4(\mathcal{V}_M, g) \to \mathbb{C} \), its Itô formula \( d_+ f \) is characterized by

\[ d_+ f(\hat{x}(\tau, \omega)) = \partial_\mu f(\hat{x}(\tau, \omega)) d_+ \hat{x}^\mu (\tau, \omega) \pm \frac{\lambda^2}{2} \partial_\mu \partial^\mu f(\hat{x}(\tau, \omega)) d\tau \text{ a.s.} \]

The probability density of \( \hat{x}(\cdot, \bullet) \), i.e., \( p(x, \tau) \) satisfies the following Fokker-Planck equation:

\[ \partial_\tau p(x, \tau) + \partial_\mu [\hat{V}_{\pm}^\mu (x)p(x, \tau)] \pm \frac{\lambda^2}{2} \partial^\mu \partial_\mu p(x, \tau) = 0 \]

For \((\Omega, \mathcal{F}, \mathcal{P})\), let \( \mathbb{E}[\hat{X}(\bullet)] \) be the expectation of the random variable \( \hat{X}(\bullet) := \{ \hat{X}(\cdot, \omega) | \omega \in \Omega \} \), namely, \( \mathbb{E}[\hat{X}(\bullet)] := \int_{\Omega} d\mathcal{P}(\omega) \hat{X}(\omega) \). The conditional probability of \( B \) given \( A \) is denoted by \( \mathcal{P}_B(A) \). For \( \{A_n\}_{n=1}^\infty \) of a countable decomposition of \( \Omega \), its minimum \( \sigma \)-algebra \( \mathcal{C} = \sigma(\{A_n\}_{n=1}^\infty) \) is introduced. \( \mathbb{E}[\hat{X}(\bullet)|\mathcal{C}](\omega) := \sum_{n=1}^\infty \int_{\Omega} d\mathcal{P}_{A_n}(\omega') \hat{X}(\omega') 1_{A_n}(\omega) \) is defined as the conditional expectation of \( \hat{X}(\bullet) \) given \( \mathcal{C} \subset \mathcal{F} ; 1_{A_n}(\omega) \) satisfies \( 1_{A_n}(\omega \in A_n) = 1 \) and \( 1_{A_n}(\omega \notin A_n) = 0 \). Since a D-progressive \( \hat{x}(\tau, \omega) \) is \( \{\mathcal{P}_\tau \cap \mathcal{F}_\tau\}\)-adapted, the following complex differential and velocity can be defined;

**Definition 7** (*Complex differential and velocity*). Let \( \hat{d} := (1-i)/2 \times d_+ + (1+i)/2 \times d_- \) be the complex differential on a given D-progressive \( \hat{x}(\cdot, \bullet) \) for \( f : \mathbb{A}^4(\mathcal{V}_M, g) \to \mathbb{C} \):

\[ \hat{d} f(\hat{x}(\tau, \omega)) = \partial_\mu f(\hat{x}(\tau, \omega)) \hat{d} \hat{x}^\mu (\tau, \omega) + \frac{i\lambda^2}{2} \partial^\mu \partial_\mu f(\hat{x}(\tau, \omega)) d\tau \text{ a.s.} \]

[1]:#
Then, consider the conditional expectation of the derivative given \( \gamma_\tau := \mathcal{P}_\tau \cap \mathcal{F}_\tau \subset \mathcal{F} \) is denoted by

\[
\mathbb{E} \left[ \frac{df}{d\tau}(\hat{x}(\tau, \bullet)) \mid \gamma_\tau \right](\omega) = \mathcal{V}^{\mu}(\hat{x}(\tau, \omega)) \partial_\mu f(\hat{x}(\tau, \omega)) + \frac{i\lambda^2}{2} \partial^\mu \partial_\mu f(\hat{x}(\tau, \omega)) \, .
\]  

(18)

Especially when \( f(\hat{x}(\tau, \omega)) = \hat{x}(\tau, \omega) \), it derives the complex velocity \( V \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4 \),

\[
\mathbb{V}^{\mu}(\hat{x}(\tau, \omega)) := E_t \hat{d}f_d(\hat{x}(\tau, \bullet)) \mid \gamma_\tau \right)(\omega) = \frac{1}{2} \mathcal{V}^{\mu}(\hat{x}(\tau, \omega)) + \frac{1}{2} \mathcal{V}^{\mu}(\hat{x}(\tau, \omega)) \, .
\]  

(19)

The following signatures are employed for the simple description:

\[
\hat{V}^{\mu}(x) := \mathcal{V}^{\mu}(x) + \frac{i\lambda^2}{2} \partial^\mu
\]
\[
\mathcal{D}_\tau = \hat{V}^{\mu}(x) \partial_\mu
\]  

(20)

(21)

**Theorem 8** (Dynamics [1]). The dynamics of a stochastic scalar electron and a field coupled with a kinematics of \( \mathbf{14} \) is as follows:

\[
m_0 \mathcal{D}_\tau \hat{V}^{\mu}(\hat{x}(\tau, \omega)) = -e \hat{V}_\nu(\hat{x}(\tau, \omega)) F^{\mu\nu}(\hat{x}(\tau, \omega))
\]
\[
\partial_\mu [F^{\mu\nu}(x) + \delta f^{\mu\nu}(x)] = \mu_0 \times E \left[ -ee \int_{\mathbb{R}} d\tau' \Re \left\{ V^{\nu}(x) \right\} \delta^4(x - \hat{x}(\tau', \bullet)) \right]
\]  

(22)

(23)

Where, (22) is equivalent to the Klein-Gordon equation via \( \mathcal{V}^{\mu}(x) := i\lambda^2 \times \partial^\mu \ln \phi(x) + e/m_0 \times A^\mu(x) \) and (23) is the Maxwell equation. Let \( F \in \mathbb{V}_M^4 \oplus \mathbb{V}_M^4 \) be a homogeneous solution of (22) and \( \delta f \in \mathbb{V}_M^4 \oplus \mathbb{V}_M^4 \) be a singularity of the field. The dynamics of (22) and (23) satisfy the \( U(1) \)-gauge symmetry.

