Period Integrals, Quantum Numbers and Confinement in SUSY QCD

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We present a direct computation of the period integrals on degenerate Seiberg-Witten curves for supersymmetric QCD, and show how these periods determine the changes in the quantum numbers of the states, when passing from the weak to the strong-coupling domains in the mass moduli space of the theory. The confinement of monopoles at strong coupling is discussed, and we demonstrate that the ambiguities in choosing the way in the moduli space do not influence to the physical conclusions on confinement of monopoles in the phase with the condensed light dyons.

1 Introduction

Supersymmetric QCD serves for a while as a laboratory for testing confinement. Not being enough realistic to describe real nature, it can be considered nevertheless as model for quantum theory, which allows non-perturbative analysis, and therefore can prolong our horizon to understand, at least in principle, what happens with gauge theory at strong coupling.

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Figure 1: Roots $\alpha$ and fundamental weights $\mu$ for the Lie algebra of the $SU(3)$ gauge group in its Cartan plane, the roots are canonically normalized to $\alpha^2 = 2$. The notations are chosen for the roots to be orthogonal $\mu_I \cdot \alpha_J = \delta_{IJ}$ ($I, J = 1, 2$) with the weights $\mu_1$ and $\mu_2$ of the fundamental representations $3$ (the weights of the dual fundamental representation $\bar{3}$ are depicted with dashed lines). Generic duality $\mu \cdot \alpha \in \mathbb{Z}$ for the arbitrary vectors $\mu$ from the weight lattice and $\alpha$ (from its root sub-lattice) turn into the Dirac quantization condition for the (chromo) electric and magnetic charges.

Below I am going to present some details of studying the properties of confinement in supersymmetric QCD along the program of [1, 2], and complete the discussion of few technical issues, arising along these lines in underlying complex geometry. The main idea of this scenario is to start with an obvious confinement of monopoles of the (dual) Meissner type at the quark vacuum in weakly coupled supersymmetric gauge theory [1], and then move this picture by adjusting the mass parameters of the theory towards the strongly coupled domain [2]. One ends up in this way with the effective theory of light dyons instead of original quarks, since the BPS-states change their quantum numbers due to nontrivial monodromies in mass-moduli space. Investigating of these monodromies is a nontrivial problem (see e.g. [3, 4]) and can be performed in the most transparent way by computing the period integrals and studying the perturbation of ramification points on (almost) singular Seiberg-Witten curves [5, 6, 7, 8].

Following [1, 2], below the supersymmetric QCD with the $SU(N_c)$ gauge group and large number $N_c \leq N_f \leq 2N_c$ of the fundamental flavors is taken as a basic model: the first nontrivial (and the main in this text) example is the $SU(3)$ gauge theories with $N_f = 4$ and $N_f = 5$. The quantum numbers of the light states can be seen on fig. 1, where the quark color charges (the fundamental weights) and monopole charges (the roots) for the $SU(3)$ gauge group are depicted. In the context of Seiberg-Witten theory [5, 6], which is necessary to study the exact properties of supersymmetric QCD around strongly-coupled vacua, the Dirac quantization condition $\mu_i \cdot \alpha_j = \delta_{ij}$ turns into the intersection form of the cycles $A_i \circ B_j = \delta_{ij}$ on spectral curve, see fig. 2 for the $SU(3)$ gauge group. In what follows we shall use only the “homological
Figure 2: $A$- and $B$- cycles and their intersection form $A_i \circ B_j = \delta_{ij}$ for the genus $N_c - 1 = 2$ Riemann surface of the $SU(3)$ gauge theory. Elementary quark’s charges $\mu_{1,2} \leftrightarrow A_{1,2}$ correspond to the $A$-cycles, while the monopole’s ones are $\alpha_{1,2} \leftrightarrow B_{1,2}$ - to the $B$-cycles.

normalization” of the charges [6], where they are measured by the cycles on Seiberg-Witten curve, and therefore are always integer.

Duality transformations do not change the complex structure on the Seiberg-Witten curves, but exchanges electric $A$-cycles with the magnetic $B$-cycles, and hence correspond to electric-magnetic duality [5, 6]. It is important, that the periods

$$a_i = \frac{1}{2\pi i} \oint_{A_i} dS, \quad a^D_i = \frac{1}{2\pi i} \oint_{B_i} dS$$

(1)

entering (together with the residues $m_A = \text{res}_{P_A} dS$) the BPS mass-formula

$$\text{Mass}(\mu, \alpha, b) \sim |\mu \cdot a + \alpha \cdot a^D + b_A m_A|$$

(2)

are never simultaneously real, except for singular or degenerate cases, when some of these period(s) vanish, giving rise to the extra massless states in the spectrum. These massless states lead to possible decays, causing change of the quantum numbers for the light states, and therefore in different domains of the moduli space the condensates acquire different charges. In order to determine these charges one has to consider the Seiberg-Witten theory for supersymmetric QCD in the vicinity of singular curves, corresponding to $\mathcal{N} = 1$ vacua.

