Integrability of a discrete Yajima–Oikawa system

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Abstract

A space discretization of an integrable long wave–short wave interaction model, called the Yajima–Oikawa system, was proposed in the recent paper arXiv:1509.06996 using the Hirota bilinear method. In this paper, we propose a Lax-pair representation for the discrete Yajima–Oikawa system as well as its multicomponent generalization also considered in arXiv:1509.06996 and prove that it has an infinite number of conservation laws. We also derive the next higher flow of the discrete Yajima–Oikawa hierarchy, which generalizes a modified version of the Volterra lattice. Relations to two integrable discrete nonlinear Schrödinger hierarchies, the Ablowitz–Ladik hierarchy and the Konopelchenko–Chudnovsky hierarchy, are clarified.

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1 Introduction

The system of two coupled partial differential equations

\[
\begin{aligned}
\begin{cases}
i S_t &= S_{xx} + L S, \\L_t &= 2 (|S|^2)_x,
\end{cases}
\end{aligned}
\tag{1.1a}
\]

where the subscripts denote the partial differentiation, describes the interaction between the complex-valued short-wave component \(S\) and the real-valued long-wave component \(L\) and is completely integrable. We call \((1.1)\) the Yajima–Oikawa system, because a system equivalent to \((1.1)\), up to a simple transformation, was first solved by Yajima and Oikawa using the inverse scattering method \([1]\). In analogy with the multicomponent generalization of the cubic nonlinear Schrödinger equation \([2–6]\), a multicomponent generalization of the Yajima–Oikawa system \([7, 8]\),

\[
\begin{aligned}
\begin{cases}
i S_t &= S_{xx} + L S, \\L_t &= 2 \langle S, \Sigma S^* \rangle_x,
\end{cases}
\end{aligned}
\tag{1.2a}
\]

\[
\begin{aligned}
\begin{cases}
\log \left(1 + \frac{1}{2} L_n \right) \bigg|_t &= \frac{1}{2} \Delta^+_n \left( \langle S_n, \Sigma S^*_{n-1} \rangle + \langle S_{n-1}, \Sigma S^*_n \rangle \right),
\end{cases}
\end{aligned}
\tag{1.2b}
\]

can be considered. Here, \(S\) is a column vector composed of short-wave components, \(\langle \cdot, \cdot \rangle\) represents the standard scalar product, the asterisk denotes the complex conjugation and \(\Sigma\) is a diagonal matrix with each diagonal entry \(+1\) or \(−1\). In this paper, the product of a scalar \(s\) and a column vector \(V\) is expressed as either \(sV\) or \(Vs\), i.e., the order in the product is irrelevant. Note that any nonsingular Hermitian matrix \(\Sigma\) can be reduced to this diagonal form using a linear transformation acting on the vector variable \(S\). The multicomponent Yajima–Oikawa system \((1.2)\) is also completely integrable and possesses exact multisoliton solutions \([7–10]\).

In the recent paper \([11]\), a space discretization of the multicomponent Yajima–Oikawa system \((1.2)\),

\[
\begin{aligned}
\begin{cases}
i S_{n,t} &= \left(1 + \frac{1}{2} L_n \right) (S_{n+1} + S_{n-1}), \\
\log \left(1 + \frac{1}{2} L_n \right) \bigg|_t &= \frac{1}{2} \Delta^+_n \left( \langle S_n, \Sigma S^*_{n-1} \rangle + \langle S_{n-1}, \Sigma S^*_n \rangle \right),
\end{cases}
\end{aligned}
\tag{1.3a}
\]

where \(\Delta^+_n\) is the forward difference operator, i.e., \(\Delta^+_n f_n := f_{n+1} - f_n\), was proposed using the Hirota bilinear method and multisoliton solutions in terms of pfaffians were constructed. We set \(1 + \frac{1}{2} L_n =: v_n\) and consider a slightly
generalized version of (1.3):

\[
\begin{align*}
S_{n,t} & = v_n (S_{n+1} + S_{n-1}) - cS_n, \\
\bar{S}_{n,t} & = -v_n (\bar{S}_{n+1} + \bar{S}_{n-1}) + c\bar{S}_n, \\
v_{n,t} & = \frac{1}{2}v_n \Delta_n^+ \left( (S_n, \overline{S}_{n-1}) + (S_{n-1}, \overline{S}_n) \right),
\end{align*}
\]

(1.4a)

(1.4b)

(1.4c)

where \(c\) is an arbitrary constant and \(\overline{S}_n\) may or may not be related to the complex conjugate of \(S_n\).

The main objective of this paper is to demonstrate the integrability of the discrete multicomponent Yajima–Oikawa system (1.4). In section 2, we prove that (1.4) admits a Lax-pair representation [12] and construct the next higher flow of the discrete multicomponent Yajima–Oikawa hierarchy that can be reduced to a modified version of the Volterra lattice. We also discuss the continuous limit of space. In section 3, we show that (1.4) possesses an infinite set of conservation laws. In section 4, we show how (1.4) can be reduced to elementary flows of two integrable discrete nonlinear Schrödinger hierarchies: the Ablowitz–Ladik hierarchy [13] and the Konopelchenko–Chudnovsky hierarchy [14–16]. Section 5 is devoted to concluding remarks.

2 Lax pair

The main result of this paper is the following proposition, which provides an auxiliary linear problem for (1.4), generally called the Lax pair [12].

