Symmetry approach in boundary value problems

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Abstract

The problem of construction of the boundary conditions for nonlinear
equations is considered compatible with their higher symmetries. Boundary
conditions for the sine-Gordon, Jiber-Shabat and KdV equations are
discussed. New examples are found for the JS equation.

1 Introduction

The subject of applications of classical Lie symmetries to boundary value
problems is well studied (see the monography [1]). In contrast, the question of
involving higher symmetries to the same problem has received much less atten-
tion, unlike say the Cauchy problem. However, one should stress that nowadays
the higher symmetries’ approach becomes the basis of the modern integrabil-
ity theory [2]. A number of attempts to apply the inverse scattering method
(ISM) to the initial boundary value problem has been undertaken. It turned
out that if both initial data and boundary value are chosen arbitrary then the
ISM loses essentially its efficiency. On the other hand side the investigation by
E.Sklyanin [3] based on the $R-$ matrix approach demonstrated that there is a
kind of boundary conditions, compatible completely with the integrability. The
analytical aspects of such kind problems were studied in [4], [5]. After [6] it be-
comes clear that boundary value problems found can effectively be investigated
with the help of the Bäcklund transformation.

Below we will discuss a higher symmetry test, proposed in [7], [8] to ver-
ify whether the boundary condition given is compatible with the integrability
property of the equation. It is worthwhile to note that all known classes of
boundary conditions, compatible with integrability occur to pass this symmetry
test. Boundary conditions involving explicit time dependance for the Toda lat-
tice compatible with higher symmetries has recently been studied in [9]. It was
established there that finite dimensional systems obtained from the Toda lattice
by imposing at both ends boundary conditions consistent with symmetries were nothing else but Painlevé type equations.

Let us consider the evolution type equation

$$u_t = f(u, u_1, u_2, \ldots, u_n)$$

(1)

and a boundary condition of the form

$$p(u, u_1, u_2, \ldots, u_k)|_{x=0} = 0,$$

(2)

imposed at the point $x = 0$. Here $u_i$ stands for the partial derivative of the order $i$ with respect to the variable $x$. Suppose that the equation given possesses a higher symmetry

$$u_\tau = g(u, u_1, \ldots, u_m).$$

(3)

We call the problem (1)-(2) compatible with the symmetry (3) if for any initial data prescribed at the point $t = 0$ a common solution to the equations (1), (3) exists satisfying the boundary condition (2). Let us explain more exactly what we mean. Evidently one can differentiate the constraint (2) only with respect to the variables $t$ and $\tau$, (but not respect to $x$). For instance, it follows from (2) that

$$\sum_{i=0}^{n} \frac{\partial p}{\partial u_i} (u_i)_\tau = 0,$$

(4)

where one should replace $\tau$-derivatives by means of the equation (3). The boundary value problem (1)-(2) be compatible with the symmetry (3) if the equation (4) holds identically by means of the condition (2) and its consequences obtained by differentiation with respect to $t$.

To formulate an effective criterion of compatibility of the boundary value problem with a symmetry it’s necessary to introduce some new set of dynamical variables consisting of the vector $v = (u, u_1, u_2, \ldots u_{n-1})$ and its $t$-derivatives: $v_t, v_{tt}, \ldots$. Passing to this set of variables allows one really to exclude the dependance on the variable $x$. In terms of these variables the symmetry (3) and the constraint (2) take the form

$$v_\tau = G(v, v_t, v_{tt}, \ldots \frac{\partial^{m_1} v}{\partial t^{m_1}}),$$

(5)

$$P(v, \frac{\partial v}{\partial t}, \ldots \frac{\partial^{k_1} v}{\partial t^{k_1}}) = 0.$$  

(6)

The following criterion of compatibility was established in ([8]).

**Theorem.** The boundary value problem (1)-(2) is compatible with the symmetry (3) if and only if the differential connection (3) is consistent with the system (3).
We call the boundary condition \( (2) \) compatible with the integrability property of the equation \( (1) \), if the problem \( (1)-(2) \) is compatible with infinite series of linearly independent higher order symmetries.

The problem of the classification of integrable boundary conditions is solved completely for the Burgers equation (see \[8\])

\[
\frac{\partial u}{\partial t} = u^2 + 2uu_1, \quad (7)
\]

**Theorem.** If the boundary condition \( p(u, u_1)|_{x=0} = 0 \) is compatible at least with one higher symmetry of the Burgers equation \( (7) \) then it is compatible with all even order homogeneous symmetries and is of the form \( c_1(u_1+u^2) + c_2u + c_3 = 0 \).

In the Burgers case the boundary conditions of the general form \( (2) \) can also be described completely with the help of the ”recursion operator for the boundary conditions” \( L = \frac{\partial}{\partial x} + u \), which acts on the set of integrable boundary conditions (see \[10\]). For instance, the boundary condition \( L(c_1(u_1+u^2) + c_2u + c_3) = c_1(u_2 + 3uu_1 + u^3) + c_2(u_1 + u^2) + c_3 u = 0 \) is also integrable.

Let us describe boundary value problems of the form

\[
\alpha(u, u_x)|_{x=0} = 0 \tag{8}
\]

\[
u_{tt} - u_{xx} + \sin u = 0 \tag{9}
\]

for the sine-Gordon equation compatible with the third order symmetry.

