A Visual-inertial Navigation Method for High-Speed Unmanned Aerial Vehicles

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Abstract

This paper investigates the localization problem of high-speed high-altitude unmanned aerial vehicle (UAV) with a monocular camera and inertial navigation system. And it proposes a navigation method utilizing the complementarity of vision and inertial devices to overcome the singularity which arises from the horizontal flight of UAV. Furthermore, it modifies the mathematical model of localization problem via separating linear parts from nonlinear parts and replaces a nonlinear least-squares problem with a linearly equality-constrained optimization problem. In order to avoid the ill-condition property near the optimal point of sequential unconstrained minimization techniques (penalty methods), it constructs a semi-implicit continuous method with a trust-region technique based on a differential-algebraic dynamical system to solve the linearly equality-constrained optimization problem. It also analyzes the global convergence property of the semi-implicit continuous method in an infinity integrated interval other than the traditional convergence analysis of numerical methods for ordinary differential equations in a finite integrated interval. Finally, the promising numerical results are also presented.

Keywords: vision odometry, monocular camera, unmanned aerial vehicle, differential-algebraic gradient flow, semi-implicit continuation method, trust-region technique

AMS subject classifications. 65H17, 65J15, 65K05, 65L05

1. Introduction

Localization is essential for autonomous navigation of unmanned aerial vehicles. In terms of aircraft navigation, the aircraft usually uses an inertial integrated navigation method to guide flight due to the unsatisfactory effect of pure inertial navigation system (INS) (see [33]). The visual-based navigation method has received widespread
attention in recent years for its great performance in this field. Therefore, we utilize
visual odometer which is complementary to inertial measurement to assist INS.

Visual-based methods are often applied in low speed and low height situations,
resulting from such drawback as motion blur. Being different from others, our work
tries to solve the localization problem of high-speed unmanned aerial vehicles (UAVs)
in high altitude. In order to overcome the motion blur and scale ambiguity that arise in
this challenging practical problem, a novel visual based method combining the inertial
navigation system is designed.

In the case of challenging camera dynamics, the imaging of landmarks inevitably
appears blurred. According to the principle of pinhole imaging, we consider an addi-
tional error to the angle of view for the landmarks imaging and the optical center of
camera lens, which is determined by the property of the camera. Note that the angular
error has a great impact on the position estimation of the aircraft due to the extremely
high flight altitude. For the scale ambiguity, rather than assume the homography, we
use the altitude difference of the aircraft in a short time interval measured by altimeter
to determine the height of aerial vehicle.

For a real engineering problem, a UAV flight trajectory is usually relatively simple.
In particular, when a UAV flies on a horizontal plane, the flight altitude difference
with reading error within a short time interval will be intensely small. Generally, that
scenario leads to the singularity even using an altimeter to help determine scale. The
singularity results in rapid accumulation of errors. In order to overcome its singularity,
we add an inertial distance between two sequential frames to assist visual localization.

Furthermore, we modify the mathematical model of visual-inertial localization
problem via separating linear parts from nonlinear parts and replace a nonlinear least-
squares problem with a linearly equality-constrained optimization problem. In order to
avoid the ill-condition property near the optimal point of sequential unconstrained min-
imization techniques (penalty methods [12, 30]), we construct a semi-implicit continu-
ous method with a trust-region technique based on a differential-algebraic dynamical
system to solve the linearly equality-constrained optimization problem. We also ana-
lyze the global convergence property of the proposed semi-implicit continuous method
in an infinity integrated interval other than the traditional convergence analysis of nu-
merical methods for ordinary differential equations in a finite integrated interval.

Finally, in order to validate the effectiveness of our proposed method, we adopt real
parameters provided by China Aerospace Science and Industry Corporation to mimic
the real flight environment and compare it with the pure inertial navigation method
[36]. The simulation results show that our proposed method has better performance,
and it meets the required accuracy in a long-term flight.

The rest of the paper is organized as follows. Firstly, we present the related work
in section 2. Then, applied environment and sensors fusion architecture of our visual-
inertial navigation method are described in section 3. In section 4, we modify the
mathematical model and give a semi-implicit continuous method with a trust-region
 technique to solve that optimization problem. The simulation results of our method in
comparison to the pure inertial navigation method are presented in section 5. Finally,
we give some discussions in section 6.

2. Related Work

In recent years, numerous methods have been applied to improve the precision of navigation. Among of navigation systems, the strap-down inertial navigation system (SINS) has great performance on pose estimation for its advantages of complete autonomy, strong anti-interference ability and high short-term precision \[32, 14, 35\]. However, the inertial measurement unit (IMU) which is the main component of SINS has an unavoidable cumulative error caused by sensor drifts \[1, 11\]. Therefore, many aerial vehicles utilize the position signal from GPS which fuses data generated by IMU to implement high navigation accuracy.

On the other hand, in many scenarios, GPS is difficult to play its role. In civil applications, GPS becomes inaccurate due to multipath effect as close to buildings and obstacles. In military field, such as ballistic missile vehicles, the position applications generally do not rely on GPS by reason of jamming and spoof \[9\]. In GPS denied environments, vision-based approach is an available and effective method, and visual-inertial odometer is ubiquitously applied on robots and unmanned aerial vehicles (UAVs) for its great performance of pose estimation and the complementarity between cameras and IMU \[6, 7, 9, 10, 24, 26, 29, 38\].

For the high altitude problem, we choose a monocular vision odometer to assist INS rather than the binocular camera, since the binocular camera reduces to a monocular camera when vehicles fly at high altitude as a result of the extremely small baseline-to-depth ratio \[34\]. If there is no additional information, scale ambiguity of a monocular camera can not be cleared up generally. Anwar et al. design a new depth-independent Jacobian matrix by relating the depth information with the area of region of interest \[2\]. In \[7\], Conte and Doherty consider the ground as flat and horizontal for aircraft flying at a relatively high altitude. Caballero et al. also assume the local ground flat but not level \[8\]. Both of them utilize planer homography to tackle the vehicle motion. In this paper, we give a method which does not require the local ground flat and does not rely on planar homography.