2 Seiberg-Witten theory for supersymmetric QCD

Generic curve for $\mathcal{N} = 2$ supersymmetric QCD with $N_c$ colors and $N_f$ flavors can be written in the form [7, 9]

$$y^2 = P(x)^2 - 4Q(x)$$

(3)
where
\[ P(x) = \prod_{i=1}^{N_c} (x - \phi_i), \quad \sum_{i=1}^{N_c} \phi_i = -\Lambda \delta_{N_f,2N_c-1} \]
\[ Q(x) = \Lambda^{2N_c-N_f} \prod_{A=1}^{N_f} (x + m_A) \]

are two polynomials of powers \( N_c \) and \( N_f \) respectively. In semiclassical regime the roots \( \{ \phi_i \} \), \( i = 1, \ldots, N_c \) of the first one coincide with the eigenvalues of matrix \( \Phi \) of the condensate of the complex scalar from the vector multiplet of \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theory, but being computed exactly they (or, better, their symmetric functions) are got corrected in (dependent upon \( \Lambda \) and \( m_A \) way) due to the instanton effects.

The hyperelliptic representation (3) can be also re-written in the form
\[ w + \frac{Q(x)}{w} = P(x) \] (5)
or
\[ W + \frac{1}{W} = \frac{P(x)}{\sqrt{Q(x)}} \] (6)

with \( y = w - \frac{Q(x)}{w} = (W - \frac{1}{W}) \sqrt{Q(x)} \).

The curves (3), (5) or (6) are endowed with a generating differential
\[ dS \sim x \frac{dw}{w} = x \frac{dW}{W} + \frac{1}{2} x \frac{dQ}{Q} = \frac{x dP}{y} - x \frac{P}{2y} \frac{dQ}{Q} + \frac{1}{2} x \frac{dQ}{Q} \] (7)

whose periods (1) enter the mass formula (2), as well as its residues
\[ \text{res}_{P_A^\pm} dS = m_A \cdot \frac{P}{2y} \bigg|_{x=-m_A} - \frac{m_A}{2} = -m_A \] (8)
at the points \( P_A \) with \( x(P_A^\pm) = -m_A \) at one of the sheets of (3). The residues (8) must disappear in the limit of vanishing masses [6], and for \( N_f = 2N_c - 1 \) this requirement causes nonvanishing term of the order of \( \Lambda \) in the r.h.s. of the second equality in (4).

The variation of (7) at constant \( W \) gives rise to
\[ \delta dS \sim \frac{dx}{y} \left( \delta P(x) - \frac{1}{2} P \delta Q(x) Q(x) \right) \] (9)

1 Sometimes the asymmetric representation with the factorized \( Q(x) = Q_+(x)Q_-(x) \) and \( w = wQ_+(x) \), so that equation (5) turns into \( w^2 Q_+(x) - P(x)w + Q_-(x) = 0 \) is more adequate from the point of view of brane-construction [10] and possible relations to the matrix models [11] and quantum integrable systems.
In the case of $SU(3)$ gauge group it is convenient to introduce explicitly

$$P(x) = (x - \phi_1)(x - \phi_2)(x - \phi_3) = x^3 - ux - v$$  \hspace{1cm} (10)$$

with, for $N_f < 5$ and $\phi_3 = -\phi_1 - \phi_2$, so that

$$u = \phi_1^2 + \phi_2^2 + \phi_1 \phi_2$$

$$v = -\phi_1 \phi_2 (\phi_1 + \phi_2)$$  \hspace{1cm} (11)$$

Now we turn to particular cases of these formulas in the vicinity of singular curves.

3 Period integrals on degenerate curves

3.1 $N_c = 2$ and $N_f = 2$ case

Starting originally with the $SU(3)$ gauge theory around vacuum with two condensed flavors ($r = 2$ in terms of [1]) at weak coupling, and moving it towards the strongly coupled domain, one gets the picture, effectively described by $SU(2)$ gauge theory with $N_f = 2$ light flavors, when $r = 2$ vacuum collides with the vacuum, containing also massless monopole [2]. In this warmup example with $N_c = 2$ and $N_f = 2$, taken for simplicity with the coinciding masses $m_1 = m_2 = m$, one gets for the curve (3)

$$y^2 = (x^2 - u)^2 - 4 \Lambda^2 (x + m_1)(x + m_2) = (x^2 - u)^2 - 4 \Lambda^2 (x + m)^2$$  \hspace{1cm} (12)$$

while the generating differential (7) turns for the pairwise coinciding masses into

$$dS \sim \frac{xdP}{y} - \frac{xP}{2y} \frac{dQ}{Q} + \frac{1}{2} \frac{xdQ}{Q} = \frac{xdP}{y} - \frac{xP}{y} \frac{dq}{q} + \frac{xdq}{q}$$

$$q(x) = \prod_{B=1}^{N_f/2} (x + m_B)$$  \hspace{1cm} (13)$$

where we have chosen $m_{B+N_f/2} = m_B$, $B = 1, \ldots, N_f/2$ for even number of flavors $N_f$. The curve (12) just corresponds to $P(x) = x^2 - u$ and $q(x) = x + m$.

