Quantum phase transitions and string orders in the spin-1/2 Heisenberg–Ising alternating chain with Dzyaloshinskii–Moriya interaction

Guang-Hua Liu, Wen-Long You, Wei Li and Gang Su

1 Department of Physics, Tianjin Polytechnic University, Tianjin 300387, People’s Republic of China
2 College of Physics, Optoelectronics and Energy, Soochow University, Suzhou, Jiangsu 215006, People’s Republic of China
3 Physics Department, Arnold Sommerfeld Center for Theoretical Physics and Center for NanoScience, Ludwig-Maximilians-Universität, 80333 Munich, Germany
4 Department of Physics, Beihang University, Beijing 100191, People’s Republic of China
5 Theoretical Condensed Matter Physics and Computational Materials Physics Laboratory, College of Physical Sciences, University of Chinese Academy of Sciences, PO Box 4588, Beijing 100049, People’s Republic of China

E-mail: W.Li@physik.lmu.de

Received 10 December 2014, revised 22 February 2015
Accepted for publication 25 February 2015
Published 30 March 2015

Abstract
Quantum phase transitions (QPTs) and the ground-state phase diagram of the spin-1/2 Heisenberg–Ising alternating chain (HIAC) with uniform Dzyaloshinskii–Moriya (DM) interaction are investigated by a matrix-product-state (MPS) method. By calculating the odd- and even-string order parameters, we recognize two kinds of Haldane phases, i.e. the odd- and even-Haldane phases. Furthermore, doubly degenerate entanglement spectra on odd and even bonds are observed in odd- and even-Haldane phases, respectively. A rich phase diagram including four different phases, i.e. an antiferromagnetic (AF), AF stripe, odd- and even-Haldane phases, is obtained. These phases are found to be separated by continuous QPTs: the topological QPT between the odd- and even-Haldane phases is verified to be continuous and corresponds to conformal field theory with central charge $c = 1$; while the rest of the phase transitions in the phase diagram are found to be $c = 1/2$. We also revisit, with our MPS method, the exactly solvable case of HIAC model with DM interactions only on odd bonds and find that the even-Haldane phase disappears, but the other three phases, i.e. the AF, AF stripe and odd-Haldane phases, still remain in the phase diagram. We exhibit the evolution of the even-Haldane phase by tuning the DM interactions on the even bonds gradually.

Keywords: iTEBD, quantum phase transitions, string order

(Some figures may appear in colour only in the online journal)
characterize the ‘dilute’ antiferromagnetic (AF) order in the Haldane phase [5]. Further study shows that the topological long-range order in the SO(2n + 1) symmetric matrix product state (MPS) can be fully identified and characterized by a set of nonlocal string order parameters [6]. The SO(2n + 1) symmetric MPS contains diluted AF orders in n channels and a hidden (parameter protected topological (SPT) order [15] and the string order bond-centered inversion, etc) are intact in the Hamiltonian long-range order in the SO phase, e.g. large-D phase (D represents single-ion anisotropy), by a phase transition [12–14], as long as symmetries (SO(3), bond-centered inversion, etc) are intact in the Hamiltonian space (path). So to speak, the Haldane phase has a symmetry-protected topological (SPT) order [15] and the string order parameter Oi can thus be regarded as a quantification of the topological order of Haldane phase. Moreover, a nonlocal unitary transformation was constructed to uncover explicitly the hidden Z2 × Z2 symmetry breaking in the Haldane phase [16–19].

On the other hand, SPT phases have also been reported in spin-1/2 multi-period chains. A spin-1/2 ferromagnetic-antiferromagnetic alternating chain has been shown to have nonzero string order, which changes continuously through the Haldane and dimer phases and shows a crossover behavior between them [20]. In addition, Kohmoto and Tasaki indicated that the spin-1/2 ferromagnetic-antiferromagnetic alternating chain fully breaks the hidden Z2 × Z2 symmetry not only in the strongly coupled (Haldane) phase but also in the decoupled (dimer) phase, supporting that the spin-1/2 dimer state belongs to a Haldane-like phase. Recently, Wang et al [21] associated the phase transition between odd- and even-dimer states as topological quantum phase transition (TQPT) between two kinds of Haldane phases (even- and odd-Haldane), which have different string order parameters.

