A counterexample for equivalence result between tails behavior and Grand Lebesgue Spaces norms

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Abstract

We bring in this short report (counter-)examples on order to show a difference between tails behavior and Grand Lebesgue Spaces norms for some classes of random variables.

Key words and phrases: Random variable and random vector (r.v.), centered (mean zero) r.v., saddle-point method, examples and counterexamples, tail and bilateral tail estimates, rearrangement invariant Banach space of random variables, tail of distribution, Lorentz norms and spaces, moments, Lebesgue-Riesz, Orlicz and Grand Lebesgue Spaces (GLS); slowly varying functions, Tchebychev-Markov inequality, tail function, Young-Fenchel transform, theorem and inequality of Fenchel-Moreau, Young-Orlicz function, norm, Markov-Tcher nov’s estimate, non-asymptotical estimates, Cramer’s condition.

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1 Definitions. Notations. Previous results. Statement of problem.

A. $B(\phi)$ spaces.

Let $(\Omega = \{\omega\}, F, P)$ be certain sufficiently rich probability space.
Let also $\phi = \phi(\lambda), \lambda \in (-\lambda_0, \lambda_0), \lambda_0 = \text{const} \in (0, \infty]$ be certain even strong convex which takes positive values for positive arguments twice continuously differentiable function, briefly: Young-Orlicz function, such that

$$\phi(0) = \phi'(0) = 0, \phi''(0) > 0, \lim_{\lambda \to \lambda_0} \phi(\lambda)/\lambda = \infty.$$ (1.1)

For instance: $\phi(\lambda) = 0.5\lambda^2, \lambda_0 = \infty$; is the so-called subgaussian case.
We denote the set of all these Young-Orlicz function as $\Phi$; $\Phi = \{ \phi(\cdot) \}$. 
**Definition 1.1.** (See [20], [7].)

We say by definition that the centered random variable (r.v) \( \xi = \xi(\omega) \) belongs to the space \( B(\phi) \), if there exists certain non-negative constant \( \tau \geq 0 \) such that

\[
\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \max_{\pm} \mathbb{E} \exp(\pm \lambda \xi) \leq \exp[\phi(\lambda \tau)]. \tag{1.2}
\]

The minimal non-negative value \( \tau \) satisfying (1.2) for all the values \( \lambda \in (-\lambda_0, \lambda_0) \), is named a \( B(\phi) \) norm of the variable \( \xi \), write

\[
||\xi||_{B(\phi)} \overset{df}{=} \inf\{\tau, \tau > 0 : \forall \lambda : |\lambda| < \lambda_0 \Rightarrow \max_{\pm} \mathbb{E} \exp(\pm \lambda \xi) \leq \exp(\phi(\lambda \tau))\}. \tag{1.3}
\]

These spaces are very convenient for the investigation of the r.v. having a exponential decreasing tail of distribution, for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous and weak compactness of random fields, study of Central Limit Theorem in the Banach space etc. The detail investigation of these spaces may be found in [2], [7]-[8], [9], [12]-[13], [15], [17], [18], [19], [20] etc.

The space \( B(\phi) \) with respect to the norm \( || \cdot ||_{B(\phi)} \) and ordinary algebraic operations is a rearrangement invariant Banach space in the classical sense [1], chapters 1,2; which is in turn isomorphic to the subspace consisting on all the centered variables of Orlicz’s space \( (\Omega, F, P), N(\cdot) \) with \( N - \) function

\[
N(u) = \exp(\phi^*(u)) - 1, \tag{1.4}
\]

where

\[
\phi^*(u) \overset{df}{=} \sup_{\lambda \in \text{Dom}(\phi)} (\lambda u - \phi(\lambda)). \tag{1.5}
\]

The transform \( \phi \rightarrow \phi^* \) is called Young-Fenchel, or Legendre transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Moreau:

\[
\phi^{**} = \phi.
\]