When exactly at quark vacuum - i.e. at the point $u = u_Q = m^2$, the curve (12) degenerates further to

$$y^2 = (x + m)^2 ((x - m)^2 - 4 \Lambda^2) \equiv (x + m)^2 Y^2$$

$$Y^2 = (x - m)^2 - 4 \Lambda^2$$  \hspace{1cm} (14)$$

and the Seiberg-Witten differential (7) turns into

$$dS = \frac{xdx}{Y} + \frac{xdx}{x + m}$$  \hspace{1cm} (15)$$
where the first term in the r.h.s. coincides with the generating differential for the formal pure \( \mathcal{N} = 2 \) SUSY \( U(1) \) gauge theory, with the only VEV given by \( m \) [12, 14].

Due to (8) and to the fact, that on degenerate curve the position of the mass pole at \( x = -m \) (for both flavors) coincides with the degenerate cut, the differential (15) is normalized by

\[
\frac{1}{2\pi i} \oint_{x = -m} dS_+ = \frac{1}{2\pi i} \oint_{A^+} dS_+ = a = -m
\]

\[
\frac{1}{2\pi i} \oint_{x = -m} dS_- = \text{res}_{x = -m} dS_- = \frac{1}{2\pi i} \oint_{A^-} dS_- = -2m + a = -m
\]

obviously true for (15). Since the curve (14) is rational, the differential (15) can be easily integrated, giving rise to

\[
S = Y + m \log(x - m + Y) + x - m \log(x + m)
\]

In order to compute the desired \( B \)-period (the monopole mass), one has to take the difference

\[
S_+ |_{x = -m} - S_- |_{x = -m}
\]

gen of the values of (17) on two different sheets of the Riemann surface (14).

This is not possible to do by direct substitution of \( x = -m \) into (17) due to the logarithmic singularity, i.e. the curve (14) is “too degenerate”\(^2\). Let us then slightly regularize it, and denote the distance between the position of the pole and the nearest end of the shrunken cut by \( \epsilon^\pm \), dependently on the sheet \( Y = Y^\pm \) of (14). The values of these \( \epsilon^\pm = \epsilon^\pm(m, \Lambda) \) can be determined as follows (see e.g. [14]): the differential

\[
d\phi = \frac{dw}{w} \equiv \frac{dx}{Y} + \frac{dx}{x + m}
\]

should have constant periods [15], moreover, its \( B \)-periods on (14) can be just chosen vanishing. Integrating (18) up to

\[
\phi = \log(x - m + Y) + \log(x + m)
\]

and putting \( \phi_+ |_{x = -m} - \phi_- |_{x = -m} \equiv \phi_+ |_{x = -m + \epsilon^+} - \phi_- |_{x = -m + \epsilon^-} = 0 \), one gets (at \( \epsilon^\pm \to 0 \))

\[
\log \frac{\epsilon^+}{\epsilon^-} = \log \frac{m + \sqrt{m^2 - \Lambda^2}}{m - \sqrt{m^2 - \Lambda^2}}
\]

Therefore

\[
S_+ |_{x = -m} - S_- |_{x = -m} = \frac{m}{\epsilon^-} \log \frac{\epsilon^+}{\epsilon^-} + 2Y |_{x = -m} + m \log \frac{-2m + Y |_{x = -m}}{-2m - Y |_{x = -m}}
\]

\[
= 4\sqrt{m^2 - \Lambda^2} + 2m \log \frac{m - \sqrt{m^2 - \Lambda^2}}{m + \sqrt{m^2 - \Lambda^2}}
\]

\(^2\)For example, already the period matrix of (14) is not a well-defined object. One can verify, that at the Argyres-Douglas point, when \( m = \Lambda \), we get \( \tau_\ast = i \), as follows from the self-duality requirement [3, 16], since the curve (14) degenerates to \( y^2 \sim (x + m)^3 \), giving rise to the desired limiting value, see e.g. [13].
Hence, evaluating $B$-period on degenerated curve (14) gives rise to explicit formula (cf. the result with [17], where similar structures have been obtained by indirect methods from two-dimensional approach)

$$a^D = \frac{1}{2\pi i} \left( S_+|_{x=-m} - S_-|_{x=-m} \right) = -\frac{i}{\pi} \left( 2\sqrt{m^2 - \Lambda^2} + m \log \frac{m - \sqrt{m^2 - \Lambda^2}}{m + \sqrt{m^2 - \Lambda^2}} \right)$$

(22)

showing in particular, that $\text{Im}(a^D)|_{m=\pm\Lambda} = 0$.