By using those ideas, a new parameter is found in Larmor’s formula instead of \( q(\chi) \) of (9).11

In Section 2 the retarded field \( \mathcal{F}_{(+)} \) and the advanced field \( \mathcal{F}_{(-)} \) are derived from the Maxwell equation (23) by its direct calculation. Then, the effective radiation reaction field \( \mathfrak{F}(\hat{x}(\tau, \omega)) \) corresponding to \( F_LAD \) of (5) is defined. Where, the key issue is the treatment of the current density in (23);

\[
\mathfrak{j}_{\text{stochastic}}(x) = E \left[ -ee \int_{\mathbb{R}} d\tau' \Re \left\{ V(x) \right\} \delta^4(x - \hat{x}(\tau', \bullet)) \right] \in \mathbb{V}_M^4.
\]

Then, the characteristics of the present model is discussed in Section 3 as the summary. Ehrenfest’s theorem is applied to explain the transition between quantum dynamics to classical dynamics and the Landau-Lifshitz approximation for the real calculation. By resulting it, we can find a simple radiation formula corresponding to (6):

\[
\frac{dW}{dt}(\mathbb{E}[\hat{x}(\tau, \bullet)]) = \int_{\omega \in \Omega_{\text{typ}}} d\mathcal{P}(\omega) \frac{dW_{\text{classical}}(\hat{x}(\tau, \omega))}{dt}
\]

9
2 Radiation fields

Let us consider how to formalize a radiation field by solving the Maxwell equation (23). By recalling Theorem 1 and Lemma 2, the retarded field $F_{\text{ret}}$ and advanced field $F_{\text{adv}}$ are derived from the Maxwell equation in classical physics. $F_{\text{ret}}$ is separated into its homogeneous solution and its singularity at $x = x(\tau)$:

$$\frac{F_{\text{ret}}(x(\tau)) - F_{\text{adv}}(x(\tau))}{2} = -\frac{m_0 \tau_0}{ec^2} [\frac{d^2v}{d\tau^2}(\tau) \otimes v(\tau) - v(\tau) \otimes \frac{d^2v}{d\tau^2}(\tau)]$$  \hspace{1cm} (24)

$$\frac{F_{\text{ret}}(x(\tau)) + F_{\text{adv}}(x(\tau))}{2} = \frac{3}{4} \frac{m_0 \tau_0}{ec^2} [\frac{dv}{d\tau}(\tau) \otimes v(\tau) - v(\tau) \otimes \frac{dv}{d\tau}(\tau)] \int_{\mathbb{R}} d\tau' \frac{\delta(\tau' - \tau)}{|\tau' - \tau|}$$  \hspace{1cm} (25)

The singularity of $\delta f := (F_{\text{ret}} + F_{\text{adv}})/2$ is known as the self-interaction of an electron due to its electromagnetic mass $m_{EM} := 3/4 \times m_0 \tau_0 \int_{\mathbb{R}} d\tau' \delta(\tau' - \tau)/|\tau' - \tau| = \infty$ not to depend on an external field $F_{\text{ex}}$ [8, 36]. Thus, let us regard $\delta f$ as a Coulomb field attached to an electron. The field $F_{\text{LAD}} := (F_{\text{ret}} - F_{\text{adv}})/2$ of its rest denotes the LAD field (the effective field), such that $\partial_\mu F_{\text{LAD}}^\mu = 0$ for all $x \in \mathbb{R}^4(V_{M}^4, g)$. Consider the one in stochastic dynamics of a scalar electron. Theorem 8 and following Lemma 9 are ones in the present model corresponding to "Theorem 1 and Lemma 2".

2.1 Derivation of fields: general idea

**Lemma 9** (Radiation fields). Consider the Maxwell equation

$$\partial_\mu F_{\mu \nu}(x) = \mu_0 \times \mathbb{E} \left[ -ec \int_{\mathbb{R}} d\tau \Re \{V^\nu(x)\} \delta^4(x - \hat{x}(\tau, \bullet)) \right],$$  \hspace{1cm} (26)

let $F_{(\pm)} \in \mathbb{V}^4_M \otimes \mathbb{V}^4_M$ and $F_{(-)} \in \mathbb{V}^3_M \otimes \mathbb{V}^3_M$ be the retarded and advanced fields as the linear independent solutions of (26). The particular solution of (26) is derived as follows:

$$F(x) = F_{\text{ex}}(x) + F_{(\pm)}(x)$$

$$= + \left[ F_{\text{ex}}(x) + \mathfrak{G}(x) \right] + \delta f(x)$$  \hspace{1cm} (27)

Where, $F_{(\pm)}$ is separated into the effective radiation (reaction) field $\mathfrak{G}$ such that $\partial_\mu \mathfrak{G}^{\mu \nu} = 0$ and its singularity $\delta f \in \mathbb{V}^3_M \otimes \mathbb{V}^3_M$.

[26] is analyzed by the following decomposition:

$$\partial_\mu F_{\pm}^{\mu \nu}(x) = \mu_0 j_{\text{stochastic}}^\nu(x)$$  \hspace{1cm} (28)

$$F_{(\pm)}^{\mu \nu}(x) := \partial^\mu A_{(\pm)}^\nu(x) - \partial^\nu A_{(\pm)}^\mu(x)$$  \hspace{1cm} (29)

$$j_{\text{stochastic}}^\nu(x) := -ec \int_{\mathbb{R}} d\tau' \mathbb{E} \left[ \Re \{V^\nu(x)\} \delta^4(x - \hat{x}(\tau, \bullet)) \right]$$  \hspace{1cm} (30)

By selecting the Lorenz gauge $\partial_\mu A_{(\pm)}^\mu(x) = 0$,

$$\partial_\mu \partial^\mu A_{(\pm)}^\nu(x) = \mu_0 j_{\text{stochastic}}^\nu(x).$$  \hspace{1cm} (31)
Let us introduce the Green function $G_{(\pm)}(x,x') = \theta(\pm \Delta x^0)/2\pi \times \delta(\Delta x_\mu \Delta x^\mu)$ satisfying $\partial_\mu \partial^{\mu} G_{(\pm)}(x,x') = \delta^4(x-x')$ with the help of the notation $\Delta x := x - x'$ and the unit step function $\theta$; $\theta(x \geq 0) = 1$ and $\theta(x < 0) = 0$. Thus, the solution of (31) is

\[ A^\nu_{(\pm)}(x) = -e \mu_0 \int_{\mathbb{R}} d\tau' \mathbb{E} \left[ \text{Re} \left\{ V^\nu(\hat{x}(\tau', \bullet)) \right\} G_{(\pm)}(x, \hat{x}(\tau', \bullet)) \right] \]  

(32)

and its field strength $F^{\mu\nu}_{(\pm)}$ by (29) becomes

\[ F^{\mu\nu}_{(\pm)}(x) = -\frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \mathbb{E} \left[ \text{Re} \left\{ V^\nu(\hat{x}(\tau', \bullet)) \right\} \partial^\mu - \text{Re} \left\{ V^\nu(\hat{x}(\tau', \bullet)) \right\} \partial^\mu \right] G_{(\pm)}(x, \hat{x}(\tau', \bullet)) \right] . \]  

(33)

Radiation reaction requires the formulation of $F_{(\pm)}(\hat{x}(\tau, \omega))$. We consider a decomposition of its first two leading-order terms for an easier analysis of $G_{(\pm)}(x, \hat{x}(\tau', \bullet))|_{x=\hat{x}(\tau, \omega)}$:

\[ F_{(\pm)}(\hat{x}(\tau, \omega)) = F_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + \delta \hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha F_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + O(\varepsilon^2 \delta \hat{x}(\tau, \omega)) \]  

(34)

The details of the each terms of $F_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ and $\delta \hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha F_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ are discussed in the following small section.