Let \( F = \{\xi(s)\}, s \in S, S \) is an arbitrary set, be the family of somehow dependent mean zero random variables. The function \( \phi(\cdot) \) may be constructive introduced by the formula

\[
\phi(\lambda) = \phi_F(\lambda) \overset{df}{=} \max_{s \in S} \ln \sup_{s \in S} \mathbb{E} \exp(\pm \lambda \xi(s)), \tag{1.6}
\]

if obviously the family \( F \) of the centered r.v. \( \{\xi(s), s \in S\} \) satisfies the so-called uniform Cramer’s condition:

\[
\exists \mu \in (0, \infty), \sup_{s \in S} T_{\xi(s)}(y) \leq \exp(-\mu y), \ y \geq 0. \tag{1.7}
\]
Hereafter the symbol $T_\xi = T_\xi(x)$ will be denote the so-called tail function for the r.v. $\xi$:

$$T_\xi(x) \overset{\text{def}}{=} \max(P(\xi \geq x), P(\xi \leq -x)), \ x \geq 0.$$ 

In this case, i.e. in the case the choice the function $\phi(\cdot)$ by the formula (1.6), we will call the function $\phi(\lambda) = \phi_0(\lambda)$ as a natural function, and correspondingly the function

$$\lambda \mapsto E e^{\lambda \xi}$$ 

is named often as a moment generating function (briefly: MGF) for the r.v. $\xi$, if of course there exists in some non-trivial neighborhood of origin.

If for example $\phi = \phi_2(\lambda) = 0.5 \lambda^2$, $\lambda \in R$, then the r.v. from the space $B\phi_2$ are named subgaussian. This notion was introduced by J.P.Kahane; see [2]; see also [7] - [8], [12] - [13], [17]. One can consider also the case when

$$\phi_m(\lambda) = m^{-1} |\lambda|^m, |\lambda| \geq 1, \ m = \text{const} > 1,$$

as well as a more general case

$$\phi_{m,L}(\lambda) = m^{-1} |\lambda|^m L(\lambda), \ |\lambda| \geq 1, \ m = \text{const} > 1,$$

where $L(\cdot)$ is arbitrary positive continuous slowly varying as $\lambda \to \infty$ function, see [7] - [8].

**B. Grand Lebesgue $G(\psi)$ spaces.**

Let now $\psi = \psi(p), \ p \in [1,b), \ b = \text{const} \in (1,\infty]$ be certain bounded from below: $\inf \psi(p) > 0$ continuous inside the semi-open interval $p \in [1,b)$ numerical valued function. We can and will suppose without loss of generality

$$\inf_{p \in [1,b)} \psi(p) = 1 \quad (1.8)$$

and $b = \sup\{p, \psi(p) < \infty\}$, so that $\text{supp} \psi = [1,b)$ or $\text{supp} \psi = [1,b]$. The set of all such a functions will be denoted by $\Psi(b) = \{\psi(\cdot)\}; \ \Psi := \Psi(\infty)$.

**Definition 1.2.** (See [20], [7]-[9].)

By definition, the (Banach) Grand Lebesgue Space (GLS) $G\psi = G\psi(b)$ consists on all the real (or complex) numerical valued measurable functions (random variables, r.v.) $f : \Omega \to R$ defined on our probability space and having a finite norm

$$|| f || = ||f||_{G\psi} \overset{\text{def}}{=} \sup_{p \in [1,b)} \left[ \frac{|f|_p}{\psi(p)} \right]. \quad (1.9)$$

Here and in what follows the notation $|f|_p$ denotes an ordinary Lebesgue-Riesz $L_p(\Omega)$ norm for the r.v. (measurable function) $f$:
\[ |f|_p \overset{\text{def}}{=} [\mathbb{E}|f|^p]^{1/p}, \quad p \geq 1. \]

The function \( \psi = \psi(p) \) is said to be (also) the \textit{generating function} for this space.

If for instance \( \psi(p) = \psi_r(p) = 1, \quad p \in [1, r] \), where \( r = \text{const} \in [1, \infty) \), (an extremal case), then the correspondent \( G\psi_r(p) \) space coincides with the classical Lebesgue - Riesz space

\[ ||\xi||_{G\psi_r} = |\xi|_r, \quad r \in [1, \infty). \]

Furthermore, let now \( \eta = \eta(z), \quad z \in S \) be arbitrary family of random variables defined on any set \( z \in S \) such that

\[ \exists b = \text{const} \in (1, \infty], \quad \forall p \in [1, b) \Rightarrow \psi_S(p) := \sup_{z \in S} |\eta(z)|_p < \infty. \]

The function \( p \to \psi_S(p) \) is named as a \textit{natural} function for the family of random variables \( S \).