However, to analyze the difference between massless monopoles and dyons one should also consider carefully the real part of (22), taking into account the logarithmic cut in $m$-plane. One can fix this real part to vanish at $m = \Lambda$, then

$$\text{Re}(a^D)|_{m=\Lambda} = 0$$

$$\text{Re}(a^D)|_{m=-\Lambda} = 2m = -2a$$

(23)

It means, that when quark singularity $u_Q$ collides with $u_M$ we get massless monopole with $|a^D| = 0$, while when $u_Q$ collides with $u_D$, one gets the vanishing mass of the $(1,1) = [2,1]$ dyon, $|a^D + 2a| = 0$. Here, following [6, 2], we specially point out, that the “homological” electric charge $[2,1]$ of this dyon is different from conventional physical one $(1,1)$, while the magnetic charges coincide in both normalizations. The electric charge of this dyon coincides with the charge of W-boson and doubles the charge of a quark: the last one is $(\frac{1}{2},0)$ in physical normalization, but corresponds to single $A$-cycle (i.e. equals to $[1,0]$) in homological one, therefore the electric charge of this dyon (and of the W-boson) corresponds to two $A$-cycles.

Asymptotically for (22) one has

$$a^D \underset{m \rightarrow \infty}{\sim} \frac{i}{\pi} 2m \left( \log \frac{2m}{\Lambda} - 1 \right) + \ldots$$

(24)

For the mass-derivatives of (22) one gets

$$\frac{\partial a^D}{\partial m} = -\frac{i}{\pi} \log \frac{m - \sqrt{m^2 - \Lambda^2}}{m + \sqrt{m^2 - \Lambda^2}}$$

(25)

and

$$\frac{\partial^2 a^D}{\partial m^2} = \frac{2i}{\pi \sqrt{m^2 - \Lambda^2}}$$

(26)

When the second derivative diverges, one can write the fractional-power expansions:

$$a^D \underset{m \rightarrow \Lambda}{\sim} \frac{4i}{3\pi} \sqrt{\frac{2}{\Lambda}} \cdot (m - \Lambda)^{3/2} + \ldots$$

(27)

at $m = \Lambda$ and, similarly

$$a^D \underset{m \rightarrow -\Lambda}{\sim} 2m + \frac{4}{3\pi} \sqrt{\frac{2}{\Lambda}} \cdot (m + \Lambda)^{3/2} + \ldots$$

(28)
at $m = -\Lambda$. Expressions (27) and (28) can be easily analyzed for the presence of nontrivial monodromies in the mass plane.

In particular, one finds from (27), that being pure imaginary at $t = m - \Lambda > 0$, when continued to negative value $t = m - \Lambda < 0$ the value of the period $a^D$ is either positive or negative dependently on the chosen way in the mass plane, or the sign of the angle variable in $t = \varepsilon e^{\pm i\varphi}$ at $t \approx 0$. This formula says, that

$$a^D \approx \frac{4i}{3\pi} \sqrt{\frac{2}{\Lambda}} t^{3/2} = \pm \frac{4}{3\pi} \sqrt{\frac{2}{\Lambda}} \varepsilon^{3/2}$$  (29)

Suppose one now takes $u = m^2 + \delta$ for $m \gg |\delta| > 0$ in the vicinity of quark vacuum. Then $a = -\sqrt{u} \approx -m - \frac{i}{2m}$, or $a + m \simeq -\frac{i}{2m}$. When $m$ approaches $\Lambda$ at $t \to 0$ the mass of the light quark is

$$|a + m| \approx \left| \frac{\delta}{2m} \right| \simeq \left| \frac{\delta}{2\Lambda} \right| \gtrsim 0$$  (30)

Going to negative $t$ formula (29) says that, dependently on the chosen way in $m$-plane, after crossing the critical line the mass of one of the $(\frac{1}{2}, \pm 1) = [1, \pm 1]$ dyons becomes less than the mass of the quark. These conditions can be formalized as

$$\text{sign(Im } t) = \text{sign}(\delta) : \quad |a + m + a^D| < |a + m|$$

$$\text{sign(Im } t) = -\text{sign}(\delta) : \quad |a + m - a^D| < |a + m|$$  (31)

If it happens, say, for the positive magnetic charge (i.e. the positive sign in front of $a^D$) in the inequality $|a + m + a^D| < |a + m|$, the quark can emit the massless anti-monopole and turn into the $(\frac{1}{2}, 1) = [1, 1]$ dyon with the charge conservation law $[1, 0] + [0, 1] = [1, 1]$, in the opposite case the sign of the monopole charge has to be changed, i.e. generally one gets

$$[1, 0] + [0, q] = [1, q], \quad q = \pm 1$$  (32)

and the particular choice of the sign $q$, as we see below, is not observable. Similarly, at another critical line, at $m = -\Lambda$, one gets the process with the conservation law $[1, q] = [2, q] + [-1, 0]$, with $q = \pm 1$ again, or the dyon $[1, q]$ decouples into the $[2, q]$-dyon, massless at $m = -\Lambda$ due to (23), and the quark $[-1, 0]$, which can be treated as “dual to” $[1, 0]$-quark after exchange the sign of the mass $m \leftrightarrow -m$, see [2].

The analysis of this section can be supplemented by direct consideration of the process of permutation of the branch points of the Seiberg-Witten curve (3) in the vicinity of the singularities [2, 18], which leads, basically, to the same conclusion.