**Proposition 2.1.** The discrete multicomponent Yajima–Oikawa system (1.4) is equivalent to the compatibility condition of the overdetermined linear equations for \(\psi_n\), \(\phi_n\) and \(\chi_n\):

\[
\begin{align*}
v_{n,t} (\psi_{n+1} + \psi_{n-1}) & = \lambda \psi_n - \langle S_n, \phi_n \rangle - \langle \chi_n, \overline{S}_n \rangle, \\
\phi_{n+1} - \phi_n & = \frac{i}{2} \overline{S}_n (\psi_{n+1} + \psi_{n-1}), \\
\chi_{n+1} + \chi_n & = \frac{i}{2} S_n (\psi_{n+1} + \psi_{n-1})
\end{align*}
\]

(2.1a)

(2.1b)

(2.1c)

and

\[
\begin{align*}
\psi_{n,t} & = v_n (\psi_{n+1} + \psi_{n-1}) - c\psi_n, \\
\phi_{n,t} & = \frac{1}{2}v_n \overline{S}_{n-1} (\psi_{n+1} + \psi_{n-1}) - \frac{1}{2}v_{n-1} \overline{S}_n (\psi_n + \psi_{n-2}), \\
\chi_{n,t} & = \frac{1}{2}v_n S_{n-1} (\psi_{n+1} + \psi_{n-1}) + \frac{1}{2}v_{n-1} S_n (\psi_n + \psi_{n-2}) + 2ic\chi_n
\end{align*}
\]

(2.2a)

(2.2b)

(2.2c)

where \(\lambda\) is a constant spectral parameter.
Proposition 2.1 can be proved by a straightforward calculation; we operate with $\partial_t$ on (2.1a), (2.1b) and (2.1c) and use (2.2) and (2.1a) to remove the time derivative and the spectral parameter. The resulting three equations are

$$
\begin{align*}
1 \left[ v_{n,t} - \frac{1}{2} v_n \Delta_n^+ \left( \langle S_n, S_{n-1} \rangle + \langle S_{n-1}, S_n \rangle \right) \right] (\psi_{n+1} + \psi_{n-1}) \\
+ \langle i S_{n,t} - v_n (S_{n+1} + S_{n-1}) + c S_n, \phi_n \rangle \\
+ \langle \chi_n, i \bar{S}_{n,t} + v_n \left( \bar{S}_{n+1} + \bar{S}_{n-1} \right) - c \bar{S}_n \rangle = 0, \\
\left[ i \bar{S}_{n,t} + v_n \left( \bar{S}_{n+1} + \bar{S}_{n-1} \right) - c \bar{S}_n \right] (\psi_{n+1} + \psi_{n-1}) = 0, \\
[i \bar{S}_{n,t} - v_n (S_{n+1} + S_{n-1}) + c S_n] (\psi_{n+1} + \psi_{n-1}) = 0,
\end{align*}
$$

which are indeed equivalent to (1.4).

The scalar $\psi_n$ and the column vectors $\phi_n$ and $\chi_n$ constitute the linear eigenfunction in this Lax-pair representation. Note that the time-evolution equation (2.2a) for $\psi_n$ takes the same form as the original time-evolution equation (1.4a) for $S_n$, which is a peculiar feature of the Lax-pair representations for Yajima–Oikawa-type systems.

By changing the time evolution of the linear eigenfunction appropriately, we obtain the next higher flow of the discrete multicomponent Yajima–Oikawa hierarchy.

**Proposition 2.2.** The compatibility condition of the overdetermined linear equations for $\psi_n$, $\phi_n$ and $\chi_n$:

$$
\begin{align*}
v_n (\psi_{n+1} + \psi_{n-1}) &= \lambda \psi_n - \langle S_n, \phi_n \rangle - \langle \chi_n, \bar{S}_n \rangle, \\
\phi_{n+1} - \phi_n &= \frac{i}{2} \bar{S}_n (\psi_{n+1} + \psi_{n-1}), \\
\chi_{n+1} + \chi_n &= \frac{i}{2} S_n (\psi_{n+1} + \psi_{n-1}),
\end{align*}
$$

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and
\[
\begin{align*}
\psi_{n+1} = & v_nv_{n+1} (\psi_{n+2} + \psi_n) - v_nv_{n-1} (\psi_n + \psi_{n-2}) \\
& + \frac{i}{2} v_n \left( \langle S_{n+1}, S_n \rangle + \langle S_n, S_{n+1} \rangle + \langle S_{n-1}, S_{n-1} \rangle \right) (\psi_{n+1} + \psi_{n-1}), \\
\phi_{n+1} = & v_nv_{n+1} (\psi_{n+2} + \psi_n) + \frac{i}{2} v_{n-1} (\psi_{n-1} + \psi_{n-3}) \\
& + \frac{i}{2} v_n \left( \langle S_{n+1}, S_n \rangle + \langle S_n, S_{n+1} \rangle + \langle S_{n-1}, S_{n-1} \rangle \right) \bar{S}_{n-1} (\psi_{n+1} + \psi_{n-1}) \\
& + \frac{i}{4} v_{n-1} \left( \langle S_{n+1}, S_n \rangle + \langle S_{n+1}, S_{n+1} \rangle + \langle S_{n+1}, S_{n+1} \rangle \right) \bar{S}_n (\psi_{n+1} + \psi_{n-1}), \\
\chi_{n+1} = & v_nv_{n+1} (\psi_{n+2} + \psi_n) - \frac{i}{2} v_n \left( \psi_{n+2} + \psi_n \right) \\
& + \frac{i}{2} v_{n-1} \left( \langle S_{n+1}, S_n \rangle + \langle S_{n+1}, S_{n+1} \rangle + \langle S_{n+1}, S_{n+1} \rangle \right) \bar{S}_{n-1} (\psi_{n+1} + \psi_{n-1}) \\
& + \frac{i}{4} v_{n-1} \left( \langle S_{n+1}, S_n \rangle + \langle S_{n+1}, S_{n+1} \rangle + \langle S_{n+1}, S_{n+1} \rangle \right) \bar{S}_n (\psi_{n+1} + \psi_{n-1}),
\end{align*}
\]