Obviously,

\[ \sup_{z \in S} ||\eta(z)||_{G\Psi_S} = 1. \]

The family \( S \) may consists on the unique r.v., say \( \Delta : \)

\[ \psi_\Delta(p) := |\Delta|_p, \]

if of course the last function is finite for some value \( p = p_0 > 1 \).

\section*{C. Tail inequalities for both the considered spaces.}

1. Let \( \xi \) be non-zero random variable from the space \( B(\phi), \quad \phi \in \Phi : \quad K := ||\xi||B(\phi) \in (0, \infty) \). It follows immediately by virtue of Tchebychev-Tchernov inequality

\[ T_\xi(x) \leq \exp(-\phi^*(x/K)), \quad x > 0, \quad (1.10) \]

see [7], [8], [12].

2. Assume that \( \psi(\cdot) \in \Psi(b), \quad b = \text{const} \in (1, \infty] \), and that the non-zero r.v. \( \eta \) belongs to the space \( G(\psi) : \quad V = ||\eta||G\psi \in (0, \infty) \). Denote

\[ h(p) = h[\psi](p) \overset{\text{def}}{=} p \ln \psi(p) \quad (1.11) \]

Then

\[ T_\eta(x) \leq \exp(-h^*(x/V)), \quad x > 0. \quad (1.12) \]
Both the inequalities (1.10) and (1.12) may be rewritten in the terms of (generalized) Lorentz spaces as follows. Let $S(x), x \geq 0$ be some tail function, i.e. left continuous numerical valued decreasing function $S = S(x)$ such that

$$S(0+) = 1, \quad S(\infty) = 0.$$  

A particular cases:

$$S[\phi](x) := \exp(-\phi^*(x)), \quad x > 0;$$

$$S_\psi(x) := \exp(-h[\psi]^*(x)), \quad x > 0.$$  

Define following, e.g., [1], chapter 4; [5], [10]-[11] the generalized Lorentz quasi-norm $||\zeta||_{L(S)}$ for arbitrary r.v. $\zeta$ as follows.

$$||\zeta||_{L(S)} \overset{def}{=} \sup_{x \geq 0} \left[ \frac{T_\zeta(x)}{S(x)} \right].$$  

(1.13)

We have the following copies of (1.10) and (1.12)

$$||\xi||_{L(S[\phi])} \leq ||\xi||_{B(\phi)}$$  

(1.14)

and

$$||\eta||_{L(S_\psi)} \leq ||\eta||_{G\psi}.$$  

(1.15)

It emerges the following natural question: to what extent is the converse inequalities to ones (1.14), (1.15), i.e. when

$$||\xi||_{B(\phi)} \leq C_1(\phi) ||\xi||_{L(S[\phi])}$$  

(1.16)

and

$$||\eta||_{G\psi} \leq C_2(\psi)||\eta||_{L(S_\psi)}$$  

(1.17)

for some finite constants $C_1(\phi), C_2(\psi)$.  

Throughout this paper, the letters $C, C_j(\cdot)$ etc. will denote a various positive finite constants which may differ from one formula to the next even within a single string of estimates and which does not depend on the essentially variables $p, x, \lambda, \eta, u$ etc.

We make no attempt to obtain the best values for these constants.

We represent for beginning some results concerning both the inverse estimates (1.6), (1.17).

It is proved in particular in the article [8], Theorem 4.1, that if

$$R[\psi] := \sup_{p \in [1, \infty)} \left[ h^*[\psi](p) \right]^{1/p} < \infty,$$  

(1.18)
then
\[ ||\eta||G\psi \leq 2 R[\psi] e^{1/e} ||\eta||L(S\psi) < \infty, \] (1.19)
i.e. the inequality (1.17).