Among others, the Heisenberg–Ising alternating chain (HIAC), originally proposed by Lieb et al [22] and re-examined subsequently by Yao et al [23], constitutes an interesting spin-1/2 chain model. The Hamiltonian of the HIAC is given by

\[ \hat{H} = \sum_{i=1}^{N/2} J_1 \hat{S}_{2i-1} \cdot \hat{S}_{2i} + J_2 \hat{S}_{2i}^z \hat{S}_{2i+1}^z, \]

where \( \hat{S} \) denotes a spin-1/2 operator (\( S^z \) is the z-component) and N (an even number) is the total number of spins. \( J_1 \) and \( J_1, J_2 > 0 \) are the Heisenberg and Ising couplings on the odd and even bonds, respectively. This model can be solved exactly and a phase transition from a quantum paramagnetic phase to the AF phase was found to occur at \( J_1/J_2 = 2 \). When the Dzyaloshinskii–Moriya (DM) interactions are switched on odd bonds of the HIAC [24], the model is still exactly soluble; it turns out that the ground-state energy exhibits an interesting non-analytic behavior accompanied by a gapless excitation spectrum along the line \( J_1 = 2\sqrt{D^2 + J_2^2} \), where D denotes the strength of DM interaction (see equation (2) below). This critical line was found to separate a quantum paramagnetic phase (\( J_1 < 2\sqrt{D^2 + J_2^2} \)) and an AF one. However, in both cases, the property of the SPT order has not been discussed in the paramagnetic phase.

In this paper, we study an extended HIAC model, which has uniform DM interactions on both even- and odd-bonds (equation (2) below), which is thus no longer exactly soluble. We here adopt an MPS based numerical method, i.e. the infinite time-evolving block decimation (iTEBD) algorithm [25, 26], to determine the ground state with high accuracy. We obtain a rich ground-state phase diagram which includes an AF phase, an antiferromagnetic stripe phase (AFSP) and two Haldane phases. In particular, the two Haldane phases, namely, the odd- and even-Haldane phases, can be characterized (and distinguished) by two different nonlocal string order parameters and doubly degenerate entanglement spectra on the odd and even bonds, respectively. Moreover, a continuous TQPT between the odd- and even-Haldane phases has been found.

The rest of this paper is arranged as follows. In section 2, the model Hamiltonian and the exact phase diagram are present. In section 3, the numerical results are shown and analyzed in detail. Lastly, we devote section 4 to a discussion and summary.

2. Model Hamiltonian and the phase diagram

The spin-1/2 HIAC with a uniform DM interaction is described by

\[ \hat{H} = \sum_{i} (J_1 \hat{S}^z_{2i-1} \cdot \hat{S}^z_{2i} + D \hat{S}^x_{2i-1} \hat{S}^x_{2i} - \hat{S}^y_{2i-1} \hat{S}^y_{2i}) + J_2 \hat{S}^z_{2i} \hat{S}^z_{2i+1} + D(\hat{S}^x_{2i} \hat{S}^x_{2i+1} + \hat{S}^y_{2i} \hat{S}^y_{2i+1})]. \]  

(2)

Here \( \hat{D} \) is called the DM vector, with \( \hat{D} = D\vec{c} \), adopted in this paper. The DM interaction is an antisymmetric spin–spin coupling, due to the spin–orbit coupling effect. Recently, such DM-like spin–orbit interaction in the single-crystal yttrium iron garnet was experimentally verified and found amenable to manipulations [27]. It is worth noting the DM interaction can be eliminated by a canonical spin rotation [24, 28] and then equation (2) can be mapped into a period-two XXZ model consequently. In contrast to equation (1), the transformed XXZ model is no longer exactly soluble.

