Distributions of Time Headways in Particle-hopping Models of Vehicular Traffic

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Abstract

We report the first analytical calculation of the distribution of the time headways in some special cases of a particle-hopping model of vehicular traffic on idealized single-lane highways and compare with the corresponding results of our computer simulation. We also present numerical results for the time-headway distribution in more general situations in this model. We compare our results with the empirical data available in the literature.

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Significant progress has been made in recent years in understanding the dynamical phases of vehicular traffic using particle–hopping models [1]; these are analogous to the "microscopic" models of interacting classical particles studied in statistical mechanics. The time interval between the arrivals at (or departures from) a detector site of two successive vehicles is defined as the time headway between this pair of vehicles [2]. The distribution of time headways is one of the most important characteristics of vehicular traffic and the aim of this letter is to calculate this distribution, for a wide range of densities of vehicles, in the steady state of the Nagel-Schreckenberg (NS) model [3] of traffic on highways.

In the NS model a lane is represented by a one-dimensional lattice of $L$ sites. Each of the lattice sites can be either empty or occupied by at most one "vehicle". If periodic boundary condition is imposed, the density $c$ of the vehicles is $N/L$ where $N(\leq L)$ is the total number of vehicles.

In the NS model [3] the speed $V$ of each vehicle can take one of the $V_{\text{max}} + 1$ allowed integer values $V = 0, 1, \ldots, V_{\text{max}}$. At each discrete time step $t \rightarrow t + 1$, the arrangement of $N$ vehicles is updated in parallel according to the following "rules". Suppose, $V_n$ is the speed of the $n$-th vehicle at time $t$.

*Step 1: Acceleration.* If, $V_n < V_{\text{max}}$, the speed of the $n$-th vehicle is
increased by one, i.e., \( V_n \rightarrow V_n + 1 \).

**Step 2: Deceleration (due to other vehicles).** If \( d \) is the gap in between the \( n \)-th vehicle and the vehicle in front of it, and if \( d \leq V_n \), the speed of the \( n \)-th vehicle is reduced to \( d - 1 \), i.e., \( V_n \rightarrow d - 1 \).

**Step 3: Randomization.** If \( V_n > 0 \), the speed of the \( n \)-th vehicle is decreased randomly by unity (i.e., \( V_n \rightarrow V_n - 1 \)) with probability \( p \) (\( 0 \leq p \leq 1 \)); \( p \), the random deceleration probability, is identical for all the vehicles and does not change during the updating.

**Step 4: Vehicle movement.** Each vehicle is moved forward so that \( X_n \rightarrow X_n + V_n \) where \( X_n \) denotes the position of the \( n \)-th vehicle at time \( t \).

We shall present our analytical calculations for \( V_{\text{max}} = 1 \) and compare these with the corresponding results of our computer simulation. As analytical calculations are too complicated to carry through for \( V_{\text{max}} > 1 \), we have computed the time-headway distributions for all \( V_{\text{max}} > 1 \) only through computer simulation.

For the convenience of our analytical calculations, we change the order of the steps in the update rules in such a manner that does not influence the steady-state properties of the model [4]. We assume the sequence of steps
$2 - 3 - 4 - 1$, instead of $1 - 2 - 3 - 4$; the advantage is that there is no vehicle with $V = 0$ immediately after the acceleration step. Consequently, if $V_{\text{max}} = 1$, we can then use a binary site variable $\sigma$ to describe the state of each site; $\sigma = 0$ represents an empty site and $\sigma = 1$ represents a site occupied by a vehicle whose speed is unity.

We label the position of the detector by $j = 0$, the site immediately in front of it by $j = 1$, and so on. The detector clock resets to $t = 0$ everytime a vehicle leaves the detector site. We begin our analytical calculations for $V_{\text{max}} = 1$ by assuming that the probability of a time headway $t$, $\mathcal{P}(t)$, between a ”leading” vehicle (LV) and the ”following” vehicle (FV) is given by

$$\mathcal{P}(t) = \sum_{t_1=1}^{t-1} P(t_1)Q(t - t_1)$$ (1)

where $P(t_1)$ is the probability that there is a time interval $t_1$ between the departure of the LV and the arrival of the FV at the detector site and $Q(t_2)$ is the probability that the FV halts at the detector site for $t_2$ time steps.