Let us turn our attention on the relation (1.16). We show here briefly the result from the aforementioned article [8].

In order to carry out this, we assume here \( \lambda_0 = \lambda_0[\phi] = \infty \), and define for any positive finite constant value \( C_1 \) the function
\[ \theta[\phi](\lambda) = \theta(\lambda) \overset{def}{=} \frac{C_1}{\lambda \phi^*(\lambda)} \] (1.20)
for all the sufficiently greatest values \( \lambda : \lambda \geq e \), (say). We introduce also the following integral
\[ Z[\phi](\lambda) = Z(\lambda) := \int_0^\infty e^{-\theta(\lambda)} \phi^*(x) \, dx. \] (1.21)

Suppose
\[ \exists C_3 = C_3(C_1) = \text{const} < \infty, \ \forall \lambda > e \Rightarrow Z[\phi](\lambda) \leq \exp \phi^*(C_3\lambda). \] (1.22)

Then the inequality (1.16) holds true.

**Examples.**

**Example 1.** Let \( L = L(\lambda), \ x > 0 \) be positive twice continuous differentiable slowly varying at infinity regular in the following sense
\[ \lim_{\lambda \to \infty} \frac{L(\lambda/L(\lambda))}{L(\lambda)} = 1 \]
function. Define also for sufficiently greatest values \( \lambda \), say for \( |\lambda| \geq 1 \), the function of the form
\[ \phi_{m,L}(\lambda) \overset{def}{=} m^{-1} |\lambda|^m L^{1/q}(|\lambda|^m), \]
m = const > 1, q = m/(m − 1); and as usually
\[ \phi_{m,L}(\lambda) \overset{def}{=} C(m, L) \lambda^2, |\lambda| < 1. \]

It is proved in [8] in particular that for these \( \Phi \) – function the equality (1.16) there holds. Namely, the inclusion
\[ \xi \in B(\phi_{m,L}) < \infty, \ \xi \neq 0 \]
is quite equivalent to the following tail estimate
\[ \exists C_2 \in (0, \infty) \Rightarrow T_\xi(x) \leq \exp \left( -C_2 \ q^{-1} x^q L^{(q-1)}(x^{q-1}) \right), \ x \geq 1. \]
Example 2. Define the other Grand Lebesgue Space space $G_{\psi, C, \beta}$ of random variables with correspondent generating function

$$\psi_{C, \beta}(p) \overset{def}{=} e^{C p^\beta}, \quad p \in [1, \infty),$$

where $\beta, C = \text{const} > 0$. It is proved in [8] in particular that for these $\Psi$ function the equality (1.17) again there holds.

In detail, the following implication there holds

$$\xi \in \bigcup_{C > 0} G_{\psi, C, \beta} \iff$$

$$\exists K \in (0, \infty), \quad T_\xi(x) \leq \exp \left( -K \left( \ln(x + 1)^{1+1/\beta} \right) \right), \quad x \geq 0.$$

Note that in general case the MGF for arbitrary r.v. $\nu$ with

$$||\eta||_{G_{\psi, C, \beta}} \in (0, \infty)$$

does not exists; on the other words this variable does not satisfy the Cramers condition.

Example 3. Define the following $\Psi$ function

$$\psi_m(p) \overset{def}{=} p^{1/m}, \quad p \in [1, \infty); \quad m = \text{const} > 0.$$

The non-zero r.v. $\xi$ belongs to the space $G_{\psi_m}$:

$$||\xi||_{G_{\psi_m}} = \sup_{p \geq 1} \left[ \frac{||\xi||_p}{p^{1/m}} \right] < \infty$$

if and only if

$$\exists C_3 \in (0, \infty) \Rightarrow T_\xi(x) \leq \exp \left( -C_3 x^m \right), \quad x \geq 1.$$

Our purpose in this short article is to show by means of building of suitable counterexamples that both the estimates (1.16) and (1.17) are not true if the conditions correspondingly relations (1.18) and (1.21) are not satisfied.

Obtained here results generalized ones in [8].

