Existence, continuation and dynamics of solutions for the generalized 0-Holm-Staley equation

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Abstract

In this paper we consider a one-parameter family of nonlocal evolution equations whose nonlinearities are controlled by the parameter. We prove that if the initial momentum of the equation is compactly supported, then this property is inherited by the momentum of the solution in the set of existence. Among the members of the equation under investigation is the 0—Holm-Staley equation, or 0—equation. It is the only member of the family, with positive parameter, for which we have a conserved quantity. For this equation we establish a unique continuation result as well as the global existence of its solutions. This last property is proved based on lower bounds of one of its first order derivatives, as well as we prove that its only compactly supported solution is identically null. Returning to the original family, we made an in-depth investigation of the dynamics of some peculiar solutions of the equation, namely, peakons and cliffs. One interesting case is a member with singularities, which corresponds to negative values of the parameter. For this equation we show that the $H^1(\mathbb{R})$—norm of its solutions with enough decaying at infinity is conserved. In particular, we present a description of the dynamics of 2-peakon solutions for this singular case. More generally, we are able to provide a fairly detailed description of the peakon-antipeakon dynamics for members of the family considered when the power non-linearity is an odd integer. We also discuss the dynamics of some kink-type solutions for these equations.

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Keywords Compactly supported solutions · Unique continuation of solutions · Global existence of solutions · Dynamics of solutions · Camassa-Holm type equations

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Contents

1 Introduction ........................................... 3

2 Preliminaries and technical results ............ 7

3 Continuation and compactly supported data .... 9
   3.1 Proof of the Theorem 1.1 ................. 9
   3.2 Proof of Theorem 1.2 .................. 10
   3.3 Proof of the Theorem 1.3 .......... 11

4 Global existence of solutions .................. 12

5 Dynamics of solutions .......................... 13
   5.1 $N$–peakons for $k \in \mathbb{N}$ ............ 14
      5.1.1 2-peakon dynamics ................. 15
   5.2 $N$–peakons for $k = -n, n \in \mathbb{N}$ .... 16
      5.2.1 2-peakon dynamics for the case $k = -1$ . . . 16
   5.3 Kink-type solutions for $k \in \mathbb{N}$ ...... 19

6 Discussion and conclusion ....................... 21
1 Introduction

In [1] the equation (up to notation)

\[ u_t - u_{txx} + au^k u_x - bu^{k-1} u_x u_{xx} - cu^k u_{xxx} = 0, \]  

(1.1)

where \((a, b, c) \neq (0, 0, 0)\) and \(k \neq 0\), was considered from different perspectives, such as conserved currents, point symmetries and peakon solutions. With these restrictions on the parameters, equation (1.1) is invariant under translations in \(t, x\), scalings \((t, x, u) \mapsto (\lambda^{-k} t, x, \lambda u), \lambda > 0\), and if \(k = 1\) and \(a = c\) we also have invariance under the Galilean boost \((t, x, u) \mapsto (t, x + \epsilon at, u + \epsilon)\), see [1, Proposition 1.1]. On the other hand, if we take a careful look on symmetry properties of (1.1), we can easily observe that the translations and the scaling \((t, x, u) \mapsto (t, x, \lambda u), \lambda > 0\), are still symmetries of (1.1) whenever \(k = 0\), a case that was not considered in [1].

In the same paper, conserved currents for (1.1) were also considered, see [1, Theorem 2.1]. Two of them are important to the present work, namely,

\[ C^0 = u, \]

(1.2)

\[ C^1 = \frac{a}{k+1} u^{k+1} + \frac{k c - b}{2} u_x^2 - cu^k u_{xx} - u_{tx}, \]

for \(k = 1\) or \(b = kc\), and

\[ C^0 = \frac{u^2 + u_x^2}{2}, \]

(1.3)

\[ C^1 = \left( \frac{a}{k+2} u - cu_{xx} \right) u^{k+1} - uu_{tx}, \]

if and only if \(b = (k+1)c\).

The relevance of the conserved currents is the following: if \((C^0, C^1)\) is a conserved current for (1.1), then

\[ \left( \partial_t C^0 + \partial_x C^1 \right)|_{on (1.1)} \equiv 0, \]

(1.4)

meaning that the divergence of the conserved currents vanishes identically on the solutions of the equation. This implies that the functional

\[ u \mapsto \mathcal{H}[u] = \int_{\mathbb{R}} C^0 dx \]

is a constant (of motion), or a conserved quantity, for the equation. Very often the last integral is also referred by analysts as conservation law for the equation. We do not follow this terminology in the present paper because for us a conservation law for (1.1) is the divergence expression (1.4) of conserved currents taken on the solutions of (1.1). For further details, see [1, 6] and references thereof.

Taking \(a = (b + c)\) in (1.1) we obtain

\[ u_t - u_{txx} + (b + c) u^k u_x = bu^{k-1} u_x u_{xx} + cu^k u_{xxx}, \]  

(1.5)

which was considered in [23]. We observe that (1.1) with \(a = c = 1\) and \(b = 0\) gives (note that if \(b = 0\) and \(a = c \neq 0\) we can always proceed with a scaling in \(t\) and take \(c = 1\))

\[ m_t + u^k m_x = 0, \quad m = u(t, x) - u_{xx}(t, x), \quad t > 0, \]  

(1.6)
and for \( k = 1 \) equation (1.6) is reduced to the equation \( m_t + um_x = 0 \), which is a very particular case of the \( b \)–equation

\[
m_t + um_x + bu_xm = 0
\]  

(1.7)

introduced in [9], later investigated in [17][18] by Holm and Staley, and sometimes referred as Holm-Staley (HS) equation. For this reason we shall refer to (1.6) with arbitrary power as generalized 0-Holm-Staley equation, or simply \( g0-HS \) equation for short.

It is worth mentioning that (1.6) with \( k = 1 \) can also be obtained from shallow water elevation equations via Kodama transformation, see [10,11], which shows its relevance in the study of shallow water models. Solutions of (1.6) with \( k = 1 \) (or (1.7) with \( b = 0 \)) were considered in [17, 18, 25].

More recently, wave-breaking and global existence of solutions for (1.5) were considered in [23]. However, some of the results proved there were done with the restriction \( b \neq 0 \). Also, in [15] ill-posedness for the \( b \)–equation (1.7) was also considered when \( b > 1 \).

We note that the results in [1, 15, 23] suggest that the cases \( b = 0 \) or \( k / \in \{−1, 0, 1\} \) make (1.1) very peculiar. This is also reinforced by the results of [26], where the solutions of (1.7) were studied and the case \( b = 0 \) was excluded in some analysis, such as in theorems 2.1 and 2.2.

It is also intriguing that (1.1) does not have conserved currents up to second order for \( b = 0 \) and \( k / \in \{−1, 0, 1\} \) (see [1, Theorem 2.1]), a fact also observed in [17] when (1.6) was considered with \( k = 1 \). All of these results make us conjecture that no further conservation laws can be obtained to (1.6) beyond those reported in [1].

For equations of the type (1.1), the conserved quantities provide qualitative information about its solutions subject to an initial condition \( u(0, x) = u_0(x) \). For example, if \( b = 0 \) and \( a = c = k = 1 \), then (1.1) has the conserved quantity

\[
\mathcal{H}_0[u] = \int_R udx.
\]  

(1.8)

It means that if \( u \) does not change its sign, then the \( L^1(\mathbb{R}) \)–norm of the rapidly decaying solutions of (1.1) with \( b = 0 \) and \( a = c = k = 1 \) is conserved (this will be better explored in Theorem 3.1 in Subsection 3.1). On the other hand, if \( k = −1 \), then the equation (1.6) has the conserved quantity

\[
\mathcal{H}[u] = \frac{1}{2} \int_R (u^2 + u_x^2)dx,
\]  

(1.9)

which is essentially the square of the \( H^1(\mathbb{R}) \)–norm of the solution \( u \) of the equation and, therefore, for solutions decaying to 0 as \( x \to \pm \infty \), their \( H^1(\mathbb{R}) \)–norms are conserved.