2.2 Radiation fields

Consider $A_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ as the leading order term of $A_{(\pm)}(\hat{x}(\tau, \omega))$.

**Lemma 10.** For $\Omega_{\text{ave}} := \{ \omega | \hat{x}(\tau, \omega) = \mathbb{E}[\hat{x}(\tau, \bullet)] \} \subset \Omega$,

\[ p(x, \tau) = \mathcal{P}(\Omega_{\text{ave}}) \times \delta^4(x - \mathbb{E}[\hat{x}(\tau, \bullet)]) + \int_{\Omega \setminus \Omega_{\text{ave}}} d\mathcal{P}(\omega) \delta^4(x - \hat{x}(\tau, \omega)). \]  

(35)

In order to this lemma, $A_{(\pm)}(x)$ is separated into two terms:

\[ A^\mu_{(\pm)}(x) = -\frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \mathcal{P}(\Omega_{\text{ave}}) \text{Re} \left\{ V^\nu(\mathbb{E}[\hat{x}(\tau', \bullet)]) \right\} G_{(\pm)}(x, \mathbb{E}[\hat{x}(\tau', \bullet)]) \]  

- \[ -\frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \int_{\Omega \setminus \Omega_{\text{ave}}} d\mathcal{P}(\omega') \text{Re} \left\{ V^\nu(\hat{x}(\tau', \omega')) \right\} G_{(\pm)}(x, \hat{x}(\tau', \omega')) \]  

(36)

$G_{(\pm)}(x, \hat{x}(\tau', \omega'))$ includes the light-cone condition via $\delta (\{ x_\mu - \hat{x}_\mu(\tau', \omega') \} \cdot [x^\mu - \hat{x}^\mu(\tau', \omega')]$ in its definition. Therefore, the substitution $x = \mathbb{E}[\hat{x}(\tau, \bullet)]$ for the derivation of $A_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ imposes

\[ \left[ \mathbb{E}[\hat{x}_\mu(\tau, \bullet)] - \hat{x}_\mu(\tau', \omega') \right] \cdot \left[ \mathbb{E}[\hat{x}^\mu(\tau, \bullet)] - \hat{x}^\mu(\tau', \omega') \right] = 0 . \]  

(37)

We regard this equation as an electron emits a field at $\hat{x}(\tau', \omega')$ and the both of the electron and the field interact at $\mathbb{E}[\hat{x}(\tau, \bullet)]$ after their propagations. Consider the case $\omega' \in \Omega \setminus \Omega_{\text{ave}}$. In this case, $\hat{x}(\tau', \omega')$ and $\mathbb{E}[\hat{x}(\tau, \bullet)]$ locate at the separated points in the Minkowski spacetime even if $\tau' = \tau$. Some of $(\tau', \omega')$ may hold the light-cone condition of (37). It expresses the self-interaction with its finite size correction.\footnote{In this diagram, the wiggly line and double line represent a photon and an electron dressing an external field like a laser. The Green function regulates the propagation of a photon and its two indexes are the positions on a trajectory of an electron.}
(self-interaction) =

The finite size correction is denoted by the blob as the set of \( \{ \hat{x}(\tau', \omega') | \omega' \in \Omega \setminus \Omega^{\text{ave}} \} \). This condition appears with the probability of \( \mathcal{P}(\Omega \setminus \Omega^{\text{ave}}) \), without any energy-momentum transfer on an electron between at the initial and at final states. Namely,\(^2\)

\[
- \frac{e}{c \varepsilon_0} \int_{\mathbb{R}} d\tau' \int_{\Omega \setminus \Omega^{\text{ave}}} d\mathcal{P}(\omega') \text{Re} \left\{ \mathcal{V}^\nu(\hat{x}(\tau', \omega')) \right\} G_{(\pm)}(x, \hat{x}(\tau', \omega')) \bigg|_{x = \mathbb{E}[\hat{x}(\tau, \bullet)]}
\]

has to be renormalized as a part of its Coulomb field attaching to an electron. Let us recall \(^1\) of the naive idea of an observable energy-momentum transfer via radiation. Such radiation can be found with the probability of \( \mathcal{P}(\Omega^{\text{ave}}) = 1 - \mathcal{P}(\Omega \setminus \Omega^{\text{ave}}) \) in the present model. Hence, \( \omega' \in \Omega^{\text{ave}} \) represents the sample paths for the derivation of radiation reaction\(^1\).

\[
A_{\text{effective,}(\pm)}^\nu \left( \mathbb{E}[\hat{x}(\tau, \bullet)] \right) := - \frac{e}{c \varepsilon_0} \int_{\mathbb{R}} d\tau' \mathcal{P}(\Omega^{\text{ave}}) \text{Re} \left\{ \mathcal{V}^\nu(\mathbb{E}[\hat{x}(\tau', \bullet)]) \right\} G_{(\pm)}(x, \mathbb{E}[\hat{x}(\tau', \bullet)]) \bigg|_{x = \mathbb{E}[\hat{x}(\tau, \bullet)]} \tag{38}
\]

**Lemma 11.** Consider a D-progressive \( \hat{x}(\alpha, \bullet) \) and \( \delta \hat{x}(\tau, \omega) := \hat{x}(\tau, \omega) - \mathbb{E}[\hat{x}(\tau, \bullet)] \). Then, the following relations are fulfilled:

\[
\text{Re} \left\{ \mathcal{V}^\nu(\mathbb{E}[\hat{x}(\tau, \bullet)]) \right\} - \frac{d \mathbb{E}[\hat{x}^\alpha(\tau, \bullet)]}{d\tau} = O(\mathbb{E}[\hat{x}^2(\tau, \bullet)]) \tag{39}
\]

\[
\frac{d \mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau}, \frac{d \mathbb{E}[\hat{x}^\alpha(\tau, \bullet)]}{d\tau} - c^2 = O(\mathbb{E}[\hat{x}^2(\tau, \bullet)]) \tag{40}
\]