where \( \lambda \) is a constant spectral parameter, is equivalent to the system:
\[
\begin{align*}
S_{n+1} = & v_nv_{n+1} (S_{n+2} + S_n) - v_nv_{n-1} (S_n + S_{n-2}) \\
& + \frac{i}{2} v_n \left( \langle S_{n+1}, S_n \rangle + \langle S_n, S_{n+1} \rangle + \langle S_{n-1}, S_{n-1} \rangle \right) (S_{n+1} + S_{n-1}), \\
\bar{S}_{n+1} = & v_nv_{n+1} (\bar{S}_{n+2} + \bar{S}_n) - v_nv_{n-1} (\bar{S}_n + \bar{S}_{n-2}) \\
& - \frac{i}{2} v_n \left( \langle S_{n+1}, S_n \rangle + \langle S_n, S_{n+1} \rangle + \langle S_{n-1}, S_{n-1} \rangle \right) (\bar{S}_{n+1} + \bar{S}_{n-1}), \\
v_{n+1} = & 2v_n^2 (v_{n+1} - v_{n-1}) \\
& + \frac{i}{2} v_nv_{n+1} \left( \langle S_{n+2}, \bar{S}_n \rangle - \langle S_n, \bar{S}_{n+2} \rangle \right) - \frac{i}{2} v_nv_{n-1} \left( \langle S_n, \bar{S}_{n-2} \rangle - \langle S_{n-2}, \bar{S}_n \rangle \right) \\
& - \frac{i}{4} v_n \left( \langle S_{n+1}, \bar{S}_n \rangle + \langle S_n, \bar{S}_{n+1} \rangle + \langle S_n, \bar{S}_{n-1} \rangle + \langle S_{n-1}, \bar{S}_n \rangle \right) \\
& \times \left( \langle S_{n+1}, \bar{S}_n \rangle + \langle S_n, \bar{S}_{n+1} \rangle - \langle S_{n+1}, \bar{S}_{n-1} \rangle - \langle S_n, \bar{S}_n \rangle \right).
\end{align*}
\]

We leave it to the reader to verify directly that the two time evolutions
indeed commute, i.e.,
\[
\frac{\partial^2 S_n}{\partial t \partial r} = \frac{\partial^2 S_n}{\partial r \partial t}, \quad \frac{\partial^2 \overline{S}_n}{\partial t \partial \tau} = \frac{\partial^2 \overline{S}_n}{\partial \tau \partial t}, \quad \frac{\partial^2 v_n}{\partial t \partial \tau} = \frac{\partial^2 v_n}{\partial \tau \partial t}.
\]

In the absence of $S_n$ and $\overline{S}_n$, the higher discrete multicomponent Yajima–Oikawa system \([2.3]\) reduces to a modified version of the Volterra lattice \([17]\):
\[
v_{n,\tau} = 2v_n^2(v_{n+1} - v_{n-1}) ,
\]
which is related to the Volterra lattice
\[
u_{n,\tau} = u_n(u_{n+1} - u_{n-1}) ,
\]
by the transformation $u_n := 2v_nv_{n+1}$. This implies that the discrete Yajima–Oikawa hierarchy generalizes a modified version of the Volterra lattice hierarchy in the same way as the continuous Yajima–Oikawa hierarchy generalizes the KdV hierarchy \([18]\).

The three-component spectral problem \([2.1]\) for $\psi_n$, $\phi_n$ and $\chi_n$ can be rewritten as a nonlocal spectral problem for the single scalar component $\psi_n$. That is, from \([2.1b]\) and \([2.1c]\), we obtain
\[
\phi_n = \frac{i}{2} \sum_{j=\infty}^{n-1} \overline{S}_j (\psi_{j+1} + \psi_{j-1}) ,
\]
and
\[
\chi_n = \frac{i}{2} \sum_{j=\infty}^{n-1} (-1)^{n-j-1} S_j (\psi_{j+1} + \psi_{j-1}) ,
\]
under the assumption that $\phi_n$ and $\chi_n$ approach zero as $n \to -\infty$. Thus, we arrive at the following proposition.

**Proposition 2.3.** If $S_n$, $\overline{S}_n$ and $v_n$ satisfy the discrete multicomponent Yajima–Oikawa system \([1.1]\), the overdetermined linear equations for $\psi_n$:
\[
v_n (\psi_{n+1} + \psi_{n-1}) = \lambda \psi_n - \frac{i}{2} \sum_{j=\infty}^{n-1} \left( S_n, \overline{S}_j \right) + (-1)^{n-j-1} (S_j, \overline{S}_n) \right) (\psi_{j+1} + \psi_{j-1}) ,
\]
and
\[
i \psi_{n,t} = v_n (\psi_{n+1} + \psi_{n-1}) - c \psi_n ,
\]
are compatible, where $\lambda$ is a constant spectral parameter.
A straightforward calculation shows that the compatibility condition for (2.5) and (2.6) is given by

\[ i \left[ v_{n,t} - \frac{1}{2} v_n \Delta_n^+ \left( \langle S_n, \overline{S}_{n-1} \rangle + \langle S_{n-1}, \overline{S}_n \rangle \right) \right] (\psi_{n+1} + \psi_{n-1}) \]

\[ + \frac{i}{2} \sum_{j=-\infty}^{n-1} \left[ \langle i S_{n,t} - v_n (S_{n+1} + S_{n-1}), S_j \rangle + \langle S_n, i S_{j,t} + v_j (\overline{S}_{j+1} + \overline{S}_{j-1}) \rangle \right] \]