To calculate $P(t_1)$, we have to consider spatial configurations at $t = 0$ for which the FV can reach the detector site within $t_1$ steps. This means considering configurations up to the maximum separation of $t_1$ sites from the detector site so that even the farthest vehicle can reach the detector site with
$t_1$ hops. The configurations of interest are thus of the form

\[ \underbrace{100 \cdots 0}_{n} | 0 \]  

\[ n = 1, 2, \cdots t_1. \]

The underlined zero implies that we have to find the conditional steady-state probability for the given configuration subject to the condition that the underlined site (detector site) is empty. This probability, $\Pi(n)$, in the 2-cluster approximation is given by [4,5]

\[ \Pi(n) = C(1|0) \{C(0|0)\}^{n-1}, \quad (2) \]

where, $C$ gives the 2-cluster steady-state configurational probability for the argument configuration and the underlined imply the conditional, as usual.

The expressions for the various $C$s are given by [4,5]

\[ C(0|0) = C(0|0) = 1 - \frac{y}{d} \quad (3) \]

\[ C(0|1) = C(1|0) = \frac{y}{c} \quad (4) \]

\[ C(1|0) = C(0|1) = \frac{y}{d} \quad (5) \]

\[ C(1|1) = C(1|1) = 1 - \frac{y}{c} \quad (6) \]

where

\[ y = \frac{1}{2q} \left( 1 - \sqrt{1 - 4qcd} \right), \quad (7) \]
\( q = 1 - p \) and \( d = 1 - c \).

For all configurations with \( t_1 > n \), \( t_1 - 1 \) time steps elapse in crossing \( n - 1 \) bonds (as the last bond is crossed certainly at the last time step). Thus

\[
P(t_1) = \sum_{n=1}^{t_1} \Pi(n)q^n p^{t_1-n} t_1^{-1} C_{n-1} \tag{8}
\]

Using the expression (2) for \( \Pi(n) \) in equation (8) we get

\[
P(t_1) = C(1|0)q [C(0|0)q + p]^{t_1-1} \tag{9}
\]

Next we calculate \( Q(t_2) \) by considering all configurations of the form

\[
\ldots \ldots \underline{1} 1 0
\]

at \( t = t_1 \). Here, the underlined part implies that the detector site is occupied by the FV and we consider configurations with a queue of \((m-1)\) vehicles ahead of it and the foremost vehicle of this queue has an empty site ahead.

The steady-state probability of this configuration is given by

\[
\Pi'(m) = C(1 \underline{1} \ldots \underline{1} 0) = \{C(1|1)\}^{m-1} C(1|0) \tag{10}
\]

in 2-cluster approximation. To find all possible configurations of this type that contribute to the waiting time of \( t_2 \) time-steps, we have to consider queue sizes upto \( t_2 \) \((m \leq t_2)\). Thus, using expression (10) for \( \Pi'(m) \) in

\[
Q(t_2) = \sum_{m=1}^{t_2} \Pi'(m)q^m p^{t_2-m} t_2^{-1} C_{m-1}
\]
we get

\[ Q(t_2) = C(\|0)q \left[ qC(\|1) + p \right]^{t_2 - 1}. \]  

(11)

The time-gap distribution, \( P(t) \), can now be calculated using equations (1), (9) and (11) and is given by

\[ P(t) = \frac{q^2C(\|0)C(\|0) \left[ \frac{[C(0\|1)q + p]^{t-1} - [qC(\|1) + p]^{t-1}}{[C(0\|1)q + p] - [qC(\|1) + p]} \right]}{c - d} \left( 1 - \frac{yq}{c} \right)^{t-1} \left( 1 - \frac{yq}{d} \right)^{t-1} \]  

(12)

where the final result (eq.(12)) was obtained using equations (3)-(6) and \( y \) is given by eq. (7).