**Proof.** By Nelson’s partial integral formula (Lemma 31 in Ref.\(^1\)), \( d/d\tau \mathbb{E}[\hat{x}^\alpha(\tau, \bullet)] = \mathbb{E}[\text{Re} \{ \mathcal{V}^\nu(\hat{x}(\tau, \bullet)) \}] \) is satisfied. Since \( \mathbb{E}[\delta \hat{x}(\tau, \omega)] = 0 \), the first relation (39) is derived as follows:

\[
\text{Re} \left\{ \mathcal{V}^\nu(\mathbb{E}[\hat{x}(\tau, \bullet)]) \right\} - \frac{d \mathbb{E}[\hat{x}^\alpha(\tau, \bullet)]}{d\tau} = \mathbb{E}[\text{Re} \{ \mathcal{V}^\nu(\mathbb{E}[\hat{x}(\tau, \bullet)]) \}] - \mathbb{E}[\text{Re} \{ \mathcal{V}^\nu(\hat{x}(\tau, \bullet)) \}]
\]

\[
= \mathbb{E}[\text{Re} \{ \mathcal{V}^\nu(\mathbb{E}[\hat{x}(\tau', \bullet)]) \}] - \mathbb{E}[\text{Re} \{ \mathcal{V}^\nu(\hat{x}(\tau', \bullet)) \}]
\]

\[
= \mathbb{E}[\delta \hat{x}(\tau, \bullet) \cdot \partial_\beta \text{Re} \{ \mathcal{V}^\nu(\mathbb{E}[\hat{x}(\tau, \bullet)]) \} + O(\delta \hat{x}(\tau, \bullet))]
\]

\[
= O(\mathbb{E}[\hat{x}^2(\tau, \bullet)])
\]

The second relation (40) is demonstrated by

\[
\frac{d}{d\tau} \mathbb{E}[\hat{x}_\alpha(\tau, \bullet)], \frac{d}{d\tau} \mathbb{E}[\hat{x}^\alpha(\tau, \bullet)] - \mathbb{E}\left[ \mathcal{V}_\alpha^\nu(\hat{x}(\tau, \bullet)) \mathcal{V}^\nu(\hat{x}(\tau, \bullet)) \right] = O(\mathbb{E}[\hat{x}^2(\tau, \bullet)]) \tag{42}
\]

with the help of the Lorentz invariant \( \mathbb{E}[\mathcal{V}_\alpha^\nu(\hat{x}(\tau, \bullet)) \mathcal{V}^\nu(\hat{x}(\tau, \bullet))] = c^2 \) of Lemma 29 in \(^1\). \( \square \)

Transforming \( A_{\text{effective,}(\pm)}^\nu \left( \mathbb{E}[\hat{x}(\tau, \bullet)] \right) \) by the above,

\[
A_{\text{effective,}(\pm)}^\nu \left( \mathbb{E}[\hat{x}(\tau, \bullet)] \right) := - \frac{e}{c \varepsilon_0} \int_{\mathbb{R}} d\tau' \mathcal{P}(\Omega^{\text{ave}}) \frac{d \mathbb{E}[\hat{x}^\nu(\tau', \bullet)]}{d\tau'} G_{(\pm)}(x, \mathbb{E}[\hat{x}(\tau', \bullet)]) \bigg|_{x = \mathbb{E}[\hat{x}(\tau, \bullet)]} \tag{43}
\]

\(^2\)Consider \( A_{\text{effective,}(\pm)}^\nu \left( \mathbb{E}[\hat{x}(\tau, \bullet)] \right) \) as \( O(\mathbb{E}[\hat{x}^2(\tau, \bullet)]) \) in \( A_{\text{effective,}(\pm)}^\nu (\hat{x}(\tau, \omega)) \).
Thus, its field strength is
\[
\mathcal{F}^{\mu\nu}_{\text{(effective, } \pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)]) = -\frac{e}{c^2} \int d\tau' \mathcal{P}(\Omega_{\tau'}^{\text{ave}}) \left( \frac{d\mathbb{E}[\hat{x}'(\tau', \bullet)]}{d\tau'} - \frac{d\mathbb{E}[\hat{x}'(\tau', \bullet)]}{d\tau} \right) \cdot \partial^\mu - \frac{d\mathbb{E}[\hat{x}'(\tau', \bullet)]}{d\tau} \cdot \partial^\nu \bigg|_{x=\mathbb{E}[\hat{x}(\tau, \bullet)]}.
\] (44)

Let us follow the method in Ref.\[36\]. By Lemma \[11\] the Green function \(G_{(\pm)}\) is transformed as
\[
G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) = \frac{\theta(\pm\mathbb{E}[\hat{x}^0(\tau, \bullet)] - \hat{x}^0(\tau', \bullet)) \times \delta(\tau - \tau')}{4\pi c^2 \times |\tau' - \tau|} \times \left[ 1 + O \left((\tau' - \tau)^2, \mathbb{E}[\hat{x}(\tau, \omega)]\right) \right].
\] (45)

Where, the causality is introduced by \(\theta(\pm\mathbb{E}[\hat{x}^0(\tau, \bullet)] - \hat{x}^0(\tau', \bullet)) = \theta(\pm(\tau - \tau'))\) and \(40\) is employed. In addition,
\[
\partial^\mu \bigg|_{x=\mathbb{E}[\hat{x}(\tau, \bullet)]} = -\frac{\mathbb{E}[\hat{x}'(\tau, \bullet)]}{c^2 \times (\tau' - \tau)} \frac{d}{d\tau} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \times \left[ 1 + O \left((\tau' - \tau)^2, \mathbb{E}[\hat{x}(\tau, \omega)]\right) \right].
\] (46)

\(\mathcal{F}_{\text{(effective, } \pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)])\) of \(44\) becomes
\[
\mathcal{F}_{\text{(effective, } \pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)]) = \frac{e}{c^2} \int d\tau' \left[ \frac{\mathcal{P}(\Omega_{\tau'}^{\text{ave}})}{2} \cdot \mathbb{E}[\hat{x}'(\tau, \bullet)] - \frac{d\mathbb{E}[\hat{x}'(\tau, \bullet)]}{d\tau} \right] \cdot \mathbb{E}[\hat{x}'(\tau', \bullet)] - \frac{d\mathbb{E}[\hat{x}'(\tau', \bullet)]}{d\tau} \times \left[ 1 + O \left((\tau' - \tau)^2\right) \right].
\] (47)