We employ the iTEBD method to obtain the ground state \( |\psi_g\rangle \) of model equation (2). The details of the iTEBD are referred to in the original papers [25, 26]. The iTEBD algorithm is a numerical approach based on the MPS ansatz, which accurately describes quantum many-body
wavefunctions in one dimension. In the framework of MPS, the wavefunction of 1D quantum system can be written as

\[
|\psi\rangle = \text{Tr}(\prod_{i=1}^{N/2} \Lambda^a_i \Gamma^a_i \Lambda^b_i \Gamma^b_i |m_{2i-1}, m_{2i}, \ldots\rangle). 
\] (3)

The \(\Gamma^a\) and \(\Gamma^b\) represent two three-indexed tensors and \(\Lambda^a\) and \(\Lambda^b\) are two \(\chi \times \chi\) (\(\chi\) is the bond dimension) diagonal matrices. An imaginary-time evolution operator \(\exp(-\tau \hat{H})\) is acted on an arbitrary initial state \(|\psi_0\rangle\). In the limit \(\tau \to \infty\), \(\exp(-\tau \hat{H})|\psi_0\rangle\) will converge to the ground state \(|\psi_g\rangle\) of Hamiltonian \(\hat{H}\). For small enough \(\delta\tau\), the operator \(\exp(-\delta\tau \hat{H})\) can be expanded through a second-order Suzuki-Trotter decomposition as a sequence of two-site gates \(U^{[i,i+1]}\). In practice, we start from \(\delta\tau = 10^{-1}\) and gradually reduce it to smaller values (eventually down to \(\delta\tau = 10^{-8}\)), with total evolution steps of number \(10^{4-5}\), given the variational MPS wavefunction is well converged in the course of imaginary-time evolutions. Regarding the retained bond dimension \(\chi\) of MPS used in practice, we found for this specific model, \(\chi \sim 30\) can already provide rather accurate and converged results (for various gapped phases of the model). Remarkably, distinct from other finite-size algorithms such as quantum Monte Carlo or finite-size density matrix renormalization group method, iTEBD exploits the translation invariance of the one-dimensional chain and makes the thermodynamic limit directly accessible. According to the ground-state wavefunction obtained by iTEBD, some physical expected values can be evaluated by \(\hat{O} = \langle \psi_g | \hat{O} | \psi_g \rangle\).

From the numerical results, we conclude a rich ground-state phase diagram of the spin-1/2 HIAC model with DM interactions, which is shown in figure 1. One can find four different phases: an AFSP, an AF phase and two kinds of Haldane (odd- and even-Haldane) phases, all separated by critical lines. Two magnetic ordered phases can be identified by local order parameters (stripe and Néel orders, respectively), while the non-magnetic odd- and even-Haldane phases are distinguished by two nonzero string order parameters. In figure 1, besides the QPTs between different magnetic ordered phases and those between ordered and Haldane phases, we also observe a TQPT between the odd- and even-Haldane phases, which cannot be explained via symmetry breaking (Landau paradigm). This TQPT can be well captured by the singular property of the bipartite entanglement, as well as even- and odd-string order parameters.

In addition, we are also interested in the critical properties on the phase boundaries. We select several representative points (P1 to P5, see figure 1) and calculate their central charges \(c\) by fitting the block entanglement entropies. We find \(c = 1/2\) at the magnetic order–disorder transition points (P1, P2 and P3); while \(c = 1\) at the TQPT points (P4 and P5) between two kinds of Haldane phases.

3. Numerical results and discussions

In the following, we study the phase transitions along two selected paths, by fixing \(D = 0.7\) or 2.0 and tuning Ising couplings \(J_i\). In section 3.1, we study the line \(D = 0.7\) crossing four different phases (see figure 1), i.e. the AFSP, odd-Haldane, AF and even-Haldane phases and focus on the magnetic order–disorder QPTs. In section 3.2, we focus on the path along \(D = 2.0\) line, along which the odd- and even-Haldane phases touch each other and a QPT was found between them. Since both the even- and odd-Haldane phases have the same symmetry properties (no symmetry breaking), this exotic phase transition between two phases can be regarded as a TQPT. In section 3.3, the block entanglement entropies and the fitted central charges of several critical points are evaluated. In our calculations, the Heisenberg coupling \(J_H = 1\) (see equation (2)) is set as an energy scale and the retained bond dimension of MPS is set as \(\chi = 30\). In order to check the reliability of our results, we reinvestigate the Heisenberg–Ising chain with odd-bond DM interactions in section 3.4 and find that the disordered phase corresponds to the odd-Haldane phase. Next we study the effect of moderate strengths of even-bond DM interactions and the evolution of the even-Haldane phase are shown pictorially in section 3.5.