The aim of the present paper is to consider the Cauchy problem

\[
\begin{align*}
\begin{cases}
    u_t - u_{txx} + u^k u_x - u^k u_{xxx} = 0, \\
    u(0, x) = u_0(x),
\end{cases}
\end{align*}
\]  

(1.10)

and determine properties and behaviour of its solutions, such as peakons and cliffs.

In [23, Theorem 2.1] it was established the local well-posedness of the equation (1.5) with initial data \( u(0, x) = u_0(x) \), where \( u_0 \in B^s_{p,r}(\mathbb{R}) \) (\( B^s_{p,r}(\mathbb{R}) \) denotes a Besov space, see [23] for further details). Taking \( p = r = 2 \), we can ensure local well-posedness to (1.5) with \( u(0, \cdot) = u_0(\cdot) \in H^s(\mathbb{R}), \)
Therefore the local well-posedness for (1.10) is proved directly by invoking these results and, therefore, is omitted.

Our first results regarding (1.10) are:

**Theorem 1.1.** Given an initial data $u_0 \in H^3(\mathbb{R})$, let $u$ be the corresponding solution of (1.10), where $k$ is a positive integer.

1. If $m_0$ does not change sign, then $m$ does not as well. Moreover, $\text{sgn}(m) = \text{sgn}(m_0)$.

2. The momentum $m$ is compactly supported if and only if $m_0$ is compactly supported.

For $k = 1$, on the other hand, we are able to prove a continuation result for solutions of (1.10), as stated in the next result.

**Theorem 1.2.** Let $T > 0$, $I \subseteq \mathbb{R}$ a non-empty open interval, $\Omega := (0, T) \times I$, and assume that $u \in C^0((0, T) \times H^s(\mathbb{R}))$, $s > 3/2$, is a solution of the equation

$$u_t - u_{txx} + uu_x - uu_{xxx} = 0.$$  (1.11)

If $u|_\Omega = 0$, then $u \equiv 0$ on $[0, T) \times \mathbb{R}$ and, moreover, $u$ can be extended globally.

Moreover, it is possible to relax the condition that $u$ vanishes on $\Omega = (0, T) \times I$, for some open set $I \subseteq \mathbb{R}$, in Theorem 1.2. In fact, we can prove a similar result on an arbitrary, non-empty set $\Omega \subseteq (0, T) \times \mathbb{R}$, but the price, however, is to impose that the solution does not change its sign.

**Theorem 1.3.** Let $t_0, t_1, T \in \mathbb{R}$ such that $0 < t_0 < t_1 < T$, $I \subseteq \mathbb{R}$ a non-empty open interval, and $\Omega := (t_0, t_1) \times I$. Suppose that $u \in C^0((0, T) \times H^s(\mathbb{R}))$, $s > 3/2$, is a solution of (1.11). If $u$ does not change its sign and $u|_\Omega = 0$, then $u \equiv 0$ on $[0, T) \times \mathbb{R}$ and, moreover, $u$ can be extended globally.

We have a very strong consequence of Theorem 1.3

**Corollary 1.1.** Assume that $u_0 \in H^3(\mathbb{R})$ is a non-vanishing compactly supported data for

$$\begin{cases}
  u_t - u_{txx} + uu_x - uu_{xxx} = 0, \\
  u(0, x) = u_0(x),
\end{cases}$$

such that $m_0$ does not change sign. Then the corresponding solution $u$ is not compactly supported.

Theorems 1.1–1.3 only request that the solution exists locally. We also observe that the local well-posedness assured by the results in [14, 23] guarantees the existence of a solution $u \in C^0([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$, for a certain $T > 0$ and $s > 3/2$. Two questions of capital importance are: Does this solution exist for $T = \infty$? Does this solution develop any singularity for $T < \infty$? The first question deals with problem of global existence, whereas the second is related to the question of blow up in finite time, meaning that the solution becomes unbounded for finite values of $T$.

We note that in [23] the question of global existence of solutions to (1.5) with a certain initial data is addressed, but not for equation (1.6), meaning that while in [23] we have the local existence for (1.10), its global existence is not considered, see [23, Theorem 4.1]. Also, in the same reference the problem of blow up is considered. In fact, it was shown that the first blow up of (1.5) occurs only as a wave-breaking. Likewise in the case of global existence, the results for wave-breaking proved in [23, Theorem 5.1] are not applicable to (1.10).
We can improve Yan’s achievements [23] regarding the global existence of solutions of (1.1) with the following global existence result, in which (1.8) is of vital importance:

**Theorem 1.4.** Given $u_0 \in H^3(\mathbb{R}) \hookrightarrow H^2(\mathbb{R})$, let $u(t, \cdot) \in H^3(\mathbb{R})$ be the unique solution of (1.12). If $m_0 := u_0 - u_0'' \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ and it does not change sign, then the solution $u$ exists globally in $C^0([0, \infty); H^3(\mathbb{R})) \cap C^1([0, \infty); H^2(\mathbb{R}))$.

We recall that for dealing with both global existence and wave-breaking problems, we usually need:

- Local existence results established;
- Qualitative properties of the solutions, quite often manifested through conserved quantities or, which is the same, conserved currents for the equation must be known. In the absence of suitable conserved currents, other similar information, such as estimates on the solutions should be at our disposal.

Apart from the cases $k = \pm 1$, (1.6) does not have other known conservation laws, which means that we do not have enough information to determine whether the local results can be extended to a global property for general $k$ using the conserved quantities.

Even in the case $k = 1$, the only known conservation law for the equation in (1.12) is useless for extending the local solution to global one. Therefore, in order to prove Theorem 1.4 we show that the solution of (1.12) is bounded from above by the $H^2(\mathbb{R})$-norm of the solutions, provided that the $x-$derivatives of the solution is bounded from below.

We note that for negative values of $k$ in (1.12) the equation is singular and, therefore, the local existence results proved in [23] cannot be extended to this case. A natural way to overcome this problem would be invoking Kato’s machinery [21], but the singularity does not allow application of such approach. For the particular case $k = -1$ we have a nice conserved quantity for the solutions, given by (1.9), but we are unable to use it to investigate global existence of its solutions because we cannot even assure the existence of a unique solution to the local level.

**Organization of the paper.** In Section 2 we prove several technical results that will be useful in the demonstration of our main results. Next, in Section 3 we study the behavior of compactly supported data and provide the continuation of solutions for the case $k = 1$ to prove theorems 1.1–1.3. In Section 4 we prove Theorem 1.4 Finally, in Section 5 we study solutions of the equation (1.6). More precisely, we investigate (multi)peakons and other wave solutions of (1.6) for any integer $k$. For the cases $k = 1$ and $k = -1$ we also use the conserved quantities (1.8) and (1.9), respectively, to construct solutions compatible with them. Our discussions and conclusions are presented in Section 6.

**Challenges and novelties of the paper.** The unique continuation result for equation (1.12) is based on some ideas introduced in [20] and the use of the conserved quantity (1.8), as observed in [12]. The fact that the integrand in (1.8) is not necessarily positive neither negative brings some complications in applying the use of (1.8). In order to overcome this problem we then find conditions for the solutions of (1.12) does not change its sign, which then implies that the integral kernel in (1.8) is either non-positive or non-negative. As a consequence of this fact we show that the $L^1(\mathbb{R})$-norm of the solution and the corresponding momentum are conserved, see Theorem 3.1, which will be of great relevance to prove Theorem 1.4 that guarantees the global existence of solutions of the problem (1.12). We observe that the Cauchy problem (1.12) has very little structure and the only structural property known for the equation in (1.12) is the invariant (1.8), which make the proof of global existence quite challenging. In order to prove it, we show that if the initial data is
in $H^3(\mathbb{R})$ and its momentum does not change sign, then the $x$ derivatives of the solution $u$ of (1.12) is bounded from below by the $L^1(\mathbb{R})$—norm of the initial momentum. This is enough to assure that the $H^2(\mathbb{R})$—norm of the solution is bounded, for each $t \in \mathbb{R}$.