Since
\[
\left( \frac{\mathcal{P}(\Omega_{\tau}^{\text{ave}})}{2} \cdot \frac{d\mathbb{E}[\hat{x}'(\tau, \bullet)]}{d\tau} \right) \cdot \mathbb{E}[\hat{x}'(\tau', \bullet)] - \frac{d\mathbb{E}[\hat{x}'(\tau, \bullet)]}{d\tau} \times \left[ 1 + O \left((\tau' - \tau)^2\right) \right] = -\frac{d\mathbb{E}[\hat{x}'(\tau', \bullet)]}{d\tau} \times \left[ 1 \times \frac{d\mathbb{E}[\hat{x}'(\tau, \bullet)]}{d\tau} \right] - \frac{d\mathbb{E}[\hat{x}'(\tau, \bullet)]}{d\tau} \times \left[ \frac{1}{2} \times \frac{d\mathbb{E}[\hat{x}'(\tau', \bullet)]}{d\tau} \right] \times \left[ \frac{1}{3} \times \frac{d\mathbb{E}[\hat{x}'(\tau, \bullet)]}{d\tau} \right] \times \left[ \frac{1}{3} \times \frac{d\mathbb{E}[\hat{x}'(\tau', \bullet)]}{d\tau} \right] \times \left[ \frac{1}{3} \times \frac{d\mathbb{E}[\hat{x}'(\tau, \bullet)]}{d\tau} \right]
\] (48)
the field \( F_{\text{effective,\pm}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \) is obtained:

\[
F_{\text{effective,\pm}}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) = \frac{3 m_0 \tau_0 \mathcal{P}(\Omega_{\text{ave}})}{4 e c^2} \times \left( \begin{array}{c}
\frac{d^2 \mathbb{E}[\hat{\dot{x}}^\mu(\tau, \bullet)]}{d \tau^2} - \frac{d \mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d \tau} \frac{d \mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d \tau} \\
\frac{d \mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d \tau} \frac{d \mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d \tau} - \frac{d^2 \mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d \tau^2}
\end{array} \right) \int_{\mathbb{R}} d\tau' \frac{\delta(\tau' - \tau)}{|\tau' - \tau|}
\]

Where, \( \tau_0 := e^2 / 6 \pi \varepsilon_0 m_0 c^3 \) and \( \dot{a}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \in \mathbb{V}_M^4 \) denotes

\[
\dot{a}(\mathbb{E}[\hat{x}(\tau, \bullet)]) := \frac{d^3 \mathbb{E}[\hat{x}(\tau, \bullet)]}{d \tau^3} + \frac{3}{2} \frac{d \ln \mathcal{P}(\Omega_{\text{ave}})}{d \tau} \frac{d^2 \mathbb{E}[\hat{x}(\tau, \bullet)]}{d \tau^2}.
\]

Therefore, the effective radiation reaction field on \( \mathbb{E}[\hat{x}(\tau, \bullet)] \) is the following:

**Lemma 12** (Effective radiation reaction field 1). The effective radiation reaction field \( \mathcal{F}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4 \) is defined as follows:

\[
\mathcal{F}(\mathbb{E}[\hat{x}(\tau, \bullet)]) := \frac{F_{\text{effective,\pm}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) - F_{\text{effective,\pm}}(\mathbb{E}[\hat{x}(\tau, \bullet)])}{2}
\]

\[
= - \frac{m_0 \tau_0 \mathcal{P}(\Omega_{\text{ave}})}{ec^2} \times \left( \dot{a}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \otimes \mathcal{E}[\mathbb{E}[\hat{x}(\tau, \bullet)] + \mathcal{E}[\mathbb{E}[\hat{x}(\tau, \bullet)] | \otimes \dot{a}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \right)
\]

Consider the following transformation of (51), the wider relation is found.

\[
\int_{\Omega_{\text{ave}}} d\mathcal{P}(\omega) \mathcal{F}(\hat{x}(\tau, \omega)) = \int_{\Omega_{\text{ave}}} d\mathcal{P}(\omega) \left[ - \frac{m_0 \tau_0}{ec^2} \times \left( \dot{a}(\hat{x}(\tau, \omega)) \otimes \mathcal{E}[\mathbb{E}[\hat{x}(\tau, \omega)] - \mathcal{E}[\mathbb{E}[\hat{x}(\tau, \omega)] \otimes \dot{a}(\hat{x}(\tau, \omega)) \right) \right]
\]

**Lemma 13** (Effective radiation reaction field 2). The field \( \mathcal{F}(\hat{x}(\tau, \omega)) \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4 \) is derived as follows:

\[
\mathcal{F}(\hat{x}(\tau, \omega)) = - \frac{m_0 \tau_0}{ec^2} \times \left[ \dot{a}(x) \otimes \mathcal{E}[\mathbb{E}[\hat{x}(\tau, \omega)] - \mathcal{E}[\mathbb{E}[\hat{x}(\tau, \omega)] \otimes \dot{a}(x) \right]_{x=\hat{x}(\tau, \omega)}
\]

Where, \( \dot{a}(x) \) satisfies

\[
\dot{a}(\mathbb{E}[\hat{\dot{x}}(\tau, \bullet)]) := \frac{d^3 \mathbb{E}[\hat{x}(\tau, \bullet)]}{d \tau^3} + \frac{3}{2} \frac{d \ln \mathcal{P}(\Omega_{\text{ave}})}{d \tau} \frac{d^2 \mathbb{E}[\hat{x}(\tau, \bullet)]}{d \tau^2} + O(\mathbb{E}[\otimes \mathbb{E}[\hat{x}(\tau, \omega)])
\]

**Lemma 12** is derived from **Lemma 13** by

\[
\mathcal{F}(\mathbb{E}[\hat{x}(\tau, \bullet)]) = \int_{\Omega_{\text{ave}}} d\mathcal{P}(\omega) \mathcal{F}(\hat{x}(\tau, \omega))
\]
Since
\[
\partial_\alpha \tilde{\mathcal{G}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) = -\frac{m_0 \tau_0}{\epsilon_0^2} \int_{\mathcal{O}_{\tau}^{\text{ave}}} d\mathcal{P}(\omega) \partial_\alpha \left[ \hat{a}(x) \otimes \text{Re} \{ \mathcal{V}(x) \} - \text{Re} \{ \mathcal{V}(x) \} \otimes \hat{a}(x) \right]_{x=\hat{x}(\tau, \omega)}
\]

\[
= -\frac{m_0 \tau_0 \mathcal{P}(\mathcal{O}_{\tau}^{\text{ave}})}{\epsilon_0^2} \partial_\alpha \left[ \hat{a}(x) \otimes \text{Re} \{ \mathcal{V}(x) \} - \text{Re} \{ \mathcal{V}(x) \} \otimes \hat{a}(x) \right]_{x=\hat{x}(\tau, \omega)},
\]

(56)

the leading-order terms of
\[
\tilde{\mathcal{G}}(\hat{x}(\tau, \omega)) = \tilde{\mathcal{G}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + \delta \hat{\alpha}(\tau, \omega) \cdot \partial_\alpha \tilde{\mathcal{G}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + \mathcal{O}(\delta^2) \delta \hat{x}(\tau, \omega)
\]

(57)
can be estimated. By putting \( F_{\text{ex}} \) a non-trivial solution of (26) such that \( \partial_\mu F_{\text{ex}}^\mu(x) = 0 \), the particular solution \( \mathcal{F} \) becomes as follows:

\[
\mathcal{F}(\hat{x}(\tau, \omega)) = F_{\text{ex}}(\hat{x}(\tau, \omega)) + \tilde{\mathcal{G}}(\hat{x}(\tau, \omega)) + \delta f(\hat{x}(\tau, \omega))
\]