3.1. The QPTs along the \(D = 0.7\) line

In this subsection, we investigate the QPTs along the line \(D = 0.7\). The local magnetizations on site \(i\), \(M_i^\sigma = \langle S_i^\sigma \rangle (\sigma = \{x, z\})\) are evaluated. We find that the transverse magnetization \(M_i^z\) vanishes completely along the line. However, local magnetizations \(M_i^x\) are nonzero in two regions: In the region of \(J_f < -2.54\), the spin configuration is \(\cdots + + - - \cdots \) or \(\cdots - - + + \cdots \) (+ and − denote spin up and down in terms of \(M_i^z\), respectively), constitutes a period-four stripe-like AF ordered phase (dubbed as AFSP); another symmetry broken phase is between \(2.0 < J_f < 2.6\), where we observe an conventional AF Néel order, i.e. \(\cdots + + - - \cdots \) or \(\cdots - - + + \cdots \). It is convenient to define the stripe order parameter

![Figure 1. Magnetic phase diagram of the spin-1/2 HIAC with uniform DM interactions (equation (2)). It includes four different phases: an AFSP, an AF phase and two kinds of Haldane (odd- and even-Haldane) phases. The filled square ■ denotes a tricritical point and the vicinity of it is subject to substantial uncertainty. Five representative points (P1, P2, P3, P4, P5) on the phase boundaries are selected and the corresponding central charges will be discussed below.](image-url)
Figure 2. The stripe order parameter \( \mathcal{M}^{\text{stripe}} \) and the Néel order parameter \( \mathcal{M}^{\text{Néel}} \) versus Ising coupling \( J_2 \) are shown (along the \( D = 0.7 \) line).

\[
\mathcal{M}^{\text{stripe}} = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} (-1)^i (M_{i}^{\uparrow} + M_{i+1}^{\downarrow}),
\]

and

\[
\mathcal{M}^{\text{Néel}} = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} (-1)^i M_{i}^{\uparrow},
\]

where \( \mathcal{N} \) is the system size.

The entanglement entropy has been shown to be an efficient tool to detect QPTs \([29-31]\). For continuous QPTs, owing to the quantum criticality at the transition point, the bipartite von Neumann entanglement entropy

\[
S_0 = -\text{Tr}(\rho \log_2 \rho),
\]

diverges at the critical point. In the MPS framework, \( S_0 \) can be evaluated by

\[
S_0 = -\sum_{i=1}^{\mathcal{N}} \Lambda_i^2 \log_2 \Lambda_i^2,
\]

where \( \Lambda_i \)'s are (normalized) singular values obtained through bond singular value decomposition, i.e. square root of the eigenvalues of the half-infinite-chain reduced density matrix. In figure 3, we show the results of the bipartite entanglement entropies \( S_{2,-1,2i} \) on the odd bond and \( S_{2,2i+1} \) on the even bond. \( S_0 \) shows three sharp peaks, indicating three continuous QPT points. Besides, we have also calculated the ground-state energy per site \( e_i \) and its first-order derivative \( (de_i/dJ_2) \) and (b), respectively, the second-order derivative \( (d^2e_i/dJ_2^2) \) exhibits divergent peaks at three critical points (figure 4(b)).

The singularities in \( d^2e_i/dJ_2^2 \) confirm the conclusion that these three magnetic order–disorder QPTs are of second order \([32]\).