Beyond the qualitative properties given in theorems [1.1–1.4] we also consider peakon and cliff solutions of the equation (1.6). We show that such solutions may exist for any integer $k \neq 0$ (the case $k = 0$ is not considered because the resulting equation is linear). We pay considerable attention to equation (1.10) with $k = -1$, which brings a considerable singularity to the problem, but has the $H^1(\mathbb{R})$—norm of the solutions as a conserved quantity. We found explicit solutions showing peakon-peakon and peakon-antipeakon dynamics, in particular, their collision. As far as we know, this is the first time that a singular non-evolution equation of the type (1.6) has peakon solutions of the same shape of the Camassa-Holm equation [3]. Moreover, although some evolution equations having peakon type solutions are known, e.g., see [5], we have no information of any other singular equation having multi-peakon solutions.

**Notation.** Throughout this paper $\mathbb{Z}$ denotes the set of the integer numbers, while $\mathbb{N}$ means the set of positive integers. We denote the usual Sobolev space by $H^s(\mathbb{R})$, for each $s \in \mathbb{R}$, with corresponding Sobolev norm denoted by $\| \cdot \|_{H^s}$. Given two functions $f$ and $g$, their convolution is denoted by $f * g$. If $u = u(t, x)$, we denote by $u_0(x)$ the function $x \mapsto u(0, x)$, $m = u - u_{xx}$ and $m_0 = u - u'_0$. Note that $m = (1 - \partial_x)^2 u$ and then $u = g * m$, where $g(x) = e^{-|x|}/2$. Of great importance for us is the fact that if $s \geq t$, then $H^s(\mathbb{R}) \hookrightarrow H^t(\mathbb{R})$. In particular, $H^3(\mathbb{R}) \hookrightarrow H^2(\mathbb{R})$, which, in particular, implies $\|u\|_{H^3} \leq c\|u\|_{H^2}$, for some $c > 0$.

### 2 Preliminaries and technical results

In this section we prove some technical results that will be relevant in the proof of theorems [1.1–1.4].

We begin with the following:

**Lemma 2.1.** Let $u = u(t, x)$ be a solution of

$$u_t - u_{txx} + uu_x - uu_{xxx} = 0$$

such that $u(0, x) =: u_0(x)$ and $u(t, \cdot)$, $u_x(t, \cdot)$, $u_{xx}(t, \cdot)$ and $u_{tx}(t, \cdot)$ are integrable and vanish at $x = \pm \infty$ for all values of $t$ such that the solution exists. Then

$$\int_{\mathbb{R}} u \, dx = \int_{\mathbb{R}} u_0 \, dx = \int_{\mathbb{R}} m \, dx = \int_{\mathbb{R}} m_0 \, dx,$$

where $m_0 := u_0 - u'_0$.

**Proof.** We note that (1.2) is a conserved vector for (2.1), which means that

$$\partial_t(u) + \partial_x(u^2 + u_x^2/2 - uu_x + u_{tx}) = 0.$$  

(2.3)

Let $\mathcal{H}_0[u]$ be given by (1.8), where $u$ is the solution of (2.1) with the initial datum $u_0$. From (2.3) we have

$$\frac{d}{dt} \mathcal{H}_0[u] = \frac{d}{dt} \int_{\mathbb{R}} u = \int_{\mathbb{R}} \partial_t u = - (u^2 + u_x^2/2 - uu_x - u_{tx})|_{-\infty}^\infty = 0,$$

which implies that $\mathcal{H}[u] := \mathcal{H}[u(t, x)] = \mathcal{H}[u(0, x)] =: \mathcal{H}[u_0]$. This proves the first equality.

Now we observe that

$$\int_{\mathbb{R}} m \, dx = \int_{\mathbb{R}} (u - u_{xx}) \, dx = \mathcal{H}[u] - \int_{\mathbb{R}} u_{xx} \, dx = \mathcal{H}[u] - u_x|_{-\infty}^\infty = \mathcal{H}[u],$$
which is enough to prove (2.2). □

Our next lemma is similar to [4, Theorem 3.1] and, actually, it reduces to the same result whenever $k = 1$.

**Lemma 2.2.** Given $u_0 \in H^2(\mathbb{R})$, let $u \in C^1([0, T), H^2(\mathbb{R}))$ be the corresponding unique solution of (1.1). Then the initial value problem

\[
\begin{cases}
\partial_t y(t, x) = u^k(t, y), \\
y(0, x) = x,
\end{cases}
\]

(2.4)

where $k$ is a positive integer, has a unique solution $y(t, x)$ such that $y_x(t, x) > 0$ for any $(t, x) \in [0, T) \times \mathbb{R}$. Moreover, for each $t \geq 0$ fixed, $y(t, \cdot)$ is an increasing diffeomorphism on the line.

**Proof.** Since $u \in C^1([0, T) \times \mathbb{R})$, then both $u(t, \cdot)$ and $u_x(t, \cdot)$ are bounded and Lipschitz, while $u(\cdot, x)$ and $u_x(\cdot, x)$ are $C^1$. For each fixed $x \in \mathbb{R}$, the Picard-Lindelöf Theorem [2, page 10] assures the existence of a unique solution $y(\cdot, x)$ satisfying the problem (2.4) and defined on $[0, T)$, for some $T > 0$.

If we let $x$ varies, we can then differentiate (2.4) and obtain

\[
\begin{cases}
\partial_t y_x(t, x) = ku^{k-1}(t, y(t, x))u_x(t, y(t, x))y_x(t, x), \\
y_x(0, x) = 1.
\end{cases}
\]

(2.5)

Fixing $x$ and defining $T_x := \sup\{t \in [0, T), y_x(t, x) > 0\}$. Therefore, for each $t \in [0, T_x)$, we have

\[
y_x(t, x) = \exp\left(ku^{k-1}(t, y(t, x))u_x(t, y(t, x))\right) > 0.
\]

(2.6)

We claim that $T_x = T$. Actually, if it were not true, for some $\overline{x} \in \mathbb{R}$ we would have $y_x(t, \overline{x}) = 0$, which is a clear contradiction with (2.6).

We observe the continuity of $y(t, \cdot)$ implies that $J_t := \{y(t, x), x \in \mathbb{R}\}$ is an open interval. Again, by (2.6) we are forced to conclude that $y(t, \cdot)$ is a diffeomorphism between $\mathbb{R}$ and $J_t$. To conclude the demonstration we need to show that $J_t = \mathbb{R}$.