In the approximate calculation of \( Q(t_2) \) we ignored the fact that, for \( V_{max} = 1 \), the LV is certainly present at \( j = 1 \) at \( t = 0 \). If the LV is still at site \( j = 1 \) at \( t > t_1 \), when the FV is at \( j = 0 \), it hinders the forward movement of the FV; therefore, we now recalculate \( P(t) \) incorporating leading corrections. The LV itself may be hindered by its predecessor halting at site \( j = 2 \) and we get a hierarchy of terms. We assume that the probability of a \( m \)-sized queue ahead of, and including the LV, is given by an expression identical to equation (10). Using arguments similar to the one used earlier for deriving equation (11), one can show that the probability of the LV hopping out of site \( j = 1 \) at \( t = t_3 \) is given by \( Q(t_3) \) (equation (11)).
(9) for $P(t_1)$ remains unaffected but the equation (1) gets modified by this correction. The expression for the total probability (i.e. $P(t)$) is now given by

$$P(t) = \sum_{t_1=1}^{t-1} \left\{ qp^{t-t_1-1} P(t_1) S(t_1) + q P(t_1) R(t_1, t - t_1) \right\}$$

(13)

where

$$S(t_1) = Q(1) + Q(2) + \cdots + Q(t_1) = \sum_{t_3=1}^{t_1} Q(t_3)$$

(14)

and

$$R(t_1, t_2) = Q(t_1+1)p^{t_2-2} + Q(t_1+2)p^{t_2-3} + \cdots + Q(t_1+t_2-1) = \sum_{t_3=t_1+1}^{t_1+t_2-1} Q(t_3)p^{t_1+t_2-t_3-1}$$

(15)

The first series on the RHS of equation (13) accounts for the situation where the LV hops out of the site $j = 1$ before the FV arrives at the site $j = 0$ and the second series accounts for the situation where the FV arrives at $j = 0$ before the LV hops out of the site $j = 1$. The general term in equation (15) means that the LV halts for $t_3$ time-steps ($t_3 > t_1$), the FV being blocked at site $j = 0$ for ($t_3 - t_1$) time-steps, contributing only a factor of $Q(t_3)$. After the LV hops out of site $j = 1$ (at $t = t_3$), the FV continues to halt at site $j = 0$ up to $t = t_1 + t_2$ by braking, picking up a factor $p^{t_1+t_2-t_3-1}$. 
Using the expression (11) for $Q(t_3)$ in (14) and (15) we get

$$S(t_1) = 1 - [C(1|1)q + p]^{t_1}$$

(16)

and

$$R(t_1, t_2) = \frac{C(1|0)}{C(1|1)} [C(1|1)q + p]^{t_1} \left[ p^{t_2-1} - [C(1|1)q + p]^{t_2-1} \right]$$

(17)

So, finally, using (16) and (17) in (13) the total probability distribution $\mathcal{P}(t)$ can be written as

$$\mathcal{P}(t) = \left[ q \frac{C(1|0)}{C(0|0)} \right] B^{t-1} + \left[ q \frac{C(1|0)}{C(1|1)} \right] A^{t-1}$$

$$+ \left[ \frac{q^2 A C(1|0)}{p - AB C(1|1)} - q \frac{C(1|0)}{C(1|1)} \right] (AB)^{t-1}$$

$$- \left[ \frac{q^2 A C(1|0)}{p - AB C(1|1)} + q \frac{C(1|0)}{C(0|1)} \right] p^{t-1}$$

(18)

where $A = 1 - qC(1|0)$, $B = 1 - qC(1|0)$ and equation (18) was obtained using equations (3)–(6) and $y$ is given by eq. (7).

The equation (18) is in excellent agreement with the corresponding results of our computer simulation (see figs.1 (a) and (b)). Moreover, the distribution for the densities $c$ and $1 - c$ are identical; this is a consequence of the particle-hole symmetry in the problem when $V_{\text{max}}$ is unity.