(58)

This is the suggestion of (27) on Lemma 9. Where, the singularity \( \delta f(x) \), namely, the part of non-effective radiation, is defined by

\[
\delta f^{\mu\nu}(x) := -\frac{e}{c \epsilon_0} \int_{\mathcal{R}} d\sigma' \int_{\mathcal{O}_{\tau}^{\text{ave}}} d\mathcal{P}(\omega') \begin{pmatrix} \text{Re} \{ \mathcal{V}(\hat{x}(\tau', \omega')) \} \partial^\mu & \partial^\nu \\
\text{Re} \{ \mathcal{V}(\hat{x}(\tau', \omega')) \} \partial^\nu & \text{Re} \{ \mathcal{V}(\hat{x}(\tau', \omega')) \} \partial^\mu 
\end{pmatrix} \frac{G(\hat{x}(\tau', \omega')) - G(\hat{x}(\tau', \omega'))}{2}
\]

\[
-\frac{e}{c \epsilon_0} \int_{\mathcal{R}} d\sigma' \mathbb{E} \begin{pmatrix} \text{Re} \{ \mathcal{V}(\hat{x}(\tau', \bullet)) \} \partial^\mu & \partial^\nu \\
\text{Re} \{ \mathcal{V}(\hat{x}(\tau', \bullet)) \} \partial^\nu & \text{Re} \{ \mathcal{V}(\hat{x}(\tau', \bullet)) \} \partial^\mu 
\end{pmatrix} \frac{G(\hat{x}(\tau', \bullet)) + G(\hat{x}(\tau', \bullet))}{2}
\]

(59)

The appearance of the integral \(-e/c \epsilon_0 \times \int_{\mathcal{O}_{\tau}^{\text{ave}}} d\mathcal{P}(\omega') \cdot \) in this equation is the difference from classical dynamics.

### 2.3 Probability: \( \mathcal{P}(\mathcal{O}_{\tau}^{\text{ave}}) \)

Let us consider an equation of continuity for a D-progressive \( \hat{x}(\phi, \bullet) \) on \((\mathcal{O}, \mathcal{F}, \mathcal{P})\), namely,

\[
\partial_\tau p(x, \tau) + \partial_\mu \left[ \text{Re} \{ \mathcal{W}(x) \} p(x, \tau) \right] = 0, \quad x \in \hat{x}(\tau, \Omega) \text{ for each } \tau
\]

(60)
derived from the "\( \hat{\phi} \)"-Fokker-Planck equations [16] [11]. We want to evaluate the probability density \( p(\tau, x) \) by using this equation. By transforming it,

\[
\partial_\tau \ln p(x, \tau) + \text{Re} \{ \mathcal{W}(x) \} \cdot \partial_\mu \ln p(x, \tau) + \partial_\mu \text{Re} \{ \mathcal{W}(x) \} = 0, \quad x \in \hat{x}(\tau, \Omega) \text{ for each } \tau
\]

(61)
is found. Since

\[
\mathcal{W}(x) := i \lambda^2 \partial^\alpha \phi(x) + \frac{e}{m_0} A^\alpha(x), \quad x \in \hat{x}(\tau, \Omega) \text{ for each } \tau,
\]

(62)

(60) becomes

\[
\partial_\tau \ln p(x, \tau) + \text{Re} \{ \mathcal{W}(x) \} \cdot \partial_\mu \ln p(x, \tau) = \text{Re} \{ \mathcal{W}(x) \} \cdot \partial_\mu \ln |\phi(x)|^2.
\]

(63)

Where, \( \text{Re} \{ \mathcal{W}(x) \} = [\phi^*(x) \phi(x)]^{-1} \times J_{K-G}(x) \) and \( \partial_\mu J_{K-G}(x) = 0 \) are employed (see Ref. [11]). The operator \( \partial_\tau + \text{Re} \{ \mathcal{W}(x) \} \cdot \partial_\mu \) denotes the Lagrangian representation in fluid dynamics and \( \partial_\mu [\phi^*(x) \phi(x)] \neq 0 \) on
Let us proceed this idea aggressively. Consider it on \( x \in \{ \mathbb{E}[\hat{x}(\tau, \bullet)] | \tau \in \mathbb{R} \} \), then,
\[
\frac{d}{d\tau} \ln p(\mathbb{E}[\hat{x}(\tau, \bullet)], \tau) = \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \cdot \partial_\mu \ln |\phi(\mathbb{E}[\hat{x}(\tau, \bullet)])|^2 + O(\mathbb{E}[\hat{x}(\tau, \bullet)])
\]
\[
= \frac{d}{d\tau} \ln |\phi(\mathbb{E}[\hat{x}(\tau, \bullet)])|^2 + O(\mathbb{E}[\hat{x}(\tau, \bullet)]). \tag{64}
\]
When the RHS of (64) is well-defined,
\[
\ln p(\mathbb{E}[\hat{x}(\tau, \bullet)], \tau) = \ln |\phi(\mathbb{E}[\hat{x}(\tau, \bullet)])|^2 + O(\mathbb{E}[\hat{x}(\tau, \bullet)]). \tag{65}
\]

**Lemma 14.** The time-evolution of the probability \( \mathcal{P}(\Omega^{\text{ave}}_\tau) := p(\mathbb{E}[\hat{x}(\tau, \bullet)], \tau) \) satisfying (66) is bellow:
\[
\mathcal{P}(\Omega^{\text{ave}}_\tau) = \mathcal{P}(\Omega_0^{\text{ave}}) \times \frac{|\phi(\mathbb{E}[\hat{x}(\tau, \bullet)])|^2}{|\phi(\mathbb{E}[\hat{x}(0, \bullet)])|^2} + O(\mathbb{E}[\hat{x}(\tau, \bullet)]) \tag{66}
\]

### 3 Conclusion and discussion

We discussed the formulation of the kinematics and the dynamics of a stochastic scalar electron equivalent to the Klein-Gordon particle interacting with a field. Especially, we focused the derivation of the radiation reaction effect as an application of **Volume I** [1]. Let us summarize the results of this article.