By the Sobolev Embedding Theorem [22, p. 317], given $t \in (0, T)$, the function $u_x(s, z)$ is uniformly bounded for each $[s, z] \in [0, t] \times \mathbb{R}$ and, consequently, there exists a positive number $k_t > 0$ such that $e^{-k_t} \leq y_x(t, x) \leq e^{k_t}$, which, after integration, yields the inequality $e^{-k_t}x \leq y(t, x) \leq e^{k_t}x$. This implies that $J_t$ cannot have either lower or upper bounds. □

**Theorem 2.1.** Let $u_0 \in H^3(\mathbb{R})$ be an initial data for (1.12), with corresponding solution $u$ and lifespan $T$, and $k$ be a positive integer. Assume that the sign of $m_0$ does not change. Then

1. $\text{sgn} (u) = \text{sgn} (u_0) = \text{sgn} (m) = \text{sgn} (m_0)$ and they do not change;

2. If $m_0 \in L^1(\mathbb{R})$, then $-u_x(t, x) \leq ||m_0||_{L^1}$, for any $(t, x) \in [0, T) \times \mathbb{R}$.

**Proof.** Let $y$ be the diffeomorphism given in Lemma 2.2. Differentiation of $m(t, y(t, x))$ with respect to $t$ yields

\[
\frac{d}{dt} m(t, y(t, x)) = m_t + y_m x = m_t + u^k m_x = 0,
\]
which means that \( m(t, y(t, x)) \) does not depend on \( t \) and, therefore, \( m(t, y(t, x)) = m_0(x) \). Since \( y \)
is a diffeomorphism, we conclude that \( \text{sgn} \left( m_0(\cdot) \right) = \text{sgn} \left( m(t, y(t, \cdot)) \right) \). Therefore, \( m_0 \) does not change sign if and only if \( m \) does not change sign. Now we observe that \( u(t, x) = g \ast m(t, x) \), where \( g(x) = e^{-|x|}/2 \). Since \( g(x) > 0 \), then \( \text{sgn} \left( u(t, x) \right) = \text{sgn} \left( m(t, x) \right) \) and \( \text{sgn} \left( u_0(x) \right) = \text{sgn} \left( m_0(x) \right) \). This proves \( 1 \). To prove the second part, let us first assume \( m_0 \geq 0 \). Then
\[
\| m_0 \|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} m_0 \, dx = \int_{\mathbb{R}} m \, dx \geq \int_{-\infty}^{\infty} m \, dx = \left( \int_{-\infty}^{x} u \right) - u_x(x, t).
\]
Since \( u \geq 0 \) and
\[
0 \leq \int_{-\infty}^{x} u \, dx \leq \int_{\mathbb{R}} u \, dx < \infty,
\]
we have \( \| m_0 \|_{L^1(\mathbb{R})} = -u_x(t, x) \).
Let us now prove the inequality whenever \( m_0 \leq 0 \). In this case, we have \(-m \geq 0 \) and \(-u \geq 0 \) as well. Then
\[
-\int_{\mathbb{R}} u \, dx = -\int_{\mathbb{R}} m \, dx = -\int_{\mathbb{R}} m_0 \, dx < \infty.
\]
Since
\[
0 \geq \int_{-\infty}^{x} m \, dx = \int_{-\infty}^{x} (u - u_{xx}) \, dx = \int_{-\infty}^{x} u \, dx - u_x(t, x),
\]
we have
\[
-u_x(t, x) \leq -\int_{-\infty}^{x} u \, dx \leq -\int_{\mathbb{R}} m_0 \, dx.
\]
As a consequence we have
\[
-u_x(t, x) \leq -\int_{-\infty}^{x} u \, dx \leq \int_{\mathbb{R}} -u \, dx = \int_{\mathbb{R}} -m_0 \, dx = \| m_0 \|_{L^1(\mathbb{R})},
\]
which proves the result.

\( \square \)

3 Continuation and compactly supported data

Here we prove theorems \( 1.1 \)-\( 1.3 \).

3.1 Proof of the Theorem \( 1.1 \)

Theorem \( 1.1 \) is an immediate recollection of results proven so far. In fact, if \( m_0 \) does not change sign, then Theorem \( 2.1 \) and the fact that \( \text{sgn} \left( m_0(\cdot) \right) = \text{sgn} \left( m(t, y(t, \cdot)) \right) \) conclude the proof of item 1. Moreover, if we assume that \( m_0 \) is compactly supported on \([a, b]\), then we conclude that \( m \) is supported on \([m(t, a), m(t, b)]\). Conversely, fix \( t > 0 \). If \( m(t, x) \) is compactly supported on \([a, b]\), then \( m_0 \) vanishes identically outside the interval \([q^{-1}(t, a), q^{-1}(t, b)]\).

Theorem \( 1.1 \) has a very strong consequence.

**Theorem 3.1.** Assume that \( u_0 \in H^3(\mathbb{R}) \) is such that \( m_0 \) does not change its sign and let \( u \) be the corresponding solution of \( (1.12) \). Then the quantities \( \| u(t, \cdot) \|_{L^1(\mathbb{R})} \) and \( \| m(t, \cdot) \|_{L^1(\mathbb{R})} \) are constant.

**Proof.** Let us prove that \( \| u(t, \cdot) \|_{L^1(\mathbb{R})} \) and \( \| m(t, \cdot) \|_{L^1(\mathbb{R})} \) are conserved for any \( t \in [0, T) \). Firstly, assume that \( u_0 \geq 0 \). Then \( u \geq 0 \) and the conserved quantity \( (1.8) \) yields
\[
\mathcal{H}_0[u] = \int_{\mathbb{R}} u(t, x) \, dx = \int_{\mathbb{R}} |u(t, x)| \, dx = \| u(t, \cdot) \|_{L^1(\mathbb{R})},
\]
which means that \( \|u(t, \cdot)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})} \). On the other hand, if \( u \leq 0 \), then
\[
\mathcal{H}_0[u] = -\int_{\mathbb{R}} (-u(t,x)) \, dx = -\int_{\mathbb{R}} |u(t,x)| \, dx = -\|u(t, \cdot)\|_{L^1(\mathbb{R})},
\]
and, again, \( \|u(t, \cdot)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})} \).

By Lemma 2.1 we know that
\[
\int_{\mathbb{R}} m(t,x) \, dx = \int_{\mathbb{R}} u(t,x) \, dx,
\]
and the remaining part of the demonstration is analogous and, therefore, we omit it. \( \square \)

### 3.2 Proof of Theorem 1.2

We begin by recalling that if \( u_0 \in H^s(\mathbb{R}) \), with \( s > 3/2 \), then the Cauchy problem of (1.7) with \( u \) satisfying \( u(0, x) = u_0(x) \) has a unique local solution \( u \in C^0([0, T), H^s(\mathbb{R})) \cap C^1([0, T), H^{s-1}(\mathbb{R})) \), see [23, Corollary 2.1] and [14, Theorem 1.1]. Instead of proving Theorem 1.2 directly, we shall prove the following stronger result regarding the \( b \)-equation (1.7).

**Theorem 3.2.** Let \( b \in [0, 3] \), \( s > 3/2 \), \( u_0 \in H^s(\mathbb{R}) \), \( u \) be the corresponding solution of (1.7) satisfying \( u(0, x) = u_0(x) \), \( I \subseteq \mathbb{R} \) be a non-empty open interval and \( \Omega = (0, T) \times I \), where \( T \) is the lifespan of the solution \( u \). If \( u|_{\Omega} \equiv 0 \), then \( u \equiv 0 \) on \([0, T) \times \mathbb{R} \). Moreover, the solution can be extended globally is immediate once we prove the result, since we take \( u(t, x) = 0 \), for all \( (t, x) \).

**Proof.** The fact that \( u \) can be extended globally is immediate once we prove the result, since we take \( u(t, x) = 0 \), for all \((t, x)\).

Note that (1.7) can be rewritten as
\[
u_t + uu_x + \partial_x \Lambda^{-2} \left( \frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right) = 0. \tag{3.1}
\]

Fix \( t_0 \in (0, T) \) and define \( F : \mathbb{R} \to \mathbb{R} \) by
\[
F(x) := \partial_x \Lambda^{-2} \left( \frac{b}{2} u(t_0, x)^2 + \frac{3-b}{2} u_x(t_0, x)^2 \right). \tag{3.2}
\]

We observe that

- The conditions on \( u \) and the definition of \( F \) implies that \( F \in C^1(\mathbb{R}) \);
- If \( x \in I \), by (3.1) and (3.2) we have
  \[
  F(x) = \partial_x \Lambda^{-2} \left( \frac{b}{2} u(t_0, x)^2 + \frac{3-b}{2} u_x(t_0, x)^2 \right) = -(u_t + uu_x)(t_0, x) \equiv 0.
  \]
- By the Fundamental Theorem of Calculus, given \( x_0, x_1 \in I \), with \( x_0 < x_1 \), we have
  \[
  0 = F(x_1) - F(x_0) = \int_{x_0}^{x_1} F'(x) \, dx.
  \]
- We have the identity \( \partial_x^2 \Lambda^{-2} = \partial_x^2 \Lambda^{-2} - 1 \).
We observe that Theorem 1.2 is nothing but an immediate corollary of Theorem 3.2 with \( \Omega \) would be able to find an open set (and, therefore, it can be extended globally), finishing the proof of Theorem 1.3.