One of the simplest dynamical models of interacting particle systems is the so-called asymmetric simple exclusion process (ASEP) [6] which is
sometimes regarded as a caricature of vehicular traffic; in this model one
particle is picked at random and moved forward by one lattice site if the new
site is empty. In this random sequential update, it is the random picking
that introduces stochasticity (noise) into the model. Although speed of the
particles do not explicitly enter into the rules, the effective speed of a particle
in the ASEP can take only two values, namely, 0 and \( V_{\text{max}} = 1 \). Since
mean-field theory is known to give exact results for ASEP, the time-headway
distribution for ASEP can be obtained from eq.(18) by using the mean-field
approximations \( C(\downarrow|1) = C(\uparrow|1) = c = C(1|\downarrow) = C(1|\uparrow) \) and \( C(\downarrow|0) = C(\uparrow|0) = 1 - c = C(0|\downarrow) = C(0|\uparrow) \).

We present our numerical data for the time-headway distribution in the
NS model with \( V_{\text{max}} = 5 \) in fig.2 as several earlier works have demonstrated
that this particular choice of \( V_{\text{max}} \) leads to quite realistic qualitative descrip-
tions of some other features of vehicular traffic on highways. The qualitative
features of the distribution in fig.2 are similar to those in fig.1, except that
\( P(t = 0) = 0 \) but \( P(t = 1) \) need not vanish when \( V_{\text{max}} = 5 \) in contrast to the fact that \( P(t \leq 1) = 0 \), irrespective of the density of the vehicles, when
\( V_{\text{max}} = 1 \).

We have plotted the most probable time-headway \( T_{\text{mp}} \) as a function of
the density in fig.3; while the curve is symmetric on the two sides of \( c = 1/2 \) when \( V_{max} = 1 \) no such symmetry is exhibited for \( V_{max} > 1 \). The common qualitative trend of variation of \( T_{mp} \) is consistent with one’s intuitive expectation that both at very low and very high densities there are long time gaps in between the departures of two successive vehicles from a given site.

Finally, the time-headway distributions and the trend of their variation with density are in qualitative agreement with the corresponding empirical data [2]. The time headway distribution in the NS model can be approximated well with the forms \( P(t) \propto (t - \tau)^\lambda e^{-\mu t} \) with \( \tau = 1 \) for \( V_{max} = 1 \) and \( \tau = 0 \) for \( V_{max} = 5 \) where the parameters \( \lambda \) and \( \mu \) depend on the density of the vehicles [7]. Our results demonstrate that, in spite of being only a minimal model, the NS model captures the essential qualitative features of the time-headway distribution of vehicular traffic on highways.

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Figure Captions:

**Fig.1:** The time-headway distribution in the steady-state of the NS model with \( V_{\text{max}} = 1 \) \((p = 0.5)\) for the densities (a) \( c = 0.1(+), c = 0.25(\times) \) and \( c = 0.5 \) \((*)\). (b) \( c = 0.9(+), c = 0.75(\times) \) and \( c = 0.5 \) \((*)\). The continuous curves represent the analytical result while the discrete data points have been obtained from computer simulation.

**Fig.2:** The time-headway distribution in the steady-state of the NS model with \( V_{\text{max}} = 5 \) \((p = 0.5)\) for the densities \( c = 0.10(+), 0.25(\times), 0.50(*), 0.75(\square), 0.90(\blacksquare) \). The discrete data points have been obtained from computer simulation while the continuous curves are merely guides to the eye.

**Fig.3:** The most probable time-headway plotted as a function of the density of the vehicles in the steady-state of the NS model. The full line corresponds to \( V_{\text{max}} = 1 \) and has been obtained from the expression (18); the corresponding data obtained from our computer simulation have been represented by the symbol +. The discrete data points (represented by the symbol \( \times \)) correspond to \( V_{\text{max}} = 5 \) and have been obtained from computer simulation; the dotted line joining these data points serves merely as a guide to the eye.