3.2. The \( D = 2 \) line and the topological phase transition

Next, we explore the line \( D = 2 \) in figure 1, along which the bipartite entanglement entropy is calculated (see figure 5). We will focus on the QPT occurred at \( J_1^c \approx 1.43 \) (P4 point of figure 1), between two magnetic disordered phases, i.e. the odd- and even-Haldane phases. The sharp peak of the entanglement entropy curve suggests that this QPT at P4 is also of second order.

We firstly show the results of the nonlocal string order parameters, which are introduced by den Nijs and Rommel \([5]\) and are used to characterize the dilute AF order in the Haldane phase of the spin-1 Heisenberg chain. The presence of a nonzero string order parameter can be explained via a hidden \( Z_2 \times Z_2 \) symmetry breaking \([16, 17]\) and is related to a symmetry-protected topological order. For the present model, in order to distinguish the odd- and even-Haldane phases, we need to define two different string orders (odd- and even-string orders), which are defined as \([21, 32]\)

\[
O_{i,\text{odd}}^a(L) = (\sigma_{2i-1}^{a} \sigma_{2i}^{a} \cdots \sigma_{2i-1}^{a} \sigma_{2i+1}^{a}),
\]

and

\[
O_{i,\text{even}}^a(L) = (\sigma_{2i}^{a} \sigma_{2i+1}^{a} \cdots \sigma_{2i}^{a} \sigma_{2i+1}^{a}),
\]
The operator denotes a spin-1/2 Pauli matrix. The absolute values of \( |Oz| \) in which \( L \) is the length of the string and \( \sigma \) operator denotes a spin-1/2 Pauli matrix. The \( \Delta \) in even-Haldane phase; (b) \( J_H = 1, J_I = 1.6 \) and \( D = 2 \) (a point in the even-Haldane phase). \( L = |j - i| \) is the lattice distance is \( L = 2(j - i + 1) \) in both cases.

In figure 6, we show the results of both the odd-string \( O_z^{\text{odd}} \) and even-string \( O_z^{\text{even}} \) order parameters at two representative points (\( J_H = 1, J_I = 1.2 \) and \( D = 2 \)) and (\( J_H = 1, J_I = 1.6 \) and \( D = 2 \)), which locate at distinct nonmagnetic phases (figure 1). From figure 6, we find that \( O_z^{\text{odd}} \) and \( O_z^{\text{even}} \) show different behaviors at these two points: the order parameter \( O_z^{\text{even}} \) (\( O_z^{\text{odd}} \)) in the odd-Haldane (even-Haldane) phase decay quickly to zero within the lattice distance \( L = 50 \) (see figure 6); On the other hand, \( O_z^{\text{odd}} \) (\( O_z^{\text{even}} \)) in the odd-Haldane (even-Haldane) phase oscillates, whose absolute values converge to nonzero values in the large \( L \) limit (figure 6). Collecting the converged (absolute) values of \( O_s^{\text{odd}} \) and \( O_s^{\text{even}} \), shown in figure 7, we can clearly see that \( O_s^{\text{odd}} \) and \( O_s^{\text{even}} \) serve as well-defined order parameters of odd- and even-Haldane phases, respectively. In addition, in the vicinity of the critical point, we find the (converged) string orders obey a power-law scaling versus parameter \( \delta \equiv |J_I - J_s| \):

\[
O_s^z \sim \delta^{2\beta},
\]

where \( \beta = 1/12 \) (see figures 8(a) and (b)).

Subsequently, we analyze the entanglement spectrum (ES) [33], which is defined as the -\( \log_2 \rho \), where \( \rho \) is the reduced density matrix of half-infinite chain. In the MPS framework, the ES can be evaluated through \( -2 \log_2 \Lambda_i^z \) with \( \Lambda_i^z \) the geometric bond space supports a projective representation of global symmetries, including the bond centered inversion symmetry [13], the rotational SO(3) symmetry [35], etc.