Zhang and Singh combine a high-accuracy INS and vision to estimate the position of a full-scale aircraft flying at an altitude of about 300 meters. They partially eliminate the effects of INS high-frequency noise through virtually rotating the camera parallel to local ground by reparametrizing features with their depth direction perpendicular to the ground \[38\]. Unlike Zhang and Singh, our method deal with the singular problem in the special case where the position of two frames have no altitude difference. In addition, we separate the nonlinear terms from the linear terms, and convert it to a linearly constrained optimization problem, rather than directly adopt the Levenberg-Marquardt method to solve nonlinear least-square problem. Furthermore, for that linearly equality-constrained optimization subproblem, we give a semi-implicit continuous method with a trust-region technique to solve it.
3. Sensors Fusion Architecture

In this paper, we focus our attention on the issue: solving the navigation problem of high speed and high altitude aircraft under the horizontal flight scenario. The navigation simulation is illustrated by Figure 1.

![Figure 1: Navigation Simulation of Straight Line](image)

The visual-inertial odometry, which is composed by a monocular vision system, an INS and an altimeter, is aimed to estimate the aircraft position and guide the flight of aerial vehicles. We consider the camera fitting a pinhole model briefly, and ignoring the lens distortion [15]. The Camera intrinsics parameters are given. As a convention, the 3D coordinate system denotes the real world as shown in Figure 1 and the symbol \( k, k \in \mathbb{Z}^+ \) denotes image frames. Besides the image coordinate system, another coordinate system is a 2D coordinate system with its origin being perpendicular to the optical center of camera lens, as shown in Figure 3. In the relatively difficult practical issue, velocity of the aircraft is between 200 meters per second and 300 meters per second, and the aerial vehicle flies at an altitude between 1000 meters and 1500 meters. Since the aerial vehicle flies with an extremely fast speed, the motion blur should be carefully considered. In order to reduce the influence of the blur, we use a camera to assist localization and add an angular error to the angle of view between camera and landmarks, about 0.2 degrees.

The sensor fusion architecture of the visual-inertial odometry is demonstrated in Figure 2. The odometer takes the camera images, altimeter reading from the altimeter and velocity from the INS. Combining those information, our method can acquire the vehicle position with low drift in the horizontal flight. When the aerial vehicle adjusts
its orientation, the angle of rotation is obtained by the IMU. Through acquiring the navigation information, reaching the destination along the scheduled route can be achieved with required accuracy.

In order to improve the navigation accuracy of ballistic missiles, we propose a method which utilizes monocular camera to assist the inertial navigation system. A monocular vision odometer typically has scale ambiguity. This scale ambiguity can be confirmed by the barometer through measuring the flying altitude. On the other hand, a vision odometer can suppress the cumulative error caused by IMU drift. Generally, taking full advantage of the complementarity of visual odometer and inertial navigation, the accuracy of navigation is improved.

4. Mathematical Model and Algorithm Descriptions

4.1. Mathematical Model

This subsection is aimed to illustrate the mathematical model of localization which is abstracted from practical problems. In that mathematical model, the positions of camera and landmarks are in the world coordinate. The sequence of frames is presented in parallel coordinate. Let the position of optical center in the $k^{th}$ frame as $(x_k, y_k, z_k)$, and the $(k + 1)^{th}$ frame as $(x_{k+1}, y_{k+1}, z_{k+1} + \delta h_{k+1}^{k+1})$, where $\delta h_{k+1}^{k+1}$ is the height difference between two frames, obtained by an altimeter. The position of the $n$-th landmark, confirmed by ORB feature [27], is denoted as $(x_{ln}, y_{ln}, z_{ln})$. We denote the location of the corresponding pixel imaged by the $n^{th}$ landmark in the $k^{th}$ frame as $(x_{pn}, y_{pn})$. The vertical distance between the $n^{th}$ landmark and the optical center of the camera of $k^{th}$ frame is expressed as $h_{kn}$. $f_i$ is the focal length of the camera. The precedent notations are shown in Figure 3. Figure 3 presents the mathematic model of our visual odometer method appropriately.

This subsection shows the mathematical derivation of our proposed visual-inertial odometer method. In subsection 4.4, the complete algorithm is presented. From the model in Figure 3 we find the following relationships between the $n^{th}$ landmark and
the corresponding projection in two frames:

\[
\begin{align*}
\frac{x_k - x_{ln}}{h_{kn}} &= \frac{x_{pn}}{f_c}, \\
\frac{y_k - y_{ln}}{h_{kn}} &= \frac{y_{pn}}{f_c}, \\
\frac{x_{k+1} - x_{ln} - \delta h_k^{k+1}}{h_{kn}^{k+1}} &= \frac{x_{pn}}{f_c}, \\
\frac{y_{k+1} - y_{ln} - \delta h_k^{k+1}}{h_{kn}^{k+1}} &= \frac{y_{pn}}{f_c}.
\end{align*}
\]

(1)

The above relationship (1) is reformulated by

\[
\begin{align*}
x_{ln} + \frac{x_{pn}}{f_c} h_k^k &= x_k, \\
y_{ln} + \frac{y_{pn}}{f_c} h_k^k &= y_k, \\
x_{k+1} - x_{ln} - \frac{x_{pn}}{f_c} h_k^{k+1} &= \frac{\delta h_k^{k+1}}{f_c} x_{pn}, \\
y_{k+1} - y_{ln} - \frac{y_{pn}}{f_c} h_k^{k+1} &= \frac{\delta h_k^{k+1}}{f_c} y_{pn}.
\end{align*}
\]

(2)

In formula (2), the position of pixel in the camera coordinate and the position of the \(k^{th}\) frame are known, and the rest are unknown. Obviously, this is an underdetermined system and we can not determine the position of the next frame from equations (2). Therefore, we use more landmarks and more frames to determine the position of the next frame. In theory, we can obtain the solution by using only two landmarks. The
corresponding formula is shown as the following equations:

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & \frac{x_{k+1}^0}{f} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{x_{k+1}^1}{f} & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & -\frac{x_{k+1}^0}{f} & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -\frac{x_{k+1}^1}{f} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{x_{k+2}^0}{f} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{x_{k+2}^1}{f} \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & -\frac{x_{k+1}^1}{f} \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & -\frac{x_{k+1}^1}{f}
\end{bmatrix}
\begin{bmatrix}
x_{k+1} \\
y_{k+1} \\
x_1 \\
y_1 \\
x_2 \\
y_2 \\
l_1 \\
l_2
\end{bmatrix}
= \begin{bmatrix}
x_k \\
y_k \\
\delta h_k^{k+1} \\
\delta h_k^{k+1} \\
\delta h_k^{k+1} \\
\delta h_k^{k+1} \\
\delta h_k^{k+1} \\
\delta h_k^{k+1}
\end{bmatrix}, \quad (3)
\]

From linear equations (3), it is not difficult to obtain the position of the next frame for the general case. For the convenience of subsequent presentations, we represent the system of equations (3) as

\[As = b.\]

(4)

Note that the linear system of equations (3) is singular when there is no height difference between two frames. In order to analyze the singularity of the linear system of equations (5), we simplify it and obtain the following equivalent formula:

\[
(x_{k+1}^0 - x_{pn}^0) \frac{h_k^0}{f} - (x_{(n+1)}^0 - x_{pn}^0) \frac{h_{n+1}^0}{f} = (x_{k+1}^0 - x_{pn}^0) \delta h_k^{k+1},
\]

\[
(y_{k+1}^0 - y_{pn}^0) \frac{h_k^0}{f} - (y_{(n+1)}^0 - y_{pn}^0) \frac{h_{n+1}^0}{f} = (y_{k+1}^0 - y_{pn}^0) \delta h_k^{k+1}.
\]

(5)

It is not difficult to see that it does not only determine landmark height variables \(h_k^n\) and \(h_{n+1}^k\) from the linear system of equations (5) when the height difference \(\delta h_k^{k+1} = 0\).

In order to overcome its singularity, we add the distance information between two sequential frames to assist visual localization, which is provided by the accelerometer in INS. Furthermore, we take into account the constant error and the random walk error of the IMU, the angular line error caused by motion blur and the error of the barometer. Then, the visual-inertial odometer problem is modelled as a stochastic constrain optimization problem. The objective function is formulated as follows:

\[
\min \left( (x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2 \right) - \left( (d_k^{k+1})^2 - (\delta h_k^{k+1})^2 \right)^2 + \left( (x_{k+1} - x_{k-1})^2 + (y_{k+1} - y_{k-1})^2 \right) - \left( (d_{k-1}^{k+1})^2 - (\delta h_{k-1}^{k+1})^2 \right)^2.
\]

(6)

For convenience, we denote the above objective function of equation (6) as \(f(s)\).

We use the relationship about five landmarks and three frames, which has a similar form to equation (4), as the constraint condition. Due to the measurement error, the
constraint condition can be denoted as
\[ A \varepsilon s = b \varepsilon, \]  
where \( A \varepsilon \) includes the vision error and the altimeter error, \( b \varepsilon \) is the constant vector with error term. Thus, the odometer problem is reformulated as a random equality-constrained optimization problem. Note that we do not solve the following equivalent nonlinear least-square problem:
\[ \min f(s) + \| A \varepsilon s - b \varepsilon \|^2. \]  
Replacing it, we construct a semi-implicit continuation method with a trust-region technique to directly solve the linearly equality-constrained optimization problem (6)-(7) (see [13, 18, 19] and [20, 21, 22] for the semi-implicit continuation method solving an unconstrained optimization problem and the smallest eigenvalue or generalized eigenvalue problem, respectively).

4.2. Semi-implicit Continuation Method for the Optimization Subproblem

In this subsection, we firstly give a semi-implicit continuation method with a trust-region technique for the linearly equality-constrained optimization subproblem (6)-(7). According to its first-order Karush-Kuhn-Tucker condition
\[ \nabla s L(s, \lambda) = \nabla f(s) + A^T \varepsilon \lambda = 0, \]  
\[ A \varepsilon s = b \varepsilon, \]  
we construct a continuous differential-algebraic gradient flow with index 2 as follows:
\[ \frac{ds}{dt} = -\nabla L(s, \lambda) = -\left( \nabla f(s) + A^T \varepsilon \lambda \right), \]  
\[ A \varepsilon s = b \varepsilon, \]  
where the Lagrangian function is written as
\[ L(s, \lambda) = f(s) + \lambda^T (A \varepsilon s - b \varepsilon). \]  
In order to solve the continuous vector \( \lambda(t) \) in (11)-(12), via differentiating its algebraic constraint (12) in variable \( t \) and using its differential equation (11), we obtain
\[ A \varepsilon \frac{ds}{dt} = -A \varepsilon \left( \nabla f(s) + A^T \varepsilon \lambda \right) = -A \varepsilon \nabla f(s) - A \varepsilon A^T \varepsilon \lambda = 0. \]  
Assuming that matrix \( A \varepsilon \) has a full row rank, from (14), we know that \( \lambda \) satisfies
\[ \lambda = -\left( A \varepsilon A^T \varepsilon \right)^{-1} A \varepsilon \nabla f(s). \]  
Replacing \( \lambda \) in (11) with equation (15), we obtain the ordinary differential gradient flow as follows:
\[ \frac{ds}{dt} = -\left( I - A \varepsilon (A^T \varepsilon) ^+ \right) \nabla f(s), \]
where \((A^T_ε)^+ = (A_εA^T_ε)^{-1}A_ε\) is the generalized inverse of matrix \(A^T_ε\). In other words, we also obtain the continuous projection gradient flow in references [5, 31] via another approach.