### 3.2 $N_c = 3$ and $N_f = 4, 5$ theories

Let us now turn directly to the $SU(3)$ theory, with $N_f = 4$ with the pairwise coinciding masses $m_1 = m_3$ and $m_2 = m_4$, when the curve (3) becomes

$$y^2 = (x - \phi_1)^2(x - \phi_2)^2(x + \phi_1 + \phi_2)^2 - 4\Lambda^2(x + m_1)^2(x + m_2)^2$$  (33)
Putting exactly $\phi_i = -m_A \delta_{i,A}$ for $i, A = 1, 2$ for the chosen quark vacuum, the curve (33) degenerates into

$$
y^2 = (x + m_1)^2(x + m_2)^2 \left( (x - m_1 - m_2)^2 - 4\Lambda^2 \right) =$$

$$= (x + m_1)^2(x + m_2)^2 \left( (x - M)^2 - 4\Lambda^2 \right) \equiv (x + m_1)^2(x + m_2)^2 Y^2$$

$$Y^2 = (x - M)^2 - 4\Lambda^2$$

$$M = m_1 + m_2$$

The Seiberg-Witten differential (7) turns in this case into

$$dS = \frac{dx}{Y} + \frac{dx}{x + m_1} + \frac{dx}{x + m_2}$$

$$p = x - M, \quad Y^2 = p^2 - 4\Lambda^2$$

(35)

The reasoning for its normalization

$$\frac{1}{2\pi i} \oint_{x=-m_k} dS_+ = \frac{1}{2\pi i} \oint_{A^+} dS_+ = a_k = -m_k$$

$$\frac{1}{2\pi i} \oint_{x=-m_k} dS_- = \text{res}_{x=-m_k} dS_- - \frac{1}{2\pi i} \oint_{A^-} dS_- = -2m_k + a_k = -m_k$$

(36)

just literally repeats that of (16) for each mass $m_{1,2}$ in (34), (35). Clearly, the residues (36) corresponds to exact vanishing of the effective masses of the quarks $a_K + m_K = 0 (K = 1, 2)$ in the chosen quark vacuum with two condensed flavors.

Let us turn to the dual to (36) $B$-periods, corresponding via $\mu_I \cdot \alpha_J = \delta_{IJ} (I, J = 1, 2)$ to the monopole masses. Again, they are given by the differences $S_+|_{x=-m_k} - S_-|_{x=-m_k} (k = 1, 2)$ of the values of the Abelian integral of (35)

$$S = Y + M \log(x - M + Y) + 2x - m_1 \log(x + m_1) - m_2 \log(x + m_2)$$

$$M = m_1 + m_2$$

(37)

Since, this is again singular at $x + m_k = 0$ one needs to introduce the regulators $\epsilon^{\pm}_k, k = 1, 2,$ on each sheet of (34). As in (20) they are determined by

$$\phi_+|_{x=-m_k} - \phi_-|_{x=-m_k} = \log \frac{\epsilon^{+}_k}{\epsilon^-_k} + \log \frac{M + m_k - \sqrt{(M + m_k)^2 - 4\Lambda^2}}{M + m_k + \sqrt{(M + m_k)^2 - 4\Lambda^2}} = 0$$

(38)

$$k = 1, 2$$

where

$$\phi = \oint \left( \frac{dx}{Y} + \frac{dx}{x + m_1} + \frac{dx}{x + m_2} \right) =$$

$$= \log(x - M + Y) + \log(x + m_1) + \log(x + m_2)$$

(39)
Thus

\[ S_+ |_{x=-m_k} - S_- |_{x=-m_k} = -m_k \log \frac{e_k^+}{e_k} + 2 Y |_{x=-m_k} + M \log \frac{-m_k - M + Y |_{x=-m_k}}{-m_k - M - Y |_{x=-m_k}} = \]
\[ = 2 \sqrt{(m_k + M)^2 - 4\Lambda^2} + (m_k + M) \log \frac{m_k + M - \sqrt{(m_k + M)^2 - 4\Lambda^2}}{m_k + M + \sqrt{(m_k + M)^2 - 4\Lambda^2}} \quad (40) \]

\[ k = 1, 2 \]

Hence, (cf. with (22)),

\[ a^D_1 = \frac{1}{2\pi i} \oint_{B_1} dS = \]
\[ = -\frac{i}{\pi} \left( \sqrt{(2m_1 + m_2)^2 - 4\Lambda^2} + \left( m_1 + \frac{m_2}{2} \right) \log \frac{2m_1 + m_2 - \sqrt{(2m_1 + m_2)^2 - 4\Lambda^2}}{2m_1 + m_2 + \sqrt{(2m_1 + m_2)^2 - 4\Lambda^2}} \right) \quad (41) \]

and

\[ a^D_2 = \frac{1}{2\pi i} \oint_{B_2} dS = \]
\[ = -\frac{i}{\pi} \left( \sqrt{(m_1 + 2m_2)^2 - 4\Lambda^2} + \left( \frac{m_1}{2} + m_2 \right) \log \frac{m_1 + 2m_2 - \sqrt{(m_1 + 2m_2)^2 - 4\Lambda^2}}{m_1 + 2m_2 + \sqrt{(m_1 + 2m_2)^2 - 4\Lambda^2}} \right) \quad (42) \]

which obviously obey the desired properties, similarly to \( N_c = 2, N_f = 2 \) theory, which effectively describes dynamics in the subgroups of the \( SU(3) \) gauge group, corresponding to the roots \( \alpha_{1,2} \).