To prove Theorem 1.3, we claim that if \( t_0 = \|u\|_1 \cdot x(t_0, x)^2 \), then \( f \) is non-negative, continuous, and

\[
(g \ast f)(x) = \int_{-\infty}^{+\infty} \frac{e^{-|x-y|}}{2} f(y) dy \geq 0.
\]

Moreover, the last integral vanishes if and only if \( f(x) \equiv 0 \).

From the observations above, we have

\[
0 = F(x_1) - F(x_0) = \int_{x_0}^{x_1} F'(x) dx = \int_{x_0}^{x_1} \left( \Lambda^{-2}(f)(x) - f(x) \right) dx = \int_{x_0}^{x_1} (g \ast f)(x) dx,
\]

which implies that

\[
\frac{b}{2} u(t_0, x)^2 + \frac{3 - b}{2} u_x(t_0, x)^2 = 0. \quad (3.3)
\]

If \( b \in (0, 3] \), then \((3.3)\) implies that \( u(t_0, x) = 0 \). If \( b = 0 \), we are forced to conclude that \( u(t_0, x) = c \), for some constant \( c \). Since \( u \to 0 \) as \( |x| \to \infty \), we conclude again that \( u(t_0, x) = 0 \).

This proves that for each \( t_0 \in (0, T) \), then \( x \mapsto u(t_0, x) \) vanishes. Therefore, the solution vanishes on \((0, T) \times \mathbb{R}\) and, by continuity, on \([0, T) \times \mathbb{R}\).

We observe that Theorem 1.2 is nothing but an immediate corollary of Theorem 3.2 with \( b = 0 \).

### 3.3 Proof of the Theorem 3.1

To prove Theorem 3.1, we claim that if \( u \) vanishes on \((t_0, t_1) \times I\), then it vanishes on \((t_0, t_1) \times \mathbb{R}\). In order to prove it, consider the function \((3.2)\) with \( b = 0 \). Proceeding similarly as in the demonstration of Theorem 3.2, for each \( t^* \in (t_0, t_1) \) fixed, we conclude that \( u_x(t^*, x) = 0 \), for all \( x \in \mathbb{R} \), which forces \( u(t^*, x) = 0 \), for all \( x \in \mathbb{R} \). Since this holds to all \( t^* \in (t_0, t_1) \), we conclude our claim.

For any \( t^* \in (t_0, t_1) \), we have \( u(t^*, x) = 0 \), which means that \( H_0[u(t^*, \cdot)] = 0 \), where \( H_0[u] \) is given by \((1.8)\). In view of the conservation of \( H_0[u] \), this implies that \( H_0[u(t, \cdot)] = 0 \) for every \( t \) such that the solution is defined. On the other hand, since the sign of \( u \) does not change, we note that the conserved quantity \((1.8)\) vanishes if and only if \( u(t, \cdot) \equiv 0 \), which says that \( u \equiv 0 \) on \([0, T) \times \mathbb{R}\) (and, therefore, it can be extended globally), finishing the proof of Theorem 3.1.

**Proof Corollary 1.1** Assume that \( 0 \not= u_0 \in H^3(\mathbb{R}) \) be a compactly supported initial data for the problem \((1.12)\), and \( u \) its corresponding solution. If \( u \) were compactly supported, then we would be able to find an open set \( \Omega \subseteq (0, T) \times \mathbb{R} \) such that \( u(t, x) = 0 \), for all \( (t, x) \in \Omega \). If the corresponding momentum \( m_0 \) does not change its sign, then, by Theorem 2.1, the sign of \( u \) does not change, while by Theorem 3.1 \( u \) is identically 0. Combining this with Theorem 3.1, we have \( 0 = \|u(t, \cdot)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})} > 0 \), which is a contradiction.
4 Global existence of solutions

We begin with the following result.

Lemma 4.1. Given \( u_0 \in H^3(\mathbb{R}) \hookrightarrow H^2(\mathbb{R}) \), let \( u \in C^0([0, T), H^3(\mathbb{R})) \) be the unique solution of (2.1) for some \( T > 0 \). If there exists a positive constant \( \kappa \) such that \( u_x > -\kappa \), then \( \|u\|_{H^2} \leq e^{\kappa t/2}\|u_0\|_{H^2} \).

Proof. Let \( u_0 \) and \( u \) be given as enunciated. After multiplying (2.1) by \( u \) and some manipulation, we obtain

\[
\partial_t \left( \frac{u^2 + u_x^2}{2} \right) + \partial_x \left( \frac{u^3}{3} - uu_t - u^2u_x \right) + 2uu_xu_{xx} = 0. \tag{4.1}
\]

Now calculating the \( x \) derivative of (2.1) and multiplying the result by \( u_x \) we have

\[
\partial_t \left( \frac{u_x^2 + u_{xx}^2}{2} \right) - \partial_x(u_xu_{tx} + uu_xu_{xxx})
+ u_x \partial_x(u_{xx}^2) + u \partial_x(u_x^2) + u_x^3 = 0. \tag{4.2}
\]

Summation of (4.1) and (4.2) yields

\[
\partial_t \left( \frac{u^2 + 2u_x^2 + u_{xx}^2}{2} \right) + \partial_x \left( \frac{u^3}{3} - uu_t - u^2u_x - u_xu_{tx} - uu_xu_{xxx} \right)
+ u_x \partial_x(u_{xx}^2) + \frac{3}{2}u \partial_x(u_x^2) + u_x^3 = 0.
\]

Let

\[ I = \int_{\mathbb{R}} (u^2 + 2u_x^2 + u_{xx}^2) \, dx. \]

Then \( I \geq \|u\|_{H^2}^2 \) and

\[
\frac{1}{2} \frac{d}{dt} I = \frac{3}{2} \int_{\mathbb{R}} u \partial_x(u_x^2) \, dx + \int_{\mathbb{R}} u_x^3 \, dx + \frac{1}{2} \int_{\mathbb{R}} u \partial_x(u_{xx}^2) \, dx = \frac{3}{2} \int_{\mathbb{R}} u_x^3 \, dx + \int_{\mathbb{R}} u_x^3 \, dx - \frac{1}{2} \int_{\mathbb{R}} u_x(u_{xx}^2) \, dx
= \frac{1}{2} \int_{\mathbb{R}} (-u_x)^3 \, dx + \frac{1}{2} \int_{\mathbb{R}} (-u_x)(u_{xx}^2) \, dx.
\]

Since \( u_x > -\kappa \), we have

\[
\frac{1}{2} \frac{d}{dt} I < \frac{\kappa}{2} \int_{\mathbb{R}} u_x^2 \, dx + \frac{\kappa}{2} \int_{\mathbb{R}} u_{xx}^2 \, dx = \frac{\kappa}{2} \int_{\mathbb{R}} (u_x^2 + u_{xx}^2) \, dx \leq \frac{\kappa}{2} \|u\|_{H^2}^2,
\]

that is,

\[
\frac{d}{dt} \|u\|_{H^2} \leq \frac{\kappa}{2} \|u\|_{H^2}.
\]

From the Grönwall’s inequality, we conclude that \( \|u\|_{H^2} \leq e^{\kappa t/2}\|u_0\|_{H^2} \). \( \square \)
Proof of Theorem 4.1 If \( m_0 \in L^1(\mathbb{R}) \) does not change sign, then \( \text{sgn}(u) = \text{sgn}(m) = \text{sgn}(m_0) = \text{sgn}(u_0) \) in view of Theorem 2.1. It also implies that \( u_x \) is bounded from below. Theorem 4.1 is then a consequence of Lemma 4.1.