As it is shown in \[11\] the complete algebra of higher symmetries for the equation \( (9) \) i.e. \( u_{\xi\eta} = \sin u \), where \( 2\xi = x + t \), \( 2\eta = x - t \) splits into the direct sum of two algebras consisting of symmetries of equations \( u_\tau = u_{\xi\xi\xi} + u_\xi^3/2 \), \( u_\tau = u_{\eta\eta\eta} + u_\eta^3/2 \), correspondingly, which are nothing else but potentiated MKdV equation. Particularly, the following flow commutes with the sine-Gordon equation

\[
u_\tau = c_1(u_{\xi\xi\xi} + u_\xi^3/2) + c_2(u_{\eta\eta\eta} + u_\eta^3/2). \tag{10}
\]

The symmetry \( (10) \) isn’t compatible with any boundary condition of the form \( (8) \) unless \( c_1 = -c_2 \), under this constraint the equation \( (8) \) is of one of the forms

\[
u = \text{const}, \quad v = c_1 \cos(u/2) + c_2 \sin(u/2). \tag{11}
\]

Note that the list of boundary conditions \( (11) \) coincides with that found by A.Zamolodchikov within the framework of the R matrix approach \[12\]. The latter in \( (11) \) in particular cases was studied earlier in \[3\] and \[5\]. The compatibility of the former in \( (11) \) with the usual version of ISM was declared earlier in \[3\]. But the statement was based in a mistake (see \[13\]). Our requirement of consistency is weaker than that is used in \[3\]. Applications of these and similar problems for the sine-Gordon equation and the affine Toda lattice in the quantum field theory are studied in \[14\] and \[15\].
According to the theorem above one reduces the problem of finding integrable boundary conditions to the problem of looking for differential connections admissible by the following system of equations, equivalent to (10) with \( c_1 = -c_2 \) and \( v = u_x \):

\[
\begin{align*}
  u_\tau &= 8u_{ttt} + 6u_t \cos u + 3v^2 u_t + u_t^3, \\
  v_\tau &= 8v_{ttt} + 6v_t \cos u + 6uvu_t + 3u^2 v_t + 3u_t^2 v_t.
\end{align*}
\]

(12)

One can prove that the boundary conditions (11) are compatible with rather large subclass of the sine-Gordon equation such that

\[
  u_\tau = \phi(u, u_1, \ldots, u_k) - \phi(u, \bar{u}_1, \ldots, \bar{u}_k),
\]

(13)

where \( u_j = \partial^j u / \partial \xi^j \), \( \bar{u}_j = \partial^j u / \partial \eta^j \), and the equation \( u_\tau = \phi_i(u, u_1, \ldots, u_k) \), \( i = 1, 2 \) is a symmetry of the equation \( u_\tau = u_{\xi\xi\xi} + u_\xi^3/2 \).

Another well-known integrable equation of hyperbolic type

\[
  u_{tt} - u_{xx} = \exp(u) + \exp(-2u) \tag{14}
\]

has applications in geometry of surfaces. For the first time it was found by Tzitzeica [16]. The presence of higher symmetries for this equation has been established by A.Jiber and A.Shabat [11]. The simplest higher symmetry of this equation is of the fifth order

\[
  u_\tau = u_{\xi\xi\xi\xi\xi} + 5(u_{\xi\xi}u_{\xi\xi\xi} - u_\xi^2 u_{\xi\xi} - u_\xi u_{\xi\xi}^2) + u_\xi^5. \tag{15}
\]

It is proved in the article cited that the symmetry algebra for (14) is the direct sum of the symmetry algebras of (15) and of the equation obtained from (15) by replacing \( \xi \) by \( \eta \).

Let us look for boundary conditions of the form

\[
  a(u, u_x) = 0, \tag{16}
\]

for the equation (14), compatible with the symmetry (17) (and then compatible with integrability) are either of the form \( u_x + c\exp(-u)|_{x=0} = 0 \) or \( u_x + c\exp(u/2) \pm \exp(-u)|_{x=0} = 0 \), where \( c \) is arbitrary.

**Theorem.** Boundary conditions (16) for the Jiber-Shabat equation compatible with the symmetry (17) (and then compatible with integrability) are either of the form \( u_x + c\exp(-u)|_{x=0} = 0 \) or \( u_x + c\exp(u/2) \pm \exp(-u)|_{x=0} = 0 \), where \( c \) is arbitrary.

Notice that all equations above are invariant under the reflection type symmetry \( x \rightarrow -x \). It is unexpected that equations which don’t admit any reflection symmetry admit nevertheless boundary conditions compatible with integrability. For instance, the famous KdV equation
\[ u_t = u_{xxx} + 6u_xu \]  

is consistent with the boundary condition

\[ u = 0 |_{x=0}, \quad u_{xx} |_{x=0} = 0. \]  

It implies immediately that the boundary value problem

\[ u_t = u_{xxx} + 6u_xu, \quad u = 0 |_{x=0} \]

with the Dirichlet type condition at the axis \( x = 0 \) admits an infinite dimensional set of "explicit" finite-gap solutions.

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