We denote

\[ P = I - A^T_ε (A^T_ε)^+ . \]  

Then, it is not difficult to verify \(P^2 = P\), i.e., \(P\) is a projection matrix and \(P\) is an orthogonal projector onto the null space \(\mathcal{N}(A)\). Using this property and from (16)-(17), we obtain

\[
\frac{df(s)}{dt} = -\nabla f(s)^T \frac{ds}{dt} = - (\nabla f(s))^T P \nabla f(s) = - (P \nabla f(s))^T P^2 \nabla f(s)
\]

\[
= - (P \nabla f(s))^T (P \nabla f(s)) = - \|P \nabla f(s)\|_2^2 \leq 0,
\]

namely, the objective function \(f(s)\) is decreasing along the solution \(s(t)\) of the continuous dynamical system (16). Furthermore, Tanabe [31] and Schropp [28] proves that the solution \(s(t)\) tends to \(s^*\) as \(t \to \infty\), where \(s^*\) satisfies the first-order Krasul-Kuhn-Tucker condition (9)-(10). Thus, We can expect to obtain an approximation solution of (6)-(7) via following the trajectory of the ordinary differential dynamical system (16) or the differential-algebraic dynamical system (11)-(12).

For the system of differential-algebraic equations (11)-(12), we look the algebraic equation (12) as a degenerate differential equation. Then, applying the implicit Euler method to the total system, we obtain (see [4, 16, 37] for the implicit Euler method)

\[
s_{k+1} = s_k - \Delta t_k (\nabla f(s_{k+1}) + A^T_ε \lambda_{k+1}),
\]

\[
A_ε s_{k+1} = b_ε.
\]

Replacing \(\nabla f(s_{k+1})\) with its first-order approximation \(\nabla f(s_k) + \Delta t_k \nabla^2 f(s_k)\) and \(\lambda_{k+1}\) with \(\lambda_k\) in (18), respectively, we obtain the predicted variable \(s^p_{k+1}\) of the \((k + 1)\)th iteration variable \(s_{k+1}\) as follows:

\[
\left( \frac{1}{\Delta t_k} I + G_k \right) d_k = - p_{s_k},
\]

\[
s^p_{k+1} = s_k + d_k.
\]

where \(G_k = \nabla^2 f(s_k)\) and \(p_{s_k} = \nabla f(s_k) + A^T_ε \lambda_k\). Since the predicted point \(s^p_{k+1}\) will escape from the constraint plane (12), we pull it back via a projection method as follows:

\[
\min \| s - s^p_{k+1} \|
\]

\[
\text{s.t. } A_ε s = b_ε.
\]

Using the Lagrangian multiplier method (see p. 479, [3]), it is not difficult to obtain the solution of the projection problem (22) as follows:

\[
s_{k+1} = s^p_{k+1} + A^T_ε (A_εA^T_ε)^{-1} (b_ε - A_ε s^p_{k+1}).
\]
Since $s_k$ is in the constraint plane (12), from equations (21) and (23), we have

$$s_{k+1} = s_k + d_k + A_e^T (A_e A_e^T)^{-1} (A_e s_k - A_e s_{k+1})$$
$$= s_k + d_k - A_e^T (A_e A_e^T)^{-1} A_e d_k$$
$$= s_k + \left( I - A_e^T (A_e A_e^T)^{-1} A_e \right) d_k$$
$$= s_k + P d_k,$$
(24)

where projection matrix $P$ is defined by (17).

Using the implicit relationship (15) between the Lagrangian multiplier $\lambda(t)$ and the differential variable $s(t)$, we obtain the Lagrangian multiplier at the $(k+1)^{th}$ iteration

$$\lambda_{k+1} = - (A_e A_e^T)^{-1} A_e \nabla f(s_{k+1}).$$
(25)

Replacing equation (25) into $p_{g_k} = \nabla f(s_k) + A_e^T \lambda_k$, we have

$$p_{g_k} = \nabla f(s_k) + A_e^T \lambda_k = \left( I - A_e^T (A_e A_e^T)^{-1} A_e \right) \nabla f(s_k) = Pg_k,$$
(26)

where $g_k = \nabla f(s_k)$.

Another issue is how to adaptively adjust the time-stepping length $\Delta t_k$ every iteration. We borrow it from the trust-region method for its robust global convergence property and fast local convergence property (see pp. 561-593, [30]). Since variable $s_k$ is feasible, i.e. it always stay in the constraint plane (12) every iteration, the objective function $f(s)$ is a suitable merit function for adjusting the time stepping length $\Delta t_k$ as we use a trust-region technique.

When a trust-region technique is selected, we need to construct an local approximation model around variable $s_k$. According to the traditional approach, we adopt a quadratic model as follows:

$$q_k(s) = (s - s_k)^T \nabla f(s_k) + \frac{1}{2} (s - s_k)^T \nabla^2 f(s_k) (s - s_k).$$
(27)

Now, based on the following measurement ratio

$$\rho_k = \frac{f(s_k) - f(s_{k+1})}{q_k(s_k) - q_k(s_{k+1})},$$
(28)

we give an adaptive adjustment time-stepping length formula

$$\Delta t_{k+1} = \begin{cases} 
\gamma_1 \Delta t_k, & \text{if } 0 \leq |1 - \rho_k| \leq \eta_1, \\
\Delta t_k, & \text{if } \eta_1 < |1 - \rho_k| < \eta_2, \\
\gamma_2 \Delta t_k, & \text{if } |1 - \rho_k| \geq \eta_2,
\end{cases}$$
(29)

where the constants are selected as $0 < \gamma_2 \leq 1/2$, $1 < \gamma_1 \leq 2$, $0 < \eta_1 \leq 0.25$ and $0.75 < \eta_2 < 1$ according to numerical experiments. The specific algorithm steps are shown in algorithm [1].
Algorithm 1 Semi-implicit Continuation Method with Trust-region Technique for Linearly Equality-constrained Optimization

**Input:**
- An objective function: \( f(s) \),
- and the linear constraint: \( A_{e}s = b_e \),
- and the minimum absolute gradient bound of Lagrangian function \( L(s, \lambda) = f(s) + \lambda^T (A_{e}s - b) \): \( \delta_e \).