3.3. Block entanglement entropy and central charge

From the phase diagram shown in figure 1, one can find that there exist four different phases and all the QPTs on the phase boundaries separating them are found to be critical. In order to study their criticalities, in figures 10 and 11 we show the block entanglement entropy \( S_L \), which measures the entanglement between a block of \( L \) spins and the remaining (infinite) environment. The block entanglements of gapped phases are found to be saturated when \( L \) is large enough, well...
class and they can be described by a free fermionic field theory. Among them, P1, P2 and P3 are critical points between the odd- and even-Haldane phases. The block entanglement entropies of both points are fitted well by function k + c Log2 L and their central charges (c) are determined to be about 1/2.

logarithmic divergent behaviors are also observed at P4 and P5 (see figure 11), but with central charges determined as c ≥ 1, showing that the topological phase transition between the odd- and even-Haldane phases belongs to the Gaussian universality class and may be described by a free bosonic field theory.

3.4. Revisit of the HIAC with DM interactions on odd bonds only

In this subsection, we switch off the DM interaction on even bonds in Hamiltonian (2) and thus reduce the model as

\[ \hat{H} = \sum_{i} [J_{i} S_{2i-1} S_{2i} + D (S_{2i-1}^{x} S_{2i}^{x} - S_{2i-1}^{y} S_{2i}^{y}) + J_{i} S_{2i}^{x} S_{2i+1}^{x}]. \]
As shown in [22–24], this model is exactly solvable. Recently Derzhko et al calculated the ground-state compressibility of a deformable spin-1/2 Heisenberg–Ising chain with DM interaction on odd bonds only [40]. In figure 12, we revisit its ground-state phase diagram through numerical simulations. Two phase boundaries detected by the sharp peaks of bipartite entanglement locate at $J_{c}^{f} = \pm 2\sqrt{D^2 + J_{H}^2}$, which perfectly agrees with analytical result [24]. As $|J_{1}| < 2\sqrt{D^2 + J_{H}^2}$, it is a disordered phases with nonzero $O_{s}^{c,even}$ but with vanishing $O_{s}^{c,odd}$. Furthermore, the doubly degenerate ES is observed on the odd bonds. These two facts, nonzero $O_{s}^{c,odd}$ and doubly degenerate ES, indicate that the disordered phase is an odd-Haldane phase. In the region $J_{1} < -2\sqrt{D^2 + J_{H}^2}$, an AFSP is observed; while an AF phase appears when $J_{1} > 2\sqrt{D^2 + J_{H}^2}$.

From the Hamiltonian (9), we realize that the case with positive $J_{1}$ can also be connected to that with negative $J_{1}$ by a unitary transformation. More specifically, the Hamiltonian with $-J_{1}$ and that with $J_{1}$ can be mutually transformed by the $\pi$-rotation of the spins at sites $4i-3$ and $4i-2$ (or equivalently at $4i-1$ and $4i$) around the $x$- or $y$-axis. For instance, if all the spins at sites $4i-3$ and $4i-2$ are rotated about $x$-axis by $\pi$, one can get

$$S_{4i-3}^{x} \rightarrow -S_{4i-3}^{x}, \quad S_{4i-3}^{y} \rightarrow -S_{4i-3}^{y}, \quad S_{4i-2}^{x} \rightarrow -S_{4i-2}^{x}, \quad S_{4i-2}^{y} \rightarrow S_{4i-2}^{y}.$$  

After this unitary transformation, the new Hamiltonian is identical to the old one, but with $-J_{1}$. Therefore, when all the spins at sites $4i-3$ and $4i-2$ are rotated over $x$-axis by $\pi$, the spin configuration $\cdots + - - + + - + \cdots$ of the AF phase in the region $J_{1} > 2\sqrt{D^2 + J_{H}^2}$ will become an AFSP configuration $\cdots - - - + + - + \cdots$ in the region $J_{1} < -2\sqrt{D^2 + J_{H}^2}$, in accordance with our numerical observation.