2 Main results: counterexamples.

A. Grand Lebesgue Space case.

I. Let us introduce the following $\Psi$ function
\[
\psi^{b,\gamma}(p) = (b - p)^{-\gamma/b}, \; p \in [1, b),
\]
where \(b = \text{const} > 1, \; \gamma = \text{const} > 0.\)

Suppose the non-zero r.v. \(\xi\) belongs to the space \(G^{b,\gamma}\); one can conclude without loss of generality \(||\xi||_{G^{b,\gamma}} = 1.\) We find my means of simple calculations based on the estimate (1.12)

\[
T_\xi(x) \leq C(b, \gamma) \left[ b - x \right]^{-\gamma/b} \ln x, \; x \geq e.
\]

II. Conversely, consider the r.v. \(\zeta\) with tail behavior of a type (2.2):

\[
T_\zeta(x) = x^{-b} \left( \ln x \right)^{\gamma}, \; x \geq x_0 = \text{const} \geq e.
\]

We have as \(p \to b - 0\)

\[
C_3^{-1} p^{-1} ||\zeta||_p^p \sim \int_{1}^{\infty} \frac{x^{p-1}}{x^{b-p} \left( \ln x \right)^{\gamma}} \, dx = \int_{0}^{\infty} e^{-x} \left( \frac{y^{\gamma}}{(b-p)^{\gamma+1}} \right) \, dy = \Gamma(\gamma + 1) \frac{(b-p)^{\gamma+1}}{(b-p)^{\gamma+1}};
\]

\[
||\zeta||_p \asymp (b - p)^{-(\gamma+1)/b}, \; p \in [1, b).
\]

Thus, the r.v. \(\zeta\) belongs to the space \(G^{b,\gamma+1}\), but it does not belongs to the space \(G^{b,\gamma}\):

\[
||\zeta||_{G^{b,\gamma}} = \infty.
\]

For illustration: consider the space \(G^{\psi(r)}\); if for some r.v. \(\zeta ||\zeta||_{G^{\psi(r)}} = 1, \; r = \text{const} \geq 1, \) or equally \(||\zeta||_r = 1, \) then

\[
T_\zeta(x) \leq \frac{1}{x^r}, \; x \geq 1;
\]

but the inverse conclusion is obviously not true.

B. \(B(\phi)\) space case.

I. Let us introduce the following \(\Phi\) function

\[
\phi_{b,\gamma}(\lambda) := \frac{b}{b - |\lambda|} \ln \left[ \frac{b}{b - |\lambda|} \right]; \; |\lambda| \in [1, b),
\]

where \(\gamma = \text{const} > 0; \; b = \text{const} > 1.\)

If the non-zero random variable \(\tau\) belongs to the space \(B(\phi_{b,\gamma})\), for instance when

\[
||\tau||_{B(\phi_{b,\gamma})} = 1,
\]
then

\[ T_\tau(x) \leq C x^\gamma e^{-b x}, \quad x \geq e. \tag{2.7} \]

II. Inversely, let the r.v. \( \theta \) has a tail behavior of a type (2.7):

\[ T_\theta(x) = C x^\gamma e^{-b x}, \quad x \geq e. \tag{2.8} \]

Then as \( \lambda \to b - 0, \; \lambda \in [1, b) \)

\[ \mathbb{E} e^{\lambda \theta} \sim \lambda \int_0^\infty e^{-(b-\lambda)x} x^\gamma dx = \]

\[ \frac{\lambda \Gamma(\gamma + 1)}{(b - \lambda)^{\gamma + 1}} \asymp \left( \frac{b}{b - \lambda} \right)^{\gamma + 1}. \tag{2.9} \]

Therefore, the r.v. \( \theta \) does not belongs to the space \( B(\phi_{b,\gamma}) \).

3 Concluding remarks.

A. It is interest by our opinion to obtain the generalization of results of this report into the multidimensional case, i.e. into random vectors, alike in the articles [8], [13].

B. We mention even briefly an important possible application of obtained results: a Central Limit Theorem in Banach spaces, in the spirit of [15], [14], [16], [12], section 4.1.

C. Authors have not the correspondent counterexamples in the case when \( b = \infty \). This is an open problem.

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