5 Dynamics of solutions

In this section we investigate some solutions of equation (1.6) of the form

\[
    u(t, x) = \sum_{j=1}^{N} u_j(t, x),
\]

(5.1)

where each function \( u_j(t, x) \) in (5.1) is at least continuous. Namely, we consider the following types of solutions:

- with pointed crest, in which their lateral derivatives are finite, but not equal. A typical solution is obtained by taking

\[
    u_j(t, x) = p_j(t)e^{-|x-q_j(t)|},
\]

(5.2)

where the functions \( p_j, q_j, 1 \leq j \leq N \), are functions having first order derivatives, but they are not necessarily continuously differentiable.

This sort of solution is best known as multi-peakons or \( N \)-peakons, see [1, 6, 9, 17, 18]. Of particular interest is the 1-peakon \( u(t, x) = Ae^{-|x-ct|} \), for certain constants \( A \) and \( c \), which has a jump in the derivatives of \( u \) along the curve \( t \mapsto (t, ct) \);

- with no jump in the derivatives, but anti-symmetric along certain curves. These solutions are called cliffs, see [17, 18], and are of the form (at least in our case)

\[
    u_j(t, x) = c_j(t) + b_j(t) \text{sgn}(x - p_j(t))(1 - e^{-|x-p_j(t)|}),
\]

(5.3)

for certain functions \( b_j(t), c_j(t) \) and \( p_j(t) \).

We note that all the ansatzes (5.2) and (5.3), when substituted into (5.1), will lead to a dynamical system to the corresponding unknown functions.

Let \( y = y(t) \in \mathbb{R}^n, t \in \mathbb{R} \) and \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a function. We recall that a point \( y_* \in \mathbb{R}^n \) is a critical point of the dynamical system \( y'(t) = F(y(t)) \) if \( F(y_*) \equiv 0 \). Moreover, if \( F \) is continuous and locally Lipschitz in a certain domain \( \Omega \), then the Picard-Lindelöf theorem (see [2, page 10]) assures the existence of a unique solution for the problem \( y'(t) = F(y(t)), y(t_0) = y_0 \).

Finally, we observe that some of the functions \( u = u(t, x) \) we want to consider here are continuous, and only continuous. Then, these solutions are to be understood as distributional ones. Henceforth, we shall use some facts about distributions. The Dirac delta distribution centered at a point \( x_0 \) is denoted by \( \delta(x - x_0) \), while \( \text{sgn}(x) \) means the sign distribution. It is related to the Dirac delta distribution by the relation \( \text{sgn}'(\cdot) = 2\delta(\cdot) \). For further details see [24, Chapter 2], and [19, Chapter 11]. We also guide the reader to [1, 16] for further details since these references study similar solutions following the same approach as ours. In particular, in [16] several similar calculations as those in our paper are done with enough detail.
5.1 $N$-peakons for $k \in \mathbb{N}$

Let us assume that

$$u(t, x) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|}, \quad (5.4)$$

for certain functions $p_i = p_i(t)$ and $q_i = q_i(t)$, $i = 1, \ldots, N$, is a solution of (1.6). We note that these solutions look like $N$ pulses, with amplitudes $p_i(t)$ and positions $q_i(t)$. They are called generically peakons, although sometimes peakon is referred to pulses with positive amplitudes whereas those with negative amplitude are named antipeakons, see Figure 3.

Figure 1: Function $u(t, x) = e^{-|x-t+1|} - e^{-|x+t|}$. We observe two pulses with opposite signs, one of them having positive amplitude (the one above the $x$-axis), corresponding to a peakon, and the other (below the $x$-axis), with negative amplitude, corresponding to an antipeakon.

Taking the distributional derivatives of $u$, we have

$$m = 2 \sum_{i=1}^{N} p_i \delta(x - q_i), \quad (5.5)$$

where the dependence on the variable $t$ was omitted for convenience, and

$$m_x = 2 \sum_{i=1}^{N} p_i \delta'(x - q_i), \quad m_t = 2 \sum_{i=1}^{N} p_i \delta(x - q_i) - 2 \sum_{i=1}^{N} p_i q_i' \delta'(x - q_i). \quad (5.6)$$

Substituting (5.5) and (5.6) into (1.6) and after straightforward calculations we obtain

$$2 \sum_{i=1}^{N} \left[ p_i' - k p_i u^{k-1}(t, q_i) u_x(t, q_i) \right] \delta(x - q_i)$$

$$-2 \sum_{i=1}^{N} p_i \left[ q_i' - u^k(t, q_i) \right] \delta'(x - q_i) = 0. \quad (5.7)$$

We can argue that the (5.7) must vanish identically, which would then imply that the coefficients of $\delta$ and $\delta'$ are 0. However, if we follow the steps in [6, Subsection 6.2 ] (the key idea is to use Lemma
1 of [13 page 9]), or the same steps as in [1][16], we can rigorously prove that
\[
\begin{align*}
  p'_i &= kp_i u^{k-1}(t, q_i)u_x(t, q_i), \\
  q'_i &= u^k(t, q_i),
\end{align*}
\]  
(5.8)

where
\[
\begin{align*}
  u(t, q_i) &= \sum_{j=1}^{N} p_j(t)e^{-|q_i(t) - q_j(t)|}, \\
  u_x(t, q_i) &= -\sum_{j=1}^{N} \text{sgn}(q_i(t) - q_j(t))p_j(t)e^{-|q_i(t) - q_j(t)|}.
\end{align*}
\]  
(5.9)

A single peakon solution can be easily found. In fact, for the case of 1-peakon we have \(q = \text{sgn} q_0 = \text{sgn} p_0 = 1\). Then (5.8) becomes
\[
\begin{align*}
  p'_i &= kp_i u^{k-1}(t, q_i)u_x(t, q_i), \\
  q'_i &= u^k(t, q_i),
\end{align*}
\]  
(5.12)

The critical points of the system (5.13) are
\[
\begin{align*}
  p' &= k \text{sgn}(2q)p^{k+1}(1 - e^{-2|q|})^{k-1}e^{-2|q|}, \\
  q' &= p^k(1 - e^{-2|q|})^k.
\end{align*}
\]  
(5.13)

The critical points of the system (5.13) are \(p = 0\) or \(q = 0\). Again this implies the trivial solution. Let \(\mathcal{R} := (0, \infty) \times (0, \infty), p(0) = p_0, q(0) = q_0\) and \((p_0, q_0) \in \mathcal{R}\). For \((p, q) \in \mathcal{R}\), system (5.13) becomes
We can easily use (5.14) to express \( p_k \)

We can explore the 2-peakons dynamics for

\[
(1 - 2q)^n p^n q' = 1, \quad (p_0, q_0) \in \mathcal{R}, \text{ and}
\]

\[
\mathcal{H}_0[u] = 0, \quad \text{where } \mathcal{H}_0[u] \text{ is given by (1.8). However, this is a conserved quantity for the equation only when } k = 1.
\]

5.2 \( N \)-peakons for \( k = -n, \ n \in \mathbb{N} \).

The general equations for the \( N \)-peakon solutions are obtained directly by taking \( k = -n \) in (5.8) and, therefore, we do not repeat the process again.