\[ + (-1)^{n-j-1} \langle i S_{j,t} - v_j (S_{j+1} + S_{j-1}), S_n \rangle + (-1)^{n-j-1} \langle S_j, i S_{n,t} + v_n (\overline{S}_{n+1} + \overline{S}_{n-1}) \rangle \]

\[ \times (\psi_{j+1} + \psi_{j-1}) = 0, \]

which does not contain the arbitrary constant \( c \) and is equivalent to the set of relations:

\[ v_{n,t} = \frac{1}{2} v_n \Delta_n^+ \left( \langle S_n, \overline{S}_{n-1} \rangle + \langle S_{n-1}, \overline{S}_n \rangle \right), \]

\[ \langle i S_{n,t} - v_n (S_{n+1} + S_{n-1}), S_j \rangle + \langle S_n, i S_{j,t} + v_j (\overline{S}_{j+1} + \overline{S}_{j-1}) \rangle \]

\[ + (-1)^{n-j-1} \langle i S_{j,t} - v_j (S_{j+1} + S_{j-1}), S_n \rangle + (-1)^{n-j-1} \langle S_j, i S_{n,t} + v_n (\overline{S}_{n+1} + \overline{S}_{n-1}) \rangle \]

\[ = 0, \quad j \leq n - 1. \]

Thus, (1.4) implies the compatibility of (2.5) and (2.6), but not vice versa. In short, the scalar nonlocal spectral problem (2.5) and the isospectral time-evolution equation (2.6) provide a weak Lax-pair representation for the discrete multicomponent Yajima–Oikawa system (1.4). This is evident from the fact that the scalar nonlocal spectral problem (2.5) is invariant under the transformation

\[ S_n \to g S_n, \quad \overline{S}_n \to g^{-1} \overline{S}_n, \]

where \( g \) is a nonzero \( n \)-independent scalar. In particular, the spatial Lax operator defined by (2.5) is invariant under the time evolution corresponding to the zeroth flow of the hierarchy:

\[ v_n(t_0) = v_n(0), \quad S_n(t_0) = \exp \left( i dt_0 \right) S_n(0), \quad \overline{S}_n(t_0) = \exp \left( -i dt_0 \right) \overline{S}_n(0), \]

so the value of \( c \) in the discrete multicomponent Yajima–Oikawa system (1.4) can no longer be determined from the compatibility of (2.5) and (2.6). This is an intrinsic drawback of the scalar nonlocal Lax-pair representations for Yajima–Oikawa-type systems.

To consider the continuous limit, we set

\[ v_n(t) = \frac{1}{\Delta^2} + \frac{1}{2} L(n \Delta, t), \quad S_n(t) = S(n \Delta, t), \quad \overline{S}_n(t) = \Delta \overline{S}(n \Delta, t), \]

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and
\[ x = n\Delta, \quad c = \frac{2}{\Delta^2}, \]
where \( \Delta \) is a lattice parameter. Then, in the continuous limit \( \Delta \to 0 \), the discrete multicomponent Yajima–Oikawa system (1.4) reduces to
\[
\begin{align*}
\dot{i}S_t &= S_{xx} + LS, \quad (2.7a) \\
\dot{i}\overline{S}_t &= -\overline{S}_{xx} - L\overline{S}, \quad (2.7b) \\
L_t &= 2\langle S, \overline{S} \rangle_x. \quad (2.7c)
\end{align*}
\]
Thus, we obtain the continuous multicomponent Yajima–Oikawa system (1.2) by further setting \( \overline{S} = \Sigma S^* \). The Lax-pair representation for (2.7) is obtained by setting
\[
\psi_n(t) = \psi(n\Delta, t), \quad \phi_n(t) = \phi(n\Delta, t), \quad \lambda = \frac{2}{\Delta^2} + \zeta,
\]
and considering the continuous limit \( \Delta \to 0 \) of Proposition 2.1; the third component \( \chi_n \) in the linear eigenfunction can be discarded in the continuous limit.

**Proposition 2.4.** The nonreduced multicomponent Yajima–Oikawa system (2.7) is equivalent to the compatibility condition of the overdetermined linear equations for \( \psi \) and \( \phi \):
\[
\begin{align*}
\psi_{xx} + L\psi &= \zeta\psi - \langle S, \phi \rangle, \\
\phi_x &= i\overline{S}\psi,
\end{align*}
\]
and
\[
\begin{align*}
\dot{i}\psi &= \psi_{xx} + L\psi, \\
\dot{\phi} &= \overline{S}\psi_x - \overline{S}_x\psi,
\end{align*}
\]
where \( \zeta \) is a constant spectral parameter.

A weak Lax-pair representation for (2.7) is obtained by rewriting \( \phi \) as
\[
\phi(x, t) = i \int_x^z \overline{S}(y, t)\psi(y, t)dy.
\]

**Proposition 2.5.** If \( S, \overline{S} \) and \( L \) satisfy the nonreduced multicomponent Yajima–Oikawa system (2.7), then the overdetermined linear equations for \( \psi \):
\[
\psi_{xx} + L\psi + i\langle S, \partial_x^{-1} \left( \overline{S}\psi \right) \rangle = \zeta\psi, \quad (2.8)
\]
and
\[
\dot{i}\psi = \psi_{xx} + L\psi, \quad (2.9)
\]
are compatible but not vice versa, where \( \zeta \) is a constant spectral parameter.
3 Conservation laws

In this section, we show that the discrete multicomponent Yajima–Oikawa system (1.4) possesses an infinite number of conservation laws, which can be constructed from its Lax-pair representation in a recursive manner. As is clear from the following derivation, the conserved densities are determined from the spectral problem (2.1) only, so the higher flows of the discrete multicomponent Yajima–Oikawa hierarchy such as (2.3) have the same set of conserved quantities as (1.4).