**Output:**
- The optimal approximation solution \( s^* \).

1: Initialize a point \( s_0 \) and the parameter \( \Delta t_0 \).
2: Choose constants \( \eta_a, \eta_1, \eta_2, \gamma_1, \gamma_2 \) to satisfy
   \( 0 < \eta_a < \eta_1 \leq 1/2 < \eta_2 < 1 \) and \( 0 < \gamma_2 < 1 < \gamma_1 \), such as \( \eta_a = 10^{-6} \), \( \eta_1 = 0.25 \), \( \eta_2 = 0.75 \) and \( \gamma_1 = 2 \), \( \gamma_2 = 0.5 \).
3: \( k \leftarrow 0 \)
4: Compute \( f_0 = f(s_0) \), \( g_0 = \nabla f(s_0) \), \( p_{g_0} = \nabla_s L(s_0, \lambda_0) = g_0 + A_{e}^T \lambda_0 \) and \( G_0 = \nabla^2 L(s_0, \lambda_0) = \nabla^2 f(s_0) \), where the Lagrangian multiplier \( \lambda_0 = -(A_{e}A_{e}^T)^{-1}A_{e}g_0 \).
5: while \( \| g_k \| > \delta_e \) do
6: if \( 1/\Delta t_k I + G_k \succ 0 \) and \( (1/\Delta t_k I + G_k - P^T G_k P) \succ 0 \) then
7: Compute \( d_k \) based on equation (20).
8: Let \( s_k^p = s_k + d_k \) and project \( s_k^p \) to the constraint plane \( A_{e}s = b_e \) by solving problem (22), and obtain \( s_{k+1} \) which is given by equation (23).
9: Compute \( f_{k+1} = f(s_{k+1}) \) and the measurement ratio \( \rho_k \) based on equations (27)–(28).
10: else
11: \( \rho_k = -1 \).
12: end if
13: if \( \rho_k \leq \eta_a \) then
14: \( s_{k+1} = s_k \).
15: else
16: Accept \( s_{k+1} \) and compute \( g_{k+1} = \nabla f(s_{k+1}) \), \( G_{k+1} = \nabla^2 f(s_{k+1}) \), \( \lambda_{k+1} = (A_{e}A_{e}^T)^{-1}A_{e}g_{k+1} \), and the projection gradient \( p_{g_{k+1}} = g_{k+1} + A_{e}^T \lambda_{k+1} \).
17: end if
18: Adjust the time-stepping length \( \Delta t_{k+1} \) based on the trust-region technique (29).
19: Update \( \lambda_k \leftarrow \lambda_{k+1}, s_k \leftarrow s_{k+1}, f_k \leftarrow f_{k+1}, g_k \leftarrow g_{k+1}, G_k \leftarrow G_{k+1}, p_{g_k} \leftarrow p_{g_{k+1}} \) and \( k \leftarrow k + 1 \).
20: end while
4.3. Convergence Analysis of Semi-implicit Continuation Method for optimization subproblem

In this subsection, we give the local and the global convergence properties of the semi-implicit continuation method for the linearly equality-constrained optimization subproblem (i.e. Algorithm 1). Firstly, we give an estimation of upper bounds for the quadratic model $q_k(s_k + Pd)$ which is similar result of the trust-region method for unconstrained optimization problem [25].

Lemma 4.1. Assume that the quadratic model $q_k(s)$ is defined by (27) and $d_k$ is solved by equations (20)~(23). If $(1/\Delta t_k + G_k) > 0$ and $(1/\Delta t_k + G_k - P^T G_k P) > 0$ for some $\Delta t_k > 0$, where projection matrix $P$ is given by equation (17), we have an estimation of lower bounds for the predicted reduction $Pred_k = q_k(s_k) - q_k(s_k + Pd_k)$ as follows:

$$\text{Pred}_k \geq \frac{1}{2} \| p_{rl} \| \min \left\{ \| Pd_k \|, \| p_{rl} \|/\| G_k \| \right\}$$, \hspace{1cm} (30)

where $p_{rl} = \nabla L(s_k, \lambda_k) = \nabla f(s_k) + A_k^T \lambda_k$ and the Lagrange multiplier $\lambda_k$ is determined by equation (25).

Proof. Assume that $d_k$ is the solution of equation (20). Then, we have

$$q_k(s_k) - q_k(s_k + Pd_k) = -\frac{1}{2} d_k^T P^T G_k Pd_k - g_k^T Pd_k$$

$$\geq -\frac{1}{2} d_k^T P^T G_k Pd_k + g_k^T (\mu_k I + G_k)^{-1} p_{rl}$$

$$\geq \frac{1}{2} P_{rl}^T (\mu_k I + G_k)^{-1} p_{rl} + \frac{1}{2} d_k^T (-P^T G_k P + \mu_k I + G_k) d_k, \hspace{1cm} (31)$$

where we denote $\mu_k = 1/\Delta t_k$. From the above equality (31), $(\mu_k I + G_k) > 0$ and selecting a constant $\mu_{lb}$ such that $\mu_{lb} = \min \{ 0, -\lambda_{min} (G_k - P^T G_k P) \}$, where $\lambda_{min} (G_k - P^T G_k P)$ is the smallest eigenvalue of matrix $(G_k - P^T G_k P)$, we obtain

$$q_k(s_k) - q_k(s_k + 1) \geq \frac{1}{2} P_{rl}^T (\mu_k I + G_k)^{-1} p_{rl} + \frac{1}{2} (\mu_k - \mu_{lb}) \| d_k \|^2$$

$$\geq \frac{1}{2} \left( \frac{1}{\mu_k + \| G_k \|} \| p_{rl} \|^2 + (\mu_k - \mu_{lb}) \| d_k \|^2 \right). \hspace{1cm} (32)$$

Now we consider the properties of the function

$$\varphi(\mu) \equiv \mu \| d_k \|^2 + \frac{1}{\mu + \mu_{lb} + \| G_k \|} \| p_{rl} \|^2.$$ \hspace{1cm} (33)