In this case we can interpret equation (1.1) as

\[
u^n m_t = m_x. \quad (5.16)
\]

Regarding the 2-peakon dynamics, likewise the previous subsection, if \( n \) is even we would only have \( u(t, x) \equiv 0 \) (note that this solution is admitted by (5.16)). However, for \( n \) odd, proceeding similarly as before, we would obtain

\[
p(t) = p_0 \left( \frac{1 - e^{-2q_0}}{1 - e^{-2q(t)}} \right)^{n/2},
\]

where \( (1 - 2q)^n p^n q' = 1, \ (p_0, q_0) \in \mathcal{R}, \) and

\[
u(t, x) = p_0 \left( \frac{1 - e^{-2q_0}}{1 - e^{-2q(t)}} \right)^{n/2} \left( e^{-|x-q(t)|} - e^{-|x+q(t)|} \right).
\]

5.2.1 2-peakon dynamics for the case \( k = -1 \)

We can explore the 2-peakons dynamics for \( k = -1 \) by using the conserved quantity (1.9).
Let us assume again \( u(t, x) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|} \). If we impose that such solution has (1.9) as a conserved quantity, then we have

\[
\mathcal{H}[u] = p_1^2 + 2p_1p_2 e^{-|q_1-q_2|} + p_2^2. \tag{5.17}
\]

Let \( p_{10} = p_1(0), \ p_{20} = p_2(0) \) and \( q_0 := |q_1(0) - q_2(0)| \), which is nothing but the initial separation of the pulses. The conservation of (1.9) yields

\[
\mathcal{H}[u(0, x)] = p_{10}^2 + 2p_{10}p_{20} e^{-q_0} + p_{20}^2 =: \mathcal{H}_0. \tag{5.18}
\]

Then, equations (5.18) and (5.17) read

\[
p_1^2 + 2p_1p_2 e^{-|q_1-q_2|} + p_2^2 = \mathcal{H}_0.
\]

If we assume that \( (p_{10}, p_{20}) \neq (0, 0) \), then \( \mathcal{H}_0 > 0 \). We observe that from (5.17) we have the estimates

\[
0 \leq e^{-|q_1-q_2|} = \frac{\mathcal{H}_0 - p_1^2 - p_2^2}{2p_1p_2} \leq 1. \tag{5.19}
\]

Let us define

\[
p_1 + p_2 e^{-|q_1-q_2|} = \frac{\mathcal{H}_0 + p_1^2 - p_2^2}{2p_1} =: A_1,
\]

\[
p_1 e^{-|q_1-q_2|} + p_2 = \frac{\mathcal{H}_0 - p_1^2 + p_2^2}{2p_2} =: A_2.
\]

Using \( A_1 \) and \( A_2 \) given above, system (5.12) with \( k = -1 \) reads

\[
p_1' = \frac{1}{2} \text{sgn} (q_1 - q_2) \frac{\mathcal{H}_0 - p_1^2 - p_2^2}{2A_1^2}, \quad q_1' = \frac{1}{A_1},
\]

\[
p_2' = -\frac{1}{2} \text{sgn} (q_1 - q_2) \frac{\mathcal{H}_0 - p_1^2 - p_2^2}{2A_2^2}, \quad q_2' = \frac{1}{A_2}. \tag{5.20}
\]

System (5.20) does not have critical points. The last two equations cannot vanish, which implies that if we have solutions of the form (5.11) then they either have two pulses or degenerate into a 1-peakon solution. On the other hand, we may have \( p_1' = p_2' = 0 \). This corresponds to one of the following situations:

- \( q_1 = q_2 \). We have the superposition of the two peakons into a single one, meaning that the solution degenerates into a (one-)peakon

\[
u(t, x) = \frac{1}{c} e^{-|x-ct-q_0|},
\]

or an antipeakon

\[
u(t, x) = -\frac{1}{c} e^{-|x+ct-q_0|},
\]

where, in any case, \( c > 0 \).

- \( \mathcal{H}_0 = p_1^2 + p_2^2 \), where \( p_1 \) and \( p_2 \) are two constants. If one of them is 0 then we have again a degenerated 1-peakon solution. However, in case both are not 0, we can define \( p_1 = 1/c_1 \) and \( p_2 = 1/c_2 \), where \( c_1 \) and \( c_2 \) are two constants. Then

\[
\mathcal{H} = \frac{1}{c_1^2} + \frac{1}{c_2^2}, \quad q_1(t) = c_1 t + q_0, \quad q_2 = c_2 t,
\]
where $q_0$ is a constant of integration (and the corresponding constant to $q_2$ is conveniently taken as 0).

We note that $q_0 = q_1(0) - q_2(0)$ is the separation of the pulses at $t = 0$, and we have the solution

$$u(t, x) = \frac{1}{c_1} e^{-|x - c_1 t - q_0|} + \frac{1}{c_2} e^{-|x - c_2 t|}.$$  \hspace{1cm} (5.21)

If both $c_1$ and $c_2$ are positive or negative, we have, respectively, 2-peakons or 2-anti-peakons. They collide only when $\text{sgn}(c_2 - c_1) = \text{sgn}(q_0)$ and at the time

$$t = \frac{q_0}{c_2 - c_1}. \hspace{1cm} (5.22)$$

In case $\text{sgn}(c_2 - c_1) \neq \text{sgn}(q_0)$, then we do not have interactions among the pulses. If $c_1$ and $c_2$ have different signs and $\text{sgn}(c_2 - c_1) = q_0$, we have a solution that necessarily collide at (5.22).

Let us explore (5.19) once more. Let $u_0(x) := u(0, x)$ and $\mathcal{H}_0 := \mathcal{H}[u_0]$, where $\mathcal{H}[u]$ is given by (5.17). We have two extreme situations: $e^{-|q_1 - q_2|} \approx 1$ and $e^{-|q_1 - q_2|} \approx 0$, or, equivalently, $q_1 \approx q_2$ and $|q_1 - q_2| \to \infty$, respectively.

- In the first case, that is $e^{-|q_1 - q_2|} \approx 1$, and we come back to the discussion $q_1 = q_2$.
- In the second case the solutions asymptotically degenerate into a 1-peakon and their amplitudes asymptotically become constant and the pulses are infinitely separated. In fact, equation (5.19) then let us infer that $\mathcal{H}[u] = p_1^2 + p_2^2$, meaning that $p_1$ and $p_2$ in (5.17) describe a circle of radius $\sqrt{\mathcal{H}_0} = \|u_0\|_{H^1(\mathbb{R})}/\sqrt{2}$, where $u_0 = u(0, x)$ is the solution at $t = 0$. 

![Figure 2: Interaction peakon-peakon of the solution (5.21) with $c_1 = 1$, $c_2 = 2$ and $q_0 = 1$. They collide at $t = 1$, while $t = 0$ and $t = 2$ show the pulses before and after the collision, respectively.](image-url)


**Figure 3:** Interaction peakon-antipeakon of the solution (5.21) with $c_1 = 1$, $c_2 = -1.5$ and $q_0 = -2$. They collide at $t = 0.8$ and the value $t = 0$ shows the solution prior the collision, whereas $t = 1$ and $t = 2$ show the solution after the interaction.

### 5.3 Kink-type solutions for $k \in \mathbb{N}$

Let us assume that

$$u_j(t, x) = c_j(t) + b_j(t) \text{sgn}(x - p_j(t))(1 - e^{-|x - p_j(t)|})$$

(5.23)

in (5.1), for some functions $c_j$, $b_j$ and $p_j$. We again omit the dependence with respect to the independent variables for convenience.