We start with the trivial identity:

\[
\log \left( \frac{\psi_n}{\psi_{n-1}} \right) \left( \frac{\psi_{n-1}}{\psi_{n-2}} \right) = \Delta_n^+ \left[ \frac{\psi_{n-1,t}}{\psi_{n-1}} \right],
\]

(3.1)

and use the quantity \( \log (\psi_n/\psi_{n-1}) = \log \psi_n - \log \psi_{n-1} \) as a generating function of the conserved densities [19, 20]. We set

\[
\begin{align*}
\psi_n & =: 1 + \frac{1}{\lambda} v_n F_n, \\
\phi_n & =: -i \frac{1}{2} S_{n+1} + \frac{1}{\lambda} v_n G_n, \\
\chi_n & =: i \frac{1}{2} S_{n+1} + \frac{1}{\lambda} v_n H_n,
\end{align*}
\]

(3.2a-b-c)

where \( F_n \) is a scalar and \( G_n \) and \( H_n \) are column vectors, and rewrite the spectral problem (2.1) as

\[
\begin{align*}
F_n & = 1 + \frac{1}{\lambda^2} v_n v_{n+1} F_{n+1} + \frac{1}{\lambda} (S_n, G_n) + \frac{1}{\lambda} (H_n, \bar{S}_n), \\
G_n & = F_n \left( -i \frac{1}{2} S_{n+1} + \frac{1}{\lambda} v_{n+1} G_{n+1} \right) - \frac{i}{2 \lambda} v_{n+1} F_n F_{n+1} S_n, \\
H_n & = -F_n \left( i \frac{1}{2} S_{n+1} + \frac{1}{\lambda} v_{n+1} H_{n+1} \right) + \frac{i}{2 \lambda} v_{n+1} F_n F_{n+1} S_n.
\end{align*}
\]

(3.3a-b-c)

The trivial identity (3.1) can be rewritten with the aid of (3.2a) and (2.2a) as a nontrivial conservation law:

\[
[\log v_n + \log F_n]_t = \Delta_n^+ \left[ -i \left( \frac{1}{\lambda} v_{n-1} F_n + \frac{1}{\lambda} \frac{1}{F_{n-1}} \right) \right].
\]

(3.4)

The set of relations (3.3) allows us to express the three quantities \( F_n \), \( G_n \), and \( H_n \) as power series in \( 1/\lambda \):

\[
\begin{align*}
F_n & = \sum_{j=0}^{\infty} \frac{1}{\lambda^j} F_n^{(j)}, \\
G_n & = \sum_{j=0}^{\infty} \frac{1}{\lambda^j} G_n^{(j)}, \\
H_n & = \sum_{j=0}^{\infty} \frac{1}{\lambda^j} H_n^{(j)}.
\end{align*}
\]

(3.5)
where the coefficients \( \{F_n^{(j)}\}_{j \geq 0}, \{G_n^{(j)}\}_{j \geq 0} \) and \( \{H_n^{(j)}\}_{j \geq 0} \) are local functions of the dependent variables \( v_n, S_n \) and \( \bar{S}_n \). Substituting (3.5) into (3.3), we obtain the recurrence relations for the coefficients \( \{F_n^{(j)}\}_{j \geq 0}, \{G_n^{(j)}\}_{j \geq 0} \) and \( \{H_n^{(j)}\}_{j \geq 0} \):

\[
\begin{align*}
F_n^{(j)} &= v_nv_{n+1} \sum_{k=0}^{j-2} F_n^{(k)} F_{n+1}^{(j-k-2)} + \langle S_n, G_n^{(j-1)} \rangle + \langle H_n^{(j-1)}, \bar{S}_n \rangle, \quad j \geq 1, \\
G_n^{(j)} &= -\frac{i}{2} F_n^{(j)} \bar{S}_{n+1} + v_{n+1} \sum_{k=0}^{j-1} F_n^{(k)} G_{n+1}^{(j-k-1)} - \frac{i}{2} v_{n+1} \sum_{k=0}^{j-1} F_n^{(k)} F_{n+1}^{(j-k-1)} \bar{S}_n, \\
H_n^{(j)} &= -\frac{i}{2} F_n^{(j)} S_{n+1} - v_{n+1} \sum_{k=0}^{j-1} F_n^{(k)} H_{n+1}^{(j-k-1)} + \frac{i}{2} v_{n+1} \sum_{k=0}^{j-1} F_n^{(k)} F_{n+1}^{(j-k-1)} S_n,
\end{align*}
\]

(3.6a) (3.6b) (3.6c)

where \( F_n^{(0)} = 1 \). Explicit expressions for the first few coefficients are as follows:

\[
\begin{align*}
F_n &= 1 + \frac{1}{\lambda} \left( -\frac{i}{2} \langle S_{n+1}, \bar{S}_n \rangle - \frac{i}{2} \langle S_n, \bar{S}_{n+1} \rangle \right) \\
&\quad + \frac{1}{\lambda^2} \left[ v_nv_{n+1} + \frac{i}{2} v_{n+1} \left( \langle S_{n+2}, \bar{S}_n \rangle - \langle S_n, \bar{S}_{n+2} \rangle \right) \right] - \frac{1}{4} \langle (S_{n+1}, \bar{S}_n) + (S_n, \bar{S}_{n+1}) \rangle^2 \\
&\quad + O \left( \frac{1}{\lambda^3} \right), \\
G_n &= -\frac{i}{2} \bar{S}_{n+1} + \frac{1}{\lambda} \left[ -\frac{i}{2} v_{n+1} \left( \bar{S}_{n+2} + S_n \right) \right] - \frac{1}{4} \langle (S_{n+1}, \bar{S}_n) + (S_n, \bar{S}_{n+1}) \rangle \bar{S}_{n+1} \\
&\quad + O \left( \frac{1}{\lambda^2} \right), \\
H_n &= -\frac{i}{2} S_{n+1} + \frac{1}{\lambda} \left[ \frac{i}{2} v_{n+1} \left( S_{n+2} + S_n \right) \right] - \frac{1}{4} \langle (S_{n+1}, \bar{S}_n) + (S_n, \bar{S}_{n+1}) \rangle S_{n+1} \\
&\quad + O \left( \frac{1}{\lambda^2} \right).
\end{align*}
\]

Now, we can state the main result of this section.