In the \( N_f = 5 \) theory, the derivation is quite similar, though the formulas are slightly more complicated. The degenerate curve is again

\[ y^2 = (x + m_1)^2(x + m_2)^2Y^2 \quad (43) \]

where now

\[ Y^2 = p^2 - 4\Lambda(x + m_5) \]
\[ p = x - M + \Lambda = x - m_1 - m_2 + \Lambda \quad (44) \]

and it is endowed with the Seiberg-Witten differential (7) to be now

\[ dS = \frac{xdx}{Y} - \frac{xp}{2Y(x + m_5)} \frac{dx}{x + m_5} + \frac{1}{2} \frac{xdx}{x + m_5} + \frac{xdx}{x + m_1} + \frac{xdx}{x + m_2} \quad (45) \]
Computation of the $B$-periods of the differential (45), similar to the $N_f = 4$ case, leads to the result

$$a_k^D = \frac{1}{2\pi i} \oint_{B_k} dS = -\frac{i}{\pi} \left( Y_k + \left( M + \frac{m_k + m_5}{2} \right) \log \frac{M + m_k + \Lambda - Y_k + m_k - m_5}{M + m_k + \Lambda + Y_k} \right)$$

$$+ \frac{m_k - m_5}{2} \log \frac{a + b_+ m_k - b_- Y_k}{a + b_+ m_k + b_- Y_k}, \quad k = 1, 2$$

(46)

$$a = (M - \Lambda)^2 + m_5(M - 3\Lambda), \quad b_\pm = M + m_5 \pm \Lambda$$

with two parabolic curves $Y_{1,2}$ in the $(m_1, m_2)$-mass plane, being defined by

$$Y_1^2 = (2m_1 + m_2)^2 - \Lambda(2m_5 + m_2) + \Lambda^2 = 0$$

(47)

and

$$Y_2^2 = (m_1 + 2m_2)^2 - \Lambda(2m_5 + m_1) + \Lambda^2 = 0$$

(48)

These parabolic curves, appearing naturally in the process of the period computation, play the role of the borders of the deformed strongly coupled domain in the $N_f = 5$ case, compare to the $N_f = 4$ case, where these borders are just straight lines $m_1 + 2m_2 = \pm 2\Lambda$ and $2m_1 + m_2 = \pm 2\Lambda$.

It is easy to check, that (46) obeys all desired properties. Note also, that this derivation has nothing in common with that of [17, 18], where the indirect methods, based on parallels between four-dimensional and two-dimensional approaches have been exploited.

Formulas (41), (42) and (46) immediately lead to simple physical conclusions [2]. Vanishing of $a_k^D = \alpha_k \cdot a^P$, $k = 1, 2$, corresponds to vanishing of masses of the monopoles with the charges

$$M_k = \sqrt{2} \left( n_e \oplus n_m \right)_k = 0 \oplus \alpha_k, \quad k = 1, 2$$

(49)

in terms of the roots of the $SU(3)$ gauge group, see fig. 1. This happens for the period (41) at $2m_1 + m_2 = 2\Lambda$, and for (42) at $m_1 + 2m_2 = 2\Lambda$.

Clearly, the imaginary part $\text{Im}(a_k^D) = 0$ vanishes also at $2m_1 + m_2 = -2\Lambda$, and similarly $\text{Im}(a_2^D) = 0$ if $m_1 + 2m_2 = -2\Lambda$. However, the real parts of expression (41) at $2m_1 + m_2 = -2\Lambda$ equals to $2m_1 + m_2 = -\alpha_1 \cdot a$, or to the mass of the W-boson with the charge $\alpha_1$. Hence, at $2m_1 + m_2 = -2\Lambda$ one gets the massless dyon with the charge

$$D_1 = \sqrt{2} \left( n_e \oplus n_m \right)_1 = \alpha_1 \oplus \alpha_1$$

(50)

and similarly, the massless dyon with the charge

$$D_2 = \sqrt{2} \left( n_e \oplus n_m \right)_2 = \alpha_2 \oplus \alpha_2$$

(51)

at $m_1 + 2m_2 = -2\Lambda$. Formulas (46) show that at $Y_k = 0$, $k = 1, 2$ (i.e. for each parabola (47), (48) in the mass plane) the imaginary part of the corresponding period $\text{Im}(a_k^D)$ vanishes,
while the real part Re(\(a_k^D\)) jumps when passing from the positive to negative branch of the corresponding \(k\)-th parabola.