If we denote $m_j = u_j - \partial_x^2 u_j$, we conclude that

$$m_j = c_j + b_j \text{sgn}(x - p_j), \quad \partial_t m_j = c'_j + b'_j \text{sgn}(x - p_j) - 2b_j p'_j \delta(x - p_j),$$

$$\partial_x m_j = 2b_j \delta(x - \partial_j).$$

Substituting the expressions above into (1.6), we obtain

$$\sum_{j=1}^{N} \left[ c'_j + b'_j \text{sgn}(x - p_j) + 2b_j \left(u(t, p_i)^k - p'_j \right) \delta(x - p_i) \right] = 0. \quad (5.24)$$

Proceeding similarly as in the previous subsection, (5.24) results, for $j = 1, \cdots, N$,

$$\left\{ \begin{array}{l}
  c'_j = 0, \quad b'_j = 0, \\
  p'_j = \left[ \sum_{j=1}^{N} (c_j + b_j \text{sgn}(p_i - p_j)(1 - e^{-|p_i - p_j|})) \right]^k.
\end{array} \right. \quad (5.25)$$

In view of (5.23) we make the technical hypothesis that if for some $k \in \{1, \cdots, N\}$, $b_k(t) \equiv 0$, then $p_k(t) \equiv 0$.

System (5.25) directly implies that $c_j = \text{const}$ and $b_j = \text{const}$, $1 \leq j \leq N$. If we assume that $N = 2$, $p_1 = p_2 = p$, $0 < c_1 = c_2 = c^{1/k}/2$, $b_1 = b_2 = b \neq 0$, we conclude that $p(t) = ct + p_0$ and
we have the solution
\[ u(t, x) = c^{1/k} + b \operatorname{sgn}(x - ct - p_0)(1 - e^{-|x - ct - p_0|}). \] (5.26)

Figure 4: Behavior of the solution (5.26) with \( c = 2, b = 2 \) and \( q_0 = -1 \).

Figure 5: Behavior of the solution (5.29) with \( p_0 = 1 \).

Yet taking \( N = 2 \), but \( p_1 = -p_2 = p \), from (5.25) we obtain two equations:
\[ p' = \left[ c_1 + c_2 + b_1 \operatorname{sgn}(2p)(1 - e^{-2|p|}) \right]^k, \]
\[ p' = -\left[ c_1 + c_2 - b_2 \operatorname{sgn}(2p)(1 - e^{-2|p|}) \right]^k. \]

If we take \( c_1 + c_2 = 0 \) and \( b_2^2 = (-1)^{k+1}b_1^k, p_0 > 0 \), up to scaling in \( t \), we have the following PVI in the region \( \mathcal{R} := (0, \infty) \times (0, \infty) \):
\[
\begin{align*}
p' &= (1 - e^{-2p})^k, \\
p(0) &= p_0. \quad (5.27)
\end{align*}
\]

We observe that \( p' > 0 \) in (5.27), which means that it a local increasing diffeomorphism. This implies that the solution of (5.27) will make (5.23) a monotonic and bounded function, which is nothing but a kink solution. Figures 4 and 5 show the typical behaviour of a kink solution.

It is worth mentioning that (5.27) has a unique local solution in \( \mathcal{R} \) and, in particular, the solution of (5.27) can be implicitly given in terms of the hypergeometric function, since
\[
\int \frac{dp}{(1 - e^{-2p})^k} = (1 - e^{-2p})^{-k}(1 - e^{2x})^k F_2(k, k; k + 1, e^{2p}) + \text{const.} \quad (5.28)
\]

For the case \( k = 1 \) we can find the solution explicitly, namely,
\[
\int \frac{dp}{1 - e^{-2p}} = \frac{1}{2} \ln(e^{2p} - 1) + \text{const.}
\]

From this and (5.27) we conclude that
\[
p(t) = \frac{1}{2} \ln \left[ 1 + (e^{2p_0} - 1)e^{2t} \right].
\]
For convenience, let us assume that $b_1 = b_2 = 1$. Then our solution is

$$u(t,x) = \text{sgn} \left( x - \frac{1}{2} \ln \left[ 1 + (e^{2p_0} - 1)e^{2t} \right] \right) \left( 1 - e^{-\frac{1}{2} \ln \left[ 1 + (e^{2p_0} - 1)e^{2t} \right]} \right)$$

$$+ \text{sgn} \left( x + \frac{1}{2} \ln \left[ 1 + (e^{2p_0} - 1)e^{2t} \right] \right) \left( 1 - e^{-\frac{1}{2} \ln \left[ 1 + (e^{2p_0} - 1)e^{2t} \right]} \right).$$

(5.29)

### 6 Discussion and conclusion

In this paper we considered the problem of global existence of the Cauchy problem (1.10). We firstly used the results proved in [14, 23] to assure the local existence of solutions to the problem for $k > 0$. The next question addressed in our work is the extension of the local results to global existence. Unfortunately we are only able to prove it to the case $k = 1$. We looked for the possibility to establish local existence results of solutions when $k < 0$, but we are unable to do it. In fact, Kato’s approach [21] cannot be applied to this case in view of the singularity, as well as the results proved in [23]. While in the present work we are successful in extending global existence results established in [23] to (1.10) with $k = 1$, the problem of wave-breaking of (1.10) is unclear and we are unable to give an answer to it, and therefore, further investigation is needed to give a complete description of the blow-up scenario of (1.6), if any.

Although the study of (1.10) for arbitrary $k$ is rather difficult in view of its lack of structure, we were able to show that if the initial momentum is compactly supported so is $m$, as shown in Theorem 1.1. Moreover, the same result also shows that the sign of the momentum is invariant, provided that the sign of the initial momentum does not change.

In the case $k = 1$ we can go further in the information we can extract from the equation. Indeed, in Theorem 3.2 we prove a unique continuation result for the $b-$equation (1.7). As a consequence of this fact, if a solution of (1.7), with $b \in [0,3]$, defined on $[0,T) \times \mathbb{R}$, for some $T > 0$, vanishes on $(0,T) \times I$, where $I \subseteq \mathbb{R}$ is an open interval, then the solution vanishes everywhere. To prove this we used some ideas recently presented in [20]. Therefore, Theorem 1.2 is a direct consequence of Theorem 3.2.

We can improve the results of Theorem 1.2 for the equation (1.12), in the following sense: While in Theorem 1.2 we requested that the solution $u$ vanishes on an open set of the type $(0,T) \times I$, in Theorem 1.3 we requested that $u$ would vanish on $(t_0,t_1) \times I \subseteq (0,T) \times \mathbb{R}$, for some open interval $I$. The price paid to relax the condition in Theorem 1.2 is the imposition that the initial momentum does not change its sign. As a consequence of this hypothesis, Theorem 3.1 assures the conservation of both $\|u(t,\cdot)\|_{L^1(\mathbb{R})}$ and $\|m(t,\cdot)\|_{L^1(\mathbb{R})}$. The proof of Theorem 1.3 has two main pillars that consists on the use of the ideas introduced in [20], combined with the use of a conserved quantity, as pointed in [12], see also [7] for further discussions and geometrical meaning of this approach. It is worth mentioning that recently one of us has studied (1.10) in Gevrey spaces, see [8].

We also studied some solutions of the equation (1.6), namely, multi-peakon and kink-type solutions. We describe the dynamics of 2-peakon solutions for odd values of $k$. A very interesting result reported here is the case $k = -1$, when the have the conservation of the $H^1(\mathbb{R})-$norm of the solutions of (1.6) with $k = -1$ is used to give a better description of the 2-peakon dynamics. We similarly make a detailed description of the peakon/antipeakon dynamics when $k = 1$ compatible with the conserved quantity (1.8).
Regarding kink-type solutions, we presented a picture of their dynamics, found some explicit solutions and also described the 2-kink solutions of the system (5.27). Although the general solution is given in terms of the hypergeometric function, see (5.28), for the case \( k = 1 \) we find the 1-parameter explicit solution (5.29), where the parameter is nothing but the initial condition of the Cauchy problem (5.27). Indeed, we recover the results due to Xia and Qiao [25] for the equation \( m_t + um_x = 0 \) to (1.6) with \( k \in \mathbb{Z} \).

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