**Proposition 3.1.** Let \( \{F_n^{(j)}\}_{j \geq 0} \) together with \( \{G_n^{(j)}\}_{j \geq 0} \) and \( \{H_n^{(j)}\}_{j \geq 0} \) be defined by the recurrence relations (3.6) with \( F_n^{(0)} = 1 \). Then, the discrete
The multicomponent Yajima–Oikawa system \((1,4)\) possesses an infinite number of conservation laws, which can be obtained by comparing the coefficients of different powers of \(1/\lambda\) on both sides of the equality:

\[
\left[ \log v_n + \log \left( 1 + \sum_{j=1}^{\infty} \frac{1}{\lambda^j} F_n^{(j)} \right) \right]_t = \Delta^+_n \left[ -i \frac{F_{n-1}}{v_{n-1}} \left( 1 + \sum_{j=1}^{\infty} \frac{1}{\lambda^j} F_n^{(j)} \right) - i \lambda \frac{1}{1 + \sum_{j=1}^{\infty} \frac{1}{\lambda^j} F_n^{(j)}} \right].
\]

(3.7)

Noting that

\[
\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots,
\]

\[
\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots,
\]

the first four conservation laws obtained from (3.7) are as follows:

\[
(\log v_n)_t = \Delta^+_n \left[ \frac{1}{2} \left( \langle S_n, \overline{S}_{n-1} \rangle + \langle S_{n-1}, \overline{S}_n \rangle \right) \right],
\]

\[
\left( \langle S_{n+1}, \overline{S}_n \rangle + \langle S_n, \overline{S}_{n+1} \rangle \right)_t = \Delta^+_n \left[ -i v_n \left( \langle S_{n+1}, \overline{S}_{n-1} \rangle - \langle S_{n-1}, \overline{S}_{n+1} \rangle \right) \right],
\]

\[
\left[ v_n v_{n+1} + \frac{i}{2} v_{n+1} \left( \langle S_{n+2}, \overline{S}_n \rangle - \langle S_n, \overline{S}_{n+2} \rangle \right) - \frac{1}{8} \left( \langle S_{n+1}, \overline{S}_n \rangle + \langle S_n, \overline{S}_{n+1} \rangle \right)^2 \right]_t
\]

\[
= \Delta^+_n \left[ \frac{1}{2} v_n v_{n+1} \left( \langle S_{n+2}, \overline{S}_{n-1} \rangle + \langle S_{n-1}, \overline{S}_{n+2} \rangle + \langle S_n, \overline{S}_{n-1} \rangle + \langle S_{n-1}, \overline{S}_n \rangle \right)
\]

\[
+ \frac{i}{4} v_n \left( \langle S_{n+1}, \overline{S}_n \rangle + \langle S_n, \overline{S}_{n+1} \rangle \right) \left( \langle S_{n+1}, \overline{S}_{n-1} \rangle - \langle S_{n-1}, \overline{S}_{n+1} \rangle \right) \right],
\]

\[
\left[ v_n v_{n+1} \left( \langle S_{n+2}, \overline{S}_{n+1} \rangle + \langle S_{n+1}, \overline{S}_{n+2} \rangle + \langle S_{n+1}, \overline{S}_n \rangle + \langle S_n, \overline{S}_{n+1} \rangle \right)
\]

\[
+ v_{n+1} v_{n+2} \left( \langle S_{n+3}, \overline{S}_n \rangle + \langle S_n, \overline{S}_{n+3} \rangle + \langle S_{n+1}, \overline{S}_n \rangle + \langle S_n, \overline{S}_{n+1} \rangle \right)
\]

\[
+ \frac{i}{2} v_{n+1} \left( \langle S_{n+2}, \overline{S}_{n+1} \rangle + \langle S_{n+1}, \overline{S}_{n+2} \rangle + \langle S_{n+1}, \overline{S}_n \rangle + \langle S_n, \overline{S}_{n+1} \rangle \right)
\]

\[
\times \left( \langle S_{n+2}, \overline{S}_n \rangle - \langle S_n, \overline{S}_{n+2} \rangle \right) - \frac{1}{12} \left( \langle S_{n+1}, \overline{S}_n \rangle + \langle S_n, \overline{S}_{n+1} \rangle \right)^3 \right]_t
\]

= \Delta^+_n [...] .

Here, the flux in the fourth conservation law is lengthy and omitted.