The analysis of the computed periods (41), (42) and (46) leads to the following picture of changing of the quantum numbers due to monodromies (see fig. 3): starting with the condensate of two quarks at weak coupling one ends up in strongly-coupled phase with the condensate of two light dyons. Formula (32) of the previous section describes the projection of fig. 3 to the horizontal line: in terms of the weight and root vectors of fig. 1 one finds that each quark with the weight-like electric charge pickups up the root-like magnetic charge and turns into a dyon of different nature from (50) and (51): in the strongly-coupled phase the light dyons \(\Psi_{1,2}\) with the charges\(^3\)

\[
\mathcal{D}_K = \mu_K \oplus \alpha_K = \sqrt{2}(n_e \oplus n_m)_K^D \equiv \sqrt{2}n^K, \quad K = 1, 2
\]

condense instead quarks [2].

4 Confinement

Since in the weak coupling regime of the original theory at large \(m\) in \(r = 2\) vacuum the quarks are in the Higgs phase, they confine the monopoles. Two of three \(SU(3)\) elementary monopoles with the magnetic charges \(\alpha_{1,2}\) (see (49)) are attached to the ends of the elementary strings, while the third one with the charge \(\alpha_{12} = \alpha_1 - \alpha_2\) becomes a string junction of two elementary strings [1]. At strong coupling one can repeat these reasoning for the dual theory of light dyons (52), where the fundamental strings can be constructed from the effective Lagranian:

\(^3\)Let us point out here, that by calligraphic \(\mathcal{D}\)-letters we denote the dyons with the weight-like electric charges, in contrast to the dyon-relatives of the monopoles with the charges (50) and (51), given entirely in terms of the root vectors.
\[ \mathcal{L} \sim \sum_{K=1,2} |\nabla \Psi_K|^2 + \ldots \], where the minimal interaction
\[ \nabla \Psi_K = (\partial - i D_K \cdot (A \oplus A^D)) \Psi_K = (\partial - i (\mu_K \cdot A + \alpha_K \cdot A^D)) \Psi_K \] (53)
is determined by the dyon charges (52).

The elementary charges \( S_{1,2} \) of the \( \mathbb{Z}_2 \)-strings follow from the dyon charges (52) via behavior of the gauge potentials at spatial infinity,
\[ S_I : \quad D_K \cdot (A \oplus A^D) = \mu_I \cdot A + \alpha_I \cdot A^D \sim \delta_{IK} d\theta, \quad I, K = 1, 2 \] (54)
where \( \theta \) is angle in the plane, transverse to the direction of string. This implies for the Cartesian projections \( A_3 \sim \alpha_{12} \cdot A \sim (\mu_1 - \mu_2) \cdot A \) and \( A_8 \sim (\alpha_1 + \alpha_2) \cdot A \sim -\mu_{12} \cdot A \) (the horizontal and vertical directions at fig. 1, while the indices \( \{3, 8\} \) come from the diagonal Gell-Mann matrices)
\[ A_3 + A^D_3 \sim d\theta, \quad \frac{A_8}{\sqrt{3}} + \sqrt{3} A^D_8 \sim d\theta \] (55)
The combinations orthogonal to (55) should vanish at infinity
\[ A_3 - A^D_3 \sim 0, \quad A^D_8 - 3A_8 \sim 0 \] (56)
As a result [2] one gets for (54) in terms of the fluxes\(^4\)
\[ \oint dx \cdot (A^D \oplus A) = \oint dx \cdot (A^D_3, A_3; A_8^D, A_8) = 4\pi (-n^e_3, n^m_3; -n^e_8, n^m_8) = 4\pi (-n^e \oplus n^m) \] (57)
that the charge of the \( S_1 \)-string is
\[ n_{S_1} = \begin{pmatrix} -\frac{1}{4}, \frac{1}{4}; -\frac{3\sqrt{3}}{20}, \frac{\sqrt{3}}{20} \end{pmatrix} \equiv \begin{pmatrix} S_1 \\ \sqrt{2} \end{pmatrix} \] (58)
while the charge of the \( S_2 \)-string, arising due to winding at spatial infinity of the phase of the second dyon, equals to
\[ n_{S_2} = \begin{pmatrix} \frac{1}{4}, -\frac{1}{4}; -\frac{3\sqrt{3}}{20}, \frac{\sqrt{3}}{20} \end{pmatrix} \equiv \begin{pmatrix} S_2 \\ \sqrt{2} \end{pmatrix} \] (59)
Now one can easily check that each of three \( SU(3) \) monopoles can be indeed confined by these two strings [2]. For the monopoles with the charges \( 0 \oplus \alpha_{1,2} \) or \( (0, \pm \frac{1}{2}, 0, \frac{\sqrt{3}}{2}) \) one has the following decompositions
\[ \frac{1}{\sqrt{2}} (0 \oplus \alpha_1) = \begin{pmatrix} 0, \frac{1}{2}; 0, \frac{\sqrt{3}}{2} \end{pmatrix} = n_{S_1} + \frac{7}{10} n^1 + \frac{2}{10} n^2, \] (60)
\[ \frac{1}{\sqrt{2}} (0 \oplus \alpha_2) = \begin{pmatrix} 0, -\frac{1}{2}; 0, \frac{\sqrt{3}}{2} \end{pmatrix} = n_{S_2} + \frac{2}{10} n^1 + \frac{7}{10} n^2 \]
\(^4\)This definition ensures that the string has the same charge as a trial dyon which can be attached to the string endpoint (not necessarily being present in the spectrum of the theory).
where \( n^K = \frac{D^K}{\sqrt{2}} \), \( K = 1, 2 \) are the normalized charges of the dyons (52). Formula (60) shows, that only a part of the monopole-anti-monopole flux is confined to the interior of the string world-sheet, while the remaining part is just screened by the dyon condensate.