It is not difficult to know that the function $\varphi(\mu)$ is convex when $(\mu + \| G_k \|) > 0$, since $\varphi'(\mu) = 2 \| p_{rl} \|^2 / (\mu + \mu_{lb} + \| G_k \|)^3 \geq 0$. Thus, the function $\varphi(\mu)$ attains its minimizer $\varphi(\mu_{min})$ when $\mu_{min}$ satisfies $\varphi'(\mu_{min}) = 0$ and $\mu \geq -\mu_{lb} + \| G_k \|$, i.e.

$$\varphi(\mu_{min}) = 2 \| p_{rl} \| \| d_k \| - (\mu_{lb} + \| G_k \|) \| d_k \|^2, \hspace{1cm} (34)$$
\[ \mu_{\min} = \| p_{s_k} \| / \| d_k \| - \mu_{ib} - \| G_k \|, \text{ and } \mu_{\min} > -(\mu_{ib} + \| G_k \|). \]  

(35)

We prove the property (30) by distinguishing two cases separately, namely \( \mu_{\min} \) is nonnegative or negative. When \( \| p_{s_k} \| / \| d_k \| \geq (\mu_{ib} + \| G_k \|) \), from (35), we have \( \mu_{\min} \geq 0 \). For this case, combining \( \mu_k \geq \mu_{ib} \) with equations (32)–(35), we obtain

\[ q_k(s_k) - q_k(s_k + Pd_k) \geq (\mu_k - \mu_{ib}) \| d_k \|^2 + \frac{1}{\mu_k + \| G_k \|} \| g_k \|^2 = \varphi(\mu_k - \mu_{ib}) \geq \varphi(\mu_{\min}) \]

\[ = \frac{1}{2} \left( \| p_{s_k} \| \| d_k \| + \left( \| p_{s_k} \| \| d_k \| - (\mu_{ib} + \| G_k \|) \| d_k \|^2 \right) \right) \geq \frac{1}{2} \| p_{s_k} \| \| d_k \|. \]

(36)

The other case is \( \| p_{s_k} \| / \| d_k \| < (\mu_{ib} + \| G_k \|) \), which gives \( \mu_{\min} < 0 \) from (35). Since the function \( \varphi(\mu) \) is monotonically increasing for all \( \mu \geq 0 \) when \( \| p_{s_k} \| / \| d_k \| < (\mu_{ib} + \| G_k \|) \), from equations (32)–(33), we obtain

\[ q_k(s_k) - q_k(s_k + Pd_k) \geq \frac{1}{2} \varphi(\mu_k - \mu_{ib}) \geq \frac{1}{2} \varphi(0) = \frac{1}{2} \| p_{s_k} \|^2 / \| G_k \|. \]

(37)

Combining (36) and (37), we get

\[ q_k(s_k) - q_k(s_k + Pd_k) \geq \frac{1}{2} \| p_{s_k} \| \min \left\{ \| p_{s_k} \| / \| G_k \|, \| d_k \| \right\}. \]

(38)

Since \( s_{k+1} \) is the projection of \( s_{k+1}^p = s_k + d_k \) in a convex set \( C_s = \{ s : A_ks = b_k \} \), according to Projection Theorem (see Proposition 1.1.4, p. 19 [3]), we have

\[ \| Pd_k \| = \| P_{s_{k+1}}p - s_{k+1} \| \leq \| s_{k+1}^p - s_k \| = \| d_k \|. \]

(39)

Using inequality (39) in equation (38), we obtain an estimation (30), which proves the lemma.

In order to prove that \( p_{s_k} \) tends to zero, we also use the following result about the lower bound estimation of the time-stepping length \( \Delta_k \) when \( \| p_{s_k} \| \geq \delta_{p_{s_k}} > 0 \).

**Lemma 4.2.** Assume that the level set of the twice continuously differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \) in the linear constraint plane (12) is bounded, i.e. \( L_f = \{ s : f(s) \leq f(s_0), A_ks = b_k \} \) is bounded. Furthermore, assume that there exists a positive constant \( \delta_s \) such that

\[ \| p_{s_k} \| \geq \delta_{p_{s_k}} > 0, \quad k = 1, 2, \ldots \]  

(40)

are satisfied, where \( p_{s_k} \) are generated by Algorithm 7. Then, it exists a positive \( \delta_M \) such that the time-stepping length

\[ \Delta_k \geq \delta_M > 0, \quad k = 1, 2, \ldots \]  

(41)

are satisfied, where \( \Delta_k \) is adaptively adjusted by formula (29).
Proof. Since the level set \( L_f \) is bounded, according to Proposition A.7 in pp. 754-755 of reference [3], \( L_f \) is closed. Then, there exists two positive constants \( M_{p_g} \) and \( M_G \) such that

\[
\| p_{g_k} \| \leq M_{p_g}, \| G_k \| \leq M_G, k = 1, 2, \ldots \tag{42}
\]

are satisfied, respectively. Selecting a positive \( \delta_{M_0} = 1/(2M_G) \), we have \( (1/\Delta t)I + G_k \succ 0 \) and \( (1/\Delta t)I + G_k - P^T G_k P \succ 0 \) when \( \Delta t \leq \delta_{M_0} \), where projection matrix \( P \) is given by equation (17).

From equations (28), (50) and the reduction estimation (30) of the quadratic model (see Lemma 4.2), when \( \Delta t \leq \delta_{M_0} \), we obtain the estimation of the measurement ratio

\[
| \rho_k - 1 | = \frac{(f(s_k) - f(s_k + Pd_k)) - (q_k(s_k) - q_k(s_k + Pd_k))}{q_k(s_k) - q_k(s_k + Pd_k)} \leq \frac{M_G \| Pd_k \|^2}{|q_k(s_k) - q_k(s_k + Pd_k)|} \leq \frac{2M_G \| Pd_k \|^2}{\delta_{p_g}} \min \left\{ \| Pd_k \|, \| G_k \| \right\} \leq \frac{2M_G \| Pd_k \|^2}{\delta_{p_g} \min \{ \| Pd_k \|, \| \delta_{p_g}/MG \| \}}. \tag{43}
\]

In the above third inequality and the last inequality, we use the Cauchy-Schwartz inequality \( |x^Ty| \leq \|x\| \|y\| \) and the lower bound assumption (40) of the projection gradient \( p_{g_k} \).