**Conclusion 15** (Radiation reaction). For \( (\Omega, \mathcal{F}, \mathcal{P}) \), consider a kinematics of a D-progressive \( \hat{x}(\omega, \bullet) \),
\[
d\hat{x}(\tau, \omega) = \mathcal{V}_\pm(\hat{x}(\tau, \omega))d\tau + \lambda \times dW_\pm(\tau, \omega) \tag{67}
\]
on \( (\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g), \mu)) \). This is coupled with the following dynamics for \( \mathcal{V} = (1 - i)/2 \times \mathcal{V}_+ + (1 + i)/2 \times \mathcal{V}_- \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4 \):
\[
m_0 \mathcal{D}_\tau \mathcal{V}(\hat{x}(\tau, \omega)) = -eF_{\mu\nu}^{\text{ex}}(\hat{x}(\tau, \omega)) \cdot \mathcal{V}_\nu(\hat{x}(\tau, \omega)) + \frac{m_0 \tau_0}{c^2} \times \left[ \frac{\mathcal{V}_\nu(\hat{x}(\tau, \omega)) \cdot \mathcal{V}_\nu(\hat{x}(\tau, \omega))}{\mathcal{V}_\nu(\hat{x}(\tau, \omega))} \cdot \mathcal{V}_\nu(\hat{x}(\tau, \omega)) \right]. \tag{68}
\]

Where, \( \hat{a}(\hat{x}(\tau, \omega)) \) is defined by (54). This is equivalent to the Klein-Gordon equation with its radiation, i.e., the quantization of LAD equation (6).

The dynamics of a radiating stochastic particle corresponding to the LAD equation was hereby derived, however, [68] includes many higher order derivatives. Even in the case of the LAD equation, the exponential factor \( dv_\nu/d\tau \propto \exp(\tau/\tau_0) \), namely, the run-away problem remains in it [8]. This suffers us in the actual estimations and numerical simulations of its experimental designs. Let us introduce the Ford-O’Connell [38]/Landau-Lifshitz [39] schemes for it, too. In the case of the Landau-Lifshitz method, it is imposed by the following perturbation of \( m_0 \tau_0/c^2 \times (d^2v_\mu/d\tau^2 \cdot v_\nu - d^2v_\nu/d\tau^2 \cdot v_\mu)/v_\mu \) in the LAD equation (6) with respect to \( \tau_0 = O(10^{-24} \text{sec}) \):
\[
\frac{dv_\mu}{d\tau} = -\frac{e}{m_0} F_{\mu\nu}^{\text{ex}} v_\nu + O(\tau_0) \tag{69}
\]
\[
d\frac{d^2 v^\mu}{dt^2} = -\frac{e}{m_0} \partial_\alpha F^\mu_{ex} v^\alpha + \frac{e^2}{m_0^2} g_{\alpha\beta} F^\mu_{ex} F^\beta_{ex} v^\nu + O(\tau_0)
\]  

(70)

Then, the Landau-Lifshitz equation is derived:

\[
m_0 \frac{dv^\mu}{dt}(\tau) = -eF^\mu_{ex}(x(\tau))v^\nu(\tau) - eF^\mu_{LL}(x(\tau))v^\nu(\tau) + O(\tau_0^2)
\]

(71)

\[
F^\mu_{LL}(x(\tau)) := \tau_0 e^{\alpha}(\tau) \partial_\alpha F^\mu_{ex}(x(\tau)) - \frac{e\tau_0}{m_0 c^2} (\delta^\mu_\delta^\nu - \delta^\nu_\delta^\mu) g_{\alpha\beta} F^\delta_{ex}(x) F^\beta_{ex}(x)v^\gamma(\tau)v^\nu(\tau)
\]

(72)

For applying it to the present model, consider Ehrenfest’s theorem of (68) at first.

\[
m_0 \frac{d^2 \mathbb{E}[\dot{x}(\tau, \bullet)]}{dt^2} = -eF^\mu_{ex}(\mathbb{E}[\dot{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\dot{x}(\tau, \bullet)]}{dt} + \frac{m_0 \tau_0}{c^2} \left[ \delta^\mu_\alpha \frac{d\mathbb{E}[\dot{x}(\tau, \bullet)]}{dt} \cdot \frac{d\mathbb{E}[\dot{x}(\tau, \bullet)]}{dt} \right] + O(\mathbb{E}[\dot{x}(\tau, \bullet)])
\]

(73)

The readers have to compare this and the LAD equation [6]. Its perturbation for the Landau-Lifshitz scheme is imposed by

\[
\frac{d^2 \mathbb{E}[\dot{x}(\tau, \bullet)]}{dt^2} = -\frac{e}{m_0} \times F^\mu_{ex}(\mathbb{E}[\dot{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\dot{x}(\tau, \bullet)]}{dt} + O(\tau_0, \mathbb{E}[\dot{x}(\tau, \bullet)])
\]

(74)

and also

\[
\frac{d^3 \mathbb{E}[\dot{x}(\tau, \bullet)]}{dt^3} = -\frac{e}{m_0} \times F^\mu_{ex}(\mathbb{E}[\dot{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\dot{x}(\tau, \bullet)]}{dt} + \frac{e^2}{m_0^2} \times g_{\alpha\beta} F^\mu_{ex}(\mathbb{E}[\dot{x}(\tau, \bullet)]) F^\beta_{ex}(\mathbb{E}[\dot{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\dot{x}(\tau, \bullet)]}{dt} + O(\tau_0, \mathbb{E}[\dot{x}(\tau, \bullet)])
\]

(75)

Thus, the effective radiation field at \(x = \mathbb{E}[\dot{x}(\tau, \bullet)]\) becomes

\[
\mathbb{F}(\mathbb{E}[\dot{x}(\tau, \bullet)]) = \mathcal{P}(\Omega^{\text{rve}}) \times F_{LL}(\mathbb{E}[\dot{x}(\tau, \bullet)]) + \frac{3}{2} \tau_0 \frac{d \mathcal{P}(\Omega^{\text{rve}})}{dt} \times F_{ex}(\mathbb{E}[\dot{x}(\tau, \bullet)]) + O(\tau_0^2, \mathbb{E}[\dot{x}(\tau, \bullet)])
\]

(76)

where, \((x(\tau), v(\tau))\) in \(F_{LL}(x(\tau))\) of (72) is replaced by \((\mathbb{E}[\dot{x}(\tau, \bullet)], d\mathbb{E}[\dot{x}(\tau, \bullet)]/dt)\). Thus, equation of motion along \(\mathbb{E}[\dot{x}(\tau, \bullet)]\) is

\[
m_0 \frac{d^2 \mathbb{E}[\dot{x}(\tau, \bullet)]}{dt^2} = \left[ 1 + \frac{3}{2} \tau_0 \frac{d \mathcal{P}(\Omega^{\text{rve}})}{dt} \right] \times \left[ -eF^\mu_{ex}(\mathbb{E}[\dot{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\dot{x}(\tau, \bullet)]}{dt} \right] + \mathcal{P}(\Omega^{\text{rve}}) \times \left[ -eF^\mu_{LL}(\mathbb{E}[\dot{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\dot{x}(\tau, \bullet)]}{dt} \right] + O(\tau_0^2, \mathbb{E}[\dot{x}(\tau, \bullet)])
\]