3.5. Evolution of the even-Haldane phase

From the phase diagrams in figures 1 and 12, we find that an even-Haldane phase will be induced as the DM interactions on even bonds are turned on. In order to explore the evolution of such an even-Haldane phase, we introduce the following Hamiltonian with alternating DM interactions to interpolate between the two limiting cases:

$$\hat{H} = \sum_{i} J_{1} \hat{S}_{2i-1}^{x} \hat{S}_{2i}^{x} + D \left( S_{2i-1}^{z} S_{2i}^{z} - S_{2i-1}^{z} S_{2i}^{z} \right) + J_{1} S_{2i-1}^{z} S_{2i}^{z} + \gamma D \left( S_{2i-1}^{z} S_{2i}^{z} - S_{2i-1}^{z} S_{2i}^{z} \right).$$  

(11)

The $\gamma$ denotes the relevant strength of the DM interactions on even bonds compared with that on odd bonds and thus the values of interest are restricted within the range $[0,1]$. When $\gamma = 0$, it reduces to the Hamiltonian (9) with DM interactions on odd bonds only and turns into Hamiltonian (2) when $\gamma = 1$. In addition to the studied cases of $\gamma = 0$ and 1, we also select $\gamma = 0.2, 0.3, 0.5$ and 0.8 to demonstrate the evolution of the even-Haldane phase. The corresponding phase diagrams by tuning $D$ and $J_{1}$ are provided in figures 13(a)–(d), respectively. Since the effect of the even-bond DM interactions on the AFSP is negligible (see figures 1 and 12), we just focus on the region $J_{1} > 0$. We find that, an even-Haldane phase can be induced as long as the DM interactions on even bonds are taken into account. When $\gamma$ is small, the even-Haldane phase locates in the region with very large $D$ and $J_{1}$ (figures are not shown here). When $\gamma$ increases, the even-Haldane phase expands quickly while the AF and the odd-Haldane phases shrink. As $\gamma$ is large enough, this even-Haldane phase comes into the region with small $D$ and $J_{1}$ (see figures 13(a)–(d)). Because the even-Haldane phase, the odd-Haldane phase and AF phase are adjacent to each other, a tricritical point exists between them. As $\gamma$ increases, such a tricritical point moves towards the region with small $D$ and $J_{1}$ and eventually the phase diagram with small $D$ and $J_{1}$ consequently (see figures 13(c) and (d)). It is worth noting that the exact position of such a tricritical point is difficult to be determined precisely.

4. Discussions and conclusion

The ground-state phase diagram and the QPTs in the spin-1/2 HIAC with uniform DM interactions have been investigated by the MPS method. By calculating the odd- and even-string order parameters, two kinds of Haldane phases, i.e. odd- and even-Haldane phases, have been identified. Furthermore, doubly degenerate entanglement spectra on odd and even bonds are observed in odd- and even-Haldane phases, respectively. A rich phase diagram including four different phases, i.e. an AFSP, an AF phase, odd- and even-Haldane phases, have been obtained. The TQPT between the odd- and even-Haldane phases is with central charge $c = 1$ (a Gaussian type phase transition) and with critical exponent $\beta = 1/2$. The central charges on the other phase boundaries are $c = 1/2$, therefore the QPTs from nonmagnetic (two Haldane) phases to the
magnetic ordered (AFSP and AF) phases belong to the Ising universality class.

In addition, the bipartite entanglement entropy has been shown as a very powerful tool for capturing QPTs (including TQPTs). Two kinds of nonlocal string orders and the doubly degenerate ES can be used to distinguish the odd- and even-Haldane phases. The nonzero string order parameters imply the breaking of the hidden topological symmetry in the odd- and even-Haldane phases. It is worth noticing that the existence of the odd-Haldane phase has been already proved for the Heisenberg–Ising chain without Dzyaloshinskii–Moriya interaction [41].

Acknowledgments

This work is supported by the Chinese National Science Foundation under Grant No. 11347008 and No. 11474211. It is also partially supported by the National Basic Research Program of China under Grant No. 2012CB932900. W-LY acknowledges support by the Natural Science Foundation of Jiangsu Province of China under Grant No. BK20141190. WL was also supported by the DFG through SFB-TR12 and NIM.