Selecting a positive constant \( \delta_{s_k} = \min \{ \delta_{p_g}/M_G, \eta_1 \delta_{p_g}/(2M_G) \} \), when \( \Delta t \leq \delta_{M_0} \) and \( \| Pd_k \| \leq \delta_{M_1} \), from equation (43), we have

\[
| \rho_k - 1 | \leq \eta_1, \tag{44}
\]

which means that the predicted point \( s_{k+1} = s_k + Pd_k \) is accepted and the time-stepping length \( \Delta t_{k+1} \) is enlarged according to the time-stepping adjustment formula (29).

From equations (20) and (50), when \( \Delta t \leq \delta_{M_2} = \min \{ 1/\delta_{M_0}, 1/(M_{p_g}/\delta_{M_1} + M_G) \} \), we have

\[
\| d_k \| = \left\| \left( \frac{1}{\Delta t}I + G_k \right)^{-1} p_{g_k} \right\| \leq \frac{\| p_{g_k} \|}{1/\Delta t_G - \| G_k \|} \leq \frac{M_{p_g}}{1/\Delta t_G - \| G_k \|} \leq \delta_{M_2}, \tag{45}
\]

which means that inequality (44) is satisfied according to the projection property (39) (i.e., \( \| Pd_k \| \leq \| d_k \| \)).

Assume that \( K \) is the first index such that \( \Delta t_K \leq \delta_{M_2} \) is satisfied. Then, according to the projection property (39), inequalities (45) and (44), we know that \( | \rho_K - 1 | \leq \eta_1 \), which means that \( s_K + Pd_K \) is accepted and the time-stepping length \( \Delta t_{K+1} \) is enlarged according to the time-stepping adjustment formula (29). Consequently, the
Using the results of Lemma 4.1 and Lemma 4.2, we can prove the global convergence property of Algorithm 1 for a linearly equality-constrained optimization subproblem.

**Theorem 4.1.** Assume that the level set of the twice continuously differentiable function \( f(s) \) in the linear constraint plane (12) is bounded, i.e. \( L_f = \{ s : f(s) \leq f(s_0), A_k s = b_k \} \) is bounded. Then, \( \lim_{k \to \infty} \inf \| p_{R_k} \| = 0 \), where \( p_{R_k} = \nabla f(s_k) + A_k^T \lambda_k \) and \( s_k, \lambda_k \) are generated by Algorithm 1.

**Proof.** We will prove it by contradiction. Assume that the conclusion is not true. Then it exists a positive constant \( \delta_{p_{R_k}} \) such that \( \| p_{R_k} \| \geq \delta_{p_{R_k}} > 0, k = 1, 2, \ldots \) are satisfied. According to Algorithm 1, we know that it exists an infinite subsequent \( k_i \) such that trial step \( P_{d_k} \) are accepted, i.e., \( \rho_{k_i} \geq \eta_a \), which gives

\[
0 = f_0 - \lim_{k \to \infty} f_k = \sum_{k=0}^{\infty} (f_k - f_{k+1}) \geq \eta_a \sum_{k_i=0}^{\infty} \left( q_k(s_{k_i}) - q_k(s_{k_i} + P_{d_{k_i}}) \right),
\]

where \( P_{d_{k}} \) is computed by equations (20) and (24). Using the bounded assumption of the objective function \( f(s) \) in the level set \( L_f \) for inequality (47), we know

\[
\lim_{k \to \infty} \left( q_k(s_{k_i}) - q_k(s_{k_i} + P_{d_{k_i}}) \right) = 0.
\]

From the result of Lemma 4.1, i.e. inequality (30), and equation (48), we get

\[
\lim_{k_i \to \infty} \| p_{R_{k_i}} \| \min \left\{ \| p_{R_{k_i}} \|, \| G_{k_i} \|, \| P_{d_{k_i}} \| \right\} = 0,
\]

where \( G_{k_i} = \nabla^2 f(s_{k_i}) \).

According to the bounded assumption of the level set \( L_f \), there exists two positive constants \( M_{p_{R_k}} \) and \( M_G \) such that

\[
\| p_{R_k} \| \leq M_{p_{R_k}}, \| G_k \| \leq M_G, k = 1, 2, \ldots.
\]

are satisfied, respectively. From equation (50) and inequalities (46) and (50), for the subsequent \( \{ k_i \} \) of accepted trial steps, we obtain

\[
\lim_{k_i \to \infty} \| P_{d_{k_i}} \| = 0.
\]

According to the bounded assumption of \( p_{R_k} \) (46) and inequality (50), from the result of Lemma 4.2, we know that it exists a positive constant \( \delta_{\Delta t} \) such that

\[
\Delta t_k \geq \delta_{\Delta t} > 0, k = 1, 2, \ldots
\]
are satisfied.