For the third \( SU(3) \alpha_{12} \)-monopole the formulas are more elegant: one gets from (60), that

\[
\frac{1}{\sqrt{2}}(0 \oplus \alpha_{12}) = (0, 1; 0, 0) = n_{S_1} - n_{S_2} + \frac{1}{2} (n^1 - n^2)
\]  

and we find that it is also confined, being a junction of two elementary strings \( S_1 \) and \( S_2 \). Pictorially, dynamics in the non-Abelian direction \( \alpha_{12} \) is determined by “difference dyon” \( D_1 - D_2 = \) and “difference string” \( S_1 - S_2 = \). Formula (61) claims, that due to the total screening of the electric charge of the “difference string” by the condensate of the “difference dyon” \( S_1 - S_2 + \frac{1}{2}(D_1 - D_2) = \), the confinement concerns only the magnetic charges of the \( (SU(2)) \)-monopoles.

Let us now come back to the problem of ambiguity. Note, that when passing to the strong coupling domain, the sign of magnetic charge, being absorbed by light quark turning into a light dyon is not observable, since it depends on the choice of particular trajectory in the space of masses (see discussion in sect. 3.1). This means, that instead of the theory of light dyons with the charges (52) one could also consider the theory, where dyons \( \tilde{D}_1,2 \) have the charges

\[
\sqrt{2}\tilde{n}^1 = \mu_1 \oplus (-\alpha_1), \quad \sqrt{2}\tilde{n}^2 = \mu_2 \oplus (-\alpha_2)
\]

Such theory is in fact equivalent to have been just discussed above: basically only the signs of the components of the dual gauge fields \( A^D \) have to be changed for the opposite. This results, instead of (58) and (59), in the elementary string charges

\[
\tilde{n}_{S_1} = \left( \frac{1}{4}, \frac{1}{4}; \frac{3\sqrt{3}}{20}, \frac{\sqrt{3}}{20} \right)
\]  

and

\[
\tilde{n}_{S_2} = \left( -\frac{1}{4}, -\frac{1}{4}; \frac{3\sqrt{3}}{20}, \frac{\sqrt{3}}{20} \right)
\]

i.e. they will come with the opposite electric components. However, one can easily check, that instead of (60) one can now write the following decomposition

\[
\frac{1}{\sqrt{2}}(0 \oplus \alpha_1) = (0, \frac{1}{2}; 0, \frac{\sqrt{3}}{2}) = \tilde{n}_{S_1} - \frac{7}{10}\tilde{n}^1 - \frac{2}{10}\tilde{n}^2,
\]

\[
\frac{1}{\sqrt{2}}(0 \oplus \alpha_2) = (0, -\frac{1}{2}; 0, \frac{\sqrt{3}}{2}) = \tilde{n}_{S_2} - \frac{2}{10}\tilde{n}^1 - \frac{7}{10}\tilde{n}^2
\]

and we find that the main conclusion of [2] remains intact under the change of the sign of magnetic components of the dyon charges: the monopoles are confined by the strings in effective
theory, with the part of their fluxes being screened by the dyon condensate. It means in particular, that dependently on choosing a particular way in the mass plane when going from weak to strongly coupled domain of the original theory one gets different charges of the condensed dyons and string solutions in effective theory. However, this ambiguity does not influence to the observable physical conclusion: in any case the same magnetic monopoles are confined.

5 Conclusion

We have discussed here some details of the Seiberg-Witten theory for degenerate curves in the vicinity of $\mathcal{N} = 1$ vacua for the supersymmetric QCD. The computation of periods on these singular curves allow to understand, how the quantum numbers of light states change, when passing from the weak to strong-coupled domains by choosing some trajectory in the mass moduli space. This analysis helps to get an exact picture of confinement in supersymmetric QCD at strong coupling.

It has been shown above, that the computation of these periods can be performed directly, using the technique of integrable systems. In this framework it simply means, that the exact solution to supersymmetric gauge theory is characterized by a curve with two meromorphic differentials with the fixed periods. Fixing these periods is enough to provide regularization of the singular curve, corresponding to a $\mathcal{N} = 1$ vacuum, and therefore the masses of the light states can be exactly computed. This is one more nontrivial application of the classical integrable systems in quantum theory.

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