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Not all of the conservation laws of the discrete multicomponent Yajima–Oikawa system (1.4) can be obtained in this way. First, the second conservation law can be generalized to the componentwise form:

$$
\left( S_{n+1}^{(j)} S_{n}^{(k)} + S_{n}^{(j)} S_{n+1}^{(k)} \right)_t = \Delta_n^+ \left[ -iv_n \left( S_{n+1}^{(j)} S_{n-1}^{(k)} - S_{n-1}^{(j)} S_{n+1}^{(k)} \right) \right],
$$

where $S_{n}^{(j)}$ is the $j$th component of the column vector $S_n$ and $S_{n}^{(k)}$ is the $k$th component of the column vector $S_n$. Second, in the case where $S_n$ and $\bar{S}_n$ are scalar, namely, $S_n = S$ and $\bar{S}_n = \bar{S}$, (1.4) admits one more conservation law:

$$
\left\{ v_n (S_{n+1} + S_{n-1}) \left( \bar{S}_{n+1} + \bar{S}_{n-1} \right) + \frac{i}{4} \left[ \left( S_{n+1} S_n \right)^2 - \left( S_n S_{n+1} \right)^2 \right] \right\}_t = \Delta_n^+ \left\{ -iv_n v_{n-1} \left[ (S_{n+1} + S_{n-1}) \left( \bar{S}_n + \bar{S}_{n-2} \right) - (S_n + S_{n-2}) \left( \bar{S}_{n+1} + \bar{S}_{n-1} \right) \right] + \frac{1}{2} v_n \left[ S_{n-1} \bar{S}_{n-1} + S_n \bar{S}_n + S_{n+1} \bar{S}_{n+1} + S_{n-1} \bar{S}_{n+1} + S_{n+1} \bar{S}_{n-1} \right] \right\},
$$

which can be found with the aid of the Mathematica package “diffdens.m” [21] available from

https://inside.mines.edu/~whereman/software/diffdens/.

4 Relations to discrete nonlinear Schrödinger hierarchies

In this section, we show how the discrete multicomponent Yajima–Oikawa system (1.4) can be related to two integrable discrete nonlinear Schrödinger hierarchies: the Ablowitz–Ladik hierarchy [13] and the Konopelchenko–Chudnovsky hierarchy [14–16].

4.1 Ablowitz–Ladik hierarchy

By changing the dependent variables as $S_n = \alpha^n q_n$ and $\bar{S}_n = \alpha^{n-1} \bar{q}_n$ with a nonzero constant $\alpha$, the discrete multicomponent Yajima–Oikawa system (1.4) is transformed to the following form:

$$
\begin{align*}
\begin{cases}
i q_{n,t} &= v_n \left( \alpha q_{n+1} + \frac{1}{\alpha} q_{n-1} \right) - \alpha q_n, \\i \bar{q}_{n,t} &= -v_n \left( \frac{1}{\alpha} \bar{q}_{n+1} + \alpha \bar{q}_{n-1} \right) + \alpha \bar{q}_n, \\v_{n,t} &= \frac{1}{2} v_n \Delta_n^+ \left( \alpha \langle q_n, q_{n-1} \rangle + \frac{1}{\alpha} \langle q_{n-1}, \bar{q}_n \rangle \right).
\end{cases}
\end{align*}
$$

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Thus, by rescaling the time variable as $\partial_t =: \alpha \partial_{t_1}$ and taking the limit $\alpha \to \infty$, (4.1) reduces to
\[
\begin{cases}
i q_{n,t_1} = v_n q_{n+1}, \\i \overline{q}_{n,t_1} = -v_n \overline{q}_{n-1}, \\v_{n,t_1} = \frac{1}{2} v_n \Delta^+_n \left( \langle q_n, \overline{q}_{n-1} \rangle \right),
\end{cases}
\]
and by rescaling the time variable as $\partial_t =: \frac{1}{\alpha} \partial_{t_{-1}}$ and taking the limit $\alpha \to 0$, (4.1) reduces to
\[
\begin{cases}
i q_{n,t_{-1}} = v_n q_{n-1}, \\i \overline{q}_{n,t_{-1}} = -v_n \overline{q}_{n+1}, \\v_{n,t_{-1}} = \frac{1}{2} v_n \Delta^-_n \left( \langle q_{n-1}, \overline{q}_n \rangle \right).
\end{cases}
\]
The two systems (4.2) and (4.3) are essentially equivalent and already known in the case of scalar $q_n$ and $\overline{q}_n$ (see, e.g., (4.19) or (4.21) in [22]). Note that $\partial_t$ and $\partial_{t_{-1}}$ do not commute.

From (4.2), we have the relation
\[
\left( v_n - \frac{i}{2} \langle q_n, \overline{q}_n \rangle \right)_{t_1} = 0,
\]
which implies that
\[
v_n = k_n + \frac{i}{2} \langle q_n, \overline{q}_n \rangle,
\]
where $k_n$ is a “constant” independent of time $t_1$. Thus, (4.2) reduces to the simpler form:
\[
\begin{cases}
i q_{n,t_1} = \left( k_n + \frac{i}{2} \langle q_n, \overline{q}_n \rangle \right) q_{n+1}, \\i \overline{q}_{n,t_1} = -\left( k_n + \frac{i}{2} \langle q_n, \overline{q}_n \rangle \right) \overline{q}_{n-1}.
\end{cases}
\]
For nonzero values of $k_n$, the change of variables
\[
q_n = \left( \prod_{j=1}^{n-1} \frac{1}{k_j} \right) a_n, \quad \overline{q}_n = \left( \prod_{j=1}^{n} k_j \right) b_n,
\]
can be used to normalize $k_n$ to 1 and to remove $\frac{i}{2}$ in (4.4). Thus, we obtain the vector generalization [23] of an elementary flow of the Ablowitz–Ladik hierarchy [13]:
\[
\begin{cases}
i a_{n,t_1} = \left( 1 + \langle a_n, b_n \rangle \right) a_{n+1}, \\i b_{n,t_1} = -\left( 1 + \langle a_n, b_n \rangle \right) b_{n-1}.
\end{cases}
\]
4.2 Konopelchenko–Chudnovsky hierarchy