From equation (20) and using the property $P^2 = P$ of projection matrix $P$ which is defined by (17), we obtain

$$P_{d_k} = P \left( \frac{1}{\Delta t_k} I + G_{k_i} \right)^{-1} P_{\text{ins}},$$

which gives

$$P_{\text{ins}}^T P_{d_k} = P_{\text{ins}}^T \left( P \left( \frac{1}{\Delta t_k} I + G_{k_i} \right)^{-1} P \right) P_{\text{ins}}$$

$$= P_{\text{ins}}^T \left( \frac{1}{\Delta t_k} I + G_{k_i} \right)^{-1} P_{\text{ins}} \geq \left\| P_{\text{ins}} \right\| \frac{1}{1/\delta_M - M_G}. $$

(54)

Using the Cauchy-Schwartz inequality $|x^T y| \leq \left\| x \right\| \left\| y \right\|$, from equation (46), we have

$$\frac{1}{1/\delta_M - M_G} \left\| P_{\text{ins}} \right\|^2 \leq |P_{\text{ins}}^T P_{d_k}| \leq \left\| P_{\text{ins}} \right\| \left\| P_{d_k} \right\|$$

(55)

which gives

$$\left\| P_{\text{ins}} \right\| \leq \frac{1}{1/\delta_M - M_G} \left\| P_{d_k} \right\|.$$

(56)

From inequality (56) and equation (51), we obtain

$$\lim_{k \to \infty} \left\| P_{\text{ins}} \right\| = 0,$$

(57)

which contradicts the lower bound assumption (46). Therefore, we prove the conclusion of the theorem. □

4.4. Visual-Inertial Algorithm Descriptions

The proposed visual-inertial odometer method is described in Algorithm 2. To convention, the input INS reading has been pre-calibrated and the camera intrinsic parameters have been obtained. In algorithm 2, the positions of the $(k-1)\text{th}$ and $k\text{th}$ frame have been determined before, denoted as $P_{c_{k-1}}$ and $P_{c_k}$. And let $dist_{k-1}^{k+1}$ and $dist_{k}^{k+1}$ be the distances between the previous two frames and the $(k+1)\text{th}$ frame, respectively, which are measured by INS. Similarly, let $\delta h_{k-1}^{k+1}$ and $\delta h_{k}^{k+1}$ present the altitude differences between the previous two frames and the $(k+1)\text{th}$ frame, respectively, which are obtained by altimeter. Then, we use feature matching to obtain the landmarks’ locations in frame coordinate system. Finally, the $(k+1)\text{th}$ frame position is determined through solving the linearly equality-constrained optimization problem (6)-(7), which is solved by Algorithm 1. After a number of iterations, the aircraft trajectory is determined.

We just consider the relationships between frames and and landmarks, and the visual-inertial method have not loop closure detection. Therefore, algorithm 1 has good real-time performance. Additionly, the proposed algorithm tolerates a certain level of altitude error as we take into account the random error of each component.
Algorithm 2 Visual-inertial Odometry Algorithm

Input:
the $(k-1)^{th}$ and $k^{th}$ frames’ locations $P_{c_{k-1}}$, $P_{c_k}$, respectively; the distances $\text{dist}_{k-1}^{k+1}$, $\text{dist}_k^{k+1}$ between the previous two frames and the $(k+1)^{th}$ frame, respectively; and the altitude differences $\delta h_{k-1}^{k+1}$, $\delta h_k^{k+1}$ between the previous two frames and the $(k+1)^{th}$ frame, respectively.

Output:
The next frame location $P_{c_{k+1}}$.

1: for a number of iterations do
2: Determining the landmarks ← matching the ORB feature in the $(k+1)^{th}$ frame
   and the ORB features in the previous two frames, respectively.
3: end for
4: for a number of iterations do
5: Obtain the landmarks’ locations in the $(k+1)^{th}$ frame coordinate system and in
   previous frame coordinate system.
6: end for
7: Get matrix $A_\varepsilon$ in equation (7).
8: Solve the equality-constrained optimization problem (6)-(7) with Algorithm 1.
9: return $P_{c_{k+1}}$.

5. Simulation Results

In order to illustrate the effect of the proposed algorithm, we compare the localization accuracy of our algorithm and the pure inertial navigation on the same trajectory. The aircraft sets off at some point in the equator and then flies along the equatorial plane for an hour. According to the given condition from the industry, we assume that the aircraft flies an hour at an altitude of 1200 meters with speed 235 meters per second. We also consider the line-of-sight angular error to be less than 0.2 degrees, the random error of altimeter and the altimeter error associated with the distance. The specific key parameters are shown in table 1.

| Description                                      | Parameter value                                      |
|--------------------------------------------------|------------------------------------------------------|
| Flight altitude of the aircraft                  | 1000 ~ 1500 meters                                   |
| Flight speed of the aircraft                      | 210 ~ 260 meters per second                          |
| The line-of-sight angular error of landmarks      | $\leq 0.2^\circ$                                     |
| The random error of altimeter                     | one meter (variance $\sigma$ value)                  |
| The altimeter error related to flight distance    | $< \text{Flight distance \times 0.0001}$             |
| Horizontal attitude error of INS                  | $< 0.06^\circ$                                       |
| The heading error of INS                          | $< 0.4^\circ$                                        |
| Required accuracy of localization                 | $< 900$ meters per hour                              |
The simulation error of the proposed method is shown in figure 4. Figure 5 presents the comparison between our method and the pure inertial navigation method. The vertical axis represents the error between the real positioning location and the ideal trajectory. The horizontal axis is about the flying time. Owing to the long-time high-speed flight, the pure INS method does not work well. From Figure 5, we find that the error of inertial navigation is more than 9 kilometers per hour, and our method which combines the advantages of inertial navigation and visual odometry effectively suppress the rapid propagation of errors. The accuracy of proposed method is only slightly less than 300 meters. The effect of our proposed method has a noticeable improvement and its accuracy meets the navigation accuracy requirements.

Figure 4: The result of proposed simulation positioning errors.

Figure 5: The comparison between proposed method and pure INS.

6. Conclusion and Future Work

The proposed algorithm combining the visual odometer to assist INS effectively utilizes the complementarity of two methods. It avoids the rapid accumulation of errors in an inertial navigation method, and has no problem of scale ambiguity. Since there is no the loop closure detection, the proposed algorithm has good real-time performance compared to other vision-based methods. Currently, we only consider the horizontal flight with small variation in yaw angle. When the roll angle and the pitch angle frequently change, the proposed method do not work very well. Thus, the proposed algorithm is only applicable to the four DOF motion. In order to solve the localization in this six DOF flight scenario, we will design a more robust algorithm for full freedom navigation base on the newly proposed method in the future.
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