References

[1] Mourigal M, Enderle M, Klüppelpieper A, Caux J S, Stunault A and Ronnow H M 2013 Nat. Phys. 9 435
[2] White S R and Huse D A 1993 Phys. Rev. B 48 3844
[3] Haldane F D M 1983 Phys. Lett. A 93 464
[4] Affleck I, Kennedy T, Lieb E H and Tasaki H 1987 Phys. Rev. Lett. 59 799
[5] Nijs M den and Rommelse K 1989 Phys. Rev. B 40 4709
[6] Hong-Hao T, Guang-Ming Z and Tao X 2008 J. Phys. A: Math. Theor. 41 415201
[7] Hagiwara M, Matsunaga K, Affleck I, Halperin B I and Renard J P 1990 Phys. Rev. Lett. 65 3181
[8] Glarum S H, Geschwind S, Lee K M, Kaplan M L and Michel J 1991 Phys. Rev. Lett. 67 1614
[9] Endres M et al 2011 Science 334 200
[10] Landau L D 1937 Zh. Eksp. Teor. Fiz. 7 19
[11] Landau L D and Lifschitz E M 1958 Statistical Physics, Course of Theoretical Physics vol 5 (London: Pergamon)
[12] Gu Z C and Wen X G 2009 Phys. Rev. B 80 155131
[13] Pollmann F, Turner A M, Berg E and Oshikawa M 2010 Phys. Rev. B 81 064439
[14] Pollmann F, Berg E, Turner A M and Oshikawa M 2012 Phys. Rev. B 85 075125
[15] Chen X, Gu Z C, Liu Z X and Wen X G 2012 Science 338 1604
[16] Tasaki H 1991 Phys. Rev. Lett. 66 798
[17] Kennedy T and Tasaki H 1992 Phys. Rev. B 45 304
[18] Oshikawa M 1992 J. Phys.: Condens. Matter 4 7469
[19] Totsuka K and Suzuki M 1995 J. Phys.: Condens. Matter 7 1639
[20] Hida K 1992 Phys. Rev. B 45 2207
[21] Hai-Tao W and Cho S C 2015 J. Phys.: Condens. Matter 27 015603
[22] Lieb E, Schultz T and Mattis D 1961 Ann. Phys. 16 407
[23] Yao H, Li J and Gong C D 2002 Solid State Commun. 121 687
[24] Strečka J, Gălisovă L and Derzhko O 2010 Acta Phys. Pol. A 118 742
[25] Vidal G 2003 Phys. Rev. Lett. 91 147902
[26] Vidal G 2007 Phys. Rev. Lett. 98 070201
[27] Xufeng Z, Tianyu L, Flatté M E and Tang H X 2014 Phys. Rev. Lett. 113 037202
[28] You W L and Dong Y L 2011 Phys. Rev. B 84 174426
[29] Amico L, Fazio R, Osterloh A and Vedral V 2008 Rev. Mod. Phys. 80 517
[30] Liu G H, Wang H L and Tian G S 2008 Phys. Rev. B 77 214418
[31] You W L 2014 J. Phys. A: Math. Theor. 47 255301
[32] Liu G H, Li W, You W L, Tian G S and Su G 2012 Phys. Rev. B 85 184422
[33] Li H and Haldane F D M 2008 Phys. Rev. Lett. 101 010504
[34] Motamedifar M, Mahdavifar S and Shayesteh S F 2011 J. Supercond. Novel Magn. 24 769
[35] Li W, Weichselbaum A and von Delft J 2013 Phys. Rev. B 88 245121
[36] Eisert J, Cramer M and Plenio M B 2010 Rev. Mod. Phys. 82 277
[37] Holzhey C, Larsen F and Wilczek F 1994 Nucl. Phys. B 424 443
[38] Calabrese P and Cardy J 2009 J. Phys. A: Math. Theor. 42 504005
[39] Vidal G, Latorre J I, Rico E and Kitaev A 2003 Phys. Rev. Lett. 90 227902
[40] Derzhko O, Strečka J and Gălisovă L 2013 Eur. Phys. J. B 86 88
[41] Liu G H, Li W, Su G and Tian G S 2014 Eur. Phys. J. B 87 105