Under the boundary condition \( \lim_{n \to -\infty} \left( \langle S_n, \overline{S}_{n-1} \rangle + \langle S_{n-1}, \overline{S}_n \rangle \right) = 0 \), (4.4c) implies the relation:

\[
\left( \prod_{j=-\infty}^{n} v_j \right)_t = \left( \prod_{j=-\infty}^{n} v_j \right)^1_2 \left( \langle S_{n+1}, \overline{S}_n \rangle + \langle S_n, \overline{S}_{n+1} \rangle \right). \tag{4.5}
\]

Thus, the transformation of dependent variables:

\[
S_n = r_n \left( \prod_{j=-\infty}^{n} v_j \right)^{-\epsilon}, \quad \overline{S}_n = \overline{r}_n \left( \prod_{j=-\infty}^{n-1} v_j \right)^{-\epsilon}, \tag{4.6}
\]

preserves the locality of the equations of motion, where \( \epsilon \) is an arbitrary constant. In particular, the transformation (4.6) with \( \epsilon = -1 \) changes the discrete multicomponent Yajima–Oikawa system (1.4) to the following form:

\[
\begin{align*}
\text{i} r_{n,t} &= v_n r_{n+1} + r_{n-1} - \frac{1}{2} r_n (v_n v_{n+1} \langle r_{n+1}, \overline{r}_n \rangle + \langle r_n, \overline{r}_{n+1} \rangle) - cr_n, \tag{4.7a} \\
\text{i} \overline{r}_{n,t} &= -\overline{r}_{n+1} - v_n v_{n-1} \overline{r}_{n-1} + \frac{1}{2} \overline{r}_n (v_n v_{n-1} \langle r_n, \overline{r}_{n-1} \rangle + \langle r_{n-1}, \overline{r}_n \rangle) + c \overline{r}_n, \tag{4.7b} \\
v_{n,t} &= \frac{1}{2} v_n \Delta^+_n (v_n v_{n-1} \langle r_n, \overline{r}_{n-1} \rangle + \langle r_{n-1}, \overline{r}_n \rangle). \tag{4.7c}
\end{align*}
\]

In the more general case where \( \lim_{n \to -\infty} \left( \langle S_n, \overline{S}_{n-1} \rangle + \langle S_{n-1}, \overline{S}_n \rangle \right) \neq 0 \), we need only replace \( \prod_{j=-\infty}^{n} v_j \) in (4.5) and (4.6) with

\[
\left( \prod_{j=-\infty}^{n} v_j \right) \exp \left[ \frac{1}{2} \int_{n \to -\infty} \lim_{n \to -\infty} \left( \langle S_n, \overline{S}_{n-1} \rangle + \langle S_{n-1}, \overline{S}_n \rangle \right) \, dt \right].
\]

Note, incidentally, that (4.7) can be rewritten in a slightly simpler form if we use the new variable \( u_n := v_n v_{n+1} \) instead of \( v_n \).

The transformed system (4.7) admits two interesting reductions. First, by setting \( v_n = 0 \), (4.7) reduces to

\[
\begin{align*}
\text{i} r_{n,t} &= r_{n-1} - \frac{1}{2} r_n \langle r_n, \overline{r}_{n+1} \rangle - cr_n, \\
\text{i} \overline{r}_{n,t} &= -\overline{r}_{n+1} + \frac{1}{2} \overline{r}_n \langle r_{n-1}, \overline{r}_n \rangle + c \overline{r}_n,
\end{align*}
\]

which is essentially an elementary flow of a discrete integrable nonlinear Schrödinger hierarchy [23]; this discrete nonlinear Schrödinger hierarchy is
closely related to an elementary auto-Bäcklund transformation for the continuous nonlinear Schrödinger hierarchy as clarified by Konopelchenko [14] and D. V. and G. V. Chudnovsky [15, 16]. Second, by setting
\[ r_n = r, \quad \bar{r}_n = \bar{r}, \quad \langle r, \bar{r} \rangle = -2i, \quad c = 0, \]
(4.7) reduces to the modified version of the Volterra lattice [17]:
\[ v_{n,t} = -iv_n^2 (v_{n+1} - v_{n-1}), \]
which coincides with (2.4) up to a rescaling of the variables.

5 Concluding remarks

In this paper, we have shown that the discrete multicomponent Yajima–Oikawa system (1.4) admits the Lax-pair representation, which then can be used to generate an infinite number of conservation laws in a recursive manner. Thus, the integrability of the discrete multicomponent Yajima–Oikawa system (1.4) is established, which provides a clear answer to the question posed in [11]. The next higher flow of the discrete multicomponent Yajima–Oikawa hierarchy is presented, which can be reduced to the modified version of the Volterra lattice. The discrete multicomponent Yajima–Oikawa system (1.4) turns out to be related to (the vector generalizations of) two integrable discrete nonlinear Schrödinger hierarchies. First, in a suitable scaling limit, (1.4) can be reduced to an elementary flow of the Ablowitz–Ladik hierarchy. Second, by applying a nonlocal transformation and imposing a reduction, (1.4) can be simplified to an elementary flow of the Konopelchenko–Chudnovsky hierarchy as well as the modified version of the Volterra lattice.

In a suitable continuous limit, the Lax-pair representation for the discrete multicomponent Yajima–Oikawa system (1.4) can be reduced to the Lax-pair representation for the continuous multicomponent Yajima–Oikawa system. However, the size of the Lax matrices in the discrete case is larger than that in the continuous case (cf. Proposition 2.1 and Proposition 2.4); thus, it is not possible to find the Lax-pair representation in the discrete case directly by discretizing the Lax-pair representation in the continuous case.

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