(77)
By using

$$
\int_{\Omega_\tau^{\text{ave}}} d\mathcal{P}(\omega) \delta(\dot{x}(\tau, \omega)) = \int_{\Omega_\tau^{\text{ave}}} d\mathcal{P}(\omega) \left[ F_{\text{LL}}(\dot{x}(\tau, \omega)) + \frac{3}{2} \tau_0 \frac{d \ln \mathcal{P}(\Omega_\tau^{\text{ave}})}{d\tau} \times F_{\text{ex}}(\dot{x}(\tau, \omega)) \right] + O(\tau_0^2), \quad (78)
$$

the following is the perturbed equation of (68):

Larmor’s radiation formula is included in \( \mathcal{P}(\Omega_\tau^{\text{ave}}) \times \left[ -e F_{\text{LL}}^\mu(\mathbb{E}[\dot{x}(\tau, \bullet)]) \right] \cdot d\mathbb{E}[\dot{x}_\nu(\tau, \bullet)]/d\tau \)

$$
\frac{dW}{dt} (\mathbb{E}[\dot{x}(\tau, \bullet)]) = -\mathcal{P}(\Omega_\tau^{\text{ave}}) \times \frac{\tau_0}{m_0} f_{\alpha}(\mathbb{E}[\dot{x}(\tau, \bullet)]) \cdot f^\alpha(\mathbb{E}[\dot{x}(\tau, \bullet)]) + O(\tau_0^2, \mathbb{E}^2 \delta \dot{x}(\tau, \bullet)) \quad (81)
$$

Where,

$$
f^\mu(\mathbb{E}[\dot{x}(\tau, \bullet)]) = -e F_{\text{ex}}^{\mu}(\mathbb{E}[\dot{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\dot{x}_\nu(\tau, \bullet)]}{d\tau}. \quad (82)
$$

Then, let us recall the classical radiation formula \( dW_{\text{classical}}/dt = -\tau_0/m_0 \times f_{\alpha}(\mathbb{E}[\dot{x}(\tau, \bullet)]) \cdot f^\alpha(\mathbb{E}[\dot{x}(\tau, \bullet)]) + O(\tau_0^2) \) (an acceleration \( dv/d\tau \) in \( (7) \) is perturbed by the parameter of \( \tau_0 \). see also \[39\]), simplify the above formula of \( dW/dt \).

**Theorem 17 (Radiation formula for a D-process).** The radiation formula for a stochastic scalar electron is expressed as follows:

$$
\frac{dW}{dt} (\mathbb{E}[\dot{x}(\tau, \bullet)]) = \int_{\omega \in \Omega_\tau^{\text{ave}}} d\mathcal{P}(\omega) \frac{dW_{\text{classical}}}{dt} (\dot{x}(\tau, \bullet))
 = \mathcal{P}(\Omega_\tau^{\text{ave}}) \times \frac{dW_{\text{classical}}}{dt} (\mathbb{E}[\dot{x}(\tau, \bullet)])
$$

(83)

Hereby, we can say \( q(\chi) \) of the factor in \( (9) \) is regarded as \( \mathcal{P}(\Omega_\tau^{\text{ave}}) \) the probability at the expectation of the particle position.

However in the real practices, the analytical derivation of \( \mathcal{P}(\Omega_\tau^{\text{ave}}) \) by solving \( (66) \) is complicated. Therefore, we assume the simplified condition for the first experiment on high-intensity field physics \[28\]. That is

---

3Larmor’s formula is found as the coefficient of the ”direct radiation term” which is proportional to the velocity \( d\mathbb{E}[\dot{x}(\tau, \bullet)]/d\tau, \) i.e., the term of \(-dW/d\tau \times d\mathbb{E}[\dot{x}(\tau, \bullet)]/d\tau\).
just an irradiation of a highly energetic electron by a long-focal and high-intensity laser. If we assume it, a plane wave condition becomes feasible. The radiation formula by the non-linear Compton scattering (the single photon emission process) of a scalar electron [21] is derived like

$$\frac{dW}{dt} = q_{\text{scalar}}(\chi) \times \frac{dW_{\text{classical}}}{dt},$$ \hspace{1cm} (84)

$$q_{\text{scalar}}(\chi) = \frac{9\sqrt{3}}{8\pi} \int_0^{\chi^{-1}} dr \int_{r/r'}^\infty dr' K_{5/3}(r'),$$ \hspace{1cm} (85)

with the definition of the non-linearity parameter,

$$\chi := \frac{3\hbar}{2m_0^2c^3} \sqrt{-f_\alpha(\mathbb{E}\[\hat{x}(\tau, \bullet\])] \cdot f_\alpha(\mathbb{E}\[\hat{x}(\tau, \bullet\])}.$$

By putting the difference $\delta\mathcal{P}(\Omega_{\text{ave}}^\tau)$ between $\mathcal{P}(\Omega_{\text{ave}}^\tau)$ and $q_{\text{scalar}}(\chi)$ in general,

$$\mathcal{P}(\Omega_{\text{ave}}^\tau) := q_{\text{scalar}}(\chi) + \delta\mathcal{P}(\Omega_{\text{ave}}^\tau).$$ \hspace{1cm} (87)

Since $r$ is characterized by the relation $r := \chi^{-1} \times \hbar\omega / m_0 c^2 \gamma$, the radiation spectrum can be introduced as follows (see also [21], [10], [39]):

$$\frac{d^2W}{dt(d\hbar\omega)} = -\frac{\tau_0}{m_0} g_{\mu\nu} f_{\text{Re}}^\mu(\mathbb{E}\[\hat{x}(\tau, \bullet\])] f_{\text{Re}}^\nu(\mathbb{E}\[\hat{x}(\tau, \bullet\]]) \times \left[ \frac{dq_{\text{scalar}}(\chi)}{d(h\omega)} + \frac{d\delta\mathcal{P}(\Omega_{\text{ave}}^\tau)}{d(h\omega)} \right]$$ \hspace{1cm} (88)

$$\frac{dq_{\text{scalar}}(\chi)}{d(h\omega)} = \frac{9\sqrt{3}}{8\pi\chi^2 (m_0 c^2 \gamma)^2} \int_{r/r'}^\infty dr' K_{5/3}(r').$$ \hspace{1cm} (89)

So, the existence of $\delta\mathcal{P}(\Omega_{\text{ave}}^\tau)$ is the difference from the non-linear Compton scatterings under the plane wave condition. Thus, plotting $\mathcal{P}(\Omega_{\text{ave}}^\tau)$ like Figure 1 is interesting in the real experiments by a high-intensity laser [28].

As the further works, the deeper analysis of this method and numerical simulations have to be expected to innovate high-intensity field physics toward together with real experiments carried out by the state-of-the-art lasers $O(10\text{PW})$ lasers [3, 4, 5, 28].

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